Kenfack – Źyczkowski indicator of nonclassicality for two non-equivalent representations of Wigner function of qutrit

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Abstract

Following Kenfack and Źyczkowski, we consider the indicator of nonclassicality of quantum states for \( N \)–level systems defined via the integral of the absolute value of the Wigner function. For these systems, remaining in the framework of Stratonovich-Weyl correspondence, one can construct a whole family of representations of the Wigner functions defined over the continuous phase-space and characterized by a set of \( (N - 2) \) moduli parameters. It is shown that the nonclassicality indicator, being invariant under the \( SU(N) \) transformations of states, turns to be sensitive to the representation of the Wigner function. We analyse this representation dependence computing the Kenfack-Życzkowski indicators for pure and mixed states of a 3-level system using a generic and two degenerate Stratonovich-Weyl kernels respectively. Our calculations reveal three classes of states: the “absolutely classical/quantum” states, which have zero and non-vanishing indicator for all values of the moduli parameters correspondingly, and the “relatively quantum-classical” states whose classicality/quantumness is susceptible to a representation of the Wigner function. Herewith, all pure states of qutrit belong to the “absolutely quantum” states.
1. Introduction

- **Negativity of Wigner Function** - The Wigner function is a famous member of a peculiar class of distributions, the so-called *quasiprobability distributions*, which have the prefix “quasi” in their names because they do not conform to the basic principle of true statistical distributions of being non-negative. This anomaly is archetypal for all quantum systems: “continuous” and “discrete”.

For the first type of systems whose states are represented by density matrices \( \rho \) acting on the space of square-integrable functions \( L^2(\mathbb{R}) \) the Wigner function is defined over a 2-dimensional phase space with canonical coordinates \((x, p) \in \mathbb{R}^2\):

\[
W_{\rho}(x, p) = \frac{1}{\pi \hbar} \int_{-\infty}^{\infty} dy \langle x + y | \rho | x - y \rangle e^{-2ipy/\hbar},
\]

(1)

with the well known bounds \([1, 2]\):

\[-\frac{1}{\pi \hbar} \leq W(x, p) \leq \frac{1}{\pi \hbar}.\]

(2)

Moreover, for the Wigner function defined on 2n-dimensional phase space, the integrals over a domain \( D \in \mathbb{R}^{2n} \) are bounded by its volume \([3]\):

\[- \frac{1}{(\pi \hbar)^n} \text{Volume}(D) \leq \int_D [dx dp]_{2n} W(x, p) \leq \frac{1}{(\pi \hbar)^n} \text{Volume}(D).\]

(3)

Similarly, for the Wigner functions associated to a “discrete” \( N \)-level quantum system, whose Hilbert space is \( \mathcal{H} = \mathbb{C}^N \), the analog of the bounds \([2, 3]\) exists. Particularly, in the framework of the Weyl-Stratonovich formalism (see, e.g., [4][11] and references therein), it was shown in [12] that the Wigner quasiprobability distributions \( W_{\rho}(\Omega_N) \) defined over the phase-space \( \Omega_N \) obey inequalities:

\[
\sum_{i=1}^{N} \pi_i r_{N-i+1} \leq W_{\rho}(\Omega_N) \leq \sum_{i=1}^{N} \pi_i r_i,
\]

(4)

where \( r_1, r_2, \ldots, r_N \) are eigenvalues of a mixed state \( \rho \) and \( \pi_1, \pi_2, \ldots, \pi_N \) denote the eigenvalues of the Stratonovich-Weyl kernel with each set arranged in a decreasing order.

The bounds \([2, 3]\), being functions of states and the Stratonovich-Weyl kernels eigenvalues, are exact in a sense that they are attainable at certain
points of the phase space. Particularly, the subset of phase space where the Wigner function acquires negative values might be not empty. [4]

- **Wigner Functions Negativity vs. Classicality** • Faced with the negativity of probability distribution, Wigner in his 1932 paper [1] wrote: “But of course this must not hinder the use of it in calculations as an auxiliary function which obeys many relations we would expect from such a probability”. This guideline turned out to be foresighted. Almost a century of history of the method of quasiprobability distributions gave us an effective tool for analyzing quantum phenomena in variety of research areas including quantum optics [13, 15], quantum theory of information and communications [16, 17]. It turns out that during this time a drastic metamorphosis happened in perception of the negative distributions: from an “auxiliary” probability function up to proclamation of quasidistributions as a basic ingredient in quantifying the degree of quantumness (cf. recent discussions in [16, 18–20]).

