

e-Positivity Results and Conjectures

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Abstract

In a 2016 ArXiv posting F. Bergeron listed a variety of symmetric functions $G[X; q]$ with the property that $G[X; 1 + q]$ is e-positive. A large subvariety of his examples could be explained by the conjecture that the Dyck path LLT polynomials exhibit the same phenomenon. In this paper we list the results of computer explorations which suggest that other examples exhibit the same phenomenon. We prove two of the resulting conjectures and propose algorithms that would prove several of our conjectures. In writing this paper we have learned that similar findings have been independently discovered by Per Alexandersson (see [1]).

Introduction

We say that the symmetric function $G[X; q]$ exhibits the e-positivity phenomenon if and only if the symmetric function $G[X; 1 + q]$ is e-positive. This only means that, in the e-basis expansion

$$G[X; 1 + q] = \sum_{\lambda} a_{\lambda}(q) e_{\lambda}[X],$$

the coefficients $a_{\lambda}(q)$ are polynomials in $q$ with positive integer coefficients. The following are four examples:

$$LLT(4, 3, [0, 1, 2])[X; 1 + q] = (q^3 + 2q^2)e_3 + (q^2 + 3q)e_2e_1 + e_1^3,$$

$$LLT(7, 4, [0, 1, 2])[X; 1 + q] = (q^3 + 2q^2)e_4 + (q^2 + 2q)e_3e_1 + qe_2^2 + e_2e_1^2,$$

$$B[3,1,1][X; 1 + q] = (q^3 + 2q^2)e_5 + (q^2 + 2q)e_4e_1 + qe_3e_2 + e_3e_1^2,$$

$$Unicell[1,4,3,2][X; q] = (s_4[4] + (2q + 1)s_3[1] + 2qs_2[2] + (q^2 + 2q)s_{[2,1,1]} + q^2s_{[1,1,1,1]}).$$

The first is the LLT polynomial of the path that alternates North steps and East steps, the second is the LLT of a rational Dyck path in the $7 \times 4$ lattice rectangle. The third is a balanced Dyck path LLT that hits the diagonal according to the partition $[3, 1, 1]$. In I.5 we have the unicellular LLT whose successive cells are in diagonals $1, 4, 3, 2$. In I.6 we see that even in the latter case the e-positivity phenomenon takes place.

The experimental evidence of widest impact we have noticed so far is that the LLT polynomials generated by Dyck paths whether classical or rational do exhibit the e-positivity phenomenon.

Since a Dyck path that alternates North and East steps is also Balanced we tested some Balanced paths and sure enough, the e-positivity phenomenon seems to occur there as well. Noticing that N-E alternating Dyck paths are also unicellular, we tested several cases of these LLT’s and discovered that the e-positivity phenomenon seems to occur there too. Our experimental data lead us to conjecture that the following families of symmetric functions exhibit the e-positivity phenomenon.

1. The Modified Macdonald polynomials at $t = 1$. That is $H_{\mu}[X; q, 1]$, for any partition $\mu$.
2. The polynomials $B_p[X; q]$ for all compositions $p$ (see section 2).
3. All unicellular LLT polynomials.
4. All column LLT polynomials, see the last section of this paper.
5. The polynomials $\nabla C_p 1$ for all compositions $p$.
6. All the polynomials $Q_{m,n}(-1)^n$, appearing in the rational Shuffle Conjecture.
7. All the polynomials $\Delta_n$, appearing in the Delta conjecture.
8. All the polynomials $\nabla(-1)^{n-1}p_n$.

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Our main results here are proofs of conjectures (1) and (2) and the construction of algorithms that would yield the asserted $e$-basis expansions for classical Dyck path, unicellular and column LLT’s.

Our presentation is divided into three sections. In the first section we prove conjecture (1). In the second section we prove conjecture (2). In the third and final section we comment on some consequences of our results and state our conjectured $e$-expansion formula for column LLT polynomials.

In fact, it will be seen that, given a classical Dyck path $D$ and its zeta image $\zeta(D)$, by means of the areaprime way of computing $LLT'$s, we can deal with classical Dyck path LLT’s, column LLT’s and unicellular LLT’s at the same time by simply marking, partially marking and not marking the removable corners of the English partition above $\zeta(D)$. This view point makes evident that the number of distinct polynomials of unicellular and column LLT’s are respectively not larger than the Catalan numbers and lower Schröder numbers.  We also show how to use the Carlsson-Mellit algorithm for constructing these LLT’s to confirm our conjectures for larger scale examples than what is achieved by purely combinatorial means.

1. **Proof of conjecture (1)**

For notation and plethystic notation we refer to [4] and [12] where the reader can also consult a Modified Macdonald polynomials “tool kit”. Our point of departure is the following specialization at $t = 1$ of the Modified Macdonald polynomial.

**Proposition 1.1**

For any partition $\mu$ we have

$$\bar{H}_\mu[X; q, 1] = \prod_{i=1}^{l(\mu)} (q; q)_\mu, h_{\mu'} \left[ \frac{X}{1-t} \right],$$  \hspace{1cm} 1.1

where for any integer $k \geq 0$ we have $(q; q)_k = (1 - q)(1 - q^2) \cdots (1 - q^k)$.

**Proof**

In [16] Chapter 8. Integral Forms (see (8.4) Remark (iii)) Macdonald proves that

$$J_\mu[X; 1, t] = (t; t)_\mu e_{\mu'}(X).$$  \hspace{1cm} 1.2

Our definition of the Modified Macdonald polynomial indexed by $\mu$ is

$$\bar{H}_\mu[X, q, t] = t^{n(\mu)} J_\mu \left[ \frac{X}{1-t/t} ; q, 1/t \right].$$  \hspace{1cm} 1.3

Thus setting $q = 1$ and using 1.2 gives

$$\bar{H}_\mu[X; 1, t] = t^{n(\mu)} (1/t; 1/t)_{\mu'} e_{\mu'} \left[ \frac{X}{1-t} \right].$$  \hspace{1cm} 1.4

Now for $\mu \vdash m$,

$$t^{n(\mu)} (1/t; 1/t)_{\mu'} = t^{n(\mu)} \prod_{i=1}^{l(\mu')} \prod_{j=1}^{\mu'_i} (1 - 1/t^j) = t^{n(\mu)} \prod_{i=1}^{l(\mu')} t^{-\mu'_i} t^{-(\mu'_i)} \prod_{j=1}^{l(\mu')} (-1)^{\mu'_i} \prod_{j=1}^{\mu'_i} (1 - t^j)$$

$$= (-t)^{-m} \prod_{i=1}^{l(\mu')} \prod_{j=1}^{\mu'_i} (1 - t^j) = (-t)^{-m} (t, t)_{\mu'},$$

and

$$e_{\mu'} \left[ \frac{X}{1-t} \right] = t^m e_{\mu'} \left[ -\frac{X}{1-t} \right] = (-t)^m h_{\mu'} \left[ \frac{X}{1-t} \right].$$  \hspace{1cm} 1.5

Combining 1.6 with 1.5 and 1.4 gives

$$\bar{H}_\mu[X; 1, t] = (t; t)_{\mu'} h_{\mu'} \left[ \frac{X}{1-t} \right].$$

Using the identity $\bar{H}_{\mu'}[X; q, 1] = \bar{H}_\mu[X; 1, q]$, we finally derive that

$$\bar{H}_{\mu'}[X; q, 1] = (q; q)_{\mu'} h_{\mu'} \left[ \frac{X}{1-t} \right].$$

But this is just another way of writing 1.1.
Since the dual of the $e$-basis with respect to the Hall scalar product is the forgotten basis, for any integer $m \geq 1$ we obtain the $e$-basis expansion

$$h_m \left[ \frac{X}{1-q} \right] = \sum_{\mu \vdash m} e_{\mu}[X] f_{\mu} \left[ \frac{1}{1-q} \right]. \quad 1.7$$

This given, in view of 1.1, to show that

$$\tilde{H}_\mu[X; 1 + q, 1] = \prod_{i=1}^{l(\mu)} (q; q)_\mu h_{\mu} \left[ \frac{X}{1-q} \right]_{q=1+q} \quad 1.8$$

is $e$-positive it is sufficient to prove the $e$-positivity of the polynomial $(q; q)_m h_m \left[ \frac{X}{1-q} \right]_{q=1+q}$ for every $m \geq 1$.

