The spherical mean Radon transform with centers on cylindrical surfaces

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Abstract

The spherical mean Radon transform maps a function in an n-dimensional space to its integrals (or averages) over spherical surfaces. Recovering a function from its spherical mean Radon transform with centers of spheres of integration restricted to some hypersurface is at the heart of several modern imaging technologies, including SONAR, ultrasound imaging, and photo- and thermoacoustic tomography. In this paper we study the inversion of the spherical mean Radon transform with centers restricted to a cylindrical domain of the form $B \times \mathbb{R}^{n_2}$, where $B$ is any bounded domain in $\mathbb{R}^{n_1}$. As our main results we derive explicit inversion formulas of the backprojection type for that transform. For that purpose we show that the spherical mean Radon transform with centers on a cylindrical surface can be decomposed into two lower dimensional spherical mean Radon transforms. For a cylindrical center set in $\mathbb{R}^3$, we demonstrate that the derived inversion formulas can be implemented by filtered-backprojection type algorithms only requiring $O(N^{4/3})$ floating point operations, where $N$ is the total number of unknowns to be recovered. We finally present some results of our numerical simulations performed in MATLAB, showing that our implementations accurately recover a 3D image of millions of unknowns in a few minutes on an ordinary notebook.

Keywords. Spherical means, Radon transform, inversion, reconstruction formula, back-projection.

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1 Introduction

In 1917, Johan Radon in [64] introduced and studied a transform that integrates a function over all hyperplanes; later this transform has become known as the (classical) Radon transform. Radon itself was inspired by work of Paul Funk, who studied the problem of recovering a function on the two-dimensional sphere from its integrals over all great circles [20]. Once many applications in PDEs, natural science, engineering and medicine of Radon’s and related transforms was perceived, these initiated the study of generalizations of Radon transforms using various different sets of integration, including spheres [2, 15, 16, 28, 33, 42, 43, 58, 60, 72], ellipsoids [23, 49, 71], cones [6, 27, 37, 48, 55] or broken rays [3, 18, 35, 38]. See also [21, 22, 32, 41, 52, 59] for monographs studying general classes of Radon transforms. In the present paper we study a particular instance of the spherical Radon transform, which integrates a function over certain spheres in Euclidian space.

1.1 The spherical mean Radon transform

Among the more recent Radon type transforms, the spherical mean Radon transform which maps a given function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) to its spherical averages

\[
(Mf)(x, r) := \frac{1}{\omega_{n-1}} \int_{S^{n-1}} f(x + r\omega) \, dS(\omega), \quad \text{for } (x, r) \in \mathbb{R}^n \times [0, \infty),
\]

is one of the most deeply studied. The increased interest in that transform is partially based on its relevance for the recently developed thermoacoustic and photoacoustic tomography (see, for example, [40, 70, 72]). However it is also relevant for other imaging technologies such as RADAR or SONAR (see, for example, [33, 47, 56, 63, 74]), SAR imaging (see, for example, [4, 65, 68]) or ultrasound imaging (see, for example, [57, 58]).

The full spherical mean Radon transform depends on the \( n + 1 \)-dimensional parameter \( (x, r) \in \mathbb{R}^n \times [0, \infty) \) and the problem of inverting the spherical mean Radon transform is therefore over-determined. In practical applications one is usually interested in the case where the set of spheres of integration is restricted to some \( n \)-dimensional sub-manifold of all spheres. Depending on the set of considered spheres, there exist various instances of the spherical mean Radon transform. For example, the spheres of integration may be centered at any point in the whole space and have a fixed radius [8, 9, 75], or the centers is located on a hypersurface and the radius is variable [4, 14–17, 43, 44, 46, 50, 57, 65, 74]. Also a spherical mean Radon transform where all spheres of integration are passing through the origin has been studied (see [13, 62, 66]). For thermoacoustic and photoacoustic tomography (as well as for SONAR, SAR and ultrasound imaging), the spherical mean Radon transforms with restricted center sets are the practically most relevant ones. Closed-form inversion formulas for the spherical mean Radon transform with restricted center sets have been known for some time. For example, such formulas exist for the case where the center set is a hyperplane [4, 7, 10, 14, 39, 50, 72] or a sphere [15, 16, 43, 53, 72]. More recently, closed-form inversions have also been derived for the cases of elliptically shaped center sets (see [5, 25, 26, 28, 51, 60, 67]).
1.2 Contributions of the present work

In this paper we study the spherical mean Radon transform with centers of the spheres of integration restricted to a cylindrically shaped hypersurface $\partial B \times \mathbb{R}^{n_2}$, where $B$ is some bounded domain in $\mathbb{R}^{n_1}$ and $n = n_1 + n_2$ is the spatial dimension. We will show that this particular spherical mean Radon transform can be written as the composition of a spherical mean Radon transform with a center set $\partial B$ in $\mathbb{R}^{n_1}$ and another spherical mean Radon transform with a planar center set in $\mathbb{R}^{n_2+1}$. Recall that inversion formulas for the spherical mean Radon transform with a planar center set are well known. Consequently, if an inversion formula is available for the center set $\partial B$, then this decomposition yields an inversion formula for the spherical mean Radon transform with centers on $\partial B \times \mathbb{R}^{n_2}$. As explicit inversion formulas are in particular known for spherical and elliptical center sets, we obtain analytic inversion formulas for any spherical or elliptical cylinder. Our decomposition approach, however, can also be combined with any other reconstruction algorithm for the spherical means, such as time reversal [12,34,69] or series expansion methods [1,44], that can be used for arbitrary bounded domain in $\mathbb{R}^{n_1}$.

In [72], the authors derived an explicit inversion formula for reconstructing the initial data $f$ from the solution of the three-dimensional wave equation

$$\begin{cases}
\partial_t^2 p(x, t) - \Delta_x p(x, t) = 0 & \text{for } (x, t) \in \mathbb{R}^3 \times [0, \infty) \\
(p(x, 0), \partial_t p(x, 0)) = (f(x), 0) & \text{for } x \in \mathbb{R}^3,
\end{cases}$$

(1.1)

restricted to points on a circular cylinder $\partial D_R \times \mathbb{R}$, with $D_R$ being a disc of radius $R$ centered at the origin in $\mathbb{R}^2$. As the solution of the wave equation (1.1) is given by $p(x, t) = (\partial_t t M f)(x, t)$, the problem of recovering the initial data $f(x)$ for $x \in D_R$ from $p(x, t)$ for $(x, t) \in \partial D_R \times \mathbb{R}$ relates to an inversion formula for a spherical mean Radon transform with centers on $\partial D_R \times \mathbb{R}$. Because in [72] the same inversion formula has been shown to be exact for the cases of spherical and planar center sets in $\mathbb{R}^3$, it is known as the universal backprojection formula (UBP). Recently, in [51] it has been shown that the UBP is also exact for the case that the center set is an ellipsoid in a three-dimensional space. Further, the UBP has been generalized to arbitrary spatial dimension in [43], where it is shown to be exact for spherical center sets. Later, in [28] the UBP has been shown to be exact for ellipsoids in arbitrary spatial dimensions. Finally, in the very recent work [29] it is shown that the UBP even provides exact reconstruction for any elliptical cylinder (and, more generally, special quadric hypersurfaces) in any spatial dimension. The approach in the present article is very different from that of [29]. Further, the results in the present paper are different from [29] and independent of a particular inversion formula: The combination of any reconstruction algorithm for a planar center set in $\mathbb{R}^{n_2+1}$ with one for a bounded domain $B \subset \mathbb{R}^{n_1}$ yields a reconstruction algorithm for the spherical mean Radon transform with centers on $\partial B \times \mathbb{R}^{n_2}$.

