Holonomy corrections in the effective equations for scalar mode perturbations in loop quantum cosmology

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Abstract

We study the dynamics of the scalar modes of linear perturbations around a flat, homogeneous and isotropic background in loop quantum cosmology. The equations of motion include quantum geometry effects and are expected to hold at all curvature scales so long as the wavelengths of the inhomogeneous modes of interest remain larger than the Planck length. These equations are obtained by including holonomy corrections in an effective Hamiltonian and then using the standard variational principle. We show that the effective scalar and diffeomorphism constraints are preserved by the dynamics. We also make some comments regarding potential inverse triad corrections.

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1. Introduction

The aim of loop quantum cosmology (LQC) is to use the methods and techniques of loop quantum gravity in order to study the dynamics of cosmological spacetimes when the spacetime curvature is of the order of the Planck scale and general relativity breaks down. For a recent comprehensive review of LQC, see [1]. There has been great progress in LQC over the last few years, especially with regard to the simplest cosmological models, the homogeneous and isotropic Friedmann–Lemaître–Robertson–Walker (FLRW) spacetimes. The dynamics of the quantum theory have been studied extensively, using both analytical and numerical methods, and it has been found that the big bang singularity is resolved. In essence, quantum geometry effects introduce a strong repulsive force, which causes a ‘bounce’: there is a quantum gravity bridge between an earlier classical contracting universe and a later classical expanding universe [2, 3].

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Now that the homogeneous sector of LQC is relatively well understood, the next step is to allow for small perturbations around a homogeneous and isotropic background and study how the presence of quantum geometry effects modify their dynamics when the spacetime curvature is near the Planck scale. In this work, we will focus on scalar perturbations; these modes are of particular interest in cosmology as they seed structure formation. With the results presented here, it will be possible to study the dynamics of scalar perturbations through the bounce for modes whose wavelengths remain larger than the Planck length.

Since these results are of interest to researchers who are not experts in LQC, we have tried to ensure that this paper is as easy to read for nonspecialists as possible. If the reader wishes to skip directly to the main result, the quantum-gravity-corrected equations for scalar perturbations around a flat homogeneous and isotropic background (presented in a notation familiar to cosmologists) are given in section 3.2.

In this paper, we will work with effective equations. Effective equations are obtained from a Hamiltonian constraint, which has been modified in an appropriate way in order to incorporate quantum gravity effects. For homogeneous and isotropic models, effective equations provide an excellent approximation to the full quantum dynamics of sharply peaked states, even at the bounce point when the quantum gravity effects are strongest [4], and we expect that this will continue to be true if small perturbations are allowed.

There are two main types of corrections due to quantum gravity effects in LQC: holonomy corrections and inverse triad corrections. Holonomy corrections arise as the connection is expressed in terms of holonomies of a minimal length and these corrections become important when the spacetime curvature nears the Planck scale. Inverse triad corrections are introduced in order to obtain well-defined operators corresponding to inverse powers of the area operator (this is necessary as 0 is in the discrete spectrum of the area operator) and they play an important role when physical length scales become as small as the Planck length. We will only consider holonomy corrections here so that the equations hold for all curvature scales, but additional inverse triad corrections are necessary if any of the physical length scales of the spacetime become comparable to the Planck length.

Previous work studying cosmological perturbations in LQC at the effective level has followed a prescription, originally presented in [5], where the background and perturbation degrees of freedom are separated at the very beginning in the Hamiltonian constraint and the symplectic structure. This is certainly a possible approach to the problem, but it seems to be an unnatural way to try to incorporate nonperturbative quantum gravity corrections. For this reason, in this work we will separate the variables into the background and perturbation parts only after the equations of motion have been determined. Once it is clear how to incorporate the holonomy corrections properly, it would of course be possible to go back and do it again using perturbed variables from the beginning, but the point is that the problem becomes less mysterious if one works with nonperturbed variables.

Vector and tensor modes were the first to be studied using effective equations in LQC [6–8]. This was done using perturbed variables as in [5] and both holonomy and inverse triad corrections have been computed. Scalar modes have also been studied in [9, 10], where inverse triad corrections were included, and in [11, 12], where holonomy corrections were considered. Finally, there have also been some preliminary results about the phenomenology of quantum gravity effects that could potentially be detected in primordial gravitational wave signatures [13, 14] and in the cosmic microwave background [15, 16]. As the subtleties of the effective equations are better understood, phenomenological studies will become more robust and, we hope, will lead to falsifiable predictions.
Scalar perturbations are the most interesting ones from an observational point of view, and therefore, it is important to understand how the equations of motion of the scalar perturbations are modified by quantum geometry effects. Since in homogeneous models, where the quantum dynamics are well understood [1–3], modifications due to holonomy corrections are significantly more important than those coming from inverse triad effects, we might expect this to continue to be true when small perturbations are allowed. This is why the study of holonomy corrections to the scalar modes of cosmological perturbations is important.

While there has been some previous work studying holonomy corrections for scalar perturbations [11, 12], there remain many open problems. In [11], only large wavelength modes were considered, and therefore all spatial derivative terms in the equations of motion were neglected. However, it is not clear that the wavelength of the modes observed today in the cosmic microwave background would necessarily have been large when the holonomy corrections are important (i.e. when the curvature nears the Planck scale). In some scenarios, such as inflation, this assumption clearly fails. Because of this, the results presented in [11] provide only partially corrected equations of motion for scalar perturbations. More recently, a further study has appeared where an effective Hamiltonian, including holonomy corrections, is studied [12]. However, the equations obtained from the effective Hamiltonian constraint are not consistent and one must add an ad hoc, non-unique modification to one of them in order to ensure that the constraints are preserved by the dynamics. It is not clear why the chosen modification should be preferred over any other. In this work, we will rectify the shortcomings of the previous studies of holonomy corrections to scalar perturbations by working with all wavelength modes and by introducing an effective Hamiltonian whose equations of motion automatically preserve the constraints.

As mentioned above, we will work with nonperturbed variables in the effective Hamiltonian constraint and symplectic structure. As we shall see, by doing this, it will become clearer how to incorporate holonomy corrections from an LQC point of view. However, in order to work with nonperturbed variables in LQC, it is essential to have a diagonal metric. If the metric is not diagonal, a key mathematical simplification is lost: in homogeneous models, it is enough to only consider a small class of holonomies, called almost periodic functions. This is a great simplification that is not available in general, but can be generalized in a straightforward way to include spacetimes where the metric can be put in a diagonal form. Therefore, we will work in the longitudinal gauge where the metric is diagonal. This gauge choice is absolutely necessary if one only wishes to consider almost periodic holonomies: if one works in a different gauge (or without choosing a gauge), then one must work with a much larger class of holonomies and the problem becomes much more complicated. We will discuss this further in section 3.

The outline of this paper is as follows: in section 2, we present a first-order Hamiltonian formalism for cosmological perturbation theory, which gives the standard equations for the scalar modes in general relativity. Then, in section 3, we modify the Hamiltonian in a suitable fashion in order to include holonomy corrections; this gives the quantum-gravity-corrected equations of motion for scalar mode perturbations. We also make some comments about inverse triad corrections in section 4 before ending with a discussion in section 5.

In this paper, we will work in units where $c = 1$ but we will keep $G$ and $\hbar$ explicit so that it will be possible to see whether a contribution is due to gravitational effects or quantum effects, or both. We will define the Planck length as $\ell_{Pl} = \sqrt{G\hbar}$. Since we are only considering linear perturbations, all terms that are of second order (or higher) in the perturbations
will be dropped, and thus, all of the equations are understood to hold up to first order in perturbations.

2. First-order Hamiltonian framework

In order to study small fluctuations around the flat FLRW cosmological background on a 3-torus, one allows small departures from homogeneity. A nice coordinate choice is the longitudinal gauge in which case the metric is given by

\[ ds^2 = -(1 + 2\psi)\, dt^2 + a^2(1 - 2\psi)\, d\vec{x}^2, \]

where \( a(t) \) is the scale factor and depends only on time, while \( \psi(x, t) \) encodes the fluctuations away from the mean scale factor and varies both with time and position. We have chosen the line element so that the volume of the 3-torus, with respect to the background metric given by \( ds^2 = d\vec{x}^2 \), is 1. Of course, one is free to make a different choice and one can check that this choice does not affect the results of this work.

