The general formula of entropy and similarity measures for hesitant fuzzy linguistic term sets

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Abstract. In order to quantitatively characterize the uncertainty of hesitant fuzzy linguistic information, the entropy and similarity measures of hesitant fuzzy linguistic term sets are studied. First, the axiomatic definitions of entropy and similarity measures of hesitant fuzzy linguistic term sets are given, and general formulas of entropy and similarity measures for hesitant fuzzy linguistic term sets are proposed. Moreover, the algorithms for generating entropy and similarity of hesitant fuzzy linguistic term set are given. Then the relationship between entropy and similarity of hesitant fuzzy linguistic term set is studied, and the general formula of entropy based on similarity is proposed. These lay the theoretical foundation for flexible selection of entropy and similarity in multi-attribute decision making.

1. Introduction

In decision-making, due to the complexity of the problem and the limitation of cognition, experts may use qualitative linguistic term to evaluate the plan more intuitively and easily. Zadeh [1] first proposed the fuzzy linguistic method, which uses words or sentences in natural language to express people’s qualitative decision information. However, in uncertain decision problems, experts may hesitate among different linguistic terms, and a single linguistic term cannot accurately reflect the expert’s point of view. Drawing on the idea of hesitant fuzzy sets proposed by Torra [2], Rodriguez [3] proposed the concept of hesitant fuzzy linguistic term sets. The value of linguistic variables is an ordered and coherent subset of linguistic term sets. Rodriguez et al.[4] studied the basic operational properties of hesitant fuzzy linguistic term sets, introduced how to transform natural language into hesitant fuzzy linguistic term sets; Liao et al.[5] gave the mathematical expression of hesitant fuzzy linguistic term sets and studied its correlation coefficient. Wang et al.[6] extended the concept of hesitant fuzzy linguistic term sets, removed the restriction that linguistic term items must be continuous, and proposed the concept of extended hesitant fuzzy linguistic term sets.

Hesitant fuzzy linguistic information is based on linguistic terms given by people, which can flexibly and comprehensively reflect the true preferences of decision makers. It is used in project management[7], fuzzy control [8], pattern recognition [9], medical diagnosis, multi-attribute decision making [10,11] and other fields. In order to measure the uncertainty of hesitant fuzzy linguistic information, the measures of hesitant fuzzy linguistic information are introduced into applications such as multi-attribute decision making. The measures of hesitant fuzzy linguistic information, including distance, similarity and entropy. The similarity of hesitant fuzzy linguistic term sets is mainly used to identify and evaluate different hesitant fuzzy linguistic information. The entropy of the hesitant fuzzy linguistic term sets is mainly used to measure the degree of uncertainty of the hesitant
fuzzy linguistic information, and can be used to determine the weight of each criterion under incomplete information.

At present, some scholars have studied the information measures of hesitant fuzzy linguistic term sets. Liao et al.[12] proposed a series of distance and similarity of hesitant fuzzy linguistic term sets; Farhadinia [13] studied the relationship between entropy, similarity and distance of hesitant fuzzy linguistic term sets; proposed specific hesitant fuzzy linguistic entropy, similarity, and distance formulas; Tang et al.[14] studied the inclusion measures of hesitant fuzzy linguistic term sets and their applications in clustering algorithms; Xu et al.[15] studied the relationship between hesitant fuzzy linguistic term sets and hesitant fuzzy set and their conversion functions, and proposed the entropy and cross entropy of various hesitant fuzzy linguistic term sets.

In practical applications, different types of applications need to select multiple types of entropy and similarity formulas. However, the existing literature only proposes some specific formulas of entropy and similarity of hesitant fuzzy linguistic term set. Therefore, this paper studies and proposes generation algorithm of entropy and similarity of hesitant fuzzy linguistic term set. Firstly, based on the relationship between hesitant fuzzy linguistic term sets and hesitant fuzzy sets and transformation functions in literature[15], the entropy and similarity of hesitant fuzzy linguistic term sets are defined, and the general formulas of entropy and similarity of hesitant fuzzy linguistic term sets are proposed. Furthermore, the algorithm for generating entropy and similarity of hesitant fuzzy linguistic term set are given. Lastly, the relationship between entropy and similarity of hesitant fuzzy linguistic value sets is studied, and a general formula for constructing entropy based on similarity is proposed.

2. Hesitant fuzzy linguistic term sets

Definition 2.1[16] Let \( S = \{ s_i | i = -\tau, \ldots, -1, 0, 1, \ldots, \tau \} \) is a linguistic term set, \( \tau \) is a positive integer. And the linguistic term of the middle term indicates that the evaluation value is "no bias", and the remaining linguistic term are symmetrically distributed on both sides of the middle term, \( s_m \) and \( s_{-\tau} \) are the upper and lower bounds of the linguistic term set. And the linguistic term set satisfies: if \( s_m < s_i \) and only if \( m < n \). When \( \tau = 3 \), the graph of the subscript symmetric seven-valued linguistic term set \( S \) is shown in Figure 1.

![Figure 1. The subscript symmetric seven-valued linguistic term set](image)

Definition 2.2[16] Let \( S = \{ s_i | i = -\tau, \ldots, -1, 0, 1, \ldots, \tau \} \) is a linguistic term set, \( H = \{ \langle x_j, h_s(x_j) \rangle | x_j \in X \} \) is the hesitant fuzzy linguistic term set (HFLTS) on the universe \( X = \{ x_1, x_2, \ldots, x_N \} \). Among them \( h_s(x_j) = \{ s_{i_0}(x_j) | s_{i_0}(x_j) \in S, i = 1, 2, \ldots, l_j \} \), \( l_j \) is the number of medium linguistic term \( h_s(x_j) \), and \( h_s(x_j) \) is a continuous set of linguistic term in \( S \). \( h_s(x_j) \) is called hesitant fuzzy linguistic term element (HFLE).

