Regularization of the restricted \((n + 1)\)-body problem on curved spaces

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Abstract We consider \((n + 1)\) bodies moving under their mutual gravitational attraction in spaces with constant Gaussian curvature \(\kappa\). In this system, \(n\) primary bodies with equal masses form a relative equilibrium solution with a regular polygon configuration; the remaining body of negligible mass does not affect the motion of the others. We show that the singularity due to binary collision between the negligible mass and the primaries can be regularized locally and globally through suitable changes of coordinates (Levi-Civita and Birkhoff type transformations).

Keywords Curved \(n\)-body problem · Relative equilibria · Local and global regularization

1 Introduction

The classical restricted three-body problem (zero curvature) was first proposed by Euler in the 18th century. It concerns the planar motion of a massless particle moving under the influence of the Newtonian attraction of two positive masses which revolve around each other in a circular orbit with uniform velocity. Since then, many authors have studied this problem; we are particularly interested in the work of Levi-Civita (Szebehely 1967) and Birkhoff (Birkhoff 1915). One of the main obstacles in the analysis of these problems is the presence of singularities due to collision. By fixing the position of the primaries we avoid the collision between them, but the problem of collision of the massless particle with one or more of the primaries still persists.

The Levi-Civita technique (Szebehely 1967) is useful to regularize just one singularity, that is, the collision of the massless particle with just one of the primaries; we call it local regularization. There are several generalizations of this technique which are applied to the restricted three-body problem, see for instance Roman and Szücs-Csillik (2014), this kind of regularization and its generalizations are very helpful to analyze the dynamics of a satellite in an orbit close to another massive body, or in general in spatial missions to explore a single planet. However, sometimes it is necessary to have a global view of the solutions, for instance to study escapes of particles or a possible connection among the equilibrium points; then it is necessary to have a global regularization of all singularities due to collision. In this latter case we use the Birkhoff technique. Both techniques consist of a suitable change of coordinates followed by a time reparametrization. As one of the important applications of the classical Newtonian restricted \((n + 1)\)-body problem we mention the study of Saturn’s rings, where, as a first approximation, some authors consider each ring as a relative equilibrium with \(n\) equal masses forming a regular \(n\)-gon, with a big mass at the center (Saturn) (Maxwell 1890, 1983). Of course, in this case the relativistic effects are dropped because they are insignificant; however, in the last years some researchers have observed that the bright star Fomalhaut, in the constellation of Piscis Austrinus, shows a dust ring well defined around it, where relativistic effects cannot be negligible (Atacama Large Milimiter/sub-mm Array (ALMA)-reports, 2012); see also MacGregor et al. (2017) and the references therein. We believe that the analysis of this and other
similar problems could be analyzed in the framework of the restricted \( n + 1 \)-body problem on curved spaces.

The curved \( n \)-body problem has its roots in the ideas about non-Euclidean geometries proposed by Lovachevski and Bolyai in the 19th century (Bolyai and Bolyai 1913; Lovachevski 1949); the idea to generalize the classical Kepler problem to spaces of constant non-zero curvature has been tackled by many authors since then. The Russian school has played an important role in the development of these ideas; see for instance Borisov et al. (2004, 2016) and Diacu (2012) and the references therein for more details of the history of this fascinating problem. In this paper we will follow the framework proposed by Diacu et al. (2012a,b).

It is well known that, through a suitable change of coordinates followed by a rescaling of time, we can focus our analysis on the cases of curvature \( \kappa = 1 \) and \( \kappa = -1 \) (see Diacu 2012 for details). In this way, we consider the unit sphere embedded in \( \mathbb{R}^3 \) with the Euclidean metric, which we denote by \( S^2 \), as a model for positive curvature. For negative curvature we consider the upper part of the hyperboloid \( x^2 + y^2 - z^2 = -1 \) embedded in \( \mathbb{R}^3 \) with the Lorentzian metric \((\mathbb{R}^2, 1)\), which can be seen as a sphere of imaginary radius \( i \) or pseudo-sphere, denoted by \( \mathbb{H}^2 \). We will study both cases in an unified way.

In this paper we generalize the Newtonian restricted \((n+1)\)-body problem to \( S^2 \) and \( \mathbb{H}^2 \). The generalization is to consider \( n \) particles with equal masses moving on a fixed plane parallel to the \( xy \)-plane forming a regular polygon configuration. We observe that when we analyze the local regularization, it does not matter if the masses of the primaries are equal or not, but when we study the global regularization, it is only possible if all masses are equal. In order to have an unified analysis, we have assumed that in both cases all masses are equal, but this is only to facilitate notations. For the global regularization, only in this case we were able to obtain analytical results. To re-force this argument, we know that in the restricted three-body problem on curved spaces, if the two primaries have different masses, then when they generate relative equilibria, its motion is on different planes, and the dynamics of the massless particle is really complicated; the difficulty of the problem increases considerably and, in this case, we only can give numerical results; see for instance Andrade et al. (2017, 2018), Martínez and Simó (2017) and the references therein.

The goal of this paper is to obtain the regularization due to collisions between the primaries and the remaining body of negligible mass. It results that the regularization transformations are similar to the classical Newtonian problem, but here we are using the different metrics coming from the respective non-Euclidean geometries. In the last section of this paper, we perform a local regularization showing a whole family of transformations, which we think might be useful for numerical aspects. The reason to introduce this family is that for any special problem (or initial conditions), just one transformation of this family gives more information as regards the respective trajectory (see Roman and Szücs-Csillik 2012 for more details). We believe that these problems may have important applications in the dynamics of the components of an atom and in new astronomical discoveries, usually studied only through semiclassical physics; see for instance Richter et al. (1993). Nevertheless we let the analysis of the possible applications for a forthcoming paper and we will concentrate here just on the theoretical aspects of the problem.

After the introduction the paper is organized as follows. In Sect. 2, we show the existence of polygonal relative equilibria given by the primaries \((n \) bodies with equal masses). In Sect. 3, we set the problem and present the equations in a convenient form (a suitable rotating frame). In Sect. 4, we present the main results of the work; in this way it is necessary to use the stereographic projection to avoid the constraints. This allows us to work with complex variables. Then we obtain the local and global regularization of the binary collisions. In Sect. 5, we show a whole family of transformations that allow us to regularize locally the collisions. Finally, in Sect. 6 we present some simulations.

## 2 Relative equilibria

In this section we introduce the equations of motion for the curved \( n \)-body problem, we define the concept of relative equilibrium solution and we will show that, for equal masses, the regular \( n \)-gon moving on a fixed plane around the \( z \)-axis forms, under certain conditions, a relative equilibrium solution.

In the curved \( n \)-body problem, the motion of the particles is due to the action of the cotangent potential

\[
U(q) = \sum_{i < j} m_i m_j \cot n(d(q_i, q_j)),
\]

where \( d \) is the geodesic distance on the corresponding space \( S^2 \) or \( \mathbb{H}^2 \), and \( \cot n \) means the usual cotangent function or the hyperbolic cotangent function, respectively.

