Momentum Scale Expansion of Sharp Cutoff Flow Equations

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Abstract
We show how the exact renormalization group for the effective action with a sharp momentum cutoff, may be organised by expanding one-particle irreducible parts in terms of homogeneous functions of momenta of integer degree (Taylor expansions not being possible). A systematic series of approximations – the $O(p^M)$ approximations – result from discarding from these parts, all terms of higher than the $M$th degree. These approximations preserve a field reparametrization invariance, ensuring that the field’s anomalous dimension is unambiguously determined. The lowest order approximation coincides with the local potential approximation to the Wegner-Houghton equations. We discuss the practical difficulties with extending the approximation beyond $O(p^0)$.

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1. Introduction.

In ref.[1], Wegner and Houghton formulated an exact renormalization group equation for the Wilsonian effective action $S_\Lambda[\varphi]$ where the associated momentum cutoff $\Lambda$ is taken to be sharp: only momentum modes $p$ satisfying $p < \Lambda$ are kept in the functional integral. The flow equation corresponds to computing the induced change in $S_\Lambda$ as $\Lambda$ is lowered to $\Lambda - \delta \Lambda$, by integrating out the modes with $\Lambda - \delta \Lambda < p < \Lambda$. In the same paper, the authors solved for the Wilson fixed point (i.e. massless case) of the large $N$ limit of three dimensional $O(N)$ scalar field theory, using the exact renormalization group language. This solution involved approximating the effective action by just a general effective potential $V_\Lambda(\varphi)$:

$$S_\Lambda[\varphi] = \int d^D x \left( \frac{1}{2} (\partial_\mu \varphi)^2 + V_\Lambda(\varphi) \right).$$

(1.1)

In the large $N$ limit this approximation effectively becomes exact. In ref.[4], Nicoll, Chang and Stanley proposed to use such a local potential approximation even when there is no justification in terms of the smallness of $1/N$. Such a local potential turns out to be one-particle irreducible and may equivalently be regarded as an approximation to the sharp cutoff flow equations for the Legendre effective action $S_\Lambda$. This approximation has proved to be very robust and has been employed many times since[4–12]. Obviously, it is very important to be able to establish such approximations, because there are many situations in quantum field theory where no genuinely small parameter exists to control the approximation.

One of the main reasons for the present paper is to provide further theoretical justification for the local potential approximation by demonstrating that it may effectively be regarded simply as the start of a certain expansion of $S_\Lambda$ in powers of momenta. This is by no means as trivial as it might appear at first sight. Quite apart from the fact that a direct expansion in small momenta of the one-particle reducible parts of $S_\Lambda$ vanishes to all orders, a Taylor expansion in momentum components $p^\mu$ cannot any way be implemented: the sharp cutoff in the flow equations induces non-analyticity at the origin of momentum space, reflecting the non-local behaviour in position space. Instead, the expansion is performed

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1 a.k.a. Hamiltonian in statistical mechanics language
2 i.e. the large $N$ limit $N$-vector model, equivalent to the exactly solved spherical model.
3 a.k.a. course-grained Helmholtz free energy. This argument is reviewed later.
4 and sometimes rediscovered. For derivations see e.g. refs.[1,4,7,8].
via the one-particle irreducible parts, and the best that one can achieve is an expansion in momentum scale\cite{13}, \( p \sim \sqrt{p^\mu p_\mu} \), the \( M \)th approximation resulting from dropping all terms beyond \( O(p^M) \). This raises the spectre that the lowest order approximation – \( O(p^0) \), in which the momentum scale dependence is completely discarded, nevertheless still involves Green functions with non-trivial dependence on the \textit{angles} between their momenta. It turns out that this dependence cancels however, and it is in this way that we recover the local potential approximation\cite{4,12}.

Our second main motivation is unfortunately more negative. We wish to provide the evidence for the claim in ref.\cite{14}: that the momentum scale expansion of the sharp cut-off flow equations is not a practical technique, i.e. beyond \( O(p^0) \), because only certain truncations (of the field dependence) are calculable in closed form, and this is not good enough\cite{8}. It is not good enough because, in contrast to a momentum scale expansion – which might be expected to work\cite{5} since it corresponds to an expansion in (an appropriately defined) ‘localness’ of the effective Lagrangian, truncations of the field dependence have limited accuracy and reliability\cite{8}. We want to emphasise here that our claim that there are practical difficulties in extending the momentum scale expansion beyond \( O(p^0) \), merely amounts to an admission of our own lack of ability in extending it in a substantive way. It would certainly be worthwhile to improve on these attempts (presented in sect.6), because a simple two-loop example – and general arguments (c.f. ref.\cite{13} and appendix B), suggest that the sharp cutoff momentum scale expansion should yield the fastest convergence (i.e. faster than the smooth cutoff momentum expansions\cite{14,15}).

Despite the increasing interest in utilising the exact renormalization group to develop non-perturbative approximations\cite{19}, there have so far been relatively few works concerned with going beyond the local potential approximation\cite{11,13}. Part of the point of sect.6 is also pedagogical: we provide simple concrete illustrations of the discussions in earlier sections, but also reemphasise\cite{11,14,16,20} in a simple context the important issue of field reparametrization invariance. It has been understood in the condensed matter literature for quite some time\cite{20,16} that it is important\footnote{6} to preserve a field reparametrization invariance, e.g. simply

\[ \varphi \mapsto \varphi/\lambda \quad , \]  

and certainly seems to so far\cite{11}.
in the flow equations. For example, if field reparametrization invariance is broken by the approximation scheme, the field’s anomalous dimension can no longer be determined uniquely – the value depends on at least one unphysical parameter. Approximations by expansion in momenta (equivalently derivative expansion in the smooth cutoff case) generally break reparametrization invariance. To date, only two forms of cutoff function are known that allow a field reparametrization invariance after approximation by momentum expansion: the sharp cutoff studied here, and power-law smooth cutoffs.

Our formulation at first closely follows that of ref., but we will organise things differently. Therefore we feel it is preferable to give as much as possible a self-contained account, indicating the similarities or differences to ref. as they arise. In sect.2, we rederive the sharp cutoff flow equations, taking due care of the limit implied, but also incorporate explicitly the possibility of an anomalous dimension, and display the field reparametrization invariance. In sect.3, the momentum scale expansion, and the approximations which it suggests, are defined – and we stress again the fact that Taylor expansions in momenta cannot be implemented. This means that the important property of locality needs to be recovered, and we do this in sect.4; we also explain why, and how, the uniqueness of the expansion then follows. In sect.5 we work out the approximation to \( O(p^0) \) and demonstrate that this coincides with the local potential approximation. Sect.6 addresses by example the new issues that arise in going beyond \( O(p^0) \), computing several examples, in particular we indicate why we seem from the practical point of view to be limited to truncations of the field dependence, and discuss – with reference to a simple truncation – the issue of field reparametrization invariance. Finally, in sect. 7 we present our summary, making some general observations on the momentum expansion presented here.

Before embarking on the paper proper, we review below the arguments that lead to expressing the flow in terms of a Legendre effective action \( \Gamma_\Lambda[\varphi] \) with I.R. (infra-red) cutoff \( \Lambda \). The reason that this, or an equivalent formulation, must be used is that the Polchinski (or equivalently, the Wilson) effective action \( S_\Lambda[\varphi] \), for the effective theory with U.V. (ultraviolet) cutoff \( \Lambda \), has a tree structure, composed of full propagators with I.R. cutoff \( \Lambda \), and one-particle irreducible parts which are those generated by \( \Gamma_\Lambda[\varphi] \). This structure is already manifest in the graphical form of the Polchinski (equivalently, Wilson, or in the limit of sharp cutoff, Wegner’s) equation for the vertices of \( S_\Lambda \), which is displayed in fig. 1. (These trees take account of the purely classical fluctuations that are integrated out as \( \Lambda \) is reduced, and would be there even in an effective action for the classical field...
theory, while the one-particle irreducible parts summarise the integrated out quantum field fluctuations).

![Diagram](image)

**Fig.1.** The Polchinski equation for the vertices of $S_\Lambda$. The vertices are drawn as open circles. In the limit of sharp cutoff, the black dot represents a delta-function restriction of the momentum $q$, in the propagator, to $q = \Lambda$.

To resolve the limiting procedure implicit in taking a sharp cutoff, it is necessary to take this structure into account, which has the effect of reducing the equations to those (or closely equivalent to those) for $\Gamma_\Lambda[\varphi]$ (c.f. [13,23]). Yet again, we must preserve the tree structure on taking a momentum expansion. The momentum expansion naively corresponds to Taylor expanding $S_\Lambda$ in the scale of the external momenta (to the vertices of $S_\Lambda$), regarding this as small compared to the cutoff $\Lambda$. In the sharp cutoff limit, this would cause all tree terms with internal propagators to vanish however, since the internal propagator is furnished with a sharp I.R. cutoff $\Lambda$ and the momentum flowing through this propagator is of the same scale as the external momenta by momentum conservation. Clearly this is too great a mutilation of the theory, in particular not only do all tree level corrections get discarded in the process, but all loop diagrams with more than one vertex get discarded also (since these arise from substituting the tree parts of $S_\Lambda$ into the second term in fig. 1). Instead we apply a momentum expansion *only to the one-particle irreducible vertices*, equivalently to $\Gamma_\Lambda$. In these cases all internal propagators are integrated over momenta greater than $\Lambda$ (as follows from integrating fig. 1 between the overall U.V. cutoff $\Lambda_0^I$ and $\Lambda$), and the momentum expansion corresponds to expanding the external momenta, regarded as small compared to these internal momenta. (Actually this oversimplifies a little: the structure is hierarchical, with ‘inner’ loops containing momenta larger than those

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7 such as that of the first term in fig. 1
of the ‘outer’ loops and inner propagators expanded in the outer loops momenta\(^\text{[13]}\). In this way no propagators are lost, the diagrammatic structure is respected, and we might reasonably hope that the momentum expansion leads to a convergent numerical series. The model two-loop calculation presented in ref.\(^\text{[13]}\) (and reviewed in appendix B) is encouraging in this respect since it results in a rapidly convergent numerical series.

