UNIVERSITY IN FERMIONIC FIELD THEORIES AT FINITE TEMPERATURE

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We discuss the critical properties of the three-dimensional NJL model at nonzero temperature. We show that the $Z_2$-symmetric model undergoes a second order phase transition with $2d$ Ising exponents and its critical region is suppressed by a factor $1/\sqrt{N_f}$. We also provide numerical evidence that the $U(1)$-symmetric model undergoes a BKT transition in accordance with the dimensional reduction scenario.

Phase transitions in QCD at nonzero temperature have been studied intensively over the last decade. However, since the problem of chiral symmetry breaking and its restoration is intrinsically non-perturbative, the number of available techniques is limited and most of our knowledge about this phenomenon comes from lattice simulations.

Although there is little disagreement that the chiral phase transition in QCD with two massless quarks is second order, no quantitative work or simulations have been done that decisively determine its universality class. Universality arguments are very appealing due to their beauty and simplicity and they can be phrased as follows: At finite $T$ phase transitions, the correlation length diverges in the transition region and the long range behavior is that of the $(d-1)$ classical spin model with the same symmetry, because the IR region of the system is dominated by the zero mode of the bosonic field. The contribution of non-zero modes does not affect the critical singularities but can be absorbed into non-universal aspects of the transition. Consequently, fermions which satisfy anti-periodic boundary conditions and do not have zero modes are expected to decouple from the scalar sector. A possible loophole to this standard scenario is that the mesons are composite $f\bar{f}$ states and their size and density increase as $T \to$

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Therefore, if the transition region can be described as a system of highly overlapping composites the violation of the bosonic character of the mesons may be maximal and the fermions become essential degrees of freedom irrespective of how heavy they are.

The NJL model has been proved to be an interesting and tractable laboratory to study phase transitions both numerically by means of lattice simulations and analytically in the form of large-\(N_f\) expansions. The lagrangian density of the \(U(1)\)-symmetric model is

\[
\mathcal{L} = \bar{\psi}_i (\partial \psi + \sigma + i\gamma_5 \pi) \psi_i + \frac{N_f}{2g^2} (\sigma^2 + \pi^2),
\]

where the index \(i\) runs over \(N_f\) fermion flavors. There are several features which make this model interesting for the modelling of strong interactions: (i) The spectrum of excitations contains both “baryons” and mesons, namely the elementary fermions \(f\) and the composite \(f\bar{f}\) states; (ii) for sufficiently strong coupling \(g^2 > g_c^2\) it exhibits spontaneous chiral symmetry breaking implying dynamical generation of a fermion mass; (iii) for \(2 < d < 4\) there is an interacting continuum limit at a critical coupling, which for \(d = 3\) has a numerical value \(g_c^2 \approx 1.0 / a\) in the large-\(N_f\) limit if a lattice regularization is employed. At leading order in \(1/N_f\) the model undergoes a second order transition at \(T_c = M_f^2 \ln 2 / 3\) (\(M_f\) is the dynamical fermion mass at \(T = 0\)) with Landau-Ginzburg mean field scaling. This result is valid for both continuum and discrete chiral symmetries.

We studied the critical behavior of the \(Z_2\)-symmetric model with finite \(N_f\) by performing lattice simulations near \(T_c\). We simulated the model exactly for \(N_f = 12\) with the Hybrid Monte Carlo method. The staggered fermion lattice action and further details concerning the algorithm can be found in. The temporal lattice size was \(L_t = 6\) and the spatial size varied from \(L_s = 18\) to 50. The expectation value of the auxiliary sigma field serves as a convenient order parameter for the theory’s critical point. By using finite size scaling (FSS) techniques we extracted the critical exponents. The results which are summarized in Table 1 support the dimensional reduction scenario, because the values of the exponents are in good agreement with those of the 2d Ising model rather than the mean field theory ones. This implies that the role of composite mesons is relevant near \(T_c\) even when \(N_f\) is large but finite. Simulations with the meron-cluster algorithm of the 4d NJL model gave 3d Ising exponents with very high accuracy.