Let us justify the choice of the longitudinal gauge. This gauge choice is particularly useful as the resulting metric is diagonal and this considerably simplifies the situation as it will be possible to make a direct analogy with the homogeneous case without first perturbing the Hamiltonian constraint. In effect, a diagonal metric allows us to restrict our attention to holonomies that are almost periodic in the connection. It is precisely this simplification that has allowed the vast amount of progress in LQC over the last few years. This is also why it is difficult to include perturbations. Typically, this means working with off-diagonal terms in the metric, which must either be treated in a perturbative manner, or one must consider a more general class of holonomies than almost periodic functions of the connection. However, as we shall see in section 3, the longitudinal gauge allows us to avoid these two problems in the case of scalar perturbations.

2.1. Elementary variables

In order to use a first-order formalism, it is necessary to work with triads and co-triads instead of a metric. In the longitudinal gauge, the co-triads are given by

\[ \omega_i^a = a(1 - \psi)\, \delta_i^a, \]

and the spatial metric is then given by \( q_{ab} = \omega_i^a \omega_i^b \). The spacetime indices \( a, b, c, \ldots \) are raised and lowered by \( q_{ab} \), while the internal indices \( i, j, k, \ldots \) are raised and lowered by \( \delta_{ij} = \text{diag}(1, 1, 1) \). Similarly, their inverse triads are given by

\[ e_i^a = \frac{1}{a}(1 + \psi)\, \delta_i^a. \]

Note that since \( \psi \ll 1 \), we can drop all terms of the order of \( \psi^2 \) or higher. The fiducial triads and co-triads used in the equations above are defined as

\[ \hat{e}_i^a = \left( \frac{\partial}{\partial x^i} \right)^a, \quad \hat{\omega}_i^a = (dx^i)_a. \]

There exists a derivative operator \( D \) compatible with the triads and co-triads:

\[ D_a e_i^b = \hat{\omega}_a e_i^b + \Gamma^b_{ac} e_i^c + \epsilon_{ij}^k \Gamma^c_{dk} e_i^h = 0, \]

In this work, we consider the case of a massless scalar field. In this case (and for all other perfect fluids), there is no anisotropic stress, and therefore, the variable \( \phi \) encoding the fluctuations around the lapse is necessarily equal to the variable \( \psi \), which describes the deviations away from the scale factor. In some cases, this fails to hold in a quantum-corrected effective setting [10], but as we shall see the equations we obtain at the end of the following analysis are consistent and it does not appear to be necessary to allow for departures from \( \phi = \psi \) here.
where $\Gamma^b_{ac}$ is the usual Christoffel connection and $\Gamma^i_a$ is the spin connection:

$$\Gamma^i_a = -\epsilon^{ijk}\epsilon^b_j(\partial_a \omega_{bk} + \frac{1}{2} \epsilon^{c}_{jk} \partial_c \omega_{bh}).$$ (2.6)

The Christoffel symbols will not be used in this work, but it is necessary to calculate the spin connection:

$$\Gamma^2_3 = -\Gamma^3_2 = \epsilon \partial_x \psi, \quad \Gamma^3_3 = -\Gamma^1_3 = \epsilon \partial_y \psi, \quad \Gamma^1_3 = -\Gamma^2_3 = \epsilon \partial_z \psi.$$ (2.7)

Here, $\epsilon = e^{123}$, which can be $\pm 1$, and therefore, $\epsilon^2 = 1$.

Now, in order to study the perturbations in the (classical) loop gravity framework, it is necessary to use the Ashtekar connection and densitized triads as our basic variables. Since the metric is diagonal, we can parametrize the densitized triads by

$$K^i = (\dot{a} - 2a \dot{\psi} - a \psi) \dot{a}_i.$$ (2.9)

A useful property of the Ashtekar connection in this context is that its diagonal terms solely come from $K^i_a$, while its off-diagonal terms solely come from $\Gamma^i_a$ (of course, this is not the case in general). Since the densitized triads are diagonal, only the diagonal part of the Ashtekar connection will appear in the induced symplectic structure, and therefore, it is convenient to parametrize the Ashtekar connection by

$$A_i = c \tau_1 - \epsilon (\partial_x \psi) \tau_2 + \epsilon (\partial_y \psi) \tau_3, \quad A_j = \epsilon (\partial_x \psi) \tau_1 + c \tau_2 - \epsilon (\partial_y \psi) \tau_3, \quad A_z = -\epsilon (\partial_z \psi) \tau_1 + \epsilon (\partial_z \psi) \tau_2 + c \tau_3,$$ (2.10)

where $A_a = A_i \tau_i$ and $\tau_i$ denotes a basis of the Lie algebra of $SU(2)$, such that $\tau_i \tau_j = \frac{1}{2} \epsilon_{ijk} \tau_k - \frac{1}{2} \delta_{ij} I$. Following this definition, we find that the induced symplectic structure on our phase space gives the following nonzero Poisson bracket:

$$\{c(\vec{x}), p(\vec{y})\} = \frac{8 \pi \gamma G}{3} \delta^3(\vec{x} - \vec{y}).$$ (2.11)

2.2. Massless scalar field

Since we are primarily interested in the effects due to quantum gravity in the early universe, we will take the simplest matter field possible, a massless scalar field. The action for a massless scalar field is

$$S = -\frac{1}{2} \int_{\mathcal{M}} \sqrt{|g|} g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi),$$ (2.12)

and it is easy to show that the conjugate momentum to $\phi$ is given by

$$\pi_\phi = N \sqrt{|g|} \dot{\phi},$$ (2.13)

where the dot represents derivation with respect to time and $N$ is the lapse function. The Poisson bracket is given by

$$\{\phi(\vec{x}), \pi_\phi(\vec{y})\} = \delta^3(\vec{x} - \vec{y}).$$ (2.14)
In this work, we will be using the Hamiltonian formulation of general relativity, and therefore, the stress energy tensor will not appear. Since a lot of the literature in the field of cosmological perturbation theory starts from the Einstein equations (and thus uses the stress energy tensor), the following dictionary can be useful in order to compare the results given here with those available in the literature.

For a massless scalar field, the stress energy tensor is given by

\[ T_{\mu\nu} = g^{\mu\lambda} (\partial_\lambda \phi)(\partial_\nu \phi) - \frac{1}{2} \delta_{\mu\nu} (\vec{\nabla} \phi)^2. \]  (2.15)

For small perturbations around a flat FLRW background, we have \( \psi = \bar{\psi} + \delta \psi \), where the bar denotes the homogeneous background and \( \delta \psi \) is the inhomogeneous perturbation. One can immediately see that the second term in \( T_{\mu\nu} \) is negligible as it is second order in \( \delta \psi \).

