A Lagrange–Galerkin scheme with a locally linearized velocity for the Navier–Stokes equations

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Abstract We present a Lagrange–Galerkin scheme free from numerical quadrature for the Navier–Stokes equations. Our idea is to use a locally linearized velocity and the backward Euler method in finding the position of fluid particle at the previous time step. Since the scheme can be implemented exactly as it is, the theoretical stability and convergence results are assured. While the conventional Lagrange–Galerkin schemes may encounter the instability caused by numerical quadrature errors, the present scheme is genuinely stable. For the $P_2/P_1$- and $P_1+/P_1$-finite elements optimal error estimates are proved in $\ell^\infty(H^1) \times \ell^2(L^2)$ norm for the velocity and pressure. We present some numerical results, which reflect these estimates and also show the genuine stability of the scheme.

Keywords Lagrange–Galerkin scheme · Finite element method · Navier–Stokes equations · Exact integration

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1 Introduction

The purpose of this paper is to present a Lagrange–Galerkin scheme free from numerical quadrature for the Navier–Stokes equations and to prove the convergence. The Lagrange–Galerkin method, which is also called characteristics finite element method or Galerkin-characteristics method, is a powerful numerical method for flow...
problems, having such advantages that it is robust for convection-dominated problems and that the resultant matrix to be solved is symmetric. It has, however, a drawback that it may lose the stability when numerical quadrature is employed to integrate composite function terms that characterize the method. Our scheme presented here overcomes this drawback.

Lagrange–Galerkin schemes for the Navier–Stokes equations have been developed in [14,15,13,14,15,17,19]; see also bibliography therein. After convergence analysis was done successfully by Pironneau [15] in a suboptimal rate, the optimal convergence result was obtained by Suli [19]. Optimal convergence results by Lagrange–Galerkin schemes were extended to the multi-step method by Boukir et al. [5], to the projection method by Achdou–Guermond [1] and to the pressure-stabilized method by Notsu-Tabata [14]. All these results of the stability and convergence are proved under the condition that the integration of the composite function terms is computed exactly. Since it is difficult to perform the exact integration in real problems, numerical quadrature is usually employed. It is, however, reported that instability may occur caused by numerical quadrature error for convection-diffusion problems in [12,17,20,23]. We observe such instability occurs for the Navier-Stokes equations by numerical examples in this paper.

Several methods have been studied to avoid the instability in [4,12,16,17,23]. The map of a fluid particle from the present position to the position a time increment $\Delta t$ before (the position is often called foot along the trajectory) is simplified. To find the foot of a particle is nothing but to solve a system of ordinary differential equations (ODEs). Morton et al. [12] solved the ODEs only at the centroids of the elements, and Priestley [17] did only at the vertices of the elements. The map of the other points is approximated by linear interpolation of those values. It becomes possible to perform the exact integration of the composite function terms with the simplified map. Bermejo et al. [4] used the same simplified map as [17] to employ a numerical quadrature of high accuracy to the composite function terms for the Navier-Stokes equations. Tanaka et al. [23] and Tabata–Uchiumi [22] approximated the map by a locally linearized velocity and the backward Euler approximation to solve the ODEs for convection-diffusion problems. The approximate map makes possible the exact integration of the composite function terms.

In this paper we prove the convergence of a Lagrange–Galerkin scheme with the same approximate map as [22,23] in the $P_2/P_1$- or $P_1+/P_1$-element for the Navier–Stokes equations. Since we neither solve the ODEs nor use numerical quadrature, our scheme can be precisely implemented to realize the theoretical results. It is, therefore, a genuinely stable Lagrange–Galerkin scheme. Our convergence results are best possible for the velocity and pressure in $\ell^\infty(\mathcal{H}^1) \times \ell^2(L^2)$-norm for both elements as well as for the velocity in $\ell^\infty(L^2)$-norm in the $P_1+/P_1$-finite element.

The contents of this paper are as follows. In the next section we describe the Navier–Stokes problem and some preparation. In Section 3 after recalling the conventional Lagrange–Galerkin scheme, we present our Lagrange–Galerkin scheme with a locally linearized velocity. In Section 4 we show convergence results, which are proved in Section 5. In Section 6 we show some numerical results, which reflect the theoretical convergence orders and the robustness of the scheme for high Reynolds number problems. In Section 7 we give the conclusions.
2 Preliminaries

We state the problem and prepare the notation used throughout this paper.

Let \( \Omega \) be a polygonal or polyhedral domain of \( \mathbb{R}^d \) (\( d = 2, 3 \)) and \( T > 0 \) a time. We use the Sobolev spaces \( L^p(\Omega) \) with the norm \( \| \cdot \|_{0,p} \), \( W^{s,p}(\Omega) \) and \( W_0^{s,p}(\Omega) \) with the norm \( \| \cdot \|_{s,p} \) and the semi-norm \( | \cdot |_{s,p} \) for \( p = 1, 2, \infty \) and a positive integer \( s \). When \( p = 2 \), we write \( H^s(\Omega) = W^{s,2}(\Omega) \) simply and drop the subscript 2 in the corresponding norms. For the vector-valued function \( w \in W^{1,\infty}(\Omega)^d \) we define the semi-norm \( |w|_{1,\infty} \) by

\[
|w|_{1,\infty} = \left\{ \left( \sum_{i,j=1}^{d} \left( \frac{\partial w_i}{\partial x_j} \right)^2 \right)^{1/2} \right\}_{0,\infty}.
\]

\( L^2_0(\Omega) \) is the subspace of \( L^2(\Omega) \) with the zero mean. The parenthesis \( \langle \cdot, \cdot \rangle \) shows the \( L^2(\Omega)^d \)-inner product for \( i = 1, d \) or \( d \times d \). For \( w \in L^2(\Omega)^d \), \( \|w\|_{-1} \) stands for the dual norm

\[
\sup_{v \in H^1_0(\Omega)^d \setminus \{0\}} \frac{\langle w, v \rangle}{\|v\|_1}.
\]

For a Sobolev space \( X(\Omega) \) we use the abbreviations \( H^m(X) = H^m(0,T;X(\Omega)) \) and \( C(X) = C([0,T];X(\Omega)) \).

\[
\begin{align*}
Z^m(t_1,t_2) &\equiv \left\{ f \in H^i(t_1,t_2;H^{m-i}(\Omega)^d) ; j = 0, \ldots, m, \|f\|_{Z^m(t_1,t_2)} < \infty \right\}, \\
\|f\|_{Z^m(t_1,t_2)} &\equiv \left\{ \sum_{j=0}^{m} \|f\|_{H^i(t_1,t_2;H^{m-i}(\Omega)^d)}^2 \right\}^{1/2}
\end{align*}
\]

and denote \( Z^m(0,T) \) by \( Z^m \).

We consider the Navier–Stokes equations: find \((u,p) : \Omega \times (0,T) \to \mathbb{R}^d \times \mathbb{R} \) such that

\[
\begin{align*}
\frac{Du}{Dt} - \nu \Delta u + \nabla p &= f & \text{in } \Omega \times (0,T), \\
\nabla \cdot u &= 0 & \text{in } \Omega \times (0,T), \\
u \cdot u &= 0 & \text{on } \partial \Omega \times (0,T), \\
u u &= u^0 & \text{in } \Omega \text{ at } t = 0,
\end{align*}
\]

where \( \partial \Omega \) is the boundary of \( \Omega \), \( \frac{Du}{Dt} \equiv \frac{du}{dt} + (u \cdot \nabla)u \) is the material derivative and \( \nu > 0 \) is a viscosity. Functions \( f \in C(L^2) \) and \( u^0 : \Omega \to \mathbb{R}^d \) are given.

We define the bilinear forms \( a \) on \( H^1_0(\Omega)^d \times H^1_0(\Omega)^d \) and \( b \) on \( H^1_0(\Omega)^d \times L^2_0(\Omega) \) by

\[
a(u,v) \equiv \nu \langle \nabla u, \nabla v \rangle, \quad b(v,q) \equiv -\langle \nabla \cdot v, q \rangle.
\]

Then, we can write the weak form of (1) as follows: find \((u,p) : (0,T) \to H^1_0(\Omega)^d \times L^2_0(\Omega) \) such that for \( t \in (0,T) \),

\[
\left( \frac{Du}{Dt}(t), v \right) + a(u(t),v) + b(v,p(t)) = (f(t),v), \quad \forall v \in H^1_0(\Omega)^d, \quad \forall t \in (0,T),
\]

(2a)
with \( u(0) = u^0 \).

Let \( u \) be smooth. The characteristic curve \( X(t;x,s) \) is defined by the solution of the system of the ordinary differential equations,

\[
\frac{dX}{dt}(t;x,s) = u(X(t;x,s),t), \quad t < s, \quad X(s;x,s) = x.
\]

(3a)

(3b)

Then, we can write the material derivative term \((\frac{d}{dt} + \nabla \cdot u)u\) as follows:

\[
\left( \frac{\partial u}{\partial t} + (\nabla \cdot u)u \right)(X(t),t) = \frac{d}{dt}v(X(t),t).
\]

Let \( \Delta t > 0 \) be a time increment. For \( w : \Omega \rightarrow \mathbb{R}^d \) we define the mapping \( X_1(w) : \Omega \rightarrow \mathbb{R}^d \) by

\[
(X_1(w))(x) \equiv x - w(x)\Delta t.
\]

(4)

**Remark 1** The image of \( x \) by \( X_1(u(\cdot,t)) \) is nothing but the approximate value of \( X(t - \Delta t;x,t) \) obtained by solving (3) by the backward Euler method.

