Baryon Chiral Perturbation Theory
in
Partially Quenched Large-$N_c$ QCD

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abstract

Baryons of lower spin in partially quenched large-$N_c$ QCD are studied with particular emphasis to interpolation between standard unquenched and fully quenched limits. In large $N_c$ limit of partially quenched QCD, we calculate $\Delta m$, the chiral one-loop correction to baryon masses. We find that leading order contribution to $\Delta m$ is independent of number of ghost quarks introduced. For finite $N_c$, $\Delta m$ does satisfy the Bernard–Golterman’s third theorem and has no infrared quenching singularity except for fully quenched limit. At large $N_c$ limit, however, we show that the third theorem is bypassed and non-trivial quenched chiral corrections do arise. In unquenched limit, we also show that standard chiral perturbation theory results are reproduced in which $\eta'$ loop contributions are explicitly taken into account.

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1 Introduction

With impressive advance in numerical simulation of QCD on a lattice, the goal of first-principle calculation for hadron spectrum as well as other physical quantities is achievable in the near future. At present, however, nearly all QCD simulations are based on various degrees of approximation. Therefore, understanding error and deviation incurred by such approximations to the calculated physical quantity is of prime theoretical importance. Among the approximations, adopted most frequently is the quenched approximation \[1, 2\]. By quenched approximation, which has been introduced in order to accelerate generation of the gauge field configurations, one drops out quark determinants and keeps only the valence quarks. While the approximation is well suited when quark masses are taken sufficiently heavy, extrapolation of quark masses to chiral limit might lead to potentially significant effects. In fact, Sharp [3, 4] and Bernard and Golterman [5, 6] have pointed out that the quenched approximation leads to sizable errors in the chiral limit. If this is the case for a generic physical quantity, then one needs to understand better the errors incurred by the quenched approximation before a prediction or a conclusion is drawn.

To investigate systematically an error introduced by the quenched approximation in the chiral limit, Sharpe [3, 4] and Bernard and Golterman [5] have developed chiral perturbation theory for quenched QCD. This so-called quenched chiral perturbation theory (Q\(\chi\)PT) utilizes the trick by Morel [7] of introducing ghost quarks, which have the same masses as the original quarks but carries opposite statistics. Thus, the chiral flavor symmetry is extended to graded, super-flavor symmetry and it is this symmetry which organizes the Q\(\chi\)PT. Moreover, diagrams with quark loops are cancelled by the same diagrams but with ghost quark loops. Since these pioneering works, using the Q\(\chi\)PT, chiral behavior of quenched QCD has been studied extensively for pseudo-scalar mesons [3, 4, 5], baryons [3], vector and tensor mesons [3, 10], heavy mesons [11, 12], weak matrix elements \(B_K\) [4, 13], baryon axial charge [14], heavy baryons [15], and pion scattering length [16]. Numerous lattice data have been compared to results from Q\(\chi\)PT: see, reviews by Sharpe [17], Gottlieb [18] and Okawa [19] and references therein.

Through these extensive study, it has been concluded that main effect of quenched approximation is due to peculiarity of super-\(\eta'\) meson (direct counterpart of “anomalous” \(\eta'\)). This super-\(\eta'\) meson has the same mass as the other Goldstone bosons (same single-pole position of propagators) and has to be retained in the Q\(\chi\)PT, unlike the standard \(\chi\)PT. Peculiar to super-\(\eta'\) is the extra double-pole term present in the propagator, which does not permit a particle interpretation at all. Most notably, it has been found that the double-pole term in the super-\(\eta'\) propagator leads to new infrared singular, non-analytic chiral corrections that have
no counterpart in the full QCD. In order to understand better such pathological behavior, it should be desirable to be able to interpolate, if possible at all, QCD between unquenched and fully quenched limits and examine changes of physics in the chiral limit. Indeed, with such motivation, Bernard and Golterman have developed partially quenched QCD, in which only a subset of quarks are paired with ghost quarks. By formulating partially quenched chiral perturbation theory (PQχPT), Bernard and Golterman have shown that change of chiral behavior for Goldstone mesons can be indeed understood from interpolation between standard QCD and fully quenched QCD.

In this paper, we study chiral dynamics of large $N_c$ baryons in partially quenched QCD. Previously, in the fully quenched QCD, baryons have been studied by Labrenz and Sharpe. They have studied quenching effect to the behavior of baryon mass spectrum in the chiral limit and have observed, most notably, that pattern of the correction is substantially different from that in the unquenched case. The leading order chiral correction in QχPT scales like $m_q^{1/2}$, which is more singular than the standard $m_q^{3/2}$ χPT correction. Prompted by such interesting results, in this paper, by formulating partially quenched chiral perturbation theory for baryons, we examine change in chiral behavior of baryon mass spectrum as the QCD interpolates between standard and fully quenched limits. In our investigation, we also invoke large $N_c$ limit and associated planar symmetry. Combined application of chiral perturbation theory and large $N_c$ expansion are expected to constrain the low-energy interactions of baryons with the Goldstone bosons more effectively than either method alone. We have found that the large $N_c$ limit affects the chiral dynamics in two interesting ways. First, the positions (on the complex plane) of the poles of the $\eta'$ propagator depend explicitly on the value of $N_c$. In PQχPT, the $\eta'$ meson propagator has two simple poles: pion pole at $p^2 = m^2$ and a shifted pole at $p^2 = m^2 + M_0^2$. The “pole shift” $M_0^2$ originates from the hairpin diagram, which is $1/N_c$ suppressed. As $N_c \to \infty$, the shifted pole will merge to the pion pole. As we will show below, there arises an interesting cancellation between contributions from each of these two poles in the $\eta'$ propagator. Second, the size of $N_c$ also controls the form of axial current couplings in the baryon sector. The $\eta'$ coupling to a single quark line inside a baryon is related to its pion counterpart, as $\pi$ and $\eta'$ are related by the “planar symmetry”, which is exact in the large $N_c$ limit. This symmetry, however, is not manifest at the hadron level. While pion coupling to different quark lines interfere constructively, the $\eta'$ coupling has huge cancellation. As a result, chiral loops involving $\pi$ and $\eta'$ appear at different order in large $N_c$ expansion.

This paper is organized as follows. In section 2, after recapitulation of baryons in the large-$N_c$ limit, we examine changes in chiral behavior of baryon mass spectrum in partially quenched QCD. Then, in section 3, we study chiral dynamics of large $N_c$ baryons in two other limits: unquenched QCD and fully quenched QCD. Finally, in section 4, we summarize our results and discuss the implications for lattice QCD with staggered fermions and Wilson valence quarks, respectively.
$N_c$ limit we formulate partially quenched chiral perturbation theory for large $N_c$ baryons. We pay particular attention to the mass corrections of nucleons and $\Delta$ in the large $N_c$ and chiral limits. In section 3, we calculate infrared singular, non-analytic chiral corrections to the baryon masses. Moreover, we provide anatomy of chiral one-loop corrections and identify contributions of $\eta'$ at leading order in $1/N_c$ expansion. In section 4, we investigate the effect of next leading order corrections in $1/N_c$ expansions, followed by some discussions.

