Stringent Statistical Fluctuation Analysis for Quantum Key Distribution Considering After-pulse Contributions

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(Dated: July 7, 2015)

Statistical fluctuation problems are faced by all quantum key distribution (QKD) protocols under finite-key condition. Most of the current statistical fluctuation analysis methods work based on independent random samples, however, the precondition can be always satisfied on account of different choice of samples and actual parameters. As a result, proper statistical fluctuation methods are required to figure out this problem. Taking the after-pulse contributions into consideration, we give the expression of secure key rate and the mathematical model for statistical fluctuations, focusing on a decoy-state QKD protocol (Sci Rep. 3, 2453, 2013) with biased basis choice. On this basis, a classified analysis of statistical fluctuation is represented according to the mutual relationship between random samples. First for independent identical relations, we make a deviation comparison between law of large numbers and standard error analysis. Secondly, we give a sufficient condition that Chernoff bound achieves a better result than Hoeffdings inequality based on independent relations only. Thirdly, by constructing the proper martingale, for the first time we represent a stringent way to deal with statistical fluctuation issues upon dependent ones through making use of Azumas inequality. In numerical optimization, we show the impact on secure key rate, the ones and respective deviations under various kinds of statistical fluctuation analyses.

PACS numbers: 03.67.Dd, 42.81.Gs, 03.67.Hk

I. INTRODUCTION

Quantum key distribution (QKD) protocols [1-3] can provide unconditional security for telecommunications based on the physical law of quantum mechanics [4-8]. However, in real-life QKD systems, there is a large gap between the ideal conditions and the practical ones. In order to figure out the problem, Gottesman et al. [9] gave an intensive study aiming at the real system imperfections, and provided a key rate formula called GLLP. Based on GLLP formula, decoy-state protocols [10-13] were proposed so as to improve secure key rate and detect photon-number-splitting (PNS) attack [14, 15]. For the first 20 years since the proposal of BB84 protocol [1], QKD postprocessing procedure had been mainly analyzed in the asymptotic assumption that the length of transmission code is infinite. Nevertheless, in real-world situations, the two parties often communicate within a limited time. As a result, many efforts [4, 13, 16, 20] have been spent on the finite-key effect in QKD postprocessing schemes.

The security of QKD under finite-key condition was first considered by Mayers [4], and soon afterwards, by using standard error analysis, Ma [13] gave a detailed analysis concerning five kinds of statistical fluctuations. Based on the work in [13, 27], Wei et al. [21] proposed an efficient decoy-state QKD protocol with biased basis choice and adopted standard error analysis to tackle statistical fluctuation problems. More recently, Zhou et al. [28] proposed an improved statistical fluctuation method in measurement-device-independent (MDI) QKD also based on standard error analysis, which raises the key rate of the recent Shanghai experimental set-up [28] by more than 600 times.

Under universal composable (UC) [29] security which is accepted and considered as the most stringent security standard in classical and quantum fields, Scarani and Renner [18] presented the finite-key QKD security analysis using Law of large numbers [17, 18, 30]. By adopting Law of large numbers to deal with statistical fluctuation issues, Cai and Scarani [18] discussed the secure key generation rate in their decoy-state protocol and obtain the shortest sending code length for generating secure key. Based on their research, Lucamarini et al. [24] used beta distribution and CP test [31] to handle the finite-key effects and gave an experimental verification for their decoy-state QKD protocol.

By taking advantage of uncertainty relationship for smooth entropies, Tomamichel et al. [19] provided an upper bound of the final key. Based on their work, through utilizing Hoeffding’s inequality [32] to analyze statisti-
cal fluctuations in decoy-state QKD protocol, Ref. \[25\] achieved a good estimation result. Since then large deviation theories \[32, 33\] have attracted lots of attentions in dealing with statistical fluctuation problems. Applying Chernoff bound \[21, 33\] to perform parameter estimation, Curty et al. demonstrated the feasibility of long-distance implementations of MDIQKD in finite-code length.

