Light-cone Superstring in AdS Space-time

R.R. Metsaev,$^{a,b,*}$ C. B. Thorn$^{c,†}$ and A.A. Tseytlin$^{a,‡}$

$^a$ Department of Physics, The Ohio State University
Columbus, OH 43210-1106, USA

$^b$ Department of Theoretical Physics, P.N. Lebedev Physical Institute,
Leninsky prospect 53, Moscow 117924, Russia

$^c$ Institute for Fundamental Theory
Department of Physics, University of Florida, Gainesville, FL 32611, USA

Abstract

We consider fixing the bosonic light-cone gauge for string in AdS space in the phase space framework, i.e. by choosing $x^+ = \tau$, and by choosing $\sigma$ so that $P^+$ is distributed uniformly (its density $P^+$ is independent of $\sigma$). We discuss classical bosonic string in $AdS_d$ and superstring in $AdS_5 \times S^5$. In the latter case the starting point is the action found in hep-th/0007036 where the $\kappa$-symmetry is fixed by a fermionic light-cone type gauge. We derive the light-cone Hamiltonian in the $AdS_5 \times S^5$ case and in the case of superstring in $AdS_3 \times S^3$. We also obtain a realization of the generators of the basic superalgebra $psu(2,2|4)$ in terms of $AdS_5 \times S^5$ superstring coordinate 2-d fields in the light-cone gauge.

$^*$ E-mail: metsaev@lpi.ru, metsaev@pacific.mps.ohio-state.edu
$^†$ E-mail address: thorn@phys.ufl.edu
$^‡$ Also at Imperial College, London and Lebedev Institute, Moscow. E-mail: tseytlin.1@osu.edu
1 Introduction

To better understand string theory duals of gauge theories with various amounts of supersymmetry [1] it is important to make progress in what should be the simplest case – (large \( N \)) \( \mathcal{N}=4 \) Super Yang-Mills theory dual to (weakly coupled) type IIB superstring theory in \( AdS_5 \times S^5 \) space with Ramond-Ramond flux [2]. Though this background has a lot of symmetry, solving the corresponding string theory appears to be a complicated problem. The commonly used procedure, in the known exactly solvable cases, is to start with a string action, solve the classical string equations, then quantize the theory, find the string spectrum, vertex operators, scattering amplitudes, etc. Even the first steps in this program are nontrivial in the \( AdS_5 \times S^5 \) string case.

To start with, the classical equations of bosonic string in \( AdS \) space, while completely integrable, are not explicitly solvable (in contrast to, e.g., the case of string on a group manifold described by the WZW model). The presence of a curved R-R background indicates that one should use the manifestly supersymmetric Green-Schwarz [3] description of superstring. Finding an explicit expression for the curved space GS action is difficult in general (one needs to know the component expansion of the background supergravity superfields). In the present \( AdS_5 \times S^5 \) case, this technical problem has a nice geometrical solution based on viewing string as moving on the supercoset \( \text{PSU}(2,2|4)/\text{SO}(1,4) \times \text{SO}(5) \) [4]. The resulting action, though explicitly known [4–6], is highly nonlinear containing terms of many orders in \( \theta \). The fermionic part of the action simplifies dramatically in proper \( \kappa \)-symmetry gauges – it becomes quadratic and quartic in \( \theta \) only [7–9]. Its structure is similar to that of the flat space GS action in a covariant \( \kappa \)-symmetry gauge which is also quartic in fermions (see e.g.[10]).

1.1 Review

To illustrate this point, let us consider the 4-d Lorentz covariant “S-gauge” [11] which leads to an action equivalent to the one in [7, 8] but uses the fermionic parametrization adopted in [11] and here, which is useful for comparison with the light-cone gauge actions below (see Appendix C of [11] for details). If one interprets the \( AdS_5 \times S^5 \) supergroup \( \text{PSU}(2,2|4) \) as the \( \mathcal{N}=4 \) superconformal group in 4 dimensions, it is natural to split the fermionic generators into 4 standard supergenerators \( Q_i \) and 4 special conformal supergenerators \( S_i \) (we suppress the 4-d spinor indices). The associated superstring coordinates will be denoted as \( \theta_i \) and \( \eta_i \). The covariant “S-gauge” corresponds to setting all \( \eta_i \) to zero. The resulting superstring Lagrangian written in the “4+6” parametrization in which the metric of \( AdS_5 \times S^5 \) is \( ds^2 = Y^2 dx^a dx^a + Y^{-2} dY^M dY^M \) \((a = 0, ..., 3; \; M = 1, ..., 6)\) has the following simple structure [11]${}^1$

\[
L = -\frac{1}{2}\sqrt{g} g^{\mu\nu} \left( Y^2 L_\mu^a L_\nu^a + Y^{-2} \partial_\mu Y^M \partial_\nu Y^M \right) - (i\epsilon^{\mu\nu} \partial_\mu Y^M \bar{\theta}^\alpha \gamma^a_M \partial_\nu \theta_3^\alpha + h.c.) ,
\]

\[
L_\mu^a \equiv \partial_\mu x^a - (i\theta_2 (\sigma^a)_{ab} \partial_\mu \theta_1^b + h.c.) ,
\]

\[{}^1\text{The actions in [7, 8] have an isomorphic form, corresponding to a specific choice of the 10-d Dirac matrix representation.}\]
where $a, \hat{b}$ are the $sl(2, C)$ (4-d spinor) indices and $\theta^{\dagger}_{ai} = -\theta^{\dagger}_{\hat{a}i}$, $\theta^{\dagger}_{ai} = \theta^{\dagger}_{\hat{a}i}$. The $\sigma^a$ are 4 Pauli matrices (off-diagonal blocks of 4-d Dirac matrices in Weyl representation), and the $\rho^M_{ij}$ are similar off-diagonal blocks of $SO(6)$ Dirac matrices in chiral representation (see Appendix A). As in most of the discussion below we set the radius $R$ of $AdS_5$ and $S^5$ to 1.

This covariant $\kappa$-symmetry gauge fixed action is well-defined and useful for developing perturbation theory near classical “long” string configurations ending at the boundary of $AdS_5$ [9, 12, 13], e.g., the ones appearing in the Wilson loop computations [14]. However, the kinetic term of the fermions which has the structure $\partial x^+ \theta \partial \theta$, i.e. it involves only one combination $x^+$ of 4-d coordinates, and then (ii) choose the light-cone bosonic gauge $x^+ = \tau$.

In the previous paper [11] it was shown how the light-cone type $\kappa$-symmetry gauge in which the fermion kinetic term becomes $\partial x^+ \partial \theta$, and then (ii) choose the light-cone bosonic gauge $x^+ = \tau$.

In order to avoid this degeneracy problem, it is natural to try to follow the same approach which worked remarkably well in flat space [3, 16]: (i) find a light-cone type $\kappa$-symmetry gauge fixing in the light-cone type string action of [9, 12, 13], e.g., the ones appearing in the Wilson loop computations [14]. However, this action is not directly applicable for computing the spectrum of closed string in the bulk of $AdS_5 \times S^5$.

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This action contains $x^-$ only in the bosonic part and only linearly, and the fermionic kinetic terms are multiplied by the derivative of $x^+$ only. One expects, therefore, that after one fixes the bosonic light-cone gauge, it should lead to a well defined starting point for quantizing the theory in the “short” string sector. As was pointed out in [11, 17, 18] (see also Section 2), in the case of the $AdS$ type curved spaces, the bosonic light-cone gauge $x^+ = \tau$ in the Polyakov string action can not be combined with the standard conformal gauge $\sqrt{g^{\mu\nu}} = \eta^{\mu\nu}$: one needs to impose a condition on $g_{\mu\nu}$ that breaks the 2-d Lorentz symmetry and leads to a rather non-standard string action, with all terms coupled to the radial function $\phi = \log Y$ of $AdS_5$ space.

1.2 Summary

The absence of manifest 2-d Lorentz symmetry suggests that it is natural to use the phase space formulation of the light-cone gauge fixed theory. To develop such a formulation is the aim of the present paper. We shall explain how the original phase space GGRT approach [19] to light-cone gauge fixing ($x^+ = \tau$, $P^+ = 1$) can be directly applied to the present case of a curved background.

Most of our discussion will be rather formal, with possible applications depending on insights into the structure of the resulting $AdS_5 \times S^5$ string light-cone Hamiltonian. To summarize, the light-cone phase space Lagrangian, corresponding to (1.3) that we obtain below, is given by (see (C.5))

$$L = P_\perp \dot{x}_\perp + P_M \dot{Y}^M + \frac{i}{2} p^+(\dot{\theta}^i \dot{\eta}_i + \dot{\theta}_i \dot{\eta}^i + \theta^i \eta^i) + P^- ,$$

(1.4)

where $P_\perp, P_M$ are the canonical momenta for $x_\perp = (x, \bar{x})$ and $Y^M$, and the light-cone Hamiltonian density is

$$H = -P^- = \frac{1}{2p^+} \left[ P_\perp^2 + |Y|^4 P_M P_M + |Y|^4 \dot{x}_\perp^2 + \dot{Y}^M \dot{Y}^M + Y^2 (p^+ (\eta^2)^2 + 2i p^+ \eta \rho^{MN} \eta Y_M P_N) \right]$$

$$- \left[ |Y| \eta^i \eta^j M^M (\dot{\theta}^j - i \sqrt{2} |Y| \eta^j \dot{x}) + h.c. \right] .$$

(1.5)

As usual for light-cone string, the coordinate $x^-$ does not appear in the Hamiltonian, but is determined in terms of the coordinates that do appear via the reparametrization constraint

$$p^+ \dot{x}^- + P_\perp \dot{x}_\perp + P_M \dot{Y}^M + \frac{i}{2} p^+(\dot{\theta}^i \dot{\eta}_i + \eta^i \eta^i + \theta^i \eta^i) = 0 .$$

(1.6)

For a closed piece of string, the integral of this constraint over $\sigma$ constrains the state space to the subspace invariant under $\sigma$ translations.

Restoring the dependence on $\alpha'$ and the scale $R$ of $AdS_5 \times S^5$ one can then analyze various limits, e.g., (i) particle theory limit ($\alpha' \to 0$), (ii) null string limit ($\alpha' \to \infty$), \(^3\) (iii) $AdS_5 \times S^5$ supergravity or strong ‘t Hooft coupling SYM limit ($\alpha'/R^2 \to 0$), \(^3\) (vi) weak ‘t Hooft coupling SYM limit ($\alpha'/R^2 \to \infty$). In the particle theory limit, the string

\(^3\)As in the flat space case [20], the null string limit is obtained by dropping terms with derivatives with respect to the world sheet coordinate $\sigma$. 

3
Hamiltonian (1.5) reduces to the light-cone Hamiltonian for a superparticle in $AdS_5 \times S^5$ found in [36]. This implies that the “massless” (zero-mode) spectrum of the superstring coincides indeed with the spectrum of type IIB supergravity compactified on $S^5$.

Further progress depends on the possibility of making a transformation to some new variables which may allow one to solve for the string theory spectrum. It would also be interesting to make connections to other related ideas and approaches in the literature, such as the one in [21]. For example, introducing twistor-like variables in the present $AdS_5 \times S^5$ context may turn out to be useful (see [24, 25] for some previous discussions of twistors in $AdS$ space). Another potentially promising direction is to apply the methods of integrable systems. It was demonstrated in [26–28] that the set of the classical equations of bosonic string in $AdS$ geometry can be interpreted as a completely integrable system (for related discussions see also [29, 30]). The integrability property should be true also for the full system of the classical equations of $AdS_5 \times S^5$ superstring. It should be crucial to include the fermions from the beginning since their coupling to the $AdS_5$ and $S^5$ sets of bosons via the R-R interaction terms insures the conformal invariance of the 2-d string theory at the quantum level [4]. Finally, knowing the light-cone description of string in $AdS$ space-time provides string theory guidance for identifying the dynamics that should emerge from the “brute force” approach to a field theory/string duality based on directly summing the planar diagrams of ’t Hooft’s large $N_c$ limit [31]. This program is most definitively carried out using a light-cone parametrization [32–34], so its comparison to the results of the present paper should be particularly instructive. For example, there should be a more or less direct link to the fishnet diagrams [35] in the strong ’t Hooft coupling limit [32, 34].

The rest of the paper is organized as follows. We start with the example of the classical bosonic string in $AdS$-type space in Section 2. We illustrate our phase-space approach to fixing the light-cone gauge by deriving the light-cone Hamiltonian for the bosonic string in $AdS_3$ space with NS-NS flux described by $SL(2, R)$ WZW model. In Section 3 we show how some known properties of the classical “long” bosonic string solutions can be directly understood in the light-cone Hamiltonian framework of Section 2. In Section 4 we review the light-cone Hamiltonian description of a superparticle in $AdS_5 \times S^5$ developed in [36], which is the zero slope limit of the string theory case. In Section 5 we derive the phase space analog of the $AdS_5 \times S^5$ superstring Lagrangian of [11] and the corresponding light-cone gauge Hamiltonian. There we use the form of the action based on the “5+5” parametrization of $AdS_5 \times S^5$, while the light-cone phase space counterpart (1.4),(1.5) of the action (1.3) is given in Appendix C. As an aside, we also present the light-cone Hamiltonian for superstring in $AdS_3 \times S^3$ with a R-R 3-form background. In Section 6 we obtain a realization of the generators of the basic symmetry superalgebra $psu(2,2|4)$ as Noether charges expressed in terms of 2-d fields which are the coordinates of the $AdS_5 \times S^5$ superstring in the light-cone gauge.