Finally we briefly mention an immediate consequence of the tree structure: Setting the external momenta for the vertices of \(S_\Lambda\) to zero, kills all the tree terms. Therefore the Wilson effective potential (supplied with U.V. cutoff \(\Lambda\)) coincides with the effective potential in the Legendre effective action, if the latter is computed with I.R. cutoff \(\Lambda\)\(^\text{[13]}\). This explains our statement in the first paragraph.

**2. The flow equations.**

Thus we take as our starting point the partition function defined as

\[
\exp W_\Lambda[J] = \int \mathcal{D}\varphi \exp\{-\frac{1}{2}\varphi.C^{-1}.\varphi - S_{\Lambda_0}[\varphi] + J.\varphi\} ,
\]

regularised by an overall U.V. momentum cutoff \(\Lambda_0^f\). The notation is essentially the same as previously\(^\text{[13,14]}\), so two-point functions are often regarded as matrices in position or momentum (\(q\)) space, one-point functions as vectors, and contractions indicated by a dot. Momentum conserving \(\delta\)-functions are factored out when appropriate. We work in \(D\) Euclidean dimensions with a single real scalar field \(\varphi\). The definition differs from ref.\(^\text{[13]}\) only in that here we include the kinetic term \(\frac{1}{2}(\partial_\mu \varphi)^2\) in \(S_{\Lambda_0}\) (and will do likewise in \(\Gamma_\Lambda\)) so that \(S_{\Lambda_0}\) is the full bare action for the theory, while \(C^{-1}(q, \Lambda) = (1/\theta_\varepsilon(q, \Lambda) - 1)\) \(q^2\) is now an ‘additive’ I.R. cutoff. Here, \(\theta_\varepsilon(q, \Lambda)\) is a smooth regularisation of the Heaviside \(\theta\) function, of width \(2\varepsilon\), satisfying \(0 < \theta_\varepsilon(q, \Lambda) < 1\) for all (positive) \(\Lambda\) and \(q\), but with \(\theta_\varepsilon(q, \Lambda) \to \theta(q - \Lambda)\) as \(\varepsilon \to 0\). The additive form for the cutoff will make manifest the field reparametrization invariance: that is invariance of the flow equations under rescaling of the field (\(1.2\)). (This fact was already used in ref.\(^\text{[12]}\). The same effect would be obtained by redefining \(\Gamma_\Lambda \mapsto \Gamma_\Lambda - \frac{1}{2} \int d^Dx (\partial_\mu \varphi)^2\) at the end of the calculation.) The reason for this can be understood as follows: For any finite \(\varepsilon\) the invariance is broken by the induced change in the cutoff term \(C^{-1} \mapsto \lambda^2 C^{-1}\), but in the limit of sharp cutoff (i.e. \(\varepsilon \to 0\)),

\[^8\text{It is not in fact an expansion in a small parameter because the external momenta } p \text{ are integrated out over the range } p < \Lambda.\]
\( \lambda^2 C^{-1} \) has the same effect for any \( \lambda \), since it is either zero (for momenta \( q > \Lambda \)) or forces the integrand of (2.1) to vanish. From (2.1) we have

\[
\frac{\partial}{\partial \Lambda} W_\Lambda[J] = -\frac{1}{2} \left\{ \frac{\delta W_\Lambda}{\delta J} \frac{\partial C^{-1}}{\partial \Lambda} \frac{\delta W_\Lambda}{\delta J} + \text{tr} \left( \frac{\partial C^{-1}}{\partial \Lambda} \frac{\delta^2 W_\Lambda}{\delta J \delta J} \right) \right\},
\]

which on rewriting in terms of the Legendre effective action \( \Gamma_\Lambda \) gives (as in ref. [13]),

\[
\frac{\partial}{\partial \Lambda} \Gamma_\Lambda[\varphi] = -\frac{1}{2} \int \frac{d^Dq}{(2\pi)^D} \frac{\partial C^{-1}(q, \Lambda)}{\partial \Lambda} \left[ C^{-1} + \frac{\delta^2 \Gamma_\Lambda}{\delta \varphi \delta \varphi} \right]^{-1}(q, -q). \tag{2.2}
\]

\( \Gamma_\Lambda \) is defined by

\[
\Gamma_\Lambda[\varphi] + \frac{1}{2} \varphi . C^{-1} . \varphi = -W_\Lambda[J] + J . \varphi,
\]

where now \( \varphi = \delta W_\Lambda / \delta J \) is the classical field. In the limit \( \varepsilon \to 0 \), the \( \partial C^{-1}(q, \Lambda) / \partial \Lambda \) term restricts the momentum integral to the spherical shell \( q = \Lambda \). Therefore any other terms in (2.2) containing \( C^{-1}(q, \Lambda) \) or \( C^{-1}(\Lambda, \Lambda) \), become ambiguous in this limit, since they in turn contain \( \theta(0) \), which is ill-defined.

To properly resolve the sharp cutoff limit we must therefore isolate all such terms and treat these more carefully [13]. Thus, following ref. [13], we separate from the two-point function the field independent full inverse propagator \( \gamma(p, \Lambda) \):

\[
\frac{\delta^2 \Gamma_\Lambda[\varphi]}{\delta \varphi \delta \varphi}(p, p') = \gamma(p, \Lambda)(2\pi)^D \delta(p + p') + \hat{\Gamma}[\varphi](p, p'; \Lambda), \tag{2.3}
\]

so that \( \hat{\Gamma}[0] = 0 \), and drop from both sides of (2.2) the field independent vacuum energy contribution. The subtracted form of (2.2) can then be written

\[
\frac{\partial}{\partial \Lambda} \Gamma_\Lambda = -\frac{1}{2} \text{tr} \left\{ \frac{\partial C^{-1}}{\partial \Lambda} (C^{-1} + \gamma)^{-2} \hat{\Gamma} \left( 1 + [C^{-1} + \gamma]^{-1} \hat{\Gamma} \right)^{-1} \right\}. \tag{2.4}
\]

We will assume that the square-bracketed \([C^{-1} + \gamma]\) term, buffered as it is on both sides by \( \hat{\Gamma}[\varphi] \), ‘almost never’ carries a momentum \( p = \Lambda \), i.e. that such points form a set of zero measure for the integrations in (2.4). (The validity of this ‘zero measure assumption’ will be addressed in sect.5.) Then, from the definition of \( C^{-1}(p, \Lambda) \) we may safely take the limit

\[
\lim_{\varepsilon \to 0} \frac{1}{C^{-1}(p, \Lambda) + \gamma(p, \Lambda)} = G(p, \Lambda), \tag{2.5}
\]

where \( G(p, \Lambda) = \theta(p - \Lambda) / \gamma(p, \Lambda) \) is the I.R. cutoff full Green function. To complete the sharp cutoff limit of (2.4) we need to note

\[
\lim_{\varepsilon \to 0} \frac{\partial C^{-1}(q, \Lambda)}{\partial \Lambda} \left[ C^{-1}(q, \Lambda) + \gamma(q, \Lambda) \right]^{-2} = -\frac{\partial}{\partial \Lambda} \left\{ \lim_{\varepsilon \to 0} \frac{1}{C^{-1}(q, \Lambda) + \gamma(q, \Lambda)} \right\} \bigg|_{\Lambda = \Lambda} \delta(q - \Lambda) / \gamma(q, \Lambda),
\]

\[\theta(0)\] is by no means simply given by \( \theta(0) = \frac{1}{2} \) [13]!
where the last line results from taking the limit in a similar way to (2.5). Substituting this and (2.3) in (2.4) we obtain the sharp cutoff flow equation:

$$
\frac{\partial}{\partial \Lambda} \Gamma_{\Lambda} = \frac{1}{2} \int \frac{d^D q}{(2\pi)^D} \frac{\delta(q - \Lambda)}{\gamma(q, \Lambda)} \left[ \hat{\Gamma}_{\Lambda}(1 + G_{\Lambda})^{-1} \right](q, -q).
$$

(2.6)

Under the field rescaling (1.2) we have that \( \hat{\Gamma} \rightarrow \lambda^2 \hat{\Gamma}, \gamma \rightarrow \lambda^2 \gamma \) and \( G \rightarrow G/\lambda^2 \), as trivially follows from (2.3). We see that the sharp cutoff flow equation (2.6) indeed has manifest invariance under field rescaling (1.2), as promised. This equation may now be expanded as a power series in the field \( \varphi \) to yield an infinite set of coupled non-linear flow equations for the \((n\text{-point})\) one particle irreducible Green functions \( \Gamma(p_1, \cdots, p_n; \Lambda) \). As usual, in momentum space we factor out the momentum conserving \( \delta \)-function \( \Gamma(p, -p; \Lambda) \equiv \gamma(p, \Lambda) \):

$$
(2\pi)^D \delta(p_1 + \cdots + p_n) \Gamma(p_1, \cdots, p_n; \Lambda) = \frac{\delta^n \Gamma_{\Lambda}[\varphi]}{\delta \varphi(p_1) \cdots \delta \varphi(p_n)}.
$$

(2.7)

Our interest in using these equations is to investigate continuum limits, which are found by approaching a fixed point as \( \Lambda \rightarrow 0 \). This justifies dropping all reference to the overall cutoff \( \Lambda_0^f \), which we do implicitly (explicitly in ref.[13]), since all momenta can be restricted to be very much less than \( \Lambda_0^f \), as is clear from the trivial boundedness of the momentum integral in (2.6). At the fixed point there is only one mass scale, namely \( \Lambda \), and therefore it is helpful to write all dimensionful quantities in terms of this. We need only observe that the scaling dimension \( d_\varphi \) of the field \( \varphi \), at the fixed point, is generally anomalous: \( d_\varphi = \frac{1}{2}(D - 2 + \eta) \), where \( \eta \) is the anomalous scaling dimension. We could scale \( \Lambda \) out of the infinite set of flow equations for the \( \Gamma(p_1, \cdots, p_n; \Lambda) \), deducing their scaling dimensions from (2.7), but it is more elegant to scale \( \Lambda \) directly out of the functional eqn.(2.6). Therefore, similarly to refs.[14,8], we write

