The fundamental form of almost-quaternionic Hermitian manifolds

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Abstract: We prove that if the fundamental 4-form $\Omega$ of an almost-quaternionic Hermitian manifold $(M,Q,g)$ of dimension $4n \geq 8$ satisfies the conformal-Killing equation, then $(M,Q,g)$ is quaternionic-Kähler.

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1 Introduction

Conformal-Killing (respectively, Killing) 1-forms are dual to conformal-Killing (respectively, Killing) vector fields. More generally, a $p$-form $\psi$ ($p \geq 1$) on a Riemannian manifold $(M^n, g)$ is conformal-Killing, if it satisfies the conformal-Killing equation

$$\nabla_X \psi = \frac{1}{p+1} i_X d\psi - \frac{1}{m-p+1} X \wedge \delta \psi, \quad \forall X \in TM,$$

(1)
where $\nabla$ is the Levi-Civita connection and (like everywhere in this note) we identify tangent vectors with 1-forms by means of the Riemannian duality. Co-closed conformal-Killing forms are called Killing. Note that $\psi$ is Killing if and only if its covariant derivative is totally skew, or, equivalently, $(\nabla_X \psi)(X, \cdot) = 0$ for any vector field $X$.

Conformal-Killing forms exist on spaces of constant curvature, on Sasaki manifolds \cite{6} and on some classes of Kähler manifolds, like Bochner-flat Kähler manifolds and conformally-Einstein Kähler manifolds \cite{1, 4}. On compact quaternionic-Kähler manifolds of dimension at least eight, there are no non-parallel conformal-Killing 2-forms, unless the quaternionic-Kähler manifold is isomorphic to the standard quaternionic projective space, in which case the space of conformal-Killing 2-forms is naturally isomorphic to the space of Killing vector fields \cite{3}.

Conformal-Killing forms exist also on manifolds which admit twistor spinors \cite{6}. Recall that a twistor spinor on a Riemannian spin manifold $(M^m, g)$ is a section $\rho$ of the spinor bundle, which satisfies the equation $\nabla_X \rho = -\frac{1}{m} X \cdot D\rho$, where $X$ is any vector field, $D$ is the Dirac operator and "$\cdot\cdot\cdot$" denotes the Clifford multiplication. If $\rho_1$ and $\rho_2$ are twistor spinors, then the $p$-form

$$\omega_p(X_1, \cdots, X_p) = \langle (X_1 \wedge \cdots \wedge X_p) \cdot \rho_1, \rho_2 \rangle$$

is conformal-Killing (for any $p \geq 1$). For a survey on conformal-Killing forms, see for example \cite{6}.

The starting point of this note is a result proved in \cite{6}, which states that if the Kähler form of an almost-Hermitian manifold is conformal-Killing, then the almost-Hermitian manifold is nearly Kähler. Our main Theorem is an analogue of this result in quaternionic geometry and is stated as follows:

**Theorem 1.** Let $(M^{4n}, Q, g)$ be an almost-quaternionic Hermitian manifold, of dimension $4n \geq 8$. Suppose that the fundamental 4-form $\Omega$ of $(M, Q, g)$ is conformal-Killing. Then $(M, Q, g)$ is quaternionic-Kähler.

Theorem \cite{1} generalizes a result proved in \cite{8}, namely that in dimension at least eight, a nearly quaternionic-Kähler manifold (i.e. an almost-quaternionic Hermitian manifold for which the fundamental 4-form is a Killing form) is necessarily quaternionic-Kähler.

The paper is organized as follows: in Section \cite{2} we recall basic facts on quaternionic Hermitian geometry. Section \cite{3} is devoted to the proof of our main result, which is based on a representation theoretic argument. Similar arguments were already employed in \cite{7} and \cite{8}.
2 Quaternionic Hermitian geometry

Let $M$ be a manifold of dimension $4n \geq 8$ (in all our considerations the dimension of the manifold will be at least eight). An almost-quaternionic structure on $M$ is a rank-three vector sub-bundle $Q \subset \text{End}(TM)$, locally generated by three anti-commuting almost complex structures $\{J_1, J_2, J_3\}$ which satisfy $J_1 \circ J_2 = J_3$. Such a triple of almost complex structures is usually called a (local) admissible basis of $Q$. An almost-quaternionic Hermitian structure on $M$ consists of an almost-quaternionic structure $Q$ and a Riemannian metric $g$ compatible with $Q$, which means that

$$g(JX, JY) = g(X, Y), \quad \forall J \in Q, \quad J^2 = -\text{Id}, \quad \forall X, Y \in TM.$$ 