But that will be true if and only if we have

$$(q; q)_m f_{\mu} \left[ \frac{1}{1-q} \right]_{q=1+q} \in \mathbb{N}[q] \quad \text{(for all } \mu \vdash m). \quad 1.9$$

Remarkably computer data revealed that this fact is due to the general validity of the following identity

**Proposition 1.1**

*For any partition $\mu \vdash m$ we have*

$$f_{\mu} \left[ \frac{1}{1-q} \right] (q; q)_m = \prod_{i=1}^{l(\mu)} (q; q)_\mu \left( q-1 \right)^{m-l(\mu)} \Pi_\mu(q) \in \mathbb{N}[q].$$

Thus our final goal in this section will be the proof of this result. It develops that to do this we need auxiliary identities some of which are well known. For sake of completeness, we will give complete proofs of all the needed identities.

We will start by dealing with the factor $f_{\mu} \left[ \frac{1}{1-q} \right]$. To this end recall that the Jacobi Trudi identity gives $h_m[X] = \det \{e_j-i+1\}_{i,j=1}^m$. Since this matrix has 1’s in the subdiagonal and nothing but zeros below them, the only non vanishing determinantal terms are as indicated in the adjacent figure. We see there that each subset of the sub-diagonal is broken up into a union of strings of adjacent elements. Each string determines a cycle of the corresponding non vanishing term. The cycles are of the form $(i, i+1, i+2, \ldots, j)$ and contribute to the determinantal term the factor $(-1)^{j-i} e_{j-i+1}^\mu$.

This example produces the term $(-1)^{13-5} e_1 e_4 e_2 e_1 e_5$. In particular, it follows that the coefficient of the $e$-basis element $e_5 e_4 e_2 e_1^2$ is equal to the number of distinct rearrangements of the cycles yielding this product. Thus the general result may be written in the form

$$h_m[X] = \sum_{\mu \vdash m} (-1)^{m-l(\mu)} |DR(\mu)| e_{\mu}[X],$$

where $DR(\mu)$ is the set of distinct rearrangements $\alpha_1, \alpha_2, \ldots, \alpha_{l(\mu)}$ of $\mu_1, \mu_2, \ldots, \mu_{l(\mu)}$. On the other hand we have

$$h_m[X \cdot 1] = \sum_{\mu \vdash m} f_{\mu}[1] e_{\mu}[X].$$

Therefore, we have for any monomial $\gamma$,

$$f_{\mu}[\gamma] = \gamma^m (-1)^{m-l(\mu)} |\hat{R}(\mu)| = (-1)^{m-l(\mu)} \sum_{\rho \in \hat{R}(\mu)} \gamma^{|\rho|}. \quad 1.10$$
To use this identity, we need the following

**Proposition 1.2**

\[ f_\mu[X + Y] = \sum_{\alpha \cup \beta = \mu} f_\alpha[X] f_\beta[Y], \]

where \( \alpha \) as well as \( \beta \) are allowed to be empty partitions

**Proof**

We have

\[ e_\mu[X(Y + Z)] = \sum_{\mu \vdash n} h_\mu[X] f_\mu[Y + Z] = \sum_{k=0}^{n} e_{n-k}[X Y] e_k[X Z] = \sum_{k=0}^{n} \sum_{\alpha \vdash n-k} \sum_{\beta \vdash k} h_\alpha[X] f_\alpha[Y] h_\beta[X] f_\beta[Z]. \]

Equating coefficients of \( h_\mu[X] \) gives 1.11.

For a sequence of partitions \( \nu = (\nu^1, \nu^2, \ldots) \) whose parts rearrange to \( \mu \) we will write \( \nu \in PR(\mu) \). Analogously, if \( p = (p^1, p^2, \ldots) \) is a sequence of compositions whose parts rearrange to \( \mu \) we will write \( p \in CR(\mu) \). In both cases we must allow the parts to be empty. In particular, 1.11 may be rewritten in the form

\[ f_\mu[X + Y] = \sum_{(\alpha,\beta) \in PR(\mu)} f_\alpha[X] f_\beta[Y]. \]

Iterating this relation we obtain, for arbitrary \( n \)

\[ f_\mu[x_1 + x_2 + \cdots + x_n] = \sum_{(\nu_1, \nu_2, \ldots, \nu^n) \in PR(\mu)} \prod_{i=1}^{n} f_{\nu_i}[x_i]. \]

Using 1.10 this may be rewritten as

\[ f_\mu[x_1 + x_2 + \cdots + x_n] = \sum_{(\nu_1, \nu_2, \ldots, \nu^n) \in PR(\mu)} \prod_{i=1}^{n} (-1)^{|\nu_i| - l(\nu^i)} \sum_{\rho^i \in R(\nu^i)} x_{i}^{\rho^i} \]

\[ = (-1)^{|\nu| - l(\mu)} \sum_{(\nu_1, \nu_2, \ldots, \nu^n) \in PR(\mu)} \prod_{i=1}^{n} \sum_{\rho^i \in R(\nu^i)} x_{i}^{\rho^i} \]

\[ = (-1)^{|\nu| - l(\mu)} \sum_{p = (p_1, p_2, \ldots, p_n) \in CR(\mu)} x_{1}^{p_1} x_{2}^{p_2} \cdots x_{n}^{p_n}. \]

Now let \( n \to \infty \) to get

\[ f_\mu[x_1 + x_2 + x_3 + \cdots] = (-1)^{|\nu| - l(\mu)} \sum_{p = (p_1, p_2, \ldots) \in CR(\mu)} x_{1}^{p_1} x_{2}^{p_2} x_{3}^{p_3} \cdots. \]

To compute \( f_\mu \left[ \frac{1}{1-q} \right] \) we need only make the replacement \( x_i \to q^{i-1} \) obtaining

\[ f_\mu \left[ \frac{1}{1-q} \right] = (-1)^{|\nu| - l(\mu)} \sum_{p = (p_1, p_2, \ldots) \in CR(\mu)} (q^0)^{p_1} (q^{1})^{p_2} \cdots (q^{i-1})^{p_i} \cdots. \]

Now in the case \( \mu = (4, 3, 3, 2, 1, 1) \) one of the possible summands is

\[ p = ((1, 3), (2, 4, 1), (\phi), (3)) \in CR((4, 3, 3, 2, 1, 1)). \]

The corresponding term in the sum is the monomial

\[ (q^0)^{(4,3)} (q^{1})^{(2,4,1)} (q^{2})^{(3)} (q^{3})^{(3)} = (q^0)^{1} (q^{0})^{3} \cdot (q^{1})^{2} (q^{1})^{4} (q^{1})^{1} \cdot (q^{3})^{3}. \]

We clearly will obtain this case with the specialization \( a = (1, 3, 2, 4, 1, 3) \) and

\[ i_1 = i_2 = 0, \ i_3 = i_4 = i_5 = 1, \ \text{and} \ i_6 = 3 \]
components of $\mu$ and rewrite 1.15 in the form

Thus we can rewrite 1.17 as

Thus from 1.14 it follows that, in full generality, the construction we gave for our particular example can be extended to obtain a bijection between the terms in the right hand sides of 1.13 and 1.14. This assures the equality of the resulting rational functions of $q$.