2 Baryons in Partially Quenched Large-$N_c$ QCD

2.1 Baryon Dynamics in Large $N_c$ QCD

Large $N_c$ meson dynamics has been first studied in Ref. [20]. Baryon dynamics in large-$N_c$ QCD has been studied originally in Ref. [21] and, more recently, in Refs. [22, 23, 24, 25, 26]. We recapitulate essential aspects of their results that will become relevant for baryons in partially quenched QCD (PQ-QCD). For simplicity, we will mainly focus on the case with $n = 2$ ($u$ and $d$). Since the $N_c$ quarks inside a baryon should form a color singlet, which is completely antisymmetric, the spin-flavor part of the wave function must be completely symmetric. As a result, the lowest lying baryons have $I = J = \frac{1}{2}, \frac{3}{2}, \ldots$, which are usually identified as the observed states $N, \Delta, \ldots$.

The interaction of baryons with Goldstone bosons are simplified in the large $N_c$ limit as the dynamical symmetries are enlarged both in the meson and the baryon sector. In the Goldstone boson sector, there arises the planar symmetry (also called “nonet symmetry” in the literature), decreeing that the $\eta'$ meson should be combined with the other Goldstone bosons into a $n \times n$ representation of U($n$) flavor symmetry group, where $n$ is the number of light flavors [20, 21]. More explicitly, for $n = 2$, Goldstone boson fields are represented by the $2 \times 2$ matrix:

$$\phi = \pi^a T^a + \eta' \frac{1}{\sqrt{2}} = \phi^a T^a = \left(\frac{\eta'}{\sqrt{2}} + \frac{\pi^0}{\sqrt{2}}, \frac{\pi^-}{\sqrt{2}}, \frac{\pi^+}{\sqrt{2}} - \frac{\eta'}{\sqrt{2}}\right),$$

where $T^a$ ($a = 1, 2, 3$) are the SU(2) generators, 1 is the identity, and $T^\alpha$ ($\alpha = 0, 1, 2, 3$) are the U(2) generators, i.e., $\{T^a\} = \{T^a\} \cup \{1\}$. As we will see below, the $N_c \to \infty$ planar symmetry implies that the same set of coupling constants will control both the $\pi$ and the $\eta'$ interactions.

See Ref. [27] for detailed discussion on validity of such an identification.
In the large $N_c$ limit, the Goldstone boson kinetic term is

$$\mathcal{L}_0 = \frac{f^2}{8} \Tr (\partial^\mu \sigma \partial_\mu \sigma) + \ldots,$$

(2)

where

$$\sigma \equiv \exp(2i\phi/f),$$

(3)

and ellipses denote higher derivative interactions that are irrelevant for low-energy dynamics. The degeneracy between $\pi$ and $\eta'$ persist even if one endows a small mass to the light quarks and breaks the chiral symmetry. The quark mass leads to a non-zero mass

$$m_\pi^2 = m_{\eta'}^2 = m^2.$$ 

The $U(1)_A$ anomaly, which breaks the planar symmetry and renders a heavy mass to $\eta'$, shows up as a $1/N_c$ suppressed correction:

$$\mathcal{L}_1 = \frac{1}{2} \left( m_0^2 (\Tr \phi)^2 + A_0 (\partial_\mu \Tr \phi)^2 \right) = \frac{n}{2} \left( m_0^2 \eta'^2 + A_0 \partial_\mu \eta' \partial^\mu \eta' \right),$$

(4)

where $n$ is the number of light flavors, and $A_0$ and $m_0^2$ are of order $\mathcal{O}(N_c^{-1})$. In the standard, unquenched $\chi$PT, the effect of $\mathcal{L}_1$ can be resummed:

$$\sum_{k=0}^{\infty} \frac{1}{p^2 - m^2} \left( \left( n A_0 p^2 + n m_0^2 \right) \frac{1}{p^2 - m^2} \right)^k = \frac{1}{1 - n A_0 \frac{m^2}{p^2} - \frac{n m_0^2}{p^2 - m^2}}.$$

(5)

One finds that the new position of the single pole at $m_{\eta'}^2 = m^2 + n m_0^2 / (1 - n A_0)$. Thus, in the chiral limit, $m_{\eta'}^2 \sim \mathcal{O}(N_c^{-1})$, as expected.

On the other hand, the baryon sector exhibits an $SU(2n)$ spin-flavor symmetry. For simplicity, we will restrict ourselves with 2 light flavors for foregoing discussions, in which case the spin-flavor $SU(4)$ symmetry will be generated by:

$$J^i = \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes 1) q_k, \quad I^a = \sum_{k=1}^{N_c} q_k^\dagger (1 \otimes T^a) q_k, \quad G^{ia} = \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes T^a) q_k.$$

(6)

Here, the spin and isospin operators on individual quark lines are denoted by $S^i$ ($i = x, y$ and $z$ are three spacelike directions perpendicular to the baryon velocity) and $T^a$ respectively, and $1$ is the identity operator in spin and isospin spaces. Note that there are $3 J$’s, $3 I$’s and $9 G^{ia}$.

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4In the literature, $m_0^2$ and $A_0$ are often written as $\mu^2/3$ and $\alpha/3$ so the for $n = 3, m_{\eta'}^2 = m^2 + \mu^2/(1 - \alpha)$. In this paper, however, we naturally opted for the notation without the factors of $1/3$ as we are mainly working with just two light flavors.

5Our description of the spin-flavor symmetry for large $N_c$ baryons follow closely Ref. [23, 24, 25]. Part of the reason is that the formalism in Ref. [24], which has been widely employed in the literature, assumes unitarity, hence, cannot be directly applicable to non-unitary, (partially) quenched QCD we study presently. However, the physical predictions of any of these formalisms should be identical.
G’s, making up the correct number of generators Eq. (15) for the spin-flavor SU(4) algebra. In what follows, we will adopt a useful notation

\[ G^{i\alpha} = \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes T^\alpha) q_k, \]  

(7)

which is equal to \( J^i \) and \( G^{i\alpha} \) for zero and non-zero \( \alpha \) respectively.

The axial current couplings of large \( N_c \) baryons are simplified by the fact that multi-quark operators are suppressed by powers of \( 1/N_c \). In the leading order of the large \( N_c \) expansion, the axial currents couple through single-quark operators. Hence, interaction of baryons to Goldstone bosons can be expressed as [23, 26]:

\[ \mathcal{L}_{\text{quark}} = i g f \left( \partial^i \phi^\alpha G^{i\alpha} \right) = i g f \left( \partial^i \pi^a \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes T^a) q_k + \partial^i \eta' \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes 1) q_k \right) + \cdots, \]  

(8)

where \( f = f_\pi = f_{\eta'} \) denotes Goldstone boson decay constant, and ellipses denote higher-order interactions involving more than one Goldstone bosons. Note that the \( \pi \) and \( \eta' \) couplings to baryons are taken by the same coupling constant \( g \), as decreed by planar symmetry in the large-\( N_c \) limit.

One can read off from Eq. (8) the meson-baryon-baryon couplings by summing up contributions from each individual quark lines. The \( \eta' \) meson couples to individual quark through \( S^i \otimes 1 \). Hence, \( \eta' \) meson coupling to a baryon obtained by summing over all constituent quarks is given by spin \( J^i \) of the baryon. That is,

\[ \mathcal{L}_{\eta'BB} = i g f \left( \partial^i \eta' \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes 1) q_k \right) = \left( \partial^i \eta' \right) J^i. \]  

(9)

Since we are interested only in states with \( I = J \sim \mathcal{O}(N_c^0) \), we will find that the \( \eta'BB \) coupling is of order \( \mathcal{O}(N_c^{-1/2}) \) (as \( f \sim \mathcal{O}(N_c^{1/2}) \)). Note that \( \mathcal{L}_{\eta'BB} \) contains sum over \( N_c \) quark terms, each of them being of order \( 1/f = \mathcal{O}(N_c^{-1/2}) \). Hence, one would naively suspect that the sum to be of order \( (N_c/f) \approx \mathcal{O}(N_c^{1/2}) \). This reasoning is fallacious, however, as for lower spin states, there exists huge cancellations among the individual quark spins.

