ON RANDOM PRIMITIVE SETS, DIRECTABLE NDFAs AND THE GENERATION OF SLOWLY SYNCHRONIZING DFAs

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ABSTRACT

We tackle the problem of the randomized generation of slowly synchronizing deterministic automata (DFAs) by generating random primitive sets of matrices. We show that when the randomized procedure is too simple the exponent of the generated sets is $O(n \log n)$ with high probability, thus the procedure fails to return DFAs with large reset threshold. We extend this result to random nondeterministic automata (NDFAs) by showing, in particular, that a uniformly sampled N DFA has both a 2-directing word and a 3-directing word of length $O(n \log n)$ with high probability. We then present a more involved randomized algorithm that manages to generate DFAs with large reset threshold and we finally leverage this finding for exhibiting new families of DFAs with reset threshold of order $\Omega(n^2/4)$.

Keywords: Synchronizing automaton, random automaton, Černý conjecture, directing nondeterministic automaton, random matrix set, primitive set.

1. Introduction

A complete deterministic finite state automaton (DFA) is directable or synchronizing if it admits a word that brings the automaton from every state to the same fixed state; a word of this kind is called a directing or synchronizing word. More formally, a DFA is a triple $\mathcal{A} = \langle Q, \Sigma, \delta \rangle$ where $Q$ is a finite set of states, $\Sigma$ is a finite set of input symbols called the alphabet and $\delta : Q \times \Sigma \to Q$ is the transition function. A synchronizing word $w$ is a finite sequence of letters of $Q$ for which there exists $v \in Q$ such that $\delta(q, w) = v$ for every $q \in Q$, where $\delta$ as been extended to $\delta : Q \times \Sigma^* \to Q$ in the usual way. Synchronizing DFAs appear in different research fields; for example they are often used as models of error-resistant systems \cite{13,9} and in symbolic dynamics

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For a brief account on synchronizing DFAs and their other applications we refer the reader to [37]. One of the most longstanding open problems in this field concerns the length of the shortest synchronizing word of a synchronizing DFA $\mathcal{A}$, called the reset threshold of the automaton and indicated by $rt(\mathcal{A})$:

**Conjecture 1 (The Černý conjecture [35]).** Any synchronizing DFA $\mathcal{A}$ on $n$ states has a synchronizing word of length at most $(n-1)^2$, so $rt(\mathcal{A}) \leq (n-1)^2$.

If the conjecture is true, the bound cannot be improved as there exists a family of automata having reset threshold of exactly $(n-1)^2$, known as the Černý’s automata [35]. Despite great effort, for long time the best upper bound known for the reset threshold of an $n$-state synchronizing DFA was $(n^3-n)/6$ [14, 29], recently improved to $(15617n^3 + 7500n^2 + 9375n - 31250)/93750$ [33]. Exhaustive search has confirmed the conjecture for small values of $n$ [3, 11] while quadratic upper bounds have been obtained for certain classes of DFAs [4, 19, 24, 32, 36]. The search for synchronizing DFAs attaining quadratic reset threshold (called extremal or slowly synchronizing automata) has been the subject of several contributions in recent years, partially due to the fact that they are hard to detect and few families are known (see [3, 11, 12, 25, 34] for examples). The great majority of these extremal DFAs is two-letter and has a quite regular structure; in particular, the action of their letters is very much similar to the ones of Černý’s. It is natural to wonder whether a randomized procedure to generate automata could obtain less structured synchronizing DFAs with possibly larger reset thresholds; this approach can be rooted back to the 60s with Erdős and his Probabilistic Method, where the existence of a structure with certain desired properties is proved by defining a suitable probabilistic space in which to embed the problem (for an account on the probabilistic method we refer the reader to [1]). This randomized procedure cannot be too simple: indeed, Berlinkov [5] and Nicaud [28] showed that an uniformly generated 2-letter DFA is synchronizing with high probability (i.e. the probability that it is synchronizing tends to 1 as the number of states $n$ tends to infinity) and it also has a synchronizing word of length $O(n \log^3 n)$ with high probability. It follows that:

- slowly synchronizing DFAs are almost surely never generated by a uniform distribution;
- synchronizing DFAs with more than two letters that need every letter to synchronize (called proper automata) are hard to find, as usually two letters are enough to make the automaton be synchronizing. As proper automata do not appear often in the literature, they are especially of interest since the behavior of their reset threshold is still unclear.

With this in mind, we decided to approach the randomized generation of (slowly synchronizing) DFAs by enforcing them to be proper; to accomplished this, we make use of the concept of primitive sets, described in the next paragraph.

The notion of synchronization can be generalized to nondeterministic finite automata (NDFA) in several ways (see for example [22]); here we will focus on the 2-directability and the 3-directability properties, that we now describe. A NDFA is defined as a triple $\mathcal{N} = (Q, \Sigma, \delta)$ where $Q$ is a finite set of states, $\Sigma$ is a finite set of input symbols and
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δ ⊆ Q × Σ × Q is the transition function; this time the transition from one state to another by a letter may be not defined or not uniquely defined. An N DFA is 2-directable if there exists a word w (called a 2-directing word) such that δ(q, w) = δ(p, w) for every p, q ∈ Q, where δ has been extended to δ ⊆ Q × Σ∗ × Q in the usual way; in other words, an N DFA is 2-directable if the set of states that can be reached by applying the word w is independent of the initial state. An N DFA is 3-directable if there exist v ∈ Q and a word w (called a 3-directing word) such that v ∈ δ(q, w) for every q ∈ Q; in other words, an N DFA is 3-directable if there exists a state that is reachable from any other state by applying the word w. Similarly to synchronizing DFAs, we can define d2(𝑁) the length of the shortest 2-directable word of an N DFA 𝑁, d3(𝑁) the length of its shortest 3-directable word and d𝑖(𝑛) = max{d𝑖(𝑁) : 𝑁 is an 𝑖-directable N DFA on 𝑛 states}, for 𝑖 = 2, 3. It is known that d2(n) = Θ(2𝑛) [16], d3(n) = O(4n/3n2) [16] and d3(n) = Ω(3n/3) [26] but, to the best of our knowledge, it is still not clear what is the average behavior of an N DFA; more precisely, we wonder what is the probability that a random N DFA 𝑁 is 2- or 3-directable as 𝑛 → ∞ and what is the expected magnitude of d2(𝑁) and d3(𝑁). Our primitive sets approach will also provide an answer this question.

1.1. The primitive set approach

A finite set of nonnegative matrices 𝑀 = {𝑀1, . . . , 𝑀𝑚} is called primitive if there exists a product 𝑀 = 𝑀𝑖1 · · · 𝑀𝑖𝑛 > 0 entrywise, for some 𝑖1, . . . , 𝑖𝑛 ∈ {1, . . . , 𝑚}; in this case 𝑀 is called a positive product. The exponent of a primitive set (exp(𝑀)) is the length of its shortest positive product. The concept of primitive set has been introduced by Protasov and Voynov in [31] as an extension of the notion of primitive matrix due to Frobenius in 1912 and has found application in different fields as in stochastic switching systems [30], consensus for discrete-time multi-agent systems [10], time-inhomogeneous Markov chain [21] and, finally, automata theory [6, 17].

Remark 2. Any N DFA 𝑁 = (𝑄, Σ, δ) with 𝑄 = {1, . . . , 𝑛} and Σ = {𝑎1, . . . , 𝑎𝑚} can be uniquely represented by the matrix set {𝐴1, . . . , 𝐴𝑚} where, for all 𝑖 = 1, . . . , 𝑚, 𝐴𝑖[𝑘, 𝑙] = 1 if 𝑘 ∈ δ(𝑙, 𝑎𝑖), 𝐴𝑖[𝑘, 𝑙] = 0 otherwise. Equivalently, any set of binary matrices is an N DFA. In this context, 𝑁 is 2-directable iff there exists a product 𝑀 = 𝐴𝑖1 · · · 𝐴𝑖𝑛 for some 𝑖1, . . . , 𝑖𝑛 ∈ {1, . . . , 𝑚} such that every column of 𝑀 is either entrywise positive, or entrywise equal to 0. Similarly, 𝑁 is 3-directable iff there exists a product 𝑀 = 𝐴𝑖1 · · · 𝐴𝑖𝑛 with an entrywise positive column, for some 𝑖1, . . . , 𝑖𝑛 ∈ {1, . . . , 𝑚}. Note that in case of DFAs, the matrices 𝐴1, . . . , 𝐴𝑚 are row-stochastic and the DFA is synchronizing iff it admits a product with an all-ones column (while all the other columns are made of zeros).

1A matrix is binary if it has entries in {0, 1}.
2A nonnegative matrix is row-stochastic if the entries of each row sum up to 1. In the case of a letter of a DFA, it means that each row of the matrix has exactly one 1.
3A column whose entries are all equal to 1.
In the rest of the paper we will mostly use this matrix representation of DFAs and NDFAs. It is clear that a primitive set \( \mathcal{N} \) of binary matrices is both a 2-directable and a 3-directable N DFA and it holds that

\[
\max\{d_2(\mathcal{N}), d_3(\mathcal{N})\} \leq \exp(\mathcal{N}).
\]

A less obvious connection between synchronizing DFAs and primitive sets is due to the following Definition \[3\] and Theorem \[4\]; we call a nonnegative matrix \( NZ \) if it has at least one positive entry in every row and in every column.

**Definition 3.** Let \( \mathcal{M} \) be a set of binary \( NZ \)-matrices. The DFA associated to the set \( \mathcal{M} \) is the automaton \( A(\mathcal{M}) = \{ A : A \) is binary row-stochastic, \( \exists M \in \mathcal{M} \) s.t. \( A[i,j] \leq M[i,j], \forall i,j \} \).

For an example of a set \( \mathcal{M} \) and its associated DFA \( A(\mathcal{M}) \) see Example \[5\].

**Theorem 4** \([6\], Theorems 16-17 and [17], Theorem 8\). Let \( \mathcal{M} = \{ M_1, \ldots, M_m \} \) be a set of \( n \times n \) binary \( NZ \)-matrices and let \( \mathcal{M}^T = \{ M_1^T, \ldots, M_m^T \} \). The set \( \mathcal{M} \) is primitive if and only if \( A(\mathcal{M}) \) (equiv. \( A(\mathcal{M}^T) \)) is synchronizing. If \( \mathcal{M} \) is primitive, it also holds that:

\[
\max\{rt(A(\mathcal{M})), rt(A(\mathcal{M}^T))\} \leq \exp(\mathcal{M}) \leq rt(A(\mathcal{M}))+rt(A(\mathcal{M}^T))+n-1. \tag{2}
\]

**Example 5.** We here present a primitive set \( \mathcal{M} \) and the DFAs \( A(\mathcal{M}) \) and \( A(\mathcal{M}^T) \).

\[
\mathcal{M} = \{ (\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}), (\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}) \}, \quad A(\mathcal{M}) = \{ (\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}), (\begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}) \} = \{ a, b, c \},
\]

\[
\mathcal{M}^T = \{ (\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}), (\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}) \}, \quad A(\mathcal{M}^T) = \{ (\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}), (\begin{array}{ccc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}) \} = \{ a, b, c' \}.
\]

One can verify that \( \exp(\mathcal{M}) = 8 = \exp(\mathcal{M}^T) \), \( rt(A(\mathcal{M})) = 4 \) and \( rt(A(\mathcal{M}^T)) = 2 \).

