Convergence-property improvement of GMRES in shielding current analysis of cracked high-temperature superconducting film

Atsushi Kamitani¹,*, Terou Takayama¹, Ayumu Saitoh¹, Soichiro Ikuno²

¹Graduate School of Science and Engineering, Yamagata University
²School of Computer Science, Tokyo University of Technology

¹kamitani@yz.yamagata-u.ac.jp

Abstract. A high-performance method is proposed for solving a linear system in the shielding current analysis of a cracked high-temperature superconducting film. After discretized with respect to time and space, the initial-boundary-value problem of the shielding current density reduces to a linear system at each iteration cycle of the Newton method. Although the linear system can be easily solved with GMRES, both crack length and the number of cracks affect convergence property significantly. In order to improve convergence property of GMRES, other variables than corrections of the current vector potential are all eliminated from the linear system. As a result, convergence property of GMRES is hardly influenced by either crack length or the number of cracks.

Keywords: Convergence of numerical methods, Finite element method, Newton method, Superconducting films

1. Introduction

Recently, a high-temperature superconducting (HTS) film has been used for numerous engineering applications: magnet, energy storage system, power cable and so on. Since evaluation of the shielding current density is indispensable in designing such engineering applications, several numerical methods [1–4] for analyzing the shielding current density have been so far developed on the basis of the current vector potential method.

After discretized with respect to time, an initial-boundary-value problem of the shielding current density is transformed to a nonlinear boundary-value problem at each time step. However, the solution of the nonlinear problem by the Newton method is extremely time-consuming because a linear system with a dense matrix has to be solved at each iteration cycle of the Newton method. Recently, the authors showed that, if GMRES [5] is adopted as a linear-system solver, the computational cost will be reduced to $O(n^2)$ at each time step [4].
Here, $n$ is the number of nodes. However, for the case with a cracked superconducting film, convergence property of GMRES will be influenced by cracks and, hence, the computational cost will be also raised.

The purpose of the present study is to propose a method for improving convergence property of GMRES in the shielding current analysis of an HTS film with cracks. In addition, by applying the proposed method to the scanning permanent-magnet method (SPM) [6, 7], its performance is numerically investigated.

2. Shielding current analysis of cracked HTS film

2.1. Initial-boundary-value problem

We first assume that an HTS film has the same cross section $\Omega$ over the thickness and that it is exposed to the time-varying magnetic field. By taking its thickness direction as $z$-axis and choosing its centroid as the origin, we use the Cartesian coordinate system $(O : e_x, e_y, e_z)$. Furthermore, the HTS film is assumed to contain $m$ cracks whose cross sections are line segments in the $xy$ plane. Note that the boundary $\partial \Omega$ of $\Omega$ is composed of not only the outer boundary $C_0$ but also crack surfaces $C_1, C_2, \cdots, C_m$. In the following, $t$ and $n$ are a unit tangent vector and a unit normal vector on $\partial \Omega$, respectively, and $x$ and $x'$ denote position vectors of two points in the $xy$ plane.

In HTS films, the electric field $E$ and the shielding current density $j$ are closely related to each other through the $J$-$E$ constitutive relation:

$$E = E(|j|) \frac{j}{|j|},$$  \hspace{1cm} (1)$$

where the function $E(j)$ characterizes the electromagnetic behavior of HTS films. As the function $E(j)$, we assume the following power law [2–4, 7–10]:

$$E(j) = E_C \left( \frac{j}{j_C} \right)^N,$$  \hspace{1cm} (2)$$

where $j_C$ and $E_C$ denote the critical current density and the critical electric field, respectively, and $N$ is a positive constant.

