A REMARK ON EXACT FORMULAS FOR THE RIESZ ENERGY OF THE NTH ROOTS OF UNITY

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Abstract. In the paper the author, Hardin and Saff [Bull. Lond. Math. Soc. 41 (2009), no. 4] we provided the complete asymptotic expansions of the Riesz s-energy of the Nth roots of unity. In this note the exact formulas (valid for all N ≥ 2) are obtained for s is an even integer. Explicit representations of the coefficients of the powers of N are derived by direct computations. In the case of positive s a continuous modified energy approximation of the Riesz energy is used. Stirling numbers of the first kind, Eulerian numbers and special values of partial Bell polynomials play a central role. A number of identities between these quantities are shown.

The Riesz s-energy (real s ≠ 0) of an N-point configuration \(X_N = \{z_1, \ldots, z_N\}\) in the complex plane (identified with the Euclidean space \(\mathbb{R}^2\)) is given by

\[E_s(X_N) = \sum_{j=1}^{N} \sum_{k=1}^{N} \frac{1}{|z_j - z_k|^{s+1}}.\]

It can be shown (see [4]) that the Nth roots of unity (or any rotation thereof) maximize (if \(-2 < s < 0\)) and minimize (if \(s > 0\)) the above energy functional over all N-point configuration on the unit circle. We remark that for \(s < -2\) an optimal configuration on the unit circle consists of half of the points at one of two opposite locations. (This follows from results of G. Björck [3].)

The Riesz s-energy if \(s ≠ 0\) is real of the N-th roots of unity can be evaluated by means of

\[L_s(N) = 2^{-s}N \sum_{k=1}^{N-1} \left( \sin \frac{\pi k}{N} \right)^{-s}, \quad N ≥ 2, s ∈ \mathbb{C}, s ≠ 0.\]

It has the following general complete asymptotic expansion of \(L_s(N)\) ([4 Thm. 1.1])

\[L_s(N) = V_s N^2 + \frac{2}{(2\pi)^s} \sum_{n=0}^{p} \alpha_s(s) \zeta(s-2n) N^{1+s-2n} + O_s,p(N^{-1+\Re s-2p}), \quad \text{as } N \to \infty,\]

provided \(s\) is not a positive odd integer. (These exceptional cases are considered in [4 Thm. 1.2].)

The Riemann zeta function has its trivial zeros at negative even integers. Thus, the sum in (2) vanishes for all \(N ≥ 2\) if \(s\) is a negative even integer or can have at most \(m\) terms if \(s = 2m\) is a positive even integer.

The case \(s = -2m\) and \(m\) a positive integer. In [4 Remark 1, Eq. (1.19)] we observed that

\[L_s(N) = V_s N^2, \quad s = -2, -4, -6, \ldots,\]

\(V_s\) given in [4], which is correct up to a remainder term of order \(O_{s,p}(N^{-1-s-2p})\), where \(p\) can be any positive integer. The precise result is as follows.

Proposition 1. Let \(m\) be a positive integer and set \(s = -2m\). Then

\[L_s(N) = V_s N^2 + 2N^2 \sum_{k=1}^{m} (-1)^k \left( \frac{2m}{m-k} \right), \quad N = 2, 3, 4, \ldots,\]

where \(N|k\) denotes that the integer \(N\) divides the integer \(k\). The sum above vanishes if \(N > m\).
For the sake of completeness we include here the proof.

**Proof.** Using the identity ([8 Eq. I.1.9])

\[
(sin \phi)^{2m} = 2^{1-2m} \sum_{k=0}^{m-1} (-1)^{m-k} \binom{2m}{k} \cos[2(m-k)\phi] + 2^{-2m} \binom{2m}{m},
\]

in [1] we get

\[
\mathcal{L}_{-2m}(N) = 2^{2m}N \sum_{k=0}^{N-1} \left( \frac{\sin \pi k}{N} \right)^{2m} = 2N \sum_{\ell=1}^{m} (-1)^{\ell} \left( \frac{2m}{m-\ell} \right) \left( \frac{N-1}{\sum_{k=0}^{N-1} \cos 2\pi k \ell \frac{N}{\ell}} \right) + \left( \frac{2m}{m} \right) N^2
\]

a representation in terms of trigonometric sums satisfying the identity ([8 Eq.s 4.4.6–8])

\[
\sum_{k=0}^{N-1} \cos(2\pi \ell k/N) = \begin{cases} N & \text{if } N \text{ divides } \ell \text{ (that is } N|\ell), \\ 0 & \text{otherwise.} \end{cases}
\]

Also observe that [4 Eq. (1.8)] can be written as

\[
V_s = 2^{-s} \frac{\Gamma((1-s)/2)}{\sqrt{\pi} \Gamma((1-s)/2)} = 2^{2m} \frac{\Gamma((1+2m)/2)}{\sqrt{\pi} \Gamma(1+2m)} = 2^{2m} \frac{\Gamma(m+1/2) \Gamma(1+m)}{\sqrt{\pi} \Gamma(1+m) \Gamma(1+m)}
\]

by the duplication formula for the gamma function. The result follows. \(\square\)

**The case** \(s = 2m\) **and** \(m\) **a positive integer.** In [4 Remark 1, Eq. (1.20)] it was observed that

\[
\mathcal{L}_s(N) = \frac{2}{(2\pi)^s} \sum_{n=0}^{m} \alpha_n(s) \zeta(s-2n)N^{1+s-2n}, \quad s = 2m, m = 1, 2, 3, \ldots,
\]

by arguing that in the general asymptotic expansion [2] the constant \(V_s\) given in [4] vanishes if \(s\) is a positive even integer and the Riemann zeta function part \(\zeta(s-2n)\) assumes the value 0 for \(s-2n\) is an even negative integer. One should add that [5] is true up to a remainder term of order \(O_{s,p}(N^{-1-s-2p})\), where \(p\) can be any positive integer. In fact, direct computations suggest that relation [5] holds for all \(N \geq 2\). The coefficients \(\alpha_n(s)\) are defined by the following generating function relation and can be expressed explicitly in terms of generalized Bernoulli polynomials \(B_k^{(s)}(x)\); that is

\[
\left( \frac{\sin \pi z}{\pi z} \right)^{-s} = \sum_{n=0}^{\infty} \alpha_n(s) z^{2n}, \quad |z| < 1, \quad s \in \mathbb{C}, \quad \alpha_n(s) = \frac{(-1)^n B_{2n}^{(s)}(s/2)}{(2n)!} (2\pi)^{2n}, \quad n \geq 0.
\]

For non negative even integers the Riemann zeta function can be expressed in terms of Bernoulli numbers by means of \(\zeta(2m) = 2^{2m-1}(1-1)^{m-1}B_{2m}\pi^{2m}/(2m)!\), \(m = 0, 1, 2, \ldots\).

The validity of relation [5] for all \(N \geq 2\) follows by means of a limit process from the corresponding result for a modified (and thus non-singular) form of the Riesz energy of the \(N\)th roots of unity.

The distance between a point \(x\) on the unit circle \((|x| = 1)\) and a point \(y\) inside the unit circle \((r = |y|) < 1)\) is given by

\[
|x - y|^2 = 1 - 2r \langle x, y \rangle + r^2 = (1 - r)^2 + 4r [\sin(\phi/2)]^2, \quad \cos \phi = \langle x, y \rangle.
\]

This motivates the following modification of the the \(s\)-energy of the \(N\)-th roots of unity:

\[
\mathcal{M}_s(N; r) := N \sum_{k=1}^{N-1} \left( 1 - 2r \cos \frac{2\pi k}{N} + r^2 \right)^{-s/2}, \quad 0 \leq r < 1.
\]
Observe that $\mathcal{L}_s(N) = \lim_{r \to 1^-} \mathcal{M}_s(N; r)$. It should be mentioned that Chu considered sums of the form

$$f_m(z) = \sum_{k=0}^{n-1} \frac{1 - z^2}{[1 - 2z \cos(2\pi k/n) + z^2]^m},$$

where $m$ and $n$ are positive integers and $z$ is complex. The following recurrence relation is given:

$$f_{m+1}(z) = \frac{f_m(z)}{1 - z^2} + \frac{z}{m} \frac{d}{dz} \left\{ \frac{f_m(z)}{1 - z^2} \right\}.$$

Chu’s main interest was in the case when $z = \sqrt{-1}$.

We make use of the Gauss hypergeometric function

$$(8) \quad _2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} \frac{z^n}{n!}, \quad |z| < 1,$$

where $(a)_n = \Gamma(n + a)/\Gamma(a)$ denotes the Pochhammer symbol.