We collect various technical details in four appendices. In Appendix A we summarize the notation used in this paper. In Appendix B we describe some relations between two different “5+5” parametrizations of $S^5$ (in terms of 5 Cartesian coordinates and in terms

\footnote{Twistors are very helpful in the construction of the theory of interacting massless higher spin fields in $AdS$ space [22, 23], which has certain similarities with string theory.}
of unit 6-d vector) which are useful for translation between the corresponding forms of the $AdS_5 \times S^5$ superstring action. In Appendix C we review the two forms of the \( \kappa \)-symmetry light-cone gauge fixed action which use the "4+6" parametrizations of $AdS_5 \times S^5$ [11] and write down the corresponding phase space Lagrangians. In Appendix D we present some details of the construction of conformal supercharges in Section 6.

2 Bosonic string in curved space: light-cone gauge approach

Let us start with a review of some previous discussions of light-cone gauge fixing for bosonic strings in curved space. In flat space in BDHP formulation [37, 38] one starts by fixing the conformal gauge

\[
\gamma^{\mu\nu} = \eta^{\mu\nu}, \quad \gamma^{\mu\nu} \equiv \sqrt{g} g^{\mu\nu}, \quad g \equiv -\det g^{\mu\nu}, \quad \det \gamma^{\mu\nu} = -1,
\]

and then fixes the residual conformal diffeomorphism symmetry on the plane by choosing \( x^+(\tau, \sigma) = \tau \). An alternative (equivalent) approach is to use the original GGRT [19] formulation based on writing the Nambu action in the first order form and fixing the diffeomorphisms by the two conditions – on one coordinate and on one canonical momentum:

\[
x^+ = \tau, \quad \mathcal{P}^+ = \text{const}.
\]

The obvious requirement for being able to choose the light-cone gauge in a curved space is the existence of a null Killing vector. The first approach based on fixing the conformal gauge for the 2-d metric does not in general apply in curved spaces with null Killing vectors which are not of the direct product form $R^{1,1} \times M^{d-2}$. The exception is the case when the null Killing vector is covariantly constant [40]. One is thus forced to give up the standard conformal gauge (2.1) and fix the diffeomorphisms by imposing instead, e.g., \( \gamma^{00} = -1, \ x^+ = \tau \). This gauge choice is consistent provided the background metric satisfies \( G^{+-} = 1, G_{-} = G_{+} = 0, \ \partial_- G_{mn} = 0 \) [17].

The above conditions do not apply in the case of the AdS-type metric (\( x_\perp \) stands for all \( d - 3 \) transverse coordinates):

\[
ds^2 = G(\phi)dx^a dx^a + d\phi^2 = G(\phi)(-dt^2 + dx_1^2 + dx_3^2) + d\phi^2.
\]

Indeed, the two null Killing vectors here are not covariantly constant and also \( G_{+-} = G(\phi) \neq 1 \). However, a slight modification of the above gauge conditions of [17] on \( \gamma^{00}, \ x^+ \) represents a consistent gauge choice [11]\(^6\):

\[
G(\phi)\gamma^{00} = -1, \quad x^+ = \tau.
\]

There is a potential complication that the norm of the Killing vector may vanish if there is a horizon where \( G = 0 \). We shall adopt a formal approach, assuming that the degeneracy

5For a discussion of various ways of fixing the light-cone gauge in the case of flat target space see, e.g., [39].

6A closely related light-cone gauge choice \( \gamma^{\mu\nu} = \text{diag}(G^{-1}, G), \ x^+ = \tau \) was originally suggested by A. Polyakov [18].
of the light-cone description reflected in the $G \to 0$ singularity of the resulting light-cone gauge fixed action should have some physical resolution, e.g., the form of the light-cone Hamiltonian may suggest how the wave functions should be defined in this region.

The case of (2.3) with $G = e^{2\phi}$ corresponds to the $AdS$ space in a Poincaré coordinate patch. One needs to use the Poincaré coordinates to have a null isometry in the bulk and at the boundary (the boundary should have $R^{1,3}$ topology). The AdS/CFT duality suggests that since the boundary SYM theory in $R^{1,3}$ has a well-defined light-cone gauge description [43], it should be possible to fix some analog of a light-cone gauge for the dual string theory as well. This is the motivation behind our light-cone gauge fixing approach, in spite of the fact that there is no globally well-defined null Killing vector in $AdS$ space (its norm proportional to $e^{2\phi}$ vanishes at the horizon $\phi = -\infty$ of the Poincaré patch).

Some further comments on the coordinate space BDHP approach using (2.4) may be found in [11]. Here we shall follow the phase space GGRT approach [19] based on fixing the diffeomorphisms in curved space by the same gauge (2.2) as in flat space. Starting with the string Lagrangian corresponding to (2.3) (we set $2\pi\alpha' = 1$)

\[
L = -\frac{1}{2} h^{\mu
u} \left[ \partial_\mu x^a \partial_\nu x^a + G^{-1}(\phi) \partial_\mu \phi \partial_\nu \phi \right] + h^{\mu \nu} \equiv \gamma^{\mu \nu} G(\phi) ,
\]

we get the canonical momenta for $x^a = (x^+, x^-, x^\perp)$ and $\phi$ (dot and prime are derivatives over $\tau$ and $\sigma$, see Appendix A for definitions)

\[
P_a = -h^{00} \dot{x}_a - h^{01} \dot{x}_a , \quad \Pi = -h^{00} G^{-1} \dot{\phi} - h^{01} G^{-1} \dot{\phi} .
\]

The phase space Lagrangian is then

\[
\mathcal{L} = \dot{x}_+ \mathcal{P}_+ + \dot{\phi} \Pi + \dot{x}^- \mathcal{P}^- + \dot{x}^- \mathcal{P}^+ + \frac{1}{2h^{00}} \left[ (\mathcal{P}_+^2 + 2\mathcal{P}^+ \mathcal{P}^-) + G^2(\phi)(\dot{x}_+^2 + 2\dot{x}_+ \dot{x}^-) + G(\phi)(\Pi^2 + \dot{\phi}^2) \right] + \frac{h^{01}}{h^{00}}(\dot{x}_+ \mathcal{P}_+ + \dot{\phi} \Pi + \dot{x}^+ \mathcal{P}^- + \dot{x}^- \mathcal{P}^+) ,
\]

where $1/h^{00}$ and $h^{01}/h^{00}$ play the role of the Lagrange multipliers. As in flat space, $x^+$ is a natural choice for the evolution parameter. This suggests choosing the light-cone gauge as in (2.2)

\[
x^+ = \tau , \quad \mathcal{P}^+ = p^+ l^{-1} = \text{const} .
\]

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7The Poincaré parametrization was used in the light-cone formulation of field dynamics in $AdS$ spacetime [41, 42].

8This subtlety and the issue of fixing a global diffeomorphism gauge for $AdS$ string was discussed in [44].

9In [41, 42] it was demonstrated that in field theory in $AdS$ space in Poincaré coordinates $x^+$ can be treated as a light-cone evolution parameter. Therefore, it is reasonable to consider $x^+$ as a light-cone evolution parameter in $AdS$ string theory too.
For flexibility, we introduced the length $l$ of the string parameter $\sigma$. $l$ may be chosen to be equal to $p^{+}$ (which is useful in the study of interactions) or to $\sqrt{2\pi\alpha^{'}} = 1$ (which is useful in the study of the spectrum). In Section 3 we shall set $l = p^{+}$, while in Sections 4-6 we shall use $l = 1$. To transform the expressions in Sections 4-6 into the “$l = p^{+}$” scheme ($p^{+} = 1$) one is simply to put $p^{+} = 1$ in the Lagrangian and Hamiltonian densities.

Integrating over $\mathcal{P}^{-}$ we get the relation

$$h^{00} = -p^{+}l^{-1}.$$  \hfill (2.9)

Note that this is equivalent (for $l = p^{+}$) to the gauge condition on the 2-d metric in (2.4). The expression for $\mathcal{P}^{-}$ follows from the $1/h^{00}$ constraint. Integration over the non-constant part of $h^{01}/h^{00}$ leads to

$$p^{+}l^{-1}\dot{x}^{-} + \dot{x}_{\perp}\mathcal{P}_{\perp} + \dot{\phi}\Pi = 0,$$  \hfill (2.10)

and integration over its constant part gives

$$\int_{0}^{l} d\sigma (\dot{x}_{\perp}\mathcal{P}_{\perp} + \dot{\phi}\Pi) = 0.$$  \hfill (2.11)

The resulting light-cone Hamiltonian is given (up to the sign, $H = -P^{-} = -\int d\sigma \mathcal{P}^{-}$) by

$$P^{-} = -\frac{l}{2p^{+}} \int_{0}^{l} d\sigma \left[ \mathcal{P}_{\perp}^{2} + G^{2}(\phi)\dot{x}_{\perp}^{2} + G(\phi)(\Pi^{2} + \dot{\phi}^{2}) \right].$$  \hfill (2.12)

Note that the equation for $x^{-}$ implies that $(h^{01}/h^{00})' = 0$. Since it also implies $h^{01} = 0$ at open string ends, it follows that $h^{01} = 0$ in the case of open string. But we can only conclude that $h^{01} = \text{const}$ in the case of closed string. This constant Lagrange multiplier imposes the integrated diffeomorphism constraint (2.11), necessary for consistency in the closed string case.

Let us note that fixing the light-cone gauge in the action, before obtaining the equations of motion, results in lost equations of motion which would be obtained by varying $x^{+}$ and $\mathcal{P}^{+}$. It is easy to check that these equations (obtained before gauge fixing) determine $\dot{x}^{-}$ and $\mathcal{P}^{-}$, the time derivatives of variables that have been eliminated by the diffeomorphism constraints. Indeed, the expressions they give for $\mathcal{P}^{-}$ and $\partial_{\sigma}\dot{x}^{-}$ are equivalent to ones obtained by differentiating the constraints with respect to time. Thus the only new information contained in the lost equations is information about the zero mode of $x^{-}$, namely, $\int_{0}^{l} d\sigma \dot{x}^{-} = \frac{l}{p^{+}} P^{-}$. Since $x^{-}$ does not enter the gauge-fixed Hamiltonian this new information is unnecessary for subsequent analysis of the dynamics of the system.

The bosonic string in $AdS$ space is not consistent (conformally invariant) at the quantum level. To make it conformally invariant one may add extra couplings to an NS-NS 3-form background or add fermions and consider superstring coupled to an R-R background. The simplest example is bosonic string propagating on the $SL(2, R)$ group manifold described by the WZW model. It is instructive to demonstrate how the phase space light-cone gauge approach described above applies in this case. In the standard
Gauss parametrization, the Lagrangian of the $SL(2,R)$ WZW model is the same as (2.5) (with $G = e^{2\phi}$) plus a WZ term

$$\mathcal{L} = -\sqrt{g} g^{\mu \nu} \left( e^{2\phi} \partial_\mu x^+ \partial_\nu x^- + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi \right) + e^{\mu \nu} e^{2\phi} \partial_\mu x^+ \partial_\nu x^- .$$

(2.13)

The corresponding phase space Lagrangian is found to be ($h_{\mu \nu} = \sqrt{g} g_{\mu \nu} e^{2\phi}$)

$$\mathcal{L} = \mathcal{P}^+ \dot{x}^- + \mathcal{P}^- \dot{x}^+ + \Pi \dot{\phi} + \frac{1}{2h_{00}} \left[ 2 \mathcal{P}^+ \mathcal{P}^- + e^{2\phi} (2 \mathcal{P}^- \dot{x}^+ - 2 \mathcal{P}^+ \dot{x}^- + \Pi^2 + \dot{\phi}^2) \right]$$

$$+ \frac{h_{01}}{h_{00}} (\mathcal{P}^+ \dot{x}^- + \mathcal{P}^- \dot{x}^+ + \Pi \dot{\phi}) .$$

(2.14)

Compared to the $AdS_d$ model (2.7) in the present $d = 3$ case there is no transverse degrees of freedom, and because of the WZ coupling the Lagrangian (2.14) does not contain $\dot{x}^+ \dot{x}^-$ term. Fixing the light-cone gauge as in (2.8) and integrating over $\mathcal{P}^-$ (getting again (2.9)) we find ($l = 1$)

$$\mathcal{L} = \Pi \dot{\phi} - \frac{e^{2\phi}}{2p^+} (\Pi^2 + \dot{\phi}^2) + e^{2\phi} \dot{x}^- - \frac{h_{01}}{p^+} (p^+ \dot{x}^- + \Pi \dot{\phi}) .$$

(2.15)

Using the expression for $\dot{x}^-$ which is implied by the constraint we can rewrite this Lagrangian as

$$\mathcal{L} = \Pi \dot{\phi} + \mathcal{P}^- - \frac{h_{01}}{p^+} (p^+ \dot{x}^- + \Pi \dot{\phi}) ,$$

(2.16)

where the (minus) Hamiltonian density is

$$\mathcal{P}^- = -\frac{e^{2\phi}}{2p^+} (\Pi + \dot{\phi})^2 .$$

(2.17)

This is to be compared with the pure metric case (2.12) which for the $AdS_3$ gives

$$\mathcal{P}^- = -\frac{e^{2\phi}}{2p^+} (\Pi^2 + \dot{\phi}^2) .$$

(2.18)

### 3 Some classical string solutions

To illustrate the utility of the bosonic light-cone Hamiltonian derived in Section 2, we shall demonstrate how the (classical) discussion of the simplest Wilson loops in $AdS_5 \times S^5$ (straight line and parallel lines) given in [14] can be phrased in the present light-cone gauge setting. Since we shall consider only the classical string approximation it is sufficient to ignore the fermions, i.e. to start with the bosonic light-cone Hamiltonian (2.12) (here we set $l = p^+$)

$$H = \frac{1}{2} \int_0^{p^+} d\sigma \left[ \mathcal{P}_\perp^2 + e^{2\phi/\gamma} \dot{x}_\perp^2 + e^{\phi/\gamma} (\Pi^2 + \dot{\phi}^2) \right] ,$$

(3.1)

where $4\gamma^2 = R^2 T_0 = R^2 / 2\pi \alpha'$. 

