CHAINING OF NUMERICAL BLACK-BOX ALGORITHMS: 
INVESTIGATING THE IMPACT OF WARM-STARTING AND THE 
SWITCHING POINT

A PREPRINT

Dominik Schröder
LIACS, Leiden University
Leiden, The Netherlands
dominik.schroeder94@gmail.com

Diederick Vermetten
LIACS, Leiden University
Leiden, The Netherlands
d.l.vermetten@liacs.leidenuniv.nl

Hao Wang
LIACS, Leiden University
Leiden, The Netherlands
h.wang@liacs.leidenuniv.nl

Carola Doerr
Sorbonne Université, CNRS, LIP6
Paris, France
Carola.Doerr@lip6.fr

Thomas Bäck
LIACS, Leiden University
Leiden, The Netherlands
t.h.w.baeck@liacs.leidenuniv.nl

ABSTRACT

Dynamic algorithm selection can be beneficial for solving numerical black-box problems, in which we implement an online switching mechanism between optimization algorithms. In this approach, we need to decide when a switch should take place and which algorithm to pick for the switching. Intuitively, this approach chains the algorithms for combining the well-performing segments from the performance profile of the algorithms. To realize efficient chaining, we investigate two important aspects - how the switching point influences the overall performance and how to warm-start an algorithm with information stored in its predecessor. To delve into those aspects, we manually construct a portfolio comprising five state-of-the-art optimization algorithms and only consider a single switch between each algorithm pair. After benchmarking those algorithms with the BBOB problem set, we choose the switching point for each pair by maximizing the theoretical performance gain. The theoretical gain is compared to the actual gain obtained by executing the switching procedure with the corresponding switching point. Moreover, we devise algorithm-specific warm-starting methods for initializing the algorithm after the switching with the information learned from its predecessor. Our empirical results show that on some BBOB problems, the theoretical gain is realized or even surpassed by the actual gain. More importantly, this approach discovers a chain that outperforms the single best algorithm on many problem instances. Also, we show that a proper warm-starting procedure is crucial to achieving high actual performance gain for some algorithm pairs. Lastly, with a sensitivity analysis, we find the actual performance gain is hugely affected by the switching point, and in some cases, the switching point yielding the best actual performance differs from the one computed from the theoretical gain.

1 Introduction

Numerical black-box optimization has important applications across numerous scientific and industrial applications. To address the various challenges practitioners are faced with, a plethora of numerical black-box optimization algorithms
Within the field of optimization, it is well understood that algorithms often need to adjust their search behavior and their hyperparameters during the optimization process in order to achieve the peak performance. Several strategies have been devised to enhance the empirical performance, for example the idea of self-adaptation in evolution strategies. However, most of these methods fall under the umbrella of hyper-heuristics [BGH+13] and related topics such as parameter control [KHE15, DD20], which only aim to change the internal settings of the algorithm, rather than the algorithm itself. While those methods are simple to apply in practice, their strengths are inherently limited by the overarching algorithm class used. In contrast, dynamically switching between completely different algorithms could potentially benefit from vast differences between widely varying algorithm techniques, at the cost of more complex development needed to design the required procedures to effectively perform the switch.

Still, even on a single problem instance, it is often the case that different algorithms will show different behaviours during the different stages of the search [SHSV17, VWBD20a], e.g., on a multi-modal objective function, a greedy search algorithm would exhibit a fast initial convergence followed by a stagnation behavior while a global search algorithm will make a slow progress in the beginning and is much less prone to stagnation in local optima. Ideally, one would like to combine well-performing segments of the optimization process from each algorithm in a portfolio, by dynamically switching from one to another and possibly warm-starting an optimizer with the information learned on the objective function from its predecessor. Such a consideration leads to the dynamic algorithm selection (dynAS) task [VWBD20a], which has the potential to assemble a switching optimizer that surpasses the performance of the single best solver for a given problem instance.

It is worth noting that the dynAS task can be understood as a special case of the dynamic algorithm configuration (dynAC) task [SBH+21, BBE+20], which aims to selecting algorithms for the switching and configuring their hyperparameters online. While many techniques exist to take advantage of different search behaviour of algorithms, most of these are focused on configuring the underlying hyperparameters of algorithms, rather than investigating the impact of switching between different algorithms, e.g., adjusting the learning rate of training a deep neural network [DTN16] and controlling the step-size of evolutionary algorithms [SBA+20]. In this paper, we contemplate the problem of choosing which algorithms to switch between, when to perform this switch, and how to transfer useful information from one algorithm to the next. For this aim, we conduct a comprehensive empirical study on a portfolio of five optimization algorithms. We investigate several strategies for warm-starting the algorithms based on the information recorded and learned during the execution of the preceding algorithm. We analyze the potential of these warm-starting procedures on the single-objective Black-Box Optimization Benchmarking (BBOB) problem set from the Comparing Continuous Optimizers (COCO) framework [HAR+21].

Structure of the paper: We provide background on the dynamic algorithm selection (dynAS) problem and relevant terminology in Section 2. We then use available performance data of static algorithms in Section 3 to illustrate the potential benefits of dynAS on the selected function suite. In Section 4 we describe our portfolio of five algorithms for which we study the dynamic switching in more detail. In particular, we focus on the warmstarting procedures in Section 5. The results of this process are shown in Section 6 and concluded in Section 7 where we also discuss future work and challenges towards realizing dynAS.

Availability of Code and Data: The code used for generating the data used in this project is available on GitHub[1] while the full data set of data analyzed here can be found on Zenodo [Sch21]. The data of the static algorithm portfolio (described in Section 3) is also available on the public repository of IOHanalyzer [WVY+20], which can be found at https://iohanalyzer.liacs.nl/ (the data set of each algorithm can be obtained by first selecting the “IOH” repository and then choosing the “bbob” data source).

2 Background and Experimental Setup

2.1 Dynamic Algorithm Selection

Within the field of optimization, it is well understood that algorithms often need to adjust their search behavior and their hyperparameters during the optimization process in order to achieve the peak performance. Several strategies have been devised to enhance the empirical performance, for example the idea of self-adaptation in evolution strategies. However, most of these methods fall under the umbrella of hyper-heuristics [BGH+13] and related topics such as parameter control [KHE15, DD20], which only aim to change the internal settings of the algorithm, rather than the algorithm itself. While those methods are simple to apply in practice, their strengths are inherently limited by the overarching algorithm class used. In contrast, dynamically switching between completely different algorithms could potentially benefit from vast differences between widely varying algorithm techniques, at the cost of more complex development needed to design the required procedures to effectively perform the switch.

https://github.com/Schroedo1994/Realizing_dynAS
Recently, the idea of dynamic algorithm selection, and the inherently related dynamic algorithm configuration, where the algorithms to switch between have their own parameters, has been formalized as a Contextual Markov Decision Process [BBE+20]. This formalization is part of a trend towards applying Reinforcement learning techniques to the dynamic adaptation of algorithms [EBR+21]. From a more data-driven approach, a study on the Configurable CMA-ES has shown that there is potential for performance improvement when switching between algorithms, even when the algorithms considered are closely related [VvRBD19].

