Abstract

The hydrodynamics equations of binary mixtures of Bose gases, Fermi gases, and mixtures of Bose and Fermi gases in the presence of external potentials are derived. These equations can be applied to current experiments on mixtures of atomic gases confined in magnetic traps.

The recent discoveries of Bose-Einstein condensation of atomic gases [1–3] have generated much interest in studying dilute Bose gases trapped in external potentials. While most of the experiments are performed on trapped atoms of the same kind, the current traps can also hold different kinds of atoms at the same time, providing opportunities to study mixtures of Bose gases, mixtures of Fermi gases and mixtures of Bose and Fermi gases. In a recent paper, we have shown [4] that there is a great variety of ground state structures in the binary mixtures of Bose gases. These structures can be experimentally accessed by varying the number of trapped Bosons of each kind. The two Bosons can be different alkalis, different isotopes of the same alkali, or different hyperfine spin (F) states of the same alkali isotope. Recently, a mixture of different hyperfine states of 87Rb (F = 1 and F = 2) has been produced by the JILA group [5], creating a system of interpenetrating Bose superfluids for the first time.

In the case of superfluid 4He, its superfluid properties are contained in the well known two-fluid hydrodynamic equations of Landau [6] that incorporate the effect of broken gauge symmetry into hydrodynamics. In order to understand the properties of mixtures, it is important to derive the corresponding hydrodynamic equations, which is the goal of this paper. Our derivation is a standard one. It follows from general principles such as conservation laws, broken symmetries, thermodynamic relations and the second law of thermodynamics [6,7]. It will be clear from the derivation that the form of the hydrodynamic equations is independent of the density of the gases, as long as the symmetry of the system remains unchanged. What distinguishes a dilute and a dense system is the explicit form of the coefficients entering these equations, which are related to the thermodynamic potential of the system. In this paper, we will focus only on the derivation of the hydrodynamic equations of mixtures. The solutions of these equations require the evaluation of the thermodynamic potentials which will be discussed elsewhere. (We have recently solved the hydrodynamic equations of single component dilute Bose gases in spherical traps and have obtained all the first and second sound modes (classified by radial and angular quantum numbers) as a function of temperature [8]).

Before proceeding, it is necessary to comment on the broken symmetry of the alkali gases in magnetic traps. Since the alkali atoms carry spin, the order parameter in the Bose
condensed phase should be a spinor field instead of a scalar field. However, in a magnetic
trap, only the spin states locally aligned with the magnetic field $B(r)$ are trapped \[^9\]. The
resulting quantum gas therefore has the spin field “frozen” along the local field, leaving
particle motion as the only degree of freedom. As a result, the spin field reduces to a scalar.
On the other hand, even though the spin field is frozen, the spinor nature is not completely
erased. In a recent paper \[^9\], we have shown that the spatial variations of the spin field will
give rise to a Berry phase which can lead to a number of dramatic effects. While the Berry
phase effects in current traps are small, they can be magnified by changing the geometry of
the trap. For simplicity, we shall only consider those traps where the Berry phase effects are
negligible. One should, however, keep in mind that the hydrodynamics equations derived
here are limited to traps of this kind.

Since both scalar Bose superfluids and s-wave Fermion superfluids have identical broken
$U(1)$ gauge symmetry, and since the hydrodynamics of a system depends entirely on the na-
ture of its broken symmetry, the hydrodynamic equations of these two systems are identical.
The hydrodynamics of binary mixtures of quantum gases (which can be either Fermions or
Bosons) therefore fall in three categories, namely, mixtures of superfluids, mixtures consist-
ing of a normal fluid and a superfluid, and mixtures of normal fluids. The superfluids can be
the scalar condensates of Bosons or the s-wave BCS condensates of Fermions. The normal
fluid can be the either Bosons or Fermions in their normal state.

Among all the cases considered, the hydrodynamics of binary superfluid mixtures in a
trap is the most complex. We shall focus on this case. As we shall see, once we derive the
hydrodynamic equations for this system, extensions to other mixtures is easy. For a binary
mixture of Bosons, in general, there will be regions in the trap where the two species coexist
and regions where there is no coexistence. Furthermore, in the region of coexistence, each
Boson species may be in their normal or superfluid states. Since the hydrodynamic modes
in each of different regions are connected to one another, not only does one need to know the
hydrodynamic equations of interpenetrating superfluids, but also those of the surrounding
mixtures of normal and superfluids.

The organization of this paper is as follows. In Section I, we will discuss the hydrody-
namic variables of binary mixtures of scalar superfluids and their Galilean transformation
properties. The hydrodynamic equations for this system of interpenetrating scalar super-
fluids will be derived in Section II. The explicit form of the hydrodynamic equations (with
and without dissipation) will be presented. Section III is a summary of the hydrodynamic
equations for other mixture systems.

I : The hydrodynamic variables of a binary superfluid mixtures, conservation
laws, and their Galilean transformation properties :

Before studying hydrodynamics, it is necessary to study the equilibrium thermodynamics
of homogeneous systems, which is described by the entropy function $S$. $S$ is a function of
a set of thermodynamic variables $\{X_i\}$, which are quantities specifying the physical state
of the system, $S = S(\{X_i\})$. To study binary mixtures, it is useful to consider the entropy
functions of the following sequence of systems with increasing complexity :

(i) Single component normal fluids : In this case, $\{X_i\} = (E, P, N, V)$, $S = S(E, P, N, V)$,
where $E, P, N$ and $V$ are the total energy, momentum, particle number and volume of the
system respectively. The meaning of $E, P$ and $V$ will remain unchanged in the following
examples (ii) to (v).

