Stable Sets and Graphs with no Even Holes

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May 7, 2014

Abstract

We develop decomposition/composition tools for efficiently solving maximum weight stable sets problems as well as for describing them as polynomially sized linear programs (using “compact systems”). Some of these are well-known but need some extra work to yield polynomial “decomposition schemes”.

We apply the tools to graphs with no even hole and no cap. A hole is a chordless cycle of length greater than three and a cap is a hole together with an additional node that is adjacent to two adjacent nodes of the hole and that has no other neighbors on the hole.

1 Introduction

A vast literature about efficiently solvable cases of the stable set problem focuses on “perfect graphs”. Based on the ellipsoid method, Grötschel, Lovász, and Schrijver [15] have developed a polynomial-time algorithm that computes a stable set of maximum weight in a perfect graph. Perfect graphs have no odd holes. (A hole is a chordless cycle of length greater than three.) It is conceivable that the stable set problem is polynomially solvable for all graphs without odd holes, and this may even extend to graphs with all holes having the same parity, so either all even or all odd. To our knowledge the case that all holes are odd has not received much attention and in this paper we take a first step in exploring this topic by considering “cap-free” graphs with no even holes. A cap is a hole together with an additional node that is adjacent to two adjacent nodes of the hole and that has no other neighbors on the hole.

Theorem 1. The stable set problem for cap-free graphs with no even holes is polynomially solvable.

The stable set polytope of a graph is the convex hull of the characteristic vectors of the stable sets of the graph. Linear descriptions of stable set polytopes require in the worst case exponentially many inequalities and arbitrarily large coefficients (in minimum integer form), even for cap-free graphs with no even hole. However, for those graphs we can tame the descriptions by allowing some extra variables. An extended formulation for a polytope $P$ in $\mathbb{R}^n$ is a system of inequalities $Ax + By \leq d$ such that

$$P = \{ x \in \mathbb{R}^n : \exists y \ [Ax + By \leq d] \}.$$

An extended formulation for $P$ is compact if its encoding has polynomial size in $n$. 

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Theorem 2. Stable set polytopes of cap-free graphs without even hole admit compact extended formulations.

We develop decomposition/composition tools for solving maximum weight stable sets problems. The working of such tools is that, when a graph is decomposable into smaller parts in a way according to the tool’s specifications, then the tool can be used to efficiently construct a solution for the whole from solutions for the parts. Some of these tools are well-known but need some extra work to make them suitable as a component in polynomial-time algorithms. We develop similar tools for combining polynomially sized linear programs for stable set problems on parts of a decomposition into such linear program for the whole.

We apply these results to cap-free graphs with no even holes. Conforti, Cornujois, Kapoor, and Vušković [6, 7, 10] give a decomposition theorem for graphs with no even hole and use that to find even holes in polynomial time [11]. The following theorem is a simplified variant of the main result in [6].

Theorem 3 ([6, Theorem 4.1]). Every cap-free graph with a triangle either admits an amalgam decomposition or a clique cutset decomposition (both defined in Section 2) or contains a node adjacent to all other nodes.

So cap-free graphs with no even holes can be built from triangle-free graphs with no even holes. These can be further decomposed into as simple graphs as “fans” and the 1-skeleton of the three dimensional cube. This is Theorem 10; as that result is a bit technical, we explain its details later, in Section 2.4. As we will see, all decompositions coming up in Theorems 3 and 10 fall in our framework and thus, taking all together, we get Theorems 1 and 2.

Notation. Let $G = (V, E)$ be a graph. If $X, Y \subseteq V$ are disjoint and $x \in X$ has a neighbor in $Y$, then $X$ and $Y$ are adjacent and $x$ and $Y$ are adjacent. The set of nodes outside $U$ that are adjacent to $U$ is denoted by $N_G(U)$. The subgraph of $G$ induced by $U \subseteq V$ is $G_U$. Moreover, $G - U = G_{V \setminus U}$ and $B_G(U) = N_G(V \setminus U)$. If $u \in V$, we write $G - u$ for $G - \{u\}$, $N_G(u)$ for $N_G(\{u\})$, etc.

The collection of stable sets in a graph $G = (V, E)$ is denoted by $\text{Stab}(G)$. The stability number $\alpha(G)$ of $G$ is the size of the largest stable set in $G$. If $U \subseteq V$, then $\alpha(U) = \alpha(G_U)$. If $w = (w_v : v \in V)$, then $w(U) = \sum_{v \in U} w_v$.

2 Solving the stable set problem by decomposition

Given a graph $G = (V, E)$ and a weighting $(w_v : v \in V)$, we consider the following problem:

Find in $G$, a stable set $S$, that maximizes $w(S)$. \hspace{1cm} (Stable set problem on $G$)

Consider a partition $(V_1, U, V_2)$ of $V$ such that $V_1$ and $V_2$ are not adjacent. Then the stable set problem is equivalent to the problem:

Find in $G_{V_1 \cup U}$, a stable set $S$, that maximizes $w(S) + c_{S \cap U}$. \hspace{1cm} (Master for $G$)

Here the correction terms $c_{S \cap U}$ come as the values $c_T = w(S_T \setminus U)$ of the stable sets $S_T$ determined by solving the following problem:

Find in $G_{V_1 \cup U}$, for each stable set $T$ in $U$,
a stable set $S_T$ with $S_T \cap U = T$, that maximizes $w(S_T \setminus U)$. \hfill \text{(Servant)}$

So, to solve the stable set problem for $G$, we first solve the servant to find the correction terms $c_T$ with $T \in S[U]$ and with those we solve the master. Once we found a solution $S$ of the master, we select, among all stable sets $S_T$ found for the servant, the one with $T = S \cap U$. With that choice, $S \cup S_T$ is solution to the stable set problem for $G$.

The servant is a stable set problem on a rooted graph; which in general is formulated as: Given a graph $G = (V, E)$, a root $Z \subseteq V$, a weighting $(w_v : v \in V)$:

Find in $G$, for each stable set $R$ in $Z$,

a stable set $S_R$ with $S_R \cap Z = R$, that maximizes $w(S_R)$.

\hfill \text{(Stable set problem on $(G, Z)$)}

We call a pair $(G, Z)$ with $Z \subseteq V$ a rooted graph. Note that the servant is special in that the weighting is 0 on the root.

The stable set problem on a rooted graph is a collection of stable set problems, one for each stable set in the root. That the servant has this “multiple-problem” feature becomes an issue when we further decompose these subgraphs of the servant as if they were totally unrelated. We easily run into exponential explosion then; even if the root has as few as two stable sets, which is always the case when $U$ is nonempty. That means that we can hardly iterate decomposing “on the servant side”. A standard way to address the issue is to try to avoid servant-graphs that can be further decomposed, for instance by taking $V_2$ inclusion-wise minimal. Clique cutsets with $V_2$ inclusion-wise minimal have that property and can be found efficiently (Whitesides [17]). But for the amalgam decompositions used in Theorem 3 this will not work. Cornuéjols and Cunningham [12] gave a polynomial-time algorithm to find an amalgam separation with minimal servant, but as illustrated by Figure 1 that does not guarantee that the amalgam blocks will have no amalgams. So, what then?

Just forbidding to decompose “on the servant side” and ignore occasions that arise, limits the applicability of the approach too much—at least for our purposes. There is a way out: the reduction of the stable set problem to the master and the servant extends easily to rooted graphs $(G, Z)$ with $Z \subseteq V_1 \cup U$. The reason is that the reduction applies to each single stable set problem in the master—with the same servant and the same correction terms. Hence, the stable set problem on $(G, Z)$ reduces to:

Find in $G_{V_1 \cup U}$, for each stable set $R$ in $Z$,

a stable set $S_R$ with $S_R \cap Z = R$, that maximizes $w(S_R) + c_{S_R \cap U}$.

\hfill \text{(Master for $(G, Z)$)}

Now this is crucial to our approach. The stable set problem on $G$ is the same as the stable set problem on the rooted graph $(G, \emptyset)$. Starting from that the rooted graph perspective, we reduce a stable set problem on a rooted graph into one stable set problem on the rooted master-graph and one stable set problem on the rooted servant-graph. When we iterate this by applying this reduction to a similar partition $(V_1', U', V_2)$ for the master-graph or for the servant-graph, we reduce he initial rooted graph $(G, \emptyset)$ with $G = (V, E)$ into a collection of rooted induced subgraphs such that at each time during this procedure the “non-root”-parts of these subgraphs partition $V$. This is a promising feature for an attempt to utilize this
reduction to the stable set problem on a graph $G$ —and, the rooted graph approach to keep the stable set problems on the “servant side” in a “rooted-graph-bundle” gives that for free. There are now two aspects left that affect the efficiency of the approach above. Both require a closer look into a single master/server reduction-step.

2.1 Preparing a single decomposition step

When applying a master/servant reduction we have to get over two hurdles:

- the servant-root can have exponentially many stable set problems,
- the master problem has a nonlinear objective function.

Both aspects puts severe limitations on the availability of the master/servant approach above for solving stable set problems.

The number of stable sets in the servant-root

The servant has to carry over the correction terms to the master. That means that the servant-roots that are generated during actual runs of a polynomial-time algorithm must have only polynomially many stable sets. Even if the master/servant reduction is applied only once. Since each stable set $S$ contains $2^{|S|}$ stable sets, this means that servant-roots used in a polynomial-time algorithm have bounded stability number. So any such algorithm has to come (explicitly or implicitly) with a bound on the stability number of the servant-root. The good thing is that such bound is not only needed but also sufficient to control the number of correction terms: an $n$-node graph with stability number $\alpha$, has at most $\binom{n}{\alpha} 2^\alpha$ stable sets.