In the language of $G$-structures, an almost-quaternionic Hermitian structure on a $4n$-dimensional manifold is an $Sp(n)Sp(1)$-structure. Therefore, on an almost-quaternionic Hermitian manifold $(M^{4n}, g, Q)$ there are two locally defined complex vector bundles $E$ and $H$, of rank $2n$ and $2$ respectively, associated to the standard representations of $Sp(n)$ and $Sp(1)$ on $E = \mathbb{C}^{2n}$ and $\mathbb{H} = \mathbb{C}^2$. Let $\omega_E \in \Lambda^2(E^*)$ and $j_E : E \to E$ be the standard symplectic form and quaternionic structure of the bundle $E$, defined by the $Sp(n)$-invariant complex symplectic form and quaternionic structure of $E$. We shall often identify $E$ with $E^*$ by means of the map $e \to \omega_E(e, \cdot)$, so that $\omega_E$ will sometimes be considered as a bivector on $E$. For any $r \geq 2$ we shall denote by $\Lambda^r_0 E \subset \Lambda^r E$ the kernel of the natural contraction

$$\omega_E \bullet : \Lambda^r E \to \Lambda^{r-2} E$$

with the symplectic form $\omega_E$, defined by

$$\omega_E \bullet (e_1 \wedge \cdots \wedge e_r) = \sum_{i<j} (-1)^{i+j+1} \omega_E(e_i, e_j) e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge \hat{e}_j \wedge \cdots \wedge e_r$$

where the hat denotes that the term is omitted. By means of contraction and wedge product with $\omega_E$ we can decompose $\Lambda^r E$ as

$$\Lambda^r E = \Lambda^r_0 E \oplus \omega_E \wedge \Lambda^{r-2}_0 E \oplus \omega^2_E \wedge \Lambda^{r-4}_0 E \oplus \cdots$$

(3)

The map $j_E$ is complex anti-linear and

$$j_E^2 = -\text{Id}, \quad \omega_E(j_Eu, j_Ev) = \overline{\omega_E(u, v)}, \quad \omega_E(e, j_Ee) > 0,$$

for any $u, v \in E$ and $e \in E \setminus \{0\}$. To simplify notations, for a vector $e \in E$ we shall often denote $\tilde{e} := j_E(e)$ its image through the quaternionic structure of $E$. Similar conventions will be used for the standard symplectic form...
\[ \omega_H \in \Lambda^2(H^*) \] and quaternionic structure \( j_H : H \to H \) of the bundle \( H \).

The bundles \( E \) and \( H \) play the role of spin bundles from conformal geometry. In particular,

\[ T_{\mathbb{C}}M = E \otimes_{\mathbb{C}} H \quad (4) \]

and the complex bilinear extension of the Riemannian metric \( g \) to \( T_{\mathbb{C}}M \) is the tensor product \( \omega_E \otimes \omega_H \). Decomposition \([\text{H}]\) induces decompositions of the form bundles in any degree. In particular, the bundles of 2 and 3-forms decompose as (see [5])

\[
\Lambda^2(T_{\mathbb{C}}M) = S^2H \oplus S^2E \oplus S^2H \Lambda_0^2E \quad (5)
\]

\[
\Lambda^3(T_{\mathbb{C}}M) = H(E \oplus K) \oplus S^3H(\Lambda_0^3E \oplus E). \quad (6)
\]

(In (5) and (6), and often in this note, we omit the tensor product signs). In (5) \( S^2H \) and \( S^2E \) are complexifications of the bundle \( Q \) and, respectively, of the bundle of \( Q \)-Hermitian 2-forms, i.e. 2-forms \( \psi \in \Lambda^2(T^*M) \) which satisfy

\[
\psi(JX, JY) = \psi(X, Y), \quad \forall J \in Q, \quad J^2 = -\text{Id}, \quad \forall X, Y \in TM.
\]

In (6) \( K \) denotes the vector bundle associated to the \( Sp(n) \)-module \( K \), which arises into the irreducible decomposition

\[
E \otimes \Lambda_0^2E \cong \Lambda_0^2E \oplus \Lambda_0^2E \oplus K \quad (7)
\]

under the action of \( Sp(n) \). A vector from \( E \otimes \Lambda_0^2E \) has non-trivial component on \( K \) if and only if it is not totally skew.

**Notations 2.** We shall identify bundles with their complexification, without additional explanations. For example, in (5) \( S^2H \Lambda_0^2E \) is a complex sub-bundle of \( \Lambda^2(T_{\mathbb{C}}M) \). We shall use the same notation for its real part, which is a sub-bundle of \( \Lambda^2(TM) \).