Let \( N_T \equiv \lfloor T/\Delta t \rfloor \), \( t^n \equiv n\Delta t \) and \( \psi^n \equiv \psi(\cdot,t^n) \) for a function \( \psi \) defined in \( \Omega \times (0,T) \). For a set of functions \( \psi = \{ \psi^n \}_{n=0}^\infty \) and a Sobolev space \( X(\Omega) \), two norms \( \| \cdot \|_{1,\Omega} \) and \( \| \cdot \|_{2,\Omega} \) are defined by

\[
\| \psi \|_{1,\Omega} \equiv \max \left\{ \| \psi^n \|_{X(\Omega)} : n = 0, \ldots, N_T \right\},
\]

\[
\| \psi \|_{2,\Omega} \equiv \left( \Delta t \sum_{n=n_1}^{n_2} \| \psi^n \|_{X(\Omega)}^2 \right)^{1/2},
\]

and \( \| \psi \|_{\ell^2(1,N_T;X)} \) is denoted by \( \| \psi \|_{\ell^2(\Omega)} \). The backward difference operator \( \mathcal{D}_{\Delta t} \) is defined by

\[
\mathcal{D}_{\Delta t} \psi^n \equiv \frac{\psi^n - \psi^{n-1}}{\Delta t}.
\]

Let \( \mathcal{T}_h \) be a triangulation of \( \Omega \) and \( h \equiv \max_{K \in \mathcal{T}_h} \text{diam}(K) \) the maximum element size. Throughout this paper we consider a regular family of triangulations \( \{ \mathcal{T}_h \}_{h>0} \). Let \( V_h \times Q_h \subset H_0^1(\Omega)^d \times L_0^2(\Omega) \) be the \( P_2/P_1 \) or \( P_1+/P_1 \)-finite element space, which is called Hood-Taylor element or MINI element \([9,2]\). Let

\[
\Pi_h^1 : C(\Omega)^d \cap H_0^1(\Omega)^d \rightarrow V_h
\]

be the Lagrange interpolation operator to the \( P_1 \)-finite element space. Let \( (\mathcal{w}_h,\mathcal{r}_h) \equiv \Pi_h^1(w,r) \in V_h \times Q_h \) be the Stokes projection of \( (w,r) \in H_0^1(\Omega)^d \times L_0^2(\Omega) \) defined by

\[
a(\mathcal{w}_h, v_h) + b(v_h, \mathcal{r}_h) = a(w, v_h) + b(v_h, r), \quad \forall v_h \in V_h, \quad (5a)
\]

\[
b(\mathcal{w}_h, q_h) = b(w, q_h), \quad \forall q_h \in Q_h. \quad (5b)
\]

We denote by \( (\Pi_h^2(w,r))_1 \) the first component \( \mathcal{w}_h \) of \( \Pi_h^2(w,r) \).

The symbol \( \circ \) stands for the composition of functions, e.g., \( (g \circ f)(x) \equiv g(f(x)) \).
3 A Lagrange–Galerkin scheme with a locally linearized velocity

The conventional Lagrange–Galerkin scheme, which we call Scheme LG, is described as follows.

**Scheme LG** Let \( u^n_h = (\Pi^L_1(u^n, 0))_1 \). Find \( \left\{ (u^n_h, p^n_h) \right\} \in V_h \times Q_h \) such that

\[
\left( \frac{u^n_h - u^{n-1}_h \circ X_1(u^{n-1}_h)}{\Delta t}, v_h \right) + a(u^n_h, v_h) + b(v_h, p^n_h) = (f^n, v_h), \quad \forall v_h \in V_h,
\]

\[
b(u^n_h, q_h) = 0, \quad \forall q_h \in Q_h,
\]

for \( n = 1, \ldots, N_T \).

**Remark 2** Süli [19] used the exact solution \( X_h^{n-1} \) of the system of ordinary differential equations,

\[
\begin{align*}
\frac{dX_h^{n-1}}{dt}(t; x, t^n) &= u^{n-1}_h(X_h^{n-1}(t; x, t^n), t), \quad t^{n-1} < t < t^n, \tag{7a} \\
X_h^{n-1}(t^n, x, t^n) &= x \tag{7b}
\end{align*}
\]

instead of \( X_1(u^{n-1}_h) \).

By a similar way to [19] combined with [3], error estimates

\[
\begin{align*}
\|u_h - u\|_{L^2(H^1)} + \|p_h - p\|_{L^2(\mathbb{V})} &\leq c(h^k + \Delta t), \tag{8a} \\
\|u_h - u\|_{L^2(\mathbb{V})} &\leq c(h^{k+1} + \Delta t), \tag{8b}
\end{align*}
\]

can be proved, where \( k = 2 \) for \( P_2/P_1 \)-element and \( k = 1 \) for \( P_1+/P_1 \)-element. In the estimate above, the composite function term \( (u^{n-1}_h \circ X_1(u^{n-1}_h), v_h) \) is assumed to be exactly integrated.

Although the function \( u^{n-1}_h \) is a polynomial on each element \( K \), the composite function \( u^{n-1}_h \circ X_1(u^{n-1}_h) \) is not a polynomial on \( K \) in general since the image \( X_1(u^{n-1}_h) \) of an element \( K \) may spread over plural elements. Hence, it is hard to calculate the composite function term \( (u^{n-1}_h \circ X_1(u^{n-1}_h), v_h) \) exactly. In practice, the following numerical quadrature has been used. Let \( g : K \to \mathbb{R} \) be a continuous function. A numerical quadrature \( I_h[g; K] \) of \( \int_K g \, dx \) is defined by

\[
I_h[g; K] \equiv \text{meas}(K) \sum_{i=1}^{N_q} w_i g(a_i),
\]

where \( N_q \) is the number of quadrature points and \( (w_i, a_i) \in \mathbb{R} \times K \) is a pair of the weight and the point for \( i = 1, \ldots, N_q \). We call the practical scheme using numerical quadrature Scheme LG'.
Scheme LG Let $u^n_h = (\Pi_h^i(u^n, 0))_1$. Find $\{(a^n_h, p^n_h)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that
\[
\frac{1}{\Delta t}(u^n_h, v_h) - \frac{1}{\Delta t} \sum_{K \in \mathcal{T}_h} I_h[(u^{n-1}_h \circ X_1(u^{n-1}_h)) \cdot v_h; K] + a(u^n_h, v_h) + b(v_h, p^n_h) = (f^n, v_h), \quad \forall v_h \in V_h,
\]
\[
b(u^n_h, q_h) = 0, \quad \forall q_h \in Q_h,
\]
for $n = 1, \ldots, N_T$.

For convection-diffusion equations it has been reported that numerical quadrature causes the instability [12,17,20,21,22]. For the Navier-Stokes equations we present numerical results showing the instability of Scheme LG in Section 5.

We now present our Lagrange-Galerkin scheme with a locally linearized velocity. It is free from quadrature and exactly computable. We call it Scheme LG-LLV.

**Proposition** 1 Let $u_h, v_h \in V_h$ and $w \in W_0^{1,\infty}(\Omega)^d$. Suppose $\alpha_{20} \Delta t |w|_{1,\infty} < 1$, where $\alpha_{20}$ is the constant defined in (12a) below. Then, $\int_{\Omega} (u_h \circ X_1(\Pi_h^{(1)} w)) \cdot v_h \, dx$ is exactly computable.

**Outline of the proof.** When $u_h$ and $v_h$ are scalar functions, the result on the exact computability has been proved in [23] and [22, Proposition 1]. Here, we do not repeat the proof but show only the outline. It is necessary that the inclusion $(X_1(\Pi_h^{(1)} w)) \subset \Omega$ holds to execute the integration of $u_h \circ X_1(\Pi_h^{(1)} w) \cdot v_h$ over $\Omega$. The condition $\alpha_{20} \Delta t |w|_{1,\infty} < 1$ is sufficient for it by virtue of Lemma 2(i) and (12a) below. The mapping $X_1(\Pi_h^{(1)} w)$ is linear on each element. When a mapping $F$ is linear, we have the following general result for any two elements $K_0$ and $K_1$ and any polynomial $\phi_h$ of any order $k$ defined on $K_1$. Proposition 1 is proved by applying the following lemma, whose proof is easy, cf. [22, Lemma 1].

**Lemma 1** Let $K_0, K_1 \in \mathcal{T}_h$ and $F : K_0 \rightarrow \mathbb{R}^d$ be linear and one-to-one. Let $E_1 \equiv K_0 \cap F^{-1}(K_1)$ and $\text{meas}(E_1) > 0$. Then, the following hold.

(i) $E_1$ is a polygon ($d = 2$) or a polyhedron ($d = 3$).

**Scheme LG-LLV** Let $u^n_h = (\Pi_h^i(u^n, 0))_1$. Find $\{(a^n_h, p^n_h)\}_{n=1}^{N_T} \subset V_h \times Q_h$ such that
\[
\frac{u^n_h - u^{n-1}_h \circ X_1(\Pi_h^{(1)} u^{n-1}_h)}{\Delta t} + a(u^n_h, v_h) + b(v_h, p^n_h) = (f^n, v_h), \quad \forall v_h \in V_h, (9a)
\]
\[
b(u^n_h, q_h) = 0, \quad \forall q_h \in Q_h, (9b)
\]
for $n = 1, \ldots, N_T$.

In the above scheme the locally linearized velocity $\Pi_h^{(1)} u^{n-1}_h$ is used in place of the original velocity $u^{n-1}_h$. The error caused by the introduction of the approximate velocity $\Pi_h^{(1)} u^{n-1}_h$ is evaluated properly in Theorems 1 and 2 in the next section. The following proposition assures that the integration $(u^n_h - u^{n-1}_h \circ X_1(\Pi_h^{(1)} u^{n-1}_h), v_h)$ can be calculated exactly.
(ii) $\phi_h \circ F_{\mathcal{K}_i} \in P_k(E_i), \quad \forall \phi_h \in P_k(K_1)$.

**Remark 3** In the case of $d = 2$, Priestley [17] approximated $X_h^{n-1}(\tau^{n-1}; x, t^n)$ in (7) by

$$\tilde{X}_h(x) = B_1 \lambda_1(x) + B_2 \lambda_2(x) + B_3 \lambda_3(x), \quad x \in K_0$$

on each $K_0 \in \mathcal{T}_h$, where $B_i = X_h^{n-1}(\tau^{n-1}; A_i, t^n)$, $\{A_i\}_{i=1}^3$ are vertices of $K_0$ and $\{\lambda_i\}_{i=1}^3$ are the barycentric coordinates of $K_0$ with respect to $\{A_i\}_{i=1}^3$. Since $\tilde{X}_h(x)$ is linear in $K_0$, the decomposition

$$\int_{K_0} (u_h^{n-1} \circ \tilde{X}_h) \cdot v_h \, dx = \sum_{I \in \Delta(K_0)} \int_{E_I} (u_h^{n-1} \circ \tilde{X}_h) \cdot v_h \, dx,$$

$$\Lambda(K_0) \equiv \left\{ I ; K_0 \cap \tilde{X}_h^{-1}(K_I) \neq \emptyset \right\}, \quad E_I \equiv K_0 \cap \tilde{X}_h^{-1}(K_I)$$

makes the exact integration possible. However, $B_i = X_h^{n-1}(\tau^{n-1}; A_i, t^n)$ are the solutions of a system of ordinary differential equations and it is not easy to solve it exactly in general since $\tilde{u}_h^{n-1}$ is piecewise polynomial. In practice, some numerical method, e.g., Runge–Kutta method, is required, which introduces another error.

