Geometric inequalities on Closed Surfaces with
\[ \lambda_1(-\Delta + \beta K) \geq 0 \]

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Abstract

Let \( \Sigma \) be a closed orientable surface satisfying the eigenvalue condition \( \lambda_1(-\Delta + \beta K) \geq \lambda \geq 0 \), where \( \beta \) is a positive constant and \( K \) is the Gaussian curvature of \( \Sigma \). This eigenvalue condition naturally arises for stable minimal surfaces and \( \mu \)-bubbles in 3-manifolds with positive scalar curvature (with \( \beta = 1 \) there). We prove that when \( \beta > 1/2 \), the Cheeger constant of \( \Sigma \) is bounded from below by \( C(\beta)/\text{diam}(\Sigma) \). When \( \beta > 1/4 \), the homogeneous isoperimetric ratio of \( \Sigma \) is bounded from below by \( C(\beta, \epsilon)\left(\frac{\text{Area}(\Sigma)}{\text{diam}(\Sigma)^2}\right)^{1/4}\beta^{-1/2} (\forall \epsilon > 0) \). We also show a weak Bonnet-Myers’ theorem \( \text{diam}(\Sigma) \leq C(\beta)\lambda^{-1/2} \) and a total volume comparison theorem \( \text{Area}(\Sigma) \leq C(\beta)\text{diam}(\Sigma)^2 \), when \( \beta > 1/4 \). Generalizing Schoen-Yau’s observation, associated to each positive supersolution \( \Delta \varphi \leq \beta K\varphi \) we make a conformal change \( \tilde{g} = \varphi^{2/\beta} g \) with \( K_{\tilde{g}} \geq 0 \). Some of the main theorems are proved by studying the geometric inequalities for \( \tilde{g} \) and transferring them back to \( g \).

1 Introduction

In this article, we study closed orientable surfaces that satisfy the eigenvalue condition

\[ \lambda_1(-\Delta + \beta K) \geq \lambda \geq 0, \tag{1.1} \]

or equivalently

\[ \int_{\Sigma} \left( |\nabla \varphi|^2 + \beta K \varphi^2 \right) dA \geq \lambda \int_{\Sigma} \varphi^2 dA, \quad \forall \varphi \in C^\infty(\Sigma), \tag{1.2} \]

or

\[ \text{there exists } \varphi > 0 \text{ such that } \Delta \varphi \leq (\beta K - \lambda)\varphi, \tag{1.3} \]

where \( K \) denotes the Gaussian curvature of \( \Sigma \), and \( \beta > 0 \) is a fixed number. By setting \( \varphi = 1 \) in (1.2), we obtain \( 2\pi \beta \chi(\Sigma) \geq \lambda|\Sigma| \), hence \( \Sigma \) is topologically either a sphere or a torus. In fact, \( \Sigma \) is either topologically a sphere or a flat torus (see Lemma 4.1). As flat tori are geometrically less interesting, we assume for the rest of this article that \( \Sigma \) has the topology of a sphere.

One motivation for Condition (1.1) is the study of positive scalar curvature (PSC) and general relativity. Let \( \Sigma \) be a stable minimal surface in a 3-dimensional manifold \( M \) with scalar curvature \( R_M \geq R_0 \geq 0 \). The second variation formula of volume gives

\[ \int_{\Sigma} \left( |\nabla \varphi|^2 + \frac{1}{2}(2K_{\Sigma} - R_M - |h|^2)\varphi^2 \right) dA_{\Sigma} \geq 0, \]
and implies (1.2) with $\beta = 1$ and $\lambda \geq R_0/2$. The study of stable minimal surfaces is a key step for many important results in positive scalar curvature, such as the Geroch conjecture [26] and the positive mass theorem [27]. As black hole horizons can be mathematically described as minimal surfaces, condition (1.1) is also related to topics in general relativity, such as the Bartnik mass [19, 21]. For background studies in positive scalar curvature and relativity, we refer the reader to the book of Lee [17]. Many new results in metrics with positive scalar curvature are recently proved using Gromov’s $\mu$-bubbles (see [6, 12, 13, 14, 18]), and we note that a stable $\mu$-bubble in a 3-manifold with uniformly positive scalar curvature also satisfy (1.1), with $\beta = 1$, $\lambda > 0$.

We briefly list some known results in the study of condition (1.1). The moduli space of metrics satisfying $\lambda_1(-\Delta + K) \geq 0$ is path-connected [21]. Based on this observation, Mantoulidis-Schoen [21] proves that the Bartnik mass of a black hole $\Sigma$ satisfying $\lambda_1(-\Delta + \beta R) \geq 0$ where $R$ is the scalar curvature (see Li-Mantoulidis [19] and references therein). Non-compact surfaces with $\lambda_1(-\Delta + \beta K) \geq 0$ are studied in [9] by Fischer-Colbrie and Schoen, and also in Gromov-Lawson [15].

It is tempting to understand (1.1) as a global positivity condition on curvature. Obviously, the pointwise condition $K \geq \beta^{-1}\lambda$ implies (1.1). However, the converse is not true. A construction in [21] shows that we can have $\int K_- dA$ arbitrarily large (where $K_-$ is the negative part of Gauss curvature) while having $\lambda_1(-\Delta + K) \geq 0$ satisfied simultaneously. The construction in [21] is local in nature: a large amount of negative curvature can be created by distorting the metric at small scales. Also, note that (1.1) is stronger for greater $\beta$: if $\beta' < \beta$ and $\Delta \varphi \leq \beta K \varphi$ ($\varphi > 0$), then $\Delta \varphi^\beta/\beta \leq \beta' K \varphi^\beta/\beta$.

It is then natural to ask to what extent does (1.1) reveal geometric characteristic of positive curvature. We approach this question by studying geometric inequalities on surfaces satisfying (1.1). In particular, we study isoperimetric inequalities, Bonnet-Myer theorems, and volume comparison theorems.

**Definition 1.1.** Given a closed Riemannian manifold $M$ of dimension $n$, the (Neumann) isoperimetric ratio is defined as

$$\text{IN}(M) := \inf_{\Omega \subset M} \frac{|\partial \Omega|^n}{\min \{|\Omega|, |\Omega^c|\}^{n-1}}.$$ 

The Cheeger constant is defined as

$$\text{Ch}(M) := \inf_{\Omega \subset M} \frac{|\partial \Omega|}{\min \{|\Omega|, |\Omega^c|\}}.$$ 

For a manifold with non-empty boundary, the (Dirichlet) isoperimetric ratio is defined as

$$\text{ID}(M) := \inf_{\Omega \subset M} \frac{|\partial \Omega|^n}{|\Omega|^{n-1}}.$$ 

In analytic aspects, it is known that the isoperimetric ratio (resp. Cheeger constant) is related to the optimal constant in the Sobolev inequality (resp. Poincaré inequality).

A curvature lower bound usually gives a certain version of isoperimetric inequality. On a closed surface with $K \geq 0$, we have for example Burago-Zalgaller’s inequality $\text{Ch}(\Sigma) \geq \frac{1}{\text{diam}(\Sigma)}$ [1]. In higher dimensions, given Ricci curvature lower bound $\text{Ric} \geq -K_0$ we have...
Berger-Kazdan-Croke and Buser’s isoperimetric inequality \( \text{Ch}(M) \geq C(n, K_0, \text{diam}(M)) \) \cite{2, 5, 7}, and Li-Yau’s first eigenvalue estimate \( \lambda_1(-\Delta) \geq C(n, K, \text{diam}(M)) \) \cite{20}. In the case of positive Ricci curvature \( \text{Ric} \geq K_0 \), we have Levy-Gromov’s comparison inequality \cite{11, Appendix C} and Lichnerowicz’s inequality \( \lambda_1(\Delta) \geq nK_0 n^{-1} \) (obtained from Bochner’s formula). One also obtains isoperimetric inequalities under curvature lower bounds in integral sense, see Gallot \cite{10}, Dai-Wei-Zhang \cite{8}.

It turns out that condition (1.1) does reveal properties of positive curvature (from geometric inequalities point of view), but subject to certain bounds for \( \beta \). When \( \beta > \frac{1}{2} \) we obtain almost as strong isoperimetric inequality as Burago-Zalgaller’s. For \( \frac{1}{2} \geq \beta > \frac{1}{4} \) we obtain a weaker form of isoperimetric inequality, while for \( \beta \leq \frac{1}{4} \) we have no control.

**Theorem 1.2.** Let \( \Sigma \) be a closed surface that satisfies \( \lambda_1(-\Delta + \beta K) \geq 0 \).

(1) When \( \beta > \frac{1}{2} \), we have

\[
\text{Ch}(\Sigma) \geq \frac{2\beta - 1}{2\beta} \frac{1}{\text{diam}(\Sigma)}. \quad (1.4)
\]

Moreover, we have the following local result: let \( \gamma \) be a closed loop in \( \Sigma \), and \( N_\rho \) be its collar neighborhood of distance \( \rho \). Assume \( N_\rho \) is compact. If the first Dirichlet eigenvalue bound \( \lambda_D^\rho(-\Delta + \beta K) \geq 0 \) is satisfied on \( N_\rho \), then we have

\[
|N_\rho| \leq \frac{4\beta}{2\beta - 1} \rho |\gamma|.
\]

(2) When \( \frac{1}{2} \geq \beta > \frac{1}{4} \), for any \( \epsilon > 0 \) we have

\[
\text{IN}(\Sigma) \geq C(\beta, \epsilon) \left( \frac{|\Sigma|}{\text{diam}(\Sigma)^2} \right)^{\frac{1}{1-\epsilon} + \frac{\epsilon}{4\beta - 1}}. \quad (1.5)
\]

The result of (1) is stronger than (2): knowing the information in the neighborhood \( N_\rho \) gives an isoperimetric inequality in \( N_\rho \). In contrast, (2) is a global result, and we need the closedness of \( \Sigma \) in an essential way.

Theorem 1.2 can be proved by a rather direct method (another indirect proof is introduced later). Given a domain \( \Omega \), the desired inequalities are obtained by testing (1.2) with functions that depend only on \( d(-, \partial \Omega) \). For (1) we try functions with linear decay, while for (2) we try functions with polynomial decay. We need to consider the possible singular behavior of the equi-distance set to \( \partial \Omega \), for which an effective argument was given in Gromov-Lawson \cite{15}.

We also have a weaker version of Bonnet-Myers’ theorem (in the case of uniformly positive curvature) and volume comparison.

**Theorem 1.3** (weak Bonnet-Myers’ Theorem). When \( \beta > \frac{1}{4} \) and \( \lambda > 0 \), for a complete surface \( \Sigma \) satisfying \( \lambda_1(-\Delta + \beta K) \geq \lambda \) we have

\[
\text{diam}(\Sigma) \leq \frac{2\pi \beta}{\sqrt{\lambda(4\beta - 1)}}. \quad (1.6)
\]

In particular, such surface must be compact.
The idea for proving Theorem 1.3 is already present in the literature [6, 14, 28, 31]. For the sake of completeness, as well as to reveal the critical bound \( \beta > \frac{1}{4} \), we present its proof in Appendix A.

**Theorem 1.4** (total volume comparison). For \( \beta > \frac{1}{4} \), let \( \Sigma \) be a closed surface that satisfies \( \lambda_1(-\Delta + \beta K) \geq 0 \). We have \(|\Sigma| \leq C(\beta) \text{diam}(\Sigma)^2\).

Theorem 1.4 is proved in Section 4 (see Theorem 4.10). Note that we only bound the total area in terms of the total diameter, under the condition that \( \Sigma \) is closed.

All the bounds for \( \beta \) (i.e., \( \frac{1}{4} \) and \( \frac{1}{2} \)) that appear in the main theorems are critical. In Appendix B we will discuss counterexamples when \( \beta \) is not within the required bounds, and explicitly construct a counterexample to Theorem 1.2(2) when \( \beta < \frac{1}{4} \).

Finally, we introduce a conformal warping technique, which is the main novelty of this paper. Observe that if \( \varphi > 0 \) satisfies \( \Delta \varphi \leq \beta K \varphi \), and we let \( u = \frac{1}{\beta} \log \varphi \), then the Gaussian curvature of the new metric \( \tilde{g} = e^{2u} g \) is computed to be

\[
\tilde{K} \geq \beta |\nabla u|^2.
\] (1.7)

Qualitatively this is analogous to the Schoen-Yau observation that minimal surfaces in PSC manifolds are Yamabe positive. The new idea here is that we can quantitatively study the geometric inequalities for \( \tilde{g} \) and transfer them back to the original metric \( g \). The following anti-Harnack inequality is a simple example. In particular, it applies to any counterexample to Theorem 1.2(1) when \( \beta \leq \frac{1}{2} \).

**Theorem 1.5.** Suppose \( \Sigma \) is a closed surface, and \( \varphi > 0 \) satisfies \( \Delta \varphi \leq \beta K \varphi \). Then

\[
\frac{\max_{\Sigma}(\varphi)}{\min_{\Sigma}(\varphi)} \geq \left[ \text{diam}(\Sigma) \cdot \text{Ch}(\Sigma) \right]^{-\beta/2}.
\]

**Proof.** Suppose \( \max_{\Sigma}(\varphi) = A \min_{\Sigma}(\varphi) \). Applying Burago-Zalgaller’s inequality to the new metric \((\Sigma, \tilde{g})\) defined above, we have

\[
\inf_{u \subset \Sigma} \frac{\int_{\partial \Omega} \varphi^{1/\beta} dl}{\min \left\{ \int_{\Omega} \varphi^{2/\beta} dA, \int_{R} \varphi^{2/\beta} dA \right\}} \geq \frac{1}{\text{diam}(\Sigma, \tilde{g})}.
\]

Note that the left hand side is \( \leq \text{Ch}(\Sigma) A^{2/\beta} (\max_{\Sigma} \varphi)^{-1/\beta} \) while the right hand side is \( \geq (\max_{\Sigma} \varphi)^{-1/\beta} \text{diam}(\Sigma) \) \( \left[ \text{diam}(\Sigma) \cdot \text{Ch}(\Sigma) \right]^{-\beta/2} \). Hence

\[
A^{2/\beta} \geq \frac{1}{\text{Ch}(\Sigma) \cdot \text{diam}(\Sigma)}.
\]

In Section 4 we will prove the following diameter bound for \( \tilde{g} \):

\[
\frac{\text{diam}(\Sigma, \tilde{g})^2}{|\Sigma|_{\tilde{g}}} \leq C(\beta) \left( \frac{\text{diam}(\Sigma, g)^2}{|\Sigma|_g} \right)^{2\beta/\beta - 1}.
\] (1.8)

Note that the bound \( \beta > \frac{1}{4} \) appears in the exponent on the right hand side. This bound is not immediately seen from condition (1.7). Rather, it emerges when we carefully analyze the geometry of \( \tilde{g} \) (see for example (4.20) and its context). One may understand Theorem 1.4 as a consequence of (1.8) in addition with the usual volume comparison theorem.
\[ |\Sigma|_g \leq \text{diam}(\Sigma, g)^2. \] (This is different from how the results are actually proved. We first prove a weaker version of (1.8), then derive the volume comparison theorem, and finally improve (1.8) to its strongest form. See Theorem 4.3, 4.10 and Corollary 4.12 for more details. We remind the reader that we are using different notations in Section 4; see subsection 4.1 for declarations.)

