Explanation of quantum paradoxes as a natural feature of the joint statistics of physical properties related by half-periodic transformations

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Abstract. Quantum paradoxes show that quantum statistics can exceed the limits of positive joint probabilities for physical properties that cannot be measured jointly. It is therefore impossible to describe the relations between the different physical properties of a quantum system by assigning joint realities to their observable values. Recent progress in quantum measurements suggests that the fundamental relations between non-commuting physical properties can instead be described by complex probabilities, where the phase is an expression of the action of transformations between the non-commuting properties (Hofmann H F 2011 New J. Phys. 13 103009). In these relations, negative probabilities necessarily emerge whenever the physical properties involved are related to each other by half-periodic transformations, since such transformations are characterized by action phases of $\pi$ in their complex probabilities. It is therefore possible to trace the failure of realist assumptions back to a fundamental and universally valid relation between statistics and dynamics that associates half-periodic transformations with negative probabilities.

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1. Introduction

The main obstacle that prevents a satisfactory explanation of quantum theory is that the Hilbert space formalism describes the relation between the outcomes of different measurements in terms of mathematical concepts without any clear physical meaning. This problem is highlighted by a number of quantum paradoxes which demonstrate that the quantum state cannot be interpreted as a positive valued probability distribution of the potential measurement outcomes \[1, 2, 3, 4, 5, 6\]. Since quantum paradoxes are predicted by the standard formalism, one might expect that there should be a well explained reason for the failure of simple realist models. However, most discussions of quantum paradoxes treat the formalism as a black box that does not relate to any known aspects of physics - almost as if the quantum states were miraculous descriptions of disembodied knowledge, rather than actual physical conditions produced in a laboratory.

Recently, there have been a number of experimental breakthroughs in the attempts to lift the veil behind which the physics of quantum paradoxes is hiding. By using weak measurements, researchers in a number of laboratories have demonstrated that it is possible to obtain non-positive joint probabilities from the experimental data \[8, 9, 10, 11, 12, 13, 14, 15\]. Oddly, these experiments have not led to a more thorough analysis of the physics described by those negative probability results, even though it should be the main purpose of quantum theory to explain experimental results, not just to simulate them by otherwise obscure mathematical formulas. The problem might be that weak measurement results themselves are usually perceived as paradoxical, as illustrated by the emphasis on the strangeness of weak values in their historical introduction \[16, 17\]. However, the outcomes of weak measurements are a natural part of quantum mechanics, and their strangeness is merely an indication that we have misunderstood the physics described by the standard formalism.

In this paper, I will show that the experimentally observable negative probabilities can be explained by the structure of transformations in quantum systems. Specifically, the probabilities obtained in weak measurements are generally complex, where the complex phase expresses the action of transformations between the three physical properties defined by initial state, final state, and weak measurement \[18, 19, 20\]. Negative probabilities emerge whenever the action phases that describe the transformations between the physical properties are larger than \(\pi/2\). Maximal negative probabilities are observed whenever the action phases of the complex probabilities are \(\pi\), which is typical for half-periodic transformations. Quantum paradoxes are therefore naturally observed in the correlations between physical properties that are related to each other by half-periodic transformations such as spin flips and swap operations.

Although negative probabilities may seem to be counterintuitive, it is comparatively easy to visualize the relation between physical properties in terms of transformation dynamics. The explanation of non-positive statistics by action phases can therefore provide a much clearer picture of the conditions under which the naive assumption of a joint reality fails. It may also be worth noting that it is the dynamics of measurement
back-actions that prevents a joint observation of non-commuting properties in actual experiments [15], and that non-commutativity itself describes the dynamics generated by the physical properties as the imaginary part of an operator-valued correlation [21]. The relation between the action of transformations and the non-classical correlations expressed by complex probabilities is therefore consistent with the general formalism of quantum theory and provides deeper insights into the actual physics described by it.

In the following, I will first show that non-positive probabilities emerge naturally from the conventional operator algebra that is commonly used to describe the statistics of quantum systems. I will then identify the fundamental relation between the dynamics of transformations and the joint probabilities of non-commuting properties. With this relation, it is possible to explain all quantum paradoxes, as will be discussed in detail in the main part of the paper. Finally, I consider the implication of the results for our fundamental understanding of quantum physics, pointing out that all paradoxes can be resolved by recognizing that there is no reality without the dynamics induced by measurement interactions. The explanation of quantum paradoxes by half-periodic transformations between the physical properties may thus close a significant gap in our present understanding of quantum physics.

2. The structure of quantum paradoxes

Quantum paradoxes concern the relation between physical properties that cannot be measured jointly. Standard quantum mechanics limits actual physical situations to the preparation of a state $a$ with a well defined physical property $A$, and the subsequent measurement of an outcome $m$ for a different physical property $M$, where the relation between $a$ and $m$ is described by the conditional probability $P(m|a)$ of the measurement outcomes $m$. Alternatively, it is possible to measure a third property $B$ with outcomes of $b$ distributed according to the conditional probability $P(b|a)$. However, the standard formalism does not permit a joint measurement of $m$ and $b$ if the properties $M$ and $B$ do not have any common eigenstates. It is therefore impossible to measure a joint probability of the form $P(m,b|a)$ to directly characterize the relation between the three physical properties $A$, $M$ and $B$.