So, the sets $U$ should have bounded stability number: they all become servant-roots, that is how the master/servant approach is set up. But there is space for improvement here. The bottleneck is the stability number of the servant-root at the time the correction terms are actually determined; before that there is no issue: at the time $(V_1, U, V_2)$ is found, constructing the servant takes only polynomial time. That gives the opportunity to toggle the servant once it is constructed.

A group of a graph is a nonempty set $X$ of nodes such that all nodes of $X$ have the same neighbors outside $X$. Suppose that $U$ has a partition $\mathcal{U}$ into groups of $G_{V_2\cup U}$ (we call that a grouping). Then correction term $c_T$ depends only on which members of $\mathcal{U}$ are met by $T$, not on how they are met. This means that we can replace $G_{V_2\cup U}$ in the servant by the graph $G_{V_2\cup \tilde{U}}$, obtained by shrinking each member $X \in \mathcal{U}$ to a single (new) node $\tilde{X}$. (For $T \subseteq U$, define $\tilde{T} = \{ \tilde{x} \in \tilde{U} : X \in \mathcal{U}, X \cap T \neq \emptyset \}.)$ Doing so, we get that, for each stable set $T \subseteq U$, the value $c_T$ in the servant above, is equal to the value $\tilde{c}_T = w(S_{\tilde{T}} \setminus \tilde{U})$ of the stable set $S_{\tilde{T}}$ determined by the solution $S_L$ with $L = \tilde{T}$ of following problem:

Find in $G_{V_2\cup \tilde{U}}$ for each stable set $L$ in $\tilde{U}$, a stable set $S_L$ with $S_L \cap \tilde{U} = L$, that maximizes $w(S_L \setminus \tilde{U})$. (Servant with $\tilde{U}$)

If $\tilde{U}$ is smaller than $U$, it has fewer stable sets and the servant has fewer correction terms to carry over to the master. So, involving a grouping relaxes the bounded-stability requirement.
on the sets $U$, by replacing it by such requirement on the servant-root. At first sight, the gain may appear incremental. After all, $|\bar{U}| < |U|$ is a very special property. But for us the gain is crucial: the decompositions in Theorem 3 come from node cutset separations with unbounded $\alpha(U)$, but with $\alpha(\bar{U}) = 1$. Actually, the stable set literature shows many graph decompositions for which—in our master/servant terminology—the stability number of the servant-root comes out a lot smaller than that of the set $U$. Extreme examples are 1- and 2-joins. They come with a grouping with one resp. two groups. Hence, the servant-roots of 1- and 2-joins have at most one or two nodes. Another example is the 6-join introduced by Conforti, Cornuéjols, Kapoor and Vušković [8,9] for the decomposition of balanced matrices. For combinatorial applications, shrinking groups of the servant root is a significant improvement.

There is a unique coarsest grouping and we can find it in polynomial time. We give an algorithmic proof of that. Clearly, if $U$ has groupings $\mathcal{U}$ with $\alpha(\mathcal{U}) < \alpha(U)$ then the coarsest grouping is among them.

Here is the algorithm: For $X \subseteq U$, let $\mathcal{N}(X)$ the collection of equivalence classes of the equivalence relation on $X$ defined by “having, in $G_{V \cup \bar{U}}$, the same neighbors outside $X$”.

Determine for $i = 0, \ldots, |U|$, the partitions $\mathcal{N}^i(U)$ of $U$ given by $\mathcal{N}^0(U) = \{U\}$ and $\mathcal{N}^{i+1}(U) = \bigcup \{\mathcal{N}(X) : X \in \mathcal{N}^i(U)\}$. Clearly we can do this in polynomial time. Note that $\mathcal{N}^{|U|+1}(U) = \mathcal{N}^{|U|}(U)$. Since for any set of nodes $X$ is a group if and only if $\mathcal{N}(X) = \{X\}$, this implies that $\mathcal{N}^{|U|}(U)$ is a grouping of $U$ in $G_{V \cup \bar{U}}$. It is easy to see that this is indeed the coarsest grouping.

**Linearizing the objective function of the master**

With a grouping $\mathcal{U}$, we can reformulate the master in terms of the root $\bar{U}$ of the servant as follows:

Find in $G_{V \cup \bar{U}}$, for each stable set $R$ in $Z$,

a stable set $S_R$ with $S_R \cap Z = R$, that maximizes $w(S_R) + \bar{c}_{\bar{R} \cap \bar{U}}$.

(Master with $\mathcal{U}$ for $(G,Z)$)

The objective function of the master is (in general) not linear. We linearize the master by adding nodes to the master-graph.

We first give a general definition. Consider a function $X \mapsto d_X$ from $S[\bar{U}]$ to $\mathbb{R}$. We say that a triple $[H, \gamma, \sigma]$ linearizes $d$, if $H = (U \cup L, F)$ is a graph with $H_U = G_U$, $\gamma = (\gamma_v : v \in U \cup L)$ is a weighting of $U \cup L$, and $\sigma$ is a number, such that for each $T \in S[U]$ the maximum value $\gamma(T)$ of stable set in $H$ with $S_T \cap U = T$ is equal to $d_T - \sigma$.

If $[H, \gamma, \sigma]$ with $H = (U \cup L, F)$ linearizes $\bar{c}$ and $L \cap (V_1 \cup U) = \emptyset$, then we can reformulate the master as a stable set problem with a linear objective function on the graph $G \cup H = (V_1 \cup U \cup L, E \cup F)$:

Find in $G \cup H$, for each stable set $R$ in $Z$,

a stable set $S_R$ with $S_R \cap Z = R$, that maximizes $w(S_R \setminus L) + \gamma(S_R \setminus V_1)$.

(Linearized master with $\mathcal{U}$ for $(G,Z)$)
Replacing the master by this linearized master changes the optimal values by a constant $\sigma$. Other than that, the optimal stable sets in the solution of the linearized master and the master match in that they coincide on $G_{V_1 \cup U}$.

Note that the correction terms $\tilde{c}$ are non-increasing with respect to set inclusion, that is $\tilde{c}_{X'} \geq \tilde{c}_X$ for each $X, X' \in S[\tilde{U}]$ with $X' \subseteq X$. We say that a graph $H$ linearizes $G_U$ with grouping $U$, or shorter, that $H$ linearizes the node cutset, if each nonnegative non-increasing function $d$ on $S[\tilde{U}]$ is linearized by some triple $[H, \gamma, \sigma]$. If $U$ is linearized by $H = (U \cup L, F)$ with $|L| \leq \tau$ we call the node cutset a $\tau$-linearizable cutset.

A canonical graph linearizing $G_U$ with $U$ is obtained by adding a record of $S[\tilde{U}]$. This means adding to $G_U$ a clique $U^{\text{record}}$ consisting of new nodes $r_S$, one for each $S \subseteq U$ with $\tilde{S} \in S[\tilde{U}]$, and connect each such $r_S$ to all nodes in $U \setminus (\bigcup S)$. We denote this new graph by $G_{V_1 \cup U}(U)$ and call it the record graph of $U$ and $G_{V_1 \cup U}$. (In case $U$ consist of singletons, we write $G_{V_1 \cup U}(U)$ instead of $G_{V_1 \cup U}(U)$.) To linearize the master, we let $\sigma = 0$ and take as weights $\gamma_r = \tilde{c}_S$, with $S \subseteq U, \tilde{S} \in S[\tilde{U}]$ and $\gamma_u = 0$ if $u \in U$.

To see this indeed linearizes the master, take $T \in S[U]$. The stable sets $S$ in $H$ with $S \cap U = T$ are: set $T$ and all sets $T \cup \{r_S\}$ with $S \subseteq U, \tilde{S} \in S[\tilde{U}]$, and $T \subseteq \bigcup S$. So, since $\gamma_r = \tilde{c}_S$ and since $\tilde{c}$ is non-increasing and nonnegative, the maximum of all values $\gamma(S_T)$ of stable sets $S$ in $G_{V_1 \cup U}(U)$ with $S_T \cap U = T$ is $\gamma(T) + \max \{0, \tilde{c}_T\} = \tilde{c}_T$, as claimed.

Since we only used that $\tilde{c}$ is nonnegative and non-increasing, we see that $G_U(U)$ linearizes $G_U$ with $U$. This in turn shows that any nonnegative non-increasing function $d$ can come up as correction terms for a master.

In Section 2.4 we consider node cutsets that induce 3-node paths. A 3-node path has 5 stable sets, so we get records on 5 nodes then. Five-node records are already quite big, but for the graphs considered in Section 2.4 they work out fine.

Besides these “3-node paths inducing” cutsets, all node cutsets we use are 1-linearizable. We analyze those in Section 2.3 and our usage of node cutsets inducing 3-node paths in Section 2.4.

### 2.2 Decomposition

We now formally state what constitutes a node cutset and a node cutset separation of a rooted graph.