While the majority of the work related to dynamic algorithm behaviour has focused on just one family of algorithms, it is clear that the benefits of dynamic algorithm selection could potentially be far greater when the algorithmic diversity is increased. Additionally, the diversity of the algorithms can lead to inherently different search behaviours, which can increase our understanding of the optimization procedure itself. Eventually, these types of insights could be combined with techniques like adaptive landscape analysis to design algorithms which can fully take advantage of their local environment [JED21].

2.2 The BBOB Problem Set

For our experiments, we take the widely used noiseless Black-Box Optimization Benchmarking (BBOB) problem set [HFRA09], comprising 24 real-valued test functions to be minimized within the default search domain $[-5, 5]^d$. The functions are chosen to cover a broad spectrum of problem characteristics that are expected to occur frequently in the continuous domain, e.g., separability and multi-modality. The dimensionality of the search domain can be chosen freely by the user. In this paper, we set the dimensionality to $d \in \{2, 3, 5, 10, 20\}$. For each BBOB problem, different instances can be generated via transformations in the domain and range of the test functions that preserve the overall problem structure [HFRA09]. In this paper, we take the first five instances of each function for the experimentation, i.e., the instance number $i \in \{1, 2, 3, 4, 5\}$. As the optimal function value $f_{\text{opt}}$ is known for each BBOB function, we commonly consider the gap between the best-so-far function value and the optimal one, $f_{\text{opt}}$, for measuring the progress. That is, we are mostly interested in the target precision $\phi = f_{\text{best-so-far}} - f_{\text{opt}}$.

The BBOB test suite has been used for benchmarking workshops at academic conferences since 2009. As of March 2022, 232 different optimization algorithms and the corresponding empirical performance data have been submitted to the BBOB platform[2]. The availability of this data makes the BBOB platform particularly useful for research on dynamic algorithm selection as it not only provides us with a diverse set of algorithms, but also allows for a more detailed analysis of alternative algorithmic search behaviors based on the large amount of available data.

2.3 Empirical Performance Measure

In this work, we take a fixed-target perspective to assess the empirical performance, where we measure the number of function evaluations that are required to reach a certain target precision. The hitting time $T(A, f, d, \phi)$ is defined by the number of function evaluations that are performed by algorithm $A$ to reach a target precision $\phi$, where $f$ denotes the objective function, and $d$ denotes the dimensionality of the problem domain. If the algorithm did not find the target within the allocated budget, the hitting time is set to $T(A, f, d, \phi) = \infty$. Note that, all algorithms considered in this study are stochastic, meaning the hitting time $T(A, f, d, \phi)$ is a random variable, for which some descriptive statistics are needed to quantify its performance (see below).

Since the optimization algorithms may not always reach the defined target precision, previous work on continuous black-box optimization commonly refers to the Expected Running Time (ERT). Assuming algorithm restarts for unsuccessful runs, the ERT estimates the expected hitting time of this restarting strategy by taking the ratio between the total number of function evaluations taken in all runs and the number of successful runs:

$$\text{ERT}(A, f, d, B, \phi) = \frac{\sum_{i=1}^\infty \min \{(T_i(A, f, d, \phi)), B\}}{\sum_{i=1}^\infty \mathbb{1}(T_i(A, f, d, \phi) < \infty)},$$

where $i$ denotes the $i$-th run of the algorithm on the problem, $B$ denotes the maximal budget for function evaluations, and $\mathbb{1}$ stands for the characteristic function. The ERT is a commonly applied metric when benchmarking continuous optimizers [HAR+21][HAR+10][KT19]. However, it must be noted that it only captures one particular aspect of the algorithm performance. It has been criticized, among other things, for unrealistically relaxed budget allocations [BBP11], using an absolute precision for varying objective function values [KT19], and the lack of distinction between different problem instances [KT19]. Nonetheless, to ensure comparability with our previous work, this work will focus on ERT as the performance measure.

[2]The data sets are available on [https://numbbo.github.io/data-archive/bbob/](https://numbbo.github.io/data-archive/bbob/) and through the IOHanalyzer webinterface at [https://iohanalyzer.liacs.nl](https://iohanalyzer.liacs.nl) (as "bbob" under "dataset source").
Figure 1: The theoretical performance gain of VBS\textsubscript{dyn} over VBS\textsubscript{static} for the portfolio as described in Sec. 4, i.e., \((\text{ERT}(\text{VBS\textsubscript{static}}) − \text{ERT}(\text{VBS\textsubscript{dyn}}))/ \text{ERT}(\text{VBS\textsubscript{static}}))\), in which the best dynamic algorithm is the \((A_1, A_2)\)-pair that maximized the theoretical gain for each combination of dimension and function. We compute this gain values on the BBOB test functions in several different dimensions. Please see Sec. 4.2 for the detail regarding the generation procedure for the data set used here.

3 Theoretical Performance Gain

Our previous work on dynamic algorithm selection [VWBD20b] has identified significant potential for performance improvement, using a data-driven technique applied on a large set of performance data. The data from more than 100 algorithms submitted to the above-mentioned BBOB data repository was analyzed by defining a so-called “theoretical performance”, which is defined based on the performance of its component algorithms. Assume we start the optimization process with an algorithm \(A_1\) and switch to a different algorithm \(A_2\) when \(A_1\) hits a pre-defined target precision \(\tau\) (the switching point, which is larger than the final target value \(\phi\)). Also, we assume that after the switching, \(A_2\) would have the same internal states as if we ran \(A_2\) from the beginning to hit \(\tau\). Under such assumptions, the theoretical performance of switching from \(A_1\) to \(A_2\) w.r.t. the switching point \(\tau\) and the final target \(\phi\) is:

\[
\text{ERT}_t(f, d, A_1, A_2, \tau, \phi) = \text{ERT}(A_1, f, d, \tau) + \text{ERT}(A_2, f, d, \phi) - \text{ERT}(A_2, f, d, \tau),
\]

