We consider a dynamo wave in the solar convective shell for the kinematic $\alpha\omega$-dynamo model. The spectrum and eigenfunctions of the corresponding equations are derived analytically with the aid of the WKB method. Our main aim here is to investigate the dynamo wave behavior in the vicinity of the solar poles. Explicit expressions for the incident and reflected waves are obtained. The reflected wave is shown to be relatively weak in comparison to the incident wave. The phase shifts and the ratio of amplitudes of the two waves are found.

1 Introduction

The solar cycle is a well-known manifestation of the magnetic activity of the Sun. The physical mechanism responsible for this activity is thought to be the dynamo generation of the large-scale solar magnetic field (see e.g. Zel’’dovich, Ruzmaikin and Sokoloff, 1983). There are a number of numerical models of the solar and stellar dynamo waves which reproduce many features of the observed solar magnetic activity (Brandenburg, 1994; Rüdiger and Brandenburg, 1995; Tobias, 1997 etc.).

It is important to support these numerical investigations by approximate analytical solutions of the dynamo equations. Recently, new analytical methods of solving the mean field dynamo equations have been developed for the case of a very intensive generation. However, analytical solutions were obtained only in the simplest cases. Kuzanyan and Sokoloff (1995) derived the asymptotic solution of the Parker migratory dynamo equations (Parker, 1955). Meunier et al. (1997) and Bassom et al. (1997) have considered asymptotic solutions of the Parker migratory dynamo in the nonlinear regime.

These results, as well as the Parker’s equations themselves, have a limited domain of applicability. In particular, they do not reflect the details of the dynamo wave behavior in the very vicinity of the poles. In the present paper, we consider dynamo effects in the framework of a more general model, taking
into account the convective shell curvature. In the limiting case of very large dynamo numbers, we derive consistently the spectrum and eigenfunctions of the dynamo equations and describe the dynamo wave near the solar pole.

According to the modern observational results, the dynamo wave and the wave of sunspots, observed in the butterfly-diagrams, propagate equator-wards within the main spatial domain (i.e. far from the solar equator and solar poles). Besides, there is a weak dynamo wave in the vicinity of the solar poles which was observed by Makarov and Sivaraman (1983). This wave propagates pole-wards. In this paper, we explore this polar dynamo wave by solving the generalized equations of the Parker migratory dynamo. We show that the incident dynamo wave reflects from the pole. The reflected wave is relatively weak compared to the incident wave. We demonstrate also that the process of dynamo wave reflection reduces to a phase jump and the reflection is not accompanied by any sharp changes of the wave amplitude.

Our paper is structured as follows: In Sec. 2, we derive the equations in the Parker’s model from the general equations of the mean field electrodynamics. In Sec. 3, we solve these equations asymptotically with the aid of the WKB method, using the fact that the dynamo number is very large for the Sun. We obtain the spectrum and the eigenfunctions describing the dynamo wave far from the pole. In Sec. 4, we reduce the equations to a simplified form valid in the vicinity of a pole. In Sec. 5, we solve these asymptotic equations and show that our solution describes two waves: the incident wave and the reflected wave. In Sec. 6, we match two asymptotics found in Secs. 3 and 5 and derive the ratio of the amplitudes of the incident and reflected waves and the corresponding phase shifts. In summary section, we discuss our results and make some simple numerical estimates.

2 Basic Equations

Here we obtain a generalization of the well-known Parker migratory dynamo equations which takes into account the curvature of the convective shell. The large-scale magnetic field generation in a turbulent flow of a differentially rotating electrically conducting fluid is governed by the following equation (see Krause and Rädler, 1980):

\[ \]
\[ \frac{\partial B}{\partial t} = \nabla \times (\alpha B) + \nabla \times (\mathbf{v} \times B) + \beta \Delta B, \]  

where \( B \) and \( \mathbf{v} \) are the large-scale (mean) magnetic and velocity fields correspondingly, \( \alpha \) is the helicity coefficient and \( \beta \) is the turbulent diffusivity.

Looking for a kinematic axisymmetric eigensolution of Eq. (1), we present the magnetic field as a superposition of the poloidal and toroidal fields as follows:

\[ B(r, t) = [B_p(r) + B_t(r)] e^{\gamma t}, \]  

where \( B_t = (0, 0, \tilde{B}) \) is the azimuthal component of the magnetic field, \( B_p = R \, \text{rot} \ (0, 0, \tilde{A}) \) is the radial component of the magnetic field (\( R \) is the solar radius) and \( \gamma \) is the eigenvalue to be found. Thus, \( \text{Re} \, \gamma \) is the magnetic field growth rate and \( 2\pi(\text{Im} \, \gamma)^{-1} \) is the dynamo wave period.