A node cutset separation with grouping $U$ of a rooted graph $(G, Z)$ is a partition $(V_1, U, V_2)$ such that the following hold: $V_1$ and $V_2$ are not adjacent, and $Z \subseteq V_1 \cup U$, each member of $U$ is a group in $G_{V_1 \cup U}$, $|V_1 \cup U| > |U|$, $|V_2| > 0$, and each node in $U$ is adjacent to $V_2$. The last condition is only a non-degeneracy condition and easily achievable by moving elements from $U$ to $V_1$. We call $U$ a node cutset (with grouping $U$). The corresponding node cutset decomposition consists of two rooted graphs: the rooted master-graph $(G_{V_1 \cup U}, Z)$ and the rooted servant-graph $(G_{V_2 \cup \tilde{U}, U})$.

The set $U$ of nodes in the master-graph, the grouping $U$, and the root $\tilde{U}$ of the rooted servant-graph link the master to the servant Together with the rooted-master-graph and the rooted
servant-graph the triple $U, \mathcal{U}, \tilde{U}$ encodes the original rooted graph $(G, Z)$. We call the set $U$ in the master-graph the mark of the decomposition. This “mark/grouping/servant-root” link also encodes the correction terms $S \mapsto \tilde{c}_S$ on $S[\tilde{U}]$. To facilitate decomposing rooted graphs that carry marks of an earlier decomposition of which they were master-graphs, we introduce “templates”.

A template is a triple $(G, Z, \Omega)$, where $(G, Z)$ is a rooted graph and $\Omega$ is a (possibly empty) collection of node sets, called marks. Each $W \in \Omega$ comes with a partition $W$. We call $Z$ the root of the template.

The node cutset separation of a template $(G, Z, \Omega)$ is a node cutset separation $(V_1, U, V_2)$ of $G$ (with a grouping $\mathcal{U}$) such that $Z \subseteq V_1 \cup U$, $|V_1| + |U| > |\mathcal{U}|, |V_2| > 0$, and such that no member of $\Omega$ meets both $V_1$ and $V_2$. We call $U$ a node cutset of $(G, Z, \Omega)$.

The node cutset separation decomposes $(G, Z, \Omega)$ into the servant-template $(G_{V_2 \cup \tilde{U}}, \tilde{U}, \Omega_2)$ where $\Omega_2$ consists of the marks in $\Omega$ that meet $V_2$, and the master-template $(G_{V_1 \cup U}, Z, \Omega_1 \cup \{U\})$ where $\Omega_1$ consists of the marks in $\Omega$ that lie in $V_1 \cup U$. If both the master-template and the servant-template are in some class $\mathcal{C}$ of templates, we say that the decomposition is in $\mathcal{C}$.

If $(G, Z, \Omega)$ occurs in an ordered list $\mathcal{L}$ of templates, then we call the list

$$\mathcal{L}' = (\mathcal{L} \setminus \{(G, Z, \Omega)\}) \cup \{(G_{V_2 \cup \tilde{U}}, Z, \Omega_1), (G_{V_1 \cup U}, \tilde{U}, \Omega_2)\}$$

with $(G_{V_1 \cup U}, Z, \Omega_1)$ and $(G_{V_2 \cup \tilde{U}}, \tilde{U}, \Omega_2)$ placed in that order on the position of $(G, Z, \Omega)$ in $\mathcal{L}$, a decomposition of $(G, Z, \Omega)$ in $\mathcal{L}$. A decomposition-list of $(G_1, Z_1, \Omega_1)$ is either $\{(G_1, Z_1, \Omega_1)\}$, or (recursively) defined as the result of a decomposition of a member $(G, Z, \Omega)$ in a decomposition-list of $(G_1, Z_1, \Omega_1)$.

Let $G = (V, E)$ be a graph and $Z \subseteq V$. We reduce a stable set problem on the rooted graph $(G, Z)$ by decomposing the initial list $\{(G, Z, \emptyset)\}$ into a decomposition-list

$$\{(G_1, Z_1, \Omega_1), (G_2, Z_2, \Omega_2), \ldots, (G_k, Z_k, \Omega_k)\},$$

and then solve, for each $i = 1, \ldots, k$, the corresponding stable set problems on the rooted graphs $(G_i, Z_i)$ with correction functions on $S[\tilde{W}]$ with $W \in \Omega_i$. At the time we do that, for a particular $i$, we need the solution of the stable set problem on the templates $(G_j, Z_j, \Omega_j)$ with $j$ in $I(i) = \{j = 1, \ldots, k : Z_j = \tilde{W}, W \in \Omega_i\}$. Since, by the ordering of the decomposition-list, we have $I(i) \subseteq \{i + 1, \ldots, k\}$, a simple rule that achieves that is tracking the list from right to left.

Since the non-root nodes of the members in the list partition $V \setminus Z$, we get the following results

**Lemma 4.** Let $G = (V, E)$ be a graph, $(G, Z, \Omega)$ a template, and let $\mathcal{L}$ be a decomposition-list of $(G, Z, \Omega)$. If each member of $\mathcal{L}$ has a non-root node, then $|\mathcal{L}| \leq |V \setminus Z|$.

If $(G, Z, \Omega)$ is a template, then $G(\Omega)$ is the graph obtained by adding records $W_{\text{record}}$ for each $W \in \Omega$. If $\mathcal{C}$ is a class of templates, then $\mathcal{C}_{\text{record}}$ denotes all rooted graphs $(G(\Omega), Z)$ with $(G, Z, \Omega) \in \mathcal{C}$.

**Theorem 5.** Let $\mathcal{C}$ and $\mathcal{P}$ be classes of templates, such that there exists a polynomial-time algorithm that finds a node cutset decomposition in $\mathcal{C}$ for each input from $\mathcal{C} \setminus \mathcal{P}$. If the stable
set problem on rooted graphs in $\mathcal{P}_{\text{record}}$ can be solved in polynomial time, then the stable set problem on rooted graphs in $\mathcal{C}_{\text{record}}$ can be solved in polynomial time.

The linearized cutset decomposition corresponding to a node cutset separation $(V_1, U, V_2)$ with grouping $\mathcal{U}$, consists of the rooted linearized master-graph $(G \cup H, Z)$ where $H = (U \cup L, F)$ is a graph with $L \cap V_1 = \emptyset$ that linearizes $G_U$ with $\mathcal{U}$. A linearized decomposition-list of $(G_1, Z_1)$ is either $\{(G_1, Z_1)\}$, or (recursively) defined as the result of a linearized decomposition of a member in a linearized decomposition-list of $(G_1, Z_1)$.

Note that the members of a linearized decomposition-list can be larger than $G$. This is not the case for $\tau$-linearized decompositions, for these are linearized decompositions such that $|L| \leq \tau$ and $|V_2| > \tau$. We always take $L$ inclusion-wise minimal in a $\tau$-linearized decomposition. Members of a linearized decomposition-list need not be subgraphs of $G$.

Mind that Lemma 4 refers to templates and (when suppressing marks or roots) to rooted graphs and graphs, and to $0$-linearized decompositions, but not to $\tau$-linearized decompositions with $\tau \geq 1$.

### 2.3 0/1-linearized decompositions

Lemma 4 implies that a decomposition along $0$-linearizable cutsets $U$ will generate only a linear number of graphs. Note that, for $[G_U, \gamma, \sigma]$ to linearize a non-increasing function $d$ on $\mathcal{U}$ forces the values $\sigma = d_\emptyset$ and $\gamma_u = d_{\{u\}} - d_\emptyset$ for all $u \in U$. So $[G_U, \gamma, \sigma]$ linearizes $d$ when $U$ is a clique, but not otherwise: if $a$ and $b$ were nonadjacent nodes in $U$, then the function $S \mapsto d_S$ that takes value 1 if $S = \emptyset$ and 0 otherwise, would have $-1 = d_{\{a,b\}} - \sigma = \gamma_a + \gamma_b = -2$; which is absurd. If $U$ is a clique and $V_1$ and $V_2$ are both nonempty, then $U$ is a clique cutset. If $U$ is a clique and $V_1 = \emptyset$ and, so, $|U| < |U|$, then $U$ contains pair of adjacent twins $u, v$ ($u, v$ are twins if they have the same neighbors in $V \setminus \{u, v\}$). Actually if $u, v$ are adjacent twins, then $(\emptyset, \{u, v\}, V \setminus \{u, v\})$ is a $0$-linearizable separation. So, the $0$-linearizable cutsets are the clique cutsets and (more or less) the adjacent twins.

Since clique cutsets can be found in polynomial time (Whitesides [17]), we get the following consequence of Lemma 4.

**Corollary 6** (Whitesides [17]). Let $\mathcal{G}$ be a class of graphs closed under clique cutset decomposition. If $\mathcal{P} \subseteq \mathcal{G}$ contains all members of $\mathcal{G}$ without clique cutsets, then the stable set problem on graphs in $\mathcal{G}$ is solvable in polynomial time if and only if then the stable set problem on graphs in $\mathcal{P}$ is solvable in polynomial time.

So $0$-linearized decompositions of a rooted graphs are well-understood: the node cutsets are cliques, they can be found in polynomial time, the $0$-linearized decomposition-lists have only linearly many members and use only proper induced subgraphs.

The same is true for $1$-linearized decompositions, with two exceptions: the linear bound is only quadratic and it is not the node cutset but only the servant-root that is guaranteed to be a clique.

