Universality of power law correlations in gravitational clustering

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Abstract. – We present an analysis of different sets of gravitational N-body simulations, all describing the dynamics of discrete particles with a small initial velocity dispersion. They encompass very different initial particle configurations, different numerical algorithms for the computation of the force, with or without the space expansion of cosmological models. Despite these differences we find in all cases that the non-linear clustering which results is essentially the same, with a well-defined simple power-law behaviour in the two-point correlations in the range from a few times the lower cut-off in the gravitational force to the scale at which fluctuations are of order one. We argue, presenting quantitative evidence, that this apparently universal behaviour can be understood by the domination of the small scale contribution to the gravitational force, coming initially from nearest neighbor particles.

N-body simulations (NBS) have been widely used in cosmology to study gravitational many-body dynamics in the non-linear regime (where fluctuations in the density become large). The principal goal of such studies, and indeed of the theories of large scale structure formation, is to understand how from some initial conditions (IC), specified as a continuous density field with correlated Gaussian fluctuations, the large-scale structures observed in the distributions of galaxies and clusters can arise. Specifically the problem is to relate the very small-amplitude fluctuations in the Universe at very early times, detected indirectly by observations of fluctuations in the cosmic microwave background, to the large-amplitude fluctuations with power-law correlations observed today in galaxy red-shift surveys. In this letter we discuss the nature and origin of the power-law clustering observed in cosmological NBS in the non-linear regime. We do this by placing them in the context of a broader class of gravitational NBS, identifying in these a universal behaviour in the non-linear clustering which develops, characterised by the exponent of the two-point power-law correlation function. Viewing the problem in this wider context, we argue that the nature of clustering in the non-linear regime has little to do with the initial fluctuations, or with the Universe being in expansion. Rather it is associated with what is common to all these simulations: their
evolution in the non-linear regime is dominated by fluctuations at small scales, which are similar in all cases at the time this clustering develops. This corresponds to domination by nearest neighbor interactions when the first non-linear structures are formed.

In cosmological NBS what one wants to model is (usually) the evolution of the cold dark matter (CDM) particles of current standard cosmological theories. In practice numerical limitations mean that the mass represented by the “particles” in NBS (which must simulate a significant portion of the Universe) is typically of order that of a galaxy i.e. many orders of magnitude larger than the microscopic mass of a CDM particle. Thus a “particle” should represent a collisionless fluid element rather than a physical particle. In practice however what one simulates in almost all cases is effectively a system of point particles: the smoothing scale $\epsilon$ introduced to cut-off the gravitational force is much smaller than the initial interparticle distance. At most certain algorithms (e.g. ‘particle-mesh’ type) make this scale comparable to the initial interparticle distance.

In a series of papers Melott & collaborators [2–4] have discussed the effects of discretization in NBS, showing discrepancies in the dynamical evolution described by different algorithms. In particular they have questioned the capacity of high resolution NBS (HRNBS) to describe correctly the evolution of a collisionless self-gravitating fluid. In a recent article [5] we have developed the central point touched on in this work further. Rather than considering the problem of whether, or in what range, such codes can describe fluid-like evolution, we have concentrated on what is actually described by these HRNBS. On the basis of an analysis with real space statistics of the Virgo consortium’s cosmological HRNBS [6] we make the case that discreteness is not only important because it introduces physical effects which should be absent in the fluid-like growth of perturbations, but that it is an essential element in the formation of power-law correlated structures. This is one of the points for which we produce further evidence in this paper, making use of our analysis of a wider class of HRNBS simulations.