1.3 Notations

For any domain $U \subset \mathbb{R}^n$ we denote by $C^\infty_c(U)$ the set of all functions $f : \mathbb{R}^n \to \mathbb{R}$ that are smooth (that is $C^\infty$) and have compact support in $U$. We use $S^{n-1} := \{ \omega \in \mathbb{R}^n : |\omega| = 1 \} \subset \mathbb{R}^n$ to denote the $n-1$-dimensional unit sphere. Here $| \cdot |$ stands for the Euclidian norm in $\mathbb{R}^n$; the associated
inner product will be denoted by \((x, y) \leftrightarrow x \cdot y\). By \(dS\) we denote the standard \(n - 1\)-dimensional surface measure, and by \(\omega_{n-1} := \int_{S_{n-1}} 1 \, dS(\omega)\) the total surface area of the unit sphere.

Let \(\Lambda\) be a set and \(\Phi: \Lambda \times \mathbb{R} \to \mathbb{R}\) be a function. If \(\Phi\) is integrable in the second component, we define the (partial) Hilbert transform \(H_s \Phi: \Lambda \times \mathbb{R} \to \mathbb{R}\) by

\[
(H_s \Phi)(\lambda, s) = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\Phi(\lambda, s')}{s - s'} \, ds
\]

for \((\lambda, s) \in \Lambda \times \mathbb{R}\). (1.2)

Likewise we denote by \(\partial_s \Phi\) the partial derivative of \(\Phi\) in the variable \(s\), and by \(D_s = (2s)^{-1} \partial_s\) the differentiation operator with respect to \(s\). Finally, by some abuse of notation we use \(s\) to denote the multiplication operator mapping the function \(\Phi\) to the function \((\lambda, s) \mapsto s \Phi(\lambda, s)\).

1.4 Outline

The rest of this paper is organized as follows. In Section 2.1 we show that the spherical mean Radon transform with centers on \(\partial B \times \mathbb{R}^{n_2}\) can be decomposed in two partial spherical mean Radon transforms, one with centers on \(\partial B\) and one with a planar center set in \(\mathbb{R}^{n_2+1}\). This decomposition will be used to derive inversion strategies in Section 2.2 and Section 3. In Section 3.1 we derive an inversion formula for the case where \(\partial B \times \mathbb{R}^{n_2}\) is an elliptical cylinder. In Section 3.2 we derive a different inversion formula for the special case where \(B\) is a disc in \(\mathbb{R}^2\). This inversion formula will be local for odd spatial dimension. In Section 4 we derive filtered backprojection algorithms based on the inversion formula for the spherical mean Radon transform with centers on a circular cylinder in a three-dimensional space. There we also present some numerical results. The paper concludes with a short discussion in Section 5.

2 The spherical mean Radon transform with centers on a cylinder

Throughout this paper, let \(n_1 \geq 2, n_2 \geq 1\) be two given natural numbers and denote by \(n := n_1 + n_2\) the spatial dimension. We study the inversion of the spherical (mean) Radon transform with centers on a cylinder \(\partial B \times \mathbb{R}^{n_2}\), where \(B \subset \mathbb{R}^{n_1}\) is any bounded domain. It will be convenient to identify \(\mathbb{R}^n = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\) and to write any point in \(\mathbb{R}^n\) in the form \(x = (x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}\) and any point on the cylinder \(\partial B \times \mathbb{R}^{n_2}\) in the form \(p = (p_1, p_2) \in \partial B \times \mathbb{R}^{n_2}\).

**Definition 2.1** (Spherical mean Radon transform with centers on \(\partial B \times \mathbb{R}^{n_2}\)).

For \(f \in C^\infty_c(\partial B \times \mathbb{R}^{n_2})\) we define the spherical (mean) Radon transform \(M f: \partial B \times \mathbb{R}^{n_2} \times [0, \infty) \to \mathbb{R}\) of \(f\) with centers on \(\partial B \times \mathbb{R}^{n_2}\) by

\[
(M f)(p_1, p_2, r) := \frac{1}{\omega_{n-1}} \int_{S_{n-1}} f((p_1, p_2) + r \omega) \, dS(\omega)
\]

for \((p_1, p_2, r) \in \partial B \times \mathbb{R}^{n_2} \times [0, \infty)\).

It is clear that the spherical mean Radon transform is an element of \(C^\infty(\partial B \times \mathbb{R}^{n_2} \times [0, \infty))\), and that \(M\) vanishes in a neighbourhood of \(r = 0\). In the following we show that \(M\) can be written
as the product of two partial spherical mean Radon transforms, one with a center set $\partial B$ in $\mathbb{R}^{n_1}$ and one with a planar center set in $\mathbb{R}^{n_2+1}$. This decomposition will be used to derive new inversion formulas for the spherical mean Radon transform with cylindrical center sets; see Section 3.

### 2.1 Decomposition

As already mentioned, our first goal is to write the spherical mean Radon transform $Mf$ as the decomposition of two partial spherical mean Radon transforms. These partial spherical mean Radon transforms are defined as follows.

**Definition 2.2 (Partial spherical mean Radon transforms).**

(a) For $f \in C_c^\infty(B \times \mathbb{R}^{n_2})$ we define the partial spherical mean Radon transform $M_1f : \partial B \times \mathbb{R}^{n_2} \times [0, \infty) \to \mathbb{R}$ in the horizontal direction by

$$
(M_1f)(p_1, x_2, s) := \frac{1}{\omega_{n_1-1}} \int_{S^{n_1-1}} f(p_1 + s\omega_1, x_2) \, dS(\omega_1) \quad \text{for } (p_1, x_2, s) \in \partial B \times \mathbb{R}^{n_2} \times [0, \infty).
$$

(b) For $\Phi \in C_c^\infty(\partial B \times \mathbb{R} \times \mathbb{R}^{n_2})$ we define the partial spherical mean Radon transform $M_2\Phi : \partial B \times \mathbb{R}^{n_2} \times [0, \infty) \to \mathbb{R}$ in the vertical direction by

$$
(M_2 \Phi)(p_1, p_2, r) := \frac{1}{\omega_{n_2}} \int_{S^{n_2}} \Phi(p_1, (p_2, 0) + r\sigma) \, dS(\sigma) \quad \text{for } (p_1, p_2, r) \in \partial B \times \mathbb{R}^{n_2} \times [0, \infty).
$$

The functions $M_1f \in C^\infty(\partial B \times \mathbb{R}^{n_2} \times [0, \infty))$ and $M_2\Phi \in C^\infty(\partial B \times \mathbb{R}^{n_2} \times \mathbb{R})$ can be considered as partial spherical mean Radon transforms, since $M_1f$ equals the spherical mean Radon transform with a center set $\partial B$ in $\mathbb{R}^{n_1}$ when $x_2 \in \mathbb{R}^{n_2}$ is considered as a fixed parameter, and $M_2\Phi$ equals the spherical mean Radon transform with a planar center set in $\mathbb{R}^{n_2+1}$ when $p_1 \in \partial B$ is considered as a fixed parameter.

