Fractional $L$-intersecting families

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Abstract

Let $L = \{\alpha_1, \ldots, \alpha_r\}$, where for every $i \in [s]$, $\alpha_i \in [0,1)$ is an irreducible fraction. Let $F = \{A_1, \ldots, A_m\}$ be a family of subsets of $[n]$. We say $F$ is a fractional $L$-intersecting family if for every distinct $i, j \in [m]$, there exists an $\alpha_i \in L$ such that $|A_i \cap A_j| \in \{\alpha_i |A_i|, \alpha_j |A_j|\}$. In this paper, we introduce and study the notion of fractional $L$-intersecting families.

1 Introduction

Let $[n]$ denote $\{1,\ldots,n\}$ and let $L = \{l_1,\ldots,l_s\}$ be a set of $s$ non-negative integers. A family $F = \{A_1, \ldots, A_m\}$ of subsets of $[n]$ is $L$-intersecting if for every $A_i, A_j \in F$, $A_i \neq A_j$, $|A_i \cap A_j| \in L$. In 1975, it was shown by Ray-Chaudhuri and Wilson in [13] that if $F$ is $t$-uniform, then $|F| \leq \binom{n}{s}$. Setting $L = \{0, \ldots, s-1\}$, the family $F = \binom{[n]}{s}$ is a tight example to the above bound, where $\binom{[n]}{s}$ denotes the set of all $s$-sized subsets of $[n]$. In the non-uniform case, it was shown by Frankl and Wilson in the year 1981 (see [7]) that if we don’t put any restrictions on the cardinalities of the sets in $F$, then $|F| \leq \binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$. This bound is tight as demonstrated by the set of all subsets of $[n]$ of size at most $s$ with $L = \{0, \ldots, s-1\}$. The proof of this bound was found using the method of higher incidence matrices. Later, in 1991, Alon, Babai, and Suzuki in [2] gave an elegant linear algebraic proof to this bound. They showed that if the cardinalities of the sets in $F$ belong to the set of integers $K = \{k_1, \ldots, k_r\}$ with every $k_i > s-r$, then $|F|$ is at most $\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{s-r+1}$. The collection of all the subsets of $[n]$ of size at least $s-r+1$ and at most $s$ with $K = \{s-r+1, \ldots, s\}$ and $L = \{0, \ldots, s-1\}$ forms a tight example to this bound. In 2002, this result was extended by Grodlusz and Sudakov [8] to $k$-wise $L$-intersecting families. In 2003, Snevily showed in [14] that if $L$ is a collection of $s$ positive integers then $|F| \leq \binom{n}{s-1} + \binom{n}{s-2} + \cdots + \binom{n}{0}$. See [11] for a survey on $L$-intersecting families and their variants.

In this paper, we introduce a new variant of $L$-intersecting families called the fractional $L$-intersecting families. Let $L = \{\alpha_1, \ldots, \alpha_r\}$, where for every $i \in [s]$, $\alpha_i \in [0,1)$ is an irreducible fraction. Let $F = \{A_1, \ldots, A_m\}$ be a family of subsets of $[n]$. We say $F$ is a fractional $L$-intersecting family if for every distinct $i, j \in [m]$, there exists an $\alpha_i \in L$ such that $|A_i \cap A_j| \in \{\alpha_i |A_i|, \alpha_j |A_j|\}$. When $F$ is $t$-uniform, it is an $L'$-intersecting family where $L' = \{\alpha_i |A_i|, \ldots, \alpha_i |A_i|\}$
and therefore (using the result in [13]), \(|F| \leq \binom{n}{s}\). A tight example to this bound is given by the family \(F = \binom{[n]}{s}\) where \(L = \{\frac{0}{1}, \ldots, \frac{1}{1}\}\). So what is interesting is finding a good upper bound for \(|F|\) in the non-uniform case. Unlike in the case of the classical \(L\)-intersecting families, it is clear from the above definition that if \(A\) and \(B\) are two sets in a fractional \(L\)-intersecting family, then the cardinality of their intersection is a function of \(|A|\) or \(|B|\) (or both).

In Section 2.1, we prove the following theorem which gives an upper bound for the cardinality of a fractional \(L\)-intersecting family in the general case. We follow the convention that \(\binom{s}{i}\) is 0, when \(b > a\).

**Theorem 1.** Let \(n\) be a positive integer. Let \(L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\}\), where for every \(i \in [s]\), \(\frac{a_i}{b_i} \in [0, 1)\) is an irreducible fraction. Let \(F\) be a fractional \(L\)-intersecting family of subsets of \([n]\). Then, \(|F| \leq 2\binom{n}{s}g^2(t, n)\ln(g(t, n)) + (\sum_{i=1}^{s-1} \binom{n}{i}) g(t, n)\), where \(g(t, n) = \frac{2(2t + \ln n)}{\ln(2t + \ln n)}\) and \(t = \max_s \max_{i \in [s]}(b_i : i \in [s])\). Further,

(a) if \(s \leq n + 1 - 2g(t, n)\ln(g(t, n))\), then \(|F| \leq 2\binom{n}{s}g^2(t, n)\ln(g(t, n))\), and

(b) if \(t > n - c_1\), where \(c_1\) is a positive integer constant, then \(|F| \leq 2c_1\binom{n}{s}g(t, n)\ln(g(t, n)) + c_1 \sum_{i=1}^{s-1} \binom{n}{i}\).

