COMPOSITION OPERATORS ON MODEL SPACES

YURI I. LYUBARSKII AND EUGENIA MALINNIKOVA

Dedicated to Nikolai K. Nikolski on the occasion of his 70th birthday

Abstract. Let \( \varphi : \mathbb{D} \rightarrow \mathbb{D} \) be a holomorphic function, \( \vartheta : \mathbb{D} \rightarrow \mathbb{D} \) be an inner function and \( K_\vartheta(\mathbb{D}) = H^2(\mathbb{D}) \ominus \vartheta H^2(\mathbb{D}) \) be the corresponding model space. We study the composition operator \( C_\varphi \) on \( K_\vartheta \) and give a necessary and sufficient condition for \( C_\varphi : K_\vartheta \rightarrow H^2 \) to be compact. The condition involves an interplay between \( \vartheta \) and the Nevanlinna counting function of \( \varphi \). For a one-component \( \vartheta \) a characterization of compact composition operators \( C_\varphi \) in terms of the Aleksandrov-Clark measures of \( \varphi \) and the spectrum of \( \vartheta \) is also given.

1. Introduction

Let \( \mathbb{D} \) be the unit disk in the complex plane. Given a holomorphic function \( \varphi : \mathbb{D} \rightarrow \mathbb{D} \), denote by

\[ C_\varphi : f \mapsto f \circ \varphi \]

the composition operator defined on holomorphic functions in \( \mathbb{D} \). This operator is bounded on the Hardy space \( H^2(\mathbb{D}) \) (see e.g. [15]). One of the intensively studied questions is when \( C_\varphi \) is a compact operator on various spaces of analytic functions. We refer the reader to the monographs [9, 16] for the history and basic results on composition operators.

Loosely speaking \( C_\varphi \) is compact on \( H^2(\mathbb{D}) \) if \( \varphi(z) \) does not approach the unit circle \( \mathbb{T} \) too fast as \( z \to \mathbb{T} \). J. Shapiro [15] quantified this idea by using the Nevanlinna counting function

\[ N_\varphi(w) = \sum_{\varphi(z) = w} -\log |z|. \]

He proved in particular that \( C_\varphi \) is compact on \( H^2(\mathbb{D}) \) if and only if

\[ \lim_{|w| \to 1^-} N_\varphi(w)/(-\log |w|) = 0. \]
The basic tool in his argument is the Stanton formula
\[
\|C_{\varphi}f\|^2 = 2 \int_{\mathbb{D}} |f'(z)|^2 N_{\varphi}(z) dA(z) + |f(\varphi(0))|^2,
\]
where \(A\) is the normalized area measure. It is obtained from the identity
\[
\|f\|^2 = 2 \int_{\mathbb{D}} |f'(z)|^2 \log \frac{1}{|z|^2} dA(z) + |f(0)|^2, \quad f \in H^2(\mathbb{D}),
\]
by substituting \(f \circ \varphi\) in place of \(f\).

Another way to describe the compactness property of \(C_{\varphi}\) is related to the Aleksandrov-Clark measures of \(\varphi\). These are the positive measures \(\mu_\alpha\) on \(\mathbb{T}\) defined by the relation
\[
\Re \frac{\alpha + \varphi(z)}{\alpha - \varphi(z)} = \int_{\mathbb{T}} P_z d\mu_\alpha,
\]
where \(P_z\) is the Poisson kernel, \(\alpha \in \mathbb{T}\). We refer the reader to the surveys \[13, 18\] for more details. In \[14\] D. Sarason showed how \(C_{\varphi}\) can be treated as an integral operator on the spaces \(L^1(\mathbb{T})\) and \(M(\mathbb{T})\) and proved that \(C_{\varphi}\) is compact on these spaces if and only if each \(\mu_\alpha\) is absolutely continuous. Due to \[17\], it is further equivalent to \(C_{\varphi}\) being compact on \(H^2(\mathbb{D})\) as well as on other Hardy spaces \(H^p(\mathbb{D})\), see also \[5\].