For the following we further use

$$
[M_1f]_\pm(p_1, x_2, s) := \begin{cases} 
-(M_1f)(p_1, x_2, |s|) & \text{if } n_1 \text{ is even} \\
(M_1f)(p_1, x_2, |s|) & \text{if } n_1 \text{ is odd}
\end{cases}
$$

to denote the odd extension of $M_1f$ in the last variable if $n_1$ is even, and the even extension of $M_1f$ if $n_1$ is odd. In particular, the function $s^{n_1-1}[M_1f]_\pm$ is always even in the last variable $s \in \mathbb{R}$.

**Theorem 2.3 (Decomposition of the spherical mean Radon transform).**

*For every $f \in C_c^\infty(B \times \mathbb{R}^{n_2})$ and every $(p_1, p_2, r) \in \partial B \times \mathbb{R}^{n_2} \times [0, \infty)$, we have*

$$
(r^{n_1-1}Mf)(p_1, p_2, r) = \frac{\omega_{n_2}\omega_{n_1-1}}{2\omega_{n_1}} (M_2 s^{n_1-1}[M_1f]_\pm)(p_1, p_2, r). \quad (2.1)
$$
Proof. Note that any \( \omega \in S^{n-1} \) can be written in the form \( \omega = (\omega_1 \sin \alpha_1, \ldots, \omega_n) \), where \( (\alpha_1, \ldots, \alpha_n) \in [0, 2\pi) \times [0, \pi]^{n-2} \) and

\[
\omega_1 := \begin{pmatrix}
\sin \alpha_1 \sin \alpha_2 \cdots \sin \alpha_{n-1} \\
\cos \alpha_1 \sin \alpha_2 \cdots \sin \alpha_{n-1} \\
\vdots \\
\cos \alpha_{n-2} \sin \alpha_{n-1} \\
\cos \alpha_{n-1}
\end{pmatrix}, \quad \omega_2 := \begin{pmatrix}
\cos \alpha_{n_1} \sin \alpha_{n_1+1} \cdots \sin \alpha_{n-1} \\
\cos \alpha_{n_1+1} \sin \alpha_{n_2} \cdots \sin \alpha_{n-1} \\
\vdots \\
\cos \alpha_{n-2} \sin \alpha_{n-1} \\
\cos \alpha_{n-1}
\end{pmatrix} \in \mathbb{R}^n.
\]

By the transformation rule we can express the spherical mean Radon transform in terms of an integral over the parameter set \([0, 2\pi) \times [0, \pi]^{n-2}\),

\[
Mf(p_1, p_2, r) = \frac{1}{\omega_{n-1}} \int_{[0, 2\pi) \times [0, \pi]^{n-2}} f((p_1, p_2) + r(\omega_1 \sin \alpha_1, \ldots, \sin \alpha_{n-1}, \omega_2)) \\
\times \sin \alpha_2 \sin^2 \alpha_3 \cdots \sin^{n-2} \alpha_{n-1} \, d\alpha_1 \, d\alpha_2 \cdots d\alpha_{n-1}
\]

\[
= \frac{1}{\omega_{n-1}} \int_{[0, \pi]^{n-2}} \int_{[0, 2\pi) \times [0, \pi]^{n-2}} f(p_1 + r \sin \alpha_1, \ldots, \sin \alpha_{n-1}, \omega_1, p_2 + r \omega_2) \\
\times \sin \alpha_2 \sin^2 \alpha_3 \cdots \sin^{n-2} \alpha_{n-1} \, d\alpha_1 \, d\alpha_2 \cdots d\alpha_{n-1}
\]

\[
= \frac{\omega_{n-1}}{\omega_{n-1}} \int_{[0, \pi]^{n-2}} M_1 f(p_1, p_2 + r \omega_2, r \sin \alpha_1, \ldots, \sin \alpha_{n-1}) \\
\times \sin^{n-1} \alpha_1 \sin^{n-2} \alpha_{n-1} \, d\alpha_1 \cdots d\alpha_{n-1}
\]

Next we multiply the above expression by \( r^{n-1} \). Together with the definition of the multiplication operator \( s \), performing some elementary manipulations shows

\[
r^{n-1}Mf(p_1, p_2, r) = \frac{\omega_{n-1}}{\omega_{n-1}} \int_{[0, \pi]^{n-2}} (M_1 f)(p_1, p_2 + r \omega_2, r \sin \alpha_1, \ldots, \sin \alpha_{n-1}) \\
\times r^{n-1} \sin^{n-1} \alpha_1 \sin^{n-2} \alpha_{n-1} \, d\alpha_1 \cdots d\alpha_{n-1}
\]

\[
= \frac{\omega_{n-1}}{\omega_{n-1}} \int_{[0, \pi]^{n-2}} (s^{n-1}M_1 f) (p_1, p_2 + r \omega_2, r \sin \alpha_1, \ldots, \sin \alpha_{n-1}) \\
\times \sin \alpha_{n+1} \sin^{n-2} \alpha_{n-1} \, d\alpha_1 \cdots d\alpha_{n-1}
\]

The used extension of \([M_1 f]_\pm(s, x)\) as an even or odd function in the last variable implies that the function \( s^{n-1}[M_1 f]_\pm(x, s) \) is even in the variable \( s \). We therefore get

\[
r^{n-1}Mf(p_1, p_2, r) = \frac{\omega_{n-1}}{2\omega_{n-1}} \int_{[0, 2\pi) \times [0, \pi]^{n-2}} (s^{n-1}M_1 f)(p_1, p_2 + r \omega_2, r \sin \alpha_1, \ldots, \sin \alpha_{n-1})
\]

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\[ \times \sin \alpha_{n1+1} \cdots \sin^{n2-1} \alpha_{n-1} \, \sin \alpha_{n1} \cdots \sin \alpha_{n-1} \]
\[ = \frac{\omega_{n2} \omega_{n1-1}}{2 \omega_{n-1}} M_2(s^{n1-1}[M_1 f])(p_1, p_2, r), \]

which is the desired identity. \( \square \)

According to Theorem 2.3 we can recover any function \( f \in C^\infty_c(B \times \mathbb{R}^{n2}) \) from \( Mf \) by first inverting \( M_2 \) and subsequently inverting \( M_1 \). Formulas for inverting \( M_2 \), the spherical mean Radon transform with centers on a hyperplane, are well known and have been derived in [4,7,10,14,39,50]. Formulas for inverting \( M_1 \), the spherical mean Radon transform with centers on the boundary of a bounded domain, are known for special domains, including spheres [15, 16, 43, 53, 72] and ellipsoids [5, 25, 26, 28, 51, 60, 67]. For bounded domains \( B \subset \mathbb{R}^{n2} \) efficient reconstruction algorithms are known, such as time reversal [12, 34, 69] or series expansion methods [1, 44].

### 2.2 Reduction to \( M_1 \)

Since \( M_2 f \) equals the spherical mean Radon transformation with centers on a hyperplane (by considering \( p_1 \in \partial B \) as a fixed parameter), several methods for inverting \( M_2 \) are well known. In this section we will apply a particular inversion formula, so that, according to Theorem 2.3, the reconstruction problem of inverting \( M \) reduces the inversion of the partial spherical mean Radon transform \( M_1 \) in the horizontal direction.

**Definition 2.4** (Partial spherical backprojection in the vertical direction).
For \( \Psi \in C^\infty(\partial B \times \mathbb{R}^{n2} \times [0, \infty)) \) we define the partial spherical backprojection \( M_2^\Psi : \partial B \times \mathbb{R}^{n2} \times \mathbb{R} \rightarrow \mathbb{R} \) in the vertical direction by
\[
(M_2^\Psi)(p_1, x_2, s) := \int_{\mathbb{R}^{n2}} \Psi(p_1, p_2, \sqrt{|x_2 - p_2|^2 + s^2}) \, dp_2 \quad \text{for} \quad (p_1, x_2, s) \in \partial B \times \mathbb{R}^{n2} \times \mathbb{R}, \quad (2.2)
\]

provided the integral is convergent.