One the other hand, pion coupling is proportional to \( S^i \otimes T^a \), and acts coherently over the individual quarks [23, 24, 25, 26, 27]. That this is so can be seen clearly using the operator identity [23, 27]:

\[ \sum_i \left( \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes 1) q_k \right)^2 + \sum_a \left( \sum_{k=1}^{N_c} q_k^\dagger (1 \otimes T^a) q_k \right)^2 + 4 \sum_{i,a} \left( \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes T^a) q_k \right)^2 \]
\[ J^2 + I^2 + 4G^2 = (\frac{3}{4}N_c^2 + 3N_c) \mathbf{1}, \]  

which is nothing but the Casimir identity for the spin-flavor SU(4). Since both \( J^2 \) and \( I^2 \) are of order \( \mathcal{O}(N_c^0) \), the operator \( G^a = \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes T^a) q_k \) must be of order \( \mathcal{O}(N_c) \) so that the sum is of order \( \mathcal{O}(N_c^2) \). As a result, \( \eta' \) coupling is one order in \( 1/N_c \) smaller than the pion counterpart, and we can see immediately that \( \eta' \) loops will be suppressed by \( 1/N_c^2 \) with respect to pion loops. Since all infrared quenched singularities arise from \( \eta' \) loops, we expect these infrared quenched singularities are subleading in orders of \( 1/N_c \). Below, we will show explicitly that this is indeed the case.

### 2.2 PQ\( \chi \)PT for Large-\( N_c \) Baryons

Now we are ready to (partially) quench the theory. Following the example of Ref. [6], one includes in the theory, in addition to the \( n \) quarks, \( k \) ghosts (spin-\( \frac{1}{2} \) objects with bosonic statistics), where \( k \leq n \). The \( n \times n \) Goldstone boson matrix \( \phi \) gets enlarged to an \((n+k) \times (n+k)\) supermatrix \( \Phi \),

\[ \Phi \equiv \begin{pmatrix} \phi & \chi^\dagger \\ \chi & \tilde{\phi} \end{pmatrix}, \]

where \( \chi \) is the \( n \times k \) matrix of quark-antighost fermionic “mesons”, its charge conjugate \( \chi^\dagger \) the \( k \times n \) matrix of ghost-antiquark “mesons”, and \( \tilde{\phi} \) the \( k \times k \) matrix of ghost-antighost bosonic mesons. The field \( \Phi \) transforms as a \((n+k) \times (n+k)\) representation of the enlarged symmetry algebra \( U(n+k) \) (or more exactly, the \( U(n|k) \) graded algebra). We will denote the generators of this \( U(n+k) \) by \( T^A \) and

\[ \Phi = \Phi^A T^A. \]  

Note that \( \{T^a\} \subset \{T^a\} \subset \{T^A\} \).

The lagrangians \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) are generalized to [3]

\[ \mathcal{L}_0 = \frac{f^2}{8} \text{Str}(\partial^\mu \Sigma \partial_\mu \Sigma) + \ldots, \]

\[ \mathcal{L}_1 = \frac{1}{2} \left( m_0^2 (\text{Str} \Phi)^2 + A_0 (\partial_\mu \text{Str} \Phi)^2 \right) \]

where

\[ \Sigma \equiv \exp(2i\Phi/f), \]

and “Str” denotes the supertrace, the sum of the first \( n \) entries on the diagonal subtracted by the sum of the last \( k \) entries. The propagator for flavored states in \( \Phi \) has a simple pole at
\( p^2 = m^2 \), while for the flavor-neutral states (states on the diagonal of \( \Phi \)) the propagator is

\[
G_{ij} = \left[ \frac{\delta_{ij} \epsilon_i}{p^2 - m^2} + \frac{1}{\Delta n} \left( \frac{1}{(1 + \Delta n A_0)} \frac{1}{p^2} \right) - \frac{1}{p^2} \right].
\]  

(16)

The grading index \( \epsilon_i \) is such that \( \epsilon(q_i) = +1, \epsilon(\bar{q}_j) = -1 \). Note that the second term (which proportional to \( 1/\Delta n \)) mixes quark-antiquark meson with ghost-antighost mesons. Both the first and the second terms have a simple pole at \( p^2 = m^2 \), but the second term has an additional pole, which we will call the “shifted pole”. Also note that, as \( n \to k \), the shifted pole moves back to the pion pole. The propagator will then have a double pole at \( m^2 \), which is a well-known result in Q\( \chi \)PT \( [5] \).

The partial quenching of the baryon chiral lagrangian is straightforward. The more general PQ\( \chi \)PT lagrangian in the large \( N_c \) limit is

\[
\mathcal{L} = i \frac{1}{f} \left( g \left( \bar{q} \Phi A \right) G^{iA} + h \left( \bar{q} \text{Str} \Phi \right) J^i \right).
\]  

(17)

The first term is a simple generalization of the unquenched lagrangian, with

\[
G^{iA} = \sum_{k=1}^{N_c} q_k^i (S^i \otimes T^A) q_k,
\]  

(18)

and \( g \) an undetermined coupling constant. The second term describes the “hairpin coupling” which couples flavor-neutral states in \( \Phi \) to the baryon operator \( J^i \), defined by

\[
J^i = \sum_{k=1}^{N_c} q_k^i (S^i \otimes 1_\epsilon) q_k,
\]  

(19)

where \( 1_\epsilon \) is the identity of U\((n|k)\), with \( n \) 1’s and \( k \) −1’s on the diagonal. In this paper, however, we will set the hairpin coupling constant \( h \) to zero to simplify the physics. A discussion of the possible effects of a non-zero \( h \) will be given wherever appropriate in later sections.

### 3 Non-analytic Chiral Correction to Baryon Masses

#### 3.1 Chiral One-loop Correction

We are now ready to calculate the chiral one-loop correction to the baryon masses in PQ\( \chi \)PT. While our theory has \( n \) quarks and \( k \) ghosts, all assumed to be degenerate, we will focus on baryons with just \( u \) or \( d \) quarks, though the non \((u, d)\) quarks (which will be collectively referred
as strange quarks) may appear in loops. All the diagrams involved have the form as shown in Fig. 1 and yield the same one-loop Feynman integral:

$$\mathcal{I}_1(m^2) \equiv -\frac{1}{12\pi f^2} m^3$$

(20)
as those in the standard $\chi$PT. We will restrict, in this paper, to the choice of hairpin coupling $h = 0$, the ghost-antighost meson does not contribute, and the only relevant coupling is the Lagrangian Eq. (8).

Depending on whether the Goldstone boson couples to the same quark line at the two vertices or not, there are two different types of contribution to the mass correction $\Delta m$.