(ii) Binary mixtures of normal fluids: Each component in the mixture can be either a Boson or a Fermion. In this case $S = S(E, P, N_1, N_2, V)$, where $N_1$ and $N_2$ are the particle numbers of the two normal fluids labelled as 1 and 2 respectively. The reason that $S$ is a function of $N_1$ and $N_2$ but not a function separate momenta $P_1$ and $P_2$ is because while Boson 1 does not turn into Boson 2, they can exchange momenta through particle collisions. As a result, the equilibrium state of the system is characterised by a single momentum.

(iii) Single component scalar superfluid: This is the well known case of $^4$He. The system now acquires a broken gauge symmetry $\langle \hat{\psi}(r) \rangle = | \langle \hat{\psi}(r) \rangle | \ e^{i\theta(r)}$, where $\hat{\psi}$ is the field operator. In this case, $S = S(E, P, N, V; v_s)$, where $v_s = (\hbar/m) \nabla \theta$ is the superfluid velocity describing the phase winding in the system, and $m$ is the mass of the Boson.

(iv) Binary mixtures consisting of a scalar superfluid and a normal fluid: Denoting the superfluid and the normal fluid components as 1 and 2, and their chemical potential (per unit mass) $\mu$ respectively, we have for this case $S = S(E, P, N_1, N_2, V; v_{s1}, v_{s2})$. The reason that $S$ is a function of total momentum $P$ instead of separate momenta $P_1$ and $P_2$ is the same as that for normal fluid mixtures.

(v) Binary mixtures of superfluids: We denote the two superfluids as 1 and 2, and their masses by $m_1$ and $m_2$ respectively. From the discussion in cases (i) to (iv) above, we have

$$S = S(E, P, N_1, N_2, V; v_{s1}, v_{s2}),$$

with

$$v_{s\alpha} = \frac{\hbar}{m_\alpha} \nabla \theta_\alpha, \quad \alpha = 1, 2.$$  

From this point onwards, we shall consider only case (v). The hydrodynamic equations of cases (i) to (iv) can be obtained from the equations derived for case (v) by setting the appropriate superfluid densities (defined below) to zero. (See section (III)).

The extensive property of $S$ implies

$$S(E, P, N_1, N_2, V; v_{s1}, v_{s2}) = V s(\epsilon, g, \rho_1, \rho_2, v_{s1}, v_{s2}),$$

where $s = S/V$, $\epsilon = E/V$, $g = P/V$ are the entropy density, the energy density, the momentum density, and $\rho_\alpha = m_\alpha N_\alpha/V$ is the mass density of the species labelled $\alpha$ in the mixture, ($\alpha = 1, 2$). The arguments $\{\epsilon, g, \rho_1, \rho_2, v_{s1}, v_{s2}\}$ of the entropy density $s$ will be referred to as the ”hydrodynamic variables” of the system, $\{x_i\}$. Defining temperature $T$, chemical potential (per unit mass) $\mu_\alpha$, “normal fluid velocity” $v_n$, and the vector $b_\alpha$ as $\partial s/\partial \epsilon \equiv 1/T, \partial s/\partial \rho_\alpha \equiv -\mu_\alpha/T, \partial s/\partial g \equiv -v_n/T$, and $\partial s/\partial v_{s\alpha} \equiv -b_\alpha/T$ respectively, we can write

$$ds = \frac{1}{T} d\epsilon - \frac{v_n}{T} \cdot dv_n - \sum_{\alpha=1}^{2} \frac{\mu_\alpha}{T} d\rho_\alpha - \sum_{\alpha=1}^{2} \frac{b_\alpha}{T} \cdot dv_{s\alpha}. $$

Defining the pressure as $P = T(\partial S/\partial V)_{E, P, N}$, and using the extensivity relation given by Eq.(3), we have

$$P = Ts - \epsilon + \mu_1 \rho_1 + \mu_2 \rho_2 + v_n \cdot g,$$

and the Gibbs-Duham relation

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\[ dP = sdT + \sum_{\alpha=1}^{2} \rho_{\alpha} d\mu_{\alpha} + \mathbf{g} \cdot d\mathbf{v}_n - \sum_{\alpha} b_{\alpha} \cdot d\mathbf{v}_{\alpha}, \]  \hspace{1cm} (6)

which shows the pressure is a function of the intensive variables \((T, \mu_{\alpha}, \mathbf{v}_n, \mathbf{v}_{\alpha})\). In analogy to binary mixtures of classical fluids, it is useful to introduce the concentration variable

\[ c = \rho_1/\rho, \hspace{0.5cm} 1 - c = \rho_2/\rho, \]  \hspace{1cm} (7)

where \(\rho = \rho_1 + \rho_2\). If we define

\[ \mu \equiv \mu_1 c + \mu_2 (1 - c), \]  \hspace{1cm} (8)

then Eq.(4) can be written as

\[ Tds = d\epsilon - \mathbf{v}_n \cdot d\mathbf{g} - \mu d\rho - (\mu_1 - \mu_2) \rho dc - \sum_{\alpha} b_{\alpha} \cdot d\mathbf{v}_{\alpha}. \]  \hspace{1cm} (9)

