Research Article

The Classification, Automorphism Group, and Derivation Algebra of the Loop Algebra Related to the Nappi–Witten Lie Algebra

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Set $L := H_4 \otimes R$, $R := \mathbb{C}[t^{\pm 1}]$, and $S := \mathbb{C}[t^{\pm 1/m}](m \in \mathbb{Z}_+)$. Then, $L$ is called the loop Nappi–Witten Lie algebra. $R$-isomorphism classes of $S/R$ forms of $L$ are classified. The automorphism group and the derivation algebra of $L$ are also characterized.

1. Introduction

The conformal field theory (CFT) plays a significant role in many different areas of mathematics and physics. Wess–Zumino–Novikov–Witten (WZNW) models [1] provide interesting examples of CFTs. WZNW models were first studied in the setting of semisimple groups. Later, since the WZNW models based on non-abelian nonsemisimple Lie groups were found to be closely related to the construction of exact string backgrounds, the special types received some attention [2, 3]. Nappi and Witten showed in [3] that the WZNW model (NW model) based on a central extension of the two-dimensional Euclidean group describes the homogeneous four-dimensional space-time corresponding to a gravitational plane wave. The related Lie algebra is called the Nappi–Witten Lie algebra, which is neither abelian nor semisimple. More results on the NW model were given in [4–7].

The Nappi–Witten Lie algebra $H_4$ is a four-dimensional Lie algebra over $\mathbb{C}$ generated by $\{a, b, c, d\}$ with the following Lie brackets:

\[
\begin{align*}
[a, b] &= c, \\
[d, a] &= a, \\
[d, b] &= -b, \\
[c, H_4] &= 0.
\end{align*}
\]

There is a nondegenerate invariant symmetric bilinear form $(, )$ on $H_4$ defined by

\[
(a, b) = 1, \\
(c, d) = 1, \quad \text{otherwise, } (, ) = 0.
\]

Set

\[
\begin{align*}
L &= H_4 \otimes R, \\
R &= \mathbb{C}[t^{\pm 1}], \\
S &= \mathbb{C}[t^{\pm 1/m}], \quad (m \in \mathbb{Z}_+).
\end{align*}
\]

Then, $L$ is a Lie algebra over $\mathbb{C}$ under the bracket as follows:

\[
[x \otimes t^{n_1}, y \otimes t^{n_2}] = [x, y] \otimes t^{n_1+n_2}, \quad \text{for } x, y \in H_4, \quad n_1, n_2 \in \mathbb{Z}.
\]

We call $L$ the loop Nappi–Witten Lie algebra. Recently, the representation theory of Lie algebras related to $L$ was studied in [8, 9].

Reference [10] studied the automorphism groups of vertex operator algebras associated with the affine Nappi–Witten algebra $\hat{H}_4$. The isomorphism of loop algebras was considered in [11] from the perspective of non-abelian Galois cohomology. Reference [12] gave an explicit description of the algebra of derivations for a large class of...
Lemma 1 (see [10]). Let \( \theta \in \text{Aut}(H_4) \). Then,
\[
\theta(a, b, c, d) = (a, b, c, d) \quad \text{or} \quad \theta(a, b, c, d) = (a, b, c, d),
\]
where \( s, n \in \mathbb{C}^* \), \( \alpha, \beta, \xi, \rho \in \mathbb{C} \).

Define a linear map \( \tau \) of \( H_4 \) by
\[
\tau(a) = b, \\
\tau(b) = a, \\
\tau(c) = -c, \\
\tau(d) = -d.
\]
It is clear that \( \tau^2 = \text{id}_{H_4} = e \) and \( \tau \in \text{Aut}(H_4) \). Set \( G_0 = \{\tau, e\} \). Then, \( G_0 \cong \mathbb{Z}/2\mathbb{Z} \).

\[
G_1 = \mathbb{C}^* \times (\mathbb{C} \times \mathbb{C}), \\
G = (\mathbb{C}^* \times (\mathbb{C} \times \mathbb{C})) \times \mathbb{Z}/2\mathbb{Z}.
\]

The multiplication in group \( G \) is defined as follows:
\[
(s, a, \beta, e) \cdot (s', a', \beta', e) = (ss', aa', \beta(s'+1 + \beta'), e), \\
(n, \xi, \rho, \tau) \cdot (n', \xi', \rho', \tau) = (n(n')^{-1}, \rho(n')^{-1} - \xi', \xi(n') - \rho', e), \\
(n, \xi, \rho, \tau) \cdot (s, a, \beta, e) = (sn, bn^{-1} + \xi, na + \beta, \tau),
\]
where \( s, s', n, n' \in \mathbb{C}^* \), \( \alpha, \beta, \alpha', \beta', \xi, \rho, \xi', \rho' \in \mathbb{C} \).

The identity element is \( (1, 0, 0, e) \), and
\[
(s, a, \beta, e)^{-1} = (s^{-1}, -as^{-1}, -\beta, s), \\
(n, \xi, \rho, \tau)^{-1} = (n, pn^{-1}, 1, n, \xi, \rho, \tau).
\]

Since
\[
(s, 0, 0, e) \cdot (1, \alpha, \beta, e) \cdot (s, 0, 0, e)^{-1} = (1, as^{-1}, \beta, s), \\
(1, 0, 0, \tau) \cdot (s, a, \beta, e) \cdot (1, 0, 0, \tau)^{-1} = (s^{-1}, -\beta, -a, e),
\]
\( \mathbb{C} \times \mathbb{C} \) is a normal subgroup of \( G_1 \) and \( G_1 \) is a normal subgroup of \( G \). Thus,
\[
G_1 = \mathbb{C}^* \times (\mathbb{C} \times \mathbb{C}), \\
G = (\mathbb{C}^* \times (\mathbb{C} \times \mathbb{C})) \times \mathbb{Z}/2\mathbb{Z}.
\]

Theorem 1 (see [10]). \( \text{Aut}(H_4) \cong G = (\mathbb{C}^* \times (\mathbb{C} \times \mathbb{C})) \times \mathbb{Z}/2\mathbb{Z} \).

By Theorem 1, any element of \( G \) can be viewed as an automorphism of \( H_4 \).

Lemma 2. Let \( \phi = (s, a, \beta, e) \) and \( \psi = (n, \xi, \rho, \tau) \). Then,
\[
\phi^k = (s^k, a(s^{k-1} + s^{k-2} + \ldots + s + 1), \beta(s^{-k+1} + s^{-k+2} + \ldots + s^{-1} + 1), e), \\
\psi^{2k} = (1, k(\rho n^{-1} - \xi), k(\xi n - \rho), e), \\
\psi^{2k+1} = (n, (k + 1)\xi - k\rho n^{-1}, (k + 1)\rho - kn\xi, \tau),
\]
where \( s, n \in \mathbb{C}^* \), \( \alpha, \beta, \xi, \rho \in \mathbb{C} \), and \( k \in \mathbb{Z}_+ \).

Proof. It follows from (8) immediately. \( \square \)

Lemma 3

(1) If \( (n, \xi, \rho, \tau)^{2k} = (1, 0, 0, e) \) for \( k \in \mathbb{Z}_+ \), then \( (n, \xi, \rho, \tau) \) is conjugate to \( (1, 0, 0, \tau) \) in \( \text{Aut}(H_4) \).

(2) Let \( (s, \alpha, \beta, e) \neq (s, 0, 0, e) \). Then, \( (s, \alpha, \beta, e) \) is conjugate to \( (s, 0, 0, e) \) in \( \text{Aut}(H_4) \) if and only if \( s \neq 1 \).