The characteristics of the NBS we analyse are summarised in Table 1. The parameters characterising the simulations vary greatly: the number of particles N varies by almost five orders of magnitude, while the ratio of the smoothing length in the gravitational force $\epsilon$ to the initial mean interparticle distance $\langle A_i \rangle$ varies by a factor of ten (and is much smaller than unity for all of our chosen simulations, as discussed above). The first set are simulations of the purely Newtonian case, without any cosmological space expansion [7, 8]. The second set, which are NBS of the Virgo Consortium [6] are of the evolution in specific cosmological models, characterised by different values of the various free parameters in currently studied models. These sets differ also in the algorithms which are used for the calculation of the

| Simulation  | N   | L   | $\langle A_i \rangle$ | $\epsilon$ | Code   | Ref. |
|------------|-----|-----|----------------------|-----------|--------|------|
| POISSON NE | $32^3$ | 32  | 1                    | 0.01      | Tree code [7] |
| POISSON NE | $8^3$  | 8   | 1                    | 0.01      | Tree code |
| SHUF.LAT NE | $8^3$  | 8   | 1                    | 0.1       | Tree code |
| SHUF.LAT NE | $32^3$ | 32  | 1                    | 0.1       | Tree code |
| SCDM E    | $256^3$ | 239.5 | 0.94                 | 0.036     | AP$^3$M [6] |
| SCDM E    | $256^3$ | 85.4  | 0.33                 | 0.036     | AP$^3$M [6] |
| $\tau$CDM E | $256^3$ | 239.5 | 0.94                 | 0.036     | AP$^3$M [6] |

**Table I – Details of the NBS analysed. (N)E indicates (no) expansion. See text for explanation**
gravitational force in each case, the former using a tree code\(^{(1)}\) and the latter an adaptive P\(^3\)M code. Note that while the lengths characterising the initial configurations in Table 1 are given in terms of an arbitrary length unit, in the cosmological sets the length \(L\) corresponds to the side of the simulated box in \(\text{Mpc}\). In this context one must choose a physical scale for the box, as the IC fix a length scale (prescribing an amplitude for density fluctuations at a certain scale at the initial time); the mass density of the Universe then fixes the mass of the simulated particles, and the simulation is run always for a time corresponding roughly to the age of the Universe. While in the first set the simulations are simply run until the clustering becomes affected by the box size, in the cosmological context the study is limited to a (small, for most of the NBS) part of the range of time evolution which could be correctly described by the simulation.

The other important difference between these simulations is in their IC. The first (non-cosmological) set starts with two very different configurations: a Poisson distribution of points and a ‘shuffled lattice’ distribution. The latter is produced by applying a stochastic uncorrelated displacement to each point of a perfect cubic lattice with unitary lattice constant, the displacement vector being random both in orientation and length, the latter being sampled from a uniform probability distribution up to a maximum displacement \(|\vec{\eta}| \ll \langle \Lambda_i \rangle\). In the cosmological simulations IC are generated in a very particular way: to represent the small initial fluctuations (typically \(\delta \rho/\rho \sim 10^{-2}\)) in the CDM fluid correlated displacements are applied to an initially “uniform” distribution (either a ‘glassy’ configuration\([6]\), obtained by evolving first with the sign of gravity reversed, or, more often, a simple lattice). The nature of the correlation in the displacements is determined by the power spectrum, which varies from model to model (the \(\tau\)CDM model in Table 1, for example, differs from the SCDM only in this point). We note that these different IC cover a very wide range in terms of their power spectra \(P(k)\). For the Poisson IC we have a constant \(P(k)\), while the shuffled lattice\([10]\) has \(P(k) \sim k^2\) for \(2\pi/L < k < 2\pi/\langle \Lambda_i \rangle\). The cosmological IC on the other hand have, in this range, \(P(k) \sim k^n\) with \(-1 < n < -3\). Note also that all these IC, except Poisson, also have non-trivial higher order correlation properties. We will return to this point below.

