Mixed convection flow of Oldroyd-B nano fluid with Cattaneo-Christov heat and mass flux model with third order slip

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ABSTRACT
This study reports the three-dimensional mixed convection flow of an Oldroyd-B nanofluid past a bidirectional stretching surface. Nonlinear partial differential equations obtained from the flow problem were converted into nonlinear ordinary differential equations using similarity transformation, and then, the numerical solutions of these ODEs with corresponding boundary conditions were obtained by employing a bvp4c solver. The effect of governing parameters on nondimensional velocity along x- and y-directions, temperature, particle concentration, local Nusselt number, and Sherwood number was presented through graphs and tables. It can be seen that the increasing values of the Brownian motion parameter Nb and thermophoresis parameter Nt lead to an increase in the temperature field and thermal boundary layer thickness, while the opposite behavior is observed for concentration field and concentration boundary layer thickness. As the Deborah number $\beta_1$ increases, the concentration profile as well as concentration boundary layer thickness increase. However, the effects of $\beta_2$ on the concentration profile are opposite to those of $\beta_1$. An increase in $\delta_t$, Pr, and $\alpha$ results in a decrease in temperature. It is reported that the local Nusselt number increases when $\beta$ and Pr increase, whereas it decreases when $\lambda$, Nb, Nt, and Sc increase. An increase in Biot number Bi results in an increase in the Nusselt number, and an increase in Nt results in an increase in the Sherwood number. The results of the present analysis were compared with the available works in particular situations, and more agreement has been noted.

I. INTRODUCTION
Mixed convection is a combination of imposed flow (forced) and natural (free) convections that act together with the heat transfer process. How much each form of convection contributes to heat transfer is determined by the flow, temperature, orientation, and geometry. In addition, heat is lost to the ambiance as a consequence of wind (forced convection) and buoyancy (natural convection), as described by Batchelor. Heat transfer is a widespread phenomenon that occurs due to the difference in temperature. Whenever there are assorted temperatures between two objects, heat propagates from the higher to lower temperature fields. With this fact, three-dimensional mixed convection occurs when the buoyant motion acts perpendicular to the forced motion. Sharada and Shankar dealt with the joint impact of mixed convection flow past an extending surface since conventional heat transfer fluids have inherently poor thermal conductivity, which makes them have inadequate high cooling applications. As a result, scientists have tried to enhance the thermal conductivity of conventional heat transfer fluids using solid additives, following the classical effective medium theory of Yu and Choi. In addition, Chen examined the mixed convection flow of a Newtonian fluid past a vertical stretching surface.

The importance of using nanofluids is to attain the highest possible thermal properties of nanoparticles with uniform dispersion and stable suspension in a base fluid. Generally, nanofluids are applied in daily life because of their importance, and they are used frequently in many engineering works and industrial works and are also used in microelectronics, heat exchangers, nuclear reactors, space technology, plastic industries,
biomedical technology, defense, and ships. Nanofluid is a new kind of heat transfer medium containing nanoparticles (1–100 nm) that are uniformly distributed in a base fluid, as discussed by Paul et al. Besides, Benkhedda et al. examined 3D forced and mixed convection heat transfer of Al2O3-water nanofluid flow through a pipe.

The phenomenon of heat transfer has widespread in industrial and biomedical applications such as cooling of electronic devices, nuclear reactor cooling, power generation, heat conduction in tissues, and many others. The heat flux model coined first by Fourier is the most successful model for understanding the heat transfer mechanism in different situations but with certain limitations. For this reason, Christov upgraded his contribution by adding Oldroyd’s upper convected derivatives. Shehzad et al. also applied the Cattaneo-Christov model to study thermal convection in a horizontal layer of incompressible Newtonian fluid under the influence of gravity. However, Giurina and Straughan proved the uniqueness and stability of the solutions for the Cattaneo-Christov equation. Svanadze also investigated the uniqueness of solutions for an incompressible flow problem by using the Cattaneo-Christov model.

Furthermore, Hayat et al. probed the flow and heat transfer of viscoelastic fluid over a stretching sheet using the Cattaneo-Christov heat flux model. In addition, Khan et al. gave the numerical solutions for viscoelastic flow over an exponentially stretching surface by considering the Cattaneo-Christov model. Meanwhile, Hayat et al. analyzed the influence of hall current on the rotating flows of an Oldroyd-B fluid in a porous medium, and Hayat et al. focused on the three-dimensional flow of non-Newtonian viscoelastic and Maxwell fluid with nanoparticles past bidirectional surfaces.

