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Concentration bounds for quantum states with finite correlation length on quantum spin lattice systems

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Abstract
We consider the problem of determining the energy distribution of quantum states that satisfy exponential decay of correlation and product states, with respect to a quantum local Hamiltonian on a spin lattice. For a quantum state on a $D$-dimensional lattice that has correlation length $\sigma$ and has average energy $e$ with respect to a given local Hamiltonian (with $n$ local terms, each of which has norm at most 1), we show that the overlap of this state with eigenspace of energy $f$ is at most $\exp(-((e-f)^2)\sigma^{1/2}/n^{1/2}D\sigma)$. This bound holds whenever $|e-f| > 2^D \sqrt{n}\sigma$. Thus, on a one-dimensional lattice, the tail of the energy distribution decays exponentially with the energy. For product states, we improve above result to obtain a Gaussian decay in energy, even for quantum spin systems without an underlying lattice structure. Given a product state on a collection of spins which has average energy $e$ with respect to a local Hamiltonian (with $n$ local terms and each local term overlapping with at most $m$ other local terms), we show that the overlap of this state with eigenspace of energy $f$ is at most $\exp(-(e-f)^2/nm^2)$. This bound holds whenever $|e-f| > m\sqrt{n}$.

1. Introduction

A question of primary interest for local Hamiltonian spin systems is to determine the energy distribution of natural class of states with respect to a given local Hamiltonian. The knowledge of energy distribution reveals a lot of information about the nature of the state itself. As we shall discuss below, a gaussian distribution of energy can be associated to a product state. On the other hand, the well known entangled state $\frac{1}{\sqrt{2}} |0\rangle^n + \frac{1}{\sqrt{2}} |1\rangle^n$ (also termed as the ‘cat state’) has energy distribution peaking at opposite ends of the spectrum of the Hamiltonian: $\sum_{i=1}^n |1\rangle \langle 1|$. Moreover, the knowledge of energy distribution plays an important role in the study of thermalization of quantum systems.

The aforementioned question has been well studied in classical setting, important examples of which are the Chernoff bound [Che52] and the Central Limit theorem (which applies to asymptotic regime). Chernoff bound can be informally stated as follows. Let $X_1, X_2 \ldots X_n$ be independent and identically distributed random variables taking values in $[0, 1]$ and each having average value $A$. Then $\Pr(|X_1 + X_2 + \ldots + X_n - nA| > \varepsilon) \leq e^{-\varepsilon^2/n}$, where $c$ is a constant that depends on $A$.

One interpretation of this bound (which was the original motivation in [Che52]) is that it provides a recipe for distinguishing between two probability distributions $P \overset{\text{def}}{=} \sum_x P(x)|x\rangle \langle x|$ and $Q \overset{\text{def}}{=} \sum_x Q(x)|x\rangle \langle x|$ with expectation values $A$ and $B$ respectively. Given $n$ independent samples $x_1, x_2 \ldots x_n$ from either of these distributions, the sum $\sum_i x_i$ is highly likely to be concentrated around $nA$ if the underlying distribution is $P$ and around $nB$ if the underlying distribution is $Q$. A more precise formulation of this idea requires characterizing the trace distance between $P^{\otimes n}$ and $Q^{\otimes n}$ as $n$ becomes large, and it has been generalized to the quantum setting in [ACMnT10].

Another interpretation of the Chernoff bound, which is the focus of present work, lies in the setting of ‘classical’ local Hamiltonian systems. Consider a product state $\rho^{\otimes n}$ on $n$ sites, where $\rho \overset{\text{def}}{=} \sum_x P(x)|x\rangle \langle x|$. Let $H$
be a 1-local Hamiltonian $H \overset{\text{def}}{=} \sum_i h_i$, such that $h_i = \sum_x x \, |x\rangle \langle x|$ acts non-trivially only on the site $i$ and is same for each site. If $A \overset{\text{def}}{=} \text{Tr}(\rho h_i)$ is the expectation value of $\rho$ with respect to $h_i$, then $nA$ is average energy of $\rho^{\otimes n}$ with respect to the Hamiltonian $H$. Let $\Pi_{\geq nA+\varepsilon}$ be the projector onto eigenspaces of $H$ with energy at least $nA + \varepsilon$. Then the Chernoff bound implies that $\text{Tr}(\rho^{\otimes n}\Pi_{\geq nA+\varepsilon}) \leq e^{-\varepsilon^2/n}$. Thus, the energy distribution of $\rho^{\otimes n}$ is highly concentrated around the average energy $nA$.

The energy distribution of a product state for quantum lattice system with infinitely many sites was considered in [GV89] (for translationally invariant systems) and in [HMH04a] (for non-translationally invariant systems). These results can be regarded as a generalization of the central limit theorem to quantum systems. A non-asymptotic version of central limit theorem is the Berry–Esseen theorem ([Ber41, Ess45]), which provides an upper bound on trace distance between energy distribution of product state and the normal distribution as a function of lattice size. This upper bound goes to zero as lattice size approaches infinity, thus recovering the central limit theorem. For quantum states with finite correlation length (which includes product states) on finite sized lattice, a quantum version of Berry–Esseen theorem was recently shown to hold in [BC15, BCG15].

These results give a strong indication that states satisfying exponential decay of correlation behave similar to product states, even when their energy distributions are measured with respect to the eigenspectrum of a non-commuting (but local) Hamiltonian. The work [KLW15] goes even further to show that non-commuting local Hamiltonians themselves have energy spectrum that resemble that of a 1-local Hamiltonian (although, quite curiously, the same work shows that almost all eigenvectors of non-commuting local Hamiltonians are highly entangled, in contrast with the eigenvectors of 1-local Hamiltonians).

Above mentioned results have added to the growing body of research on general properties of local Hamiltonian systems, such as the Lieb–Robinson bound [LR72], exponential decay of correlation [Has04], the area laws [AKLV13, ALV12, Has07] and local reversibility [KAAV15], to name a few. They have also found several applications in the problem of thermalization of many body systems. To start with, one of the first steps towards the problem of locality of temperature was taken in [HMH04b]. Crucially using the central limit theorem obtained in [HMH04a], the authors characterized a set of conditions under which a given thermal state of a quantum local Hamiltonian on a lattice would be close to a tensor product of thermal states on local subsystems on the lattice.

The work [Cra12] considered the problem of thermalization under random Hamiltonians, where the Hamiltonian was generated via a random unitary on a fixed local Hamiltonian $H$. One of the main technical problems in this work was the study of the characteristic function $\text{Tr}(e^{iH\xi})$, where $\xi$ is the maximally mixed state (which is also a product state on the lattice). The techniques were inspired from the proof of central limit theorem in the works [GV89, HMH04a], where the characteristic function $\text{Tr}(e^{iH\rho})$ of a product state $\rho$ had been investigated in detail.

The quantum version of Berry–Esseen theorem [BCG15] was used to show in [BC15] that Gibbs state of a local Hamiltonian $H$ at sufficiently high temperature (high enough to ensure a clustering of correlation) is indistinguishable, over sufficiently large regions of lattice (that scale sub-linearly with lattice size), from the microcanonical ensemble of eigenspaces of $H$ which have eigenvalues close to the average energy of the Gibbs state. This result bears close resemblance to the eigenstate thermalization hypothesis [Dev91, Sre94], which is a stronger conjecture stating that every eigenstate of $H$ with eigenvalue close to the average energy of Gibbs state of $H$ is locally indistinguishable from this Gibbs state.