Let us now turn to our results. To characterise the clustering observed in the simulations we consider simply the behaviour of the conditional density \([9]\) defined as \(\Gamma(r) = \langle n(r) n(0) \rangle / \langle n \rangle\), where \(n(r)\) is the microscopic number density. We have then considered its volume average \(\Gamma^*(r)\) \([9]\), which represents the mean density of points in a ball of radius \(r\) about an occupied point. The regime of small fluctuations corresponds in terms of it to \(\Gamma^*(r) \approx \langle n \rangle\), where \(\langle n \rangle = N/\Lambda^3\), the mean particle density in the simulation box. Within each simulation we observe the same qualitative behaviour of this quantity observed by \([7,8]\) in the Poisson NE simulation, and by ourselves\([5]\) in the Virgo simulations. Strong clustering (corresponding to \(\Gamma^*(r) > \langle n \rangle\)) develops first at scales well below the initial mean interparticle separation \(\langle \Lambda_i \rangle\), with a characteristic power-law form from a little above the smoothing scale \(\epsilon\). Subsequently the evolution is observed to be self-similar, the same power-law form simply translating to larger and larger scales, until (in the case that the evolution is continued that far) boundary effects become important. In Figure 1 we show in a single plot the \(\Gamma^*(r)\) for each of the simulations in Table 1. For this comparison we have normalised in each case \(\Gamma^*(r)\) to \(\langle n \rangle\) (so that \(\Gamma^*(r)/\langle n \rangle \rightarrow 1\) at large distances) and we have performed an arbitrary normalization on the \(x\)-axis, as here we are interested in the slopes of the power-law decay and not in the range of scale where clustering develops. A simple power-law fits \(\Gamma^*(r) \sim r^{-\gamma}\) the data in each case, in the range between a few times the smoothing scale \(\epsilon\) up to a scale around \(\gamma \approx 2\langle n \rangle\), corresponding to a range of scales between one and two decades. Performing fits in this range

\(^{(1)}\)http://www.mpa-garching.mpg.de/gadget/
we estimate $\gamma = 1.6 \pm 0.1$ for the cosmological NBS, $\gamma = 1.5 \pm 0.2$ and $\gamma = 1.4 \pm 0.2$ respectively for the Poisson and shuffled lattice NBS. The larger estimated error bars in the latter reflect the more limited range of non-linearity in these much smaller NBS.

From our results we now argue for three conclusions about the nature of clustering in the non-linear regime observed in these NBS. With respect to cosmological NBS, we conclude that the exponent characterising the non-linear clustering observed has essentially nothing to do with (i) the expansion of the Universe, or (ii) the nature of the small initial fluctuations imposed in the IC. We further present evidence for the qualitative description of the dynamics driving this clustering given in [7, 8] based on the Poisson case, and in [5] based on a similar analysis of the Virgo simulations: (iii) The non-linear clustering develops from the large fluctuations intrinsic to the particle distribution at small scales (specifically around the smallest resolved scale $\epsilon$). In particular we show here that the exponent characterising it can be seen to emerge at early times in the simulations when the evolution is well approximated as being due only to the interactions between nearest neighbour (NN) particles.

The first point can be understood easily. Given the (observed) self-similarity in the evolution in the non-expanding case, it is natural that it survives, essentially unmodified, in the cosmological case: The expansion of the Universe, for which the characteristic time is long compared to that of the non-linear clustering, is simply an adiabatic rescaling of the physical scales. This can modify the amplitude of the correlation function, but should not change the exponent if the latter is indeed determined by the fluctuations at small scales.

Before discussing the second and third point it is useful to describe in a little detail the nature of the fluctuations in the IC of these different simulations. The initial configuration of particles may be characterised by its correlation functions. The reduced two-point correlation function, defined as $\tilde{\xi}(r) = \langle n(\mathbf{r})n(0)\rangle/\langle n \rangle^2 - 1$, for any point distribution can be written as [10]:

$$\tilde{\xi}(r) = \delta(r)/\langle n \rangle + \xi(r).$$  \hspace{1cm} (1)

The variance in the number of points $N$ in a volume $V$ is given by

$$\Sigma^2 = \langle (\Delta N)^2 \rangle = \langle n \rangle V + \langle n \rangle^2 \int_V d^3r_1 \int_V d^3r_2 \xi(|\mathbf{r}_1 - \mathbf{r}_2|)$$  \hspace{1cm} (2)
where $\Delta N = N - \langle N \rangle$. The first term, which comes from the ‘diagonal’ ($r = 0$) term in $\tilde{\xi}(r)$, describes the fluctuations intrinsic to any point distribution, while the second the fluctuations associated with whatever non-trivial spatial correlation there is in the distribution. Note that this first term is specific to point distributions, in which fluctuations are never absent, and have large amplitude (of order one) at small scale; in a continuous distribution, instead, only the second term is present in both expressions and fluctuations, which can be arbitrarily small at all scales, are uniquely associated with correlations.