Definition 2.3[15] Let \( S = \{ s_i | i = -\tau, \ldots, -1, 0, 1, \ldots, \tau \} \) is a linguistic term set, \( h_s = \{ s_i | i \in [-\tau, \tau] \} \) is a HFLE, and \( h_\gamma = \{ \gamma | \gamma \in [0,1] \} \) is a hesitant fuzzy element(HFE). The
membership $\gamma$ of information equivalent to linguistic variables $s_i$ is obtained by the following function:

$$g : [-\tau, \tau] \rightarrow [0, 1], \quad g(s_i) = \frac{t}{2\tau} \ 	ext{for} \ t \geq 0, \ g(s_i) = \frac{t + 1}{2\tau} \ 	ext{for} \ t < 0.$$  

In addition, the linguistic variables $s_i$ equivalent to membership $\gamma$ are obtained by the following functions:

$$g^{-1} : [0, 1] \rightarrow [-\tau, \tau], \quad g^{-1}(\gamma) = s_i.$$  

**Definition 2.4** Let $S = \{s_i|=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ is a linguistic term set, $h_i(x) = \{s_{(i)}|i=1,\ldots,l_s\}$ is an HFLs, $x \mapsto h_i(x)$ is called complement.

The definition of extended hesitant fuzzy linguistic term set (EHFLTS) is the same as that of HFLTS, except that the restriction of terminology items of HFLTS must be continuous. EHFLTS also meets definitions 2.3 and 2.4.

**Note:** In order to facilitate the calculation of entropy and similarity, referring to [17], this article makes the following assumptions:

1. An EHFLE is $h_i(x_j)$, and the linguistic term in $h_i(x_j)$ are arranged in ascending order, and $s_{(i)}(x_j)$ represents the $i$ th linguistic term in $h_i(x_j)$ from small to large;

2. The two EHFLEs are $h^i(x)$ and $h^j(x)$. If $\forall x \in X$, $l(h^i(x)) \neq l(h^j(x))$, record $l_s = \max\{l(h^i(x)), l(h^j(x))\}$, where $l(h^i(x))$ is the number of linguistic term in $h^i(x)$. For EHFLS with a small number of linguistic term, the maximum linguistic term is repeatedly added until the number of linguistic term is $l_s$.

**Definition 2.5** Let $S = \{s_i|=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ is the linguistic term set, $H^1_s$ and $H^2_s$ are the two EHFLTS on the universe $X = \{x_1, x_2, \ldots, x_j\}$, $\forall x \in X$, $l(h^1_s(x)) = l(h^2_s(x)) = l_s$, and give the following definitions:

1. $H^1_s \leq H^2_s \iff \forall x \in X, h^1_s(x) \leq h^2_s(x) \iff \forall x \in X, s^1_{(i)}(x) \leq s^2_{(i)}(x), \ i = 1, \ldots, l_s$;

2. $H^1_s = H^2_s \iff \forall x \in X, h^1_s(x) = h^2_s(x) \iff \forall x \in X, s^1_{(i)}(x) = s^2_{(i)}(x), \ i = 1, \ldots, l_s$.

Moreover, referring to [13], the following special EHFLTS are defined:

1. $H^t_s = \{<x_j, h^t_s(x_j)>|x_j \in X\} = \{<x_j, s_{(i)}(x_j)>|s_{(i)}(x_j) = s_{(i)}, \forall i = 1, \ldots, l_j \}$

2. $H^{t-1}_s = \{<x_j, h^{t-1}_s(x_j)>|x_j \in X\} = \{<x_j, s_{(i)}(x_j)>|s_{(i)}(x_j) = s_{(i)}, \forall i = 1, \ldots, l_j \}$

3. $H^{t+1}_s = \{<x_j, h^{t+1}_s(x_j)>|x_j \in X\} = \{<x_j, s_{(i)}(x_j)>|s_{(i)}(x_j) = s_{(i)}, \forall i = 1, \ldots, l_j \}$

The hesitant fuzzy linguistic term set is a special case of the extended hesitant fuzzy set. The above assumptions and definitions 2.5 also apply to hesitant fuzzy linguistic term sets.

3. The entropy for hesitant fuzzy linguistic term sets

3.1. Axiomatic definition of entropy for hesitant fuzzy linguistic term set

According to the relation function between the hesitant fuzzy linguistic term set and the hesitant fuzzy set in Definition 2.3, this article gives an axiomatic definition of the entropy of fuzzy linguistic term set.

