EXAMPLES ON LOEWY FILTRATIONS AND K-STABILITY OF FANO
VARIETIES WITH NON-REDUCTIVE AUTOMORPHISM GROUPS

ATSUSHI ITO

Abstract. It is known that the automorphism group of a K-polystable Fano manifold is reductive. Codogni and Dervan construct a canonical filtration of the section ring, called Loewy filtration, and conjecture that the Loewy filtration destabilizes any Fano variety with non-reductive automorphism group. In this note, we give a counterexample to their conjecture.

1. Introduction

For a Fano manifold $X$ over $\mathbb{C}$, it is known that $X$ admits Kähler-Einstein metrics if and only if $X$ is K-polystable [Ti1, Do2, CT, St, Be, CDS1, CDS2, CDS3, Ti2]. The K-polystability of $X$ is defined by using the Donaldson-Futaki invariant $DF(X, L)$ of a test configuration $(X, L)$ of $X$. Roughly, $X$ is called $K$-polystable if $DF(X, L) \geq 0$ for any test configuration of $X$, and equality holds only for a special type of test configurations, called of product type. On the other hand, Matsushima [Ma] shows that if $X$ admits Kähler-Einstein metrics then the automorphism group $\text{Aut}(X)$ of $X$ is reductive. Hence if $\text{Aut}(X)$ is not reductive, $X$ is not K-polystable. Then there exists a test configuration $(X, L)$ of $X$ which destabilizes $X$, i.e. $DF(X, L) < 0$, or $DF(X, L) = 0$ and $(X, L)$ is not of product type.

By this observation, Codogni and Dervan [CD1] consider the following question:

Question 1.1. If $\text{Aut}(X)$ is not reductive, can we find a (canonical) destabilizing test configuration $(X, L)$ of $X$ related to $\text{Aut}(X)$?

A test configuration $(X, L)$ can be interpreted as a suitable finitely generated decreasing filtrations $\mathcal{F}_*R = \{\mathcal{F}_iR\}_{i \in \mathbb{Z}}$ of the section ring $R = \bigoplus_{d \geq 0} H^0(X, -dK_X)$ by

$$(X, L) = \left( \text{Proj}_{A^1} \bigoplus_i (\mathcal{F}_iR)t^{-i}, \mathcal{O}(1) \right) \to A^1 = \text{Spec} \mathbb{C}[t],$$

where $\mathcal{F}_*R$ is called finitely generated if $\bigoplus_i (\mathcal{F}_iR)t^{-i}$ is a finitely generated $\mathbb{C}[t]$-algebra.

Using the action of $\text{Aut}(X)$, Codogni and Dervan construct a canonical filtration of $R$, called the Loewy filtration of $X$. Note that we do not know whether or not the Loewy filtration is finitely generated in general [CD1, CD2].

The following is a special case of [CD1, Conjecture B], i.e. the case when the Loewy filtration is finitely generated:

Conjecture 1.2. Let $X$ be a Fano manifold with non-reductive automorphism group. Assume that the Loewy filtration of $X$ is finitely generated. Then the induced test configuration $(\mathcal{X}_{\text{Loe}}, L_{\text{Loe}})$ destabilizes $X$.

We note that they state the conjecture [CD1, Conjecture B] not only for Fano manifolds but also for polarized varieties.

The purpose of this note is to give a counterexample to Conjecture 1.2, and hence to [CD1, Conjecture B] as follows:

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This note is organized as follows. In Section 2, we recall K-stability and Loewy filtrations. In Section 3, we explain some known results about toric varieties. In Section 4, we give a counterexample to Conjecture 1.2. In Appendix, we show a property of Socle filtrations. Throughout this note, we work over \( \mathbb{C} \). We denote by \( \mathbb{N} \) the set of all non-negative integers.

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2. K-stability, test configurations, and filtrations

Throughout this section, \( X \) is a \( \mathbb{Q} \)-Fano variety, that is, \( X \) is a normal projective variety with at most klt singularities such that the anti-canonical divisor \(-K_X\) is \( \mathbb{Q} \)-Cartier and ample.

2.1. K-stability.

Definition 2.1. A test configuration \((X, L)\) of \( X \) consists of the following data:

- a variety \( X \) with a projective morphism \( \pi : X \to \mathbb{A}^1 \),
- a \( \mathbb{Q} \)-line bundle \( L \) on \( X \) which is ample over \( \mathbb{A}^1 \),
- a \( \mathbb{G}_m \)-action on \((X, L)\) such that \( \pi \) is \( \mathbb{G}_m \)-equivariant and \((X \setminus X_0, L|_{X \setminus X_0})\) is \( \mathbb{G}_m \)-equivariantly isomorphic to \((X \times (\mathbb{A}^1 \setminus \{0\}), p_1^*(-K_X))\), where \( \mathbb{G}_m \) acts on \( \mathbb{A}^1 \) multiplicatively and \( X_0 \) is the fiber over \( 0 \in \mathbb{A}^1 \).

For a test configuration \((X, L)\) of \( X \), we can define a rational number \( DF(X, L) \), called the Donaldson-Futaki invariant of \((X, L)\). See [Do1] for the definition of \( DF(X, L) \).

Definition 2.2. A \( \mathbb{Q} \)-Fano variety \( X \) is called

1. K-semistable if for any test configuration \((X, L)\) of \( X \), we have \( DF(X, L) \geq 0 \).
2. K-polystable if \( X \) is K-semistable and, if \( DF(X, L) = 0 \) for a test configuration \((X, L)\) of \( X \), then \( X \) is isomorphic to a test configuration of product type outside a codimension two subset, i.e. \( X \) is isomorphic to \( X \times \mathbb{A}^1 \) over \( \mathbb{A}^1 \) outside a codimension two subset.

Let

\[
R = \bigoplus_{d \geq 0} R_d = \bigoplus_{d \geq 0} H^0(X, -dK_X)
\]

be the section ring of \( X \). In this note, a decreasing filtration \( F_\bullet R \) of \( R \) is a sequence of vector subspaces

\[
\cdots \supset F_i R \supset F_{i+1} R \supset \cdots
\]

of \( R \) for \( i \in \mathbb{Z} \) such that \( F_i R = \bigoplus_{d \geq 0} (F_i R \cap R_d) \) holds for any \( i \in \mathbb{Z} \) and \( \bigcup_{i \in \mathbb{Z}} F_i R = R \).

A decreasing filtration \( F_\bullet R \) is called

- multiplicative if \( F_i R \cdot F_j R \subset F_{i+j} R \) for any \( i, j \). We note that if \( F_\bullet R \) is multiplicative, \( \bigoplus_{i \in \mathbb{Z}} (F_i R) t^{-i} \) has a natural \( \mathbb{C}[t] \)-algebra structure.
- finitely generated if it is multiplicative and the \( \mathbb{C}[t] \)-algebra \( \bigoplus_{i \in \mathbb{Z}} (F_i R) t^{-i} \) is finitely generated.

