EXPANSION AND UNIFORM RESONANCE FREE REGIONS FOR
CONVEX COCOMPACT HYPERBOLIC SURFACES

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Abstract. Let $X = \Gamma \backslash \mathbb{H}^2$ be a convex cocompact hyperbolic surface with critical
exponent $\delta_\Gamma$. For each family of finite regular covers $X_n = \Gamma_n \backslash \mathbb{H}^2$ of $X$ we let $\mathcal{G}_n$ be
the family of Cayley graphs of the covering groups $G_n = \Gamma / \Gamma_n$ with respect to the
Schottky generators of $\Gamma$. Motivated by the work of Brooks [13], Burger [14, 15] and
Bourgain–Gamburd–Sarnak [7], we conjecture that the surfaces $X_n$ have a
uniform resonance-free strip if and only if the graphs $G_n$ form a family of expanders. Among
other things, we prove a new upper bound for the number of resonances for covers
of $X$ near the vertical line $\text{Re}(s) = \delta_\Gamma$, allowing us to prove the conjecture when the
groups $G_n$ are “strongly” quasirandom.

1. Introduction

1.1. Spectral gap and expanders. The relation between spectral gaps of the Laplacian on
compact Riemannian manifolds and expander graphs is well-known in the
literature, mainly through the work of Brooks [13] and Burger [14, 15].

There is a vast literature on expanders and we refer the reader to [22, 29, 42, 27] for a
comprehensive discussion. Expanders are highly-connected sparse graphs with many
applications in computer science and pure mathematics. They are usually defined in
terms of the Cheeger (isoperimetric) constant. For an undirected graph $\mathcal{G}$ with vertex
set $V$ and a subset $A \subset V$, the boundary $\partial A$ of $A$ is defined to be the set of edges
with one extremity in $A$ and the other in $V \setminus A$. Then the Cheeger constant of $\mathcal{G}$ is
defined as

$$h(\mathcal{G}) \mathrel{\overset{\text{def}}{=}} \min \left\{ \frac{|\partial A|}{|A|} : A \subset V, 1 \leq |A| \leq \frac{|V|}{2} \right\},$$

and a family of $k$-regular graphs $\mathcal{G}_n$ is said to form a family of expanders if $h(\mathcal{G}_n)$ is
bounded below by some uniform positive constant.

Given a compact Riemannian manifold $X$, a family of finite coverings $X_n$ of $X$ and
a set of generators of the fundamental group $\pi_1(X)$, we can naturally build a family
of graphs $\mathcal{G}_n$ as follows. Note that $\pi_1(X_n)$ can be viewed as a subgroup of $\pi_1(X)$ for
each $n$, so we define the vertices of $\mathcal{G}_n$ to be the cosets of $\pi_1(X_n)/\pi_1(X_n)$ (of which
there are finitely many) and we join two vertices of $\mathcal{G}_n$ by an edge if and only if the
corresponding cosets differ by left multiplication by an element in $S$. If we assume
furthermore that the $X_n$’s are regular coverings of $X$, then we can view each $\pi_1(X_n)$ as a normal
subgroup of $\pi_1(X)$. In this case the quotient $G_n = \pi_1(X)/\pi_1(X_n)$ forms a
group and the graphs $\mathcal{G}_n$ are the Cayley graphs of the group $G_n$ with respect to the set
$S_n \subset G_n$, where $S_n$ is the image of $S$ under the natural projection map $\pi_1(X) \to G_n$.

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The aforementioned result of Brooks [13] and Burger [14, 15] says that the Laplacians acting on the $X_n$’s have a uniform positive spectral gap if and only if the graphs $G_n$ form a family of expanders. Bourgain–Gamburd–Sarnak [7] extended this result to the case where $X$ is a geometrically finite hyperbolic surface with cusps. In these papers, the $X_n$’s are said to have uniform spectral gap if there exists some constant $c > 0$ such that
\[ \lambda_1(X_n) \geq \lambda_0(X_n) + c, \]
holds true for each $n$, where
\[ 0 \leq \lambda_0(X_n) < \lambda_1(X_n) \leq \cdots \]
is the sequence of $L^2$-eigenvalues of the positive Laplacian on $X_n$.

In the present paper, we focus on the case where $X = \Gamma \setminus \mathbb{H}^2$ is a convex cocompact hyperbolic surface, that is, a non-compact hyperbolic surface with a finite number of funnel ends and no cusps (see [3]). We conjecture that a similar relation between expanders and spectral gap remains true, under a suitable reformulation of the notion of spectral gap in terms of the resonances of $X$.

To properly state the conjecture, let us briefly review some aspects of the spectral theory of infinite-area hyperbolic surfaces, referring the reader to Borthwick’s book [3] for more details. For the remainder of this introduction we assume that $\Gamma$ is a finitely generated, non-cofinite Fuchsian group and we let $X = \Gamma \setminus \mathbb{H}^2$ be the associated hyperbolic surface. The limit set $\Lambda_\Gamma$ is defined as the set of accumulation points of orbits of $\Gamma$ acting on the hyperbolic plane $\mathbb{H}^2$. It is a subset of the boundary of $\mathbb{H}^2$ at infinity, which we identify with $\mathbb{R} = \mathbb{R} \cup \{\infty\}$. The limit set is a Cantor-like fractal and its Hausdorff dimension, which we denote by $\delta_\Gamma$, coincides with the exponent of convergence of Poincaré series for $\Gamma$.

The $L^2$-spectrum of the positive Laplacian $\Delta_X$ on $X$ has been described completely by Lax–Phillips [28]. The continuous spectrum of $\Delta_X$ coincides with the interval $[1/4, \infty)$, which does not contain any embedded $L^2$-eigenvalues. The pure point spectrum is empty if $\delta_\Gamma \leq \frac{1}{2}$ and finite and starting at $\delta_\Gamma (1 - \delta_\Gamma)$ if $\delta_\Gamma > \frac{1}{2}$. This description of the spectrum shows that the resolvent
\[ R_X(s) = (\Delta_X - s(1 - s))^{-1} : L^2(X) \to L^2(X) \]
is well-defined and analytic on the half-plane $\text{Re}(s) > \frac{1}{2}$ except at the finite set of poles corresponding to the pure point spectrum of $\Delta_X$. The resonances of $X$ are the poles of the meromorphic continuation of
\[ R_X(s) : C_0^\infty(X) \to C^\infty(X) \]
to the whole complex plane. This continuation can be deduced using the analytic Fredholm theorem together with an adequate parametrix provided by Guillopé–Zworski [21]. In the sequel, we denote by $\mathcal{R}_X$ the set of resonances of $X$. The resolvent operator (1) has a simple pole at $s = \delta_\Gamma$ and no other poles in the half-plane $\text{Re}(s) \geq \delta_\Gamma$, by a result of Patterson [36]. Put differently, $s = \delta_\Gamma$ is the resonance of $X$ with the largest real part.

The Selberg zeta function is defined for $\text{Re}(s) \gg 1$ by the infinite product
\[ Z_\Gamma(s) \overset{\text{def}}{=} \prod_{k=0}^{\infty} \prod_{[\gamma] \in [\Gamma]_{\text{prim}}} \left( 1 - e^{-(s+k)\ell(\gamma)} \right), \]
where the inner product is taken over the set $[\Gamma]_{\text{prim}}$ of conjugacy classes of primitive hyperbolic elements of $\Gamma$. By the work of Patterson–Perry [37] resonances appear as
zeros of the Selberg zeta function and the multiplicity of s as a zero of $Z_{\Gamma}(s)$ is equal to the rank of the residue operator of the resolvent $R_X(s)$. In particular, $s = \delta_{\Gamma}$ is a simple zero of $Z_{\Gamma}(s)$. Throughout this work we will call $s = \delta_{\Gamma}$ the “trivial” zero of $Z_{\Gamma}(s)$ and the “trivial” resonance of $X$.

From the work of Naud [30] we know that every non-elementary, convex cocompact hyperbolic surface $X$ has a spectral gap in the following sense: there exists some $\eta > 0$ such that there are no non-trivial resonances for $X$ in the half-plane $\text{Re}(s) > \delta_{\Gamma} - \eta$ (see also the more recent work of Bourgain–Dyatlov [4]). Finding resonance-free regions has a long history and applications in many settings. In this paper we are interested in uniform resonance-free regions in infinite families of finite degree covers of $X$.

By Button’s work [16] every convex cocompact hyperbolic surface $X$ is isometric to a quotient $\Gamma \backslash \mathbb{H}^2$ where $\Gamma$ is a Schottky group. Schottky groups stand out, among other Fuchsian groups, by their simple geometric construction (see §2.1 for definitions and properties).

In what follows, we fix a non-elementary Schottky group $\Gamma$ and a family of covers $X_n = \Gamma_n \backslash \mathbb{H}^2$ where each $\Gamma_n$ is a finite-index normal subgroup of $\Gamma$. We let $G_n = \Gamma / \Gamma_n$ be the (finite) covering groups and we let $G_n$ be the associated family of Cayley graphs of $G_n$ with respect to the fixed set of Schottky generators (this is a special set of generators which arise from the geometric construction of $\Gamma$). We can now formulate the following conjecture:

**Conjecture 1.1.** Let assumptions and notations be as above. Then the graphs $G_n$ form a family of expanders if and only if there exists $\eta > 0$ such that for each $n$ the Selberg zeta function $Z_{\Gamma_n}(s)$ has no non-trivial zeros in the half-plane

$$\text{Re}(s) \geq \delta_{\Gamma} - \eta.$$ 

Equivalently, the $(G_n)_n$’s form a family of expanders if and only if the surfaces $X_n = \Gamma_n \backslash \mathbb{H}^2$ have a uniform resonance-free strip, in the sense that for each $n$,

$$\mathcal{R}_{X_n} \cap \{\text{Re}(s) \geq \delta_{\Gamma} - \eta\} = \{\delta_{\Gamma}\}.$$ 

**Remark 1.2.**

1. A Fuchsian group $\Gamma$ is said to be “non-elementary” if it is generated by more than one element. Conjecture 1.1 fails for elementary groups $\Gamma = \langle \gamma \rangle$. For elementary groups the quotient $\langle \gamma \rangle \backslash \mathbb{H}^2$ is a hyperbolic cylinder, in which case the resonances are given by an explicit formula [3]. It is clear from this formula that there is no spectral gap.

2. For every finite-index, normal subgroup $\Gamma'$ of $\Gamma$ we have $\Lambda_{\Gamma'} = \Lambda_{\Gamma}$. In particular, for any given family $\Gamma_n$ satisfying the assumptions of Conjecture 1.1, we have $\delta_{\Gamma_n} = \delta_{\Gamma}$ and the point $s = \delta_{\Gamma}$ is a common simple resonance of the surfaces $X_n$ and therefore also a common simple zero of the Selberg zeta function $Z_{\Gamma_n}(s)$. Note also that every zero of $Z_{\Gamma}(s)$ is a zero of $Z_{\Gamma_n}(s)$. This follows directly from the Venkov–Zograf factorization formula in §2.3.

3. For $\delta_{\Gamma} > \frac{1}{2}$ the statement in Conjecture 1.1 follows from [7, Theorem 1.2], since each resonance $s$ in the half-plane $\text{Re}(s) > \frac{1}{2}$ gives an $L^2$-eigenvalue $\lambda = s(1-s)$ of the Laplacian. On the other hand, if $\delta_{\Gamma} \leq \frac{1}{2}$ then $X = \Gamma \backslash \mathbb{H}^2$ must be convex cocompact by the results of Beardon [1, 2]. Hence the case $\delta_{\Gamma} \leq \frac{1}{2}$ comes under the purview of Conjecture 1.1.

4. We finally point out that the methods used by Brooks [13], Burger [14, 15] and Bourgain–Gamburd–Sarnak [7] rely solely on $L^2$-methods. These methods are no longer available in the range $\delta_{\Gamma} \leq \frac{1}{2}$. 


1.2. Statement of results. The purpose of this paper is to prove some results towards Conjecture 1.1. For the rest of this introduction we fix

a Schottky group $\Gamma$ with Schottky generators $\gamma_1^\pm, \ldots, \gamma_m^\pm$ with $m \geq 2$, and we set $X = \Gamma \backslash \mathbb{H}^2$.

While expanders are usually defined in terms of the Cheeger constant, it will be more convenient for us to use the spectral notion of expansion. Let $G$ be a finite group, let $\varepsilon > 0$ and let $g_1, \ldots, g_m \in G$. Define the measure

$$\mu = \frac{1}{2m} \sum_{j=1}^{m} \left( \delta_{g_j} + \delta_{g_j^{-1}} \right)$$

(2)
on

on $G$, where $\delta_x$ is the Dirac measure at $x$. Consider the convolution operator

$$T: f \mapsto f \ast \mu$$

on the Hilbert space $L^2(G)$ of functions $f: G \to \mathbb{C}$ with norm

$$\|f\|_{L^2(G)} \overset{\text{def}}{=} \left( \frac{1}{|G|} \sum_{x \in G} |f(x)|^2 \right)^{1/2},$$

where convolution is defined by

$$(f \ast \mu)(x) \overset{\text{def}}{=} \sum_{y \in G} f(y) \mu(y^{-1}x) = \frac{1}{2m} \sum_{j=1}^{m} \left( f(xg_j) + f(xg_j^{-1}) \right).$$

We say that the set $\{g_1, \ldots, g_m\}$ is $\varepsilon$-expanding if one has

$$\|Tf\|_{L^2(G)} \leq (1 - \varepsilon)\|f\|_{L^2(G)}$$

for all functions $f: G \to \mathbb{C}$ of mean zero. Equivalently, $\{g_1, \ldots, g_m\}$ is $\varepsilon$-expanding if all the eigenvalues of the self-adjoint operator $T$, other than the trivial eigenvalue 1, lie in the interval $[-1 + \varepsilon, 1 - \varepsilon]$. This notion of expansion is equivalent with the notion of two-sided expanders (which is closely related to but slightly stronger than the definition of expanders in terms of the Cheeger constant). In particular, if $\{g_1, \ldots, g_m\}$ is $\varepsilon$-expanding then the Cayley graph $Cay(G, \{g_1^\pm, \ldots, g_m^\pm\})$ is an expander in the sense that its Cheeger constant is bounded below by some constant depending on $\varepsilon$.

Note that if $\pi: \Gamma \to G$ is a surjective homomorphism, then its kernel $\Gamma' \overset{\text{def}}{=} \ker(\pi)$ is a finite-index, normal subgroup of $\Gamma$ with $\Gamma/\Gamma' \cong \con$. In other words, the hyperbolic surface $X' \overset{\text{def}}{=} \Gamma' \backslash \mathbb{H}^2$ is a finite, regular $G$-cover of $X = \Gamma \backslash \mathbb{H}^2$. Our first main result states that the size of the resonance-free neighbourhood for $X'$ around $s = \delta_\Gamma$ depends solely on the expansion constant of the elements $\pi(\gamma_1), \ldots, \pi(\gamma_m) \in G$ and the Schottky data of $\Gamma$.

Theorem 1.3 (First main theorem). Let $G$ be a finite group and let $\Gamma' = \ker(\pi)$ be the kernel of a surjective homomorphism $\pi: \Gamma \to G$. Assume that there exists $\varepsilon > 0$ such that $\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \subseteq G$ is $\varepsilon$-expanding. Then there exists $\eta = \eta(\varepsilon, \Gamma) > 0$ such that Selberg zeta function $Z_{\Gamma'}(s)$ has no non-trivial zero with $|s - \delta_\Gamma| < \eta$.

For our next result we will consider the following generalization of the Selberg zeta function: given a finite-dimensional representation $\rho, V$ of the group $\Gamma$, the $\rho$-twisted Selberg zeta function is defined by the infinite product

$$Z_\Gamma(s, \rho) \overset{\text{def}}{=} \prod_{k=0}^{\infty} \prod_{[\gamma] \in [\Gamma]_{\text{prim}}} \det \left( I - \rho(\gamma) e^{-(s+k)\ell(\gamma)} \right).$$

(3)
We refer to §2.2 and §2.3 for more details. We define the zero-counting functions

$$N_{\Gamma'}(\sigma, T) \overset{\text{def}}{=} \# \{ s \in \mathbb{C} : Z_{\Gamma'}(s) = 0 \text{ with } \Re(s) \geq \sigma \text{ and } \Im(s) \in [T, T+1] \}$$

and

$$N_{\Gamma}(\sigma, T, \rho) \overset{\text{def}}{=} \# \{ s \in \mathbb{C} : Z_{\Gamma}(s, \rho) = 0 \text{ with } \Re(s) \geq \sigma \text{ and } \Im(s) \in [T, T+1] \}$$

where the zeros are counted with multiplicities. Note that the number $N_{\Gamma'}(\sigma, T)$ is equal to the number of resonances of the surface $X'$ in the rectangular region of the complex plane $[\sigma, \delta_{\Gamma}] + i[T, T+1]$.

We have the following:

**Theorem 1.4** (Second main theorem). Let $G$ be a finite group, let $\pi : \Gamma \to G$ be a surjective homomorphism and let $\Gamma' = \ker(\pi)$. Suppose that $\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \subseteq G$ is an $\varepsilon$-expanding set for some $\varepsilon > 0$. Then there exist constants $C = C_i(\varepsilon, \Gamma) > 0$ ($i = 1, 2$) such that

\[ N_{\Gamma'}(\sigma, T) \leq C_1 \log(2|G|)^\delta |G|^{\min\{C_2(\delta-\sigma), 1\}} (T^\delta) \]

and for every irreducible representation $\rho$ of $G$ we have

\[ N_{\Gamma}(\sigma, T, \rho) \leq C_1 \log(2|G|)^\delta |G|^{\min\{C_2(\delta-\sigma), 1\}} \frac{(T^\delta)}{\dim(\rho)}. \]

**Remark 1.5.**

1. We stress that the constants in Theorem 1.4 depend only on the Schottky data arising from the construction of the group $\Gamma$ and the expansion constant $\varepsilon > 0$. We made no effort to effectively compute them. Without the aspect of coverings and expansion, Theorem 1.4 follows immediately from the main result of Guillopé–Lin–Zworski [20]. The emphasis of Theorem 1.4 is in understanding the behaviour of the resonance counting function of covers $X' \to X$ as the size $|G|$ of the covering group tends to infinity.

2. We know from [26] that large abelian covers of $X$ have an abundance of resonances close to $s = \delta_{\Gamma}$. In fact, for every $\sigma < \delta_{\Gamma}$ we can construct covers $X'$ with arbitrarily large abelian covering groups $G$ such that $N_{\Gamma'}(\sigma, T) \gg |G|$. This speaks to the well-known fact that Cayley graphs of abelian groups can never form a family of expanders.

Combining Theorem 1.4 above with the high-frequency result of Magee–Naud [32, Theorem 1.8], we obtain as a corollary a resonance-free strip in the complex plane if, in addition to expansion, we assume that the group $G$ is *quasirandom* in a strong sense or, more generally, if it is a product of such groups. To adequately state this result, we let $d(G)$ be the quasirandom degree of $G$,

\[ d(G) \overset{\text{def}}{=} \min_{\rho \in \text{irrep}_0(G)} \dim(\rho), \]

that is, the minimum possible dimension of a non-trivial, irreducible linear representation of $G$. We then have:

**Corollary 1.6** (Main corollary). Let $G$ be a finite group, let $\pi : \Gamma \to G$ be a surjective homomorphism, and let $\Gamma' = \ker(\pi)$. Assume that there are positive constants $\varepsilon, c$ and $A$ such that

(i) the set $\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \subseteq G$ is $\varepsilon$-expanding and
(ii) \( G = G_1 \times \cdots \times G_r \) is a direct product of groups \( G_i \) satisfying \( d(G_i) \geq c |G_i|^A \) for all \( i \in \{1, \ldots, r\} \).

Then there exists \( \eta = \eta(\Gamma, \varepsilon, c, A) > 0 \) such that the Selberg zeta function \( Z_{\Gamma^*}(s) \) has no non-trivial zeros in the half-plane \( \text{Re}(s) \geq \delta - \eta \).

**Remark 1.7.** Let us give a couple of remarks on Corollary 1.6:

1. By an argument in Tao’s notes [45, Proposition 4] we can drop the expander hypothesis in the statement of Corollary 1.6 and replace it with a more technical “flattening” assumption: there exist constants \( 0 < \beta < \frac{A}{2} \) and \( C > 0 \) such that for some \( N \leq C \log |G| \), we have
   \[
   \|\mu^{*N}\|_{L^2(G)} \leq C|G|^{-1/2+\beta}
   \]
   where \( \mu^{*N} \) is the \( N \)-fold convolution of the measure \( \mu \) defined in (2). In fact, when taken together, this bound and the assumption that each factor \( G_i \) has \( d(G_i) \geq |G_i|^{3A} \) imply that the set \( \{g_1, \ldots, g_n\} \) is \( \varepsilon \)-expanding for some \( \varepsilon = \varepsilon(A, \beta, C) > 0 \).
2. By the work of Landazuri–Seitz [47] there is an abundance of groups satisfying the second condition in Corollary 1.3. Indeed, every finite simple group of Lie type \( G \) is quasirandom in the sense that \( d(G) \geq c|G|^A \) holds true for some absolute constants \( c > 0 \) and \( A > 0 \).
3. Following several prior developments, Breuillard–Green–Guralnick–Tao [12] showed that simple groups of Lie type always give rise to two-sided expanders. More precisely, for any such group \( G \) there exists a constant \( \varepsilon > 0 \) depending only on the rank of \( G \) such that if the elements \( g_1, \ldots, g_r \in G \) are selected uniformly at random, they generate the group \( G \) and they form an \( \varepsilon \)-expanding set with probability \( 1 - o(1) \) as \( |G| \to \infty \). Note that this can be used in conjunction with Corollary 1.6 to produce \( G \)-covers of \( X \) with a resonance-free strip depending only on the rank of \( G \); indeed, given any generating subset \( \{g_1, \ldots, g_r\} \) of \( G \), we may define a homomorphism \( \phi: \Gamma \to G \) by setting \( \phi(\gamma_j) = g_j \) for all \( j \).

