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Coupled system of a fractional order differential equations with weighted initial conditions

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Abstract: Here, a coupled system of nonlinear weighted Cauchy-type problem of a diff-re-integral equations of fractional order will be considered. We study the existence of at least one integrable solution of this system by using Schauder fixed point Theorem. The continuous dependence of the uniqueness of the solution is proved.

Keywords: Riemann-Liouville differential operator; coupled system; weighted Cauchy type problem; continuous dependence

MSC 2010: 34A08, 34A12, 45D05

1 Preliminaries and introduction

The coupled system was studied by many authors (see [1] and [11]). Also, the weighted Cauchy-type problem (see [2]-[7]). In [5] the author studied the problem:

\[
\begin{aligned}
D^\alpha u(t) &= f(t, u) + \int_0^t g(t, s, u(s)) \, ds, \quad t > 0, \\
t^{1-a} u(t)|_{t=0} &= b, \quad 0 < a < 1, \quad b \in R,
\end{aligned}
\]

such that the functions \( f \) and \( g \) satisfies the following assumptions

1. \( t^{1-a} f(t, u) \) is continuous on \( R^+ \times C_{1-a}^0(R^+) \) and
   \[
   |f(t, u)| \leq t^{\mu} \varphi(t)|u|^{m_1}, \quad \mu \geq 0, \quad m_1 > 1,
   \]
2. \( s^{1-a} g(t, s, u(s)) \) is continuous on \( D_{R^+} \times C_{1-a}^0(R^+) \) where
   \[
   D_{R^+} = \{(t, s) \in R^+ \times R^+, \; 0 \leq s \leq t\},
   \]
   and
   \[
   |g(t, s, u(s))| \leq (t-s)^{\beta-1} s^\sigma \psi(s)|u|^{m_2}, \quad 0 < \beta < 1, \quad \sigma \geq 0, \quad m_2 > 1,
   \]
where \( \varphi(t) \) and \( \psi(s) \) are such that
3. \( \varphi(t) \) is continuous and \( t^{\mu-(1-a)m_1} \varphi(t) \) is continuous in case
   \[
   \mu - (1-a)m_1 < 0,
   \]

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(4) \( \psi(t) \) is continuous and \( t^{\sigma - (1 - a)m_2} \psi(t) \) is continuous in case
\[
\sigma - (1 - a)m_2 < 0.
\]

Under these assumptions the author proved the existence of at least one local solution in the space 
\( C^{\alpha}_{1-a}([0, h]) \), where for \( h > 0 \)
\[
C^{\alpha}_{1-a}([0, h]) = \{ v \in C^{\alpha}_{1-a}([0, h]) : \exists c \in R \text{ and } v' \in C^{\alpha}([0, h]) \text{ such that } v(t) = ct^{\alpha - 1} + I^{\alpha} v'(t) \},
\]
and
\[
C^{\alpha}_0([0, h]) = \{ v \in C^{\alpha}_0((0, h)) : \lim_{t \to 0^+} t' v(t) \text{ exists and finite} \},
\]
(the space \( C^{\alpha}_0((0, h)) \) is the usual space of continuous functions on \([0, h])\).

In comparison with earlier results, we study the coupled system of weighted Cauchy-type problems of differential equations of fractional order
\[
\begin{align*}
D^\alpha u(t) &= f_1(t, v(t), \int_0^t g_1(t, s, v(s)) ds), \\
D^\beta v(t) &= f_2(t, u(t), \int_0^t g_2(t, s, u(s)) ds),
\end{align*}
\]
where \( t \in I = [0, 1] \) and \( \alpha, \beta \in (0, 1) \) with the initial conditions
\[
t^{1-\alpha} u(t)|_{t=0} = k_1 \text{ and } t^{1-\beta} v(t)|_{t=0} = k_2,
\]
such that the functions \( f_i \) and \( g_i, i = 1, 2 \) satisfy the following assumptions:

(i) \( f_i : I \times R \times R \to R \) be a function with the following properties:
(a) for each \( t \in I, f_i(t, \cdot, \cdot) \) is continuous,
(b) for each \( (u, v) \in R \times R, f_i(\cdot, u, v) \) is measurable,
(c) there exist a real function \( t \to a(t), a \in L_1(I) \) and a positive constants \( b_1 \) and \( b_2 \) such that
\[
|f_i(t, u, v)| \leq a(t) + b_1 |u| + b_2 |v|, \quad \text{for each } t \in I, (u, v) \in R \times R;
\]

(ii) \( g_i : I \times I \times R \to R \) be a function with the following properties:
(a) for each \( (t, s) \in I \times I, g_i(t, s, \cdot) \) is continuous,
(b) for each \( u \in R, g_i(\cdot, s, u) \) is measurable,
(c) there exist a real function \( (t, s) \to k(t, s), k \in L_1(I) \) and a positive constant \( b_3 \) such that
\[
|g_i(t, s, u)| \leq k(t, s) + b_3 |u|, \quad \text{for each } (t, s) \in I \times I, u \in R;
\]

(iii) \( b_1 + b_2 b_3 < \Gamma(1 + \alpha) \) and \( b_1 + b_2 b_3 < \Gamma(1 + \beta) \).

