ON THE LOCATION OF CONCENTRATION POINTS FOR SINGULARLY PERTURBED ELLIPTIC EQUATIONS

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Abstract. By exploiting a variational identity of Pohožaev-Pucci-Serrin type for solutions of class $C^1$, we get some necessary conditions for locating the peak-points of a class of singularly perturbed quasi-linear elliptic problems in divergence form. More precisely, we show that the points where the concentration occurs, in general, must belong to what we call the set of weak-concentration points. Finally, in the semi-linear case, we provide a new necessary condition which involves the Clarke subdifferential of the ground-state function.

1. Introduction

Let $\varepsilon > 0$, $n \geq 3$, and $1 < p < n$. In this paper we consider the following class of singularly perturbed quasi-linear elliptic problems in divergence form

\[
(P_\varepsilon) \quad \begin{cases} -\varepsilon^p \text{div}(\alpha(x) \nabla \beta(\nabla u)) + V(x) u^{p-1} = K(x) f(u) & \text{in } \mathbb{R}^n \\ u > 0 & \text{in } \mathbb{R}^n. \end{cases}
\]

We assume that the functions $\alpha$, $V$, $K: \mathbb{R}^n \to \mathbb{R}$ are positive, of class $C^1$ with bounded derivatives, and $\alpha, K \in L^\infty(\mathbb{R}^n)$. Moreover, let

\[
\inf_{x \in \mathbb{R}^n} \alpha(x) > 0 \quad \text{and} \quad \inf_{x \in \mathbb{R}^n} V(x) > 0.
\]

The function $\beta: \mathbb{R}^n \to \mathbb{R}$ is of class $C^1$, strictly convex, and positively homogeneous of degree $p$, namely, $\beta(\lambda \xi) = \lambda^p \beta(\xi)$ for every $\lambda > 0$ and $\xi \in \mathbb{R}^n$. Moreover, there exist $\nu > 0$ and $c_1, c_2 > 0$ such that

\[
\begin{align*}
(1.1) & \quad \nu |\xi|^p \leq \beta(\xi) \leq c_1 |\xi|^p, \\
(1.2) & \quad |\nabla \beta(\xi)| \leq c_2 |\xi|^{p-1},
\end{align*}
\]

for every $\xi \in \mathbb{R}^n$. The nonlinearity $f: \mathbb{R}^+ \to \mathbb{R}$ is of class $C^1$ and such that

\[
\begin{align*}
\lim_{s \to 0^+} \frac{f(s)}{s^{p-1}} = 0 \quad \text{and} \quad \lim_{s \to +\infty} \frac{f(s)}{s^{q-1}} = 0,
\end{align*}
\]

for some $p < q < p^*$, with $p^* = np/(n-p)$. Moreover,

\[
0 < \vartheta F(s) \leq f(s)s, \quad \text{for every } s > 0,
\]

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for some \( \vartheta > p \), where we have set \( F(s) = \int_0^s f(t) \, dt, \ s \in \mathbb{R}^+ \).

Let us define the space \( W_V(\mathbb{R}^n) \) by setting
\[
W_V(\mathbb{R}^n) := \left\{ u \in W^{1,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)|u|^p \, dx < \infty \right\},
\]
endowed with the natural norm \( \|u\|_{W_V} = \int_{\mathbb{R}^n} |\nabla u|^p \, dx + \int_{\mathbb{R}^n} V(x)|u|^p \, dx \). For \( p = 2 \), we write \( H_V(\mathbb{R}^n) \) in place of \( W_V(\mathbb{R}^n) \). Under the previous assumptions, if \( K = 1 \), it has been recently proved in [12] (see also [24]) that if for some compact subset \( \Lambda \subset \mathbb{R}^n \), and \( K \equiv 1 \), we write
\[
V(z) = \min_{z \in \partial \Lambda} \inf_{\alpha > 0} \alpha(z) = \min_{z \in \Lambda} \alpha(z),
\]
then, for every \( \varepsilon \) sufficiently small, there exists a solution \( u_\varepsilon \in W_V(\mathbb{R}^n) \) of \( (P_\varepsilon) \) which has a maximum point \( z_\varepsilon \in \Lambda \), with
\[
\lim_{\varepsilon \to 0} V(z_\varepsilon) = \min_\Lambda V \quad \text{and} \quad \lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\Omega \setminus B_\rho(z_\varepsilon))} = 0, \quad \text{for every } \rho > 0.
\]

In the semi-linear case, the construction of solutions concentrating at critical points (or minima) of the potential \( V(x) \) or other finite dimensional driven functions has been deeply investigated in the last decade and also stronger results can be found in the literature (see e.g. [11, 16, 17, 18, 19] and references therein).

The goal of this paper is to establish some necessary conditions for a sequence of solutions \( (u_\varepsilon, \varepsilon) \) of \( (P_\varepsilon) \) to concentrate around a given point \( z_0 \in \mathbb{R}^n \), in the sense of Definition 2.8. If \( \beta(\xi) = \xi \), we will prove (see Theorem 3.6) that if \( z_0 \) is a concentration point for a sequence \( (u_\varepsilon, \varepsilon) \subset H_V(\mathbb{R}^n) \) of solutions of the problem then there exists a locally Lipschitz function \( \Sigma : \mathbb{R}^n \to \mathbb{R} \), the ground-state function, which has, under suitable assumptions, a critical point in the sense of the Clarke subdifferential at \( z_0 \), that is,
\[
0 \in \partial \Sigma(z_0).
\]
Under more stringent assumptions, it turns out that \( \Sigma \) admits all the directional derivatives at \( z_0 \) and \( \nabla \Sigma(z_0) = 0 \). In the general case, as a first necessary condition, the gradient vectors
\[
\nabla \alpha(z_0), \ \nabla V(z_0), \ \nabla K(z_0)
\]
must be linearly dependent. Moreover, in Theorem 2.6 (see also Theorem 2.11), we show that the concentration points for problem \( (P) \) must belong to a set \( \mathcal{C} \) (which has a variational structure) that we call the set of weak-concentration points (see Definition 2.1). To the authors’ knowledge, this kind of necessary conditions in terms of generalized gradients seem to be new. Quite interestingly, the lack of uniqueness (up to translations) for the limiting problem (namely the rescaled problem with frozen coefficients)
\[
(P_z)
\]
induces a lack of regularity for \( \Sigma \). Some conditions ensuring uniqueness of solutions for \( (P_z) \) can be found in [5, 23, 26]. For instance, for \( 1 < p \leq 2 \), \( \beta(\xi) = |\xi|^{p-2}\xi \), and \( f(u) = u^{p-1} \) with \( p < q < p^* \), we have uniqueness and \( \Sigma \) admits all the directional derivatives.
We stress that some necessary conditions for the location of concentration points were previously obtained by Ambrosetti et al. in [1], and by Wang and Zeng in [26, 27] in the case $p = 2$ and $\beta(\xi) = \xi$. Their approach is based on a repeated use of the Divergence Theorem. With respect to those papers we prove our main results by means of a locally Lipschitz variant of the celebrated Pucci-Serrin variational identity [19]. In our possibly degenerate setting, classical $C^2$ solutions might not exist, the highest general regularity class being $C^{1,\beta}$ (see [25]). Therefore, the classical identity is not applicable in our framework. However, it has been recently shown in [7] that, under minimal regularity assumptions, the identity holds for locally Lipschitz solutions (see Theorem 2.5), provided that the operator ($\beta$, in our case) is strictly convex in the gradient, which, from our viewpoint, is a very natural requirement.