All quantum paradoxes are based on the assumption that the measurement statistics of separate measurements should be consistent with a positive joint probability $P(m,b|a)$, even if it is impossible to obtain any joint measurement outcomes $(m,b)$. It is then shown that the combination of measurement statistics observed in separate measurements of the marginal probabilities defined by $P(m,b|a)$ exceeds the limits obtained by assuming that all $P(m,b|a)$ must be positive real numbers. In general, quantum paradoxes therefore demonstrate that the statistics of quantum states cannot be reproduced by positive valued joint probabilities of pairs of measurement outcomes $(m,b)$.

To some extend, this is not a surprising result. If it was so simple to replace the rather unintuitive rules of quantum mechanics with a conventional joint probability, it
would seem to be preferable to explain quantum physics in terms of the fundamental
realities \((m, b)\), and not by some mysterious “superpositions” of mutually exclusive
alternatives. However, the quantum formalism is not a black box, and there are well
defined rules that determine the circumstances under which quantum statistics will
exceed the limits imposed by positivity. In the following, I will show that these rules
can be expressed in terms of non-positive joint probabilities, where the non-positive
values are obtained from a simple rule that expresses the relation between physical
properties in terms of the action of transformations between them \cite{18, 19, 20, 22}.

3. Operator algebra and the physical meaning of complex probabilities

How does the quantum formalism describe the relation between non-commuting
properties? In the conventional formulation, the outcomes of each measurement can
be represented by projection operators, e.g. \(|a \rangle \langle a|\) for the outcome \(a\) of the property
\(A\), etc. The conditional probabilities \(P(m|a)\) are then obtained from the product trace
of the projection operators,

\[
P(m|a) = \text{Tr} \left( |m \rangle \langle m| a \rangle \langle a| \right) = P(a|m).
\]

In this relation, state preparation and measurement are expressed by the same operators,
so that it is possible to exchange the roles of the two to obtain \(P(a|m) = P(m|a)\). State
preparation therefore corresponds to the selection of a specific physical property that
characterizes the initial conditions in the experimental setup.

Eq.\((1)\) also shows that the relation between the two properties \(A\) and \(M\) includes
an element of randomness. In general, each state \(a\) can produce every possible outcome
\(m\), so there is no fundamental relation that determines \(m\) as a function of \(a\). However,
it is possible to express any operator (and hence any physical property) as a function
of two mutually overlapping basis sets \(a\) and \(b\) by using the weak values of the operator
for these states \cite{19, 23, 24, 25}. The quantum formalism therefore defines deterministic
relations between any three physical properties with mutually overlapping eigenstates,
and these relations are directly observed in weak measurements \cite{19}. In particular, the
relation between the measurement outcomes (or state preparations) \(a, m,\) and \(b\) can be
expressed by a complex-valued joint probability,

\[
P(m, b|a) = \text{Tr} \left( |b \rangle \langle b| m \rangle \langle m| a \rangle \langle a| \right).
\]

Note that this is actually a rather natural extension of Eq.\((1)\). In fact, it can be argued
that it is the only reasonable definition of a joint probability consistent with the operator
formalism \cite{22}. As in Eq.\((1)\), each projector refers to a specific physical property,
without any distinction between initial state, final state, or weak measurements. Eq.\((2)\)
therefore expresses a fundamental relation between the physical properties that is quite
independent of the circumstances under which it is observed in experiments.

Since the projection operators do not commute with each other, the product trace
of three operators that describes the joint probabilities in Eq.\((2)\) generally results in
a complex number. Clearly, these complex numbers cannot be identified with relative
frequencies of joint measurement outcomes. However, this is not as problematic as it may seem, since it is impossible to simultaneously obtain precise measurement outcomes for \( m \) and \( b \) under the initial condition \( a \). Instead, weak measurements and related methods show that the negative and imaginary contributions will always be “covered up” by measurement uncertainties. The situation is particularly clear in sequential measurements [15, 26]: the interaction dynamics of the intermediate measurement result in a trade-off between measurement errors due to limited measurement resolution and back-action errors due to changes in the observable measured in the final measurement. In general, measurements require some kind of interaction dynamics, and the rules of quantum mechanics do not permit a clear separation between the measurement outcome and the dynamics generated by the observable in the system. The latter observation actually provides complex probabilities with a clear physical meaning: the complex phase represents the effects of transformation dynamics generated by one property on the probabilities of measurement outcomes for the other [18]. Specifically, weak measurement statistics predict the effects of a unitary transformation characterized by the action \( S(m) \) assigned to the generator outcomes \( m \), where

\[
P(b|U(a)) = |\langle b | \hat{U} | a \rangle|^2 = \frac{1}{P(b|a)} \left| \sum_m P(m, b|a) \exp \left( -i S(m) \right) \right|^2.
\]  

(3)

Importantly, this relation would not make much sense if the joint probabilities were given by real and positive numbers, since in that case, the unitary transform could only reduce the probabilities of \( b \). This is clearly not the case if the transformation can actually transform \( a \) into \( b \), so that \( P(b|U(a)) = 1 \) for a specific non-zero action function \( S_{\text{opt}}(m) \). In general, the maximal statistical overlap \( P(b|U(a)) \) is achieved whenever the action \( S_{\text{opt}}(m) \) exactly compensates the original phase of the complex joint probability,

\[
S_{\text{opt}}(m, a, b) = \hbar \text{Arg} \left( P(m, b|a) \right).
\]

(4)

We can therefore conclude that Eq. (3) requires that the joint probabilities are complex whenever transformations in \( m \) can increase the statistical overlap between \( U(a) \) and \( b \). This provides the phase of the complex probabilities with a well-defined operational meaning: \( S_{\text{opt}} \) simply describes the transformation that maps \( a \) to \( b \). It is therefore possible to understand the complex phases of the joint probabilities intuitively as an expression of the optimized transformation from \( a \) to \( b \) along \( m \) [18, 20].