Moreover, we obtain a second proof of Theorem 1.2(2) with weaker constants (see Theorem 4.13). Given the diameter bound (1.8), we can find a piecewise smooth bi-Lipschitz map from \((\Sigma, \tilde{g})\) to the standard sphere, with bounded Lipschitz norms. This allows us to obtain a geometric inequality on \((\Sigma, \tilde{g})\) from a corresponding geometric inequality on the round sphere. To obtain the isoperimetric inequality for \(g (= e^{-2u} \tilde{g})\), the following geometric inequality is what we need.

**Theorem 1.6.** Let \(u\) be a function on the round sphere \((S^2, g_0)\) with \(\int_{S^2} |\nabla_0 u|^2 dA_0 \leq E\), where \(E\) is a constant. Then for any smooth domain \(\Omega\) we have

\[
(\int_{\partial \Omega} e^u d\ell_0)^2 \geq C(E) \min \{ \int_{\Omega} e^{2u} dA_0, \int_{\Omega^c} e^{2u} dA_0 \},
\]

where \(C(E)\) is a constant that depends only on \(E\).

Note that \(\int |\nabla u|^2 d\tilde{A} \leq 4\pi\beta^{-1}\) on \((\Sigma, \tilde{g})\), thus Theorem 1.6 implies a weaker version of Theorem 1.2(2).

This paper is organized as follows. In section 2, we prove Theorem 1.2 using direct construction of test functions. In section 3, we prove several auxiliary results needed in later sections. This includes some known facts in convex geometry, a precise form of the Moser-Trudinger inequality, and the proof of Theorem 1.6. In section 4 we discuss the conformal warping technique introduced above, and prove the related results including the diameter bound (1.8) and the volume comparison theorem 1.4. In Appendix A we give two proofs of Theorem 1.3. In Appendix B we discuss counterexamples to Theorem 1.2 when \(\beta\) does not satisfy the bounds required.

**Notations.** Throughout the paper, we fix the following notations. \(\Sigma\) will always denote a closed surface, and \(K\) denotes its Gaussian curvature. We use \(|\cdot|\) to denote the area or length of an object, whose dimension is understood from the context. We may suppress the volume form when there is no ambiguity. We will use \(N\) to denote the unit normal vector. We assume all surfaces are smooth, connected, closed and orientable (for non-orientable surface, we pass to its double cover). When considering isoperimetric problems, we assume that all domains have smooth boundaries.

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2 Proof of Theorem 1.2 Using Direct Methods

Assume
\[ \int_{\Sigma} \left( |\nabla \varphi|^2 + \beta K \varphi^2 \right) dA \geq 0 \quad (\forall \varphi \in C^\infty) \tag{2.1} \]
on a closed surface $\Sigma$. If we assume radial symmetry, i.e. $g = dr^2 + f(r)^2 d\theta^2$ (0 $\leq r \leq D$) and $\varphi = \varphi(r)$, then we would obtain
\[ \int_0^D \left( f(\varphi')^2 - \beta f''(\varphi^2) \right) dx \geq 0. \tag{2.2} \]

Testing this condition with appropriate functions yields the desired isoperimetric inequality. Without radial symmetry, we test (2.1) with functions of the form $\varphi = \phi(d)$, where $d = d(-, \partial \Omega)$ is the distance function to the boundary. We eventually obtain equation (2.7), which is essentially the same as (2.2). Note that the level set of $d$ may not be smooth. The argument for resolving this issue is due to Gromov-Lawson [15, Proposition 8.11], where they proved that a non-compact surface satisfying (2.1) has infinite area when $\beta > \frac{1}{4}$.

Suppose $\Sigma = \Omega^+ \cup \Omega^-$, with common smooth boundary $\gamma = \partial \Omega^+ = \partial \Omega^-$. We note that it suffices to prove Theorem 1.2 when $\Omega^-$ is connected. Suppose $\Omega^- = \Omega_1 \cup \Omega_2$ as disjoint union. If both $|\Omega_1|$ and $|\Omega_2|$ are less than $\frac{2}{3}|\Sigma|$, then by induction on the number of connected components, we have $|\partial \Omega_i| \geq |\Omega_i|^* C_{iso}^*$, where $C_{iso} = \text{IN}(\Sigma)$ or $\text{Ch}(\Sigma)$, and $s = 1/2$ or 1. Hence $|\partial \Omega| \geq |\Omega|^* C_{iso}^*$, which proves the isoperimetric inequality for $\Omega$. If $|\Omega_i| \geq \frac{1}{2}|\Sigma|$, we have $|\partial \Omega| \geq |\partial \Omega_i| \geq C_{iso}^* \Sigma \setminus \Omega_i| \geq C_{iso}^* \Sigma \setminus \Omega|^*$.

Therefore, we assume that both $\Omega^+$ and $\Omega^-$ are connected, thus $\gamma$ is connected (since we have assumed that $\Sigma$ is topologically $S^2$). Define the signed distance function
\[
d(x) := \begin{cases} 
  d(x, \gamma) & (x \in \Omega^+), \\
  -d(x, \gamma) & (x \in \Omega^-). 
\end{cases}
\]

We may perturb the metric so that the level sets of $d$ are piecewise smooth almost everywhere. Let $-\rho^- := \min(d)$, $\rho^+ := \max(d)$. We may further perturb the metric so that the maximum and minimum of $d$ are attained at only one point. For $s \in [-\rho^-, \rho^+]$, define the following quantities:
\[
\gamma(s) := \{d = s\}, \quad L(s) := |\gamma(s)|, \quad \Omega(s) := \begin{cases} 
  \{0 \leq d \leq s\} & (s \geq 0), \\
  \{s \leq d \leq 0\} & (s \leq 0), 
\end{cases} \\
\chi(s) := \chi(\Omega(s)), \quad K(s) := \int_{\gamma(s)} K dl, \quad \Gamma(s) := \text{total geodesic curvature of } \gamma(s).
\]

where the total geodesic curvature is understood as the integral of geodesic curvature at smooth part plus all the exterior angles at vertices. We assume the convention that the normal vector of $\gamma(s)$ is pointing inside $\{d \geq s\}$. Under this convention, by Gauss-Bonnet’s theorem we have
\[
K(s) = \begin{cases} 
  \frac{d}{ds} \left[ 2\pi \chi(s) - \Gamma(s) \right] & (s \geq 0), \\
  \frac{d}{ds} \left[ -2\pi \chi(s) - \Gamma(s) \right] & (s \leq 0), 
\end{cases} \tag{2.3}
\]
almost everywhere.

Let \( \phi : [\rho^-, \rho^+] \to \mathbb{R} \) be a function, smooth except at \( s = 0 \), such that \( \phi \geq 0 \), \( \phi' \leq 0 (\forall s > 0) \) and \( \phi' \geq 0 (\forall s < 0) \). Testing Condition (2.1) with function \( \phi(d(x)) \), we have

\[
0 \leq \int_{\rho^-}^{\rho^+} \left[ L(s)\phi'(s)^2 + \beta K(s)\phi(s)^2 \right] ds. \tag{2.4}
\]

From (2.3) and integration by part, we obtain

\[
\int_{-\rho}^{\rho} K\phi^2 \, ds = \int_{-\rho}^{\rho} \Gamma(\phi^2)' \, ds + 4\pi \int_{-\rho}^{0} \chi\phi\phi' \, ds - 4\pi \int_{0}^{\rho} \chi\phi\phi' \, ds + 2\pi \chi(\rho^+ - \epsilon) - \Gamma(\rho^+ - \epsilon) \phi(\rho^+ - \epsilon)^2
\]

\[
\leq \int_{-\rho}^{\rho} \Gamma(\phi^2)' \, ds + 2\pi \phi(\rho^-)^2 + 2\pi \phi(\rho^+)^2, \tag{2.5}
\]

where we use the fact that \( \chi(s) \leq 0 \), since \( \Omega(s) \) is connected and has more than one boundary components. We would like to do another integration by part for the term \( \Gamma(\phi^2)' \). Note that \( L'(s) = \Gamma(s) \) when \( \gamma(s) \) is smooth. When \( s > 0 \), \( \gamma \) cannot have positive angle at any vertex since it is an equidistance set to a smooth curve. At a vertex of negative angle \( -\theta < 0 \), the derivative of length is contributed by \( -2\tan \frac{\theta}{2} \) while \( \Gamma(s) \) is contributed by \( -\theta \). Hence \( L'(s) \leq \Gamma(s) \). When \( s < 0 \) we analogously have \( L' \geq \Gamma \). Hence

\[
\int_{-\rho}^{\rho} \Gamma(\phi^2)' \, ds = 2 \int_{-\rho}^{\rho} \Gamma\phi' \, ds \leq 2 \int_{-\rho}^{\rho} L'\phi' \, ds
\]

\[
= 2L(0)\phi(0)\left[ \phi_-(0) - \phi_+(0) \right] - 2 \int_{-\rho}^{\rho} L(\phi'' + (\phi')^2) \, ds. \tag{2.6}
\]

Combining (2.4) and (2.6), we obtain

\[
(2\beta - 1) \int_{-\rho}^{\rho} L(\phi')^2 \, ds + 2\beta \int_{-\rho}^{\rho} L\phi\phi'' \, ds \leq 2\beta |\gamma| \phi(0)\left[ \phi_-(0) - \phi_+(0) \right] + 2\beta \left[ \phi(\rho^+)^2 + \phi(\rho^-)^2 \right]. \tag{2.7}
\]

We are now ready to prove Theorem 1.2. To prove (1), we test equation (2.7) with

\[
\phi(s) = \begin{cases} 
\frac{(\rho^- + s)/\rho^-}{(\rho^+ - s)/\rho^+} & (s \leq 0), \\
\frac{(\rho^- - s)/\rho^-}{(\rho^+ + s)/\rho^+} & (s \geq 0).
\end{cases}
\]

We obtain

\[
(2\beta - 1) \left( \frac{|\Omega^-|}{(\rho^-)^2} + \frac{|\Omega^+|}{(\rho^+)^2} \right) \leq 2\beta |\gamma| \left( \frac{1}{\rho^-} + \frac{1}{\rho^+} \right). \tag{2.8}
\]

We may assume \( \rho^- \leq \rho^+ \). The first part of Theorem 1.2(1) now follows from

\[
|\Omega^-| \leq \frac{4\beta}{2\beta - 1} \rho^- |\gamma| \leq \frac{2\beta}{2\beta - 1} |\gamma| \text{diam}(\Sigma).
\]
For the second part, we choose \( \rho^+ = \rho^- = \rho \), and note that \( \phi \) is indeed a test function for the Dirichlet eigenvalue condition (i.e. \( \phi \) vanishes on the boundary \( \partial N_\rho \)).

To prove Theorem 1.2(2), we choose a test function of polynomial decay:

\[
\phi(s) = \begin{cases} 
(1 + \sigma)s^p 
& (s \leq 0), \\
\frac{(1 + \sigma)s^p}{(1 + \sigma)p} 
& (s > 0), 
\end{cases}
\]

where the coefficients \( \sigma > 0, p > 1 \) are to be chosen later. Equation (2.7) now gives

\[
I^- + I^+ \leq 4\pi \beta \frac{\sigma^p}{(1 + \sigma)^{2p}} + \frac{2p\beta}{1 + \sigma} |\gamma| \left( \frac{1}{\rho^-} + \frac{1}{\rho^+} \right), 
\quad \text{(2.9)}
\]

where

\[
I^- := \left[ (4\beta - 1)p^2 - 2\beta p \right] \frac{1}{(1 + \sigma)^{2p}} \int_{-\rho^-}^0 \left[ (1 + \sigma)p - s \right]^{2p-2} ds, 
\quad \text{(2.10)}
\]

and \( I^+ \) is defined analogously. The coefficient \( (4\beta - 1)p^2 - 2\beta p \) must be positive, thus \( p > \frac{2\beta}{4\beta - 1} \). Hence we choose \( p = \frac{2\beta + \epsilon}{4\beta - 1}, \epsilon > 0 \). Now (2.9) (2.10) implies

\[
C_1 \frac{\sigma^2}{(1 + \sigma)^2} \left[ \frac{1}{(\rho^-)^2} + \frac{1}{(\rho^+)^2} \right] \leq 4\pi \beta \frac{\sigma^p}{(1 + \sigma)^{2p}} + \frac{2p\beta}{1 + \sigma} |\gamma| \left( \frac{1}{\rho^-} + \frac{1}{\rho^+} \right),
\]

where \( C_1 = C_1(\beta, \epsilon) \) is a constant depending only on \( \beta, \epsilon \). For convenience, we denote \( Z := \frac{\Omega^-}{(\rho^-)^2} + \frac{\Omega^+}{(\rho^+)^2} \). Choose \( \sigma = \sqrt{\frac{C_1}{8\pi \beta}} Z \). For this choice we have

\[
\frac{2p\beta}{1 + \sigma} |\gamma| \left( \frac{1}{\rho^-} + \frac{1}{\rho^+} \right) \geq \frac{C_1}{2} \frac{\sigma^2}{(1 + \sigma)^{2p}} Z.
\]

Without loss of generality, assume \( \rho^- \leq \rho^+ \). Then we have

\[
\frac{|\gamma|}{\rho^-} \geq C(\beta, \epsilon) \frac{\sigma^{2p-1}}{(1 + \sigma)^{2p-1}} \sqrt{Z} \geq C(\beta, \epsilon) \min \left( 1, \frac{\sigma^{2p-1}}{\rho^-} \right) \sqrt{\Omega^-},
\]

or equivalently,

\[
\frac{|\gamma|^2}{|\Omega^-|} \geq C(\beta, \epsilon) \min \left( 1, \frac{\sqrt{Z^{2p-1}}}{\min(1, \frac{|\Sigma|}{\text{diam}(\Sigma)^2})^{2p}} \right).
\]

Combined with total volume comparison (Theorem 1.4), we obtain the result in Theorem 1.2(2).

3 Auxiliary Lemmas

3.1 Collection of facts in convex geometry

Lemma 3.1. (1) (Weyl’s embedding theorem) Given any metric \((S^2, g)\) with curvature \(K > 0\), there exists an isometric embedding into \(\mathbb{R}^3\), unique up to rigid motion. The image of the embedding is the boundary of a strictly convex set.
(2) Suppose \( \Omega_1 \subset \Omega_2 \) are two domains in \( \mathbb{R}^n \), with \( \Omega_1 \) smooth convex and \( \Omega_2 \) piecewise smooth. Then the orthogonal projection from \( \partial \Omega_2 \) to \( \partial \Omega_1 \) is 1-Lipschitz.

(3) For a closed surface \( \Sigma \) with curvature \( K \geq 0 \), we have \( \text{IN}(\Sigma) \geq |\Sigma|/\text{diam}(\Sigma)^2 \).

Proof. (1) is a classical theorem, see Nirenberg [23] and Pogorelov [25].

(2) is proved by computing the differential of the projection map. Let \( \Phi : \partial \Omega_2 \rightarrow \partial \Omega_1 \) be the (well-defined) orthogonal projection map. Denote by \( N, A \) the outer unit normal vector and second fundamental form of \( \partial \Omega_1 \). Let \( h(x) = d(x, \Phi(x)) \geq 0 \).

For any \( x \in \partial \Omega_2 \) we have \( x - \Phi(x) = h(x)N(\Phi(x)) \). Differentiating this identity at a tangent vector \( v \), we obtain \( v - d\Phi_x(v) = dh_x(v)N(\Phi(x)) + h(x)dN_{\Phi(x)}(d\Phi_x(v)) \).

Taking inner product with \( d\Phi_x(v) \) we obtain \( |v| \cdot |d\Phi_x(v)| - |d\Phi_x(v)|^2 \geq (v - d\Phi_x(v)) \cdot d\Phi_x(v) = h(x)A(d\Phi_x(v), d\Phi_x(v)) \geq 0 \). This shows \( d\Phi \) is non-expanding.