Now, by a very simple trick, we can obtain an explicit formula for the rational function

The standard step is to simply make the change of variables

and rewrite 1.15 in the form

Thus from 1.14 it follows that

This given we are now ready to give our

**Proof of Proposition 1.1**

Since the components of $a = (a_1, a_2, \cdots, a_{l(\mu)})$ are only a rearrangement of the components of $\mu \vdash m$, the integers

are distinct and therefore form a subset of $\{1, 2, 3, \ldots, m\}$. Let us then set

Thus we can rewrite 1.17 as

This completes the proof of our proposition and the proof of conjecture (1).
2. Proof of conjecture (2)

Letting \( \epsilon \) denote the variable which takes the value \(-1\) outside the plethystic bracket, the modified Hall-Littlewood operator \( B_\alpha \) used in the statement of the compositional shuffle conjecture [13] is defined by setting, for any symmetric function \( F[\underline{X}] \),

\[
B_\alpha F[\underline{X}] = F[\underline{X} + \epsilon \frac{1-2}{\epsilon} \sum_{r \geq 0} z^r e_r[\underline{X}]_{\mid_{z=\epsilon}}].
\]  

In Haglund’s book [12] it is shown that the symmetric polynomial

\[
B_\mu[\underline{X}, q] = B_\mu 1 = B_{\mu_1} B_{\mu_2} \cdots B_{\mu_{(\mu)}} 1
\]

indexed by any partition \( \mu \) is, up to a factor, the LLT polynomial of a Balanced path indexed by \( \mu \). A Dyck path \( D \) is said to be balanced if and only if every North segment of \( D \) is immediately followed by an East segment of equal length.

Our goal in this section is to prove the following

**Theorem 2.1**

For any composition \( p \) we have

\[
B_p 1 \bigg|_{q=1+q} = \sum_{\mu \vdash |p|} e_\mu P_\mu(q)
\]

for some polynomial \( P_\mu(q) \in N[q] \).

We must mention that this \( e \)-positivity is quite surprising since for some compositions, the polynomial \( B_p 1 \) is not even Schur positive. We will derive this result from the following auxiliary fact.

**Proposition 2.1**

For any integer \( a \geq 1 \) and partition \( \mu \) we have

\[
B_a e_\mu \bigg|_{q=1+q} = \sum_{\nu \vdash |\mu|+a} e_\nu Q_{\nu,\mu,a}(q)
\]

for some polynomial \( Q_{\nu,\mu,a}(q) \in N[q] \).

In fact, since the definition in 2.1 gives \( B_a 1 = e_a \), we can proceed by induction on the number of components of \( p \) and assume that 2.2 is valid with \( P_\mu(q) \in N[q] \). Thus we may write

\[
B_p 1 = \sum_\mu e_\mu P_\mu(q - 1).
\]

Then for any integer \( a \) we derive that

\[
B_a B_p 1 = \sum_\mu B_a e_\mu P_\mu(q - 1),
\]

and 2.3 gives

\[
B_a B_p 1 = \sum_\mu \left( \sum_{\nu \vdash |\mu|+a} e_\nu Q_{\nu,\mu,a}(q - 1) \right) P_\mu(q - 1) = \sum_\nu e_\nu \sum_\mu Q_{\nu,\mu,a}(q - 1) P_\mu(q - 1).
\]

Since \( Q_{\nu,\mu,a}(q) P_\mu(q) \in N[q] \) for all \( \mu, \nu \) and \( a \), the identity in 2.4 completes the induction. This shows that we only need to prove Proposition 2.1.
To this end let us recall that for any expression $E$ we have
\[ s_\lambda[X + E] = \sum_{\mu \subseteq \lambda} s_{\lambda/\mu}[X] s_\mu[E]. \]

In the case that $E = y$ (a monomial) or $E = -y$ we obtain
\[ s_\lambda[X + y] = \sum_{k \geq 0} s_{\lambda/[k]}[X] y^k = \sum_{k \geq 0} h_k^\perp s_\lambda[X] y^k \]
and
\[ s_\lambda[X - y] = \sum_{k \geq 0} s_{\lambda/[k^\perp]}[X] (-y)^k = \sum_{k \geq 0} c_k^\perp s_\lambda[X](-y)^k. \]

and since the Schur functions are a basis, for any symmetric function $F[X]$ we can write
\[ F[X + y] = \sum_{k \geq 0} y^k h_k^\perp F[X] \quad \text{and} \quad F[X - y] = \sum_{k \geq 0} (-y)^k c_k^\perp F[X]. \]

Therefore, we derive
\[ F[X + e^{\frac{1-a}{z}}] = F[X + \frac{q}{z} - \frac{eq}{z}] = \sum_{r,s \geq 0} (-1)^s q^r e_{a+r+s}^\perp h_s^\perp F[X]. \]

The operator in 2.1 can then be rewritten as
\[ B_a = \sum_{r,s \geq 0} (-1)^s q^r e_{a+r+s}^\perp h_s^\perp. \]

To compute $B_a e_\mu$ we will depict $e_\mu$ as the skew Schur function obtained by juxtaposing, corner to corner and on top of each other, columns of lengths $\mu_1, \mu_2, \ldots, \mu_{\ell(\mu)}$. For instance the $e$-basis element $e_3 e_2 e_1$ will be depicted as the leftmost skew diagram in the following display. Given $r \geq 0$ and $s \geq 0$, we now construct a set $T_{\mu,r,s}$ of labeled tableaux of shape $\mu$ as drawn on the left of the following display. Each element $S \in T_{\mu,r,s}$ has a weight $wt(S)$ which will give
\[ B_a e_\mu \mid_{q = 1+q} = \sum_{0 \leq r+s \leq \ell(\mu)} e_{a+r+s} \sum_{S \in T_{\mu,r,s}} wt(S). \]

To construct $T_{\mu,r,s}$, first select $s$ cells which are on the top of their columns and inscribe the cells with “$-1$”. For instance, if $s = 2$, we have the following three choices for filling the example on the left:

Next choose $r$ cells so that they form a skew column in the remaining shape, and for each cell choose whether to inscribe it with a “$1$” or “$q$”. One example with $s = 2$ and $r = 3$ is given by the left member of the rightmost pair. Let $\lambda(S)$ be the partition whose parts are the numbers of empty cells in the columns of $S$. The above example would then produce the partition $(1)$ since there is one empty cell in column $2$. Let $|S|$ be the product of the entries in the cells of $S$. The weight of this object is computed by taking
\[ wt(S) = |S| \cdot e_{\lambda(S)}. \]

The example above would give $wt(S) = (-1) \cdot q \cdot 1 \cdot q \cdot (-1) \cdot e_1 = q^2 e_1$ and $wt(S') = (-1) \cdot q \cdot 1 \cdot q \cdot 1 \cdot e_1 = -q^2 e_1$. 
We now show that \( \sum_{S \in T^{r,s}_\mu} wt(S) \) is a positive polynomial by a sign-reversing involution. Given \( S \), scan from left to right for the first top cell in a column that is either inscribed with a 1 or a \(-1\). Switch the 1 into a \(-1\) in the first case, and switch the \(-1\) to a 1 in the second case. If no such entry exists, leave the tableaux fixed. This is clearly an involution, and it is sign-reversing since we are negating the value of \(|S|\), yet preserving the number of \(q\)'s. This involution pairs off the two labeled diagrams in the above display.

Let \( U^{r,s}_\mu \) be the subset of \( T^{r,s}_\mu \) with the condition that if the top cell of a column is labeled, then it contains a \(q\). Thus we have

\[
B_a e_\mu \bigg|_{q=1+q} = \sum_{r+s \leq |\mu|} e_{a+r+s} \sum_{S \in U^{r,s}_\mu} wt(S),
\]

which is a positive polynomial, completing our proof of Proposition 2.1.

Theorem 2.1 has a beautiful application. To state it we need some auxiliary facts and notation.

In a recent posting in the ArXiv Mike Zabrocki [18] states a general conjecture asserting that a certain \( S_n\) module has the symmetric function appearing in the Delta conjecture [14] as Frobenius Characteristic. Our application of Theorem 2.1 is that Zabrocki’s Conjecture implies that a submodule of Zabrocki’s module exhibits the \(e\)-positivity phenomenon.