**Proposition 2.** Let $s > 0$. Then for integers $N \geq 2$

$$\mathcal{M}_s(N; r) = N^2 G_0(s; r) - N (1 - r)^{-s} + 2N^2 \sum_{\nu=1}^{\infty} G_{\nu N}(s; r), \quad 0 \leq r < 1,$$

where the series is uniformly convergent with respect to $r$ on compact subsets of $[0, 1)$ and

$$G_n(s; r) = (1 - r^2)^{1-s} \frac{(s/2)_n r^n}{n!} _2F_1 \left\{ 1 - s/2, n + 1 - s/2; r^2 \right\}.$$

For $s$ a positive even integer, that is $s = 2m$, we can write (cf. [11, Eq. 15.8.7])

$$(9) \quad G_n(2m; r) = (1 - r^2)^{1-2m} \frac{2^{2m-2} \Gamma(m - 1/2)}{\sqrt{\pi} \Gamma(m)} r^n _2F_1 \left\{ 1 - m, n + 1 - m; 1 - r^2 \right\}.$$

The hypergeometric function above reduces to a polynomial of degree $m - 1$.

For $s = 2$, that is $m = 1$, one simply has $G_n(2; r) = r^n/(1 - r^2)$ which leads to

$$\mathcal{M}_2(N; r) = \frac{N^2}{1 - r^2} - \frac{N}{(1 - r)^2} + 2 \frac{N^2}{1 - r^2} \sum_{\nu=1}^{\infty} r^{\nu N} = \frac{N^2}{1 - r^2} - \frac{N}{(1 - r)^2} + \frac{2N^2 r^N}{(1 - r^2) (1 - r^N)}.$$

Using Mathematica, a limit process $r \to 1^-$ yields that

$$\mathcal{L}_2(N) = \lim_{r \to 1^-} \mathcal{M}_2(N; r) = \frac{N^3}{12} - \frac{N}{12} \quad \text{and by [5]:} \quad \mathcal{L}_2(N) = \frac{2}{4\pi^2} \left( \frac{\pi^2}{6} N^3 - \frac{\pi^2}{6} N \right).$$

It follows that [5] is correct for $s = 2$ and all $N \geq 1$.

In the general result we make use of the Stirling numbers of the first kind $s(n, k)$ defined by the expansion ([11, Eq. 26.8.7])

$$(10) \quad \sum_{k=0}^{n} s(n, k) x^k = x (x - 1) \cdots (x - n + 1) = (x + 1 - n)_n,$$

where $(a)_n$ is the Pochhammer symbol, and the Eulerian numbers $\langle n \rangle_k$ given by (cf. [11, Eq.s 26.14.5 and 26.14.6])

$$(11) \quad \sum_{k=0}^{n-1} \langle n \rangle_k \langle x + k \rangle_n = x^n; \quad \langle n \rangle_k = \sum_{j=0}^{k} (-1)^j \begin{pmatrix} n + 1 \cr j \end{pmatrix} (k + 1 - j)^n \quad n \geq 1.$$

There holds $\langle 0 \rangle_0 = 1$ and $\langle n \rangle_{n-1} = 1$. By convention we set $\langle n \rangle_k = 0$ for $k \geq n$. 
Proposition 3. Let \( m \) and \( N \) be positive integers. Then
\[
\mathcal{M}_{2m}(N; r) = \frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{\Gamma(2m - k - 1)}{\Gamma(m - k)k!} \left[ -(1 - m)_k + 2 \sum_{q=0}^{k} g(k, q; N, 1 - m) \left( 1 - r^N \right)^{-q-1} \right] (1 - r^2)^{k+1-2m} - N (1 - r)^{-2m}, \quad 0 \leq r < 1,
\]
where
\[
g(k, q; a, b) = \sum_{p=0}^{k-q} (-1)^p s(k, p + q; b) b(p + q, p) a^{p+q}, \quad b(p + q, p) = \sum_{j=0}^{q} \binom{p+q}{j} \binom{p+q-j}{p},
\]
\[
s(n, \ell; y) = \sum_{k=\ell}^{n} \binom{k}{\ell} s(n, k) (y + n - 1)^{k-\ell}.
\]

We are interested in the limit as \( r \to 1^- \). Clearly, \( \mathcal{M}_{2m}(N) := \lim_{r \to 1} \mathcal{M}_{2m}(N; r) = \mathcal{L}_{2m}(N) \).

Proposition 4. Let \( m \) and \( N \) be positive integers. Then
\[
\mathcal{M}_{2}(N) = \frac{N^3}{12} - \frac{N}{12}, \quad \mathcal{M}_{2m}(N) = \sum_{\nu=0}^{2m-1} \beta_{2m-\nu}(m) N^1+2m-\nu, \quad m \geq 2,
\]
where the coefficients \( \beta_{\nu}(m) \) (\( 0 \leq \nu \leq 2m \)) do not depend on \( N \).

Explicit expressions for the coefficients \( \beta_{\nu}(m) \) are given in the Proof of Prop. 4 (see Eq. (12)). The expansion (12) holds for every integer \( N \geq 2 \), whereas the expansion (5) (derived from an asymptotic result as \( N \to \infty \)) representing the same quantity holds for sufficiently large \( N \). Since the coefficients of the powers of \( N \) in either expansion do not depend on \( N \), by comparing (12) and (5) we obtain for \( s = 2m \) and \( m \) a positive integer the connection formulas
\[
\beta_{2n+1}(m) = 0 \quad (0 \leq n < m), \quad \beta_{2m-2n}(m) = \frac{2 \zeta(s-2n)}{(2\pi)^s} \alpha_n(s) \quad (0 \leq n \leq m).
\]
The coefficients \( \alpha_n(2m) \) can be expressed in terms of generalized Bernoulli polynomials and the Riemann zeta function for nonnegative even integers can be expressed in terms of Bernoulli numbers (cf. 6 and text there). This leads to
\[
\beta_{2m-2n}(m) = \frac{(-1)^{m-n-1}B_{2m-2n}(-1)^nB_{2m}(m)}{(2m-2n)!}, \quad 0 \leq n \leq m.
\]

For \( n = 0 \) one gets the following identity.

Proposition 5. For \( m = 1, 2, \ldots \) on has
\[
\frac{(-1)^{m-1}B_{2m}}{(2m)!} = \sum_{\nu=0}^{m-1} \sum_{p=0}^{\nu} (-1)^p \frac{\zeta^{2+\nu-2m} \Gamma(2m - 1 - \nu)}{(m)^{\nu} \Gamma(m - \nu)} b(\nu, p) G(2m - p, 2m - p, \nu - p + 1),
\]
where the numbers \( b(\nu, p) \) and \( G(n, n, q) \) are given by
\[
b(k, p) = \sum_{j=0}^{k-p} \binom{k}{j} \binom{k-j}{p}, \quad \sum_{n=1}^{\infty} (-1)^n G(n, n, q) x^n = \left( \frac{e^x - 1}{x} \right)^{-q} - 1.
\]

1. Proofs

The Gegenbauer polynomials \( C_n^{(\lambda)} \) of degree \( n \) with index \( \alpha > 0 \) may be defined by means of the generating function relation
\[
\sum_{n=0}^{\infty} z^n C_n^{(\lambda)}(\cos \phi) = \frac{1}{(1 - 2z \cos \phi + z^2)^{\lambda}}, \quad |z| < 1, \lambda > 0.
\]
They admit a representation through trigonometric functions (cf. [2, Section 22]) as follows

\begin{equation}
C_n^{(\lambda)}(\cos \phi) = \sum_{\ell=0}^{n} \frac{(\lambda)_\ell}{\ell!} C_{n-\ell}^{(\lambda)} \cos[(n-2\ell)\phi], \quad \lambda \neq 0.
\end{equation}

We need the following auxiliary result.

**Lemma 6.** Let \( s > 0 \). Then

\[
A_n(s; N) := N \sum_{k=1}^{N-1} C_{n}^{(s/2)}(\cos \frac{2\pi k}{N}) = N^2 \sum_{\ell=0}^{n} \frac{(s/2)_\ell}{\ell!} \frac{(s/2)_{n-\ell}}{(n-2\ell)!} - N C_{n}^{(s/2)}(1).
\]

For even and odd \( n \) one has

\[
A_{2\nu}(s; N) = \frac{(s/2)_\nu}{|\nu|!} N^2 - C_{2\nu}^{(s/2)}(1) N + 2N^2 \sum_{\ell=0}^{\nu} \frac{(s/2)_{\nu-\ell}(s/2)_{\nu+\ell}}{(\nu-\ell)! (\nu+\ell)!},
\]

\[
A_{2\nu+1}(s; N) = 2N^2 \sum_{\ell=0}^{\nu} \frac{(s/2)_{\nu-\ell}(s/2)_{\nu+1+\ell}}{(\nu-\ell)! (\nu+1+\ell)!} - C_{2\nu+1}^{(s/2)}(1) N.
\]

**Proof.** By (14) and (9) we have

\[
A_n(s; N) = \sum_{\ell=0}^{n} \frac{(s/2)_\ell}{\ell! (n-\ell)!} N \sum_{k=1}^{N-1} \cos[(n-2\ell)\frac{2\pi k}{N}] = N^2 \sum_{\ell=0}^{n} \frac{(s/2)_\ell}{\ell! (n-\ell)!} - N C_{n}^{(s/2)}(1).
\]

Since \(|n-2\ell|\) if and only if \(|2\ell-n|\), where \(2\ell-n = -(n-2(\ell-n))\), we can use symmetry to write

\[
\sum_{\ell=0}^{2\nu} \frac{(s/2)_\ell}{\ell!(2\nu-\ell)!} = 2 \sum_{\ell=0}^{\nu-1} \frac{(s/2)_\ell}{\ell!(2\nu-\ell)!} + \frac{(s/2)_\nu}{\nu!},
\]

\[
\sum_{\ell=0}^{2\nu+1} \frac{(s/2)_\ell}{\ell!(2\nu+1-\ell)!} = 2 \sum_{\ell=0}^{\nu} \frac{(s/2)_\ell}{\ell!(2\nu+1-\ell)!}.
\]

The results for even and odd \( n \) follow by reversing the order of terms on the right-hand sides above and substituting the expressions into the result for general \( n \).

