The magnetoelectric effect due to a semispherical capacitor surrounded by a spherical topologically insulating shell

Daniel G Velázquez and L F Urrutia

Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México, 04510 México, Distrito Federal, México
E-mail: danielgv@ciencias.unam.mx and urrutia@nucleares.unam.mx

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Abstract
We calculate the magnetic field produced by a capacitor formed by two semispherical perfectly conducting plates of radius $a$, subjected to a potential difference and surrounded by a spherical shell of a topologically insulating material having internal radius $r_1$ and external radius $r_2$. Fixing $r_2 = 1 \mu m$ and considering the case where the shell touches the metallic plates, i.e. when $r_1 = a$, we find that the maximum magnetic field occurs at the external surface of the shell when $r_1 \leq 0.75 \mu m$. For these cases we examine the effect in the two remaining variables: the thickness of the shell and axial angular dependence of the magnetic field. For $r_1 = 0.5 \mu m$ we find a highly isotropic angular distribution with an average field of magnitude $|B| \approx 0.2 G$. The angular anisotropy increases with $r_1$, yielding $0.17 G \lesssim |B| \lesssim 0.35 G$ for $r_1 = 0.62 \mu m$ and $0.1 G \lesssim |B| \lesssim 0.6 G$ for $r_1 = 0.75 \mu m$. These magnitudes fall within the sensitivities of magnetometers based upon nitrogen-vacancy centers in diamond, as well as of devices using scanning SQUID magnetometry. In the latter case we obtain fluxes in the range of $(6 - 10) \times 10^{-10} G \cdot cm^2$ for a pickup loop of radius $10 \mu m$ centered at the axial symmetry axis and located parallel to the equatorial plane at distances ranging from 2 to $6 \mu m$ from the center of the capacitor.

Keywords: Non dynamical Axion Electrodynamics, Topological insulators, The magnetoelectric effect, Measurable magnetic field

(Some figures may appear in colour only in the online journal)

1. Introduction

The magnetoelectric (ME) effect, consisting either in the generation of a magnetization by the action of an electric field or in the production of a polarization due to a magnetic field, was predicted to occur in $Cr_2O_3$ by Dzyaloshinskii in 1959 [1] and closely followed by its observation by Astrov in 1960 [2]. Both authors gave credit to Landau and Lifshitz [3] for anticipating the existence of this property in antiferromagnets, of which $Cr_2O_3$ is an example. For a detailed account of the advances in this field prior to 1959 see [4]. After the discovery, the early stages of the theoretical and experimental research on the electrodynamics of the ME effect were summarized in [5]. A recent update of the revival of the effect, including new methods for designing ME materials, new experimental techniques and novel theoretical concepts to understand the ME behavior is also reported in [4].

Magnetoelectrics, i.e. materials exhibiting the ME effect, can be found among the class of ferroic materials. A ferroic phase is characterized by been able to adopt a spontaneous, switchable internal alignment of some property. The two cases relevant for our work are: (i) ferromagnets, where the internal spins can be switched by an external magnetic field, and (ii) ferroelectrics where a similar manipulation of the electric dipole moments can be achieved by means of an electric field. A magnetoelectric multiferroic combines those

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1 Author to whom any correspondence should be addressed.
two primary ferroic orderings in single or composite phases and is susceptible to produce a larger ME effects. Nevertheless, magnetoelectrics need not necessarily arise from ferroic phases but can be present in either magnetically and/or electrically polarizable media [6, 7].

The signature of the linear ME effect is the appearance of the term $\alpha_{ij} E_i B_j$ in the expansion of the free energy density $F(\mathbf{E}, \mathbf{B})$ of the material, which Requires that both parity and time reversal symmetries are violated separately in the crystal structure. Then, the polarization $P_i = -\partial F / \partial E_i$ and the magnetization $M_i = -\partial F / \partial B_i$ acquire additional terms $\alpha_{ij} B_j$ and $\alpha_{ij} E_j$, respectively. The tensor $\alpha_{ij} = \partial P_i / \partial B_j = \partial M_i / \partial E_j$, dubbed the magnetoelectric coefficient, is symmetric because of the equality of mixed partial derivatives which gives raise to the Maxwell relations in thermodynamics [8, 9]. Symmetry considerations, based on the magnetic group classification of crystals have played a fundamental role in the identification of their ME properties [10]. In general the coefficient $\alpha_{ij}$ is very weak being of the order of $4.13 \text{ ps m}^{-1}$ (SI units) for $\text{SrO}_3$, for example. This weakness is vividly illustrated in [4] by observing that applying an electric field of $10^5 \text{ V cm}^{-1}$ would induce a magnetization obtained after reversal of only five of every $10^5$ spins in this antiferromagnetic lattice. Higher couplings of the order of $735 \text{ ps m}^{-1}$ have been achieved in $\text{TbPO}_4$ and is the search for giant ME couplings continues mainly in composite multiferroics [11]. These materials offer significant advantages and improvements including greater flexibility in designing specific applications, room temperature operation and optimization of the coupling properties [12]. The first composite magnetoelectric material was created from the ferroelectric $\text{BaTiO}_3$ and the ferromagnetic $\text{CoFe}_2\text{O}_4$ [4]. These composite materials have shown more intense couplings than the monophasic multiferroics.

Although remarkable improvements to the properties of nanoscales devices have been achieved through the current-induced magnetic field control of the ME effect, further reduction in power operation is envisaged by using electric field (voltage) sources which greatly cut down the Joule heating associated to the current required to produce the source magnetic field [6, 13, 14]. This has drawn much attention in the electric field manipulation of the ME effect which is an exiting new area of research with the potential to impact magnetic data storage, spintronics and high frequency magnetic devices, leading to smaller and more energy efficient devices.

A new family of materials that also enable the ME effect has been recently discovered. These are the lately disclosed time-reversal three dimensional topological insulators (TIs), which exhibit remarkable properties originating from their peculiar band structure [15, 16]. From an effective macroscopic perspective, which corresponds to the approach we consider in this work, they can be characterized as insulators in the bulk, but conductors on their surfaces due to the presence of quantized Hall currents. The recognition of topological phenomena in condensed matter dates back to [17] which identified the conductivity of the quantum Hall effect [18] with the first Chern number of a Berry curvature in the reciprocal space. Since then, several investigations [19–24] led to a more profound understanding of these topological phases, both theoretically and experimentally. In particular, Bernevig [25] predicted the existence of two-dimensional TIs in quantum wells of HgTe, and very Konig confirmed it experimentally [26]. In subsequent years, the phenomenon became generalized to three dimensions, again starting with theoretical predictions [27–32] followed by experimental confirmation [33]. Different manifestations of the ME effect have been studied in TIs, among which we mention some representatives. A universal topological ME effect has recently been measured in TIs [34]. Electric charges close to the interface between two TIs are shown to induce image magnetic monopoles [35–37]. The propagation of electromagnetic waves across a boundary between TIs have been studied finding that a non trivial Faraday rotation of the polarization appears [38, 39]. The shifting of the spectral lines in hydrogen-like ions placed in front of a planar TI, as well as the modifications to the Casimir–Polder potential in the non-retarded approximation were studied in [40]. The classical dynamics of a Rydberg hydrogen atom near a planar TI has also been investigated [41]. In this work we will consider the case when the ME effect is produced by TIs, which manifest itself through the isotropic contribution to the ME coefficient, i.e. when $\alpha_{ij} = \Theta \delta_{ij}$.

In the static regime, the production of electrically generated magnetic field in magnetoelectrics has already been explored in [35, 36] by locating point-like charges in front of a planar semi-infinite magnetoelectric medium. Nevertheless, point-like charges are difficult to manipulate and it would be difficult to fix and localize them above the surface, as already pointed out in [36]. Given the relevance that the topic of electric control of the magnetoelectric effect has acquired, the first motivation for our work arises from the necessity of having more manageable devices to achieve this goal in TIs. Certainly the use of macroscopic sources would be a realistic advance in this direction, together with a voltage control of the electric field through the use of capacitors. In [42] point like charges were replaced by metallic spheres at a given potential, with radii $r$ about $0.4–2.0 \mu m$ located at optimal distances $D$ of the order of $\mu m$ also, with $D > r$, in front of a semi-infinite planar TI. Even when measurable magnetic fields were obtained, the optimal regions for the measurement are between the metallic sphere and the planar medium, which might difficult the insertion of the magnetometer. Thus, a second motivation is to produce a compact device that would facilitate the insertion of the magnetometer at the points of maximum magnetic field, accompanied with the substitution of the semi-infinite medium yielding the ME effect. Also, an important aspect of the proposal is to ensure that the magnetic field produced is measurable according to the sensitivity of present day precise magnetometers. This naturally leads to a third motivation, which is the use of such devices to determine the magnetoelectric coefficient of the TI, or of other magnetoelectrics, by measuring the resulting magnetic field. The device we propose consists of a capacitor formed by two metallic semisheres (or two semispherical plates), charged by a potential difference and covered with a uniform film of the magnetoelectric material, which constitutes a new
realization of the ME effect in TIs. Recent advances in the fabrication of electrically manipulable magnetoelectric materials [13, 14] and the new developments in coating techniques for conductors [43], might lend feasibility to this compact setup, which solves some of the previously mentioned challenges. A first difference that shows up is that we consider, for the first time to our knowledge, a dipole as the source for the static ME effect, instead of the charged sources used in the previous cases. An interesting possibility that emerges is that the voltage between the plates can be made time-dependent thus giving additional motivation to the study of radiation in the electrodynamics describing the electromagnetic response of magnetoelectrics. The details of the proposed device are given in section 3.

The paper is organized as follows. Section 2 deals with a basic review of non-dynamical Axion Electrodynamics, which describes the effective macroscopic electromagnetic response of TIs. In section 3 we establish our general setup and adapt the general equations to this case. The boundary conditions at the interfaces are also written, allowing us to match the solutions to the standard Maxwell equations in each bulk region. Since the general solution depends on the isotropic contribution Θ of the ME coefficient α_{ij}, in a way that is hard to handle, we take advantage of the small value of Θ and consider a perturbative expansion in this parameter. The detailed procedure is found in the appendix A. In section 4, we select some particular configurations (limiting cases) to simplify the problem on one side and, on the other, to confer more physical relevance to the results. Also we show several plots of the streamlines of the fields E and B for some configurations that are representative of the selected cases mentioned above. In section 5 we show some estimations of the magnetic field produced by the ME effect in this semi-spherical system, which turn out to be detectable within the current experimental possibilities. Nowadays, it is possible to measure magnetic fields of order 10–100 mG [44, 45] and magnetic fluxes of the order of 10^{−13} G cm² [46]. This bestows phenomenological relevance to the problem. Our results suggest two different empirical approaches to detect the magnetic field: of the two most relevant configurations, one generates highly isotropic but weak fields (though detectable), while the other gives rise to more intense fields in particular directions.