To define Zabrocki’s submodule we consider the vector space \( R_n[X, \Theta] = Q[x_1, \ldots, x_n; \theta_1, \ldots, \theta_n] \), with the \(x_i\) commuting variables, the \(\theta_j\) anti-commuting and commuting with the \(x_i\). This space is itself an \( S_n \) module under the diagonal action. The latter is simply defined by letting a permutation \( \sigma \in S_n \) send \( x_i \) into \( x_{\sigma(i)} \) and \( \theta_i \) into \( \theta_{\sigma(i)} \). The Zabrocki submodule is none other than the analogue of the Diagonal Harmonics module when \( R_n[X;Y] = Q[x_1, \ldots, x_n; y_1, \ldots, y_n] \) with the \(x_i, y_j\) commuting variables. This given, Zabrocki’s submodule is the quotient of \( R_n[X; \Theta] \) by the ideal \( I_n \) generated by the diagonal invariants in \( R_n[X, \Theta]^{S_n} \) with vanishing constant term. Let us call them the \( x, \theta\)-Coinvariants. Or equivalently, by taking the the orthogonal complement of \( I_n \), the \( x, \theta\)-Diagonal Harmonics. Now it follows from Zabrocki’s conjecture that the Frobenius Characteristic of this module is the symmetric function

\[
DH_{x,\theta}[X; q] = \sum_{k=1}^{n} (-t/q)^{n-k} \omega E_{n,k}[X; 1/q].
\]

The polynomials \( E_{n,k}[X; q] \) were originally defined in [6] by the Pochhammer expansion

\[
e_n[X^{1-z}_{1-q}] = \sum_{k=1}^{n} \binom{z}{q}^k \omega E_{n,k}[X; q].
\]

In [13] it is shown that

\[
E_{n,k}[X, q] = \sum_{\alpha=1}^{l(\alpha)=n} C_{\alpha_1} C_{\alpha_2} \cdots C_{\alpha_k} 1, \quad (l(\alpha) = k),
\]

where the operators \(C_\alpha\) are defined by setting for any symmetric function \(F[X]\),

\[
C_\alpha F[X] = (-\frac{1}{q})^{|\alpha|-1} F[X - \frac{1}{z} - \frac{1}{z}] \sum_{r \geq 0} z^r h_r[X].
\]

It turns out that we can derive an expression similar to 2.8 for the polynomial

\[
\tilde{E}_{n,k}[X, q] = (-1/q)^{n-k} \omega E_{n,k}[X, 1/q].
\]

To see this, notice first that the operators \(B_a\) defined in 2.1 may also be defined by setting \(B_a = \omega B_a \omega\) with

\[
\tilde{B}_a F[X] = F[X - \frac{1}{z}] \sum_{r \geq 0} z^r h_r[X].
\]
This given, notice that replacing $q$ by $1/q$ in 2.9 we obtain
\[ C_1^{1/q} F[X] = (-q)^{n-1} F[X - \frac{1-k}{z}] \sum_{r \geq 0} z^r h_r[X] \bigg|_{z=1} = (-q)^{n-1} \tilde{B}_n F[X], \]
and 2.8 becomes
\[ E_{n,k}[X, 1/q] = (-q)^{n-k} \sum_{\alpha | n} \tilde{B}_{\alpha_1} \tilde{B}_{\alpha_2} \cdots \tilde{B}_{\alpha_k}. \]
Since $\omega 1 = 1$, the equality in 2.13 can also be rewritten as
\[ \omega E_{n,k}[X, 1/q] = (-q)^{n-k} \sum_{\alpha | n} \omega \tilde{B}_{\alpha_1} \omega \tilde{B}_{\alpha_2} \omega \cdots \omega \tilde{B}_{\alpha_k} \omega 1, \]
and 2.10 becomes
\[ \tilde{E}_{n,k}[X, q] = \sum_{\alpha | n} B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} 1. \]
Thus the Zabrocki conjecture in 2.6 may be also computed by the formula
\[ DH_{x, \theta}[X; q] = \sum_{k=1}^{n} t^{n-k} \sum_{\alpha | n} B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} 1. \]
Since Theorem 2.1 states that all the symmetric polynomials $B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} 1$ exhibit the $e$-positivity phenomenon, it follows from 2.16 that so does the polynomial $x \theta DH[X; q, t]$.
Since computer data shows that the summands in 2.16 are not necessarily Schur positive when $\alpha$ is not a partition, the question remains as to what mechanism causes their sum to be Schur positive. The answer is quite simple. In fact, let us recall that in Haglund’s book [12] it is shown that the polynomial $B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} 1$ is the LLT polynomial of a Dyck path only when the $\alpha_i$’s are the components of a partition. However, it follows by combining Zabrocki’s conjecture in [18], the conjecture in 2.6 and the Delta conjecture at $t = 0$ (see [7]) that the left-hand side 2.6 can be rewritten in the two forms
\[ DH_{x, \theta}[X; q] = \sum_{k=1}^{n} z^{n-k} \sum_{\alpha | n} B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} 1 = \sum_{D \in \mathcal{D}_n} \mu_{\text{area}(D)} LLT_D[X; q] Hag(D; z) \bigg|_{z=1} \]
where $Hag(D; z)$ is the Haglund factor of $D$. If the component $D_i$ is the co-area of the $i$th North step of $D$ and $a_i$ denotes the number of area cells in the $i$th row, then we can write
\[ Hag(D; z) = \prod_{i=2}^{n} (1 + z/t^{a_i})^{\chi(D_i = D_{i-1})}, \quad \text{thus} \quad \mu_{\text{area}(D)} Hag(D; z) = \prod_{i=2}^{n} t^{a_i} \chi(D_i \neq D_{i-1}) \prod_{i=2}^{n} (t^{a_i} + z)^{\chi(D_i = D_{i-1})}. \]
In order that the latter factor contributes to the sum in 2.17 it must be that whenever $D_i \neq D_{i-1}$ then $a_i = 0$. That means the path $D$ must hit the diagonal every time $D_i \neq D_{i-1}$. This forces $D$ to be a Balanced Dyck path and the equality in 2.17 to be none other than
\[ DH_{x, \theta}[X; q] = \sum_{k=1}^{n} z^{n-k} \sum_{\alpha | n} B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} 1 = \sum_{D \in \mathcal{D}_n \text{ balanced}} LLT_D[X; q] z^{\sum_{i=2}^{n} \chi(D_i = D_{i-1})}. \]
Since $\sum_{i=2}^{n} \chi(D_i = D_{i-1}) = n - k$ if $D$ hits the diagonal in $k$ places, it follows that we also have the equality
\[ \sum_{\alpha | n} B_{\alpha_1} B_{\alpha_2} \cdots B_{\alpha_k} 1 = \sum_{D \in \mathcal{D}_n \text{ balanced with } k \text{ hits}} LLT_D[X; q]. \]
This clearly explains why the left hand side ends up being Schur positive.
3. Some consequences and conjectures

First and foremost, the $e$-positivity phenomenon suggests that an action of $S_n$ is involved, yet so far none of our proofs uses anything of the sort. Let us recall that any finite group action breaks up into a direct sum of transitive group actions. Moreover, transitive submodules of group actions are none other than the orbits of the action. However orbit actions are equivalent to left coset actions. Thus the character of an orbit action, as an element of the acting group algebra, can be simply expressed in terms of the stabilizer of any element of the orbit. In the case of $S_n$, if all these stabilizers happen to be Young subgroups, then the Frobenius characteristic of the character of the action must be $h$-positive or $e$-positive.

These ideas suggest a simple computer exploration. Namely, finding out what is being acted upon. We can do this by computing the Hilbert series of the conjectured module and set $t, q = 1$. In those cases where we obtain a sequence of integers, the resulting data is a good candidate for the encyclopedia of integer sequences. The simplest case is $\nabla e_n$ which is the symmetric function side of the shuffle conjecture, now a theorem [2]. Since the combinatorial side is obtained as a sum of all $n \times n$ Dyck paths LLT’s, $\nabla e_n$ itself should also exhibit the $e$-positivity phenomenon. Computer data strongly confirms that it does. Thus it seems worthwhile to find out what is the cardinality of the set of objects that $S_n$ should be acting upon in this case. Doing this exploration with $\nabla e_n \big|_{q=2}$ for $n = 1, 2, 3, 4, 5, 6, 7$ we get

$$1, 4, 38, 728, 26704, 1866256, 251548592, \ldots$$

Entering this sequence in the encyclopedia returns an avalanche of hits. The immediate answer is

The number of connected graphs on $n + 1$ nodes.