We first show that $\tilde{\mathcal{U}}$ is a clique. Linearizing a nonnegative non-increasing function $d$ on $S[\tilde{\mathcal{U}}]$
with only one extra node \( r \) forces the values:

\[
\sigma = d_{\emptyset} - \max\{0, \gamma_r\} \quad \text{and} \quad \gamma_u = \begin{cases} 
\max\{0, \gamma_r\} - 1 & \text{if } u \in U \text{ is not adjacent to } r, \\
0 & \text{if } u \in U \text{ is adjacent to } r.
\end{cases}
\]

For the function that takes 1 for \( \emptyset \) and 0 otherwise, this means

\[
\sigma = 1 - \max\{0, \gamma_r\} \quad \text{and} \quad \gamma_u = \begin{cases} 
\max\{0, \gamma_r\} - 1 & \text{if } u \in U \text{ is not adjacent to } r, \\
-1 & \text{if } u \in U \text{ is adjacent to } r.
\end{cases}
\]

Hence for a nonadjacent pair \( a, b \) that are both not adjacent to \( r \), we get:

\[
\max\{0, \gamma_r\} - 1 = d_{\{a,b\}} - \sigma = \gamma(\{a,b\}) + \max\{0, \gamma_r\} = \max\{0, \gamma_r\} - 2,
\]

which is impossible. For a nonadjacent pair \( a, b \) with \( b \) adjacent to \( r \), we get:

\[
\max\{0, \gamma_r\} - 1 = d_{\{a,b\}} - \sigma = \gamma(\{a,b\}) = \gamma_a + \max\{0, \gamma_r\} - 1.
\]

This can only be the case when \( \gamma_a = 0 \neq -1 \). Hence each nonadjacent pair \( a, b \) lies in \( N_H(r) \).

If \( \{a\} \neq \{b\} \), then the function that takes the values \( d_{\emptyset} = 3, d_{\{a\}} = d_{\{b\}} = 2 \) and 0 otherwise (so also \( d_{\{a,b\}} = 0 \)), would give: \(-\sigma = d_{\{a\}} - \sigma = \gamma(\{a,b\}) = 4 - 2\sigma \). Hence, then \( \sigma = 4 \), so \( \max\{0, \gamma_r\} = 1 \), which again is impossible. So, \( \{a\} = \{b\} \). So we see that each nonadjacent pair in \( U \) lies in some member of \( \mathcal{U} \), in other words: \( \mathcal{U} \) is a clique, as claimed.

We have also seen in our analysis that all nonadjacent pairs in \( U \) lie in \( N_H(r) \). But we get more out of it: Let \( A_1 \) and \( A_2 \) be members of \( \mathcal{U} \) that both contain a nonadjacent pair, \( a_1, b_1 \) resp. \( a_2, b_2 \) say. Then, if \( [G_{U \cup \{r\}}, \gamma, \sigma] \) linearizes function \( d \), we get:

\[
d_{\{a_i, b_i\}} - \sigma = d_{\{a_i, b_i\}} - \gamma(\{a_i, b_i\}) = d_{\{a_i\}} + d_{\{b_i\}} - 2\sigma = 2d_{\{a_i\}} - 2\sigma,
\]

for both \( i = 1 \) and \( i = 2 \). So \( d_{A_1} = d_{A_2} \). And that, for each nonnegative non-increasing function \( d \) on \( \mathcal{U} \). Thus \( A_1 = A_2 \): in other words, \( \mathcal{U} \) has only one member that is not a clique. This indicates the following result.

**Lemma 7.** If \( (G, Z) \) has a 1-linearized decomposition with node cutset \( U \) and grouping \( \mathcal{U} \), then \( (G, Z) \) has a 1-linearized decomposition such that the servant-root \( \tilde{U} \) is a clique and the linearized master-graph is a proper induced subgraph of \( G \).

**Proof.** We may assume that \( U \) not is a clique. By the deduction above, \( \mathcal{U} \) has a member, \( A \) say, that contains all 2-element stable sets in \( U \). Since \( U \) is not a clique, the set \( A \) is nonempty. Take \( r \in V_2 \) adjacent to \( A \). Since \( A \) is a group, \( A \) is contained \( N_G(r) \).

Let \( d \) be nonnegative and non-increasing on \( \mathcal{U} \). The values of \( d \) are \( d_{\emptyset}, d_{\{u\}} \) with \( u \in U \setminus A \), and \( d_{\bar{A}} \). Define \( \sigma \) and \( \gamma \) as follows:

\[
\sigma = d_{\bar{A}}, \quad \gamma_r = d_{\emptyset} - d_{\bar{A}}, \quad \text{and, for } u \in U, \quad \gamma_u = \begin{cases} 
\max\{0, \gamma_r\} - 1 & \text{if } u \not\in N_G(r) \\
0 & \text{if } u \in N_G(r)
\end{cases}
\]

and

\[
\gamma_u = \begin{cases} 
\max\{0, \gamma_r\} - 1 & \text{if } u \not\in N_G(r) \\
0 & \text{if } u \in N_G(r)
\end{cases}
\]
that have a clique consisting of all but two of the nodes. We make as of now the convention that we will not decompose near-cliques, these are graphs that have a clique consisting of all but two of the nodes.

Next we analyze normal decompositions in $G$, so when $\text{load}(G, Z) \geq 0$. By (2) and because $\text{load}(G, Z) \geq 2$, at least one of $(G_1, Z)$ and $(G_2, \bar{U})$ has positive load. So a normal decomposition in $G$ gives a list $\mathcal{H}$ with:

$$\text{load}(\mathcal{H}, Z) = \text{load}(\mathcal{G}, Z), \quad |\mathcal{H}| = |\mathcal{G}|, \quad \text{root-size}(\mathcal{H}, Z) < \text{root-size}(\mathcal{G}, Z).$$

Moreover:

$$\text{load}(\mathcal{H}, Z) < \text{load}(\mathcal{G}, Z) \quad \text{or} \quad |\mathcal{H}| > |\mathcal{G}|.$$ 

(Note that $\bar{u} = \bar{A}$ if $u \in A$, so then $\gamma_u = 0$). Since $d_{\emptyset} \geq d_{\bar{A}}$, it is straightforward to see that $[G_{V \cup \{r\}}, \gamma, \sigma]$ linearizes $d$. So the master can be formulated as a stable set problem on $G_{V \cup U \cup \{r\}}$, which is an induced subgraph of $G$. The definition of 1-linearized decomposition in Section 2.2 implies that this containment is proper. 

We make as of now the convention that we will not decompose near-cliques, these are graphs that have a clique consisting of all but two of the nodes.

Consider a member $(G, Z)$ in $\mathcal{G}$ with 1-linearized decomposition $(G_1, Z)$, $(G_2, \bar{U})$, coming from a separation $(V_1, U, V_2)$ with a grouping $\bar{U}$. Recall from the definition of $\tau$-linearized decompositions that $G_1 = G_{V_1 \cup U \cup \{r\}}$ for some node $r \in V_2$, if $U$ is not a clique, and that $G_1 = G_{V_1 \cup U}$, otherwise. By our convention, $G$ is not a near-clique, so $\text{load}(G, Z) \geq 2$.

We call a 1-linearized decomposition special if $\text{load}(G_1, Z) = -1$ and normal if $\text{load}(G_1, Z) \geq 0$. Let $\mathcal{G}_{>0}$ consist of the members of $\mathcal{G}$ with positive load. We analyze the impact of a single decomposition in $\mathcal{G}$ on the following parameters:

1. The total $\text{load}(\mathcal{G}_{>0}) = \sum_{(G', Z') \in \mathcal{G}_{>0}} \text{load}(G', Z')$ of the positive loads in $\mathcal{G}$.
2. The number $|\mathcal{G}_{>0}|$ of members of $\mathcal{G}$ with positive load.
3. The total root-size($\mathcal{G}_{>0}$) = $\sum_{(G', Z') \in \mathcal{G}_{>0}} |Z'|$ of the members of $\mathcal{G}$ with positive load.

Clearly, these numbers satisfy:

$$|\mathcal{G}_{>0}| \leq \text{load}(\mathcal{G}_{>0}).$$

We use the following identity:

$$\text{load}(G_1, Z) + \text{load}(G_2, \bar{U}) = \begin{cases} \text{load}(G, Z) - 1 & \text{if } U \text{ is a clique,} \\ \text{load}(G, Z) & \text{if } U \text{ is not a clique}. \end{cases}$$

First we analyze special decompositions. For those, $V_1 \cup U = Z$, so $U$ is a clique. Then (2) gives: $\text{load}(G_2, \bar{U}) = \text{load}(G, Z)$, which is positive. Since $|V_1 \cup U| > |\bar{U}|$ and $Z = V_1 \cup U$, the root of $(G_2, \bar{U})$ is smaller than the root of $(G, Z)$. So a special decomposition in $\mathcal{G}$ gives a list $\mathcal{H}$ with:

$$\text{load}(\mathcal{H}, Z) = \text{load}(\mathcal{G}, Z), \quad |\mathcal{H}| = |\mathcal{G}|, \quad \text{root-size}(\mathcal{H}, Z) < \text{root-size}(\mathcal{G}, Z).$$

Next we analyze normal decompositions in $\mathcal{G}$, so when $\text{load}(G_1, Z) \geq 0$. By (2) and because $\text{load}(G, Z) \geq 2$, at least one of $(G_1, Z)$ and $(G_2, \bar{U})$ has positive load. So a normal decomposition in $\mathcal{G}$ gives a list $\mathcal{H}$ with:

$$\text{load}(\mathcal{H}, Z) \leq \text{load}(\mathcal{G}, Z), \quad |\mathcal{H}| \geq |\mathcal{G}|, \quad \text{root-size}(\mathcal{H}, Z) \leq \text{root-size}(\mathcal{G}, Z) + n - 1.$$
Indeed, if \( \text{load}(\mathcal{H}_{>0}) = \text{load}(\mathcal{G}_{\geq 0}) \), then \( \text{load}(G_1, Z) + \text{load}(G_2, \tilde{U}) = \text{load}(G, Z) \), so by (2), \( U \) is not a clique. Then \( \text{load}(G_2, \tilde{U}) = |V_2| - 1 \geq 1 \) and \( U \) contains a node that is not in \( Z \). Since the extra node in \( G_1 \) is also not in \( Z \), we see that also \( \text{load}(G_1, Z) \geq 1 \). So \( |\mathcal{H}_{>0}| > |\mathcal{G}_{>0}| \). This proves (5).