(3) Let \( \Omega \) be any domain. An inequality of Burago-Zalgaller (see [24, (4.26)]) states that \( \rho |\partial \Omega| \geq |\Omega| + (\pi - \frac{1}{2} \int_{\Omega} K) \rho^2 \), where \( \rho \) is the radius of the largest metric ball contained in \( \Omega \). By possibly switching between \( \Omega \) and \( \Omega^c \), we may assume \( \pi - \frac{1}{2} \int_{\Omega} K \geq 0 \), hence \( |\Omega| \leq \rho |\partial \Omega| \). By Bishop-Gromov relative volume comparison we obtain \( |\partial \Omega|^2 \geq |\Omega| |\Omega|/\rho^2 \geq |\Omega| \cdot |\Omega|/\text{diam}(\Sigma)^2 \). \( \square \)

3.2 A precise form of the Moser-Trudinger’s inequality

The classical Moser-Trudinger Sobolev inequality [22] states that for a domain \( \Omega \subset \mathbb{R}^2 \) and a function \( u \in H_0^1(\Omega) \) we have

\[
\int_{\Omega} \exp \left( \frac{4\pi u^2}{\int_{\Omega} |\nabla u|^2} \right) \leq C|\Omega| \tag{3.1}
\]

for a universal constant \( C \). For general domains in smooth surfaces, (3.1) continue to hold with the same critical exponent \( 4\pi \), while the constant \( C \) depends on the domain under consideration. When the surface has a conic singularity of cone angle \( \theta < 2\pi \), the critical exponent becomes \( 2\theta \) [3, 32].

We need to know precisely how the constants in (3.1) depend on the geometry of the surface. The following estimate is sufficient for our needs in the next section.

**Theorem 3.2.** Let \( (\Omega, g) \) be a smooth domain with non-empty boundary, such that \( \text{ID}(\Omega) \geq \delta \). Then for any function \( u \in H_0^1(\Omega) \) we have

\[
\int_{\Omega} \exp \left( \frac{\delta u^2}{\int_{\Omega} |\nabla u|^2} \right) \leq C(\delta) \cdot |\Omega|, \tag{3.2}
\]

where \( C(\delta) \) means a constant depending on \( \delta \).

**Proof.** We can replace \( u \) by \( |u| \) and therefore assume \( u \geq 0 \). Since (3.2) is scale-invariant, we may assume \( |\Omega| = 1 \). We apply a Pólya-Szegö symmetrization procedure, comparing \( \Omega \) to a model space that has smaller isoperimetric ratio. Let \( (r, \theta) \) be the polar coordinates on a disk \( D^2 \). Equip \( D^2 \) with the cone metric \( g_0 = dr^2 + \epsilon^2 r^2 d\theta^2 \) \( (0 \leq r \leq L, 0 \leq \theta \leq 2\pi) \), where \( \epsilon \) is determined by requiring \( |\partial \Omega|^2 = \delta|\Omega| \) for concentric cones \( \Omega \), and \( L \) is determined such that \( |D^2|_{g_0} = 1 \). Consider the radial function \( u_0(r) \in H_0^1(D^2) \) uniquely determined by \( du_0/dr \leq 0 \) and the condition

\[
\{ y \in D^2 : u_0(y) > t \}_{g_0} = \{ x \in \Sigma : u(x) > t \}_g, \quad \forall t \geq 0.
\]
From isoperimetric ratio comparison we have
\[ \{ y \in D^2 : u_0(y) = t \} \bigg|_{g_0} \leq \{ x \in \Sigma : u(x) = t \} \bigg|_{g}. \]
The standard argument (see for example [29, Section III.1]), using the coarea formula, implies \( \int_{D^2} |\nabla_{g_0} u_0|^2 \, dA_0 \leq \int_{\Omega} |\nabla u|^2 \, dA. \) The desired result follows from the Moser-Trudinger inequality with conic singularity, applied to \((D^2, g_0)\) and \(u_0.\)

\[ \text{Corollary 3.3.} \] For the same conditions as in Theorem 3.2, we have
\[ \int_{\Omega} e^{2u} \leq C(\delta) |\Omega| \exp \left( \frac{1}{\delta} \int_{\Omega} |\nabla u|^2 \right). \]

\[ \text{Proof.} \] Combine Theorem 3.2 with Young’s inequality \( 2u \leq \frac{4u^2}{|\nabla u|^2} + \frac{1}{\delta} \int |\nabla u|^2. \)

Clearly, the same inequality applies to functions that are non-positive on the boundary.

### 3.3 A 2-dimensional weighted isoperimetric inequality

The goal of this subsection is to prove Theorem 1.6. We first prove the Euclidean version, then derive Theorem 1.6 as a consequence.

\[ \text{Theorem 3.4.} \] Let \( u \) be a smooth function on \( \mathbb{R}^2 \) with \( \int_{\mathbb{R}^2} |\nabla u|^2 \leq E. \) Then for any smooth domain \( \Omega \) we have
\[ (\int_{\partial\Omega} e^{u} \, dl)^2 \geq C e^{-E/4\pi} \int_{\Omega} e^{2u} \, dA \]
where \( C \) is a universal constant.

\[ \text{Proof.} \] It suffices to prove the statement for connected domain \( \Omega. \) Since filling in the holes decrease the isoperimetric ratio, it suffices to prove the theorem for simply-connected domains.

Let \( v \) be the harmonic function on \( \Omega \) such that \( v|_{\partial\Omega} = u|_{\partial\Omega}. \) Let \( u_1 = u - v. \) The new metric \( \tilde{g} = e^{2v} g \) is also flat. Since \( \Omega \) is simply-connected, the flat domain \((\Omega, \tilde{g})\) has a globally defined injective development map \( \Omega \hookrightarrow \mathbb{R}^2. \) Therefore, the Euclidean isoperimetric inequality holds for \((\Omega, \tilde{g})\).

Doing integration by part we can show that
\[ \int_{\Omega} |\tilde{\nabla} u_1|^2 \, d\tilde{A} = \int_{\Omega} |\nabla u_1|^2 \, dA = \int_{\Omega} (|\nabla u|^2 - |\nabla v|^2) \, dA \leq E. \]

Applying the Moser-Trudinger inequality to \( u_1 \in H^1_0(\Omega), \) as well as the planar isoperimetric inequality to \((\Omega, \tilde{g}),\) we obtain
\[ \int_{\Omega} e^{2u} \, dA = \int_{\Omega} e^{2u_1} \, d\tilde{A} \leq C |\Omega| \tilde{g} e^{E/4\pi} \leq C |\partial\Omega|^2 \tilde{g} e^{E/4\pi} = C e^{E/4\pi} (\int_{\partial\Omega} e^{u} \, dl)^2. \]
Proof of Theorem 1.6. For convenience, we denote $\gamma = \partial \Omega$, $\Omega_1 = \Omega$, and $\Omega_2 = \Omega^c$. If one of the domains $\Omega_i$, say $\Omega_1$, is contained in one hemisphere, then we consider the stereographic projection $f : \Omega_1 \to f(\Omega_1) \subset \mathbb{R}^2$ from the pole of the opposite hemisphere of $\Omega_1$. Let $\Omega' = f(\Omega_1)$ and $u' = u \circ f^{-1}$. By conformal invariance, $u'$ has the same energy as $u$. Since $f$ is uniformly bi-Lipschitz, we have $\int_{\Omega_1} e^{2u} \leq C \int_{\Omega'} e^{2u'}$ and $\int_{\partial \Omega_1} e^u \geq C' \int_{\partial \Omega'} e^{u'}$. The desired result then follows from Theorem 3.4.

Now we assume that $\gamma$ is not contained in any hemisphere. Equivalently, every equator must intersect with $\gamma$. For $p \in S^2$ we denote $p^\perp = \{ q \in S^2 : p \perp q \}$, where we regard $p, q$ as unit vectors in $\mathbb{R}^3$. Recall Crofton's counting formula:

$$\int_\gamma e^u \, dl = \frac{1}{4} \int_{S^2} \left( \sum_{q \in \gamma \cap p^\perp} e^{u(q)} \right) dp.$$ (3.3)

Applying coarea formula to the projection map from $\{ p \perp q \} \subset S^2 \times S^2$ onto both factors, we obtain the following identity:

$$E = \int_{S^2} |\nabla u|^2 \, dA = \frac{1}{2\pi} \int_{S^2} dp \int_{p^\perp} |\nabla u|^2 \, dl.$$ (3.4)

Denote by $\tilde{L} = \int_\gamma e^u \, dl$ the weighted boundary length. By (3.3) and (3.4), there is a positive measure of points $p \in S^2$ such that $\int_{p^\perp} |\nabla u|^2 \, dl \leq 6\pi E$, and that there exists a point $q \in \gamma \cap p^\perp$ with $e^{u(q)} \leq 12\tilde{L}$. For any such point $p$, these conditions combine to give

$$e^b := \max_{p^\perp} e^u \leq Ce^{C'\sqrt{E}\tilde{L}}.$$ (4.1)

Applying the Moser-Trudinger inequality individually on the two hemispheres separated by $p^\perp$, we obtain

$$\int_{S^2} e^{2u} \leq e^{2b} \int_{S^2} e^{\max\{u-b,0\}} \leq Ce^{C'\sqrt{E}\tilde{L}^2} \cdot Ce^{E/4\pi} = C(E)\tilde{L}^2,$$

which implies the statement of the theorem. □

4 A Conformal Warping Technique

4.1 Preparation of data

New convention on notations. To make the expressions more concise in this section, we use different notations than in other sections. We use $(\Sigma, \overline{g})$ to denote the original surface, i.e., a sphere with $\lambda_1(-\Delta + \beta K) \geq 0$. We denote

$$\overline{\text{diam}} := \text{diam}(\Sigma, \overline{g}), \quad \overline{\text{Area}} = \text{Area}(\Sigma, \overline{g}).$$ (4.1)

The notation $\overline{g}$ is reserved to denote the conformally changed metric to be defined. We will be analyzing objects in the Euclidean space $\mathbb{R}^3$, where we use dot product to denote the inner product of spatial vectors.
Let \( \varphi = e^{\beta u} \) satisfy (1.3) on \((\Sigma, \overline{g})\), with \( \beta > \frac{1}{4} \). We have
\[
e^{\beta u}(\beta \Delta u + \beta^2 |\nabla u|^2_{\overline{g}}) = \Delta e^{\beta u} \leq \beta K e^{\beta u}.
\] (4.2)

Consider the conformal change \( g = e^{2u} \overline{g} \). We may normalize \( \varphi \) so that \( |\Sigma|_g = \int_{\Sigma} e^{2u} dA = 1 \). The Gauss curvature of \( g \) is computed to be
\[
K_g = e^{-2u}(\overline{K} - \overline{\Delta} u) \geq e^{-2u}\beta|\nabla u|^2_{\overline{g}} = \beta|\nabla u|^2_g.
\] (4.3)

For technical reasons (e.g. (4.13)), in this section we also assume \( \beta \leq 1 \). When \( \beta > 1 \) we obviously have \( K \geq \beta |\nabla u|^2 \Rightarrow K \geq |\nabla u|^2 \), hence the \( \beta = 1 \) version of all the results applies to \( \beta > 1 \) as well.

Our argument relies on Weyl’s embedding theorem applied to \((\Sigma, g)\), hence we need strictly positive curvature \( K > 0 \). However, we only obtain \( K \geq 0 \) from (4.3). By the following lemma, we can always do a small perturbation and assume that (4.2) is a strict inequality, hence \( K > \beta |\nabla u|^2 \) strictly. In addition, it shows that \( \Sigma \) is either a sphere or a flat torus, as mentioned at the beginning of this paper.

**Lemma 4.1.** Let \((\Sigma, \overline{g})\) be a closed surface with \( \lambda_1(-\Delta + \beta \overline{K}) \geq 0 \). If \( \Sigma \) is not a flat torus, then for any \( \epsilon > 0 \) there exists another metric \( \overline{g}' \) such that \( ||\overline{g} - \overline{g}'||_{C^2(\overline{g})} < \epsilon \) and \( \lambda_1(-\Delta_{\overline{g}'} + \beta \overline{K}) > 0 \).

**Proof.** We can assume \( \lambda_1(-\Delta + \beta \overline{K}) = 0 \). Let \( \varphi \) be the first eigenfunction: \( \Delta \varphi = \beta \overline{K} \varphi \).
Consider a smooth family of metrics \( g(t) = e^{2f(t)} \overline{g}, f(0) = 0 \), with initial variation \( \frac{dg}{dt}|_{t=0} = h\overline{g} \). By [21, Lemma A.1], the first eigenfunctions \( \varphi_t \) of \( -\Delta_{g(t)} + \beta K_{g(t)} \) constitute a smooth family. We may normalize so that \( ||\varphi_t||_{L^2(g_t)} = 1 \). We compute
\[
\frac{d\lambda_1}{dt} \bigg|_{t=0} = \frac{d}{dt} \bigg|_{t=0} \int_{\Sigma} \left( |\nabla_t \varphi_t|^2 + \beta K_t \varphi_t^2 \right) dA_t
= \frac{d}{dt} \bigg|_{t=0} \int_{\Sigma} \left( |\nabla_t \varphi|^2 + \beta K_t \varphi^2 \right) dA_t
= -\beta \int_{\Sigma} \Delta h \cdot \varphi^2 dA
\] (4.4)

If (4.4) is zero for all \( h \), then we have \( \Delta \varphi = 0 \Rightarrow \varphi \) is constant, which implies that \( \Sigma \) is a flat torus. Hence (4.4) is nonzero for some \( \varphi \). We may change the sign of \( \varphi \) to make it positive. Now the perturbed metric \( g(\delta) \) satisfies our requirement for small \( \delta \). \( \square \)

**Conditions 4.2.** To summarize, we have prepared the following data for the rest of this section: a closed surface \((\Sigma, \overline{g})\), a function \( \varphi \) with \( \overline{\Delta} \varphi \leq \beta \overline{K} \varphi \), \( \frac{1}{4} < \beta \leq 1 \), and a conformally warped metric \( g = e^{2u} \overline{g} \) where \( u = \frac{1}{\beta} \log \varphi \). We normalize \( \varphi \) so that \( |\Sigma|_g = 1 \). The Gauss curvature of \( g \) satisfies
\[
K > \beta |\nabla u|^2.
\] (4.5)

**Convention on universal constants.** We denote universal constants by \( C \), and assume implicitly that they depend on \( \beta \). Also, they vary from term to term. We always assume \( 1 \geq \beta > \frac{1}{4} \). All the geometric quantities (integrals, distances, gradients) on \( \Sigma \) are with respect to \( g \), unless marked with \( \overline{g} \) explicitly. We may suppress the length and area forms in the integration when there is no ambiguity.
4.2  Bounding the diameter of \((\Sigma, g)\)

The goal of this subsection is to prove the following weaker diameter upper bound. The strongest estimate, where the “1” term is not present, will be proved in Corollary 4.12.

**Theorem 4.3.** Assume Condition 4.2. We have

\[
\text{diam}(\Sigma) \leq C \left( 1 + \frac{\text{diam}^2}{\text{Area}} \right)^{\frac{1}{\sqrt{2} - 1}}. \tag{4.6}
\]

**Step 1.** Denote \(\text{diam}(\Sigma) =: 2D\). Since we aim at finding an upper bound for \(D\), we may assume \(D \gg 1\). By (4.5) and Weyl’s embedding theorem, we embed \(\Sigma\) isometrically into \(\mathbb{R}^3\) as a convex figure. We identify \(\Sigma\) with its image of embedding. We show that \(\Sigma\) has the shape of a long thin needle when \(D \gg 1\).