In this article we study the compactness of the operator \(C_{\varphi} : K_\vartheta \to H^2(\mathbb{D})\), where \(\vartheta\) is an inner function in \(\mathbb{D}\) and \(K_\vartheta = H^2(\mathbb{D}) \ominus \vartheta H^2(\mathbb{D})\) is the corresponding model space. Consider the canonical factorization of \(\vartheta\)
\[
\vartheta(z) = B_\Lambda(z) \exp \left( \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} d\omega(\xi) \right),
\]
where \(\Lambda\) is the zero set of \(\vartheta\), \(B_\Lambda\) is the corresponding Blachke product, and \(\omega\) is a singular measure on \(\mathbb{T}\). Functions in \(K_\vartheta\) admit analytic continuation through \(\mathbb{T} \setminus \Sigma(\vartheta)\), where
\[
\Sigma(\vartheta) = (\mathbb{T} \cap \text{Clos}(\Lambda)) \cup \text{supp}(\omega)
\]
is the spectrum of \(\vartheta\) (see \[12\], Lecture 3). Therefore the compactness property of \(C_{\varphi}\) does not suffer as the values of \(\varphi\) approach points in \(\mathbb{T} \setminus \Sigma(\vartheta)\). We quantify this idea below and give a condition that is necessary and sufficient for the compactness of \(C_{\varphi} : K_\vartheta \to H^2(\mathbb{D})\).

**Acknowledgments.** This work was started when the authors visited the Mathematics Department of the University of California, Berkeley. It is our pleasure to thank the Department for the hospitality and Donald Sarason for useful discussions.

The authors will also thank Anton Baranov for his comments on the preliminary version of this note and for showing us the inequality in \[8\] that is crucial for the proof of Theorem 1.
2. Nevanlinna Counting Function

In this section we give a counterpart of the condition (1) for the operator $C_{\varphi} : K_\varphi \to H^2(\mathbb{D})$. The proof follows the ideas of [15]. The main new ingredients are estimates for the reproducing kernels and its derivatives given in Lemma 1 below.

Let $\kappa_w$ be the reproducing kernel for $K_\varphi$,

$$\kappa_w(\zeta) = \frac{1 - \overline{\varphi(w)} \varphi(\zeta)}{1 - \overline{w} \zeta}, \quad \|\kappa_w\|^2 = \frac{1 - |\varphi(w)|^2}{1 - |w|^2},$$

and let $\tilde{\kappa}_w$ be its normalized version

$$\tilde{\kappa}_w(\zeta) = \left( \frac{1 - |w|^2}{1 - |\varphi(w)|^2} \right)^{1/2} \frac{1 - \overline{\varphi(w)} \varphi(\zeta)}{1 - \overline{w} \zeta}.$$

By

$$k_w(\zeta) = \frac{1}{1 - \overline{w} \zeta}, \quad \tilde{k}_w(\zeta) = \frac{(1 - |w|^2)^{1/2}}{1 - \overline{w} \zeta}$$

we denote the reproducing kernel for $H^2$ and its normalized version.

**Lemma 1.** Let $\{w_n\} \subset \mathbb{D}$, $|w_n| \to 1$ be such that

$$|\varphi(w_n)| < a,$$

for some $a \in (0, 1)$. Then

(i) $\tilde{\kappa}_{w_n} \overset{w^*}{\to} 0$ as $n \to \infty$;

(ii) there exist $\epsilon > 0$, $c > 0$ and $n_0$ such that

$$|\kappa'_{w_n}(\zeta)| > \frac{c}{(1 - |w_n|^2)^2}, \quad \zeta \in D_\epsilon(w_n)$$

holds for any $n > n_0$, where $D_\epsilon(w) = \{\zeta; |\zeta - w| < \epsilon |1 - \overline{w} \zeta|\}$ is a hyperbolic disk with center at $w$.

**Proof.** (i) It suffices to show that

$$\frac{1 - |w_n|^2}{1 - \overline{w} \zeta} (1 - |\varphi(w_n)|^2) \overset{w^*}{\to} 0$$

in $L^2(\mathbb{T})$ as $|w_n| \to 1$.