**Definition 3.1** Let $S = \{s_i|=-\tau,\ldots,-1,0,1,\ldots,\tau\}$ is the linguistic term set, $H^t_s = \{<x_j, h^t_s(x_j)>\}$ is the HFLTS on the domain $X$, where $h^t_s(x) = \{s_{(i)}(x)|s_{(i)}(x) \in S, i = 1, \ldots, l_s\}$ is the HFLTS of $H^t_s$, $l_s$ is the number of linguistic term in $h_s(x)$, and $H^t_s$ is the complement of $H^t_s$. If $E(H^t_s)$ satisfies:
(1) \( E(H_s) = 0 \) if and only if \( \forall x \in X \text{, } g(h_s(x)) = \{0\} \text{ or } g(h_s(x)) = \{1\}; \)
(2) \( E(H_s) = 1 \) if and only if \( \forall x \in X \text{, } g(s_0(x)) + g(s_{(l_i+1)}(x)) = 1, i = 1, 2, ..., l; \)
(3) \( E(H_s^1) \leq E(H_s^2) \), if \( g(s_{(l_i)}(x)) \leq g(s_{(l_i)}(x)), g(s_{(l_i)}(x)) + g(s_{(l_i+1)}(x)) \leq 1 \text{ or } g(s_{(l_i)}(x)) \geq g(s_{(l_i)}(x)) \), \( g(s_{(l_i)}(x)) \geq g(s_{(l_i)}(x)), g(s_{(l_i)}(x)) + g(s_{(l_i+1)}(x)) \geq 1, i = 1, 2, ..., l; \) \( I_s^2 = \max \{I(h_s^1(x)), I(h_s^2(x))\} \);
(4) \( E(H_s) = E(\tilde{H}_s) \).

Then \( E(H_s) \) is called the entropy of the hesitant fuzzy linguistic term set \( H_s \).

In addition, referring to the literature [13], another definition of entropy of fuzzy linguistic term set is given as:
(1) \( E(H_s) = 0 \) if and only if \( H_s = H_s^{[\tau]} \) or \( H_s = H_s^{[0]} \);
(2) \( E(H_s) = 1 \) if and only if \( H_s = H_s^0 \);
(3) \( E(H_s^1) \leq E(H_s^2) \), if \( H_s^1 \leq H_s^2 \) or \( H_s^1 \leq H_s^0 \) or \( H_s^0 \leq H_s^2 \);
(4) \( E(H_s) = E(\tilde{H}_s) \).

Then \( E(H_s) \) is called the entropy of the hesitant fuzzy linguistic term set \( H_s \).

3.2. General formula of entropy for hesitant fuzzy linguistic term set

According to the axiomatic definition of entropy of fuzzy linguistic term set, general formula and generation algorithm of hesitant of fuzzy linguistic term set entropy are proposed.

**Theorem 3.1** Let \( S = \{s_t|t = -\tau, ..., -1, 0, 1, ..., \tau\} \) is the linguistic term set, \( H_s \) is the HFLTS on the domain \( X = \{x_1, x_2, ..., x_n\} \), and \( h_s(x_j) = \{s_i(x_j) \text{ if } i = 1, ..., l\} \) is the HFLE of \( H_s \),

\[
E(H_s) = d \left[ \sum_{j=1}^{n} c_j \sum_{i=1}^{l_j} f_i \left( \frac{g(s_i(x_j)) + g(s_{(l_i+1)}(x_j))}{2} \right) \right] \quad (1)
\]

Where \( c_j \) is a positive real number and \( f_i : [0, 1] \rightarrow [0, +\infty) \) satisfies:
(1) \( \forall x \in X, f_i(x) = f_i(1-x) \);
(2) \( f_i(0) = 0 \);
(3) \( f_i(x) \) is strictly increasing on \([0, 0.5]\).

Let \( a = \sum_{j=1}^{n} c_j \sum_{i=1}^{l_j} f_i(0.5), \ d : [0, a] \rightarrow [0, 1] \) is strictly increasing, and \( d(0) = 0, d(a) = 1, \) then \( E(H_s) \) is the entropy of the hesitant fuzzy linguistic term \( H_s \).

**Proof** 1) if \( \forall x \in X, g(h_s(x)) = \{0\}, \) then \( g(s_i(x_j)) + g(s_{(l_i+1)}(x_j)) = 0; \) or \( g(h_s(x)) = \{1\}, \) then \( g(s_i(x_j)) + g(s_{(l_i+1)}(x_j)) = 2. \) Since \( f_i(0) = f_i(1) = 0, \) there is \( f_i \left( \frac{g(s_i(x_j)) + g(s_{(l_i+1)}(x_j))}{2} \right) = 0. \)

By the definition of \( E(H_s) \), and \( d(0) = 0, \) there is \( E(H_s) = 0. \)

Conversely, when \( E(H_s) = 0, \) there is \( d \left[ \sum_{j=1}^{n} c_j \sum_{i=1}^{l_j} f_i \left( \frac{g(s_i(x_j)) + g(s_{(l_i+1)}(x_j))}{2} \right) \right] = 0, \)

Since \( d \) is strictly increasing on \([0, a], \) \( d(0) = 0, \) there is \( \frac{1}{l_j} \sum_{i=1}^{l_j} f_i \left( \frac{g(s_i(x_j)) + g(s_{(l_i+1)}(x_j))}{2} \right) = 0, \)

Since \( f_i \) is strictly increasing on \([0, 0.5], \) \( f_i(0) = f_i(1) = 0, \) there is \( g(s_i(x_j)) = 0, \) \( i = 1, 2, ..., l; \)
that is \( g(h_s(x_j)) = \{0\}. \) Or \( g(s_i(x_j)) = 1, \) \( i = 1, 2, ..., l; \) that is \( g(h_s(x_j)) = \{1\}. \)

2) When \( g(s_i(x_j)) + g(s_{(l_i+1)}(x_j)) = 1, \) \( i = 1, 2, ..., l, \) there is \( \frac{g(s_i(x_j)) + g(s_{(l_i+1)}(x_j))}{2} = \frac{1}{2}. \)
Since \( a = \sum_{j=1}^{N} c_j \frac{N}{L_j} \sum_{i=1}^{L_j} f_i(0.5) \) and \( d(a) = 1 \), there is \( E(H_a) = d(a) = 1 \).