The degrees of freedom of a perfect fluid are given by the energy density \( \rho \), the pressure \( P \) and the velocity four-vector \( u^\mu \). The energy density and pressure can be split into background quantities and perturbations, \( \rho = \bar{\rho} + \delta \rho \) and \( P = \bar{P} + \delta P \), while the only scalar mode of the perturbation of the four-velocity at first order is given by \( \delta u_a = \partial_a (\delta \psi) \). In terms of these variables, the stress–energy tensor is given by (see, e.g., [17, 18])

\[ T_0^0 = -\bar{\rho} - \delta \rho, \]  (2.16)

\[ T_0^a = (\bar{\rho} + \bar{P})\partial_a (\delta \psi), \]  (2.17)

\[ T^a_b = \delta_b^a (\bar{P} + \delta P), \]  (2.18)

and then, by equation (2.15), the relations between the ‘standard’ variables and the ones used in the Hamiltonian framework are the following:

\[ \bar{\rho} = \bar{P} = \frac{1}{2} \dot{\psi}^2, \]  (2.19)

\[ \delta \rho = \delta P = \dot{\psi} (\ddot{\psi} - \dot{\psi}^2), \]  (2.20)

\[ (\bar{\rho} + \bar{P})\delta u = -\dot{\psi} \delta \psi. \]  (2.21)

Another useful relation can be obtained by adding equations (2.19) and (2.20), this gives

\[ \rho = P = (1 - 2 \psi) \dot{\psi}^2. \]  (2.22)

### 2.3. The Hamiltonian constraint and dynamics

Before writing the Hamiltonian constraint, it is necessary to express \( \psi \) in terms of the conjugate variables \( (c, p) \). In order to do this, we will assume that \( \int_M \psi = 0 \) (i.e. the zero mode of the Fourier decomposition of \( \psi \) is zero), and then, it follows that

\[ \psi = \frac{\bar{p} - p}{2\bar{p}}, \quad \text{where} \quad \bar{p} = \int_M p. \]  (2.23)

This requires the introduction of the nonlocal \( \bar{p} \), but it is necessary in order to express the perturbation \( \psi \) in terms of the elementary variable \( p \). As we shall see, the resulting nonlocal Hamiltonian system provides the usual local equations of motion for linear cosmological perturbations.

The Hamiltonian constraint is given by

\[ C_H = \int_M [NH + N^0 \mathcal{H}_a + N' \mathcal{G}], \]  (2.24)
where, for a massless scalar field, the scalar constraint is

\[ \mathcal{H} = \frac{-E_i E_j}{16\pi G}\gamma^{ij} e^{1/4}(F_{ab} \,^k - (1 + \gamma^2)\Omega_{ab} \,^k) + \frac{1}{2\sqrt{|q|}}(\pi_\psi^2 + E_i E^\beta (\partial_\psi \phi)(\partial_\beta \phi)) \approx 0, \]

the diffeomorphism constraint is

\[ \mathcal{H}_a = \frac{E_i}{4\pi G}\gamma^i \,^b F_{ab} \,^i + \pi_\psi \partial_\psi \phi \approx 0 \]

and the Gauss constraint is

\[ G_i = D_\psi E_i = \partial_\psi E_i + \epsilon_{ij} A_j^l E_l \approx 0. \]

\( F_{ab} \,^k \) appearing above is the field strength of the Ashtekar connection:

\[ F_{ab} \,^k = 2\partial_\psi A^k_\alpha + \epsilon_{ij} A^l_\alpha A^j_\beta. \]

Similarly, \( \Omega_{ab} \,^k \) is the field strength of the spin connection.

\( \approx 0 \) indicates that \( \mathcal{H}, \mathcal{H}_a \) and \( G_i \) are constraints and must vanish for physical solutions. A solution to these three constraints at an initial time can then be evolved to later times by taking Poisson brackets with the Hamiltonian constraint \( C_H \). An important point is that any solution that initially satisfies the scalar, diffeomorphism and Gauss constraints will continue to do so under the evolution determined by \( C_H \).

For linear perturbations around the flat FLRW spacetime in the longitudinal gauge, we find that the Gauss constraint is automatically satisfied, while the scalar constraint becomes

\[ \mathcal{H} = \sqrt{q} \left[ \frac{-\sqrt{|p|}}{8\pi G} \left( \frac{3c^2}{\gamma^2} + 2\bar{\nabla}^2 \left( \bar{\psi} - \frac{\bar{\rho} - \rho}{2\bar{\rho}} \right)^2 \right) + \frac{\pi_\psi^2}{2\bar{\rho}^{3/2}q} + \frac{\sqrt{\bar{p}}}{\bar{\rho}} (\bar{\nabla} \phi)^2 \right] \approx 0, \]

and the diffeomorphism constraint is given by

\[ \mathcal{H}_a = \sqrt{q}\bar{p} \left[ \partial_\psi c + c \partial_\psi \left( \frac{\bar{\rho} - \rho}{2\bar{\rho}} \right) \right] + \pi_\psi \partial_\psi \phi \approx 0. \]

Finally, since \( N = 1 + \psi = 1 + \frac{\bar{\rho} - \rho}{2\bar{\rho}} \) and \( N^a = 0 \) in the longitudinal gauge, the Hamiltonian constraint is given by

\[ C_H = \int_\mathcal{M} \left( 1 + \frac{\bar{\rho} - \rho}{2\bar{\rho}} \right) \mathcal{H}. \]

Although one might be tempted to drop terms in \( \mathcal{H}_a \), like \( (\bar{\nabla} \phi)^2 \), which are of second order in the perturbations, they contribute first-order terms in the evolution equations (in this case for \( \pi_\psi \) after the derivative is integrated by parts).

This Hamiltonian formalism is of course well defined for all \( c(\bar{x}), p(\bar{x}), \phi(\bar{x}) \) and \( \pi_\psi(\bar{x}) \), but the only case we are interested in is a homogeneous background with small perturbations in which case the resulting system is equivalent to linear cosmological perturbations in general relativity. Therefore, we will only consider equations for linear perturbations around a homogeneous background in what follows.

It is now possible to obtain the standard results for linear perturbations in cosmology from the two constraints and the Hamiltonian. Using the relations (but only after the equations of motion have been obtained by the variational principle)

\[ p = a^2(1 - 2\psi), \quad \bar{p} = a^2, \quad c = \bar{c} + \delta c, \quad \phi = \bar{\phi} + \delta \phi, \quad \pi_\psi = \bar{\pi}_\psi + \delta \pi_\psi, \]

(2.32)
where as usual the ‘barred’ quantities correspond to the unperturbed background quantities, we find that the scalar constraint $\mathcal{H} = 0$ implies that

$$\frac{-ac^2}{\gamma^2}(1 - \psi) - \frac{2a}{3}\sqrt{2}\psi + \frac{4\pi G}{3}\frac{\pi^2}{\bar{q}} = 0,$$  \hspace{1cm} (2.33)

where we have only kept terms linear in the perturbations. The diffeomorphism constraint

$$\sqrt{q} \frac{a^2}{\gamma^2}[\delta_c(\delta c) + \delta_c \delta_c] + 4\pi G\pi_\psi \delta_c(\delta \psi) = 0.$$  \hspace{1cm} (2.34)

The dynamics are obtained from the Hamiltonian constraint. Starting with the equations of motion for the matter degrees of freedom since they are simpler,

$$\dot{\psi} = \psi (\mathcal{C}_H) = \frac{\delta\mathcal{C}_H}{\delta \pi_\psi} = (1 + \psi) \frac{\pi_\psi}{p^{1/2}/\sqrt{q}},$$  \hspace{1cm} (2.35)

$$\dot{\pi}_\psi = [\pi_\psi, \mathcal{C}_H] = -\frac{\delta\mathcal{C}_H}{\delta \psi} = \sqrt{q} a \frac{\pi_\psi}{p^{1/2}/\sqrt{q}}.$$  \hspace{1cm} (2.36)

Note that the first relation is equivalent to equation (2.13), as one should expect and it also implies, together with equation (2.22), that

$$\rho = \frac{\pi^2}{2p^{1/2}/\sqrt{q}},$$  \hspace{1cm} (2.37)

which will be a useful relation later.

The equation of motion for $\rho$ also has a simple form,

$$\dot{\rho} = [\rho, \mathcal{C}_H] = -\frac{8\pi G}{3}\frac{\delta\mathcal{C}_H}{\delta \psi}(1 + \psi) \frac{\sqrt{q}}{p^{1/2}/\sqrt{q}},$$  \hspace{1cm} (2.38)

which, solving for $c$, gives

$$c = \gamma (\dot{a} - 2\dot{a} \psi - a \dot{\psi}),$$  \hspace{1cm} (2.39)

which is exactly what is given in equation (2.9) (recall that the diagonal portion of $A'_i$ is given by $\gamma K_i^i$ as the diagonal part of $\Gamma_i^j$ is zero).

