Conventional Quantum Mechanics Without Wave Function and Density Matrix

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Abstract
The tomographic invertable map of the Wigner function onto the positive probability distribution function is studied. Alternatives to the Schrödinger evolution equation and to the energy level equation written for the positive probability distribution are discussed. Instead of the transition probability amplitude (Feynman path integral) a transition probability is introduced. A new formulation of the conventional quantum mechanics (without wave function and density matrix) based on the “probability representation” of quantum states is given. An equation for the propagator in the new formulation of quantum mechanics is derived. Some paradoxes of quantum mechanics are reconsidered.

Introduction
During more than 70 years of existence of quantum mechanics, there was a dream to reduce misterious and intuitively very unusual notions of this theory to the well-known and intuitively acceptable classical notions. There was a common prejudice that it is impossible to describe the notion of quantum state in the framework of the conventional quantum mechanics in terms of the probability density and one is obliged to use either complex wave function or density matrix in different representations. Fortunately, it turns out that it is possible to associate with a quantum state the usual
probability density and the use of the wave function or density matrix is not mandatory in the conventional quantum mechanics. This course of lectures is devoted to the new formulation of quantum mechanics.

Quantum mechanics is based on the concept of a complex wave function which satisfies the Schrödinger equation \[1\]. Several attempts to give classical-like interpretations of the wave function were done in \[2, 3, 4\] (see also \[5\]). These attempts \[2, 3, 4\] and related constructions of quasidistribution functions in the phase space of the system \[3, 4, 5, 6\] give the idea that for quantum mechanics it is impossible to describe the state of the quantum system in terms of measurable positive probability analogously to the case of classical statistical mechanics, where the state of the system is described by the positive probability distribution due to the presence of classical fluctuations.

Nevertheless, it was shown recently \[10, 11, 12, 13, 14, 15\] that in the framework of the symplectic tomography scheme \[16, 17\], which generalizes the optical tomography scheme \[18, 19\], it is possible to introduce a classical-like description of a quantum state using the measurable positive probability (instead of the complex probability amplitude).

This result was obtained because, in addition to considering a measured physical observable in a fixed reference frame in the phase space of the quantum system, different reference frames in the phase space were considered. In the spirit of methodology, it is close to special relativity theory, where to get unusual effects due to motion with high velocities, different reference frames connected by Lorentz transform must be used. In the quantum case, the extra parameters distinguishing different reference frames replace the information coded by a phase of the wave function. This approach can be considered as introducing a new representation in quantum mechanics which can be called the “probability representation” \[20, 21\].

The description of the quantum state in terms of positive probability was obtained not only for continuous observables like the position \[14, 17, 17\], but also for pure quantum observables like spin \[22, 23\] (see also \[24\]).

A classical formulation of quantum evolution was suggested, and for the marginal distribution a new quantum evolution equation was found \[10\] which is an alternative to the time-dependent Schrödinger equation. This equation gives the classical-like description of quantum evolution in terms of a normalized positive distribution containing complete information on the state of the system. Examples of free motion and some excited states of the harmonic osc-
cillator were also considered. The evolution of even and odd coherent states of a particle in a Paul trap was investigated in the framework of the classical-like description. The even and odd coherent states of a trapped ion were discussed in recent papers. Experimentally these states were realized. A review of the method of integrals of motion and its application to oscillator’s models used in the Paul trap problem is given in Ref.

The aim of the course of lectures is to discuss, following the notion of a quantum state in the new formulation of quantum mechanics. We review the classical-like description of transition probabilities between stationary states (energy levels) of quantum systems and obtain analogs of the orthogonality and the completeness relations. We show that, if the evolution equation describing the dynamics of a quantum system is determined by the imaginary part of the system’s potential energy considered as a function of a complex coordinate, the energy states of the system are determined by the real part of the potential energy. The energy levels of the harmonic oscillator are rederived in the framework of classical-like alternatives to Schrödinger evolution and stationary equations. A new type of eigenvalue problem is formulated for the positive and normalized marginal distributions.

Classical Statistical Mechanics and Tomography Map

In quantum mechanics, the tomography methods of measuring quantum states gave the possibility to introduce new approach to the notion of quantum states. It turned out that the tomography methods can be used in classical statistical mechanics. Following we start from introducing the positive probability distribution function for a state of a classical system. In the course of lectures, we will show that both classical and quantum states are described by the same probability distribution function (called the marginal distribution function).

Main expressions for marginal distributions in the optical tomography method as well as in the symplectic tomography method are based on a theorem which connects the characteristic function with the probability distribution function. This connection is valid for quantum states described
by a density matrix \([34]\). It is obvious that the same connection also exists for classical systems in the framework of classical statistical mechanics. One can prove that the Fourier transform of the characteristic function (calculated by means of a classical probability distribution) is a positive distribution function. We illustrate this statement by an example of a one-dimensional system.

“States” in classical statistics are described by the function \(f(q, p)\), which is the probability distribution function in the phase space, i.e.,

\[
f(q, p) \geq 0, \quad \int f(q, p) \, dp = P(q), \quad \int f(q, p) \, dq = \tilde{P}(p),
\]

with \(P(q)\) and \(\tilde{P}(p)\) probability distributions for position and momentum, respectively, (marginals).

Let the nonnegative function \(f(q, p)\) be a distribution function of the classical system in the phase space. The coordinates \(-\infty < q < \infty\) and \(-\infty < p < \infty\) are the position and momentum of the system, respectively. The function \(f(q, p)\) is taken to be normalized

\[
\int f(q, p) \, dq \, dp = 1. \tag{1}
\]

We consider an observable \(X(q, p)\) which is a function on the phase space of the system under study. For the case of classical statistical mechanics, the characteristic function for the observable \(X(q, p)\)

\[
\chi(k) = \langle e^{ikX} \rangle \tag{2}
\]

is given by the relation

\[
\chi(k) = \int e^{ikX(q,p)} f(q, p) \, dq \, dp. \tag{3}
\]

The Fourier transform of the characteristic function

\[
w(X) = \frac{1}{2\pi} \int \chi(k) e^{-ikX} \, dk \tag{4}
\]

is a real nonnegative function which is normalized

\[
\int w(X) \, dX = 1. \tag{5}
\]
In fact, due to Fourier representation of Dirac delta-function, one has

$$w(X) = \int f(q, p) \delta (X(q, p) - X) \, dq \, dp. \quad (6)$$

The distribution function is nonnegative and the delta-function is also nonnegative. So we integrate the product of two nonnegative functions over the phase space. The result of the integration $w(X)$ is obviously a nonnegative function.

Let us now check the normalization of the function $w(X)$. We have

$$\int w(X) \, dX = \int f(q, p) \delta (X(q, p) - X) \, dq \, dp \, dX. \quad (7)$$

In view of the definition of delta-function, one has

$$\int \delta (X(q, p) - X) \, dX = 1. \quad (8)$$

This means that

$$\int w(X) \, dX = \int f(q, p) \, dq \, dp. \quad (9)$$

Since the distribution function $f(q, p)$ satisfies the normalization condition (5), we have shown that the Fourier transform of the characteristic function $w(X)$ given by (4) is normalized too, i.e., it satisfies the normalization condition (5). As a classical analog of the quantum symplectic-tomography observable introduced in Ref. [16] we consider the classical observable which is a linear function on the phase space of the system,

$$X(q, p) = \mu q + \nu p, \quad (10)$$

where the real parameters $\mu$ and $\nu$ are interpreted as the parameters of symplectic transform of the position and momentum of the system under study. (We discuss only one variable —the position $X(q, p)$— and do not take into account the conjugate momentum.)

The variable $X(q, p)$ can be considered from two equivalent points of view. It can be interpreted as a canonically transformed position which is a linear combination of position and momentum in a fixed reference frame in the phase space of the system. Another equivalent interpretation of the variable $X(q, p)$ given by Eq. (10) is that it is a position of the system measured
in the rotated and scaled reference frame in the classical phase space of the system.