Theorem 1.3. There exists a smooth toric Fano 3-fold \( X \) with non-reductive automorphism group such that the Loewy filtration is finitely generated and the Donaldson-Futaki invariant \( DF(\Lambda_{\text{Loe}}, \mathcal{L}_{\text{Loe}}) \) is positive. In particular, \((\Lambda_{\text{Loe}}, \mathcal{L}_{\text{Loe}})\) does not destabilize \( X \).

In a preliminary version [CD0] of [CD1], they also mention Socle filtrations, which are “dual” of Loewy filtrations. We also study Socle filtrations.
An increasing filtration $G \cdot R$ of $R$ is a sequence of vector subspaces
\[
\cdots \subset G_i R \subset G_{i+1} R \subset \cdots
\]
of $R$ for $i \in \mathbb{Z}$ such that $G_i R = \bigoplus_{d \geq 0} (G_i R \cap R_d)$ holds for any $i \in \mathbb{Z}$ and $\bigcup_{i \in \mathbb{Z}} G_i R = R$.

In this paper, we can reduce the general case to the unipotent case by taking the unipotent radical
\[
\text{Definition 2.6.}
\]

Let $G \cdot R$ be a finite dimensional $U$-module.

(1) The **Loewy filtration** $F^L_i V = \{F^L_i V\}_{i \in \mathbb{N}}$ is a decreasing filtration of $U$-modules defined by
\[
\begin{align*}
(i) & \quad F^L_0 V = V, \\
(ii) & \quad \text{for } i > 0, ~ F^L_i V \text{ is the minimal } U\text{-submodule of } F^L_{i-1} V \text{ such that the quotient } F^L_{i-1} V/F^L_i V \text{ is semisimple, i.e. the action on } F^L_{i-1} V/F^L_i V \text{ is trivial.}
\end{align*}
\]

(2) The **Socle filtration** $G^S_i V = \{G^S_i V\}_{i \in \mathbb{N}}$ is an increasing filtration of $U$-modules defined by
\[
\begin{align*}
(i) & \quad G^S_0 V = V^U, \text{ the invariant part of } V \text{ by the action of } U, \\
(ii) & \quad \text{for } i > 0, ~ G^S_i V/G^S_{i-1} V = (V/G^S_{i-1} V)^U.
\end{align*}
\]

**Remark 2.4.**

Loewy filtrations can be defined for not necessarily unipotent algebraic groups. However, we can reduce the general case to the unipotent case by taking the unipotent radical [CD1, Lemma 2.3].

Since $U$ is unipotent and $V$ is finite dimensional, $F^L_i V = \{0\}$ and $G^S_i V = V$ for $i \gg 0$.

We also note that the indexes of the Socle filtration in Definition 2.3 is shifted by one from those in [CD0]. More precisely, it is defined as $G^S_0 V = \{0\}, G^S_1 V = V^U, \ldots$ in [CD0].

**Example 2.5.**

Fix $N \in \mathbb{N}$ and set $V_N = \{f \in \mathbb{C}[x] \mid \deg(f) \leq N\} \subset \mathbb{C}[x]$. Let $U$ be the additive unipotent algebraic group $\mathbb{C}$, and consider the action of $U$ on $V_N$ by $\alpha \cdot x := x + \alpha$ for $\alpha \in U = \mathbb{C}$. In this case, it holds that
\[
F^L_i V_N = \{f \in V \mid \deg(f) \leq N - i\}, \quad G^S_i V_N = \{f \in V \mid \deg(f) \leq i\}.
\]

**Definition 2.6.**

Let $X$ be a $\mathbb{Q}$-Fano variety and $U$ be the unipotent radical of the automorphism group $\text{Aut}(X)$ of $X$. Then $U$ acts on $R_d = H^0(X, -dK_X)$ for each $d \geq 0$.

(1) The **Loewy filtration** $F^L \cdot R$ of $X$ is a decreasing filtration of $R$ defined by
\[
\begin{align*}
& \bullet \quad F^L_i R = R \quad \text{for } i < 0, \\
& \bullet \quad F^L_i R := \bigoplus_{d \geq 0} F^L_d R_d \quad \text{for } i \geq 0, \text{ where } F^L_d R_d \text{ is the Loewy filtration of the } U\text{-module } R_d.
\end{align*}
\]
(2) The **Socle filtration** $G^S_i R$ of $X$ is an increasing filtration of $R$ defined by
\[ G^S_0 R = \{0\} \text{ for } i < 0, \]
\[ G^S_i R := \bigoplus_{d=0}^{\infty} G^S_d R_d \text{ for } i \geq 0, \]
where $G^S_d R_d$ is the Socle filtration of the $U$-module $R_d$.

(3) If the Loewy filtration $F^L_i R$ (resp. Socle filtration $G^S_i R$) is finitely generated, we denote by $(\mathcal{X}_{\text{Loe}}, \mathcal{L}_{\text{Loe}})$ (resp. $(\mathcal{X}_{\text{Soc}}, \mathcal{L}_{\text{Soc}})$) the induced test configuration of $X$.

**Remark 2.7.** We note that $\bigcup_{i \in \mathbb{Z}} F^L_i R = \bigcup_{i \in \mathbb{Z}} G^S_i R = R$ holds by Remark 2.4. It is not known whether or not the Loewy filtration of a $\mathbb{Q}$-Fano variety is multiplicative in general [CD2]. On the other hand, we will show that the Socle filtration is multiplicative in Appendix.

**Example 2.8.** Let $S \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2 = \text{Proj} \mathbb{C}[X, Y, Z]$ at $[1 : 0 : 0]$ and $[0 : 1 : 0]$. The Loewy filtration of $S$ is computed in [CD1, Subsection 3.2] as follows.

The unipotent radical of $\text{Aut}(S)$ consists of matrices of the form
\[
\begin{pmatrix}
1 & 0 & \alpha \\
0 & 1 & \beta \\
0 & 0 & 1
\end{pmatrix}
\text{ for } \alpha, \beta \in \mathbb{C},
\]
which acts on $\mathbb{C}[X, Y, Z]$ by
\[
X \mapsto X + \alpha Z, \quad Y \mapsto Y + \beta Z, \quad Z \mapsto Z.
\]

Since $-K_S = 3H - E_1 - E_2$, where $H$ is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$ and $E_1, E_2$ are the exceptional divisors, we have
\[ R_d = H^0(S, -dK_S) = (X^a Y^b Z_3^{3d-a-b} | 0 \leq a, b \leq 2d, a + b \leq 3d). \]

In [CD1], it is shown that $F^L_i R = (X^a Y^b Z_3^{3d-a-b} | 0 \leq a, b \leq 2d, a + b \leq 3d - i)$ for $i \geq 0$, and hence $F^L_i R$ is finitely generated. In this example, $\text{DF}((\mathcal{X}_{\text{Loe}}, \mathcal{L}_{\text{Loe}}) < 0$ holds as computed in [CD1].