Conversely, when \( E(H_a) = 1 \), there is \( d(\sum_{j=1}^{N} c_j \frac{N}{L_j} \sum_{i=1}^{L_j} f_i(g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j))) \frac{1}{2}) = 1 \).

Since \( d \) is strictly increasing on \([0,a]\), \( d(a) = 1 \), there is \( \sum_{j=1}^{N} c_j \frac{N}{L_j} \sum_{i=1}^{L_j} f_i(g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j))) = a \);

Since \( f(x) \) is strictly increasing on \([0,0.5]\), \( a = \sum_{j=1}^{N} c_j \frac{N}{L_j} \sum_{i=1}^{L_j} f_i(0.5) \), there is
\[
g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j)) = \frac{1}{2}, \quad i = 1,2,...,L_j; \quad \text{that is } \ g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j)) = 1.\]

3) When \( g(s_{i(j)}(x_j)) \leq g(s_{i(j)}(x_j)), g(s_{i(j+1)}(x_j)) \leq 1 \), there is
\[
g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j)) \leq g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j)) \leq \frac{1}{2} \text{. Since } d \text{ is strictly increasing on } [0,a], \text{ there is }
\[
f(\frac{g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j))}{2}) \leq f(\frac{g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j))}{2}) \leq f(\frac{1}{2}) \text{,Since } f(x) \text{ is strictly increasing on } [0,0.5], \text{ there is }
\[
E(H_{i(j)}) = E(H_{i(j+1)}).\]

Similarly, when \( g(s_{i(j)}(x_j)) \geq g(s_{i(j)}(x_j)), g(s_{i(j+1)}(x_j)) \geq 1 \), there is \( E(H_{i(j)}) = E(H_{i(j+1)}) \).

4) \( E(H_a) = d(\sum_{j=1}^{N} c_j \frac{N}{L_j} \sum_{i=1}^{L_j} f_i(1 - g(s_{i(j+1)}(x_j)) + g(s_{i(j)}(x_j))) \frac{1}{2}) \)
\[
= d(\sum_{j=1}^{N} c_j \frac{N}{L_j} \sum_{i=1}^{L_j} f_i(g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j))) \frac{1}{2}) \]
\[
= d(\sum_{j=1}^{N} c_j \frac{N}{L_j} \sum_{i=1}^{L_j} f_i(g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j))) \frac{1}{2}) = E(H_a). \]

Based on the general formula of entropy for hesitant fuzzy linguistic term set, a generation algorithm of entropy for hesitant fuzzy linguistic term set is given. The steps are as follows:

1) First, select the specific functions \( f_i(x) \) and \( d(x) \);

2) Verify that functions \( f_i(x) \) and \( d(x) \) meet the conditions of Theorem 3.1;

3) From the general entropy formula of hesitant fuzzy linguistic term set of Theorem 3.1, a specific entropy formula of hesitant fuzzy linguistic term set can be generated. Different functions \( f_i \) and \( d \) are selected. Then a specific entropy formula of the hesitant fuzzy linguistic term set can be generated. And the entropy formula of the hesitant fuzzy linguistic term sets proposed by Xu[14] is a special case of satisfying Theorem 3.1.

\[
E_1(H_a) = \frac{1}{N(\sqrt{2}-1)} \sum_{j=1}^{N} \frac{L_j}{\sum_{i=1}^{L_j} \frac{1}{2} \left( \sum_{i=1}^{L_j} \frac{1}{4} \sin(g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j))) \right) + \sin(\frac{2 - g(s_{i(j)}(x_j)) - g(s_{i(j+1)}(x_j)))}{4} - 1 \right) \}

E_2(H_a) = \frac{1}{N(\sqrt{2}-1)} \sum_{j=1}^{N} \frac{L_j}{\sum_{i=1}^{L_j} \frac{1}{2} \left( \sum_{i=1}^{L_j} \frac{1}{4} \cos(g(s_{i(j)}(x_j)) + g(s_{i(j+1)}(x_j))) \right) + \cos(\frac{2 - g(s_{i(j)}(x_j)) - g(s_{i(j+1)}(x_j)))}{4} - 1 \right) \}

When \( f_i(x) = |0.5-|0.5-x||^p \) and \( d(x) = 2\left(\frac{x}{n}\right)^{1/p} \), there are
\[
E_3(A) = \frac{2}{n^p} \left( \sum_{j=1}^{n} \sum_{i=1}^{L_j} \frac{1}{2} \left( |0.5-|0.5-x||^p \right) \right) , \quad p > 0
\]
And \( E_3(A) \) is the the new entropy formula of hesitant fuzzy linguistic term sets constructed in this paper.
Theorem 3.2 Let \( S = \{ s_j | \tau = -\ldots, -1, 0, 1, \ldots, \tau \} \) is the linguistic term set, \( H_s \) is the HFLTS on the domain \( X = \{ x_1, x_2, \ldots, x_N \} \), and \( h_i(x_j) = \{ s_{i_0}(x_j) | i = 1, \ldots, I_j \} \) is the HFLE of \( H_s \),

\[
E(H_s) = d\left[ \sum_{j=1}^{N} c_j \max\left\{ \frac{g(s_{i_0}(x_j)) + g(s_{i_{j-1}+1}(x_j))}{2} \right\} \right] \tag{5}
\]

Where \( c_j \) is a positive real number and \( f_i : [0, 1] \rightarrow [0, +\infty) \) satisfies:

1. \( \forall x \in X, f_i(x) = f_i(1-x) \);
2. \( f_i(0) = 0 \);
3. \( f_i(x) \) is strictly increasing on \([0, 0.5]\).

