Geometrical Formulation of Quantum Mechanics

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Abstract

States of a quantum mechanical system are represented by rays in a complex Hilbert space. The space of rays has, naturally, the structure of a Kähler manifold. This leads to a geometrical formulation of the postulates of quantum mechanics which, although equivalent to the standard algebraic formulation, has a very different appearance. In particular, states are now represented by points of a symplectic manifold (which happens to have, in addition, a compatible Riemannian metric), observables are represented by certain real-valued functions on this space and the Schrödinger evolution is captured by the symplectic flow generated by a Hamiltonian function. There is thus a remarkable similarity with the standard symplectic formulation of classical mechanics. Features—such as uncertainties and state vector reductions—which are specific to quantum mechanics can also be formulated geometrically but now refer to the Riemannian metric—a structure which is absent in classical mechanics. The geometrical formulation sheds considerable light on a number of issues such as the second quantization procedure, the role of coherent states in semi-classical considerations and the WKB approximation. More importantly, it suggests generalizations of quantum mechanics. The simplest among these are equivalent to the dynamical generalizations that have appeared in the literature. The geometrical reformulation provides a unified framework to discuss these and to correct a misconception. Finally, it also suggests directions in which more radical generalizations may be found.

I. INTRODUCTION

Quantum mechanics is probably the most successful scientific theory ever invented. It has an astonishing range of applications—from quarks and leptons to neutron stars and white dwarfs—and the accuracy with which its underlying ideas have been tested is equally impressive. Yet, from its very inception, prominent physicists have expressed deep reservations about its conceptual foundations and leading figures continue to argue that it is
incomplete in its core. Time and again, attempts have been made to extend it in a non-trivial fashion. Some of these proposals have been phenomenological (see, e.g., [1–3]), aimed at providing a ‘mechanism’ for the state reduction process. Some have been more radical, e.g., invoking hidden variables (see, e.g., [4]). Yet others involve non-linear generalizations of the Schrödinger equation [5–7]. Deep discomfort has been expressed at the tension between objective descriptions of happenings provided by the space-time geometry of special relativity and the quantum measurement theory [8,9]. Further conceptual issues arise when one brings general relativity into picture, issues that go under the heading of ‘problem of time’ in quantum gravity [10–12]. Thus, while there is universal agreement that quantum mechanics is an astonishingly powerful working tool, in the ‘foundation of physics circles’ there has also been a strong sentiment that sooner or later one would be forced to generalize it in a profound fashion [13–15].

It is often the case that while an existing theory admits a number of equivalent descriptions, one of them suggests generalizations more readily than others. Furthermore, typically, this description is not the most familiar one, i.e., not the one that seems simplest from the limited perspective of the existing theory. An example is provided by Cartan’s formulation of Newtonian gravity. While it played no role in the invention of the theory (it came some two and a half centuries later!) at a conceptual level, Cartan’s framework provides a deeper understanding of Newtonian gravity and its relation to general relativity. A much more striking example is Minkowski’s geometric reformulation of special relativity. His emphasis on hyperbolic geometry seemed abstract and abstruse at first; at the time, Einstein himself is said to have remarked that it made the subject incomprehensible to physicists. Yet, it proved to be an essential stepping stone to general relativity.

The purpose of this article is to present, in this spirit, a reformulation of the mathematical framework underlying standard quantum mechanics (and quantum field theory). The strength of the framework is that it is extremely natural from a geometric perspective and succinctly illuminates the essential difference between classical and quantum mechanics. It has already clarified certain issues related to the second quantization procedure and semi-classical approximations [16]. It also serves to unify in a coherent fashion a number of proposed generalizations of quantum mechanics; in particular, we will see that generalizations that were believed to be distinct (and even incompatible) are in fact closely related. More importantly, this reformulation may well lead to viable generalizations of quantum mechanics which are more profound than the ones considered so far. Finally, our experience from seminars and discussions has shown that ideas underlying this reformulation lie close to the heart of geometrically oriented physicists. It is therefore surprising that the framework is not widely known among relativists. We are particularly happy to be able to rectify this situation in this volume honoring Engelbert and hope that the role played by the Kähler geometry, in particular, will delight him.

Let us begin by comparing the standard frameworks underlying classical and quantum mechanics. The classical description is geometrical: States are represented by points of a symplectic manifold $\Gamma$, the phase space. The space of observables consists of the (smooth) real-valued functions on this manifold. The (ideal) measurement of an observable $f$ in a state $p \in \Gamma$ yields simply the value $f(p)$ at the point $p$; the state is left undisturbed. These outcomes occur with complete certainty. The space of observables is naturally endowed with the structure of a commutative, associative algebra, the product being given simply by
pointwise multiplication. Thanks to the symplectic structure, it also inherits a Lie-bracket—the Poisson bracket. Finally to each observable $f$ is associated a vector field $X_f$ called the Hamiltonian vector field of $f$. Thus, each observable generates a flow on $\Gamma$. Dynamics is determined by a preferred observable, the Hamiltonian $\hat{H}$; the flow generated by $X_H$ describes the time evolution of the system.

The arena for quantum mechanics, on the other hand, is a Hilbert space $\mathcal{H}$. States of the system now correspond to rays in $\mathcal{H}$, and the observables are represented by self-adjoint linear operators on $\mathcal{H}$. As in the classical description, the space of observables is a real vector space equipped with with two algebraic structures. First, we have the the Jordan product—i.e., the anti-commutator—which is commutative but now fails to be associative. Second, we have $(1/2i$ times) the commutator bracket which endows the space of observables with the structure of a Lie algebra. Measurement theory, on the other hand, is strikingly different. In the textbook description based on the Copenhagen interpretation, the (ideal) measurement of an observable $\hat{A}$ in a state $\Psi \in \mathcal{H}$ yields an eigenvalue of $\hat{A}$ and, immediately after the measurement, the state is thrown into the corresponding eigenstate. A specific outcome can only be predicted probabilistically. As in the classical theory, each observable $\hat{A}$ gives rise to a flow on the state space. But now, the flow is generated by the 1-parameter group $\exp i\hat{A}t$ and respects the linearity of $\mathcal{H}$. Dynamics is again dictated by a preferred observable, the Hamiltonian operator $\hat{H}$.

Clearly, the two descriptions have several points in common. However, there is also a striking difference: While the classical framework is geometric and non-linear, the quantum description is intrinsically algebraic and linear. Indeed, the emphasis on the underlying linearity is so strong that none of the standard textbook postulates of quantum mechanics can be stated without reference to the linear structure of $\mathcal{H}$.

From a general perspective, this difference seems quite surprising. For linear structures in physics generally arise as approximations to the more accurate non-linear ones. Thus, for example, we often encounter non-linear equations which correctly capture a physical situation. But, typically, they are technically difficult to work with and we probe properties of their solutions through linearization. In the present context, on the other hand, it is the deeper, more correct theory that is linear and the non-linear, geometric, classical framework is to arise as a suitable limiting case.

However, deeper reflection shows that quantum mechanics is in fact not as linear as it is advertised to be. For, the space of physical states is not the Hilbert space $\mathcal{H}$ but the space of rays in it, i.e., the projective Hilbert space $\mathcal{P}$. And $\mathcal{P}$ is a genuine, non-linear manifold. Furthermore, it turns out that the Hermitian inner-product of $\mathcal{H}$ naturally endows $\mathcal{P}$ with the structure of a Kähler manifold. Thus, in particular, like the classical state space $\Gamma$, the correct space of quantum states, $\mathcal{P}$, is a symplectic manifold! We will therefore refer to $\mathcal{P}$ as the quantum phase space. Given any self-adjoint operator $\hat{H}$, we can take its expectation value to obtain a real function on $\mathcal{H}$. It is easy to verify that this function admits an unambiguous projection $h$ to the projective Hilbert space $\mathcal{P}$. Recall, now, that every phase space function gives rise to a flow through its Hamiltonian vector field. What then is the interpretation of the flow $X_h$? It turns out [17] to be exactly the (projection to $\mathcal{P}$ of the) flow defined by the Schrödinger equation (on $\mathcal{H}$) of the quantum theory. Thus, Schrödinger evolution is precisely the Hamiltonian flow on the quantum phase space!

As we will see, the interplay between the classical and quantum ideas stretches much
further. The overall picture can be summarized as follows. Classical phase spaces \( \Gamma \) are, in general, equipped only with a symplectic structure. Quantum phase spaces, \( \mathcal{P} \), on the other hand, come with an additional structure, the Riemannian metric provided by the Kähler structure. Roughly speaking, features of quantum mechanics which have direct classical analogues refer only to the symplectic structure. On the other hand, features—such as quantum uncertainties and state vector reduction in a measurement process—refer also the Riemannian metric. This neat division lies at the heart of the structural similarities and differences between the (mathematical frameworks underlying the) two theories.

Section II summarizes this geometrical reformulation of standard quantum mechanics. We begin in II A by showing that the quantum Hilbert space can be regarded as a (linear) Kähler space and discuss the roles played by the symplectic structure and the Kähler metric. In II B, we show that one can naturally arrive at the quantum state space \( \mathcal{P} \) by using the Bergmann-Dirac theory of constrained systems. (This method of constructing the quantum phase space will turn out to be especially convenient in section III while analyzing the relation between various generalizations of quantum mechanics.) Section II C provides a self-contained treatment of the various issues related to observables—associated algebraic structures, quantum uncertainty relations and measurement theory—in an intrinsically geometric fashion. These results are collected in section II D to obtain a geometric formulation of the postulates of quantum mechanics, a formulation that makes no reference to the Hilbert space \( \mathcal{H} \) or the associated linear structures. In practical applications, except while dealing with simple cases such as spin systems, the underlying quantum phase space \( \mathcal{P} \) is infinite-dimensional (since it comes from an infinite-dimensional Hilbert space \( \mathcal{H} \)). Our mathematical discussion encompasses this case. Also, in the discussion of measurement theory, we allow for the possibility that observables may have continuous spectra.

In section III, we consider possible generalizations of quantum mechanics. These generalizations can occur in two distinct ways. First, we can retain the original kinematic structure but allow more general dynamics, e.g., by replacing the Schrödinger equation by a suitable non-linear one (see, e.g., [5]). In section III A we show that the geometrical reformulation of section I naturally suggests a class of such extensions which encompasses those proposed by Birula and Mycielski [6] and by Weinberg [7]. In section III B we consider the possibility of more radical extensions in which the kinematical set up itself is changed. Although (to our knowledge) there do not exist interesting proposals of this type, such generalizations would be much more interesting. In particular, it is sometimes argued that the linear structure underlying quantum mechanics would have to be sacrificed in a subtle but essential way to obtain a satisfactory quantum theory of gravity and/or to cope satisfactorily with the ‘measurement problem’ [13]. To implement such ideas, the underlying kinematic structure will have to be altered. A first step in this direction is to obtain a useful characterization of the kinematical framework of standard quantum mechanics. Section III B provides a ‘reconstruction theorem’ which singles out quantum mechanics from its plausible generalizations. The theorem provides powerful guidelines: it spells out directions along which one can proceed to obtain a genuine extension.

Section IV is devoted to semi-classical issues. Consider a simple mechanical system, such as a particle in \( \mathbb{R}^3 \). In this case, the classical phase space \( \Gamma \) is six-dimensional while the quantum phase space \( \mathcal{P} \) is infinite-dimensional. Is there a relation between the two? In section IV A we show that the answer is in the affirmative: \( \mathcal{P} \) is a bundle over \( \Gamma \). Furthermore,
the bundle is trivial. Thus, through each quantum state \( p \in \mathcal{P} \), there is a cross-section, i.e., a copy of \( \Gamma \). It turns out that the quantum states that lie on any one cross-section are precisely the generalized coherent states \([18, 20, 21]\). In the remainder of section \([V]\), we use this interplay between \( \mathcal{P} \) and \( \Gamma \) to discuss the relation between classical and quantum dynamics. Section \([VIII]\) is devoted to the correspondence in terms of Ehrenfest’s theorem while \([VC]\) discusses the problem along the lines of the WKB approximation. Somewhat interestingly, it turns out that WKB dynamics is an example of generalized dynamics of the Weinberg type \([7]\).

Our conventions are as follows. If the manifold under consideration is infinite-dimensional, we will assume that it is a Hilbert manifold. (Projective Hilbert spaces are naturally endowed with this structure; see, e.g., \([16]\).) Riemannian metrics and symplectic structures on these manifolds will be assumed to be everywhere defined, smooth, strongly non-degenerate fields. (Thus, they define isomorphisms between the tangent and cotangent spaces at each point). In detailed calculations we will often use the abstract index notation of Penrose’s \([22, 23]\). Note that, in spite of the appearance of indices, this notation is well-defined also on infinite-dimensional manifolds. (For example, if \( V^a \) denotes a contravariant vector field; the subscript \( a \) does not refer to its components but is only a label telling us that \( V \) is a specific type of tensor field, namely a contravariant vector field. Similarly, \( V^a \omega_a \) is the function obtained by the action of the 1-form \( \omega \) on the contravariant vector field \( V \).) Finally, due to space limitation, we have not included detailed proofs of several technical assertions; they can be found in \([16]\). Our aim here is only to provide a thorough overview of the subject.

This work was intended to be an extension of a paper by Kibble \([17]\) which pointed out that the Schrödinger evolution can be regarded as an Hamiltonian flow on \( \mathcal{H} \). However, after completing this work, we learned that many of the results contained in sections \([I] \) and \([III]\) were obtained independently by others (although the viewpoints and technical proofs are often distinct.) In 1985, Heslot \([24]\) observed that quantum mechanics admits a symplectic formulation in which the phase space is the projective Hilbert space. That discussion was, however, restricted to the finite-dimensional case and did not include a discussion of the role of the metric, probabilistic interpretation and quantum uncertainties. Anandan and Aharonov \([25]\) rediscovered some of these results and also discussed some of the probabilistic aspects. This work was also restricted to finite-dimensional systems and focussed on the issue of evolution. Similar observations were made by Gibbons \([26]\) who also discussed density matrices (which are not considered here) and raised the issue of characterization of quantum mechanics (which is resolved in section \([VIII]\)). An essentially complete treatment of the finite-dimensional case was given by Hughston \([27]\). (This work was done in parallel to ours. However, it also contains some proposals for mechanisms for state reduction \([28]\) which are not discussed here.) The only references (to our knowledge) which treat the infinite-dimensional case are \([29, 30]\), which also discuss the issue of characterization of standard quantum mechanics. Finally, since the geometric structures that arise here are so natural, it is quite possible that they were independently discovered by other authors that we are not aware of.
II. GEOMETRIC FORMULATION OF QUANTUM MECHANICS

The goal of this section is to show that quantum mechanics can be formulated in an intrinsically geometric fashion, without any reference to a Hilbert space or the associated linear structure. We will assume that the reader is familiar with basic symplectic geometry.

A. The Hilbert space as a Kähler space

Let us begin with the standard Hilbert space formulation of quantum mechanics. In this sub-section we will view the Hilbert space $\mathcal{H}$ as a Kähler space and examine the role played by the associated symplectic structure and the Riemannian metric. This discussion will serve as a stepping stone to the analysis of the quantum phase space $\mathcal{P}$ in section II B.

The similarities between classical and quantum mechanics can be put in a much more suggestive form with an alternative, but equivalent, description of the Hilbert space. We view $\mathcal{H}$ as a real vector space equipped with a complex structure $J$. The complex structure is a preferred linear operator which represents multiplication by $i$; hence $J^2 = -I$. Initially, this change of notation seems rather trivial; the element which is typically written $(a + ib)\Psi$ is now denoted $a\Psi + bJ\Psi$ and (external) multiplication of vectors by complex numbers is not permitted. However, this slight change of viewpoint will come with a reward—a symplectic formulation of quantum mechanics.