\( q \mapsto \Lambda q, \varphi(\Lambda q) \mapsto \Lambda^{d_\varphi - D} \varphi(q)/\zeta, \Gamma_{\Lambda}[\zeta^{-1} \Lambda^{d_\varphi} \varphi(x)] \mapsto \zeta^{-2} \Gamma_t[\varphi], \gamma(\Lambda p, \Lambda) \mapsto \Lambda^{2-\eta} \gamma(p, t), \hat{\Gamma}[\zeta^{-1} \Lambda^{d_\varphi} \varphi](\Lambda p, \Lambda p'; \Lambda) \mapsto \Lambda^{2-D-\eta} \hat{\Gamma}[\varphi](p, p'; t) \) and \( t = \ln(\Lambda_0/\Lambda) \). Here \( \Lambda_0 \) is the energy scale at which the ‘bare’ \( \Gamma_0[\varphi] \) will be defined \( (\Lambda_0 \ll \Lambda_0^f) \), and the number \( \zeta = (4\pi)^{D/4} \sqrt{\Gamma(D/2)} \) is chosen for convenience. (The physical meaning of \( \Gamma_0[\varphi] \) is discussed in sect.4). In addition we perform the radial part of the \( q \) integral in (2.6). The result is:

$$
\left( \frac{\partial}{\partial t} + d_\varphi \Delta_\varphi + \Delta_p \right) \Gamma_t[\varphi] = \frac{1}{\gamma(1, t)} \left\langle \left[ \hat{\Gamma}_{\Lambda}(1 + G_{\Lambda})^{-1} \right](q, -q) \right\rangle.
$$

(2.8)
In here, $\Delta \varphi = \varphi \delta \delta \varphi$ is the ‘phi-ness’ counting operator: it counts the number of occurrences of the field $\varphi$ in a given vertex. $\Delta_p$ may be expressed as
\begin{equation}
\Delta_p = \int \frac{d^D p}{(2\pi)^D} \varphi(p) p^\mu \frac{\partial}{\partial p^\mu} \frac{\delta}{\delta \varphi(p)} \tag{2.9}
\end{equation}
and is the momentum scale counting operator. The angular brackets stand for an average (i.e. normalised angular integration) over all directions of the unit vector $q = 1$. The full propagator $G(p, t)$ is now given by
\begin{equation}
G(p, t) = \theta(p - 1)/\gamma(p, t) \tag{2.10}
\end{equation}
Using $(1 + G.\hat{\Gamma})^{-1} = 1 - G.\hat{\Gamma} + (G.\hat{\Gamma})^2 - \cdots$ in (2.8) and expanding in $\varphi$ we obtain the infinite set of scaled flow equations for the $n$-point Green functions:
\begin{equation}
\left( \frac{\partial}{\partial t} + \sum_{i=1}^n p_i^\mu \frac{\partial}{\partial p_i^\mu} + nd_\varphi - D \right) \Gamma(p_1, \cdots, p_n; t) = \frac{1}{\gamma(1, t)} \langle \Gamma(q, -q, p_1, \cdots, p_n; t) \rangle - \frac{2}{\gamma(1, t)} \sum_{\{I_1, I_2\}} \langle \Gamma(q, -q - P_1, I_1; t) G(|q + P_1|, t) \Gamma(q - P_2, -q, I_2; t) \rangle \nonumber
\end{equation}
\begin{equation}
+ \frac{2}{\gamma(1, t)} \sum_{\{I_1, I_2, I_3\}} \langle \Gamma(q, -q - P_1, I_1; t) G(|q + P_1|, t) \times \Gamma(q + P_1, -q + P_2, I_3; t) G(|q - P_2|, t) \Gamma(q - P_2, -q, I_2; t) \rangle + \cdots \tag{2.11}
\end{equation}
These are illustrated in fig. 2. $P_i = \sum_{p_k \in I_i} p_k$, and $\sum_{\{I_1, I_2, I_3, \ldots I_m\}}$ is a sum over all disjoint subsets $I_i \cap I_j = \emptyset \ (\forall i, j)$ such that $\bigcup_{i=1}^m I_i = \{p_1, \cdots, p_n\}$. The symmetrization $\{I_1, I_2\}$ means this pair is counted only once i.e. $\{I_1, I_2\} \equiv \{I_2, I_1\}$. Evidently the expansion stops at the term where all vertices have their minimum number of legs, i.e. at the $n$th term in general, or the $(n/2)$th term in the $\varphi \leftrightarrow -\varphi$ invariant theory. Their structure is identical to that given in ref.[13], except for scaling out of $\Lambda$ and the redefinitions to exhibit field reparametrization invariance (not discussed in [13]). The latter now appears, by (2.7)(2.10), simply as ‘$n$-point’ invariance:
\begin{equation}
\Gamma(p_1, \cdots, p_n; t) \mapsto \lambda^n \Gamma(p_1, \cdots, p_n; t) \quad n \geq 2 \tag{2.12}
\end{equation}
Of course (2.8) and (2.11) are still exact, however these are now in a particularly helpful form for considering the momentum expansion.
Fig. 2. The sharp cutoff flow equations for one particle irreducible vertices. Internal lines are full propagators. The black dot now represents restriction to momentum \( q = 1 \); the other propagators have an I.R. momentum cutoff so that \( p > 1 \). The symbol \( \Delta \) represents the operator on the LHS (Left Hand Side) of eqn.(2.11).

3. The momentum scale expansion.

The momentum expansion for the Green functions \( \Gamma(p_1, \cdots, p_n; t) \) is in terms of homogeneous functions of non-negative integer degree [13]:

\[
\Gamma(p_1, \cdots, p_n; t) = \sum_{m=0}^{\infty} \Gamma^{(m)}(p_1, \cdots, p_n; t)
\]

such that \( \Gamma^{(m)}(\rho_p p_1, \cdots, \rho_p p_n; t) = \rho^m \Gamma^{(m)}(p_1, \cdots, p_n; t) \).
Or, before expanding in $\varphi$, we may write equivalently:

$$\Gamma_t[\varphi] = \sum_{m=0}^{\infty} \Gamma_t^{(m)}[\varphi]$$

where

$$\Delta_p \Gamma_t^{(m)}[\varphi] = (m - D) \Gamma_t^{(m)}[\varphi] \quad (3.2)$$

[hence the name “momentum scale counting operator” (2.9). The extra factor of $D$ arises from the momentum conserving $\delta$-function in (2.7).] All other external momentum dependence in (2.11) may be expanded as a series in integer degree homogeneous functions by introducing the “momentum scale” expansion parameter $\rho$, and expanding as a power series in $\rho$, e.g. the one particle irreducible Green functions generally are reexpanded in $\rho$ via $\Gamma(m)(q + \rho \hat{P}, -q - \rho \hat{P}', \cdots)$, after which $\rho$ may be reset to one. The I.R. cutoffs thus have expansions as:

$$\theta(|P + q| - 1) = \theta(q \cdot \hat{P} + P/2) = \theta(q \cdot \hat{P}) + \sum_{m=1}^{\infty} \frac{1}{m!} (P/2)^m \delta^{(m-1)}(q \cdot \hat{P}) \quad (3.3)$$

where we have introduced the unit vector $\hat{P} = \frac{P}{P}$ (and used $\theta(x - 1) \equiv \theta(x^2 - 1)$. $\delta^{(m)}(x)$ means the $m^{th}$ derivative of the $\delta$-function with respect to $x$.) Of course the fact that all external momentum dependence in (2.11) can be expanded in this way in terms of homogeneous functions of non-negative integer degree, guarantees that such an integral momentum scale expansion is self-consistent (as opposed to needing for example homogeneous functions of fractional degree). A systematic sequence of approximations – the $O(p^n)$ approximations – now results from replacing the sum $\sum_{m=0}^{\infty}$ in (3.1) (3.2) by $\sum_{m=0}^{M}$, a sum up to some maximum power of momentum scale $M = 0, 1, 2, \cdots$, substituting this into the flow equations (2.8) (2.11), and expanding the RHS (Right Hand Side) of these equations up to the same order in momentum scale.

If the $\Gamma(p_1, \cdots, p_n; t)$ were analytic functions of momenta at the origin $p_i = 0$, then they would have a Taylor expansion in the momenta, equivalent to a derivative expansion in position space, and therefore odd $m$ $\Gamma^{(m)}$ would vanish and even $m$ $\Gamma^{(m)}$ would be sums of Lorentz invariant products of $m$ momenta. Such an expansion is not however possible for sharp cutoff$^{[22,13]}$, a point we reemphasise here. The sharp cutoff induces non-analyticity at the origin of momentum space. This is already evident in the expansion (3.3): the odd $m$ terms do not vanish and the $O(p^0)$’th term, $\theta(q \cdot \hat{P})$, is still a function of the angle between between $P$ and $q$ and not simply a constant as would be required by analyticity. Remarkably, this non-analytic dependence cancels out at $O(p^0)$, as we will see in sect.5. For an expansion in small momenta beyond $O(p^0)$, there is no such serendipity,
and the best that one can arrange is an expansion in momentum scale of integer degree as in (3.1)(3.2). This can be readily confirmed by computing the momentum expansion beyond lowest order, on perturbatively evaluated Green functions (with I.R. cutoff internal propagators) e.g. the 4-point function to one loop in $\lambda \phi^4$ theory (as was done in effect in ref. [13] and reproduced here in appendix A, for completeness). At this point we should also recall that this non-analyticity is only a technical problem, and does not indicate that there is anything fundamentally wrong with a sharp cutoff [13].