For a function \( \Phi(p_1, x_2, s) \) defined on \( \partial B \times \mathbb{R}^{n2} \times \mathbb{R} \) we denote by \( \Delta \) the Laplacian operator in the last two variables \((x_2, s)\). Then, the following inversion formula has been derived in [4].

**Lemma 2.5** (Inversion formula for \( M_2 \)).

Suppose that \( \Phi \in C^\infty_c(\partial B \times \mathbb{R}^{n2} \times \mathbb{R}) \) is even in the last component. Then, for every \( (p_1, x_2, s) \in \partial B \times \mathbb{R}^{n2} \times \mathbb{R} \), we have
\[
\Phi(p_1, x_2, s) = \frac{\omega_{n2}}{2(2\pi)^{n2}} (-\Delta)^{(n2-1)/2} H_s \partial_\delta M_2^\Psi \Phi(p_1, x_2, s). \quad (2.3)
\]

**Proof.** The inversion formula (2.3) has essentially been derived in [4]. However, as pointed out in [39] a slightly wrong constant has been given in [4]; see [39] for the correct constant used for the inversion formula (2.3). Closely related formulas have been derived in [14, 52]. \( \square \)
Remark 2.6 (Well-definedness of the inversion formula (2.3)).
As pointed out in [39], the integral (2.2) defining in the spherical backprojection in the vertical
direction is not absolutely convergent when applied to $\Psi = M_2\Phi$ for a physically reasonable function
$\Phi$. Therefore, strictly taken, the inversion formula (2.3) requires extending $M_2$ to a more general
class containing all functions of the form $M_2\Phi$ with $\Phi \in C_c^\infty(\partial B \times \mathbb{R}^{n_2} \times \mathbb{R})$. Actually, in [4]
the operator $M_2^{\ast}$ has been extended to an appropriate space of distributions containing every function
$M_2\Phi$ with $\Phi \in C_c^\infty(\partial B \times \mathbb{R}^{n_2} \times \mathbb{R})$. By using such an extension the inversion formula (2.3) is well
defined, but the backprojection $M_2^{\ast}\Psi$ is not defined via classical integrals.

A different approach for rigorously defining (2.3) that avoids the use of distributions has been proposed in [39]. There the author introduces the modified back-projection operation

$$\left(M_2^{\ast,\partial_s}\Psi\right)(p_1, x_2, s) := \int_{\mathbb{R}^{n_2}} \partial_s \Psi(p_1, p_2, \sqrt{\|x_2 - p_2\|^2 + s^2}) \, dp_2,$$

which formally appears in the inversion formula (2.3) by interchanging the order of the operators
$\partial_s$ and $M_2^{\ast}$. It has been shown in [39] that the modified back-projection in (2.4) is absolutely
convergent when applied to any $\Psi = M_2\Phi$ with $\Phi \in C_c^\infty(\partial B \times \mathbb{R}^{n_2} \times \mathbb{R})$, and that the inversion formula (2.3) holds true if the operator $\partial_s M_2^{\ast}$ is replaced by the modified back-projection $M_2^{\ast,\partial_s}$. We
shall therefore use $M_2^{\ast,\partial_s}$ for rigorously defining $\partial_s M_2^{\ast}$ (and therefore the inversion formula (2.3)),
but keep the formal notation $\partial_s M_2^{\ast}$.

By combining the decomposition result of Theorem 2.3 with the inversion formula for the partial
spherical mean Radon transform $M_2$ given in Lemma 2.5, we obtain the following result reducing
the inversion of $M$ to the inversion of the $n_1$-dimensional partial spherical mean Radon transform
$M_1$ with centers of spheres of integration on $\partial B$.

Theorem 2.7 (Reduction to the inversion of $M_1$).
For any $f \in C_c^\infty(B \times \mathbb{R}^{n_2})$ and every $(p_1, x_2, s) \in \partial B \times \mathbb{R}^{n_2} \times \mathbb{R}$ we have

$$[M_1 f]_\pm(p_1, x_2, s) = \frac{\omega_{n-1}}{(2\pi)^{n_2}\omega_{n_1-1}} \left(s^{1-n_1}(-\Delta)^{(n_2-1)/2}H_s\partial_s M_2^{\ast,\partial_s}r^{n_1-1}Mf\right)(p_1, x_2, s).$$

Proof. According to Theorem 2.3 we have $M_2\left(s^{n_1-1}[M_1 f]_\pm\right) = 2\omega_{n-1}/(\omega_{n_2}\omega_{n_1-1})r^{n_1-1}Mf$. Application of the inversion formula of Lemma 2.5 followed by multiplication with $s^{1-n_1}$ yields (2.5).

Using Theorem 2.7, the spherical mean Radon transform with centers on $\partial B \times \mathbb{R}^{n_2}$ can be inverted
by means of any reconstruction algorithm for the spherical mean Radon transform with centers on $\partial B$. For bounded domains this can be done by time reversal [12, 34, 69] or series expansion
methods [1, 44]. For spherical and elliptical domains explicit inversion formulas exist [5, 15, 16, 25,
26, 28, 43, 51, 53, 56, 60, 67, 72]. Then these yield to explicit inversion formulas for spherical and elliptical
cylinders. Some inversion formulas will be derived in the following section.
3 Explicit inversion formulas

In the special case that $B$ is a ball or a solid ellipsoid, $M_1$ can be inverted by means of explicit inversion formulas yielding explicit inversion formulas of the backprojection type for $M$.

3.1 Inversion formula for elliptical cylinders

Let $A \in \mathbb{R}^{n_1 \times n_1}$ denote a diagonal matrix with positive diagonal entries $a_1, a_2, \ldots, a_{n_1}$, and let

$$E_A := \{x_1 \in \mathbb{R}^{n_1} : |A^{-1}x_1| < 1\}$$

(3.1)
de note the corresponding solid ellipsoid. Note that any point on the boundary $\partial E_A$ can be uniquely written in the form $p_1 = A\sigma$ for $\sigma \in S^{n_1 - 1}$.

Definition 3.1 (Partial spherical backprojection in the horizontal direction).

For $\Phi \in C^\infty(\partial E_A \times \mathbb{R}^{n_2} \times [0, \infty))$ we define the partial spherical backprojection $M_1^s \Phi : E_A \times \mathbb{R}^{n_2} \rightarrow \mathbb{R}$ in the horizontal direction by

$$\left(M_1^s \Phi\right)(x_1, x_2) := \int_{S^{n_1 - 1}} \Phi(A\sigma, x_2, |x_1 - A\sigma|) \, dS(\sigma) \quad \text{for} \quad (x_1, x_2) \in E_A \times \mathbb{R}^{n_2}.$$  

(3.2)

For the spherical means on $\partial E_A$ we have the following inversion formula.

Lemma 3.2 (Inversion of $M_1$ for centers on ellipsoids).