For the Socle filtration $G^S_i R$, we need to compute the invariant part of the action of $U$. By (2.1), an element in $R_d = (X^a Y^b Z_3^{3d-a-b} | 0 \leq a, b \leq 2d, a + b \leq 3d)$ is invariant only if $a = 0$ holds. Hence, we have
\[
G^S_i R_d = R^U_d = (Z_3^d).
\]

For $G^S_i R_d$, we need to consider the action on
\[ R_d/G^S_0 R_d = (X^a Y^b Z_3^{3d-a-b} | 0 \leq a, b \leq 2d, a + b \leq 3d)/(Z_3^d). \]

Since $(R_d/G^S_0 R_d)^U = (X Z_3^{3d-1}, Y Z_3^{3d-1}, Z_3^d)/(Z_3^d)$, we have $G^S_i R_d = (X Z_3^{3d-1}, Y Z_3^{3d-1}, Z_3^d)$. Inductively, it holds that
\[ G^S_i R_d = (X^a Y^b Z_3^{3d-a-b} | 0 \leq a, b \leq 2d, a + b \leq \min\{i, 3d\}) \]

In this example, the Socle filtration is essentially the same as the Loewy filtration. More precisely, $G^S_i R_d = F^L_{3d-i} R_d$ holds for any $i, d$ and hence $(\mathcal{X}_{\text{Soc}}, \mathcal{L}_{\text{Soc}})$ coincides with $(\mathcal{X}_{\text{Loe}}, \mathcal{L}_{\text{Loe}})$.

3. **Toric varieties**

Let $M \cong \mathbb{Z}^n$ be a lattice of rank $n$, and $N$ be the dual lattice of $M$. An $n$-dimensional lattice polytope $P \subset M_\mathbb{R} := M \otimes \mathbb{R}$ is called **reflexive** if $P$ contains $0 \in M$ in its interior and the dual polytope
\[ P^* := \{v \in N_\mathbb{R} := N \otimes \mathbb{R} | \langle u, v \rangle \geq -1 \text{ for any } u \in P\} \]
is a lattice polytope as well. A reflexive polytope $P \subset M_\mathbb{R}$ defines an $n$-dimensional Gorenstein toric Fano variety $X$ by
\[ (X, -K_X) = (\text{Proj} \mathbb{C}[\Gamma_P], \mathcal{O}(1)). \]
where $\Gamma_P = \{(d,u) \in \mathbb{N} \times M \mid u \in dP\}$ and $\mathbb{C}[\Gamma_P] = \bigoplus_{(d,u) \in \Gamma_P} \mathbb{C} \chi^{(d,u)}$ is the semigroup ring graded by $\mathbb{N}$. In particular, it holds that
\[
H^0(X, -dK_X) = \bigoplus_{u \in dP \cap M} \mathbb{C} \chi^{(d,u)}.
\]

In the rest of this section, $P \subset M_\mathbb{R}$ is a reflexive polytope and $X$ is the corresponding Gorenstein toric Fano variety.

3.1. Toric test configurations. Let $f : P \to \mathbb{R}$ be a piecewise linear concave function with rational coefficients. As is well known, $f$ induces a test configuration of $X$ as follows.

Consider a decreasing filtration $\mathcal{F}^i R$ of the section ring $R = \mathbb{C}[\Gamma_P]$ by
\[
\mathcal{F}^i R = \langle \chi^{(d,u)} \mid (d,u) \in \Gamma_P, f(u/d) \geq i/d \rangle.
\]
This filtration $\mathcal{F}^i R$ is multiplicative by the concavity of $f$, and finitely generated since $f$ is piecewise linear with rational coefficients. Hence $\mathcal{F}^i R$ induces a test configuration $(\mathcal{X}_f, \mathcal{L}_f)$ of $X$.

Similarly, a piecewise linear convex function $g : P \to \mathbb{R}$ with rational coefficients induces a finitely generated increasing filtration $\mathcal{G}^i R$ by
\[
\mathcal{G}^i R = \langle \chi^{(d,u)} \mid (d,u) \in \Gamma_P, g(u/d) \leq i/d \rangle.
\]
In particular, $\mathcal{G}^i R$ induces a test configuration $(\mathcal{X}_g, \mathcal{L}_g)$ of $X$.

We note that $(\mathcal{X}_f, \mathcal{L}_f) = (\mathcal{X}_g, \mathcal{L}_g)$ holds if $g = C - f$ for some rational number $C$.

Other than the Donaldson-Futaki invariant $\text{DF}(\mathcal{X}, \mathcal{L})$, there exists another invariant $\text{Ding}(\mathcal{X}, \mathcal{L})$ introduced in [Be], called the Ding invariant of $(\mathcal{X}, \mathcal{L})$, which also can be used to define K-stability.

For toric test configurations, the following formulas are known:

**Theorem 3.1** ([Do1],[Ya, Theorem 5, Proposition 7]). Under the above notation, it holds that
\[
\text{DF}(\mathcal{X}_f, \mathcal{L}_f) = n \left( \frac{1}{\text{vol}(P)} \int_P f(u)du - \frac{1}{\text{vol}(\partial P)} \int_{\partial P} f(u)d\sigma \right),
\]
\[
\text{Ding}(\mathcal{X}_f, \mathcal{L}_f) = f(0) - \frac{1}{\text{vol}(P)} \int_P f(u)du,
\]
\[
\text{DF}(\mathcal{X}_g, \mathcal{L}_g) = n \left( -\frac{1}{\text{vol}(P)} \int_P g(u)du + \frac{1}{\text{vol}(\partial P)} \int_{\partial P} g(u)d\sigma \right),
\]
\[
\text{Ding}(\mathcal{X}_g, \mathcal{L}_g) = -g(0) + \frac{1}{\text{vol}(P)} \int_P g(u)du,
\]
where $du$ is the Euclidean measure on $M_\mathbb{R}$ and $d\sigma$ is the boundary measure on $\partial P$ induced by the lattice $M$. The volumes $\text{vol}(P), \text{vol}(\partial P)$ are with respect to $du, d\sigma$ respectively.

Furthermore, it holds that
\[
\text{DF}(\mathcal{X}_f, \mathcal{L}_f) \geq \text{Ding}(\mathcal{X}_f, \mathcal{L}_f) \quad (\text{resp. } \text{DF}(\mathcal{X}_g, \mathcal{L}_g) \geq \text{Ding}(\mathcal{X}_g, \mathcal{L}_g)),
\]
and the equality holds if and only if $f$ (resp. $g$) is radically affine, where we say that a function $\varphi : P \to \mathbb{R}$ is radically affine if $\varphi(tu) = t(\varphi(u) - \varphi(0))$ for any $t \in [0,1]$ and $u \in P$.

3.2. Automorphism groups. The automorphism group of toric varieties are studied by [De, Co1, Co2, SMS], etc. For simplicity, we only consider the Gorenstein Fano case here.

Let $v_1, \ldots, v_N \in N_\mathbb{R}$ be all the vertices of $P$*. Then we have
\[
P = \{ u \in M_\mathbb{R} \mid \langle u, v_i \rangle \geq -1 \text{ for all } i \}.
\]
We denote by $D_i$ the torus invariant prime divisor on $X$ corresponding to $v_i$.