8
3.1 Straight string: isolated “quark” source

A quark source is represented as a static open string stretched from the horizon \( \phi = -\infty \) to the boundary \( \phi = +\infty \) of \( AdS \) space. For static solutions, \( \mathcal{P}_\perp = \Pi = 0 \) and the classical equations reduce to those extremizing the Hamiltonian:

\[
\left(e^{2\phi/\gamma}x_\perp\right)' = 0 , \tag{3.2}
\]

\[
\frac{1}{\gamma}e^{2\phi/\gamma}x_\perp^2 + \frac{1}{2\gamma}e^{\phi/\gamma} - (e^{\phi/\gamma})' = 0 . \tag{3.3}
\]

The first equation implies

\[
x_\perp = T_\perp e^{-2\phi/\gamma} , \tag{3.4}
\]

where \( T_\perp \) is an integration constant. Note that for static solutions, the constraint (2.10), i.e. \(-\dot{x}^- = \dot{x}_\perp \mathcal{P}_\perp + \dot{\phi} \Pi \), implies that there is no extension in \( x^- \). For an isolated quark, we also want no extension in \( x_\perp \), so, in this case, we have \( T_\perp = 0 \).

If we define \( \rho = e^{\phi/\gamma} \), the second equation (3.3) implies

\[
\frac{\dot{\rho}^2}{2\rho} + \frac{T_\perp^2}{2\gamma^2 \rho^2} = C , \tag{3.5}
\]

where \( C \) is a constant of integration. The left hand side is \( 1/\gamma^2 \) times the density of \( P^- \), so we may identify \( C = p^-/\gamma^2 p^+ \). For \( T_\perp = 0 \), one can trivially integrate the equation to determine

\[
\sqrt{\rho} = \frac{\sigma}{2\gamma} \sqrt{|2p^-/p^+|} , \tag{3.6}
\]

where we have fixed the integration constant so that the end at \( \sigma = 0 \) is on the horizon (\( \rho = 0 \)). The other end is at \( \sigma = p^+ \), so we find

\[
m_q \equiv \sqrt{2|p^+p^-|} = 2\gamma \sqrt{\rho_{\text{max}}} . \tag{3.7}
\]

If that end reaches the boundary, \( \rho_{\text{max}} \to \infty \), implying an infinite mass for the quark source, to be expected for a point charge.

3.2 String ending on the boundary: “quark-antiquark” source at separation \( L \)

A quark-antiquark source corresponds to an open string with both ends on the boundary separated by the distance \( L \). In this case \( T_\perp \neq 0 \) and there is a minimum value \( \rho_{\text{min}} > 0 \) of \( \rho \) for an interior point of the string. For reasons of symmetry we shift the \( \sigma \) range to mark this point by \( \sigma = 0: -p^+/2 < \sigma < p^+/2 \). \( \rho_{\text{min}} \) occurs at the point where \( \dot{\rho} = 0 \), so \( C = T_\perp^2/2\gamma^2 \rho_{\text{min}}^2 \). Thus we have

\[
m_{qq} \equiv \sqrt{2|p^+p^-|} = \frac{T_\perp p^+}{\rho_{\text{min}}} . \tag{3.8}
\]
The separation between the ends of string is obtained by integrating $\dot{x}_\perp$:

$$L \equiv \left| \int d\sigma \dot{x}_\perp \right| = T_\perp \int_{-p^\perp/\rho^2}^{p^\perp/\rho^2} \frac{d\sigma}{\rho^2} = 2\gamma \int_{\rho_{\min}}^{\rho_{\max}} \frac{d\chi}{\rho^2 \sqrt{\chi^2 / \rho_{\min}^2 - 1/\chi}}$$

$$= \frac{2\gamma}{\sqrt{\rho_{\min}}} \int_{1}^{\rho_{\max}/\rho_{\min}} \frac{d\chi}{\chi^2 \sqrt{\chi - 1/\chi}} \to \frac{4\gamma}{\sqrt{\rho_{\min}}} \frac{\Gamma(3/4) \Gamma(1/2)}{\Gamma(1/4)}, \quad (3.9)$$

where the last form is the limit as $\rho_{\max} \to \infty$. Similarly, we obtain $\sigma$ as a function of $\rho$ by direct integration leading to

$$\frac{p^+}{2} = \frac{\gamma (\rho_{\min})^{3/2}}{T_\perp} \int_{1}^{\rho_{\max}/\rho_{\min}} \frac{d\chi}{\sqrt{\chi - 1/\chi}}, \quad (3.10)$$

$$\sqrt{2|p^+ p^-|} = 2\gamma \sqrt{\rho_{\min}} \int_{1}^{\rho_{\max}/\rho_{\min}} \frac{d\chi}{\sqrt{\chi - 1/\chi}}$$

$$\to 4\gamma \sqrt{\rho_{\max}} - 4\gamma \sqrt{\rho_{\min}} \frac{\Gamma(3/4) \Gamma(1/2)}{\Gamma(1/4)}, \quad (3.11)$$

where the last form gives the non-vanishing terms of the behavior as $\rho_{\max} \to \infty$. We finally eliminate $\rho_{\min}$ in favor of $L$ to reach the final result:

$$\sqrt{2|p^+ p^-|} \to 4\gamma \sqrt{\rho_{\max}} - 4\gamma \sqrt{\rho_{\max}} \frac{\Gamma(3/4) \Gamma(1/2)}{\Gamma(1/4)}$$

$$= 4\gamma \sqrt{\rho_{\max}} - 4\gamma \sqrt{\rho_{\max}} \frac{\Gamma(3/4) \Gamma(1/2)}{\Gamma(1/4)} \frac{2(2\pi)^3}{L \Gamma(1/4)^4}. \quad (3.12)$$

To compare with the known result [14], recall that $4\gamma^2 = R^2 T_0 = R^2 / 2\pi \alpha' = \sqrt{\lambda} / 2\pi$, where $\lambda$ is the 't Hooft coupling. The first divergent term is just twice the isolated quark source mass, so the second finite term is the predicted interaction energy between quark and antiquark.

### 4 Superparticle in $AdS_5 \times S^5$: light-cone Hamiltonian

Before discussing superstring it is instructive to consider first a superparticle in $AdS_5 \times S^5$ space. The covariant $\kappa$-symmetric action for a superparticle in $AdS_5 \times S^5$ can be obtained from the superstring action of [4] by taking the zero slope limit $\alpha' \to 0$. By applying the light-cone gauge fixing procedure described here one could then obtain the superparticle light-cone gauge fixed action. One the other hand, there is a method [45] which reduces the problem of finding a new (light-cone gauge) dynamical system to the problem of finding a new solution of the commutation relations of the defining symmetry algebra (in our case $psu(2,2|4)$ superalgebra). This method was applied to the $AdS$ superparticle case in [36]. In the notation of the present paper the quantum (operator-ordered) light-cone Hamiltonian ($H = -\mathcal{P}^-$) for the superparticle found there has the following form

$$\mathcal{P}^- = -\frac{1}{2p^+}(\mathcal{P}_\perp^2 + e^{\phi}\Pi e^{\phi}\Pi + e^{2\phi}A), \quad (4.1)$$
\[ A \equiv X - \frac{1}{4}, \quad X \equiv l^2 + (p^+ \eta^2 - 2)^2 + 4p^+ \eta^j l^j, \quad (4.2) \]

where \( \eta^2 \equiv \eta^i \eta_i \) and \( l^j \) is the angular momentum operator of the \( su(4) \) algebra (for details see [36]). The \( \mathcal{P}_\perp = (\mathcal{P}, \bar{\mathcal{P}}) \) and \( \Pi \) are the bosonic conjugate momentum operators as in (2.7). The odd part of the phase space is represented by \( \theta^i, \eta^j \) considered as fermionic coordinates and \( \theta_i, \eta_i \) considered as fermionic momenta. Note that the Hamiltonian does not depend on the fermionic variables \( \theta^i \) and \( \theta_i \) (present in the light-cone gauge formulation of 4-dimensional \( \mathcal{N} = 4 \) SYM theory [43]) but they will appear in the phase space Lagrangian as \( \theta^i \dot{\theta}_i + \theta_i \dot{\theta}^i \).

The canonical operator commutation relations are

\[ [\mathcal{P}, \bar{x}] = -i, \quad [\bar{\mathcal{P}}, x] = -i, \quad [\Pi, \phi] = -i, \quad (4.3) \]
\[ \{\theta^i, \theta_j\} = \frac{1}{p^+} \delta^i_j, \quad \{\eta^i, \eta_j\} = \frac{1}{p^+} \delta^i_j. \quad (4.4) \]

The operator \( A \) is equal to zero only for massless representations which are irreducible representations of the conformal algebra [42, 46] (so(5, 2) in the case of \( AdS_5 \) space). The important property of the operator \( X \) (4.2) is that its eigenvalues are equal to squares of integers for all states of type IIB supergravity compactified on \( S^5 \). This fact plays an important role in formulating the AdS/CFT correspondence for chiral primary states [36]. The relation (4.2) then implies that the operator \( A \) is never equal to zero and thus the scalar fields [47] as well as all other modes [36] of \( S^5 \) compactified type IIB supergravity have equations of motion which are not conformally invariant. We expect that this property of the operator \( X \) should have a string-theory analog.

The light-cone gauge phase space Lagrangian for the superparticle in \( AdS_5 \times S^5 \) is obtained from the Hamiltonian (4.1) in the usual way

\[ \mathcal{L} = \mathcal{P}_\perp \dot{x}_\perp + \Pi \dot{\phi} + \mathcal{P}_M u^M + \frac{1}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) + \mathcal{P}^-. \quad (4.5) \]

\( u^M \) is a unit 6-d vector used to parametrize \( S^5 \). Note that here (and in the string case) we treat \( x^- \), \( p^+ = p^+ \) separately from the rest of the phase space variables; \( p^+ \) is conserved, while \( x^- \) satisfies the equation

\[ \dot{x}^- = \frac{1}{p^+} \mathcal{P}^- . \quad (4.6) \]

5 Light-cone Hamiltonian approach to superstring in \( AdS_5 \times S^5 \)

5.1 Review of \( \kappa \)-symmetry gauge fixed action

Let us first recall the form of the superstring action in the \( \kappa \)-symmetry light-cone gauge [11]. It is formulated in terms of 10 bosonic coordinates \((x^\pm, x, \bar{x}; \phi, y^A) \) (\( y^A \) are 5 independent coordinates of \( S^5 \)) and 16 fermionic coordinates \((\theta^i, \bar{\theta}_i, \eta^i, \bar{\eta}_i) \) which transform

\[ ^{10} \text{Note that in the rest of the paper the brackets will stand for the classical Poisson brackets, and thus there will be no i's in similar relations.} \]
in the fundamental representations of \( SU(4) \). The Lagrangian (equivalent to the one in \( (1.3) \)) is given by the sum of the “kinetic” and “Wess-Zumino” terms (see Appendices A and B for notation)

\[
\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{WZ}} ,
\]

\[
\mathcal{L}_{\text{kin}} = -\sqrt{g} g^{\mu \nu} \left[ e^{2\phi} (\partial_\mu x^+ \partial_\nu x^- + \partial_\mu x^\pm \partial_\nu x^\pm) + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} G_{\alpha \beta}(y) D_\mu y^\alpha D_\nu y^\beta \right]
- \frac{1}{2} \sqrt{g} g^{\mu \nu} e^{2\phi} \partial_\mu x^+ \left[ \theta^i \partial_\nu \theta_i + \theta_i \partial_\nu \theta^i + \eta^i \partial_\nu \eta_i + \eta_i \partial_\nu \eta^i + i e^{2\phi} \partial_\nu x^+(\eta^2)^2 \right] ,
\]

\[
\mathcal{L}_{\text{WZ}} = e^{\mu \nu} e^{2\phi} \partial_\mu x^+ \eta^i C_{ij} \left( \partial_\nu \theta^j - i \sqrt{2} e^{\phi} \eta^j \partial_\nu x^+ \right) + \text{h.c.},
\]

where

\[
D_\mu y^\alpha = \partial_\mu y^\alpha - 2i \eta_i (V^A)^i_j y^\alpha D_\mu y^\beta .
\]

Here \( G_{\alpha \beta} \) and \( (V^A)^i_j \) are the metric tensor and the Killing vectors of \( S^5 \) respectively, i.e. this Lagrangian corresponds to the following parametrization of the metric of \( AdS_5 \times S^5 \)

\[
ds^2 = e^{2\phi} dx^a dx^a + d\phi^2 + G_{\alpha \beta} dy^\alpha dy^\beta .
\]

This form of the superstring action (which we shall call “intermediate”) is most convenient for deriving other forms which differ by the way one chooses the bosonic coordinates that parametrize \( AdS_5 \times S^5 \) (see [11] and Appendices B and C).

