KERNEL THEOREM FOR THE SPACE OF BEURLING - KOMATSU TEMPERED ULTRADISTRIBUTIONS

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Abstract

We give a simple proof of the Kernel theorem for the space of tempered ultradistributions of Beurling - Komatsu type, using the characterization of Fourier-Hermite coefficients of the elements of the space. We prove in details that the test space of tempered ultradistributions of Beurling - Komatsu type can be identified with the space of sequences of ultrapolynomial falloff and its dual space with the space of sequences of ultrapolynomial growth. As a consequence of the Kernel theorem we have that the Weyl transform can be extended on a space of tempered ultradistributions of Beurling - Komatsu type.

1 Introduction

Tempered ultradistributions spaces, as good spaces for harmonic analysis, have appeared in papers of many authors, among others we mention Björk [2], Wloka [29], Grudzinski [6], De Roever [26], Kahspirovskij [13] and Pilipović [23], Matsuzawa, [5], Chung, [3], [4], Budinčević, Perišić, Lozanov-Crvenković [1] and Yoshino [30]. These spaces have been developed in the framework of several ultradistribution theories: Komatsu’s theory, Beurling-Björk’s theory, Cioranesku-Zsidó’s and Braun-Meise-Taylor’s theory.

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*Thanks.
In the paper we study tempered ultradistributions following Komatsu’s approach to ultradistribution theory, introduced and developed in his fundamental and inspiring papers \cite{komatsu14}, \cite{komatsu15}, \cite{komatsu16}. We study the space $S^{(M_p)'}$ of tempered ultradistributions of Beurling-Komatsu type, which was introduced and investigated in \cite{beurling18} and \cite{beurling19}. It is the generalization of the space of tempered distributions $S'$.

Beurling-Björk space $S_\omega$, $\omega \in M_c$, introduced in \cite{beurling2}, is equal to the space $S^{(M_p)}(\mathbb{R}^d)$, where

$$M_p = \sup_{\rho > 0} \rho^p e^{-\omega(\rho)}.$$  

The sequence satisfies the conditions (M.1) and (M.3)', and it is in general different from a Gevrey sequence. If we assume additionally that $\omega(\rho) \geq C(\log \rho)^2$ for some $C > 0$, then (M.2) is satisfied.

The Pilipović space $\Sigma'_\alpha$, (see \cite{pilipovic23}), is an example of the space of tempered ultradistributions, where $\{M_p\}$ is a Gevrey sequence $p^{\alpha p}$, $p \in \mathbb{N}$, $\alpha > 1$, which appeared to be interesting for the development of pseudodifferential operator (ΨDO) theory in the framework of Time-Frequency analysis, see the papers of Pilipović, Teofanov, \cite{pilipovic24}, \cite{pilipovic25}.

After introduction, in Section 2. we present without proof the basic identification of the spaces $S^{(M_p)}$ and $S^{(M_p)'}$ with the sequence spaces. In the sequence representation, the space $S^{(M_p)}$ represents the space of sequences of ultrapolynomial falloff and $S^{(M_p)'}$ represents the space of sequences of ultrapolynomial growth. We use these results to prove the Kernel theorem for tempered ultradistributions. As a consequence of the Kernel theorem we have that the Weyl transform can be extended on a space of tempered ultradistributions of Beurling-Komatsu type and this gives the possibility to introduce the ΨDO theory in the framework of tempered ultradistributions, where $\{M_p\}$ is not necessarily a Gevrey sequence, but can be, for example

$$M_p = (p^\nu(\log p)^\mu)^p, \quad p \in \mathbb{N},$$

where $\nu > 1$, and $\mu \in \mathbb{R}$, or $\nu = 1$ and $\mu > 1$.

In Section 3. we give the detailed proofs of characterizations of Fourier-Hermite coefficients of elements of the spaces $S^{(M_p)}$ and $S^{(M_p)'}$ which are only stated in Section 2. This characterization appeared in several papers, \cite{beurling18}, \cite{komatsu10}, \cite{pilipovic23} but in different forms and without a proof. Here we give complete and detailed proofs, based on the ideas from B. Simon’s paper \cite{simon27} where as a basic tool we use the harmonic oscillator wave functions - Hermite functions.

### 1.1 Notations

Let $\{M_p, \; p \in \mathbb{N}_0\}$ be a sequence of positive numbers, where $M_0 = 1$. An infinitely differentiable function $\varphi$ on $\mathbb{R}^d$ is ultradifferentiable function of class $(M_p)$ if on each
compact set $K$ in $\mathbb{R}^d$ its derivatives satisfy

$$||\varphi^{(\alpha)}||_{C(K)} \leq C h^{||\alpha||} M_{||\alpha||}, \quad |\alpha| = 0, 1, \ldots$$

where we use multi-index notation: for $d \in \mathbb{N}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, we put

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d, \quad \alpha! = \alpha_1! \alpha_2! \cdots \alpha_d!,$$

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}, \quad |x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

For $x \in \mathbb{R}^d$, $\varphi(x) \in C^\infty(\mathbb{R}^d)$,

$$\varphi^{(\alpha)}(x) = (\partial/\partial x)^\alpha \varphi(x) = (\partial/\partial x_1)^{\alpha_1} (\partial/\partial x_2)^{\alpha_2} \cdots (\partial/\partial x_d)^{\alpha_d} \varphi(x).$$

We impose the conditions on the sequence $\{M_p, \ p \in \mathbb{N}_0\}$ which are standard in ultradistribution theory (for their detailed analysis see, for example Komatsu’s paper [14]).