Consider the following examples for a fractional \(L\)-intersecting family.

**Example 1.** Let \(L = \{\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots, \frac{n-1}{n}\}\), where we omit fractions, like \(\frac{2}{1}\), which are not irreducible. The collection of all the non-empty subsets of \([n]\) is a fractional \(L\)-intersecting family of cardinality \(2^n - 1\). Here, \(|L| = s = \Theta(n^2)\). Since \(t \geq s\), we can apply Statement (b) of Theorem 1 to get an upper bound of \(c_1(2^n - 1)\) which is asymptotically tight. In general, when \(L = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots, \frac{1}{n-c}, \ldots, \frac{n-1}{n-c}\}\), where \(c \geq 0\) is a constant, the set of all the non-empty subsets of \([n]\) of cardinality at most \(n - c\) is an example which demonstrates that the bound given in Statement (b) of Theorem 1 is asymptotically tight.

**Example 2.** Let us now consider another example where \(s = |L|\) is a constant. Let \(L = \{\frac{0}{s}, \frac{1}{s}, \ldots, \frac{s-1}{s}\}\). The collection of all the \(s\)-sized subsets of \([n]\) is a fractional \(L\)-intersecting family of cardinality \(\binom{n}{s}\). In this case, the bound given by Theorem 1 is asymptotically tight up to a factor of \(\ln^2 n / \ln m\). We believe that if \(F\) is a fractional \(L\)-intersecting family of maximum cardinality, where \(s = |L|\) is a constant, then \(|F| \in \Theta(n^s)\).

Coming back to the classical \(L\)-intersecting families, it is known that when \(F\) is an \(L\)-intersecting family where \(|L| = s = 1\), the Fisher’s Inequality (see Theorem 7.5 in [9]) yields \(|F| \leq n\). Study of such intersecting families was initiated by Ronald Fisher in 1940 (see [3]).

This fundamental result of design theory is among the first results in the field of \(L\)-intersecting families. Analogously, consider the scenario when \(L = \{\frac{a}{b}\}\) is a singleton set. Can we get a tighter (compared to Theorem 1) bound in this case? We show in Theorem 2 that if \(b\) is a constant prime we do have a tighter bound.

**Theorem 2.** Let \(n\) be a positive integer. Let \(G\) be a fractional \(L\)-intersecting families of subsets of \([n]\), where \(L = \{\frac{a}{b}\}, \frac{a}{b} \in [0, 1)\) and \(b\) is a prime. Then, \(|G| \leq (b-1)(n+1)\left[\frac{\ln n}{\ln b}\right] + 1\).

Assuming \(L = \{\frac{1}{b}\}\), Examples 3 and 4 in Section 3 give fractional \(L\)-intersecting families on \([n]\) of cardinality \(\frac{2n}{2b} - 2\) thereby implying that the bound obtained in Theorem 2 is asymptotically tight up to a factor of \(\ln n\) when \(b\) is a constant prime. We believe that the cardinality of such families is at most \(cn\), where \(c > 0\) is a constant.

The rest of the paper is organized in the following way: In Section 2.1, we give the proof of Theorem 1 after introducing some necessary lemmas in the beginning. In Theorem 6 in Section 2.2, we give an upper bound of \(n\) for fractional \(L\)-intersecting families on \([n]\) whose
member sets are ‘large enough’. In Section 3 we consider the case when \( L \) is a singleton set and give the proof of Theorem 2. Later in this section, in Theorem 5, we consider the case when the cardinalities of the sets in the fractional \( L \)-intersecting family are restricted. Finally, we conclude with some remarks, some open questions, and a conjecture.

2 The general case

2.1 Proof of Theorem 1

Before we move to the proof of Theorem 1, we introduce a few lemmas that will be used in the proof.

2.1.1 Few auxiliary lemmas

The following lemma is popularly known as the ‘Independence Criterion’ or ‘Triangular Criterion’.

**Lemma 3** (Lemma 13.11 in \[8\], Proposition 2.5 in \[3\]). For \( i = 1, \ldots, m \) let \( f_i : \Omega \to F \) be functions and \( v_i \in \Omega \) elements such that

(a) \( f_i(v_i) \neq 0 \) for all \( 1 \leq i \leq m \);

(b) \( f_i(v_j) = 0 \) for all \( 1 \leq j < i \leq m \).

Then \( f_1, \ldots, f_m \) are linearly independent members of the space \( F^\Omega \).

**Lemma 4.** Let \( p \) be a prime; \( \Omega = \{0,1\}^n \). Let \( f \in F^\Omega_p \) and let \( i \in \mathbb{F}_p \). For any \( A \subseteq [n] \), let \( V_A \in \{0,1\}^n \) denote its 0-1 incidence vector and let \( x_A = \Pi_{j \in A} x_j \). Assume \( f(V_A) \neq 0 \), for every \( |A| \not\equiv i \pmod{p} \). Then, the set of functions \( \{ x_A f : |A| \not\equiv i \pmod{p} \mbox{ and } |A| < p \} \) is linearly independent in the vector space \( F_p^{(0,1)^n} \) over \( F_p \).

