**Pseudo-potential of a power-law decaying interaction in two-dimensional systems**

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We analytically derive the general pseudo-potential operator of an arbitrary isotropic interaction for particles confined in two-dimensional (2D) systems, using the frame work developed by Huang and Yang for 3D scattering. We also analytically derive the low energy dependence of the scattering phase-shift for an arbitrary interaction with a power-law decaying tail, \(V_{2D}(\rho) \propto \rho^{-\alpha}\) (for \(\alpha > 2\)). We apply our results to the 2D dipolar gases (\(\alpha = 3\)) as an example, calculating the momentum and dipole moment dependence of the pseudo-potential for both \(s\)- and \(p\)-wave scattering channels if the two scattering particles are in the same 2D layer. Results for the \(s\)-wave scattering between particles in two different (parallel) layers are also investigated. Our results can be directly applied to the systems of dipolar atoms and/or polar molecules in a general 2D geometry.

**Introduction:** Low-dimensional strongly correlated systems have been one of the most important subjects in condensed matter physics in the last few decades. From the many-body point of view, the standard meanfield approximation for the 3D system is usually broken down by thermal fluctuation at finite temperature, while from the two-body point of view, the widely used first Born approximation for a 3D weak potential totally fails in lower dimensional systems \([1]\). As a result, a proper effective approximation for a 3D weak potential becomes essential for \(s\)-wave scattering channels of dipolar interaction (or called pseudo-potential) becomes essential upto a single non-universal parameter to be determined by the short-ranged details of \(V_{2D}\). (3) Finally we apply our results to the study of \(s\)- and \(p\)-wave scattering channels of dipolar interaction (\(\alpha = 3\)), and numerically evaluate the non-universal parameter for a model interaction. The \(s\)-wave scattering for the two scattering particles confined in two different (parallel) 2D layers are also investigated, showing a Feshbach-like resonance even at zero dipole moment limit. Our results can therefore be applied to the many-body physics of magnetic dipolar atoms \([12]\), cold polar molecules \([13]\), or indirect excitons in a semi-conductor based double-well system \([14]\).

**General pseudo-potential for 2D scattering:** We start from solving the two-particle scattering problem of the following 2D Schrödinger equation with total energy \(E\):

\[
-\frac{\hbar^2}{2\mu} \nabla_\perp^2 \psi(r_\perp) + V_{2D}(\rho) \psi(r_\perp) = E \psi(r_\perp),
\]

where \(\nabla_\perp^2 \equiv \frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2}\) in cylindrical coordinate, \(\mu\) is the reduced mass, and \(\psi(r_\perp)\) is the scattered wavefunction in the relative coordinate, \(r_\perp \equiv (x, y)\). \(V_{2D}(\rho)\) is the effective 2D interaction, obtained by integrating out the transverse degree of freedom \((z)\), and is assumed to be isotropic about the \(z\)-axis here. Note that we also have assumed that the transverse confinement potential is so strong that no confinement-induced resonance \([8]\) has to be considered here.

Since \(V_{2D}(\rho)\) is assumed to be isotropic and decays faster than \(\rho^{-2}\) in large \(\rho\), the wavefunction, \(\psi(r)\), can be always expanded by noninteracting eigenstates in large distance: \(\psi(\rho, \phi) = \sum_{m=0}^{\infty} u_m(k, \rho) \sum_{\sigma=\pm} C_m^\sigma(k) e^{im\phi}\), where \(k = \sqrt{2\mu E/\hbar^2}\), and \(u_m(k, \rho) \equiv A_m(k) J_m(k \rho) + B_m(k) N_m(k \rho)\) is the radial wavefunction with \(J_m(x)/N_m(x)\) being the Bessel function of the first/second kind. Here \(A_m(k)\), \(B_m(k)\), and \(C_m^\sigma(k)\) are coefficients to be determined by boundary conditions. Similar to the 3D case \([10]\), we first investigate the short-distance behavior of such noninteracting solution in the leading order terms:

\[
u_0(k, \rho) \sim A_0(k) + \frac{2B_0(k)}{\pi} \ln \left(\frac{k \rho}{2\beta_0}\right),
\]

\[
u_m(k, \rho) \sim \frac{A_m(k)}{m!} \left(\frac{k \rho}{2}\right)^m - \frac{B_m(k)(m-1)!}{\pi} \left(\frac{2}{k \rho}\right)^m \left[2B_m(k) \left(\frac{k \rho}{2}\right)^m \ln \left(\frac{k \rho}{2\beta_m}\right)\right] (m \neq 0).
\]
Here we define $\beta_m \equiv e^{-\gamma}H_m/2$ with $\gamma \approx 0.57722$ being the Euler's constant and $H_m \equiv \sum_{k=1}^{\infty} k^{-1}$ being the Harmonic number [15] (here $H_0 \equiv 0$). We note that the third term in the right hand side of Eq. (4), resulted from the irregular solution, $N_m(k\rho)$, is of the same order (upto a logarithmic function) as the first term if $A_m(k) \sim B_m(k)$.