(3) Let \( s \neq 1 \) and \( s \neq s' \). Then, \( (s, 0, 0, e) \) is conjugate to \( (s', 0, 0, e) \) in \( \text{Aut}(H_4) \) if and only if \( ss' = 1 \).

Proof

(1) If \( (n, \xi, \rho, \tau)^{2k} = (1, k(\rho n^{-1} - \xi), k(\xi n - \rho), e) = (1, 0, 0, e) \), then \( \rho = \xi n. \) Choose \( (s, \alpha, \beta, e) \) such that \( s' = 1/n \) and \( na + \beta = -\rho \). Then,
Thus, \((n, \xi, \rho, r)\) is conjugate to \((1, 0, 0, r)\) in \(\text{Aut}(H_4)\).

(2) We assume that there exists \((s', a', b', e)\) such that
\[(s', a', b', e)(s, a, b, e)(s', a', b', e)^{-1} = (s, 0, 0, e).\] (14)

By (8) and (9), we have
\[
\alpha' (s - 1) + \alpha = 0,
\beta'(s^{-1} - 1) + \beta = 0. \tag{15}
\]

\[(n, \xi, \rho, r)(s, 0, 0, e)(n, \xi, \rho, r)^{-1} = (s^{-1}, (s^{-1} - 1)\rho n^{-1}, (s-1)\xi n, e),\]
\[(s'', a'', b'', e)(s, 0, 0, e)(s'', a'', b'', e)^{-1} = \left(s, (s^{-1} - 1)\alpha'' s^{-1}, (s^{-1} - 1)\beta'' s'' - 1, e\right).\] (16)

Suppose \(s = 1\). Then, \(\alpha = \beta = 0\) by (15) is a contradiction. Thus, \(s \neq 1\). Conversely, if \(s \neq 1\), \(s' = s(1 - s)\) and \(\beta' = \beta/(1 - s^{-1})\). Then, there exists \((s', a', b', e)\) such that (14) holds. So, \((s, a, b, e)\) is conjugate to \((0, 0, 0, e)\) in \(\text{Aut}(H_4)\).

(3) Since

it is clear that \((s, 0, 0, e)\) is conjugate to \((s', 0, 0, e)\) if and only if \(ss' = 1\).

The aim of this paper is to study structures of the loop Nappi-Witten Lie algebra \(L = H_4 \otimes \mathbb{R}\). Some ideas we use come from [10–12]. This paper is organized as follows. In Section 2, we classify \(R\)-isomorphism classes of \(S/R\) forms of \(L\) by the first cohomology set. In Section 3, we determine the automorphism group of \(L\). Finally, the derivation algebra of \(L\) is also characterized in Section 4.

Throughout the paper, the sets of the complex numbers, the nonzero complex numbers, the nonnegative integers, the integers, and the positive integers are denoted by \(\mathbb{C}, \mathbb{C}^*, \mathbb{N}, \mathbb{Z}\), and \(\mathbb{Z}_+\), respectively.

2. The First Cohomology

Set \(H^1(\Gamma, \text{Aut}_5(H_4(S)))\)

We first review some basic notions and assumptions in [11] and then apply them in the class of \(H_4\).

Let \(k\) be an algebraically closed field of characteristic 0. For each \(m \in \mathbb{Z}_+\), we choose a primitive \(m\)-th root of unity \(\zeta_m\) such that these primitive roots of unity are compatible in the sense that \(\zeta_{lm} = \zeta_m^l\) for all \(l, m \in \mathbb{Z}_+\), and fix the sequence \(\{\zeta_m\}_{m \in \mathbb{Z}_+}\) from now on. Suppose that \(\mathfrak{g}\) is a Lie algebra over \(k\) and that \(\sigma\) is an automorphism of \(\mathfrak{g}\) of period \(m\) (the order of \(\sigma\) might not be equal to \(m\), but of course it is a divisor of \(m\)).

Let
\[
\Gamma := \frac{\mathbb{Z}}{m\mathbb{Z}} = \{i : i \in \mathbb{Z}\}, \tag{17}
\]
be the group of integers modulo \(m\), where \(i\) denotes the congruence class represented by \(i\). Let
\[
R = k[z^{1}],
S = k[z^{1/m}], \tag{18}
\]
be two algebras of Laurent polynomials over \(k\) and \(R \subset S\) is a \(k\)-subalgebra. If we set \(z = t^{1/m}\), then \(t = z^m\) and
\[
R = k[z^{\pm m}],
S = k[z^{\pm 1}]. \tag{19}
\]

It is easy to see that \(S\) is an \(R\)-algebra.

There is an isomorphism of \(\Gamma\) onto \(\text{Aut}_S(S)\) (the group of automorphisms of \(S\) which fixes \(R\)) such that
\[
\overline{i} \mapsto i, \quad i(z) = \zeta_i^m z. \tag{20}
\]

Let \(K \in \{R, S\}\). Set
\[
\mathfrak{g}(K) := \mathfrak{g} \otimes_k K, \tag{21}
\]
where \(\mathfrak{g}(K)\) is a \(K\)-module with \(K\)-action given by \(f(x \otimes h) = x \otimes f h\) for \(x \in \mathfrak{g}\) and \(f, h \in K\). Note that \(\mathfrak{g}(K)\) is also a Lie algebra over \(K\) under the bracket as follows:
\[
[x \otimes f, y \otimes h] = [x, y] \otimes f h, \quad \text{for } x, y \in \mathfrak{g}, f, h \in K. \tag{22}
\]

We now recall the definition of the loop algebra of \(\mathfrak{g}\) relative to \(\sigma\). Let
\[
\mathfrak{g}_i := \{x \in \mathfrak{g} | \sigma(x) = \zeta_i^m x\}. \tag{23}
\]

The loop algebra of \(\mathfrak{g}\) relative to \(\sigma\) is the subalgebra
\[
L(\mathfrak{g}, \sigma) := \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \otimes_k z^i, \tag{24}
\]
of the Lie algebra
Note that \( L(g, \sigma) \) is the set of fixed points in \( g(S) \) of the \( R \)-algebra automorphism \( \sigma \circ \text{id}^{-1} \), and then \( L(g, \sigma) \) is an \( R \)-subalgebra of \( g(S) \). Thus, \( L(g, \sigma) \) is an \( R \)-algebra and hence also a \( k \)-algebra. Furthermore, the \( R \)-isomorphism class of \( L(g, \sigma) \) does not depend on the choice of the period \( m \) for \( \sigma \) (see Section 2.3 in [11]).

To interpret loop algebras as \( S/R \) forms of the \( R \)-algebra \( g(R) \), we review the definitions of the \( S/R \) form of \( g(R) \) and a certain (non-abelian) first cohomology set and relations between them.

An \( S/R \) form of \( g(R) \) is a Lie algebra \( \mathcal{F} \) over \( R \) such that
\[
\mathcal{F} \otimes_p R \cong g(S),
\]
as \( S \)-algebras.

We now recall some concepts of a (non-abelian) cohomology set.

Let \( G \) be a group. We say that \( \Gamma \)-acts on \( G \) if there is a map \( (i, x) \to i \cdot x \) of \( \Gamma \times G \to G \) satisfying \( i(x \cdot y) = (i \cdot x) \cdot (i \cdot y) \) for \( i \in \Gamma, x, y \in G \).

If \( \Gamma \) acts on a group \( G \), the map \( u: \Gamma \to G, i \mapsto u_i \) satisfying
\[
u_i = f^{-1} u_i \bar{f},
\]
is called \( 1 \)-cocyclic, where \( u_i \) is the \( \Gamma \)-action on \( G \). Two \( 1 \)-cocycles \( u, v \) are said to be cohomologous if there exists some \( f \in G \) such that
\[
u_i = f^{-1} u_i \bar{f},
\]
for all \( i \in \Gamma \). It is an equivalence relation. For every \( 1 \)-cocycle \( u \), we denote by \( [u] \) the equivalence class containing \( u \). The set of equivalence classes (called cohomology classes) of \( 1 \)-cocycles from \( \Gamma \) to \( G \) is denoted by \( H^1(\Gamma, G) \), and we call it the (non-abelian) first cohomology set.