In present work, we provide further details on energy distribution of states satisfying exponential decay of correlation and product states. Our main results can be seen as an analogue of the Chernoff bound for quantum lattice systems.

Our first result concerns states that satisfy exponential decay of correlation on a $D$-dimensional lattice. Well known examples of such states include the ground states of gapped local Hamiltonians [Has04] and Gibbs state above a finite temperature [KGK’13]. In fact, it has been shown in [FWB’15] that for local Hamiltonians exhibiting many body localization and having non-degenerate energy spectrum, all eigenvectors satisfy exponential decay of correlation. Thus our result provides information about structure of eigenvectors of such Hamiltonians and may have applications in the phenomena of many body localization.

Consider a local Hamiltonian $H$ on a $D$-dimensional lattice with $n$ local interaction terms, such that each local term has operator norm at most 1 (see section 2 and figure 2 for a detailed description of $H$).

**Theorem 1.1 (Informal).** Let $\rho$ be a quantum state with correlation length $\sigma$ and $\langle H \rangle_\rho \overset{\text{def}}{=} \text{Tr}(\rho H)$ be the average energy of $\rho$. Let $\Pi_{\leq f}$ ($\Pi_{\geq f}$) be the projection onto subspace which is union of eigenspaces of $H$ with eigenvalues $\geq f$ ($\leq f$).

---

1 Which is roughly the problem of assigning a temperature to reduced density matrix of thermal state of a local Hamiltonian, detailed discussion can be found in [KGK’13].
For a \( \geq \sqrt{\frac{2^{|D|}}{n^2}} \) it holds that

\[
\text{Tr}(\rho \Pi_{>\langle H \rangle + \frac{a}{2}}) \leq O(\sigma) e^{-\frac{n^2 \sigma}{\log(|D|)}} \quad \text{and} \quad \text{Tr}(\rho \Pi_{<\langle H \rangle - \frac{a}{2}}) \leq O(\sigma) e^{-\frac{n^2 \sigma}{\log(|D|)}}.
\]

Formal statement of the theorem is given in theorem 4.2. Thus in one-dimensional spin chain (with \( D = 1 \)), our upper bound decays exponentially with energy, rather than as a gaussian. The bound becomes weaker with higher dimensions and is depicted in figure 1.

Our second result concerns product states over a collection of spins and does not require any underlying lattice arrangement of these spins. It does impose, however, a locality constraint on the Hamiltonian that acts on these spins. Consider a Hamiltonian \( H \) which is a sum of \( n \) terms, each term being \( k \)-local (that is, it acts non-trivially on at most \( k \) spins) and having operator norm at most 1. Let \( m \) be the maximum number of neighbours of any local term, where two local terms are neighbours if there is a spin on which both act non-trivially (See section 5 and figure 3 for detailed description of \( H \)). We show the following.

**Theorem 1.2 (Informal).** Consider a product state \( \rho \) with average energy \( \langle H \rangle_{\rho} \overset{\text{def}}{=} \text{Tr}(\rho H) \). Fix a real number \( a \geq \sqrt{\frac{2^{|D|}}{n^2}} \). Let \( \Pi_{\geq f} (\Pi_{\leq f}) \) be the projection onto subspace which is union of eigenspaces of \( H \) with eigenvalues \( \geq f \) (\( \leq f \)). It holds that

\[
\text{Tr}(\rho \Pi_{\geq \langle H \rangle + \frac{a}{2}}) \leq e^{-\frac{n^2 \sigma}{O(|D|)}}
\]

and

\[
\text{Tr}(\rho \Pi_{\leq \langle H \rangle - \frac{a}{2}}) \leq e^{-\frac{n^2 \sigma}{O(|D|)}}.
\]

Formal statement of the theorem is given in theorem 5.3. The energy distribution is depicted in figure 1. The bound is not only independent of any underlying lattice structure, but is also independent of the locality \( k \). This is not surprising, since the quantity \( n \) that appears in the bound is the number of local terms in \( H \), rather than number of spins on which \( H \) acts. Following corollary is a restatement of above bound, in terms of number of spins (which we call \( N \)) and the maximum number of local terms that act on any given spin (which we call \( g \)). In the following, we also assume that each local term is exactly \( k \)-local.

**Corollary 1.3.** Consider a product state \( \rho \) with average energy \( \langle H \rangle_{\rho} \overset{\text{def}}{=} \text{Tr}(\rho H) \). Fix a real number \( \varepsilon \geq \sqrt{O(g^3 k N)} \). Let \( \Pi_{\geq f} (\Pi_{\leq f}) \) be the projection onto subspace which is union of eigenspaces of \( H \) with eigenvalues \( \geq f \) (\( \leq f \)). It holds that

\[
\text{Tr}(\rho \Pi_{\geq \langle H \rangle + \frac{a}{2}}) \leq e^{-\frac{n^2 \sigma}{O(|D|)}}
\]
and

\[ \text{Tr}(\rho \Pi_{\xi(H)-n}) \leq e^{-\frac{1}{2} \sigma^2/n}. \]

Formal statement of the corollary appears as corollary 5.4. It shows a gaussian decay for tail of energy distribution of product states in the scenario where \( g, k \) are constants independent of \( N \).

**Related recent works**

A recent work [Kuw16] has obtained a similar concentration result for product states (lemma 4 therein). The key idea is to split the Hamiltonian \( H = H_1 + H_2 + \ldots \), where each \( H_i \) is composed of local terms that are non-overlapping. Then from classical Chernoff bound, the product state exhibits a Gaussian decay in energy distribution for each of the Hamiltonians \( H_1, H_2, \ldots \). Final step (which is also the main argument of the paper) is to combine these tails bounds to obtain a final bound for energy distribution with respect to the original Hamiltonian \( H \). Unfortunately, the techniques do not extend to states satisfying exponential decay of correlation. To establish a bound for energy distribution with respect to \( H \), one needs the knowledge of bounds for energy distribution with respect to each of the 'classical Hamiltonians' \( H_1, H_2, \ldots \). But even for these classical Hamiltonians, no bound is known for states that satisfy exponential decay of correlation (apart from theorem 1.1, to the best of author's knowledge). We have provided further comparison of the bound in [Kuw16] and theorem 1.2 in section 5.1.

A concentration result has been noted in [KAAV15] (section 5 in this reference) for ground states of gapped local Hamiltonians on finite dimensional quantum lattice systems, which also exhibit exponential decay of correlation ([Has04]). In this work, the probability distribution has been shown to be concentrated about the median of the distribution with the weight of the distribution above energy \( \varepsilon \) decaying as \( e^{-1/f(1)/\sqrt{\ln n}} \) (\( f \) being the median of the distribution, \( n \) being the number of local terms in the local Hamiltonian and \( \sigma \) being the correlation length of the ground state). In comparison, we show a concentration about the mean of the distribution for all states satisfying exponential decay of correlation. While our bounds are weaker than those of [KAAV15] in higher dimensions, it may be noted that we have considered a larger class of states that might possess weaker properties than the ground states of gapped local Hamiltonians. This behaviour appears in the context of area laws as well: ground states of gapped local Hamiltonians are known to have very good scaling of area laws with correlation length [AKL13]; whereas a recent observation of Hastings [Has15] suggests that states satisfying exponential decay of correlation may have much weaker dependence of area law with correlation length [BH13a].