Under the thin-plate approximation, there exists a scalar function $T(x, t)$ such that $j = (2/b) \nabla \times (Te_z)$, and its time evolution is governed by the following equation [1–4, 7]:

$$\mu_0 \frac{\partial}{\partial t} (\hat{W}T) + (\nabla \times E) \cdot e_z = -\frac{\partial}{\partial t} \langle B \cdot e_z \rangle.$$  \hspace{1cm} (3)$$

Here, $B$ and $\mu_0$ are the applied magnetic flux density and the permeability of vacuum, respectively, and $b$ denotes the film thickness. In addition, $T(x, t)$ is $z$-component of the current vector potential and $\langle \rangle$ denotes an average operator over the thickness. Furthermore, the operator $\hat{W}$ is defined by

$$\hat{W}T = \frac{2T(x, t)}{b} + \iint_{\Omega} Q(|x - x'|) T(x', t) d^2x',$$
where the integration kernel \( Q(r) \) is given by

\[
Q(r) = -\frac{1}{\pi b^2} \left( \frac{1}{r} - \frac{1}{\sqrt{r^2 + b^2}} \right).
\]

Also, the two-dimensional integral of function \( f(x) \) over \( \Omega \) is denoted by \( \int_\Omega f(x') \, d^2x' \).

The initial and boundary conditions to (3) can be written as follows:

\[
T = 0 \text{ at } t = 0; \quad (4)
\]

\[
T \in H(\bar{\Omega}) \equiv \left\{ w(x) : w = 0 \text{ on } C_0, \frac{\partial w}{\partial s} = 0 \text{ on } C_i \ (i = 1, 2, \cdots, m) \right\}; \quad (5)
\]

\[
h_i(E) \equiv \int_{C_i} E \cdot t \, ds = 0 \ (i = 1, 2, \cdots, m). \quad (6)
\]

Here, \( s \) is an arc length along crack surfaces. By solving (3) together with the initial and boundary conditions, we can analyze the time evolution of the shielding current density \( j \).

Incidentally, (5) is derived from \( j \cdot n = 0 \) on \( \partial \Omega \), whereas (6) is required for uniqueness of the solution for the above initial-boundary-value problem.

2.2. Scanning permanent magnet method

In the present study, the above analysis is applied to the SPM that is one of contactless methods for detecting cracks in HTS films. A schematic view of the SPM is shown in Fig. 1. In the SPM, a cylindrical permanent magnet of radius \( R \) and height \( H \) is moved along the film surface and, simultaneously, an electromagnetic force acting on the film is monitored. During the movement of the magnet, the distance \( L \) between the magnet bottom and the film surface is kept constant. In the following, the cross section \( \Omega \) of the film is assumed as a rectangle of width \( w \) and length \( Aw \). Furthermore, the longitudinal direction of the film is taken as \( x \)-axis. Also, the symmetry axis of the magnet is denoted by \((x, y) = (x_A, y_A)\), and its movement is assumed as \( x_A = vt - Aw/2 \) and \( y_A = \text{const} \). Here, \( v \) is a scanning speed.

Throughout the present study, the physical and geometrical parameters are fixed as follows: \( j_C = 1 \text{ MA/cm}^2, \ E_C = 1 \text{ mV/m}, \ N = 20, \ b = 1 \mu m, \ w = 12 \text{ mm}, \ A = 11, \ R = 0.8 \text{ mm}, \ H = 2 \text{ mm}, \ L = 0.5 \text{ mm}, \ y_A = 0 \text{ mm}, \) and \( v = 10 \text{ cm/s} \).

Two types of cracked HTS films are used in the present study: type-A film and type-B film (see Figs. 2(a) and 2(b)). Only one crack is contained in type-A film and its cross
Figure 2: Cross sections of (a) type-A film and (b) type-B film. In these figures, red line segments denote cracks.

section is assumed to be a line segment connecting two points, \((\pm L_c/2, 0 \text{ mm})\). In contrast, \(m\) cracks are contained in type-B film and the cross section of the \(p\)th crack is assumed to be a line segment connecting two points, \((-Aw/2 + p\Delta x, \pm L_c/2)\) \((p = 1, 2, \ldots, m)\). For type-B film, \(\Delta x\) and \(L_c\) are fixed as \(\Delta x = 8.25 \text{ mm}\) and \(L_c = 3 \text{ mm}\).