An almost-quaternionic Hermitian manifold \((M, g, Q)\) has a canonical 4-form, defined, in terms of an arbitrary admissible basis \( \{J_1, J_2, J_3\} \) of \( Q \), by

\[
\Omega = \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3,
\]

where \( \omega_i := g(J_i \cdot, \cdot) \) are the Kähler forms corresponding to \((g, J_i)\). As proved in [7] and [8], the covariant derivative \( \nabla \Omega \) with respect to the Levi-Civita connection \( \nabla \) of \( g \) is a section of \( T^*M \otimes (S^2H \Lambda_0^2E) \), where \( S^2H \Lambda_0^2E \) is embedded into \( \Lambda^2(T^*M) \) (identified with \( \Lambda^2(TM) \) using the Riemannian metric), in the following way. Note first that \( \Lambda^2(S^2H) \) is canonically isomorphic to \( S^2H \) (this is because \( S^2H \) is the complexification of \( Q \), which has a natural metric
and orientation, for which any admissible basis \( \{J_1, J_2, J_3\} \) is orthonormal and positively oriented). The map
\[
S^2 H \Lambda_0^3 E \cong \Lambda^2(S^2 H) \Lambda_0^3 E \to \Lambda^4_C(TM)
\]
defined by
\[
(s_1 \wedge s_2)\beta \to s_1\beta \wedge s_2\omega_E - s_2\beta \wedge s_1\omega_E, \quad \forall s_1, s_2 \in S^2 H, \quad \forall \beta \in \Lambda_0^2 E
\]
is the promised embedding of \( S^2 H \Lambda_0^3 E \) into \( \Lambda^4(TM) \).

An almost-quaternionic Hermitian manifold \((M, Q, g)\) is quaternion-Kähler if the Levi-Civita connection \( \nabla \) of \( g \) preserves the bundle \( Q \), or, equivalently, the fundamental 4-form \( \Omega \) is parallel with respect to \( \nabla \). In fact, as already mentioned in the Introduction, according to Theorem 1.2 of [8] the weaker condition \( (\nabla_X \Omega)(X, \cdot) = 0 \), for any vector field \( X \), implies that \((M, Q, g)\) is quaternionic-Kähler.

3 Proof of the main result

In this Section we prove our main result. Let \((M, Q, g)\) be an almost-quaternionic Hermitian manifold, whose fundamental 4-form \( \Omega \) is conformal-Killing. In order to prove that \( \Omega \) is parallel with respect to the Levi-Civita connection \( \nabla \), it is enough to show that it is co-closed (being conformal-Killing, \( \Omega \) is co-closed if and only if it is Killing, if and only if it is parallel, by Theorem 1.2 of [8] already mentioned before). Recall now that \( \nabla \Omega \) is a section of \( T^* M \otimes (S^2 H \Lambda_0^3 E) \), which decomposes into irreducible sub-bundles as
\[
T^*_C M \otimes (S^2 H \Lambda_0^3 E) = HE \oplus H \Lambda_0^3 E \oplus HK \oplus (S^3 H)E \oplus S^3 H \Lambda_0^3 E \oplus (S^3 H)K.
\]
Decomposition (10) follows from (7), together with the irreducible decomposition
\[
\mathbb{H} \otimes S^2 \mathbb{H} \cong S^3 \mathbb{H} \oplus \mathbb{H}
\]
of \( \mathbb{H} \otimes S^2 \mathbb{H} \) under \( Sp(1) \). While \( H \Lambda_0^3 E \) and \( (S^3 H)K \) are irreducible sub-bundles of \( T^*_C M \otimes (S^2 H \Lambda_0^3 E) \), see (10), they are not irreducible sub-bundles of \( \Lambda^3(T_C M) \), see (6). These observations readily imply that if \( \nabla \Omega \) is a section of \( H \Lambda_0^3 E \oplus (S^3 H)K \), then \( \Omega \) is co-closed: just write \( \delta \Omega = - \sum_i (\nabla_{E_i} \Omega)(E_i, \cdot) \), where \( \{E_i\} \) is a local orthonormal frame of \( TM \), and use the fact that an invariant linear map between non-isomorphic irreducible representations is identically zero. (Actually, by Theorem 2.3 of [8], also the converse is true: if \( \delta \Omega = 0 \) then \( \nabla \Omega \) is a section of \( H \Lambda_0^3 E \oplus (S^3 H)K \).
Therefore, we aim to show that $\nabla \Omega$ is a section of $H\Lambda^3_0 E \oplus (S^3 H)K$. For this, we define the algebraic conformal-Killing operator

$$\mathcal{T} : T^* M \otimes \Lambda^4(TM) \to T^* M \otimes \Lambda^4(TM),$$

by

$$\mathcal{T}(\gamma \otimes \alpha)(X) = \frac{4}{5} \gamma(X)\alpha + \frac{1}{5} \gamma \wedge i_X \alpha - \frac{1}{4n-3} X \wedge i_\gamma \alpha$$  \quad \text{(11)}$$

where $\gamma \in T^* M$ (is identified with a vector using the Riemannian metric), $\alpha \in \Lambda^4(TM)$ and $X \in TM$. Note that, for any 4-form $\psi \in \Omega^4(M)$,

$$\mathcal{T}(\nabla \psi)(X) = \nabla_X \psi - \frac{1}{5} i_X d\psi + \frac{1}{4n-3} X \wedge \delta \psi, \quad \forall X \in TM.$$  \quad \text{(12)}$$