Since $\mathcal{H}$ is now viewed as a real vector space, the Hermitian inner-product is slightly unnatural. We therefore decompose it into real and imaginary parts,

$$\langle \Phi, \Psi \rangle := \frac{1}{2\hbar} G(\Phi, \Psi) + \frac{i}{2\hbar} \Omega(\Phi, \Psi). \quad (2.1)$$

(The reason for the factors of $1/2\hbar$ will become clear shortly.) Properties of the Hermitian inner-product imply that $G$ is a positive definite, real inner-product and that $\Omega$ is a symplectic form, both of which are strongly non-degenerate. Moreover, since $\langle \Phi, J\Psi \rangle = i\langle \Phi, \Psi \rangle$, one immediately observes that the metric, symplectic structure and complex structure are related as

$$G(\Phi, \Psi) = \Omega(\Phi, J\Psi). \quad (2.2)$$

That is, the triple $(J, G, \Omega)$ equips $\mathcal{H}$ with the structure of a Kähler space. Therefore, every Hilbert space may be naturally viewed as a Kähler space.

Next, by use of the canonical identification of the tangent space (at any point of $\mathcal{H}$) with $\mathcal{H}$ itself, $\Omega$ is naturally extended to a strongly non-degenerate, closed, differential 2-form $\mathcal{H}$, which we will denote also by $\Omega$. Any Hilbert space is therefore naturally viewed as the simplest sort of symplectic manifold, i.e., a phase space. The inverse of $\Omega$ may be used to define Poisson brackets and Hamiltonian vector fields. As we are about to see, these notions are just as relevant in quantum mechanics as in classical mechanics.
1. The symplectic form

In classical mechanics, observables are real-valued functions, and to each such function is associated a corresponding Hamiltonian vector field. In quantum mechanics, on the other hand, the observables themselves may be viewed as vector fields, since linear operators associate a vector to each element of the Hilbert space. However, the Schrödinger equation, which in our language is written as $\dot{\Psi} = -\frac{i}{\hbar} \hat{H} \Psi$, motivates us to associate to each quantum observable $\hat{F}$ the vector field

$$Y_{\hat{F}}(\Psi) := -\frac{1}{\hbar} J \hat{F} \Psi.$$  \hspace{1cm} (2.3)

This Schrödinger vector field is defined so that the time-evolution of the system corresponds to the flow along the Schrödinger vector field associated to the Hamiltonian operator. (Note that, if the Hamiltonian is unbounded, it is only densely defined and so is the vector field. The (unitary) flow, however, is defined on all of $\mathcal{H}$. See, e.g., [31].)

Natural questions immediately arise. Let $\hat{F}$ be any bounded, self-adjoint operator on $\mathcal{H}$. Is the corresponding vector field, $Y_{\hat{F}}$, Hamiltonian on the symplectic space $(\mathcal{H}, \Omega)$? If so, what is the real-valued function which generates this vector field? What is the physical meaning of the Poisson bracket? In particular, how is it related to the commutator Lie algebra?

The answers to these questions are remarkably simple. As we know from standard quantum mechanics, $\hat{F}$ generates a one-parameter family of unitary mappings on $\mathcal{H}$. By definition, $Y_{\hat{F}}$ is the generator of this one-parameter family and therefore preserves both the metric $G$ and symplectic form $\Omega$. It is therefore locally Hamiltonian, and, since $\mathcal{H}$ is a linear space, also globally Hamiltonian! In fact, the function which generates this Hamiltonian vector field is of physical interest; it is simply the expectation value of $\hat{F}$.

Let us see this explicitly. Denote by $F : \mathcal{H} \rightarrow \mathbb{R}$ the expectation value function,

$$F(\Psi) := \langle \Psi, \hat{F} \Psi \rangle = \frac{1}{2\hbar} G(\Psi, \hat{F} \Psi).$$ \hspace{1cm} (2.4)

We will continue to use this notation; expectation value functions will be denoted by simply “un-hatting” the corresponding operators. Now, if $\eta$ is any tangent vector at $\Psi$, then

$$(dF)(\eta) = \frac{d}{dt} \langle \Psi + t\eta \hat{F}(\Psi + t\eta) \rangle \big|_{t=0} = \langle \Psi, \hat{F} \eta \rangle + \langle \eta, \hat{F} \Psi \rangle = \frac{1}{\hbar} G(\hat{F} \Psi, \eta) = \Omega(Y_{\hat{F}} \eta, \eta) = (i_{Y_{\hat{F}}} \Omega)(\eta),$$ \hspace{1cm} (2.5)

where we have used the self-adjointness of $\hat{F}$, Eq. (2.2) and the definition of $Y_{\hat{F}}$. Therefore, the Hamiltonian vector field $X_F$ generated by the expectation value function $F$ coincides with the Schrödinger vector field $Y_{\hat{F}}$ associated to $\hat{F}$. As a particular consequence, the

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1With our conventions for symplectic geometry, the Hamiltonian vector field $X_f$ generated by the function $f$ satisfies the equation $i_{X_f} \Omega = df$, and the Poisson bracket is defined by $\{f, g\} = \Omega(X_f, X_g)$. 

7
time evolution of any quantum mechanical system may be written in terms of Hamilton’s equation of classical mechanics; the Hamiltonian function is simply the expectation value of the Hamiltonian operator. Schrödinger’s equation is Hamilton’s equation in disguise!

Next, let $\hat{F}$ and $\hat{K}$ be two quantum observables, and denote by $F$ and $K$ the respective expectation value functions. It is natural to ask whether the Poisson bracket of $F$ and $K$ is related in a simple manner to an algebraic operation involving the original operators. Performing a calculation as simple as Eq. (2.5), one finds that

$$\{F, K\}_{\Omega} = \Omega(X_F, X_K) = \left\langle \frac{1}{\hbar} [\hat{F}, \hat{K}] \right\rangle.$$  

(2.6)

Notice that the quantity inside the brackets on the right side of Eq. (2.6) is precisely the quantum Lie bracket of $\hat{F}$ and $\hat{K}$. The algebraic operation on the expectation value functions, which is induced by the commutator bracket is exactly a Poisson bracket! Note that this is not Dirac’s correspondence principle; the Poisson bracket here is the quantum one, determined by the imaginary part of the Hermitian inner-product.

The basic features of the classical formalism appear also in quantum mechanics. The Hilbert space, as a real vector space, is equipped with a symplectic form. To each quantum observable is associated a real-valued function on $\mathcal{H}$, and the time-evolution is determined by the Hamiltonian vector field associated to a preferred function. Moreover, the Lie bracket of two quantum observables corresponds precisely to the Poisson bracket of the corresponding functions.

2. Uncertainty and the real inner-product

Let us now examine the role played by the metric $G$. Clearly, $G$ enables us to define a real inner-product, $G(X_F, X_K)$ between any two Hamiltonian vector fields $X_F$ and $X_K$. One may expect that this inner-product is related to the Jordan product in much the same way that the symplectic form corresponds to the commutator Lie bracket. It is easy to verify that this expectation is correct. Operating just as in Eq. (2.6), we obtain

$$\{F, K\}_+ := \frac{\hbar}{2} G(X_F, X_K) = \left\langle \frac{1}{2} [\hat{F}, \hat{K}]_+ \right\rangle.$$  

(2.7)

The operation defined by the first equation above will be called the Riemann bracket of $F$ and $K$. Up to the factor of $\hbar/2$, the Riemann bracket of $F$ and $K$ is simply given by the (real) inner-product of their Hamiltonian vector fields, and corresponds precisely to the Jordan product of the respective operators.

Since the classical phase space is, in general, not equipped with a Riemannian metric, the Riemann product does not have an analogue in the classical formalism; it does, however, admit a physical interpretation. In order to see this, note that the uncertainty of the observable $\hat{F}$ at a state with unit norm is given by

$$\langle \Delta \hat{F} \rangle^2 = \langle \hat{F}^2 \rangle - \langle \hat{F} \rangle^2 = \{F, F\}_+ - F^2.$$  

(2.8)

Thus, the uncertainty of an operator $\hat{F}$, when written in terms of the expectation value function $F$, involves the Riemann bracket. Moreover, this expression for the uncertainty is quite simple.
In fact Heisenberg’s famous uncertainty relation also assumes a nice form when expressed in terms of the expectation value functions. It is very well-known that the familiar uncertainty relation between two quantum observables may be written in a slightly stronger form (see, e.g., [32]):

\[
(\Delta \hat{F})^2 (\Delta \hat{K})^2 \geq \left( \frac{1}{2i} [\hat{F}, \hat{K}] \right)^2 + \left( \frac{1}{2} [\hat{F}_\perp, \hat{K}_\perp]_+ \right)^2,
\]

(2.9)

where \(\hat{F}_\perp\) is the non-linear operator defined by

\[
\hat{F}_\perp(\Psi) := \hat{F}(\Psi) - F(\Psi)
\]

so that \(\hat{F}_\perp(\Psi)\) is orthogonal to \(\Psi\), if \(\|\Psi\| = 1\).

Using the above results, we may immediately rewrite Eq. (2.9) in the form

\[
(\Delta \hat{F})^2 (\Delta \hat{K})^2 \geq \left( \frac{\hbar}{2} \{F, K\}_\Omega \right)^2 + (\{F, K\}_+ - FK)^2.
\]

(2.10)

Incidentally, the last expression in Eq. (2.10) may be interpreted as the “quantum covariance” of \(\hat{F}\) and \(\hat{K}\); see Ref. [16] for an explanation.

B. The quantum phase space

We have seen that to each quantum observable is associated a smooth real-valued (expectation value) function on the Hilbert space. Further, the familiar operations involving quantum operators correspond to simple “classical looking” operations on the corresponding functions. These observations suggest a formulation of standard quantum mechanics in the language of classical mechanics. However, there are two difficulties. First, although the Hilbert space is a symplectic space, because two state vectors related by multiplication by any complex number define the same state, it is not the space of physical states, i.e., the quantum analog of the classical phase space. Second, the description of the measurement process in a manner intrinsic to the Kähler structure on \(\mathcal{H}\) turns out not to be natural.

The true state space of the quantum system is the space of rays in the Hilbert space, i.e., the projective Hilbert space, which we shall denote \(\mathcal{P}\). It should not be surprising that \(\mathcal{P}\) is a Kähler manifold, and hence, in particular, a symplectic manifold. After all, for the special case in which \(\mathcal{H}\) is \(\mathbb{C}^{n+1}\), \(\mathcal{P}\) is the complex projective space \(\mathbb{C}P^n\)—the archetypical Kähler manifold. In this section, we present a particularly useful description of the projective Hilbert space which is valid for the infinite-dimensional case and which illuminates the role of its symplectic structure. These developments will enable us to handle the above complications and allow an elegant geometric formulation of quantum mechanics.

1. Gauge reduction

The standard strategy to handle the ambiguity of the state vector is to consider only those elements of the Hilbert space which are normalized to unity. We will adopt this
approach by insisting that the only physically relevant portion of the Hilbert space is that on which the constraint function,

\[ C(\Psi) := \langle \Psi, \Psi \rangle - \frac{1}{2\hbar} G(\Psi, \Psi) - 1, \quad (2.11) \]

vanishes. The attitude adopted here is one in which the constraint surface—the unit sphere, with respect to the Hermitian inner-product—is the only portion of the Hilbert space which is accessible to the system. The rest of the Hilbert space is often quite convenient, but is viewed as an artificial element of the formalism.

Let us consider the above restriction from the point of view of the Bergmann-Dirac theory of constrained systems (see, e.g., [33]). In other words, we will pretend, for a moment, that we are dealing with a classical theory with the constraint \( C = 0 \). First notice that since the time-evolution preserves the constraint surface, no further (secondary) constraints arise. Since we have only a single constraint, it is trivially of first-class in Dirac’s terminology; i.e., the constraint generates a motion which preserves the constraint surface \( \{ \mathcal{L}_{X_{C}}, C \} = 0 \).

Recall that to every first-class constraint on a Hamiltonian system is associated a gauge degree of freedom; the associated gauge transformations are defined by the flow along the Hamiltonian vector field generated by the constraint function. In our case, the gauge directions are simply given by

\[ X_{C}^{a} = \frac{1}{\hbar} \omega^{ab} \Psi_{b} = -\frac{1}{\hbar} J^{a}_{b} \Psi^{b}, \quad (2.12) \]

where \( D_{a} \) denotes the Levi-Civita derivative operator. For later convenience, let us define

\[ J^{a} := \hbar X_{C}^{a} |_{S} = -J^{a}_{b} \Psi^{b} |_{S}. \quad (2.13) \]

Notice that \( J^{a} \) is the generator of phase rotations on \( S \). Therefore, the gauge transformations generated by the constraint are exactly what they ought to be; they represent the arbitrariness in our choice of phase!

Thus, we now see the relevance of the description in terms of constrained Hamiltonian systems. By taking the quotient of the constraint surface of any constrained system by the action of the gauge transformations, one obtains the true phase space of the system—often called the reduced phase space. The projective Hilbert space may therefore be interpreted as the “reduced phase space” of our constrained Hamiltonian system. In order to emphasize both its physical role and geometric structure, we will refer to the projective Hilbert space \( P \) as the quantum phase space. One can explicitly show that if \( H \) is infinite-dimensional, \( P \) is an infinite-dimensional Hilbert manifold [16].

As the terminology suggests, any reduced phase space is equipped with a natural symplectic structure. This fact may be seen as follows. Denote by \( i : S \to H \) and \( \pi : S \to P \) the inclusion mapping and projection to the quantum phase space, respectively. By restricting the symplectic structure \( \Omega \) to the constraint surface, one obtains a closed 2-form \( i^{*}\Omega \) on \( S \). This 2-form is degenerate, but only in the gauge direction. Fortunately, since gauge transformations are defined by the Hamiltonian vector field generated by the constraint function, \( i^{*}\Omega \) is constant along its directions of degeneracy. As a result, there exists a symplectic form \( \omega \) on \( P \) whose pull-back via \( \pi \) agrees precisely with \( i^{*}\Omega \). This is a standard construction in
the theory of systems with first class constraints; the only novelty here lies in its application to ordinary quantum mechanics. Finally, we note that this result applies to the typical case of interest, in which the original Hilbert space is infinite-dimensional \([16]\). The symplectic structure \(\omega\) is then a smooth, strongly non-degenerate field (i.e., defines an isomorphism from the tangent space to the cotangent space at each point.)

Before discussing the geometry the quantum phase space, we should point out that the viewpoint adopted in this section in fact generalizes quantum mechanics, but in a very trivial way. In section II A we observed that each quantum observable defines a real-valued function on the entire Hilbert space, and that the quantum evolution is given by the Hamiltonian flow defined by a preferred function. In viewing a quantum system as a constrained Hamiltonian system, we must concede that it is only the constraint surface \(S\) that is of physical relevance. In particular, the restriction \(F|_S\) of the expectation value function \(F\) contains all gauge-invariant information about the observable; one may extend \(F\) off the constraint surface in any desirable manner without affecting the corresponding flow on the projective space. The particular extensions defined by expectation values of (bounded) self-adjoint operators may be viewed as mere convention. We will make use of this point in section II A.

2. Symplectic geometry

Our method of arriving at the quantum phase space \(P\) by the reduced phase space construction of constrained systems immediately suggests further definitions and constructions.