Define an action of \( \Gamma \) on \( \text{Aut}_S(g(S)) \) by
\[
\tau \cdot \psi = (\text{id} \otimes \bar{j}) \tau (\text{id} \otimes j)^{-1},
\]
for \( \tau \in \Gamma \) and \( \psi \in \text{Aut}_S(g(S)) \), where \( \text{Aut}_S(g(S)) \) is the group of automorphisms of \( g(S) \) as a Lie \( S \)-algebra.

The following result comes from Proposition 3.4 of [11].

**Proposition 1** (see [11]). \( R \)-isomorphism classes of \( S/R \) forms of \( g(R) \) are in one-to-one correspondence with cohomology classes in \( H^1(\Gamma, \text{Aut}_S(g(S))) \). Explicitly this correspondence is as follows: if \( u \) is a \( 1 \)-cocycle on \( \Gamma \) in \( \text{Aut}_S(g(S)) \), then the cohomology class \([u]\) corresponds to the \( R \)-isomorphism classes of the \( S/R \) form:
\[
g(S)_u = \{ x \in g(S); (u_i (\text{id} \otimes \bar{j}) \cdot x = x \text{ for } i \in \Gamma) \},
\]
of \( g(S) \).

Furthermore, Theorem 3.6 in [11] gives our description of loop algebras as \( S/R \) forms of \( g(R) \). In view of Theorem 3.6 in [11], the loop algebra \( L(g, \sigma) \) is an \( S/R \)-form of \( g(R) \) and is determined by the \( 1 \)-cocycle:
\[
u_i = \sigma_i \otimes \text{id}, \quad \text{for } i \in \Gamma.
\]

In the following, set \( k = C \) and \( g = H_4 \). Let \( S \) and \( R \) be the algebras of Laurent polynomials over \( C \) as in (19). We will classify loop algebras of \( H_4 \) from the view of \( S/R \) forms of \( H_4(R) \).

**Lemma 4.** Let \( \sigma, \tau \in \text{Aut}(H_4) \) and \( \sigma^m = \tau^m = \text{id}_{H_4} \). We define \( 1 \)-cocycles \( u \) and \( v \) on \( \Gamma \) in \( \text{Aut}_S(H_4(S)) \) by
\[
u_i = \sigma_i \otimes \text{id}, \quad \text{and } v_i = \tau_i \otimes \text{id}, \quad \forall i \in \Gamma.
\]
If \( \sigma \) is conjugate to \( \tau \) in \( \text{Aut}(H_4) \), then \([u] = [v]\).

**Proof.** Suppose that there exists \( h \in \text{Aut}(H_4) \) such that
\[
h^{-1} \sigma h = \tau \text{, and } h^{-1} \sigma^i h = \tau^i, \forall i \in \mathbb{Z}. \quad \text{Set } f = h \otimes \text{id}, \text{then } f \in \text{Aut}_S(H_4(S)). \text{For any } i \in \Gamma \text{ and } x \otimes z^n \in H_4(S), \text{we have}
\]
\[
f^{-1} \nu_i f (x \otimes z^n) = (h^{-1} \otimes \text{id})(\sigma^{-i} \otimes \text{id})(\text{id} \otimes \bar{j})(h \otimes \text{id})(\text{id} \otimes j)^{-1}(x \otimes z^n)
\]
\[
= h^{-1} \sigma^{-i} h(x) \otimes z^n
\]
\[
= \tau^{-i}(x) \otimes z^n
\]
\[
= v_i(x \otimes z^n).
\]
Thus, \([u] = [v]\). \qed

**Theorem 2.** Let \( \Gamma = \mathbb{Z}/m\mathbb{Z} \).

(i) Then, there are automorphisms

\[
\begin{align*}
\phi_{m, r} & = \{(m, 0, 0, e)|0 \leq r \leq \left\lfloor \frac{m}{2} \right\rfloor \}, \\
\psi & = (1, 0, 0, r).
\end{align*}
\]

of period \( m \) of \( H_4 \) in the sense of conjugation. Furthermore, if \( m \in 2\mathbb{Z} + 1, \) \( \psi \) does not exist.

(ii) If we define \( 1 \)-cocycles \( u_{m, r}, v \) on \( \Gamma \) in \( \text{Aut}_S(H_4(S)) \) by
In this section, we study the automorphisms of $L$. The Automorphism Group $\text{Aut}_L$ will be used later. Let $R$ be a unital commutative associative algebra over $K$ and $T$ be a Lie algebra over $K$.

We first recall the definition of the centroid in [11] that will be used later. Let $K$ be a unital commutative associative algebra over $C$ and $T$ be a Lie algebra over $K$. Set

$$\text{Ctd}_K(T) = \{ \chi \in \text{End}_{K\text{-mod}}(T) | \chi([x,y]) = [\chi(x), y] \}$$

where $\text{Ctd}_K(T)$ is called the centroid of $T$ over $K$. Define $\lambda_T: K \rightarrow \text{End}_{K\text{-mod}}(T)$ by

$$\lambda_T(r)(x) = rx, \quad \text{for } r \in K \text{ and } x \in T.$$  

$$\text{}$$

It is clear that $\lambda_T(K) \subseteq \text{Ctd}_K(T)$. We say that $T$ is central over $K$ if $\text{Ctd}_K(T) = \lambda_T(K)$ (see [11]).

Lemma 5. Let $\chi \in \text{Ctd}_C(H_4)$. Then,

then the corresponding $R$-isomorphism classes of the $S/R$ form of $H_4(R)$ are as follows:

(i) Let $\psi = (n, \xi, \rho, r) \in \text{Aut}(H_4)$ such that $\psi^m = (1, 0, 0, e)$. Then, $m \in 2\mathbb{Z}$, by Lemma 2, and $\psi$ is conjugate to $(1, 0, 0, r)$ in $\text{Aut}(H_4)$ by Lemma 3(1).

Let $\varphi = (s, a, \beta, e) \in \text{Aut}(H_4)$ such that $\varphi^m = (1, 0, 0, e)$.

Then,

$$\varphi^{m} = (s^m, a(s^{m-1} + s^{m-2} + \ldots + s + 1), b(s^{-(m-1)} + s^{-(m-2)} + \ldots + s^{-1} + 1), e).$$

Proof. Let $\chi \in \text{Ctd}_C(H_4)$. We can assume that

$$\chi(a) = \lambda_1 a + \mu_1 b,$$

$$\chi(b) = \lambda_2 a + \mu_2 b,$$

where $\lambda_j, \mu_j \in C, j = 1, 2$. Since

$$\chi \in \text{Ctd}_C(H_4),$$

we have $\mu_1 = 0$ and $\lambda_1 = \mu_2$. Thus,

$$\chi(x) = \lambda_1 x,$$

$$\chi(y) = \lambda_1 y + \rho c,$$

$$\chi(z) = \lambda_1 z + \rho c, \quad \lambda_1, \rho \in C.$$  

Theorem 3. If $\chi \in \text{Ctd}_C(L)$, then

$$\chi(x \otimes t^n) = \lambda_0^x(t)(x \otimes t^n),$$

$$\chi(d \otimes t^m) = \lambda_0^d(t)(d \otimes t^n) + \alpha_0^n(t)(c \otimes t^0),$$

where $x \in \{a, b, c\}$, $\lambda_0^x(t), \alpha_0^n(t) \in R$, and $n \in \mathbb{Z}$.