We use the second interpretation, according to which the real parameters $\mu$ and $\nu$ determine the reference frame in the phase space of the system in which the position is measured. For the position in the transformed reference frame, we get from Eq. 3) the distribution function (the tomography map)

$$w(X, \mu, \nu) = \frac{1}{2\pi} \int e^{-ik(X-\mu q-\nu p)} f(q, p) \, dq \, dp \, dk.$$  \hspace{1cm} (11)

Another form for the probability distribution is given by Eq. 6)

$$w(X, \mu, \nu) = \int f(q, p) \delta(\mu q + \nu p - X) \, dq \, dp.$$  \hspace{1cm} (12)

One can see that the marginal distribution is a homogenous function, i.e.,

$$w(\lambda X, \lambda \mu, \lambda \nu) = |\lambda|^{-1} w(X, \mu, \nu).$$  \hspace{1cm} (13)

We introduced the notation $w(X, \mu, \nu)$ for the probability distribution of the position of the classical system in the transformed reference frame in the phase space to point out the dependence of the distribution on the parameters $\mu$ and $\nu$ determining the reference frame. Due to the dependence of the distribution $w(X, \mu, \nu)$ on these parameters, we call the distribution a marginal distribution function.

The partial case of the canonical transform is a rotation in the phase space

$$X = q \cos \varphi + p \sin \varphi.$$  \hspace{1cm} (14)

This means that we choose the parameters of the symplectic transform

$$\mu = \cos \varphi, \quad \nu = \sin \varphi.$$  \hspace{1cm} (15)

By introducing the notation for the marginal distribution of the rotated position

$$w(X, \varphi) = w(X, \mu = \cos \varphi, \nu = \sin \varphi),$$  \hspace{1cm} (16)

we get, in view of 12),

$$w(X, \varphi) = \int f(q, p) \delta(q \cos \varphi + p \sin \varphi - X) \, dq \, dp.$$  \hspace{1cm} (17)

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Using Eq. (11) we have another representation for the marginal distribution of the rotated position, namely,

$$w(X, \varphi) = \frac{1}{2\pi} \int e^{-ik(X-q\cos \varphi-p\sin \varphi)} f(q, p) \, dq \, dp \, dk.$$  

(18)

Introducing the transformed position and momentum

$$Q = q\cos \varphi + p\sin \varphi; \quad P = -q\sin \varphi + p\cos \varphi,$$  

(19)

in view of the invariance of the volume in the phase space

$$dq \, dp = dQ \, dP,$$  

(20)

we get, using the Fourier representation of delta-function,

$$w(X, \varphi) = \int \delta(X - Q) f(Q\cos \varphi - P\sin \varphi, Q\sin \varphi + P\cos \varphi) \, dQ \, dP,$$  

(21)

or

$$w(X, \varphi) = \int f(X\cos \varphi - P\sin \varphi, X\sin \varphi + P\cos \varphi) \, dP.$$  

(22)

Formula (22) is mathematically identical to Eq. 12) of Ref. [19] where the marginal distribution for the homodyne observable was considered. But in (22), the positive classical distribution $f(q, p)$ in the phase space is used instead of the Wigner function $W(q, p)$ elaborated in Eq. 12) of Ref. [19].

It is worth noting that the form of the expression for the marginal distribution $w(X, \varphi)$ is invariant. The only difference between the quantum and classical statistics in the context of the expression for the marginal distribution $w(X, \varphi)$ is in the difference between the classical distribution in the phase space and the Wigner function. The Wigner function $W(q, p)$ can take negative values. The classical distribution function $f(q, p)$ takes only nonnegative values. Nevertheless, the result of integration in both cases gives the nonnegative marginal distribution $w(X, \varphi)$.

Formula (11) has the inverse

$$f(q, p) = \frac{1}{4\pi^2} \int w(X, \mu, \nu) \exp [-i(\mu q + \nu p - X)] \, dX \, d\mu \, d\nu.$$  

(23)
In classical statistical mechanics, the admissible marginal distributions in formula (23) always satisfy the condition that the result of convolution \( f(q, p) \) is a nonnegative function.

We have shown that instead of the distribution function \( f(q, p) \) the state of the classical system in the framework of classical statistical mechanics can be determined by the marginal distribution function \( w(X, \mu, \nu) \), in complete analogy with the quantum case where the symplectic tomography procedure is used [14]. Since the map

\[
 f(q, p) \implies w(X, \mu, \nu)
\]

is invertible, the information contained in the distribution function \( f(q, p) \) is equivalent to the information contained in the marginal distribution \( w(X, \mu, \nu) \).

For \( \mu = \cos \varphi \) and \( \nu = \sin \varphi \), we have an analog of the optical tomography procedure developed for the quantum case in [19]. We have to invert formula (22). The inverse is given by the Radon transform (see, for example, Eq. 13) in [19] and also [35]). For example, if one introduces the distribution function in the form

\[
 f(q, p) = \delta(q - x_0) \delta(p - p_0), \quad (24)
\]

the marginal distribution takes the form

\[
 w(X, \mu, \nu) = \delta(X - \mu x_0 - \nu p_0) \quad (25)
\]

and

\[
 w(X, \varphi) = \delta(X - x_0 \cos \varphi - p_0 \sin \varphi). \quad (26)
\]

For classical statistical mechanics, the tomography maps discussed connect the positive distributions, and in this context our understanding of the notion of the classical state for systems with fluctuations is unchanged.

The evolution equation for the classical distribution function for a particle with mass \( m = 1 \) and potential \( U(q) \),

\[
 \frac{\partial f(q, p, t)}{\partial t} + p \frac{\partial f(q, p, t)}{\partial q} - \frac{\partial U(q)}{\partial q} \frac{\partial f(q, p, t)}{\partial p} = 0 \quad (27)
\]

can be rewritten in terms of the marginal distribution \( w(X, \mu, \nu, t) \)

\[
 \dot{w} - \mu \frac{\partial}{\partial \nu} w - \frac{\partial U(q)}{\partial q} \left[ \nu \frac{\partial}{\partial X} w \right] = 0, \quad (28)
\]
where the argument of the function $\frac{\partial U}{\partial q}$ is replaced by the operator

$$\tilde{q} = - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu}.$$  \hfill (29)

For the harmonic oscillator with frequency $\omega = 1$, the potential energy term $U(q) = \frac{q^2}{2}$ gives in Eq. 28 the following evolution equation

$$\dot{w} - \mu \frac{\partial w}{\partial \nu} + \nu \frac{\partial w}{\partial \mu} = 0.$$  \hfill (30)

We used the equality

$$\frac{1}{2} \frac{\partial q^2}{\partial q} (\tilde{q}) = - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu}$$  \hfill (31)

and the property

$$\left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} \nu \frac{\partial}{\partial X} = \nu \frac{\partial}{\partial \mu}.$$  \hfill (32)

For the mean value of position in classical statistics, we have

$$\langle q \rangle = \int f(q, p) q dq dp = i \int w(X, \mu, \nu) e^{i X} \delta'(\mu) \delta(\nu) \, dX d\mu d\nu.$$  \hfill (33)

One can see that in classical statistical mechanics there exists a function associated to the position, and by means of this function one can calculate the mean value of position using the marginal distribution $w(X, \mu, \nu)$.