Recall that to each bounded, self-adjoint operator \(\hat{F}\) on \(\mathcal{H}\), we have associated the function \(F(\Psi) := \langle \Psi, \hat{F}\Psi \rangle\) on the Hilbert space. In fact, we may go a short step further. First, let us restrict the expectation value function \(F\) to the constraint surface, thereby obtaining the function \(i^*F : S \to \mathbb{R}\). \(i^*F\) is clearly gauge-invariant (i.e. independent of phase), and therefore defines the function \(f : P \to \mathbb{R}\) for which \(\pi^*f = i^*F\). Therefore, to each quantum observable is associated a smooth, real-valued function on the quantum phase space. The functions obtained in this manner will represent the observables in the geometric formulation of quantum mechanics. Let us therefore make

**Definition II.1** Let \(f : P \to \mathbb{R}\) be a smooth function on \(P\). If there exists a bounded, self-adjoint operator \(\hat{F}\) on \(\mathcal{H}\) for which \(\pi^*f = \langle \hat{F} \rangle|_S\), then \(f\) is said to be an observable function.

Note that we consider the set of quantum observables to consist of the bounded self-adjoint operators on \(\mathcal{H}\). At first sight this appears to be a severe restriction. However, further reflection shows that it is not. In any actual experiment, one deals only with a finite range of relevant parameters and hence in practice one only measures observables of the type considered here. Thus, there is by definition a one-to-one correspondence between quantum observables and the observable functions on \(P\). As we will see below, the set of observable functions is a very small subset of the entire function space.

A natural question arises: What is the relationship between the Hamiltonian vector fields \(X_F\) (on \(\mathcal{H}\)) and \(X_f\) (on \(P\))? Given any point \(\Psi \in S\), we may push-forward the vector \(X_F|_\Psi\) to obtain a tangent vector at \(\pi(\Psi)\). Since \(F\) is gauge-invariant, it commutes with \(C\); therefore
As a consequence, $X_F$ is “constant along the integral curves of $\mathcal{J}$”. Thus, by pushing-forward $X_F$ at each point of $S$, one obtains a well-defined (smooth) vector field on all of $\mathcal{P}$. As is known to those familiar with the analysis of constrained systems, this vector field is also Hamiltonian; in fact, it agrees precisely with $X_f$. The flow on $\mathcal{P}$, which is induced by the Schrödinger vector field of $\hat{F}$ corresponds exactly to the Hamiltonian flow determined by the observable function $f$.

Next, consider the Poisson bracket $\{,\}_\omega$ defined by the reduced symplectic structure $\omega$. Let $F, K : \mathcal{H} \rightarrow \mathbb{R}$ be expectation value functions of two quantum observables and denote by $f, k : \mathcal{P} \rightarrow \mathbb{R}$ the corresponding observable functions on $\mathcal{P}$. As a consequence of the above result,

$$\pi^*\{f, k\}_{\omega} = \pi^*(\omega(X_f, X_k)) = \omega(\pi_*X_F, \pi_*X_K) = \Omega(X_F, X_K)|_S = \{F, K\}_\Omega|_S.$$ (2.14)

Therefore, the Poisson bracket defined by $\omega$ exactly reflects the commutator bracket on the space of quantum observables.

In summary, to each quantum observable $\hat{F}$ is associated a real-valued function $f : \mathcal{P} \rightarrow \mathbb{R}$ on the quantum phase space. The Schrödinger vector field determined by $\hat{F}$ determines a flow on $\mathcal{P}$; this flow is generated by the Hamiltonian vector field associated to the observable function $f$. Further, the mapping $\hat{F} \mapsto f$ is one-to-one and respects the Lie algebraic structures provided by the commutator and Poisson bracket on $\mathcal{P}$, respectively.

C. Riemannian geometry and measurement theory

Any quantum mechanical system may be described as an infinite-dimensional Hamiltonian system. However, the structure of the quantum phase space is much richer than that of classical mechanics. $\mathcal{P}$ is also equipped with a natural Riemannian metric. As we will see, the probabilistic features of quantum mechanics are conveniently described by the Riemannian structure.

The quantum metric may be described in much the same way as the symplectic structure. The restriction $i^*G$ of $G$ to the unit sphere is a strongly non-degenerate Riemannian metric on $S$. Recall that the gauge generator $\mathcal{J}$ is (up to the constant factor of $\hbar$) the Schrödinger vector field associated to the identity operator. Since any Schrödinger vector field preserves the Hermitian inner-product, it preserves both the symplectic structure $\Omega$ and the metric $G$. As a consequence, $\mathcal{J}$ is a Killing vector field on $S$;

$$\mathcal{L}_\mathcal{J}(i^*G) \equiv 0.$$ (2.15)

Therefore, $\mathcal{P}$ may also be described as the Killing reduction [34] of $S$ with respect to the Killing field $\mathcal{J}$.

A manifold which arises in this way is always equipped with a Riemannian metric of its own. Although $i^*G$ is “constant” on the integral curves of $\mathcal{J}$, it is not degenerate in that

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1One must require that the integral curves of the Killing vector field do not come arbitrarily close to one another. This condition is satisfied in our case.
direction. However, by subtracting off the component in the direction of $\mathcal{J}$,

$$
\tilde{g} := \left[ G - \frac{1}{2\hbar}(\Psi \otimes \Psi + \mathcal{J} \otimes \mathcal{J}) \right]_S,
$$

we obtain a symmetric tensor field which agrees with $i^*G$ when acting on vectors orthogonal to $\mathcal{J}$, is constant along $\mathcal{J}$ and is degenerate only in the direction of $\mathcal{J}$. Therefore $\tilde{g}$ defines a strongly non-degenerate Riemannian metric $g$ on $\mathcal{P}$. It is a simple matter to verify that $g$, when combined with the symplectic structure $\omega$, equips the quantum phase space with the structure of a Kähler manifold.

1. Quantum observables

According to Def. II.1, quantum mechanical observables may be represented by real-valued functions on the quantum phase space. Unfortunately, these functions have still been defined in terms of self-adjoint operators on the Hilbert space. Our goal, however, is a formulation of quantum mechanics which is intrinsic to the projective space; we wish to avoid any explicit reference to the underlying Hilbert space. We now explain how this deficiency may be overcome.

Since the Schrödinger vector field $Y_F = X_F$ generates a one-parameter family of unitary transformations on $\mathcal{H}$, $X_F$ preserves not only the symplectic structure $\Omega$, but the metric $G$ as well; $X_F$ is also a Killing vector field. This fact also holds for the corresponding observable function $f$; the Hamiltonian vector field $X_f$ associated to any observable function $f$ is also a Killing vector field on $(\mathcal{P}, g)$. We will see that it is this property which characterizes the set of observable functions on the quantum phase space.

Let us begin by recalling a general property of Killing vector fields. Since the calculations are somewhat involved, we will now use Penrose's abstract index notation (which, as already pointed out, is meaningful also on infinite-dimensional Hilbert manifolds.) Let $X^\alpha$ be any Killing vector field on $\mathcal{P}$. Then, by definition, $\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0$, where $\nabla$ denotes the (Levi-Civitá) derivative operator associated to the metric $g$. Therefore, $K_{\alpha\beta} := \nabla_\alpha X_\beta$ is necessarily skew-symmetric. As one can easily verify, $K$ satisfies the identity

$$
\nabla_\alpha K_{\beta\gamma} = R_{\gamma\beta\alpha}^\delta X_\delta.
$$

(Our conventions are such that for any 1-form $k_\gamma$, $R_{\alpha\beta\gamma}^\delta k_\delta = (\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) k_\gamma$.) As a consequence, the Killing vector field $X$ is completely determined by its value and first covariant derivative at a single point. (See [35] or Appendix C of Ref. [35] for a discussion of this useful fact.)

Now suppose that the above Killing vector field is generated by the observable function $f$. (Below, it will be understood that $X = X_f$.) Since

$$
\omega_{\alpha}^\gamma K_{\gamma\beta} = -\omega_{\alpha}^\gamma \nabla_\beta X_\gamma = \nabla_\alpha \nabla_\beta f,
$$

$K$ satisfies the additional property that $\omega_{\alpha}^\gamma K_{\gamma\beta}$ is symmetric. (It then defines a bounded, skew-self-adjoint operator on each tangent space.) By considering the coupled differential equations:
\[ \nabla_\alpha f = \omega_\gamma \alpha X^\gamma, \quad (2.19) \]
\[ \nabla_\alpha X^\beta = K^\alpha_\beta, \quad (2.20) \]
\[ \nabla_\alpha K^\beta_\gamma = R^\gamma_\delta^\beta_\alpha X^\delta, \quad (2.21) \]

we see [16] that any observable function is completely determined by its value and first two derivatives at a single point! This fact motivates

**Definition II.2** For each point \( p \in \mathcal{P} \), let \( \mathcal{S}_p \) consist of all triples, \((\lambda, X_\alpha, K_{\alpha\beta})\), where \( \lambda \) is a real number, \( X_\alpha \) is a covector at \( p \), and \( K_{\alpha\beta} \) is a 2-form at \( p \) for which \( \omega^\gamma_\alpha K_\gamma^\beta_\alpha = \omega^\gamma_\beta K_\gamma^\alpha_\alpha \). We call \( \mathcal{S}_p \) the algebra of symmetry data at \( p \).

Thus, any observable function \( f \) determines an element \((\lambda, X, K) \in \mathcal{S}_p \) (\( \lambda \) provides the value of \( f \) at \( p \)), and \( f \) is completely determined by this symmetry data. The algebra of quantum observables is then isomorphic to a subset of \( \mathcal{S}_p \). The converse to this statement is provided by [16]:

**Theorem II.1** For any element \((\lambda, X, K) \) of \( \mathcal{S}_p \), there exists an observable function \( f : \mathcal{P} \to \mathbb{R} \) such that \( f(p) = \lambda \), \((X_f)_\alpha = X_\alpha \) and \( \nabla_\alpha (X_f)_\beta = K_{\alpha\beta} \).

Therefore, the space of observable functions is isomorphic to the algebra of symmetry data \( \mathcal{S}_p \) for any \( p \in \mathcal{P} \). We will utilize this result in section III B. For the purposes of this section, however, the main use of the above theorem is the following. Recall that the Hamiltonian vector field of any observable function is a Killing vector field. Conversely, let \( f \) be any smooth, real-valued function on \( \mathcal{P} \) for which \( X_f \) is also a Killing vector field. Of course, the value and first two covariant derivatives of \( f \) at \( p \in \mathcal{P} \) determine an element of \( \mathcal{S}_p \). Therefore, by Thm. II.1, \( f \) is an observable function. This important result is expressed in

**Corollary 1** A smooth function \( f : \mathcal{P} \to \mathbb{R} \) is observable if and only if its Hamiltonian vector field is also Killing.

Therefore, as in classical mechanics, the space of quantum observables is isomorphic to the space of smooth functions on the phase space whose Hamiltonian vector fields are infinitesimal symmetries of the available structure. However, now the available structure includes not just the symplectic structure but also the metric. Hence, unlike in classical mechanics, this function space is an extremely small subset of the set of all smooth functions on \( \mathcal{P} \). This should not be terribly surprising; as an example consider the finite-dimensional case, for which the function space is infinite-dimensional while the algebra of Hermitian operators is finite-dimensional.

2. Quantum uncertainty

In analogy with our considerations of the symplectic geometry of \( \mathcal{P} \), let us consider the Riemann bracket defined by the quantum metric \( g \). We denote the \( g \)-Riemann bracket of \( f, k : \mathcal{P} \to \mathbb{R} \) by

\[
(f, k) := \frac{\hbar}{2} g(X_f, X_k) = \frac{\hbar}{2} (\nabla_\alpha f) g^{\alpha\beta} (\nabla_\beta k). \quad (2.22)
\]
If $f$ and $k$ correspond to the expectation value functions $F, K : \mathcal{H} \to \mathbb{R}$, then, by Eq. (2.16),

$$\{F, K\}_+ = \frac{\hbar}{2} G(X_F, X_K) = \frac{\hbar}{2} g(X_F, X_K) + \frac{1}{4} G(J, X_F) G(J, X_K) = \pi^*[(f, k) + fk], \quad (2.23)$$

where we have used the fact that for any observable $\hat{F}$,

$$G(J, X_F)|_\Psi = -G(J\Psi, X_F) = \Omega(X_F, \Psi) = \Psi \circ (dF) = 2F. \quad (2.24)$$

Therefore, the observable function which corresponds to the Jordan product of $\hat{F}$ and $\hat{K}$ is given not by the $g$-Riemann bracket of $f$ and $k$, but by the quantity

$$\{f, k\}_+ := (f, k) + fk, \quad (2.25)$$

where we have utilized a minor notational abuse to emphasize the relationship with Eq. (2.7).

The above quantity will be called the symmetric bracket of $f$ and $k$.

Note that the $g$-Riemann bracket of two observables, while not necessarily an observable itself, is a physically meaningful quantity; it is simply the quantum covariance function (see the comment following Eq. (2.10)). In particular, $(f, f)(p)$ is exactly the squared uncertainty of $f$ at the quantum state labeled by $p$;

$$(\Delta f)^2(p) := (\Delta \hat{F})^2(\pi^{-1} p) = (f, f)(p). \quad (2.26)$$

In order to obtain a feeling for the physical meaning of the quantum covariance, notice that the uncertainty relation of Eq. (2.10) assumes the form

$$(\Delta f)(\Delta k) \geq \left(\frac{\hbar}{2}\{f, k\}_\omega\right)^2 + (f, k)^2. \quad (2.27)$$

As one can easily show, the standard uncertainty relation

$$(\Delta f)(\Delta k) \geq \left(\frac{\hbar}{2}\{f, k\}_\omega\right)^2 \quad (2.28)$$

is saturated at the state $p \in \mathcal{P}$ if and only if $(f, k)(p) = 0$ and $X_f \propto j(X_k)$, where $j$ is the complex structure on $\mathcal{P}$ (compatible with $\omega$ and $g$). The quantum covariance $(f, k)(p)$ therefore measures the “coherence” of the state $p$ with respect to the observables $f$ and $k$.

Let us conclude this sub-section by reproducing a result of Anandan and Aharanov [23]. Suppose that the Hamiltonian operator of a quantum system is bounded and let $\hbar$ be the corresponding observable function. By definition of the Riemann bracket, the uncertainty of $\hbar$ is given by $(\Delta \hbar)^2 = \frac{\hbar^2}{2} g(X_h, X_h)$. Therefore, apart from the constant coefficient, the uncertainty in the energy is exactly the length of the Hamiltonian vector field which generates the time-evolution. Thus, the energy uncertainty can be thought of as ‘the speed with which the system moves through the quantum phase space’; during its evolution, the system passes quickly through regions where the energy uncertainty is large and spends more time in states where it is small.
3. The measurement process

Some of the most significant aspects of quantum mechanics involve the measurement process and, without a complete geometric description of these issues, our program would be incomplete. In this sub-section we will sketch the desired geometric description, including the case when the spectrum of the operator is continuous. A more complete discussion may be found in [16].

Let \( \Psi_0 \in S \) be an arbitrary normalized element of the Hilbert space, and let \( p_0 = \pi(\Psi_0) \) be its projection to \( \mathcal{P} \). Of obvious interest, in the context of measurement, is the function

\[
\tilde{\delta}_{\Psi_0} : \Psi \in S \mapsto |\langle \Psi_0, \Psi \rangle|^2.
\]

Since \( \tilde{\delta}_{\Psi_0} \) is independent of the phase of \( \Psi \) it defines a function \( \delta_{p_0} \) on \( \mathcal{P} \) via

\[
\delta_{p_0}(p) := \tilde{\delta}_{\Psi_0}(\pi^{-1}(p)) = |\langle \pi^{-1}(p_0), \pi^{-1}(p) \rangle|^2.
\] (2.29)

If the quantum system is in the state labeled by \( p_0 \) when a measurement is performed, the relevant quantum mechanical probability distribution is determined by the function \( \delta_{p_0} \). We therefore desire a description of this function which does not rely explicitly on the underlying Hilbert space. This is provided by

**Theorem II.2** Given arbitrary points \( p_0, p \in \mathcal{P} \) there exists a (closed) geodesic which passes through \( p_0 \) and \( p \). Further, \( d_{p_0}(p) = \cos^2 \left( \frac{\sigma(p_0, p)}{\sqrt{2\hbar}} \right) \), where \( \sigma(p_0, p) \) denotes the geodesic separation of \( p_0 \) and \( p \).

