Coherent States of $gl_q(2)$-covariant Oscillators

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Abstract

In this paper two types of coherent states of $gl_q(2)$-covariant oscillators are investigated.

1

Quantum groups or q-deformed Lie algebra implies some specific deformations of classical Lie algebras.

From a mathematical point of view, it is a non-commutative associative Hopf algebra. The structure and representation theory of quantum groups have been developed extensively by Jimbo [1] and Drinfeld [2].

The q-deformation of Heisenberg algebra was made by Arik and Coon [3], Macfarlane [4] and Biedenharn [5]. Recently there has been some interest in more general deformations involving an arbitrary real functions of weight generators and including q-deformed algebras as a special case [6-10].

Recently V. Spiridonov [11] found the new coherent states of the q-Weyl algebra $aa^\dagger - qa^\dagger a = 1, 0 < q < 1$ which are defined as eigenstates of the operator $a^\dagger$. 
The purpose of this paper is to find such coherent states for $gl_q(2)$-covariant oscillator algebra.

The $gl_q(2)$-covariant oscillator algebra is defined as

\[ a_1^\dagger a_2^\dagger = \sqrt{qa_2^\dagger a_1^\dagger} \]
\[ a_1 a_2 = \frac{1}{\sqrt{q}} a_2 a_1 \]
\[ a_1 a_2^\dagger = \sqrt{qa_2^\dagger a_1} \]
\[ a_2 a_1^\dagger = \sqrt{qa_1^\dagger a_2} \]
\[ a_1 a_1^\dagger = 1 + qa_1^\dagger a_1 + (q-1)a_2^\dagger a_2 \]
\[ a_2 a_2^\dagger = 1 + qa_2^\dagger a_2 \] (1)

Throughout, $(\cdot)^\dagger$ denotes the hermitian conjugate of $(\cdot)$. By the $gl_q(2)$-covariance of the system, it is meant that the linear transformations

\[ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} \]
\[ a_1^\dagger \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} a_2^\dagger = (a_1')^\dagger (a_2')^\dagger \] (2)

leads to the same commutation relations (1) for $(a_1', (a_1^\dagger)')$ and $(a_2', (a_2^\dagger)')$ when the matrix $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$ belongs to $gl_q(2)$. The relation between the entries of this matrix obeys the following commutation relation

\[ ab = \sqrt{q}ba \quad cd = \sqrt{q}dc \]
\[ ac = \sqrt{q}ca \quad bd = \sqrt{q}db \]
\[ bc = cb \quad ad - da = (\sqrt{q} - \frac{1}{\sqrt{q}})bc \] (3)
It should be noted that the particular coupling between the two modes is completely dictated by the required $gl_q(2)$-covariance.

The Fock space representation of the algebra (1) can be easily constructed by introducing the hermitian number operators $\{N_1, N_2\}$ obeying

$$[N_i, a_j] = -\delta_{ij} a_j \quad [N_i, a_j^\dagger] = \delta_{ij} a_j^\dagger, \quad (i, j = 1, 2)$$

(4)

Let $|0, 0\rangle$ be the unique ground state of this system satisfying

$$N_i|0, 0\rangle = 0, \quad a_i|0, 0\rangle = 0, \quad (i, j = 1, 2)$$

(5)

and $\{|n, m\rangle | n, m = 0, 1, 2, \cdots \}$ be the set of the orthogonal number eigenstates

$$N_1|n, m\rangle = n|n, m\rangle, \quad N_2|n, m\rangle = m|n, m\rangle$$

$$<n, m|n', m'\rangle = \delta_{nn'}\delta_{mm'}$$

(6)

From the algebra (1) the representation is given by

$$a_1|n, m\rangle = \sqrt{[n]!} |n - 1, m\rangle, \quad a_2|n, m\rangle = \sqrt{[m]!} |n, m - 1\rangle$$

$$a_1^\dagger|n, m\rangle = \sqrt{[n + 1]!} |n + 1, m\rangle, \quad a_2^\dagger|n, m\rangle = \sqrt{[m + 1]!} |n, m + 1\rangle$$

(7)

where the q-number $[x]$ is defined as

$$[x] = \frac{q^x - 1}{q - 1}$$

The general eigenstates $|n, m\rangle$ is obtained by applying $a_2^\dagger$ m times after applying $a_1^\dagger$ n times.

$$|n, m\rangle = \frac{(a_2^\dagger)^m (a_1^\dagger)^n}{\sqrt{[n]! [m]!}} |0, 0\rangle$$