Finally, the equation of motion for $\dot{c}$ is the most complicated as one must vary $\mathcal{C}_H$ with respect to $\rho$. There are two contributions that must be integrated by parts (since the topology is $T^3$, there are no boundary terms) and there are also terms that depend on $\dot{\rho}$ which could potentially contribute. Since $\delta \mathcal{P}/\delta \rho(\bar{x}) = 1$—note that there is no delta function—all of the terms that are obtained by varying $\dot{\rho}$ with respect to $\rho(\bar{x})$ are inside an integral over $\mathcal{M}$ and would contribute a nonlocal term to the dynamics; this would disagree with the well-known general relativity results. However, all of the nonlocal terms vanish and one is left with a local equation of motion. In order to see this, we start with

$$\dot{c}(\bar{x}) = [c(\bar{x}), \mathcal{C}_H] = \frac{8\pi G}{3} \int_{\mathcal{M}} d^3y \left[ \frac{\delta \mathcal{P}(\bar{y})}{\delta p(\bar{x})} \mathcal{H}(\bar{y}) + [1 + \psi(\bar{y})] \frac{\delta \mathcal{H}(\bar{y})}{\delta p(\bar{x})} \right],$$  \hspace{1cm} (2.40)

where the first term can be dropped as it is multiplied by the constraint $\mathcal{H}$, which vanishes on-shell. A straightforward calculation then gives

$$\dot{c}(\bar{x}) = -\frac{[1 + \psi(\bar{x})]c(\bar{x})^2}{2\gamma \sqrt{p(\bar{x})}} - \frac{\gamma}{5a} \nabla^2 \psi(\bar{x}) - \frac{2\pi G}{3} \frac{\pi_\psi(\bar{x})^2}{p(\bar{x})^{1/2}}$$

$$- \frac{2\gamma a}{3} \int_{\mathcal{M}} d^3y \nabla^2 \frac{\delta \psi(\bar{y})}{\delta p(\bar{x})} + \frac{2\gamma a}{3} \int_{\mathcal{M}} d^3y [\nabla \psi(\bar{y})] \nabla \left( \frac{\delta \psi(\bar{y})}{\delta p(\bar{x})} \right).$$  \hspace{1cm} (2.41)
The fourth term can dropped as it is a total divergence, while the fifth term requires a little more work. From equation (2.23), it follows that
\[ \frac{\delta \psi (\vec{y})}{\delta \rho (\vec{x})} = \frac{p(\vec{y})}{2\rho^2} - \frac{\delta^3 (\vec{x} - \vec{y})}{2\rho}, \]
where the delta function only appears in one of the two terms. Using this relation, integrating by parts and dropping terms that are of second order in \( \psi \), the fifth term in equation (2.41) becomes
\[ \frac{2\gamma a}{3} \int_{\mathcal{M}} d^3 \vec{y} \left[ (\nabla \psi (\vec{y})) \nabla \left( \frac{\delta \psi (\vec{y})}{\delta \rho (\vec{x})} \right) \right] = \frac{Y}{3a} \nabla^2 \psi (\vec{x}) - \frac{Y}{3a} \int_{\mathcal{M}} d^3 \vec{y} \nabla^2 \psi (\vec{y}). \]
The second term vanishes as it is a total divergence and one is left with the local relation
\[ \dot{c} = -\frac{1}{2\gamma a} (1 + 2\psi) c^2 - 2\pi G \gamma a \frac{\pi^2}{|p|^4} \dot{\psi}. \]
manner. One example of a quantum theory that could generate these quantum-corrected equations is a form of lattice LQC, where each cell is described by a homogeneous and isotropic geometry but where the geometries vary from one cell to another so that inhomogeneities are present at scales larger than the cell size [19, 20]. In this setting, the Hamiltonian would be separated into an ultralocal term (the ‘homogenous’ part) and all of the terms containing derivatives would be treated as interactions between different cells. This would of course constitute an approximation but would be valid so long as the volumes of the individual cells remain larger than the Planck volume and that there are enough cells in the lattice in order to properly describe the inhomogeneities up to a prescribed maximum wave number.

In the longitudinal gauge, the homogeneous metric in each cell is isotropic, and thus, the symplectic structure in each cell would be that of an FLRW spacetime. This indicates that one might be able to quantize each cell following the methods used in homogeneous models [1–3] and then turn on interactions in the Hamiltonian. It is worth pointing out that the diagonal form of the metric in the longitudinal gauge will greatly simplify the quantization process. This is mainly because, as one can easily check, when the metric is nondiagonal, the field strength obtained by taking a holonomy around a square loop of nonzero area is generically not almost periodic in the connection (which is proportional to the extrinsic curvature in the spatially flat homogeneous case where the spin connection vanishes). However, in this lattice LQC quantization, due to the diagonal form of the metric, it is possible to restrict our attention to almost periodic functions of the connection, just as in homogeneous spacetimes.

Once the quantum theory has been constructed, the effective theory can be obtained by taking a sharply peaked wavefunction and studying the evolution of the expectation values of certain observables under the action of the Hamiltonian operator. Some observables, such as $p$, can be obtained directly from the expectation values of the fundamental operators in the quantum theory while others—such as $c$ that one could associate with the expectation value of an operator corresponding to a short holonomy (minus the identity) divided by the length of the holonomy—would require a little more care. In any case, we will not construct the quantum theory here but leave it for future research. Instead, we will build the effective Hamiltonian by applying the heuristic methods following from (and well supported by) homogeneous LQC.

In the first part of this section, we implement holonomy corrections in the system studied in the previous section, that of a massless scalar field. Then, in the second part, we present a conjectured generalization of these results for all perfect fluids.

3.1. Effective equations for a massless scalar field

One of the key parts in the process of the LQC quantization of the Hamiltonian constraint operator is that the field strength $F_{ab}^k$ of the Ashtekar connection must be expressed in terms of a holonomy of the connection $A^k_a$ around the smallest possible loop. It is posited that the smallest loop has an area given by $\Delta c_{\text{Pl}}^2 = 4\sqrt{3}\pi \gamma c_{\text{Pl}}^2$, the smallest eigenvalue of the area operator in loop quantum gravity.

For homogeneous and isotropic cosmologies, it has been found that the full LQC dynamics of states, which at some initial time are sharply peaked around a classical solution, remain sharply peaked as they are evolved by the action of the Hamiltonian constraint operator [2]. In addition, the dynamics of the wave packet are extremely well approximated by a modified classical Hamiltonian constraint, where one includes holonomy corrections by determining the field strength by taking a holonomy around the minimal loop as prescribed above [4]. In this sense, it seems that the most important quantum gravity modifications to the Einstein equations in this very simple setting come from the holonomy corrections. These corrections become
important when the spacetime curvature approaches the Planck scale, but are completely negligible when the matter energy density is less than $0.01\rho_P$.

Unfortunately, a priori it is not entirely clear how to incorporate holonomy corrections for inhomogeneous spacetimes. In previous works [6–8, 12], generic holonomy correction terms were added to the effective equations by hand and then the form of the correction terms was restricted by the condition that the constraint algebra must continue to close. However, so far it has proven to be impossible to complete this program for holonomy corrections to scalar perturbations.