(M.1) $M_p^2 \leq M_{p-1} M_{p+1}, \quad p = 1, 2, \ldots$

(*logarithmic convexity*)

(M.2) There exist constants $A, H > 0$ such that

$$M_p \leq AH^p \min_{0 \leq q \leq p} M_q M_{p-q}, \quad p = 0, 1, \ldots$$

(*stability under ultradifferential operators*)

(M.3) There exists a constant $A > 0$ such that

$$\sum_{q=p+1}^\infty \frac{M_{q-1}}{M_q} \leq Ap \frac{M_p}{M_{p+1}}, \quad p = 1, 2, \ldots$$

(*strong non-quasi-analyticity*)

Some results remain valid, however, when (M.2) and (M.3) are replaced by the following weaker conditions:

(M.2)’ There are constants $A, H > 0$ such that

$$M_{p+1} \leq AH^p M_p, \quad p = 0, 1, \ldots$$

(*stability under differential operators*)

(M.3)’ $B := \sum_{p=1}^\infty \frac{M_{p-1}}{M_p} < \infty.$

(*non-quasi-analyticity*)
We will often use the inequality

\[ M_p M_q \leq M_0 M_{p+q}, \quad p, q = 0, 1, 2, \ldots, \]

which follows from the condition (M.1), (see [14, p. 45]).

The associated function for the sequence \( \{M_p, p \in \mathbb{N}_0\} \) is

\[ M(\rho) = \sup_{p \in \mathbb{N}_0} \log \frac{\rho^p}{M_p}, \quad \rho > 0. \]

In the paper we will assume that the conditions (M.1), (M.2) and (M.3)' are satisfied. The Gevrey sequence \( \{p^s p\}_{p \in \mathbb{N}_0} \), for \( s > 1 \), satisfies all the above conditions and in that case \( M(\rho) \sim \rho^{1/s} \). The examples of such sequences are also

\[ M_p = (p^\nu (\log p)^\mu)^p, \quad p \in \mathbb{N}, \]

where \( \nu > 1 \), and \( \mu \in \mathbb{R} \), or \( \nu = 1 \) and \( \mu > 1 \). In this special case \( M(\rho) = \rho^{\frac{\mu}{s}}(\log \rho)^{-\frac{\mu}{s}} \), \( \rho \gg 0 \). There is a real need to consider that more general case. In the study of the spaces of admissible data in the Cauchy problems it is necessary to consider ultradifferential classes wider than any Gevrey class. For example Matsumoto considered, see [21] and references therein, among others the case (2).

Following Komatsu, we denote by \( \mathcal{E}^{(M_p)} \) the space of all ultradifferentiable functions on \( \mathbb{R}^d \) of class \( (M_p) \) and by \( \mathcal{D}^{(M_p)} \) the subspace of \( \mathcal{E}^{(M_p)} \), of all ultradifferentiable functions with compact support.

If the sequence \( \{M_p, p \in \mathbb{N}_0\} \) satisfies conditions (M.1) and (M.3)', by Denjoy-Carleman-Mandelbrojt theorem \( \mathcal{D}^{(M_p)} \) is not a trivial space. In this paper we will always assume that these conditions are satisfied.

We denote by \( \mathcal{D}^{(M_p)'} \) the strong dual space of \( \mathcal{D}^{(M_p)} \), and call its elements ultradistributions of Beurling-Komatsu type.

### 1.2 Space of tempered ultradistributions of Beurling-Komatsu type

The space \( \mathcal{S}^{(M_p)'} \) of tempered ultradistributions of type \( (M_p) \), is a subspace of the space of Beurling-Komatsu ultradistributions. The \( \mathcal{D}^{(M_p)} \) is dense in \( \mathcal{S}^{(M_p)} \) and set \( \mathcal{S}^{(M_p)} \backslash \mathcal{D}^{(M_p)} \) is nonempty. Moreover: In the following diagram the arrows denote continuous inclusions:

\[
\begin{array}{cccc}
\mathcal{D} & \hookrightarrow & \mathcal{S} & \hookrightarrow & \mathcal{E} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{D}^{(M_p)} & \hookrightarrow & \mathcal{S}^{(M_p)} & \hookrightarrow & \mathcal{E}^{(M_p)} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{G} & \hookrightarrow & \mathcal{S}^{(M_p)'} & \hookrightarrow & \mathcal{E}^{(M_p)'}
\end{array}
\]
where $\mathcal{S}'$ is the Schwartz space tempered distributions and $\mathcal{G}'$ is the space of extended Fourier hyperfunctions (defined as in [3]). For the proof see [20].

Let us now give the precise definitions.

By $S^{M_p, m}$, $m > 0$, we denote the Banach space of smooth functions $\varphi$ on $\mathbb{R}^d$, such that for some constant $C > 0$

$$||x^\beta \varphi^{(\alpha)}||_2 \leq C M|\alpha|M|\beta|, \text{ for every } \alpha, \beta \in \mathbb{N}_0^d,$$

where $|| \cdot ||_2$ is the standard norm in space $L^2(\mathbb{R}^d)$, equipped with the norm

$$s_m(\varphi) = \sup_{\alpha, \beta \in \mathbb{N}_0^d} M|\alpha|M|\beta| ||x^\beta \varphi^{(\alpha)}(x)||_2.$$

The test space for the space of tempered ultradistributions is defined by

$$S^{(M_p)} = \lim_{m \to \infty} S^{M_p, m}.$$

It is a Frechet space.

The strong dual $S^{(M_p)'}$ of the space $S^{(M_p)}$ is the space of tempered ultradistributions of Beurlinger - Komatsu type. The space of tempered ultradistributions is a good space for Harmonic analysis, since the Fourier transform is an isomorphism of $S^{(M_p)'}$ onto itself, (see [11]). It is defined on $S^{(M_p)}$ by

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx, \quad \varphi \in S^{(M_p)},$$

and on $S^{(M_p)'}$ by

$$\langle \mathcal{F} f, \varphi \rangle = \langle f, \mathcal{F} \varphi \rangle,$$

for $f \in S^{(M_p)'}$ and $\varphi \in S^{(M_p)}$.

