A Stochastic Volatility Model with Mean-reverting Volatility Risk Premium

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Abstract. The volatility risk premium (VRP) has long been the core issue in option pricing and risk management. The VRP is usually defined as a linear function of volatility which ignores the time-varying property of VRP and limits the degree of freedom of the model. In this paper, we adopt a CIR process in the stochastic volatility model (VRP-CIR-SV) to incorporate the mean-reverting and time-varying properties of VRP. We show that the decomposition of VRP is consistent to investor’s behaviour. Our Monte Carlo simulation results show that, compared with the traditional linear VRP model, the VRP-CIR-SV model can better depict the rich shapes of implied volatility curve. Our paper innovatively models the time-varying VRP with mean-reverting property, which may provide new thoughts for VRP estimation.

1. Introduction

The canonical B-S option pricing model assumes that the return of the underlying asset follows normal distribution and the volatility is constant, but these assumptions are contrary to reality. Scholars have proposed stochastic volatility (SV) models such as the Hull-White (1987) model and the Heston (1993) model. The Heston model with affine structure cannot well describe the nonlinear characteristics of financial time series, while the non-affine GARCH model which can better describe the volatility and distribution of asset prices to obtain more accurate option prices has received more attention in recent years (Kaeck, 2012; Durham, 2013). When price option, it is necessary to convert the underlying asset price process under the objective probability measure to risk-neutral measure, and the key to the conversion is the reasonable determination of volatility risk premium (VRP). The traditional SV model sets the VRP as a linear function of variance. However, the real relation may be nonlinear and time-varying. Risk aversion of investors is the main determinant of VRP (Bollerslev et al. 2010) which is found time varying in literature. For example, Wu (2019) shows that the setting of constant coefficient risk aversion does not conform to actual market conditions empirically. Rosenberg and Engle (2001) and Grith et al. (2013) find that investor risk aversion has the characteristics of autocorrelation, mean reversion and counter-cyclicality. Ma (2020) shows that the VRP is related to the degree of trading activity, and the impact of trading volume on risk premium is asymmetric.

The time-varying VRP has been used on the prediction of degree of risk aversion (Cotter and Hanly, 2010), future excess returns (Yoon, 2017) and optimizing investment portfolios (Diaz and Esparcia, 2021) etc. But few researches have studied the impact of time-varying risk aversion on option prices. Kiesel and Rahe (2017) construct a two-factor option pricing model and find that models with time-varying risk aversion can better predict risk. Wu (2019) demonstrates that a non-affine time-varying...
risk aversion SV (TVRA-SV) option pricing model can significantly improve the pricing performance of SSE 50ETF options.

Inspired by Rosenberg and Engle (2001) and Grith et al. (2013), we adopt the CIR (Cox-Ingersoll-Ross) process into VRP modelling and construct a stochastic volatility (VRP-CIR-SV) model with mean reversion and time-varying properties, then use Monte Carlo simulation to demonstrate the flexible ability of the model to capture the rich shapes of implied volatility curve in reality. Our paper innovatively models the time-varying VRP with mean-reverting property, which may provide new thoughts for VRP estimation. This paper is organized as follows: Section 2 is the model construction; Section 3 is the numerical simulation; Section 4 concludes the paper.

2. Model Construction

2.1. GARCH Diffusion Model

Let \( S \) be the price of the underlying asset, \( V \) be the volatility of the return rate of the underlying asset. GARCH diffusion model proposed by Nelson (1990) assumes that: under the objective probability measure, \( S \) and \( V \) obey the following stochastic processes:

\[
\frac{dS}{S} = \mu dt + \sqrt{V} d\tilde{W}_s\tag{1}
\]

\[
dV = \kappa_V (\theta_V - V) dt + \sigma_V \left( \rho_V d\tilde{W}_s + \sqrt{1 - \rho^2_v} d\tilde{W}_v \right)\tag{2}
\]

\( \mu, \kappa_V, \theta_V, \sigma_V, \rho_V \) are constants and \( \kappa_V, \theta_V, \sigma_V \) are greater than zero, and \( \mu \) is the drift rate of asset price return, \( \kappa_V \) is the mean recovery speed of volatility, \( \theta_V \) is the long-term average of volatility, \( \sigma_V \) is the volatility of the underlying asset return volatility, \( \rho_V \) is the correlation coefficient between volatility and asset return, \( \tilde{W}_s \) and \( \tilde{W}_v \) are independent standard Brownian motions.

