An Improved Exact Algorithm for the Exact Satisfiability Problem

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Abstract. The Exact Satisfiability problem, XSAT, is defined as the problem of finding a satisfying assignment to a formula \( \varphi \) in CNF such that exactly one literal in each clause is assigned to be “1” and the other literals in the same clause are set to “0”. Since it is an important variant of the satisfiability problem, XSAT has also been studied heavily and has seen numerous improvements to the development of its exact algorithms over the years.

The fastest known exact algorithm to solve XSAT runs in \( O(1.1730^n) \) time, where \( n \) is the number of variables in the formula. In this paper, we propose a faster exact algorithm that solves the problem in \( O(1.1674^n) \) time. Like many of the authors working on this problem, we give a DPLL algorithm to solve it. The novelty of this paper lies on the design of the nonstandard measure, to help us to tighten the analysis of the algorithm further.

Keywords: XSAT; Measure and Conquer; Exponential Time Algorithms.

1 Introduction

Given a propositional formula \( \varphi \) in conjunctive normal form (CNF), a common question to ask would be if there is a satisfying assignment to \( \varphi \). This is known as the satisfiability problem, or SAT. SAT is seen to be a problem that is at the center of computational complexity because it has been commonly used as a framework to solve other combinatorial problems. In addition, SAT has found many uses in practice as well. Some of these examples include: AI-planning, software model checking, etc[3].

Because of its importance, many other variants of the satisfiability problem have also been explored. One such important variant is the Exact Satisfiability problem, XSAT, where it asks if one can find a satisfying assignment such that exactly one of the literal in each clause is assigned the value “1” and all other literals in the same clause are assigned “0”. All the mentioned problems, SAT and XSAT, are both known to be NP-complete [12].

In this paper, we will focus on the XSAT problem and in particular, exact algorithms to solve it. XSAT is a well-studied problem and has seen numerous improvements [15,17] to it, with the fastest solving it in \( O(1.1730^n) \) time.
In this paper, we will propose an algorithm to solve XSAT in $O(1.1674^n)$ time, using polynomial space. Like most of the earlier authors, we will design a Davis-Putnam-Logemann-Loveland (DPLL) [11] style algorithm to solve this problem. We build our work upon the works of the earlier authors. While the earlier authors all used the standard measure, which is the number of variables $n$, we propose the use of a nonstandard measure to help us to tighten the analysis of the algorithm further.

2 Preliminaries

In this section, we will introduce some definitions and also the techniques needed to understand the analysis of DPLL algorithm.

2.1 Branching factor and vector

Our algorithm is a DPLL style algorithm, or also known as a branch and bound algorithm. DPLL algorithms are recursive in nature and have two kinds of rules associated with them: Simplification and Branching rules. Simplification rules help us to simplify a problem instance or to act as a case to terminate the algorithm. Branching rules on the other hand, help us to solve a problem instance by recursively solving smaller instances of the problem. To help us to better understand the execution of a DPLL algorithm, the notion of a search tree is commonly used. We can assign the root node of the search tree to be the original problem, while the child nodes are assigned to be the smaller instances of the problem whenever we invoke a branching rule. For more information of this area, one may refer to the textbook written by Fomin and Kratsch [9].

Let $\mu$ be our measure of complexity. To analyse the running time of DPLL algorithms, one just needs to bound the number of leaves generated in the search tree. This is because the complexity of such algorithm is proportional to the number of leaves, modulo polynomial factors, that is, $O(\text{poly}(|\varphi|, \mu) \times \text{number of leaves in the search tree}) = O^*(\text{number of leaves in the search tree})$, where the function $\text{poly}(|\varphi|, \mu)$ is some polynomial dependent on $|\varphi|$ and $\mu$, and $O^*(f(\mu))$ is the class of all function $g$ bounded by some polynomial $p(\cdot) \times f(\mu)$.

Then we let $T(\mu)$ denote the maximum number of leaf nodes generated by the algorithm when we have $\mu$ as the parameter for the input problem. Since the search tree is only generated by applying a branching rule, it suffices to consider the number of leaf nodes generated by that rule (as simplification rules take only polynomial time). To do this, we use techniques in [10]. Suppose a branching rule has $r \geq 2$ children, with $t_1, t_2, \ldots, t_r$ decrease in measure for these children. Then, any function $T(\mu)$ which satisfies $T(\mu) \geq T(\mu-t_1) + T(\mu-t_2) + \ldots + T(\mu-t_r)$, with appropriate base cases, would satisfy the bounds for the branching rule. To solve the above linear recurrence, one can model this as $x^{-t_1} + x^{-t_2} + \ldots + x^{-t_r} = 1$. Let $\beta$ be the unique positive root of this recurrence, where $\beta \geq 1$. Then any $T(\mu) \geq \beta^\mu$ would satisfy the recurrence for this branching rule. In addition, we denote the branching factor $\tau(t_1, t_2, \ldots, t_r)$ as $\beta$. If there are $k$ branching rules in
the DPLL algorithm, then the overall complexity of the algorithm is the largest branching factor among all \( k \) branching rules; i.e. \( c = \max\{\beta_1, \beta_2, \ldots, \beta_k\} \), and therefore the time complexity of the algorithm is bounded above by \( O^*(c^k) \).

Next, we will introduce some known results about branching factors. If \( k < k' \), then we have that \( \tau(k', j) < \tau(k, j) \), for all positive \( k, j \). In other words, comparing two branching factor, if one eliminates more weights, then this will result in a smaller branching factor. Suppose that \( i + j = 2\alpha \), for some \( \alpha \), then \( \tau(\alpha, \alpha) \leq \tau(i, j) \). In other words, a more balanced tree will result in a smaller branching factor.

Finally, the correctness of DPLL algorithms usually follows from the fact that all cases have been covered.

### 2.2 Definitions

**Definition 1.** A clause is a disjunction of literals. We also say that a clause is a multiset of literals. A \( k \)-literal clause is a clause \( C \) with \( |C| = k \). Let \( C \) be a clause, then \( \delta \) is a subclause of \( C \) if \( \delta \subset C \).

Suppose we have \( C = (a \lor b \lor c \lor d) \), then \( C \) is a 4-literal clause. In addition, \( \delta = (a \lor b \lor c) \) is a subclause of \( C \). We may also write \( C = (\delta \lor d) \). For now, we define a clause as a multiset of literals as the same literal may appear twice in a clause. When no simplification rules\(^1\) can be applied, we may then think of a clause as a set of literals instead.

**Definition 2.** Two clauses are called neighbours if they share at least a common variable. Two variables are called neighbours if they appear in some clause together. Let \( C_1 \) and \( C_2 \) be two clauses that are neighbours. Now if \(|C_1 \cap C_2| = k \geq 2\), we say that \( C_1 \) and \( C_2 \) have \( k \) overlapping variables. In addition, the variables in \( C_1 - C_2 \) and \( C_2 - C_1 \) are known as outside variables. Let \(|C_1 - C_2| = i \) and \(|C_2 - C_1| = j \), \( i, j \geq 1 \). Then we say that there are \( i + j \) outside variables, in an \( i-j \) orientation.

Note that this definition \((i-j \text{ orientation})\) is strictly used for the case when we have \( k \geq 2 \) overlapping variables between any two clauses\(^2\). We only consider \( i, j \geq 1 \) because if \( i \) or \( j \) is 0, then one of the clause must be a subclause of the other. Consider the following example.

**Example 1.** Let \( C_1 = (a \lor b \lor c \lor d \lor e) \) and \( C_2 = (d \lor e \lor f \lor g \lor h) \). Then in this case, since \( C_1 \cap C_2 = \{d, e\} \), there are 2 literals in the intersection and we say that \( C_1 \) and \( C_2 \) have 2 overlapping variables. In addition, \( C_1 - C_2 = \{a, b, c\} \) and \( C_2 - C_1 = \{f, g, h\} \). Now, we say \( C_1 \) and \( C_2 \) have 6 outside variables in a 3-3 orientation.