2. Axion electrodynamics

The equations that encode the electromagnetic response of ordinary matter can be derived from the standard Lagrangian density in electrodynamics $L_{\text{em}} = (1/8\pi) (\varepsilon E^2 - (1/\mu) B^2 + J \cdot A - \rho \Phi)$, where the fields are expressed in terms of the electromagnetic potentials A, Φ. Here ε and μ are the permittivity and the permeability of the medium while ρ and J stand for the external charge density and current density, respectively.

The electromagnetic response of TIs is captured by adding the term $\mathcal{L}_\Delta = (\alpha/4\pi^2) \partial_\tau (r, t) E \cdot B$ to the Lagrangian density $L_{\text{em}}$, where $\alpha = e^2/(\hbar c)$ is the fine structure constant and $\partial_\tau$ describes an additional property of the medium called the magnetoelectric polarizability (MEP) [47]. The major modification that results from adding this term is precisely the appearance of the ME effect.

A simple way to show the properties of $\mathcal{L}_\Delta$ is to introduce the Faraday tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and the Levi-Civita symbol $\varepsilon_{\mu\nu\rho\sigma}$, in terms of which we rewrite $E \cdot B = -\frac{1}{8} \varepsilon_{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \equiv -\frac{1}{8} \mathcal{P}$, where $\mathcal{P}$ is the abelian Pontryagin density, which is a topological invariant [48]. The Bianchi identity yields $E \cdot B = -\frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} \partial_\rho (A_\sigma F_{\mu\nu})$, which implies that when $\partial_\rho$ is constant, the term $\mathcal{L}_\Delta$ is a total derivative, so it does not affect the equations of motion.

The topological properties induced by the coupling to the MEP are most clearly appreciated by defining the action of the system as $S = \frac{\hbar}{4e} \int d^4x (L_{\text{em}} + \mathcal{L}_\Delta)$ in the CGS system. Assuming periodic boundary conditions and a manifold without boundaries, the contribution $S_\partial$ results

$$ S_\partial = \frac{\partial}{32\pi^2} \int d^4x e^{\alpha/j} \varepsilon_{\mu\nu\rho\sigma} \frac{1}{e^2} F_{\alpha\beta} F_{\mu\nu} \equiv \partial C_2, $$

where the integer $C_2$ is the second Chern number of the manifold. Recalling that the relevant physical object is $e^{\alpha/j}$ and $\hbar$ we realize that the extended electrodynamics is invariant under the transformation $\partial \rightarrow \partial + 2\pi n$ [37, 49]. Moreover, imposing time reversal invariance $\partial \rightarrow -\partial$ yields the two possible values $\partial = 0$ and $\partial = \pi$, which satisfy $e^{\alpha/j} = e^{-i\sigma} [47]$. This produces the Z2 classification of TIs where $\partial = 0$ characterizes normal insulators, whereas $\partial = \pi$ defines the topological phase. To detect the ME effect in TIs we need at least an interface between two media with different values $\partial_1$ and $\partial_2$ such that $\partial_1 \partial_2 \neq 0$ there. Also, this effect becomes observable only if the surfaces and interfaces of the sample are consistently gapped by some time-reversal breaking perturbation, which can be achieved by covering them with a thin magnetic layer of the order of a few nanometers. In a real experimental situation the effects due to the magnetic coating of the TI are usually negligible. Since the values of $\partial$ are determined modulo $2\pi$, we will have $\partial_1 - \partial_2 = \pi + 2m$, with $n$ by determined by the specific time reversal symmetry breaking mechanism at the interface. This freedom might provide a way to enhance the ME coupling in TIs. The non-trivial coefficient $(2n + 1)\pi$ is related to a semi-quantized conductivity on the surface of the material $\sigma = (\partial_1 - \partial_2) e \ell^2 = (n + 1/2)e^2$ which arises due to the boundary terms in $L_\Delta$ [35]. Here $\hbar = 2\pi\hbar$ is the Planck constant. The physical origin of this phenomenon in topological materials is the quantum Hall effect [50].

Let us observe that the equations which arise from the Lagrangian density $L_{\text{em}} + \mathcal{L}_\Delta$ may describe different physical phenomena according to the choice of $\partial (r, x)$ and not only the electromagnetic response of TIs. For example, the electrodynamics of metamaterials when $\partial \in \mathbb{C}$ [51, 52] and the response of Weyl semimetals when $\partial(x, t) = 2b \cdot x - 2\theta_0t$ [53, 54]. The term $L_\Delta$ also describes the interaction of the hypothetical axionic field with the electromagnetic field in elementary particle physics [55–57].

To obtain the modified Maxwell equations from the Lagrangian density $L_{\text{em}} + \mathcal{L}_\Delta$ we introduce the electromagnetic
potentials $\Phi$, $\mathbf{A}$ such that
\[ \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi, \quad \mathbf{B} = \nabla \times \mathbf{A}, \]
which leads to the standard homogeneous Maxwell equations $\nabla \cdot \mathbf{B} = 0$ and $c \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$. The resulting inhomogeneous equations are
\[ \nabla \cdot (\varepsilon \mathbf{E}) = 4\pi \rho = \frac{\partial \rho}{\partial t}, \]
\[ \nabla \times \left( \mu \frac{\partial \mathbf{E}}{\partial t} \right) - \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{J} - \frac{\alpha \varepsilon}{c \pi} \mathbf{B}. \]
The above equations can also be understood as those of electrodynamics in a material medium having the constitutive relations
\[ \mathbf{D} = \varepsilon \mathbf{E} - \frac{\partial \rho}{\partial t}, \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B} + \frac{\partial \mathbf{E}}{\partial t}. \]
From the above equations we read $\mathbf{P} = -\frac{\partial \rho}{\partial t} (4\pi^2) \mathbf{B}$ and $\mathbf{M} = -\partial \mathbf{E} / (4\pi^2) \mathbf{B}$ in non-rationalized Gaussian units, yielding $\rho_{\Sigma} = \alpha \varepsilon / (4\pi^2) = \mu \Phi / (4\pi)$ such that for a TI ($\varepsilon = \pi$) the magnitude of isotropic magnetoelectric coefficient is $\Theta = \alpha / (4\pi)$ and takes the value 1.9 ps/m.

We choose the simplest case where $\theta(x)$ takes constant values, $\theta_1$ and $\theta_2$, in two regions $\Omega_1$ and $\Omega_2$ separated by an interface $\Sigma$ parametrized with the equation $F(x) = 0$, the effective sources arising from equations (3) are
\[ \rho_{\Sigma} = \frac{\alpha}{4\pi^2} (\theta_2 - \theta_1) \delta (F(x)) \mathbf{n}_\Sigma \cdot \mathbf{B}, \]
\[ \mathbf{J}_\Sigma = \frac{\alpha}{4\pi^2} (\theta_2 - \theta_1) \delta (F(x)) \mathbf{n}_\Sigma \times \mathbf{E}, \]
where $\mathbf{n}_\Sigma$ is a vector perpendicular to the interface. That is to say, corrections to the dynamics arise only at the interface, while the bulk regions satisfy the unmodified Maxwell equations. A particularly interesting case occurs when $\Sigma$ coincides with the surface of a perfect conductor, since then $\mathbf{E}$ is normal to the interface, while $\mathbf{B}$ is tangential, yielding that both effective sources are zero, in spite of the presence of a gradient of $\theta(x)$. It is important to note that this result is general: if a $\theta$ interface matches a perfect conducting surface, there is no contribution from that interface to the Maxwell equations (3).

When $\theta$ is piecewise constant, it is important to emphasize that equations (3) yield the electromagnetic response of an arbitrary isotropic magnetoelectric media. This happens when the material violates both time reversal and space inversion. In this general case the MEP will have arbitrary values as compared with the fixed value $\theta = \pi$ characteristic of TIs. Even though they are theoretically allowed, no canonical magnetic insulators with this ME response has yet been identified experimentally [36]. An interesting consequence of the ME effect in magnetoelectric media is that an electric charge near its surface can generate not only an image electric charge as usual, but also a magnetic image monopole. This image monopole provides an alternative mathematical interpretation of the magnetic field produced by the surface current $\mathbf{J}_\Sigma$ in equation (5).

3. The magnetoelectric effect due to a semispherical capacitor surrounded by a spherical shell made of a TI

One of the simplest way to explore the ME effect in TIs is to locate an electric source in front of the material and determine/mesure the induced magnetic field. Implementation of this idea has been reported for the following cases: (i) a pointlike charge in front of a planar TI [35, 36] and (ii) a sphere of finite radius in front of a planar TI [42]. In this work we propose to replace point-like charges and semi-infinite TIs by macroscopic spherical capacitors producing a voltage controlled magnetic field in the presence of a TI uniformly deposited in the spherical surface, thus producing a very compact setup which can be easily accessed by the required magnetometers. It is interesting to observe that the simplest possibility in this direction would be a metallic sphere at a given potential covered by the TI. Nevertheless, the spherical symmetry of this setup prevents the generation of the ME effect due to the effective current density $\mathbf{J}_\Sigma$ would be zero at the interfaces of the material, thus yielding a zero magnetic field.

In other words, we need to break spherical symmetry, which we do by considering a capacitor formed by two metallic semispherical plates of radius $a$ that are kept at different potentials ($V$ and $-V$), and which do not touch at the equator. We call this device a semispherical capacitor. The region 1 ($a < r < r_1$) is filled with a media with permittivity $\varepsilon_1$ and MEP $\theta_1$. At a distance $r_1$ from its center, such that $r_1 \geq a$, there is a thick TI shell of width $r_2 - r_1$ defining the region 2 ($r_1 < r < r_2$) and having permittivity $\varepsilon_2$ and MEP $\theta_2$. The region 3 ($r_2 < r < \infty$) has the same parameters as the region 1. We take regions 1 and 3 to be the vacuum, such that $\varepsilon_1 = 1, \theta_1 = 0$, and we consider non-magnetic materials taking $\mu = 1$ everywhere. The setup is shown in figure 1.

