PSEUDO-DIFFERENTIAL OPERATORS WITH SYMBOLS IN THE HÖRMANDER CLASS $S^0_{\alpha,\alpha}$ ON $\alpha$-MODULATION SPACES

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Abstract. In this paper, we study the boundedness of pseudo-differential operators with symbols in the Hörmander class $S^0_{\rho,p}$ on $\alpha$-modulation spaces $M^s_{p,q}$, and consider the relation between $\alpha$ and $\rho$.

1. Introduction

In Gröbner’s Ph.D. thesis [8], $\alpha$-modulation spaces $M^{s,\alpha}_{p,q}$ were introduced as intermediate spaces between modulation spaces $M^{s}_{p,q}$ and Besov spaces $B^{s}_{p,q}$. The parameter $\alpha \in [0,1)$ determines how the frequency space is decomposed. Modulation spaces which are constructed by the frequency uniform decomposition correspond to the case $\alpha = 0$ and Besov spaces which are constructed by the dyadic decomposition can be regarded as the limiting case $\alpha \to 1$. See the next section for the precise definition of $\alpha$-modulation spaces.

Let $b \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$, $\delta < 1$, $1 < p, q < \infty$ and $s \in \mathbb{R}$. It is known that all operators of class $\text{Op}(S^{b}_{\rho,\delta})$ are bounded on $L^p$ if and only if $b \leq -|1/p - 1/2|(1 - \rho)n$ ([14, Chapter VII, Section 5.12]), and the same condition assures the $B^{s}_{p,q}$-boundedness, namely the boundedness of operators of class $\text{Op}(S^{b}_{\rho,\delta})$, $b = -|1/p - 1/2|(1 - \rho)n$, on $B^{s}_{p,q}$ holds (see, e.g., Bourdaud [5], Gibbons [7] and Sugimoto [15]). It should be remarked that the boundedness of operators of class $\text{Op}(S^{0}_{\delta,1})$ on $B^{s}_{p,q}$ also holds for $s > 0$ (see the references above). On the other hand, as a difference between boundedness on Besov and modulation spaces, it is known that all operators of class $\text{Op}(S^{0}_{0,0})$ are bounded on $M^s_{p,q}$ (see, e.g., Gröchenig and Heil [9], Tachizawa [18] and Toft [19]). Moreover, Sugimoto and Tomita proved that the boundedness of operators of class $\text{Op}(S^{0}_{1,\delta})$, $0 < \delta < 1$, on $M^0_{p,q}$, $q \neq 2$, does not hold in general ([16, Theorem 2.1]), and also

Theorem A ([17, Theorem 1]). Let $1 < q < \infty$, $b \in \mathbb{R}$, $0 \leq \delta \leq \rho \leq 1$ and $\delta < 1$. Then, all pseudo-differential operators with symbols in $S^{b}_{\rho,\delta}$ are bounded on $M^0_{2,q}(\mathbb{R}^n)$ if and only if $b \leq -1/q - 1/2|\delta n$.

In this paper, we discuss the $M^{s,\alpha}_{p,q}$-boundedness of pseudo-differential operators with symbols in the so-called exotic class $S^{0}_{\rho,\rho}$, and try to clarify the relation between $\alpha$ and $\rho$. Borup [1] proved that all operators of class $\text{Op}(S^{0}_{\rho,\rho})$ are bounded from $M^{s,\alpha}_{p,q}$ to $M^{s-1\alpha,\alpha}_{p,q}$ for the space dimension $n = 1$. Borup and Nielsen [3] also obtained the boundedness of operators of class $\text{Op}(S^{0}_{\rho,\rho})$ on $M^{s,\alpha}_{p,q}$. Our purpose is to improve the result of [1] by removing the loss of the smoothness $1 - \alpha$ and that of [3] by replacing $\delta = 0$ with $\delta = \alpha$ in the full range $0 < p, q \leq \infty$. Our main result is the following:

Theorem 1.1. Let $0 \leq \alpha < 1$, $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then, all pseudo-differential operators with symbols in $S^{0}_{\alpha,\alpha}$ are bounded on $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$, that is, there exists a positive integer $N$ such that the estimate

$$\|\sigma(X,D)f\|_{M^{s,\alpha}_{p,q}} \lesssim \|\sigma;S^{0}_{\alpha,\alpha}\|_{N} \|f\|_{M^{s,\alpha}_{p,q}} \tag{1.1}$$

holds for all $\sigma \in S^{0}_{\alpha,\alpha}$ and $f \in S(\mathbb{R}^n)$.

Recalling the relation $S^{0}_{\rho,\delta_1} \subset S^{0}_{\rho,\delta_2}$ for $\delta_1 \leq \delta_2$, we see that the class $S^{0}_{\alpha,\alpha}$ in Theorem 1.1 is wider than $S^{0}_{\alpha,\alpha}$ in [3]. More generally, we have the following:

Corollary 1.1. Let $0 \leq \alpha < 1$, $0 < p, q \leq \infty$, $s, b \in \mathbb{R}$ and $0 \leq \delta \leq \alpha \leq \rho \leq 1$. Then, all pseudo-differential operators with symbols in $S^{b}_{\rho,\delta}$ are bounded from $M^{s,\alpha}_{p,q}(\mathbb{R}^n)$ to $M^{s-b,\alpha}_{p,q}(\mathbb{R}^n)$, that is, there exists a positive integer $N$ such that the estimate

$$\|\sigma(X,D)f\|_{M^{s-b,\alpha}_{p,q}} \lesssim \|\sigma;S^{b}_{\rho,\delta}\|_{N} \|f\|_{M^{s,\alpha}_{p,q}}$$

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holds for all $\sigma \in S^b_{\rho,\delta}$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

As a direct consequence of Theorem 1.1, Theorem A and inclusion relations between modulation and $\alpha$-modulation spaces, we immediately have the following statement (see Remark 4.1).

**Corollary 1.2.** Let $1 < q < \infty$, $q \neq 2$, $s \in \mathbb{R}$, and $0 \leq \delta, \alpha < 1$. Then, all pseudo-differential operators with symbols in $S^b_{\rho,\delta}$ are bounded on $M^{s,\alpha}_{2,q}(\mathbb{R}^n)$ if and only if $\delta \leq \alpha$.

By the same argument as in Remark 4.1, we can prove that Corollary 1.2 with $S^b_{\rho,\delta}$ replaced by $S^0_{\rho,\delta}$ is still valid for $\alpha \leq \rho \leq 1$. We shall explain the “optimality” of the symbol class in Theorem 1.1. Corollary 1.2 implies that $\text{Op}(S^0_{\alpha,\alpha}) \not\subset \mathcal{L}(M^{s,\alpha}_{2,q})$, $q \neq 2$, for any $\varepsilon > 0$. On the other hand, $\text{Op}(S^0_{\alpha-\varepsilon,\alpha-\varepsilon}) \not\subset \mathcal{L}(M^{s,\alpha}_{p,q})$, $0 < p < 1$, for any $\varepsilon > 0$ (see Remark 4.2). Therefore, the class $S^0_{\alpha,\alpha}$ in Theorem 1.1 seems to be optimal to obtain the $M^{s,\alpha}_{p,q}$-boundedness.