4. Uniqueness and locality.

So far we have indicated how to form the $O(p^M)$ approximations to the flow equations (2.11). From their structure as first order differential equations in $t$, it is clear that they serve to determine uniquely the $O(p^M)$ approximation to the Green functions $\Gamma^{(m)}$, $m = 0, \cdots, M$, at ‘time’ $t + \delta t$, from their values at time $t$, and hence uniquely from the ‘initial’ values of the $\Gamma^{(m)}$ at time $t = 0$ (i.e. $\Lambda = \Lambda_0$). It is here (without loss of generality) that we must recover the notion of locality. The point is that we want the full theory (2.1), in the limit $\Lambda \to 0$, not to suffer from these spurious non-analyticities at zero momenta (which correspond to spurious non-local behaviour in position space). In the continuum limit, this behaviour should be purely a consequence of inserting a sharp low energy I.R. cutoff in the path integral, and cancel out once the momentum modes with $p < \Lambda$ are also included. Locality would usually be implemented by insisting that the bare action $S_{\Lambda_0}$ be local (that is, be expressible in terms of a derivative expansion). However this would require reintroducing the overall cutoff $\Lambda_0$, complicating the flow equations somewhat [13], and this we wish to avoid. Therefore we will impose instead, at $\Lambda = \Lambda_0$, that the ‘bare’ Legendre action $\Gamma_0[\varphi]$ be local [13]. By universality (of the continuum limits) we do not expect that this change matters for renormalized quantities, since it amounts to some minor alterations in how the U.V. cutoff is imposed. [Feynman diagrams vanish if any internal momentum is larger than $\Lambda_0$, while otherwise these diagrams vanish only if all internal momenta are larger than $\Lambda_0$ [13]. The reason for this difference is simple: the latter is the complement to the requirement that is imposed by the infrared cutoff when $\Lambda = \Lambda_0$ namely that the contribution would be non-vanishing only if all internal momenta are greater than $\Lambda_0$. That the weaker U.V. constraint at $\Lambda_0$ already fully regularises the theory, is clearest from the boundedness of the momentum integrals in (2.6)(2.8)(2.11).] Using (2.11) to evolve away from $t = 0$, non-analyticity will be generated by the terms in
(3.3), however this non-analyticity is now fixed, up to the usual arbitrariness in the choice of local bare action, and acceptable in the sense that it can be expected to disappear as \( \Lambda/\mu \to 0 \) (where \( \mu \ll \Lambda_0 \) is some typical scale of low energy physics). This is a full ‘in principle’ solution to fixing the momentum dependence of the \( \Gamma^{(m)} \), but, providing the angular averages on the RHS of (2.11) can be performed, we find in practice that much more can be done, as follows.

One method is simply to ansatz the general momentum dependence of the \( \Gamma^{(m)} \) (with appropriate \( t \) dependent coefficients). If the ansatz solves the equations (2.11), in the sense that the momentum dependence of both sides of the equation can be made to agree (within the \( O(p^M) \) approximation) by appropriate choice of the flow equations for the coefficients, and if the ansatz is sufficiently general that it matches the local bare action [up to \( O(p^M) \)] at \( t = 0 \) (by appropriate choice of boundary conditions for the coefficients\(^{10}\)), then it is the solution, by the uniqueness properties of the first order in \( t \) differential equation (as discussed above).

Another method is to determine the solution systematically as follows. We have that at \( t = 0 \) the \( \Gamma^{(m)} \) are non-vanishing only for \( m \) even, and the latter are generally arbitrary sums of products of \( m \) momenta [subject only to the general constraints of Lorentz invariance, permutation invariance, momentum conservation etc. as follows from e.g. (2.7)]. Substituting these general expressions for \( \Gamma = \sum_{m=0}^{M} \Gamma^{(m)} \) into (2.11), and performing the momentum scale expansion, we can determine the general form of the non-analytic parts in \( \Gamma^{(m)} \) at some small time later \( t = \delta t \). From this we can deduce at the ‘linearised level’ the general \( O(p^M) \) form of \( \Gamma \) at any time \( t \) by replacing those coefficients of order \( \delta t \) by general coefficients. Now we substitute this expression into the RHS of (2.11) and hence deduce the general form of the \( O(p^M) \) \( \Gamma \) at the ‘quadratic level’. Iterating, we find in practice that this procedure converges and in this way we deduce the general \( O(p^M) \) form of the \( \Gamma^{(m)} \) as a linear superposition of a finite set of ‘basis’ functions of momenta. Only the coefficients carry the \( t \) dependence and of course those multiplying non-analytic momentum functions, must vanish at \( t = 0 \). In the next two sections we give examples of \( O(p^M) \) approximations.

\(^{10}\) Of course any terms which are then zero for all \( t \), are dropped.
5. The local potential approximation.

We will now work out the $O(p^0)$ approximation. Since truncations of the field dependence tend to have limited accuracy and reliability, we make no further approximation.

It is here that we must discuss the validity of the zero measure assumption, made below (2.4), which is equivalent to assuming that the ill-defined $p = 1$ propagator $G(1, t) = \theta(0)/\gamma(1, t)$ [from (2.10)] appears only at a set of points of zero measure in (2.8). [If this is the case then the result is the same whatever finite value one assigns to these $\theta(0)$’s, after performing the integrals in (2.8).] Actually this assumption is plainly wrong when $\varphi(x)$ obtains a non-zero spatially independent vacuum expectation value $\varphi(x) = \langle \varphi \rangle$ [13], since then $\hat{\Gamma}[\varphi]$ has a non-zero ‘diagonal’ part i.e. a part proportional to $\delta(p + p')$. In ref. 13 we handled this by redefining $\gamma$ in (2.3) to absorb the diagonal part. One of the main results of this paper however will be to show that this procedure is not necessary. Instead, we will show that at $O(p^0)$, the vanishing external momentum limit of the sharp cutoff equations (i.e. considering $\varphi(x) \rightarrow \langle \varphi \rangle$, after having taken $\varepsilon \rightarrow 0$), is unique and equal to the result of taking the $\varepsilon \rightarrow 0$ limit of the equations where all momentum dependence is discarded first (as was done in refs. [13,8,12]). In other words we will show that these two limits commute. As we will see, this fact is by no means trivial however: the agreement is obtained only at the end of the computation and in very different ways in the two methods. What is more, the $O(p^0)$ approximation exactly coincides with the local potential approximation (1.1), providing further justification for the ubiquitous use of the latter [4–12] by demonstrating that it is simply the start of the systematic sequence of $O(p^M)$ approximations. Where derivations of the local potential approximation appear in refs. [4–12], they are all of the same type as that in refs. [1,4,7]: utilising a finite width momentum shell (with sharp edges) and discarding the momentum dependence first, before allowing the width to tend to zero. This is equivalent in effect to the derivation in refs. [13,8] if one specializes to a cutoff function that simply linearly interpolates between constant values outside the shell i.e. to $\theta_\varepsilon(q, \Lambda) = (q - \Lambda)/\varepsilon$ for $|q - \Lambda| < \varepsilon$. The important point is that in all these derivations it is not possible even in principle to improve on the approximation of discarding all the momentum dependence, while the latter is done before letting the width $\varepsilon \rightarrow 0$. This is simply because any non-zero external momentum $p$ will satisfy $p \gg \varepsilon$ as $\varepsilon \rightarrow 0$, but proper consideration of this regime precisely corresponds to interchanging the limits. Therefore any attempt to go beyond the simplest approximation of discarding all momentum dependence, must first address the question of whether this exchange of limits
still yields the same answer for the simplest approximation. As we have already stated, this turns out to be non-trivial but true.

At $O(p^0)$ the local bare action can only include interactions in a bare potential $V_0(\varphi)$ which however is taken to be completely general to begin with. By either of the methods discussed in sect.4, this suggests that we start by ansatzing

$$\Gamma_t[\varphi] = \int d^Dx \left\{ \frac{1}{2} (\partial_\mu \varphi)^2 + V_t(\varphi) \right\} . \quad (5.1)$$

This will turn out to be the complete solution to the momentum dependence at $O(p^0)$, i.e. the vertices $\Gamma(p_1, \ldots, p_n; t)$ with $n > 2$, which are in principle still functions of the angles between the external momenta, may be taken to be simply constants. Note that the kinetic term will not get corrected at $O(p^0)$ [or $O(p^1)$] so carries no $t$ dependence. The normalization for the kinetic term is conventional, but would require (and could at this order be given) separate justification here if field reparametrization invariance had been broken, since then the normalization in general affects the results\[20,14,12,11\]. However in this case we have immediately that all normalizations are equivalent, by the invariance (1.2). Equation (5.1) implies that

$$\gamma(p, t) = p^2 + V''_t(0) \quad (5.2)$$

and

$$\hat{\Gamma}[\varphi](p, -p - P; t) \equiv \hat{\Gamma}(P) = \int d^Dx \left[ V''_t(\varphi(x)) - V''_t(0) \right] e^{i P \cdot x} . \quad (5.3)$$

Here, prime refers to partial differentiation with respect to $\varphi$. The latter formula follows directly from $\delta^2 \Gamma/\delta \varphi(p) \delta \varphi(p') \equiv \int d^Dx d^Dy e^{-i (p \cdot x + p' \cdot y)} \delta^2 \Gamma/\delta \varphi(x) \delta \varphi(y)$. We have introduced the notation $\hat{\Gamma}(P)$ to emphasise that $\hat{\Gamma}$ depends only on the sum of its two momentum arguments; this is a consequence of dropping all momentum dependence in the vertices. Substituting this, (5.1) and (2.10) into (2.8), expanding the RHS of (2.8) via $(1 + G \hat{\Gamma})^{-1} = 1 - G \hat{\Gamma} + (G \hat{\Gamma})^2 - \cdots$, and dropping all but the $O(p^0)$ terms in $\gamma$ and $\theta$ in the RHS (c.f. (3.2),(3.3) and the discussion inbetween) one obtains:

$$\frac{\eta}{2} \int d^Dx (\partial_\mu \varphi)^2 = 0 ,$$

11 Higher terms in momentum scale can be added, but at $O(p^0)$ there is no feedback into these terms from the RHS of the flow equations (2.8)(2.11). The LHS of these equations then constrain these terms to be singular in $\varphi$ (which is not allowed) or vanish. See ref.\[14\] or ref.\[15\] for further explanation.
implying $\eta = 0$ for this approximation, as expected, and:

$$\int d^Dx \left\{ \frac{\partial}{\partial t} V_i(\phi) + \frac{1}{2}(D - 2)\phi V'_i(\phi) - DV_i(\phi) \right\} =$$

$$\frac{\hat{\Gamma}(0)}{\gamma(1, t)} - \sum_{r=2}^{\infty} \frac{(-1)^r}{\gamma(1, t)^r} \int \frac{d^DP_1d^DP_2 \cdots d^DP_{r-1}}{(2\pi)^D(r-1)} \hat{\Gamma}(P_1)\hat{\Gamma}(P_2) \cdots \hat{\Gamma}(P_r)$$

$$\times \langle \theta(q.P_1) \theta(q.[P_1 + P_2]) \cdots \theta(q.[P_1 + P_2 + \cdots + P_{r-1}]) \rangle,$$

where $P_r = -\sum_{k=1}^{r-1} P_k$. The $r$th contribution can be represented diagrammatically as in fig. 3, and alternatively follows from the $O(p^0)$ parts of contributions in fig. 2, after integration over the fields $\phi(p_i)$ and resummation over $n$.