![Figure 1. System composed by a semispherical capacitor of radius $a$ surrounded by a thick TI shell of width $r_2 - r_1$.](Image 331x581 to 524x773)
In the case that concerns us, the MEP \( \vartheta(x) \) displays spherical symmetry and is given by
\[
\vartheta(r) = \vartheta_1 H(r-a)H(r_1-r) + \vartheta_2 H(r-r_1)H(r_2-r) + \vartheta_3 H(r-r_2),
\]
where \( H(r) \) is the Heaviside function. The two inhomogeneous Maxwell equations in equation (3) become
\[
\nabla \cdot (\epsilon \mathbf{E}) = \alpha \delta(r-r_1) - \delta(r-r_2) \mathbf{B} \cdot \hat{r},
\]
\[
\nabla \times \left( \frac{\mathbf{B}}{\mu} \right) - \frac{1}{c} \frac{\partial (\epsilon \mathbf{E})}{\partial t} = \tilde{\alpha} \delta(r-r_1) \mathbf{E} \times \hat{r},
\]
where we have introduced the quantity
\[
\tilde{\alpha} = (\vartheta_2 - \vartheta_3) \alpha / \pi = (2n + 1) \alpha,
\]
which parametrizes the magnetoelectric effect in TIs. As explained before, the contribution at \( \delta(r-a) \) does not appear in the above equations because the capacitor plates are perfect conductors. Here \( \hat{r} \) is the unit vector in the radial direction. Since the MEP’s are constant in each bulk region, the dynamical modifications only arise at the spherical interfaces \( \Sigma_1, \Sigma_2 \) located at \( r = r_1, r = r_2 \), respectively, where a discontinuity in the MEP arises. They are coded in the boundary conditions at the interfaces, which are obtained from equations (7) and (8). Alternatively, using the constitutive relations (4), one can determine them by imposing at the interfaces the continuity of the normal components of \( \mathbf{D} \) and \( \mathbf{B} \) together with the continuity of the parallel components of \( \mathbf{H} \) and \( \mathbf{E} \), in the absence of external sources. The results at the interface \( \Sigma_1 \) located at \( r = r_1 \) are
\[
[e \mathbf{E} \cdot \hat{r}]_{r=r_1} = \alpha (\mathbf{B} \cdot \hat{r})_{r=r_1},
\]
\[
[\mathbf{B} \times \hat{r}]_{r=r_1} = -\tilde{\alpha} (\mathbf{E} \times \hat{r})_{r=r_1},
\]
\[
[\mathbf{B} \cdot \hat{r}]_{r=r_1} = 0, \quad [\mathbf{E} \times \hat{r}]_{r=r_1} = 0.
\]
The results at the interface \( \Sigma_2 \) at \( r = r_2 \) are obtained from the above just making \( \tilde{\alpha} \rightarrow -\tilde{\alpha} \). Here \( [Q]_{r=b} \equiv \lim_{b \rightarrow 0} (Q(r=b + \delta) - Q(r=b - \delta)) \) denotes the discontinuity of \( Q \) at the spherical interface located at \( r = b \), while \( (S)_{r=b} \equiv S(r=b) \) is the evaluation of a continuous function \( S \) at the corresponding interface.

As equation (8) indicates, the ME effect is physically realized by the generation of surface currents \( \mathbf{K} \) at the interfaces, due to the magnetization \( \mathbf{M} = -\frac{\tilde{\alpha}}{4\pi} \mathbf{E} \), such that
\[
\mathbf{K}_{s,i} = -\frac{\alpha}{4\pi} \mathbf{E} \times \hat{n} |_{r=r_i},
\]
where \( i = 1, 2 \) denotes the corresponding interfaces \( \Sigma_1 \) and \( \Sigma_2 \) respectively. Here \( \hat{n} \) is the unit normal exterior to the surface of the TI. These currents are the physical sources of the magnetic field.

For the resolution of the equations in the bulk we use scalar electric and magnetic potentials \( \Phi \) and \( \Psi \), a choice that is allowed by the conditions \( \nabla \times \mathbf{E} = 0 \) and \( \nabla \times \mathbf{B} = 0 \) in each bulk, such that
\[
\mathbf{E} = -\nabla \Phi, \quad \mathbf{B} = -\nabla \Psi.
\]
Since the electric and magnetic fields satisfy the homogeneous Maxwell equations \( \nabla \cdot \mathbf{E} = 0 \) and \( \nabla \cdot \mathbf{B} = 0 \), the aforementioned potentials satisfy Laplace equation in the regions 1, 2 and 3 previously defined. As the system possesses azimuthal symmetry with respect to the axis \( (z-\text{axis}) \) perpendicular to the plane \( (x-y) \) plane that separates the two semispherical plates, it is enough to express the general solution in spherical coordinates \( (x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \theta) \) as
\[
\Phi_{i}(r, \theta) = \sum_{l=0}^{\infty} (A_{i}^{l} r^{l} + B_{i}^{l} r^{-(l+1)}) P_{l}(\cos \theta),
\]
\[
\Psi_{i}(r, \theta) = \sum_{l=0}^{\infty} (C_{i}^{l} r^{l} + D_{i}^{l} r^{-(l+1)}) P_{l}(\cos \theta),
\]
where the index \( i \) can take the values 1, 2, and 3, referring to the regions defined before. This reduces the whole inquiry to the determination of the coefficients \( A_{i}^{l}, B_{i}^{l}, C_{i}^{l}, D_{i}^{l} \) according to the boundary conditions in equation (10), together with the similar ones at \( r = r_2 \). The appendix A contains a detailed discussion of the solution for the potentials \( \Phi_{i} \) and \( \Psi_{i} \). Let us emphasize that the boundary condition \( \Phi(\theta) = -\Phi(-\theta) \) on the surface of the capacitor implies that all the coefficients with even values of \( l \) are zero. There are twelve coefficients for every odd value of \( l \) and twelve equations that relate them linearly. From now on we introduce the notation
\[
\sum_{l=1}^{\infty} \sum_{l=1,3,5,\ldots}.
\]
Let us emphasize that our setup has a discontinuity of the electric potential at the equator of the capacitor, i.e. at \( r = a, \theta = \pi/2 \), so that approaching to these particular points in any calculation has to proceed as a limiting process. Since the electric potential is the source of the ME effect, we expect that similar care is required when dealing with the magnetic field at these points.

4. The Solution in power series of \( \tilde{\alpha} \) for a time-reversal invariant topological insulator

The full system of equations (A40)–(A51) for the coefficients is very involved and their solution is not to much illuminating. We will consider an approximate solution motivated by time reversal invariance (TRI) together with the fact that \( \tilde{\alpha} \) for TIs is of the order of the fine structure constant for \( n \leq 5 \). TRI in topological insulators imposes some restrictions on the \( \tilde{\alpha} \) dependence of the coefficients according to whether their origin is electric or magnetic. In fact, under time reversal symmetry \( (T) \) the electric and magnetic fields transform according to \( 7 \mathbf{E} = \mathbf{E} \) and \( 7 \mathbf{B} = -\mathbf{B} \), while the charge and current densities behave as \( 7 \rho = \rho \) and \( 7 \mathbf{J} = -\mathbf{J} \). Also \( T \) takes \( \tilde{\alpha} \) into \( -\tilde{\alpha} \) in agreement with the \( \mathbb{Z}_2 \) classification of time-reversal invariant TIs. In our case, both the electric and magnetic fields are sourced by charge densities which create the potential on the capacitor plates. Then, the linear relation between sources and fields, here constructed as gradients of the potentials, demand
that $\Phi(\tilde{a}) = \Phi(-\tilde{a})$ while $\Psi(\tilde{a}) = -\Psi(-\tilde{a})$ to guarantee the correct behavior of the corresponding fields under TRI [42]. Our strategy is to look for an expansion of the coefficients in powers of $\tilde{a}$, which we expect to converge rapidly due to the smallness of the expansion parameter. Then, the previous reasoning requires that the electric (magnetic) coefficients $A^i_f$, $B^i_f$ ($C^i_f$, $D^i_f$) include only even (odd) powers of $\tilde{a}$. We keep the approximation to the lowest order in $\tilde{a}$, retaining only the first order in the magnetic contributions, together with the zeroth order in the electric potential. Even in this simplified situation, the solutions for $A^i_f$, $B^i_f$, $C^i_f$ and $D^i_f$ in equations (A64)–(A75) are not easy to handle, so it is convenient to discuss some particular configurations of the general setup in figure 1 to obtain more accessible results from the theoretical point of view, which in turn will allow to interpret their physical consequences more easily. In the subsequent examples we take medium 1 and medium 3 as the vacuum ($\epsilon_3 = \epsilon_1 = 1$, $\theta_1 = \theta_3 = 0$), and medium 2 as a TI with $\epsilon_2 = 4$ and $\theta_2 = \pi$.

Before dealing with some particular cases we plot the streamlines of the fields for the configuration $a = 1$, $r_1 = 2$, $r_2 = 3$ in figure 2, which is intended to represent the general case. The shape of the electric field is explained by the fact that the permittivity of the magnetoelectric medium has been taken as distinct from that of the vacuum. In contrast, the magnetic field shows a more variable behavior, and it is worth noting that within the medium it is almost completely tangential, except for the vicinity of $\theta = 0$ and $\theta = \pi$. Notice that in spite of the discontinuity of the source at $r = a$, $\theta = \pi/2$, the fields are well behaved everywhere outside the capacitor.

**4.1. Case $\varepsilon = 1$, $\tilde{a} = 0$**

This first case consists in replacing region 2, between $r_1$ and $r_2$, by the vacuum. This should imply null magnetic coefficients, since the magnetoelectric effect cannot manifest with a zero MEP. This problem corresponds to that of finding the electric field produced by a sphere with a northern hemisphere at potential $+V$ and a southern hemisphere at potential $-V$, which is solved in many textbooks (e.g. [58]), and that we will call the trivial configuration henceforth. Under these conditions we verify that the system of equations (A64)–(A75) yields the only non-zero coefficients

$$B^1_f = B^2_f = B^3_f = a^{i+1}V_i,$$

with $V_i$ given by equation (A5). Therefore, the magnetic field is identically zero throughout the space due to the absence of MEP gradients, while the electric potential matches the results found in the literature [58]

$$\Phi(r, \theta) = \sum_i \left(\frac{a}{r}\right)^{i+1}V_iP_i(\cos \theta).$$

A plot of the streamlines of the electric field produced by the trivial configuration is shown in figure 3. It is interesting to observe that the result (17) is independent of $\varepsilon$ since the boundary conditions at the capacitor are stated in terms of the potential instead of the charges at the plates.

**4.2. Case $r_1 = r_2$**

This situation corresponds to the concurrence of the interfaces of the TI, which is now absent. Again we should expect to obtain zero magnetic field. To verify this we focus on the
induced currents $K_{a,k}$. In the limit we have

$$K_{a,1} + K_{a,2} = -\frac{\partial}{4\pi r_{a\rightarrow r_2}} \left( e \times \mathbf{E} \big|_{r=r_1} - e \times \mathbf{E} \big|_{r=r_2} \right) = 0.$$  

(18)

due to the continuity of the tangential component of the electric field. That is, the surface currents are canceled and no magnetic field is produced so that $\Psi = 0$. Therefore, the solution is reduced to the trivial case.