### 4 Main results

We present the main results of error estimates for Scheme LG-LLV, which are proved in the next section. We first state the result when the $P_2/P_1$-element is employed.

**Hypothesis 1** The solution of (1) satisfies

$$u \in Z^2 \cap H^1(\Omega), \quad p \in H^1(\Omega).$$

**Remark 4** Hypothesis 1 implies $(u, p) \in C(H^3 \times H^2)$, which yields $\nabla \cdot u^0 = 0$.

**Hypothesis 2** The sequence $\{\mathcal{T}_h\}_{h > 0}$ satisfies the inverse assumption. In addition, for each $h$, $\forall K \in \mathcal{T}_h$ has at least one vertex in $\Omega$.

**Theorem 1** Let $V_h \times Q_h$ be the $P_2/P_1$-finite element space. Suppose Hypotheses 7 and 2. Then, there exist positive constants $c_0$ and $h_0$ such that if $h \in (0, h_0)$ and $\Delta t \leq c_0 h^{1/4}$, the solution $(u_h, p_h) \equiv \left\{ (u_h^n, p_h^n) \right\}_{n=0}^{N_T}$ of Scheme LG-LLV exists and the estimates

$$\|u_h - u\|_{L^2(\Omega)}, \quad \|p_h - p\|_{L^2(\Omega)} \leq c_1 (h^2 + \Delta t)$$

hold, where $c_1$ is a positive constant independent of $h$ and $\Delta t$.

Next, we state the result when the $P_1+/P_1$-element is employed.

**Hypothesis 1’** The solution of (1) satisfies

$$u \in Z^2 \cap H^1(\Omega), \quad p \in H^1(\Omega).$$

**Remark 5** Hypothesis 1’ implies $(u, p) \in C(H^2 \times H^1)$, which yields $\nabla \cdot u^0 = 0$. 
Hypothesis 3 The Stokes problem is regular, that is, for all $g \in L^2(\Omega)^d$ the solution $(w, r) \in H^1_0(\Omega)^d \times L^2(\Omega)$ of the Stokes problem,

$$-\nu \Delta w + \nabla r = g, \quad x \in \Omega,$$
$$\nabla \cdot w = 0, \quad x \in \Omega$$

belongs to $H^2(\Omega)^d \times H^1(\Omega)$ and the estimate

$$\|(w, r)\|_{H^2 \times H^1} \leq c \|g\|_0.$$

holds, where $c$ is a positive constant independent of $g, w$ and $r$.

Remark 6 Hypothesis 3 holds, for example, if $d = 2$ and $\Omega$ is convex [9].

Theorem 2 Let $V_h \times Q_h$ be the $P_1+/P_1$-finite element space. Suppose Hypotheses 1 and 2. Then, there exist positive constants $c_0$ and $h_0$ such that if $h \in (0, h_0]$ and $\Delta t \leq \text{O}(h^{d/4})$, the solution $(u_h, p_h) \equiv \{(u^n_{h, p^n_{h}})_{n=0}^{\infty}\}$ of Scheme LG-LLV exists, and the estimates

$$\|u_h - u\|_{\ell^\infty(\mathcal{M})}, \|p_h - p\|_{\ell^2(\mathcal{L})} \leq c_2(h + \Delta t)$$

hold, where $c_2$ is a positive constant independent of $h$ and $\Delta t$. Moreover, under Hypothesis 3 the estimate

$$\|u_h - u\|_{\ell^\infty(\mathcal{L})} \leq c_3(h^2 + \Delta t)$$

holds, where $c_3$ is a positive constant independent of $h$ and $\Delta t$.

Remark 7 The convergence proof is easily extended for any pairs satisfying the inf-sup condition. However, the convergence order with respect to the space discretization is bounded by $O(h^2)$ caused by the locally linearized approximation of the velocity. In fact, in the case of the $P_2/P_1$-element the estimate (8b) with $k = 2$ does not hold in Scheme LG-LLV, cf., Example 1 in Section 6.

5 Proofs of the main theorems

We prove Theorem 1 in Subsections 5.2–5.4 and Theorem 2 in Subsection 5.5.

5.1 Some lemmas

We recall some results used in proving the main theorems. For proofs of Lemmas 2–6 we refer to the cited bibliography.

Lemma 2 (Poincaré’s inequality [6]) There exists a positive constant $\alpha_1(\Omega)$ such that

$$\|v\|_0 \leq \alpha_1 \|v\|_1, \quad \forall v \in H^1_0(\Omega)^d.$$
Lemma 3 (the Lagrange interpolation [6]) Suppose \( \{ T_h \}_{h>0} \) is a regular family of triangulations of \( \tilde{\Omega} \). Let \( X_h \) be the \( P_2 \)- or \( P_1+ \)-finite element space and \( \Pi_h^{(1)} \) be the Lagrange interpolation operator to the \( P_1 \)-finite element space. Then, it holds that
\[
\| \Pi_h^{(1)} v \|_{0,\infty} \leq \| v \|_{0,\infty}, \quad \forall v \in C(\tilde{\Omega})^d,
\]
and there exist positive constants \( \alpha_{20} \geq 1, \alpha_{21} \) and \( \alpha_{22} \) such that
\[
\| \Pi_h^{(1)} v \|_{1,\infty} \leq \alpha_{20} \| v \|_{1,\infty}, \quad \forall v \in W^{1,\infty}(\Omega)^d, \tag{12a}
\]
\[
\| \Pi_h^{(1)} v - v \|_s \leq \alpha_{21} h^{2-s} \| v \|_2, \quad s = 0, 1, \forall v \in H^s(\Omega)^d, \tag{12b}
\]
\[
\| \Pi_h^{(1)} v \|_0 \leq \alpha_{22} \| v \|_0, \quad \forall v \in X_h. \tag{12c}
\]

Remark 8 The inequality \( \| v \|_0 \leq \alpha_{22} \| v \|_0 \) holds since \( X_h \) is finite-dimensional. If we replace \( \Pi_h^{(1)} \) by the Clément interpolation operator [7], this inequality holds for all \( v \in L^2(\Omega)^d \).

Lemma 4 (the inverse inequality [6][19]) Suppose \( \{ T_h \}_{h>0} \) satisfies the inverse assumption. Let \( X_h \) be the \( P_2 \)- or \( P_1+ \)-finite element space. Then, there exist positive constants \( \alpha_{30} \) and \( \alpha_{31} \) such that
\[
\| v_h \|_{0,\infty} \leq \alpha_{30} h^{-d/6} \| v_h \|_1, \quad \forall v_h \in X_h,
\]
\[
\| v_h \|_{1,\infty} \leq \alpha_{31} h^{-d/2} \| v_h \|_1, \quad \forall v_h \in X_h.
\]

Lemma 5 (the inf-sup condition [2][4][24]) Suppose Hypothesis 2. Let \( V_h \times Q_h \subset H^1_0(\Omega)^d \times L^2_0(\Omega) \) be the \( P_2/P_1- \) or \( P_1+P_1- \)finite element space. Then, there exists a positive constant \( \alpha_5 \) independent of \( h \) such that
\[
\inf_{\mathbf{q}_h \in Q_h \setminus \{0\}} \sup_{v_h \in V_h \setminus \{0\}} \frac{b(v_h, \mathbf{q}_h)}{\| v_h \|_1 \| \mathbf{q}_h \|_0} \geq \alpha_5.
\]

Lemma 6 ([9]) (i) Suppose Hypothesis 2, and that \( V_h \times Q_h \subset H^1_0(\Omega)^d \times L^2_0(\Omega) \) is the \( P_2/P_1- \) or \( P_1+P_1- \)finite element space. Let \( (\hat{w}_h, \hat{r}_h) \) be the Stokes projection of \( (w, r) \) defined in [9]. Then, there exists a positive constant \( \alpha_{50} \) independent of \( h \) such that
\[
\| \hat{w}_h - w \|_1, \| \hat{r}_h - r \|_0 \leq \alpha_{50} h^k \| (w, r) \|_{H^{k+1} \times H^k},
\]
where \( k = 2 \) for the \( P_2/P_1- \)element and \( k = 1 \) for the \( P_1+P_1- \)element.

(ii) Moreover, suppose Hypothesis 3. Then, there exists a positive constant \( \alpha_{51} \) such that
\[
\| \hat{w}_h - w \|_0 \leq \alpha_{51} h^{k+1} \| (w, r) \|_{H^{k+1} \times H^k},
\]
where \( k = 2 \) for the \( P_2/P_1- \)element and \( k = 1 \) for the \( P_1+P_1- \)element.

Lemma 7 (i) Let \( w \in W^{1,\infty}(\Omega)^d \) and \( X_1(\omega) \) be the mapping defined in [4]. Then, under the condition \( \Delta t \| w \|_{1,\infty} < 1, X_1(\omega) : \Omega \to \Omega \) is bijective.

