(Regular) pseudo-bosons versus bosons

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Received 13 September 2010, in final form 7 November 2010
Published 6 December 2010
Online at stacks.iop.org/JPhysA/44/015205

Abstract

We discuss in which sense the so-called regular pseudo-bosons, recently introduced by Trifonov and analyzed in some detail by the author, are related to ordinary bosons. We repeat the same analysis for pseudo-bosons, and analyze the role played by certain intertwining operators, which may be bounded or not.

PACS number: 05.30.Jp

1. Introduction

In a series of recent papers [1–4], we have investigated some mathematical aspects of the so-called pseudo-bosons (PB), originally introduced by Trifonov in [5]. They arise from the canonical commutation relation \([a, a^\dagger] = \mathbb{1}\) upon replacing \(a^\dagger\) by another (unbounded) operator \(b\) not (in general) related to \(a\): \([a, b] = \mathbb{1}\). We have shown that, under suitable assumptions, \(N = ba\) and \(N^\dagger = a^\dagger b^\dagger\) both can be diagonalized, and that their spectra coincide with the set of natural numbers (including 0), \(\mathbb{N}_0\). However, the sets of related eigenvectors are not orthonormal (o.n.) bases but, nevertheless, they are automatically biorthogonal. In most of the examples considered so far, they are bases of the Hilbert space of the system, \(\mathcal{H}\), and, in some cases, they turn out to be Riesz bases.

In [6] and [7], some physical examples arising from concrete models in quantum mechanics have been discussed. These examples suggested introducing the difference between regular pseudo-bosons (RPB) and PB: the RPB, see section 2, arise when the two sets of eigenvectors of \(N\) and \(N^\dagger\) are mapped one onto the other by a bounded operator with bounded inverse. If this operator is unbounded, then we have to make do with PB. PB have also been considered by other authors recently, see [8] for instance, without using this name. These PB have been shown to be associated with the so-called pseudo-Hermitian quantum mechanics, which in recent years have become more and more appealing since it considers the possibility of having non-self-adjoint Hamiltonians with real spectra, showing that this possibility is related to some commutativity conditions between the Hamiltonian itself and the parity and...
the time reversal operators [9]. The same feature, more from a mathematical side, has been analyzed for instance in [10, 11]. Of course, these references should just be considered a starting point for a deeper analysis.

In this paper, we consider the relation between PB, RPB, and ordinary bosons, proving two similar theorems, one for PB and the other for RPB. In the next section, we introduce and discuss some features of \(d\)-dimensional PB. In section 3 we prove our main theorem for RPB, while section 4 contains an analogous result for PB, together with some physical examples; we will see that the techniques of unbounded operators are the natural tools in that case. We give our conclusions in section 5.

2. \(d\)-dimensional PB and RPB

In this section we construct a \(d\)-dimensional (dD) version of that originally proposed in [1], to which we refer for further comments on the 1D situation.

Let \(\mathcal{H}\) be a given Hilbert space with scalar product \(\langle \cdot, \cdot \rangle\) and related norm \(|\cdot|\). We introduce \(d\) pairs of operators, \(a_j\) and \(b_j\), \(j = 1, 2, \ldots, d\), acting on \(\mathcal{H}\) and satisfying the following commutation rules:

\[
[a_j, b_j] = 1, \tag{2.1}
\]

where \(j = 1, 2, \ldots, d\), all the other commutators being trivial. Of course, they collapse to the CCR’s for \(d\) independent modes if \(b_j = a_j\), \(j = 1, 2, \ldots, d\). It is well known that \(a_j\) and \(b_j\) are unbounded operators, so they cannot be defined on all of \(\mathcal{H}\). Following [1], and writing \(D^\infty(X) := \cap_{p \geq 0}D(X^p)\) (the common domain of all the powers of the operator \(X\)), we consider the following.

Assumption 1. There exists a non-zero \(\varphi_0 \in \mathcal{H}\) such that \(a_j \varphi_0 = 0\), \(j = 1, 2, \ldots, d\), and \(\varphi_0 \in D^\infty(b_1) \cap D^\infty(b_2) \cap \cdots \cap D^\infty(b_d)\).

Assumption 2. There exists a non-zero \(\Psi_0 \in \mathcal{H}\) such that \(b_j^\dagger \Psi_0 = 0\), \(j = 1, 2, \ldots, d\), and \(\Psi_0 \in D^\infty(a_1) \cap D^\infty(a_2) \cap \cdots \cap D^\infty(a_d)\).

Under these assumptions, we can introduce the following vectors in \(\mathcal{H}\):

\[
\begin{align*}
\varphi_n &:= \varphi_{n_1, n_2, \ldots, n_d} = \frac{1}{\sqrt{n_1! n_2! \cdots n_d!}} b_1^{n_1} b_2^{n_2} \cdots b_d^{n_d} \varphi_0 \\
\Psi_n &:= \Psi_{n_1, n_2, \ldots, n_d} = \frac{1}{\sqrt{n_1! n_2! \cdots n_d!}} b_1^{n_1} b_2^{n_2} \cdots b_d^{n_d} \Psi_0,
\end{align*}
\tag{2.2}
\]

\(n_j = 0, 1, 2, \ldots, \) for all \(j = 1, 2, \ldots, d\). Let us now define the unbounded operators \(N_j := b_j a_j\) and \(\Omega_j := N_j^\dagger = a_j^\dagger b_j^\dagger\), \(j = 1, 2, \ldots, d\). It is possible to check that \(\varphi_n\) belongs to the domain of \(N_j\), \(D(N_j)\), and that \(\Psi_n \in D(\Omega_j)\), for all possible \(n\). Moreover,

\[
N_j \varphi_n = n_j \varphi_n, \quad \Omega_j \Psi_n = n_j \Psi_n. \tag{2.3}
\]