For a typical short-range interaction, however, this term can be neglected because $B(k)/A(k) \propto k^{2m}$ in the long wavelength limit. Here, since we want to derive a general pseudo-potential for a power-law decaying potential (see below) at a small (but finite) scattering energy, we will still keep this term for the most general application. Such hybridization between the regular and irregular solutions of noninteracting partial waves does not exist in the 3D case [10]. All other terms can be shown irrelevant to the derivation of the pseudo-potential below.

To derive the proper pseudo-potential, we have to apply the noninteracting Hamiltonian on the asymptotic wavefunction above [10][11] and integrating over a small spherical area of radius $\rho$ by using Green' theorem [11]. Separating contributions from the $s$-wave and the non-$s$-wave parts, we obtain

\[ -\frac{\hbar^2}{2\mu} \left( \nabla^2 + k^2 \right) \psi(r_\perp) = -\frac{\hbar^2}{2\mu} \delta(r_\perp) \left[ 4B_0(k) \right. \]
\[ + \sum_{m=1}^{\infty} \frac{B_m(k)2m+2m!}{(k\rho)^m} \sum_{s=\pm} C_m^s(k)e^{ism\phi} \left. \right] . \] (4)

The next step is to rewrite the right hand side to be a function of $P_m(k) \equiv B_m(k)/A_m(k)$, which is the only quantity related to the phase shift of the $m$th partial wave, $\delta_m(k)$. (In fact, $P_m(k) = -\tan \delta_m(k)$). In order to get an expression of $A_m(k)$, we have to take certain derivatives on the wavefunction and let $\rho \to 0$, as in the 3D case [10]. After some simple calculation we obtain

\[ A_0(k) = \lim_{\rho \to 0} - \left( \frac{k\rho}{2\beta_0} \right)^2 \frac{\partial}{\partial \rho} \left[ \frac{u_0(k\rho)}{\ln(k\rho/2\beta_0)} \right] , \] (5)

\[ A_m(k) = - \frac{2B_m(k)}{m!} \left[ \frac{2}{k} \right]^m \frac{\partial^m}{\partial \rho^m} \left[ \rho^m u_m(k, \rho) \right] , \] (6)

where we have used the following identity: $\partial^2m(x^2 \ln(x/b)) = (2m)!((H_{2m} + \ln(x/b))$ [13].

As a result, by combining Eqs. (4), (5) and (6), we find

\[ -\frac{\hbar^2}{2\mu} \nabla^2 \psi(r_\perp) + \sum_{m=0}^{\infty} \hat{V}_m \psi(r_\perp) = E\psi(r_\perp) , \] (7)

where the pseudo-potential operator, $\hat{V}_m$, is

\[ \hat{V}_0 = \delta(r_\perp) \frac{4\hbar^2}{2\mu} P_0(k) \left( \frac{k\rho}{2\beta_0} \right)^2 \frac{\partial}{\partial \rho} \frac{1}{\ln(k\rho/2\beta_0)} \] (8)

\[ \hat{V}_m = \delta(r_\perp) \frac{\hbar^2}{2\mu P_m(k)k^m} \frac{4(m!)^2}{(2m)!} \left( [H_{2m} + \ln(k\rho/2\beta_m)] \right. \]
\[ \times \frac{2^{2m} k^{2m} \rho^m}{\hbar^2 m!} \frac{1}{\partial \rho^m} \left. \right] \] (9)

Eqs. (7)-(9) can be interpreted as the effect equation of Eq. (1) with the same boundary condition at origin ($r_\perp = 0$) in the low energy limit ($E, k\rho \to 0$), and hence is the 2D version of Huang and Yang's result in Ref. [10] (see Eq. (12) therein). We note that one can show that the pseudo-potentials derived above are equivalent to results of earlier work both in the $s$-wave channel [2, 3, 5, 6] and in the higher angular momentum channels [6, 7].