The plan of this paper is as follows. In Section 2, we will state basic notations which will be used throughout this paper, and then introduce the definition and some basic properties of $\alpha$-modulation spaces. After stating and proving some lemmas needed to show the main theorem in Section 3, we will actually prove it in Section 4.

We end this section by mentioning a remark on arguments to give a proof of the boundedness. If we prove estimate (1.1) for all Schwartz functions $\sigma$ on $\mathbb{R}^{2n}$, then the same estimate holds for all $\sigma \in S^0_{\rho,\delta}$ by a limiting argument (see, e.g., the beginning of the proof of [14, Chapter VII, Section 2.5, Theorem 2]). Hence, in the following statements, we will prove Theorem 1.1 for symbols $\sigma$ belonging to the good class.

2. Preliminaries

2.1. Basic notations. In this section, we collect notations which will be used throughout this paper. We denote by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{Z}_+$ the sets of reals, integers and non-negative integers, respectively. The notation $a \lesssim b$ means $a \leq Cb$ with a constant $C > 0$ which may be different in each occasion, and $a \sim b$ means $a \lesssim b$ and $b \lesssim a$. We write $\langle x \rangle = (1 + |x|)$.

The Fourier transform is given by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi \cdot x} f(x) dx,$$

and the inverse Fourier transform is given by

$$\mathcal{F}^{-1}f(x) = \hat{f}(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} f(\xi) d\xi.$$

We next recall the symbol class $S^b_{\rho,\delta} = S^b_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ for $b \in \mathbb{R}$ and $0 \leq \delta \leq \rho \leq 1$, which consists of all functions $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying

$$|\partial_\beta^\rho \partial_\gamma^\sigma(x,\xi)| \leq C_{\beta,\gamma} |\xi|^{b+|\beta|-\rho|\gamma|}$$

for all multi-indices $\beta, \gamma \in \mathbb{Z}_+^n$, and set

$$\|\sigma; S^b_{\rho,\delta}\|_N = \max_{|\beta|+|\gamma| \leq N} \left( \sup_{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n} |(\xi)^{-(b+|\beta|-\rho|\gamma|)} |\partial_\beta^\rho \partial_\gamma^\sigma(x,\xi)| | \right)$$

for $N \in \mathbb{Z}_+$. Note that $S^b_{\rho_1,\delta_1} \subset S^b_{\rho_2,\delta_2}$ holds if $b_1 \leq b_2$, $\rho_1 \geq \rho_2$ and $\delta_1 \leq \delta_2$. For $\sigma \in S^b_{\rho,\delta}$, the pseudo-differential operator $\sigma(X,D)$ is defined by

$$\sigma(X,D)f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \sigma(x,\xi) \hat{f}(\xi) d\xi.$$

We also write the Fourier multiplier operator as

$$g(D)f = \mathcal{F}^{-1}g \mathcal{F}f = \mathcal{F}^{-1}[g \cdot \mathcal{F}f]$$

and the Bessel potential as $(I - \Delta)^{s/2}f = \mathcal{F}^{-1}(1 + |\cdot|^2)^{s/2} \mathcal{F}f$ for $s \in \mathbb{R}$. We denote by $\text{Op}(S^b_{\rho,\delta})$ the class of all pseudo-differential operators with symbols in $S^b_{\rho,\delta}$. 
We will use some function spaces. We denote the Schwartz space of rapidly decreasing smooth functions on \(\mathbb{R}^n\) by \(S = S(\mathbb{R}^n)\) and its dual, the space of tempered distributions, by \(S' = S'(\mathbb{R}^n)\). The Lebesgue space \(L^p = L^p(\mathbb{R}^n)\) is equipped with the (quasi)-norm
\[
\|f\|_{L^p} = \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{1/p}
\]
for \(0 < p < \infty\). If \(p = \infty\), \(\|f\|_\infty = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|\). Moreover, for a compact subset \(\Omega \subset \mathbb{R}^n\), \(L^p_\Omega = L^p_\Omega(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) \cap S'(\mathbb{R}^n) : \text{supp} (\mathcal{F}f) \subset \Omega \}\). For \(0 < q \leq \infty\), we denote by \(\ell^q\) the set of all complex number sequences \(\{a_k\}_{k \in \mathbb{Z}^n}\) such that
\[
\|a_k\|_{\ell^q} = \left( \sum_{k \in \mathbb{Z}^n} |a_k|^q \right)^{1/q} < \infty,
\]
with the usual modification for \(q = \infty\). For a function space \(X\), we denote by \(\mathcal{L}(X)\) the space of all bounded linear operators on \(X\). We end this subsection with stating the following lemmas from [21].

**Lemma 2.1** ([21, Section 1.5.3]). Let \(\Omega = \{x : |x - x_0| \leq R\}\) and \(0 < p \leq 1\). Then we have
\[
\|f * g\|_{L^p} \lesssim R^{n(1/p - 1)} \|f\|_{L^p} \|g\|_{L^p}
\]
for any \(f, g \in L^p_\Omega\), where the implicit constant is independent of \(x_0\) and \(R\).

**Lemma 2.2** ([21, Theorem 1.4.1 (i) and Theorem 1.6.2]). Let \(\Omega = \{x : |x - x_0| \leq R\}\) and \(0 < p \leq \infty\). If \(0 < r < p\), then we have
\[
\left\| \sup_{y \in \mathbb{R}^n} \left| \frac{f(x - y)}{1 + |Ry|^{n/r}} \right| \right\|_{L^p(\mathbb{R}^n)} \lesssim \|f\|_{L^p}
\]
for any \(f \in L^p_\Omega\), where the implicit constant is independent of \(x_0\) and \(R\).

### 2.2. \(\alpha\)-modulation spaces

We give the definition of \(\alpha\)-modulation spaces and their basic properties. Let \(c < 1\) and \(C > 1\) be positive constants which depend on space dimensions. Suppose that a sequence of Schwartz functions \(\{\eta_k^\alpha\}_{k \in \mathbb{Z}^n}\) satisfies that
- \(\inf \{ |\eta_k^\alpha(\xi)| : |\xi - (k)\alpha/((1 - \alpha)k| \leq c(k)\alpha/((1 - \alpha), k \in \mathbb{Z}^n \} > 0\),
- \(\text{supp} \eta_k^\alpha \subset \{\xi \in \mathbb{R}^n : |\xi - (k)\alpha/((1 - \alpha)| \leq C(k)\alpha/((1 - \alpha))\},
- \(|\partial^\beta \eta_k^\alpha(\xi)| \leq C_{\beta} (k)^{-|\beta|\alpha/((1 - \alpha))}\) for every multi-index \(\beta \in \mathbb{Z}^n\),
- \(\sum_{k \in \mathbb{Z}^n} \eta_k^\alpha(\xi) = 1\) for any \(\xi \in \mathbb{R}^n\).