There is even a connection with Novak’s Free probability notes [17] where we can find a list of all the 38 connected graphs on 4 nodes. A further search more closely connected with the replacement $q \rightarrow 2$ yields the papers of Kreweras [15] and Gessel-Wang [10] who now appear to have hit the tip of an iceberg.

Finding a bi-graded $S_n$-Module with Frobenius characteristic the $e$-basis expansion of the polynomial $\nabla e_n \big|_{q=1+q}$ would make an interesting research problem indeed. Likewise, the conjecture that the LLT polynomials of Dyck paths exhibit the $e$-positivity phenomenon suggests that such a module should exist also in these cases. But before we focus more closely on Dyck path LLT’s it will be good to recall the definition of the ingredients that enter in their construction.

In the adjacent display we have our depiction of a Parking Function. To begin we have drawn a Dyck path $D$ in the $6 \times 6$ lattice square $R_6$. This is a path that goes from $(0, 0)$ to $(6, 6)$ by unit North and East steps always remaining weakly above the lattice diagonal (the yellow cells). We have also labeled the cells adjacent to the North steps of $D$ by the numbers 1, 2, 3, 4, 5, 6, usually referred to as “cars” in a column increasing manner. We have two statistics of a parking function called $area(PF)$ and $dinv(PF)$. The statistic $area(PF)$ is actually the area of $D$ which is the number of cells between the path and the lattice diagonal. The statistic $dinv(PF)$ is obtained as the total number of “primary” and “secondary dinvs. Two cars in the same diagonal yield a primary dinv if the one on the left is smaller than the one on the right. A secondary dinv is yielded by two cars when the one on the left is on a higher diagonal but adjacent to the diagonal of the car on the right, and the car on the left is larger than the car on the right. In the above example we have two primary dinvs 3, 4 and 5, 6 and the secondary dinv 5, 4. The word of the parking function, denoted $\sigma(PF)$, is the permutation obtained by reading the cars by diagonals from right to left starting from the highest and ending with the lowest. Thus for our example $\sigma(PF) = 165432$. The largest dinv is obtained when $\sigma(PF) = 654321$. This is the dinv of the Dyck path.
This given, the following identity gives us precise information as to the number of orbits (or $e$-basis elements) and their weight.

**Proposition 3.1**

*Suppose for a Dyck path $D$ in the $n \times n$ lattice square we have the expansion*

$$LLT_D[X; 1 + q] = \sum_{\mu \vdash n} e_\mu[X] P_\mu(q),$$  \hspace{1cm} 3.1

*then*

$$\sum_{\mu \vdash n} P_\mu(q) = (1 + q)^{dinv(D)}.$$  \hspace{1cm} 3.2

**Proof**

Recall that by definition the LLT polynomial of a Dyck path $D$ is given by the formula

$$LLT_D[X; q] = \sum_{D(PF) = D} q^{dinv(PF)} F_{\text{pides}(PF)}[X]$$  \hspace{1cm} 3.3

where the sum is over all Parking Functions supported by $D$. The last factor here is the Gessel quasi-symmetric function basis element indexed by $\text{pides}(PF)$, the composition giving the descent set of the inverse of the word $\sigma(PF)$. Since LLT polynomials are symmetric it follows from a theorem of Gessel [9] and 3.3 that

$$\langle LLT_D[X; q], s_1^n[X] \rangle = \sum_{D(PF) = D} q^{dinv(PF)} \chi(\text{pides}(PF) = 1^n),$$  \hspace{1cm} 3.4

where the left hand side of this identity is a Hall scalar product of two symmetric functions. On the right hand side the equality $\text{pides}(PF) = 1^n$ can only happen when $\sigma(PF) = n \cdots 31$. This reduces the sum in 3.4 to a single term and in that case we have $dinv(PF) = inv(D)$. Thus 3.4 may also be rewritten as

$$\langle LLT_D[X; 1 + q], s_1^n[X] \rangle = (1 + q)^{dinv(D)}.$$  \hspace{1cm} 3.5

But then 3.2 follows from 3.1 since for any $\mu \vdash n$ we have $\langle e_\mu[X], s_1^n[X] \rangle = 1$. This completes our proof.

Guided by the identity in 3.2 and supported by computer data we have been led to the following

**Conjecture 3.1**

Given a Dyck path $D$ in the $n \times n$ lattice square $R_n$, the following algorithm constructs the $e$-basis expansion of the polynomial $LLT_D[X; 1 + q]$. Draw the parking function $PF$ with $\sigma(PF) = n \cdots 31$ and determine the set of pairs of cars $(a, b)$ with $a < b$ producing a dinv. Call this “$\text{dinvset}(D)$”. Constructs the set of pairs of cars $a < b$ placed one above the other in one of the columns of $D$ and call it “$\text{forced}(D)$”. Then our final product can be written in the form

$$LLT_D[X; 1 + q] = \sum_{S \subseteq \text{dinvset}(D)} q^{|S|} e_{\mu(S)}.$$  \hspace{1cm} 3.5

To construct the terms of this sum repeat the following 4 steps until all the subsets of $\text{dinvset}(D)$ have been processed. Begin by setting $out := 0$.

1. **Choose a subset $S$ of $\text{dinvset}(D)$ and set $temp := q^{|S|}$.

2. **Using all the pairs in $S$ and $\text{forced}(D)$ to construct the poset $\Pi = (\{1, 2, \ldots, n\}, \preceq)$ (here each pair $(a, b) \in S \cup \text{forced}(D)$ must be interpreted as $a < b$).

3. **Recursively start by setting $\Pi' := \Pi$ and $\text{max}(\Pi') := n$ and repeat the following commands until $\Pi'$ has no more elements:

   (i) Determine the downset of $\text{max}(\Pi')$, (the set of elements of $\Pi'$ that are $\preceq \text{max}(\Pi')$)
   (ii) if the size of this downset is $k$ do $temp := temp \times e_k$.
   (iii) Remove from $\Pi'$ all the elements of $\text{downset}(\text{max}(\Pi'))$ and let the result be the new $\Pi'$.

4. **Save the result of steps (1), (2), (3) by the command $out := out + temp$.**

We conjecture that at the completion of this algorithm $out$ will give the right hand side of 3.5.
This is best illustrated by an example. In the display on the right we have depicted in $R_6$ the same Dyck path $D$ we had in our previous display. However we have labeled the cells adjacent to the North steps of $D$ by the permutation $654321$. More precisely, in diagonal 2 we placed 654, in diagonal 1 we placed 32 and in diagonal 0 we placed 1. This labeling gives the Parking Function with the highest dinv. It causes a total of 5 dinvs which include primary pairs $(2, 3), (4, 5), (4, 6), (5, 6)$ and a secondary pair $(3, 4)$. Thus we obtain

$$a) \text{ dinvset}(D) = \{(2, 3), (4, 5), (4, 6), (5, 6), (3, 4)\} \quad \text{and} \quad b) \text{ forced}(D) = \{(1, 2), (2, 4), (3, 5)\}. \quad \text{3.6}$$

Since the number of subsets of $\text{dinvset}(D)$ is $2^5$ we will only carry out steps $(1), (2), (3)$ of the above algorithm in the special cases $S = \{(4, 6)\}, S = \{(2, 3), (4, 5), (4, 6)\}, S = \{(3, 4), (4, 6)\}, S = \{(3, 4)\}$ and obtain the following four posets.

