On the Constantin-Lax-Majda Model with Convection

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Abstract

The famous Constantin-Lax-Majda (CLM) model, an important toy of the 3D Euler equations without convection, can develop finite time singularities [4]. De Gregorio modified the CLM model by adding a convective term [5], which is known important for fluid dynamics [8, 10]. Presented are two results on the De Gregorio model. The first one is the global well-posedness of such a model for general initial data with non-negative (or non-positive) vorticity which is based on a new discovered conserved quantity. This verifies the numerical observations for such class of initial data. The second one is a new proof of the exponential stability result of ground states in the recent significant work of Jia, Steward and Sverak [9] (here the zero mean constraint on the initial data can be removed). The novelty of the method is the introduction of the new solution space $H_{DW}$ together with a new basis and a magic inner product of $H_{DW}$.

1 Introduction

The classical Constantin-Lax-Majda (CLM) model is

$$\partial_t \omega = \omega H\omega,$$

where $\omega : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ with $\Omega$ being the whole real line $\mathbb{R}$ or the circle $S^1$, and $H$ is the Hilbert transform. It is one of the famous models which are proposed to analyze the potential singularities of 3D Euler equations [4], mimicking the essence of the 3D mechanism and, at the same time, being feasible for mathematical analysis. On one hand, the blowup mechanism for the CLM model has been well understood by experts (see, for instance, [4]). On the other hand, as being pointed out by Okamoto [10], the CLM model ignores the role of convection, which we now know is important, see [8, 10].

De Gregorio [5] suggested to include a convection term to the CLM model which reads

$$\partial_t \omega + u\omega' = \omega u', \quad u' = H\omega.$$  \hfill (1.1)

where $f'$ denotes the space derivative of a function $f(t, \cdot)$ and

$$H\omega(\theta) = \frac{1}{2\pi} \text{P.V.} \int_{-\pi}^{\pi} \cot \frac{\theta - \phi}{2} \omega(\phi) d\phi.$$
if $\omega$ is defined on the circle $S^1$ and
\[
H\omega(x) = \frac{1}{\pi} \text{P.V.} \int_{-\infty}^{\infty} \frac{\omega(y)}{x - y} dy
\]

if $\omega$ is defined on the whole line $\mathbb{R}$. Numerical calculations seem to suggest that there is no blow-up from smooth initial data for the De Gregorio’s modification of the CLM model [5]. In this paper, we will verify this numerical conjecture theoretically for general initial data with non-negative vorticity $\omega(0, x) \geq 0$ (or non-positive) but without any smallness assumptions.

**Theorem 1.1.** Let $\Omega$ be the whole line $\mathbb{R}$ or the circle $S^1$. Let the initial data satisfy $\omega_{in} \geq 0$, $\omega_{in} \in L^1(\Omega)$ and $(\sqrt{\omega_{in}})' \in L^2(\Omega)$. Suppose that $u$ satisfies $\int_{S^1} u(t, \theta) d\theta \equiv 0$ if $\Omega = S^1$ or $u(t, 0) \equiv 0$ if $\Omega = \mathbb{R}$.

- The De Gregorio’s modification of the CLM model \((\text{1.1})\) is globally well-posed and

\[
\frac{d}{dt} \| (\sqrt{\omega(t, \cdot)})' \|_{L^2(\Omega)} = 0, \quad \| \omega(t, \cdot) \|_{H^1(\Omega)} \leq C_0
\]

for some positive constant $C_0$ and all $t \geq 0$. Here $C_0$ only depends on $\| \omega_{in} \|_{L^1(\Omega)}$ and $\| (\sqrt{\omega_{in}})' \|_{L^2(\Omega)}$.

- Let $s \geq 1$. If $\omega_{in} \in H^s(\Omega)$, then $\omega(t, \cdot) \in H^s(\Omega)$ for all $t > 0$.

**Remark 1.2.** Our existence result holds true for different choices of gauge for the velocity $u$. For $\Omega = S^1$, one could replace the condition $\int_{S^1} u(t, \theta) d\theta \equiv 0$ with $u(t, 0) \equiv 0$. The latter gauge will be convenient for our stability result in Section 3. Solutions under different gauges are equivalent up to translations, see [9].

**Remark 1.3.** It is clear that $(\sqrt{\omega_{in}})' \in L^2$ is a reasonable assumption on the initial data $\omega_{in}$ if $\omega_{in}$ is strictly positive or degenerates at its zeros at an order $\gamma > 1$. Our key observation here is the conservation of the quantity $\| (\sqrt{\omega(t, \cdot)})' \|_{L^2}$ in time (see Section 2), which seems totally new in the literatures (see [9] for other known conserved quantities). Note that
\[
\partial_t \omega' + u \omega'' = \omega u'\tag{1.2}
\]

which has been observed by Jia, Steward and Sverak [9]. Thus zeros of $\omega(t, \cdot)$ and the values of $\omega'(t, \cdot)$ at these zeros are transported by $u$, which makes sense of $\| (\sqrt{\omega})' \|_{L^2}$. It remains an open problem whether one could remove the non-negative assumption imposed on the initial data.

**Remark 1.4.** Theorem \([\text{1.1}]\) also holds true for non-positive initial data $\omega_{in}$, in which case we require that $\omega_{in} \in L^1(\Omega)$ and $\sqrt{-\omega_{in}} \in L^2(\Omega)$. In fact, we only need to consider $\bar{\omega} = -\omega$, which satisfies:
\[
- \partial_t \bar{\omega} + \bar{u} \bar{\omega}' = \bar{\omega} \bar{u}', \quad \bar{u}' = H\bar{\omega}. \tag{1.2}
\]

Note that equation \((\text{1.2})\) is simply a time reversed version of \((\text{1.1})\) and the proof of Theorem \([\text{1.1}]\) in Section 2 still works.
Our second result concerns the recent fantastic work of Jia, Steward and Sverak [9] in which the authors proved, under the zero mean constraint on the initial data, the exponential stability of ground states of the De Gregorio modification of the CLM model (1.1) on the circle for initial data \( \theta - \gamma \eta_n \in L^2, \frac{3}{2} < \gamma < 2 \), where \( \eta_n = \omega_{in} + \sin \theta \). Their proof involves some deep spectral theories and complex variable methods, together with many fantastic novel observations on the structure of the De Gregorio model formulated as dynamical systems. Here we prove a similar exponential stability result to theirs which implies Theorem 1.1 in [9], using a direct energy method (the definition of the space \( H_{DW} \) will be introduced after stating the theorem).