This identity has also turned out to be useful in characterizing the exact energy level of the least energy solutions of the problem ($P_z$). Indeed in [12, Theorem 3.2] it was proved that ($P_z$) admits a least energy solution $u_z \in W^{1,p}(\mathbb{R}^n)$ having the Mountain-Pass energy level. This is precisely the motivation that led us to define the ground-state function $\Sigma$ also in a degenerate setting.

2. The Quasi-linear Case

The aim of this section is the study of some necessary conditions for the concentration of the solutions at a point $z_0$ to occur, in the quasi-linear framework.

2.1. Some preliminary definitions and properties. If $z$ is fixed in $\mathbb{R}^n$, we consider the limiting functional $I_z : W^{1,p}(\mathbb{R}^n) \to \mathbb{R}$,

$$I_z(u) := \alpha(z) \int_{\mathbb{R}^n} \beta(\nabla u) \, dx + \frac{V(z)}{p} \int_{\mathbb{R}^n} |u|^p \, dx - K(z) \int_{\mathbb{R}^n} F(u) \, dx.$$ 

It follows from our assumptions on $\beta$ and $f$ that $I_z$ is a $C^1$ functional, and its critical points are solutions of the limiting problem ($P_z$). We define the minimax value $c_z$ for $I_z$ by setting

$$c_z := \inf_{\gamma \in P_z} \sup_{t \in [0,1]} I_z(\gamma(t)), \quad \text{(2.1)}$$

$$P_z := \{ \gamma \in C([0,1], W^{1,p}(\mathbb{R}^n)) : \gamma(0) = 0, \quad I_z(\gamma(1)) < 0 \}.$$ 

Throughout the rest of the paper, we will denote by $G(z)$ the set of all the nontrivial solutions, up to translations, of the limiting problem ($P_z$) (the set of bound-states). Under our assumptions on $f$, $G(z) \neq \emptyset$ for every $z \in \mathbb{R}^n$. Finally, $\cdot$ will always stand for the usual inner product of $\mathbb{R}^n$.

We now introduce two functions $\partial \Gamma^-, \partial \Gamma^+$ that will be useful in the sequel.

**Definition 2.1.** For every $z, w \in \mathbb{R}^n$ we define $\partial \Gamma^-(z; w)$ and $\partial \Gamma^+(z; w)$ by setting

$$\partial \Gamma^-(z; w) := \sup_{v \in G(z)} \nabla_z I_z(v) \cdot w,$$

$$\partial \Gamma^+(z; w) := \inf_{v \in G(z)} \nabla_z I_z(v) \cdot w,$$
where $\nabla_z$ denotes the gradient with respect to $z$. Explicitly, for every $z, w \in \mathbb{R}^n$,

$$
\begin{align*}
\partial \Gamma^-(z; w) &= \sup_{v \in G(z)} \left[ \nabla \alpha(z) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v) \, dx \right. \\
&\quad + \nabla V(z) \cdot w \int_{\mathbb{R}^n} \frac{|v|^p}{p} \, dx - \nabla K(z) \cdot w \int_{\mathbb{R}^n} F(v) \, dx \left. \right], \\
\partial \Gamma^+(z; w) &= \inf_{v \in G(z)} \left[ \nabla \alpha(z) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v) \, dx \right. \\
&\quad + \nabla V(z) \cdot w \int_{\mathbb{R}^n} \frac{|v|^p}{p} \, dx - \nabla K(z) \cdot w \int_{\mathbb{R}^n} F(v) \, dx \left. \right].
\end{align*}
$$

Finally, we define a set $\mathcal{C} \subset \mathbb{R}^n$ by

$$
\mathcal{C} := \left\{ z \in \mathbb{R}^n : \partial \Gamma^-(z, w) \geq 0 \text{ and } \partial \Gamma^+(z, w) \leq 0, \text{ for every } w \in \mathbb{R}^n \right\}.
$$

We say that $\mathcal{C}$ is the set of weak-concentration points for problem $(P_\varepsilon)$.

The motivations that lead us to introduce the functions $\partial \Gamma^-$, $\partial \Gamma^+$ and the set of weak-concentration points will be clear in the course of the investigation.

For the sake of completeness, we recall the following

**Definition 2.2.** We define the ground-state function $\Sigma : \mathbb{R}^n \to \mathbb{R}$ by setting

$$
\Sigma(z) := \min_{u \in G(z)} I_z(u), \text{ for every } z \in \mathbb{R}^n.
$$

We now collect a few useful properties of the function $\Sigma$.

**Lemma 2.3.** Assume that (2.2) the map $s \in \mathbb{R}^+ \mapsto s^\frac{p}{p-1}$ is increasing.

Then, the following facts hold:

(i) the map $\Sigma$ is well defined, continuous, and

$$
\Sigma(z) = c_z, \quad \text{for every } z \in \mathbb{R}^n;
$$

(ii) the map $\Sigma$ can be written as

$$
\Sigma(z) = \inf_{u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\}} \max_{\vartheta \geq 0} I_z(\vartheta u) = \inf_{u \in \mathcal{N}_z} I_z(u), \quad \text{for every } z \in \mathbb{R}^n,
$$

where $\mathcal{N}_z$ is the Nehari manifold, defined as

$$
\mathcal{N}_z := \left\{ u \in W^{1,p}(\mathbb{R}^n) \setminus \{0\} : I_z'(u)[u] = 0 \right\}.
$$