Under the above assumptions, and if we choose the normalization of \(\Psi_0\) and \(\varphi_0\) in such a way that \(\langle \Psi_0, \varphi_0 \rangle = 1\), we find that

\[
\langle \Psi_n, \varphi_m \rangle = \delta_{n,m} = \prod_{j=1}^d \delta_{n_j, m_j}. \tag{2.4}
\]
This means that the sets $\mathcal{F}_\psi = \{ \psi_n \}$ and $\mathcal{F}_\phi = \{ \phi_n \}$ are biorthogonal and, because of this, the vectors of each set are linearly independent. If we now call $\mathcal{D}_\phi$ and $\mathcal{D}_\psi$ respectively the linear span of $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$, and $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ their closures, then

$$f = \sum_n \langle \psi_n, f \rangle \psi_n, \quad \forall f \in \mathcal{H}_\psi, \quad h = \sum_n \langle \phi_n, h \rangle \psi_n, \quad \forall h \in \mathcal{H}_\psi. \quad (2.5)$$

What is not in general ensured is that the Hilbert spaces introduced so far all coincide, i.e. that $\mathcal{H}_\phi = \mathcal{H}_\psi = \mathcal{H}$. Indeed, we can only state that $\mathcal{H}_\phi \subseteq \mathcal{H}$ and $\mathcal{H}_\psi \subseteq \mathcal{H}$. However, motivated by the examples discussed in the literature, we make

**Assumption 3.** The above Hilbert spaces all coincide: $\mathcal{H}_\phi = \mathcal{H}_\psi = \mathcal{H}$,

which was introduced in [1]. This means, in particular, that both $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$ are bases of $\mathcal{H}$, so that the following resolutions of the identity, written in bra-ket notation, hold:

$$\sum_n |\psi_n\rangle \langle \psi_n| = \sum_n |\phi_n\rangle \langle \phi_n| = \mathbb{1}. \quad (2.6)$$

Let us now introduce the operators $S_\phi$ and $S_\psi$ via their action respectively on $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$:

$$S_\phi \Psi_n = \phi_n, \quad S_\psi \phi_n = \Psi_n, \quad (2.7)$$

for all $n$, which in particular imply that $\Psi_n = (S_\psi S_\phi) \Psi_n$ and $\phi_n = (S_\psi S_\phi) \phi_n$, for all $n$. Hence

$$S_\psi S_\phi = S_\phi S_\psi = \mathbb{1} \quad \Rightarrow \quad S_\psi = S_\phi^{-1}. \quad (2.8)$$

In other words, both $S_\psi$ and $S_\phi$ are invertible and one is the inverse of the other. Furthermore, we can also check that they are both positive, well defined and symmetric, [1]. Moreover, it is possible to write these operators as

$$S_\phi = \sum_n |\phi_n\rangle \langle \phi_n|, \quad S_\psi = \sum_n |\Psi_n\rangle \langle \Psi_n|. \quad (2.9)$$

These expressions are only formal, at this stage, since the series may or may not converge in the uniform topology and the operators $S_\phi$ and $S_\psi$ could be unbounded. Indeed we know, [12], that two biorthogonal bases are related by a bounded operator, with bounded inverse, if and only if they are Riesz bases\(^1\). This is why in [1] we have also considered

**Assumption 4.** $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$ are Bessel sequences. In other words, there exist two positive constants $A_\phi, A_\psi > 0$ such that, for all $f \in \mathcal{H}$,

$$\sum_n |\phi_n, f\rangle |^2 \leq A_\phi \| f \|^2, \quad \sum_n |\Psi_n, f\rangle |^2 \leq A_\psi \| f \|^2. \quad (10)$$

This assumption is equivalent to requiring that $\mathcal{F}_\phi$ and $\mathcal{F}_\psi$ are both Riesz bases, and implies that $S_\phi$ and $S_\psi$ are bounded operators: $\|S_\phi\| \leq A_\phi$, $\|S_\psi\| \leq A_\psi$. Moreover, $\frac{1}{A_\phi} \mathbb{1} \leq S_\phi \leq A_\phi \mathbb{1}$ and $\frac{1}{A_\psi} \mathbb{1} \leq S_\psi \leq A_\psi \mathbb{1}$. Hence the domains of $S_\phi$ and $S_\psi$ can be taken to be all of $\mathcal{H}$. While assumptions 1, 2 and 3 are quite often satisfied, as the examples contained in our previous papers and in the recent review [13] show, it is quite difficult to find physical examples satisfying assumption 4 also. On the other hand, it is rather easy to find mathematical examples satisfying all the assumptions, see section 2.1 below. Hence, as announced, we introduce the following difference: we call PB those excitations satisfying the first three

\(^1\) Recall that a set of vectors $\phi_1, \phi_2, \phi_3, \ldots$ is a Riesz basis of a Hilbert space $\mathcal{H}$ if there exists a bounded operator $V_j$ with bounded inverse, on $\mathcal{H}$, and an orthonormal basis of $\mathcal{H}$, $\psi_1, \psi_2, \psi_3, \ldots$, such that $\phi_j = V_j \psi_j$, for all $j = 1, 2, 3, \ldots$.\n
assumptions, while, if assumption 4 is also satisfied, these will be called RPB. Clearly, RPB are PB, but the converse is false, in general.

Generalizing what already discussed in [1, 7], these $d$-dimensional PB give rise to interesting intertwining relations among non-self-adjoint operators, see also [3] and references therein. In particular, it is easy to check that

$$S_\Psi N_j = \Omega_j S_\Psi \quad \text{and} \quad N_j S_\Psi = S_\Psi \Omega_j,$$

(2.11)

$j = 1, 2, \ldots, d$. This is related to the fact that the spectra of, say, $N_1$ and $\Omega_1$ coincide and that their eigenvectors are related by the operators $S_\Psi$ and $S_\Psi$ (see equations (2.3) and (2.7)) in agreement with the literature on intertwining operators, [14, 15], and on pseudo-Hermitian quantum mechanics, see [9–11] and references therein.