(ii) Furthermore, under the condition \( \Delta t \| w \|_{1,\infty} \leq 1/4 \), the estimate
\[
\frac{1}{2} \leq \text{det} \left( \frac{\partial X_1(\omega)}{\partial x} \right) \leq \frac{3}{2}
\]
holds, where \( \text{det}(\partial X_1(\omega)/\partial x) \) is the Jacobian.
Proof The former is proved in [18] Proposition 1. We prove the latter only in the case \( d = 3 \) since the proof in \( d = 2 \) is much easier. Let \( I \) be the \( 3 \times 3 \) identity matrix, \( A = (a_{ij}) \) and \( a_j = (a_{1j}, a_{2j}, a_{3j})^T \), where \( a_{ij} = \Delta t \frac{\partial w_j}{\partial x_i} \) for \( i, j = 1, 2, 3 \). The notation \( | \cdot | \) stands for the absolute value, or the Euclidean norm in \( \mathbb{R}^3 \) or \( \mathbb{R}^{3 \times 3} \). From the condition
\[
|a| = (|a_1|^2 + |a_2|^2 + |a_3|^2)^{1/2} \leq 1/4,
\]
we obtain
\[
det(A) \leq |a_1| |a_2| |a_3| \leq \left( \frac{1}{4\sqrt{3}} \right)^3.
\]
Then, we have
\[
\left| \det \left( \frac{\partial X_1(w)}{\partial x} \right) - 1 \right| = |\det(I - A) - 1|
\]
\[
= |-(a_{11} + a_{22} + a_{33}) + a_{11}a_{22} + a_{33}a_{11} - a_{12}a_{21} - a_{23}a_{32} - a_{31}a_{13} - det(A)|
\]
\[
\leq |a_{11} + a_{22} + a_{33}|
\]
\[
+ |a_{11}a_{22} + a_{23}a_{31} + a_{32}a_{13} - a_{12}a_{21} - a_{23}a_{32} - a_{31}a_{13}| + |\det(A)|
\]
\[
\leq \sqrt{3}|a| + |a|^2 + |\det(A)| \leq 1/2,
\]
which implies the result. \( \square \)

Lemma 8 Let \( 1 \leq q < \infty, 1 \leq p < \infty, 1/p + 1/p' = 1 \) and \( w_i \in W_0^{1,\infty}(\Omega)^d \), \( i = 1, 2 \). Under the condition \( \Delta t |w_i|_{1,\infty} \leq 1/4 \), it holds that, for \( \psi \in W^{1,qp'}(\Omega)^d \),
\[
\|\psi \circ X_1(w_1) - \psi \circ X_1(w_2)\|_{0,q} \leq 2^{1/(qp')} \Delta t \|w_1 - w_2\|_{0,pq} \|\nabla \psi\|_{0,qp'},
\]
where \( X_1(\cdot) \) is defined in (4).

Lemma 8 is a direct consequence of [11] Lemma 4.5] and Lemma [7(ii).

Lemma 9 Let \( w \in W_0^{1,\infty}(\Omega)^d \). Under the condition \( \Delta t |w|_{1,\infty} \leq 1/4 \), there exists a positive constant \( \alpha_0 \) such that, for \( \psi \in L^2(\Omega)^d \),
\[
\|\psi - \psi \circ X_1(w)\|_{-1} \leq \alpha_0 \Delta t \|w\|_{1,\infty} \|\psi\|_0,
\]
where \( X_1(\cdot) \) is defined in (4).

Lemma 9 is obtained from [8] Lemma 1] and Lemma [7(ii).

5.2 Estimates of \( e_h^n \) under some assumptions

Let
\[
(e_h^n, e_h^n) = (u_h^n - \tilde{u}_h^n, p_h^n - \tilde{p}_h^n), \quad \eta(t) \equiv u(t) - \tilde{u}_h(t),
\]

(13)
where \((u, p)\) is the solution of \((1)\), \((\vec{u}_h(t), \vec{p}_h(t))\) is the Stokes projection of \((u(t), p(t))\) defined in (5) and \((u^n_h, p^n_h)\) is the solution of Scheme LG-LLV at the step \(n\). From (2), (5) and (9) we have the error equations in \((e^n_h, e^n_{h^\alpha})\):

\[
(D_{\Delta t} e^n_h, v_h) + a(e^n_h, v_h) + b(v_h, e^n_h) = \sum_{i=1}^4 \left( R^n_i, v_h \right), \quad \forall v_h \in V_h, \tag{14a}
\]

\[
b(e^n_h, q_h) = 0, \quad \forall q_h \in Q_h, \tag{14b}
\]

for \(n = 1, \ldots, N_T\), where

\[
R^n_1 \equiv \frac{Du^n}{\Delta t} - \frac{u^n - u^{n-1} \circ X_1(u^{n-1})}{\Delta t},
\]

\[
R^n_2 \equiv \frac{\eta^n - \eta^{n-1} \circ X_1(\Pi_h^{(1)} u_h^{n-1}) - u^{n-1} \circ X_1(u^{n-1})}{\Delta t},
\]

\[
R^n_3 \equiv \frac{\eta^n - \eta^{n-1} \circ X_1(\Pi_h^{(1)} u_h^{n-1})}{\Delta t}, \quad R^n_4 \equiv \frac{\hat{e}_h^{n-1} - e^{n-1}_h \circ X_1(\Pi_h^{(1)} u_h^{n-1})}{\Delta t}.
\]

**Lemma 10** Suppose Hypotheses 2 and 2 Under the condition

\[
\Delta t |u^{n-1}|_{1, \infty}, \Delta t |\Pi_h^{(1)} u_h^{n-1}|_{1, \infty} \leq 1/4,
\]

it holds that

\[
\| R^n_1 \|_0 \leq \beta_1 \sqrt{\Delta t} \| u \|_{W^2(p^n-1, p^n)}, \tag{17a}
\]

\[
\| R^n_2 \|_0 \leq \beta_2 \| e^{n-1}_h \|_0 + \beta_3 \Delta t \| (u, p)^{n-1} \|_{H^3 \times H^2} + |u^{n-1}|_{1, 2}, \tag{17b}
\]

\[
\| R^n_3 \|_0 \leq \beta_4 \frac{\Delta t}{\sqrt{\Delta t}} \left( (\| u, p \|_{H^1(p^n-1, p^n, H^3 \times H^2)}^2 + \| u^{n-1} \|_{1, \infty}^2) \right), \tag{17c}
\]

\[
\| R^n_4 \|_0 \leq \beta_5 \| e^{n-1}_h \|_{1, \infty} |e^{n-1}_h|_1, \tag{17d}
\]

for \(n = 1, \ldots, N_T\), where \(\beta_1 = \beta_1 (\| u \|_{C(W^1, \cdot)}, \| u \|_{C(W^1, \cdot), \alpha_22}), \beta_2 = \beta_2 (\| u \|_{C(W^1, \cdot), \alpha_22}, \alpha_21, \alpha_22, \alpha_50), \beta_3 = \beta_4(\alpha_50), \beta_5 = \sqrt{\Delta t}\) and the notation \(\beta(A)\) means that a positive constant \(\beta\) depends on a set of parameters \(A\).

**Proof** We prove (17a). We decompose \(R^n_1\) as follows:

\[
R^n_1(x) = \left\{ \frac{\partial u^n}{\partial t}(x) + (u^{n-1}(x) \cdot \nabla)u^n(x) - \frac{u^n - u^{n-1} \circ X_1(u^{n-1})}{\Delta t} \right\}.
\]

Setting

\[
y(x, s) = x + (s - 1)\Delta t u^{n-1}(x), \quad t(s) = s^{n-1} + s\Delta t,
\]

we have

\[
\frac{u^n - u^{n-1} \circ X_1(u^{n-1})}{\Delta t} = \frac{1}{\Delta t} \left[ u(y(\cdot, s), t(s)) \right]_{s=0}^1.
\]
which implies that

\[ R_{11} = \frac{\partial u^n}{\partial t} + (u^{n-1} \cdot \nabla)u^n - \int_0^1 \left\{ (u^{n-1}(\cdot) \cdot \nabla)u + \frac{\partial u}{\partial t} \right\} (y(\cdot, s), t(s)) ds \]

\[ = \Delta t \int_0^1 ds \int_s^1 \left\{ (u^{n-1}(\cdot) \cdot \nabla + \frac{\partial}{\partial t})^2 u \right\} (y(\cdot, s), t(s)) ds_1 \]

\[ = \Delta t \int_0^1 s_1 \left\{ (u^{n-1}(\cdot) \cdot \nabla + \frac{\partial}{\partial t})^2 u \right\} (y(\cdot, s), t(s)) ds_1. \]

Hence, we have

\[ \| R_{11} \|_0 \leq \Delta t \int_0^1 s_1 \left\{ \left( (u^{n-1}(\cdot) \cdot \nabla + \frac{\partial}{\partial t})^2 u \right) (y(\cdot, s), t(s)) \right\} ds_1 \]

\[ \leq \beta'(\| u \|_{C^1(\Omega)}) \sqrt{\Delta t} \| u \|_{L^2(\Omega^1, x)}, \]

where we have used the transformation of independent variables from \( x \) to \( y \) and \( s_1 \) to \( t \) and the estimate \( |\det(\partial x / \partial y)| \leq 2 \) by virtue of Lemma 7(ii). It is easy to show

\[ \| R_{12} \|_0 \leq \sqrt{\Delta t} \| u' \|_{L^1(\Omega)} \frac{\partial u}{\partial t} \|_{L^2(\Omega^1, x, \bar{\Omega}^2)}. \]

From the triangle inequality we get (17a).