Now (1)-(5) tell that will make at most \( n \) normal decompositions, and that we will, over time, create no more than \( n(n-1) \) root nodes. So we do at most \( n(n-1) \) special decompositions and at most \( n^2 \) in total. So \( |\mathcal{G}| \leq n^2 \), as claimed.

Theorem 8. Let \( \mathcal{R} \) and \( \mathcal{P} \) be classes of rooted graphs where the roots are cliques such that exists a polynomial-time algorithm that for each input from \( \mathcal{R} \setminus \mathcal{P} \) finds a 1-linearized decomposition into \( \mathcal{R} \). If the stable set problem on rooted graphs in \( \mathcal{P} \) is solvable in polynomial time, then the stable set problem on rooted graphs in \( \mathcal{R} \) is solvable in polynomial time.

Amalgam decomposition

A partition \((V_1, A_1, K, A_2, V_2)\) of \( V \) is an amalgam separation for \( G = (V, E) \), with amalgam \((A_1, K, A_2)\), if \( K \) is a (possible empty) clique and \( A_1, A_2 \) are nonempty subsets of \( N_G(K) \) such that: all nodes of \( A_1 \) are adjacent with all nodes in \( A_2 \), \( V_1 \cup A_1 \) is not adjacent with \( V_2 \), \( V_2 \cup A_2 \) is not adjacent with \( V_1 \), and \(|V_1 \cup A_1| |V_2 \cup A_2| \geq 2\). Amalgams were introduced by Burlet and Fonlupt [2] to design a polynomial-time algorithm to recognize Meyniel graphs, a special class of perfect graphs. Cunningham and Cornuésjols [12] designed a polynomial-time algorithm that finds an amalgam separation or decides that none exists. In [2, 12], a graph with an amalgam separation \((V_1, A_1, K, A_2, V_2)\) is decomposed into two amalgam blocks \( G_1 \) and \( G_2 \), were \( G_i \) is obtained from \( G_{V_i \cup A_i \cup K} \) by adding a single new node adjacent to all nodes in \( A_i \cup K \), for \( i = 1, 2 \). We call the pair \( G_1, G_2 \) an amalgam decomposition of \( G \).

Observe that if \((A_1, K, A_2)\) is an amalgam, then \( A_1 \cup K \) is a 1-linearizable cutset of \((G, \emptyset)\) with grouping \( \mathcal{U} = \{A_1\} \cup \{u\} : u \in K \). And, if \((G_1, \emptyset), (G_2, \tilde{U})\) is the 1-linearized decomposition of \((G, \emptyset)\) corresponding to that cutset, then \( G_1, G_2 \) is the amalgam decomposition of \( G \) corresponding to amalgam \((A_1, K, A_2)\). This means that if \( \mathcal{G} \) is an amalgam decomposition-list of \( G = (V, E) \), then there exists a set of nodes \( Z' \) in each \( G' \in \mathcal{G} \) so that the list \( \{(G', Z') : G' \in \mathcal{G}\} \) is a 1-linearized decomposition-list of \((G, \emptyset)\). This 1-linearized decomposition-list has at most \(|V|^2 \) rooted graphs, so \( \mathcal{G} \) contains at most \(|V|^2 \) graphs.

Theorem 3 uses amalgams to describe the structure of cap-free graphs with no even holes. When we wanted to use that to design a maximal stable set problem on these graphs, we ran into the multiple-problem aspect of the servant. Cornuésjols and Cunningham [12] can find an amalgam separation with minimal servant, but as illustrated by Figure I, that does not guarantee that the amalgam blocks will have no amalgams. This lead us to the rooted graph approach of this paper; it is essential here. To our knowledge it is new.

Theorem 9. Let \( \mathcal{G} \) be a class of graphs closed under amalgam decomposition. If \( \mathcal{P} \subseteq \mathcal{G} \) contains all members of \( \mathcal{G} \) without amalgams, then the stable set problem on graphs in \( \mathcal{G} \) is solvable in polynomial time if and only if then the stable set problem on graphs in \( \mathcal{P} \) is solvable in polynomial time.
Proof. Let \( \mathcal{R} \) be the class of rooted graphs \((G, Z)\) such that \( G \in \mathcal{G} \) and \( Z \) is a clique. Let \( \mathcal{Q} \) consist of all members of \( \mathcal{R} \) and all near-cliques. Then we can solve the stable set problem on \( \mathcal{Q} \) in polynomial-time. So by Theorem 8 we only need to design a polynomial-time algorithm to find 1-linearized decompositions for all rooted graphs \((G, Z)\) with \( G \in \mathcal{G} \setminus \mathcal{Q} \) and were \( Z \) is a clique.

This is this algorithm: If \( G \) is a near-clique, \((G, Z)\) is in \( \mathcal{Q} \). Otherwise, use the algorithm of Cornuéjols and Cunningham [12] to search for amalgams in \( G \). If none is found: \( G \in \mathcal{P} \), so in \( \mathcal{Q} \). If an amalgam separation \((V_1, A_1, K, A_2, V_2)\) is found, proceed as follows to find a 1-linearized decomposition for \((G, Z)\).

If the clique \( Z \) meets \( V_1 \cup V_2 \), the root \( Z \) is contained in \( V_1 \cup A_1 \cup K \) or in \( V_2 \cup A_2 \cup K \), so then one of the node cutsets \( A_1 \cup K \) and \( A_2 \cup K \) yields a 1-linearized decomposition of \((G, Z)\).

If \( Z \) does not meet \( V_1 \cup V_2 \), it lies in \( A_1 \cup K \cup A_2 \) and thus \( K \cup Z \) is a clique. Since \( G \) is not a near-clique, it has at least 3 nodes outside \( K \cup Z \). Assume two of those lie in \( A_2 \cup V_2 \). Then the node cutset \( A_1 \cup K \cup Z \) yields a 1-linearized decomposition of \((G, Z)\). \( \square \)

![Figure 1: The amalgam of the graph on the left is unique (in this case the clique \( K \) is empty). Its blocks have amalgams, for instance \( \{a_1, a_2\}, \{u\}, \{b\} \).](image)

2.4 Decomposing into fan-templates—Proof of Theorem 1

To prove Theorem 1 we use Theorem 3 which decomposes cap-free graphs into triangle-free graphs, and Theorem 10 which further decomposes triangle-free “odd-signable” graphs into “the cube” and “fan-templates”.

A graph is odd-signable if it contains a set \( F \) of edges such that \(|F \cap C|\) is odd for each chordless cycle \( C \). Triangle-free graphs with no even holes are clearly odd-signable.

The cube is the unique 3-regular bipartite graph on 8 vertices, so that is the 1-skeleton of the three dimensional cube. The cube is odd-signable.

A fan with base \((u, c, v)\) consists of an \( uv \)-path \( P \) together with a node \( c \) adjacent to a subset of nodes of \( P \) including \( u \) and \( v \). If \( Z \) is a subset of the base of fan \( G = (V, E) \) and \( \Omega \) is a collection of triples in \( V \), then we call \((G, Z, \Omega)\) a fan-template. If the triples in \( \Omega \) each induce a subpath of one of the holes of \( G \), we call the fan-template good. The following
results say that good fans do come up in decomposing cap-free odd-signable graphs and that they are well tractable.

**Theorem 10** ([7, Theorems 2.4 and 6.4]). If $G = (V, E)$ is a triangle-free odd-signable graph that is not isomorphic to the cube and has no clique cutset, then the template $(G, \emptyset, \emptyset)$ can in polynomial time be decomposed into list of at most $|V|$ good fan-templates.

**Proof.** By [7, Theorem 2.4], $G$ has no induced subgraph isomorphic to the cube. Now [7, Theorem 6.4] says that $G$ can be obtained from the hole by a sequence of “good ear additions” (defined in [7, Definition 6.1]). An ear addition is the reverse of a node cutset decomposition where the servant graph is a fan and the node cutset is the base of the fan. So reversing the sequence of good ear additions amounts to a decomposition of $G$ into fan-templates. The marks of these templates are the locations where the ears are added and the goodness of these ear additions means that the fan-templates are good. Since adding an ear increases the size of the graph, we obtain at most at most $|V|$ good fan-templates. As the node cutsets needed for this decomposition are triples, we can find them in polynomial time, by enumeration.

**Lemma 11.** If $(G, Z, \Omega)$ is a good fan-template with base $(u, c, v)$ and $n$ nodes, then $G(\Omega) - c$ can be decomposed along 2-node clique cutsets into a decomposition-list of at most $|\Omega| + n$ graphs, each with at most 8 nodes.

