A Problem of Coefficient Determination in Parabolic Equations Solved as Moment Problem

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Abstract

The problem is to find \( a(t) \) and \( w(x,t) \) such that

\[
\dot{w}_i = a(t)(w_x)_x + r(x,t)
\]

under the initial condition

\[
w(x,0) = \varphi(x)
\]

and the boundary conditions

\[
w(0,t) = 0, \quad w_x(0,t) = w_x(1,t) + \alpha w(1,t)
\]

about a region \( D = \{(x,t), \ 0 < x < 1, \ t > 0\} \). In addition it must be fulfilled

\[
\int_0^1 w(x,t)dx = E(t)
\]

where \( \varphi(x) \), \( r(x,t) \) and \( E(t) \) are known functions and \( \alpha \) is an arbitrary real number other than zero.

We also assume that the underlying space is \( L^2(D) \).

This problem is studied in [1]. Citing the abstract of this work: “this paper investigates the inverse problem of simultaneously determining the time-dependent thermal diffusivity and the temperature distribution in a parabolic equation in the case of nonlocal boundary conditions containing a real parameter and integral overdetermination conditions, and under some consistency conditions on the input data the existence, uniqueness and continuously dependence upon the data of the classical solution are shown by using the generalized Fourier method.”

In general the methods applied to solve the problem are varied. Other works that solve the parabolic equation but under different conditions are [2, 3, 4].

There is a great variety of inverse problems in which a parabolic equation must be solved and additionally we must determine an unknown parameter [5, 6, 7], to name a few examples.

The objective of this work is to show that we can solve the problem using the techniques of inverse moment problem. First deduce an exact expression for \( a(t)w(1,t) \). Then, we wrote \( w'(x,t) = a(t)w(x,t) \), and is resolved in a first step in numerical form the integral equation

\[
\iint_D G(x,t)w^{m}Exp(-(m+1)t)dxdt = \psi_1(m)
\]

where \( \psi_1(m) \) is written in terms of known expressions, and

\[
G(x,t) = -xw^x_0(x,t) - w^t_1(x,t)
\]

it is the function to be determined.

In a second step the following integral equation is solved in

1. Introduction

We want to find \( a(t) \) y \( w(x,t) \) such that

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under the initial condition

\[
w(x,0) = \varphi(x)
\]

and the boundary conditions

\[
w(0,t) = 0, \quad w_x(0,t) = w_x(1,t) + \alpha w(1,t)
\]

about a region \( D = \{(x,t), \ 0 < x < 1, \ t > 0\} \). In addition it must be fulfilled

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\int_0^1 w(x,t)dx = E(t)
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where \( \varphi(x) \), \( r(x,t) \) and \( E(t) \) are known functions and \( \alpha \) is an arbitrary real number other than zero.

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\[
\iint_D G(x,t)x^mExp(-(m+1)t)dxdt = \psi_1(m)
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where \( \psi_1(m) \) is written in terms of known expressions, and

\[
G(x,t) = -xw^x_0(x,t) - w^t_1(x,t)
\]

it is the function to be determined.

In a second step the following integral equation is solved in
numerical form

\[ \int_D w^*(x,t)K(m,n,x,t)dxdt = \psi_2(m,n) \]

where \( w^*(x,t) \) is the unknown function, \( \psi_2(m,n) \) is an expression in function of \( G(x,t) \) with \( K(m,n,x,t) \) known.

Both integral equations are solved numerically by applying the moment problems techniques.

Then we find an approximation for \( r(x,t) \), this approximation we write \( w_{Ap}(x,t) \), using the solution found in the second step and condition (3).

Finally we find an approximation for \( a(t) \) using \( a(t)w(1,t) \) and \( w_{Ap}(x,t) \).

2. Inverse Generalized Moment Problem

The d-dimensional generalized moment problem [8, 9, 10] and [11, 12] can be posed as follows: find a function \( f(x) \) on a domain \( \Omega \subset \mathbb{R}^d \) satisfying the sequence of equations

\[ \int_{\Omega} f(x)g_i(x)dx = \mu_i \quad i \in \mathbb{N} \]  

(4)

where \( (g_i) \) is a given sequence of functions lying in \( L^2(\Omega) \) linearly independent, and the sequence of real numbers \( \{\mu_i\}_{i \in \mathbb{N}} \) are the known data.

The moments problem of Hausdorff is a classic example of moment problems techniques.

Then we find an approximation for \( r(x,t) \), this approximation we write \( w_{Ap}(x,t) \), using the solution found in the second step and condition (3).