In the new terms, Eq. (1) takes on the following dimensionless form:

\[ \gamma \tilde{A} = R_\alpha \alpha \tilde{B} + \frac{1}{r} \frac{\partial^2}{\partial r^2} (\tilde{A} r) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\tilde{A} \sin \theta) \right], \]

\[ \gamma \tilde{B} = R_\omega G \frac{\partial}{\partial \theta} (\tilde{A} \sin \theta) + R'_\omega G' \frac{\partial}{\partial r} (\tilde{A} \sin \theta) - R_\alpha \frac{1}{r} \frac{\partial}{\partial r} \left[ \alpha \frac{\partial}{\partial r} (\tilde{A} r) \right] - R_\alpha \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{\alpha}{\sin \theta} \frac{\partial}{\partial \theta} (\tilde{A} \sin \theta) \right] + \frac{1}{r} \frac{\partial^2}{\partial r^2} (\tilde{B} r) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\tilde{B} \sin \theta) \right], \]

where we have introduced the following dimensionless constants:

\[ R_\alpha = \frac{\alpha_{\max} R}{\beta}, \quad R_\omega = \frac{R^2}{\beta} G_{\max}, \quad R'_\omega = \frac{R^2}{\beta} G'_{\max}. \]

Here, \( G = \frac{1}{r} \frac{\partial}{\partial r} \) is the radial gradient of the mean angular velocity and \( G' = \frac{1}{r} \frac{\partial}{\partial \theta} \) is its meridional gradient. Note, that lengthes are measured in units of the solar radius, time in units of the diffusion time \( \tau_{\text{diff}} = R^2 / \beta \), \( \alpha \) and \( G \) in units of their maximal values.

We suppose that the differential rotation is more intensive than the mean helicity and that \( G(r, \theta) \) weakly depends on the latitude. Thus, the terms in Eq. (1) which contain \( R_\alpha \) and \( G' \) can be omitted. This approximation is known as \( \alpha \omega \)-dynamo model.
We also suppose that the process of intensive magnetic field generation takes place in a thin shell. Parker (1955) has noted, that in this case, two-dimensional dynamo equations could be reduced to effectively one-dimensional equations by their averaging over $r$ (see also Kuzanyan and Sokoloff, 1996). Below, we explore the corresponding one-dimensional problem.

All the assumptions, made above, yield a significant simplification of Eqs. (3, 4) and we have:

$$\gamma A = \alpha(\theta) B + \frac{d}{d\theta} \left[ \frac{1}{\sin \theta} \frac{d}{d\theta} (A \sin \theta) \right], \quad (5)$$

$$\gamma B = DG(\theta) \frac{d}{d\theta} (A \sin \theta) + \frac{d}{d\theta} \left[ \frac{1}{\sin \theta} \frac{d}{d\theta} (B \sin \theta) \right]. \quad (6)$$

Here $D = R_\alpha R_\omega$ is the dimensionless dynamo number, which characterizes the intensity of the generation sources, function $A(\theta) = R^{-1}_\alpha \langle \tilde{A}(\theta, r) \rangle$ is proportional to the averaged azimuthal component of the vector-potential and $B(\theta) = \langle \tilde{B}(\theta, r) \rangle$ is the averaged azimuthal component of the magnetic field ($\langle \ldots \rangle$ means averaging over $r$), $\theta$ is the latitude measured from the solar pole. Helicity coefficient $\alpha(\theta)$ is also averaged over the shell section. We suppose that $\alpha(0) \neq 0$. Note, that after averaging, the diffusive terms take the following form (see also Proctor & Spiegel, 1991):

$$\left\langle \frac{1}{r} \frac{\partial^2}{\partial r^2} \left( \tilde{B} r \right) \right\rangle = -\mu^2 B, \quad \left\langle \frac{1}{r} \frac{\partial^2}{\partial r^2} \left( \tilde{A} r \right) \right\rangle = -\mu^2 A$$

and they can be taken into account by a redefinition of eigenvalue $\gamma$.

Mention, that a typical estimate of the dynamo-number for the Sun is $|D| \approx -10^4$ or even more. Below, we investigate Eqs. (5, 6) analytically taking into account the large value of the dynamo number. We derive the asymptotical spectrum corresponding to these equations and the asymptotic behavior of the dynamo wave. We consider two spatial domains: The first one is the main domain which is arranged far both from the poles and from the equator and the second one is the domain near a solar pole ($\theta \ll 1$).

Let us note that the simplest approximate form of the one-dimensional dynamo equations, was obtained phenomenologically by Parker in 1955 (see also Stix, 1989). Those equations, known as Parker’s equations follow from
equation system (5, 6). They appear as a first approximation for the case of very short dynamo waves:

\[ \gamma A = \alpha(\theta)B + \frac{d^2A}{d\theta^2}; \quad (7) \]

\[ \gamma B = DG(\theta)\sin\theta \frac{dA}{d\theta} + \frac{d^2B}{d\theta^2}. \quad (8) \]

Let us mention that using the Parker’s equations one can obtain the correct expressions for eigenvalues \( \gamma \) in the leading approximation. However, the eigenfunctions do not coincide, even in the main approximation, with the corresponding solution of Eqs.(5, 6). Note also that equations similar to (7, 8) were solved numerically in the nonlinear regime by Jennings (1991) in the main domain.