It has a lot of nice properties, for example, (see [11] and [19] for more details), if

$$P(x, D) = \sum a_{\mu, \nu}(-1)^\nu D^\nu x^\mu$$

where $a_{\mu, \nu}$ are complex numbers such that there exists $L > 0$ and $C > 0$ and

$$|a_{\mu, \nu}| \leq S \frac{L^{\mu+\nu}}{M_\mu M_\nu}, \quad \mu, \nu \in \mathbb{N},$$

then

$$\mathcal{F}(P(\cdot, D)f(\cdot))(\xi) = P(-D, \xi)(\mathcal{F} f)(\xi), \quad \xi \in \mathbb{R},$$

for each $f \in S^{(M_p)'}$. 

5
2 Main Results

Komatsu proved, in his papers, [15] and [16] the Kernel theorem for ultradistributions and ultradistributions with compact support which are an analogue of L. Schwartz Kernel theorem for distributions.

The main result of the paper is the Kernel theorem for tempered ultradistributions. Applying the theorem we show that the Weyl transform, which is a representation of the convolution algebra on \( L^1 \) functions over the reduced Heisenberg group \( H^\text{red}_d \), extends uniquely to a bijection from \( \mathcal{S}'(M_p) \) to \( \mathcal{S}'(\mathbb{R}^l) \).

The simple proof of the Kernel theorem for tempered ultradistributions, which will be given in this section, depend on the characterization of the Fourier-Hermite coefficients of elements of \( \mathcal{S}(M_p) \) and \( \mathcal{S}'(M_p) \). In this section we will state the theorems which give these characterizations and prove the Kernel theorem. We follow the ideas from the paper [27] of B. Simon.

The elements of the space \( \mathcal{S}(M_p) \) are in \( L^2 \), thus they have in the space the Hermite expansion \( \sum a_n(\varphi)h_n \) where \( h_n \) are Hermite functions:

\[
h_n(x) = (-1)^n\pi^{-\frac{1}{4}}2^{-\frac{1}{4}n!(n!)^{-\frac{1}{4}}}e^{\frac{1}{2}x^2}\left(\frac{d}{dx}\right)e^{-x^2}, \quad n \in \mathbb{N},
\]

which are harmonic oscillator wave functions, since

\[
\left(-\frac{d}{dx^2} + x^2\right)h_n = (2n+1)h_n.
\]

Let \( \varphi \in \mathcal{S}(M_p) \), the Hermite coefficients of \( \varphi \) are

\[
a_n(\varphi) = \int_{\mathbb{R}} \varphi(x)h_n(x)dx, \quad n \in \mathbb{N}_0.
\]

The sequence of the Fourier-Hermite coefficients \( \{a_n\}_{n \in \mathbb{N}_0} \) of \( \varphi \) we call the Hermite representation of \( \varphi \).

The first main result of the paper is the Hermite representation of the space \( \mathcal{S}(M_p)(\mathbb{R}) \):

**Theorem 2.1.** (a) If \( \varphi \in \mathcal{S}(M_p)(\mathbb{R}) \), then for every \( \theta > 0 \)

\[
\|\varphi\|_\theta = \sum_{n=0}^{\infty} |a_n(\varphi)|^2 \exp[2M(\theta\sqrt{2n+1})] < \infty.
\]

(b) Conversely, if a sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) satisfy that for every \( \theta > 0 \),

\[
\sum_{n=0}^{\infty} |a_n|^2 \exp[2M(\theta\sqrt{2n+1})] < \infty,
\]

then the series \( \sum a_nh_n \) converges in the space \( \mathcal{S}(M_p)(\mathbb{R}) \) to some \( \varphi \), and the sequence \( \{a_n\}_{n \in \mathbb{N}_0} \) is the Hermite representation of \( \varphi \).
From the proof of Theorem 2.1 which will be given in the next section, it follows that that the test space of tempered ultradistributions of Beurling - Komatsu type $S^{(M_p)}(\mathbb{R})$ can be identified with the space $s^{(M_p)}$ of sequences of ultrapolynomial falloff, where we define it by

$$s^{(M_p)} = \{ \{a_n\}_{n \in \mathbb{N}_0} \mid \forall \theta > 0, \sum_{n=0}^{\infty} |a_n|^2 \exp[2M(\theta \sqrt{2n+1})] < \infty \},$$

and equip it with the topology induced by the family of seminorms

$$||\{a_n\}||_\theta = \sum_{n=0}^{\infty} |a_n|^2 \exp[2M(\theta \sqrt{2n+1})], \quad \theta > 0.$$ 

More precisely, the map

$$\mathcal{S}^{(M_p)}(\mathbb{R}) \to s^{(M_p)}, \quad \varphi \mapsto \{a_n(\varphi)\}_{n \in \mathbb{N}_0}, \quad a_n(\varphi) = \langle \varphi, h_n \rangle$$

is a topological isomorphism. It is easy to prove that the space $s^{(M_p)}$ is nuclear and therefore space $\mathcal{S}^{(M_p)}(\mathbb{R})$ is nuclear.

If $f \in \mathcal{S}^{(M_p)'}(\mathbb{R})$, then $a_n(f) = \langle f, h_n \rangle, n = 0, 1, 2, \ldots$, are the Fourier-Hermite coefficients of $f$, and the sequence $\{a_n(f)\}_{n \in \mathbb{N}_0}$, is the **Hermite representation** of tempered ultradistribution $f$.

Hermite representation of the space $\mathcal{S}^{(M_p)'}(\mathbb{R})$ is given by:

**Theorem 2.2.** (a) Let $f \in \mathcal{S}^{(M_p)'}(\mathbb{R})$. Then for some $\theta > 0$

$$\sum_{n=0}^{\infty} |a_n(f)|^2 \exp[-2M(\theta \sqrt{2n+1})] < \infty$$

and $\langle f, \varphi \rangle = \sum_{n=0}^{\infty} a_n(f) a_n(\varphi)$, where the sequence $\{a_n(\varphi)\}_{n \in \mathbb{N}_0}$ is the Hermite representation of $\varphi \in \mathcal{S}^{(M_p)}(\mathbb{R})$.