Proof. We assume that

$$\text{}$$
By Lemma 5, we can define

\[ \chi(a \otimes t^n) = \lambda_n^i(t)(a \otimes t^n) + \mu_n^i(t)(b \otimes t^n), \quad (47) \]

\[ \chi(b \otimes t^n) = \delta_n^i(t)(a \otimes t^n) + \rho_n^i(t)(b \otimes t^n), \]

where \( \lambda_n^i(t), \mu_n^i(t), \delta_n^i(t), \rho_n^i(t) \in R \). By

we have

\[ \rho_n^i(t) = \delta_n^i(t) = 0, \lambda_n^i(t) = \lambda_n^i(t), \rho_n^i(t) = \rho_n^i(t), \lambda_n^i(t) = \rho_n^i(t). \quad (49) \]

So,

\[ \chi(a \otimes t^n) = \lambda_n^i(t)(a \otimes t^n), \]
\[ \chi(b \otimes t^n) = \lambda_n^i(t)(b \otimes t^n), \]
\[ \chi(c \otimes t^n) = \lambda_n^i(t)(c \otimes t^n), \]
\[ \chi(d \otimes t^n) = \lambda_n^i(t)(d \otimes t^n) + \alpha_n^i(t)(c \otimes t^n), \quad \lambda_n^i(t), \alpha_n^i(t) \in R. \quad (50) \]

**Corollary 1.** \( \text{Ctd}_C(L) \equiv R \times \text{Hom}(L/[L, L], Z(L)) \) as vector spaces over \( C \), where \( Z(L) \) is the center of \( L \).

**Proof.** Define the map

\[ r: \text{Ctd}_C(L) \longrightarrow R \times \text{Hom}\left( \frac{L}{[L, L]}, Z(L) \right), \]
\[ \chi \mapsto (\lambda_n^i(t), \varphi^i), \quad (51) \]

\[ \chi_{1, \chi_2}(d \otimes t^n) = \chi_1(\lambda_n^i(t)(d \otimes t^n)) = \lambda_n^i(t)(\lambda_n^i(t)(d \otimes t^n)) = \lambda_n^i(t)(\lambda_n^i(t)(d \otimes t^n)) = \chi_2(\lambda_n^i(t)(d \otimes t^n)). \quad (54) \]

**Corollary 3.** \( \text{Ctd}_R(L) = \text{Ctd}_R(H_4 \otimes_C R) \equiv \text{Ctd}_C(H_4) \otimes_C R \) as algebras over \( R \).

**Proof.** By Lemma 5, we can define \( \varphi, \psi \in \text{Ctd}_C(H_4) \) by

\[ \varphi(x) = \lambda x, \ x \in \{a, b, c, d\}, \]
\[ \psi(y) = 0, \quad y \in \{a, b, c\}, \]
\[ \psi(d) = \rho c, \quad \lambda, \rho \in \mathbb{C}. \quad (55) \]

Define the map

\[ \eta: \text{Ctd}_R(L) \longrightarrow \text{Ctd}_C(H_4) \otimes_C R, \]
\[ \chi \mapsto \varphi \otimes \frac{\lambda_n^i(t)}{\lambda} + \psi \otimes \frac{\alpha_n^i(t)}{\rho}, \quad (56) \]

where \( \lambda, \rho \in \mathbb{C}, \lambda_n^i(t), \alpha_n^i(t) \in R \). It is easy to check that \( \eta \) is an \( R \)-algebra isomorphism.
\textbf{Remark 1.} Corollary 3 can also follow from Lemma 5 in [12].

\textbf{Lemma 6.} If \( \theta \in \text{Aut}_c(L) \), then
\begin{align*}
\theta(x \otimes t^k) &= \lambda_0 \lambda_k(t) \theta(x \otimes t^0), \quad x \in \{a, b, c\}, \\
\theta(d \otimes t^k) &= \lambda_0 \lambda_k(t) \theta(d \otimes t^0) + \left( \delta_k(t) - \lambda_0 \lambda_k(t) \delta_0(t) \right) (c \otimes t^0),
\end{align*}
(57)
where \( \lambda_k(t), \delta_k(t) \in R, k \in \mathbb{Z} \), and \( \lambda_{k_1+k_2}(t) = \lambda_0 \lambda_{k_1}(t) \lambda_{k_2}(t) \), \( \lambda_0 = \lambda_0(t) \in \{ \pm 1 \} \).

\textbf{Proof.} Let \( \theta \in \text{Aut}_c(L) \). We can assume that
\begin{align*}
\theta(a \otimes t^k) &= s_k(t) (a \otimes t^0) + m_k(t) (b \otimes t^0) + a_k(t) (c \otimes t^0), \\
\theta(b \otimes t^k) &= n_k(t) (a \otimes t^0) + l_k(t) (b \otimes t^0) + \beta_k(t) (c \otimes t^0), \\
\theta(d \otimes t^k) &= \lambda_k(t) (d \otimes t^0) + \mu^1_k(t) (a \otimes t^0) + \mu^2_k(t) (b \otimes t^0) \\
&\quad + \delta_k(t) (c \otimes t^0),
\end{align*}
(58)
where \( s_k(t), m_k(t), a_k(t), n_k(t), l_k(t), \beta_k(t), \lambda_k(t), \mu^1_k(t), \mu^2_k(t), \delta_k(t) \in R \). By
\begin{align*}
\left[ \theta(d \otimes t^0), \theta(a \otimes t^k) \right] &= \theta(a \otimes t^k) = \left[ \theta(d \otimes t^0), \theta(a \otimes t^0) \right], \\
\left[ \theta(d \otimes t^0), \theta(b \otimes t^k) \right] &= -\theta(b \otimes t^k) = \left[ \theta(d \otimes t^0), \theta(b \otimes t^0) \right],
\end{align*}
(59)
we deduce that
\begin{align*}
\lambda_0(t) &= 1, \\
\mu^2_k(t) &= \lambda_k(t) \mu^2(t), \\
\lambda_0(t) &= -1,
\end{align*}
(60)
or
\begin{align*}
\lambda_0(t) &= -1, \\
\mu^2_k(t) &= -\lambda_k(t) \mu^2(t), \\
\lambda_0(t) &= 1.
\end{align*}
(61)
By (60) and (61) and \( \theta(c \otimes t^k) = [\theta(a \otimes t^k), \theta(b \otimes t^0)] \), we have
\begin{align*}
\lambda_0(t) &= 1, \\
\theta(a \otimes t^k) &= \lambda_k(t) (s_0(t) (a \otimes t^0) + a_0(t) (c \otimes t^0)) = \lambda_k(t) \theta(a \otimes t^0), \\
\theta(b \otimes t^k) &= \lambda_k(t) (n_0(t) (a \otimes t^0) + \beta_0(t) (c \otimes t^0)) = \lambda_k(t) \theta(b \otimes t^0), \\
\theta(c \otimes t^k) &= \lambda_k(t) (n_0(t) (a \otimes t^0) + \beta_0(t) (c \otimes t^0)) = \lambda_k(t) \theta(c \otimes t^0), \\
\theta(d \otimes t^k) &= \lambda_k(t) (d \otimes t^0) + \lambda_k(t) \mu^0_0(t) (a \otimes t^0) + \lambda_k(t) \mu^0_0(t) (b \otimes t^0) + \delta_k(t) (c \otimes t^0) \\
&= \lambda_k(t) \theta(d \otimes t^0) + (\delta_k(t) - \lambda_k(t) \delta_0(t)) (c \otimes t^0),
\end{align*}
(62)
or
\begin{align*}
\lambda_0(t) &= -1, \\
\theta(a \otimes t^k) &= -\lambda_k(t) (m_0(t) (b \otimes t^0) + a_0(t) (c \otimes t^0)) = -\lambda_k(t) \theta(a \otimes t^0), \\
\theta(b \otimes t^k) &= -\lambda_k(t) (n_0(t) (a \otimes t^0) + \beta_0(t) (c \otimes t^0)) = -\lambda_k(t) \theta(b \otimes t^0), \\
\theta(c \otimes t^k) &= \lambda_k(t) (m_0(t) (b \otimes t^0) + \beta_0(t) (c \otimes t^0)) = -\lambda_k(t) \theta(c \otimes t^0), \\
\theta(d \otimes t^k) &= \lambda_k(t) (d \otimes t^0) - \lambda_k(t) \mu^0_0(t) (a \otimes t^0) - \lambda_k(t) \mu^0_0(t) (b \otimes t^0) + \delta_k(t) (c \otimes t^0) \\
&= -\lambda_k(t) \theta(d \otimes t^0) + (\delta_k(t) + \lambda_k(t) \delta_0(t)) (c \otimes t^0).
\end{align*}
(63)
Since

