Analytical Solutions Of The Schrödinger Equation For The Hulthén Potential Within SUSY Quantum Mechanics

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(Dated:)

Abstract

The analytical solution of the modified radial Schrödinger equation for the Hulthén potential is obtained within ordinary quantum mechanics by applying the Nikiforov-Uvarov method and supersymmetric quantum mechanics by applying the shape invariance concept that was introduced by Gendenshtein method by using the improved approximation scheme to the centrifugal potential for arbitrary $l$ states. The energy levels are worked out and the corresponding normalized eigenfunctions are obtained in terms of orthogonal polynomials for arbitrary $l$ states.

PACS numbers: 03.65.Ge

Keywords: Nikiforov-Uvarov method, Hulthén potential, Supersymmetric Quantum Mechanics

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I. INTRODUCTION

As known, one of the main objectives in theoretical physics since the early years of quantum mechanics (QM) is to obtain an exact solution of the Schrödinger equation for some special physically important potentials. Since the wave function contains all necessary information for full description of a quantum system, an analytical solution of the Schrödinger equation is of high importance in non-relativistic and relativistic quantum mechanics \[1, 2\]. There are few potentials for which the Schrödinger equation can be solved explicitly for all \(n\) and \(l\) quantum states.

The Hulthén potential is one of the important short-range potentials in physics. The potential has been used in nuclear and particle physics, atomic physics, solid-state physics, and its bound state and scattering properties have been investigated by a variety of techniques. General wave functions of this potential have been used in solid-state and atomic physics problems. It should be noted that, Hulthén potential is a special case of Eckart potential \[3\].

The Hulthén potential is defined by \[4, 5\]

\[
V(r) = -\frac{Ze^2\delta e^{-\delta r}}{(1 - e^{-\delta r})}
\]  

where \(Z\) is a constant and \(\delta\) is the screening parameter, dimensionless parameters. It should be noted that, the radial Schrödinger equation for the Hulthén potential can be solved analytically for only the states with zero angular momentum \[4–8\]. For any \(l\) states a number of methods have been employed to evaluate bound-state energies numerically \[9–19\].

At small values of the radial coordinate \(r\), the Hulthén potential behaves like a Coulomb potential, whereas for large values of \(r\) it decreases exponentially so that its influence for bound state is smaller than, that of Coulomb potential.

Because of these results, in this article we have used a method within the frame of supersymmetric quantum mechanics (SUSYQM) using an effective Hulthén potential for any \(l \neq 0\) angular momentum states, which can be solved analytically. In Ref. \[20\] authors used SUSY QM Hamiltonian hierarchy method for analytically solving radial Schrödinger equation for the Hulthén potential for any \(l\) states.

In contrast to the Hulthén potential, the Coulomb potential is analytically solvable for any \(l\) angular momentum. Take into account of this point will be very interesting and
important solving Schrödinger equation for the Hulthén potential for any \( l \) states within ordinary and SUSY QM and also to compare and analyze.

In this study, we obtain the energy eigenvalues and corresponding eigenfunctions for arbitrary \( l \) states by solving the Schrödinger equation for the Hulthén potential using Nikiforov-Uvarov (NU) method \[21\] and the shape invariance concept that was introduced by Gendenshtein \[22\].

It is known that, using for this potential the Schrödinger equation can be solved exactly for s-wave (\( l = 0 \)) \[6\].

Unfortunately, for an arbitrary \( l \)-states (\( l \neq 0 \)), the Schrödinger equation does not get an exact solution. But many papers show the power and simplicity of NU method in solving central and noncentral potentials \[23–27\] for arbitrary \( l \) states. This method is based on solving the second-order linear differential equation by reducing to a generalized equation of hypergeometric-type which is a second-order type homogeneous differential equation with polynomials coefficients of degree not exceeding the corresponding order of differentiation.

In this study, we obtain the energy eigenvalues and corresponding eigenfunctions for arbitrary \( l \) states by solving the radial Schrödinger equation for the Hulthén potential within ordinary and SUSY QM.

It should be noted that the same problem have been studied within SUSY QM in Ref. \[28\] as well, but our results disagree with the result obtained.

The structure of this work is as follows. Bound-state Solution of the radial Schrödinger equation for Hulthén potential by NU method within ordinary quantum mechanics is provided in Section II. The Solution of Schrödinger equation for Hulthén potential within SUSY QM \[III\] and the numerical results for energy levels and the corresponding normalized eigenfunctions are presented in Section IV. Finally, some concluding remarks are stated in Section V.

II. BOUND STATE SOLUTION OF THE RADIAL SCHRODINGER EQUATION FOR HULTHÉN POTENTIAL WITHIN ORDINARY QUANTUM MECHANICS.

The Schrödinger equation in spherical coordinates is given as
\[ \nabla^2 \psi + \frac{2\mu}{\hbar^2} [E - V(r)] \psi = 0. \quad (2.1) \]

Considering this equation, the total wave function is written as

\[ \psi(r, \theta, \phi) = R(r) Y_{l,m}(\theta, \phi), \quad (2.2) \]

Thus, for radial Schrödinger equation with Hulthén potential is

\[ R''(r) + \frac{2}{r} R'(r) + \frac{2\mu}{\hbar^2} \left[ E + Z e^2 \delta \frac{e^{-\delta r}}{1 - e^{-\delta r}} - \frac{l(l + 1)\hbar^2}{2\mu r^2} \right] R(r) = 0, \quad (2.3) \]

respectively.