Likewise, Eq.(5) can be written as

\[ P = Ts - \epsilon + \mu \rho + \mathbf{v}_n \cdot \mathbf{g}, \]  \hspace{1cm} (10)

while the Gibbs-Duham relation can be cast in the form

\[ \delta P = s\delta T + \mathbf{g} \cdot \delta \mathbf{v}_n + \rho \delta \mu - (\mu_1 - \mu_2) \rho \delta c - \sum_{\alpha} b_{\alpha} \cdot \delta \mathbf{v}_{\alpha}. \]  \hspace{1cm} (11)

**The Galilean transformation properties of the hydrodynamic variables**: As we shall see, the derivation of the hydrodynamic equations can be simplified by using the Galilean transformation (GT) properties of the hydrodynamic variables \(X_i\) and their derivatives \(\partial S/\partial X_i\). A Galilean transformation is a space-time transformation \((\mathbf{r} \rightarrow \mathbf{r}' = \mathbf{r} + \mathbf{u}t, \hspace{0.2cm} t \rightarrow t' = t)\), where \(\mathbf{u}\) is the relative velocity of the two inertial frames. Under this transformation, a scalar function \(F\) of \(\mathbf{r}\) and \(t\) will change to \(F'\), which is a function of \(\mathbf{r}'\) and \(t'\). In particular, we have

\[ \rho'_{\alpha} = \rho_{\alpha}, \hspace{0.5cm} \alpha = 1, 2 \]  \hspace{1cm} (12)

\[ \mathbf{g}' = \mathbf{g} + \rho \mathbf{u}, \hspace{0.5cm} \text{where} \hspace{0.2cm} \rho = \rho_1 + \rho_2, \]  \hspace{1cm} (13)

\[ \epsilon' = \epsilon + \mathbf{g} \cdot \mathbf{u} + \rho u^2/2, \]  \hspace{1cm} (14)

and

\[ \mathbf{v}'_{\alpha} = \mathbf{v}_{\alpha} + \mathbf{u}. \]  \hspace{1cm} (15)

While the validity of Eqs.(13) and (14) is easy to see, Eq.(13) follows from the Galilean transformation properties of the field operator of the species \(\alpha\).
\begin{equation}
\hat{\psi}'(r', t') = \hat{\psi}(r, t) \exp \left( \frac{im_{\alpha} \cdot r}{\hbar} + \frac{im_{\alpha} u^2 t}{2\hbar} \right). \tag{16}
\end{equation}

The phase $\theta_{\alpha}$ of $<\hat{\psi}_{\alpha}(r, t)>$ transforms as
\begin{equation}
\theta'_{\alpha} = \theta_{\alpha} + \frac{m_{\alpha} \cdot r}{\hbar} + \frac{m_{\alpha} u^2 t}{2\hbar}, \tag{17}
\end{equation}
hence Eq.(13), which says that $v_s$ indeed transforms like a velocity field. Finally, we also note that the momentum density of the species $\alpha$, written as $g_{\alpha}$, transforms as
\begin{equation}
g_{\alpha} \rightarrow g'_{\alpha} = g_{\alpha} + \rho_{\alpha} u, \quad \text{and} \quad g = g_1 + g_2. \tag{18}
\end{equation}

Since the entropy involves counting of states, it is a Galilean invariant; we then have
\begin{equation}
s' \left( (\epsilon + g \cdot u + \frac{1}{2} \rho u^2), (g + \rho u), \rho_1, \rho_2, (v_{s1} + u), (v_{s2} + u) \right) = s (\epsilon, g, \rho_1, \rho_2, v_{s1}, v_{s2}). \tag{19}
\end{equation}
Differentiating both sides of Eq.(19) with respect to $\epsilon$ and $v_{s\alpha}$, we find that both $T$ and $b_{\alpha}$ are Galilean invariant,
\begin{equation}
T' = T, \quad b'_{\alpha} = b_{\alpha}. \tag{20}
\end{equation}
Differentiating both sides of Eq.(19) with respect to $g$ and $\rho_{\alpha}$, we get
\begin{equation}
v'_{n} = v_{n} + u, \quad \text{and} \quad \mu'_{\alpha} = \mu_{\alpha} - v_{n} \cdot u - \frac{1}{2} u^2, \quad \alpha = 1, 2. \tag{21}
\end{equation}
This shows that $v_{n}$ also transforms like a velocity field. Finally, differentiating Eq.(19) with respect to $u$ at $u = 0$ gives
\begin{equation}
b_1 + b_2 = g - \rho v_{n} \equiv g^{0}, \tag{22}
\end{equation}
where the superscript $^{0}$ refers to quantities evaluated in the $v_{n} = 0$ frame. From Eq.(21), we then have
\begin{equation}
\mu_{\alpha} = \mu_{\alpha}^{0} - \frac{1}{2} v_{n}^2, \quad \alpha = 1, 2, \tag{23}
\end{equation}
and from Eqs.(13), (14), and (21), it is easy to see that the pressure $P$ is a Galilean invariant and can also be written as
\begin{equation}
P = Ts - \epsilon^{0} + \mu_{\alpha}^{0} \rho_{\alpha} + \mu_{\alpha}^{0} \rho_{\alpha}, \tag{24}
\end{equation}