**Proof.** The graph $G(\Omega) - c$ consists of the path $G-c$ together with all the records. As each mark of a fan-template is a 3-node subpath of one of the holes of $G$, each record has 5 nodes and is attached in $G(\Omega) - c$ to a 2- or 3-node subpath of $G(\Omega) - c$. Note that each edge of the path $G(\Omega) - c$ forms a 2-node clique cutset of $G(\Omega) - c$ (except maybe the first or the last edge of $G(\Omega) - c$). If we decompose $G(\Omega) - c$ along all these 2-node clique cutsets, we obtain collection of graphs, each consisting of a 2- or 3-node subpath of $G(\Omega) - c$ together with at most one of the records. Such graphs have at most 8 nodes.

**Lemma 12.** The stable set problem on rooted record graphs of good fan-templates is solvable in polynomial time.

**Proof.** Let $(u, c, v)$ be the base of a good fan-template $(G, Z, \Omega)$. By Corollary 6 and Lemma 11 for each of the (at most 5) stable sets $T$ in $Z$, we can find in polynomial time, a stable set $S_T$ in $G(\Omega)$ that has maximum weight among those that intersect $Z$ in $T$. Among these stable sets $S_T$, we choose the best one.

**Theorem 13.** The stable set problem on cap-free odd-signable graphs is solvable in polynomial time.

**Proof.** If node $u$ in graph $G$ is adjacent to all other nodes, then we can set it aside to compare it with a maximum weight stable set in $G - u$, once we found that. The stable set problem on the cube can be found by enumeration. So the result follows from Theorems 3, 9, 5, 10, Corollary 6 and Lemma 12.
3 Stable set polytopes

We examine extended formulations for the stable set polytope of a graph that admits certain decompositions into smaller graphs and combine formulations for these smaller parts to one for the whole graph. We apply this to cap-free odd-signable graphs and thus prove Theorem 2.

Notation. We denote the convex hull of characteristic vectors of stable sets in a graph $G = (V, E)$ by $P[G]$. If $\mathcal{L}$ is a collection of cliques in $G$, we denote the collection of stable sets in $G$ that intersect each member of $\mathcal{L}$ by $S[G, \mathcal{L}]$ and the convex hull of the characteristic vectors of these stable sets by $P[G, \mathcal{L}]$. So $P[G] = P[G, \emptyset]$ and $P[G, \mathcal{L}]$ is the face of $P[G]$ obtained by setting at equality all clique constraints associated to the cliques in $\mathcal{L}$. If $x \in \mathbb{R}^V$ and $H = G_U$ with $U \subseteq V$, we denote the restriction of $x$ to $U$ by $x_H$, or by $x_U$.

3.1 Extended formulations and records

Extra variables used in extended formulations here mostly come from records. If $U$ is a set of nodes in $G$, we denote by $\mathcal{L}(U)$ the collection consisting of the clique $U^{\text{record}}$ together with all the cliques $\{v\} \cup \{r_T : v \notin T \in S[G_U]\}$ with $v \in U$. The following result says that any (extended) formulation for $P[G(U), \mathcal{L}(U)]$ is an extended formulation for $P[G]$.

Lemma 14. Let $U$ be a set of nodes in a graph $G$. Then each stable set $S$ in $G$ has a unique extension to a member of $S[G(U), \mathcal{L}(U)]$, namely $S \cup \{r_{S \cap U}\}$, and

$$P[G] = \{x_S : x \in P[G(U), \mathcal{L}(U)]\}.$$  

If, moreover, $\mathcal{L}$ is a collection of cliques in $G$, then

$$P[G, \mathcal{L}] = \{x_S : x \in P[G(U), \mathcal{L} \cup \mathcal{L}(U)]\}.$$  

Proof. Proving that $S \cup \{r_{S \cap U}\}$ is the only extension of $S$ in $S[G(U), \mathcal{L}(U)]$ is straightforward. Moreover, if $S$ meets $\mathcal{L}$, then so does $S \cup \{r_{S \cap U}\}$. The rest now follows as each of $P[G], P[G(U), \mathcal{L}(U)], P[G, \mathcal{L}]$, and $P[G(U), \mathcal{L} \cup \mathcal{L}(U)]$ are convex hulls of stable sets.

3.2 Composing across node cutsets

Lemma 14 expresses the stable set polytope of a graph $G$ as a particular face of the stable set polytope of the record graph of $U$ and $G$. Our next result is that those particular faces admit a simple composition rule when $U$ is a node cutset.

Theorem 15. Let $(V_1, U, V_2)$ be a node cutset separation of graph $G = (V, E)$. Moreover, let $\mathcal{L}_1$ be a collection of cliques in $G_1 = G_{V_1 \cup U}$ and let $\mathcal{L}_2$ be a collection of cliques in $G_2 = G_{V_2 \cup U}$. Then, each $x \in \mathbb{R}^{|U|^{\text{record}}}$ satisfies:

$$x \in P[G(U), \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}(U)] \text{ if and only if } x_{G_1(U)} \in P[G_1(U), \mathcal{L}_1 \cup \mathcal{L}(U)] \text{ and } x_{G_2(U)} \in P[G_2(U), \mathcal{L}_2 \cup \mathcal{L}(U)].$$

Hence

$$P[G] = \{x \in \mathbb{R}^{|V|} : \exists y \in \mathbb{R}^{|U|^{\text{record}}} \left[ (x_{G_1}, y) \in P[G_1(U), \mathcal{L}(U)] \text{ and } (x_{G_2}, y) \in P[G_2(U), \mathcal{L}(U)] \right] \}.$$
Proof. By Lemma \[14\] the second assertion follows from the first one. The “only if” direction of the first assertion is obvious. Consider the “if” direction. Let \( x \in \mathbb{R}^{|V_1 \cup L_{\text{record}}|} \) be such that \( x_{G_1(U)} \in P[G_1(U), L_1 \cup L(U)] \) and \( x_{G_2(U)} \in P[G_2(U), L_2 \cup L(U)] \). Then, for \( i = 1, 2 \), there exists a positive integer \( n_i \), so that \( n_ix_{G_i(U)} \) is the sum of the characteristic vectors of a collection of (not necessarily distinct) stable sets \( S_1^i, \ldots, S_n^i \in G_i(U) \). By replicating members in these two collections of stable sets (if necessary), we may assume that \( n_1 = n_2 \); let \( n = n_1 = n_2 \).

Since \( x_{G_i(U)} \in P[G_i(U), L(U)] \), each \( S_1^i, \ldots, S_n^i \) meets each clique in \( L(U) \) exactly once. So, for each stable set \( S \) in \( U \), the number of sets among \( S_1^i, \ldots, S_n^i \) that intersect \( U \) in \( S \) is equal to \( n(x_{G_i(U)})_{rs} = nx_{rs} \). As this applies to both \( i = 1, 2 \), we can renumber \( S_1^2, \ldots, S_n^2 \) so that \( S_1^2 \cap U = S_2^2 \cap U \) for \( j = 1, \ldots, n \). Doing so, \( x \) is a convex combination of the characteristic vectors of the stable sets \( S_1^1 \cup S_1^2, \ldots, S_n^1 \cup S_n^2 \). Since \( S_j^1 \in P[G_1(U), L_1 \cup L(U)] \) and \( S_j^2 \in P[G_2(U), L_2 \cup L(U)] \) for all \( j \), each \( S_j^1 \cup S_j^2 \) is in \( S[G(U), L_1 \cup L_2 \cup L(U)] \). Hence, \( x \in P[G(U), L_1 \cup L_2 \cup L(U)] \), as claimed.

Consider Theorem \[15\] in case \( U \) is a clique cutset. Then \( U_{\text{record}} = \{r_2\} \cup \{r_{(v)} : v \in U\} \). Moreover, for \( (x, y) \in \mathbb{R}^V \times \mathbb{R}^{U_{\text{record}}} \), we have that \( (x, y) \in P[G(U), L \cup L(U)] \) if and only if:

\[
x_{G} \in P[G, L], \quad y_{r_2} = 1 - \sum_{v \in U} x_v, \quad \text{and} \quad y_{r_{(v)}} = x_v \ (v \in U).
\]

Applying this to each of the three graphs \( G, G_1, G_2 \) in Theorem \[15\] we obtain the following result of Chvátal.

**Corollary 16** (Chvátal \[5\]). Let \( (V_1, U, V_2) \) be a clique cutset separation of a graph \( G = (V, E) \) and let \( G_1 = G_{V_1 \cup U} \) and \( G_2 = G_{V_2 \cup U} \). Then:

\[
P[G] = \{ x \in \mathbb{R}^V : x_{G_1} \in P[G_1] \text{ and } x_{G_2} \in P[G_2] \}.
\]

If, moreover, \( L_1 \) is a collection of cliques in \( G_1 \) and \( L_2 \) is a collection of cliques in \( G_2 \), then each \( x \in \mathbb{R}^V \) satisfies:

\[
x \in P[G, L_1 \cup L_2] \text{ if and only if } x_{G_1} \in P[G_1, L_1] \text{ and } x_{G_2} \in P[G_2, L_2].
\]

In Corollary \[16\] we can not drop the condition that \( U \) is a clique. Indeed, let \( u \) and \( v \) be two nonadjacent nodes in \( U \) and suppose \( G_1 \) has a chordless even \( uv \)-path \( Q_1 \) and \( G_2 \) has a chordless odd \( uv \)-path \( Q_2 \). Consider the vector \( x \in \mathbb{R}^V \) with \( x_v = 1/2 \) if \( v \) lies on \( Q_1 \cup Q_2 \) and \( x_v = 0 \) otherwise. Then \( x \notin P[G] \), but \( x_{G_1} \in P[G_1] \) and \( x_{G_2} \in P[G_2] \).