**Theorem 1.5.** Let \( 0 < \beta < \frac{3}{2} \) be a given constant and \( \omega_{in}(\theta) = -\sin \theta + \eta_n(\theta) \) with \( \theta \in S^1 \). Suppose that \( \eta_n \in H_{DW} \) and \( \int_{S^1} \eta_n d\theta = 0 \). There exists an absolute small constant \( \delta_0 > 0 \) such that if \( \| \eta_n \|_{H_{DW}} < \delta_0 \), then the De Gregorio modification of the CLM model (1.1) under the gauge \( u(t, 0) \equiv 0 \) with initial data \( \omega(0, \theta) = \omega_{in}(\theta) \) is globally well-posed and \( \| \omega(t, \cdot) + \sin \theta \|_{H_{DW}} \lesssim e^{-\beta t} \| \eta_n \|_{H_{DW}} \) for all \( t \geq 0 \).

There are two main ingredients in our proof. For simplicity, let us take the odd perturbations as an example (the generic perturbations are a little bit more complicated and are treated in Section 3). Firstly, we define a new magic Hilbert space by

\[
H_{DW} = \{ \eta \in H^1(S^1) \mid \eta(0) = 0, \int_{-\pi}^{\pi} \frac{|\eta'|^2}{\sin^2 \frac{\theta}{2}} d\theta < \infty \}. \tag{1.3}
\]

The inner product of \((H_{DW}, g)\) is defined to be

\[
\langle \xi, \eta \rangle_g = \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\xi' \eta'}{\sin^2 \frac{\theta}{2}} d\theta. \tag{1.4}
\]

Secondly, by introducing the following new vectors

\[
e^{(o)}_k = \frac{\sin((k+1)\theta)}{k+1} - \frac{\sin(k\theta)}{k}, \quad k \geq 1, \tag{1.5}
\]

we find that

\[
\langle e^{(o)}_k, e^{(o)}_l \rangle = \delta_{kl}, \quad k, l \geq 1. \tag{1.6}
\]

We remark that the inner product in (1.4) and the exact form of the basis vectors in (1.5) are still mysterious to us even though we are fortunate enough to find them accidentally. With the above accidental discoveries, we are able to obtain the exponential decay for the linearized equations by using a direct energy estimate, providing a different angle for the highly non-trivial but fantastic method of Jia, Steward and Sverak [9].

By removing the constraint \( \int_{S^1} \omega_{in} d\theta = 0 \), Theorem 1.5 can be generalized as follows, which will be proved based on the same ideas.

**Theorem 1.6.** Let \( 0 < \beta < \frac{3}{2} \) be a given constant and \( \omega_{in}(\theta) = -\sin \theta + \zeta_{in}(\theta) \) with \( \theta \in S^1 \). Suppose that \( \zeta_{in} \in H_{DW} \). There exists an absolute small constant \( \delta_1 > 0 \) such that if \( \| \zeta_{in} \|_{H_{DW}} < \delta_1 \), then the De Gregorio modification of the CLM model (1.1) under the gauge \( u(t, 0) \equiv 0 \) with initial data \( \omega(0, \theta) = \omega_{in}(\theta) \) is globally well-posed and \( \| \omega(t, \cdot) + \sin \theta + \alpha(\cos \theta - 1) \|_{H_{DW}} \lesssim e^{-\beta t} \| \zeta_{in} \|_{H_{DW}} \) for all \( t \geq 0 \), where \( \alpha = \int_{S^1} \zeta_{in} d\theta. \)
For the first excited state \( \sin(2\theta) \), we have the following result on the linearized level, for even initial data.

**Theorem 1.7.** The linearized equation of (1.1) at \(-\sin 2\theta\) reads

\[
\partial_t \eta = L_2 \eta,
\]

with

\[
L_2 \eta = -\frac{1}{2} \sin(2\theta) \eta' + \cos(2\theta) \eta - \sin(2\theta) v' + 2 \cos(2\theta) v,
\]

where \( v \) satisfies \( v' = H \eta \) and the gauge \( \int_{S_1} v \, d\theta \equiv 0 \). For even initial data \( \eta(0, \theta) = \sum_{k \geq 1} \eta_k \cos(k\theta) \), (1.7) is globally well-posed and satisfies

\[
\frac{d}{dt} \| \eta \|_X^2 + \frac{3}{2} | \eta_1^{(e)} |^2 = 0, \quad \text{for} \quad \eta = \sum_{k \geq 1} \eta_k^{(e)} \cos(k\theta).
\]

Here \( \| \eta \|_X^2 \triangleq \sum_{k \neq 2} g_k^{(e)} | \eta_k^{(e)} |^2 \) with \( g_1^{(e)} = 1 \) and \( g_k^{(e)} \sim k^3 \). \( \eta_2^{(e)} \) grows at most linearly.

**Remark 1.8.** This result is analogous to an observation in [9] that the linearized equation of (1.1) at \(-\sin \theta\) has a conserved (semi)norm. The exact forms of \( g_k^{(e)} \) will be given in Section 4. We did not find identities like (1.9) for linearization of (1.1) at higher excited states, i.e. \( \sin k\theta \) with \( k \geq 3 \). Besides, theorem 1.7 doesn’t hold for general odd initial data either.

There are some other aspects on the studies of the De Gregorio modification of the CLM model, see for instance, [1, 2, 3, 6, 7, 11, 12]. The remaining part of this paper is organized as follows: In Section 2, we derive an identity for the new conserved quantity \( \| (\sqrt{\omega(t, \cdot)})' \|_{L^2} \) and prove Theorem 1.1. We introduce a new basis of functions in Section 3, which leads to the linear stability of the ground state \(-\sin \theta\). Then we prove nonlinear stability as stated in Theorem 1.5 and Theorem 1.6. The last section is devoted to a careful analysis of the linearized equation at the excited state \(-\sin 2\theta\) for both odd and even data and proving Theorem 1.7.

## 2 Global Wellposedness with Non-negative Initial Vorticity

According to numerical experiments conducted in [11], smooth solutions for (1.1) exist globally for smooth initial data and converge to equilibriums in general. Prior to our work, only local well-posedness theory in \( H^s \) with \( s > \frac{1}{2} \) is known (for experts). Here we show global existence for nonnegative initial vorticity and prove that smoothness of such initial data can be preserved by the flow.