### 3 Asymptotic solution in the main domain

To obtain the asymptotic solution of Eqs.(3, 4) in the main domain, we use the WKB approach (see e.g. Maslov and Fedorjuk, 1981). Similar method was applied by Kuzanyan and Sokoloff (1995) to solve the Parker’s equations. In this section, we follow their treatment to obtain the solution of the more general equations under discussion.

Let us rewrite Eqs.(3, 4) in the spectral form explicitly:

\[ \hat{H} \left( \begin{array}{c} A(\theta) \\ B(\theta) \end{array} \right) = \gamma \left( \begin{array}{c} A(\theta) \\ B(\theta) \end{array} \right), \quad (9) \]

where \( \hat{H} \) is a linear differential operator:

\[ \hat{H} = \left( \begin{array}{cc} \frac{d}{d\theta} \frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta & \alpha(\theta) \\ DG(\theta) \frac{d}{d\theta} \sin \theta & \frac{d}{d\theta} \frac{1}{\sin \theta} \frac{d}{d\theta} \sin \theta \end{array} \right). \quad (10) \]

Let us present the eigenvector and eigenvalues in the following form:

\[ \left( \begin{array}{c} A \\ \varepsilon^2 B \end{array} \right) = \left( \begin{array}{c} \mu \\ \nu \end{array} \right) + \varepsilon \left( \begin{array}{c} \mu_1 \\ \nu_1 \end{array} \right) + \ldots \exp \left[ \frac{iS}{\varepsilon} \right], \text{ where } \varepsilon = |D|^{-(1/3)}, \quad (11) \]
Here $\mu$, $\mu_1$, $\nu$, $\nu_1$ and $S$ are complex functions of the latitude and $\Gamma_0$, $\Gamma_1$ are complex constants. Emphasize that value $\varepsilon = |D|^{-\frac{1}{4}}$ is the true parameter of our asymptotic expansion and its smallness is the main condition of applicability of our solution.

To find the spectrum of Eq. (9), we have to formulate boundary conditions. Worledge et al. (1997) have emphasized the role of these conditions. They showed that the solution in the linear regime is very sensitive to the changes in the boundary conditions (see Tobias et al., 1997 for the nonlinear regime). Our asymptotic WKB expansion is not applicable near the boundary, i.e. in the very vicinity of the poles and the solar equator. Thus, we can not formulate boundary conditions in the explicit form in the framework of the asymptotic theory. However, we can match the asymptotic WKB solution applicable in the main domain with the solution applicable near the pole. The boundary condition for the latter is that the magnetic field is limited everywhere including the pole. To perform the matching one must require that the asymptotic solution decays for $\theta \to 0$ and $\theta \to \frac{\pi}{2}$. Otherwise, the field grows exponentially in the vicinity of the boundary and the matching is impossible. The other requirement is that the asymptotic solution is a smooth function. These two conditions are found to be sufficient to obtain an asymptotic spectrum.

Note, that the dynamo number can have any sign. The sign determines the direction of the dynamo wave propagation. We choose the most realistic case in our solution. Using the observational fact that the dynamo wave propagates equator-wards in the main domain, we accept the negative sign for $D$ that corresponds to this behavior.

Equations of the first approximation can be obtained by substituting Eqs. (11) and (12) into Eq. (9) and equating the terms of the minimal power of $\varepsilon$. This yields:

$$
\mu(\theta) = \frac{1}{\varepsilon^2} \Gamma_0 + \frac{1}{\varepsilon} \Gamma_1 + \ldots
$$

(A2)

A non-trivial solution of this equation exists only if the determinant of the matrix in the left-hand side of Eq. (13) vanishes. This condition leads to so-called Hamilton-Jakobi equation (we use the standard terminology of the
WKB-method, see e.g. Landau and Lifshitz, 1958):

\[
\left( \Gamma_0 + k^2 \right)^2 + i\dot{\alpha}k = 0, \tag{14}
\]

where \( k(\theta) = \frac{dS}{d\theta} \) is the generalized momentum (by analogy with the semi-classical approximation in quantum mechanics). We have also introduced a new function \( \dot{\alpha}(\theta) = G(\theta)\alpha(\theta)\sin \theta \).

Note that the Hamilton-Jakobi equation in our problem coincides with the one for the Parker’s equations. Its solution is described in the paper of Kuzanyan and Sokoloff (1995). Here, we will recall shortly the reasonings that allow to calculate spectral parameter \( \Gamma_0 \) from the Hamilton-Jakobi equation.