Let $S = \mathbb{C}[x_1, \ldots, x_N]$, which is called the Cox ring of $X$, be the polynomial ring whose variables correspond to the prime divisors $D_1, \ldots, D_N$ on $X$. Hence a torus invariant effective
Weil divisor $D = \sum a_i D_i$ corresponds to the monomial $x_1^{a_1} x_2^{a_2} \ldots x_N^{a_N} \in S$, which is denoted by $x^D$.

Under this notation, $S$ is the direct sum of $\mathbb{C} x^D$ for all torus invariant effective Weil divisors $D$. Hence the Cox ring is graded by the Chow group $A^1(X)$ of $X$ by

$$S = \bigoplus_{\alpha \in A^1(X)} S_{\alpha} = \bigoplus_{\alpha \in A^1(X)} \left( \bigoplus_{[D]=\alpha} \mathbb{C} x^D \right),$$

where $[D]$ is the class of $D$ in $A^1(X)$.

We note that the monomial $\chi^{(d,u)}(\alpha) \in H^0(X, -dK_X)$ defines an effective torus invariant divisor $\sum_i (\langle u, v_i \rangle + d)D_i \in | -dK_X|$. Thus we can naturally identify $H^0(X, -dK_X)$ with $S_{[-dK_X]}$. Hence the section ring $R = \mathbb{C}[\Gamma_P]$ can be identified with the subring of $S$

$$\bigoplus_{d \geq 0} S_{[-dK_X]} \subset S.$$

**Definition 3.2.** An element $m \in M$ is called a root of $P$ if there exists some $i$ such that $\langle m, v_i \rangle = -1$ and $\langle m, v_j \rangle \geq 0$ for any $j \neq i$. In other words, $m \in M$ is a root if and only if $m$ is contained in the relative interior of a facet $F$ of $P$.

A root $m$ is called semisimple if $-m \in M$ is a root as well. Otherwise, $m$ is called unipotent.

We note that $-m$ is called a root in [De, Co1] for a root $m$ in Definition 3.2. We follow the notation in [Ni].

**Example 3.3.** The reflexive polytope in Figure 1 has two semisimple roots ▲ and two unipotent roots ⋆.

![Figure 1](image-url)

For each root $m \in M$, we have a corresponding one-parameter subgroup $y_m : \mathbb{C} \to \text{Aut}(X)$, and the unipotent radical $U$ of $\text{Aut}(X)$ is generated by $\bigcup_m y_m(\mathbb{C})$, where we take the union over all the unipotent roots of $P$.

Recall the definition of $y_m$. Let $i$ be the unique index with $\langle m, v_i \rangle = -1$ as in Definition 3.2. We note that $D_i$ is linearly equivalent to the effective Weil divisor $D = \sum_{j \neq i} \langle m, v_j \rangle D_j$. For each $\alpha \in \mathbb{C}$, we have an automorphism of $S$ defined by

$$x_i \mapsto x_i + \alpha x^D, \quad x_j \mapsto x_j \quad \text{for} \quad j \neq i,$$

which preserves the $A^1(X)$-grading. This induces an automorphism $y_m(\alpha) \in \text{Aut}(X)$.

4. **Examples**

Let $P \subset M_\mathbb{R}$ be a reflexive polytope and $X$ be the corresponding Gorenstein toric Fano variety. In this section, we only consider examples with the simplest non-trivial unipotent radical, that is, we assume that there exists a unique unipotent root $m$ of $P$ throughout this section. Hence the unipotent radical $U$ of $\text{Aut}(X)$ is isomorphic to $\mathbb{C}$ via the one-parameter subgroup $y_m$. In this case, the Loewy and Socle filtrations and the Donaldson-Futaki invariants of them are described as follows.
Let $F$ be the unique facet of $P$ containing $m$. Without loss of generality, we may assume $M = M' \times \mathbb{Z}$ for $M' \simeq \mathbb{Z}^{n-1}$, $m = (0, -1) \in M' \times \mathbb{Z}$, and $F = F' \times \{-1\}$ for a lattice polytope $F' \subset M'_R$. By [Ni, Lemma 5.9], there exists a piecewise linear concave function $h : F' \to \mathbb{R}$ such that

$$P = \{(u', t) \in F' \times \mathbb{R} \mid -1 \leq t \leq h(u')\}. \tag{4.1}$$

**Example 4.1.** For the reflexive polytope $P \subset \mathbb{R}^2$ in Figure 2, $F' = [-1, 1] \subset \mathbb{R}$ and $h(u') = 1 - |u'|$. 

![Figure 2.](image)

For $u' \in dF' \cap M'$, set

$$R_d^{u'} = \langle \chi^{(d,u)} \mid u = (u', l) \in dP \cap M \rangle = \langle \chi^{(d,u)} \mid u = (u', l) \text{ with } l \in \mathbb{Z}, -d \leq l \leq |dh(u'/d)| \rangle \subset R_d.$$ 

By (4.1), we have a decomposition

$$R_d = \bigoplus_{u' \in dF' \cap M'} R_d^{u'}$$

as vector spaces. In fact, this is a decomposition as $U$-modules by the following lemma:

**Lemma 4.2.** For any $u' \in dF' \cap M'$, $R_d^{u'}$ is a $U$-submodule of $R_d$. Furthermore $R_d^{u'}$ is isomorphic to $V_{dh(u'/d)}$ in Example 2.5 as $U$-modules.

**Proof.** As in §3.2, let $D_1, \ldots, D_N$ be all the torus invariant prime divisors on $X$. We may assume that the facet $F \ni m = (0, -1)$ corresponds to $D_1$. Recall that $\alpha \in \mathbb{C} = U \subset \text{Aut}(X)$ acts on the Cox ring $S$ by

$$x_1 \mapsto x_1 + \alpha x^D, \quad x_i \mapsto x_i \quad \text{for } i \geq 2,$$

where $D = \sum_{i \geq 2} (m, v_i) D_i$.

Fix $u' \in dF' \cap M'$. For simplicity, set $\chi_l = \chi^{(d,u)} \in R_d^{u'}$ for $u = (u', l)$ with $-d \leq l \leq |dh(u'/d)|$. By the identification of $R = \mathbb{C}[\Gamma_P] \oplus \bigoplus_{d \geq 0} S_{[-dK_X]}$ in §3.2, $\chi_l \in R_d^{u'} \subset R$ is identified with

$$X_l := \prod_{i=1}^N x_i^{(u, v_i)+d} \in S.$$ 

Since $v_1 = (0, 1) \in N' \times \mathbb{Z}$, where $N'$ is the dual lattice of $M'$, we have $\langle u, v_1 \rangle + d = l + d$. Hence

$$X_l = x_1^{l+d} \prod_{i=2}^N x_i^{(u, v_i)+d} \in S,$$

which is mapped to

$$(x_1 + \alpha x^D)^{l+d} \prod_{i=2}^N x_i^{(u, v_i)+d}$$
by the action of $\alpha \in \mathbb{C}$. Since $x^D = \prod_{i=2}^{N} x_i^{(m_i, v_i)}$, 
\[(x_1 + \alpha x^D)^{l+d} = \sum_{j=0}^{l+d} \binom{l+d}{j} \alpha^j x_1^{l+d-j} x^D \]
\[= \sum_{j=0}^{l+d} \binom{l+d}{j} \alpha^j x_1^{l+d-j} \prod_{i=2}^{N} x_i^{(m_i, v_i)}.\]