In particular, since $\Omega$ is conformal-Killing,

$$\mathcal{T}(\nabla \Omega) = 0.$$  \quad \text{(13)}$$

The operator $\mathcal{T}$ is $Sp(n)Sp(1)$-invariant and we extend it, by complex linearity, to $T^*_C M \otimes \Lambda^4(T_C M)$. Define

$$S := T^*_C M \otimes (S^2 H\Lambda^2_0 E) \oplus (H\Lambda^3_0 E \oplus (S^3 H)K).$$

From (10), the irreducible sub-bundles of $S$ are

$$HE, \quad HK, \quad (S^3 H)E, \quad S^3 H\Lambda^3_0 E.$$  \quad \text{(14)}$$

For any irreducible sub-bundle $W$ of $S$, we will determine an $Sp(n)Sp(1)$-invariant linear map

$$T_W : T^*_C M \otimes \Lambda^4(T_C M) \to W$$

which factors through $\mathcal{T}$ (i.e. $T_W = \text{pr}_W \circ \mathcal{T}$ is the composition of $\mathcal{T}$ with an $Sp(n)Sp(1)$-invariant linear map $\text{pr}_W$ from $T^*_C M \otimes \Lambda^4(T_C M)$ to $W$) such that the restriction of $T_W$ to $T^*_C M \otimes (S^2 H\Lambda^2_0 E)$ is non-zero. An easy argument which uses (13), Schur’s Lemma and the fact that irreducible sub-bundles of $T^*_C M \otimes (S^2 H\Lambda^2_0 E)$ are pairwise non-isomorphic, would then imply that $\nabla \Omega$ has trivial component on $W$ and therefore that $\nabla \Omega$ is a section of $H\Lambda^3_0 E \oplus (S^3 H)K$, as needed.

In order to define the maps $T_W$, we apply several suitable contractions to the algebraic conformal-Killing operator $\mathcal{T}$. We first define $T_{HE}$ and $T_{HK}$ as follows. For a section $\eta$ of $T^*_C M \otimes \Lambda^4(T_C M)$, define $\omega_E \bullet T(\eta)$, a 1-form with values in $(S^2 H)\Lambda^2(T_C M)$, by

$$\omega_E \bullet (T(\eta))(X) := \omega_E \bullet (T(\eta)(X)), \quad \forall X \in TM,$$  \quad \text{(15)}$$

where $\omega_E$ is a 1-form with values in $(S^2 H)\Lambda^2(T_C M)$, defined by

$$\omega_E(X) := \frac{1}{5} \nabla_X \psi - \frac{1}{5} i_X d\psi + \frac{1}{4n-3} X \wedge \delta \psi, \quad \forall X \in TM,$$  \quad \text{(16)}$$

and $\psi$ is a 4-form with values in $(S^2 H)\Lambda^2(T_C M)$, defined by

$$\psi(X) := \frac{1}{5} \nabla_X \psi - \frac{1}{5} i_X d\psi + \frac{1}{4n-3} X \wedge \delta \psi, \quad \forall X \in TM.$$  \quad \text{(17)}$$

For any irreducible sub-bundle $W$ of $S$, we will determine an $Sp(n)Sp(1)$-invariant linear map

$$T_W : T^*_C M \otimes \Lambda^4(T_C M) \to W$$

which factors through $\mathcal{T}$ (i.e. $T_W = \text{pr}_W \circ \mathcal{T}$ is the composition of $\mathcal{T}$ with an $Sp(n)Sp(1)$-invariant linear map $\text{pr}_W$ from $T^*_C M \otimes \Lambda^4(T_C M)$ to $W$) such that the restriction of $T_W$ to $T^*_C M \otimes (S^2 H\Lambda^2_0 E)$ is non-zero. An easy argument which uses (13), Schur’s Lemma and the fact that irreducible sub-bundles of $T^*_C M \otimes (S^2 H\Lambda^2_0 E)$ are pairwise non-isomorphic, would then imply that $\nabla \Omega$ has trivial component on $W$ and therefore that $\nabla \Omega$ is a section of $H\Lambda^3_0 E \oplus (S^3 H)K$, as needed.
where in (15) $T(\eta)(X)$ belongs to $\Lambda^4(T_C M)$ (is the value of the $\Lambda^4(T_C M)$-valued 1-form $T(\eta)$ on $X \in T_C M$) and