**Proof.** To prove (ii), it suffices to argue as in [18, Proposition 2.5]. We now come to assertion (i). By [12, Theorem 3.2], for every $z \in \mathbb{R}^n$, problem $(P_\varepsilon)$ admits a solution $v_z \in W^{1,p}(\mathbb{R}^n)$, $v_z \neq 0$, such that

$$
I_z(v_z) = \Sigma(z) = c_z,
$$
where \( c_z \) is defined as in (2.1). The continuity of \( \Sigma \) then follows from the continuity of the map \( z \mapsto c_z \), which we now prove directly using an argument envisaged by Rabinowitz [21]. For \( \alpha, V, K \in \mathbb{R} \), define the functional \( I_{\alpha,V,K} : W^{1,p}(\mathbb{R}^n) \rightarrow \mathbb{R} \) by

\[
I_{\alpha,V,K}(u) := \alpha \int_{\mathbb{R}^n} \beta(\nabla u) \, dx + \frac{V}{p} \int_{\mathbb{R}^n} |u|^p \, dx - K \int_{\mathbb{R}^n} F(u) \, dx.
\]

Let us set:

\[
c(\alpha, V, K) := \inf_{\gamma \in \mathcal{P}_{\alpha,V,K}} \max_{t \in [0,1]} I_{\alpha,V,K}(\gamma(t)),
\]

\[
\mathcal{P}_{\alpha,V,K} := \{ \gamma \in C([0,1], W^{1,p}(\mathbb{R}^n)) : \gamma(0) = 0, \ I_{\alpha,V,K}(\gamma(1)) < 0 \}.
\]

Claim: For every \( (\alpha, V, K) \in \mathbb{R}^3 \) we have

\[
\lim_{\eta \to 0} c(\alpha + \eta, V + \eta, K - \eta) = c(\alpha, V, K).
\]

We first observe that a simple adaptation of the argument of [21, Lemma 3.17] yields

\[
(2.3) \quad \alpha_1 > \alpha_2, \ V_1 > V_2, \ K_1 < K_2 \implies c(\alpha_1, V_1, K_1) \geq c(\alpha_2, V_2, K_2).
\]

The proof of the claim will be accomplished indirectly. By virtue of (2.3), we get

\[
\lim_{\eta \to 0^-} c(\alpha + \eta, V + \eta, K - \eta) := c^- \leq c(\alpha, V, K).
\]

Suppose that \( c^- < c(\alpha, V, K) \). For the sake of brevity, we define

\[
J_\eta(u) := I_{\alpha+\eta, V+\eta, K-\eta}(u).
\]

Let \( \eta_h \to 0^- \) as \( h \to \infty \), and \( \delta_j \to 0^+ \) as \( j \to \infty \). For each \( h \in \mathbb{N} \), by assertion (ii), there is a sequence \((u_{hj})\) in \( W^{1,p}(\mathbb{R}^n) \), \( u_{hj} \neq 0 \), such that

\[
(2.4) \quad \alpha \int_{\mathbb{R}^n} \beta(\nabla u_{hj}) \, dx + V \int_{\mathbb{R}^n} |u_{hj}|^p \, dx = 1
\]

and

\[
(2.5) \quad \max_{\varphi \geq 0} J_{\eta_h}(\varphi u_{hj}) \leq c(\alpha + \eta_h, V + \eta_h, K - \eta_h) + \delta_j.
\]

Notice that we can choose the sequence \((u_{hj})\) satisfying (2.4), since the position

\[
u \mapsto \alpha \int_{\mathbb{R}^n} \beta(\nabla u) \, dx + V \int_{\mathbb{R}^n} |u|^p \, dx
\]
defines on $W^{1,p}(\mathbb{R}^n)$ a norm equivalent to the natural one, as follows from (1.1). Take now $j = h$, and set $u_h = u_{hh}$. Hence, in view of (2.5), we have
\[
c(\alpha, V, K) = \max_{\vartheta \geq 0} \int \frac{|h|^p}{p} dx - \eta_h \phi(u_h)^p \int \beta(\nabla u_h) dx
\]
\[
\leq c(\alpha + \eta_h, V + \eta_h, K - \eta_h) + \eta_h \phi(u_h)^p \int \beta(\nabla u_h) dx + \eta_h \phi(u_h)^p \int F(\phi(u_h) u_h) dx
\]
\[
\leq c^{-} + \delta_h - \eta_h \phi(u_h)^p \int \beta(\nabla u_h) dx + \eta_h \phi(u_h)^p \int \beta(\nabla u_h) dx
\]
\[
- \eta_h \phi(u_h)^p \int \frac{|h|^p}{p} dx - \eta_h \phi(u_h)^p \int \beta(\nabla u_h) dx
\]
\[
- \eta_h \phi(u_h)^p \int F(\phi(u_h) u_h) dx.
\]
At this point, one can show exactly as in [21, pp. 281-282] that there exists a constant $C > 0$ such that $\phi(u_h) \leq C$, for every $h \in \mathbb{N}$ sufficiently large. Therefore, recalling the properties of $F$ and the Sobolev embedding, the above chain of inequalities contradicts $c^{-} < c(\alpha, V, K)$, at least for every $h \in \mathbb{N}$ large enough. We conclude that $c^{-} < c(\alpha, V, K)$ is impossible. In a completely similar fashion one can prove that the inequality
\[
c(\alpha, V, K) < \lim_{\eta \to 0^+} c(\alpha + \eta, V + \eta, K - \eta)
\]
leads to a contradiction. Therefore the claim is proved.

Let now $(z_h)$ be a sequence in $\mathbb{R}^n$ such that $z_h \to z$ as $h \to \infty$. Observe that, given $\eta > 0$, for large $h \in \mathbb{N}$, we have
\[
V(z) + \eta \geq V(z) + |V(z_h) - V(z)|
\]
\[
\geq V(z) \geq V(z) - |V(z_h) - V(z)|
\]
\[
\geq V(z) - \eta,
\]
and similar relations hold for $\alpha$ and $K$. Therefore the continuity of $z \mapsto c_z$ follows from the previous claim, applied with $\alpha = \alpha(z)$, $V = V(z)$, and $K = K(z)$. This completes the proof of assertion (i).

\[\Box\]

**Remark 2.4.** As we have already pointed out in the introduction, we believe that the lack of regularity of the ground-state map $\Sigma$ is essentially inherited by the lack of uniqueness assumptions on the limiting equation [12]. From this viewpoint, in the degenerate case $p \neq 2$, the problem of establishing the regularity of $\Sigma$ seems
quite a difficult matter. On the contrary, if $p = 2$ and, for instance, $\beta(\xi) = \xi$, it is known that $\Sigma$ is always at least locally Lipschitz continuous (cf. Lemma 3.1). If, additionally, $f(u)$ is exactly the power $u^{p-1}$ (in which case equation $(P)$ has in fact a unique solution [3]), then $\Sigma$ is smooth and it also admits an explicit representation formula (see Remark 3.2).

Let now $L: \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a function of class $C^1$ such that
\[
\text{the function } \xi \mapsto L(x, s, \xi) \text{ is strictly convex,}
\]
for every $(x, s) \in \mathbb{R}^n \times \mathbb{R}$, and let $\varphi \in L^\infty_{\text{loc}}(\mathbb{R}^n)$.

