A lower bound for the tree-width of planar graphs with vital linkages

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Abstract

The disjoint paths problem asks, given an graph $G$ and $k + 1$ pairs of terminals $(s_0, t_0), \ldots, (s_k, t_k)$, whether there are $k + 1$ pairwise disjoint paths $P_0, \ldots, P_k$, such that $P_i$ connects $s_i$ to $t_i$. Robertson and Seymour have proven that the problem can be solved in polynomial time if $k$ is fixed. Nevertheless, the constants involved are huge, and the algorithm is far from implementable. The algorithm uses a bound on the tree-width of graphs with vital linkages, and deletion of irrelevant vertices. We give single exponential lower bounds both for the tree-width of planar graphs with vital linkages, and for the size of the grid necessary for finding irrelevant vertices.

1 Introduction

The disjoint paths problem is the following problem.

**Input:** Graph $G$, terminals $(s_0, t_0), \ldots, (s_k, t_k) \in V(G)^{2(k+1)}$

**Question:** Are there $k + 1$ pairwise vertex disjoint paths $P_0, \ldots, P_k$ in $G$ such that $P_i$ has endpoints $s_i$ and $t_i$?

It is a classic problem in algorithmic graph theory and it has many applications, e.g. in routing problems, PCB design and VLSI layout [1-9]. It is NP-hard [7], and it remains NP-hard on planar graphs [10]. Robertson and Seymour proved that for fixed $k$ it is decidable in polynomial time [14]. More precisely, they showed that it can be decided in FPT time $f(k) \cdot |V(G)|^3$, where $f$ is a computable function. For planar graphs, Reed et al. [11] gave an algorithm that solves the problem in time $g(k) \cdot |V(G)|$ for some computable function $g$. Both functions $f$ and $g$ are not really made explicit. They are huge towers of exponentiations, and the algorithms are far from being implementable, even for small $k$. Hence an important task is to improve the algorithms towards a better dependency on the parameter $k$.

We address this problem, exhibiting a lower bound on the parameter dependency arising from an essential technique used in both algorithms [14, 11].
This technique is finding irrelevant vertices [12], i.e. vertices in the input graph $G$, such that their deletion does not affect the answer to the problem. This is closely related to the theorem on vital linkages [15], and it is a main source of the impractical parameter dependency. The irrelevant vertices can be guaranteed if $G$ contains a sufficiently large subdivided grid as a subgraph (depending on $k$). Hence an interesting open question is to determine tight bounds on the size of the subdivided grid, necessary for $G$ to contain an irrelevant vertex. In this paper we give a lower bound by showing that a $(2^k + 1) \times (2^k + 1)$ grid may not suffice – even in planar graphs. Despite recent progress [8], the quest for good upper bounds is still open.

Indeed, grids occur naturally in many graphs that are relevant in practical applications of the disjoint paths problem, such as PCB and VLSI design. For many classes of NP-hard graph problems, attempts have been made to reduce the computational complexity by restricting the class of input graphs by an upper bound on a width parameter, such as tree-width and rank-width [5]. However, grids remain notoriously hard to deal with [13, 4, 6].

2 Preliminaries

Graphs are finite, undirected and simple. We denote the vertex set of a graph $G$ by $V(G)$ and the edge set by $E(G)$. Every edge is a two-element subset of $V(G)$. A graph $H$ is a subgraph of a graph $G$, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$ and
$E(H) \subseteq E(G)$. A path in a graph $G$ is a sequence $P = v_1, \ldots, v_n$ of pairwise distinct vertices of $G$, such that $\{v_i, v_{i+1}\} \in E(G)$ for all $1 \leq i \leq n - 1$. We use standard graph terminology as in [3].

**Definition 1 (Grid).** Let $m, n \geq 1$. The $(m \times n)$ grid is the graph $G_{m,n}$ given by

\[
V(G_{m,n}) := \{1, \ldots, m\} \times \{1, \ldots, n\}, \quad \text{and} \quad E(G_{m,n}) := \{(i_1, i_2), (j_1, j_2)\mid (i_1 = j_1 \text{ and } |j_2 - i_2| = 1) \text{ or } (|j_1 - i_1| = 1 \text{ and } i_2 = j_2)\}
\]

A subdivided grid is a graph obtained from a grid by replacing some edges of the grid by pairwise internally vertex disjoint paths of length at least one. **Embeddings** of graphs in the plane, **planar graphs** and **faces** are defined in the usual way.

**Definition 2 (Inner vertex).** Let $G$ be a grid embedded in the plane. An inner vertex of $G$ is a vertex that does not lie on the outer face of $G$.

**Definition 3 (Crossing).** We say that a path crosses the grid $G$ if it contains an inner vertex of $G$ and its endpoints are not inner vertices of $G$. For $k \in \mathbb{N}$ we say that a path $P = p_0, p_1, \ldots, p_n$ crosses $k$ times, if it can be split into $k$ paths $P_0 = p_0, p_1, \ldots, p_i, P_1 = p_i, p_{i+1}, \ldots, p_{i+2}, \ldots, P_{k-1} = p_{i-k}, p_{i-k+1}, \ldots, p_n$ with each $P_i, i = 0, \ldots, k-1$ crossing $G$.