(8)
where
\[ [n]! = [n][n-1] \cdots [2][1], \quad [0]! = 1 \]
The coherent states for algebra (1) are usually defined as
\[ a_1|z_1, z_2 >_z = z_1|z_1, \sqrt{q}z_2 >_z \]
\[ a_2|z_1, z_2 >_z = z_2|z_1, z_2 >_z \quad (9) \]
It can be easily checked that the coherent state satisfies \( a_1a_2 = \frac{1}{\sqrt{q}}a_2a_1 \).
Solving the relation (9) we have
\[ |z_1, z_2 >_z = c(z_1, z_2)\sum_{n,m=0}^{\infty} \frac{z_1^n z_2^m}{\sqrt{[n]![m]!}} |n, m > \quad (10) \]
Using the eq.(8) we can rewrite eq.(9) as
\[ |z_1, z_2 >_z = c(z_1, z_2)e_q(z_1a_1^\dagger)e_q(z_2a_2^\dagger)|0, 0 > \quad (11) \]
In order to obtain the normalized coherent states, we should impose the condition \( <z_1, z_2|z_1, z_2 >_z = 1 \). Then the normalized coherent states are given by
\[ |z_1, z_2 >_z = \frac{1}{\sqrt{e_q(|z_1|^2)}e_q(|z_2|^2)}} e_q(z_1a_1^\dagger)e_q(z_2a_2^\dagger)|0, 0 > \quad (12) \]
where
\[ e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!} \]
is a q-deformed exponential function.
The purpose of this section is to obtain another type of coherent states for algebra (1). In order to do so, it is convenient to introduce the two commuting operators as follows

\[ H = a_1^\dagger a_1 + a_2^\dagger a_2 - \nu, \quad T = a_2^\dagger a_2 - \nu \]

where

\[ \nu = \frac{1}{1 - q} \]

and \( H \) is a hamiltonian and \( T \) is an integral motion for \( H \), i.e. an independent operator commuting with the hamiltonian \( H \). From the algebra (1) we can easily check that \([H,T] = 0\). Then the commutation relation between two commuting operators and mode operators are given by

\[ Ha_1^\dagger = qa_1^\dagger H, \quad Ta_2^\dagger = qa_2^\dagger T \]

\[ Ha_2^\dagger = qa_2^\dagger H, \quad Ta_1^\dagger = a_1^\dagger T \]

Acting the two commuting operators on the eigenstates gives

\[ H|n,m> = E_{nm}|n,m> \quad T|n,m> = t_m|n,m> \quad (13) \]

where

\[ E_{nm} = -\frac{q^{m+n}}{1 - q}, \quad t_m = -\frac{q^m}{1 - q} \quad (14) \]

It is worth noting that the energy spectrum is degenerate and negative for \( 0 < q < 1 \). The degenerated states are split by the eigenvalues of operator \( T \), but the physical interpretation of operator \( T \) is not clear to the author.
As was noticed in ref[11], for the positive energy states it is not \( a_1(a_2) \) but \( a_1^\dagger(a_2^\dagger) \) that play a role of the lowering operator:

\[
H|\lambda q^n, \mu q^m >, \; n, m = 0, \pm 1, \pm 2, \cdots = \lambda q^{n+m}|\lambda q^n, \mu q^m >
\]

\[
T|\lambda q^n, \mu q^m > = \mu q^m|\lambda q^n, \mu q^m >
\]

\[
a_1^\dagger|\lambda q^n, \mu q^m > = \sqrt{\lambda q^{n+1} - \mu q^m}|\lambda q^{n+1}, \mu q^m >
\]

\[
a_2^\dagger|\lambda q^n, \mu q^m > = \sqrt{\mu q^m + \nu}|\lambda q^{n+1}, \mu q^m >
\]

\[
a_1|\lambda q^n, \mu q^m > = \sqrt{\lambda q^n - \mu q^m}|\lambda q^{n-1}, \mu q^m >
\]

\[
a_2|\lambda q^n, \mu q^m > = \sqrt{\mu q^m + \nu}|\lambda q^{n-1}, \mu q^{m-1} >
\]

where \( \lambda, \mu > 0 \) are arbitrary chosen eigenvalues of \( H \) and \( T \). This representation is a nonhighest weight representation of the \( gl_q(2) \)-covariant oscillator algebra. And this representation has no classical analogue because it is not defined for \( q \to 1 \). Due to this fact, it is natural to define coherent states corresponding to the representation (15) as the eigenstates of \( a_1^\dagger \) and \( a_2^\dagger \):

\[
a_1^\dagger|z_1, z_2 > _+ = z_1|z_1, \frac{1}{\sqrt{q}}z_2 > _+ , \quad a_2^\dagger|z_1, z_2 > _+ = z_2|z_1, z_2 > _+ \quad (16)
\]

Because the representation (15) depend on two free parameter \( \lambda \) and \( \mu \), the coherent states \(|z_1, z_2 > _+ \) can take different forms.