![Figure 1: The automata \( A(\mathcal{M}) \) and \( A(\mathcal{M}^T) \) of Example 5.](image)

Theorem \[4\] more generally holds for any set of \( NZ \)-matrices with nonnegative entries, due to the fact that the property of being primitive is not influenced by the actual values of the positive entries of the matrices of the set. In this case the automaton \( A(\mathcal{M}) \) of Definition \[3\] should be defined as \( A(\mathcal{M}) = \{ A : A \) is binary row-stochastic, \( \exists M \in \mathcal{M} \) s.t. \( \forall i,j, M[i,j] = 0 \Rightarrow A[i,j] = 0 \} \). Equation \[2\] shows that primitive sets can be used for generating synchronizing DFAs; a primitive set with large exponent implies that the associated DFA has large reset threshold. In particular, an \( NZ \)-primitive set with exponent greater than \( 2(n - 1)^2 - n + 1 \) would disprove the Černý conjecture.
1.2. Our contribution

The present article aims to answer the two following questions:

**Q1** Is it possible to randomly generate NZ-primitive sets with large exponent thus leading to synchronizing DFAs with large reset threshold?

**Q2** What is the probability that a random NDFA is 2- or 3-directable? What is the expected length of its shortest 2-directing and 3-directing words?

In Section 3 we give a negative answer to question **Q1** in case the randomized generation is too simple and we answer question **Q2**. In Subsection 3.1 we show that a uniformly generated perturbed permutation set (see Definition 9, Section 3) is primitive and has exponent of order $O(n \log n)$ with high probability, which implies that its associated DFA has reset threshold of order $O(n \log n)$ with high probability. This result leads us to the main theorem of Section 3, presented in Subsection 3.2: we show that a random binary set of $n \times n$ matrices generated by setting each entry of each matrix to 1 with probability $p$ and to 0 with probability $1 - p$, independently of each others, is primitive and has exponent of order $O(n \log n)$ with high probability when $p \geq (1 + \alpha)(\log n + c)/n$ for some $c \in \mathbb{R}$ and $\alpha > 0$, while it is almost surely never primitive when $p \leq (1 - \alpha)(\log n + c)/n$ for some $c \in \mathbb{R}$ and $\alpha > 0$. In the case $p = (\log n + c)/n$ for some $c \in \mathbb{R}$, the set exhibits an intermediate behavior and we show that its exponent is of order $O(n \log^3 n)$ with high probability under some conditions. In other words, $p = (\log n + c)/n$ is a *sharp threshold* for the property of these sets to be primitive and this result show that their associated DFAs have small reset threshold most of the times. As corollaries, in Subsection 3.3 we show that any NDFA randomly generated as described above is 2-directable and has a 2-directing word of length $O(n \log n)$ with high probability when $p \geq (1 + \alpha)(\log n + c)/n$ for some $c \in \mathbb{R}$ and $\alpha > 0$, and that the 3-directability property of these sets has the same *threshold* behavior described for primitivity. In particular, a random NDFA generated according to the uniform distribution ($p = 1/2$) has both a 2-directing word and a 3-directing word of length $O(n \log n)$ with high probability.

In Section 4 we present a more involved randomized algorithm that manages to generate NZ-primitive sets with quadratic exponent, thus providing a positive answer to question **Q1**. The algorithm generates proper primitive perturbed permutation sets (see Definition 9) of cardinality greater than two by exploiting a combinatorial characterization theorem of NZ-primitive sets (Theorem 7, Section 2), and from them we obtain proper synchronizing DFAs with more than two letters. To the best of our knowledge, this is the first time where a constructive procedure for finding proper synchronizing DFAs is presented. Finally, in Section 5 we present the new families of slowly synchronizing automata found by our algorithm: they are 3-letter proper synchronizing DFAs that do not resemble the Černý’s family and with reset threshold of order $\Omega(n^2/4)$. This last result improves the state of the art in the direction initiated by Gonze et. al. in [18]: they prove that the diameter of the square graph (see Definition 24, Section 4.2) of any $n$-state DFA made of $m \geq 2$ permutation matrices is lower bounded by $n^2/4 + o(n^2)$ when $n$ is odd. We prove that this lower bound holds

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4 A primitive set is *proper* if it needs all its matrices to be primitive.
for any \( n \) and any \( n \)-state synchronizing DFA containing \( m \geq 2 \) permutation matrices.

2. Definitions and notation

In this section we briefly go through some definitions and results that will be needed in the rest of the paper. Sometimes we will refer to synchronizing deterministic finite automata just as synchronizing automata; when we will consider nondeterministic finite automata it will always be specified. We indicate with \([n]\) the set \( \{1, \ldots, n\} \) and with \( S_n \) the set of permutations over \( n \) elements; with a slight abuse of notation \( S_n \) will also denote the set of the \( n \times n \) permutation matrices, where a permutation matrix is a binary matrix having exactly one 1 in every row and in every column. We indicate with \( I_{i,j} \) the matrix such that \( I_{i,j}[i,j] = 1 \) and all the other entries are equal to 0; \( M^T \) denotes the transpose of a matrix \( M \). The set of all the binary row-stochastic matrices of size \( n \times n \) is indicated by \( R_n \), \( C_n := \{R^T : R \in R_n\} \) is the set of all the binary column-stochastic matrices and \( NZ \) represents the set of all the binary NZ-matrices. Given two sequences \( \{a_n\}, \{b_n\}, n \in \mathbb{N} \), we say that \( a_n = O(b_n) \) if there exist \( C > 0 \) and \( N \in \mathbb{N} \) such that \( a_n \leq C b_n \) for every \( n > N \), that \( a_n = \Omega(b_n) \) if there exist \( C > 0 \) and \( N \in \mathbb{N} \) such that \( a_n \geq C b_n \) for every \( n > N \) and that \( a_n = \Theta(b_n) \) if \( a_n = O(b_n) \) and \( a_n = \Omega(b_n) \). If \( P \) is a probability distribution over a finite space \( \Omega \) and \( A, B \subset \Omega \) two events, we indicate with \( P(A|B) \) the conditional probability of \( A \) given \( B \).

We remind that a matrix \( M \) is irreducible if there does not exist a permutation matrix \( P \) such that \( PMP^T \) is block-triangular; a set \( \{M_1, \ldots, M_m\} \) is said to be irreducible if the matrix \( \sum_{i=1}^{m} M_i \) is irreducible. The directed graph associated to an \( n \times n \) nonnegative matrix \( M \) is the digraph \( D_M \) on \( n \) vertices with an edge from \( i \) to \( j \) iff \( M[i,j] > 0 \). A matrix \( M \) is irreducible if and only if \( D_M \) is strongly connected, i.e. if and only if there exists a directed path between any two given vertices in \( D_M \). Irreducibility is a necessary but not sufficient condition for a matrix set to be primitive ([31], Section 1). Primitive sets of NZ-matrices can be characterized as follows:

**Definition 6.** Let \( \Omega = \bigcup_{i=1}^{k} \Omega_i \) be a partition of \([n]\) with \( k \geq 2 \). We say that an \( n \times n \) matrix \( M \) has a block-permutation structure on the partition \( \Omega \) if there exists a permutation \( \sigma \in S_k \) such that \( \forall l = 1, \ldots, k \) and \( \forall i \in \Omega_l \), if \( M[i,j] > 0 \) then \( j \in \Omega_{\sigma(l)} \). We say that a set of matrices has a block-permutation structure if there exists a partition on which all the matrices of the set have a block-permutation structure.

**Theorem 7 ([31], Theorem 1).** An irreducible set of NZ-matrices is not primitive if and only if the set has a block-permutation structure.

We say that a matrix \( A \) dominates a matrix \( B \) (\( A \geq B \)) if \( A[i,j] \geq B[i,j], \forall i, j \).

**Proposition 8.** Consider an irreducible set \( \{M_1, \ldots, M_m\} \) in which every matrix dominates a permutation matrix. If the set has a block-permutation structure, then all the blocks of the partition must have the same size.
Proof. Let $Q_i$ be the permutation matrix dominated by $M_i$; if $M_i$ has a block-permutation structure on a given partition, so does $Q_i$ on the same partition. Theorem 2 in [18] states that if a set of permutation matrices has a block-permutation structure then all the blocks of the partition must have the same size, so we conclude. □

3. Primitivity and small exponent with high probability

3.1. Random perturbed permutation sets

In this section we focus on perturbed permutation sets (see the following definition) and we show that in case of uniform distribution these sets have small exponent most of the times, which implies that their associated DFAs (see Definition 3) have almost surely small reset threshold.

Definition 9. A perturbed permutation set is a matrix set made of permutation matrices where a 0-entry of one of the matrices is changed into a 1.

Perturbed permutation sets are particularly of interest as they have the following properties:

- they have the least number of positive entries that an NZ-primitive set can have, which intuitively should lead to sets with large exponent;
- their associated DFAs are easily computable;
- if they are primitive and proper, their associated DFAs are synchronizing and proper (or they can be made proper by removing one known letter, as shown in Section 4, Proposition 20).

For these reasons they will play a significant role in the randomized generation of slowly synchronizing automata in Section 4.

We call random perturbed permutation set a perturbed permutation set of $m \geq 2$ matrices constructed with the following randomized procedure:

Procedure 10. (i) $m$ permutation matrices $\{P_1, \ldots, P_m\}$ are sampled independently and uniformly from the set $S_n$;
(ii) a matrix $P_i$ is uniformly chosen from the set $\{P_1, \ldots, P_m\}$ and one of its 0-entry is uniformly selected among its 0-entries and changed into a 1. It becomes then a perturbed permutation matrix $\bar{P}_i$;
(iii) The final set is the set $\{P_1, \ldots, P_{i-1}, \bar{P}_i, P_{i+1}, \ldots, P_m\}$.

This procedure is equivalent to choosing independently and uniformly $m - 1$ permutation matrices from $S_n$ and one perturbed permutation matrix from $\bar{S}_n = \{\bar{P} : P = P + \delta_{i,j}, P \in S_n, \exists i \neq i : P[i', j] = 1\}$, the set of the perturbed permutation matrices.

We say that a property $X$ holds for a random matrix set with high probability if the probability that property $X$ holds tends to 1 as the matrix dimension $n$ tends to infinity.
There exists a product $O \tilde{W}$ since it is still a perturbed permutation set with length $O$ and property $(\text{uniformly})$ from the set $P$. Notice that Corollary 12 also holds in case some of the matrices $P_i$ are sampled uniformly and independently at random from $S_n$. We remind that the diameter of a (strongly connected) directed graph $D = (V, E)$ is equal to $\max_{u,v \in V} d(u, v)$ where $d(u, v)$ is the length of the shortest path connecting $u$ to $v$.

**Corollary 12 (Friedman et al. [15], Theorem 2.1 and Theorem 2.2).** Let $m \geq 2$ and $r \geq 1$ be two integers and let $\{P_1, \ldots, P_m\}$ be a set of $m$ permutation matrices sampled uniformly and independently at random from $S_n$. Let $D_r$ be the directed graph with vertex set the set of the $r$-tuples of distinct elements of $[n]$, having an edge from $(u_1, u_2, \ldots, u_r)$ to $(v_1, v_2, \ldots, v_r)$ if there exists an $i \in [n]$ such that $P_1[u_k, v_i] = 1$ for all $k = 1, \ldots, r$. Then $D_r$ has diameter of order $O(\log n)$ with high probability.