**Proof.** Arrange every subset of \([n]\) of cardinality less than \( p \) in a linear order, say \( \prec \), such that \( A \prec B \) implies \( |A| \leq |B| \). For any two distinct sets \( A \) and \( B \), we know that \( x_A(V_B)f(V_B) = 0 \) when \( |B| \leq |A| \), where \( x_A(V_B) \) denote the evaluation of the function \( x_A \) at \( V_B \). Suppose

\[
\sum_{A : |A| \not\equiv i \pmod{p}, |A| < p} \lambda_A x_A f = 0
\]

has a non-trivial solution. Then, identify the first set, say \( A_0 \), in the linear order \( \prec \) for which \( \lambda_{A_0} \) is non-zero. Evaluate the functions on either side of the above equation at \( V_{A_0} \) to get \( \lambda_{A_0} = 0 \) which is a contradiction to our assumption.

The following lemma is from \[3\] (see Lemma 5.38).

**Lemma 5** (Lemma 5.38 in \[3\]). Let \( p \) be a prime; \( \Omega = \{0,1\}^n \). Let \( f \in F_p^\Omega \) be defined as \( f(x) = \sum_{i=1}^n x_i - k \). For any \( A \subseteq [n] \), let \( V_A \in \{0,1\}^n \) denote its 0-1 incidence vector and let \( x_A = \Pi_{j \in A} x_j \). Assume \( 0 \leq s,k \leq p - 1 \) and \( s + k \leq n \). Then, the set of functions \( \{ x_A f : |A| \leq s - 1 \} \) is linearly independent in the vector space \( F_p^\Omega \) over \( F_p \).

2.1.2 The proof

**Proof.** Let \( p \) be a prime and let \( p > t \). We partition \( \mathcal{F} \) into \( p \) parts, namely \( \mathcal{F}_0, \ldots, \mathcal{F}_{p-1} \), where \( \mathcal{F}_i = \{ A \in \mathcal{F} : |A| \equiv i \pmod{p} \} \).
Estimating $|\mathcal{F}_i|$, when $i > 0$.

Let $\mathcal{F}_i = \{A_1, \ldots, A_m\}$ and let $V_1, \ldots, V_m$ denote their corresponding 0-1 incidence vectors. Define $m$ functions $f_1$ to $f_m$, where each $f_j \in \mathbb{F}_p^{(0,1)^n}$, in the following way.

$$f_j(x) = \left\{ \begin{array}{cl} 1, & \text{if } x = V_j \\ 0, & \text{otherwise} \end{array} \right.$$  

Note that since $|A_j| \equiv i \pmod{p}$, $(V_j, V_j') \equiv i \pmod{p}$. Since $p > t$, for every $l \in [s]$, $i \not\equiv b_l i \pmod{p}$ unless $i \equiv 0 \pmod{p}$. So,

$$f_j(x) \left\{ \begin{array}{cl} \neq 0, & \text{if } x = V_j \\ 0, & \text{otherwise} \end{array} \right. \quad (1)$$

So, $f_j$’s are linearly independent in the vector space $\mathbb{F}_p^{(0,1)^n}$ over $\mathbb{F}_p$ (by Lemma 3). Since $x = (x_1, x_2, \ldots, x_n) \in \{0,1\}^n$, $x_i^r = x_i$ for any positive integer $r$. Each $f_j$ is thus an appropriate linear combination of distinct monomials of degree at most $s$. Therefore, $|\mathcal{F}_i| = m \leq \sum_{j=0}^{s} \binom{n}{j}$.

We can improve this bound by using the “swallowing trick” in a way similar to the way it is used in the proof of Theorem 1.1 in [2]. Let $f \in \mathbb{F}_p^{(0,1)^n}$ be defined as $f(x) = \sum_{j \in [n]} x_j - i$. From Lemma 4, we know that the set of functions $\{x_A f : |A| \neq i \pmod{p} \text{ and } |A| < s\}$ is linearly independent in the vector space $\mathbb{F}_p^{(0,1)^n}$ over $\mathbb{F}_p$.

**Claim 5.1.** $\{f_j : 1 \leq j \leq m\} \cup \{x_A f : |A| \neq i \pmod{p} \text{ and } |A| < s\}$ is a collection of functions that is linearly independent in the vector space $\mathbb{F}_p^{(0,1)^n}$ over $\mathbb{F}_p$.

In order to prove the claim, assume $\sum_{j=1}^{m} \lambda_j f_j + \sum_{A: |A| \leq s-1, |A| \neq i \pmod{p}} \mu_A x_A f = 0$ for some $\lambda_j, \mu_A \in \mathbb{F}_p$. Evaluating at $V_j$, all terms in the second sum vanish (since $f(V_j) = 0$) and by Equation 1 only the term with subscript $j$ remains of the first sum. We infer that $\lambda_j = 0$, for every $j$. It then follows from Lemma 4 that every $\mu_A$ is zero thus proving the claim.