As usual the kinetic energy is defined as

\[
T = \frac{1}{2} \sum_i m_i \dot{q}_i \circ \dot{q}_i,
\]

where the symbol \( \circ \) represents the usual inner product if we consider \( S^2 \), or the Lorentzian inner product if \( \mathbb{H}^2 \) is considered (in this case for \( a, b \in \mathbb{R}^3 \) we have \( a \circ b = a_x b_x + a_y b_y - a_z b_z \)).
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From the Euler–Lagrange equations, the equations of motion take the form

\[
\ddot{q}_i = \sum_{j=1,j \neq i}^n \frac{m_j(q_j - \sigma(q_i \circ q_j)q_j)}{\left[\sigma - \sigma(q_i \circ q_j)j^2\right]^{3/2}} - \sigma (\ddot{q}_i \circ \dot{q}_i)q_i,
\]

where \(q_i \circ q_j = 1\), \(i = 1, \ldots, n\),

\[
(2)
\]

where \(\sigma\) stands for 1 if we analyze \(\mathbb{S}^2\) or \(-1\) for \(\mathbb{H}^2\).

### 2.1 Relative equilibria on \(\mathbb{S}^2\)

Consider the group of isometries \(\text{SO}(3)\) acting on \(\mathbb{R}^3\). It is well known that it consists of all uniform rotations. The relative equilibria are invariant solutions of the equations of motion under the action of the group \(\text{SO}(3)\). Now, since the principal theorem axis states that any \(A \in \text{SO}(3)\) can be written, in some orthonormal basis, as a rotation about a fixed axis, we take without loss of generality the \(z\)-axis as the rotation axis. Hence we can characterize the relative equilibria as follows.

**Proposition 1** A solution \(q_i\), \(i = 1, \ldots, n\), of the equations of motion on \(\mathbb{S}^2\) is a relative equilibrium if and only if \(q_i = (x_i, y_i, z_i)\), with \(x_i = r_i \cos(\Omega t + \alpha_i)\), \(y_i = r_i \sin(\Omega t + \alpha_i)\), and \(z_i = \sqrt{1 - r_i^2}\), where \(\Omega\), \(\alpha_i\) and \(r_i\), \(i = 1, \ldots, n\), are constants.

**Proof** The result follows directly by straightforward computations; we omit the details here. \(\square\)

We observe that the characterization of relative equilibria on \(\mathbb{S}^2\) given in the above proposition is independent of the value of the masses; the only restriction is that if \(n\) masses form a relative equilibrium, then each mass must be rotating on a circle parallel to the equator. Only in the case of equal masses we can ensure that all masses are on the same circle. In Andrade et al. (2017), Martínez and Simó (2017), Pérez-Chavela and Reyes-Victoria (2012), the reader can find some kinds of relative equilibria with different masses. However, for the purposes of this article we focus our analysis on the case on \(n\) equal masses.

Using Proposition 1 we state and prove the following result.

**Theorem 2** For \(n\) equal masses on \(\mathbb{S}^2\) located on a regular polygon initial configuration centered at the origin of a fixed plane orthogonal to the \(z\)-axis with the \(z\)-coordinate \(\neq 0\), there exist a positive and a negative value for the angular velocity such that the solution is a relative equilibrium.

**Proof** Consider \(n\) particles of equal masses, \(m = 1\), with a regular polygon initial configuration.

The position for the \(i\)th body at a given time \(t\) is \(q_i(t) = (x_i(t), y_i(t), z(t))\) where

\[
x_i(t) = r \cos \left[ \Omega t + (i - 1) \frac{2\pi}{n} \right],
\]

\[
y_i(t) = r \sin \left[ \Omega t + (i - 1) \frac{2\pi}{n} \right], \quad z(t) = c,
\]

where \(c \neq 0\) is a constant \((c \in (-1, 0) \cup (0, 1))\).

We will show that there exist values of \(\Omega\) such that the above functions satisfy (2). By symmetry we can consider only the equations of motion for the \(x_i(t)\) coordinates. We have

\[
x_{i+k} = r \cos \left[ \Omega t + (i + k - 1) \frac{2\pi}{n} \right],
\]

\[
x_{i-k} = r \cos \left[ \Omega t + (i - k - 1) \frac{2\pi}{n} \right].
\]

(3)

Let be \(A = \Omega t + (i - 1) \frac{2\pi}{n}\). For \(n\) odd, we get

\[
\dot{x}_i = \sum_{j=1,j \neq i}^{n} \frac{x_j - (q_i \cdot q_j) x_i}{\left[1 - (q_i \cdot q_j)^2\right]^{3/2}} - (\dot{q}_i \cdot \dot{q}_i) x_i.
\]

(4)

For each \(i\), we enumerate the particles as \((- \frac{n+1}{2} + i + 1 \cdots, i - 1, i, i + 1, \ldots, \frac{n+1}{2} + i - 1\). Hence

\[
\dot{x}_i = \sum_{j=1}^{\frac{n+1}{2}+i-1} \frac{x_j - (q_i \cdot q_j) x_i}{\left[1 - (q_i \cdot q_j)^2\right]^{3/2}} - \frac{\frac{n+1}{2}+i+1}{\sum_{j=1}^{\frac{n+1}{2}+i+1} x_j - (q_i \cdot q_j) x_i - (\dot{q}_i \cdot \dot{q}_i) x_i}
\]

\[
= \sum_{j=1}^{\frac{n+1}{2}-1} \frac{x_{j+i} - (q_i \cdot q_{j+i}) x_i}{\left[1 - (q_i \cdot q_{j+i})^2\right]^{3/2}} + \frac{\frac{n+1}{2}}{\sum_{j=1}^{\frac{n+1}{2}+1} x_{j+i} - (q_i \cdot q_{j+i}) x_i - (\dot{q}_i \cdot \dot{q}_i) x_i}
\]

\[
= \sum_{j=1}^{\frac{n+1}{2}-1} \frac{x_{j+i} - (q_i \cdot q_{j+i}) x_i}{\left[1 - (q_i \cdot q_{j+i})^2\right]^{3/2}} + \frac{\frac{n+1}{2}}{\sum_{j=1}^{\frac{n+1}{2}+1} x_{j+i} - (q_i \cdot q_{j+i}) x_i - (\dot{q}_i \cdot \dot{q}_i) x_i}
\]

(5)
Notice that \( \dot{x}_i = -\Omega^2 x_i \), \( x_{i+j} = x_i \cos(2j\pi/n) \mp y_i \sin(2j\pi/n) \), \( q_i \cdot q_{i-j} = q_i \cdot q_{i+j} = r^2 \cos(2j\pi/n) + 1 - r^2 \)
and \( \dot{q}_i \cdot \dot{q}_i = r^2 \Omega^2 \). With these facts we obtain
\[
\Omega^2 = 2 \sum_{j=1}^{n+1-1} \frac{1 - \cos(2j\pi/n)}{[1 - (r^2 \cos(2j\pi/n) + 1 - r^2)^2]^{3/2}}. 
\] (6)