3. Virtual-voltage method

In this section, superscript \((j)\) denotes a value at time \(t = j\Delta t\), where \(\Delta t\) is a time-step size. If the initial-boundary-value problem of (3) is discretized with respect to time by using the complete implicit method, \(T^{(j)}(x) \equiv T(x, j\Delta t)\) becomes a solution of the following nonlinear boundary-value problem:

\[
G(T) \equiv \mu_0 \hat{W} T + \Delta t e_z \cdot (\nabla \times E) - u = 0 \quad \text{in } \Omega, \tag{7}
\]

\[
T \in H(\bar{\Omega}), \tag{8}
\]

\[
h_i(E) = 0 \quad (i = 1, 2, \ldots, m), \tag{9}
\]

where \(u \equiv \mu_0 \hat{W} T^{(j-1)} - \langle (B^{(j)} - B^{(j-1)}) \cdot e_z \rangle\). As is apparent from the definition of \(h_i(E)\) in the previous section, \(h_i(E)\) is a voltage around the crack surface, \(C_i\).

Although the above boundary-value problem can be easily solved with the finite element method (FEM), the accuracy of its numerical solution will be degraded with a decrease in the film thickness. Especially, if \(h_i(E)\) is numerically evaluated by means of the Gauss-Legendre quadrature and the resulting value is denoted by \(N_i(E)\), \(N_i(E)\) does not always become negligibly small [3]. This difficulty is attributable to the fact that (9) is treated as the natural boundary condition. In other words, (9) is completely incorporated into the weak
form, that is the starting point of the FEM. As a result, \( N_i(E) \) is never computed before the numerical solution of (7)–(9) is obtained.

In order to resolve the above difficulty, the authors proposed the virtual-voltage method [3,4]. In the method, (9) is replaced with \( h_i(E) = \phi_i \) and \( N_i(E) = 0 \) so as to have \( N_i(E) = 0 \) exactly satisfied \((i = 1, 2, \cdots, m)\). Here, \( \phi_1, \phi_2, \cdots, \phi_m \) are unknown parameters and they are controlled so that numerically evaluated voltages, \( N_1(E), N_2(E), \cdots, N_m(E) \), around cracks may vanish. Furthermore, \( h_i(E) = \phi_i \) and \( N_i(E) = 0 \) are treated as the natural and essential boundary conditions, respectively. The resulting boundary-value problem is solved for \((T, \{\phi_i\}_{i=1}^m)\) by means of the Newton method.

At each iteration cycle of the Newton method, the following linear boundary-value problem is solved for \((\delta T, \{\delta \phi_i\}_{i=1}^m)\):

\[
\begin{align*}
\delta G &= -G(T), \\
\delta T &\in H(\Omega), \\
\delta h_i - \delta \phi_i &= -[h_i(E) - \phi_i] \quad (i = 1, 2, \cdots, m), \\
\delta N_i &= -N_i(E) \quad (i = 1, 2, \cdots, m).
\end{align*}
\]

Here, \( \delta T \) and \( \delta \phi_i \) are corrections of \( T \) and \( \phi_i \), respectively, whereas \( \delta G \), \( \delta h_i \), and \( \delta N_i \) denote Fréchet derivatives of \( G(T) \), \( h_i(E) \) and \( N_i(E) \), respectively.