2.1. Construction of RPB

We will show here that each Riesz basis produces some RPB. Let $\mathcal{F}_\Psi := \{\varphi_n\}$ be a Riesz basis of $\mathcal{H}$ with bounds $A$ and $B$, $0 < A \leq B < \infty$. The associated frame operator $S := \sum_n \langle \varphi_n | \varphi_n \rangle$ is bounded, positive and admits a bounded inverse. Also, the set $\mathcal{F}_\Psi := \{\varphi_n := S^{-1/2} \varphi_n\}$ is an o.n. basis of $\mathcal{H}$. Therefore, we can define $d$ lowering operators $a_{j, \Psi}$ on $\mathcal{F}_\Psi$ as $a_{j, \Psi} \varphi_n = \sqrt{n_j} \varphi_{n-1}$, and their adjoints, $a_{j, \Psi}^\dagger$, as $a_{j, \Psi}^\dagger \varphi_n = \sqrt{n_j} + 1 \varphi_{n+1}$. Here $\varphi_{n-1} = (n_1, \ldots, n_j - 1, \ldots, n_d)$ and $\varphi_{n+1} = (n_1, \ldots, n_j + 1, \ldots, n_d)$. Hence $[a_{j, \Psi}, a_{j, \Psi}^\dagger] = \delta_{j,k} \mathbb{1}$. If we now define $a_j := S^{1/2} a_{j, \Psi} S^{-1/2}$, this acts on the Riesz basis $\mathcal{F}_\Psi$ as a lowering operator. However, since $\mathcal{F}_\Psi$ is not an o.n. basis in general, $a_j^\dagger$ is not a raising operator, so that $[a_j, a_j^\dagger] \neq \delta_{j,k} \mathbb{1}$. However, if we now define the operator $b_j := S^{1/2} a_{j, \Psi}^\dagger S^{-1/2}$, it is clear that in general $b_j \neq a_j^\dagger$, and $b_j$ acts on $\varphi_0$ as a raising operator: $b_j \varphi_0 = \sqrt{n_j} + 1 \varphi_n$, for all $n$. Then we have $[a_j, b_j] = \delta_{j,k} \mathbb{1}$. So we have constructed two sets of operators satisfying (2.1) and which are not related by a simple conjugation. This is not the end of the story. Indeed:

1. Assumption 1 is verified since $\varphi_0$ is annihilated by $a_j$ and belongs to the domain of all the powers of $b_j$.
2. As for assumption 2, it is enough to define $\Psi_0 = S^{-1} \varphi_0$. With this definition $b_j^\dagger \Psi_0 = 0$, and $\Psi_0$ belongs to the domain of all the powers of $a_j^\dagger$.
3. Since $\mathcal{F}_\Psi$ is a Riesz basis of $\mathcal{H}$ by assumption, $\mathcal{H}_\Psi = \mathcal{H}$. Note now that the vectors $\Psi_n$ can be written as $\Psi_n = S^{-1} \varphi_n$, for all $n$. Hence $\mathcal{F}_\Psi$ is in duality with $\mathcal{F}_\Psi$ and therefore is a Riesz basis of $\mathcal{H}$ as well. Hence $\mathcal{H}_\Psi = \mathcal{H}$. This proves assumption 3.
4. As for assumption 4, this is equivalent to the hypothesis originally assumed here, i.e. $\mathcal{F}_\Psi$ is a Riesz basis.

Explicit examples arising from this general construction can be found in [4].

2.2. Coherent states

It is well known that there exists several different, and not always equivalent, ways to define coherent states, [16, 17]. In this paper, following [1], we will adopt the following definition: let $z_j, j = 1, 2, \ldots, d$, be $d$ complex variables, $z_j \in \mathcal{D}$ (some domain in $\mathbb{C}$), and let us introduce the following operators:

$$
\begin{align*}
U_j(z_j) &= e^{z_j b_j - \frac{1}{2} |z_j|^2} e^{\tau_j a_j}, \\
V_j(z_j) &= e^{\tau_j a_j^\dagger} e^{-\frac{1}{2} |z_j|^2} e^{\tau_j a_j^\dagger} e^{-\tau_j b_j^\dagger},
\end{align*}
$$

(2.12)
\begin{equation}
\begin{aligned}
&j = 1, 2, \ldots, d,
&\begin{cases}
U(z_1, z_2, \ldots, z_d) := U_1(z_1)U_2(z_2)\cdots U_d(z_d), \\
V(z_1, z_2, \ldots, z_d) := V_1(z_1)V_2(z_2)\cdots V_d(z_d),
\end{cases}
\end{aligned}
\tag{2.13}
\end{equation}

and the following vectors:
\begin{equation}
\begin{aligned}
\varphi(z_1, z_2, \ldots, z_d) &= U(z_1, z_2, \ldots, z_d)\varphi_0, \\
\Psi(z_1, z_2, \ldots, z_d) &= V(z_1, z_2, \ldots, z_d)\Psi_0.
\end{aligned}
\tag{2.14}
\end{equation}

Remarks.

(1) Due to the commutation rules for the operators \(b_j\) and \(a_j\), we clearly have \([U_j(z_j), U_k(z_k)] = [V_j(z_j), V_k(z_k)] = 0\), for \(j \neq k\).

(2) Since the operators \(U\) and \(V\) are, for generic \(z_j\), unbounded, definition (2.14) makes sense only if \(\varphi_0 \in D(U)\) and \(\Psi_0 \in D(V)\), a condition which will be assumed here. In [1] it was proven that, for instance, this is so when \(F_\varphi\) and \(F_\Psi\) are Riesz bases.