**Proof of Theorem 1.1** First of all, let us assume that \( \omega \in L^\infty(0,T;L^1(\Omega)) \) and \( (\sqrt{\omega})' \in L^\infty((0,T);L^2(\Omega)) \). Suppose that \( \int_{S_1} u(t,\theta) \, d\theta \equiv 0 \) if \( \Omega = S^1 \) or \( u(t,0) \equiv 0 \) if \( \Omega = \mathbb{R} \). We are going to derive some a priori estimates for \( \omega \).

It is clear that \( \sqrt{\omega} \in L^\infty((0,T);H^1(\Omega)) \). By Sobolev imbedding, one has \( \sqrt{\omega} \in L^\infty((0,T) \times \Omega) \), which gives that \( \omega \in L^\infty((0,T) \times \Omega) \). As a consequence, one has \( \omega \in L^\infty((0,T);H^1(\Omega)) \). By the anti-symmetry property of the Hilbert transform, it is clear that

\[
\partial_t \int_{\Omega} \omega \, dx = \int_{\Omega} (-u \omega' + H \omega) \, dx = 2 \int_{\Omega} H \omega \, dx = 0.
\]
Hence,
\[ \| \omega(t, \cdot) \|_{L^1(\Omega)} = \| \omega_{in} \|_{L^1}. \]

Next, using
\[ \partial_t \sqrt{\omega} = -u(\sqrt{\omega})' + \frac{1}{2} \sqrt{w} H \omega \quad \text{on } \{ \theta : \omega > 0 \}, \]
we can take derivative on both sides of the above equation to derive that
\[ \partial_t (\sqrt{\omega})' = -u(\sqrt{\omega})'' - u'(\sqrt{\omega})' + \frac{1}{2} (\sqrt{w})' H \omega (\sqrt{\omega})' + \frac{1}{2} \sqrt{w} (H \omega)' (\sqrt{\omega})' \quad \text{on } \{ \theta : \omega > 0 \}. \]

Further calculations give that
\[ \frac{1}{2} \partial_t ((\sqrt{w})')^2 = -\frac{1}{2} u((\sqrt{w})')^2 - u'(\sqrt{w})'(\sqrt{w})' + \frac{1}{2} (\sqrt{w})' H \omega (\sqrt{\omega})' + \frac{1}{2} \sqrt{w} (H \omega)' (\sqrt{\omega})' \]
\[ = -\frac{1}{2} u((\sqrt{w})')^2 - \frac{1}{2} H \omega (\sqrt{\omega})'^2 + \frac{1}{4} (H \omega)' \omega'. \]

Note that the above equation is also true on \( \{ \theta : \omega = 0 \} \). Integrating over \( \Omega \) and using the fact \( \int H \omega' \omega' = 0 \), we finally arrive at
\[ \partial_t \int_{\Omega} ((\sqrt{w})')^2 dx = 0. \]

Hence, we also have
\[ \|(\sqrt{\omega(t, \cdot)})'\|_{L^2(\Omega)} = \|(\sqrt{\omega_{in}})'\|_{L^2(\Omega)}. \]

The above argument certainly implies that \( \sup_{t>0} \| \omega \|_{H^1(\Omega)} \leq C_0 \) for some constant \( C_0 \) depending only on \( \| \omega_{in} \|_{L^1} \) and \( \| (\sqrt{\omega_{in}})' \|_{L^2} \).

We point out that the higher regularity of solution is obvious if the data has higher order regularity once one has the \( H^1 \) estimate of \( \omega \).

To finish the proof of the theorem, it remains to establish a local well-posedness theory of (1.1) for initial data satisfying the constraints stated in the theorem, which is more or less standard and thus omitted (for instance, in the periodic case, solving (1.1) with the initial data \( \omega_{in} = \omega_{in} + \epsilon \) and in the whole space case, solving (1.1) with the initial data \( \omega_{in} = \omega_{in} + \epsilon e^{-\theta^2} \), one can obtain a sequence of approximate solutions \( \omega^\epsilon \) which have desired bounds in \( \epsilon \). Then one can solve the equation (1.1) with initial data \( \omega_{in} \) by taking \( \epsilon \to 0 \). \( \square \)

3 Stability of the Ground State

As has been observed in [9, 11], (1.1) has an infinite number of stationary solutions, of the form \( \sin k \theta \), \( \forall k \geq 1 \) (up to trivial translations and multiplication by constants). We call \( \sin \theta \) the ground state and \( \sin k \theta \) (\( k \geq 2 \)) the excited states (by translation \( \cos k \theta \) are ground state for \( k = 1 \) and excited states for \( k \geq 2 \)).

3.1 Linearized equation at \( \omega = -\sin \theta \)

Consider solutions to (1.1) of the form
\[ \omega = -\sin \theta + \eta, \quad u = \sin \theta + v. \]
Clearly, one has
\[
\begin{align*}
\partial_t \eta + \sin \theta (\eta + v)' - (\eta + v) \cos \theta = \eta v' - v\eta' \triangleq -[v, \eta], \\
H \eta = v'.
\end{align*}
\] (3.1)

In [9], J. Hao et al. carefully studied the linearized equation for \( \eta \), which reads
\[
\partial_t \eta = L \eta,
\] (3.2)

where \( L \) is the linear operator defined by
\[
L \eta = -[\sin \theta, \eta + v] = -\sin \theta \eta' + \cos \theta \eta - \sin \theta H \eta + \cos \theta v.
\] (3.3)

They worked with the gauge \( v(t, 0) = 0 \) and under the assumption
\[
\int_{S^1} \eta d\theta = 0.
\] (3.4)

This assumption is reasonable since \( \int_{S^1} \eta d\theta \) is an invariant both for the linear problem (3.2) and for the nonlinear problem (1.1). We do not need (3.4) for now but it will be important in Section 3.3. The representation of \( L \) on the Fourier side has been computed in [9]. For odd data one has
\[
L \tilde{e}_k^{(o)} = A_k e_{k+1}^{(o)} + B_k e_k^{(o)}, \quad k \geq 2,
\] (3.5)

where
\[
e_k^{(o)} \triangleq \sin(k\theta),
\]
and
\[
A_k = -\frac{1}{2}(k - 1)(1 - \frac{1}{k}), \quad B_k = \frac{1}{2}(k + 1)(1 - \frac{1}{k}), \quad k \geq 2.
\] (3.6)

For \( k = 1 \) one has
\[
L \tilde{e}_1^{(o)} = 0.
\]