![Diagram](image)

Processing these posets gives the following four terms of the polynomial in 3.5 for our choice of $D$:

$$q e_4 e_2, \quad q^3 e_4 e_2, \quad q^2 e_5 e_1, \quad q e_3 e_2 e_1. \quad \text{3.7}$$

Using the definition in 3.3 and replacing each Gessel Fundamental by the Schur function indexed by the same composition yields the $e$-basis expansion (see [3] and [8] for this operation)

$$LLT_D[X; 1 + q] = (q^5 + 4q^4 + 5q^3 + 2q^2)e_6 + (q^4 + 4q^3 + 4q^2 + q)e_5 e_1 + (q^3 + 2q^2 + q)e_4 e_2 + (q^2 + q)e_4 e_1^2 + (q + 1)e_3 e_2 e_1$$

and we can verify the terms in 3.7 do occur in the $e$-basis expansion of our polynomial.

It turns out that this conjecture has a surprising consequence that can actually be proved. This occurs for the Dyck path $D$ with no area as illustrated in the adjacent display. It easily seen that in this case we have no forced pairs and

$$\text{dinvset}(D) = \{(a, b) : 1 \leq a < b \leq n\}. \quad \text{3.8}$$

Let us see what are the possible selections of subsets $S$ of $\text{dinvset}(D)$ yielding

$$\text{downset}(6) = \{t_1 < t_2 < t_3 < 6\}. \quad \text{3.9}$$

Recall that in this case the first factor of the final $e$-basis element is $e_4$. In order that we will be able to recognize other factors it will be necessary to work with the original polynomial $LLT_D[X; q]$. Thus rewriting $q = q - 1 + 1$ we can interpret “$q - 1$” as including a given pair in $S$ and “1” as not including it. So the question is how can we guarantee that our choices will result in $\text{downset}(6)$ to be as in 3.9. We also must recall that each new edge $(a, b)$ we add to $S$ must satisfy $(a < b)$. Thus to assure that $t_3$ is in $\text{downset}(6)$ we need only add to $S$ the edge $(t_3, 6)$. Recursively, to assure that $t_2$ is in $\text{downset}(6)$ we need only add to $S$ at least one of the edges $(t_2, t_3)$ or $(t_2, 6)$. Finally to assure that $t_1$ is in $\text{downset}(6)$ we need at least one of the three edges $(t_1, t_2), (t_1, t_3)$, and $(t_1, 6)$.

This given, the contribution to the $q$ factor of the final polynomial resulting from these three choices is $(q - 1)(q^2 - 1)(q^3 - 1)$. The $-1$ in each case results from the fact that in each case we are required to pick at least one of the given choices, i. e. picking none is not permitted.
But we are not done yet with powers of $q$. Set

$$R = \{1, 2, 3, 4, 5, 6\} - T$$

with $T = \text{downset}(6)$. Without affecting $\text{downset}(6)$ we can add to $S$ any edges $(t, r)$ with $t \in T$ and $r \in R$ with $t < r$. For instance if $\{t_1, t_2, t_3\} = \{2, 4, 5\}$ then $R = \{1, 3\}$. The insertion or not insertion of such an edge results in an additional $q$ for each of the available choices. For this particular instance of $t_1, t_2, t_3$ we have only one choice $(2, 3)$. Picking or not picking this pair yields a $q$ factor. In the display below we have listed all possible choices of $\{t_1, t_2, t_3\}$ (vertically), their remainder $R$ and the power of $q$ they contribute.

We can thus easily see that the $q$-factor that accounts for all the choices that do not affect the size of $\text{downset}(6)$, is none other than the polynomial that $q$-counts by area the partitions that are contained in a $3 \times 2$ rectangle, that is the $q$-binomial coefficient

$$\left[\frac{6 - 1}{4 - 1}\right]_q.$$  \hspace{1cm} (3.10)

To complete the contribution due to this $\text{downset}$ we need to observe that when $D$ is the no area path in the lattice square $R_n$ we may as well use the notation

$$\text{LLT}_D[X; q] = \text{LLT}_n[X; q].$$ \hspace{1cm} (3.11)

The idea is that the choices we were forced to make to assure our particular $\text{downset}$ will not affect the remaining construction. More precisely, at this point it is natural to assume that whatever must be added to complete the contribution of this $\text{downset}$ will be recursively provided by the construction of the polynomial $\text{LLT}_{6-4}[X; q]$.

In summary, the contribution to the polynomial $\text{LLT}_6[X; q]$ due to all $\text{downsets}$ of size 4 should be

$$e_4[X] (q - 1)(q^2 - 1)(q^3 - 1)\left[\frac{6 - 1}{4 - 1}\right]_q \text{LLT}_{6-4}[X; q].$$ \hspace{1cm} (3.12)

It turns out that the validity of the idea that suggested 3.12 is confirmed by the following recursion we can actually prove.

**Theorem 3.1**

For any $n \geq 2$ we have

$$\text{LLT}_n[X; q] = \sum_{k=1}^n e_k[X] (-1)^k(q; q)_{k-1} \left[\frac{n - 1}{k - 1}\right]_q \text{LLT}_{n-k}[X; q].$$ \hspace{1cm} (3.13)

**Proof**

To begin we must observe that in this particular case the LLT polynomials have explicit expressions. In fact, a moment’s reflection reveals that for each $n \geq 1$ we have

$$\text{LLT}_n[X; q] = \tilde{H}_{[n]}[X; q, 1] = (q; q)_n h_n\left[n\right]_{1-q}.$$ \hspace{1cm} (3.14)
Using this identity in 3.13 the definition of the $q$-binomial coefficient gives

$$(q; q)_n h_n \left[ \frac{X}{1-q} \right] = \sum_{k=1}^{n} e_k [X] (-1)^{k-1} (q; q)_{k-1} \frac{(q; q)_{n-1}}{(q; q)_{k-1}} (q; q)_{n-k} h_{n-k} \left[ \frac{X}{1-q} \right].$$

Carrying out all the obvious cancellations we get

$$(1 - q^n) h_n \left[ \frac{X}{1-q} \right] = - \sum_{k=1}^{n} e_k [X] (-1)^k h_{n-k} \left[ \frac{X}{1-q} \right].$$

However this is the same as

$$(1 - q^n) h_n \left[ \frac{X}{1-q} \right] = - \sum_{k=0}^{n} h_k [-X] h_{n-k} \left[ \frac{X}{1-q} \right] + h_n \left[ \frac{X}{1-q} \right].$$

Or better

$$-q^n h_n \left[ \frac{X}{1-q} \right] = - \sum_{k=0}^{n} h_k \left[ -\frac{X+qX}{1-q} \right] h_{n-k} \left[ \frac{X}{1-q} \right] = -h_n \left[ \frac{qX}{1-q} \right].$$

Completing our proof.

In trying to prove the recursion of Theorem 3.1 for general LLT’s it was discovered that some of the terms recursively constructed were column LLT’s. These findings resulted in further discoveries.