For $f \in C^\infty_c(E_A \times \mathbb{R}^{n_2})$ and every $(x_1, x_2) \in E_A \times \mathbb{R}^{n_2}$, we have

$$f(x_1, x_2) = \frac{2^{n_1 - 3} \det(A)}{\omega_{n_1 - 2}(n_1 - 2)!} \left(\Delta_{A^{x_1}} M_1^s B M_1 f\right)(x_1, x_2).$$

(3.3)

Here $\Delta_{A^{x_1}} := \sum_{i=1}^{n_1} (1/a_i^2)\partial_{x_i}^2$ is the Laplacian with respect to $Ax_1$ and

$$\left(B_s \Phi\right)(p_1, x_2, s) := \begin{cases} \frac{2}{\pi} \int_0^\infty s' \Phi(p_1, x_2, s') \log |s'^2 - s^2| \, ds' & n_1 = 2 \\ (-1)^{(n_1-1)/2} \left(s D_s^{n_1 - 3} s_{n_1 - 2} \Phi\right)(p_1, x_2, s) & n_1 \geq 3 \text{ odd} \\ (-1)^{(n_1-2)/2} \left(H_s D_s^{n_1 - 3} s_{n_1 - 2} [\Phi]_\sigma\right)(p_1, x_2, s) & n_1 \geq 4 \text{ even} \end{cases}$$

(3.4)

where $[\Phi]_\sigma(p_1, x_2, s)$ is the odd extension of $\Phi(p_1, x_2, s)$ with respect to $s$.

Proof. For $n_1 = 2, 3$ the inversion formula (3.3) has been derived in [67] and for odd $n_1 \geq 3$ in [26]. For even $n_1 \geq 4$ in [26] the following inversion formula has been shown:

$$f(x_1, x_2) = \frac{2^{n_1 - 3} \det(A)}{\omega_{n_1 - 2}(n_1 - 2)!}(-1)^{(n_1-2)/2}$$
After writing \(2s(s^2 - |A\sigma - x_1|^2)^{-1} = (s - |A\sigma - x_1|)^{-1} + (s + |A\sigma - x_1|)^{-1}\), the inner integral in (3.5) can be written in the form

\[
\frac{1}{\pi} \int_0^\infty \left( D^{n-2} \right) (A\sigma, x_2, s) \log |x_1 - A\sigma|^2 - s^2 | 2s \, ds
\]

Inserting the last displayed equation for the inner integral in (3.5) finally shows the inversion formula (3.3) for the case of even \(n_1 \geq 4\).

Theorem 2.7 and Lemma 3.2 then imply the following inversion formula for the spherical Radon transform on elliptical cylinders.

**Theorem 3.3 (Inversion formula for M for elliptical cylinders).**

Let \(E_A \subset \mathbb{R}^{n_1}\) be an elliptical domain as in (3.1). Then, for every \(f \in C^\infty_0(E_A \times \mathbb{R}^{n_2})\) and every \((x_1, x_2) \in E_A \times \mathbb{R}^{n_2}, we have

\[
f(x_1, x_2) = \frac{2^{n_1-3} \det(A)}{(n_1 - 2)!(2\pi)^{n_2} \omega_{n_1-1} \omega_{n_2-2}} \left( \Delta_{Ax_1} M_1^2 B s^{1-n_1} (-\Delta)^{(n_2-1)/2} H_{x_2} \partial_{x_2} M_2 s^{n_1-1} Mf \right) (x_1, x_2).
\]

**Proof.** This immediately follows from Theorem 2.7 and Lemma 3.2.

As we shall discuss in Section 4, the inversion formula (3.6) can be implemented efficiently by a filtered-backprojection type algorithm, using only \(O(N^{4/3})\) floating point operations for \(n = 3\), where \(N\) is the number of unknowns to be recovered.

**Remark 3.4 (Non-locality of the inversion formula (3.6)).**

The inversion formula (3.6) is nonlocal for every spatial dimension: Recovering \(f\) at some reconstruction point \((x_1, x_2) \in E_A \times \mathbb{R}^{n_2}\) requires the integrals of \(f\) over all spheres. A local inversion
formula instead would only use integrals over spheres which are passing through an arbitrarily small neighbourhood of the reconstruction point. The non-locality of the inversion is due to several nonlocal operations: The Hilbert transform is nonlocal, the fractional Laplacian \((-\Delta)^{(n_2-1)/2}\) is nonlocal if and only if \(n_2\) is even and the filtration \(B_s\) is nonlocal if and only if \(n_1\) is even.

For Radon type transforms (such as the classical Radon transform) it often happens that inversion formulas are local for odd spatial dimension and nonlocal in even spatial dimension. Therefore one might conjecture that there should also exist a local inversion formula for the spherical mean Radon transform with centers on \(\partial E_A \times \mathbb{R}^{n_2}\). In fact, the universal backprojection (UBP) for elliptical cylinders in odd space dimension (see [29])

\[
\left( M_1^s \Phi \right) (x_1, x_2) = \frac{1}{R} \int_{\partial D_R} \Phi(p_1, x_2, |x_1 - p_1|) \, dS(p_1),
\]

where \(\nu_p\) is the outward pointing unit normal to \(\partial E_A \times \mathbb{R}^{n_2}\), is local and theoretically exact. It would be interesting to find out if also our inversion formula (3.6) can, for \(n\) odd, be rewritten in a local form (by combining some nonlocal operations). We currently don’t know if such a reformulation is possible.

In the following section we derive a different inversion formula for the spherical Radon transform with centers on circular cylinders where \(n_1 = 2\). This formula will be local if \(n\) is odd and is different from the inversion formula of [29].

### 3.2 Alternative inversion formula when \(D_R \subset \mathbb{R}^2\) is a disk

In the following let \(B = D_R \subset \mathbb{R}^2\) be a disc of radius \(R\) centered at the origin in the plane. Several inversion formulas for recovering a function from spherical means with a center set \(\partial D_R\) have been derived in [15]. In order to obtain a local inversion formula from the spherical mean Radon transform with centers on \(\partial D_R \times \mathbb{R}^{n_2}\), in this section we derive a slightly different version of the formulas in [15]. Our formula differs from the filtered back-projection type formulas in [15, Corollary 1.2] with respect to the order of application of the operators \(H_s, \partial_s\) and the multiplication operator \(s\).

For the following recall that the definition of the spherical backprojection \(M_1^s\) is in the vertical direction as defined in (3.2). For the special case where the the ellipsoid \(E_A\) is a two-dimensional disc we have

\[
\left( M_1^s \Phi \right) (x_1, x_2) = \int_{\partial D_R} \Phi(p_1, x_2, |x_1 - p_1|) \, dS(p_1).
\]

We will now derive the following inversion formula for \(M_1\).