Note that if \( \alpha = \beta, f_1 = f_2, g_1 = g_2, k_1 = k_2 = b \) and \( u(t) = v(t) \), then problem (2) will take the form
\[
\begin{align*}
D^\alpha u(t) &= f \left( t, u(t), \int_0^t g(t, s, u(s)) ds \right), \\
t^{1-\alpha} u(t)|_{t=0} &= b,
\end{align*}
\]
which is the generalization of problem (1).
2 Main results

2.1 Integral representation

**Lemma 2.1.** Let the assumptions (i-iii) be satisfied. If the solution of the coupled system (2) exists, then it can be represented by the coupled system of nonlinear integral equations of fractional order

\[ u(t) = k_1 t^{α-1} + \int_0^t \frac{(t - s)^{α-1}}{Γ(α)} f_1(s, v(s), \int_0^s g_1(s, θ, v(θ)) dθ) ds, \]

\[ v(t) = k_2 t^{β-1} + \int_0^t \frac{(t - s)^{β-1}}{Γ(β)} f_2(s, u(s), \int_0^s g_2(s, θ, u(θ)) dθ) ds. \]

**Proof.** Let \( u(t) \) be a solution of

\[ D^α u(t) = \frac{d}{dt} I^{1-α} u(t) = f_1(t, v(t), \int_0^t g_1(t, s, v(s)) ds). \]

Integrate both sides, we get

\[ I^{1-α} u(t) - I^{1-α} u(0) = f_1(t, v(t), \int_0^t g_1(t, s, v(s)) ds). \]

Operating by \( I^α \) on both sides of the last equation, we get

\[ I^α u(t) - I^α C = I^{1+α} f_1(t, v(t), \int_0^t g_1(t, s, v(s)) ds). \]

Differentiate both sides, we get

\[ u(t) - C_1 t^{α-1} = I^α f_1(t, v(t), \int_0^t g_1(t, s, v(s)) ds). \]

From the initial condition, we find that \( C_1 = k_1 \), then

\[ u(t) = k_1 t^{α-1} + \int_0^t \frac{(t - s)^{α-1}}{Γ(α)} f_1(s, v(s), \int_0^s g_1(s, θ, v(θ)) dθ) ds. \]

Similarly, we can get

\[ v(t) = k_2 t^{β-1} + \int_0^t \frac{(t - s)^{β-1}}{Γ(β)} f_2(s, u(s), \int_0^s g_2(s, θ, u(θ)) dθ) ds. \]

Therefore, the solution \( (u, v) \) of system (2) can be represented by system (3).

2.2 Existence of solution

Let \( L_1(I) \) be a class of Lebesgue integrable functions on the interval \( I \), with the norm \( ||x|| = \int_I |x(t)| dt \).

Define the operator \( T \) by

\[ T(u, v)(t) = (T_1 v(t), T_2 u(t)), \]
where
\[ T_1 v(t) = k_1 t^{a-1} + \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta))d\theta \right) ds, \]
\[ T_2 u(t) = k_2 t^{\beta-1} + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2 \left( s, u(s), \int_0^s g_2(s, \theta, u(\theta))d\theta \right) ds. \]

It is clear that the fixed point of the operator \( T \) is the solution of system (3).

**Theorem 2.1.** Assume that \( f_i \) and \( g_i \) satisfy the assumptions (i-iii). Then the coupled system of weighted Cauchy-type problems (2) has at least one solution \((u, v) \in L_1 \times L_1\).

**Proof.** Define
\[ X = \left\{ (u(t), v(t)) \mid (u(t), v(t)) \in L_1 \times L_1 \text{ and } \| (u, v) \|_{L_1 \times L_1} = \| u \|_{L_1} + \| v \|_{L_1} \leq r \right\}. \]