For \( n \) even we obtain
\[
\dot{x}_i = \sum_{j=1}^{n} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} - (\dot{q}_i \cdot \dot{q}_i)x_i. 
\] (7)

For each \( i \) we enumerate the particles as \((-\frac{n}{2} + i + 1 \cdots, i-1, i+1, \ldots, \frac{n}{2} + i - 1\)\). Hence
\[
\dot{x}_i = \sum_{j=1}^{i-1} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} + \sum_{j=1}^{i+1} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} \\
+ \sum_{j=1}^{\frac{n}{2} - i} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} + \sum_{j=1}^{\frac{n}{2} - i} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} \\
+ \sum_{j=1}^{\frac{n}{2} - i} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} + \sum_{j=1}^{\frac{n}{2} - i} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} \\
= \sum_{j=1}^{\frac{n}{2} - i} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} + \sum_{j=1}^{\frac{n}{2} - i} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} \\
+ \sum_{j=1}^{\frac{n}{2} - i} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2} + \sum_{j=1}^{\frac{n}{2} - i} \frac{x_j - (q_i \cdot q_j)x_i}{1 - (q_i \cdot q_j)^2}.
\] (8)

We have
\[
\Omega^2 = 2 \sum_{j=1}^{1} \frac{1 - \cos(2j\pi/n)}{[1 - (r^2 \cos(2j\pi/n) + 1 - r^2)^2]^{3/2}} \\
+ \frac{1}{4r^3(1 - r^2)^{3/2}}.
\] (9)

The right-hand-side of Eqs. (6) and (9) are positive, hence we conclude that there exist a positive and a negative value of \( \Omega \) that generate relative equilibria.

Remark 3 We observe that the value of the angular velocity which generates relative equilibria, Eqs. (6) and (9), depends strongly on the coordinate \( r \) and therefore on the height of the fixed parallel plane orthogonal to the \( z \)-axis where the primaries are rotating.

If the \( n \) particles with equal masses form a regular polygon and are located on the equator of \( S^2 \), then we have the following (Diacu et al. 2018): if \( n \) is odd, for any constant angular velocity \( \Omega \), then the positions and velocities given by Eq. (3) generate a solution of relative equilibrium.

### 2.2 Relative equilibria on \( \mathbb{H}^2 \)

In this case the group of isometries of \( \mathbb{R}^3 \) which leave \( \mathbb{H}^2 \) invariant is the group of orthogonal transformations with determinant \( \pm 1 \). This group is called the Lorentz group and it is denoted by \( \text{Lor}(\mathbb{R}^{2,1}, \circ) \). The principal axis theorem in this case states that every \( G \in \text{Lor}(\mathbb{R}^{2,1}, \circ) \) has one of the following canonical forms:

\[
A = P \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1},
B = P \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh s & \sinh s \\ 0 & \sinh s & \cosh s \end{pmatrix} P^{-1},
\]

or
\[
C = P \begin{pmatrix} 1 & -t & t \\ t & 1 - \frac{t^2}{2} & \frac{t^2}{2} \\ t & \frac{t^2}{2} & 1 + \frac{t^2}{2} \end{pmatrix} P^{-1},
\]

where \( \theta \in [0, 2\pi) \), \( s, t \in \mathbb{R} \), and \( P \in \text{Lor}(\mathbb{R}^{2,1}, \circ) \). The above transformations are, respectively, called elliptic, hyperbolic and parabolic.

In Diacu et al. (2012a) the authors proved the non-existence of parabolic relative equilibria. The solutions generated by elliptic or hyperbolic transformations are called elliptic or hyperbolic relative equilibria.

In this work we are interested in solutions where the \( n \) primaries form a relative equilibrium with a regular polygon configuration. Recently, in Pérez-Chavela and Sánchez-Cerritos (2018b), the authors proved the non-existence of polygonal hyperbolic relative equilibria, in particular the non-existence of Lagrange hyperbolic relative equilibria. Hence we will focus only on elliptic relative equilibria.

Analogously, as in \( S^2 \), we can have relative equilibria with different masses, but in order to ensure that the \( n \) masses form a regular polygon, it is necessary that all masses are equal (see for instance Pérez-Chavela and Sánchez-Cerritos 2018a and the references therein). So, assuming that all masses are equal we have the following result.

**Theorem 4** For \( n \) equal masses on \( \mathbb{H}^2 \) with a regular polygon initial configuration with the bodies at a fixed constant height \( z > 1 \), there exist a positive and a negative value of the angular velocity that generates an elliptic relative equilibrium.

**Proof** The proof is similar to the positive curvature case by noticing that \( q_i \cap q_{i-j} = q_i \cap q_{j-i} = r^2 \cos(\frac{2\pi j}{n}) - 1 - r^2 \).
3 The restricted curved \((n + 1)\)-body problem

In order to unify our analysis we introduce the notation \(M^2\) without distinction for \(S^2\) or \(H^2\). The restricted curved \((n + 1)\)-body problem refers to the motion of a system of \(n + 1\) particles moving under the action of the cotangent potential given by Eq. (1) on \(M^2\), where \(n\) particles of equal masses (called primaries), with positions \(q_i\), forming a regular polygon, are rotating on a circle parallel to the \(xy\) plane with velocities (6), (9), (10) or (11). The remaining body, located at position \(q\) has a negligible mass and its motion is given by

\[
\ddot{q} = \sum_{i=1}^{n} \frac{q_i - \sigma(q_i \odot q)q}{|\sigma - \sigma(q \odot q)|^2} - \sigma(q \odot q)\dot{q}.
\]

The equal masses of the primaries are taken as \(m_i = 1\). As in the classical case, we introduce rotating coordinates. Let \(q = RQ\) with \(Q = (\xi, \eta, \vartheta)^T\), where \(R\) is the rotation matrix

\[
R = \begin{pmatrix}
\cos \Omega & -\sin \Omega & 0 \\
\sin \Omega & \cos \Omega & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

After a straightforward computation the new equation of motion in the new variables is

\[
\ddot{Q} - 2\dot{Q} J \dot{Q} + \left[\sigma(\dot{\xi} - \Omega \eta)^2 + \sigma(\dot{\eta} + \Omega \xi)^2 + \dot{\vartheta}^2\right]Q = \nabla Q \left(\frac{\Omega^2}{2} (\xi^2 + \eta^2) + \sum_{i=1}^{n} \frac{Q_i \odot Q}{(\sigma - \sigma(Q_i \odot Q)^2)^{1/2}}\right).
\]

with

\[
J = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

In these coordinates the positions of the primaries take the form

\[
Q_i = \left[r \cos \left(\frac{2\pi}{n} (i - 1)\right), r \sin \left(\frac{2\pi}{n} (i - 1)\right), z\right].
\]