**a) The “areaprime” construction of column LLT’s**

We will start by recalling the construction of the “areaprime” image of a Parking function. In the adjacent display we have a parking function supported by a Dyck path $D$ and its areaprime image. To obtain this image the first step is to construct the permutation on the diagonal. This is done by reading the cars in $PF$ by diagonals from left to right starting from the lowest and ending with the highest. This done, we determine the positions of the Blue crosses. Each cross is determined by a pair $(a, b)$ of cars $a < b$ with $a$ directly below $b$ in one of the North segments of $D$. These pairs are $(2, 4)$, $(4, 5)$, $(1, 6)$, $(3, 7)$. This gives us the positions of the crosses in the areaprime image. Once we draw the crosses we can easily obtain the sweep map image $\zeta(D)$ of $D$ by drawing the English partition whose removable corners are the cells that contain the crosses. The Blue squares in the area cells of $\zeta(D)$ are caused by the increasing diagonal pairs $(3, 5)$, $(3, 8)$, $(5, 8)$, $(5, 7)$, $(1, 7)$. In fact, these pairs of cars are precisely those producing the dinv of $PF$. We will show that column LLT’s generalize Dyck paths LLT’s by constructing them from the areaprime image of their parking functions. On the right in the above display we have the sweep map image $\zeta(D)$. The English partition above $\zeta(D)$ contains 4 crosses. If we remove the crosses created by the forced pairs $(4, 5)$ and $(1, 6)$ we are left with the left portion of this adjacent display. If we remove the 2 remaining crosses and replace the North and East unit steps touching each removed cross by a single diagonal step the Dyck path $\zeta(D)$ becomes a Shroeder path. Since these diagonal steps can never occur on the diagonal, the number of Shroeder paths thus obtained is one half of the Shroeder number. In the next display we will transform this areaprime with two removed crosses into what we will call a “column parking function”. The figure on the right of the above display gives an intermediate step.
To obtain it we simply insert in the original parking function, between cars 4, 5 and 1, 6, two separating blue dashes. In the left figure of the adjacent display we have simply reproduced only the columns with their blue dashes. In the right figure we have separated the cells containing 5 and 6 from their columns. This yields the right portion of the adjacent display. In summary, we have identified an areaprime with missing Blue crosses with a column parking function. In the display below we show how to construct the areaprime of a column parking function. We have labeled the cars by letters but we assume that the cars they represent are column increasing. To construct its areaprime we start by placing $\sigma(CPF)$ in the diagonal as shown. Next we insert a blue dot at the center of any cell defined by a potential dinv. For instance the dot in the 3rd row is the potential secondary dinv created when $b < c$. The dot in the 4th row is due to the primary dinv caused by $c < d$. Every one of the blue dots is caused by a potential dinv. The next step is to enclose all the blue dots by a Dyck path. Finally we add the blue crosses corresponding to the “forced” pairs of cars. We see that not all the removable corners of the English partition above the path have crosses. Confirming the fact that areaprimes of CPF are none other than areaprimes of PF with missing Blue crosses.

The LLT polynomial of a column LLT may the be written in the form

$$LLTC_{D,T}[X; q] = \sum_{\sigma \in CPF \subseteq PF_T} q^{dinv(\sigma)} F_{pides(\sigma)}[X]$$

where $T \subseteq forced(D)$, $PF_T$ is the family of column parking functions restricted to increase across the pairs of cars in $T$, and the composition $pides(\sigma)$ as usual gives the descent set of $\sigma(\sigma)^{-1}$. It also follows from our construction that the cardinality of these polynomials is given by the lower Schroeder number.

For later purposes it will be necessary to construct the same polynomial $LLTC_{D,T}[X; q]$ by starting from a general Dyck path $Z = \zeta(D)$ in the $n \times n$ lattice square $R_n$ and a subset of pairs $T \subseteq forced(D)$, by following the following rules.

1) **Draw the path Z where Z[j] gives the number of coarea cells in the jth row of Rn.**

2) A permutation $\sigma \in S_n$ is called $Z,T$-compatible if and only if for every pair $(r, s) \in T$ we have $\sigma_{n+1-r} < \sigma_{n+1-s}$.

3) For a given $Z,T$-compatible $\sigma$, a pair $1 \leq i < j \leq n$ contributes a unit of dinv if and only if $[i, j]$ is in the dinvset of $Z$ and $\sigma_{n+1-j} > \sigma_{n+1-i}$.

This given, we have the Schur expansion

$$LLTC_{D,T}[X; q] = \sum_{\sigma \in Z,F_T} q^{dinv(\sigma)} s_{pides(\sigma)}[X].$$

where $F_Z,T$ is the family of all the $Z,T$-compatible $\sigma \in S_n$, and “ $pides(\sigma)$” denotes the composition that gives the descent set of $\sigma^{-1}$.

It turns out that this formula can be used only for moderately small $n \leq 7$. To confirm the validity of our $e$-positivity conjectural expansions, we will use the Carlsson-Mellit super-fast manipulatorial way of computing the same symmetric polynomials. These new formulas have a complexity which is only linear in the number of steps of $Z$. We will present them here in full detail since they are somewhat difficult to extract out of the original paper [2].
These formulas are in terms of operators acting on the family $\Lambda[X; q, Y]$ of symmetric functions in the infinite alphabet $X$ with coefficients polynomials in $q$ and a variable alphabet $Y = \{y_1, y_2, y_3, \ldots\}$. The parameter $k$ will denote the size of the $Y$-alphabet.

The operators are $d_+, d-$ and the bracket $[d-, d+]$. They are all expressed in terms of the operators $T_i$ with $T_0$ acting as identity and $T_i$ (for $i \geq 1$) acting on $F \in \Lambda[X; q, Y]$ precisely as follows

$$T_i F = \frac{(q-1)y_i F + (y_{i+1} - qy_i) s_i F}{y_{i+1} - y_i},$$

where $s_i$ denotes the transposition that interchanges $y_i$ with $y_{i+1}$. Using 3.17, we set for $F \in \Lambda[X; q, y_1, \ldots, y_k]$ 

$$
\begin{align*}
 d_k^+ F &= T_1 T_2 \cdots T_k F [X + (q-1)y_k+1]. \\
 d_k^- F &= -F [X - (q-1)y_k] \sum_{i \geq 0} (-1/y_k)^i e_i \bigg|_{y_k^{-1}}. \\
 [d-, d^+]^k F &= \frac{d_{k+1}^+ d_k^- F - d_k^+ d_{k-1}^- F}{q-1}.
\end{align*}
$$

These operators are used in a very simple manner to obtain Dyck path LLT’s, column LLT’s and unicellular LLT’s. We need only carry out the details in a special case. In the display above we have the areaprime image of a typical column LLT. Here $Z = \{0, 0, 0, 2, 4, 4, 4, 6\}$. Notice that the English partition above $Z$ has three removable corners $[[2, 4], [4, 5], [6, 8]]$ but only the first two are marked. So in this case $T = [[2, 4], [4, 5]]$. The permutation we placed in the diagonal corresponds to the maximal column parking function, the one whose $dinv$ is the number of blue dots under $Z$. Using a “0” for each North step and a “1” for each East step, the sequence of steps of $Z$ can be represented by the following word

$$0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1.$$  \hspace{1cm} 3.21

We can immediately identify the three removable corners by simply locating the East steps followed by a North step. We purposely framed the marked ones. The final word we will use to guide the construction of the polynomial $LLTC_{D,T}[X; q]$ where $Z = \zeta(D)$ is obtained by replacing the framed pairs by a “2” obtaining the compressed word

$$0 \ 0 \ 0 \ 1 \ 2 \ 1 \ 2 \ 0 \ 0 \ 1 \ 1 \ 0 \ 1 \ 1.$$  \hspace{1cm} 3.22

Starting with the symmetric function $F = 1$, and proceeding from right to left, we apply a “$d_+$” for each 1, a “$d_-$” for each 0, and a “$[d-, d^+]$” for each 2, according to the following sequence of commands based on $W$ being the reverse of the word in 3.22:

1) set $k = 0$; set out $= 1$
2) for $i$ from 1 to 14 do
3)  if $W[i]=1$ then out $= d_k^+ out$; $k = k + 1$
4)  else if $W[i]=2$ then out $= [d-, d^+]^k out$;
5)  else out $= d_k^- out$; $k = k - 1$
6)  end if; end do;

The general case is easy to derive from this example. What is remarkable about this algorithm, is not only that it is of linear complexity but that it can be used in all three cases: Dyck path LLT’s ($T = forced(Z)$, all corners marked), column LLT’s ($T \subseteq forced(Z)$, some corners marked), unicellular LLT’s ($T = \phi$, no corners marked).

We will see later how this Carlsson-Mellit way of obtaining the column LLT polynomials can be used to check, for relatively large $n$, the recursive way of constructing our conjectured $e$-expansions.
b) The extension to column LLT’s of Conjecture 3.1.