Then, by construction, the kernel \( \Gamma_{\phi} \) of \( \phi \) is a normal subgroup of \( \Gamma \) and the quotient \( X_{\phi} = \Gamma_{\phi} \backslash \mathbb{H}^2 \) is a \( G \)-cover of \( X \). (One can show that every \( G \)-cover \( X' \) of \( X \) is isometric to such an \( X_{\phi} \).) If additionally \( \{g_1, \ldots, g_r\} \) is assumed to be \( \varepsilon \)-expanding, then \( X_{\phi} \) has no non-trivial resonances in \( \text{Re}(s) \geq \delta_{\Gamma} - \eta \), with \( \eta = \eta(\varepsilon, \Gamma) > 0 \).
4. Breuillard–Gamburd [11] showed that there exists \( \varepsilon > 0 \) such that every generating set of \( G = \text{SL}_2(\mathbb{F}_p) \) is \( \varepsilon \)-expanding, whenever \( p \) stays outside a set of primes \( \mathcal{P}_0 \) of density zero. A classical result of Frobenius says that \( d(G) \geq \frac{2^{r-1}}{2} \gg |G|^{1/3} \). Hence, arguing as in the previous item, we obtain the following deterministic statement: there exists \( \eta > 0 \) such that every \( \text{SL}_2(\mathbb{F}_p) \)-cover \( X' \) of \( X \) with \( p \notin \mathcal{P}_0 \) has no non-trivial resonances \( s \) with \( \text{Re}(s) \geq \delta_{\Gamma} - \eta \).

Another consequence of Theorem 1.4 is a uniform resonance-free strip for the family of congruence covers of convex cocompact hyperbolic surfaces. We recall that for a group \( \Gamma \subset \text{SL}_2(\mathbb{Z}) \) and an integer \( q \geq 2 \), the congruence subgroup of level \( q \) of \( \Gamma \) is defined as

\[ \Gamma(q) \overset{\text{def}}{=} \{ \gamma \in \Gamma : \gamma \equiv I \pmod{q} \} \]

We let

\[ X(q) \overset{\text{def}}{=} \Gamma(q) \backslash \mathbb{H}^2 \]
be the corresponding congruence covers of $X$. Oh–Winter [34] proved that $X(q)$ has a resonance-free strip uniform in $q$, provided $q$ is a square-free integer with no small prime divisors. Later, Bourgain–Kontorovich–Magee [24] were able to remove the square-free assumption. In this paper we show how to recover this result (for arbitrary level $q$), using Theorem 1.4, the work on expanders by Bourgain–Varjú [8] and an inductive argument on the sum of the exponents of the prime factors of $q$.

**Theorem 1.8** (Resonance-free strip for $X(q)$, see [34], [24]). Let $\Gamma \subset SL_2(\mathbb{Z})$ be a convex cocompact Fuchsian group. Then there exists an integer $q_0 = q_0(\Gamma)$ and a constant $\eta = \eta(\Gamma) > 0$ such that for all integers $q \geq 2$ co-prime to $q_0$ the Selberg zeta function $Z_{\Gamma(q)}(s)$ has no non-trivial zeros in the half-plane $\Re(s) \geq \delta_{\Gamma} - \eta$. In particular, the surfaces $X(q)$ have no non-trivial resonance in $\Re(s) \geq \delta_{\Gamma} - \eta$ for all integers $q \geq 2$ co-prime to $q_0$.

**Remark 1.9.** The statement of Theorem 1.8 is true for all non-elementary, finitely generated Fuchsian groups $\Gamma \subset SL_2(\mathbb{Z})$. For the modular group $\Gamma = SL_2(\mathbb{Z})$ (in which case we have $\delta_{\Gamma} = 1$) Selberg’s famous $3/16$-theorem [43] gives a more precise statement with the explicit spectral gap $\eta = \frac{1}{4}$. For $\delta_{\Gamma} > \frac{3}{8}$, Theorem 1.8 follows from Gamburd’s thesis [18] with $\eta = \delta_{\Gamma} - \frac{3}{8}$. For $\delta_{\Gamma} > \frac{1}{2}$, the statement was proven by Bourgain–Gamburd–Sarnak [7]. The more difficult case $\delta_{\Gamma} \leq \frac{1}{2}$ is covered by Theorem 1.8, since $\delta_{\Gamma} > \frac{1}{2}$ implies that $\Gamma$ has at least one cusp by [1, 2] (see also Remark 1.7 above). Similarly to [34, 24], our method does not yield explicit bounds on $\eta$, due to its reliance on the expander theory developed among others by Bourgain–Gamburd [6] and Bourgain–Varjú [8].

1.3. **Acknowledgments.** I would like to thank Michael Magee who alerted me to his joint work with Jean Bourgain and Alex Kontorovich [24]. Although not explicitly stated in their work, the arguments in [24] may be generalized to give an alternative proof of Corollary 1.6.

1.4. **Organization and overview of proofs.** The aim of this subsection is to give an overview on the main ideas and the structure of this paper, without delving into technical details or elaborating on the notations. In §2 we introduce the three main ingredients of this work:

(a) The construction of Schottky groups $\Gamma$.

(b) Twisted Selberg zeta functions $Z_{\Gamma}(s, \rho)$ and their realization as Fredholm determinants of transfer operators. The family of transfer operators in question, denoted by $L_{s,\rho}$, is parametrized by the spectral variable $s \in \mathbb{C}$ and they act on vector-valued Bergman spaces.

(c) The Venkov–Zograf factorization formula, according to which the Selberg zeta function of the subgroup $\Gamma' < \Gamma$ is the product of twisted Selberg zeta functions $Z_{\Gamma'}(s, \rho)$ over all the irreducible representations $\rho$ of $G = \Gamma/\Gamma'$.

In §3 we show how to deduce both Corollary 1.6 and Theorem 1.8 from our second main theorem, Theorem 1.4. The rest of the paper is devoted to proving Theorems 1.3 and 1.4. The construction of Schottky groups gives rise to a dynamical system on the limit set $\Lambda_{\Gamma}$, sometimes called the Bowen–Series map, whose properties we briefly recall in §4. We record some properties of the topological pressure associated to this dynamical system and the Ruelle–Perron–Frobenius theorem. Moreover, we develop some a priori estimates for the transfer operators acting on vector-valued Bergman spaces.
In §5 we show that expansion of \( \{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \subseteq G \) implies exponential decay in certain sums over words \( W_N \) of length \( N \) (Proposition 5.1). To give a first overview on these estimates, let us point out that the elements of \( \Gamma \) can be indexed by finite words in the alphabet \([2m] \defeq \{1, \ldots, 2m\}\), where \( m \) is the number of Schottky generators of \( \Gamma \) (details will be explained in §4.1). Letting \( \gamma_a \) denote the element in \( \Gamma \) corresponding to the word \( a \), we then have for each non-trivial, irreducible representation \((\rho, V)\) of \( G \) the bound

\[
\left\| \sum_{a \in W_N} \rho(\gamma_a) \right\|_{\text{End}(V)} \ll e^{-cN|W_N|} \quad \text{as } N \to \infty,
\]

where \( c > 0 \) depends only on the expansion rate and \( \Gamma \). We then use a decoupling argument to generalize this purely combinatorial statement to more general sums over words of length \( N \), where the summands are now weighted by functions \( g: \Lambda \Gamma \to V \) on the limit set \( \Lambda \Gamma \) (Proposition 5.3). More precisely, we prove

\[
\left\| \sum_{a \in W_N} \rho(\gamma_a)^{-1} g(\gamma_a(x)) \right\|_V \ll e^{-cN|W_N| \cdot \mathcal{N}(g)}
\]

for all \( x \in \Lambda \Gamma \), where \( \mathcal{N}(\cdot) \) is a certain norm defined for functions \( g: \Lambda \Gamma \to V \). The key feature of this estimate is that the exponential decay rate \( c > 0 \) depends only on the expansion constant of the set \( \{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \subseteq G \).

In §6 we build on (6) and we use some rather subtle arguments to establish exponential decay estimates for the norm of the \( \rho \)-twisted transfer operator at \( s = \delta \) (Proposition 6.1). These estimates take the form

\[
\|L^N_{\delta, \rho} f\| \ll e^{-cN}\|f\|
\]

for every non-trivial, irreducible representation \((\rho, V)\) of the group \( G \) and for all functions \( f \) belonging to the \( V \)-valued Bergman space. This improves upon the “trivial” estimate

\[
\|L^N_{\delta, \rho} f\| \ll \|f\|.
\]

In §7 we finalize the proofs of the main theorems. Theorem 1.3 follows quickly from the estimate (7), thanks to the relation between zeros of the Selberg zeta function and \( 1 \)-eigenfunctions of the transfer operator. The proof of Theorem 1.4 is more complicated. First, using (7), we derive an effective equidistribution result (Proposition 7.1). In fact, we will only use a weaker “flattening” result (Corollary 7.2), which when combined with a general bound on resonances for \( G \)-covers (Proposition 7.3), concludes the proof of the second main theorem. The rest of the paper is devoted to the proof of Proposition 7.3, which is a combination of ideas by Guilloupolé–Lin–Zworski [20], Naud [31] and Pohl–Soares [38].

Finally, we point out that the estimate

\[
\|L^N_{\delta, \rho+it} f\| \ll e^{-cN}\|f\| \quad \text{as } N \to \infty
\]

for all real \( t \) would suffice to remove the quasirandomness assumption in Corollary 1.1. (Note that the key estimate (7) is a special case of (8) with \( t = 0 \).) In the large frequency regime \(|t| \gg 1\), (8) follows essentially from the methods developed in [32, 34]. Unfortunately, in the intermediate case \( 0 < |t| \ll 1 \), establishing this estimate seems to be more complicated.
1.5. **Notation.** We write \( f(x) \ll g(x) \) or \( f(x) = O(g(x)) \) interchangeably to mean that there exists an implied constant \( C > 0 \) such that \( |f(x)| \leq C|g(x)| \) for all \( x \geq C \) and we write \( f(x) \ll_y g(x) \) or \( f(x) = O_y(g(x)) \) to indicate that \( C \) depends on \( y \). We write \( f(x) \asymp g(x) \) to mean \( g(x) \ll_y f(x) \). Whenever the Schottky group \( \Gamma \) is fixed we will drop it from the subscripts, writing \( \delta = \delta_\Gamma \) and \( \Lambda = \Lambda_\Gamma \). We write \( f(x) \asymp g(x) \) or \( f(x) = O(x) \) to indicate that \( C \) depends on \( y \). We write \( f(x) \asymp g(x) \) or \( f(x) = O(x) \) to mean \( g(x) \ll_y f(x) \). Whenever the Schottky group \( \Gamma \) is fixed we will drop it from the subscripts, writing \( \delta = \delta_\Gamma \) and \( \Lambda = \Lambda_\Gamma \).

2. **Preliminaries**

2.1. **Schottky groups.** Let us briefly review some properties of Schottky groups, referring the reader to Borthwick’s book [3, §15] for a comprehensive discussion. The group \( \text{SL}_2(\mathbb{R}) \) acts on the extended complex plane \( \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\} \) by Möbius transformations

\[
\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{R}), \quad z \in \overline{\mathbb{C}} \implies \gamma(z) = \frac{az + b}{cz + d}.
\]

A Schottky group is a convex cocompact subgroup \( \Gamma \subset \text{SL}_2(\mathbb{R}) \) constructed in the following way:

- Fix \( m \geq 1 \) and non-intersecting euclidean disks \( D_1, \ldots, D_{2m} \subset \mathbb{C} \) with centers on the real line,
- For every \( j \in \{1, \ldots, m\} \) let \( \gamma_j \in \text{SL}_2(\mathbb{R}) \) be the isometry that maps the exterior of \( D_j \) to the interior of \( D_{j+m} \), for every \( j \in \{m+1, \ldots, 2m\} \) set

\[
\gamma_j \overset{\text{def}}{=} \gamma_{j-m}^{-1}
\]

and extend this definition cyclically to all \( j \in \mathbb{Z} \) by setting

\[
\gamma_j \overset{\text{def}}{=} \gamma_j \mod 2m.
\]

- Let \( \Gamma \subset \text{SL}_2(\mathbb{R}) \) be the free group generated by the elements \( \gamma_1, \ldots, \gamma_{2m} \).

One of the standard models for the hyperbolic plane is the Poincaré half-plane

\[
\mathbb{H}^2 = \{x + iy \in \mathbb{C} : y > 0\}
\]

endowed with its standard metric of constant curvature \(-1\),

\[
ds^2 = \frac{dx^2 + dy^2}{y^2}.
\]

By Button’s result [16] every convex cocompact hyperbolic surface \( X \) can be realized as the quotient of \( \mathbb{H}^2 \) by a Schottky group \( \Gamma \), see also [3, Theorem 15.3]. Notice that the complement \( \mathbb{H}^2 \setminus \bigcup_{j=1}^{2m} D_j \) provides a fundamental domain for the action of \( \Gamma \) on \( \mathbb{H}^2 \). Given a convex cocompact hyperbolic surface \( X \) we fix a Schottky group \( \Gamma \) such that \( X \cong \Gamma \backslash \mathbb{H}^2 \) and we refer to \( D_1, \ldots, D_{2m} \) and \( \gamma_1, \ldots, \gamma_{2m} \) as the **Schottky data** of \( X \) (or \( \Gamma \)) and we refer to \( \gamma_1, \ldots, \gamma_{2m} \) as the **Schottky generators** of \( \Gamma \).
2.2. Selberg zeta function and transfer operator. Twisted Selberg zeta functions play a crucial role in this work. For a finitely generated Fuchsian group $\Gamma < \text{SL}_2(\mathbb{R})$ it is well-known that the set of prime periodic geodesics on $X = \Gamma \backslash \mathbb{H}^2$ is bijective to the set $[\Gamma]_{\text{prim}}$ of $\Gamma$-conjugacy classes of the primitive hyperbolic elements in $\Gamma$ (see for instance [3]). We denote by $\ell(\gamma)$ the length of the geodesic corresponding to $[\gamma] \in [\Gamma]_{\text{prim}}$.

Let $V$ be finite-dimensional complex vector space endowed with a hermitian inner product $\langle \cdot , \cdot \rangle_V$ and induced norm $\|v\|_V = \sqrt{\langle v, v \rangle_V}$ and let $\rho: \Gamma \to \text{U}(V)$ be a unitary representation of $\Gamma$. Then the twisted Selberg function is defined for $\text{Re}(s) \gg 1$ by the infinite product

$$Z_{\Gamma}(s, \rho) \overset{\text{def}}{=} \prod_{k=0}^{\infty} \prod_{[\gamma] \in [\Gamma]_{\text{prim}}} \det \left( I_V - \rho(\gamma)e^{-(s+k)\ell(\gamma)} \right).$$

Notice that (9) reduces to the classical Selberg zeta function when $\rho$ is the trivial one-dimensional representation of $\Gamma$.

Now let $\Gamma$ be a Schottky group with Schottky data $D_1, \ldots, D_{2m}$ and $\gamma_1, \ldots, \gamma_{2m}$, which we assume to be fixed. Let $D \overset{\text{def}}{=} \bigcup_{j=1}^{2m} D_j$ be the union of the Schottky disks and consider the Hilbert space $H^2(D, V)$ of square-integrable, holomorphic functions on $D$,

$$H^2(D, V) \overset{\text{def}}{=} \{ f: D \to V \text{ holomorphic} \mid \|f\| < \infty \}.$$ 

Here $\|f\|$ is the $L^2$-norm given by

$$\|f\|^2 \overset{\text{def}}{=} \int_D \|f(z)\|_V^2 \, \text{dvol}(z),$$

where $\text{dvol}$ denotes the Lebesgue measure on the complex plane. Finally, we define the twisted transfer operator

$$\mathcal{L}_{s, \rho}: H^2(D, V) \to H^2(D, V)$$

by the formula

$$\mathcal{L}_{s, \rho} f(z) \overset{\text{def}}{=} \sum_{i=1}^{2m} \sum_{i \neq j}^{2m} \gamma_i(z)^s \rho(\gamma_i)^{-1} f(\gamma_i(z)) \text{ if } z \in D_j$$

for all $f \in H^2(D, V)$. In the one-dimensional case, $V = \mathbb{C}$, the functional space (10) reduces to the classical Bergman space $H^2(D)$ and the operator (11) reduces to the

![Diagram of Schottky disks and isometries with $m = 3$](image-url)
well-known transfer operator which can be found for instance in Chapter 14 of [3]). The twisted Selberg zeta function is related to the transfer operator $L_{s,\rho}$ through the following result:

**Proposition 2.1** (Fredholm determinant identity). For every $s \in \mathbb{C}$ the operator (11) is trace class and we have the identity

$$Z_{\Gamma}(s, \rho) = \det(1 - L_{s,\rho}).$$

Identities such as (13) are well-known in thermodynamic formalism, a subject going back to Ruelle [41], at least in the case where $\rho$ is the trivial one-dimensional representation. The relation between the Selberg zeta function and transfer operators has been studied by a number of different authors. For the convex cocompact setting (no cusps) we refer to [39, 40, 20]. The extension to non-trivial twists $\rho$ can be found in the more recent papers [17, 38, 33, 32]. A proof of Proposition 2.1 can be found in [26].

Proposition 2.1 has many remarkable corollaries. In particular, since $L_{s,\rho}$ depends holomorphically on the variable $s$, it follows directly that $Z_{\Gamma}(s, \rho)$ is an entire function when $\Gamma$ is a Schottky group, which is far from obvious from its definition in (9) as an infinite product over conjugacy classes.

### 2.3. Finite covers and Venkov–Zograf formula

Twisted Selberg zeta functions are extremely helpful when studying resonances in families of covers of surfaces $X = \Gamma \backslash \mathbb{H}^2$. Fix a finite degree, regular cover $X' = \Gamma' \backslash \mathbb{H}^2$ of $X$. We may assume without loss of generality that $\Gamma'$ is a normal subgroup of $\Gamma$, in which case the quotient $G = \Gamma/\Gamma'$ is a group, called the covering group (sometimes also called the Galois group of the covering). The (left) regular representation of $G$ is the representation $R_G: G \to U(L^2(G))$ defined by

$$R_G(g)\varphi(x) = \varphi(xg^{-1}).$$

for all $g \in G$, $x \in G$ and $\varphi \in L^2(G)$. This representation can be extended in a natural way to a representation $R_G: \Gamma \to U(L^2(G))$ by setting

$$R_G(\gamma) = R_G(\pi(\gamma))$$

for all $\gamma \in \Gamma$, where $\pi: \Gamma \to G$ is the natural projection map. We then have the relation

$$Z_{\Gamma'}(s) = Z_{\Gamma}(s, R_G),$$

which was proven by Venkov–Zograf [49] in the case where $\Gamma$ is a cofinite Fuchsian group (see also [48]). For an extension of this formula to the non-cofinite case we refer to [17].

Twisted Selberg functions enjoy the following nice property: for any two finite-dimensional unitary representations $(\rho_1, V_1)$ and $(\rho_2, V_2)$ of $\Gamma$, we have the factorization

$$Z_{\Gamma}(s, \rho_1 \oplus \rho_2) = Z_{\Gamma}(s, \rho_1)Z_{\Gamma}(s, \rho_2),$$

where the symbol $\oplus$ stands for the direct sum of representations. As is well-known, the regular representation decomposes as the direct sum

$$R_G = \bigoplus_{(\rho, V) \in \text{irrep}(G)} \dim(\rho)\rho,$$

where $\text{irrep}(G)$ denotes the finite set of all irreducible representations $(\rho, V)$ of $\Gamma$ (up to equivalence of representations). Keeping in mind that each representation $\rho$ of $G$
extends to a representation of \( \Gamma \) by composing it with \( \pi \) similarly to (14), we obtain the identity

\[
Z_{\Gamma'}(s) = \prod_{\rho \in \text{irrep}(G)} Z_{\Gamma}(s, \rho)^{\dim(\rho)},
\]

which we will refer to as the Venkov–Zograf (factorization) formula.

3. Deducing Corollary 1.6 and Theorem 1.8 from Theorem 1.4

3.1. Deducing Corollary 1.6 from Theorem 1.4. Fix a Schottky group \( \Gamma \) and assume that it is generated by \( m \geq 2 \) Schottky generators \( \gamma_1, \ldots, \gamma_m \). Let \( r > 0 \) be a positive integer, let \( c \) and \( A \) be positive constants, and let \( G_1, \ldots, G_r \) be a collection of finite groups with \( d(G_i) \geq c|G_i|^A \), where for a finite group \( H \), \( d(H) \) denotes its quasirandom degree, that is, the smallest dimension of a non-trivial linear representation of \( H \). Set

\[
G = G_1 \times \cdots \times G_r
\]

and let \( \pi: \Gamma \to G \) be a surjective homomorphism. Assume furthermore that the elements \( \pi(\gamma_1), \ldots, \pi(\gamma_m) \in G \) form an \( \varepsilon \)-expanding subset of \( G \). The aim of this subsection is to prove Corollary 1.6, which says that the Selberg zeta function of \( \Gamma' = \ker(\pi) \) has no non-trivial zeros in the half-plane \( \Re(s) \geq \delta - \eta \) for some \( \eta > 0 \) depending only on \( \varepsilon, c, A \) and the Schottky data of \( \Gamma \).

First we recall the high-frequency result of Naud–Magee [32, Theorem 1.8]: there are constants \( \eta_\Gamma > 0 \) and \( T_\Gamma > 0 \) such that for every finite cover \( \Gamma' \) of \( \Gamma = \Gamma \setminus \mathbb{H}^2 \), there are no resonances \( s \) for \( \Gamma' \) with \( \Re(s) > \delta_\Gamma - \eta_\Gamma \) and \( |\Im(s)| > T_\Gamma \).