3. Resolution of the Parabolic Partial Differential Equation

We consider the equation \( w_t = a(t)(w_x)_x + r(x,t) \). If we integrate with respect to \( x \) between 0 and 1 we obtain

\[ \int_0^1 w_t dx = a(t)[w_x(1,t) - w_x(0,t)] + \int_0^1 r(x,t)dx \]

If we write \( r^*(t) = \int_0^1 r(x,t)dx \) and \( E'(t) = \frac{d}{dt}E(t) \) then

\[ E'(t) = a(t)(-aw(1,t)) + r^*(t) \]

Thus

\[ a(t)w(1,t) = \frac{r^*(t) - E'(t)}{\alpha} \]

(6)

On the other hand we consider the vector field

\[ F^* = (a(t)w_x, -a(t)w) = (w^*_x, -w^*) \]

Let \( u(i,z,x,t) \) be the auxiliary function

\[ u(i,z,x,t) = x^ie^{-(z+1)t} \]

Then

\[ \text{div}(uF^*) = (ua(t)w_x)_x - (ua(t)w)_t = \]

\[ = u_xa(t)w_x + ua(t)w_x - u_t a(t)w - ua'(t)w - ua(t)w \]

Also

\[ u\text{div}(F^*) = ua(t)w_x - ua'(t)w - ua(t)w_i \]

Moreover, as

\[ u\text{div}(F^*) = \text{div}(uF^*) - F^* \cdot \nabla u \]

\[ \int_D u\text{div}(F^*)dA = \int_D \int_D \text{div}(uF^*)dA - \int_D F^*\nabla u dA \]

(7)

where \( \nabla u = (u_x, u_t) \) besides

\[ \int_D \int_D \text{div}(uF^*)dA = \int_D ((aw^*_x)_x - (aw^*)_t)dA = \]

\[ \int_D u\text{div}(F^*)dA + \int_D ((uw^*_x)_x - (uw^*)_t)dA \]

(8)

Then of (7) and (8)

\[ \int_E (u_xw_x^* - u_tw^*)dA = \int_E F^*\nabla u dA \]

(9)

Can be proven that, after several calculations, (9) is written as...
We wrote
\[ G \]
We approach the solution \( \phi \)
We consider the base
We can apply the truncated expansion method detailed in \[ \text{International Journal of Applied Mathematical Research} \ 3 \]
previously calculated
necessary functions until reaching an orthonormal basis.

\[ \int_0^1 w^*(x,0)u(i,i,x,0)dx - \int_0^1 \frac{z+1}{i+1} \int_0^\infty (w^*(1,t)u(i,z,1,t) - w^*(0,t)u(i,z,0,t))dt = \]
\[ = - \int_0^1 \frac{z+1}{i+1} \int_0^\infty \int_0^\infty w^*_i(x,t)udtdx - \int_0^1 \int_0^\infty w^*_i(x,t)udtdx \]
and if \( z = i \) then
\[ \int_0^1 \int_0^\infty (-xw^*_i(x,t) - w^*_i(x,t))u(i,i,x,t)dt = \]
\[ \int_0^1 \int_0^\infty w^*(x,0)u(i,i,x,0) - \int_0^\infty (w^*(1,t)u(i,i,1,t) = \phi(i) \]

Note that
\[ w^*(x,0) = a(0)w(x,0) = a(0)\varphi(x) \]
\[ a(0) = \frac{r^*(0) - E'(0)}{\alpha \varphi(1)} \]
and
\[ w^*(1,t) = a(t)w(1,t) \]
previously calculated

We wrote \( G(x,t) = -xw^*_i(x,t) - w^*_i(x,t) \)
We solve the integral equation numerically
\[ \int_0^1 \int_0^\infty G(x,t)u(i,i,x,t)dt = \]
\[ = \int_0^1 \int_0^\infty G(x,t)H_i(x,t)dt = \varphi(i) = \mu_i \]
and we will obtain an approximate solution for \( G(x,t) \)
We can apply the truncated expansion method detailed in \[ 11 \] and generalized in \[ 12, 13 \] to find an approximation \( p_{i\mu}(x,t) \) for \( G(x,t) \) for the corresponding finite problem with \( i = 0, 1, \ldots, n \) where \( n \) is the number of moments \( \mu_i \).

We consider the base \( \phi_i(x,t) \) \( i = 0, 1, 2, \ldots \) obtained by applying the Gram-Schmidt orthonormalization process on \( H_i(x,t) \) \( i = 0, 1, 2, \ldots, n \) and adding to the resulting set the necessary functions until reaching an orthonormal basis.