Let \( a = \sum_{j=1}^{N} c_j \max f_i(0.5) \) , \( d : [0, a] \rightarrow [0, 1] \) is strictly increasing, and \( d(0) = 0, d(a) = 1 \), then \( E(H_s) \) is the entropy of the hesitant fuzzy linguistic term set \( H_s \).

Proof The proof process of Theorem 3.2 is similar to the proof process of Theorem 3.1.

The entropy formula generation algorithm is used to select different functions and verify whether it meets Theorem 3.2. From the general formula of Theorem 3.2, a variety of entropy formulas for hesitant fuzzy linguistic term sets that are not available in the literature can be constructed.

\[
E_2(H_s) = \frac{1}{n(\sqrt{2}-1)} \sum_{i=1}^{n} \max_{j \in [\mathbb{I}_{s,i}]} \left\{ \frac{\sin(g(s_{i_0}(x_j)) + g(s_{i_{j-1}+1}(x_j)))}{4} + \frac{\sin(2 - g(s_{i_0}(x_j)) - g(s_{i_{j-1}+1}(x_j)))}{4} - 1 \right\} \tag{6}
\]

\[
E_2(A) = \frac{2}{n^{p+1}} \left( \sum_{i=1}^{n} \max_{j \in [\mathbb{I}_{s,i}]} \left| 0.5 - \frac{g(s_{i_0}(x_j)) + g(s_{i_{j-1}+1}(x_j))}{2} \right|^p \right)^p, p > 0 \tag{7}
\]

4. The similarity measure for hesitant fuzzy linguistic term sets

4.1. Axiomatic definition of similarity measure for hesitant fuzzy linguistic term sets

The similarity measure is an important information measure of hesitant fuzzy linguistic term sets. According to Definition 2.3, this paper gives the axiomatic definition of the similarity of the hesitant fuzzy linguistic term set.

Definition 4.1 Let \( S = \{ s_j | \tau = -\ldots, -1, 0, 1, \ldots, \tau \} \) is the linguistic term set, \( H_1^s \) and \( H_2^s \) are the two HFLTS on domain \( X = \{ x_1, x_2, \ldots, x_N \} \). \( h_1^s(x_j) \) and \( h_2^s(x_j) \) are the HFLEs of \( H_1^s \) and \( H_2^s \) respectively. If \( S(H_1^s, H_2^s) = 0 \) if and only if \( \forall x \in X, g(h_1^s(x_j)) = 0 \) or \( g(h_2^s(x_j)) = 0 \) if and only if \( g(h_1^s(x_j)) = g(h_2^s(x_j)) \);

2. \( S(H_1^s, H_2^s) = 1 \) if and only if \( \forall x \in X, g(h_1^s(x_j)) = g(h_2^s(x_j)) \);

3. \( S(H_1^s, H_2^s) \leq S(H_1^s, H_3^s), S(H_1^s, H_2^s) \leq S(H_1^s, H_3^s) \) if \( \forall x \in X, g(s_{i_0}(x_j)) \leq g(s_{i_{j-1}+1}(x_j)) \) or \( g(s_{i_0}(x_j)) \geq g(s_{i_{j-1}+1}(x_j)) \) or \( g(s_{i_0}(x_j)) = g(s_{i_{j-1}+1}(x_j)) \); i = 1, 2, ..., I_s; \( I_s = \max\{ l(h_1^s(x_j)), l(h_2^s(x_j)), l(h_3^s(x_j)) \} \);

4. \( S(H_1^s, H_2^s) = S(H_2^s, H_1^s) \).

Then \( S(H_1^s, H_2^s) \) is called the similarity measure between hesitant fuzzy linguistic term sets \( H_1^s \) and \( H_2^s \).

In addition, referring to literature[13], another definition of similarity measure of hesitant fuzzy linguistic term sets is given as:

1. \( S(H_1^s, H_2^s) = 0 \) if and only if \( H_1^s = H_1^{s_0}, H_2^s = H_2^{s_1} \); or \( H_1^s = H_2^{s_1}, H_2^s = H_1^{s_0} \);

2. \( S(H_1^s, H_2^s) = 1 \) if and only if \( H_1^s = H_2^s \);

3. \( S(H_1^s, H_2^s) \leq S(H_1^s, H_3^s), S(H_2^s, H_1^s) \leq S(H_2^s, H_3^s) \) if \( H_1^s \leq H_2^s \) or \( H_1^s \geq H_2^s \) or \( H_1^s \geq H_2^s \) or \( H_1^s \geq H_2^s \).
(4) \( S(H^1_s, H^2_s) = S(H^2_s, H^1_s) \).

Then \( S(H^1_s, H^2_s) \) is called the similarity measure between hesitant fuzzy linguistic term sets \( H^1_s \) and \( H^2_s \).

4.2. General formula of similarity measure for hesitant fuzzy linguistic term sets

Based on the axiomatic definition of the similarity of hesitant fuzzy linguistic term sets, general formula and generating algorithm for the similarity of hesitant fuzzy linguistic term set are proposed.