Non-Newtonian fluids cannot be described by a linear relationship; the shear stress and rate of strain are very complicated in comparison to Newtonian fluids, as explored by Hayat et al. Thus, due to the diversity of fluids in nature, several models have been suggested in the literature. Among them, rate type fluids have gained much popularity, and one of the most popular models for the rate type fluids is known as the Oldroyd-B fluid model. The Oldroyd-type model is often referred to as the Johnson-Segalman model. A class of generalized Oldroyd-B fluids is studied with reference to properties that are relevant in obtaining energy estimates. This model is an extension of the upper convected Maxwell model and equivalent to fluid filled with elastic bead and spring dumbbells, named after its creator James G. Oldroyd. Many studies were investigated: for example, Shehzad et al. examined the three-dimensional flow of an Oldroyd-B fluid with variable thermal conductivity and heat generation on a stretching sheet. Hayat et al. also explored the three-dimensional boundary layer flow of an Oldroyd-B nanofluid, which was induced by a bidirectional stretching surface. The Oldroyd-B fluid can describe the stress-relaxation and the normal stress differences, but it cannot describe either shear thinning or shear thickening, a phenomenon that is exhibited by many polymer materials. Accordingly, Oldroyd was the first to systematically develop rate-type models that met the requirements of frame indifference. This study ensures the concept behind three-dimensional flows. The attention of this article is to further investigate this regime. To our knowledge, no study has been reported on 3D mixed convection flow taking into account of the Oldroyd-B nanofluid with the Cattaneo-Christov heat and mass flux model.

II. MATHEMATICAL FORMULATION

This study considers steady, laminar three-dimensional mixed convection flow of the incompressible Oldroyd-B fluid over the stretching sheet with third order slip, convective boundary conditions, and passive control of nanoparticles. The flow is characterized in the presence of linear mixed convection. The Cartesian coordinate system is chosen in such a way that x- and y-axes are taken along the stretching surface and the z-axis is normal to the sheet, as shown in Fig. 1. The surface at z = 0 in the x- and y-directions has velocities \( U_w(x) = ax \) and \( V_w(y) = by \), respectively, with \( a, b > 0 \) being the constants, and the flow is in the domain of \( z > 0 \). The velocity components along the x, y, and z directions are \( u, v, \) and \( w \), respectively. The Cattaneo-Christov heat and mass flux model is implemented to examine the features of heat and mass transfer; convective boundary conditions are considered.

Generally, the tensor form of the governing equations is as follows:

\[
\nabla \mathbf{V} = 0, \quad (1)
\]

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla \mathbf{P} + \nabla \tau + \rho \mathbf{b}, \quad (2)
\]

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla \mathbf{P} + \nabla \tau + \rho \mathbf{b}, \quad (3)
\]

\[
\rho \left( \frac{\partial \mathbf{V}}{\partial t} + \mathbf{V} \cdot \nabla \mathbf{V} \right) = -\nabla \mathbf{P} + \nabla \tau + \rho \mathbf{b}, \quad (4)
\]

\[
q + \lambda \varepsilon \left[ \frac{\partial q}{\partial t} + \mathbf{V} \cdot \nabla q - q \cdot \nabla \mathbf{V} + \left( \nabla \cdot \mathbf{V} \right) q \right] = -k_f \nabla T, \quad (5)
\]

\[
J + \lambda \varepsilon \left[ \frac{\partial J}{\partial t} + \mathbf{V} \cdot \nabla J - J \cdot \nabla \mathbf{V} + \left( \nabla \cdot \mathbf{V} \right) J \right] = -D_h \nabla C, \quad (6)
\]
where $b_x$, $b_y$, and $b_z$ are body forces along $x$, $y$, and $z$ directions, $V = (u, v, w)$ are the three-dimensional velocity of the viscous fluid, $q$ is the heat flux, $J$ is the mass flux, $T$ is the temperature of the fluid, $C$ is the concentration of the fluid, $k$ is the thermal conductivity of the fluid, and $\lambda_1$ and $\alpha_C$ are the relaxation time parameter for the heat flux and mass flux, respectively.