Then, for \(0 < p, q \leq \infty\), \(s, r \in \mathbb{R}\), and \(\alpha \in [0, 1)\), we denote the \(\alpha\)-modulation space \(M_{p,q}^{s,\alpha}(\mathbb{R}^n)\) by
\[
M_{p,q}^{s,\alpha}(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) : \|f\|_{M_{p,q}^{s,\alpha}} = \left\| \langle k \rangle^{s/(1 - \alpha)} \|\eta_k^\alpha(D)f\|_{L^p}\right\|_{L^q(\mathbb{Z}^n)} < +\infty \right\}.
\]
See Borup and Nielsen [2, 3] for the abstract definition including the end point case \(\alpha = 1\).

We remark that \(M_{p,q}^{s,\alpha}\) is a quasi-Banach space (Banach space if \(1 \leq p, q \leq \infty\)). Moreover, \(S \subset M_{p,q}^{s,\alpha} \subset S'\), and especially, \(S\) is dense in \(M_{p,q}^{s,\alpha}\) for \(0 < p, q \leq \infty\) (see Borup and Nielsen [4]). The definition of \(\alpha\)-modulation spaces is independent of the choice of the sequence \(\{\eta_k^\alpha\}_{k \in \mathbb{Z}^n}\). Next, we recall some basic properties of \(\alpha\)-modulation spaces.

**Proposition 2.1.** Let \(0 < p, q \leq \infty\), \(s, t \in \mathbb{R}\) and \(0 \leq \alpha < 1\). Then the mapping \((I - \Delta)^{t/2} : M_{p,q}^{s,\alpha} \hookrightarrow M_{p,q}^{s-t,\alpha}\) is isomorphic.

The proof of Proposition 2.1 is similar to that for Besov spaces in [21, Section 2.3.8]. One can find the explicit proof in [11, Appendix A].

**Lemma 2.3** ([10, Proposition 6.1]). Let \(0 < p, q \leq \infty\), \(s, r \in \mathbb{R}\) and \(0 \leq \alpha < 1\). Let a smooth radial bump function \(g\) satisfy that \(g(\xi) = 1\) on \(|\xi| < 1\), and \(g(\xi) = 0\) on \(|\xi| \geq 2\). Then, we have for all \(f \in M_{p,q}^{s,\alpha}\)
\[
\|f\|_{M_{p,q}^{s,\alpha}} \sim \left\| \langle k \rangle^{s/(1 - \alpha)} \|\eta_k^\alpha(D)f\|_{L^p}\right\|_{L^q(\mathbb{Z}^n)}^{1/\alpha},
\]
where
\[ \vartheta^a_k(\xi) = \vartheta \left( \frac{\xi - \langle k \rangle^{\alpha/(1-\alpha)}k}{C\langle k \rangle^{\alpha/(1-\alpha)}} \right). \]

Here, the constant \( C > 1 \) is the same as in the definition of the sequence \( \{ \eta_k^a \}_{k \in \mathbb{Z}^n} \).

**Lemma 2.4.** Let \( 0 < p \leq \infty \) and \( 0 \leq \alpha < 1 \). If \( 0 < r < p \), then we have
\[ \left( \sup_{y \in \mathbb{R}^n} \left\| \frac{|\eta_k^a(D)(f)(x-y)| y}{1 + \langle | k \rangle^{\alpha/(1-\alpha)} | y \rangle^{1/r}} \right\|_{L^p(\mathbb{R}^n)} \right) \lesssim \| \eta_k^a(D)f \|_{L^p}, \]
for all \( k \in \mathbb{Z}^n \).

**Proof of Lemma 2.4.** It follows from the definition of the decomposition \( \{ \eta_k^a \}_{k \in \mathbb{Z}^n} \) that
\[ \text{supp} \mathcal{F}[\eta_k^a(D)f] \subset \left\{ \xi : |\xi - \langle k \rangle^{\alpha/(1-\alpha)}k| \leq C\langle k \rangle^{\alpha/(1-\alpha)} \right\}, \]
so that Lemma 2.4 holds from Lemma 2.2. \( \square \)

**Remark 2.1.** Taking \( q \)-th power to the both sides of (2.1) and summing over \( k \in \mathbb{Z}^n \), we have for \( 0 \leq \alpha < 1, \) \( 0 < p, q \leq \infty \) and \( 0 < r < p \)
\[ \left\| \left( \sup_{y \in \mathbb{R}^n} \left\| \frac{|\eta_k^a(D)(f)(x-y)| y}{1 + \langle | k \rangle^{\alpha/(1-\alpha)} | y \rangle^{1/r}} \right\|_{L^p(\mathbb{R}^n)} \right) \right\|_{L^q} \lesssim \| \mathcal{F}[\eta_k^a(D)f] \|_{L^q}. \]

We end this subsection by stating the definition of modulation spaces, which is the special case of \( \alpha \)-modulation spaces. The precise definition and properties can be found in \( [6, 12, 13, 22] \). Let a sequence of Schwartz functions \( \{ \varphi_k \}_{k \in \mathbb{Z}^n} \) satisfy that
\[ \text{supp} \varphi \subset \{ \xi : |\xi| \leq \sqrt{n} \} \quad \text{and} \quad \sum_{k \in \mathbb{Z}^n} \varphi_k(\xi) \equiv 1 \quad \text{for any} \quad \xi \in \mathbb{R}^n, \]
where \( \varphi_k = \varphi(\cdot - k) \). Then, for \( 0 < p, q \leq \infty \) and \( s \in \mathbb{R} \), we denote the modulation space \( M^s_{p,q} \) by
\[ M^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \| f \|_{M^s_{p,q}} = \| (\langle k \rangle^{a}) \mathcal{F}[\varphi_k(D)f] \|_{L^p} \right\} \}

3. Lemmas

For the sake of simplicity, we will denote \( A = A(\alpha) = \frac{1}{1-\alpha} \) in the following argument.

In this section, we prepare some lemmas to use in the proof of Theorem 1.1. As mentioned in the end of Section 1, we may assume \( \sigma \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n) \) in the following statements. Remark that for the partition of unity \( \{ \varphi_{\ell} \}_{\ell \in \mathbb{Z}^n} \) constructing modulation spaces
\[ \sum_{\ell \in \mathbb{Z}^n} \varphi_{\ell}(\xi) \equiv 1 \quad \text{for any} \quad \xi \in \mathbb{R}^n \quad \Longrightarrow \quad \sum_{\ell \in \mathbb{Z}^n} \varphi_{\ell} \left( \frac{\xi}{\langle m \rangle^A} \right) \equiv 1 \quad \text{for any} \quad \xi \in \mathbb{R}^n \quad \text{and} \quad m \in \mathbb{Z}^n. \]

Then, we can decompose the symbols \( \sigma \) as
\[ \sigma(x, \xi) = \sum_{m \in \mathbb{Z}^n} \sigma(x, \xi) \cdot \eta_m(\xi) = \sum_{\ell, m \in \mathbb{Z}^n} \left( \varphi_{\ell} \left( \frac{D_x}{\langle m \rangle^A} \right) \right) (x, \xi) \cdot \eta_m(\xi), \]
where \( \{ \eta_m \}_{m \in \mathbb{Z}^n} \) is the decomposition constituting \( \alpha \)-modulation spaces. By setting
\[ \sigma_{\ell, m}(x, \xi) = \left( \varphi_{\ell} \left( \frac{D_x}{\langle m \rangle^A} \right) \right) (x, \xi) \cdot \eta_m(\xi), \]
the left hand side of (1.1) can be expressed from Lemma 2.3 as
\[ \left\| \sigma(X, D)f \right\|_{M^s_{p,q}} \sim \left\| (\langle k \rangle^{\alpha}) \mathcal{F}[\sigma(X, D)f] \right\|_{L^p} \right\|_{\ell^q(\mathbb{Z}^n)} \]
\[ = \left\| (\langle k \rangle^{\alpha}) \sum_{\ell, m \in \mathbb{Z}^n} \vartheta^a_k(\xi) \right\|_{L^p} \right\|_{\ell^q(\mathbb{Z}^n)}. \]
In the following, we investigate some properties of \( \mathcal{U}_{k}^{m}(D)[\sigma_{\ell,m}(X,D)f] \).