The effective Hulthén potential is

\[ V_{\text{eff}}(r) = V_H + V_I = -Ze^2 \delta \frac{e^{-\delta r}}{1 - e^{-\delta r}} + \frac{l(l + 1)\hbar^2}{2\mu r^2}, \quad (2.4) \]

As we know, Eq.(2.3) is the radial Schrödinger equation for Hulthén potential. In order to solve Eq.(2.3) for \( l \neq 0 \), we must make an approximation for the centrifugal term. When \( \delta r \ll 1 \), we use an improved approximation scheme \[29\] to deal with the centrifugal term,

\[ \left[ C_0 + \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2} \right] \approx \frac{1}{\delta^2 r^2} + \left( C_0 - \frac{1}{12} \right) + O(\delta^2 r^2), \quad C_0 = \frac{1}{12}, \quad \frac{1}{r^2} \approx \delta^2 \left[ C_o + \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2} \right], \quad (2.5) \]

Now, the effective potential becomes

\[ \tilde{V}_{\text{eff}}(r) = -Ze^2 \delta \frac{e^{-\delta r}}{1 - e^{-\delta r}} + \frac{l(l + 1)\hbar^2}{2\mu} \left( C_0 + \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2} \right), \quad (2.6) \]

where the parameter \( C_0 = \frac{1}{12} \) (Ref. \[29\]) is a dimensionless constant. However, when \( C_0 = 0 \), the approximation scheme becomes the convectional approximation scheme suggested by Greene and Aldrich \[30\]. It should be noted that this approximation, is only valid for small \( \delta r \) and it breaks down in the high screening region. After using this approximation radial Schrödinger equation is solvable analytically.

We assume \( R(r) = \frac{1}{r} \chi(r) \) in Eq.(2.3) and the radial Schrödinger equation becomes

\[ \chi''(r) + \frac{2\mu}{\hbar^2} \left[ \varepsilon + Ze^2 \delta \frac{e^{-\delta r}}{1 - e^{-\delta r}} - \frac{\hbar^2 l(l + 1)}{2\mu} \left( C_0 + \frac{e^{-\delta r}}{(1 - e^{-\delta r})^2} \right) \right] \chi(r) = 0. \quad (2.7) \]
In order to transform Eq.(2.7), the equation of the generalized hypergeometric-type which
is in the form [21]
\[\chi''(s) + \frac{\tilde{\tau}}{\sigma} \chi'(s) + \frac{\tilde{\sigma}}{\sigma^2} \chi(s) = 0,\]
we use the following ansatz in order to make the differential equation more compact,
\[-\varepsilon^2 = \frac{2\mu}{\hbar^2 \delta^2} E, \quad E < 0, \quad \alpha^2 = \frac{2\mu Z e^2}{\hbar^2 \delta}, \quad s = e^{-\delta r}.\]

Hence, we obtain
\[\chi''(s) + \frac{\chi'(s)}{s} + \frac{1}{s^2(1-s)^2} \left[ -\varepsilon^2 (1-s)^2 - l(l+1)(C_0(1-s)^2 + s) + \alpha^2 s(1-s) \right] \chi(s) = 0.\]

(2.10)

Now, we can successfully apply NU method for defining eigenvalues of energy. By com-
paring Eq.(2.10) with Eq.(2.8), we can define the following:
\[\tilde{\tau}(s) = 1 - s, \quad \sigma(s) = s(1-s),\]
\[\tilde{\sigma}(s) = -\varepsilon^2 (1-s^2) - l(l+1)(C_0(1-s)^2 + s) + \alpha^2 s(1-s).\]

(2.11)

We change \(\lambda = l(l+1)\), then we obtain:
\[\tilde{\sigma}(s) = -\varepsilon^2 (1-s^2) - \lambda(C_0(1-s)^2 + s) + \alpha^2 s(1-s).\]

(2.12)

If we take the following factorization,
\[\chi(s) = \phi(s)y(s),\]
for the appropriate function \(\phi(s)\), Eq.(2.10) takes the form of the well-known hypergeometric-
type equation. The appropriate \(\phi(s)\) function must satisfy the following condition:
\[\frac{\phi'(s)}{\phi(s)} = \frac{\pi(s)}{\sigma(s)},\]

(2.14)

where function \(\pi(s)\) is defined as
\[\pi(s) = \frac{\sigma' - \tilde{\tau}}{2} \pm \sqrt{\left(\frac{\sigma' - \tilde{\tau}}{2}\right)^2 - \tilde{\sigma} + k\sigma}.\]

(2.15)
Finally, the equation, where \( y(s) \) is one of its solutions, takes the form known as hypergeometric-type,

\[
\sigma(s)y''(s) + \tau(s)y'(s) + \bar{\lambda}y(s) = 0, \quad (2.16)
\]

where

\[
\bar{\lambda} = k + \pi'
\]  
(2.17)

and

\[
\tau(s) = \tilde{\tau}(s) + 2\pi(s). \quad (2.18)
\]

For our problem, the \( \pi(s) \) function is written as

\[
\pi(s) = \frac{-s}{2} \pm \sqrt{s^2[a - k] - s[b - k] + c}, \quad (2.19)
\]

where the values of the parameters are

\[
a = \frac{1}{4} + \epsilon^2 + \lambda C_0 + \alpha^2, \\
b = 2\epsilon^2 + 2\lambda C_0 + \alpha^2 - \lambda, \\
c = \epsilon^2 + \lambda C_0.
\]

The constant parameter \( k \) can be found under the condition that the discriminant of the expression under the square root is equal to zero. Hence, we obtain

\[
k_{1,2} = (b - 2c) \pm 2\sqrt{c^2 + c(a - b)}. \quad (2.20)
\]

Now, we can find four possible functions for \( \pi(s) \):

\[
\pi(s) = \frac{-s}{2} \pm \begin{cases} 
(\sqrt{c} - \sqrt{c + a - b})s - \sqrt{c} & \text{for } k = (b - 2c) + 2\sqrt{c^2 + c(a - b)}, \\
(\sqrt{c} + \sqrt{c + a - b})s - \sqrt{c} & \text{for } k = (b - 2c) - 2\sqrt{c^2 + c(a - b)}. 
\end{cases} \quad (2.21)
\]