Let \(p^\pm\) be a pair of points in \(\Sigma\) with the largest extrinsic distance. After a rigid motion, we may assume \(p^\pm = (0, 0, \pm R_1)\). \tag{4.7}

Clearly \(R_1 \leq D\). The tangent planes at \(p^\pm\) are parallel to the \(xy\)-plane. Let \(Q\) be the projection image of \(\Sigma\) onto the \(xy\)-plane, which is a convex planar domain. Thus, \(\Sigma\) is contained in the cylinder \(Q \times [-R_1, R_1]\). Define

\(R_2 := \text{diam}(Q)\).

Clearly \(R_1 \geq R_2\). By Lemma 3.1(2) we have

\[2D = \text{diam}(\Sigma) \leq \text{diam}(Q \times [-R_1, R_1]) = \sqrt{4R_1^2 + R_2^2},\]

and

\[1 = |\Sigma| \leq |\partial(Q \times [-R_1, R_1])| \leq 4\pi R_1 R_2 + 2\pi R_2^2.\]

Let \(y_1, y_2\) be points in the \(xy\)-plane that realizes the diameter of \(Q\). We make an orthogonal projection from \(\Sigma\) to the plane spanned by \(y_1 y_2\) and the \(z\)-axis. The projection image is convex and has area less than \(|\Sigma|\), hence

\[1 = |\Sigma| \geq R_1 R_2.\]

Combining what we obtained, we have

\[4R_1^2 \leq 4D^2 \leq 4R_1^2 + R_2^2 \leq 4R_1^2 + \frac{1}{R_1^2} \leq \frac{1}{R_1^2} + \frac{1}{R_2^2}.\]

Therefore, as \(D \gg 1\) we obtain

\[R_1 = (1 - o(1))D, \quad C_1 D^{-1} \leq R_2 \leq C_2 D^{-1}, \tag{4.8}\]

where \(o(1)\) is with respect to \(D \to \infty\).

**Step 2.** We have shown that \(\Sigma\) has length \(\sim D\) in the \(z\) direction and has size \(O(1/D)\) in the \(x, y\) direction. The next step is to identify the two tip regions of \(\Sigma\) where the
It is convenient to introduce a cylindrical coordinate \((z, \theta)\) on \(\Sigma\):
\[
\Sigma = \{(h(z, \theta) \cos \theta, h(z, \theta) \sin \theta, z) : 0 \leq \theta \leq 2\pi, -R_1 \leq z \leq R_1\}
\]
where \(h\) is the radius function. Define the radial and angular unit vector field in \(\mathbb{R}^3\):
\[
v_r = (\cos \theta, \sin \theta, 0), \quad v_\theta = (-\sin \theta, \cos \theta, 0).
\]
Denote by \(N\) the outward unit normal vector of \(\Sigma\). It is not hard to compute
\[
N = \frac{v_r - h_z \partial_z - h^{-1} h_\theta v_\theta}{\sqrt{1 + h_z^2 + h^{-2} h_\theta^2}} \tag{4.9}
\]
Since \(\Sigma\) is a convex surface, its intersection with any constant \(\theta\) half-plane is also convex. Therefore \(h_{zz} < 0\). Note that \(h(R_1) = h(-R_1) = 0\), hence
\[
-\frac{h(z, \theta)}{R_1 - z} \leq h_z(z, \theta) \leq \frac{h(z, \theta)}{R_1 + z}. \tag{4.10}
\]
Let \(c_1\) be a small constant (independent of \(D\)) to be determined. Define
\[
\zeta^+ := \min\{z : \text{there exists } p = (h \cos \theta, h \sin \theta, z) \in \Sigma \text{ such that } \frac{h(z, \theta)}{R_1 - z} \geq c_1\} \tag{4.11}
\]
and
\[
\Omega^+ := \{(x, y, z) \in \Sigma : z \geq \zeta^+\}.
\]
Denote
\[
r^+ := R_1 - \zeta^+,
\]
thus \(\Omega^+ = \{z \geq R_1 - r^+\}\). Since \(\max_\Sigma(h) \leq CD^{-1}\), we have \(r^+ \leq CD^{-1}\) and \(\zeta^+ = (1 - o(1))D\). Define \(\zeta^-, \Omega^-\) and \(r^-\) analogously for the negative tip region. Finally, let \(\Omega^m = \Sigma \setminus (\Omega^+ \cup \Omega^-)\). Now we have decomposed \(\Sigma\) into two tip regions \(\Omega^\pm\) and a central cylindrical region \(\Omega^m\).

From (4.9), (4.10), we know that
\[
|h_z| \leq c_1, \quad |N \cdot \partial_z| \leq c_1 \quad \text{on } \Omega^m. \tag{4.12}
\]

**Step 3.** Choose any pair of points \(x^\pm \in \partial \Omega^\pm\). Let \(\eta : [0, L] \to \Sigma, \eta(0) = x^+, \eta(L) = x^-\) be a unit speed minimal geodesic between \(x^\pm\). Without loss of generality, we can assume \(\eta \subset \Omega^m\). Note that
\[
2D \geq L \geq 2R_1 - 2r^+ = 2D(1 - o(1)).
\]
For convenience, for any function \(f\) on \(\Sigma\) we denote \(f(t) = f(\eta(t))\) and \(f'(t) = \frac{\partial f}{\partial \eta'}(\eta(t))\).

The following technical lemma is to be used in coarea formulas.

**Lemma 4.4.** If \(L\) is sufficiently large and \(c_1\) is sufficiently small, then we have \(|\eta'(t) \cdot \partial_z| \geq \frac{1}{4}\) for all \(t\).
Proof. The idea for the proof is that: if $|\eta \cdot \partial_z|$ is small for some $t$, then $\eta$ will spiral around $\Omega^m$ and violate the minimizing assumption.

Suppose $|\eta' \cdot \partial_z| \leq \frac{1}{4}$ at time $t_1$. We may assume $t_1 \leq \frac{1}{4}L$. Let $\tilde{\eta}$ be the orthogonal projection of $\eta$ onto the $xy$-plane. We have $|\tilde{\eta}^\prime|^2 + |\eta' \cdot \partial_z|^2 = 1$, thus $\tilde{\eta}$ is a smooth immersed planar curve near $t_1$. Let $\tilde{N}$ be the outer normal vector of $\tilde{\eta}$ in $\mathbb{R}^2$. Let $\tau$ be the unit speed parametrization of $\tilde{\eta}$. The abbreviated derivative $f'$ always denotes $df/\mathrm{d}\tau$, while $df/d\tau$ will be written down explicitly.

Let $t_2 > t_1$ be the smallest time such that $|\eta^\prime \cdot \partial_z| \geq \frac{3}{4}$. We have $\int_{t_1}^{t_2} |\eta'' \cdot \partial_z| \geq \frac{1}{2}$ (where $\eta''$ is understood as a spatial vector). Since $\eta$ is a geodesic on $\Sigma$, we have $\eta'' = -|\eta''|N$. Combined with (4.12) this gives $|\eta'' \cdot \partial_z| \leq c_1|\eta''|$.

To show that $\eta$ is spiraling, we compute how much angle has the direction of $\eta$ changed. We switch to the unit speed parametrization:

$$\frac{d^2 \tilde{\eta}^\prime}{dt^2} = \frac{dt}{d\tau} \frac{d}{dt} \left( \frac{dt}{d\tau} \frac{d\tilde{\eta}^\prime}{d\tau} \right) = \frac{\tilde{\eta}^\prime}{|\tilde{\eta}|^2} + \frac{1}{|\tilde{\eta}|} \left( \frac{d}{dt} \frac{1}{|\tilde{\eta}|} \right) \tilde{\eta}^\prime.$$ 

where $d\tau/\mathrm{d}t = |	ilde{\eta}| \in [\frac{\sqrt{2}}{4}, 1]$. Therefore,

$$\frac{d^2 \tilde{\eta}^\prime}{dt^2} \cdot \tilde{N} = \frac{\tilde{\eta}'' \cdot \tilde{N}}{|\tilde{\eta}|^2}.$$

To derive a lower bound for this, we decompose $N = (N \cdot \partial_z) \partial_z + N_1$. In $[t_1, t_2]$ we have $|N_1 \cdot \tilde{\eta}| = |(N \cdot \partial_z)(\eta' \cdot \partial_z)| \leq \frac{2}{3} c_1$, hence $(N_1 \cdot \tilde{N})^2 = |N_1|^2 - (N \cdot \tilde{\eta})^2 \geq 1 - \frac{16}{9} c_1^2$, hence $|N_1 - \tilde{N}|^2 \leq 2 - c_2^2 - 2\sqrt{1 - \frac{16}{9} c_1^2}$, which is small when $c_1$ is small. Next, we have

$|\tilde{\eta}'' \cdot \tilde{N}| \geq |\tilde{\eta}'' \cdot N_1| - |N_1 - \tilde{N}||\eta''| \quad \text{and} \quad |\tilde{\eta}| \leq |\eta''| \quad \text{and} \quad |\eta'' \cdot N| \leq |\tilde{\eta}'' \cdot N|_1 + |\eta'' \cdot \partial_z| \cdot |N_1 - \partial_z| \leq |\eta'' \cdot N|_1 + c_1^2 |\eta''|$. These combine to give

$$|\eta'' \cdot \tilde{N}| \geq C(c_1)|\eta''|,$$

where $C(c_1)$ is close to 1 when $c_1$ is small. Now since $\frac{d^2 \tilde{\eta}^\prime}{dt^2} \cdot \tilde{N}$ is nonzero everywhere, $\tilde{\eta}$ is spiraling in one direction. That is, $\frac{d\tilde{\eta}}{d\tau} = e^{i\alpha(\tau)}$ where $\alpha$ is monotone in $t$. We have

$$|\alpha(\tau_2) - \alpha(\tau_1)| = \int_{\tau_1}^{\tau_2} \frac{d^2 \tilde{\eta}}{dt^2} \cdot \tilde{N} \, d\tau \geq C(c_1) \int_{\tau_1}^{\tau_2} |\eta''| \, dt \geq C(c_1)c_1^{-1},$$

where $C(c_1)$ is close to 1 when $c_1$ is small. By choosing $c_1$ sufficiently small, we obtain $\alpha(\tau_2) - \alpha(\tau_1) \geq 4\pi$. Recall that $\theta$ is the angle parameter in the cylindrical coordinate system. We denote $\theta(t) = \theta(\eta(t))$ and ask it be continuous with respect to $t$. We claim that $\frac{d\theta}{dt} \neq 0$ when $t \in [t_1, t_2]$. If $\frac{d\theta}{dt} = 0$ at time $t$, then $\eta'(t)$ is tangent to the constant $\theta$ half-plane. Therefore $|\eta'(t) \cdot \partial_r| = |\frac{d\theta}{dt}(\eta(t))| \leq c_1$. But we also have $|\eta'(t) \cdot \partial_z| \leq \frac{3}{4}$, which leads us to a contradiction if $c_1$ is sufficiently small. Now $\frac{d\theta}{dt} \neq 0$ implies that $\tilde{\eta}''$ is never parallel to $\tilde{\eta}$, equivalently, $\theta(t) - \alpha(t)$ can never attain the value $k\pi$ ($k \in \mathbb{Z}$). By continuity, we can assume $\theta(t) - \alpha(t) \in (k\pi, (k + 1)\pi)$ for all $t \in [t_1, t_2]$.

Since $\alpha(t_2) \geq \alpha(t_1) + 4\pi$, we have $\theta(t_2) \geq \theta(t_1) + 3\pi$. Therefore, there exists $t_3 \in (t_1, t_2)$ such that $\theta(t_3) = \theta(t_2) + 2\pi$, that is, $\eta(t_3)$ has the same $\theta$ coordinate as $\eta(t_1)$. Let $\gamma$ be the constant $\theta$ path from $\eta(t_1)$ to $\eta(t_3)$. Since $\eta' \cdot \partial_z < \frac{3}{4}$, we have $|z(\eta(t_3)) - z(\eta(t_1))| \leq \frac{3}{4}|t_3 - t_1|$. In addition that $|t_2| \leq c_1$ on $\eta$, we obtain $|\gamma| \leq \frac{3}{4} \sqrt{1 + c_1^2}|t_3 - t_1| < |t_3 - t_1|$ if $c_1$ is small. This contradicts the fact that $\eta$ is a minimizing geodesic. \qed
Next, we note that there is an embedded curve $\eta_1$ from $x^+$ to $x^-$, namely the $\overline{\gamma}$-minimal geodesic, such that $\int_{\eta_1} e^{-u} \, dl \leq \text{diam}$. The lemma below transfers this diameter information to $\eta$.

**Lemma 4.5.** We have $\int_{\eta} e^{-u} \, dl \leq C(\text{diam} + \text{Area}^{1/2})$.

**Proof.** By coarea formula we have

$$\int_{\eta} e^{-u} \geq \int_{\zeta} e^{-u(p)} \, d\zeta.$$ 

By Lemma 4.4 and coarea formula we have

$$\int_{\eta} e^{-u} \leq 4 \int_{\zeta} e^{-u(p)} \, d\zeta \leq 4 \int_{\eta_1} e^{-u} + 4 \int_{\zeta} d\zeta \int_{\{z=\zeta\}\cap \Sigma} e^{-u|\nabla u|}$$

$$\leq 4 \text{diam} + 4 \left[ \int_{\zeta} e^{-2u} \right]^{1/2} \left[ \int_{\zeta} |\nabla u|^2 \right]^{1/2}$$

$$\leq 4 \text{diam} + 4 \cdot \text{Area}^{1/2} \cdot \sqrt{4\pi \beta^{-1}},$$

where $p_\zeta$ is the unique intersection point of $\eta$ and the horizontal slice $\{z = \zeta\}$. \qed

By Young’s inequality, we have

$$\int_{\eta} e^{-\sqrt{\beta}u} \leq C(\text{diam} + \text{Area}^{1/2}) \sqrt{\beta} L^{1-\sqrt{\beta}} \quad (0 < \beta < 1). \quad (4.13)$$

Next, we utilize the condition that $\eta$ is a minimal geodesic. The second variation of length gives

$$0 \leq \int_{\eta} [((\varphi')^2 - K \varphi^2)] \, dl \leq \int_{0}^{L} \left[ (\varphi')^2 - \beta (u')^2 \varphi^2 \right] \, dx \quad (4.14)$$

for any $\varphi$ with $\varphi(0) = \varphi(L) = 0$. From this we are able to find a pointwise lower bound for $u$ on $\eta$.

**Lemma 4.6.** Assume (4.14) with any $\beta > 0$. Denote $B = \int_{0}^{L} e^{-\sqrt{\beta}u} \, dx$. We have

$$e^{2\sqrt{\beta}u(r)} \geq \min \left\{ \frac{Lr}{16B^2}, \frac{L(L-r)}{16B^2} \right\}. \quad (4.15)$$

**Proof.** We substitute $\varphi = e^{\sqrt{\beta}u} \psi$ in (4.14), and obtain

$$0 \leq \int_{0}^{L} e^{2\sqrt{\beta}u} [((\psi')^2 + 2\sqrt{\beta}u' \psi \psi')] \, dx. \quad (4.16)$$

Then apply the substitution of variable $d\tilde{x} = e^{-\sqrt{\beta}u(x)} \, dx$, $\tilde{u}(\tilde{x}) = u(x)$, to (4.16). We obtain

$$0 \leq \int_{0}^{B} \left[ e^{\sqrt{\beta}u}(\psi')^2 + 2(e^{\sqrt{\beta}u})' \psi \psi' \right] \, d\tilde{x}, \quad (4.17)$$

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where the derivatives in (4.17) are now with respect to $\tilde{x}$. Note that (4.17) is exactly the same as condition (2.2) with $\beta = 1$ and $f = e^{\sqrt{3}u}$ there. Thus, the pointwise bound (4.15) is just an isoperimetric inequality obtained in the same way as in Section 2.