First, we consider the relations between the indices \( k, \ell, m \in \mathbb{Z}^n \):

**Lemma 3.1.** For any \( k, \ell, m \in \mathbb{Z}^n \),

\[
\text{supp } \mathcal{U}_{k}^{m} \cap \text{supp } \mathcal{F}[\sigma_{\ell,m}(X,D)f] \neq \emptyset \quad \Rightarrow \quad |k - m| \lesssim \langle \ell \rangle,
\]

where the implicit constant is independent of \( k, \ell, m \in \mathbb{Z}^n \).

**Remark 3.1.** Lemma 3.1 implies that the function \( \mathcal{U}_{k}^{m}(D)[\sigma_{\ell,m}(X,D)f] \) always vanishes unless \( |k - m| \lesssim \langle \ell \rangle \) is satisfied.

Before beginning with the proof of Lemma 3.1, we prepare one lemma.

**Lemma 3.2.** For any \( k, m \in \mathbb{Z}^n \), we have

\[
|\langle k \rangle^A + \langle m \rangle^A| |k - m| \lesssim \langle \langle k \rangle^A \rangle |k|,
\]

and

\[
|\langle k \rangle^A k - \langle m \rangle^A m| \geq \langle \langle k \rangle^A \rangle |k| - \left(1 + \frac{|k|}{2}\right)^A \cdot \frac{|k|}{2} \gtrsim \langle \langle k \rangle^A \rangle |k|,
\]

we have \( |\langle k \rangle^A + \langle m \rangle^A| |k - m| \lesssim \langle \langle k \rangle^A \rangle |k - m| \).

**Case 1:** \( |k| \geq 2|m| \). Since

\[
|\langle k \rangle^A + \langle m \rangle^A| |k - m| \lesssim \langle \langle k \rangle^A \rangle |k|,
\]

and

\[
|\langle k \rangle^A k - \langle m \rangle^A m| \geq \langle \langle k \rangle^A \rangle |k| - \left(1 + \frac{|k|}{2}\right)^A \cdot \frac{|k|}{2} \gtrsim \langle \langle k \rangle^A \rangle |k|,
\]

we have \( |\langle k \rangle^A + \langle m \rangle^A| |k - m| \lesssim \langle \langle k \rangle^A \rangle |k - m| \).

**Case 2:** \( |k| = |m| \). Obviously, \( |\langle k \rangle^A + \langle m \rangle^A| |k - m| = 2|\langle k \rangle^A k - \langle m \rangle^A m| \).

**Case 3:** \( |m| < |k| < 2|m| \). Note that

\[
|k - m| \leq \left|\frac{\langle k \rangle^A}{\langle m \rangle^A} k - m\right|.
\]

holds in this case. In fact, if \( A = 0 \) \( (\Leftrightarrow \alpha = 0) \), then (3.3) holds obviously true. Assume that \( 0 < A < \infty \) \( (\Leftrightarrow 0 < \alpha < 1) \). Since \( \langle k \rangle^A / \langle m \rangle^A > 1 \),

\[
|k - m| \leq \left|\frac{\langle k \rangle^A}{\langle m \rangle^A} k - m\right| \quad \Leftrightarrow \quad 2 \left(\frac{\langle k \rangle^A}{\langle m \rangle^A} - 1\right) k \cdot m \leq \left(\frac{\langle k \rangle^{2A}}{\langle m \rangle^{2A}} - 1\right) |k|^2 \quad \Leftrightarrow \quad 2k \cdot m \leq \left(\frac{\langle k \rangle^A}{\langle m \rangle^A} + 1\right) |k|^2.
\]

The last statement is justified from the facts \( 2 < \langle \langle k \rangle^A \rangle / \langle m \rangle^A + 1 \) and \( k \cdot m < |k|^2 \). Therefore, it follows that

\[
|\langle k \rangle^A + \langle m \rangle^A| |k - m| \sim \langle m \rangle^A |k - m|
\]

\[
\leq \langle m \rangle^A \left|\frac{\langle k \rangle^A}{\langle m \rangle^A} k - m\right| \quad = \langle \langle k \rangle^A k - \langle m \rangle^A m|.
\]

Gathering all the cases, we obtain the desired estimate. \( \square \)

Now, we start the proof of Lemma 3.1.
Proof of Lemma 3.1. We first determine the support of $F[\sigma_{\ell,m}(X,D)f]$. By the Fubini-Tonelli theorem and the definition of $\sigma_{\ell,m}$ in (3.1), we have
\[
F[\sigma_{\ell,m}(X,D)f](\zeta) = (2\pi)^{-n} \int_{\mathbb{R}^n_\zeta} \hat{f}(\xi) \int_{\mathbb{R}^n_\zeta} e^{-ix(\zeta-\xi)} \sigma_{\ell,m}(x,\xi) dxd\xi
\]
\[
= (2\pi)^{-n} \int_{\mathbb{R}^n_\zeta} \eta_m^\alpha(\xi) \cdot \hat{f}(\xi) \int_{\mathbb{R}^n_\zeta} e^{-ix(\zeta-\xi)} \left( \phi_{\ell} \left( \frac{D_x}{\langle m \rangle^A} \right) \sigma \right) (x,\xi) dxd\xi
\]
\[
= (2\pi)^{-n} \int_{\mathbb{R}^n_\zeta} \eta_m^\alpha(\xi) \cdot \phi \left( \frac{\zeta-\xi}{\langle m \rangle^A} - \ell \right) \cdot F_x[\sigma](\zeta-\xi) \cdot \hat{f}(\xi) d\xi,
\]
where $F_x[\sigma]$ is the partial Fourier transform of $\sigma(x,\xi)$ with respect to the $x$-variable. Hence, the facts
\[
supp \eta_m^\alpha \subset \{ \xi \in \mathbb{R}^n : |\xi - \langle m \rangle^A m| \leq C(\langle m \rangle^A) \};
\]
\[
supp \phi \left( \frac{\zeta-\xi}{\langle m \rangle^A} - \ell \right) \subset \{ \xi \in \mathbb{R}^n : |\xi - \langle m \rangle^A \ell| \leq \sqrt{n} \cdot \langle m \rangle^A \}
\]
yield that
\[
(3.4) \quad supp F[\sigma_{\ell,m}(X,D)f] \subset \{ \zeta \in \mathbb{R}^n : |\zeta - \langle m \rangle^A (\ell + m)| \leq (C + \sqrt{n}) \langle m \rangle^A \}.
\]
On the other hands, we have
\[
(3.5) \quad supp g_k^\alpha \subset \{ \zeta \in \mathbb{R}^n : |\zeta - \langle k \rangle^A k| \leq 2C \langle k \rangle^A \}.
\]
Combining (3.4) with (3.5), we obtain
\[
(3.6) \quad |\langle k \rangle^A k - \langle m \rangle^A (\ell + m)| \lesssim \langle m \rangle^A + \langle k \rangle^A.
\]
Since (3.6) implies that
\[
|\langle k \rangle^A k - \langle m \rangle^A m| \lesssim (\langle m \rangle^A + \langle k \rangle^A) \ell,
\]
recalling Lemma 3.2, we conclude
\[
|k - m| \lesssim \frac{1}{\langle k \rangle^A + \langle m \rangle^A}, \quad |\langle k \rangle^A k - \langle m \rangle^A m| \lesssim \ell,
\]
which completes the proof.