The action itself can usually be expressed by a product of a generator with a measure of transformation distance: energy times time, angular momentum times angle, momentum times distance, and so on. In the context of quantum paradoxes, it is often convenient to consider spin systems, where the angular momentum around the z-axis is given by \( L_z = \hbar m_z \) and the action of a rotation by an angle of \( \phi \) around the z-axis is given by \( S(m_z) = \hbar m_z \phi \). If \( b \) is obtained from \( a \) by a rotation of angle \( \phi \) around the z-axis, the action phases of the complex joint probabilities \( P(m_z, b|a) \) are given by \( S_{\text{opt}} = \hbar m_z \phi + S_{\text{Norm}} \), where \( S_{\text{Norm}} \) is needed to ensure that the marginal probability \( P(b|a) = \sum_{m_z} P(m_z, b|a) \) is a positive number.
Although the imaginary parts of complex probabilities do not appear in the observable statistics of \(a\), \(b\) and \(m\), negative real parts appear in the individual statistics as a reduction of probabilities below the values required to keep joint probabilities positive. Eq. (3) shows that these contributions to the statistics of \(a\), \(b\) and \(m\) will be negative whenever the action phases associated with the specific contribution are larger than \(\pi/2\) \[18\]. In particular, action phases of zero and \(\pi\) will result in a probability distribution with negative and positive real values. Such action phases describe a transformation that returns to its origin when it is applied twice, so that the transformation itself is half-periodic. We can therefore conclude that quantum mechanics predicts negative joint probabilities for the physical properties \(a\), \(b\) and \(m\) if \(a\) can be approximately transformed into \(b\) by a half-periodic transformation generated by \(m\). Quantum paradoxes are an immediate result of these negative statistical weights. It is therefore possible to explain quantum paradoxes by replacing the assumption of joint realities and positive probabilities with the correct relation between potential realities given by Eq. (3). According to this relation, joint probabilities are expected to be non-positive, and the origin of negative probabilities can be traced directly to transformations between the physical properties involved in the paradox. In the following, I will apply this analysis to the various quantum paradoxes. In each case, it can be shown that the negative probabilities that result in the paradox originate from the action of half-periodic transformations that describe the relation between the physical properties involved in the paradox. Quantum paradoxes can therefore be explained by the transformation dynamics that defines the fundamental relations between the different properties of a physical system.

4. Negative probabilities in violations of Leggett-Garg inequalities

Perhaps the most direct connection between joint probabilities and inequality violations is given by the Leggett-Garg inequalities. Originally, these inequalities were formulated as a temporal equivalent to Bell’s inequalities. However, the dynamics of a two-level system always corresponds to a spin precession, so correlations of the same observable at different times correspond to correlations between different spin directions at the same time. Therefore, Leggett-Garg inequalities essentially describe the limits that realism imposes on the values of three different spin directions of a two-level system, given by \(\sigma_a = \pm 1\), \(\sigma_m = \pm 1\) and \(\sigma_b = \pm 1\) \[3\]. Fig. 1 illustrates this relation between three different spin directions for the symmetric case of equal angles between \(\sigma_a\) and \(\sigma_m\) and between \(\sigma_m\) and \(\sigma_a\).

If a specific value of \(\sigma_a\) is chosen as initial condition, the joint probabilities of \(\sigma_m\) and \(\sigma_b\) can be determined from the operator algebra. Specifically, it is possible to derive joint probabilities of \(\sigma_m\) and \(\sigma_b\) by using the expectation values of \(\sigma_m\), \(\sigma_b\) and their operator product, e.g.

\[
P(\sigma_m = -1, \sigma_b = +1|\sigma_a) = \frac{1}{4} (1 + \langle \sigma_b \rangle_{\sigma_a} - \langle \sigma_m \rangle_{\sigma_a} - \langle \sigma_m \sigma_b \rangle_{\sigma_a} ). \tag{5}
\]
In a two level system, the expectation value of $\sigma_m$ for an initial spin of $\sigma_a = +1$ is given by $\cos(\theta)$, where $\theta$ is the angle between the directions of the spins. If the three spins are equally spaced and lie in the same plane, so that the angle between $\sigma_a$ and $\sigma_b$ is $2\theta$ and $\sigma_m$ is exactly in the middle between $\sigma_a$ and $\sigma_b$, the joint probability predicted from separate measurements of the expectation values is

$$P(\sigma_m = -1, \sigma_b = +1 | \sigma_a = +1) = \frac{1}{4} (1 + \cos(2\theta) - \cos(\theta) - \cos(\theta)) = \frac{1}{2} \cos(\theta) (\cos(\theta) - 1).$$

This probability is negative for all values of $\theta < \pi/2$, so that the Leggett-Garg inequality for the correlations in Eq. (5) is violated whenever the positive spins are in the upper half of the Bloch sphere. On the other hand, the specific value of the negative probability is a function of the angle between the spins, with a maximal violation of Legett-Garg inequalities at $\cos(\theta) = 1/2$, where the joint probability is $-1/8$.