**Proof of Proposition [3] Using (13) in (7) and Lemma [5] we obtain**

\[
M_s(N; r) = \sum_{n=0}^{\infty} r^n N \sum_{k=1}^{N-1} C_{n}^{(s/2)}(\cos \frac{2\pi k}{N}) = N^2 \sum_{n=0}^{\infty} r^n \sum_{\ell=0}^{n} \frac{(s/2)_\ell}{\ell! (n-\ell)!} - N \sum_{n=0}^{\infty} r^n C_{n}^{(s/2)}(1).
\]

The last series has the closed form \((1-r)^{-s}\) (cf. [13]). One has

\[
\sum_{\nu=0}^{\infty} r^{2\nu} \sum_{\ell\in|2\nu-2\ell|} \frac{(s/2)_\ell}{\ell!(2\nu-\ell)!} = \sum_{\nu=0}^{\infty} \frac{(s/2)_\nu}{\nu!} r^{2\nu} + 2 \sum_{\ell=1}^{\infty} r^{2\nu} \sum_{\ell=0}^{\nu} \frac{(s/2)_{\nu-\ell}(s/2)_{\nu+\ell}}{(\nu-\ell)! (\nu+\ell)!} = \sum_{\nu=0}^{\infty} \frac{(s/2)_\nu}{\nu!} r^{2\nu} + 2 \sum_{\ell=1}^{\infty} \sum_{\nu=\ell}^{\infty} \frac{(s/2)_{\nu-\ell}(s/2)_{\nu+\ell}}{(\nu-\ell)! (\nu+\ell)!} r^{2\nu} = 2 \text{F}_1\left(s/2, 1, r^2\right) + 2 \sum_{\ell=1}^{\infty} \frac{(s/2)_{2\ell}}{(2\ell)!} \text{F}_1\left(s/2, 2\ell + s/2, r^2\right).
\]
In the last step (8) was used. By the same token
\[ \sum_{\nu=0}^{\infty} r^{2\nu+1} \sum_{t=0}^{2\nu} \frac{(s/2)_e(s/2)_{2\nu+1-t}}{t!(2\nu+1-t)!} = 2 \sum_{t=0}^{\infty} \frac{(s/2)_{2t+1}}{(2t+1)!} r^{2t+1} \text{F}_1 \left( \frac{s/2}{2}, \frac{2t+1+s/2}{2t+2} ; r^2 \right). \]

Putting everything together we obtain
\[ \mathcal{M}_s(N;r) = N^2 G_0(s;r) - N (1 - r)^{a+2N^2} \sum_{n=1}^{\infty} G_n(s;r), \quad G_n(s;r) = \frac{(s/2)_n}{n!} r^n \text{F}_1 \left( \frac{s/2}{n+1} \right). \]

The series \( \sum_{n=1}^{\infty} G_{\nu N}(s;r) \) converges uniformly with respect to \( r \) on compact subsets of \([0,1)\), which can be seen from the integral representation (cf. \cite[Eq. 15.6.1]{1}) and estimate
\[ G_n(s;r) = r^n \frac{1}{[\Gamma(s/2)]^2} \frac{\Gamma(n + s/2)}{\Gamma(n + 1 - s)/2} \int_0^1 \frac{t^{s/2-1} (1 - t)^{n-s/2}}{(1-r^2 t)^{n+s/2}} \, dt \leq c_s(r) \frac{\Gamma(n + s/2)}{\Gamma(n + 1 - s/2)} r^n, \]
valid for \( n \geq n_0 > s/2 \), where \( c_s(r) \) can be uniformly bounded on compact sets in \([0,1)\). Hence
\[ 0 < \sum_{n=1}^{\infty} G_n(s;r) \leq C + \sum_{n=n_0}^{\infty} G_n(s;r) \leq C + c_s(r) \sum_{n=n_0}^{\infty} \frac{\Gamma(n + s/2)}{\Gamma(n + 1 - s)} r^n \quad \text{for some} \ C > 0 \]
and the right-most series above converges uniformly on compact sets in \([0,1)\) for \( n_0 > s/2 > 0 \).

Application of the last linear transformations in \cite[Eqs. 15.8.1]{1} yields
\[ G_n(s;r) = (1 - r^2)^{1-s} \frac{(s/2)_n}{n!} r^n \text{F}_1 \left( \frac{1 - s}{n+1}, \frac{n + 1 - s/2}{2} ; r^2 \right). \]

For \( s \) a positive even integer, that is \( s = 2m \), we can write (cf. \cite[Eq. 15.8.7]{1})
\[ G_n(2m;r) = (1 - r^2)^{1-2m} \frac{(2m)_n}{n!} \left( \frac{2m-2}{m-1} \right)^{m-1} r^n \text{F}_1 \left( \frac{1 - m}{2 - 2m}, \frac{n + 1 - m}{2} ; 1 - r^2 \right), \]
where the ratios can be simplified further. This shows (9).

**Proof of Proposition**

The series expansion of the hypergeometric polynomial in (9) yields
\[ G_n(2m;r) = \frac{2^{2m-2} \Gamma(m-1/2)}{\sqrt{\pi} \Gamma(m)} r^n \sum_{k=0}^{m-1} \frac{(1 - m)_k (n + 1 - m)_k}{(2 - 2m)_k} \left( 1 - r^2 \right)^{k+1-2m}. \]

Since
\[ \frac{(1 - m)_k}{(2 - 2m)_k} = \frac{m-1}{k} \left( \frac{2m-2}{k} \right) = \frac{\Gamma(m) \Gamma(2m - 1 - k)}{\Gamma(m-k) \Gamma(2m - 1)} \]
and \( \Gamma(2m - 1) = 2^{2m-2} \Gamma(m-1/2) \Gamma(m)/\sqrt{\pi} \) (duplication formula for the gamma function) we obtain
\[ G_n(2m;r) = \frac{1}{\Gamma(m)} r^n \sum_{k=0}^{m-1} \frac{\Gamma(2m - 1 - k) (n + 1 - m)_k}{(m-k)!} (1 - r^2)^{k+1-2m}. \]

By Proposition (2) and the last representation we get
\[ \mathcal{M}_{2m}(N;r) = \sum_{k=0}^{m-1} \frac{1}{\Gamma(m)} r^n \left( \frac{2m - k - 1}{m - k} \Gamma(m) \right) \Gamma(m-k)! \left[ \sum_{|\nu|=0}^{\infty} \left( \frac{-1}{k} + 2 \sum_{q=0}^{k} g(k, q; N, 1 - m) (1 - r^N)^{-q-1} \right) \right] (1 - r^2)^{k+1-2m} \]

By Lemma (13) the infinite series reduces to a rational function in \( r^N \). That is
\[ \mathcal{M}_{2m}(N;r) = \sum_{k=0}^{m-1} \frac{1}{\Gamma(m)} \left( \frac{2m - k - 1}{m - k} \Gamma(m) \right) \Gamma(m-k)! \left[ -(1 - m)_k + 2 \sum_{q=0}^{k} g(k, q; N, 1 - m) (1 - r^N)^{-q-1} \right] (1 - r^2)^{k+1-2m}. \]
where \( g(k, q; N, b) \) is defined in Lemma 15.

