Toeplitz algebras in quantum Hopf fibrations

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Abstract

The paper presents applications of Toeplitz algebras in Noncommutative Geometry. As an example, a quantum Hopf fibration is given by gluing trivial U(1) bundles over quantum discs (or, synonymously, Toeplitz algebras) along their boundaries. The construction yields associated quantum line bundles over the generic Podleś spheres which are isomorphic to those from the well-known Hopf fibration of quantum SU(2). The relation between these two versions of quantum Hopf fibrations is made precise by giving an isomorphism in the category of right U(1)-comodules and left modules over the C*-algebra of the generic Podleś spheres. It is argued that the gluing construction yields a significant simplification of index computations by obtaining elementary projections as representatives of K-theory classes.

1 Introduction

In Noncommutative Geometry, the Toeplitz algebra has a fruitful interpretation as the algebra of continuous function on the quantum disc [10]. In this picture, the description of the Toeplitz algebra as the C*-algebra extension of continuous functions on the circle by the compact operators corresponds to an embedding of the circle into the quantum disc. Analogous to the classical case, one can construct “topologically” non-trivial quantum spaces by taking trivial fibre bundles over two quantum discs and gluing them along their boundaries. Here, the gluing procedure is described by a fibre product in an appropriate category (C*-algebras, finitely generated projective modules, etc.). This approach has been applied successfully to the construction of line bundles over quantum 2-spheres [2, 8, 17] and to the description of quantum Hopf fibrations [1, 3, 7, 9]. One of the advantages of the fibre product approach is that it provides an effective tool for simplifying index computations. This has been discussed in [17] on the example of the Hopf fibration of quantum SU(2) over the generic Podleś spheres [14]. Whilst earlier index computations for quantum 2-spheres relied heavily on the index theorem

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4, 6, the fibre product approach in 17 allowed to compute the index pairing directly by producing simpler representatives of K-theory classes.

The description of quantum line bundles in 17 bears a striking analogy to the classical case: the same transition functions are used to glue the trivial bundles over the (quantum) disc along their boundaries. However, the link between the fibre product approach of quantum line bundles and the Hopf fibration of quantum SU(2) has been established only at a “K-theoretic level”, i. e., it has been shown that the corresponding projective modules are Murray-von Neumann equivalent. The present work will give a more geometrical picture of the quantum Hopf fibration. Analogous to the classical case, we will construct a non-trivial U(1) quantum principal bundle over the generic Podleś spheres such that the associated line bundles are the previously obtained quantum line bundles. Here, a quantum principal bundle is described by a Hopf-Galois extension (see the preliminaries). It turns out that our U(1) quantum principal bundle is isomorphic to a quantum 3-sphere from 3. As an application of the fibre product approach, we will show that the associated quantum line bundles are isomorphic to projective modules given by completely elementary 1-dimensional projections which leads to a significant simplification of index computations.

It is known that the Hopf fibration of quantum SU(2) over the generic Podleś spheres is not given by a Hopf-Galois extension but only by a so-called coalgebra Galois extension (that is, U(1) is only considered as a coalgebra). In the present paper, we will establish a relation between both versions of a quantum Hopf fibration by describing an explicit isomorphism in the category of right U(1)-comodules and left modules over the C*-algebra of the generic Podleś spheres. Clearly, this isomorphism cannot be turned into an algebra isomorphism of quantum 3-spheres since otherwise the Hopf fibration of quantum SU(2) over the generic Podleś spheres would be a Hopf-Galois extension.

2 Preliminaries

2.1 Coalgebras and Hopf algebras

A coalgebra is a vector space $C$ over a field $K$ equipped with two linear maps $\Delta : C \rightarrow C \otimes C$ and $\varepsilon : S \rightarrow K$, called the comultiplication and the counit, respectively, such that

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta, \quad (\varepsilon \otimes \text{id}) \circ \Delta = \text{id} = (\text{id} \otimes \varepsilon) \circ \Delta.$$  \hfill (2.1)  
(2.2)

A (right) corepresentation of a coalgebra $C$ on a $K$-vector space $V$ is a linear mapping $\Delta_V : V \rightarrow V \otimes C$ satisfying

$$(\Delta_V \otimes \text{id}) \circ \Delta_V = (\text{id} \otimes \Delta) \circ \Delta_V, \quad (\text{id} \otimes \varepsilon) \circ \Delta_V = \text{id}.$$  \hfill (2.3)

We then refer to $V$ as a right $C$-comodule. The corepresentation is said to be irreducible if $\{0\}$ and $V$ are the only invariant subspaces. A linear mapping $\phi$ between right $C$-comodules $V$ and $W$ is called colinear, if $\Delta_W \circ \phi = (\phi \otimes \text{id}) \circ \Delta_V$. 

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A Hopf algebra \( A \) is a unital algebra and coalgebra such that \( \Delta \) and \( \varepsilon \) are algebra homomorphism, together with a linear mapping \( \kappa : A \to A \), called the antipode, such that

\[
m \circ (\kappa \otimes \text{id}) \circ \Delta(a) = \varepsilon(a) = m \circ (\text{id} \otimes \kappa) \circ \Delta(a), \quad a \in A,
\]

where \( m : A \otimes A \to A \) denotes the multiplication map.

We say that \( C \) and \( A \) are a \(*\)-coalgebra and a \(*\)-Hopf algebra, respectively, if \( C \) and \( A \) carry an involution such that \( \Delta \) becomes a \(*\)-morphism. This immediately implies that \( \varepsilon(x^*) = \overline{\varepsilon(x)} \). A finite dimensional corepresentation \( \Delta_V : V \to V \otimes A \) is called unitary, if there exists a linear basis \( \{ e_1, \ldots, e_n \} \) of \( V \) such that

\[
\Delta_V(e_i) = \sum_{j=1}^n e_j \otimes v_{ji} \quad \text{and} \quad \sum_{j=1}^n v_{ji}^* v_{ji} = \delta_{ij},
\]

where \( \delta_{ij} \) denotes the Kronecker symbol. The elements \( v_{ij} \) are called matrix coefficients. A Hopf \(*\)-algebra \( A \) is called a compact quantum group algebra if it is the linear span of all matrix coefficients of irreducible finite dimensional unitary corepresentations. It can be shown that then \( A \) admits a \( C^*\)-algebra completion \( H \) in the universal \( C^*\)-norm (that is, the supremum of the norms of all bounded irreducible Hilbert *-representations).

We call \( H \) also a compact quantum group and refer to the dense subalgebra \( A \) as its Peter-Weyl algebra. The counit of \( A \) has then a unique extension to \( H \), and \( \Delta \) has a unique extension to a \(*\)-homomorphism \( \Delta : H \to H \otimes H \), where \( H \otimes H \) denotes the least \( C^*\)-completion of the algebraic tensor product.

The main example in this paper will be \( H = C(S^1) \), the \( C^*\)-algebra of continuous functions on the unit circle \( S^1 \). It is a compact quantum group with comultiplication \( \Delta(f)(p,q) = f(pq) \), counit \( \varepsilon(f) = f(1) \) and antipode \( \kappa(f)(p) = f(p^{-1}) \). Note that \( \Delta, \varepsilon \) and \( \kappa \) are given by pullbacks of the group operations of \( S^1 = U(1) \). Let \( U \in C(S^1), \ U(e^{i\phi}) = e^{i\phi} \), denote the unitary generator of \( C(S^1) \). Then the Peter-Weyl algebra of \( H \) is given by \( O(U(1)) = \text{span}\{ U^N : N \in \mathbb{Z} \} \) with \( \Delta(U^N) = U^N \otimes U^N \), \( \varepsilon(U^N) = 1 \) and \( \kappa(U^N) = U^{-1} \). Note also that the irreducible unitary corepresentations of \( O(U(1)) \) are all 1-dimensional and are given by \( \Delta_C(1) = 1 \otimes U^N \).

From the previous paragraph, it becomes clear why noncommutative compact quantum groups are regarded as generalizations of function algebras on compact groups. We give now the definition for a quantum analogue of principal bundles. First we remark that a group action on a topological space corresponds to a coaction of a quantum group or, more generally, to a coaction of a coalgebra. Now let \( A \) be a Hopf algebra, \( P \) a unital algebra, and \( \Delta_P : P \to P \otimes A \) a corepresentation which is also an algebra homomorphism (one says that \( P \) is a right \( A \)-comodule algebra). Then the space of coinvariants

\[
P^{\text{co}A} := \{ b \in P : \Delta(b) = b \otimes 1 \}
\]

is an algebra considered as a function algebra on the base space, and \( P \) plays the role of a function algebra on the total space. If \( A \) is a Hopf \(*\)-algebra and \( P \) is a \(*\)-algebra, we require \( \Delta \) to be a \(*\)-homomorphism so that \( B \) becomes a unital \(*\)-subalgebra of \( P \).