Another useful form is found by using a 6-d unit vector \( u^M \) to parametrize \( S^5 \), i.e. by replacing \( (5.4) \) by

\[
ds^2 = e^{2\phi} dx^a dx^a + d\phi^2 + du^M du^M , \quad u^M u^M = 1 .
\]

Then \( (5.1),(5.2) \) are replaced by

\[
\mathcal{L}_{\text{kin}} = -\sqrt{g} g^{\mu \nu} \left[ e^{2\phi} (\partial_\mu x^+ \partial_\nu x^- + \partial_\mu x^\pm \partial_\nu x^\pm) + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} D_\mu u^M D_\nu u^M \right]
- \frac{1}{2} \sqrt{g} g^{\mu \nu} e^{2\phi} \partial_\mu x^+ \left[ \theta^i \partial_\nu \theta_i + \theta_i \partial_\nu \theta^i + \eta^i \partial_\nu \eta_i + \eta_i \partial_\nu \eta^i + i e^{2\phi} \partial_\nu x^+(\eta^2)^2 \right] ,
\]

\[
\mathcal{L}_{\text{WZ}} = e^{\mu \nu} e^{2\phi} \partial_\mu x^+ \eta^i \rho^M_{ij} u^M (\partial_\nu \theta^j - i \sqrt{2} e^{\phi} \eta^j \partial_\nu x^+) + \text{h.c.},
\]

where

\[
D_\mu u^M = \partial_\mu u^M - 2i \eta_i (R^M)^j_i \eta^j u^M (\partial_\nu \theta^j - i \sqrt{2} e^{\phi} \eta^j \partial_\nu x^+) , \quad R^M = -\frac{1}{2} \rho^M_{\alpha \beta} u^\alpha u^\beta .
\]

The parametrization using \( u^M \) is the most convenient one for the discussion of the superparticle in \( AdS_5 \times S^5 \) [36, 48], and is well-suited for the harmonic decomposition of the light-cone superfield of type IIB supergravity into the Kaluza-Klein modes [36]. Other forms of the superstring action using 4+6 Cartesian coordinates for \( AdS_5 \times S^5 \) (see \( (1.3) \) and Appendix C) are directly related to \( (5.6),(5.7) \) by a coordinate transformation \( Y^M = e^{\phi} u^M , \ |Y| = e^{\phi} \).
The superstring Lagrangian (5.1), (5.2) and all of its other forms mentioned above can be represented in the following way

\[ \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3, \]  

(5.9)

where the three parts are

\[ \mathcal{L}_1 = -h^{\mu\nu} \partial_\mu x^+ \partial_\nu x^- + \partial_\mu x^+ A^\mu + \frac{1}{2} h^{\mu\nu} \partial_\mu x^+ \partial_\nu x^+ B - \frac{1}{2} h^{\mu\nu} g_{AB} D_\mu y^A D_\nu y^B, \]  

(5.10)

\[ \mathcal{L}_2 = -\frac{1}{2} h^{\mu\nu} \partial_\mu x^\perp \partial_\nu x^\perp + \partial_\mu x^\perp C^\mu_{\perp}, \]  

(5.11)

\[ \mathcal{L}_3 = -\frac{1}{2} h^{\mu\nu} e^{-2\phi} \partial_\mu \phi \partial_\nu \phi + T. \]  

(5.12)

Here \( x^\perp = (x, \bar{x}) \),

\[ g_{AB} \equiv e^{-2\phi} G_{AB}, \quad D_\mu y^A \equiv \partial_\mu y^A + F^A \partial_\mu x^+, \]  

(5.13)

and \( h^{\mu\nu} \) is defined as in (2.5), i.e.

\[ h^{\mu\nu} \equiv \sqrt{g^{\mu\nu}} e^{2\phi}, \quad h^{00} h^{11} - (h^{01})^2 = -e^{4\phi}. \]  

(5.14)

The decomposition (5.9) is made so that the functions \( A^\mu, B, C^\mu_{\perp}, F^A \) depend on (i) the anticommuting coordinates and their derivatives with respect to both worldsheet coordinates \( \tau \) and \( \sigma \), and (ii) the bosonic coordinates and their derivatives with respect to the worldsheet spatial coordinate \( \sigma \) only. The reason for this decomposition is that we shall use the phase space description with respect to the bosonic coordinates only, i.e. we shall not make the Legendre transformation with respect to the fermionic coordinates.

In the case of the “intermediate” form of the action (5.1), (5.2) these functions take the following form

\[ A^\mu = -\frac{i}{2} h^{\mu\nu}(\theta^i \partial_\nu \eta_i + \eta^i \partial_\nu \theta_i) + \epsilon^{\mu_1} e^{2\phi} \eta^i C^{i\nu}_{j\nu}(\dot{\theta}^j - i\sqrt{2} e^\phi \eta^j \dot{x}) + h.c., \]  

(5.15)

\[ B = e^{2\phi}(\eta^2)^2, \]  

(5.16)

\[ C^{\mu\nu} = i\sqrt{2} \epsilon^{\mu_1} e^{3\phi} \dot{x}^+ \eta^i C_{ij}^{\nu}(\dot{\theta}^j - i\sqrt{2} e^\phi \eta^j \dot{x}) + h.c., \]  

(5.17)

\[ F^A = -2i e^{2\phi} \eta_i (V^A)^{i\nu} \eta^\nu, \]  

(5.18)

\[ T = -e^{2\phi} \dot{x}^+ \eta^i C^{i\nu}_{ij}(\dot{\theta}^j + h.c.. \]  

(5.19)

### 5.2 Phase space Lagrangian

Computing the canonical momenta for the bosonic coordinates

\[ \mathcal{P}_a = \frac{\partial \mathcal{L}}{\partial \dot{x}^a}, \quad \Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}, \quad \mathcal{P}_\lambda = \frac{\partial \mathcal{L}}{\partial \dot{y}^\lambda}, \]  

(5.20)
we get

\[
\Pi = -h^{00}e^{-2\phi}\dot{\phi}^+ - h^{01}e^{-2\phi}\dot{\phi}^+ ,
\]

\[
\mathcal{P}^+ = -h^{00}\dot{x}^+ - h^{01}\dot{x}^+ ,
\]

\[
\mathcal{P}^\perp = -h^{00}\dot{x}^\perp - h^{01}\dot{x}^\perp + C^0 ,
\]

\[
\mathcal{P}^A = -h^{00}y^A - h^{01}y^A + F^A\mathcal{P}^+ ,
\]

\[
\mathcal{P}^- = -h^{00}\dot{x}^- - h^{01}\dot{x}^- + A^0 - B\mathcal{P}^+ + \mathcal{P}_A F^A .
\]

where \( \mathcal{P}^\pm \equiv \mathcal{P}_\mp \), \( \mathcal{P}^A \equiv g^{AB}\mathcal{P}_B \). By applying the same procedure as in the bosonic case we find then the following phase space Lagrangian \( \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 \) (cf. (2.7))

\[
\mathcal{L}_1 = \mathcal{P}^+ \dot{x}^- + \mathcal{P}^- \dot{x}^+ + \mathcal{P}_A \dot{y}^A + \frac{1}{2h^{00}} \left[ 2\mathcal{P}^+ \mathcal{P}^- + 2e^{4\phi}\dot{x}^+ \dot{x}^- \right.
\]

\[
+ g^{AB}\mathcal{P}_A \mathcal{P}_B + e^{4\phi} g_{AB} D_1 y^A D_1 y^B + (\mathcal{P}^{+2} - e^{4\phi}\dot{x}^{+2})B - 2F^A \mathcal{P}_A \mathcal{P}^+
\]

\[
+ \frac{h^{01}}{h^{00}}(\mathcal{P}^+ \dot{x}^- + \mathcal{P}^- \dot{x}^+ + \mathcal{P}_A \dot{y}^A) - \frac{1}{h^{00}}(\mathcal{P}^+ + h^{01}\dot{x}^+ + A^0 + \dot{x}^+ A^1) ,
\]

\[
\mathcal{L}_2 = \mathcal{P}_\perp \dot{x}_\perp + \frac{1}{2h^{00}}((\mathcal{P} - C^0)^2 + e^{4\phi}\dot{x}_\perp^2) + \frac{h^{01}}{h^{00}}(\mathcal{P} - C^0) \dot{x}_\perp + \dot{x}_\perp C^1 ,
\]

\[
\mathcal{L}_3 = \Pi \dot{\phi} + \frac{1}{2h^{00}}e^{2\phi}(\Pi^2 + \dot{\phi}^2) + \frac{h^{01}}{h^{00}}\Pi \dot{\phi} + T .
\]

Next, we impose the light-cone gauge

\[
x^+ = \tau , \quad \mathcal{P}^+ = p^+ .
\]

Using these gauge conditions in the action and integrating over \( \mathcal{P}^- \) we get the expression for \( h^{00} \)

\[
h^{00} = -p^+ .
\]

Inserting this into (5.26), (5.27) (5.28) we get the following general form of the phase space light-cone Lagrangian

\[
\mathcal{L}_1 = \mathcal{P}_A \dot{y}^A - \frac{1}{2p^+} \left[ g^{AB}\mathcal{P}_A \mathcal{P}_B + e^{4\phi} g_{AB} \dot{y}^A \dot{y}^B + p^{+2}B - 2p^+ F^A \mathcal{P}_A \right]
\]

\[
- \frac{h^{01}}{p^+}(p^+ \dot{x}^- + \mathcal{P}_A \dot{y}^A) + A^0 ,
\]

\[
\mathcal{L}_2 = \mathcal{P}_\perp \dot{x}_\perp - \frac{1}{2p^+}(\mathcal{P}_\perp^2 + e^{4\phi}\dot{x}_\perp^2) - \frac{h^{01}}{p^+} \mathcal{P}_\perp \dot{x}_\perp ,
\]

\[
\mathcal{L}_3 = \Pi \dot{\phi} - \frac{1}{2p^+}e^{2\phi}(\Pi^2 + \dot{\phi}^2) - \frac{h^{01}}{p^+}\Pi \dot{\phi} .
\]
In deriving these expressions we used the fact the functions $C^{\mu \perp}$, $T$ given in (5.17),(5.19) are equal to zero in the light-cone gauge (5.29). This general form of the phase space Lagrangian can be used to derive explicit forms of Lagrangians corresponding to different choices of bosonic coordinates: one needs only to insert the appropriate functions $A^0$, $B$, and $F^A$. For the “intermediate” case (5.1),(5.2) these functions are given by (5.15),(5.16),(5.18) so that we get

$$\mathcal{L} = \mathcal{P}_\perp \dot{x}_\perp + \Pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i)$$

$$- \frac{1}{2 p^+} \left[ \mathcal{P}_\perp^2 + e^{2\phi} \dot{x}_\parallel^2 + e^{2\phi} \frac{1}{2} L^2 + \dot{l}^i_j + \dot{u}^M \dot{u}^M + p^{+2}(\eta^2)^2 + 4 p^+ \eta_i \dot{l}^i_j \eta^j \right]$$

$$+ e^{2\phi} \eta^j \eta^j \dot{y}_j (\dot{\theta}^i - i \sqrt{2} e^{\phi} \eta^i \dot{x}) + e^{2\phi} \eta_j \eta^j \dot{y}_j (\dot{\theta}_j + i \sqrt{2} e^\phi \eta_j \dot{x})$$

$$- \frac{h_{01}}{p^+} \left[ p^+ \dot{x}^+ + \mathcal{P}_\perp \dot{x}^\perp + \Pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) \right].$$

Here $C^{ij} = -(C^{ij}_U)^*$, and we introduced the notation

$$l^i_j \equiv i (V^A)^i_j \mathcal{P}_A,$$

and used the relation

$$G^{AB} \mathcal{P}_A \mathcal{P}_B = l^i_j, \quad l^2_j \equiv l^i_j l^i_j.$$

By applying the coordinate transformation the above Lagrangian can be rewritten in the form corresponding to the case (5.6),(5.7) in which the $S^5$ part is parametrized by the unit 6-d vector $u^M$

$$\mathcal{L} = \mathcal{P}_\perp \dot{x}_\perp + \Pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i)$$

$$- \frac{1}{2 p^+} \left[ \mathcal{P}_\perp^2 + e^{2\phi} \dot{x}_\parallel^2 + e^{2\phi} \frac{1}{2} L^2 + \dot{l}^i_j + \dot{u}^M \dot{u}^M + p^{+2}(\eta^2)^2 + 4 p^+ \eta_i \dot{l}^i_j \eta^j \right]$$

$$+ e^{2\phi} \eta^j \eta^j \dot{y}_j (\dot{\theta}^i - i \sqrt{2} e^{\phi} \eta^i \dot{x}) + e^{2\phi} \eta_j \eta^j \dot{y}_j (\dot{\theta}_j + i \sqrt{2} e^\phi \eta_j \dot{x})$$

$$- \frac{h_{01}}{p^+} \left[ p^+ \dot{x}^+ + \mathcal{P}_\perp \dot{x}^\perp + \Pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+(\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i) \right],$$

where $\mathcal{P}_M$ is the canonical momentum for $u^M$ and (see Appendix B)

$$y_{ij} \equiv \rho_{ij}^M u^M = C_{ij}, \quad y^{ij} \equiv (\rho^M)^{ij} u^M = C^{ij}, \quad l^i_j = \frac{i}{2} (\rho^{MN})^i_j u^M \mathcal{P}^N.$$

Taking into account the constraint $u^M \mathcal{P}^M = 0$ (see (5.59)) we get $l^i_k l^k_j = \frac{i}{2} \mathcal{P}^M \mathcal{P}^M \delta^i_j$. The above Lagrangian gives the Hamiltonian
\[ H = -P^- , \quad P^- = \int d\sigma \, P^- , \quad (5.40) \]

where the Hamiltonian density \( P^- \) is

\[
P^- = -\frac{1}{2p^+} \left[ p_{\perp}^2 + e^{4\phi} \dot{x}_\perp^2 + e^{2\phi} \left( \Pi^2 + \dot{\phi}^2 + l^2 \right) + \dot{u}^M \dot{\bar{u}}^M + p^{+2} (\eta^2)^2 + 4p^+ \eta_i \dot{l}_i \dot{\eta}^j \right] 
+ e^{2\phi} \eta^i y_{ij} (\dot{\theta}^j - i\sqrt{2} e^{\phi} \eta^i \dot{x}) + e^{2\phi} \eta_i y^{ij} (\dot{\theta}_j + i\sqrt{2} e^{\phi} \eta^j \dot{x}) . \quad (5.41)\]

It should be supplemented by the constraint

\[
p^+ \dot{x}^- + p_{\perp} \dot{x}^- + \Pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+ (\theta^i \dot{\theta}^i + \eta^i \dot{\eta}^i + \theta \dot{\theta}^i + \eta \dot{\eta}^i) = 0 . \quad (5.42)\]

As usual, this constraint allows one to express the non-zero modes of the bosonic coordinate \( x^- \) in terms of the transverse physical ones.