where \(f\) is a test function supported on a \(d\)-dimensional manifold and the \(\text{ERT}_t\) indicates that it is a theoretical measure. By calculating this theoretical value for all combinations of \(A_1, A_2,\) and \(\tau\), we can find, for each function-dimension pair, theoretically the best dynamic algorithm switching procedure, which we refer to as the dynamic virtual best solver (VBS\textsubscript{dyn}). In the following discussion, when the \((A_1, A_2)\)-pair of a VBS\textsubscript{dyn} is clear from the context, we shall denote the theoretical gain alternatively by \(\text{ERT}_t(\text{VBS\textsubscript{dyn}})\). Furthermore, we can compare the ERT value of VBS\textsubscript{dyn} to the static virtual best solver VBS\textsubscript{static}, i.e., the best single algorithm for the corresponding function-dimension pair in terms of the ERT measure, resulting in the theoretical performance gain of the dynamic switching approach w.r.t. the best single algorithm, e.g., taking a relative measure \((\text{ERT}(\text{VBS\textsubscript{static}}) − \text{ERT}(\text{VBS\textsubscript{dyn}}))/ \text{ERT}(\text{VBS\textsubscript{static}})\). Such a performance gain is illustrated in Fig. 1 for the five selected optimization algorithms of our algorithm portfolio (see next section for details), showing that for most functions there is a significant theoretical gain brought by the dynamic switching approach. Also, this theoretical advantage of VBS\textsubscript{dyn} varies hugely across different functions and dimensions with a diminishing trend when the dimensionality increases. This seems to be caused by the fact that very few algorithms manage to reach the final target precision (10\(^{-8}\)) in high dimensions, and in the cases where they do, it is usually the case that one algorithm completely dominates all the others in terms of the ERT measure.

Given a benchmarking data set, we find the best switching point \(\tau\) for a specific algorithm pair \((A_1, A_2)\) by performing a line search of \(\tau\). Afterwards, for each function-dimension pair, we determine the theoretical VBS\textsubscript{dyn} as the \((A_1, A_2)\) pair that produces the smallest theoretical performance defined in Eq. (2). We then check whether the actual performance of a VBS\textsubscript{dyn} would catch or be comparable to the theoretical performance thereof, if we execute the VBS\textsubscript{dyn} on the corresponding function-dimension pair, with the switching point determined in the theoretical study. In the following experiments, we shall denote by \(\text{ERT}_t(\text{VBS\textsubscript{dyn}})\) the performance of the dynamic virtual best solver for a function-dimension pair. Similarly, the actual performance gain over VBS\textsubscript{static} is defined as \((\text{ERT}(\text{VBS\textsubscript{static}}) − \text{ERT}_t(\text{VBS\textsubscript{dyn}}))/ \text{ERT}(\text{VBS\textsubscript{static}})\).
4 Algorithm Portfolio

We focus on a small algorithm portfolio rather than including the entirety of available solvers on the BBOB platform, which allows us to scrutinize each pair of \( A_1 \) and \( A_2 \) algorithm closely and to study in-depth the impact of a warm-starting method on \( A_2 \). To ensure diversity within our portfolio, we include algorithms from different categories according to the taxonomy introduced in [SEB18], such as population-based and trajectory algorithms. Specifically, we select the algorithms that appear most frequently as either \( A_1 \) or \( A_2 \) of a VBS dyn on some test function, based on the data set generated in [VWBD20b] (see Fig. 2). This is based on the assumption that findings on the warm-starting procedure for the overall algorithm family will presumably be transferable to its variants as well, as long as they maintain similar internal parameters. Concretely, our approach suggests the following five algorithms for our portfolio:

- the Broyden-Fletcher-Goldfarb-Shanno (BFGS) algorithm [Bro70, Fle70, Gol70, Sha70], where the partial derivatives of the test function are approximated via the finite difference method.
- the Multi-Level Single Linkage (MLSL) algorithm [KT87], which is based on the clustering of the search population.
- The Particle Swarm Optimization (PSO) [KE95] algorithm with a fully-connected neighbourhood topology (a.k.a. global best) for the particles.
- The standard Covariance Matrix Adaption Evolution Strategy (CMA-ES) [HO96, HO01] without restarting heuristics.
- A Differential Evolution (DE) [SP97] algorithm using binomial crossover and the mutation that uses the current best solution in the population (a.k.a. DE/best/1).

As for the implementations, we orientate ourselves towards existing algorithm submissions on the BBOB platform. In cases where the authors do not provide the code, we fall back on a different implementation while employing algorithm settings and parameters similar to the ones outlined in the respective submissions. Details are provided in the next subsection.

4.1 Implementation Details

For BFGS, we use the implementation from Scipy’s optimize module [VGO+20]. Similar to [Bau14], we keep the algorithm’s default settings. The major difference to their implementation is that we refrain from basin hopping as a
We run all five algorithms on all 24 functions within the BBOB test suite. We set the dimensionality to $d$. We take the modular CMA-ES framework [vRWvLB16] for the implementation of the standard CMA-ES algorithm, where we use the default setting for the population size, i.e., $N = 50$. The hyperparameters are set according to the BBOB submission [Pál13], with a population size of $50d$ and $10\%$ of the function evaluation budget being allocated to the local search. The major difference in our Python implementation is that we use the Powell method from Scipy’s optimize module for the local search routine rather than MATLAB’s fmincon interior-point method because the Powell method yields the least numerical issues among other optimizers in Scipy, based on our initial testing on BBOB. We use the default settings to run the Powell method, except for a reduced fitness tolerance parameter $f_{tol} = 10^{-8}$.

For PSO, we also use our own implementation as we cannot find the original code from the submission [EK09]. In our implementation, we use the global best neighbourhood topology for the particles. The swarm size accounts for 40 particles, which are sampled uniformly at random in $[-5, 5]^d$. Also, we initialize the velocity randomly in $[-1, 1]^d$ to favour exploiting search behaviour. The learning rate for updating the velocity are set to $1.4944$, as recommended in [BM17]. In terms of boundary handling, particles that violate a constraint are clipped to the corresponding boundary, where their velocities are set to zero. Also, the velocities of particles are always clipped to $[-5, 5]$. The inertia weight $\omega$, which dampens the velocity update, is essential for preventing premature convergence, is adjusted using a linearly decreasing schedule: $\omega = 0.9 - 0.8t$, where $t$ is the iteration counter.

We take the modular CMA-ES framework [vRWvLB16] for the implementation of the standard CMA-ES algorithm, where we use the default setting for the population size, i.e., $\lambda = 4 + [3 \cdot \log d]$. Also, we took the setting for hyperparameters as suggested in [Han06]. The modules for additional features, such as an increasing population size and random restarts, are all turned off. The step size $\sigma$ is initially set to 0.5. Lastly, the center of mass of the population is sampled uniformly at randomly in $[0, 1]^d$.

As DE implementation, we employ the version that is available via Scipy’s optimize module [VGO+20]. Most parameters are left in their default settings: most importantly, we use the binomial crossover operator with a crossover rate of 0.7 and the so-called “DE/best/1” mutation operator which only uses the current best solution in the population for the mutation. We set the population size to $5d$ and reduce the convergence tolerance from $10^{-2}$ to $10^{-12}$ to prevent the DE algorithm from terminating too early before the depletion of the function evaluation budget. The mutation scaling factor is sampled uniformly from the interval $[0.5, 1]$ at each generation.