In classical statistical mechanics, one can introduce the propagator

$$\Pi_{\text{cl}}(X, \mu, \nu, X', \mu', \nu', t_2, t_1)$$

that connects the two marginal distributions given for times $t_1$ and $t_2$ ($t_2 > t_1$)

$$w(X, \mu, \nu, t_2) = \int \Pi_{\text{cl}}(X, \mu, \nu, X', \mu', \nu', t_2, t_1) \, w(X', \mu', \nu', t_1) \, dX' d\mu' d\nu'.$$  \hfill (34)

The propagator satisfies the following equation

$$\frac{\partial \Pi_{\text{cl}}}{\partial t_2} - \mu \frac{\partial}{\partial \nu} \Pi_{\text{cl}} - \frac{\partial U(q)}{\partial q} (\tilde{q}) \nu \frac{\partial}{\partial X} \Pi_{\text{cl}}$$

$$= \delta(t_2 - t_1) \delta(X - X') \delta(\mu - \mu') \delta(\nu - \nu'),$$  \hfill (35)
which follows from the evolution equation \[28\].

Any integral of motion \(I(q, p, t)\) in classical statistical mechanics satisfies the equation

\[
\frac{dI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial I}{\partial p} \frac{\partial H}{\partial q} = 0 ,
\]

where \(H(q, p, t)\) is the Hamiltonian of the classical system.

Equation \[36\] coincides for

\[
H = \frac{p^2}{2} + U(q)
\]

with the equation for the classical distribution function \[27\],

\[
\frac{\partial I(q, p, t)}{\partial t} + p \frac{\partial I(q, p, t)}{\partial q} - \frac{\partial U(q)}{\partial q} \frac{\partial I(q, p, t)}{\partial p} = 0 .
\]

This follows from the fact that the distribution function itself is the integral of motion.

If one introduces the map \[11\] for the integrals of motion

\[
\mathcal{I}(X, \mu, \nu, t) = \frac{1}{2\pi} \int e^{-ik(X-\mu q-\nu p)} I(q, p, t) \, dq \, dp \, dk ,
\]

the integral of motion \(\mathcal{I}(X, \mu, \nu, t)\) satisfies Eq. \[28\] in which one has to make the replacement \(w \to \mathcal{I}\).

In classical statistical mechanics, the distribution function \(f(q, p, t)\) is a function of the integrals of motion, and the propagator that determines the evolution of the distribution function has the form

\[
P(q, p, q', p', t) = \delta(q' - q_0(q, p, t)) \delta(p' - p_0(q, p, t)) ,
\]

where \(q_0(q, p, t)\) and \(p_0(q, p, t)\) are integrals of motion which have the following property:

\[
q_0(q, p, 0) = q , \quad p_0(q, p, 0) = p .
\]

Using Eq. \[39\] one can find the propagator for the marginal distribution function.

For example, the initial distribution \[24\] takes the form

\[
f_0(q, p, t) = \delta(p - p_0) \delta(q - tp - x_0)
\]

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and the initial marginal distribution reads
\[ w_0(X, \mu, \nu, t) = \delta(X - \mu t p_0 - \mu x_0 - \nu p_0). \] (42)

In quantum and classical statistical mechanics, the forms of the propagators determining the evolution of the marginal distributions \( w(X, \mu, \nu, t) \) are identical for linear systems like an oscillator or free motion.

**New Notion of Quantum State**

We consider now a new approach to the notion of quantum state. It was shown \[16\] that for the generic linear combination of quadratures which is a measurable observable \((\hbar = 1)\)
\[ \hat{X} = \mu \hat{q} + \nu \hat{p}, \] (43)
where \( \hat{q} \) and \( \hat{p} \) are the position and momentum, respectively; the marginal distribution \( w(X, \mu, \nu) \) (normalized with respect to the variable \( X \)), depending on the two extra real parameters \( \mu \) and \( \nu \), is related to the state of the quantum system expressed in terms of its Wigner function \( W(q, p) \) as follows:
\[ w(X, \mu, \nu) = \int \exp \left[ -ik(X - \mu q - \nu p) \right] W(q, p) \frac{dk dq dp}{(2\pi)^2}. \] (44)

We use the same notation as in the classical case. If one has a pure state with the wave function \( \Psi(y) \), the marginal distribution has the form found in Ref. \[36\]
\[ w(X, \mu, \nu) = \frac{1}{2\pi |\nu|} \left| \int \Psi(y) \exp \left( \frac{i\mu y^2}{2\nu} - \frac{i\nu X}{\nu} \right) dy \right|^2. \] (45)

The physical meaning of the parameters \( \mu \) and \( \nu \) is that they describe an ensemble of rotated and scaled reference frames in which the position \( X \) is measured. For \( \mu = \cos \varphi \) and \( \nu = \sin \varphi \), the marginal distribution \[14\] is the distribution for the homodyne-output variable used in optical tomography \[19\]. Formula \[14\] can be inverted and the Wigner function of the state can be expressed in terms of the marginal distribution \[10\]:
\[ W(q, p) = \frac{1}{2\pi} \int w(X, \mu, \nu) \exp \left[ -i(\mu q + \nu p - X) \right] d\mu d\nu dX. \] (46)
Since the Wigner function determines completely the quantum state of a system and, on the other hand, this function itself is completely determined by the marginal distribution, one can understand the notion of the quantum state in terms of the classical marginal distribution for squeezed and rotated quadrature.

So, we say that the quantum state is given if the position probability distribution \( w(X, \mu, \nu) \) in an ensemble of rotated and squeezed reference frames in the classical phase space is given.

It is worth noting, that the information contained in the marginal distribution \( w(X, \mu, \nu) \) is overcomplete. To determine the quantum state completely, it is sufficient to give the function for arguments with the constraints \((\mu^2 + \nu^2 = 1)\) which corresponds to the optical tomography scheme \([19, 37]\), i.e., \( \mu = \cos \varphi \) and the rotation angle \( \varphi \) labels the reference frame in the classical phase space.

So, we formulate also the notion of quantum states as follows:

We say that the quantum state is given if the position probability distribution \( w(X, \varphi) \) in an ensemble of rotated reference frames in the classical phase space is given.

Since the quantum state is defined by the position distribution, one could associate an entropy with the state using the standard relation known in classical probability theory, i.e., the entropy \( S(\mu, \nu) \) is given by the formula

\[
S(\mu, \nu) = - \int dX w(X, \mu, \nu) \ln[w(X, \mu, \nu)].
\] (47)

If we use the distribution \( w(X, \varphi) \), the entropy \( S(\varphi) \) depends only on the rotation angle.

The description of quantum states by the probability function gives the possibility to formulate quantum mechanics without using the wave function or density matrix. These ingredients of the quantum theory can be considered as objects which are not mandatory ones since they are not directly measurable. The marginal probability distribution function \( w(X, \mu, \nu) \), which can be measured directly, replaces the wave function in the new formulation of quantum mechanics. Since the quantum mechanics formalism is reduced to the formalism of classical probability theory, well-known results of the probability theory can be used to get new results in quantum theory (including quantum computing, teleportation, and quantum cryptography).
One can also use the introduced formulation of the notion of quantum states to describe situations in which the states are either close or essentially different. We say that two states are close if their distributions are close, i.e., all the highest momenta of the distributions differ very slightly. We also say that two states are substantially different if their distributions differ substantially, i.e., there are highest momenta for the two distributions with large corresponding differences. The notion of distance in quantum mechanics using the tomography map was discussed in [38].