A few comments regarding Thm. II.2 are in order. First, if \( \pi^{-1}(p_0) \) and \( \pi^{-1}(p) \) are non-orthogonal, then the above-mentioned geodesic is unique, up to re-parameterization. Next, since all geodesics on \( \mathcal{P} \) are closed, the geodesic distance \( \sigma(p_0, p) \) is, strictly speaking, ill-defined. Due to the periodicity of the cosine function, however, the “transition amplitude function”, \( d_{p_0} \), is insensitive to this ambiguity. For the sake of precision, by \( \sigma(p_0, p) \), we will mean the minimal geodesic distance separating \( p_0 \) and \( p \).

Suppose one is dealing with the measurement of an operator \( \hat{F} \) with discrete, non-degenerate spectrum, and let \( f \) be the corresponding observable function. Each eigenspace of \( \hat{F} \) is one complex-dimensional, and therefore determines a single point of \( \mathcal{P} \). Denote these eigenstates by \( p_i \). Suppose the system is in the state labeled by the point \( p_0 \) when an ideal measurement of \( f \) is performed. We know that the system will ‘collapse’ to one of the states \( p_i \). Theorem II.2 provides the corresponding probabilities. It is interesting to notice that the probability of collapse to an eigenstate \( p_i \) is a monotonically decreasing function of the geodesic separation of \( p_0 \) and \( p_i \); the system is more likely to collapse to a nearby state than a distant one.

We now have a description of the probabilities associated with the measurement process, but there are two deficiencies to be remedied. The eigenstates \( p_i \) above have been defined in terms of the algebraic properties of the operator \( \hat{F} \). For our program to be complete, we require a definition of these eigenstates which does not refer to the Hilbert space explicitly. Next, the above discussion was limited to the measurement of an observable with discrete, non-degenerate spectrum. We will describe the generic situation in two steps. First, we consider the measurement of observables with discrete, but possibly degenerate, spectra.
This will require the aforementioned description of the eigenstates. We will then be prepared to consider measurement of observables with continuous spectra.

Let us first examine the notions of eigenstates and eigenvalues of an observable operator \(\hat{F}\). Let \(F : \mathcal{H} \to \mathbb{R}\) and \(f : \mathcal{P} \to \mathbb{R}\) be the expectation value and corresponding observable function, respectively. A vector \(\Psi \in \mathcal{H}\) is an eigenstate of \(\hat{F}\) iff \(\hat{F} \Psi = \lambda \Psi\), for some (real) \(\lambda\). Alternatively, by Eq. \(2.3\),

\[
X_F|\Psi = Y_F|\Psi = (\lambda/\hbar)J|\Psi.
\]

That is, \(\Psi\) is an eigenstate of \(\hat{F}\) if and only if the Hamiltonian vector field, \(X_F\), is vertical (i.e., purely gauge) at \(\Psi\). This will be the case if and only if \(X_f\) vanishes at \(\pi(\Psi)\); \(\pi(\Psi)\) is then a critical point of the function \(f\). Evidently, the corresponding eigenvalue is exactly the (critical) value of \(f\) at \(\pi(\Psi)\). In summary:

**Definition II.3** Let \(f : \mathcal{P} \to \mathbb{R}\) be an observable function. Critical point of \(f\) are called eigenstates of \(f\). The corresponding critical values are called eigenvalues.

We now consider the measurement of an observable \(f\) whose spectrum is discrete but possibly degenerate (of course, the “spectrum of \(f\)” coincides, by definition, with the spectrum of the corresponding operator \(\hat{F}\)). Let \(\lambda\) be a degenerate eigenvalue of \(\hat{F}\), and denote by \(\mathcal{E}_\lambda\) the associated eigenspace of \(\mathcal{H}\). Associated to this eigenspace is a sub-manifold, \(\mathcal{E}_\lambda\), of \(\mathcal{P}\), which we shall call the *eigenmanifold* associated to \(\lambda\). Suppose that the system is prepared in the state labeled by \(\Psi_0\) and let \(\Psi_0 \in \pi^{-1}p_0\). The postulates of ordinary quantum mechanics assert that measurement of \(f\) will yield the value \(\lambda\) with probability \(\langle P_\lambda \Psi_0, P_\lambda \Psi_0 \rangle = \langle \Psi_0, P_\lambda \Psi_0 \rangle = ||\langle \Psi_0, P_\lambda \Psi_0 / ||P_\lambda \Psi_0|| \rangle||^2\), where \(P_\lambda\) is the projector onto the relevant eigenspace of \(\hat{F}\). From the above considerations, we know that this probability may be expressed in terms of the geodesic separation of the points \(p_0\) and \(\pi(\Psi_0 / ||P_\lambda \Psi_0||) \in \mathcal{E}_\lambda\). We will denote the latter point by \(P_\lambda(p_0)\).

What sets the point \(P_\lambda(p_0)\) apart from all other elements of \(\mathcal{E}_\lambda\)? We need only notice that for any point \(\Phi \in \mathcal{E}_\lambda \cap \mathcal{S}\),

\[
|\langle \Psi_0, \Phi \rangle|^2 = |\langle \Psi_0, P_\lambda \Phi \rangle|^2 = |\langle P_\lambda \Psi_0, \Phi \rangle|^2 \leq ||P_\lambda \Psi_0||^2.
\]

Therefore, of all elements \(\Phi \in \mathcal{E}_\lambda\) with unit normalization, that which maximizes the quantity \(|\langle \Psi_0, \Phi \rangle|^2\) is simply \(P_\lambda \Psi_0 / ||P_\lambda \Psi_0||\), i.e., that to which the state \(\Psi_0\) will ‘collapse’ in the event that measurement of \(\hat{F}\) yields the value \(\lambda\). Therefore, by Thm. II.2, \(P_\lambda(p_0)\) is simply that point of \(\mathcal{E}_\lambda\) which is *nearest* \(p_0\!\!\).!

We may now describe the measurement of an observable with discrete spectrum as follows. Suppose that immediately prior to measurement of \(f\), the system is in the state \(p_0 \in \mathcal{P}\). Denote by \(\lambda_i\) the critical values of \(f\) and, by \(\mathcal{E}_{\lambda_i}\) the corresponding eigenmanifolds. Interaction with the measurement device causes the system to be projected to one of the eigenmanifolds, say \(\mathcal{E}_{\lambda_0}\). “Realizing that it collapsed” to the state \(P_{\lambda_0}(p_0)\), the system returns what is known to be the value of the observable under consideration, i.e., \(f(P_{\lambda_0}(p_0)) = \lambda_0\). Of course, the probability that measurement causes reduction to \(\mathcal{E}_{\lambda_0}\) is given by \(\cos^2 \left(\sigma(p_0, \mathcal{E}_{\lambda_0})/\sqrt{2} \hbar\right)\), where, \(\sigma(p_0, \mathcal{E}_{\lambda_0})\) denotes the *minimal* geodesic separation of \(p_0\) and the sub-manifold \(\mathcal{E}_{\lambda_0}\).

Now let us study the generic case. Let \(\hat{F}\) be any observable operator on \(\mathcal{H}\), the spectrum of which is allowed to be continuous. We first need a definition of the spectrum of \(\hat{F}\) in terms
of the corresponding observable function \( f : \mathcal{P} \to \mathbb{R} \). Recall the standard definition \[37\]: 
\( \lambda \) is an element of the spectrum \( \text{sp}(\hat{F}) \) if and only if the operator \( \hat{F} - \lambda \) is not invertible. Equivalently, \( \lambda \in \text{sp}(\hat{F}) \) iff given any positive \( \varepsilon \in \mathbb{R} \), \( \exists \Psi \in S \) such that \( \| \hat{F} \Psi - \lambda \Psi \| < \varepsilon \).

This condition guarantees that, to arbitrary precision, \( \lambda \) is an approximate eigenvalue of \( \hat{F} \).

Using Eqs. (2.25) and (2.26), we may write the (square of the) above quantity as
\[
\| \hat{F} \Psi - \lambda \Psi \|^2 = \{ f - \lambda, f - \lambda \}_{+|\pi(\Psi)} = \left[ (\Delta f)^2 + (f - \lambda)^2 \right]_{\pi(\Psi)}.
\]

This equation allows us to define the spectrum of \( \hat{F} \) in terms of the function \( f : \mathcal{P} \to \mathbb{R} \):

**Definition II.4** The spectrum \( \text{sp}(f) \) of an observable \( f \) consists of all real numbers \( \lambda \) for which the function \( n_\lambda : \mathcal{P} \to \mathbb{R} \cup \{ \infty \} \), 
\[ n_\lambda : p \mapsto \left[ (\Delta f)^2(p) + (f(p) - \lambda)^2 \right]^{-1} \]

is unbounded.

Of course, a point at which \( n_\lambda = \infty \) corresponds to an eigenstate of \( f \).

The next step is a description of the spectral projection operators. Let \( \Lambda \) be a closed subset of the spectrum \( \text{sp}(f) \) of \( f \), and denote by \( P_{\hat{F}, \Lambda} \) the projection operator associated to \( \hat{F} \) and \( \Lambda \). In analogy with the above, put \( \tilde{\mathcal{E}}_{\hat{F}, \Lambda} = \{ P_{\hat{F}, \Lambda} \Psi | \Psi \in \mathcal{H} - \{ 0 \} \} \), and let \( \mathcal{E}_{f, \Lambda} \) denote the projection of \( \tilde{\mathcal{E}}_{\hat{F}, \Lambda} \) to \( \mathcal{P} \). Note that the set \( \tilde{\mathcal{E}}_{\hat{F}, \Lambda} \)—the analogue of the eigenspace above—actually is the eigenspace of \( P_{\hat{F}, \Lambda} \) corresponding to the eigenvalue \( 1 \). Therefore, we have \( \mathcal{E}_{f, \Lambda} \) consists of the critical points of an expectation value function, associated to the critical value \( 1 \). Unfortunately, this expectation value function is not directly expressible in terms of \( f \) and \( \Lambda \). We must look for an alternative description of the sub-manifold \( \mathcal{E}_{f, \Lambda} \).

In the representation defined by the operator \( \hat{F} \), elements of \( \tilde{\mathcal{E}}_{\hat{F}, \Lambda} \) have support on \( \Lambda \). Therefore, \( \Psi \in \tilde{\mathcal{E}}_{\hat{F}, \Lambda} \) iff \( \langle \hat{F} \rangle_{\Psi} \in \Lambda^n \ \forall n > 0 \), where \( \Lambda^n \) denotes the image of \( \Lambda \) under the map \( \lambda \mapsto \lambda^n \). Recall that \( \{ f, f \}_+ \) is the (projection to \( \mathcal{P} \) of the) expectation value of \( \hat{F}^2 \). In general, the expectation value of \( \hat{F}^n \) projects to the \( n \)-fold symmetric product \( \{ f, \{ f, \ldots \}_+ \}_+ \). Therefore, we have
\[
\mathcal{E}_{f, \Lambda} = \{ q \in \mathcal{P} | \{ f, \{ f, \{ f, \ldots \}_+ \}_+ \}_+ | q \in \Lambda^n \ \forall n > 0 \},
\]

where there are \( n \) factors of \( f \) occurring above.

Having obtained a description of \( \mathcal{E}_{f, \Lambda} \), we may now define the spectral projections in a manner intrinsic to the projective space. By precisely the same reasoning surrounding Eq. (2.31), \( P_{f, \Lambda} \) maps a point \( p \in \mathcal{P} \) to that element of \( \mathcal{E}_{f, \Lambda} \) which is nearest \( p \).

The measurement process may then be described as follows. Suppose the quantum system is in the state labeled by the point \( p_0 \in \mathcal{P} \) at the instant an experimenter performs a measurement of the observable \( f \). Following the rules of quantum mechanics, she “asks the system” whether the value of \( f \) lies in \( \Lambda \)—a closed subset of \( \text{sp}(f) \), which she is free to choose. The experimental apparatus drives the system to one of two states—either \( P_{f, \Lambda}(p_0) \) or \( P_{f, \Lambda^c}(p_0) \), where \( \Lambda^c \) is the (closure of the) complement, in \( \text{sp}(f) \), of \( \Lambda \). The system is reduced to the former with probability
\[
\cos^2 \left( \frac{\sigma(p_0, P_{f, \Lambda}(p_0))}{\sqrt{2\hbar}} \right);
\]
in this event, the experiment yields the positive result \((f \in \Lambda)\). Having precisely prepared the system in the state \(p_0\), the experimenter may then infer the value \(f(P_{f\Lambda}(p_0))\) of the observable \(f\). The probability of reduction to the latter state is obtained by replacing \(\Lambda\) by \(\Lambda^c\) above.

Note that this description encompasses all measurement situations. In the event that the spectrum of \(f\) is discrete and non-degenerate, the experimenter may choose to let \(\Lambda_i\) contain the single eigenvalue \(\lambda_i\). Moreover, she may measure all of the projections simultaneously. In this way, one recovers the first familiar description of the measurement process. Note also that, while the above discussion of the spectral projections may seem complicated and somewhat unnatural at first, the definition of the spectral projection operators on the Hilbert space has the same features. (Indeed, most text books simply skip this technical discussion.) This is simply one of the technical complications that the geometric formalism inherits from the Hilbert space framework.

To conclude, we wish to emphasize that the topic of our discussion has been ordinary quantum mechanics. We have just restated the well-known quantum mechanical formalism in a language intrinsic to the true space of states—the quantum phase space, \(\mathcal{P}\); no new ingredients have been added to the physics. A particularly attractive feature of the formalism, however, is the fact that slight modifications of the standard picture naturally present themselves. For example, using many of these geometric ideas, Hughston [28] has explored a novel approach to the measurement problem in terms of stochastic evolution.

### D. The postulates of quantum mechanics

Let us collect the results obtained in the first three sub-sections.

We have formulated ordinary quantum mechanics in a language which is intrinsic to the true space of quantum states—the projective Hilbert space \(\mathcal{P}\). As in classical mechanics, observables are smooth, real-valued functions which preserve the kinematic structure. Being a Kähler manifold, \(\mathcal{P}\) is a symplectic manifold. The role of the quantum symplectic structure is precisely that of classical mechanics; it defines both the Lie algebraic structure on the space of observables and generates motions including the time-evolution.

There are, however, two important features of quantum mechanics which are not shared by the classical description. First, the phase space is of a very particular nature; it is a Kähler manifold and, as we will see in section [I1], one of a rather special type—namely one of constant holomorphic sectional curvature. The second difference lies in the probabilistic aspects of the formalism, which is itself intimately related to the presence of the Riemannian metric. More generally, this metric describes those quantum mechanical features which are absent in the classical theory—namely, the notions of uncertainty and state reduction. For example, the transition probabilities which arise in quantum mechanics are determined by a simple function of the geodesic distance between points of the phase space.

These results are most easily summarized by stating the postulates in the geometric language:

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3In this sense, the quantum framework is actually a special case of the classical one!
Physical states: Physical states of the quantum system are in one-to-one correspondence with points of a Kähler manifold $\mathcal{P}$, which is a projective Hilbert space.\(^4\)

Kähler evolution: The evolution of the system is determined by a flow on $\mathcal{P}$, which preserves the Kähler structure. The generator of this flow is a densely defined vector field on $\mathcal{P}$.

Observables: Physical observables are represented by real-valued, smooth functions $f$ on $\mathcal{P}$ whose Hamiltonian vector fields $X_f$ preserve the Kähler structure.