If \( \Delta : P \to P \otimes C \) is a corepresentation of a coalgebra \( C \), then we set

\[
P^{\text{co}C} := \{ b \in P : \Delta(bp) = b \Delta(p) \text{ for all } p \in P \}
\]

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with multiplication $b(p \otimes c) = bp \otimes c$ on the left tensor factor. Again, $P^{\text{co}C}$ is a subalgebra of $P$. In our examples, there will be a group like element $e \in C$ (that is, $\Delta(e) = e \otimes e$) such that $\Delta(1) = 1 \otimes e$ and

$$P^{\text{co}C} = B := \{ b \in P : \Delta(b) = b \otimes e \}.$$ 

If $P$ and $C$ carry an involution, $\Delta_P$ is a $*$-morphism and $e^* = e$, then $B$ is a $*$-subalgebra of $P$.

Analogous to right corepresentations, one defines left corepresentations $\nu \Delta : V \to C \otimes V$. The associated (quantum) vector bundles are given by the cotensor product $P \boxdot_C V$, where

$$P \boxdot_C V := \{ x \in P \otimes V : (\Delta_P \otimes \text{id})(x) = (\text{id} \otimes \nu \Delta)(x) \}.$$ 

Obviously, $P \boxdot_C V$ is a left $P^{\text{co}C}$-module. For the 1-dimensional representation $\varepsilon(1) = U^N \otimes 1$, this module is equivalent to

$$P_N := \{ p \in P : \Delta_P(p) = p \otimes U^N \}$$

and is considered as a (quantum) line bundle.

### 2.2 Pullback diagrams and fibre products

The purpose of this section is to collect some elementary facts about fibre products. For simplicity, we start by considering the category of vector spaces. Let $\pi_0 : A_0 \to A_{01}$ and $\pi_1 : A_1 \to A_{01}$ be vector spaces morphisms. Then the fibre product $A := A_0 \times_{(\pi_0, \pi_1)} A_1$ is defined by the pullback diagram

$$A \xrightarrow{pr_1} A_1 \xleftarrow{pr_0} A_0 \xrightarrow{\pi_1} A_{01}.$$  \hspace{1cm} (2.5)

Up to a unique isomorphism, $A$ is given by

$$A = \{ (a_0, a_1) \in A_0 \times A_1 : \pi_0(a_0) = \pi_1(a_1) \},$$  \hspace{1cm} (2.6)

where the morphisms $pr_0 : A \to A_0$ and $pr_1 : A \to A_1$ are the left and right projections, respectively. In this paper, we will consider fibre products in the following categories:

- If $\pi_0 : A_0 \to A_{01}$ and $\pi_1 : A_1 \to A_{01}$ are morphisms of $*$-algebras, then the fibre product $A_0 \times_{(\pi_0, \pi_1)} A_1$ is a $*$-algebra with componentwise multiplication and involution.

- If we consider the pullback diagram (2.5) in the category of unital $C^*$-algebras, then $A_0 \times_{(\pi_0, \pi_1)} A_1$ will be a unital $C^*$-algebra.
• If \( B \) is an algebra and \( \pi_0 : A_0 \rightarrow A_{01} \) and \( \pi_1 : A_1 \rightarrow A_{01} \) are morphisms of left \( B \)-modules, then the fibre product \( A := A_0 \times_{(\pi_0, \pi_1)} A_1 \) is a left \( B \)-module with left action \( b.(a_0, a_1) = (b.a_0, b.a_1) \), where \( b \in B \) and the dot denotes the left action.

• If we consider the pullback diagram in the category of right \( C \)-comodules (or right \( H \)-comodule algebras), then \( A := A_0 \times_{(\pi_0, \pi_1)} A_1 \) will be a right \( C \)-comodule (or a right \( H \)-comodule algebra) with the coaction given by \( \Delta_A(a_1, a_2) = (\Delta_A_1(a_1), 0) + (0, \Delta_A_2(a_2)) \).

Finally we remark that if \( B_0, B_1 \) and \( B_{01} \) are dense subalgebras of \( C^* \)-algebras \( A_0, A_1 \) and \( A_{01} \), respectively, and \( \pi_0 \) and \( \pi_1 \) restrict to morphisms \( \pi_0 : B_0 \rightarrow B_{01} \) and \( \pi_1 : B_1 \rightarrow B_{01} \), then \( B_0 \times_{(\pi_0, \pi_1)} B_1 \) is not necessarily dense in \( A_0 \times_{(\pi_0, \pi_1)} A_1 \). A useful criterion for this to happen can be found in [9, Theorem 1.1]. It suffices that \( \pi_1 |_{B_1} : B_1 \rightarrow B_{01} \) is surjective and \( \ker(\pi_1) \cap B_1 \) is dense in \( \ker(\pi_1) \).

2.3 Disc-type quantum 2-spheres

From now on we will work over the complex numbers and \( q \) will denote a real number from the interval \((0,1)\).

The \(*\)-algebra \( \mathcal{O}(D^2_q) \) of polynomial functions on the quantum disc is generated by two generators \( z \) and \( z^* \) with relation

\[
\text{zz}^* - qzz^* = 1 - q. \tag{2.7}
\]

A complete list of bounded irreducible \(*\)-representations of \( \mathcal{O}(D^2_q) \) can be found in [10]. First, there is a faithful representation on the Hilbert space \( \ell^2(N_0) \). On an orthonormal basis \( \{e_n : n \in N_0\} \), the action of the generators reads as

\[
\begin{align*}
ze_n &= \sqrt{1-q^{n+1}}Se_n, \\
\text{z}^*e_n &= \sqrt{1-q^n}S^*e_n,
\end{align*} \tag{2.8}
\]

where

\[
Se_n = e_{n+1},
\]

denotes the shift operator on \( \ell^2(N_0) \).

Next, there is a 1-parameter family of irreducible \(*\)-representations \( \rho_u \) on \( C \), where \( u \in S^1 = \{x \in \mathbb{C} : |x| = 1\} \). They are given by assigning

\[
\rho_u(z) = u, \quad \rho_u(z^*) = \bar{u}.
\]

The set of these representations is considered as the boundary \( S^1 \) of the quantum disc consisting of "classical points".

The universal \( C^* \)-algebra of \( \mathcal{O}(D^2_q) \) is well known. It has been discussed by several authors (see, e.g., [10, 12, 16]) that it is isomorphic to the Toeplitz algebra \( \mathcal{T} \). Here, it is convenient to view the Toeplitz algebra \( \mathcal{T} \) as the universal \( C^* \)-algebra generated by \( S \) and \( S^* \) in \( B(\ell^2(N_0)) \). Then above \(*\)-representation on \( \ell^2(N_0) \) becomes simply an embedding.

Another characterization is given by the \( C^* \)-extension

\[
0 \longrightarrow \mathcal{K}(\ell^2(N_0)) \longrightarrow \mathcal{T} \overset{\sigma}{\longrightarrow} C(S^1) \longrightarrow 0,
\]
where $\sigma : T \to C(S^1)$ is the so-called symbol map and corresponds, in the classical case, to an embedding of $S^1$ into the complex unit disc. Let again $U(e^{i\phi}) = e^{i\phi}$ denote the unitary generator of $C(S^1)$. Then the symbol map is completely determined by setting $\sigma(z) = U$. We can now construct a quantum 2-sphere $C(S^2_q)$ by gluing two quantum discs along their boundaries. The gluing procedure is described by the fibre product $T \times_{(\sigma, \sigma)} T$, where $T \times_{(\sigma, \sigma)} T$ is defined by the following pullback diagram in the category of C*-algebras:

\[
\begin{array}{ccc}
T \times_{(\sigma, \sigma)} T & \xrightarrow{pr_1} & T \\
\downarrow \sigma & & \downarrow \\
T & \xrightarrow{\sigma} & C(S^1) \\
\end{array}
\]  
(2.9)

Up to isomorphism, the C*-algebra $C(S^2_q) := T \times_{(\sigma, \sigma)} T$ is given by

\[
C(S^2_q) = \{(a_1, a_2) \in T \times T : \sigma(a_1) = \sigma(a_2)\}.  
\]  
(2.10)

In the classical case, complex line bundles with winding number $N \in \mathbb{Z}$ over the 2-sphere can be constructed by taking trivial bundles over the northern and southern hemispheres and gluing them together along the boundary via the map $U^N : S^1 \to S^1$, $U^N(e^{i\phi}) = e^{i\phi N}$. In [17], the same construction has been applied to the quantum 2-sphere $C(S^2_q)$. The roles of the northern and southern hemispheres are played by two copies of the quantum disc, and the transition function along the boundaries remains the same. This construction can be expressed by the following pullback diagram:

\[
\begin{array}{ccc}
T \times_{(U^N \sigma, \sigma)} T & \xrightarrow{pr_1} & T \\
\downarrow \sigma & & \downarrow \\
C(S^1) & \xrightarrow{f \mapsto U^N f} & C(S^1). \\
\end{array}
\]  
(2.11)

So, up to isomorphism, we have

\[
T \times_{(U^N \sigma, \sigma)} T \cong \{(a_0, a_1) \in T \times T : U^N \sigma(a_0) = \sigma(a_1)\}.  
\]  
(2.12)

It follows directly from Equation (2.11) that $T \times_{(U^N \sigma, \sigma)} T$ is a $C(S^2_q)$-(bi)module. This can also be seen from the general pullback construction by equipping $T$ and $C(S^1)$ with the structure of a left $C(S^2_q)$-module. Explicitly, for $(a_0, a_1) \in C(S^2_q)$, one defines $(a_0, a_1).a = a_0 a$ for $a \in T$ on the left side, $(a_0, a_1).a = a_1 a$ for $a \in T$ on the right side, and $(a_0, a_1).a = \sigma(a_0) b = \sigma(a_1) b$ for $b \in C(S^1)$.