### 4.2 Benchmarking Setup

We run all five algorithms on all 24 functions within the BBOB test suite. We set the dimensionality to $d \in \{2, 3, 5, 10, 20\}$. Similar to a large part of existing research, the algorithms run on the first five problem instances, $i \in \{1, 2, 3, 4, 5\}$. We perform 5 independent runs on each instance, resulting in a total of 25 runs per function-dimension pair. Runs are stopped when the final target precision $\phi = 10^{-8}$ is hit or when the budget for function evaluations $B = 100000d$ is depleted.

### 4.3 Identifying promising $(A_1, A_2)$-pairs

Taking the data set generated by the procedure outlined above, we compute the theoretical performance gain defined in Eq. (2) to determine the best possible switching combinations for each pair of test function and dimension. In Table 4, we summarize the number of cases where a combination of $A_1$ and $A_2$ is VBS$_{dyn}$ for a function-dimension pair. We observe that the combinations of CMA-ES and BFGS constitute the majority of all detected VBS$_{dyn}$. Also, it is also promising to switch from PSO to CMA-ES or from MSL to BFGS.

### 5 Warm-starting Procedures

When switching from one algorithm to another, it is often wasteful to start the new one completely from scratch without considering valuable information already learned by its predecessor, especially when the algorithm contains some type of adaptive control mechanism. Hence, in this work, we contemplate a warm-starting procedure to inherit helpful information from the algorithm that precedes the switching point. While it seems straightforward to initialize the new search algorithm around the best point found by its predecessor, it is more challenging to configure its free parameters (e.g., the step-size in CMA-ES) from the optimization trajectory. Notably, we take all previous function evaluations (pairs of evaluated points and corresponding objective values) and estimate the initial setting of some critical internal parameters used by the subsequent algorithm.
Table 1: The number of use cases where a \((A_1, A_2)\)-pair from our algorithm portfolio leads to the VBS\(_{\text{dyn}}\) for each function-dimension pair. The final target precision is set to \(\phi = 10^{-8}\). The most promising \((A_1, A_2)\)-pairs are marked in boldface.

| \(A_1\)   | \(A_2\)   | Number of use cases | Functions and dimensions                  |
|-----------|-----------|---------------------|-------------------------------------------|
| CMA-ES    | DE        | 3                   | F16, F19, F23 [5D]                        |
| CMA-ES    | PSO       | 0                   | -                                          |
| CMA-ES    | MLSL      | 0                   | -                                          |
| CMA-ES    | BFGS      | 12                  | F1 [10D, 20D], F8, F9 [all dimensions]    |
| PSO       | CMA-ES    | 8                   | F3, F4, F15-F17 [2D], F19 [2D, 3D], F5 [5D]|
| PSO       | DE        | 5                   | F3, F7, F17, F22 [2D], F7 [3D]            |
| PSO       | MLSL      | 0                   | -                                          |
| PSO       | BFGS      | 1                   | F22 [3D]                                  |
| MLSL      | CMA-ES    | 4                   | F7 [3D], F16 [2D], F23 [2D, 3D]           |
| MLSL      | DE        | 1                   | F21 [2D]                                  |
| MLSL      | PSO       | 3                   | F18, F24 [2D], F20 [5D]                   |
| MLSL      | BFGS      | 6                   | F1 [5D], F2 [20D], F15 [2D], F20 [2D, 3D], F21 [3D] |
| DE        | CMA-ES    | 3                   | F5, F13 [2D], F18 [3D]                    |
| DE        | PSO       | 4                   | F3, F4, F17 [5D], F24 [3D]                |
| DE        | MLSL      | 0                   | -                                          |
| DE        | BFGS      | 2                   | F1 [3D], F4 [2D]                          |
| BFGS      | CMA-ES    | 30                  | F5, F6, F10-F14, F21, F22 [various dimensions] |
| BFGS      | DE        | 0                   | -                                          |
| BFGS      | PSO       | 0                   | -                                          |
| BFGS      | MLSL      | 4                   | F2, F10, F11 [2D], F21 [20D]              |

For switching between CMA-ES and BFGS, the warmstarting task is relatively straightforward since the covariance matrix in CMA-ES is shown to learn the local Hessian matrix of smooth landscapes, which is also approximated by BFGS. Also, the initial step-size \(\sigma\) of CMA-ES can be set directly to the last step-size of BFGS, or calculated as the average displacement of the search point in the last few iterations of BFGS (please see Sec. 6.1 for details). For warm-starting PSO or DE from MLSL, we draw its first swarm of particles/points uniformly at random in the hyperbox \([\mathbf{x}_{\text{MLSL}} - \eta \mathbf{1}, \mathbf{x}_{\text{MLSL}} + \eta \mathbf{1}]\), where \(\eta = 0.1\) is chosen via preliminary experiments and \(\mathbf{x}_{\text{MLSL}}^*\) is the best point found by MLSL. For switching from MLSL to CMA-ES, we only initialize CMA-ES’s mean to the best point of MLSL and use the default values for the step-size and covariance matrix.

6 Results

In Fig. 3 we show the actual performance gain calculated after executing the VBS\(_{\text{dyn}}\) (determined by the theoretical performance gain) on each function in 2D, in which we see several cases that actual performance gain \(\hat{ERT}(\text{VBS}_{\text{dyn}})\) reaches or even exceeds the theoretical one \(ERT(\text{VBS}_{\text{dyn}})\) (cf. Fig. 3), e.g., on \(F3, F6-F9, F14, F17,\) and \(F24\). However, there is also a large number of cases where the actual performance is worse than the theoretical one or even the ERT value of VBS\(_{\text{static}}\), e.g., \(F1, F4, F5, F15,\) and \(F20\). For the function-dimension pairs where we observe a large gain over the VBS\(_{\text{static}}\), it is often the case that the ERT curve of \(A_1\) is continued with that of \(A_2\) after shifting the latter down to the switching point, for instance, switching from BFGS to CMA-ES as \(\tau = 3.98 \cdot 10^{-6}\) on function \(F14\) in 2D (see Fig. 6).

In all, the VBS\(_{\text{dyn}}\) outperforms the VBS\(_{\text{static}}\) on 45 out of the 87 identified use cases (see Fig. 4). The highest actual performance gain (92\% relative reduction of the ERT w.r.t. the VBS\(_{\text{static}}\)) is observed when switching from PSO to DE on \(F22\) in 2D. Also, switching from BFGS to CMA-ES on functions \(F10-F14\) produces a high actual performance gain while the reverse switching scheme, i.e., from CMA-ES to BFGS, still achieves significant gain on functions \(F8\) and \(F9\). A notable observation from our experiments is that the actual performance gain can only be realized when \(A_2\) is properly warm-started. We will investigate the impact of the warm-starting procedure in the following sections.