4.3. Case $r_1 = a$

This corresponds to eliminate the vacuum region between the capacitor and the TI, such that the interior of the shell touches the plates of the capacitor. Here the region $1$ shrinks to the surface of a sphere. In order to consistently eliminate the region $1$ we have to verify that the correct boundary conditions on the surface of the capacitor are now satisfied in terms of the coefficients $A_i^2$, $B_i^2$, $C_i^2$, $D_i^2$ when $\Phi_2$ and $\Psi_2$ are evaluated at $r = a$. We establish these conditions after restricting the coefficients in equations (A64)–(A73) to $r_1 = a$, which yields

$$A_i^2 = \left( l + 1 \right) (\varepsilon - 1) \lambda_i^i,$$

$$B_i^2 = r_{a}^{2i+1} \left( \varepsilon + l + 1 \right) \lambda_i^i,$$

$$B_i^3 = \left( 2l + 1 \right) \varepsilon r_{a}^{2i+1} \lambda_i^i,$$

$$C_i^2 = -\alpha \left( l + 1 \right) \varepsilon \lambda_i^i,$$

$$D_i^2 = -\alpha \varepsilon a^{2i+1} \lambda_i^i,$$

$$D_i^3 = -\alpha \varepsilon \left( a^{2i+1} - r_{a}^{2i+1} \right) \lambda_i^i,$$

$$\lambda_i^i = \frac{a^{i+1} \phi_i}{a^{i+1} \left( l + 1 \right) (\varepsilon - 1) + r_{a}^{2i+1} \left( l + 1 \right)}$$  

(19)

The previous condition (A3) for the potential in the surface of the capacitor has to be replaced by requiring $A_i^2 d^2 + B_i^2 a^{-i-l} = V_l$ in terms of the coefficients of the region $2$. From the results in equation (19) we verify that this relation is indeed satisfied. The boundary condition (A39) for the magnetic field at the capacitor should now reads $ld^{-1}C_i^2 = (l + 1) a^{-i-l} D_i^2$. Again, the values in equation (19) show that this condition is satisfied. In this way we can safely forget about the region $1$. The streamlines for this case are plotted in figure 4 which, as will be seen in section 5, is of particular importance because it maximizes the field in the $z$-axis direction for a wide range of values of $a$.

4.4. Case $r_2 \to \infty$

Here we keep the vacuum region between $a$ and $r_1$ but extend the TI to infinity. Going back to the system of equations we observe that (A36), (A28), (A31) and (A38) reduce to

$$\varepsilon \left( A_i^2 r_{a}^{i-l-1} \right) = 0, \quad A_i^2 r_{a}^{i-l} = 0, \quad C_i^2 r_{a}^{i-l} = 0, \quad lC_i^2 r_{a}^{i-l-1} = 0,$$

(20)

since $r_{a}^{n} = 0$ for all $n \geq 1$ in the limit $r_2 \to \infty$. So the only way that the above conditions are satisfied is to take $A_i^2 = C_i^2 = 0$. On the other hand, since region $3$ is now equivalent to $r \to \infty$ the coefficients $B_i^2$ and $D_i^3$ can be set equal to zero such that the potentials vanish. By further eliminating these four variables $A_i^2, C_i^2, B_i^2, D_i^3$ in equations (A52)–(A63), a system of six equations is obtained. The solution is given by

$$A_i^1 = \left( l + 1 \right) (\varepsilon - 1) \lambda_i^i,$$

$$B_i^1 = -r_{a}^{2i+1} \left( l + \varepsilon + l \varepsilon \right) \lambda_i^i,$$

$$C_i^1 = -\alpha \left( l + 1 \right) \lambda_i^i,$$

$$D_i^1 = -\alpha l a^{2i+1} \lambda_i^i,$$

$$D_i^2 = -\alpha l \left( a^{2i+1} - r_{a}^{2i+1} \right) \lambda_i^i,$$

$$\lambda_i^i = \frac{a^{i+1} \phi_i}{a^{i+1} \left( l + 1 \right) (\varepsilon - 1) - r_{a}^{2i+1} \left( l + \varepsilon \right)}.$$  

(21)

The streamlines for this case are shown in figure 5. Its relation to the case C will be discussed in the next subsection.

4.5. The relation between the cases $C$ and $D$

After taking the corresponding limits, the two interfaces of the setups $C$ and $D$ are determined by the radii $\left( a, r_2 \right)$ and $\left( a, r_1 \right)$, as shown in figures 4 and 5 respectively. Such plots suggest that both cases could be related by the transformations

$$\theta_1 \leftrightarrow \theta_2, \quad (\alpha \leftrightarrow -\alpha), \quad r_1 \leftrightarrow r_2, \quad \varepsilon_1 \leftrightarrow \varepsilon_2, \quad (\varepsilon \to 1/\varepsilon),$$

(22)

where $\varepsilon = \varepsilon_1/\varepsilon_2$. Within our approximation, let us verify this assertion starting from Case $C$, where we perform the above transformations (22) in equations (19), denoting the resulting coefficients with a bar superindex. Explicitly we obtain

$$\lambda_i^i \to \bar{\lambda}_i^i = -\varepsilon \lambda_i^i$$  

(23)

$$A_i^2 \to \bar{A}_i^2 = \left( l + 1 \right) \left( 1/\varepsilon - 1 \right) \left( -\varepsilon \lambda_i^i \right) = A_i^1,$$

$$B_i^2 \to \bar{B}_i^2 = r_{a}^{2i+1} \left( 1/\varepsilon + l + 1 \right) \left( -\varepsilon \lambda_i^i \right) = B_i^1,$$

$$B_i^3 \to \bar{B}_i^3 = \left( 2l + 1 \right) \left( 1/\varepsilon \right) r_{a}^{2i+1} \left( -\varepsilon \lambda_i^i \right) = B_i^2$$  

(24)

(25)

(26)
where the coefficients in the right hand side of the above equations are those of the Case D, given in equation (21).

However, the following point in the comparison still requires clarification: in the $r_1 = a$ configuration there are two MEP gradients, while in the $r_2 \to \infty$ configuration there is only one. In the first case, one of them appears at the interface $r = r_2$, while the second corresponds to the surface of the conductor $r = a$. However, the interface at the perfect conductor does not play any role as previously stated.

The established relations in equations (23) and (29) provides a simple quantitative explanation for the change in the direction of the magnetic field in the region external to the spherical shell that takes place in the configuration C with respect to the configuration D and which is apparent in figures 4(b) and 5(b). Let us recall that the magnetic potential is determined by the coefficients $D_l^n$ ($D_l^0$) in the case C (D), respectively. We have to look at the ratio

\[
\frac{D_l^n}{D_l^0} = \frac{D_l^1}{D_l^0}
\]

\[
= \frac{1}{\varepsilon} \frac{a^{2l+1}(l+1)(\varepsilon - 1) + r_1^{2l+1}(\varepsilon + l + 1)}{a^{2l+1}(l+1)(\varepsilon - 1) - r_1^{2l+1}(\varepsilon + l + 1)} \equiv \frac{N_l}{\varepsilon D_l^0}
\]

Figure 4. Streamlines for $a = r_1 = 1 \mu m$, $r_2 = 3 \mu m$ and $\varepsilon = 4$. Panel (a): the electric field and Panel (b): the magnetic field.

Figure 5. Streamlines for $r_2 \to \infty$, with $a = 1 \mu m$, $r_1 = 2 \mu m$ and $\varepsilon = 4$. Panel (a): the electric field. Panel (b): the magnetic field.
and determine its sign. Since $\varepsilon > 1$ and the numerator $N$ is positive we are left only with finding the sign of the numerator $D$. But $r_1 > a$ so that $D < a^{2l+1} ((l + 1)(\varepsilon - 1) - (l \varepsilon + l + \varepsilon)) = -a^{2l+1}(2l + 1) < 0$. In other words $N(\varepsilon D)$ is negative, which yields a relative minus sign between $\Psi_C$ and $\Psi_D$. This explains that the configurations C and D produce magnetic fields in the opposite directions in that region. The analogous situation occurs with the electric potential, where the relevant coefficients are $B_1^2$ and $B_1^3 = B_2^3$. Their ratio is now $B_1^2/B_1^3 = -N/(\varepsilon D)$ which tells us that the relative sign between $\Phi_C$ and $\Phi_D$ is plus, thus yielding electric fields in the same direction for each case. This behavior can be seen in the external regions of figures 4(a) and 5(a).

4.6. $r_1 = a$ and $r_2 \to \infty$

The solution is obtained taking $r_1 = a$ in equation (21) and it is

$$
A_1^0 = -\frac{a^{-l}(1 + l)(\varepsilon - 1)V_1}{2l + 1}, \quad B_1^1 = \frac{a^{l+1}(1 + \varepsilon + l\varepsilon)V_1}{2l + 1},
$$

$$
A_2^0 = 0, \quad B_2^1 = a^{l+1}V_1,
$$

$$
C_1^l = \frac{a^{l+1}(l + 1)V_1}{2l + 1}, \quad D_1^l = \frac{a^{l+1}lV_1}{2l + 1}, \quad C_2^l = 0,
$$

$$
D_2^l = 0.
$$

(31)

Note that in the region $a < r < \infty$ the magnetic potential is null. This is what is expected for a homogeneous medium with constant $\varepsilon$, after recalling that the discontinuity of the MEP in the perfectly conducting interface at $r_1 = a$ does not contribute to the current producing the magnetic field. On the other hand, although the coefficients $A_1^0$ and $B_1^1$ are not zero, the potential in the region 1 only makes sense at $r = a$. It is expected, however, that $\Phi_1(a, \theta) = \Phi_2(a, \theta)$ satisfying the correct boundary conditions (A3). In fact, substituting the equations (31) we verify that

$$
\Phi_1(a, \theta) = \sum_l \left[ \frac{a^{-l}(1 + l)(\varepsilon - 1)V_1}{2l + 1} \right] a^l + \left[ \frac{a^{l+1}(1 + \varepsilon + l\varepsilon)V_1}{1 + 2l} \right] a^{-l+1} P_l(\cos \theta)
$$

$$
= \sum_l V_l P_l(\cos \theta).
$$

(32)

This is an anticipated result because the above boundary condition was imposed from the very beginning to determine $A_1^0$ and $B_1^1$. At the same time,

$$
\Phi_2(a, \theta) = \sum_l (B_2^l a^{-l+1}) P_l(\cos \theta) = \sum_l V_l P_l(\cos \theta).
$$

(33)

The electric potential in the bulk ($a < r < \infty$) is given by

$$
\Phi_2(r, \theta) = \sum_l (B_2^l a^{-l+1}) P_l(\cos \theta)
$$

$$
= \sum_l \left( \frac{a}{r} \right)^{l+1} V_l P_l(\cos \theta),
$$

(34)

that once again coincides with the trivial case.