To determine the K-theory and K-homology of $C(S^2_q)$, we may use the results of [12]. There it is shown that $K_0(C(S^2_q)) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $K^0(C(S^2_q)) \cong \mathbb{Z} \oplus \mathbb{Z}$. The two
generators of the $K_0$-group can be chosen to be the class $[1]$ of the unit element of $C(S^3_2)$, and the class $[(0, 1 - SS^*)]$.

Describing an even Fredholm module by a pair of representations on the same Hilbert space such that the difference is a compact operator, one generator of $K^0(C(S^3_2))$ is obviously given by the class $[(pr_1, pr_0)]$ on the Hilbert space $\ell^2(N_0)$. A second generator is $[(\pi_+ \circ \sigma, \pi_- \circ \sigma)]$, where $\sigma$ denotes the symbol map and $\pi_{\pm} : C(S^3) \to B(\ell^2(\mathbb{Z}))$ is given by

$$
\begin{align*}
\pi_+(U)e_n &= e_{n+1}, \quad n \in \mathbb{Z}, \\
\pi_-(U)e_n &= e_{n+1}, \quad n \in \mathbb{Z}\setminus\{-1, 0\}, \\
\pi_-(U)e_{-1} &= e_1, \quad \pi_-(U)e_0 &= 0
\end{align*}
$$

(2.13)
on an orthonormal basis $\{e_n : n \in \mathbb{Z}\}$ of $\ell^2(\mathbb{Z})$. Note that the representation $\pi_-$ is non-unital: $\pi_-(1)$ is the projection onto span$\{e_n : n \in \mathbb{Z}\setminus\{0\}\}$.

### 2.4 Quantum 3-spheres and quantum Hopf fibrations

First we follow [1] and introduce the coordinate ring of a Heegaard-type quantum 3-sphere $O(S^3_{pq})$, $p, q \in (0, 1)$ as the $*$-algebra generated by $a, a^*, b, b^*$ subjected to the relations

$$
\begin{align*}
a^*a - qaa^* &= 1 - q, \\
b^*b - pbb^* &= 1 - p, \\
(1 - aa^*)(1 - bb^*) &= 0, \\
ab &= ba, \\
a^*b &= ba^*.
\end{align*}
$$

(2.14)
Its universal C*-algebra (i.e., the closure of $O(S^3_{pq})$ in the universal C*-norm given by the supremum over all bounded Hilbert space representations) will be denoted by $C(S^3_{pq})$.

One can easily verify that the coaction $\Delta_{O(S^3_{pq})} : O(S^3_{pq}) \to O(S^3_{pq}) \otimes O(U(1))$ given by

$$
\Delta_{O(S^3_{pq})}(a) = a \otimes U^*, \quad \Delta_{O(S^3_{pq})}(b) = b \otimes U
$$
turns $O(S^3_{pq})$ into a $O(U(1))$-comodule $*$-algebra. Its $*$-subalgebra of $O(U(1))$-coinvariants $O(S^3_2) := O(S^3_{pq})^{coO(U(1))}$ is generated by

$$
A := 1 - aa^*, \quad B := 1 - bb^*, \quad R := ab
$$
with involution $A^* = A$, $B^* = B$ and commutation relations

$$
R^*R = 1 - qA - pB, \quad RR^* = 1 - A - B, \quad AR = qRA, \quad BR = pRB, \quad AB = 0.
$$

Note that $O(S^3_{pq})$ can also be considered as a $*$-subalgebra of $C(S^3_2)$ from (2.10) by setting

$$
A = (1 - zz^*, 0), \quad B = (0, 1 - yy^*), \quad R = (z, y),
$$
where $y$ and $z$ denote the generators of the quantum discs $O(D^2_p)$ and $O(D^2_q)$, respectively, satisfying the defining relation (2.7). Using the fact that $O(D^2_q)$ is dense in the Toeplitz algebra $T$ for all $q \in (0, 1)$, and the final remark of Section 2.2 one easily proves that $C(S^3_2) = T \times_{(\sigma, \sigma)} T$ is the universal C*-algebra of $O(S^3_{pq})$.

For $N \in \mathbb{Z}$, let

$$
L_N := \{p \in O(S^3_{pq}) : \Delta_{O(S^3_{pq})}(p) = p \otimes U^N\}
$$

(2.15)
denote the associated quantum line bundles. It has been shown in [7] that $L_N$ is isomorphic to $O(S^2_{pq})^{\mathbb{N}+1}E_N$, where

$$E_N = X_N Y_N^\ast \in \text{Mat}_{\mathbb{N}+1,\mathbb{N}+1}(O(S^2_{pq}))$$

(2.16)

and, for $n \in \mathbb{N}$,

$$X_n = (b^n, a^{n+1}, \ldots, a^n)^t, \quad X_n = (a^n, b^{n+1}, \ldots, b^n)^t,$$

$$Y_n = \left(\binom{n}{0}\right)_n p^n a^n, \left(\binom{n}{1}\right)_n p^{n-1} a^{n-1} b^n, \ldots, \left(\binom{n}{n}\right)_n a^n,$$

$$Y_n = \left(\binom{n}{0}\right)_q q^n a^n, \left(\binom{n}{1}\right)_q q^{n-1} B^n a^{n+1} b^n, \ldots, \left(\binom{n}{n}\right)_q b^n,$$

with

$$\binom{n}{0} = \binom{n}{n}_x := 1, \quad \binom{n}{k}_x := \binom{(1-x)(1-x^n)}{(1-x)(1-x^n)(1-x^n)}, \quad 0 < k < n, \quad x \in (0, 1).$$

That $E_N$ is indeed an idempotent follows from $Y_N^2 X_N = 1$ which can be verified by direct computations.

Now we consider a much more prominent example of a quantum Hopf fibration. The *-algebra $O(SU_q(2))$ of polynomial functions on the quantum group $SU_q(2)$ is generated by $\alpha, \beta, \gamma, \delta$ with relations

$$\alpha \beta = q \beta \alpha, \quad \alpha \gamma = q \gamma \alpha, \quad \beta \delta = q \delta \beta, \quad \gamma \delta = q \delta \gamma, \quad \beta \gamma = \gamma \beta,$$

$$\alpha \delta - q \beta \gamma = 1, \quad \delta \alpha - q^{-1} \beta \delta \gamma = 1,$$

and involution $\alpha^* = \delta, \beta^* = -q \gamma$. This is actually a Hopf *-algebra with the Hopf structure $\Delta, \varepsilon, \kappa$. Here, we will only need explicit formulas for the homomorphism $\varepsilon : O(SU_q(2)) \to \mathbb{C}$ given by

$$\varepsilon(\alpha) = \varepsilon(\delta) = 1, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0.$$

For $s \in (0, 1]$, the *-subalgebra generated by

$$\eta_s := (\delta + q^{-1} s \beta)(\beta - s \delta), \quad \zeta_s := 1 - (\alpha - q s \gamma)(\delta + s \beta).$$

is known as the generic Podleś sphere $O(S^2_{qs})$ [14]. Its generators satisfy the defining relations

$$\zeta_s \eta_s = q^2 \eta_s \zeta_s, \quad \eta_s^2 \eta_s = (1 - \zeta_s)(s^2 + \zeta_s), \quad \eta_s^2 \eta_s^2 = (1 - q^{-2} \zeta_s)(s^2 + q^{-2} \zeta_s),$$

and $\zeta_s^* = \zeta_s$. For all $s \in (0, 1)$ and $q \in (0, 1)$, the universal C*-algebra of $O(S^2_{qs})$ is isomorphic to $C(S^2_q)$ [12 16]. With $x$ the generator of $O(D^2_q)$, set $t := 1 - xx^* \in T$. An embedding of $O(S^2_{qs})$ into $C(S^2_q)$ as a dense *-subalgebra is given by

$$\zeta_s = (-s^2 q^2 t, q^2 t), \quad \eta_s = \left( s \sqrt{1 - q^2 t}(1 + s^2 q^2 t) S, \sqrt{1 - q^2 t}(s^2 + q^2 t) S \right).$$

(2.17)