(b) Conversely, if a sequence $\{b_n\}_{n \in \mathbb{N}_0}$ satisfy

$$\sum_{n=0}^{\infty} |b_n|^2 \exp[-2M(\theta \sqrt{2n+1})] < \infty \quad \text{for some} \quad \theta > 0,$$

then the sequence $\{b_n\}_{n \in \mathbb{N}_0}$ is the Hermite representation of some $f \in \mathcal{S}^{(M_p)'}(\mathbb{R})$ and Parseval equation holds $\langle f, \varphi \rangle = \sum_{n=0}^{\infty} a_n(f) a_n(\varphi)$.

Although we have stated the results for $\mathcal{S}^{(M_p)}(\mathbb{R})$ and $\mathcal{S}^{(M_p)'}(\mathbb{R})$ in one dimensional case, analogous results hold in the multidimensional case, for $\mathcal{S}^{(M_p)}(\mathbb{R}^l)$ and $\mathcal{S}^{(M_p)'}(\mathbb{R}^l)$.

The Hermite functions in multidimensional case are defined by

$$h_n(x) = h_{n_1}(x_1)h_{n_2}(x_2) \cdots h_{n_d}(x_d), \quad x = (x_1, x_2, \ldots x_d) \in \mathbb{R}^d,$$

where $n = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$. By $h_{(n,k)}$ we denote

$$h_{(n,k)} = h_{n_1}(x_1)h_{n_2}(x_2) \cdots h_{n_l}(x_l)h_{k_1}(x_{l+1})h_{k_2}(x_{l+2}) \cdots h_{k_s}(x_{l+s}), \quad (n, k) \in \mathbb{N}_0^l \times \mathbb{N}_0^s.$$


2.1 The Kernel theorem

The Kernel theorem for tempered ultradistributions states essentially that every continuous linear map $\mathcal{K}$ of the space $(\mathcal{S}(\mathcal{M}_p)(\mathbb{R}^l))_x$ of test functions in some variable $x$, into the space $(\mathcal{S}(\mathcal{M}_p)'(\mathbb{R}^s))_y$ of tempered ultradistributions in a second variable $y$, is given by a unique tempered ultradistributions $K$ in both variables.

**Theorem 2.3 (Kernel theorem).** Every jointly continuous bilinear functional $K$ on $\mathcal{S}(\mathcal{M}_p)(\mathbb{R}^l) \times \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^s)$ defines a linear map $\mathcal{K} : \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^s) \to \mathcal{S}(\mathcal{M}_p)'(\mathbb{R}^l)$ by

\[ \langle \mathcal{K}\phi, \psi \rangle = K(\psi \otimes \phi), \quad \text{for } \phi \in \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^s), \psi \in \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^l). \]  

and $(\psi \otimes \phi)(x, y) = \phi(x)\psi(y)$, which is continuous in the sense that $\mathcal{K}\phi_j \to 0$ in $\mathcal{S}(\mathcal{M}_p)'(\mathbb{R}^l)$ if $\phi_j \to 0$ in $\mathcal{S}(\mathcal{M}_p)(\mathbb{R}^l)$.

Conversely, for linear map $\mathcal{K} : \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^s) \to \mathcal{S}(\mathcal{M}_p)'(\mathbb{R}^l)$ there is unique tempered ultradistribution $K \in \mathcal{S}(\mathcal{M}_p)'(\mathbb{R}^{l+s})$ such that $[7]$ is valid. The tempered ultradistribution $K$ is called the kernel of $\mathcal{K}$.

Proof. If $K$ is a jointly continuous bilinear functional on $\mathcal{S}(\mathcal{M}_p)(\mathbb{R}^l) \times \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^s)$, then (7) defines a tempered ultradistribution $(\mathcal{K}\phi) \in \mathcal{S}(\mathcal{M}_p)'(\mathbb{R}^l)$ since $\psi \mapsto K(\psi \otimes \phi)$ is continuous. The mapping $\mathcal{K} : \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^s) \to \mathcal{S}(\mathcal{M}_p)'(\mathbb{R}^l)$ is continuous since the mapping $\phi \mapsto K(\psi \otimes \phi)$ is continuous.

Let us prove the converse. To prove the existence we define a bilinear form $B$ on $\mathcal{S}(\mathcal{M}_p)'(\mathbb{R}^l) \otimes \mathcal{S}(\mathcal{M}_p)'(\mathbb{R}^s)$ by

\[ B(\varphi, \psi) = \langle \mathcal{K}\psi, \phi \rangle, \quad \psi \in \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^l), \phi \in \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^s). \]

The form $B$ is a separately continuous bilinear form on the product $\mathcal{S}(\mathcal{M}_p)(\mathbb{R}^l) \times \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^s)$ of Frechet spaces and therefore it is jointly continuous, see [28].

Let $C > 0, \theta \in \mathbb{R}_+, \nu \in \mathbb{R}_+$ be chosen so that

\[ |B(\varphi, \psi)| \leq C||\varphi||_\theta||\psi||_\nu, \]

and let

\[ t_{(n,k)} = B(h_n, h_k), \quad n \in \mathbb{N}^l, k \in \mathbb{N}^s. \]

Since $B$ is jointly continuous on $\mathcal{S}(\mathcal{M}_p)(\mathbb{R}^l) \times \mathcal{S}(\mathcal{M}_p)(\mathbb{R}^s)$, for $\varphi = \sum a_n h_n$ and $\psi = \sum b_k h_k$ we have that

\[ B(\varphi, \psi) = \sum t_{(n,k)} a_n b_k. \]

On the other hand, for $(n, k) \in \mathbb{N}^l \times \mathbb{N}^s$ and $(\theta, \nu) \in \mathbb{R}^l \times \mathbb{R}^s$, by (8) we have

\[ |t_{(n,k)}| \leq C||h_n||_{\theta}||h_k||_{\nu} = \]

\[ \exp[2 \sum_{i=1}^{l} M(\theta_i \sqrt{n_i})] \exp[2 \sum_{j=1}^{s} M(\nu_j \sqrt{k_j})]. \]
Thus, from Theorem 2.2 it follows that the sequence \( \{ t_{(n,k)} \}_{(n,k)} \) is a Hermite representation of a tempered ultradistribution \( K \in \mathcal{S}^{(M_p)'}(\mathbb{R}^l \times \mathbb{R}^s) \). Thus

\[
\langle K, \varphi \rangle = \sum t_{(n,k)} c_{(n,k)},
\]

for \( \varphi = \sum c_{(n,k)} h_{n,k} \in \mathcal{S}^{(M_p)}(\mathbb{R}^{l+s}) \).