As explained in the previous section, the appearance of negative joint probabilities can be traced back to half-periodic transformations between the physical properties in question. As indicated in Fig. 1, $\sigma_a = +1$ can be transformed into $\sigma_b = +1$ by a 180 degree rotation around the axis given by $\sigma_m$. Since the action of the spin-flip around the $m$-axis can be given by $S_{+1} = 0$ and $S_{-1} = \pi h$, the sum over $m$ in Eq. (3) is equal to the difference between the two joint probabilities for
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(σ_m = +1, σ_b = +1) and (σ_m = -1, σ_b = +1). Since the overlap after the transformation is \( P(σ_b = +1|U(σ_a = +1)) = 1 \), the difference between the joint probabilities is

\[
P(σ_m = +1, σ_b = +1|σ_a = +1) - P(σ_m = -1, σ_b = +1|σ_a = +1) = \sqrt{P(σ_b = +1|σ_a = +1)}.
\]  
(7)

By definition, the sum of the joint probabilities is given by the marginal probability \( P(σ_b = +1|σ_a = +1) \). The joint probability of \( σ_m = -1 \) and \( σ_b = +1 \) can therefore be derived from the directly observable probability \( P(σ_b = +1|σ_a = +1) \) by

\[
P(σ_m = -1, σ_b = +1|σ_a = +1) = \frac{1}{2} \left( P(σ_b = +1|σ_a = +1) - \sqrt{P(σ_b = +1|σ_a = +1)} \right).
\]  
(8)

Since \( P(σ_b = +1|σ_a = +1) = \cos^2(θ) \), this is equal to the result derived from the expectation values in Eq.(6).

The essential advantage of the derivation from Eq.(3) is that it derives the negative probability from the familiar physics of rotations around the \( m \)-axis. In close analogy to classical physics, this action is basically given by the product of angular momentum and angle, with an additional offset that is needed to ensure that the marginal probabilities are positive and real. However, the laws of quantum physics require that this action also appears as a phase in the complex probabilities that describe the statistics of the spins, and it is this phase difference of \( π \) that explains the negative probabilities responsible for Leggett-Garg inequality violations.

5. Three box paradox and quantum Cheshire cats

In the Leggett-Garg scenario, the limitation to a single two-level system means that we can easily understand the physics of half-periodic transformations. In multi-level systems, there is a much wider range of half-periodic transformations, so there is a much wider variety of quantum paradoxes that can be constructed from the corresponding negative probabilities. A particularly, illustrative example is the three box paradox, where the initial state is defined in terms of a superposition of three possible paths represented by numbered boxes [4],

\[
| a \rangle = \frac{1}{\sqrt{3}} (| 1 \rangle + | 2 \rangle + | 3 \rangle).
\]  
(9)

After the system passes through the three boxes, it is measured in a different superposition given by

\[
| b \rangle = \frac{1}{\sqrt{3}} (| 1 \rangle + | 2 \rangle - | 3 \rangle).
\]  
(10)

The paradox is based on the observation that \( | 2 \rangle - | 3 \rangle \) is orthogonal to \( | a \rangle \) and \( | 2 \rangle + | 3 \rangle \) is orthogonal to \( | b \rangle \). If we assume that we can independently eliminate possibilities that contradict either \( a \) or \( b \), it seems that only box 1 is left as a possible path between \( a \) and \( b \). However, the same argument can be made to exclude box 1 and box 3, leaving only box 2 as a possible path between \( a \) and \( b \).
Here, the resolution of the paradox by Eq.(3) is particularly direct, since the formulation of the problem defines the relation between \(a\) and \(b\) in terms of a half-periodic transformation with actions of \(S(1) = S(2) = 0\) and \(S(3) = \hbar \pi\). Using \(P(b|a) = 1/9\) and \(P(b|U(a)) = 1\), Eq.(3) determines the joint probabilities as

\[
P(1,b|a) = \frac{1}{9}, \quad P(2,b|a) = \frac{1}{9}, \quad P(3,b|a) = -\frac{1}{9}.
\]

The probability of zero for the superpositions of box 2 and box 3 can be confirmed, but it originates from a cancellation of positive and negative joint probabilities. The assignment of paths fails because the relation between the physical properties \(a\), \(b\) and the boxes is given by the transformation between \(a\) and \(b\) generated by applying a box-dependent action.

The logic of the three box paradox nicely illustrates the role of half-periodic transformations, since the initial and the final state are defined only in terms of the phase relations of their components in the intermediate basis. In fact, it is possible to construct a large number of quantum paradoxes in this manner. If a single photon travels through a multi-path interferometer from an input port \(a\) to an output port \(b\) such that the initial probability of finding the photon in \(b\) is 1, phase shifts of \(\pi\) induced in a specific selection of intermediate paths will not only reduce the output probability in \(b\), but actually result in negative joint probabilities for the intermediate paths \(m\) and the output \(b\).