(i) 1 quark line chiral correction: this class of corrections arises when in which the Goldstone meson couples to the same quark line at both vertices. The quark level diagrams are Fig. 2a and 2b. Notice that Fig. 2b coupling is possible only when the intermediate Goldstone meson state is $\eta'$. The contribution of this class is given by

$$\Delta m = g^2 \sum_{k=1}^{N_c} \left[ \mathcal{I}_1(m^2) \cdot \left( q_k^\dagger (S^i \otimes T^A) q_k \right)^2 
+ \frac{2}{\Delta n} \cdot \left( \mathcal{I}_1(M^2) - \mathcal{I}_1(m^2) \right) \cdot \left( q_k^\dagger (S^i \otimes 1) q_k \right)^2 \right].$$

(21)
The first line comes from the chiral loops with flavored Goldstone mesons, and the flavor-neutral mesons propagating with the first term in propagator Eq. (10). Note that the sum over $U(n|k)$ flavor generators $T^A$ originates from the internal loop in Fig. 2a, which can be either a quark or a ghost. The second term line originates from the second term in Eq. (10), with

$$\mathcal{I}_1(M^2) = \frac{1}{1 + \Delta n \cdot A_0} \cdot \mathcal{I}_1 \left( \frac{m^2 + \Delta n \cdot m_0^2}{1 + \Delta n \cdot A_0} \right).$$

(22)
Figure 2: Quark line diagram for chiral one-loop correction to baryon two-point function. (a) Goldstone boson coupling to the same quark line, (b) hairpin propagator to the same quark line, (c) Goldstone boson coupling to the different quark lines, (d) hairpin propagator to the different quark lines.
(ii) 2 quark line chiral correction: this is the class of corrections in which the Goldstone meson couples to different quark lines at each vertices. The quark level diagrams are depicted in Fig. 2c and 2d. The contribution is given by

\[
\Delta m = g^2 \sum_{k,\ell=1}^{N_c} \left[ \mathcal{I}_1(m^2) \cdot \left( q_k^\dagger (S^i \otimes T^\alpha) q_k \right) \cdot \left( q_\ell^\dagger (S^i \otimes T^\alpha) q_\ell \right) + \frac{2}{\Delta n} \cdot \left( \mathcal{I}_1(M^2) - \mathcal{I}_1(m^2) \right) \cdot \left( q_k^\dagger (S^i \otimes 1) q_k \right) \cdot \left( q_\ell^\dagger (S^i \otimes 1) q_\ell \right) \right].
\]

The prime on the summation sign denotes the condition \( k \neq \ell \). Interpretation of different terms is identical with that of (i). Note that in the first line, even though the sum should be in principle over all possible Goldstone bosons in \( \Phi \), there is actually no internal quark/ghost loops in Fig. 2c, hence, the ghosts do not contribute. As a result, we can just sum over intermediate states in \( \phi \) instead, leading to a sum over \( T^\alpha \) instead of \( T^A \).

Now it is time to perform the sums over the quark lines. First, we have terms involving the SU(2) generators \( T^a \), i.e., the pion loop contributions.

\[
\Delta m_{\pi-\text{loop}} = g^2 \mathcal{I}_1(m^2) \cdot \sum_{k,\ell=1}^{N_c} \left( q_k^\dagger (S^i \otimes T^\alpha) q_k \right) \cdot \left( q_\ell^\dagger (S^i \otimes T^\alpha) q_\ell \right) = g^2 \mathcal{I}_1(m^2) \left( \frac{3}{8} N_c^2 + \frac{3}{2} N_c - \frac{1}{2} J^2 \right),
\]

where we have used the identity [25]

\[
\sum_{k,\ell=1}^{N_c} \left( q_k^\dagger (S^i \otimes T^\alpha) q_k \right) \cdot \left( q_\ell^\dagger (S^i \otimes T^\alpha) q_\ell \right) = \left( \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes T^\alpha) q_k \right)^2 = \frac{3}{8} N_c^2 + \frac{3}{2} N_c - \frac{1}{2} J^2.
\]

Second, we have terms involving the flavor identity \( 1 \), i.e., the \( \eta' \) loop contributions.

\[
\Delta m_{\eta'-\text{loop}} = g^2 \left[ \mathcal{I}_1(m^2) + \frac{2}{\Delta n} \left( \mathcal{I}_1(M^2) - \mathcal{I}_1(m^2) \right) \right] \cdot \sum_{k,\ell=1}^{N_c} \left( q_k^\dagger (S^i \otimes 1) q_k \right) \cdot \left( q_\ell^\dagger (S^i \otimes 1) q_\ell \right) = g^2 \left[ \mathcal{I}_1(m^2) + \frac{2}{\Delta n} \left( \mathcal{I}_1(M^2) - \mathcal{I}_1(m^2) \right) \right] J^2,
\]

where we have used another identity, again from [25]

\[
\sum_{k,\ell=1}^{N_c} \left( q_k^\dagger (S^i \otimes 1) q_k \right) \cdot \left( q_\ell^\dagger (S^i \otimes 1) q_\ell \right) = \left( \sum_{k=1}^{N_c} q_k^\dagger (S^i \otimes 1) q_k \right)^2 = J^2.
\]

Finally, there are terms involving \( T^A \), those \( T^A \) which do not belongs to the set of \( T^\alpha \). Physically these come from Fig. 2a, with a ghost or a strange quark running around the internal loop. There are \( n - 2 \) strange quarks and \( k \) ghosts, so the contribution is

\[
\Delta m_{K/\chi-\text{loop}} = (\Delta n - 2) \Delta m_{K-\text{loop}} \text{ with 1 strange quark.}
\]
The combinatorics factor of the latter has been calculated, in Ref. [22, 23] to be \( \frac{3}{4}N_c \). Hence,

\[ \Delta m_{K/\chi\text{-}loop} = g^2 I_1(m^2)(\Delta n - 2)\frac{3}{4}N_c. \] (29)

Combining all the contributions, we have chiral correction to baryons in partially quenched large \( N_c \) QCD:

\[ \Delta m = g^2 \left[ \left( \frac{3}{8}N_c^2 + \frac{3}{4}\Delta nN_c + \frac{1}{2}J^2 \right) \cdot I_1(m^2) + 2 \cdot \frac{1}{\Delta n} \cdot J^2 \cdot \left( I_1(M^2) - I_1(m^2) \right) \right]. \] (30)

Note that the final result depends only on \( \Delta n \), which counts the difference in the number of physical quarks \( n \) and the number of ghost quarks \( k \), but not on \( n \) and \( k \) separately. Physically this reflects the fact that physical quantities should not be changed upon introduction of an extra set of degenerate quark and ghost quark pair, as the contribution should cancel out completely. To sum up, Eq. (30) is the mass correction to large \( N_c \) baryons in PQ\( \chi \)PT. Note that this single expression works for all states with spin \( J \) as long as \( J \ll N_c \).

Before we move on, we note that our result can be expressed in terms of just \( I_1 \), the same functional form as standard \( \chi \)PT results. The “new chiral singularities” or “quenched infrared divergences” in Q\( \chi \)PT do not appear except the limit \( \Delta n \to 0 \). This is in agreement with the Bernard–Golterman’s third theorem [6], which states that quenched infrared divergences appear if and only if one or more of the valence quarks are fully quenched. Since the theory we are considering is only partially quenched, we do not see the new chiral singularities.

### 3.2 The Large-\( N_c \) Decomposition of Chiral Corrections

So far, we have analyzed the chiral one-loop correction to baryon mass spectrum without explicit reference to large-\( N_c \) counting. In this section, we will decompose the chiral one-loop correction explicitly as a function of three distinct parameters: chiral symmetry breaking parameter \( m_\pi \), planar symmetry parameter \( 1/N_c \) and quenching parameter \( \Delta n \).