This in turn follows from the known fact that the normalized reproducing kernels $\tilde{k}_{w_n}$ for the Hardy space $H^2(\mathbb{D})$ tend weakly to 0 as $|w_n| \to 1$, see e.g. [15].

(ii) We start with the following well-known estimate

$$|\varphi'(\zeta)| \leq \frac{1 - |\varphi(\zeta)|^2}{1 - |\zeta|^2}, \quad \zeta \in \mathbb{D}.$$

Together with (5) it readily yields

$$|\varphi(\zeta)| < b, \quad \zeta \in \cup_n D_\epsilon(w_n),$$

for some $b < 1$ and $\epsilon > 0$. 
We claim now that for sufficiently large \( n_0 \)

\[
|\kappa'_\zeta(\zeta)| > \frac{\text{const}}{(1 - |\zeta|^2)^2}, \quad \zeta \in \bigcup_{n > n_0} D_\epsilon(w_n).
\]

Indeed,

\[
\kappa'_\zeta(\zeta) = -\frac{\vartheta'(\zeta)\vartheta(\zeta)}{1 - |\zeta|^2} + \frac{1 - |\vartheta(\zeta)|^2}{(1 - |\zeta|^2)^2} = A_1 + A_2.
\]

It follows from (8) that

|\( A_1 | < q|A_2| \) for some \( q \in (0, 1) \),

when \( \zeta \in \bigcup_{n > n_0} D_\epsilon(w_n) \). The relation (7) yields

\[
|A_1| \leq |\vartheta(\zeta)| \frac{1 - |\vartheta(\zeta)|^2}{(1 - |\zeta|^2)^2} < \frac{b}{|\zeta|} |A_2|, \quad \zeta \in \bigcup_{n} D_\epsilon(w_n).
\]

Since \( b < 1 \) and \( \inf\{|\zeta| : \zeta \in \bigcup_{n > m} D_\epsilon(w_n)\} \to 1 \) as \( m \to \infty \), the required estimate follows.

The inequality (9) proves (6) for the special case \( \zeta = w_n \). In order to complete the proof consider the function

\[
g(w, \zeta) = \kappa'_{\omega_n}(\zeta) = -\frac{\vartheta'(\zeta)\vartheta(w)}{1 - \zeta w} + w \frac{1 - \vartheta(\zeta)\vartheta(w)}{(1 - \zeta w)^2}.
\]

We have \( |g(w_n, \zeta)| = |\kappa'_{\omega_n}(\zeta)| \). On the other hand

\[
|g(w_n, \zeta) - g(\zeta, \zeta)| < |g'(\tilde{w}, \zeta)||\zeta - w_n|,
\]

for some point \( \tilde{w} \in [\zeta, w_n] \), where the derivative is taken with respect to the first variable. A straightforward estimate shows

\[
|g'(\tilde{w}, \zeta)| < \frac{\text{const}}{(1 - |w_n|^2)^2}, \quad \tilde{w}, \zeta \in D_\epsilon(w_n),
\]

the constant being independent of \( n \). Now, given any \( \eta > 0 \) we can choose \( \epsilon' < \epsilon \) such that the right-hand side in (11) does not exceed \( \eta(1 - |\zeta|^2)^{-2} \) when \( \zeta \in D_{\epsilon'}(w_n) \). Taking \( \eta \) sufficiently small we obtain (6). \( \square \)

In what follows we assume for simplicity that \( \varphi(0) = 0 \).

**Theorem 1.** The following statements are equivalent

(C) \( C_\varphi : K_\theta \to H^2 \) is a compact operator.