We approach the solution \( G(x,t) \) with \[ 12, 13 \]:
\[ p_{i\mu}(x,t) = \sum_{i=0}^n \lambda_i \phi_i(x,t) \]
\[ \lambda_i = \sum_{j=0}^i C_{ij} \mu_j \ i = 0, 1, 2, \ldots, n \]

And the coefficients \( C_{ij} \) verifies
\[ C_{ij} = \left( \sum_{k=0}^{i-1} (1 - \frac{1}{k+1}) \frac{H_k(x,t)}{\| \phi_k(x,t) \|^2} C_{kj} \right) \cdot \| \phi(x,t) \|^{-1} \]
\[ 1 < i \leq n; 1 \leq j < i \]
The terms of the diagonal are
\[ C_{ii} = \| \phi(x,t) \|^{-1} \ i = 0, 1, \ldots, n. \]
The proof of the following theorem is in \[ 14, 15 \]. In \[ 15 \] he proof is done for \( b_2 = \infty \) instead of taking polynomials the Legendre are taken polynomials of Laguerre. In \[ 16 \] the demonstration is done for the one-dimensional case.

**Theorem.** Let \( \{\mu_i\}_{i=0}^n \) be a set of real numbers and suppose that \( f(x,t) \in L^2((a_1, b_1) \times (a_2, \infty)) \) verify for some \( \varepsilon \) and \( E \) (two positive numbers)
\[ \sum_{i=0}^n \int_{a_2}^{b_1} \int_{a_1}^{b_1} H_i(x,t)f(x,t)dxdt - \mu_i \leq \varepsilon^2 \]
\[ \int_{a_2}^{b_1} \int_{a_1}^{b_1} (x^2 + t^2) \varepsilon^{x+t} dxdt \leq E^2 \]
then
\[ \int_{a_2}^{b_1} \int_{a_1}^{b_1} |f(x,t)|^2 dxdt \leq \min \left\{ \|CC^T\| \varepsilon^2 + \frac{E^2}{8 \sum_{i=0}^{n+1}} \right\} \ i = 0, 1, \ldots, n \]
where \( C \) is the triangular matrix with elements \( C_{ij} \) \( 1 < i \leq n; 1 \leq j < i \).

and
\[ \int_{a_2}^{b_1} \int_{a_1}^{b_1} |p_{i\mu}(x,t) - f(x,t)|^2 dxdt \leq \|CC^T\| \varepsilon^2 + \frac{E^2}{8 \sum_{i=0}^{n+1}} \]

It must be fulfilled that
\[ f(x,t) \rightarrow 0 \quad \text{if} \quad t \rightarrow \infty \ \forall \epsilon \in \mathcal{N} \]
If we apply the truncated expansion method to solve equation \( 10 \) we obtain an approximation \( p_{i\mu}(x,t) \) for \( G(x,t) = -xw^*_i(x,t) - w^*_i(x,t) \).

Then we have an equation in first order partial derivatives
\[ -xw^*_i(x,t) - w^*_i(x,t) = p_{i\mu}(x,t) \]
of the form
\[ A_1(x,t)w^*_i(x,t) + A_2(x,t)w^*_i(x,t) = p_{i\mu}(x,t) \]
where $A_1(x,t) = -x$ and $A_2(x,t) = -1$. It is solved as in [15],
to prove that solving this equation is equivalent to solving
the integral equation
\[
\int_0^1 \int_0^w w^*(x,t)K(i,z,x,t)dtdx = \varphi 2(i,z) = \mu_i
\]
where
\[
K(i,z,x,t) = [(x(A_i)_1(x,t) + (A_2)_1(x,t) + iA_1(x,t) - x(z+1)A_2(x,t)]x^{-1}e^{-(z+1)t} =
\]
\[
(z-i)xte^{-(z+1)t}
\]
that is
\[
\int_0^1 \int_0^w u(i,z,x,t)w^*(x,t)(z-i)dtdx = \varphi 2(i,z) = \mu_i
\]
with
\[
\varphi 2(i,z) = \int_0^w [u(i,z,x,t)w^*(x,t)] dt -
\]
\[
- \int_0^1 \int_0^w u(i,z,x,0)w^*(x,0)dx - \int_0^1 \int_0^w u_{12}dtdx
\]
Again we consider the base
\[
\phi_i(z) = i = 0, 1, 2, ..., z = i + 1, \ldots \text{ obtained by applying the Gram-Schmidt orthonormalization process on u(i,z,x,t)(z-i) = K_i(x,t) = 0, 1, 2, ..., z = i + 1, \ldots \text{ and then the above equation can be transformed into a generalized moment problem}
\]
\[
\int_0^1 \int_0^w w^*(x,t)K_i(x,t)dtdx = \mu_i
\]
Applying again the techniques of generalized moments problem
to the corresponding finite problem, we found an approx-
imate solution $p_{2n}(x,t)$ for $w^*(x,t)$.
To find a numerical approximation for $w(x,t)$ we use condition
(3):
\[
\int_0^1 a(t)w(x,t)dx \approx \int_0^1 p_{2n}(x,t)dx = p_3(t)
\]
\[
\therefore a(t)E(t) \approx p_3(t)
\]
Then
\[
w(x,t) \approx \frac{p_{2n}(x,t)}{p_3(t)} = wAp(x,t)
\]
And a numerical approximation for $a(t)$ will be
\[
a(t) \approx \frac{a(t)w(1,t)}{wAp(1,t)} = aAp(t)
\]
We can measure the accuracy of the approximation (13) using
the previous theorem, where $\mu_i$ would be the $i$th generalized
moment of $wAp(x,t)$, that is, we consider the moments of $w(x,t)$ measured with error.
An analogous argument is used to measure the accuracy of the approximation $aAp(t)$