In most of the above QKD postprocessing schemes \[lim2014concise, 13, 18, 21, 22, 24, 34, 35\], the main precondition of statistical fluctuation analysis is almost the same, which requires an independent mutual relation between random samples. Renner \[34\] pointed out that symmetry of large physical systems implies independence of subsystems, thus the independent relationship is reasonable as long as the systems physical properties are robust under small disturbances. However, when it comes to after-pulse contributions, the detection event might be caused by the after-pulse of the prior one. Many efforts have been spent on investigating after-pulse \[37, 38\] and its effect \[39, 40\], concentrating on the correlations of neighbouring pulses and the security analysis. Somma et al. \[39\] demonstrated that if the independent condition is broken by after-pulse contributions, then the security parameters of decoy-state implementations could be worse, and they also pointed out that the eavesdropper possesses the ability to correlate her PNS attack which breaks the independent condition, so that the corresponding security analysis would be invalid in the above mentioned protocols. Furthermore, the after-pulse effect will lower the secure key rate, which is even worse at higher gate frequencies because of a large number of after-pulse-related errors \[40\]. As a result, the after-pulse contributions cant be neglected, and we need to find a stringent way to reconsider the security analysis, especially statistical fluctuation issues based on dependent random samples, which is the motivation behind our analysis.

In this paper, we focus on the mathematical models, classifications and deviation comparisons for various kinds of statistical fluctuation analyses, including standard error analysis, Law of large numbers, Hoeffdings inequality and Chernoff bound. However, these methods are not the accurate ways to deal with statistical fluctuation issues when taking the after-pulse contribution into consideration. In order to solve the problem, for the first time we provide a new way called Azumas inequality \[33\] to deal with dependent random samples. Since the efficiency of a practical QKD system needs to care about the total quantum resource consumption, it is important to take the sending qubits as the total random samples.

The article is organized as follows. Section \[II\] gives an analysis of after-pulse contributions based on Weis decoy-state protocol \[21\]. \[III\] we set up mathematical models for standard error analysis and Law of large numbers based on independent identically distributed (i.i.d.) random samples, and present a deviation comparison. For independent random samples only, we give a similar analysis for large deviation theories in section \[IV\]. In section \[V\] we bring a stringent way to solve the statistical fluctuation issues on dependent samples. Simulations for secure key rates and respective deviations of the five kinds of estimation methods are given in section \[VI\]. At last, some conclusions are presented in section \[VII\].

II. ANALYSIS OF AFTER-PULSE CONTRIBUTIONS

The after-pulse effect is due to carrier traps, and the uncontrollable clicks will contribute to the quantum bit error rate(QBER). To make a detailed analysis, here we choose vacuum+weak decoy-state QKD protocol with biased basis choice \[21\]. Taking after-pulse contributions into consideration, we need to reconsider the expected detection rate, the yield of vacuum decoy-state yield and the observed QBER.

A. Parameter description

Denoting \( p_{ap} \) as the after-pulse probability, here we suppose \( p_{ap} = 4 \times 10^{-2} \). Then the parameters mentioned above change their expressions as follows.

\[
\begin{align*}
D_\mu & = Q_\mu (1 + p_{ap}) = (1 - e^{-\eta p})(1 - Y_0) (1 + p_{ap}) \\
D_\nu & = Q_\nu (1 + p_{ap}) = (1 - e^{-\eta v}(1 - Y_0)) (1 + p_{ap}) \\
Y_0' & = 2p_{dc} (1 + p_{ap}) \\
E_\mu' & = \frac{(e_0 Y_0' + e_0 (1 - e_0)(1 - Y_0') + p_{ap} Q_\mu/2)}{2p_{dc}}
\end{align*}
\]

where \( D_\mu, D_\nu, Y_0' \) and \( E_\mu' \) respectively denote the overall gain of signal pulses, the weak decoy-state pulses, the yield of vacuum decoy-state pulses and the observed QBER under after-pulse impacts; \( \eta, p_{dc}, e_0 \) and \( e_d \) represent the transmittance of the channel, the background count rate, the background error rate and the probability that a photon hits the wrong detector.