Combining this result with the Venkov–Zograf factorization formula (18), reduces Corollary 1.6 to the following statement: there exists \( \eta > 0 \) depending only on \( \Gamma, \varepsilon, c \) and \( A \) such that for every non-trivial, irreducible representation \( \rho \) of \( G \), the twisted Selberg function \( Z_{\Gamma}(s, \rho) \) does not vanish in the region

\[
R_\eta \overset{\text{def}}{=} \{ \Re(s) \geq \delta - \eta \text{ and } |\Im(s)| \leq T_\Gamma \}.
\]

We will argue by induction on the number of factors \( r \) of \( G \) in (19). Let us first address the case \( r = 1 \). In this case, by assumption, every non-trivial, irreducible representation \( \rho \) of \( G = G_1 \) has dimension at least \( \dim(\rho) \geq c|G|^A \). Then by Theorem 1.4 there exists \( C = C(\varepsilon, \Gamma) > 0 \) such that

\[
\#\{ \text{zeros of } Z_{\Gamma}(s, \rho) \text{ in } R_\eta \} \ll_{\Gamma, \varepsilon} \frac{|G|^C_\eta}{\dim(\rho)} \ll_{\Gamma, \varepsilon, c} |G|^{-A + C_\eta}.
\]

Clearly, if we take \( \eta = \frac{A}{2C} \), the right-hand side becomes smaller than 1 whenever \( G \) is large enough, that is, larger than some constant \( K = K(\Gamma, c, \varepsilon, A) \). Because the number of zeros in \( R_\eta \) has to be a non-negative integer, we obtain

\[
\#\{ \text{zeros of } Z_{\Gamma}(s, \rho) \text{ in } R_\eta \} = 0,
\]

as claimed. If, on the other hand, \( G \) has order less than \( K \), we use the following argument: there are only finitely many coverings of \( X = \Gamma \setminus \mathbb{H}^2 \) of degree less than \( K \), up to isometry. Invoking Naud’s result [30] gives for each of these coverings a positive “spectral gap”. Letting \( \eta_0 > 0 \) denote the smallest of these spectral gaps (of which there are only finitely many), we can replace the value \( \eta \) obtained above by \( \min\{\eta_0, \eta\} \) to obtain the same conclusion for every size of \( G \). This concludes the proof of Corollary 1.6 in the case \( r = 1 \).
Now suppose that \( r > 1 \) and let \( \rho \) be a non-trivial, irreducible representation of \( G = G_1 \times \cdots \times G_r \). Then \( \rho \) is the tensor product of representations
\[
\rho = \rho_1 \otimes \cdots \otimes \rho_r.
\]
where each \( \rho_i \) is an irreducible (but not necessarily non-trivial) representation of \( G_i \). Assume first that each \( \rho_i \) is non-trivial. In this case we can give a lower bound for the dimension of \( \rho \), because for each \( i \in [r] \) we have
\[
\dim(\rho_i) \geq c|G_i|^A.
\]
Without loss of generality we may assume that \( |G_i| > 1 \) for each factor (for if otherwise \( G_i \) would be the trivial group and we could simply exclude it from the product (19)). Thus we may assume that
\[
\dim(\rho) \geq |G|^{A'}
\]
where \( A' > 0 \) is sufficiently small depending only on \( c \) and \( A \), so that
\[
\dim(\rho) \geq \prod_{i=1}^r \dim(\rho_i) \geq \prod_{i=1}^r |G_i|^{A'} = |G|^{A'}.
\]
Applying Theorem 1.4 once more shows that
\[
\# \{\text{zeros of } Z_{\Gamma}(s, \rho) \text{ in } R_\eta \} \ll_{\Gamma, \varepsilon} |G|^{-A' + C\eta}.
\]
We can argue as above to conclude that \( Z_{\Gamma}(s, \rho) \) has no zeros in \( R_\eta \) provided \( \eta > 0 \) is chosen small enough depending only on \( \Gamma, \varepsilon, A' \).

Now suppose alternatively that for some \( i \in [r] \), the \( i \)-th factor in the tensor product (21) is trivial. Without loss of generality we may assume that \( i = r \) so that \( \rho \) is equivalent to the representation
\[
\tilde{\rho} = \rho_1 \otimes \cdots \otimes \rho_{r-1}
\]
of the group \( \tilde{G} = G_1 \times \cdots \times G_{r-1} \). Let \( \tilde{\rho}: G \to \tilde{G} \) be the projection onto the first \( r-1 \) coordinates and note that the map
\[
\tilde{\pi} \overset{\Delta}{=} \pi \circ \tilde{\rho}: \Gamma \to \tilde{G}
\]
is a surjective homomorphism. Moreover, since \( \{\pi(\gamma_1), \ldots, \pi(\gamma_r)\} \subseteq G \) is an \( \varepsilon \)-expanding set, so is the set \( \{\tilde{\pi}(\gamma_1), \ldots, \tilde{\pi}(\gamma_r)\} \subseteq \tilde{G} \) (this follows easily from the definition of \( \varepsilon \)-expanding sets.) Finally, since \( Z_{\Gamma}(s, \rho) = Z_{\Gamma}(s, \tilde{\rho}) \), the claim follows by induction.

3.2. Deducing Theorem 1.8 from Theorem 1.4. For the remainder of this section, let \( \Gamma \) be a convex cocompact subgroup of \( \text{SL}_2(\mathbb{Z}) \) with Schottky generators \( \gamma_1, \ldots, \gamma_m \). Recall that for every integer \( q \geq 2 \) the level \( q \) congruence subgroup \( \Gamma(q) \) of \( \Gamma \) is defined as the kernel of the reduction modulo \( q \) map
\[
\pi_q: \Gamma \to G_q \overset{\Delta}{=} \text{SL}(\mathbb{Z}/q\mathbb{Z}), \quad \gamma \mapsto \gamma \pmod{q}.
\]
The goal is to use Theorem 1.4 to deduce Theorem 1.8, the latter of which says that there exists \( q_0 \) and \( \eta > 0 \) (both depending only on the Schottky data of \( \Gamma \)), such that for all \( q \) co-prime to \( q_0 \) the Selberg zeta function of \( \Gamma(q) \) has no zeros in \( \text{Re}(s) \geq \delta - \eta \).

The main additional input for our proof is the well-known and deep result of Bourgain–Varjú [8]. It says that the set \( \{\pi_q(\gamma_1), \ldots, \pi_q(\gamma_m)\} \) is \( \varepsilon \)-expanding in the group \( \pi_q(\Gamma) \leq G_q \) for some constant \( \varepsilon = \varepsilon(S) > 0 \). Moreover, it says that the map (22) is surjective for all integers \( q \geq 2 \) co-prime to some integer \( q_0 = q_0(S) \), so that \( G_q = \pi_q(\Gamma) \) whenever \( (q, q_0) = 1 \). (Here \( (a, b) \) denotes the largest common divisor of \( a \) and \( b \).)
We will argue similarly as in Subsection 3.1 above. First note that we can reduce Theorem 1.8 to the following statement: for all \(q\) with \((q,q_0) = 1\) and for every non-trivial, irreducible representation \(\rho\) of \(G_q\), the Selberg zeta function \(Z_{\Gamma}(s,\rho)\) has no zeros in \(R_q\), where \(R_q\) is the region in (20), with some constant \(\eta > 0\) depending only on the Schottky data of \(\Gamma\). Now note that if
\[
q = p_1^{r_1} \cdots p_k^{r_k}
\]
is the prime factorization of \(q\), then \(G_q\) is isomorphic to the direct product
\[
G_q \cong G_{p_1^{r_1}} \times \cdots \times G_{p_k^{r_k}}
\]
where \(G_{p_i^{r_i}} = \text{SL}_2(\mathbb{Z}/p_i^{r_i}\mathbb{Z})\) for each \(i \in [k]\). We will use induction over the sum of prime powers
\[
\ell(q) = r_1 + \cdots + r_k.
\]
Let us record here the well known formula for the number of elements of \(G_{p^r}\) with \(p\) prime and \(r\) a positive integer:
\[
|G_{p^r}| = p^{3r} - p^r.
\]
Thus the order of \(G_q\) is given by
\[
|G_q| = q^3 \prod_{j=1}^{k} \left(1 - \frac{1}{p_j^{2r_j}}\right).
\]
Note that the argument of Subsection 3.1 goes through if \(q\) is square-free, because in this case each prime power \(r_i\) in the factorization of \(q\) equals 1 and \(G_q\) is a direct product of the groups \(G_i \overset{\text{def}}{=} G_{p_i}\), which are strongly quasirandom in the sense that
\[
d(G_p) \gg p \gg |G_p|^{1/3},
\]
by a classical result of Frobenius. However, to prove the more general case where \(q\) is only assumed to be co-prime to \(q_0\), we need the following ingredient pertaining to the representation theory of the groups \(G_q\):

**Lemma 3.1** (Dichotomy for representations of \(G_q\)). *For all integers \(q \geq 2\) and for each non-trivial, irreducible representation \(\rho\) of \(G_q\) one of the following statements holds:*

1. *the dimension of \(\rho\) is bounded below by \(c|G_q|^{1/4}\) for some absolute constant \(c > 0\), or*
2. *\(\rho\) “descends” to a representation \(\rho'\) of \(G_{q'}\) with \(q'|q, q' < q\). More precisely, \(\rho = \rho' \circ \pi_{q',q}\) where \(\pi_{q',q}: G_q \to G_{q'}\) is the natural inclusion map.*

Here, we write \(q'|q\) to mean that \(q'\) is a divisor of \(q\).

Let us see how this can be used to finish the proof of Theorem 1.8. Recall that we need to show that for every non-trivial, irreducible representation \(\rho\) of \(G_q\) with \((q,q_0) = 1\), the function \(Z_{\Gamma}(s,\rho)\) does not vanish in \(R_q \subset \mathbb{C}\).

By Theorem 1.4 and the formula for the order of \(G_q\) in (23) we have
\[
\#\{\text{zeros of } Z_{\Gamma}(s,\rho) \text{ in } R_q\} \leq \frac{C' |G_q|^\eta}{\dim(\rho)} \leq \frac{C' q^{3\eta}}{\dim(\rho)}
\]
for some positive constants \(C = C(\Gamma)\) and \(C' = C(\Gamma)\). Hence for all representations \(\rho\) whose dimension is bounded below by \(c|G_q|^{1/4}\) and for all \(q\) large enough, we have
\[
\#\{\text{zeros of } Z_{\Gamma}(s,\rho) \text{ in } R_q\} \leq C'' q^{3\eta - \frac{1}{4}} < 1
\]
Thus, increasing \( q_0 \) if necessary and taking \( \eta = \frac{1}{100c} \) (say), we obtain the following intermediate result: for all \( (q, q_0) = 1 \) and all irreducible representations \( \rho \) of \( G_q \) with \( \dim(\rho) \geq c|G_q|^{1/4} \), the function \( Z_\Gamma(s, \rho) \) has no zeros in \( R_\eta \).

Now we can use induction over the sum of the powers of the prime factors \( \ell(q) \). The case where \( \ell(q) = 1 \) (i.e. when \( q \) is prime) is clear from this intermediate result and the quasirandomness bound from (24).

Hence we may assume that \( q \) is a positive integer with \( \ell = \ell(q) > 1 \) and we may assume that the statement of Theorem 1.8 is known to be true with \( \eta \) taken as above for all positive integers \( q' \) with \( (q', q_0) = 1 \) and \( \ell(q') < \ell \).

Then by Lemma 3.1 we have either \( \dim(\rho) \geq c|G_q|^{1/4} \), in which case we know from the intermediate result above that \( Z(s, \rho) \) has no zeros in \( R_\eta \), or \( \rho \) descends to a representation \( \rho' \) of \( G_{q'} \) with \( q' \) a proper divisor of \( q \). In the latter case we clearly have \( \ell(q') < \ell \) and \( Z_\Gamma(s, \rho) = Z_\Gamma(s, \rho') \), so the result follows by induction. Theorem 1.8 is thus finished, assuming Theorem 1.3 and Lemma 3.1. The latter will be proven here.

**Proof of Lemma 3.1.** If \( q = p \) is prime we have \( d(G_p) \gg |G_p|^{1/3} \) from the discussion preceding Lemma 3.1. Hence we can bound the dimension of each non-trivial representation of \( G_p \) from below by \( c|G_p|^{1/4} \) for some small absolute constant \( c > 0 \).

Now assume that \( q > 2 \) is not a prime and let \( q = p_1^{r_1} \cdots p_k^{r_k} \) be its prime factorization. Then \( G_q \) factorizes as

\[
G_q \cong G_{p_1}^{r_1} \times \cdots \times G_{p_k}^{r_k},
\]

and each irreducible representation \( \rho \) of \( G_q \) is equivalent to a tensor product

\[
\rho = \rho_1 \otimes \cdots \otimes \rho_k
\]

where each \( \rho_j \) is an irreducible representation of \( G_{p_j}^{r_j} \).

We now divide the proof into two complementary cases. In the first case we assume that each \( \rho_j \) in (26) is faithful. (Recall that a representation \( (\rho, V) \) of a group \( G \) is called faithful if \( \rho : G \to \mathrm{U}(V) \) is an injective homomorphism.) Then, for all \( j \in [k] \) we have the estimate

\[
\dim(\rho_j) \geq \begin{cases} \frac{1}{2}p_j^{r_j-1}(p^2 - 1) & \text{if } r_j \geq 2 \\ \frac{1}{2}(p - 1) & \text{if } r_j = 1, \end{cases}
\]

which is due to Bourgain–Gamburd [5, Lemma 7.1]. To simplify the argument we will only use the weaker bound \( \dim(\rho_j) \geq \frac{1}{2}p_j^{r_j} \) from which we obtain

\[
\dim(\rho) = \prod_{j=1}^k \dim(\rho_j) \geq \prod_{j=1}^k \frac{1}{3}p_j^{r_j} = 3^{-k}q^{1-\varepsilon}
\]

for all \( \varepsilon > 0 \). (The rightmost bound is justified because the number \( k \) of distinct primes dividing \( q \) satisfies \( k = o(\log q) \).

Recalling that the order of \( G_q \) is bounded by \( |G_q| < q^3 \), we deduce

\[
\dim(\rho) \gg \varepsilon |G_q|^{1/3-\varepsilon} \geq c|G_q|^{1/4}
\]

for some sufficiently small absolute constant \( c > 0 \). This shows that \( \rho \) has dimension larger than \( c|G_q|^{1/4} \) if each factor in (26) is faithful.

Now suppose that we are in the second case: there exists some index \( j \in [k] \) such that \( \rho_j \) is not faithful. Then \( \ker(\rho_j) \) is a non-trivial normal subgroup of \( G_{p_j}^{r_j} \). We may
assume that \( j = k \) for notational convenience. For a prime \( p \) and integers \( 1 \leq r' < r \), let \( H^p_{r,r'} \) denote the kernel of the surjective homomorphism

\[
\pi^p_{r,r'} : G^p_{r'} \to G^p_{r'}
\]

The set \( \{ H^p_{r,r'} : 1 \leq r' \leq r \} \) is a complete list of the normal subgroups of \( G^p_{r'} \) up to isomorphism. Hence, because \( \rho_{r_k} \) is non-faithful, its kernel \( \ker(\rho_{r_k}) \) is isomorphic to one of the groups \( H^p_{r_{k'},r_k'} \) for some \( 1 \leq r_{k'} < r_k \), showing that \( \rho_{r_k} \) descends to a homomorphism

\[
\rho'_{r_k} : G^p_{r_{k'}} / H^p_{r_{k'},r_k'} \cong G^p_{r_{k'}} \to U(V).
\]

Consequently \( \rho \) descends to the representation

\[
\rho' = \rho_1 \otimes \cdots \otimes \rho_{k-1} \otimes \rho_k
\]

of the group \( G_{q'} \) with \( q' = p_{r_1}^{r_1} \cdots p_{r_k}^{r_k} \). Clearly \( q'|q \) is a proper divisor of \( q \), which completes the proof of Lemma 3.1. \( \square \)

4. A PRIORI ESTIMATES

4.1. Reduced words and bounds for derivatives. From now on, we assume that \( X = \Gamma \setminus \mathbb{H}^2 \) is a fixed convex cocompact hyperbolic surface and that \( \Gamma \) is a Schottky group with Schottky data \( D_1, \ldots, D_{2m} \) and \( \gamma_1, \ldots, \gamma_{2m} \). We will henceforth drop \( \Gamma \) from the subscripts, writing \( \delta = \delta_\Gamma \) and \( \Lambda = \Lambda_\Gamma \). Moreover, we denote by \( \mathcal{L}_{s,p} \) the transfer operator introduced in §2.2 above.

In this section we will introduce some additional notations which will be used throughout this work. Given a finite word

\[
a = a_1 \cdots a_N
\]

with \( a_1, \ldots, a_N \in [2m] \) def \( \{1, \ldots, 2m\} \), we set

\[
\gamma_a \overset{\text{def}}{=} \gamma_{a_1} \circ \cdots \circ \gamma_{a_n}
\]

Recall that \( \Gamma \) is freely generated by the elements \( \gamma_1, \ldots, \gamma_{2m} \), that is, there are no relations among the elements \( \gamma_1, \ldots, \gamma_{2m} \) other than the trivial relations \( \gamma_i^{-1} \gamma_i = \gamma_i \gamma_i^{-1} = e \). Recall also that \( \gamma_i^{-1} = \gamma_{i+m} \), where the indices are defined modulo \( 2m \). We say that the word \( a \) is reduced if the corresponding element \( \gamma_a \) is reduced when viewed as a word in the alphabet \( \{\gamma_1, \ldots, \gamma_{2m}\} \). We then denote by \( W_N \) the set of reduced words of length \( N \),

\[
W_N \overset{\text{def}}{=} \{ a = a_1 \cdots a_N : a_i \neq a_{i+1} + m \text{ (mod } 2m) \text{ for all } i = 1, \ldots, N-1 \}.
\]

Notice that the set \( W_N \) corresponds one-to-one to the elements of reduced word length \( N \) in \( \Gamma \) via the map \( a \mapsto \gamma_a \). Moreover, we define for all \( j \in [2m] \) the set of words

\[
W_j^N \overset{\text{def}}{=} \{ a = a_1 \cdots a_N \in W_N : a_N \neq j \}.
\]

Observe that if \( a = a_1 \cdots a_N \in W_j^N \), then \( \gamma_a \) maps the closure of \( D_j \) into the interior of \( D_{a_1+m} \). With these notations in place, we have for all \( N \geq 1 \) and all \( j \in [2m] \) the formula

\[
\mathcal{L}_{s,p}^N f(z) = \sum_{a \in W_j^N} \gamma_a^* f(\rho(\gamma_a)^{-1} f(\gamma_a(z))) \text{ if } z \in D_j.
\]

Finally, we record two distortion estimates (see [31] for the proofs) which are crucial in this work:
• (Uniform hyperbolicity) There exist constants $c_1, c_2 > 0$ and $0 < \theta_1 < \theta_2 < 1$ such that for all $N$, all $j \in [2m]$ and all $a \in W^j_N$, we have
\begin{equation}
    c_1 \theta_1^N < \sup_{z \in D_j} |\gamma'_a(z)| < c_2 \theta_2^N.
\end{equation}

• (Bounded distortion) There exists a constant $c_3 > 0$ such that for all $N$, all $j \in [2m]$, all $a \in W^j_N$ and all $z \in D_j$ we have
\begin{equation}
    \sup_{z \in D_j} \frac{\gamma''_a(z)}{\gamma'_a(z)} \leq c_3.
\end{equation}

The bounded distortion estimate has the following important consequence: there exists a constant $c_5 > 0$ such that for all words $a \in W^j_N$ and all pair of points $z_1, z_2 \in D_j$, we have
\begin{equation}
    \frac{\gamma'_a(z_1)}{\gamma'_a(z_2)} \leq c_5.
\end{equation}

### 4.2. Topological pressure and Ruelle-Perron-Frobenius theorem.

Let $D_1, \ldots, D_{2m}$ and $\gamma_1, \ldots, \gamma_{2m}$ be the Schottky data of $\Gamma$ and for each $j \in [2m]$ set $I_j \overset{\text{def}}{=} D_j \cap \mathbb{R}$. Notice that the $I_j$’s are mutually disjoint real intervals. Let $I = \bigcup_{j=1}^{2m} I_j$ denote its union. It turns out that the map
\begin{equation}
    T: I \to I
\end{equation}
given by
\begin{equation}
    T(x) = \gamma_j(x) \text{ if } x \in I_j,
\end{equation}
sometimes called the Bowen-Series map, encodes the dynamics of the full group $\Gamma$. The origins of this type of coding go back to the work of Bowen–Series [10]. The limit set $\Lambda$ can be re-interpreted as the non-wandering set of the map $T$ as
\begin{equation}
    \Lambda = \bigcap_{N=1}^{\infty} T^{-N}(I).
\end{equation}

Given any continuous function $\varphi: I \to \mathbb{R}$, we define the topological pressure in terms of weighted sums over periodic orbits through the formula
\begin{equation}
    P(\varphi) \overset{\text{def}}{=} \lim_{N \to \infty} \frac{1}{N} \log \left( \sum_{T^n x = x} e^{\varphi^{(n)}(x)} \right),
\end{equation}
where
\begin{equation}
    \varphi^{(n)}(x) \overset{\text{def}}{=} \varphi(x) + \varphi(Tx) + \ldots + \varphi(T^{n-1}x).
\end{equation}

By the variational formula we have
\begin{equation}
    P(\varphi) = \sup_{\mu} \left( h_\mu(T) - \int_\Lambda \varphi d\mu \right),
\end{equation}
where $\mu$ ranges over the set of $T$-invariant probability measures and $h_\mu(T)$ is the measure theoretic entropy. We refer the reader to [35] for general facts on topological pressure and thermodynamical formalism. More important for us is Bowen’s celebrated result [9] which says that the map
\begin{equation}
    \mathbb{R} \to \mathbb{R}, \quad \sigma \mapsto P(\sigma) \overset{\text{def}}{=} P(-\sigma \log |T'|)
\end{equation}
is convex, strictly decreasing and vanishes precisely at \( \sigma = \delta \), the Hausdorff dimension of the limit set. The relevance of the topological pressure stems from the Ruelle-Perron-Frobenius theorem which we recall here:

**Proposition 4.1** (Ruelle-Perron-Frobenius). Set \( \mathcal{L}_\sigma = \mathcal{L}_{\sigma, \text{id}} \) where \( \sigma \in \mathbb{R} \) is real and \( \text{id} \) is the one-dimensional trivial representation. Then the following statements hold true:

- The spectral radius of \( \mathcal{L}_\sigma \) on \( C^1(I) \) is equal to \( e^{P(\sigma)} \) which is a simple eigenvalue associated to a strictly positive eigenfunction \( \varphi_\sigma > 0 \) in \( C^1(I) \).
- The operator \( \mathcal{L}_\sigma \) on \( C^1(I) \) is quasi-compact with essential spectral radius smaller than \( \kappa(\sigma)e^{P(\sigma)} \) for some \( \kappa(\sigma) < 1 \).
- There are no other eigenvalues on \( |z| = e^{P(\sigma)} \).
- Moreover, the spectral projector \( \mathbb{P}_\sigma \) on \( \{e^{P(\sigma)}\} \) is given by
  \[
  \mathbb{P}_\sigma(f) = \varphi_\sigma \int_{\Lambda} f d\mu_\sigma,
  \]
  where \( \mu_\sigma \) is the unique \( T \)-invariant probability measure on \( \Lambda \) that satisfies
  \[
  \mathcal{L}_\sigma^*(\mu_\sigma) = e^{P(\sigma)}\mu_\sigma.
  \]

For a proof, we refer to [35, Theorem 2.2] (see also [26, Proposition 4.4]). One consequence of the Ruelle-Perron-Frobenius theorem is the following estimate due to Naud [31], which we will use multiple times in subsequent parts of this paper:

**Lemma 4.2** (Pressure estimate). For every \( M > 0 \) there exists a constant \( C > 0 \) such that for all \( \sigma \in [0, M] \) and all \( N \geq 1 \) we have

\[
(37) \quad \sum_{j=1}^{2m} \sum_{a \in W_N^j} \| \gamma_a^j \|_{\infty,D_j} \leq C e^{NP(\sigma)},
\]

where we use the notation
\[
\| g \|_{\infty,D_j} \overset{\text{def}}{=} \sup_{z \in D_j} |g(z)|,
\]
and \( P(\sigma) \) is the pressure function defined in (36).