Let $O(S^2_{qs})^\ast := \{ x \in O(S^2_{qs}) : \varepsilon(x) = 0 \}$. It has been shown in [13] that the quotient space $O(SU_q(2))/O(S^2_{qs})^\ast O(SU_q(2))$ with coaction $(pr_s \otimes pr_s) \circ \Delta$ is a
coalgebra isomorphic to $\mathcal{O}(U(1))$. Here $\text{pr}_x$ denotes the canonical projection and $\Delta$ the coaction of $\mathcal{O}(SU_q(2))$. We emphasize that this isomorphism holds only in the category of coalgebras, that is, $\mathcal{O}(SU_q(2))/\mathcal{O}(S^2_{qs}) \cong \mathcal{O}(SU_q(2))$ is a linear space (not an algebra!) spanned by basis elements $U^N$, $N \in \mathbb{Z}$, with coaction $\Delta(U^N) = U^N \otimes U^N$. The composition $(\text{id} \otimes \text{pr}_x) \circ \Delta$ turns $\mathcal{O}(SU_q(2))$ into an $\mathcal{O}(U(1))$-comodule and the associated line bundles are given by

$$M_N := \{ p \in \mathcal{O}(SU_q(2)) : (\text{id} \otimes \text{pr}_x) \circ \Delta(p) = p \otimes U^N \}, \quad N \in \mathbb{Z}.$$  

Moreover, $\mathcal{O}(S^2_{qs}) = M_0 = \mathcal{O}(SU_q(2))^{\text{co} \mathcal{O}(U(1))}$ and $\mathcal{O}(SU_q(2)) = \oplus_{N \in \mathbb{Z}} M_N$. In contrast to quantum line bundles $L_N$ defined above, $M_N$ is only a left $\mathcal{O}(S^2_{qs})$-module but not a bimodule. This is also due to the fact that $\mathcal{O}(SU_q(2))$ with above coaction is only an $\mathcal{O}(U(1))$-comodule but not an $\mathcal{O}(U(1))$-comodule algebra.

Explicit descriptions of idempotents representing $M_N$ have been given in [6, 15]. Analogous to $L_N$, there are elements $v^N_0, v^N_1, \ldots, v^N_{|N|} \in \mathcal{O}(SU_q(2))$ such that $M_N \cong \mathcal{O}(S^2_{qs})^{\mid N\rangle + 1} P_N$, where

$$P_N := (v^N_0, v^N_1, \ldots, v^N_{|N|})^t (v^N_0, v^N_1, \ldots, v^N_{|N|}) \in \text{Mat}_{\mid N\rangle + 1,\mid N\rangle + 1}(\mathcal{O}(S^2_{qs}))$$

(2.18) with

$$(v^N_0, v^N_1, \ldots, v^N_{|N|}) (v^N_0, v^N_1, \ldots, v^N_{|N|})^t = 1.$$  

(2.19)

For a definition of $v^N_N$, see [15].

A description of the universal C*-algebra $\mathcal{C}(SU_q(2))$ of $\mathcal{O}(SU_q(2))$ as a fibre product can be found in [9]. There it is shown that $\mathcal{C}(SU_q(2))$ is isomorphic to the fibre product C*-algebra of the following pullback diagram:

$$\begin{array}{ccc}
\mathcal{T} \otimes \mathcal{C}(S^1) & \xrightarrow{\chi} & \mathcal{C}(S^1) \\
pr_1 \downarrow & & \downarrow \text{id} \\
\mathcal{T} \otimes \mathcal{C}(S^1) & \xrightarrow{\sigma \otimes \text{id}} & \mathcal{C}(S^1) \\
pr_2 \downarrow & & \downarrow \pi_2 \\
\mathcal{C}(S^1) \otimes \mathcal{C}(S^1) & \xrightarrow{W} & \mathcal{C}(S^1) \otimes \mathcal{C}(S^1).
\end{array}$$

(2.20)

Here, $\pi_2 : \mathcal{C}(S^1) \to \mathcal{C}(S^1) \otimes \mathcal{C}(S^1)$ is defined by $\pi_2(f)(x, y) = f(y)$, and

$$W : \mathcal{C}(S^1) \otimes \mathcal{C}(S^1) \to \mathcal{C}(S^1) \otimes \mathcal{C}(S^1), \quad (f(x, y) = f(x, xy),$$

(2.21)

is the so-called multiplicative unitary. In the next section, we will frequently use that $W(g \otimes U^N)(x, y) = g(x)x^N y^N = (g U^N \otimes U^N)(x, y)$, that is,

$$W(g \otimes U^N) = g U^N \otimes U^N$$

(2.22)

for all $g \in \mathcal{C}(S^1)$ and $N \in \mathbb{Z}$. As above, $U$ denotes the unitary generator of $\mathcal{C}(S^1)$ given by $U(e^{i\phi}) = e^{i\phi}$ for $e^{i\phi} \in S^1$. 

9
3 Fibre product approach to quantum Hopf fibrations

3.1 C*-algebraic construction of a quantum Hopf fibration

The aim of this section is to construct a U(1) quantum principal bundle over a quantum 2-sphere such that the associated quantum line bundles are given by (2.11). Our strategy will be to start with trivial U(1)-bundles over two quantum discs and to glue them together along their boundaries by a non-trivial transition function. Working in the category of C*-algebras, an obvious quantum analogue of a trivial bundle $D \times S^1$ is given by the completed tensor product $T \bar{\otimes} C(S^1)$, where we regard $T$ as the algebra of continuous functions on the quantum disc.

Since $C(S^1)$ is nuclear, there is no ambiguity about the tensor product completion. Recall from Section 2.1 that a group action on a principal bundle gets translated to a Hopf algebra coaction (or, slightly weaker, coalgebra coaction). As our group is $U(1) = S^1$, we take the Hopf *-algebra $C(S^1)$ introduced in Section 2.1.

On the trivial bundle $T \bar{\otimes} C(S^1)$, we consider the “trivial” coaction given by applying the coproduct of $C(S^1)$ to the second tensor factor. The gluing of the trivial bundles $T \bar{\otimes} C(S^1)$ will be accomplished by a fibre product over the “boundary” $C(S^1) \bar{\otimes} C(S^1)$. To obtain a non-trivial fibre bundle, we impose a non-trivial transition function. From the requirement that the associated quantum line bundles should be given by (2.11), the transition function is easily guessed: We use the multiplicative unitary $W$ from (2.21). The result is described by the following pullback diagram.

$$
\begin{align*}
T \bar{\otimes} C(S^1) 
\times_{(W \circ \pi_1, \pi_2)} T \bar{\otimes} C(S^1) \\
\downarrow_{\pi_1 := \sigma \bar{\otimes} \text{id}} & \quad \quad \quad \quad \quad \downarrow_{\pi_2 := \sigma \bar{\otimes} \text{id}} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \qm
Proof. Let $z$ and $y$ be the generators of the quantum discs $\mathcal{O}(D_q^2)$ and $\mathcal{O}(D_q^3)$, respectively. Consider the $\ast$-algebra homomorphism $\iota : \mathcal{O}(S_{pq}^3) \to C(S_q^3)$ given by
\[ \iota(a) = (z \otimes U^*, 1 \otimes U^*), \quad \iota(b) = (1 \otimes U, y \otimes U). \]