Quantum Evolution and Energy Levels

As was shown in [10], for systems with the Hamiltonian

\[ H = \frac{\hat{p}^2}{2} + V(\hat{q}), \] (48)

the marginal distribution satisfies the quantum time-evolution equation, being the integral equation determined by the imaginary part of the potential energy considered as a function of a complex coordinate. The evolution equation reads

\[ \dot{w} - \mu \frac{\partial}{\partial \nu} w - i \left[ V \left( -\frac{1}{\partial/\partial X} \frac{\partial}{\partial \mu} - i \frac{\nu}{2} \frac{\partial}{\partial X} \right) - V \left( -\frac{1}{\partial/\partial X} \frac{\partial}{\partial \mu} + i \frac{\nu}{2} \frac{\partial}{\partial X} \right) \right] w = 0. \] (49)

This equation is alternative to the Schrödinger equation

\[ i\dot{\Psi} = H\Psi \] (50)

and it can be obtained from the equation for density matrix

\[ \dot{\rho} + i [H, \rho] = 0, \] (51)

in view of the following formulas:

\[ q W (q, p) \rightarrow - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} w (X, \mu, \nu). \]
\[
\frac{\partial}{\partial q} W (q, p) \rightarrow \mu \frac{\partial}{\partial X} w (X, \mu, \nu),
\]

\[
p W (q, p) \rightarrow - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \nu} w (X, \mu, \nu),
\]

\[
\frac{\partial}{\partial p} W (q, p) \rightarrow \nu \frac{\partial}{\partial X} w (X, \mu, \nu)
\]

and

\[
\frac{\partial}{\partial X} \rho (X, X') \rightarrow \left( \frac{1}{2} \frac{\partial}{\partial q} + i p \right) W (q, p),
\]

\[
\frac{\partial}{\partial X'} \rho (X, X') \rightarrow \left( \frac{1}{2} \frac{\partial}{\partial q} - i p \right) W (q, p),
\]

\[
X \rho (X, X') \rightarrow \left( q + \frac{i}{2} \frac{\partial}{\partial p} \right) W (q, p),
\]

\[
X' \rho (X, X') \rightarrow \left( q - \frac{i}{2} \frac{\partial}{\partial p} \right) W (q, p).
\]

Equation (49) can be considered as a Fokker–Planck-like equation of classical probability theory. The measurable position is a cyclic variable for the evolution equation.

In order to compare the classical and quantum evolution equations, let us rewrite the quantum evolution equation (49) in the form of a series,

\[
\dot{w} - \mu \frac{\partial w}{\partial \nu} + 2 \sum_{n=0}^{\infty} \frac{V^{2n+1} (\hat{q})}{(2n+1)!} \left( \nu \frac{\partial}{\partial X} \right)^{2n+1} (-1)^{n+1} w = 0.
\]

Here

\[
V^{2n+1} (\hat{q}) = \frac{\partial^{2n+1} V}{\partial q^{2n+1}} (q = \hat{q}),
\]

where the operator \( \hat{q} \) is given by Eq. (29).

Equation (53) can also be presented in the form

\[
\dot{w} - \mu \frac{\partial w}{\partial \nu} - \frac{\partial V}{\partial q} (\hat{q}) \nu \frac{\partial}{\partial X} w + 2 \sum_{n=1}^{\infty} \frac{V^{2n+1} (\hat{q})}{(2n+1)!} \left( \nu \frac{\partial}{\partial X} \right)^{2n+1} (-1)^{n+1} w = 0.
\]

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The three first terms give the $\hbar \to 0$ classical Boltzman equation. It is important that both classical and quantum evolution equations are written for the same function $w(X, \mu, \nu)$.

Let us rewrite, following [13], the Schrödinger equation for the stationary state density matrix $\rho_E$ of the quantum system with Hamiltonian $48)\]

$$H\rho_E = \rho_E H = E\rho_E$$

in terms of the time-independent marginal distribution $w_E(X, \mu, \nu)$ of the squeezed and rotated quadrature introduced in [16]. We have

$$\frac{1}{2} \left( \frac{\partial}{\partial X} \right)^{-2} \frac{\partial^2}{\partial \nu^2} w_E - \frac{1}{8} \mu^2 \frac{\partial^2}{\partial X^2} w_E + \text{Re} V \left[ \frac{i}{2} \nu \frac{\partial}{\partial X} - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} \right] w_E = E w_E. \quad (58)$$

The positive marginal distribution (eigendistribution) satisfies this eigenvalue equation and also the equation

$$-\mu \frac{\partial}{\partial \nu} w_E = 2 \text{Im} V \left[ \frac{i}{2} \nu \frac{\partial}{\partial X} - \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} \right] w_E. \quad (59)$$

Equation [59] follows from the evolution equation [49] for the marginal distribution of the quantum system (see Ref. [10]), if the marginal distribution does not depend on time. Thus, the normalized marginal distributions of stationary states of quantum systems satisfy the system of two equations [58] and [59].

We consider an example of the quantum harmonic oscillator since it is one of the most important quantum systems. For this case, using the Hamiltonian

$$H = \frac{\hat{p}^2}{2} + \frac{\hat{q}^2}{2}, \quad (60)$$

we reduce Eq. [49] to the following one (in view of Ref. [10]):

$$\dot{w} - \mu \frac{\partial}{\partial \nu} w + \nu \frac{\partial}{\partial \mu} w = 0. \quad (61)$$
The marginal distribution of the oscillator’s ground state is

\[
 w^{(\text{OS})}_0(X, \mu, \nu) = \frac{1}{\sqrt{\pi (\mu^2 + \nu^2)}} \exp\left[-\frac{X^2}{\mu^2 + \nu^2}\right].
\]  

(62)

The marginal distribution must be consistent with the uncertainty relation, i.e.,

\[
 \left[ \int w(X, 1, 0) X^2 dX - \left\{ \int w(X, 1, 0) X dX \right\}^2 \right] \\
\times \left[ \int w(X, 0, 1) X^2 dX - \left\{ \int w(X, 0, 1) X dX \right\}^2 \right] \geq \frac{1}{4}.
\]

(63)

**Propagator**

In Ref. [11], the classical transition-probability density from an initial position \(X'\) measured at time \(t = 0\) in the reference frame in the classical phase space labeled by the parameters \(\mu'; \nu'\) to the position \(X\) measured at time \(t\) in the reference frame in the classical phase space labeled by the parameters \(\mu; \nu\) was introduced. This classical transition-probability density is the propagator for the evolution equation (49) for the marginal distribution and the propagator is the kernel of the integral relation

\[
 w(X, \mu, \nu, t) = \int \Pi(X, \mu, \nu, X', \mu', \nu', t) w(X', \mu', \nu', 0) dX' d\mu' d\nu'.
\]

(64)

The classical propagator has a specific feature, it takes into account that the transition probability is considered in different reference frames in the phase space. In view of this fact, parameters of reference frames \(\mu\) and \(\nu\) are present in the evolution equation. Due to this, the equation for the propagator slightly differs from the Smoluchowski–Chapman–Kolmogorov equation elaborated in the classical probability theory.

The classical propagator can be related to a quantum propagator (the Green function) for the density matrix \(\rho(X, X', t)\) in the coordinate representation. For a pure state with the wave function \(\Psi(X, t)\), we have

\[
 \rho(X, X', t) = \Psi(X, t) \Psi^*(X', t).
\]

(65)
The Green function of the Schrödinger equation $G (X, X', t)$ connects the wave functions at the initial time moment $t = 0$ and at time $t$

$$\Psi (X, t) = \int G (X, X', t) \Psi (X') dX'. \quad (66)$$

We have for the density matrix (65), in view of relation (66), the following expression:

$$\rho (X, X', t) = \int K (X, X', Y, Y', t) \rho (Y, Y', t = 0) dY dY', \quad (67)$$

where the propagator $K (X, X', Y, Y', t)$ is expressed in terms of the Green function (for unitary evolution)

$$K (X, X', Y, Y', t) = G (X, Y, t) G^* (X', Y', t). \quad (68)$$

Since the relation of the density matrix to the marginal distribution is known for any time $t$ (given by (65) and (66)), it is possible to obtain

$$K (X, X', Z, Z', t) = \frac{1}{(2 \pi)^2} \int \frac{1}{|\nu'|} \exp \left\{ i \left( Y - \mu \frac{X + X'}{2} \right) 
- i \frac{Z - Z'}{\nu'} Y' + i \frac{Z^2 - Z'^2}{2 \nu'} \mu' \right\}
\times \Pi (Y, \mu, X - X', Y', \mu', \nu', t) d\mu d\mu' dY dY' dv' dv'. \quad (69)$$

Thus, given the classical propagator for the classical marginal distribution, the propagator for the density matrix is also given. Formula (69) can be converted.