Since each function in the collection of functions in Claim 5.1 can be obtained as a linear combination of distinct monomials of degree at most $s$, we can infer that $m + \sum_{j=0}^{s-1} \binom{n}{j} \leq \sum_{j=0}^{s} \binom{n}{j}$. We thus have

$$|\mathcal{F}_i| \leq \left\{ \begin{array}{cl} \binom{n}{s} + \binom{n}{i}, & \text{if } i < s \\ \binom{n}{s}, & \text{otherwise} \end{array} \right. \quad (2)$$

Observe that $i \leq p - 1$. We will shortly see that the prime $p$ we choose is always at most $2g(t, n) \ln(g(t, n))$, where $g(t, n) = \frac{(2t+\ln n)}{\ln(2t+\ln n)}$. So if $s \leq n + 1 - 2g(t, n) \ln(g(t, n))$, the condition $s+i \leq n$ (here $i$ stands for the symbol $k$ in Lemma 3) given in Lemma 3 is satisfied and therefore the more powerful Lemma 5 can be used instead of Lemma 4 while applying the swallowing trick. We can then claim that (proof of this claim is similar to the proof of Claim 5.1 and is therefore omitted) $\{f_j : 1 \leq j \leq m\} \cup \{x_A f : |A| < s\}$, where $f(x) = \sum_{j=0}^{n} x_j - i$ is a collection of functions that is linearly independent in the vector space $\mathbb{F}_p^{(0,1)^n}$ over $\mathbb{F}_p$ which can be obtained as a linear combination of distinct monomials of degree at most $s$. It then follows that $|\mathcal{F}_i| \leq \binom{n}{s}$.

In the rest of the proof, we shall assume the general bound for $|\mathcal{F}_i|$ given by Inequality 2. (Using the $\binom{n}{s}$ upper bound for $|\mathcal{F}_i|$ in place of Inequality 2 when $s \leq n + 1 - 2g(t, n) \ln(g(t, n))$ in the rest of the proof will yield the tighter bound for $|\mathcal{F}|$ given in Statement (a) in the theorem.)

Observe that we still do not have an estimate of $|A_0|$ since $i \equiv 2^d i \pmod{p}$ when $i \equiv 0 \pmod{p}$. To overcome this problem, consider the collection $P = \{p_q+1, \ldots, p_r\}$ of $r-q$
smallest primes with \( p_{q+1} < \cdots < p_r \) \((p_j \text{ denotes the } j\text{-th prime; } p_1 = 2, p_2 = 3, \text{ and so on})\) such that for every \( A \in \mathcal{F} \), there exists a prime \( p \in P \) with \( p \nmid |A| \). Note that if we repeat the steps done above for each \( p \in P \), we obtain the following upper bound.

\[
|\mathcal{F}| \leq (p_{q+1} + \cdots + p_r - (r - q)) \binom{n}{s} + (r - q) \sum_{j=1}^{s-1} \binom{n}{j}
\]

\[
< (r - q) \left( p_r \binom{n}{s} + \sum_{j=1}^{s-1} \binom{n}{j} \right)
\]

To obtain a small cardinality set \( P \) of the desired requirement, we choose the minimum \( r \) such that \( p_{q+1} p_{q+2} \cdots p_r > n \). If \( t > n - c_1 \), for some positive integer constant \( c_1 \), then \( P = \{p_{q+1}, \ldots, p_{q+c_1}\} \) satisfies the desired requirements of \( P \). We thus have,

\[
|\mathcal{F}| < \begin{cases} 
  c_1 \left( p_r \binom{n}{s} + \sum_{j=1}^{s-1} \binom{n}{j} \right), & \text{if } t > n - c_1 \text{ (here } c_1 \text{ is a positive integer constant)} \\
  r \left( p_r \binom{n}{s} + \sum_{j=1}^{s-1} \binom{n}{j} \right), & \text{otherwise}
\end{cases}
\]

(3)

The product of the first \( k \) primes is the **primorial function** \( p_k \# \) and it is known that \( p_k \# = e^{(1+o(1))k \ln k} \).

Given a natural number \( N \), let \( N \# \) denote the product of all the primes less than or equal to \( N \) (some call this the primorial function). It is known that \( N \# = e^{(1+o(1))N} \). Since \( \frac{\frac{p_{r+1} \#}{t \#}} = p_{k+1} p_{k+2} \cdots p_r \), setting \( \frac{e^{(1+o(1))r \ln r}}{e^{(1+o(1))t \ln t}} > n \), we get, \( r \leq \frac{2(2t + \ln t)}{\ln (2t + \ln t)} = g(t, n) \). Using the prime number theorem, the \( r \)th prime \( p_r \) is at most \( 2r \ln r \). Thus, we have \( p_r \leq 2g(t, n) \ln(g(t, n)) \).

Substituting for \( r \) and \( p_r \) in Inequality (3) gives the theorem.

\[ \square \]

### 2.2 When the sets in \( \mathcal{F} \) are ‘large enough’

In the following theorem, we show that when the sets in a fractional \( L \)-intersecting \( \mathcal{F} \) are ‘large enough’, then \( |\mathcal{F}| \) is at most \( n \).

**Theorem 6.** Let \( n \) be a positive integer. Let \( L = \{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\} \), where for every \( i \in [s] \), \( \frac{a_i}{b_i} \in [0, 1) \) is an irreducible fraction. Let \( \frac{a}{b} = \max\{\frac{a_1}{b_1}, \ldots, \frac{a_s}{b_s}\} \). Let \( \mathcal{F} \) be a fractional \( L \)-intersecting family of subsets of \([n]\) such that for every \( A \in \mathcal{F} \), \( |A| > \alpha n \), where \( \alpha = \max(\frac{1}{2}, \frac{a - b}{2b}) \). Then, \(|\mathcal{F}| \leq n\).