![Figure 2. Explanation of the Cheshire cat paradox as an extension of the three box paradox to four boxes. The transformation from \(a\) to \(b\) is achieved by an action of \(S(p2,V) = \pi \hbar\). The “cat” appears to be in \(p1\), but the “smile” given by the probability difference \(P(H) - P(V)\) of the polarizations is found in \(p2\).](image)

An example of a quantum paradox that can be constructed in this manner is the Cheshire cat scenario, where the polarization of a photon appears to be separated from the spatial path that the photon took [6]. The paradox is easy to explain once the complete statistics of polarization and spatial paths is considered. Fig. 2 illustrates the relation with the three box paradox by assigning a separate “box” to each combination.
of path and polarization inside the interferometer. The photon can be found in the spatial paths $p_1$ or $p_2$ with a horizontal polarization $H$ or a vertical polarization $V$. If the initial state is an equal superposition of all four possibilities, and the final state is generated by a phase shift of $\pi$ in the $V$-polarized component of path $p_2$ followed by a detection of an equal superposition in the output, the joint probabilities of paths and polarizations inside the interferometer are given by

$$P(p_1, H; b|a) = \frac{1}{8}, \quad P(p_1, V; b|a) = \frac{1}{8},$$

$$P(p_2, H; b|a) = \frac{1}{8}, \quad P(p_2, V; b|a) = -\frac{1}{8}. \quad (12)$$

It is tempting to consider the polarization and the paths separately, in which case it seems that the probability of $p_1$ is one and that of $p_2$ is zero, while the probability of $H$-polarization is 1 and that of $V$ polarization is zero. However, the correlation between polarization and path then seems to indicate that the $H$-polarization propagates only along $p_2$, even though the photon only propagates along path $p_1$.

Importantly, there is no direct measurement of path or polarization, since the initial and final states are not eigenstates of the corresponding properties. Similar to the three box paradox, information about the path and the polarization is induced based on the relations of the intermediate eigenstates with either the initial state or the final state, one of which now expresses a particular correlation between path and probability. To resolve the paradox, it is important to realize that this induction is based on false assumptions about the relations between the initial and final conditions $a, b$ and the intermediate properties of path and polarization.

6. The Hardy paradox

Like the Cheshire cat paradox, the Hardy paradox is based on two-path interferometers [5]. Two particles are sent through two different interferometers set up in parallel, with outer paths $O$ and inner paths $I$. However, the inner paths cross each other, so that an interaction effectively eliminates the particles if they are both in the inner paths. As a result of this interaction, the interference between the two paths is disturbed in both interferometers, and the output port can switch from the original output port to the opposite port.

The situation after the interaction has eliminated the combination $(I_1, I_2)$ can be described by an equal superposition of the three remaining combinations of paths,

$$|a⟩ = \frac{1}{\sqrt{3}} (|O_1, O_2⟩ + |O_1, I_2⟩ + |I_1, O_2⟩). \quad (13)$$

The detection of a photon in the port opposite to the original output port is described by a negative superposition of the paths,

$$|b_i⟩ = \frac{1}{\sqrt{2}} (|O_i⟩ - |I_i⟩). \quad (14)$$

The probability of finding both particles in the opposite output ports is $P(b_1, b_2|a) = 1/12$. However, it seems that a switch to the opposite port requires the presence of
the other particle in the inner path, as shown by the directly observable probabilities $P(b_1, O_2|a) = 0$ and $P(O_1, b_2|a) = 0$. Assuming a measurement independent joint reality of paths and output ports, one would think that the outcome $(b_1, b_2)$ is only possible if both particles took the inner path, $(I_1, I_2)$. However, this option was clearly eliminated in the preparation of $a$.

The probability $P(b_1, b_2|a)$ can be expressed as a sum of joint probabilities,

$$P(b_1, b_2) = P(O_1, O_2; b_1, b_2|a) + P(O_1, I_2; b_1, b_2|a) + P(I_1, O_2; b_1, b_2|a) + P(I_1, I_2; b_1, b_2|a).$$

Because of the preparation condition, we know that $P(I_1, I_2; b_1, b_2|a) = 0$. In addition, we can impose the condition that $P(b_1, O_2|a) = 0$ and $P(O_1, b_2|a) = 0$ by requiring that the corresponding contributions of the joint probabilities sum up to zero,

$$P(O_1, O_2; b_1, b_2|a) + P(O_1, I_2; b_1, b_2|a) = 0,$$
$$P(O_1, O_2; b_1, b_2|a) + P(I_1, O_2; b_1, b_2|a) = 0.$$  \tag{16}

With these conditions, it is possible to derive the values of the joint probabilities in the Hardy paradox,

$$P(O_1, O_2; b_1, b_2|a) = -\frac{1}{12}, \quad P(O_1, I_2; b_1, b_2|a) = \frac{1}{12},$$
$$P(I_1, O_2; b_1, b_2|a) = \frac{1}{12}, \quad P(I_1, I_2; b_1, b_2|a) = 0. \tag{17}$$

Thus the negative probability of $(O_1, O_2)$ ensures that the individual probabilities of $(O_1)$ and $(O_2)$ can be zero even though $(O_1, I_2)$ and $(I_1, O_2)$ contribute positively to the probability of the observed output port combination $(b_1, b_2)$.

As in the previous examples, the negative probability in the Hardy paradox can be explained in terms of the transformation dynamics in the paths $m$ that relates the initial state $a$ to the final state $b$. According to Eq.\,[13],

$$P(O_1, O_2; b_1, b_2|a) - P(O_1, I_2; b_1, b_2|a) - P(I_1, O_2; b_1, b_2|a)$$
$$= -\sqrt{P(b_1, b_2|a)P(b_1, b_2|U(a))}, \tag{18}$$

where the maximal overlap of $P(b|U(a)) = 3/4$ is achieved by applying a phase shift of $\pi$ between the arms of each interferometer. Together with the condition given by Eq.\,[15], this relation determines the negative value of $P(O_1, O_2; b_1, b_2|a)$ that is responsible for the paradoxical statistics observed in separate measurements of paths and outputs. Thus the Hardy paradox can also be traced back to a half-periodic transformation that relates the initial conditions $a$ with the final conditions $b$, represented in this case by a phase flip in both of the two interferometers.