### 4.3. Bergman space estimates.

We will also need some estimates for functions in the Bergman space \( H^2(D,V) \) defined in (10). Recall that \( D \) is the union of the Schottky disks \( D_1, \ldots, D_{2m} \) of \( \Gamma \) and that \( V \) is a finite-dimensional vector space endowed with a hermitian inner product \( \langle \cdot, \cdot \rangle_V \) and induced norm \( \| v \|_V = \sqrt{\langle v, v \rangle_V} \).

Bergman spaces enjoy the following property which we will use throughout this work: for every function \( f \in H^2(D,V) \) and every point \( z_0 \in D \) bounded away from the boundary \( \partial D \), the sizes of both \( f(z_0) \) and the derivative \( f'(z_0) \) are controlled by the \( L^2 \)-norm of \( f \). Recall that the norm of \( f \in H^2(D,V) \) is given by

\[
(38) \quad \| f \|^2 \overset{\text{def}}{=} \int_D \| f(z) \|^2_V \text{dvol}(z).
\]

More precisely, we have the following:

**Lemma 4.3** (Bergman space estimates). For every \( \kappa > 0 \) there exists a constant \( C = C(\kappa) > 0 \) such that for every point \( z_0 \in D \) with \( \text{dist}(z_0, \partial D) \geq \kappa \) and every \( f \in H^2(D,V) \), we have

\[
(39) \quad \| f(z_0) \|_V \leq C \| f \|
\]
and

(40) \[ \|f'(z_0)\|_V \leq C\|f\|. \]

Here \( \text{dist}(z_0, \partial D) \) denotes the euclidean distance of the point \( z_0 \) to the boundary of \( D \),

\[ \text{dist}(z, A) \overset{\text{def}}{=} \inf_{w \in A} |z - w|. \]

**Remark 4.4.** For any given basis \( e_1, \ldots, e_d \) of the vector space \( V \) we can write the function \( f \in H^2(D, V) \) as

(41) \[ f(z) = f_1(z)e_1 + \cdots + f_d(z)e_d \]

where each \( f_k \) belongs to the classical Bergman space \( H^2(D) \). We can then define the derivative of \( f \) componentwise:

(42) \[ f'(z) \overset{\text{def}}{=} f'_1(z)e_1 + \cdots + f'_d(z)e_d. \]

**Proof.** Let \( f \in H^2(D, V) \). Let \( d = \text{dim}(V) \) be the dimension of \( V \), fix an orthonormal basis \( e_1, \ldots, e_d \) for \( V \) and decompose \( f \) as in (41). Then we have

\[
\|f(z)\|_V^2 = \sum_{k=1}^{d} \|f_k(z)\|_V^2, \quad \|f'(z)\|_V^2 = \sum_{k=1}^{d} \|f'_k(z)\|_V^2
\]

for all \( z \in D \), and by integrating the first equation over \( z \in D \), we obtain

\[
\|f\|_V^2 = \sum_{k=1}^{d} \|f_k\|_V^2,
\]

so we can assume without loss of generality that \( d = 1 \) and \( f \in H^2(D) \).

Now let \( z_0 \in D \) be a point with \( \text{dist}(\partial D, z_0) \geq \kappa \). Then we have \( \overline{D(z_0, \frac{\kappa}{2})} \subset D \), where \( D(z_0, r) \) denotes the open disk in the complex plane with center \( z_0 \) and radius \( r \). Since \( f \) is holomorphic on \( D \), we have

\[
f(z_0) = \frac{1}{\text{vol}(D(z_0, \frac{\kappa}{2}))} \int_{D(z_0, \frac{\kappa}{2})} f(z) \, \text{dvol}(z).
\]

Applying the Cauchy–Schwarz inequality yields

\[
|f(z_0)|^2 \leq \frac{1}{\text{vol}(D(z_0, \frac{\kappa}{2}))} \int_{D(z_0, \frac{\kappa}{2})} |f(z)|^2 \, \text{dvol}(z) \leq \frac{1}{\text{vol}(D(z_0, \frac{\kappa}{2}))} \|f\|^2.
\]

Inserting the formula

\[
\text{vol}(D(z_0, \frac{\kappa}{2})) = \pi \left( \frac{\kappa}{2} \right)^2
\]

into this estimate, we obtain

(43) \[ |f(z_0)| \leq 2\sqrt{\pi} \kappa^{-1} \|f\|, \]

which proves (39). To prove (40) we use the Cauchy integral formula

\[
f'(z_0) = \frac{1}{2\pi i} \int_{\partial D(z_0, \frac{\kappa}{2})} \frac{f(z)}{(z - z_0)^2} \, dz
\]

to estimate

(44) \[ |f'(z_0)| \leq 2\kappa^{-1} \max_{w \in \partial D(z_0, \frac{\kappa}{2})} |f(w)|. \]
Using the triangle inequality, we have \( \text{dist}(w, \partial D) \geq \frac{\pi}{2} \) for all \( w \in \partial D(z_0, \frac{\pi}{2}) \). Hence, combining (43) with (44) yields

\[
|f'(z_0)| \leq 8\sqrt{\pi} \kappa^{-2} \| f \|
\]

completing the proof of Lemma 4.3.

**Lemma 4.5.** There exists \( C > 0 \) such that for all \( f \in H^2(D, V) \) we have

\[
\|f\|_{\infty, \Lambda} \leq C\|f\|
\]

where

\[
\|f\|_{\infty, \Lambda} \overset{\text{def}}{=} \sup_{x \in \Lambda} \|f(x)\|_V.
\]

**Proof.** By Lemma 4.3 it suffices to show that each point in the limit set is uniformly bounded away from the boundary \( \partial D \). To see this we use (32) to rewrite the limit set in terms of reduced words as

\[
\Lambda = \bigcup_{N \geq 1} \left( \bigcup_{j=1}^{2m} \bigcup_{\alpha \in W_N^j} \gamma_\alpha(I_j) \right),
\]

where \( I_j = D_j \cap \mathbb{R} \) is the interval spanned by the Schottky disk \( D_j \). For all \( j \in [2m] \) and for every \( \alpha = a_1 \ldots a_N \in W_j^2 \), the map \( \gamma_\alpha \) maps the closure of \( I_j \) into the interior of \( I_{a_1+m} \). Thus we have

\[
\text{dist}(\gamma_\alpha(I_j), \partial D) = \text{dist}(\gamma_\alpha(I_j), \partial I_{a_1+m}) \geq \kappa
\]

for some \( \kappa = \kappa(\Gamma) > 0 \). Using (45) we deduce that for every point \( x \in \Lambda \),

\[
\text{dist}(x, \partial D) \geq \kappa.
\]

Lemma 4.5 now follows from Lemma 4.3.

**Lemma 4.6** (Mean value estimate). There exists a constant \( C > 0 \) such that for each \( j \in [2m], N \geq 1, \alpha \in W_N^j, \) all points \( z, z' \in D_j \) and each function \( f \in H^2(D, V) \), we have

\[
\|f(\gamma_\alpha(z)) - f(\gamma_\alpha(z'))\|_V \leq C\|\gamma_\alpha\|_{D_j, \infty} \|f\|.
\]

Moreover, we have

\[
\sum_{\alpha \in W_N^j} \|f(\gamma_\alpha(z)) - f(\gamma_\alpha(z'))\|_V \leq Ce^{P(1)N} \|f\|
\]

where \( P(\cdot) \) is the pressure defined in (36).

**Proof.** The second estimate follows from first the one and the pressure estimate in Lemma 4.2. Thus it suffices to prove the first estimate. We can reduce the proof to the one-dimensional case with the same argument used in the proof of Lemma 4.3. Thus we may assume that \( V = \mathbb{C} \) and \( f \in H^2(D) \).

Fix points \( z, z' \in D \). By the mean value theorem there exists a point \( z'' \) on the line segment connecting \( z \) and \( z' \) such that

\[
f(\gamma_\alpha(z)) - f(\gamma_\alpha(z')) = \gamma'_\alpha(z'') f'(\gamma_\alpha(z''))(z - z').
\]

We deduce that

\[
|f(\gamma_\alpha(z)) - f(\gamma_\alpha(z'))| \leq \|\gamma_\alpha\|_{\infty, D_j} \text{diam}(D_j)|f'(\gamma_\alpha(z''))|.
\]
Noticing that $\gamma_a(z'')$ is bounded away from the boundary $\partial D$ uniformly in $j, N, a, z, z'$, we can invoke Lemma 4.3 to obtain
\[ |f(\gamma_a(z)) - f(\gamma_a(z'))| \leq C \|\gamma_a\|_{\infty, D_j} \|f\| \]
for some constant $C = C(\Gamma) > 0$, completing the proof.

4.4. A priori estimates for the transfer operator. In this section we prove a
priori estimates for the norm of $L^N_{\delta, \rho}$ in the space $H^2(D, V)$. Recall that for every
function $f: D \to V$, we set
\[ \|f\|_{\infty, \Lambda} \overset{\text{def}}{=} \sup_{x \in \Lambda} \|f(x)\|_V, \]
whereas $\|f\|$ denotes the $L^2$-norm defined in (38). The next estimate, which will be
crucial at a later stage of this work, says that the $L^2$-norm of $L^N_{\delta, \rho} f$ is essentially
controlled by the maximum norm of $f$ on the limit set.

Lemma 4.7 (A priori estimates). There exist constants $c > 0$ and $C > 0$ depending
only on the Schottky data of $\Gamma$ such that for every finite-dimensional unitary representation $(\rho, V)$ of $\Gamma$, all $f \in H^2(D, V)$ and all $N$,
\[ \|L^N_{\delta, \rho} f\|^2 \leq C (\|f\|^2_{\infty, \Lambda} + e^{-cN} \|f\|^2). \]  
In particular, there exists $C > 0$ depending only on the Schottky data of $\Gamma$ such that
\[ \|L^N_{\delta, \rho} f\| \leq C \|f\|. \]

Proof. The second estimate follows from the first one and Lemma 4.5. Thus it is
enough to establish the first estimate. Fix an index $j \in [2m]$ and a point $z \in D_j$ and
recall from (28) that
\[ L^N_{\delta, \rho} f(z) = \sum_{a \in W^j_N} \gamma_a(z)^{\delta} \rho(\gamma_a)^{-1} f(\gamma_a(z)). \]
Now fix a point $x \in \Lambda \cap D_j$ and write
\[ L^N_{\delta, \rho} f(z) = \sum_{a \in W^j_N} \gamma_a(z)^{\delta} \rho(\gamma_a)^{-1} f(\gamma_a(x)) + \sum_{a \in W^j_N} \gamma_a(z)^{\delta} \rho(\gamma_a)^{-1} (f(\gamma_a(z)) - f(\gamma_a(x))). \]
The limit set $\Lambda$ is invariant under the action of $\Gamma$, so we have $\gamma_a(x) \in \Lambda$ for all words $a$. Thus, using the triangle inequality (together with the fact that $\rho$ is unitary) and the mean value estimate in Lemma 4.6, we obtain
\[ \|L^N_{\delta, \rho} f(z)\|_V \leq \sum_{a \in W^j_N} \|\gamma_a^{\delta} \rho(\gamma_a)^{-1} f(\gamma_a(x))\|_V + \sum_{a \in W^j_N} \|\gamma_a\|_{\infty, D_j} \|f(\gamma_a(z)) - f(\gamma_a(x))\| \]
\[ \ll \sum_{a \in W^j_N} \|\gamma_a\|_{\infty, D_j} \|f\|_{\infty, \Lambda} + \sum_{a \in W^j_N} \|\gamma_a\|_{\infty, D_j} \|f\|. \]
Using the pressure estimate from Lemma 4.2 for each summand in the previous line, we get
\[ \|L^N_{\delta, \rho} f(z)\|_V \ll \|f\|_{\infty, \Lambda} + e^{P(\delta + 1)N} \|f\|. \]
Squaring and integrating this estimate over all $z \in D$ yields
\[ \|L^N_{\delta, \rho} f\|^2 = \int_D \|L^N_{\delta, \rho} f(z)\|^2 \, \text{dvol}(z) \ll \|f\|^2_{\infty, \Lambda} + e^{2P(\delta + 1)N} \|f\|^2. \]
This proves (48) with $c = -2P(\delta + 1) > 0$. \qed
5. Exploiting the expansion property

5.1. Exponential decay estimate for $W_N$. Let us briefly recall some notations. Given a Schottky group $\Gamma$ with Schottky data $\gamma_1, \ldots, \gamma_m$ and $D_1, \ldots, D_{2m}$, a finite group $G$ and a homomorphism $\pi: \Gamma \to G$, we let $\Gamma' = \ker(\pi)$ be the kernel of $\pi$. We assume that $\pi: \Gamma \to G$ is surjective (in which case the quotient $\Gamma/\Gamma'$ is isomorphic to $G$) and that $\Gamma$ is non-elementary, which is equivalent with saying that $m \geq 2$. For elements $g_1, \ldots, g_m \in G$ we define the operator $T: L^2(G) \to L^2(G)$ by

$$T f(x) = \frac{1}{2m} \sum_{j=1}^{m} \left( f(xg_j) + f(xg_j^{-1}) \right)$$

for $x \in G$ and $f \in L^2(G)$. Recall that $g_1, \ldots, g_m \in G$ are said to form an $\varepsilon$-expanding set if

$$\|Tf\|_{L^2(G)} \leq (1 - \varepsilon)\|f\|_{L^2(G)}$$

for all functions $f \in L^2_0(G)$, where $L^2_0(G) \subset L^2(G)$ is the subspace of functions $f$ with

$$\sum_{x \in G} f(x) = 0.$$

Equivalently, $\{g_1, \ldots, g_m\}$ is $\varepsilon$-expanding if all the eigenvalues of $T$, other than the trivial eigenvalue 1, lie in the interval $[-1 + \varepsilon, 1 - \varepsilon]$.

We denote by $\text{irrep}(G)$ the finite set of irreducible representations of $G$ and we let

$$\text{irrep}_0(G) \overset{\text{def}}{=} \text{irrep}(G) \setminus \{\text{id}\}$$

be the set of non-trivial, irreducible representations of $G$. It is important to notice for every representation $(\rho, V)$ of $G$, we can endow its representation space $V$ with a hermitian inner product $\langle \cdot, \cdot \rangle_V$ with respect to which $\rho(g)$ is unitary for every $g \in G$. We will always tacitly assume that $V$ is endowed with such an inner product and we let

$$\|v\|_V \overset{\text{def}}{=} \sqrt{\langle v, v \rangle_V}$$

be the induced norm. Notice furthermore that every representation $(\rho, V)$ of $G$ extends naturally to a representation $(\rho, V)$ of $\Gamma$ by setting

$$\rho(\gamma) \overset{\text{def}}{=} \rho(\pi(\gamma)), \quad \gamma \in \Gamma.$$

For every finite set $Z$ of words in the alphabet $[2m] = \{1, \ldots, 2m\}$ and every finite-dimensional representation $(\rho, V)$ of $\Gamma$, we define the operator $Z(\rho): V \to V$ by

$$Z(\rho) = \sum_{a \in Z} \rho(\gamma_a),$$

where for $a = a_1 \cdots a_N$ we write $\gamma_a = \gamma_{a_1} \circ \cdots \circ \gamma_{a_N}$. Recall from (27) that $W_N$ denotes the set of reduced words of length $N$ in the alphabet $[2m]$ which corresponds one-to-one to the set elements of $\Gamma$ of reduced word length $N$ via the map $a \mapsto \gamma_a$.

Finally, for a linear operator $A: V \to V$ we define its operator norm as

$$\|A\|_{\text{End}(V)} \overset{\text{def}}{=} \max_{\|v\|_V=1} \|Av\|_V.$$
Proposition 5.1 (Exponential decay norm estimate for $W_N(\rho)$). Assume that $$\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\}$$ is an $\varepsilon$-expanding set in $G$. Then for every $(\rho, V) \in \text{irrep}_0(G)$ and for every positive integer $N$ we have $$\|W_N(\rho)\|_{\text{End}(V)} \leq Ce^{-cN}|W_N|,$$ where both $C > 0$ and $c > 0$ are positive constants depending only on $\varepsilon$ and the number $m$ of Schottky generators.

Proof. The group $G$ acts by linear automorphisms on $L^2(G)$ through the left regular representation $R_G(g)\varphi(x) = \varphi(xg^{-1})$ for all $g \in G$, $x \in G$ and $\varphi \in L^2(G)$, giving a unitary representation $R_G : G \to U(L^2(G))$.

The regular representation can be decomposed into the irreducible representations of $G$,

$$R_G = \bigoplus \dim(\rho) \rho.$$  

This allows us to decompose the operator $T$ in (50) accordingly as

$$(51) \quad T = \bigoplus_{\rho \in \text{irrep}(G)} \dim(\rho)T_\rho = \text{id} \oplus \bigoplus_{\rho \in \text{irrep}_0(G)} \dim(\rho)T_\rho$$

where for each $\rho$, $T_\rho$ is the operator

$$T_\rho : L^2(V) \to L^2(V), \quad T_\rho \overset{\text{def}}{=} \frac{1}{2m} W_1(\rho) = \frac{1}{2m} S(\rho) = \frac{1}{2m} \sum_{s \in S} \rho(s).$$

For the remainder of this proof we fix $(\rho, V) \in \text{irrep}_0(G)$ and we let $\lambda_1, \ldots, \lambda_d$ denote the eigenvalues of the operator $W_1(\rho)$ (we suppress the dependence on $\rho$ in the eigenvalues for notational convenience). In light of (51) we deduce that

$$(52) \quad \max_{1 \leq k \leq d} |\lambda_k| \leq 2m(1 - \varepsilon),$$

since $\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\}$ is $\varepsilon$-expanding. We will use this bound later in the proof.

Now we set $W_0(\rho) \overset{\text{def}}{=} I_V$, where $I_V$ denotes the identity operator on $V$ and we consider the following formal power series with coefficients in the endomorphism ring $\text{End}(V)$ of $V$:

$$(53) \quad G_\rho(z) \overset{\text{def}}{=} \sum_{N=0}^\infty W_N(\rho)z^N.$$  

An elementary argument shows that

$$(54) \quad |W_N| \asymp (2m - 1)^N,$$

where the implied constants are independent of $N$. Note that the triangle inequality gives the trivial estimate

$$\|W_N(\rho)\|_{\text{End}(V)} \leq |W_N|,$$

from which we deduce that (53) is absolutely convergent for $|z| < \frac{1}{2m-1}$ with respect to the operator norm $\| \cdot \|_{\text{End}(V)}$. 

The next step is to derive a closed-form expression for $G_\rho(z)$. To that effect, note that we have the elementary but essential recursion formulas$^1$:

\[(55) \quad W_1(\rho)^2 = W_2(\rho) + 2m \cdot I_V\]

and for all $N \geq 2$,

\[(56) \quad W_1(\rho)W_N(\rho) = W_{N+1}(\rho) + (2m - 1)W_{N-1}(\rho).\]

We can now multiply $G_\rho(z)$ by $zW_1(\rho)$ on the right and apply the recursion formulas (55) and (56) to obtain

\[
zW_1(\rho)G_\rho(z) = W_1(\rho)z + W_1(\rho)^2z^2 + \sum_{N=2}^{\infty} W_1(\rho)W_N(\rho)z^{N+1}
\]

\[
= W_1(\rho)z + (W_2(\rho) + 2m \cdot I_V)z^2
\]

\[
+ \sum_{N=2}^{\infty} W_{N+1}(\rho)z^{N+1} + (2m - 1) \sum_{N=2}^{\infty} W_{N-1}(\rho)z^{N+1}.
\]

Inserting

\[
\sum_{N=2}^{\infty} W_{N+1}(\rho)z^{N+1} = G_\rho(z) - W_2(\rho)z^2 - W_1(\rho)z - I_V
\]

and

\[
\sum_{N=2}^{\infty} W_{N-1}(\rho)z^{N+1} = z^2 (G_\rho(z) - I_V)
\]

into the above equation yields

\[
zW_1(\rho)G_\rho(z) = (z^2 - 1)I_V + G_\rho(z) \left( 1 + (2m - 1)z^2 \right).
\]

Rearranging gives the closed-form expression

\[
G_\rho(z) = \left( 1 - z^2 \right) \left( I_V - W_1(\rho)z + (2m - 1)z^2 \right)^{-1}.
\]

Using this formula we can now derive an explicit formula for $W_N(\rho)$. To do so, observe that $W_1(\rho)$ is a self-adjoint operator by virtue of the fact that $\rho$ is a unitary representation and $S$ is symmetric, i.e., $S^{-1} = S$. Hence, we can find a basis of $V$ with respect to which $W_1(\rho)$ acts diagonally, i.e., we can write

\[
W_1(\rho) = \text{diag}(\lambda_1, \ldots, \lambda_d)
\]

where $\lambda_1, \ldots, \lambda_d$ are the eigenvalues of $W_1(\rho)$. For each $k \in [d]$, let $\omega_k^\pm$ be the pair of complex numbers satisfying

\[
1 - \lambda_k z + (2m - 1)z^2 = (1 - \omega_k^+ z)(1 - \omega_k^- z).
\]

With respect to the diagonalizing basis, we then have the expression

\[
G_\rho(z) = \text{diag} \left( \frac{1 - z^2}{(1 - \omega_k^+ z)(1 - \omega_k^- z)} \right)_{k=1}^d.
\]

Expanding each diagonal entry as a power series in $z$ gives

\[
\frac{1 - z^2}{(1 - \omega_k^+ z)(1 - \omega_k^- z)} = \sum_{N=0}^{\infty} (\xi_{k,N} - \xi_{k,N-2}) z^N,
\]

\[\footnotesize{\text{References}}\]

$^1$These recursion formulas were used by Ihara [23] in his derivation of a closed-form expression of what came to be known as the Ihara zeta function.
where $\xi_{k,-1} = \xi_{k,-2} = 0$ and

\begin{equation}
(57) \quad \xi_{k,N} = \sum_{l=0}^{N} (\omega_k^+)^l (\omega_k^-)^{N-l}
\end{equation}

for all $N \geq 0$. This gives

$$W_N(\rho) = \text{diag} (\xi_{k,N} - \xi_{k,N-2})_{k=1}^d.$$  

It follows that

\begin{equation}
(58) \quad \|W_N(\rho)\|_{\text{End}(V)} = \max_{1 \leq k \leq d} |\xi_{k,N} - \xi_{k,N-2}|.
\end{equation}

A direct computation shows that

\begin{equation}
(59) \quad \omega_k^\pm = \frac{\lambda_k \pm \sqrt{\lambda_j^2 - 4(2m-1)}}{2}.
\end{equation}

Suppose that $|\lambda_k| \leq 2\sqrt{2m-1}$. In this case it follows from (59) that

$$|\omega_k^\pm| = \sqrt{2m-1}.$$

Now suppose $|\lambda_j| > 2\sqrt{2m-1}$. In this case we recall from (52) that

$$|\lambda_j| \leq 2m(1-\varepsilon),$$

which we use to deduce the bound

$$|\omega_k^\pm| \leq \frac{|\lambda_k| + \sqrt{(2m)^2 - 4(2m-1)}}{2} \leq (2m - 1) \left(1 - \frac{m}{2m-1} \varepsilon\right).$$

Therefore, setting

$$\varepsilon' = \min \left\{ \frac{m}{2m-1} \varepsilon, 1 - \frac{1}{\sqrt{2m-1}} \right\} > 0,$$

we obtain

\begin{equation}
(60) \quad \max_{1 \leq k \leq d} |\omega_k^\pm| \leq (2m - 1)(1 - \varepsilon') < (2m - 1)e^{-\varepsilon'}.\end{equation}

Using (57), (60) and (54) we obtain

$$\max_{1 \leq k \leq d} |\xi_{k,N}| \leq N \left( \max_{1 \leq k \leq d} |\omega_k^\pm| \right)^N \leq Ne^{-\varepsilon'N}(2m-1)^N \ll e^{-\varepsilon''N}|W_N|,$$

where we have set $\varepsilon'' = \varepsilon'/2$. Thus, returning to (58), we obtain

$$\|W_N(\rho)\|_{\text{End}(V)} \leq \max_{1 \leq k \leq d} |\xi_{k,N}| + \max_{1 \leq k \leq d} |\xi_{k,N-2}| \ll e^{-\varepsilon''N}|W_N|.$$

This completes the proof of Proposition 5.1. □

For every positive integer $N$ and for every pair of indices $i, j \in [2m]$ consider the following subset of reduced words of length $N$:

$$A_{N}^{ij} = \{ a = a_1 \cdots a_N \in W_N : a_1 = i, a_N = j \}.$$  

It turns out that in our application it is more useful to have a bound on the operator norm of $A_{N}^{ij}(\rho)$ rather than $W_N(\rho)$.
Corollary 5.2 (Norm estimate for $A_{N}^{i,j}(\rho)$). Assume that
\[ \{ \pi(\gamma_1), \ldots, \pi(\gamma_m) \} \]
is an $\varepsilon$-expanding set in $G$. Then for every $(\rho, V) \in \text{irrep}_0(G)$, for every $N$ and for every pair of indices $i, j \in [2m]$, we have the bound
\[ \| A_{N}^{i,j}(\rho) \|_{\text{End}(V)} \leq C e^{-cN} |W_N| \]
where $C > 0$ and $c > 0$ depend only on $\varepsilon$ and $m$.