Next, we recall a Pucci-Serrin variational identity for locally Lipschitz continuous solutions of a general class of Euler equations, recently proved in [7]. As we have already remarked in the introduction, the classical identity [19] is not applicable here, since it requires the $C^{2}$ regularity of the solutions, while the maximal regularity for degenerate equations is $C^{1, \beta}$ (see e.g. [25]).

**Theorem 2.5.** Let $u: \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz solution of
\[
-\text{div}(\partial_\xi L(x, u, \nabla u)) + \partial_s L(x, u, \nabla u) = \varphi \text{ in } D'(\mathbb{R}^n).
\]

Then,
\[
(2.6) \quad \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \partial_i h^j \partial_\xi L(x, u, \nabla u) \partial_j u \, dx + \int_{\mathbb{R}^n} \left[ (\text{div } h) L(x, u, \nabla u) + h \cdot \partial_x L(x, u, \nabla u) \right] dx = \int_{\mathbb{R}^n} (h \cdot \nabla u) \varphi \, dx,
\]
for every $h \in C^1_c(\mathbb{R}^n, \mathbb{R}^n)$.

### 2.2. Necessary conditions for locating peak-points.
We now state and prove the main results of this section.

**Theorem 2.6.** Let $z_0 \in \mathbb{R}^n$, and assume that $(u_{\varepsilon_h})$ is a sequence of solutions of problem $(P)$ such that
\[
(2.7) \quad u_{\varepsilon_h} = v_0 \left( \cdot - \frac{z_0}{\varepsilon_h} \right) + o(1), \quad \text{strongly in } W_1(\mathbb{R}^n),
\]
for some $v_0 \in W_1(\mathbb{R}^n) \setminus \{0\}$. Then, the following facts hold:
(a) the vectors
\[
\nabla \alpha(z_0), \ \nabla V(z_0), \ \nabla K(z_0)
\]
are linearly dependent;
(b) $z_0 \in \mathcal{C}$, that is $z_0$ is a weak-concentration point for $(P)$;
(c) if $G(z_0) = \{v_0\}$, then all the partial derivatives of $\Sigma$ at $z_0$ exist, and
\[
\nabla \Sigma(z_0) = 0,
\]
that is $z_0$ is a critical point of $\Sigma$. 
\begin{proof}
We write \( u_h \) in place of \( u_{\varepsilon,h} \), and we define
\begin{equation}
\label{eq:2.8}
v_h(x) := u_h(z_0 + \varepsilon_h x).
\end{equation}
Therefore, \( v_h \) satisfies the rescaled equation
\begin{equation}
- \text{div}(\alpha(z_0 + \varepsilon x)\nabla \beta(\nabla v_h)) + V(z_0 + \varepsilon x) v_h^{p-1} = K(z_0 + \varepsilon x)f(v_h) \quad \text{in } \mathbb{R}^n.
\end{equation}

By (2.7), we have \( v_h \to v_0 \) strongly in \( W_V(\mathbb{R}^n) \). We now prove that \( v_h \to v_0 \) in the \( C^1 \) sense over the compact sets of \( \mathbb{R}^n \) and that \( v_0 \) is a nontrivial positive solution of the equation
\begin{equation}
\label{eq:2.9}
-\alpha(z_0)\text{div}(\nabla \beta(\nabla v)) + V(z_0) v^{p-1} = K(z_0)f(v) \quad \text{in } \mathbb{R}^n.
\end{equation}

Let us set
\begin{align*}
d_h(x) & := \begin{cases} V(z_0 + \varepsilon_h x) - K(z_0 + \varepsilon_h x) & \text{if } v_h(x) \neq 0 \\
0 & \text{if } v_h(x) = 0,
\end{cases} \\
A(x, s, \xi) & := \alpha(z_0 + \varepsilon_h x)\nabla \beta(\xi), \\
B(x, s, \xi) & := d_h(x)s^{p-1},
\end{align*}
for every \( x \in \mathbb{R}^n, s \in \mathbb{R}^+, \text{ and } \xi \in \mathbb{R}^n \). Taking into account (1.2), and the strict convexity of \( \beta \), we get
\begin{equation*}
A(x, s, \xi) \cdot \xi \geq \nu|\xi|^p \quad \text{and} \quad |A(x, s, \xi)| \leq c_2|\xi|^{p-1}.
\end{equation*}

Notice that, in view of the growth assumptions on \( f \), there exists \( \delta > 0 \) sufficiently small such that \( d_h \in L^{n/(p-\delta)}(B_{2\rho}) \) for every \( \rho > 0 \) and
\begin{equation*}
S = \sup_{h \in \mathbb{N}} \|d_h\|_{L^{n/(p-\delta)}(B_{2\rho})} \leq D_\rho \left(1 + \sup_{h \in \mathbb{N}} \|v_h\|_{L^p(B_{2\rho})}\right) < \infty,
\end{equation*}
for some \( D_\rho > 0 \). Since we have \( \text{div}(A(x, v_h, \nabla v_h)) = B(x, v_h, \nabla v_h) \) for every \( h \in \mathbb{N} \), by exploiting [22, Theorem 1] there exists a radius \( \rho > 0 \) and a positive constant \( M = M(\nu, c_2, S^p) \) such that
\begin{equation*}
\sup_{h \in \mathbb{N}} \max_{x \in B_\rho} |v_h(x)| \leq M(2\rho)^{-N/p} \sup_{h \in \mathbb{N}} \|v_h\|_{L^p(B_{2\rho})} < \infty,
\end{equation*}
so that \( (v_h) \) is uniformly bounded in \( B_\rho \). Then, by virtue of [22, Theorem 8], up to a subsequence \( (v_h) \) converges uniformly to \( v_0 \) in a small neighborhood of zero. Similarly one shows that \( v_h \to v_0 \) in \( C^1_{\text{loc}}(\mathbb{R}^n) \). Therefore, it is easily seen that \( v_0 \) is a nontrivial positive solution of (2.9), that is \( v_0 \in G(z_0) \). Since the map \( \beta \) is strictly convex, we can use Theorem 2.3 by choosing in (2.6) \( \varphi = 0 \) and
\begin{align*}
\mathcal{L}(x, s, \xi) & := \alpha(z_0 + \varepsilon_h x)(\xi) + V(z_0 + \varepsilon_h x)s^p - K(z_0 + \varepsilon_h x)F(s), \\
h(x) & = h_{\varepsilon,k}(x) := \sum_{k=1}^{n-1} \sum_{j=1}^{n-k} T(\varepsilon x, 0, \ldots, 0), \quad \text{for } \varepsilon > 0 \text{ and } k = 1, \ldots, n,
\end{align*}
for every \( x \in \mathbb{R}^n, s \in \mathbb{R}^+ \text{ and } \xi \in \mathbb{R}^n \), the function \( T \in C^1_c(\mathbb{R}^n) \) being chosen so that \( T(x) = 1 \) for \( |x| \leq 1 \), and \( T(x) = 0 \) for \( |x| \geq 2 \). In particular, \( h_{\varepsilon,k} \in C^1_c(\mathbb{R}^n, \mathbb{R}^n) \) and
\begin{equation*}
\partial_i h_{\varepsilon,k}^j(x) = \varepsilon \partial_i T(\varepsilon x)\delta_{kj}, \quad \text{for every } x \in \mathbb{R}^n, \varepsilon > 0, \text{ and } i, j, \text{ and } k.
\end{equation*}
Then, it follows from (2.6) that