**Proof of Proposition 5** For the Proof of Proposition 5 we need some preparations. We define the following two functions

\[
f_q(r) = \left(\frac{1 - r^N}{1 - r}\right)^{-q}, \quad h(r) = \frac{1 - r^N}{1 - r} = \sum_{\ell=0}^{N-1} r^\ell
\]

and get a series expansion of \( f_q(r) \) at \( r = 1 \). For that we use Faà di Bruno’s differentiation formula

\[
\{f(g(x))\}^{(n)} = \sum_{k_1! \cdots k_n!} \frac{n!}{k_1! \cdots k_n!} f^{(k)}(g(x)) \prod_{\nu=1}^{n} \left( \frac{g^{(\nu)}(x)}{\nu!} \right)^{k_\nu} = \sum_{k=1}^{n} f^{(k)}(g(x)) B_{n,k}(g'(x), g''(x), \ldots),
\]

where \( k = k_1 + \cdots + k_n \) in the first sum and this sum is extended over all partition of \( n \), that is integers \( k_1, \ldots, k_n \geq 0 \) such that \( k_1 + 2k_2 + \cdots + nk_n = n \). The polynomials \( \text{B}_{n,k}(x_1, x_2, \ldots) \) in the second sum are the (partial) Bell polynomials, explicitly given by

\[
B_{n,k}(x_1, x_2, \ldots) = \sum_{k_1+k_2+\cdots+k_n = n \atop k_1+2k_2+\cdots+n} \frac{n!}{k_1!k_2!\cdots} \left(\frac{x_1}{1!}\right)^{k_1} \left(\frac{x_2}{2!}\right)^{k_2} \cdots,
\]

which satisfy the generating function relation (see, for example, [6])

\[
\sum_{n=0}^{\infty} B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) t^n/n! = \frac{1}{k!} \left( \sum_{m=1}^{\infty} x_m t^m/m! \right)^k.
\]

(For the polynomial \( B_{n,k} \) only the variables \( x_1, x_2, \ldots, x_{n-k+1} \) will be effective.) We record that

\[
\begin{align*}
B_{n,k}(kx_1, \alpha x_2, \ldots) &= \alpha^k B_{n,k}(x_1, x_2, \ldots), \quad \alpha \neq 0, \\
B_{n,k}(\alpha^x x_1, \alpha^2 x_2, \ldots) &= \alpha^n B_{n,k}(x_1, x_2, \ldots), \quad \alpha \neq 0.
\end{align*}
\]

**Lemma 7.** Let \( n \) be an integer \( \geq 0 \). Then

\[
h^{(n)}(1) = \frac{N(N-1)\cdots(N-n)}{n+1} = n! \binom{N}{n+1}.
\]

**Proof.** One has

\[
h^{(n)}(r) = \left(\frac{1 - r^N}{1 - r}\right)^{(n)} = \left(\sum_{\ell=0}^{N-1} r^\ell\right)^{(n)} = \sum_{\ell=0}^{N-1} \left(\sum_{\ell=0}^{N-1} \ell(\ell - 1)\cdots(\ell-n) r^{\ell-n}\right)^{(n)}
\]

with the understanding that a sum with upper index is smaller than the lower index is empty and therefore set to zero. If \( r = 1 \), then

\[
h^{(n)}(1) = \sum_{\ell=n}^{N-1} \ell(\ell - 1)\cdots(\ell-n) = \frac{N-N}{1+1} = \frac{(N-1)\cdots(N-n)}{n+1}.
\]

The last step follows by complete induction in \( N \).

**Lemma 8.** Let \( q \) be a positive real number and \( N \) a positive integer. Then for \( m = 1, 2, \ldots \)

\[
f_q(r) = N^{-q} + \sum_{n=1}^{m} \frac{f_q^{(n)}(1)}{n!} (r - 1)^n + R_m(r), \quad 0 < r < 1,
\]

where the coefficients \( f_q^{(n)}(1) \) can be written as

\[
f_q^{(n)}(1) = \sum_{k_1} (-1)^k q_k N^{-q-k} B_{n,k}(1! \binom{N}{2}, 2! \binom{N}{3}, 3! \binom{N}{4}, \ldots)
\]

and the remainder can be estimated by

\[
|R_m(r)| \leq \frac{(1-r)^m}{(m+1)!} \sum_{k=1}^{m+1} (q_k f_{q+k}(r) B_{m+1,k}(1! \binom{N}{2}, 2! \binom{N}{3}, 3! \binom{N}{4}, \ldots)).
\]
Proof. Let $m \geq 1$. By Taylor’s theorem
\[ f_q(r) = \left( \frac{1 - r^N}{1 - r} \right)^{-q} = \sum_{n=0}^{m} \frac{f_q^{(n)}(1)}{n!} (r - 1)^n + R_m(r), \quad R_m(r) = \frac{f_q^{(m+1)}(\rho)}{(m+1)!} (r - 1)^{m+1} \]
for some $\rho$ with $r < \rho < 1$. By Faà di Bruno’s differentiation formula
\[ f_q^{(n)}(r) = \left\{ \left( \frac{1 - r^N}{1 - r} \right)^{-q} \right\}^{(n)} = \sum_{k=1}^{n} (-1)^k q^k (h(r))^k \mathcal{B}_{n,k}(h'(r), h''(r), \ldots). \]

The remainder $R_m(r)$ is estimated next. Observe that the function $h(1) = N$ and Lemma 8.

The specific Bell polynomials appearing in Lemma 8 can be represented as follows.

The representation of $f_q^{(n)}(1)$ follows from the fact $h(1) = N$ and Lemma 8.

The remainder $R_m(r)$ is estimated next. Observe that the function $h^{(\nu)}(r)$ is positive and strictly monotonically increasing on $(0, 1)$ for each $\nu \geq 0$. Thus, by the definition of Bell polynomials (15) one gets
\[ \left| f_q^{(m+1)}(\rho) \right| \leq \sum_{k=1}^{m+1} (q)^k (h(\rho))^{-q-k} |\mathcal{B}_{m+1,k}(h'(r), h''(r), \ldots)| \]
\[ \leq \sum_{k=1}^{m+1} (q)^k (h(\rho))^{-q-k} \mathcal{B}_{m+1,k}(h'(1), h''(1), \ldots). \]

The estimate follows. \( \square \)

Let $\binom{a_1+\cdots+a_k}{a_1,\ldots,a_k}$ denote the multinomial coefficient.

The specific Bell polynomials appearing in Lemma 8 can be represented as follows.

Lemma 9. Let $n$, $k$ and $N$ be positive integers. Then
\[ \mathcal{B}_{n,k}(1! \binom{N}{2}, 2! \binom{N}{3}, 3! \binom{N}{4}, \ldots) = \frac{n!}{k!} \sum_{n_1+\cdots+n_k=n}^{N} \sum_{n_1=1}^{N-1} \sum_{n_2=1}^{N-1} \binom{N}{n_1+1} \cdots \binom{N}{n_k+1} \]
\[ = \frac{n!}{k!(n+k)!} \sum_{\ell=k}^{k+n} (-1)^{n-k} H_\ell(n,k) N^\ell, \]

where the coefficients
\[ H_\ell(n,k) = \sum_{n_1=1}^{n} \cdots \sum_{n_k=1}^{n} \binom{n+k}{n_1+1, n_2+1, \ldots, n_k+1} \sum_{\ell_1=1}^{n_1+1} \cdots \sum_{\ell_k=1}^{n_k+1} |s(n_1+1, \ell_1)| \cdots |s(n_k+1, \ell_k)|. \]

vanish for $\ell = 0, 1, \ldots, k-1$ and do not depend on $N$.

Proof. By the generating function relation for partial Bell polynomials (16)
\[ \sum_{n=k}^{\infty} \mathcal{B}_{n,k}(1! \binom{N}{2}, 2! \binom{N}{3}, 3! \binom{N}{4}, \ldots) \frac{t^n}{n!} = \frac{1}{k!} \left( \sum_{m=1}^{\infty} \left( \frac{N}{m+1} \right)^t \right)^k. \]
The right-hand side above evaluates as

\[
\frac{1}{k!} \left[ \frac{(1 + t)^N - 1 - Nt}{t} \right]^k = \frac{1}{k!} \left[ \sum_{\nu=2}^{N} \binom{N}{\nu} t^{\nu-1} \right]^k = \frac{1}{k!} \left[ \sum_{\nu=1}^{N-1} \binom{N}{\nu + 1} t^\nu \right]^k = \frac{1}{k!} \left[ \sum_{\nu_1=1}^{N-1} \cdots \sum_{\nu_k=1}^{N-1} \binom{N}{\nu_1 + 1} \cdots \binom{N}{\nu_k + 1} t^{\nu_1 + \cdots + \nu_k} \right] = \frac{k^{(N-1)}}{\nu!} \left\{ \sum_{\nu=0}^{N-1} \sum_{\nu_1=0}^{N-1} \cdots \sum_{\nu_k=0}^{N-1} \binom{N}{\nu_1 + 1} \cdots \binom{N}{\nu_k + 1} \right\} t^\nu.
\]