Notice that Corollary 12 also holds in case some of the matrices $P_i$ are sampled (uniformly) from the set $S_n$.

**Proof of Corollary 12**

It suffices to prove the theorem for $m = 2$. Let $\mathcal{M} = \{P_1, P_2\}$ be a random perturbed permutation set with $P_2 = P_2 + I_{i,j}$ and let $i' \neq i$ be the integer such that $P_2[i', j] = 1$. Corollary 12 with $r = 2$ and $m = 2$ implies that, with high probability, for any indices $v_1, v_2, w_1, w_2 \in [n]$ there exists a product $Q$ of elements of $\mathcal{M}$ of length $O(\log n)$ such that $Q[v_1, w_1] > 0$ and $Q[v_2, w_2] > 0$; we call this property $F_2$. We now construct a product of elements of $\mathcal{M}$ whose $j$-th column is entrywise positive; to do so we proceed recursively by constructing at each step a product that has one more positive entry in the $j$-th column than in the previous step. We will then construct from it a positive product.

The matrix $P_2$ has two ones in its $j$-th column; let $a_1$ and $b_1$ be two indices such that $P_2[a_1, j] = 0$ and $P_2[b_1, b_1] = 1$ (they do exist as the matrices are NZ). By property $F_2$ there exists a product $Q_1$ of elements in $\mathcal{M}$ such that $Q_1[j, i'] > 0$ and $Q_1[b_1, i'] > 0$; then the product $P_2Q_1P_2 := K_1$ has at least three positive entries in its $j$-th column. Let now $a_2$ and $b_2$ be two indices such that $K_1[a_2, j] = 0$ and $K_1[a_2, b_2] > 0$; by property $F_2$ there exists a product $Q_2$ such that $Q_2[j, i'] > 0$ and $Q_2[b_2, i'] > 0$ and so the product $K_1Q_2P_2 := K_2$ has at least four positive entries in its $j$-th column. By iterating this procedure, it is clear that $K_{n-2}$ has a positive column in position $j$. As each product $Q_i$ has length $O(\log n)$, $K_{n-2}$ has length $O(n \log n)$. The same reasoning can be applied to the set $\mathcal{M}^T = \{P_1^T, P_2^T\}$ since it is still a perturbed permutation set with $P_2^T = P_2^T + I_{j,i}$: there exist products $T_1, T_2, \ldots, T_{n-2}$ of elements in $\mathcal{M}^T$ of length $O(\log n)$ such that, by setting $W_1 = P_2^T T_1 P_2^T$ and $W_s = W_{s-1} T_1 P_2^T$ for $s = 2, \ldots, n - 2$, the final product $W_{n-2}$ has length $O(n \log n)$ and its $i$-th column is entrywise positive. Finally, by property $F_2$ there exists a product $S$ of elements in $\mathcal{M}$ of length $O(\log n)$ such that $S[i, i] > 0$. Then $K_{n-2}SW_{n-2}$ is a positive product of elements in $\mathcal{M}$ of length $O(n \log n)$.  \[ \Box \]
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Theorem 11 shows that Procedure 10 will (almost surely) never lead to automata with large reset threshold (see Definition 3 and Theorem 1). In the following section we present a similar result for random binary sets.

3.2. Random sets of binary matrices

We here rephrase some standard notions used in random graph theory (see [23], Section 1.5-1.6) in terms of sets of binary matrices; we refer the reader to [23] for a detailed review on random graphs. Given a property $\mathcal{P}$ and a set $\mathcal{B} = \{B_1, \ldots, B_m\}$ of binary matrices, we write $\mathcal{B} \in \mathcal{P}$ to indicate that the set $\mathcal{B}$ has the property $\mathcal{P}$. A property $\mathcal{P}$ is said to be increasing if for any matrix sets $\mathcal{B} = \{B_1, \ldots, B_m\}$ and $\mathcal{B}' = \{B'_1, \ldots, B'_m\}$ such that $\forall i = 1, \ldots, m$, $B'_i \leq B_i$, $\mathcal{B}' \in \mathcal{P}$ implies $\mathcal{B} \in \mathcal{P}$.

We denote with $B(n, p)$ an $n \times n$ random binary matrix where each entry is independently set to 1 with probability $p$ and to 0 with probability $1-p$; we denote with $B_m(n, p) = \{B_1(n, p), \ldots, B_m(n, p)\}$ a set of $m \geq 2$ matrices obtained independently in this way. The parameter $p$ may depend on the matrix size $n$, so it has to be intended as a sequence of real numbers $p(n) \in [0, 1]$, $n \in \mathbb{N}$; to ease the notation, we will sometimes avoid to explicit the dependency of $p$ on $n$, so we will write $B(n, p)$ instead of $B(n, p(n))$ and $B_m(n, p)$ instead of $B_m(n, p(n))$.

**Definition 13.** Given an increasing property $\mathcal{P}$, a sequence $\hat{p}(n) \in [0, 1]$, $n \in \mathbb{N}$, is called a threshold for the random binary set $B_m(n, p)$ with respect to $\mathcal{P}$ if, for any sequence $p(n) \in [0, 1]$, $n \in \mathbb{N}$:

$$\lim_{n \to \infty} \mathbb{P}(B_m(n, p(n)) \in \mathcal{P}) = \begin{cases} 1 & \text{if } p \gg \hat{p}, \\ 0 & \text{if } p \ll \hat{p}, \end{cases}$$

where $p \gg \hat{p}$ if and only if $\lim_{n \to \infty} p(n)/\hat{p}(n) = 0$. Furthermore, a sequence $\hat{p}(n) \in [0, 1]$, $n \in \mathbb{N}$, is said to be a sharp threshold for the random binary set $B_m(n, p)$ with respect to $\mathcal{P}$ if for any sequence $p(n) \in [0, 1]$, $n \in \mathbb{N}$, and for every fixed $\alpha > 0$:

$$\lim_{n \to \infty} \mathbb{P}(B_m(n, p(n)) \in \mathcal{P}) = \begin{cases} 1 & \text{if } \exists N \in \mathbb{N} : \forall n > N, p(n) \geq (1 + \alpha)\hat{p}(n) \\ 0 & \text{if } \exists N \in \mathbb{N} : \forall n > N, p(n) \leq (1 - \alpha)\hat{p}(n). \end{cases}$$

A (sharp) threshold thus represents a phase transition for $B_m(n, p)$ from not having property $\mathcal{P}$ with high probability to having property $\mathcal{P}$ with high probability.

**Remark 14.** Note that thresholds are in general defined up to the asymptotic relation $\hat{p}' = \Theta(\hat{p})$; in other words, if $\hat{p}$ is a threshold, then so is every sequence $\hat{p}'(n) \in [0, 1]$, $n \in \mathbb{N}$, for which there exist $C, c > 0$ and $N \in \mathbb{N}$ such that $\forall n \geq N$, $c\hat{p}'(n) \leq \hat{p}(n) \leq C\hat{p}'(n)$. This implies that a threshold is never uniquely defined, despite it is customary to call it the threshold (see for example [7, 23]). The same can be said about a sharp threshold $\hat{p}$; in this case, any sequence $\hat{p}'(n) \in [0, 1]$, $n \in \mathbb{N}$, such that $\lim_{n \to \infty} \hat{p}'(n)/\hat{p}(n) = 1$ is as well a sharp threshold.
We denote with $\mathcal{PR}$ the property for a binary matrix set to be primitive; it is easy to prove that it is an increasing property. The following theorem establishes a sharp threshold for $\mathcal{B}_m(n,p)$ to be primitive and provides an asymptotic estimate of the expected exponent of $\mathcal{B}_m(n,p)$ when it is a primitive NZ-set.

**Theorem 15.** Let $m \geq 2$ be an integer, $c \in \mathbb{R}$ and $\hat{p}(n) = (\log n + c)/n$. Then the sequence $\hat{p}$ is a sharp threshold for $\mathcal{B}_m(n,p)$ with respect to $\mathcal{PR}$. Moreover,

$$a(m,c) \leq \lim_{n \to \infty} \Pr\left(\mathcal{B}_m(n,\hat{p}(n)) \in \mathcal{PR}\right) \leq 1 - (1 - e^{-\hat{c}})^m, \quad (3)$$

where $a(m,c) = 1 - (1 - e^{-2\hat{c}})^m - me^{-\hat{c}}(1 - e^{-2\hat{c}})^{m-1}$.

In addition:

(i) If $p(n) \in [0,1]$, $n \in \mathbb{N}$, is such that $\exists\alpha > 0, N \in \mathbb{N} : \forall n > N, p(n) \geq (1 + \alpha)\hat{p}(n)$, then $\exp(\mathcal{B}_m(n,p)) = O(n\log n)$ with high probability;

(ii) $\exp(\mathcal{B}_m(n,\hat{p})) = O(n\log^3 n)$ with high probability, under the condition that $\mathcal{B}_m(n,\hat{p})$ is an NZ-primitive set.

Note that Theorem 15 implies that for any constant sequence $p(n) \equiv q \in [0,1]$, the set $\mathcal{B}_m(n,q)$ has a positive product of length $O(n\log n)$ with high probability; in particular, $q = 1/2$ induces the uniform distribution over the set of the binary matrix sets of cardinality $m$. Before proving Theorem 15 we need two preliminary results, the following Lemma 16 and Theorem 17, the latter presented by Nicaud in 23.

**Lemma 16.** Let $\mathcal{C}$ be a finite set of $n \times n$ binary matrices such that one of the following properties hold:

(i) for all $P, Q \in S_n$, $\mathcal{C} = \{PCQ : C \in \mathcal{C}\} := P\mathcal{C}Q$ and for all $C, D \in \mathcal{C}$, there exist $T_1, T_2 \in S_n$ such that $C = T_1 DT_2$;

(ii) $\mathcal{C} = \mathcal{R}_n$;

(iii) $\mathcal{C} = \mathcal{C}_n$.