$$\omega_E \cdot : \Lambda^4(T_C M) \to (S^2 H) \Lambda^2(T_C M)$$

(16)
denotes the contraction with $\omega_E$, which on decomposable multi-vectors takes value

$$\beta = h_1 e_1 \wedge \cdots \wedge h_4 e_4 \in \Lambda^4(T_C M)$$

takes value

$$\omega_E(\beta) = \sum_{i<j} (-1)^{i+j+1} \omega_E(e_i, e_j)(h_i h_j + h_j h_i) h_1 e_1 \wedge \cdots \wedge \hat{h}_i e_i \wedge \cdots \wedge \hat{h}_j e_j \wedge \cdots \wedge h_4 e_4.$$

Next, we define $\omega_H \cdot \omega_E \cdot T(\eta)$, by contracting $\omega_E \cdot T(\eta)$, which is a section of $HE \otimes (S^2 H) \Lambda^2(T_C M)$, with $\omega_H$ in the first two $H$-variables. Therefore, $\omega_H \cdot \omega_E \cdot T(\eta)$ is a section of $EH \Lambda^2(T_C M)$. Considering $EH \Lambda^2(T_C M)$ naturally embedded into $EH(HHEE)$, we contract further $\omega_H \cdot \omega_E \cdot T(\eta)$ with $\omega_H$ again in the first two $H$-variables. The result is a section $\omega_H^2 \cdot \omega_E \cdot T(\eta)$ of $HHEE$. Applying suitable projections to $\omega_H^2 \cdot \omega_E \cdot T(\eta)$ we finally obtain $T_{HE}(\eta)$ and $T_{HK}(\eta)$, as follows.

The contraction of $\omega_H^2 \cdot \omega_E \cdot T(\eta)$ with $\omega_E$ in the first two $E$-variables defines

$$T_{HE}(\eta) := \omega_E \cdot \omega_H^2 \cdot \omega_E \cdot T(\eta).$$

(17)

Similarly, we can project $\omega_H^2 \cdot \omega_E \cdot T(\eta)$ to $H \otimes E \Lambda^2 E$ and then to $HK$, by means of the decomposition (7) (translated to vector bundles). The result of this projection is the value of $T_{HK}$ on $\eta$. More precisely,

$$T_{HK}(\eta) := \text{pr}_{HK} \left( \omega_H^2 \cdot \omega_E \cdot T(\eta) \right).$$

(18)

**Proposition 3.** The operators $T_{HE}$ and $T_{HK}$ defined by (17) and (18) are non-trivial on $\tau_{C M}^* \otimes (S^2 H \Lambda^2 E)$.

In order to prove Proposition 3 we will show that $T_{HE}$ and $T_{HK}$ take non-zero value on $\gamma_0 \alpha_0$, where

$$\gamma_0 := \hat{e}_1 h, \quad \alpha_0 := e_1 h \wedge e_2 h \wedge \hat{e}_i \hat{h} \wedge \hat{e}_i \hat{h} = e_1 h \wedge e_2 h \wedge e_i h \wedge \hat{e}_i h$$

(19)

was already considered in [8]. In (19) $\{e_1, \cdots, e_{2n}\}$ is a unitary basis of (local) sections of $E$, with respect to the (positive definite) Hermitian metric $g_E := \omega_E(\cdot, j_E \cdot)$, chosen such that $e_{n+j} = \hat{e}_j$ for any $1 \leq j \leq n$, and $\{h, \hat{h}\}$ is a unitary basis of (local) sections of $H$, with respect to $g_H := \omega_H(\cdot, j_H \cdot)$. In
order to simplify notations, in (19) and below we omit the summation sign over $1 \leq i \leq 2n$. The symplectic forms of $E$ and $H$ can be written as

$$
\omega_E = \frac{1}{2} \hat{e}_i \wedge \hat{e}_i \in \Lambda^2 E, \quad \omega_H = h \wedge 
hat{h} \in \Lambda^2 H.
$$

(20)

From (9) and (20), $\alpha_0$ is a section of the sub-bundle $S^2H\Lambda^3_0E$ of $\Lambda^4(T^*_C M)$ and $\gamma_0\alpha_0$ is a section of $T^*_C M \otimes (S^2H\Lambda^3_0E)$.

We divide the proof of Proposition 3 into the following two Lemmas.

**Lemma 4.** The section $\text{pr}_{H\Lambda^3_0E}(\omega_H^2 \bullet \omega_E \bullet T(\gamma_0\alpha_0))$ is not totally skew in the $E$-variables. In particular, $T_{HK}(\gamma_0\alpha_0) \neq 0$.

