Cournot duopoly games with isoelastic demands and diseconomies of scale

Xiaoliang Li∗

School of Finance and Trade, Dongguan City College, Dongguan, China, 523419

March 21, 2022

Abstract
In this discussion draft, we investigate five different models of duopoly games, where the market is assumed to have an isoelastic demand function. Moreover, quadratic cost functions reflecting decreasing returns to scale are considered. The games in this draft are formulated with systems of two nonlinear difference equations. Existing equilibria and their local stability are analyzed by symbolic computations. In the model where a gradiently adjusting player and a rational (or a boundedly rational) player compete with each other, diseconomies of scale are proved to have an effect of stability enhancement, which is consistent with the similar results found by Fisher for homogeneous oligopolies with linear demand functions.

1 Models
Let us consider a market served by two firms producing homogeneous products. We use $q_i(t)$ to denote the output of firm $i$ at period $t$. Moreover, the cost function of firm $i$ is supposed to be quadratic, i.e., $C_i(q_i) = c_i q_i^2$ with $c_i > 0$. At each period $t$, firm $i$ first estimates the possible price $p^e_i(t)$ of the product, then the expected profit of firm $i$ would be

$$\Pi^e_i(t) = p^e_i(t)q_i(t) - c_i q_i^2(t), \quad i = 1, 2.$$ 

In order to maximize the expected profit, at period $t$ each firm would decide the quantity of the output by solving

$$q_i(t) = \arg \max_{q_i(t)} \Pi^e_i(t) = \arg \max_{q_i(t)} \left[ p^e_i(t)q_i(t) - c_i q_i^2(t) \right], \quad i = 1, 2.$$

Furthermore, assume that the demand function of the market is isoelastic, which is founded on the hypothesis that the consumers have the Cobb-Douglas utility function. Hence, the real (not expected) price of the product should be

$$p(Q) = \frac{1}{Q} = \frac{1}{q_1 + q_2},$$

where $Q = q_1 + q_2$ is the total supply. Five types of players with distinct rationality degrees are involved in this draft, which are described in detail as follows.

A rational player not only knows clearly the form of the price function, but also has complete information of the decision of its rival. If firm $i$ is a rational player, at period $t + 1$ we have

$$p^e_i(t + 1) = \frac{1}{q_i(t + 1) + q^e_{-i}(t + 1)},$$

where $q^e_{-i}(t + 1)$ is the expectation of the output of the rival. Due to the assumption of complete information, which means that $q^e_{-i}(t + 1) = q_{-i}(t + 1)$, it is acquired that the expected profit of firm $i$ would be

$$\Pi^e_i(t + 1) = \frac{q_i(t + 1)}{q_i(t + 1) + q_{-i}(t + 1)} - c_i q_i^2(t + 1).$$

∗Corresponding author: xiaoliangbuaa@gmail.com
Then firm rational player in 

there are no dynamics in the system. In order to tackle this problem, Puu introduced the bounded both rational players, the equilibrium (the best decision of output) would be arrived in a shot and

where

is a boundedly rational player, then it naively expects its competitor to produce the same quantity of output as the last period, i.e.,

If firm is less rational than a boundedly rational player. Specifically, if firm would be

It is easy to verify that there exists only one positive solution for if solving [1], but the expression could be quite complex.

For simplicity, we temporarily denote (2) by

where

For simplicity, we denote the above map as

which is simply denoted as \( F_i(q_i(t+1), q_{-i}(t+1)) = 0 \) in the sequel. The player could maximize its profit by solving the above equation. It is easy to verify that there exists only one positive solution for if solving [1], but the expression could be quite complex.

By solving the first order condition, the best response for firm would be

For simplicity, we denote the above map as

The first order condition for profit maximization gives rise to a cubic polynomial equation. To be exact, the condition for the reaction function of firm \( i \) would be

which is simply denoted as \( F_i(q_i(t+1), q_{-i}(t+1)) = 0 \) in the sequel. The player could maximize its profit by solving the above equation. It is easy to verify that there exists only one positive solution for if solving [1], but the expression could be quite complex.

For simplicity, we temporarily denote [2] by

where \( R_i \) is called the reaction function of firm \( i \). It is evident that if the two firms in the market are both rational players, the equilibrium (the best decision of output) would be arrived in a shot and there are no dynamics in the system. In order to tackle this problem, Puu introduced the bounded rational player in [5].