We prove (17b). Using Lemma 8 with \( q = 2, p = 1, p' = \infty, w_1 = \Pi_h^{(1)} u^{n-1}_h, w_2 = u^{n-1} \) and \( x = u^{n-1} \), we have

\[ \| R_v \|_0 \leq \| u^{n-1} \|_{1, \infty} \| \Pi_h^{(1)} u^{n-1}_h - u^{n-1} \|_0 \]

\[ \leq \| u^{n-1} \|_{1, \infty} (\| \Pi_h^{(1)} u^{n-1}_h - \Pi_h^{(1)} u^{n-1} \|_0 + \| \Pi_h^{(1)} u^{n-1} - u^{n-1} \|_0). \]

From Lemmas 3 and 6(i) we evaluate the first term as follows:

\[ \| \Pi_h^{(1)} u^{n-1}_h - \Pi_h^{(1)} u^{n-1} \|_0 = \| \Pi_h^{(1)} (u^{n-1}_h - \Pi_h^{(1)} u^{n-1}) \|_0 \]

\[ \leq \alpha_22 \| u^{n-1}_h - \Pi_h^{(1)} u^{n-1} \|_0 \]

\[ \leq \alpha_22 \| u^{n-1}_h - \Pi_h^{(1)} u^{n-1} \|_0 + \| u^{n-1} - \Pi_h^{(1)} u^{n-1} \|_0 + \| u^{n-1} - \Pi_h^{(1)} u^{n-1} \|_0 \]

\[ \leq \alpha_22 (\| e_h^{(n-1)} \|_0 + \alpha_29 h^2 \| (u, p) \|^{(n-1)}_{H^1 \times H^2} + \alpha_21 h^2 |u^{n-1}|_2^2). \]

The second term is evaluated as follows:

\[ \| \Pi_h^{(1)} u^{n-1} - u^{n-1} \|_0 \leq \alpha_21 h^2 |u^{n-1}|_2^2. \]

Thus, we have

\[ \| R_v \|_0 \leq \| u^{n-1} \|_{1, \infty} \left\{ \alpha_22 (\| e_h^{(n-1)} \|_0 + \alpha_29 h^2 \| (u, p) \|^{(n-1)}_{H^1 \times H^2}) \right. \]

\[ + \left. \alpha_21 (1 + \alpha_22) h^2 |u^{n-1}|_2^2 \right\}. \]
which implies (17b).

We prove (17c). Let

\[ y(x) = x + (s - 1)\Delta t \Pi_h^{(1)} u_h^{n-1}(x), \quad t(s) = t^n + s\Delta t. \]

Since \( R^3 \) is rewritten as

\[ R^3 = \int_0^1 \left\{ \frac{\partial \eta}{\partial t} + (\Pi_h^{(1)} u_h^{n-1} \cdot \nabla) \eta \right\} (\eta(t), t(s)) ds, \]

we have, by using the change of the variable and Lemma 7(ii),

\[
\| R^3 \|_0 \leq \left\| \int_0^1 \left| \frac{\partial \eta}{\partial t}\right| (\eta(t), t(s)) ds \right\|_0 + \left\| \int_0^1 \left| (\Pi_h^{(1)} u_h^{n-1} \cdot \nabla) \eta \right| (\eta(t), t(s)) ds \right\|_0 \\
\leq \frac{\sqrt{2}}{\sqrt{\Delta t}} \left( \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2((p-1, r^2) \times [0, T])} + \| \Pi_h^{(1)} u_h^{n-1} \|_{L^2((p-1, r^2) \times [0, T])} \right) \\
\leq \frac{\sqrt{2} \alpha_D h^2}{\sqrt{\Delta t}} \left( \left\| (u, p) \right\|_{H^1((p-1, r^2, H^3 \times H^2))} + \| u_h^{n-1} \|_{L^2((p-1, r^2, H^3 \times H^2))} \right),
\]

which implies (17c).

The inequality (17d) is obtained from Lemma 8 with \( q = 2, p = 3, p' = 1, w_1 = 0, w_2 = \Pi_h^{(1)} u_h^{n-1} \) and \( \psi = e_h^{n-1} \).

**Lemma 11** Suppose Hypotheses 7 and 2. Let \( n \in \{1, \cdots, N_T\} \) be any integer and let \( u_h^{n-1} \in V_h \) be known. Suppose that \( u_h^{n-1} \) satisfies

\[ b(u_h^{n-1}, q_h) = 0, \quad \forall q_h \in Q_h. \]  

(19)

Under the condition (16), there exists a solution \((u_h^n, p_h^n)\) of (9) and it holds that

\[
\| D_{\Delta t} e_h^n \|_0^2 + D_{\Delta t} \left( \nabla |e_h^n|^2 \right) \\
\leq \beta_{21} \left( \| u_h^{n-1} \|_{L^2((p-1, r^2))}^2 \right) \| \Delta t \|_{L^2((p-1, r^2))} + h^4 \left( \| (u, p) \|_{H^3 \times H^2}^2 + \| u^{n-1} \|_{L^2((p-1, r^2))}^2 \right),
\]

where \( e_h^n \) is defined in (13), and \( \beta_{21}(\xi) \) and \( \beta_{22}(\xi) \) are the functions defined in (21) below.

**Proof** Since it holds that \( \| \Pi_h^{(1)} u_h^{n-1} \|_{1, \infty} \leq 1/4 \), the mapping \( X_1(\Pi_h^{(1)} u_h^{n-1}) : \Omega \rightarrow \Omega \) is bijective from Lemma 7(i). Hence, there exists a solution \((u_h^n, p_h^n)\) of (9). Substituting \( v_h = D_{\Delta t} e_h^n \) in (14a), we have

\[
\| D_{\Delta t} e_h^n \|_0^2 + D_{\Delta t} \left( \frac{1}{2} \| \nabla e_h^n \|_0^2 \right) + b(D_{\Delta t} e_h^n, e_h^n) \leq \sum_{i=1}^4 \langle R^3_i, D_{\Delta t} e_h^n \rangle .
\]  

(20)
From (19) and (9) the term \( b(D_{\Delta}e_h^n, e_h^n) \) of the left-hand side vanishes. Using Schwarz’ and Young’s inequalities and Lemma 10 we have
\[
\| D_{\Delta}e_h^n \|_0^2 + D_{\Delta}(v | e_h^n |^2) \leq 2 \left\{ \beta_1^2 \Delta t \| u \|_{Z^2(\rho-1, \rho^*)}^2 + \beta_2 \| \phi^{-1} \|_0 + \beta_3 \| u_0 \|_{\infty}^2 \| \omega^{-1} \|_2^2 \right\}
\]
\[
+ \beta_4 \| D_t (\| (u, p) \|_{H^1(\rho-1, \rho; H^3 \times H^2)} + \| u^{-1} \|_{0, \infty} \| (u, p) \|_{L^2(\rho-1, \rho; H^3 \times H^2)})^2 \]
\]
\[
+ \beta_5 \| u^{-1} \|_{0, \infty}^2 | e_h^{-1} |_1^2 \right\} + \frac{1}{2} \| D_{\Delta}e_h^n \|_0^2,
\]
which implies that
\[
D_{\Delta}(v | e_h^n |^2) + \| D_{\Delta}e_h^n \|_0^2 \leq \beta_{11} \left( \| e^{-1} \|_1^2 + \| u^{-1} \|_{0, \infty}^2 \| e^{-1} \|_1^2 \right) + \beta_{12} \left( \Delta t \| u \|_{Z^2(\rho-1, \rho^*)}^2 \right)
\]
\[
+ \frac{1}{\Delta t} \left( \| (u, p) \|_{H^1(\rho-1, \rho; H^3 \times H^2)}^2 + \| u^{-1} \|_{0, \infty} \| (u, p) \|_{L^2(\rho-1, \rho; H^3 \times H^2)}^2 \right)
\]
\[
+ \frac{1}{\Delta t} \left( \| (u, p) \|_{L^2(\rho-1, \rho; H^3 \times H^2)}^2 + \| u^{-1} \|_2^2 \right),
\]
where \( \beta_{11} \) and \( \beta_{12} \) are constants depending only on \( \beta_1, \ldots, \beta_5 \). Using Poincaré’s inequality \( \| e_h^{-1} \|_0 \leq \alpha_1 \| e_h^{-1} \|_1 \) and defining the functions \( \beta_{21} \) and \( \beta_{22} \) by
\[
\beta_{21}(\xi) = \frac{\beta_1^2}{\nu} (\alpha_1^2 + \xi^2), \quad \beta_{22}(\xi) = \beta_1 (1 + \xi^2),
\]
we have the conclusion.

5.3 Definitions of constants \( c_\ast, c_0 \) and \( h_0 \)

We first define constants \( \beta_{21}^2 \) and \( \beta_{22}^2 \) by
\[
\beta_{21}^2 \equiv \beta_{21}(\| u \|_{C(\Lambda)} + 1), \quad \beta_{22}^2 \equiv \beta_{22}(\| u \|_{C(\Lambda)} + 1).
\]

We define two positive constants \( c_\ast \) and \( c_0 \) by
\[
c_\ast \equiv \left\{ \nu^{-1} \left( 1 + \alpha_0^2 \right) \exp(\beta_{21}^2 T) \beta_{22}^2 \right\}^{1/2} \max \left\{ \| u \|_{L^2} \left( \| (u, p) \|_{H^1(\rho; H^3 \times H^2)} \times T \right), \left( \| (u, p) \|_{C(H^1 \times H^2)} + \nu \| \omega \|_{L^2} \right) \right\} \]
\[
+ T \left( \| (u, p) \|_{C(H^1 \times H^2)} + \| u \|_{C(\Lambda)} \right) + \nu \| \omega \|_{L^2} \right\}^{1/2}
\]
and
\[
c_0 \equiv \frac{1}{4} \sqrt{\frac{1}{\alpha_{20} \alpha_{31} c_\ast}}.
\]

Let a positive constant \( h_0 \) be small enough to satisfy that
\[
\alpha_{0} h_0^{\frac{d-6}{2}} \left( c_0 h_0 + \alpha_{50} h_0 \| (u, p) \|_{C(H^1 \times H^2)} + \alpha_{21} \| u \|_{C(H^1 \times H^2)} \right)
\]
which are possible since all the powers of $h_0$ are positive.

5.4 Induction

For $n = 0, \ldots, N_T$ we define the property $P(n)$ by

$$P(0)-(a)$$

$$\text{or}$$

$$P(0)-(b)$$

and (c) below, we omit the proofs.

When $n$ isfied trivially when $n \geq 2$. When $n = 1$, from the choice of $u^0_h$ and Remark 4.4 we have

$$b(\hat{u}_h^0, q_h) = b(u^0_h, q_h) - b(\hat{u}_h^0, q_h) = 0 - 0 = 0, \quad \forall q_h \in Q_h. \quad (24)$$

We consider the condition (25). The former condition follows from $\Delta t \leq c_0 h^{d/4}$ and (23b) by the inequality

$$\Delta t |u|^{n-1}_{1,\infty} \leq c_0 h^{d/4} |u|_{C(W^{1,-})} \leq \frac{3}{16 \alpha_{20}} \leq 1/4,$$

and the latter condition $\Delta t |\Pi_h^{(1)} u_h^{n-1}|_{1,\infty} \leq 1/4$ follows from $P(n - 1)-(c)$. Hence, there exists a solution $(u^0_h, p^0_h)$ at the step $n$.