Conjecture 3.2

Given a Dyck path $D$ in the $n \times n$ lattice square $R_n$ and a subset $T \subseteq \text{forced}(D)$, the following algorithm constructs the e-basis expansion of the polynomial $LLTC_{D,T}[X; 1 + q]$. Draw the parking function $CPF$ with $\sigma(CPF) = n \cdots 321$ and determine the set of pairs of cars $(a, b)$ with $a < b$ producing a dinv. Call this “dinvset($D$)”. Then our final product can be written in the form

$$LLTC_{D,T}[X; 1 + q] = \sum_{S \subseteq \text{dinvset}(D)} q^{|S|} e_\mu(S,T),$$

3.16

where the polynomial

$$\sum_{S \subseteq \text{dinvset}(D)} q^{|S|} e_\mu(S,T)$$

is obtained by repetitions of the 4 steps stated in Conjecture 3.1 except that in the second step the “forced($D$)” set of pairs must be replaced by $T$.

c) The conjecture that extends to all column LLT’s the recursion in Theorem 3.2

Our point of departure, as before, is a Dyck path $D$ in $R_n$. However, here it will be convenient to work entirely with the areaprime image. Thus we let $Z = \zeta(D)$ and fill, from bottom to top, all the diagonal cells with “cars” 1, 2, $\ldots$, $n$. Again, it will suffice to carry out our algorithm in a special case. Our choice is $Z = \{0, 1, 1, 1, 2, 2, 6, 6\}$, we also choose $T = [[2, 5], [6, 7]]$. These two pairs are represented by the blue crosses in our display. The blue dots below $Z$ represent, in the same manner, the pairs in dinvset($Z$). The first step in the algorithm is to determine the possible downsets of $n$. In our case, $n = 8$. This requires two important properties. A subset $S = \{i_1 < i_2 < \cdots < i_k = n\} \subseteq \{1, \ldots, n\}$ is “possible” if and only if every pair of elements $(i_j, i_{j+1})$ in $S$ is in the union of pairs $T \cup \text{dinvset}(Z)$; moreover, whenever $(i, j) \in T$ and $j \in S$ then $S$ must also contain $i$. These two properties are easy consequences of the manner our downsets are constructed. Our implementation of this algorithm yields only 12 possible downsets. The next step of our algorithm is to compute the weight of each possible subset $S$ of cars. This consists of a polynomial weight$_S(q)$. The computation of this weight is the most delicate step of our algorithm. We illustrate our procedures by displaying the output in the two cases $S_1 = \{2, 3, 5, 6, 7, 8\}$ and $S_2 = \{3, 4, 6, 7, 8\}$.

- $S_1 = \{2, 3, 5, 6, 7, 8\} \rightarrow e_6[X](q^2 - 1)(q - 1)^3 q^9$
- $1 \rightarrow 2, 3, 5, 6, 7, 8 \rightarrow q$
- $1 \rightarrow 2, 3, 5, 6, 7, 8 \rightarrow q$
- $1 \rightarrow 2, 3, 5, 6, 7, 8 \rightarrow q$
- $1 \rightarrow 2, 3, 5, 6, 7, 8 \rightarrow q$

- $S_2 = \{3, 4, 6, 7, 8\} \rightarrow e_6[X](q^2 - 1)(q - 1)^2 q^2$
- $1 \rightarrow 2, 3, 4, 6, 7, 8 \rightarrow q$
- $2 \rightarrow 1, 3, 4, 6, 7, 8 \rightarrow q - 1$
- $1, 3, 4, 6, 7, 8 \rightarrow q - 1$
- $1, 3, 4, 6, 7, 8 \rightarrow q - 1$

Our procedures yield the conjectured e-expansion of the polynomial $LLTC_{D,T}[X; q]$ in terms of $Z = \zeta(D)$ and the pairs in $T$. The basic principle is to use the split $q = q - 1 + 1$ and take $q - 1$ or 1 as the case may be. We take both when the result does not affect downset($n$), this yields a contribution of $q$ to its weight.

For instance, $S_1 - \{8\} = \{2, 3, 5, 6, 7\}$ that leaves $\{1, 4\}$. Since the additional edges $[2, 4]$ and $[3, 4]$ do not affect downset(8) we can add $q^2$ to its weight. The analogous argument holds for $S_2$, with the additional edges $[3, 5]$ and $[4, 5]$, as indicated in the adjacent display. To obtain the total weight of downset(8) for $S_1$, each of its elements are processed one at the time, as indicated in the left display. In the column of 2 there is only a blue dot in the row of 3 that accounts for the 1 in $[2, 3, 1]$. In the same column we see a blue cross in the row of 5 (see above display). This accounts for the 2 in $[2, 5, 2]$. Now since blue crosses do not contribute the weight, the weight of the column of 2 reduces to $q$. The reason for this is that we may or may not add the edge $[2, 3]$. When we process the column of 3 whether we add at least one of the edges $[3, 5]$ or
[3, 6] by connecting 3 to 5 or 6 we will guarantee the addition of 3 to downset(8). This accounts for the contribution of a $q^2 - 1$ of the column of 3. The reader should have no difficulty interpreting the outputs of the two cases in the previous display.

Once all the possible downsets are processed, we obtain

$$area'_{Z,T}[X, q] = \sum_{S \in \text{possible}_{Z,T}} e_{|S|}[X] \text{weight}_S(q) area'_{Z(S), T(S)}[X, q],$$

where the Dyck path $Z(S)$ and the residual $T(S) = \text{forced}(Z(S))$ are both obtained by simply deleting each element of $S$ in the diagonal of $R_n$ along with every cell of $R_n$ that is in the same row and column as that element. Notice that the validity of our conjectured $e$-expansion can now be checked for relatively large examples by computing all symmetric polynomials $area'_{Z,T}[X, q]$, and $area'_{Z(S), T(S)}[X, q]$ by the fast algorithm of Carlsson-Mellit.

It may be good to see that the deletions of rows and columns invariably yield that all the resulting $Z(S)$ are Dyck paths. By induction the following argument should be sufficient. Notice first that since the row and column we are deleting meet at a diagonal cell, except in the case this cell is the first or the last, the Dyck path $Z$ is broken up into three pieces. Leaving aside that limit case, in the left of the above display, we have depicted the first and the last of these three pieces in red, the middle piece in blue and the two deleted steps in pink. Now the new path $nZ$ consists of all the steps of $Z$ except the two deleted steps. To construct the resulting path $nZ$, we start by the steps of $Z$ up to the deleted North step. That is the first red piece of $Z$ in our display. Now $nZ$ will continue by the steps of the middle piece, in blue in our display. To compensate for the deleted North step, the middle piece of $Z$ must move vertically down one unit. Next comes the deleted East step. To compensate for the deleted steps $nZ$ must follow the steps of $Z$ shifted diagonally by one cell. To prove that $nZ$ is also a Dyck path, we need only show that it remains weakly above the diagonal. Since first piece of $Z$ and $nZ$ are identical and the last piece of $Z$ moves diagonally, neither can cross the diagonal. However, the middle part of $nZ$ cannot cross the diagonal either. In fact, the middle part of $Z$ must all be strictly above the diagonal, due to the center of deletion being a diagonal cell.

The marginal cases we left aside can be dealt with in the same way since in the first case the first piece is missing and in the second case the last piece is missing. The remaining pieces can be dealt with exactly as in the above argument.

It must be mentioned the finding of Kreweras in [15] that hit the tip of an iceberg is the surprising relation between $\nabla e_n$ and the Kreweras polynomials defined in [15] as the family satisfying the recursion and base case

$$P_{n+1}(q) = \sum_{i=0}^{n} \binom{n}{i} [i + 1]_q P_i(q) P_{n-i}(q), \quad P_0(q) = 1.$$ 

The precise relation between the Kreweras polynomials and $\nabla e_n$, that follows from our conjectured $e$-positivity phenomenon of Dyck path LLT’s, is

$$P_n(1 + q) = \partial^n_{p_1} (\nabla e_n[X; 1, 1 + q]).$$

It is clear that all these $e$-positivities create a variety of new problems in Algebraic Combinatorics. We might even say that the Dyck paths LLT conjecture by itself creates a major upheaval in this field.
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