The chiral mass correction Eq. (30) contains three terms with different \( N_c \) and \( \Delta n \) dependences:

\[ \Delta m = \Delta m_{(+1,0)} + \Delta m_{(0,+1)} + \Delta m_{(-1,0)} + \Delta m_{(-1,-1)}, \] (31)

with

\[ \Delta m_{(+1,0)} = \frac{3}{8} \cdot g^2 \cdot N_c^2 \cdot I_1(m^2) \sim O\left( N_c^1, \Delta n^0 \right), \] (32)

\[ \Delta m_{(0,+1)} = \frac{3}{4} \cdot g^2 \cdot \Delta n \cdot N_c \cdot I_1(m^2) \sim O\left( N_c^0, \Delta n^1 \right), \] (33)
\[ \Delta m_{(-1,0)} = \frac{1}{2} \cdot g^2 J^2 \cdot \mathcal{I}_1(m^2) \sim \mathcal{O}\left(N_c^{-1}, \Delta n^0\right) \] (34)

\[ \Delta m_{(-1,-1)} = g^2 J^2 \cdot \frac{2}{\Delta n} \cdot (\mathcal{I}_1(M^2) - \mathcal{I}_1(m^2)) \sim \mathcal{O}\left(N_c^{-1}, \Delta n^{-1}\right). \] (35)

where we have pulled out implicit $1/N_c$ dependence through $1/f^2$ in our definition of $\mathcal{I}_1$. We immediately observe that the terms that depend on $J^2$, which contribute to $\Delta-N$ mass splitting, appear only at order $\mathcal{O}(N_c^{-1})$. This is in accordance with the large $N_c$ counting given in Ref. \[22, 23, 24, 25, 26, 27\].

We can get a better understanding by identifying the diagrams behind these four terms. The first term $\Delta m_{(+1,0)}$, is the dominant contribution in the large $N_c$ limit. It comes from Fig. 2c, where a pion attaches to different quark lines at the two vertices. There is a combinatoric factor $N_c$ at each vertex. As a result the sum is proportional to $N_c^2/f^2 \sim N_c$. Since there is no quark loop in these diagrams, they are completely unaffected by the introduction of ghosts. Hence we come to the conclusion that, at leading order of $1/N_c$, chiral one-loop corrections to baryon masses are independent of the degree of quenching. Fig. 2c contributes at both the fully quenched and the unquenched limits, in sharp constrast to the PQ$\chi$PT for mesons, where the diagrams appear in the fully quenched limit always vanish in the unquenched limit, and vice versa. In passing, we note that the higher order correction $\Delta m_{(-1,0)}$ also comes from the Fig. 2c.

The term $\Delta m_{(0,+1)}$ comes from Fig. 2a, where the pion attaches to the same quark line at both vertices. As a result, the combinatoric factor is $N_c$, and the whole term is of the order $N_c/f^2 \sim N_c^0$. Fig. 2a has an internal loop, which may be a quark or a ghost. As a result, it is sensitive to quenching, hence, proportional to $\Delta n$. Lastly, $\Delta m_{(-1,-1)}$ is the $\eta'$ loop contribution. It is $1/N_c^2$ suppressed with respect to the leading pion loop correction as discussed before, and the $1/\Delta n$ factor comes from the hairpin term in the propagator. Actually, $\Delta m_{(-1,-1)}$ contains implicit suppression factors as the hairpin diagrams are OZI, hence, $1/N_c$ suppressed. We will study this in detail in section 3.5.

### 3.3 Quenching-Senstivity of Leading Order Chiral Corrections

In the previous section, we have shown that at leading order of $1/N_c$, chiral one-loop corrections to baryon masses are independent of $\Delta n$. A casual reader may suggest that this result is a rather trivial observation. Isn’t it true that, since internal quark loops are suppressed by $1/N_c$ \[20\], the effect of quenching must vanish in the large $N_c$ limit uniformly for any degree of quenching, hence, our result trivially follows?
The above line of reasoning actually turns out fallacious. While it is true that quark loops are suppressed by $1/N_c$, it is definitely not true that the chiral corrections are independent of the degree of quenching at leading order correction of $1/N_c$ expansion. There are numerous counter-examples that exhibit large $N_c$ corrections that depend sensitively on the degree of quenching. As mentioned above, the mass of the $\rho$ meson receives a chiral correction which is proportional to $m^3$ in unquenched chiral perturbation theory, but this piece is absent in the quenched theory [9], and is in fact proportional to $\Delta n$ in the partially quenched theory [10]. Similarly, the chiral correction to heavy meson mass is non-zero in the unquenched theory but vanish in the fully quenched limit [11, 12].

To see flaws in the above argument, note that chiral corrections always come from one Goldstone boson loop level. For the case of the $\rho$ meson and heavy mesons, all Goldstone boson loop diagrams always contain a quark loop. Hence the quark loop enter at leading order of the $1/N_c$ expansion, and hence the chiral corrections are sensitive to the degree of quenching. On the other hand, there are Goldstone boson loop diagrams contributing the baryon mass correction which does not contain any internal quark loop (Fig. 2c), which is also the leading order contribution in large $N_c$. As a result, the leading order result $\Delta m_{(+1,0)}$ is $\Delta n$-independent.

From the above discussion, it should be clear that the question whether the leading order (in $1/N_c$) chiral correction of a given matter field is quenching-sensitive is highly non-trivial. It simply is not true that for the chiral corrections for mesons are always quenching-sensitive while those for baryons are quenching-independent. In fact one can show that the chiral corrections of flavor non-singlet mesons and baryons with two or more light quarks are always quenching-sensitive [28].

Lastly, we will discuss the implication of the quenching-independence of $\Delta m_{(+1,0)}$. We have shown that the effects of quenching are $1/N_c$ suppressed. While we cannot prove that these $1/N_c$ suppressed effects are actually negligible (they may come with huge coefficients), it is a well-known rule of thumb in hadron phenomenology that $1/N_c$ corrections are usually small in comparison with the leading order result if the latter is non-vanishing. In this case, this rule of thumb provides circumstantial evidence that the chiral corrections to baryon masses are small, and gives us more confidence in the utility of quenched QCD as an approximation of the real world of QCD.

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3.4 Fully Quenched QCD Limit: $\Delta n = 0$

As recalled earlier, the partially quenched QCD can interpolate and bridge between the two extreme limits, viz. fully quenched and unquenched QCD theories by dialing the analytic parameter $\Delta n$ between zero and $n$. For example, one would expect that by setting $\Delta n = 0$ the previously known Q$\chi$PT results should be recovered. In particular, strongly infrared singular non-analytic corrections should reappear. In this section, we will study the fully quenched limit carefully to understand how these quenched chiral singularities arise.

First of all, we note that, in the fully quenched limit, $\Delta n \to 0$, the contributions $\Delta m_{(+1,0)}$ and $\Delta m_{(-1,0)}$ remain intact and $\Delta m_{(0,+1)}$ vanishes identically. On the other hand, the contribution $\Delta m_{(-1,-1)}$ apparently diverges in this limit. Note, however, that $\Delta m_{(-1,-1)}$ can be re-expressed as

$$\Delta m_{(-1,-1)} = g^2 J^2 \cdot \frac{2}{\Delta n} \cdot \left( \frac{1}{1 + \Delta n \cdot A_0} \cdot \mathcal{I}_1 \left( \frac{m^2 + \Delta n \cdot m_0^2}{1 + \Delta n \cdot A_0} \right) - \mathcal{I}_1(m^2) \right). \quad (36)$$

As $\Delta n \to 0$, the contribution can be re-expressed as a derivative with respect to the interpolating parameter $\Delta n$:

$$\Delta m_{(-1,-1)} = 2g^2 J^2 \frac{d}{d\Delta n} \left[ \frac{1}{1 + \Delta n \cdot A_0} \cdot \mathcal{I}_1 \left( \frac{m^2 + \Delta n \cdot m_0^2}{1 + \Delta n \cdot A_0} \right) \right]_{\Delta n=0} \quad (37)$$