We begin the proof of $P(n)-(a)$. By putting

$$x_n \equiv v |\hat{e}_h^0|^2, \quad y_n \equiv \|\hat{D}_{\Delta t} e_h^0\|^2_0,$$

we obtain

$$\Delta t \sum_{i=1}^n y_i \leq \exp(\beta_2 |n| \Delta t) \frac{3}{16 \alpha_{20}} \leq 1/4.$$

P(n)-(a) is rewritten as

$$x_n + \Delta t \sum_{i=1}^n y_i \leq \exp(\beta_2 |n| \Delta t) \frac{3}{16 \alpha_{20}} \leq 1/4.$$
On the other hand, Lemma [11] implies that
\[ x_n + \Delta t y_n \leq (1 + \beta_{21}^1 \Delta t) x_{n-1} + \beta_{22}^1 \Delta t b_n, \]
where we have used the inequalities \( \beta_{2i}(\|u^{i-1}_h\|_{0,\infty}) \leq \beta_{2i}^1, \ i = 1, 2, \) obtained from P(n - 1)-(b). Using the inequalities \( 1 \leq 1 + x \leq \exp(x) \) for \( x \geq 0 \) and P(n - 1)-(a) rewritten by (25), we have
\[
\begin{align*}
&x_n + \Delta t \sum_{i=1}^{n} y_i = x_n + \Delta t y_n + \Delta t \sum_{i=1}^{n-1} y_i \\
&\leq (1 + \beta_{21}^1 \Delta t) x_{n-1} + \beta_{22}^1 \Delta t b_n + \Delta t \sum_{i=1}^{n-1} y_i \\
&\leq (1 + \beta_{21}^1 \Delta t) \exp(\beta_{21}^1 (n-1) \Delta t) \beta_{22}^1 \left( x_0 + \Delta t \sum_{i=1}^{n-1} b_i \right) + \beta_{22}^1 \Delta t b_n \\
&\leq \exp(\beta_{21}^1 n \Delta t) \beta_{22}^1 \left( x_0 + \Delta t \sum_{i=1}^{n} b_i \right),
\end{align*}
\]
which is nothing but P(n)-(a).

Since \( u_h^n \) is the first component of \( \Pi_h^3(u^0, 0) \), we have
\[
e_h^0 = u_h^n - \tilde{u}^0_h = (\Pi_h^3(0, -p^0))_1 - ((\Pi_h^3 - I)(0, -p^0))_1,
\]
which implies \( \| e_h^0 \|_1 \leq \alpha_{50} h^2 \| p^0 \|_2 \). From P(0)-(a) and the definition of \( c_* \), we have
\[
\| e_h^0 \|_1 \leq c_*(h^2 + \Delta t). \tag{26}
\]

P(n)-(b) is proved as follows:
\[
\begin{align*}
&\| u_h^n - \Pi_h^{(1)} u^p \|_{0,\infty} \\
&\leq \| u_h^n - \Pi_h^{(1)} u^p \|_{0,\infty} + \| \Pi_h^{(1)} u^p \|_{0,\infty} \\
&\leq \alpha_{50} h^{-d/6} \| u_h^n - \Pi_h^{(1)} u^p \|_1 + \| u^p \|_{0,\infty} \tag{by Lemmas 4 and 3} \\
&\leq \alpha_{50} h^{-d/6} \left( c_*(h^2 + \Delta t) + \alpha_{50} h^{2} \| (u, p)^p \|_{H^1} \right) + \| u^p \|_{0,\infty} \tag{by (25), Lemma 5(i) and Lemma 3} \\
&\leq \alpha_{50} h^{1-d/6} \left( c_* + \alpha_{50} \| (u, p)^p \|_{H^2} + \alpha_{21} h \| u^p \|_2 \right) + \| u^p \|_{0,\infty} \tag{since \( \Delta t \leq c_0 h^{d/4} \)} \\leq & 1 + \| u \|_{C(L^\infty)} \tag{since \( h \leq h_0 \) and by (23a)}
\end{align*}
\]

We prove P(n)-(c). We can estimate \( | u_{h,n}^n |_{1,\infty} \Delta t \) as follows:
\[
\begin{align*}
&| u_{h,n}^n |_{1,\infty} \Delta t \\
&\leq (| u_h^n - \Pi_h^{(1)} u^p |_{1,\infty} + | \Pi_h^{(1)} u^p |_{1,\infty}) \Delta t
\end{align*}
\]
\[
\begin{align*}
&\leq \alpha_3 h^{-d/2} (|u_h^n - \Pi_h^{(1)} u^n|_1) + \alpha_20 |u^n|_{1,\infty} |\Delta t| \\
&\leq \alpha_3 h^{-d/2} (|u_h^n - \tilde{u}_h^n|_1 + |\tilde{u}_h^n - u^n|_1 + |u^n - \Pi_h^{(1)} u^n|_1) + \alpha_20 |u^n|_{1,\infty} |\Delta t| \\
&\leq \alpha_3 h^{-d/2} (c_s (h^2 + |\Delta t|) + \alpha_50 h^2 \| (u, p)^n \|_{H^3} + \alpha_21 h |u^n|_2) \\
&\quad + \alpha_20 |u^n|_{1,\infty} |\Delta t| \\
&\quad + \alpha_3 h^{-d/4} (c_s h + \alpha_50 h \| (u, p)^n \|_{H^3}) + \alpha_21 h |u^n|_2 + \alpha_20 h^{d/4} |u^n|_{1,\infty} \\
&\leq c_0 \left\{ \alpha_3 h^{-d/4} (c_s h + \alpha_50 h \| (u, p)^n \|_{H^3}) + \alpha_21 h |u^n|_2 + \alpha_20 h^{d/4} |u^n|_{1,\infty} \right\} \\
&\quad + \frac{3}{16} \alpha_{20} + \frac{1}{16} \alpha_{20} = \frac{1}{4} \alpha_{20}. 
\end{align*}
\]

From this estimate and the definition of \( \alpha_{20} \), we have \( \Delta t |\Pi_h^{(1)} u_h^n|_{1,\infty} \leq 1/4 \).

Thus, we have proved that \( P(n) \) holds for \( n = 0, \ldots, N_T \).

From \( P(n)-(a) \), we obtain

\[
\| e_h \|_{\varepsilon_1(H^3)} + \| \mathcal{D}_{\Delta t} e_h \|_{\varepsilon_1(N_T; L^2)} \leq c_s (h^2 + |\Delta t|).
\]

Using the triangle inequality \( \| u_h - u \|_{\varepsilon_1(H^3)} \leq \| e_h \|_{\varepsilon_1(H^3)} + \| \eta \|_{\varepsilon_1(H^3)} \), we get

\[
\| u_h - u \|_{\varepsilon_1(H^3)} \leq c_1 (h^2 + |\Delta t|).
\]

We now prove the estimate on the pressure. We can evaluate \( e_h^p \) as follows:

\[
\| e_h^p \|_0 \leq \frac{1}{\alpha_4} \sup_{v_h \in V_h} \frac{b(v_h, e_h^p)}{\| v_h \|_1} \\
= \frac{1}{\alpha_4} \sup_{v_h \in V_h} \frac{1}{\| v_h \|_1} \left( \sum_{i=1}^4 \| R_i^n v_h \|_0 + \| \mathcal{D}_{\Delta t} e_h^p \|_0 + \| \mathcal{D}_{\Delta t} e_h^p \|_0 + \| \mathcal{D}_{\Delta t} e_h^p \|_0 \right) \\
\leq c \left( \| \mathcal{D}_{\Delta t} e_h^p \|_0 + \| \mathcal{D}_{\Delta t} e_h^p \|_0 + \| \mathcal{D}_{\Delta t} e_h^p \|_0 + \| \mathcal{D}_{\Delta t} e_h^p \|_0 \right) \\
\quad + \frac{h^2}{\sqrt{\Delta t}} \| (u, p) \|_{H^3} + \| e_h^p \|_{1,\infty} + \| e_h^p \|_{1,\infty} + \| e_h^p \|_{1,\infty} + \| e_h^p \|_{1,\infty} \\
\quad + \| (u, p) \|_{H^3} + \| e_h^p \|_{1,\infty} + \| e_h^p \|_{1,\infty} + \| e_h^p \|_{1,\infty}.
\]

\[
(\text{by Lemma } 10 \text{ and } P(n-1)-(b))
\]

which implies that, from \( (26) \),

\[
\| e_h \|_{\varepsilon_1(L^2)} \leq c \left( \| \mathcal{D}_{\Delta t} e_h \|_{\varepsilon_1(L^2)} + h^2 + |\Delta t| \right) \leq c (h^2 + |\Delta t|),
\]

where \( c \) is a positive constant independent of \( h \) and \( |\Delta t| \).

Using the triangle inequality

\[
\| p_h - p \|_{\varepsilon_1(L^2)} \leq \| e_h \|_{\varepsilon_1(L^2)} + \| p - \hat{p}_h \|_{\varepsilon_1(L^2)},
\]

we obtain \( \| p_h - p \|_{\varepsilon_1(L^2)} \leq c_1 (h^2 + |\Delta t|) \).
5.5 Proof of Theorem\[2\]

In this subsection we prove the result on the $P_1+/P_1$-element. At first we replace the estimates of $R_2^e$ and $R_3^e$ in Lemma [10]

Lemma 10' Suppose Hypotheses 1' and 2. Under the condition (16) it holds that

\[
\|R_2^e\|_0 \leq \beta_2 \|e_h^{n-1}\|_0 + \beta_3 \|u\|_{H^2} + h^2\|u^{n-1}\|_2,
\]

\[
\|R_3^e\|_0 \leq \beta_4 \frac{h}{\sqrt{M}} \left( \|u\|_{H^1(\Omega_1, \Omega_2)} + \|u^{n-1}\|_{L^2(\Omega_1, \Omega_2)} \right)
\]

for $n = 1, \ldots, N_T$.

The proof is similar to Lemma [10] by replacing the order $k = 2$ by $k = 1$ in Lemma 6(i).