Probabilistic interpretation: Let $\Lambda \subset \mathbb{R}$ be a closed subset of the spectrum of an observable $f$, and suppose the system is in the state corresponding to the point $p \in \mathcal{P}$. The probability that measurement of $f$ will yield an element of $\Lambda$ is given by

$$\delta_p(\Lambda) = \cos^2 \left( \frac{\sigma(p, P_{f,\Lambda}(p))}{\sqrt{2\hbar}} \right),$$

(2.34)

where $P_{f,\Lambda}(p)$ is the point, closest to $p$, in the space $\mathcal{E}_{f,\Lambda}$, defined by Eq. (2.33).

Reduction, discrete spectrum: Suppose the spectrum of an observable $f$ is discrete. This spectrum provides the set of possible outcomes of the ideal measurement of $f$. If measurement of $f$ yields the eigenvalue $\lambda$, the state of the system immediately after the measurement is given by the associated projection, $P_{f,\lambda}(p)$, of the initial state $p$.

Reduction, continuous spectrum: A closed subset $\Lambda$ of the spectrum of $f$ determines an ideal measurement that may be performed on the system. This measurement corresponds to inquiring whether the value of $f$ lies in $\Lambda$. Immediately after this measurement, the state of the system is given by $P_{f,\Lambda}(p)$ or $P_{f,\Lambda^c}(p)$, depending on whether the result of the measurement is positive or negative, respectively.

In the last postulate, $\Lambda^c$ is the closure of the complement, in the spectrum of $f$ of the set $\Lambda$. Although the first “reduction postulate” is a special case of the second, both have been included for comparison with standard textbook presentations.

To conclude, although it is not obvious from textbook presentations, the postulates of quantum mechanics can be formulated in an intrinsically geometric fashion, without any reference to the Hilbert space. The Hilbert space and associated algebraic machinery provides convenient technical tools. But they are not essential. Mathematically, the situation is similar to the discussion of manifolds of constant curvature. In practice, one often establishes their properties by first embedding them in $\mathbb{R}^n$ (equipped with a flat metric of appropriate signature). However, the embedding is only for convenience; the object of interest is the manifold itself. There is also a potential analogy from physics, alluded to in the

\(^4\)We will see in section 1113 that quantum phase spaces $\mathcal{P}$ can be alternatively singled out as Kähler manifolds which admit maximal symmetries. This provides an intrinsic characterization without any reference to Hilbert spaces.
Introduction. Perhaps the habitual linear structures of quantum mechanics are analogous to the inertial rest frames in special relativity and the geometric description summarized here, analogous to Minkowski’s reformulation of special relativity. Minkowski’s description paved the way to general relativity. Could the geometric formulation of quantum mechanics lead to a more complete theory one day?

III. A UNIFIED FRAMEWORK FOR GENERALIZATIONS OF QUANTUM MECHANICS

There are three basic elements of the quantum mechanical formalism which may be considered for generalization: the state space, the algebra of observables and the dynamics. The framework developed in section II suggests avenues for each of these. First, while it is not obvious how one might “wiggle” a Hilbert space, one may generalize the quantum phase space by considering, say, the class of all Kähler manifolds \( \{M, g, \omega\} \). The geometric language also suggests an obvious generalization of the space of quantum observables: one might consider the space of all smooth, real-valued functions on the phase space. Finally, whether or not one chooses either of these extensions, one may consider generalized dynamics which, as in classical mechanics, preserves only the symplectic structure. Thus, one might require the dynamical flow to preserve only the symplectic structure, and not necessarily the metric.

While each of these structures may be extended separately, they are intimately related and construction of a complete, consistent framework is a highly non-trivial task. Thus, for example, if one allows all Kähler manifolds as possible quantum phase spaces, it seems very difficult to obtain consistent probabilistic predictions for outcomes of measurements. More generally, the problem of systematically analyzing viable, non-trivial generalizations of the kinematic structure would be a major undertaking, although the pay-off may well be exceptional. Modification of dynamics, on the other hand, is easier at least in principle. Therefore, we will first consider these in section III A and return to kinematics in section III B.

A. Generalized dynamics

Let us then suppose that we continue to use a projective Hilbert space for the quantum phase space, and let dynamics be generated by a preferred Hamiltonian function. However, let us only require that time-evolution should preserve the symplectic structure \( \omega \) (as in classical mechanics), and not necessarily the metric \( g \). From the viewpoint of the geometric formulation, this is the simplest and most obvious generalization of the standard quantum dynamics.\(^5\) The idea is reminiscent of the “non-linear Schrödinger equations” that have been

\(^5\)Recall that the Hamiltonian need not be an observable function in the sense of section II B. We could extend the kinematical set up as well and regard any smooth function on \( P \) as an observable function. (This would be analogous to Weinberg’s proposal which, however, was made at the
considered in the past. Therefore, it is natural to ask if these there is a relation between
the two. We will see that the answer is in the affirmative. Furthermore, the geometric
framework provides a unified treatment of these proposals and makes the relation between
them transparent, thereby enabling one to correct a misconception.

Let us begin by defining the the class $C_H$ of Hamiltonians we now wish to consider. $C_H$
will consist of densely defined functions $f$ on $P$ satisfying the following properties: i) $f$ is
smooth on its domain of definition; and ii) the Hamiltonian vector field it defines generates
a flow on all of $P$. In particular, The Hamiltonian functions we considered in section II—
expectation values of a possibly unbounded self-adjoint operator—belong to $C_H$ but they
constitute only a ‘small subset’ of $C_H$.

The existing proposals of non-linear dynamics refer to flows in the full Hilbert space $H$
rather than in the quantum phase space $P$. To compare the two, we need to lift our flows to
$H$. Let us begin by recalling that it is natural to regard $P$ as a reduced phase space, resulting
from the first class constraint $C(\Psi) := \langle \Psi, \Psi \rangle - 1 = 0$ on $H$ (see Eq. 2.11). Therefore, a
function $f$ on $P$ admits a natural lift $F$ to $S$, the unit sphere in $H$. This function $F$ on $S$
is constant along the integral curves of the vector field $J$ which generates phase rotations;
$L_J f = 0$. Denote the space of these lifts by $\tilde{C}_H$.

To discuss dynamics on $H$, we need to extend these functions $F$ off $S$. From the reduced
phase space viewpoint, the extension is completely arbitrary. For, we can construct the
Hamiltonian vector field on $H$ generated by any extension $F_{ext}$. The restriction to $S$ of this
vector field does depend on the extension but the the horizontal part of the restriction—i.e.,
the part orthogonal to $J$—does not. Hence, the projection of the vector field to $P$ agrees
with $X_f$, irrespective of the choice of the initial extension. Thus, the generalized dynamics
generated by a given Hamiltonian function $f$ on $P$ can be lifted to a whole family of flows
on $H$, all of which, however, evolve the physical quantum states—elements of $P$—in the
same way. Because of this, apparently distinct proposals for non-linear evolutions on $H$ can
in fact be physically equivalent. This point is rather trivial from the viewpoint of geometric
quantum mechanics. The reason for our elaboration is that—as we will see below—it has
not been appreciated in the Hilbert space formulations.

While the extension off $S$ of elements of $\tilde{C}_H$ is completely arbitrary, one can use the
standard quantum mechanical framework to select a specific rule. Consider, to begin with,
a bounded, self-adjoint operator $\hat{F}$ on $H$, and let $F$ be the restriction to $S$ of the corre-
spanding expectation value function. There is then an obvious extension of $F$ to all of $H$:
set $F_{ext}(\Psi) := \langle \Psi, \hat{F}\Psi \rangle$ (which we denoted by $F$ in section IIA). One can restate this rule

level of the Hilbert space $H$ rather than the quantum phase space $P$.) We have refrained from
doing this because the required extension of measurement theory is far from obvious.

6Strictly speaking, we only need to extend the dynamical flow off $S$. However, to compare our
results with Weinberg’s [8] we need to consider extensions of Hamiltonians. In discussions on
generalized dynamics [7,9], the issue of domains of definition of operators is generally ignored.
Our treatment will be at the same level of rigor. In particular, we will ignore the fact that our
Hamiltonian functions and vector fields are only densely defined.
as:

\[ F_{\text{ext}}(\Psi) := \|\Psi\|^2 F(\Psi/\|\Psi\|). \]  

(3.1)

The advantage is that this equation may now be used to extend any element \( F \) of \( \hat{C}_H \) to all of \( \mathcal{H}^\times = \mathcal{H} - \{0\} \). Note that, with this preferred extension, the Hamiltonian vector field \( X_{F_{\text{ext}}} \) is homogeneous of degree one on \( \mathcal{H}^\times \):

\[ X_{F_{\text{ext}}}(c\Psi) = cX_{F_{\text{ext}}} (\Psi) \quad \forall c \in \mathbb{C}. \]  

(3.2)

Hence, the flow on \( \mathcal{H}^\times \) which is generated by \( F_{\text{ext}} \) is homogeneous, but fails to be linear unless \( F \) is the restriction to \( S \) of the expectation value function defined by a self-adjoint operator. Next, it is easy to verify that these \( F_{\text{ext}} \) strongly commute with the constraint function \( C(\Psi) \). Hence the flow along \( X_{F_{\text{ext}}} \) preserves the constraint. Therefore, the specific extension considered above has the property that the corresponding flow preserves not only the symplectic structure \( \Omega \) but also the norm on \( \mathcal{H}^\times \). However, unless \( f \) is an observable function, it does not preserve the metric \( G \).

Note that, even if we consider just these preferred extensions, the set of possible Hamiltonian functions on \( \mathcal{H}^\times \) has been extended quite dramatically. To see this, consider the case when \( \mathcal{H} \) is finite-dimensional. Then, the class of Hamiltonian functions on \( \mathcal{H} \) allowed in standard quantum mechanics forms a finite-dimensional real vector space; it is just the space of expectation value functions constructed from self-adjoint operators. The space \( \hat{C}_H \), on the other hand, is infinite-dimensional since its elements are in one-to-one correspondence with smooth functions on \( \mathcal{P} \). And each element of \( \hat{C}_H \) admits an unique extension to \( \mathcal{H}^\times \) via Eq. (3.1). It is natural to ask if one can do something ‘in-between’. Can we impose more stringent requirements to select a class of permissible Hamiltonians which is larger than that of observable functions of section [4] but smaller than \( C_H \)? For example, one might imagine looking for the class of functions on \( \mathcal{H}^\times \) whose Hamiltonian flows preserve not just the norms but also the inner-product. It turns out, however, that this class consists precisely of the expectation value functions defined by self-adjoint operators; there is no such thing as a non-linear unitary flow on \( \mathcal{H} \). Despite the magnitude of our generalization, it seems to be the only available choice.

We are now ready to discuss the relation between this generalization and those that have appeared in the literature. Note first that Eq. (3.1) implies that there is a one-to-one correspondence between elements of \( C_H \) and smooth functions on \( \mathcal{H}^\times \) which are gauge-invariant (i.e. insensitive to phase) and homogeneous of degree two. (If \( \mathcal{H} \) is viewed as a vector space over complex numbers, this corresponds to homogeneity of degree one in both \( \Psi \) and \( \bar{\Psi} \).) It turns out that this is precisely the class of permissible Hamiltonians that Weinberg [7] was led to consider while looking for a general framework for non-linear generalizations of quantum mechanics. Let us therefore call functions on \( \mathcal{H}^\times \) satisfying Eq. (3.1) Weinberg functions, and denote the space of these functions by \( \mathcal{O}_W \). Our discussion shows that there is a one-to-one correspondence between smooth functions on the projective Hilbert space \( \mathcal{P} \) and Weinberg functions on the punctured Hilbert space \( \mathcal{H}^\times \); the homogeneity restriction simply serves to eliminate the freedom in the extension of the function on \( \mathcal{H}^\times \). Thus the extension of quantum dynamics that is immediately suggested by the geometrical framework reproduces key features of Weinberg’s proposal.
There are, however, considerable differences in the motivations and general viewpoints of the two treatments. In particular, Weinberg works with the Hilbert space \( \mathcal{H} \) (and, without explicitly saying so, sometimes with \( \mathcal{H}^\times \)). However, he does assume at the outset that elements \( \Psi \) and \( c\Psi \) of \( \mathcal{H} \) define the same physical state of the quantum system for all complex numbers \( c \). Thus, although it is not explicitly stated, his space of physical states is also \( \mathcal{P} \). Therefore, it is possible to translate his constructions to the geometric language. As we will see below, the geometric viewpoint is often clarifying.

Next, let us consider two specific examples of non-linear dynamics that have been considered in the literature. Each of these involves the non-relativistic mechanics of a point particle moving in \( \mathbb{R}^n \) and the Hilbert space consists of square-integrable functions on \( \mathbb{R}^n \). Therefore, it will be useful to reinstate a complex notation for the remainder of this subsection. In this notation, an element of \( \mathcal{O}_W \) is a real-valued function \( F(\Psi, \bar{\Psi}) \) of both \( \Psi \) and its conjugate, which is homogeneous of degree one in each argument. The Hamiltonian vector field generated by such a function corresponds to

\[
X_F[\Psi](x) = \frac{1}{i\hbar} \frac{\delta F}{\delta \bar{\Psi}}(x).
\]

The simplest example of this type is provided by the so-called “non-linear Schrödinger equation”. This equation is given by

\[
i\hbar \frac{\partial \Psi}{\partial t}(x, t) = (\hat{H}_0 \Psi)(x, t) + \epsilon |\Psi(x, t)|^2 \Psi(x, t),
\]

where \( \hat{H}_0 = \hat{P}^2/2m + \hat{V} \) is the standard Hamiltonian operator describing a non-relativistic particle under the influence of a conservative force with potential \( \hat{V} \). Note that the quantity \( |\Psi(x, t)| \) is the modulus of \( \Psi(x, t) \), not the norm \( \|\Psi\| \) of the state-vector \( \Psi \). It is easy to verify that (3.4) induces a flow on \( \mathcal{P} \). This flow is Hamiltonian and the generating function is the projection to \( \mathcal{P} \) of the function \( \langle \Psi, H_0 \Psi \rangle + H_\epsilon(\Psi) \) on \( \mathcal{H} \), where,

\[
H_\epsilon(\Psi) := \frac{\epsilon}{2} \int d^n x \langle \Psi^*(x, t) \Psi(x, t) \rangle^2.
\]

Thus, Eq. (3.4) is indeed a specific example of our generalized dynamics.

Note, however, that the dynamical vector field \( X \) on \( \mathcal{H} \) defined directly by the non-linear Schrödinger equation is given by

\[
X[\Psi](x) = X_{H_0}[\Psi](x) + X_\epsilon[\Psi](x),
\]

where \( X_\epsilon[\Psi](x) = (\epsilon/i\hbar)|\Psi(x, t)|^2 \Psi(x, t) \). Clearly, it is not homogeneous in the sense of Eq. (3.2), and hence is not generated by a Weinberg function. Therefore, Weinberg was led to state that the “results obtained by the mathematical studies of this equation are unfortunately of no use” to the generalization he considered. However, we just saw that the non-linear Schrödinger equation does correspond to generalized dynamics on \( \mathcal{P} \) and is therefore of ‘Weinberg type’. Hence, the statement in quotes is somewhat misleading.

Let us elaborate on this point. If we first focus on physical states, what matters is just the projected flow on \( \mathcal{P} \). This in turn is completely determined by the restriction \( H_\epsilon|_S \) of \( H_\epsilon \) to \( S \). Therefore, we may feel free to ignore the behavior of \( H_\epsilon \) off of \( S \). Of course,
to construct a vector field on $S$ which projects to the relevant one on $P$, we can extend $H_\epsilon|_S$ arbitrarily and compute the associated Hamiltonian vector field. In particular, we may extend it in the way which Weinberg would suggest:

$$H'_\epsilon(\Psi) := \|\Psi\|^2 H_\epsilon|_S(\Psi/\|\Psi\|). \quad (3.7)$$

Thus, we have seen explicitly that the flow on $P$ which is defined by $H_\epsilon$ may also be described by a Weinberg function! The emphasis on the true space of states $P$ of the geometric treatment clarifies this point which seems rather confusing at first from the Hilbert space perspective.