2.2. Volatility Risk Premium

We define \( d\tilde{W}_s \) and \( d\tilde{W}_v \) following the Girsanov theorem (Wu, 2019):

\[
d\tilde{W}_s + \gamma_s dt = dW_s\tag{3}
\]

\[
d\tilde{W}_v + \gamma_v dt = dW_v\tag{4}
\]

\( \gamma_s \) represents the market price of asset risk (corresponds to Brown shock \( d\tilde{W}_s \)), \( \gamma_v \) represents the market price of volatility risk (corresponds to Brown shock \( d\tilde{W}_v \)). \( W_s \) and \( W_v \) are independent standard Brownian motions. Since the discounting process of asset prices is a martingale under the risk-neutral measure, where the underlying asset return is risk-free rate of return \( r \), we have:

\[
\gamma_s = \frac{\mu - r}{\sqrt{V}}\tag{5}
\]

We substitute formula (3)-(5) into (1)-(2) and let \( \ln S = X \). According to Ito’s lemma, we get the process of the underlying asset price under the risk-neutral measure, where \( \lambda \) represents the VRP.

\[
dX = \left(r - \frac{1}{2}V\right) dt + \sqrt{V} dW_s\tag{6}
\]

\[
dV = \left[\kappa_V (\theta_V - V) - \lambda\right] dt + \sigma_V \left( \rho_V dW_s + \sqrt{1 - \rho^2_v} dW_v \right)\tag{7}
\]

\(
\lambda = \sigma_V \rho_V \gamma_s + \sigma_V \sqrt{1 - \rho^2_v} \gamma_v V\tag{8}
\)
In practice, $\rho_\ell$ is usually less than zero, indicating that there is a negative correlation between the return on assets and changes in volatility, i.e., the so called "leverage effect" (Black, 1976; Bakshi, 2003). Leverage effect and negative systematic volatility risk are inherently consistent. Falling stock prices lead to an increase in the company's debt ratio, leading to the increasing risks and then volatility increases correspondingly. Mature financial markets usually present a strong leverage effect, that is $\rho_\ell \rightarrow -1$ (Christie A, 1982), then VRP is $\lambda = \sigma_\ell V_\rho \gamma_\ell = \sigma_\ell \rho_\ell (\mu - r)\sqrt{V}$ . VRP and the leverage effect ($\rho_\ell$) no longer show a strong correlation. Anti-leverage effect happens when $\rho_\ell$ is positive, which has been documented in immature markets (Chen, 2019; Wang, 2019).

As there is no unified conclusion in leverage effect in the literature, the process of $\gamma_\ell$ in formula (8) is difficult to specify. Previous literature usually assumes the VRP follows a linear form (Heston, 1993), which cannot capture the variation of VRP in reality. Scholars have found that the VRP is closely related to the degree of risk aversion of investors (Bollerslev et al. 2011), and risk aversion presents characteristics such as time-varying, autocorrelation and mean reversion (Rosenberg, 2002; Grith, 2013). CIR process can better summarize the above characteristics. According to such characteristics, we propose that the VRP obeys the CIR process as follows:

$$d\lambda = \kappa_\lambda (\delta\theta_\lambda - \lambda)dt + \sigma_\lambda \lambda (\rho_\ell \lambda dW_\ell + \sqrt{1 - \rho_\ell^2} dW_\ell)$$ \hspace{1cm} (9)

$$\theta_\lambda = \sigma_\lambda \rho_\ell (\mu - r)\sqrt{V}$$ \hspace{1cm} (10)

Where $\kappa_\lambda$ is the mean recovery speed of the VRP, $\delta\theta_\lambda$ is the long-term average of the VRP, $\delta$ is the adjustment factor, $\sigma_\lambda$ is the volatility coefficient of the VRP, $\rho_\ell$ is the correlation coefficient between the VRP and the asset price.

2.3. Theoretical Analysis

Our specification of VRP has the following features: (1) In the long term, the VRP is mainly related to the systematic volatility risk $\theta_\Delta$ . (2) In the short term, the changes in the VRP caused by non-systematic risks are characterized by stochastic process $\sigma_\lambda \lambda (\rho_\ell \lambda dW_\ell + \sqrt{1 - \rho_\ell^2} dW_\ell)$ . (3) The larger the $\kappa_\lambda$ and the smaller the $\sigma_\lambda$, the more stable the VRP will be, and the less it will be affected by other factors in the short term. (4) $\rho_\ell$ the process incorporates both the leverage effect and the anti-leverage effect. The VRP can be regarded as the difference between the volatility estimated under the objective probability measure and the volatility estimated under the risk-neutral measure, i.e., the realized volatility of the underlying asset return minus the implied volatility of the option price (Carr and Wu, 2009).

Considering a financial market that exhibits leverage effect (vice versa for the anti-leverage effect), we can observe three different performances of VRP incorporated in the model: (1) $\theta_\Delta$ represents the long-term mean value of VRP associated with the risk hedgers, who hedge in opposite directions in cash market and derivatives market. When volatility rises, gains made from derivative market can make up losses from cash market caused by falling asset prices, so investors are willing to buy derivatives while expected volatility rises. We will see when the implied volatility is greater than the realized volatility the VRP becomes negative. (2) $\sigma_\lambda \lambda (\rho_\ell \lambda dW_\ell + \sqrt{1 - \rho_\ell^2} dW_\ell)$ represents VRP associated with speculators, who operate in the same direction in cash market and derivatives market. Losses caused by falling asset prices will be accompanied by derivative losses caused by rising volatility. Therefore, investors are only willing to spend lower prices to buy derivatives related to volatility. We will see when implied volatility falls and the VRP rises. (3) $\sigma_\lambda \sqrt{1 - \rho_\ell^2} dW_\ell$ represents the VRP associated
with noise traders, who have no specific trading strategies in the options market. Therefore, the VRP has nothing to do with the rise or fall of asset prices, which is a purely random process.