**Our technique and organisation**

The idea behind our approach is straightforward, to compute the moment generating function \( \text{Tr}(H'\rho) \) of the energy distribution and then use Markov’s inequality to upper bound the desired probability. Without loss of generality, we can assume that \( H = \sum_w w \), where \( w \) is a label for local terms and \( \langle w \rangle_{H} = \text{Tr}(\rho_h w) = 0 \). Our key technical contribution is the combinatorial lemma (lemma 3.1) which answers the following question: if we expand \( H' = \sum_{w_1, w_2, \ldots} h_{w_1} h_{w_2} \ldots h_{w_n} \) how many terms \( h_{w_1} h_{w_2} \ldots h_{w_n} \) make non-negligible (or non-zero) contribution to the moment generating function? We observe that make non-negligible contribution possess a common property: there is no \( h_{w_i} \) which is supported 'far' from all of \( h_{w_1}, h_{w_2}, \ldots, h_{w_{n-1}}, h_{w_{n+1}}, \ldots h_{w_n} \). Making the notion of 'far' precise, we compute the number of such terms in lemma 3.1 and use it to bound the moment generating function.

The paper is organized as follows. We state basic facts and describe our physical set-up needed for theorem 1.1 in section 2. We prove our combinatorial lemma in section 3. In section 4 we prove our bounds for states satisfying exponential decay of correlation. In section 5, we introduce the physical set-up required for theorem 1.2 and provide the proof of the theorem. This proof also requires a variant of the combinatorial lemma (lemma 3.1) which we prove in the appendix. We conclude in section 6 and address some questions left open by this work.

**2. Physical set-up and basic facts**

In this section, we introduce the physical-set up required for theorem 1.1. For simplicity of the presentation, we shall assume that the spins are arranged on a square lattice, with a local interaction term acting between only those spins that are the vertices of a common ‘unit-hypercube’. We shall introduce the notion of a ‘dual lattice’ below, to formally and concisely represent these local interactions between the spins. It can be observed that more general local interactions on a square lattice can be put in this form by sufficient coarse-graining of lattice sites. The physical set-up for theorem 1.2 is relatively simple, and shall be introduced directly in section 5.

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2 An interesting class of local Hamiltonian system with constant \( g, k \) is the family of Hamiltonians defined on an expander graph, which has recently been a subject of interest with reference to the quantum PCP conjecture [AAV13, BH13b, AE15, Eld15, EH15, FH14].
Consider a $D$-dimensional real vector space $\mathbb{R}^D$. For a vector $v \in \mathbb{R}^D$, let $v_i$ represent its $i$th component. For two vectors $v, v' \in \mathbb{R}^D$, define the ‘1-norm distance’ as

$$\|v - v'\| \overset{\text{def}}{=} \sum_i |v_i - v'_i|.$$ 

It satisfies the triangle inequality: given $v, v', v'' \in \mathbb{R}^D$, we have

$$\|v - v''\| \leq \|v - v'\| + \|v' - v''\|.$$ 

For brevity, we shall refer to 1-norm distance simply as distance.

For an integer $L > 0$, define a lattice $\mathbb{L}_{D,L}$ as the set of all vectors $v \in \mathbb{R}^D$ that satisfy the following: for all $i \in \{1, 2 \ldots D\}$, it holds that $v_i$ is an integer and $0 \leq v_i \leq L$. Two vectors $v, v' \in \mathbb{L}_{D,L}$ are neighbours if $\|v - v'\| = 1$. Henceforth, the vectors belonging to $\mathbb{L}_{D,L}$ shall be referred to as sites.

For each site $v \in \mathbb{L}_{D,L}$, we associate a $d$-dimensional Hilbert space $\mathcal{H}_v$ and define the full Hilbert space as $\mathcal{H} = \bigotimes_{v \in \mathbb{L}_{D,L}} \mathcal{H}_v$. Local Hamiltonian system is conveniently represented using the notion of dual lattice. Let $\mathbb{L}_{D,L}$ be the set of vectors $w$ such that for all $i \in \{1, 2 \ldots D\}$, $0 < w_i < L$ and $w_i$ is a half integer (that is, $w_i = k + \frac{1}{2}$, for $k$ an integer). For a fixed $w \in \mathbb{L}_{D,L}$, let $\mathcal{S}(w)$ be the set of all sites $v \in \mathbb{L}_{D,L}$ such that: for all $i \in \{1, 2 \ldots D\}$, $v_i = w_i + \frac{1}{2}$. For a quantum state $\rho \in \mathcal{H}$, the reduced density matrix of $\rho$ on a set $T$ of sites is represented as $\rho_{T}$. We define the average energy of $\rho$ to be $\text{Tr}(\rho \mathcal{H})$, and represent it as $\langle \mathcal{H} \rangle_\rho$. For every local term $h_w$ let $\langle h_w \rangle_\rho \overset{\text{def}}{=} \text{Tr}(\rho_S h_w)$. Then we have

$$\langle \mathcal{H} \rangle_\rho = \sum_w \langle h_w \rangle_\rho.$$ 

A state $\rho \in \mathcal{H}$ satisfies $(C, l_0, \sigma)$-decay of correlation if for any two operators $A, B$ such that distance between $A, B$ is $1 \geq l_0$, it holds that

$$|\text{Tr}(\rho A \otimes B) - \text{Tr}(\rho A)\text{Tr}(\rho B)| \leq C\|A\|\|B\|e^{-\sigma}.$$ 

Define $\Pi_f$ to be the projector onto eigenspace of $\mathcal{H}$ with eigenvalue (energy) equal to $f$. Let $\Pi_{f<} (\Pi_{f>})$ be the projection onto the subspace which is union of eigenspaces of $\mathcal{H}$ with eigenvalues greater (less) than $f$. The following fact follows from Markov’s inequality.

**Fact 2.1.** For every $t, a > 0$ and $r$ even

$$\text{Tr}(\rho \Pi_{f>(\mathcal{H})_{\Pi} + a}) \leq \frac{\text{Tr}(\rho (H - \langle \mathcal{H} \rangle_\rho) \Pi_f)}{(a)^r}.$$ 

**Proof.** We have $(H - \langle \mathcal{H} \rangle_\rho) \Pi_f = \sum_r (f - \langle \mathcal{H} \rangle_\rho) \Pi_f$, which gives

---

**Figure 2.** A lattice $\mathbb{L}_{2,4}$ in dimension $D = 2$ with $L = 4$. Black dots represent the lattice sites. Blue dots represent the vectors of the dual lattice $\mathbb{L}_{2,4}$, which we call interactions. For the interaction $w$, the set $\mathcal{S}(w)$ is $\{v_1, v_2, v_3, v_4\}$. The distance between sites $v$ and $v_3$ is $|v - v_3| = 3$. The distance between interactions $w$ and $w'$ is $|w - w'| = 3$. 


Tr(ρ(H - ⟨H⟩)} = ρ Tr(ρ)}). This implies

\[ \text{Tr}(\rho(H - \langle H \rangle)') = \sum_{j}(f - \langle H \rangle)') \text{Tr}(\rho \Pi_j). \]

\[ \text{Tr}(\rho(H - \langle H \rangle)') \geq (a)^r \sum_{f>(\langle H \rangle) + a} \text{Tr}(\rho \Pi_j). \]

3. A combinatorial lemma

In this section, we shall prove a combinatorial lemma, which we shall use in section 4 to prove theorem 1.1. A slight variant of this lemma shall be proved in the appendix and used in section 5 to prove theorem 1.2. We recall the definition of the set \( \hat{\mathcal{L}}_{D,L} \) from section 2 and let the number of interactions in \( \hat{\mathcal{L}}_{D,L} \) be \( n \). It is easily seen that \( n = (L - 1)^D \).