Proof of Theorem\[2\] We only show the outline of the proof for the existence of $(u_h, p_h)$ and the inequality (10) since the proof is similar to that of Theorem[1]. We replace the definition of $c_*$ by

\[
c_* = \left\{ \nu^{-1} (1 + \alpha^2_t) \exp(\beta_2 T) \beta_2 \right\}^{1/2} \max \left\{ \|u\|_{L^2} \left( \|u\|_{H^1} + \|u\|_{C(H^2)} + \nu \|u\|_{C(H^2)} \right) \right\}^{1/2},
\]

redefine $c_0$ by (22), with the new $c_*$, and replace the condition (23) on $h_0$ by

\[
a_0 h_0^{1-\delta/4} \left( c_* + a_50 \|u\|_{C(H^2)} + a_211 \|u\|_{C(H^2)} + a_50 |c|_{C(H^2)} \right) \leq 1,
\]

(27a)

\[
c_0 \left\{ a_31 h_0^{1-\delta/4} (c_* + a_50 \|u\|_{C(H^2)} + a_211 \|u\|_{C(H^2)} + a_50 |c|_{C(H^2)} \right) \right\} \leq \frac{3}{16 a_20}.
\]

We also replace P(n)-(a) by

\[
\nu \|e_h\|_{L^2}^2 + \|D_{\Delta t} e_h\|_{L^2(\Omega_1, \Omega_2)}^2 \leq \exp(\beta_2 n \Delta t) \beta_2 \left\{ \Delta t^2 \max \left\{ \|u\|_{L^2} \left( \|u\|_{C(H^2)} + \nu \|u\|_{C(H^2)} \right) \right\}^2 \right\} + \|e_h\|_{L^2(\Omega_1, \Omega_2)}^2.
\]

P(n)-(a) implies the estimate

\[
\|e_h\|_1 \leq c_0 (h + \Delta t).
\]

(28)

The choice (27) is sufficient to derive P(n)-(b) and (c). Hence, the existence of the solution and the estimate (11) are obtained similarly.

We now prove the estimate [11], following [19] except the introduction of $X_1(P_1^{n-1} u_h^{n-1})$. Substituting $(v_h, q_h) = (e_h^n, e_h^n)$ in (14), we have

\[
\frac{1}{2} D_{\Delta t} \|e_h^n\|_0^2 + \frac{1}{2} \Delta t \|e_h^n - e_h^{n-1}\|_0^2 + \nu \|e_h^n\|_1^2 = \sum_{i=1}^4 \left( R_i, e_h^n \right).
\]

(29)
where $R_i, i = 1, \cdots, 4$, are defined in (15). The term $(R_7^i, e_h^p)$ is evaluated by (17g). From Lemma 6(ii) we have

$$\|\tilde{g}^n_{h} - u^{n-1}\|_0 \leq \alpha_5 h^2 \|(a, p)^{n-1}\|_{H^2 \times H^1}.$$  

Using this estimate in the last line in (18), we have

$$(R_2^i, e_h^p) \leq \|R_2^i\|_0 \|e_h^p\|_0 \leq \{\beta_2 \|e_h^p\|_0 + \beta_3 h^2 \|(a, p)^{n-1}\|_{H^2 \times H^1} + |a^{n-1}|_2\} \|e_h^p\|_0.$$  

We divide the term $(R_3^i, e_h^p)$ as follows:

$$(R_3^i, e_h^p) = \frac{1}{\Delta t} (\eta^n - \eta^{n-1}, e_h^p) + \frac{1}{\Delta t} (\eta^{n-1} - \eta^{n-1} \circ X_1(\Pi_{h}^{(1)} u^{n-1}), e_h^p) + \frac{1}{\Delta t} (\eta^{n-1} \circ X_1(\Pi_{h}^{(1)} u^{n-1}) - \eta^{n-1} \circ X_1(\Pi_{h}^{(1)} u_h^{n-1}), e_h^p) \equiv I_1 + I_2 + I_3.$$  

The first term $I_1$ is evaluated as

$$I_1 \leq \frac{1}{\sqrt{\Delta t}} \left\| \frac{\partial \eta}{\partial x} \right\|_{L^2(p^n-1, p^n; L^2)} \|e_h^p\|_0 \leq \alpha_5 h^2 \sqrt{\Delta t} \|(a, p)^{n-1}\|_{H^1(p^n-1, p^n; H^2 \times H^1)} \|e_h^p\|_0.$$  

By Lemma 9 the second term $I_2$ is evaluated as

$$I_2 \leq \alpha_6 \alpha_2 \|u\|_{C(W^{1, \infty})} \|\eta^{n-1}\|_0 \|e_h^p\|_1 \leq \alpha_6 \alpha_2 \|u\|_{C(W^{1, \infty})} \alpha_5 h^2 \|(a, p)^{n-1}\|_{H^2 \times H^1} \|e_h^p\|_1.$$  

In order to evaluate $I_3$ we prepare the estimate

$$\alpha_5 h^{-d/6} |\eta^{n-1}|_1 \leq \alpha_5 h^{-d/6} \alpha_{50} \|(a, p)^{n-1}\|_{H^2 \times H^1} \leq 1,$$

where we have used Lemma 6(i) and (27h).

Using Lemma 8 with $q = 1$, $p = p', w_1 = \Pi_{h}^{(1)} u^{n-1}$, $w_2 = \Pi_{h}^{(1)} u_h^{n-1}$ and $\eta = \eta^{n-1}$, Lemma 7 the above estimate and (18), we can evaluate $I_3$ as follows:

$$I_3 \leq \frac{1}{\Delta t} \|\eta^{n-1} \circ X_1(\Pi_{h}^{(1)} u^{n-1}) - \eta^{n-1} \circ X_1(\Pi_{h}^{(1)} u_h^{n-1})\|_{0, 1} \|e_h^p\|_{0, \infty} \leq \sqrt{2} \alpha_{50} h^{-d/6} \|\eta^{n-1}\|_1 \|\Pi_{h}^{(1)} u^{n-1} - \Pi_{h}^{(1)} u_h^{n-1}\|_0 \|e_h^p\|_1 \leq \sqrt{2} \alpha_{50} (\|e_h^p\|_1 + \alpha_5 h^2 \|(a, p)^{n-1}\|_{H^2 \times H^1} + \alpha_5 h^2 \|a^{n-1}\|_2) \|e_h^p\|_1.$$  

In order to evaluate $(R_4^i, e_h^p)$ we prepare the estimate

$$\alpha_{50} h^{-d/6} |\eta^{n-1}|_1 \leq \alpha_{50} h^{-d/6} c_h (h + \Delta t) \leq \alpha_{50} c_h h^{-d/6} (h + c_0 h^{d/4}) \leq 1,$$
where we have used (28) and (27). Using Lemma 9 the above inequality and a similar estimate to (31) in (\(R^h_i, e^h_k\)), we can evaluate (\(R^h_i, e^h_k\)) as follows:

\[
\begin{align*}
(R^h_i, e^h_k) &= -\frac{1}{\Delta t} (e^{n-1}_h - e^{n-1}_h \circ X_1(\Pi^h_1) u^{n-1}, e^h_k) \\
&\quad - \frac{1}{\Delta t} (e^{n-1}_h \circ X_1(\Pi^h_1) u^{n-1} - e^{n-1}_h \circ X_1(\Pi^h_1) u^{n-1}_h, e^h_k) \\
&\quad \leq \alpha_0 \alpha_1 \|u\|_{C(W^{1,1})} \|e^{n-1}_h\|_0 \|e^h_k\|_1 \\
&\quad + \sqrt{2} \alpha_2 \|e^{n-1}_h\|_0 + \alpha_3 \|u\|_{H^2} ||H^2 \times H^3 + \alpha_2 h^2 \|u^{n-1}\|_1 \|e^h_k\|_1.
\end{align*}
\]

Combining (29) with these estimates and using Young’s inequality and Poincaré’s inequality \(\|e^h_k\|_0 \leq \alpha_1 \|e^h_k\|_1\), we have,

\[
\begin{align*}
\frac{1}{2} T_{\Delta t} \|e^h_k\|_0^2 + \nu \|e^h_k\|_1^2 &\leq \beta_1 \left( \|e^{n-1}_h\|_0 + \sqrt{\Delta t} \|u\|_{L^2(\rho^{-1}, \rho^1)} \right) \\
&\quad + \frac{h^2}{\sqrt{\Delta t}} \|u\|_{H^1(\rho^{-1}, \rho^1, H^2 \times H^3)} + \frac{h^2}{\Delta t} \|u\|_{H^2 \times H^3} + \frac{h^2}{\Delta t} \|u^{n-1}\|_2 \|e^h_k\|_1 \\
&\quad \leq \nu \|e^h_k\|_0^2 + \beta_2 \|e^{n-1}_h\|_0^2 + \beta_3 \left( \|u\|_{L^2(\rho^{-1}, \rho^1)} + \frac{h^4}{\Delta t} \|u\|_{H^1(\rho^{-1}, \rho^1, H^2 \times H^3)} + \frac{h^4}{\Delta t} \|u^{n-1}\|_2 \right).
\end{align*}
\]

where \(\beta_1, \beta_2, \beta_3\) are positive constants independent of \(h\) and \(\Delta t\). Applying Gronwall’s inequality, we obtain (11). \(\Box\)

6 Numerical results

We show numerical results in \(d = 2\) for the \(P_2/P_1\)-element. We compare the conventional Scheme LG with the present Scheme LG-LLV. For the triangulation of the domain the FreeFem++ [11] is used. In Scheme LG we employ numerical quadrature of seven-point formula of degree five [10]. The relative error \(E_x\) is defined by

\[
E_x(\phi) = \frac{\|I_0 \phi - \phi_x\|_X}{\|I_0 \phi\|_X},
\]

for \(\phi = u\) in \(X = L^\infty(H^1_0)\) and \(L^\infty(L^2)\), and for \(\phi = p\) in \(X = L^2(L^2)\).