In summary, our stochastic volatility model based on the mean recovery process of the VRP (VRP-CIR-SV Model) can well describe the behaviour of VRP in real market condition.

3. Numerical Simulation

There are various forms of implied volatility curves, e.g., "volatility smile/smirk/sadness". A model which conforms to market shall well capture the various shapes of implied volatility. We test the ability of the VRP-CIR-SV model to portray the implied volatility curves, setting up three sets (Set I/II/III) of modelling experiments using Monte Carlo method (the number of simulations is 500,000). We simulate the European call option price based our model, and then interpolate the implied volatility curve. The Euler dispersion formula of the VRP-CIR-SV model are as follows:

\[
\ln S_{t+\Delta t} = \ln S_t + \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sqrt{\sigma^2 \Delta t} \sqrt{N(0,1)}
\]

\[
V_{t+\Delta t} = V_t + \left[ \kappa_V \left( \theta_V - V_t \right) - \lambda \right] \Delta t + \sigma_V \sqrt{\Delta t} \left( \rho_V N(0,1) + \sqrt{1 - \rho_V^2} N(0,1) \right)
\]

\[
\lambda_{t+\Delta t} = \lambda_t + \kappa_\lambda \left( \delta \sigma_\lambda \rho_\lambda \left( \mu - r \right) \sqrt{V_t - \lambda_t} \right) \Delta t + \sigma_\lambda \sqrt{\Delta t} \left( \rho_\lambda N(0,1) + \sqrt{1 - \rho_\lambda^2} N(0,1) \right)
\]

The option pricing formula is:

\[
C = \frac{1}{N} \sum_{i=1}^{N} \max \left( S_{t+\Delta t} - K, 0 \right)
\]

Further, we select the stochastic volatility model with linear VRP (LVRP-SV) for comparison:

\[
dX = \left( r - \frac{1}{2} \sigma^2 \right) dt + \sqrt{\sigma^2 } dW_t
\]

\[
dV = \left[ \kappa_V \left( \theta_V - V \right) - \lambda V \right] dt + \sigma_V \sqrt{\rho_\lambda \sqrt{V + \lambda \Delta t} + \sqrt{1 - \rho_\lambda^2} N(0,1)}
\]

The parameter refers to Wu (2019). The fixed parameters are shown in Table 1.

| Parameter | $S_0$ | $V_0$ | $\lambda_0$ | $r$ | $\Delta t$ | $\tau$ | $\mu$ |
|-----------|-------|-------|-------------|-----|-----------|--------|------|
| Value     | 100   | 0.1   | -0.1        | 0.03| 1/250     | 0.2    | 0.1  |

The varied parameters are shown in Table 2.

| Parameter | $\kappa_V$ | $\theta_V$ | $\rho_V$ | $\kappa_\lambda$ | $\delta$ | $\rho_\lambda$ | $\lambda$ |
|-----------|------------|------------|----------|-------------------|------|-----------------|--------|
| Value     | 1          | 0.1        | -0.8     | 1                 | 5    | -0.8            | -1     |

The implied volatility curves of the three sets of modelling experiments are shown in Figure 1 to Figure 3. Obviously, no matter how the $\sigma_V$ changes, the implied volatility under the LVRP-SV model gradually decreases as the virtual value of the call option increases, and the shapes of the implied volatility curve that can be described are relatively simple.

The VRP-CIR-SV model can describe implied volatility curves more versatile. When the $\sigma_\lambda$ is large:

1. A higher $\sigma_V$ can describe the "volatility smirk", i.e., the implied volatility of in-the-money (ITM)
options and out-of-the-money (OTM) options are higher than that of at-the-money (ATM) options, and the implied volatility of ITM options is much higher than that of OTM options; (2) A lower $\sigma_p$ can describe the "volatility smile", i.e., the implied volatility of ITM options and OTM options are higher than that of ATM options, and the implied volatility of OTM options is much higher than that of ITM options. (3) When the $\sigma_s$ is small, a lower $\sigma_v$ can describe the "volatility sad", i.e., the implied volatility gradually decreases as OTM option becomes deep out-of-the-money.

Figure 1. Implied volatility curve of the experiment set I.

Figure 2. Implied volatility curve of the experiment set II.

Figure 3. Implied volatility curve of the experiment set III.

4. Conclusion
In this paper we incorporate a CIR process into the VRP modelling process innovatively, and constructs a stochastic volatility model (VRP-CIR-SV) with mean reversion feature of VRP. Our model overcomes the drawbacks of the traditional SV model where the VRP is defined as a linear function of volatility, which does not match the reality. The model can well capture the behaviour of risk hedging investors, speculators and noise traders and the jointly effects on the VRP. Monte Carlo simulation results shows that when applied on option pricing, our model can capture the rich shapes of the implied volatility curve much better than the traditional LVRP-SV model. We have not obtained the analytical pricing formula of options under this model in this paper, and this will be our future research direction.
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