Here, due to the choice of both the gauge and the variables, the situation is luckily quite simple. In the classical scalar constraint, we find that the term with the difference between the field strengths of the Ashtekar connection and the spin connection can be simplified:

$$E_i^a E_j^b \epsilon^{ij}_k [F_{ab}^k - (1 + \gamma^2) \Omega_{ab}^k] = E_i^a E_j^b \epsilon^{ij}_k [F_{ab}^{(iso)}^k - \gamma^2 \Omega_{ab}^k], \quad (3.1)$$

where we have defined

$$F_{ab}^{(iso)} = c^2 \epsilon^{ij}_k \hat{\omega}_{ab}^i \hat{\omega}_b^j, \quad (3.2)$$

where of course $c$ depends on position. Although this ‘field’ strength is also inhomogeneous, it nonetheless has a very similar form to the field strength in homogeneous settings. Note that this provides a natural separation of $F_{ab}^k$ and $\Omega_{ab}^k$ terms into an ultralocal term $F_{ab}^{(iso)}$ and an interaction term $\Omega_{ab}^k$. Therefore, following the arguments introduced at the beginning of this section, we can introduce holonomy corrections to the ultralocal term $F_{ab}^{(iso)}$ as in homogeneous models, while $\Omega_{ab}^k$ is treated as an interaction term that only depends on $p$ and therefore will not be modified. Thus, in order to incorporate holonomy corrections in the scalar constraint, we shall mimic the procedure in the homogeneous case and replace

$$F_{ab} = c^2 \epsilon^{ij}_k \hat{\omega}_{ab}^i \hat{\omega}_b^j \rightarrow \frac{\sin^2 \bar{\mu} c}{\bar{\mu}^2} e^{ij}_k \hat{\omega}_{ab}^i \hat{\omega}_b^j, \quad (3.3)$$

where

$$\bar{\mu} = \sqrt{\Delta \bar{E}_P^2 / |p|}, \quad (3.4)$$

just as in the homogeneous case [1, 2], except that $p = a^2(1 - 2\psi)$ is no longer homogeneous.

Thus, the scalar constraint becomes

$$\mathcal{H}^{eff} = \sqrt{q} \left[ \frac{-1}{8\pi G} \left( \frac{3|p|^3}{\gamma^2 \Delta \bar{E}_P^2} \right) \sin^2 \bar{\mu} c + 2\sqrt{|p|} \nabla^2 \left( \bar{\mu} \frac{\bar{E}_P}{2|p|} \right) - \sqrt{|p|} \left( \bar{\mu} \frac{\bar{E}_P}{2|p|} \right)^2 \right] + \frac{\pi \bar{\mu}^2}{2|p|^{3/2}} \left( \frac{\bar{\mu} c}{|p|} \right)^2 \approx 0, \quad (3.5)$$

where $\cos 2\bar{\mu} c$ in the last term is added in order to ensure that the effective scalar constraint is preserved by the dynamics, i.e. $\mathcal{H}^{eff} \approx 0$.

3 The Belinskii, Khalatnikov, Lifshitz (BKL) conjecture—which suggests that spatial derivatives can be ignored when the spatial curvature becomes very large [21, 22]—also supports this way of introducing the holonomy corrections. Since holonomy corrections are only important when the curvature approaches the Planck regime, if the BKL conjecture holds near the Planck scale, then spatial derivatives and, thus, $\Omega_{ab}^k$ can be ignored when one incorporates holonomy corrections. Then, only $F_{ab}^{(iso)}$ remains and the holonomy corrections are introduced as in equation (3.3).

Note that although the BKL conjecture says nothing about isotropy, $F_{ab}^{(iso)}$ is locally isotropic due to the locally isotropic form of the metric in the longitudinal gauge.
On the other hand, the diffeomorphism constraint is not modified:

\[
\mathcal{H}_a^{\text{eff}} = \frac{\sqrt{\bar{q}p}}{4\pi G\gamma} \left[ \partial_{\mu}c + \frac{c}{\sqrt{3}} \left( \frac{\bar{p} - p}{2\bar{p}} \right) \right] + \pi_{\psi} \partial_{\mu}\phi \approx 0. 
\]  

(3.6)

The Gauss constraint is not modified either and continues to hold due to the choice of the longitudinal gauge, and therefore, we do not need to consider it any further. As an aside, we point out that it is not immediately clear how to incorporate an unchanged diffeomorphism constraint in lattice LQC; we leave this problem for future work.

As before, the Hamiltonian constraint provides the dynamics and is simply given by

\[
C_H^{\text{eff}} = \int_{\mathcal{M}} \left(1 + \frac{\bar{p} - p}{2\bar{p}}\right) \mathcal{H}_a^{\text{eff}}. 
\]  

(3.7)

The scalar and diffeomorphism constraints must be satisfied by the initial data, and thus,

\[
- \frac{|p|^{3/2}}{\gamma^3 \Delta \ell_p^2} \sin^2 \bar{\mu}c - \frac{2a}{3} \nabla^2 \psi + \frac{4\pi G}{3|p|^{3/2}\bar{q}} \pi_\psi^2 = 0, 
\]  

(3.8)

\[
\sqrt{\bar{q}} \frac{p}{4\pi G\gamma} \left[ \partial_{\mu}(\delta c) + c \partial_{\mu}\psi \right] + \pi_\psi \partial_{\mu}(\delta \phi) = 0. 
\]  

(3.9)

Finally, the equations of motion generated by \(C_H^{\text{eff}}\) are the following:

\[
\dot{\psi} = (1 + \psi) \frac{\pi_\psi}{p^{3/2}}. 
\]  

(3.10)

\[
\dot{\pi}_\psi = \sqrt{\bar{q}} a \cos 2\bar{\mu}c \nabla^2 \psi. 
\]  

(3.11)

\[
\dot{\bar{p}} = \frac{2p(1 + \psi)}{\gamma^2 \sqrt{\Delta \ell_p}} \sin \bar{\mu}c \cos \bar{\mu}c, 
\]  

(3.12)

\[
\dot{\bar{c}} = - \frac{3a}{2\gamma \Delta \ell_p} \sin^2 \bar{\mu}c + \frac{c(1 + \psi)}{\gamma^2 \sqrt{\Delta \ell_p}} \sin \bar{\mu}c \cos \bar{\mu}c - \frac{2\pi G\gamma}{a^2} (1 + 6\psi) \pi_\psi^2. 
\]  

(3.13)

Once again all nonlocal contributions due to a variation of \(\bar{p}\) vanish, as one should expect. Note that while classically \(c\) is proportional to \(\bar{p}\), this is not the case in the effective theory, and the relation between them, as seen in equation (3.12), is considerably more complicated. Finally, the relationship between \(\pi_\psi\) and \(\rho\) given in equation (2.37) continues to hold.

It is interesting to solve for the Hubble rate by squaring equation (3.12) and using equation (3.8); this gives the modified Friedmann equation (including perturbations)

\[
H^2 = \left( \frac{\bar{p}}{2\bar{p}} \right)^2 = (1 + 2\psi) \frac{8\pi G}{3} \rho \left( 1 - \frac{\rho}{\rho_c} \right) - 2 \frac{2\rho}{3a^2} \left( 1 - \frac{2\rho}{\rho_c} \right) \sqrt{\gamma^2 \psi}, 
\]  

(3.14)

where the critical energy density is \(\rho_c = \frac{3}{8\pi G^2 \Delta \ell_p^2}\). Clearly, the classical result is obtained so long as the local matter energy density remains well below the critical energy density. It is also clear that when the matter energy density approaches the Planck scale, the Hubble rate diminishes due to the repulsive nature of the quantum gravity corrections and reaches zero close to the critical density (depending on the local strength of the fluctuations in \(\psi\)) at which point there is a bounce where the Hubble rate begins to grow again, and as soon as the matter energy density drops below the Planck regime, contributions due to quantum gravity are completely negligible.