**Lemma 3.5** (Inversion of \(M_1\) for centers on a circle).

For every \(f \in C^\infty_c(D_R \times \mathbb{R}^{n-2})\) and every \((x_1, x_2) \in D_R \times \mathbb{R}^{n-2}\), we have

\[
f(x_1, x_2) = \frac{1}{2} \left( M_1^s H_s \partial_s s[M_1 f]_- \right) (p_1, x_2, |x_1 - p_1|)(x_1, x_2).
\]
Proof. The proof of (3.7) is based on the first inversion formula in [15, Corollary 1.2] and a range condition for \( M_R \) derived in [17]. In fact, the first inversion formula of [15, Corollary 1.2] states that for \((x_1, x_2) \in D_R \times \mathbb{R}^{n-2}\) we have

\[
f(x_1, x_2) = \frac{1}{2R} \int_{\partial D_R} H_s s \partial_s [M_1 f - (p_1, x_2, |x_1 - p_1|)] dS(p_1).
\]  

(3.8)

Next, using the range condition for the spherical mean Radon transform on the disc of [17], we have the following range condition satisfied by the spherical means,

\[
0 = \int_{\partial D_R} \text{PV} \int_0^\infty \frac{2s}{s^2 - |x_1 - p_1|^2} M_1 f(p_1, x_2, s) ds dS(p_1)
\]

\[
= \int_{\partial D_R} \text{PV} \int_0^\infty M_1 f(p_1, x_2, s) \frac{ds dS(p_1)}{s - |x_1 - p_1|} + \int_{\partial D_R} \int_0^\infty M_1 f(p_1, x_2, s) \frac{ds dS(p_1)}{s + |x_1 - p_1|}
\]

\[
= \int_{\partial D_R} \text{PV} \int \frac{[M_1 f - (p_1, x_2, s)] ds dS(p_1)}{s - |x_1 - p_1|}
\]

\[
= \pi \int_{\partial D_R} H_s[M_1 f - (p_1, x_2, |x_1 - p_1|)] dS(p_1).
\]  

(3.9)

Here the first identity is the range condition of [17], the second identity follows after writing

\[
2s(s^2 - |x_1 - p_1|^2)^{-1} = (s - |x_1 - p_1|)^{-1} + (s + |x_1 - p_1|)^{-1},
\]

for the third identity we changed variables \( s \to -s \) in the second integral, and the last identity follows from the definition of the Hilbert transform.

Now, by applying the product rule for one-dimensional differentiation, the right hand side in (3.7) can be written in the form

\[
\frac{1}{2R} \int_{\partial D_R} H_s[M_1 f - (p_1, x_2, |x_1 - p_1|)] dS(p_1) + \frac{1}{2R} \int_{\partial D_R} H_s s \partial_s [M_1 f - (p_1, x_2, |x_1 - p_1|)] dS(p_1).
\]

According to the modified range condition (3.9), the first term in the last displayed equation equals zero, and according to the inversion formula (3.8) the the second term is equal to \( f(x_1, x_2) \). This yields the inversion formula (3.7).

By combining Theorem 2.7 and Lemma 3.5 we can derive the following inversion formula for the spherical mean Radon transform with centers on \( \partial D_R \times \mathbb{R}^{n-2} \).

**Theorem 3.6** (Inversion formula for \( M \) for centers on \( \partial D_R \times \mathbb{R}^{n-2} \).

*Let \( D_R \subset \mathbb{R}^2 \) denote a disc of radius \( R \). Then, for every \( f \in C^\infty_c(D_R \times \mathbb{R}^{n-2}) \) and every \((x_1, x_2) \in B \times \mathbb{R}^{n-2} \), we have

\[
f(x_1, x_2) = -\frac{\omega_{n-1}}{2(2\pi)^{n-1}} \left( M_1^f(-\Delta)^{(n-3)/2} \partial_s^2 M_{2^f} M f \right)(x_1, x_2).
\]  

(3.10)
Proof. According to Theorem 2.7 we have

\[ [M_1 f] - (p_1, x_2, s) = \frac{\omega_{n-1}s^{-1}}{(2\pi)^{n-2}\omega_1} \left( (-\Delta)^{(n-3)/2}H_s \partial_s M_2^r Mf \right) (p_1, x_2, s). \]

Together with the inversion formula of Lemma 3.5 for inverting the spherical Radon transform with centers on a circle this yields the desired inversion formula

\[
\begin{align*}
  f(x_1, x_2) &= \frac{\omega_{n-1}}{2(2\pi)^{n-1}} \left( M_1^2 H_s \partial_s (-\Delta)^{(n-3)/2}H_s \partial_s M_2^r Mf \right) (x_1, x_2) \\
  &= -\frac{\omega_{n-1}}{2(2\pi)^{n-1}} \left( M_1^2 (-\Delta)^{(n-3)/2} \partial_s^2 M_2^r Mf \right) (x_1, x_2).
\end{align*}
\]

For the last identity we used the commutation relation \( H_s \partial_s (-\Delta)^{(n-3)/2} = (-\Delta)^{(n-3)/2} \partial_s H_s \) and the identity \( H_s H_s \Phi = -\Phi \).

In the case that \( n \geq 3 \) is an even natural number, then \((-\Delta)^{(n-3)/2}\) is a differential operator and therefore local. Consequently, in such a situation, the inversion formula (3.10) is local: Recovering the function \( f \) at any point \((x_1, x_2) \in D_R \times \mathbb{R}^{n-2}\) uses only integrals over those spheres which pass through an arbitrarily small neighbourhood of \((x_1, x_2)\). In the case that \( n \geq 4 \) is an even natural number, then \((-\Delta)^{(n-3)/2} = (-\Delta)^{(n-4)/2}\sqrt{-\Delta}\) is a non-integer power of the Laplacian. Consequently, in such a situation, the inversion formula (3.10) is non-local. Such a different behaviour for even and odd dimension is also well known for the standard inversion formulas of the classical Radon transform, as well as for known inversion formulas for the spherical mean Radon transform with centers on spheres or ellipses.

**Remark 3.7** (Inversion formula for three spatial dimensions). Let us consider the important special case of \( n = 3 \). In such a situation \( D_R \times \mathbb{R} \) is a standard circular cylinder in \( \mathbb{R}^3 \) of radius \( R \). Formula (3.10) then becomes

\[
\begin{align*}
  f(x_1, x_2) &= -\frac{1}{2\pi} \left( M_1^2 \partial_s^2 M_2^r Mf \right) (x_1, x_2) \\
  &= -\frac{1}{2\pi R} \int_{\partial D_R} \left[ \partial_s^2 \int_{\mathbb{R}} \left( \frac{1}{r Mf} \left( p_1, p_2, \sqrt{|x_2 - p_2|^2 + s^2} \right) dp_2 \right) \right] \quad dS(p_1). \tag{3.11}
\end{align*}
\]

The inversion formula (3.11) has a simple structure and can be implemented numerically by combining two backprojection algorithms for the circular Radon transform (the two-dimensional spherical mean Radon transform) with centers on a hyperplane and the circular Radon transform with centers on a circle; compare Section 4.

Interchanging the order of differentiation and inner integration further yields

\[
\begin{align*}
  f(x_1, x_2) &= -\frac{1}{2\pi R} \int_{\partial D_R \times \mathbb{R}} \left( \frac{|x_1 - p_1|^2 r^{-1} \partial_r r^{-1} \partial_r r + r^{-1} \partial_r r}{|x_1 - p_1|^2 + |x_2 - p_2|^2} \right) \quad dS(p_1, p_2) \quad \text{for} \quad (x_1, x_2) \in D_R \times \mathbb{R}. \tag{3.12}
\end{align*}
\]

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This inversion formula (3.12) is again of the filtered backprojection type: \( |\mathbf{x}_1 - \mathbf{p}_1|^2 r^{-1} \partial_r r^{-1} \partial_r r + r^{-1} \partial_r r \) can be interpreted as a filtration operation applied to the spherical means, and the integration as spherical backprojection, which integrates the filtered data over the set of all spheres centered on \( \partial D_R \times \mathbb{R} \) and passing through the reconstruction point \( (\mathbf{x}_1, x_2) \).