If \( \varphi = \sum a_n h_n \in \mathcal{S}^{(M_p)}(\mathbb{R}^l) \) and \( \psi = \sum b_k h_k \in \mathcal{S}^{(M_p)}(\mathbb{R}^s) \) then \( \varphi \otimes \psi \) has the Hermite representation \( \{ a_n b_k \}_{(n,k)} \) and we have that for tempered ultradistribution \( K \) defined by (9)

\[
K(\varphi \otimes \psi) = \sum_{(n,k)} t_{(n,k)} a_n b_k = B(\varphi, \psi),
\]

so \( K = B \). This proves the existence.

The uniqueness follows from the fact that \( K \) is completely determined by its Hermite representation \( \{ \langle K, h_{(n,k)} \rangle \}_{(n,k)} \) and the fact that for every \( (n,k) \in \mathbb{N}^l \times \mathbb{N}^s \)

\[
\langle K, h_{(n,k)} \rangle = \langle K, h_n \otimes h_k \rangle = B(h_n, h_k) = t_{(n,k)}.
\]

QED

The unitary representation \( \rho \) of the reduced Heisenberg group \( H_n^{red} \) might be considered as a representation of \( L^1(\mathbb{R}^2) \), with a nonstandard convolution structure. Accordingly, for \( F \in L^1(\mathbb{R}^2) \) let us define (see [7]) a bounded operator \( \rho(F) \) on \( L^2(\mathbb{R}^l) \) (it is sometimes called the Weyl transform of \( F \)) as

\[
\rho(F) \varphi(x) = \int \int F(y - x, q) e^{\pi i q(x + y)} \varphi(y) dy dq.
\]

In other words \( \rho(F) \) is an integral operator with kernel

\[
K_F(x, y) = \mathcal{F}_2^{-1} F(y - x, \frac{y + x}{2}),
\]

where \( \mathcal{F}_2 \) denotes Fourier transform in the second variable and \( \mathcal{F}_2^{-1} \) its inverse transform. Calling \( \rho(F) \) the Weyl transform is historically inaccurate. In fact, the Weyl transform of \( F \) should be \( \rho(\mathcal{F}(F)) \).

**Corollary 2.4.** The map \( \rho \) from \( L^1(\mathbb{R}^2) \) to the space of bounded operators on \( L^2(\mathbb{R}^l) \), defined by (10), extends uniquely to a bijection from \( \mathcal{S}^{(M_p)'}(\mathbb{R}^2) \) to the space of continuous linear maps from \( \mathcal{S}^{(M_p)}(\mathbb{R}^l) \) to \( \mathcal{S}^{(M_p)'}(\mathbb{R}^l) \).

**Proof.** The kernel \( K_F \) is well defined when \( F \) is an arbitrary ultradistribution and belongs to the space \( \mathcal{S}^{(M_p)'} \). So the Weyl transform extends to a bijection from \( \mathcal{S}^{(M_p)'}(\mathbb{R}^2) \) to the space of continuous linear maps from \( \mathcal{S}^{(M_p)}(\mathbb{R}^l) \) to \( \mathcal{S}^{(M_p)'}(\mathbb{R}^l) \).

The uniqueness of the extension follows from the kernel theorem since every continuous linear map from \( \mathcal{S}^{(M_p)}(\mathbb{R}^l) \) to \( \mathcal{S}^{(M_p)'}(\mathbb{R}^l) \) is of the form (7). \( \square \)
3 Proofs of Theorems 2.1 and 2.2

In the proof of Theorem 2.1 we will use the following facts:

(i) Since the sequence \( m_n = M_n/M_{n-1}, n = 1, 2, \ldots \) is increasing (which is equivalent to the condition (M.1), see [14, p. 50]), we have

\[
\frac{k}{m_k} \leq \sum_{k=0}^{\infty} \frac{1}{m_k} =: B < \infty,
\]

and

\[
\frac{k! M_n}{n! M_k} = \frac{k!}{m_k m_{k-1}} \cdots \frac{n+1}{m_{n+1}} \leq B^{k-n}, \quad n, k \in \mathbb{N}, n \leq k.
\]

From above and Stirling formula

\[
n^n = n! \frac{1}{\sqrt{2\pi n}} e^{g(n)} \leq C e^n n!, \quad |g(n)| \leq \frac{1}{12n},
\]

it follows that

\[
\frac{k^k M_n}{n^n M_k} \leq C e^k \sqrt{n} B^{k-n}, \quad k, n \in \mathbb{N}, n \leq k.
\]

(ii) For \( n, m \in \mathbb{N} \)

\[
x^m h_n(x) = 2^{-m/2} \sum_{k=0}^{m} \alpha^{(n)}_{k,m} h_{n-m+2k}(x), \quad x \in \mathbb{R},
\]

where

\[
0 \leq |\alpha^{(n)}_{k,m}| \leq \binom{m}{k} ((2n+1)^{m/2} + m^{m/2})
\]

(iii) If we denote \( R = \left( -\frac{d}{dx^2} + x^2 \right) \), then for \( \varphi \in \mathcal{S} \) and \( k \in \mathbb{N} \) it holds

\[
R^k \varphi = \sum_{n=0}^{k} \sum_{p=0}^{2n} C_{p;2n-p}^{(k)} x^{(2n-p)} \varphi^{(2n-p)}, \quad |C_{p;2n-p}^{(k)}| \leq 10^k k^{k-n}.
\]

The (ii) and (iii) were proved by mathematical induction in [13].