The accepted decay condition (the solution should be small near the boundary) can be rewritten in the terms of the generalized momenta as follows:

\[
\text{Im} \, k|_{\theta \to \frac{\pi}{2}} > 0, \quad \text{Im} \, k|_{\theta \to 0} < 0.
\]

These expressions play the role of boundary conditions.
Equation (14) is the algebraic equation of the fourth order and thus it possesses four branches of roots. However, none of these branches satisfies the boundary conditions. It means that we should construct our solution by matching two (or more) branches. To explore this case, we denote

\[ H(k, \theta) = (\Gamma_0 + k^2)^2 + ik\hat{\alpha}(\theta). \]

In terms of function \( H(k, \theta) \), it is easy to formulate the conditions of crossing of two branches at a point \( \theta' \):

\[
\begin{align*}
H[k(\theta'), \theta'] &= 0; \\
\frac{\partial H}{\partial k}[k(\theta'), \theta'] &= 0.
\end{align*}
\]  

These equations can be solved explicitly and we have

\[ \Gamma_0^{(1,2)} = \frac{3\hat{\alpha}(\theta')^2e^{\pm i\pi}}{2^\frac{3}{8}}, \quad \Gamma_0^{(3)} = -\frac{3\hat{\alpha}(\theta')^2}{2^\frac{3}{8}}. \]

As one can see, the real part of \( \Gamma_0^{(3)} \) is negative, which corresponds to a decaying solution. The two other conjugated values correspond to growing solutions of our interest. The leading mode is the most important from the physical point of view. Thus, we choose \( \theta' \) to be the point where function \( \hat{\alpha}(\theta) \) reaches its maximum (see Fig.1). It was shown (Galitski and Sokoloff, 1998), that only this choice of \( \Gamma \) leads to smooth eigenfunctions of the Parker equations.

From Eq.(13), we have:

\[
\begin{pmatrix}
\mu(\theta) \\
\nu(\theta)
\end{pmatrix} = \begin{pmatrix}
\Gamma_0 + k^2 \\
-ik\sin\theta
\end{pmatrix} \sigma(\theta),
\]

where \( \sigma(\theta) \) is to be defined from the equation of the second approximation. To obtain this equation, we substitute Eqs.(11) and (12) into Eq.(9) and equate the terms of the first power of \( \varepsilon \). This yields:

\[
\begin{pmatrix}
\Gamma_0 + k^2 & -\alpha \\
-ik\sin\theta & \Gamma_0 + k^2
\end{pmatrix} \begin{pmatrix}
\mu_1 \\
\nu_1
\end{pmatrix} = \begin{pmatrix}
[ik' + ik\text{ctg}\theta - \Gamma_1] \mu + 2ik\mu' \\
[ik' + ik\text{ctg}\theta - \Gamma_1] \nu + 2ik\nu' - [\mu\sin\theta]' \end{pmatrix}.\]

\[
(17)
\]
Here, for the sake of simplicity we neglect the term containing $G'(\theta)$, supposing that the angular rotation weakly depends on the latitude in the main domain. Note, that (17) differs from the corresponding expression for the Parker’s equations.

We see that the matrixes in the left-hand sides of Eqs. (13) and (17) are identical. As we have seen the corresponding matrix is degenerate. Thus, Eq. (17) possesses solutions only if the Fredholm resolvability condition is satisfied, i.e. if the vector in the right-hand side of Eq. (17) is orthogonal to the eigenvector of the adjoint equation. It yields so-called transport equation:

$$\Gamma_1 - ik' \left(1 + \frac{2k^2}{\Gamma_0 + k^2}\right) - \text{ctg} \theta \left(2ik - \frac{\dot{\alpha}}{2(\Gamma_0 + k^2)}\right) \sigma(\theta) =$$

$$= \left[2ik - \frac{\dot{\alpha}}{2(\Gamma_0 + k^2)}\right] \sigma(\theta').$$

(18)

Using the fact that the function in the brackets in the right-hand side of equation (18) vanishes at $\theta'$ (recall that $\theta'$ is the point at which $\dot{\alpha}$ is maximal) and following the standard procedure of the WKB theory (see Maslov and Fedorjuk, 1981) we can find the spectrum. First, let us denote the following functions:

$$P(\theta) = 2ik - \frac{\dot{\alpha}}{2(\Gamma_0 + k^2)};$$

(19)

and

$$Q(\theta) = ik' \left(1 + \frac{2k^2}{\Gamma_0 + k^2}\right) + P(\theta) \text{ctg} \theta.$$

(20)

We suppose that $P(\theta)$, $Q(\theta)$, $\sigma(\theta)$ are analytical functions in the vicinity of $\theta'$. Furthermore, we present all these functions in the form of Taylor series:

$$P(\theta) = [P_0 + P_1(\theta - \theta')](\theta - \theta'),$$

where $P_0 = 3ik'(\theta')$;

$$Q(\theta) = Q_0 + Q_1(\theta - \theta'),$$

where $Q_0 = \frac{3ik'(\theta')}{2}$.