\[ 0 = \sum_{i=1}^{n} \int_{\mathbb{R}^n} \varepsilon \partial_i T(\varepsilon x) \alpha(z_0 + \varepsilon h x) \partial_{z_i} \beta(\nabla v_h) \partial_k v_h \, dx \]

\[ - \int_{\mathbb{R}^n} \varepsilon \partial_k T(\varepsilon x) \left[ \alpha(z_0 + \varepsilon h x) \beta(\nabla v_h) + V(z_0 + \varepsilon h x) \frac{v_h^p}{p} - K(z_0 + \varepsilon h x) F(v_h) \right] \, dx \]

\[ - \int_{\mathbb{R}^n} \varepsilon h T(\varepsilon x) \left[ \frac{\partial \alpha}{\partial x_k}(z_0 + \varepsilon h x) \beta(\nabla v_h) + \frac{\partial V}{\partial x_k}(z_0 + \varepsilon h x) \frac{v_h^p}{p} - \frac{\partial K}{\partial x_k}(z_0 + \varepsilon h x) F(v_h) \right] \, dx \]

for every \( \varepsilon > 0, \, h \in \mathbb{N}, \, \text{and} \, k = 1, \ldots, n. \) Since the sequence \((v_h)\) is bounded in \( W_{V}(\mathbb{R}^n) \), by (1.1), (1.2) and the boundedness of \( \alpha \) and \( K \), we have

\[ \left| \sum_{i=1}^{n} \int_{\mathbb{R}^n} \partial_i T(\varepsilon x) \alpha(z_0 + \varepsilon h x) \partial_{z_i} \beta(\nabla v_h) \partial_k v_h \, dx \right| \leq C, \]

\[ \left| \int_{\mathbb{R}^n} \partial_k T(\varepsilon x) \left[ \alpha(z_0 + \varepsilon h x) \beta(\nabla v_h) + V(z_0 + \varepsilon h x) \frac{v_h^p}{p} - K(z_0 + \varepsilon h x) F(v_h) \right] \, dx \right| \leq C', \]

for some positive constants \( C, C' \). Therefore, letting first \( \varepsilon \to 0 \) yields

\[ \int_{\mathbb{R}^n} \left[ \frac{\partial \alpha}{\partial x_k}(z_0 + \varepsilon h x) \beta(\nabla v_h) + \frac{\partial V}{\partial x_k}(z_0 + \varepsilon h x) \frac{v_h^p}{p} \right. \]

\[ \left. - \frac{\partial K}{\partial x_k}(z_0 + \varepsilon h x) F(v_h) \right] \, dx = 0, \]

for every \( h \in \mathbb{N}, \, \text{and} \, k = 1, \ldots, n. \) Letting now \( h \to \infty, \) by (2.7), we find

\[ \frac{\partial \alpha}{\partial x_k}(z_0) \int_{\mathbb{R}^n} \beta(\nabla v_h) \, dx + \frac{\partial V}{\partial x_k}(z_0) \int_{\mathbb{R}^n} \frac{v_h^p}{p} \, dx - \frac{\partial K}{\partial x_k}(z_0) \int_{\mathbb{R}^n} F(v_h) \, dx = 0, \]

for every \( k = 1, \ldots, n, \) which yields

\[ \nabla \alpha(z_0) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v_h) \, dx + \nabla V(z_0) \cdot w \int_{\mathbb{R}^n} \frac{v_h^p}{p} \, dx = \nabla K(z_0) \cdot w \int_{\mathbb{R}^n} F(v_h) \, dx, \]

for every \( w \in \mathbb{R}^n. \) Then, since \( v_0 \neq 0, \) assertion (a) immediately follows. Moreover, since \( v_0 \in G(z_0), \) by the definition of \( \partial \Gamma^-, \) we obtain

\[ \partial \Gamma^-(z_0; w) = \sup_{v \in G(z_0)} \left[ \nabla \alpha(z_0) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v) \, dx \right. \]

\[ + \nabla V(z_0) \cdot w \int_{\mathbb{R}^n} \frac{|v|^p}{p} \, dx - \nabla K(z_0) \cdot w \int_{\mathbb{R}^n} F(v) \, dx \]

\[ \geq \nabla \alpha(z_0) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v_0) \, dx \]

\[ + \nabla V(z_0) \cdot w \int_{\mathbb{R}^n} \frac{v_0^p}{p} \, dx - \nabla K(z_0) \cdot w \int_{\mathbb{R}^n} F(v_0) \, dx = 0, \]
for every $w \in \mathbb{R}^n$. Analogously, by the definition of $\partial \Gamma^+$, we have

$$
\partial \Gamma^+(z_0; w) = \inf_{v \in G(z_0)} \left[ \nabla \alpha(z_0) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v) \, dx 
+ \nabla V(z_0) \cdot w \int_{\mathbb{R}^n} \frac{|v|^p}{p} \, dx - \nabla K(z_0) \cdot w \int_{\mathbb{R}^n} F(v) \, dx \right]
\leq \nabla \alpha(z_0) \cdot w \int_{\mathbb{R}^n} \beta(\nabla v_0) \, dx
+ \nabla V(z_0) \cdot w \int_{\mathbb{R}^n} \frac{v_0^p}{p} \, dx - \nabla K(z_0) \cdot w \int_{\mathbb{R}^n} F(v_0) \, dx = 0,
$$

for every $w \in \mathbb{R}^n$. Therefore $z_0 \in \mathcal{C}$ and assertion (b) is proved. If $G(z_0) = \{ v_0 \}$, then clearly $\Sigma$ admits all the directional derivatives at $z_0$, and

$$
\frac{\partial \Sigma}{\partial w}(z_0) = \partial \Gamma^-(z_0; w) = \partial \Gamma^+(z_0; w) = 0, \quad \text{for every } w \in \mathbb{R}^n,
$$

by virtue of (b). This proves assertion (c). \hfill \Box

The strong convergence required by (2.7) allows us to take the limit as $h \to \infty$ in equation (2.10). In the semi-linear case one can construct uniform exponential barriers for the family $(v_h)$, and therefore the strong convergence of $(v_h)$ follows easily from the Lebesgue Convergence Theorem (see [18, 26, 27]). The well-known loss of regularity for solutions of quasi-linear equations is usually an obstruction to pointwise concentration. However, if the solutions belong to a suitable space, then a pointwise concentration suffices (see Corollary 2.9).