This shows that the infinite series at the left-hand side above is, in fact, a finite series and comparison of coefficients yields the first identity of our result

\[
B_{n,k}(1! \binom{N}{2}, 2! \binom{N}{3}, 3! \binom{N}{4}, \ldots) = \frac{n!}{k!} \sum_{n_1 + \cdots + n_k = n} \frac{[N]_{n_1+1}}{(n_1 + 1)!} \cdots \frac{[N]_{n_k+1}}{(n_k + 1)!}.
\]

Let \([x]_n\) denote the falling factorial \([x]_n = x(x-1) \cdots (x-n+1)\). Taking into account that \((N)_{\nu+1} = [N]_{\nu+1}/(\nu + 1)!\) vanishes if \(\nu + 1 > N\) for the positive integer \(N\), we may write

\[
B_{n,k}(1! \binom{N}{2}, 2! \binom{N}{3}, 3! \binom{N}{4}, \ldots) = \frac{n!}{k!} \sum_{n_1 + \cdots + n_k = n} \frac{[N]_{n_1+1}}{(n_1 + 1)!} \cdots \frac{[N]_{n_k+1}}{(n_k + 1)!}.
\]

By the definition of Stirling numbers of the first kind and using that \(n_1 + \cdots + n_k = n\) one has

\[
\frac{[N]_{n_1+1}}{(n_1 + 1)!} \cdots \frac{[N]_{n_k+1}}{(n_k + 1)!} = \prod_{\nu=1}^{k} \left[ \frac{1}{(\nu + 1)!} \sum_{\ell=0}^{n} s(n_{\nu} + 1, \ell) N^\ell \right] = \sum_{\ell_1=0}^{n_1+1} \cdots \sum_{\ell_k=0}^{n_k+1} \frac{s(n_1 + 1, \ell_1) \cdots s(n_k + 1, \ell_k)}{(n_1 + 1)! \cdots (n_k + 1)!} N^\ell_1 + \cdots + \ell_k = \sum_{\ell=0}^{k+n} \left\{ \sum_{\ell_1=0}^{n_1+1} \cdots \sum_{\ell_k=0}^{n_k+1} \frac{s(n_1 + 1, \ell_1) \cdots s(n_k + 1, \ell_k)}{(n_1 + 1)! \cdots (n_k + 1)!} \right\} N^\ell.
\]

When expanding the left-hand side above in powers of \(N\) we infer that the lowest power appearing at the right-hand side above is \(N^k\). That is

\[
\sum_{\ell_1=0}^{n_1+1} \cdots \sum_{\ell_k=0}^{n_k+1} \frac{s(n_1 + 1, \ell_1) \cdots s(n_k + 1, \ell_k)}{(n_1 + 1)! \cdots (n_k + 1)!} = 0 \quad \text{for } \ell = 0, 1, \ldots, k - 1.
\]

Putting everything together we get the identity

\[
B_{n,k}(1! \binom{N}{2}, 2! \binom{N}{3}, 3! \binom{N}{4}, \ldots) = \frac{n!}{k!} \sum_{\ell=k}^{k+n} \mu_\ell(n, k; N) N^\ell
\]

and the representation

\[
\mu_\ell(n, k) = \sum_{n_1+1}^{n} \cdots \sum_{\ell_k=0}^{n_k+1} \sum_{\ell_1=0}^{n_1+1} \cdots \sum_{\ell_k=0}^{n_k+1} \frac{s(n_1 + 1, \ell_1) \cdots s(n_k + 1, \ell_k)}{(n_1 + 1)! \cdots (n_k + 1)!}.
\]
It is well-known that \((-1)^{n-k} s(n, k) > 0\) for \(n \geq k\). Thus

\[
\prod_{\nu=1}^{k} s(n\nu + 1, \ell\nu) = \prod_{\nu=1}^{k} (-1)^{n\nu+1-\ell\nu} |s(n\nu + 1, \ell\nu)| = (-1)^{n_1+\cdots+n_k+k-\ell_1-\cdots-\ell_k} \prod_{\nu=1}^{k} s(n\nu + 1, \ell\nu)
\]

and from \(n_1 + \cdots + n_k = n\) and \(\ell_1 + \cdots + \ell_k = \ell\) it follows that

\[
B_{n,k}(1! \left\(\begin{array}{c} N \\ 2 \end{array}\right\}, 2! \left\(\begin{array}{c} N \\ 3 \end{array}\right\}, 3! \left\(\begin{array}{c} N \\ 4 \end{array}\right\}, \ldots) = \frac{n!}{k!} \sum_{\ell=k}^{k+n} (-1)^{n-\ell+k} |\mu_\ell(n, k)| N^\ell.
\]

By definition \((-1)^{n-k} s(n, k)\) counts the permutations of \(1, 2, \ldots, n\) with exactly \(k\) cycles. With this combinatorial interpretation in mind we can write

\[
B_{n,k}(1! \left\(\begin{array}{c} N \\ 2 \end{array}\right\}, 2! \left\(\begin{array}{c} N \\ 3 \end{array}\right\}, 3! \left\(\begin{array}{c} N \\ 4 \end{array}\right\}, \ldots) = \frac{n!}{k!(n+k)!} \sum_{\ell=k}^{k+n} (-1)^{n-\ell+k} H_\ell(n, k) N^\ell,
\]

where the coefficients \(H_\ell(n, k)\) (which do not depend on \(N\)) are given in the lemma.

Special values of the numbers \(H_\ell(n, k)\) are as follows.

**Lemma 10.** Let \(n \geq 1\) and \(k \geq 1\) be integers. Then

\[
H_{n+k}(n, n) = 2^{-n}(2n)! \left(\begin{array}{c} n \\ k \end{array}\right),
\]

\[
H_{n+k}(n, k) = \sum_{n_1=2}^{n+1} \cdots \sum_{n_k=2}^{n+1} \left(\begin{array}{c} n+k \\ n_1, n_2, \ldots, n_k \end{array}\right) = \frac{k!(n+k)!}{n!} B_{n,k}(\frac{1}{2}, \frac{1}{3}, \ldots).
\]

**Proof.** By Lemma 9

\[
H_{n+k}(n, n) = \sum_{n_1=1}^{n} \cdots \sum_{n_k=1}^{n} \left(\begin{array}{c} 2n \\ n_1+1, n_2+1, \ldots, n_k+1 \end{array}\right) \sum_{\ell_1=1}^{n_1+1} \cdots \sum_{\ell_k=1}^{n_k+1} |s(n_1+1, \ell_1)| \cdots |s(n_k+1, \ell_k)|.
\]

It follows that \(n_1 = \cdots = n_k = 1\). Hence, the multinomial coefficient evaluates as \(2^{-n}(2n)!\) and

\[
H_{n+k}(n, n) = 2^{-n}(2n)! \prod_{\ell_1=0}^{1} \cdots \prod_{\ell_n=0}^{1} |s(2, \ell_1+1)| \cdots |s(2, \ell_n+1)|.
\]

Since \(s(2, 1) = -1\) and \(s(2, 2) = 1\), the sum counts the number of \(n\)-tuples \((\ell_1, \ldots, \ell_n)\) with precisely \(k\) of the entries being \(1\) which is equal to \(\binom{n}{k}\). The result follows.

Similarly

\[
H_{n+k}(n, k) = \sum_{n_1=1}^{n} \cdots \sum_{n_k=1}^{n} \left(\begin{array}{c} n+k \\ n_1+1, n_2+1, \ldots, n_k+1 \end{array}\right) \sum_{\ell_1=1}^{n_1+1} \cdots \sum_{\ell_k=1}^{n_k+1} |s(n_1+1, \ell_1)| \cdots |s(n_k+1, \ell_k)|.
\]

Here, the inner sum reduces to a single term with \(\ell_1 = n_1+1, \ldots, \ell_k = n_k+1\) which evaluates to \(1\). Thus

\[
H_{n+k}(n, k) = \sum_{n_1=1}^{n} \cdots \sum_{n_k=1}^{n} \left(\begin{array}{c} n+k \\ n_1+1, n_2+1, \ldots, n_k+1 \end{array}\right) = \sum_{n_1=2}^{n+1} \cdots \sum_{n_k=2}^{n+1} \left(\begin{array}{c} n+k \\ n_1, n_2, \ldots, n_k \end{array}\right).
\]

On the other hand, by Lemma 9

\[
\frac{n!}{k!(n+k)!} H_{n+k}(n, k)
\]

is the coefficient of the highest power of \(N\) in

\[
B_{n,k}(1! \left\(\begin{array}{c} N \\ 2 \end{array}\right\}, 2! \left\(\begin{array}{c} N \\ 3 \end{array}\right\}, 3! \left\(\begin{array}{c} N \\ 4 \end{array}\right\}, \ldots).
\]
From \[
\nu^! \binom{N}{\nu + 1} = \frac{1}{\nu + 1} N^{\nu + 1} \left(1 + O(N^{-1})\right) \quad \text{as } N \to \infty
\]
and the identities \([17]\) we obtain
\[
B_{n,k}(1! \binom{N}{2}, 2! \binom{N}{3}, 3! \binom{N}{4}, \ldots) = N^{n+k} B_{n,k}(1 + O(N^{-1}), 1 + O(N^{-1}), \ldots) \quad \text{as } N \to \infty.
\]
It follows that
\[
\frac{n!}{k!(n + k)!} H_{n+k}(n,k) = B_{n,k}(\frac{1}{2}, \frac{1}{3}, \ldots), \quad n \geq k \geq 1.
\]
Lemma \([8]\) allows us to recast the representation of \(f_q^{(n)}(1)\) given in Lemma \([3]\).