(21)

(22)

For $\sigma_n$, we have

$$\sigma_n(\theta) = (\theta - \theta')^n \left[C_n + C_{n+1}(\theta - \theta')\right],$$

(23)
Where $n$ is an integer parameter, which classifies the eigenvalues. Substituting (21), (22) and (23) into (18) and setting $\theta = \theta'$, we have

$$P_0 n = \Gamma_{1,n} - Q_0.$$  \hfill (24)

From here we can easily obtain the spectrum:

$$\Gamma_{1,n} = 3ik'(\theta') \left( n + \frac{1}{2} \right), \quad n = 0, 1, 2, \ldots$$  \hfill (25)

The explicit expression for $\sigma$ is

$$\sigma(\theta) = \frac{1}{\sin \theta} \exp \left\{ \int \frac{\Gamma_1 - ik' \left( 1 + \frac{2k^2}{\Gamma_0 + k^2} \right)}{2ik - \frac{\alpha}{2(\Gamma_0 + k^2)}} d\theta \right\}. \hfill (26)$$

We emphasize that Eqs.(25) and (26), in contrast to the Hamilton-Jakobi equation, differ from the corresponding expressions obtained by Kuzanyan and Sokoloff (1995). Surprisingly, our solution has a simpler structure despite the fact that the initial equations in our problem are more complicated than the Parker equations.

4 Equations Near the Pole

We now proceed to investigate equations (3, 4) in the second domain near the solar pole. In the following, we have two small values: $\theta$ and $\varepsilon$. Since $\theta \ll 1$, we can rewrite equation Eq.(3) in the following form:

$$\gamma A = \alpha(0) B + \frac{d^2 A}{d\theta^2} + \frac{1}{\theta} \frac{dA}{d\theta} - \frac{1}{\theta^2} A.$$  \hfill (27)

Here we keep only the terms of the minimal power of $\theta$.

Taking into account results of Sec. 3, it is reasonable to assume that $|A| \sim |B| \varepsilon^2$ and each differentiation of the field multiplies it by $\varepsilon^{-1}$ (formally we can write $\frac{d}{d\theta} \sim \frac{1}{\varepsilon}$). Our assumptions will be confirmed by the results obtained below. This yields the following estimates of the terms in (3):

$$B'' \sim \frac{1}{\varepsilon^4} |A|, \quad \gamma B \sim \frac{1}{\varepsilon^4} |A|, \quad \text{ctg} \theta B' \sim \frac{1}{\theta \varepsilon^3} |A|, \quad \frac{1}{\sin \theta^2} B \sim \frac{1}{\theta^2 \varepsilon^2} |A|.$$  \hfill (28)
\[ DG \sin \theta A' \sim \frac{\theta}{\varepsilon^4} |A|, \quad DG \cos \theta A \sim \frac{1}{\varepsilon^3} |A|. \]

The terms corresponding to the last two estimates can be neglected and Eq. (6) reads:

\[ \gamma B = \frac{d^2 B}{d\theta^2} + \frac{1}{\theta} \frac{dB}{d\theta} - \frac{1}{\theta^2} B. \quad (28) \]

Equations (27, 28) can be rewritten in the following compact form:

\[ \hat{J}_1(x) A(x) = \frac{\alpha(0)}{\gamma} B(x); \quad (29) \]

\[ \hat{J}_1(x) B(x) = 0, \quad (30) \]

where we have introduced a new variable \( x = \sqrt{-\gamma \theta} \) and a differential operator

\[ \hat{J}_1(x) = \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + \left(1 - \frac{1}{x^2}\right). \]

Let us note that in Eqs. (29, 30) \( \gamma \) plays the role of an external parameter. Its value was obtained in Sec. 3.

5 Dynamo Wave in The Vicinity of The Solar Pole.

In this section, we solve equations Eqs. (29, 30) obtained above. Equation (30) is the homogeneous Bessel’s equation of the first order for \( B \) and Eq. (29) is the inhomogeneous Bessel’s equation of the first order for \( A \). The magnetic field must be finite for all \( x \). It means that we should take Bessel function of the first order \( J_1(x) \) as a solution of Eq. (30). Hence,

\[ B(x) = 2C_B J_1(x) = C_B \left[H^{(1)}_1(x) + H^{(2)}_1(x)\right]. \quad (31) \]

Here \( C_B \) is a constant and \( H^{(1,2)}_1(x) \) are Hankel functions of the first type and the second type of the first order. Note, that Hankel functions diverge at the pole (\( \theta = 0 \)), but their sum is finite. As we shall see below, \( H^{(1)}_1(x) \) describes the incident wave and \( H^{(2)}_1(x) \) describes the reflected wave.
The general solution of Eq. (29) is a sum of general solution of the corresponding homogeneous equation and a particular solution of the inhomogeneous equation. To find particular solution $A_1(x)$, note that the Wronski determinant of Hankel functions is

$$H_1^{(1)}(x)H_1^{(2)}(x)' - H_1^{(1)}(x)'H_1^{(2)}(x) = -\frac{4i}{\pi x},$$  \hfill (32)

hence,

$$A_1(x) = -\frac{\pi \alpha(0)}{4i\gamma} C_B \left[ H_1^{(2)}(x) \int \left[ H_1^{(1)}(x)^2 + H_1^{(1)}(x)H_1^{(2)}(x) \right] xdx - 
+ H_1^{(1)}(x) \int \left[ H_1^{(2)}(x)^2 + H_1^{(1)}(x)H_1^{(2)}(x) \right] xdx \right] .$$ \hfill (33)