Choosing a Poincaré-Birkhoff-Witt basis of $\mathcal{O}(S_{pq}^3)$, for instance all ordered polynomials in $a, a^*, b, b^*$, and using the embedding $\mathcal{O}(D_q^2) \subset \mathcal{T}$, one easily verifies that $\iota$ is injective. Moreover, since the operators $\pi(a)$ and $\pi(b)$ satisfy the quantum disc relation (3.3), for any bounded representation $\pi$, the $\ast$-representation $\iota$ is actually an isometry if we equip $\mathcal{O}(S_{pq}^3)$ with the universal $\ast$-norm. Therefore it suffices to prove that $\iota(\mathcal{O}(S_{pq}^3))$ is dense in $C(S_q^3)$. For this, consider the image of $\iota(\mathcal{O}(S_{pq}^3))$ under the projections $\text{pr}_1$ and $\text{pr}_2$. Since $1 \otimes U = \text{pr}_1(\iota(b)) = \text{pr}_1(\iota(\mathcal{O}(S_{pq}^3)))$ and $z \otimes 1 = \text{pr}_1(\iota(ab)) \in \text{pr}_1(\iota(\mathcal{O}(S_{pq}^3)))$, we get $\text{pr}_1(\iota(\mathcal{O}(S_{pq}^3))) = \mathcal{O}(D_q^2) \otimes \mathcal{O}(U(1))$, and similarly $\text{pr}_2(\iota(\mathcal{O}(S_{pq}^3))) = \mathcal{O}(D_q^3) \otimes \mathcal{O}(U(1))$. Note that the latter is a dense $\ast$-subalgebra of $\mathcal{T} \otimes C(S_q^3)$. Moreover, $(\sigma \otimes \text{id})(\mathcal{O}(D_q^2) \otimes \mathcal{O}(U(1))) = \mathcal{O}(U(1)) \otimes \mathcal{O}(U(1))$ is dense in $C(S_q^3) \otimes C(S_q^3)$, and $W : \mathcal{O}(U(1)) \otimes \mathcal{O}(U(1)) \to \mathcal{O}(U(1)) \otimes \mathcal{O}(U(1))$ is an isometry. Since $W(U^n \otimes U^m) = U^{n+m} \otimes U^m$ for all $n, m \in \mathbb{Z}$ by (2.22), it is a bijection of $\mathcal{O}(U(1)) \otimes \mathcal{O}(U(1))$ onto itself. From the foregoing, it follows that $\iota(\mathcal{O}(S_{pq}^3)) = \mathcal{O}(D_q^2) \otimes \mathcal{O}(U(1)) \times (\sigma \otimes \text{id})_{\mathcal{T}}(\mathcal{O}(D_q^2) \otimes \mathcal{O}(U(1)))$. By considering the ideal generated by the compact operator $1 - zz^* \in \mathcal{O}(D_q^3)$ (or $1 - yy \in \mathcal{O}(D_q^2)$), one easily shows that $\ker(\sigma \otimes \text{id}) \cap (\mathcal{O}(D_q^2) \otimes \mathcal{O}(U(1)))$ is dense in $\ker(\sigma \otimes \text{id})$. From the final remark in Section 2.2 we conclude that $\iota(\mathcal{O}(S_{pq}^3))$ is dense in $C(S_q^3)$. To determine $C(S_q^3)_N$, recall that the coaction is given by the coproduct on the second tensor factor $C(S_q^3)$. Assume that $f \in C(S_q^3)$ satisfies $\Delta(f) = f \otimes U^N$. Then it follows from $f = (\varepsilon \otimes \text{id}) \circ \Delta(f) = f(1) U^N$ that $\Delta(\mathcal{O}(D_q^2) \otimes \mathcal{O}(U(1))) = \mathcal{O}(D_q^2) \otimes \mathcal{O}(U(1)) \otimes \mathcal{O}(U(1))$ if and only if $x = t \otimes U^N$ with $t \in \mathcal{T}$. Since the morphisms in the pullback diagram (3.3) are right colinear, we get $p \in C(S_q^3)_N$ if and only if $p = (t_1 \otimes U^N, t_2 \otimes U^N)$ and $(W \circ \sigma \otimes \text{id})(t_1 \otimes U^N) = (\sigma \otimes \text{id})(t_2 \otimes U^N)$. By (2.22), $W(\sigma(t_1) \otimes U^N) = \sigma(t_1) U^N \otimes U^N$. Therefore $(t_1 \otimes U^N, t_2 \otimes U^N) \in C(S_q^3)_N$ if and only if $\sigma(t_1) U^N = \sigma(t_2)$. This shows that an isomorphism between $C(S_q^3)_N$ and $\mathcal{T} \times_{(U^N, \sigma, \sigma)} \mathcal{T}$ is given by
\[ C(S_q^3)_N \ni (t_1 \otimes U^N, t_2 \otimes U^N) \mapsto (t_1, t_2) \in \mathcal{T} \times_{(U^N, \sigma, \sigma)} \mathcal{T}. \]

From (3.3) and $\Delta(U^N) = U^N \otimes U^N$, it follows that $\Delta C(S_q^3)(\iota(a)) = \iota(a) \otimes U^*$ and $\Delta C(S_q^3)(\iota(b)) = \iota(b) \otimes U$. Hence $\iota$ is right colinear. Since $\iota$ is also an isometry, we can view $\mathcal{O}(S_{pq}^3)$ as a subalgebra of $C(S_q^3)$. Then $L_N \subset C(S_q^3)_N$ by the definitions of $L_N$ and $C(S_q^3)_N$ in (2.15) and (3.2), respectively. \hfill \Box

We remark that the universal $\ast$-algebra of $\mathcal{O}(S_{pq}^3)$ has been studied in [7], the $\text{K}$-theory of $C(S_q^3)$ has been determined in [1]; and from the last example in [5], it follows that $C(S_q^3)$ behaves well under the $C(S_q^3)$-coaction (it is a principal Hopf-Galois extension).

### 3.2 Index computation for quantum line bundles

The aim of this section is to illustrate that the fibre product approach may lead to a significant simplification of index computations. First we remark that, in
(algebraic) quantum group theory, algebras are frequently defined by generators and relations similar those in (2.14) for \( \mathcal{O}(S^3_{pq}) \) (more examples can be found, e.g., in [11]). A pair of *-representations on the same Hilbert space such that the difference yields compact operators gives rise to an even Fredholm module and can be used for index computations by pairing it with \( K_0 \)-classes. If we want to compute for instance the index pairing with the \( K_0 \)-class of the projective modules \( L_N \) from (2.15) by using the idempotents given in (2.10), then we face difficulties because of the growing size of the matrices. It is therefore desirable to find simpler representatives of K-theory classes of the projective modules \( L_N \). This section shows that the fibre product approach provides us with an effective tool for obtaining more suitable projections. In our example, the index pairing will reduce to its simplest possible form: it remains to calculate a trace of a projection onto a finite dimensional subspace.

We start by proving that the projective modules \( \mathcal{C}(S^3_{pq})_N \) can be represented by elementary 1-dimensional projections. Because of the isomorphism between \( \mathcal{C}(S^3_{pq})_N \) and \( \mathcal{T} \times_{(\mathcal{U}\cup\sigma,\mathcal{O})} \mathcal{T} \) in Proposition 3.1 this result has already been obtained in [17]. For the convenience of the reader, we include here the proof. It uses essentially the same “bra-ket” argument that was used in [6, 15] to prove \( M_N \cong \mathcal{O}(S^2_{pq})^{N+1}P_N \) for the Hopf fibration of \( \mathcal{O}(\mathrm{SU}_q(2)) \).

**Proposition 3.2.** For \( N \in \mathbb{Z} \), define

\[
\chi_N := (1, S^{|N|}S^{|N|}) \in \mathcal{C}(S^2_q), \quad \text{for } N < 0, \tag{3.5}
\]

\[
\chi_N := (S^N, S^N, 1) \in \mathcal{C}(S^2_q), \quad \text{for } N \geq 0. \tag{3.6}
\]

Then the left \( \mathcal{C}(S^3_{pq})_N \) and \( \mathcal{C}(S^2_q)\chi_N \) are isomorphic.

**Proof.** Since \( \sigma(S^nS^m) = U^nU^m = 1 \) for all \( n \in \mathbb{N} \), the projections \( \chi_N \) belong to \( \mathcal{C}(S^2_q) = \mathcal{T} \times_{(\mathcal{U},\mathcal{O})} \mathcal{T} \). We will use the isomorphism of Proposition 3.1 and prove that \( \mathcal{C}(S^2_q)\chi_N \) is isomorphic to \( \mathcal{E}_N := \mathcal{T} \times_{(U\cup\sigma,\mathcal{O})} \mathcal{T} \).

Let \( N \geq 0 \). From (2.10) and (2.12), it follows that \((fS^N, g) \in \mathcal{C}(S^2_q)\) for all \((f, g) \in \mathcal{E}_N\). Therefore we can define a \( \mathcal{C}(S^2_q) \)-linear map \( \Psi_N : \mathcal{E}_N \to \mathcal{C}(S^2_q)\chi_N \) by

\[
\Psi_N(f,g) := (fS^N, g)\chi_N = (fS^N, g), \tag{3.7}
\]

where we used \( S^*S = id \) in the second equality. Since \( S^* \) is right invertible, we have \((fS^N, g) = 0 \) if and only if \((f, g) = 0 \), hence \( \Psi_N \) is injective.

Now let \((f,g)\chi_N \in \mathcal{C}(S^2_q)\chi_N \). Then one has \((fS^N, g) \in \mathcal{E}_N \) and \( \Psi_N(fS^N, g) = (fS^N, g)\chi_N \), thus \( \Psi_N \) is also surjective. This proves the claim of Proposition 3.2 for \( N \geq 0 \). The proof for \( N < 0 \) runs analogously with \( \Psi_N \) defined by \( \Psi_N(f,g) := (f, gS^N)\chi_N \).

Clearly, the (left) multiplication by elements of the \( \mathcal{C}^* \)-algebra \( \mathcal{C}(S^2_q) \) turns \( L_N \cong \mathcal{O}(S^2_q)^{|N|+1}E_N \) into a (left) \( \mathcal{C}(S^2_q) \)-module. With a slight abuse of notation, we set \( \mathcal{C}(S^2_q)L_N := \langle xv : x \in \mathcal{C}(S^2_q), v \in L_N \rangle \). (Later it turns out that this module is generated by one element in \( L_N \) so that the notation is actually correct.) If we show that \( \mathcal{C}(S^3_{pq})L_N \) is isomorphic to \( \mathcal{C}(S^3_{pq})_N \), then the elementary projections \( \chi_N \) and the \((|N|+1) \times (|N|+1)\)-matrices \( E_N \) define the same \( K_0 \)-class.
The desired isomorphism will be established in the next proposition by using the embedding \( L_N \subset C(S_q^3)_N \) from Proposition 3.1.

**Proposition 3.3.** The left \( C(S_q^2) \)-modules \( C(S_q^2)L_N \cong C(S_q^2)^{N+1}E_N \) and \( C(S_{pq}^3)_N \) are isomorphic.