A boundedly rational player knows the form of the price function, but do not know the rival’s decision of the production. If firm \( i \) is a boundedly rational player, then it naively expects its competitor to produce the same quantity of output as the last period, i.e., \( q_{-i}^{-}(t+1) = q_{-i}(t) \). Thus,

Then the best response for firm \( i \) would be \( q_i(t + 1) = R_i(q_{-i}(t)) \).

A local monopolistic approximation (LMA) player, which even does not know the exact form the price function, is less rational than a boundedly rational player. Specifically, if firm \( i \) is an LMA player, then it just can observe the current market price \( p(t) \) and the corresponding total supply \( Q(t) \), and is able to correctly estimate the slope \( p'(Q(t)) \) of the price function around the point \( (p(t), Q(t)) \). Then firm \( i \) uses such information to conjecture the demand function and expect the price at period \( t + 1 \) to be

where \( Q_i^e(t + 1) = q_i(t + 1) + q_{-i}^{-}(t + 1) \) represents the expected aggregate production of firm \( i \) at period \( t + 1 \). Moreover, an LMA player do not know the decision of its rival either, and is assumed to use the naive expectation, i.e., \( q_{-i}^{-}(t+1) = q_{-i}(t) \). Thus,

The expected profit would be

By solving the first order condition, the best response for firm \( i \) would be

For simplicity, we denote the above map as \( q_i(t + 1) = S_i(q_i(t), q_{-i}(t)) \).
In addition, we have an adaptive player that decides the quantity of production according to its output of the previous period as well as the expectation of its rival. Specifically, if firm $i$ is an adaptive player, then at period $t+1$ this firm naively expects its competitor would produce the same quantity of output as the last period, i.e., $q_{-i}(t+1) = q_{-i}(t)$. Then the best response for firm $i$ would be $q_i(t+1) = R_i(q_{-i}(t))$. The adaptive decision mechanism for firm $i$ is that it choose the output $q_i(t+1)$ proportionally to be

$$q_i(t+1) = (1 - L)q_i(t) + LR_i(q_{-i}(t)),$$

where $L \in (0, 1)$ is a parameter controlling the proportion. It should be noticed that an adaptive player degenerate into a boundedly player if we suppose $L = 1$.

Furthermore, we consider a gradiently adjusting player, which increases/decreases its output according to the information given by the marginal profit of the last period. Specifically, if firm $i$ is a gradiently adjusting player, then at period $t+1$ this firm is supposed to know its own profit function at period $t$, that is

$$\Pi_i(t) = \frac{q_i(t)}{q_i(t) + q_{-i}(t)} - c_iq_i^2(t).$$

Hence, firm $i$ could adjust its output at period $t+1$ with a gradient mechanism as

$$q_i(t+1) = q_i(t) + Kq_i(t)\frac{\partial \Pi_i(t)}{\partial q_i(t)},\tag{3}$$

where

$$\frac{\partial \Pi_i(t)}{\partial q_i(t)} = \frac{q_{-i}(t)}{(q_i(t) + q_{-i}(t))^2} - 2c_iq_i(t)$$

is the marginal profit of firm $i$ as period $t$, and $K > 0$ is a parameter controlling the adjustment speed. It is worth noting that the adjustment speed depends upon not only the parameter $K$ but also the size of the firm $q_i(t)$. One may observe from the above iteration map that a gradient adjusting player does not need to expect or guess the production output of its rival at the current period. Denote

$$G_i(q_i(t), q_{-i}(t)) = q_i(t)\frac{\partial \Pi_i(t)}{\partial q_i(t)} = \frac{q_i(t)q_{-i}(t)}{(q_i(t) + q_{-i}(t))^2} - 2c_iq_i^2(t).$$

Naturally, the following models could be considered.

**Model 1** (GR).

$$M_{GR}(q_1, q_2) : \begin{cases} q_1(t+1) = q_1(t) + KG_1(q_1(t), q_2(t)), \\ q_2(t+1) = R_2(q_1(t+1)), \end{cases} \tag{4}$$

where $K > 0$.

**Model 2** (GB).

$$M_{GB}(q_1, q_2) : \begin{cases} q_1(t+1) = q_1(t) + KG_1(q_1(t), q_2(t)), \\ q_2(t+1) = R_2(q_1(t)), \end{cases} \tag{5}$$

where $K > 0$.