**Corollary 11.** Let \(q, n\) and \(N\) be positive integer. Then
\[
\frac{f_q^{(n)}(1)}{n!} = \sum_{\ell=0}^{n} (-1)^{\ell} G(n, \ell, q) N^{\ell - q}, \quad G(n, \ell, q) = \sum_{k=1}^{n} (-1)^{n-k} q_k H_{\ell+k}(n,k),
\]
where the numbers \(H_{\ell+k}(n,k)\) are given in Lemma \([9]\).

**Proof.** From Lemmas \([8]\) and \([9]\)
\[
\frac{f_q^{(n)}(1)}{n!} = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{k} q_k N^{-q-k} B_{n,k}(1! \binom{N}{2}, 2! \binom{N}{3}, 3! \binom{N}{4}, \ldots)
\]
\[
= \frac{1}{n!} \sum_{k=1}^{n} (-1)^{k} q_k N^{-q-k} \frac{n!}{k!(n + k)!} \sum_{\ell=k}^{k+n} (-1)^{n-\ell+k} H_{\ell}(n,k) N^\ell
\]
\[
= \sum_{\ell=0}^{n} (-1)^{\ell} q_k N^{-q-k} \frac{1}{k!(n + k)!} \sum_{k=1}^{n} (-1)^{n-\ell} H_{\ell+k}(n,k) N^{\ell+k}
\]
\[
= \sum_{\ell=0}^{n} (-1)^{\ell} \sum_{k=1}^{n} (-1)^{n-k} q_k H_{\ell+k}(n,k) N^{\ell - q}.
\]
The result follows. \(\square\)

**Lemma 12.** Let \(q\) be a positive integer. Then one has the generating function relation
\[
\sum_{n=1}^{\infty} (-1)^{n} G(n,n,q) x^n = \left(\frac{e^x - 1}{x}\right)^{-q} - 1.
\]

**Proof.** By Corollary \([11]\) and Lemma \([11]\)
\[
(-1)^{n} G(n,n,q) = \sum_{k=1}^{n} \frac{(-1)^{k} q_k}{k!(n + k)!} H_{n+k}(n,k) = \frac{1}{n!} \sum_{k=1}^{n} (-1)^{k} q_k B_{n,k}(1/2, 1/3, \ldots).
\]
Application of the identities (cf. Faà di Bruno’s differentiation formula)
\[
g(f(x)) = \sum_{n=1}^{\infty} \sum_{k=1}^{n} b_k B_{n,k}(a_1, a_2, \ldots) x^n, \quad f(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n, \quad g(y) = \sum_{n=1}^{\infty} \frac{b_n}{n!} y^n,
\]
to the functions
\[
f(x) = \sum_{n=1}^{\infty} \frac{x^n}{(n + 1)!} = \frac{e^x - 1 - x}{x}, \quad g(y) = \sum_{n=1}^{\infty} \frac{(-1)^{n} q_n y^n}{n!} = (1 + y)^{-q} - 1
\]
yields
\[
\sum_{n=1}^{\infty} (-1)^{n} G(n,n,q) x^n = g(f(x)) = \left(\frac{e^x - 1}{x}\right)^{-q} - 1.
\] \(\square\)
With these preparations we are ready to give the following proof.

Proof of Proposition 4. The formula of $\mathcal{M}_{2m}(N; r)$ in Prop. is rearranged w.r.t. powers of $(1-r)$. That is

$$
\mathcal{M}_{2m}(N; r) + N(1-r)^{-2m} = -\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{\Gamma(2m-k-1)}{\Gamma(m-k)k!} (1-m)_k (1+r)^{k+1-2m} (1-r)^{k+1-2m} \\
+ 2 \frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \sum_{q=0}^{k} \frac{\Gamma(2m-k-1)}{\Gamma(m-k)k!} g(k, q; N, 1-m) \left( \frac{1-r^N}{1-r} \right)^{-q-1} (1+r)^{k+1-2m} (1-r)^{k-q-2m}.
$$

Note that $k+1-2m < 0$. Using Taylor expansion (cf. Lemma since $f_q(r) = (1+r)^{-q}$ if $N = 2$)

$$(1+r)^{-q} = \sum_{n=0}^{\infty} \frac{(q)_n}{n!} 2^{-q-n} (1-r)^n + \mathcal{R}_m(r), \quad |\mathcal{R}_m(r)| \leq \frac{(q)_{m+1}}{(m+1)!} (1-r)^{m+1},$$

one gets ($N$ is assumed to be fixed)

$$
\mathcal{M}_{2m}(N; r) + N(1-r)^{-2m} = -\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{\Gamma(2m-k-1)}{\Gamma(m-k)k!} (1-m)_k \\
\times \sum_{n=0}^{2m-k-1} \frac{(2m-1-k)!}{n!} 2^{k-n+1-2m} (1-r)^{k+n+1-2m} + O((1-r)) \\
+ 2 \frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \sum_{q=0}^{k} \frac{\Gamma(2m-k-1)}{\Gamma(m-k)k!} g(k, q; N, 1-m) \left( \frac{1-r^N}{1-r} \right)^{-q-1} \\
\times \sum_{n=0}^{2m+q-k} \frac{(2m-1-k)!}{n!} 2^{k-n+1-2m} (1-r)^{n+k-q-2m} + O((1-r)) \\
= -\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \sum_{n=0}^{2m-k-1} \frac{\Gamma(n+2m-k-1)}{\Gamma(m-k)k!n!} (1-m)_k 2^{k+1-2m-n} (1-r)^{n+k+1-2m} \\
+ 2 \frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \sum_{q=0}^{k} \sum_{n=0}^{2m+q-k} \frac{\Gamma(n+2m-k-1)}{\Gamma(m-k)k!n!} g(k, q; N, 1-m) \left( \frac{1-r^N}{1-r} \right)^{-q-1} \\
\times 2^{k+1-2m-n} (1-r)^{n+k-q-2m} + O((1-r)), \quad \text{as } r \to 1^-.
$$

By Lemma

$$
\mathcal{M}_{2m}(N; r) + N(1-r)^{-2m} = -\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \sum_{n=0}^{2m-k-1} \frac{\Gamma(n+2m-k-1)}{\Gamma(m-k)k!n!} (1-m)_k 2^{k+1-2m-n} (1-r)^{n+k+1-2m} \\
+ 2 \frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \sum_{q=0}^{k} \sum_{n=0}^{2m+q-k} \frac{\Gamma(n+2m-k-1)}{\Gamma(m-k)k!n!} g(k, q; N, 1-m) \left( \frac{1-r^N}{1-r} \right)^{-q-1} \\
\times 2^{k+1-2m-n} (1-r)^{n+k-q-2m} + O((1-r)), \quad \text{as } r \to 1^-.
$$

The power $(1-r)^{-2m}$ appears only in the triple sum when $q = k$ and $n = 0$. Its coefficient is

$$
c_{-2m} := 2 \frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{\Gamma(2m-k-1)}{\Gamma(m-k)k!} g(k, q; N, 1-m) N^{-1-q} 2^{k+1-2m-n} (1-r)^{n+k-q-2m}.
$$
By the formulas in Prop. 3 and the properties of the Stirling and Eulerian numbers (cf. (23))

\[ g(k; k; N, 1 - m) = s(k, k; 1 - m)b(k, 0)N^k = s(k, k)N^\sum_{j=0}^{k} \binom{k}{j} = s(k, k)k!N^k = k!N^k. \]

Thus, by [3] Eq. 4.2.3.10,

\[ c_{-2m} = \sum_{k=0}^{m-1} \frac{\Gamma(2m - k - 1)}{\Gamma(m - k)\Gamma(m)}2^k = \sum_{k=0}^{m-1} \frac{(2m - 2 - k)}{m - 1}2^k = N \]

and, therefore, the term \( N(1 - r)^{-2m} \) could be removed from both sides of the identity above. First, we reorder the terms

\[
\mathcal{M}_{2m}(N; r) + N(1 - r)^{-2m} = -\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \sum_{n=0}^{2m-1-k} \frac{\Gamma(2(2m - 1 - k) - n)}{(m - k)!2m - 1 - k - n}!(1 - m)_k 2^{n-2(2m-1-k)} (1 - r)^{-n}
\]

\[
+ 2\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \sum_{n=0}^{2m-1-k} \frac{\Gamma(2(2m - 1 - k) + q - n + 1)}{(m - k)!2m + q - k - n}!(1 - m)_k 2^{n-2(2m-1-k)} (1 - r)^{-n}
\]

\[
+ 2\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \sum_{n=0}^{2m-1-k} \frac{\Gamma(2(2m - 1 - k) + q - n + 1)}{(m - k)!2m + q - k - n}!(1 - m)_k 2^{n-2(2m-1-k)} (1 - r)^{-n}
\]

