Pricing Exchange Options under Stochastic Correlation

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January 14, 2020

Abstract

In this paper we study the pricing of exchange options when underlying assets have stochastic volatility and stochastic correlation. An approximation using a closed-form approximation based on a Taylor expansion of the conditional price is proposed. Numerical results are illustrated for exchanges between WTI and Brent type oil prices.

1 Introduction

In this paper we study the pricing of exchange options when the underlying assets have stochastic volatility and correlation. Its main contribution is the proposal of a approximated closed-form formula under this framework. The exchange of two assets is used to hedge against the changes in price of underling assets by betting on the difference between both. The price of these instruments has been first considered in [12] under a standard bivariate Black-Scholes model, where a closed-form formula for the pricing is provided. The results have been extended in [5, 6] to the case of a jump-diffusion model, while in [3] it has been considered the pricing of the derivative under stochastic interest rates.

It is well known that constant correlation and volatilities assumed in the context of a Black-Scholes model are not supported by empirical evidence. In the seminal paper of Heston, see [9], the pricing of option contracts under stochastic volatility is studied. The idea is extended to stochastic correlation in [1], while still considering constant volatilities.

As an alternative view to correlation, models for the covariance process have been proposed. The pricing of exchanges under stochastic covariance is adopted in Olivares and Villamor(2018), see [13]. See for example [8] for the Wishart model and [14] for an Ornstein-Uhlenbeck Levy type model.

We consider a bivariate continuous-time GARCH process to model the correlation combined with a pricing method based on a Taylor expansion of the conditional Margrabe price. Continuous-time GARCH processes as limits of the embedded discrete-time counterpart have been proposed, for example, in [7] or [4]. See also [11].
The organization of the paper is the following:

In section 2 we introduce the model, discuss the approximated pricing formula and compute the first and second order moment of the underlying assets, their volatilities and their correlations, whose proofs are deferred to the appendix. In section 3 we discuss the numerical results for the pricing of exchange options between WTI and Brent type oil futures. Finally, we present the conclusions.

2 Pricing exchange options in models with stochastic correlation

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)\) be a filtered probability space. We denote by \(Q\) a risk-neutral equivalent martingale measure (EMM) and \(E_Q\) the expected value with respect to the measure \(Q\). For a process \((X_t)_{t \geq 0}\), the integrated process associated with it is denoted by \((X^+_t)_{t \geq 0}\) and defined as:

\[
X^+_t = \int_0^t X_s \, ds
\]

The functions \(f_X(x)\) and \(f_{X/Y}(x/y)\) are respectively the probability density function (p.d.f.) of the random vector \(X\) and the conditional p.d.f. of the random vector \(X\) on the random vector \(Y\).

A two-dimensional adapted stochastic process \((S_t)_{t \geq 0} = (S^{(1)}_t, S^{(2)}_t)_{t \geq 0}\), where their components are prices of certain underlying assets, is defined on the filtered probability space above.

We assume that the process of prices has a dynamic under \(Q\) given by:

\[
\begin{align*}
    dS^{(1)}_t &= r S^{(1)}_t dt + \sigma^{(1)}_t S^{(1)}_t dZ^{(1)}_t \\
    dS^{(2)}_t &= r S^{(2)}_t dt + \sigma^{(2)}_t \sqrt{1 - \rho^2} S^{(2)}_t dZ^{(2)}_t + \sigma^{(2)}_t \rho_t S^{(2)}_t dZ^{(1)}_t
\end{align*}
\]

where the \((\sigma_t)_{t \geq 0} = (\sigma^{(1)}_t, \sigma^{(2)}_t)_{t \geq 0}\) is the volatility process and \(\rho_t\) is the instantaneous correlation coefficient, which in our models are going to be stochastic.

The payoff of a European exchange option, with maturity at time \(T > 0\) is:

\[
h(S_T) = (c S^{(1)}_T - m S^{(2)}_T)_+
\]

where \(m\) is the number of assets of type two exchanged against \(c\) assets of type one. To simplify we assume \(c = m = 1\).

The volatilities are modeled as an Ornstein-Ulenbeck processes:

\[
d\sigma^{(j)}_t = -\alpha_j \sigma^{(1)}_t + \beta_j dW^{(j)}_t, \quad j = 1, 2
\]

The Brownian motions \((W^{(1)}_t)_{t \geq 0}\) and \((W^{(2)}_t)_{t \geq 0}\) have instantaneous correlation \(\rho_V\).

By Ito formula:

\[
\begin{align*}
    dV^{(j)}_t &= c_1 (V^{(j)}_L - V^{(j)}_t) dt + \xi_j \sigma^{(j)}_t dW^{(j)}_t, \quad j = 1, 2 \\
    d\rho_t &= \bar{\gamma} (\bar{\Gamma}_L - \rho_t) dt + \bar{\alpha} \sqrt{1 - \rho^2} d\bar{W}_t
\end{align*}
\]
Remark 2.1. Parameters in models \(\alpha_j\) and \(\beta_j\) are related by \(\frac{\beta_j^2}{2\sigma_j^2}\) and \(\xi_j = 2\beta_j\).

Next, we find an expression for the price of the exchange contract. Notice that the price of this contract at time \(t\), \(0 \leq t \leq T\) with maturity at \(T\) is given by:

\[
C_t = e^{-r(T-t)}E_Q[h(S_T)]
\]

(7)

Its terminal value is \(C_T = h(S_T)\).

The price of the exchange contract at time \(t\), \(t < T\), depends on the behavior of the processes \((V_t, \rho_t)_{t \leq T}\) described by equations (5)–(6) and integrated on the interval \([t, T]\). It depends also on the spot prices, volatilities and correlation at time \(t\). For simplicity in the notations we explicitly drop this last dependence.

For the same reason, we analyze only the case \(t = 0\). Hence:

\[
C_0 = e^{-rT} \int_{R^3} h(x) f_{S_T, V_T^+, \rho_T^+}(x) \, dx
\]

(8)

where \(x = (x', x'') \in \mathbb{R}^5\).

The function \(C_M(x'') = e^{-r(T-t)} \int_{R^2} C_T(x', x'') f_{S_T/V_T^+, \rho_T^+}(x'/x'') \, dx'\) is the Margrabe price conditionally on \((V_T^+, \rho_T^+) = x''\). After conditioning it equals the Margrabe price, see [12]. A closed-form for the latter is given by:

\[
C_M(V_T^+, \rho_T^+) = e^{-rT} S_t^{(1)} N(d_1(v_T^+)) - e^{-rT} S_t^{(2)} N(d_2(v_T^+))
\]

(9)

with:

\[
d_1(v_T^+) = \log \left( \frac{S_t^{(1)}}{S_t^{(2)}} \right) + \frac{1}{2} v_T^+
\]

\[
d_2(v_T^+) = \log \left( \frac{S_t^{(1)}}{S_t^{(2)}} \right) - \frac{1}{2} v_T^+ = d_1(v_T^+) - \sqrt{v_T^+}
\]
where:
\[ v_T^+ = V_T^{1, +} + V_T^{2, +} - 2\sqrt{V_T^{1, +} V_T^{2, +}} \rho_T^+ \]

and \((V_t^+)_{t \geq 0} = (V_t^{1, +}, V_t^{2, +})_{t \geq 0}\).