**Model 3** (GL).

$$M_{GL}(q_1, q_2) : \begin{cases} q_1(t+1) = q_1(t) + KG_1(q_1(t), q_2(t)), \\ q_2(t+1) = S_2(q_1(t), q_2(t)), \end{cases} \tag{6}$$

where $K > 0$.

**Model 4** (GA).

$$M_{GL}(q_1, q_2) : \begin{cases} q_1(t+1) = q_1(t) + KG_1(q_1(t), q_2(t)), \\ q_2(t+1) = (1 - L)q_2(t) + LR_2(q_1(t)), \end{cases} \tag{7}$$

where $K > 0$ and $0 < L < 1$. 


Model 5 (GG).

\[ M_{GG}(q_1, q_2) : \begin{cases} 
q_1(t + 1) = q_1(t) + K_1 G_1(q_1(t), q_2(t)), \\
q_2(t + 1) = q_2(t) + K_2 G_2(q_2(t), q_1(t)),
\end{cases} \tag{8} \]

where \( K_1 > 0 \) and \( K_2 > 0 \).

It is easy to verify that all the above models has one unique Nash equilibrium, of which the closed-form expression is

\[ E = \left[ \frac{\sqrt{c_2}}{\sqrt{c_1} + \sqrt{c_2}}, \frac{1}{2\sqrt{c_1 c_2}}, \frac{\sqrt{c_1}}{\sqrt{c_1} + \sqrt{c_2}} \right]. \]

2 Local Stability

In this section, we study the local stability of the unique Nash equilibrium of all the models. Indeed, for each model, this problem could be transformed into determining the existence of real solutions of a system formulated by polynomial equations and inequalities. Afterward, the symbolic approach proposed by the author and his coworker in [4] is used to systematically address the resulting systems. It should be noticed that the processes of computations in this paper are similar to [3], but the considered models are different.

2.1 Model GR

This model could be equivalently described by a one-dimensional iteration map as follows.

\[ M_{GR}(q_1) : q_1(t + 1) = q_1(t) + K q_1(t) G_1(q_1(t), R_2(q_1(t))). \tag{9} \]

Hence, the stable equilibria are the solutions of the following system.

\[ \begin{cases} 
K \cdot G_1(q_1, q_2) = 0, \\
F_2(q_2, q_1) = 0, \\
q_1 > 0, \ q_2 > 0, \\
1 + \left( 1 + K \frac{d[q_1 G_1(q_1, R_2(q_1))]}{dq_1} \right) > 0, \\
1 - \left( 1 + K \frac{d[q_1 G_1(q_1, R_2(q_1))]}{dq_1} \right) > 0, \\
c_1 > 0, \ c_2 > 0, \ K > 0.
\] \tag{10} \]

For the above system, the squarefree part of the border polynomial is

\[ SP_{GR} = c_1 c_2 K (c_1 - c_2) (c_1 + c_2) (c_1 - 1/9 c_2) (c_1^3 c_2 K^4 - 3/2 c_1^2 c_2 K^2 - 81/64 c_1^2 + 9/32 c_1 c_2 - 1/64 c_2^2). \]

We select the sample points as

\( (1, 1/2, 2), \ (1, 2, 1), \ (1, 2, 2), \ (1, 10, 1), \ (1, 10, 2). \)

By checking the number of real solution of (10) at these sample points, the following results are finally acquired.

**Theorem 1.** For Model GR, there exists a unique equilibrium with \( q_1, q_2 > 0 \). Moreover, this equilibrium is locally stable if \( R_{GR} < 0 \), where

\[ R_{GR} = 64 c_1^3 c_2 K^4 - 96 c_1^2 c_2 K^2 - 81 c_1^2 + 18 c_1 c_2 - c_2^2. \]

If we consider the counterpart of this model with quadratic costs replaced by linear costs, the following theorem is obtained.
Figure 1: The 3-dimensional \((c_1, c_2, K)\) parameter space of Model GR. The red surface is \(R_{GR}^1 = 0\), and the blue surface is \(R_{GR}^2 = 0\).
Theorem 2. For Model GR, if \( C_1(q_1) = c_1q_1 \) and \( C_2(q_2) = c_2q_2 \), there exists a unique equilibrium with \( q_1, q_2 > 0 \). Moreover, this equilibrium is locally stable if \( R_{GR}^2 < 0 \), where
\[
R_{GR}^2 = c_1K + c_2K - 4 < 0.
\]

The following proposition is consistent with the stability enhancement effect of diseconomies of scale found by Fisher in \[2\].