Further, we obtain the secure key rate

\[
\begin{align*}
R & \geq q \{ Q_0 + Q_0^* [1 - H (e_1^{p_{ec}})] - leak_{EC} \} \\
q & = \frac{N_{tot} p_{s} N_{opt} }{N_{opt}} \\
Q_0 & = Y_0' e^{-\mu}
\end{align*}
\]

where \( q \) denotes the received signal state ratio; \( Q_0 \) and \( Q_0^* \) represent the background and the single photon gains, \( e_1^{p_{ec}} \) is the phase error rate of single photon signal state; \( leak_{EC}, f_{EC}, Q_{0} \) and \( E_{\mu} \) separately denote the cost of error correction, the bilateral error correction inefficiency, the overall gain of signal state and quantum bit error rate (QBER); \( H (x) = -x \log_2 (x) - (1 - x) \log_2 (1 - x) \) is the binary Shannon entropy function; \( N_{tot} \), \( N_{\mu}, p_{s} \) and \( Y_0 \) are the length of the sending qubits, the signal pulses in the basis, the probability that Bob chooses the basis, and the background count rate.

In security analysis, we need to estimate the lower bound of single photon yield denoted by \( Y_1 \) and the upper bound of its relevant error rate \( e_1 \) measured in the \( Z \)

basis. Their expressions are

\[ Y_1 \geq Y_1^L = \frac{\mu}{\mu v - v^2} \left( Q_v e^v - Q_v e^v \frac{v^2}{\mu^2} - \frac{\mu^2 - v^2}{\mu^2} Y_0' \right) \]

and

\[ e_1 \leq e_1' = \frac{E_v Q_v e^v - e_0 Y_0'}{Y_1^L v}. \]

**B. Mathematical model**

Mathematical model for the corresponding statistical fluctuation analysis considering after-pulse contributions is discussed as follows. Let \( X \) denote the observed value, which is obtained by detection events of \( m \) random samples denoted by \( X_1, X_2, \ldots, X_m \), detected with the value 1 and 0 otherwise, satisfying \( \sum_{i=1}^{m} X_i \). \( E \) is the expected value of \( \frac{1}{m} \sum_{i=1}^{m} X_i \) when \( m \) tends to infinity. Because of after-pulse contributions, we have the following probability relations

\[ \Pr \{ X_{m+1} = s | X_m = t, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1, X_0 = x_0 \} = \Pr \{ X_{m+1} = s | X_m = t \} \neq \Pr \{ X_{m+1} = s \} \]

which shows that the random samples are Markov chain. Then for any and , we need to find a proper inequality satisfying

\[ \Pr (|X - E| \geq \xi) \leq \varepsilon \]

where the parameter can be seen as a function of . That is to say the expected value is contained in the interval

\[ [X - \xi, X + \xi] \]

with a failure probability \( \varepsilon \).

**III. ANALYSIS OF I.I.D RANDOM SAMPLES**

The purpose of this section is to make a deviation comparison between two kinds of analysis methods based on i.i.d. random samples.

**A. Standard error analysis**

The original protocol adopts standard error analysis which is the application of central-limit theorem using Gaussian distribution to deal with the statistical fluctuation problems. The application precondition is that the random samples must be i.i.d. and the sum of them has to obey Gaussian distributions. We denote \( \xi_1(S_{obs}) \) as the deviation when using standard error analysis, here \( S_{obs} \) refers to the respective random samples. There are four kinds of deviations corresponding to Wei’s protocol defined in Appendix [A].

**B. Law of large numbers**

Law of large numbers [30] mainly describes the stable equivalence relation between occurrence frequency and probability distribution when the test number is large enough. Its application precondition only requires that the random samples are i.i.d.. And the mathematical model is stated as follows.

The only difference from the above mentioned one is that the observed value \( X \) is obtained by detection events of \( m \) i.i.d. random samples. Then for any \( \xi > 0 \) and \( \varepsilon \geq 0 \), a quantum version of law of large numbers yields the following statement

\[ \Pr \left( \left| \frac{1}{m} \sum_{i=1}^{m} X_i - \mu \right| \geq \xi \right) \leq \varepsilon, \]

where \( \xi = \sqrt{\frac{2 \ln \left( \frac{\mu}{\varepsilon} \right) + 2 \ln (m+1)}{m}} \). Applying law of large numbers in dealing with statistical fluctuation problems in Wei’s protocol, we obtain the relevant deviations denoted by \( \xi_2(S_{obs}) \), which can be found in Appendix [B].