Now, it should become clear how the quenched infrared singularity arises. While the contributions from both the pion pole and the shifted pole have the same functional form as the one denoted as $\mathcal{I}_1$, in the fully quenched limit, the shifted pole returns back to the unshifted position and gives rise to a derivative contribution, which is not of the form $\mathcal{I}_1$. In fact, this yields the precise statement of Bernard–Golterman’s third theorem for baryons in the fully quenched limit. Expanding the derivative in Eq. (37) explicitly, one finds that

$$\frac{d}{d\Delta n} \left[ \frac{1}{1 + \Delta n \cdot A_0} \cdot \mathcal{I}_1 \left( \frac{m^2 + \Delta n \cdot m_0^2}{1 + \Delta n \cdot A_0} \right) \right]_{\Delta n=0} = \frac{1}{12\pi f^2} \cdot \left( \frac{3}{2} \cdot m_0^2 \cdot m - \frac{5}{2} \cdot A_0 \cdot m^3 \right) \equiv \mathcal{I}_2(m^2). \quad (38)$$

Note the appearance of the term linear in $m$. This is the anticipated contribution of non-analytic infrared singularity in quenched QCD. To summarize, we conclude that the mass correction in the fully quenched limit is reproduced correctly and is given by

$$\Delta m = g^2 \left( \frac{3}{8} N_c^2 + \frac{1}{2} J^2 \right) \cdot \mathcal{I}_1(m^2) + 2J^2 \cdot \mathcal{I}_2(m^2). \quad (39)$$
3.5 Large-\(N_c\) Suppression of \(\Delta m_{(-1,-1)}\)

As mentioned before, the hairpin diagram is OZI suppressed. As a result, the hairpin parameters, \(A_0\) and \(m_0^2\), are of order \(1/N_c\). In light of this piece of new information, we will reanalyze the \(1/N_c\) and \(\Delta n\) dependence of \(\Delta m_{(-1,-1)}\), the term which originates from the hairpin propagators.

To make the \(N_c\) dependences explicit, rescale the hairpin parameters as

\[
m_0^2 = \bar{m}_0^2/N_c, \quad A_0 = \bar{A}_0/N_c,
\]

where \(\bar{m}_0^2\) and \(\bar{A}_0\) have smooth non-trivial large \(N_c\) limit. Then the mass correction becomes

\[
\Delta m_{(-1,-1)} = g^2 J^2 \cdot \left( \frac{2}{\Delta n} \right) \cdot \left[ \left( \frac{1}{1 + (\Delta n/N_c)A_0} \right) \cdot I_1 \left( \frac{m^2 + (\Delta n/N_c)\bar{m}_0^2}{1 + (\Delta n/N_c)A_0} \right) - I_1(m^2) \right]. \tag{41}
\]

This can be expanded in a Taylor series of \(\Delta n/N_c\),

\[
\Delta m_{(-1,-1)} = 2g^2 J^2 \sum_{k=1}^{\infty} \frac{(\Delta n/N_c)^k}{k!} \cdot \frac{d^k}{d(\Delta n/N_c)^k} \left[ \frac{1}{1 + (\Delta n/N_c)A_0} \cdot I_1 \left( \frac{m^2 + (\Delta n/N_c)\bar{m}_0^2}{1 + (\Delta n/N_c)A_0} \right) \right]_{(\Delta n/N_c)=0} = \frac{2g^2 J^2}{N_c} \left[ \bar{I}_2(m^2) + \frac{\Delta n}{N_c} \bar{I}_3(m^2) + \cdots \right], \tag{42}
\]

where \(\bar{I}_{2,3}\) are defined as

\[
\bar{I}_2(m^2) \equiv \frac{d}{d(\Delta n/N_c)} \left[ \frac{1}{1 + (\Delta n/N_c)A_0} \cdot I_1 \left( \frac{m^2 + (\Delta n/N_c)\bar{m}_0^2}{1 + (\Delta n/N_c)A_0} \right) \right]_{(\Delta n/N_c)=0} = \frac{1}{12\pi f^2} \left( \frac{3}{2} \cdot \bar{m}_0^2 \cdot m - \frac{5}{2} \cdot \bar{A}_0 \cdot m^3 \right), \tag{43}
\]

\[
\bar{I}_3(m^2) \equiv \frac{d^2}{d(\Delta n/N_c)^2} \left[ \frac{1}{1 + (\Delta n/N_c)A_0} \cdot I_1 \left( \frac{m^2 + (\Delta n/N_c)\bar{m}_0^2}{1 + (\Delta n/N_c)A_0} \right) \right]_{(\Delta n/N_c)=0} = \frac{1}{12\pi f^2} \left( \frac{3}{4} \cdot \bar{m}_0^4 \cdot \frac{1}{m} - \frac{15}{2} \cdot \bar{A}_0 \cdot m^2 \cdot m - \frac{35}{4} \cdot \bar{A}_0^2 \cdot m^3 \right) \tag{44}
\]

From Eq. (42) one can see clearly that the leading \(q'\) correction is of order \(g^2/f^2N_c \sim 1/N_c^2\), viz. suppressed by one higher order in \(1/N_c\) than we have naively expected. Moreover, all the higher order corrections are in positive powers of \(\Delta n\), which vanish identically in the fully quenched limit. The real significance of this result is that, in the leading order of an \(1/N_c\) expansion which keeps only the first term of Eq. (42), \(\Delta m_{(-1,-1)}\) is of the form of \(\bar{I}_2(m_{\pi}^2)\), a form which leads to new chiral divergences such as \(m_q^{1/2}\), even when the theory is far from
the fully quenched limit $\Delta n \to 0!$ In other word, through taking the large $N_c$ limit, one can bypass the Bernard–Golterman’s third theorem, and have quenched chiral singularities even if the theory is only partially quenched. Note that our result is not in contradiction with the Bernard–Golterman’s third theorem: the analysis of Ref. [6] is for $N_c = 3$. In fact, the third theorem holds for any fixed value of $N_c$ but fails only when $N_c$ is taken strictly to infinity. It is important to understand that these quenched infrared singularities appear when the shifted pole coincide with the pion pole. Since the shift of the pole is proportional to $\Delta n / N_c$, these new chiral divergences appear both at the fully quenched limit and also at large (but not infinite) $N_c$ limit.

We note in passing that the higher order terms in the Taylor expansion Eq. (42) are more singular in the chiral limit. In the chiral limit, while $I_1$ itself goes like $m^3$, $\tilde{I}_2$ and $\tilde{I}_3$ behave like $1/m$ respectively. It is easy to verify that each term will be more singular by a factor of $1/m^2$. If we attempt to take the chiral limit in expansion Eq. (42) before taking the large $N_c$ limit, all the terms (except $\tilde{I}_2$) will diverge, and the series will be a sum of infinitely many infinities. However, all these divergences are not physical, as the whole series can be exactly resummed into the closed form expression Eq. (41). The infinities just assert that the true expansion parameter in the Taylor series is $(\Delta n / N_c)(\tilde{m}_0^2 / m^2)$, which diverges in the chiral limit. As a result, the Taylor expansion has behaved badly. In other words, the chiral divergences of $\tilde{I}_k (k > 2)$ are merely an artifact of applying an expansion outside its radius of convergence (zero in this case), while all physically relevant quantities can be obtained from Eq. (41), which has indeed a well-defined chiral limit.