We next prove that $\sigma_{\ell,m}(X,D)f$ has large decay rate with respect to $|\ell|$. For technical purposes to prove our main theorem, we slightly change the formulation of $\sigma_{\ell,m}(X,D)f$ as follows. Choose a function $\kappa \in \mathcal{S}(\mathbb{R}^n)$ satisfying that $\kappa(\xi) = 1$ on $|\xi| \leq 1$ and $\kappa(\xi) = 0$ on $|\xi| \geq 2$, and set
\[
(3.7) \quad \kappa_m^\alpha(\xi) = \kappa \left( \frac{\xi - \langle m \rangle^A m}{C(\langle m \rangle^A)} \right)
\]
with the constant $C > 1$ in the definition of $\alpha$-modulation spaces (see Section 2.2). Then, $\kappa_m^\alpha = 1$ on the support of $\eta_m^\alpha$, and thus
\[
[\sigma_{\ell,m}(X,D)f](x) = (2\pi)^{-n} \int_{\mathbb{R}^n_\zeta} e^{ix\xi} \left( \phi_{\ell} \left( \frac{D_x}{\langle m \rangle^A} \right) \sigma \right) (x,\xi) \cdot \eta_m^\alpha(\xi) \cdot \hat{f}(\xi) d\xi
\]
\[
= (2\pi)^{-n} \int_{\mathbb{R}^n_\zeta} e^{ix\xi} \left( \phi_{\ell} \left( \frac{D_x}{\langle m \rangle^A} \right) \sigma \right) (x,\xi) \cdot \eta_m^\alpha(\xi) \kappa_m^\alpha(\xi) \cdot \hat{f}(\xi) d\xi
\]
\[
= [\tilde{\sigma}_{\ell,m}(X,D)\eta_m^\alpha(D_x)f](x),
\]
where $\tilde{\sigma}_{\ell,m}(x,\xi) = \left( \phi_{\ell} \left( \frac{D_x}{\langle m \rangle^A} \right) \sigma \right) (x,\xi) \cdot \kappa_m^\alpha(\xi)$. For the symbol $\tilde{\sigma}_{\ell,m}$, we have the following lemma.

Lemma 3.3. Let $0 < p \leq \infty$. Suppose that $\sigma \in S_{\alpha,\alpha}(\mathbb{R}^n_\zeta \times \mathbb{R}^n_\xi)$. Then, we have for any $N \geq 0$
\[
\|\tilde{\sigma}_{\ell,m}(X,D)\eta_m^\alpha(D_x)f\|_{L^p} \lesssim \langle \ell \rangle^{-N} \|\eta_m^\alpha(D_x)f\|_{L^p}.
\]
Lemma 3.4. Suppose that $\sigma \in S^0_{\alpha,\alpha}(\mathbb{R}_x \times \mathbb{R}_\xi)$. Then, we have for any $M, N \geq 0$
$$\left| \int_{\mathbb{R}_\xi^n} e^{i\mathbf{x} \cdot \mathbf{\xi}} \sigma_{\ell,m}(x, \mathbf{\xi}) d\mathbf{\xi} \right| \lesssim \langle \ell \rangle^{-N} \frac{\langle m \rangle^{\alpha n}}{(1 + \langle m \rangle^A |y|)^M},$$
where the implicit constant is independent of $y \in \mathbb{R}^n$ and $\ell, m \in \mathbb{Z}^n$.

Proof of Lemma 3.4. In order to obtain the decay of $|y|$, we use integration by parts with respect to the $\xi$-variable, so that we first observe the derivatives of $\sigma_{\ell,m}$.

**Step 1:** For any multi-index $\gamma \in \mathbb{Z}^n_+$ with $|\gamma| = M$ and $N \geq 0$, we have
$$\left| \partial_{\mathbf{x}}^\gamma (\sigma_{\ell,m}(x, \mathbf{\xi})) \right| \lesssim \langle \ell \rangle^{-N} \langle m \rangle^{-A |\gamma|} \chi_{\langle \xi - (\ell) A \rangle m \leq 2C(\langle m \rangle^A)}.$$

Note that $(x) \sim (m)^{\frac{1}{1-n}}$ if $\xi \in \text{supp} \left( \partial^{\gamma - \beta}_z \varphi \left( \frac{\xi}{\langle m \rangle^A} - \ell \right) \right)$ in (3.8). Then, using the Fubini-Tonelli theorem, we have
$$\left| \left( \varphi_{\ell} \left( \frac{D_z}{\langle m \rangle^A} \right) \partial_{\mathbf{x}}^\sigma (x, \mathbf{\xi}) \right) \right| = \left| (2\pi)^{-n} \int_{\mathbb{R}_\xi^n} \partial_{\mathbf{x}}^\sigma \left( \varphi_{\ell} \left( \frac{D_z}{\langle m \rangle^A} \right) \partial_{\mathbf{x}}^\sigma (x, \mathbf{\xi}) \right) d\mathbf{\xi} \right|$$

$$= \langle m \rangle^{\alpha n} \int_{\mathbb{R}_\xi^n} \partial_{\mathbf{x}}^\sigma \left( \varphi_{\ell} \left( \frac{D_z}{\langle m \rangle^A} \right) \partial_{\mathbf{x}}^\sigma (x, \mathbf{\xi}) \right) d\mathbf{\xi} dz$$

where, in the second identity, we used the changes of variables: $\zeta \rightarrow (m)^A \zeta$ and $\zeta - \ell \rightarrow \zeta$. If $\ell \neq 0$, we have by the $N$-times integration by parts with respect to the $z$-variable
$$\left| \int_{\mathbb{R}_\xi^n} e^{-i(m)^A z} \left( \partial_{\mathbf{x}}^\sigma \varphi_{\ell} \left( (m)^A (x - z) \right) \right) d\mathbf{\xi} \right|$$

$$\lesssim \langle (m)^A |\ell| \rangle^{-N} \sum_{\beta \leq \gamma \atop \gamma = N} \langle m \rangle^{-A |\gamma - \beta|} \int_{\mathbb{R}_\xi^n} \left| \partial_{\mathbf{x}}^\gamma \partial_{\mathbf{\xi}}^\beta \sigma \right| (x, \mathbf{\xi}) \right| \left| \partial_{\mathbf{\xi}}^\gamma \varphi_{\ell} \left( (m)^A (x - z) \right) \right| d\mathbf{\xi}$$