**C. Deviation comparison**

Here we should point out that the quantile denoted by \( u_\alpha \) in standard error analysis is determined by the failure probability when applying the statistical fluctuation analysis method. Thus, before making a deviation comparison between the two methods, we need to set up a fair comparison standard first. We set a fixed failure probability no matter which method is used. Suppose that the failure probability is \( \varepsilon = 10^{-10} \), which is equivalent to taking the quantile as \( u_\alpha = 6.4 \). So we have to change the quantile from \( u_\alpha = 5 \) in the original protocol into \( u_\alpha = 6.4 \).

Given that the length of sending qubits is long enough so that the sum of them satisfies Gaussian distribution. Now we just take \( Q_\mu \) as an example and the respective deviations are

\[ \xi_1(Q_\mu) = \frac{0.4}{\sqrt{N_{\mu} P_\mu Q_\mu}} \] \[ \text{and} \] \[ \xi_2(Q_\mu) = \sqrt{\frac{2 \ln (10^{10} + 2 \ln (N_{\mu} P_\mu + 1))}{N_{\mu} P_\mu Q_\mu}}. \]

By some reduction, it equals to compare \( 6.4 \cdot \sqrt{Q_\mu} \) and \( \sqrt{46.06 + 4 \ln (N_{\mu} P_\mu + 1)} \).

Since \( \sqrt{Q_\mu} < 1 \) and \( 4 \ln (N_{\mu} P_\mu + 1) > 0 \), then we can obtain the following inequalities

\[ \sqrt{46.06 + 4 \ln (N_{\mu} P_\mu + 1)} > \sqrt{46.06} > 6.4 > 6.4 \cdot \sqrt{Q_\mu}. \]

As a result, it can be safely concluded that the relationship between relevant deviations satisfies \( \xi_1 < \xi_2 \). In other words, standard error analysis can achieve a better estimation result than law of large numbers. The simulation result is shown in [IV].
IV. ANALYSIS OF INDEPENDENT RANDOM SAMPLES

Large deviation theory [32, 33] describes the remote tails’ asymptotic behavior of probability distribution sequences. That is to say, it concerns the exponential decline of the probability measures of certain kinds of tail events. It is a precise form of law of large numbers and the main application precondition only requires the relation of observed samples is independent. Here we take two basic theorems to deal with the statistical fluctuation issues concerning after-pulse contributions.

A. Hoeffding’s inequality

The theorem introduced first is called Hoeffding’s inequality [32], which provides an upper bound on the probability that the sum of independent random variables deviates from its expected value. A quantum version of Hoeffding’s inequality yields the following statement.

Let $X$ denote the observed value, which is obtained by detection events of $m$ independent random samples. And $X_i$ are almost surely bounded, that is $\Pr (X_i \in [a_i, b_i]) = 1, 1 \leq i \leq m$. Then for any $\xi > 0$ and $\varepsilon \geq 0$, the following inequality holds

$$\Pr (|X - E[X]| \geq \xi) \leq 2e^{-\frac{2m^2 \varepsilon^2}{\xi^2}}, \quad \xi = \varepsilon, \quad (10)$$

where $E[X]$ refers to the expected value of $X$.

Using Hoeffding’s inequality, we get the relevant deviations denoted by $\xi_3(S_{obs})$, which can be found in Appendix A.

B. Chernoff bound

The second theorem introduced here is called Chernoff bound [32], also providing an upper bound on the probability that the sum of independent random variables deviates from its expected value. To apply the method in Wei’s protocol, we change the form mentioned in [20] into the following claim.

Let $X$ denote the observed value, which is obtained by detection events of $m$ independent random samples, satisfying $\Pr (X_i = 1) = p_i$, $X = \frac{1}{m} \sum_{i=1}^{m} X_i$, and $a = E[X] = \frac{1}{m} \sum_{i=1}^{m} p_i$, where $E[\cdot]$ denotes the mean value. Let $x$ be the observed outcome of $X$ and $a_L = x - \frac{1}{2m} \ln \left( \frac{1}{\varepsilon_1} \right)$ for certain $\varepsilon_1 > 0$. For certain $\varepsilon_2$, $\varepsilon_3 \geq 0$, we have that $x$ satisfies

$$x = a + \delta,$$

except with error probability $\gamma = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$, where $\delta \in [\Delta, \Delta']$. Here $\varepsilon_2(\varepsilon_3)$ refers to the probability that $x < a - \Delta (x > a + \Delta')$. The limitations and corresponding values of $\Delta$ and $\Delta'$ are given in Appendix C.