Proposition 1. For Model GR, if \( c_1 > 4 \) or \( c_2 > 3 \), the stable region for the linear costs \( C_i(q_i) = c_iq_i \) is strictly contained in that for the quadratic costs \( C_i(q_i) = c_iq_i^2 \).

Proof. The proof is tedious and we leave it to the readers.

2.2 Model GB

The Jacobian matrix is
\[
J_{GB} = \begin{bmatrix}
1 + K \cdot \partial G_1 / \partial q_1 & K \cdot \partial G_1 / \partial q_2 \\
\frac{dR_2}{dq_1} & 0
\end{bmatrix}
\]

Hence, the stable equilibria can be described by
\[
\begin{array}{l}
K \cdot G_1(q_1, q_2) = 0, \\
F_2(q_2, q_1) = 0, \\
q_1 > 0, \ q_2 > 0, \\
1 + \text{Tr}(J_{GB}) + \text{Det}(J_{GB}) > 0, \\
1 - \text{Tr}(J_{GB}) + \text{Det}(J_{GB}) > 0, \\
1 - \text{Det}(J_{GB}) > 0, \\
c_1 > 0, \ c_2 > 0, \ K > 0.
\end{array}
\]

(12)

Afterward, we acquire the following theorem.

Theorem 3. For Model GB, there exists a unique equilibrium with \( q_1, q_2 > 0 \). Moreover, this equilibrium is locally stable if \( R_{GB}^1 < 0 \), where
\[
R_{GB}^1 = 4c_1^2c_2K^4 - 272c_1^6c_2^3K^4 + 4632c_1^5c_2^3K^4 - 272c_1^4c_2^4K^4 + 4c_1^3c_2^5K^4 + 264c_1^3c_2K^2 \\
- 2464c_1^5c_2^2K^2 - 6096c_1^4c_2^2K^2 + 96c_1^3c_2^4K^2 + 8c_1^2c_2^5K^2 - 81c_1^6 + 342c_1^5 + 559c_1^4c_2^2 + 436c_1^4c_2 - 159c_1^2c_2^4 + 22c_1c_2^5 - c_2^6.
\]

The linear case was first studied in \[6\] and is restated as follows.

Proposition 2. For Model GB, if \( C_1(q_1) = c_1q_1 \) and \( C_2(q_2) = c_2q_2 \), there exists a unique equilibrium with \( q_1, q_2 > 0 \). Moreover, this equilibrium is locally stable if \( R_{GB}^2 > 0 \) and \( R_{GB}^3 < 0 \), where
\[
R_{GB}^2 = c_1^2K - 6c_1c_2K + c_2^2K + 4c_1 + 4c_2, \\
R_{GB}^3 = c_1^3K - 2c_1c_2K + c_2^3K - 2c_1 - 2c_2.
\]

The following result is similar to the paper by Fisher \[2\].

Proposition 3. For Model GB, if \( c_1 > 13 \) or \( c_2 > 7 \), the stable region for the linear costs \( C_i(q_i) = c_iq_i \) is strictly contained in that for the quadratic costs \( C_i(q_i) = c_iq_i^2 \).

Proof. The proof is tedious and we leave it to the readers.
Figure 2: The 3-dimensional \((c_1, c_2, K)\) parameter space of Model GB. The red surface is \(R_{GB}^1 = 0\), the blue surface is \(R_{GB}^2 = 0\), and the green surface is \(R_{GB}^3 = 0\).
2.3 Model GL

The Jacobian matrix is

\[
J_{GL} = \begin{bmatrix}
1 + K \cdot \partial G_1/\partial q_1 & K \cdot \partial G_1/\partial q_2 \\
\partial S_2/\partial q_1 & \partial S_2/\partial q_2
\end{bmatrix}
\]  (13)

Hence, the stable equilibria can be described by

\[
\begin{align*}
K \cdot G_1(q_1, q_2) &= 0, \\
S_2(q_2, q_1) &= 0, \\
q_1 &> 0, \quad q_2 > 0, \\
1 + \text{Tr}(J_{GL}) + \text{Det}(J_{GL}) &> 0, \\
1 - \text{Tr}(J_{GL}) + \text{Det}(J_{GL}) &> 0, \\
1 - \text{Det}(J_{GL}) &> 0, \\
c_1 &> 0, \quad c_2 > 0, \quad K > 0.
\end{align*}
\]  (14)