Relations between nonclassicality and Wigner function negativity become highly intricate taking into consideration existence of infinitely many quasiprobability distributions for a given quantum state. Nowadays this aspect has drawn a wide attention especially in connection with a special class of the so-called discrete Wigner functions [21, 22]. Particularly, very important findings has been done by R.W. Spekkens about the interplay between negativity and nonclassicality. It was demonstrated that the negativity is “neither a necessary nor a sufficient condition for the failure of classical explanation” [23]. Moreover, it was proved that for the discrete Wigner functions the negativity is equivalent to the quantum contextuality and the role of the contextualization of negativity of the Wigner function in a resource theoretical framework of non-Gaussianity was intensively discussed (see e.g., discussions in [23–28]). Below, having in mind these results known for the discrete Wigner functions, we are going to discuss

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1Definitely, there are states such that the bounds (2) and (3) are not optimal and the lower bound of the Wigner function for certain states can be positive. However, in view of the well-known Hudson’s theorem [13], a positive definiteness occurs only for very special classes of states, e.g., a Gaussian wave function is the only pure state corresponding to a positive Wigner function.
similar issues for the Wigner quasidistributions defined on a continuous phase-space. More precisely, an analysis of how the variety of the Wigner functions representations affects a special measure of nonclassicality will be given.

We focus here mainly on the three level quantum systems since the smallest system revealing contextuality is the qutrit [29–31].

- **Degree of Negativity as Measure of Quantunness** - Nonclassicality measures based on the violation of the Wigner function semi-positivity can be divided into two different types. If one takes states with positive Wigner functions as the reference ”classical states” then the measures of nonclassicality are based either on the distance from the set of “classical states” [14, 32, 33] or on the volume of a phase space region where the Wigner function is negative [12, 34]. In the present note, following the second approach, we will make use of the volume indicator of nonclassicality introduced by A. Kenfack and K. Życzkowski [34] for an $N$-dimensional system admitting a Wigner function defined over compact continuous phase space $\Omega_N$:

$$\delta(\varrho) = \int_{\Omega_N} d\Omega_N |W_\varrho(\Omega_N)| - 1.$$  \hspace{1cm} (5)

In definition (5), the notation $| \cdot |$ stands for the absolute value (modulus) of the Wigner function. Hereinafter, the function $\delta(\varrho)$ will be termed as the KZ-indicator.

In the next sections, the results of calculation of the KZ-indicators of nonclassicality (5) for two- and three-level systems, qubits and qutrits respectively, will be given. In the case of qutrit we will analyze a functional dependence of the KZ-indicator on the choice of representation of the Wigner function. Our computations are based on the description of the admissible representations of the Wigner function using the set of the so-called moduli parameters introduced in [11]. The analyses shows that there is a special subset of “absolutely classical” states, such that their KZ-indicator is vanishing independently of the Wigner function representation. There is also a class of “absolutely quantum” states whose KZ-indicator is non-zero for all values of the Wigner function moduli parameter. Particularly, we will show that all pure states of qutrit belong to
the class of “absolutely quantum” states.

There are also “relatively quantum-classical” states whose classicality/quantumness depends on the representation of the Wigner function.

2. Generalities on Wigner function of N-level system

In this section, following presentations of [4, 5, 11, 35], we collect all necessary notations and definitions from the Stratonovich-Weyl approach to the Wigner quasiprobability distribution of a finite-dimensional system.

- The Stratonovich-Weyl principles
- Consider an $N$-level quantum system in a mixed state characterized by a density matrix $\rho$. Its expansion over the Hermitian basis $\lambda = \{ \lambda_1, \cdots, \lambda_{N^2-1} \}$, of $\mathfrak{su}(N)$ algebra with the orthonormality conditions, $\text{tr}(\lambda_\mu \lambda_\nu) = 2 \delta_{\mu\nu}$, reads

$$\rho = \frac{1}{N} I_N + \sqrt{\frac{N-1}{2N}} (\xi, \lambda), \quad (6)$$

where $\xi$ is $(N^2 - 1)$-dimensional Bloch vector.

The Wigner distribution $W_\rho(\Omega_N)$ of an $N$-dimensional quantum system as a function on symplectic space $\Omega_N$ is defined by pairing of a density matrix $\rho$ and the Stratonovich-Weyl kernel $\Delta(\Omega_N)$,

$$W_\rho(\Omega_N) = \text{tr}(\rho \Delta(\Omega_N)). \quad (7)$$

The kernel $\Delta(\Omega_N)$ in (7) obeys the following set of postulates, known under the name of Stratonovich-Weyl correspondence [4, 5]:

1. **Reconstruction**: a state $\rho$ is reconstructed from the Wigner function (7) via the integral over a phase space:

$$\rho = \int_{\Omega_N} d\Omega_N \Delta(\Omega_N) W_\rho(\Omega_N); \quad (8)$$

2. **Hermicity**: $\Delta(\Omega_N) = \Delta(\Omega_N)^\dagger$;

3. **Finite Norm**: a state norm is given by the integral of the Wigner distribution:

$$\text{tr}[\rho] = \int_{\Omega_N} d\Omega_N W_\rho(\Omega_N), \quad \int_{\Omega_N} d\Omega_N \Delta(\Omega_N) = 1; \quad (9)$$
4. **Covariance**: the adjoint unitary transformations $U(\alpha) \in SU(N)$ of a density matrix results in the kernel change:

$$\Delta'(\Omega) = U(\alpha)^\dagger \Delta(\Omega) U(\alpha),$$

(10)

via symplectic transformations $T_\alpha \in \text{Sp}(d_N)$ on $\Omega_N$.

$$\Delta'(z) = \Delta(z'), \quad z' = T_\alpha z.$$  

(11)

Here $z = \{z_1, z_2, \ldots z_{d_N}\} \in \Omega_N$ denote $d_N$ symplectic coordinates.