Employing Chernoff bound in Wei’s protocol, we obtain the corresponding deviations denoted by $\xi_4(S_{obs})$ in Appendix A.

C. Deviation comparison

In the definition of Chernoff bound, it can be easily concluded that if the three conditions are not satisfied, then Chernoff bound changes its form into Hoeffding’s inequality. Here we only consider the situation that the three conditions in Chernoff bounds are all fulfilled, and the comparisons corresponding to the other results are shown in Appendix B.

For convenience, we suppose $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 10^{-10}$. Still we take $Q_\mu$ as an example and the respective deviations are $\xi_3 (Q_\mu) = \sqrt{\frac{2Q_\mu}{N_{\mu} p_z}} \ln \frac{2}{\varepsilon_3}$ and $\Delta = \sqrt{\frac{2Q_\mu}{N_{\mu} p_z}} \ln \frac{2}{\varepsilon_3}$ and $\Delta' = \sqrt{\frac{2Q_\mu}{N_{\mu} p_z}} \ln \frac{1}{\varepsilon_3^2}$, here $\xi_3$, $\Delta$ and $\Delta'$ refer to the deviations when using Hoeffding’s inequality and Chernoff bound.

Then if both $\sqrt{\frac{2Q_\mu}{N_{\mu} p_z}} \ln \frac{2}{\varepsilon_3} \leq \sqrt{\frac{1}{2N_{\mu} p_z}} \ln \frac{1}{\varepsilon_3}$ and $\sqrt{\frac{2Q_\mu}{N_{\mu} p_z}} \ln \frac{1}{\varepsilon_3^2} \leq \sqrt{\frac{1}{2N_{\mu} p_z}} \ln \frac{1}{\varepsilon_3}$ hold, we can safely say that Chernoff bound can achieves a better result than Hoeffding’s inequality.

Simplify the above inequalities, we obtain the sufficient condition is

$$Q_\mu \leq 0.06. \quad (12)$$

V. ANALYSIS OF DEPENDENT RANDOM SAMPLES

Concerning the worst case that the after-pulse (maybe caused by eavesdropper) exists in every detection event, the mutual relationship between adjacent samples doesn’t satisfy the independent condition anymore. Therefore, we need a stringent way to deal with the problem based on dependent observed samples. Here we try to solve the problem by constructing martingale [34] and bring Azuma inequality [35] into statistical fluctuation analysis. Detailed proof is shown in Appendix C. Here the quantum version of Azumas inequality is presented as follows.

Let $X$ denote the observed value, which is obtained by detection events of $m$ independent random samples, $M_n = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $M_0 = E$, here $E$ is the value of $\frac{1}{n} \sum_{i=1}^{n} X_i$ when $n$ tends to infinity. Then for any $\xi > 0$ and $\varepsilon \geq 0$, the following inequality holds

$$\Pr (|M_n - M_0| \geq \xi) = \Pr (|\frac{1}{n} \sum_{i=1}^{n} X_i - E| \geq \xi) \leq 2e^{-\frac{\xi^2}{2n}}, \quad (13)$$

with $\xi = \sqrt{\frac{2 \ln \frac{2}{\varepsilon}}{n}}$. Applying Azuma’s inequality in solving the statistical fluctuation problems based on depen-
dent samples, we get the relevant deviations denoted by \( \xi_5(S_{obs}) \) in Appendix A.

VI. SIMULATION

To obtain the impact on secure key rate of after-pulse contributions, we take Wei’s protocol as an example to see the comparisons with and without after-pulse effect.

![Graph showing secure key rate vs total transmission loss](image1)

**FIG. 1.** (Color online) Plot of secure key rate \( R \) versus total transmission loss \( t \) with and without after-pulse contributions. The blue solid line denotes the secure key rate under after-pulse impact, and the blue dotted line stands for the secure key rate without after-pulse contributions. Here we take \( N = 6 \times 10^9, \alpha = 5 \).