The non-linear Schrödinger equation is a fairly simple example since the generating function $H_\epsilon$ is itself homogeneous (but of the “wrong” degree to be a Weinberg function). Let us now consider an example which is more sophisticated.

In an effort to address the problem of combining systems which are subject to a non-linear equation of motion, Bialynicki-Birula and Mycielski were led to a logarithmic equation 

$$i\hbar \frac{\partial \Psi}{\partial t}(x, t) = (\hat{H}_0 \Psi)(x, t) + \alpha(||\Psi(x, t)||^2)\Psi(x, t), \quad (3.8)$$

and showed that physical considerations, particularly the requirement that $\alpha$-term should not introduce interactions between otherwise non-interacting subsystems, imposes severe restrictions the functional form of $\alpha$. The only possibility is to have: $\alpha(\rho) = -b \ln(a\rho)$ for some constants $a$ and $b$. (For details see [5]). Choosing units with respect to which $a = 1$, the vector field along which the system evolves may be written as

$$X[\Psi](x) = X_{H_0}[\Psi](x) + X_1[\Psi](x), \quad (3.9)$$

where $X_1[\Psi](x) = -(b/i\hbar) \ln(||\Psi||^2)\Psi(x)$. Again, the extra term may be seen to be Hamiltonian; $X_1 = X_{H_1}$, where

$$H_1(\Psi) = b \int d^n x \Psi^\ast(x)\Psi(x)[1 - \ln(\Psi^\ast(x)\Psi(x))]. \quad (3.10)$$

Since $H_1$ is phase-invariant, this is also an example of our generalized dynamics.

Again, we can carry out the procedure used for the non-linear Schrödinger equation to see that the corresponding motion on the projective space may be described by use of a Weinberg function. It is not difficult to show that the corresponding homogeneous function is given by

$$H'_1(\Psi) = H_1(\Psi) + b||\Psi||^2 \ln(||\Psi||^2). \quad (3.11)$$

Therefore, the logarithmic equation induces a flow on $P$ which may be described by a function of the Weinberg type.

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The constant $a$ is a length scale of no physical significance, since it may be altered by addition of a constant to the Hamiltonian operator.
In retrospect, essentially any generalized dynamics of the form specified by Eq. (3.8) may be written as a Hamiltonian flow on $H^\times$ by “integrating” the function $F$. The resulting Hamiltonian function is not likely to satisfy Weinberg’s homogeneity condition. However, one may restrict it to the unit sphere, then extend this restriction as in Eq. (3.1). The resulting function will generate a Hamiltonian vector field which differs from the original one in general, but agrees with it on the unit sphere. Hence, both will generate the same flows on the projective space. This may be done with any Hamiltonian function on $H^\times$ which preserves the unit sphere.

B. Characterization of the standard quantum kinematics

Let us now turn to the issue of generalizations of the kinematic framework itself. As noted above, these extensions may well turn out to be profound. However, they also appear to be much more difficult to carry out. Therefore, as a first step, it is natural to ask what sets ordinary quantum mechanics apart from its natural generalizations. The purpose of this sub-section is provide such a characterization of the standard kinematic framework.

Let us consider an arbitrary Kähler manifold $\{M, g, \omega\}$, which is to represent the phase space of a (radically!) generalized quantum theory. (As before, $M$ will be assumed to be a Hilbert manifold, and $g$ and $\omega$ to be strongly non-degenerate.) The question is then: Are there natural conditions which will single out ordinary quantum mechanics from this class of generalizations? As we saw in section II, the observables of ordinary quantum mechanics are smooth functions on the phase space whose Hamiltonian vector fields satisfy Killing’s equation; we therefore let the observables of the generalized framework consist of all smooth functions on $M$ whose Hamiltonian flows are isometries. Denote this set as

$$\mathcal{O} := \{f : M \rightarrow \mathbb{R} \mid \mathcal{L}_X g = 0\}.$$ (3.12)

(Note that while we call elements of $\mathcal{O}$ observables, we do not claim to possess a complete and consistent formalism incorporating measurement. Indeed, this seems an impossible feat at the present level of generality.)

It should be emphasized that there are Kähler manifolds which do not admit a single observable function (other than constants); the torus is an example. Let us amplify this point briefly. Since the discussion leading to Def. II.2 is valid for an arbitrary Kähler manifold, any observable is completely determined by its value and first two derivatives at a single point. As before, we will denote by $\mathcal{S}_p$ the set of symmetry data at the point $p \in M$ (if $\lambda$, $X$ and $K$ are 0-, 1- and 2-forms at $p$ for which $\omega^{\alpha\gamma} K_{\gamma\beta}$ is symmetric, then $(\lambda, X, K) \in \mathcal{S}_p$).

Given an arbitrary $p \in M$, each observable function then determines an element of $\mathcal{S}_p$. According to Thm. II.1, the converse holds for ordinary quantum mechanics: there, the algebra of symmetry data at each point $p$ is integrable. In this sense, the phase space of standard quantum mechanics admits “as many observables as possible.” This useful idea is captured in

**Definition III.1** If, for each $p \in M$, every element $(\lambda, X, K) \in \mathcal{S}_p$ is integrable, we say that $\mathcal{O}$ is maximal.
The set of observables of ordinary quantum mechanics is maximal. As we will see, it is essentially this property of the quantum observables which allows us to recover the standard formalism.

In order to illustrate the importance of the maximality of the space of observables, it is necessary to introduce one mathematical concept. It is fairly easy to verify that the Riemann curvature tensor of a projective Hilbert space is of the special form

$$R_{\alpha\beta\gamma\delta} = C^2 g_{\gamma\delta} + \omega_{\alpha\beta} \omega_{\delta\gamma} - \omega_{\gamma[\alpha} \omega_{\beta]\delta],$$

(3.13)

where $C = \hbar/2$ and $[\ldots]$ denotes skew-symmetrization. If the curvature tensor of our generalized phase space $M$ satisfies this equation at a point $p$, it is said to be of constant holomorphic sectional curvature (CHSC) at $p$. In that case, the real number $C$ is the value of the holomorphic sectional curvature at $p$. If for some real $C$, the Riemann tensor assumes the form written in Eq. (3.13) (for all $p \in M$), $M$ is called a manifold of CHSC = $C$. It may be useful to note that if a Kähler manifold $M$ is of CHSC at each point, then it must be of overall CHSC; that is, if $R$ satisfies Eq. (3.13) at each $p$, then $C$ must be a constant [38].

Holomorphic sectional curvature, in the context of complex manifolds, is analogous to the scalar curvature of real manifolds. Since there is a very strong relationship between the number of independent Killing vector fields on a real manifold and the form of the Riemann curvature tensor [36], one may expect that the maximality of the space of quantum observables to be strongly related to Eq. (3.13). We will therefore use an approach analogous to that presented in Ref. [36] to consider more closely the interplay between the geometry of the phase space and the algebra of observables. The results in this section are stated without proof; for details, see Ref. [16].

Since the commutator of two Killing vector fields is another Killing vector field, the set of observables on an arbitrary Kähler manifold is closed under the Poisson bracket $\{ f, k \}$. The Poisson bracket also satisfies the Jacobi identity; this equips $\mathcal{O}$ with the structure of a Lie algebra. On the entire function space, we may also define the commutative operation

$$\{ f, k \}_+ := (f, k) + fk, \quad (f, k) := \frac{\hbar}{2} (\nabla_\alpha f) g^{\alpha\beta} (\nabla_\beta g),$$

(3.14)

which we call the symmetric bracket. The symmetric bracket is not necessarily closed on the space of observables on an arbitrary Kähler manifold.

Suppose that $f_1$ and $f_2$ are any two observable functions on $M$, and let $f_3 = \{ f_1, f_2 \}$. Since each $f_i$ generates a Killing vector field, it determines an element $(f_i, X_i, K_i)|_p \in S_p$ for any $p \in M$, where $X_i = X_{f_i}$ and $K_{i\alpha\beta} = \nabla_\alpha (X_{f_i})_\beta$. Of course, $f_3 = \omega(X_1, X_2)$ and $X_3 = -g_{\alpha\beta} [X_{f_1}, X_{f_2}]^\beta |_p = X_2^\beta K_{1\beta\alpha} - X_1^\beta K_{2\beta\alpha}$, where $[,]$ denotes the commutator of vector fields. It is straightforward to derive the corresponding expression for the 2-form $K_3$; the result is

$$K_{3\alpha\beta} = K_{2\alpha} \gamma K_{1\gamma\beta} - K_{1\alpha} \gamma K_{2\gamma\beta} + X_{1\mu} X_{2\nu} R_{\alpha\beta}^{\mu\nu}.$$
\[ [(f_1, X_1, K_1), (f_2, X_2, K_2)]_p := \left( \begin{array}{l}
\omega(X_1, X_2), \\
X_2^\alpha K_{1\beta\alpha} - X_1^\beta K_{2\alpha\beta}, \\
K_{2\alpha\gamma} K_{1\gamma\beta} - K_{1\alpha\gamma} K_{2\gamma\beta} + X_1^\mu X_2^\nu R_{\alpha\beta\mu\nu} \end{array} \right). \tag{3.16} \]

This bracket on \( S_p \) is defined only to mirror the Poisson bracket on \( O \). That is, to obtain the symmetry data corresponding to the Poisson bracket of two observables, one may simply apply the above “bracket” operation to the symmetry data of the initial observables.

For an arbitrary point \( p \) on an arbitrary Kähler manifold, \( S_p \) is closed under \([\cdot, \cdot]_p\). However, the Jacobi identity will, in general, fail. It is natural to ask for circumstances under which \( S_p \) forms a Lie algebra. The answer to this question is provided by

**Lemma III.1** \([\cdot, \cdot]_p\) is a Lie bracket on \( S_p \) if and only if the Riemann tensor is of CHSC at \( p \).

If \( O \) is maximal, then \([\cdot, \cdot]_p\) is a Lie bracket on \( S_p \) for any \( p \in M \). As a consequence of Lemma [III.2], the Riemann tensor is of CHSC at each point of \( M \). Therefore \( M \) is a manifold of CHSC; this is summarized by

**Corollary 2** Suppose \( O \) is maximal. Then \( M \) is a manifold of CHSC.

Let us make the analogous construction for the symmetric bracket. Again, let \( f_1, f_2 \in O \) and let \( f_4 \) denote the symmetric bracket, \( f_4 = \{f_1, f_2\} = f_1 f_2 + (\hbar/2)g(X_1, X_2) \). One can easily show the following:

\[
\begin{align*}
\omega_\alpha^{\cdot \beta} \nabla_\beta f_4 &= f_1 X_2^\alpha + f_2 X_1^\alpha + (\hbar/2) \omega_\alpha^{\cdot \beta} (K_{1\beta\gamma} X_2^\gamma + K_{2\beta\gamma} X_1^\gamma), \\
\nabla_\alpha X_4^\beta &= f_1 K_{2\alpha\beta} + f_2 K_{1\alpha\beta} + \hbar K_{1\gamma[a} \omega^{\gamma\beta} K_{2\beta]b} + X_1^\mu X_2^\nu \left[ h R_{\alpha(\mu\nu)} - 2 g_{\alpha(\mu} g_{\nu)\gamma} \right] \omega_\beta^{\gamma}. \tag{3.18} 
\end{align*}
\]

Therefore it is natural to define the following commutative operation on \( S_p \):

\[ ((f_1, X_1, K_1), (f_2, X_2, K_2))_p := \left( \begin{array}{l}
f_1 f_2 + \hbar/2 g(X_1, X_2), \\
f_1 X_2^\alpha + f_2 X_1^\alpha + (\hbar/2) \omega_\alpha^{\cdot \beta} (K_{1\beta\gamma} X_2^\gamma + K_{2\beta\gamma} X_1^\gamma), \\
f_1 K_{2\alpha\beta} + f_2 K_{1\alpha\beta} + \hbar K_{1\gamma[a} \omega^{\gamma\beta} K_{2\beta]b} + X_1^\mu X_2^\nu \left[ h R_{\alpha(\mu\nu)} - 2 g_{\alpha(\mu} g_{\nu)\gamma} \right] \omega_\beta^{\gamma}. \tag{3.19} \end{array} \right) \]

We know that if \( M \) is a projective Hilbert space this operation produces the symmetry data determined by the symmetric (i.e. Jordan) bracket of the corresponding observables. For the generic case however, \( O \) will not be closed under the symmetric bracket; hence we do not expect \( S_p \) to be closed under its symmetric bracket. Therefore, the symmetric bracket defined by Eq. (3.19) should be viewed as an operation on the space of all triples \((f, X, K)\), without the symmetry condition imposed on \( K \).

The condition that \( S_p \) be closed under \((\cdot, \cdot)_p\) is even stronger than that found in Lemma [III.1]:

**Lemma III.2** The set \( S_p \) of symmetry data at \( p \) is closed under the symmetric bracket \((\cdot, \cdot)_p\) if and only if the Riemann tensor is of CHSC \(= 2/\hbar \) at \( p \).
The difference between this condition and that specified in Lemma III.1 is that here the actual value of the holomorphic sectional curvature is determined by the coefficient appearing in the definition of the symmetric bracket. Lemma III.1 states that the holomorphic sectional curvature be constant, but does not specify its value.

Note that each of these lemmas involves only a single point of $\mathcal{M}$. If the set of observables is maximal, then by definition its elements are in one-to-one correspondence with elements of $\mathcal{S}_p$ for any $p \in \mathcal{M}$. Therefore, the closure of a maximal set of observables under the symmetric bracket is equivalent to the closure of $\mathcal{S}_p$ under $(\cdot)_p$ for any $p \in \mathcal{M}$. If we apply Lemma III.2 to each point of $\mathcal{M}$, we immediately obtain

**Corollary 3** Suppose $\mathcal{O}$ is maximal. Then $\mathcal{O}$ is closed under the symmetric bracket if and only if the Riemann tensor is of CHSC = $2/\hbar$.

Corollary 2 does not specify the value of the holomorphic sectional curvature, but merely states that the Riemann tensor assumes the special form written in Eq. (3.13) for some constant $C$. However, combining it with Lemma III.2, one obtains

**Theorem III.1** Suppose $\mathcal{O}$ is maximal and that $\mathcal{S}_p$ is closed under $(\cdot)_p$ for a single point $p \in \mathcal{M}$. Then $\mathcal{M}$ is a manifold of CHSC = $2/\hbar$.

Of course, we would like to go one step further to say that $\mathcal{M}$ must be a projective Hilbert space. Now, it is known that any two finite-dimensional Kähler manifolds which are complete, simply-connected and of CHSC = $2/\hbar$ are isomorphic [38]. Therefore, in the finite-dimensional case, we have obtained a characterization of the structure that picks out the standard quantum kinematics from possible generalized frameworks: If the generalized quantum phase space is a complete, simply-connected Kähler manifold and the set of observables is maximal and closed under $\{\cdot\}$+, then one is dealing with the structure of ordinary quantum mechanics. This characterization should be useful in the search for genuine generalizations. In a generalized theory, one or more of the conditions must be violated. Thus, the characterization systematizes the search for generalizations and suggests concrete directions to proceed.