3.1 Stereographic projection

In our analysis we consider the stereographic projection from the point \((0, 0, -1)\) to \(\mathbb{R}^2\), \(\Pi : M^2 \to \mathbb{R}^2\). This function maps \(Q \mapsto (u, v)\), with

\[
u = \frac{\xi}{1 + \vartheta}, \quad v = \frac{\eta}{1 + \vartheta}.
\]

The inverse function \(\Pi^{-1}\) maps \((u, v) \mapsto Q\) where

\[
\xi = \frac{2u}{1 + \sigma(u^2 + v^2)}, \quad \eta = \frac{2v}{1 + \sigma(u^2 + v^2)},
\]

\[
\vartheta = \frac{1 - \sigma(u^2 + v^2)}{1 + \sigma(u^2 + v^2)}.
\]

From differential geometry, when \(M^2 = S^2\), it is well known that \(\Pi\) maps \(S^2\) onto the whole plane \(\mathbb{R}^2\), with the metric \(ds^2 = \frac{4}{1 - u^2 - v^2}\), the above plane with this metric is known by some authors as the curved plane.

In the case of \(M^2 = H^2\), this space is projected onto the open unitary disk with the metric \(ds^2 = \frac{4}{1 - u^2 - v^2}\); this space is the classical model of hyperbolic geometry called the Poincaré disk.

Under \(\Pi\), the primaries, originally on \(M^2\), now are located at \(w_i = \frac{1}{1 + r(z_i, h_i)}\), with \(k_i = r \cos \left(\frac{2\pi}{n} (i - 1)\right)\) and \(h_i = r \sin \left(\frac{2\pi}{n} (i - 1)\right)\).

The right part of (13) is known as the effective potential; it can be written as

\[
\frac{\Omega^2}{2} (\xi^2 + \eta^2) + \sum_{i=1}^{n} \frac{Q_i \odot Q}{(\sigma - \sigma(Q_i \odot Q)^2)^{1/2}} = \frac{2\Omega^2(u^2 + v^2)}{(1 + \sigma(u^2 + v^2))} + \sum_{i=1}^{n} \frac{2k_i u + 2h_i v + \sigma z(1 - \sigma(u^2 + v^2))}{(\sigma(1 + \sigma(u^2 + v^2)))^2 - \sigma(2k_i u + 2h_i v + \sigma z(1 - \sigma(u^2 + v^2)))^2}^{1/2} \psi(u, v) + U(u, v).
\]

4 Regularization

We first write the equation of motion (13) written in the above coordinates as a Hamiltonian system, with Hamilto-
nian function given by
\[ H(u, v, p_u, p_v) = \frac{(1 + \sigma(u^2 + v^2))^2}{8} (p_u^2 + p_v^2) + \Omega(vp_u - up_v) - U(u, v), \tag{15} \]
where \( U \) is the corresponding potential getting from the stereographic projection of \( \mathbb{S}^2 \) or \( \mathbb{H}^2 \) onto \( \mathbb{R}^2 \). If no confusion arises, we will keep denoting the positions of the primaries on \( \mathbb{C} \) as \( w_i \).

In order to analyze the regularization of the binary collisions between the negligible mass with the primaries, we consider complex variables through the following change of coordinates:
\[ z = u + iv, \quad Z = p_u + ip_v. \]

Then (15) takes the form
\[ H = \frac{(1 + \sigma |z|^2)^2}{4} |Z|^2 + 2\Omega \text{Im}(z\overline{Z}) - 2U(z, \overline{Z}), \tag{16} \]
with
\[ U(z, \overline{Z}) = \sum_{j=1}^{n} \frac{k_j(z + \overline{Z}) - ih_j(z - \overline{Z}) + \sigma z(1 - \sigma |z|^2)}{r|z - w_j||z - \overline{w}_j|}, \]
where \( \overline{w}_j = \frac{1}{1 - z}(k_j, h_i) \).

If we consider \( \mathbb{S}^2 \), then \( \Pi^{-1}(\overline{w}_i) \) corresponds to the antipodal point of the primary \( Q_i \), but if we consider \( \mathbb{H}^2 \), then \( \overline{w}_i \) is a point such that \( \Pi^{-1}(\overline{w}_i) \) does not belong to \( \mathbb{H}^2 \).

### 4.1 Local regularization

In this section we state the first main result of this paper.

**Theorem 6** The transformation \( z = g(w) = w\overline{w} + w_k \) (i = 1, \ldots, n), with the time transformation \( \frac{dt}{ds} = |w|^2 \), regularizes the singularity of (16) due to collision between the negligible mass and the kth primary body.

**Proof** First, we consider the space transformation \( z = g(w) \) with \( Z = W/\overline{g}(w) \), and the new time \( s \) such that \( \frac{dt}{ds} = |g'(w)|^2 \).

Take a constant energy level \( H = -\frac{C}{2} \). We define a new Hamiltonian \( \hat{H} = |g'(w)|^2 |H + \frac{C}{2}| \). It is easy to verify that the flows generated by the previous Hamiltonian and this one are equivalent, hence we will consider the flow generated by \( \hat{H} \) at zero energy level.

Performing the change of variables we see that the Hamiltonian \( \hat{H} \) takes the form
\[ \hat{H} = \frac{(1 + \sigma |g(w)|^2)^2}{4} |W| \]
\[ + 2|g'(w)|^2 \Omega \text{Im}(g(w)\overline{W}/g'(w)) \]
\[ - 2|g'(w)|^2 U(w, \overline{w}) + |g'(w)|^2 \frac{C}{2}. \tag{17} \]

Taking \( z = g(w) = w\overline{w} + w_k \), we obtain \( g'(w) = \overline{w} \) and \( |g'(w)|^2 = |\overline{w}|^2 = |w|^2 = w\overline{w} = |w\overline{w}| \). We will check that this transformation regularizes the singularity due to collision between the negligible mass and the primary \( k \).