Figure 1 shows that the secure key rate under after-pulse impact is 75% lower than the one without after-pulse contribution. As a result, the influence of after-pulse on secure key rate can’t be neglected.

Take the after-pulse contribution into consideration, we obtain the secure key rates under different statistical fluctuation analysis methods in figure 2.

And the corresponding deviation comparisons are presented in figure 3.

From the above analysis, we get a clear comparison of the estimation accuracy using different methods. And for different relations of the observed samples, we should choose the proper way to deal with statistical fluctuation problems so as to get a precise result.

![Graph showing deviations vs total transmission loss](image2)

**FIG. 2.** (Color online) Plot of secure key rate \( R \) versus total transmission loss \( t \) under different estimation methods. The red lines denote the respective secure key rates with \( N = 5 \times 10^{10} \), and the blue lines refer to the ones with \( N = 10^{12} \). The abbreviations respectively stands for Law of large numbers (LLN), standard error analysis (SEA), Hoeffding’s inequality (HI), Chernoff bound (CB) and Azuma’s inequality (AI). Taking \( N = N_{\text{total}} \in \{ 5 \times 10^{10}, 10^{12} \} \), \( \varepsilon = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 10^{-10}, \) and \( u_\alpha = 6.4 \).

![Graph showing deviations comparisons vs total transmission loss](image3)

**FIG. 3.** (Color online) Plot of deviations comparisons versus total transmission loss \( t \) under various estimation methods. The blue lines from top to bottom respectively stands for the deviations of Law of large numbers, Azuma’s inequality, Hoeffding’s inequality, Hoeffding’s inequality, Chernoff bound and standard error analysis with \( N = 10^{12} \).

VII. CONCLUSION

In this paper, we give a detailed analysis in dealing with statistical fluctuation issues, based on independent and dependent random samples. We set up respective mathematical models for each estimation method and give a fair comparison standard, together with the corresponding deviation comparisons. For the first time we put forward a stringent way in handling the statistical fluctuations based on dependent samples. From the result of numerical simulation, we conclude that the after-pulse impact can’t be neglected since it greatly lowers the secure key rate. And if the number of the total estimation accuracy of standard error analysis is the best one in the four methods. However, the large deviation theory possesses more generality than standard error analysis. Although the secure key rate using Azuma’s inequality is a little lower than Hoeffdings inequality, its
a strict tool dealing with fluctuation problems when taking after-pulse contributions into consideration, and can be extended in other QKD protocols.

We emphasize that Azuma inequality is a stringent way in dealing with statistical fluctuations based on dependent samples, but not an efficient one. So there is still more room for accuracy improvement in finding other proper ways.

Appendix A: Respective deviations

Deviations under standard error analysis are represented as follows.

\[
\xi_1 (Q_\mu) = \frac{u_\alpha}{\sqrt{N_\mu P_z Q_\mu}} \\
\xi_1 (Q_v) = \frac{u_\alpha}{\sqrt{N_\mu P_z Q_v}} \\
\xi_1 (Y_0') = \frac{u_\alpha}{\sqrt{N_0 Y_0}} \\
\xi_1 (Q_0) = \frac{u_\alpha}{\sqrt{2N_0 Q_0}}
\]

where \(Q_\mu, Q_v, Y_0\) and \(Q_0\) are the observed values instead of probabilities.

Deviations under Law of large numbers are represented as follows.

\[
Q_\mu^U = Q_\mu (1 + \xi_2 (Q_\mu)) = Q_\mu + \sqrt{\frac{2 \ln (\frac{1}{\epsilon}) + 2 \ln (N_\mu P_z + 1)}{N_\mu P_z}} \\
Q_v^L = Q_v (1 - \xi_2 (Q_v)) = Q_v - \sqrt{\frac{2 \ln (\frac{1}{\epsilon}) + 2 \ln (N_\mu P_z + 1)}{N_\mu P_z}} \\
Y_0^L = Y_0' (1 - \xi_2 (Y_0')) = Y_0' - \frac{2 \ln (\frac{1}{\epsilon}) + 2 \ln (N_0 + 1)}{N_0} \\
Q_0^L = Q_0 (1 - \xi_2 (Q_0)) = Q_0 - \frac{2 \ln (\frac{1}{\epsilon}) + 2 \ln (N_0 + 1)}{N_0}
\]