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\(^1\) More details later in Section 3, when the algorithm is given

\(^2\) Mainly in Section 4.3
Definition 3. Let x be a literal. Now the degree of a variable, \( \text{deg}(x) \), denotes the total number of times that the literal \( x \) and \( \neg x \) appears in \( \varphi \). If \( \text{deg}(x) \geq 3 \), then we say that the variable \( x \) is heavy. Further, for a heavy variable \( x \) that appears in clauses \( C_1, C_2, \ldots, C_k \), \( k \geq 3 \), we say that \( x \) is in \( (l_1, l_2, \ldots, l_k) \), where \( |C_i| = l_i \), \( 1 \leq i \leq k \). Adding on to this,

1. if \( \neg x \) appears in \( C_1 \), then we say \( x \) is in \((l_1, l_2, \ldots, \neg l_i, \ldots, l_k)\).
2. if \( |C_i| \geq l_i \), then we say \( x \) is in \((l_1, l_2, \ldots, \geq l_i, \ldots, l_k)\).

Note that if \( x \) is a heavy variable, we will only use this definition that \( x \) is in \((l_1, l_2, \ldots, l_k)\), whenever given any two clauses that \( x \) is in, they have at most 1 overlapping variable between them.

Example 2. Suppose we have the following clauses: \((x \lor a \lor b \lor c \lor d), (\neg x \lor e \lor f \lor g)\), \((x \lor b \lor i \lor j \lor k)\). Then in this case, we have \( x \) in \((5, \neg 4, 5)\). We can also say that \( x \) is in \((\geq 4, \neg 4, 5)\) and we use \( \geq i \) whenever we just need to know that the clause length is at least \( i \). Note that the order in which the clause length is presented here does not matter, i.e. \((5, \neg 4, 5)\) can also be written as \((-4, 5, 5)\).

Definition 4. We say that two variables, \( x \) and \( y \), are linked when we can deduce either \( x = y \) or \( x = \neg y \). When this happens, we can proceed to remove one of the linked variable, either \( x \) or \( y \), and replace by the other.

Suppose we have a 3-literal clause \((0 \lor x \lor y)\), by definition of being exact satisfiable, we can deduce that \( x = \neg y \) in this case, and proceed to remove one variable, say \( x \), by replacing all instances of \( x \) by \( \neg y \) and \( \neg x \) by \( y \) respectively.

Definition 5. Given a formula \( \varphi \) and \( \delta \) a multiset of literals,

1. If \(|\delta| = 1\), then let \( x \) be the only literal in \( \delta \). Now \( \varphi[x = 1] \) and \( \varphi[x = 0] \) denotes the new formula obtained after assigning \( x = 1 \) and \( x = 0 \) respectively.
2. If \(|\delta| \geq 2\), then we only allow the following when \( \delta \subset C \), for some clause \( C \) in \( \varphi \). \( \varphi[\delta = 1] \) denotes the new formula obtained after assigning all the \( C - \delta \) to be 0. By definition of being exact-satisfiable, this is saying that the “1” must only appear in one of the literals in \( \delta \). Therefore, all the literals in \( C - \delta \) are assigned 0. On the other hand, \( \varphi[\delta = 0] \) denotes the new formula obtained after assigning all the literals in \( \delta \) to be 0.

Similarly, given two literals \( x \) and \( y \), we say that \( \varphi[x = y] \) is the new formula obtained by replacing all occurrences of \( x \) by \( y \).

Example 3. Suppose \( \varphi = (a \lor b \lor c \lor d) \) and \( \delta = (a \lor b \lor c) \). Then \( \varphi[\delta = 1] = (a \lor b \lor c \lor 0) \) since we are saying that the “1” appears in either \( a, b, \) or \( c \). On the other hand, \( \varphi[\delta = 0] = (0 \lor 0 \lor 0 \lor d) \).

Definition 5.1 is used whenever we are branching a variable. On the other hand, Definition 4.2 is used when we want to branch a subclause, especially when we deal with \( k \geq 2 \) overlapping variables between two clauses. In addition, when
we have a subclause $\delta$ such that $|\delta| = 2$, then let $x$ and $y$ be the literals in $\delta$. Saying that $\varphi[\delta = 1]$ is the same as saying $\varphi[x = \neg y]$, linking $x = \neg y$.

A common technique used by the earlier authors is known as resolution. If there are clauses $C_1 = (C \lor x)$ and $C_2 = (C' \lor \neg x)$, where $x$ is a literal, $C$ and $C'$ are subclauses of $C_1$ and $C_2$ respectively, then we can replace every clause $(x \lor \alpha)$ by $(C' \lor \alpha)$, and every clause $(\neg x \lor \beta)$ by $(C \lor \beta)$, for some subclause $\alpha, \beta$. In addition, every literal in $C \cap C'$ can be assigned 0. This can help us to remove literals appearing as $x$ and $\neg x$ in different clauses.

2.3 A nonstandard measure

Instead of using the number of variables as our measure, we will design a nonstandard measure to help us to improve the worst case time complexity of our algorithm. Let $\{x_1, x_2, ..., x_n\}$ be the set of variables in $\varphi$. For $1 \leq i \leq n$, we define the weight $w_i$ for $x_i$ as:

$$w_i = \begin{cases} 0.8823, & \text{if } x_i \text{ is on a 3-literal clause such that all 3 variables in that clause do not have the same neighbour} \\ 1, & \text{otherwise} \end{cases}$$

We then define our choice of measure as $\mu = \sum_i w_i$, where $\mu \leq n$ by definition. This value of 0.8823 is chosen by a linear search program to bring down the overall runtime of the algorithm to as low as possible. Therefore, we have $O(\mu^n) \subseteq O(c^n)$, for some constant $c \geq 1$ by definition.

**Example 4.** Suppose we have the following clauses : $(x \lor y \lor z \lor a), (x \lor u \lor w \lor v), (x \lor r \lor s \lor t), (a \lor v \lor t)$ and the clause $(y \lor e \lor f)$. The variables $x, z, u, w, r$ and $s$ have weight 1. By definition, variables $a, v$ and $t$ are assigned the weight 1 because these variables have $x$ as their neighbour. Variables $y, e, f$ have weights 0.8823 because these 3 variables do not have the same neighbour.

3 Algorithm

All of our simplification rules and branching rules are designed to ensure that the overall measure does not increase after applying them. That is, the measure before applying any of the rule, $\mu$, and the measure after applying any of the rule, $\mu'$, is always $\mu' \leq \mu$. We call our DPLL algorithm XSAT(.). Note that if every variable $x$ has $deg(x) \leq 2$, then we can solve XSAT in polynomial time [5].

With this in mind, we’ll design our algorithm by branching all heavy variables. Note that each line of the algorithm has decreasing priority; Line 1 has higher priority than Line 2, Line 2 than Line 3 etc. Let $\alpha, \beta, \delta$ be subclauses.