**Proof.** A straightforward computation shows that

$$
i_{\gamma_0}\alpha_0 = e_i h \wedge \hat{e}_i h \wedge e_2 \hat{h} - 2e_1 h \wedge e_2 h \wedge \hat{e}_1 \hat{h}.
$$

Therefore, using (11), we can write

$$
T(\gamma_0\alpha_0) = \frac{4}{5} \gamma_0\alpha_0 + \frac{1}{5} \gamma_0 \wedge \alpha_0(\cdot) - \frac{1}{4n - 3}(F - 2G),
$$

(21)

where $\gamma_0 \wedge \alpha_0(\cdot)$ is a 1-form with values in $\Lambda^4(T^*_C M)$, whose natural contraction with a vector $X \in T^*_C M$ is $\gamma_0 \wedge i_X \alpha_0$. Similarly, $F$ and $G$ are defined by

$$
F(X) := X \wedge e_i h \wedge \hat{e}_i h \wedge e_2 \hat{h}
$$

and

$$
G(X) := X \wedge e_1 h \wedge e_2 h \wedge \hat{e}_1 \hat{h}.
$$

Now, it is straightforward to check that

$$
\omega_H^2 \bullet \omega_E \bullet (\gamma_0\alpha_0) = -4\eta h(\hat{e}_1 e_1 e_2 - \hat{e}_1 e_2 e_1)
$$

$$
\omega_H^2 \bullet \omega_E \bullet (\gamma_0 \wedge \alpha_0(\cdot)) = 2h(-e_i \hat{e}_i e_1 e_2 - \hat{e}_i e_1 e_2 - \hat{e}_2 e_1 e_2 + e_i e_2 e_1)
$$

$$
+ h(-e_2 e_i \hat{e}_i + e_2 \hat{e}_i e_1 + e_i e_2 \hat{e}_1 - \hat{e}_i e_2 e_1)
$$

$$
+ (4n + 2)h(e_2 \hat{e}_1 e_1 - e_1 \hat{e}_1 e_2)
$$

$$
+ 4h(e_1 e_2 \hat{e}_1 - e_2 e_1 \hat{e}_1)
$$

and also

$$
\omega_H^2 \bullet \omega_E \bullet F = -(4n - 4)he_i e_2 \hat{e}_i + 3h(e_2 e_i \hat{e}_i - e_2 e_i \hat{e}_i)
$$

$$
\omega_H^2 \bullet \omega_E \bullet G = 3h(e_1 \hat{e}_1 e_2 - e_2 \hat{e}_1 e_1 - \hat{e}_1 e_2 e_1 + \hat{e}_1 e_2 e_1) - he_i e_2 \hat{e}_i.
$$
These relations combined with \((21)\) readily imply that
\[
\omega^2_H \cdot \omega_E \cdot T(\gamma_0 \alpha_0) = \lambda_1 h \tilde{e}_1(e_1 \wedge e_2) + \lambda_2 h(e_2 \tilde{e}_1e_1 - e_1 \tilde{e}_1e_2) + \lambda_3 h e_2 \tilde{e}_i + \lambda_4 h e_i \tilde{e}_2 e_1 + \frac{h}{5} (4(e_1 \wedge e_2) \tilde{e}_1 + 2(e_i \wedge e_i) e_2 - \tilde{e}_1 e_2 e_i),
\]
with constants
\[
\lambda_1 = \frac{8(-8n^2 + 7n + 3)}{5(4n - 3)}, \quad \lambda_2 = \frac{4(4n^2 - n - 9)}{5(4n - 3)}, \quad \lambda_3 = -\frac{4(n + 3)}{5(4n - 3)}
\]
and
\[
\lambda_4 = \frac{24n - 33}{5(4n - 3)}.
\]
Projecting the expression for \(\omega^2_H \cdot \omega_E \cdot T(\gamma_0 \alpha_0)\) obtained above onto \(HE\Lambda^2_0 E\) we get
\[
\text{pr}_{HE\Lambda^2_0 E} \left( \omega^2_H \cdot \omega_E \cdot T(\gamma_0 \alpha_0) \right) = 2\lambda_1 h \tilde{e}_1(e_1 \wedge e_2) + \left( \lambda_2 + \frac{4}{5} \right) h e_2(\tilde{e}_1 \wedge e_1) - \left( \lambda_2 + \frac{4}{5} \right) h e_1(\tilde{e}_1 \wedge e_2) + \left( \lambda_4 + \frac{2}{5} \right) h e_i(\tilde{e}_1 \wedge e_2) + \frac{3}{5} h \tilde{e}_i(e_1 \wedge e_2) + \frac{1}{2n} \left( \lambda_2 - \lambda_4 - \frac{1}{5} \right) h e_2(e_i \wedge \tilde{e}_i),
\]
which is not totally skew in the \(E\)-variables. Our claim follows.