Finally, the form of the equations of motion given above is a little unwieldy, and for some applications it will be useful to expand these equations and separate the background terms from the perturbed quantities. For the scalar field, this is easy,

\[
\dot{\psi} = \frac{\pi_\psi}{a^3 \sqrt{\bar{q}}}. 
\]  

(3.15)
\[
\dot{\psi} = \frac{4\psi \pi_\psi}{a^3 \sqrt{\dot{a}}} + \frac{\delta \pi_\psi}{a^3 \sqrt{\dot{q}}}, \tag{3.16}
\]

\[
\dot{\pi}_\psi = 0, \tag{3.17}
\]

\[
(\delta \pi_\psi) = \sqrt{\dot{q}} a \cos 2\tilde{\mu} c \nabla^2 \delta \varphi, \tag{3.18}
\]

but it becomes a little more involved for the geometric degrees of freedom. For \( p \), since \( p = a^2 (1 - 2\psi) \), we find that

\[
\dot{\psi} = \frac{\dot{a}}{a} (1 - 2\psi) - \frac{\dot{\rho}}{2a^2}. \tag{3.19}
\]

It is also necessary to expand the trigonometric functions of \( \tilde{\mu} c \). This gives, e.g.,

\[
\sin \tilde{\mu} c = \sin \sqrt{\Delta \ell_{Pl}^2} \left( \tilde{c} \psi + \delta c \right) \cos \sqrt{\Delta \ell_{Pl}^2} \frac{\pi_\psi}{a}. \tag{3.20}
\]

From these equations, it follows that

\[
\dot{\rho} = 2a \dot{a} = \frac{a^2}{\gamma \sqrt{\Delta \ell_{Pl}}} \sin 2\sqrt{\Delta \ell_{Pl}^2}, \tag{3.21}
\]

\[
\dot{\psi} = \frac{-\psi}{2a} \sin \frac{2\sqrt{\Delta \ell_{Pl}^2}}{a} - \frac{\tilde{c} \psi + \delta c}{\gamma a} \cos \frac{2\sqrt{\Delta \ell_{Pl}^2}}{a}, \tag{3.22}
\]

\[
\dot{c} = \frac{-c(\tilde{c} \psi + \delta c)}{\gamma a} \cos \frac{2\sqrt{\Delta \ell_{Pl}^2}}{a} + \frac{\tilde{c}}{2a} \sin \frac{2\sqrt{\Delta \ell_{Pl}^2}}{a} - \frac{2\pi G \rho}{a^5} \frac{\pi_\psi}{a^3}, \tag{3.23}
\]

\[
(\dot{\delta} c) = \frac{c(\tilde{c} \psi + \delta c)}{\gamma a} \cos \frac{2\sqrt{\Delta \ell_{Pl}^2}}{a} - \frac{\tilde{c} \psi + \delta c}{\gamma \sqrt{\Delta \ell_{Pl}^2}} \sin \frac{2\sqrt{\Delta \ell_{Pl}^2}}{a} - \frac{12 \pi G \rho \psi}{a^5} \frac{\pi_\psi}{a^3} \frac{\pi_\psi}{a^3} - \frac{4 \pi G \rho}{a^5} \frac{\pi_\psi}{a^3} \delta \pi_\psi. \tag{3.24}
\]

The trigonometric identities \( 2 \sin \theta \cos \theta = \sin 2\theta \) and \( \cos^2 \theta - \sin^2 \theta = \cos 2\theta \) have been used in order to shorten the expressions above.

The dynamics, both of the background and of the perturbations around it, are given by the equations above and one can check that, on-shell, \( \dot{T}^{\text{eff}} = T_w^{\text{eff}} = 0 \), just as one would expect. Note that the \( \cos 2\tilde{\mu} c \) term that was added to the effective scalar constraint in equation (3.5) is essential for \( T^{\text{eff}} = 0 \) to hold. This result shows that the equations of motion and the constraints are consistent with each other.

### 3.2. Conjectured generalization to other perfect fluids

In this part, we provide a conjectured extension of our results to all other matter fields that can be treated as perfect fluids. It would be possible to use a Hamiltonian framework for perfect fluids [23], but we will show that this is not necessary: the generalization is straightforward enough that it can be done by hand. There is a possibility that a strong quantum backreaction from the matter field could spoil the validity of the effective equations, but we will not consider this potential effect here.

It is worth pointing out that it would be significantly more difficult to generalize these equations in order to include matter fields that allow anisotropic stress since at the very beginning it was assumed that the perturbation in the lapse \( \phi \) was equal to the perturbation in the scale factor \( \psi \). If this assumption is not made, then it is not immediately clear how to
construct a Hamiltonian formulation with the symplectic structure given in equation (2.11), without expanding it in terms of the background and perturbations, which would ruin the simple substitution used in equation (3.3).

Nonetheless, this generalization to all perfect fluids will be very useful, especially since many matter fields behave like radiation at very high energies, which are precisely the conditions to be expected when the curvature is near the Planck scale.

For generic perfect fluids, the effective scalar constraint becomes

\[
\rho - \frac{3}{8\pi G} \left[ \frac{1}{\gamma^2 E_{\gamma}^2} \sin^2 \mu c + \frac{2}{3a^2} \nabla^2 \psi \right] = 0, \tag{3.25}
\]

which, as \(\sin^2 \theta \leq 1\), shows that the matter energy density is bounded above by the critical density

\[
\rho_c = \frac{3}{8\pi G\gamma^2 E_{\gamma}^2}, \tag{3.26}
\]

(modulo some small corrections from \(\nabla^2 \psi\)), which is consistent with equation (3.14).

As before, the diffeomorphism constraint is unchanged, and therefore,

\[
\frac{1}{\gamma^2} \left[ \delta_u \psi + \partial_u (\delta c) \right] - 4\pi G a(\rho + P) \delta_u (\delta u) = 0. \tag{3.27}
\]

Note, however, that since \(c\) is not proportional to \(\dot{\rho}\) when holonomy corrections are present, the diffeomorphism constraint cannot be simplified as much as it usually is when there are no holonomy corrections.

The equations of motion for the matter degrees of freedom are easy to generalize. Using equations (2.37) and (3.11), along with the scalar constraint in order to solve for the \(\cos 2\bar{\mu}c\) term, we find that

\[
\dot{\rho} + \frac{a}{\rho^2} (\rho + P) + \delta \rho + \frac{a}{\rho^2} (\delta \rho + \delta P) - 3(\rho + P) \psi + \frac{1}{a^2} (\rho + P) \nabla^2 (\delta u) \left( 1 - \frac{2\rho}{\rho_c} \right) = 0, \tag{3.28}
\]

where the critical density \(\rho_c\) appears once more. The equation for \(\delta u\),

\[
\delta_u [(\rho + P) \delta u] = -\delta P - (\rho + P) \psi - \frac{a}{\rho} (\rho + P) \delta u, \tag{3.29}
\]

continues to hold.

Moving on to the geometrical degrees of freedom, since there are no matter terms in the equation for \(\dot{c}\), it is trivially generalized:

\[
\dot{c} = \frac{2\rho(1 + \psi)}{\gamma^2 E_{\gamma}^2} \sin \mu c \cos \mu c, \tag{3.30}
\]

while the generalization of equation (3.13) is a little trickier. A careful analysis of the equation of motion for \(\dot{c}\) at the classical level shows that \(\pi^e_c\) appearing there is related to the pressure of the massless scalar field rather than its energy density. Therefore, it follows that for an arbitrary perfect fluid,

\[
\dot{c} = -\frac{3a}{2\gamma^2 E_{\gamma}^2} \sin^2 \mu c + \frac{c(1 + \psi)}{\gamma^2 E_{\gamma}^2} \sin \mu c \cos \mu c - 4\pi G a P. \tag{3.31}
\]

Depending on the situation, it may be useful to expand these equations as was done in the previous subsection in equations (3.15)–(3.24); this can be done by following the same procedure that was used there. Also, one can check that for these effective equations, generalized in order to include all perfect fluids, the scalar and diffeomorphism constraints are again preserved by the dynamics.
We can obtain the modified Friedmann equation by squaring equation (3.30) and using equation (3.25):

$$\frac{\ddot{a}}{a^2} = \frac{8\pi G}{3} \rho \left( 1 - \frac{\rho}{\rho_c} \right) + 2 \frac{\dot{a}^2}{a^2} \dot{\psi} + 2 \frac{\ddot{a}}{a} \dot{\psi} - \frac{2}{3a^2} \nabla^2 \dot{\psi} \left( 1 - \frac{2\rho}{\rho_c} \right),$$

(3.32)

where it is understood that $\rho = \bar{\rho} + \delta \rho$ and the critical density $\rho_c$ appears again. Similarly, the modified Raychaudhuri equation can be obtained by taking the time derivative of equation (3.30) and then using equations (3.25), (3.30) and (3.31) in order to remove all the trigonometric functions. This gives

$$\frac{\ddot{a}}{a} = \frac{\dot{a}^2}{a^2} + 2 \frac{\ddot{a}}{a} \dot{\psi} - \frac{\dot{a}^2}{a^2} \dot{\psi} + \frac{\ddot{a}}{a} \dot{\psi} + \dot{\psi} - \left[ 4\pi G (\rho + P) - \frac{1}{a^2} \nabla^2 \dot{\psi} \right] \left( 1 - \frac{2\rho}{\rho_c} + \frac{1}{2\pi G \rho_c a^2} \nabla^2 \dot{\psi} \right).$$

(3.33)

These equations can be compared with the usual general relativity results obtained in section 2.3; the classical results are obtained in the limit $\rho_c \to \infty$.