Proof. Consider first the set of reduced words of length $N$ starting with the letter $i$,
\[ B_N^i = \{ a = a_1 \cdots a_N \in W_N : a_1 = i \}. \]
and note that
\[ B_N^i(\rho) = \rho(\gamma_i) \left( W_{N-1}(\rho) - B_{N-1}^{i+m}(\rho) \right). \]
Applying the triangle inequality and using the fact that $\rho$ is unitary gives
\[ \| B_N^i(\rho) \|_{\text{End}(V)} \leq \| W_{N-1}(\rho) \|_{\text{End}(V)} + \| B_{N-1}^{i+m}(\rho) \|_{\text{End}(V)}. \]
Hence, setting
\[ b_N(\rho) \overset{\text{def}}{=} \max_{1 \leq i \leq 2m} \| B_N^i(\rho) \|_{\text{End}(V)}, \]
we obtain
\[ b_N(\rho) \leq \| W_{N-1}(\rho) \|_{\text{End}(V)} + b_{N-1}(\rho). \]
Iterating this inequality gives
\[ b_N(\rho) \leq \sum_{k=0}^{N-1} \| W_k(\rho) \|_{\text{End}(V)}. \]
By Proposition 5.1 we have
\[ \| W_k(\rho) \|_{\text{End}(V)} \ll e^{-cNk} |W_k| \ll e^{-cN}(2m-1)^k, \]
where the implied constant and $c > 0$ depend only on the expansion constant $\varepsilon$ and $m$. By inserting this bound into (62), and using a geometric series summation and the fact that $|W_N| \asymp (2m-1)^N$, we obtain
\[ b_N(\rho) \ll \sum_{k=0}^{N-1} e^{-cN(2m-1)^k} \ll e^{-cN}(2m-1)^N \ll e^{-cN}|W_N|. \]
Notice that we also have the relation
\[ A_{N}^{i,j}(\rho) = \left( B_{N-1}^i(\rho) - A_{N}^{i,j+m}(\rho) \right) \rho(\gamma_j). \]
Now we set
\[ a_N(\rho) \overset{\text{def}}{=} \max_{1 \leq i, j \leq 2m} \| A_{N}^{i,j}(\rho) \|_{\text{End}(V)}. \]
and we repeat the argument above (with the necessary modifications). From (64) we get the estimate
\[ a_N(\rho) \leq \| B_{N-1}^{i}(\rho) \|_{\text{End}(V)} + a_{N-1}(\rho) \leq b_{N-1}(\rho) + a_{N-1}(\rho), \]
which we can iterate to obtain
\[ a_N(\rho) \leq \sum_{k=0}^{N-1} b_k(\rho). \]
Combining this with the bound in (63), we obtain
\[ a_N(\rho) \ll e^{-cN}|W_N|, \]
concluding the proof of Corollary 5.2.

5.2. Decoupling estimate. Recall that for every function \( g: \Lambda \to V \) we set
\[
\|g\|_{\infty, \Lambda} \overset{\text{def}}{=} \max_{x \in \Lambda} \|g(x)\|_V.
\]

Moreover, we introduce the semi-norm
\[
[g]_\Lambda \overset{\text{def}}{=} \max_{1 \leq j \leq 2m} \max_{x,y \in \Lambda \cap I_j, x \neq y} \left\| \frac{g(x) - g(y)}{x - y} \right\|_V,
\]
where \( I_j = D_j \cap \mathbb{R} \) are the real intervals spanned by the Schottky disks \( D_1, \ldots, D_{2m} \) of \( \Gamma \). The aim of this subsection is to prove the following result:

**Proposition 5.3** (Decoupling estimate). Suppose \( \{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \) is \( \epsilon \)-expanding. Then for every \( n \), for every \( (\rho, V) \in \text{irrep}_0(G) \), for every function \( g: \Lambda \to V \), for every index \( j \in [2m] \) and for every point \( x \in \Lambda \cap I_j \), we have the bound
\[
\frac{1}{|W_N^j|} \left\| \sum_{a \in W_N^j} \rho(\gamma_a)^{-1}g(\gamma_a(x)) \right\|_V \leq Ce^{-cN} (\|g\|_{\infty, \Lambda} + [g]_\Lambda),
\]
where the constants \( c > 0 \) and \( C > 0 \) depend only on \( \epsilon \) and the Schottky data of \( \Gamma \).

**Remark 5.4.** By specializing Proposition 5.3 to the constant 1 function \( g = 1 \) we recover the purely combinatorial statement in Proposition 5.1.

**Proof.** For each pair \( j, l \in [2m] \) we define
\[
Z_N^{l,j} = \{a = a_1 \cdots a_N \in W_N : a_1 = l + m \text{ and } a_N \neq j\}.
\]

This set corresponds precisely to the set of reduced words \( a \in W_N \) for which the element \( \gamma_a \) maps the disk \( D_j \) into the interior of \( D_l \). We can write
\[
Z_N^{l,j}(\rho) = \sum_{1 \leq j' \leq 2m} A_N^{l+m,j'}(\rho)
\]
where
\[
A_N^{l,j} = \{a = a_1 \cdots a_N \in W_N : a_1 = l, a_N = j\}.
\]

Corollary 5.2 gives
\[
\|Z_N^{l,j}(\rho)\|_{\text{End}(V)} \leq \sum_{1 \leq j' \leq 2m, j' \neq j} \|A_N^{l+m,j'}(\rho)\|_{\text{End}(V)} \ll e^{-cN}|W_N|,
\]
where both \( c > 0 \) and the implied constant depend only on \( \epsilon \) and the Schottky data of \( \Gamma \). Note that the sets \( Z_N^{l,j} \) and \( W_N \) are of comparable size, that is, we have
\[
|Z_N^{l,j}| \asymp |W_N|
\]
with some absolute implied constants. Combining this with (66) yields the estimate
\[
\|Z_N^{l,j}(\rho)\|_{\text{End}(V)} \ll e^{-cN}|Z_N^{l,j}|,
\]
which will be used later in the proof.

We will use a decoupling argument similar to the one contained in [7, §8]. Write \( N = N_1 + N_2 \) where \( N_1 = \lfloor N/2 \rfloor \) and \( N_2 = \lceil N/2 \rceil \) and notice that for every \( j \in [2m] \) and every word \( a \in W_N^j \) there exists a unique index \( l \in [2m] \) and unique words
\(a_1 \in W_{N_1}^l\) and \(a_2 \in Z_{N_2}^{l,j}\) with \(a = a_1a_2\). (Here, \(a_1a_2\) stands for the concatenation of the words \(a_2\) and \(a_2\).) With this in mind, we write
\[
(68) \quad \sum_{a \in W_{l_i}^l} \rho(\gamma_a)^{-1}g(\gamma_a(x)) = \sum_{l=1}^{2m} S_l,
\]
where for each \(l \in [2m]\) we have set
\[
S_l \overset{\text{def}}{=} \sum_{a_1 \in W_{N_1}^l, a_2 \in Z_{N_2}^{l,j}} \rho(\gamma_{a_1a_2})^{-1}g(\gamma_{a_1a_2}(x)).
\]
We can now split each \(S_l\) into a sum
\[
S_l = S_l^1 + S_l^2
\]
where
\[
S_l^1 \overset{\text{def}}{=} \sum_{a_1 \in W_{N_1}^l, a_2 \in Z_{N_2}^{l,j}} \rho(\gamma_{a_1a_2})^{-1}\left(g(\gamma_{a_1a_2}(x)) - \frac{1}{|Z_{N_2}^{l,j}|} \sum_{b_2 \in Z_{N_2}^{l,j}} g(\gamma_{a_1b_2}(x))\right)
\]
and
\[
S_l^2 \overset{\text{def}}{=} \sum_{a_1 \in W_{N_1}^l, a_2 \in Z_{N_2}^{l,j}} \rho(\gamma_{a_1a_2})^{-1}\left(\frac{1}{|Z_{N_2}^{l,j}|} \sum_{b_2 \in Z_{N_2}^{l,j}} g(\gamma_{a_1b_2}(x))\right)
\]
and we consider these two parts separately. Notice that we can rewrite \(S_l^1\) as
\[
S_l^1 = \frac{1}{|Z_{N_2}^{l,j}|} \sum_{a_2, b_2 \in Z_{N_2}^{l,j}} \sum_{a_1 \in W_{N_1}^l} \rho(\gamma_{a_1a_2})^{-1}(g(\gamma_{a_1a_2}(x)) - g(\gamma_{a_1b_2}(x)))
\]
so we can use the triangle-inequality and the unitarity of \(\rho\) to estimate
\[
\|S_l^1\|_V \leq \frac{1}{|Z_{N_2}^{l,j}|} \sum_{a_2, b_2 \in Z_{N_2}^{l,j}} \sum_{a_1 \in W_{N_1}^l} \|g(\gamma_{a_1a_2}(x)) - g(\gamma_{a_1b_2}(x))\|_V
\]
\[
\leq \frac{1}{|Z_{N_2}^{l,j}|} \sum_{a_2, b_2 \in Z_{N_2}^{l,j}} \sum_{a_1 \in W_{N_1}^l} [g]_{\Lambda} |\gamma_{a_1} (\gamma_{a_2}(x)) - \gamma_{a_1} (\gamma_{b_2}(x))|.
\]
By construction of the sets \(Z_{N_2}^{l,j}\), the following holds true: for all \(a_2, b_2 \in Z_{N_2}^{l,j}\) and each point \(x \in \Lambda \cap I_j\), the images \(\gamma_{a_2}(x)\) and \(\gamma_{b_2}(x)\) are contained in \(D_l\). Hence, we can use Lemma 4.6 to estimate the inner sum as follows:
\[
\sum_{a_1 \in W_{N_1}^l} |\gamma_{a_1} (\gamma_{a_2}(x)) - \gamma_{a_1} (\gamma_{b_2}(x))| \ll e^{N_1 P(1)}.
\]
Here, \(P(\cdot)\) is the pressure defined in (36) and the implied constant depends only on the Schottky data of \(\Gamma\). Inserting this into the above bound for \(\|S_l^1\|_V\) gives
\[
(69) \quad \|S_l^1\|_V \ll \frac{1}{|Z_{N_2}^{l,j}|} \sum_{a_2, b_2 \in Z_{N_2}^{l,j}} [g]_{\Lambda} e^{N_1 P(1)} \ll |Z_{N_2}^{l,j}| e^{N_1 P(1)} [g]_{\Lambda}.
\]
Let us now focus on $S_i^2$. Because $\rho$ is a representation, we have $\rho(\gamma_{a_1 a_2}) = \rho(\gamma_{a_1}) \rho(\gamma_{a_2})$, which we can use to interchange the order of summation in $S_i^2$:

$$S_i^2 = \sum_{a_1 \in W_{N_1}^l, a_2 \in Z_{N_2}^{i,j}} \rho(\gamma_{a_1 a_2})^{-1} \left( \frac{1}{|Z_{N_2}^{i,j}|} \sum_{b_2 \in Z_{N_2}^{i,j}} g(\gamma_{a_1} \gamma_{b_2}(x)) \right)$$

$$= \frac{1}{|Z_{N_2}^{i,j}|} \sum_{a_1 \in W_{N_1}^l, b_2 \in Z_{N_2}^{i,j}} \left( \sum_{a_2 \in Z_{N_2}^{i,j}} \rho(\gamma_{a_2})^{-1} \rho(\gamma_{a_1})^{-1} g(\gamma_{a_1} \gamma_{b_2}(x)) \right).$$

Since $\rho$ is unitary, we have $\rho(\gamma)^{-1} = \rho(\gamma)^*$ for all $\gamma \in \Gamma$, where $A^*$ denotes the adjoint of $A$. Thus, we have

$$\sum_{a_2 \in Z_{N_2}^{i,j}} \rho(\gamma_{a_2})^{-1} = \sum_{a_2 \in Z_{N_2}^{i,j}} \rho(\gamma_{a_2})^* = Z_{N_2}^{i,j}(\rho)^*,$$

which we insert above to obtain

$$S_i^2 = \frac{1}{|Z_{N_2}^{i,j}|} \sum_{a_1 \in W_{N_1}^l, b_2 \in Z_{N_2}^{i,j}} Z_{N_2}^{i,j}(\rho)^* \rho(\gamma_{a_1})^{-1} g(\gamma_{a_1} \gamma_{b_2}(x)).$$

Applying the triangle inequality and using the fact that the operator norm $\|A\|_{\text{End}(V)}$ satisfies

$$\|A^*\|_{\text{End}(V)} = \|A\|_{\text{End}(V)}$$

for each $A \in \text{End}(V)$, gives

$$\|S_i^2\|_V \leq \frac{1}{|Z_{N_2}^{i,j}|} \sum_{a_1 \in W_{N_1}^l, b_2 \in Z_{N_2}^{i,j}} \|Z_{N_2}^{i,j}(\rho)^* \rho(\gamma_{a_1})^{-1} g(\gamma_{a_1} \gamma_{b_2}(x))\|_V$$

$$\leq \frac{\|Z_{N_2}^{i,j}(\rho)\|_{\text{End}(V)}}{|Z_{N_2}^{i,j}|} \sum_{a_1 \in W_{N_1}^l, b_2 \in Z_{N_2}^{i,j}} \|g(\gamma_{a_1} \gamma_{b_2}(x))\|_V.$$

Invoking the bound in (67) yields

$$\|S_i^2\|_V \leq e^{-c N_2} \sum_{a_1 \in W_{N_1}^l, b_2 \in Z_{N_2}^{i,j}} \|g(\gamma_{a_1} \gamma_{b_2}(x))\|_V \leq e^{-c N_2} \|W_N^j\| \|g\|_{\infty, \Lambda}.$$

Hence, putting together (69) and (70) gives

$$\|S_i\|_V \leq |S_i^1|_V + \|S_i^2\|_V \leq e^{-c N_2} \|W_N^j\| \|g\|_{\infty, \Lambda} + |Z_{N_2}^{i,j}| e^{-P(1) N_1} |g|_{\Lambda}.$$

Recalling that $N_1 = \lfloor N/2 \rfloor$, $N_2 = \lfloor N/2 \rfloor$ and $P(1) < 0$ and noticing the trivial bound $|Z_{N_2}^{i,j}| < |W_N^j|$, we obtain

$$\|S_i\|_V \leq e^{-c_0 N} \|W_N^j\| (\|g\|_{\infty, \Lambda} + [g]_{\Lambda}),$$

with

$$c_0 = \min \left\{ \frac{c}{2}, \frac{-P(1)}{2} \right\} > 0.$$

Thus, returning to (68), we get

$$\left\| \sum_{a \in W_N^j} \rho(\gamma_a)^{-1} g(\gamma_a(x)) \right\|_V \leq \sum_{l=0}^{2m} \|S_i\|_V \leq e^{-c_0 N} \|W_N^j\| (\|g\|_{\infty, \Lambda} + [g]_{\Lambda})$$
which concludes the proof.

6. Norm Estimate for $\mathcal{L}^N_{\delta,\rho}$

6.1. Statement of Main Estimate. We will keep all the notations and assumptions of the previous sections. The goal of this section is to show that the norm of $\mathcal{L}^N_{\delta,\rho}$ is exponentially decaying as $N \to \infty$, with a decay rate depending solely on the expansion constant of $\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\}$ and the Schottky data of $\Gamma$.

**Proposition 6.1** (Norm estimate for $\mathcal{L}^N_{\delta,\rho}$). Assume that $\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\}$ is an $\varepsilon$-expanding set in $G$. Then there exists a positive integer $N_0 = N_0(\varepsilon, \Gamma)$ such that for every $(\rho, V) \in \text{irrep}_0(G)$ and for every $f \in H^2(D, V)$, we have

$$\|\mathcal{L}^N_{\delta,\rho} f\| \leq \frac{1}{2} \|f\|.$$

Moreover, for every positive integer $N$ we have

$$(71) \quad \|\mathcal{L}^N_{\delta,\rho} f\| \leq C_1 e^{-c_1 N} \|f\|$$

and

$$(72) \quad \|\mathcal{L}^N_{\delta,\rho} f\|_{\infty, \Lambda} \leq C_2 e^{-c_2 N} \|f\|$$

where $c_i > 0$ and $C_i > 0$ ($i = 1, 2$) are constants depending solely on $\varepsilon$ and the Schottky data of $\Gamma$.

**Remark 6.2.** Here and always, $\|\cdot\|$ denotes the $L^2$-norm given in (38) while $\|f\|_{\infty, \Lambda} = \sup_{x \in \Lambda} |f(x)|_V$ is the maximum norm of $f$ on the limit set.

6.2. The Argument in a Nutshell. The rather lengthy proof of Proposition 6.1 will be divided into several lemmas. Let us first provide a quick overview of the argument, glossing over the technical details. Fix a representation $(\rho, V) \in \text{irrep}_0(G)$ and a function $f \in H^2(D)$, and assume that the elements $\pi(\gamma_1), \ldots, \pi(\gamma_m)$ form an $\varepsilon$-expanding set for some $\varepsilon > 0$. We will initially work with the normalized operator

$$(73) \quad \tilde{\mathcal{L}}_{\delta,\rho} \overset{\text{def}}{=} \varphi^{-1} \mathcal{L}_{\delta,\rho} \varphi,$$

where $\varphi \overset{\text{def}}{=} \varphi_\delta$ is the 1-eigenfunction of $\mathcal{L}_\delta$ furnished by the Ruelle–Perron–Frobenius theorem (Proposition 4.1). The operator $\tilde{\mathcal{L}}_{\delta,\rho}$ is “normalized” in the sense that

$$(74) \quad \tilde{\mathcal{L}}_{\delta,\rho}(1) = 1,$$

where $1$ is the trivial one-dimensional representation and $1$ is the constant 1 function on $\Lambda$. Since $\varphi$ is positive on the limit set $\Lambda$, (73) defines an operator

$$\tilde{\mathcal{L}}_{\delta,\rho} : C^0(\Lambda, V) \to C^0(\Lambda, V)$$

on the set of functions $C^0(\Lambda, V) = \{g : \Lambda \to V\}$. Important for us are the subsets $\mathcal{F}_{c,V} \overset{\text{def}}{=} \{g : \Lambda \to V : |g|_\Lambda \leq c \|f\|_{\infty, \Lambda} \} \subset C^0(\Lambda, V)$ with $c > 0$, where $|g|_\Lambda$ is the semi-norm introduced in (65). Using a convexity argument and the decoupling estimate of Proposition 5.3, we show first that if there is some $c > 0$ such that $f \in \mathcal{F}_{c,V}$, then there exists $N_0 = N_0(\varepsilon, c)$ and $\nu = \nu(\varepsilon, c) > 0$ such that

$$(75) \quad \|\tilde{\mathcal{L}}_{\delta,\rho}^N f\|_{\infty, \Lambda} \leq (1 - \nu) \|f\|_{\infty, \Lambda}.$$
Using an inductive argument, we can strengthen this statement as follows: for any $c > 0$ and any (arbitrarily small) $\eta > 0$ there exists $N_0 = N_0(\varepsilon, c, \eta)$ such that for all $\tilde{f} \in F_{c,V}$,

$$\|\tilde{L}_{\delta,\rho}^{N_0} \tilde{f}\|_{\infty,\Lambda} \leq \eta \|\tilde{f}\|_{\infty,\Lambda}.$$  

We can now return to the Bergman space setting. We will use the estimates

$$\|\tilde{L}_{\delta,\rho}^{N_0} \tilde{f}\|_{\infty,\Lambda} \asymp \|L_{\delta,\rho}^{N_0} f\|_{\infty,\Lambda} \text{ and } \|\tilde{f}\|_{\infty,\Lambda} \asymp \|f\|_{\infty,\Lambda}$$

in conjunction with

$$[f]_\Lambda \ll \|f\|$$

for all $f \in H^2(D, V)$, and the a priori bounds in Lemma 4.7 to prove the following statement: for all $f \in H^2(D, V)$ whose restriction $f|_\Lambda$ to the limit set lies in $F_{c,V}$, there exists $N = N(\varepsilon, c, \eta)$ such that

$$\|L_{\delta,\rho}^{N_0} f\|_{\infty,\Lambda} \leq \eta \|f\|_{\infty,\Lambda} \text{ (77)}$$

for all $N \geq 1$, we distinguish two cases: first we assume that

$$\|L_{\delta,\rho}^{N_0} f\|_{\infty,\Lambda} \leq \frac{1}{2C} \|f\|_{\infty,\Lambda} \text{ (76)}$$

where $N_1$ is chosen beforehand. In the second case we assume alternatively that

$$\|L_{\delta,\rho}^{N_0} f\|_{\infty,\Lambda} > \frac{1}{2C} \|f\|_{\infty,\Lambda} \text{ (78)}$$

We then show that (78) forces $f|_\Lambda$ to lie in $F_{c,V}$ for some constant $c > 0$ uniform in $f$, $G$ and $\rho$. Hence, in the second case we can apply the statement in (76) (with $\eta = \frac{1}{2C}$) to find $N_2$ such that

$$\|L_{\delta,\rho}^{N_2} f\|_{\infty,\Lambda} \leq \frac{1}{2C} \|f\|_{\infty,\Lambda} \text{ (79)}$$

Thus, both cases taken together, one of the following two statements must hold:

$$\|L_{\delta,\rho}^{N_1} f\|_{\infty,\Lambda} \leq \frac{1}{2C} \|f\|_{\infty,\Lambda} \text{ or } \|L_{\delta,\rho}^{N_2} f\|_{\infty,\Lambda} \leq \frac{1}{2C} \|f\|_{\infty,\Lambda} \text{ (80)}$$

But now we can set $N_0 = N_1 + N_2$ and use (77) to finally obtain

$$\|L_{\delta,\rho}^{N_0} f\|_{\infty,\Lambda} \leq \frac{1}{2} \|f\|_{\infty,\Lambda} \text{ (81)}$$

which is precisely the first statement in Proposition 6.1. The bound in (71) follows from this one by writing the integer $N$ as $N = kN_0 + m$ with $k = \left\lfloor \frac{N}{N_0} \right\rfloor$ and $0 \leq m < N_0$ and using (77) again. The bound in (72) is a combination of (71) and the first bound in Lemma 4.7.