![Diagram](image)

**Fig.3.** The $r$th contribution to eq.(5.4), represented diagrammatically. Recall that the small dot indicates averaging over all directions of the propagators momentum $q$, which is restricted to be of unit norm.

The above average over the unit vector $q$ is indeed a non-trivial function of the angles between the momenta $P_k$, but by utilising the cyclic symmetry of fig. 3 or (5.4) with respect to the external momenta, we can effectively replace its contribution by $1/r$, as follows. We rewrite the RHS of (5.4) as:

$$-\sum_{r=1}^{\infty} \frac{(-1)^r}{\gamma(1, t)^r} \int \prod_{k=1}^{r} \frac{d^DP_k}{(2\pi)^D} \hat{\Gamma}(P_k) \left(2\pi\right)^D \delta \left(\sum_{k=1}^{r} P_r\right) \langle \theta(q.Q_2) \theta(q.Q_3) \cdots \theta(q.Q_r) \rangle,$$

(5.5)
where we have introduced the partial sums $Q_k = \sum_{j=1}^{k-1} P_j$ for $2 \leq k \leq r$, $Q_1 = 0$, and defined the angular average to be unity for $r = 1$. Taking the average of the $r$ integrals obtained by cyclically relabelling the integration variables $P_1, \cdots, P_r$ (geometrically corresponding to the $r$ equivalent diagrams obtained from fig. 3 by placing the small dot on each of the $r$ propagators) we obtain:

$$-\sum_{r=1}^{\infty} \frac{(-1)^r}{r} \int \left( \prod_{k=1}^{r} \frac{d^D P_k}{(2\pi)^D \gamma(1,t)} \right) (2\pi)^D \delta \left( \sum_{k=1}^{r} P_r \right) \left\langle \sum_{j=1}^{r} \prod_{k=1, k\neq j}^{r} \theta(q_{[Q_k - Q_j]}) \right\rangle .$$

Now, generically for any given vector $q$, one of the $q\cdot Q_k$, say $q\cdot Q_i$, will be less than all the others i.e. $q\cdot Q_i < q\cdot Q_k \quad \forall k \neq i \in (1, \cdots, r)$. It follows that for this $q$, the $j = i$ contribution to the above $j$-sum is unity, and the $j \neq i$ contributions vanish. Therefore, except for a set of points of zero measure for the $q = 1$ angular average (defined by those vectors $q$ such that $q\cdot Q_k = q\cdot Q_i$ for some $k$), $\sum_{j=1}^{r} \prod_{k=1, k\neq j}^{r} \theta(q_{[Q_k - Q_j]}) = 1$. Substituting this, and (5.3), and writing $(2\pi)^D \delta(\sum_{k=1}^{r} P_r) = \int d^D x \exp -i(x\cdot P_1 + \cdots + x\cdot P_r)$, we have immediately that (5.6) equals $-\int d^D x \left\{ \ln \left[ 1 + V''t(0) - V''t(\varphi) \right]^{r} / \gamma(1,t)^r \right\}$. Using (5.2), this may be resummed to $\int d^D x \left\{ \ln \left[ 1 + V''t(\varphi) \right] - \ln \gamma(1,t) \right\}$. Here, the second term, which corresponds to a vacuum energy, may be renormalised away by redefining the field independent part of the potential as $V_t \mapsto V_t + E(t)$, where, by (5.4), we require $\partial E / \partial t - D E = -\ln \gamma(1,t)$. It is neater to notice however, that a $\gamma$ dependent term was discarded in (2.4) as a result of dropping from (2.2) the field independent vacuum energy contribution. If this $\gamma$ dependent term is kept, it is straightforward to show that it precisely cancels the $-\int d^D x \ln \gamma(1,t)$ above. Hence finally, we have from (5.4):

$$\frac{\partial}{\partial t} V_t(\varphi) + \frac{1}{2} (D - 2) \varphi V'_t(\varphi) - D V_t(\varphi) = \ln \left[ 1 + V''t(\varphi) \right] .$$

This coincides with the local potential approximation to the Wegner-Houghton equations.

### 6. Beyond lowest order.

We now proceed to the $O(p^1)$ approximation. Following the iterative method described in sect.4 we try to proceed by initially ansatzing a momentum independent form for $\Gamma_t$ as in (5.1), (5.2) and (5.3). We then substitute these expressions into (2.8) to see what form
the $O(p^1)$ $\Gamma_t$ takes at the ‘linearised level’. We see from expanding $\gamma(p,t)$ in (2.10) and using (3.3), that amongst other terms a set of averages of $\mathbf{q}\cdot \mathbf{Q}_k$ ($k = 2, \ldots, r$) of the form

$$\langle \theta(\mathbf{q}\cdot \mathbf{Q}_2) \cdots \theta(\mathbf{q}\cdot \mathbf{Q}_k) \mathbf{q} \cdot \mathbf{Q}_k \theta(\mathbf{q}\cdot \mathbf{Q}_{k+1}) \cdots \theta(\mathbf{q}\cdot \mathbf{Q}_r) \rangle ,$$

have to be computed [where we have used the notation introduced in (5.5)]. While we were able to replace the analogous expression in (5.5) by a constant, using (cyclic) symmetry arguments, we have found the permutation symmetry of the diagrams fig. 3 to be insufficient, when $r > 3$, to allow a simple replacement for the above expression. Of course the above expression can be evaluated exactly, in principle, but the result will be a very complicated $O(p^1)$ function, so that further insight is necessary to make progress at this level.

For this reason we now restrict the discussion of momentum dependence to cases where $r \leq 3$. This may be ‘justified’ by approximating the flow equations by truncations of the field dependence\cite{13} of the $O(p^{m>0})$ terms to six-point functions or less (in a $\varphi \leftrightarrow -\varphi$ invariant theory), i.e. discarding $\sim \varphi^8$ terms and higher in non-zero momentum scale pieces. However, we already know that truncations of the field dependence yield results that do not converge (beyond a certain order) and generally also result in spurious fixed points\cite{8}. Nevertheless, we will discuss these truncations in order to provide explicit examples of the general considerations of the previous sections. As we will see, the results are not impressive. In particular they are much worse than the results of the momentum expansion with smooth cutoff in ref.\cite{14}. We attribute this to the truncations of the field dependence, because this was not employed in ref.\cite{14}, the sharp cutoff momentum scale expansion results in a rapidly convergent (numerical) series in a model two-loop calculation (\cite{13} and app.B), and intuitively one would expect a better convergence for sharp cutoff than for smooth\cite{13}. Although we cannot at this stage rule out the possibility that the sharp cutoff momentum scale expansion is ‘at fault’ at the non-perturbative level, we suspect that truncations of the field dependence of the kinetic term $K(\varphi)$ in the smooth case\cite{14} would produce similarly unimpressive results, which would then strongly support our contention that it is also only the truncations that are at fault here.

Because we will be working to $O(p^m)$ with $m > 0$, only for certain $n$-point functions, we must generalise the discussion of previous sections to the cases where $m$ can depend on $n$, i.e. where the equations (2.11) are expanded to different order in momentum scale depending on the value of $n$. This generalisation is however straightforward and will be explained within the examples below.
We concentrate again on the critical exponents for the non-perturbative Wilson fixed point in three dimensions: the results will provide a simple measure of how good these approximations are, in a non-perturbative setting. In three dimensions, averages of functions \( f \) that depend on only one component of \( q \), say \( q \hat{P} \) as in (3.3), satisfy:

\[
\langle f(q \hat{P}) \rangle = \frac{1}{2} \int_{-1}^{1} dz f(z) .
\]

(6.2)

It is of course straightforward to give the analogous formula in any other dimension, four for example, and all averages we consider can be reduced to this (or its analogue) by symmetry arguments, in any dimension.