**Theorem 4.** For Model GL, there exists a unique equilibrium with \(q_1, q_2 > 0\). Moreover, this equilibrium is locally stable if \(R_{GL} < 0\), where

\[
R_{GL}^1 = 64 c_1^7 c_2 K^4 - 672 c_1^6 c_2^2 K^4 + 1796 c_1^5 c_2^3 K^4 - 168 c_1^4 c_2^4 K^4 + 4 c_1^3 c_2^5 K^2 - 400 c_1^2 c_2^6 K^2 + 2136 c_1^1 c_2^7 K^2 + 384 c_1^0 c_2^8 K^2
\]

\[
- 256 c_1^6 + 544 c_1^5 c_2 - 353 c_1^4 c_2^2 - 38 c_1^3 c_2^4 + 4 c_1^2 c_2^6 - 6 c_1^1 c_2^8.
\]

The linear case was first studied in [1], which is restated here.

**Proposition 4.** For Model GL, if \(C_1(q_1) = c_1 q_1\) and \(C_2(q_2) = c_2 q_2\), there exists a unique equilibrium with \(q_1, q_2 > 0\). Moreover, this equilibrium is locally stable if \(R_{GL}^2 > 0\) and \(R_{GL}^3 < 0\), where

\[
R_{GL}^2 = 3 c_1 K - c_2 K + 2,
\]

\[
R_{GL}^3 = 7 c_1 c_2 K - c_2^2 K - 8 c_1 - 4 c_2.
\]

**Proposition 5.** For Model GL, even if \(c_1 > 10^{100}\) and \(c_2 > 10^{100}\), the stable region for the linear costs \(C_i(q_i) = c_i q_i\) is not strictly contained in that for the quadratic costs \(C_i(q_i) = c_i q_i^2\).

2.4 Model GA

The Jacobian matrix is

\[
J_{GA} = \begin{bmatrix}
1 + K \cdot \partial G_1/\partial q_1 & K \cdot \partial G_1/\partial q_2 \\
L \cdot dR_2/\partial q_1 & L - L
\end{bmatrix}
\]  (15)

Hence, the stable equilibria can be described by

\[
\begin{align*}
K \cdot G_1(q_1, q_2) &= 0, \\
F_2(q_2, q_1) &= 0, \\
q_1 &> 0, \quad q_2 > 0, \\
1 + \text{Tr}(J_{GA}) + \text{Det}(J_{GA}) &> 0, \\
1 - \text{Tr}(J_{GA}) + \text{Det}(J_{GA}) &> 0, \\
1 - \text{Det}(J_{GA}) &> 0, \\
c_1 &> 0, \quad c_2 > 0, \quad K > 0, \quad L > 0, \quad 1 - L > 0.
\end{align*}
\]  (16)

**Theorem 5.** For Model GG, there exists a unique equilibrium with \(q_1, q_2 > 0\). Moreover, this equilibrium is locally stable if \(R_{GG}^1 < 0\), where

\[
R_{GA}^1 = 64 c_1^7 c_2 K^4 L^4 - 256 c_1^6 c_2^2 K^4 L^4 + 384 c_1^5 c_2^3 K^4 L^4 - 256 c_1^4 c_2^4 K^4 L^4 + 64 c_1^3 c_2^5 K^4 L^3 - 384 c_1^2 c_2^6 K^4 L^3 - 256 c_1^2 c_2^6 K^4 L^3 - 384 c_1^1 c_2^7 K^4 L^3 + 256 c_1^1 c_2^7 K^4 L^3 - 4352 c_1^0 c_2^8 K^4 L^3 + 2560 c_1^0 c_2^8 K^4 L^3 - 384 c_1^0 c_2^8 K^4 L^3 + 864 c_1^0 c_2^8 K^4 L^2
\]
Figure 3: The 3-dimensional \((c_1, c_2, K)\) parameter space of Model GL. The red surface is \(R_{GL}^1 = 0\), the blue surface is \(R_{GL}^2 = 0\), and the green surface is \(R_{GL}^3 = 0\).
Theorem 6. For Model GA, if $C_1(q_1) = c_{1q_1}$ and $C_2(q_2) = c_{2q_2}$, there exists a unique equilibrium with $q_1, q_2 > 0$. Moreover, this equilibrium is locally stable if $R_{GA}^2 > 0$ and $R_{GA}^2 < 0$, where

$$R_{GA}^2 = c_{1}^2 KL + 2 c_{1} c_{2} KL + c_{2}^2 KL - 4 c_{1} c_{2} K - 2 c_{1} L - 2 c_{2} L,$$

$$R_{GA}^2 = c_{1}^2 KL + 2 c_{1} c_{2} KL + c_{2}^2 KL - 8 c_{1} c_{2} K - 4 c_{1} L - 4 c_{2} L + 8 c_{1} + 8 c_{2}.$$