6.3. The actual proof. We now turn to the actual proof of Proposition 6.1 which will occupy the remainder of this section. From the formula (28) for the iterates of $L_{s,\rho}$ and the definition of the normalized operator in (73), we have

$$\tilde{L}_{\delta,\rho}^{N_0} g(x) = \varphi^{-1} L_{\delta,\rho}^{N_0} \varphi g(x) = \sum_{a \in W^j_k} w_a(x) \rho(\gamma_a)^{-1} g(\gamma_a(x)) \text{ if } x \in \Lambda \cap I_j$$

for all $j \in [2m]$ and $g \in C^0(\Lambda, V)$, where the $w_a$’s are positive weights given by

$$w_a(x) = \frac{\varphi(\gamma_a(x))}{\varphi(x)} \gamma_a^\delta(x).$$
From the normalization condition in (74) we have
\[ \sum_{a \in W_j^N} w_a(x) = 1 \]
for all \( j \in [2m] \). Thus for every \( x \in \Lambda \cap I_j \) the vector \( \tilde{L}_N^{\delta, \rho} g(x) \) (note that this can be viewed as a vector in \( V \)) is a convex linear combination of the vectors \( \rho(\gamma_a)^{-1} g(\gamma_a(x)) \) with \( a \in W_j^N \). We will use the following elementary statement, which is a quantitative way of saying that the unit sphere in \( V \) is strictly convex:

**Lemma 6.3** (Strict convexity). Let \( V \) be a complex vector space endowed with the hermitian inner product \( \langle \cdot, \cdot \rangle_V \) and induced norm \( \| v \|_V = \sqrt{\langle v, v \rangle_V} \) and let \( M > 0 \). Consider the convex linear combination in \( V \),
\[(81) w = \sum_{i=1}^{k} a_i v_i,\]
where all the vectors \( v_1, \ldots, v_k \) satisfy \( \| v_i \| \leq M \) and \( a_1, \ldots, a_k \) are positive real numbers summing up to 1. Assume that
\[ \| w \|_V \geq M(1 - \nu) \]
for some \( 0 < \nu < 1 \). Then we have
\[(82) \| w - v_i \|_V \leq \frac{2M \sqrt{D}}{\min_{1 \leq j \leq k} a_j} \]
for all \( i = 1, \ldots, k \) and in particular
\[(83) \| w \|_V \leq \left( \frac{1}{k} \sum_{i=1}^{k} v_i \right) + \frac{2M \sqrt{D}}{\min_{1 \leq j \leq k} a_j}. \]

**Remark 6.4.** In the above statement, \( \nu = 0 \) forces \( w = v_1 = \cdots = v_k \), which is to say that the sphere \( \{ v \in V \mid \| v \|_V = M \} \) is strictly convex.

**Proof.** Squaring the norm of (81), we obtain
\[(84) \| w \|^2_V = \sum_{1 \leq i, j \leq k} a_i a_j \text{Re} \langle v_i, v_j \rangle_V. \]
Using the relation
\[ \text{Re} \langle v_i, v_j \rangle_V = \frac{1}{2}(\| v_i \|^2_V + \| v_j \|^2_V - \| v_i - v_j \|^2_V). \]
and the assumption that \( \| v_i \|_V \leq M \), we obtain
\[(85) \text{Re} \langle v_i, v_j \rangle_V \leq M^2 - \frac{1}{2} \| v_i - v_j \|^2_V. \]
Inserting (85) into (84) and using the fact that \( a_1, \ldots, a_k \) are positive reals that sum up to 1 gives
\[ \| w \|^2_V \leq M^2 - \frac{1}{2} \sum_{1 \leq i, j \leq k} a_i a_j \| v_i - v_j \|^2_V, \]
which we can rearrange to give
\[(86) \sum_{1 \leq i, j \leq k} a_i a_j \| v_i - v_j \|^2_V \leq 2(M^2 - \| w \|^2). \]
Using the assumption that \( \|w\|_V \geq M(1 - \nu) \) gives
\[
\sum_{1 \leq i,j \leq k} a_i a_j \|v_i - v_j\|_V^2 \leq 2M^2(2\nu - \nu^2) < 4M^2\nu.
\]
Thus for all \( 1 \leq i, j \leq k \) we obtain the bound
\[
\|v_i - v_j\|_V < \frac{2M\sqrt{\nu}}{\min_{1 \leq j \leq k} a_j}.
\]
Consequently, writing
\[
w - v_i = \sum_{j=1}^k a_j (v_i - v_j)
\]
and applying the triangle inequality, we obtain
\[
(87) \quad \|w - v_i\|_V \leq \sum_{j=1}^k a_j \|v_i - v_j\|_V < \frac{2M\sqrt{\nu}}{\min_{1 \leq j \leq k} a_j}
\]
which is the bound we claimed in (82). The bound in (83) now follows by applying this bound and the triangle inequality to
\[
w = \frac{1}{k} \sum_{i=1}^k v_i + \frac{1}{k} \sum_{i=1}^k (w - v_i).
\]
This completes the proof of Lemma 6.3. \( \square \)

Recall that for every function \( g: \Lambda \to V \) we define
\[
\|g\|_{\infty,\Lambda} = \max_{x \in \Lambda} \|g(x)\|_V.
\]
and
\[
[g]_{\Lambda} = \max_{1 \leq j \leq 2m} \max_{x,y \in \Lambda \cap I_j, x \neq y} \frac{\|g(x) - g(y)\|_V}{\|x - y\|_V}.
\]
For any \( c > 0 \) we introduce the set of functions
\[
\mathcal{F}_{c,V} \overset{\text{def}}{=} \{ g: \Lambda \to V : [g]_{\Lambda} \leq c\|g\|_{\infty,\Lambda} \}.
\]
We can now state the next result.

**Lemma 6.5.** Assume that \( \{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \) is \( \varepsilon \)-expanding and let \( c > 0 \) be a positive constant. Then there are constants \( \nu = \nu(\Gamma, \varepsilon, c) > 0 \) and \( N = N(\Gamma, \varepsilon, c) \in \mathbb{N} \) such that for every \( (\rho, V) \in \text{irrep}_0(G) \) and every \( g \in \mathcal{F}_{c,V} \), we have
\[
\|\tilde{L}_{\delta,\rho}^N g\|_{\infty,\Lambda} \leq (1 - \nu)\|g\|_{\infty,\Lambda}.
\]

**Proof.** Assume that \( \{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \) is \( \varepsilon \)-expanding and fix a non-trivial representation \( (\rho, V) \) of \( G \) and a function \( g \in \mathcal{F}_{c,V} \). Recall from Proposition 5.3 that there are positive constants \( c_1, c_2 \) depending solely on \( \varepsilon \) and the Schottky data of \( \Gamma \) such that for all positive integers \( N \),
\[
(88) \quad \frac{1}{|W_N|} \left\| \sum_{a \in W_N} \rho(\gamma_a)^{-1} g(\gamma_a(x)) \right\|_V \leq c_1 e^{-c_2 N(\|g\|_{\infty,\Lambda} + [g]_{\Lambda})}.
\]
Using the assumption that $g \in \mathcal{F}_{c,V}$, this gives

$$
\left(89\right) \frac{1}{|W_N^j|} \left\| \sum_{a \in W_N^j} \rho(\gamma_a)^{-1} g(\gamma_a(x)) \right\|_V \leq c_3 e^{-c_2 N} \|g\|_{\infty,\Lambda},
$$

with $c_3 = c_1 (1 + c)$. For the remainder of this proof, let

$$
N = \left\lceil \frac{\log(2c_3 + 1)}{c_2} \right\rceil,
$$

so that

$$
\left(90\right) \frac{1}{|W_N^j|} \left\| \sum_{a \in W_N^j} \rho(\gamma_a)^{-1} g(\gamma_a(x)) \right\|_V \leq \frac{1}{2} \|g\|_{\infty,\Lambda}.
$$

Set

$$
c_4 = \left( \min_{1 \leq j \leq 2m} \min_{a \in W_N^j} \min_{x \in \Lambda \cap I_j} w_a(x) \right)^{-1} > 0,
$$

where the $w_a$'s are the positive weights in (80) and set

$$
\nu = \frac{1}{9(1 + c_4)^2}.
$$

Notice that both $N$ and $\nu$ depend only on $\varepsilon$, $c$ and the Schottky data of $\Gamma$. We want to show that

$$
\left(92\right) \|\hat{L}_{\delta,\rho} g\|_{\infty,\Lambda} \leq (1 - \nu) \|g\|_{\infty,\Lambda}.
$$

Let $x_0 \in \Lambda$ be the point which maximizes the function $\|\hat{L}_{\delta,\rho} g(x)\|_V$, i.e., choose $j$ and $x_0 \in \Lambda \cap I_j$ such that

$$
\|\hat{L}_{\delta,\rho} g\|_{\infty,\Lambda} = \|\hat{L}_{\delta,\rho} g(x_0)\|_V.
$$

Suppose by contradiction that we have

$$
\left(93\right) \|\hat{L}_{\delta,\rho} g(x_0)\|_V > (1 - \nu) \|g\|_{\infty,\Lambda}.
$$

Recall from (79) that $\hat{L}_{\delta,\rho} g(x_0)$ is a convex linear combination of the vectors

$$
v_a = \rho(\gamma_a)^{-1} g(\gamma_a(x_0)) \in V,
$$

given by

$$
\hat{L}_{\delta,\rho} g(x_0) = \sum_{a \in W_N^j} w_a(x_0) v_a.
$$

Notice that since $\rho$ is unitary, we have $\|v_a\| \leq \|g\|_{\infty,\Lambda}$. Thus we can apply Lemma 6.3 to obtain

$$
\|\hat{L}_{\delta,\rho} g(x_0)\|_V \leq \frac{1}{|W_N^j|} \left\| \sum_{a \in W_N^j} v_a \right\|_V + c_4 \sqrt{\nu} \|g\|_{\infty,\Lambda}.
$$

By the choice of $N$ and $\nu$ this gives

$$
\|\hat{L}_{\delta,\rho} g(x_0)\|_V \leq \left( \frac{1}{2} + c_4 \sqrt{\nu} \right) \|g\|_{\infty,\Lambda} < \frac{5}{6} \|g\|_{\infty,\Lambda} < (1 - \nu) \|g\|_{\infty,\Lambda},
$$

contradicting (93). This concludes the proof. $\square$
Lemma 6.6. There exists a constant $C > 0$ such that for every finite-dimensional, unitary representation $(\rho, V)$ of $\Gamma$, every function $g: \Lambda \to V$ and every positive integer $N$, we have
\[
[\mathcal{L}_{\delta, \rho}^N g] \leq C(\|g\|_{\infty, \Lambda} + [g]_\Lambda).
\]

Proof. Recall from (79) that for all $j \in [2m]$ and all $x \in \Lambda \cap I_j$,
\[
\mathcal{L}_{\delta, \rho}^N g(x) = \sum_{a \in W_j^N} w_a(x) \rho(\gamma_a)^{-1} g(\gamma_a(x)).
\]

Now let $x, y \in \Lambda \cap I_j$ and write
\[
\tilde{\mathcal{L}}_{\delta, \rho}^N g(x) - \tilde{\mathcal{L}}_{\delta, \rho}^N g(y) = \sum_{a \in W_j^N} \rho(\gamma_a)^{-1} (w_a(x)g(\gamma_a(x)) - w_a(y)g(\gamma_a(y))).
\]

By the unitarity of $\rho$ we get
\[
\|\tilde{\mathcal{L}}_{\delta, \rho}^N g(x) - \tilde{\mathcal{L}}_{\delta, \rho}^N g(y)\|_V \leq \sum_{a \in W_j^N} \|w_a(x) (\gamma_a(x)) - w_a(y) (\gamma_a(y))\|_V.
\]

Notice also that
\[
\|w_a(x) (\gamma_a(x)) - w_a(y) (\gamma_a(y))\|_V \leq \|w_a\|_{\infty, I_j} \|g(\gamma_a(x)) - g(\gamma_a(y))\|_V + \|g\|_{\infty, \Lambda} \|w_a(x) - w_a(y)\|
\]
\[
\leq \|w_a\|_{\infty, I_j} \|g\|_{\infty, \Lambda} |x - y| + \|g\|_{\infty, \Lambda} \|w_a(x) - w_a(y)\|
\]

We claim that we have the bounds
\[
\|w_a\|_{\infty, I_j} \ll \|\gamma_a'\|_{\infty, I_j}^\delta \quad \text{and} \quad \|w_a(x) - w_a(y)\| \ll \|\gamma_a'\|_{\infty, I_j}^\delta.
\]

Using these bounds we get
\[
\|w_a(x) (\gamma_a(x)) - w_a(y) (\gamma_a(y))\|_V \ll \|\gamma_a'\|_{\infty, I_j}^\delta \|g\|_{\infty, \Lambda} |x - y|.
\]

Inserting this into (94) and using Lemma 4.2, we get for all $x, y \in I_j$ the bound
\[
\frac{\|\tilde{\mathcal{L}}_{\delta, \rho}^N g(x) - \tilde{\mathcal{L}}_{\delta, \rho}^N g(y)\|_V}{|x - y|} \ll (\|g\|_{\infty, \Lambda} + [g]_\Lambda) \sum_{a \in W_j^N} \|\gamma_a'\|_{\infty, I_j}^\delta \ll \|g\|_{\infty, \Lambda} + [g]_\Lambda,
\]

with some implied constant depending solely on the Schottky data of $\Gamma$. Since $x$ and $y$ were arbitrary points in $I_j$, the statement follows. Now let us prove the two bounds we claimed in (95). Recall from (80) that the weights $w_a$ are given by
\[
w_a(x) = \frac{\varphi(\gamma_a(x))}{\varphi(x)} \gamma_a(x)^\delta,
\]

where $\varphi = \varphi_\delta \in C^1(I)$ is the 1-eigenfunction of $\mathcal{L}_\delta$ provided by Proposition 4.1. The first bound in (95) immediately follows because $\varphi$ is bounded above and below by some constants depending only on the Schottky data of $\Gamma$. Now let us prove the second bound. Differentiating the right hand side of (96), using the chain and product rules, we obtain
\[
w_a'(x) = \delta \frac{\varphi(\gamma_a(x))}{\varphi(x)} \gamma_a'(x) \gamma_a(x)^\delta + \frac{\varphi(\gamma_a(x))}{\varphi(x)} \gamma_a'(x) (\gamma_a(x)^\delta + 1) - \frac{\varphi(\gamma_a(x)) \varphi'(x)}{\varphi(x)^2} \gamma_a(x)^\delta.
\]

We deduce that
\[
\|w_a'\|_{\infty, I_j} \ll \|\gamma_a'\|_{\infty, I_j}^\delta \cdot \|\gamma_a'\|_{\infty, I_j} + \|\gamma_a'\|_{\infty, I_j}^\delta + \|\gamma_a'\|_{\infty, I_j}^\delta.
\]
By the bounded distortion estimate (30) we have
\[ \left\| \frac{\gamma'_{\alpha}}{\gamma_{\alpha}} \right\|_{\infty,J} \ll 1, \]
thus we obtain
\[ \| w'_{\alpha} \|_{\infty,J} \ll \| \gamma'_{\alpha} \|_{\infty,J} \]
proving the second bound in (95).

The next result is a strengthening of Lemma 6.5.

**Lemma 6.7.** With the same assumptions and notations as in Proposition 6.1 and Lemma 6.5, let \( \eta > 0 \) and \( c > 0 \) be fixed constants. Then there exists a positive integer \( N_0 \) depending only on \( \varepsilon, \eta, c \) and the Schottky data of \( \Gamma \) such that for all \( (\rho, V) \in \text{irrep}_0(G) \) and all \( g \in F_{c,V} \) we have
\[ \| \tilde{L}_{\delta,\rho}^N g \|_{\infty,\Lambda} \leq \eta \| g \|_{\infty,\Lambda}. \]

**Proof.** Fix a non-trivial, irreducible representation \((\rho, V)\) of \( G \), a function \( g \in F_{c,V} \) and a constant \( \eta > 0 \). Suppose by contradiction that
\[ \| \tilde{L}_{\delta,\rho}^N g \|_{\infty,\Lambda} > \eta \| g \|_{\infty,\Lambda} \]
for all positive integers \( N \). The goal is to show that (98) is false for some \( N = N(\varepsilon, \eta, c, \Gamma) \). In what follows the \( c_k \)'s \((k \in \mathbb{N})\) are positive constants which are only allowed to depend on \( \varepsilon, \eta, c \) and the Schottky data of \( \Gamma \). Applying Lemma 6.6 and using the assumption that \( g \in F_{c,V} \) gives
\[ \| \tilde{L}_{\delta,\rho}^N g \|_{\infty,\Lambda} \leq c_2 \| g \|_{\infty,\Lambda} \]
for all \( N \). Combining (98) and (99), we obtain for all \( N \),
\[ \| \tilde{L}_{\delta,\rho}^N g \|_{\infty,\Lambda} \leq c_2 \| \tilde{L}_{\delta,\rho}^N g \|_{\infty,\Lambda} \]
with \( c_3 = c_2 \eta^{-1} \). Thus the assumption in (98) forces
\[ \tilde{L}_{\delta,\rho}^N g \in F_{c_3} \]
for all positive integers \( N \). We set \( c_4 = \max\{c_3, c\} \) so that
\[ g, \tilde{L}_{\delta,\rho}^N g, \tilde{L}_{\delta,\rho}^N g, \tilde{L}_{\delta,\rho}^3 g, \cdots \in F_{c_4}. \]
By Lemma 6.5 there exist \( \nu = \nu(\varepsilon, c, \Gamma) > 0 \) and \( k = k(\varepsilon, c, \Gamma) \in \mathbb{N} \) such that for all \( h \in F_{c_4}, \)
\[ \| \tilde{L}_{\delta,\rho}^k h \|_{\infty,\Lambda} \leq (1 - \nu) \| h \|_{\infty,\Lambda}. \]
In particular, for all integers \( m \geq 2 \) we have
\[ \| \tilde{L}_{\delta,\rho}^{m+1} g \|_{\infty,\Lambda} \leq \| \tilde{L}_{\delta,\rho}^{m+1} \tilde{L}_{\delta,\rho}^{(m-1)k} g \|_{\infty,\Lambda} \leq (1 - \nu) \| \tilde{L}_{\delta,\rho}^{(m-1)k} g \|_{\infty,\Lambda}. \]
We can now iterate this to give
\[ \| \tilde{L}_{\delta,\rho}^m g \|_{\infty,\Lambda} \leq (1 - \nu)^m \| g \|_{\infty,\Lambda} \leq e^{-\nu m} \| g \|_{\infty,\Lambda}. \]
But this contradicts (98) for \( m > \nu^{-1} \log \eta^{-1} \). Hence, Lemma 6.7 follows with
\[ N_0 = k \left( \nu^{-1} \log \eta^{-1} \right), \]
which depends only on the constants \( \varepsilon, \eta, c \) and the Schottky data of \( \Gamma \), as desired. \( \square \)
Lemma 6.8. With the same assumptions and notations as in Proposition 6.1, let \( \eta > 0 \) and \( c > 0 \) be fixed constants. Then there exists a positive integer \( N = N(\varepsilon, \eta, c) \) such that for all \((\rho, V) \in \text{irrep}_0(G)\) and all functions \( f \in H^2(D, V) \) with \( f|_\Lambda \in \mathcal{F}_{c,V} \), we have

\[
\|L_{\delta,\rho}^N f\| \leq \eta \|f\|.
\]

Here, \( f|_\Lambda \) denotes the restriction of \( f \) to the limit set \( \Lambda \).