Any initial data satisfying equations (3.25) and (3.27) can then be evolved by using equations (3.28), (3.29), (3.32) and (3.33); these four equations are written in a form similar to the standard cosmological notation. Although the constraints are written in terms of different variables, one can use the standard classical constraints given in equations (2.45) and (2.46) in order to constrain the initial data so long as the energy density $\rho$ is far from the critical density $\rho_c$ at the time the initial conditions are set. If the initial conditions are chosen at a time where $\rho$ is within two orders of magnitude of $\rho_c$ or less, then the holonomy-corrected constraints (3.25) and (3.27) must be used.

4. Some comments on inverse triad corrections

In this section, we make some comments regarding the type of inverse triad corrections that might be expected in the effective equations for cosmological perturbations. This section stands apart from the remainder of the paper and can be skipped on a first reading.

There are two main inputs from loop quantum gravity that are used in order to construct the Hamiltonian constraint operator in LQC. First, the field strength of the Ashtekar connection must be expressed in terms of holonomies, and second, inverse triad operators must be introduced carefully as the operator $\hat{\rho}$ includes zero in its discrete spectrum. In the previous section, we showed how it is possible to treat holonomy corrections in an effective action; in this section, we comment on some of the properties that we expect inverse triad corrections to have. This has already been studied in LQC (see, e.g., the appendix of [24], where similar considerations arise in a slightly different setting) but there are some important new aspects that appear in the context of lattice LQC which provide further insights into this problem.

As is well known, there are many different ways to build inverse triad operators in LQC, all of which have the correct semiclassical limit but behave differently at the Planck scale. Because of this ambiguity, it is not clear which inverse triad operator to choose. Although there are some heuristic arguments indicating that inverse triad corrections might be comparable to holonomy corrections [15], all of the inverse triad operators defined in LQC so far vanish in noncompact spacetimes. This indicates that our current understanding of inverse triad operators is flawed as these operators should be local, rather than global, operators. However, the inverse triad operators that naturally arise in the setting of lattice LQC have different properties and may explain what is the correct way to implement inverse triad operators in a cosmological setting.
In order to gain an idea of the form inverse triad corrections should take, we will consider a relatively simple lattice LQC model [19, 20]. In this setting, we discretize the 3-torus into $N^3$ cells, which are each taken to be homogeneous, but as the gravitational field and matter fields can vary from one cell to the next, inhomogeneities are present at large scales. The number of cells $N^3$ gives the largest wave number that perturbations may have in this discretization:

$$k = \frac{N}{2}.$$  

(4.1)

Gravitational field is taken to be homogeneous in each cell, the induced symplectic structure from the full theory changes and the Poisson brackets become

$$\{\epsilon(\vec{n}), p(\vec{m})\} = \frac{8\pi G \gamma N^3}{3} \delta_{\vec{n}, \vec{m}},$$  

(4.2)

$$\{\varphi(\vec{n}), \pi_{\varphi}(\vec{m})\} = N^3 \delta_{\vec{n}, \vec{m}},$$  

(4.3)

where $\vec{n}$ and $\vec{m}$ are the vectors that label the cells in the three-dimensional lattice and $\delta_{\vec{n}, \vec{m}}$ is a Kronecker delta. Note the explicit dependence of $N^3$ in the symplectic structure; this will reappear in the inverse triad operators.

Following the full theory, inverse triad operators are constructed by starting with a Poisson bracket (which is equal to $1/p$ when it is evaluated) between two functions on the phase space that can easily be promoted to operators, and then, the inverse triad operator is given by the commutator between the two operators, divided by $i\hbar$ [25]. For lattice LQC, a simple choice for the inverse triad operator corresponding to $1/p$ acting on a particular cell is given by

$$\frac{1}{p(\vec{n})} |p(\vec{n})\rangle = \frac{3}{4 \pi \gamma \sqrt{\Delta \ell_{pl}^3 N^3}} \left[ |p(\vec{n})\rangle^{3/2} + 2 \pi \gamma \sqrt{\Delta \ell_{pl}^3 N^3} |p(\vec{n})\rangle^{1/3} \right].$$

(4.4)

Clearly, this operator is always well defined, even on the state $|p(\vec{n})\rangle = 0$, which it annihilates. Also, the eigenvalue approximates $1/p$ for large $p$.

In order to incorporate inverse triad corrections for the operator chosen above, one would simply replace all occurrences of $1/p$ in the scalar constraint with the right-hand side of equation (4.4). Since we know that the singularity is avoided due to the holonomy corrections, it follows that $p \neq 0$ and then inverse triad corrections can be written as

$$\frac{1}{p} \rightarrow \frac{1}{p} \times F(p, N),$$

(4.5)

where the corrections are encoded in the function $F(p, N)$:

$$F(p, N) = \frac{3}{2 f(p, N)} \left[ |1 + f(p, N)|^{1/3} - |1 - f(p, N)|^{1/3} \right],$$

(4.6)

and $f(p, N)$, in turn, is given by

$$f(p, N) = \frac{2 \pi \gamma \sqrt{\Delta \ell_{pl}^3 N^3}}{p^{3/2}}.$$  

(4.7)

Since the inverse triad correction depends on the number of cells $N^3$, this type of correction is sometimes viewed as being unphysical. However, since the physical wavelength of the

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4 This can be derived by noting that $p^{-1} = -(3i/4\pi G \gamma \sqrt{\Delta \ell_{pl}^3 N^3}) e^{-i\varphi(\vec{n})} e^{i\varphi(\vec{m})} e^{-i\varphi(\vec{n})}$ (where we have suppressed the label $\vec{n}$ in order to simplify the notation) and then promoting the right-hand side to be an operator by replacing $\langle \cdot \rangle$ with $\langle \cdot \rangle / \hbar$. The action of this operator is most easily seen by changing variables to $v \propto p^{3/2}$ so that $e^{i\varphi(\vec{n})}$ acts as a simple shift operator on states $|v\rangle$. Then, changing variables back to $p$, one obtains the result given in equation (4.4). See [1, 2] for further information on inverse triad operators in the $\mu$ scheme in homogeneous and isotropic LQC.
Fourier mode in a compact, (approximately) homogeneous, space of volume $p^3/2$ divided into $N^3$ cells is $\lambda_{\text{phy}} = 2\sqrt{p}/N$ (see equation (4.1)), it follows that $F$ and $f$ should not be viewed as the functions of $p$ and $N$ but rather of the physical wavelength of the Fourier mode studied $\lambda_{\text{phy}}$. Then,

$$F(\lambda_{\text{phy}}) = \frac{3}{2f(\lambda_{\text{phy}})}[|1 + f(\lambda_{\text{phy}})|^{1/3} - |1 - f(\lambda_{\text{phy}})|^{1/3}],$$

$$f(\lambda_{\text{phy}}) = \frac{16\pi \gamma \sqrt{\Delta\ell_{Pl}^3}}{\lambda_{\text{phy}}^3}. \quad (4.8)$$

The inverse triad correction to a given Fourier mode depends on the ratio of the wavelength of that mode to the Planck length. For modes with a wavelength considerably larger than $\ell_{Pl}$, inverse triad corrections are completely negligible, even if the spacetime curvature is large.