The constitutive equation satisfied by an Oldroyd-B fluid is:

$$M = -\rho \dot{\gamma} + \tau,$$

$$\tau = \lambda_1 \frac{d\tau}{dt} - L \tau - \left(\tau L^T\right),$$

where $M$ is the Cauchy stress tensor, $P$ is the fluid pressure, $I$ is the identity tensor, $\tau$ is the extra stress tensor, $\mu$, $\lambda_1$, and $\lambda_2$ are the viscosity coefficients, relaxation time, and retardation time parameters, respectively, $\frac{d\tau}{dt}$ is the material time derivative, $L$ is the velocity gradient, and $L^T$ is the transpose of the tensor. $A_1$ is defined as $A_1 = \nabla V + \left(\nabla V\right)^T$ and $L = \nabla V$, where $V$ is the velocity.

Christov\textsuperscript{9} stated that Oldroyd’s indifferent formulation of the upper-convected derivative material frame of the Maxwell-Cattaneo model can be written as follows, which is valid in any coordinate system:

$$\frac{DA}{Dt} = \frac{\partial A}{\partial t} + V \cdot \nabla A - A \cdot \nabla V + (\nabla \cdot V),$$

where $V$ represents the velocity vector and $A$ is a vector that can be replaced by the heat flux vector or the mass flux vector.

Then, after applying the usual boundary layer analysis, the continuity equation becomes:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

The momentum equation along the $x$-direction is as follows:

$$\frac{u}{u} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \lambda_1 \left[ \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 u}{\partial y^2} + w \frac{\partial^2 u}{\partial z^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} + 2uv \frac{\partial^2 u}{\partial x \partial z} + 2uw \frac{\partial^2 u}{\partial y \partial z} \right]$$

$$= \left[ \frac{\partial^2 u}{\partial x^2} + \lambda \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) \right],$$

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \lambda_1 \left[ \frac{\partial^2 u}{\partial x^2} + v \frac{\partial^2 u}{\partial y^2} + w \frac{\partial^2 u}{\partial z^2} + 2uv \frac{\partial^2 u}{\partial x \partial y} + 2uv \frac{\partial^2 u}{\partial x \partial z} + 2uw \frac{\partial^2 u}{\partial y \partial z} \right]$$

Similarly, the momentum equation along the $y$-direction becomes:

$$\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} + \lambda_1 \left[ \frac{\partial^2 v}{\partial x^2} + v \frac{\partial^2 v}{\partial y^2} + w \frac{\partial^2 v}{\partial z^2} + 2uv \frac{\partial^2 v}{\partial x \partial y} + 2uv \frac{\partial^2 v}{\partial x \partial z} + 2uw \frac{\partial^2 v}{\partial y \partial z} \right]$$

$$= \left[ \frac{\partial^2 v}{\partial y^2} + \lambda \left( \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} \right) \right].$$

The three-dimensional energy and concentration equations are given, respectively, as follows:

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} + \lambda_5 \Phi_E$$

$$= \frac{\partial^2 T}{\partial x^2} + \gamma \left[ \frac{\partial C}{\partial y} \frac{\partial T}{\partial x} + \frac{\partial C}{\partial z} \frac{\partial T}{\partial x} \right],$$

(11)

$$u \frac{\partial C}{\partial x} + v \frac{\partial C}{\partial y} + w \frac{\partial C}{\partial z} + \lambda_5 \Phi_C = \frac{\partial^2 C}{\partial x^2} + \gamma \left( \frac{\partial C}{\partial y} \frac{\partial C}{\partial x} + \frac{\partial C}{\partial z} \frac{\partial C}{\partial x} \right),$$

(12)

where

$$\Phi_E = u^2 \frac{\partial^2 T}{\partial x^2} + v^2 \frac{\partial^2 T}{\partial y^2} + w^2 \frac{\partial^2 T}{\partial z^2} + 2uv \frac{\partial^2 T}{\partial x \partial y} + 2uw \frac{\partial^2 T}{\partial x \partial z} + 2vw \frac{\partial^2 T}{\partial y \partial z},$$

(13)

$$\Phi_C = u^2 \frac{\partial^2 C}{\partial x^2} + v^2 \frac{\partial^2 C}{\partial y^2} + w^2 \frac{\partial^2 C}{\partial z^2} + 2uv \frac{\partial^2 C}{\partial x \partial y} + 2uw \frac{\partial^2 C}{\partial x \partial z} + 2vw \frac{\partial^2 C}{\partial y \partial z}.$$
and Brownian motion of the relaxation time parameter for the heat flux and mass flux.