We observe that on \( \mathbb{M}^2 \):
\[ |g'(w)|^2 U(w, \overline{w}) = |w|^2 \left[ \sum_{j=1}^{n} \frac{(k_j(g(w) + \overline{g}(w)) - ih_j(g(w) - \overline{g}(w)) + \sigma z(1 - \sigma |g(w)|^2))}{r|w\overline{w} + w_k - w_j||w\overline{w} + w_k - \overline{w}_j|} \right] \]
\[ = |w|^2 \left[ \sum_{j=1,j\neq k}^{n} \frac{(k_j(g(w) + \overline{g}(w)) - ih_j(g(w) - \overline{g}(w)) + \sigma z(1 - \sigma |g(w)|^2))}{r|w\overline{w} + w_k - w_j||w\overline{w} + w_k - \overline{w}_j|} \right] \]
\[ + |w|^2 \frac{(k_j(g(w) + \overline{g}(w)) - ih_j(g(w) - \overline{g}(w)) + \sigma z(1 - \sigma |g(w)|^2))}{r|w\overline{w} + w_k - w_j||w\overline{w} + w_k - \overline{w}_j|} \]
\[ = |w|^2 \left[ \sum_{j=1,j\neq k}^{n} \frac{(k_j(g(w) + \overline{g}(w)) - ih_j(g(w) - \overline{g}(w)) + \sigma z(1 - \sigma |g(w)|^2))}{r|w\overline{w} + w_k - w_j||w\overline{w} + w_k - \overline{w}_j|} \right] \]
\[ + \frac{(k_j(g(w) + \overline{g}(w)) - ih_j(g(w) - \overline{g}(w)) + \sigma z(1 - \sigma |g(w)|^2))}{r|w|^2 + w_i - \overline{w}_j|.} \tag{18} \]

With the above computations we have verified that the singularities of Eq. (16) do not appear in Eq. (18); this concludes the proof of Theorem 6. \( \square \)

**Remark 7** Notice that, since the character of Theorem 6 is local, we do not require a regular polygon configuration of the primaries. So, the result still holds for any relative equi-
libria formed by \( n \) different masses (the primaries), in this way the result is the same by replacing \( r \) by \( r_i \), \( z \) by \( z_i \) and considering the values of the masses \( m_i \) in the above proof.

### 4.2 Global regularization

The second main result of this work is the following.

**Theorem 8** The transformation \( \mathbf{z} = g(w) = \frac{n-1}{n} w + \frac{w^n}{n|w-w_1|^2 |w-w_2|^2 \cdots |w-w_n|^2} \) and the time transformation \( \frac{dt}{ds} = \frac{(w-1)^2}{n^2} \times |g'(w)| \) regularize the \( n \) binary collision-singularities of (16), between the negligible mass and each one of the primaries.

**Proof** Consider the transformation \( \mathbf{z} = g(w) := \alpha w + \frac{\beta}{w^n}, \ Z = W/g'(w), \) and the time \( s \) such that \( \frac{dt}{ds} = |g'(w)|^2 \frac{(n-1)^2 |w-w_1|^2 |w-w_2|^2 \cdots |w-w_n|^2}{n^2 |w|^2} \). As we did in the previous case, we will consider a fixed energy level \( -C_T \) and a new Hamiltonian defined as \( \tilde{H} = |g'(w)|^2 (H + C_T) \). We will be working using this new Hamiltonian at zero energy level.

Performing the change of variables we see that the Hamiltonian takes the form

\[
\tilde{H} = \frac{(1 + |g(w)|^2)^2 |W|}{4} + 2 |g'(w)|^2 \Omega \text{ Im} (g(w)\overline{W}/g'(w)) - 2 |g'(w)|^2 U(w, \overline{w}) + |g'(w)|^2 \frac{C}{2}.
\]

In order to remove the singularities we will find \( \alpha \) and \( \beta \) with the following properties:

The first one is that the primaries remain fixed, meaning on \( \mathbb{S}^2 \):

\[ g(w_i) = w_i, \ \text{g}(\hat{w}_i) = \hat{w}_i; \]

and on \( \mathbb{H}^2 \):

\[ g(w_i) = w_i. \]

The second condition is that the functions \( g(w) \) allow us to remove all the collision singularities, meaning \( g'(w_i) = 0 \), or

\[
g'(w) = \alpha + \frac{\beta(n-1)}{w^n} = \frac{\alpha}{w^n} (w^n - \beta (n-1)) = \frac{\alpha}{w^n} (w-w_1)(w-w_2) \cdots (w-w_n).
\]

The last equality is satisfied if the following occurs:

\[
0 = \sum_{j=1}^{n} w_j,
\]

\[
0 = w_1 \sum_{j=2}^{n} w_j + w_2 \sum_{j=3}^{n} w_j + \cdots + w_{n-1} \sum_{j=n}^{n} w_j + w_n - 1
\]

\[
0 = w_1 w_2 \sum_{j=3}^{n} w_j + w_3 \sum_{j=4}^{n} w_j + \cdots + w_{n-1} w_n,
\]

\[
0 + w_n - 2 w_n - \frac{n-1}{n} w_n = 0,
\]

\[
\vdots
\]

\[
0 = \sum_{i=1}^{n} \prod_{j=1 \atop j \neq i}^{n} w_j,
\]

\[
(-1)^n \prod_{j=1}^{n} w_j = -\frac{\beta}{\alpha} (n-1).
\]

The first \( (n-1) \) conditions are satisfied if the first one occurs, and it is true since it is the sum of the \( n \)th roots of the unity.

Since \( w_{j+1} = e^{i 2 \pi j / n} w_j \), we have \( w_n / w_{n-1} = e^{i \pi / n} \). We also see that \( g(w_1) = w_1 \) implies \( \frac{\beta}{\alpha} = \frac{w_1}{w_{n-1}} \). With these facts we conclude \( \alpha = \frac{n-1}{n} \) and \( \beta = \frac{\alpha}{n} \).

Notice that

\[
g(w) - w_i = \frac{n-1}{n} w + \frac{w^n}{n w^{n-1}} - w_i = \frac{(w-w_i)^2}{w^{n-1}} G(w),
\]

where

\[
G(w) = \sum_{k=0}^{n-2} \frac{n-k-1}{n} w^{n-2-k} w_i^k.
\]

We have \( G(w_i) \neq 0 \), for \( i = 1, \ldots, n \).

Finally, we will verify that the singularities of (19) due to collisions have been removed. We can write

\[
|g'(w)|^2 U(w, w) = \left[ \sum_{j=1}^{n} (k_j (g(w) + g(w)) - i h_j (g(w) - g(w)) + \sigma z (1 - \sigma |g(w)|^2)) \right] r |g(w) - w_j||g(w) - \hat{w}_j| + \left[ \frac{(n-1)^2}{n^2 w^{2n}} (w-w_1)(w-w_2) \cdots (w-w_n)^2 \right].
\]
In this section we will present a family of transformations that locally regularize the restricted \((n + 1)\)-body problem. This family includes the well-known Levi-Civita transformation, which corresponds to \(n = 2\).

First we re-write the Hamiltonian function (15) in the form

\[
H(u, v, p_u, p_v) = \frac{\Delta^2}{8} (p_u^2 + p_v^2) + \Omega (v p_u - u p_v) - U(u, v),
\]

(22)

and \(\Delta = 1 + \sigma (u^2 + v^2)\).