It is easy to see that in the particle theory limit the superstring Hamiltonian (5.41) reduces to the superparticle one in (4.1). The latter was found in [36] by applying the direct method of constructing relativistic dynamics [45] based on the symmetry algebra.\(^{11}\)

The present discussion thus provides a self-contained derivation of the light-cone gauge superparticle action from the covariant one.

### 5.3 Equations of motion

The equations of motion corresponding to the phase space superstring Lagrangian (5.37) take the following form

\[
\dot{x} = \frac{1}{p^+} P , \quad \dot{\bar{x}} = \frac{1}{p^+} \bar{P} , \quad \dot{\phi} = \frac{e^{2\phi}}{p^+} \Pi , \quad (5.43)\]

\[
\dot{\Pi} = \frac{1}{p^+} \partial_x (e^{4\phi} \dot{x}) - i\sqrt{2} \partial_x (e^{3\phi} \eta^i y_{ij} \dot{\eta}^j) , \quad (5.44)\]

\[
\dot{p} = \frac{1}{p^+} \partial_x (e^{4\phi} \dot{x}) + i\sqrt{2} \partial_x (e^{3\phi} \eta^i y_{ij} \dot{\eta}^j) , \quad (5.45)\]

\[
\dot{p}_\perp = \frac{1}{p^+} \partial_x (e^{2\phi} \dot{x}) - \frac{2}{p^+} e^{4\phi} \dot{x}_\perp^2 - \frac{e^{2\phi}}{p^+} \left( \Pi^2 + \dot{\phi}^2 + l^2 \right) + \dot{u}^M \dot{\bar{u}}^M + p^{+2} (\eta^2)^2 + 4p^+ \eta_i \dot{l}_i \dot{\eta}^j \right) 
+ e^{2\phi} \eta^i y_{ij} (2\dot{\theta}^j - 3i\sqrt{2} e^{\phi} \eta^i \dot{x}) + e^{2\phi} \eta_i y^{ij} (2\dot{\theta}_j + 3i\sqrt{2} e^{\phi} \eta^j \dot{x}) , \quad (5.46)\]

\[
\dot{u}^M = \frac{e^{2\phi}}{p^+} \bar{P} - ie^{2\phi} \eta^i (p^{MN})_j \eta^j u^N \quad (5.47)\]

\(^{11}\)Strictly speaking, the string Hamiltonian (5.41) reduces to (4.1) modulo terms proportional to \( \eta^2 \) and some constant. This difference is related to the fact that here we are considering the classical string Hamiltonian, i.e. ignore operator ordering, while the particle Hamiltonian (4.1) is defined in terms of quantum operators.
\[ \dot{P}^M = -\frac{e^{2\phi}}{p^+} u^M P^N P^N + \frac{1}{p^+} v^{MN} \partial_\sigma (e^{2\phi} \bar{u}^N) - i e^{2\phi} \eta_i (\rho^M)^i_j \eta^j P^N \]  
(5.48)

\[ + \frac{e^{2\phi} v^{MN} \eta_j p^N_j (\dot{\theta}^j - i \sqrt{2} e^{\phi} \eta^j \dot{x}) + e^{2\phi} v^{MN} \eta_j (\rho^N)^{ij} (\dot{\theta}_j + i \sqrt{2} e^{\phi} \eta_j \dot{x})} \]  
(5.49)

\[ \dot{\eta}^i = -\frac{i}{p^+} \partial_\sigma (e^{2\phi} y^{ij} \eta_j), \quad \dot{\theta}_i = -\frac{i}{p^+} \partial_\sigma (e^{2\phi} y_{ij} \eta^j), \]  
(5.50)

\[ \dot{\eta}_i = e^{2\phi} [-i \eta^2 \eta_i + \frac{2i}{p^+} (\eta \eta_i) + \frac{i}{p^+} y_{ij} (\dot{\theta}^j + i \sqrt{2} e^{\phi} \eta^j \dot{x})] \]  
(5.51)

Here we defined

\[ v^{MN} \equiv \delta^{MN} - u^M u^N, \]  
(5.53)

and, as previously, do not distinguish between the upper and lower indices \( M, N \), i.e. use the convention \( P_M = P^M \). These equations can be written in the Hamiltonian form. Introducing the notation \( X \) for the phase space variables \( (P_\perp, x_\perp, \Phi, P^M, u^M, \theta^i, \eta^i, \eta_i) \) one has the Hamiltonian equations

\[ \dot{X} = [X, \mathcal{P}^-], \]  
(5.54)

where the phase space variables satisfy the (classical) Poisson-Dirac brackets

\[ [\mathcal{P}(\sigma), \bar{x}(\sigma')] = \delta(\sigma, \sigma'), \quad [\mathcal{P}(\sigma), x(\sigma')] = \delta(\sigma, \sigma'), \quad [\Phi(\sigma), \phi(\sigma')] = \delta(\sigma, \sigma'), \]  
(5.55)

\[ [\mathcal{P}^M(\sigma), u^N(\sigma')] = v^{MN} \delta(\sigma, \sigma'), \quad [\mathcal{P}^M(\sigma), \mathcal{P}^N(\sigma')] = (u^M \mathcal{P}^N - u^N \mathcal{P}^M) \delta(\sigma, \sigma'), \]  
(5.56)

\[ \{\theta_i(\sigma), \theta^j(\sigma')\} = \frac{i}{p^+} \delta^j_i \delta(\sigma, \sigma'), \quad \{\eta_i(\sigma), \eta^j(\sigma')\} = \frac{i}{p^+} \delta^j_i \delta(\sigma, \sigma'), \]  
(5.57)

\[ [x^0, \theta^i] = \frac{1}{2p^+} \theta^i, \quad [x^0, \theta_i] = \frac{1}{2p^+} \theta_i, \quad [x^0, \eta^i] = \frac{1}{2p^+} \eta^i, \quad [x^0, \eta_i] = \frac{1}{2p^+} \eta_i, \]  
(5.58)

where \( x^0 \) is the zero mode of \( x^- \). All the remaining brackets are equal to zero (with exception of \([p^+, x^0] = 1\)). The structure of (5.56) reflects the fact that in the Hamiltonian formulation the constraint \( u^M u^N = 1 \) should be supplemented by the constraint

\[ u^M \mathcal{P}^M = 0. \]  
(5.59)

These are second class constraints and the Dirac procedure leads then to the classical Poisson-Dirac brackets (5.56). To derive (5.57),(5.58) one is to take into account that the Lagrangian (5.37) has the following second class constraints

\[ p^{\theta^i} + \frac{i}{2} p^+ \theta_i = 0, \quad p_{\theta_i} + \frac{i}{2} p^+ \theta^i = 0, \]  
(5.60)

where \( p^{\theta^i}, p_{\theta_i} \) are the canonical momenta of fermionic coordinates. The same constraints are found for the fermionic coordinates \( \eta^i, \eta_i \). Starting with the Poisson brackets
\{p^\theta_t, \tilde{\theta}^j\}_{p,B} = \delta^i_j, \quad \{p_{\theta_t}, \theta^j\}_{p,B} = \delta^i_j, \quad [p^+, x^-]_{p,B} = 1, \quad (5.61)

one gets then the Poisson-Dirac brackets given in (5.57),(5.58).

### 5.4 Light-cone Hamiltonian for superstring in AdS$_3 \times S^3$

Finally, let us note that our results for the AdS$_5 \times S^5$ string can be generalized in a rather straightforward way to the case of superstring in AdS$_3 \times S^3$ space with RR 3-form background (for the corresponding covariant GS action see [49]). Our light-cone gauge action is written in the form which allows a straightforward generalization to the AdS$_3 \times S^3$ case: one is just to do a dimensional reduction.

To get the Lagrangian and Hamiltonian $\mathcal{P}^-$ in AdS$_3 \times S^3$ case we are to set $x_- = 0, \mathcal{P}_\perp = 0$ in (5.37), (5.41). Now instead of $su(4) \sim so(6)$ we have $so(4)$ which is decomposed into $su(2)$ and $\tilde{su}(2)$. The fermionic variables $\eta$ and $\theta$ are now transforming in the fundamental representation of $SU(2)$ and $\tilde{SU}(2)$ respectively (i.e. the indices $i, j$ now take values 1, 2). The charge conjugation matrix $C'_{ij}$ is now given by $C' = c\sigma_2, |c| = 1$.

The AdS$_3 \times S^3$ superstring Lagrangian then takes the form

$$\mathcal{L} = \Pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+ (\theta^i \dot{\bar{\theta}}_i + \eta^i \dot{\bar{\eta}}_i + \theta_i \dot{\bar{\theta}}^i + \eta_i \dot{\bar{\eta}}^i) + \mathcal{P}^-$$

$$- \frac{\hbar}{p^+} \left[ p^+ \dot{x}^- + \Pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+ (\theta^i \dot{\bar{\theta}}_i + \eta^i \dot{\bar{\eta}}_i + \theta_i \dot{\bar{\theta}}^i + \eta_i \dot{\bar{\eta}}^i) \right]. \quad (5.63)$$

The (minus) Hamiltonian density is given by (cf. (2.18))

$$\mathcal{P}^- = -\frac{e^{2\phi}}{2p^+} \left[ \Pi^2 + \phi^2 + 2\eta^2 + \dot{u}^M \dot{u}^M + p^{+2} (\eta^2)^2 + 4p^+ \eta_i \dot{\eta}_j \right] + e^{2\phi} (\eta^i \eta_j \dot{\theta}^j + h.c.), \quad (5.64)$$

where

$$y_{ij} = C'_{ik}(\sigma^M)^{kj} u^M, \quad \sigma^M = (\sigma^1, \sigma^2, \sigma^3, \sigma^4). \quad (5.65)$$

and $M, N = 1, 2, 3, 4$. The expression for orbital part of SU(2) generators $l^i_\j$ takes the form

$$l^i_\j = \frac{1}{2}(\sigma^{MN})^{ij}_k u^M \mathcal{P}^N, \quad (\sigma^{MN})^{ij}_k = \frac{1}{2}(\sigma^M)^{ik}(\sigma^N)^{kj} - (M \leftrightarrow N), \quad (5.66)$$

where $\sigma^M = (\sigma^1, \sigma^2, \sigma^3, i\sigma^4)$. Note that in this case one has the relation $2l^2_\j = \mathcal{P}^M \mathcal{P}_M$, and this explains the factor of 2 in front of $l^2_\j$ in (5.64). The variation over $\hbar \partial_0$ gives the constraint

$$p^+ \dot{x}^- + \Pi \dot{\phi} + \mathcal{P}_M \dot{u}^M + \frac{i}{2} p^+ (\theta^i \dot{\bar{\theta}}_i + \eta^i \dot{\bar{\eta}}_i + \theta_i \dot{\bar{\theta}}^i + \eta_i \dot{\bar{\eta}}^i) = 0. \quad (5.67)$$

An interesting feature of the Hamiltonian (5.64) is that $e^{2\phi}$ factors out (cf. (5.41)). Note also that the dependence on matrix $C'_{ij}$ can be eliminated by the redefinition of the fermionic coordinates $\eta_i \rightarrow C'_{ij} \eta^j$. 

18
6 Noether charges as generators of supersymmetry algebra $psu(2,2|4)$

The Noether charges play an important role in the analysis of the symmetries of dynamical systems. The choice of the light-cone gauge spoils manifest global symmetries, and in order to demonstrate that these global invariances are still present one needs to find the Noether charges which generate them. These charges play a crucial role in formulating superstring field theory in the light-cone gauge (see [50, 51]).

The Noether charges for a superparticle in $AdS_5 \times S^5$ were found in [36]. These charges are helpful in establishing a correspondence between the bulk fields of type IIB supergravity and the chiral primary operators of the boundary theory in a manifestly supersymmetric way. Superstring Noether charges should thus be useful for the study of the AdS/CFT correspondence at the full string-theory level.

In the light-cone formalism the generators (charges) of the basic $psu(2,2|4)$ superalgebra can be split into two groups:

$$P^+, P, \bar{P}, J^{+x}, J^{+\bar{x}}, K^+, K, \bar{K}, Q^{+i}, Q^{+i}, S^{+i}, S^{+i}, D, J^{+-}, J^{x\bar{x}}, \quad (6.1)$$

which we shall refer to as kinematic generators, and

$$P^-, J^{-x}, J^{-\bar{x}}, K^-, Q^{-i}, Q^{-i}, S^{-i}, S^{-i}, \quad (6.2)$$

which we shall refer to as dynamical generators (see also [11]). The kinematic generators have positive or zero $J^{+-}$ charges, while dynamical generators have negative $J^{+-}$ charges.

For $x^+ = 0$ the kinematic generators in the superfield realization are quadratic in the physical string fields (i.e. they have the structure $J = J_1 + x^+ J_2 + x^{+2} J_3$ where $J_1$ is quadratic but $J_2, J_3$ contain higher order terms in second-quantized fields), while the dynamical generators receive higher-order interaction-dependent corrections. The first step in the construction of superstring field theory is to find a free (quadratic) superfield representation of the generators of the $psu(2,2|4)$ superalgebra. The charges we obtain below can be used to obtain (after quantization) these free superstring field charges.