For \((u, v) \in X\), we have
\[
\| T_1 v \| = \int_0^1 \| T_1 v(t) \| dt \\
\leq \int_0^1 \left\| k_1 t^a \right\| dt + \int_0^1 \left\| \frac{(t-s)^{a-1}}{\Gamma(a)} f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta))d\theta \right) \right\| ds \| dt \\
\leq \left( \frac{k_1 t^a}{a} \right)_0^1 + \int_0^1 \frac{(t-s)^{a-1}}{\Gamma(a)} ds \| f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta))d\theta \right) \| ds \\
= \frac{k_1}{a} + \int_0^1 \frac{(1-s)^a}{\Gamma(1+a)} ds \| f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta))d\theta \right) \| ds \\
\leq \frac{k_1}{a} + \frac{1}{\Gamma(1+a)} \int_0^1 \left| f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta))d\theta \right) \right| ds \\
\leq \frac{k_1}{a} + \frac{1}{\Gamma(1+a)} \int_0^1 \left[ a(s) + b_1 |v(s)| + b_2 \int_0^s |g_1(s, \theta, v(\theta))| d\theta \right] ds \\
\leq \frac{k_1}{a} + \frac{1}{\Gamma(1+a)} \| a \| + \frac{b_1}{\Gamma(1+a)} \| v \|_{L_1} + \frac{b_2}{\Gamma(1+a)} \int_0^1 \left[ k(s, \theta) + b_3 |v(\theta)| \right] d\theta ds \\
\leq \frac{k_1}{a} + \frac{1}{\Gamma(1+a)} \| a \| + \frac{b_1}{\Gamma(1+a)} \| v \|_{L_1} + \frac{b_2}{\Gamma(1+a)} \| v \|_{L_1} + \frac{b_2 b_3}{\Gamma(1+a)} \| v \|_{L_1} \\
\leq \frac{k_1}{a} + \frac{1}{\Gamma(1+a)} \| a \| + \frac{b_1 + b_2 b_3}{\Gamma(1+a)} r_1 + \frac{b_2 k^*}{\Gamma(1+a)} \| v \|_{L_1} \leq r_1,
\]
where \( k^* = \int_0^1 \int_0^s k(s, \theta) d\theta ds \) and
\[
r_1 = \frac{k_1}{a} + \frac{\| a \|}{\Gamma(1+a)} + \frac{b_2 k^*}{\Gamma(1+a)\Gamma(1+a)}.
\]
Similarly, we get

\[ ||T_2 u|| \leq \frac{k_2}{\beta} + \frac{1}{\Gamma(1+\beta)} ||a|| + \frac{b_1 + b_2 b_3}{\Gamma(1+\beta)} r_2 + \frac{b_2 k^*}{\Gamma(1+\beta)} \leq r_2, \]

where

\[ r_2 = \frac{k_2}{\beta} + \frac{||a||}{\Gamma(1+\beta)} + \frac{b_1 k^*}{\Gamma(1+\beta)} \frac{1}{1 - \frac{b_1 + b_2 b_3}{\Gamma(1+\beta)}}. \]

Let

\[ r = \max \{r_1 + r_2\}. \]

Then,

\[ ||T(u, v)(t)|| = ||T_1 v(t), T_2 u(t)|| = ||T_1 v(t)|| + ||T_2 u(t)|| \leq r_1 + r_2 \leq r. \]

Therefore, for \( (u, v) \in X \), we get \( T(u, v) \in X \) and hence \( TX \in X \). Now, from the assumptions (i-a) - (ii-a), we deduce that \( T \) maps \( X \) into \( L_1 \times L_1 \) continuously. Moreover, we have

\[ ||f_i|| = \int_0^1 \left| f_i \left( t, w(t), \int_0^t g_i(t, s, w(s)) ds \right) \right| dt \]

\[ \leq \int_0^1 \left( a(t) + b_1 |w(t)| + b_2 \int_0^t \left| g_i(t, s, w(s)) \right| ds \right) dt \]

\[ \leq ||a|| + b_1 ||w|| + b_2 \int_0^1 \left[ k(t, s) + b_3 |w(s)| \right] ds dt \]

\[ \leq ||a|| + b_1 ||w|| + b_2 k^* + b_2 b_3 ||w||. \]

This estimation shows that \( f_i \) in \( L_1(I) \).

Now, we will use Kolmogorov compactness criterion (see [8]) to show that \( T \) is compact. So, let \( \mathcal{K} \) be a bounded subset of \( L_1 \). Then \( T(\mathcal{K}) \) is bounded in \( L_1(I) \). Now we show that \( (T_1 v)_h \to T_1 v \) in \( L_1(I) \) as \( h \to 0 \), uniformly with respect to \( T_1 v \in T \mathcal{K} \).

Indeed:

\[ ||(T_1 v)_h - T_1 v|| = \int_0^1 \left| (T_1 v)_h(t) - (T_1 v)(t) \right| dt \]

\[ = \int_0^1 \left| \frac{1}{h} \int_t^{t+h} (T_1 v)(s) ds - (T_1 v)(t) \right| dt \]

\[ \leq \int_0^1 \left( \frac{1}{h} \int_t^{t+h} \left| (T_1 v)(s) - (T_1 v)(t) \right| ds \right) dt \]

\[ \leq \frac{1}{h} \int_0^1 \left[ k_1 s^{\alpha - 1} - k_1 t^{\alpha - 1} \right] ds dt \]

\[ + \frac{1}{h} \int_0^1 \left[ -t^\alpha f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta)) d\theta \right) \right. \]

\[ \left. - t^\alpha f_1 \left( t, v(t), \int_0^t g_1(t, s, v(s)) ds \right) \right| ds dt. \]
since $f_1 \in L_1(I)$ we get that $I^{a} f_1(.) \in L_1(I)$. Moreover $t^{a-1} \in L_1(I)$. So, we have (see [10])

$$
\frac{1}{h} \int_{t}^{t+h} | k_1 s^{a-1} - k_1 t^{a-1} | \, ds \to 0
$$

and

$$
\frac{1}{h} \int_{t}^{t+h} | I^a f_1 \left( s, v(s), \int_{0}^{s} g_1(s, \theta, v(\theta)) d\theta \right) - I^a f_1 \left( t, v(t), \int_{0}^{t} g_1(t, s, v(s)) ds \right) | \, ds \to 0
$$

for a.e. $t \in I$. Therefore, $T_1(\mathbb{I})$ is relatively compact, that is, $T_1$ is a compact operator, similarly $T_2$ is a compact operator. Hence $T$ is a compact operator.