$$\lesssim \langle (m)^A |\ell| \rangle^{-N} \sum_{\beta \leq \gamma \atop \gamma = N} \langle m \rangle^{-A |\gamma - \beta|} \cdot \langle m \rangle^{-A |\beta - A |} \cdot \langle m \rangle^{-An}$$

$$\sim \langle \ell \rangle^{-N} \langle m \rangle^{-A |\beta|} - An.$$
Hence, we obtain

\[(3.9) \quad \left| \varphi_{\ell} \left( \frac{D_x}{(m)^A} \right) \left( \partial_{\xi}^{\beta} \sigma \right) (x, \xi) \right| \lesssim (\ell)^{-N} (m)^{-A|\beta|} \]

for any \( \ell \in \mathbb{Z}^n \). Substituting (3.9) into (3.8), it holds that

\[
\left| \partial_{\xi}^{\beta} \left( \sigma_{\ell,m}(x, \xi) \right) \right| \lesssim \sum_{|\gamma| \leq |\beta|} (\ell)^{-N} (m)^{-A|\beta|} \cdot \left| \partial_{\xi}^{\gamma-\beta} \kappa \left( \frac{(m)^A \xi - (m)^A m}{C(m)^A} \right) \right| \cdot (m)^{-A|\gamma - \beta|}
\]

where we used the \(|\beta| + |\gamma - \beta| = M\) to obtain the last inequality. This concludes the result in this step.

**Step 2:** Next, we actually investigate the decay of \(|y|\) and obtain the desired estimate for any \(M, N \geq 0\).

Obviously, we have

\[(3.10) \quad \left| \int_{\mathbb{R}^n} e^{iy \cdot \xi} \sigma_{\ell,m}(x, \xi) d\xi \right| \lesssim (\ell)^{-N} \int_{\mathbb{R}^n} \chi_{\{|\xi - (m)^A m| \leq 2C(m)^A\}} d\xi \sim (\ell)^{-N} (m)^{A_n}. \]

On the other hand, by using the \(M\)-times integration by parts with respect to the \(\xi\)-variable, we have

\[(3.11) \quad \left| \int_{\mathbb{R}^n} e^{iy \cdot \xi} \sigma_{\ell,m}(x, \xi) d\xi \right| \leq |y|^{-M} \sum_{|\gamma| = M} \left| \partial_{\xi}^{\gamma} \left( \sigma_{\ell,m}(x, \xi) \right) \right| d\xi \lesssim |y|^{-M} \sum_{|\gamma| = M} (\ell)^{-N} (m)^{-A|\gamma|} \int_{\mathbb{R}^n} \chi_{\{|\xi - (m)^A m| \leq 2C(m)^A\}} d\xi \sim (\ell)^{-N} (m)^{A_n} (|m|^A |y|)^{-M} \]

for \(y \neq 0\). Combining (3.10) and (3.11), we obtain

\[
\left| \int_{\mathbb{R}^n} e^{iy \cdot \xi} \sigma_{\ell,m}(x, \xi) d\xi \right| \lesssim (\ell)^{-N} \frac{(m)^{A_n}}{(1 + (m)^A |y|)^{M}}
\]

for any \(y \in \mathbb{R}^n\) and \(\ell, m \in \mathbb{Z}^n\).

We are now in a position to prove Lemma 3.3.

**Proof of Lemma 3.3.** Choose \(M = (n + 1) + n/r\) for \(0 < r < p \leq \infty\) in Lemma 3.4. Then we have by the Fubini-Tonelli theorem and Lemma 3.4

\[
\|\sigma_{\ell,m}(X, D) \eta_m(D) f \|_{L^p}(x) \leq (2\pi)^{-n} \int_{\mathbb{R}^n} |\eta_m(D) f(y)| \cdot \left| \int_{\mathbb{R}^n} e^{i(x-y) \cdot \xi} \sigma_{\ell,m}(x, \xi) d\xi \right| dy \lesssim \int_{\mathbb{R}^n} |\eta_m(D) f(y)| \cdot (\ell)^{-N} \frac{(m)^{A_n}}{(1 + (m)^A |x-y|)^{(n+1)+n/r}} dy \lesssim (\ell)^{-N} \sup_{y \in \mathbb{R}^n} |\eta_m(D) f(x-y)| \frac{(m)^{A_n}}{1 + ((m)^A |y|)^{n/r}} \int_{\mathbb{R}^n} (m)^{A_n} (1 + (m)^A |y|)^{n+1} dy \sim (\ell)^{-N} \cdot \sup_{y \in \mathbb{R}^n} |\eta_m(D) f(x-y)| \frac{(m)^{A_n}}{1 + ((m)^A |y|)^{n/r}}.
\]

Then, taking the \(L^p\) (quasi-)norm to the both sides and applying Lemma 2.4, we obtain

\[
\|\sigma_{\ell,m}(X, D) \eta_m(D) f\|_{L^p} \lesssim (\ell)^{-N} \|\eta_m(D) f\|_{L^p},
\]

which is the desired result.  \(\square\)
4. Proof of the main theorem

As noted in the beginning of Section 3, we will write $A = A(\alpha) = \frac{\alpha}{1 - \alpha}$.
In this section, we prove Theorem 1.1 and Corollary 1.2. We first prepare one lemma.

**Lemma 4.1.** Let $0 < p \leq \infty$. Then we have
\[
\|q_k^\alpha(D) [\tilde{\sigma}_{\ell,m}(X,D)\eta_m^\alpha(D)f]\|_{L^p} \lesssim \langle \ell \rangle^{An(\frac{1}{\min(1,p)} - 1)} \|\tilde{\sigma}_{\ell,m}(X,D)\eta_m^\alpha(D)f\|_{L^p},
\]
where the implicit constant is independent of $k, \ell, m \in \mathbb{Z}^n$.