**Algorithm : XSAT**

Input : A formula $\varphi$
Output : 1 if $\varphi$ is exact satisfiable, else 0
1. If there is a clause that is not exact-satisfiable, then return 0.
2. If there is a clause $C = (1 \lor \delta)$ or $C = (x \lor \neg x \lor \delta)$, for some variable $x$, then set all literals in $\delta$ to 0 and drop the clause $C$. Return $\text{XSAT}(\varphi[\delta = 0])$.
3. If there exist clauses $C = (0 \lor \delta)$, then update $C = \delta$. Update $\varphi'$ as the new formula and return $\text{XSAT}(\varphi')$.
4. If there exist a 1-literal clause containing the literal $l$, then drop that clause. Return $\text{XSAT}(\varphi[l = 1])$.
5. If there exist a 2-literal clause containing the literal $l$ and $l'$, then drop that clause. Return $\text{XSAT}(\varphi[l = \neg l'])$.
6. If there exist a clause $C$ with a literal $l$ appearing at least twice, then return $\text{XSAT}(\varphi[l = 0])$.
7. If there exist clauses of the type $(\alpha \lor x \lor y)$ and $(\beta \lor x \lor \neg y)$, for some literal $x$ and $y$, then return $\text{XSAT}(\varphi[x = 0])$.
8. If there exist clauses of the type $(\alpha \lor x \lor y)$ and $(\beta \lor \neg x \lor \neg y)$, then return $\text{XSAT}(\varphi[x = \neg y])$.
9. If there are clauses $C$ and $C'$ such that $C \subseteq C'$, then set all literals in $\delta = C' - C$ as 0, remove the clause $C'$ and return $\text{XSAT}(\varphi[\delta = 0])$.
10. If there is a variable $x$ appearing in at least three 3-literal clauses, then we either simplify it or branch $x$. If we simplify it, let $\varphi'$ be the new formula after simplifying. Return $\text{XSAT}(\varphi')$. If we branch $x$, return $\text{XSAT}(\varphi[x = 1]) \lor \text{XSAT}(\varphi[x = 0])$.
11. If there are clauses $C_1$ containing $x$ and $C_2$ containing $\neg x$, for some literal $x$. Then we apply resolution and let $\varphi'$ be the new formula. Return $\text{XSAT}(\varphi')$.
12. If there are clauses $C_1$ and $C_2$ such that they have $k \geq 2$ overlapping variables, then check if the outside variables are in a 1-j orientation, $j \geq 1$. If yes, then let $\varphi'$ be the new formula after applying some changes then return $\text{XSAT}(\varphi')$. Else, let $\delta = C_1 \cap C_2$ and we branch the subclause $\delta$. Return $\text{XSAT}(\varphi[\delta = 1]) \lor \text{XSAT}(\varphi[\delta = 0])$.
13. If there is a heavy variable $x$, then branch $x$. Return $\text{XSAT}(\varphi[x = 1]) \lor \text{XSAT}(\varphi[x = 0])$.
14. If all the variables $x$ have $\text{deg}(x) \leq 2$, then solve the problem in polynomial time. Return 1 if exact-satisfiable, else return 0.

Lines 1 to 9, 11 are simplification rules, while Lines 10, 12 and 13 are branching rules. Line 14 takes only polynomial time to decide if there is an exact-satisfiable assignment to $\varphi$ when $\text{deg}(x) \leq 2$ for all variable $x$. Line 1 says that if any clause is found not to be exact-satisfiable, then we can return 0. Line 2 says if a clause contains a “1”, then the other literals appearing in the clause must be assigned 0. Line 3 says that if we have a clause containing “0”, then we can update that clause by dropping off the constant “0”. Line 4 says that if we encounter a 1-literal clause, then that literal must be assigned 1. Line 5 says that if there are any 2-literal clause containing some literals $x$ and $y$, then we can just link the two literals $x = \neg y$ together. After Line 5 of the algorithm, every clause in $\varphi$ must be at least a 3-literal clause.

Footnote: Full details given in the Section 4.3.
Line 6 deals with clauses containing the same literals that appear at least twice. After Line 6, every clause can only contain any literal at most once. Lines 7 and 8 deals with two clauses that have at least two variables in common, in different permutations. After Line 8, if any two clauses have at least two variables in common, then this implies that they have share at least two literals in common. After Line 9, no clause is a subclause of a larger clause in $\varphi$.

In Line 10, we deal with variables that appears in at least three 3-literal clauses. We deal with this case early on because it helps us to reduce the number of cases that we need to handle later on while branching in Section 4.3 and 4.4. In Line 11, we deal with clauses $C_1$ containing the literal $x$ and $C_2$ containing $\neg x$. Line 12 deals with two clauses having $k \geq 2$ overlapping variables. First, we deal with such cases in a $1-j$ orientation, $j \geq 1$, followed by such cases in an $i-j$ orientation, $i, j \geq 2$. After which, any two clauses must have only at most one variable in common. Line 13 deals with heavy variables. After that, no heavy variables exist in the formula $\varphi$ and we can proceed to solve the problem in polynomial time in Line 14. We have therefore covered all cases in our algorithm.

4 Analysis of Algorithm

In this section, we will analyze the overall runtime of the algorithm given in the previous section. Note that simplification rules only take polynomial time. Therefore, we will analyse from Lines 10 to 13 of the algorithm.

Due to the way we design our measure, if a $k$-literal clause drops to a 3-literal clause, $k > 3$, we can factor in the change of measure of $1 - 0.8823 = 0.1177$ for each of the variables in the 3-literal clause, if there is no common neighbour. Whenever we are dealing with a 3-literal clause, for simplicity, we will treat all the variables in it as having a weight of 0.8823 instead of 1. This gives us an upper bound on the branching factor without the need to consider all kinds of cases.

In addition, when we are dealing with 3-literal clause, sometimes we have to increase the measure after linking. For example, suppose we have the clause $(0 \lor x \lor y)$, for some literals $x$ and $y$. Now we can link $x = \neg y$ and proceed to remove one variable, say $x$. This means that the 3-literal clause is removed and the surviving variable $y$, may no longer be appearing in any other 3-literal clause. Therefore, the weight of $y$ increases from 0.8823 to 1. This increase in weight means that we increase our measure and therefore, we have to factor in “-0.1177” whenever we are linking variables in a 3-literal clause together.

4.1 Line 10 of the Algorithm

Line 10 of the algorithm deals with a variable appearing in at least three 3-literal clauses. We can either simplify the case further, or branch $x$. At this point in time, Lines 11 and 12 of the algorithm has not been called. This means that we have to deal with literals appearing as $x$ and $\neg x$, and that given any two clause, it is possible that they have $k \geq 2$ overlapping variables.
Lemma 1. If $x$ appears in at least three 3-literal clauses, we either simplify this case further or we branch $x$, incurring at most $O(1.1664^n)$ time.

Proof. Now let $x$ be appearing in two 3-literal clauses. We first deal with the case that for any two 3-literal clauses, there are $k \geq 2$ overlapping variables. Since simplification rules do not apply anymore, the only case we need to handle here is $(x \lor y \lor z)$ and $(x \lor y \lor w)$, for some literals $w, y, z$. In this case, we can link $w = z$ and drop one of these clauses.

For the remaining cases, $x$ must appear in three 3-literal clause and there are no $k \geq 2$ overlapping variables between any two of the 3-literal clause. Therefore, for the remaining case, $x$ must be in $(3, 3, 3)$ or $(3, 3, -3)$.

For the $(3, 3, 3)$ case, let the clauses be $(x \lor v_1 \lor v_2)$, $(x \lor v_3 \lor v_4)$ and $(x \lor v_5 \lor v_6)$, where $v_1, \ldots, v_6$ are unique literals. We branch $x = 1$ and $x = 0$ here. When $x = 1$, we remove the variables $v_1, \ldots, v_6$ and $x$ itself. This gives us a change of measure of $7 \times 0.8823$. When $x = 0$, we remove $x$, and link $v_1 = \neg v_2$, $v_3 = \neg v_4$ and $v_5 = \neg v_6$. This gives us a change of measure of $4 \times 0.8823 - 3 \times 0.1177$. This gives us a branching factor of $\tau(7 \times 0.8823, 4 \times 0.8823 - 3 \times 0.1177) = 1.1664$. The case for $(3, -3, -3)$ is symmetric. Therefore, this takes at most $O(1.1664^n)$ time.