Consider first the simplest possible truncation that will allow non-trivial momentum dependence: discarding all but the two-point and four-point Green functions. By (2.11) this gives

\[
\left( \frac{\partial}{\partial t} + p \frac{\partial}{\partial p} - 2 + \eta \right) \gamma(p, t) = \frac{1}{\gamma(1, t)} \langle \Gamma(q, -q, p, -p; t) \rangle
\]

(6.3a)

\[
\left( \frac{\partial}{\partial t} + \sum_{i=1}^{4} p_i^\mu \frac{\partial}{\partial p_i^\mu} + 1 - 2\eta \right) \Gamma(p_1, p_2, p_3, p_4; t) =
\]

\[
- \frac{2}{\gamma(1, t)} \langle \Gamma(q, -r_{12}, p_1, p_2; t) G(r_{12}, t) \Gamma(r_{12}, -q, p_3, p_4; t) + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \rangle .
\]

(6.3b)

where \( r_{ij} = q + p_i + p_j \), and in the second average the last two terms are the same as the first term but with indices swapped as indicated. We have thrown away the six point contribution \( \sim \langle \Gamma(q, -q, p_1, p_2, p_3, p_4; t) \rangle \) to (6.3b). Working to \( O(p^1) \) we start again by ansatzing first that the four-point function is momentum independent. The analogue of (6.1) is now, using (6.2), simply

\[
\langle \theta(q, p) q \hat{P} \rangle = P/4 ,
\]

where \( P = p_1 + p_2, p_1 + p_3 \) or \( p_1 + p_4 \) depending on the term on the RHS of (6.3b). Similarly the \( O(p^1) \) term in (3.3) produces these \( P \)'s. Therefore at ‘linearised level’ we have that

\[
\Gamma(p_1, p_2, p_3, p_4; t) = \alpha_0(t) + \alpha_1(t) \left\{|p_1 + p_2| + |p_1 + p_3| + |p_1 + p_4| \right\} ,
\]

(6.4)

where flow equations for the \( \alpha_i \) are to be determined. Substituting this into eqn.(5.3a) results in \( p \) dependence of the form \( \langle |q + p| \rangle = 1 + p^2/3 \) [by (5.2)], therefore a quadratic

\begin{footnote}
This is closely similar to a model computation in ref.13. Were we also to throw away all momentum dependence it would be a differential version of the “\( S^4 \) model” 22.
\end{footnote}
ansatz remains sufficient for $\gamma(p, t)$ at this order. Substituting (6.4) back into (6.3b) results in contributions e.g. of the form

$$\Gamma(q + p_1 + p_2, -q, p_3, p_4; t) = \alpha_0 + \alpha_1\{2 + |p_1 + p_2| + q.(p_1 + p_2)\} + O(p^2) \quad ,$$

where we have used momentum conservation and $q = 1$ (and similarly for the other such contributions). Therefore the RHS of (6.3b) reproduces the general form of (6.4); our iterative procedure has closed at the linearised level. To obtain an approximation for $\eta$ we will work to $O(p^2)$ with $\gamma(p, t)$, writing

$$\gamma(p, t) = a(t)p^2 + \sigma(t) \quad ,$$

and expanding (6.3c) to $O(p^2)$, but keep to $O(p^1)$ with $\Gamma(p_1, \cdots, p_4; t)$. Substituting (6.4) and (6.6) into (6.3) we obtain:

$$\frac{\partial \sigma}{\partial t} + \eta \sigma = \alpha_0 + 2\alpha_1 \frac{\sigma + a}{\sigma + a} \quad (6.7a)$$

$$\frac{\partial a}{\partial t} + \eta a = \frac{2\alpha_1}{\sigma + a} \quad (6.7b)$$

$$\frac{\partial \alpha_0}{\partial t} + (2\eta - 1)\alpha_0 = -3 \left(\frac{\alpha_0 + 2\alpha_1}{\sigma + a}\right)^2 \quad (6.7c)$$

$$\frac{\partial \alpha_1}{\partial t} + 2\eta \alpha_1 = -\frac{1}{2} \frac{(\alpha_0 + 2\alpha_1)(\sigma \alpha_0 + \alpha_0 - a\alpha_0 + 8\sigma \alpha_1 + 4a\alpha_1)}{(\sigma + a)^3} \quad . \quad (6.7d)$$

By the locality requirements of sect.4 we have that the bare $\sigma(0), a(0)$ and $\alpha_0(0)$ are a priori arbitrary, but $\alpha_1(0) = 0$. By the field reparametrization invariance (1.2), equivalently $n$-point invariance (2.12), we can impose the conventional wavefunction renormalization condition $a(\infty) = 1$. (The full renormalized quantities are given by their $t = \infty$ values.)

Had field reparametrization invariance been broken (as happens generally under such approximations [20][16][11]), one would discover the following problem: the approximate physical quantities would differ for different values of $a(\infty)$, even though they do not in the exact formulation [as follows by using (1.2)]. This problem only potentially arises when we seek to go beyond the local potential approximation, and is not yet generally appreciated in the corresponding literature, largely because $a(\infty)$ (or its analogue) is set to unity anyway (in effect by the tuning of bare parameters or the form of the cutoff), even when the justification for this specialization is missing. To date, only two forms of cutoff function are known that allow a field reparametrization invariance after the momentum expansion approximation: the sharp cutoff studied here, and power-law smooth
cutoffs[14,13,11] $C(q, \Lambda) \sim (q/\Lambda)^{2\kappa}$[13]. The problem is particularly clear if we study the Wilson fixed point[22]. We assume that choices can be found for $\sigma(0), a(0)$ and $\alpha_0(0)$ that lie on the critical surface. Then as $t \to \infty$, the four quantities $\sigma(t), a(t), \alpha_0(t)$ and $\alpha_1(t)$ become $t$ independent and eqns.(3,4) collapse to four simultaneous algebraic equations. The problem is however, that with $\eta$, there are five quantities to be determined. Without field reparametrization invariance, this allows $\eta$ to be determined only as a function of $a(\infty)$ (say). With field reparametrization invariance, these equations correspond to non-linear eigenvalue equations[1] for the field’s anomalous dimension $\eta$, which therefore (generically) can have only certain discrete values – as it should be.

Solving the eqns.(6,7) at a fixed point $\sigma(t) \equiv \sigma, a(t) \equiv a$, etc, where they reduce to algebraic equations, we obtain that $\alpha_0, \alpha_1$ and $\eta$ can be written as polynomials in $\sigma$, which itself satisfies the quintic

$$3\sigma^5 + 92\sigma^4 - 154\sigma^3 - 159\sigma^2 + 74\sigma + 8 = 0,$$

except for the Gaussian solution $\sigma = \alpha_i = \eta = 0$. By the scaling (1.2)(2.12), we have set (in all cases) $a \equiv 1$. The quintic has five real solutions $\sigma = -32.2, -0.953, -0.0917, 0.441$ and $2.15$, however only $\sigma = -0.0917$ gives a positive $\alpha_0$ (and hence a physically stable potential). We assume that the domains of attraction of the other fixed points do not include physically sensible bare actions with $a(0) > 0$, $\alpha_0(0) > 0$ and $\alpha_1(0) = 0$.[14] Using $\sigma = -0.0917$ we obtain $\alpha_0 = .103, \alpha_1 = .0307$ and $\eta = .0225$. This last value, being universal, can be compared to the best determinations: $\eta = .032(3)$ or $.038(3)$, where the first derives from perturbation theory in fixed dimension and the second from $\varepsilon$ expansions[25]. Computing the linearised perturbations of (6,7) about this fixed point, we find a numerical matrix eigenvalue equation[13] whose eigenvalues are $\lambda = 1.88, 0$, and $-3.90 \pm .630i$. The most positive eigenvalue yields $\nu = 1/\lambda = .532$ by standard arguments[22], differing little from the $S^4$ model[22] ($\nu = .527$), and should be compared to the best determinations $\nu = .631(2)$[23]. The zero eigenvalue corresponds to the

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13 It can be shown that for other choices the field reparametrizations[24] only leave the flow equations invariant if they can carry non-trivial momentum dependence, and this is then broken by momentum expansion to finite order.

14 In general however, there are no reliable ways to reject the spurious answers resulting from truncation of the field dependence[8]. We note that in this case, the other values of $\sigma$ also result in wildly large values for the other parameters.

15 N.B. $a(t) = 1 + \delta a(t)$ where $\delta a(t)$ is not fixed.
reparametrization symmetry (1.2) and is therefore redundant [24,14]. Finally, the irrelevant complex eigenvalues represent the (first) corrections to scaling for this truncation \( \omega = -\lambda = 0.390 \pm 0.630i \), which should be compared to the best determinations \( \omega = 0.80(4) [24] \). This eigenvalue is the most affected by the approximation, as should be suspected. These results could probably be improved by making the truncation around the minimum of the fixed point effective potential [17,9]. Our point here however, is to begin with the simplest possible non-trivial example of the general method of momentum scale expansions.

We turn next to more sophisticated truncations, although we are limited to truncations of the field dependence in the momentum dependent sector, as we pointed out above. First let us generalise the above example by avoiding the truncation of the potential, that is we include all the higher \( n \)-point functions \( \Gamma(p_1, \ldots, p_n; t) \) with \( n \geq 6 \), at \( O(p^0) \), with the corresponding \( n \geq 6 \) flow equations (2.11) also evaluated to \( O(p^0) \). In this case it is clear that the ansatz that these higher point functions are independent of momenta, must satisfy these equations, since the equations differ from those of sect.5 only in the following ways: \( a(t) \neq 1, \eta \neq 0 \), and the four-point function appears in \( O(p^0) \) expressions as \( a_0(t) + 2a_1(t) \), as follows from (5.3) (and its generalisations). By sect.4, this is then the unique answer for this truncation, up to the choice of bare potential \( V_0(\varphi) \), and bare kinetic term \( a(0) \), for the local bare action. The resulting flow equations are the higher momentum scale equations (6.7b, d) and the flow for the potential which may be resummed to:

\[
\frac{\partial}{\partial t} V_t(\varphi) + \frac{1}{2} (1 + \eta) \varphi V'_t(\varphi) - 3V_t(\varphi) = \ln \left[ a(t) + a_1(t) \varphi^2 + V''_t(\varphi) \right]. \tag{6.8a}
\]

The changes between (5.7) and this equation just follow from the differences with sect.5, listed above. At the fixed point we again set \( a(t) \equiv a = 1 \), using (1.2). Equations (5.7a, b, d) are unmodified by the inclusion of the \( O(p^0) \) \( n \geq 6 \)-point functions [(6.7a) being subsumed in (6.8a)], and may be solved in terms of \( \sigma \) as \( a_0 = (1 + \sigma)([\sigma - 3] \eta - 2\sigma), \var_1 = \frac{3}{2}(1 + \sigma) \eta \) and

\[
\eta = \frac{9 + 7\sigma + 2\sigma^2 \pm \sqrt{3(1 + \sigma)\sqrt{35 - 8\sigma}}}{6 + 21\sigma + 14\sigma^2 + \sigma^3}. \tag{6.8b}
\]