Fix an integer \( l \). An ordered set \( \{w_1, w_2, \ldots, w_r\} \) of interactions in \( \hat{\mathcal{L}}_{D,L} \) is said to satisfy a property \( P(l) \) if the following holds: for all \( w_i \), there exists a \( w_j \) such that \( \|w_i - w_j\| \leq l \). Let the number of such ordered sets be \( N_D(n, r, l) \).

Rest of the section is devoted to the proof of following lemma.

Lemma 3.1. It holds that \( N_D(n, r, l) \leq (44l)^{Dn}r! \).

We start with the following definition that we shall extensively use.

Definition 3.2. A selection is an ordered set \( \{(b_1, x_1), (b_2, x_2), \ldots, (b_r, x_r)\} \), where \( b_i \in \{0, 1\} \) and \( x_i \in \hat{\mathcal{L}}_{D,L} \), that satisfies the following constraints:

1. If \( b_i = 0 \), then \( x_i \) can be any interaction in \( \hat{\mathcal{L}}_{D,L} \) and if \( b_i = 1 \), \( x_i \) has to satisfy \( \|x_i - x_j\| \leq 2 \cdot l \) for some \( j < i \).
2. Number of \( i \) for which \( b_i = 0 \) is at most \( \frac{r}{2} \).

We show the following lemma from which the proof of lemma 3.1 shall follow immediately.

Lemma 3.3. Every ordered set \( \{w_1, w_2, \ldots, w_r\} \) that satisfies property \( P(l) \) can be mapped to a selection in such a way that for any two distinct sets satisfying \( P(l) \), the corresponding selections are distinct.

Proof. We assign a selection to an ordered set \( \{v_1, v_2, \ldots, v_r\} \) satisfying \( P(l) \) using the algorithm below.

**Initialization**

- Set \( i = 1 \) and \( b_i = 0 \), \( x_i = v_i \).
- While \( i \leq r \), do:
  - \( i \leftarrow i + 1 \). Set \( x_i = v_i \), \( b_i = 0 \) if there is no \( j < i \) such that \( \|v_i - v_j\| \leq l \). Else set \( b_i = 1 \).
- End while.

**Pointer creation**

- Define a relation \( R : \{1, 2, \ldots, r\} \rightarrow \{0, 1, 2, \ldots, r\} \) as follows.
- Set \( i = 1 \). While \( i \leq r \), do:
  - If \( b_i = 1 \), set \( R(i) = 0 \). If \( b_i = 0 \), find the smallest \( j > i \) such that \( b_j = 1 \) and \( \|x_j - x_i\| \leq l \) (such a \( j \) exists due to property \( P(l) \)). Set \( R(i) = j \). Set \( i \leftarrow i + 1 \).
- End while.

**Update**

- Let \( S \) be the set of all subsets of \( \{1, 2, \ldots, r\} \) which have cardinality at least 2.
- For each element \( S \in S \), do:
  - Let \( s \) be the cardinality of \( s \) and \( i_1, i_2, \ldots, i_s \) be its elements arranged in increasing order. If it holds that \( b_{i_1} = b_{i_2} = \ldots b_{i_s} = 0 \) and \( R(i_1) = R(i_2) = \ldots R(i_s) > 0 \); set \( b_{i_1} = b_{i_2} = \ldots b_{i_s} = 1 \).
- End For.
We show that above algorithm terminates and assigns a selection to each ordered set satisfying property \( P(l) \).

1. Consider the running of algorithm during the step Initialization. Condition 1 of a selection holds: for every \( i \) for which there is a \( j < i \) such that \( \|x_i - x_j\| \leq l \), we have set \( b_i = 1 \). But we haven’t constructed a selection yet, since condition 2 may not be satisfied.

2. After the step Pointer creation, it may be possible that there exist indices \( i_1, i_2, \ldots, i_s \) (for some \( s < r \)) such that \( b_{i_1} = b_{i_2} = \ldots b_{i_s} = 0, R(i_1) = R(i_2) = \ldots R(i_s) > 0 \) and \( i_s < i_{s-1} \ldots < i_1 \). In this case, we find using triangle inequality that
   \[
   \|w_{i_1} - w_{i_2}\| \leq \|w_{i_1} - w_{R(i_1)}\| + \|w_{R(i_1)} - w_{i_2}\| = \|w_{i_1} - w_{R(i_1)}\| + \|w_{R(i_1)} - w_{i_2}\| \leq 2l.
   \]
   Similarly,
   \[
   \|w_{i_2} - w_{i_3}\| \leq 2l, \|w_{i_3} - w_{i_4}\| \leq 2l, \ldots \|w_{i_s} - w_{i_1}\| \leq 2l.
   \]
   Thus, the step Update sets \( b_{i_1} = b_{i_2} = \ldots b_{i_s} = 1 \), recognizing the fact that each of the points \( w_{i_1}, w_{i_2}, \ldots, w_{i_s} \)
   are at a lattice distance of at most \( 2l \) from \( w_{i_1} \). This ensures that condition 1 of selection is still satisfied.

3. After the step Update terminates, condition 2 of selection is now satisfied as well. We now have that for every \( i \) with \( b_i = 0 \), there is no other \( i' \) such that \( R(i) = R(i') \) and \( b_i = b_{i'} = 0 \). Furthermore, \( b_{R(i)} = 1 \). Thus, number of \( i \) with \( b_i = 0 \) is at most as large as the number of \( j \) with \( b_j = 1 \).

Lemma follows as two distinct ordered sets satisfying \( P(l) \) are not assigned the same selection. \( \square \)

Now we prove lemma 3.1.

**Proof of Lemma 3.1.** For \( r \leq r(4l)^D \), we clearly have \( N_0(n, r, l) \leq n^r \leq ((4l)^D nr)^r < (4l)^D nr) \).

So we assume \( n > r(4l)^D \). We bound the number of selections, which gives the desired upper bound on \( N_0(n, r, l) \) using lemma 3.3.

Consider those selections for which number of \( i \) such that \( b_i = 0 \) is \( u \). For each \( i \) with \( b_i = 0 \), number of possible choices of \( x_i \) is \( n \). For each \( i \) with \( b_i = 1 \), number of possible choices of \( x_i \) is at most \((4l)^D \) (as there are at most \((4l)^D \) points \( x_i \in \mathbb{L}_{D, L} \) that satisfy \( \|x_i - x_j\| \leq 2l \) for a given \( x_i \)). Hence total number of such selections is at most
\[
\binom{r}{u} n^u ((4l)^D r)^{r-u} \leq \sum_{u=0}^{\lfloor r/2 \rfloor} \binom{r}{u} n^u ((4l)^D r)^{r-u} < 2^r ((4l)^D nr)^r.
\]

This proves the lemma. \( \square \)

4. **Energy distribution of states that satisfy an exponential decay of correlation**

Consider a state \( \rho \) that satisfies \((C, l_0, \sigma)\)-decay of correlation and the Hamiltonian \( H = \sum_{w \in \mathbb{L}_{D, L}} h_w \). Let
\[
g_w \overset{\text{def}}{=} h_w - \text{Tr}(\rho h_w)I.
\]
We prove following bound on \( r \)th moment.