**Theorem 4.1** Let \( S = \{ s_i \mid i = -\tau, \ldots, -1, 0, 1, \ldots, \tau \} \) is the linguistic term set, \( H^1_s \) and \( H^2_s \) are the two HFLTS on domain \( X = \{ x_1, x_2, \ldots, x_N \} \). \( h^1_i(x_j) \) and \( h^2_i(x_j) \) are the HFLEs of \( H^1_s \) and \( H^2_s \) respectively.

\[
S(H^1_s, H^2_s) = 1 - G\left[ \frac{1}{n} \sum_{j=1}^{n} \sum_{l,j} \frac{c^j_l}{c} f_i(g(s^1_i(x_j))) - g(s^2_i(x_j)) \right] \quad (8)
\]

Where \( c^j_l \) is a positive real number and \( f_i : [0,1] \rightarrow [0,1] \) satisfies:

1) \( \forall x \in [0,1], f_i(x) = f_i(-x) \);
2) \( f_i(0) = 0, f_i(1) = 1 \);
3) \( f_i(x) \) is strictly increasing on \([0,1]\).

Let \( a = \frac{1}{n} \sum_{j=1}^{n} c^j \), \( G : [0,a] \rightarrow [0,1] \) is strictly increasing, and \( G(0) = 0, G(a) = 1 \), then \( S(H^1_s, H^2_s) \) is the similarity measure between hesitant fuzzy linguistic term sets \( H^1_s \) and \( H^2_s \).

**Proof**

1) If \( \forall x \in X \), \( g(h^1_i(x)) = \{0\}, g(h^2_i(x)) = \{1\} \), then \( g(s^1_i(x_j)) - g(s^2_i(x_j)) = -1 \); or \( g(h^1_i(x)) = \{1\}, g(h^2_i(x)) = \{0\} \), then \( g(s^1_i(x_j)) - g(s^2_i(x_j)) = 1 \). Since \( f_i(-1) = f_i(1) = 1 \), there is

\( f_i(g(s^1_i(x_j)) - g(s^2_i(x_j))) = 1 \). Since \( a = \frac{1}{n} \sum_{j=1}^{n} c^j \) and \( G(a) = 1 \), there is \( S(H^1_s, H^2_s) = 1 - G(a) = 0 \).

Conversely, when \( S(H^1_s, H^2_s) = 0 \), there is \( G\left[ \frac{1}{n} \sum_{j=1}^{n} \sum_{l,j} \frac{c^j_l}{c} f_i(g(s^1_i(x_j))) - g(s^2_i(x_j)) \right] = 1 \);

Since \( f_i \) is strictly increasing on \([0,a]\), \( G(a) = 1 \), there is \( \frac{1}{n} \sum_{j=1}^{n} \sum_{l,j} \frac{c^j_l}{c} f_i(g(s^1_i(x_j))) - g(s^2_i(x_j)) = a \);

Since \( f_i(x) \) is strictly increasing on \([0,1]\), \( a = \frac{1}{n} \sum_{j=1}^{n} c^j \), there is \( f_i(g(s^1_i(x_j))) - g(s^2_i(x_j)) = 1 \);

Since \( f_i(-1) = f_i(1) = 1 \), there is \( g(s^1_i(x_j)) = 0, g(s^2_i(x_j)) = 1 \), or \( g(s^1_i(x_j)) = 1, g(s^2_i(x_j)) = 0 \), \( j = 1,2,\ldots,n \), \( i = 1,2,\ldots,l_j \).

3) If \( \forall x \in X \), \( g(h^1_i(x)) = g(h^2_i(x)) \), so \( g(s^1_i(x_j)) - g(s^2_i(x_j)) = 0 \); Since \( f_i(0) = 0 \), there is

\( f_i(g(s^1_i(x_j)) - g(s^2_i(x_j))) = 0 \). By the definition of \( S(H^1_s, H^2_s) \), there is \( S(H^1_s, H^2_s) = 1 - G(0) = 1 \).

Conversely, when \( S(H^1_s, H^2_s) = 1 \), there is \( G\left[ \frac{1}{n} \sum_{j=1}^{n} \sum_{l,j} \frac{c^j_l}{c} f_i(g(s^1_i(x_j))) - g(s^2_i(x_j)) \right] = 0 \);

Since \( f_i \) is strictly increasing on \([0,a]\), \( G(0) = 0 \), there is \( \frac{1}{n} \sum_{j=1}^{n} \sum_{l,j} \frac{c^j_l}{c} f_i(g(s^1_i(x_j))) - g(s^2_i(x_j)) = 0 \);

Since \( c^j \) is a positive real number and \( 0 \leq f_i \leq 1 \), there is \( f_i(g(s^1_i(x_j)) - g(s^2_i(x_j))) = 0 \); Since \( f_i(x) \) is strictly increasing on \([0,1]\) and \( f_i(0) = 0 \), thus \( g(s^1_i(x_j)) = g(s^2_i(x_j)) \), that is \( H^1_s = H^2_s \).
4) When \( g(s_{(i)}^{1}(x)) \leq g(s_{(i)}^{2}(x)) \leq g(s_{(i)}^{3}(x)) \), there is
\[
g(s_{(i)}^{1}(x)) - g(s_{(i)}^{2}(x)) \leq g(s_{(i)}^{2}(x)) - g(s_{(i)}^{3}(x)) \leq 0 \quad \text{and} \quad g(s_{(i)}^{1}(x)) - g(s_{(i)}^{2}(x)) \leq g(s_{(i)}^{2}(x)) - g(s_{(i)}^{3}(x)) \leq 0;
\]

Since \( f_i(x) \) is strictly increasing on \([0,1]\), there is
\[
f_i(g(s_{(i)}^{1}(x))) - f_i(g(s_{(i)}^{2}(x))) \geq f_i(g(s_{(i)}^{2}(x))) - f_i(g(s_{(i)}^{3}(x))) \geq 0
\]
and \( f_i(g(s_{(i)}^{1}(x))) - f_i(g(s_{(i)}^{3}(x))) \geq f_i(g(s_{(i)}^{2}(x))) - f_i(g(s_{(i)}^{3}(x))) \geq 0 \); Since \( G : [0,a] \rightarrow [0,1] \) is strictly increasing, there is
\[S(H^1_s, H^3_s) \leq S(H^1_s, H^2_s), S(H^1_s, H^3_s) \leq S(H^2_s, H^1_s)\,.