If the above linear boundary-value problem is spatially discretized by means of the FEM with \( n \) nodes, we get

\[
\begin{bmatrix} A(T) & C & H \\ C^T & O & O \end{bmatrix} \begin{bmatrix} \delta T \\ \lambda \end{bmatrix} = \begin{bmatrix} G(T, \phi) \\ 0 \end{bmatrix}
\]

\[
\begin{bmatrix} D^T(T) & O & O \end{bmatrix} \begin{bmatrix} \delta \phi \end{bmatrix} = \begin{bmatrix} N(T) \end{bmatrix}
\]

Here, \( T \) and \( \delta T \) are nodal vectors corresponding to \( T \) and \( \delta T \), respectively, \( \delta \phi \) is defined by \( \delta \phi \equiv [\delta \phi_1, \delta \phi_2, \cdots, \delta \phi_m]^T \), and \( \lambda \) is a \( k \)-dimensional unknown vector. In addition, \( A(T) \in \mathbb{R}^{n \times n}, \ C \in \mathbb{R}^{n \times k} \) and \( H \in \mathbb{R}^{n \times m} \) are derived from the weak form equivalent to (10) and (12). Also, \( D^T(T) \delta T = N(T) \) is a discretized form of (13). Furthermore, \( G(T, \phi) \in \mathbb{R}^n \) and \( N(T) \in \mathbb{R}^m \) are vector-valued functions. Also, \( 2k \) denotes the number of elements adjacent to crack surfaces and it is assumed to satisfy \( n \gg k \). In the following, \( k \) is called the number of crack elements. Note that \( A(T) \) is a symmetric dense matrix. In contrast, both \( H, D(T) \in \mathbb{R}^{n \times m} \) and \( C \in \mathbb{R}^{n \times k} \) are non-square matrices. Incidentally, (12) and (13) are treated as the natural and essential boundary conditions, respectively.

By solving the linear system (14), we can determine corrections, \( \delta T \) and \( \delta \phi \), in the iteration cycle of the Newton method.

### 4. Linear-system solver

The linear system (14) changes depending both on the iteration cycle of the Newton method and on time. Hence, throughout this section, two linear-system solvers are applied to (14) for the 1st iteration cycle at \( t = Aw/(1200v) \) and their performances are compared with each other. Numerical computations were carried out on FUJITSU PRIMEHPC FX100 of LHD Numerical Analysis Server in National Institute of Fusion Science.
4.1. Conventional method

Since the numerical solution of (14) is a rate-determining step in the virtual-voltage method, it is desirable to apply a fast and stable linear-system solver to (14). For this reason, the authors have applied GMRES to the solution of (14) [4].

As is well known, the restart technique is usually effective in reducing the operation count and the memory requirement in GMRES. For the purpose of investigating the effect of the restart technique on GMRES, the restarted version of GMRES is applied to the solution of (14). In Appendix A, how the restart affects convergence property of GMRES is explained in detail. The results of computations show that convergence property of GMRES is remarkably degraded with a decrease in the restart parameter. In other words, it is very difficult to properly determine the restart parameter in solving (14) with GMRES. Hence, the restart technique is not employed in GMRES throughout the present study.

In general, convergence property of GMRES might be improved by using the preconditioning. On the other hand, most elements of the coefficient matrix in (14) are occupied by the dense submatrix $A(T)$ and, hence, the coefficient matrix in (14) is nearly a dense matrix. Thus, if the incomplete $LU$ decomposition is used as the preconditioning for GMRES, its computational cost amounts to $O(n^3)$. In contrast, the computational cost per each iteration of GMRES is $O(n^2)$. Therefore, in solving (14), the incomplete $LU$ decomposition is inappropriate as the preconditioning. For this reason, the preconditioning is not incorporated to GMRES throughout the present study.

As mentioned above, neither the restart technique nor the preconditioning is applied to GMRES throughout the present study. In the following, the numerical solution of (14) with GMRES is called the conventional method.

Let us first investigate how residual histories of the conventional method are affected either by the number $k$ of crack elements or by the number $m$ of cracks. To this end, residual histories are determined for various values of $k$ or $m$, and they are depicted in the insets of Figs. 3(a) and 3(b). These figures suggest that an increase in $k$ or $m$ will degrade convergence property of GMRES remarkably.