Next, to approximate the price in equation (7) we consider a second order Taylor expansion of the conditional Margrabe price \(C_M(x), x \in \mathbb{R}^3\) around the average values given by \(x_0 = (E_Q(V_T^{1, +}), E_Q(V_T^{2, +}), E_Q(\rho_T^+))\). It leads to:

\[
\hat{C}_M(x) = C_M(x_0) + \frac{\partial C_M(x_0)}{\partial x_1}(x_1 - x_{0,1}) + \frac{\partial C_M(x_0)}{\partial x_2}(x_2 - x_{0,2}) + \frac{\partial C_M(x_0)}{\partial x_3}(x_3 - x_{0,3})
+
\frac{1}{2} \frac{\partial^2 C_M(x_0)}{\partial x_1^2}(x_1 - x_{0,1})^2 + \frac{1}{2} \frac{\partial^2 C_M(x_0)}{\partial x_2^2}(x_2 - x_{0,2})^2
+
\frac{1}{2} \frac{\partial^2 C_M(x_0)}{\partial x_3^2}(x_3 - x_{0,3})^2 + \frac{\partial^2 C_M(x_0)}{\partial x_1 x_2}(x_1 - x_{0,1})(x_2 - x_{0,2})
+
\frac{\partial^2 C_M(x_0)}{\partial x_1 x_3}(x_1 - x_{0,1})(x_3 - x_{0,3}) + \frac{\partial^2 C_M(x_0)}{\partial x_2 x_3}(x_2 - x_{0,2})(x_3 - x_{0,3})
\]

(10)

Combining equations (8) and (10) we have the price \(C_0\) is approximated by:

\[
\hat{C}_0 = C_M(x_0) + \frac{1}{2} \frac{\partial^2 C_M(x_0)}{\partial x_1^2} Var_Q(V_T^{1, +}) + \frac{1}{2} \frac{\partial^2 C_M(x_0)}{\partial x_2^2} Var_Q(V_T^{2, +})
+
\frac{1}{2} \frac{\partial^2 C_M(x_0)}{\partial x_3^2} Var_Q(\rho_T^+) + \frac{\partial^2 C_M(x_0)}{\partial x_1 x_2} cov_Q(V_T^{1, +}, V_T^{2, +})
\]

(11)

Notice that the Margrabe price \(C_M(x) \in C^\infty(\mathbb{R}^3)\) except in a set of zero Lebesgue measure.

We substitute equation (10) into (8). Noticing that:

\[
\int_{\mathbb{R}^3} (x_1 - x_{0,1}) f_{V_T^{1, +}, \rho_T^+}(x) \, dx = \int_{\mathbb{R}} (x_1 - x_{0,1}) \left[ \int_{\mathbb{R}^2} f_{V_T^{1, +}, \rho_T^+}(x) \, dx_2 \, dx_3 \right] x_1
\]

= \int_{\mathbb{R}} (x_1 - x_{0,1}) f_{V_T^{1, +}}(x_1) \, dx_1 = E_Q(V_T^{1, +} - E_Q(V_T^{1, +})) = 0

\[
\int_{\mathbb{R}^3} (x_2 - x_{0,2}) f_{V_T^{1, +}, \rho_T^+}(x) \, dx = \int_{\mathbb{R}} (x_2 - x_{0,2}) \left[ \int_{\mathbb{R}^2} f_{V_T^{1, +}, \rho_T^+}(x) \, dx_1 \, dx_3 \right] x_2
\]

= \int_{\mathbb{R}} (x_2 - x_{0,2}) f_{V_T^{1, +}}(x_2) \, dx_2 = E_Q(V_T^{2, +} - E_Q(V_T^{2, +})) = 0

\[
\int_{\mathbb{R}^3} (x_3 - x_{0,3}) f_{V_T^{1, +}, \rho_T^+}(x) \, dx = \int_{\mathbb{R}} (x_3 - x_{0,3}) \left[ \int_{\mathbb{R}^2} f_{V_T^{1, +}, \rho_T^+}(x) \, dx_1 \, dx_2 \right] x_3
\]

= \int_{\mathbb{R}} (x_3 - x_{0,3}) f_{\rho_T^+}(x_3) \, dx_3 = E_Q(\rho_T^+ - E_Q(\rho_T^+)) = 0

Hence, we have equation (11).
Remark 2.2. Sensitivities with respect to the parameters in the contract can be computed in a similar way. For example, an approximation of the deltas in the exchange contract are obtaining by differentiating equation (11) with respect to the price of the underlying assets.

Computing derivatives of the Margrabe price, given by equation (9), with respect to the volatilities and correlation is straightforward. This aspect is addressed in appendix B.

In order to estimate the option pricing function above we need to compute the moments of \((V_T^1, V_T^2, \rho_T^+).\) To this end we introduce the following notations:

\[
\begin{align*}
mr_j(t) &= E[\rho_t^j], \quad mr^+_j(t) = E[(\rho_t^j)^+] \quad j = 1, 2 \\
mv_{j,k}(t) &= E[(V_t^k)^j], \quad mv^+_{j,k}(t) = E[(V_t^k)^+] \\
mv_{12}(t) &= E[V_t^1 V_t^2], \quad mv^+_{12}(t) = E[V_t^1 V_t^2]^+
\end{align*}
\]

Results are given in the propositions below, while proofs are deferred to appendix A.

**Proposition 2.3.** Let the correlation process \((\rho_t)_{t \geq 0}\) satisfy equation (6). Then:

\[
\begin{align*}
E_Q(\rho^+_t) &= \bar{\Gamma}_L t + \left(\frac{\rho_0 - \bar{\Gamma}_L}{\bar{\gamma}}\right) (1 - e^{-\bar{\gamma} t}) \\
Var_Q(\rho^+_t) &= b_0 + \left(\frac{\rho_0 - \bar{\Gamma}_L}{\bar{\gamma}}\right)^2 + \left(b_1 + 2\bar{\Gamma}_L \left(\frac{\rho_0 - \bar{\Gamma}_L}{\bar{\gamma}}\right)\right) t \\
&\quad + \left(b_2 + \bar{\Gamma}_L^2\right) t^2 + \left(b_3 - 2\bar{\Gamma}_L \left(\frac{\rho_0 - \bar{\Gamma}_L}{\bar{\gamma}}\right)\right) t e^{-\bar{\gamma} t} + b_4 e^{-(2\bar{\gamma} + \alpha^2) t} \\
&\quad - \left(b_0 + b_4 + 2 \left(\frac{\rho_0 - \bar{\Gamma}_L}{\bar{\gamma}}\right)^2\right) e^{-\bar{\gamma} t} + \left(\frac{\rho_0 - \bar{\Gamma}_L}{\bar{\gamma}}\right)^2 e^{-2\bar{\gamma} t}
\end{align*}
\]

where:

\[
\begin{align*}
a_1 &= \frac{2\bar{\gamma} \bar{\Gamma}_L^2 + \alpha^2}{2\bar{\gamma}^2 + \alpha^2}, \quad a_2 = \frac{2\bar{\gamma} \bar{\Gamma}_L (\rho_0 - \bar{\Gamma}_L)}{2\bar{\gamma}^2 + \alpha^2} \\
b_0 &= \frac{1}{\bar{\gamma}^2} \left(-a_1 + \rho_0^2 - 2\bar{\Gamma}_L + \frac{2}{\bar{\gamma}^2}\right) \\
&\quad - \frac{\alpha^2}{\bar{\gamma}^2} \left(1 + \frac{a_1}{\bar{\gamma}} + a_2\right) - \frac{\alpha^2 (\rho_0^2 - a_1 - a_2)}{\bar{\gamma}^2 (2\bar{\gamma} + \alpha^2)} \\
b_1 &= \frac{1}{\bar{\gamma}} \left(-a_2 + 2\bar{\Gamma}_L - \frac{2}{\bar{\gamma}^2} + \frac{\alpha^2}{\bar{\gamma}} - a_1 \alpha^2\right) \\
b_2 &= 1, \quad b_3 = \frac{a_2 \alpha^2}{\bar{\gamma}^3} \\
b_4 &= -\frac{\alpha^2}{\bar{\gamma}^2} \frac{(\rho_0^2 - a_1 - a_2)}{(2\bar{\gamma} + \alpha^2)(\bar{\gamma} + \alpha^2)}
\end{align*}
\]
Second order moments and covariance of the integrated squared volatility are given in the propositions above:

**Proposition 2.4.** Let the process $(V_t)_{t \geq 0}$ satisfy equations (6)-(9). Then:

\[
\begin{align*}
mv_{1,j}^+(t) &= V_L^{(j)} t + \frac{V_0^{(j)} - V_L^{(j)}}{c_j} (1 - e^{-c_j t}) \\
mv_{2,j}^+(t) &= P_1(t) + ce^{-c_j t} + g_0e^{-2c_j t} + g_1e^{-3c_j t} + g_2te^{-c_j t} \\
\text{Var}_Q[V_t^{+,j}] &= mv_{2,j}^+(t) - [mv_{1,j}^+(t)]^2
\end{align*}
\]

with:

\[
\begin{align*}
P_1(t) &= \frac{1}{c_j^3} (V_L^{(j)})^2 t^2 + \frac{1}{c_j^3} \left( 2 - \frac{V_L^{(j)}}{c_j} \right) V_0^{(j)} + \xi_j^2 V_L^{(j)} t \\
&\quad + \frac{1}{c_j^3} \left( (V_0^{(j)})^2 + \frac{2(V_L^{(j)})^2}{c_j^2} - \xi_j^2 V_L^{(j)} \right) t \\
c &= \frac{1}{c_j^4} \left( \frac{1}{c_j} \xi_j^2 V_L^{(j)} + d_0 + \frac{d_1}{2} - \frac{2(V_L^{(j)})^2}{c_j^2} \right) \\
g_0 &= -\frac{1}{c_j^2} \left( d_0 + (V_0^{(j)})^2 - d_1 \right) \\
g_1 &= -\frac{1}{c_j^3}, \quad g_2 = \frac{1}{c_j^3} \left[ \xi_j^2 (V_0^{(j)} - V_L^{(j)}) \right]
\end{align*}
\]

**Proposition 2.5.** Let the process $(V_t)_{t \geq 0}$ satisfy equations (6)-(9). Then:

\[
\text{cov}(V_t^{(1)}, V_t^{(2)}) = mv_{1,2}^+ - mv_{1,1}^+(t)mv_{1,2}^+(t)
\]

where:

\[
\begin{align*}
mv_{1,2}^+(t) &= E_Q[V_t^{+,1}V_t^{+,2}] = \frac{1}{c_1 c_2} \left[ P_3(t) - (V_0^{(1)} + c_1 V_0^{(1)})mv_{1,2}(t) - (V_0^{(2)} + c_2 V_L^{(2)})mv_{1,1}(t) \right] \\
&\quad + \text{ms}_{12}(t) - \xi_1 \xi_2 \rho v e^{-c_1 t} B_1(t) - \xi_1 \xi_2 \rho v e^{-c_2 t} B_2(t) + \xi_1 \xi_2 \rho v A(t)
\end{align*}
\]

where:

\[
\begin{align*}
P_3(t) &= V_0^{(1)} V_0^{(2)} + c_2 V_0^{(1)} V_L^{(2)} t + c_1 V_0^{(2)} V_L^{(1)} t + c_1 c_2 V_L^{(1)} V_L^{(2)} t^2 \\
A(t) &= \frac{\xi_1 \xi_2 \rho v}{2(c_1 + c_2)} \left( t - \frac{2}{c_1 + c_2} (1 - e^{-\frac{1}{2}(c_1 + c_2) t}) \right) + \frac{2(\sigma_0^{(1)} \sigma_0^{(2)})}{c_1 + c_2} \left( 1 - e^{-\frac{1}{2}(c_1 + c_2) t} \right) \\
B_j(t) &= \frac{\xi_1 \xi_2 \rho v}{2(c_1 + c_2)} \left( \frac{1}{c_j} (e^{c_j t} - 1) - \frac{2(-1)^j}{c_2 - c_1} (e^{\frac{1}{2}(-1)^j(c_2 - c_1)t} - 1) \right) \\
&\quad + \frac{\sigma_0^{(1)} \sigma_0^{(2)}}{c_2 - c_1} \left( 2(-1)^j (e^{\frac{1}{2}(-1)^j(c_2 + c_1)t} - 1) \right)
\end{align*}
\]
The functions $m^+_{1,j}(t)$ are given by equation (14) while:

$$m_{12}(t) = \frac{\xi_1 \xi_2 \rho \nu}{2(c_1 + c_2)} \left(1 - \exp\left(-\frac{1}{2}(c_1 + c_2)t\right)\right) + \sigma_0^{(1)} \sigma_0^{(2)} \exp(-\frac{1}{2}(c_1 + c_2)t)$$

$$m_{1,j}(t) = V_{L}^{(j)} + (V_{0}^{(j)} - V_{L}^{(j)}) e^{-c_j t}$$

Figure 1: Left: Fifty days moving window correlation coefficient between WTI and Brent daily future prices. Right: Same window for the log-returns

3 Numerical results

We consider the series of daily closure prices per barrel in US dollars in NYSE of types WTI (blue) and Brent (red), period Dec 2013 to Jan 2019 and the corresponding log-returns. Both series of prices exhibit similar patterns and, as it is expected, are highly correlated. The overall correlation of the series of prices is equal to 98% while the correlation of the log-returns is 3.81%. However, when the correlation is computed on a sliding windows of 50 days it exhibits notable random variations. See figures 1a) and b).

A summary of the first forth moments of the log-return series is shown in table 1. A high kurtosis indicates the presence of heavy-tailed distribution in both commodities.

| Asset | Mean     | Standard deviation | Skewness | Kurtosis |
|-------|----------|--------------------|----------|----------|
| WTI   | -0.0003  | 0.0211             | 0.1089   | 6.0696   |
| Brent | -0.0004  | 0.0201             | 0.1473   | 5.9818   |

Table 1: First four moments of log-returns WTI, Brent, US/Can

To illustrate the behavior of the components in the model we take the following set of parameters in table 2. As initial prices of both assets values $S_{0}^{(1)} =$
Figure 2: Counterclockwise from the top left figure a simulated series of prices, while the top right figure shows a realization of the squared volatilities. The series in the bottom is a simulated trajectory of the correlation process.

Figure 3: A change in the prices of an exchange contract with respect to squared volatilities(left) and the correlation(right).
100, \( S_0^{(2)} = 100 \) dollars are taken, initial squared volatilities \( V_0 = (0.3, 0.3) \), the initial correlation \( \rho_0 = 0.8 \), correlation between the Brownian motions in the volatility \( \rho_v = 0.80 \), the mean-reverting levels and rates of the volatility processes are \( V_L = (1, 1) \) respectively while analogous parameters in the correlation processes are \( \Gamma = 0.8 \) and \( \gamma = 0.8 \). The annual interest rate is \( r = 4\% \), and the simulation time is one year. Parameters were chosen for illustrative proposes. Other parameters are shown in table 2.

| Asset Component | WTI sqr. vol. | Brent sqr. vol. | Correlation |
|-----------------|--------------|----------------|-------------|
| MR level        | \( V_L^{(1)} = 1 \) | \( V_L^{(2)} = 1 \) | \( \Gamma = 0.8 \) |
| MR rate         | \( c_1 = 1 \) | \( c_2 = 1 \) | \( \gamma = 0.8 \) |
| vol.            | \( x_{11} = 1 \) | \( x_{12} = 1 \) | \( \alpha = 1 \) |
| Initial values  | \( V_0^{(1)} = 0.3 \) | \( V_0^{(2)} = 0.3 \) | \( \rho_0 = 0.7 \) |

Table 2: Parametric set for the squared volatilities and correlation sets.

The results of the simulation are shown in figure 2. The top left graph represents the series of prices, while the top right figure shows a realization of the squared volatilities. The series in the bottom is a simulated trajectory of the correlation process.

A change in the prices of an exchange contract with respect to squared volatilities and the correlation are shown in figure 3. The remaining parameters are held constant. Prices are calculated according to a Monte Carlo procedure with \( 10^5 \) realizations.