Since \( \lim_{r \to 1^-} \mathcal{M}_{2m}(N; r) = \mathcal{M}_{2m}(N) \) and \( \mathcal{M}_{2m}(N) \) exists and is finite for all positive integers \( m \) and \( N \) with \( N \geq 2 \) (and \( N(1 - r)^{-2m} \) can be removed from both sides of the identities above) it follows that the cumulative coefficient of a negative power of \( (1 - r) \) is zero and we are left with

\[
\mathcal{M}_{2m}(N) = -\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{\Gamma(2(m + k))}{(m - k)!k!(m + k)!}(1 - m)_k 2^{2(m+k)}
\]

\[
+ 2\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{\Gamma(2(2m - 1 - k) + q + 1)}{(m - k)!k!(m + k)!}g(k; q; N, 1 - m)N^{-1-q}2^{-q-1-2(2m-1-k)}
\]

\[
+ 2\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{\Gamma(2(2m - 1 - k) + q - n + 1)}{(m - k)!k!(m + k)!}g(k; q; N, 1 - m)(-1)^n f^{(n)}_{q+1}(1) \frac{1}{n!}
\]

Substituting the expressions for \( g(k; q; N, 1 - m) \) (cf. Prop. 3) and \( f^{(n)}_{q+1}(1) \) (cf. Cor. 11) we get

\[
\mathcal{M}_{2m}(N) = -\frac{N^2}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{\Gamma(2(m + k))}{(m - k)!k!(m + k)!}(1 - m)_k 2^{2(m+k)}
\]

\[
+ N \sum_{k=0}^{m-1} \sum_{q=0}^{k-q} \sum_{p=0}^{q} X(k, q, p; m)N^p + N \sum_{k=0}^{m-1} \sum_{q=0}^{2m-q-k-q} \sum_{n=0}^{q} \sum_{p=0}^{n} \sum_{\ell=0}^{p} X(k, q, n, \ell; m) N^{\ell+p},
\]

(18)
where the primary coefficients are given by

$$X(k, q, p; m) := \frac{2}{\Gamma(m)} \frac{\Gamma(2(2m - 1 - k) + q + 1)}{\Gamma(m - k)! (2m + q - k)!} \prod_{p=0}^{2q + 1 + 2(2m - 1)} \frac{s(k, p + q; 1 - m)b(p + q, p)}{2^{q + 1} + 2(2m - 1)}$$

$$X(k, q, p, n, \ell; m) := \frac{2}{\Gamma(m)} \frac{\Gamma(2(2m - 1 - k) + q + n + 1)}{\Gamma(m - k)! (2m + q - k - n)!} \prod_{p=0}^{2q + 1 + 2(2m - 1 - k) - n} \frac{s(k, p + q; 1 - m)b(p + q, p)}{2^{q + 1} + 2(2m - 1 - k) - n} G(n, \ell, q + 1)$$

which depend on the secondary coefficients

$$s(n, \ell; y) = \sum_{k=\ell}^{n} \binom{k}{\ell} s(n, k) (y + n - 1)^{k-\ell}, \quad b(p + q, p) = \sum_{j=0}^{q} \binom{p + q}{j} \binom{p + q - j}{p}$$

and $G(n, \ell, q)$ is given in Cor. 11 (cf. Lemma 9).

We reorder the terms in (18) w.r.t. powers of $N$. The first part is

$$\sum_{k=0}^{m-1} \sum_{q=0}^{k} \sum_{p=0}^{k-q} X(k, q, p; m) N^p = \sum_{k=0}^{m-1} \sum_{q=0}^{k} \sum_{p=0}^{k-q} X(k, q, p; m) N^p = \sum_{p=0}^{m-1} \sum_{k=0}^{m-1 - k} \sum_{q=0}^{k} \sum_{p=0}^{k-q} X(k, q, p; m) N^p.$$

The second part is

$$\sum_{k=0}^{m-1} \sum_{q=0}^{k} \sum_{p=0}^{k-q} X(k, q, p, n, \ell; m) N^{\ell+p} = \sum_{k=0}^{m-1} \sum_{q=0}^{k} \sum_{p=0}^{k-q} X(k, q, p, n, \ell; m) N^{\ell+p} = \sum_{p=0}^{m-1} \sum_{q=0}^{k} \sum_{p=0}^{k-q} X(k, q, p, n, \ell; m) N^{\ell+p}.$$

Putting everything together we arrive at

$$\mathcal{M}_{2m}(N) = -N^2 \sum_{k=0}^{m-1} \frac{\Gamma(2(m + k))}{\Gamma(m + k)!} (1 - m)_{m-1-k} 2^{2(m+k)} + N \sum_{p=0}^{m-1} \sum_{q=0}^{m-1 - k} X(k, q, p; m) N^p$$

$$+ N \sum_{p=0}^{m-1} \sum_{q=0}^{m-1 - k} \sum_{n=0}^{p} \sum_{\ell=0}^{p} X(k, q, p, n, \ell; m) N^{\ell+p}$$

$$+ N \sum_{p=0}^{m-1} \sum_{q=0}^{m-1 - k} \sum_{n=0}^{p} \sum_{\ell=0}^{p} X(k, q, p, n, \ell; m) N^{\ell+p}.$$
Direct computations (with the help of Mathematica) give the expected results (cf. [5])

\[
\mathcal{M}_2(N) = \frac{N^3}{12} - \frac{N}{12},
\]

\[
\mathcal{M}_4(N) = -\frac{N^2}{8} - N \left( \frac{N}{4} - \frac{3}{8} \right) + N \left( \frac{3}{8}N - \frac{281}{720} \right) - N^3 \left( \frac{N^3}{16} - \frac{N^2}{72} \right) + N \left( \frac{N^4}{720} + \frac{N^3}{16} \right)
\]

\[
= \frac{N^5}{720} + \frac{10}{720} N^3 - \frac{11}{720} N.
\]

In general, one has for \( m \geq 2 \)

\[
\mathcal{M}_{2m}(N) = \sum_{\nu=0}^{2m} \beta_\nu(m) N^{1+\nu},
\]

where for \( \nu = 0, 1, \ldots, m-1 \) and \( \nu \neq 1 \)

\[
(19a) \quad \beta_\nu(m) := \sum_{k=\nu}^{m-1} \sum_{k=\nu}^{m-1} \sum_{k=\nu}^{m-1} \sum_{k=\nu}^{m-1} \sum_{k=\nu}^{m-1} X(k, q, p, n, m - p; m),
\]

for \( \nu = m \) and \( m \neq 1 \)

\[
(19b) \quad \beta_m(m) := \sum_{p=0}^{m-1} \sum_{q=p}^{m-1} \sum_{k=q}^{m-1} \sum_{k=q}^{m-1} \sum_{k=q}^{m-1} X(k, q, p, n, m - p; m),
\]

for \( \nu = m + 1, \ldots, 2m - 1 \) (and \( m \geq 2 \))

\[
(19c) \quad \beta_\nu(m) := \sum_{p=\nu}^{m-1} \sum_{q=p}^{m-1} \sum_{k=q}^{m-1} \sum_{k=q}^{m-1} \sum_{k=q}^{m-1} X(k, q, p, n, \nu - p; m)
\]

\[
+ \sum_{p=0}^{m-1} \sum_{q=p}^{m-1} \sum_{k=q}^{m-1} \sum_{k=q}^{m-1} \sum_{k=q}^{m-1} X(k, q, p, n, \nu - p; m),
\]

for \( \nu = 2m \) (and \( m \geq 2 \))

\[
(19d) \quad \beta_{2m}(m) := \sum_{p=0}^{m-1} \sum_{k=p}^{m-1} X(k, k - p, p, 2m - p, 2m - p; m)
\]

and for \( \nu = 1 \) (and \( m \geq 2 \))

\[
(19e) \quad \beta_1(m) = -\frac{1}{\Gamma(m)} \sum_{k=0}^{m-1} \frac{\Gamma(2(m + k))}{\Gamma(m - k)k!(m + k)!} (1 - m)_{m-1-k} 2^{-2(m+k)}
\]

\[
+ \sum_{k=1}^{m-1} \sum_{q=0}^{m-1} X(k, q, 1; m) + \sum_{p=0}^{m-1} \sum_{q=p}^{m-1} \sum_{k=q}^{m-1} \sum_{k=q}^{m-1} X(k, q, p, n, 1 - p; m).
\]

The result follows. \( \square \)

Next the coefficients \( \beta_\nu(m) \) are studied in more detail.