Similarly, when \( g(s_{(i)}^{1}(x)) \geq g(s_{(i)}^{2}(x)) \geq g(s_{(i)}^{3}(x)) \), there is
\[S(H^2_s, H^1_s) \leq S(H^1_s, H^2_s), S(H^1_s, H^3_s) \leq S(H^2_s, H^1_s).

Based on the general formula of the similarity for hesitant fuzzy linguistic term sets, an algorithm for generating the similarity of hesitant fuzzy linguistic term sets is given. The steps are as follows:

1) First, select the specific functions \( f_i(x) \) and \( G(x) \);

2) Verify that functions \( f_i(x) \) and \( G(x) \) meet the conditions of Theorem 3.1;

3) From the general formula of similarity for hesitant fuzzy linguistic term set of Theorem 3.1, a specific formula for similarity for hesitant fuzzy linguistic term set can be generated.

Different functions \( f_i \) and \( G \) are selected, then specific similarity formulas of the hesitant fuzzy linguistic term set can be generated.

\[
S_i(H^1_s, H^2_s) = 1 - \left[ \frac{1}{2} \sum_{j=1}^{n} \left( 1 - \cos \left( \frac{\pi}{2} \left( g(s_{(i)}^{1}(x)) - g(s_{(i)}^{2}(x)) \right) \right) \right)^p \right]^{1/p}, p > 0
\]

\[
S_2(H^1_s, H^2_s) = 1 - \frac{1}{n} \sum_{j=1}^{n} \left( g(s_{(i)}^{1}(x)) - g(s_{(i)}^{2}(x)) \right) \]

\[
S_3(H^1_s, H^2_s) = 1 - \frac{1}{n} \sum_{j=1}^{n} \left( g(s_{(i)}^{1}(x)) - g(s_{(i)}^{2}(x)) \right)^2
\]

**Theorem 4.2** Let \( S = \{s_i| \in \tau, \cdots, -1, 0, 1, \cdots, \tau \} \) is the linguistic term set, \( H^1_s \) and \( H^2_s \) are the two HFLTS on domain \( X = \{x_1, x_2, \cdots, x_N \} \). \( h^1_s(x_i) \) and \( h^2_s(x_i) \) are the HFLEs of \( H^1_s \) and \( H^2_s \) respectively.

\[
S(H^1_s, H^2_s) = 1 - G \left( \frac{1}{n} \sum_{j=1}^{n} c_j \max f_i(g(s_{(i)}^{1}(x)) - g(s_{(i)}^{2}(x))) \right)
\]

Where \( c_j \) is a positive real number and \( f_i : [-1,1] \rightarrow [0,1] \) satisfies:

(1) \( \forall x \in [-1,1], f_i(x) = f_i(-x) \);

(2) \( f_i(0) = 0, f_i(1) = 1 \);

(3) \( f_i(x) \) is strictly increasing on \([0,1]\).

Let \( a = \frac{1}{n} \sum_{j=1}^{n} c_j \), \( G : [0,a] \rightarrow [0,1] \) is strictly increasing, and \( G(0) = 0, G(a) = 1 \), then \( S(H^1_s, H^2_s) \) is the similarity measure between hesitant fuzzy linguistic term sets \( H^1_s \) and \( H^2_s \).

**Proof** The proof process of Theorem 4.2 is similar to the proof process of Theorem 4.1.
By the similarity formula generation algorithm, different functions $f$ and $G$ are selected to verify whether it meets Theorem 4.2, and specific similarity formulas for the hesitant fuzzy linguistic term set can be generated.

$$S_3(H^1_s, H^2_s) = 1 - \frac{1}{n} \sum_{j=1}^{n} \max \{ |f_j(g(s^1_{ij}(x_j)) - g(s^2_{ij}(x_j)))| \}$$  \hspace{1cm} (13)

$$S_4(H^1_s, H^2_s) = 1 - \frac{1}{n} \sum_{j=1}^{n} \max \{ |g(s^1_{ij}(x_j)) - g(s^2_{ij}(x_j))|^2 \}^{1/2}$$  \hspace{1cm} (14)

In multi-attribute decision making, the importance of each hesitant fuzzy linguistic elements in a hesitant fuzzy linguistic term set are usually different. A general formula for weighted similarity of hesitant fuzzy linguistic term sets is constructed in this paper.

**Theorem 4.3** Let $S = \{s_t|t=\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ is a linguistic term set, $H^1_s$ and $H^2_s$ are the two HFLTS on domain $X = \{x_1, x_2, \ldots, x_n\}$. $h^1_s(x_j)$ and $h^2_s(x_j)$ are the HFLEs of $H^1_s$ and $H^2_s$ respectively. And $W = (w_1, w_2, \ldots, w_n)^T$ is the weight vector of hesitant fuzzy linguistic term elements in $H^1_s$ and $H^2_s$, satisfying $w_j > 0$, $j = 1, 2, \ldots, n$, and $\sum_{j=1}^{n} w_j = 1$,

$$S(H^1_s, H^2_s) = 1 - G\left( \frac{1}{n} \sum_{j=1}^{n} w_j c_j \sum_{i=1}^{l_j} f_i(g(s^1_{ij}(x_j)) - g(s^2_{ij}(x_j))) \right)$$  \hspace{1cm} (15)

Where $c_j$ is a positive real number and $f_i: [-1,1] \rightarrow [0,1]$ satisfies:

1. $\forall x \in [-1,1], f_i(x) = f_i(-x)$;
2. $f_i(0) = 0, f_i(1) = 1$;
3. $f_i(x)$ is strictly increasing on $[0,1]$.