**Proof.** Let \( \mathcal{F} = \{A_1, A_2, \ldots, A_m\} \). For every \( A_i \in \mathcal{F} \), we define its \((+1, -1)\)-incidence vector as:

\[
X_{A_i}(j) = \begin{cases} 
 +1, & \text{if } j \in A_i \\
 -1, & \text{if } j \notin A_i.
\end{cases}
\]

(4)

We prove the theorem by proving the following claim.

**Claim 6.1.** \( X_{A_1}, \ldots, X_{A_m} \) are linearly independent in the vector space \( \mathbb{R}^n \) over \( \mathbb{R} \).

Assume for contradiction that \( X_{A_1}, \ldots, X_{A_m} \) are linearly dependent in the vector space \( \mathbb{R}^n \) over \( \mathbb{R} \). Then, we have some reals \( \lambda_{A_1}, \ldots, \lambda_{A_m} \) where not all of them are zeroes such that

\[
\lambda_{A_1} X_{A_1} + \cdots + \lambda_{A_m} X_{A_m} = 0.
\]

(5)

It is given that, for every \( A_i \in \mathcal{F} \), \( |A_i| > \frac{a}{b} \).

Let \( u = (1, 1, \ldots, 1) \in \mathbb{R}^n \) be the all ones vector. Then, \( \langle X_{A_i}, u \rangle > 0 \), for every \( A_i \in \mathcal{F} \). Therefore, if all non-zero \( \lambda_{A_i} \)'s in Equation (5) are of the same sign, say positive, then the inner product of \( u \) with the L.H.S of Equation (5) would be
non-zero which is a contradiction. Hence, we can assume that not all $\lambda_A, s$ are of the same sign.

We rewrite Equation (5) by moving all negative $\lambda_A, s$ to the R.H.S. Without loss of generality, assume $\lambda_A, \ldots, \lambda_k$ are non-negative and the rest are negative. Thus, we have

$$v = \lambda_A X_A + \cdots + \lambda_k X_k = - (\lambda_{k+1} X_{k+1} + \cdots + \lambda_m X_m),$$

where $v$ is a non-zero vector.

For any two distinct sets $A, B \in F$, \( \exists \frac{a_i}{b_i} \in L \) such that

$$\langle X_A, X_B \rangle = \begin{cases} n - 2|A| + \frac{4\alpha - 2b_i}{b_i}|B|, & \text{if } |A \cap B| = \frac{a_i}{b_i}|B|, \\ n - 2B| + \frac{4\alpha - 2b_i}{b_i}|A|, & \text{otherwise (that is, if } |A \cap B| = \frac{a_i}{b_i}|A|.\)$$

(6)

Since \( \frac{a}{b} = \max(\frac{a_1}{b_1}, \ldots, \frac{a_k}{b_k}) \), we have \( \langle X_A, X_B \rangle \leq n - 2|A| + \frac{4\alpha - 2b_i}{b_i}|B| \) or \( \langle X_A, X_B \rangle \leq n - 2|B| + \frac{4\alpha - 2b_i}{b_i}|A| \). Applying the fact that the cardinality of every set $S$ in $F$ satisfies $\alpha n < |S| \leq n$, where $\alpha = \max(\frac{1}{b}, \frac{4\alpha - 2b_i}{b_i})$, we get \( \langle X_A, X_B \rangle < 0 \). This implies that \( \langle v, v \rangle = \langle \lambda_A X_A + \cdots + \lambda_k X_k, -(\lambda_{k+1} X_{k+1} + \cdots + \lambda_m X_m) \rangle < 0 \) which is a contradiction. This proves the claim and thereby the theorem.

\[ \square \]

3 \hspace{1cm} \text{L is a singleton set}

As explained in Section 1, the Fisher’s Inequality is a special case of the classical $L$-intersecting families, where $|L| = 1$. In this section, we study fractional $L$-intersecting families with $|L| = 1$; a fractional variant of the Fisher’s inequality.

3.1 \hspace{1cm} \text{Proof of Theorem 2}

Statement of Theorem 2: Let $n$ be a positive integer. Let $G$ be a fractional $L$-intersecting families of subsets of $[n]$, where $L = \{\frac{a}{b}\}, \frac{a}{b} \in [0, 1),$ and $b$ is a prime. Then, $|G| \leq (b - 1)(n + 1)\left[\frac{\ln n}{\ln b}\right] + 1$.

Proof. It is easy to see that if $a = 0$, then $|G| \leq n$ with the set of all singleton subsets of $[n]$ forming a tight example to this bound.

So assume $a \neq 0$. Let $F = G \setminus H$, where $H = \{A \in G : b \not| |A|\}$. From the definition of a fractional $\frac{a}{b}$-intersecting family it is clear that $|H| \leq 1$. The rest of the proof is to show that $|F| \leq (b - 1)(n + 1)\left[\frac{\ln n}{\ln b}\right]$.

