THE LEAST COMMON MULTIPLE OF SEQUENCE OF PRODUCT
OF LINEAR POLYNOMIALS

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ABSTRACT. Let $f(x)$ be the product of several linear polynomials with integer coefficients. In this paper, we obtain the estimate: $\log \lcm(f(1), \ldots, f(n)) \sim An$ as $n \to \infty$, where $A$ is a constant depending on $f$.

1. Introduction

The first significant attempt for proving the prime number theorem was made by Chebyshev in 1848-1852. In fact, Chebyshev [2] introduced the following two functions:

$$\vartheta(x) := \sum_{p \leq x} \log p = \log \prod_{p \leq x} p$$

and

$$\psi(x) := \sum_{p^k \leq x} \log p = \log \lcm_{1 \leq i \leq \lfloor x \rfloor} \{i\},$$

where $p$ denotes a prime number and $\lfloor x \rfloor$ denotes the greatest integer no more than $x$. The prime number theorem asserts that $\vartheta(n) \sim \psi(n) \sim n$. Thus the asymptotic formula $\log \lcm_{1 \leq i \leq n} \{i\} \sim n$ is equivalent to the prime number theorem. From then on, the topic of estimating the least common multiple of any given sequence of positive integers become prevalent and important. Hanson [6] and Nair [9] got the upper bound and lower bound of $\lcm_{1 \leq i \leq n} \{i\}$ respectively. Farhi [3] investigated the least common multiple of arithmetic progression while Farhi and Kane [4] and Hong and Yang [8] studied the least common multiple of consecutive positive integers. Recently, Hong and Qian [7] got some results on the least common multiple of consecutive arithmetic progression terms. In 2002, Bateman, Kalb and Stenger [1] proved that for any integers $a$ and $b$ such that $a \geq 1$ and $a + b \geq 1$ and $\gcd(a, b) = 1$, one has

$$\log \lcm_{1 \leq i \leq n} \{ai + b\} \sim \frac{an}{\varphi(a)} \sum_{\gcd(r, a) = 1} \frac{1}{r},$$

as $n \to \infty$, where $\varphi(a)$ denotes the number of integers relatively prime to $a$ between 1 and $a$.

Let $h$ and $l$ be any two relatively prime positive integers. The renowned Dirichlet’s theorem says that there are infinitely many prime numbers in the arithmetic progression $\{hm + l\}_{m \in \mathbb{N}}$. Furthermore, if we define

$$\vartheta(x; h, l) := \sum_{\substack{p \leq x \atop p \equiv l \pmod{h}}} \log p,$$

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then the prime number theorem for arithmetic progressions says that
\[
\vartheta(x; h, l) = \frac{x}{\varphi(h)} + o(x),
\]
(1.1)

For an analytic proof, see Davenport [5]. Moreover, Selberg [10] gave an elementary proof of this result.

In this paper, we concentrate on the least common multiple of sequence of product of linear polynomials with integer coefficients. Throughout this paper, for any polynomial \( g(x) = a_n x^n + \ldots + a_0 \) with integer coefficients, we define
\[
L_n(g) := \text{lcm}(g(1), \ldots, g(n)).
\]

If \( \gcd(a_0, \ldots, a_n) = d \), then
\[
\log L_n(g) = \log(dL_n(g_1)) = \log L_n(g_1) + \log d = \log L_n(g_1) + O(1),
\]
where \( g_1(x) = \frac{a_n}{d} x^n + \ldots + \frac{a_0}{d} \) is a primitive polynomial. Thus it suffices to give the estimate for primitive polynomials. As usual, let \( \mathbb{Q} \) and \( \mathbb{N} \) denote the field of rational numbers and the set of nonnegative integers. Define \( \mathbb{N}^* := \mathbb{N} \setminus \{0\} \). For any prime number \( p \), we let \( v_p \) be the normalized \( p \)-adic valuation of \( \mathbb{N}^* \), i.e., \( v_p(a) = s \) if \( p^s \parallel a \). For any two positive integers \( a \) and \( b \), let \( \langle b \rangle_0 \) denote the least nonnegative integer congruent to \( b \) modulo \( a \) between 0 and \( a - 1 \). We can now state the main result of this paper.