Thus by the action of $\alpha \in \mathbb{C}$, $X_l$ is mapped to 
\[(x_1 + \alpha x^D)^{l+d} \prod_{i=2}^{N} x_i^{(u_i, v_i)+d} = \left( \sum_{j=0}^{l+d} \binom{l+d}{j} \alpha^j x_1^{l+d-j} \prod_{i=2}^{N} x_i^{(u_i+v_i, v_i)+d} \right) \prod_{i=2}^{N} x_i^{(u_i, v_i)+d} \]
\[= \sum_{j=0}^{l+d} \binom{l+d}{j} \alpha^j X_l^{l-j},\]

where the last equality follows from $u + jm = (u', l) + j(0, -1) = (u', l - j)$. In particular, 
\[\langle X_l | -d \leq l \leq \lfloor dh(u'/d) \rfloor \rangle \subset S \text{ is closed under the action of } U. \text{ Hence so is } R_d^u = \langle \chi_l | -d \leq l \leq \lfloor dh(u'/d) \rfloor \rangle \subset R_d, \text{ i.e. } R_d^u \text{ is a } U\text{-submodule.}

By the above argument, 
\[R_d^u \rightarrow V_{\lfloor dh(u'/d) \rfloor + d} : \chi_l \mapsto x^{l+d}\]
is an isomorphism as $U$-modules. \hfill \Box

**Lemma 4.3.** Under the above setting, for $u = (u', l) \in dP \cap M$, $\chi^{(d,u)} \in R_d$ is contained in $F^l_i R_d$ if and only if $l \leq dh(u'/d) - i$.

On the other hand, $\chi^{(d,u)} \in R_d$ is contained in $G^S_i R_d$ if and only if $l \leq i - d$.

**Proof.** By (4.1), this lemma holds for $i < 0$. Hence we may assume $i \geq 0$.

We use the notation of the proof of Lemma 4.2. By Lemma 4.2, we have a decomposition 
\[R_d = \bigoplus_{u' \in dP \cap M'} R_d^{u'}\]
as $U$-modules. Hence $F^l_i R_d = \bigoplus_{u' \in dP \cap M'} F^l_i R_d^{u'}$ holds.

Since $F^l_i V_{\lfloor dh(u'/d) \rfloor + d} = \langle x^l | 0 \leq j \leq \lfloor dh(u'/d) \rfloor + d - i \rangle$ by Example 2.5, we have
\[F^l_i R_d^{u'} = \langle \chi_l | -d \leq l \leq \lfloor dh(u'/d) \rfloor - i \rangle\]
by (4.2). Thus $\chi_l = \chi^{(d,u)}$ for $u = (u', l)$ is contained in $F^l_i R_d$ if and only if $l \leq \lfloor dh(u'/d) \rfloor - i$, which is equivalent to $l \leq dh(u'/d) - i$ since $l$ and $i$ are integers.

Similarly, we have $G^S_i R_d = \bigoplus_{u' \in dP \cap M'} G^S_i R_d^{u'}$ and 
\[G^S_i R_d^{u'} = \langle \chi_l | -d \leq l \leq -d + i \rangle.\]
Hence $\chi_l = \chi^{(d,u)}$ is contained in $G^S_i R_d$ if and only if $l \leq -d + i$. \hfill \Box

**Proposition 4.4.** Under the above setting, the Loewy filtration $F^l_i R$ of $X$ coincides with the decreasing toric filtration $F^l_i R$ induced by the concave function $f$ defined as 
\[f : P \rightarrow \mathbb{R}, \quad (u', t) \mapsto h(u') - t.\]

On the other hand, the Socle filtration $G^S_i R$ of $X$ coincides with the increasing toric filtration $G^S_i R$ induced by the affine (hence convex) function $g$ defined as 
\[g : P \rightarrow \mathbb{R}, \quad (u', t) \mapsto t + 1.\]
Proof. By the definition of toric filtrations, for \( u = (u', l) \in dP \cap M \), \( \chi^{(d,u)} \in R_d \) is contained in \( F'_l R_d \) if and only if
\[
i/d \leq f(u/d) = f(u'/d, l/d) = h(u'/d) - l/d,
\]
which is equivalent to \( l \leq dh(u'/d) - i \). Hence \( F'_l R \) coincides with the Loewy filtration \( F'_l R \) by Lemma 4.3.

Similarly, \( \chi^{(d,u)} \in R_d \) is contained in \( G^s_l R_d \) if and only if
\[
i/d \geq g(u/d) = g(u'/d, l/d) = l/d + 1,
\]
which is equivalent to \( l \leq i - d \). Hence \( G^s_l R \) coincides with the Socle filtration \( G^s_l R \) by Lemma 4.3. \( \square \)

Since \( P = \{(u', t) \in F' \times \mathbb{R} \mid -1 \leq t \leq h(u')\} \), roughly Proposition 4.4 states that the Loewy (resp. Socle) filtration is determined by the distance from the top facets of \( P \) defined by \( h \) (resp. the distance from the bottom facet \( F = F' \times \{-1\} \)).

By Proposition 4.4, both \( F'_l R \) and \( G^s_l R \) are finitely generated, and hence induce test configurations \((X_{\text{Loc}}, L_{\text{Loc}})\) and \((X_{\text{Soc}}, L_{\text{Soc}})\), respectively. By Theorem 3.1, we can compute the Donaldson-Futaki invariant and the Ding invariant of these test configurations as follows:

**Corollary 4.5.** It holds that
\[
\text{Ding}(X_{\text{Loc}}, L_{\text{Loc}}) = \frac{1}{\text{vol}(P)} \int_P (h(0) - h(u') + t) du' dt,
\]
\[
\text{DF}(X_{\text{Soc}}, L_{\text{Soc}}) = \text{Ding}(X_{\text{Soc}}, L_{\text{Soc}}) = \frac{1}{\text{vol}(P)} \int_P t du' dt.
\]
If \( h : F' \to \mathbb{R} \) is radically affine, \( \text{DF}(X_{\text{Loc}}, L_{\text{Loc}}) = \text{Ding}(X_{\text{Loc}}, L_{\text{Loc}}) \) holds.

**Proof.** This follows from Theorem 3.1 and Proposition 4.4. We note that \( g(u', t) = t + 1 \) is affine, in particular, radically affine. On the other hand, \( f(u', t) = h(u') - t \) is radically affine if and only if so is \( h \). \( \square \)

In all the following examples, \( h \) is radically affine and hence \( \text{DF}(X_{\text{Loc}}, L_{\text{Loc}}) = \text{Ding}(X_{\text{Loc}}, L_{\text{Loc}}) \) holds.

4.1. A singular toric del Pezzo surface. In this subsection, we give a counterexample to Conjecture 1.2 with singular \( X \).