Balas \[11\] has shown how to obtain an extended formulation for the convex hull of polytopes \( P_1, \ldots, P_k \), whose size is approximately the sum of the sizes of the descriptions for these polytopes. If \( A^i x + B^i y \leq d^i \) is an extended formulation for \( P_i (i = 1, \ldots, k) \), then Balas’s formulation for the convex hull reads:

\[
x = x^1 + \cdots + x^k, \quad \lambda_1 + \cdots + \lambda_k = 1; \quad A^i x^i + B^i y^i - \lambda_i d^i \leq 0, \quad \lambda_i \geq 0 \ (i = 1, \ldots, k).
\]

This can be used to obtain a linear description of \( P[G(U)] \) from such description of \( P[G] \) as follows: For every stable set \( S \) in \( U \), a description of the face of \( P[G(U)] \) given by \( x_{rs} = 1 \)
can be inferred from any linear description of the face \( \{ x \in P[G] : x_v = 1 (v \in S) \} \) of \( P[G] \). Since \( P[G(U)] \) is the convex hull of these faces, Balas’s formula \(^7\) gives an extended formulation of \( P[G(U)] \) whose size is in the order of \( |U^{\text{record}}| = |S(G(U))| \) times the size of the linear description of \( P[G] \). Combined with Theorem \(^15\) this means that we can use records to construct compact formulations of \( P[G] \) from such formulations for parts of a node cutset decomposition. So we get the following result.

**Theorem 17.** Let \( G \) be a graph and \( \{(G_1, Z_1, \Omega_1), \ldots, (G_k, Z_k, \Omega_k)\} \) be a decomposition-list of \((G, \emptyset, \emptyset)\). Assume we are given for each \( i = 1, \ldots, k \) an extended formulation with size \( m_i \) for \( P[G_i(\{Z_i \cup \Omega_i\})] \). Then there exists an extended formulation for \( P[G] \) with size at most \( O(k) + m_1 + \cdots + m_k \).

**Proof.** Recursively apply the following immediate corollary of Theorem \(^15\) if \((V_1, U, V_2)\) is a cutset separation of template \((G, Z, \Omega)\) with master template \((G_1, Z, \Omega_1)\) and servant template \((G_2, U, \Omega_2)\), then a vector \( x \) lies in \( P[G(\{Z \cup U \} \cup \Omega), \mathcal{L}(U)] \) if and only if \( x_{G_1(\{Z \cup U \} \cup \Omega_1)} \in P[G_1(\{Z \cup U \} \cup \Omega_1), \mathcal{L}(U)] \) and \( x_{G_2(\{U \cup \Omega_2\})} \in P[G_2(\{U \cup \Omega_2\}, \mathcal{L}(U)] \). \( \square \)

An alternative for adding a record to a graph \( G \) is lifting a node set \( U \) to a clique. This amounts to deleting \( U \) from \( G \) and replacing it by a clique with node set \( U^{\text{record}} \setminus \{r_8\} \), and connecting each \( r_S \in U^{\text{record}} \setminus \{r_8\} \) with each node in \( N_G(S) \setminus U \). We call the new graph the **clique lift** of \( U \) from \( G \). An advantage of clique lifts over records is that clique lifts yield extended formulations for stable sets that do not involve “tight clique constraints”: \( x(K) = 1 (K \in \mathcal{L}(U)) \).

**Lemma 18.** Let \( G^+ \) be the clique lift of \( U \subseteq V \) from a graph \( G = (V, E) \). Then the stable set polytope \( P[G] \) is the image of \( P[G^+] \) under the projection \( p : \mathbb{R}^{(V \setminus U) \cup (U^{\text{record}} \setminus \{r_8\})} \to \mathbb{R}^V \) defined by

\[
p_v(x) = \begin{cases} 
\sum_{S \subseteq S[G_U], S \ni v} x_{r_S} & \text{if } v \in U \\
x_v & \text{otherwise}.
\end{cases}
\]

Lifting a node cutset to a clique turns it into a clique cutset. So we get the following consequence of Corollary \(^16\)

**Corollary 19.** Let \((V_1, U, V_2)\) be a node cutset separation of graph \( G = (V, E) \). Moreover, let \( G^+, G^+_1, \) and \( G^+_2 \) be the clique lifts of \( U \) from \( G, G_{V_1 \cup U}, \) respectively \( G_{V_2 \cup U} \). Then each \( x \in \mathbb{R}^{(V \setminus U) \cup (U^{\text{record}} \setminus \{r_8\})} \) satisfies:

\[
x \in P[G^+] \text{ if and only if } x_{G^+_1} \in P[G^+_1] \text{ and } x_{G^+_2} \in P[G^+_2].
\]

### 3.3 Generalized amalgams

We give a decomposition rule for stable set polytopes of graphs \( G = (V, E) \) that admit a **generalized amalgam separation**, this is a pair \((U, W)\) where \( U \subseteq V \) and \( W \) is a partition of \( V \setminus U \) into nonempty sets \( W \) so that each node in \( B_{G-U}(W) \) is adjacent to each node in \( U \).

Generalized amalgam separation unifies a great variety of known separations. Clique cutset separation and amalgam separation are obvious special cases. A notable other example is the “strip-structure for trigraphs” introduced by Chudnovsky and Seymour \(^1\).
Faenza, Oriolo, and Stauffer [13] used strip-structures to obtain extended formulations and polynomial-time algorithms for stable sets problems in “claw-free” graphs. The “2-clique-bonds” that Galluccio, Gentile, and Ventura [14] use to compose linear formulations of stable set problems are generalized amalgam separations as well.

Before actually decomposing a graph along a generalized amalgam separation \((U, W)\), we first lift \(U\) to a clique. The generalized amalgam separation fully carries over to the clique lift, with the same \(W\) and all other structure, except for \(U\) internally. Lemma [18] explains the effect of clique lifts to the stable set polytope. So it is enough to discuss generalized amalgams separations \((K, W)\) for the case that \(K\) a clique.

For \(W \subseteq W\), we denote by \(A_W\) the collection of equivalence classes in \(B_{G-K}(W)\) of the relation “having the same neighbors outside \(W\)”. Related to \(A_W\) we will consider a clique \(A_W^{\operatorname{power}}\) consisting of new nodes \(r_X\), one for each subcollection \(X\) of \(A_W\).

The generalized amalgam decomposition of \(G\) along \((K, W)\) consists of a collection of graphs \(G(K, W)\), one for each \(W \subseteq W\), together with a “connecting” graph \(G^{\operatorname{connect}}(K, W)\). Each graph \(G(K, W)\) is obtained from the disjoint union of \(G_{K,W} \subseteq G\) and \(A_W^{\operatorname{power}}\) by connecting each \(r_X \in A_W^{\operatorname{power}}\) to all nodes in \(B_{G-K}(W) \cup X\) and to all nodes in \(K\). The graph \(G^{\operatorname{connect}}(K, W)\) is obtained from the disjoint union of the clique \(K\) and all cliques \(A_W^{\operatorname{power}}\) with \(W \subseteq W\), by adding edges from each node in \(K\) to all nodes in all cliques \(A_W^{\operatorname{power}}\) and by adding all edges \(r_XX'\) such that \(\cup X\) and \(\cup X'\) are adjacent in \(G\) and \(X \subseteq A_W, X' \subseteq A_W\). We also define \(L(G, K, W) = \{K \cup A_W^{\operatorname{power}} : W \subseteq W\}\).

**Theorem 20.** Let \((K, W)\) be a generalized amalgam separation of a graph \(G = (V, E)\) such that \(K\) is a clique. Moreover, let \(L = L(G, K, W)\). Then \(P[G]\) consists of the restrictions \(x_G\) of those \(x \in \mathbb{R}^V \cup (\cup(A_W^{\operatorname{power}} : W \subseteq W))\) with

\[
x_{G^{\operatorname{connect}}(K, W)} \in P[G^{\operatorname{connect}}(K, W), L] \text{ and } x_{G(K, W)} \in P[G(K, W)] \text{ for all } W \subseteq W.
\]

**Proof.** Let \(H\) be the graph with node set \(V \cup (\cup(A_W^{\operatorname{power}} : W \subseteq W))\) and with as edge set the union of the edge set of \(G^{\operatorname{connect}}(K, W)\) and the edge set of all \(G(K, W)\) with \(W \subseteq W\). Since each member of \(L\) is a clique cutset of \(H\), it follows from Corollary [16] that \(x \in P[H, L]\) if and only if \(x\) satisfies (8). Hence, it suffices to prove that \(P[G] = \{x_G : x \in P[H, L]\}\). For that it suffices to prove that the function \(S \mapsto S_G\) maps \(S[H, L]\) onto \(S[G]\).