H. Jia et al. proved exponential decay of \( e^{itL} \) in a weighted \( L^2 \) space based on a study of spectral properties of \( L \), see [9] for more details. We take a different approach from [9], by introducing a sequence of new basis functions. Denote
\[
\tilde{c}_k^{(o)} = \frac{e_{k+1}^{(o)}}{k + 1} - \frac{e_k^{(o)}}{k}, \quad k \geq 1.
\]

Then
\[
L \tilde{c}_k^{(o)} = \frac{A_{k+1} e_{k+2}^{(o)}}{k + 1} + \frac{B_{k+1} e_{k+2}^{(o)}}{k + 1} - \frac{A_k e_{k+1}^{(o)}}{k} e_k^{(o)} - \frac{B_k e_k^{(o)}}{k} e_{k-1}^{(o)}
\]
\[
= -\frac{k^2}{2(k + 1)^2} e_k^{(o)} + \frac{k(k + 2)}{2(k + 1)^2} e_{k+2}^{(o)} + \frac{(k - 1)^2}{2k^2} e_{k+1}^{(o)} - \frac{(k + 1)(k - 1)}{2k^2} e_{k+1}^{(o)} - \frac{(k + 1)(k - 1)}{2k^2} e_{k-1}^{(o)}
\]
\[
= -\frac{k^2(k + 2)}{2(k + 1)^2} \left( \frac{e_{k+2}^{(o)}}{k + 2} - \frac{e_{k+1}^{(o)}}{k + 1} \right) + \left( \frac{(k - 1)^2(k + 1)}{2k^2} - \frac{k^2(k + 2)}{2(k + 1)^2} \right) \left( \frac{e_{k+1}^{(o)}}{k + 1} - \frac{e_k^{(o)}}{k} \right)
\]
\[
= -\frac{k^2(k + 2)}{2(k + 1)^2} e_{k+1}^{(o)} + \left( \frac{(k - 1)^2(k + 1)}{2k^2} - \frac{k^2(k + 2)}{2(k + 1)^2} \right) \tilde{c}_k^{(o)} + \frac{(k - 1)^2(k + 1)}{2k^2} \tilde{c}_{k-1}^{(o)}
\]
i.e.

\[ L e_k^{(e)} = -d_{k+1} \tilde{e}_{k+1}^{(e)} - (d_{k+1} - d_k) \tilde{e}_k^{(e)} + d_k \tilde{e}_{k-1}^{(e)}, \quad d_k = \frac{(k - 1)^2 (k + 1)}{2k^2}. \]

Note that \( d_1 = 0 \). The above equality holds true for all \( k \geq 1 \). We can also include even perturbations by introducing

\[ e_k^{(e)} = \cos k \theta - 1, \quad k \geq 1, \]

and

\[ \tilde{e}_k^{(e)} = \cos (k + 1) \theta - 1 = \cos k \theta - 1. \]

The constant \(-1\) in the definitions are added to make sure that \( e_k^{(e)}(0) = \tilde{e}_k^{(e)}(0) = 0 \). Set \( \tilde{e}_0^{(e)} = \cos \theta - 1 = e_1^{(e)} \). Similarly we have

\[ L e_k^{(e)} = A_k e_{k+1}^{(e)} + B_k e_{k-1}^{(e)} - (1 - \frac{1}{k}) e_1^{(e)}, \quad k \geq 2 \]

and \( L e_0^{(e)} = L e_1^{(e)} = 0 \). It follows that for \( k \geq 2 \),

\[ L \tilde{e}_k^{(e)} = \frac{A_{k+1}}{k+1} \tilde{e}_{k+2}^{(e)} + \frac{B_{k+1}}{k+1} \tilde{e}_k^{(e)} - \frac{A_k}{k} \tilde{e}_{k+1}^{(e)} - \frac{B_k}{k} \tilde{e}_{k-1}^{(e)} + \left( -\frac{k}{(k+1)^2} + \frac{k-1}{k^2} \right) \tilde{e}_1^{(e)} \]

\[ = -d_{k+1} \tilde{e}_{k+1}^{(e)} - (d_{k+1} - d_k) \tilde{e}_k^{(e)} + d_k \tilde{e}_{k-1}^{(e)} + \frac{k^2 - k - 1}{k^2 (k+1)^2} \tilde{e}_0^{(e)}. \]

Direct computation shows the above holds for \( k = 1 \) as well:

\[ L \tilde{e}_1^{(e)} = -\frac{3}{8} \tilde{e}_2^{(e)} - \frac{3}{8} \tilde{e}_1^{(e)} - \frac{1}{4} \tilde{e}_0^{(e)}. \]

Hence if we write \( \eta = \sum_{k \geq 1} \tilde{\eta}_k^{(o)} \tilde{e}_k^{(o)} + \sum_{k \geq 0} \tilde{\eta}_k^{(e)} \tilde{e}_k^{(e)} \), then (5.2) can be written as the following infinite dimensional ODE system

\[
\begin{align*}
\partial_t \tilde{\eta}_k^{(o)} &= -d_k \tilde{\eta}_{k-1}^{(o)} - (d_{k+1} - d_k) \tilde{\eta}_k^{(o)} + d_{k+1} \tilde{\eta}_{k+1}^{(o)}, \quad k \geq 1, \\
\partial_t \tilde{\eta}_k^{(e)} &= -d_k \tilde{\eta}_{k-1}^{(e)} - (d_{k+1} - d_k) \tilde{\eta}_k^{(e)} + d_{k+1} \tilde{\eta}_{k+1}^{(e)}, \quad k \geq 1,
\end{align*}
\]

where \( d_1 \tilde{\eta}_0^{(o)} \) is understood to be 0. For the "0th mode", we have

\[ \partial_t \tilde{\eta}_0^{(e)} = \sum_{k \geq 1} \frac{k^2 - k - 1}{k^2 (k+1)^2} \tilde{\eta}_k^{(e)}. \]

Hence formally we deduce

\[ \frac{1}{2} \partial_t \sum_{k \geq 1} (\tilde{\eta}_k^{(o)})^2 \leq \sum_{k \geq 1} -d_k \tilde{\eta}_{k-1}^{(o)} \tilde{\eta}_k^{(o)} - (d_{k+1} - d_k) (\tilde{\eta}_k^{(o)})^2 + d_{k+1} \tilde{\eta}_{k+1}^{(o)} \tilde{\eta}_{k+1}^{(o)} \]

\[ = \sum_{k \geq 1} - (d_{k+1} - d_k) (\tilde{\eta}_k^{(o)})^2 \]

\[ \leq -\frac{3}{8} \sum_{k \geq 1} (\tilde{\eta}_k^{(o)})^2. \]
For the even part, we also have
\[
\frac{1}{2} \partial_t \sum_{k \geq 1} (\tilde{\eta}_k^{(e)})^2 = \sum_{k \geq 1} -(d_{k+1} - d_k)(\tilde{\eta}_k^{(e)})^2 \\
\leq -\frac{3}{8} \sum_{k \geq 1} (\tilde{\eta}_k^{(e)})^2.
\] (3.8)