For $\tilde{r} \in (0, \frac{1}{2}B)$ we test (4.17) with the linear bump function

$$\psi(\tilde{x}) = \begin{cases} \tilde{x} & \tilde{x} \in [0, \tilde{r}] \\ \frac{\tilde{r}B - \tilde{x}}{B - r} & \tilde{x} \in [\tilde{r}, B] \end{cases}$$

and we obtain

$$0 \leq -\int_0^{\tilde{r}} e^{\sqrt{3}\tilde{u}} d\tilde{x} - \frac{\tilde{r}^2}{(B - \tilde{r})^2} \int_\tilde{r}^B e^{\sqrt{3}\tilde{u}} d\tilde{x} + 2\tilde{r}(1 + \frac{\tilde{r}}{B - r})e^{\sqrt{3}\tilde{u}(\tilde{r})}. \quad (4.18)$$

Suppose that $e^{2\sqrt{3}\tilde{u}(\tilde{r})} = e \int_0^{\tilde{r}} e^{\sqrt{3}\tilde{u}} d\tilde{x}$. Inequality (4.18) implies

$$0 \leq -\int_0^{\tilde{r}} e^{\sqrt{3}\tilde{u}} d\tilde{x} - \frac{\tilde{r}^2}{B^2} \int_\tilde{r}^B e^{\sqrt{3}\tilde{u}} d\tilde{x} + 4\tilde{r}e^{\sqrt{3}\tilde{u}(\tilde{r})}. \quad (4.19)$$

Hence $\epsilon^{-1}e^{2\sqrt{3}\tilde{u}(\tilde{r})} = \int_0^{\tilde{r}} e^{\sqrt{3}\tilde{u}} \leq 4\tilde{r}e^{\sqrt{3}\tilde{u}(\tilde{r})} \Rightarrow e^{\sqrt{3}\tilde{u}(\tilde{r})} \leq 4\tilde{r} \epsilon$. By (4.19) again we have

$$16\epsilon \tilde{r}^2 \geq 4\tilde{r} e^{\sqrt{3}\tilde{u}(\tilde{r})} \geq \frac{\tilde{r}^2}{B^2} \int_0^B e^{\sqrt{3}\tilde{u}} d\tilde{x} = \frac{\tilde{r}^2}{B^2} L,$$

implying $\epsilon \geq \frac{L}{16B^2}$. We obtain $e^{2\sqrt{3}\tilde{u}(\tilde{r})} \geq \frac{L}{16B^2} \int_0^{\tilde{r}} e^{\sqrt{3}\tilde{u}} dx, \forall \tilde{r} \leq \frac{1}{2}B$. Changing the variables back to $x$ we obtain

$$e^{2\sqrt{3}u(r)} \geq \frac{Lr}{16B^2}, \quad \forall r \leq r_*( \int_0^{r_*(} e^{-\sqrt{3}u} dx = \frac{1}{2}B).$$

When $r \geq r_*$, we exchange $\tilde{x}$ and $B - \tilde{x}$. The same argument gives $e^{2\sqrt{3}u(r)} \geq \frac{L(L-r)}{16B^2}$. \hfill \square

Combining (4.13) and Lemma 4.6, we obtain

$$e^{-2u(\eta(t))} \leq C(\text{diam}^2 + \text{Area}) L^{1/\sqrt{3} - 2} t^{-1/\sqrt{3}}, \quad 0 < \beta \leq 1, \quad \forall t < \frac{2}{3}L. \quad (4.20)$$

Note the exponent on $L$; this is one place where the bound $\beta > \frac{1}{4}$ appears.

**Step 4.** The next step is to derive an upper bound for $\int_\Sigma e^{-2u} (= \text{Area})$ from condition (4.20). In the case of radial symmetry, this bound is easily obtained by integrating (4.20) with respect to the $z$-coordinate (see the end of this subsection). Without radial symmetry, there can be small curvature bumps where $\Sigma$ is close to a cone, such that $e^{-2u}$ is large. The idea here is to cut $\Sigma$ into pieces for which $e^{-2u}$ is controlled on the boundary, and apply Moser-Trudinger inequality to obtain an estimate of $\int e^{-2u}$ on each piece. Finally, we take summation of the estimates on all pieces. This process resembles a Riemann sum. The argument below applies to the $z \geq 0$ part of the surface, and the $z \leq 0$ part follows analogously.

Denote by

$$\Omega_{\zeta, \zeta'} := \{(x, y, z) \in \Sigma : \zeta \leq z \leq \zeta'\}$$

...
a section of the central cylinder, where \(0 \leq \zeta < \zeta' \leq \zeta^+\). Also, denote by

\[
\gamma_\zeta := \{(x, y, z) \in \Sigma : z = \zeta\}, \\
H(\zeta) := \max_{p \in \gamma(\zeta)} h(p)
\]
a horizontal slice and the maximal radius on a slice, where \(0 \leq \zeta \leq \zeta^+ = R_1 - r^+\).

(Recall that \(h\) is the radius function in the cylindrical coordinate, namely, the distance to the \(z\)-axis.) From (4.8), (4.11) and the fact that \(h\) is convex in \(z\), we have the following inequality:

\[
H(\zeta) \leq \min \left\{ CD^{-1}, c_1(R_1 - \zeta) \right\}, \quad \forall \zeta \in [0, \zeta^+]. \tag{4.21}
\]

We choose a sequence of \(z\) coordinates \(\zeta_0 = R_1, \zeta^+ \geq \zeta_1 > \zeta_2 > \cdots > \zeta_n = 0\) by the following process. \(\zeta_i\) is chosen in \([\zeta^+ - 2r^+, \zeta^+ - r^+]\) such that \(\int_{\gamma_{\zeta_i}} \|\nabla u\|^2 \leq C(r^+)^{-1}\). The existence of \(\zeta_i\) is guaranteed by coarea formula

\[
4\pi\beta^{-1} \geq \int_{\Sigma} \|\nabla u\|^2 \geq \int_{R_1 - 2r^+} \int_{\gamma_{\zeta}} \|\nabla u\|^2.
\]

By (4.21) and Lemma 3.1(2), we have \(|\gamma_{\zeta_i}| \leq Cr^+\), therefore,

\[
\max_{\gamma_{\zeta_1}} (u) - \min_{\gamma_{\zeta_1}} (u) \leq \int_{\gamma_{\zeta_1}} \|\nabla u\| \leq \left( |\gamma_{\zeta_1}| \cdot \int_{\gamma_{\zeta_1}} \|\nabla u\|^2 \right)^{1/2} \leq C.
\]

Combined with (4.20) this gives

\[
e^{-2u} \leq C(\text{diam}^2 + \text{Area}) L^{1/\sqrt{\pi}} - 2 (\zeta^+ - \zeta_1)^{-1/\sqrt{\pi}} \quad \text{on} \quad \gamma_{\zeta_1}, \tag{4.22}
\]

where we note that the time \(t\) for which \(z(\eta(t)) = \zeta_1\) satisfies \(t \geq \zeta^+ - \zeta_1\).

Suppose that we have chosen \(\zeta_1, \zeta_2, \cdots, \zeta_{i-1}\). We choose \(\zeta_i\) so that

\[
\zeta_{i-1} - \zeta_i \in [H(\zeta_{i-1}), 2H(\zeta_{i-1})], \tag{4.23}
\]

and

\[
\int_{\gamma_{\zeta_i}} \|\nabla u\|^2 \leq CH(\zeta_{i-1})^{-1}.
\]

The purpose of the first condition is to guarantee that the set \(\Omega_{\zeta_i, \zeta_{i-1}}\) has an isoperimetric constant lower bound. Again, the existence of \(\zeta_i\) is guaranteed by coarea formula. The length upper bound of \(\gamma_{\zeta_i}\) is given by Lemma 4.7 (2) and (4.24) below, from which we see that inequality (4.22) actually holds for all \(\zeta_i, i \geq 1\).

Once \(\zeta_i < 0\), we stop the selection process. Denote \(\Omega_i := \Omega_{\zeta_i, \zeta_{i-1}}\). The following technical lemma controls the geometry of \(\Omega_i\).

**Lemma 4.7.** Suppose \(0 \leq \zeta \leq \zeta' \leq \zeta^+\) and \(\zeta' - \zeta \leq CD^{-1}\). Then we have

1. \(\frac{1}{2}H(\zeta') \leq H(\zeta) \leq \frac{R_1 - \zeta}{R_1 - \zeta'} H(\zeta')\).
2. For any \(\zeta\) we have \(2H(\zeta) \leq |\gamma_\zeta| \leq 2\pi H(\zeta)\).
3. For the regions \(\Omega_i\) defined above, we have \(|\Omega_i| \leq C(\zeta_{i-1} - \zeta_i)H(\zeta_i)\).
4. We have \(\text{ID}(\Omega_i) \geq C\) when \(i \geq 2\) and \(\text{ID}(\Omega_1) \geq Cc_1\), where the universal constant \(C\) is independent of \(i\).
Proof. (1): Let $p \in \gamma'_c$ realizes the maximum in the definition of $H(\zeta')$. Connect $p$ with the point $(0,0,-R_1)$ by a line segment, which is contained in the interior of $\Sigma$ by convexity. Hence on $\gamma_c$ there is a point $q$ with radius $h(q) \geq H(\zeta')\frac{R_1+\xi}{R_1+\eta} > \frac{1}{2}H(\zeta')$. This prove the lower bound for $H(\zeta)$. If the upper bound is violated, then we find a point $q \in \gamma'_c$ with $h(q) = H(\zeta)$. Connect it with $(0,0,R_1)$ by a line segment and intersect this segment with the plane $\{z = \zeta\}$, then we can find a point $p \in \gamma_c$ with $h(p) \geq \frac{R_1-\xi}{R_1-\eta}h(q) > H(\zeta')$, which is a contradiction.

(2): Note that $\gamma_c$ is a convex planar circle. The upper bound on $|\gamma_c|$ follows from Lemma 3.1(2). For the lower bound, we find a point $p$ with radius $h(p) = H(\zeta)$ and connect it to $(0,0,\zeta)$, and observe that $|\gamma_c|$ is at least twice the length of this segment.

(3): First consider the case $i = 1$, where $\zeta_0 = R_1$ so $\Omega_1$ is not an annulus but a disk at the tip region. By (1) and the selection process of $\zeta_1$, we have $\frac{1}{2}c_1r^+ \leq H(\zeta_1) \leq 3c_1r^+$. Consider the cylinder $\Sigma' = B(0,3c_1r^+) \times [R_1 - 3r^+, R_1]$ which contains $\Omega_1$ in its interior. By Lemma 3.1(2) we have $|\Omega_i| \leq |\Sigma'| \leq 2\pi(9c_1 + 6)c_1(r^+)^2 \leq C(R_1 - \zeta_1)H(\zeta_1).

Next we consider $i \geq 2$. By (1) and (4.21), we have
\[
H(\zeta_i) \leq \frac{R_1 - \zeta_i + 2H(\zeta_i)}{R_1 - \zeta_i}H(\zeta_i - 1) \leq (1+2c_1)H(\zeta_i - 1).
\]
Consider the cylinder $\Sigma' = B(0,(1+2c_1)H(\zeta_i - 1)) \times [\zeta_i, \zeta_i - 1]$. We have
\[
|\Omega_i| \leq |\Sigma'| \leq CH(\zeta_i - 1)(\zeta_i - 1 - \zeta_i) + CH(\zeta_i - 1)^2 \leq C' H(\zeta_i - 1)(\zeta_i - 1 - \zeta_i).
\]

(4): First consider the case $i \geq 2$. Let $\Lambda'$ be the cone with base $\gamma_{\zeta_i}$ and vertex $(0,0,2\zeta_i - \zeta_i - 1)$, and let $\Lambda''$ be the cone with base $\gamma_{\zeta_i - 1}$ and vertex $(0,0,2\zeta_i - 1 - \zeta_i)$. Both cones are contained in the interior of $\Sigma$ when $c_1$ is sufficiently small. The surface $\Sigma' := \Omega''_i \cup \Lambda' \cup \Lambda''$ is thus convex. Although $\Sigma'$ is not smooth, we can always approximate it by smooth convex surfaces. By (1) (2) we have
\[
|\Lambda'| \geq |\gamma_{\zeta_i}|(\zeta_i - 1 - \zeta_i) \geq H(\zeta_i - 1)(\zeta_i - 1 - \zeta_i),
\]
while $\text{diam}(\Sigma') \leq 3(\zeta_i - 1 - \zeta_i) + 4\pi H(\zeta_i - 1)$. By Lemma 3.1(3) and (4.23) we have
\[
\text{IN}(\Sigma') \geq \frac{|\Lambda'|}{\text{diam}(\Sigma')^2} \geq C.
\]
Thus for any domain $P \subset \subset \Omega_i$ we have
\[
|\partial P|^2 \geq \text{IN}(\Sigma')\min\{|P|, |\Lambda'|\} \geq C|P|,
\]
where we used $|P| \leq |\Omega_i| \leq C|\Lambda'|$.

For the case $i = 1$, we only need one cone $\Lambda'$. The result follows from the same argument and the fact that $H(\zeta_1) \leq Cc_1r^+$, $\text{diam}(\Sigma') \leq Cr^+$, and $|\Omega_i| \leq Cc_1(r^+)^2 \leq C'|\Lambda'|$.