The dependence of the inverse triad correction on the wavelength of each mode is completely natural. In homogeneous models in LQC, inverse volume effects are only relevant in compact spacetimes, and then, the strength of the correction depends on the ratio of the volume of the spacetime to the Planck volume. This indicates that for inverse triad effects to be present, there must be length scales in the spacetime, which can be compared to the Planck length. In homogeneous spacetimes, there is only the size of the entire spacetime, which provides a length scale\(^5\), but when perturbations are present there are additional length scales provided by the wavelengths of the Fourier modes of the perturbations. Therefore, this property of the inverse triad corrections should not be surprising.

As a final note, we point out that much of the previous work studying inverse triad corrections in the cosmology perturbation equations for scalar, vector and tensor modes does not allow for the inverse triad corrections to depend on the wavelength of the mode, instead the corrections typically depend only on the volume of the background homogeneous spacetime. We suggest that these results should be generalized.

5. Discussion

We have shown how it is possible to incorporate holonomy-type corrections into the equations of motion for perturbations around a flat FLRW background. This was first done in the case of a massless scalar field, and then, we presented a conjectured generalization for all perfect fluid matter fields.

Since holonomy corrections are nonperturbative, we decided to use elementary variables that are nonperturbative. This is to say that they are not separated into background and perturbation terms; rather it is after the equations of motion are obtained that this split is performed. It was then necessary to work in the longitudinal gauge so that we could restrict our attention to holonomies that are almost periodic in the connection. These two choices greatly simplified the problem and the correct way to implement holonomy corrections in the effective Hamiltonian constraint for this particular gauge became more obvious.

It would be nice to extend this analysis in order to include tensor and vector modes, but this may not be as trivial as one would hope. The problem is that a lot of simplifications, which made this problem tractable at a nonperturbative level—such as a diagonal metric that allowed us to work with a relatively small phase space and restrict our attention to holonomies that are almost periodic functions of the connection—are not possible once tensor and vector perturbations are allowed. Therefore it seems that a combination of the techniques used here

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\(^5\) This is why inverse volume corrections in homogeneous, noncompact spacetimes vanish in LQC.
and perturbative methods used in [6–10, 12] will be necessary in order to be able to treat scalar, vector and tensor perturbations all together. Despite these remaining open issues, a lot can be learnt from this treatment of scalar perturbations.

First, the diffeomorphism and Gauss constraints are unchanged, and therefore, the symmetries they encode hold exactly at all scales, including the Planck scale. On the other hand, the scalar constraint and the dynamics of the gravitational field are modified by the holonomy corrections. As we saw in section 3, the holonomy corrections are introduced in precisely the same way as in the homogeneous models and the resulting constraints are shown to be preserved by the dynamics. As usual in LQC, we found that the holonomy corrections are parametrized by the critical energy density \( \rho_c \); these corrections vanish in the limit of \( \rho_c \to \infty \), and then, the standard cosmological perturbation equations are recovered.

Although it is not yet clear how to include inverse triad operators—which are essential if one wants to construct the quantum theory—it is possible to learn some lessons from a discretization of LQC. The main point is that the inverse triad corrections should vary from one Fourier mode of the perturbation to the next. To be more precise, the inverse triad corrections, for a given Fourier mode of the perturbation, should explicitly depend upon the ratio of the wavelength of the mode to the Planck length. Thus, the shorter the wavelength, the more important the inverse triad correction will be. This is natural as the only physical length scale provided by perturbations is the wavelength of the modes. Because of this, inverse triad corrections should be expected to be completely negligible so long as the wavelengths of the perturbative modes of interest stay an order of magnitude or two away from the Planck scale. However, if the wavelengths of interest approach the Planck scale, then inverse triad corrections cannot be ignored. For example, these corrections could be important in inflationary scenarios, but not in ekpyrotic or matter bounce models.

This shows that holonomy corrections and inverse triad corrections have a completely different nature: holonomy corrections become important when the spacetime curvature approaches the Planck scale, while inverse triad operators become important when physical length scales are comparable to the Planck length. Thus, depending upon the situation, one of the corrections may be important, while the other is negligible. In particular, we expect that the equations derived in this paper hold at all curvature scales so long as the wavelengths of the scalar modes of interest remain large compared to \( \ell_{Pl} \).

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References

[1] Ashtekar A and Singh P 2011 Loop quantum cosmology: a status report Class. Quantum Grav. 28 213001
[2] Ashtekar A, Pawłowski T and Singh P 2006 Quantum nature of the big bang: improved dynamics Phys. Rev. D 74 084003
[3] Ashtekar A, Corichi A and Singh P 2008 Robustness of key features of loop quantum cosmology Phys. Rev. D 77 024046
[4] Taveras V 2008 Corrections to the Friedmann equations from LQG for a universe with a free scalar field Phys. Rev. D 78 064072
[5] Langlois D 1994 Hamiltonian formalism and gauge invariance for linear perturbations in inflation Class. Quantum Grav. 11 389
[6] Bojowald M and Hossain G M 2007 Cosmological vector modes and quantum gravity effects Class. Quantum Grav. 24 4801
[7] Bojowald M and Hossain G M 2008 Loop quantum gravity corrections to gravitational wave dispersion *Phys. Rev.* D **77** 023508
[8] Mielczarek J, Cailleteau T, Barrau A and Grain J 2011 Anomaly-free vector perturbations with holonomy corrections in loop quantum cosmology arXiv:1106.3744 [gr-qc]
[9] Bojowald M, Hossain G M, Kagan M and Shankaranarayanan S 2008 Anomaly freedom in perturbative loop quantum gravity *Phys. Rev.* D **78** 063547
[10] Bojowald M, Hossain G M, Kagan M and Shankaranarayanan S 2009 Gauge invariant cosmological perturbation equations with corrections from loop quantum gravity *Phys. Rev.* D **79** 043505
[11] Arzonymski M, Lalaz Z and Szulc L 2009 Loop quantum cosmology corrections to inflationary models *J. Cosmol. Astropart. Phys.* JCAP01(2009)004
[12] Wu J-P and Ling Y 2010 The cosmological perturbation theory in loop quantum gravity with holonomy corrections *J. Cosmol. Astropart. Phys.* JCAP05(2010)026
[13] Copeland E J, Mirlene D J, Nunes N J and Shaeri M 2009 The gravitational wave background from super-inflation in loop quantum cosmology *Phys. Rev.* D **79** 023508
[14] Grain J, Cailleteau T, Barrau A and Gorecki A 2010 Fully loop-quantum-cosmology-corrected propagation of gravitational waves during slow-roll inflation *Phys. Rev.* D **81** 024040
[15] Bojowald M, Calcagni G and Tsujikawa S 2011 Observational test of inflation in loop quantum cosmology *J. Cosmol. Astropart. Phys.* JCAP11(2011)046
[16] Agullo I, Ashok A and Nelson W unpublished
  Agullo I 2011 Observational signatures in LQC International Loop Quantum Gravity Seminar (March 2011) pp 1–23 (http://relativity.phys.lsu.edu/ilqgs/)
[17] Weinberg S 2008 *Cosmology* (Oxford: Oxford University Press)
[18] Mukhanov V 2005 *Physical Foundations of Cosmology* (Cambridge: Cambridge University Press)
[19] Bojowald M 2006 Loop quantum cosmology and inhomogeneities *Gen. Rel. Grav.* **38** 1771
[20] Wilson-Ewing E unpublished
[21] Belinski V A, Khalatnikov I M and Lifshitz E M 1970 Oscillatory approach to a singular point in relativistic cosmology *Adv. Phys.* **19** 525
[22] Belinski V A, Khalatnikov I M and Lifshitz E M 1982 A general solution of the Einstein equations with a time singularity *Adv. Phys.* **31** 639
[23] Schutz B F 1971 Hamiltonian theory of a relativistic perfect fluid *Phys. Rev.* D **4** 3559
[24] Bojowald M, Cartin D and Khanna G 2007 Lattice refining loop quantum cosmology, anisotropic models and stability *Phys. Rev.* D **76** 064018
[25] Thiemann T 1998 Quantum spin dynamics (QSD) *Class. Quantum Grav.* **15** 839