### 7. Correlations and contextuality

The reason why quantum paradoxes come as a surprise is that we have no intuitive understanding of the quantum formalism. In the standard formalism, it is completely unclear why physical properties can be “represented” by operators, and we are forced
to accept that the mathematical properties of the operators somehow correspond to the outcomes of measurements obtained in actual experiments. One example that illustrates these difficulties can be expressed by the odd structure of correlations between two spin-1/2 systems. If the orthogonal spin components of spin $i$ are given by $X_i$, $Y_i$, $Z_i$ with values of $\pm 1$, it is possible to describe the correlations between two spin components of different spin systems by their products, which will also have values of $\pm 1$. Even though it is not possible to measure different spin components jointly, it is possible to measure a complete set of correlations, since the products of the operators commute even when the individual operators do not. For example, the well known Bell states are typically defined as eigenstates of the correlations $(X_1 X_2)$, $(Y_1 Y_2)$, and $(Z_1 Z_2)$. If the three correlations were independent of each other, there should be eight possible combinations of eigenvalues. However, there are only four, and all four of them satisfy the relation

$$ (X_1 X_2)(Y_1 Y_2)(Z_1 Z_2) = -1. \quad (19) $$

Taken all by itself, the value of this product seems to suggest a rather strong deterministic relation between the spin components $X_i$, $Y_i$, $Z_i$. However, it is also possible to define a different set of eigenstates by simply exchanging $X_2$ and $Y_2$ in the correlations. In that case, the value of the product changes its sign,

$$ (X_1 Y_2)(Y_1 X_2)(Z_1 Z_2) = 1. \quad (20) $$

Clearly, any measurement independent assignment of values to the six spin components can only satisfy one of these two relations. It is therefore impossible to explain the simultaneous assignment of correlations in terms of the individual properties that define the correlation.

One way to analyze this problem further is to look at the relation between the local properties defined by $X_1$ and $Y_2$, and the local properties defined by $Y_1$ and $X_2$. It is then possible to ask how the correlation $(Z_1 Z_2)$ is determined by the values of the local spins. Classical realism would require that either Eq. (19) or Eq. (20) should be satisfied. However, quantum mechanics actually allows both relations to be correct by identifying them with different measurement contexts. Specifically, it makes a difference whether we chose to measure $X_1$ and $X_2$ jointly, or if we combine $X?1$ with $Y_2$ in the individual measurements. The correlations between two spin-1/2 systems are therefore an example of the contextuality paradox, where it is shown that the measurement results of the two correlation measurements and the four possible combinations of local measurements cannot be explained by any measurement independent assignment of local spin values [2, 27].

In principle, all contextuality paradoxes can be analyzed using the negative conditional probabilities associated with half-periodic transformations. However, the case of spin correlations is of particular interest, because it can be explained in terms of transformations that act on joint properties of the two particles. For example, we can choose the initial condition $a = (X_1 = +1, Y_2 = +1)$ and the final condition $b = (Y_1 = +1, X_2 = +1)$. These two conditions are related by a swap of system 1
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and system 2. Since a double swap restores the original situation, the swap operation is a half-periodic transformation. Moreover, the swap operation leaves the correlations \((X_1X_2), (Y_1Y_2),\) and \((Z_1Z_2)\) unchanged. All we need to do to apply Eq.\((3)\) to the problem is to identify the combination of correlations that is associated with the action phase of \(\pi\), and hence with the negative probability responsible for the paradox. For this purpose, we need to classify the correlations permitted by Eq.\((19)\). If all three correlations are \(-1\), there is no preferred spin direction and we can label this state by \(S\), where the letter reflects the historical fact that this is the state of the spin singlet. Otherwise, only one of the three correlations will be \(-1\), so the three states form the triplet \(T_x, T_y, T_z\). The swap operation assigns a phase of \(\pi\) to the singlet \(S\), so the joint probabilities derived from Eq.\((3)\) are

\[
P(S; b|a) = -\frac{1}{8}, \quad P(T_z; b|a) = \frac{1}{8},
\]

\[
P(T_x; b|a) = \frac{1}{8}, \quad P(T_z; b|a) = \frac{1}{8}.
\]

(21)

It is now possible to explain the contradiction between Eq.\((19)\) and Eq.\((20)\) and the contextuality paradox associated with it. If we consider the relation between \(a = (X_1, Y_2)\) and \(b = (Y_1, X_2)\), they are related by a swap along generator-states \(m\) defined by the correlations \((X_1X_2), (Y_1Y_2),\) and \((Z_1Z_2)\). Eq.\((19)\) describes the allowed correlations in \(m\), but the conditional average of each correlation is determined by the negative joint probability in Eq.\((21)\). Incidentally, these conditional averages are all equal to +1, so that the product of the averages actually satisfies Eq.\((19)\).

Importantly, it is not correct to derive the value of \((X_1X_2)\) from the results of two separate non-commuting measurements that happen to include the value of \(X_1\) in measurement \(a\) and the value of \(X_2\) in measurement \(b\). The correct relation between three different measurements is determined by the action of transformations between them, and the classical product relation does not apply when \(X_1, X_2,\) and \((X_1X_2)\) belong to three different measurements that cannot be performed jointly. Thus, Eq.\((3)\) can give a more explicit meaning to the notion of contextuality in quantum mechanics.