4.2 Line 11 of the Algorithm

Line 11 of the algorithm applies resolution. One may note that our measure is designed in terms of the length of the clause. Therefore, it is possible that the measure may increase from 0.8823 to 1 after applying resolution. Applying resolution on $k$-literal clauses, $k \geq 4$, is fine because doing so will not increase the measure. On the other hand, applying on 3-literal clauses will increase the length of the clause and hence, increase the weights of the other variables in the clause, and finally, the overall measure. Therefore to apply resolution on such cases, we have to ensure that the removal of the variable $x$, is more than the increase of the weights of from 0.8823 to 1 ($1 - 0.8823 = 0.1177$). To give an upper bound, we assume that $x$ has weight 0.8823. Taking $0.8823 \div 0.1177 = 7.5$. Therefore, if there are more than 7.5 variables increasing from 0.8823 to 1, then we refrain from doing so. This translates to $x$ appearing in at least four 3-literal clauses. However, this has already been handled by Line 10 of the algorithm. Hence, when we come to Line 11 of the algorithm, we can safely apply resolution.
4.3 Line 12 of the algorithm

In this section, we deal with Line 12 of the algorithm. Since simplification rules do not apply anymore when this line is reached, we may then think of clauses as sets (instead of multiset) of literals, since the same literal can no longer appear more than once in the clause. In addition, from the previous line of the algorithm, we know that we will not have \( x \) and \( \neg x \) appearing in the formula, for any literal \( x \). Now, we fix the following notation for the rest of this section. Let \( C_1 \) and \( C_2 \) be any clauses given such that \( \lvert C_1 \cap C_2 \rvert = \delta \), with \( \lvert \delta \rvert \geq 2 \) overlapping variables, in an \( i \)-\( j \) orientation, where \( \lvert C_1 \setminus C_2 \rvert = i \) and \( \lvert C_2 \setminus C_1 \rvert = j \), where \( i, j \geq 1 \).

We divide them into 3 parts, let \( L = C_1 \setminus C_2 \) (left), \( R = C_2 \setminus C_1 \) (right) and \( \delta \) (middle). For example, in Example 1, we have \( L = \{a, b, c\} \) and \( R = \{f, g, h\} \).

We first deal with the cases \( i = 1, j \geq 1 \).

**Lemma 2.** The time complexity of dealing with two clauses with \( k \geq 2 \) overlapping variables, having \( 1 \)-\( j \) orientation, \( j \geq 1 \), is at most \( O(1.1664^n) \).

**Proof.** If \( j = 1 \), then let \( x \in L \) and \( y \in R \). Then we can just link \( x = y \) and this case is done. If \( j \geq 2 \), then let \( C_1 = (x \lor \delta) \) and \( C_2 = (\delta \lor R) \). From \( C_1 \), we know that \( \neg x = \delta \). Therefore, \( C_2 \) can be rewritten as \((\neg x \lor R)\). With the clauses \( C_1 = (x \lor \delta) \) and \( C_2 = (\neg x \lor R) \), we can apply Line 11 of the algorithm which either uses resolution to remove the literals \( x \) and \( \neg x \), or to apply branching to get a complexity of \( O(1.1664^n) \).

Now, we deal with the case of having \( k \geq 2 \) overlapping variables in an \( i \)-\( j \) orientation, \( i, j \geq 2 \). Note that during the course of branching \( \delta = 0 \), when a longer clause drops to a 3-literal clause \( L \) (or \( R \)), then we can factor in the change of measure of \( 1 - 0.8823 = 0.1177 \) for each of the variable in \( L \) (Normal Case). However, there are situations when we are not allowed to factor in this change. Firstly, when there is a common neighbour to the variables in \( L \) (Case 1). Secondly, when some or all variables in \( L \) already have weights 0.8823, which means the variable appears in further 3-literal clauses prior to the branching (Case 2).

Instead of enumerating every single case, we show that some cases can be avoided by upper bounding them from a different case. We first show how to deal with Case 1.

**Case 1:** The variables in \( L \), with \( \lvert L \rvert = 3 \), (similarly for \( R \)) have a common neighbour.

- When there is a clause \( L' \), such that \( L \subset L' \) and therefore every variable in \( L' - L \) is a neighbour to \( C \). However, if this case happens. Then by our simplification rule, we can set the literals in \( L' - L \) to 0. We can remove at least one such variable, and the weight of such a variable is at least 0.8823.
- Let the literals in \( L \) be \( a, b, c, \alpha, \beta, \gamma \) be subclauses.
  1. \((s \lor \beta \lor a \lor b)\) and \((s \lor \alpha \lor c)\)
  2. \((s \lor \alpha \lor a), (s \lor \beta \lor b)\) and \((s \lor \gamma \lor c)\)
Then in the above 2 cases, \( s = 0 \) and the weight of \( s \) is at least 0.8823.

In all 3 possible cases in Case 1, we are able to factor in an additional measure of 0.8823. Now let \( \Delta\mu_{\delta=1} \) be the change of measure when we branch \( \delta = 1 \) and \( \Delta\mu_{\delta=0} \) when we branch \( \delta = 0 \) for the Normal Case. Note that when \( \delta = 0 \), we remove all the variables in \( \delta \) and we can also factor in the change of measure for at most 6 variables (in a 3-3 orientation). In Case 1, we can remove an additional variable that has weight at least 0.8823, which means 0.8823 \(- 6 \times 0.1177 = 0.1761 \), allowing us to factor in additional change of measure of 0.1761 in the worst case. Therefore, we have \( \tau(\Delta\mu_{\delta=1}, \Delta\mu_{\delta=0} + 0.1761) < \tau(\Delta\mu_{\delta=1}, \Delta\mu_{\delta=0}) \), being upper bounded by the branching factor in the Normal Case. Hence, it suffices to just show the Normal Case.

For Case 2, we pay special attention to the outside variables in an \( i-j \) orientation, \( i \leq 3 \) or \( j \leq 3 \). This is because when \( i, j \geq 4 \), and while branching \( \delta = 0 \), we can only remove the variables in \( \delta \) and not factor in other changes in measure from the variables in \( L \) or \( R \). On the other hand, when \( \delta = 1 \), we can remove additional variables not in \( L, R \) and \( \delta \), whenever we have a variable having weight 0.8823. Let \( s \) be a variable not appearing in \( L, R \) or \( \delta \). We show all the possibilities below.

**Case 2:** The variables in \( L \) (or \( R \)) appear in further 3-literal clauses.

1. **Case 2.1:** A pair of 3-literal clauses containing \( s \), with the neighbours of \( s \) appearing in \( L \) and \( R \). For example, if we have \((l_1 \lor l_2 \lor \delta)\) and \((\delta \lor r_2 \lor r_1)\), and the two 3-literal clauses \((s \lor l_1 \lor r_1)\) and \((s \lor l_2 \lor r_2)\).

   In such a case, we branch \( s = 1 \) and \( s = 0 \). Now when we branch \( s = 1 \), we remove at least \( s, l_1, l_2, r_1, r_2 \). When \( s = 0 \), we link \( l_1 = \neg r_1 \) and \( l_2 = \neg r_2 \). Then, the new clauses will be \((\neg r_1, \neg r_2 \lor \delta)\) and \((\delta \lor r_2 \lor r_1)\). Then, by our simplification rule, we must have that \( \neg r_1 = r_2 \), and we can remove \( \delta \). To upper bound this branching factor, we treat all the variables as having weight 0.8823. This gives us a branching factor of \( \tau(5 \times 0.8823, (4 + |\delta|) \times 0.8823) \). Since \(|\delta| \geq 2 \), our branching factor is bounded above by 1.1541.