## 4. Numerical Examples

To obtain an approximation $p_{1n}(x,t)$ for $G(x,t) = -w^*_2(x,t) - w^*_1(x,t)$, we consider the base \(\phi_i(x,t) = i = 0, 1, 2, \ldots, n\) obtained by applying the Gram-Schmidt orthonormalization process on $H_i(x,t) = x^*e^{-(i+1)t} = i = 0, 1, 2, \ldots, n$.
In other words, it applies the Gram-Schmidt orthonormalization process on \(\{e^{-i, x}e^{-2i}, x^2e^{-3i}, \ldots, x^{n-i-1}e^{-ni}\}\), and is taken as a measure $\int_D e^{-i} dt dx$.
We will obtain, by applying the truncated expansion method,
$p_{1n}(x,t)$ so that $e^n p_{1n}(x,t) = p_{1n}(x,t)$.
Analogously to obtain $p_{2n}(x,t)$, we consider the base
$\phi_i(x,t) = i = 0, 1, 2, \ldots, n_1, z = i + 1, \ldots, n_2$ obtained by applying the Gram-Schmidt orthonormalization process on $u(i,z,x,t)(z-i) = K_i(x,t) = i = 0, 1, 2, \ldots, n_1, z = i + 1, \ldots, n_2$, and is taken as a measure $\int_D e^{-i} dt dx$.
We will obtain, by applying the truncated expansion method,
$p_{2n}(x,t)$ so that $e^n p_{2n}(x,t) = p_{2n}(x,t)$.
To apply the method must be $w(1,0) \neq 0$.
It may happen that (13) or (14) have discontinuities because the
denominator is overridden for certain values of $t$.
In this case we can vary the number of moments that are taken so
that the denominator does not have real roots that cancel it.

### 4.1. Example 1

We consider the equation
\[
w_i = a(t)w_i(x,t) + e^{-\frac{\alpha}{4}t}\left(4 - r^2 + e^{\frac{\alpha}{2}(2 + r^3)}\right)
\]
\[
0 < x < 1 ; \quad t > 0
\]
and conditions
\[
E(t) = (-2 + \sqrt{\alpha})e^{-\frac{\alpha}{4}t} ; \quad \alpha = \frac{1}{2} ; \quad w(x,0) = e^{-\frac{x}{2}} - 1
\]
The following conditions are met:
\[
w(0,t) = 0 ; \quad w_x(0,t) - w_x(1,t) = \frac{1}{2}w(1,t)
\]
the solution is
\[
w(x,t) = \left(e^{-\frac{t}{2}} - 1\right)e^{-\frac{\alpha}{2}t}
\]
and $a(t) = \frac{1}{2 + t^2}$.
We calculate $p_{1n}(x,t)$ with $n = 10$ moments and $p_{2n}(x,t)$ with
$n = n_1 \times n_2 = 4 \times 4 = 16$ moments.
Approximates $w(x,t)$ with $wAp(x,t)$
Accuracy is $\int_0^1 \int_0^w |w(x,t) - wAp(x,t)|^2 dt dx = 0.127967$.
And approximates $a(t)$ with $aAp(t)$
Accuracy is $\int_0^1 |a(t) - aAp(t)|^2 dt dx = 0.0684834$. In Fig. 1 and Fig. 2 the exact solution and the approximate solution are compared.
4.2. Example 2

