DETERMINING CYCLICITY OF FINITE MODULES

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Abstract. We present a deterministic polynomial-time algorithm that determines whether a finite module over a finite commutative ring is cyclic, and if it is, outputs a generator.

1. Introduction

If \( R \) is a commutative ring, then an \( R \)-module \( M \) is cyclic if there exists \( y \in M \) such that \( M = Ry \).

**Theorem 1.1.** There is a deterministic polynomial-time algorithm that, given a finite commutative ring \( R \) and a finite \( R \)-module \( M \), decides whether there exists \( y \in M \) such that \( M = Ry \), and if there is, finds such a \( y \).

We present the algorithm in Algorithm 4.1 below. The inputs are given as follows. The ring \( R \) is given as an abelian group by generators and relations, along with all the products of pairs of generators. The finite \( R \)-module \( M \) is given as an abelian group, and for all generators of the abelian groups \( R \) and all generators of the abelian group \( M \) we are given the module products in \( M \).

Our algorithm depends on \( R \) being an Artin ring, and should generalize to finitely generated modules over any commutative Artin ring that is computationally accessible.

**Theorem 1.1** is one of the ingredients of our work [4, 5] on lattices with symmetry, and a sketch of the proof is contained in [4]. Previously published algorithms of the same nature appear to restrict to rings that are algebras over fields. Subsequently to [4], I. Ciocânea-Teodorescu [2], using different and more elaborate techniques, greatly generalized our result, dropping the commutativity assumption on the finite ring \( R \) and finding, for any given finite \( R \)-module \( M \), a set of generators for \( M \) of smallest possible size.

See Chapter 8 of [1] for commutative algebra background. For the purposes of this paper, commutative rings have an identity element 1, which may be 0.

2. Lemmas on commutative rings

If \( R \) is a commutative ring and \( \mathfrak{a} \) is an ideal in \( R \), let \( \text{Ann}_R \mathfrak{a} \) denote the annihilator of \( \mathfrak{a} \) in \( R \). We will use that every finite commutative ring is an Artin ring,

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that every Artin ring is isomorphic to a finite direct product of local Artin rings, and that the maximal ideal in a local Artin ring is always nilpotent.

**Lemma 2.1.** If $A$ is a local Artin ring, $a$ is an ideal in $A$, and $a^2 = a$, then $a$ is 0 or $A$.

*Proof.* If $a$ contains a unit, then $a = A$. Otherwise, $a$ is contained in the maximal ideal $m$, which is nilpotent. Thus there is an $r \in \mathbb{Z}_{>0}$ such that $m^r = 0$. Now $a = a^2 = \cdots = a^r \subset m^r = 0$. $\square$

**Lemma 2.2.** Suppose that $A$ is a finite commutative ring, $a$ is an ideal in $A$, $b = \text{Ann}_A a$, and $a \cap b = 0$. Then:

(i) $a^2 = a$;

(ii) there is an idempotent $e \in A$ such that $a = eA$, $b = (1 - e)A$, and $A = (1 - e)A \oplus eA = b \oplus a$;

(iii) if $b = 0$ then $a = A$.

*Proof.* Write $A$ as a finite direct product of local Artin rings $A_1 \times \cdots \times A_s$. Then $a$ is a direct product $a_1 \times \cdots \times a_s$ of ideals $a_i \subset A_i$. Assume $a^2 \neq a$. Then there is an $i$ such that $a_i^2 \neq a_i$. Let $b_i = \text{Ann}_{A_i} a_i$. Since $a \cap b = 0$, it follows that $a_i \cap b_i = 0$. Since $A_i$ is a local ring, $a_i$ is contained in the maximal ideal of $A_i$, so $a_i$ is nilpotent. Let $r$ denote the smallest positive integer such that $a_i^r = 0$. Since $a_i \neq 0$ we have $r \geq 2$. Then $a_i^{r-1}$ is contained in $a_i$ and kills $a_i$, so $0 \neq a_i^{r-1} \subset a_i \cap b_i = 0$, a contradiction. This gives (i).