**Proof.** Using embedding (3.3) and the inclusion from Proposition 5.1, we can view \( C(S_q^2)L_N = \text{span}\{xv : x \in C(S_q^2), \ v \in L_N\} \subset C(S_{pq}^3)_N \) as a submodule of \( C(S_{pq}^3)_N \). Let \( N \in \mathbb{N}_0 \). It follows from the isomorphism \( \Psi_N \) defined in (3.7) that the left \( C(S_q^2) \)-module \( C(S_{pq}^3)_N \) is \( \{((fS_q^N, g) : (f, g) \in C(S_q^2))\} \) generated by the element \((S^N, 1)\). Therefore, to prove \( C(S_q^2)L_N = C(S_{pq}^3)_N \), it suffices to show that \((S^N, 1) \in C(S_q^2)L_N\). Since \( \sigma(z^n) = U^{-N} \), we have \((z^N, 1) \in T \times (U^N, 1) \). Since \((z^N, 1)\) is the image of \( t(a^N) = (z^N \otimes U^N, 1 \otimes U^N) \) under the isomorphism (3.3), we can view \((z^N, 1)\) as an element of \( L_N \). Let \( t := 1 - z \in T \). Note that \( t \) is a self-adjoint operator with spectrum \( \text{spec}(t) = \{q^n : n \in \mathbb{N}_0\} \cup \{0\} \) (see Equation (2.38)). Applying the commutation relations (2.27), one easily verifies that \( z^n z^N = \Pi_{k=1}^{n} (1 - q^k t) \). Since \( \text{spec}(t) \subset [0, 1] \), the operator \( z^n z^N \) is strictly positive. Hence \( |z^n|^1 = (z^n z^N)^{-1/2} \) belongs to the \( C^* \)-algebra \( T \). Moreover, \( \sigma(|z^n|^1) = 1 \) since \( \sigma(z^n z^N) = 1 \). Therefore \( (z^n z^N)^{-1} \in T \times (\sigma, \sigma) = C(S_q^2) \) and thus \((S^N, 1) = (z^n z^N, 1) \in C(S_q^2)L_N \). This completes the proof for \( N \geq 0 \). The case \( N < 0 \) is treated analogously.

Recall that an (even) Fredholm module of an \( * \)-algebra \( A \) can be given by a pair of \( * \)-representations \((\rho_+, \rho_-)\) of \( A \) on a Hilbert space \( H \) such that the difference \( \rho_+(a) - \rho_-(a) \) yields a compact operator. In this case, for any projection \( P \in \text{Mat}_{n,n}(A) \), the operator \( \rho_+(P)\rho_-(P) : \rho_-(P)H^n \to \rho_+(P)H^n \) is a Fredholm operator and its Fredholm index does neither depend on the \( K_0 \)-class of \( P \) nor on the class of \((\rho_+, \rho_-)\) in \( K \)-homology. This pairing between \( K \)-theory and \( K \)-homology is referred to as index pairing. If it happens that \( \rho_+(a) - \rho_-(a) \) yields trace class operators, then the index pairing can be computed by a trace formula, namely

\[
\langle \rho_+, \rho_- \rangle, \langle P \rangle = \text{tr}_H(\text{tr}_{\text{Mat}_{n,n}(\rho_+ - \rho_-)}(P)) \quad (3.8)
\]

In general, the computation of the traces gets more involved with increasing size of the matrix \( P \). This will especially be the case if one works only with the polynomial algebras \( O(S_q^2) \) and \( O(S_{pq}^2) \), and uses the the \( (|N| + 1) \times (|N| + 1) \)-projections \( E_N \) from (2.10) with entries in belonging to \( O(S_{pq}^2) \). In our example, the \( C^* \)-algebraic fibre product approach improves the situation considerably since Propositions 3.2 and 3.3 provide us with the equivalent 1-dimensional projections \( \chi_N \). As the index computation is one of our main objectives, we state the result in the following theorem.

**Theorem 3.4.** Let \( N \in \mathbb{Z} \). The isomorphic projective left \( C(S_q^2) \)-modules \( C(S_q^3)_{pq}L_N \), \( C(S_q^2)^{N+1}E_N \) and \( C(S^2)_{\chi_N} \) define the same class in \( K_0(C(S_q^2)) \), say \( [\chi_N] \), and the pairing with the generators of the \( K \)-homology \( K^0(C(S_q^2)) \) from the end of Section 2.3 is given by

\[
\langle \chi_N \rangle = N, \quad \langle \pi_+ \circ \sigma, \pi_- \circ \sigma \rangle, [\chi_N] = 1. \quad (3.9)
\]
Proof. The equivalences of the left $C(S^2_q)$-modules has been shown in Propositions 3.2 and 3.3. In particular, we are allowed to choose $\chi_N$ as a representative.

For all $N \in \mathbb{Z}$, the operator $\pi_+ \circ \sigma(\chi_N) - \pi_- \circ \sigma(\chi_N) = \pi_+(1) - \pi_-(1)$ is the projector onto the 1-dimensional subspace $C_0$, see Equation (2.13). In particular, it is of trace class so that Equation (3.8) applies. Since the trace of a 1-dimensional projector is 1, we get

$$\langle [(\pi_+ \circ \sigma, \pi_- \circ \sigma)], [\chi_N] \rangle = \text{tr}_{E(N_0)}(1 - \pi_-(1)) = 1.$$ 

Now let $N \geq 0$. Then $(pr_1, pr_0)(\chi_N) = (pr_1 - pr_0)(S^N S^{*N}, 1) = 1 - S^N S^{*N}$ is the projection onto the subspace $\text{span}\{e_0, \ldots, e_{n-1}\}$. Since it is of trace class with trace equal to the dimension of its image, we can apply Equation (3.8) and get

$$\langle [(pr_1, pr_0)], [\chi_N] \rangle = \text{tr}_{E(N_0)}(1 - S^N S^{*N}) = N.$$ 

Analogously, for $N < 0$,

$$\langle [(pr_1, pr_0)], [\chi_N] \rangle = \text{tr}_{E(N_0)}(S^{|N|} S^{*|N|} - 1) = -|N| = N,$$

which completes the proof. 

Since the C*-algebra $C(S^2_q)$ is isomorphic to the universal C*-algebra of the Podleś spheres $O(S^2_q)$, the indices in Equation (3.9) have also been obtained in [6] and [17]. In the first paper, the computations relied heavily on the index theorem, whereas in [17] and Theorem 3.4 the traces were computed directly by using elementary projections.

Note that Equation (3.9) has a geometrical interpretation: The pairing with the K-homology class $[(\pi_+ \circ \sigma, \pi_- \circ \sigma)]$ detects the rank of the projective module, and the pairing $\langle [(pr_1, pr_0)], [\chi_N] \rangle = N$ coincides with the power of $U$ in (2.11) and thus computes the “winding number”, that is, the number of rotations of the transition function along the equator.

3.3 Equivalence to the generic Hopf fibration of quantum $SU(2)$

Recall from Section 2.4 that $O(SU_q(2)) = \oplus_{N \in \mathbb{Z}} M_N$, where

$$M_N := \{ p \in O(SU_q(2)) : \Delta_{O(SU_q(2))}(p) = p \otimes U^N \} \cong O(S^2_{pq})^{N+1} P_N$$

with $P_N \in \text{Mat}_{|N|+1,|N|+1}(O(S^2_{pq}))$ given in Equation (2.13). For the definition of the $O(U(1))$-coaction $\Delta_{O(SU_q(2))} = (\text{id} \otimes pr_q) \circ \Delta$, see Section 2.4. Since $O(U(1))$ can be embedded into its universal C*-algebra, which is isomorphic to $C(S^2_q)$, we can turn $M_N$ into a left $C(S^2_q)$-module by considering $\overline{M}_N := C(S^2_q)^{N+1} P_N$. It has been shown in [17], that this left $C(S^2_q)$-module is isomorphic to $C(S^2_q)\chi_N$, and therefore to $C(S^3_{pq})\chi_N$.

The aim of this section is to define a left $C(S^2_q)$-module and right $O(U(1))$-comodule $P$ such that, for all $N \in \mathbb{Z}$, the line bundle associated to the 1-dimensional left corepresentation $\varphi \Delta(1) = U^N \otimes 1$ is isomorphic to $\overline{M}_N$. A natural idea
would be to consider the embedding of \( \mathcal{O}(SU_q(2)) \) into \( \mathcal{C}(SU_q(2)) \) and to extend the right coaction \( \Delta_{\mathcal{O}(SU_q(2))} \) to the \( \mathcal{C}^* \)-algebra closure. But then we face the problem that \( \Delta_{\mathcal{O}(SU_q(2))} \) is merely a coaction and not an algebra homomorphism. If we impose at \( \mathcal{O}(U(1)) \) the obvious multiplicative structure given by \( U^N U^K = U^{N+K} \), and turn \( \oplus_{N \in \mathbb{Z}} M_N \) into a \( \mathcal{C}^* \)-algebra such that the right \( \mathcal{O}(U(1)) \)-coaction becomes an algebra homomorphism, then the \( \mathcal{C}^* \)-closure of \( \oplus_{N \in \mathbb{Z}} M_N \cong \oplus_{N \in \mathbb{Z}} \mathcal{C}(S^3_{pq})_N \) would be isomorphic to \( \mathcal{C}(S^3_{pq}) \) and not to \( \mathcal{C}(SU_q(2)) \). Note that there cannot be an isomorphism between \( \mathcal{C}(S^3_{pq}) \) and \( \mathcal{C}(SU_q(2)) \), otherwise, by the pullback diagrams \( (3.10) \) and \( (3.11) \), \( \mathcal{C}(S^1) \cong \ker(\pi_1) \cong T \otimes \mathcal{C}(S^1) \), a contradiction.