Rotational symmetry : Under an infinitesimal rotation $a$, a vector such as $g$ changes to $g + a \times g$. Since entropy of the system is invariant under such a rotation, we have
\begin{equation}
s (\epsilon, (g + a \times g), \rho_{\alpha}, (v_{s1} + a \times v_{s1}), (v_{s2} + a \times v_{s2})) = s (\epsilon, g, \rho_{\alpha}, v_{s1}, v_{s2}). \tag{25}
\end{equation}
Differentiating both sides with respect to $a$, we have
\begin{equation}
v_{n} \times g + b_1 \times v_{s1} + b_2 \times v_{s2} = 0, \tag{26}
\end{equation}
which implies $b_1 \times v_{s1}^0 + b_2 \times v_{s2}^0 = 0$. The general form of $b_\alpha$ satisfying Eq.(20) can be written as

$$b_1 = \rho_{s1} v_{s1}^0 + \bar{\rho}(v_{s1}^0 - v_{s2}^0), \quad b_2 = \rho_{s2} v_{s2}^0 + \bar{\rho}(v_{s2}^0 - v_{s1}^0),$$

(27)

where $\rho_{s1}, \rho_{s2}$, and $\bar{\rho}$ are parameters which will be referred to as superfluid densities. With this parameterization, using Eq.(22), we have

$$g_1^0 = \rho_{n1} v_{n1} + \rho_{s1} v_{s1} + \bar{\rho}(v_{s1} - v_{s2}) + J^D, \quad g_2^0 = \rho_{n2} v_{n2} + \rho_{s2} v_{s2} + \bar{\rho}(v_{s2} - v_{s1}) - J^D,$$

(28)

$$g = \rho_n v_n + \rho_{s1} v_{s1} + \rho_{s2} v_{s2}, \quad \rho_n \equiv \rho - \rho_{s1} - \rho_{s2},$$

(29)

where $\rho_n$ will be referred to as the normal fluid density. For later discussions, it is convenient to define the “diffusion” current $J^D$ as

$$g_1^0 \equiv b_1 + J^D, \quad g_2^0 \equiv b_2 - J^D.$$

(30)

Using the above definition of $J^D$, $g_\alpha$ can be written as

$$g_1 = \rho_{n1} v_n + \rho_{s1} v_{s1} + \bar{\rho}(v_{s1} - v_{s2}) + J^D,$$

(31)

$$g_2 = \rho_{n2} v_n + \rho_{s2} v_{s2} + \bar{\rho}(v_{s2} - v_{s1}) - J^D,$$

(32)

where the normal density of the species $\alpha$, $\rho_{n\alpha}$ is defined as

$$\rho_{n\alpha} \equiv \rho_\alpha - \rho_{s\alpha}, \quad \rho_n = \rho_{n1} + \rho_{n2}.$$

(33)

From Eqs.(31) and (32), it is clear that the $\bar{\rho}$ term describes the backflow created by one superfluid component on the other in the mixture.

### Continuity equations and symmetry relations

The time evolution of the hydrodynamic variables $\{x_i\}$ are governed by continuity equations and conservation laws, which are

$$\partial_t \rho_\alpha = -\nabla \cdot g_\alpha, \quad \alpha = 1, 2,$$

(34)

$$\partial_t g_i = -\rho_1 \nabla_1 \phi_1 - \rho_2 \nabla_2 \phi_2 - \nabla j \Pi_{ij},$$

(35)

$$\partial_t \epsilon = -\nabla \cdot Q - g_1 \cdot \nabla \phi_1 - g_2 \cdot \nabla \phi_2,$$

(36)

$$\partial_t v_{s\alpha} = -\nabla \chi_\alpha, \quad \alpha = 1, 2,$$

(37)

where $\chi_\alpha = -\frac{\hbar}{m} \partial_t \theta_\alpha$, and $\phi_1$ and $\phi_2$ are the external potential experienced by Bosons 1 and 2 respectively. Eqs.(34) to (37) are continuity equations for density, momentum and energy respectively. $\Pi_{ij}$ is the stress tensor and $Q$ is the energy current. An argument of Martin et al., shows that $\Pi_{ij}$ can always be taken to be a symmetric tensor [11]. The quantities
\{Q_i, \Pi_{ij}, g_\alpha, \chi_\alpha\} \equiv \{J_i\}, referred to as "currents", control the flow of the hydrodynamic variables \{x_i\}.

To complete the hydrodynamic description, we need to express the currents \{J_i\} in terms of the hydrodynamic variables \{x_i\} so as to form a closed set of equations. As explained in the next section, this can be done by studying the entropy production of the system. The derivation can be simplified considerably by using the Galilean transformation properties of \{J_i\}. By noting that the partial derivatives \(\partial_t\) and \(\nabla\) transform as \(\partial_t \to \partial_t' = \partial_t - u \cdot \nabla\), and \(\nabla \to \nabla' = \nabla\) under a Galilean transformation, we have

\[
Q'_i = Q_i + \Pi_{ij} u_j + u_i' + \frac{1}{2}(g_{1i} + g_{2i})u^2
\]

\[
\Pi'_{ij} = \Pi_{ij} + u_i(g_{1j} + g_{2j}) + (g_{1j} + g_{2j})u_j + \rho u_i u_j
\]

\[
\chi'_\alpha = \chi_\alpha + u \cdot v_s + u^2 / 2.
\]

For example, Eq.(40) is obtained by noting

\[
\chi'_\alpha = -\frac{\hbar}{m_\alpha} (\partial_t - u \cdot \nabla)(\theta_\alpha + m_\alpha u \cdot \mathbf{r} / m_\alpha + \rho u^2 t / 2\hbar)
\]

\[
= \chi_\alpha - u^2 / 2 + u \cdot v_s + u^2.
\]