\[
\begin{align*}
\left[ \theta(d \otimes t^k), \theta(a \otimes t^k) \right] &= \theta(a \otimes t^{k^2}) = \lambda_0(t) \lambda_{k_1} \lambda_{k_2}(t) \theta(a \otimes t^0) \\
&= \left[ \lambda_0(t) \lambda_{k_1}(t) \theta(d \otimes t^0) + \left( \delta_{k_1}(t) - \lambda_0(t) \lambda_{k_1}(t) \delta_0(t) \right)(c \otimes t^0), \lambda_0(t) \lambda_{k_2}(t) \theta(a \otimes t^0) \right] \\
&= \lambda_0(t) \lambda_{k_1}(t) \lambda_{k_2}(t) \theta(a \otimes t^k),
\end{align*}
\]

we have \( \lambda_{k_1+k_2}(t) = \lambda_0(t) \lambda_{k_1}(t) \lambda_{k_2}(t) \). The lemma is proved. \( \Box \)

Define a map \( \pi: R \times \text{Hom}(L/[L, L], Z(L)) \rightarrow R \) by

\[
\pi(r, \varphi) = r
\]

for \( r \in R, \varphi \in \text{Hom}(L/[L, L], Z(L)) \). Furthermore, \( (r, 0)(x \otimes t^k) = x \otimes rt^k \) for \( x \otimes t^k \in L \).

**Lemma 7**

1. Let \( \theta \in \text{Aut}_C(L) \). Then, \( \theta(t^n, 0)\theta^{-1} \in \text{Ctd}_C(L) \) and \( \pi(\theta(t^n, 0)\theta^{-1}) = \lambda_0 \lambda_n(t) \);

2. Given \( \theta \in \text{Aut}_C(L) \). The linear map \( \land: \text{Aut}_C(L) \rightarrow \text{Aut}_C(R) \) is defined by the following identity:

\[
\theta(t^n, 0)\theta^{-1}(\theta(x \otimes t^k)) = \theta(t^n, 0)(x \otimes t^k)
\]

\[
= \theta(x \otimes t^{nk})
\]

\[
= \lambda_0 \lambda_n(t) \theta(x \otimes t^0)
\]

\[
= \lambda_0 \lambda_n(t) \lambda_k(t) \theta(x \otimes t^0)
\]

\[
= \lambda_0 \lambda_n(t) \theta(x \otimes t^k),
\]

\[
\theta(t^n, 0)\theta^{-1}(\theta(d \otimes t^k)) = \theta(d \otimes t^{nk})
\]

\[
= \lambda_0 \lambda_n(t) \theta(d \otimes t^0) + \left( \delta_{nk}(t) - \lambda_0 \lambda_n(t) \delta_0(t) \right)(c \otimes t^0)
\]

\[
= \lambda_0 \lambda_n(t) \lambda_k(t) \theta(d \otimes t^0) + \left( \delta_{nk}(t) - \lambda_0 \lambda_n(t) \lambda_k(t) \delta_0(t) \right)(c \otimes t^0)
\]

\[
= \lambda_0 \lambda_n(t) \theta(d \otimes t^k) + \left( \delta_{nk}(t) - \lambda_0 \lambda_n(t) \delta(t) \right)(c \otimes t^0).
\]

Then, by (62)-(63) and Theorem 3, we deduce that \( \pi(\theta(t^n, 0)\theta^{-1}) = \lambda_0 \lambda_n(t) \).

2. By the definition of \( \theta_c \) and Lemma 6, for \( x \in \{a, b, c\} \), we have

\[
\theta(t^n + t^m, 0)\theta^{-1}(\theta(x \otimes t^k)) = \theta(t^n + t^m, 0)(x \otimes t^k)
\]

\[
= \theta(x \otimes (t^n + t^m))
\]

\[
= \lambda_0(\lambda_n(t) + \lambda_n(t)) \theta(x \otimes t^k)
\]

\[
= (\theta_c(t^n) + \theta_c(t^m)) \theta(x \otimes t^k),
\]
Thus, \( \theta_c(t^{n_1} + t^{n_2}) = \theta_c(t^{n_1}) + \theta_c(t^{n_2}) \). Similarly, we can get \( \theta_c(\mu t^n) = \mu \theta_c(t^n) \) for any \( \mu \in C \). Furthermore, we have \( \theta_c(t^{n_1} \cdot t^{n_2}) = \lambda_0 \lambda_{n_1} n_2 (t) = \lambda_0 \lambda_{n_1} n_2 (t) \cdot \theta_c(t^{n_1}) \cdot \theta_c(t^{n_2}) \).

(70)

\[
\theta(t^{n_1} + t^{n_2}, 0) \theta^{-1}(\theta(d \otimes t^k)) = \theta(d \otimes t^{n_1+k}) + \theta(d \otimes t^{n_2+k})
\]

\[
= \lambda_0 \lambda_{n_1} (t) \theta(d \otimes t^k) + \left( \delta_{n_1+k} (t) - \lambda_0 \lambda_{n_1} (t) \delta_k (t) \right) (c \otimes t^0) + \lambda_0 \lambda_{n_1} (t) \theta(d \otimes t^k)
\]

\[
+ \left( \delta_{n_2+k} (t) - \lambda_0 \lambda_{n_1} (t) \delta_k (t) \right) (c \otimes t^0) + \sum_{i=1}^2 \left( \delta_{n_i+k} (t) - \lambda_0 \lambda_{n_1} (t) \delta_k (t) \right) (c \otimes t^0).
\]

(69)

In the following, we check that \( (\theta_c)^{-1} = (\theta^{-1})_c \). By \( \theta \in \text{Aut}_C(L) \) and Lemma 6, we can assume that

\[
(\theta^{-1})_c \in C\text{Id}_C(L)
\]

and \( \pi(\theta^{-1}(t^n), 0) = \lambda_0 \lambda_{n} \). Then,

\[
(\theta^{-1})_c(t^n) = \lambda_0 \lambda_{n}.
\]

(72)