**Proof.** (Theorem 2.1) We give a detailed proof in one dimensional case.

a) Let us first prove that if \( \varphi \in \mathcal{S}^{(M_p)} \), then for every \( \theta > 0 \)

\[
\sum_{n=0}^{\infty} |a_n|^2 \exp[2M(\theta \sqrt{2n+1})] < \infty.
\]
Let $\varphi \in S^{(M_p)}$. From (17), (3) (M.1), (1), Stirling formula, (M.2), (12), (14) it follows that there exists $C$ such that for each $m > 0$ and each $k \in \mathbb{N}_0$,

$$\left( \sum_{n \in \mathbb{N}_0} |a_n|^2 (2n + 1)^{2k} \right)^{1/2} = ||R^k \varphi||_2 \leq \sum_{n=0}^{k} \sum_{p=0}^{2n} C_{p, 2n-p}^{(k)} ||x^p \varphi^{(2n-p)}||_2$$

$$\leq C \sum_{n=0}^{k} \sum_{p=0}^{2n} 10^k k^{-n} m^{-2n} M_p M_{2n-p}$$

$$\leq C \sum_{n=0}^{k} \sum_{p=0}^{2n} 10^k \frac{k!}{n!} e^{k-n + \frac{1}{12n}} \sqrt{n} m^{-2n} M_{2n}$$

$$\leq C \sum_{n=0}^{k} \sum_{p=0}^{2n} 10^k \frac{k!}{n!} e^{k-n + \frac{1}{12n}} \sqrt{n} m^{-2n} M_{2n}$$

$$\leq C 20^k e^k (1 + H^2)^k m^{2k} M_{2k} \sum_{n=0}^{k} \frac{1}{2n} B^{-k-n} M_n \left( \frac{1}{m^2} \right)^{-n}$$

$$\leq C 80^k e^k (1 + H^2)^k B^k (1 + \frac{1}{B})^k m^{2k} M_{2k} \times$$

$$\times \left( \sum_{n=0}^{\frac{1}{m^2}} H^{k+1} \frac{M_n}{M_{k+1}} (k+1)^{k+1} + \sum_{n=\frac{1}{m^2}+1}^{k} \frac{H^{n+1} M_{n-1}}{M_{k+1}} (n-1)^{n-1} \right)$$

$$\leq C 80^k e^k (1 + H^2)^k B^k (1 + \frac{1}{B})^k m^{2k} M_{2k} \times$$

$$\times \left( \sum_{n=0}^{\frac{1}{m^2}} H^{k+1} \frac{e^{k+1}}{2k} \sqrt{n} B^{k+1-n} + \sum_{n=\frac{1}{m^2}+1}^{k} \frac{H^{n+1} e^{k+1}}{2n} \sqrt{n-1} B^{k+1-n} \right)$$

$$\leq C 160^k e^{2k} (1 + H^2)^k (1 + H)^{2k} B^{2k} (1 + \frac{1}{B})^{2k} m^{2k} M_{2k}$$

$$\leq C m_1^{2k} M_{2k}$$

where $m_1 = \sqrt{160(1 + H^2)} e(1 + H) B (1 + \frac{1}{B}) m$.

It follows that, for $2k = \alpha + 2$

$$|a_n|^2 (2n + 1)^{\alpha+2} \leq C m_1^{\alpha+2} M_\alpha^2 .$$

Applying (M.2) we obtain

$$|a_n|^2 (2n + 1)^{\alpha+2} \leq C m_1^{\alpha+2} A^2 H^{2(\alpha+2)} M_\alpha^2$$
which implies that for each $\alpha$, $n \in \mathbb{N}_0$ and $\theta = m_1H$, there exists $C$, such that
\[
\frac{|a_n|^2\theta^{2\alpha}(2n + 1)^\alpha}{M_\alpha^2} \leq \frac{C}{(2n + 1)^2},
\]
which implies
\[
|a_n|^2\exp[2M(\theta\sqrt{2n + 1})] = |a_n|^2 \sup_{\alpha \in \mathbb{N}_0} \frac{\theta^{2\alpha}(2n + 1)^\alpha}{M_\alpha^2} \leq \frac{C}{(2n + 1)^2}.
\]
Therefore
\[
\sum_{n \in \mathbb{N}_0} |a_n|^2\exp[2M(\theta\sqrt{2n + 1})] < \infty,
\]
for every $\theta > 0$.

b) Let us now prove that if
\[
(19) \sum_{n \in \mathbb{N}_0} |a_n|^2\exp[2M(\theta\sqrt{2n + 1})] < \infty,
\]
for every $\theta > 0$, then the series $\sum a_n h_n$ converges in the space $S^{(M_p)}$.

Suppose that inequality (19) holds for every $\theta > 0$. Applying (15), (16), Cauchy-Schwartz inequality and
\[
(20) \exp[-M(\theta\sqrt{2n + 1})] \leq \frac{M_4}{\theta^4(2n + 1)^2},
\]
which follows from the definition of the associated function. We get that for every $\theta > 0$
there exist a constant $C$ such that for each $p \in \mathbb{N}_0$

\[ \|x^p\varphi\|_2 \leq 2^{-p/2} \| \sum_{n \in \mathbb{N}_0} a_n \left( \sum_{k \leq p} \alpha_{n,k} \right) \|_2 \]

\[ \leq 2^{-p/2} \sum_{n \in \mathbb{N}_0} a_n ((2n + 1)^{p/2} + p^{p/2}) \left( \sum_{k \leq p} \left( \frac{p}{k} \right) \right) \| h_{n,p} \|_2 \]

\[ \leq 2^{p/2} \sum_{n \in \mathbb{N}_0} a_n \exp[M(\theta \sqrt{2n + 1}) - M(\theta \sqrt{2n + 1})((2n + 1)^{p/2} + p^{p/2})] \]

\[ \leq 2^{p/2} \left( \sum_{n \in \mathbb{N}_0} |a_n|^2 \exp[2M(\theta \sqrt{2n + 1})] \right)^{1/2} \times \]