Using these properties, we can write Eqs.(38) to (40) as

\[
Q_i = Q'_i + \Pi_{ij}^0 v_{nj} + v_n^i \epsilon + \frac{1}{2}(g_{1i}^0 + g_{2i}^0)v_n^2,
\]

\[
\Pi_{ij} = \Pi_{ij}^0 + v_{ni}(g_{1j}^0 + g_{2j}^0) + (g_{1j}^0 + g_{2j}^0)v_{nj} + \rho v_{ni} v_{nj},
\]

\[
\chi_\alpha = \chi_\alpha^0 + v_n \cdot v_s^0 + \frac{1}{2}v_n^2.
\]

For later use, we note that

\[
(\nabla \cdot g)^0 = \nabla \cdot g^0 + \rho \nabla \cdot v_n,
\]

\[
(\nabla \cdot Q)^0 = \nabla \cdot Q^0 + \Pi_{ij}^0 \partial_j v_{ni} + \epsilon^0(\nabla \cdot v_n),
\]

\[
(\nabla \chi_\alpha)^0 = \nabla \chi_\alpha^0 + (v_{sa}^0) v_{ni}.
\]

II. Derivation of the hydrodynamic equations for binary mixtures of scalar superfluids:

According to the second law of thermodynamics, the entropy production of the system must be positive, \(\frac{d}{dt} \int d^3 x s \geq 0\). Note that

\[
\partial_t s = -\frac{1}{T} \nabla \cdot Q + v_{ni} \frac{1}{T} \partial_j \Pi_{ij} + \sum_{\alpha = 1, 2} \left( -\frac{1}{T} g_{i\alpha} \cdot \nabla \phi_\alpha + \frac{\mu_{\alpha}}{T} \nabla \cdot g_\alpha + \frac{1}{T} b_{\alpha} \cdot \nabla \chi_\alpha + \frac{1}{T} \rho_{\alpha} v_n \cdot \nabla \phi_\alpha \right).
\]

(47)
Evaluating Eq. (47) in the \( v_n = 0 \) frame, we have
\[
T(\partial_t s)^0 = - \left( \nabla \cdot Q^0 + \Pi_{ij}^0 \partial_i v_{nj} + \epsilon^0 \nabla \cdot v_n \right) + \\
+ \sum_{\alpha=1,2} \left[ -g^0_\alpha \cdot \nabla \phi_\alpha + \mu^0_\alpha \rho_\alpha \nabla \cdot v_n + \mu^0_\alpha \nabla \cdot g^0_\alpha + b_\alpha \cdot \nabla \chi^0_\alpha \right] \\
+ (v^0_{sa})_i(b_\alpha \cdot \nabla \cdot v_{ni}].
\]
(48)

Using Eq. (24), and Eqs. (44) to (46), Eq. (48) can be written as
\[
T(\partial_t s)^0 = - \nabla \cdot [Q^0 + \sum_{\alpha=1,2} (g^0_\alpha \phi_\alpha - b_\alpha \chi^0_\alpha)] + \\
- \left[ \Pi_{ij}^0 + (Ts - P) \delta_{ij} - \sum_{\alpha} (v^0_{sa})_j b_{ai} \right] \partial_i v_{nj} \\
+ \sum_{\alpha=1,2} \left[ (\phi_\alpha + \mu^0_\alpha) \nabla \cdot g^0_\alpha - \chi^0_\alpha \nabla \cdot b_\alpha \right].
\]
(49)

By noting that \( \partial_t s + \nabla \cdot (s v_n) = (\partial_t s)^0 + s \nabla \cdot v_n \) is Galilean invariant and using Eq. (30), Eq. (49) can be written as
\[
T[\partial_t s + \nabla \cdot (s v_n)] = - \nabla \cdot Q^D - \Pi_{ij}^D \partial_i v_{nj} - \sum_{\alpha=1,2} \chi^D_\alpha \nabla \cdot b_\alpha + \nu \nabla \cdot J^D,
\]
(50)
where \( J^D \) is defined in Eq. (30). We introduce \( \Pi^D, Q^D, \chi^D, J^D, \) and \( \nu \) such that
\[
\Pi_{ij}^0 = \delta_{ij} P + \sum_{\alpha=1,2} b_{ai}(v^0_{sa})_j + \Pi_{ij}^D,
\]
(51)
\[
Q^0 = \sum_{\alpha=1,2} \left( b_\alpha \chi^0_\alpha - \phi_\alpha g^0_\alpha \right) + Q^D,
\]
(52)
\[
\chi^0_\alpha = \phi_\alpha + \mu^0_\alpha + \chi^D_\alpha,
\]
(53)
\[
\nu = (\mu^0_1 + \phi_1) - (\mu^0_2 + \phi_2).
\]
(54)