6.1 Switching between CMA-ES and BFGS

To illustrate the impact of the warm-starting procedure on the optimization process, we can zoom in on the cases where we switch between CMA-ES and BFGS, where a basic warm-starting procedure is implemented: the best point in the CMA-ES population can be selected as the starting point for BFGS, and similarly the current point in BFGS can be selected as the center for the new CMA-ES population. Of course, considering only the search point ignores a large
fraction of the information learned during the search, resulting in poor performance. In Fig. 5 (the leftmost sub-plot), we show the covariance matrix of CMA-ES is initialized from only the best-so-far point of BFGS, which shows that information about the direction of the search and the step-sizes are not taken into account.

To initialize the covariance matrix, we can use the fact that for convex-quadratic functions, the covariance matrix directly relates to the inverse Hessian of the objective function, which is approximated during the BFGS procedure, and can thus be used during warm-starting [GK20, Han16, SY20]. While this assumption of convex-quadratic functions does not hold for all BBOB test functions, it is likely that the local landscape at the time of algorithm switching resembles a quadratic shape when the step-size is sufficiently small. As such, we will use the approximated inverse Hessian matrix of BFGS for estimating the covariance matrix of CMA-ES, as well as the reverse (with some scaling factor $\beta$).

We will proceed to warm-start CMA-ES’s step-size $\sigma$ from the internal states of BFGS. While the initial value of $\sigma$ can be set directly to its counterpart in BFGS - the length of steps obtained from BFGS’s internal line search procedure, this setting usually leads to unsatisfactory performance since BFGS’s line search procedure sometime exhibits unstable behavior, thereby making the initial $\sigma$ of CMA-ES improper. Instead, we can consider averaging the step length of
Figure 5: Contour plot of the covariance matrix of CMA-ES immediately after warm-starting from a BFGS-run (F14 in 2D, instance 1, with $\tau = 10^{-5.4}$). The blue points are created by sampling 10 000 points from the CMA-ES state after switching. Only the best-so-far point is used in warm-starting from BFGS. The left-most figure shows the case where covariance and step-size are not warm-started, the two right images show the result when incorporating the warm-starting. The right-most figure is a zoomed-in version of the middle one.

Figure 6: ERT chart for BFGS, CMA-ES, and switching from BFGS to CMA-ES with different warm-starting routines on function F14 in 2D. The switching point is set to $\tau = 3.98 \cdot 10^{-6}$. ERT values are computed from 25 independent runs.

BFGS over the $n$ most recent iterations, i.e., $\sigma = (n - 1)^{-1} \sum_{j=0}^{n-2} \| x_j - x_{j+1} \|$, where $x_j$ is the $j$-th recent search point ($x_0$ is the last point of BFGS prior to the switching) in the trajectory of BFGS.

We compare the effect of these different parts of the warm-starting procedure on the performance of a dynamic combination of BFGS and CMA-ES. In Fig. 6, we show the resulting ERT-curves on F14 in 2D. From this figure, it
Table 2: ERT values for switching between BFGS and CMA-ES on selected function-dimension pairs. The target precision has been set to $\phi = 10^{-8}$. Performance gain values are calculated as $(\text{ERT}(\text{VBS}_{\text{static}}) - \text{ERT}(\text{VBS}_{\text{dyn}}))/\text{ERT}(\text{VBS}_{\text{static}})$ for the theoretical measure and similarly for the actual one. We mark the cases in boldface where the actual performance $\text{ERT}_i(\text{VBS}_{\text{dyn}})$ is even better (lower) than the theoretical one $\text{ERT}_i(\text{VBS}_{\text{dyn}})$ or the static virtual vest solver $\text{ERT}(\text{VBS}_{\text{static}})$.

| $A_i$ | $A_j$ | $F$ | $d$ | $\tau$ | ERT($\text{VBS}_{\text{static}}$) | Theoretical ERT($\text{VBS}_{\text{dyn}}$) | Actual ERT($\text{VBS}_{\text{dyn}}$) | Actual performance gain | Theoretical performance gain |
|-------|-------|-----|-----|-------|-----------------|---------------------------|-----------------|------------------|-------------------|
| BFGS  | CMA-ES| 6   | 2   | 6.31 $10^{-8}$ | 589.56 | 324.33 | 362.72 | 45% | -1% |
| BFGS  | CMA-ES| 6   | 3   | 1.58 $10^{-7}$ | 942.56 | 672.32 | 1018.36 | 8% | -51% |
| BFGS  | CMA-ES| 10  | 3   | 1.58 $10^{-7}$ | 1135.92 | 267.18 | 267.36 | 76% | 0% |
| BFGS  | CMA-ES| 10  | 20  | 2.51 $10^{-6}$ | 19825.48 | 3096.4 | 3341 | 83% | -8% |
| BFGS  | CMA-ES| 11  | 10  | 2.51 $10^{-6}$ | 5797.32 | 1072.12 | 1226.2 | 79% | -14% |
| BFGS  | CMA-ES| 11  | 20  | 1.58 $10^{-6}$ | 15457 | 2540.52 | 2553.4 | 83% | -1% |
| BFGS  | CMA-ES| 12  | 2   | 1.58 $10^{-6}$ | 1257.07 | 791.6 | 1052.04 | 17% | -30% |
| BFGS  | CMA-ES| 12  | 20  | 6.31 $10^{-6}$ | 26771.84 | 16854.47 | 15785.28 | 41% | 6% |
| BFGS  | CMA-ES| 13  | 5   | 2.51 $10^{-5}$ | 3938.08 | 2794.00 | 2111.24 | 46% | 8% |
| BFGS  | CMA-ES| 13  | 10  | 2.51 $10^{-5}$ | 15987.36 | 11193.2 | 9209.24 | 42% | 18% |
| BFGS  | CMA-ES| 14  | 2   | 3.98 $10^{-6}$ | 705 | 364.72 | 271.64 | 61% | 26% |
| BFGS  | CMA-ES| 14  | 10  | 3.98 $10^{-6}$ | 6750.84 | 3414.2 | 2999.08 | 64% | 30% |
| BFGS  | CMA-ES| 21  | 5   | 1    | 49843.62 | 2258.5 | 118401.5 | -2276% | -52236% |
| BFGS  | CMA-ES| 21  | 10  | 1    | 294066.29 | 9024.00 | 732071 | -149% | -7237% |
| BFGS  | CMA-ES| 22  | 10  | 1.58 | 31849.05 | 12056.5 | 563100.88 | 77% | 4571% |
| CMA-ES| BFGS  | 8   | 10  | 175.08 | 131.68 | 87.52 | 44% | 34% |
| CMA-ES| BFGS  | 8   | 3   | 3.98 | 277.32 | 242.16 | 204.92 | 26% | 15% |
| CMA-ES| BFGS  | 8   | 10  | 100  | 500.83 | 472.11 | 580.26 | -16% | -23% |
| CMA-ES| BFGS  | 8   | 10  | 100  | 1965.2 | 1771.28 | 1234.21 | 31% | 30% |
| CMA-ES| BFGS  | 8   | 10  | 100  | 4724.16 | 4135.44 | 3513.82 | 10% | -3% |
| CMA-ES| BFGS  | 9   | 2   | 1.58 | 175.24 | 93.92 | 84.64 | 52% | 10% |
| CMA-ES| BFGS  | 9   | 3   | 2.51 | 259.24 | 184.8 | 168.36 | 3% | 12% |
| CMA-ES| BFGS  | 9   | 5   | 100  | 551.8 | 428.56 | 301.42 | 45% | 30% |
| CMA-ES| BFGS  | 9   | 10  | 100  | 1635.9 | 1270.73 | 1080.77 | 34% | 15% |
| CMA-ES| BFGS  | 9   | 20  | 6.31 | 4994.96 | 3745.76 | 3502.26 | 30% | 7% |
| CMA-ES| BFGS  | 10  | 10  | 100  | 35.76 | 22.00 | 47.52 | -33% | -116% |
| CMA-ES| BFGS  | 1   | 20  | 100  | 74.92 | 30.84 | 117.4 | -57% | -281% |