Let \( P \subset \mathbb{R}^2 \) be the reflexive polytope in Figure 2. We note that the corresponding \( X \) is a singular del Pezzo surface of degree 6 with one ordinary double point. In this case, \( F' = [-1, 1] \) and \( h : F' \to \mathbb{R} \) is defined by \( h(x) = 1 - |x| \) as stated in Example 4.1. Since \( h \) is radically affine, we have
\[
\text{DF}(X_{\text{Loc}}, L_{\text{Loc}}) = \frac{1}{\text{vol}(P)} \int_P (|x| + t) dx dt = \frac{2}{9} > 0,
\]
\[
\text{DF}(X_{\text{Soc}}, L_{\text{Soc}}) = \frac{1}{\text{vol}(P)} \int_P t dx dt = -\frac{2}{9} < 0.
\]
by Corollary 4.5. Hence the Loewy filtration does not destabilize \( X \), but the Socle filtration does.

4.2. A smooth toric Fano 3-fold. In this subsection, we show Theorem 1.3, i.e. we give a counterexample to Conjecture 1.2 with smooth \( X \).

The reflexive polytope
\[
F' = \text{Conv}((1, 1), (0, 1), (-2, -1), (1, -1)) \subset \mathbb{R}^2
\]
in Figure 3 corresponds to the Hirzebruch surface \( \Sigma_1 = \mathbb{P}^1(\mathcal{O} \oplus \mathcal{O}(-1)) \). Let \( X \) be the smooth toric Fano 3-fold obtained as the blow-up of \( \Sigma_1 \times \mathbb{P}^1 \) along \( C \times \{p\} \), where \( C \subset \Sigma_1 \) is the
torus invariant section with \((C^2) = 1\) and \(p \in \mathbb{P}^1\) is a torus invariant point. Since \(\Sigma_1 \times \mathbb{P}^1\) corresponds to \(F' \times [-1,1]\), the polytope \(P\) corresponding to \(X\) is written as

\[
P = \left\{ (x,y,t) \in F' \times \mathbb{R} \subset \mathbb{R}^3 \mid -1 \leq t \leq h(x,y) := \min\{1, 1 + y\} \right\}.
\]

We note that \(P\) has two semisimple roots and one unipotent root \(m = (0,0,-1)\).

\[\begin{align*}
F' & \quad F' \times [-1,1] & P
\end{align*}\]

**Figure 3.**

Since \(h\) is radically affine, we have

\[
DF(\mathcal{X}_{\text{Loe}}, \mathcal{L}_{\text{Loe}}) = \frac{1}{\text{vol}(P)} \int_P (\max\{0,-y\} + t) \, dx \, dy \, dt = \left(\frac{20}{3}\right)^{-1} \cdot \left(\frac{7}{8}\right) = \frac{21}{160},
\]

\[
DF(\mathcal{X}_{\text{Soc}}, \mathcal{L}_{\text{Soc}}) = \frac{1}{\text{vol}(P)} \int_P t \, dx \, dy \, dt = \left(\frac{20}{3}\right)^{-1} \cdot \left(-\frac{7}{8}\right) = -\frac{21}{160}.
\]

**Proof of Theorem 1.3.** The above \(X\) satisfies the conditions in the theorem. \(\square\)

### 4.3. A singular toric Fano 3-fold

For examples in Subsections 4.1 and 4.2, the invariant \(DF(\mathcal{X}_{\text{Soc}}, \mathcal{L}_{\text{Soc}})\) is negative, and hence the Socle filtration destabilizes \(X\).

As we will see in Appendix, the Socle filtration is the filtration induced from a valuation on the function field of \(X\), and hence multiplicative in general. Thus we might expect that the Socle filtration destabilizes any \(\mathbb{Q}\)-Fano varieties.

However, the answer is no, at least for singular \(X\). The following is an example of a Gorenstein toric Fano 3-fold with non-reductive automorphism group such that \(DF(\mathcal{X}_{\text{Soc}}, \mathcal{L}_{\text{Soc}}) = \text{Ding}(\mathcal{X}_{\text{Soc}}, \mathcal{L}_{\text{Soc}}) > 0\).

Let \(F' \subset \mathbb{R}^2\) be the hexagon with vertexes \((1,0),(0,1),(-1,1),(-1,0),(0,-1),(1,-1)\). We define a function \(h : F' \to \mathbb{R}\) by

\[
h(x,y) = \begin{cases} 1 - 2x & (x \geq 0) \\ 1 - x & (x \leq 0) \end{cases}
\]

for \((x,y) \in F'\). The polytope \(P \subset \mathbb{R}^3\) in Figure 4 is defined by \(h\) and (4.1). We can check that \(P\) is reflexive, and \(m = (0,0,-1) \in P\) is the unique unipotent root. By Corollary 4.5, we can compute

\[
DF(\mathcal{X}_{\text{Loe}}, \mathcal{L}_{\text{Loe}}) = \left(\frac{16}{3}\right)^{-1} \cdot \left(\frac{3}{8}\right) = -\frac{9}{128} < 0,
\]

\[
DF(\mathcal{X}_{\text{Soc}}, \mathcal{L}_{\text{Soc}}) = \left(\frac{16}{3}\right)^{-1} \cdot \frac{3}{8} = \frac{9}{128} > 0.
\]
Appendix A. On Socle filtrations

Let \( R = \bigoplus_{d=0}^{\infty} R_d \) be a finitely generated graded integral \( \mathbb{C} \)-algebra and set \( X = \text{Proj} \, R \). We do not assume that \( X \) is Fano. Let \( U \) be a unipotent algebraic group which acts on \( R \) as a graded \( \mathbb{C} \)-algebra. By exactly the same definition as Definition 2.3, we can define the Socle filtration \( G_c^SR \) of \( R \).

Recall that an increasing filtration \( \mathcal{G}_c \) is multiplicative if and only if \( \mathcal{G}_c R \cdot \mathcal{G}_j R \subset \mathcal{G}_{i+j} R \) holds for any \( i, j \).

**Lemma A.1.** Under the above setting, the Socle filtration \( G_c^SR \) is multiplicative.

To show this lemma, we use the Lie algebra \( u \) of \( U \). Since \( U \) acts on \( R \) as a \( \mathbb{C} \)-algebra, any \( D \in u \) acts on \( R \) as a \( \mathbb{C} \)-derivation, i.e. \( Dc = 0 \) for any \( c \in \mathbb{C} \) and

\[
D(xy) = (Dx)y + x(Dy)
\]

holds for any \( x, y \in R \). By induction, for any \( D_1, \ldots, D_N \in u \) it holds that

\[
D_N \cdots D_1(xy) = \sum_{(\varepsilon_1, \ldots, \varepsilon_N) \in \{0,1\}^N} (D_N^{\varepsilon_N} \cdots D_1^{\varepsilon_1} x)(D_1^{1-\varepsilon_1} \cdots D_N^{1-\varepsilon_N} y),
\]

where \( D^0x = x \) by convention.