Next we tried to understand how the large-\(N_f\) mean field theory prediction reconciles with the dimensional reduction and universality arguments. The answer is that the large-\(N_f\) description has its applicability region. We
studied this in detail for a Yukawa theory and we showed that the phenomenon which leads to an apparent contradiction is the suppression of the width of the non-mean field critical region by a power of $1/N_f$. The 2d Ising critical behavior sets in when $T \gg M_\sigma(T)$ ($M_\sigma(T)$ is the thermal mass of the $\sigma$ meson). If the cutoff $\Lambda \gg T$ the renormalized self-interaction coupling $\lambda(T)$ in the large-$N_f$ limit is close to the IR fixed point of the Yukawa theory and is given by $\lambda(T) \sim T^{4-d}/N_f$ for $2 < d < 4$. The mean field approximation breaks down because of self-inconsistency when the value of the coupling of the $(d-1)$ dimensional scalar theory $\lambda_{d-1} \sim T\lambda(T)$ on the scale $M_\sigma^{3-d}(T)$ (the power $d-5$ comes from dimensional analysis) is not small anymore. Therefore, for $d = 3$ the Ginzburg criterion for the applicability of the mean field scaling is given by $M_\sigma(T) \gg T/\sqrt{N_f}$. This scenario was also verified numerically by measuring $\langle \sigma \rangle = M_f(T) = M_\sigma(T)/2$ at the crossover into the mean field region.

Additional evidence in favor of the dimensional reduction scenario was produced in studies of the $U(1)$-symmetric NJL$_3$ model. Both analytical and numerical results showed that its phase structure at $T \neq 0$ is the same as that of the 2d $XY$ model. As we already mentioned, at leading order in $1/N_f$ this model undergoes a second order chiral phase transition. This conclusion, however, is expected to be valid only when $N_f$ is strictly infinite, i.e. when the fluctuations of the bosonic fields are neglected, otherwise it runs foul of the Coleman-Mermin-Wagner (CMW) theorem, which states that in two-dimensional systems the continuous chiral symmetry must be manifest for all $T > 0$. Next-to-leading order calculations demonstrated that the model undergoes a Berezinskii-Kosterlitz-Thouless (BKT) transition, in accordance with the dimensional reduction scenario. It is easy to visualize the BKT scenario if we use the “modulus-phase” parametrization: $\sigma + i\pi \equiv \rho e^{i\theta}$. In two spatial dimensions logarithmically divergent infrared fluctuations do not allow the phase $\theta$ to take a fixed direction and therefore prevent spontaneous symmetry breaking via $\langle \theta \rangle \neq 0$. The critical temperature $T_{\text{BKT}}$ is expected to separate two different chiral symmetry phases: a low $T$ phase, which is characterized by power law phase correlations $\langle e^{i\theta(x)}e^{-i\theta(0)} \rangle \sim x^{-\eta(T)}$ at distances $x \gg 1/T$ but

| Exponents | FSS | $Z_2$ | MF |
|-----------|-----|------|----|
| $\nu$     | 1.00(3) | 1    | 0.5 |
| $\beta_{m}/\nu$ | 0.12(6) | 0.125 | 1   |
| $\gamma/\nu$ | 1.66(9) | 1.75 | 2   |
no long range order (i.e. a spinwave phase where chiral symmetry is “almost but not quite broken”), and a high $T$ phase which is characterized by exponentially decaying phase correlations with no long range order. In other words for $T \leq T_{\text{BKT}}$ there is a line of critical points characterised by $\eta(T) = \frac{f(T)}{T^{\frac{9}{2}}}$ for $T \ll T_{\text{BKT}}$ and at $T = T_{\text{BKT}}$ $\eta(T)$ is expected to be 0.25.

In order to study the behavior of the chiral symmetry at $T > 0$ in the absence of a fermion bare mass in the Lagrangian the best thing to measure is an effective “order parameter” $|\Phi| \equiv \sqrt{\sigma^2 + \pi^2}$, which is a projection onto the direction of $\Phi^\alpha \equiv (\sigma, \pi)$ separately for each configuration. In Fig.1 we plot $|\Phi|$ versus $\beta \equiv 1/g^2$ for fixed lattice temporal size $L_t = 4$ and different lattice spatial sizes together with the results from simulations of the model with a $Z_2$ chiral symmetry. In both cases $N_f = 4$. At the bulk critical coupling $\beta_{\text{bulk}} \equiv 1/g^2 = 0.86$ the lattice spacing becomes zero and $T \to \infty$.