According to NU method, from the four possible forms of the polynomial \( \pi(s) \), we select the one for which the function \( \tau(s) \) has the negative derivative. Therefore, the appropriate function \( \pi(s) \) and \( \tau(s) \) are

\[
\pi'(s) = -\frac{1}{2} - \left[ \sqrt{c} + \sqrt{c + a - b} \right], \quad (2.22)
\]
\[ \pi(s) = \sqrt{c} - s \left[ \frac{1}{2} + \sqrt{c + \sqrt{c + a - b}} \right], \quad (2.23) \]

\[ \tau(s) = 1 + 2\sqrt{c} - 2s \left[ 1 + \sqrt{c + \sqrt{c + a - b}} \right], \quad (2.24) \]

for

\[ k = (b - 2c) - 2\sqrt{c^2 + c(a - b)}. \quad (2.25) \]

Also by Eq.(2.17), we can define the constant \( \bar{\lambda} \) as

\[ \bar{\lambda} = b - 2c - 2\sqrt{c^2 + c(a - b)} - \left[ \frac{1}{2} + \sqrt{c + \sqrt{c + a - b}} \right]. \quad (2.26) \]

Given a nonnegative integer \( n \), the hypergeometric-type equation has a unique polynomial solution of degree \( n \) if and only if

\[ \bar{\lambda} = \bar{\lambda}_n = -n\tau' - \frac{n(n - 1)}{2}\sigma'', \quad (n = 0, 1, 2...) \quad (2.27) \]

and \( \bar{\lambda}_m \neq \bar{\lambda}_n \) for \( m = 0, 1, 2, ..., n - 1 \), then it follows that

\[ \bar{\lambda}_{n_r} = b - 2c - 2\sqrt{c^2 + c(a - b)} - \left[ \frac{1}{2} + \sqrt{c + \sqrt{c + a - b}} \right] = 2n_r \left[ 1 + \left( \sqrt{c + \sqrt{c + a - b}} \right) \right] + n_r(n_r - 1). \quad (2.28) \]

We can solve Eq.(2.28) explicitly for \( c \) and by using the relation \( c = \varepsilon^2 + \lambda C_0 \), which brings

\[ \varepsilon^2 = \left[ \frac{\lambda + 1/2 + (l + \frac{1}{2})(2n + 1) + 2n + n^2 - n - \alpha^2}{2(l + \frac{1}{2}) + 2n + 1} \right]^2 - \lambda C_0, \quad (2.29) \]

Finally, we can found for \( \varepsilon^2 \)

\[ \varepsilon^2 = \left[ \frac{l + n + 1}{2} - \frac{\alpha^2}{2(l + n + 1)} \right]^2 - l(l + 1)C_0. \quad (2.30) \]

We substitute \( \varepsilon^2 \) into Eq.(2.9) with \( \lambda = l(l + 1) \), which identifies

\[ E_{nl} = -\frac{\hbar^2}{2\mu} \left[ \frac{(l + n + 1)}{2} - \frac{\alpha^2}{l + n + 1} \right]^2 - \hbar^2\delta^2 l(l + 1)C_0. \quad (2.31) \]
If we take $C_0 = 0$ in the Eq.(2.31), then we obtain result [32].

Now, using NU method we can obtain the radial eigenfunctions. After substituting $\pi(s)$ and $\sigma(s)$ into Eq.(2.14) and solving first-order differential equation, it is easy to obtain

$$\phi(s) = s^{\sqrt{c}}(1-s)^K,$$  \hspace{1cm} (2.32)

where $K = \frac{1}{2} + \sqrt{c+a-b} = l + 1$

Furthermore, the other part of the wave function $y(s)$ is the hypergeometric-type function whose polynomial solutions are given by Rodrigues relation

$$y_n(s) = \frac{B_n}{\rho(s)} d^n \left[ \sigma^n(s) \rho(s) \right],$$  \hspace{1cm} (2.33)

where $B_n$ is a normalizing constant and $\rho(s)$ is the weight function which is the solution of the Pearson differential equation. The Pearson differential equation and $\rho(s)$ for our problem is given as

$$(\sigma \rho)' = \tau \rho,$$ \hspace{1cm} (2.34)

$$\rho(s) = (1-s)^{2k-1} s^{2\sqrt{c}},$$ \hspace{1cm} (2.35)

respectively.

Substituting Eq.(2.35) in Eq.(2.33) we get

$$y_n(s) = B_n (1-s)^{1-2K-s^{2\sqrt{c}}} \frac{d^{n_r}}{ds^{n_r}} \left[ s^{2\sqrt{c}+n_r} (1-s)^{2K-1+n_r} \right].$$ \hspace{1cm} (2.36)

Then, by using the following definition of the Jacobi polynomials [33]:

$$P_n^{(a,b)}(s) = \frac{(-1)^n}{n! 2^n (1-s)^a (1+s)^b} \frac{d^n}{ds^n} \left[ (1-s)^{a+n}(1+s)^{b+n} \right],$$ \hspace{1cm} (2.37)

we can write

$$P_n^{(a,b)}(1-2s) = \frac{C_n}{s^{a}(1-s)^{b}} \frac{d^n}{ds^n} \left[ s^{a+n}(1-s)^{b+n} \right],$$  \hspace{1cm} (2.38)
and
\[
\frac{d^n}{ds^n} [s^{a+n} (1-s)^{b+n}] = C_n s^a (1-s)^b P_n^{(a,b)}(1-2s). (2.39)
\]

If we use the last equality in Eq.(2.36), we can write
\[
y_{n_r}(s) = C_{n_r} P_{n_r}^{(2\sqrt{c},2K-1)}(1-2s). (2.40)
\]