Proposition 6. For Model GG, if $K_1 = K_2$, the stable region for the linear costs $C_i(q_i) = c_{iq_i}$ is strictly contained in that for the quadratic costs $C_i(q_i) = c_{iq_i}^2$.

2.5 Model GG

The Jacobian matrix is

$$J_{GG} = \begin{bmatrix} 1 + K_1 \cdot \partial G_1/\partial q_1 & K_1 \cdot \partial G_1/\partial q_2 \\ K_2 \cdot \partial G_2/\partial q_1 & 1 + K_2 \cdot \partial G_2/\partial q_2 \end{bmatrix}$$

(17)

Hence, the stable equilibria can be described by

$$\begin{align*}
K_1 \cdot G_1(q_1, q_2) &= 0, \\
K_2 \cdot G_2(q_2, q_1) &= 0, \\
q_1 &> 0, ~ q_2 > 0, \\
1 + \text{Tr}(J_{GG}) + \text{Det}(J_{GG}) &> 0, \\
1 - \text{Tr}(J_{GG}) + \text{Det}(J_{GG}) &> 0, \\
1 - \text{Det}(J_{GG}) &> 0, \\
c_1 &> 0, ~ c_2 > 0, ~ K_1 > 0, ~ K_2 > 0.
\end{align*}$$

(18)

Theorem 7. For Model GG, there exists a unique equilibrium with $q_1, q_2 > 0$. Moreover, this equilibrium is locally stable if $R_{GG}^1 > 0$ and $R_{GG}^2 < 0$, where

$$R_{GG}^1 = -1024 c_{1}^4 c_{2} K_{1}^4 K_{2}^4 + 384 c_{1}^2 c_{2} K_{1}^4 K_{2}^2 + 384 c_{1}^3 c_{2} K_{1}^3 K_{2}^2 + 384 c_{1}^3 c_{2} K_{1}^2 K_{2}^2 + 384 c_{1}^2 c_{2} K_{1} K_{2}^2 + c_{1} K_{1} - 18 c_{1}^4 c_{2} K_{1}^4 K_{2} - 32 c_{1}^4 c_{2} K_{1}^3 K_{2} - 18 c_{1}^4 c_{2} K_{1}^2 K_{2} + 81 c_{1}^4 c_{2} K_{1} K_{2} + 288 c_{1}^3 c_{2} K_{1} K_{2} + 420 c_{1}^2 c_{2} K_{1} K_{2}^2 + 288 c_{1}^3 c_{2} K_{1} K_{2} + 81 c_{1}^3 c_{2} K_{1} K_{2}^2 - 18 c_{1}^3 c_{2} K_{1} K_{2} - 32 c_{1}^3 c_{2} K_{1} K_{2} - 18 c_{1}^3 c_{2} K_{1} K_{2} + c_{1} K_{1},
$$

and

$$R_{GG}^2 = -64 c_{1}^4 c_{2} K_{1}^4 K_{2}^4 + 96 c_{1}^3 c_{2} K_{1}^4 K_{2}^2 - 32 c_{1}^3 c_{2} K_{1}^3 K_{2}^2 - 32 c_{1}^3 c_{2} K_{1}^2 K_{2}^2 + 96 c_{1}^3 c_{2} K_{1} K_{2}^2 + c_{1} K_{1}.$$
Figure 4: The parameter space of Model GA. $R_{GA}^1 = 0$, $R_{GA}^2 = 0$ and $R_{GA}^3 = 0$ are marked in red, blue and green, respectively.
For Model GG, even if Proposition 8, strictly contained in that for the quadratic costs Proposition 7.