Instead of extending the coaction \( \Delta_{\mathcal{O}(SU_q(2))} \) to some closure of \( \mathcal{O}(SU_q(2)) \), we turn \( \mathcal{O}(SU_q(2)) \) into a left \( \mathcal{C}(S^2) \)-module by setting \( P = \mathcal{C}(S^2) \otimes_{\mathcal{O}(SU_q(2))} \mathcal{O}(SU_q(2)) \) and keeping the \( \mathcal{O}(U(1)) \)-coaction, now acting on the second tensor factor. Then it follows immediately that

\[
P = \oplus_{N \in \mathbb{Z}} \mathcal{C}(S^2) \otimes_{\mathcal{O}(SU_q(2))} M_N \quad \text{and} \quad \mathcal{C}(S^2) \otimes_{\mathcal{O}(SU_q(2))} M_N = \{ p \in P : \Delta_P(p) = p \otimes U^N \}.
\]

Thus our aim will be achieved if we show that \( \overline{M}_N \cong \mathcal{C}(S^2) \otimes_{\mathcal{O}(SU_q(2))} M_N \). For this, we prove that \( P \), as a left \( \mathcal{C}(S^2) \)-module and right \( \mathcal{O}(U(1)) \)-comodule, is isomorphic to the following fibre product

\[
\begin{array}{ccc}
\mathcal{T} \otimes \mathcal{O}(U(1)) & \times & \mathcal{T} \otimes \mathcal{O}(U(1)) \\
\downarrow \pi_1 \otimes \text{id} & & \downarrow \pi_2 \otimes \text{id} \\
\mathcal{C}(S^1) \otimes \mathcal{O}(U(1)) & \xrightarrow{\Phi} & \mathcal{C}(S^1) \otimes \mathcal{O}(U(1)).
\end{array}
\]

Here \( \Phi \) is defined by \( \Phi(f \otimes U^N) = f U^N \otimes U^N \). Then, by comparing the pullback diagrams \( (3.1) \) and \( (3.10) \) in the category of left \( \mathcal{C}(S^2) \)-modules and right \( \mathcal{C}(S^1) \)-comodules, it follows that

\[
\overline{M}_N \cong \mathcal{C}(S^2) \otimes_{\mathcal{O}(SU_q(2))} M_N \cong P \square_{\mathcal{C}(S^2)} \mathcal{C} \cong \mathcal{C}(S^2) \otimes_{\mathcal{O}(SU_q(2))} M_N
\]

with the 1-dimensional corepresentation \( c(S^3) \Delta(1) = U^N \otimes 1 \) on \( \mathcal{C} \).

For simplicity of notation, we set

\[
\mathcal{A} := \mathcal{O}(SU_q(2)), \quad \mathcal{B} := \mathcal{O}(S^2_{pq}), \quad \overline{\mathcal{A}} := \mathcal{T} \times \mathcal{T} \cong \mathcal{C}(S^2_{pq}), \quad \mathcal{C} := \mathcal{O}(U(1)).
\]

Recall that \( \mathcal{B} \) can be embedded in \( \mathcal{A} \) as well as in \( \overline{\mathcal{B}} \), so both are \( \mathcal{B} \)-bimodules with respect to the multiplication. Moreover, the pullback diagram \( \overline{2.5} \) provides us with \( * \)-algebra homomorphism \( \text{pr}_q : \overline{\mathcal{B}} \to \mathcal{T} \) and \( \text{pr}_1 : \overline{\mathcal{B}} \to \mathcal{T} \) by projecting onto the left and right component, respectively. Perhaps it should here also be mentioned that \( \mathcal{C} \) is only considered as a coalgebra, not as an algebra.
Let \( v_j^N, v_N^N, \ldots, v_M^N \in \mathcal{A} \) denote the matrix elements from the definition of \( P_N \) in (2.18). Since the entries of \( P_N \) belong to \( \mathcal{B} \), we have \( v_j^N, v_k^N \in \mathcal{B} \) for all \( j, k = 0, \ldots, |N| \). The following facts are proven in [15, Lemma 6.5].

**Lemma 3.5.** Let \( l \in \mathbb{Z} \) and \( k, m \in \{0, \ldots, |l| \} \).

(i) If \( l \geq 0 \), the elements \( \text{pr}_1(v_l^l v_l^*) \) and \( \text{pr}_0(v_l^l v_l^*) \) are invertible in \( \mathcal{T} \).

(ii) If \( l < 0 \), the elements \( \text{pr}_1(v_0 v_0^*) \) and \( \text{pr}_0(v_0 v_0^*) \) are invertible in \( \mathcal{T} \).

(iii) \( \text{pr}_1(v_l^l v_l^*) \) is invertible, and thus \( \text{pr}_0(v_l^l v_l^*)^{-1} \) are invertible in \( \mathcal{T} \).

(iv) \( \text{pr}_1(v_l^l v_l^*) \) is invertible, and thus \( \text{pr}_0(v_0 v_0^*)^{-1} \) are invertible in \( \mathcal{T} \).

We can turn \( \mathcal{T} \) into a \( \mathcal{B} \)-bimodule by setting \( a.t.b := \text{pr}_0(a) t \text{pr}_0(b) \) and \( a.t.b = \text{pr}_1(a) t \text{pr}_1(b) \), where \( a, b \in \mathcal{B} \) and \( t \in \mathcal{T} \). To distinguish between both bimodules, we denote \( \mathcal{T} \) equipped with the first action by \( \mathcal{T}_- \), and write \( \mathcal{T}_+ \) if we use the second action. Clearly, as left or right \( \mathcal{B} \)-module, both are generated by \( 1 \in \mathcal{T} \).

The next proposition is the key in proving (3.11).

**Proposition 3.6.** The left \( \mathcal{B} \)-modules \( \mathcal{T}_\pm \) and \( \mathcal{T}_\pm \otimes_B M_l \) are isomorphic. The corresponding isomorphisms are given by

\[
\psi_{l,+}: \mathcal{T}_+ \to \mathcal{T}_+ \otimes_B M_l, \quad \psi_{l,+}(t) = t \text{pr}_0(v_l^l v_l^*)^{-1/2} \otimes_B v_0^l, \quad l \geq 0, \\
\psi_{l,-}: \mathcal{T}_- \to \mathcal{T}_- \otimes_B M_l, \quad \psi_{l,-}(t) = t \text{pr}_1(v_l^l v_l^*)^{-1/2} \otimes_B v_l^l, \quad l \geq 0, \\
\psi_{l,+}: \mathcal{T}_+ \to \mathcal{T}_+ \otimes_B M_l, \quad \psi_{l,+}(t) = t \text{pr}_0(v_l^l v_l^*)^{-1/2} \otimes_B v_l^l, \quad l < 0, \\
\psi_{l,-}: \mathcal{T}_- \to \mathcal{T}_- \otimes_B M_l, \quad \psi_{l,-}(t) = t \text{pr}_1(v_l^l v_l^*)^{-1/2} \otimes_B v_0^l, \quad l < 0.
\]

The inverse isomorphisms satisfy, for all \( k = 0, 1, \ldots, |l| \),

\[
\psi_{l,+}^{-1}(1 \otimes_B v_k^l) = \text{pr}_0(v_k^l v_k^*) \text{pr}_0(v_0^l v_0^*)^{-1/2}, \quad l \geq 0, \quad \tag{3.12}
\]
\[
\psi_{l,-}^{-1}(1 \otimes_B v_k^l) = \text{pr}_1(v_k^l v_k^*) \text{pr}_1(v_l^l v_l^*)^{-1/2}, \quad l \geq 0, \quad \tag{3.13}
\]
\[
\psi_{l,+}^{-1}(1 \otimes_B v_k^l) = \text{pr}_1(v_k^l v_k^*) \text{pr}_1(v_l^l v_l^*)^{-1/2}, \quad l < 0, \quad \tag{3.14}
\]
\[
\psi_{l,-}^{-1}(1 \otimes_B v_k^l) = \text{pr}_0(v_k^l v_k^*) \text{pr}_0(v_0^l v_0^*)^{-1/2}, \quad l < 0. \quad \tag{3.15}
\]

**Proof.** We prove the proposition for \( \psi_{l,+} \) with \( l \geq 0 \), the other cases are treated analogously. Since \( \text{pr}_0(v_l^l v_l^*)^{-1/2} \) is positive and invertible, \( \text{pr}_0(v_l^l v_l^*)^{-1/2} \in \mathcal{T} \) is invertible, and thus \( \psi_{l,+} \) is injective. The left \( \mathcal{B} \)-module \( \mathcal{T}_+ \otimes_B M_l \) is generated by \( 1 \otimes_B v_k^l, k = 0, 1, \ldots, l \) (cf. [15, Theorem 4.1]). As \( \psi_{l,+} \) is left \( \mathcal{B} \)-linear, it suffices to prove that the elements \( 1 \otimes_B v_k^l \) belong to the image of \( \psi_{l,+} \). Applying (2.19)
and Lemma 3.5(iii), we get

\[ 1 \otimes_B v_k^i = \sum_j 1 \otimes_B v_k^i v_j^i \]
\[ = \sum_j \text{pr}_0(v_k^i v_j^i) \otimes_B v_j^i \]
\[ = \sum_j \text{pr}_0(v_k^i v_j^i) \otimes_B v_j^i \]
\[ = \text{pr}_0(v_k^i v_j^i) \otimes_B v_j^i \]
\[ = \text{pr}_0(v_k^i v_j^i) \otimes_B v_j^i = \psi_{l,+} \left( \text{pr}_0(v_k^i v_j^i) \right) \otimes_B v_j^i. \]

This proves the surjectivity of \( \psi_{l,+} \) and Equation 3.12.