is clear to see that switching from BFGS to CMA-ES is beneficial to avoid the numerical issues near the end of the optimization procedure, but the details of the switching procedure have a large effect on the overall performance. We can see that only warm-starting the initial point is not sufficient to beat the CMA-ES itself, as the step-size will be much too large (Fig. 5 (the leftmost sub-plot)). Utilizing the warm-starting procedures for both step-size and covariance matrix, we observe 61% actual performance gain over the VBSstatic.

This warm-starting procedure can be applied to all use-cases where the data-driven analysis from Section 4.3 found a combination of BFGS and CMA-ES be optimal. This corresponds to a total of 42 function-dimension pairs. The resulting ERT values for selected function-dimension pairs are summarized in Table 2. From this table, we see that most of the dynamic use-cases do indeed improve over the static algorithms. However, there are several functions where this is not the case. Notably, for test functions $F21$ and $F22$, the switch from BFGS to CMA-ES does not perform well, with barely any successful runs. This highlights a disadvantage of the ERT-based calculations to select use-cases: if there is only one successful run of an algorithm, the ERT-curve after a particular target can be quite misleading.

### 6.2 Impact of Switching Point

In the final experiment, we check how the actual performance of a VBSdyn is sensitive to the switching point. To conduct this sensitivity analysis, we take one of the case switching from BFGS to CMA-ES on $F10$ in 5D, where we observe an significant actual performance gain when using the theoretically derived switching point. We investigate the sensitivity of the switching point $\tau$ by varying its value in a wide range $\{10^i : i \in \{-7.8, -7.6, ..., 1.8, 2\}\}$. The result is depicted in Fig. 7 where it is obvious that the best actual performance of switching from BFGS to CMA-ES obtained with $\tau = 10^{-3.6}$. Interestingly, when the switch point is smaller than $10^{-2}$, the empirical distribution of the hitting time of the VBSdyn is substantially different from those with a much larger the switch point, which implies the existence of a transition phase of the behavior of the VBSdyn.

This behaviour seems to indicate that while a dynamic algorithm selection approach can be beneficial, there is yet more potential to be gained by using run-specific information to determine when to switch between configurations. This
could be achieved by using both the internal algorithm state and local algorithm features to create a much more granular dynamic algorithm selection/configuration policy.

7 Conclusions and Future Work

While we have shown that dynamic algorithm selection can be highly beneficial in certain scenarios, it is clear that much more research is needed to be able to generalize these results to more complex settings. In particular, we have shown that the procedures used to warm-start the algorithms are critical to achieving the potential improvements seen in the performance data of the individual algorithms. However, extending this to more algorithms will require an extendable way to effectively make use of the information collected during the first part of the optimization procedure.

Another major challenge in generalizing dynamic algorithm selection is the determination of the split point \cite{RY21}. In this work, we use a fixed-target perspective, which necessarily breaks the black-box assumption which is present in a lot of practical optimization problems. As such, the determination of when to switch between algorithms should be studied in more detail, for example from the area of dynamic landscape analysis \cite{JED21}. In this way, the limitation of using a single switch point could also be relaxed, allowing the resulting algorithm to more effectively exploit local search behaviour.

Dynamic algorithm selection can be seen as one part of the dynamic algorithm configuration problem, where we include hyperparameters of the algorithms in the design process. This inclusion will allow for even further specialization of the individual algorithms, leading to even more potential differences to exploit in the dynamic context. However, this comes at a cost of increased computational complexity, since a data-driven approach as used in this work is computationally infeasible as soon as continuous hyperparameters are introduced.
References

[Bau14] Petr Baudis. Cocopf: An algorithm portfolio framework. *CoRR*, abs/1405.3487, 2014.

[BBE+20] André Biedenkapp, H. Furkan Bozkurt, Theresa Eimer, Frank Hutter, and Marius Lindauer. Dynamic Algorithm Configuration: Foundation of a New Meta-Algorithmic Framework. In Giuseppe De Giacomo, Alejandro Catalá, Bistra Dilkina, Michela Milano, Senén Barro, Alberto Bugarin, and Jérôme Lang, editors, *ECAI 2020 - 24th European Conference on Artificial Intelligence, 29 August-8 September 2020, Santiago de Compostela, Spain, August 29 - September 8, 2020 - Including 10th Conference on prestigious Applications of Artificial Intelligence (PAIS 2020)*, volume 325 of *Frontiers in Artificial Intelligence and Applications*, pages 427–434. IOS Press, 2020.

[BBP11] Thomas Bartz-Beielstein and Mike Preuss. *Experimental Analysis of Optimization Algorithms: Tuning and Beyond*. 01 2011.

[BGH+13] Edmund K. Burke, Michel Gendreau, Matthew R. Hyde, Graham Kendall, Gabriela Ochoa, Ender Özcan, and Rong Qu. Hyper-heuristics: a survey of the state of the art. *JORS*, 64(12):1695–1724, 2013.

[BM17] Mohammad Reza Bonyadi and Zbigniew Michalewicz. Particle Swarm Optimization for Single Objective Continuous Space Problems: A Review. *Evol. Comput.*, 25(1):1–54, 2017.

[Bro70] Charles G. Broyden. The Convergence of a Class of Double-rank Minimization Algorithms: 2. The New Algorithm. *IMA Journal of Applied Mathematics*, 6(3):222–231, 09 1970.