4 Conclusions

Taylor approximation offers a suitable method to price exchanges contracts beyond the classic framework developed originally by Margrabe. In the parametric set considered it produces accurate results with less computational effort than a traditional Monte Carlo approach.

5 Appendix

5.1 Appendix A: Moments of the volatility and correlation

Proof of proposition 2.3

Proof. For the first moment notice that:

\[
\rho_t = \rho_0 + \gamma \Gamma_L t - \bar{\gamma} \int_0^t \rho_s \, ds + \bar{\alpha} \int_0^t \sqrt{1 - \rho_s^2} \, dW_s
\]  

(16)
Taking expected value on both sides:

\[ mr_1(t) := E_Q(\rho_t) = \rho_0 + \bar{\gamma} \bar{\Gamma}_L t - \bar{\gamma} \int_0^t mr_1(s) \, ds \]

Differentiating we get:

\[ mr'_1(t) = \bar{\gamma} \bar{\Gamma}_L - \bar{\gamma}mr_1(t) \]

whose solution is:

\[ mr_1(t) = \bar{\Gamma}_L + (\rho_0 - \bar{\Gamma}_L)e^{-\bar{\gamma}t} \]

Similarly, for the integrated process:

\[ E_Q(\rho_t^+) = \int_0^t \bar{\Gamma}_L + (\rho_0 - \bar{\Gamma}_L)e^{-\bar{\gamma}s} \, ds \]

\[ = \bar{\Gamma}_Lt + \left(\frac{\rho_0 - \bar{\Gamma}_L}{\bar{\gamma}}\right)(1 - e^{-\bar{\gamma}t}) \]

To compute the second moment we first apply Ito formula to \( f(x) = x^2 \) and the correlation process. Hence:

\[ \rho_t^2 = \rho_0^2 + 2 \int_0^t \rho_s d\rho_s + <\rho_t> \]

\[ = \rho_0^2 + 2\bar{\gamma}\bar{\Gamma}_L \int_0^t \rho_s \, ds - 2\bar{\gamma} \int_0^t \rho_s^2 \, ds + 2\bar{\alpha} \int_0^t \rho_s \sqrt{1 - \rho_s^2} \, d\bar{W}_s \]

\[ + \bar{\alpha}^2 \int_0^t (1 - \rho_s^2) \, ds \]

\[ = \rho_0^2 + \bar{\alpha}^2 t + 2\bar{\gamma}\bar{\Gamma}_L \int_0^t \rho_s \, ds - (\bar{\alpha}^2 + 2\bar{\gamma}) \int_0^t \rho_s^2 \, ds \]

\[ + 2\bar{\alpha} \int_0^t \rho_s \sqrt{1 - \rho_s^2} \, d\bar{W}_s \]

Taking expected value:

\[ E_Q(\rho_t^2) = \rho_0^2 + \bar{\alpha}^2 t + 2\bar{\gamma}\bar{\Gamma}_L \int_0^t E_Q(\rho_s) \, ds - (\bar{\alpha}^2 + 2\bar{\gamma}) \int_0^t E_Q(\rho_s^2) \, ds \]

or after differentiating:

\[ mr'_2(t) + (2\bar{\gamma} + \bar{\alpha}^2)mr_2(t) = 2\bar{\gamma}\bar{\Gamma}_Lmr_1(t) + \bar{\alpha}^2 \]

\[ mr_2(0) = \rho_0^2 \]

its solution is:

\[ mr_2(t) = a_1 + a_2 e^{-\bar{\gamma}t} + (\rho_0^2 - a_1 - a_2)e^{-(2\bar{\gamma} + \bar{\alpha}^2)t} \quad (17) \]

Next, notice that we have:

\[ \frac{mr'_2}{dt} = 2E_Q[\rho_t^+ \rho_t] \]
From equation (16):

\[ E_Q(\rho_t + \bar{\gamma}\rho_t^+)^2 = E_Q(\rho_0 + \bar{\gamma}\Gamma_L t + \bar{\alpha} \int_0^t \sqrt{1 - \rho_s^2} d\bar{W}_s)^2 \]

Expanding both sides in the equation above we have:

\[ LHS = E_Q(\rho_t + \bar{\gamma}\rho_t^+)^2 = E_Q(\rho_t^2) + 2\bar{\gamma}E_Q(\rho_t \rho_t^+) + \bar{\gamma}^2 E_Q(\rho_t^+)^2 \]

\[ = mr_2(t) + \bar{\gamma}\frac{mr_2^+}{dt} + \bar{\gamma}^2 mr_2^+(t) \]

and

\[ RHS = (\rho_0 + \bar{\gamma}\Gamma_L t)^2 + 2(\rho_0 + 2\bar{\gamma}\Gamma_L t)\bar{\alpha}E_Q(\int_0^t \sqrt{1 - \rho_s^2} d\bar{W}_s) \]

\[ + \bar{\alpha}^2 E_Q \left( \int_0^t \sqrt{1 - \rho_s^2} d\bar{W}_s \right)^2 \]

\[ = (\rho_0 + \bar{\gamma}\Gamma_L t)^2 + \bar{\alpha}^2 E_Q \left( \int_0^t \sqrt{1 - \rho_s^2} d\bar{W}_s \right)^2 \]

\[ = (\rho_0 + \bar{\gamma}\Gamma_L t)^2 + \bar{\alpha}^2 E_Q \left( \int_0^t (1 - \rho_s^2) ds \right) \]

\[ = (\rho_0 + \bar{\gamma}\Gamma_L t)^2 + \bar{\alpha}^2 (t - \int_0^t mr_2(s) ds) \]

From equation (19):

\[ \int_0^t mr_2(s) ds = \int_0^t (a_1 + a_2 e^{-\bar{\gamma}s} + (\rho_0^2 - a_1 - a_2) e^{-(2\bar{\gamma}\bar{\alpha} + \bar{\gamma})s}) ds \]

\[ = a_1 t + \frac{a_2}{\bar{\gamma}} (1 - e^{-\bar{\gamma}t}) + \frac{\rho_0^2 - a_1 - a_2}{2\bar{\gamma} + \bar{\alpha}^2} (1 - e^{-(2\bar{\gamma} + \bar{\alpha}^2)t}) \]

Hence,

\[ \frac{mr_2^+}{dt} + \bar{\gamma}mr_2^+(t) = b(t) \]

where:

\[ b(t) = \frac{1}{\bar{\gamma}} \left( -mr_2(t) + (\rho_0 + \bar{\gamma}\Gamma_L t)^2 + \bar{\alpha}^2 t - \frac{\bar{\alpha}^2}{\bar{\gamma}} \int_0^t mr_2(s) ds \right) \]

\[ = \frac{1}{\bar{\gamma}} \left( -mr_2(t) + (\rho_0 + \bar{\gamma}\Gamma_L t)^2 + \bar{\alpha}^2 t \right) \]

\[ - \frac{1}{\bar{\gamma}} \left( a_1 t + \frac{a_2}{\bar{\gamma}} (1 - e^{-\bar{\gamma}t}) + \frac{\rho_0^2 - a_1 - a_2}{2\bar{\gamma} + \bar{\alpha}^2} (1 - e^{-(2\bar{\gamma} + \bar{\alpha}^2)t}) \right) \]

and initial condition \( mr_{2+}(0) = 0 \).