Substituting \(\phi(s)\) and \(y_{n_r}(s)\) into Eq.(2.13), we obtain
\[
\chi_{n_r}(s) = C_{n_r} s^{\sqrt{c}} (1-s)^K P_{n_r}^{(2\sqrt{c},2K-1)}(1-2s). (2.41)
\]

Using the following definition of the Jacobi polynomials \[33\]:
\[
P_n^{(a,b)}(s) = \frac{\Gamma(n+a+1)}{n!\Gamma(a+1)} \binom{-n, a+b+n+1, 1+a; 1-s}{2}, (2.42)
\]
we are able to write Eq.(2.41) in terms of hypergeometric polynomials as
\[
\chi_{n_r}(s) = C_{n_r} s^{\sqrt{c}} (1-s)^K \frac{\Gamma(n_r+2\sqrt{c}+1)}{n_r!\Gamma(2\sqrt{c}+1)} \binom{-n_r, 2\sqrt{c}+2K+n_r, 1+2\sqrt{c}; s}{2}. (2.43)
\]

The normalization constant \(C_{n_r}\) can be found from normalization condition
\[
\int_0^\infty |R(r)|^2 r^2 dr = \int_0^\infty |\chi(r)|^2 dr = b \int_0^1 \frac{1}{s} |\chi(s)|^2 ds = 1, (2.44)
\]
by using the following integral formula \[34\]:
\[
\int_0^1 (1-s)^{2(\delta+1)} s^{2\lambda-1} \left\{ \binom{-n_r, 2(\delta+\lambda+1) + n_r, 2\lambda+1; s}{2} \right\}^2 dz =
\frac{(n_r+\delta+1)n_r!\Gamma(n_r+2\delta+2)\Gamma(2\lambda)\Gamma(2\lambda+1)}{(n_r+\delta+\lambda+1)\Gamma(n_r+2\lambda+1)\Gamma(2(\delta+\lambda+1)+n_r)}, (2.45)
\]
for \(\delta > \frac{-3}{2}\) and \(\lambda > 0\). After simple calculations, we obtain normalization constant as
\[
C_{n_r} = \frac{n_r!2\sqrt{c}(n_r+K+\sqrt{c})\Gamma(2(K+\sqrt{c})+n_r)}{b(n_r+K)\Gamma(n_r+2\sqrt{c}+1)\Gamma(n_r+2K)}. (2.46)
\]
III. THE SOLUTION OF SCHR"{O}DINGER EQUATION FOR HULTH"{E}N POTENTIAL WITHIN SUSY QUANTUM MECHANICS

In the Supersymmetric QM, it is necessary to define nilpotent operators, namely $Q$ and $Q_+$, satisfying the algebra

\[
Q = \begin{pmatrix}
0 & 0 \\
A^- & 0
\end{pmatrix}, \quad (3.1)
\]

\[
Q_+ = \begin{pmatrix}
0 & A^+ \\
0 & 0
\end{pmatrix}, \quad (3.2)
\]

where $A^+$ and $A^-$ are bosonic operators.

The Hamiltonian, $H$ in terms of these operators is given by

\[
H = \begin{pmatrix}
A^+ A^- & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
H_+ & 0 \\
0 & H_-
\end{pmatrix}. \quad (3.3)
\]

Supersymmetric algebra allows us to write Hamiltonians as [35, 36]

\[
H_{\pm} = -\frac{\hbar^2}{2\mu} \frac{d^2}{dx^2} + V_{\pm}(x), \quad (3.4)
\]

where the SUSY partner potentials $V_{\pm}$ in terms of the superpotential $W(x)$

\[
V_{\pm} = W^2 \pm \frac{\hbar}{2\mu} \frac{dW}{dx}. \quad (3.5)
\]

The superpotential has a definition

\[
W(x) = -\frac{\hbar}{\sqrt{2}\mu} \left( \frac{d\ln \psi_0^{(0)}(x)}{dx} \right), \quad (3.6)
\]

where $\psi_0^{(0)}(x)$ denotes the ground-state wave function that satisfies the relation

\[
\psi_0^{(0)}(x) = N_0 \exp \left( -\sqrt{\frac{2\mu}{\hbar}} \int x W(x') dx' \right). \quad (3.7)
\]

The Hamiltonian $H_{\pm}$ can also be written in terms of the bosonic operators $A^+$ and $A^-$

\[
H_{\pm} = A^+ A^\pm \quad (3.8)
\]
where
\[ A^\pm = -\frac{\hbar}{\sqrt{2\mu}} \frac{d}{dx} + W(x) \quad (3.9) \]

It is a remarkable result that the energy eigenvalues of \( H_- \) and \( H_+ \) are identical except for the ground-state. In the case of unbroken supersymmetry, the ground-state energy of the Hamiltonian \( H_- \) is zero \( E_0^0 = 0 \) \[35, 36\]. In the factorization of the Hamiltonian, Eqs.(3.4), (3.8) and (3.9) are used, respectively. Hence, we obtain
\[ H_1 = -\frac{\hbar^2}{2\mu} \frac{d^2}{dx^2} + V_1(x) = A_1^+ A_1^- + E_0^{(1)} \quad (3.10) \]

Thus, comparing each side of Eq.(3.10), term by term, we receive the Riccati equation for the superpotential \( W_1 \)
\[ W_1^2(x) - W_1'(x) = \frac{2\mu}{\hbar^2} (V_1(x) - E_0^{(1)}) \quad (3.11) \]

Let us now construct the SUSY partner Hamiltonian \( H_2 \) as
\[ H_2 = -\frac{\hbar^2}{2\mu} \frac{d^2}{dx^2} + V_2(x) = A_2^+ A_2^- + E_0^{(2)} \quad (3.12) \]

and Riccati equation takes the form
\[ W_2^2(x) - W_2'(x) = \frac{2\mu}{\hbar^2} (V_2(x) - E_0^{(2)}) \quad (3.13) \]