(3) The set \(D\) could be, in principle, a proper subset of \(C\).

It is possible to write the vectors \(\varphi(z_1, z_2, \ldots, z_d)\) and \(\Psi(z_1, z_2, \ldots, z_d)\) in terms of the vectors of \(F_\varphi\) and \(F_\Psi\) as
\begin{equation}
\begin{aligned}
\varphi(z_1, z_2, \ldots, z_d) &= e^{-(|z_1|^2+|z_2|^2+\ldots+|z_d|^2)/2} \sum_n \frac{z_1^{n_1}z_2^{n_2}\cdots z_d^{n_d}}{\sqrt{n_1!n_2!\cdots n_d!}} \varphi_n, \\
\Psi(z_1, z_2, \ldots, z_d) &= e^{-(|z_1|^2+|z_2|^2+\ldots+|z_d|^2)/2} \sum_n \frac{\sqrt{\frac{n_1!n_2!\cdots n_d!}}{z_1^{n_1}z_2^{n_2}\cdots z_d^{n_d}}} \Psi_n.
\end{aligned}
\tag{2.15}
\end{equation}

These vectors are called coherent since they are eigenstates of the lowering operators. Indeed we can check that
\begin{equation}
\begin{aligned}
a_j\varphi(z_1, z_2, \ldots, z_d) &= \varphi_j(z_1, z_2, \ldots, z_d), \\
b_j\Psi(z_1, z_2, \ldots, z_d) &= \varphi_j(z_1, z_2, \ldots, z_d),
\end{aligned}
\tag{2.16}
\end{equation}

for \(j = 1, 2, \ldots, d\) and \(z_j \in \mathcal{D}\). It is also a standard exercise, putting \(z_j = r_j e^{i\theta_j}\), to check that the following operator equalities hold:
\begin{equation}
\begin{aligned}
&\frac{1}{\pi^d} \int_C dz_1 \int_C dz_2 \cdots \int_C dz_d \varphi(z_1, z_2, \ldots, z_d)\varphi(z_1, z_2, \ldots, z_d) = S_\varphi, \\
&\frac{1}{\pi^d} \int_C dz_1 \int_C dz_2 \cdots \int_C dz_d \Psi(z_1, z_2, \ldots, z_d)\Psi(z_1, z_2, \ldots, z_d) = S_\Psi,
\end{aligned}
\tag{2.17}
\end{equation}

as well as
\begin{equation}
\begin{aligned}
&\frac{1}{\pi^d} \int_C dz_1 \int_C dz_2 \cdots \int_C dz_d \varphi(z_1, z_2, \ldots, z_d)\Psi(z_1, z_2, \ldots, z_d) = \sum_n |\varphi_n\rangle \langle \Psi_n| = 1,
\end{aligned}
\tag{2.18}
\end{equation}

which are written in convenient bra-ket notation. It should be said that these equalities are, most of the time, only formal results. Indeed, extending an analogous result given in [7] for \(d = 2\), we can prove the following.

**Theorem 1.** Let \(a_j, b_j, F_\varphi, F_\Psi, \varphi(z_1, z_2, \ldots, z_d)\) and \(\Psi(z_1, z_2, \ldots, z_d)\) be as above. Let us assume that (1) \(F_\varphi, F_\Psi\) are Riesz bases; (2) \(F_\varphi, F_\Psi\) are biorthogonal. Then (2.18) holds true.

Suppose therefore that the above construction gives coherent states that do not satisfy a resolution of the identity (see [2] for such an example). Then, since \(F_\varphi\) and \(F_\Psi\) are automatically biorthogonal, they cannot be Riesz bases (neither one of them).
3. RPB versus bosons

In this section we prove the following theorem, given in $d = 1$ for simplicity, establishing a sort of equivalence between RPB and ordinary bosons. This equivalence is related to the existence of a bounded operator $T$ with bounded inverse and of a pair of conjugate operators $c$ and $c^\dagger$ satisfying the canonical commutation rule $[c, c^\dagger] = \mathbb{1}$, which are related to the original pair of operators $a$ and $b$. In detail we have

**Theorem 2.** Let $a$ and $b$ be two operators on $\mathcal{H}$ satisfying $[a, b] = \mathbb{1}$, and for which assumptions 1, 2, 3 and 4 of section 2 are satisfied. Then an unbounded, densely defined, operator $c$ on $\mathcal{H}$ exists, together with a positive bounded operator $T$ with bounded inverse $T^{-1}$, such that $[c, c^\dagger] = \mathbb{1}$. Moreover,

$$a = TcT^{-1}, \quad b = Tc^\dagger T^{-1}.$$  \hfill (3.1)

Vice versa, given an unbounded, densely defined, operator $c$ on $\mathcal{H}$ satisfying $[c, c^\dagger] = \mathbb{1}$ and a positive bounded operator $T$ with bounded inverse $T^{-1}$, two operators $a$ and $b$ can be introduced for which $[a, b] = \mathbb{1}$, and for which equations (3.1) and assumptions 1, 2, 3 and 4 of section 2 are satisfied.