We do this by partitioning $F$ into $(b - 1)\lceil \frac{n}{\ln b} \rceil$ parts and then showing that each part is of size at most $n + 1$. We define $F_i$ as

$$F_i = \{A \in F||A| \equiv j \text{ (mod } i)\}.$$ 

Since $b$ divides $|A|$, for every $A \in F$, under this definition $F$ can be partitioned into families $F_{bi}^{k-1}$, where $2 \leq k \leq \lceil \log_b n \rceil$ and $1 \leq i \leq b - 1$. We show that, for every $i \in [b - 1]$ and for every $2 \leq k \leq \lceil \log_b n \rceil$, $|F_{bi}^{k-1}| \leq n + 1$.

In order to estimate $|F_{bi}^{k-1}|$, for each $A \in F_{bi}^{k-1}$, create a vector $X_A$ as follows:

$$X_A(j) = \begin{cases} \frac{1}{\sqrt{b^{k-2}}}, & \text{if } j \in A; \\ 0, & \text{otherwise}. \end{cases}$$

Note that, for $A, B \in F_{bi}^{k-1}$

$$\langle X_A, X_B \rangle \equiv \begin{cases} b \text{ (mod } b^2), & \text{if } A = B, \\ ai \text{ (mod } b), & \text{if } A \neq B, \end{cases}$$

(7)
Let $|F_{b^k-1}^{b^k}| = m$. Let $M_{k,i}$ denote the $m \times n$ matrix formed by taking $X_{A,i}$ as rows for each $A \in F_{b^k-1}^{b^k}$. Then, $|F_{b^k-1}^{b^k}| \leq n + 1$ can be proved by considering $B = M_{k,i} \times M_{k,i}^T$ and showing that $B - aiJ$, (where $J$ is the $m \times m$ all 1 matrix,) has full rank; determinant of $B - aiJ$ is non-zero since the only term not divisible by the prime $b$ in the expansion of its determinant comes from the product of all the diagonals (note that $a < b$, $i < b$, and since $b$ is a prime, we have $b \nmid a$).

We shall call $F$ a bisection closed family if $F$ is a fractional $L$-intersecting family where $L = \{\frac{1}{2}\}$. We have two different constructions of families that are bisection closed and are of cardinality $\frac{3n}{2} - 2$ on $[n]$.

**Example 3.** Let $n$ be an even positive integer. Let $B$ denote the collection of 2-sized sets that contain only 1 as a common element in any two sets, i.e. $\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}$; and let $C$ denote collection of 4-sized sets that contain only 1 as a common element, i.e. $\{1, 2, 3, 4\}, \{1, 2, 5, 6\}, \ldots, \{1, 2, n - 1, n\}$. It is not hard to see that $B \cup C$ is indeed bisection closed.

**Example 4.** The second example of a bisection closed family of cardinality $\frac{3n}{2} - 2$ comes from Recursive Hadamard matrices. A Recursive Hadamard matrix $H(k)$ of size $2^k \times 2^k$ can be obtained from $H(k - 1)$ of size $2^{k-1} \times 2^{k-1}$ as follows

$$H(k) = \begin{bmatrix} H(k - 1) & H(k - 1) \\ H(k - 1) & -H(k - 1) \end{bmatrix},$$

where $H(0) = 1$. Now consider the matrix:

$$M(k) = \begin{bmatrix} H(k - 1) & H(k - 1) \\ H(k - 1) & -H(k - 1) \\ H(k - 1) & J(k - 1) \end{bmatrix},$$

where $J(k - 1)$ denotes the $2^{k-1} \times 2^{k-1}$ all 1s’ matrix.

Let $M'(k)$ be the matrix obtained from $M(k)$ by removing the first and the $(2^k + 1)$th rows and replacing the -1’s by 1’s and 1’s by 0’s. $M'(k)$ is clearly bisection closed and has cardinality $\frac{3n}{2} - 2$, where $n = 2^k$.

### 3.2 Restricting the cardinalities of the sets in $F$

When $L = \{\frac{1}{2}\}$, where $b$ is a prime, Theorem 2 yields an upper bound of $O(\frac{b \log n \log n}{b^{\log n}})$ for $|F|$. However, we believe that when $|L| = 1$, the cardinality of any fractional $L$-intersecting family on $[n]$ would be at most $cn$, where $c > 0$ is a constant. To this end, we show in Theorem 8 that when the sizes of the sets in $F$ are restricted, we can achieve this.

The following lemma is crucial to the proof of Theorem 8.

**Lemma 7.** \[ [4] \] Let $A$ be an $m \times m$ real symmetric matrix with $a_{i,i} = 1$ and $|a_{i,j}| \leq \epsilon$ for all $i \neq j$. Let $tr(A)$ denote the trace of $A$, i.e., the sum of the diagonal entries of $A$. Let $rk(A)$ denote the rank of $A$. Then,