Next, we translate the origin of the coordinates to the position of the first primary through the change of coordinates

\[
u = q_1 + \gamma, \quad v = q_2, \quad p_u = p_1, \quad p_v = p_2, \quad \text{where} \quad \gamma = -\frac{r}{1 + z}.
\]

(24)

The new Hamiltonian is given by

\[
H(q_1, q_2, p_1, p_2) = \frac{\Delta^2}{8} (p_1^2 + p_2^2) + \Omega (q_2 p_1 - (q_1 + \gamma) p_2)
\]

\[
- \sum_{i=1}^{n} \left[ \sigma (1 + \sigma((q_1 + \gamma)^2 + q_2^2)) \right]
\]

\[
= \frac{2\sigma (k_i u + h_i v) + z(1 - \sigma(u^2 + v^2))}{\sigma (1 + \sigma((q_1 + \gamma)^2 + q_2^2))} - \sigma (2\sigma (k_i u + h_i v) + z(1 - \sigma(u^2 + v^2))\right)\frac{1}{2},
\]

(25)

with \(\Delta = 1 + \sigma ((q_1 + \gamma)^2 + q_2^2)\).

The equations of motion are

\[
\dot{q}_1 = \frac{\partial H}{\partial p_1} = \frac{\Delta^2}{4} p_1 + \Omega q_2,
\]

\[
\dot{q}_2 = \frac{\partial H}{\partial p_2} = \frac{\Delta^2}{4} p_2 - \Omega (q_1 + \gamma),
\]

\[
\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -\sigma \frac{\Delta (q_1 + \gamma)}{2} (p_1^2 + p_2^2) + \Omega p_2 + U_{q_1},
\]

\[
\dot{p}_2 = -\frac{\partial H}{\partial q_2} = -\sigma \frac{\Delta q_2}{2} (p_1^2 + p_2^2) - \Omega p_1 + U_{q_2}.
\]

(26)

In the next step, we introduce new coordinates \(Q_1\) and \(Q_2\), and a new generating function \(S = -p_1 f(Q_1, Q_2) - p_2 g(Q_1, Q_2)\). The functions \(f\) and \(g\) should be harmonic and conjugate (further details of the generating function \(S\))
and the conditions on $f$ and $g$ can be found on Stiefel and Scheifele 1971), meaning

$$\frac{\partial f}{\partial Q_1} = \frac{\partial g}{\partial Q_2} \quad \text{and} \quad \frac{\partial f}{\partial Q_2} = -\frac{\partial g}{\partial Q_1}. \quad (27)$$

The generating equations are

$$q_1 = -\frac{\partial S}{\partial p_1}, \quad q_2 = -\frac{\partial S}{\partial p_2}, \quad (28)$$

$$p_1 = -\frac{\partial S}{\partial Q_1}, \quad p_2 = -\frac{\partial S}{\partial Q_2}.$$ 

Here $P_1$ and $P_2$ are the new momenta. Then we have

$$q_1 = -\frac{\partial S}{\partial p_1} = f(Q_1, Q_2),$$

$$q_2 = -\frac{\partial S}{\partial p_2} = g(Q_1, Q_2).$$

The Hamiltonian (25) takes the form

$$H(Q_1, Q_2, P_1, P_2) = \frac{\Delta^2}{8D} (P_1^2 + P_2^2) + \frac{\Omega}{2D} \left( P_1 \frac{\partial (f^2 + g^2)}{\partial Q_2} - P_2 \frac{\partial (f^2 + g^2)}{\partial Q_1} \right) - \frac{\Omega \gamma}{D} \left( -P_1 \frac{\partial f}{\partial Q_2} + P_2 \frac{\partial f}{\partial Q_1} \right) - \sum_{i=1}^{n} \frac{2\sigma(k_i(f + \gamma) + h_i g) + z(1 - \sigma((f + \gamma)^2 + g^2))}{\Delta^2 - \sigma(2\sigma(k_i(f + \gamma) + h_i g) + z(1 - \sigma((f + \gamma)^2 + g^2)))^2}} \right)^{1/2}, \quad (31)$$

with $\Delta = 1 + \sigma((f + \gamma)^2 + g^2)$.

The equations of motion are

$$\dot{Q}_1 = \frac{\partial H}{\partial P_1} = \frac{\Delta^2 P_1}{4D} + \frac{\Omega}{2D} \frac{\partial (f^2 + g^2)}{\partial Q_2} + \frac{\Omega \gamma}{D} \frac{\partial f}{\partial Q_1},$$

$$\dot{Q}_2 = \frac{\partial H}{\partial P_2} = \frac{\Delta^2 P_2}{4D} - \frac{\Omega}{2D} \frac{\partial (f^2 + g^2)}{\partial Q_1} - \frac{\Omega \gamma}{D} \frac{\partial f}{\partial Q_2},$$

$$\dot{P}_1 = -\frac{\partial H}{\partial Q_1} = -\sigma \frac{\Delta}{2D} (P_1^2 + P_2^2) \left[ (f + \gamma) \frac{\partial f}{\partial Q_1} + g \frac{\partial g}{\partial Q_1} \right] - \frac{\Omega}{2D} \left[ P_1 \frac{\partial^2 (f^2 + g^2)}{\partial Q_1 \partial Q_2} - P_2 \frac{\partial^2 (g^2 + f^2)}{\partial Q_1^2} \right]$$

$$+ \frac{\Omega \gamma}{D} \left[ -P_1 \frac{\partial^2 f}{\partial Q_1^2} + P_2 \frac{\partial^2 f}{\partial Q_1^2} \right]$$

$$+ \frac{\partial}{\partial Q_1} \left[ \sum_{i=1}^{n} \frac{2\sigma(k_i(f + \gamma) + h_i g)}{\Delta^2 - \sigma(2\sigma(k_i(f + \gamma) + h_i g) + z(1 - \sigma((f + \gamma)^2 + g^2)))^2}} \right],$$

$$\dot{P}_2 = -\frac{\partial H}{\partial Q_2} = -\sigma \frac{\Delta}{2D} (P_1^2 + P_2^2) \left[ (f + \gamma) \frac{\partial f}{\partial Q_2} + g \frac{\partial g}{\partial Q_2} \right] - \frac{\Omega}{2D} \left[ P_1 \frac{\partial^2 (f^2 + g^2)}{\partial Q_2^2} - P_2 \frac{\partial^2 (g^2 + f^2)}{\partial Q_2^2} \right]$$

$$+ \frac{\Omega \gamma}{D} \left[ -P_1 \frac{\partial^2 f}{\partial Q_2^2} + P_2 \frac{\partial^2 f}{\partial Q_2^2} \right]$$

$$+ \frac{\partial}{\partial Q_2} \left[ \sum_{i=1}^{n} \frac{2\sigma(k_i(f + \gamma) + h_i g)}{\Delta^2 - \sigma(2\sigma(k_i(f + \gamma) + h_i g) + z(1 - \sigma((f + \gamma)^2 + g^2)))^2}} \right].$$
We observe that we may have many harmonic conjugate functions $f$ and $g$ which satisfy the desired conditions (harmonic and conjugate functions). In order to obtain a family of polynomial functions with these characteristics, we define $z = Q_1 + i Q_2$ and a new function $h(z) = h(Q_1, Q_2) := f(Q_1, Q_2) + ig(Q_1, Q_2)$. If $h(z)$ is a complex function, we know that its real and imaginary parts are harmonic functions (see for instance Marsden 1973), meaning
\[
\frac{\partial^2 f}{\partial Q_1^2} + \frac{\partial^2 f}{\partial Q_2^2} = 0 = \frac{\partial^2 g}{\partial Q_1^2} + \frac{\partial^2 g}{\partial Q_2^2}.
\]

Taking $h(z) = z$, we have $h(z^m) = z^m$, $m \in \mathbb{N}$ and $z^m = (Q_1 + i Q_2)^m$. The variable $z$ can be written as $z = \sqrt{Q_1^2 + Q_2^2} \cos(t) + i \sin(t)$.