Therefore, Schauder fixed point Theorem (see [9]) implies that $T$ has a fixed point $(u, v)$ which is a solution of the coupled system (3).

To complete the proof, let $(u(t), v(t))$ be a solution of

$$
\begin{align*}
    u(t) &= k_1 t^{a-1} + I^a f_1 \left( t, v(t), \int_{0}^{t} g_1(t, s, v(s)) ds \right), \\
    v(t) &= k_2 t^{\beta-1} + I^\beta f_2 \left( t, u(t), \int_{0}^{t} g_2(t, s, u(s)) ds \right),
\end{align*}
$$

which gives

$$
t^{1-a} u(t)_{|_{t=0}} = k_1, \quad t^{1-\beta} v(t)_{|_{t=0}} = k_2.
$$

Operating on both sides of the first and second equations in (4) by $I^{1-a}$ and $I^{1-\beta}$ respectively, we get

$$
\begin{align*}
    I^{1-a} u(t) &= k_1 + I^{1-a} I^a f_1 \left( t, v(t), \int_{0}^{t} g_1(t, s, v(s)) ds \right), \\
    I^{1-\beta} v(t) &= k_2 + I^{1-\beta} I^\beta f_2 \left( t, u(t), \int_{0}^{t} g_2(t, s, u(s)) ds \right).
\end{align*}
$$

Differentiate both sides, we obtain

$$
\begin{align*}
    D^a u(t) &= f_1 \left( t, v(t), \int_{0}^{t} g_1(t, s, v(s)) ds \right), \\
    D^\beta v(t) &= f_2 \left( t, u(t), \int_{0}^{t} g_2(t, s, u(s)) ds \right).
\end{align*}
$$

2.3 Uniqueness of the solution

For the uniqueness of the solution we have the following theorem:

**Theorem 2.2.** Suppose that the functions $f_i$ and $g_i$ satisfy conditions (i-b), (ii-b) and (iii) of Theorem 2.1 in addition to the following assumptions:

$$
| f_i(t, u_1, v_1) - f_i(t, u_2, v_2) | \leq b_1 | u_1 - u_2 | + b_2 | v_1 - v_2 |, \quad i = 1, 2 \tag{5}
$$

and

$$
| g_i(t, s, v_1) - g_i(t, s, v_2) | \leq b_3 | v_1 - v_2 |, \quad i = 1, 2. \tag{6}
$$

Then the coupled system of weighted Cauchy-type problems (2) has a unique solution.
Proof. From assumption (5), we get
\[ |f_i(t, u, v) - f_i(t, 0, 0)| \leq b_1 |u| + b_2 |v|. \]
But since
\[ |f_i(t, u, v)| - |f_i(t, 0, 0)| \leq |f_i(t, u, v) - f_i(t, 0, 0)| \leq b_1 |u| + b_2 |v|, \]
therefore
\[ |f_i(t, u, v)| \leq |f_i(t, 0, 0)| + b_1 |u| + b_2 |v|, \]
i.e. assumptions (i - a) and (i - c) of theorem 2.1 are satisfied, similarly assumptions (ii - a) and (ii - c) of Theorem 2.1 are satisfied.

Then from Theorem 2.1 the solution exists. Now we prove the uniqueness of this solution:

Let \((u_1, v_1)\) and \((u_2, v_2)\) be two solutions of (3). Then
\[
\begin{align*}
\int_0^t \left( u_2(t) - u_1(t) \right) dt &\leq \int_0^t \int_0^s \frac{(t - s)^{a-1}}{I(a)} \left( b_1 |v_2(s) - v_1(s)| + b_2 \left| g_1(s, \theta, v_2(\theta)) - g_1(s, \theta, v_1(\theta)) \right| d\theta \right) ds dt, \\
\| u_2 - u_1 \|_{L_1} &\leq \int_0^1 \int_0^s \frac{(1-s)^{a}}{I(1+a)} \left( b_1 |v_2(s) - v_1(s)| + b_2 \left| g_1(s, \theta, v_2(\theta)) - g_1(s, \theta, v_1(\theta)) \right| d\theta \right) ds dt, \\
\| v_2 - v_1 \|_{L_1} &\leq \frac{b_1 + b_2 b_3}{I(1+a)} \| v_2 - v_1 \|_{L_1}. 
\end{align*}
\]
Similarly
\[
\| u_2 - u_1 \|_{L_1} \leq \frac{b_1 + b_2 b_3}{I(1+a)} |u_2 - u_1|_{L_1}.
\]
Therefore
\[
\| (u_2, v_2) - (u_1, v_1) \| = \| u_2 - u_1 \|_{L_1} + \| v_2 - v_1 \|_{L_1} \leq \frac{b_1 + b_2 b_3}{I(1+a)} \| v_2 - v_1 \|_{L_1} + \frac{b_1 + b_2 b_3}{I(1+a)} \| u_2 - u_1 \|_{L_1} \leq \| u_2 - u_1 \|_{L_1} + \| v_2 - v_1 \|_{L_1},
\]
which implies
\[
\| (u_2, v_2) - (u_1, v_1) \| = 0 \Rightarrow (u_2, v_2) = (u_1, v_1).
\]
This completes the proof.
2.4 Continuous dependence on initial data