$$rk(A) \geq \frac{(tr(A))^2}{tr(A^2)} \geq \frac{m}{1 + (m - 1)\epsilon^2}.$$

**Proof.** Let $\lambda_1, \ldots, \lambda_m$ denote the eigenvalues of $A$. Since only $rk(A)$ eigenvalues of $A$ are non-zero, $(tr(A))^2 = (\sum_{i=1}^{m} \lambda_i)^2 = (\sum_{i=1}^{rk(A)} \lambda_i)^2 \leq rk(A) \sum_{i=1}^{rk(A)} \lambda_i^2 = rk(A) tr(A^2)$, where the inequality follows from the Cauchy-Schwartz Inequality. Thus, $rk(A) \geq \frac{(tr(A))^2}{tr(A^2)}$. Substituting $tr(A) = m$ and $tr(A^2) = m + m(m - 1)\epsilon^2$ in the above inequality proves the theorem. \[ \square \]
Theorem 8. Let $n$ be a positive integer and let $\delta > 1$. Let $\mathcal{F}$ be a fractional $L$-intersecting family of subsets of $[n]$, where $L = \left\{ \frac{a}{b} \right\}$, $\frac{a}{b} \in [0,1)$ is an irreducible fraction and for every $A \in \mathcal{F}$, $|A|$ in an integer in the range $\left[ \frac{b}{4(\delta-a)} n - \frac{b}{4\delta} \sqrt{n}, \frac{b}{4(\delta-a)} n + \frac{b}{4\delta} \sqrt{n} \right]$. Then, $|\mathcal{F}| < \frac{\delta^2}{\delta^2-1} n$.

Proof. For any $A \in \mathcal{F}$, let $Y_A \in \mathbb{R}^n$ be a vector defined as:

$$Y_A(j) = \begin{cases} \frac{1}{\sqrt{n}}, & \text{if } j \in A \\ -\frac{1}{\sqrt{n}}, & \text{if } j \not\in A. \end{cases}$$

Clearly, $\langle Y_A, Y_A \rangle = 1$. For any two distinct sets $A, B \in \mathcal{F}$, we have

$$\langle Y_A, Y_B \rangle = \begin{cases} \frac{n-2|A|+\frac{2b}{n}|B|}{n-2|B|+\frac{2b}{n}|A|}, & \text{if } |A \cap B| = \frac{\theta}{\delta}|B|, \\ \frac{n-2|A|+\frac{2b}{n}|B|}{n-2|B|+\frac{2b}{n}|A|}, & \text{otherwise (that is, if } |A \cap B| = \frac{\theta}{\delta}|A|). \end{cases}$$

(8)

Suppose $\mathcal{F} = \{A_1, \ldots, A_m\}$. Let $B$ be the $m \times n$ matrix with $Y_{A_1}, \ldots, Y_{A_m}$ as its rows. Then, from Equation (8) it follows that $BB^T$ is an $m \times m$ real symmetric matrix with the diagonal entries being 1 and the absolute value of any other entry being at most $\frac{1}{\sqrt{n}}$. Applying Lemma 7 we have $n \geq rk(BB^T) \geq \frac{m}{1+\frac{m}{2n}} \geq \frac{m}{1+\frac{m}{2n}}$. Thus, $n + \frac{m}{\delta^2} > m$ or $m < \frac{\delta^2}{\delta^2-1} n$. \qed

4 Discussion

In Theorem 1 we gave a general upper bound for $|\mathcal{F}|$, where $\mathcal{F}$ is a fractional $L$-intersecting family. In Section 3 we also gave an example to show that this bound is asymptotically tight up to a factor of $\frac{\log n}{m^2/n^2}$ when $s (= |L|)$ is a constant. However, when $s$ is a constant, we believe that $|\mathcal{F}| \in \Theta(n^s)$.

Consider the following special case for a fractional $L$-intersecting family $\mathcal{F}$, where $L = \{\frac{1}{2}\}$. We call such a family a bisection-closed family (see definition in Section 3).

Conjecture 9. If $\mathcal{F}$ is a bisection-closed family, then $|\mathcal{F}| \leq cn$, where $c > 0$ is a constant.

We have not been able to find an example of a bisection-closed family of size $2n$ or more.

The problem of determining a linear sized upper bound for the size of any bisection-closed family leads us to pose the following question:

Open problem 10. Suppose $0 < a_1 \leq \cdots \leq a_n$ are $n$ distinct reals. Let $\mathcal{M}_n(a_1, \ldots, a_n)$ denote the set of all symmetric matrices $M$ satisfying $m_{ij} \in \{a_i, a_j\}$ for $i \neq j$ and $m_{ii} = 0$ for all $i$. Then, does there exist an absolute constant $c > 0$ such that $rk(M) \geq cn$, for all $M \in \mathcal{M}_n(a_1, \ldots, a_n)$?

To see how this question ties in with our problem, suppose that a family $\mathcal{F} \subset \mathcal{P}([n])$ is a bisection-closed family, i.e., for $A, B \in \mathcal{F}$ and $A \neq B$ then $|A \cap B| \in \{|A|/2, |B|/2\}$. For simplicity, let us write $\mathcal{F} = \{A_1, \ldots, A_n\}$ and denote $|A_i| = a_i$ where the $a_i$ are arranged in ascending order. We say $A$ bisects $B$ if $|A \cap B| = |B|/2$. For each $A \in \mathcal{F}$, let $u_A \in \mathbb{R}^n$ where $u_A(i) = 1$ if $i \in A$ and $-1$ if $i \not\in A$. Then note that

$$\langle u_A, u_B \rangle = n - 2|A| \quad \text{if } A \text{ bisects } B,$$

$$= n - 2|B| \quad \text{if } B \text{ bisects } A,$$

$$\| u_A \|^2 = n.$$
If $X = \frac{1}{2}(nJ - M)$, where $J$ is the all ones matrix of order $m$, then $rk(X) \leq n + 1$. But note that $X \in M(a_1, \ldots, a_m)$. So, if the answer to the aforementioned open problem is ‘yes’, then $rk(X) \geq cm$. This gives $cm \leq r(X) \leq n + 1$ which in turn gives $m \leq c^{-1}(n + 1)$.