Deriving formula (69) we used the relations

$$W (q, p) = \frac{1}{2 \pi} \int w (X, \mu, \nu) \exp [-i (\mu q + \nu p - X)] d\mu d\nu dX,$$

$$\rho (X, X') = \frac{1}{2 \pi} \int w (Y, \mu, X - X') \exp \left[ i \left( Y - \mu \frac{X + X'}{2} \right) \right] d\mu dY,$$

and

$$w (X, \mu, \nu) = \frac{1}{2 \pi |\nu'|} \int \rho (Z, Z') 
\times \exp \left[ -i \frac{Z - Z'}{\nu'} \left( X - \mu \frac{Z + Z'}{2} \right) \right] dZ dZ'. \quad (70)$$
The last formulas give some relationships between the marginal distribution \( w (X, \mu, \nu) \), the Wigner function, and the density matrix in the coordinate representation.

In Ref. [6], the Wigner function was introduced in terms of the density matrix

\[
W(q, p) = \int \rho\left(q + \frac{u}{2}, q - \frac{u}{2}\right) e^{-ipu} du,
\]

which can be rewritten as

\[
W(q, p) = \int \rho(Z, Z') \delta\left(Z - q - \frac{u}{2}\right) \delta\left(Z' - q + \frac{u}{2}\right) e^{-ipu} du dZ dZ',
\]
or

\[
W(q, p) = 2 \int \rho(Z, Z') e^{-2ip(Z - q)} \delta(Z' + Z - 2q) dZ dZ'.
\]

Comparing formulas (72) and (70) one can conclude that the Wigner quasidistribution function \( W(q, p) \) and the classical probability distribution \( w(X, \mu, \nu) \), the latter being a positive and normalized function, are obtained using similar integral transforms of the density matrix.

The difference between the two functions is determined by the difference in the kernels of the integral transforms. In the case of the Wigner transform, the kernel reads

\[
K_W(Z, Z', q, p) = 2 e^{-2ip(Z - q)} \delta(Z' + Z - 2q).
\]

In the case of the symplectic tomography transform suggested in Ref. [16], the kernel reads

\[
K_M(Z, Z', X, \mu, \nu) = \frac{1}{2\pi|\nu|} \exp\left[-i \frac{Z - Z'}{\nu} \left(X - \mu - \frac{Z + Z'}{2}\right)\right].
\]

Due to the difference of the kernels, the Wigner function takes negative values and the marginal probability distribution is nonnegative function.

If one writes the classical propagator as a function of the initial time moment \( t_1 \) and the final time moment \( t_2 \) (i.e., \( t_1 \neq 0 \)), relation (64) can be rewritten as

\[
w(X, \mu, \nu, t_2) = \int \Pi(X, \mu, \nu, X', \mu', \nu', t_2, t_1) w(X', \mu', \nu', t_1) dX' d\mu' d\nu'.
\]
From the physical meaning of the classical propagator, the nonlinear integral relation follows

\[
\Pi (X, \mu, \nu, X', \mu', \nu', t_2, t_1) = \quad \int \Pi \left( X, \mu, \nu, X'', \mu'', \nu'', t_2, t' \right) \\
\times \Pi \left( X'', \mu'', \nu'', X', \mu', \nu', t', t_1 \right) \, dX''\, d\mu''\, d\nu''.
\] (76)

This relation means that if the system is initially located at the point \( X' \) at time \( t_1 \) in the reference frame in the phase space labeled by the parameters \( \mu'; \nu' \), the probability for the system to arrive at the point \( X \) in the reference frame in the phase space labeled by the parameters \( \mu; \nu \) at time \( t_2 \) is equal to the probabilities to arrive at an intermediate point \( X'' \) in the reference frame in the phase space labeled by the parameters \( \mu''; \nu'' \) at time \( t' \) integrated over all the intermediate positions and all the intermediate reference frames.

The above integral equation (76) is an analog of the Smoluchowski–Chapman–Kolmogorov relation which in the approach introduced in Ref. [11] is generalized to the case of families of conditional probabilities if different reference frames in the phase space (parameters \( \mu \) and \( \nu \)) are taken into account. Also the propagator satisfies the differential equation (see Ref. [11])

\[
\frac{\partial \Pi}{\partial t_2} - \mu \frac{\partial}{\partial \nu} \Pi - i \left[ V \left( -\frac{1}{\partial X} \frac{\partial}{\partial \mu} - i \frac{\nu}{2} \frac{\partial}{\partial X} \right) \right] \Pi \\
- V \left( -\frac{1}{\partial X} \frac{\partial}{\partial \mu} + i \frac{\nu}{2} \frac{\partial}{\partial X} \right) \Pi \\
= \delta (t_2 - t_1) \delta (X - X') \delta (\mu - \mu') \delta (\nu - \nu').
\] (77)

The classical propagator satisfies the initial condition

\[
\Pi (X, \mu, \nu, X', \mu', \nu', t, t) = \delta (X - X') \delta (\mu - \mu') \delta (\nu - \nu').
\] (78)

The relation that could be used to express the classical propagator in terms of the functional integral can be also written

\[
\Pi \left( X, \mu, \nu, X', \mu', \nu', t_1, t_\text{in} \right) \quad = \quad \int \prod_{k=1}^{N-1} \left\{ \Pi \left( X_{k+1}, \mu_{k+1}, \nu_{k+1}, X_k, \mu_k, \nu_k, t_{k+1}, t_k \right) \right\} \, dX_k \, d\mu_k \, d\nu_k.
\] (79)

where the time interval \( t_1 - t_\text{in} = N \tau; \quad t_k = t_\text{in} + k \tau; \quad k = 1, 2, \ldots, N. \)

Taking in relation (79) the limit \( \tau \to 0; \quad N \to \infty \), one obtains the expression for the classical propagator in terms of the functional integral.

19
Quantum Transition Probabilities

We express quantum-transition probabilities in terms of an overlap integral of classical-like marginal distributions describing the initial and final quantum states.

If the initial pure state of a quantum system is described by the marginal distribution
\[ w_{\text{in}} = w_1 (X, \mu, \nu) \] (80)
and the final state of the quantum system is described by the marginal distribution
\[ w_{\text{f}} = w_2 (X, \mu, \nu) , \] (81)
the probability of the quantum transition \( P_{12} (1 \Rightarrow 2) \) can be obtained using the known expression for the probability in terms of an overlap integral of the Wigner functions \( W_1 (q, p) \) and \( W_2 (q, p) \) of the initial and final states (see, for example, Ref. [39])
\[ P_{12} = \frac{1}{2\pi} \int W_1(q, p) W_2(q, p) \, dq \, dp . \] (82)

For the transition probability, one can obtain, in view of relation (82), the expression in terms of the marginal distributions
\[ P_{12} = \int w_1(X, \mu, \nu) w_2(Y, -\mu, -\nu) \exp[i(X + Y)] \frac{d\mu \, d\nu \, dX \, dY}{2\pi} . \] (83)

As follows from relation (83), any pure normalized quantum state is described by the marginal distribution \( w_p (X, \mu, \nu) \), which satisfies the additional condition
\[ \int w_p(X, \mu, \nu) w_p(Y, -\mu, -\nu) \exp[i(X + Y)] \frac{d\mu \, d\nu \, dX \, dY}{2\pi} = 1 . \] (84)

The complex wave functions, which belong to different energy levels of a quantum system, are orthogonal. This orthogonality condition is expressed in terms of the classical marginal distribution as the relation
\[ \int w_n(X, \mu, \nu) w_m(Y, -\mu, -\nu) \exp[i(X + Y)] \frac{d\mu \, d\nu \, dX \, dY}{2\pi} = \delta_{mn} , \] (85)
where the labels $m, n$ correspond to the energy levels $E_m, E_n$. The pure states $|n\rangle$ satisfy the completeness relation

$$\sum_n |n\rangle\langle n| = \hat{1}. \quad (86)$$

This relation can be rewritten as the condition for the marginal distributions of the pure states with the energies $E_n$

$$\sum_n w_n (X, \mu, \nu) = w^{(\text{wn})} (X, \mu, \nu), \quad (87)$$

where the distribution $w^{(\text{wn})}$ describes the white noise,

$$w^{(\text{wn})} (X, \mu, \nu) = \int \frac{dx \, dy \, dk}{2\pi} \exp \left[ ik (\mu x + \nu y) - ikx - ik^2 \mu \nu \right]. \quad (88)$$

Thus, the classical marginal distributions describing the energy levels of quantum systems are positive solutions to the system of equations (88) and (89) and these solutions satisfy the orthogonality condition (85) and the analog of the completeness relation (87). The distributions form an interesting mathematical set that differs substantially from the usual Hilbert space of states described by the normalized complex wave functions. Of course, the structure of the Hilbert space can be traced using the map, which connects the states expressed in terms of the density matrix and the states expressed in terms of the marginal distribution functions.