**Lemma A.2.** For any \( i, d \geq 0 \), it holds that

\[
G_i^SR_d = \{ x \in R_d \mid D_{i+1} \cdots D_1 x = 0 \text{ for any } D_1, \ldots, D_{i+1} \in u \}.
\]

**Proof.** For \( i = 0 \), \( x \in R_d \) is contained in the invariant part \( G_0^SR_d = (R_d)^U \) if and only if \( Dx = 0 \) for any \( D \in u \). Hence the statement holds for \( i = 0 \). By the induction on \( i \), this lemma follows. \( \square \)

**Proof of Lemma A.1.** Take \( x \in G_i^SR \) and \( y \in G_i^SR \) for \( i, j \in \mathbb{Z} \). We need to show \( xy \in G_{i+j}^SR \). Since \( G_k^SR = \{0\} \) for \( k < 0 \) by definition, \( xy = 0 \in G_{i+j}^SR \) holds if \( i \) or \( j \) is negative. Hence we may assume \( i, j \geq 0 \).

Set \( N = i + j + 1 \) and take any \( D_1, \ldots, D_N \in u \). It suffices to show \( D_N \ldots D_1(xy) = 0 \) by Lemma A.2. For each \( (\varepsilon_1, \ldots, \varepsilon_N) \in \{0,1\}^N \), one of \( \sum \varepsilon_k \geq i + 1 \) or \( \sum (1-\varepsilon_k) \geq j + 1 \) must hold. Hence \( D_N^{\varepsilon_N} \cdots D_1^{\varepsilon_1} x = 0 \) or \( D_1^{1-\varepsilon_1} \cdots D_N^{1-\varepsilon_N} y = 0 \) holds by Lemma A.2. By (A.1), we have \( D_N \ldots D_1(xy) = 0 \). \( \square \)

In fact, we can show the following proposition, which refines Lemma A.1.

**Proposition A.3.** Let \( x \in G_i^SR \setminus G_{i-1}^SR \) and \( y \in G_j^SR \setminus G_{j-1}^SR \) for \( i, j \geq 0 \). Then \( xy \in G_{i+j}^SR \setminus G_{i+j-1}^SR \) holds.
Proof. Since \( xy \in G_{i+j}^S \) by Lemma A.1, what we need to show is \( xy \not\in G_{i+j-1}^S R \). By Lemma A.2, it is enough to find \( D_1, \ldots, D_{i+j} \in R \) such that \( D_{i+j} \cdots D_1(xy) \neq 0 \).

Consider the set \( \Phi \) which consists of sequences of non-negative integers \( (a_k)_{k=1}^\infty \) satisfying

- \( \sum_{k=1}^\infty a_k = i \). In particular, there exists \( m \) such that \( a_k = 0 \) for any \( k \geq m + 1 \).
- For the above \( m \), there exist \( D_1, \ldots, D_m \in R \) such that \( D_m^{a_m} \cdots D_1^{a_1} x \neq 0 \).

We note that \( a_k \) could be zero even if \( k \leq m \). For simplicity, we denote \( (a_k)_{k=1}^\infty \) by \( (a_1, \ldots, a_m) \) if \( a_k = 0 \) for any \( k \geq m + 1 \).

Since \( x \not\in G_{i+1}^S R \), \( D_i \cdots D_1(x) \neq 0 \) for some \( D_1, \ldots, D_i \in R \). Hence \((1, \ldots, 1)\) is contained in \( \Phi \). In particular, \( \Phi \neq \emptyset \).

Let \( (a_1, \ldots, a_m) = (a_1, \ldots, a_m, 0, 0, \ldots) \in \Phi \) be the maximum element with respect to the lexicographical order. Take and fix \( D_1, \ldots, D_m \in R \) with \( D_m^{a_m} \cdots D_1^{a_1} x \neq 0 \).

Consider another set \( \Phi' \subset \mathbb{N}^m \) defined as follows: \( (a'_1, \ldots, a'_m) \in \mathbb{N}^m \) is contained in \( \Phi' \) if and only if

- \( n := j - (a'_1 + \cdots + a'_m) \geq 0 \) and \( D'_n \cdots D'_1 D_m^{a_m} \cdots D_1^{a_1} y \neq 0 \) for some \( D'_1, \ldots, D'_n \in R \).

Since \( y \not\in G_{j-1}^S R \), \( D'_j \cdots D'_1 y \neq 0 \) for some \( D'_1, \ldots, D'_j \). Hence \((0, \ldots, 0)\) is contained in \( \Phi' \).

In particular, \( \Phi' \neq \emptyset \).

Let \( (a'_1, \ldots, a'_m) \in \Phi' \) be the maximum element with respect to the lexicographical order. Take and fix \( D'_1, \ldots, D'_n \in R \) with \( D'_n \cdots D'_1 D_m^{a_m} \cdots D_1^{a_1} y \neq 0 \) for \( n = j - (a'_1 + \cdots + a'_m) \).

To prove \( xy \not\in G_{i+j-1}^S R \), it suffices to show

\[
(D'_n \cdots D'_1 D_m^{a_m} \cdots D_1^{a_1} (xy)) \neq 0
\]

since \( \sum_{i=1}^m (a_i + a'_i) + n = \sum_{i=1}^m a_i + (n + \sum_{i=1}^m a'_i) = i + j \).

By (A.1), \( D'_n \cdots D'_1 D_m^{a_m} \cdots D_1^{a_1} (xy) \) is equal to

\[
\sum_{\alpha, \varepsilon} c_{\alpha, \varepsilon} (D'_n^{\varepsilon_n} \cdots D'_1^{\varepsilon_1} D_m^{a_m} \cdots D_1^{a_1} x)(D'_n^{1-\varepsilon_n} \cdots D'_1^{1-\varepsilon_1} D_m^{a_m-a_m} \cdots D_1^{a_1+a'_1-\alpha_1} y)
\]

where the sum is taken over all \( (\alpha, \varepsilon) = (\alpha_1, \ldots, \alpha_m, \varepsilon_1, \ldots, \varepsilon_n) \) with

\[
\alpha_i \in \{0, 1, \ldots, a_i + a'_i\}, \quad \varepsilon \in \{0, 1\}^n
\]

and the coefficient \( c_{\alpha, \varepsilon} \in \mathbb{N} \) is

\[
c_{\alpha, \varepsilon} = \prod_{i=1}^m \binom{a_i + a'_i}{a_i}.
\]

If \( \sum_{i=1}^m a_i + \sum_{j=1}^n \varepsilon_j > i \), it holds that \( D'_n^{\varepsilon_n} \cdots D'_1^{\varepsilon_1} D_m^{a_m} \cdots D_1^{a_1} x \) is \( 0 \) by \( x \in G_i^S R \). If \( \sum_{i=1}^m a_i + \sum_{j=1}^n \varepsilon_j < i \), \( D'_n^{1-\varepsilon_n} \cdots D'_1^{1-\varepsilon_1} D_m^{a_m-a_m} \cdots D_1^{a_1+a'_1-\alpha_1} y \) is \( 0 \) by \( y \in G_i^S R \).