2. **Case 2.2:** Not Case 2.1. In other words, there is no such \( s \) that appears in two 3-literal clauses, where the neighbours of \( s \) are the variables in \( L \) and \( R \). In this case, we have 3-literal clauses, each containing a variable from \( L \), a variable from \( R \), and another variable not from \( L, R \), and \( \delta \).

By Line 10 of the algorithm, \( s \) cannot appear in a third 3-literal clause. Therefore, we must either have Case 2.1 or Case 2.2.

For Case 2.1, we have shown that if such a case arises, then 1.1541 acts as an upper bound for all such cases of \( k \geq 2 \) overlapping variables in an \( i-j \) orientation, \( i, j \geq 2 \). Therefore, in the Lemma below, we will not deal with such cases.

Case 2.2 arises when it is not Case 2.1; when there is no such \( s \), appearing in two 3-literal clauses, with the neighbours of \( s \) appearing in \( L \) and \( R \). Case 2.2 represents the case where we can have \((l \lor s \lor r)\), where \( l \in L, r \in R \) (s only appears in exactly one 3-literal case in Case 2.2).
Note that, apart from such a scenario in Case 2.2, we can of course have a variable appearing in a further 3-literal clause, containing a variable from \( L \) or \( R \), and then containing two variables not from \( L, R \) and \( \delta \) (Standalone 3-literal). For example, we have \( C_1 = (a \lor b \lor c \lor \delta) \) and \( C_2 = (\delta \lor d \lor e \lor f) \). So Case 2.2 has 3-literal clauses like \((c \lor s \lor d)\). However, we can also have Standalone 3-literal clauses like \((f \lor g \lor h)\), where \( g, h \) does not appear in \( L, R \) and \( \delta \).

We can show that Case 2.2 upper bounds the case of having Standalone 3-literal. Given any case of \( k \geq 2 \) overlapping variables, in an \( i-j \) orientation, \( i, j \geq 2 \), let our clauses be \((\alpha \lor x \lor \delta)\) and \((\delta \lor y \lor \beta)\), for some subclause \( \alpha, \beta \) and \( \delta \). We will now compare Case 2.2 with the case of having Standalone 3-literal clauses. We can have two Standalone 3-literal clause, on the variables \( x, y \), or we can have only one Standalone 3-literal clause, on either \( x \) or \( y \). Now let
\[
\Delta \mu_{\delta=1} \quad \text{(Case 2.2)}
\]
be the change of measure when branching \( \delta = 1 \). Therefore, we will always get a better branching factor when \( \delta = 1 \) (\( \delta = 0 \)).

- We can have a single Standalone 3-literal clause \((x \lor v_1 \lor v_2)\), where \( v_1, v_2 \) is not from \( \alpha, \beta \) and \( \delta \). Then this case gives us a branching factor of \( \tau(\Delta \mu_{\delta=1} + 1 + 0.8823 + 0.7646, \Delta \mu_{\delta=0}) \leq \tau(\Delta \mu_{\delta=1} + 3 \times 0.8823, \Delta \mu_{\delta=0}) \) (Case 2.2). We have 1 from the removal of \( y \), 0.8823 from \( x \) and 0.7646 (0.8823 – 0.1177) from linking \( v_1 \) and \( v_2 \).
- We can have a clause \((x \lor v_1 \lor v_2)\) and \((y \lor v_3 \lor v_4)\), where \( v_1, \ldots, v_4 \) are not from \( \alpha, \beta, \delta \). Then this gives us \( \tau(\Delta \mu_{\delta=1} + 2 \times (0.8823 + 0.7646), \Delta \mu_{\delta=0}) \leq \tau(\Delta \mu_{\delta=1} + 1 + 3 \times 0.8823, \Delta \mu_{\delta=0}) \) (Case 2.2).

Therefore, we see that the branching factor in Case 2.2 acts as an upper bound for the Standalone 3-literal case. Finally, we show that we can just treat all the variables in \( \delta \) as having weight 1 instead of 0.8823. Suppose a variable in \( \delta \) appears in a 3-literal clause. Then the same 3-literal clause cannot contain another variable from \( L, R \) or \( \delta \) it would be a 1-j orientation that would have already be handled earlier. So this 3-literal clause must be a Standalone. Let
\[
\Delta \mu_{\delta=1} \quad \text{be the change of measure when branching } \delta = 1 \quad \text{and } \Delta \mu_{\delta=0} \quad \text{be the change of measure when branching } \delta = 0
\]
for any case when the weight of variables in \( \delta \) is 1. For \( |\delta| \geq 3 \), the variable in \( \delta \) appears in a 3-literal clause, then the branching will give us \( \tau(\Delta \mu_{\delta=1}, \Delta \mu_{\delta=0} + 0.6469) < \tau(\Delta \mu_{\delta=1}, \Delta \mu_{\delta=0}) \). Note that when we have a Standalone 3-literal clause, we have a change of measure of 0.8823 + 0.7646 when \( \delta = 0 \). Now the difference between this and when the weight is 1 is 0.8823 + 0.7646 – 1 = 0.6469. When \( |\delta| = 2 \), we apply linking when we branch \( \delta = 1 \). This gives us \( \tau(\Delta \mu_{\delta=1} + 0.8823, \Delta \mu_{\delta=0} + 0.6469) < \tau(\Delta \mu_{\delta=1} + 1, \Delta \mu_{\delta=0}) \). Therefore, we will always get a better branching factor because the search tree becomes more balanced. Therefore it suffices to just deal with the case that the variables in \( \delta \) have weight 1 for our analysis below.

For the Lemma below, we will only show the Normal Case and Case 2.2 since these two cases upper bounds the other cases as shown above.

**Lemma 3.** The time complexity of dealing with two clauses with \( k \geq 2 \) overlapping variables, having \( i-j \) orientation, \( i, j \geq 2 \), is at most \( O(1.1674^n) \) time.
Proof. Let any two clauses be given with \( k \geq 2 \) overlapping variables and have at least 4 outside variables in a 2-2 orientation. We will show the Normal Case first, followed by Case 2.2 (only for outside variables \( i \leq 3 \) or \( j \leq 3 \)). For Case 2.2, and the appearance of each 3-literal clause, note that when branching \( \delta = 1 \), we can remove all the variables in the 3-literal clause, giving us \( 3 \times 0.8823 \) per 3-literal clause that appears in this manner. Let \( h \) denote the number of further 3-literal clauses for Case 2.2 encountered below. In addition, for Case 2.2 having odd number of outside variables, we treat the variable not in any 3-literal clause as having weight 1, acting as an upper bound to our cases.

For \( k = 2 \), and we have 4 outside variables in a 2-2 orientation. When \( \delta = 1 \), we remove all 4 outside variables and another 1 from linking the variables in \( \delta \). This gives us a change of measure 5. When \( \delta = 0 \), we remove 2 variables in \( \delta \) and another 2 from linking the variables in \( L \) and \( R \). This gives us a change of measure of 4. Therefore, we have a branching factor of \( \tau(5, 4) = 1.1674 \). For Case 2.2, we can have at most two 3-literal clauses here. This gives us \( \tau(h \times (3 \times 0.8823) + 2 \times (2 - h) + 1, 2 + 2 \times 0.8823), h \in \{1, 2\} \), which is at max branching factor of 1.1612, when \( h = 1 \). This completes the case for 4 outside variables.

Suppose we have 5 outside variables in 2-3 orientation. Branching \( \delta = 1 \) will remove all outside variables, and 1 of the linked variable in \( \delta \). This gives us a change of measure of 6. On the other hand, branching \( \delta = 0 \) will allow us to remove all the variables in \( \delta \), link the 2 variables in \( L \), and factor in the change of measure for the remaining variables in \( R \). This gives us a change of measure of \( \tau(6, 3 + 3 \times 0.1177) = 1.1648 \). For Case 2.2, we have at most two 3-literal clauses appearing in both \( L \) and \( R \). Then we have \( \tau(h \times (3 \times 0.8823) + 2 \times (2 - h) + 2, 2 + 0.8823 + 0.1177), h \in \{1, 2\} \), which is at max branching factor of 1.1636 when \( h = 1 \). This completes the case for 5 outside variables.