**Propagator for Systems with Quadratic Hamiltonians**

As an example, we consider the system with the quadratic Hermitian Hamiltonian

$$H = \frac{1}{2} (Q B Q) + C Q, \quad (89)$$

where one has the vector-operator $Q = (p, q)$. The symmetric $2\times2$ matrix $B$ and real $2$-vector $C$ depend on time. The system has linear integrals of motion (see Ref. [39, 40]):

$$I (t) = \Lambda (t) Q + \Delta (t). \quad (90)$$
Here the real symplectic $2 \times 2$ matrix $\Lambda (t)$ and the real vector $\Delta (t)$ satisfy the equations
\[
\dot{\Lambda} = i \Lambda B \sigma_y, \quad \dot{\Delta} = i \Lambda \sigma_y C,
\]
and the initial conditions
\[
\Lambda (0) = 1; \quad \Delta (0) = 0.
\]

As follows from relation (44) and from the property of the Wigner function (see Ref. [39]), the classical propagator is
\[
\Pi (X, \mu, \nu, X', \mu', \nu', t) = \delta (X - X' + N \Lambda^{-1} \Delta) \delta (N' - N \Lambda^{-1}),
\]
where the vectors $N$ and $N'$ are
\[
N = (\nu, \mu), \quad N' = (\nu', \mu').
\]

For the quadratic systems without linear terms ($C = 0$), the classical propagator is
\[
\Pi (X, \mu, \nu, X', \mu', \nu', t) = \delta (X - X') \delta (N' - N \Lambda^{-1}).
\]

Thus, if one knows the linear integrals of motion, i.e., the matrix $\Lambda (t)$ and the vector $\Delta (t)$, one knows the classical propagator.

For free motion with the Hamiltonian
\[
H = \frac{p^2}{2},
\]
there are two linear invariants found in Ref. [39, 40]
\[
p_0(t) = p, \quad q_0(t) = q - pt.
\]

This means that $\Delta (t) = 0$, and the symplectic $2 \times 2$ matrix reads
\[
\Lambda (t) = \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix}.
\]

Thus, we have
\[
N \Lambda^{-1}(t) = (\nu + \mu t, \mu).
\]
Consequently, the classical propagator of free motion has the form
\[
\Pi^{(f)}(X, \mu, \nu, X', \mu', \nu', t) = \delta(X - X') \delta(\nu' - \nu - \mu t) \delta(\mu - \mu'). \tag{99}
\]
For the harmonic oscillator with the Hamiltonian (90), linear invariants are known (see Ref. [39, 40]), and the matrix \( \Lambda(t) \) is
\[
\Lambda(t) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \tag{100}
\]
This means that for the harmonic oscillator
\[
\mathcal{N} \Lambda^{-1}(t) = (\nu \cos t - \mu \sin t, \nu \sin t + \mu \cos t). \tag{101}
\]
Consequently, the classical propagator of the harmonic oscillator is
\[
\Pi^{(os)}(X, \mu, \nu, X', \mu', \nu') = \delta(X - X') \delta(\nu' - \nu \cos t + \mu \sin t) \delta(\mu' - \nu \sin t - \mu \cos t). \tag{102}
\]

**Energy Levels of the Harmonic Oscillator**

The marginal distribution of the coherent state of the harmonic oscillator has the form obtained in Ref. [11]:
\[
w_\alpha(X, \mu, \nu) = [\pi (\mu^2 + \nu^2)]^{-1/2} \exp \left[ -|\alpha|^2 - \frac{X^2}{\mu^2 + \nu^2} + \frac{\alpha^2(\nu + i\mu)^2}{2(\mu^2 + \nu^2)} + \frac{\alpha^2(\nu - i\mu)^2}{2(\mu^2 + \nu^2)} - \frac{i\sqrt{2}\alpha X(\nu + i\mu)}{\mu^2 + \nu^2} + \frac{i\sqrt{2}\alpha^* X(\nu - i\mu)}{\mu^2 + \nu^2} \right]. \tag{103}
\]
The eigendistribution function for the energy level of the harmonic oscillator satisfies the eigenvalue equation
\[
\left\{ \frac{1}{2} \left[ \left( \frac{\partial}{\partial \nu} \right)^2 + \left( \frac{\partial}{\partial \mu} \right)^2 \right] \left( \frac{\partial}{\partial X} \right)^2 - \frac{1}{8} (\mu^2 + \nu^2) \left( \frac{\partial}{\partial X} \right)^2 \right\} w_E(X, \mu, \nu) = E w_E(X, \mu, \nu). \tag{104}
\]
This equation can be rewritten for the Fourier component of the marginal distribution
\[
\tilde{w}_E(k, \mu, \nu) = \frac{1}{2\pi} \int w_E(X, \mu, \nu) \exp(-ikX) dX. \tag{105}
\]
in the form

\[
\left\{- \frac{1}{2} k^2 \left[ \left( \frac{\partial}{\partial \nu} \right)^2 + \left( \frac{\partial}{\partial \mu} \right)^2 \right] + \frac{1}{8} k^2 (\mu^2 + \nu^2) \right\} \tilde{w}_E(k, \mu, \nu) = E \tilde{w}_E(k, \mu, \nu).
\]  

(106)

Since the marginal distribution of the stationary state of the harmonic oscillator must satisfy the stationarity condition found in Ref. [12],

\[
\left( \mu \frac{\partial}{\partial \nu} - \nu \frac{\partial}{\partial \mu} \right) w_E(X, \mu, \nu) = 0, \tag{107}
\]

Eq. (106) is equivalent to the equation for axially symmetric wave functions of a two-mode harmonic oscillator with mass \( m = k^2 \), frequency \( \omega = 1/2 \), and angular momentum \( M = 0 \). The wave function corresponding to zero angular momentum is expressed in terms of the Laguerre polynomials

\[
\tilde{w}_n(k, \mu, \nu) = \frac{1}{2\pi} \exp \left[ -\frac{k^2 (\mu^2 + \nu^2)}{4} \right] L_n \left( \frac{k^2 \mu^2 + k^2 \nu^2}{2} \right). \tag{108}
\]

The main quantum number \( n \) of the one-dimensional harmonic oscillator under discussion is equal to the integer radial quantum number \( n_r \) of the artificial two-mode oscillator

\[
n = n_r, \quad n_r = 0, 1, 2, \ldots. \tag{109}
\]

The energy level of the artificial symmetric two-mode oscillator labeled by the radial quantum number \( n_r \) and the angular momentum \( M \) as

\[
E_{n_r, M} = \omega (|M| + 1 + 2 n_r) \tag{110}
\]

for \( \omega = 1/2, \ M = 0, \ n_r = n \) gives exactly the spectrum of the one-dimensional oscillator

\[
E_n = E_{n_r, M} = n + \frac{1}{2}, \quad n = 0, 1, 2, \ldots \tag{111}
\]