As it was shown in [11], the above axioms are fulfilled if the Hermitian kernel $\Delta(\Omega)$ in (7) satisfies the following set of algebraic “master equations”:

$$\text{tr}(\Delta(\Omega)) = 1, \quad \text{tr}(\Delta(\Omega)^2) = N.$$  

(12)

These equations determine the spectrum of Stratonovich-Weyl kernel $\text{spec}(\Delta_N) = \{\pi_1(\nu), \pi_2(\nu), \ldots, \pi_N(\nu)\}$ non-uniquely and thereby cause the existence of the variety of representations for the Wigner functions.

The corresponding moduli space of solutions to master equations [12] represents a spherical polyhedron on $(N - 2)$–dimensional sphere $S_{N-2}(1)$ of radius one. We denote the coordinates on moduli space by $\nu = (\nu_1, \cdots, \nu_{N-2})$ and hereafter will point to the corresponding functional dependence of the Stratonovich-Weyl kernel and the Wigner function explicitly. See more on the moduli space of the Stratonovich-Weyl kernel in [35].

The phase space $\Omega_N$ is determined by the symmetries of the Stratonovich-Weyl kernel. Assuming that the Stratonovich-Weyl kernel has a spectrum with the algebraic multiplicities $k = \{k_1, k_2, \ldots, k_s\}$ then the phase-space can be identified with a complex flag variety $\Omega_{N,k} \simeq F_N^k = U(N)/H_k$, where the isotropy group of the Stratonovich-Weyl kernel $H_k \in U(N)$ is of the form $^3$ $H_k = U(k_1) \times U(k_2) \times \cdots \times U(k_{s+1})$.

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2The rule [11] is an analogue of the well-known transformations generated by the metaplectic group $\text{Mp}(n)$ of operators acting on $L^2(\mathbb{R})$, cf. [29].

3The volume form on $\Omega_{N,k}$ is given by the bi-invariant normalised Haar measure $d\mu_{SU(N)}$ on $SU(N)$ group: $d\Omega_{N,k} = N \text{Vol}(H_k) d\mu_{SU(N)}/d\mu_{H_k}$, where $d\mu_{H_k}$ is the induced measure on the isotropy group $H_k$. 

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Finalising this section, we give the expression for the Wigner function of an $N$-dimensional quantum system in terms of the Bloch vector $\xi$ and a unit $(N^2 - 1)$-dimensional vector $n$ characterizing representative Stratonovich-Weyl kernel. Using (6) and the SVD decomposition of the Stratonovich-Weyl kernel $\Delta(\Omega_N|\nu)$:

$$\Delta(\Omega_N|\nu) = \frac{1}{N} U(\Omega_N) \left( I_N + \kappa \sum_{\lambda_s \in K} \mu_s(\nu) \lambda_s \right) U(\Omega_N)^\dagger.$$  \hspace{1cm} (13)

In (13) $\kappa = \sqrt{\frac{N(N^2-1)}{2}}$ is the normalization constant, $K \in su(N)$ is the Cartan subalgebra of $su(N)$ algebra and $(N - 1)$ real coefficients $\mu_s(\nu)$ are coordinates of points on a unit sphere $S_{N-2}(1)$

$$\mu_3^2(\nu) + \mu_8^2(\nu) + \cdots + \mu_{N^2-1}^2(\nu) = 1. \hspace{1cm} (14)$$

In these terms the Wigner function $W^{(\nu)}_{\xi}(\Omega)$ can be represented as (see details in [35])

$$W^{(\nu)}_{\xi}(\Omega) = \frac{1}{N} \left( 1 + \frac{N^2 - 1}{\sqrt{N + 1}} (\xi, n) \right), \hspace{1cm} (15)$$

where $(N^2 - 1)$-dimensional vector $n = \mu_3 n^{(3)} + \mu_8 n^{(8)} + \cdots + \mu_{N^2-1} n^{(N^2-1)}$ is a superposition of $(N - 1)$ orthonormal vectors whose components are determined by diagonalizing matrix,

$$n^{(s)} = \frac{1}{2} \text{tr} \left( U \lambda_s U^\dagger \lambda_{s} \right), \hspace{1cm} s = 3, 8, \ldots, N^2 - 1. \hspace{1cm} (16)$$

3. KZ-indicator as unitary invariant

Now we will discuss $SU(N)$ invariance of the indicator of nonclassicality originated from the unitary symmetry of a quantum system.