We consider the equation

$$w_t = a(t) \left(w_x\right)_x + \frac{-2e^{-t}(2+t^2+e^t x^2)}{(2+t^2)^2}$$

$$0 < x < 1 \quad ; \quad t > 0$$

and conditions

$$E(t) = \frac{1}{6+\pi^2} \quad ; \quad \alpha = -2 \quad ; \quad w(x,0) = \frac{x^2}{2}$$

The following conditions are met:

$$w(0,t) = 0 \quad ; \quad w_x(0,t) - w_x(1,t) = -2w(1,t)$$

the solution is

$$w(x,t) = \frac{x^2}{2+t^2} \quad and \quad a(t) = e^{-t}$$

We calculate $p_{1n}(x,t)$ with $n = 7$ moments and $p_{2n}(x,t)$ with $n = n_1 \times n_2 = 3 \times 3 = 9$ moments.

Approximates $w(x,t)$ with $wAp(x,t)$ . Accuracy is $\int_0^1 \int_0^T \left| w(x,t) - wAp(x,t) \right|^2 dt dx = 0.0176301$.

And approximates $a(t)$ with $aAp(t)$ . Accuracy is $\int_0^T \left| a(t) - aAp(t) \right|^2 dt dx = 0.0195688$. In Fig.3 and Fig.4 the exact solution and the approximate solution are compared.

4.3. Example 3

We consider the equation

$$w_t = a(t) \left(w_x\right)_x + \left( -\frac{1}{4} e^{-3t} (4e^{2t} - \pi^2 \sin \left( \frac{\pi x}{2} \right) ) \right)$$

$$0 < x < 1 \quad ; \quad t > 0$$

and conditions

$$E(t) = \frac{2e^{-t}}{\pi} \quad ; \quad \alpha = \frac{\pi}{2} \quad ; \quad w(x,0) = \frac{x\pi}{2}$$

The following conditions are met:

$$w(0,t) = 0 \quad ; \quad w_x(0,t) - w_x(1,t) = \frac{\pi}{2}w(1,t)$$

the solution is

$$w(x,t) = \frac{x\pi}{2} e^{-t} \quad and \quad a(t) = e^{-2t}$$

We calculate $p_{1n}(x,t)$ with $n = 7$ moments and $p_{2n}(x,t)$ with $n = n_1 \times n_2 = 3 \times 3 = 9$ moments.

Approximates $w(x,t)$ with $wAp(x,t)$ . Accuracy is $\int_0^1 \int_0^T \left| w(x,t) - wAp(x,t) \right|^2 dt dx = 0.0916382$.

And approximates $a(t)$ with $aAp(t)$ . Accuracy is $\int_0^T \left| a(t) - aAp(t) \right|^2 dt dx = 0.188199$. In Fig.5 and Fig.6 the exact solution and the approximate solution are compared.
We consider the problem of finding \( w(x,t) \) such that

\[
\frac{\partial w}{\partial t} = a(t) \frac{\partial^2 w}{\partial x^2} + r(x,t)
\]

under the initial condition \( w(x,0) = \varphi(x) \) and the boundary conditions \( w(0,t) = w(L,t) = 0 \) about a region \( D = \{(x,t) \ 0 < x < L, \ t > 0\} \). In addition it must be fulfilled \( \int_D w(x,t)dx = E(t) \) where \( \varphi(x), r(x,t), \) and \( E(t) \) are known functions and \( \alpha \) is an arbitrary real number other than zero. We also assume that the underlying space is \( L^2(D) \).

First deduce an exact expression for \( a(t)w(1,t) \). Then, we wrote \( w(x,t) = \alpha(t)w(x,t) \), and is resolved in a first step in numerical form the integral equation

\[
\int_D G(x,t)w(x,t)dx = \psi\lambda(t)
\]

where

\[
G(x,t) = -xw_0^*(x,t) - w_0^*(x,t)
\]

it is the function to be determined.

In a second step the following integral equation is solved in numerical form

\[
\int_D w(x,t)K(m,n,x,t)dxdt = \psi_2(m,n)
\]

where \( w(x,t) \) it’s the unknown function, \( \psi_2(m,n) \) is an expression in function of \( G(x,t) \) with \( K(m,n,x,t) \) known.

Both integral equations are solved numerically by applying the moment problems techniques.

Then we find an approximation for \( w(x,t) \), this approximation we write \( w(t,x) \), using the solution found in the second step and condition \( \int_D w(x,t)dx = E(t) \).

Finally we find an approximation for \( a(t) \) using \( a(t)w(1,t) \) and \( w(t,x) \).

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