It is clear that the order parameter of the $Z_2$ model is independent of the lattice size until just before the transition at $\beta_{Z_2}^c = 0.565(3)$, whereas in the $U(1)$ model $|\Phi|$ has a strong size dependence for a large range of values of $\beta$, i.e. it decreases rapidly as the spatial volume increases in accordance with the expectation that chiral symmetry should be restored for $T > 0$.

The finite spatial extent $L_s$ provides a cut-off for the divergent correlation length and according to the BKT scenario the slow decay of the correlation function $\langle e^{i\theta(x)} e^{-i\theta(0)} \rangle$ for $T < T_{\text{BKT}}$ ensures a non-zero magnetization even in a system with very large size. In Fig.1 we plot the susceptibility of the order parameter $\chi = V(\langle |\Phi|^2 \rangle - \langle |\Phi| \rangle^2)$, measured on lattices with different spatial size. We observe from this figure that: (a) there is a phase transition since the peak of $\chi$ diverges as we increase $L_s$; and (b) in the low $T$ phase the susceptibility has a stronger size dependence than in the high $T$ phase and the errors at low $T$ are much larger than at high $T$. This is strong evidence that the system is critical in the low $T$ phase in accordance with the BKT scenario. It is instructive to compare the results for the $U(1)$ model with results extracted from recent simulations of the $SU(2) \times SU(2)$-symmetric NJL$_3$ model. In the latter case the correlation length is finite for $T > 0$ and the effective order parameter goes to zero more rapidly as we increase $L_s$ than in the $U(1)$ case. It was also shown that both models have a pseudogap phase where the fermions remain massive in the absence of long range correlations. Recently, simulations of an extension of the attractive Hubbard model with the meron-cluster algorithm confirmed with high accuracy that $0 \leq \eta \leq 0.25$.

In summary, we have shown that the dimensional reduction scenario describes correctly the phase transition of simple fermionic field theories...
with composite mesons. We plan to extend our work to the four-dimensional NJL model and study the dependence of the width of the critical region on the degree of chiral symmetry. It has already been shown that in the (3 + 1)d $O(N)$ sigma model the critical region is much wider for $N \geq 2$ than for the discrete symmetry case \textsuperscript{13}. Finally, it will be interesting to understand the issue of universality in two-flavor QCD. Simulations with massless quarks of QCD with a four-fermi interaction \textsuperscript{14} near the continuum limit may provide decisive results in this direction.

References

1. B. Rosenstein, B.J. Warr and S.H. Park, Phys. Rep. \textbf{205} (1991) 59.
2. S.J. Hands, A. Kocic and J.B. Kogut, Ann. Phys. \textbf{224} (1993) 29.
3. K.G. Klimenko, Z. Phys. \textbf{C37} (1988) 457.
4. J. Kogut, M. Stephanov and C. Strouthos, Phys. Rev. \textbf{D58} (1998) 096001.
5. S. Chandrasekharan, et al., Nucl. Phys. \textbf{B576} (2000) 481.
6. E. Babaev, Phys. Lett. \textbf{B497} (2001) 323; Int. J. Mod. Phys. \textbf{A16} (2001) 1175.
7. S.J. Hands, J.B. Kogut and C.G. Strouthos, Phys. Lett. \textbf{B515}, (2001) 407.
8. H. Yamamoto and I. Ichinose, Nucl. Phys. \textbf{B370}, (1992) 695.
9. T. Appelquist and M. Schwetz, Phys. Lett. \textbf{B491} (2000) 367.
10. V.L. Berezinskii, Zh. Eksp. Teor. Fiz. \textbf{59} (1970) 907 [Sov. Phys., JETP \textbf{32} (1971) 493]; J.M. Kosterlitz and D.J. Thouless, J. Phys. \textbf{C6} (1973) 1181.
11. S. Chandrasekharan and J.C. Osborn, \texttt{cond-mat/0109424}.
12. C.G. Strouthos and D.N. Walters, \texttt{hep-lat/0212002}.
13. C.G. Strouthos and I.N. Tziligakis \texttt{hep-lat/0212040}.
14. J.B. Kogut, J.F. Lagae and D.K. Sinclair, Phys. Rev. \textbf{D58} (1998) 054504; J.B. Kogut and D.K. Sinclair, Phys. Rev. \textbf{D64} (2001) 034508; \textit{ibid.}, \texttt{hep-lat/0211008}.