Below it will be argued that the KZ-indicator is a scalar function which depends only on the $SU(N)$ group invariants built out of a density matrix $\varrho$ and Stratonovich-Weyl kernel $\Delta(\Omega_N)$. This statement follows from the $SU(N)$ covariance properties of states and SW kernels. Indeed, according to the covariance axiom (10), the rule (11) ensures the following relation:

$$W_{U \varrho U^{-1}}(V^{-1} \Delta(z) V) = W_{\varrho}(\Delta(T_{V^{-1}})). \hspace{1cm} (17)$$
for all $U, V \in SU(N)$. Now, in order to prove the invariance of the KZ-indicator, it is convenient to consider the phase space $\Omega_N$ as an embedded subspace, $\Omega_N = SU(N)/H \subset SU(N)$, with some isotropy subgroup $H$. Then, since the Wigner function depends only on the coset coordinates $z \in \Omega_N$, one can extend the integration in (5) to the whole $SU(N)$ group as follows:

$$
\delta (\varrho | \Delta) = Z_N^{-1} \int_{SU(N)} d\mu_{SU(N)} \left| W_\varrho (\Omega_N) \right| - 1.
$$

Furthermore, identifying the measure $d\mu_{SU(N)}$ in (18) with the normalized bi-invariant Haar measure, one can fix the normalization constant, $Z_N = 1/N$. Hence, using the property (17) and representation (18), one can get convinced that the effect of the $SU(N)$ group action, $\varrho' = U\varrho U^{-1}$ and $\Delta' = V^{-1}\Delta V$, leaves the KZ-indicator unchanged,

$$
\delta (\varrho' | \Delta') = Z_N^{-1} \int_{SU(N)} d\mu_{SU(N)} \left| W_{\varrho'} (T_{VU}z) \right| - 1 = \delta (\varrho | \Delta).
$$

In the last equality the inverse transformation $z \to z' = T_{VU}^{-1}z$ has been performed taking into account the $SU(N)$ invariance of the Haar measure.

Finally, we present two additional observations on KZ indicator. The essence of the indicator reveals itself the best for multiparticle systems. Indeed, if one considers such a system described by density matrix Eq. (6) than Eq. (15) implies that for $\xi \leq \xi^* \leq \xi_{mb}$, where $\xi^* = \frac{1}{(N-1)\sqrt{N+1}}$, the Wigner function is non-negative. Here, $\xi_{mb}$ is the radius of the maximal ball inscribed into the set of mixed states [37]. It has been proven that all states lying in this ball are absolutely positive partial transpose states and moreover, they are absolutely separable (they can not be entangled by global unitary transformations) [38, 39]. Thus, all the states with Bloch vector less than $\xi^*$ are guaranteed to be absolutely separable. There is no doubt, that the issue of negativity of the Wigner function for multiparticle systems and its relation to quantum correlations is very involved and requires a separate consideration.

On the other hand, an upper boundary for the linear entropy through $\delta(\varrho)$ may be given. Immediately from Hölders inequality and the definition of $\delta$ it
follows that

\[ \delta(\varrho) + 1 \leq \left( N \int W_\varrho(\Omega_N)^2 d\mu_{SU(N)} \right)^{\frac{1}{2}} \left( N \int d\mu_{SU(N)} \right)^{\frac{1}{2}}, \quad (20) \]

where the integration is performed with respect to the SU(N) invariant normalized Haar measure. Since the linear entropy is

\[ S(\varrho) = 1 - \text{tr}[\varrho^2] = 1 - N \int W_\varrho(\Omega_N)^2 d\mu_{SU(N)}, \quad (21) \]

then the inequality (20) provides

\[ S(\varrho) \leq 1 - \frac{\delta(\varrho) + 1}{\sqrt{N}}. \quad (22) \]

4. KZ-indicator of a single qubit

For \( N = 2 \) the master equations determine the spectrum of a qubit Stratonovich-Weyl kernel uniquely:

\[ \text{spec}(\Delta(\Omega_2)) = \left\{ \frac{1 + \sqrt{3}}{2}, \frac{1 - \sqrt{3}}{2} \right\}. \quad (23) \]

If the unitary factor \( U(\Omega_2) \) in SVD decomposition of the Stratonovich-Weyl kernel is given in the symmetric 3-2-3 Euler parameterization:

\[ U(\Omega_2) = \exp\left\{ \frac{\alpha}{2} \sigma_3 \right\} \exp\left\{ \frac{\beta}{2} \sigma_2 \right\} \exp\left\{ \frac{\gamma}{2} \sigma_3 \right\}, \quad (24) \]

with \( \alpha \in [0, 2\pi], \beta \in [0, \pi], \gamma \in [0, 4\pi] \), then the Euler angles \( \alpha \) and \( \beta \) are coordinates of 2-dimensional symplectic manifold \( \Omega_2 = SU(2)/U(1) \) and the Wigner function of qubit reads

\[ W_{\xi}(\Omega_2) = \frac{1}{2} + \frac{\sqrt{3}}{2} (\xi, n). \quad (25) \]

Here, the unit vector \( n = (-\cos \alpha \sin \beta, \sin \alpha \sin \beta, \cos \beta) \) parameterizes \( \Omega_2 \), and \( \xi = (r \sin \psi \cos \phi, r \sin \psi \sin \phi, r \cos \psi) \) is the Bloch vector of qubit in a mixed state,

\[ \varrho = \frac{1}{2} \left( \mathbb{1}_2 + (\xi, \sigma) \right). \quad (26) \]
Figure 1: KZ-indicator for a single qubit is zero for the Bloch radius \( r \in [0, 1/\sqrt{3}] \).