6.1 Currents for $\kappa$-symmetry light-cone gauge fixed superstring action

As usual, symmetry generating charges can be obtained from conserved currents. Since currents themselves may be helpful in some applications, we shall first derive them starting with the $\kappa$-symmetry gauge fixed Lagrangian in the form given in (5.6),(5.7).

---

12 In what follows “currents” and “charges” will mean both bosonic and fermionic ones, i.e. will include supercurrents and supercharges.

13 Let us note in passing that the development of the light-cone string field theory approach in the case of the $AdS_5 \times S^5$ background may be useful in the context of AdS/CFT correspondence. One striking feature of the light-cone closed superstring field theory actions in the flat space case is that their interaction part contains only cubic vertices. Same may happen to be true also in the case of the $AdS_5 \times S^5$ background, and that may have important implications for establishing correspondence with the SYM theory in the light-cone gauge framework.
To obtain the currents we shall use the standard Noether method (see, e.g., [52]) based on the localization of the parameters of the associated global transformations. Let \( \epsilon \) be a parameter of some global transformation which leaves the action invariant. Replacing it by a function of worldsheet coordinates \( \tau, \sigma \), the variation of the action takes the form

\[
\delta S = \int d^2 \sigma \, G^\mu \partial_\mu \epsilon ,
\]

(6.3)

where \( G^\mu \) is the corresponding current. Making use of this formula, we shall find below those currents which are related to symmetries that do not involve compensating \( \kappa \)-symmetry transformation. The remaining currents will be found in the next subsection starting from the action (5.37) where both the \( \kappa \)-symmetry and the bosonic light-cone gauges are fixed.

Let us start with the translation invariance \( \delta x^a = \epsilon^a \). Applying (6.3) to the Lagrangian (5.6),(5.7) gives the translation currents

\[
P^{+\mu} = -\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\nu x^+, \quad (6.4)
\]

\[
P^\mu = -\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\nu x - i\sqrt{2}e^{3\phi}e^{\mu\nu}\eta_{ij}\eta_{j}\partial_\nu x^+, \quad (6.5)
\]

\[
\bar{P}^\mu = -\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\nu \bar{x} + i\sqrt{2}e^{3\phi}e^{\mu\nu}\eta_{ij}\eta_{j}\partial_\nu x^+, \quad (6.6)
\]

\[
P^{-\mu} = -\sqrt{g}g^{\mu\nu}(e^{2\phi}\partial_\nu x^- + F^M D_\nu u^M)
\]

\[
- \frac{i}{2} \sqrt{g}g^{\mu\nu}e^{2\phi}(\theta^i \partial_\nu \theta_i + \theta_i \partial_\nu \theta^i + \eta^i \partial_\nu \eta_i + \eta_i \partial_\nu \eta^i + 2ie^{2\phi}\partial_\nu x^+(\eta^2)^2)
\]

\[
+ e^{\mu\nu}e^{2\phi}\eta^i y_{ij}(\partial_\nu \theta^j - i\sqrt{2}e^{\phi}\eta^j \partial_\nu x) + e^{\mu\nu}e^{2\phi}\eta_i y^j(\partial_\nu \theta^j + i\sqrt{2}e^{\phi}\eta_j \partial_\nu x), \quad (6.7)
\]

where

\[
F^M \equiv i\hbar (\rho^{MN})^i_j \eta^i e^{2\phi}u^N/n. \quad (6.8)
\]

Invariance of the action (5.6),(5.7) with respect to rotations in \((x^-, x)\) and \((x^-, \bar{x})\) planes

\[
\delta \bar{x} = \epsilon_j x^+, \quad \delta x^- = -\epsilon_j x, \quad \delta x = \epsilon_j x^+, \quad \delta x^- = -\epsilon_j \bar{x}, \quad (6.9)
\]

gives the following conserved currents

\[
J^{+x} = -\sqrt{g}g^{\mu\nu}e^{2\phi}(x^+ \partial_\nu x - x \partial_\nu x^+) - ix^+ e^{\mu\nu}e^{3\phi}\partial_\nu x^+ \eta_{ij} y_{ij} \eta_j, \quad (6.10)
\]

\[
J^{+\bar{x}} = -\sqrt{g}g^{\mu\nu}e^{2\phi}(x^+ \partial_\nu \bar{x} - \bar{x} \partial_\nu x^+) + ix^+ e^{\mu\nu}e^{3\phi}\partial_\nu x^+ \eta_i y_{ij} \eta^j. \quad (6.11)
\]

Making use of (6.4)-(6.6) we get

\[
J^{+x\mu} = x^+P^\mu - xP^{+\mu}, \quad J^{+\bar{x}\mu} = x^+\bar{P}^\mu - \bar{x}P^{+\mu}. \quad (6.12)
\]

Some of the remaining bosonic currents can be expressed in terms of supercurrents. The invariance with respect to the super-transformations

\[
\delta \theta^i = \epsilon^i, \quad \delta \bar{\theta}_i = \epsilon_i, \quad \delta x^- = -\frac{i}{2} \epsilon^i_\theta \bar{\theta}_i - \frac{i}{2} \epsilon_i \theta^i, \quad (6.13)
\]

gives the following supercurrents.
\[ Q^{+\mu} = -\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\nu x^+ - ie^{\mu\nu}e^{2\phi}y_i^j\eta_j\partial_\nu x^+ , \quad (6.14) \]
\[ Q_i^{+\mu} = -\sqrt{g}g^{\mu\nu}e^{2\phi}\theta_i\partial_\nu x^+ - ie^{\mu\nu}e^{2\phi}y_i^j\eta_j\partial_\nu x^+ . \quad (6.15) \]

The invariance of the action (5.6), (5.7) with respect to the rotation of (super)coordinates in the \((x^+, x^-)\) plane

\[ \delta x^\pm = e^{\pm\epsilon}x^\pm , \quad \delta(\theta^i, \theta_i, \eta^i, \eta_i) = e^{-\epsilon/2}(\theta^i, \theta_i, \eta^i, \eta_i) , \quad (6.16) \]
gives the following conserved current

\[ \mathcal{J}^{+-} = -\sqrt{g}g^{\mu\nu}e^{2\phi}(x^+\partial_\nu x^- - x^-\partial_\nu x^+) \]
\[ - \frac{i}{2}\sqrt{g}g^{\mu\nu}e^{2\phi}x^+ \left( \theta^i\partial_\nu \theta_i + \theta_i\partial_\nu \theta^i + \eta^i\partial_\nu \eta_i + \eta_i\partial_\nu \eta^i + 2ie^{2\phi}\partial_\nu x^+(\eta^2)^2 \right) \]
\[ + x^+e^{\mu\nu}e^{2\phi}\eta_i y^j \eta_j \left( \partial_\nu \theta_i - i\sqrt{2}e^{\phi}y^j \partial_\nu x \right) + x^+e^{\mu\nu}e^{2\phi}\eta_i y^j \eta_j \left( \partial_\nu \theta_j + i\sqrt{2}e^{\phi}y^j \partial_\nu \bar{x} \right) \]
\[ - \frac{1}{2}e^{\mu\nu}e^{2\phi}\partial_\nu x^+ \eta_i y^j \eta_j - \frac{1}{2}e^{\mu\nu}e^{2\phi}\partial_\nu x^+ \eta_i y^j \eta_j . \quad (6.17) \]

This current can be represented in terms of the translation currents as follows

\[ \mathcal{J}^{+-} = x^+\mathcal{P}^- - x^-\mathcal{P}^+ + \frac{i}{2}\theta^i Q_i^{+\mu} + \frac{i}{2}\theta_i Q_i^{+\mu} . \quad (6.18) \]

The invariance with respect to the rotation of (super)coordinates in \((x, \bar{x})\) plane

\[ \delta x = e^{i\epsilon}x , \quad \delta \bar{x} = e^{-i\epsilon}x , \quad \delta \theta^i = e^{\frac{\epsilon}{2}}\theta^i , \quad \delta \theta_i = e^{-\frac{\epsilon}{2}}\theta_i , \quad \delta \eta^i = e^{-\frac{\epsilon}{2}}\eta^i , \quad \delta \eta_i = e^{\frac{\epsilon}{2}}\eta_i , \]
leads to the following conserved current

\[ \mathcal{J}^{\pm} = -\sqrt{g}g^{\mu\nu}e^{2\phi}(x\partial_\nu \bar{x} - \bar{x}\partial_\nu x) + \frac{i}{2}\sqrt{g}g^{\mu\nu}\partial_\nu x^+(\theta^i\theta_i - \eta^i\eta_i) \]
\[ - \frac{1}{2}e^{\mu\nu}e^{2\phi}\partial_\nu x^+ \eta_i y^j \left( \theta^j - i2\sqrt{2}\eta^j x \right) + \frac{1}{2}e^{\mu\nu}e^{2\phi}\partial_\nu x^+ \eta_i y^j \left( \theta_j + i2\sqrt{2}\eta_j \bar{x} \right) \quad (6.19) \]

which can be rewritten in terms of translation currents as follows

\[ \mathcal{J}^{\pm} = x\mathcal{P}^\mu - \bar{x}\mathcal{P}^\mu - \frac{i}{2}\theta^i Q_i^{\pm\mu} + \frac{i}{2}\theta_i Q_i^{\pm\mu} + \frac{i}{2}(\theta^i\theta_i + \eta^i\eta_i)\mathcal{P}^{\pm\mu} \quad (6.20) \]

The invariance with respect to the dilatations

\[ \delta x^a = e^{\epsilon}x^a , \quad \delta \phi = -\epsilon , \quad \delta(\theta^i, \theta_i, \eta^i, \eta_i) = e^{-\epsilon/2}(\theta^i, \theta_i, \eta^i, \eta_i) , \quad (6.21) \]
leads to the dilatation current
This current can be rewritten as

\[ D^\mu = -\sqrt{g} g^{\mu\nu} \left( e^{2\phi} x^\alpha \partial_\nu x^\alpha + F^M D_\nu u^M - \partial_\nu \phi \right) \]

\[ - \frac{i}{2} \sqrt{g} g^{\mu\nu} e^{2\phi} x^+ \left( \theta^i \partial_\nu \theta_i + \theta_i \partial_\nu \theta^i + \eta^i \partial_\nu \eta_i + \eta_i \partial_\nu \eta^i + 2ie^{2\phi} \partial_\nu x^+ (\eta^i)^2 \right) \]

\[ + \frac{e^{2\phi}}{2} \partial_\nu x^+ \eta^i y_{ij} (\theta_j - i2\sqrt{2} e^{\phi} \eta^i x) - \frac{e^{2\phi}}{2} \partial_\nu x^+ \alpha^i y_{ij} (\theta_j + i2\sqrt{2} e^{\phi} \eta^i x) . \] (6.22)

This current can be rewritten as

\[ D^\mu = x^\alpha \partial_\alpha \phi - \frac{1}{2} \theta^i \theta_i^\mu - \frac{1}{2} \theta_i \theta^{+i\mu} . \] (6.23)

The invariance with respect to the SU(4) rotations \( (\epsilon^i_i = 0) \)

\[ \delta y^{ij} = \epsilon^i_i y^{ij} + \epsilon^j_j y^{ij} , \quad \text{i.e.} \quad \delta u^M = -\frac{1}{2} \epsilon^i_j (\rho^{MN})^i_j u^N , \] (6.24)

\[ \delta \theta^i = \epsilon^i_j \theta^j , \quad \delta \theta_i = -\theta_j \epsilon^j_i , \quad \delta \eta^i = \epsilon^i_j \eta^j , \quad \delta \eta_i = -\eta_j \epsilon^j_i , \] (6.25)

gives the following SU(4) current

\[ \mathcal{J}^i_j = - \frac{i}{2} \sqrt{g} g^{\mu\nu} \left( \rho^{MN} \right)^i_j u^M u^N + \left[ \theta^i \theta_j + \eta^i \eta_j - \frac{1}{4} \delta^i_j (\theta^i \theta_i + \eta^i \eta_i) \right] \mathcal{P}^{\mu} \]

\[ - \frac{ie^{2\phi}}{2} \partial_\nu x^+ (\theta^j y_{ij}) \left( 1 - \frac{1}{4} \delta^i_j \theta^k y_{kl} \eta^l \right) + i e^{2\phi} \partial_\nu x^+ (\theta_j y^i \eta^i - \frac{1}{4} \delta^j_i \theta_k y^i \eta_k) . \] (6.26)

### 6.2 Charges for bosonic and \( \kappa \)-symmetry light-cone gauge fixed superstring action

In the previous section we have found (super)currents starting with the \( \kappa \)-symmetry light-cone gauge fixed action given in (5.6),(5.7). These currents can be used to find currents for the action where both the fermionic \( \kappa \)-symmetry and the bosonic reparametrization symmetry are fixed by the light-cone type gauges (5.37). To find the components of currents in the world-sheet time direction \( G^0 \) one needs to use the relations (5.21)–(5.25) for the canonical momenta and to insert the light-cone gauge conditions (5.29) and (5.30) into the expressions for the currents given in the previous subsection. The charges are then given by

\[ G = \int d\sigma \ G^0 . \] (6.27)

Let us start with the kinematic generators (charges) (6.1). The results for the currents imply the following representations for some of them

\[ P = \int \mathcal{P} , \quad \bar{P} = \int \bar{\mathcal{P}} , \quad P^+ = \bar{P}^+ , \] (6.28)
\[ J^{+x} = \int x^+ P - x p^+ , \quad J^{+\bar{x}} = \int x^+ \bar{P} - \bar{x} p^+ , \quad (6.29) \]
\[ Q^{++i} = \int p^+ \theta^i , \quad Q^{++}_{i\bar{j}} = \int p^+ \theta_{i\bar{j}} . \quad (6.30) \]