As an additional check of our results we verify that the non zero values of $C_1^l$ and $D_1^l$ in equation (31) reproduce the boundary condition on the perfect conducting plates imposing that the normal component of the magnetic field is zero. This is given by equation (A34) evaluated at $r = a$, which after substitution yields

$$
\left( \frac{\partial \Phi_l}{\partial r} \right)_{r=a} = \sum_l \left[ \left( \frac{\alpha a^{l+1} l V_1}{2l + 1} \right) a^{l-1} \right. \right.
$$

$$
- \left. \left( \frac{\alpha a^{l+1} l V_1}{2l + 1} \right) (l + 1) a^{-l-2} \right] P_l(\cos \theta) = 0.
$$

(35)

As it was done in the previous limiting cases, the coefficients of region 1 can be safely ignored, because those in the region 2 satisfy the boundary conditions at the plates of the spherical capacitor. Thus, it is clear that the present case is equivalent to the trivial one.

In this section we have highlighted three particular cases, each representing a limiting case of the general setup. However, two of them ($r_1 = a$ and $r_2 = \infty$) are closely related, while the case 4.6 is experimentally difficult to achieve. Despite the usefulness of the graphic representations presented in this section, it should be noticed that they do not intend to make explicit the magnitude of the fields produced, since they only show streamlines. With this in mind, in the next section we calculate the magnitudes of the fields and explore how to find the configurations that maximize them, bringing the problem closer to a possible experimental consideration.

The determination of the field streamlines is made by taking the potentials up to $l = 9$. As will be seen in section 5 this approximation is more than enough for the configurations considered here. The configuration with $a = r_1$ will turn out to be relevant from the phenomenological point of view and figure 4 shows this case, in which $r_1 = \frac{1}{2} r_2$, a value that is within the domain that does not need large $l$ approximations.

5. Numerical calculations of the magnetic field

Having in mind a lowest attainable value for the magnetic field of the order of $10^{-2}$ G, we fix the parameters that determine the magnetic field (MEP, permittivity, radii), and estimate its magnitude at different points in space. Let us recall that we have taken the regions 1 and 3 to be the vacuum, so that the magnetoelectric effect arises only from the TI in the region 2. It is important to focus our attention close to the external interface of the TI, because this region is
accessible to measurement devices and also takes advantage of its proximity with the plates of the semispherical capacitor which source the magnetic field. As for the direction, we analyze the cases $\theta = 0, \pi/4, \pi/2$. For all the numerical estimations in the following we take the external interface of the TI to be fixed at $r_1 = 1.0 \mu m$.

In particular we will pay special attention to the case where the spherical shell touches de capacitor plates (i.e. $r_1 = a$) and when the magnetic field is measured at the external interface (i.e. at $r = r_2$). Under these conditions the corresponding fields can be written as

$$[r_2 B_2]_{r_1=a, r=r_2} = \hat{\alpha} \sum_i F_i(s) V_i P_i(\cos \theta),$$

$$[r_2 B_2]_{r_1=a, r=r_2} = -\hat{\alpha} \sum_i F_i(s) V_i \frac{dP_i(\cos \theta)}{d\theta}$$

with

$$F_i(s) = \varepsilon \frac{s^{l+1}(1 - s^{2j+1})}{[(\varepsilon - 1)(l + 1)s^{2j+1} + (\varepsilon + l + 1)]}, \quad s = \frac{a}{r_2}$$

Dimensional reasons indicate that the potentials $[\Phi]_{r_1=a, r=r_2}$ and $[\Psi]_{r_1=a, r=r_2}$, which are linear in $V_i$, are only functions of $s$.

In the following numerical estimations we set $\hat{\alpha} = \alpha \approx 1/137$, the minimum value for this parameter in a TI which we take as the compound TlBiSe$_2$, with $\varepsilon \approx 4$. Also we set $V = 3 \text{ V}$; $a$ and $r_1$ of the order of $\mu m = 10^{-4} \text{ cm}$ and we fix $r_2 = 1 \mu m$. This choice of the TI together with the characteristics of the setup are motivated by [42].

We work in the non-rationalized CGS system where the electric field and the magnetic field are measured in statV/cm and G, respectively.

### 5.1. Optimal configuration for $\theta = 0$

Examining the behavior of the magnetic field in the direction $\theta = 0$, where $B = B_2 \hat{r}$, provides a general notion of the problem. As a first approximation we assume that the magnetic field is maximized near the external interface for any configuration. This hypothesis is reasonable because the field must decrease with distance.

Now we study the effect of keeping a vacuum region between the metallic plates and the interior interface of the TI. To this end, we determine how the magnetic field behaves as a function of the distance $r > r_2$, for different choices of the thickness of the TI (parametrized by $r_1$) and of the radius $a$ of the capacitor. Some cases are shown in figure 6. Each panel corresponds to a fixed thickness of the TI, and their different curves are labeled by the radius $a$ of the capacitor ($a < r_1 < r_2$). Notice that the choice $a = r_1$, i.e. when the internal interface of the TI is touching the metallic plates, yields a maximum value for $|B(r_2, 0)|$ for values of $r_1 \lesssim 0.6 \mu m$, as shown in figures 6(a) and (b). In contrast, for larger values of $r_1$ there is an optimal choice of $a < r_1$ for each particular case (figures 6(c) and (d)). To find this optimal value, one must solve the equation $d|B(r_2, 0)|/da = 0$ at $r = r_2$, for fixed $r_1$ and $r_2$.

Continuing with this analysis, we observe that choosing $a = r_1$, it is possible to determine the value of $r_{1m}$ that maximizes $|B(r_2, 0)|$. One must simply solve the equation $d|B(r_2, 0)|/dr_1 = 0$. Figure 7 shows a plot of $|B(r_2, 0)|$ as a function of $r_1$, that reaches its maximum at $r_{1m} \approx 0.5 \mu m$. Regarding these observations, some comments can be made. Firstly, despite that in the domain $r_1 \geq 0.6 \mu m$ the hypothesis that $a = r_1$ maximizes $|B|$ at the external interface is not satisfied, in general (for any value of $a$) the maximum of $|B|$ in that domain is much smaller than $0.2 \text{ G}$, which is reached in the case where $r_{1m} \approx 0.5 \mu m$. Secondly, it is easier to make a coating of the magnetoelectric material on the surface of the conducting sphere than to leave an empty space between the capacitor plates and the TI shell. In this way we consider the configurations for which $a$ equals $r_1$ to be the most relevant and focus only on them in the following. Since we do not want the magnetoelectric effect to disappear we keep the TI shell with a minimum thickness of $0.05 \mu m$. In other words we will explore $r_1$ in the range $0 < a = r_1 < 0.95 \mu m$. Also this constraint will force us to stay away from the dangerous points $a = r_1 \rightarrow r_2$, when observing at $r = r_2$ and $\theta = \pi/2$.

From the previous discussion, we conclude that the condition $a = r_1 \approx 0.5 \mu m$ gives rise to the most intense magnetic field in the direction of the $z$-axis. As seen in the figures 6(b) and 7, this field would be of order $0.2 \text{ G}$ at the interface $r = r_2$.

For completeness, in the case where $a = r_1 = 0.5 \mu m$ it is pertinent to know how the magnetic field behaves in other directions. Figure 8 shows a plot of $|B|$ in the directions $\theta = 0, \pi/4, \pi/2$ as a function of $r \geq r_2 = 1.0 \mu m$. It should be noted that the magnitude of the field is highly isotropic in this case.

Finally, we comment on the precision required in the calculations to adequately describe the physics associated with $r_2 = 1 \mu m$ and large values of $r_1$ (like $r_1 \approx 0.9 \mu m$ for example), where it is necessary to consider a broad number of terms for the magnetic potential. The figure 9 shows the ratio between the approximation of $\Psi$ to order $l = 7$ and the analogous approximation to order $l = 1000$, for $r_1 = a$ at $\theta = 0$ as a function of $r_1 / r_2$. For $r_1 \approx 0.7 r_2$, $l = 7$ is no longer a good cut-off value to describe the system. For greater values of $r_1 / r_2$, the approximation requires higher values of $l$. It is important to clarify that to plot figures 6(a) and (b) it was enough to take $l = 7$, while for figures 6(c) and (d) it was necessary to increase the number of terms.

### 5.2. Optimal configuration for $\theta = \pi/2$

At these points the total magnetic field is in the direction of $\hat{\Theta}$. Making an analysis similar to that of the previous section, in figure 10 we show that, unlike the case in $\theta = 0$ (figure 7), the magnetic field at $\theta = \pi/2$ and $r = r_2$ increases with $r_1$ when $a = r_1$. When $a = r_1 = 0.95 \mu m$ we have $|B(r_2, \pi/2)| = 4.17 \text{ G}$.

Since the source potential in the capacitor is discontinuous at the equatorial plane it is convenient to stay away from the points $a = r_1 \rightarrow r_2$ when $\theta = \pi/2$ in numerical calculations. We safely avoid this situation with the constraint $a = r_1 < 0.95 \mu m < r_2 = 1 \mu m$ and moreover we choose the optimal configuration for this case to be at $a = r_1 = 0.75 \mu m$, where we still get measurable magnetic fields. Fixing this parameter, we proceed in analogous way to
the previous case. The figure 11 shows a plot of $|\mathbf{B}|$ in this optimal configuration for three angles ($\theta = 0, \pi/4, \pi/2$) as a function of distance, suggesting a highly anisotropic response of the magnetic field near the exterior interface of the TI.

5.3. Optimal configuration for $\theta = \pi/4$

Similarly to the case (5.1), it is possible to find the condition that maximizes the magnetic field in the direction $\theta = \pi/4$.

The figure 12 is a plot of $|\mathbf{B}|$ at $r_2 = 1 \mu m$, as a function of $a = r_1$, for $\theta = \pi/4$. The maximum appears at $r_{1m} \approx 0.62 \mu m$. This case is a hybrid between the two previous ones, because it gives rise to fields nor as intense as in the case where $r_{1m} = 0.75 \mu m$, neither as isotropic as in the case where $r_{1m} = 0.5 \mu m$. The figure 13 shows the magnetic field as a function of $r$ produced by this optimal configuration in the directions $\theta = 0, \pi/4, \pi/2$. 