Let $f_m = \Re(z^m)$ and $g_m = \Im(z^m)$, $m \in \mathbb{N}$; hence the first polynomials functions $f_m$ and $g_m$ are
\[
m = 1: \quad f_1 = Q_1, \quad g_1 = Q_2, \\
m = 2: \quad f_2 = Q_1^2 - Q_2^2, \quad g_2 = 2Q_1 Q_2, \\
m = 3: \quad f_3 = Q_1^3 - 3Q_1 Q_2^2, \quad g_3 = 3Q_1^2 Q_2 - Q_2^3, \\
m = 4: \quad f_4 = Q_1^4 - 6Q_1^2 Q_2^2 + Q_2^4, \quad g_4 = 4Q_1^3 Q_2 - 4Q_1 Q_2^3, \\
m = 5: \quad f_5 = Q_1^5 - 10Q_1^3 Q_2^2 + 5Q_1 Q_2^4, \quad g_5 = 5Q_1^4 Q_2 - 10Q_1^2 Q_2^3 + Q_2^5, \\
m = 6: \quad f_6 = Q_1^6 - 15Q_1^4 Q_2^2 + 15Q_1^2 Q_2^4 - Q_2^6, \quad g_6 = 6Q_1^5 Q_2 - 20Q_1^3 Q_2^3 + 6Q_1 Q_2^5.
\]

We introduce the expression $D_m$ by
\[
D_m = \left(\frac{\partial f_m}{\partial Q_1}\right)^2 + \left(\frac{\partial f_m}{\partial Q_2}\right)^2 = \left(\frac{\partial}{\partial Q_1} \Re(z^m)\right)^2 + \left(\frac{\partial}{\partial Q_2} \Im(z^m)\right)^2 = \left(\frac{\partial}{\partial Q_1} (Q_1^2 + Q_2^2)^{m/2} \cos(mt)\right)^2 + \left(\frac{\partial}{\partial Q_2} (Q_1^2 + Q_2^2)^{m/2} \sin(mt)\right)^2 = m^2 (Q_1^2 + Q_2^2)^{m-1}.
\]

Using the functions $f_m$ and $g_m$ instead of $f$ and $g$ in Eq. (31) we get
\[
H(Q_1, Q_2, P_1, P_2) = \frac{\Delta^2}{8D_m} \left(P_1^2 + P_2^2\right) + \frac{\Omega}{2D_m} \left(P_1 \frac{\partial ((Q_1^2 + Q_2^2)^m)}{\partial Q_2} - P_2 \frac{\partial ((Q_1^2 + Q_2^2)^m)}{\partial Q_1}\right) \\
- \frac{\Omega \gamma}{D_m} \left(-P_1 \frac{\partial f_m}{\partial Q_2} + P_2 \frac{\partial f_m}{\partial Q_1}\right) \\
- \sum_{i=1}^{n} \left[\sigma \Delta^2 - \sigma (2\sigma (k_i (f_m + \gamma) + h_i g_m) + z(1 - \sigma ((f_m + \gamma)^2) + g_m^2))\right]^{1/2},
\]
with $\Delta = 1 + \sigma ((f_m + \gamma)^2 + g_m^2)$.

Using $D_m$ to simplify the above equation we obtain
\[
H(Q_1, Q_2, P_1, P_2) = \frac{\Delta^2}{8D_m} \left(P_1^2 + P_2^2\right) + \frac{\Omega}{m} (P_1 Q_2 - P_2 Q_1) \\
- \frac{\Omega \gamma}{D_m} \left(-P_1 \frac{\partial f_m}{\partial Q_2} + P_2 \frac{\partial f_m}{\partial Q_1}\right) \\
- \sum_{i=1}^{n} \left[\sigma \Delta^2 - \sigma (2\sigma (k_i (f_m + \gamma) + h_i g_m) + z(1 - \sigma ((f_m + \gamma)^2) + g_m^2))\right]^{1/2},
\]
with $\Delta = 1 + \sigma ((f_m + \gamma)^2 + g_m^2)$. 

$\square$ Springer
Finally we define

\[ r_{i1} = |(f_m + \gamma, g_m) - \hat{w}_i|, \]
\[ r_{i2} = |(f_m + \gamma, g_m) - \hat{w}_i|, \]

where \( \hat{w}_i = (\hat{w}_{i1}, \hat{w}_{i2}) := (\frac{k_i}{\tau^2}, \frac{h_i}{\tau^2}) \) and \( \hat{w}_i = (\hat{w}_{i1}, \hat{w}_{i2}) := -\sigma(\frac{k_i z_i}{\tau^2}, \frac{h_i}{\tau^2}) \).

The Hamiltonian in the new coordinates takes the form

\[
H(Q_1, Q_2, P_1, P_2) = \frac{\Delta^2}{8D_m}(P_1^2 + P_2^2) + \frac{\Omega}{m}(P_1Q_2 - P_2Q_1) - \frac{\Omega \gamma}{D_m} \left( P_1 \frac{\partial f_m}{\partial Q_2} + P_2 \frac{\partial f_m}{\partial Q_1} \right) - \sum_{i=1}^{n} \frac{\sigma(k_i(f_m + \gamma) + h_ig_m)}{rr_{i1}r_{i2}} \left( P_1 \frac{\partial f_m}{\partial Q_i Q_2} + P_2 \frac{\partial f_m}{\partial Q_i Q_1} \right) + \frac{\Omega \gamma}{D_m} \left( P_1 \frac{\partial f_m}{\partial Q_1} - P_2 \frac{\partial f_m}{\partial Q_2} \right) + \sum_{i=1}^{n} \left[ 2\sigma(k_i(f_m + \gamma) + h_ig_m) + z(2 - \Delta) \right] \]

\[
\times \left[ (f_m - \hat{w}_{i1} + \gamma) \frac{\partial f_m}{\partial Q_2} + (g_m - \hat{w}_{i2}) \frac{\partial g_m}{\partial Q_1} \right] \frac{1}{rr_{i1}r_{i2}} \left[ (f_m - \hat{w}_{i1} + \gamma) \frac{\partial f_m}{\partial Q_1} + (g_m - \hat{w}_{i2}) \frac{\partial g_m}{\partial Q_2} \right] \frac{1}{rr_{i1}r_{i2}} + 2\sigma(k_i \frac{\partial f_m}{\partial Q_1} + h_i \frac{\partial g_m}{\partial Q_1}) - \sigma((f_m + \gamma) \frac{\partial f_m}{\partial Q_1} + g_m \frac{\partial g_m}{\partial Q_1}) \right),
\]