(N) The Nevanlinna counting function of \( \varphi \) satisfies

\[
N_\varphi(w) \frac{1 - |\vartheta(w)|^2}{1 - |w|^2} \to 0 \text{ as } |w| \searrow 1.
\]

**Proof (N) ⇒ (C).** Since \( N_\varphi(w)(1 - |w|^2)^{-1} \) and \( 1 - |\vartheta(w)|^2 \) are bounded, the condition (N) means that for any \( a < 1 \)

\[
\lim_{|\vartheta(w)| < a, |w| \to 1} N_\varphi(w)(1 - |w|^2)^{-1} = 0.
\]
In particular, for any $p > 0$

$$N_ϕ(w) \frac{(1 - |ϕ(w)|)^p}{1 - |w|} \to 0 \text{ as } |w| \nearrow 1.$$  

We use the following inequality, see [8, page 187] and [3],

$$(12) \quad \|f\|_2^2 \geq C_p \int_D |f'(z)|^2 \frac{1 - |z|}{(1 - |ϕ(z)|)^p} dA(z) + |f(0)|^2, \quad f \in K_ϕ,$$

which is valid for some $p \in (0, 1)$.

In our setting this formula replaces (3). We follow the argument of J. Shapiro; a similar argument for compactness of the composition operator in some weighted spaces of analytic functions can be found in [9, Ch. 3.2].

Let $K_ϕ^{(n)} = \{f \in K_ϕ; f \text{ has zero of order } n \text{ at the origin}\}$, and let $Π^{(n)} : K_ϕ \to K_ϕ^{(n)}$ be the corresponding orthogonal projection. We will prove that

$$\|C_ϕ Π^{(n)}\|_{K_ϕ \to H^2} \to 0, \quad n \to \infty.$$

Thus $C_ϕ$ is compact as it can be approximated by the finite-rank operators $C_ϕ(I - Π^{(n)})$.

Indeed, given $f \in K_ϕ$, $\|f\| = 1$, denote $g_n = Π^{(n)} f$. We have $\|g_n\| \leq 1$ and, for each $R < 1$, $κ > 0$ we can choose $n(κ, R)$ independent of $f$ and such that

$$|g_n(w)| < κ, \quad |g_n'(w)| < κ, \quad \text{for all } n > n(κ, R), \quad \text{and } |w| < R.$$

It follows from (12) that

$$\int_D |g_n'(z)|^2 \frac{1 - |z|}{(1 - |ϕ(z)|)^p} dA(z) < C,$$

with $C$ independent of $f, n$. Next, by (2) we have

$$\|C_ϕ Π^{(n)} f\|^2 = \int_D |g_n'(z)|^2 N_ϕ(z) dA(z) \leq \int_{|z| < R} + \int_{R < |z| < 1} \leq$$

$$\leq \max_{|z| < R} \{|g_n'(z)|^2\} \int_{|z| < R} N_ϕ(z) dA(z) + \max_{R < |z| < 1} \left\{N_ϕ(z) \frac{(1 - |ϕ(z)|)^p}{1 - |z|}\right\} \int_{R < |z| < 1} \frac{|g_n'(z)|^2}{1 - |ϕ(z)|^p} dA(z).$$

Choosing first $R$ such that the second summand is small, and then $n$ large enough to provide smallness of the first summand we can make the whole expression arbitrary small for all $f \in K_ϕ$, $\|f\| = 1$. □

**Proof (C) ⇒ (N).** Assume that $C_ϕ$ is compact but (11) does not hold. Then there exists a sequence $\{w_n\} \subset \mathbb{D}$, $|w_n| \to 1$, satisfying

$$N_ϕ(w_n) \frac{1 - |ϕ(w_n)|^2}{1 - |w_n|^2} > κ > 0.$$
By the Littlewood subordination principle, which implies that $N_\varphi(w) \leq \log \frac{1}{|w|}$, there exists $a < 1$ such that (5) holds. Applying Lemma 1 (i) and the compactness of $C_\varphi$, we get $\|C_\varphi \tilde{k}_{w_n}\|^2 \to 0$ as $n \to \infty$. On the other hand, (5), Lemma 1 (ii) and the subharmonicity inequality for $N_\varphi$ (see [15]) imply

\begin{align}
\|C_\varphi \tilde{k}_{w_n}\|^2 &\geq \int_D |\tilde{k}'_{w_n}(\zeta)|^2 N_\varphi(\zeta) dA(\zeta) \\
&\geq c_1 \int_D |\kappa'_{w_n}(\zeta)|^2 (1 - |w_n|^2) N_\varphi(\zeta) dA(\zeta) \\
&\geq \frac{c_2}{(1 - |w_n|^2)^3} \int_{D_\epsilon(w_n)} N_\varphi(\zeta) dA(\zeta) \\
&\geq \frac{c_3 N_\varphi(w_n)}{1 - |w_n|^2}.
\end{align}