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**Figure 6.** Plot of $|\mathbf{B}(r_2, 0)|$ as a function of distance, for different values of $a$. The values for $r_1$ are: Panel (a) $r_1 = 0.3 \mu m$, Panel (b) $r_1 = 0.5 \mu m$, Panel (c) $r_1 = 0.7 \mu m$ and Panel (d) $r_1 = 0.9 \mu m$. In the panels (a) and (b), the choice $a = r_1$ maximizes the magnitude of the field near $r = r_2$. For values greater that $r_1 \approx 0.6 \mu m$ this is no longer the case, as shown in the panels (c) and (d).

**Figure 7.** Plot of $|\mathbf{B}(r_2, 0)|$ as a function of $r_1$, ($0 < r_1 < 0.95 \mu m$), when $a = r_1$. The magnetic field reaches its maximum value (0.2G) at $r_{1m} \approx 0.5 \mu m$.

**Figure 8.** Plot of the magnitude of the magnetic field in the directions $\theta = 0, \pi/4$ and $\pi/2$, as a function of distance. The parameters are $r_2 = 1 \mu m$ and $a = r_1 = 0.50 \mu m$. 

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Summarizing, in section 5 we have chosen three configurations dictated by the maximum magnitude of the magnetic field in each of the directions $\theta = 0, \pi/4$ and $\pi/2$, which determined an optimal thickness of the magnetoelectric shell. The choice of each configuration does not say anything about the magnitude of the field in other directions different from the one that determined the value of $a = r_1$ for the maximum magnitude. In fact, for each of those configurations, the figures 8, 11 and 13 indicate an interesting angular behavior which we now made explicit at the external interface of the TI. The figure 14 gives a detailed account of such anisotropy showing that is possible to generate large magnetic fields in the vicinity of $\theta = \pi/2$ that decrease dramatically in other directions. A first conclusion that suggests two distinct and even opposite empirical approaches is that the $a = r_1 = 0.75 \mu m$ configuration would generate intense but anisotropic fields, while the configuration $a = r_1 = 0.5 \mu m$ would generate almost isotropic fields but of lesser magnitude.
We close this section with a general comment regarding the scaling of our setup as a whole, assuming that $a, r_1, r_2$ are each multiplied by a factor $\Lambda > 1$. Let us focus on the exterior region 3, where $D_3^l \rightarrow N^l+1D_3^l$ yielding

$$\begin{align*}
\tilde{B}_i(r) &= \sum_{l} N^l \left( \frac{D_3^l}{r^{l+2}} \right) (l+1) P_l(\cos \theta), \\
\tilde{B}_0(r) &= -\sum_{l} N^l \left( \frac{D_3^l}{r^{l+2}} \right) \frac{dP_l(\cos \theta)}{d\theta}.
\end{align*}$$

Here $\tilde{B}$ denotes the magnetic field after scaling the setup. If we now evaluate the fields produced by the scaled setup at the scaled point $r = \Lambda r$ we obtain $B(\Lambda r) = B(r)/\Lambda$. In particular this means that the magnetic field measured at the exterior interface of the scaled setup is reduced by a factor $\Lambda$ with respect to that measured in the same interface before the scaling. It is notable that an increase in system dimensions up to ten times, for example, starting with an initial optimum value of $|B| \approx 0.2$ G at the exterior interface, would maintain the possibility of experimentally measuring a detectable field of $\approx 0.02$ G at the new interface, for any of the two most significant configurations.

6. Summary and conclusions

The electromagnetic response of TIs is described by a modification of Maxwell equations that includes the production of polarization (magnetization) due to the presence of magnetic (electric) fields. This phenomenon, called the magnetoelectric effect, occurs due to charges or currents at the interfaces of the material, induced by gradients of the magnetoelectric polarizability $\theta$, an additional parameter characterizing these media, besides their permittivity and permeability. Although for several decades the magnetoelectric effect was known to occur in some ferroelectric and ferromagnetic materials, it was not until recently that its study gained a remarkable strength, both theoretically and experimentally, due to the advances in the understanding and in the fabrication of new multifunctional materials, together with the recognition that this effect is also present in topological phases. Since then, significant advances have been made that allow technological applications in subsequent years fostered by the production of composite materials with stronger magnetoelectric couplings.

In this work, we study the magnetoelectric effect produced by a capacitor with semispherical plates at potentials $+V$ and $-V$, respectively, placed in vacuum ($\theta_2 = 0$) and surrounded by a thick shell of a time reversal three-dimensional topological insulator with $\theta_1 = \pi$, as shown in figure 1. Using spherical coordinates appropriate for the symmetry of the problem, the static magnetic and electric fields were determined in the absence of additional free sources. The bulk regions are governed by the standard Maxwell equations for material media, while the effects of the topological insulator show up only at the interfaces through the boundary conditions (10) that result from the modified constitutive relations (4). Our main concern is with the magnetic field produced by the electric configuration. The general solution, resulting from equations (A40)–(A51), is difficult to handle, so we consider a series expansion in the parameter $\tilde{\alpha} = (\theta_2 \alpha)/\pi = \alpha$, where $\alpha$ is the fine structure constant. Even in this approximation we do not present a detailed discussion of the electromagnetic fields in the most general configuration, with the exception of the streamlines in figures 2 and the results in figure 6 which motivate our restriction to the cases where $a = r_1$.

Time reversal invariance in the bulk of the TI ($\theta = \pi$), together with the electric origin of the electromagnetic fields demand that the expansion of $\Phi(p)$ must contain only even (odd) powers of $\tilde{\alpha}$. Nevertheless, a closer look at the equations (A40)–(A51) reveals that such a choice for the powers of $\tilde{\alpha}$ is just an algebraic consequence of the equations. In this way, our expressions for the coefficients $A_0, B_0, C_0, D_0$, which we solve up to first order in $\tilde{\alpha}$, are also valid when time reversal invariance $\psi \rightarrow -\psi$ does not hold, thus extending the application of our results to the case of a linear isotropic magnetoelectric media with arbitrary values of $\theta$ such that $\tilde{\alpha} \ll 1$.

Subsequently, the problem was reduced to some limiting cases, allowing a more direct interpretation of the results. Those configurations that produce the most intense fields were sought. In particular, by taking $a = r_1$ we single out the cases where the TI is in direct contact with the capacitor thus avoiding a vacuum region between them. This allows a stronger magnetoelectric effect as well as simplifies the construction of the setup. Since the magnetic field decreases with distance we also concentrate in determining the magnetic field at the external interface of the TI located at $r_2$. Fixing $r_2 = 1 \mu$m we are left with two variables to explore: the thickness of the TI parametrized by $a = r_1$ and the angular isotropy of the field at the external interface.

As repeatedly mentioned along the text, the discontinuity of the source potential at $r = a$ and $\theta = \pi/2$ makes these points unreliable in a numerical approximation. This is particularly noticeable when considering $a = r_1$ and observing the magnetic field at $r_2$ in the limit of a thin TI shell, i.e. when $r_1 \rightarrow r_2$. The calculation becomes particularly involved in this limit and requires more sophisticated numerical techniques which are out of the scope of the present work. However, according to the physics of the problem we expect that in the limit $a = r_1 \rightarrow r_2$, that is to say when the TI disappears, the magnetic field should be zero everywhere, in particular at the external interface. This behavior, which is evident in the figures 7 and 12, is consistent with the cases when $\theta = \pi/4, 5\pi/16, 3\pi/8, 7\pi/16$, approaching $\pi/2$, shown in the figure 15. Nevertheless, this is not the case in the figure 10 for $\theta = \pi/2$. Thus, our calculations are not trustworthy in this particular limiting case, which we avoid by restricting ourselves to $a = r_1 \leq 0.95 \mu$m in all the relevant configurations.

Choosing the directions $\theta = 0, \pi/4, \pi/2$ and plotting the magnetic field as a function of $a = r_1 < 0.95 \mu$m we find a maximum which increases in value and occurs for larger values of $r_1$ as $\theta$ approaches $\pi/2$. These optimal configurations occur when $a = r_1$ takes the values $0.50, 0.62$ and
of the order of $10^{-9}$ G cm$^2$, which comfortably fall within the measurement capabilities of SQUID devices.

A comparison of our results for the electromagnetic fields with those obtained by other static methods implemented to study the linear ME effect, as for example those in [35, 36, 42], is not immediate because such works use a monopolar electric source to trigger the ME effect, while we consider, for a first time to the best of our knowledge, a dipolar source for that purpose. Anyway, in all reported cases, including ours, the resulting magnetic effects turn out to be measurable within present day sensitivities. Still, along this work we have performed many internal consistency checks concerning the validity of our general results: (i) by establishing appropriate limits yielding well known results, (ii) by verifying some expected symmetry properties and finally (iii) by checking that the elimination of some region in a limiting process consistently translates the required boundary conditions to the remaining coefficients.

In conclusion, we propose a new compact spherically-shaped setup to study the linear isotropic magnetoelectric effect triggered by an electric dipolar source, which yields measurable magnetic fields according to the precision of present-day magnetometers. Such measurements can be employed to determine the magnetoelectric coupling of the different materials that can be handled to build up the external shell of the device.

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Appendix A. The solution of the system of equations

The objective is to find the algebraic expressions that relate the coefficients $A_i, B_i, C_i, D_i$ in equations (13) and (14) to the source of the electric excitation given by the potential difference $2V$. The first step is to rewrite the boundary conditions at the interfaces in terms of the electric and magnetic potentials.

Let us start with the metallic surface of the capacitor. The particular form of the electric potential $\Phi$ in $r = a$ is given by

$$\Phi(r) = \frac{c}{r} + D.$$
the equation
\[
\Phi_i(a, \theta) = \begin{cases} 
+V & \text{if } (0 \leq \theta < \pi/2) \\
-V & \text{if } (\pi/2 < \theta \leq \pi).
\end{cases}
\] (A1)

Let us observe that the source potential is discontinuous in the equator of the capacitor \( r = a, \theta = \pi/2 \). In this way, the calculations yielding the fields at points close to this ring, resulting from different choices of our setup, require a careful limiting process.