Proof. Fix a function \( f \in H^2(D, V) \) with \( f|_\Lambda \in \mathcal{F}_{c,V} \) and notice that on \( \Lambda \), we have

\[
L_{\delta,\rho}^N f = \varphi \tilde{L}_{\delta,\rho}^N \tilde{f}
\]

where

\[
\tilde{f} = \frac{f|_\Lambda}{\varphi}.
\]

One easily verifies that \( \tilde{f} \in \mathcal{F}_{c_0} \) for some \( c_0 > 0 \) depending only on \( c \) and \( \varphi \). Hence, applying Lemma 6.7, we obtain for any \( \eta_0 > 0 \) a positive integer \( N' \) depending only on \( \varepsilon, \eta_0, c \) and the Schottky data of \( \Gamma \), such that

\[
\|\tilde{L}_{\delta,\rho}^{N'} \tilde{f}\|_{\infty,\Lambda} \leq \eta_0 \|\tilde{f}\|_{\infty,\Lambda}.
\]

This bound will be used further below. For the rest of the proof the \( c_n \)'s are positive constants depending only on \( \varepsilon, \eta, c \) and the Schottky data of \( \Gamma \). By (102) we have

\[
\|L_{\delta,\rho}^{N'} f\|_{\infty,\Lambda} \leq c_1 \|\tilde{L}_{\delta,\rho}^{N'} \tilde{f}\|_{\infty,\Lambda},
\]

where \( c_1 = \max_\Lambda |\varphi| \) and by (103) we have

\[
\|\tilde{f}\|_{\infty,\Lambda} \leq c_2 \|f\|_{\infty,\Lambda}
\]

where \( c_2 = (\min_\Lambda |\varphi|)^{-1} \). By combining (104), (105) and (106), we get

\[
\|L_{\delta,\rho}^{N'} f\|_{\infty,\Lambda} \leq c_1 c_2 \eta_0 \|f\|_{\infty,\Lambda}.
\]

Recall from Lemma 4.7 that there exist positive constants \( c_3 \) and \( c_4 \) such that for all positive integers \( N \),

\[
\|L_{s,\rho}^N f\|^2 \leq c_3 (\|f\|^2_{\infty,\Lambda} + e^{-c_4 N} \|f\|^2).
\]

Recall from Lemma 4.5 that there exists \( c_5 > 0 \) such that

\[
\|f\|_{\infty,\Lambda} \leq c_5 \|f\|.
\]

Now we choose \( N'' = \lceil c_4^{-1} \log(1 + \eta_0^{-2}) \rceil \) so that

\[
e^{-c_4 N''} < \eta_0^2
\]

and we set \( N_0 = N' + N'' \). Notice that \( N_0 \) depends only on \( \varepsilon, \eta_0, c \) and the Schottky data of \( \Gamma \). The quantity \( \eta_0 \) will be specified below in terms of \( \eta \). Applying the above estimates, we obtain

\[
\|L_{\delta,\rho}^{N_0} f\|^2 = \|L_{\delta,\rho}^{N''} (L_{\delta,\rho}^{N'} f)\|^2
\]

\[
\leq c_3 \left( \|L_{\delta,\rho}^{N'} f\|^2_{\infty,\Lambda} + e^{-c_4 N''} \|L_{\delta,\rho}^{N'} f\|^2 \right) \quad \text{by (108)}
\]

\[
\leq c_3 \left( \|L_{\delta,\rho}^{N'} f\|^2_{\infty,\Lambda} + \eta_0^2 \|L_{\delta,\rho}^{N'} f\|^2 \right) \quad \text{by (109)}
\]

\[
\leq c_3 \left( c_2^2 \eta_0^2 \|f\|^2_{\infty,\Lambda} + c_2^2 \eta_0^2 \|f\|^2 \right) \quad \text{by (107)}
\]

\[
\leq c_1^2 c_3 (c_2^2 c_3^2 + 1) \eta_0^2 \|f\|^2 \quad \text{by (109)}.
\]
We can now choose $\eta_0 > 0$ so that 
\[
c_1^2 c_3 (c_2^2 c_0^2 + 1) \eta_0^2 = \eta^2,
\]
giving 
\[
\|L_{h, \rho}^{N_0} f\| \leq \eta \|f\|.
\]
This completes the proof of Lemma 6.8.

Lemma 6.9. There exists $C > 0$ depending only on the Schottky data of $\Gamma$ such that for all $f \in H^2(D, V)$, we have 
\[
[f]_{\Lambda} \leq C \|f\|.
\]

Proof. Let $d = \dim(V)$ be the dimension of $V$. Fix an orthonormal basis $e_1, \ldots, e_d$ of $V$ and decompose $f$ as 
\[
f = f_1 e_1 + \cdots + f_d e_k
\]
where $f_1, \ldots, f_d \in H^2(D)$, so that 
\[
\|f\|^2 = \|f_1\|^2 + \cdots + \|f_d\|^2.
\]
Notice that 
\[
[f]_{\Lambda}^2 \leq [f_1]_{\Lambda}^2 + \cdots + [f_d]_{\Lambda}^2,
\]
so it suffices to show that 
\[
[f_k]_{\Lambda} \ll \|f_k\|^2
\]
for all $k \in [d]$. To that effect, fix $j \in [2m]$ and two points $x, y \in \Lambda \cap I_j$. Thanks to the mean value theorem, there exists a point $x'$ in the line segment connecting $x$ and $y$ such that 
\[
\frac{f_k(y) - f_k(x)}{y - x} = f_k'(x').
\]
Since the line segment line segment connecting $x$ and $y$ is bounded away from $\partial D_j$ uniformly in $x, y$, Lemma 4.3 gives 
\[
\left| \frac{f_k(y) - f_k(x)}{y - x} \right| \ll \|f_k\|,
\]
where the implied constant depends only on the Schottky data of $\Gamma$. Since $x, y$ were taken arbitrarily from $\Lambda \cap I_j$, the bound in (112) follows, completing the proof of Lemma 6.9.

We are now ready for the proof of Proposition 6.1.

Proof of Proposition 6.1. Assume that 
\[
\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \subseteq G
\]
is an $\varepsilon$-expanding set. Fix a non-trivial, irreducible representation $(\rho, V)$ of $G$ and a function $f \in H^2(D, V)$. We want to show that there exists a positive integer $N_0$ depending only on $\varepsilon$ and the Schottky data of $\Gamma$ such that 
\[
\|L_{h, \rho}^{N_0} f\| \leq \frac{1}{2} \|f\|.
\]
The bounds (71) and (72) in Proposition 6.1 follow directly from this bound by the discussion at the end of §6.2, so we only need to establish (113). Recall from Lemma 4.7 that there exists positive constants $c_1, c_2, c_3$ such that for all $N \geq 1$, 
\[
\|L_{h, \rho}^N f\| \leq c_1 \|f\|,
\]
and
(115) \[ \|L_{s,\rho}^N f\|^2 \leq c_2 \left( \|f\|_{\infty,\Lambda}^2 + e^{-c_3 N} \|f\|^2 \right). \]

We choose
\[ N_1 = \left\lceil \frac{\log(8c_1^2 + 1)}{c_3} \right\rceil \]
so that
\[ 4c_1^2 c_2 e^{-c_3 N_1} < \frac{1}{2} \]

We now distinguish the following two opposite cases:

Case 1: assume that
\[ \|L_{s,\rho}^N f\| \leq \frac{1}{2c_1} \|f\|. \]

Case 2: assume alternatively that
\[ \|L_{s,\rho}^N f\| > \frac{1}{2c_1} \|f\|, \]
or equivalently,

(116) \[ \|f\| < 2c_1 \|L_{s,\rho}^N f\|. \]

We claim that in the second case, the restriction \( f|_{\Lambda} \) of \( f \) to the limit set belongs to \( F_{c,V} \) for some \( c > 0 \) uniform in \( f, G \) and \( \rho \). To see this, we combine (116) and (115) to get

(117) \[ \|f\|^2 < 4c_1^2 \|L_{s,\rho}^N f\|^2 \leq 4c_1^2 c_2 (\|f\|_{\infty,\Lambda}^2 + e^{-c_3 N_1} \|f\|^2). \]

By the choice of \( N_1 \) this implies
\[ \|f\|^2 < 4c_1^2 c_2 \|f\|_{\infty,\Lambda}^2 + \frac{1}{2} \|f\|^2, \]
which can be rearranged to give

(118) \[ \|f\| < \sqrt{8c_2 c_1} \|f\|_{\infty,\Lambda}. \]

By Lemma 6.9,
\[ [f]_{\Lambda} \leq c_4 \|f\|, \]
which when combined with (118) gives
\[ [f]_{\Lambda} \leq \sqrt{8c_2 c_1} c_4 \|f\|_{\infty,\Lambda}. \]

Hence, in Case 2 we have \( f|_{\Lambda} \in \mathcal{F}_{c_5} \) with \( c_5 = \sqrt{8c_2 c_1} c_4 \). But then we can appeal to Lemma 6.8 (with \( \eta = \frac{1}{2c_1} \)) which implies that there exists a positive integer \( N_2 \) depending only on \( \varepsilon \) and the Schottky data of \( \Gamma \) such that
\[ \|L_{s,\rho}^{N_2} f\| \leq \frac{1}{2c_1} \|f\|. \]

In conclusion, in either case we have
\[ \min\{\|L_{s,\rho}^{N_1} f\|, \|L_{s,\rho}^{N_2} f\|\} \leq \frac{1}{2c_1} \|f\|. \]

Set \( N_0 = N_1 + N_2 \) and notice that \( N_0 \) depends only on \( \varepsilon \) and the Schottky data of \( \Gamma \). Using (114) again, this gives
\[ \|L_{s,\rho}^{N_0} f\| \leq \frac{1}{2} \|f\|, \]
completing the proof. \( \square \)
7. Proof of Theorem 1.3 and Theorem 1.4

7.1. Proof of Theorem 1.3. We are now ready to prove the first main result of this paper, Theorem 1.3. Let $\Gamma$ be a non-elementary Schottky group, let $G$ be a finite group and let $\pi: \Gamma \to G$ a surjective homomorphism. Suppose that $\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \subseteq G$ is an $\varepsilon$-expanding set. Set $\Gamma' = \ker(\pi)$. We want to show that there exists $\eta = \eta(\varepsilon, \Gamma) > 0$ such that $Z_{\Gamma'}(s)$ has no non-trivial zeros in the disk $|s - \delta| < \eta$, the trivial zero being the point $s = \delta$.

Recall the Venkov–Zograf factorization formula in (18):

$$Z_{\Gamma'}(s) = \prod_{\rho \in \text{irrep}(G)} Z_{\Gamma'}(s, \rho)^{\text{dim}(\rho)} = Z_{\Gamma'}(s) \prod_{\rho \in \text{irrep}_0(G)} Z_{\Gamma'}(s, \rho)^{\text{dim}(\rho)}.$$ 

In light of this formula, establishing Theorem 1.3 amounts to proving that for all non-trivial, irreducible representations $\rho \in \text{irrep}_0(G)$, the twisted Selberg zeta function $Z_{\Gamma'}(s, \rho)$ has no zeros in $|s - \delta| < \eta$.

Fix $\rho \in \text{irrep}_0(G)$. Suppose that $Z_{\Gamma'}(s, \rho)$ vanishes at some point $s$ with $|s - \delta| < \eta$. By the Fredholm determinant identity in Proposition 2.1 and the general theory of Fredholm determinants, the operator $L_{s, \rho}$ must have a non-zero 1-eigenfunction $f \in H^2(D, V)$, implying that for all integers $N \geq 1$,

$$L_{s, \rho}^N f = f$$

and in particular

$$\|L_{s, \rho}^N f\| \geq \|f\|.$$ 

We will show that this leads to a contradiction, provided $\eta > 0$ is taken to be sufficiently small depending only on $\varepsilon$ and the Schottky data of $\Gamma$. From Proposition 6.1 we know that

$$\|L_{s, \rho}^{N_0} f\| \leq \frac{1}{2} \|f\|.$$ 

for some integer $N_0 = N_0(\varepsilon, \Gamma) \geq 1$. Recall from (28) that

$$L_{s, \rho}^{N_0} f(z) = \sum_{a \in W_{N_0}^j} \gamma_a(z)^s \rho(\gamma_a)^{-1} f(\gamma_a(z))$$

for all $z \in D_j$.

For the remainder of this proof we assume that $s \in \mathbb{C}$ satisfies $|s - \delta| < 1$. Then there exists a constant $C = C(\varepsilon, \Gamma) > 0$ such that for all $j \in [2m]$, all $a \in W_{N_0}^j$ and all $z \in D_j$ we have

$$|\gamma'_a(z)^s - \gamma'_a(z)^\delta| < C|s - \delta|.$$ 

Combining this with Lemma 4.3 shows that

$$\|L_{s, \rho}^{N_0} f(z) - L_{s, \rho}^{N_0} f(z)\| \leq \sum_{a \in W_{N_0}^j} \|\gamma'_a(z)^s - \gamma'_a(\gamma_a(z))\| \leq C' |s - \delta| \|f\|,$$

for some constant $C' = C'(\varepsilon, \Gamma) > 0$. It follows that

$$\|L_{s, \rho}^{N_0} f - L_{s, \rho}^{N_0} f\| \leq C'' |s - \delta| \|f\|$$

for some constant $C'' = C''(\varepsilon, \Gamma) > 0$. We can now set $\eta = \min \left\{ 1, \frac{1}{3C''} \right\}$ such that

$$|s - \delta| < \eta \implies \|L_{s, \rho}^{N_0} f - L_{s, \rho}^{N_0} f\| < \frac{1}{3} \|f\|.$$
Thus for all $|s - \delta| < \eta$ the triangle inequality gives
\[
\|L_{s,\rho}^{N_{\delta}} f\| < \|L_{s,\rho}^{N_0} f\| + \frac{1}{3}\|f\| + \frac{1}{3}\|f\| = \frac{5}{6}\|f\|,
\]
contradicting (119). The proof of Theorem 1.3 is finished.

7.2. **Effective equidistribution estimate.** Heuristically, we expect the sums
\[
\sum_{a \in W_N^j, \pi(\gamma_a) = g} \gamma_a(x)^{\delta} w(\gamma_a(x))
\]
to be uniformly distributed among the elements $g \in G$ for every suitable function $w$. For the proof of Theorem 1.4 we need an effective equidistribution statement that controls the dependence of the error term on $N$ and the group $G$:

**Proposition 7.1** (Effective equidistribution). Suppose that $\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \subseteq G$ is an $\varepsilon$-expanding set and let $w \in H^2(D)$. Then for every $g \in G$, for every $j \in [2m]$ and for every $x \in \Lambda \cap I_j$ we have
\[
\sum_{a \in W_N^j, \pi(\gamma_a) = g} \gamma_a(x)^{\delta} w(\gamma_a(x)) = \frac{1}{|G|} \sum_{a \in W_N^j} \gamma_a(x)^{\delta} w(\gamma_a(x)) + O(e^{-cN}\|w\|)
\]
where both $c > 0$ and the implied constant in the error term depend solely on $\varepsilon$ and the Schottky data of $\Gamma$.

We will actually only use a weaker estimate on the following quantity which will appear repeatedly in the sequel:
\[
E_T^N(\pi, G) \overset{\text{def}}{=} \sum_{j=1}^{2m} \sum_{a \in W_N^j, \pi(\gamma_a) = \pi(\gamma_b)} \|\gamma_a\|_{\infty, D_j}^\delta \|\gamma_b\|_{\infty, D_j}^\delta.
\]

**Corollary 7.2** (Flattening). Suppose that the set $\{\pi(\gamma_1), \ldots, \pi(\gamma_m)\} \subseteq G$ is $\varepsilon$-expanding. Then there are constants $C_i = C_i(\varepsilon, \Gamma) > 0$ ($i = 1, 2$) such that for all $N \geq C_1 \log(2|G|)$,
\[
E_T^N(\pi, G) \leq C_2|G|^{-1}.
\]

**Proof of Corollary 7.2.** Applying Proposition 7.1 to the constant function $w \equiv 1$ and setting $C_1 = 1/c$ where $c$ is the constant in the statement of that proposition, we obtain for all $N \geq C_1 \log(2|G|)$ and all $g \in G$,
\[
\sum_{a, b \in W_N^j, \pi(\gamma_a) = \pi(\gamma_b)} \|\gamma_a\|_{\infty, D_j}^\delta \leq C|G|^{-1}.
\]
with some constant $C = C(\Gamma) > 0$. Thus, after rewriting $E_T^N(\pi, G)$, we obtain for all $N \geq C_1 \log(2|G|)$,
\[
E_T^N(\pi, G) = \sum_{j=1}^{2m} \sum_{g \in G} \left( \sum_{a \in W_N^j, \pi(\gamma_a) = g} \|\gamma_a\|_{\infty, D_j}^\delta \right)^2 \leq C^2|G|^{-1}
\]
which proves the claim.
Proof of Proposition 7.1. Given a representation \((\rho, V)\) of \(G\), we let \(\chi_\rho\) be the associated character,

\[
\chi_\rho(g) \overset{\text{def}}{=} \text{tr}_V \rho(g), \quad g \in G.
\]

We claim that for all \(j \in [2m]\), all \(x \in \Lambda \cap I_j\) and all \((\rho, V) \in \text{irrep}_0(G)\) we have

\[
S_{w, N, \rho, g}(x) \overset{\text{def}}{=} \sum_{a \in W^j_N} \chi_\rho(g \gamma_a^{-1}) \gamma'_a(x) \delta w(\gamma_a(x)) = O(\dim(\rho) e^{-cN \|w\|}),
\]

where both \(c > 0\) and the implied constant in the error term depend only on the expansion constant \(\varepsilon\) and the Schottky data of \(\Gamma\). Here we slightly abuse notation to write

\[
\chi_\rho(g \gamma_a^{-1}) \overset{\text{def}}{=} \chi_\rho(g \cdot \pi(\gamma_a)^{-1}).
\]

Before we prove this bound, let us see how it implies (120). Decomposing the characteristic function \(1_{\pi(\gamma_a) = g}\) into irreducible characters as

\[
1_{\pi(\gamma_a) = g} = \frac{1}{|G|} \sum_{\rho \in \text{irrep}(G)} \dim(\rho) \chi_\rho(g \gamma_a^{-1}) = \frac{1}{|G|} \sum_{\rho \in \text{irrep}_0(G)} \dim(\rho) \chi_\rho(g \gamma_a^{-1}),
\]

we can write

\[
\sum_{a \in W^j_N, \pi(\gamma_a) = g} \gamma'_a(x) \delta w(\gamma_a(x)) + \frac{1}{|G|} \sum_{\rho \in \text{irrep}_0(G)} \dim(\rho) S_{w, N, \rho, g}(x).
\]

Thus, assuming the bound in (123), we obtain

\[
\sum_{a \in W^j_N, \pi(\gamma_a) = g} \gamma'_a(x) \delta w(\gamma_a(x)) = \frac{1}{|G|} \sum_{a \in W^j_N} \gamma'_a(x) \delta w(\gamma_a(x)) + O \left( \frac{1}{|G|} \sum_{\rho \in \text{irrep}_0(G)} \dim(\rho)^2 e^{-cN \|w\|} \right)
\]

\[
= \frac{1}{|G|} \sum_{a \in W^j_N} \gamma'_a(x) \delta w(\gamma_a(x)) + O \left( e^{-cN \|w\|} \right),
\]

as claimed, where in the previous line we have used the well-known dimension formula

\[
\sum_{\rho \in \text{irrep}(G)} \dim(\rho)^2 = |G|.
\]

Now let us prove (123). Fix some \((\rho, V) \in \text{irrep}_0(G)\). Set \(d = \dim(\rho)\) and fix an orthonormal basis \(e_1, \ldots, e_d\) for the representation space \(V\). Since \(\chi_\rho(g \gamma_a^{-1})\) is the trace of the endomorphism \(\rho(g \gamma_a^{-1})\), we have

\[
\chi_\rho(g \gamma_a^{-1}) = \sum_{k=1}^d \langle \rho(g \gamma_a^{-1}) e_k, e_k \rangle_V = \sum_{k=1}^d \langle \rho(\gamma_a)^{-1} e_k, \rho(g)^{-1} e_k \rangle_V,
\]

which when inserted into (123) gives

\[
S_{w, N, \rho, g}(x) = \sum_{a \in W^j_N} \left( \sum_{k=1}^d \langle \rho(\gamma_a)^{-1} e_k, \rho(g)^{-1} e_k \rangle_V \right) \gamma'_a(x) \delta w(\gamma_a(x))
\]

\[
= \sum_{k=1}^d \left( \sum_{a \in W^j_N} \gamma'_a(x) \delta w(\gamma_a(x)) \rho(\gamma_a)^{-1} e_k, \rho(g)^{-1} e_k \right)_V.
\]
Writing $f_k(x) = w(x)e_k$ and regarding $f_k$ as a function in $H^2(D, V)$, we can rewrite the last line as

$$S_{w,N,ρ,g}(x) = \sum_{k=1}^{d} \langle L_{δ,ρ}f_k(x), ρ(g)^{-1}e_k \rangle_V$$

Using the triangle inequality and the Cauchy–Schwarz inequality $|\langle v_1, v_2 \rangle_V| \leq \|v_1\|_V\|v_2\|_V$ and noticing that $ρ(g)^{-1}e_k$ is a unit vector in $V$ for all $k \in [d]$, we obtain

$$|S_{w,N,ρ,g}(x)| \leq \sum_{k=1}^{d} \|L_{δ,ρ}f_k(x)\|_V \leq \sum_{k=1}^{d} \|L_{δ,ρ}f_k\|_{∞,Λ}.$$ 

Invoking Proposition 6.1 and noticing that $\|f_k\| = \|w\|$ for all $k \in [d]$ yields

$$|S_{w,N,ρ,g}(x)| \ll \dim(ρ)e^{-cN}\|w\|,$$

which is the bound we claimed in (123). □

7.3. **Proof of Theorem 1.4.** Recall the zero counting functions defined in the introduction of this paper

$$N_Γ(σ, T) = \#\{s ∈ C : Z_Γ(s) = 0 \text{ with } \Re(s) ≥ δ − σ \text{ and } \Im(s) ∈ [T, T + 1]\}$$

and

$$N_Γ(σ, T, ρ) = \#\{s ∈ C : Z_Γ(s, ρ) = 0 \text{ with } \Re(s) ≥ δ − σ \text{ and } \Im(s) ∈ [T, T + 1]\},$$

where the zeros are counted with multiplicities. Theorem 1.4 is a consequence of the following result:

**Proposition 7.3** (resonance count for $G$-covers). Let $Γ$ be a non-elementary, convex cocompact Fuchsian group with Schottky data $γ_1, . . . , γ_m$ and $D_1, . . . , D_{2m}$. Let $G$ be a finite group, let $π : Γ → G$ be a surjective homomorphism and let $Γ' = \ker(π)$ be its kernel. Then there are positive constants $C$ and $N_0$ depending only on the Schottky data of $Γ$ such that for all integers $N ≥ N_0$ we have

$$N_Γ(σ, T) ≤ CN^3\langle T \rangle^{δ} |G|e^{C(N(δ−σ))}E_Γ^N(π, G) + C|G|e^{−C−1N^2},$$

where

$$E_Γ^N(π, G) = \sum_{j=1}^{2m} \sum_{a,b \in W_N'} \|γ_a\|_{∞,D_j}\|γ_b\|_{∞,D_j}$$

is the quantity introduced in (121).