We combine the last estimate with (13) to get a contradiction. □

3. Aleksandrov-Clark measures

For $\alpha \in \mathbb{T}$ let as before $\mu_\alpha$ be the Aleksandrov-Clark measure of $\varphi$ corresponding to $\alpha$ and let

$$d\mu_\alpha = h_\alpha dm + d\sigma_\alpha$$

be its decomposition into absolutely continuous and singular parts, where $m$ is the normalized Lebesgue measure on $\mathbb{T}$. Then

$$h_\alpha(\zeta) = \frac{1 - |\varphi(\zeta)|^2}{|\alpha - \varphi(\zeta)|^2}$$

for almost every $\zeta$ on $\mathbb{T}$. As above, we assume for simplicity that $\varphi(0) = 0$, then $\|\mu_\alpha\| = 1$.

We give a condition in terms of the Aleksandrov-Clark measures that is sufficient for the compactness, it is also necessary if $\vartheta$ is a one-component inner function, i.e. the set $\{z \in \mathbb{D} : |\vartheta(z)| < r\}$ is connected for some $r \in (0, 1)$. The one-component inner functions were introduced by W. S. Cohn in [7], see also [2] for a number of equivalent characterizations of one-component inner functions.

**Theorem 2.** Let $\vartheta$ be a one-component inner function. The following statements are equivalent

(C) $C_\varphi : K_\vartheta \to H^2$ is a compact operator.

(S) $\sigma_\alpha = 0$ for all $\alpha \in \Sigma(\vartheta)$.

Moreover, the implication (S) $\Rightarrow$ (C) holds for any inner function $\vartheta$.

The proof mainly follows the pattern as described in [18] section 7, see [14] for the original approach and also [5]. We need the following description of the spectrum of a one-component inner function.

**Lemma A.** Let $\vartheta$ be a one-component inner function and $\alpha \in \mathbb{T}$. The following statements are equivalent

(a) $\alpha \in \Sigma(\vartheta)$; (b) $\liminf_{w \to \alpha} |\vartheta(w)| < 1$; (c) $\liminf_{r \to 1-} |\vartheta(r\alpha)| < 1$. 

The implications $(c) \Rightarrow (a) \Rightarrow (b) \Rightarrow (a)$ are straightforward and hold for any inner function, see [12], Lecture 3; $(a) \Rightarrow (c)$ is true when $\vartheta$ is one-component, it follows from [19], Section 5, see also Theorem 1.11 in [2].

Proof $(C) \Rightarrow (S)$. Fix $\alpha \in \Sigma(\vartheta)$ and chose a sequence $r_n \to 1$ so that $|\vartheta(\alpha r_n)| < a < 1$. By Lemma 1, we have
\[
\|C_{\varphi} k_{\vartheta} \|^2 \geq \int_{|\varphi - \alpha| < 1} \frac{1 - r_n^2}{|\varphi - \alpha|^2} |d\xi| \to 0, \text{ as } n \to \infty.
\]
Since $|\vartheta(\alpha r_n)| < a < 1$, this yields
\[
\|C_{\varphi} k_{\vartheta} \|^2 = \int_{|\varphi - \alpha| < 1} \frac{1 - r_n^2}{|\varphi - \alpha|^2} |d\xi| \to 0, \text{ as } n \to \infty,
\]
where $k_{\vartheta}$ is the normalized reproducing kernel for $H^2$, see (4).