3.3 Comparison with the UBP (universal back-projection) formula

For three spatial dimensions, in [72] it is shown that the UBP formula

\[
f(\mathbf{x}) = \frac{2}{\Omega_0} \int_{\partial U} \left[ \nu_\mathbf{p} \cdot \frac{\mathbf{x} - \mathbf{p}}{|\mathbf{x} - \mathbf{p}|} \right] (\partial_r r^{-1} \partial_r r \mathbf{M} f)(\mathbf{p}, |\mathbf{x} - \mathbf{p}|) \, dS(\mathbf{p}) \quad \text{for } \mathbf{x} \in U,
\]

(3.13)
gives theoretically exact reconstruction for the cases that \( \partial U \subset \mathbb{R}^3 \) is a plane, a sphere, or a circular cylinder, where \( dS(\mathbf{p}) \) is the measure of the boundary of \( U \), \( \nu_\mathbf{p} \) is the outward pointing unit normal to \( \partial U \), and \( \Omega_0 \) equals \( 2\pi \) for the case of a half space and equals \( 4\pi \) for the case of spheres and cylinders. Here \( f \) is supposed to be supported in the domain \( U \). As mentioned in the introduction the UBP formula has been generalized to spheres, elliptical cylinders and other quadric hypersurfaces in arbitrary spatial dimension in [28,29,43,51]. For a planar center set the UBP is closely related to the inversion formulas of [7,10,50].

In the case of a circular cylinder \( U = D_R \times \mathbb{R} \subset \mathbb{R}^3 \) and writing any point in \( \mathbb{R}^3 \) in the form \( \mathbf{x} = (\mathbf{x}_1, x_2) \), the UBP can be rewritten as follows:

\[
f(\mathbf{x}_1, x_2)
= \frac{1}{2\pi} \int_{\partial D_R \times \mathbb{R}} \left[ \frac{(\mathbf{p}_1, 0) \cdot (\mathbf{x}_1, x_2) - (\mathbf{p}_1, p_2)}{|\mathbf{p}_1|} \right] \frac{\left( \mathbf{x}_1, x_2 \right) - (\mathbf{p}_1, p_2)}{\left| \left( \mathbf{x}_1, x_2 \right) - (\mathbf{p}_1, p_2) \right|}
\times (\partial_r r^{-1} \partial_r r \mathbf{M} f)(\mathbf{p}_1, p_2, \left| (\mathbf{x}_1, x_2) - (\mathbf{p}_1, p_2) \right|) \, dS(\mathbf{p}_1, p_2)
\]

\[
= \frac{1}{2\pi} \int_{\partial D_R \times \mathbb{R}} \frac{\mathbf{p}_1 \cdot (\mathbf{x}_1 - \mathbf{p}_1) (\partial_r r^{-1} \partial_r r \mathbf{M} f)(\mathbf{p}_1, p_2, \sqrt{\mathbf{x}_1 - \mathbf{p}_1^2 + |x_2 - p_2|^2})}{|\mathbf{p}_1|} \sqrt{\mathbf{x}_1 - \mathbf{p}_1^2 + |x_2 - p_2|^2} \, dS(\mathbf{p}_1, p_2)
\]

\[
= \frac{1}{2\pi R} \int_{\partial D_R} \mathbf{p}_1 \cdot (\mathbf{x}_1 - \mathbf{p}_1) \left[ \int_{\mathbb{R}} \frac{(\partial_r r^{-1} \partial_r r \mathbf{M} f)(\mathbf{p}_1, p_2, \sqrt{|x_2 - p_2|^2 + s^2}) \, dp_2}{\sqrt{|x_2 - p_2|^2 + s^2}} \right]_{s = |\mathbf{x}_1 - \mathbf{p}_1|} \, dS(\mathbf{p}_1).
\]

The last expression is obviously different from our three-dimensional inversion formulas (3.11) and (3.12). However, it can be again written as the decomposition of a local filtration operator, one vertical backprojection (the inner integral) and one backprojection (the outer integral) along the horizontal direction.

4 Numerical implementation

In the previous section we derived several inversion formulas for recovering a function supported in a cylinder. In this section we report on the numerical implementation of these formulas and
present some numerical results. We restrict our attention to the important special case of the circular cylinder \( D_R \times \mathbb{R} \subset \mathbb{R}^3 \), where \( D_R \) is a disc of radius \( R \) centered at the origin in \( \mathbb{R}^2 \).

\[ p = (R \theta[k], H m/L) \]

Figure 4.1: Discrete measurement geometry. Suppose the function \( f \) to be recovered is supported in the finite height cylinder \( D_R \times [-H, H] \). The data \( Mf \) are uniformly sampled data at discrete points \( p = (R \theta[k], H m/L) \) on \( \partial D_R \times [-H, H] \) (for \( k \in \{0, \ldots, K - 1\} \) and \( m \in \{-L, \ldots, L\} \)) and discrete radii \( r = r_0 l/M \) for \( l \in \{0, \ldots, M\} \).

4.1 Discrete reconstruction algorithm

In the numerical implementation the centers have to be restricted to a finite number of discrete samples on the truncated cylinder \( \partial D_R \times [-H, H] \) of finite height (see Figure 4.1). Further the radii are restricted to discrete values in \([0, r_0] \). Here \( 2H \) is the height of the detection cylinder and the maximal radius \( r_0 \) is chosen in such a way that \((Mf)(p, r) = 0\) for \( p \in D_R \times [-H, H] \) and \( r > r_0 \). We model these discrete data by

\[ g[k, m, l] := (Mf) \left( R \theta[k], H m L, r_0 \frac{1}{M} \right) \text{ for } (k, m, l) \in \{0, \ldots, K - 1\} \times \{-L, \ldots, L\} \times \{0, \ldots, M\}, \]

where \( \theta[k] := (\cos(2\pi k/K), \sin(2\pi k/K)) \) are the angular detector positions. Our aim is to derive a discrete algorithm based on the inversion formulas derived in the previous section, that outputs an approximation

\[ f[n_1, n_2, n_3] \simeq f \left( \frac{R n_1}{N_x}, \frac{R n_2}{N_x}, \frac{H n_3}{L} \right) \text{ for } (n_1, n_2, n_3) \in \{-N_x, \ldots, N_x\}^2 \times \{0, \ldots, L\}, \]

to the original phantom, where \( (R n_1/N_x, R n_2/N_x, H n_3/L) \) are the discrete reconstruction points.

For our numerical simulations we implement all inversion formulas (3.11), (3.12) as well as the UBP (3.13) by replacing any filtration operations and any partial backprojection by some discrete
approximation. We shall outline this for the inversion formula (3.11). The other formulas are discretized in an analogous manner. For that purpose recall that the inversion formula (3.11) is given by
\[ f(x_1, x_2) = -\frac{1}{(2\pi R)}(\mathbf{M}_1^\sharp \partial_s^2 \mathbf{M}_2^\sharp r \mathbf{M} f)(x_1, x_2) \]
with \( \mathbf{M}_2^\sharp \) and \( \mathbf{M}_1^\sharp \) denoting the partial spherical backprojection in the vertical and in the horizontal direction, respectively. For the numerical implementation, all operators \( \mathbf{M}_1^\sharp, \partial_s^2, \mathbf{M}_2^\sharp \) and \( r \) are replaced by a finite-dimensional approximation and \( \mathbf{M} f \) is replaced by the discrete data \( g \). We therefore define
\[ f := -\frac{1}{2\pi R} \left( \mathbf{M}_1^\sharp D_s^2 \mathbf{M}_2^\sharp r \right) g. \] as the approximate reconstruction, where \( r \) is a discrete multiplication operator, and \( D_s^2 \) is a finite difference approximation of \( \partial_s^2 \). Further, the discrete backprojection operators \( \mathbf{M}_1^\sharp \) and \( \mathbf{M}_2^\sharp \) are obtained by discretizing \( \mathbf{M}_1 \) and \( \mathbf{M}_2 \) with the composite trapezoidal rule and linear interpolation as described in [11,15]. The discrete operators \( \mathbf{M}_2 \) and \( \mathbf{M}_1 \) are two-dimensional discrete backprojection operators for fixed \( k \) and \( n_3 \), respectively. Assuming that \( n_x \sim k \sim L \sim n_3 \), both of these back-projection operators can be implemented using \( \mathcal{O}(n_x^3) \) floating point operations (see [11,31]). Therefore the numerical effort of evaluating (4.1) is \( \mathcal{O}(N^{5/3}) \), where \( N := n_x^3 \) is the total number of reconstructed unknowns. Note that the direct implementation of a three-dimensional spherical back-projection operator requires \( \mathcal{O}(n_x^5) \) floating point operations, see [11,46]. Typically we have \( n_x \geq 100 \), and therefore our implementations are faster than standard 3D spherical back-projections by at least two orders of magnitude.