The determination of the “currents” \( \{J_i\} = \{Q, \Pi_{ij}, g_\alpha, \chi_\alpha\} \) now reduces to the determination of their “dissipative” components \( \{J_i^D\} \equiv \{Q^D, \Pi_{ij}^D, J^D, \chi^D\} \). Introducing \( R_1 \) and \( R_2 \) defined below, Eq. (51) becomes
\[
\partial_t s + \nabla \cdot \left( s v_n + \frac{Q^D}{T} - \frac{\nu J^D}{T} \right) = R \equiv R_1 + R_2,
\]
(55)
\[
R_1 = -\frac{1}{T} \left[ \chi^D_1 \nabla \cdot b_1 + \chi^D_2 \nabla \cdot b_2 + \Pi_{ij}^D \partial_i v_{nj} \right],
\]
(56)
\[
R_2 = Q^D \cdot \nabla \left( \frac{1}{T} \right) - J^D \cdot \nabla \left( \frac{\nu}{T} \right) = - (Q^D - \nu J^D) \cdot \left( \frac{\nabla T}{T^2} \right) - J^D \cdot \left( \frac{\nabla \nu}{T} \right).
\]
(57)
The right hand side of Eq.(55) expresses the entropy generated at \( \mathbf{r} \) by the “external forces” \( \{f_i\} \equiv \{-\nabla T, \partial_i v_{nj}, -\nabla \nu, \nabla \cdot \mathbf{b}_a\} \). The dissipative “currents” \( \{\mathcal{J}_i^D\} \) are responses to the disturbances \( \{f_i\} \) as the system tries to relax to equilibrium. In the linear response limit, \( \{\mathcal{J}_i^D\} \) is linear in \( \{f_i\} \), and \( R \) becomes a quadratic form in the forces, which must be positive in order to satisfy the second law of thermodynamics. Once the relations between the dissipative currents \( \{\mathcal{J}_i^D\} \) and the driving forces \( \{f_i\} \) are determined, the hydrodynamic equations, given by Eqs.(34) to (37) will form a closed set. Using Eqs.(31) and (32) for \( \mathbf{g}_\alpha \), Eqs.(42) and (51) for \( \Pi_{ij} \), Eqs.(33) and (53) for \( \chi_\alpha \), and Eqs.(34) and (52) for \( \mathbf{Q} \), we can write Eqs.(34) to (37) as

I. \[ \partial_t \rho_\alpha = -\nabla \cdot \mathbf{g}_\alpha, \quad \alpha = 1, 2, \] (58)
\[ \mathbf{g}_1 = \rho_{n1} \mathbf{v}_n + \rho_{s1} \mathbf{v}_{s1} + \bar{\rho} (\mathbf{v}_{s1} - \mathbf{v}_{s2}) + \mathbf{J}_1^D, \] (59)
\[ \mathbf{g}_2 = \rho_{n2} \mathbf{v}_n + \rho_{s2} \mathbf{v}_{s2} + \bar{\rho} (\mathbf{v}_{s2} - \mathbf{v}_{s1}) - \mathbf{J}_2^D, \] (60)

II. \[ \partial_t (\mathbf{g}_1 + \mathbf{g}_2)_i = -\rho_1 \nabla_i \phi_1 - \rho_2 \nabla_i \phi_2 - \nabla_j (\Pi_{ij}^D + \Pi_{ij}^D), \] (61)
\[ \Pi_{ij}^D = \delta_{ij} P + (\rho_{n1} + \rho_{n2}) v_{n1} v_{n2} + \sum_{\alpha=1,2} \rho_{s\alpha} (v_{s\alpha})_i (v_{s\alpha})_j \]
\[ + \bar{\rho} (v_{s1} - v_{s2})_i (v_{s1} - v_{s2})_j, \] (62)

III. \[ \partial_t \mathbf{v}_{s\alpha} = -\nabla \left( \phi_\alpha + \mu_\alpha + \mathbf{v}_n \cdot \mathbf{v}_{s\alpha} + \chi_\alpha^D \right), \quad \alpha = 1, 2, \] (63)

IV. \[ \partial_t s + \nabla \cdot (s \mathbf{v}_n) + \nabla \cdot \left( \frac{\mathbf{Q}^D}{T} - \frac{\nu \mathbf{J}^D}{T} \right) = \mathcal{R}, \] (64)

where we have replaced the energy equation Eq.(36) by the entropy equation Eq.(54). For hydrodynamic processes with weak dissipation, the dissipative currents \( \mathbf{J}^D, \Pi^D, \chi^D, \mathbf{Q}^D \) can be ignored and the resulting equations I to IV above will be referred to as “dissipation-free” hydrodynamic equations. For dissipative processes, however, it is necessary to determine the explicit form of the dissipative currents.

Determination of dissipative currents : Note that the forces in \( R_1 \) and \( R_2 \) (eq.(56) and Eq.(57) ) have different parity and time reversal symmetries. As a result, the currents \( (\chi_\alpha, \Pi_{ij}^D) \) and \( (\mathbf{Q}^D, \mathbf{J}^D) \) are only linear combinations of \( (\nabla \cdot \mathbf{b}_\alpha, \partial_i v_{nj}) \) and \( (\nabla T, \nabla \nu) \) respectively. Since \( \Pi_{ij}^D \) is symmetric, it can be decomposed as

\[ \Pi_{ij}^D = A_{ij} + \tau \delta_{ij}, \quad \text{Tr} A = 0, \] (65)

which allows \( R_1 \) to be expressed as

\[ R_1 = -\frac{1}{T} \left[ \frac{1}{2} A_{ij} \left( \nabla_i v_{nj} + \nabla_j v_{ni} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v}_n \right) + \tau \nabla \cdot \mathbf{v}_n + \sum_{\alpha=1,2} \chi_\alpha (\nabla \cdot \mathbf{b}_\alpha) \right]. \] (66)