Deviations under Hoeffding’s inequality are represented as follows.

\[
Q_\mu^U = Q_\mu (1 + \xi_3 (Q_\mu)) = Q_\mu + \frac{1}{2 \sqrt{2 N_\mu P_z}} \ln \frac{1}{\epsilon} \\
Q_v^L = Q_v (1 - \xi_3 (Q_v)) = Q_v - \frac{1}{2 \sqrt{2 N_\mu P_z}} \ln \frac{1}{\epsilon} \\
Y_0^L = Y_0' (1 - \xi_3 (Y_0')) = Y_0' - \frac{1}{\sqrt{2 N_0}} \ln \frac{1}{\epsilon} \\
Q_0^L = Q_0 (1 - \xi_3 (Q_0)) = Q_0 - \frac{1}{\sqrt{2 N_0}} \ln \frac{1}{\epsilon}
\]

Appendix B: Chernoff bound

When it comes to Chernoff bound theory, there are three conditions satisfied or not corresponding to six different results.

i. \((2 \varepsilon^{-1})^{1/\text{max}} \leq e^{(3/4 \sqrt{3})^2}\)

ii. \((\varepsilon^{-1})^{1/\text{max}} \leq e^{3/3}\)

iii. \((\varepsilon^{-1})^{1/\text{max}} \leq e^{(2e^{-1})/2^2}\)

Let \(f(x, y) = \sqrt{\frac{2x}{m}} \ln \frac{1}{y}\), then we have the following results.

1. If condition (i) and (ii) are fulfilled, then \(\Delta = f(x, \varepsilon^{3/16})\) and \(\Delta = f\left(x, \varepsilon^{3/2}\right)\).
2. If condition (i) and (iii) are fulfilled, while (ii) not, then \(\Delta = f\left(x, \varepsilon^{3/16}\right)\) and \(\Delta = f\left(x, \varepsilon^{3/2}\right)\).
3. If condition (i) is fulfilled and (iii) not, then \(\Delta = f\left(x, \varepsilon^{3/16}\right)\) and \(\Delta = f\left(m/4, \varepsilon\right)\).
4. If condition (ii) is fulfilled and (i) not, then \(\Delta = f\left(x, \varepsilon^{3/2}\right)\) and \(\Delta = f\left(m/4, \varepsilon\right)\).
5. If condition (i) and (ii) are not fulfilled, while (iii) is, then \(\Delta = f\left(m/4, \varepsilon\right)\) and \(\Delta = f\left(x, \varepsilon^{3/2}\right)\).
6. If condition (i) , (ii) and (iii) are not fulfilled then \(\Delta = f\left(m/4, \varepsilon\right)\).

The other deviation comparison results with Hoeffding’s inequality are stated here, also taking \(Q_\mu\) as an example and \(\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 10^{-10}\).

(a) Deviation comparison between \(\frac{2Q_\mu}{N_\mu P_z} \ln \frac{16}{\varepsilon}\), \(\frac{2Q_v}{N_\mu P_z} \ln \frac{1}{\varepsilon}\) and \(\frac{2Q_\mu}{2N_\mu P_z} \ln \frac{1}{\varepsilon}\), under result (2).
The sufficient condition that Chernoff bound achieves a better result than Hoeffding’s inequality is

\[
\begin{align*}
\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{16}{\varepsilon^2} & \leq \frac{1}{2N_\mu p_\mu} \ln \frac{1}{\varepsilon}, \\
\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{1}{\varepsilon^2} & \leq \frac{1}{2N_\mu p_\mu} \ln \frac{1}{\varepsilon} 
\end{align*}
\] (B1)

This condition can be equivalently written as

\[Q_\mu \leq 0.06.\] (B2)

(b) Deviation comparison between \(\sqrt{\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{16}{\varepsilon^2}}\) and \(\sqrt{\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{1}{\varepsilon^2}}\), under result (3). Obviously, the condition is the same as (a).