Using the integrating factor \( e^{\bar{\gamma}t} \) we find that its solution is:

\[ mr_2^+(t) = e^{-\bar{\gamma}t} \int e^{\bar{\gamma}t} b(t) dt + ce^{-\bar{\gamma}t} \]
But:

\[
\int e^{\gamma t} b(t) \, dt = \frac{1}{\gamma} \int e^{\gamma t} \left( -mr_2(t) + (\rho_0 + \bar{\gamma} \Gamma_L t)^2 + \bar{\alpha}^2 t \right) \, dt
- \frac{\bar{\alpha}^2}{\gamma^2} \int e^{\gamma t} \left( a_1 t + \frac{a_2}{\gamma} (1 - e^{-\gamma t}) + \frac{\rho_0^2 - a_1 - a_2}{2\gamma + \bar{\alpha}^2} (1 - e^{-(2\bar{\gamma} + \bar{\alpha}^2)t}) \right) \, dt
\]

Moreover, from equation (19):

\[
\int e^{\bar{\gamma} t} mr_2(t) \, dt = \int e^{\bar{\gamma} t} (a_1 + a_2 e^{-\bar{\gamma} t} + (\rho_0^2 - a_1 - a_2) e^{-(2\bar{\gamma} + \bar{\alpha}^2)t}) \, dt
= \frac{a_1}{\bar{\gamma}} e^{\bar{\gamma} t} + a_2 t - \frac{\rho_0^2 - a_1 - a_2}{\bar{\gamma}} \exp(- (\bar{\gamma} + \bar{\alpha}^2)t)
\]

\[
\int (\rho_0 + \bar{\gamma} \Gamma_L t)^2 e^{\gamma t} \, dt = \frac{\rho_0^2}{\gamma} e^{\gamma t} + 2\bar{\gamma} \Gamma_L \left( \frac{1}{\gamma} e^{\gamma t} t - \frac{1}{\gamma^2} e^{\gamma t} \right)
+ \bar{\gamma}^2 \Gamma_L \left( \frac{1}{\gamma^2} e^{2\gamma t} t^2 - \frac{2}{\gamma} e^{\gamma t} + \frac{2}{\gamma^3} e^{\gamma t} \right)
= \left[ \rho_0^2 + 2\bar{\gamma} \Gamma_L (t - \frac{1}{\gamma}) + t^2 - \frac{2}{\gamma} t + \frac{2}{\gamma^3} \right] e^{\gamma t}
\]

\[
\int t e^{\gamma t} \, dt = \left[ t - \frac{1}{\gamma^2} \right] e^{\gamma t}
\]

Hence:

\[
\int e^{\gamma t} b(t) \, dt = -\frac{1}{\gamma^2} a_1 e^{\gamma t} - \left( \frac{a_2}{\gamma} \right) t + \left( \frac{\rho_0^2 - a_1 - a_2}{\gamma (\gamma + \bar{\alpha}^2)} \right) e^{-(\gamma + \bar{\alpha}^2)t}
+ \frac{1}{\gamma^2} \left( \rho_0^2 + 2\bar{\gamma} \Gamma_L (t - \frac{1}{\gamma}) + t^2 - \frac{2}{\gamma} t + \frac{2}{\gamma^3} \right) e^{\gamma t}
+ \frac{\bar{\alpha}^2}{\gamma^2} \left( t - \frac{1}{\gamma^2} \right) e^{\gamma t}
- \frac{\bar{\alpha}^2}{\gamma^2} \left( a_1 \left( t - \frac{1}{\gamma^2} \right) e^{\gamma t} + \frac{a_2}{\gamma^2} e^{\gamma t} - \frac{a_2}{\gamma} t \right)
+ \frac{\rho_0^2 - a_1 - a_2}{\gamma (2\gamma + \bar{\alpha}^2)} e^{\gamma t} + \frac{\rho_0^2 - a_1 - a_2}{(2\gamma + \bar{\alpha}^2) (\gamma + \bar{\alpha}^2)} e^{-(\gamma + \bar{\alpha}^2)t}
\]

Combining the expressions above into equation (18) we have:

\[
mr_2^+(t) = b_0 + b_1 t + b_2 t^2 + b_3 t e^{-\gamma t} + b_4 t e^{-(2\bar{\gamma} + \bar{\alpha}^2)t} + c e^{-\gamma t}
\]

From the initial conditions \( c = -b_0 - b_4 \).

Combining the first and second moments of \( \rho_i^+ \) we obtain the expression for the variance in equation (13) \( \Box \)

**Proof of proposition 2.4**
Proof. To compute the first and second moments we proceed similarly to the proof of proposition 2.3. Notice equations for squared volatilities are of mean-reverting square root type s.d.e’s as well.

Hence:

\[ m v_{1,j}(t) = V_L^{(j)} + (V_0^{(j)} - V_L^{(j)})e^{-c_jt} \]
\[ m v_{1,j}^+(t) = E_q[V_t^{+j}] = V_*^{(j)} + \frac{V_0^{(j)} - V_L^{(j)}}{c_j}(1 - e^{-c_jt}) \]

Moreover,

\[ (V_t^{(j)})^2 = (V_0^{(j)})^2 + 2 \int_0^t V_s^{(j)} dV_s^{(j)} + <V_t^{(j)}> \]
\[ = (V_0^{(j)})^2 + 2c_j V_L^{(j)} V_t^{j,*} - 2c_j \int_0^t (V_s^{(j)})^2 ds + 2\xi_j \int_0^t V_s^{(j)} \sigma_s^{(j)} dW_s^{(j)} \]
\[ + \xi_j^2 V_t^{j,*} \]
\[ = (V_0^{(j)})^2 + (2c_j V_L^{(j)} + \xi_j^2) V_t^{j,*} - 2c_j \int_0^t (V_s^{(j)})^2 ds \]
\[ + 2\xi_j \int_0^t V_s^{(j)} \sigma_s^{(j)} dW_s^{(j)} \]

Taking expected value on both sides:

\[ m v_{2,j}(t) = (V_0^{(j)})^2 + (2c_j V_L^{(j)} + \xi_j^2) \int_0^t m v_{1,j}(s) ds - 2c_j \int_0^t m v_{2,j}(s) ds \]
or

\[ m v_{2,j}'(t) + 2c_j m v_{2,j}(t) = (2c_j V_L^{(j)} + \xi_j^2) m v_{1,j}(t) \]
\[ m v_{2,j}(0) = (V_0^{(j)})^2 \]

with \( c(t) = (2c_j V_L^{(j)} + \xi_j^2) m v_{1,j}(t) \).

Its solution is:

\[ m v_{2,j}(t) = e^{-2c_jt} \int e^{2c_jt} c(t) dt + d_2 e^{-2c_jt} \]

But:

\[ \int e^{2c_jt} c(t) dt = (2c_j V_L^{(j)} + \xi_j^2) \int e^{2c_jt} m v_{1,j}(t) dt \]
\[ = (2c_j V_L^{(j)} + \xi_j^2) \int e^{2c_jt} (V_L^{(j)} + (V_0^{(j)} - V_L^{(j)})e^{-c_jt}) dt \]
\[ = (2c_j V_L^{(j)} + \xi_j^2)(V_L^{(j)} \frac{V_0^{(j)} - V_L^{(j)}}{c_j} e^{-c_jt}) \]
Then:

\[ mv_{2,j}(t) = d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t} \]  \hspace{1cm} (19)

where:

\[ d_0 = (2c_j V^{(j)} + \xi^2_j) \frac{V^{(j)}}{2c_j} \]

\[ d_1 = (2c_j + \xi^2_j) \frac{(V_0^{(j)} - V^{(j)})}{c_j} \]

\[ d_2 = (V_0^{(j)})^2 - d_0 - d_1 \]

Next, notice that we have:

\[ \frac{d mv_{2,j}^+}{dt} = 2 E_Q[V_t^{(j)} V_{t}^{(j)}] \]  \hspace{1cm} (20)