Now we show that the solution of the coupled system (2) is depending continuously on initial data.

**Theorem 2.3.** Let the assumptions of Theorem 2.2 be satisfied. Then the solution of the weighted Cauchy-type problem (2) is depending continuously on initial data.

**Proof.** Let \( (u(t), v(t)) \) be a solution of the couple

\[
\begin{align*}
u(t) = k_1 t^{a-1} + \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} f_1(s, v(s), \int_0^s g_1(s, \theta, v(\theta)) d\theta) \, ds, \\
v(t) = k_2 t^{\beta-1} + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f_2(s, u(s), \int_0^s g_2(s, \theta, u(\theta)) d\theta) \, ds
\end{align*}
\]

and let \((\bar{u}(t), \bar{v}(t))\) be a solution of the above coupled system such that \(t^{1-a} \bar{u}(t)|_{t=0} = \bar{k}_1\) and \(t^{1-\beta} \bar{v}(t)|_{t=0} = \bar{k}_2\). Then

\[
\begin{align*}
u(t) - \bar{u}(t) &= (k_1 - \bar{k}_1) t^{a-1} + \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} \left[ f_1(s, \bar{v}(s), \int_0^s g_1(s, \theta, \bar{v}(\theta)) d\theta) \right] \, ds, \\
|\nu(t) - \bar{u}(t)| &\leq |k_1 - \bar{k}_1| t^{a-1} + \int_0^t \frac{(t-s)^{a-1}}{\Gamma(a)} \left[ b_1 |v(s) - \bar{v}(s)| ight. \\
&\quad + b_2 \int_0^s g_1(s, \theta, \bar{v}(\theta)) - g_1(s, \theta, v(\theta)) \, d\theta \left. \right] \, ds.
\end{align*}
\]

Therefore

\[
\int_0^1 |\nu(t) - \bar{u}(t)| \, dt \leq \frac{1}{a} |k_1 - \bar{k}_1| + \int_0^1 \frac{(t-s)^{a-1}}{\Gamma(a)} \left[ b_1 |v(s) - \bar{v}(s)| ight. \\
&\quad + b_2 \int_0^s g_1(s, \theta, \bar{v}(\theta)) - g_1(s, \theta, v(\theta)) \, d\theta \left. \right] \, ds, \\
||\nu - \bar{u}||_{L_1} &\leq \frac{1}{a} |k_1 - \bar{k}_1| + \int_0^1 \frac{(t-s)^{a-1}}{\Gamma(a)} \left[ b_1 |v(s) - \bar{v}(s)| ight. \\
&\quad + b_2 \int_0^s g_1(s, \theta, \bar{v}(\theta)) - g_1(s, \theta, v(\theta)) \, d\theta \left. \right] \, ds
\]

\[
= \frac{1}{a} |k_1 - \bar{k}_1| + \int_0^1 \frac{(1-s)^a}{\Gamma(1+a)} \left[ b_1 |v(s) - \bar{v}(s)| ight. \\
&\quad + b_2 \int_0^s g_1(s, \theta, \bar{v}(\theta)) - g_1(s, \theta, v(\theta)) \, d\theta \left. \right] \, ds
\]