**Theorem 1.1.** Let \( \{t_i\}_{i=0}^k \) be an increasing sequence of integers with \( t_0 = 0 \), and let \( \{d_j\}_{j=1}^n \) be a sequence of positive integers such that \( d_1 = \ldots = d_{t_1} > d_{t_1+1} = \ldots = d_{t_2} > \ldots > d_{t_{k-1}+1} = \ldots = d_{t_k} \). Let
\[
f(x) := \prod_{i=1}^k \prod_{j=t_{i-1}+1}^{t_i} (a_j x + b_j)^{d_j},
\]
where \( a_j, b_j \in \mathbb{N}^* \) and \( \gcd(a_j, b_j) = 1 \) for each \( 1 \leq j \leq t_k \) and \( a_{j_1} b_{j_2} \neq a_{j_2} b_{j_1} \) for any two integers \( 1 \leq j_1 \neq j_2 \leq t_k \). Then we have
\[
\log L_n(f) \sim \frac{n}{\varphi(q)} \sum_{r=1}^q \sum_{v \mid \gcd(r, q)} \prod_{i=1}^k (d_i - d_{i+1}) \max_{1 \leq j \leq t_i} \frac{a_j}{\langle b_j \rangle_{v} a_j} \quad (1.2)
\]
as \( n \to \infty \), where \( q = \text{lcm}_{1 \leq j \leq t_k} \{a_j\} \) and \( d_{t_k+1} := 0 \).

Note that if some \( b_j \) in Theorem 1.1 are negative integers, then (1.2) is still true. Evidently, if one picks \( k = t_1 = d_{t_1} = 1 \), then Theorem 1.1 reduces to the Bateman-Kalb-Stenger theorem [1]. The proof of Theorem 1.1 will be given in the second section.

### 2. Proof of Theorem 1.1

In this section, we prove the main theorem. For convenience, in what follows we let \( g_j(x) := a_j x + b_j \) for all \( 1 \leq j \leq t_k \). Then \( f(x) = \prod_{i=1}^k \prod_{j=t_{i-1}+1}^{t_i} g_j(x)^{d_j} \). We can now show Theorem 1.1 as follows.

**Proof of Theorem 1.1.** Let \( q = \text{lcm}_{1 \leq j \leq t_k} \{a_j\} \), and let \( R(q) = \{r \in \mathbb{N}^* \mid 1 \leq r \leq q, \gcd(r, q) = 1\} \) be the set of positive integers relatively prime to \( q \) and not exceeding \( q \). In the following, we let \( P_n(f) \) be the set of the prime factors of \( L_n(f) \), and let \( P_{n,1}(f) \) denote the set of the elements in \( P_n(f) \) which divide either \( \text{lcm}_{1 \leq j \leq t_k} \{(a_j, b_j) -
Define \( P_{n,2}(f) := P_n(f) \setminus P_{n,1}(f) \) to be the complementary set of \( P_{n,1}(f) \) in \( P_n(f) \). Obviously we have

\[
L_n(f) = \left( \prod_{p \in P_{n,1}(f)} p^{v_p(L_n(f))} \right) \left( \prod_{p \in P_{n,2}(f)} p^{v_p(L_n(f))} \right),
\]
equivalently,

\[
\log L_n(f) = \sum_{p \in P_{n,1}(f)} v_p(L_n(f)) \log p + \sum_{p \in P_{n,2}(f)} v_p(L_n(f)) \log p. \tag{2.1}
\]

We claim that if \( p \in P_{n,2}(f) \) and \( p \mid f(m) \) for some positive integer \( m \), then there is a unique integer \( j_0 \) with \( 1 \leq j_0 \leq t_k \) such that \( p \mid g_{j_0}(m) \) and \( p \nmid g_j(m) \) for all other integers \( j \) between 1 and \( t_k \). In fact, we suppose that \( p \mid (a_{j_1} m + b_{j_1}) \) and \( p \mid (a_{j_2} m + b_{j_2}) \) for some positive integers \( j_1 \) and \( j_2 \) with \( 1 \leq j_1 \neq j_2 \leq t_k \). Then we have

\[
p|a_{j_1}(a_{j_2} m + b_{j_2}) - a_{j_2}(a_{j_1} m + b_{j_1}) = a_{j_1}b_{j_2} - a_{j_2}b_{j_1}.
\]

It follows that

\[
p|\text{lcm}_{1 \leq j \leq t_k} \{a_j b_j - a_i b_i\},
\]

which means that \( p \in P_{n,1}(f) \). This is a contradiction. So the claim is proved. Thus for any \( p \in P_{n,2}(f) \), by the claim we have

\[
v_p(L_n(f)) = \max_{1 \leq m \leq n} \{v_p(f(m))\} = \max_{1 \leq m \leq n} \{d_j v_p(g_j(m))\}
= \max_{1 \leq j \leq t_k} \{d_j v_p(g_j(m))\} = \max_{1 \leq j \leq t_k} \{d_j v_p(L_n(g_j))\}. \tag{2.2}
\]