Now:

\[ E_Q(V_t^{(j)} + c_j V_t^{(j)+})^2 = E_Q[V_0^{(j)} + c_j V_L^{(j)} t + \xi_j \int_0^t \sigma_s^{(j)} dW_s^{(j)}]^2 \]

\[ = (V_0^{(j)} + c_j V_L^{(j)} t)^2 + 2(V_0^{(j)} + c_j V_L^{(j)} t) \xi_j E_Q \left( \int_0^t \sigma_s^{(j)} dW_s^{(j)} \right) + \xi_j^2 E_Q \left( \int_0^t \sigma_s^{(j)} dW_s^{(j)} \right)^2 \]

\[ = (V_0^{(j)} + c_j V_L^{(j)} t)^2 + \xi_j^2 \int_0^t \int_0^t \sigma_s^{(j)} \sigma_t^{(j)} dW_s^{(j)} dW_t^{(j)} \]  \hspace{1cm} (21)

On the other hand, after expanding the expression above and taking into account equation (20):

\[ E_Q(V_t^{(j)} + c_j V_t^{(j)+})^2 = mv_{2,j}(t) + c_j \frac{d mv_{2,j}^+}{dt} + c_j^2 mv_{2,j}^+ \]  \hspace{1cm} (22)

Hence, equating equations (21) and (20) we have that \( mv_{2,j}^+ \) satisfies:

\[ \frac{d mv_{2,j}^+}{dt} + c_j mv_{2,j}^+(t) = d(t) \]  \hspace{1cm} (23)

\[ mv_{2,j}^+(0) = 0 \]
with:

\[ d(t) = \frac{(V_0^{(j)} + c_j V_L^{(j)} t)^2}{c_j} + \frac{\xi_j^2}{c_j} \int_0^t m v_{1,j}(s) \, ds - \frac{1}{c_j} m v_{2,j}(t) \]

\[ = \frac{(V_0^{(j)} + c_j V_L^{(j)} t)^2}{c_j} + \frac{\xi_j^2}{c_j} \int_0^t V_L^{(j)} + (V_0^{(j)} - V_L^{(j)}) e^{-c_j t} \, ds \]

\[ - \frac{1}{c_j} (d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t}) \]

\[ = \frac{(V_0^{(j)} + c_j V_L^{(j)} t)^2}{c_j} + \frac{\xi_j^2}{c_j} V_L^{(j)} t - \frac{\xi_j^2}{c_j} (V_0^{(j)} - V_L^{(j)}) e^{-c_j t} \]

\[ - \frac{1}{c_j} (d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t}) \]

The solution of equation (23) is:

\[ m v_{2,j}^+(t) = e^{-c_j t} \int e^{c_j t} d(t) \, dt + c e^{-c_j t} \]

\[ = e^{-c_j t} \int e^{c_j t} \left[ \frac{(V_0^{(j)} + c_j V_L^{(j)} t)^2}{c_j} + \frac{\xi_j^2}{c_j} V_L^{(j)} t - \frac{\xi_j^2}{c_j} (V_0^{(j)} - V_L^{(j)}) e^{-c_j t} \right] \]

\[ - \frac{1}{c_j} (d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t}) \, dt + c e^{-c_j t} \]

\[ = \frac{e^{-c_j t}}{c_j} \int e^{c_j t} \left[ (V_0^{(j)} + c_j V_L^{(j)} t)^2 + \frac{\xi_j^2}{c_j} V_L^{(j)} t - \frac{\xi_j^2}{c_j} (V_0^{(j)} - V_L^{(j)}) e^{-c_j t} \right] \]

\[ - (d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t}) \, dt + d_3 e^{-c_j t} \]  

(24)

Moreover:

\[ \int e^{c_j t} (V_0^{(j)} + c_j V_L^{(j)} t)^2 \, dt = \frac{(V_0^{(j)})^2}{c_j} V_0^{(j)} e^{c_j t} + 2c_j V_L^{(j)} \int t e^{c_j t} \, dt + (V_L^{(j)})^2 c_j \int t^2 e^{c_j t} \, dt \]

\[ = \frac{(V_0^{(j)})^2}{c_j} e^{c_j t} + 2c_j V_0^{(j)} V_L^{(j)} \left( \frac{t}{c_j} e^{c_j t} - \frac{1}{c_j^2} e^{c_j t} \right) + c_j^2 (V_L^{(j)})^2 \left( \frac{t^2}{c_j} e^{c_j t} - \frac{2t}{c_j^2} e^{c_j t} + \frac{2}{c_j^3} e^{c_j t} \right) \]

\[ = \frac{e^{c_j t}}{c_j} \left[ c_j^2 (V_L^{(j)})^2 t^2 + 2c_j V_0^{(j)} V_L^{(j)} - c_j (V_L^{(j)})^2 t + (V_0^{(j)})^2 + 2(V_L^{(j)})^2 \right] \]
\[
\int e^{c_j t} \xi_j^2 V_L^{(j)} t \, dt = \frac{\xi_j^2 V_L^{(j)}}{c_j} e^{c_j t} (t - \frac{1}{c_j})
\]
\[
\int e^{c_j t} \frac{\xi_j^2}{c_j} (V_0^{(j)} - V_L^{(j)}) e^{-c_j t} \, dt = \frac{\xi_j^2}{c_j} (V_0^{(j)} - V_L^{(j)}) t
\]
\[
\int e^{c_j t} (d_0 + d_1 e^{-c_j t} + d_2 e^{-2c_j t}) \, dt = \frac{d_0}{c_j} e^{c_j t} + d_1 t - d_2 e^{-c_j t}
\]
\[
\quad = -\frac{e^{-c_j t}}{c_j} [d_0 + (V_0^{(j)})^2 - d_1 + \frac{d_1}{2} e^{-c_j t}]
\]
Therefore substituting in equation (24):
\[
mv_{2,j}^+(t) = \frac{1}{c_j} \left[ (V_L^{(j)})^2 t^2 + (2c_j - \frac{V_L^{(j)}}{c_j}) V_L^{(j)} t + (V_0^{(j)})^2 \frac{2(V_L^{(j)})^2}{c_j^2} \right]
\]
\[
\quad + \frac{\xi_j^2 V_L^{(j)}}{c_j} (t - \frac{1}{c_j}) + \frac{\xi_j^2}{c_j} e^{-c_j t} (V_0^{(j)} - V_L^{(j)}) t
\]
\[
\quad - \frac{e^{-2c_j t}}{c_j} [d_0 + (V_0^{(j)})^2 - d_1 + \frac{d_1}{2} e^{-c_j t}] + d_3 e^{-c_j t}
\]
Where, from the initial conditions:
\[
d_3 = \frac{1}{c_j} \left[ \frac{1}{c_j} \xi_j^2 V_L^{(j)} + d_0 + \frac{d_1}{2} - \frac{2(V_L^{(j)})^2}{c_j^2} \right]
\]

**Proof of proposition 2.5**

Proof. To compute the covariance of the integrated squared volatilities we start noticing that \(<\sigma_t^{(1)} \sigma_t^{(2)}> = \beta_1 \beta_2 \rho_V t\). Therefore by integration by parts formula:
\[
\sigma_t^{(1)} \sigma_t^{(2)} = \sigma_0^{(1)} \sigma_0^{(2)} + \int_0^t \sigma_s^{(1)} d\sigma_s^{(2)} + \int_0^t \sigma_s^{(2)} d\sigma_s^{(1)} + <\sigma_t^{(1)} \sigma_t^{(2)}>
\]
\[
\quad = \sigma_0^{(1)} \sigma_0^{(2)} - \alpha_2 \int_0^t \sigma_s^{(1)} \sigma_s^{(2)} ds + \beta_2 \int_0^t \sigma_s^{(1)} dW_t^{(2)}
\]
\[
\quad - \sigma_0^{(1)} \sigma_0^{(2)} - \alpha_1 \int_0^t \sigma_s^{(1)} \sigma_s^{(2)} ds + \beta_1 \int_0^t \sigma_s^{(2)} dW_t^{(1)} + \beta_1 \beta_2 \rho_V t
\]
Taking expected value on both sides:
\[
E_Q[\sigma_t^{(1)} \sigma_t^{(2)}] = \sigma_0^{(1)} \sigma_0^{(2)} - (\alpha_1 + \alpha_2) \int_0^t E_Q[\sigma_s^{(1)} \sigma_s^{(2)}] ds + \beta_1 \beta_2 \rho_V t
\]
The expression above leads to the differential equation:

\[ ms_{12}(t) + (\alpha_1 + \alpha_2)ms_{12}(t) - \beta_1\beta_2\rho_V = 0 \]

with \( ms_{12}(t) = E_Q[\sigma_t^{(1)}\sigma_t^{(2)}] \).