\[
= \frac{1}{a} |k_1 - \bar{k}_1| + \int_0^1 \frac{(1-s)^a}{\Gamma(1+a)} \left[ b_1 |v(s) - \bar{v}(s)| ight. \\
&\quad + b_2 \int_0^s g_1(s, \theta, \bar{v}(\theta)) - g_1(s, \theta, v(\theta)) \, d\theta \left. \right] \, ds
\]
Therefore
\[ ||(u, v) - (\tilde{u}, \tilde{v})||_{L_1} = ||u - \tilde{u}||_{L_1} + ||v - \tilde{v}||_{L_1} \]
\[ \leq \frac{1}{\alpha} |k_1 - \tilde{k}_1| + \frac{1}{\Gamma(1 + \alpha)} \int_0^s \left( b_1 |v(s) - \tilde{v}(s)| + b_2 b_3 \int_0^s |v(\theta) - \tilde{v}(\theta)| d\theta \right) ds \]
\[ \leq \frac{1}{\alpha} |k_1 - \tilde{k}_1| + \frac{b_1 + b_2 b_3}{\Gamma(1 + \alpha)} ||v - \tilde{v}||_{L_1}, \]
Similarly
\[ ||v - \tilde{v}||_{L_1} \leq \frac{1}{\beta} |k_2 - \tilde{k}_2| + \frac{b_1 + b_2 b_3}{\Gamma(1 + \beta)} ||u - \tilde{u}||_{L_1}. \]
Therefore
\[ ||(u, v) - (\tilde{u}, \tilde{v})||_{L_1} \leq \frac{1}{\alpha} |k_1 - \tilde{k}_1| + \frac{b_1 + b_2 b_3}{\Gamma(1 + \alpha)} ||v - \tilde{v}||_{L_1} + \frac{1}{\beta} |k_2 - \tilde{k}_2| + \frac{b_1 + b_2 b_3}{\Gamma(1 + \beta)} ||u - \tilde{u}||_{L_1} \]
\[ \leq \max \left\{ \frac{1}{\alpha}, \frac{1}{\beta} \right\} \left( |k_1 - \tilde{k}_1| + |k_2 - \tilde{k}_2| \right) \]
\[ + (b_1 + b_2 b_3) \max \left\{ \frac{1}{\Gamma(1 + \alpha)}, \frac{1}{\Gamma(1 + \beta)} \right\} \left( ||v - \tilde{v}||_{L_1} + ||u - \tilde{u}||_{L_1} \right) \]
\[ \leq M^{*} \left( |k_1 - \tilde{k}_1| + |k_2 - \tilde{k}_2| \right) \]
\[ + N^{*} (b_1 + b_2 b_3) \left( ||v - \tilde{v}||_{L_1} + ||u - \tilde{u}||_{L_1} \right) \]
\[ = M^{*} \left( |k_1 - \tilde{k}_1| + |k_2 - \tilde{k}_2| \right) \]
\[ + N^{*} (b_1 + b_2 b_3) ||(u, v) - (\tilde{u}, \tilde{v})||_{L_1}, \]
where $M^{*} = \max \left\{ \frac{1}{\alpha}, \frac{1}{\beta} \right\}$ and $N^{*} = \max \left\{ \frac{1}{\Gamma(1 + \alpha)}, \frac{1}{\Gamma(1 + \beta)} \right\}$.

Therefore, if $|k_1 - \tilde{k}_1| < \frac{\delta_{(\alpha)}}{2}$ and $|k_2 - \tilde{k}_2| < \frac{\delta_{(\alpha)}}{2}$, then $||(u, v) - (\tilde{u}, \tilde{v})||_{L_1} < \varepsilon$. Now from the equivalence we get that the solution of the weighted Cauchy-type problem (2) is depending continuously on initial data.

3 Solution in $C_{1-a}([0, T])$

Now, define the space $C_{1-a}([0, T])$ by
\[ C_{1-a}([0, T]) = \left\{ u : t^{1-a}u(t) \in C([0, T]) \right\}, \]
with norm
\[ ||u||_{C_{1-a}} = ||t^{1-a} u||_{C} \]
and $C([0, T])$ is the space of continuous functions defined on $[0, T]$ with norm
\[ ||u||_{C} = \sup_{t \in [0, T]} |u(t)|. \]
Corollary 3.1. Let the assumptions of Theorem 2.1 satisfied. Then the coupled system of weighted Cauchy-type problems (2) has a solution \((u, v) \in C_{1-a}([0, T]) \times C_{1-a}([0, T])\).

Proof. Define

\[ Y = \{(u(t), v(t))(u(t), v(t)) \in C_{1-a} \times C_{1-a} : \|(u, v)||c_{1-a} \leq r'\}, \]

and define the subset \(Q_r\) by

\[ Q_r = \{(u(t), v(t))(u(t), v(t)) \in Y : \|(u(t), v(t)||_Y \leq r'\}, \]

where \(r' = \max\{r_1', r_2'\}, (r_1' \text{ and } r_2' \text{ will be indicated in the proof})\). The set \(Q_r\) is nonempty, closed and convex. Let \(T : Q_r \to Q_r\). For \((u, v) \in Q_r\), \(T\) is a continuous operator: indeed, if \(\{(u_n(t), v_n(t))\}\) is a sequence in \(Q_r\) which converges to \((u(t), v(t))\) for every \(t \in [0, T]\). Then

\[
\lim_{n \to \infty} T_1v_n(t) = k_1 t^{a-1} + \lim_{n \to \infty} \int_0^t \frac{(t-s)^{a-1}}{I(a)} f_1(s, v_n(s), \int_0^s g_1(s, \theta, v_n(\theta))d\theta) ds,
\]

from the assumptions and Lebesgue dominated convergence theorem, we get that

\[
\lim_{n \to \infty} T_1v_n(t) = T_1v(t).
\]

Similarly

\[
\lim_{n \to \infty} T_2u_n(t) = T_2u(t).
\]

Then

\[
\lim_{n \to \infty} T(u_n, v_n)(t) = T(u, v)(t).
\]