**Pseudo-potential for $\rho^{\alpha}$ interaction:** After deriving the most general form of pseudo-potential for 2D scattering, we further derive an analytical closed form of the momentum dependence of $\mathcal{P}_m(k)$ for a power-law decaying potential: $V_{2D}(\rho) \approx U/\rho^\alpha$ as $\rho \to \infty$. Here $U$ measures the strength of interaction, and $\alpha > 2$ is the decay exponent. We start from the zero energy scattering ($E = k = 0$) of Eq. (1) and the radial wavefunction, $u_m(0, \rho) = u_m(\rho)$, can be calculated analytically: $u_m(\rho) = \hat{A}_m I_{\frac{\alpha}{2}} \left( \frac{\Delta_m}{\rho} \right) + \hat{B}_m K_{\frac{\alpha}{2}} \left( \frac{\Delta_m}{\rho} \right)$ with $\hat{A}_m$ and $\hat{B}_m$ being the coefficients to be determined by the short-distance behavior of $V_{2D}(\rho)$. Here $\xi \equiv \alpha/2 - 1$, $\Delta_x \equiv (\frac{UX}{\xi})^\frac{1}{\alpha}$, and $I_m(x)/K_m(x)$ is the modified Bessel function of the first/second kind. In the limit of long distance (or weak interaction, $\Delta_\alpha^2/m^2 \ll 1$), we have

\[ u_0(\rho) \sim \hat{A}_0 + \hat{B}_0 \left[ \ln(2\beta_0\xi) + \xi \ln \left( \frac{\rho}{\Delta_\alpha} \right) \right] \]

\[ u_m(\rho) \sim \frac{\hat{A}_m(2\xi)^{\frac{\alpha}{2}} \Delta_\alpha^m}{\Gamma \left( \frac{\alpha}{\xi} + 1 \right) \rho^m} + \frac{\hat{B}_m(2\xi)^{\frac{\alpha}{2}}}{\Gamma \left( \frac{\alpha}{\xi} + 1 \right) \rho^m} \]

which should be also reproducible by taking the zero energy limit ($k \to 0$) of Eqs. (2) and (3) (the last term of Eq. (3) can be neglected in this limit). Therefore the relationship between $\mathcal{P}_m(k)$ and $\hat{P}_m \equiv \hat{B}_m/\hat{A}_m$ can be easily derived to be

\[ \mathcal{P}_0(k) = \frac{\pi \xi/2}{\mathcal{P}_0^{-1} + \ln(2\beta_0\xi) - \xi \ln(k\Delta_\alpha/2\beta_0)} \] (10)

\[ \mathcal{P}_m(k) = \frac{-2\pi(2\xi)^{\frac{\alpha}{2}}}{m!(m-1)!} \frac{(k\Delta_\alpha/2\beta_0)^2}{\Gamma \left( \frac{\alpha}{\xi} + 1 \right) \Gamma \left( \frac{\alpha}{\xi} + 1 \right)} \mathcal{P}_0^{-1} \] (11)

where $\hat{P}_m \equiv \hat{B}_m/\hat{A}_m$ is the only non-universal parameter, depending on the detailed shape of $V_{2D}(\rho)$ in the short-distance regime. Note that above results apply only in the low energy and/or weak interaction limit, i.e. $k \ll \rho^{-1} \ll \xi^{1/\xi}/\Delta_\alpha$.

For $s$-wave scattering channel, we can define an effective scattering length, $a_\alpha \equiv \Delta_\alpha(2\beta_0\xi)^{\frac{1}{\alpha}} e^{\pi^2/\alpha}$, so that the scattered wavefunction can be approximated by a smooth function at origin after cross-graining the short-ranged fluctuation (for example, the manifold
or fermions in two parallel layers with layer separation $d$.

Case C: scattering between identical fermions in the same layer, hard-disk potential [2,4,5] with an effective "radius", resulting pseudo-potential above becomes the same as a k $\rightarrow$ $\alpha$ or $\Delta_3$ $\rightarrow$ 0 as a function of dipolar strength, $\Delta_3$. One can see that when $\Delta_3$ is small, $(\mathcal{P}_0^{(0)})^{-1}$ can be quiet large, leading to a very small scattering length, $a_{3}^{(0)}$ for $\Delta_3 < 0.05$ $\mu$m. However, for larger $\Delta_3$, $a_{3}^{(0)}$ becomes proportional to $\Delta_3$, which is the only relevant length scale in this regime (i.e. the short-ranged details of the dipolar interaction becomes negligible). In Fig. 1(b), we show the calculated strength of pseudo-potential, $\mathcal{P}_1^{(0)}(k)$, for different values of incident wavevector, $k$.