Using the last proposition and the decomposition \( A = \oplus_{N \in \mathbb{Z}} M_N \), we can define left \( \overline{B} \)-linear, right \( C \)-colinear isomorphisms

\[ \Psi_- : \mathcal{T}_- \otimes_B A \rightarrow \bigoplus_{N \in \mathbb{Z}} \mathcal{T}_- \otimes N, \quad \Psi_+ : \mathcal{T}_+ \otimes_B A \rightarrow \bigoplus_{N \in \mathbb{Z}} \mathcal{T}_+ \otimes N \]

by setting

\[ \Psi_{\pm}(t \otimes_B m_N) = \psi_{N,\pm}^{-1}(t \otimes_B m_N) \otimes N, \quad t \in \mathcal{T}, \ m_N \in M_N. \quad (3.16) \]

Next we define left \( \overline{B} \)-linear, right \( C \)-colinear surjections

\[ \text{pr}_{\pm} : \overline{B} \otimes_B A \rightarrow \mathcal{T}_\pm \otimes_B A, \]

by

\[ \text{pr}_-((t_1, t_2) \otimes_B a) := t_1 \otimes_B a, \quad \text{pr}_+(t_1, t_2) \otimes_B a) := t_2 \otimes_B a. \quad (3.17) \]

Furthermore, we turn \( C(S^1) \) into a left \( \overline{B} \)-module by defining \( b f := \sigma(b) f \) for all \( b \in \overline{B} \) and \( f \in C(S^1) \). Now consider the following diagram in the category of left \( \overline{B} \)-modules, right \( C \)-comodules:

\[ \overline{B} \otimes_B A \xrightarrow{\Psi_- \circ \text{pr}_-} \mathcal{T}_- \otimes C \xrightarrow{\sigma \circ \text{id}} C(S^1) \otimes C, \quad (3.18) \]

where \( \Phi \) is the same as in (3.10).

**Lemma 3.7.** The diagram (3.18) is commutative, \( \Psi_- \circ \text{pr}_- \) and \( \Psi_+ \circ \text{pr}_+ \) are surjective and \( \ker(\Psi_- \circ \text{pr}_-) \cap \ker(\Psi_+ \circ \text{pr}_+) = \{0\} \).

**Proof.** Since all maps are left \( \overline{B} \)-linear, it suffices to prove the lemma for generators of the left \( \overline{B} \)-module \( \overline{B} \otimes_B A \). Moreover, since \( A = \oplus_{N \in \mathbb{Z}} M_N \), we can restrict ourselves to the generators of the left \( B \)-modules \( M_N \).

Let \( l \geq 0 \). Since \( \sigma(\text{pr}_0(f)) = \sigma(f) = \sigma(\text{pr}_1(f)) \) for all \( f \in \overline{B} \) by (2.10), we get from Equation (3.10) and Lemma 3.6

\[ (\sigma \circ \text{id}) \circ \Psi_+ \circ \text{pr}_- (1 \otimes_B v_k^i) = \sigma(v_k^i v_0^i) \sigma(v_0^i v_0^i)^{-1/2} \otimes U^l, \quad (3.19) \]

\[ \phi \circ (\sigma \circ \text{id}) \circ \Psi_- \circ \text{pr}_+ (1 \otimes_B v_k^i) = \sigma(v_k^i v_0^i) \sigma(v_0^i v_0^i)^{-1/2} U^l \otimes U^l. \quad (3.20) \]
By Lemma 3.6 (iii) (with \( m = 0 \)), we have
\[
\sigma(v_k^l v_0^s) = \sigma(v_k^l v_0^s) \sigma(v_k^l v_0^s)^{-1} \sigma(v_k^l v_0^s).
\tag{3.21}
\]
Inserting the latter equation into (3.19) and comparing with (3.20) shows that it suffices to prove
\[
\sigma(v_k^l v_0^s)^{-1/2} \sigma(v_k^l v_0^s) \sigma(v_k^l v_0^s)^{-1/2} = U_l.
\tag{3.22}
\]
It follows from [17, Lemma 2.2] (with \( v_k^l \sim u_l \) and \( v_0^s \sim w_s \)), or can be computed directly by using explicit expressions for \( v_0^s \) and \( v_k^l \), that \( v_k^l v_0^s \sim \eta^l_k \). From the embedding (2.17), we deduce that \( \eta^l_k \) has polar decomposition \( \eta^l_k = (S^l, S^l)|\eta^l_k| \).

Therefore we can write \( v_k^l v_0^s = (S^l, S^l)|\eta^l_k^0| \) which implies
\[
\sigma(v_k^l v_0^s) = \sigma(|\eta^l_k^0|) U_l.
\]
By comparing with (3.22), we see that it now suffices to verify
\[
\sigma(v_k^l v_0^s)^{-1/2} \sigma(|\eta^l_k^0|) \sigma(v_k^l v_0^s)^{-1/2} = 1.
\tag{3.23}
\]
Multiplying both sides of Equation (3.21) with \( \sigma(v_k^l v_0^s) \) gives
\[
\sigma(v_k^l v_0^s) \sigma(v_k^l v_0^s) = \sigma(v_k^l v_0^s) \sigma(v_k^l v_0^s).
\]
Thus
\[
\sigma(|\eta^l_k^0|) = \sigma((v_k^l v_0^s v_k^l v_0^s)^{-1/2}) = (\sigma(v_k^l v_0^s) \sigma(v_k^l v_0^s))^{-1/2} = (\sigma(v_k^l v_0^s) \sigma(v_k^l v_0^s))^{1/2},
\]
which proves (3.23). This concludes the proof of the commutativity of \( \Psi_{-0} \) for \( l \geq 0 \). The case \( l < 0 \) is treated analogously.

The surjectivity of \( \Psi_{-0} \circ pr_{-0} \) and \( \Psi_{+0} \circ pr_{+0} \) follows from the bijectivity of \( \Psi_{+0} \) and the surjectivity of \( pr_{+0} \).

Suppose that \( \sum_{k=1}^n (r_k, s_k) \otimes_B a_k \in \ker(\Psi_{-0} \circ pr_{-0}) \cap \ker(\Psi_{+0} \circ pr_{+0}) \). Since \( \Psi_{+0} \) is an isomorphism, we get \( \sum_{k=1}^n r_k \otimes_B a_k = 0 \) and \( \sum_{k=1}^n s_k \otimes_B a_k = 0 \) by (3.17).

Hence \( \sum_{k=1}^n (r_k, s_k) \otimes_B a_k = \sum_{k=1}^n (r_k, 0) \otimes_B a_k + \sum_{k=1}^n (0, s_k) \otimes_B a_k = 0 \) which proves last claim of the lemma.

We are now in a position to prove the main theorem of this section.

**Theorem 3.8.** There is an isomorphism of left \( C(S^2_q) \)-modules and right \( O(U(1)) \)-comodules between the fibre product \( T \otimes O(U(1)) \times (\Phi_{0,1}, \pi_2) T \otimes O(U(1)) \) from (3.10) and \( C(S^2_q)^M \otimes_{C(S^2_q)} O(SU_q(2)) \). Moreover, the chain of isomorphisms in (3.11) holds.

**Proof.** Lemma 3.7 states that \( C(S^2_q) \otimes_{C(S^2_q)} O(SU_q(2)) \) is an universal object of the pull back diagram (3.15). Comparing (3.18) and (3.10) shows that both pullback diagrams define up to isomorphism the same universal object which proves the first part of the theorem.

The first isomorphism in (3.11) follows from the Murray-von Neumann equivalence of the corresponding projections, see [17]. The second isomorphism follows from the above equivalence of pullback diagrams, and the last one from fact that all mappings in (3.18) are right \( O(U(1)) \)-colinear. 

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