For \((u, v) \in Q_r\), we have

\[
|t^{1-a} T_1v(t)| \leq k_1 + t^{1-a} \int_0^t \frac{(t-s)^{a-1}}{I(a)} \left| f_1(s, v(s), \int_0^s g_1(s, \theta, v(\theta))d\theta) \right| ds
\]

\[
\leq k_1 + t^{1-a} \int_0^t \frac{(t-s)^{a-1}}{I(a)} \left[ a(s) + b_1 |v(s)| + b_2 \int_0^s \left| g_1(s, \theta, v(\theta)) \right| d\theta \right] ds
\]

\[
\leq k_1 + t^{1-a} I^{a} a(t) + b_1 t^{1-a} \int_0^t \frac{(t-s)^{a-1}}{I(a)} |v(s)| ds
\]

\[
+ b_2 t^{1-a} \int_0^t \frac{(t-s)^{a-1}}{I(a)} \int_0^s \left[ k(s, \theta) + b_3 |v(\theta)| \right] d\theta ds
\]

\[
= k_1 + t^{1-a} I^{a} a(t) + b_1 t^{1-a} \int_0^t \frac{(t-s)^{a-1}}{I(a)} s^{a-1} \left| v(s) \right| ds
\]

\[
+ b_2 t^{1-a} \int_0^t \frac{(t-s)^{a-1}}{I(a)} \int_0^s \left| k(s, \theta) \right| d\theta ds
\]

\[
+ b_2 b_3 t^{1-a} \int_0^t \frac{(t-s)^{a-1}}{I(a)} \int_0^s \left| \theta^{a-1} \theta^{1-a} \right| |v(\theta)| d\theta ds
\]

\[
\leq k_1 + t^{1-a} I^{a} a(t) + b_1 t^{1-a} \left| v \right|_{C_{1-a}} \int_0^t \frac{(t-s)^{a-1}}{I(a)} s^{a-1} ds
\]
where \( a' = \sup_{t \in [0, 1]} |\Gamma'| a(t) \). Then

\[
||T_1 v||_{C_{1-a}} \leq r'_1,
\]

where

\[
r'_1 = \frac{k_1 + \frac{a'}{\Gamma(a - \gamma + 1)} T^{1-\gamma} + \frac{b_1 k'}{\Gamma(a)} T^{1-a}}{1 - \left( \frac{b_1 (\Gamma(\alpha))}{\Gamma(2\alpha)} T^\alpha + \frac{b_2 (\Gamma(\alpha))}{\Gamma(2\alpha + 1)} T^{\alpha + 1} \right)}, \quad 0 < \gamma < \alpha.
\]

Similarly

\[
|T_2 u(t)| \leq k_2 + \frac{a'}{\Gamma(\beta - \gamma + 1)} T^{1-\gamma} + \frac{b_1 r'_2}{\Gamma(2\beta)} T^\beta + \frac{b_2 k'}{\Gamma(\beta)} T^{1-\beta} + \frac{b_2 r'_2}{\Gamma(2\beta + 1)} T^{\beta + 1},
\]

\[
||T_2 u||_{C_{1-a}} \leq r'_2,
\]

where

\[
r'_2 = \frac{k_2 + \frac{a'}{\Gamma(\beta - \gamma + 1)} T^{1-\gamma} + \frac{b_1 r'_2}{\Gamma(2\beta)} T^\beta + \frac{b_2 k'}{\Gamma(\beta)} T^{1-\beta} + \frac{b_2 r'_2}{\Gamma(2\beta + 1)} T^{\beta + 1}}{1 - \left( \frac{b_1 (\Gamma(\beta))}{\Gamma(2\beta)} T^\beta + \frac{b_2 (\Gamma(\beta))}{\Gamma(2\beta + 1)} T^{\beta + 1} \right)}, \quad 0 < \gamma < \beta.
\]

Then \( T_1 v(t) \) is uniformly bounded in \( Q_t \), similarly \( T_2 u(t) \) is uniformly bounded in \( Q_T \).

Since

\[
||T(u, v)(t)|| = ||T_1 v(t), T_2 u(t)|| = \max \left( ||T_1 v||_{C_{1-a}}, ||T_2 u||_{C_{1-a}} \right) \leq \max(r'_1, r'_2) \leq r'.
\]

Therefore, \( T \) is uniformly bounded in \( Q_T \).

Now, we show that \( T \) is a completely continuous operator.