Let $a = \frac{1}{n} \sum_{j=1}^{n} c_j$, $G: [0,a] \rightarrow [0,1]$ is strictly increasing, and $G(0) = 0, G(a) = 1$, then $S(H^1_s, H^2_s)$ is the weighted similarity between hesitant fuzzy linguistic term sets $H^1_s$ and $H^2_s$.

**Proof** The proof process of Theorem 4.3 is similar to the proof process of Theorem 4.1.

By the similarity formula generation algorithm, different functions $f_i$ and $G$ are selected to verify whether the theorem 4.3 is satisfied, and specific weighted similarity formulas for hesitant fuzzy linguistic term sets can be generated.

$$S_5(H^1_s, H^2_s) = 1 - \frac{1}{\sum_{j=1}^{l_j} \sum_{i=1}^{l_j} \cos(g(s^1_{ij}(x_j)) - g(s^2_{ij}(x_j))) \frac{\pi}{2}}$$  \hspace{1cm} (16)

$$S_6(H^1_s, H^2_s) = 1 - \frac{1}{\sum_{j=1}^{l_j} \sum_{i=1}^{l_j} |g(s^1_{ij}(x_j)) - g(s^2_{ij}(x_j))|^p}^{1/p}, p > 0$$  \hspace{1cm} (17)

4.3. The relationship between entropy and similarity of hesitant fuzzy linguistic term set

This paper further studies the relationship between entropy and similarity of hesitant fuzzy linguistic term sets, and proposes a general formula for entropy based on similarity.

**Theorem 4.4** Let $S = \{s_t|t=\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ is the linguistic term set, $H_s$ is the hesitant fuzzy linguistic term set on the domain $X = \{x_1, x_2, \ldots, x_n\}$, then $S(H_s, \bar{H}_s)$ is the entropy of the hesitant fuzzy linguistic term set $H_s$.

**Theorem 4.5** Let $S = \{s_t|t=\tau, \ldots, -1, 0, 1, \ldots, \tau\}$ is the linguistic term set, $H_s$ is the HFLTS on the domain $X = \{x_1, x_2, \ldots, x_n\}$, and $h_s(x_j) = \{|s_{ij}(x_j)| i=1, \ldots, l_j\}$ is the HFLE of $H_s$. 
\[
S(H_s, \bar{H}_s) = 1 - G \left[ \frac{1}{n} \sum_{j=1}^{n} \sum_{l_i=1}^{l_j} f_j(g(s_{(i)}(x_j)) + g(s_{(l_i,-i+1)}(x_j)) - 1) \right]
\]

(18)

Where \( c_j \) is a positive real number and \( f_j : [-1,1] \to [0,1] \) satisfies:
(1) \( \forall x \in [-1,1], f_j(x) = f_j(-x) \);
(2) \( f_j(0) = 0, f_j(1) = 1 \);
(3) \( f_j(x) \) is strictly increasing on \([0,1]\).

Let \( a = \frac{1}{n} \sum_{j=1}^{n} c_j \), \( G : [0,a] \to [0,1] \) is strictly increasing, and \( G(0) = 0, G(a) = 1 \), then
\[
S(H_s, \bar{H}_s) = E(H_s)
\]

is the entropy of the hesitant fuzzy linguistic term set \( H_s \).

**Proof** From Theorem 4.1 and Theorem 4.4, it is easy to prove that Theorem 4.5 is true. Then specific formulas for entropy based on similarity for hesitant fuzzy linguistic term sets can be generated.

\[
E_k(H_s) = 1 - \left[ \frac{1}{n} \sum_{j=1}^{n} \sum_{l_i=1}^{l_j} \left( 1 - \cos \left( g(s_{(i)}(x_j)) + g(s_{(l_i,-i+1)}(x_j)) - 1 \right) \right) \right]^{\frac{1}{2}}
\]

(19)

\[
E_l(H_s) = 1 - \left[ \frac{1}{n} \sum_{j=1}^{n} \sum_{l_i=1}^{l_j} \left| g(s_{(i)}(x_j)) + g(s_{(l_i,-i+1)}(x_j)) - 1 \right|^2 \right]^{\frac{1}{2}}
\]

(20)

\[
E_p(H_s) = 1 - \left[ \frac{1}{n} \sum_{j=1}^{n} \sum_{l_i=1}^{l_j} \left| g(s_{(i)}(x_j)) + g(s_{(l_i,-i+1)}(x_j)) - 1 \right|^p \right]^{1/p}, p > 0
\]

(21)

5. Conclusion
In this paper, we gave the axiomatic definitions of the entropy and similarity measures for hesitant fuzzy linguistic term sets, and proposed the general formulas of entropy and similarity for hesitant fuzzy linguistic term sets. We gave the algorithm for generating entropy and similarity of hesitant fuzzy linguistic term set. We studied the relationship between entropy and similarity of hesitant fuzzy linguistic value sets. These lay the theoretical foundation for flexible selection of entropy and similarity in multi-attribute decision making.

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