Similarly, one can write, in general, the Riccati equation and Hamiltonians by iteration as
\[ W_n^2(x) - W_n'(x) = \frac{2\mu}{\hbar^2} (V_n(x) - E_0^{(n)}) = A_n^+ A_n^- + E_0^{(n)} \quad (3.14) \]

and
\[ H_n = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V_n(x) = A_n^+ A_n^- + E_0^{(n)}, n = 1, 2, 3...... \quad (3.15) \]

Because of the SUSY unbroken case, the partner Hamiltonians satisfy the following expressions \[35, 36\]
\[ E_0^{(n+1)} = E_1^{(n)}, n = 1, 2, 3......; E_0^{(0)} = 0 \quad (3.16) \]

In SUSY QM, the ground-state eigenfunction \( \psi_0(x) \) can be written as Eq.(3.7). Through the superalgebra, we make following ansatz for the superpotential:
\[ W_1(r) = -\frac{\hbar}{\sqrt{2\mu}} \left( A + \frac{Be^{-\delta r}}{1 - e^{-\delta r}} \right) \]  

(3.17)

and having inserted this expression into Eq.(3.11), we obtain

\[ W_1^2(r) - \frac{\hbar}{\sqrt{2\mu}} W_1'(r) = \frac{\hbar^2}{2m} \left( A^2 + \frac{2ABe^{-\delta r}}{1 - e^{-\delta r}} + \frac{B^2e^{-2\delta r}}{(1 - e^{-\delta r})^2} - \frac{\delta Be^{-\delta r}}{(1 - e^{-\delta r})^2} \right) \]  

(3.18)

If take into account Eqs.(2.7) and (3.18), then we obtain:

\[ \frac{\hbar^2}{2\mu} \left( A^2 + \frac{2ABe^{-\delta r}}{1 - e^{-\delta r}} + \frac{B^2e^{-2\delta r}}{(1 - e^{-\delta r})^2} - \frac{\delta Be^{-\delta r}}{(1 - e^{-\delta r})^2} \right) = \left[ -\varepsilon - \frac{Ze^2\delta e^{-\delta r}}{1 - e^{-\delta r}} + \frac{h^2l(l+1)}{2\mu} \delta^2C_0 + \frac{h^2l(l+1)}{2\mu} \frac{\delta^2e^{-\delta r}}{(1 - e^{-\delta r})^2} \right]. \]  

(3.19)

After small manipulations, we obtain

\[ A^2 + \frac{2ABe^{-\delta r}}{1 - e^{-\delta r}} + \frac{B^2e^{-2\delta r}}{(1 - e^{-\delta r})^2} - \frac{\delta Be^{-\delta r}}{(1 - e^{-\delta r})^2} = \frac{2\mu}{\hbar^2} \left[ -E - \frac{Ze^2\delta e^{-\delta r}}{1 - e^{-\delta r}} + \frac{h^2l(l+1)}{2\mu} \delta^2C_0 + \frac{h^2l(l+1)}{2\mu} \frac{\delta^2e^{-\delta r}}{(1 - e^{-\delta r})^2} \right]. \]  

(3.20)

where it satisfies the associated Riccati equation, so we can obtain the following identity. With comparison of the each side of the Eq.(3.20), we obtain

\[ A^2 = -\frac{2\mu E}{\hbar^2} + \delta^2C_0 l(l+1) = \varepsilon^2\delta^2 + \delta^2C_0 l(l+1), \]  

(3.21)

\[ 2AB - \delta B = \delta^2l(l+1) - \frac{2\mu}{\hbar^2} Ze^2\delta = \delta^2l(l+1) - \delta^2\alpha^2, \]  

(3.22)

\[ B^2 - \delta B = \delta^2l(l+1). \]  

(3.23)

After inserting Eq.(3.17) into (3.7), the eigenfunction for ground-state in terms of \( r \) will be obtained as

\[ \chi(r) = N_0 e^{Ar} (1 - e^{-\delta r}) \frac{B^2}{\pi} \]  

(3.24)

Considering extremity conditions to wave functions, we obtain \( B > 0 \) and \( A < 0 \).
Solving Eq.(3.23) yields

$$ B = \frac{\delta \pm \sqrt{\delta^2 + 4\delta^2 l(l+1)}}{2} = \frac{\delta \pm \delta \sqrt{(2l+1)^2}}{2} = \delta \pm \delta(2l+1), \quad (3.25) $$

and considering $B > 0$ from Eqs.(3.22) and (3.23), we find

$$ 2AB - B^2 = -\delta^2 \alpha^2, \quad (3.26) $$
or

$$ A = \frac{B}{2} - \frac{\delta^2 \alpha^2}{2B}, \quad (3.27) $$

From Eqs.(2.9) and (3.21), we find

$$ -\frac{2\mu E_0}{\hbar^2 \delta^2} = \frac{1}{\delta^2} \left( \frac{B}{2} - \frac{\delta^2 \alpha^2}{2B} \right)^2 - C_0 l(l+1). \quad (3.28) $$

Finally, for energy eigenvalue, we obtain

$$ E_0 = \frac{\hbar^2 l(l+1)C_0 \delta^2}{2\mu} - \frac{\hbar^2}{2\mu} \left( \frac{l + 1}{2} - \frac{\alpha^2 \delta}{2(l+1)} \right)^2, \quad (3.29) $$