**Remark 2.7.** We wish to point out that Theorem 2.6 holds true also for the more general class of quasi-linear equations

$$
-\varepsilon^p \operatorname{div}(\alpha(x) \partial_z \beta(u, \nabla u)) + \varepsilon^p \alpha(x) \partial_s \beta(u, \nabla u) + V(x) u^{p-1} = K(x) f(u),
$$

under suitable assumptions on $\partial_z \beta(s, \xi)$ and $\partial_s \beta(s, \xi)$ (see [12]). On the other hand, although the ground-state function $\Sigma$ can be defined exactly as in Definition 2.2 and $\Sigma(z) = c_2$ (cf. [12] Theorem 3.2), the presence of $u$ itself in the function $\beta$ makes the problems of the regularity of $\Sigma$ and of the decay at infinity for the rescaled family of solutions very complicated, even in the nondegenerate case $p = 2$.

**Definition 2.8.** Let $z_0 \in \mathbb{R}^n$. We say that a sequence $(u_{\varepsilon h})$ of solutions of (P) concentrates at $z_0$ if $u_{\varepsilon h}(z_0) \geq \ell > 0$ for some $\ell > 0$ and for every $\eta > 0$ there exist $\varepsilon > 0$ and $h_0 \in \mathbb{N}$ such that

$$
u_{\varepsilon h}(x) \leq \eta, \quad \text{for every } h \geq h_0 \text{ and } |x - z_0| \geq \varepsilon h_0.
$$

This is precisely the notion of concentration adopted in [26, 27].

**Corollary 2.9.** Let $(u_{\varepsilon h})$ be a family of solutions of (P) which concentrates at a point $z_0 \in \mathbb{R}^n$. Suppose that, for every $h \in \mathbb{N}$ sufficiently large,

$$
u_{\varepsilon h} \in C^1_d(\mathbb{R}^n) \cap W^{2,n}(\mathbb{R}^n),
$$

where

$$
C^1_d(\mathbb{R}^n) := \left\{ u \in C^1(\mathbb{R}^n) : \lim_{|x| \to \infty} u(x) = 0 \text{ and } \lim_{|x| \to \infty} \nabla u(x) = 0 \right\}.
$$
Then, all the conclusions of Theorem 2.6 hold true.

Proof. If \( u_\varepsilon \in C^1_d(\mathbb{R}^n) \cap W^{2,n}(\mathbb{R}^n) \), then one can apply the results contained in [20] to show that the rescaled sequence \( v_\varepsilon \) decays exponentially fast at infinity, uniformly with respect to \( h \), together with all its partial derivatives. Hence we can pass to the limit in equation (2.10), and complete the proof as in Theorem 2.6. \( \square \)

For the particular, but important, case \( \alpha(x) = 1, \beta(\xi) = |\xi|^{p-2}\xi, \) and \( f(s) = s^{q-1}, p < q < p^* \), we can still prove a fast-decay at infinity for the solutions.

**Lemma 2.10.** Let \( (u_\varepsilon) \) be a sequence of solutions of the problem

\[
\begin{cases}
-\varepsilon^p \Delta_p u + V(x)u^{p-1} = K(x)u^{q-1} & \text{in } \mathbb{R}^n \\
u > 0 \quad & \text{in } \mathbb{R}^n
\end{cases}
\]

which concentrates at \( z_0 \in \mathbb{R}^n \). Then, if we set

\[ v_h(x) := u_\varepsilon(z_0 + \varepsilon x), \]

for each \( \eta > 0 \) there exist \( R_\eta, C_\eta > 0 \) independent of \( h \) such that

\[ |v_h(x)| \leq C_\eta \exp \left\{ -\left( \frac{\eta}{p-1} \right)^{1/p} |x| \right\}, \]

for every \( |x| \geq R_\eta \) and every \( h \in \mathbb{N} \).

Proof. For every \( h \in \mathbb{N} \), \( v_h \) satisfies the equation

\[ -\Delta_p v_h + V(z_0 + \varepsilon_h x)v_h^{p-1} = K(z_0 + \varepsilon_h x)v_h^{q-1} \quad \text{in } \mathbb{R}^n. \]

Since \( (u_\varepsilon) \) is a concentrating sequence, it results that

\[ \lim_{|x| \to \infty} v_h(x) = 0, \quad \text{uniformly in } h \in \mathbb{N}. \]

Then, setting \( \inf_{x \in \mathbb{R}^n} V(x) = V_0 \), given \( \eta > 0 \) there exists a positive constant \( R_\eta \) independent of \( h \) such that

\[ V(z_0 + \varepsilon h x)v_h^{p-1}(x) - K(z_0 + \varepsilon h x)v_h^{q-1}(x) \geq (V_0 - \eta)v_h^{p-1}(x), \]

for every \( |x| \geq R_\eta \). It follows that the inequality

(2.11) \[ -\operatorname{div}(|\nabla v_h|^{p-2} \nabla v_h) + (V_0 - \eta)v_h^{p-1} \leq 0 \]

holds true for every \( h \in \mathbb{N} \), and \( |x| \geq R_\eta \). Define now the function

\[ \Phi(x) := C_\eta \exp \left\{ -\left( \frac{V_0 - \eta}{p-1} \right)^{1/p} |x| \right\}, \]

where

\[ C_\eta := \exp \left\{ \left( \frac{V_0 - \eta}{p-1} \right)^{1/p} R_\eta \right\} \max_{|x|=R_\eta} v_h(x). \]

Notice that, since \( v_h \) is uniformly bounded, we can assume that \( C_\eta \) is independent of \( h \). Now, exactly the same computations of [14, Theorem 2.8] entail

(2.12) \[ -\operatorname{div}(|\nabla \Phi|^{p-2} \nabla \Phi) + (V_0 - \eta)\Phi^{p-1} \geq 0. \]
Testing inequalities (2.11) and (2.12) with \( \phi = (v_h - \Phi)^+ \) on \( \{|x| \geq R_n\} \) yields:

\[
\int_{\{|x| \geq R_n\} \cap \{v_h > \Phi\}} \left( |\nabla v_h|^{p-2} \nabla v_h \cdot \nabla (v_h - \Phi) + (V_0 - \eta) v_h^{p-1} (v_h - \Phi) \right) \, dx \leq 0,
\]

\[
\int_{\{|x| \geq R_n\} \cap \{v_h > \Phi\}} \left( |\nabla \Phi|^{p-2} \nabla \Phi \cdot \nabla (v_h - \Phi) + (V_0 - \eta) \Phi^{p-1} (v_h - \Phi) \right) \, dx \geq 0.
\]

By subtracting the previous inequalities, and taking into account that the critical points of \( \Phi \), which concentrates at \( z \) of \([5, 15] \) and \([23] \), for every \( I \) such that \( u \to 0 \) as \( |x| \to \infty \). Then \( G(z_0) = \{v_0\} \), and assertion (c) follows by the corresponding assertion in Theorem 2.6.