To find the marginal distribution, we have to calculate

\[
w_n(X, \mu, \nu) = \frac{1}{2\pi} \int \exp \left[ -\frac{k^2 (\mu^2 + \nu^2)}{4} + ikX \right] L_n \left( \frac{k^2 \mu^2 + k^2 \nu^2}{2} \right) dk. \tag{112}
\]
In view of the integral
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp \left( -\frac{k^2}{4} + ikX \right) L_n \left( \frac{k^2}{2} \right) dk = \pi^{-1/2} 2^{-n} (n!)^{-1} \exp (-X^2) H_n^2(X), \tag{113}
\]
one obtains the marginal distribution
\[
w_n(X, \mu, \nu) = [\pi (\mu^2 + \nu^2)]^{-1/2} 2^{-n} (n!)^{-1}
\times \exp \left( -\frac{X^2}{\mu^2 + \nu^2} \right) H_n^2 \left( \frac{X}{\sqrt{\mu^2 + \nu^2}} \right), \tag{114}
\]
It is worth noting that the normalization condition for the marginal distribution \(w_n(X, \mu, \nu)\) implies the condition for the Fourier component
\[
\int \tilde{w}_n(k, \mu, \nu) \exp (ikX) dk dX = 2\pi \tilde{w}_n(k = 0, \mu, \nu) = 1. \tag{115}
\]
We take solutions \(108\) without using the normalization condition in terms of the variables \(\mu\) and \(\nu\) of the artificial two-mode oscillator, but using the normalization condition of the marginal distribution in terms of the variable \(X\) and the corresponding property of its Fourier component.

**Tomography of Spin States**

Let us introduce the probability distribution for the spin projection in a given direction considered in a rotated reference frame. For arbitrary values of spin, let the spin state have the density matrix
\[
\rho_{mm'}^{(j)} = \langle jm | \hat{\rho}^{(j)} | jm' \rangle, \quad m = -j, -j + 1, \ldots, j - 1, j, \tag{116}
\]
where
\[
\hat{j}_3 | jm \rangle = m | jm \rangle, \tag{117}
\]
\[
\hat{j}^2 | jm \rangle = j(j + 1) | jm \rangle,
\]
and
\[
\hat{\rho}^{(j)} = \sum_{m=-j}^{j} \sum_{m'=-j}^{j} \rho_{mm'}^{(j)} | jm \rangle \langle jm' |. \tag{118}
\]
The operator $\rho^{(j)}$ is the density operator of the state under discussion. The diagonal elements of the density matrix determine the positive probability distribution

$$\rho_{mm}^{(j)} = w_0(m),$$

which is normalized,

$$\sum_{m=-j}^j w_0(m) = 1.$$  

In Refs. [41, 42, 43], a general group construction of tomographic schemes was discussed, and this scheme was also used for spin tomography in Refs. [22, 23]. The idea is to consider the diagonal elements of the density matrix in another reference frame. The density matrix in another reference frame reads

$$\rho_{m_1 m_2}^{(j)} = (D \rho D^\dagger)_{m_1 m_2}.$$  

Here the unitary rotation transform $D$ depends on the Euler angles $\alpha, \beta, \gamma$ and, by definition, the diagonal matrix elements of the density matrix yield the positive normalized probability distribution. For the diagonal elements of the density matrix [121],

$$m_1 = m_2.$$  

We introduce new notation and rewrite equality [121] for $m_1 = m_2$ in the form

$$\bar{w}(m_1, \alpha, \beta, \gamma) = \sum_{m_1'=-j}^j \sum_{m_2'=-j}^j D_{m_1' m_1}^{(j)}(\alpha, \beta, \gamma) \rho_{m_1' m_2'}^{(j)} D_{m_2' m_2}^{(j)*}(\alpha, \beta, \gamma).$$

Here the matrix elements $D_{m_1' m_1}^{(j)}(\alpha, \beta, \gamma)$ (the Wigner function) are the matrix elements of the rotation-group representation

$$D_{m'm}^{(j)}(\alpha, \beta, \gamma) = e^{im'\gamma} d_{m'm}^{(j)}(\beta) e^{ima},$$

where

$$d_{m'm}^{(j)}(\beta) = \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{1/2} \left(\cos \frac{\beta}{2}\right)^{m'+m} \left(\sin \frac{\beta}{2}\right)^{m'-m} \times P_{m'-m}^{(j-m',m'+m)}(\cos \beta),$$

\[ 26 \]
and \( P_{n}^{(a,b)}(x) \) is the Jacobi polynomial.

Since
\[
D_{n_1}^{(j_1)}(\alpha, \beta, \gamma) = (-1)^{m_1 - m} D_{-m_1}^{(j_1)}(\alpha, \beta, \gamma),
\]
the marginal distribution depends only on two angles, \( \alpha \) and \( \beta \).

Thus, let us denote
\[
w(m_1, \alpha, \beta) = \tilde{w}(m_1, \alpha, \beta, \gamma),
\]
which satisfies
\[
\sum_{m_1=-j}^{j} w(m_1, \alpha, \beta) = 1.
\]

For a spin-1/2 state with spin projection +1/2 and wave function
\[
\psi_{+1/2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]
or with density matrix
\[
\rho_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\]
the marginal distribution is equal to
\[
w\left(\frac{1}{2}, \alpha, \beta\right) = \cos^2 \frac{\beta}{2} \quad \text{for} \quad m_1 = + \frac{1}{2},
\]
and, correspondingly,
\[
w\left(-\frac{1}{2}, \alpha, \beta\right) = \sin^2 \frac{\beta}{2} \quad \text{for} \quad m_1 = - \frac{1}{2}.
\]

In Ref. [23], by using the properties of the Wigner function and the Clebsch–Gordan coefficients, formula [22] was inverted and the density matrix was expressed in terms of the marginal distribution
\[
\sum_{j_3=0}^{2j} \sum_{m_3=-j_3}^{j_3} (2j_3 + 1)^2 \sum_{m_1=-j}^{j} \int (-1)^{m_1} w(m_1, \alpha, \beta) \ D_{0m_3}^{(j_3)}(\alpha, \beta, \gamma) \times \left( \begin{array}{ccc} j & j & j_3 \\ m_1 & -m_1 & 0 \end{array} \right) \left( \begin{array}{ccc} j & j & j_3 \\ m'_1 & -m'_1 & m_3 \end{array} \right) \frac{d\omega}{8\pi^2} = \rho_{m'_1 m'_2}^{(j)},
\]

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Here \(m, m' = -j, -j + 1, \ldots, j\) and one integrates over rotation parameters \(\alpha, \beta, \gamma\).

To derive formula (130), we used the known property of the Wigner function:

\[
\int D_{m_1 m_1}^{(j_1)}(\omega) D_{m_2 m_2}^{(j_2)}(\omega) D_{m_3 m_3}^{(j_3)}(\omega) \frac{d\omega}{8\pi^2} = \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1' & m_2' & m_3' \end{array} \right) \times \left( \begin{array}{ccc} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{array} \right),
\]

(131)

where

\[
\int d\omega = \int_0^{2\pi} d\alpha \int_0^\pi \sin \beta d\beta \int_0^{2\pi} d\gamma,
\]

(132)

along with the orthogonality property of 3\(j\)-symbols:

\[
(2j + 1) \sum_{m_1=-j_1}^{j_1} \sum_{m_2=-j_2}^{j_2} \left( \begin{array}{ccc} j_1 & j_2 & \hat{j} \\ m_1 & m_2 & -m \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 & j' \\ m_1 & m_2 & -m' \end{array} \right) = \delta_{jj'} \delta_{mm'},
\]

\[
\sum_{j=|j_1-j_2|}^{j_1+j_2} \sum_{m=-j}^{j} (2j + 1) \left( \begin{array}{ccc} j_1 & j_2 & \hat{j} \\ m_1 & m_2 & -m \end{array} \right) \left( \begin{array}{ccc} j_1 & j_2 & \hat{j} \\ m_1' & m_2' & -m' \end{array} \right) = \delta_{m_1 m_1'} \delta_{m_2 m_2'}.
\]

Formula (130), being the inverse of (122), is an analog of the Radon transform for spin states. *Given a measurable marginal distribution for arbitrary spin, one can reconstruct the state density matrix by means of this relation.*

The results obtained enable one to measure the spin state by measuring a spin projection on a given axis. One obtains the experimental probability-distribution function which depends on two angles determining the axis. Using the relationship between the probability distribution and the state density matrix, one reconstructs all the information about the quantum spin state. This means that the probability distribution can be used instead of complex spinors and density matrices for the spin-state description since it contains complete (even overcomplete) information on the state.