Indeed, let \( \tau_1, \tau_2 \in [0, T], \tau_1 < \tau_2 \) such that \( |\tau_2 - \tau_1| < \delta \), we have

\[
\tau_2^{1-a} T_1 v(\tau_2) - \tau_1^{1-a} T_1 v(\tau_1) = \tau_2^{1-a} \int_0^\tau \frac{(\tau_2 - s)^{a-1}}{\Gamma(a)} f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta)) d\theta \right) ds - \tau_1^{1-a} \int_0^\tau \frac{(\tau_1 - s)^{a-1}}{\Gamma(a)} f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta)) d\theta \right) ds
\]
\[
\begin{align*}
&= \tau_2^{1-a} \int_0^{\tau_2} \left( \frac{(\tau_2 - s)^{a-1}}{I(a)} \right) f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta)) d\theta \right) ds \\
+ & \tau_2^{1-a} \int_0^{\tau_2} \left( \frac{(\tau_2 - s)^{a-1}}{I(a)} \right) f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta)) d\theta \right) ds \\
- & \tau_1^{1-a} \int_0^{\tau_1} \left( \frac{(\tau_1 - s)^{a-1}}{I(a)} \right) f_1 \left( s, v(s), \int_0^s g_1(s, \theta, v(\theta)) d\theta \right) ds \\
\leq & \left( \frac{(\tau_2 - \tau_1)^{a-1}}{I(a)} \right) \int_0^{\tau_1} \left[ a(s) + b_1|v(s)| \right] ds \\
+ & b_2 \int_0^{\tau_1} \left| g_1(s, \theta, v(\theta)) \right| d\theta \left| v\left( u \right) \right| ds \\
+ & \tau_2^{1-a} \int_0^{\tau_2} \left( \frac{(\tau_2 - s)^{a-1}}{I(a)} \right) \left[ a(s) + b_1|v(s)| \right] ds \\
\leq & \left( \tau_2^{1-a} - \tau_1^{1-a} \right) I^{a-\gamma} I^\gamma a(\tau_1) \\
+ & b_1 \left( \frac{(\tau_2 - \tau_1)^{a-1}}{I(a)} \right) \left[ a(s) + b_1|v(s)| \right] ds \\
+ & b_2 \left( \frac{(\tau_2 - \tau_1)^{a-1}}{I(a)} \right) \left[ k(s, \theta) + b_3|v(\theta)| \right] d\theta ds \\
+ & \tau_2^{1-a} I_1^{a-\gamma} I^\gamma a(\tau_2) + b_1 \tau_1^{1-a} \int_0^{\tau_2} \left( \frac{(\tau_2 - s)^{a-1}}{I(a)} \right) \left[ a(s) + b_1|v(s)| \right] ds \\
\leq & \frac{\alpha^*}{I(a - \gamma + 1)} \left( \tau_2^{1-a} - \tau_1^{1-a} \right) \tau_1^{a-\gamma} \\
+ & b_1 \left( \frac{(\tau_2 - \tau_1)^{a-1}}{I(a)} \right) \left| v \right| C_1 \left( \frac{I(\alpha) \tau_1^{a-\gamma}}{I(2\alpha)} \right) \\
+ & b_2 k^* \left( \frac{1}{I(a)} \right) \left( \tau_2^{1-a} - \tau_1^{1-a} \right)
\end{align*}
\]
Therefore, the conditions of the Schauder fixed point Theorem hold, which implies that \( (T(u, v))(t) \) is equi-continuous. By Arzela-Ascoli Theorem then \( (T(u, v))(t) \) is relatively compact. Therefore, the conditions of the Schauder fixed point Theorem hold, which implies that \( T \) has a fixed point in \( Q_r \). Then (2) has a solution \( (u, v) \in C_{1-a}([0, T]) \times C_{1-a}([0, T]) \).

References

[1] Chen Y., Chen D., Lv Z., The existence results for a coupled system of nonlinear fractional differential equations with multi-point boundary conditions, Bull. Iranian Math. Soc., 2012, 38(3), 607–624.
[2] El-Sayed A.M.A., Abd El-Salam Sh.A., Weighted Cauchy-type problem of a functional differ-integral equation, Electron. J. Qual. Theory Differ. Equ., 2007, 30, 1–9.
[3] El-Sayed A.M.A., Abd El-Salam Sh.A., \( L_q \)-solution of weighted Cauchy-type problem of a diffre-integral functional equation, Inter. J. Nonlinear Sci., 2008, 5(3), 281–288.
[4] Gaafar F.M., Cauchy-type problems of a functional differintegral equations with advanced arguments, J. Fract. Calc. Appl., 2014, 5(2), 71–77.
[5] Furati K.M., Tatar N.E., Long time behavior for a nonlinear fractional model, J. Math. Anal. Appl., 2007, 332, 441–454.
[6] Furati K.M., Tatar N.E., Power-type estimates for a nonlinear fractional differential equation, Nonlinear Analysis, 2005, 62, 1025–1036.
[7] Furati K.M., Tatar N.E., An existence result for a nonlocal fractional differential problem, J. Fract. Calc., 2004, 26, 43–51.
[8] Dugundji J., Granas A., Fixed Point Theory, Monografie Matematyczne, PWN, Warsaw, 1982.
[9] Deimling K., Nonlinear Functional Analysis, Springer-Verlag, 1985.
[10] Swartz C., Measure, Integration and Function Spaces, World Scientific, Singapore, 1994.
[11] Su X., Boundary value problem for a coupled system of nonlinear fractional differential equations, Appl. Math. Lett., 2009, 22(1), 64–69.