Using Eq.(3.5), we can find SUSY partner potentials $V_+(r)$ and $V_-(r)$ in the form

$$ V_+(r) = W^2(r) + \frac{\hbar}{\sqrt{2\mu}} W'(r) = \frac{\hbar^2}{2\mu} \left[ A^2 + \frac{(2AB + \delta B)e^{-\delta \gamma}}{1 - e^{-\delta \gamma}} + \frac{(B^2 + \delta B)e^{-2\delta \gamma}}{(1 - e^{-\delta \gamma})^2} \right] \quad (3.30) $$

$$ V_-(r) = W^2(r) + \frac{\hbar}{\sqrt{2\mu}} W'(r) = \frac{\hbar^2}{2\mu} \left[ A^2 + \frac{(2AB - \delta B)e^{-\delta \gamma}}{1 - e^{-\delta \gamma}} + \frac{(B^2 - \delta B)e^{-2\delta \gamma}}{(1 - e^{-\delta \gamma})^2} \right] \quad (3.31) $$

The shape invariance concept that was introduced by Gendenshtein is

$$ R(B_1) = V_+(B, r) - V_-(B_1, r) = \frac{\hbar^2}{2\mu} \left[ A^2 - A_1^2 \right] = \frac{\hbar^2}{2\mu} \left[ \left( \frac{B}{2} - \frac{\delta^2 \alpha^2}{2B} \right)^2 - \left( \frac{B + \delta}{2} - \frac{\delta^2 \alpha^2}{2(B + \delta)} \right)^2 \right]. \quad (3.32) $$
If we now consider a mapping of the form

\[ B \rightarrow B_1 = B + \delta, \]

\[ B_n = B + n\delta, \]  

(3.33)

so, we have

\[ R(B_i) = V_+ [B + (i - 1)\delta, r] - V_- [B + i\delta, r] = \]

\[-\frac{\hbar^2}{2\mu} \left[ \frac{B + i\delta}{2} - \frac{\delta^2 \alpha^2}{2(B + i\delta)} \right]^2 - \frac{\hbar^2}{2\mu} \left[ \frac{B + (i - 1)\delta}{2} - \frac{\delta^2 \alpha^2}{2(B + (i - 1)\delta)} \right]^2. \]  

(3.34)

where the reminder \( R(B_1) \) is independent of \( r \). Thus, we have

\[ E_{nl} = E_0 + \sum_{i=0}^{n} R(B_i) = \]

\[ \frac{\hbar^2 l(l + 1)}{2\mu} \delta^2 C_0 - \frac{\hbar^2}{2\mu} \left( \frac{B}{2} - \frac{\delta^2 \alpha^2}{2B} \right)^2 - \frac{\hbar^2}{2\mu} \left[ \frac{B + \delta}{2} - \frac{\delta^2 \alpha^2}{2(B + \delta)} \right]^2 - \]

\[ \left( \frac{B + (n - 1)\delta}{2} - \frac{\delta^2 \alpha^2}{2(B + (n - 1)\delta)} \right)^2 - \left( B + \frac{(n - 2)\delta}{2} - \frac{\delta^2 \alpha^2}{2(B + (n - 2)\delta)} \right)^2 - \]

\[ \left( B + \frac{(n - 1)\delta}{2} - \frac{\delta^2 \alpha^2}{2(B + (n - 1)\delta)} \right)^2 = \]

\[ \frac{\hbar^2 l(l + 1)}{2\mu} \delta^2 C_0 - \frac{\hbar^2}{2\mu} \left( \frac{B + n\delta}{2} - \frac{\delta^2 \alpha^2}{2(B + n\delta)} \right)^2, \]  

(3.35)

and we obtain

\[ E_{nl} = \frac{\hbar^2 l(l + 1)}{2\mu} \delta^2 C_0 - \frac{\hbar^2}{2\mu} \left( \frac{B + n\delta}{2} - \frac{\delta^2 \alpha^2}{2(B + n\delta)} \right)^2, \]  

(3.36)

Finally, for energy eigenvalues we found

\[ E_{nl} = \frac{\hbar^2 l(l + 1)}{2\mu} \delta^2 C_0 - \frac{\hbar^2}{2\mu} \left( \frac{n + l + 1}{2} - \frac{\delta^2 \alpha^2}{2(l + n + 1)} \right^2, \]  

(3.37)
IV. NUMERICAL RESULTS AND DISCUSSION

Solution of the modified radial Schrödinger equation for the Hulthén potential are obtained within ordinary quantum mechanics by applying the Nikiforov-Uvarov method and within SUSY QM by applying the shape invariance concept that was introduced by Genden-shtein method in which we have used the improved approximation scheme to the centrifugal potential for arbitrary \( l \) states. Both ordinary and SUSY quantum mechanical energy eigenvalues and corresponding eigenfunctions have obtained for arbitrary \( l \) quantum numbers. In the Table I, we present numerical results for the energy eigenvalues of the Hulthén potential as a function of screening parameter for various state in atomic units is obtained by within ordinary (obtained by NU method) and SUSY QM(shape invariance method) methods.

For comparison, in the Table II shows that energy eigenvalues of the Hulthén potential as a function of screening parameter for various state in atomic units which are obtained of the asymptotic iteration method [37], the SUSY [20], numerical integration [8] and the variational method [8]. As it can be seen from the results presented in these tables, the numerical results obtained of the analytically solution are in good agreement with results of the other methods for the small \( \delta \) values, but in the large screening region, the agreement is poor. Analysis our calculation is shows that the main reason is simply that when the \( \delta r \) increases in the large screening region, the agreement between \( V_{eff}(r) \) and \( \tilde{V}_{eff}(r) \) potential decreases. However, this problem could be solved by making a better approach of the centrifugal term.

It should be noted, that Eqs.(2.31)and (3.37) in cases \( C_0 = 0 \) and \( l \neq 0 \) is exactly the same result obtained by other works [20, 32], also Eqs.(2.31) and (3.37) in cases \( C_0 = 0 \) and \( l = 0 \) is exactly the same result in [6].

It is shown, that energy eigenvalues and corresponding eigenfunctions are identical for both ordinary and SUSY QM.