**Theorem 2.11.** Let \( (u_{\varepsilon_h}) \) be a sequence of solutions of the problem

\[
\begin{cases}
-\varepsilon^p \Delta_p u + V(x) u^{p-1} = K(x) u^{q-1} & \text{in } \mathbb{R}^n \\
u > 0 & \text{in } \mathbb{R}^n
\end{cases}
\]

which concentrates at \( z_0 \in \mathbb{R}^n \). Then, the following facts hold:

(a) the vectors \( \nabla V(z_0) \) and \( \nabla K(z_0) \) are proportional;

(b) \( z_0 \in \mathcal{C} \), that is \( z_0 \) is a weak-concentration point for (2.13);

(c) if \( 1 < p \leq 2 \) then all the partial derivatives of \( \Sigma \) at \( z_0 \) exist, and \( \nabla \Sigma(z_0) = 0 \), that is \( z_0 \) is a critical point of \( \Sigma \).

**Proof.** By virtue of Lemma 2.10 we can pass to the limit in equation (2.10) and get assertions (a) and (b) as in Theorem 2.6. If \( 1 < p \leq 2 \), by combining the results of [5, 15] and [23], for every \( z \in \mathbb{R}^n \), problem (P) admits a unique positive \( C^1 \) solution (up to translations) such that \( u(x) \to 0 \) as \( |x| \to \infty \). Then \( G(z_0) = \{v_0\} \), and assertion (c) follows by the corresponding assertion in Theorem 2.6.

3. The Semi-linear Case

The main goal of this section is that of getting, in the particular case \( \beta(\xi) = \xi \), namely semi-linear equations, a more accurate version of Theorem 2.6 involving the Clarke subdifferential of the ground-state function \( \Sigma \). We wish to stress that we have in mind the case when \( f \) is not simply the power nonlinearity \( u^{p-1} \) (cf. Remark 3.2).

For \( z \in \mathbb{R}^n \) fixed, we consider the limiting functional \( I_z : H^1(\mathbb{R}^n) \to \mathbb{R} \),

\[
I_z(u) := \alpha(z) \int_{\mathbb{R}^n} |\nabla u|^2 \, dx + \frac{V(z)}{p} \int_{\mathbb{R}^n} |u|^p \, dx - K(z) \int_{\mathbb{R}^n} F(u) \, dx
\]

whose critical points are of course solutions of (P). The minimax levels \( c_z \) of \( I_z \) are defined according to (2.1). Throughout the rest of this section, we will denote by...
$S(z)$ the set of all the nontrivial solutions of $(P_z)$ corresponding to the energy level $\Sigma(z)$ (the set of ground-states). It is known that $S(z) \neq \emptyset$ for every $z \in \mathbb{R}^n$ (see [2]). As the next lemma shows, in this particular situation, the function $\Sigma$ has further regularity properties (and in some cases it relates to the maps $\partial \Gamma^-$ and $\partial \Gamma^+$).

**Lemma 3.1.** If $p = 2$ and condition ([27]) holds, then the following facts hold:

(i) $\Sigma$ is locally Lipschitz;

(ii) the directional derivatives from the left and the right of $\Sigma$ at $z$ along $w$, $(\frac{\partial \Sigma}{\partial w})^- (z)$ and $(\frac{\partial \Sigma}{\partial w})^+ (z)$ respectively, exist at every point $z \in \mathbb{R}^n$, and it holds

$$
(\frac{\partial \Sigma}{\partial w})^- (z) = \sup_{v \in \mathbb{R}^n} \nabla_v I_z (v) \cdot w,
$$

$$
(\frac{\partial \Sigma}{\partial w})^+ (z) = \inf_{v \in \mathbb{R}^n} \nabla_v I_z (v) \cdot w,
$$

for every $z, w \in \mathbb{R}^n$. In particular, if $G(z) = S(z)$, we have

$$
(3.1) \quad \partial \Gamma^- (z; w) = \left( \frac{\partial \Sigma}{\partial w} \right)^- (z) \quad \text{and} \quad \partial \Gamma^+ (z; w) = \left( \frac{\partial \Sigma}{\partial w} \right)^+ (z),
$$

for every $w \in \mathbb{R}^n$.

**Proof.** By the results of [27], $\Sigma$ is a locally Lipschitz map. We remark here that, since $z$ acts as a parameter, the functional $I_z$ is invariant under orthogonal change of variables. Therefore, without loss of generality, to get the formulas for the left and right directional derivatives of $\Sigma$, it suffices to show that

$$
\left( \frac{\partial \Sigma}{\partial z_i} \right)^- (z) = \sup_{v \in \mathbb{R}^n} \left[ \frac{\partial \alpha}{\partial z_i} (z) \int_{\mathbb{R}^n} \frac{|\nabla v|^2}{2} + \frac{\partial V}{\partial z_i} (z) \int_{\mathbb{R}^n} \frac{|v|^p}{p} - \frac{\partial K}{\partial z_i} (z) \int_{\mathbb{R}^n} F(v) \right],
$$

$$
\left( \frac{\partial \Sigma}{\partial z_i} \right)^+ (z) = \inf_{v \in \mathbb{R}^n} \left[ \frac{\partial \alpha}{\partial z_i} (z) \int_{\mathbb{R}^n} \frac{|\nabla v|^2}{2} + \frac{\partial V}{\partial z_i} (z) \int_{\mathbb{R}^n} \frac{|v|^p}{p} - \frac{\partial K}{\partial z_i} (z) \int_{\mathbb{R}^n} F(v) \right],
$$

for every $z \in \mathbb{R}^n$ and $i = 1, \ldots, n$. These can be obtained arguing as in [18, 27]. Finally, formulas (3.1) follow by the definition of $\partial \Gamma^+(z; w)$ and $\partial \Gamma^-(z; w)$. \hfill \Box

**Remark 3.2.** Assume that $p = 2$, $K$ is bounded from below away from zero, and $f(u) = u^{q-1}$, where $2 < q < 2^*$. Then $\Sigma$ is smooth and it can be given an explicit form (cf. [18, Remark 2.1]): there exists $C_q > 0$ such that

$$
\Sigma(z) = C_q \left[ \frac{V(z)}{K(z)} \right]^{\frac{q-2}{2}} \frac{\sqrt{\alpha(z)K(z)}}{\sqrt{\sqrt{\alpha(z)K(z)}}}, \quad \text{for every } z \in \mathbb{R}^n.
$$

Let us now recall from [4] two definitions that will be useful in the sequel.