Suppose we have 6 outside variables in 3-3 orientation. Branching \( \delta = 1 \) will remove all 6 outside variables in \( L \) and \( R \), and also remove an additional variable by linking the two variables in \( \delta \). This gives us a change of measure of 7. On the other hand, when \( \delta = 0 \), we remove all the variables in \( \delta \) and also factor in the change of measure for the variables in \( L \) and \( R \), a total of \( 2 + 6 \times 0.1177 \) for this branch. This gives us a branching factor of \( \tau(7, 2 + 6 \times 0.1177) = 1.1664 \). When Case 2.2 applies, then we can have at most three 3-literals clauses appearing. This gives us a branching factor of \( \tau(h \times (3 \times 0.8823) + 2 \times (3 - h) + 1, 2 + 2 \times (3 - h) \times 0.1177), h \in \{1, 2, 3\} \), with max branching of 1.1641 when \( h = 1 \). This completes the case for 6 outside variables.

Suppose we have 7 outside variables in a 3-4 orientation. Branching \( \delta = 1 \) will allow us to remove all 7 outside variables, and 1 variable from \( \delta \) via linking. This gives us a change of measure of 8. On the other hand, when \( \delta = 0 \), we can factor in a change of measure of \( 3 \times 0.1177 \) from the variables. This gives us a branching factor of \( \tau(8, 2 + 3 \times 0.1177) = 1.1630 \). For Case 2.2, there are at most three 3-literal clauses between \( L \) and \( R \). This gives us a branching factor of \( \tau(h \times (3 \times 0.8823) + 2 \times (3 - h) + 1, 2 + (3 - h) \times 0.1177) \), which is at max of 1.1585 when \( h = 1 \). This completes the case for 7 outside variables.
Let $p \geq 8$ be the number of outside variables. Branching $\delta = 1$ allows us to remove all $p$ outside variables, and an additional variable from linking in $\delta$, which has a change of measure of 9. For the $\delta = 0$ branch, we remove two variables. This gives us a branching factor of $\tau(p+1,2) \leq \tau(9,2) = 1.1619$. This completes the case for $k = 2$ overlapping variables.

Now we deal with $k = 3$ overlapping variables. If there are 4 outside variables in a 2-2 orientation, then branching $\delta = 1$ will allow us to remove all 4 outside variables, which is a change of measure of 4. On the other hand, branching $\delta = 0$ will allow us to remove all the variables in $\delta$, as well as link the two variables in $L$ and $R$, removing a total of 5 variables. This gives $\tau(4,5) = 1.1674$. When we have Case 2.2, then we have at most two 3-literal clauses appearing. This gives us a branching factor of at most $\tau(h \times (3 \times 0.8823) + 2 \times (2-h), 3 + 2 \times 0.8823), \ h \in \{1,2\}$, which has a max branching factor of 1.1588 when $h = 1$. This completes the case for 4 outside variables in a 2-2 orientation.

For the case of 5 outside variables, they are in a 2-3 orientation. Branching $\delta = 1$ will allow us to remove all 5 outside variables. On the other hand, branching $\delta = 0$ will allow us to remove all the variables in $\delta$, as well as an additional variable from linking the two variables in $L$, a total of 4 variables. This gives us a branching factor of $\tau(5,4) = 1.1674$. For Case 2.2, we can have at most two 3-literal clauses occurring. For simplicity, we treat the 3rd variable in $R$ as having weight 0.8823. Then we have a branching factor of $\tau(h \times (3 \times 0.8823) + 2 \times (2-h) + 1,3 + 0.8823 + 0.1177), h \in \{1,2\}$, which is at max branching factor of 1.1563 when $h = 1$. This completes the case for 5 outside variables.

For the case of 6 outside variables, they are in a 3-3 orientation. When branching $\delta = 1$, we can remove all 6 outside variables. When branching $\delta = 0$, we remove all 3 variables in $\delta$, and we can factor in the change of measure for these of 0.1177 for these 6 variables. This gives a branching factor of $\tau(6,3 + 6 \times 0.1177) = 1.1569$. In Case 2.2, we can have at most three 3-literal appearing in $L$ and $R$. Then the branching factor for this case would be $\tau(h \times (3 \times 0.8823) + 2 \times (3-h), 3 + 2 \times (3-h) \times 0.1177), h \in \{1,2,3\}$, which is at max branching factor of 1.1526 when $h = 1$. This completes the case for 6 outside variables.

Let $p \geq 7$ be the number of outside variables. Then branching $\delta = 1$ will allow us to remove at least 7 variables, and when $\delta = 0$, we remove all the variables in $\delta$. This gives us a branching factor of $\tau(p,3) \leq \tau(7,3) = 1.1586$. For Case 2.2, we can have at most $h \leq \lfloor \frac{p}{2} \rfloor$ 3-literals clauses. Then our branching factor is $\tau(h \times (3 \times 0.8823) + 2 \times (\lfloor \frac{p}{2} \rfloor - h) + 1,3)$, which is at max of 1.1503 when $h = 1$ and $\lfloor \frac{p}{2} \rfloor = 3$. This completes the case for $k = 3$ overlapping variables.

Now, we deal with the case of $k = 4$ overlapping variables. When we have 4 outside variables in a 2-2 orientation, then branching $\delta = 1$ will allow us to remove all 4 outside variables, giving us a change of measure of 4. On the other hand, when $\delta = 0$, we remove all the variables in $\delta$, and link the two variables in $L$ and $R$. This gives us a change of measure of 6. Therefore, we have a branching factor of $\tau(4,6) = 1.1510$. For Case 2.2, we can have at most two 3-literal clauses. Then our branching factor is $\tau(h \times (3 \times 0.8823) + 2 \times (2-h), 4 + 2 \times 0.8823)$,
\( h \in \{1, 2\} \), which is at max 1.1431 when \( h = 1 \). This completes the case for 4 outside variables.

If there are \( p \geq 5 \) outside variables, then branching \( \delta = 1 \) will allow us to remove at least 5 variables. On the other hand, branching \( \delta = 0 \) will remove all the variables in \( \delta \). This gives a branching factor of \( \tau(p, 4) \leq \tau(5, 4) = 1.1674 \). For Case 2.2, we can have at most \( h \leq \lfloor \frac{p}{2} \rfloor \) number of 3-literal clauses. The branching factor is \( \tau(h \times (3 \times 0.8823) + 2 \times (\lfloor \frac{p}{2} \rfloor - h), 1, 4) \), which is at max of 1.1563 when \( h = 1 \) and \( \lfloor \frac{p}{2} \rfloor = 2 \). This completes the case for 5 outside variables.

Finally for \( k \geq 5 \) overlapping variables and \( p \geq 4 \) outside variables, branching \( \delta = 1 \) will remove at least 4 variables, while branching \( \delta = 0 \) will remove at least 5 variables. This gives us \( \tau(p, k) \leq \tau(4, 5) = 1.1674 \). For Case 2.2, there can be at most \( h \leq \lfloor \frac{p}{2} \rfloor \) number of 3-literal clauses. Then our branching factor is at most \( \tau(h \times (3 \times 0.8823) + 2 \times (\lfloor \frac{h}{2} \rfloor - h), k) \leq \tau(h \times (3 \times 0.8823) + 2 \times (\lfloor \frac{h}{2} \rfloor - h), 5) \), which has max branching factor of 1.1547 when \( h = 1 \) and \( \lfloor \frac{h}{2} \rfloor = 2 \). This completes the case for \( k \geq 2 \) overlapping variables and the max branching factor while executing this line of the algorithm is 1.1674.