Select the following transform:

\[ u = axf'(\eta), \nu = ayg'(\eta), w = -\sqrt{\nu}f(\eta) + g(\eta), \]

\[ \eta = \sqrt{\frac{\nu}{\nu}}, \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty}, \phi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}. \]  

(21)

The governing equation and related boundary conditions of the Cattaneo-Christov heat flux and mass flux model for the Oldroyd-B nanofluid under the above assumptions can be summarized as follows:

\[ f'''' + (f + g)f''' - f'' + \beta_1(2(f + g)^2)f'' - (f + g)^3f'''' \]

\[ + \beta_2(f'' + g'')f'' - \beta_2(f + g)f''' + \lambda(\theta + N\phi) = 0, \]  

(22)

\[ g'''' + (f + g)g''' - g'' + \beta_1(2(f + g)g'g'' - (f + g)^2g''') \]

\[ + \beta_2(2(f' + g')g'' - (f + g)g'''') = 0, \]  

(23)

\[ \theta'' + Pr(f + g)\theta' + PrN_0\theta' + \theta + Pr(\theta' + \theta')(f + g)f' = 0, \]  

(24)

\[ \phi' + \frac{Ni}{N_0} \theta' + S_c(f + g)\phi' - \delta, S_c[(f + g)^2\phi'' + (f + g)(f' + g')\phi'] = 0, \]  

(25)

with associated boundary conditions \( f(0) = 0, f'(0) = 1 + yf''(0) + g''(0), g'(0) = 0, N_0\phi'(0) + N_0\theta'(0) = 0, \) at \( \eta = 0, \) \( g(\eta) = g''(\eta) = \beta + \alpha g''(0) + Sg'''(0) \) \( C_g''''(0), \) as \( \eta \rightarrow \infty, \)

where the governing parameters are defined by

\[ \gamma = A\sqrt{\frac{\nu}{\nu}}, \delta = B\sqrt{\frac{\nu}{\nu}}, B = c\sqrt{\frac{\nu}{\nu}}, \]

\[ \alpha = D\sqrt{\frac{\nu}{\nu}}, \beta = \frac{\beta}{\beta}, \]

\[ S = E\sqrt{\frac{\nu}{\nu}}, G = F\sqrt{\frac{\nu}{\nu}}, Pr = \frac{\nu}{\nu}, \]

\[ N_0 = (\frac{2(\nu - C_w)}{C_w - C_\infty}), N_1 = (\frac{2(T_f - T_\infty)}{T_f - T_\infty}), \]

\[ \lambda_c = \delta, \lambda_c a = \delta, S_c = P_L, \lambda_1 a = \beta_1, \lambda_2 a = \beta_2, \]  

(26)

where \( f', g', \theta, \) and \( \phi \) are dimensionless velocity along the \( x \) - and \( y \)-axes, temperature, and particle concentration, respectively. \( \eta \) is the similarity variable. \( y, \delta, B, P_r, N_0, N_1, B, \lambda, N, \) \( Bi, \beta, \) and \( \delta \) denote first order slip parameter, second order slip parameter, third order slip parameter, Prandtl number, Brownian motion parameter, thermophoresis parameter, stretching ratio parameter, mixed convection parameter, buoyancy ratio parameter, thermal Biot number, relaxation time parameter for temperature, and relaxation time parameter for particle concentration, respectively.