Thus, we have \( \theta^{-1}(c \otimes t^k) = (\theta^{-1})_c(t^k) \theta^{-1}(c \otimes t^0) \). For any \( \lambda \in \mathbb{Z} \), we have

\[
c \otimes t^k = \theta^{-1}(c \otimes t^k) = \theta\left( (\theta^{-1})_c(t^k) \theta^{-1}(c \otimes t^0) \right)
\]

\[
= \begin{cases} 
\theta(c \otimes (\theta^{-1})_c(t^k))(c \otimes t^0), & \text{if } \lambda_0 = 1, \\
-\theta(c \otimes (\theta^{-1})_c(t^k))(c \otimes t^0), & \text{if } \lambda_0 = -1,
\end{cases}
\]

\[
= \begin{cases} 
\theta(c \otimes (\theta^{-1})_c(t^k))(c \otimes t^0), & \text{if } \lambda_0 = 1, \\
-\theta(c \otimes (\theta^{-1})_c(t^k))(c \otimes t^0), & \text{if } \lambda_0 = -1,
\end{cases}
\]

\[
= \begin{cases} 
\theta(c \otimes (\theta^{-1})_c(t^k))(c \otimes t^0), & \text{if } \lambda_0 = 1, \\
-\theta(c \otimes (\theta^{-1})_c(t^k))(c \otimes t^0), & \text{if } \lambda_0 = -1,
\end{cases}
\]

\[
= \theta(c \otimes (\theta^{-1})_c(t^k))(c \otimes t^0).
\]

(73)

So, we deduce that \( \theta_c(\theta^{-1}(t^k)) = t^k \). Similarly, we have \( (\theta^{-1})_c(\theta_c(t^k)) = t^k \) for any \( k \in \mathbb{Z} \). Therefore, \( (\theta^{-1})_c = (\theta^{-1})_c \) and \( \theta_c \in \text{Aut}_C(R) \). \( \Box \)

**Theorem 4.** For the linear map \( \wedge: \text{Aut}_C(L) \rightarrow \text{Aut}_C(R) \) with \( \theta \rightarrow \theta_c \), we have

(1) \( \wedge \) is a group homomorphism

(2) \( \wedge \) is surjective
assume \( \tau(t^n) = t^n \) for any \( n \in \mathbb{Z} \). By Lemma 6, let \( \theta \in \text{Aut}_C(L) \) such that
\[
\theta(x \otimes t^n) = t^n \theta(x \otimes t^n), \quad x \in \{a, b, c\},
\]
\[
\theta(d \otimes t^n) = t^n \theta(d \otimes t^n) + \left( \delta_n(t) - t^n \delta_0(t) \right) (c \otimes t^0).
\]
(78)

Then, \( \theta_c(t^n) = t^n = \tau(t^n) \) for any \( n \in \mathbb{Z} \). Thus, \( \theta_c = \tau \). If \( \tau \) is the generator of \( \mathbb{Z}/2\mathbb{Z} \), we assume \( \tau(t) = t^{-1} \).

Let \( \theta \in \text{Aut}_C(L) \) such that
\[
\theta(x \otimes t^n) = t^{-n} \theta(x \otimes t^n), \quad x \in \{a, b, c\},
\]
\[
\theta(d \otimes t^n) = t^{-n} \theta(d \otimes t^n) + \left( \delta_n(t) - t^{-n} \delta_0(t) \right) (c \otimes t^0).
\]
(79)

So, \( \theta_c(t^n) = t^{-n} = \tau(t^n) \) for any \( n \in \mathbb{Z} \) and then \( \theta_c = \tau \). Therefore, \( \wedge \) is surjective.

(3) Let \( \theta \in \text{Ker}\lambda \). Then, \( \theta_c = \text{id}_R \) and \( t^k = \theta_c(t^k) = \lambda_0 \lambda_k(t) \) for any \( k \in \mathbb{Z} \). Thus, by Lemma 6, we have
\[
\theta(x \otimes t^k) = t^k \theta(x \otimes t^0), \quad x \in \{a, b, c\}, \quad k \in \mathbb{Z},
\]
(80)
\[
\theta(d \otimes t^k) = t^k \theta(d \otimes t^0) + \left( \delta_k(t) - t^{-k} \delta_0(t) \right) (c \otimes t^0).
\]
(81)

If \( \delta_k(t) - t^{-k} \delta_0(t) = 0 \) for any \( k \in \mathbb{Z} \), then \( \theta \in \text{Aut}_R(L) \). If \( \delta_k(t) - t^{-k} \delta_0(t) \neq 0 \), define \( \varphi \otimes \tau \in \text{Hom}(H_4/\mathbb{Z}(H_4), Z(H_4) \otimes F(R)) \) by
\[
(\varphi \otimes \tau)(d \otimes t^k) = \varphi(d) \otimes \tau(t^k) = c \otimes \left( \delta_k(t) - t^{-k} \delta_0(t) \right), \quad \forall k \in \mathbb{Z},
\]
(82)

where \( \varphi(d) = c \), \( (\varphi \otimes \tau)(d \otimes t^k) = 0 \) if \( \varphi(d) = 0 \), \( F(R) := \{ \tau \in \text{End}_C(R) \mid \tau(t^k) = \delta_k(t) - t^{-k} \delta_0(t) \} \), and \( \text{End}_C(R) \) is the set of all linear transformations of \( R \).

Let \( \overline{\theta}_1 \in \text{Aut}_R(L) \). By (62)–(63), we can assume that
\[
\overline{\theta}_1(d \otimes t^0) = \lambda_{10}(d \otimes t^0) + \mu_{10}^0(t)a \otimes t^0 + \mu_{10}^2(t)b \otimes t^0 + \delta_{10}(t)c \otimes t^0,
\]
(83)
for \( \lambda_{10} \in \{\pm 1\}, i = 1, 2 \). Define the group \( \text{Aut}_R(L) \times (\text{Hom}(H_4/\mathbb{Z}(H_4), Z(H_4) \otimes F(R)) \), whose multiplication is given by
\[
(\overline{\theta}_1, \varphi \otimes \tau_1)(\overline{\theta}_2, \varphi_2 \otimes \tau_2) = (\overline{\theta}_1 \overline{\theta}_2, \lambda_{20}(\varphi_1 \otimes \tau_1) + \overline{\theta}_1(\varphi_2 \otimes \tau_2)),
\]
(84)

Furthermore,
\[
(\overline{\theta}, \varphi \otimes \tau)(1, 0) = (\overline{\theta}, \varphi \otimes \tau),
\]
\[
(\overline{\theta}, \varphi \otimes \tau)^{-1} = \left( \overline{\theta}^{-1}, -\lambda_0 \overline{\theta}^{-1}(\varphi \otimes \tau) \right),
\]
\[
(\overline{\theta}, 0)(1, \varphi \otimes \tau)(\overline{\theta}, 0)^{-1} = (1, \lambda_0 \overline{\theta}(\varphi \otimes \tau)),
\]
(85)

where \( (\overline{\theta}, \varphi \otimes \tau_1), (\overline{\theta}, \varphi \otimes \tau) \in \text{Aut}_R(L) \times (\text{Hom}(H_4/\mathbb{Z}(H_4), \mathbb{Z}(H_4) \otimes F(R))) \), \( \lambda_{10}, \lambda_0 \in \{\pm 1\}, i = 1, 2 \).