The rest of the proof follows literally [5], see also [18], Lemma 7.6. We give it here for the sake of completeness. We have
\[
\|C_{\varphi} k_{\vartheta} \|^2 = \int_{|\varphi - \alpha| < 1} \frac{1 - r_n^2}{|\varphi - \alpha|^2} |d\xi| \to 0, \text{ as } n \to \infty,
\]
where $k_{\vartheta}$ is the normalized reproducing kernel for $H^2$, see (4).

We remark that the one-component condition was employed only in the description of the spectrum, so the following statement holds for any inner function: If $C_{\varphi} : K_{\vartheta} \to H^2$ is a compact operator then $\sigma_{\alpha} = 0$ for all $\alpha \in \mathbb{T}$ such that $\lim \inf_{r \to 1} |\vartheta(\alpha)| < 1$.

Proof $(S) \Rightarrow (N)$. We will prove this implication and refer to Theorem [11]. Suppose that $(N)$ is false, then
\[
N_{\varphi}(w_n) \frac{1 - |\varphi(w_n)|^2}{1 - |w_n|^2} > c > 0, \text{ for some } \{w_n\} \subset \mathbb{D}, w_n \to \alpha \in \mathbb{T}.
\]
Clearly $\alpha \in \Sigma(\vartheta)$ and (5) holds for some $a < 1$. Further,
\[
(1 - |w_n|)^{-1} N_{\varphi}(w_n) > c_1 > 0.
\]
We obtain a contradiction in the same way as in [5] see also [18], Theorem 7.5. We have, by a simple version of (14)
\[
\|C_{\varphi} k_{\vartheta} \|^2 \geq C N_{\varphi}(w_n) \frac{1 - |w_n|^2}{1 - |w_n|^2} > c_2 > 0.
\]
On the other hand by the Fatou lemma,
\[
\limsup_{n \to \infty} \|C_\varphi \hat{k}_{w_n}\|^2 = 1 - \liminf_{n \to \infty} \int_T |w_n|^2 \frac{1 - |\varphi(\xi)|^2}{|1 - w_n \varphi(\xi)|^2} d\xi \leq 1 - \int_T \frac{1 - |\varphi(\xi)|^2}{|\alpha - \varphi(\xi)|^2} d\xi = \|\sigma_\alpha\| = 0,
\]
which leads to a contradiction. \qed

4. Examples and concluding remarks

Inner functions with one point spectra. Consider the Paley-Wiener space $K_{\vartheta_1}$ generated by
\[
\vartheta_1(z) = e^{\frac{i\pi}{2z-1}},
\]
this space can be obtained from the classical Paley-Wiener space of entire functions by the substitution $\zeta \mapsto \frac{1}{2i\pi z - 1}$. Then $\vartheta_1$ is a one-component inner function and $\Sigma(\vartheta_1) = \{1\}$. Theorem 1 and explicit calculation show that $C_\varphi : K_{\vartheta_1} \to H^2$ is a compact operator if and only if
\[
(15) \quad \frac{N_\varphi(w)}{\max((1 - |w|^2), |1 - w|^2)} \to 0, \quad \text{as } w \to 1.
\]
Consider now $D = \{w \in \mathbb{D}; |w - 1/4| < 3/4\}$ and let $\varphi$ be a conformal mapping $\varphi : \mathbb{D} \to D$, $\varphi(0) = 0, \varphi(1) = 1$. Clearly (15) does not hold and the operator $C_\varphi : K_{\vartheta_1} \to H^2$ is not compact. Evidently, the Aleksandrov-Clark measure $\mu_1$ of $\varphi$ is not absolutely continuous. Below we give an example of a (multi-component) inner function $\vartheta$ with $\Sigma(\vartheta) = \{1\}$ and such that $C_\varphi : K_{\vartheta} \to H^2$ is a compact operator. Thus (C) does not imply (S) for general $\vartheta$.