Summarizing, we have

$$\begin{cases}
  A(x) = C_A \left[ H_1^{(1)}(x) + H_1^{(2)}(x) \right] + A_1(x), \\
  B(x) = C_B \left[ H_1^{(1)}(x) + H_1^{(2)}(x) \right].
\end{cases}$$ \hfill (34)

Two complex constants $C_A$ and $C_B$ in (34) are not independent. Really, fields $A(x)$ and $B(x)$ represent two different components of a true eigenvector for Eq. (5, 6). The norm of this eigenvector is an arbitrary quantity, but its orientation is prescribed. Our asymptotic solution (34) must satisfy this condition too. To find a connection between $C_A$ and $C_B$, one should consider a spatial domain where both asymptotics (11) and (34) are valid. This domain is characterized by the following condition:

$$\varepsilon \ll \theta \ll 1 \hfill (35)$$

Since $|x| \gg 1$, we can use the well-known asymptotic form of Hankel functions for large arguments:

$$H_1^{(1)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{i(x-\frac{3\pi}{4})},$$ \hfill (36)

$$H_1^{(2)}(x) \approx \sqrt{\frac{2}{\pi x}} e^{-i(x-\frac{3\pi}{4})},$$ \hfill (37)
i.e.

\[ B(x, t) \approx C_B \sqrt{\frac{2}{\pi x}} \left[ e^{i(x - \frac{3}{8} \pi)} + e^{-i(x - \frac{3}{8} \pi)} \right] e^{\gamma t}. \]  

(38)

Let us emphasize, that constraint (35) is not a condition of applicability of Eq. (34), but a condition of applicability of asymptotics (36, 37) and (46) (see Sec. 6). These approximate presentations allow us to perform the matching in the explicit form. However, the condition (35) can be satisfied only if \( \varepsilon \) is extremely small. Otherwise (if it is not possible to find a domain where the value \( \theta \) is both much lesser than unity and much greater than \( \varepsilon \)), one should perform the matching using the general form of our asymptotic solution.

Recalling that \( x = \sqrt{-\gamma \theta} \), one can define the phase surface as

\[ \phi(\theta, t) = \pm \frac{\text{Re}\sqrt{-\Gamma_0}}{\varepsilon} \theta + \frac{\text{Im}\Gamma_0}{\varepsilon^2} t = \text{const}. \]

The phase velocity is

\[ v_{\text{phase}} = \dot{\theta} = \pm \frac{1}{\varepsilon} \frac{\text{Im}\Gamma_0}{\text{Re}\sqrt{-\Gamma_0}}. \]  

(39)

If \( v_{\text{phase}} < 0 \) than the wave propagates pole-wards, if \( v_{\text{phase}} > 0 \) than the wave propagates equator-wards.

Evaluating \( v_{\text{phase}} \) explicitly, we readily conclude that \( H^{(1)}_1(x) \) corresponds to the incident wave and \( H^{(2)}_1(x) \) corresponds to the reflected wave.

The asymptotic form of \( A(x, t) \) is

\[ A(x, t) = \left[ \sqrt{\frac{2}{\pi x}} C_A + \frac{\alpha(0)}{\gamma} \sqrt{\frac{x}{2\pi}} C_B \right] e^{i(x - \frac{3}{8} \pi)} + \gamma t + \]

\[ + \left[ \sqrt{\frac{2}{\pi x}} C_A - \frac{\alpha(0)}{\gamma} \sqrt{\frac{x}{2\pi}} C_B \right] e^{-i(x - \frac{3}{8} \pi)} + \gamma t. \]  

(40)

6 Matching of Asymptotics

One can see from (39) that branches k_3 and k_4 (see Fig.1) correspond to the incident wave. Now, we have to find a branch of roots \( k_i(\theta) \) that corresponds to the reflected wave. We expect that the reflected wave, if exists, should
be very weak. It means that this wave decays going equator-wards. Thus, we must choose branch $k_2$ to represent the reflected wave. Only this branch is arranged in the upper part of the complex plane $k$ and so it describes a decaying wave.

Now, we should match a linear combination of asymptotics (11) for the incident and reflected waves and asymptotics (11, 18). For this aim, it is necessary to evaluate approximate expressions for $k_\pm(\theta)$, $\sigma_\pm(\theta)$, $A_\pm(\theta)$ and $B_\pm(\theta)$ for the case $\theta \ll 1$ (here and further +’s correspond to the incident wave and -’s correspond to the reflected wave). From equations (11), (14), (26) and (27) after a rather long but straightforward algebra we have following expressions (for $\theta \to 0$):

$$k_+(\theta) = k_0 - \kappa e^{-i\frac{\pi}{12}} \sqrt{\theta},$$

$$k_-(\theta) = -k_0 + \kappa e^{\frac{5i\pi}{12}} \sqrt{\theta},$$

where $\kappa$ and $k_0$

$$\kappa = \sqrt{\frac{\hat{\alpha}'(0)}{2\pi \frac{7}{3} \hat{\alpha}'(\theta')}} \quad k_0 = \frac{\sqrt{3\hat{\alpha}(\theta')}}{2\pi} e^{-i\frac{\pi}{12}},$$

where $f_\pm$

$$f_\pm = \frac{7}{2\pi \frac{7}{3} \hat{\alpha}'(\theta')} \sqrt{\frac{\hat{\alpha}'(0)}{\hat{\alpha}'(\theta')}} (1 \pm i) .$$