**Proof.** To prove the first part of the theorem we first remind that, because of assumption 4 of section 2, the operators $a$ and $b$ defined as in (2.9),

$$S_q f = \sum_{n=0}^{\infty} \langle \varphi_n, f \rangle \varphi_n, \quad S_\psi f = \sum_{n} \langle \Psi_n, f \rangle \Psi_n,$$  \hfill (3.2)

$f \in \mathcal{H}$, are well defined, bounded and positive (hence, self-adjoint). Also, $S_q = S_q^{-1}$. These are standard results in the theory of Riesz bases, [12, 18]. In particular, choosing the normalization constants in $\Psi_0$ and $\varphi_0$ in such a way that $\langle \Psi_0, \varphi_0 \rangle = 1$, we know that $\langle \Psi_n, \varphi_m \rangle = \delta_{n,m}$ and, as a consequence,

$$S_q \Psi_n = \varphi_n, \quad S_\psi \varphi_m = \varphi_m,$$  \hfill (3.3)

for all $m \geq 0$. Because of the properties of $S_q$ and $S_\psi$, their square roots surely exist and, for instance, $S_q^{-1/2} = S_q^{1/2}$. Hence we define the vectors $\hat{\varphi}_n = S_q^{-1/2} \varphi_n$, $n \geq 0$, and the related set $\mathcal{F}_q = \{ \hat{\varphi}_n, n \geq 0 \}$. It is well known that $\mathcal{F}_q$ is an o.n. basis of $\mathcal{H}$, and it coincides with the o.n. basis we would construct introducing (apparently) new vectors $\tilde{\Psi}_n = S_q^{1/2} \Psi_n$, $n \geq 0$, since it can be easily checked that, for all $n$, $\Psi_n = \tilde{\varphi}_n$.

On $\mathcal{F}_q$ we can define the ordinary bosonic lowering and raising operators:

$$\begin{cases} c \hat{\varphi}_n = \sqrt{n} \hat{\varphi}_{n-1}, \\ c^\dagger \hat{\varphi}_n = \sqrt{n+1} \hat{\varphi}_{n+1}, \end{cases}$$  \hfill (3.4)

with the convention that $c \hat{\varphi}_0 = 0$. Of course, $[c, c^\dagger] = \mathbb{1}$. Recall now, [1], that our working hypotheses also imply that $a \varphi_n = \sqrt{n} \varphi_{n-1}$ and $b \varphi_n = \sqrt{n+1} \varphi_{n+1}$, which can be rewritten as $S_q^{-1/2} a S_q^{1/2} \varphi_n = \sqrt{n} \varphi_{n-1}$, and $S_q^{-1/2} b S_q^{1/2} \varphi_n = \sqrt{n+1} \varphi_{n+1}$. Hence $a$, $b$ and $c$ are related as follows:

$$c = S_q^{-1/2} a S_q^{1/2}, \quad c^\dagger = S_q^{-1/2} b S_q^{1/2},$$

which are exactly equations (3.1), identifying $T$ with $S_q^{1/2}$.

Let us now prove the second part of the theorem. First of all, by means of $c$ and $c^\dagger$, we construct the o.n. basis $\mathcal{F}_c$ of $\mathcal{H}$, $\mathcal{F}_c = \{ \hat{\varphi}_n = \sum_{m} \delta_{n,m} \hat{\varphi}_m \}$, where $c \hat{\varphi}_0 = 0$. Then, since both $T$ and $T^{-1}$ are bounded and, therefore, everywhere defined, we can introduce two new families
of vectors: $\mathcal{F}_\psi = \{ \varphi_n = T \Phi_n, n \geq 0 \}$ and $\mathcal{F}_\psi = \{ \Psi_n = T^{-1} \Phi_n, n \geq 0 \}$. These two families are obviously biorthogonal, $(\Psi_n, \varphi_m) = \delta_{n,m}$, and they are both complete in $\mathcal{H}$: so they are two (in general different) bases of $\mathcal{H}$. We can now define on, say, $\mathcal{F}_\psi$ the two operators $a$ and $b$ which act as lowering and raising operators:

$$
\begin{align*}
  a\varphi_n &= \sqrt{n}\varphi_{n-1}, \\
  b\varphi_n &= \sqrt{n+1}\varphi_{n+1},
\end{align*}
$$

for all $n \geq 0$. In particular, the first equation implies that $a\varphi_0 = 0$. Incidentally we observe that $b^* \neq a$, since $\mathcal{F}_\psi$ is not, in general, an o.n. basis. Iterating the second equation in (3.5), we deduce that $\varphi_n = \frac{a^n}{\sqrt{n!}} \varphi_0$, which gives an alternative expression for the vector $\varphi_n$ and, moreover, shows that $\varphi_0 \in \mathcal{D}(b)$. Hence assumption 1 is satisfied.

Since $(T^\dagger T^{-1})\varphi_n = T^\dagger \Phi_n = \sqrt{n+1} T \Phi_{n+1} = \sqrt{n+1} \varphi_{n+1}$, and since $\mathcal{F}_\psi$ is a basis of $\mathcal{H}$, we deduce that $b = T^\dagger T^{-1}$. Analogously, we can prove that $a = T^\dagger T^{-1}$. It is now clear that $[a, b] = \mathbb{1}$ and that $a^1 = T^{-1} c^1 T$. To prove assumption 2 we first note that $b^1 \Psi_0 = (T^\dagger T^{-1}) (T^{-1} \varphi_0) = T^{-1} c \varphi_0 = 0$. Moreover, since for all $n \geq 0$,

$$
a^1 \Psi_n = (T^{-1} c^1 T) T^{-1} \varphi_n = T^{-1} c^1 \varphi_n = \sqrt{n+1} \Psi_{n+1},
$$

by iteration we deduce that $\Psi_n = \frac{a^n}{\sqrt{n!}} \Psi_0$, which means that $\Psi_0 \in \mathcal{D}(a^1)$. This proves assumption 2, while assumption 3 follows from our previous claim on $\mathcal{F}_\psi$ and $\mathcal{F}_\psi$: they are both bases of $\mathcal{H}$. Finally, since they are obtained by the o.n. basis $\mathcal{F}_\psi$ by acting with the bounded operators $T$ or $T^{-1}$, they are also Riesz bases.

\[ \square \]

Remarks.

(1) The proof of the above theorem recalls, at least in part, the construction given in section 2.1. This is not surprising since we are now dealing with Riesz bases. The difference will be evident in the next section.