If we assume that the positive energy states are normalizable, i.e.

\[
< \lambda q^n, \mu q^m|\lambda q^{n'}, \mu q^{m'} > = \delta_{nn'}\delta_{mm'}, \text{ and form exactly one series for some fixed } \lambda \text{ and } \mu, \text{ then we can write}
\]

\[
|z_1, z_2 > _+ = \sum_{n,m=-\infty}^{\infty} c_{nm}(z_1, z_2)|\lambda q^n, \mu q^m >
\]

(17)
Inserting eq.(17) into eq.(16), we find

\[
|z_1, z_2 >_{+} = C \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm} z_1^{m-n} z_2^{-m} |\lambda q^n, \mu q^m > + \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} z_1^{m-n} z_2^{-m} |\lambda q^n, \mu q^{-m} > + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} c_{nm} z_1^{m+n} z_2^{-m} |\lambda q^{-n}, \mu q^m > + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} z_1^{m-n} z_2^{m} |\lambda q^{-n}, \mu q^{-m} >
\]

(18)

where

\[
c_{nm}^1 = q^{\frac{m(m-1)}{2}} \frac{(-\mu)^n (\sqrt{\nu})^m}{(\nu)^m} \left[ \frac{(-\mu)^{1-m}; q)_n (-\frac{\mu}{\nu}; q)_m}{(\mu; q)_m} \right]
\]

\[
c_{nm}^2 = q^{\frac{m(m-1)}{2}} \frac{(-\mu)^n (\sqrt{-\mu})^m}{(\nu)^m} \left[ \frac{(-\mu q^{m+1}; q)_n (\frac{\mu}{\nu} q; q)_m}{(-\frac{\mu}{\nu}; q)_m} \right]
\]

\[
c_{nm}^3 = q^{\frac{m(n-1)}{2}} + \frac{m(n-1)}{2} \frac{(\sqrt{\nu})^m}{(\nu)^n (\sqrt{\nu})^m} \left[ \frac{(-\mu q^{m+1}; q)_n (\frac{\mu}{\nu}; q)_m}{(-\frac{\mu q^{m+1}; q)_n (\frac{\mu}{\nu} q; q)_m} \right]
\]

\[
c_{nm}^4 = q^{-\frac{m(n-1)}{2}} + \frac{m(n-1)}{2} \frac{(-\mu)^n (\sqrt{-\mu})^m}{(\nu)^m} \left[ \frac{(-\frac{\mu}{\nu}; q)_m}{(\nu)^n (\sqrt{-\mu})^m} \right]
\]

(19)

If we demand that \(+ < z_1, z_2 | z_1, z_2 >_{+} = 1\), we have

\[
|C|^{-2} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-m) q^{-nm+\frac{m(n-1)}{2} + \frac{m(m-1)}{2}} \frac{(\frac{\lambda}{\mu} q^m; q)_m}{(\lambda q^{-m}; q)_n (-\frac{\mu}{\nu}; q)_m} |z_1|^{2(n-m)} |z_2|^{2m}
\]

(20)

If we substitute \(n - m \rightarrow l\) in eq.(20) and use the identity

\[
(aq^{-m}; q)_n = (-a)^m q^{-\frac{m(m+1)}{2}} (q/a; q)_m (a; q)_{n-m}
\]

(21)
we can express the normalization constant $C$ in terms of the bilateral $q$-hypergeometric series [12]:

$$|C|^{-2} = \psi_1\left(-\frac{\nu}{\mu}; q, -\frac{|z_2|^2}{\mu}\right) \psi_1\left(\frac{\mu}{\lambda}; q, -\frac{|z_1|^2}{\lambda}\right)$$  \hspace{1cm} (22)

where general bilateral $q$-hypergeometric series is defined by [12]

$$r\psi_s\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \\ q, z \end{array}\right) = \sum_{n=-\infty}^{\infty} \frac{(a_1; q)_n \cdots (a_r; q)_n}{(b_1; q)_n \cdots (b_s; q)_n} (-)^n q^{n(n-1)/2} z^n$$  \hspace{1cm} (23)

We can introduce the nonunitary displacement operator $D(z_1, z_2)$

$$|z_1, z_2> = D(z_1, z_2)|\lambda, \mu>$$  \hspace{1cm} (24)

Then the operator $D(z_1, z_2)$ is given by

$$D(z_1, z_2) = \frac{\psi_1\left(-\frac{\nu}{\mu}; q, -\frac{z_2a_2}{\mu}\right) \psi_1\left(\frac{\mu}{\lambda}; q, -\frac{z_1a_1}{\lambda}\right)}{\sqrt{\psi\left(-\frac{\nu}{\mu}; q, -\frac{|z_2|^2}{\mu}\right) \psi_1\left(\frac{\mu}{\lambda}; q, -\frac{|z_1|^2}{\lambda}\right)}}$$  \hspace{1cm} (25)