**Proof of Lemma 4.1.** The case $1 \leq p \leq \infty$ follows from the Young inequality. Assume that $0 < p < 1$. Recalling (3.4) and (3.5) from the proof of Lemma 3.1,
\[
supp F[\sigma_{\ell,m}(X,D)f] \subset \{ \zeta \in \mathbb{R}^n : |\zeta - \langle m \rangle^A(\ell + m)| \leq (C + \sqrt{n})\langle m \rangle^A \};
\]
\[
supp \rho_k^\alpha \subset \{ \zeta \in \mathbb{R}^n : |\zeta - \langle k \rangle^A k| \leq 2C\langle k \rangle^A \}.
\]
Combining these, we see that $q_k^\alpha(D) [\tilde{\sigma}_{\ell,m}(X,D)\eta_m^\alpha(D)f]$ always vanishes unless
\[
|\langle m \rangle^A(\ell + m) - \langle k \rangle^A k| \leq 2C\langle k \rangle^A + (C + \sqrt{n})\langle m \rangle^A.
\]
Hence, we obtain
\[
supp F[\sigma_{\ell,m}(X,D)f] \subset \{ \zeta \in \mathbb{R}^n : |\zeta - \langle k \rangle^A k| \leq 2C\langle k \rangle^A + 2(C + \sqrt{n})\langle m \rangle^A \};
\]
\[
supp q_k^\alpha \subset \{ \zeta \in \mathbb{R}^n : |\zeta - \langle k \rangle^A k| \leq 2C\langle k \rangle^A + 2(C + \sqrt{n})\langle m \rangle^A \}.
\]
Moreover, recalling Lemma 3.1 (or Remark 3.1), we have $|k - m| \lesssim \langle \ell \rangle$, which implies $\langle m \rangle^A \lesssim \langle k \rangle^A + \langle \ell \rangle^A$. Hence, we have by Lemma 2.1
\[
\|q_k^\alpha(D) [\tilde{\sigma}_{\ell,m}(X,D)\eta_m^\alpha(D)f]\|_{L^p} \lesssim \langle \langle k \rangle^A + \langle m \rangle^A \rangle^{n(\frac{1}{p} - 1)} \|F^{-1}[\rho_k^\alpha]\|_{L^p} \cdot \|\tilde{\sigma}_{\ell,m}(X,D)\eta_m^\alpha(D)f\|_{L^p}
\]
\[
\lesssim \langle \langle k \rangle^A + \langle \ell \rangle^A \rangle^{n(\frac{1}{p} - 1)} \langle k \rangle^{-An(\frac{1}{\min(1,p)} - 1)} \cdot \|\tilde{\sigma}_{\ell,m}(X,D)\eta_m^\alpha(D)f\|_{L^p}
\]
\[
\lesssim \langle \ell \rangle^{An(\frac{1}{p} - 1)} \|\tilde{\sigma}_{\ell,m}(X,D)\eta_m^\alpha(D)f\|_{L^p},
\]
which completes the proof. \qed

Now, we begin with the proof of the main result. Note that
\[
\left\| \sum_{\ell \in \mathbb{Z}^n} f_{\ell}(x) \right\|_{L^p(\mathbb{R}^n)} \lesssim \left( \sum_{\ell \in \mathbb{Z}^n} \|f_{\ell}(x)\|_{L^p(\mathbb{R}^n)}^{\min(1,p)} \right)^{\frac{1}{\min(1,p)}};
\]
\[
\left\| \sum_{k \in \mathbb{Z}^n} a_{k,\ell} \right\|_{L^q(\mathbb{Z}^n)} \lesssim \left( \sum_{k \in \mathbb{Z}^n} \|a_{k,\ell}\|_{L^q(\mathbb{Z}^n)}^{\min(1,q)} \right)^{\frac{1}{\min(1,q)}}
\]
hold for $0 < p, q \leq \infty$.

**Proof of Theorem 1.1.** Due to Proposition 2.1 and the symbolic calculus [14, Chapter VII, Section 5.8], it suffices to prove Theorem 1.1 only for $s = 0$. In fact, if all $\sigma(X,D) \in \text{Op}(S^0_{\alpha,\alpha})$ is bounded on $M^{0,\alpha}_{p,q}$,
\[
\|\sigma(X,D)f\|_{M^{0,\alpha}_{p,q}} \sim \|J^s\sigma(X,D)J^{-s}J^s f\|_{M^{0,\alpha}_{p,q}} \lesssim \|J^s f\|_{M^{0,\alpha}_{p,q}} \sim \|f\|_{M^{0,\alpha}_{p,q}},
\]
where we set \( J = (I - \Delta)^{1/2} \) and used that \( J^s \sigma(X, D) J^{-s} \in \text{Op}(S^0_{\delta, \alpha}) \). We first estimate the \( L^p \) (quasi)-norm of \( \eta_\ell^0(D)[\sigma(X, D)f] \). Set \( p^* = \min(1, p) \). Then, we have for any \( k \in \mathbb{Z}^n \)

\[
\| \eta_\ell^0(D)[\sigma(X, D)f] \|_{L^p} \times \left( \sum_{\ell \in \mathbb{Z}^n} \sum_{m \in \mathbb{Z}^n : |k - m| \leq |\ell|} \| \eta_\ell^0(D)[\sigma_{\ell, m}(X, D)\eta_m^0(D)f] \|_{L^p} \right)^{1/p^*} \leq \sum_{\ell \in \mathbb{Z}^n} \left( \left( \sum_{m \in \mathbb{Z}^n : |k - m| \leq |\ell|} \| \eta_\ell^0(D)[\sigma_{\ell, m}(X, D)\eta_m^0(D)f] \|_{L^p} \right)^{p^*} \right)^{1/p^*},
\]

where we applied (3.2), (3.7) and Lemma 3.1 to the first equivalence. In the last inequality, we invoked following inclusion relations between modulation spaces and \( \delta \) Proposition 4.1.

Remark 4.1. \( M \) also bounded on \( 0 \leq s \leq 1 \) and 0 \( \leq \alpha < 1 \). Hence, Theorem A gives \( \| f \|_{M^0_{p, q}} \) holds, which completes the proof of the main theorem.

Remark 4.1. We prove Corollary 1.2. The “IF” part immediately follows from the relation \( S^0_{1, \delta} \subset S^0_{\alpha, \alpha} \) for \( \delta \leq \alpha \) and Theorem 1.1, so that we only consider the “ONLY IF” part. Before that, we recall the following inclusion relations between modulation spaces and \( \alpha \)-modulation spaces by [10, Theorem 4.1] and [20, Section 1].

**Proposition 4.1.** Let \( 0 < q \leq \infty \) and \( 0 \leq \alpha < 1 \).

1. \( M^s_{2,q} \subset M^0_{2,q} \) holds for \( s = n\alpha \cdot \max(0, 1/q - 1/2) \); 
2. \( M^0_{q, q} \subset M^s_{2,q} \) holds for \( s = n\alpha \cdot \min(0, 1/q - 1/2) \).

Now, we assume that all \( \sigma(X, D) \in \text{Op}(S^0_{1, \delta}) \) are bounded on \( M^s_{2,q} \). Then all \( \sigma(X, D) \in \text{Op}(S^0_{1, \delta}) \) are also bounded on \( M^0_{2,q} \). Indeed, since \( J^{-s_{1,s_{2}}} \sigma(X, D) J^{s_{1,s_{2}}} \in \text{Op}(S^0_{1, \delta}) \) for \( \sigma \in S^0_{1, \delta} \), where \( J = (I - \Delta)^{1/2} \) and \( s_{1,s_{2}} \) are the same as in Proposition 4.1, we have by Propositions 2.1 and 4.1

\[
\| \sigma(X, D)f \|_{M^0_{2,q}} \leq \| J^{-s_{1,s_{2}}} \sigma(X, D)f \|_{M^s_{2,q}} \leq \| J^{-s_{1,s_{2}}} \sigma(X, D)J^{s_{1,s_{2}}} f \|_{M^s_{2,q}} \leq \| J^{s_{1,s_{2}}} f \|_{M^s_{2,q}}.
\]

Hence, Theorem A gives \( b - s_{1,s_{2}} \leq -1/\alpha \leq -\alpha \) and thus \( b \leq -1/\alpha \). This means that \( \delta \leq \alpha \) for \( b = 0 \) and \( q \neq 2 \), which concludes the “ONLY IF” part in Corollary 1.2.