The case when $\mathcal{M}$ is infinite-dimensional is much more interesting physically. However, in this case, we do not know if we can conclude that a (complete) Kähler manifold on which $\mathcal{O}$ is maximal and closed under $\{\cdot\}$+ is isomorphic to a projective Hilbert space. Indeed there may exist many different infinite-dimensional Kähler manifolds (satisfying the above completeness requirements) of the same constant holomorphic sectional curvature. This is an important open problem. If the situation turns out to be the same as in the finite-dimensional case, we will again have a theorem characterizing ordinary quantum mechanics. If the situation is different and we have many such Kähler manifolds, it would be even more interesting. For, we would then be presented with viable generalizations of the standard quantum formalism. Such examples would be very interesting because they are likely to admit a consistent measurement theory and thus lead to physically complete generalizations of a rather subtle type.
IV. SEMI-CLASSICAL CONSIDERATIONS

One of the most striking features of the geometric approach to quantum mechanics is its resemblance to the classical formalism. One might therefore expect that the geometric framework may shed some light on the relation between quantum and classical physics and, in particular, semi-classical approximations. We will see that this expectation is correct. In section IV A we consider the relation between the two theories at a kinematical level and elucidate the special role played by coherent states. These results are then used in the remaining section to analyze dynamical issues. In IV B we consider systems such as harmonic oscillators and free fields; in IV C, we discuss dynamics in the WKB approximation.

A. Kinematics

Let us now suppose that we are given a classical phase space \((\Gamma, \alpha)\), where \(\Gamma\) is assumed to be a finite-dimensional vector space and \(\alpha\) is the symplectic form thereon. Let \((P, g, \omega)\) be the quantum phase space which results from the application of the textbook quantization procedure to \((\Gamma, \alpha)\). Thus, on the quantum Hilbert space \(H\), there are position and momentum operators \((\hat{Q}_i, \hat{P}_i, i = 1, ..., n)\) corresponding to the classical position and momentum observables. We are all accustomed to this direction of construction. Our goal now is to do things in “reverse”: Given the quantum phase space, we will provide a construction of the classical one.

We will continue with our previous convention and write the projections to \(P\) of the expectation values of these operators as \(q_i, p_i\). It should be emphasized, however, that these functions are the elementary quantum observables, not the classical ones; they are functions on the infinite-dimensional space \(P\) and not on the finite-dimensional space \(\Gamma\). Note that by standard construction the Poisson brackets of these elementary quantum observables satisfy the same “commutation relations” as do the classical variables:

\[
\{q_i, p_j\} = \delta_{ij} \quad \text{and} \quad \{q_i, q_j\} = \{p_i, p_j\} = 0 \quad (4.1)
\]

One of the most common approaches to the classical limit comes from a simple theorem of Ehrenfest’s which suggests that expectation values of quantum operators are to be approximated, in some sense, by the corresponding classical observables. The geometric formulation is particularly well-suited to implement these ideas. To each quantum state \(x \in P\), let us associate the classical state \((q_i(x), p_j(x)) \in \Gamma\). This association defines the obvious mapping \(\rho : P \to \Gamma\). In fact, one might view this as the definition of the classical phase space. (The astute reader will object, correctly pointing out that one must specify the elementary quantum observables \(q_i\) and \(p_i\) before such a “definition” of the classical phase space can be made. Recall, however, that this is just the counterpart of the fact that elementary classical observables must be specified before construction of the quantum theory.)

For convenience, let us denote the generic elementary observables by \(f_r, r = 1 \ldots 2n\) (i.e., \(\{f_r\} = \{q_i, p_i, i = 1 \ldots n\}\)). Now let us define an equivalence relation on \(P\): \(x_1 \sim x_2 \iff f_r(x_1) = f_r(x_2) \forall r\). This equivalence relation fibrates the quantum phase space, which may now view as a bundle over the classical phase space, \(\Gamma = P/\sim\).
Associated to any bundle is a “vertical” distribution; i.e., a special class of (vertical) tangent vectors at each state \( x \in \mathcal{P} \). That a tangent vector \( v \in T_x \mathcal{P} \) is vertical simply means that \( v(f_r) = 0 \forall r = 1 \ldots 2n \). Equivalently the vertical subspace may be defined as

\[
V_x := \{ v \in T_x \mathcal{P} \mid \omega(X_{f_r}|_x, v) = 0 \forall r = 1 \ldots 2n \}.
\] (4.2)

The vertical directions are simply those in which the elementary quantum observables assume constant values.

Let \( V^\perp_x \) denote the \( \omega \)-orthogonal complement of the vertical subspace at \( x \). One can show that each tangent space \( T_x \mathcal{P} \) may be written as the sum \( T_x \mathcal{P} = V_x \oplus V^\perp_x \). Let us therefore call elements of \( V^\perp_x \) horizontal. If \( Y \) is a vector field which is everywhere horizontal (resp. vertical), we will simply write \( Y \in V^\perp \) (resp. \( Y \in V \)). Note in particular that any algebraic combination of the \( X_{f_r} \) is necessarily horizontal everywhere. Let us make three preliminary observations that will be used later. First, the distribution \( V \) is integrable since for any \( v_1, v_2 \in V \), \([v_1, v_2](f_r) = 0 \Rightarrow [v_1, v_2] \in V \). Next, since \( X_{f_r}(f_s) = \{f_s, f_r\} \), we must have \([X_{f_r}, v](f_s) = 0 \forall v \in V \). Therefore, the \( X_{f_r} \) preserve the vertical distribution. Similarly, the Hamiltonian vector field of any algebraic function of the \( f_r \) also preserves the distribution. Lastly, since \([X_{f_r}, X_{f_s}] = 0 \forall r, s \) the horizontal spaces are also integrable. Therefore, there exist global horizontal sections of our quantum bundle over \( \Gamma \).

We are now prepared to reconstruct the classical symplectic structure from the geometry of the quantum phase space. Let \( \xi \) and \( \zeta \) be two vector fields on \( \Gamma \) and denote by \( \xi \) and \( \zeta \) their horizontal lifts to \( \mathcal{P} \). Then the classical symplectic structure is defined by

\[
\alpha(\xi, \zeta) := \omega(\tilde{\xi}, \tilde{\zeta}).
\] (4.3)

We have simply defined \( \alpha \) as the “horizontal part” of the quantum symplectic structure. Since the classical phase space is linear, it is obvious that Eq. (4.3) correctly defines the classical symplectic structure. However, in order to allow more general constructions, let us see why this definition of \( \alpha \) provides a symplectic structure on \( \Gamma \). First of all, it is easy to show that \( \omega(\tilde{\xi}, \tilde{\zeta}) \) is constant along the fibres of \( \mathcal{P} \). Therefore, the definition is self-consistent.

To see that \( \alpha \) is non-degenerate, one need only notice that \( \alpha(\xi, \zeta) = 0 \forall \zeta \Rightarrow \omega(\tilde{\xi}, \tilde{\zeta}) = 0 \forall \zeta \in V^\perp \Rightarrow \zeta \in V \); by construction, \( \zeta \) must therefore vanish. Finally, that \( \alpha \) is closed is obvious since it is the pull-back, via any horizontal section, of a closed form.

Let us summarize these results. Quantization of a classical theory with a linear phase space is canonical, given the elementary position and momentum observables. For this case, the quantum phase space may be naturally viewed as a bundle over the classical phase space. The classical phase space is simply the base space of this bundle. The symplectic structure on \( \mathcal{P} \) naturally defines a notion of horizontality, and the classical symplectic structure is simply the horizontal part of \( \omega \).

Incidentally, note that our arguments go a short step toward the more general theory for which the classical phase space is not necessarily linear. Suppose that \( \mathcal{V} \) is an integrable, symplectic distribution on \( \mathcal{P} \) of finite co-dimension \( 2n \) which may be specified locally by the constancy of functions \( f_r, r = 1 \ldots 2n \). Suppose also that these functions may be chosen in such a way that their Poisson algebra closes (up to constants). Then the quotient of \( \mathcal{P} \) by \( \mathcal{V} \) inherits a natural symplectic structure. It is desirable to eliminate the requirement that the co-dimension of \( \mathcal{V} \) be finite; otherwise treatment does not apply to, e.g., quantum field
theory. We believe that this assumption can be removed although a detailed analysis is yet to be carried out.

Note that our construction provides not only a projection from $P$ to $\Gamma$ but also a class of preferred embeddings of classical phase space into the quantum one. One may guess that these embeddings are related, in some way, to coherent states. After all, it is well-known that spaces of coherent states are endowed with natural symplectic structures [39,18].

Let us therefore examine the nature of the coherent state spaces from the geometric point of view. There are a number of different constructions of coherent states. We consider the attractive and fairly general approach introduced by Perelomov [19], Gilmore [20] and, to some extent much earlier, by Klauder [21]. Consider the the Heisenberg-Weyl group obtained by exponentiating the Lie algebra generated by $\hat{Q}_i, \hat{P}_j$ and the identity operator on $\mathcal{H}$. One defines Perelomov’s space of generalized coherent states by the action of this group on an arbitrary element $\Psi_0 \in \mathcal{H}$. In this manner, one obtains generalized coherent states $\Psi_{(q',p')} := \exp[-\frac{i}{\hbar} \sum_i(q'_i\hat{P}_i - p'_i\hat{Q}_i)] \Psi_0$, labeled by pairs of parameters $(q'_i,p'_i)$ with dimensions of position and momenta, respectively. If one chooses for $\Psi_0$ the ground state of the oscillator, one recovers the space of standard coherent states. However, one may apply the above construction using arbitrary $\Psi_0$ as the fiducial state. Hence, the notion of generalized coherent states is a viable one for the generic system.

Let us now return to the relation between the quantum and classical phase spaces. Given an arbitrary element $x_0 \in P$ of the quantum phase space, one can obtain a sub-manifold of generalized coherent states as follows: Choose a state $\Psi_0$ which projects to $x_0$ and construct the generalized coherent states via the standard method described above. To begin with, these states constitute a genuinely non-linear subspace of the Hilbert space. Nonetheless, we may project the entire space to $P$. An understanding of the nature of the resulting sub-manifold of $P$ is provided by the following facts. First, the expectation values of the basic operators at the coherent states are given by

$$q_i(x_{(q',p')}) = q_i(x_0) + q'_i \quad \text{and} \quad p_i(x_{(q',p')}) = p_i(x_0) + p'_i,$$ (4.4)

(so that if we choose for $\Psi_0$ the ground state of the harmonic oscillator, the expectation values are precisely $q'$ and $p'$.) Second, any two coherent states generated by the same fiducial element $x_0$ possesses the same uncertainties. Therefore, the uncertainties in $q_i$ and $p_j$ are constant on the generalized coherent state space, and hence on the sub-manifold of $P$ they define. Finally, note that the definition of the Heisenberg-Weyl group implies:

$$\frac{\partial}{\partial q'} \Psi_{(q',p')} = \frac{1}{i\hbar} \hat{P} \Psi_{(q',p')},$$

$$\frac{\partial}{\partial p'} \Psi_{(q',p')} = -\frac{1}{i\hbar} \hat{Q} \Psi_{(q',p')}.$$ (4.5, 4.6)

Projecting this result to $P$, one immediately verifies that each space of generalized coherent states is everywhere horizontal. Thus, our horizontal sections on $P$ are precisely the gen-

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8Although this group is kinematical in origin, in the literature on coherent states, it is often referred to as the \textit{dynamical group} associated with the simple harmonic oscillator.
eralized coherent state spaces! Hence, it follows in particular that the uncertainties in the elementary quantum observables are constant on the horizontal sections.

Finally, it is interesting to note that our construction of the horizontal sections did not refer to the Heisenberg group. Given a general system, Perelomov’s generalized coherent state spaces are constructed as above, but by use of a different (“dynamical”, see footnote 8) group; an entirely different set of generalized coherent state spaces will then result, which will not correspond to horizontal sections. This seems an a point worth further investigation.

B. Dynamics: Oscillators

Let us now consider dynamical issues. Since the horizontal spaces are integrable, the classical phase space may be embedded into $\mathcal{P}$ in infinitely many ways. Is there a preferred horizontal section? The answer to this question will involve the dynamics of the system. In general, we expect this issue to be far from trivial. The case of the harmonic oscillator is, as one may guess, fairly simple. In this sub-section we will restrict ourselves to this case.

Recall the elementary treatment of the classical limit in terms of Ehrenfest’s theorem. The rate of change of the expectation value $\langle \hat{F} \rangle$ of any observable operator $\hat{F}$ is given by

$$\frac{d}{dt} \langle \hat{F} \rangle = \frac{1}{i\hbar} \left\langle \left[ \hat{F}, \hat{H} \right] \right\rangle,$$

(4.7)

where $\hat{H}$ is the Hamiltonian operator. As usual, let $f$ and $h$ denote the corresponding observable functions on $\mathcal{P}$. Then as a result of Eq. (2.14) and its subsequent discussion, Eq. (4.7) directly translates to

$$\frac{df}{dt} = \{f, h\}.$$

(4.8)

The fact that this equation exactly mirrors the classical expression is, however, deceiving because the functional form of the quantum Hamiltonian $h$ is typically entirely different from that of the classical Hamiltonian. For example, in general, one may not even be able to express $h$ in terms of $q_i$ and $p_i$ alone. In fact, this is the case already for the harmonic oscillator. At first, this seems puzzling because, as is well-known, the $q_i$ and $p_i$ “follow the classical trajectories” if the Hamiltonian operator is a quadratic function of $\hat{Q}_i$ and $\hat{P}_i$.

Let us therefore pursue this a bit further. For simplicity, consider the one-dimensional case. First, we must obtain the form of the quantum Hamiltonian. This is easy. Since $\hat{H} = (1/2m)\hat{P}^2 + (m\omega^2/2)\hat{Q}^2 = (1/2m)\{\hat{P}, \hat{P}\}_+ + (m\omega^2/2)\{\hat{Q}, \hat{Q}\}_+$, the Hamiltonian function on $\mathcal{P}$ must be of the form

$$h = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 + \frac{1}{2m}(\Delta p)^2 + \frac{m\omega^2}{2}(\Delta q)^2,$$

(4.9)

where we have used only the definition and properties of the symmetric bracket. Cross-terms involving the uncertainties—and hence the total Hamiltonian $h$—can not be expressed in terms of $q_i$ and $p_i$ alone. More generally, such cross terms are by-products of the quantization process and provide one of the significant differences between the forms of the classical and quantum dynamics.
Notice that the Hamiltonian may be decomposed into two parts; hence, so may the corresponding Hamiltonian vector field. \( X_h = X_{h_0} + X_{h_\Delta} \) where \( h_0 \) takes the form of the classical Hamiltonian and \( h_\Delta = (1/2m)(\Delta p)^2 + (m\omega^2/2)(\Delta q)^2 \). Now, recall that the uncertainties \( \Delta q \) and \( \Delta p \) are constant in the horizontal directions. As a consequence, \( X_{h_\Delta} \) must be purely vertical. This should not be at all surprising since, as we already know, time-dependence of \( q \) and \( p \) is the same as for the classical observables. Since \( X_{h_\Delta} \) is everywhere vertical, the pull-back \( s^*h \) of the quantum Hamiltonian, via any horizontal section \( s : \Gamma \to \mathcal{P} \), to the classical phase space is precisely the classical Hamiltonian, up to a physically irrelevant overall constant. This is essentially the reason why the evolution of the basic quantum observables agrees with the classical evolution. However, the quantum evolution is quite different from the classical, for it does not generally preserve the horizontal sections. A natural question arises: is there a horizontal section which is preserved by the Hamiltonian evolution?