8. Violation of Bell’s inequalities by contextual correlations

The insights gained about the contextuality of correlations can also be used to shed some light on what is arguably the best known quantum paradox, the violation of Bell’s inequalities \([1]\). Essentially, Bell’s inequalities describe the limit of correlations between the local spin components \(X_i\) and \(Y_i\) when each component has a value of \(\pm 1\). The limit is given by a sum of four individual correlations,

\[
K = X_1X_2 + X_1Y_2 + Y_1X_2 - Y_1Y_2.
\]

(22)

It is easy to see that, for any combination of spin values, the value of \(K\) is either \(+2\) or \(-2\). Therefore, all positive valued joint probability distributions for \(X_i\) and \(Y_i\) satisfy the Bell’s inequality

\[
|\langle K \rangle| \leq 2.
\]

(23)
Oppositely, a violation of Bell’s inequality corresponds to the assignment of a negative joint probability to $K = -2$ for $\langle K \rangle > 2$, or to $K = +2$ for $\langle K \rangle < -2$.

Importantly, there are two non-equivalent ways to define joint probabilities for pairs of spins, depending on the combinations of spins in the conditions $m$ and $b$. We can either chose $m = (X_1, X_2)$ and $b = (Y_1, Y_2)$, or we can chose $m' = (X_1, Y_2)$ and $b' = (Y_1, X_2)$. In the former case, the correlations $(X_1X_2)$ and $(Y_1Y_2)$ are directly defined by $m$ and $b$, respectively, but the correlations $(X_1Y_2)$ and $(Y_1X_2)$ will depend on the transformation dynamics of $b$ generated by $m$. In the latter case, the transformation dynamics will show up in $(X_1X_2)$ and $(Y_1Y_2)$, while $(X_1Y_2)$ and $(Y_1X_2)$ are directly defined by $m$ and $b$.

In general, negative probabilities occur if there is a half-periodic transformation that transforms $b$ into $a$ along $m$. To explain the violation of Bell’s inequalities, we need a half-periodic transformation that transforms can transform product states into entangled states, where the eigenstates themselves should also be product states. A well-known half-periodic transformation with these properties is the quantum-controlled NOT, which essentially describes conditional spin-flips in the two systems. Specifically, a quantum-controlled NOT with eigenstates $m = (X_1, X_2)$ can transform eigenstates of $b = (Y_1, Y_2)$ into entangled states with maximal correlations of $X_1Y_2 = \pm 1$ and $Y_1X_2 = \pm 1$. We can therefore express the initial state $a(0)$ with well-defined correlations of $m = (X_1, X_2)$ and $b = (Y_1, Y_2)$ according to the transformation rules of Eq.(3),

$$
P(m = (+1, +1); b = (+1, +1)|a(0)) = \frac{1}{8},$$

$$
P(m = (+1, -1); b = (+1, +1)|a(0)) = \frac{1}{8},$$

$$
P(m = (-1, +1); b = (+1, +1)|a(0)) = \frac{1}{8},$$

$$
P(m = (-1, -1); b = (+1, +1)|a(0)) = -\frac{1}{8}. \quad (24)$$

Comparable sets of probabilities can be obtained for all other values of $b = (Y_1, Y_2)$. Although the joint probabilities already include negative values, this state does not violate Bell’s inequalities, since the negative and the positive contributions cancel out in the expectation values of $(X_1X_2)$ and $(Y_1Y_2)$, resulting in expectation values of zero for these correlations. The negative probabilities obtained from the half-periodic transformations between $a(0)$ and $b$ along $m$ are not sufficient to achieve a violation of Bell’s inequalities all by themselves. However, Eq. (3) does not require a perfect transformation with $p(b|U(a)) = 1$. Instead, it is sufficient if the half-periodic transformation optimizes the overlap between the initial condition $a$ and the final condition $b$. It is therefore possible to increase the correlation $(X_1X_2 - Y_1Y_2)$ in $a$ without eliminating the negative probabilities defined by the transformation of $(Y_1, Y_2)$ into $(X_1Y_2, Y_1X_2)$.
The complex joint probabilities defined by Eq. (3) now include both the negative correlations between spin-directions in the $XY$-plane with an angle of $\theta$ between them,

$$\cos(\theta)X_1Y_2 + \sin(\theta)X_1X_2 = 1, \quad \cos(\theta)Y_1X_2 - \sin(\theta)Y_1Y_2 = 1. \quad (25)$$

The optimization can be achieved by choosing an initial condition $a(\theta)$ defined by correlations between spin-directions in the $XY$-plane with an angle of $\theta$ between them,

$$P(m = (X_1, X_2); b = (Y_1, Y_2)) = \frac{1}{8} \cos(\theta),$$

$$P(m = (X_1, X_2); b = (Y_1, Y_2)) = \frac{1}{8} (1 - \sin(\theta)),$$

$$P(m = (X_1, X_2); b = (Y_1, Y_2)) = \frac{1}{8} (1 - \sin(\theta)),$$

$$P(m = (X_1, X_2); b = (Y_1, Y_2)) = -\frac{1}{8} \cos(\theta). \quad (26)$$

Among these probabilities, only $m = (1, +1)$ contributes to $K = +2$. Therefore, the negativity of $P(K = -2)$ can be increased by reducing the positive probabilities of $m = (1, -1)$ and $m = (-1, +1)$. The maximal Bell’s inequality violation is obtained by finding the maximal negative value of the sum of all probabilities for outcomes with $K = -2$.