[CV97] Diane J. Cook and R. Craig Varnell. Maximizing the benefits of parallel search using machine learning. In Benjamin Kuipers and Bonnie L. Webber, editors, *Proceedings of the Fourteenth National Conference on Artificial Intelligence and Ninth Innovative Applications of Artificial Intelligence Conference, AAAI 97, IAAI 97, July 27-31, 1997, Providence, Rhode Island, USA*, pages 559–564. AAAI Press / The MIT Press, 1997.

[DD20] Benjamin Doerr and Carola Doerr. Theory of parameter control mechanisms for discrete black-box optimization: Provable performance gains through dynamic parameter choices. In *Theory of Evolutionary Computation: Recent Developments in Discrete Optimization*, pages 271–321. Springer, 2020. Also available online at [https://arxiv.org/abs/1804.05650](https://arxiv.org/abs/1804.05650).

[DTN16] Christian Daniel, Jonathan Taylor, and Sebastian Nowozin. Learning Step Size Controllers for Robust Neural Network Training. In Dale Schuurmans and Michael P. Wellman, editors, *Proceedings of the Thirtieth AAAI Conference on Artificial Intelligence, February 12-17, 2016, Phoenix, Arizona, USA*, pages 1519–1525. AAAI Press, 2016.

[EBR+21] Theresa Eimer, André Biedenkapp, Maximilian Reimer, Steven Adriaensen, Frank Hutter, and Marius Lindauer. Dacbench: A benchmark library for dynamic algorithm configuration. In Zhi-Hua Zhou, editor, *Proceedings of the Thirtieth International Joint Conference on Artificial Intelligence, IJCAI 2021, Virtual Event / Montreal, Canada, 19-27 August 2021*, pages 1668–1674. ijcai.org, 2021.

[EK09] Mohammed El-Abd and Mohamed S. Kamel. Black-box optimization benchmarking for noiseless function testbed using particle swarm optimization. In Franz Rothlauf, editor, *Genetic and Evolutionary Computation Conference, GECCO 2009, Proceedings, Montreal, Québec, Canada, July 8-12, 2009, Companion Material*, pages 2269–2274. ACM, 2009.

[Fle70] Roger Fletcher. A new approach to variable metric algorithms. *The Computer Journal*, 13(3):317–322, 01 1970.

[GK20] Tobias Glasmachers and Oswin Krause. The hessian estimation evolution strategy. In Thomas Bück, Mike Preuss, André H. Deutz, Hao Wang, Carola Doerr, Michael T. M. Emmerich, and Heike Trautmann, editors, *Parallel Problem Solving from Nature - PPSN XVI - 16th International Conference, PPSN 2020, Leiden, The Netherlands, September 5-9, 2020, Proceedings, Part I*, volume 12269 of *Lecture Notes in Computer Science*, pages 597–609. Springer, 2020.

[Gol70] Donald Goldfarb. A family of variable-metric methods derived by variational means. *Mathematics of Computation*, 24(109):23–26, 1970.

[Han06] Nikolaus Hansen. The CMA evolution strategy: A comparing review. In José Antonio Lozano, Pedro Larrañaga, Iñaki Inza, and Endika Bengoetxea, editors, *Towards a New Evolutionary Computation - Advances in the Estimation of Distribution Algorithms*, volume 192 of *Studies in Fuzziness and Soft Computing*, pages 75–102. Springer, 2006.

[Han16] Nikolaus Hansen. The CMA evolution strategy: A tutorial. *CoRR*, abs/1604.00772, 2016.
Chaining of Numerical Black-box Algorithms: Warm-Starting and Switching Points

[HAR+10] Nikolaus Hansen, Anne Auger, Raymond Ros, Steffen Finck, and Petr Pošík. Comparing results of 31 algorithms from the black-box optimization benchmarking bbob-2009. In Proceedings of the 12th Annual Conference Companion on Genetic and Evolutionary Computation, GECCO ’10, page 1689–1696, New York, NY, USA, 2010. Association for Computing Machinery.

[HAR+21] Nikolaus Hansen, Anne Auger, Raymond Ros, Olaf Mersmann, Tea Tusar, and Dimo Brockhoff. COCO: a platform for comparing continuous optimizers in a black-box setting. Optim. Methods Softw., 36(1):114–144, 2021.

[HFRA09] Nikolaus Hansen, Steffen Finck, Raymond Ros, and Anne Auger. Real-Parameter Black-Box Optimization Benchmarking 2009: Noiseless Functions Definitions. Research Report RR-6829, INRIA, 2009.

[HO96] Nikolaus Hansen and Andreas Ostermeier. Adapting arbitrary normal mutation distributions in evolution strategies: The covariance matrix adaptation. In Toshio Fukuda and Takeshi Furushashi, editors, Proceedings of 1996 IEEE International Conference on Evolutionary Computation, Nayoya University, Japan, May 20-22, 1996, pages 312–317. IEEE, 1996.

[HO01] Nikolaus Hansen and Andreas Ostermeier. Completely derandomized self-adaptation in evolution strategies. Evol. Comput., 9(2):159–195, 2001.

[JED21] Anja Jankovic, Tome Eftimov, and Carola Doerr. Towards feature-based performance regression using trajectory data. In Proc. of Applications of Evolutionary Computation (EvoApplications 2021), volume 12694 of Lecture Notes in Computer Science, pages 601–617. Springer, 2021.

[KE95] James Kennedy and Russell Eberhart. Particle swarm optimization. In Proceedings of International Conference on Neural Networks (ICNN’95), Perth, WA, Australia, November 27 - December 1, 1995, pages 1942–1948. IEEE, 1995.

[KHE15] Giorgos Karafotias, Mark Hoogendoorn, and A.E. Eiben. Parameter control in evolutionary algorithms: Trends and challenges. IEEE Transactions on Evolutionary Computation, 19:167–187, 2015.

[KHNT19] Pascal Kerschke, Holger H. Hoos, Frank Neumann, and Heike Trautmann. Automated algorithm selection: Survey and perspectives. Evol. Comput., 27(1):3–45, 2019.

[KT87] Alexander H. G. Rinnooy Kan and G. T. Timmer. Stochastic global optimization methods part II: multi level methods. Math. Program., 39(1):57–78, 1987.

[KT19] Pascal Kerschke and Heike Trautmann. Automated algorithm selection on continuous black-box problems by combining exploratory landscape analysis and machine learning. Evolutionary Computation, 27(1):99–127, 03 2019.

[LHHS15] Marius Lindauer, Holger H. Hoos, Frank Hutter, and Torsten Schaub. AutoFolio: An Automatically Configured Algorithm Selector. J. Artif. Intell. Res., 53:745–778, 2015.