**Lemma 4.1.** Given the state \( \rho \) that satisfies \((C, l_0, \sigma)\)-decay of correlation, it holds that
\[
\text{Tr}(\rho (H - \langle H \rangle)^r) \leq C \sigma 4 \left( \frac{D \sigma}{2} \right)^D nr^{D+1}.
\]

**Proof.** Consider
\[
\text{Tr}\left( \rho \left( \sum_{w \in L_{D, L}} g_w \right)^r \right) = \sum_{w_1, w_2, \ldots, w_r} \text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_r}).
\]
(1)

For every ordered set \( \{w_1, w_2, \ldots, w_r\} \) define the quantity \( D(w_1, w_2, \ldots, w_r) \overset{\text{def}}{=} \max_{x \in \mathbb{L}_{D, L}} |w_i - w_j| \). This is the distance of farthest site from rest of the sites in the ordered set.

For an integer \( l > 0 \), define \( T(l) \) as the collection\(^3\) of all sets \( \{w_1, w_2, \ldots, w_r\} \) that satisfy \( D(w_1, w_2, \ldots, w_r) = l \). Now, fix a set \( \{w_1, w_2, \ldots, w_r\} \in T(l) \). Without loss of generality, suppose that \( w_1 \) is a site at the distance \( l \) from rest of the sites. Then from \((C, \sigma, l_0)\)-decay of correlation and the relation \( \text{Tr}(\rho g_{w_1}) = 0 \), it holds that

\(^3\) This is a very crude upper bound and can be found as follows. The number of non-negative integers \( \{a_0, a_1, \ldots, a_l\} \) such that \( \sum a_i \leq 2l \) is at most \((2l)^l \). Thus, number of integers \( \{a_0, a_1, \ldots, a_l\} \) such that \( \sum |a_i| \leq 2l \) is at most \( 2^{l+1} (2l)^l \).

\(^4\) To avoid confusion, we call \( T(l) \) a ‘collection’ instead of a ‘set’.
\[ \text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_n}) \leq \text{Tr}(\rho g_{w_1}) \cdot \text{Tr}(\rho g_{w_2} \cdots g_{w_n}) + Ce^{-\frac{1}{c}} = Ce^{-\frac{1}{c}}. \]

Now, the number of sets in the collection \( \mathcal{I}(l) \) is at most \( N_D(n, r, l) \), which is upper bounded by \((4(4l)^Dnr)^{\frac{r}{r+1}}\) (lemma 3.1). Thus we have

\[
\sum_{w_1, w_2, \ldots, w_n} \text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_n}) = \sum_{l=0}^{\infty} \sum_{(w_1, w_2, \ldots, w_n) \in \mathcal{I}(l)} \text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_n}) = \sum_{l=0}^{\infty} \sum_{(w_1, w_2, \ldots, w_n) \in \mathcal{I}(l)} \text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_n})
\]

\[
\leq N_D(n, r, l_0) + \sum_{l=l_0}^{\infty} N_D(n, r, l) \cdot Ce^{-\frac{1}{c}}
\]

\[
= (4(4l_0)^Dnr)^{\frac{r}{r+1}} + C\sum_{l=1}^{\infty} (4(4l)^Dnr)^{\frac{r}{r+1}}e^{-\frac{1}{c}}
\]

\[
\leq (4(4l_0)^Dnr)^{\frac{r}{r+1}} + C(4nr)^{\frac{r}{r+1}}\sum_{l=1}^{\infty} e^{-\frac{1}{c}}.
\]

Now, we evaluate

\[
\sum_{l=1}^{\infty} l^{\frac{r}{r+1}}e^{-\frac{1}{c}} \leq \int_0^\infty l^{\frac{r}{r+1}}e^{-\frac{1}{c}}dl = \frac{\sigma^{\frac{r+1}{r}}}{\sigma^{\frac{r+1}{r}+1}} \int_0^\infty l^{\frac{r+1}{r}}\sigma e^{-\frac{1}{c}}ds \leq \frac{\sigma^{\frac{r+1}{r}}}{\sigma^{\frac{r+1}{r}+1}} \left( \frac{rD}{2} \right)^{\frac{r}{r+1}}.
\]

Using this in equation (2), we obtain

\[
\sum_{w_1, w_2, \ldots, w_n} \text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_n}) \leq (4(4l_0)^Dnr)^{\frac{r}{r+1}} + C\sigma \left( \frac{D\sigma}{2} \right)^{\frac{r}{r+1}}nr^{D+1}.
\]

Now we proceed to state theorem 1.1 formally and provide its proof.

**Theorem 4.2.** Consider a quantum state \( \rho \) that satisfies \((C, l_0, \sigma, \sigma)-\)decay of correlation and has average energy \( \langle H \rangle_\rho \). For \( \sqrt{\frac{8\epsilon(4l_0)^{D+1}}{nD\sigma}} > a \geq \sqrt{\frac{8\epsilon(4l_0)^D}{n}} \) (if the range exists) it holds that

\[
\text{Tr}(\rho \Pi_{\langle H \rangle_\rho + na}) \leq (1 + C\sigma)e^{-\frac{na^2}{4k^2\sigma^2}} \text{ and } \text{Tr}(\rho \Pi_{\langle H \rangle_\rho - na}) \leq (1 + C\sigma)e^{-\frac{na^2}{4k^2\sigma^2}}.
\]

For \( a \geq \sqrt{\frac{(4l_0)^{D+1}}{Dn\sigma}} \) it holds that

\[
\text{Tr}(\rho \Pi_{\langle H \rangle_\rho + na}) \leq (1 + C\sigma)e^{-\frac{na^2}{8k^2\sigma^2}} \text{ and } \text{Tr}(\rho \Pi_{\langle H \rangle_\rho - na}) \leq (1 + C\sigma)e^{-\frac{na^2}{8k^2\sigma^2}}.
\]

**Proof.** Using fact 2.1 and lemma 4.1 we have

\[
\text{Tr}(\rho \Pi_{\langle H \rangle_\rho + na}) \leq \left( \frac{4(4l_0)^D}{na^2} \right)^{\frac{r}{r+1}} + C\sigma \left( \frac{D\sigma}{2} \right)^{\frac{r}{r+1}}nr^{D+1}.
\]

Consider the following two cases.

- \( a \geq \sqrt{\frac{(4l_0)^{D+1}}{Dn\sigma}}, \) or equivalently \( \frac{(4l_0)^{D+1}}{Dn\sigma} \) \(< 1.\)

Then we set \( r = 2 \left( \frac{na^2}{8k^2\sigma^2} \right)^{\frac{1}{r+1}} \), where \([\cdot]\) denotes the ceiling operation (rounding to the nearest larger integer) to obtain
In this section, we introduce the physical setup for theorem 1.2 and also provide its proof. We shall continue

5. Energy distribution of a product state

In this section, we introduce the physical set-up for theorem 1.2 and also provide its proof. We shall continue using the notations $H$ and $h$ for the Hamiltonian and its local term, as this notation is restricted only to this section.

Consider a collection $C$ of spins, such that a $d$-dimensional Hilbert space $\mathcal{H}_d$ is associated to each spin $s \in C$. Let full Hilbert space $\mathcal{H}$ be defined as $\mathcal{H} = \otimes C \mathcal{H}_d$. For an integer $k > 0$, let $S_k$ be the set of all subsets of $C$ of size at most $k$. For an integer $m > 0$, let $W_{k,m}$ be a subset of $S_k$ defined as follows (note that $W_{k,m}$ is also a set of subsets of $C$): for each $w \in W_{k,m}$ the number of $w' \in W_{k,m}$ such that $|w' \cap w| > 0$ is at most $m$. For each $w \in W_{k,m}$, let $\mathcal{N}(w)$ be the set of all $w' \in W_{k,m}$ such that $|w' \cap w| > 0$. Elements of $\mathcal{N}(w)$ shall be referred to as neighbours of $w$. The set-up has been depicted in figure 3.