Formally we can write also

$$D(z_1, z_2) = \sum_{n,m=0}^{\infty} \left(\frac{a_2^\dagger}{z_2}\right)^m \left(\frac{a_1^\dagger}{z_1}\right)^{n-m}$$

or

$$D(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \left(\frac{a_2^\dagger}{z_2}\right)^m \left(\frac{a_1^\dagger}{z_1}\right)^{n-m}$$

$$+ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{q^{m(n-1)}}{(-\nu; q)_m} \left(\frac{z_2a_2}{\mu}\right)^m \left(\frac{a_1^\dagger}{z_1}\right)^{n+m}$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{1/2(n+m)(n+m-1)}}{q^{m}(-\nu; q)_m} \left(\frac{z_1a_1}{\lambda}\right)^{n+m}$$

$$+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{q^{1/2m}}{q^{m}(-\nu; q)_m} \left(\frac{z_1a_1}{\lambda}\right)^{n+m}$$
For some realizations of $\mathfrak{gl}_q(2)$-covariant oscillator algebra, the states (24) belongs to a continuous spectrum. Thus it is appropriate to consider integrals over both $\lambda$ and $\mu$ for the expansion of $|z_1, z_2>_{+}$ in the basis $|\lambda, \mu>$ instead of sum:

$$|z_1, z_2>_{+} = \int_0^\infty \int_0^\infty d\lambda d\mu C(\lambda, \mu, z_1, z_2)|\lambda, \mu>$$  \hspace{1cm} (27)

Inserting this expression into eq.(16) gives

$$|z_1, z_2>_{+} = \int_0^\infty \int_0^\infty d\lambda d\mu \lambda^d \mu^\epsilon h(\lambda)|\lambda, \mu> \sqrt{(q\lambda/\mu; q)_{\infty}(-q\mu/\nu; q)_{\infty}}$$  \hspace{1cm} (28)

where

$$d = -\frac{\ln q z_1}{\ln q}, \quad \epsilon = \frac{\ln \frac{z_1}{z_2}}{\ln q}$$

where $h(\lambda)$ is an arbitrary function satisfying $h(q\lambda) = h(\lambda)$.

If we impose the normalization condition

$$<\lambda, \mu|\sigma, \eta> = \lambda \mu \delta(\lambda - \sigma) \delta(\mu - \eta)$$

then we find that the states are normalizable,

$$+ <z_1, z_2|z_1, z_2>_{+} = 1 = \int_0^\infty \int_0^\infty d\lambda d\mu \lambda^{2Re d} \mu^{2Re \epsilon} \frac{|h(\lambda)|^2}{(q\lambda/\mu; q)_{\infty}(-q\mu/\nu; q)_{\infty}}$$  \hspace{1cm} (29)

if $h(\lambda)$ is a bounded function and $Re d > -1, Re \epsilon > -1$, or $|z_1|^2 < 1, |z_2|^2 > \nu$. Expanding $h(\lambda)$ into the Fourier series, we have an infinite number of linearly independent (but not orthogonal) coherent states of the form

$$|z_1, z_2>_{+s} = C(z_1, z_2) \int_0^\infty \int_0^\infty d\lambda d\mu \frac{\lambda^d \mu^\epsilon |\lambda, \mu>}{\sqrt{(q\lambda/\mu; q)_{\infty}(-q\mu/\nu; q)_{\infty}}}$$  \hspace{1cm} (30)
where
\[ \delta_s = d + \frac{2\pi is}{\ln q}, \quad s = 0, \pm 1, \cdots \]

From the normalization condition of the coherent states, we can determine the normalization constant,
\[ |C(z_1, z_2)|^{-2} = \int_0^\infty \int_0^\infty d\lambda d\mu \frac{\tau^\xi \mu^\xi |\lambda, \mu>}{(q\lambda/\mu; q)_\infty (-q\mu/\nu; q)_\infty} \] (31)
where
\[ \tau = \frac{\ln \frac{1}{q|z_1|^2}}{\ln q}, \quad \xi = \frac{\ln \frac{|z_1|^2\nu}{q^2|z_2|^2}}{\ln q}. \]

3

To conclude, in this paper I have constructed two types of coherent states, one of which is related to the positive energy representation of the \( gl_q(2) \)-covariant oscillator system. These states are shown to be related to the product of two bilateral basic hypergeometric series. I think that this method will be applied to more general case, \( gl_q(n) \)-covariant oscillator system. In that case I guess that the new coherent states associated with the positive energy will be related to the product of \( n \) bilateral basic hypergeometric series. I hope that this problem and its related topics will become clear in the near future.

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