Let $X_{\mathcal{C}}$ be a random variable with values in $\mathcal{C} \cup \{0\}$, defined in the following way: a random binary matrix $B(n,p)$ is generated, then $X = 0$ if $B(n,p)$ does not dominate any matrix in $\mathcal{C}$, otherwise $X = C$ with $C$ sampled uniformly among the elements of $\mathcal{C}$ dominated by $B(n,p)$. Let $\mathbb{P}_{X_{\mathcal{C}}}$ be the distribution of $X_{\mathcal{C}}$. Then it holds that, for any $C, D \in \mathcal{C}$:

$$\mathbb{P}_{X_{\mathcal{C}}}(C) = \mathbb{P}_{X_{\mathcal{C}}}(D). \quad (4)$$

**Proof.** Suppose first that (i) holds. Let $\mathbb{P}$ be the distribution of $B(n,p)$; we write $\mathbb{P}(M)$ for $\mathbb{P}(B(n,p) = M)$. By definition, for any $C \in \mathcal{C}$, $\mathbb{P}_{X_{\mathcal{C}}}(C) = \sum_{M \geq C} \mathbb{P}(M) |\{|C' \in \mathcal{C} : M \geq C'\}|^{-1}$, where $M$ is taken in the set of the binary matrices. Let $C, D \in \mathcal{C}$ and $T_1, T_2 \in S_n$ such that $C = T_1 DT_2$. Observe that $\mathbb{P}(M)$ depends only on the number of positive entries of $M$ so $\mathbb{P}(M) = \mathbb{P}(T_1^{-1}MT_2^{-1})$ as $T_1$
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and $T_2$ are permutations. It follows that

$$P_{X_e}(C) = \sum_{M \geq T_1 T_2} P(T_1^{-1} M T_2^{-1} \mid \{C' \in C : M \geq C'\})^{-1} =$$

$$= \sum_{T_1^{-1} M T_2^{-1} \geq D} P(T_1^{-1} M T_2^{-1} \mid \{C' \in C : T_1^{-1} M T_2^{-1} \geq C'\})^{-1} =$$

$$= P_{X_e}(D).$$

Suppose now that (ii) holds. We show that $P_{X_e}(C)$ does not depend on $C \in \mathcal{C}$ and so $[3]$ must hold. Let $a = (a_1, \ldots, a_n)$ be a vector in $[n]^n$; we write that $M = a$ if the $i$-th row of $M$ has exactly $a_i$ positive entries. Then,

$$P_{X_e}(C) = \sum_{a_1=1}^{n} \cdots \sum_{a_n=1}^{n} \sum_{M \geq a, \ M \geq C} P(M)$$

$$= \sum_{a_1=1}^{n} \cdots \sum_{a_n=1}^{n} \prod_{i=1}^{n} a_i^{-1} \left(\frac{n-1}{a_i} - 1\right)^{p a_i (1 - p)^{n-a_i}}.$$

Case (iii) can be proved analogously. □

**Theorem 17 ([25], Theorem 3).** Let $A$ be a random $n$-state DFA of $m \geq 2$ letters where each letter is chosen independently and uniformly at random from $\mathcal{R}_n$. Then $A$ admits a synchronizing word of length $O(n \log^2 n)$ with high probability.

**Proof of Theorem [15]**

With a slight abuse of notation we denote with $P$ the distribution of $B_m(n, p)$.

Suppose first that there exists $\alpha > 0$ and $N \in \mathbb{N}$ such that $\forall n > N$, $p(n) \geq (1 + \alpha) \hat{p}(n)$.

We need to prove that

$$\lim_{n \to \infty} P \left( B_m(n, p(n)) \in \mathbb{P} \mathcal{R} \ \text{and} \ \exp \left( B_m(n, p(n)) \right) = O(n \log n) \right) = 1.$$

Without loss of generality we can just consider the case $m = 2$. We first show that $B(n, p)$ dominates a permutation matrix with high probability and it also dominates a perturbed permutation matrix with high probability. This will imply that the random set $B_2(n, p)$ dominates a perturbed permutation set with high probability.

Let $G(n, n, p)$ be a random bipartite graph with vertex set $V = V_1 \cup V_2$ where $V_1 = [n] = V_2$, and edge set $E$ such that $(i, j) \in E$ if and only if $i \in V_1$, $j \in V_2$ and $B(n, p)[i, j] = 1$. Equivalently, in the bipartite graph $G(n, n, p)$ there is an edge between vertices $i \in V_1$ and $j \in V_2$ with probability $p(n)$. A perfect matching of a

\[ \text{perturbed permutation matrix is a permutation matrix where one of its 0-entries has been changed into a 1.} \]

\[ \text{We say that a matrix set } \mathcal{M} = \{M_1, \ldots, M_m\} \text{ dominates a matrix set } \mathcal{M}' = \{M'_1, \ldots, M'_m\} \text{ if for all } i = 1, \ldots, m, M_i \geq M'_i. \]
Suppose now that there exist $E'$ of its edges such that exactly one edge in $E'$ is incident to each vertex of the graph; it is easy to see that $G(n, n, p)$ admits a perfect matching if and only if $B(n, p)$ dominates a permutation matrix. Theorem 4.1 in [23] shows that, under our hypothesis, $G(n, n, p)$ admits a perfect matching with high probability; consequently, $B(n, p)$ dominates a permutation matrix with high probability. It is now easy to prove that $B(n, p)$ dominates a perturbed permutation matrix with high probability: indeed this probability is equal to the probability that $B(n, p)$ dominates a permutation matrix minus the probability that $B(n, p)$ is a permutation matrix. The former term goes to 1 as $n$ tends to infinity as proved above while the latter term is smaller than $(1-p(n))^{n-1}$, which tends to 0 as $n$ goes to infinity.

We now use Theorem 11 on the perturbed permutation set dominated by $B_2(n, p)$. Both the sets $S_n$ and $\bar{S}_n$ satisfy the hypothesis (i) of Lemma 16; let $X = X_{\bar{S}_n}$ and $\bar{X} = X_{S_n}$ be the random variables defined in the same lemma. We assume $X$ and $\bar{X}$ to be independent. Lemma 16 implies that for every $P \in S_n$, $P_X(P) = (1-P_X(0))/n!$ and for every $\bar{P} \in \bar{S}_n$, $\bar{P}_{\bar{X}}(\bar{P}) = (1-n\bar{P}_\bar{X}(0))/n!(n(n-1))$; indeed one can verify that $|\bar{S}_n| = n!n(n-1)$. The fact that $B(n, p)$ dominates a permutation matrix with high probability implies that $P_X(0) \rightarrow 0$ as $n \rightarrow +\infty$ and the fact that $B(n, p)$ dominates a perturbed permutation matrix with high probability implies that $\bar{P}_{\bar{X}}(0) \rightarrow 0$ as $n \rightarrow +\infty$. Let $P_{X \times \bar{X}} = P_X \cdot \bar{P}_{\bar{X}}$ be the joint distribution of $X$ and $\bar{X}$ on $S_n \times \bar{S}_n$ and let $\Omega \subset S_n \times \bar{S}_n$ be the event that a perturbed permutation set of cardinality 2 is primitive and with exponent of order $O(n \log n)$. Since $\mathcal{PR}$ is an increasing property, it holds that:

$$P\left(\mathcal{B}_m(n, p) \in \mathcal{PR} \text{ and } \exp(\mathcal{B}_m(n, p)) = O(n \log n)\right) \geq P_{X \times \bar{X}}(\Omega) \quad (5)$$

and

$$P_{X \times \bar{X}}(\Omega) = (1-P_X(0))(1-P_X(0)) \sum_{\{P_1, P_2\} \in \Omega} (n!)^{-1}(n(n-1))^{-1}. \quad (6)$$

The summation in the right-hand side of (6) is the probability that a set of cardinality 2 generated by Procedure 10 is primitive and with exponent of order $O(n \log n)$, which goes asymptotically to 1 by Theorem 11. Since $P_X(0)$ and $\bar{P}_{\bar{X}}(0)$ tend to zero as $n$ goes to infinity, eq. (6) goes asymptotically to 1. In view of the inequality (5), we conclude.

Suppose now that there exist $\alpha > 0$ and $N \in \mathbb{N}$ such that $\forall n > N$, $p(n) \leq (1-\alpha)p(n)$. We need to prove that $\lim_{n \rightarrow \infty} P(\mathcal{B}_m(n, p(n)) \in \mathcal{PR}) = 0$. If every matrix of a set has a zero-row, the set cannot be primitive: we show that $\mathcal{B}_m(n, p)$ has this property with high probability. Indeed, this probability is equal to $(1-(1-(1-p(n))^n)^m)$; by hypothesis $(1-p(n))^n \rightarrow 0$ as $n \rightarrow \infty$, so $(1-(1-p(n))^n)^m \sim e^{-nc-p(n)n}$ that tends asymptotically to 0. It remains to prove (i) and (ii); we start by proving (i). The term $1-P(\mathcal{B}_m(n, p(n)) \in \mathcal{PR}) = P(\mathcal{B}_m(n, p(n)) \notin \mathcal{PR})$ is lower bounded by the probability that each matrix in $\mathcal{B}_m(n, p)$ has at least a zero row, which is equal to $(1-P(B(n, p) \text{ has no zero rows}))^{-1}$. The probability that $B(n, p)$ has exactly $k$
zero-rows is a binomial distribution of parameters $n$ and $q(n) = (1 - \hat{p}(n))^n$, which converges to a Poisson distribution of mean $\mu = e^{-c} = \lim_{n \to \infty} nq(n)$. This implies that $P(B(n, \hat{p}) \text{ has no zero rows})$ converges asymptotically to $e^{-e^{-c}}$, and so $1 - \lim_{n \to \infty} P(B(m, \hat{p}(n)) \in PR) \geq (1 - e^{-e^{-c}})^m$ which proves the upper bound in (3). For the lower bound, let $D$ be the event that there exist at least two matrices in $B_m(n, \hat{p})$ such that each of them dominates a permutation matrix; it holds that $P(B_m(n, \hat{p}) \in PR) \geq P(B_m(n, \hat{p}) \in PR \mid D)P(D)$. The term $P(B_m(n, \hat{p}) \in PR \mid D)$ tends asymptotically to 1: this can be proved similarly as in the case where $p \geq (1 + \alpha)\hat{p}$, by introducing the random variables $X = X_{S_n}$ and $\bar{X} = X_{\overline{S}_n}$ as in Lemma 16. The difference is that now $B(n, \hat{p})$ is generated conditioned to the fact that it dominates a permutation matrix so $X$ takes value in $S_n$; eq. (4) still holds and so we can apply Theorem 11. It then remains to show that $P(D)$ tends to $a(m,c)$ as $n \to \infty$: this is straightforward as the probability that $B(n, \hat{p})$ dominates a permutation matrix tends asymptotically to $e^{-2e^{-c}}$ (23, Theorem 4.1).

Finally, we prove item (ii). We can suppose $m = 2$ without loss of generality. Let $X_e = X_{R_n}$ and $X_c = X_{C_n}$ be the random variables defined in Lemma 16. By hypothesis the sampled set $B_2(n, \hat{p})$ is known to be NZ, so Lemma 16 implies that $P_{X_e}$ is the uniform distribution over $R_n$ and $P_{X_c}$ is the uniform distribution over $C_n$. By Theorem 17 we have that with high probability $B_2(n, \hat{p})$ admits a product $C$ of length $O(n \log^3 n)$ with a positive column (say in position $j$) and with high probability $B_2(n, \hat{p})$ admits a product $R$ of length $O(n \log^3 n)$ with a positive row (say in position $i$). Since $B_2(n, \hat{p})$ is primitive by hypothesis, its underlying graph is strongly connected, which means that there exists a product $L$ of elements of $B_2(n, \hat{p})$ of length at most $n - 1$ such that $L[i,j] > 0$. The product $CLR$ is a positive product of length $O(n \log^3 n)$ so (ii) follows. □

Notice that, since primitivity is not influenced by the actual values of the positive entries of the matrices, Theorem 15 is naturally extended to random nonnegative matrices. Furthermore, in the case $p = \hat{p}$ both the left-hand term and the right-hand term of eq. (3) approaches 1 as the number of matrices $m$ increases, which is reasonable to expect. We underline that the difference in the upper bounds on $\exp(B_m(n,p))$ that we get when $p = \hat{p}$ or when $p \geq (1 + \alpha)\hat{p}$ is due to the fact that it is not possible to use the same reasoning. Indeed, when $p = \hat{p}$ the probability that $B(n, \hat{p})$ dominates a permutation matrix is asymptotically equal to a constant strictly smaller than 1 (23, Theorem 4.1) and so we cannot make use of Theorem 11 anymore. Notice also that the condition that all the matrices of the set are NZ is weaker than requiring that all the matrices of the set dominate a permutation matrix: it is indeed easy to build an NZ matrix that does not dominate a permutation matrix. It is interesting to compare Theorem 15 with a result of Gerencsér et al. (17), Corollary 3); they prove that $\lim_{n \to \infty} \log(\exp(n))/n = (\log 3)/3$ where $\exp(n) = \max\{\exp(M) : \text{$M$ is a primitive set of $n \times n$ matrices}\}$. Our result shows that the sets whose exponent reaches $\exp(n)$ must be very few and that they are almost impossible to be attained by $B_m(n,p)$ and in particular from a uniform distribution; indeed the average exponent is much smaller.
Summarizing, in view of the connection between primitive sets and synchronizing DFAs established by Theorem 4, Theorem 15 suggests that there is very little hope of generating slowly synchronizing automata from $B_m(n, p)$, no matter how the sequence $p(n)$ behaves.