Let the Hamiltonian $H$ be defined as:

$$H = \sum_{w \in W_{k,m}} h_w,$$

where $h_w$ acts non-trivially only on spins in $w$ and acts trivially on rest of the spins. Further, we assume that $\|h_w\|_\infty \leq 1$.

The definition of $W_{k,m}$ thus translates to the assumption that:

1. Each `local term' $h_w$ acts non-trivially on at most $k$ particle, and hence is $k$-local.

2. For each $h_w$, the number of $h_{w'}$ such that the supports of $h_w$ and $h_{w'}$ intersect, is at most $m$.

Let $\rho \in \mathcal{H}$ be a product state, that is, $\rho = \Pi_{s \in C} \rho_s$ and support of each $\rho_s$ is exactly the spin $s$. Let the reduced density matrix of $\rho$ on a subset $T \subseteq C$ of spins be denoted in the usual way as $\rho_T$.

We bound the moment function $\text{Tr}(\rho (H - \langle H \rangle_\rho)^r)$ for an even $r$ to be chosen later and use it to prove theorem 1.2. Define $g_w \overset{\text{def}}{=} h_w - \langle h_w \rangle_\rho$.

$$\text{Tr}(\rho \Pi_0 (H)_{\rho} + \Pi_0) \leq \left( \frac{4(D_0)^D r}{n a^2} \right)^{\frac{r}{2}} + C \sigma \left( \frac{4(D_0)^D r^{D+1}}{n a^2} \right)^{\frac{r}{2}}$$

$$\leq \left( \frac{1}{e} \left( \frac{4(D_0)^{D+1}}{D n a^2} \right)^{\frac{r}{2}} \right)^{\frac{r}{2}} + C \sigma \left( \frac{1}{e} \right)^{\frac{r}{2}}$$

$$\leq (1 + C \sigma) e^{-\frac{r}{2} n a^2}.$$

The last inequality follows from the assumption: $\frac{4(D_0)^{D+1}}{D n a^2} < 1$.

- $a \leq \frac{8 e (4D)^{D+1}}{n D a^2}$ and $a \geq \frac{8 e (4D)^2}{n}$, or equivalently $\frac{n a^2}{8 e (4D)^2} \geq 1$ and $\frac{D n a^2}{8 e (4D)^{D+1}} < 1$.

We set $r = 2 \frac{n a^2}{8 e (4D)^2}$ to obtain

$$\text{Tr}(\rho \Pi_0 (H)_{\rho} + \Pi_0) \leq \left( \frac{4(D_0)^D r}{n a^2} \right)^{\frac{r}{2}} + C \sigma \left( \frac{4(D_0)^D r^{D+1}}{n a^2} \right)^{\frac{r}{2}}$$

$$\leq \left( \frac{1}{e} \right)^{\frac{r}{2}} + C \sigma \left( \frac{1}{8 e (4D)^{D+1}} \right)^{\frac{r}{2}}$$

$$\leq (1 + C \sigma) e^{-\frac{r}{2} n a^2}.$$

Last inequality follows from the assumption: $\frac{D n a^2}{8 e (4D)^{D+1}} < 1$.

For second part of the theorem, consider the Hamiltonian $H' = \sum_{w \in W_{k,m}} 1 - h_w$.

Define $\langle H' \rangle_\rho \overset{\text{def}}{=} \text{Tr}(\rho H') = n - \langle H \rangle_\rho$. Let $\Pi_0$ be the projector onto subspace with eigenvalues of $H'$ larger than $f$. Same analysis as above for $H'$ in place of $H$, along with the relation $\Pi_{\geq f} = \Pi_{< n - f}$ completes the proof. \qed

5. Energy distribution of a product state

For each $w \in W_{k,m}$ be a subset of $S_k$ defined as follows (note that $W_{k,m}$ is also a set of subsets of $C$): for each $w \in W_{k,m}$ the number of $w' \in W_{k,m}$ such that $|w' \cap w| > 0$ is at most $m$. For each $w \in W_{k,m}$, let $\mathcal{N}(w)$ be the set of all $w' \in W_{k,m}$ such that $|w' \cap w| > 0$. Elements of $\mathcal{N}(w)$ shall be referred to as neighbours of $w$. The set-up has been depicted in figure 3.

Let the Hamiltonian $H$ be defined as:

$$H = \sum_{w \in W_{k,m}} h_w,$$

where $h_w$ acts non-trivially only on spins in $w$ and acts trivially on rest of the spins. Further, we assume that $\|h_w\|_\infty \leq 1$.

The definition of $W_{k,m}$ thus translates to the assumption that:

1. Each `local term' $h_w$ acts non-trivially on at most $k$ particle, and hence is $k$-local.

2. For each $h_w$, the number of $h_{w'}$ such that the supports of $h_w$ and $h_{w'}$ intersect, is at most $m$.

Let $\rho \in \mathcal{H}$ be a product state, that is, $\rho = \Pi_{s \in C} \rho_s$ and support of each $\rho_s$ is exactly the spin $s$. Let the reduced density matrix of $\rho$ on a subset $T \subseteq C$ of spins be denoted in the usual way as $\rho_T$.

We bound the moment function $\text{Tr}(\rho (H - \langle H \rangle_\rho)^r)$ for an even $r$ to be chosen later and use it to prove theorem 1.2. Define $g_w \overset{\text{def}}{=} h_w - \langle h_w \rangle_\rho$. \qed
We shall prove the following lemma.

**Lemma 5.1.** Let \( n \) be the number of local terms. Given the product state \( \rho = \prod_{I \in \mathcal{P}} \rho_I \) it holds that

\[
\text{Tr} \left( \rho \left( \sum_{w \in \mathcal{W}_{k,m}} g_w \right) \right) \leq (4m^2nr)^2.
\]

**Proof.** Consider

\[
\text{Tr} \left( \rho \left( \sum_{w \in \mathcal{W}_{k,m}} g_w \right) \right) = \sum_{w_1, w_2 \ldots w_r} \text{Tr} (\rho g_{w_1} g_{w_2} \cdots g_{w_r}).
\]

Using \( \text{Tr}(\rho g_w) = \text{Tr}(g_w \rho) = 0 \), we observe that the term \( \text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_r}) \) is non-zero only if the ordered set \( \{ w_1, w_2 \ldots w_r \} \) satisfies the following property \( Q \): for every \( w_i \), there exists a \( w_j \) such that \( |w_i \cap w_j| > 0 \). In other words, there is a \( w_i \in \mathcal{W}_{k,m} \) such that \( w_i \in \mathcal{N}(w_j) \). Let number of ordered sets \( \{ w_1, w_2 \ldots w_r \} \) that satisfy above property be \( N_{k,m}(n, r) \). This gives us

\[
\text{Tr} \left( \rho \left( \sum_{w \in \mathcal{W}_{k,m}} g_w \right) \right) \leq N_{k,m}(n, r) \max_{w_1, w_2 \ldots w_r} |\text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_r})|.
\]

Setting the trivial bound \( \max_{w_1, w_2 \ldots w_r} |\text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_r})| \leq 1 \), and using lemma 5.2 below, the proof follows.  