\begin{lemma}
The value of \(T_{HE}\) on \(\gamma_0 \alpha_0\) is
\[
T_{HE}(\gamma_0 \alpha_0) = \frac{8n(2n + 1)}{5(4n - 3)} h e_2.
\]
\end{lemma}

In particular, \(T_{HE}(\gamma_0 \alpha_0)\) is non-zero.

\begin{proof}
The claim follows from a straightforward calculation, using the expression of \(\omega^2_H \cdot \omega_E \cdot T(\gamma_0 \alpha_0)\) determined in the proof of Lemma 4 and the definition of the operator \(T_{HE}\).
\end{proof}

Lemma 4 and Lemma 5 conclude the proof of Proposition 3.

We now define the maps \(T_{(S^3H)E}\) and \(T_{S^3H\Lambda^2_0 E}\). For a section \(\eta\) of \(T^* C M \otimes \Lambda^4(T C M)\), \(T(\eta)\) is a section of \(EH \otimes \Lambda^4(T C M)\). We consider \(\omega_H \cdot T(\eta)\), the contraction of \(T(\eta)\) with \(\omega_H\) in the first two \(H\)-variables, which is a section.
of $EE \otimes \Lambda^3(T_C M)$. Its total symmetrization $\text{sym}^H(\omega_H \bullet \mathcal{T}(\eta))$ in the $H$-variables is a section of $EE(S^3 H)\Lambda^3 E$. Leaving the first two $E$-variables of $\text{sym}^H(\omega_H \bullet \mathcal{T}(\eta))$ unchanged and contracting $\text{sym}^H(\omega_H \bullet \mathcal{T}(\eta))$ with $\omega_E$ on $\Lambda^3 E$, as in (2), we get a section $\omega_E \bullet \text{sym}^H(\omega_H \bullet \mathcal{T}(\eta))$ of $EE(S^3 H)E$.

To define $\mathcal{T}_{(S^3 H)\Lambda^3 E}(\eta)$ and $\mathcal{T}_{(S^3 H)E}(\eta)$ we project $\omega_E \bullet \text{sym}^H(\omega_H \bullet \mathcal{T}(\eta))$ on $(S^3 H)\Lambda^3 E$ and then we project the result on $(S^3 H)\Lambda^3 E$ and $(S^3 H)E$ respectively, using the decomposition (3), with $r = 3$. Therefore,

$$\mathcal{T}_{(S^3 H)\Lambda^3 E}(\eta) := \text{pr}_{(S^3 H)\Lambda^3 E}(\omega_E \bullet \text{sym}^H(\omega_H \bullet \mathcal{T}(\eta))). \tag{23}$$

Similarly,

$$\mathcal{T}_{(S^3 H)E}(\eta) := \omega_E \bullet \text{pr}_{(S^3 H)\Lambda^3 E}(\omega_E \bullet \text{sym}^H(\omega_H \bullet \mathcal{T}(\eta))) \tag{24}$$

is the contraction of $\text{pr}_{(S^3 H)\Lambda^3 E}(\omega_E \bullet \text{sym}^H(\omega_H \bullet \mathcal{T}(\eta)))$ with the symplectic form $\omega_E$.

**Proposition 6.** The operators $\mathcal{T}_{(S^3 H)\Lambda^3 E}$ and $\mathcal{T}_{(S^3 H)E}$ defined by (23) and (24) are non-trivial on $T^*_\mathcal{C} M \otimes (S^2 H)\Lambda^3 E$.

Like in the proof of Proposition 3 we will show that $\mathcal{T}_{(S^3 H)\Lambda^3 E}(\gamma_0 \alpha_0)$ and $\mathcal{T}_{(S^3 H)E}(\gamma_0 \alpha_0)$ are non-zero. This is a consequence of the next Lemma.

**Lemma 7.** The following fact holds:

$$\text{pr}_{S^3 H\Lambda^3 E}(\omega_E \bullet \text{sym}^H(\omega_H \bullet \mathcal{T}(\gamma_0 \alpha_0))) = -\frac{6(n-1)}{4n-3} \text{sym}^H(\mathcal{Hh})(e_i \wedge \tilde{e}_i \wedge e_2) - \frac{4(4n^2 - 3n + 3)}{4n-3} \text{sym}^H(\mathcal{Hh})(e_1 \wedge e_2 \wedge \tilde{e}_1).$$