Since $A$ is a finite product of local Artin rings, $a$ is generated by an idempotent $e$, by Lemma 2.1. Then $b = (1 - e)A$ and $A = (1 - e)A \oplus eA = b \oplus a$. This gives (ii) and (iii). $\square$

### 3. Preparatory Lemmas

If $R$ is a commutative ring, then a commutative $R$-algebra is a commutative ring $A$ equipped with a ring homomorphism from $R$ to $A$. Whenever $A$ is an $R$-algebra, we let $M_A$ denote the $A$-module $A \otimes_R M$.

From now on, suppose $R$ is finite commutative ring and $M$ is a finite $R$-module. Let $S$ denote the set of quadruples $(A, B, y, N)$ such that:

(i) $A$ and $B$ are finite commutative $R$-algebras for which the natural map $f : R \rightarrow A \times B$ is surjective and has nilpotent kernel,

(ii) $y \in M$ is such that the map $B \rightarrow M_B = B \otimes_R M$ defined by $b \mapsto b \otimes y$ is an isomorphism and such that $1 \otimes y = 0$ in $M_A$,

(iii) and $N$ is a submodule of $M$ such that the natural map $N \rightarrow M_A$ defined by $z \mapsto 1 \otimes z$ is onto and such that the natural map $N \rightarrow M_B$ is the zero map.

In Algorithm 4.1 below, initially we take $(A, B, y, N) = (R, 0, 0, M)$. Clearly, $(R, 0, 0, M) \in S$. Throughout that algorithm, we always have $(A, B, y, N) \in S$. While $A$ and $B$ occur in the proof of correctness of Algorithm 4.1, the $R$-algebra $B$ does not actually occur in the algorithm itself.

**Lemma 3.1.** If $(A, B, y, N) \in S$ and $M_A = 0$, then $M = Ry$.

*Proof.* Let $J$ denote the kernel of $f : R \rightarrow A \times B$, and let $I_A$ (resp., $I_B$) denote the kernel of the composition of $f$ with projection from $A \times B$ onto $A$ (resp., $B$). Since $J$ is nilpotent we have $J^r = 0$ for some $r \in \mathbb{Z}_{>0}$. Since $0 = M_A = \ldots$
Lemma 3.2. Suppose \((A, B, y, N) \in S\) and \(M_A \neq 0\). Then there exists \(x \in N\) such that \(1 \otimes x \neq 0\) in \(M_A\). Choosing \(x\) and letting \(a = \text{Ann}_A(1 \otimes x)\) and \(b = \text{Ann}_A a\), we have:

\[(i) \quad (A/(a \cap b), B, y, N) \in S;\]
\[(ii) \quad \text{If } a \cap b = 0 \text{ and } (A/a) \otimes x = M_A/a, \text{ then } (A/b, (A/a) \times B, x + y, aN) \in S, \text{ where } aN \text{ denotes } f^{-1}(a \times B)N.\]
\[(iii) \quad \text{If } a \cap b = 0 \text{ and } (A/a) \otimes x \neq M_A/a, \text{ then } M \text{ is not cyclic.}\]

Proof. Since the map \(N \to M_A, z \mapsto 1 \otimes z\) is onto, as long as \(M_A \neq 0\) there exists \(x \in N\) such that \(1 \otimes x \neq 0\) in \(M_A\).

Since \(ab = 0\), we have \((a \cap b)^2 = 0\), so \(a \cap b\) is a nilpotent ideal in \(A\). It follows that \((A/(a \cap b), B, y, N) \in S\), giving (i).