(2) Theorem 2 implies that the intertwining operators in (2.11) for RPB are bounded, with bounded inverse.

4. PB versus bosons

In this section we will not assume that $T$ and $T^{-1}$ are bounded operators, and many domain problems will arise as a consequence. This is related to the nature of the biorthogonal bases we work with, which will not be Riesz bases any longer. The relevance of this section, as widely explained in [13] and references therein, follows from the fact that all the examples arising from concrete physical models seem to give rise to PB and not to RPB. From the mathematical side, we will now formulate a different theorem which is the analog of the one proven in the previous section in these different settings and we will show that, even if part of that proof can be repeated here, most of the arguments should be changed to take care of the unboundedness of the operators. This is not just an exercise in operator theory, but it is crucial to give rigor to the results of the paper. However, it should be mentioned that in some examples discussed in the literature, see [10, 11] for instance, only finite-dimensional Hilbert spaces play a role, so that no unbounded operator can appear in the game. As in the previous section, to simplify the proof and the notation, we fix $d = 1$. Extension to $d > 1$ is straightforward.

**Theorem 3.** Let $a$ and $b$ be two operators on $\mathcal{H}$ satisfying $[a, b] = \mathbb{1}$, and for which assumptions 1, 2, and 3 (but not 4) of section 2 are satisfied. Then two unbounded, densely
defined, operators $c$ and $R$ on $\mathcal{H}$ exist, such that $[c, c^\dagger] = \mathbb{I}$ and $R$ is positive, self-adjoint and admits an unbounded inverse $R^{-1}$. Moreover,

$$a = RcR^{-1}, \quad b = Rc^\dagger R^{-1},$$

and, introducing $\hat{\phi}_n = \frac{c_n}{\sqrt{\varphi_0}} \phi_0$, $\varphi_0 = 0$, we have the following: $\hat{\phi}_n \in D(R) \cap D(R^{-1})$, for all $n \geq 0$, and the sets $\{R\hat{\phi}_n\}$ and $\{R^{-1}\hat{\phi}_n\}$ are biorthogonal bases of $\mathcal{H}$.

Vice versa, let us consider two unbounded, densely defined, operators $c$ and $R$ on $\mathcal{H}$ satisfying $[c, c^\dagger] = \mathbb{I}$ with $R$ positive, self-adjoint with unbounded inverse $R^{-1}$. Suppose that, introduced $\hat{\varphi}_n$ as above, $\hat{\varphi}_n \in D(R) \cap D(R^{-1})$, for all $n \geq 0$, and that the sets $\{R\hat{\varphi}_n\}$ and $\{R^{-1}\hat{\varphi}_n\}$ are biorthogonal bases of $\mathcal{H}$. Then two operators $a$ and $b$ can be introduced for which $[a, b] = \mathbb{I}$, and for which equations (4.1) and assumptions 1, 2, and 3 (but not 4) of section 2 are satisfied.

**Proof.** To prove the first part of the theorem we recall that the two sets $\mathcal{F}_\varphi = \{\psi_n, n \geq 0\}$ and $\mathcal{F}_\Psi = \{\Psi_n, n \geq 0\}$ defined as in section 2 are biorthogonal bases of $\mathcal{H}$ but they are not Riesz bases. Hence, defining

$$S_\varphi \Psi_n = \varphi_n, \quad S_\Psi \varphi_n = \Psi_n,$$  

for all $n \geq 0$, on the domains $D(S_\varphi) = \text{linear span} \{\Psi_n\}$ and $D(S_\Psi) = \text{linear span} \{\varphi_n\}$, it follows from general results, [12], that both these operators are unbounded, so that they are not everywhere defined. It is possible to check that $(f, S_\varphi f) \geq 0$ for all $f \in D(S_\varphi)$ and $(f, S_\Psi f) \geq 0$ for all $f \in D(S_\Psi)$. In particular, if $f \neq 0$, both these mean values are strictly positive. It is straightforward to check that, as in the previous section, $S_\varphi = S_\Psi^{-1}$, and that both operators are symmetric:

$$\langle f, S_\varphi g \rangle = \langle S_\varphi f, g \rangle, \quad \forall f, g \in D(S_\varphi),$$

$$\langle f, S_\Psi g \rangle = \langle S_\Psi f, g \rangle, \quad \forall f, g \in D(S_\Psi).$$

In these conditions it is known [19] that each of these operators admits a (not necessarily unique) self-adjoint extension, which is also positive. We call these (Friedrichs) extensions $\hat{S}_\varphi$ and $\hat{S}_\Psi$. Using standard results in functional calculus, we can now define square roots of these operators and the following holds:

$$\hat{S}_\varphi = \hat{S}_{\Psi}^{-1}, \quad \hat{S}_{\varphi}^{1/2} = \hat{S}_{\Psi}^{-1/2}, \quad \hat{S}_{\varphi}^{-1/2} = \hat{S}_{\Psi}^{1/2}.$$  

It is easy to check that, for all $n \geq 0$, $\varphi_n \in D(\hat{S}_{\varphi}^{-1/2})$, so that $D(\hat{S}_\varphi) = D(\hat{S}_{\varphi}^{-1}) \subseteq D(\hat{S}_{\varphi}^{-1/2})$. Indeed, we can check that $\|\hat{S}_{\varphi}^{-1/2} \varphi_n\| = 1$. This is a particular case of the following more general result:

$$\langle \hat{S}_{\varphi}^{-1/2} \varphi_n, \hat{S}_{\varphi}^{-1/2} \varphi_k \rangle = \langle \varphi_n, \hat{S}_{\varphi}^{-1} \varphi_k \rangle = \langle \varphi_n, \hat{S}_\Psi \varphi_k \rangle = \langle \varphi_n, \Psi_k \rangle = \delta_{n,k},$$

due to the biorthogonality of $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$. This suggests introducing a third set of vectors of $\mathcal{H}$, $\mathcal{F}_\hat{\varphi} = \{\hat{\varphi}_n := \hat{S}_{\varphi}^{-1/2} \varphi_n, n \geq 0\}$, which is made of o.n. vectors. As in section 3, defining $\hat{\Psi}_n := \hat{S}_{\Psi}^{-1/2} \Psi_n$ does not produce new vectors; again we get $\hat{\Psi}_n = \hat{\varphi}_n \forall n \geq 0$. We also deduce that $D(\hat{S}_\varphi) \subseteq D(\hat{S}_{\varphi}^{1/2})$.