Define a map \( \eta \colon \text{Ker}\lambda \longrightarrow \text{Aut}_R(L) \times (\text{Hom}(H_4/\mathbb{Z}(H_4), \mathbb{Z}(H_4) \otimes F(R))) \) by
\[
\theta \mapsto (\overline{\theta}, \varphi \otimes \tau),
\]
(86)

for \( \overline{\theta} \in \text{Aut}_R(L), \varphi \otimes \tau \in \text{Hom}(H_4/\mathbb{Z}(H_4), \mathbb{Z}(H_4) \otimes F(R)), \) and
\[
(\overline{\theta}, \varphi \otimes \tau)(x \otimes t^k) = \overline{\theta}(x \otimes t^k),
\]
\[
(\overline{\theta}, \varphi \otimes \tau)(d \otimes t^k) = \overline{\theta}(d \otimes t^k) + (\varphi \otimes \tau)(d \otimes t^k),
\]
(87)

for \( x \in \{a, b, c\} \) and any \( k \in \mathbb{Z} \). It is easy to see that \( \eta \) is well-defined and bijective. Let \( \overline{\theta}_1 \in \text{Ker}\lambda \) and \( \eta(\overline{\theta}_1) = (\overline{\theta}_1, \varphi_1 \otimes \tau_1), i = 1, 2, \) By (80)–(86), we have
\[
\theta_1 \theta_2(d \otimes t^k) = \theta_1 \left[ t^k \overline{\theta}_2(d \otimes t^0) + (\varphi_2 \otimes \tau_2)(d \otimes t^k) \right]
\]
\[
= \theta_1 \left[ \lambda_{20} t^k a \otimes t^0 + \mu_{20}^0(t) b \otimes t^0 + \delta_{20}(t) c \otimes t^0 \right] + (\varphi_2 \otimes \tau_2)(d \otimes t^k)
\]
\[
= t^k \lambda_{20}(\varphi_1 \otimes \tau_1)(d \otimes t^k) + \lambda_{20}(\varphi_1 \otimes \tau_1)(d \otimes t^k) + t^k \mu_{20}^1(t) \overline{\theta}_1(a \otimes t^0) + t^k \mu_{20}^2(t) \overline{\theta}_1(b \otimes t^0)
\]
\[
+ t^k \delta_{20}(t) \overline{\theta}_2(c \otimes t^0) + \overline{\theta}_1(\varphi_2 \otimes \tau_2)(d \otimes t^k)
\]
\[
= t^k \lambda_{20}(\varphi_1 \otimes \tau_1) + \lambda_0 \lambda_k(t) + \overline{\theta}_1(\varphi_2 \otimes \tau_2)(d \otimes t^k).
\]
Corollary 4

\[ \text{Aut}_C(L) \equiv \text{Aut}_C(R) \times \text{Ker} \Lambda \]
\[ \equiv \text{Aut}_C(R) \times (\text{Aut}_R(L) \times (C \otimes F(R))). \]  

Proof. It follows from Theorem 4. \[ \square \]

4. The Derivation Algebra \( \text{Der}_C(L) \) of \( L \)

In this section, we study the derivations of \( L = H_4 \otimes_R R \), where \( R = C[t^\pm 1] \). Denote by \( \text{Der}_R(L) \) the algebra of all \( R \)-linear derivations of \( L \).

Lemma 8. Let \( g \in \text{Der}(H_4) \). Then,
\[ g(a) = \lambda_1 a + \delta_1 c, \]
\[ g(b) = \lambda_2 b + \delta_2 c, \]
\[ g(c) = (\lambda_1 + \mu_2) c, \]
\[ g(d) = -\delta_2 a - \delta_1 b + \delta_3 c, \]
where \( \lambda_1, \mu_2, \delta_1 \in C, 1 \leq i \leq 3 \).

Proof. For \( g \in \text{Der}(H_4) \), we can assume that
\[ g(a) = \lambda_1 a + \mu_1 b + \delta_1 c, \]
\[ g(b) = \lambda_2 a + \mu_2 b + \delta_2 c, \]
\[ g(c) = a(d) + \lambda_3 a + \mu_3 b + \delta_3 c, \]
where \( \lambda_1, \mu_1, \delta_1, \alpha \in C, 1 \leq i \leq 3 \). Since
\[ g(a) = [g(d), a] + [d, g(a)], \]
\[ -g(b) = [g(d), b] + [d, g(b)], \]
\[ g(c) = [g(a), b] + [a, g(b)], \]

we have
\[ \alpha = 0, \]
\[ \mu_1 = 0, \]
\[ \lambda_2 = 0, \]
\[ \mu_3 = -\delta_1, \]
\[ \lambda_3 = -\delta_2. \]  

Thus, the lemma is proved. \[ \square \]

Let \( d_i \in \text{Der}(H_4) \) \((1 \leq i \leq 3)\) be such that
\[ d_1(a) = a, \]
\[ d_1(c) = c, \]
\[ d_1(b) = d_1(d) = 0, \]
\[ d_2(b) = b, \]
\[ d_2(c) = c, \]
\[ d_2(a) = d_2(d) = 0, \]
\[ d_3(c) = c, \]
\[ d_3(a) = d_3(b) = d_3(c) = 0. \]  

Theorem 5. \( \text{Der}(H_4) = C d_1 \oplus C d_2 \oplus C d_3 \oplus C \text{ad}(a) \oplus C \text{ad}(b). \)

Proof. For any \( g \in \text{Der}(H_4) \), by Lemma 8, we have
\[ g = \lambda_1 d_1 + \mu_2 d_2 + \delta_3 d_3 + \delta_2 \text{ad}(a) - \delta_1 \text{ad}(b). \]  

It is clear that \( d_1, d_2, d_3, \text{ad}(a), \) and \( \text{ad}(b) \) are linearly independent. \[ \square \]

Lemma 9. Let \( D \in \text{Der}_C(L) \). Then,
\[ D(x \otimes t^\alpha) = x \otimes (\lambda_1 t - \lambda_1 t t^n) + t^\alpha D(x \otimes t^0), \]
\[ D(d \otimes t^\alpha) = d \otimes (\lambda_1 t - \lambda_1 t t^n) + t^\alpha D(d \otimes t^0) + c \otimes (\delta_3 n(t) - \delta_3 n(t) t^m), \]
\[ \text{where } x \in \{a, b, c\}, \lambda_1(t), \delta_3 n(t) \in R, \text{ and } \lambda_{1, m, n}(t) = \lambda_1(t) t^n + \lambda_1(t) t^n - \lambda_1(t) t^n. \]

Proof. For \( D \in \text{Der}_C(L) \), we may assume that
\[ D(a \otimes t^\alpha) = \lambda_1(t) (a \otimes t^0) + \mu_1(t) (b \otimes t^0) + \delta_1(t) (c \otimes t^0), \]
\[ D(b \otimes t^\alpha) = \lambda_2(t) (a \otimes t^0) + \mu_2(t) (b \otimes t^0) + \delta_2(t) (c \otimes t^0), \]
\[ D(d \otimes t^\alpha) = \lambda_3(t) (d \otimes t^0) + \mu_3(t) (b \otimes t^0) + \delta_3(t) (c \otimes t^0), \]  

where \( x \in \{a, b, c\}, \lambda_1(t), \delta_3 n(t) \in R, \text{ and } \lambda_{1, m, n}(t) = \lambda_1(t) t^n + \lambda_1(t) t^n - \lambda_1(t) t^n. \)
where $\lambda_i(t), \mu_i(t), \delta_i(t), \alpha_n(t) \in R$, $i \in [1, 2, 3]$. By

\[
D(a \otimes t^n) = [D(d \otimes 1), a \otimes t^n] + [d \otimes 1, D(a \otimes t^n)] = [D(d \otimes t^n), a \otimes 1] + [d \otimes t^n, D(a \otimes 1)],
\]

\[
-D(b \otimes t^n) = [D(d \otimes 1), b \otimes t^n] + [d \otimes 1, D(b \otimes t^n)] = [D(d \otimes t^n), b \otimes 1] + [d \otimes t^n, D(b \otimes 1)],
\]

\[
D(c \otimes t^n) = [D(a \otimes 1), b \otimes t^n] + [a \otimes 1, D(b \otimes t^n)] = [D(a \otimes t^n), b \otimes 1] + [a \otimes t^n, D(b \otimes 1)],
\]

we have

\[
\begin{align*}
\alpha_0(t) &= 0, \\
\mu_1(t) &= 0, \\
\lambda_{2n}(t) &= 0, \\
\delta_{1n}(t) &= \delta_{10}(t)t^n, \\
\delta_{2n}(t) &= \delta_{20}(t)t^n, \\
\alpha_n(t) &= \lambda_{1n}(t) - \lambda_{10}(t)t^n = \mu_{2n}(t) - \mu_{20}(t)t^n, \\
\mu_{3n}(t) &= -\delta_{1n}(t), \\
\lambda_{3n}(t) &= -\delta_{2n}(t).
\end{align*}
\]