In order to quantitatively investigate convergence property, we use the number of iterations required for convergence of GMRES. Hereafter, it is called the convergent iteration number, $N_c$. Dependences of $N_c$ on $k$ or $m$ are numerically determined and they are depicted in Figs. 4(a) and 4(b). The convergent iteration number $N_c$ increases linearly with $k$, whereas it increases monotonously with $m$ until reaching a constant value. In other words, even if the number of nodes is kept constant, an increase either in crack length or in the number of cracks will raise the computational cost of the conventional method. Such an increase in the computational cost might be attributable to the fact that the linear system (14) contains $\lambda$ and $\delta\phi$.

4.2. Variable-reduction method

In order to suppress strong dependences of the computational cost for the conventional method on crack size and on the number of cracks, both $\lambda$ and $\delta\phi$ have to be eliminated
Figure 3: Residual histories of the variable-reduction method. Here, (a) and (b) are obtained for type-A film and type-B film, respectively, and the number of nodes is fixed as \( n = 11649 \). Insets of these figures show residual histories of the conventional method for the same film.

From (14). For this purpose, the QR factorization [11] of the matrix \( C \) is first computed as

\[
C = Q_C R_C P_C^T ,
\]

(15)

where \( R_C \in \mathbb{R}^{k \times k} \) and \( P_C \in \mathbb{R}^{k \times k} \) denote an upper triangular matrix and a permutation matrix, respectively, and \( Q_C \in \mathbb{R}^{n \times k} \) is a matrix such that \( Q_C^T Q_C = E \). Similarly, two matrices, \((E - Q_C Q_C^T)H\) and \((E - Q_C Q_C^T)D(T)\), are QR factorized as

\[
(E - Q_C Q_C^T)H = Q_H R_H P_H^T ,
\]

(16)

\[
(E - Q_C Q_C^T)D(T) = Q_D(T) R_D(T) P_D^T(T) .
\]

(17)
Figure 4: Dependence of the convergent iteration number $N_c$ on (a) the number $k$ of crack elements and (b) the number $m$ of cracks. Here, (a) and (b) are obtained for type-A film and for type-B film, respectively, and the number of nodes is fixed as $n = 11649$.

Besides, the projection matrices, $U_H$, $U(T)$ and $F(T)$, are defined by

$$U_H \equiv E - Q_C Q_C^T - Q_H Q_H^T,$$
$$F(T) \equiv Q_C Q_C^T + Q_H Q_H^T (T),$$
$$U(T) \equiv E - F(T).$$

After a straightforward algebra, we get the following equations equivalent to (14):

$$A^*(T) \delta T = G^*(T, \phi),$$
$$\delta \phi = P_H R_H^{-1} Q_H^T [G(T, \phi) - A(T) \delta T].$$

Here, the matrix $A^*(T) \in \mathbb{R}^{n \times n}$ and the vector $G^*(T, \phi) \in \mathbb{R}^n$ are given by

$$A^*(T) = U_H A(T) U(T) + F(T),$$
$$G^*(T, \phi) = U_H [G(T, \phi) - A(T) c(T)] + c(T).$$
where $\mathbf{c}(T) = Q_H R_D^T(T) P_D^T(T) N(T)$. Note that neither $\lambda$ nor $\delta \phi$ is contained in (18). By solving (18) for $\delta T$ and, subsequently, substituting $\delta T$ into (19), $\delta \phi$ is also calculated. In the actual calculation, (18) is numerically solved with GMRES. Throughout the present study, this method is called the variable-reduction method.

Let us assess the performance of the variable-reduction method as a linear-system solver. To this end, residual histories are determined for various values of $k$ or $m$, and they are depicted in Figs. 3(a) and 3(b). As expected, the residual histories are hardly influenced by $k$ or $m$. For the purpose of quantitatively investigating these tendencies, dependences of the convergent iteration number on $k$ or $m$ are numerically determined and are also depicted in Figs. 4(a) and 4(b). These figures indicate that both $k$ and $m$ have little effect on the convergent iteration number.