If \( p \in P_{n,2}(f) \) and \( \max_{1 \leq j \leq t_k} \{v_p(L_n(g_j))\} \geq 2 \), then we have \( p^2 | L_n(g_{j_0}) \) for some integer \( j_0 \in [1, t_k] \), which implies that \( p^2 | g_{j_0}(m) \) for some positive integers \( m \leq n \). Therefore

\[
p \leq \sqrt{g_{j_0}(m)} \leq \sqrt{g_{j_0}(n)} \leq M_n := \max_{1 \leq j \leq t_k} \{\sqrt{g_j(n)}\}. \tag{2.3}
\]

On the other hand, by the definition of \( P_{n,1}(f) \), we obtain that \( P_{n,1}(f) \) consists of only finitely many primes, and hence for all primes \( p \in P_{n,1}(f) \) and all sufficiently large \( n \), we have \( p \leq M_n \ll \sqrt{n} \). Thus for all sufficiently large \( n \), we can rewrite (2.1) as

\[
\log L_n(f) = \sum_{p \leq M_n} v_p(L_n(f)) \log p + \sum_{p > M_n} v_p(L_n(f)) \log p. \tag{2.4}
\]

It is obvious that if \( p \leq M_n \), then

\[
v_p(L_n(f)) \leq \frac{\log f(n)}{\log p} = \sum_{j=1}^{t_k} d_j \frac{\log g_j(n)}{\log p} \leq \sum_{j=1}^{t_k} d_j \frac{\log n}{\log p} \ll \frac{\log n}{\log p}.
\]

Thus we have

\[
\sum_{p \leq M_n} v_p(L_n(f)) \log p \ll \sum_{p \leq M_n} \frac{\log n}{\log p} \log p \ll \sum_{p \leq M_n} \log n \ll \pi(M_n) \log n
\ll \frac{M_n}{\log M_n} \log n \ll \frac{\sqrt{n}}{\log \sqrt{n}} \log n \ll \sqrt{n}.
\]
It then follows from (2.4) that
\[
\log L_n(f) = \sum_{p > M_n} v_p(L_n(f)) \log p + O(\sqrt{n}).
\]

Now let \( p \in P_{n,2}(f) \). Then it is easy to see that \( p \) is congruent to \( r' \) modulo \( q \) for some \( r' \in R(q) \) and gcd\((r', a_j) = 1 \) for all \( 1 \leq j \leq t_k \). For such \( r' \), there is exactly one \( r \in R(q) \) such that \( rr' \equiv 1 \pmod{q} \), and hence we have \( (b_j r')_a, p \equiv (b_j r')_a \) \( r' \equiv b_j r' \equiv b_j \pmod{a_j} \) for all \( 1 \leq j \leq t_k \). We can easily derive that \( (b_j r')_a \) \( p \) is the smallest multiple of \( p \) which is congruent to \( b_j \) modulo \( a_j \) for all \( 1 \leq j \leq t_k \). It follows that for any \( 1 \leq j \leq t_k \) and any \( p \in P_{n,2}(f) \) which is congruent to \( r' \) modulo \( q \), we have that \( p|(a_jm + b_j) \) for some \( m \leq n \) if and only if \( p \leq \frac{a_jm+b_j}{(b_j r')_a} \). Thus, for any sufficiently large \( n \) and for any prime \( p \in P_{n,2}(f) \) which is congruent to \( r' \in R(q) \) modulo \( q \) satisfying \( p > M_n \), we have by (2.2) and (2.3) that
\[
v_p(L_n(f)) = \max_{1 \leq j \leq t_k} \{ d_j v_p(L_n(g_j)) \} = \max_{1 \leq j \leq t_k} \{ e_j \},
\]
where
\[
e_j := \begin{cases} d_j, & \text{if } M_n < p \leq \frac{a_jn+b_j}{(b_j r')_a}, \\ 0, & \text{if } p > \frac{a_jn+b_j}{(b_j r')_a}. \end{cases}
\]

Since \( d_1 + \cdots + d_t > d_{t+1} = \cdots = d_{t+1} = \cdots = d_{t_k} \), we deduce that for sufficiently large \( n \),
\[
v_p(L_n(f)) = \begin{cases} d_{t_1}, & \text{if } M_n < p \leq \max_{1 \leq j \leq t_{i-1}} \{ \frac{a_jn+b_j}{(b_j r')_a} \}, \\ d_{t_i}, & \text{if } \max_{1 \leq j \leq t_{i-1}} \{ \frac{a_jn+b_j}{(b_j r')_a} \} < p \leq \max_{1 \leq j \leq t_{i-1}} \{ \frac{a_jn+b_j}{(b_j r')_a} \} \text{ for some } 2 \leq i \leq k. \end{cases}
\]