Its solution is:

\[ ms_{12}(t) = \frac{\beta_1\beta_2\rho_V}{\alpha_1 + \alpha_2} \left( 1 - e^{-(\alpha_1 + \alpha_2)t} \right) + \sigma_0^{(1)}\sigma_0^{(2)}e^{-(\alpha_1 + \alpha_2)t} \]

With the reparametrization in remark 2.1 it becomes:

\[ ms_{12}(t) = \frac{\xi_1\xi_2\rho_V}{2(c_1 + c_2)} \left( 1 - \exp\left(-\frac{1}{2}(c_1 + c_2)t\right) \right) + \sigma_0^{(1)}\sigma_0^{(2)}\exp\left(-\frac{1}{2}(c_1 + c_2)t\right) \]

Moreover, from equation (5):

\[ \text{(25)} \]

Again, taking expected value on both sides of the equation above and differentiating:

\[ mv'_{12}(t) = c_1V_{L}^{(1)} + c_2V_{L}^{(2)} - (c_1 + c_2)mv_{12}(t) + \xi_1\xi_2\rho_Vm_{12}(t) \]

whose solution is given by:

\[ mv_{12}(t) = e^{-(c_1 + c_2)t}\xi_1\xi_2\rho_V\int e^{(c_1 + c_2)s}ms_{12}(s) \, ds + ce^{-(c_1 + c_2)t} \]

\[ = e^{-(c_1 + c_2)t}\xi_1\xi_2\rho_V\int e^{(c_1 + c_2)s}\frac{\xi_1\xi_2\rho_V}{2(c_1 + c_2)} \left( 1 - e^{-\frac{1}{2}(c_1 + c_2)s} \right) \, ds \]

\[ + \sigma_0^{(1)}\sigma_0^{(2)}\xi_1\xi_2\rho_V e^{-(c_1 + c_2)t} \int e^{(c_1 + c_2)s}e^{-\frac{1}{2}(c_1 + c_2)s} \, ds + ce^{-(c_1 + c_2)t} \]

\[ = \frac{(\xi_1\xi_2\rho_V)^2}{2(c_1 + c_2)}e^{-(c_1 + c_2)t} \int e^{(c_1 + c_2)s} ds - \frac{(\xi_1\xi_2\rho_V)^2}{2(c_1 + c_2)}e^{-(c_1 + c_2)t} \int \exp\left(\frac{1}{2}(c_1 + c_2)s\right) ds \]

\[ + \frac{\sigma_0^{(1)}\sigma_0^{(2)}\xi_1\xi_2\rho_V}{c_1 + c_2}e^{-(c_1 + c_2)t} \int e^{-\frac{1}{2}(c_1 + c_2)s} \, ds + ce^{-(c_1 + c_2)t} \]

\[ = \frac{(\xi_1\xi_2\rho_V)^2}{2(c_1 + c_2)^2} - \frac{(\xi_1\xi_2\rho_V)^2}{(c_1 + c_2)^2}e^{-\frac{1}{2}(c_1 + c_2)t} + \frac{2\sigma_0^{(1)}\sigma_0^{(2)}\xi_1\xi_2\rho_V}{c_1 + c_2}e^{-\frac{1}{2}(c_1 + c_2)t} + ce^{-(c_1 + c_2)t} \]

From the initial condition \( mv_{12}(0) = V_0^{(1)}V_0^{(2)} \) we have that:

\[ c = V_0^{(1)}V_0^{(2)} + \frac{1}{2}(\xi_1\xi_2\rho_V)^2 - \frac{2\sigma_0^{(1)}\sigma_0^{(2)}\xi_1\xi_2\rho_V}{c_1 + c_2} \]
On the other hand, from equation (5):

\[
V_t^{(j)} = \frac{1}{c_t} [V_0^{(j)} + c_1 V_L^{(j)} t - V_t^{(j)} + \xi_t e^{-c_t t} \int_0^t e^{c_t s} \sigma_s^{(j)} dW_s^{(j)}]
\]

\[
V_t^{(j)} = V_0^{(j)} e^{-c_t t} + V_L^{(j)} (1 - e^{-c_t t}) + \xi_t e^{-c_t t} \int_0^t e^{c_t s} \sigma_s^{(j)} dW_s^{(j)}
\]

Hence:

\[
\begin{align*}
mv_{12}^+(t) &:= E_Q[V_t^{1,+} V_t^{2,+}] \\
&= \frac{1}{c_1 c_2} E_Q[(V_0^{(1)} + c_1 V_L^{(1)} t - V_t^{(1)} + \xi_1 \int_0^t \sigma_s^{(1)} dW_s^{(1)})(V_0^{(2)} + c_2 V_L^{(2)} t - V_t^{(2)} + \xi_2 \int_0^t \sigma_s^{(2)} dW_s^{(2)})] \\
&= \frac{1}{c_1 c_2} \left[ V_0^{(1)} V_0^{(2)} + c_2 V_0^{(2)} V_L^{(1)} t - V_{t}^{(1)} E_Q[V_t^{(2)}] + \xi_2 V_0^{(1)} E_Q[\int_0^t \sigma_s^{(2)} dW_s^{(2)}] + c_1 V_0^{(2)} V_L^{(1)} + c_1 c_2 V_L^{(1)} V_L^{(2)} t^2 - c_1 V_L^{(1)} t E_Q[V_t^{(2)}] + c_1 \xi_2 V_L^{(1)} t E_Q[\int_0^t \sigma_s^{(2)} dW_s^{(2)}] - V_0^{(2)} E_Q[V_t^{(1)}] - c_2 V_0^{(2)} t E_Q[V_t^{(1)}] + E_Q[V_t^{(1)} V_t^{(2)}] - \xi_2 E_Q[V_t^{(1)} \int_0^t \sigma_s^{(2)} dW_s^{(2)}]\right] \\
&+ \xi_1 V_0^{(2)} E_Q[\int_0^t \sigma_s^{(1)} dW_s^{(1)}] + c_2 \xi_1 V_L^{(2)} t E_Q[\int_0^t \sigma_s^{(1)} dW_s^{(1)}] - \xi_1 E_Q[V_t^{(2)} \int_0^t \sigma_s^{(1)} dW_s^{(1)}] + \xi_2 E_Q[\int_0^t \sigma_s^{(1)} dW_s^{(1)} \int_0^t \sigma_s^{(2)} dW_s^{(2)}]
\end{align*}
\]

Now, we have that:

\[
E_Q[\int_0^t \sigma_s^{(j)} dW_s^{(j)}] = 0, \ j = 1, 2
\]

\[
E_Q[\int_0^t \sigma_s^{(1)} dW_s^{(1)} \int_0^t \sigma_s^{(2)} dW_s^{(2)}] = E_Q[\int_0^t \sigma_s^{(1)} dW_s^{(1)} \int_0^t \sigma_s^{(2)} dW_s^{(2)}] = \rho_v \int_0^t m_{12}(s) \, ds
\]

\[
E_Q[V_t^{(1)} \int_0^t \sigma_s^{(2)} dW_s^{(2)}] = E_Q[(V_0^{(1)} e^{-c_1 t} + V_L^{(1)} (1 - e^{-c_1 t}) + \xi_1 e^{-c_1 t} \int_0^t e^{c_1 s} \sigma_s^{(1)} dW_s^{(1)} \int_0^t \sigma_s^{(2)} dW_s^{(2)}]
\]

\[
= (V_0^{(1)} e^{-c_1 t} + V_L^{(1)} (1 - e^{-c_1 t})) E_Q[\int_0^t \sigma_s^{(2)} dW_s^{(2)}] + \xi_1 e^{-c_1 t} E_Q[\int_0^t e^{c_1 s} \sigma_s^{(1)} dW_s^{(1)} \int_0^t \sigma_s^{(2)} dW_s^{(2)}] + \xi_1 e^{-c_1 t} E_Q[\int_0^t e^{c_1 s} \sigma_s^{(1)} dW_s^{(1)} \int_0^t \sigma_s^{(2)} dW_s^{(2)}]
\]

\[
= \xi_1 \rho_v e^{-c_1 t} \int_0^t e^{c_1 s} m_{12}(s) \, ds
\]