Hence, taking into account that \( \Omega_2 \sim S^2(1) \) with the standard induced measure, one can write the integral representation for the KZ-indicator:

\[
\delta_2(r) = \frac{2}{2\pi^2} \frac{1}{2^3} \int_0^{4\pi} d\gamma \int_0^{\pi} d\beta \int_0^{2\pi} d\alpha \left| W_{\xi}(\Omega_2) \right| \sin(\beta) - 1. \tag{27}
\]

A straightforward evaluation of the integral (27) gives:

\[
\delta_2(r) = \begin{cases} 
0, & \text{for } 0 \leq r \leq \frac{1}{\sqrt{3}}, \\
\frac{\sqrt{3}}{2} \left( r + \frac{1}{3r} \right) - 1, & \text{for } \frac{1}{\sqrt{3}} < r \leq 1.
\end{cases} \tag{28}
\]

5. KZ-indicator of a single qutrit

The three-level system in a mixed state \( \varrho \) is characterized by the 8-dimensional Bloch vector \( \xi = \{\xi_1, \ldots, \xi_8\} \):

\[
\varrho_3 = \frac{1}{3} \mathbb{I}_3 + \frac{1}{\sqrt{3}} (\xi, \lambda). \tag{29}
\]
In [29] the standard Gell-Mann basis $\lambda = \{\lambda_1, \ldots, \lambda_8\}$ of $\mathfrak{su}(3)$ algebra is used.

Based on the $SU(3)$ invariance of the KZ-indicator shown in section 3, one can pass to the basis where the density matrix $\varrho$ is diagonal, i.e., the Bloch vector is of the form $\xi = \{0, 0, \xi_3, 0, 0, 0, 0, \xi_8\}$.

$$\varrho = \text{diag}(|r_1, r_2, r_3|) = \frac{1}{3} \mathbb{1}_3 + \frac{1}{\sqrt{3}} (\xi_3 \lambda_3 + \xi_8 \lambda_8).$$  (30)

Below we will assume that the eigenvalues of a qutrit density matrix belong to the following ordered $C_2$ simplex:

$$C_2 = \left\{ r \in \mathbb{R}^3 \left| \sum_{i=1}^{3} r_i = 1, \quad 1 \geq r_1 \geq r_2 \geq r_3 \geq 0 \right. \right\}. \quad (31)$$

This simplex represented in terms of the Bloch components $\xi_3$ and $\xi_8$ is given by inequalities

$$0 \leq \xi_3 \leq \frac{\sqrt{3}}{2}, \quad \frac{\xi_3}{\sqrt{3}} \leq \xi_8 \leq \frac{1}{2}$$

and is depicted in Fig.2.

According to the master equations (12), the spectrum $\text{spec}(\Delta(\Omega_3)) = \{\pi_1, \pi_2, \pi_3\}$ of the Stratonovich-Weyl kernel (13) of qutrit can be written as

$$\pi_1 = \frac{1}{3} + \frac{2}{\sqrt{3}} \mu_3 + \frac{2}{3} \mu_8, \quad \pi_2 = \frac{1}{3} - \frac{2}{\sqrt{3}} \mu_3 + \frac{2}{3} \mu_8, \quad \pi_3 = \frac{1}{3} - \frac{4}{3} \mu_8, \quad (32)$$

where $\mu_3$ and $\mu_8$ are Cartesian coordinates of a segment of a unit circle (Fig.3) with the apex angle $\zeta$:

$$\mu_3 = \sin \zeta, \quad \mu_8 = \cos \zeta, \quad \zeta \in [0, \pi/3]. \quad (33)$$

The range of the apex angle corresponds to the descending order of the eigenvalues, $\pi_1 \geq \pi_2 \geq \pi_3$. The angle $\zeta$ serves as the moduli parameter of the unitary non-equivalent representations of the Wigner functions of qutrit. Note, that the edge points M and N of the segment in Fig.3 with apexes $\zeta = 0$ and $\zeta = \pi/3$ correspond to degenerate Stratonovich-Weyl kernels:

$$\text{spec}\left(\Delta\right)\Bigg|_{\zeta=0} = \{1, 1, -1\}, \quad \text{spec}\left(\Delta\right)\Bigg|_{\zeta=\pi/3} = \frac{1}{3} \{5, -1, -1\}. \quad (34)$$

\[4\]This kernel with the last two equal eigenvalues was found by Luis [7].
Figure 2: Triangle AOC as the orbit space of qutrit. Regular 6D orbits with $r_1 > r_2 > r_3$ correspond to $\Delta_{AOC}/\{AO, OC\}$; Degenerate 4D orbits ($r_1 = r_2 > r_3$ and $r_1 > r_2 = r_3$) correspond to the boundary segments $\{AO, OC\}/\{O\}$; maximally mixed state $O$ represents the exceptional orbit.