The boundary conditions (A1) must be implemented in
\[
\Phi_i(a, \theta) = \sum_{l=0}^{\infty} \alpha_l P_l(\cos \theta),
\] (A2)

where \( \alpha_l = A_{1l}^1 a^l + B_{1l}^l a^{-(l+1)} \) needs to be solved in terms of the given potential \( V \). According to Jackson [58], the result is
\[
\alpha_l = A_{1l}^1 a^l + B_{1l}^l a^{-(l+1)} = V_l \quad \text{(odd l)},
\] (A3)
\[
\alpha_l = A_{1l}^1 a^l + B_{1l}^l a^{-(l+1)} = 0 \quad \text{(even l)},
\] (A4)

with
\[
V_l = V \left( -\frac{1}{2} \right)^{(l-1)/2} \frac{(1+1)(l-2)!!}{2^{(l+1)/2}}.
\] (A5)

This means that only odd Legendre polynomials appear in the expression of the potentials, that is odd powers of \( \cos \theta \), which is in accordance with the symmetry of the problem. Let us recall our notation
\[
\sum_l' = \sum_{l=1,3,\ldots}^{\infty}.
\] (A6)

A second condition at the capacitor is that the normal component of the magnetic field must be zero at \( r = a \), since this is the surface of a perfect conductor:
\[
\left. \frac{\partial \Phi_i}{\partial r} \right|_{r=a} = 0.
\] (A7)

At the interfaces \( \Sigma_1 \) and \( \Sigma_2 \), the electric potentials are continuous, i.e.
\[
(\Phi_1 = \Phi_2 \equiv \Phi_{1,2})_{r=r_1} = (\Phi_2 = \Phi_3 \equiv \Phi_{2,3})_{r=r_2}.
\] (A8)

When writing two subindices, \( \Theta_i(R) \) we make explicit that \( \Theta_i(R) = \Theta_i(R) \), so it is indifferent if we use the \( i \)-th or the \( j \)-th function \( \Theta \) for this particular boundary condition. Let us recall that \( -\partial \Phi / \partial \theta \) (tangential component of \( \mathbf{E} \)) and \( -\partial \Phi / \partial r \) (normal component of \( \mathbf{B} \)) are continuous at the interfaces.

Let us now consider the boundary conditions on the derivatives of the potentials which arise from equation (10). The interface at \( r = r_1 \) yields
\[
0 = \left( \frac{\partial \Phi_1}{\partial r} - \frac{\partial \Phi_2}{\partial r} \right)_{r=r_1},
\] (A9)
\[
0 = \left( \frac{\partial \Phi_2}{\partial \theta} - \frac{\partial \Phi_1}{\partial \theta} \right)_{r=r_1},
\] (A10)
\[
0 = \left( \frac{\partial \Phi_1}{\partial r} - \frac{\partial \Phi_2}{\partial r} \right)_{r=r_2},
\] (A11)

while that at \( r = r_2 \) produces
\[
0 = \left( \frac{\partial \Phi_3}{\partial r} - \frac{\partial \Phi_2}{\partial r} \right)_{r=r_2}.
\] (A12)
\[
0 = \left( \frac{\partial \Phi_3}{\partial \theta} - \frac{\partial \Phi_2}{\partial \theta} \right)_{r=r_2},
\] (A13)
\[
0 = \left( \frac{\partial \Phi_2}{\partial r} - \frac{\partial \Phi_3}{\partial r} \right)_{r=r_2}.
\] (A14)

Finally, both potentials must be zero in infinity if physical solutions are desired. This condition translates directly into:
\[
C_1^3 = 0, \quad A_1^3 = 0.
\] (A15)

Before writing the boundary conditions in terms of the coefficients in equations (13) and (14), a clarification must be made. Take, for example, equation (A12) after grouping terms
\[
\sum_l [(A_{1l}^1 r_l + B_{1l}^l r_l^{-(l+1)}) - (A_{1l}^1 r_l + B_{1l}^l r_l^{-(l+1)})] \frac{dP_l(\cos \theta)}{d \theta} = 0.
\] (A16)

For simplicity, we define \( \gamma_l = A_{1l}^1 r_l + B_{1l}^l r_l^{-(l+1)} \), so that
\[
\sum_l [\gamma_l - \gamma_l^2] \frac{dP_l(\cos \theta)}{d \theta} = 0.
\] (A17)

Recalling that \( dP_l(\cos \theta)/d \theta \) is a combination of the basis \( P_l(\cos \theta) \) and \( P_{l-1}(\cos \theta) \) it is not immediate whether the above equation yields that \( \gamma_l^2 - \gamma_l^2 = 0 \), or a recurrence relation among these quantities instead. This question is resolved by proceeding as follows. Notice that
\[
\frac{dP_l(\cos \theta)}{d \theta} = \frac{dP_l(\cos \theta)}{d \cos \theta} = \frac{dP_l(\cos \theta)}{d \cos \theta} (-\sin(\theta)).
\] (A18)

So, by multiplying the expression (A19) by \( dP_m(\cos \theta)/d \theta \) and integrating from \(-1 \) to \( 1 \) we obtain
\[
\sum_l [\gamma_l^2 - \gamma_l^2] \int_{-1}^1 (1 - x^2) \frac{dP_l(x)P_m(x)}{dx} dx = 0,
\] (A19)

with \( x = \cos \theta \). Integrating by parts and using the orthogonality relations of the Legendre polynomials,
\[
\int_{-1}^1 P_l(x)P_m(x) dx = \frac{2}{2l+1} \delta_{lm},
\] (A20)
in addition to the property
\[
\frac{d}{dx} \left[ (1 - x^2) \frac{dP_l(x)}{dx} \right] = -l(l+1)P_l(x),
\] (A21)

we get to
\[
\int_{-1}^{1} (1 - x^2) \frac{dP_0(x)}{dx} \frac{dP_1(x)}{dx} dx
\]
\[
= (1 - x^2) \frac{dP_0(x)}{dx} \bigg|_{-1}^{1} - \int_{-1}^{1} -l(l+1) P_l(x) P_0(x) dx
\]
\[
= 2l(l+1) \delta_{0l}.
\] (A24)

Consequently, the expression (A21) reduces to
\[
(\gamma_1^l - \gamma_2^l) \frac{2l(l+1)}{2l + 1} = 0,
\] (A25)
yielding finally
\[
\gamma_1^l = \gamma_2^l
\] (A26)

for every \( l > 0 \). The case \( l = 0 \) is not a problem since the only nonzero contributions to the coefficients arise from odd values of \( l \), as a consequence of the boundary conditions in the capacitor. In terms of \( A_l^i \) and \( B_l^i \), the previous relation is
\[
A_l^1 r_1^l + B_l^1 r_1^{l-1} = A_l^2 r_2^l + B_l^2 r_2^{l-1},
\] (A27)

which is the same restriction imposed by the condition \( \Phi_1(r_1) = \Phi_2(r_1) \). Similarly, for the exterior interface \( r = r_2 \) we obtain
\[
A_l^2 r_2^l + B_l^2 r_2^{l-1} = B_l^3 r_2^{l-1}.
\] (A28)

Then we conclude that any condition that involves derivatives of the potentials with respect to \( \theta \) can be translated into a condition on the potentials themselves. Thus, equations (A12) and (A16) give rise to the same conditions as equation (A8). In the same way, the equations (A10) and (A14) provide the following boundary conditions
\[
\Psi_2(r_1) - \Psi_1(r_1) = -\tilde{\alpha} \Phi_{1,2}(r_1),
\] (A29)
\[
\Psi_2(r_2) - \Psi_2(r_2) = -\tilde{\alpha} \Phi_{2,3}(r_2),
\] (A30)

whose explicit expressions are
\[
(C_l^1 r_1^l + D_l^1 r_1^{l-1}) - (C_l^1 r_1^l + D_l^1 r_1^{l-1}) = -\tilde{\alpha}(A_l^1 r_1^l + B_l^1 r_1^{l-1}),
\] (A31)
\[
D_l^3 r_2^{l-1} - (C_l^2 r_1^l + D_l^2 r_1^{l-1}) = \tilde{\alpha} B_l^3 r_2^{l-1}. \tag{A32}
\]

Having dealt with all the relationships involving tangential derivatives, it is necessary to focus our attention on the conditions associated with radial derivatives. Notice that from (13) and (14), we obtain
\[
\frac{\partial \Phi_i}{\partial r} = \sum_l (IA_l^i r_1^{l-1} - (l + 1)B_l^i r_1^{l-2})P_l(\cos \theta),
\] (A33)
\[
\frac{\partial \Psi_i}{\partial r} = \sum_l (IC_l^i r_1^{l-1} - (l + 1)D_l^i r_1^{l-2})P_l(\cos \theta). \tag{A34}
\]

So, the expressions (A9) and (A13) give rise to
\[
\varepsilon_2 (IA_l^2 r_1^{l-1} - (l + 1)B_l^2 r_1^{l-2})
- \varepsilon_2 (IA_l^1 r_1^{l-1} - (l + 1)B_l^1 r_1^{l-2})
= \tilde{\alpha} (IC_l^2 r_1^{l-1} - (l + 1)D_l^2 r_1^{l-2}),
\] (A35)
\[
\varepsilon_3 ((l + 1)B_l^3 r_2^{l-1} - (l + 1)B_l^2 r_2^{l-2})
- \varepsilon_3 (IA_l^3 r_2^{l-1} - (l + 1)B_l^2 r_2^{l-2})
= \tilde{\alpha} ((l + 1)D_l^3 r_2^{l-1} - (l + 1)D_l^2 r_2^{l-2}),
\] (A36)

On the other hand, from the continuity of the radial derivative of the magnetic potential (A11) and (A15) it follows that
\[
IC_l^1 r_1^{l-1} - (l + 1)D_l^1 r_1^{l-2} = IC_l^2 r_1^{l-1} - (l + 1)D_l^2 r_1^{l-2},
\] (A37)
\[
IC_l^3 r_2^{l-1} - (l + 1)D_l^3 r_2^{l-2} = (l + 1)D_l^3 r_2^{l-2}. \tag{A38}
\]

The equation (A7) indicates that
\[
IC_l^3 a^{l-1} = (l + 1)D_l^3 a^{l-2}, \tag{A39}
\]