**Remark.** For the case of translationally invariant systems, where \( h_w = h \) for all \( w \), the bound \( \max_{w_1, w_2 \ldots w_r} |\text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_r})| \leq 1 \) can be improved to \( \max_{w_1, w_2 \ldots w_r} |\text{Tr}(\rho g_{w_1} g_{w_2} \cdots g_{w_r})| \leq (1 - \langle h \rangle)^r \). This gives a minor improvement over the statement of lemma 5.1.

Now we prove an upper bound on the quantity \( N_{k,m}(n, r) \) in the following lemma. Proof is very similar to lemma 3.1 and is deferred to appendix for completeness.

**Lemma 5.2.** It holds that \( N_{k,m}(n, r) \leq (4m^2nr)^2 \).

We restate theorem 1.2 formally and give its proof below.

**Theorem 5.3.** Given the product state \( \rho = \prod_{I \in \mathcal{P}} \rho_I \) with average energy \( \langle H \rangle \), consider a real number \( a \geq \sqrt{\frac{2m^2}{\pi}} \). It holds that

\[
\text{Tr}(\rho \Pi_{\geq \langle H \rangle^2 + na}) \leq e^{-\frac{na}{2m^2}}
\]
Proof. Lemma 5.1 gives the following upper bound on the r-th moment:

\[ \text{Tr}(\rho (H - \langle H \rangle)^r) \leq (4m^2nr)^\frac{r}{2}. \]  

Using fact 2.1, we have

\[ \text{Tr}(\rho \Pi_{\geq(na)^r}) \leq \frac{(4m^2nr)^\frac{r}{2}}{(na)^{r}} = \left(\frac{4m^2r}{na^2}\right)^\frac{r}{2}. \]

Choosing \( r = 2 \left\lceil \frac{na^2}{8m^2} \right\rceil \), we obtain for \( a \geq \sqrt{\frac{8m^2e}{n}} \)

\[ \text{Tr}(\rho \Pi_{\geq(na)^r}) \leq \left(\frac{8m^2 \cdot na^2}{8m^2 \cdot na^2}\right)^\frac{r}{2} \leq e^{-\frac{na^2}{8m^2}}. \]

For the second part of the theorem, consider the Hamiltonian

\[ H' = \sum_{w \in W_{\text{loc}}} I - h_w. \]

Define \( (H')_{\rho} \overset{\text{def}}{=} \text{Tr}(\rho H') = n - \langle H \rangle \). Let \( \Pi'_{\geq f} \) be the projector onto subspace with eigenvalues of \( H' \) larger than \( f \). Same analysis as above for \( H' \) in place of \( H \) gives

\[ \text{Tr}(\rho \Pi'_{\geq(na)^r}) \leq e^{-\frac{na^2}{4m^2}}. \]

This completes the proof since \( \Pi'_{\geq f} = \Pi_{\leq n-f} \).

5.1. Restatement of theorem 5.3 in terms of number of spins

We introduce a new parameter that captures the number of local terms that act on any given spin. Define

\[ g_s \overset{\text{def}}{=} \sum_{w \in W_{\text{loc}}, w \ni s} 1, \quad \text{and} \quad g \overset{\text{def}}{=} \max_s g_s, \]

where \( g_s \) is the maximum number of local terms that act non-trivially on spin \( s \).

Now, we prove corollary 1.3. Its formal statement is as follows, where we also assume that each local term is exactly \( k \)-local.

Corollary 5.4. Let the Hamiltonian \( H \) be such that each term \( h_w \) has locality equal to \( k \). Let \( N \overset{\text{def}}{=} |\mathcal{C}| \) be the number of spins. Given the product state \( \rho = \prod_{s \in \mathcal{C}} P_s \) with average energy \( \langle H \rangle \), consider a real number \( \varepsilon \geq \sqrt{8eg^3kN} \). It holds that

\[ \text{Tr}(\rho \Pi_{\geq(na)^r}) \leq e^{-\frac{\varepsilon^2}{6eg^3N}} \]

and

\[ \text{Tr}(\rho \Pi_{\leq(na)^r}) \leq e^{-\frac{\varepsilon^2}{6eg^3N}}. \]

Proof. We set \( \varepsilon \overset{\text{def}}{=} na \) as the energy with respect to \( H \). Then the bound in theorem 5.3 can be restated as:

\[ \text{Tr}(\rho \Pi_{\geq(na)^r}) \leq e^{-\frac{\varepsilon^2}{6eg^3N}}. \]

Relation between \( N \) and \( n \) can be computed as follows. To each local term \( h_w \), one can associate exactly \( k \) spins on which \( h_w \) acts non-trivially. On the other hand, to each spin \( s \), one can associate at most \( g \) local terms that contain \( s \) in their support. From the first argument, the number of associations is exactly \( k \cdot n \), whereas from the second argument, the number of associations is at most \( g \cdot N \). Thus, \( g \cdot N \geq k \cdot n \) which implies \( n \leq \frac{N}{k} \). Also, \( m \leq k \cdot g \), since each local term is supported on \( k \) spins, and each of these spins are in the support of at most \( g \) other local terms Collectively we obtain \( mn^2 \leq Ng^3k \) and our bound takes the form:

\[ \text{Tr}(\rho \Pi_{\geq(na)^r}) \leq \exp \left( -\frac{\varepsilon^2}{4eg^3kN} \right). \]

This completes the proof.

Above upper bound may be compared to theorem 7 in [Kuw16]. In this reference, the notion of \( g' \)-extensitivity has been introduced (definition 2, [Kuw16]), which is analogous to the locality parameter \( g \).
defined above. It is defined as follows: a local Hamiltonian $H$ is $g'$-extensive if for every spin $s$, we have $\sum_{w \in W, \omega \leq s} \|h_w\| \leq g'$. Using this, the following theorem has been shown in [Kuw16]:

**Theorem 5.5 (Informal version of theorem 7, [Kuw16]).** Given a $g'$-extensive local Hamiltonian with locality $k$, it holds that

$$\text{Tr}(\rho H_{s}(H_{s}+\varepsilon)) \leq O(1) \exp \left( -\frac{\varepsilon^2}{cN \log \left( \frac{\varepsilon}{\sqrt{N}} \right)} \right),$$

where $c$ is an $O(1)$ constant that depends only on $k$, $g'$.

We observe that equation (6) achieves a marginally better bound whenever the norm of each local term $h_w$, that is $\|h_w\|$, is a constant independent of $w$. In such a case, $g'$ and $g$ are same up to the norm of local terms. If the normalizations of each local term are different, it is not clear how $g$, $g'$ are related to each other. In such a case, equation (6) and theorem 5.5 may be viewed as complementary results.

### 6. Conclusion

We have shown upper bounds on tail of energy distribution of states that satisfy exponential decay of correlation and product states, with respect to a local Hamiltonian. Main technical tool we use is a combinatorial lemma that gives a non-trivial upper bound on the moments of the energy distribution. The results may have applications in the study of thermalization of many body quantum systems and also for many body localization, as noted in the Introduction. Main questions that we leave open are connected to tightness of our bounds, as we discuss below.