[LNA+03] Kevin Leyton-Brown, Eugene Nudelman, Galen Andrew, Jim McFadden, and Yoav Shoham. A Portfolio Approach to Algorithm Selection. In Georg Gottlob and Toby Walsh, editors, IJCAI-03, Proceedings of the Eighteenth International Joint Conference on Artificial Intelligence, Acapulco, Mexico, August 9-15, 2003, page 1542. Morgan Kaufmann, 2003.

[Pál13] László Pál. Benchmarking a hybrid multi level single linkage algorithm on the bbob noiseless testbed. In Christian Blum and Enrique Alba, editors, Genetic and Evolutionary Computation Conference, GECCO ’13, Amsterdam, The Netherlands, July 6-10, 2013, Companion Material Proceedings, pages 1145–1152. ACM, 2013.

[Ric76] John R. Rice. The algorithm selection problem. Adv. Comput., 15:65–118, 1976.

[RY21] Antonio Bolufé Röhler and Ye Yuan. Machine learning for determining the transition point in hybrid metaheuristics. In Proc. of IEEE Congress on Evolutionary Computation (CEC 2021), pages 1115–1122. IEEE, 2021.

[SBA+20] Gresa Shala, André Biedenkapp, Noor H. Awad, Steven Adriaensen, Marius Lindauer, and Frank Hutter. Learning Step-Size Adaptation in CMA-ES. In Thomas Bäck, Mike Preuss, Andrè H. Deutz, Hao Wang, Carola Doerr, Michael T. M. Emmerich, and Heike Trautmann, editors, Parallel Problem Solving from Nature - PPSN XVI - 16th International Conference, PPSN 2020, Leiden, The Netherlands, September 5-9, 2020, Proceedings, Part I, volume 12269 of Lecture Notes in Computer Science, pages 691–706. Springer, 2020.

[SBH+21] David Speck, André Biedenkapp, Frank Hutter, Robert Mattmüller, and Marius Lindauer. Learning Heuristic Selection with Dynamic Algorithm Configuration. In Susanne Biundo, Minh Do, Robert
Chaining of Numerical Black-box Algorithms: Warm-Starting and Switching Points

Goldman, Michael Katz, Qiang Yang, and Hankz Hankui Zhuo, editors, Proceedings of the Thirty-First International Conference on Automated Planning and Scheduling, ICAPS 2021, Guangzhou, China (virtual), August 2-13, 2021, pages 597–605. AAAI Press, 2021.

[Sch21] Dominik Schroeder. Dynamic algorithm selection - experimental data, April 2021.

[SEB18] Jörg Stork, A. E. Eiben, and Thomas Bartz-Beielstein. A new taxonomy of continuous global optimization algorithms. CoRR, abs/1808.08818, 2018.

[Sha70] David F. Shanno. Conditioning of quasi-newton methods for function minimization. Mathematics of Computation, 24(111):647–656, 1970.

[SHSV17] Ingrida Steponavice, Rob J. Hyndman, Kate Smith-Miles, and Laura Villanova. Dynamic algorithm selection for pareto optimal set approximation. J. Glob. Optim., 67(1-2):263–282, 2017.

[Smi08] Kate Smith-Miles. Cross-disciplinary perspectives on meta-learning for algorithm selection. ACM Comput. Surv., 41(1):6:1–6:25, 2008.

[SP97] Rainer Storn and Kenneth V. Price. Differential evolution - A simple and efficient heuristic for global optimization over continuous spaces. J. Glob. Optim., 11(4):341–359, 1997.

[SY20] Ofer M. Shir and Amir Yehudayoff. On the covariance-hessian relation in evolution strategies. Theor. Comput. Sci., 801:157–174, 2020.

[VGO+20] Pauli Virtanen, Ralf Gommers, Travis E. Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, Pearu Peterson, Warren Weckesser, Jonathan Bright, Stéfan J. van der Walt, Matthew Brett, Joshua Wilson, K. Jarrod Millman, Nikolay Mayorov, Andrew R. J. Nelson, Eric Jones, Robert Kern, Eric Larson, C J Carey, Ian Polat, Yu Feng, Eric W. Moore, Jake VanderPlas, Denis Laxalde, Josef Perktold, Robert Cimrman, Ian Henriksen, E. A. Quintero, Charles R. Harris, Anne M. Archibald, Antônio H. Ribeiro, Fabian Pedregosa, Paul van Mulbregt, and SciPy 1.0 Contributors. SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python. Nature Methods, 17:261–272, 2020.

[vRWvLB16] Sander van Rijn, Hao Wang, Matthijs van Leeuwen, and Thomas Bäck. Evolving the structure of evolution strategies. In 2016 IEEE Symposium Series on Computational Intelligence, SSCI 2016, Athens, Greece, December 6-9, 2016, pages 1–8. IEEE, 2016.

[VvRBD19] Diederick Vermetten, Sander van Rijn, Thomas Bäck, and Carola Doerr. Online selection of CMA-ES variants. In Anne Auger and Thomas Stützle, editors, Proceedings of the Genetic and Evolutionary Computation Conference, GECCO 2019, Prague, Czech Republic, July 13-17, 2019, pages 951–959. ACM, 2019.

[VWBD20a] Diederick Vermetten, Hao Wang, Thomas Bäck, and Carola Doerr. Towards dynamic algorithm selection for numerical black-box optimization: investigating BBOB as a use case. In Carlos Artemio Coello Coello, editor, GECCO ’20: Genetic and Evolutionary Computation Conference, Cancún, Mexico, July 8-12, 2020, pages 654–662. ACM, 2020.

[VWBD20b] Diederick Vermetten, Hao Wang, Thomas Bäck, and Carola Doerr. Towards dynamic algorithm selection for numerical black-box optimization: investigating BBOB as a use case. In Carlos Artemio Coello Coello, editor, GECCO ’20: Genetic and Evolutionary Computation Conference, Cancún, Mexico, July 8-12, 2020, pages 654–662. ACM, 2020.

[WVY+20] Hao Wang, Diederick Vermetten, Furong Ye, Carola Doerr, and Thomas Bäck. Iohanalyzer: Performance analysis for iterative optimization heuristic, 2020.

[XHHL08] Lin Xu, Frank Hutter, Holger H. Hoos, and Kevin Leyton-Brown. Satzilla: Portfolio-based algorithm selection for SAT. J. Artif. Intell. Res., 32:565–606, 2008.

[XHHL12] Lin Xu, Frank Hutter, Holger H. Hoos, and Kevin Leyton-Brown. Evaluating component solver contributions to portfolio-based algorithm selectors. In Alessandro Cimatti and Roberto Sebastiani, editors, Theory and Applications of Satisfiability Testing - SAT 2012 - 15th International Conference, Trento, Italy, June 17-20, 2012. Proceedings, volume 7317 of Lecture Notes in Computer Science, pages 228–241. Springer, 2012.