This above discussion poses an important implication on whether we can really see the $m \sim m_0^{1/2}$ quenched chiral singularities from lattice data. The $\tilde{I}_2$ term dominates the Taylor expansion only when $m^2 \geq (\Delta n / 2N_c)(\tilde{m}_0^2) = \Delta nm_0^2 / 2$, and $\Delta m_{(-1,-1)}$ is linear in $m$. With small pion mass $m^2 \leq \Delta nm_0^2 / 2$, however, $\Delta m_{(-1,-1)}$ will deviate from linearity in $m$, as the higher $\tilde{I}$ are no longer negligible. On the other hand, $\Delta m_{(0,-1)}$ is usually dominated by $\Delta m_{(+1,0)}$ except at very low pion masses, i.e., $m^2 \leq 8 J^2 m_0^2 / N_c^2$. So we can observe $\Delta m$ with a linear dependence on $m$ only over the window

$$\Delta nm_0^2 / 2 \leq m^2 \leq 8 J^2 m_0^2 / N_c^2, \quad (45)$$

and such a window exists only if

$$\Delta n \leq 16 J^2 / N_c^2. \quad (46)$$

This condition is independent of the value of the parameter $m_0^2$, and is marginally satisfied for $\Delta n = 1, N_c = 3$ and $J = 1/2$. While this is only an order of magnitude estimate, it suggests that it would be difficult to see the linear dependence; we expect that of the Bernard–Golterman’s
third theorem is bypassed only when $N_c \to \infty$, but in that limit the right hand side of Eq. (46) goes to zero, and the window just disappears. Physically it means that, when $N_c \to \infty$, the linear term is overwhelmed by the leading order term $\Delta m_{(+1,0)}$, which is larger than the linear term by a factor of $N_c^3$. We conclude by restating that, while the Bernard–Golterman’s third theorem is bypassed in the large $N_c$ limit, it would probably be difficult to observe in real lattice data.

3.6 Unquenched QCD Limit – $k = 0$

Let us now consider the conventional QCD with no ghost quarks by taking the limit $k \to 0$, viz. $\Delta n = n$. We continue working on theory with two flavors only. Thus, Eq. (30) becomes

$$\Delta m = g^2 \left[ \left( \frac{3}{8} N_c^2 + \frac{3}{4} n N_c + \left( \frac{1}{2} - \frac{2}{n} \right) J^2 \right) \cdot I_1 (m^2) + J^2 \cdot \frac{2/n}{1 + n \cdot A_0} \cdot I_1 \left( \frac{m^2 + n \cdot m_0^2}{1 + n \cdot A_0} \right) \right].$$

(47)

Note that the term proportional to $I_1 (m^2)$ is exactly the loop correction with one-pion exchange in the standard $n = 2 \chi$PT. The interpretation of the shifted pole is in the second term is that it is the reinstatement of the contribution of $\eta'$ meson. The shift of the pole just reflects the fact that, in the real world, the $\eta'$ mass is shifted from the pion mass due to the nonperturbative resummation of necklace diagrams.

To see this clearer, let us rearrange Eq. (30) in the following way:

$$\Delta m = g^2 \cdot \left( \frac{3}{8} N_c^2 + \frac{3}{4} n N_c + \left( \frac{1}{2} - \frac{2}{n} \right) J^2 \right) \cdot I_1 (m^2) + g_{\eta'BB}^2 \cdot I_1 (m_{\eta'}^2),$$

(48)

in which

$$g_{\eta'BB}^2 \equiv 2 J^2 g^2 / Z n, \quad m_{\eta'}^2 \equiv (m^2 + n \cdot m_0^2) / Z, \quad \text{where} \quad Z \equiv (1 + n \cdot A_0).$$

(49)

These represent coupling, mass and wave function renormalizations of the $\eta'$ meson. Note that the expression for $m_{\eta'}$ and $Z$ agrees with the standard $\chi$PT results in Eq. (1).

The second term in Eq. (48) represents the contribution of $\eta'$-meson to the chiral one-loop correction to the baryon mass spectrum. We thus conclude that the unquenched limit $k = 0$ does not reduce to the $\chi$PT in which only octets of Goldstone mesons are retained. Instead we have reproduced the $\chi$PT without integrating out the $\eta'$ meson. This alerts us that we have to be careful when applying the Bernard-Golterman’s first theorem, which states that in the subsector where all valence quarks are unquenched, the SU($n|k$) theory is completely equivalent to a normal, completely unquenched SU($n - k$) theory. We found that this is indeed true, but
unlike the standard $\chi$PT, this “completely unquenched SU($n-k$) theory” contains an $\eta'$ meson which may (and does) contribute to chiral loop corrections. As a result, naive comparisons between results of (P)Q$\chi$PT and their counterparts in standard $\chi$PT may be problematic or misleading. One should instead compare results of (P)Q$\chi$PT with their counterparts in $\chi$PT with $\eta'$ loops.

In passing we observe that it is impossible to compare results of standard $\chi$PT with their counterparts in (P)Q$\chi$PT with the $\eta'$ integrated out. In (P)Q$\chi$PT, the $\eta'$ propagator always has a single pole at $m^2$ except for the unquenched limit $k = 0$. Consequently, there is no $\eta'$-impossible to integrate out the $\eta'$ meson. So the scheme suggested above, namely comparing results of (P)Q$\chi$PT with their counterparts in $\chi$PT with $\eta'$ loops, is the only option which is theoretically well-justified.

4 Discussion

4.1 $1/N_c$ Corrections

So far, we have kept only contributions that are leading order in $1/N_c$ expansion. In this section we will discuss several sources of higher-order $1/N_c$ corrections we have ignored so far. First of all, there are corrections to the axial current couplings to baryons through multi-quark operator. It has been shown [26] that the leading order correction to pion coupling is through the two-quark operator

$$\frac{1}{N_c} g' f \left( \partial^i \pi^a \right) \sum_{k,l=1}^{N_c} \left( q_k^\dagger (S^i \otimes 1) q_k \right) \left( q_l^\dagger (1 \otimes T^a) q_l \right),$$

where $g'$ is an undetermined coupling constant of order $O(N_c^0)$. This will lead to new single-pion loop corrections: a $gg'/f^2$ term at the order of $O(N_c^{-1})$, and a $g^2/f^2$ term at the order of $O(N_c^{-3})$. Since these terms are sub-leading with respect to the leading pion contribution $\sim O(N_c)$, they can be safely ignored in the large $N_c$ limit. What cannot be ignored is the $\eta'$ counterpart. By planar symmetry, $\eta'$ can also couple to a baryon through the following two-quark operator:

$$\frac{1}{N_c} g' f \left( \partial^i \eta' \right) \sum_{k,l=1}^{N_c} \left( q_k^\dagger (S^i \otimes 1) q_k \right) \left( q_l^\dagger (1 \otimes 1) q_l \right) = g' f \left( \partial^i \eta' \right) \sum_{k,l=1}^{N_c} \left( q_k^\dagger (S^i \otimes 1) q_k \right),$$

which is of the same form as the leading order $\eta'$ coupling. In a consistent $1/N_c$ expansion, one has to keep the effect of $g'$, which contribute in leading order ($O(N_c^{-1})$) to the quenched
infrared divergences $\Delta m_{(-1,-1)}$:

$$
\Delta m_{(-1,-1)} = (g + g')^2 J^2 \frac{2}{\Delta n} (I_1(M^2) - I_1(m^2)) \sim \mathcal{O}(N_c^{-1} \Delta n^{-1}).
$$

(52)

This does not change the form of the new chiral singularities, but its coefficient is no longer related to the standard $\chi$PT correction ($\Delta m_{(1,0)}$).