The bounds presented in theorem 1.2 can only be improved up to constants, since classical Chernoff bound also exhibits a Gaussian decay, which is known to be tight. More interesting situation occurs with the bounds presented in theorem 1.1. In one-dimensional spin chain, our bound decays exponential with the energy. For the ground states of gapped local hamiltonians, this is very similar to the behaviour noted in [KAAV15] (section 5) using completely different techniques. This suggests that the ground states of gapped local hamiltonians (such as the ground state of Transverse field Ising model, which is exactly solvable) are strong candidates for the study of tightness of above results. Our result for higher dimensions appears to be much weaker that those obtained in [KAAV15] (section 5) for gapped ground states, and we expect further improvement using better combinatorial arguments.

An another interesting question is with respect to matrix product states (with constant bond dimension) which are defined on one-dimensional spin chain. It is well known that under reasonable assumptions (see section 5.1.1, [Oru14]) Matrix product states satisfy exponential decay of correlation. Furthermore, it has already been shown in [Oga10] that given a Matrix product state $\rho$, if $n$ is large enough and energy $\varepsilon \approx O(n)$, it holds that $\text{Tr}(\rho H_{s}(H_{s}+\varepsilon)) \leq e^{-\varepsilon O(n)}$. It is a strong indication that our bound (which applies for all energies $\varepsilon > O(\sqrt{n})$) may be considerably improved for this special, but important, class of states.

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### Appendix. Proof of lemma 5.2

We repeat most of the proof of lemma 3.1, making changes wherever necessary.

We start with the following definition.

**Definition A.1.** A selection is an ordered set $\{(b_1, x_1), (b_2, x_2) \ldots (b_r, x_r)\}$, where $b_i \in \{0, 1\}$ and $x_i \in W_{k,m}$, that satisfies the following constraints:
(1) If \( b_i = 0 \), then \( x_i \) can be any element of \( W_{\mathbb{R}_i} \) and if \( b_i = 1 \), \( x_i \) has to satisfy \( |\mathcal{N}(x_j) \cap \mathcal{N}(x_i)| > 0 \) for some \( j < i \).

(2) Number of \( i \) for which \( b_i = 0 \) is at most \( \frac{\ell}{2} \).

We show the following lemma, from which the proof of lemma 3.1 follows easily.

**Lemma A.2.** Every ordered set \( \{w_1, w_2 \ldots w_r\} \) that satisfies property \( Q \) can be mapped to a selection in such a way that for any two distinct ordered sets satisfying \( Q \), the corresponding selections are distinct.

**Proof.** We assign a selection to an ordered set \( \{w_1, w_2 \ldots w_r\} \) satisfying \( Q \) using the algorithm below.

**Initialization**

- Set \( i = 1 \) and \( b_i = 0 \), \( x_i = w_i \).
- While \( (i \leq r) \), do:
  - \( i \rightarrow i + 1 \). Set \( x_i = w_i \). Set \( b_i = 0 \) if there is no \( j < i \) such that \( w_i \in \mathcal{N}(w_j) \). Else set \( b_i = 1 \).
- End while.

**Pointer creation**

- Define a relation \( R: \{1, 2 \ldots r\} \rightarrow \{0, 1, 2 \ldots r\} \) as follows.
- Set \( i = 1 \). While \( (i \leq r) \), do:
  - If \( b_j = 1 \), set \( R(i) = 0 \). If \( b_j = 0 \), find the smallest \( j > i \) such that \( b_j = 1 \) and \( x_i \in \mathcal{N}(x_j) \) (such a \( j \) exists due to property \( Q \)). Set \( R(i) = j \). Set \( i \rightarrow i + 1 \).
- End while.

**Update**

- Let \( S \) be the set of all subsets of \( \{1, 2 \ldots r\} \) which have cardinality at least 2.
- For each element \( S \in S \), do:
  - Let \( s \) be the cardinality of \( S \) and \( i_0, i_2 \ldots i_s \) be its elements arranged in increasing order. If it holds that \( b_{i_0} = b_{i_1} = \ldots b_{i_s} = 0 \) and \( R(i_0) = R(i_2) = \ldots R(i_s) > 0 \); set \( b_{i_0} = b_{i_1} = \ldots b_{i_s} = 1 \.
- End For.

We show that above algorithm terminates and assigns a selection to each ordered set satisfying property \( Q \).

(1) Consider the running of algorithm during the step Initialization. Condition 1 of a selection holds: for every \( i \) for which there is a \( j < i \) such that \( x_i \in \mathcal{N}(x_j) \), we have set \( b_i = 1 \). But we have not constructed a selection yet, since condition 2 may not be satisfied.

(2) After the step Pointer creation, it may be possible that there exist indices \( i_0, i_2 \ldots i_s \) (for some \( s \leq r \)) such that \( b_{i_0} = b_{i_1} = \ldots b_{i_s} = 0 \), \( R(i_0) = R(i_2) = \ldots R(i_s) > 0 \) and \( i_0 > i_{s-1} > \ldots > i_s \). In this case, we find that \( w_{i_0} \in \mathcal{N}(w_{i_0}), w_{i_s} \in \mathcal{N}(w_{i_s}) \). But \( \mathcal{N}(w_{i_0}) = \mathcal{N}(w_{i_s}) \), which implies that \( |\mathcal{N}(i_0) \cap \mathcal{N}(i_s)| > 0 \). Similarly, \( |\mathcal{N}(i_0) \cap \mathcal{N}(i_2)| > 0, \ldots |\mathcal{N}(i_0) \cap \mathcal{N}(i_s)| > 0 \). Thus, the step Update sets \( b_{i_0} = b_{i_1} = \ldots b_{i_s} = 1 \), recognizing the fact that each of the points \( w_{i_0}, w_{i_1} \ldots w_{i_s} \) satisfy the property that the neighbourhood of each of them intersects with \( \mathcal{N}(w_{i_0}) \). This ensures that condition 1 of selection is still satisfied.

(3) After the step Update terminates, condition 2 of selection is now satisfied as well. We now have that for every \( i \) with \( b_i = 0 \), there is no other \( i' \) such that \( R(i) = R(i') \) and \( b_i = b_i' = 0 \). Furthermore, \( b_R(i) = 1 \). Thus, number of \( i \) with \( b_i = 0 \) is at most as large as the number of \( j \) with \( b_j = 1 \).

Lemma follows as two distinct ordered tuples satisfying \( Q \) are not assigned the same selection. □

Now we prove lemma 5.2.
Proof of lemma 5.2. For $n \leq m^r$, we clearly have $N_{k,m}(n, r) \leq n^r \leq (m^r)^r < (4m^r)^r$. So we assume $n > m^r$. We bound the number of selections, which gives the desired upper bound on $N_{k,m}(n, r)$ using lemma 3.3.

Consider the collection of those selections, for which number of $i$ such that $b_i = 0$ is $u$. For each $i$ with $b_i = 0$, number of possible choices of $x_i$ is $n$. For each $i$ with $b_i = 1$, number of possible choices of $x_i$ is at most $m^r$ (as there are at most $m^r$ number of $x_i \in \mathcal{W}_m$ that satisfy $|\mathcal{N}(x_i) \cap \mathcal{N}(x_i)| > 0$ for a given $x_i$). Hence total number of such selections is at most $r\left(\begin{array}{c} r \\\ u \end{array}\right) n^u (m^r)^{r-u}$. Since $u \leq \frac{r}{2}$, total number of selections is at most

$$\sum_{u=0}^{r/2} r\left(\begin{array}{c} r \\\ u \end{array}\right) n^u (m^r)^{r-u} \leq \sum_{u=0}^{r/2} r\left(\begin{array}{c} r \\\ u \end{array}\right) n^r (m^r)^{r} < 2^r (m^r)^r.$$

This proves the lemma. □

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