Next, we investigate the speed of the variable-reduction method. The CPU time required for solving (14) is measured both for the conventional method and for the variable-reduction method, and its dependence on the number of nodes is depicted in Figs. 5(a) and 5(b). For the case with $L_c = 6$ mm, the variable-reduction method is slightly faster than the conven-
Table 1: Ratio of $\tau_{CH}$ to $\tau_T$. Here, the total CPU time $\tau_T$ is measured while the initial-boundary-value problem of (3) is solved from $t = 0$ and $t = Aw/(6v)$.

| number of nodes, $n$ | $\tau_{CH}/\tau_T$ (%) |
|---------------------|------------------------|
| 1729                | $1.3 \times 10^{-3}$   |
| 4641                | $4.7 \times 10^{-4}$   |
| 8961                | $2.8 \times 10^{-4}$   |

tional method. In contrast, for the case with $L_c = 45$ mm, the CPU time for the variable-reduction method becomes 18–40% less than that for the conventional method. This result means that the variable-reduction method shows the speedup effect as compared with the conventional method. In addition, the speedup effect of the variable-reduction method becomes remarkable with increasing crack length.

For the purpose of quantitatively investigating the speedup effect of the variable-reduction method, we measure the speedup ratio $\tau_N/\tau_T$. Here, $\tau_N$ and $\tau_T$ denote the CPU time required for solving (14) with the conventional method and that for solving (14) with the variable-reduction method, respectively. Note that $\tau$ does not include the CPU time $\tau_{CH}$ for the $QR$ factorizations of two matrices, $C$ and $(E - QCQ^T)$. This is because the $QR$ factorizations, (15) and (16), need to be executed only once before the time evolution of $j$. In fact, the results of computations show that the ratio of $\tau_{CH}$ to the total CPU time $\tau_T$ is always less than $1.3 \times 10^{-3}$% (see Table 1). Thus, $\tau_{CH}$ is negligibly small as compared with the total CPU time $\tau_T$ and, hence, its effect does not have to be included in the speedup ratio. The speedup ratio is measured as functions of the number of nodes and is depicted in Fig. 6(a). This figure indicates that an increase either in the number of nodes or in crack length will enhance the speedup effect. Hence, we can conclude that acceleration of the variable-reduction method becomes effective with an increase either in crack length or in the number of nodes.

4.3. Acceleration by $\mathcal{H}$-matrix method

As is well known, GMRES is an iterative linear-system solver and matrix-vector multiplications are the most time-consuming at each iteration of GMRES. Thus, in order to accelerate the variable-reduction method, matrix-vector multiplication $A^*(T)v$ has to be performed with high speed.

As is apparent from the definition (20) of $A^*(T)$, the operation count for $A(T)v$ fills a large portion of that for $A^*(T)v$. On the other hand, an FEM matrix $W$ corresponding to $W$ is a dense matrix, whereas a matrix $A(T) - W$ is a sparse one. In other words, the operation count for $Wv$ fills a large portion of that for $A(T)v$. Hence, the operation count for $A^*(T)v$ is nearly equal to that for $Wv$. This means that the variable-reduction method can be accelerated by using high-speed matrix-vector multiplication $Wv$.

In the present study, high-speed matrix-vector multiplication $Wv$ is performed by means of the $\mathcal{H}$-matrix method [12–16]. In the $\mathcal{H}$-matrix method, $W$ is first transformed to a hierarchical matrix $W^*$ on the basis of the information on node positions. Next, instead of exactly evaluating matrix-vector multiplication $Wv$, its approximate value $W^*v$ is calculated.
Figure 6: Dependence of the speedup ratio $\tau_N/\tau$ on the number $n$ of nodes. As an HTS film, type-A film is adopted. Here, (a) the variable-reduction method and (b) the variable-reduction method accelerated by the $H$-matrix method.

If many submatrices of $W^*$ are expressed by the ACA decomposition, the computation speed of $W^*v$ is much higher than that of $Wv$ [12–16].