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Indeed, for the function:

\[ \text{Derivatives of the Margrabe price are computed by elementary differentiation.} \]

5.2 Appendix B: Derivatives of the Margrabe price

Similarly:

\[ E_Q[V^{(2)}_t] = \xi_2 \rho e^{-\xi_2 t} \int_0^t e^{\xi_2 s} m_{12}(s) \, ds \]

Therefore:

\[ m_{12}(t) := \frac{1}{c_{12}} \left[ P_3(t) - (V_0^{(1)} + c_1 V_L^{(1)} t) m_{1,2}(t) - (V_0^{(2)} + c_2 V_L^{(2)} t) m_{1,1}(t) \right. \]

\[ \left. + m s_{12}(t) - \xi_2 \rho e^{-\xi_2 t} B_1(t) - \xi_2 \rho e^{-\xi_2 t} B_2(t) + \xi_2 \rho A(t) \right] \]

where:

\[ P_3(t) = V_0^{(1)} V_0^{(2)} + c_2 V_0^{(1)} V_L^{(2)} t + c_1 V_0^{(2)} V_L^{(1)} t + c_1 c_2 V_L^{(1)} V_L^{(2)} t^2 \]

Moreover, from equation 25

\[ A(t) = \int_0^t m s_{12}(s) \, ds = \frac{\xi_1 \xi_2 \rho V}{2(c_1 + c_2)} \left( t - \int_0^t e^{-\frac{1}{2}(c_1 + c_2)s} \, ds \right) + \sigma_0^{(1)} \sigma_0^{(2)} \int_0^t e^{-\frac{1}{2}(c_1 + c_2)s^2} \, ds \]

\[ B_j(t) = \int_0^t e^{\xi_j s} m s_{12}(s) \, ds = \frac{\xi_1 \xi_2 \rho V}{2(c_1 + c_2)} \left( \frac{1}{c_j} (e^{c_j t} - 1) - \int_0^t e^{c_j - \frac{1}{2}(c_1 + c_2)s} \, ds \right) \]

\[ + \sigma_0^{(1)} \sigma_0^{(2)} \int_0^t e^{c_j - \frac{1}{2}(c_1 + c_2)s} \, ds \]

\[ = \frac{\xi_1 \xi_2 \rho V}{2(c_1 + c_2)} \left( \frac{1}{c_j} (e^{c_j t} - 1) - \frac{2(-1)^j}{c_j - c_1} (e^{\frac{1}{2}(c_j - c_1)t} - 1) \right) \]

\[ + \sigma_0^{(1)} \sigma_0^{(2)} \frac{2(-1)^j}{c_2 - c_1} (e^{\frac{1}{2}(c_2 - c_1)t} - 1) \]

5.2 Appendix B: Derivatives of the Margrabe price

Derivatives of the Margrabe price are computed by elementary differentiation. Indeed, for the function:

\[ M_4(x) = x_1 x_2 - 2 \sqrt{x_1} \sqrt{x_2} x_3 \]

We see that:

\[ \frac{\partial M_4(x)}{\partial x_1} = x_2 - \frac{\sqrt{x_2} x_3}{\sqrt{x_1}}, \quad \frac{\partial M_4(x)}{\partial x_2} = x_1 - \frac{\sqrt{x_1} x_3}{\sqrt{x_2}} \]

\[ \frac{\partial M_4(x)}{\partial x_3} = -2 \sqrt{x_1} \sqrt{x_2} \]

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The second derivatives of $M_4(x)$ are:

$$\frac{\partial^2 M_4(x)}{\partial x^2} = \frac{\sqrt{x_2} x_3}{2 x_1^{3/2}}, \quad \frac{\partial^2 M_4(x)}{\partial x_1 \partial x_2} = 1 - \frac{x_3}{2 \sqrt{x_1} \sqrt{x_2}}$$

$$\frac{\partial^2 M_4(x)}{\partial x_1 \partial x_3} = -\frac{\sqrt{x_2}}{\sqrt{x_1}}, \quad \frac{\partial^2 M_4(x)}{\partial^2 x_2} = \frac{\sqrt{x_1} x_3}{2 x_2^{3/2}}$$

$$\frac{\partial^2 M_4(x)}{\partial x_2 \partial x_3} = -\frac{\sqrt{x_1}}{\sqrt{x_2}}, \quad \frac{\partial^2 M_4(x)}{\partial x_3^2} = 0$$

Regarding the function:

$$d_1(x) = M_3 M_4^{-\frac{1}{2}}(x) - \frac{1}{2} M_4^{\frac{1}{2}}(x)$$

$$= \frac{M_3}{\sqrt{x_1 x_2 - 2 \sqrt{x_1} \sqrt{x_2} x_3}} - \frac{\sqrt{x_1 x_2 - 2 \sqrt{x_1} \sqrt{x_2} x_3}}{2}$$

where $M_3 = \log \left( \frac{S_{1}^{(i)}}{S_{1}^{(j)}} \right)$, the first and second derivatives of $d_1(x)$ are:

$$\frac{\partial d_1(x)}{\partial x_j} = -\frac{1}{2} M_3 M_4^{-\frac{1}{2}}(x) \frac{\partial M_4(x)}{\partial x_j} - \frac{1}{4} M_4^{\frac{1}{2}}(x) \frac{\partial^2 M_4(x)}{\partial x_j}, j = 1, 2, 3$$

$$\frac{\partial^2 d_1(x)}{\partial x_j \partial x_k} = \frac{3}{4} M_3 M_4^{-\frac{1}{2}}(x) \frac{\partial^2 M_4(x)}{\partial x_j \partial x_k} - \frac{1}{2} M_3 M_4^{-\frac{1}{2}}(x) \frac{\partial^2 M_4(x)}{\partial x_j \partial x_k}$$

$$+ \frac{1}{8} M_4^{\frac{1}{2}}(x) \frac{\partial M_4(x)}{\partial x_j} \frac{\partial M_4(x)}{\partial x_k} - \frac{1}{4} M_4^{\frac{1}{2}}(x) \frac{\partial^2 M_4(x)}{\partial x_j \partial x_k}, j, k = 1, 2, 3$$

Finally:

$$\frac{\partial C_1(x)}{\partial x_j} = M_1 f_Z(d_1(x)) \frac{\partial d_1(x)}{\partial x_j} - M_2 f_Z(d_1(x)) \frac{\partial d_1(x)}{\partial x_j}, j = 1, 2, 3$$

$$\frac{\partial^2 C_1(x)}{\partial x_j \partial x_k} = M_1 \left( \frac{\partial f_Z(d_1(x))}{\partial x_k} \frac{\partial d_1(x)}{\partial x_j} + f_Z(d_1(x)) \frac{\partial^2 d_1(x)}{\partial x_j \partial x_k} \right)$$

$$- M_2 \left( \frac{\partial f_Z(d_2(x))}{\partial x_k} \frac{\partial d_2(x)}{\partial x_j} + f_Z(d_2(x)) \frac{\partial^2 d_2(x)}{\partial x_j \partial x_k} \right)$$

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