Here we have used the fact that for all \(k \geq 1,\)
\[
d_{k+1} - d_k = \frac{k^2(k+2)}{2(k+1)^2} - \frac{(k-1)^2(k+1)}{2k^2} \\
= \frac{1}{2} + \frac{k^2 - k - 1}{2k^2(k+1)^2} \\
\geq \frac{3}{8}.
\]

It is important to note that \(\tilde{\eta}_0^{(e)}\) has no influence on the evolution of other modes. There are a number of ways to make the calculations (3.7) and (3.8) rigorous (the summations involved may not converge). For instance one may use basic linear semigroup theory as follows. Consider the real Hilbert space \(Y\) formally spanned by the basis functions \(\tilde{e}_k^{(o)}, k \geq 1\) and \(\tilde{e}_k^{(e)}, k \geq 0\) in which this basis is orthonormal, i.e.
\[
Y = \{ \eta = \sum_{k \geq 1} \tilde{\eta}_k^{(o)} e_k^{(o)} + \sum_{k \geq 0} \tilde{\eta}_k^{(e)} e_k^{(e)} \mid \{ \tilde{\eta}_k^{(o)} \}_{k \geq 1}, \{ \tilde{\eta}_k^{(e)} \}_{k \geq 0} \in l^2 \}.
\]

Then \(L\) defines an unbounded closed operator on the Hilbert space \(\tilde{Y} \equiv Y/\mathbb{R} \tilde{e}_0^{(e)}\). (3.7) and (3.8) implies, via a direct application of Hille-Yosida theorem that, \(L\) generates a strongly continuous semigroup with the desired decay estimate
\[
\| e^{tL} \eta(0) \|_{\tilde{Y}} \leq e^{-\frac{3}{8}t} \| \eta(0) \|_{\tilde{Y}}.
\]

We now deal with \(\tilde{\eta}_0^{(e)}\) separately.

\[
| \partial_t \tilde{\eta}_0^{(e)} | = \left| \sum_{k \geq 1} \frac{k^2 - k - 1}{k^2(k+1)^2} \tilde{\eta}_k^{(e)} \right| \\
\lesssim \| \eta \|_{\tilde{Y}}.
\]

Hence \(\tilde{\eta}_0^{(e)}\) converges exponentially to some limit, which we denote by \(\tilde{\eta}_0^{(e)}(\infty)\). So far we have shown that a solution to our linearized equation converges exponentially to \(\tilde{\eta}_0^{(e)}(\infty)(\cos \theta - 1)\) in \(Y\). We state it as the following proposition.

**Proposition 3.1.** For any initial data \(\eta_0 \in Y\), there exists a finite number \(\tilde{\eta}_0^{(e)}(\infty)\) such that
\[
\left\| e^{tL} \eta_0 - \tilde{\eta}_0^{(e)}(\infty)(\cos \theta - 1) \right\|_Y \lesssim e^{-\frac{3}{8}t} \| \eta_0 \|_{\tilde{Y}}.
\]
3.2 Equivalence of norms

In this section, we point out that the $Y$-norm is actually equivalent to a weighted $H^1$ norm. This observation is essential for proving nonlinear stability, since our basis functions $\tilde{e}_k^{(o)}$ and $\tilde{e}_k^{(e)}$ are not helpful for estimating the nonlinear terms. We recall the Hilbert space $\mathcal{H}_{DW}$ defined before:

$$\mathcal{H}_{DW} = \{ \eta \in H^1(S^1) | \eta(0) = 0, \int_{-\pi}^{\pi} \frac{|\eta'|^2}{\sin^2 \frac{\theta}{2}} d\theta < \infty \}.$$ 

And the corresponding inner product of $(\mathcal{H}_{DW}, g)$ is defined to be

$$\langle \xi, \eta \rangle_g = \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{\xi' \eta'}{\sin^2 \frac{\theta}{2}} d\theta.$$ 

We claim that there is an isometry between $Y$ and $\mathcal{H}_{DW}$, given by the following lemma:

**Lemma 3.1.** $\{ \tilde{e}_k^{(o)}, k \geq 1 \} \cup \{ \tilde{e}_l^{(e)}, l \geq 0 \}$ is a complete orthonormal basis for $\mathcal{H}_{DW}$.

**Proof.** First we notice that

$$\frac{(\tilde{e}_k^{(o)})'}{\sin \frac{\theta}{2}} = -2 \sin(k + \frac{1}{2})\theta, \quad \frac{(\tilde{e}_l^{(e)})'}{\sin \frac{\theta}{2}} = 2 \cos(l + \frac{1}{2})\theta, \quad \forall k \geq 1, l \geq 0.$$ 

Hence

$$\langle \tilde{e}_k^{(o)}, \tilde{e}_l^{(o)} \rangle_g = \delta_{kl}, \quad k, l \geq 1,$$

$$\langle \tilde{e}_k^{(o)}, \tilde{e}_l^{(e)} \rangle_g = 0, \quad k \geq 1, l \geq 0,$$

$$\langle \tilde{e}_k^{(e)}, \tilde{e}_l^{(e)} \rangle_g = \delta_{kl}, \quad k, l \geq 0.$$ 

It remains to show completeness. Assume that $\xi \in \mathcal{H}_{DW}$, satisfying

$$\langle \xi, \tilde{e}_k^{(o)} \rangle_g = 0, \quad \langle \xi, \tilde{e}_l^{(e)} \rangle_g = 0, \quad \forall k \geq 1, l \geq 0,$$

i.e.,

$$\int_{-\pi}^{\pi} \frac{\xi'}{\sin \frac{\theta}{2}} \sin(k + \frac{1}{2})\theta d\theta = 0, \quad \forall k \geq 1.$$ 

and

$$\int_{-\pi}^{\pi} \frac{\xi'}{\sin \frac{\theta}{2}} \cos(l + \frac{1}{2})\theta d\theta = 0, \quad \forall l \geq 0.$$ 

We note that the first equality holds for $k = 0$ as well, since

$$\int_{-\pi}^{\pi} \xi' d\theta = 0.$$ 

Since $\{\sin(k + \frac{1}{2})\theta, k \geq 0\} \cup \{\cos(l + \frac{1}{2})\theta, l \geq 0\}$ forms a complete basis of $L^2(S^1)$, thus we have $\xi' = 0$, which implies $\xi = 0$. \qed

Hence we can now identify $Y$ as $\mathcal{H}_{DW}$. Clearly there is a continuous embedding $Y = \mathcal{H}_{DW} \hookrightarrow H^1$. Under the condition \ref{condition}, proposition 3.1 can be improved using this embedding along with the invariance of $\int_{-\pi}^{\pi} \eta d\theta$. 