There are twelve coefficients for every odd value of \( l \) and twelve equations that relate them linearly, which are
\[
A_l^1 a^l + B_l^1 a^{l-1} = V_l, \tag{A40}
\]
\[
A_l^1 r_1^{l-1} + B_l^1 r_1^{l-2} = A_l^2 r_2^l + B_l^2 r_2^{l-1}, \tag{A41}
\]
\[
A_l^2 r_2^l + B_l^2 r_2^{l-1} = B_l^3 r_2^{l-1}, \tag{A42}
\]
\[
\varepsilon_2 (IA_l^2 r_1^{l-1} - (l + 1)B_l^2 r_1^{l-2})
- \varepsilon_2 (IA_l^1 r_1^{l-1} - (l + 1)B_l^1 r_1^{l-2})
= \tilde{\alpha} (IC_l^2 r_1^{l-1} - (l + 1)D_l^2 r_1^{l-2}), \tag{A43}
\]
\[
\varepsilon_3 ((l + 1)B_l^3 r_2^{l-1} - (l + 1)B_l^2 r_2^{l-2})
- \varepsilon_3 (IA_l^3 r_2^{l-1} - (l + 1)B_l^2 r_2^{l-2})
= \tilde{\alpha} ((l + 1)D_l^3 r_2^{l-1} - (l + 1)D_l^2 r_2^{l-2}), \tag{A44}
\]
\[
A_l^3 = 0, \tag{A45}
\]
\[
(C_l^2 r_1^{l-1} + D_l^2 r_1^{l-2}) - (C_l^1 r_1^{l-1} + D_l^1 r_1^{l-2}) = -\tilde{\alpha}(A_l^1 r_1^{l-1} + B_l^1 r_1^{l-2}), \tag{A46}
\]
\[
(D_l^3 r_2^{l-1} + C_l^2 r_2^{l-2}) - (C_l^1 r_1^{l-1} + D_l^1 r_1^{l-2}) = \tilde{\alpha}(B_l^3 r_2^{l-1} + D_l^3 r_2^{l-2}), \tag{A47}
\]
\[
IC_l^1 r_1^{l-1} - (l + 1)D_l^1 r_1^{l-2} = IC_l^2 r_1^{l-1} - (l + 1)D_l^2 r_1^{l-2}, \tag{A48}
\]
\[
IC_l^3 r_2^{l-1} - (l + 1)D_l^3 r_2^{l-2} = (l + 1)D_l^3 r_2^{l-2}, \tag{A49}
\]
\[
IC_l^3 a^{l-1} = (l + 1)D_l^3 a^{l-2}, \tag{A50}
\]
\[
C_l^3 = 0. \tag{A51}
\]

The coefficients \( A_l^i \) and \( B_l^i \) correspond to the electric response of the system and their source is the potential \( V \) as shown in equation (A40). On the other hand, the magnetic coefficients \( C_l^i \) and \( D_l^i \) arise due to the magnetoelectric effect and their source is \( \tilde{\alpha} V \), as shown in equations (A46) and (A47). Thus they should vanish when \( \tilde{\alpha} = 0 \). Then, it is clear from equations (A43) and (A44) that \( A_l^1 \) and \( B_l^1 \) will receive additional corrections proportional to \( \tilde{\alpha} V \). Due to the complexity of the above equations we choose to expand the solution in powers of \( \tilde{\alpha} \) only up to the first order. This means we neglect the second order contribution to \( A_l^1 \) and \( B_l^1 \). Nevertheless, in the case of a TI, \( \tilde{\alpha} \) is proportional to the fine structure constant, being small enough to justify the validity of the expansion. According to the above strategy, the zeroth
order equations for $A_l^1$, $B_l^1$ are

\[ A_l^1 a^l + B_l^1 a^{-(l+1)} = V_l, \]  

\[ A_l^1 r_1^l + B_l^1 r_1^{-(l+1)} = A_l^1 r_1^l + B_l^2 r_1^{-(l+1)}, \]  

\[ A_l^2 r_2^l + B_l^2 r_2^{-(l+1)} = B_l^1 r_2^{-(l+1)}, \]  

\[ \varepsilon (A_l^1 r_1^{-(l+1)} - (l + 1) B_l^1 r_1^{-(l+2)}) - (l A_l^1 r_1^{-(l-1)} - (l + 1) B_l^1 r_1^{-(l+2)}) = 0 \]  

\[ (-l + 1) B_l^1 r_2^{-(l+2)} - \varepsilon (l A_l^2 r_2^{-(l+2)} - (l + 1) B_l^2 r_2^{-(l+2)}) = 0 \]  

\[ A_l^3 = 0, \]  

where we have introduced the relative permittivity $\varepsilon = \varepsilon_2 / \varepsilon_1$. The first order equations in $\alpha$, for $C_l^1, D_l^1$, are

\[ (C_l^1 r_1^{-(l+1)} - (l A_l^1 r_1^{-(l+1)}) = -\alpha (A_l^1 r_1^l + B_l^1 r_1^{-(l+1)}), \]  

\[ D_l^1 r_1^{-(l+1)} - (C_l^2 r_2^{-(l+1)} + D_l^2 r_2^{-(l+1)}) = \alpha (B_l^1 r_2^{-(l+1)}), \]  

\[ lC_l^1 r_1^{l-1} - (l + 1) D_l^1 r_1^{-(l+2)} = lC_l^2 r_1^{l-1} - (l + 1) D_l^2 r_1^{-(l+2)}, \]  

\[ lC_l^2 r_2^{l-1} - (l + 1) D_l^2 r_2^{-(l+2)} = -l + 1) D_l^3 r_2^{-(l+2)}, \]  

\[ lC_l^3 a^{l-1} = (l + 1) D_l^1 a^{-(l+2)}, \]  

\[ C_l^3 = 0. \]  

The sources of the linear equations (A58) and (A59) are calculated with the zeroth-order values for $A_l^1, B_l^1$ obtained from the previous set of equations (A52)–(A57). Under the above conditions we find

\[ A_1^1 = \lambda (l + 1)(\varepsilon - 1)(\varepsilon + l + 1) (r_2^{2l+1} - r_2^{2l+1}), \]  

\[ A_2^1 = \lambda (l + 1)(2l + 1)(\varepsilon - 1) r_1^{2l+1}, \]  

\[ B_1^1 = -\lambda (l + 1)(l + 1)(\varepsilon - 1) r_1^{2l+1} - r_2^{2l+1}(\varepsilon + l + 1), \]  

\[ B_2^1 = \lambda (2l + 1) r_1^{2l+1} r_2^{2l+1}(\varepsilon + l + 1), \]  

\[ B_3^1 = \lambda (2l + 1)^2 r_1^{2l+1} r_2^{2l+1}, \]  

\[ C_1^1 = -\lambda (l + 1)(l + 1)(r_2^{2l+1} - r_2^{2l+1}), \]  

\[ C_2^1 = -\lambda (l + 1)(2l + 1) r_2^{2l+1}, \]  

\[ D_1^1 = -\lambda (l + 1)(l + 1) (r_2^{2l+1} - r_2^{2l+1}), \]  

\[ D_2^1 = -\lambda (l + 1)(l + 1) (r_2^{2l+1} - r_2^{2l+1}) + r_1^{2l+1} r_2^{2l+1}(\varepsilon + l + 1) + (l + 1)(\varepsilon - 1) r_1^{4l+2}, \]  

\[ D_3^1 = -\lambda (l + 1)(l + 1) (r_2^{2l+1} - r_2^{2l+1}) (r_2^{2l+1} + (l + 1)(\varepsilon - 1) r_1^{4l+2}, \]  

where we have defined

\[ \lambda \equiv \lambda (r_1, r_2, a) = \frac{a^{d+1} V_l}{D}. \]  

\[ D = (l + 1)(\varepsilon^2 - 1)(\varepsilon + l + 1)(r_2^{2l+1} - r_2^{2l+1}) + r_1^{2l+1} r_2^{2l+1}(\varepsilon + l + 1) + (l + 1)(\varepsilon - 1) r_1^{4l+2}, \]  

we verify that in the purely electrical case, when $\alpha = 0$, the magnetic coefficients vanish, as expected.

**Appendix B. The magnetic flux over a circular pickup loop**

We calculate the magnetic flux through a flat circular loop of radius $R$ located perpendicularly to the $z$ axis which center is at a distance $\zeta$ from the center of the semispherical capacitor, as shown in figure B1. The distance $s$ form the center of the loop to any point in the region bounded by the loop can be described by means of the polar angle $\theta$ and $\zeta$, according to

\[ s = \zeta \tan \theta, \quad r = \zeta \sec \theta. \]  

The position of the points that lie in the interior region of the loop is given by $r = \zeta \sec \theta$ with $\theta < \theta_0 = \arctan (R/\zeta)$. In region 3, the magnetic field components evaluated at the surface bounded by the loop are

\[ B_r(\theta) \equiv B_r(\zeta \sec \theta, \theta) = \sum_l \left( \frac{D_l^3}{(\zeta \sec \theta)^{l+2} P_l(\cos \theta) \sin \theta} \right), \]  

\[ B_\theta(\theta) = B_\theta(\zeta \sec \theta, \theta) = \sum_l \left( \frac{D_l^3}{(\zeta \sec \theta)^{l+2} [1 \sin \theta] P_l(\cos \theta) \sin \theta} \right), \]  

On the other hand, the differential area element of the circular loop is given by $da = \hat{sd}d\phi d\theta$, where $\hat{k}$ is the unit vector in the $z$-direction. The relation $s = \zeta \tan \theta$ allows us to express the above equation in a different and, although tangled, more useful fashion

\[ da = \zeta^2 \tan \theta \sec^2 \theta d\theta d\phi d\zeta. \]  

Recalling that in spherical coordinates $\theta = \cos \theta r - \sin \theta \hat{\phi}$, we obtain

\[ B \cdot da = (B_r(\cos \theta - B_\theta(\theta) \sin \theta))(\zeta^2 \tan \theta \sec^2 \theta d\theta d\phi d\zeta), \]  

with

\[ B_r(\theta) \cos \theta - B_\theta(\theta) \sin \theta \]  

\[ = \sum_l \left( \frac{D_l^3}{(\zeta \sec \theta)^{l+2} [2l + 1] P_l(\cos \theta) \cos \theta - LP_{l-1}(\cos \theta)], \right) \]  

according to equations (B2) and (B3). The magnetic flux $\Phi = \int \mathbf{B} \cdot da$, with $S$ being the entire surface bounded by the loop is
Figure B1. A flat circular loop positioned perpendicularly to the \( z \) axis at a distance \( \zeta \) from the center of the semispherical capacitor.

\[
\Phi = 2\pi \int_0^{\phi_0} \left( \sum_l D_l^3 \left( \frac{D_l^3}{\zeta \sec \theta} \right)^{(2l+1)} P_l(\cos \theta) \sin \theta - l P_{l-1}(\cos \theta) \tan \theta \right) d\theta. \tag{B7}
\]

Again we restrict ourselves to the case when the interior interface of the TI touches the plates of the capacitor, i.e. \( a = r_1 \). Then, the coefficients \( D_l^3 \) given by the equation (19). The evaluation of the flux \( \Phi \) proceeds by numerical integration and the results are presented in the table 2.

**ORCID iDs**

L F Urrutia @ https://orcid.org/0000-0001-9792-7344

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