Let us note that, since $D(\hat{S}_\varphi) \subseteq D(\hat{S}_{\varphi}^{-1/2}) \subseteq \mathcal{H}$ and since the closure of $D(\hat{S}_\varphi)$ returns $\mathcal{H}$, $D(\hat{S}_{\varphi}^{-1/2}) = \mathcal{H}$. Analogously, $D(\hat{S}_{\varphi}^{1/2}) = \mathcal{H}$. Moreover, $\forall n \geq 0$, $\hat{\varphi}_n \in D(\hat{S}_{\varphi}^{1/2}) \cap D(\hat{S}_{\varphi}^{-1/2})$: indeed, a straightforward computation shows that $\hat{S}_{\varphi}^{1/2} \hat{\varphi}_n = \varphi_n$ and that $\hat{S}_{\varphi}^{-1/2} \hat{\varphi}_n = \hat{S}_{\varphi}^{-1} \hat{\varphi}_n = S_\Psi \varphi_n = \Psi_n$.

Finally, if $f \in D(\hat{S}_{\varphi}^{1/2})$ is orthogonal to all $\hat{\varphi}_n, f = 0$. Hence, due to the density of $D(\hat{S}_{\varphi}^{1/2})$ in $\mathcal{H}$, we conclude that $\mathcal{F}_\hat{\varphi}$ is an o.n. basis of $\mathcal{H}$. [20]. On $\mathcal{F}_\hat{\varphi}$ we
define the standard annihilation operator $c$ as usual, $c \hat{\phi}_n = \sqrt{n+1}\hat{\phi}_{n+1}$. We can rewrite the first of these equations as $cS_{\psi}^{-1/2} \varphi_n = \sqrt{n+1}S_{\psi}^{-1/2} \varphi_{n-1}$, which implies, first of all, that $cS_{\psi}^{-1/2} \varphi_n \in D(S_{\psi}^{1/2})$. Also, $S_{\psi}^{1/2} cS_{\psi}^{-1/2} \varphi_n = \sqrt{n+1} \varphi_{n-1}$, which, compared with $a \varphi_n = \sqrt{n} \varphi_{n-1}$, shows that $a = S_{\psi}^{1/2} cS_{\psi}^{-1/2}$.

In a similar way, $c^† \hat{\phi}_n = \sqrt{n+1} \hat{\phi}_{n+1}$ can be rewritten as $c^† S_{\psi}^{-1/2} \varphi_n = \sqrt{n+1} S_{\psi}^{-1/2} \varphi_{n-1}$. Therefore, $c S_{\psi}^{-1/2} \varphi_n \in D(S_{\psi}^{1/2})$ and $S_{\psi}^{1/2} c S_{\psi}^{-1/2} \varphi_n = \sqrt{n+1} \varphi_{n+1}$ which, compared with $b \varphi_n = \sqrt{n+1} \varphi_{n+1}$, shows that $b = S_{\psi}^{1/2} c S_{\psi}^{-1/2}$. This proves (4.1), identifying $R$ with $S_{\psi}^{1/2}$. Also, since $R \hat{\phi}_n = S_{\psi}^{1/2} \hat{\phi}_n = \varphi_n$ and $R^{-1} \varphi_n = S_{\psi}^{-1/2} \varphi_n = \psi_n$, the linear spans of both $\{R \hat{\phi}_n\}$ and $\{R^{-1} \hat{\phi}_n\}$ are biorthogonal bases of $\mathcal{H}$.

Let us now prove the inverse statement. Because of our assumptions, the set $\mathcal{F}_\psi$ of vectors $\tilde{\varphi}_n = \frac{1}{\sqrt{\varphi_n}} \varphi_n c \psi_0 = 0$, is an o.n. basis in $\mathcal{H}$ and $\varphi_n \in D(R) \cap D(R^{-1})$, $\forall n \geq 0$. Then we define, for all $n \geq 0$, $\varphi_n = R \tilde{\varphi}_n$, $\psi_n = R^{-1} \tilde{\varphi}_n$, $\mathcal{F}_\varphi = \{\varphi_n, n \geq 0\}$, $\mathcal{F}_\psi = \{\psi_n, n \geq 0\}$, and $\mathcal{D}_\varphi$ and $\mathcal{D}_\psi$ their linear span, which are both dense in $\mathcal{H}$ since, by assumption, $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are (biorthogonal) bases of $\mathcal{H}$.

We can now introduce lowering and raising operators on $\mathcal{F}_\psi$ as in (3.5). In particular, iterating $b \varphi_n = \sqrt{n+1} \varphi_{n+1}$, we get $\varphi_n = \frac{b^n}{\sqrt{\varphi_0}} \varphi_0$ and we also find that $b^† \psi_n = \sqrt{n+1} \sqrt{n} \varphi_{n+1}$. The first equation, $a \varphi_n = \sqrt{n+1} \varphi_{n+1}$, produces $a \psi_n = \sqrt{n+1} \sqrt{n+1} \varphi_{n+1}$, which, again by iteration, gives $\psi_n = a^n \sqrt{\varphi_0} \varphi_0$.