From now on, suppose that \(a \cap b = 0\). By Lemma 2.2 there is an idempotent \(e \in A\) such that \(a = eA, b = (1 - e)A\), and \(A = (1 - e)A \oplus eA = b \oplus a\). It follows that \(A \twoheadrightarrow A/a \times A/b\), so \(M_A \twoheadrightarrow M_A/a \times M_A/b\). If \((x', x'')\) is the image of \(1 \otimes x\) under the latter map, then \(x'' = 0\) (we have \(bx'' = 0\) since \(x'' \in (A/b) \otimes R M\), and \(ax'' = 0\) since \(a(1 \otimes x) = 0\); thus \(Ax'' = (a + b)x'' = 0\), so \(x'' = 0\)). The map \(i_a : A/a \to M_A/a\) defined by \(i_a(t) = tx = t \otimes x\) is injective since \(\text{Ann}_A/a x' = 0\).

First suppose \((A/a) \otimes x = M_A/a\). Then the injective map \(i_a\) is an isomorphism. Since \(0 = x'' = 1_{A/b} \otimes x\), we have \(1 \otimes (x + y) = 0\) in \(M_A/b\). It is now easy to check that \((A/b, (A/a) \times B, x + y, aN) \in S\), giving (ii). Note that \(b \neq 0\) (if \(b = 0\), then \(a = A\) by Lemma 2.2 contradicting that \(1 \otimes x \neq 0\) in \(M_A\)).

Now suppose that \((A/a) \otimes x \neq M_A/a\). By way of contradiction, suppose \(M\) is a cyclic \(R\)-module. Then \(M_{A/a}\) is a cyclic \(A/a\)-module. Since the domain and codomain of \(i_a : A/a \to M_{A/a}\) are both finite, it now follows that \(i_a\) is surjective, so \((A/a) \otimes x = M_{A/a}\). This contradiction gives (iii).

The intuition behind Algorithm 4.1 is that throughout the algorithm, \(y\) generates the “non-\(A\) part” of \(M\), and the goal is to shrink the “\(A\)-part” of \(M\), namely \(N\).

4. Main algorithm

Algorithm 4.1. Input a finite commutative ring \(R\) and a finite \(R\)-module \(M\). Decide whether there exists \(y \in M\) such that \(M = Ry\), and if there is, find such a \(y\).

(i) Initially, take \(A = R, y = 0\), and \(N = M\).
(ii) If \(M_A = 0\), stop and output “yes” with generator \(y\).
(iii) Otherwise, pick \(x \in N\) such that \(1 \otimes x \neq 0\) in \(M_A\), and compute \(a = \text{Ann}_A(1 \otimes x), b = \text{Ann}_A a, \text{ and } a \cap b\).
(iv) If \(a \cap b \neq 0\), replace \(A\) by \(A/(a \cap b)\) and go back to step (ii).
(v) If \( a \cap b = 0 \), then if \((A/\mathfrak{a}) \otimes x \neq M_{A/\mathfrak{a}}\) terminate with “no”, and if \((A/\mathfrak{a}) \otimes x = M_{A/\mathfrak{a}}\) replace \( A, y, \) and \( N \) by \( A/b, x+y, \) and \( aN \), respectively, and go back to step (ii).

**Proposition 4.2.** Algorithm 4.1 runs in polynomial time, and on input a finite commutative ring \( R \) and a finite \( R \)-module \( M \), decides whether there exists \( y \in M \) such that \( M = Ry \), and if there is, finds such a \( y \).

**Proof.** Since \( A \) is a finite ring, if the algorithm does not stop with “no” then eventually \( A = 0 \) and \( M_A = 0 \). Step (ii) of the algorithm is justified by Lemma 3.1, while steps (iii), (iv), and (v) are justified by Lemma 3.2.

The computations of annihilators and of the decompositions \( A \cong A/\mathfrak{a} \times A/\mathfrak{b} \) can be done in polynomial time using linear algebra (see §14 of [3]); in particular, \( \mathfrak{a} \) is the kernel of the map \( A \rightarrow M_A \) defined by \( t \mapsto t(1 \otimes x) \). For any \( B \), compute \( M_B \) by computing \( M/I_B M \) (and analogously for \( M_A \)). Each new \( A \) is at most half the size of the \( A \) it replaces. This implies that the number of steps is at most linear in the length of the input. \( \Box \)

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