At a fixed point, \( V_t(\varphi) \equiv V(\varphi) \) therefore depends only on the choice of \( \sigma \) which may be regarded as the \( \varphi = 0 \) initial condition, together with the evenness constraint \( V'(0) = 0 \). As explained previously [8] one finds that all but a discrete set of solutions of (6.8) are singular (at some finite \( \varphi \)) and therefore unacceptable. We find only two non-singular solutions, being the trivial Gaussian solution \( V(\varphi) = 0 \), and an approximation to the Wilson fixed point. Quantitative methods for computing the latter are described in ref. [8].
See also refs. [14,15]. We used the second of the better expansion methods in [8]. It is significant that all spurious results have disappeared: this is a benefit of not truncating the field dependence in the potential [11]. We find $\sigma = -0.214$ and $\eta = 0.0660$. Solving for small perturbations about this fixed point [8] we deduce $\nu = 0.612$ and $\omega = 0.91$. As expected, this is a significant improvement over the crude truncation taken previously. These results are an improvement over those of the pure local potential approximation [6,7,8]: $\eta = 0$, $\nu = 0.690$ and $\omega = 0.595$, but are not as good as the $O(p^2)$ smooth cutoff approximation [14]. Again, we attribute this to the effects here of truncation of the field dependence in the $O(p)$ part.

Now we try to improve the results by truncating the $O(p)$ part at the maximum practical value of $n = 6$. Here the computation starts to become quite involved, but once the general form of the $O(p)$ dependence is determined, it proceeds similarly to the previous example. Since our primary purpose is to illustrate the techniques of the momentum scale expansion, we indicate here only how to determine the momentum dependence, and then present the results. Since we now wish to include the $O(p)$ dependence of the six-point function, we need to work explicitly with the $n = 6$ case of the flow equations (2.11).

We can determine the $O(p)$ dependence by iteration, substituting the previous solutions for the two-point (6.6) and four-point functions (6.4). The six-point function’s momentum dependence then arises in particular from the sum over all permutations of the legs $p_1, \ldots, p_6$ in the graph fig. 4, that is from the angular average:

$$\sum_{\mathcal{P}(p_1, \ldots, p_6)} \langle \Gamma(q, -q - P_1, p_1, p_2; t) G(|q + P_1|, t) \times \Gamma(q + P_1, -q + P_2, p_5, p_6; t) G(|q - P_2|, t) \Gamma(q - P_2, -q, p_3, p_4; t)\rangle .$$

(6.9)

Here $\mathcal{P}(p_1, \ldots, p_6)$ is a permutation of the six momenta $p_1, \ldots, p_6$. $P_1 = p_1 + p_2$, $P_2 = p_3 + p_4$ and $P_3 = p_5 + p_6$.

Expanding (6.9) to $O(p)$ we find that, as well as averages that directly follow from (6.2) or sect.5, the following averages need to be determined:

$$P_2 \langle \theta(q.P_1) \delta(-q.\hat{P}_2) \rangle \quad (6.10a)$$

$$P_1 \langle \theta(-q.P_2) \delta(q.\hat{P}_1) \rangle \quad (6.10b)$$

$$\langle \theta(q.P_1) \theta(-q.P_2) q.(P_1 - P_2) \rangle \quad , \quad (6.10c)$$

where it is always to be understood that these contributions appear summed over the permutations $\mathcal{P}(p_1, \ldots, p_6)$. First note that (6.10a) contributes the same as (6.10d), as
Fig.4. One graphical contribution to the flow of the six-point function.

can be seen by substituting $q \mapsto -q$ in the average and then permuting $P_1$ and $P_2$. (This permutation simply reorganizes the terms in the sum over permutations.) Now, substituting $q \mapsto -q$ in (6.10d) gives the equivalent expression $P_2 \langle \theta(-q.P_1) \delta(-q.P_2) \rangle$, on using the evenness of the $\delta$-function. Taking the average of this and (6.10a), and using $\theta(q.P_1) + \theta(-q.P_1) = 1$, we deduce that (6.10d) equals $\frac{1}{2} P_2 \langle \delta(q.P_2) \rangle = P_2/4$. This solves for (6.10a,b). Using $\theta(-q.P_2) = 1 - \theta(q.P_2)$ in (6.10d) we obtain the equivalent expression:

$$\langle \theta(q.P_1) q.P_1 \rangle - \langle \theta(q.P_1) q.P_2 \rangle - \langle \theta(q.P_1) \theta(q.P_2) q.(P_1 - P_2) \rangle .$$

The last term in this expression yields zero by the $P_1 \leftrightarrow P_2$ symmetry of the sum over permutations. In the second term we may replace $P_2$ by $P_3$, using permutation symmetry, and hence replace the second term by $-\frac{1}{2} \langle \theta(q.P_1) q.(P_2 + P_3) \rangle$. But this is just half the first term, by momentum conservation. Lastly, the first term is $P_1/4$, by (6.2).

Computing the momentum dependence in this way we find that the six-point function is, to linearised level:

$$\Gamma(p_1, \cdots, p_6; t) = g_0(t) + g_1(t) \sum_{\text{pairs } i,j} |p_i + p_j| ,$$

(6.11)
where the flow equations for the $g_i$ are to be determined. Substituting this, (6.4), (6.6), and constants for all $O(p^0)$ higher $n$-point functions, we find the equations again close at the linearised level. The $O(p^0)$ parts resum into a partial differential equation for the potential $V_1(\varphi)$. The bare locality conditions require now $\alpha_1(0) = g_1(0) = 0$. Determining the first few couplings at the Wilson fixed point, we find that $\eta$ can be expressed as the solution of a cubic with coefficients that depend on $\sigma$, after which non-singular solutions of the fixed point equation for $V(\varphi)$, and perturbations around it, can be studied. We find $\eta = 0.0591$, $\nu = 0.604$ and $\omega = 0.540 \pm 0.47i$. This estimate for $\eta$ has improved slightly compared with the previous truncation, but the estimates for $\nu$ and $\omega$ are actually worse.

Finally, let us mention that we have extended the approximation to $O(p^2)$ for the four-point function. The only new average that presents itself is a term of the form $\langle \theta(q.P) q^\mu q^\nu \rangle = \delta^{\mu\nu}/6$, which may be evaluated in a similar way to that of (6.10b). We find, by iteration as before, that the four-point function (6.4) acquires the new term $\alpha_2(t) \{p_1^2 + p_2^2 + p_3^2 + p_4^2\}$, otherwise all $n$-point functions have the same form as before. $\eta$ is now given implicitly as a function of $\sigma$ through solutions of a complicated sextic. Unfortunately the sextic seems to have no sensible solutions (i.e. such that $\sigma < 0$ and $\eta > 0$)! If we drop instead to an $O(p^0)$ ansatz for the six-point function [i.e. set $g_1 \equiv 0$ in (6.11)], we find that the polynomial bounds the anomalous dimension to be $\eta < 0.016$, and therefore we must obtain worse results than our previous approximations. We again attribute these negative results to the, in general, poor behaviour of truncations of the field dependence — although, as discussed above, more research is necessary to confirm this conjecture.

7. Conclusions.

We have shown how the renormalization group for the effective action with a sharp momentum cutoff[1], may be organised by expanding one-particle irreducible parts in terms of homogeneous functions of momenta of integer degree i.e. by performing a momentum scale expansion of the I.R. cutoff Legendre effective action. A systematic sequence, $M = 0, 1, 2, \cdots$, of $O(p^M)$ approximations then follow. Significantly, the $O(p^0)$ approximation coincides with the local potential approximation[4] and provides further justification for the ubiquitous use of the latter[4–12] by demonstrating how it might be systematically improved.

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16 To simplify matters, we restricted our search only to the promising region of $\sigma$ where $\sigma < 0$ and $\eta > 0$. 

24
Note that the momentum scale expansion, although it is unambiguous to perform, is not an expansion in a small parameter: although the external momenta are regarded, for the purposes of the expansion, as small compared to the cutoff $\Lambda$, they are in fact set to values $p \ll \Lambda$ when the results of the expansion are substituted into the RHS of the flow equations. It is a trivial but important point to note that this implies that the expansion is a numerical one which either converges or does not. In other words there is no hinterland of asymptotic convergence, where all of quantum field theory’s small parameter expansions lie. It is not yet known in general whether the momentum expansion actually converges (although low order results compare favourably with other methods when such methods are available\textsuperscript{19}). We might expect that it generally does, because it corresponds to an expansion in an appropriately defined ‘localness’ of the effective Lagrangian, the expansion in momentum scale $p$ corresponding in position space (for sharp cutoff) to an expansion in inverse powers of the average relative distance $r$ between any two points in a vertex. Indeed, if there are large non-local contributions to the effective Lagrangian, then the description in terms of the chosen fields is probably itself inappropriate and indicates that other degrees of freedom should be introduced. This is very different from the cases where truncations of the field dependence itself could be considered appropriate: such truncations will only be valid if the field amplitude fluctuations can be considered small – but this is only true when mean field theory is a good approximation, i.e. precisely the regime where (weak coupling) perturbation theory is valid. In this way, we can understand why truncations of the field dependence in truly non-perturbative situations are somewhat unreliable\textsuperscript{8}, while momentum expansions of the effective Lagrangian (with no other approximation) seem to be so successful\textsuperscript{19}.

Another trivial but important point is to note that universal quantities, such as critical exponents, depend, at finite order in the momentum expansion, on the ‘shape’ of cutoff taken—whether it be sharp or smooth of some form $C(p, \Lambda)$. More crucially, the sharp cutoff and power-law smooth cutoffs\textsuperscript{14} are the only forms of cutoff known that preserve a field reparametrization invariance of the flow equations after approximation by momentum expansion. This is necessary for obtaining, with a given form of cutoff, a unique value\textsuperscript{8} for the fields anomalous dimension.