The last case can be directly applied to the systems of multi-layer structure made by 1D optical lattice [10]. In the rest of the paper, we will use $V_{2D}^{(0)/(1)}$ to denote the bare intra-/inter-layer interaction, with the superscript, (0)/(1), to identify all the quantities obtained by either of them. For the convenience of numerical calculation, we further approximate the effective 2D interaction by the following analytic form: for the intra-layer interaction, we use $V_{2D}^{(0)}(\rho) = \frac{g^2}{\rho}$ for $\rho > W$ and $= \frac{\mu^2}{\rho}$ for $\rho \leq W$, where $W$ should be about the same order of the layer width and is fixed to be 0.1 $\mu$m in the following calculation. We note that different choices of the cut-off, $W$, can bring only minor quantitative difference in the results of phase shift (not shown here), because the intra-layer interaction, $V_{2D}^{(0)}(\rho)$, is assumed to be repulsive for all $\rho$, and hence no Feshbach resonance type resonance should be expected. For the inter-layer interaction, we use $V_{2D}^{(1)}(\rho) = \frac{D^2}{(\rho + d)^2} - \pi_2$, where the effect of finite layer width is expected to be smaller since $W \ll d$ in a deep optical lattice. As a result, the only length scale associate with the our represent model interaction is $\Delta_3 = MD^2/\hbar^2$ (also denoted to be $a_d$ in the literature [12]). For a typical molecule, say SrO, the fully polarized dipole moment can be $D = 8.9$ Debye, leading to $\Delta_3$ as large as 123.2 $\mu$m. However, for magnetic atoms like $^{52}$Cr, the maximum value of $\Delta_3$ is just about $1.03$ nm.

In Fig. 1, we show the numerical results for Case A and Case B together by evaluating the original two-particle Schrödinger equation of Eq. (1): In (a), we show $\mathcal{P}_0^{(0)}$ and $a_3^{(0)}$ as a function of dipolar strength, $\Delta_3$. The resulting pseudo-potential above becomes the same as a hard-disk potential [2,4,5] with an effective "radius", $a_{\alpha}$. The justification of the 2D pseudo-potential depends on the interaction strength, i.e. when $ka_{\alpha} \sim k\Delta_3 \ll 1$.

Numerical results for the dipolar gases: In the rest of this paper, we will concentrate on a physical example, say systems of polar molecules, for the results of dipolar interaction ($\alpha = 3$). An external electric field is assumed to applied perpendicular to the layer plane, inducing a field-dependent dipole moment, $D$. We consider three cases of scattering here: Case A: s-wave scattering between identical bosons in the same layer, Case B: p-wave scattering between identical fermions in the same layer, and Case C: s-wave scattering between identical bosons or fermions in two parallel layers with layer separation $d$.

In Fig. 1, we show the results for Case A: (a) and the inset are respectively the value of $a_3^{(0)}$ and $(\mathcal{P}_0^{(0)})^{-1}$ as a function of $\Delta_3$. Here we set $W = 0.1$ $\mu$m (see the text). (b) and (c) show the calculated $\mathcal{P}_0^{(0)}(k)$ and $\mathcal{P}_1^{(0)}(k)$ respectively for different values of incident wavevector, $k$.
dence is still observed.

In Fig. 2 we show the results for Case C: the s-wave scattering between particles in two different layers. In (a) and its inset, we show the calculated scattering length, \( a_s^{(1)} \), and the associated \( P_0^{(1)} \) as a function of \( \Delta_3 \). It is interesting to see that, different from the intra-layer case, \( (P_0^{(1)})^{-1} \) diverges to negative infinity and \( a_s^{(1)} \) also diverges in the regime of small \( \Delta_3 \). Such divergence originates from the fact that our dipolar interaction can always sustain an inter-layer bound state in 2D system, even when the interaction strength is infinitely small. The calculated bound state energy (also in the inset) shows a logarithmically small binding energy for the first bound state, while the second bound state appears near \( \Delta_3/d \approx 71 \). When considering the finite size effect, i.e. \( k a \) is bounded below, the first resonance will occur at a finite dipole moment as shown in (b) (also see Ref. [8]). The existence of an inter-layer bound state can lead to a strong modification of the pseudo-potential strength (similar to the Feshbach resonance), leading to some exotic many-body phases as predicted in Refs. [3][16].

In summary, we analytically derive the general form of the pseudo-potential for an arbitrary short-ranged and isotropic interaction in a uniform 2D system. The energy and interaction dependence of the pseudo-potential is also derived analytically for an arbitrary power-law interaction. Numerical results are provided for the dipolar interaction, and therefore can be applied in the study of the 2D quantum dipolar gases.

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