Example 1 [11] let \(\Omega \equiv (0, 1)^2, T = 1\). We consider the two cases, \(v = 10^{-2}\) and \(10^{-4}\). The functions \(f\) and \(u_0\) are defined so that the exact solution is

\[
\begin{align*}
u_1(x,t) &= \phi(x_1, x_2, t), \\
u_2(x,t) &= -\phi(x_2, x_1, t), \\
p(x,t) &= \sin(\pi x_1 + 2x_2) + 1 + t,
\end{align*}
\]

where \(\phi(a, b, t) \equiv -\sin(\pi a)^2 \sin(\pi b) \{\sin(\pi(a + t)) + 3 \sin(\pi(a + 2b + t))\}.)
Let $N$ be the division number of each side of $\Omega$. We set $h \equiv 1/N$. Figure 1 shows the triangulation of $\bar{\Omega}$ for $N = 16$. The time increment $\Delta t$ is set to be $\Delta t = h^2$ ($N = 16, 23, 32, 45$ and $64$) or $\Delta t = h^{3}$ ($N = 16, 19, 23, 27$ and $32$) so that we can observe the convergence behavior of order $h^2$ or $h^3$. The purpose of the choice $\Delta t = O(h^2)$ or $O(h^3)$ is to examine the theoretical convergence order, but it is not based on the stability condition, which is much weaker as shown in Theorem 1.

Table 1 Symbols used in Figs. 2 and 3, and Tables 2 and 3

| $\phi$ | $u$ | $p$ | $u$ | $u$ |
|--------|-----|-----|-----|-----|
| $X$    | $\ell^\infty(H^1_0)$ | $\ell^2(L^2)$ | $\ell^\infty(L^2)$ | $\ell^2(L^2)$ |
| $\Delta t$ | $h^2$ | $h^2$ | $h^3$ | $h^3$ |
| Scheme LG' | $\bullet$ | $\blacksquare$ | $\triangle$ | $\triangleleft$ |
| Scheme LG-LLV | $\circ$ | $\square$ | $\triangle$ | $\triangleleft$ |

Table 2 The values of relative errors and orders in Fig. 2 by Scheme LG'(top) and Scheme LG-LLV(bottom)

| $N$ | $\bullet$ | order | $\circ$ | order | $\square$ | order | $\triangle$ | order | $\triangleleft$ | order |
|-----|-----------|-------|--------|-------|--------|-------|-----------|-------|----------------|-------|
| 16  | 8.55e-2  | 1.6e-1 | 7.77e-2|       |        |       |           |       |                |       |
| 23  | 4.34e-2  | 1.87   | 4.03e-2| 1.81  |        |       |           |       |                |       |
| 32  | 2.30e-2  | 1.93   | 2.17e-2| 1.87  |        |       |           |       |                |       |
| 45  | 1.20e-2  | 1.90   | 1.13e-2| 1.93  |        |       |           |       |                |       |
| 64  | 6.02e-3  | 1.97   | 5.64e-3| 1.96  |        |       |           |       |                |       |

| $N$ | $\circ$ | order | $\square$ | order | $\triangle$ | order | $\triangleleft$ | order |
|-----|--------|-------|--------|-------|-----------|-------|----------------|-------|
| 16  | 6.45e-3|        |        |       |           |       |                |       |
| 23  | 3.73e-3| 3.19  |        |       |           |       |                |       |
| 32  | 2.10e-3| 3.02  |        |       |           |       |                |       |
| 45  | 1.29e-3| 3.02  |        |       |           |       |                |       |
| 64  | 7.57e-4| 3.15  |        |       |           |       |                |       |

Table 1 shows the symbols used in the graphs and tables. Since every graph of the relative error $E_X$ versus $h$ is depicted in the logarithmic scale, the slope corresponds to the convergence order. Figure 3 shows the graphs of $E_{\ell^\infty(H^1_0)}(u)$, $E_{\ell^2(L^2)}(p)$ and
Fig. 2 Relative errors $E_{ℓ∞(H1)}(u)$, $E_{ℓ2(L2)}(p)$, $E_{ℓ∞(L2)}(u)$ with $Δt = h^2$ (left) and $E_{ℓ∞(L2)}(u)$ with $Δt = h^3$ (right) in the case of $ν = 10^{-2}$ in Example 1.

Table 3 The values of errors and orders of the graphs in Fig. 3 with $Δt = h^2$ (top) and $Δt = h^3$ (bottom)

| $N$ | $•$ order | $■$ order | $▲$ order | $△$ order | $•$ order | $■$ order | $▲$ order | $△$ order |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 16  | 1.91e+0  | 2.14e-1  | 1.93e-1  |  |  |  |  |  |
| 23  | 1.34e+0  | 0.97     | 8.97e-2  | 3.39 | 8.81e-2 | 2.16 |  |  |
| 32  | 9.42e+0  | -5.90    | 3.48e-1  | -1.11 | 5.28e-1 | -5.43 | 23 | 3.65e-2 | 3.14 |
| 45  | 4.10e+1  | -4.31    | 1.28e+0  | -3.81 | 1.46e+0 | -2.98 | 27 | 1.92e-2 | 4.01 |
| 64  | 8.82e+1  | -2.18    | 2.77e+0  | -2.20 | 2.02e+0 | -0.93 | 32 | 1.02e-2 | 3.71 |

Table 3 (continued)

| $N$ | $□$ order | $△$ order | $▽$ order |
|-----|-----------|-----------|-----------|
| 16  | 6.72e-1  | 2.65e-1  | 2.09e-1  |
| 23  | 3.91e-1  | 1.50     | 1.36e-1  | 1.83 | 9.88e-2 | 2.07 |
| 32  | 1.85e-1  | 2.26     | 6.98e-2  | 2.02 | 4.18e-2 | 2.60 |
| 45  | 1.27e-1  | 1.10     | 3.73e-2  | 1.84 | 2.12e-2 | 1.99 |
| 64  | 7.21e-2  | 1.61     | 1.83e-2  | 2.03 | 9.78e-3 | 2.20 |
| 16  | 2.47e-1  |  |  |  |  |  |  |
| 19  | 6.64e-2  | 4.92     |  |  |  |  |  |
| 23  | 3.65e-2  | 3.14     |  |  |  |  |  |
| 27  | 1.92e-2  | 4.01     |  |  |  |  |  |
| 32  | 1.02e-2  | 3.71     |  |  |  |  |  |

$E_{ℓ∞(L2)}(u)$ versus $h$ in the case of $ν = 10^{-2}$. Their values and convergence orders are listed in Table 2. When $Δt = h^2$, the convergence orders of $E_{ℓ∞(H1)}(u)$ ($•$, $□$), $E_{ℓ2(L2)}(p)$ ($■$, $□$) and $E_{ℓ∞(L2)}(u)$ ($▲$, $△$) are almost 2 in both schemes. When $Δt = h^3$, the order of $E_{ℓ∞(L2)}(u)$ is almost 3 in Scheme LG$′$ ($▼$) and 2 in Scheme LG-LLV ($▽$). They reflect the theoretical results.

We consider a higher Reynolds number case. Figure 4 shows the graphs in the case of $ν = 10^{-4}$ and their values are listed in Table 3. When $Δt = h^3$, all errors increase abnormally at $N = 32, 45$ and 64 in Scheme LG$′$ ($•$, $□$, $▲$) while the con-
Fig. 3  Relative errors $E_{\ell^\infty} (H^1_0)$, $E_{L^2} (p)$, $E_{L^2} (u)$ with $\Delta t = h^2$ (left) and $E_{L^2} (u)$ with $\Delta t = h^3$ (right) in the case of $v = 10^{-4}$ in Example 1.

Fig. 4 The domain $\Omega$ and the boundary condition (left) and the triangulation of $\Omega$ (right) in Example 2.

Convergence is observed in Scheme LG-LLV ($\bigcirc$, $\square$, $\bigtriangleup$) but the order of $E_{\ell^\infty} (H^1_0)$ ($\bigcirc$) is less than 2. In order to obtain the theoretical convergence order $O(h^2)$ in Scheme LG-LLV, it seems that finer meshes will be necessary. When $\Delta t = h^3$, the order of $E_{L^2} (u)$ is more than 3 in Scheme LG' ($\blacktriangledown$) while it is less than 3 between $N = 19$ and 23, and $N = 23$ and 27 in Scheme LG-LLV ($\triangledown$).

We now consider a cavity problem to see that Scheme LG-LLV is robust for high Reynolds number while Scheme LG' is not. This problem is not a homogeneous
Dirichlet boundary problem, but it is often used as a benchmark problem. In order to assure the existence of the solution we deal with a regularized cavity problem, where the prescribed velocity is continuous on the boundary.

Example 2 Let \( \Omega \equiv (0, 1)^2, f = 0, u^0 = 0 \). We consider the two cases, \( \nu = 10^{-4} \) and \( 10^{-5} \). The boundary condition is described in Fig. 4 (left), where \( g_1 = 4x_1(1-x_1) \).

Figure 4 (right) shows the triangulation of \( \Omega \). Figures 5 and 6 show the stereographs of the solution \( u^n \) at \( t^n = 8 \) in the subdomain \((0.3, 0.7) \times (0.8, 1.0)\) by Scheme LG' and Scheme LG-LLV, respectively, when \( \nu = 10^{-4} \). Neither solution is oscillating although \( u_{h2} \) of Scheme LG' takes larger values than that of Scheme LG-LLV.
Figures 7 and 8 show the stereographs of the solution $u^n_h$ at $t^n = 8$ in the subdomain $(0.3, 0.7) \times (0.8, 1.0)$ by Scheme LG' and Scheme LG-LLV, respectively, when $\nu = 10^{-5}$. While oscillation is observed for both components of the solution by Scheme LG' in Figure 7, we can see that the solution by Scheme LG-LLV is solved without any oscillation in Figure 8.

7 Conclusions

We have present a Lagrange–Galerkin scheme free from numerical quadrature for the Navier–Stokes equations. By virtue of the introduction of a locally linearized velocity, the scheme can be implemented exactly and the theoretical stability and the convergence results are assured for practical numerical solutions. We have shown optimal error estimates in $\ell^\infty(H^1) \times \ell^2(L^2)$-norm for the velocity and pressure in the case of $P_2/P_1$- and $P_1+/P_1$-finite elements. Numerical results have reflected these estimates and the robustness of the scheme for high-Reynolds number problems.

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