Figure 3: The moduli space of a qutrit Stratonovich-Weyl kernel is given by the arc of a unit circle centered at the origin of $(\mu_3, \mu_8)$-plane; it is a union of the regular stratum, the arc $\overline{MN}/\{M, N\}$, and two degenerate Stratonovich-Weyl kernels correspond to the edge points $M$ and $N$ of the segment.

Depending on the degeneracy of the eigenvalues $\pi_1, \pi_2, \pi_3$, we define the corresponding phase-spaces:\footnote{Hereafter, following V.I.Arnold we adopt notations for ordered set of nonequal eigenvalues by $(12\ldots N)$ and use the sign “|” between equal eigenvalues, e.g., $(1|23|4)$.}

1. $\Omega_{(123)} = SU(3)/H_{(123)}$ with the isotropy group $H_{(123)} = U(1)^2$ for a generic Stratonovich-Weyl kernel;

2. $\Omega_{(1|23)} = SU(3)/H_{(1|23)}$ with $H_{(1|23)} \simeq SU(3)/S(U(2) \times U(1))$ for the Stratonovich-Weyl kernel with the first two equal eigenvalues, $\pi_1 = \pi_2$;

3. $\Omega_{(12|3)} = SU(3)/H_{(12|3)}$ with $H_{(12|3)} \simeq SU(3)/S(U(1) \times U(2))$ for the Stratonovich-Weyl kernel with the last two equal eigenvalues, $\pi_2 = \pi_3$;

To parameterize all these factor spaces, we will use the generalized Euler de-
composition of \(U(\Omega_3) \in SU(3)\):

\[
U = e^{i \frac{\lambda_3}{2} e^{i \frac{\lambda_2}{2} e^{i \frac{\lambda_1}{2} \theta} \lambda_3 e^{i \frac{\lambda_2}{2} \gamma} e^{i \frac{\lambda_1}{2} \phi} \lambda_3 e^{i \frac{\lambda_2}{2} \beta} e^{i \frac{\lambda_1}{2} \lambda_3} e^{i \alpha} \lambda_3} = e^{i \theta \lambda_3 e^{i \phi \lambda_3}},
\]

and the corresponding normalized Haar measure on \(SU(3)\):

\[
d\mu_{SU(3)} = \frac{1}{64 \sqrt{3} \pi^5} \cos \theta \sin^3 \theta \sin \beta \sin b \, d\alpha \wedge d\beta \wedge d\gamma \wedge d\theta \wedge da \wedge db \wedge dc \wedge d\phi.
\]

In order to cover “almost the entire” \(SU(3)\), the angles ranges are \(\alpha, a \in [0, 2\pi]\), \(\beta, b \in [0, \pi]\), \(\gamma, c \in [0, 4\pi]\), \(\theta \in [0, \pi/2]\), \(\phi \in [0, \sqrt{3}\pi]\). Gathering all the above ingredients together, the KZ-indicators for generic and two degenerate Stratonovich-Weyl kernels read

\[
\delta_{(123)}(\xi_4|\zeta) = \int_{\Omega_{(123)}} d\Omega_{(123)} \left| W_{\xi_4}^{(c)}(\Omega_{(123)}) \right| - 1,
\]

\[
\delta_{(1|23)}(\xi_4|0) = \int_{\Omega_{(1|23)}} d\Omega_{(1|23)} \left| W_{\xi_4}^{(0)}(\Omega_{(1|23)}) \right| - 1,
\]

\[
\delta_{(12|3)}(\xi_4|\pi/3) = \int_{\Omega_{(12|3)}} d\Omega_{(12|3)} \left| W_{\xi_4}^{(\pi/3)}(\Omega_{(12|3)}) \right| - 1.
\]

Here the \(\zeta\)-parametric family of the Wigner function of a qutrit state characterized by the Bloch vector \(\xi_4 = \{0, 0, \xi_3, 0, 0, 0, 0, \xi_8\}\) is

\[
W_{\xi_4}^{(c)}(\Omega_{(123)}) = \frac{1}{3} + \frac{4}{3} \left[ \sin(\zeta) (\xi_4, n^{(3)}) + \cos(\zeta) (\xi_4, n^{(8)}) \right],
\]

where \(n^{(3)}\) and \(n^{(8)}\) are defined in (16). The integration measures in (36)-(38) for each stratum is determined by the corresponding isotropy group:

\[
d\Omega_{(123)} = \frac{3 \text{Vol}(H_{(123)})}{64 \sqrt{3} \pi^5} \cos \theta \sin^3 \theta \sin \beta \sin b \, d\alpha \wedge d\beta \wedge d\gamma \wedge d\theta \wedge da \wedge db,
\]

\[
d\Omega_{(1|23)} = \frac{3 \text{Vol}(H_{(1|23)})}{64 \sqrt{3} \pi^5} \cos \theta \sin^3 \theta \sin \beta \sin b \, d\alpha \wedge d\beta \wedge d\theta \wedge db,
\]

\[
d\Omega_{(12|3)} = \frac{3 \text{Vol}(H_{(12|3)})}{64 \sqrt{3} \pi^5} \cos \theta \sin^3 \theta \sin \beta \sin b \, d\beta \wedge d\theta \wedge db.
\]