Note that these charges depend only on the zero modes of string coordinates. In (6.28)–(6.30) the integrands are \( G^0 \) parts of the corresponding currents in world-sheet time direction: \( P^0, \bar{P}^0, J^{+x0}, J^{+\bar{x}0}, Q^{++0}, Q^{++}_{i\bar{j}} \) and \( P^{+0} = p^+ \). The components of currents in the world-sheet spatial direction \( G^1 \) can be found simply by using the conservation laws, the expressions for \( G^0 \) and the equations of motion (5.43)–(5.52). In this way we obtain

\[ P^1 = -\frac{1}{p^+} e^{4\phi} \dot{x} + i \sqrt{2} e^{3\phi} \eta_i y^{ij} \eta_j , \quad \bar{P}^1 = -\frac{1}{p^+} e^{4\phi} \dot{x} - i \sqrt{2} e^{3\phi} \eta_i y^{ij} \eta_j , \quad (6.31) \]
\[ J^{+x1} = x^+ P^1 , \quad J^{+\bar{x}1} = x^+ \bar{P}^1 , \quad Q^{+1i} = \frac{i}{p^+} e^{2\phi} y^{ij} \eta_j , \quad Q^{+1}_{i\bar{j}} = \frac{i}{p^+} e^{2\phi} y_{i\bar{j}} \eta^j . \quad (6.32) \]

The remaining kinematic charges depend on non-zero string modes and are given by

\[ J^{x\bar{x}} = \int x \bar{P} - \bar{x} P - \frac{i}{2} p^+ \theta^2 + \frac{i}{2} p^+ \eta^2 , \quad (6.33) \]
\[ J^i_j = \int l^i_j + p^+ \theta^i \theta_j + p^+ \eta^i \eta_j - \frac{1}{4} \delta^i_j p^+ (\theta^2 + \eta^2) , \quad (6.34) \]
\[ J^{+\bar{-}-} = \int x^+ \bar{P}^- - x^- p^+ , \quad (6.35) \]
\[ D = \int x^+ \bar{P}^- + x^- p^+ + \bar{x} P + \bar{x} P - \Pi . \quad (6.36) \]

The derivation of the remaining charges can be found in Appendix D. The expressions for the conformal supercharges are given by

\[ S^+_i = S^+_i |_{x^+=0} - i x^+ Q^-_i , \quad (6.37) \]
where the \( x^+ = 0 \) parts (cf. (D.3)) are given by

\[ S^+_i |_{x^+=0} = \int \frac{1}{\sqrt{2}} e^{-\phi} p^+ \eta_i + i p^+ \theta_i x , \quad S^{+i} |_{x^+=0} = \int \frac{1}{\sqrt{2}} e^{-\phi} p^+ \eta^i - i p^+ \theta^i \bar{x} . \quad (6.38) \]

The Poincaré supercharges \( Q^- i, Q^- i \) are

\[ Q^-_i = \int \bar{P} \theta_i + \frac{e^\phi}{\sqrt{2}} \left( i \eta_i \Pi - p^+ \eta^2 \eta_i + 2 (\eta l)_i + y_{ij} (\dot{\theta}^j - i \sqrt{2} e^\phi y_{ij} \dot{x}) \right) , \quad (6.39) \]
\[ Q^- i = \int \bar{P} \theta^i + \frac{e^\phi}{\sqrt{2}} \left( -i \eta^i \Pi - p^+ \eta^2 \eta^i + 2 (l \eta)^i - y^{ij} (\dot{\theta}_j + i \sqrt{2} e^\phi y_{ij} \dot{x}) \right) , \quad (6.40) \]
where

\[(\eta^l)_i \equiv \eta^j l^j_i, \quad (l^{(\eta)})^i \equiv l^j_i \eta^j.\]  

(6.41)

The conformal boost charges can be represented as follows (see Appendix D)

\[K^+ = K^+_|_{x^+=0} + x^+ (D + J^+) |_{x^+=0} + x^{+2} P^-, \quad (6.42)\]

\[K = K^+_|_{x^+=0} - x^+ J^{-x}, \quad (6.43)\]

where the parts that do not depend on \(x^+\) are

\[K^+_|_{x^+=0} = \int -\frac{1}{2} (e^{-2\phi} + 2x\bar{x}) p^+, \]  

\[K|_{x^+=0} = \int -\frac{1}{2} e^{-2\phi} \mathcal{P} + x (x^- \overline{p} + x \overline{x} - \Pi + \frac{i}{2} p^+ \theta^2 + \frac{i}{2} p^+ \eta^2) - \theta^i S^{+0}_i |_{x^+=0}, \]  

\[\bar{K}|_{x^+=0} = \int -\frac{1}{2} e^{-2\phi} \overline{\mathcal{P}} + \bar{x} (x^- \overline{p} + \bar{x} \overline{p} - \Pi - \frac{i}{2} p^+ \theta^2 - \frac{i}{2} p^+ \eta^2) + \theta^i S^{+0}_i |_{x^+=0}. \]  

(6.44 - 6.46)

Here the densities of conformal supercharges \(S^{+0}_i|_{x^+=0}, S^{+0}_i|_{x^+=0}\) are given by the integrands in (6.38).

Note that the \(G|_{x^+=0}\) parts of the kinematic charges can be obtained from the superparticle ones simply by replacing the particle coordinates by the string ones. The remaining dynamical generators \(J^{-x}, J^{-\bar{x}}, S^{-i}, S^{i}, K^{-}\) can be found by using the expression found above and applying the commutation relations of \(psu(2,2|4)\) superalgebra.

The structure of the \(psu(2,2|4)\) generators we have presented in this section is, of course, more complicated than found in the flat space case. But still there are some interesting simplifications which have an algebraic origin. A remarkable feature of \(psu(2,2|4)\) as compared to the Poincaré superalgebra is that in order to find all of its dynamical generators it is sufficient to know only the kinematic ones (6.1) and the Hamiltonian \(P^-\) (5.40): the dynamical generators are obtained using commutation relations between the kinematic generators and the Hamiltonian \(P^-\).

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Appendix A  Notation

In the main part of the paper we use the following conventions for the indices:

- \(a, b, c = 0, \ldots, 3\) boundary Minkowski space indices
- \(A, B, C = 1, \ldots, 5\) \(S^5\) coordinate space indices
- \(A', B', C' = 1, \ldots, 5\) \(so(5)\) vector indices (\(S^5\) tangent space indices)
- \(M, N, K, L = 1, \ldots, 6\) \(so(6)\) vector indices
- \(i, j, k, l = 1, \ldots, 4\) \(su(4)\) vector indices
- \(\mu, \nu = 0, 1\) world sheet coordinate indices

We decompose \(x^a\) into the light-cone and 2 complex coordinates:

\[
x^\pm \equiv \frac{1}{\sqrt{2}}(x^3 \pm x^0), \quad x, \bar{x} = \frac{1}{\sqrt{2}}(x^1 \pm i x^2).
\]  

(A.1)

We suppress the flat space metric tensor \(\eta_{ab} = (-, +, +, +)\) in scalar products, i.e. \(A^a B^a \equiv \eta_{ab} A^a B^b\). The \(SO(3, 1)\) vector \(A^a\) is decomposed as \(A^a = (A^+, A^-, A^x, A^\bar{x})\) so that the scalar product is

\[
A^a B^a = A^+ B^- + A^- B^+ + A \bar{B} + \bar{A} \bar{B},
\]

(A.2)

where we use the convention

\[
A \equiv A^x = A_x, \quad \bar{A} \equiv A^\bar{x} = A_{\bar{x}}.
\]

(A.3)

We use the notation \(x^\perp\) to represent \((x, \bar{x})\) with the following summation rule

\[
x^\perp z^\perp = x \bar{z} + \bar{x} z.
\]

(A.4)

The derivatives with respect to the world-sheet coordinates \((\tau, \sigma)\) are

\[
\dot{x} \equiv \partial_{\tau} x, \quad \dot{\bar{x}} \equiv \partial_{\sigma} x.
\]

(A.5)

The world-sheet Levi-Civita \(\epsilon^{\mu\nu}\) is defined with \(\epsilon^{01} = 1\).

The six matrices \(\rho^M_{ij}\) represent the \(SO(6)\) Dirac matrices \(\gamma^M\) in the chiral representation, i.e.

\[
\gamma^M = \begin{pmatrix} 0 & (\rho^M)^{ij} \\ (\rho^M)^{ij} & 0 \end{pmatrix},
\]

(A.6)

\[
(\rho^M)^{il} \rho^N_{lj} + (\rho^N)^{il} \rho^M_{lj} = 2 \delta^M N \delta^i_j, \quad \rho^M_{ij} = -\rho^M_{ji}, \quad (\rho^M)^{ij} \equiv -(\rho^M_{ij})^*.
\]

(A.7)

The \(SO(5)\) Dirac and charge conjugation matrices can be expressed in terms of the \(\rho^M\) matrices as follows

\[
(\gamma^{A'})^i_j = i (\rho^{A'})^{il} \rho^6_{lj}, \quad C'_{ij} = \rho^6_{ij}.
\]

(A.8)

The \(\rho^M\) matrices satisfy the identities
\[ \rho^M_{ij} = \frac{1}{2} \epsilon_{ijkl} (\rho^M)^{kl}, \quad \rho^M_{ij} (\rho^M)^{kl} = 2 (\delta_i^l \delta_j^k - \delta_i^k \delta_j^l) \, . \]  
(A.9)

The matrices \( \rho^{MN} \) are defined by

\[ (\rho^{MN})^i_j \equiv \frac{1}{2} (\rho^M)^i_l n^N_l - (M \leftrightarrow N), \]  
(A.10)

so that

\[ (\rho^{MN})^i_j (\rho^{MN})^k_l = 2 (\delta_i^l \delta_j^k - 8 \delta_i^l \delta_j^k) . \]  
(A.11)

We use the following hermitian conjugation rule for the fermionic coordinates

\[ \theta^\dagger_i = \theta^i, \quad \eta^\dagger_i = \eta^i , \]  
(A.12)

and the following notation for their squares

\[ \theta^2 \equiv \theta^i \theta_i , \quad \eta^2 \equiv \eta^i \eta_i. \]  
(A.13)

### Appendix B  
Relation between different parametrizations of \( S^5 \)

In the superstring Lagrangian in (5.2), we use the 5 independent \( S^5 \) coordinates \( y^A \) in terms of which the 5-sphere interval, metric tensor and vielbein are given by

\[ ds^2_{S^5} = d|y|^2 + \sin^2|y|ds^2_{S^4} , \quad ds^2_{S^4} = dn^A dn^A' , \quad n^A n^A' = 1 , \]  
(B.1)

\[ G_{AB} = e_A^A' e_B^B' , \quad e_A^A' = \frac{\sin|y|}{|y|} (\delta_A^A' - n_A n^{A'}) + n_A n^{A'} , \]  
(B.2)

\[ G_{AB} = \frac{\sin^2|y|}{|y|^2} (\delta_{AB} - n_A n_B) + n_A n_B' , \quad n^A \equiv \frac{y^A}{|y|} , \quad |y| = \sqrt{y^A y^{A'}} . \]  
(B.3)

We use the convention \( y^A = \delta_A^A' y^{A'} \) and the same for \( n^A'. \) The \( S^5 \) Killing vectors \( V^A \) and \( V^{A'B'} \) corresponding to the five translations and ten \( SO(5) \) rotations respectively are given by

\[ V^A' = \left[ |y| \cot|y| (\delta_A^A' - n_A n^A') + n^{A'} n^A \right] \partial_y^A , \]  
(B.4)

\[ V^{A'B'} = y^{A'} \partial_{y^{A'}} - y^B \partial_y^A . \]  
(B.5)

They can be collected into the \( SU(4) \) combination

\[ (V^A)^i_j \partial_{y^A} = \frac{1}{4} (\gamma^{A'B'})^i_j V^{A'B'} + \frac{i}{2} (\gamma^A)^i_j V^A' . \]  
(B.6)

The commutation relations of \( so(6) \) algebra then include

\[ [V^A', V^{B'}] = -V^{A'B'} , \quad [V^A', V^{B'C'}] = \delta^{A'B'} V^{C'} - \delta^{A'C'} V^{B'} . \]  
(B.7)
The matrix $C^U_{ij}$ in the WZ part of the action (5.2) is given by

$$C^U_{ij} = C'_{ij} \cos |y| + i(C'_{i'A'}n^A' \sin |y|). \quad (B.8)$$

This matrix is related to the standard charge conjugation matrix $C'_{ij}$ (which has the properties: $C'^T = -C'$, $(C'_{i'A'})^T = -C'_{i'A'}$, $C'^\dagger C' = 1$) via a $y$-dependent unitarity transformation $C^U_{ij} = U^k_i C'_k l^j_j$ (the explicit form of the matrix $U(y)$ is given in eq. (5.21) in [11]).

To transform the superstring Lagrangian from the form (5.1),(5.2) into the one in (5.6),(5.7) we define the six-dimensional unit vector $u^M$

$$u^6 = \cos |y|, \quad u^{A'} = \sin |y|n^{A'}, \quad (B.9)$$

and use (A.8), (B.3)–(B.6) which imply the following relations

$$G_{AB}(\eta V^A\eta)(\eta V^B\eta) = (\eta R^M\eta)^2, \quad G_{AB}(\eta V^A\eta)dy^B = (\eta R^M\eta)du^M, \quad (B.10)$$

where the $SU(4)$ rotation matrix $R^M$ is given by

$$(R^M)^i_j = -\frac{1}{2}(\rho^{MN})^i_j u^N. \quad (B.11)$$

Note that $(R^M)^i_j P_M$ satisfies the commutation relations of the $su(4)$ algebra, i.e. plays the role of the angular momentum operator $l^i_j$.