Let us investigate the influence of the $H$-matrix method on acceleration of the variable-reduction method. To this end, the CPU time for solving (14) is measured and is also plotted in Figs. 5(a) and 5(b). These figures indicate that, by implementing the $H$-matrix method to matrix-vector multiplications in GMRES, the CPU time for the variable-reduction method can be reduced considerably. In addition, the speedup ratio is measured as functions of the number of nodes and is depicted in Fig. 6(b). We see from this figure that, like in Fig. 6(a), an increase either in crack length or in the number of nodes will enhance the speedup effect. The largest difference between Figs. 6(a) and 6(b) consists in values of the speedup ratio. For example, for the case with $n = 11649$ and $L_c = 45$ mm, the speedup ratios for the variable reduction method with and without the $H$-matrix method are 19.0 and 5.6, respectively. Thus, the $H$-matrix method further accelerates the variable-reduction method. From these results, we conclude that, when combined with the $H$-matrix method, the variable-reduction method can be a powerful tool for analyzing the shielding current density in a cracked HTS film.
Figure 7: The time dependence of the relative error for type-A film. Here, the value of $L_c$ is assumed as $L_c = 45$ mm.

Although the $\mathcal{H}$-matrix method enables high-speed matrix-vector multiplication $Wv$, it only yields an approximate vector $W^*v$ of $Wv$. Such an approximate vector is calculated only once at each iteration of GMRES, which is executed at every iteration cycle of the Newton method. Thus, an approximate vector $W^*v$ is evaluated many times at each time step. Therefore, the error between $Wv$ and $W^*v$ might accumulate so as to degrade the accuracy of $T$ at each time step. For this reason, we investigate how the $\mathcal{H}$-matrix method affects the accuracy of the variable-reduction method. As the measure of the accuracy, we employ the relative error $\epsilon = \|T_{\text{with}} - T_{\text{without}}\|_\infty/\|T_{\text{without}}\|_\infty$. Here, both $T_{\text{with}}$ and $T_{\text{without}}$ are numerical solutions of the nonlinear boundary-value problem (7)-(9) by using the variable-reduction method. The only difference between them is that $T_{\text{with}}$ and $T_{\text{without}}$ are obtained with and without the $\mathcal{H}$-matrix method, respectively. Also, $\| \cdot \|_\infty$ means the maximum norm. The relative error is calculated as a function of time and is depicted in Fig. 7. This figure indicates that, although the relative error changes violently with time, it is always less than $10^{-7}$. In this sense, the accuracy of the variable-reduction method is hardly influenced by the implementation of the $\mathcal{H}$-matrix method.

5. Conclusion

We have proposed a novel method for improving convergence property of GMRES in the shielding current analysis of a cracked HTS film. In the method, variables other than corrections of the current vector potential are all eliminated from the linear system appearing in the analysis. The proposed method is applied to the SPM and its performance is numerically evaluated. Conclusions obtained in the present study are summarized as follows:

- Convergence property of the proposed method is hardly affected by crack length or the number of cracks, whereas that of the conventional method is remarkably degraded by both of them.

- If the $\mathcal{H}$-matrix method is incorporated to matrix-vector multiplications, the proposed
Figure 8: The residual history of GMRES(l) for type-A film with $L_c = 30 \text{ mm}$. Here, the number of nodes is fixed as $n = 11649$. Also, $l = \infty$ indicates that no restart is performed.

method will be a powerful solver in the shielding current analysis of a cracked HTS film.

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Appendix A. Restart of GMRES

In GMRES, both the operation count and the memory requirement per iteration rise linearly with the iteration count. In order to resolve this problem, restarts are generally performed at every $l$ iterations. This restarted version of GMRES is called GMRES(l) and $l$ is called a restart parameter.