In order to make our presentation transparent and to simplify the analysis, below we will give only the results of evaluation of the KZ-indicator for two representative Wigner functions whose spectrum is degenerate. The evaluation
of the integral \[37\] gives
\[
\delta_{(1|23)}(\xi_3, \xi_8) = \begin{cases} 
0, & \text{if } \xi_3, \xi_8 \in \triangle OAP, \\
\frac{1}{36} \frac{(2(\sqrt{3} \xi_3 + \xi_8) - 1)^3}{\xi_3(\xi_3 + \sqrt{3} \xi_8)}, & \text{if } \xi_3, \xi_8 \in \triangle APC.
\end{cases}
\] (43)

Here the triangles \(\triangle OAP\) and \(\triangle APC\) decompose simplex \(C_2\) in a way shown in Fig.4. The triangle \(\triangle APC\) represents the domain of negativity of the Wigner function:
\[
\triangle APC := \{ \xi_3, \xi_8 \in C_2 \mid \frac{1}{8} \leq \xi_8 \leq \frac{1}{2}, \frac{1 - 2\xi_8}{2\sqrt{3}} \leq \xi_3 \leq \sqrt{3} \xi_8 \}.
\] (44)

Similarly, evaluating the integral \[38\] for the Stratonovich-Weyl kernel with \(\zeta = \pi/3\), we obtain
\[
\delta_{(12|3)}(\xi_3, \xi_8) = \begin{cases} 
0, & \text{if } \xi_3, \xi_8 \in \triangle OSQ, \\
\frac{1}{18} \frac{(1 - 4\xi_8)^3}{(\xi_3^2 - 3\xi_8^2)}, & \text{if } \xi_3, \xi_8 \in \square ARQS, \\
\frac{1}{36} \frac{(2(\sqrt{3} \xi_3 + \xi_8) + 1)^3}{\xi_3(\xi_3 + \sqrt{3} \xi_8)} - 2, & \text{if } \xi_3, \xi_8 \in \triangle CQR.
\end{cases}
\] (45)

The domains of definitions of the both KZ-indicators as well as their plots are given in Fig.4 - Fig.5 respectively.

5.1. Qutrit KZ indicator for pure states

In this section we will discuss KZ-indicator \(\delta_\zeta\) for pure states of a qutrit, while the Wigner functions moduli parameter \(\zeta \in [0, \pi/3]\) is arbitrary. Our calculations show that for all values of the moduli parameter \(\zeta\) the indicator is a monotone decreasing function, \(1/2 \leq \delta_\zeta \leq 17/54\), and it hereby reveals nonclassicality of all pure states (see Fig.6).

Each pure state of qutrit (i.e. the rank-1 state) belongs to a class of 4-dimensional \(SU(3)\) orbits being characterized either by the isotropy group \(SU(3)/S(U(1)\times \ldots U(1))\).
$U(2))$ or its conjugated group $SU(3)/S(U(2) \times U(1))$. Let us fix a density matrix $\rho_0$ as the representative of this class,

$$\rho_0 = |1\rangle\langle 1| = \text{diag}\{1, 0, 0\} = \frac{1}{3} I_3 + \frac{1}{2} \lambda_3 + \frac{1}{2\sqrt{3}} \lambda_8.$$  

(46)

Then an arbitrary pure state $\rho_\chi = |\chi\rangle\langle \chi|$ can be written as

$$\rho_\chi = V_\chi |1\rangle\langle 1| V_\chi^\dagger,$$  

(47)

with $V_\chi$ from the coset $SU(3)/S(U(1) \times U(2))$ parameterized by 4 coordinates $\chi$.

Similarly, according to (13), a generic SW kernel is given by the adjoint action of $U_\vartheta \in SU(3)/S(U(1) \times U(1) \times U(1))$ on a matrix $P_\zeta = \text{diag}\{\pi_1, \pi_2, \pi_3\}$:

$$\Delta(\vartheta | \zeta) = U_\vartheta \text{diag}\{\pi_1, \pi_2, \pi_3\} U_\vartheta^\dagger.$$  

(48)

In (48), the $6$–tuple of coordinates of points on $SU(3)/S(U(1) \times U(1) \times U(1))$ coset is denoted by $\vartheta$, and the diagonal entries $\{\pi_1, \pi_2, \pi_3\}$ are parameterized.
according to (32). With this input one can get convinced that the Wigner function of qutrit in the pure state $|\chi\rangle\langle\chi|$ is related to the WF of qutrit in the state $|1\rangle\langle1|$ by the induced transformation on phase space,

$$W_{|\chi\rangle\langle\chi|}(\vartheta|\zeta) = \text{tr}(g_0 V_\chi^T U_{\vartheta} P_{\zeta} U_{\vartheta}^T V_\chi) = \langle 1|\Delta(T^{-1}_\chi \vartheta|\zeta)|1 \rangle = W_{|1\rangle\langle1|}(T^{-1}_\chi \vartheta|\zeta). \quad (49)$$

Here, the composition law, $g_\chi^T g_\vartheta = g_{T^{-1}_\chi \vartheta}$, and covariance property of SW kernel have been used.