Using these expressions, one can obtain the approximate form of the solution of (5, 6) in the main domain:

$$\begin{pmatrix} A \\ B \end{pmatrix} = C_+ \left( i\rho \theta^{-\frac{3}{4}} \beta e^{-2\theta^{-\frac{1}{4}}} \right) e^{ix} - C_- \left( \rho \theta^{-\frac{3}{4}} \beta e^{-2\theta^{-\frac{1}{4}}} \right) e^{-ix},$$

where

$$\rho = \frac{3^{\frac{3}{4}} \alpha(\theta')^{\frac{3}{4}} \sqrt{\alpha'(0)}}{2\pi} \beta = \frac{\sqrt{3\alpha(\theta')^{\frac{3}{4}}}}{2\pi} e^{-\frac{5i\pi}{12}}.$$
From the other side we have (see (40, 38)):

\[
\begin{pmatrix}
A \\
B
\end{pmatrix} = \left( \frac{\chi_0 \sqrt{\frac{\alpha}{\pi}} C_A + \chi_1 \varepsilon \frac{3}{2} \sqrt{\theta} C_B}{\chi_0 \sqrt{\frac{\alpha}{\pi}} C_B} \right) e^{i(x - \frac{3\pi}{4})} + \\
\left( \frac{\chi_0 \sqrt{\frac{\alpha}{\pi}} C_A - \chi_1 \varepsilon \frac{3}{2} \sqrt{\theta} C_B}{\chi_0 \sqrt{\frac{\alpha}{\pi}} C_B} \right) e^{-i(x - \frac{3\pi}{4})},
\]

(47)

where

\[
\chi_0 = \left( \frac{2}{\pi \sqrt{-1}} \right)^{\frac{3}{2}}, \quad \chi_1 = \frac{\alpha(0)}{i \Gamma_0} \left( \frac{\sqrt{-1}}{2\pi} \right)^{\frac{3}{2}}.
\]

Our aim now is to express constants \( C_A, C_B \) and \( C_- \) through \( C_+ \). To find the corresponding expressions, one should match the amplitudes of the incident and reflected waves in Eqs. (46) and (47) at a point \( \theta_1 \). After some algebra, we have

\[
C_- = i C_+,
\]

(48)

\[
C_B = \beta \theta_1 \frac{\chi_1}{\chi_0 \varepsilon^{\frac{3}{2}}} C_+,
\]

(49)

\[
\frac{C_B}{C_A} = \frac{i \chi_0}{\varepsilon \theta_1 \chi_1}.
\]

(50)

The last step is to find the point \( \theta_1 \) at which the matching is being made. In solutions (46) and (47), the phase-dependences are the same. However, phase-shifts appear in the higher order approximations (see Eqs. (44) and (45) for the explicit expressions). We choose \( \theta_1 \) to be the point at which the phase-shifts for different asymptotics of the incident wave are the same. It yields

\[
\theta_1 = \frac{3 \varepsilon \text{Im} f_+}{2 \kappa \cos \frac{\pi}{12}}
\]

(51)

Mention, that our results are in agreement with all the assumptions made in Sec. 4. From Eq. (50), we see that \( A \sim \varepsilon^2 B \), from Eq. (47), we obtain \( \frac{dA}{d\theta} \sim \frac{A}{\varepsilon} \) and \( \frac{dB}{d\theta} \sim \frac{B}{\varepsilon} \), as we have supposed. However, we see that \( \theta_1 \sim \varepsilon \) and this result is on the border of applicability of asymptotics (36, 37). Strictly speaking, we should use the exact form of Hankel functions to perform the matching, instead of using their asymptotic representations. It
could have some influence on the values (49—51), but not on (48). For the case of simplicity, we avoid here the complication, connected with the exact determination of $\theta_1$. However, the physical results of our analysis remain applicable for any values of $\theta_1$. Moreover, we emphasize, that asymptotics (36, 37) reflect the behavior of Hankel function with a very good accuracy even for $|x| \sim 1$ (see Fig. 3) and hence, expressions (49–51) are well applicable as well.

7 Summary And Discussion

This paper has derived the analytical expressions for the large-scale magnetic field in the vicinity of the solar poles. These expressions can be used to compare the relative magnitudes of the incident and reflected waves.