Note that we are asking for much more than a dynamical trajectory confined to one horizontal section. The question is whether there exists an entire horizontal section of \( \mathcal{P} \) which is preserved by the quantum evolution. A section \( s_0 : \Gamma \to \mathcal{P} \) is of this special nature if and only if \( X_{h_\Delta} \) vanishes on the entire image of \( s_0 \). Equivalently, we may search for a section \( s_0 \) on which the uncertainty term \( h_\Delta \) attains a local extremum. It seems a rather natural guess that this will be the case at states which saturate the uncertainty relation between \( q \) and \( p \). This expectation is correct. One can see this by writing

\[
\frac{1}{\omega\hbar} h_\Delta = \left[ \Delta q \sqrt{m\omega/2\hbar} - \Delta p \sqrt{1/2m\omega\hbar} \right]^2 + \frac{1}{\hbar} \Delta p \Delta q \geq \frac{1}{2}, \tag{4.10}
\]

where the inequality is due to Heisenberg. Therefore, any state \( x \) at which \( h_\Delta(x) = \omega\hbar/2 \) extremizes the uncertainty term. It is now easy to see that the only values of \( \Delta p \) and \( \Delta q \) (subject, of course, to Heisenberg’s uncertainty relation) which extremize \( h_\Delta \) are given by \((\Delta q)^2 = \hbar/2m\omega \) and \((\Delta p)^2 = m\omega\hbar/2 \). There is therefore a single horizontal section which is preserved by the Hamiltonian evolution; it is the section on which the quantum evolution is “most classical”. As one might suspect, this preferred section corresponds to the standard coherent state space, generated by the oscillator’s ground state. These results clearly hold for any finite-dimensional oscillator, and likely for the infinite-dimensional case (e.g., quantum Maxwell theory).

C. Dynamics: WKB approximation

The previous discussion of dynamics has been, to some extent, in the context of the Ehrenfest approach to semi-classical dynamics. Let us also consider the problem from the point of view of Hamilton-Jacobi theory—the context in which one often introduces the WKB approximation. While the discussion of section IV B has been limited to the special case of the oscillator, we now consider more general systems and obtain two main results. First, we will present interesting condition for the validity of the WKB approximation which, to our knowledge, has not been discussed in the literature. Second, we will show that the WKB equation actually corresponds to a Hamiltonian evolution on the projective space, and therefore defines a generalized quantum dynamics of the Weinberg type.
We will consider the dynamics of a non-relativistic particle moving in $\mathbb{R}^n$ under the influence of a general conservative force (see footnote 6). The wave function is therefore assumed to satisfy the Schrödinger equation
\[ i\hbar \frac{\partial}{\partial t} \Psi(x, t) = \left( -\frac{\hbar^2}{2m} \Delta + V(x) \right) \Psi(x, t), \quad (4.11) \]
where $\Delta$ is the Laplace operator on $\mathbb{R}^n$. In the spirit of our approach, we decompose the state vector into real and imaginary parts. This defines two real fields $\phi$ and $\pi$ via
\[ \Psi = (\phi + i\pi)/\sqrt{2\hbar}. \]
In terms of these fields, the metric and symplectic structure on $\mathcal{H}$ assume the forms
\[ G((\phi_1, \pi_1), (\phi_2, \pi_2)) = \int d^n x [\phi_1(x)\phi_2(x) + \pi_1(x)\pi_2(x)], \quad (4.12) \]
\[ \Omega((\phi_1, \pi_1), (\phi_2, \pi_2)) = \int d^n x [\phi_1(x)\pi_2(x) - \phi_2(x)\pi_1(x)]. \quad (4.13) \]
The second equation may look familiar from classical field theory. It implies that the fields $\phi$ and $\pi$ are canonically conjugate; one may view them, respectively, as the ‘field’ and ‘momentum density’ of the classical field theory under consideration. This is not surprising because, after all, any quantum theory can be regarded as a field theory.

Indeed, one may calculate the expectation value of the above Hamiltonian operator,
\[ H(\phi, \pi) = \frac{1}{2\hbar} \int d^n x \left[ \frac{\hbar^2}{2m} \left( (\vec{\partial}\phi)^2 + (\vec{\partial}\pi)^2 \right) + V(x) \left( \phi^2(x) + \pi^2(x) \right) \right], \quad (4.14) \]
and verify that the fields evolve according to the canonical equations of motion:
\[ \frac{\partial \phi}{\partial t} = \frac{\delta H}{\delta \pi} \quad \text{and} \quad \frac{\partial \pi}{\partial t} = -\frac{\delta H}{\delta \phi}. \quad (4.15) \]
We are interested in the relationship between these equations of motion provided by quantum dynamics and the Hamilton-Jacobi equation.

For this it is convenient to express $\Psi$ as $\Psi = \sqrt{\rho} \exp(iS/\hbar)$ and rewrite Schrödinger dynamics as equations of motion for $\rho$ and $S$:
\[ \frac{\partial S}{\partial t} + \frac{1}{2m} (\vec{\partial} S)^2 + V(x) = \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}, \quad (4.16) \]
\[ m \frac{\partial \rho}{\partial t} + \vec{\partial} \cdot (\rho \vec{\partial} S) = 0. \quad (4.17) \]
The second equation is the conservation equation, $\partial \rho/\partial t + \vec{\partial} \cdot \vec{J} = 0$, where $\vec{J} = \rho \vec{\partial} S/m$ is the probability current density. The familiar observation is the fact that Eq. (4.10) becomes the Hamilton-Jacobi equation upon dropping the term involving $\hbar$; in a loose sense, one obtains the classical limit upon taking $\hbar \to 0$. (For details, see standard texts, e.g., [40, 41].)

Let us explore the standard approach in which one simply drops the “quantum correction” to the Hamilton-Jacobi equation. To be concrete, let us refer to the corresponding evolution as WKB evolution. First, one may express the Hamiltonian function in terms of $\rho$ and $S$:
\[ H(\rho, S) = \int d^nx \left[ \frac{\hbar^2}{8m\rho(x)}(\nabla\rho)^2 + \frac{1}{2m} \rho(x)(\nabla S)^2 + \rho(x)V(x) \right]. \] (4.18)

It may be seen that \( \rho \) and \( S \) are also canonically related and that their equations of motion may be obtained from Eqs. (4.15) by making the replacements \( \phi \rightarrow \rho \) and \( \pi \rightarrow S \). One may also verify that the WKB evolution corresponds to dropping the first term in Eq. (4.18). That is, if \( H_h \) denotes this first term, and \( H_{WKB} := H - H_h \), then the WKB evolution corresponds to integration of the Hamiltonian vector field \( X_{WKB} \) generated by \( H_{WKB} \).

Let us first examine the condition of validity of the the WKB approximation. We require conditions under which the Hamiltonian vector field generated by \( H_h \) is small compared to that generated by \( H \). To this end, we compute the functional derivatives of \( H_h \) with respect to the fields \( \phi \) and \( \pi \). We obtain

\[
\frac{\delta H_h}{\delta \phi} = -\frac{1}{8\hbar \rho^2} \left[ 2\hbar \rho (\phi \Delta \phi + \pi \Delta \pi) + (\phi \partial \phi - \pi \partial \phi)^2 \right] \phi; \quad (4.19)
\]

the corresponding expression for \( \delta H_h/\delta \pi \) is obtained by making the replacement \( \phi \leftrightarrow \pi \) above. This expression may be written in terms of quantities which are somewhat more physical. One need only notice that

\[
\langle \hat{P} \rangle = \int d^nx \ m \vec{J}(x), \quad \vec{J} = \frac{1}{2} (\phi \partial \pi - \pi \partial \phi), \quad \langle \hat{P}^2 \rangle = \int d^nx \ K(x), \quad K = \frac{\hbar}{2} (\phi \Delta \phi + \pi \Delta \pi). \quad (4.20)
\]

The quantities \( m \vec{J} \) and \( K \) which appear in Eq. (4.19) may be interpreted physically as the “density of momentum” and the “density of squared-momentum”.

In terms of these quantities, the above functional derivative may be written as

\[
\frac{\delta H}{\delta \phi} = -\frac{1}{2m\hbar \rho^2} \left[ (m \vec{J})^2 - \rho K \right] \phi. \quad (4.21)
\]

Therefore,

\[
\text{i} \hbar \left( \frac{\delta H_h}{\delta \pi} - \frac{1}{\hbar} \frac{\delta H_h}{\delta \phi} \right) = \frac{\text{i}}{2m\rho^2} \left[ (m \vec{J})^2 - \rho K \right] (\phi + \text{i} \pi). \quad (4.22)
\]

The WKB evolution is obtained by subtracting ((1/\sqrt{2\hbar} times) this term from the Schrödinger equation. The WKB equation of motion may therefore be written

\[
\text{i} \hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi + \frac{1}{2m\rho^2} \left[ (m \vec{J})^2 - \rho K \right] \Psi, \quad (4.23)
\]

where \( \hat{H} \) is the unaltered Hamiltonian operator. Hence, the condition of validity of the WKB equation is that the second term on the right side of Eq. (4.23) be small; i.e. that the “density of squared-momentum” (weighted by the probability density) be comparable in magnitude to the square of the “density of momentum”.

We have just arrived at an interesting form of the WKB equation of motion. One should compare it with the general form given by Eq. (3.8) of the non-linear Schrödinger...
equation. Note that the additional term $\alpha$ appearing in that equation is a function of only the amplitude $\rho$. Hence, the WKB equation appears to be considerably more complicated than the explicit forms of generalized dynamics that are generally considered.

However, we have already seen (c.f. the discussion following Eq. (4.18)) that the WKB evolution is generated by a Hamiltonian function on $\mathcal{H}$. Surprisingly, this evolution actually induces a Hamiltonian flow on the projective Hilbert space. To see this, in the light of section III A, we need only show that $X_{\text{WKB}}$ preserves the unit sphere. This is the case if and only if the Poisson bracket of $H_{\text{WKB}}$ and the constraint function, $C = \int d^n x \rho(x) - 1$ vanishes at each normalized state. This fact may be proven as follows: Since $H_{\bar{H}}$ is independent of $S$, $\{H_{\bar{H}}, C\} = 0$. Moreover, since $H$ generates the Schrödinger evolution, which preserves the unit sphere, $\{H, C\} = 0$. Taking the difference between these two Poisson brackets, we obtain the desired result: $\{H_{\text{WKB}}, C\} = 0$. This Poisson bracket actually vanishes strongly; therefore, the WKB evolution actually preserves the norm of all state-vectors (not just the unit vectors). Therefore, the dynamics of the WKB approximation is an example of generalized dynamics of the Weinberg type; in particular, it defines a Hamiltonian flow on the projective Hilbert space.

V. DISCUSSION

Let us begin with a summary of the main results.

We first showed that ordinary quantum mechanics can be reformulated in a geometric language. In particular, as in classical mechanics, the space of physical states—the “quantum phase space”—is a symplectic manifold and dynamics is generated by a Hamiltonian vector field. However, unlike in classical mechanics, the quantum phase space is equipped also with a Kähler structure. As one might suspect, the Riemannian metric, which is absent in the classical description, governs uncertainty relations and state vector reduction which are hallmarks of quantum mechanics. The geometric formulation shows that the linear structure which is at the forefront in text-book treatments of quantum mechanics is, primarily, only a technical convenience and the essential ingredients—the manifold of states, the symplectic structure and the Riemannian metric—do not share this linearity. Therefore, the framework can serve as a stepping stone for non-linear generalizations of quantum mechanics.

One can consider such generalizations in various directions. A “conservative” approach would retain the kinematical structure and generalize only the dynamics. This is the viewpoint that underlies the various non-linear generalizations of the Schrödinger equation that have been considered in the literature. The strategy is also natural and easy to implement in the geometric formulation: While in the standard formulation, dynamics can be generated by a very restricted class of functions on the quantum phase space, we can generalize the framework by allowing the Hamiltonian flow to be generated by any smooth function on the quantum phase space. We saw that the known proposals of generalized dynamics fall in this class. Furthermore, by concentrating on the quantum phase space—rather than the fiducial Hilbert space—one can separate essential features of dynamics from inessential ones. In particular, this naturally led to a clarification of the relation between Weinberg’s framework and non-linear Schrödinger equations and corrected some misconceptions.

The kinematical generalizations are more difficult. However, we were able to streamline
the search for such generalizations by providing a characterization of ordinary quantum mechanics using structures that have direct physical interpretation: In essence, what singles out quantum mechanics is that the underlying Kähler manifold has maximal symmetries. From the geometric viewpoint then, restricting oneself to ordinary quantum mechanics is rather analogous, in (pseudo-)Riemannian geometry, to working only with manifolds of constant curvature. Now, in (pseudo-)Riemannian geometry, a rich theory remains even if one drops the restriction of maximal symmetries. Indeed, in the context of general relativity, the manifolds of constant curvature are only the ‘vacua’ and in most physically interesting situations, there are significant departures from these geometries. Is the situation perhaps similar in the case of quantum mechanics? Have we been restricting our attention only to the most elementary of viable theories?

The geometrical formulation is also useful to probe semi-classical issues. We saw in particular, that the quantum phase space has a natural bundle structure and the horizontal cross-sections correspond precisely to families of generalized coherent states. It also provides a succinct and clear condition for validity of the WKB approximation. Furthermore, it turned out that dynamics in the WKB approximation yields a well-defined flow on the quantum phase space which corresponds to a “generalized quantum dynamics” in the sense of Weinberg.

The quantum (Hilbert space and hence) phase space is finite-dimensional only in exceptional cases, such as spin systems. Most work in the literature in the area of geometric formulations of quantum mechanics deals only with this case. By working with Hilbert manifolds, we were able to treat the generic case—such as particles moving in $\mathbb{R}^n$—where the quantum phase space is infinite-dimensional. Finally, most of our results go through also in quantum field theory (although the measurement postulates are, as is usual, geared to non-relativistic quantum mechanics.) Indeed, the geometric treatment sheds new light on the second quantization procedure. Because the space of quantum states is itself a symplectic manifold equipped with a Kähler structure, it turns out that one can use (a natural infinite dimensional generalization of) the machinery of geometric quantization to carry out quantization again. The resulting theory is precisely the second quantized one. Thus, second quantization is indeed (quantization)$^2$! (For details, see [16].)

We will conclude by listing a few of the important open problems. First, we have given an intrinsically geometric formulation of the five postulates of quantum mechanics that deal with kinematics, unitary evolution and measurements of observables (possibly with continuous spectra). However, we did not include the spin-statistics postulate. The reason is that we do not have a succinct formulation of this postulate which refers only to the essential geometric structures. Obtaining such a formulation is a key open problem. The remaining issues deal with generalizations of the standard framework. We saw that it is rather straightforward to extend dynamics by allowing the Hamiltonian to be any densely-defined function on the quantum phase space. However, unless it is an observable function in the sense of definition II.1, we may not have a consistent measurement theory for it. Whether this is a problem is not so clear. Indeed, this feature arises already for the non-linear Schrödinger equations considered in the literature and there it is generally not perceived as a problem. However, a systematic analysis of this issue should be carried out. Next, as indicated in section III.B, there is possibility that there exist infinite-dimensional Kähler manifolds with constant holomorphic sectional curvature ($= 2/\hbar$) which are not isomorphic to a projective
Hilbert space. If this does turn out to be the case, we will obtain viable, non-trivial generalizations of quantum kinematics for which even the measurement theory could be developed in detail. Therefore, it is important to settle this issue. Also, even in the finite-dimensional case, we do not know if there exist any Kähler manifolds other than projective Hilbert spaces for which a satisfactory measurement theory can be developed. Even isolated examples of such manifolds would be very illuminating. The final issue stems from the fact that the space of quantum states shares several features with the classical phase space. Is there then a quantization procedure to arrive directly at \((\mathcal{P}, \omega, g)\) without having to pass through the Hilbert space? Not only may the answer be in the affirmative but the new procedure may even be useful in cases when the standard quantization procedure encounters difficulties. That is, such a procedure may itself suggest generalizations of quantum mechanics.

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