Another possible source of correction is the inclusion of the hairpin coupling $h$, which is again of order $1/N_c$. We have not calculated $\Delta m$ in the presence of a non-zero hairpin coupling, which should be straightforward but adds significantly more complications. However, the functional form of $\Delta m$ is not affected by the inclusion of $h$, as the contributing Feynman diagrams are still of the same form as those shown in Fig. 1. Hence, we expect that only the overall factor $g^2$ will be modified:

$$
g^2 \rightarrow (g + xh)^2.
$$

(53)

Unfortunately, the variable $x$ may be different for different contributions for $\Delta m$, i.e., $x$ for $\Delta m_{(+1,0)}$ may be different from that of $\Delta m_{(-1,-1)}$. However, since $h$ is $1/N_c$ suppressed with respect to $g$, the change will only be higher order effects in a $1/N_c$ expansion. In any case, our analysis of the fully quenched and the unquenched limit depends only on the functional form taken by $\Delta m$. The coupling constants are irrelevant for this analysis, and we expect the reported results remain valid even with a non-zero hairpin coupling, even though the details have to be checked.

### 4.2 Conclusion

In this paper, we have studied baryons in partially quenched large-$N_c$ QCD with particular emphasis to the interpolation between fully unquenched and fully quenched limits. In the large $N_c$ limit of partially quenched QCD, we have calculated $\Delta m$, the chiral one-loop correction to baryon masses. The main results are:

- The leading order contribution to $\Delta m$ is of order $\mathcal{O}(N_c)$, in agreement of the large $N_c$ counting rules. This contribution does not depend on the value of $\Delta n$, which survives both in the fully quenched and the unquenched limits. In other words, the leading term of $\Delta m$ is independent of number of ghost quarks introduced. This provides circumstantial evidence that quenching correction to baryon masses is small.

- For any fixed $N_c$ our expression for $\Delta m$ does satisfy the Bernard–Golterman’s third theorem and has no quenched infrared singularities except $\Delta n \rightarrow 0$ limit. In the large $N_c$ limit,
however, the third theorem is bypassed. We indeed find appearance of the quenched chiral
corrections, which are unfortunately of order $O(N_c^{-2})$ and $1/N_c^3$ suppressed with respect to
the leading order contribution to $\Delta m$ and hence making them difficult to be observed.

- In the unquenched limit, $\chi$PT results are reproduced with $\eta'$ loop contributions.

Unfortunately, it is highly non-trivial to relate our result with that of Ref. [8], which has
studied $Q\chi$PT for baryons with $n = k = 3$ and $N_c = 3$. The reason is that it turns out that
the spin-flavor structure of large $N_c$ baryons with $n = 3$ are much more complicated than that
in $n = 2$. In particular, the large $N_c$ limit of the ratios between the SU(3) coupling constants
$F$, $D$, $C$ and $H$ in Ref. [8] are ambiguous [22]. The study and clarification of these issues are
straightforward, but is beyond the scope of the present paper. Being that the motivation and
the focus of our study in the present paper are of more theoretical issues than phenomenological
ones, detailed comparison of our results with those of Ref. [8] will be relegated in a separate
paper elsewhere.

We will end the paper by discussing possible relevance of our studies. Our motivations are
mainly theoretical and we have tried to disentangle the interplays between chiral dynamics, large
$N_c$ expansion and quenching. On the other hand, while most of the lattice QCD simulation are
performed with $N_c = 3$, there are also simulations with arbitrary $N_c$, most notably by Teper
[23]. So far, these studies have focused on pure Yang-Mills theory and have studied string
tension and glueball masses. However, new data on large $N_c$ QCD should become available in
the future. More phenomenologically, there has been a lot of investigation of baryon properties
in the large $N_c$ limit [22, 23, 24, 25, 26, 27], with many new interesting results. For example, the
$\Delta$-N splitting is expected to be of order $1/N_c$. While these results are successful in organizing
the baryon properties, one would like to directly verify these predictions with the $\Delta$-N splitting
for different values of $N_c$. This, however, is only possible with lattice simulations. Our studies
should be relevant if these simulations are really undertaken in the foreseeable future. In
conclusion, we are using the $1/N_c$ expansion as a guiding principle to help identifying and
organizing spectrums and chiral corrections when one works on quenched lattice world, hoping
that $1/N_c$ expansion is as successful in hadrons phenomenology on the lattice as it has been in
the real world.

References

[1] H. Hamber and G. Parisi, Phys. Rev. Lett. 47 1792 (1981).
[2] D. Weingarten, Phys. Lett. **109B** 57 (1982).

[3] S.R. Sharpe, Phys. Rev. **D41** 3233 (1990); *ibid.* Nucl. Phys. [Proc. Suppl.] **B17** 146 (1990).

[4] S.R. Sharpe, Phys. Rev. **D46** 3146 (1992); *ibid.* Nucl. Phys. [Proc. Suppl.] **30** 213 (1993).

[5] C.W. Bernard and M.F. Gotterman, Phys. Rev. **D46** 853 (1992).

[6] C.W. Bernard and M.F. Gotterman, Phys. Rev. **D49** 486 (1994).

[7] A. Morel, J. Physique **48** 111 (1987).

[8] J.N. Labrenz and S.R. Sharpe, Phys. Rev. **D54** 4595 (1996).

[9] M. Booth, G. Chiladze and A.F. Falk, Phys. Rev. **D55** 3092 (1997).

[10] C.K. Chow and S.-J. Rey, *hep-ph/9708432*.

[11] M. Booth, Phys. Rev. **D51** 2338 (1995).

[12] S.R. Sharpe and Y. Zhang, Phys. Rev. **D53** 5125 (1996).

[13] M. Golterman and K.C. Leung, Phys. Rev. **D56** 2950 (1997).

[14] M. Kim and S. Kim, *hep-lat/9608091*.

[15] G. Chiladze, *hep-lat/9704428*.

[16] C. Bernard and M. Golterman, Phys. Rev. **D53** 476 (1996).

[17] S. Sharpe, Nucl. Phys. [Proc. Suppl.] **53** 181 (1997).

[18] S. Gottlieb, Nucl. Phys. [Proc. Suppl.] **53** 155 (1997).

[19] M. Okawa, in the Proceedings of the XVth International Conference on Lattice Field Theory, Edinburgh, Scotland (1997).

[20] G."t Hooft, Nucl. Phys. **B72** 461, (1974).

[21] E. Witten, Nucl. Phys. **B160** 57 (1979).

[22] R.F. Dashen, E. Jenkins and A.V. Manohar, Phys. Rev. **D49** 4713 (1994), *erratum D51* 2489 (1995).

[23] C. Carone, H. Georgi and S. Osofsky, Phys. Lett. **322B** 227 (1994).

[24] M.A. Luty, J. March-Russell, Nucl. Phys. **B426** 71 (1994).
[25] R.F. Dashen, E. Jenkins and A.V. Manohar, Phys. Rev. D51 3697 (1995).

[26] E. Jenkins, Phys. Rev. D53 2625 (1996).

[27] H. Georgi, in “Chiral Dynamics: Theory and Experiment; Proceedings, Cambridge MA 1994” edited by A.M. Berstein and B.R. Holstein.

[28] C.K. Chow, hep-lat/9711375, to appear in Phys. Rev. D.

[29] M. Teper, Phys. Lett. B397 223 (1997).