The engineering aspects of this study are the skin friction coefficient \( c_f, c_f, \) and \( c_f, \) local Nusselt number \( Nu_k, \) and the local Sherwood number \( Sh_k, \)

\[ c_f = \frac{\tau_w}{\rho w_u^2}, \quad c_f = \frac{T_{\text{avg}}}{\rho w_{u_y}}, \quad Nu_k = \frac{x_{u_k}}{k(T_f - T_\infty)}, \quad Sh_k = \frac{x_{sh}}{k(C_w - C_\infty)}. \]  

(27)

where the wall shear stress \( r_w, \) wall heat flux \( q_w, \) and wall mass flux \( h_w \) are given by

\[ q_w = -k(\frac{\partial C}{\partial z})_{z=0}, \quad h_w = -D_k(\frac{\partial C}{\partial z})_{z=0}. \]  

(28)

Using Eqs. (28) and (29), we get

\[ \frac{Nu_k}{\sqrt{R_e}} = -\beta(0), \text{at } \eta = 0, \]

(29)

\[ \frac{Sh_k}{\sqrt{R_e}} = -\phi(0), \text{at } \eta = 0, \]  

(30)

where \( R_e = \frac{u_f}{\nu}, \) \( R_{\nu} = \frac{u_f}{\nu}, \) \( Nu_k, \) and \( Sh_k \) are local Reynolds numbers, local Nusselt number, and local Sherwood number, respectively.

III. NUMERICAL SOLUTION

The aforementioned four coupled ordinary differential equations with matching boundary conditions were transformed into a system of first order ODEs and then solved numerically by means of bvp4c solver in Matlab software. To solve the system of ODE, new variables can be defined as \( y_1 = f, y_2 = f'', y_3 = f''' + \beta_1(2(y_1 + y_3)y_2 - (y_1 + y_3)^2)y_4 \)

\[ y_4 = \beta_2[(y_3 + y_7)y_4 + \lambda(y_9 + y_1_1 N)]/\beta_2(y_1 + y_3) \]

\[ y_5 = y_6 = y_7 = y_8 = y_9 = y_{10} = y_{11} = y_{12} = \phi, \]  

(31)

\[ \phi = \frac{\lambda_1 a}{\lambda_2 a} \phi. \]  

(32)
and the boundary conditions are
\[
\begin{align*}
\begin{cases}
y_1(0) & = 0 \\
y_2(0) & = 1 + \gamma y_3(0) + \delta y_4(0) + B y_4'(0) \\
y_3(0) & = 0 \\
y_4(0) & = \beta + \alpha y_5(0) + S y_5(0) + G y_5'(0) \\
y_5(0) & = 1 \\
y_{inf}(1) & = y_0(10) - Bi + (y_0(9) - 1) \\
y_{inf}(2) & = Nb \times y_0(12) + Nt \times y_0(10) \\
y_{inf}(6) & = 0 \\
y_{inf}(7) & = 0 \\
y_{inf}(9) & = 0 \\
y_{inf}(11) & = 0 \\
\end{cases}
\end{align*}
\]
(33)

IV. RESULTS AND DISCUSSION

Here, outcomes of numerous parameters are explored on the dimensionless velocity along the x- and y-axes \( f'(\eta) \) and \( g'(\eta) \), temperature \( \theta(\eta) \), and concentration \( \phi(\eta) \) graphs and tables. In all cases, the bvp4c solver outcomes offered in this work were obtained using default values of \( Pr = 0.72, \lambda = 0.2, \beta_1 = 0.2, \beta = 0.4, Nt = Nb = 0.1, N = 0.1, \delta c = 0.2, Sc = 0.8, \gamma = 1, \delta = -1, \beta_1 = 0.2, G = B = 0.3, S = 0.8, \delta t = 0.2, \alpha = 1, \lambda_1 = 0.5, \) and \( Bi = 1. \)

Figures 2 and 3 reveal the impacts of the Deborah numbers \( \beta_1 \) and \( \beta_2 \) on velocity along the x-axis. An increase in the values of the Deborah number results in a decrease in the velocity along \( f'(\eta) \) near the wall. Because the Deborah number depends on relaxation time and the relaxation time scale up for the amounted number of \( \beta_1 \). Such an increase tends to lower velocity. Figure 4 shows the behavior of mixed convection parameter \( \lambda \) on the velocity field along the x-axis \( f'(\eta) \). It is noted that \( f'(\eta) \) is an ascending function of convection parameter \( \lambda \). This result physically holds because an increase in mixed convection parameter increases buoyancy forces, which accelerates fluid motion. In addition, Fig. 5 illustrates the
effect of third order slip parameter $B$ on the velocity profile along the x-axis, and it is clear that $f'(\eta)$ is decreasing at the surface wall. Physically, with an increase in third order slip parameter $B$, the resistance of fluid motion reduced, which increases the flow field and velocity boundary layer thickness. It is inversely proportional to kinematic viscosity and induces the differing influence on the velocity profile graph. Their values do not agree with the values of fluid motion because of the increase in the value of kinematic viscosity.