We are ready to apply the Moser-Trudinger inequality (Corollary 3.3) to each $\Omega_i$. For $i = 1$ we have
\[
\int_{\Omega_1} e^{-2u} \leq \max_{\partial \Omega_1} (e^{-2u}) \cdot C\text{Area}(\Omega_1) \exp\left(\frac{1}{Cc_1} \int_{\Sigma} \frac{1}{\text{Area}} \right. \leq C(\text{diam}^{-2} + \text{Area})L^{1/\sqrt{3} - 2}(r^+)^{2 - 1/\sqrt{3}}.
\]
For $\Omega_i (i \geq 2)$, by the following simple computation we can compare the boundary values of $\Omega_i$ on the two sides:

$$\zeta^+ - \zeta_i \leq \zeta^+ - \zeta_{i-1} + 2H(\zeta_{i-1})$$
$$\leq (1 + 2c_1)(\zeta^+ - \zeta_{i-1}) + 2c_1r^+$$
$$\leq (1 + 4c_1)(\zeta^+ - \zeta_{i-1}).$$

(4.27)

Applying Corollary 3.3 we obtain

$$\int_{\Omega_i} e^{-2u} \leq \max_{\partial \Omega_i} (e^{-2u}) \cdot C|\Omega_i| \exp\left(\frac{1}{C} \int_{\Sigma} |\nabla u|^2\right)$$
$$\leq C(\text{diam}^2 + \text{Area})L^{1/\sqrt{\beta} - 2}(\zeta_{i-1} - \zeta_i)(\zeta^+ - \zeta_i)^{-1/\sqrt{\beta}}H(\zeta_i)$$

(4.28)

Let $m$ be the smallest index such that $\zeta_m < \zeta^+ - 1$. By (4.21) we have

$$\sum_{2 \leq i < m} \int_{\Omega_i} e^{-2u} \leq C(\text{diam}^2 + \text{Area})L^{1/\sqrt{\beta} - 2} \times \sum_{2 \leq i < m} (\zeta_{i-1} - \zeta_i)(\zeta^+ - \zeta_i)^{-1/\sqrt{\beta}}$$
$$\leq C(\text{diam}^2 + \text{Area})L^{1/\sqrt{\beta} - 2} \times \sum_{2 \leq i < m} \int_{\zeta_{i-1}}^{\zeta_i} (\zeta^+ - x)^{-1/\sqrt{\beta}} dx$$
$$\leq C(\text{diam}^2 + \text{Area})L^{1/\sqrt{\beta} - 2}$$

(4.29)

For $i \geq m$ we have

$$\sum_{i \geq m} \int_{\Omega_i} e^{-2u} \leq C(\text{diam}^2 + \text{Area})L^{1/\sqrt{\beta} - 2} \times \sum_{i \geq m} (\zeta_{i-1} - \zeta_i)(\zeta^+ - \zeta_i)^{-1/\sqrt{\beta}}CD^{-1}$$
$$\leq C(\text{diam}^2 + \text{Area})L^{1/\sqrt{\beta} - 2}.$$ 

(4.30)

Recall that $\frac{1}{4} < \beta \leq 1$. Combining (4.26) (4.29) (4.30), and the same bounds for the $z \leq 0$ part of $\Sigma$, we obtain

$$\text{Area} = \int_{\Sigma} e^{-2u} \leq C(\text{diam}^2 + \text{Area})L^{1/\sqrt{\beta} - 2},$$

which implies

$$L \leq C\left(1 + \frac{\text{diam}^2}{\text{Area}}\right)^{\frac{\sqrt{\beta}}{2\sqrt{\beta} - 1}}.$$ 

This completes the proof of Theorem 4.3. □

Finally, we remark on various cases where the proof can be simplified. First, consider the case $\lambda_1(-\Delta + \beta K) \geq \lambda > 0$. In this case, (4.3) is strengthened to

$$K \geq \beta|\nabla u|^2 + \lambda|\nabla u|^{-2}u.$$ 

Recall that (4.12) implies $\int_{\Omega} K \leq 4\pi c_1$. Therefore, we easily obtain $\int_{\Omega} e^{-2u} \leq C\beta\lambda^{-1}$, and the only remaining task is to estimate $\int_{\Omega} e^{-2u}$. Moreover, Lemma 4.5 can be replaced by the weighted diameter bound proved in subsection A.1.

Next, consider the case of radial symmetry. In this case Lemma 4.5 becomes unnecessary since the shortest geodesics for $g$ and $\overline{g}$ coincide. Also, the estimates on $\Omega^m$ can be simplified:

$$\int_{\Omega^m} e^{-2u} \leq C(\int_{\zeta^-}^{\zeta^{-1}} + \int_{\zeta^-}^{\zeta^{-1}} + \int_{\zeta^-}^{\zeta^{-1}}) e^{-2u(z)} dz := I^- + I^m + I^+.$$ 

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By (4.21) have
\[ I^+ \leq C \text{diam}^2 L^{1/\sqrt{3}-2} \int_{\zeta = -1}^{\zeta = 1} (\zeta + \zeta)^{1-1/\sqrt{3}} d\zeta \leq C \text{diam}^2 L^{1/\sqrt{3}-2} \]
and
\[ I'^n \leq C \text{diam}^2 L^{1/\sqrt{3}-2} CD^{-1} \cdot (\zeta + \zeta^{-1}) \leq C \text{diam}^2 L^{1/\sqrt{3}-2}. \]

Note that we obtain \( L \leq C (\text{diam}^2 / \text{Area})^{2/\sqrt{3}-1} \) under radial symmetry. Without radial symmetry, this strongest bound is proved in later sections.

### 4.3 Existence of bi-Lipschitz maps to the round sphere

In this section we construct a (piecewise smooth) bi-Lipschitz map from \( \Sigma \) to the round sphere, given an upper bound on diameter.

**Lemma 4.8.** Let \( \Sigma \subset \mathbb{R}^3 \) be a convex surface, satisfying \( d_1 \leq |x| \leq d_2 \) for any \( x \in \Sigma \). Then \( \Phi : x \mapsto \frac{x}{|x|} \) is a bi-Lipschitz map to the round sphere, with Lipschitz norms \( ||\Phi||_{Lip} \leq C(d_1, d_2) \), \( ||\Phi^{-1}||_{Lip} \leq C(d_1, d_2) \).

**Proof.** We compute
\[ d\Phi_x(v) = \frac{|x|^2 v - (x \cdot v)x}{|x|^3}, \quad v \in T_x \Sigma. \]
It is not hard to see \( |d\Phi| \leq \frac{2}{|x|} \leq \frac{2}{d_1} \). Note that \( x \cdot N \geq d_1 \) by convexity, hence for any \( |v| = 1 \) we have
\[ |x \cdot v|^2 \leq |x|^2 - (x \cdot N)^2 \leq |x|^2 (1 - \frac{d_1^2}{d_2^2}). \]
Hence
\[ |d\Phi_x(v)|^2 = \frac{|x|^2 - |x \cdot v|^2}{|x|^4} \geq \frac{d_1^2}{d_2^2}. \]

We are ready to prove the main lemma:

**Lemma 4.9.** Let \( \Sigma \subset \mathbb{R}^3 \) be a convex surface with unit area and diameter \( \leq D \). Then there exists a bi-Lipschitz map to the round sphere \( \Phi : \Sigma \to (S^2, g_0) \), with Lipschitz norms \( ||\Phi||, ||\Phi^{-1}|| \) bounded in terms of \( D \).

**Proof.** Let \( W \) be the extrinsic width of \( \Sigma \), defined as
\[ W := \min \{ d : \text{we can embed } \Sigma \text{ into the region } \{-d/2 \leq z \leq d/2\} \subset \mathbb{R}^3 \}, \quad (4.31) \]
where the embedding is isometric. First we consider the case that \( W \geq \frac{1}{4\sqrt{D}} \). Let \( S \) be a largest inscribed sphere of \( \Sigma \), and \( r \) be the radius of \( S \). The width-inradius inequality gives
\[ \frac{D}{2} \geq r \geq \frac{W}{4}. \]
A proof of the two-dimensional analogue can be found in [16, p.215], while the proof there can be directly generalized to three dimensions. Now Lemma 4.8 can be applied to give us a bi-Lipschitz map.
Next we assume $W \leq \frac{1}{4\pi D}$. We will replace the nearly flat top and bottom faces of $\Sigma$ by two cones, thus increasing the inradius. We may assume that $\Sigma$ is already embedded in the region $\{-W/2 \leq z \leq W/2\} \subset \mathbb{R}^3$. Let $Q$ be the projection image of $\Sigma$ onto the $xy$-plane, which is a convex planar domain. Thus $\Sigma \subset Q \times [-W/2, W/2]$. We have

$$1 = |\Sigma| \leq |\partial(Q \times [-W/2, W/2])| \leq 2|Q| + 2W|\partial Q| \leq 2|Q| + 2\pi DW$$

Thus

$$|Q| \geq \frac{1}{4}.$$

Let $r_0$ be the inradius of $Q$ (i.e. the radius of the largest disk contained in $Q$). From [30] we know $r_0 \geq \frac{|Q|}{2\text{diam}(Q)} \geq \frac{1}{8D}$. After a translation, let

$$B := B(0, r_0) \subset Q \subset \mathbb{R}^2$$

be an inscribed disk in $Q$. Let $p^\pm = (0, 0, \pm 1)$. Let $\Lambda^+$ be the convex cone obtained by taking the union of all tangent line segments of $\Sigma$ emanating from $p^+$. We claim that the normal unit vector $N_\Lambda$ of $\Lambda^+$ satisfies $|N_\Lambda \cdot \partial_2| \geq \frac{r_0}{\sqrt{r_0^2 + 4}}$. To show this, let $P$ be any plane tangent to $\Lambda^+$. Consider the intersection line $l = P \cap \{z = -1\}$, we observe that $d(l, p^-) \geq r_0$ by convexity. Our claim then follows from a simple computation.

Let $\gamma^+$ be the set at which $\Lambda^+$ is tangent to $\Sigma$. Now $\gamma^+$ encloses a region $\Omega^+$ in $\Sigma$. Let $Q^+$ be the projection image of $\Omega^+$ onto the $xy$-plane. Consider the orthogonal projection map $F_1^+ : \Omega^+ \rightarrow Q^+$, $F_2^+ : \Lambda^+ \rightarrow Q^+$ (which are both bijective). It is not hard to prove the following general fact: if $G$ is the projection map from a surface $S$ to the $xy$-plane, then $||G^{-1}||_{\text{Lip}} \leq \sup_S |N_S \cdot \partial_2|^{-1}$, where $N_S$ is the unit normal vector of $S$. By our claim above, the map $F^+ = (F_2^+)^{-1} \circ F_1^+ : \Omega^+ \rightarrow \Lambda^+$ has controlled bi-Lipschitz norms. Analogously, we can define a bi-Lipschitz map $F^- : \Omega^- \rightarrow \Lambda^-$. Let $\Omega'' = \Sigma \setminus (\Omega^+ \cup \Omega^-)$. Combining $F^+$ with $F^-$ we obtain a bi-Lipschitz map $F$ from $\Sigma$ to $\Sigma' = \Omega''' \cup \Lambda^+ \cup \Lambda^-$. It is not hard to see $\text{diam}(\Sigma') \leq \text{diam}(\Sigma) + 2$ and $|x| > \min(1/4, r_0/4)$ on $\Sigma'$. The result then follows from Lemma 4.8.

### 4.4 Improved diameter bound, total volume comparison, and a second proof of the isoperimetric inequality

Theorem 4.3 has provided us with a bound on $\text{diam}(\Sigma, g)$. In this subsection we see obtain the volume comparison theorem from this diameter bound. Also, we give a second proof of Theorem 1.2(2) with a weaker constant.

**Theorem 4.10** (equivalent to Theorem 1.4). Assume Condition 4.2. We have $\overline{\text{Area}} \leq C\text{diam}^2$ for a constant $C = C(\beta)$.

**Proof.** We may assume $\overline{\text{Area}} \geq \text{diam}^2$. Then Theorem 4.3 implies $\text{diam}(\Sigma, g) \leq C$. We also have $\text{diam}(\Sigma, g) \geq 1$ by $|\Sigma|_g = 1$ and volume comparison. Let $p, q \in \Sigma$ be two points with $d(p, q) \geq 1$. There exists an embedded curve $\gamma$ ($\gamma(0) = p, \gamma(L) = q$) such that $\int_0^L e^{-u} dl \leq \text{diam}$. Let $U = \{p \in \gamma : e^{-u}(p) > C_1\text{diam}\}$ and $V = \gamma \setminus U = \{e^{-u} \leq C_1\text{diam}\}$, where the constant $C_1$ is to be determined below. Clearly $|U| \leq \frac{1}{C_1}$. We would like to apply the Moser-Trudinger inequality to $e^{-2u}$ with Dirichlet condition on $V$. Therefore, we need a corresponding isoperimetric inequality:
Lemma 4.11. If \( C_1 \) is sufficiently large, then for any domain \( \Omega \subset \Sigma \) with \( \partial \Omega \cap V = \emptyset \), we have \( |\partial \Omega|^2 \geq C|\Omega| \) for some universal constant \( C \).

With this lemma available, by Corollary 3.3 we have

\[
\text{Area} = \int_{\Sigma} e^{-2u} \leq C \max_{p \in V} (e^{-2u(p)}) \cdot |\Sigma| \cdot \exp \left( C \int_{\Sigma} |\nabla u|^2 \right) \leq C \text{diam}^2,
\]

which completes Theorem 4.10.

Proof of Lemma 4.11. Again, we may assume that \( \Omega \) is connected (see the beginning of Section 2). By Lemma 4.9, we have a bi-Lipschitz map \( \Phi : \Sigma \to S^2 \) with bi-Lipschitz norms at most \( M \). Let \( \Omega' = \Phi(\Omega) \), hence \( |\partial \Omega| \geq M^{-1}|\partial \Omega'|, |\Omega| \leq M^2|\Omega'| \).

Let \( p' = \Phi(p), q' = \Phi(q) \). We have \( d_{S^2}(p', q') \geq \frac{1}{M} \). By possibly truncating \( \gamma \), we can also assume \( d_{S^2}(p', q') \leq 1 \). Embed \( S^2 \) as the unit sphere in \( \mathbb{R}^3 \). Rotate the sphere so that \( p = (p_1, p_2, 0), q = (q_1, q_2, -z_0) \), with \( \frac{1}{2} < z_0 < \frac{3}{2} \). Let \( P_2 \) be the projection map onto the \( z \)-axis, and define \( I = P_2(\Phi(V)) \cap \{ -z_0 \leq z \leq z_0 \} \subset \mathbb{R} \). Since \( P_2 \) is a surjective map from \( \Phi(\gamma) \) to \( \{ z \in \mathbb{R} : -z_0 \leq z \leq z_0 \} \), we have \( |I| \geq 2z_0 - |\Phi(U)| \geq \frac{1}{2M} - \frac{M}{C_1} \).

By choosing \( C_1 > 4M^2 \), we have \( |I| \geq \frac{1}{M} \).

Let \( J = P_2(\partial \Omega') \). First consider the case \( J \subset I \). In this case we have \( |\partial \Omega| \geq |J| \geq |I| \geq \frac{1}{4M} \geq C|\Omega'| \). Next suppose \( J \not\subset I \). There exists \( z_1 \in I \) such that \( \partial \Omega' \cap \{ (x, y, z) : z = z_1 \} = \emptyset \). Therefore, we have either \( \{ (x, y, z) : z = z_1 \} \subset \Omega' \) or \( \{ (x, y, z) : z = z_1 \} \cap \Omega' = \emptyset \). The first case is not allowed since we assume \( \Omega' \cap V = \emptyset \), thus the second case must hold. Now, \( \Omega' \) is contained in either the spherical cap \( \{ z > z_1 \} \) or \( \{ z < z_1 \} \), both case yields \( |\partial \Omega|^2 \geq C|\Omega'| \) by the usual spherical isoperimetric inequality. \( \square \)

Corollary 4.12. Assume Condition 4.2. We have

\[
\text{diam}(\Sigma) \leq C \left( \text{diam}^2 / \text{Area} \right)^{\gamma}. \quad (4.32)
\]

Combining Lemma 4.3, 4.9 with Theorem 1.6, we obtain another proof of Theorem 1.2(2) with a weaker constant.

Theorem 4.13. Assuming Condition 4.2, for any smooth domain \( \Omega \subset \Sigma \) we have

\[
\frac{|\partial \Omega|^2}{\text{diam}^2} = \min \left\{ \frac{|\Omega|}{|\Omega'|}, \frac{|\Omega|^2}{|\Omega'|} \right\} \geq \frac{\left( \int_{\partial \Omega} e^{-u} \right)^2}{\min \{ \int_{\Omega} e^{-2u}, \int_{\Omega'} e^{-2u} \}} \geq C(\beta, \text{Area}/\text{diam}^2).
\]

where the constant \( C \) becomes large when \( \text{Area}/\text{diam}^2 \) is small.