**Remark 4.2.** In this remark, we find a counterexample of the inclusion \( \text{Op}(S^0_{\alpha, \alpha - \varepsilon}) \subset \mathcal{L}(M^s_{p,q}) \) for \( 0 < \varepsilon < \alpha \) and \( 0 < p < 1 \). We write \( A_{\varepsilon} = \frac{\alpha - \varepsilon}{\alpha - \varepsilon} \) for any \( \varepsilon > 0 \). Choose \( \psi, \tilde{\psi} \in S(\mathbb{R}^n) \) satisfying that
supp \psi \subset \{|\xi| \leq c\}, \tilde{\psi}(\xi) = 1 on \{|\xi| \leq c\} and \tilde{\psi}(\xi) = 0 on \{|\xi| \leq 2c\}. Here, the constant c = c(\alpha) is sufficiently small. Set
\[
\sigma(\xi) = \sum_{m \in \mathbb{Z}^n} \psi \left( \frac{\xi - \langle m \rangle A m}{\langle m \rangle A_x} \right) \text{ and } f_\ell(\xi) = \tilde{\psi} \left( \frac{\xi - \langle \ell \rangle A \ell}{\langle \ell \rangle A} \right)
\]
for all \( \ell \in \mathbb{Z}^n \). Here, we remark that it follows from the fact \( A > A_x \) that
\[
\text{supp} \psi \left( \cdot - \langle m \rangle A m \right) \cap \text{supp} \tilde{\psi} \left( \cdot - \langle \ell \rangle A \ell \right) = \emptyset \text{ if } m \neq \ell
\]
and
\[
\tilde{\psi} \left( \frac{\xi - \langle m \rangle A m}{\langle m \rangle A_x} \right) = 1 \text{ on } \text{supp} \psi \left( \cdot - \langle m \rangle A m \right).
\]
In addition, note that at most one term in the sum of \( \sigma \) is non-zero for each \( \xi \) since
\[
\text{supp} \psi \left( \cdot - \langle m \rangle A m \right) \cap \text{supp} \psi \left( \cdot - \langle m' \rangle A m' \right) = \emptyset \text{ if } m \neq m'.
\]
Then, since \( \langle \xi \rangle \sim \langle m \rangle \frac{1}{\langle m \rangle} \) if \( \xi \in \text{supp} \psi \left( \cdot - \langle m \rangle A m \right) \), we see that \( \sigma \in S^0_{\alpha-\varepsilon, \alpha-\varepsilon} \) for any \( \delta \geq 0 \). Especially, \( \sigma \in S^0_{\alpha-\varepsilon, \alpha-\varepsilon} \). Now, by using these functions \( \sigma \) and \( f_\ell \), we actually prove that \( \text{Op}(S^0_{\alpha-\varepsilon, \alpha-\varepsilon}) \subset \mathcal{L}(M_{p,q}^{\alpha}) \) does not hold for \( 0 < p < 1 \). We first estimate the \( \alpha \)-modulation space norm of \( f_\ell \). By using integration by parts, we have for any \( N \geq 0 \)
\[
\|f_\ell\|_{M_{p,q}^{\alpha}} = \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle \frac{1}{\langle k \rangle} \left\| F^{-1} \left[ \hat{\eta}_k^\prime(\xi) \cdot \hat{\psi} \left( \frac{\xi - \langle \ell \rangle A \ell}{\langle \ell \rangle A} \right) \right] \right\|_{L^q} \right)^{1/q} \leq \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle \frac{1}{\langle k \rangle} \left\| \frac{\langle \ell \rangle A_n}{(1 + \langle \ell \rangle A) \langle \ell \rangle A} \right\|_{L^p} \right)^{1/q} \sim \langle \ell \rangle^{\frac{1}{\alpha}} \cdot \langle \ell \rangle^{\frac{A_n(1-\frac{1}{p})}{\alpha}}.
\]
Here, in the second line, we used the fact \( \langle k \rangle \sim \langle \ell \rangle \) which is given by the condition \( |k - \ell| \lesssim 1 \). We next consider the \( \alpha \)-modulation space norm of \( \sigma(X, D)f_\ell \). Using (4.1), (4.2) and Lemma 2.3, we have
\[
\|\sigma(X, D)f_\ell\|_{M_{p,q}^{\alpha}} \sim \left( \sum_{k \in \mathbb{Z}^n} \langle k \rangle \frac{1}{\langle k \rangle} \left\| F^{-1} \left[ \hat{\sigma}_k^\prime(\xi) \cdot \left( \sum_{m \in \mathbb{Z}^n} \psi \left( \frac{\xi - \langle m \rangle A m}{\langle m \rangle A_x} \right) \right) \cdot \hat{\psi} \left( \frac{\xi - \langle \ell \rangle A \ell}{\langle \ell \rangle A} \right) \right] \right\|_{L^q} \right)^{1/q} \geq \langle \ell \rangle^{\frac{1}{\alpha}} \left\| F^{-1} \left[ \hat{\sigma}_k^\prime(\xi) \cdot \hat{\psi} \left( \frac{\xi - \langle \ell \rangle A \ell}{\langle \ell \rangle A} \right) \right] \right\|_{L^p} \sim \langle \ell \rangle^{\frac{1}{\alpha}} \cdot \langle \ell \rangle^{\frac{A_n(1-\frac{1}{p})}{\alpha}}.
\]
If we recall the definition of the function \( \varrho \) in Lemma 2.3, then we see that \( \varrho_k^\prime(\xi) = 1 \) on \( \text{supp} \psi \left( \cdot - \langle \ell \rangle A \ell \right) \). Hence, we obtain
\[
\|\sigma(X, D)f_\ell\|_{M_{p,q}^{\alpha}} \gtrsim \langle \ell \rangle^{\frac{1}{\alpha}} \left\| F^{-1} \left[ \hat{\psi} \left( \frac{\xi - \langle \ell \rangle A \ell}{\langle \ell \rangle A} \right) \right] \right\|_{L^p} \sim \langle \ell \rangle^{\frac{1}{\alpha}} \cdot \langle \ell \rangle^{\frac{A_n(1-\frac{1}{p})}{\alpha}}.
\]
We are now in position to prove the conclusion of this remark. We assume toward a contradiction that \( \sigma \) is bounded on \( M_{p,q}^{\alpha} \). Then, we have
\[
\langle \ell \rangle^{\frac{1}{\alpha}} \cdot \langle \ell \rangle^{\frac{A_n(1-\frac{1}{p})}{\alpha}} \lesssim \|\sigma(X, D)f_\ell\|_{M_{p,q}^{\alpha}} \lesssim \|f_\ell\|_{M_{p,q}^{\alpha}} \lesssim \langle \ell \rangle^{\frac{1}{\alpha}} \cdot \langle \ell \rangle^{\frac{A_n(1-\frac{1}{p})}{\alpha}}
\]
for all \( \ell \in \mathbb{Z}^n \). However, since \( A_x < A \) and \( 0 < p < 1 \), this is a contradiction. Therefore, \( \sigma \) belongs to \( S^0_{\alpha-\varepsilon, \alpha-\varepsilon} \), but \( \sigma(X, D) \) is not bounded on \( M_{p,q}^{\alpha} \).
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