The last equations of motion are still singular at \( r_{i1}, i = 1, \ldots, n \). Hence, we reparametrize the time via a new variable \( \tau \) with \( \frac{dt}{d\tau} = r_{i1}^3 r_{i2}^3 \cdot \cdot \cdot r_{n1}^3 \), and we obtain the regularized equations of motion given by

\[
\frac{dQ_i}{d\tau} = F_i(P_1, P_2, Q_1, Q_2),
\]
\[
\frac{dP_i}{d\tau} = G_i(P_1, P_2, Q_1, Q_2), \quad i = 1, 2.
\]

Remark 9 Notice that the family of local-regularizing functions does not require a regular polygon configuration of the primaries. So, the result still holds for any relative equilibria formed by an arbitrary masses. The proof in this case is the same, we only are required to replace \( r \) by \( r_i, z \) by \( z_i \), and consider the values of the masses \( m_i \).

### 6 Numerical experiments

The relations between the initial conditions in the non-regularized case \((m = 1)\), given by \( q_1(0), q_2(0), p_1(0), p_2(0)\), and the regularized ones, given by \( Q_1(0), Q_2(0), P_1(0), P_2(0)\), are

\[
q_1(0) = f_m(Q_1(0), Q_2(0)),
\]
\[
q_2(0) = g_m(Q_1(0), Q_2(0));
\]
\[
P_1(0) = p_1(0) \left( \frac{\partial f_m}{\partial Q_1} \right)_{t=0} + p_2(0) \left( \frac{\partial g_m}{\partial Q_1} \right)_{t=0},
\]
\[
P_2(0) = -p_1(0) \left( \frac{\partial q_m}{\partial Q_1} \right)_{t=0} + p_2(0) \left( \frac{\partial q_m}{\partial Q_1} \right)_{t=0}.
\]

The following result (Roman and Szücs-Csilik 2014) is helpful to find the initial conditions in the regularized planes, given those in the non-regularized ones.
Fig. 1 Trajectory of the test particle on the regularized planes respect to the primary on the origin on $S^2$. The filled dot on the trajectory represents the particle at $t = 0$. 
Fig. 2 Trajectory of the test particle on the regularized planes with respect to the primary at the origin on $H^2$. The filled dot on the trajectory represents the particle at $t = 0$. 

(a) $m = 1$

(b) $m = 2$

(c) $m = 3$

(d) $m = 4$

(e) $m = 5$

(f) $m = 6$
Let $A = (q_1, q_2)$ be any point of the trajectory in the non-regularized plane ($m = 1$), $S$ be one of the primaries, and $B = (Q_1, Q_2)$ the point corresponding to $A$ in the regularized plane. Then the relations between $A$ and $B$ are

$$|SA| = |SB|^m, \quad \angle ASQ_1 = m \angle BSQ_1.$$ 

The orbits were computed using Matlab employing the solver ode23.

### 6.1 Positive curvature

**Case $n = 2$** For the test particle we regularize the primary con mass $\mu = 1$, and consider for $m = 1$ (the non-regularized plane), $r_1 = r_2 = 4/5$. The angular velocity for this case is $\omega^2 = (8r_1^4(1 - r_1^2)^{3/2})^{-1}$ (for details of the angular velocity, see Andrade et al. 2018). We regularize with respect to the primary at the origin. The position of the other primary (originally at $(1, 0)$) changes to $(\cos(2j\pi/m), i\sin(2j\pi/m))$, $j = 0, \ldots, m - 1$ (see Fig. 1).

The conditions at $t = 0$ taken with $j = 0$ are

$m = 1: \quad q_1(0) = -0.8000, \quad q_2(0) = 0.25, \quad p_1(0) = -1.0000, \quad p_2(0) = 7, \quad Q_1(0) = 0.1381, \quad Q_2(0) = 0.9050, \quad P_1(0) = 12.3942, \quad P_2(0) = 1.7437,$

$m = 2: \quad Q_1(0) = 0.5513, \quad Q_2(0) = 0.7649, \quad P_1(0) = 18.5538, \quad P_2(0) = -3.372,$

$m = 3: \quad Q_1(0) = 0.7258, \quad Q_2(0) = 0.6235, \quad P_1(0) = 22.6599, \quad P_2(0) = -10.0199,$

$m = 4: \quad Q_1(0) = 0.8139, \quad Q_2(0) = 0.5191, \quad P_1(0) = 26.0374, \quad P_2(0) = -16.2615,$

$m = 6: \quad Q_1(0) = 0.8643, \quad Q_2(0) = 0.4425, \quad P_1(0) = 29.0897, \quad P_2(0) = -22.2473.$

### 6.2 Negative curvature

**Case $n = 2$** For the test particle we take the value of masses $m_1 = m_2 = 1/2$, and consider the position for the particles $r_1 = r_2 = 4/3$. The angular velocity for this case is $\omega^2 = (8r_1^4(1 + r_1^2)^{3/2})^{-1}$ (for details of the angular velocity, see Andrade et al. 2018). We regularize with respect to the primary at the origin. The position of the other primary (originally at $(1, 0)$) changes to $(\cos(2j\pi/m), i\sin(2j\pi/m))$, $j = 0, \ldots, m - 1$ (see Fig. 2).

The conditions at $t = 0$ taken with $j = 0$ are

$m = 1: \quad q_1(0) = 0.1, \quad q_2(0) = 0.2, \quad p_1(0) = 0.1, \quad p_2(0) = 0, \quad m = 2: \quad Q_1(0) = 0.2486, \quad Q_2(0) = 0.4022, \quad P_1(0) = 0.0497, \quad P_2(0) = -0.0804,$

$m = 3: \quad Q_1(0) = 0.4727, \quad Q_2(0) = 0.3808, \quad P_1(0) = 0.0235, \quad P_2(0) = -0.1080,$

$m = 4: \quad Q_1(0) = 0.6006, \quad Q_2(0) = 0.3349, \quad P_1(0) = 0.0058, \quad P_2(0) = -0.1299,$

$m = 5: \quad Q_1(0) = 0.6806, \quad Q_2(0) = 0.2933, \quad P_1(0) = -0.0086, \quad P_2(0) = -0.1506,$

$m = 6: \quad Q_1(0) = 0.7347, \quad Q_2(0) = 0.2591, \quad P_1(0) = -0.0214, \quad P_2(0) = -0.1709.$

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