\[ \left( \sum_{n \in \mathbb{N}_0} \right. \exp[-2M(\theta \sqrt{2n + 1})((2n + 1)^{p/2} + p^{p/2})] \left. \right)^{1/2} \]

\[ \leq C 2^{p/2} \left( \sum_{n \in \mathbb{N}_0} \exp[-2M(\theta \sqrt{2n + 1})((2n + 1)^{p/2} + p^{p/2})] \right)^{1/2} \]

\[ \leq C 2^{p/2} \sup_{n \in \mathbb{N}_0} \left( \left( (2n + 1)^{p/2} + p^{p/2} \right) \exp\left[ -\frac{1}{2} M(\theta \sqrt{2n + 1}) \right] \right) \times \]

\[ \left( \sum_{n \in \mathbb{N}_0} \exp[-M(\theta \sqrt{2n + 1})] \right)^{1/2} \]

\[ \leq C \left( \frac{\sqrt{2}}{\theta} \right)^{p/2} M_p \left( \theta^{p/2} \sup_{n} (2n + 1)^{p/2} \exp\left[ -\frac{1}{2} M(\theta \sqrt{2n + 1}) \right] \right) \times \]

\[ \frac{\theta^{p/2} p^{p/2}}{M_p} \leq p! e^{\theta^{p/2}} \rightarrow 0, \text{ as } p \rightarrow \infty, \]

and

\[ \sup_{n \in \mathbb{N}_0} \frac{\theta^{p/2} (2n + 1)^{p/2} \exp\left[ -\frac{1}{2} M(\theta \sqrt{2n + 1}) \right]}{M_p} = \frac{1}{M_p} \sup_{n \in \mathbb{N}_0} \left( \frac{\theta^{p/2} (2n + 1)^{p/2} \exp\left[ -\frac{1}{2} M(\theta \sqrt{2n + 1}) \right]}{M_p} \right)^{1/2} \]

\[ \leq \sqrt{\frac{M_p}{M_p}} \rightarrow 0, \text{ as } p \rightarrow \infty, \]

which follows from [14 (3.3)], we have that for each $\theta > 0$ there exists $C > 0$ such that

(21) \[ \|x^p\varphi\|_2 \leq C \left( \frac{\sqrt{2}}{\theta} \right)^{p} M_p, \quad p \in \mathbb{N}_0. \]

From above and properties of the Fourier transform we obtain that for each $\theta > 0$ there exists $C > 0$ such that

(22) \[ \|\varphi^{(q)}\|_2 = \frac{1}{\sqrt{2\pi}} \|x^q \mathcal{F}(\varphi)\|_2 = \frac{1}{\sqrt{2\pi}} \|x^q \sum_{k \in \mathbb{N}_0} a_k h_k\|_2 = \frac{1}{\sqrt{2\pi}} \|x^q \varphi\|_2 \leq C \left( \frac{\sqrt{2}}{\theta} \right)^{q} M_q. \]
If \( p, q \in \mathbb{N}_0 \) and \( \gamma = \min(q, 2p) \), by using (21), (22), (M.1), (M.3)' and (M.2) we get
\[
\left( ||x^p \varphi^{(q)}||^2 \right)^2 = (x^p \varphi^{(q)}, x^p \varphi^{(q)})_{L^2} = |((x^{2p} \varphi^{(q)(q)}), \varphi)_{L^2}|
\]
\[
\leq \sum_{k=0}^{q} \frac{(2p)!}{(2p-k)!} (x^{2p-k} \varphi^{(2q-k)}, \varphi)_{L^2} \leq \sum_{k=0}^{q} \frac{(2p)!}{k!} (2/\theta)^{(q-p-k)} M_{k}^2 \frac{M_2}{M_k} \frac{M_{2q-k} M_{2p-k}}{M_k}
\]
\[
\leq C \sum_{k=0}^{q} \frac{(2p)!}{k!} (2/\theta)^{(q-p-k)} M_{k}^2 \frac{M_2}{M_k} \frac{M_{2q-k} M_{2p-k}}{M_k}
\]
\[
\leq C \sum_{k=0}^{q} \frac{(2p)!}{k!} (2/\theta)^{(q-p-k)} M_{k}^2 \frac{M_2}{M_k} M_{2q} M_{2p}
\]
\[
\leq C \sum_{k=0}^{q} \frac{(2p)!}{k!} (2/\theta)^{(q-p-k)} M_{k}^2 \frac{M_2}{M_k} \frac{M_{2q-k} M_{2p-k}}{M_k}
\]
\[
\leq C \sum_{k=0}^{q} \frac{(2p)!}{k!} (2/\theta)^{(q-p-k)} M_{k}^2 \frac{M_2}{M_k} M_{2q} M_{2p}
\]
\[
\leq C m^{q+p} M_{q}^2 M_{p}^2,
\]
\[
(23)
\]

Let us now prove Theorem 2.2.

**Proof.** (Theorem 2.2) (b) Let us suppose that for some \( \theta > 0 \)
\[
(24)
\sum_{n=0}^{\infty} |b_n|^2 \exp[-2M(\theta \sqrt{2n+1})] < \infty.
\]

We will prove the convergence of the series \( \sum_{n=1}^{\infty} \langle b_n h_n, \varphi \rangle \) for every \( \varphi \in S^{(M_p)} \). From that it follows that the mapping \( f \mapsto \sum_{n=1}^{\infty} a_n b_n \) is an element of \( S^{(M_p)} \). Using Schwartz inequality we have
\[
\sum_{n=0}^{\infty} |\langle b_n h_n, \varphi \rangle| = \sum_{n=0}^{\infty} |b_n \langle h_n, \varphi \rangle| =
\]
\[
= \sum_{n=0}^{\infty} (|b_n| \exp[-M(\theta \sqrt{2n+1})] \cdot |\langle \varphi, h_n \rangle| \exp[M(\theta \sqrt{2n+1})])
\]
\[
\leq \left( \sum_{n=1}^{\infty} |b_n|^2 \exp[-2M(\theta \sqrt{2n+1})] \right)^{1/2} \left( \sum_{n=0}^{\infty} |\langle \varphi, h_n \rangle|^2 \exp[2M(\theta \sqrt{2n+1})] \right)^{1/2}.
\]

The first sum on the right hand side converges by supposition and the convergence of the second sum follows from Theorem 2.3 since \( \varphi \in S^{(M_p)} \).