### 4.4 Line 13 of the algorithm

Now, we deal with Line 13 of the algorithm, to branch off heavy variables in the formula. After Line 12 of the algorithm, given any two clauses \( C_1 \) and \( C_2 \), there can only be at most only 1 variable appearing in them. Cases 1 and 2 in the previous section will also apply here. In Section 4.3, we paid special attention to \( L \) and \( R \) when \( |L| = 3 \) or \( |R| = 3 \). Here, we pay special attention to \( x \) being in 4-literal clauses, because after branching \( x = 0 \), it will drop to a 3-literal clause. Since we have dealt with \((3, 3, 3)\) case earlier, here, we’ll deal with the remaining cases; cases from \((3, 3, \geq 4)\) to \((\geq 5, \geq 5, \geq 5)\).

For Case 1 (common neighbour), we will only show the analysis for the \((4, 4, 4)\) case because it is only this case where we can factor in a change of \( 9 \times 0.1177 > 0.8823 \), which is better than removing the common neighbour. For Case 2, there are some changes as well. Here, we are dealing with 3 clauses instead of 2 in the previous section, therefore, there will be more permutation of 3-literal clauses to consider. Recall previously that we dealt with a case where \( s \) appears in two 3-literal clauses in Case 2.1 of Section 4.3. Here, we deal with something similar.

Suppose there are clauses \( C_1 = (l_1 \lor l_2 \lor \delta \lor x) \), \( |C_1| \geq 3 \), \( C_2 = (r_1 \lor r_2 \lor \alpha \lor x) \), \( |C_2| \geq 4 \), for some subclause \( \delta \) and \( \alpha \), \( (s \lor l_1 \lor r_1) \) and \( (s \lor l_2 \lor r_2) \), \( s \) not appearing in the clauses \( C_1 \) and \( C_2 \). We give an upper bound for such cases. When \( s = 1 \), we remove 5 variables, with a change of measure of \( 5 \times 0.8823 \).

When \( s = 0 \), we remove \( s \), and link \( l_1 = \neg r_1 \), and \( l_2 = \neg r_2 \). After which, the remaining clauses become \((\neg r_1 \lor \neg r_2 \lor \delta \lor x)\) and \((r_1 \lor r_2 \lor \alpha \lor x)\). Then, we must have \( r_1 = \neg r_2 \), and we can remove one of the linked variable, an additional variable from \( C_2 \) and \( x \). This gives us a change of measure of \( 6 \times 0.8823 \). Note that we require that one of the two clauses to be at least length 4. When both clauses are length 3, by default, we have already treat all variables to have weight 0.8823, hence we ignore such cases. This gives us a branching factor of at most
We define Case 2.1 and Case 2.2, while keeping the Normal Case as before.

Let $\tau$ be a variable not appearing in the clauses that we are discussing about. We define Case 2.1 and Case 2.2, while keeping the Normal Case as before.

Case 2.1: If there are two clauses $C_1 = (a_1 \lor a_2 \lor ... \lor x)$, $C_2 = (b_1 \lor b_2 \lor ... \lor x)$, $C_3 = (c_1 \lor c_2 \lor ... \lor x)$, $(s \lor a_1 \lor b_1)$ and $(s \lor b_2 \lor c_1)$. Note that some clause $C_2$ has two variables as neighbours of $s$. For the proof below, we will use this (a clause having two variables in it as neighbours of $s$) notation to denote the worst case. For the other variables, it is possible to have 3-literal clauses appearing in different permutation.

Case 2.2: No such $s$ occurs where we have a clause that has two variables in it as neighbours of $s$. Therefore, for each 3-literal clause appearing, it will only contain two variables from the clauses, and a new variable not in the three clauses. For example, if we have $(x \lor v_1 \lor v_2 \lor v_3)$, $(x \lor v_4 \lor v_5 \lor v_6)$ and $(x \lor v_7 \lor v_8 \lor v_9)$. Here, we consider 3-literal clauses appearing as $(s \lor v_1 \lor v_4)$. Similar to Case 2.2 in the previous section.

In Case 2, $s$ cannot appear in the third 3-literal clause, else Line 10 of the algorithm would have already handled it. Therefore, the new variable $s$ can appear in at most two 3-literal clauses. Our cases here are complete.

**Lemma 4.** The time complexity of branching heavy variables is $O(1.1668^n)$.

**Proof.** Let $x$ be a heavy variable. Given $(l_1, l_2, l_3)$, then there are $|l_1| + |l_2| + |l_3| - 2$ unique variables in these clauses. Let $h$ denote the number of 3-literal clauses as shown in Case 2.2 above. We will give the Normal case, Case 1 (only for $(4, 4, 4)$), Case 2.1 and Case 2.2. For Case 2.1, we will treat all variables as having weight 0.8823 to lessen the number of cases we need to consider. In addition, we handle the cases in the following order: $(3, 3, \geq 4)$, then $(3, \geq 4, \geq 4)$ etc.

$(3, 3, \geq 4)$. We’ll first start with $(3, 3, 4)$. Branching $x = 1$ will allow us to remove all the variables in this case, with a change in measure of $5 \times 0.8823 + 3$. When $x = 0$, we will have a change in measure of $3 \times 0.8823 - 2 \times 0.1177$, and when the 4-literal clause drops to a 3-literal clause, another $3 \times 0.1177$. This gives $\tau(5 \times 0.8823 + 3, 3 \times 0.8823 + 0.1177) = 1.1591$. If Case 2.1 occurs, then the worst case here would be that one of the 3-literal clauses (in $(3, 3, 4)$), contain two variables that are neighbours to $s$. We branch $x = 1$ and $x = 0$ to get $\tau(7 \times 0.8823 + 3, 3 \times 0.8823) = 1.1526$. If Case 2.2 occurs, then we can have at most three 3-literal clauses. We branch $x = 1$ and $x = 0$. Then the branching factor is given as $\tau(5 \times 0.8823 + h \times (2 \times 0.8823) + (3 - h), 3 \times 0.8823 + (3 - h) \times 0.1177 - 2 \times 0.1177), h \in \{1, 2, 3\}$, which is at most of 1.1526 when $h = 1$. This completes the case for $(3, 3, \geq 4)$. Next, we deal with $(3, 3, \geq 5)$. For such a case, when $x = 1$, we remove all variables, which gives us a change of measure of $5 \times 0.8823 + 4$. When $x = 0$, we remove $x$ and link up the two variables in the 3-literal clauses. This gives us a change of $3 \times 0.8823 - 2 \times 0.1177$. The branching factor for this case would be $\tau(5 \times 0.8823 + 4, 3 \times 0.8823 - 2 \times 0.1177) = 1.1562$. If Case 2.1 or 2.2 applies here, then we give an upper bound to these cases by treating all variables in the 5-literal clause as having weight 0.8823. When $x = 1$, we remove all 9 variables, this
(3, ≥ 4, ≥ 4). We start with (3, 4, 4). Branching \( x = 1 \) will allow us to remove all the variables, this gives us a change of measure of \( 6 + 3 \times 0.8823 \). On the other hand, branching \( x = 0 \), we can factor in a change of measure of \( 2 \times 0.8823 - 0.1177 + 6 \times 0.1177 \). This gives us a branching factor of \( (6 + 3 \times 0.8823) (2 \times 0.8823 + 5 \times 0.1177) = 1.1551 \). For Case 2.1, the worst case happens when we have two variables in any of the 4-literal clauses as neighbours of \( s \). Branching \( s = 1 \) will allow us to remove 7 variables, where one of which is via linking of a variable in a 3-literal clause, giving us \( 7 \times 0.8823 - 0.1177 \). When \( s = 0 \), we remove \( x, s \) and 2 variables via linking in the 3-literal clause, giving us \( 4 \times 0.8823 - 2 \times 0.1177 \). This gives \( \tau(7 \times 0.8823 - 0.1177, 4 \times 0.8823 - 2 \times 0.1177) = 1.1653 \). For Case 2.2, we can have at most three 3-literal clauses appearing across the two 4-literal clauses. Then branching \( x = 1 \) and \( x = 0 \) gives us \( \tau(3 \times 0.8823 + h \times (3 \times 0.8823 - 2 \times 0.1177), 2 \times (3 - h) \times 2 \times 0.1177), h \in \{ 1, 2, 3 \} \). This is at max of 1.1571 when \( h = 3 \). This completes the case for (3, 4, 4). For (3, 4, ≥ 5), branching \( x = 1 \) will allow us to remove all variables, representing a change in measure of \( 3 \times 0.8823 + 7 \). On the other hand, branching \( x = 0 \) will allow us to remove \( x \), link a variable in the 3-literal clause and factor in the change in measure for the 4-literal clauses. This gives us \( \tau(3 \times 0.8823 - 0.1177, 2 \times 0.8823 + 2 \times 0.1177) = 1.1547 \). For Case 2.1 and Case 2.2, we can find a variable \( s \) that does not appear in any of the clauses. We give an upper bound for this case by treating all variables as having weight 0.8823. When \( x = 1 \), we remove all 10 variables and \( s \). This gives us \( 11 \times 0.8823 \). When \( x = 0 \), we remove \( x \) and link up the other variable in the 3-literal clause, giving us \( 2 \times 0.8823 + 0.1177 \). This gives us a branching factor of at most \( \tau(11 \times 0.8823, 1 \times 0.8823 - 0.1177) = 1.1666 \). This completes the case for (3, 4, ≥ 5). Finally, for the case of (3, ≥ 5, ≥ 5), we give an upper bound for this case by treating all the variables as having weight 0.8823, to deal with the Normal Case, Case 2.1 and 2.2 at the same time. Branching \( x = 1 \) gives us a change of measure of \( 11 \times 0.8823 \). When \( x = 0 \), this gives us \( 2 \times 0.8823 - 0.1177 \). Putting them together, we have a branching factor of at most \( \tau(11 \times 0.8823, 2 \times 0.8823 - 0.1177) = 1.1666 \) for this case. This completes the case for (3, ≥ 5, ≥ 5) and hence (3, ≥ 4, ≥ 4).