**Proof of Proposition 4** By the definition of the numbers \( X(k, q, p, n, \ell; m) \) given in the Proof of Prop. 4 we get

\[
\beta_{2m}(m) = \sum_{p=0}^{m-1} \sum_{k=p}^{m-1} X(k, k - p, p, 2m - p, 2m - p; m)
\]

\[
= \sum_{p=0}^{m-1} \sum_{k=p}^{m-1} (-1)^p 2^{2+k-2m} \Gamma(2m - 1 - k) \Gamma(m - k) \Gamma(m - k) s(k; 1 - m)b(k, p)G(2m - p, 2m - p, k - p + 1).
\]
By the definitions of \( s(n, k; x) \) (Lemma 13) and \( b(n, \ell) \) (Lemma 14) we have

\[
\begin{align*}
  s(k, k; 1 - m) &= s(k, k) = 1, \\
  b(k, p) &= \sum_{j=0}^{k-p} \binom{k}{j} \binom{k-j}{p}
\end{align*}
\]

and the numbers \( G(n, n, q) \) are defined by the generating function relation (Lemma 12)

\[
\sum_{n=1}^{\infty} (-1)^n G(n, n, q) x^n = \left( \frac{e^x - 1}{x} \right)^{-q} - 1.
\]

The result follows after interchanging the two summation signs. □

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Appendix A. Auxiliary results

**Lemma 13.** Let \( n \) be a nonnegative integer. Then

\[
(x + y)_n = \sum_{\ell=0}^{n} s(n, \ell; y) x^\ell, \quad s(n, \ell; y) := \sum_{k=\ell}^{n} \binom{k}{\ell} s(n, k) (y + n - 1)^{k-\ell}.
\]

**Proof.** Using (10) and the Binomial theorem, one gets

\[
(x + y)_n = ((x + y + n - 1) + (1 - n))_n = \sum_{k=0}^{n} s(n, k) (x + y + n - 1)^k
\]

\[
= \sum_{k=0}^{n} \sum_{\ell=0}^{k} s(n, k) \binom{k}{\ell} x^\ell (y + n - 1)^{k-\ell}.
\]

The result follows after reordering the sum. □

Note that

\[
(20) \quad s(n, 0; y) = \sum_{k=0}^{n} s(n, k) (y + n - 1)^k = (y + n - 1 + 1 - n)_n = (y)_n.
\]

Alternatively, one can use the identities (\( \binom{n}{k} \) is the unsigned Stirling number of the first kind)

\[
(a + b)_n = \sum_{j=0}^{\infty} \binom{n}{j} (a)_{n-j} (b)_j, \quad (x)_n = \sum_{k=0}^{n} \binom{n}{k} x^k
\]

to obtain the representation

\[
s(n, \ell; y) = \sum_{k=\ell}^{n} \binom{n}{k} \binom{k}{\ell} (y)_{n-k}.
\]

We need the **polylogarithm function** defined by

\[
L_\nu(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^\nu}, \quad |z| < 1.
\]

In particular, one has for \( \nu = -n \) and \( n \) a positive integer the relations (cf. [7], [9])

\[
L_{-n}(z) = \sum_{k=1}^{n} k^n z^k = \frac{1}{(1 - z)^{n+1}} \sum_{j=0}^{n-1} \binom{n}{j} z^{n-j}, \quad |z| < 1.
\]
Lemma 14. Let \( n \) be a positive integer. Then
\[
L_{-n}(z) = \frac{n! + \sum_{\ell=1}^{n} (-1)^\ell b(n, \ell) (1 - z)^\ell}{(1 - z)^{n+1}}, \quad b(n, \ell) := \sum_{j=0}^{n-\ell} \binom{n}{j} \binom{n-j}{\ell}.
\]

Proof. By (22) and the Binomial theorem
\[
L_{-n}(z) = \frac{1}{(1 - z)^{n+1}} \sum_{j=0}^{n-1} \binom{n}{j} (1 - z)^{n-j} = \frac{1}{(1 - z)^{n+1}} \sum_{j=0}^{n-1} \sum_{\ell=0}^{n-j} \binom{n}{j} \binom{n-j}{\ell} (-1)^\ell (1 - z)^\ell.
\]
Reordering of the sum yields
\[
L_{-n}(z) = \frac{1}{(1 - z)^{n+1}} \left\{ \sum_{j=0}^{n-1} \binom{n}{j} + \sum_{\ell=1}^{n} \left[ \sum_{j=0}^{n-\ell} \binom{n}{j} \binom{n-j}{\ell} \right] (-1)^\ell (1 - z)^\ell \right\}
\]
The sum over the Eulerian numbers gives \( n! \) (cf. [1] Eq. 26.14.10). The result follows. \( \square \)

Note that (using \( \binom{n}{q} = 0 \) for \( q \geq 1 \))
\[
(23) \quad b(n, 0) = \sum_{j=0}^{n} \binom{n}{j} = \sum_{j=0}^{n-1} \binom{n}{j} = n!.
\]

Lemma 15. Let \( k \) be a non-negative integer and \( a, b \) complex numbers. Then
\[
\sum_{\nu=-\infty}^{\infty} \nu^{|\nu|} a^\nu b^{|\nu|} = -(b)_{k} + 2 \sum_{q=0}^{k} g(k, q; a, b) (1 - z)^{-q-1}, \quad |z| < 1,
\]
where \( s(n, \ell; y) \) is defined in Lemma 13 and \( b(n, \ell) \) is defined in Lemma 14.
\[
g(k, q; a, b) := \sum_{p=0}^{k-q} (-1)^p s(k, p + q; b) (p + q, p) a^{p+q}.
\]

Proof. Let \( f(z) \) denote the infinite series above. For \( k = 0 \) one gets
\[
f(z) = \sum_{\nu=-\infty}^{\infty} z^{|\nu|} = 1 + 2 \sum_{\nu=1}^{\infty} z^\nu = 1 + \frac{2z}{1-z} = \frac{1+z}{1-z}.
\]

Let \( k \geq 1 \). By Lemma 13 one has
\[
f(z) = (b)_{k} + 2 \sum_{\nu=1}^{\infty} (\nu a + b)_{\nu} z^\nu = (b)_{k} + 2 \sum_{p=0}^{k} s(k, p; b) a^p \sum_{\nu=1}^{\infty} p^p z^\nu = (b)_{k} + 2 \sum_{p=0}^{k} s(k, p; b) a^p L_{-p}(z).
\]

By Lemma 14 and 10 (and using Eqs 20 and 23) one gets
\[
f(z) = (b)_{k} + 2 s(k, 0; b) L_0(z) + 2 \sum_{p=1}^{k} s(k, p; b) a^p \left[ \sum_{\ell=1}^{p} (-1)^\ell b(p, \ell) (1 - z)^\ell - (-1)^p b(p, 0) (1 - z)^p \right] + \frac{2}{1-z} \sum_{p=1}^{k} s(k, p; b) a^p (1 - z)^{-p-1} + 2 \sum_{p=1}^{k} s(k, p; b) a^p (-1)^\ell b(p, \ell) (1 - z)^{-p-1-\ell}.
\]
Simplification of the first three terms and summing up over equal powers of \((1 - z)\) in the double sum yields

\[
f(z) = -(b)_k + 2 \sum_{p=0}^{k} s(k, p; b) p a^p (1 - z)^{-p-1} + 2 \sum_{q=0}^{k-1} \left[ \sum_{p=1+q}^{k} s(k, p; b) a^p (-1)^{-q} b(p, p - q) \right] (1 - z)^{-q-1}
\]

\[
= -(b)_k + 2 s(k, k; b) k! a^k (1 - z)^{-k-1} + 2 \sum_{q=0}^{k-1} \left[ s(k, q; b) q! a^q + \sum_{p=1+q}^{k} s(k, p; b) a^p (-1)^{-q} b(p, p - q) \right] (1 - z)^{-q-1}
\]

\[
= -(b)_k + 2 s(k, k; b) k! a^k (1 - z)^{-k-1} + 2 \sum_{q=0}^{k-1} \left[ \sum_{p=q}^{k} s(k, p; b) a^p (-1)^{-q} b(p, p - q) \right] (1 - z)^{-q-1}
\]

\[
= -(b)_k + 2 \sum_{q=0}^{k} \left[ \sum_{p=q}^{k} (-1)^p s(k, p + q; b) b(p + q, p) a^{p+q} \right] (1 - z)^{-q-1},
\]

where we used that \(\langle n \rangle = 0\) for \(n \geq 1\) and \(\langle 0 \rangle = 1\). \(\square\)

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