Take a sequence $t_m \searrow 0$ such that $\{\zeta_m\} = \{(1 - i t_m^3)^{1/2} e^{it_m}\}$ is an interpolating sequence in $\mathbb{D}$. Given a sequence $\{\alpha_m\} \in l^1$, $\alpha_m \in (0, 1)$, denote $\Lambda = \{\lambda_m\} = \{(1 - \alpha_m t_m^3)^{1/2} e^{it_m}\}$, this is also an interpolating sequence. Let now $\vartheta = B_\Lambda$ be the Blaschke product corresponding to the sequence $\Lambda$. We claim that $C_\varphi : K_{\vartheta} \to H^2$ is a compact operator.

Indeed, $\|C_\varphi \tilde{k}_\zeta\| \leq 1, \zeta \in \mathbb{D}$, here $\tilde{k}_\zeta$ is the normalized reproducing kernel for $H^2$, this follows just from the fact that $C_\varphi$ is contractive. In particular
\[
t^3_m \int_T \frac{|d\xi|}{|1 - \mu \varphi(\xi)|^2} = \|C_\varphi \tilde{k}_{\zeta_m}\|^2 \leq 1.
\]
Since $|1 - \zeta_m \varphi(\xi)|^2 \leq c|1 - \lambda_m \varphi(\xi)|^2$, $\xi \in T$, we have,
\[
\|C_\varphi \tilde{k}_{\lambda_m}\|^2 \asymp \alpha_m t^3_m \int_T \frac{|d\xi|}{|1 - \lambda_m \varphi(\xi)|^2} \leq C\alpha_m.
\]
On the other hand the system $\{\tilde{k}_{\lambda_m}\}$ forms a Riesz basis in $K_{\vartheta}$ (see e.g. [12], Lecture VII). Compactness of $C_\varphi : K_{\vartheta} \to H^2$ is now straightforward, alternatively it could be deduced from Theorem 1.
**Concluding remarks.** In the classical case of $H^2(\mathbb{D})$ the essential norm of the composition operator was obtained by J. Shapiro

$$\|C_{\varphi}\|_e^2 = \limsup_{|w| \to 1^-} \frac{N_{\varphi}(w)}{-\log |w|}.$$  

For a given one-component $\vartheta$ the equivalence of the norms proved in [8] and similar arguments give

$$\|C_{\varphi} : K_{\vartheta} \to H^2 \|_e^2 \approx \limsup_{|w| \to 1^-} \frac{1 - |\vartheta(w)|^2}{1 - |w|^2}.$$  

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function and $\varphi^*$ be its radial boundary values. Define a measure $\nu_{\varphi}$ on $\mathbb{D}$ by $\nu_{\varphi}(E) = m((\varphi^*)^{-1}(E))$ for any $E \subset \mathbb{D}$, where $m$ is the Lebesgue measure on $\mathbb{T}$. The composition operator $C_{\varphi}$ on $H^2(\mathbb{D})$ is isometrically equivalent to the embedding of $H^2$ into $L^2(\overline{\mathbb{D}}, \nu_{\varphi})$, see [9][11] for details. The connecting between the Nevanlinna counting function and the measure $\nu_{\varphi}$ was recently studied in details in [10].

Respectively, the compactness of the composition operator on $K_{\vartheta}$ can be reduced to the question of the compactness of the embedding $K_{\vartheta} \hookrightarrow L^2(\overline{\mathbb{D}}, \nu_{\varphi})$. It is well-known that the embeddings are easier to study for one-component inner functions $\vartheta$, see [7][8], and subsequent works [19] and [1][2]. The compactness of the embedding $K_{\vartheta} \hookrightarrow L^2(\mathbb{D}, \mu)$ was studied by J. A. Cima and A. L. Matheson [6] and by A. D. Baranov [4]. The latter article contains in particular necessary and sufficient conditions for the compactness of the embedding for the case of one-component inner function. The approach also shows that for one-component $\vartheta$ the compactness of the composition operator $\varphi : K_{\vartheta} \to H^p$ does not depend on $p \in (1, \infty)$.

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Department of Mathematical Sciences, Norwegian University of Science and Technology, NO-7491, Trondheim, Norway

E-mail address: yura@math.ntnu.no

E-mail address: eugenia@math.ntnu.no