(4, 4, 4). When \( x = 1 \), we remove all variables. This gives us a change of measure of 10. On the other hand, when \( x = 0 \), we have a change of measure of \( 1 + 9 \times 0.1177 \). This gives us a branching factor of \( \tau(10, 1 + 9 \times 0.1177) = 1.1492 \). If Case 1 occurs, then we have at most \( \tau(10, 1 + 0.8823) = 1.1548 \). When Case 2.1 occurs, then one of the 4-literal clause must have 2 variables in it that are neighbours to \( s \). Suppose we have \( (x \lor a_1 \lor a_2 \lor a_3), (x \lor b_1 \lor b_2 \lor b_3), (x \lor c_1 \lor c_2 \lor c_3), (s \lor a_1 \lor b_1) \) and \( (s \lor b_2 \lor c_1) \). Then we branch \( b_1 = 1 \) and \( b_1 = 0 \). When \( b_1 = 1 \), then \( s = a_1 = b_2 = b_3 = x = 0 \). Since \( s = b_2 = 0 \), then \( c_1 = 1 \). Therefore, we must have \( c_2 = c_3 = 0 \) and we can link up \( a_2 = \neg a_3 \). Now, \( x \) must have weight 1, else earlier cases would have handled it. This gives a change of measure of \( 9 \times 0.8823 + 1 \). On the other hand, when \( b_1 = 0 \), we link up \( a_1 = \neg s \) (no increase
in measure here since $s$ is still in another 3-literal clause), $x$ will drop in weight, giving us a change of measure of $2 \times 0.8823 + 0.1177$. This gives a branching factor of at most $\tau(9 \times 0.8823 + 1, 2 \times 0.8823 + 0.1177) = 1.1668$. When Case 2.2 arises, then we can have at most six 3-literal clauses appearing across the 4-literal clauses. This gives an upper bound of $\tau(15 \times 0.8823, 1) = 1.1610$ when there are six such 3-literal clauses. This completes the case for $(4, 4, 4)$.

$(4, 4, 5)$. When $x = 1$, we remove all 11 variables. When $x = 0$, we remove $x$ and factor in the change of measure from the 4-literal clauses, giving us $1 + 6 \times 0.1177$. This gives us a branching factor of $\tau(11, 1 + 6 \times 0.1177) = 1.1509$. If Case 2.1 occurs, and two variables from a 4-literal clause is a neighbour to $s$, then choose the variable that is a neighbour to $s$ to branch, such that we can remove all the variables in the 5-literal clause (same technique as above). The same upper bound of 1.1668 will also apply here. If two variables from a 5-literal clause is a neighbour to $s$, then branch any of these two variables to get the same upper bound of 1.1668. If Case 2.2 applies, then there are at most two 3-literal clauses between the two 4-literal clauses. Then our branching factor is given as $\tau(h \times (3 \times 0.8823) + 5 + 2 \times (3 - h), 1 + 2 \times (3 - h) \times 0.1177), h \in \{1, 2, 3\}$, which is at max of 1.1637 when $h = 3$. This completes the case for $(4, 4, 5)$.

$(4, 5, 5)$. When $x = 1$, we remove all 12 variables. When $x = 0$, we remove $x$ and factor in the change of measure of $1 + 3 \times 0.1177$. Therefore, we have $\tau(12, 1 + 3 \times 0.1177) = 1.1551$. When Case 2.1 occurs, then follow the same technique as given in $(4, 4, 5)$ to get the upper bound of 1.1668 here. When Case 2.2 occurs, then we can have at most three 3-literal clauses appearing across the 4-literal and the 5-literal clauses. Then, we have a max branching factor of $\tau(h \times (3 \times 0.8823) + 2 \times (3 - h) + 6, 1 + (3 - h) \times 0.1177), h \in \{1, 2, 3\}$, which is at max of 1.1550 when $h = 3$. This completes the case of $(4, 5, 5)$.

$(5, 5, 5)$. When $x = 1$, we remove 13 variables and when $x = 0$, we remove only $x$. This gives us $\tau(13, 1) = 1.1632$. If Case 2.1 occurs, follow the same technique as given in $(4, 4, 5)$ when two variables in the 5-literal clause are neighbours to $s$. This gives us the same upper bound of 1.1668. For Case 2.2, then worst case occurs when every variable in $(5, 5, 5)$ has weight 1, which gives 1.1632 (Normal Case). This is because when $x = 0$, we can only remove $x$ and not factor in any other change in measure. On the other hand, when any of the variables have weight 0.8823, this means we can remove additional variables when $x = 1$. This completes the case for $(5, 5, 5, 5)$. Hence, Line 14 of the algorithm runs in $O(1.1668^n)$ time.

Therefore, putting all the lemmas together, we have the following result:

**Theorem 1.** The algorithm runs in $O(1.1674^n)$ time.

In summary, we proposed a DPLL style algorithm to solve the XSAT problem in $O(1.1674^n)$. Prior to this work, the current state of the algorithm is another DPLL style algorithm which runs in $O(1.1730^n)$. The novelty of our algorithm lies on the design of a nonstandard measure to help us to tighten our analysis further. However, this has led to some additional cases that we have to analyse. Perhaps a question for interested readers would be: Is it possible to design a
simple nonmeasurer to either improve the worst case bound further? Or to cut down the number of cases that we need to analyse.

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