**Quantum Measurements and Collapse of Wave Function**

We will review the discussion of quantum measurements done in Refs. [10, 11, 14]. It is known (see, for example, Refs. [14, 15]) that quantum mechanics
is problematic in the sense that it is incomplete and needs the notion of a classical device measuring quantum observables as an important ingredient of the theory. Due to this, one accepts that there exist two worlds: the classical one and the quantum one. In the classical world, the measurements of classical observables are produced by classical devices. In the framework of standard theory, in the quantum world the measurements of quantum observables are produced by classical devices, too. Due to this, the theory of quantum measurements is considered as something very specifically different from classical measurements.

It is psychologically accepted that to understand the physical meaning of a measurement in the classical world is much easier than to understand the physical meaning of analogous measurement in the quantum world.

As was pointed out in Refs. [10, 11], all the roots of the difficulties of quantum measurements are present in classical measurements, as well. Using the relations of the quantum states in the standard representation and in the classical one (described by classical distributions), one can conclude that complete information on a quantum state is obtained from purely classical measurements of the position of a particle made by classical devices in each reference frame of an ensemble of classical reference frames, which are scaled and rotated in the classical phase space.

These measurements do not need any quantum language if we know how to produce, in the classical world (using the notion of classical position and momentum), reference frames in the classical phase space differing from each other by rotation and scaling of the axis of the reference frame and how to measure only the position of the particle from the viewpoint of these different reference frames. So, knowing how to obtain the classical marginal distribution function \( w(X, \mu, \nu) \) which depends on the parameters \( \mu; \nu \), labeling each reference frame in the classical phase space, we reconstruct the quantum density operator.

Thus, we avoid the paradox of the quantum world which requires for its explanation measurements by a classical apparatus accepted in the framework of standard treatment of quantum mechanics. But the difficulties of the quantum approach are present, since we need to understand better the procedure of measurement in a rotated reference frame in the phase space of the classical system. The problem of wave function collapse [44, 45] reduces to the problem of a reduction of the probability distribution which occurs as soon as we “pick” a classical value of the classical random observable in
the classical framework of [10, 11]. This means that we “solved” the paradox of the wave function collapse reducing it to the problem of standard measurement of a classical random variable used in the probability theory.

The approach developed in [10, 11] enables one to transform such an unpleasant problem of standard quantum mechanics as the need of a classical device and the reduction of wave packets into the standard problem of classical measurements of classical random variables. In fact, this means that the problem of classical measurements is as difficult as the problem of quantum measurements. An important analogy with methodology of special relativity arises: It turns out that it is necessary to introduce a consideration of events in the set of moving reference frames in space–time in order to explain relativistic effects, and it is necessary to introduce a consideration of events in the set of rotated and scaled reference frames in the phase space in order to explain the nonrelativistic quantum mechanics in terms of only classical concepts of classical fluctuation theory. But these reference frames are the reference frames in the phase space (not in space–time). Possibly, a combination of these two approaches can be generalized to give a classical description of relativistic quantum mechanics.

One can conclude that the stationary states of quantum systems (for example, of a harmonic oscillator) can be obtained using classical-like alternative equations to the Schrödinger equations. A new type of eigenvalue problems for real positive marginal distributions is formulated. The analogs of orthogonality and completeness relations for the wave functions are formulated in terms of conditions for the marginal distributions as well as the transition probabilities among the energy levels. The criterion for determining the pure states of the quantum system is given in terms of the classical marginal distribution.

Thus, using the marginal distribution one can formulate the standard quantum mechanics without the complex wave function and density matrix. But the position distributions in an ensemble of classical reference frames in the phase space play an important role.

It should be pointed out that in the standard formulation of quantum mechanics there exist different representations such as the coordinate representation, the momentum representation, etc. The counterpart of this variety in the classical formulation is related to different tomography schemes like optical tomography [19, 37], symplectic tomography [10, 19], and photon number tomography [10, 46, 47]. The photon number tomography uses the
marginal distribution of a discrete variable, which corresponds to a number representation. Just as different representations in the standard formulation of quantum mechanics are related by some transformations in the Hilbert space of states, the marginal distributions of different tomography schemes can be transformed into each other. This transformation consists of two steps. First, one makes a map of the marginal distribution (in one of the tomography schemes) onto the Wigner function and then one makes another map (of the different tomography scheme) of this Wigner function onto the corresponding marginal distribution.

The construction introduced in spirit is similar to the Moyal approach which considers quantum mechanics as a statistical theory. But in Ref. [48], the quantum state was described by a quasidistribution function in the phase space that is identical to the Wigner function. Thus, the “negative probabilities” to find the system in some domain in the phase space is an unavoidable feature of the Moyal approach. The density matrix was introduced in Ref. [49, 50]. In the framework of the new formulation of quantum mechanics, the density matrix is not mandatory to be used.

In the introduced formulation of quantum mechanics, only positive probabilities of the measurable position in an ensemble of reference frames in the phase space of the system is used. It is remarkable that in the positive probability representation the states in quantum mechanics are described identically with the states in classical statistical mechanics if one uses the positive marginal distribution $w(X, \mu, \nu)$ (though the sets of the distributions in the classical and quantum cases are different).

The difference between classical statistical mechanics and quantum mechanics in the formulation introduced appears in the dynamics of the marginal distributions, since in quantum mechanics the evolution equation for the positive probability distribution has a different form from that in the classical case. It is remarkable that the relations of the propagators (conditional probabilities) in the phase space representation and in the probability representation are described by the same formula both in classical statistical mechanics and in quantum mechanics.

For linear systems (oscillators), the propagators in classical statistical mechanics and in quantum mechanics coincide. The difference for these systems in the quantum and classical cases is due to the fact that not all positive probability distributions $w(X, \mu, \nu)$ are realized for classical systems. Also not all marginal distributions are admissible in the quantum case, but only
which satisfy uncertainty relations.

We have demonstrated that spin states and states of a trapped ion [51] can be described by measurable positive probability distributions. This implies that quantum-mechanical systems can be considered in the framework of the same formalism of probability theory as classical statistical systems. Thus, the known results of classical probability theory can be applied to the study of quantum states. For example, the central limit theorem can be used for describing multimode systems. The approach developed can be elaborated for solving many problems of quantum optics [31, 52, 53] and quantum computing. It is important to study the Schrödinger uncertainty relation [54] in the framework of the new approach. Linear integrals of motion for quadratic systems [55, 56] are useful to obtain the propagator of the new evolution equation for the marginal distribution [57]. The new approach can be also applied to study nonlinear coherent states [58, 59, 60].

We have shown that quantum mechanics can be formulated without wave function and density matrix using the tomographic probability representation. The general approach to the tomographic map and relations among different tomography schemes are discussed in Refs. [61, 62]. The tomography of spin states for two particles is described in Ref. [63]. A review of the new approach to quantum mechanics is given in Ref. [64]. The generalization of the metod to the case of the field theory one can find in Ref. [65]. One can conclude that the problem of formulation of quantum theory using only probabilities both for continuous and discrete observables has the solution in the framework of the tomographic probability representation of quantum mechanics.

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