Figure 4.2: 3D PHANTOM AND 3D RECONSTRUCTION. Left: Phantom consisting of a superposition of three balls of radius \( a = 0.2 \). Right: Numerical reconstruction from its spherical means data using an inversion formula (3.11).
4.2 Numerical results

For the numerical simulations presented below we consider superpositions of radially symmetric indicator functions

\[ f_{m,a}(x) = \begin{cases} 1 & \text{if } |x - m| \leq a \\ 0 & \text{otherwise} \end{cases} \quad (4.2) \]

where \( m \in D_1 \times \mathbb{R} \) is the center and \( a > 0 \) the radius of the radially symmetric indicator phantom. For such radially symmetric phantoms, the spherical Radon transform can easily be computed analytically and is given by

\[
(Mf_{m,a})(x) = \begin{cases} \frac{a^2 - (|x - m| - r)^2}{4r|x - m|} & \text{if } ||x - m| - r| < a \\ 0 & \text{otherwise} \end{cases}
\]

In particular, we use this formula for computing the spherical means of the test phantom \( f = f_{m(1),a} + f_{m(2),a} + f_{m(3),a} \), which is a superposition of three radially symmetric indicator functions having radius \( a = 0.2 \), and centers \( m^{(1)} = (-0.5,0,-0.5) \), \( m^{(2)} = (0.3,0,0.35) \) and \( m^{(3)} = (0.3,0,-0.5) \).

Figure 4.3: Reconstruction results using simulated data. Top row (from left to right): Horizontal slices of the phantom, the reconstruction using (3.11), the reconstruction using (3.12), and the reconstruction with the UBP (3.13). Bottom row: The same for the vertical slices.

Figure 4.2 shows the considered phantom and the numerical reconstruction using the inversion formula (4.1) for \( R = 1 \) and \( H = 2 \). For the discretization we used \( K = 256 \), \( L = 200 \), \( M = 200 \) and \( N_x = 100 \). The horizontal slices of the phantom and the reconstruction with (4.1) are displayed in the top row of Figure 4.3. For comparison purpose we also display the results using a numerical implementations of formula (3.12) and the UPB formula (3.13). The bottom row of Figure 4.3 shows
the same for the horizontal slices. One can see that the reconstruction results for all inversion formulas are very good, especially in the horizontal direction. In the vertical slide the vertical boundaries of the reconstructed balls are blurred. Such artifacts are expected and arise from truncating the observation surface, see [19,45,54,61,68,73]. All computations have been performed in MATLAB. The reconstructed three dimensional data sets consisting of $N = 16,200,801$ unknowns using any of the inversion formulas (3.11), (3.12) or (3.13) have been computed in a few minutes on a MacBook Pro with 2.3 GHz Intel Core i7 processor. More precisely, the computation times using (3.11) and (3.13) are 6 min and the reconstruction time using (3.12) is 8 min.

![Figure 4.4: Reconstruction results using noisy data. Top row (from left to right): Noisy spherical means data (restricted to centers on the horizontal circle $\partial D_1 \times \{0\}$), and the reconstructions in the horizontal direction using (3.11), (3.12) and (3.13). Bottom row (from left to right): Noisy spherical means data (restricted to centers on the vertical line $\{(1,0)\} \times [-2,2]$), and the reconstructions in the vertical direction using (3.11), (3.12) and (3.13).](image)

Finally, in order to illustrate the stability of the derived discrete back-projection algorithms with respect to the noise we applied all algorithms to simulated data, where we added Gaussian white noise with variance equal to 2% of the maximal absolute value of $g$. The reconstruction results (using $R = 1$, $H = 2$, $K = 256$, $L = 200$, $M = 200$ and $N_x = 100$ as above) for noisy data are shown in Figure 4.4. As can be seen the implementations of all back-projection formulas perform quite stably with respect to noise. The slight amplification of noise is again expected due to the ill-posedness of the inversion of the spherical mean Radon transform reflected by the two derivatives in any of the inversion formulas. The sensitivity with respect to noise could easily be further reduced by applying a regularization strategy similar to [24,31] for the spherical mean Radon transform with centers on a sphere.
5 Discussion

Reconstructing a function from its spherical mean Radon transform $Mf(p, r)$ with centers of spheres of integration restricted to a hyper-surface is crucial for the success of several modern imaging technologies, including RADAR (radio detection and ranging), SONAR (sound navigation and ranging), SAR (synthetic aperture radar), photoacoustic- and thermoacoustic tomography, or ultrasound imaging. In particular, the recently invented photoacoustic- and thermoacoustic tomography initiated a lot of research on the spherical mean Radon transform. Explicit inversion formulas have been derived for planar and spherical center sets \[4,10,14-16,39,43,50,72\], and more recently for elliptical center sets \[5,25,26,28,51,60,67\].

In this paper we investigated the spherical mean Radon transform in the case where the centers of the spheres of integration are located on a cylindrical surface $\partial B_R \times \mathbb{R}^{n_2}$, and that the function to be recovered is supported in $B_R \times \mathbb{R}^{n_2}$. We showed that this particular instance of the spherical mean Radon transform can be decomposed into two lower dimensional partial spherical mean Radon transforms, one with center set $\partial B_R \subset \mathbb{R}^{n_1}$ and one with a planar center set in $\mathbb{R}^{n_2+1}$, see Theorem 2.3. This decomposition was used to derive explicit inversion formulas for elliptical cylinders (given in Theorem 3.3) and circular cylinders (given in Theorem 3.6). The latter inversion formula is local for odd spatial dimension and non-local for even spatial dimension.

In Section 4 we reported on the numerical implementation of the derived formulas for the case of three spatial dimensions. Again using the decomposition into lower dimensional transforms these algorithms require only $O(N^{4/3})$ floating point operations, where $N$ is the total number of unknowns (see Section 4.1). It has been demonstrated, that 3D phantoms having several millions of unknowns are reconstructed accurately on a standard notebook in a few minutes. Opposed to that, standard implementations of three dimensional back-projection algorithms for the spherical Radon transform as well as for the classical Radon transform require $O(N^{5/3})$ floating point operations (see \[31,46,52\]). Using such algorithms for reconstructing a 3D phantom at a reasonable resolution would require several hours, which is significantly slower than our implementations. In combination with existing Fourier algorithms for the spherical mean Radon transform with linear and circular center sets (see \[30,36,44,46\]) our decomposition approach would even yield $O(N \log N)$ algorithms for inverting the spherical mean Radon transform with centers on a circular cylinder in $\mathbb{R}^3$.

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