Proposition 3.2. For any initial data $\eta_{in} \in \mathcal{H}_{DW}$ satisfying $\int_{-\pi}^{\pi} \eta_{in} d\theta = 0$, we have
\[
\| e^{tL} \eta_{in} \|_{\mathcal{H}_{DW}} \lesssim e^{-\frac{3}{8}t} \| \eta_{in} \|_{\mathcal{H}_{DW}}.
\]

Proof. From (3.3) it is easy to see that $\int_{-\pi}^{\pi} e^{tL} \eta_{in} d\theta \equiv 0$ is conserved. Passing to limit, we obtain
\[
\int_{-\pi}^{\pi} \tilde{\eta}^{(e)}_0 (\infty) (\cos \theta - 1) d\theta = 0.
\]
Hence $\tilde{\eta}^{(e)}_0 (\infty) = 0$. \hfill \Box

The constraint $\int_S \eta d\theta = 0$ for $\eta \in \mathcal{H}_{DW}$ is equivalent to
\[
\tilde{\eta}^{(e)}_0 = \sum_{k \geq 1} \frac{1}{k(k+1)} \tilde{\eta}^{(e)}_k.
\]
(3.9)

Recall that the space $\tilde{Y}$ is defined as $Y/Re^{(e)}_0$ in which the norm is given by
\[
\| \eta \|_{\tilde{Y}}^2 = \sum_{k \geq 1} |\tilde{\eta}^{(o)}_k|^2 + \sum_{l \geq 1} |\tilde{\eta}^{(e)}_l|^2.
\]
Hence (3.9) implies that
\[
\| \eta \|_{\mathcal{H}_{DW}} \lesssim \| \eta \|_{\tilde{Y}} \leq \| \eta \|_{\mathcal{H}_{DW}}.
\]
(3.10)
This observation will be useful in the next section.

3.3 Nonlinear stability

Proof of Theorem 1.5. Consider the nonlinear equation for $\eta = \omega + \sin x$,
\[
\begin{cases}
\eta_t = L\eta + v' \eta - v \eta', \\
v' = H\eta, \quad v(t,0) = 0.
\end{cases}
\]
(3.11)

To avoid the technical difficulties caused by the evolution of $\tilde{\eta}^{(e)}_0$, we work with the natural inner product $\tilde{g}$ in $\tilde{Y}$. The discussion in Section 3.1 gives
\[
\langle L\eta, \eta \rangle \tilde{g} \leq -\frac{3}{8} \| \eta \|_{\tilde{Y}}^2.
\]
From (3.11), taking $\tilde{g}$-inner product with $\eta$, we get
\[
\frac{1}{2} \frac{d}{dt} \| \eta \|_{\tilde{Y}}^2 = \langle L\eta, \eta \rangle \tilde{g} + \langle v' \eta - v \eta', \eta \rangle \tilde{g}
\leq -\frac{3}{8} \| \eta \|_{\tilde{Y}}^2 + \langle v' \eta - v \eta', \eta - \tilde{\eta}^{(e)}_0 (\cos \theta - 1) \rangle g.
\]
where $g$ is the inner product in $\mathcal{H}_{DW}$. We estimate the second term as

$$
\langle v' \eta - v \eta', \eta - \eta_0^{(e)}(\cos \theta - 1) \rangle_g = \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{1}{\sin^2 \frac{\theta}{2}} \frac{(v'' \eta - v \eta'')(\eta' + \tilde{\eta}_0^{(e)} \sin \theta)}{\sin \frac{\theta}{2}} d\theta 
$$

$$
= \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{v'' \eta \eta'}{\sin^2 \frac{\theta}{2}} d\theta - \frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{v \eta'' \eta'}{\sin^2 \frac{\theta}{2}} d\theta + \frac{\tilde{\eta}_0^{(e)}}{2\pi} \int_{-\pi}^{\pi} \frac{(v'' \eta - v \eta'')(\cos \frac{\theta}{2})}{\sin \frac{\theta}{2}} d\theta 
$$

$$
=: I + II + III.
$$

Estimate of I:

$$
I \lesssim \|v''\|_{L^2} \left\| \frac{\eta}{\sin \frac{\theta}{2}} \right\|_{L^\infty} \left\| \frac{\eta'}{\sin \frac{\theta}{2}} \right\|_{L^2} 
$$

$$
\lesssim \|\eta\|_{\mathcal{H}_{DW}^3} \lesssim \|\eta\|_{\mathcal{H}_{DW}^3}.
$$

For the second line we have used the following estimate

$$
\left| \frac{\eta}{\sin \frac{\theta}{2}} \right| \lesssim \frac{1}{\theta} \int_0^\theta \eta' (\tau) d\tau 
$$

$$
\lesssim \frac{1}{\theta} \left\| \frac{\eta'}{\sin \frac{\theta}{2}} \right\|_{L^2} \left( \int_0^\theta \sin^2 \frac{\tau}{2} d\tau \right)^{\frac{1}{2}} 
$$

$$
\lesssim \|\eta\|_{\mathcal{H}_{DW}}.
$$

Estimate of II:

$$
II = -\frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{v(\eta^2)'}{\sin^2 \frac{\theta}{2}} d\theta 
$$

$$
= \frac{1}{8\pi} \int_{-\pi}^{\pi} \frac{v' \eta'^2}{\sin^2 \frac{\theta}{2}} d\theta + \frac{1}{8\pi} \int_{-\pi}^{\pi} v \eta'^2 \left( \frac{1}{\sin^2 \frac{\theta}{2}} \right)' d\theta 
$$

$$
\lesssim \|v'\|_{L^\infty} \left\| \frac{\eta'}{\sin \frac{\theta}{2}} \right\|_{L^2}^2 + \int_{-\pi}^{\pi} \frac{v \eta'^2 \cos \frac{\theta}{2}}{\sin^3 \frac{\theta}{2}} d\theta 
$$

$$
\lesssim \|\eta'\|_{L^2} \|\eta\|_{\mathcal{H}_{DW}^2} + \left\| \frac{\eta}{\sin \frac{\theta}{2}} \right\|_{L^\infty} \|\eta\|_{\mathcal{H}_{DW}}^2 
$$

$$
\lesssim \|\eta\|_{\mathcal{H}_{DW}^3}^3.
$$

For the last inequality we have used Sobolev embedding as follows

$$
\left\| \frac{v}{\sin \frac{\theta}{2}} \right\|_{L^\infty} \lesssim \|v'\|_{L^\infty} \lesssim \|v''\|_{L^2} \lesssim \|\eta'\|_{L^2}.
$$