3.3. Random NDFAs

The random binary set $B_m(n, p)$ can be seen as a random nondeterministic finite automaton. We here apply Theorem 15 to the 2-directability and 3-directability properties of $B_m(n, p)$.

**Corollary 18.** Let $m \geq 2$ be an integer and $\hat{p}(n) = (\log n + c)/n$ for some $c \in \mathbb{R}$. Let $p(n) \in [0, 1]$, $n \in \mathbb{N}$, be a sequence such that there exist $\alpha > 0$ and $N \in \mathbb{N}$: $\forall n > N$, $p(n) \geq (1 + \alpha)^\hat{p}(n)$. Then with high probability $B_m(n, p)$ is 2-directable and $d_2(B_m(n, p)) = O(n \log n)$. In particular, for any fixed integer $m \geq 2$, with high probability an $m$-letter N DFA generated according to the uniform distribution is 2-directable and has a 2-directing word of length $O(n \log n)$.

**Proof.** It is a straightforward consequence of Theorem 15 and (1). The uniform distribution is obtained by choosing $p(n) = 1/2$ for every $n \in \mathbb{N}$. \hfill $\square$

The following corollary shows that $\hat{p}(n) = (\log n + c)/n$ is as well a sharp threshold for $B_m(n, p)$ with respect to the 3-directability property:

**Corollary 19.** Let $m \geq 2$ be an integer, $c \in \mathbb{R}$ and $\hat{p}(n) = (\log n + c)/n$. The sequence $\hat{p}$ is a sharp threshold for $B_m(n, p)$ with respect to the 3-directability property. It also holds that

$$a(m, c) \leq \lim_{n \to \infty} \mathbb{P}(B_m(n, \hat{p}(n)) \text{ is 3-directable}) \leq 1 - (1 - e^{-e^{-\hat{p}}})^m. \quad (7)$$

where $a(m, c) = 1 - (1 - e^{-2e^{-c}})^m - me^{-2e^{-c}}(1 - e^{-2e^{-c}})^{m-1}$.

Furthermore:

(i) If $p(n) \in [0, 1]$, $n \in \mathbb{N}$, is such that $\exists \alpha > 0, N \in \mathbb{N}$: $\forall n > N$, $p(n) \geq (1 + \alpha)^\hat{p}(n)$, then $d_3(B_m(n, p)) = O(n \log n)$ with high probability;

(ii) $d_3(B_m(n, \hat{p})) = O(n \log^4 n)$ with high probability, under the condition that $B_m(n, \hat{p})$ is an NZ-primitive set.

In particular, for any fixed integer $m \geq 2$, with high probability an $m$-letter N DFA generated according to the uniform distribution is 3-directable and has a 3-directing word of length $O(n \log n)$.

**Proof.** If the sequence $p(n)$ is such that there exist $\alpha > 0$ and $N \in \mathbb{N}$ such that $\forall n > N$, $p(n) \geq (1 + \alpha)^\hat{p}(n)$, then by Theorem 15 and (1) it holds that

$$\lim_{n \to \infty} \mathbb{P}(B_m(n, p(n)) \text{ is 3-directable and } d_3(B_m(n, p(n))) = O(n \log n)) = 1.$$ 

If there exist $\alpha > 0$ and $N \in \mathbb{N}$ such that $\forall n > N$, $p(n) \leq (1 - \alpha)^\hat{p}(n)$, then $\lim_{n \to \infty} \mathbb{P}(B_m(n, p(n)) \text{ is 3-directable}) = 0$ due to the same argument used in the proof.
of Theorem 15 with high probability all the matrices of \( B_m(n, p) \) have a zero-row. Theorem 15 also trivially implies the lower bound in (7) and item (ii), since a positive product has in particular an entrywise positive column.

It remains to prove the upper bound in (7). In the proof of Theorem 15 we have seen that the asymptotic probability for \( B_m(n, \hat{p}) \) to have each matrix with a zero-row is equal to \((1 - e^{-c})^m\), in which case \( B_m(n, \hat{p}) \) is not 3-directable. Therefore, \( \lim_{n \to \infty} \mathbb{P}(B_m(n, \hat{p}(n))) \) is 3-directable \( \leq 1 - (1 - e^{-c})^m \).

The uniform distribution is obtained by choosing \( p(n) = 1/2 \) for every \( n \in \mathbb{N} \). \( \square \)

Notice again that, for any fixed \( c \in \mathbb{R} \), the right-hand term and the left-hand term of (7) both tend to 1 as the number of matrices \( m \) (the cardinality of the alphabet of the N DFA) increases.

4. A randomized algorithm for generating proper primitive sets

In this section we describe a randomized procedure to build proper\(^7\) primitive sets making use of the Protasov-Voynov characterization theorem (Theorem 7, Section 2), which describes a combinatorial property that an NZ-matrix set must have in order not to be primitive: by constructing a primitive set such that each of its proper subsets has this property, we can make it proper. In particular, we will build proper perturbed permutation sets, for the reasons presented at the beginning of Section 3.

Theorem 7 implies that a primitive set of \( m \) matrices is proper if and only if each of its subsets of cardinality \( m - 1 \) has a block-permutation structure on a certain partition, so this is the condition we will enforce. As we are dealing with perturbed permutation sets, by Proposition 8 these partitions must have blocks of the same size; if the blocks of the partition have size \( n/q \), we call it a \( q \)-partition and we say that the set has a \( q \)-permutation structure. The algorithm first generates a set of permutation matrices satisfying the requested block-permutation structures and then a 0-entry of one of the obtained matrices is changed into a 1; while doing this last step, we will make sure to preserve all the block-permutation structures of the matrix. We underline that our algorithm finds perturbed permutation sets that, if are primitive, are also proper; the construction itself does not guarantee primitivity and this property has to be verified at the end.

One of the advantages of using perturbed permutation sets is that we can easily generate proper synchronizing DFAs from them, as shown by the following proposition:

**Proposition 20.** Let \( M = \{P_1, \ldots, P_{m-1}, P_m + I_{i,j}\} \) be a proper primitive perturbed permutation set and let \( j' \neq j \) be the integer such that \( P_m[i,j'] = 1 \). The synchronizing automaton \( A(M) \) (see Definition 3) can be written as \( A(M) = \{P_1, \ldots, P_{m-1}, P_m, M\} \) with \( M = P_m + I_{i,j} - I_{i,j'} \). If \( A(M) \) is not proper, then \( A = \{P_1, \ldots, P_{m-1}, M\} \) is.

**Proof.** Suppose \( A(M) \) is not proper; the only matrix we can delete from the set without losing synchronization is \( P_m \). Indeed, we cannot delete \( M \) as all the others

\(^7\)We remind that we call a primitive set proper if it needs all its matrices to be primitive.
are permutation matrices. For \( i = 1, \ldots, m - 1 \), let \( \mathcal{M}_i \) be the set obtained from \( \mathcal{M} \) by erasing \( P_i \); by hypothesis, \( \mathcal{M}_i \) is not primitive so the automaton \( \mathcal{A}(\mathcal{M}_i) \) is not synchronizing. But \( \mathcal{A}(\mathcal{M}_i) \) is indeed the automaton obtained by erasing \( P_i \) from \( \mathcal{A}(\mathcal{M}) \), so \( \mathcal{A} \) has to be synchronizing and proper.

4.1. The algorithm

Given \( R, C \subset [n] \) and a matrix \( M \), we indicate with \( M[R, C] \) the submatrix of \( M \) with rows indexed by \( R \) and columns indexed by \( C \). For generating a set of \( m \) matrices \( \mathcal{M} = \{M_1, \ldots, M_m\} \) we choose \( m \) prime numbers \( q_1 \geq \cdots \geq q_m \geq 2 \) and we set \( n = \prod_{i=1}^{m} q_i \). For \( j = 1, \ldots, m \), we require the set \( \{M_1, \ldots, M_{j-1}, M_{j+1}, \ldots, M_m\} \) (the set obtained from \( \mathcal{M} \) by erasing matrix \( M_j \)) to have a \( q_j \)-permutation structure; this construction will ensure the set to be proper. More in detail, for all \( j = 1, \ldots, m \) we enforce the existence of a \( q_j \)-partition \( \Omega^{(j)} = \bigcup_{i=1}^{q_j} \Omega_i^{(j)} \) of \([n]\) on which, for all \( k \neq j \), the matrix \( M_k \) has to have a block-permutation structure. This request means that for every \( k = 1, \ldots, m \) and for every \( j \neq k \) there must exist a permutation \( \sigma_j^k \in S_{q_j} \) such that for all \( i = 1, \ldots, q_j \) and \( l \neq \sigma_j^k(i) \), \( M_k[\Omega_i^{(j)}, \Omega_l^{(j)}] \) is a zero-matrix (see Definition 6).

The main idea of the algorithm is to initialize every entry of each matrix to 1 and then, step by step, to set to 0 the entries that are not compatible with the conditions that we are requiring. As our final goal is to have a set of permutation matrices with the desired properties, at every step we need to make sure that each matrix dominates at least one permutation matrix, despite the increasing number of zeros among their entries.