First consider \(S \in S[H, L]\). We prove that \(S_G \in S[G]\). If \(S \cap K \neq \emptyset\), then \(S \subseteq V \setminus (\cup(B_G(W) : W \subseteq W))\), so \(S \in S[G]\). Hence we may assume that \(S \cap K = \emptyset\). Then there exists, for each \(W \subseteq W\), a collection \(x_W \subseteq A_W\) with \(S \cap A_W^{\operatorname{power}} = \{r_X\}\). Since \(S\) is a stable set in \(H\), we have that \(S \cap W \subseteq B_{G-K}(W) \setminus N_H(r_X) = \bigcup x_W\). Now consider \(W, W' \subseteq W\) with \(W' \neq W\). Then in \(H\), node \(r_X\) is not adjacent to node \(r_X'\). Hence \(\bigcup x_W\) and \(\bigcup x_W'\) are not adjacent in \(G\). From this it follows that \(S_G\) is a stable set in \(G\), as claimed.

Next consider \(S' \in S[G]\). We prove that there exists an \(S \in S[H, L]\) with \(S' = S_G\). If \(S' \cap K \neq \emptyset\), we just take \(S = S'\). Indeed, in that case, \(S' \subseteq V \setminus (\cup(B_G(W) : W \subseteq W))\), so \(S' \in S[H, L]\). Hence, we may assume \(S' \cap K = \emptyset\). For each \(W \subseteq W\), let \(x_W\) be the members of \(A_W\) that contain an element of \(S'\). Define \(S = S' \cup \{r_X : W \subseteq W\}\). Then \(S \in S[H, L]\) and \(S_G = S'\), as required.
Amalgams

If graph $G = (V, E)$ has an amalgam separation $(V_1, A_1, K, A_2, V_2)$, then $(K, \{V_1 \cup A_1, V_2 \cup A_2\})$ is a generalized amalgam separation and $K$ is a (possibly empty) clique. By Theorem 21, $P[G]$ consists of the restrictions $x_G$ of all vectors $x \in \mathbb{R}^{V \cup \{r_{A_1}, r_{A_2}\}}$ with

\begin{align}
    x_{V_1 \cup A_1 \cup K \cup \{r_{A_1}, r_{A_2}\}} & \in P[G(K, V_1 \cup A_1)], \\
    x_{V_2 \cup A_2 \cup K \cup \{r_{A_1}, r_{A_2}\}} & \in P[G(K, V_2 \cup A_2)], \\
    x_{K \cup \{r_{A_1}, r_{A_2}\}} & \in P[G_{\text{connect}}(K, \{V_1 \cup A_1, V_2 \cup A_2\}), \mathcal{L}],
\end{align}

where $\mathcal{L}$ consists of the two cliques $K \cup \{r_{A_1}, r_{A_2}\}$ and $K \cup \{r_{A_2}, r_{A_1}\}$.

For $x \in \mathbb{R}^{K \cup \{r_{A_1}, r_{A_2}\}}$, condition (11) is equivalent to

\begin{align}
    x(K) + x_{r_{A_1}} + x_{r_{A_2}} = 1, & \quad x(K) + x_{r_{A_1}} + x_{r_{A_2}} = 1, & \quad x(K) + x_{r_{A_1}} + x_{r_{A_2}} \leq 1,
\end{align}

so, with

\begin{align}
    x_{r_{A_1}} = 1 - x(K) - x_{r_{A_2}}, & \quad x_{r_{A_2}} = 1 - x(K) - x_{r_{A_1}}, & \quad x(K) + x_{r_{A_1}} + x_{r_{A_2}} \geq 1.
\end{align}

We now eliminate $x_{r_{A_1}}$ and $x_{r_{A_2}}$. In (9), this amounts to deleting $r_{A_1}$ from $G(K, V_1 \cup A_1)$, In (10), this amounts to deleting $r_{A_2}$ from $G(K, V_2 \cup A_2)$. Since $G(K, V_1 \cup A_1) - r_{A_1}$ and $G(K, V_2 \cup A_2) - r_{A_2}$ are the blocks of the amalgam decomposition of $G$, we get the following result.

**Theorem 21.** If $G_1$ and $G_2$ are the blocks of an amalgam decomposition of $G = (V, E)$ using the amalgam separation $(V_1, A_1, K, A_2, V_2)$, then the stable set polytope $P[G]$ of $G$ satisfies:

\[ P[G] = \{x_G \in \mathbb{R}^{V \cup \{r_{A_1}, r_{A_2}\}}, x_{G_1} \in P[G_1], x_{G_2} \in P[G_2], x(K) + x_{r_{A_1}} + x_{r_{A_2}} \geq 1\} \]

If we have original space descriptions for $P[G_1]$ and $P[G_2]$, Theorem 21 yields an extended formulation for $P[G]$ with $x_{r_{A_1}}$ and $x_{r_{A_2}}$ as the only extra variables. With Fourier-Motzkin elimination it easy to remove $x_{r_{A_1}}$ and $x_{r_{A_2}}$ from that extended formulation. This leads to a new proof of the following result of Burlet and Fonlupt (see [16] for an extension).

**Corollary 22** (Burlet and Fonlupt [3]). Let the stable set polytopes of the blocks of an amalgam decomposition of $G$ be described by the following systems:

\begin{align}
    x & \geq 0, & \quad D^1x & \leq \delta^1, & x_{r_{A_1}} & \geq 0, & x_{r_{A_1}} + c^{1,i}x & \leq \gamma^{1,i} (i = 1, \ldots, n_1),
\end{align}

and

\begin{align}
    x & \geq 0, & \quad D^2x & \leq \delta^2, & x_{r_{A_2}} & \geq 0, & x_{r_{A_2}} + c^{2,i}x & \leq \gamma^{2,i} (i = 1, \ldots, n_2),
\end{align}

where $r_{A_1}$ and $r_{A_2}$ are the nodes that are not in $G$. Then $P[G]$ is given by the following system:

\begin{align}
    x & \geq 0, & \quad D^1x & \leq \delta^1, & D^2x & \leq \delta^2, \\
    [c^{1,i} + c^{2,j}]x - x(K) & \leq \gamma^{1,i} + \gamma^{2,j} - 1 (i = 1, \ldots, n_1, j = 1, \ldots, n_2),
\end{align}

where $K$ is the clique in the amalgam separation.
Proof. Let \( G_1 \) and \( G_2 \) be the blocks of the amalgam decomposition, where [13] describes \( P[G_1] \) and [14] describes \( P[G_2] \). By Theorem 21, \( P[G] \) consists of all \( x \) for which there exists \( x_{r_1} \) and \( x_{r_2} \) such that \((x, x_{r_1}, x_{r_2}) \) satisfies: [13], [14], and "\( x(K) + x_{r_1} + x_{r_2} \geq 1 \)". Since [13] describes \( P[G_1] \), we get that [13] implies "\( x(K) + x_{r_1} \leq 1 \)". Subtracting that inequality from "\( x(K) + x_{r_1} + x_{r_2} \geq 1 \)" yields "\( x_{r_2} \geq 0 \)". In other words: the constraint "\( x_{r_2} \geq 0 \)" is redundant in the system of linear inequalities given by [13], [14] and "\( x(K) + x_{r_1} + x_{r_2} \geq 1 \)". By symmetry, the same applies to "\( x_{r_1} \geq 0 \)". So that system is equivalent to the system consisting of [15] together with:

\[
\begin{align*}
    x_{r_1} &\leq c^1 - c^{1j}x \quad (i = 1, \ldots, n_1) \\
    x_{r_2} &\leq c^2 - c^{2j}x \quad (j = 1, \ldots, n_2) \\
    1 - x(K) - x_{r_1} &\leq x_{r_2}
\end{align*}
\]

Eliminating \( x_{r_2} \), replaces [17]-[19] by

\[
\begin{align*}
    x_{r_1} &\leq c^1 - c^{1j}x \quad (i = 1, \ldots, n_1) \\
    1 - x(K) + c^{2j}x - c^2 &\leq x_{r_1} \quad (j = 1, \ldots, n_2)
\end{align*}
\]

Eliminating \( x_{r_1} \), replaces [20] and [21] by [16]. \( \square \)

3.4 Proof of Theorem 2

Lemma 23. Stable set polytopes of record graphs of fan-templates have compact extended formulations that can be constructed in polynomial time.

Proof. Let \( H \) be the record graph of a fan-template with base \((u, c, v)\). Then \( P[H] \) is the convex hull of \( P[H - c] \) and of a face of the convex hull of the characteristic vector of \( \{c\} \) and \( P[H - N_H(c) - c] \). Since, by Lemma 11, the graphs \( H - c \) and \( H - N_H(c) - c \) are decomposable by 2-node clique sets into a decomposition-list of at most \( |V| \) graphs, each with at most 8 nodes, the lemma follows from Corollary 16 and Balas’s formula [7]. \( \square \)

Theorem 24. The stable set polytopes of cap-free odd-signable graphs have a compact extended formulation that can be constructed in polynomial time.

Proof. When graph \( G \) has as a node \( u \) adjacent to all other nodes, \( P[G] \) is the convex hull of the characteristic vector of \( \{u\} \) and \( P[G - u] \). Hence in that case it follows from [7], that \( P[G] \) has an extended formulation with only three more variables than any such formulation for \( P[G - u] \). Recall from Section 2.3, that clique cutset separations and amalgam separations are 1-linearizable and that a 1-linearized decomposition-list of a graph \( G = (V, E) \) can have at most \( |V|^2 \) members. Hence, by Lemma 23, the result follows from the decomposition results Theorem 3, 10 and the polyhedral composition results Corollary 16 and Theorems 17 and 21. \( \square \)

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