V. CONCLUSION

It is well know that the Hulthén potential is one of the important exponential potential, and it has been a subject of interest in many fields of physics and chemistry. The main results of this paper are the explicit and closed form expressions for the energy eigenvalues
and the normalized wave functions. The method presented in this paper is a systematic one and in many cases it is more than the other ones.

Analytical solution of the modified radial Schrödinger equation for the Hulthén potential are obtained within ordinary quantum mechanics by applying the Nikiforov-Uvarov method and within SUSY QM by applying the shape invariance concept that was introduced by Gel’den’shtein method in which we used the improved approximation scheme to the centrifugal potential for arbitrary $l$ states. The energy eigenvalues and corresponding eigenfunctions are obtained for arbitrary $l$ quantum numbers. It is shown that energy eigenvalues and corresponding eigenfunctions are the same for both ordinary and SUSY QM.

Consequently, studying of analytical solution of the modified Schrödinger equation for the Hulthén potential within framework ordinary and SUSY QM could provide valuable information on the QM dynamics at atomic and molecules physics and opens new window.

We can conclude that our results are not only interesting for pure theoretical physicist but also for experimental physicist because of the exact and more general the results.

Acknowledgments

The authors thanks to Dr. A.I.Ahmadov and Dr. V.H.Badalov for fruitful discussions and useful comments.
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| state | $\delta$ | Present work, $NU C_0 = 0$ | Present work, $NU C_0 \neq 0$ | Present work, SUSY $C_0 = 0$ | Present work, SUSY $C_0 \neq 0$ |
|-------|--------|---------------------|---------------------|---------------------|---------------------|
| 2p    | 0.025  | -0.1128125          | -0.1127604          | -0.1128125          | -0.1127604          |
|       | 0.050  | -0.1012500          | -0.10104166         | -0.1012500          | -0.10104166         |
|       | 0.075  | -0.0903125          | -0.08984375         | -0.0903125          | -0.08984375         |
|       | 0.10   | -0.080000           | -0.07916666         | -0.080000           | -0.07916666         |
|       | 0.150  | -0.0612500          | -0.059375           | -0.0612500          | -0.059375           |
|       | 0.200  | -0.45000            | -0.416666           | -0.45000            | -0.416666           |
|       | 0.250  | -0.0312500          | -0.02604166         | -0.0312500          | -0.02604166         |
|       | 0.300  | -0.02000            | -0.012500           | -0.02000            | -0.012500           |
|       | 0.350  | -0.01125            | -0.00104166         | -0.01125            | -0.00104166         |
| 3p    | 0.025  | -0.04375868         | -0.04370659         | -0.04375868         | -0.04370659         |
|       | 0.050  | -0.03336805         | -0.03315972         | -0.03336805         | -0.03315972         |
|       | 0.075  | -0.02438737         | -0.0239149305       | -0.02438737         | -0.0239149305       |
|       | 0.100  | -0.01680555         | -0.015972222         | -0.01680555         | -0.015972222         |
|       | 0.150  | -0.00586805         | -0.003993055        | -0.00586805         | -0.003993055        |
| 3d    | 0.025  | -0.04375868         | -0.04370659         | -0.04375868         | -0.04370659         |
|       | 0.050  | -0.03336805         | -0.03315972         | -0.03336805         | -0.03315972         |
|       | 0.075  | -0.02438737         | -0.0239149305       | -0.02438737         | -0.0239149305       |
|       | 0.100  | -0.01680555         | -0.015972222         | -0.01680555         | -0.015972222         |
|       | 0.150  | -0.00586805         | -0.003993055        | -0.00586805         | -0.003993055        |
| 4p    | 0.025  | -0.02000            | -0.0199478          | -0.02000            | -0.0199478          |
|       | 0.050  | -0.01125            | -0.011041666        | -0.01125            | -0.011041666        |
|       | 0.075  | -0.00500            | -0.00453125         | -0.00500            | -0.00453125         |
|       | 0.100  | -0.00125            | -0.00041666         | 0.00125             | -0.00041666         |
| 4d    | 0.025  | -0.02000            | -0.0199478          | -0.02000            | -0.0199478          |
|       | 0.050  | -0.01125            | -0.011041666        | -0.01125            | -0.011041666        |
|       | 0.075  | -0.00500            | -0.00453125         | -0.00500            | -0.00453125         |
| state | $\delta$ | Present work $\nu C_0 = 0$ | Present work $\nu C_0 \neq 0$ | Present work SUSY $C_0 = 0$ | Present work SUSY $C_0 \neq 0$ |
|-------|---------|-----------------|-----------------|-----------------|-----------------|
| 4f    | 0.025   | -0.02000        | -0.0199478      | -0.02000        | -0.0199478      |
|       | 0.050   | -0.01125        | -0.011041666    | -0.01125        | -0.011041666    |
|       | 0.075   | -0.00500        | -0.00453125     | -0.00500        | -0.00453125     |
| 5p    | 0.025   | -0.009453125    | -0.009401       | -0.009453125    | -0.009401       |
|       | 0.050   | -0.0028125      | -0.00260416     | -0.0028125      | -0.00260416     |
| 5d    | 0.025   | -0.009453125    | -0.009401       | -0.009453125    | -0.009401       |
|       | 0.050   | -0.0028125      | -0.00260416     | -0.0028125      | -0.00260416     |
| 5f    | 0.025   | -0.009453125    | -0.009401       | -0.009453125    | -0.009401       |
|       | 0.050   | -0.0028125      | -0.00260416     | -0.0028125      | -0.00260416     |
| 5g    | 0.025   | -0.009453125    | -0.009401       | -0.009453125    | -0.009401       |
|       | 0.050   | -0.0028125      | -0.00260416     | -0.0028125      | -0.00260416     |
| 5f    | 0.025   | -0.009453125    | -0.009401       | -0.009453125    | -0.009401       |
|       | 0.050   | -0.0028125      | -0.00260416     | -0.0028125      | -0.00260416     |
| 5g    | 0.025   | -0.009453125    | -0.009401       | -0.009453125    | -0.009401       |
|       | 0.050   | -0.0028125      | -0.00260416     | -0.0028125      | -0.00260416     |
| 6p    | 0.025   | -0.00420138     | -0.004149305    | -0.00420138     | -0.004149305    |
| 6d    | 0.025   | -0.00420138     | -0.004149305    | -0.00420138     | -0.004149305    |
| 6g    | 0.025   | -0.00420138     | -0.004149305    | -0.00420138     | -0.004149305    |