Linear response implies

\[ A_{ij} = -\eta \left( \nabla_i v_{nj} + \nabla_j v_{ni} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v}_n \right), \] (67)
$$\begin{pmatrix} \tau \\ \chi_1^D \\ \chi_2^D \end{pmatrix} = - \begin{pmatrix} \zeta_{00} & \zeta_{01} & \zeta_{02} \\ \zeta_{01} & \zeta_{11} & \zeta_{12} \\ \zeta_{02} & \zeta_{21} & \zeta_{22} \end{pmatrix} \begin{pmatrix} \nabla \cdot \mathbf{v}_n \\ \nabla \cdot \mathbf{b}_1 \\ \nabla \cdot \mathbf{b}_2 \end{pmatrix}, \quad (68)$$

where $\eta$ will be referred to as the “first” viscosity, and $\zeta_{ij}$ as the “second” viscosity matrix. The fact that $\zeta_{ij}$ is symmetric is a consequence of the general Onsager relations for kinetic coefficients. More explicitly, we have

$$\Pi_{ij}^D = - \eta \left( \nabla_i v_{nj} + \nabla_j v_{ni} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v}_n \right) - \zeta_{00} \nabla \cdot \mathbf{v}_n - \zeta_{01} \nabla \cdot \mathbf{b}_1 - \zeta_{02} \nabla \cdot \mathbf{b}_2, \quad (69)$$

and

$$\chi_{\alpha}^D = - \zeta_{0\alpha} \nabla \cdot \mathbf{v}_n - \sum_{\sigma=1,2} \zeta_{\alpha\sigma} \nabla \cdot \mathbf{b}_\sigma. \quad (70)$$

Recall from Eq.(27) that $\mathbf{b}_1 = \rho_{s1} \mathbf{v}_{s1}^0 + \tilde{\rho} (\mathbf{v}_{s1}^0 - \mathbf{v}_{s2}^0)$ and $\mathbf{b}_2 = \rho_{s2} \mathbf{v}_{s2}^0 + \tilde{\rho} (\mathbf{v}_{s2}^0 - \mathbf{v}_{s1}^0)$. Clearly, in order to have $R_1 \geq 0$, we need to have $\eta > 0$, and $\text{det}|\zeta| > 0$.

To find the dissipative currents $\mathbf{Q}^D$ and $\mathbf{J}^D$, we note that the expression for $R_2$ is identical to the entropy production of a classical binary mixture [11]. The expressions of these currents are therefore identical to those of classical fluids and are of the form

$$\mathbf{J}^D = - \alpha T \left( \frac{\nabla \nu}{T} \right) - \beta T^2 \left( \frac{\nabla T}{T^2} \right), \quad (71)$$

$$\mathbf{Q}^D - \nu \mathbf{J}^D = - \beta T^2 \left( \frac{\nabla \nu}{T} \right) - \gamma T^2 \left( \frac{\nabla T}{T^2} \right), \quad (72)$$

where we have again used the Onsager relation to identify the off-diagonal kinetic coefficients in Eqs.(71) and (72). Eq.(57) then becomes $R_2 = \gamma (\nabla T)^2 / T^2 + \alpha (\nabla \nu)^2 / T + 2 \beta (\nabla T \cdot \nabla \nu) / T$, or,

$$R_2 = \kappa (\nabla T)^2 / T^2 + \frac{1}{\alpha T} (\mathbf{J}^D)^2, \quad (73)$$

where $\kappa \equiv \gamma - \beta^2 T / \alpha$. The positivity of $R_2$ implies $\kappa \geq 0$, and $\alpha \geq 0$. Eqs.(69), (71), (72), and (73) completely determine the dissipative currents and complete the full dissipative hydrodynamic equations, given by Eqs.(58) to (64) together with Eqs.(69) to (73).

While we shall not be solving the hydrodynamic equations here, we remind the readers that the sound modes of the system are contained in the so-called linearized hydrodynamic equations, which are obtained by linearizing Eqs.(58) to (64) about the equilibrium configuration. In the single component superfluid case, Landau [6] has pointed out that it is useful to express all the hydrodynamic variables in terms of externally controllable quantities such as temperature $T$ and pressure $P$. (Also see reference [8]). In the mixture case, there is another controllable quantity, which is the concentration $c = \rho_1 / \rho$ defined earlier in Section I. Expressing $\mathbf{J}^D$ and $\mathbf{Q}^D$ in terms of $T$, $P$ and $c$ as in Landau and Lifshitz [11], we have

$$\mathbf{J}^D = - \rho D \left( \nabla c + \frac{k_T}{T} \nabla T + \frac{k_P}{P} \nabla P \right), \quad (74)$$

Recall from Eq.(27) that $b_1 = \rho_{s1} v_{s1}^0 + \tilde{\rho} (v_{s1}^0 - v_{s2}^0)$ and $b_2 = \rho_{s2} v_{s2}^0 + \tilde{\rho} (v_{s2}^0 - v_{s1}^0)$. Clearly, in order to have $R_1 \geq 0$, we need to have $\eta > 0$, and $\text{det}|\zeta| > 0$. To find the dissipative currents $\mathbf{Q}^D$ and $\mathbf{J}^D$, we note that the expression for $R_2$ is identical to the entropy production of a classical binary mixture [11]. The expressions of these currents are therefore identical to those of classical fluids and are of the form

$$\mathbf{J}^D = - \alpha T \left( \frac{\nabla \nu}{T} \right) - \beta T^2 \left( \frac{\nabla T}{T^2} \right), \quad (71)$$

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where we have again used the Onsager relation to identify the off-diagonal kinetic coefficients in Eqs.(71) and (72). Eq.(57) then becomes $R_2 = \gamma (\nabla T)^2 / T^2 + \alpha (\nabla \nu)^2 / T + 2 \beta (\nabla T \cdot \nabla \nu) / T$, or,

$$R_2 = \kappa (\nabla T)^2 / T^2 + \frac{1}{\alpha T} (\mathbf{J}^D)^2, \quad (73)$$

where $\kappa \equiv \gamma - \beta^2 T / \alpha$. The positivity of $R_2$ implies $\kappa \geq 0$, and $\alpha \geq 0$. Eqs.(69), (71), (72), and (73) completely determine the dissipative currents and complete the full dissipative hydrodynamic equations, given by Eqs.(58) to (64) together with Eqs.(69) to (73).