Figure 6 reveals the influence of stretching ratio parameter $\beta$ on $g'\eta$ and deliberates an increase near the wall. As the values of $\beta$ become greater than before, the ratio of the retardation to the relaxation time increases and a higher boundary layer thickness is observed. A finer value of $\beta$ can be found by increasing the retardation time and keeping relaxation time fixed. Figure 7 shows the effect of buoyancy ratio parameter $N$ on the temperature graph. As shown in this figure, fluid temperature increases when decreasing the values of $N$, but the fluid temperature decreases when increasing the values of $N$. The explanation for this phenomenon is that the thermal buoyancy parameter is defined as the ratio of buoyancy to viscous forces in the boundary layer; therefore, amplifying its value reduces the viscosity of the fluid, which results in an enhanced flow velocity. Figure 8 shows the impact of thermal relaxation parameter $\delta_t$ on the temperature profile graph. It is illustrated that an augmentation in the thermal relaxation parameter causes less transfer of heat from the surface of the sheet to the fluid. Hence, temperature distribution decreases with an increase in the values of the thermal relaxation parameter. For $\delta_t = 0$, the heat transfers rapidly throughout the material. Hence, temperature distribution is higher for $\delta_t = 0$. That is, temperature for the Cattaneo-Christov heat flux model is less than that of the classical Fourier law of the heat conduction model. Figure 9 shows the effect of

![FIG. 6. Velocity graph $g'\eta$ for different values of $\beta$.](image1)

![FIG. 7. Temperature graph $\theta\eta$ for different values of $N$.](image2)

![FIG. 8. Temperature graph $\theta\eta$ for different values of $\delta_t$.](image3)

![FIG. 9. Concentration graph $\phi\eta$ for different values of $\delta_c$.](image4)
the relaxation time parameter for particle concentration parameter $\delta_i$ on the concentration profile graph. It shows that an augmentation in the relaxation time parameter for the particle concentration parameter near the wall tends to increase the contraction profile, but the opposite trend is observed after some distance away from the wall.

Figures 10 and 11 are plotted for the concentration profile by considering various values of Deborah numbers $\beta_1$ and $\beta_2$, respectively. As the values of the Deborah numbers $\beta_1$ are large, the concentration profile and concentration boundary layer thickness increase. However, the effects of $\beta_2$ on the concentration profile are quite opposite to those of $\beta_1$. In addition, an increase in the Deborah number leads to an increase in the concentration of nanoparticles and their related boundary layer thickness. It shows that the nanoparticle concentration profile is enhanced with an increase in Deborah number $\beta_1$ and larger values of Deborah number $\beta_2$ decrease the nanoparticle concentration profile. Figure 12 shows the variation of thermophoresis parameter $N_t$ on the concentration graph. It can be noted that both the concentration boundary layer thickness and concentration profile improve with increasing $N_t$. Moreover, as the values of $N_t$ engorged, more nanoparticles are pushed away from the hot surface. As a result, the nanoparticle concentration is increased. Furthermore, it is noted that the thermophoresis parameter $N_t$ affects the concentration profile than Brownian motion parameter $N_b$.

Figure 13 depicts the effect of Biot number $\text{Bi}$ on the concentration graph. An increase in Biot number $\text{Bi}$ corresponds to a poor Brownian diffusion coefficient, which leads to short dissemination of concentration at the reduced surface, with increased influence of the Biot number. Figure 14 shows the effect of buoyancy ratio parameter $N$ on the skin friction coefficient along the $x$-axis $-f''(0)$ graph. It can be seen from the graph that as the values
of $N$ increase, the graph moves downward and as the values of $N$ decrease, the graph moves upward which leads to an increase in buoyancy ratio parameter $N$ as well as a decrease in the skin friction coefficient along the x-axis. Figure 15 depicts the variation of the local Nusselt number $Nu_x = -\theta'(0)$ graph. This figure describes that as the values of Prandtl $Pr$ increase from 0.6 to 1.0, the graph moves upward, where temperature decreases in case of heat absorption. Moreover, heat is lost from the sheet as the strength of heat absorption amplifies and condenses in thermal boundary layer thickness. Figure 16 shows the characteristic of $\delta_l$ on the local Sherwood number $Sh_x = -\phi'(0)$ graph. As shown in this figure, the local Sherwood number increases and concentration boundary layer thickness decreases for a large value of thermal relaxation.