Hence it suffices to take the sum in (A.3) over \( (\alpha, \varepsilon) \) with

\[
\sum_{i=1}^m a_i + \sum_{j=1}^n \varepsilon_j = i.
\]

Assume that \( (\alpha, \varepsilon) \) satisfies (A.4). By the definition of \( \Phi' \), \( D'_n^{\varepsilon_n} \cdots D'_1^{\varepsilon_1} D_m^{a_m} \cdots D_1^{a_1} x \) is \( 0 \) if \( (\alpha_1, \ldots, \alpha_m, \varepsilon_1, \ldots, \varepsilon_n) \not\in \Phi' \). Since \((a_1, \ldots, a_m) \in \Phi' \) is the maximum element, it suffices to take the sum in (A.3) over \( (\alpha, \varepsilon) \) with (A.4) and

\[
(\alpha_1, \ldots, \alpha_m, \varepsilon_1, \ldots, \varepsilon_n) \leq (a_1, \ldots, a_m).
\]
By the definition of $\Phi'$, $D_n^{-1} \cdots D_1^{-1} D_m^a + a'_m - a_m \cdots D_1^{a_1 + a'_1 - a_1} = 0$ if $(a_1 + a'_1 - a_1, \ldots, a_m + a'_m - a_m) \in \Phi'$. Since $(a_1', \ldots, a_m')$ is the maximum element, it suffices to take the sum in (A.3) over $(\alpha, \varepsilon)$ with (A.4), (A.5) and (A.6)

\[
(\alpha_1 + a'_1 - \alpha_1, \ldots, a_m + a'_m - a_m) \leq (a_1', \ldots, a_m').
\]

Assume that the index $(\alpha, \varepsilon)$ satisfies (A.4), (A.5), and (A.6). Then $\alpha_1 \leq a_1$ and $a_1 + a'_1 - \alpha_1 \leq a'_1$ hold. Hence $\alpha_1$ must be $a_1$.

Since $\alpha_1 = a_1$, we have $a_2 \leq a_2$ and $a_2 + a'_2 - \alpha_2 \leq a'_2$, which imply $\alpha_2 = a_2$. Repeating this, $(\alpha_1, \ldots, \alpha_m)$ must coincide with $(a_1, \ldots, a_m)$. By (A.4) and $\sum_{i=1}^m a_i = i$, we have $\varepsilon = (0, \ldots, 0)$.

After all, the index $(\alpha, \varepsilon)$ which we need to take is only $((a_1, \ldots, a_m), (0, \ldots, 0))$. Hence

\[
D_1' \cdots D_n' D_m^a + a'_m \cdots D_1^{a_1 + a'_1} (xy) = c_{(a_1, \ldots, a_m), (0, \ldots, 0)} (D_m^a \cdots D_1^{a_1} x)(D_n' \cdots D_1') y,
\]

which is nonzero since both $D_m^a \cdots D_1^{a_1} x$ and $D_n' \cdots D_1'$ are non-zero elements in the integral domain $R$, and $c_{(a_1, \ldots, a_m), (0, \ldots, 0)} \neq 0$. Thus $xy \notin G^{S}_{i+j-1} R$ follows.

Proposition A.3 implies that the Socle filtration induces a valuation on the function field of $X$ as follows.

**Definition A.4.** For $x \in R$, we set

\[
\iota(x) = \inf \{i \in \mathbb{Z} \mid x \in G_i^S R \} \in \{-\infty\} \cup \mathbb{N}.
\]

We note that $\{i \in \mathbb{Z} \mid x \in G_i^S R \} \neq \emptyset$ for any $x \in R$ since $\bigcup G_i^S R = R$, and $\iota(x) = -\infty$ if and only if $x = 0$, and $\iota(c) = 0$ for $c \neq 0 \in \mathbb{C} \subset R_0$. For $x, y \in R$, $\iota(xy) = \iota(x) + \iota(y)$ holds by Proposition A.3.

**Definition A.5.** Let $K(X)$ be the function field of $X$. We define a function $v : K(X) \to \mathbb{Z} \cup \{\infty\}$ by

\[
v\left(\frac{x}{y}\right) = -\iota(x) + \iota(y)
\]

for $d \geq 0, x, y \in R_d, y \neq 0$.

**Corollary A.6.** The above function $v$ is well-defined. Furthermore, $v$ is a valuation which is trivial on $\mathbb{C}$.

**Proof.** For the well-definedness, we need to check

1. For $x, y \in R_d, y \neq 0$, $-\iota(x) + \iota(y)$ is an integer.
2. If $x/y = x'/y' \in K(X)$, it holds that $-\iota(x) + \iota(y) = -\iota(x') + \iota(y')$.

As in Definition A.4, $\iota(y) \in \mathbb{N}$ if $y \neq 0$. Since $-\iota(x)$ is in $\mathbb{Z} \cup \{\infty\}$, we have $-\iota(x) + \iota(y) \in \mathbb{Z} \cup \{\infty\}$. Thus (1) follows.

For (2), if $x/y = x'/y' \in K(X)$, we have $xy' = x'y \in R$. Then

\[
\iota(x) + \iota(y') = \iota(xy') = \iota(x'y) = \iota(x') + \iota(y)
\]

by Proposition A.3. Hence $-\iota(x) + \iota(y) = -\iota(x') + \iota(y')$ holds. Thus $v$ is well-defined.

By Lemma A.1, Lemma A.2 and Proposition A.3, we can check that $v$ satisfies the definition of valuation, i.e.

- $v(0) = 0$ and $v(x) \neq 0$ for $x \in K(X) \setminus 0$.
- $v(x + y) \geq \min\{v(x), v(y)\}$ for $x, y \in K(X)$, with equality if $v(x) \neq v(y)$.
- $v(xy) = v(x) + v(y)$ for $x, y \in K(X)$.
- $v(a) = 0$ for $a \in \mathbb{C} \setminus 0$.

\[\square\]
Example A.7. Let $P$ be a reflexive polytope with a unique unipotent root $m = (0, -1) \in \mathbb{M}' \times \mathbb{Z}$ such that

$$P = \{(u', t) \in F' \times \mathbb{R} | -1 \leq t \leq h(u')\}$$

for some $F', h$ as in §4. In this case, the valuation induced by the Socle filtration $G^*_S R$ is the toric valuation corresponding to $(0, -1) \in \mathbb{N}' \times \mathbb{Z}$. We note that this is not the divisorial valuation $\text{ord}_D$, which corresponds to $(0, 1) \in \mathbb{N}' \times \mathbb{Z}$, for the prime divisor $D \subset X$ corresponding to the facet $F = F' \times \{-1\}$ of $P$.

For example, for the singular del Pezzo surface in §4.1, the valuation $v$ is nothing but the divisorial valuation $\text{ord}_E$, where $E$ is the exceptional divisor of the blow-up of the ordinary double point in $X$.

Recall that the function $g$ in §4 corresponding to the Socle filtration is not only concave but also affine, contrary to the convex function $f$. The affineness is due to Corollary A.6.

The author does not know whether or not the valuation $v$ is the divisorial valuation for some prime divisor over $X$ in general.

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Graduate School of Mathematics, Nagoya University, Nagoya, Japan
E-mail address: atsushi.itoh@math.nagoya-u.ac.jp