\textsuperscript{17} Since the equations to be solved are non-linear, it is also possible that solutions fail to exist beyond a certain $M$.

\textsuperscript{18} i.e. universal, in the sense of being independent of the details of the bare action.
Note that, while Taylor expansions in momenta (a.k.a. derivative expansions) are possible for any finite width $\varepsilon$ of the (smooth) cutoff function, as $\varepsilon \to 0$ the dominant terms in these expansions are positive powers of ratios $p/\varepsilon$, and diverge. The alternative expansion in momentum scale must be used in the sharp cutoff limit. As $\varepsilon \to 0$, the flow equations become independent of the shape of cutoff and considerably simplify compared to their smooth cutoff cousins. As discussed in appendix B, the momentum scale expansion also appears to be rapidly convergent. There is a price to pay however: the homogeneous functions of momenta, which appear at each order of the momentum scale expansion, carry in principle much more information than Taylor expansions. Therefore it seems that one ‘pays’ for the better convergence by absorbing more complication into the expansion at each order. Indeed, we did not succeed here in computing beyond $O(p^0)$ without truncating the field dependence. In the smooth cutoff case, it is clear that the expansion of the flow equations can be organised in such a way as to give differential equations only but here there is not such a simple action on the coefficient functions of the field. It is quite possible for sharp cutoff, beyond $O(p^0)$, that the resulting flow equations, if the field dependence is not truncated, would only be castable as integro-differential equations. One also needs (in principle) to check that a given fixed point behaviour can be reached from a local bare action: otherwise the behaviour may be the result of non-local physics that does not disappear as the I.R. cutoff becomes much less than typical low energy scales.

Finally, we noted that the results of the $O(p^0)$ approximation are well-defined for exceptional momenta, i.e. the limit $P_i \to 0$ is well-defined for any set of exceptional momenta – despite the fact that simply putting $P_i = 0$ results in $\theta(0)$’s which are ill-defined. In our examples beyond $O(p^0)$, in sect.6, the limits are again well-defined. Let us mention that in all these cases, the results obtained are the same as would be obtained by taking the exceptional case first and then letting $\varepsilon \to 0$. We do not have a proof that these properties are enjoyed to all orders in $p^n$. If an ambiguous result was obtained at some $O(p^n)$, in the sense that the limit for exceptional momenta depended on the direction in momentum space in which this limit was taken, then one could presumably proceed by defining these quantities (if necessary) by the value obtained when the limit $\varepsilon \to 0$ is taken after taking the exceptional case.

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Appendix A. Momentum scale expansion to one loop.

In these two appendices we review some aspects of model results obtained in ref. [13]. We rearrange them a little, to suit the present purposes. Working in four dimensions and with massless ($\varphi \leftrightarrow -\varphi$ invariant) $\lambda \varphi^4$ theory, one obtains to one-loop that the one-particle irreducible four-point function is given by

$$-\lambda_0^2 \sum_{i=2}^{4} \int_{\Lambda}^{\Lambda_0} \frac{d^4 q}{(2\pi)^4} \frac{\theta(|q + P_i| - q)}{q^2(q + P_i)^2}, \quad (A.1)$$

where $\lambda_0$ is the bare coupling, at tree level given by $\Gamma_{\Lambda_0}[\varphi] = \int d^4 x \frac{1}{2} (\partial \varphi)^2 + \lambda_0 \varphi^4 / 4!$. The momentum integral is restricted to $\Lambda < q < \Lambda_0$, as indicated. $P_i = p_1 + p_i$, and $p_1, \cdots, p_4$ are the four external momenta. The above expression corresponds to the standard $s, t$ and $u$ channel one-loop contributions. It may be obtained from one iteration of the flow eqn. (2.6), and takes the form expected – with a sharp I.R. cutoff multiplying each propagator – after a little rearrangement (c.f. sect.4 and ref. [13]). The momentum expansion of the flow equations corresponds, at this order in perturbation theory, to expanding (A.1) directly in small $P_i$. We immediately see that a Taylor expansion is not possible, as a result of the $\theta$-function, whose expansion may be taken from (3.3). In fact using this, and Taylor expanding the denominator in (A.1), one obtains:

$$-\frac{3\lambda_0^2}{(4\pi)^2} \ln \left( \frac{\Lambda_0}{\Lambda} \right) + \lambda_0^2 \sum_{i=2}^{4} \left\{ \frac{P_i}{6} \left( \frac{1}{\Lambda} - \frac{1}{\Lambda_0} \right) + \frac{P_i^3}{720} \left( \frac{1}{\Lambda^3} - \frac{1}{\Lambda_0^3} \right) + O(P_i^5) \right\} . \quad (A.2)$$

The essential point is that this is an expansion in momentum scale $P_i$, and not a Taylor expansion in momentum components $P_i^\mu$. In fact, the even powers of momentum scale are missing, in this example, to all higher orders.

Appendix B. Convergence in a two loop example.

In this appendix we go on to investigate the convergence of the momentum scale expansion in perturbation theory. It is at two loops where one first has to confront the fact that the momentum scale expansion is not an expansion in a small parameter. In fact the

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19 This is the less favourable but crucial case: the massive case has better convergence properties.
only contribution to the four-point function that does not trivially converge, is shown in fig. 5.

It arises by iterating (2.6) twice, and corresponds to the sum of two contributions, one where (say) the $t$ channel contribution from (A.1) has two of its legs tied together by replacing $P_1$ and $\Lambda$ by $p$, weighting by the two outer propagators i.e. by $\sim 1/p^4$, and then integrating over $\Lambda < p < \Lambda_0$, and another similar contribution constructed from the one-loop six-point function. Since the two contributions behave similarly under momentum expansion, we only investigate the former. Furthermore, we restrict our attention to the divergent parts, and can therefore set all external momenta to zero. (The finite parts in fact converge somewhat faster.) Therefore we are left to investigate a contribution of the form

$$6\lambda_0^3 \int_\Lambda^{\Lambda_0} \frac{d^4 p}{(2\pi)^4} \frac{1}{p^4} \int_p^{\Lambda_0} \frac{d^4 q}{(2\pi)^4} \frac{\theta(|q+p|-q)}{q^2(q+p)^2}.$$ 

In the momentum scale expansion we are to expand the inner one-loop integral as we did in appendix A. Therefore we can simply substitute the result corresponding to (A.2). After performing the $p$ integral one obtains the exact leading divergence as expected [13], while the subleading divergence is given as a rapidly convergent numerical series:

$$-\beta_2 \frac{\lambda_0^3}{(4\pi)^4} \ln \frac{\Lambda_0}{\Lambda},$$

where $\beta_2 = \frac{1}{\pi}(8 + \frac{1}{15} + \frac{9}{2800} + \cdots)$ is a contribution to the two-loop $\beta$-function coefficient. The ratios of the partial sums to the exact answer $\beta_2 = 2.568818$, are $r = .99130, .99956, .99996, \cdots$, corresponding to the $O(p^M)$ contributions where $M = 1, 3, 5, \cdots$, and converge to 3sf already at $O(p^3)$, after which approximately an extra decimal place in accuracy is added with each new term. We expect that in this way, the sharp cutoff momentum scale expansion can be shown to converge at any desired order of the loop expansion.
Now we briefly compare with smooth cutoffs. The convergence of the momentum expansion with power law smooth cutoffs is rather subtle\cite{14}, in that the polynomial corrections to the full inverse propagator play a crucial rôle in the convergence of the momentum integrals. Their momentum expansion convergence properties will be discussed elsewhere. Here we consider only a simpler example of an exponential infrared cutoff \(1/q^2 \mapsto \theta(q, \Lambda)/q^2\), where \(\theta(q, \Lambda) = 1 - \exp(-q^2/\Lambda^2)\), which can directly be compared with the above example. We might expect at least as good a convergence of the momentum expansion using this cutoff as with power law cutoffs. The smooth cutoff equivalent of (B.1) is

\[
12\lambda_0^3 \int_\Lambda^{\Lambda_0} d\Lambda_1 \int \frac{d^4p}{(2\pi)^4} \frac{\theta(p, \Lambda_1)}{p^4} \frac{\partial}{\partial \Lambda_1} \theta(p, \Lambda_1) \int_\Lambda^{\Lambda_0} d\Lambda_2 I_q ,
\]

where the one-loop integral \(I_q\) is given by

\[
I_q = \int \frac{d^4q}{(2\pi)^4} \frac{\theta(|q + p|, \Lambda_2)}{q^2(q + p)^2} \frac{\partial}{\partial \Lambda_2} \theta(q, \Lambda_2) = \frac{1}{\Lambda_2} \left( \frac{\Lambda_2}{p} \right)^2 \left\{ 1 - \exp \left( -\frac{p^2}{2\Lambda_2^2} \right) \right\} .
\]

The momentum scale expansion of \(I_q\) results in convergent momentum integrals and gives of course the second line expanded as a power series in \(p^2/\Lambda_2^2\). Performing the momentum expansion and the remaining integrals, one obtains the following closed expression for the corresponding \(\beta_2\):

\[
\beta_2 = 12 \sum_{n=1}^{\infty} \frac{(-1)^n+1}{n(n+1)} \left( \frac{1}{2} \right)^n \left[ 1 - \left( \frac{1}{2} \right)^{n+1} \right] .
\]

The partial sums up to \(n = M\) correspond to the \(O(p^{2M})\) approximations. It is clear that this sequence converges. \(\beta_2\) is different from its sharp cutoff value because, as described above, we have not included the term that follows from the one-loop six point function. We can compare ratios (of the partial sums to the full sum) however. In this case, these are \(r = 1.18269, 0.95273, 1.01432, \cdots\), corresponding to the \(O(p^M)\) contributions where \(M = 2, 4, 6, \cdots\). We see that convergence is about twice as slow as for the sharp cutoff momentum scale expansion, in this example. Slower convergence for smooth cutoffs is expected, on general grounds\cite{13}.
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