(c) Deviation comparison between \(\sqrt{\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{1}{\varepsilon}}\) and \(\sqrt{\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{1}{\varepsilon^2}}\), under result (4).

Here the sufficient condition is

\[\sqrt{\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{1}{\varepsilon}} \leq \sqrt{\frac{1}{2N_\mu p_\mu} \ln \frac{1}{\varepsilon}}\] (B3)

And we have

\[Q_\mu \leq 0.162.\] (B4)

(d) Deviation comparison between \(\sqrt{\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{1}{\varepsilon}}\) and \(\sqrt{\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{1}{\varepsilon^2}}\), under result (5).

Here the sufficient condition is

\[\sqrt{\frac{2Q_\mu}{N_\mu p_\mu} \ln \frac{1}{\varepsilon}} \leq \sqrt{\frac{1}{2N_\mu p_\mu} \ln \frac{1}{\varepsilon}}\] (B5)

And we have

\[Q_\mu \leq 0.122.\] (B6)

(e) Chernoff bound changes its form into Hoeffding’s inequality under result (6).

Next we construct the martingale through analyzing the mutual relationships between dependent samples. If the observed value \(X\) are obtained by detection events of \(m\) dependent random samples denoted by \(X_1, X_2, \cdots, X_m\), detected with the value 1 and 0 otherwise, satisfying \(X = \frac{1}{m} \sum_{i=1}^{m} X_i\). Considering after-pulse contributions we have the following probability relations

\[
\begin{align*}
\Pr \{X_{m+1} = s | X_m = t, X_{n-1} = x_{n-1}, \cdots, X_0 = x_0\} &= \Pr \{X_{m+1} = s | X_m = t\} \\
&\neq \Pr \{X_{m+1} = s\}\] (C2)

(D2) shows that the relationship of the samples satisfies a Markov chain. And take signal pulse as an example, we have \(P \{X_{m+1} = 1\} = Q_{m+1}\) and \(P \{X_{m+1} = 1 | X_m = t\} = D_{m+1}\), here \(Q_m\) and \(D_m\) are the measured values of overall signal state gain with and without considering after-pulse contributions, when the number of observed samples is \(m\), satisfying \(D_m = Q_m (1 + p_{\text{op}})\). Let \(S_n = \sum_{i=1}^{n} X_i = k\), then \(D_{m+1} = \frac{S_{m+1}}{m+1} = \frac{k}{m+1}\). And we have the following relationship.

\[
\begin{align*}
E(S_n | S_n = k) &= k \cdot P(S_{n+1} = k | S_n = k) \\
&\quad + (k+1) \cdot P(S_{n+1} = k+1 | S_n = k) \\
&= k \cdot P(X_{m+1} = 0 | X_m = t) \\
&\quad + (k+1) \cdot P(X_{m+1} = 1 | X_m = t) \\
&= k \cdot (1 - D_{m+1}) + (k+1) \cdot D_{m+1} \\
&= k + D_{m+1} \\
&= S_n + D_{m+1}
\end{align*}
\] (C3)

Let \(M_n = \frac{S_n}{n}, M_0 = E\), here \(E\) is the value of \(\frac{1}{n} \cdot \sum_{i=1}^{n} X_i\) when \(n\) tends to infinity. We have

\[
\begin{align*}
E(M_{n+1} | S_1, \cdots, S_n) &= E(M_{n+1} | S_n) \\
&= E(S_{n+1} | S_n) \\
&= \frac{1}{n+1} E(S_{n+1} | S_n) \\
&= \frac{E(S_n + D_{m+1})}{n+1} \\
&= \frac{S_{n+1}}{n+1} = M_n
\end{align*}
\] (C4)

From the above analysis we can conclude that \(M_n\) is a martingale which can be used in Azuma’s inequality.

**ACKNOWLEDGEMENTS**

This work is supported by the National High Technology Research and Development Program of China Grant No.2011AA010803, the National Natural Science Foundation of China Grants No.61472446 and No.U1204602 and the Open Project Program of the State Key Laboratory of Mathematical Engineering and Advanced Computing Grant No.2013A14.
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