Taking into account the fact that generalized momenta $k$ for the incident and reflected waves differ only by sign at the pole [see Eq.(43)], we can estimate the ratio of the absolute values of the amplitudes of the reflected and incident waves at point $\theta$ as follows (for the exact dependence see Fig. 3):

$$R(\theta) \approx \exp\left(-\frac{2}{\varepsilon} |\text{Im} k_0| \theta\right),$$

where for $k_0$ see (43). Obviously, this ratio depends on $\theta$ and, as one can see, the farther from the pole, the weaker the reflected wave is compared to the incident wave. Let us assume, that the dynamo-number is $D = -10^4$ (or, equivalently, $\varepsilon \approx 0.05$) and the helicity coefficient is $\alpha(\theta) = \cos \theta$. Evaluating $\theta_1$ explicitly we have $\theta_1 \approx 2.3^\circ$. This means that for $\theta < 2.3^\circ$ the solution (11) is applicable, for $\theta > 2.3^\circ$ one should use expressions (34). Let $\theta = 10^\circ$, then we obtain

$$R(10^\circ) \approx 0.03.$$

It follows that the reflected wave is about 30 times weaker than the incident wave. However, the incident wave itself is rather weak near the pole against the background of the dynamo wave in the main domain. For example, the incident wave at $\theta = 10^\circ$ is approximately 10 times weaker than the dynamo wave in the domain of generation (see Kuzanyan and Sokoloff, 1995). It follows, that the reflected wave is 300 times weaker than the main wave.

Summarizing, we see that the solar magnetic field in the accepted model is described with the aid of the three dynamo waves. The most intensive
Figure 2: The ratio of magnitudes of the incident and reflected waves as a function of latitude $\theta$. The solid line represents dependence $R(\theta)$, obtained with use of (14). The dashed line corresponds to the approximate dependence $R(\theta)$, calculated with the aid of asymptotics (16, 17). Both curves are extrapolated for large $\theta$. Asterisks represent some values $R(\theta)$ calculated using the WKB solution (11). The value $D = -10^4$ for all curves is accepted.

wave propagates from high latitudes equator-wards. This wave, of course, is well-known both from observations and from numerical simulations. The second wave propagates in high latitudes pole-wards. It has essentially lesser intensity than the first one. However, it has been detected in the observations of the solar subpolar flares (see Makarov and Sivaraman, 1983). In this paper we predict the existence of another dynamo wave, which reflects from the pole and propagates equator-wards. We show that the reflected wave is very weak and it decays exponentially propagating from the pole. It is possible however that the reflected wave could be detected in the future specialized observations of the solar magnetic activity in the subpolar domain.

Using the solution obtained, we can also estimate the generation threshold as a dynamo number, for which the generation of magnetic field commences:

$$\gamma_n \left( D_n^{(\alpha)} \right) = 0.$$
Hence, for the leading mode we have ($n=0$):

$$
\left| D_{0}^{(cr)} \right| \approx -\left( \frac{\text{Re} \Gamma_1}{\text{Re} \Gamma_0} \right)^3 = \frac{2^{11/2}}{\alpha_{\text{max}}}. \quad (52)
$$

For $\alpha(\theta) = \cos(\theta)$ we have $\left| D_{0}^{(cr)} \right| \approx 90.5$. Note, that this value is about two times larger than the critical dynamo number for the Parker’s equations. Evaluating similar expressions $D_{n}^{(cr)}$ for $n = 1, 2, 3, \ldots$ and taking into consideration existing estimates of $D$ for the Sun, one can conclude that only one additional mode (with $n = 1$) is excited. One should mention, that even provided that $\left| D^{(cr)} \right| = 100$, we see, that the true parameter of our asymptotic expansion is not very small and Eq.(52) should be considered as a lower estimation for the critical dynamo number.

As it was shown in Sec. 3, the value of $\Gamma_0$ depends on the point $\theta'$ at which the matching of branches $k_3$ and $k_4$ is made. We have chosen this point to be the point of maximum of $\hat{\alpha}(\theta)$. The general case for the Parker’s equations was considered by Galitski and Sokoloff (1998). It was shown that the corresponding eigensolutions can not be chosen to be smooth functions for any other values of $\theta'$. The same result can be obtained for Eqs.(4, 1). It means, that operator $\hat{H}$ [see (10)] does not possess any other discrete eigenvalues except the ones obtained above (24).

It is interesting also to compare our asymptotic theory with so-called Maximally-Efficient Generation Approach (MEGA), see Ruzmaikin et al. (1990). The latter suggests that the generated magnetic field is localized in a small vicinity of the point at which the magnetic field generation sources have maximum. Far from this point, the magnetic field appears due to the diffusion mechanism only. As one can see, the point of maximum of our solution is shifted from the point of maximum of function $\hat{\alpha}(\theta)$ (this property of the solution was emphasized by Kuzanyan and Sokoloff, 1995). It means, that in our case the MEGA does not work quite well and one should be careful in application of this approach.

Here, we considered dynamo equations in the kinematic approximation. Let us note that in the recent paper of Meunier et al. (1997), it was shown that in some nonlinear dynamo models, one can use the kinematic approximation to describe the spatial profile of the solution. Certainly, in the nonlinear case, spectral parameter $\gamma$ differs from eigenvalues (23) (in the steady nonlinear regime, $\gamma$ has only an imaginary part which determines the period of
the solar cycle). However, our analysis of the dynamo wave behavior near the pole remains applicable.

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