**Proof.** The proof goes as in Lemma 4. Applying definitions, we get:

$$\omega_E \bullet \text{sym}^H(\omega_H \bullet (\gamma_0 \alpha_0)) = 2n \text{sym}^H(\mathcal{Hh})(\tilde{e}_1 e_2 e_1 - \tilde{e}_1 e_1 e_2)$$

$$\omega_E \bullet \text{sym}^H(\omega_H \bullet (\gamma_0 \land \alpha_0(\cdot))) = (2n-4) \text{sym}^H(\mathcal{Hh})(e_1 \tilde{e}_1 e_2 - e_2 \tilde{e}_1 e_1)$$

$$+ 4 \text{sym}^H(\mathcal{Hh})(\tilde{e}_1 e_2 e_1 - \tilde{e}_1 e_1 e_2)$$

$$- 2 \text{sym}^H(\mathcal{Hh})(\tilde{e}_1 e_2 e_1 + e_2 \tilde{e}_1 e_1)$$

$$+ 2 \text{sym}^H(\mathcal{Hh})(e_2 e_1 \tilde{e}_1 + e_1 e_2 \tilde{e}_1)$$

$$\omega_E \bullet \text{sym}^H(\omega_H \bullet F) = \text{sym}^H(\mathcal{Hh})((4n-5)e_1 e_2 e_1 - (2n-3)e_1 e_2 e_1)$$

$$+ \text{sym}^H(\mathcal{Hh})(\tilde{e}_1 e_2 e_1 - 2e_2 e_1 \tilde{e}_1 + e_2 \tilde{e}_1 e_1 - \tilde{e}_2 \tilde{e}_1 e_1)$$

$$\omega_E \bullet \text{sym}^H(\omega_H \bullet G) = \text{sym}^H(\mathcal{Hh})(-2e_1 \tilde{e}_2 e_2 + e_2 e_1 \tilde{e}_1 - \tilde{e}_1 e_2 e_1 - e_1 e_2 \tilde{e}_1)$$

$$+ \text{sym}^H(\mathcal{Hh})(\tilde{e}_1 e_2 e_1 + e_2 \tilde{e}_1 e_1 + e_1 e_2 e_2 - e_2 \tilde{e}_1 e_1).$$
Combining (21) with these relations we get
\[
\omega_E \cdot \text{sym}^H (\omega_H \cdot T(\gamma_0 \alpha_0)) = \text{sym}^H (hh\tilde{h})(\beta_1 \tilde{e}_1 (e_1 \wedge e_2) + \beta_2 \tilde{e}_i e_2 e_i) \\
+ \text{sym}^H (hh\tilde{h})(\beta_3 e_2 (\tilde{e}_i \wedge e_i) + \beta_4 e_1 e_2 \tilde{e}_i) \\
+ \beta_5 \text{sym}^H (hh\tilde{h})(e_2 \tilde{e}_1 e_1 - e_1 \tilde{e}_1 e_2) \\
- \frac{\text{sym}^H (hh\tilde{h})}{4n-3}((4n-1)e_1 \tilde{e}_i e_2 + \tilde{e}_i e_1 e_2) \\
- \frac{2\text{sym}^H (hh\tilde{h})}{4n-3}(e_1 \wedge e_2)\tilde{e}_1,
\]
where the constants $\beta_i$ are defined by
\[
\beta_1 = -\frac{2(16n^2 - 4n - 1)}{5(4n-3)}, \quad \beta_2 = -\frac{8n - 11}{5(4n-3)}, \quad \beta_3 = -\frac{8n - 1}{5(4n-3)}, \quad \beta_4 = \frac{18n - 11}{5(4n-3)}
\]
and
\[
\beta_5 = -\frac{2(4n^2 - 11n + 11)}{5(4n-3)}.
\]
Skew-symmetrizing $\omega_E \cdot \text{sym}^H (\omega_H \cdot (\gamma_0 \alpha_0))$ in the $E$-variables we obtain our claim.

\[\square\]

**Corollary 8.** Both $T_{(S^3H)\Lambda^3E}(\gamma_0 \alpha_0)$ and $T_{(S^3H)E}(\gamma_0 \alpha_0)$ are non-zero.

**Proof.** Since $\text{pr}_{S^3H \Lambda^3E} (\omega_E \cdot \text{sym}^H (\omega_H \cdot T(\gamma_0 \alpha_0)))$ is not a multiple of $\omega_E$, $T_{(S^3H)\Lambda^3E}(\gamma_0 \alpha_0)$ is non-zero. On the other hand, using Lemma 7, it is easy to check that
\[
T_{(S^3H)E}(\gamma_0 \alpha_0) = \frac{4n(n+3)}{4n-3} \text{sym}^H (hh\tilde{h}) e_2.
\]

\[\square\]

Corollary 8 implies Proposition 6. Proposition 3 and Proposition 6 conclude the proof of our main result.

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