Obviously, we have that for sufficiently large \( n \), \( v_p(L_n(f)) = 0 \) for any prime \( p > M_n \) and \( p \notin P_{n,2}(f) \). Thus we obtain by (2.5) that
\[
\log L_n(f) = \sum_{p > M_n} v_p(L_n(f)) \log p + O(\sqrt{n})
\]
\[
= \sum_{r' \in R(q)} \sum_{p \in P_{n,2}, \ p \equiv r' \pmod{q} \ p > M_n} v_p(L_n(f)) \log p + O(\sqrt{n})
\]
\[
= \sum_{r' \in R(q)} \left( \sum_{M_n < p \leq \max_{1 \leq j \leq t_i} \{ \frac{a_jn+b_j}{(b_j r')_a} \}} d_{t_i} \log p 
\right)
\]
\[
+ \sum_{i=2}^k \sum_{\max_{1 \leq j \leq t_{i-1}} \{ \frac{a_jn+b_j}{(b_j r')_a} \} < p \leq \max_{1 \leq j \leq t_{i-1}} \{ \frac{a_jn+b_j}{(b_j r')_a} \}} d_{t_i} \log p 
\right) + O(\sqrt{n})
\]
\[
= \sum_{r' \in R(q)} \left( d_{t_1} \left( \vartheta\left( \max_{1 \leq j \leq t_i} \{ \frac{a_jn+b_j}{(b_j r')_a} \}; q, r' \right) - \vartheta(M_n; q, r') \right) + \sum_{i=2} F_i(n) \right) + O(\sqrt{n}),
\]
where
\[
F_i(n) := \vartheta\left( \max_{1 \leq j \leq t_i} \{ \frac{a_jn+b_j}{(b_j r')_a} \}; q, r' \right) - \vartheta\left( \max_{1 \leq j \leq t_{i-1}} \{ \frac{a_jn+b_j}{(b_j r')_a} \}; q, r' \right).
\]
Now, applying the prime number theorem for arithmetic progressions (i.e. (1.1)), we obtain that
\[
\log L_n(f) = \frac{n}{\varphi(q)} \sum_{r R(q)} \sum_{i=1}^{k} d_t \left( \max_{1 \leq j \leq t_i} \left\{ \frac{a_j}{(b_j r)_{a_j}} \right\} - \max_{1 \leq j \leq t_{i-1}} \left\{ \frac{a_j}{(b_j r)_{a_j}} \right\} \right) + o(n),
\]
where \( \max_{1 \leq j \leq t_i} \left\{ \frac{a_j}{(b_j r)_{a_j}} \right\} := 0 \) and \( rr' \equiv 1 \pmod q \).

Since \( r \) runs over \( R(q) \) as \( r' \) does, it follows immediately that
\[
\log L_n(f) = \frac{n}{\varphi(q)} \sum_{r R(q)} \left( d_{t_k} \max_{1 \leq j \leq t_k} \left\{ \frac{a_j}{(b_j r)_{a_j}} \right\} - \sum_{i=1}^{k-1} (d_{t_i} - d_{t_{i+1}}) \max_{1 \leq j \leq t_i} \left\{ \frac{a_j}{(b_j r)_{a_j}} \right\} \right) + o(n).
\]
So we get (1.2) and Theorem 1.1 is proved.

In particular, we have the following two consequences.

**Corollary 2.1.** Let \( l \geq 1 \) be an integer and \( \{s_i\}_{i=1}^{l} \) be a decreasing sequence of positive integers, and let \( g(x) = \prod_{i=1}^{l} (a_i x + b_i)^{s_i} \), where \( a_i, b_i \in \mathbb{N}^* \) and \( \gcd(a_i, b_i) = 1 \) for each \( 1 \leq i \leq l \) and \( a_i b_j \neq a_j b_i \) for any \( 1 \leq i \neq j \leq l \). Then we have
\[
\log L_n(g) \sim \frac{n}{\varphi(q)} \sum_{\gcd(r,q)=1} \sum_{i=1}^{l} (s_i - s_{i+1}) \max_{1 \leq j \leq l} \left\{ \frac{a_j}{(b_j r)_{a_j}} \right\},
\]
where \( q = \text{lcm}_{1 \leq i \leq l} \{a_i\} \) and \( s_{l+1} := 0 \).

**Corollary 2.2.** Let \( l, d \geq 1 \) be integers and \( g(x) = \prod_{i=1}^{l} (a_i x + b_i)^{d} \), where \( a_i, b_i \in \mathbb{N}^* \) and \( \gcd(a_i, b_i) = 1 \) for each \( 1 \leq i \leq l \) and \( a_i b_j \neq a_j b_i \) for any two integers \( 1 \leq i \neq j \leq l \). Then we have
\[
\log L_n(g) \sim \frac{dn}{\varphi(q)} \sum_{\gcd(r,q)=1} \max_{1 \leq i \leq l} \left\{ \frac{a_i}{(b_i r)_{a_i}} \right\},
\]
where \( q = \text{lcm}_{1 \leq i \leq l} \{a_i\} \).

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