**Definition 3.3.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function near a given point $z \in \mathbb{R}^n$. The generalized derivative of the function $f$ at $z$ along the direction $w \in \mathbb{R}^n$ is defined by

$$
f^0 (z; w) := \limsup_{\xi \to z} \frac{f(\xi + \lambda w) - f(\xi)}{\lambda}.
$$
Definition 3.4. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function near a given point $z \in \mathbb{R}^n$. The Clarke subdifferential (or generalized gradient) of $f$ at $z$ is defined by
\[
\partial f(z) := \{ \eta \in \mathbb{R}^n : f^0(z, w) \geq \eta \cdot w, \text{ for every } w \in \mathbb{R}^n \}.
\]

By [4, Proposition 2.3.1] we learn that

Proposition 3.5. For every $z \in \mathbb{R}^n$, the set $\partial f(z)$ is nonempty and convex, and $\partial(-f)(z) = -\partial f(z)$.

The next is the main result of this section.

Theorem 3.6. Assume that $(u_{\varepsilon_h})$ is a sequence of solutions of the problem
\[
\begin{aligned}
-\varepsilon^2 \text{div}(\alpha(x) \nabla u) + V(x)u &= K(x)f(u) \quad \text{in } \mathbb{R}^n \\
u &> 0 \quad \text{in } \mathbb{R}^n
\end{aligned}
\]
which concentrates at $z_0$. Then, the following facts hold:

(a) the vectors
\[
\nabla \alpha(z_0), \quad \nabla V(z_0), \quad \nabla K(z_0)
\]
are linearly dependent;
(b) $z_0 \in \mathcal{C}$, that is $z_0$ is a weak-concentration point for (3.2);
(c) if either $G(z_0) = S(z_0)$ or
\[
\varepsilon_h^{-n} J_{\varepsilon_h}(u_{\varepsilon_h}) \to c_{z_0},
\]
where
\[
J_{\varepsilon}(v) = \frac{\varepsilon^2}{2} \int_{\mathbb{R}^n} \alpha(x)|\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^n} V(x)|v|^2 \, dx - \int_{\mathbb{R}^n} K(x)F(v) \, dx,
\]
we have
\[
0 \in \partial \Sigma(z_0),
\]
that is $z_0$ is a critical point of $\Sigma$ in the sense of the Clarke subdifferential;
(d) if $S(z_0) = \{ v_0 \}$, then all the partial derivatives of $\Sigma$ at $z_0$ exist, and
\[
\nabla \Sigma(z_0) = 0,
\]
that is $z_0$ is a critical point of $\Sigma$.

Proof. For problem (3.2) it is possible to prove the existence of uniform exponentially decaying barriers. Then we can pass to the limit in equation (2.10), to get assertions (a) and (b) as in Theorem 2.6. If $G(z_0) = S(z_0)$, by combining formulas (3.1) of Lemma 3.1 with (b) of Theorem 2.6 we have
\[
\left( \frac{\partial \Sigma}{\partial w} \right)^-(z_0) \geq 0 \quad \text{and} \quad \left( \frac{\partial \Sigma}{\partial w} \right)^+(z_0) \leq 0,
\]
for every $w \in \mathbb{R}^n$. In particular, it holds
\[
\left( \frac{\partial(-\Sigma)}{\partial w} \right)^+(z_0) \geq 0, \quad \text{for every } w \in \mathbb{R}^n.
\]
Then, by the definition of \((-\Sigma)^0(z_0; w)\) we get
\[
(-\Sigma)^0(z_0; w) \geq \left( \frac{\partial (-\Sigma)}{\partial w} \right) (z_0) \geq 0, \quad \text{for every } w \in \mathbb{R}^n.
\]
By the definition of \(\partial (-\Sigma)(z_0)\) we immediately get \(0 \in \partial (-\Sigma)(z_0)\), which, together with Proposition 3.3 yields assertion (c). To prove the same conclusion when (3.3) holds, we simply remark that \(c_{z_0} = \Sigma(z_0)\). Therefore, if \(v_0\) is the limit of the sequence \((v_h)\) defined in (2.8), then \(v_0 \in S(z_0)\) because we can exploit again some exponential barrier to pass to the limit. As a consequence, arguing as in Theorem 2.6 it follows that inequalities (3.5) hold and we are reduced to the previous case. Finally, if \(S(z_0) = \{v_0\}\), the map \(\Sigma\) admits all the directional derivatives at \(z_0\) and, by virtue of (3.3) they are equal to zero, which proves (d).

We would like to remark that a different definition of concentration has been used in [13]. We recall it here, suitably adapted to our purposes.

**Definition 3.7.** Assume that \(u_\varepsilon \in C^2(\mathbb{R}^n)\) is a family of solutions of (3.2) and let \(J_\varepsilon\) be as in (3.4). Moreover, let \(x_\varepsilon \in \mathbb{R}^n\) be such that \(\max_{x \in \mathbb{R}^n} u_\varepsilon = u_\varepsilon(x_\varepsilon)\). We say that \(u_\varepsilon\) concentrates at \(z_0 \in \mathbb{R}^n\) if the following facts hold:

1. \(\lim_{\varepsilon \to 0} x_\varepsilon = z_0\);
2. \(\lim_{\varepsilon \to 0} \varepsilon^{-n} J_\varepsilon(u_\varepsilon) = c_{z_0}\).

It is not difficult to check that if \((u_\varepsilon)\) is a sequence as in the above definition, then \((u_\varepsilon)\) concentrates at \(z_0\) in the sense of Definition 2.8 vanishing at an exponential rate away from \(z_0\) (cf. [13, Lemma 4.2]). In particular, according to (c) of Theorem 3.6 we have \(0 \in \partial \Sigma(z_0)\).

We finish the paper with an open problem. Assume that \((u_h)\) is a sequence of solutions of problem (3.2). Suppose that these solutions concentrate at \(z_0 \in \mathbb{R}^n\), and \(S(z_0) = \{v_0\}\). Is it possible to prove that \(z_0\) is a \(C^1\)-stable critical point of \(\Sigma\), according to the definition of Yanyan Li [16]? 

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