**Definition 21.** Given a matrix \( M \) and a \( q \)-partition \( \Omega = \bigcup_{i=1}^{q} \Omega_i^{(q)} \), we say that a permutation \( \sigma \in S_q \) is compatible with \( M \) and \( \Omega \) if for all \( i = 1, \ldots, q \), there exists a permutation matrix \( Q_i \) such that

\[
M[\Omega_i^{(q)}, \Omega_{\sigma(i)}^{(q)}] \geq Q_i.
\]  

The algorithm itself is formally presented in Listing 1; we here describe in words how it operates. Each entry of each matrix is initialized to 1. The algorithm has two for-loops: the outer one on \( j = 1, \ldots, m \), where a \( q_j \)-partition \( \Omega^{(j)} = \bigcup_{i=1}^{q_j} \Omega_i^{(j)} \) of \([n]\) is uniformly randomly sampled, and the inner one on \( k = 1, \ldots, m \) with \( k \neq j \) where we verify whether there exists a permutation \( \sigma_j^k \in S_{q_j} \) that is compatible with \( M_k \) and \( \Omega^{(j)} \). If it does exist, we choose one among all the compatible permutations and the algorithm moves to the next step \( k+1 \). If such permutation does not exist, then the algorithm exits the inner for-loop and it selects another \( q_j \)-partition of \([n]\); it then repeats the inner for-loop for \( k = 1, \ldots, m \) with \( k \neq j \) with this new partition. If after \( T_1 \) steps it is choosing a different \( q_j \)-partition \( \Omega^{(j)} \), the existence for each \( k \neq j \) of a permutation \( \sigma_j^k \in S_{q_j} \) that is compatible with \( M_k \) and \( \Omega^{(j)} \) is not established, we stop the algorithm and we say that it did not converge. If the inner for-loop is completed, then for each \( k \neq j \) the algorithm modifies the matrix \( M_k \) by keeping unchanged each block \( M_k[\Omega_i^{(j)}, \Omega_{\sigma_j^k(i)}^{(j)}] \) for \( i = 1, \ldots, q_j \) and by setting to zero all the other entries.
of $M_k$, where $\sigma^k_j$ is the selected compatible permutation; the matrix $M_k$ has now a block-permutation structure over the partition $\Omega^q$. The algorithm then moves to the next step $j + 1$. If it manages to finish the outer for-loop, we have a set of binary matrices with the desired block-permutation structures. We then just need to select a permutation matrix $P_k \leq M_k$ for every $k = 1, \ldots, m$ and then to randomly change a $0$-entry of the matrices into a $1$ without modifying the block-permutation structures of the matrix: this is always possible as the blocks of the partitions are nontrivial and a permutation matrix has just $n$ positive entries. We finally check whether the set is primitive.

Here below we present the procedures that the algorithm uses:

(i) $[p, P] = ExtractPerm(M, met)$

This is the key function of the algorithm, formally presented in Listing 2. It returns $p = 1$ if the matrix $M$ dominates a permutation matrix, it returns $p = 0$ and $P = M$ otherwise. In the former case it also returns a permutation matrix $P$ selected among the ones dominated by $M$ according to $met$: if $met = 2$ the matrix $P$ is sampled uniformly at random, if $met = 3$ we make the choice of $P$ deterministic. More in detail, the procedure works as follows: we first count the numbers of ones in each column and in each row of the matrix $M$. We then consider the row or the column with the least number of ones; if this number is zero we stop the procedure and we set $p = 0$, as in this case $M$ does not dominate a permutation matrix. Otherwise, we choose one of the $1$-entries of the row or the column attaining this minimum: if $met = 2$ (method 2) the entry is chosen uniformly at random while if $met = 3$ (method 3) we take the first $1$-entry in the lexicographic order. Suppose that the chosen entry is in position $(i, j)$: we set to zero all the other entries in row $i$ and column $j$ and we iterate the procedure on the submatrix obtained from $M$ by erasing row $i$ and column $j$. We can prove that this procedure is well-defined and in at most $n$ steps it produces the desired output. Method 3 will play an important role in our numerical experiments in Section 4.2 and in the discovery of new families of automata with quadratic reset threshold in Section 5.

(ii) $[a, A] = DomPerm(M, \Omega, met)$

It returns $a = 1$ if there exists a permutation compatible with the matrix $M$ and the partition $\Omega = \bigcup_{i=1}^q\Omega^q_i$, it returns $a = 0$ and $A = M$ otherwise. In the former case it chooses one of the compatible permutations $\sigma$ according to $met$ and returns the $n \times n$ matrix $A$ such that $A[\Omega^q_i, \Omega^q_{\sigma(i)}] = M[\Omega^q_i, \Omega^q_{\sigma(i)}]$ for all $i = 1, \ldots, q$, and all the other entries of $A$ are equal to zero. $A$ has then a block-permutation structure on $\Omega$. More precisely, $DomPerm$ acts in two steps: it first defines a $q \times q$ matrix $B$ such that, for all $i, k = 1, \ldots, q$, $B[i, k] = \begin{cases} 1 & \text{if } M[\Omega^q_i, \Omega^q_k] \text{ dominates a permutation matrix} \\ 0 & \text{otherwise} \end{cases}$; this is done by calling $ExtractPerm$ with input $M[\Omega^q_i, \Omega^q_k]$ and $met$ for all $i, k = 1, \ldots, q$. Notice that there exists a permutation compatible with $M$ and $\Omega$ if and only if $B$ dominates a permutation matrix, so the second step of the procedure is to call again $[p, P] = ExtractPerm(B, met)$: if $p = 0$ we set $a = 0$.
and $A = M$, while if $p = 1$ we set $a = 1$ and $A$ as described before with $\sigma = P$
(i.e. $\sigma(i) = j$ iff $P[i,j] = 1$).

(iii) $\textit{Mset} = \textit{Addone}(P_1, \ldots, P_m)$
It changes a 0-entry of one of the matrices $P_1, \ldots, P_m$ into a 1 preserving all its
block-permutation structures. The matrix and the entry are chosen uniformly
at random and the procedure iterates the choice till it finds a compatible entry
(which always exists); it then returns the final perturbed permutation set $\textit{Mset}$.

(iv) $\textit{pr} = \textit{Primitive}(\textit{Mset})$
It returns $\textit{pr} = 1$ if the matrix set $\textit{Mset} = \{M_1, \ldots, M_m\}$ is primitive and $\textit{pr} = 0$
otherwise. It first verifies if the set is irreducible by checking the strong
connectivity of the digraph $D_N$ where $N = \sum_{i=1}^{k} M_i$ (see Section 2) via breadtth-
first search on every node, then primitivity is checked by the Protasov-Voynov
algorithm ([31], Section 4).

All the above routines have polynomial time complexity in $n$, apart from routine
$\textit{Primitive}$ that has time complexity of $O(mn^2)$.

\textbf{Remark 22.} (i) In all our numerical experiments the algorithm always converged,
\textit{i.e.} it always ended before reaching the stopping value $T_1$, for $T_1$ large enough.
This is probably due to the fact that the matrix dimension $n$ grows exponen-
tially as the number of matrices $m$ increases, which produces enough degrees of
freedom. We leave the proof of this fact for future work.

(ii) A recent work of Alpin and Alpina ([2], Theorem 3) generalizes Theorem 7 for
the characterization of primitive sets to sets that are allowed to be reducible
and the matrices to have zero columns (but not zero rows). Clearly, DFAs fall
within this category. Our algorithm could leverage this recent result in order
to directly construct proper synchronizing DFAs. We also leave this for future
work.

\begin{verbatim}
Listing 1: Algorithm for generating proper primitive sets.
Input : q_1 ,... ,q_m,T1,met
Initialize M_1,... ,M_m as all–ones matrices
for j:=1 to m do
    t1=0
    while t1<T1 do
        t1=t1+1
        choose a q_j–partition Omega_j
        for k=1 to m and k!=j do
            [a,A_k]=DomPerm(M_k,Omega_j,met)
            if a==0, exit inner for–loop end
        end
        if a==1, exit while–loop end
    end
    if t1==T1
        display ‘does not converge’, exit procedure
end
\end{verbatim}
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else
    set \( M_k = A_k \) for all \( k = 1, \ldots, m \) and \( k \neq j \)
end
for \( i = 1 \) to \( m \) do
    \([p_i, P_i] = \text{Extractperm}(M_i, \text{met})\)
end
Mset = Addone\((P_1, \ldots, P_m)\)
pr = Primitive\((\text{Mset})\)
return Mset, pr

Listing 2: Procedure for extracting a permutation matrix from a binary one.

Input: \( M \), \( \text{met} \)
n = size of \( M \)
P = \( M \), \( p = 1 \), \( I = [1, 2, \ldots, n] \), \( J = [1, 2, \ldots, n] \)
for \( i = 1 \) to \( n \) do
    \( v_1 = \) vector of the number of 1s in the rows of \( P \) indexed by \( I \)
    \( v_2 = \) vector of the number of 1s in the columns of \( P \) indexed by \( J \)
    \( v = [v_1, v_2] \)
    sort \( v \) in ascending order
    if \( v(1) = 0 \)
        \( p = 0 \), \( P = M \), exit procedure
    else
        if \( v(1) \) belongs to \( v_1 \)
            choose a 1-entry in row \( v(1) \) according to \( \text{met} \)
            \( j = \) column index of the 1-entry chosen
            set to 0 all the other entries in row \( v(1) \) and column \( j \)
            delete \( v(1) \) from \( I \), delete \( j \) from \( J \)
        else
            choose a 1-entry in column \( v(1) \) according to \( \text{met} \)
            \( i = \) row index of the 1-entry chosen
            set to 0 all the other entries in column \( v(1) \) and row \( i \)
            delete \( v(1) \) from \( J \), delete \( i \) from \( I \)
        end
    end
end
return \( p, P \)

4.2. Numerical results

We here compare four methods of generating random primitive sets with respect to the magnitude of the reset threshold of their associated synchronizing DFAs and we show that our randomized procedure manages to generate synchronizing DFAs with quadratic reset threshold.

We call method 1 the sets generated by Procedure \textbf{[10]} with \( m = 2 \) (two matrices);
method 2 and method 3, already introduced in the previous paragraph, refer to our randomized construction where, respectively, a permutation matrix is extracted from a binary one uniformly at random or deterministically. Finally, we call method 4 a set generated by the following procedure:

Procedure 23. (i) Two permutation matrices $P_1$ and $P_2$ are sampled uniformly and independently at random from $S_n$; 

(ii) A 1-entry of $P_1$ is selected uniformly at random. Suppose this entry is in row $i$ and column $j$; we select uniformly an index $j \in [n] \setminus \{j\}$ and we set $P_1[i, j] = 0$ and $P_1[i, j] = 1$; 

(iii) Let $i' \neq i$ be the other index such that $P_1[i', j] = 1$ (it always exists as $P_1$ is a permutation matrix). We select uniformly an index $\bar{i} \in [n] \setminus \{i, i'\}$ and we set $P_1[\bar{i}, j] = 1$.

The matrix $P_1$ generated by Procedure 23 does not dominate a permutation matrix and it has the least number of positive entries that an NZ-matrix that does not dominate a permutation matrix can have. Procedure 23 has been developed because Theorem 11 and Theorem 15 show that when all the matrices of the set dominate a permutation matrix can have. Procedure 23 has been made by taking into account two facts: on one hand, it is desirable to keep constant the rate $it(n)/k_m(n)$ between the number of sampled sets $it(n)$ and the cardinality $k_m(n)$ of the state space. Since $k_m(n + 1)/k_m(n)$ grows approximately as $n^m$, we have that $k_m(n)$ explodes very fast and so we also have to deal with the limited computational speed of our computers. The choice of $it(n) = 50n^2$ comes as a compromise between these two issues, at least when $n \leq 70$. Among the $it(n)$ generated sets, we select the primitive ones and we generate their associated DFAs (Definition 3); we then check which ones are not proper synchronizing and we make them proper by using Proposition 20 (it is easy to prove a similar result for method 4). We set $T_1=1000$ for method 2 and 3. Due to the fact that computing the reset threshold of thousands of generated instances is prohibitive (computing the reset threshold of an automaton is an NP-hard problem [19]), we use a proxy for the reset threshold, the so called diameter of the square graph, which is introduced here below. The square graph diameter is computable in polynomial time, namely $O(mn^2)$ with $m$ the number of letters of the automaton and $n$ its number of states.