The problem of determining the maximum size of a fractional $L$-intersecting family is far from robust in the following sense. Suppose $L = \{1/2\}$ and we consider the problem of determining the size of an ‘$\varepsilon$-approximately fractional $L$-intersecting family,’ i.e., for any $A \neq B$ we have that at least one of $|A \Delta B|$, $|A \cap B|$ $\in (1/2 - \varepsilon, 1/2 + \varepsilon)$ for some $\varepsilon > 0$, then such families can in fact be exponentially large in size. Let each set $A_i$ be chosen uniformly and independently at random from $P([n])$. Then since each $|A_i|$ and $|A_i \cap A_j|$ are independent binomial $B(n, 1/2)$ and $B(n, 1/4)$ respectively, by standard Chernoff bounds (see [12], chapter 5), it follows (by straightforward computations) that one can get such a family of cardinality at least $e^{2\varepsilon^2 n/75}$. In fact this same construction gives super-polynomial sized families even if $\varepsilon = n^{-1/2+\delta}$ for any fixed $\delta > 0$.

Another interesting facet of the fractional intersection notion is the following extension of $l$-avoiding families [6, 10]. A set $B$ bisects another set $A$ if $|A \cap B| = \frac{|A|}{2}$. A family $F$ of even subsets of $[n]$ is called fractional ($\frac{1}{2}$)-avoiding (or bisection-free) if for every $A, B \in F$, neither $B$ bisects $A$ nor $B$ bisects $B$ (if we allow odd subsets in the definition of a fractional ($\frac{1}{2}$)-avoiding family, then the set of all the odd-sized subsets on $[n]$ is an example of one such family). Let $\vartheta(n)$ denote the maximum cardinality of a fractional ($\frac{1}{2}$)-avoiding family on $[n]$.

Let $A, B \subseteq [n]$ such that $|A| > \frac{2n}{3}$ and $|B| > \frac{2n}{3}$. It is not very hard to see that $|A \cap B| > n/3$ whereas $|A \cap (([n] \setminus B)| < n/3$. So, neither $A$ can bisect $B$ nor $B$ can bisect $A$. Therefore, if we construct a family $F = \{A \subseteq [n]|A > \frac{2n}{3}, |A| \text{ is even.}\}$, $F$ is fractional ($\frac{1}{2}$)-avoiding. Moreover, $|F| = \sum_{2i+1}^{\frac{n}{2}} \binom{n}{i} > 1.88^n$, for sufficiently large $n$ (using Stirling’s formula). Let us now try to find an upper bound to the cardinality of a fractional ($\frac{1}{2}$)-avoiding family. An application of a result of Frankl and Rödl [6, Corollary 1.6] gives the following theorem for the cardinalities of $l$-avoiding families as a corollary (see [10, Theorem 1.1]).

**Theorem 11.** [6, 10] Let $\alpha, \varepsilon \in (0, 1)$ with $\varepsilon \leq \frac{\alpha}{2}$. Let $k = \lfloor \alpha n \rfloor$ and $l \in [\max(0, 2k - n) + \varepsilon n, k - \varepsilon n]$. Then any $l$-avoiding family $A \subseteq \binom{[n]}{k}$ satisfies $|A| \leq (1 - \delta)\alpha^n$ where $\delta = \delta(\alpha, \varepsilon) > 0$.

For any fractional ($\frac{1}{2}$)-avoiding family $F$, any $F' \subseteq F$ consisting of sets of cardinality $l$ is $\frac{1}{2}$-avoiding. So, given any fractional ($\frac{1}{2}$)-avoiding family $F$, split $F$ into families $F_{\leq \frac{i}{2} - 1}$, $F_{\frac{i}{2}}$, $F_{\frac{i}{2} + 1}$, $F_{\geq \frac{i}{2} + 1}$. From Theorem 11 we know that each $F_i$ has a cardinality at most $(1 - \delta_i)\alpha^n$ for $\frac{i}{2} \leq l \leq 2\frac{n}{2}$. Let $\delta = \min(\delta_2, \ldots, \delta_{2\frac{n}{2}})$. Then $\sum_{i=2}^{2\frac{n}{2}} |F_i| \leq ((1 - \delta)2)^n$. Further, $|F_{\frac{i}{2} - 1}| \leq \sum_{i=0}^{\frac{i}{2}-1} \binom{n}{i}$ and $|F_{\frac{i}{2} + 1}| \leq \sum_{i=0}^{\frac{i}{2}+1} \binom{n}{i} < 2^{nH(\frac{1}{2})} < 1.89^n$, where $H(\nu) = -\nu \log_2 \nu - (1 - \nu) \log_2 (1 - \nu)$ is the binary entropy function. Thus, for sufficiently large values of $n$, $1.88^n \leq \vartheta(n) \leq ((1 - \varepsilon)2)^n$, for some $0 < \varepsilon \leq 0.06$.

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1A family $F$ is called $l$-avoiding if for each $A, B \in F$, $|A \cap B| \neq l$ for some $l \in [n]$. 
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