The Poisson distribution is the uncorrelated point distribution, with $\xi = 0$, in which only the first term in (2) is non-zero. Both the perfect lattice and the ‘glassy’ distribution used as ‘pre-initial’ distributions (in both the ‘shuffled lattice’ simulation and all the cosmological NBS) are in a specific class of discrete distributions: they are extremely ordered distributions, with a $\xi(r)$ non-zero at all scales, describing delicately balanced correlations and anti-correlations, with a variance which grows in proportion to the surface of the volume [10]. This is in fact the slowest possible growth of the fluctuations in any point distribution i.e. these distributions are the most uniform possible point distributions. It is essential, however, to note that they are not uniform. At the scales of the interparticle distance there are large amplitude fluctuations ($\delta \rho/\rho \sim 1$), like in a Poisson distribution, which then decay rapidly ($\delta \rho/\rho \propto r^{-2}$ in a sphere of radius $r$, compared to $\delta \rho/\rho \propto r^{-3/2}$ in a Poisson distribution).

When a displacement field is superimposed on these latter distributions the two-point correlation properties, and fluctuations, are modified. Given that the displacement fields are small we can write $\Sigma^2(r) = \Sigma_P^2(r) + \Sigma_D^2(r)$, where the subscripts denote the ‘pre-initial’ and ‘displacement’ contributions. The new term is sub-dominant on small scales, but can dominate at larger scales if the small amplitude fluctuations superimposed decay less rapidly than those in the ‘pre-initial’ distribution.

The universality of the non-linear exponent for which we have produced evidence above clearly can be explained only by something which is common to all these distributions. We will argue here that it is essentially the first term in the variance in (2), which is common to all these distributions, which at the root of this universality. This interpretation is in contrast to the standard one given of cosmological NBS [6], in which it is supposed that only the variance introduced by the displacement field which is dynamically relevant. Thus the evolution is understood to depend only on the small amplitude correlated fluctuations of the continuous CDM fluid of the corresponding theoretical model (but see [11]). Note, however, we are not claiming that these latter fluctuations play no role in the evolution of the system, but only that the exponent in the non-linear clustering does not significantly depend on them.

What is important for the evolution under gravity is evidently the relation between these intrinsic and imposed fluctuations and the gravitational force which acts on the particles. Let us consider first the gravitational force and the evolution of the system in the Poisson case. In these IC the gravitational force on a point is dominated by that due to its nearest neighbour (NN) [13]. The contribution from points further away cancels very efficiently due to statistical isotropy and the trivial higher order correlation properties of the distribution. To study the role of these large NN interactions in the evolution of clustering, we have modified an NBS to include only this NN contribution to the gravitational force. The result is shown in Figure 2. We see that the evolution of the non-linear clustering at early times is very well described, and in particular the exponent describing the clustering.

The other two classes of IC show very different qualitative behavior. For the shuffled lattice the (modulus of the) force on a particle fluctuates around a non-zero but small (compared to the NN contribution) value. In the cosmological IC we have a different behaviour: the correlation of the displacements leads to a growth of the force as function of scale, so that the latter is in fact dominated by the (small amplitude) perturbations at large scales [11].
These very different qualitative behaviours of the initial force distribution reflect the very specific correlation properties of the “pre-initial” lattice or glass configurations. They are configurations in which the force on a point is zero because there are non-trivial correlations, in particular at small scales (at the order of the interparticle distance) which mean that the forces of nearby particles exactly cancel. These are not, however, as we noted in [5], properties that are preserved by the gravitational evolution. Once particles start to move, due to the initial fluctuations and on a time scale determined by these, on a scale comparable to the interparticle separation, we expect the NN contribution to the force to become very important. Further, given that strong correlation develops starting from scales well below the initial interparticle separation in all these simulations, this contribution inevitably is important at the time when these structures form. Let us consider this point more quantitatively. In Figure 3 we show the average value of the modulus of the difference in the (vector) forces acting on two randomly chosen particles with a separation of less than $\langle \Lambda_i \rangle$. It is given as a function of the radius of a sphere centred on each particle over which we integrate to calculate the forces. We see that in all cases the difference in the forces - which will determine the relative motion of the particles - is dominated by small scales, around the NN distance.