In this appendix, we investigate the influence of the restart parameter $l$ on convergence property of GMRES(l). To this end, residual histories are determined for various values of $l$ and are depicted in Fig. 8. For the cases with $l \leq 160$, no converged solutions are obtained even after 1000 iterations. In contrast, for the case with no restart, GMRES will converge in no more than 200 iterations. Furthermore, convergence property of GMRES(l) will be degraded with decreasing restart parameter. From these results, we can conclude that, in solving (14) with GMRES(l), it is very difficult to select a reasonable value of the restart parameter $l$. For this reason, the restart technique is not used in the present study.
References

[1] Y. Yoshida, M. Uesaka, K. Miya: Magnetic-field and force analysis of high-$T_c$ superconductor with flux-flow and creep, IEEE. Trans. Magn., 30:5 (1994), 3503-3506.

[2] A. Kamitani, T. Takayama, S. Ikuno: High-speed method for analyzing shielding current density in high-temperature superconductor, IEEE. Trans. Magn., 47:5 (2011), 1138-1141.

[3] A. Kamitani, T. Takayama, S. Ikuno: Virtual voltage method for analyzing shielding current density in high-temperature superconducting film with cracks/holes, IEEE Trans. Magn., 49:5 (2013), 1877-1880.

[4] A. Kamitani, T. Takayama, A. Saitoh: High-speed shielding current analysis in high-temperature superconducting film with cracks, IEEE. Trans. Magn., 52:3 (2016), Art. no. 7202404.

[5] Y. Saad, M. H. Schultz: GMRES - A generalized minimal residual algorithm for solving nonsymmetric linear-systems, SIAM J. Sci. Stat. Comput., 7:3 (1986), 856-869.

[6] K. Hattori, A. Saito, Y. Takano, T. Suzuki, et al.: Detection of smaller $J_c$ region and damage in YBCO coated conductors by using permanent magnet method, Physica C, 471:21 (2011), 1033-1035.

[7] A. Kamitani, T. Takayama, A. Saitoh: Numerical investigations on applicability of permanent magnet method to crack detection in HTS film, Physica C, 504 (2014), 57-61.

[8] A. Kameni, M. Boubekeur, L. Alloui, F. Bouillault, et al.: A 3-D semi-implicit method for computing the current density in bulk superconductors,” IEEE. Trans. Magn., 50:2 (201), Art. no. 7009204.

[9] L. Makong, A. Kameni, P. Masson, J. Lambrechts, et al.: 3-D modeling of heterogeneous and anisotropic superconducting media, IEEE. Trans. Magn., 52:3 (2016), Art. no. 7205404.

[10] B. Dianati, H. Heydari, SA. Afsari: Analytical computation of air-gap magnetic field in a viable superconductive magnetic gear, IEEE Trans. Appl. Supercond., 26:6 (2016), Art. no. 5205612.

[11] G. H. Golub and C. F. Van Loan: Matrix Computations, 4th ed., Johns Hopkins University Press, MD, USA, 2013.

[12] S. Kurz, O. Rain, S. Rjasanow: The adaptive cross-approximation technique for the 3-D boundary-element method, IEEE Trans. Magn., 38:2 (2002), 412-424.

[13] Y. Takahashi, C. Matsumoto, S. Wakao: Large-scale and fast nonlinear magnetostatic field analysis by the magnetic moment method with the adaptive cross approximation, IEEE Trans. Magn., 43:4 (2007), 1277-1280.
[14] M. Bebendorf: *Hierarchical Matrices: A Means to Efficiently Solve Elliptic Boundary Value Problems*, Springer, Berlin, Germany, 2008.

[15] P. Alotto, P. Bettini, R. Specogna: Sparsification of BEM matrices for large-scale eddy current problems, *IEEE Trans. Magn.*, 52:3 (2016), Art. no. 7203204.

[16] A. Ida, T. Iwashita, T. Mifune, Y. Takahashi: Variable preconditioning of Krylov subspace methods for hierarchical matrices with adaptive cross approximation, *IEEE Trans. Magn.*, 52:3 (2016), Art. no. 7205104.