It is now a simple exercise to check that:

1. $\psi_0 = 0$ and $\psi_0 \in D(\delta(b))$. Hence assumption 1 is satisfied.
2. $b \psi_0 = 0$ and $\psi_0 \in D(\delta(a^†))$. Hence assumption 2 is satisfied.
3. With similar techniques as in the first part of the proof, we deduce that $b = R c R^{-1}$ and $a = R c R^{-1}$, which could also be checked by computing directly their action on the vectors $\tilde{\varphi}_n$.
4. $\mathcal{D}_\varphi^{-1} = \mathcal{D}_\psi^{-1} = \mathcal{H}$. Hence assumption 3 is satisfied.
5. Since $\mathcal{F}_\varphi$ and $\mathcal{F}_\psi$ are obtained from the o.n. basis $\mathcal{F}_\psi$ via the action of an unbounded, invertible operator with unbounded inverse, they cannot be Riesz bases, [12]. Hence assumption 4 is violated.

This concludes the proof. \hfill \square

4.1. Physical examples

We conclude this section with some examples, arising from quantum mechanics, in which the operators $\hat{S}_{\psi}$ and $\hat{S}_{\phi}$ can be explicitly identified. These examples are reviewed in [13], where the original references and more examples (even in $d > 1$) can be found.

4.1.1. The extended quantum harmonic oscillator. The Hamiltonian of this model, introduced in [21], is the non-self-adjoint operator $H_\beta = \frac{\beta}{2}(p^2 + x^2) + i\sqrt{2}p$, where $\beta$ is a positive parameter and $[x, p] = i$. Introducing the standard bosonic operators $a = \frac{1}{\sqrt{2}}(x + \frac{d}{dx})$, $a^† = \frac{1}{\sqrt{2}}(x - \frac{d}{dx})$, $[a, a^†] = 1$, and the number operator $N = a^† a$, we can write $H_\beta = \beta N + (a - a^†) + \frac{\beta}{2} \mathbb{1}$, which, introducing further the operators

$$\hat{A}_\beta = a - \frac{1}{\beta}, \quad \hat{B}_\beta = a^† + \frac{1}{\beta},$$

(4.4)
can be written as

\[ H_\beta = \beta (\hat{B}_\beta \hat{A}_\beta + \gamma_\beta \mathbb{1}) , \]  

(4.5)

where \( \gamma_\beta = \frac{2i\beta^2}{2\theta} \). It is clear that, for all \( \beta > 0 \), \( \hat{A}_\beta \neq \hat{B}_\beta \) and that \([\hat{A}_\beta, \hat{B}_\beta] = \mathbb{1}\). Hence we have to do with pseudo-bosonic operators which, as proved in [6], satisfy assumptions 1, 2 and 3 but not assumption 4. Indeed, we have deduced that \( \hat{S}_\theta = e^{2i(\alpha + \alpha^*)/\beta} \), which is unbounded with unbounded inverse. We have PB which are not regular.

4.1.2. The Swanson Hamiltonian. The starting point is the following non-self-adjoint Hamiltonian, [21]:

\[ H_\theta = \frac{1}{2} (p^2 + x^2) - \frac{i}{2} \tan(2\theta) (p^2 - x^2) , \]

where \( \theta \) is a real parameter taking value in \( \left( -\frac{\pi}{4}, \frac{\pi}{4} \right) \setminus \{0\} =: I \). It is clear that \( H_\theta \neq H_\theta^* \), for all \( \theta \in I \). Introducing the annihilation and creation operators \( a \) and \( a^\dagger \) as usual, we write

\[ H_\theta = N + \frac{i}{2} \tan(2\theta) (a^2 + (a^\dagger)^2) + \frac{1}{2} \mathbb{1} , \]

where \( N = a^\dagger a \). This Hamiltonian can still be rewritten, by introducing the operators

\[ \begin{align*}
A_\theta &= \cos(\theta) a + i \sin(\theta) a^\dagger , \\
B_\theta &= \cos(\theta) a^\dagger + i \sin(\theta) a ,
\end{align*} \]

as

\[ H_\theta = \omega_\theta (B_\theta A_\theta + \frac{1}{2} \mathbb{1}) , \]

where \( \omega_\theta = \cos(2\theta) \) is well defined since \( \cos(2\theta) \neq 0 \) for all \( \theta \in I \). It is clear that \( A_\theta \neq B_\theta \) and that \([A_\theta, B_\theta] = \mathbb{1}\). In [6] we have proven that these operators satisfy assumptions 1, 2 and 3 but not assumption 4. In particular, we have deduced that \( \hat{S}_\theta = |\alpha|^2 e^{i\theta (a^2 - a^\dagger)^2} \), where \( \alpha \in \mathbb{C} \) is arbitrary but fixed, which is unbounded with unbounded inverse. Again, we find PB which are not regular.

5. Conclusions

In this paper we have discussed the relation between RPB and PB with ordinary bosons. As the two theorems proven here clearly show that there is a strong connection between these excitations, at least under suitable assumptions. The assumptions which are relevant are clarified by the theorems: for instance, if we just consider operators satisfying \([a, b] = \mathbb{1}\), this is not enough to get any relevant functional structure. If, as an example, we take \( a = \frac{d}{dx} \), \( b = x \) and \( \mathcal{H} = L^2(\mathbb{R}) \), then no square integrable function \( \psi_0(x) \) exists with the required properties. So assumptions 1 and 2 are not satisfied. So we cannot introduce, starting from \( a \) and \( b \), a basis of \( \mathcal{H} \). This suggests that, while assumption 4 can be avoided, and assumption 3 could be weakened by considering the relevant subspaces of \( \mathcal{H} \), assumptions 1 and 2 are absolutely necessary.

Further analysis on these operators is in progress.

Acknowledgment

The author acknowledges M.I.U.R. for financial support.
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