**Proof of Theorem 1.4 assuming Proposition 7.3.** Let $N_0$ be as in Proposition 7.3. Taking $N = N_0$ in (124) and using the “trivial” estimate $E_Γ^N(π, G) = O(1)$ gives

$$N_Γ(σ, T) \ll |G|\langle T \rangle^δ.$$ 

Now suppose that $\{π(γ_1), . . . , π(γ_m)\}$ is an $ε$-expanding set in $G$ with some $ε > 0$. From Corollary 7.2 we know that there exists $C' = C'(ε, Γ) > 0$ such that for

$$N = \lfloor C' \log(2|G|) \rfloor$$

we have

$$E_Γ^N(π, G) \ll |G|^{−1}.$$ 

If necessary we increase the constant $C'$ so that $N ≥ N_0$. Inserting this bound into (124) gives

$$N_Γ(σ, T) \ll N^3\langle T \rangle^{δ}|G|e^{C(N(δ−σ))}E_Γ^N(π, G) + C|G|e^{−C−1N^2}$$




\[
\ll \log(2|G|)^3 e^{C'|C'\log(2|G|)|\gamma} \langle T \rangle^\delta
\]
\[
\ll \log(2|G|)^3 |G|^{C''(\delta-\sigma)} \langle T \rangle^\delta
\]
for some \( C'' = C''(\varepsilon, \Gamma) > 0 \). Taken together, this estimate and (125) show that
\[
N_{\Gamma'}(\sigma, T) \ll \log(2|G|)^3 |G|^{\min\{1, C''N(\delta-\sigma)\}} \langle T \rangle^\delta,
\]
as claimed. Finally, using the Venkov–Zograf formula (18) we get
\[
N_{\Gamma'}(\sigma, T) = \sum_{\rho \in \text{irrep}(G)} \dim(\rho) N_{\Gamma}(\sigma, T, \rho),
\]
so we obtain for each irreducible representation \( \rho \) of \( G \) the estimate
\[
N_{\Gamma}(\sigma, T, \rho) \leq N_{\Gamma'}(\sigma, T) \ll \frac{\log(2|G|)^3 |G|^{\min\{1, C''N(\delta-\sigma)\}} \langle T \rangle^\delta}{\dim(\rho)}.
\]
The proof of Theorem 1.4 is finished. \( \square \)

The remainder of this section is devoted to the proof of Proposition 7.3.

7.4. Refined function spaces and Hilbert–Schmidt norm. The proof of Proposition 7.3 relies on the approach of Guillopé–Lin–Zworski [20]. We introduce a (small) parameter \( h > 0 \) and we consider the real \( h \)--neighbourhood of the limit set
\[
\Lambda(h) \overset{\text{def}}{=} \Lambda + (0, h) \subset \mathbb{R}.
\]
The set \( \Lambda(h) \) is covered by \( n(h) = O(h^{-\delta}) \) open intervals \( \tilde{I}_1, \ldots, \tilde{I}_{n(h)} \) of size \( |\tilde{I}| = O(h) \) (see [20]). For each of these intervals, let \( \tilde{D}_l \subset \mathbb{C} \) be the open euclidean disk of the complex plane centered on \( \mathbb{R} \) and satisfying \( \tilde{I}_l = \tilde{D}_l \cap \mathbb{R} \). Let \( \Omega(h) \subset \mathbb{C} \) be the union of these disks,
\[
\Omega(h) \overset{\text{def}}{=} \bigcup_{l=1}^{n(h)} \tilde{D}_l.
\]
For \( h > 0 \) sufficiently small we have \( \Omega(h) \subset D \) and the key idea is to restrict the operator \( L_{s, \rho}^N \) (given by (28)) to the “refined” Bergman space \( H^2(\Omega(h), V) \), consisting of all square-integrable \( V \)--valued holomorphic functions on \( \Omega(h) \).

The following lemma summarizes the key properties of refined function spaces which are well-known from the papers [31, 25]:

**Lemma 7.4** (Basic facts on refined function spaces). There exist constants \( h_0 > 0 \), \( N_0 \in \mathbb{N} \) and \( C > 0 \) depending only on the Schottky data of \( \Gamma \) such that the following statements hold true for all \( j \in [2m] \), whenever \( h \in (0, h_0) \) and \( N \geq N_0 

(i) We have \( \overline{\Omega(h)} \subset D \). Moreover, for all \( z \in D_j \cap \Omega(h) \) and \( a \in W_N^2 \), we have
\[
\text{dist}(\gamma_a(z), \partial\Omega(h)) \geq \frac{1}{2} h
\]
where \( \text{dist} \) is the euclidean distance and \( \partial\Omega(h) \) denotes the boundary of \( \Omega(h) \).

(ii) The Lebesgue measure of the set \( \Omega(h) \) is bounded from above by
\[
\text{vol}(\Omega(h)) \leq C h^{2-\delta}.
\]

(iii) For \( z \in D_j \cap \Omega(h) \) and \( a \in W_N^3 \), and for all \( s \in \mathbb{C} \) with \( \sigma = \text{Re}(s) \geq 0 \), we have
\[
|\gamma'_a(z)^s| \leq C e^{C h \text{Im}(s)} \|\gamma'_a\|_{\text{L}^\infty(D)}.
\]
Let us recall the definition of the Bergman kernel in the last part of Lemma 7.4 above. By Lemma 4.3 (applied to \(\Omega(h)\) instead of \(D\)), the evaluation map
\[
ev_z : H^2(\Omega(h)) \to \mathbb{C}, f \mapsto f(z)
\]
is a continuous linear functional for every \(z\) in the interior of \(\Omega(h)\). The Riesz representation theorem guarantees the existence of a unique \(B_{\Omega(h)}(z, \cdot) \in H^2(\Omega(h))\) such that for all \(f \in H^2(\Omega(h))\),
\[
(126) \quad f(z) = \ev_z(f) = \int_{\Omega(h)} B_{\Omega(h)}(z, w)f(w) \, d\text{vol}(w).
\]
Understanding the properties of \(B_{\Omega(h)}(z, w)\), called the Bergman (reproducing) kernel, will be useful below.

The properties in Lemma 7.4 can be used to show that for every \(s \in \mathbb{C}\) and every finite-dimensional unitary representation \((\rho, V)\) of \(\Gamma\), the operator
\[
(127) \quad \mathcal{L}^N_{s, \rho} : H^2(\Omega(h), V) \to H^2(\Omega(h), V)
\]
is well-defined and trace-class, whenever \(h > 0\) is small enough. Moreover, its Fredholm determinant
\[
(128) \quad Z_{\Gamma}^{(N)}(s, \rho) \overset{\text{def}}{=} \det(1 - \mathcal{L}^N_{s, \rho})
\]
is well-defined, independent of \(h\) and a holomorphic multiple of the Selberg zeta function \(Z_{\Gamma}(s, \rho)\). As a consequence, the zeros of \(Z_{\Gamma}^{(N)}(s, \rho)\) are contained (with multiplicities) in the zero set of \(Z_{\Gamma}(s, \rho)\).

The next result is an estimate on the Hilbert–Schmidt norm of (127). Recall that the Hilbert–Schmidt norm of a compact operator \(A\) on a separable Hilbert space \(H\) is defined as the sum
\[
\|A\|_{\text{HS}}^2 \overset{\text{def}}{=} \sum_{k=1}^{\infty} \mu_k(A)^2,
\]
where \(\mu_k(A)\) are the singular values of \(A\), that is, the eigenvalues of the positive self-adjoint compact operator \(|A| \overset{\text{def}}{=} \sqrt{A^* A}\).

**Lemma 7.5** (Hilbert–Schmidt norm). Let \(N_0\) and \(h_0\) be as Lemma 7.4. Then for all \(N \geq N_0, h \in (0, h_0)\) and every finite-dimensional unitary representation \((\rho, V)\) of \(\Gamma\), the Hilbert–Schmidt norm of (127) is given explicitly by
\[
(129) \quad \|\mathcal{L}^N_{s, \rho}\|_{\text{HS}}^2 = |G| \sum_{j=1}^{2m} \sum_{a, b \in W_N^j} \chi_{\rho}(\gamma b \gamma^{-1}_a) \int_{\Omega(h) \cap D_j} \gamma'_a(z)^s \gamma'_b(z)^s B_{\Omega(h)}(\gamma a(z), \gamma b(z)) \, d\text{vol}(z),
\]
where \(\chi_{\rho}(\cdot) \overset{\text{def}}{=} \text{tr}_V(\rho(\cdot))\) is the character associated to \(\rho\). In particular, for the regular representation we have
\[
(130) \quad \|\mathcal{L}^N_{s, \rho_G}\|_{\text{HS}}^2 = |G| \sum_{j=1}^{2m} \sum_{a, b \in W_N^j} \int_{\Omega(h) \cap D_j} \gamma'_a(z)^s \gamma'_b(z)^s B_{\Omega(h)}(\gamma a(z), \gamma b(z)) \, d\text{vol}(z).
\]
Moreover, for \( s = \sigma + i \) with \( \sigma = \Re(s) \geq 0 \) we have the estimate

\[
\| \mathcal{L}^N_{s,R_G} \|_{\text{HS}}^2 \lesssim h^{-\delta} |G| e^{Ch|t| + CN(\delta - \sigma)} E^N_{\Gamma}(\pi, G),
\]

where both the implied constant and \( C > 0 \) depend only on the Schottky data of \( \Gamma \).

**Proof.** Identity (129) is well-known from the papers [38, 32]. Let us quickly sketch an alternative but essentially equivalent argument. Recalling the defining property of the Bergman kernel in (126), we can write the operator (127) as an integral operator given by

\[
\mathcal{L}^N_{s,\rho} f(z) = \int_{\Omega(h)} K(z, w) f(w) \, d\nu(w)
\]

for all \( z \in \Omega(h) \) and \( f \in H^2(\Omega(h), V) \), with kernel

\[
K(z, w) \overset{\text{def}}{=} \sum_{a \in W^j_N} \gamma'_a(z) \rho(\gamma_a)^{-1} B_{\Omega(h)}(\gamma_a(z), w) \text{ for } z \in \Omega(h) \cap D_j.
\]

The Hilbert–Schmidt norm can then be computed as

\[
\| \mathcal{L}^N_{s,\rho} \|_{\text{HS}}^2 = \int_{\Omega(h)} \int_{\Omega(h)} \| K(z, w) \|_2^2 \, d\nu(z) \, d\nu(w),
\]

where for every endomorphism \( A \in \text{End}(V) \), we let \( \| A \|_2 \overset{\text{def}}{=} \sqrt{\text{tr}(A^*A)} \) be its Frobenius norm. Squaring the Frobenius norm of \( K(z, w) \) in (132) and using the reproducing properties of the Bergman kernel leads to (129), as claimed. Specializing this identity to the regular representation \( \rho = R_G \) leads to (130). Using (130) together with the bounds in Lemma 7.4, we obtain

\[
\| \mathcal{L}^N_{s,R_G} \|_{\text{HS}}^2 \leq |G| \sum_{j=1}^{2m} \sum_{a,b \in W^j_N \atop \pi(\gamma_a) = \pi(\gamma_b)} \int_{\Omega(h) \cap D_j} |\gamma'_a(z)|^s |\gamma'_b(z)|^s |B_{\Omega(h)}(\gamma_a(z), \gamma_b(z))| \, d\nu(z)
\]

\[
\lesssim |G| \sum_{j=1}^{2m} \sum_{a,b \in W^j_N \atop \pi(\gamma_a) = \pi(\gamma_b)} e^{Ch|t|} |\gamma'_a|_{\pi,D_j}^{\sigma} |\gamma'_b|_{\pi,D_j}^{\sigma} h^{-2} \, \text{vol}(\Omega(h) \cap D_j)
\]

\[
\lesssim h^{-2} \, \text{vol}(\Omega(h)) |G| e^{Ch|t| + CN(\delta - \sigma)} \sum_{j=1}^{2m} \sum_{a,b \in W^j_N \atop \pi(\gamma_a) = \pi(\gamma_b)} |\gamma'_a|_{\pi,D_j}^{\delta} |\gamma'_b|_{\pi,D_j}^{\delta}
\]

\[
\lesssim h^{-\delta} |G| e^{Ch|t| + CN(\delta - \sigma)} E^N_{\Gamma}(\pi, G),
\]

proving (131). The proof is complete. \( \square \)

For the proof of Proposition 7.3 it is crucial to control the growth of the Fredholm determinant \( Z^{(N)}(s, \rho) \) in (128). We will prove the following result:

**Proposition 7.6** (Fredholm determinant bounds). Write \( s \in \mathbb{C} \) in Cartesian coordinates \( s = \sigma + i \). Then, assumptions and notations being as in Lemma 7.4, there are positive constants \( C_i \) \( (i = 1, 2) \) depending only on the Schottky data of \( \Gamma \), such that for all \( N \geq N_0 \) and \( \sigma > 0 \),

\[
\log |Z^{(2N)}_{\Gamma}(s, R_G)| \leq C_2(t)^{\delta} |G| e^{C_1 N(\delta - \sigma)} E^N_{\Gamma}(\pi, G).
\]
Moreover, for $\sigma > \delta$, we have

$$- \log |Z^{(2N)}_\Gamma(s, R_G)| \leq C_2 N |G| \frac{e^{-C_1 N(\sigma - \delta)}}{1 - e^{-C_1 N(\sigma - \delta)}}.$$  

**Proof.** Let us recall a few well-known facts from functional analysis, referring the reader to Gohberg–Krein [19] and Simon [44]. For an operator $A : H \to H$ on a separable Hilbert space $H$ we let

$$\|A\|_1 \overset{\text{def}}{=} \sum_{k=1}^\infty \mu_k(A)$$

be its trace norm, that is, the sum of the singular values of $A$. By Weyl's inequality we have

$$\log |\det (1 - A)| \leq \|A\|_1. \hspace{1cm} (133)$$

Moreover, for all trace-class operators $A, B : H \to H$ we have the inequality

$$\|AB\|_1 \leq \|A\|_{HS} \|B\|_{HS}. \hspace{1cm} (134)$$

In this proof, the implied constants depend only on the Schottky data of $\Gamma$. Applying (133) and (134) yields

$$\log |Z^{(2N)}_\Gamma(s, R_G)| \leq \log |\det (1 - \mathcal{L}^{2N}_{s,\rho})| \leq \|\mathcal{L}^{2N}_{s,\rho}\|_1 \leq \|\mathcal{L}^N_{s,\rho}\|_{HS}^2.$$ 

Applying Lemma 7.5 gives for all $0 < h < h_0$ the bound

$$\log |Z^{(2N)}_\Gamma(s, R_G)| \ll h^{-\delta}|G| e^{Ch|t| + C N(\delta - \sigma)} E^N_{\Gamma}(\pi, G).$$

We can now take $h = h_0(t)^{-1}$ (notice that $0 < h < h_0$) to obtain

$$\log |Z^{(2N)}_\Gamma(s, R_G)| \ll |t|\delta |G| e^{C N(\delta - \sigma)} E^N_{\Gamma}(\pi, G),$$

as claimed. To prove the second estimate we use the expansion of the Fredholm determinant

$$Z^{(2N)}_\Gamma(s, R_G) = \det (1 - \mathcal{L}^N_{s,\rho}) = \exp \left(- \sum_{k=1}^\infty \frac{1}{k} \text{tr}(\mathcal{L}^{2Nk}_{s,\rho,\Gamma}) \right),$$

and we recall that the sum on the right is absolutely convergent when $\sigma > \delta$. Taking absolute values gives

$$|Z^{(2N)}_\Gamma(s, R_G)| = \exp \left(- \sum_{k=1}^\infty \frac{1}{k} \text{Re}(\text{tr}(\mathcal{L}^{2Nk}_{s,\rho,\Gamma})) \right).$$

Taking logarithms and using $\text{Re}(z) \leq |z|$ shows that

$$- \log |Z^{(2N)}_\Gamma(s, R_G)| \leq \sum_{k=1}^\infty \frac{1}{k} |\text{tr}(\mathcal{L}^{2Nk}_{s,\rho,\Gamma})|. \hspace{1cm} (135)$$

Following the computations in the proof of [26, Proposition 2.2], we can evaluate the trace of $\mathcal{L}^m_{s,\rho,\Gamma}$ as a sum over conjugacy classes in $\Gamma$:

$$\text{tr}(\mathcal{L}^m_{s,\rho,\Gamma}) = |G| \sum_{d|m} \sum_{\{\gamma\} \in [\Gamma] \text{ s.t. } \pi(\gamma) \in G \text{ and } \text{WL}(\gamma) = d} \frac{de^{-s\ell(\gamma)\frac{m}{d}}}{1 - e^{-\ell(\gamma)\frac{m}{d}}},$$

where $\text{WL}(\gamma)$ denotes the minimal word length with respect to the generating set $S$ among all the elements in the conjugacy class of $\gamma$,

$$\text{WL}(\gamma) \overset{\text{def}}{=} \min_{\tilde{\gamma} \in [\gamma]} |\tilde{\gamma}|.$$
This formula is not very important for our purposes other than it allows us to deduce the following upper bound which is uniform the imaginary part of $s$: \begin{equation} |\text{tr}(L^m_{s,R_G})| \leq \text{tr}(L^m_{s,R_G}). \end{equation}

For any trace class operator $A : H \to H$ we have the inequality \begin{equation} |\text{tr}(A)| \leq \|A\|_1, \end{equation}
so we can use (136), (137) and the relation between the trace norm and Hilbert–Schmidt norm in (134) to obtain
\[
|\text{tr}(L^{2Nk}_{s,R_G})| \leq |\text{tr}(L^m_{s,R_G})| \leq \|L^{2Nk}_{\sigma,R_G}\|_1 \leq \|L^{Nk}_{\sigma,R_G}\|_{HS}^2.
\]

Applying Lemma 7.5 with $h = h_0/2$ (so that $0 < h < h_0$) and using the “trivial” bound $E^N_T(\pi, G) = O(1)$ gives
\[
|\text{tr}(L^{2Nk}_{s,R_G})| \ll N|G|e^{-Ckn(\sigma - \delta)}E^N_T(\pi, G) \ll N|G|e^{-Cn(\sigma - \delta)}.
\]

Inserting this bound back into (135), we obtain
\[
-\log |Z^{(2N)}_T(s, R_G)| \ll N|G| \sum_{k=1}^{\infty} e^{-Ckn(\sigma - \delta)} \ll N|G| \frac{e^{-CN(\sigma - \delta)}}{1 - e^{-CN(\sigma - \delta)}},
\]
which completes the proof. \hfill \Box

7.5. **Finishing the proof of Proposition 7.3.** To finish the proof of Proposition 7.3, we use essentially the same argument as in [25, 38] which relies on Jensen’s formula for holomorphic functions, or rather a weaker variant thereof. We refer the reader to Titchmarsh’s classical book [46] for a proof of Jensen’s formula and its relatives. By the discussion in §7.4, the number $N_T^\pi(\sigma, T)$ is bounded above by the number of zeros of $Z^{(2N)}(s, R_G)$ within the complex box $[\sigma, \delta] + i[T, T + 1] \subset \mathbb{C}$.

Let $s_0 = \sigma_0 + iT$ be a point in the complex plane where $\sigma_0$ is a positive real and consider the pair of concentric disks $D_i = D_C(s_0, r_i) \quad (i = 1, 2)$ with radii $r_2 > r_1 > 0$. Assume that the parameters $\sigma_0, r_1, r_2$ are chosen in such a way that \begin{equation} [\sigma, \delta] + i[T, T + 1] \subset \overline{D_1} \subset D_2. \end{equation}

From Jensen’s formula we obtain the bound \begin{equation} N_T^\pi(\sigma, T) \leq \frac{1}{\log(r_2/r_1)} \max_{|s - s_0| = r_2} \left( \log |Z^{(2N)}_T(s, R_G)| - \log |Z^{(2N)}_T(s_0, R_G)| \right). \end{equation}

We introduce a large parameter $K \geq 1$ to be chosen below and we set $\sigma_0 = K, \quad r_1 = \sqrt{(\sigma_0 - \sigma)^2 + 1} \approx K$ and $r_2 = r_1 + 1/N$.

One verifies that these parameters meet the condition in (138). We have
\[
r_2/r_1 - 1 \approx \frac{1}{KN}.
\]

Also note that
\[
\sigma_0 - \sigma \ll K \quad \text{and} \quad \sqrt{1 + \frac{1}{(\sigma_0 - \sigma)^2}} = 1 + O\left(\frac{1}{K^2}\right)
\]
and
\[
r_1 = \sqrt{(\sigma_0 - \sigma)^2 + 1} = (\sigma_0 - \sigma)\sqrt{1 + \frac{1}{(\sigma_0 - \sigma)^2}} = (\sigma_0 - \sigma) + O\left(\frac{1}{K}\right).
\]
These estimates show that every point \( s \in C \) on the circle \(|s - s_0| = r_2\) satisfies
\[
\text{Re}(s) \geq \sigma_0 - r_2 = \sigma_0 - r_1 - \frac{1}{N} = \sigma - O\left( \frac{1}{K} \right) - \frac{1}{N}
\]
and
\[
|\text{Im}(s)| \leq |T| + O(K).
\]
Thus, using (139) together with the bounds in Proposition 7.6, we obtain
\[
N_{\Gamma'}(\sigma, T) \ll \frac{1}{\log(1 + \frac{1}{KN})} \langle |T| + K \rangle^{\delta} |G| e^{CN(\delta - \sigma + \frac{1}{K} + \frac{1}{N})} E^{N}(\pi, G) + |G| Ne^{-C'N^2}.
\]
We can now simplify this estimate by inserting the bounds
\[
\log(1 + \frac{1}{KN}) \gg \frac{1}{KN} \quad \text{and} \quad \langle |T| + K \rangle \ll K\langle T \rangle
\]
and choosing \( K = N \). This gives
\[
N_{\Gamma'}(\sigma, T) \ll N^{3} \langle |T| \rangle^{\delta} |G| e^{CN(\delta - \sigma)} E^{N}(\pi, G) + |G| e^{-C'N^2},
\]
which is what we claimed in Proposition 7.3.

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