Thus,

\[
\rho(D)(t^n) = \lambda_{1n}(t) - \lambda_{10}(t)t^n \Longleftrightarrow \pi([D, (t^n, 0)]) = \lambda_{1n}(t) - \lambda_{10}(t)t^n.
\]

Since

\[
D(a \otimes t^{n,m}) = \lambda_{1,m+n}(t)(a \otimes t^m) = [D(d \otimes t^m), a \otimes t^n] + [d \otimes t^n, D(a \otimes t^m)] = \lambda_{1n}(t) - \lambda_{10}(t)t^n + \lambda_{1m}(t)t^n - \lambda_{10}(t)t^{m+n}.
\]

we have $\lambda_{1,m+n}(t) = \lambda_{1n}(t)t^m + \lambda_{1m}(t)t^n - \lambda_{10}(t)t^{m+n}$. The lemma is proved.

\[\square\]
Then, \([D, (t^n, 0)] \in \text{Ctd}_c(L)\). By Lemma 9, for \(x \in \{a, b, c\}\) and \(m \in \mathbb{Z}\), we have

\[
[D, (t^n, 0)](x \otimes t^m) = D(x \otimes t^{n+m}) - (t^n, 0)D(x \otimes t^m)
\]

\[
= x \otimes (\lambda_1 t^m - \lambda_{1n} (t)t^n)
\]

\[
= (\lambda_1 t - \lambda_{10} (t)t^n) (x \otimes t^m),
\]

\[
[D, (t^n, 0)](d \otimes t^m) = D(d \otimes t^{n+m}) - (t^n, 0)D(d \otimes t^m)
\]

\[
= d \otimes (\lambda_1 t^m - \lambda_{1n} (t)t^n) + c \otimes (\delta_3 t^m - \delta_3 t^m)(c \otimes t^m).
\]

(103)

Therefore, by Theorem 3 and the definition of \(\pi\) in (65), we have \(\pi([D, (t^n, 0)]) = \lambda_1 t - \lambda_{10} (t)t^n\).

(2) Since \(\text{Der}_c(R) = \text{Der}_c(\mathbb{C}[t^{\pm 1}])\) is a Witt algebra and \(\text{Der}_c(R) = \text{Span}_c[\delta_t = -t^{-1}(d/dt)[i \in \mathbb{Z}]]\), for any \(\delta_t = -t^{-1}(d/dt) \in \text{Der}_c(R), i \in \mathbb{Z}\), there exists \(D_t = \text{id}_{H_1} \otimes \delta_t \in \text{Der}_c(L)\) such that \(\rho(D_t) = \delta_t\).

(3) For any \(D \in \text{Ker}(\rho)\), we have \(\rho(D) = 0\) and by Lemma 9,

\[
D(x \otimes t^m) = t^m D(x \otimes t^m), \quad x \in \{a, b, c\},
\]

\[
D(d \otimes t^m) = t^m D(d \otimes t^m) + c \otimes (\delta_{3n} (t) - \delta_{30} (t)) t^n.
\]

(107)

If \(\delta_{3n} (t) - \delta_{30} (t)t^n = 0\) for any \(n \in \mathbb{Z}\), then \(D \in \text{Der}_c(L)\).

If \(\delta_{3n} (t) - \delta_{30} (t)t^n \neq 0\), define \(\varphi \otimes \tau \in \text{Hom}(H_1/[H_1, H_1], Z(H_1)) \otimes E(R)\) by

\[
(\varphi \otimes \tau)(d \otimes t^n) = \varphi(d) \otimes \tau(t^n) = c \otimes (\delta_{3n} (t) - \delta_{30} (t)) t^n.
\]

(108)

where \(\varphi(d) = c\), \(\varphi \otimes \tau)(d \otimes t^n) = 0\) if \(\varphi(d) = 0\), \(E(R) = \{ \tau \in \text{End}_c(R) | \tau(t^n) = \delta_{3n} (t) - \delta_{30} (t)t^n, \quad \text{for} \ \delta_{30} (t) \in R\}\), and any \(n \in \mathbb{Z}\), \(\text{End}_c(R)\) is the set of all linear transformations of \(R\). So, \(\text{Ker}(\rho) \subseteq \text{Der}_c(L) + \text{Hom}(H_1/[H_1, H_1], Z(H_1)) \otimes E(R)\). Conversely, for any \(D \in \text{Der}_c(L)\), \(y \in H_1\), we have \(D(y \otimes t^n) = t^n D(y \otimes t^n)\). By Lemma 9 and (101), it follows that \(\rho(D) = 0\) and \(\text{Der}_c(L) \subseteq \text{Ker}(\rho)\).

For any \(\varphi \otimes \tau \in \text{Hom}(H_1/[H_1, H_1], Z(H_1)) \otimes E(R)\), by the definition of \(\varphi \otimes \tau\), it is clear that \(\varphi \otimes \tau \in \text{Der}_c(L)\) and \(\varphi \otimes \tau \in \text{Ker}(\rho)\). Therefore, \(\text{Der}_c(L) + \text{Hom}(H_1/[H_1, H_1], Z(H_1)) \otimes E(R) = \text{Ker}(\rho)\). Moreover, since \(H_1/[H_1, H_1] = C d\) and \(Z(H_1) = C c\), it is easy to see that \(\text{Hom}(H_1/[H_1, H_1], Z(H_1)) \otimes E(R)\) is isomorphic to \(C\) as vector spaces over \(C\). Thus, (3) is proved.
Corollary 5
\[ \text{Der}_c (L) \cong \text{Der}_c (R) + \text{Ker} (\rho) \]
\[ \cong id_{H_4} \otimes \text{Der}_c (R) + \text{Der}_R (L) + C \otimes E (R) \]
\[ \cong \text{Ctd}_c (H_4) \otimes \text{Der}_c (R) \oplus \text{Der}_c (H_4) \otimes R \oplus C \otimes \text{Der}_c (R) \]
\[ \cong \text{id}_{H_4} \otimes \text{Der}_c (R) + C \otimes \text{Der}_c (R). \]

(109)

Proof. The first and second isomorphism relations follow from Theorem 6. The third one follows from \[ \text{Der}_R (L) = \text{Der}_R (H_4 \otimes c R) \cong \text{Der} (H_4 \otimes 1, H_4 \otimes R) \]
\[ = \text{Der} (H_4) \otimes R. \]

(110)

See also Theorem 7.1 in [13], Theorem 1.1 in [14], and Lemmas 2.7 and 2.8 in [15]. Furthermore, by Lemma 5, we have
\[ \text{Ctd}_c (H_4) \otimes \text{Der}_c (R) = id_{H_4} \otimes \text{Der}_c (R) + C \otimes \text{Der}_c (R). \]

(111)

Then, the corollary is proved. □

Remark 2. The last isomorphism relation in Corollary 5 overlaps with Theorem 7.1 in [13], Theorem 1.1 in [14], Corollary 2.2 in [16], Theorem 2.9 in [15], and Theorem 4.2 in [12].

Data Availability
The data of the Lie algebra relations used to support the findings of this study are included within the article.

Conflicts of Interest
The author declares that there are no conflicts of interest.

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