While we shall not be solving the hydrodynamic equations here, we remind the readers that the sound modes of the system are contained in the so-called linearized hydrodynamic equations, which are obtained by linearizing Eqs.(58) to (64) about the equilibrium configuration. In the single component superfluid case, Landau [6] has pointed out that it is useful to express all the hydrodynamic variables in terms of externally controllable quantities such as temperature $T$ and pressure $P$. (Also see reference [8]). In the mixture case, there is another controllable quantity, which is the concentration $c = \rho_1 / \rho$ defined earlier in Section I. Expressing $\mathbf{J}^D$ and $\mathbf{Q}^D$ in terms of $T$, $P$ and $c$ as in Landau and Lifshitz [11], we have

$$\mathbf{J}^D = - \rho D \left( \nabla c + \frac{k_T}{T} \nabla T + \frac{k_P}{P} \nabla P \right), \quad (74)$$
\[ \mathbf{Q}^D = \left[ k_T \left( \frac{\partial \nu}{\partial c} \right)_{p,T} - T \left( \frac{\partial \nu}{\partial T} \right)_{p,c} + \nu \right] \mathbf{J}^D - \kappa \nabla T, \]  

(75)

where \( \kappa, D, k_T, k_P \) are the thermal conductivity, the diffusion coefficient, the coefficient of thermal diffusion, and the coefficient of baro-diffusion respectively; they can be written as

\[ \kappa = \gamma - \beta^2 T / \alpha, \quad D = \frac{\alpha}{\rho} \left( \frac{\partial \nu}{\partial T} \right)_{T,P} \]  

(76)

\[ \rho D k_T / T = \alpha \left( \frac{\partial \nu}{\partial T} \right)_{P,c} + \beta, \quad \rho D k_P / P = \alpha \left( \frac{\partial \nu}{\partial P} \right)_{T,c}. \]  

(77)

Using the relations above, the full expression for entropy production is

\[ R = \frac{\eta}{2T} \left( \nabla_i v_{nj} + \nabla_j v_{ni} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{v} \right)^2 + \frac{\zeta_{00}}{T} (\nabla \cdot \mathbf{v})^2 + 2 \sum_{\alpha=1,2} \frac{\zeta_{0\alpha}}{T} (\nabla \cdot \mathbf{v}) (\nabla \cdot \mathbf{b}_\alpha) \]

\[ + \sum_{\alpha=1,2} \frac{\zeta_{\alpha\alpha}}{T} (\nabla \cdot \mathbf{b}_\alpha)^2 + 2 \sum_{\alpha,\sigma=1,2} \frac{\zeta_{\alpha\sigma}}{T} (\nabla \cdot \mathbf{b}_\alpha)(\nabla \cdot \mathbf{b}_\sigma) \]

\[ + \frac{\eta}{T} \left( \frac{\partial \nu}{\partial c} \right)_{p,c} \left( \nabla \cdot \mathbf{v} \right)^2 + \frac{\kappa_T}{T} \nabla T + \frac{\kappa_P}{P} \nabla P \right)^2. \]  

(78)

(79)

III. Hydrodynamic equations of other mixture systems :

(a) Binary mixture of a normal fluid and a superfluid : Let component 1 be the superfluid and component 2 be the normal fluid. Since 2 does not have broken gauge symmetry, \( \mathbf{v}_{s2} \) is absent in the hydrodynamic equations. Rotational symmetry (Eq.(20)) yields \( \mathbf{b}_1 \times \mathbf{v}_{s1} = 0 \), which implies \( \mathbf{b}_1 = \rho_{s1} \mathbf{v}_{s1} \), or \( \rho_2 = \bar{\rho} = 0 \). Following exactly the same steps as before, it is easy to see that the hydrodynamic equations of this system are still given by Eqs.(58) to (64) with the quantities \( (\rho_{s1}, \rho_{s2}, \bar{\rho}, \mathbf{v}_{s1}, \mathbf{v}_{s2}) \) all set to zero. Moreover, the dissipative part of the stress tensor \( \Pi^D \) as well as \( \chi^D \) are given by Eqs.(69) and (70) with \( \zeta_{02} \) and \( \mathbf{b}_2 \) set to zero, while the expressions of the diffusion current \( \mathbf{J}^D \) and the heat current \( \mathbf{Q}^D \) are still given by Eqs.(74) and (75).

(b) Binary mixture of normal fluids : In this case, both \( \mathbf{v}_{s1} \) and \( \mathbf{v}_{s2} \) are absent from the equations. The hydrodynamic equations are given by Eqs.(58) to Eq.(74) with the quantities \( (\rho_{s1}, \rho_{s2}, \bar{\rho}, \mathbf{v}_{s1}, \mathbf{v}_{s2}) \) set to zero. The dissipative component \( \Pi^D \) is given by Eq.(69) with all \( \zeta_{0\beta} = 0 \) except for \( \zeta_{00} \), and \( \chi^D = 0 \). The diffusion current and heat current are still given by Eqs.(74) and (75).

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