(3.13)
Estimate of III:

\[
\begin{align*}
\text{III} & \lesssim \|\eta\|_{H_{DW}} \int_{S^1} \left| \frac{(\eta H \eta' + \eta' H \eta)}{\sin \frac{\theta}{2}} \cos \frac{\theta}{2} \right| + \frac{v|\eta'|}{\sin^2 \frac{\theta}{2}} \, d\theta \\
& \lesssim \|\eta\|_{H_{DW}} \left( \left\| \frac{\eta}{\sin \frac{\theta}{2}} \right\| L^\infty \left\| H\eta' \right\|_2 + \|\eta\|_{H_{DW}} \left\| H \eta \right\|_\infty \right) + \frac{v}{\sin \frac{\theta}{2}} \left\| \eta \right\|_{H_{DW}} \\
& \lesssim \|\eta\|_{H_{DW}}^3.
\end{align*}
\]

Combining the above estimates we arrive at

\[\frac{d}{dt} \|\eta\|^2_Y \leq -\frac{3}{4} \|\eta\|^2_Y + C \|\eta\|^3_{H_{DW}},\]  

for some constant \(C > 0\). Remembering that we are working under the condition \(\int_{S^1} \eta d\theta = 0\), (3.10) is valid. Theorem 1.6 now follows easily from the above energy estimate. \(\square\)

**Proof of Theorem 1.6.** Let \(\eta(t, \theta) = \zeta(t, \theta) + \alpha(\cos \theta - 1) = \omega(t, \theta) + \sin \theta + \alpha(\cos \theta - 1)\), where \(\alpha = \int_{S^1} \zeta_i d\theta\). Then we have \(\int_{S^1} \eta d\theta = 0\) and \(\eta\) satisfies the following evolution equations

\[
\begin{align*}
\eta_t &= L\eta + \alpha((1 - \cos \theta)(\eta' + v') - \sin \theta(\eta + v)) + v' \eta - \eta'v', \\
v' &= H\eta, \quad v(t, 0) = 0.
\end{align*}
\]

Taking \(\tilde{g}\)-inner product with \(\eta\) and applying the estimates in the previous proof, we get

\[
\begin{align*}
\frac{d}{dt} \|\eta\|^2_Y &= \langle L\eta, \eta \rangle_{\tilde{g}} + \alpha \langle (1 - \cos \theta)(\eta' + v') - \sin \theta(\eta + v), \eta \rangle_{\tilde{g}} + \langle v' \eta - \eta'v', \eta \rangle_{\tilde{g}} \\
& \leq -\frac{3}{4} \|\eta\|^2_Y + \alpha \langle (1 - \cos \theta)(\eta' + v') - \sin \theta(\eta + v), \eta - \eta_0^{(e)}(\cos \theta - 1) \rangle_{\tilde{g}} \\
& \quad + C \|\eta\|_{H_{DW}}^3.
\end{align*}
\]

Denote the second term on the right hand side by \(S\). We handle it using integration by parts, (3.12) and (3.13),

\[
\begin{align*}
S &= \frac{\alpha}{4\pi} \int_{S^1} \left[ (1 - \cos \theta)(\eta'' + v'') - \cos \theta(\eta + v) \right] \frac{\eta' + \eta_0^{(e)}}{\sin \frac{\theta}{2}} \sin \frac{\theta}{2} \, d\theta \\
&= \frac{\alpha \eta_0^{(e)}}{2\pi} \int_{S^1} (\eta'' + v'') \sin \theta d\theta - \frac{\alpha}{4\pi} \int_{S^1} (\eta + v) \frac{\eta' \cos \theta}{\sin \frac{\theta}{2}} d\theta - \frac{\alpha \eta_0^{(e)}}{2\pi} \int_{S^1} (\eta + v) \frac{\cos \frac{\theta}{2}}{\sin \frac{\theta}{2}} d\theta \\
& \lesssim |\alpha| \eta_0^{(e)}(\|\eta''\|_L^2 + \|H \eta''\|_L^2) + (|\alpha| \|\eta\|_{H_{DW}} + |\alpha| \eta_0^{(e)})(\|\eta\|_{L^\infty} + \left\| \eta \frac{\sin \theta}{\sin \frac{\theta}{2}} \right\|_{L^\infty} + \left\| \frac{v}{\sin \frac{\theta}{2}} \right\|_{L^\infty}) \\
& \lesssim |\alpha| \|\eta\|_{H_{DW}}^2.
\end{align*}
\]

It remains to use the estimate \(|\alpha| + \|\eta(0, \cdot)\|_{H_{DW}} \lesssim \|\zeta_i\|_{H_{DW}}\) to finish the proof. \(\square\)

**Remark 3.2.** In our proof of Theorem 1.6, the smallness of \(\alpha\) played an important role. Based on the numerical behaviour of (1.1), we conjecture that the shifted ground states \(-\sin \theta - \alpha(\cos \theta - 1)\) are exponentially stable for all \(\alpha \in \mathbb{R}\) in some suitable spaces. For now, we only know this is true in \(H_{DW}\) for \(|\alpha|\) small.
4 Well-posedness of the Linearized Equation at \(-\sin 2\theta\)

Consider solutions of the form
\[
w = -\sin 2\theta + \eta, \quad u = \frac{1}{2}\sin 2\theta + v.
\]

Then the linearized equation for \(\eta\) reads
\[
\begin{aligned}
\eta_t + \frac{1}{2}\sin 2\theta \eta' - \cos 2\theta \eta + \sin 2\theta H \eta - 2\cos 2\theta v &= 0, \\
v' &= H \eta.
\end{aligned}
\] (4.1)