TABLE I: Energy eigenvalues of the Hultheén potential as a function of the screening parameter for 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states in atomic units ($\hbar = m = e = 1$) and for $Z = 1$. 


| state | $\delta$ | AIM [27] | SUSY [20] | Numerical [8] | Variational [8] |
|-------|---------|----------|-----------|---------------|-----------------|
| 2p    | 0.025   | 0.1128125| 0.1127605 | 0.1127605     | 0.1127605       |
|       | 0.050   | 0.1012500 | 0.1010425 | 0.1010425     | 0.1010425       |
|       | 0.075   | 0.0903125 | 0.0898478 | 0.0898478     | 0.0898478       |
|       | 0.10    | 0.0800000 | 0.0791794 | 0.0791794     | 0.0791794       |
|       | 0.150   | 0.0612500 | 0.0594415 | 0.0594415     | 0.0594415       |
|       | 0.200   | 0.0450000 | 0.0418854 | 0.0418860     | 0.0418860       |
|       | 0.250   | 0.0312500 | 0.0266060 | 0.0266111     | 0.0266108       |
|       | 0.300   | 0.0200000 | 0.0137596 | 0.0137900     | 0.0137878       |
|       | 0.350   | 0.0112500 | 0.0036146 | 0.0037931     | 0.0037734       |
| 3p    | 0.025   | 0.0437590 | 0.0437068 | 0.0437069     | 0.0437069       |
|       | 0.050   | 0.0333681 | 0.0331632 | 0.0331645     | 0.0331645       |
|       | 0.075   | 0.0243837 | 0.0239331 | 0.0239397     | 0.0239397       |
|       | 0.100   | 0.0168056 | 0.0160326 | 0.0160537     | 0.0160537       |
|       | 0.150   | 0.0058681 | 0.0043599 | 0.0044663     | 0.0044660       |
| 3d    | 0.025   | 0.0437587 | 0.0436030 | 0.0436030     | 0.0436030       |
|       | 0.050   | 0.0333681 | 0.0327532 | 0.0327532     | 0.0327532       |
|       | 0.075   | 0.0243837 | 0.0230306 | 0.0230307     | 0.0230307       |
|       | 0.100   | 0.0168055 | 0.0144832 | 0.0144842     | 0.0144842       |
|       | 0.150   | 0.0058681 | 0.0132820 | 0.0013966     | 0.0013894       |
| 4p    | 0.025   | 0.0200000 | 0.0199480 | 0.0199489     | 0.0199489       |
|       | 0.050   | 0.0112500 | 0.0110430 | 0.0110582     | 0.0110582       |
|       | 0.075   | 0.0050000 | 0.005385  | 0.0046219     | 0.0046219       |
|       | 0.100   | 0.0012500 | 0.000434  | 0.0007550     | 0.0007532       |
| 4d    | 0.025   | 0.0200000 | 0.0198460 | 0.0198462     | 0.0198462       |
|       | 0.050   | 0.0112500 | 0.0106609 | 0.0106674     | 0.0106674       |
|       | 0.075   | 0.0050000 | 0.0037916 | 0.0038345     | 0.0038344       |
| state | $\delta$ | AIM [37] | SUSY [20] | Numerical [8] | Variational [8] |
|-------|---------|----------|-----------|--------------|-----------------|
| 4f    | 0.025   | 0.0200000| 0.0196911 | 0.0196911    | 0.0196911       |
|       | 0.050   | 0.0112500| 0.0100618 | 0.0100620    | 0.0100620       |
|       | 0.075   | 0.0050000| 0.0025468 | 0.0025563    | 0.0025557       |
| 5p    | 0.025   | 0.0094531| 0.0094011 | 0.0094036    |                 |
|       | 0.050   | 0.0028125| 0.0026058 | 0.0026490    |                 |
| 5d    | 0.025   | 0.0094531| 0.0092977 | 0.0093037    |                 |
|       | 0.050   | 0.0028125| 0.0022044 | 0.0023131    |                 |
| 5f    | 0.025   | 0.0094531| 0.0091507 | 0.0091521    |                 |
|       | 0.050   | 0.0028125| 0.0017421 | 0.0017835    |                 |
| 5g    | 0.025   | 0.0094531| 0.0089465 | 0.0089465    |                 |
|       | 0.050   | 0.0028125| 0.0010664 | 0.0010159    |                 |
| 6p    | 0.025   | 0.0042014| 0.0041493 | 0.0041548    |                 |
| 6d    | 0.025   | 0.0042014| 0.0040452 | 0.0040606    |                 |
| 6f    | 0.025   | 0.0042014| 0.0038901 | 0.0039168    |                 |
| 6g    | 0.025   | 0.0042014| 0.0036943 | 0.0037201    |                 |

TABLE II: Energy eigenvalues of the Hultheén potential as a function of the screening parameter for 2p, 3p, 3d, 4p, 4d, 4f, 5p, 5d, 5f, 5g, 6p, 6d, 6f and 6g states in atomic units ($\hbar = m = e = 1$) and for $Z = 1$. 