This analysis applies clearly only up to the time at which clustering develops at scales of the order of the initial interparticle separation. At larger times what is observed is that the non-linear clustering which develops first at these scales develops in a self-similar manner at larger and larger scales. The self-similarity refers to the fact that the exponent of the correlation function remains approximately the same. The fact that this is so suggests very strongly that the dynamics at play is the same as that at early times, which is essentially that of particles interacting by NN forces. The evolution of the system would then be described as defining a coarse-graining to new “particles” as a function of time. This is intrinsically
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A dynamics of a discrete system in which the fluctuations at the smallest (inter-“particle”) distance are those which are dominant in the gravitational evolution. Note that this does not mean that the small fluctuations at large scales imposed by the IC are irrelevant in the evolution of the system. We have noted that they will play a role in fixing the time scales for the evolution, and thus in fixing the amplitude of the correlations (and non-linearity scale) as a function of time. This may be in keeping with what is envisaged in the cosmological context (through the linear amplification of power at large scales). The point we have made here is that the fluctuations at small scales (at the NN scale at early times) appear to be those essential to the formation of the power-law correlation functions.

A more quantitative description of this dynamics is evidently needed, with the principal goal of understanding the specific value observed of the exponent. In the cosmological literature (see e.g. [1]) the idea is widely dispersed that the exponents in non-linear clustering are related to that of the initial power-spectrum of the small fluctuations in the CDM fluid, and even that the non-linear two-point correlation can be considered an analytic function of the initial two-point correlations [14] (although, see [15] where more emphasis is put on the tendency for IC to be washed out in the non-linear regime). The models used to explain the behaviour in the non-linear regime usually involve both the expansion of the Universe, and a description of the clustering in terms of the evolution of a continuous fluid. We have argued that the exponent is universal in a very wide sense, being common to the non-linear clustering observed in the non-expanding case. It would appear that the framework for understanding the non-linear clustering must be one in which discreteness (and hence intrinsically non-analytical behaviour of the density field) is central, and that the simple context of non-expanding models should be sufficient to elucidate the essential physics.

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REFERENCES

[1] P. J. E. Peebles, Principles Of Physical Cosmology (Princeton University Press, NJ) 1993.
[2] Melott A.L., Comments Astrophys., 15, (1990) 1.
[3] Splinter R.J., Melott A.L., Shandarin S.F. and Suto Y., Astrophys.J., 497, (1998) 38.
[4] Kuhlman B., Melott A.L. & Shandarin S.F., Astrophys.J., 470, (1996) L41.
[5] Baertschiger T., Joyce M. & Sylos Labini F., Astrophys.J., 581, L63, 2002.
[6] Jenkins A. et al., Astrophys.J., 499, (1998) 20.
[7] Bottaccio M. et al., Europhys. Lett., 57, (2002) 315.
[8] Bottaccio M., Ph.D. Thesis, University of Rome “La Sapienza”, Italy (2002).
[9] Sylos Labini F., Montuori M. and Pietronero L., Phys.Rep., 293, (1998) 66.
[10] Gabrielli A., Joyce M. and Sylos Labini F., Phys.Rev., D65, (2002) 083523.
[11] Baertschiger T., & Sylos Labini F., Europhys.Lett., 57, (2002) 322.
[12] Gabrielli A., Sylos Labini F. and Pellegrini S., Europhys. Lett., 46, (1999) 127.
[13] Chandrasekhar S., Rev. Mod. Phys., 15, (1943) 1.
[14] Peacock J. and Dodds S., Mon. Not. R. Astron. Soc., 280, (1996) L19.
[15] Saslaw W.C., The Distribution of the Galaxies (Cambridge University Press, Cambridge, England) 2000.