Denote
\[
L\eta = -[\sin 2\theta, \frac{\eta}{2} + v],
\]
then
\[
\eta_t = L\eta.
\]

As before, we calculate the linearized operator \(L\) on the Fourier side. For odd data:
\[
Le_k^{(o)} = A_k e_k^{(o)} + B_k e_{k-2}^{(o)}, \quad e_k^{(o)} = \sin k\theta, \quad k \geq 2,
\]
and
\[
Le_1^{(o)} = -\frac{1}{4}e_3^{(o)} + \frac{3}{4}e_1^{(o)},
\]
where the coefficients
\[
A_k = -\frac{(k-2)^2}{4k}, \quad B_k = \frac{(k+2)(k-2)}{4k}
\]
are different from those in Section 3.1. Similarly, for the even data we have
\[
Le_k^{(e)} = A_k e_k^{(e)} + B_k e_{k-2}^{(e)}, \quad e_k^{(e)} = \cos k\theta, \quad k \geq 2,
\]
and
\[
Le_1^{(e)} = -\frac{1}{4}e_3^{(e)} - \frac{3}{4}e_1^{(e)}.
\]

Assume \(\eta = \sum_{k \geq 1} \eta_k^{(o)} e_k^{(o)} + \eta_k^{(e)} e_k^{(e)}\) which satisfies \(\int_{S^1} \eta d\theta = 0\), then we have
\[
\partial_t \eta_1^{(o)}(t) = \frac{3}{4} \eta_1^{(o)} + \frac{5}{12} \eta_3^{(o)}, \quad \partial_t \eta_2^{(o)}(t) = \frac{3}{4} \eta_4^{(o)},
\]
\[
\partial_t \eta_k^{(o)}(t) = A_{k-2} \eta_{k-2}^{(o)} + B_{k+2} \eta_{k+2}^{(o)}, \quad k \geq 3.
\]
and
\[
\partial_t \eta_1^{(e)}(t) = -\frac{3}{4} \eta_1^{(e)} + \frac{5}{12} \eta_3^{(e)}, \quad \partial_t \eta_2^{(e)}(t) = \frac{3}{4} \eta_4^{(e)},
\]
\[
\partial_t \eta_k^{(e)}(t) = A_{k-2} \eta_{k-2}^{(e)} + B_{k+2} \eta_{k+2}^{(e)}, \quad k \geq 3.
\]

Define \(X \subseteq L^2\) to be the real Hilbert space with the following inner product
\[
\langle \eta, \xi \rangle_X = \sum_{k \geq 1} g_k^{(o)} \eta_k^{(o)} \xi_k^{(o)} + g_k^{(e)} \eta_k^{(e)} \xi_k^{(e)},
\]
where \( g^{(o)}_k, g^{(e)}_k \) are to be determined later. For odd data, one has

\[
\langle L\eta, \eta \rangle_X = \frac{3}{4} g^{(o)}_1 \eta^{(o)}_1 + \frac{5}{12} g^{(o)}_3 + \frac{3}{4} g^{(o)}_2 \eta^{(o)}_2 \eta^{(o)}_4 + \sum_{k \geq 3} g^{(o)}_k \eta^{(o)}_k (A_{k-2}\eta^{(o)}_{k-2} + B_{k+2}\eta^{(o)}_{k+2})
\]

\[
= \frac{3}{4} g^{(o)}_1 (\eta^{(o)}_1)^2 + \sum_{k \geq 1} (g^{(o)}_k B_{k+2} + g^{(o)}_k A_k)\eta^{(o)}_k \eta^{(o)}_{k+2}.
\]

Unless we set \( g^{(o)}_{2k-1} = 0, \forall k \geq 1 \) and

\[
g^{(o)}_{2k} B_{2k+2} + g^{(o)}_{2k} A_{2k} = 0, \quad k \geq 1,
\]

\( \langle L\eta, \eta \rangle_X \) changes sign in general. Hence there seems no natural conserved (or decreasing) norm for odd perturbation at \(-\sin 2\theta\). This accounts for the numerically observed nonlinear instability of \(-\sin 2\theta\), as mentioned in [9].

We remark that if we only consider initial data of the form \( \eta = \sum_{k \geq 1} \eta^{(e)}_k e^{(e)}_k \), a conserved (semi)norm can be found using (4.2). Furthermore, our methods in Section 3 can be easily adapted to prove exponential decay for such data. More precisely we set

\[
e^{(e)}_{2k} = \frac{e^{(o)}_{2k+2}}{2k+2} - \frac{e^{(o)}_{2k}}{2k}, \quad k \geq 1,
\]

and compute \( L \) for such basis vectors. This will lead to exponential decay in the corresponding weighted \( H^1 \) space

\[
\mathcal{H}_{DW} = \left\{ \eta = \sum_{k \geq 1} \eta^{(0)}_{2k} e^{(o)}_{2k} \left| \int_{S^1} \frac{\eta^2}{\sin \theta} d\theta \right| \right\}.
\]

Now we turn to consider even data \( \eta = \sum_{k \geq 1} \eta^{(e)}_k e^{(e)}_k \), for which the situation is quite different. We prove Theorem 1.7 as follows.

\[
\langle L\eta, \eta \rangle_X = -\frac{3}{4} g^{(e)}_1 (\eta^{(e)}_1)^2 + \sum_{k \geq 1} (g^{(e)}_k B_{k+2} + g^{(e)}_k A_k)\eta^{(e)}_k \eta^{(e)}_{k+2}.
\]

Let \( g^{(e)}_1 = g^{(e)}_4 = 1, g^{(e)}_2 = 0 \) and set

\[
g^{(e)}_k B_{k+2} + g^{(e)}_k A_k = 0, \quad k \geq 1.
\]

It is easy to check that

\[
g^{(e)}_k \sim k^2, \quad k \to \infty.
\]

The (semi)norm defined by \( g^{(e)}_k \) will be decreasing for even data. More precisely from (4.3) we have

\[
\langle L\eta, \eta \rangle_X = -\frac{3}{4} (\eta^{(e)}_1)^2.
\]

(4.4) clearly implies Theorem 1.7.
Remark 4.1. Here we are not able to prove exponential decay, so nonlinear stability cannot be deduced. However, according to our own numerical experiments, solution to (1.1) with initial data of the form $-\sin 2\theta + \epsilon \cos \theta$ converges to some multiple of $-\sin 2\theta$ instead of ground states. This interesting phenomenon remains to be investigated in future works.

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