Definition 24. The square graph $S(A)$ of an $n$-state DFA $A$ is the labeled directed graph with vertex set $V = \{(i, j) : 1 \leq i \leq j \leq n\}$ and edge set $E$ such that $e = \{(i, j), (i', j')\} \in E$ if there exists a letter $A[i,i'] > 0$ and $A[j, j'] > 0$, or $A[i, j'] > 0$ and $A[j, i'] > 0$. In this case, we label the edge $e$ by $A$ (multiple labels are allowed). A vertex of type $(i, i)$ is called a singleton.

The diameter of $S(A)$, indicated by $diam(S(A))$, is the maximum of $d(u, s)$ on any non-singleton vertex $u$ and any singleton $s$, where $d$ denotes the length of the shortest path from $u$ to $s$. 

Theorem 11 and Theorem 15 show that when all the matrices of the set dominate a permutation matrix can have. Procedure 23 has been developed because Theorem 11 and Theorem 15 show that when all the matrices of the set dominate a permutation matrix with high probability we expect low exponents. 

For each method and each choice of $n$ we run the algorithm $it(n) = 50n^2$ times, thus producing each time $50n^2$ sets. This choice for $it(n)$ has been made by taking into account two facts: on one hand, it is desirable to keep constant the rate $it(n)/k_m(n)$ between the number of sampled sets $it(n)$ and the cardinality $k_m(n)$ of the state space. Since $k_m(n + 1)/k_m(n)$ grows approximately as $n^m$, we have that $k_m(n)$ explodes very fast and so we also have to deal with the limited computational speed of our computers. The choice of $it(n) = 50n^2$ comes as a compromise between these two issues, at least when $n \leq 70$. Among the $it(n)$ generated sets, we select the primitive ones and we generate their associated DFAs (Definition 3); we then check which ones are not proper synchronizing and we make them proper by using Proposition 20 (it is easy to prove a similar result for method 4). We set $T_1=1000$ for method 2 and 3. Due to the fact that computing the reset threshold of thousands of generated instances is prohibitive (computing the reset threshold of an automaton is an NP-hard problem [19]), we use a proxy for the reset threshold, the so called diameter of the square graph, which is introduced here below. The square graph diameter is computable in polynomial time, namely $O(mn^2)$ with $m$ the number of letters of the automaton and $n$ its number of states. 

Definition 24. The square graph $S(A)$ of an $n$-state DFA $A$ is the labeled directed graph with vertex set $V = \{(i, j) : 1 \leq i \leq j \leq n\}$ and edge set $E$ such that $e = \{(i, j), (i', j')\} \in E$ if there exists a letter $A[i,i'] > 0$ and $A[j, j'] > 0$, or $A[i, j'] > 0$ and $A[j, i'] > 0$. In this case, we label the edge $e$ by $A$ (multiple labels are allowed). A vertex of type $(i, i)$ is called a singleton.

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The diameter of $S(A)$, indicated by $diam(S(A))$, is the maximum of $d(u, s)$ on any non-singleton vertex $u$ and any singleton $s$, where $d$ denotes the length of the shortest path from $u$ to $s$. 

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The diameter of $S(A)$, indicated by $diam(S(A))$, is the maximum of $d(u, s)$ on any non-singleton vertex $u$ and any singleton $s$, where $d$ denotes the length of the shortest path from $u$ to $s$. 

Theorem 11 and Theorem 15 show that when all the matrices of the set dominate a permutation matrix with high probability we expect low exponents.
A well-known result ([37], Proposition 1) states that a DFA is synchronizing if and only if in its square graph there exists a path from any non-singleton vertex to a singleton one; the proof of this fact also implies that

\[ \text{diam}(S(A)) \leq rt(A) \leq n \cdot \text{diam}(S(A)). \] (9)

The diameter of the square graph thus represents a lower bound on the reset threshold of an automaton and can be hence used as a proxy.

Figure 2 reports on the y axis the maximal square graph diameter found among the associated automata of the sets generated by methods 1, 2, 3 and 4 for each matrix dimension \( n \) when \( n \) is the product of three prime numbers. Figure 3 reports the same but when \( n \) is the product of four prime numbers. We can see that our randomized construction manages to reach higher values of the square graph diameter than the mere random generation; in particular, method 3 reaches quadratic diameters in case of three matrices. We also report in Figure 4 the behavior of the average diameter of the proper synchronizing automata generated on 50 \( n \) iterations when \( n \) is the product of three prime numbers: we can see that in this case method 2 does not perform better than method 1 and 4, while method 3 performs just slightly better. This behavior could have been expected since our primary goal was to randomly generate at least one slowly synchronizing automata, which is indeed what happens with method 3 that manages to reach quadratic reset thresholds most of the times.

A remark can be done on the percentage of the generated sets that are not primitive; this is reported in Figure 5, where we divide nonprimitive sets into two categories: reducible sets and imprimitive sets, i.e. irreducible sets that are not primitive. We can see that the percentage of nonprimitive sets generated by method 1 and 4 goes to 0 as \( n \) increases, behavior that we partially expected (see Section 3, Theorem 11), while method 2 seems to always produce a non-negligible percentage of nonprimitive sets, although quite small. The behavior is reversed for method 3: most of the generated sets are not primitive. This can be interpreted as a good sign. Indeed, nonprimitive sets can be seen as sets with infinite exponent; as we are generating a lot of them with method 3, we intuitively should expect that, when a primitive set is generated, it has high chances to have large diameter.

The slowly synchronizing automata found by our randomized construction are presented in the following section.

5. New families of synchronizing DFAs with quadratic reset threshold

We present here four new families of slowly synchronizing automata with square graph diameter of order \( \Omega(n^2/4) \), which represents a lower bound for their reset threshold; they have been found via the randomized algorithm described in Section 4 via method 3. Our families are made of proper synchronizing automata with three letters: two symmetric permutation matrices and a matrix that fixes all the states but one. This characteristic makes our families differ from the Černý automaton and the other known families of extremal automata (e.g. [5, 11, 20, 12, 25]) to the extent that, if we set \( r(A) = \min\{k \in \mathbb{N} : a^k = a, \ \forall a \in \Sigma\} \) where \( \Sigma \) is the alphabet of the DFA \( A \),
Figure 2: Comparison between methods 1, 2, 3 and 4 with respect to the maximal diameter found on $50n^2$ iterations when $n$ is the product of three prime numbers; the $y$ axis is in logarithmic scale.

Figure 3: Comparison between methods 1, 2, 3 and 4 with respect to the maximal diameter found on $50n^2$ iterations when $n$ is the product of four prime numbers.
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Figure 4: Average diameter obtained by methods 1, 2, 3 and 4 when $n$ is the product of three prime numbers.

Figure 5: Percentage of nonprimitive sets (divided into reducible and imprimitive sets) generated by methods 1, 2, 3 and 4 (indicated above each bar) when $n$ is the product of three prime numbers. For instance, on sets of dimension $n = 20$, method 1 generates 0.35% of nonprimitive sets (0.35% reducible, 0% imprimitive), method 2 generates 6.15% of nonprimitive sets (5.18% reducible, 0.97% imprimitive), method 3 generates 84.5% of nonprimitive sets (77.9% reducible, 6.6% imprimitive) and method 4 generates 5.88% of nonprimitive sets (5.88% reducible, 0% imprimitive).
then \( r(A) = 3 \) for any automaton \( A \) that belongs to our families while \( r(C_n) = n + 1 \) for the Černý automaton on \( n \) states \( C_n \). Our families belong to the class of automata with simple idempotents introduced by Rystsov in \cite{Rystsov1977}, who proved an upper bound of \( 2(n - 1)^2 \) on their reset threshold, and they are the associated DFAs of primitive sets made of a perturbed identity matrix and two symmetric permutations. The following proposition shows that primitive sets of this kind must have a very specific shape. With a slight abuse of notation we identify a permutation matrix \( M \) under its underlying permutation, that is we say that \( Q(i) = j \) if and only if \( Q[i,j] = 1 \); the identity matrix is denoted by \( I \). Note that a permutation matrix is symmetric if and only if its cycle decomposition is made of fixed points and cycles of length 2.

**Proposition 25.** Let \( M_{ij} = \{I_{ij}, Q_1, Q_2\} \) be a matrix set of \( n \times n \) matrices where \( I_{ij} = I + I_{ij}, j \neq i, \) is a perturbed identity and \( Q_1 \) and \( Q_2 \) are two symmetric permutations. If \( M \) is irreducible then, up to a relabeling of the vertices, \( Q_1 \) and \( Q_2 \) have the following form:
- if \( n \) is even
  \[
  Q_1(i) = \begin{cases} 
  1 & \text{if } i = 1 \\
  i + 1 & \text{if } i \text{ even}, 2 \leq i \leq n - 2 \\
  i - 1 & \text{if } i \text{ odd}, 3 \leq i \leq n - 1 \\
  n & \text{if } i = n
  \end{cases}, \quad Q_2(i) = \begin{cases} 
  i - 1 & \text{if } i \text{ even} \\
  i + 1 & \text{if } i \text{ odd}
  \end{cases}
  \tag{10}
  \]
  or
  \[
  Q_1(i) = \begin{cases} 
  n & \text{if } i = 1 \\
  i + 1 & \text{if } i \text{ even}, 2 \leq i \leq n - 2 \\
  i - 1 & \text{if } i \text{ odd}, 3 \leq i \leq n - 1 \\
  1 & \text{if } i = n
  \end{cases}, \quad Q_2(i) = \begin{cases} 
  i - 1 & \text{if } i \text{ even} \\
  i + 1 & \text{if } i \text{ odd}
  \end{cases}
  \tag{11}
  \]
- if \( n \) is odd
  \[
  Q_1(i) = \begin{cases} 
  1 & \text{if } i = 1 \\
  i + 1 & \text{if } i \text{ even} \\
  i - 1 & \text{if } i \text{ odd}, 3 \leq i \leq n
  \end{cases}, \quad Q_2(i) = \begin{cases} 
  i - 1 & \text{if } i \text{ even} \\
  i + 1 & \text{if } i \text{ odd}, 1 \leq i \leq n - 2 \\
  n & \text{if } i = n
  \end{cases}.
  \tag{12}
  \]