### 3 The lower bound

From now on we will consider the case of a planar graph $G$ containing a grid $G$ with all $s_i$ and $t_i$ lying on the edge of or outside the grid $G$ (Figure 1). Intuitively, we construct our example from a grid $G$ of sufficient size. We add endpoints $s_0$ and $t_0$ on the boundary of the grid, mark the areas opposite to the grid as not part of the graph and connect $s_0$ to $t_0$ without crossing the grid.

Now we continue to mark vertices by $s_i$ and $t_i$ in such a way that $P_i$ has to cross $G$ as often as possible (in order to avoid crossing $P_j, j < i$). Once $s_i$ and $t_i$ have been added we remove the area opposite to the grid from $s_i$ to $t_i$ from the graph. Figure 2(a) shows the situation after doing this for $i$ up to 2. In this construction $P_0$ does not cross the grid at all, while $P_1$ crosses it once and $P_{i+1}$ crosses it twice as often as $P_i$ for $i > 0$: Let $k_i$ be the number of times $P_i$ crosses the grid. $k_0 = 0, k_1 = 1, k_{i+1} = 2 k_i, k_i = 2^{i-1}, i > 0$. After the last $P_i$ has been added, the areas opposite to the grid from both $s_i$ and $t_i$ are removed from the graph as seen in Figure 2(c).

Formally, to construct problem and graph with $k + 1$ terminals, we use a $(2^k + 1) \times (2^k + 1)$ grid. Let the vertices on the left border of the grid be $n_0, \ldots, n_{2^k}$. Terminals are assigned as follows: $t_0$ is the topmost vertex on the left border on the grid, $t_1$ the middle vertices on the right border. For all other terminals: $s_i := n_{2^k-i}, t_i := n_{3 \cdot 2^{k-i}}$. Then add edges going around the $t_i$ to the
For $i > 1$, $t_i = n_j - n_{j+1}, n_{j-2}, \ldots, n_{j-2^i+1} - n_{j+2^{i-1}-1}$, and on the right border of $G$ do the analogue for $t_1$. See Figure 2(d) for a graph constructed this way.

**Theorem 1.** There is only one solution to the constructed DPP, all vertices of the graph lie on paths of the solution and the grid is crossed $2^k-1$ times by such paths.

**Proof.** To connect $s_k$ to $t_k$ we need to cross $G$ at least once (Figure 3(a)). However, a solution in which $P_k$ crosses $G$ only once would block $P_{k-1}$. To connect $s_{k-1}$ to $t_{k-1}$ $P_k$ has to be routed around $t_{k-1}$, which requires leaving and reentering $G$ (Figure 3(b)). Thus inductively constructing a solution each $P_i$, $i > 0$ requires a crossing of $G$ and doubles the number of crossings in each $P_j$, $j > i$. The solution uses all edges on the left side of $G$ and uses all but one of the edges on the right side. Thus the only way to connect $s_0$ to $t_0$ is without crossing the grid. q. e. d.

In particular, $G$ has no irrelevant vertex in the sense of [12].

**Corollary 1.** There is a planar graph $G$ with $k+1$ pairs of terminals such that

- $G$ contains a $(2^k+1) \times (2^k+1)$ grid as a subgraph,
- the disjoint paths problem on this input has a unique solution,
- the solution uses all vertices of $G$; in particular, no vertex of $G$ is irrelevant.

**Conjecture.** There is a function $f \in 2^{O(k)}$ such that for every planar input $G$ together with $k+1$ pairs of terminals: if $G$ contains a subdivided $f(k) \times f(k)$ grid as a subgraph, then $G$ contains an irrelevant vertex.

### 4 Vital linkages and tree-width

We refer the reader to [2] for the definitions of tree-width and path-width.

**Definition 4** (Vital linkage). Let $L$ be a subgraph of $G$ such that every component of $L$ is a path and all vertex of $G$ are vertices of $L$. The pattern of $L$ is the set of vertices of degree 1 in $L$. $L$ is a vital linkage in $G$ if there is no other such $L$ that has the same pattern.

**Theorem 2** (Robertson and Seymour [15]). There are functions $f$ and $g$ such that if $G$ has a vital linkage with $k$ components then $G$ has tree-width at most $f(k)$ and path-width at most $g(k)$.

Recall that the $n \times n$ grid has path-width $n$ and tree-width $n$. Our example yields a lower bound for $f$ and $g$:

**Corollary 2.** Let $f$ and $g$ be as in Theorem 2. Then $2^{k-1} + 1 \leq f(k)$ and $2^{k-1} + 1 \leq g(k)$.
Figure 2: Construction of graph and solution
Proof. Looking at the graph $G$ and DPP constructed above the solution to the DPP is, due to its uniqueness, a vital linkage for the graph $G$. $G$ contains a $(2^k + 1) \times (2^k + 1)$ grid as a minor. The tree-width of such a grid is $2^k + 1$, its path-width $2^k + 1$. Thus we get lower bounds $2^{k-1} + 1 \leq f(k), g(k)$ for the functions $f$ and $g$. q. e. d.
Figure 3: Optimality of solution
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