To relate the WZ parts of the Lagrangians (5.2) and (5.7) we use of the representation for the $SO(5)$ Dirac and $C'$ matrices given in (A.8) and thus find that [11]

$$C^U_{ij} = \rho^{M}_{ij} u^M \quad (B.12)$$

which implies the desired transformation of the WZ parts of the actions.

It is often useful to replace the unit six dimensional vector $u^M$ by the selfdual $SU(4)$ tensor $y_{ij}$ (or its inverse $y^{ij}$) defined by

$$y_{ij} \equiv \rho^M_{ij} u^M, \quad y^{ij} \equiv (\rho^M)^{ij} u^M, \quad y^{i} y_{ij} = \delta^i_j. \quad (B.13)$$

The relations (A.9) for the $\rho^M$ matrices imply that

$$y_{ij} = \frac{1}{2} \epsilon_{ijkl} y^{kl}, \quad y^*_{ij} = -y^{ij}, \quad y_{ij} = -y_{ji}. \quad (B.14)$$

**Appendix C  Manifestly $SU(4)$ invariant forms of phase space $AdS_5 \times S^5$ Lagrangian**

The light-cone gauge superstring action of [11] can be written in several equivalent forms corresponding to different parametrizations of $AdS_5 \times S^5$ space. The two forms corresponding to “5+5” parametrizations (5.1),(5.2) and (5.6),(5.7) were discussed in Section 5. Here we shall review the Lagrangians [11] for the two standard “4+6” coordinate choices and present their phase space counterparts.
Choosing the 10 Cartesian coordinates \((x^a, Y^M)\) such that the metric of \(AdS_5 \times S^5\) takes the form
\[
ds^2 = Y^2 dx^a dx^a + Y^{-2} dY^M dY^M , \quad Y^2 = |Y|^2 = Y^M Y^M , \quad (C.1)
\]
one can transform the bosonic coordinates in the “intermediate” form of the superstring Lagrangian (5.1),(5.2) to obtain its “4+6” form [11]
\[
\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{ZW} ,
\]
\[
\mathcal{L}_{\text{kin}} = -\sqrt{g} g^{\mu \nu} \left[ Y^2 (\partial_\mu x^+ \partial_\nu x^- + \partial_\mu x \partial_\nu \bar{x}) + \frac{1}{2} Y^{-2} D_\mu Y^M D_\nu Y^M \right] - \frac{1}{2} \sqrt{g} g^{\mu \nu} Y^2 \partial_\mu x^+ \left[ \theta^i \partial_\nu \theta_i + \theta_i \partial_\nu \theta^i + \eta^i \partial_\nu \eta_i + \eta_i \partial_\nu \eta^i + i Y^2 \partial_\nu x^+ (\eta^2) \right] , \quad (C.2)
\]
\[
\mathcal{L}_{ZW} = e^{\mu \nu} |Y| \partial_\mu x^+ \eta^i \rho^M_{ij} Y^M (\partial_\nu \theta^i - i\sqrt{2} |Y| \eta^j \partial_\nu x) + h.c. . \quad (C.3)
\]
Here
\[
DY^M = dY^M - 2i\eta_i (R^M)^i j \eta^j Y^2 dx^+ , \quad R^M = -\frac{1}{2} \rho^{MN} Y^N , \quad (C.4)
\]
and the matrices \(\rho^M, \rho^{MN}\) were defined in (A.6),(A.10).

It is easy to see that this Lagrangian can be represented in the same form as (5.9). Applying the final result for the phase space Lagrangian given in (5.31)–(5.33) we get
\[
\mathcal{L} = \mathcal{P}_\perp \dot{x}_\perp + \mathcal{P}_M \dot{Y}^M + \frac{i}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i )
- \frac{1}{2} p^+ \left[ \mathcal{P}_\perp^2 + |Y|^4 \mathcal{P}_M \mathcal{P}_M + |Y|^4 \dot{x}_\perp^2 + \dot{Y}^M \dot{Y}^M + Y^2 (p^+ (\eta^2)^2 + 4i p^+ \eta R^M \eta \mathcal{P}_M) \right]
+ \left[ |Y| \eta^i \rho^M_{ij} Y^M (\dot{\theta}^i - i\sqrt{2} |Y| \eta^j \dot{x}) + h.c. \right]
- \frac{h^0}{p^+} \left[ p^+ \dot{x}^- + \mathcal{P}_\perp \dot{x}_\perp + \mathcal{P}_M \dot{Y}^M + \frac{i}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \theta_i \dot{\theta}^i + \eta_i \dot{\eta}^i ) \right] . \quad (C.5)
\]
The closely related choice of 4+6 coordinates is the one which makes explicit the hidden spacetime conformal symmetry of \(AdS_5 \times S^5\) geometry, i.e. the one in which the metric takes the conformally flat form
\[
ds^2 = Z^{-2} (dx^a dx^a + dZ^M dZ^M ) , \quad Z^M = \frac{Y^M}{Y^2} . \quad (C.6)
\]
The superstring Lagrangian expressed in terms of these “conformally flat” coordinates is readily obtained from (C.2),(C.3) [11]
\[
\mathcal{L}_{\text{kin}} = -\sqrt{g} g^{\mu \nu} Z^{-2} \left[ \partial_\mu x^+ \partial_\nu x^- + \partial_\mu x \partial_\nu \bar{x} + \frac{1}{2} D_\mu Z^M D_\nu Z^M \right] - \frac{1}{2} \sqrt{g} g^{\mu \nu} Z^{-2} \partial_\mu x^+ \left[ \theta^i \partial_\nu \theta_i + \theta_i \partial_\nu \theta^i + \eta^i \partial_\nu \eta_i + \eta_i \partial_\nu \eta^i + i Z^{-2} \partial_\nu x^+ (\eta^2) \right] , \quad (C.7)
\]
\[
\mathcal{L}_{ZW} = e^{\mu \nu} |Z|^{-3} \partial_\nu x^+ \eta^i \rho^M_{ij} Z^M (\partial_\nu \theta^i - i\sqrt{2} |Z|^{-1} \eta^j \partial_\nu x) + h.c. . \quad (C.8)
\]
where
\[ DZ^M = dz^M - 2\eta_i (R^M)_{ij} \dot{x}^j Z^{-2} dx^+ , \quad R^M = -\frac{1}{2} \rho^{MN} Z^N . \] (C.9)

This Lagrangian has again the same form as in (5.9)–(5.12), and its phase space counterpart is thus found from (5.26)–(5.28)

\[
\mathcal{L} = \mathcal{P}_\perp \dot{x}_\perp + \mathcal{P}_M \dot{Z}^M + \frac{i}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \bar{\theta}^i \dot{\bar{\theta}}_i + \bar{\eta}^i \dot{\bar{\eta}}_i ) \\
- \frac{1}{2 p^+} [\mathcal{P}_\perp^2 + \mathcal{P}_M \mathcal{P}_M + Z^{-4} (\dot{x}_\perp^2 + \dot{Z}^M \dot{Z}^M) + Z^{-2} (p^+ \eta^2)^2 + 4 i p^+ \eta^M \eta^N \mathcal{P}_M ] \\
+ \left[ |Z|^{-3} \eta^i \rho^M_{ij} Z^M (\dot{\theta}^j - i \sqrt{2} |Z|^{-1} \eta^j \dot{x}) + h.c. \right] \\
- \frac{h_{01}}{p^+} [p^+ \dot{x}^+ + \mathcal{P}_\perp \dot{x}_\perp + \mathcal{P}_M \dot{Z}^M + \frac{i}{2} p^+ (\theta^i \dot{\theta}_i + \eta^i \dot{\eta}_i + \bar{\theta}^i \dot{\bar{\theta}}_i + \bar{\eta}^i \dot{\bar{\eta}}_i ) ] . \] (C.10)

An interesting feature of this Lagrangian is that the squares of the AdS$_5$ and S$^5$ momenta enter exactly as in the flat space case.

### Appendix D  Conformal supercharges

In order to find the conformal supercharges supplementing the generators given in Section 6, one needs to know the contribution of the compensating $\kappa$-symmetry transformation given in [4]. Finding these compensating transformations directly is rather complicated. Here we suggest an indirect method based on exploiting the known equations of motion given in (5.43)–(5.52) and the commutation relations of the psu$(2, 2|4)$ superalgebra. Let us demonstrate this procedure for the case of the conformal supercharges $S^{\pm i}$.

In general, the charges depend on the time variables $x^+ = \tau$ explicitly and also through the dynamical variables, i.e. $G = G(\tau, \mathcal{X}(\tau))$. Because our equations of motion (5.43)–(5.52) have a Hamiltonian form (5.54) we can rewrite the conservation law $dG/dx^+ = 0$ as follows (as in (5.54)–(5.58) here $[\ , ]$ stand for the classical Poisson bracket)

\[
\frac{\partial G}{\partial x^+} + [G, P^-] = 0 . \] (D.1)

This determines the explicit dependence on $x^+$, namely,

\[
G_{x^+} = G_{x^+ = 0} + x^+ [P^-, G_{x^+ = 0}] + \frac{x^{+2}}{2} [P^-, [P^-, G_{x^+ = 0}]] , \] (D.2)

where we used the notation

\[
G|_{x^+ = 0} \equiv G(0, \mathcal{X}(\tau)) , \] (D.3)

and took into account that the expansion in $x^+$ terminates at $(x^+)^2$ order because the psu$(2, 2|4)$ superalgebra does not have generators with absolute value of the $J^{+-}$ charge higher than 1 (so that triple and higher commutators of $P^-$ with any generator vanish).
For example, eq. (D.2) implies that the charges $J^{+x}$, $J^{-}$, $D$ depend on $x^+$ as follows

$$J^{+x} = J^{+x}|_{x^+ = 0} + x^+ P, \quad J^{\mp} = J^{+\mp}|_{x^+ = 0} + x^+ \tilde{P}, \quad D = D|_{x^+ = 0} + x^+ P^-. \quad (D.4)$$

These expressions match the ones given in (6.29),(6.34),(6.36). We also learn from (D.2) and the commutation relations of $psu(2,2|4)$ that the dynamical generators and $J^i_j$, $J^{\pm \mp}$ do not explicitly depend on $x^+$ (cf. (6.33),(6.34)).

Let us now turn to the conformal supercharges. From (D.2) and commutation relations of $psu(2,2|4)$ we find

$$S^+_i = S^+_i|_{x^+ = 0} - i x^+ Q^-_i, \quad S^{+i} = S^{+i}|_{x^+ = 0} + i x^+ Q^{-i}, \quad (D.6)$$

and thus

$$Q^-_i|_{x^+ = 0} = Q^-_i, \quad Q^{-i}|_{x^+ = 0} = Q^{-i}, \quad (D.7)$$

As a result, in order to determine the conformal supercharges $S^+_i$ we have to find only their $S^+_i|_{x^+ = 0}$ part. In the flat space case this $x^+ = 0$ part of the kinematic generators is obtainable simply from the particle charges by replacing the particle coordinates by the ones of the string. The same can be done in the case of $AdS$ space. Here we start with the superparticle expressions for the conformal supercharges $S^+_i$ given in [36] and replace the superparticle coordinates by the superstring ones

$$S^{+0}_i|_{x^+ = 0} = \frac{1}{\sqrt{2}} e^{-\phi} p^+ \eta_i + i p^+ \theta_i x. \quad (D.9)$$

Using the analog of (D.6) for the currents

$$S^{+0}_i = S^{+0}_i|_{x^+ = 0} - i x^+ Q^{-0}_i, \quad (D.10)$$

and the conservation law for $S^{+\mu}_i$ and $Q^{+\mu}_i$ one finds the following expressions for the conformal supercurrent $S^{+1}_i$ and the Poincaré supercurrent $Q^{+\mu}_i$

$$S^{+1}_i = e^{2\phi} y_{ij} y^j x - i x^+ Q^{-1}_i, \quad S^{+1} = -e^{2\phi} y^{ij} \eta_j \tilde{x} + i x^+ Q^{-1}, \quad (D.11)$$

$$Q^{-1}_i = \mathcal{P} \theta_i + \frac{e^\phi}{\sqrt{2}} \left( i \eta_i \Pi - p^+ \eta^2 \eta_i + 2(\eta_I) + y_{ij} (\delta^j_i - i \sqrt{2} e^\phi y^j \dot{x}) \right), \quad (D.12)$$

$$Q^{+1}_i = e^{2\phi} \theta_i \left( -\frac{e^\phi}{p^+} \tilde{x} + i \sqrt{2} \eta_k y^k \eta_i \right) + \frac{i e^{2\phi}}{p^+} y_{ij} y^j \mathcal{P} + \frac{i e^{2\phi}}{\sqrt{2} p^+} \partial_\sigma (\epsilon^\phi \eta_i) + 2 \sqrt{2} e^\phi \eta_j (\dot{l}^j_i)^1, \quad (D.13)$$

where

$$(\dot{l}^j_i)^1 \equiv -\frac{i}{2 p^+} e^{2\phi} (\rho^M N)^{j_1} u^M_i u^N_i. \quad (D.14)$$

These relations determine the supercharges. Finally, one can check that these charges satisfy the commutation relations of the $psu(2,2|4)$ superalgebra. The corresponding commutation relations are obtainable from the ones given in Section 3 of [11] by making the following substitutions there: $J^i_j \rightarrow -iJ^i_j$, $P^a \rightarrow -P^a$, $K^a \rightarrow -K^a$, $S \rightarrow -S$. 

30
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