The complete set of joint probabilities is shown in table 1. Probabilities of $(1 + \sin(\theta))/8$ are assigned to all combinations with $X_1X_2 = +1$ and $Y_1Y_2 = -1$. For all of these combinations, $K = +2$. Likewise, probabilities of $(1 - \sin(\theta))/8$ are assigned to combinations with $X_1X_2 = -1$ and $Y_1Y_2 = +1$, which have $K = -2$. For the remaining eight combinations, positive probabilities of $\cos(\theta)/8$ are assigned to combinations with $X_1Y_2 = +1$ and $Y_1X_2 = +1$, where $K = +2$, and negative probabilities of $-\cos(\theta)/8$ are assigned to combinations with $X_1Y_2 = -1$ and $Y_1X_2 = -1$, where $K = -2$. In summary, probabilities are low but positive if the value of $K = -2$ can be obtained directly from $m = (X_1, X_2)$ and $B = (Y_1, Y_2)$, while the probabilities are negative if the value of $K = -2$ originates from correlations between $m$ and $b$. Bell’s inequality is violated because the total probabilities corresponding to correlation sums of $K = -2$
have a negative value of

\[ P(K = -2) = \frac{1}{2} (1 - \sin(\theta) - \cos(\theta)) < 0. \]  

(27)

Thus, the violation of Bell’s inequality is achieved by a combination of two local contexts in the initial correlations \( a \), where one context is given by the choice of \( m \) and \( b \) and the other context is given by the correlations between \( m \) and \( b \) that are determined by the dynamics of transformations. Negative probabilities appear in the correlations between \( m \) and \( b \) as a result of the half-periodic transformations that describe the relation between the initial condition \( a \) and the properties \( m \) and \( b \), while the positive probabilities of contributions to \( K = -2 \) that are determined individually by \( m \) and \( b \) are sufficiently lower, resulting in an overall violation of the positive probability limit of \( \langle K \rangle \leq 2 \).

9. Why non-positive probabilities make sense

All quantum paradoxes can be resolved if we can understand why the fundamental relations between the physical properties involved in the paradoxes must be expressed in terms of complex joint probabilities, where the complex phase is determined by the action of transformations between the different properties as given by Eq.(3). Importantly, this means that the idea of a simultaneous reality of these physical properties needs to be abandoned. Obviously, this is not a trivial matter - we normally think of objects in terms of a complete set of physical properties, regardless of whether these properties are observed in a measurement or not. How can we explain the limitation to properties that are actually measured?

Here, it is essential that complex probabilities relate to transformations that describe the actual dynamics generated by a physical property. We need to remind ourselves that physical reality is only known from interactions, and these interactions necessarily involve transformations of the type that define the phases of complex probabilities. In the classical limit, we can approximately separate the effect of the object from the changes of the object that are inadvertently caused by the interactions, even though light scattering or physical touch exert forces on the object that do change its shape and motion. The quantum formalism suggests that these changes cannot be separated from the effects that define the object as a thing that we can see and touch.

In direct measurements of a physical property, the interaction completely randomizes the unobserved properties, thereby eliminating the possibility of joint measurements. However, weak measurements can be used to obtain statistical evidence of the unobserved properties before the strong measurement randomized them. It is therefore possible to experimentally confirm the negative probabilities predicted by Eq.(3), and corresponding experiments have already been reported for most of the paradoxes analyzed above [8, 9, 10, 11, 12, 13, 14, 15]. The discussion presented here is intended as an explanation of the general physical principles that are revealed by these experimentally confirmed results. Both the theory and the experiment strongly
indicate that the paradoxes only arise because we insist on an artificial separation of the physical reality of an object from the dynamics of the interactions by which we know the object. However, we should realize that quantum physics integrates the structure of the dynamics that characterizes all measurement interactions into the statistics of the measurement outcomes. As shown in section 3, the non-classical correlations described by the standard formalism correspond to complex probabilities, which naturally express this integration of dynamics into the statistics of measurement outcomes by their complex phases.

Complex probabilities make sense because their complex phases have a well-defined operational meaning: they describe the relations between physical properties in terms of the dynamics of transformations between them. This relation replaces the assumption of a joint reality, which has no foundation in observable fact and fails to explain the statistics observed in quantum paradoxes. By expressing the relation between transformations and statistics in terms of complex probabilities, Eq. (3) predicts that negative real parts must appear in the joint probabilities of non-commuting properties as a direct consequence of the transformations between them. Quantum paradoxes can therefore be resolved by properly understanding that the quantum formalism defined the relations between physical properties by the dynamics of their transformations, and not by a hypothetical joint reality that is never observed in any experiment.

10. Conclusions

In physics, all paradoxes are the consequence of wrong assumptions about the laws of nature. Once the physics is properly understood, paradoxes can be resolved by a correct explanation of the fundamental relations between the actual phenomena. Up to now, quantum paradoxes remained mysterious because the formalism that actually predicts them was treated as a mathematical black box without any physical meaning. In this paper, I have shown that a better understanding of the physics is possible. Quantum paradoxes can all be explained by a single fundamental relation between the physical properties involved in the paradox, and this fundamental relation predicts negative joint probabilities whenever three physical properties are related to each other by half-periodic transformations. Importantly, this fundamental relation also explains why the expectation of a measurement independent reality fails: the reality of an object only emerges in interactions, so it is entirely possible that the interactions themselves are an inseparable part of this reality. By describing the relation between physical properties in terms of complex probabilities, quantum mechanics objectively identifies the correct relation between dynamics and reality valid for all phenomena at the quantum level [20]. Quantum paradoxes can thus be resolved by a fundamental explanation of the actual physics described by the quantum formalism.
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