Proof. By Lemma 4.3 and 4.9, there is a map \( \Phi : \Sigma \to (S^2, g_0) \) with Lipschitz norm \( \max(\|\Phi\|_{Lip}, \|\Phi^{-1}\|_{Lip}) := M \leq C(\beta, \text{Area}/\text{diam}^2) \). Denote \( u' = u \circ \Phi^{-1} \in H^1(S^2) \) and \( \Omega' = \Phi(\Omega) \subset S^2 \). The following inequalities are not hard to show:

\[
\int_{S^2} |\nabla u|^2 dA_0 \leq M^4 \int_{\Sigma} |\nabla u|^2 dA \leq 4\pi \beta^{-1} M^4,
\]

\[
\int_{\partial \Omega} e^{-u} dl \geq M^{-1} \int_{\partial \Omega'} e^{-u'} d\ell', \quad \int_{\Omega} e^{-2u} dA \leq M^2 \int_{\Omega'} e^{-2u'} dA'.
\]

The result now follows from Theorem 1.6. \( \square \)
A Two Proofs of the weak Bonnet-Myers’ Theorem

In this section we present two proofs of Theorem 1.3, using the methods of weighted geodesics and weighted \( \mu \)-bubbles.

Let \( u > 0 \) satisfies

\[
\Delta u^\beta \leq (\beta K - \lambda)u^\beta, \tag{A.1}
\]
equivalently,

\[
\Delta u \leq (K - \lambda \beta^{-1})u + (1 - \beta)u^{-1}|\nabla u|^2. \tag{A.2}
\]

Note that the function \( u \) defined here is different than in Section 4.

A.1 Proof using weighted geodesics

The argument here is similar to the ones in Schoen-Yau \cite{28} and Shen-Ye \cite{31}.

Let \( p, q \in \Sigma \) be two points with the largest distance. If \( \Sigma \) is non-compact, then choose \( p, q \) with large enough distance to obtain a contradiction below. Let \( \gamma : [0, L] \rightarrow \Sigma \), \( \gamma(0) = p, \gamma(L) = q \), be a minimizer of the weighted length functional \( \int_\gamma u \, dl \). Parametrize \( \gamma \) with unit speed. The second variational formula gives the following inequality:

\[
0 \leq \int_\gamma \left[ u(\varphi')^2 + (\Delta u - u'')\varphi^2 - Ku\varphi^2 - u^{-1}(u_N)^2\varphi^2 \right] \, dl \tag{A.3}
\]

for any function \( \varphi : \varphi(0) = \varphi(L) = 0 \), where we denote \( f' := \partial f/\partial \gamma' \) for a function \( f \), and denote \( u_N := \partial u/\partial N \). Substituting \( \varphi = u^{-1/2}\psi \) into (A.3) and using equation (A.2), we obtain

\[
0 \leq \int_\gamma \left[ u\left( -\frac{1}{2}u^{-3/2}u'\psi + u^{-1/2}\psi' \right)^2 + u'(-u^{-2}u'\psi^2 + 2u^{-1}\psi\psi') - \lambda\beta^{-1}\psi^2 + (1 - \beta)u^{-2}(u')^2 \right] \, dl
\]
\[
\leq \int_\gamma \left[ \left( \frac{1}{4} - \beta \right)u^{-2}(u')^2 + \psi'^2 + u^{-1}u'\psi\psi' - \lambda\beta^{-1}\psi^2 \right] \, dl
\]
\[
\leq \int_\gamma \left[ \left( 1 + \frac{1}{4}(\beta - \frac{1}{4} \lambda)^{-1} \right)\psi'^2 - \lambda\beta^{-1}\psi^2 \right] \, dl.
\]

Theorem 1.3 follows by letting \( \psi(t) = \sin(\pi t/L) \).

A.2 Proof using weighted \( \mu \)-bubbles

A \( \mu \)-bubble is a hypersurface with prescribed mean curvature \( H = h \), where \( h \) is a given function on the ambient manifold. The variational characterization of \( \mu \)-bubbles is given by the energy functional (A.4). By choosing an appropriate function \( h \), we can extract information about the ambient manifold from the stability inequality. The argument here closely follows Chodosh-Li \cite{6}.

Let \( \Omega^+, \Omega^- \) be two disjoint domains in \( \Sigma \). Let \( h \) be a Lipschitz function (whose conditions will be determined later) on \( \Sigma \setminus (\Omega^- \cup \Omega^+) \), such that \( h|_{\partial \Omega^\pm} = \pm \infty \). Let \( \Omega^0 \) be
a domain containing $\Omega^+$ and disjoint from $\Omega^-$. (This set serves the role of renormalizing.) Consider the following functional acting on all open sets $\Omega$ with $\Omega \Delta \Omega^0 \subset \subset \Sigma \setminus (\Omega^- \cup \Omega^+)$:

\[ E_u(\Omega) := \int_{\partial \Omega} u \, dl - \int_M (\chi_\Omega - \chi_{\Omega^0}) hu \, dA. \]  
(A.4)

A critical point of $E(\Omega)$ (or its boundary) is called a $\mu$-bubble. Since $h$ is infinite on $\partial \Omega^\pm$, any $\mu$-bubble must lie between $\Omega^+$ and $\Omega^-$. Note that we have added a weight $u$ into the functional. The unweighted version is $E(\Omega) = |\partial \Omega| - \int_M (\chi_\Omega - \chi_{\Omega^0}) h \, dA$, whose critical point satisfies $H = h$. By geometric measure theory, a $\mu$-bubble is always a $C^{2,\alpha}$ hypersurface.

In [6] is was shown in detail that a global minimizer of (A.4) always exists. The first variation of (A.4) gives

\[ \kappa = h - u^{-1}u_N, \]  
(A.5)

on a $\mu$-bubble, where $\kappa$ is the geodesic curvature of $\partial \Omega$. The second variation of (A.4) at a global minimizer gives the following stability inequality:

\[ 0 \leq \int_\gamma \left[ (u(\varphi')^2 - K w \varphi^2 - h^2 w \varphi^2 + hu_N \varphi^2 - u^{-1}u_N^2 + (\Delta u - u'') \varphi^2 - hu_N w \varphi^2 \right] \, dl. \]  
(A.6)

Testing (A.6) with $\varphi = u^{-1}$, we obtain

\[ 0 \leq \int_\gamma \left[ - \beta u^{-3} (u')^2 + ( - h^2 - \lambda \beta^{-1} + |\nabla h|) u^{-1} + hu_N - \beta u^{-3}u_N^2 \right] \, dl \]
\[ \leq \int_\gamma \left[ \left( \frac{1}{4\beta} - 1 \right) h^2 - \lambda \beta^{-1} + |\nabla h| \right] u^{-1} \, dl. \]

For the last line we used

\[ hu_N u^{-2} \leq \beta u^{-3}u_N^2 + \frac{1}{4\beta} h^2 u^{-1}. \]  
(A.7)

Therefore, we would reach a contradiction if $h$ satisfies

\[ |\nabla h| < (1 - \frac{1}{4\beta}) h^2 + \lambda \beta^{-1}. \]  
(A.8)

Suppose $\beta > \frac{1}{4}$, the function

\[ h(x) = \sqrt{\frac{\lambda}{\beta(1 - \frac{1}{4\beta} - \epsilon) - \epsilon}} \cot \left[ \sqrt{\lambda \beta^{-1}(1 - \frac{1}{4\beta} - \epsilon)} d(x, \Omega^+) \right] =: C_1 \cot \left[ C_2 d(x, \Omega^+) \right] \]

satisfies (A.8). If diam($\Sigma$) > $\pi/C_2$, then we can let $p, q$ be points with maximal distance, and choose $\Omega^+ = B_\epsilon(p)$, $\Omega^- = \{ x \in \Sigma : d(x, \Omega^+) \geq \pi/C_2 \}$. For these choices we obtain a contradiction with the stability inequality. This proves diam($M$) $\leq \pi/C_2$.

Inspired by the $\mu$-bubble proof, here we describe a method to construct non-compact surfaces satisfying (1.1) when $\beta \leq \frac{1}{4}$. Consider the metric $g = dr^2 + f(r)^2 d\theta^2$ and $\varphi = \varphi(r)$.
be the first eigenfunction. Let \( u = \varphi^{1/\beta} \) and \( h = h(r) \). Consider the following constraint equations: first of all, we want all constant \( r \) slices to be \( \mu \)-bubbles (see (A.5)):

\[
\frac{f'}{f} = h - \frac{u'}{u}. \tag{A.9}
\]

Second, we want Young’s inequality (A.7) to be equality at all time:

\[
\beta \left( \frac{u'}{u} \right)^2 = \frac{h^2u}{4\beta} \quad \text{or} \quad \frac{u'}{u} = \frac{h}{2\beta}. \tag{A.10}
\]

Finally, we let \( h \) to satisfy the ODE corresponding to (A.8):

\[
h' = \left( \frac{1}{4\beta} - 1 \right) h^2 - \lambda \beta^{-1} \tag{A.11}
\]

We can check by elementary computation that the solution to (A.9) (A.10) (A.11) satisfies the identity

\[
u'' + \frac{f'}{f} u' + \frac{f''}{f} u + \lambda \beta^{-1} u - (1 - \beta) \left( \frac{u'}{u} \right)^2 = 0,
\]

which is equivalent to \( \Delta_g (u^3) = (\beta K_g - \lambda) u^3 \).

When \( \beta < \frac{1}{4} \), we obtain the following solution:

\[
f(r) = \cosh \left[ r \sqrt{\lambda \beta^{-1} \left( \frac{1}{4\beta} - 1 \right)^2 \left( \frac{2}{1 - 4\beta} \right)} \right], \quad \varphi(r) = \cosh \left[ r \sqrt{\lambda \beta^{-1} \left( \frac{1}{4\beta} - 1 \right)} \right]^{-\frac{2i}{1 - 4\beta}}. \tag{A.12}
\]

When \( \beta = \frac{1}{4} \), we obtain the following solution:

\[
f(r) = e^{2\lambda r^2}, \quad \varphi(r) = e^{-\lambda r^2}.
\]

**B Discussions on Counterexamples**

In this section, we discuss counterexamples to Theorem 1.2 when \( \beta \) does not satisfy the required bounds, thus showing that these bounds are optimal. The counterexamples are obtained by a gluing process. We first discuss the possible building pieces for the gluing construction, then explicitly work out an example.

We first consider Theorem 1.2(2), where the required bound is \( \beta > \frac{1}{4} \). Consider the metric \( g = dr^2 + e^{pr} d\theta^2 \) and the eigenfunction \( \varphi = e^{qr} \). It is not hard to show that \( \Delta \varphi \leq \beta K \varphi \) is equivalent to

\[
\beta p^2 + pq + q^2 \leq 0,
\]

which is possible only when \( \beta < \frac{1}{4} \). Note that \( g \) is the quotient space of the hyperbolic plane by a parabolic Mobius transform. In fact, the hyperbolic plane itself has \( \lambda_1(-\Delta) = \frac{1}{4} \) [4]. Note that a good counterexample candidate for isoperimetric inequalities should contain a neck. Therefore, we choose to work with the metric (A.12) constructed in the last section. We will glue two caps to a truncation of (A.12), thus obtain a closed surface satisfying (1.1) and containing a thin neck.
Next, we consider Theorem 1.2(1), for which the bound is $\beta > \frac{1}{4}$. This time we try the metric $g = dr^2 + r^{-p}d\theta^2$ ($p > 0$, $r > 0$) and eigenfunction $\varphi = r^q$. A short computation shows that $\Delta \varphi \leq \beta K \varphi$ is equivalent to

$$q(q - 1) - pq + \beta p(p + 1) \leq 0. \quad (B.1)$$

When $\beta > \frac{1}{4}$, the largest achievable value of $p$ under (B.1) is $p = \frac{1}{4\beta - 1}$. Let $D \gg 1$; the thin neck in our potential counterexample is obtained at $r = D$. Similar to what we did above, we truncate $g$ at a fixed $r$ and glue a cap to obtain a closed surface. Finally, we double the resulting surface to obtain a bone-like surface $\Sigma$ with diameter $\sim 2D$. This surface is not smooth at the middle, but we can further glue a catenoid at $r = D$ before doubling. Let $\Omega$ be one half of the resulting surface; for $\Sigma$ to be a counterexample, we need $\frac{|\Omega|}{|\Sigma|} \cdot \text{diam}(\Sigma)$ to approach zero when $D \to \infty$. Since $|\partial \Omega| = O(D^{-p})$ and

$$|\Omega| = \begin{cases} O(D^{1-p}) & (p < 1), \\ O(\log D) & (p = 1), \\ O(1) & (p > 1), \end{cases}$$

we can achieve our goal if and only if $p \geq 1$. Under the constraint (B.1), $p \geq 1$ is possible only when $\beta \leq \frac{1}{2}$. Furthermore, we have $\text{IN}(\Sigma) = O(D^{-2p}) = O(D^{-\frac{2}{4\beta - 1}})$, $\frac{|\Sigma|}{\text{diam}(\Sigma)^2} = O(D^{-2})$, therefore the exponent in Theorem 1.2(2) is almost optimal.

If we only close the surface at small $r$ and leave open the end at $r = \infty$, then we obtain a non-compact surface satisfying (1.1) and has finite area. Therefore the condition $\beta > \frac{1}{2}$ in Gromov-Lawson’s theorem [15, Proposition 8.11] is also critical.

Here we choose to present the technical construction of the first counterexample, i.e. for $\beta < \frac{1}{4}$. The other examples could be constructed similarly. To summarize, the resulting surface has the property that: (1) is closed, (2) satisfies $\lambda_1(-\Delta + \beta K) \geq \lambda > 0$ strictly, and (3) $y = \text{IN}(\Sigma)$ is related to $x = \text{Area} / \text{diam}^2$ by the rate $y = O(e^{-C/x})$.

**Declaration of constants.** $\beta$ ($0 < \beta < \frac{1}{4}$) and $\lambda > 0$ are fixed numbers. $\epsilon$ is a sufficiently small number independent of $\beta$. The $o(1)$ quantities are with respect to $\epsilon \to 0$.