(a) Suppose that \( f \in S^{(M_p)'} \) and let \( b_n = \langle f, h_n \rangle \). Then for every \( \varphi = \sum_{n} a_n h_n \in S^{(M_p)} \), we have that the series \( \sum a_n b_n \) converges. We will prove that for some \( \theta > 0 \)
\[
(25)
\sum_{n \in \mathbb{N}_0} |b_n|^2 \exp[-2M(\theta \sqrt{2n+1})] < \infty.
\]
Let us first prove that the sequence
\[
\{ |b_n| \exp[-M(\theta \sqrt{2n + 1})] \}_{n \in \mathbb{N}}
\]
is bounded for some \( \theta \), say \( \theta_0 \). If it is not so, then for every \( q \in \mathbb{N} \) there exist \( n_q \geq q \) such that
\[
|b_{n_q}| \exp[-M(\sqrt{2n_q + 1})] \geq 1.
\]
Let \( \varphi = \sum a_m b_m \) where
\[
|a_m| = \exp[-M(\sqrt{2n_q + 1})]q^{-1}
\]
for \( a_m = n_q \), and \( a_m = 0 \) for \( m \neq n_q \), and \( a_m \) satisfies that \( a_m b_m = |a_m b_m| \). We will prove that \( \varphi \in S^{(M_p)} \). Using [14, Proposition 3.4, p.50],
\[
M(k\rho) - M(\rho) \geq \frac{\log(\rho/A) \log k}{\log H} \geq \frac{\log(\rho/A) \log k}{\log(1 + H)}
\]
for \( \rho = \mu \sqrt{2n_q + 1}, k = q/\mu \), we have that
\[
\sum_{m=1}^{\infty} |a_m|^2 \exp[2M\mu \sqrt{2n + 1}] = \sum_q q^{-2} \exp[-2M(\sqrt{2n_q + 1}) + 2M(\mu \sqrt{2n_q + 1})] \\
\leq \sum_q q^{-2} \exp[-2 \frac{\log(\mu \sqrt{2n_q + 1}) \log(\mu)}{\log(1 + H)}].
\]
The above sum converges since for large enough \( q \)
\[
\exp[-2 \frac{\log(\mu \sqrt{2n_q + 1}) \log(\mu)}{\log(1 + H)}]
\]
is bounded. From Theorem ?? it follows that \( \varphi \in S^{(M_p)} \), and therefore
\[
\sum_{n=0}^{\infty} a_n b_n = \langle f, \varphi \rangle < \infty,
\]
which is in contradiction with
\[
\sum_{n=0}^{\infty} a_n b_n = \sum_{n=0}^{\infty} |a_n b_n| \geq \sum_{q=1}^{\infty} q^{-1} = \infty.
\]
Therefore, sequence \( (26) \) is bounded for some \( \theta_0 \).

Using inequality \( (27) \) for \( \rho = \theta_0 \sqrt{2n + 1} \) and \( k = \theta/\theta_0 \), we have that
\[
\exp[-M(\theta \sqrt{2n + 1})] \leq \exp[-M(\theta_0 \sqrt{2n + 1})] \exp[-\frac{\log(\theta_0 \sqrt{2n + 1}) \log(\theta_0)}{\log(1 + H)}]
\]

which implies that the sequence

\[ b_n \exp[-M(\theta\sqrt{2n+1})] \to 0, \quad n \to \infty. \]

for some \( \theta > \theta_0 \).

And finally, we will prove that the series (6) converges for some \( \theta > \theta_0 \). If we suppose that the sequence diverges for every integer \( \theta = q > \theta_0 \), then there exists an increasing sequence of integers \( k_q \) such that

\[ 1 \leq \sum_{n=k_{q-1}}^{k_q-1} |b_n| \exp[-M(q\sqrt{2n+1})] < 2, \quad q = \theta_0 + 1, \theta_0 + 2, \ldots \]

Put

\[ |a_n| = |b_n| \exp[-2M(q\sqrt{2n+1})]q^{-1} \]

for \( k_{q-1} \leq n, k_q, q \theta_0 \). Then for every fixed \( \theta > 0 \) we have

\[
\sum_{n=k_{q-1}}^{k_q-1} |a_n|^2 \exp[2M(\theta\sqrt{2n+1})] = \\
\sum_{n=k_{q-1}}^{k_q-1} |b_n| \exp[-2M(q\sqrt{2n+1})] \exp[2M(\theta\sqrt{2m+1}) - 2M(q\sqrt{2n+1})].
\]

Using once again the inequality (27) we see that

\[ \exp[-2M(q\sqrt{2n+1}) + 2M(\theta\sqrt{2n+1})] \leq 1, \]

and therefore, for every \( \theta > 0 \),

\[
\sum_{n=k_{q-1}}^{k_q-1} |a_n|^2 \exp[2M(\theta\sqrt{2n+1})] \leq 2q^{-2}.
\]

It implies that the sum

\[ \sum_{n=0}^{\infty} |a_n| \exp[2M(\theta\sqrt{2n+1})] \]

converges for every \( \theta > 0 \). From theorem ?? it follows that \( \varphi = \sum a_n h_n \) belongs to \( S^{(M_p)} \) which contradict the fact that the sum \( \sum a_n b_n \) diverges since

\[
\sum_{n=k_{q-1}}^{k_q-1} |a_n b_n| = \sum_{n=k_{q-1}}^{k_q-1} |b_n|^2 \exp[-2M(q\sqrt{2n+1})]q^{-1} \geq q^{-1}.
\]
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