Using this, the computation of KZ-indicator of qutrit gives:

$$\delta_\zeta = \begin{cases} 
\frac{(-1 + 4 \cos(\zeta))^3}{18 (1 + 2 \cos(2\zeta))}, & \text{if } 0 \leq \zeta \leq 2 \arctan\left(\frac{\sqrt{3}}{2 + \sqrt{5}}\right), \\
\frac{(4 \sin(\zeta + \frac{\pi}{6}) + 1)^3}{18 (1 - 2 \cos(2(\zeta + \frac{\pi}{6})))} - 2, & \text{if } 2 \arctan\left(\frac{\sqrt{3}}{2 + \sqrt{5}}\right) \leq \zeta \leq \frac{\pi}{3}.
\end{cases} \quad (50)$$

Note that the expressions (50) evaluated for the moduli parameters corresponding to the degenerate SW kernels, $\zeta = 0$ and $\pi/3$, coincide with the corresponding limits of KZ-indicator (43) and (45),

$$\lim_{\xi_3 \to \sqrt{3}/2, \xi_8 \to 1/2} \delta_{\zeta=0}(\xi_d) = \frac{1}{2}, \quad \lim_{\xi_3 \to \sqrt{3}/2, \xi_8 \to 1/2} \delta_{\zeta=\pi/3}(\xi_d) = \frac{17}{54}. \quad (51)$$

6. Conclusion

In the present note we rise the question of dependence of the KZ-indicators of nonclassicality on the representation of the Wigner functions. This issue was analysed by constructing the KZ-indicator for two so-called degenerate Stratonovich-Weyl kernels, which are special representatives of $\zeta$-parametric family of the Wigner function of qutrit. Our calculations show that despite the quantitative distinction of these indicators, there are interesting common features between both indicators of nonclassicality:

- If we identify the boundary of a quantum-classical transition as the locus of vanishing Wigner function, then it turns out that quantum-classical
transitions are smooth. Namely, both KZ-indicators, (43) and (45), are smooth functions on these boundaries.

- The isometries of the state space induce certain symmetry of the nonclassicality indicators. To find out the roots of this symmetry, note that the triangles $\triangle OAP$ and $\triangle OSQ$ where the Wigner function is positive are congruent. By performing rotation of the triangle $\triangle OSQ$ on $\pi/3$ around the point $O$ with subsequent reflection over $OC$, one can superpose them. This symmetry is a reminiscent of the existence of the Weyl group acting on the eigenvalues of a qutrit density matrix by discrete rotations and reflection. As a result of the Weyl symmetry, one can expect that there are characteristics of the nonclassicality of qubit which are equal modulo $\pi/3$. From the geometrical reasoning, it is easy to find such characteristics. Indeed, one can get convinced that for both, $\zeta = 0$ and $\zeta = \pi/3$, the Euclidean areas of the domain where the Wigner function is positive are
equal, $S_{\Delta OAP} = S_{\Delta OSQ} = 2^{-5}\sqrt{3}$. Therefore, assuming that the eigenvalues of qutrit are uniformly distributed, the geometric probability to find a random qutrit state with positive Wigner function is the same for both degenerate Stratonovich-Weyl kernels with $\zeta = 0$ and $\pi/3$:

\[
\frac{\text{Euclidean Area of WF Positive Part of Orbit Space}}{\text{Total Euclidean Area of Orbit Space}} = \frac{1}{4}.
\]

(52)

It is clear that the above argumentation can be extended to the case of metrics possessing the Weyl symmetry. As an example, one can consider the flat Hilbert-Schmidt metric on a qutrit state space. For this case, the volume form on the orbit space reads

\[
w_3 = \frac{8}{9\sqrt{3}} \xi_3^2 \left( \frac{\xi_3^2}{3} - \xi_8^2 \right)^2 \, d\xi_3 \land d\xi_8,
\]

(53)

and evaluation of the integrals over $\Delta OAP$ and $\Delta OSQ$ gives the same results:

\[
\int_{\Delta OAP} w_3 = \int_{\Delta OSQ} w_3 = \frac{1}{2580480}.
\]

Hence, noting that $S_{\Delta OAC} = \int_{\Delta OAC} w_3 = \frac{1}{10080}$, we conclude that for both representative WFs the ratio is

\[
\frac{\text{Hilbert-Schmidt Area of WF Positive Part of Orbit Space}}{\text{Total Hilbert-Schmidt Area of Orbit Space}} = \frac{1}{256}.
\]

(54)

- Finally, the indicator of nonclassicality points to the existence of the following three classes of states:
  
  • the “absolutely classical” states, which have zero KZ-indicator for all values of the moduli parameters $\zeta$;
  
  • the “absolutely quantum” states, whose KZ-indicator depends on the moduli parameter but is always non-vanishing;
  
  • the “relatively quantum-classical” states whose classicality/quantumness is susceptible to a representation of the Wigner function.

Furthermore, all pure states of qutrit belong to “absolutely quantum” states.
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