As shown in the end of Appendix A, the metric $g_1 = dr^2 + f_1(r)^2d\theta^2$ and function $\varphi_1$ defined as follows satisfies $\Delta \varphi_1 = (\beta K - \lambda)\varphi_1$:

$$f_1(x) = \epsilon \cosh \left[ x \sqrt{\lambda \beta^{-1}(\frac{1}{4\beta} - 1)} \right]^{\frac{2\beta - 4\epsilon}{2\epsilon}} \quad (B.2)$$

$$\varphi_1(x) = \cosh \left[ x \sqrt{\lambda \beta^{-1}(\frac{1}{4\beta} - 1)} \right]^{-\frac{2\beta}{4\epsilon}} \quad (B.3)$$

Define the following constants (to simplify the expressions, we choose not to eliminate the possibly redundant ones):

$$p = \frac{2}{1 - 4\beta} \sqrt{\lambda \beta^{-1}(\frac{1}{4\beta} - 1)}, \quad q = \frac{2\beta}{1 - 4\beta} \sqrt{\lambda \beta^{-1}(\frac{1}{4\beta} - 1)},$$

$$d_1 = 2^{-\frac{4\epsilon}{2\beta - 4\epsilon}}, \quad d_2 = 2^{\frac{4\epsilon}{2\beta - 4\epsilon}}, \quad d_3 = \frac{7}{4}, \quad \eta = \cot^{-1}(d_3 \sqrt{\beta}),$$

$$c_3 = 4096\beta^{-2}, \quad c_2 = \frac{256}{d_3 \sqrt{\beta}}, \quad e^{-2c_1} = \frac{c_3 - 1}{c_3} (1 + c_2^2 d_1^2 p^4 \cos^2 \eta),$$

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\[ \kappa = 16p^2\beta^2, \quad A = c_2p^2, \]
\[ x_1 = \frac{1}{p}(\log(\frac{p}{\epsilon}) + c_1), \quad x_2 = \frac{1}{\sqrt{\kappa}} \sinh^{-1}\left(\frac{1}{c_3 - 1}\right), \]
\[ \theta = \frac{\sqrt{\beta}}{4}(\pi - \eta - \tan^{-1}(c_2)), \quad R = \frac{4\theta}{\sqrt{A\beta}} + x_1 + x_2. \]

Note that all the constants are independent of \( \epsilon \), except for \( x_1 \). For a fixed \( \beta \), we have \( x_1 \to \infty \) when \( \epsilon \to 0 \), hence \( f_1 \) and \( \varphi_1 \) have the following asymptotic behaviors:
\[ f_1(x_1) = [1 + o(1)]\epsilon d_1 e^{px_1} = [1 + o(1)]d_1 e^{c_1}p, \]
\[ f_1'(x_1) = [1 + o(1)]\epsilon p d_1 e^{px_1} = [1 + o(1)]d_1 e^{c_1}p^2, \]
\[ \varphi_1'(x_1) = -[1 + o(1)]q\varphi(x_1). \quad (B.4) \]

Now we consider the metric \( g_0 = dr^2 + f(r)^2d\theta^2 \), where
\[ f(x) = \begin{cases} 
    f_1(x) & (0 \leq x \leq x_1), \\
    d_1 e^{c_1}p \cos[\sqrt{A}(x - x_1)] + d_1 e^{c_1}p^2 \sin[\sqrt{A}(x - x_1)] & (x_1 \leq x \leq R - x_2), \\
    \frac{1}{\sqrt{\kappa}} \sinh[\sqrt{\kappa}(R - x)] & (R - x_2 \leq x \leq R), 
\end{cases} \quad (B.5) \]

and the function \( \varphi = \varphi(r) \), where
\[ \varphi(x) = \begin{cases} 
    \varphi_1(x) & (0 \leq x \leq x_1), \\
    \varphi_1(x_1) \cosh[\frac{\sqrt{A\beta}}{4}(x - x_1)] \\
    \quad + \varphi_1(x_1) \frac{1 - \cosh \theta}{\sinh \theta} \sinh[\frac{\sqrt{A\beta}}{4}(x - x_1)] & (x_1 \leq x \leq R - x_2), \\
    c_4 e^{-\sqrt{\kappa}(R-x)/2} & (R - x_2 \leq x \leq R), 
\end{cases} \quad (B.6) \]

and \( c_4 \) is determined by requiring \( \varphi(x_1) = \varphi(R - x_2) \). It is not hard to show that \( \varphi > 0 \).

The choice of \( x_1 \) is intended to guarantee an area lower bound independent of \( \epsilon \). Indeed,
\[ \frac{1}{2} |\Sigma| > 2\pi \int_0^{x_1} f_0(r) \, dr > 2\pi \epsilon \int_0^{x_1} e^{px} \, dr = \frac{2\pi \epsilon}{p} (e^{px_1} - 1) = 2\pi e^{c_1} - o(1). \]

Note that \( \text{IN}(\Sigma) = O(\epsilon^2) \) whereas \( \frac{|\Sigma|}{\text{diam}(\Sigma)} = O((\log \frac{1}{\epsilon})^{-2}) \).

The remaining task is to show that \( \varphi \) is a supersolution:
\[ \Delta \varphi \leq (\beta K - \lambda)\varphi \quad (\text{in distributional sense}), \quad (B.7) \]

For convenience, we label \( f_2 := f\big|_{[x_1,R-x_2]} \), \( f_3 := f\big|_{[R-x_2,R]} \), and \( \varphi_2 := \varphi\big|_{[x_1,R-x_2]} \), \( \varphi_3 := \varphi\big|_{[R-x_2,R]} \). It is easy to check that \( \varphi \) is continuous everywhere, and one may utilize the definition of constants to show that \( f \) is \( C^1 \) at \( R - x_2 \). Note that \( f \) is only differentiable at \( x_1 \) up to an \( o(1) \) factor, due to the non-precise asymptotes \( (B.4) \). To make \( f \) precisely differentiable, we have to adjust by replacing \( d_1 \) with \( d_1 [1 + o(1)] \) everywhere in
the definition of the constants. We do not write down this adjustment explicitly for the simplicity of expressions. All the inequalities proved below are strict and independent of \( \epsilon \), hence still hold after the adjustment if \( \epsilon \ll 1 \).

First, note that \( \varphi_1 \) satisfies (B.7) by construction. On \((R - x_2, R)\) we can compute
\[ \Delta \varphi_3 < -\frac{3}{4} \varphi_3 = [\beta K - (\frac{1}{4} - \beta)\kappa] \varphi_3. \]
At the point \( x = R \) the function \( \varphi_3 \) is semi-concave, thus is a supersolution in distributional sense. To guarantee Condition (B.7) for \( \varphi_3 \), we therefore only need
\[ \lambda < (\frac{1}{4} - \beta) \kappa. \]  
(B.8)

However, since \( \kappa = 16p^2 \beta^2 = \frac{16(1-2\beta)^2 \lambda}{1-4\beta} \), this condition is automatically satisfied. We still need to verify that \( \varphi_2 \) satisfies condition (B.7), and \( \varphi'_1(x_1) > \varphi'_2(x_1), \varphi'_2(R - x_2) > \varphi'_2(R - x_2) \), and they are proved in the three lemmas below.

**Lemma B.1.** \( \varphi_2 \) satisfies (B.7).

**Proof.** Observe that we can simplify \( f_2 \) as
\[ f_2(x) = d_1 c^3 p \sqrt{1 + \frac{1}{c_2^2}} \sin \left[ \sqrt{A}(x - x_1) + \tan^{-1}(c_2) \right]. \]
Thus
\[ \frac{f'_2}{f_2} = \sqrt{A} \cot \left[ \sqrt{A}(x - x_1) + \tan^{-1}(c_2) \right]. \]
By the definition of the constants we have \( \sqrt{A}(x - x_1) + \tan^{-1}(c_2) \in [\tan^{-1}(c_2), \pi - \eta] \subset \left[ \frac{\pi}{2} - \tan^{-1}(d_3 \sqrt{\beta}), \frac{\pi}{2} + \tan^{-1}(d_3 \sqrt{\beta}) \right] \), hence \( |f'_0/f_0| \leq d_3 \sqrt{A \beta} \). It is not hard to verify \( \lambda < \frac{1}{7} A \beta \) (which is an inequality in \( \beta \) only). Also, on \([x_1, R - x_2]\) the Gauss curvature is equal to \( A \). Thus it suffices to show that \( \varphi_2 \) satisfies
\[ \varphi'' + \frac{f'_2}{f_2} \varphi' < \frac{2A\beta}{3} \varphi_2. \]
(B.9)

Note that \( \varphi_2 \) has the form \( \varphi_2 = b_1 \cosh \left[ \frac{\sqrt{A \beta}}{4} x + b_2 \right] \) where \( b_1 > 0 \) and \( b_2 \) are some values. Thus we can check for \( h = \cosh(\frac{\sqrt{A \beta}}{4} x) \):
\[ h'' + \frac{f'_2}{f_2} h' - \frac{2}{3} A \beta h < -\frac{7}{16} A \beta \cosh(\frac{\sqrt{A \beta}}{4} x) + \frac{d_3}{4} A \beta \cdot |\sinh(\frac{\sqrt{A \beta}}{4} x)| < 0 \]

**Lemma B.2.** \( \varphi'_1(x_1) > \varphi'_2(x_1) \).

**Proof.** We have \( \varphi'_2(x_1) = \frac{\sqrt{A \beta}}{4} \frac{1 - \cosh \theta}{\sinh \theta} \varphi_1(x_1), \) and \( \varphi'_1(x_1) = -q \varphi_1(x_1) \) up to an \( o(1) \) factor. Therefore it suffices to show
\[ \frac{4q}{\sqrt{A \beta}} < \frac{\cosh \theta - 1}{\sinh \theta}. \]
From the definitions we have \( \frac{4q}{\sqrt{A \beta}} = \frac{d_3 \beta}{64(1 - 2\beta)} < \frac{d_3 \beta}{32} \). Note that
\[ \theta = \frac{\sqrt{3}}{4} (\tan^{-1}(\frac{1}{c_2}) + \tan^{-1}(d_3 \beta)) \in \left( \frac{d_3 \beta}{4 \sqrt{2}}, \frac{d_3 \beta}{2} \right), \]

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where we use $x > \tan^{-1}(x) > x/\sqrt{2}$ when $0 < x < 1$. Using the fact that $x \mapsto \frac{\cosh x - 1}{x \sinh x}$ is decreasing, we can estimate

$$\frac{\cosh \theta - 1}{\sinh \theta} > \frac{1}{2\sqrt{2}} > \frac{d_3 \beta}{16},$$

thus completes the proof.

Lemma B.3. $\varphi_2(R - x_2) > \varphi_3(R - x_2)$.

Proof. We have

$$\varphi'_2(R - x_2) = \frac{\sqrt{A \beta}}{4} \varphi_1(x_1)(\sinh \theta + \frac{1 - \cosh \theta}{\sinh \theta} \cosh \theta)$$

$$= \frac{\sqrt{A \beta}}{4} \frac{\cosh \theta - 1}{\sinh \theta} \varphi_1(x_1),$$

and

$$\varphi'_3(R - x_2) = \frac{\sqrt{\kappa}}{2} \varphi_1(x_1).$$

Hence, it suffices to show

$$\frac{2\sqrt{\kappa}}{\sqrt{A \beta}} < \frac{\cosh \theta - 1}{\sinh \theta}.$$  

But this follows from (B.10) and $\frac{2\sqrt{\kappa}}{\sqrt{A \beta}} = \frac{8p^3}{c^2 p \sqrt{\beta}} = d_3 \beta / 32$.  

The metric constructed here is not smooth. However, after small $C^1$ perturbation it becomes smooth. The Laplacian depends $C^1$ continuously on the metric, hence (B.7) is preserved after the perturbation.

References

[1] Yu. D. Burago and V. A. Zalgaller, Geometric inequalities, Translated from the Russian by A. B. Sosinski˘ı, Springer Series in Soviet Mathematics, Springer-Verlag, Berlin, 1988, xiv+331 pp.

[2] P. Buser, A note on the isoperimetric constant, Ann. Sci. École Norm. Sup. (4) 15 (1982), no. 2, 213–230.

[3] A. S.-Y. Chang and P. C. Yang, Conformal deformation of metrics on $S^2$, J. Differential Geom. 27 (1988), no. 2, 259–296.

[4] I. Chavel, Eigenvalues in Riemannian geometry, Pure and Applied Mathematics, 115. Academic Press, Inc., Orlando, FL, 1984, xiv+362 pp.

[5] I. Chavel, Riemannian geometry. A modern introduction, 2nd.ed, Cambridge Studies in Advanced Mathematics, 98. Cambridge University Press, Cambridge, 2006. xvi+471 pp.

[6] O. Chodosh and C. Li, Generalized soap bubbles and the topology of manifolds with positive scalar curvature, https://arxiv.org/abs/2008.11888 (2020).
[7] C. B. Croke, Some isoperimetric inequalities and eigenvalue estimates, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 4, 419–435.

[8] X. Dai and G. Wei and Z. Zhang, Local Sobolev constant estimate for integral Ricci curvature bounds, Adv. Math. 325 (2018), 1–33.

[9] D. Fischer-Colbrie and R. Schoen, The Structure of Complete Stable Minimal Surfaces in 3-Manifolds of Non-Negative Scalar Curvature, Comm. Pure Appl. Math. 33 (1980), 199-211.

[10] S. Gallot, Isoperimetric inequalities based on integral norms of Ricci curvature, Astérisque No. 157-158 (1988), 191–216.

[11] M. Gromov, Metric structures for Riemannian and non-Riemannian spaces, Progress in Mathematics, 152. Birkhäuser Boston, 1999.

[12] M. Gromov, Metric inequalities with scalar curvature, GAFA 28, 6 (2018), 645–726.

[13] M. Gromov, Four lectures on scalar curvature, https://arxiv.org/abs/1908.10612 (2020).

[14] M. Gromov, No metrics with Positive Scalar Curvatures on Aspherical 5-Manifolds, https://arxiv.org/abs/2009.05332 (2020).

[15] M. Gromov and H. B. Lawson, Jr., Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Inst. Hautes Études Sci. Publ. Math. No. 58 (1983), 83–196.

[16] I. M. Jaglom and V. G. Boltjanski˘ı, Convex figures, Translated by Paul J. Kelly and Lewis F. Walton Holt, Rinehart and Winston, New York 1960, xv+301 pp.

[17] D. A. Lee, Geometric Relativity, Graduate Studies in Mathematics, 201. American Mathematical Society, Providence, RI, 2019. xii+361 pp.

[18] M. Lesourd and R. Unger and S.-T. Yau, The Positive Mass Theorem with Arbitrary Ends, https://arxiv.org/abs/2103.02744 (2021).

[19] C. Li and C. Mantoulidis, Metrics with $\lambda_1(-\Delta + kR) \geq 0$ and flexibility in the Riemannian Penrose Inequality, https://arxiv.org/abs/2106.15709 (2021).

[20] P. Li and S.-T. Yau, Estimates of eigenvalues of a compact Riemannian manifold, Proc. Sympos. PureMath. 36, 1980, 205-239.

[21] C. Mantoulidis and R. Schoen, On the Bartnik mass of apparent horizons, Classical Quantum Gravity 32 (2015), no. 20, 205002, 16 pp.

[22] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 20 (1970/71), 1077–1092.

[23] L. Nirenberg, The Weyl and Minkowski problems in differential geometry in the large, Comm. Pure Appl. Math. 6 (1953), 337-394.

[24] R. Osserman, The isoperimetric inequality, Bull. Amer. Math. Soc. 84 (1978), no. 6, 1182–1238.
[25] A. V. Pogorelov, *Extrinsic geometry of convex surfaces*, Translated from the Russian by Israel Program for Scientific Translations. Translations of Mathematical Monographs, Vol. 35. American Mathematical Society, Providence, R.I., 1973, vi+669 pp.

[26] R. Schoen and S.-T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. 28 (1979), no. 1-3, 159–183.

[27] R. Schoen and S.-T. Yau, *On the Proof of the Positive Mass Conjecture in General Relativity*, Comm. Math. Phys. 65 (1979), 45-76.

[28] R. Schoen and S.-T. Yau, *The existence of a black hole due to condensation of matter*, Comm. Math. Phys. 90 (1983), no. 4, 575–579.

[29] R. Schoen and S.-T. Yau, *Lectures on Differential Geometry*, Int’l Press Boston, 2010.

[30] P. R. Scott and P. W. Awyong, *Inequalities for convex sets*, J. Inequal. Pure Appl. Math. 1 (2000), no. 1, Article 6, 6 pp.

[31] Y. Shen and R. Ye, *On stable minimal surfaces in manifolds of positive bi-Ricci curvatures*, Duke Math J. 85 (1996), no.1, 109-116.

[32] M. Troyanov, *Prescribing curvature on compact surfaces with conical singularities*, Trans. Amer. Math. Soc. 324, no.2 (1991), 793-821.

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