\[\begin{align*}
&-18 c_1^4 c_2^2 K_1^1 + 32 c_1^3 c_2^3 K_1^2 - 18 c_1^4 c_2^2 K_1^1 K_2^0 + 81 c_1^3 c_2^3 K_1^2 K_2^n + 96 c_1^3 c_2^3 K_1^2 K_2^0 - 92 c_1^3 c_2^3 K_1^2 K_2^n \\
&+ 96 c_1^3 c_2^3 K_1^2 + 81 c_1^3 c_2^3 K_1^2 K_2^0 - 18 c_1^2 c_2^4 K_1^3 K_2^n + 32 c_1^2 c_2^4 K_1^3 K_2^0 - 18 c_1^2 c_2^4 K_1^3 - 8 c_1^2 c_2^4 K_1^2 + 8 c_1^2 c_2^4 K_1^n \\
&- 8 c_1^2 c_2 K_1 K_2 - 16 c_1^2 c_2^2 K_1^2 - 120 c_1^3 c_2^3 K_1 K_2 - 120 c_1^3 c_2^3 K_1^2 - 120 c_1^3 c_2^3 K_1^2 K_2^n - 120 c_1^3 c_2^3 K_1 K_2 \\
&- 16 c_1^2 c_2^2 K_2^0 - 8 c_1^2 c_2 K_1 K_2 + 8 c_1^2 c_2^2 K_1^2 - 4 c_1^4 + 16 c_1^3 c_2 - 24 c_1^2 c_2^2 + 16 c_1 c_2^2 - 4 c_2^2.
\end{align*}\]

**Theorem 8.** For Model GG, if \(C_1(q_1) = c_1 q_1\) and \(C_2(q_2) = c_2 q_2\), there exists a unique equilibrium with \(q_1, q_2 > 0\). Moreover, this equilibrium is locally stable if \(R_{GG}^3 > 0\) and \(R_{GG}^4 < 0\), where

\[\begin{align*}
R_{GG}^3 &= c_1 K_1 K_2 + c_2 K_1 K_2 - 2 K_1 - 2 K_2, \\
R_{GG}^4 &= c_1^2 c_2 K_1 K_2 + c_1^2 c_2 K_1 K_2 - 4 c_1 c_2 K_1 - 4 c_1 c_2 K_2 + 4 c_1 + 4 c_2.
\end{align*}\]

**Proposition 7.** For Model GG, if \(K_1 = K_2\), the stable region for the linear costs \(C_i(q_i) = c_i q_i\) is strictly contained in that for the quadratic costs \(C_i(q_i) = c_i q_i^2\).

**Proposition 8.** For Model GG, even if \(c_1 > 10^{100}\) and \(c_2 > 10^{100}\), the stable region for the linear costs \(C_i(q_i) = c_i q_i\) is not strictly contained in that for the quadratic costs \(C_i(q_i) = c_i q_i^2\).

**References**

[1] F. Cavalli, A. Naimzada, and F. Tramontana. Nonlinear dynamics and global analysis of a heterogeneous Cournot duopoly with a local monopolistic approach versus a gradient rule with endogenous reactivity. *Communications in Nonlinear Science and Numerical Simulation*, 23(1-3):245–262, 2015.

[2] F. M. Fisher. The stability of the Cournot oligopoly solution: The effects of speeds of adjustment and increasing marginal costs. *The Review of Economic Studies*, 28(2):125–135, 1961.

[3] X. Li. Dynamics of cournot duopoly games with quadratic costs and distinct rationality degrees. ArXiv, https://doi.org/10.48550/arXiv.2112.05948.

[4] X. Li and D. Wang. Computing equilibria of semi-algebraic economies using triangular decomposition and real solution classification. *Journal of Mathematical Economics*, 54:48–58, 2014.

[5] T. Puu. Chaos in duopoly pricing. *Chaos, Solitons & Fractals*, 1(6):573–581, 1991.

[6] F. Tramontana. Heterogeneous duopoly with isoelastic demand function. *Economic Modelling*, 27(1):350–357, 2010.
(a) $K_1 = K_2$.

(b) $K_1 = K_2$.

(c) $K_1 = K_2, c_1 = c_2$.

(d) $K_1 = K_2 = 1$.

(e) $c_1 = 1, c_2 = 1$.

(f) $c_1 = 1, c_2 = 1/2$.

Figure 5: The parameter space of Model GG. The surfaces $R_{GG}^1 = 0$, $R_{GG}^2 = 0$, $R_{GG}^3 = 0$ and $R_{GG}^4 = 0$ are marked in magenta, red, blue and green, respectively.