**Proof.** The set \( M \) is irreducible if and only if the digraph \( D \) induced by matrix \( I_{ij} + Q_1 + Q_2 \) is strongly connected (see Section 2). If \( D \) is strongly connected, then the digraph induced by \( Q_1 + Q_2 \) must be strongly connected as \( Q_1 \) and \( Q_2 \) are symmetric and the matrix \( I_{ij} \) adds just a single edge that is not a selfloop in \( D \). Consider vertex 1: there must exist a matrix in the set \( \{Q_1, Q_2\} \) that links it to another vertex; let this matrix be \( Q_2 \) (wlog) and label this vertex with 2. As \( Q_2 \) is symmetric, we have \( Q_2(1) = 2 \) and \( Q_2(2) = 1 \). This implies that \( Q_1 \) needs to link vertex 2 to some vertex other than 1 as otherwise the digraph would not be strongly connected; we label this vertex with 3 and so we have \( Q_1(2) = 3 \) and \( Q_1(3) = 2 \). By

\[ ^8 \text{And similarly } r(A) \text{ is linear in } n \text{ for most of the known extremal automata.} \]
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![Figure 6: The automata $A_{1,6}$ with $n = 8$; $rt(A_{1,6})=31$. Dashed arrows refer to matrix $Q_2$, normal arrows to matrix $Q_1$ and bold arrows to matrix $I_{1,6}$, where its selfloops have been omitted.](image)

iterating this reasoning, it follows that $Q_1$ and $Q_2$ must be as in (10) or (11) if $n$ is even or as in (12) if $n$ is odd. \qed

**Proposition 26.** A matrix set $M_{ij} = \{I_{ij}, Q_1, Q_2\}$ of type (11) is never primitive.

**Proof.** Due to the symmetry of digraph $D_{Q_2+Q_1}$, up to a relabeling of the vertices we can assume without loss of generality that $i = 1$. If $j$ is odd, all the three matrices have a block-permutation structure over the partition $\{\{1, 3, \ldots, n-1\}, \{2, 4, \ldots, n\}\}$, while if $j$ is even they have a block-permutation structure over the partition $\{\{1, k\}, \{2, k-1\}, \ldots, \{\frac{k}{2}, \frac{k}{2}+1\}, \{k+1, n\}, \{k+2, n-1\}, \ldots, \{\frac{n+k}{2}, \frac{n+k}{2}+1\}\}$. By Theorem 7, the set cannot be primitive. \qed

We now present our new families of slowly synchronizing automata, prove closed formulas for their square graph diameter and finally state a conjecture on their reset thresholds.

**Definition 27.** Let $M_{ij} = \{I_{ij}, Q_1, Q_2\}$ where $I_{ij} = I + \mathbb{1}_{ij}$ for $j \neq i$ and $Q_1$ and $Q_2$ are as in eq. (10) if $n$ is even and as in eq. (12) if $n$ is odd. We define $A_{ij} = \{I_{ij}, Q_1, Q_2\}$ to be the associated DFA (see Definition 3) of $M_{ij}$, where $L_{ij} = I + \mathbb{1}_{ii} - \mathbb{1}_{ij}$.

Figure 6 represents the automaton $A_{1,6}$ with $n = 8$. We set $E_n = A_{1, n-2}$ for $n = 4k$ and $\geq 2$, $E'_n = A_{1, n-4}$ for $n = 4k+2$ and $k \geq 2$, $O_n = A_{\frac{2k+1}{2}, \frac{2k+1}{2}+1}$ for $n = 4k+1$ and $k \geq 1$, $O'_n = A_{\frac{2k+1}{2}-1, \frac{2k+1}{2}+1}$ for $n = 4k+3$ and $k \geq 1$. The following theorem holds:

**Theorem 28.** The automaton $E_n$ has square graph diameter (SGD) of $(n^2 + 2n - 4)/4$, $E'_n$ has SGD of $(n^2 + 2n - 12)/4$, $O_n$ has SGD of $(n^2 + 3n - 8)/4$ and $O'_n$ has SGD of $(n^2 + 3n - 6)/4$. Therefore all the families $E_n$, $E'_n$, $O_n$ and $O'_n$ have reset threshold of $\Omega(n^2/4)$.

**Proof.** We prove the theorem just for the family $E_n$; the other square graph diameters can be obtained by a similar reasoning. We set $I = I_{1, n-2}$ to ease the notation. In the following we describe the shape of $S(A_{1,n-2})$ with $n = 4k$ in order to compute its diameter, i.e. the maximal distance between a non-singleton vertex and the singleton $(n - 2, n - 2)$, as it is the only singleton that has an in-going edge starting from a non-singleton vertex; we invite the reader to refer to Figure 8 during the proof.
The digraph $S(A_{1,n-2} \setminus \{I\})$, without considering the singletons, is disconnected and has $n/2$ strongly connected components: $C_0$ of size $n/2$ and $C_1, \ldots, C_{n/2-1}$ of size $n$. The component $C_0$ is made of the vertices $\{(1+s, n-s): s = 0, \ldots, n/2-1\}$ while component $C_i$ is made of the vertices $\{(i, i+1), (i-1, i+2), \ldots, (1, 2i), (1, 2i+1), (2, 2i+2), \ldots, (n-i, n-i+1)\}$ for $1 \leq i \leq n/2-1$: these components look like “chains” due to the symmetry of $Q_1$ and $Q_2$ (see Figure 8). In particular, the vertices $(1, n)$ and $(3, n-2)$ belong to $C_0$, the vertices $(1, 2i)$ and $(1, 2i+1)$ belong to $C_i$ for $1 \leq i \leq n/2-1$, the vertices $(n-4, n-2)$ and $(n-2, n)$ belong to $C_1$, the vertices $(1, n-2)$ and $(4, n-2)$ belong to $C_{n/2-1}$ and the vertices $(n-2i-2, n-2)$ and $(n-2i+3, n-2)$ belong to $C_i$ for $2 \leq i \leq n/2-2$. The matrix $\bar{I}$ connects the components $\{C_i\}_i$ by linking vertex $(1, a)$ to vertex $(a, n-2)$ for every $a = 2, \ldots, n$ in such a way that the $\{C_i\}_i$ can be ordered from the farthest to the closest to the singleton $(n-2, n-2)$ (see Figure 8). Indeed, the diagram in Figure 7 shows how the components $\{C_i\}_i$ are linked together for $2 \leq i \leq n/2-1$: an arrow between two vertices means that there exists a word mapping the first vertex to the second one, a number next to the arrow represents the length of such word if the two vertices belong to the same component while arrows connecting vertices from different components are labeled by $\bar{I}$; bold vertices represent the ones that are linked by $\bar{I}$ to other chains.

How $C_0$ is connected to $C_1$ is directly shown in Figure 8. It follows that the digraph $S(A_{1,n-2})$ is formed by “layers” represented by the components $\{C_i\}_i$ where

$$C_0, C_1, C_{2-4}, C_3, C_{2-5}, C_5, C_{2-12}, \ldots$$ (13)

is the sequence of components from the farthest to the closest to the singleton $(n-2, n-2)$. In order to compute the diameter we need to measure the length of the shortest path from vertex $(n/2, n/2+1)$ to vertex $(n-2, n-2)$, which is colored in red in Figure 8. This means that for $0 \leq i \leq n/2-1$ we have to compute the distance $d_i$ in $C_i$ between vertices $(2i, n-2)$ and $(1, n-2i-1)$ if $i$ is odd or between vertices $(2i+1, n-2)$ and $(1, n-2i+2)$ if $i$ is even. In view of (13), we have the following sequence for the $d_i$:

$$d_0 = \frac{n}{2} - 1, \quad d_1 = n - 2, \quad d_{2-4} = 1, \quad d_3 = n - 3, \quad d_{2-5} = 5, \quad d_5 = n - 7, \quad d_{2-12} = 9, \ldots$$

Since the number of edges labeled by $\bar{I}$ that appear in the path is $n/2$, the diameter is equal to

$$\text{diam}(S(A_{1,n-2})) = \frac{n}{2} + \sum_{k=0}^{\frac{n}{2}-1} d_k = \frac{n^2}{4} + \frac{n}{2} - 1.$$
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\[ C_{\frac{n-2i-2}{2}} : \quad C_1 : \quad C_{\frac{n-2i-2}{2}} : \]

\[
\begin{align*}
(1, 2i) & \mapsto (2i, n - 2) \\
1 & \mapsto 2i - 1 \\
(2i + 1, n - 2) & \mapsto (1, 2i + 1) \quad (1, n - 2i - 1) \\
2i & \mapsto (n - 2i - 2, n - 2) \quad 1 \\
5 & \mapsto (1, n - 2i + 3) \quad (n - 2i + 3, n - 2) \\
1 & \mapsto (1, n - 2i + 2)
\end{align*}
\]

Figure 7: Diagram on how the components \( \{ C_i \} \) in the proof of Theorem 28 are linked together. Vertices in the same column belong to the same component (indicated above the column).
Conjecture 29. The automaton $E_n$ has reset threshold of $(n^2 - 2)/2$, $E_n'$ has reset threshold of $(n^2 - 10)/2$ and $O_n$ and $O_n'$ have reset threshold of $(n^2 - 1)/2$. Furthermore, they represent the automata with the largest possible reset threshold among the family $\{A_{ij}\}_{i \neq j}$ for respectively $n = 4k$, $n = 4k + 2$, $n = 4k + 1$ and $n = 4k + 3$.

Notice that, although the randomized construction for proper primitive sets presented
in Section 4 is defined just when the matrix size \( n \) is the product of at least three prime numbers, we here presented an extremal \( n \)-state automaton of quadratic reset threshold for \( \text{any} \) value of \( n \). Theorem 28 can also be seen as an improvement in the direction initiated by Gonze et. al. in [18], where they show that the square graph diameter of any \( n \)-state automaton made of \( m \geq 2 \) permutation matrices is lower bounded by \( n^2/4 + o(n^2) \) when \( n \) is odd. We have proved that this lower bound holds for any \( n \) and any \( n \)-state synchronizing automaton containing \( m \geq 2 \) permutation matrices.

6. Conclusion

In this paper we have exploited the connection between primitive sets of NZ-matrices and synchronizing DFAs to propose a randomized construction for generating slowly synchronizing automata. We have first shown that random perturbed permutation sets have small exponent most of the times, thus producing fast synchronizing DFAs. The same behavior applies to random binary sets (alias random NDFAs) where each entry of each matrix is independently set to \( 1 \) with probability \( p \); we have also shown that \( p(n) = (\log n + c)/n \) is a threshold for the property of these random sets to be primitive and to be 3-directable. In particular, an uniformly sampled NDFA of at least two letters has both a 2-directing word and a 3-directing word of length \( O(n \log n) \) with high probability. Secondly, we have proposed a more involved randomized construction for primitive sets based on a recent characterization of NZ-primitive sets (Theorem 7) and we have shown via Theorem 4 that it is able to generate some synchronizing DFAs with quadratic reset threshold. Finally, we have presented four new families of DFAs with simple idempotents with reset threshold of order \( \Omega(n^2/4) \); to the best of our knowledge, this is one of the few cases where an extremal family of automata does not resemble the Černý’s one. The primitive set approach to synchronizing DFAs seems promising and we believe that some parameters of our construction, as the way a permutation matrix is extracted from a binary one or the way the partitions of \( [n] \) are selected, could be further improved in order to generate new families of slowly synchronizing automata; for example, we could think about selecting the ones in the procedure Extractperm according to a given distribution. As mentioned at the end of Section 4, one can also apply the construction directly to automata by leveraging the recent result of Alpin and Alpina (2, Theorem 3). Finally, it would be of interest to determine how the exponent of \( B_m(n,p) \) behaves when \( p \) is chosen differently for each matrix of the set, e.g. when \( B_m(n,p) = \{B_1(n,p_1),\ldots,B_m(n,p_m)\} \) for \( p = (p_1(n),\ldots,p_m(n)) \in [0,1]^m, n \in \mathbb{N} \).

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