Table I describes the comparison of the present result with the previous result in the open literature in the limiting condition. This table also shows that the present result agrees with a similar paper in terms of the limiting case. Table II portrays the comparison of the present result with the previous result in the open literature in the limiting condition. This table also shows that the present result agrees with a similar paper in terms of the limiting case. Table III shows the influence of no slip, first order slip, second order slip, and third order slip on the local skin-friction

| $N_t$ | $Pr$ | $\beta_1$ | $\beta_2$ | $\lambda$ | $S_c$ | $\delta_l$ | $\phi(0)$ | $\theta'(0)$ |
|-------|------|-----------|-----------|-----------|-------|------------|-----------|-------------|
| 0.0   | 0.0  | 0.0       | 0.0       | 0.0       | 1.0   | 0.0        | 0.0       | 0.0         |
| 0.1   | 1.02026 | 1.02026  | 0.00665   | 0.00665   | 1.02026 | 0.00665    | 0.00665   | 1.02026     |
| 0.2   | 1.03949 | 1.03949  | 0.14874   | 0.14874   | 1.03949 | 0.14874    | 0.14874   | 1.03949     |
| 0.3   | 1.05795 | 1.05795  | 0.24336   | 0.24336   | 1.05795 | 0.24336    | 0.24336   | 1.05795     |

| $N_t$ | $N_b$ | $\beta_1$ | $\beta_2$ | $\lambda$ | $S_c$ | $\delta_l$ | $\phi(0)$ | $\theta'(0)$ |
|-------|-------|-----------|-----------|-----------|-------|------------|-----------|-------------|
| 0.0   | 0.0   | 0.0       | 0.0       | 0.0       | 1.0   | 0.0        | 0.0       | 0.0         |
| 0.1   | 0.0   | 0.0       | 0.0       | 0.0       | 1.0   | 0.0        | 0.0       | 0.0         |
| 0.2   | 0.0   | 0.0       | 0.0       | 0.0       | 1.0   | 0.0        | 0.0       | 0.0         |
| 0.3   | 0.0   | 0.0       | 0.0       | 0.0       | 1.0   | 0.0        | 0.0       | 0.0         |

**Table I.** Comparison of the present result for the local skin-friction along the x-axis $f''(0)$ and along the y-axis $g''(0)$ with the previous results from the literature for different values of $\beta$ when $N_t = 0.1$, $Pr = 6$, $\beta_1 = \beta_2 = 0.0$, $\lambda = 0.0$, and $S_c = 1$, but other parameters are zero. Note: $Pr = \text{present value.}$
coefficients along both x- and y-directions. An increase in the magnitude of second order slip and third order slip parameters results in an increase in the values of skin-friction coefficients along both x- and y-directions. However, an increase in the magnitude of first order slip decreases the skin-friction coefficient along the x-direction, and in the case of no-slip, the result shows that the magnitude of second order slip decreases the skin-friction coefficient along the x-direction, and in the case of no-slip, the result shows that the magnitude of skin-friction coefficients along both x- and y-directions is higher than other values. Table IV shows the influence of γ, δ, B, β₁, and β₂ on the local skin-friction coefficients along both x- and y-directions. An increase in the magnitude of δ, B, and β₁ results in an increase in the values of the local skin-friction coefficients along both x- and y-directions. However, an increase in the magnitude of γ and β₂ decreases the local skin-friction coefficient along the x-direction.

Table V presents the impacts of α, Re, s, y, and δ on the local skin-friction coefficients along both x- and y-directions. An increase in the value of the stretching ratio parameter (β = \frac{t}{t_0}) results in an increase in the values of local skin-friction coefficients along x- and y-directions. Moreover, an increase in first order parameters (γ and α) decreases the local skin-friction coefficients along both directions. However, a decrease in second order slip parameters (δ and S)
results in a decrease in the local skin-friction coefficients along both x- and y-directions. Table VI presents the effect of N, Nt, Nb, Pr, δt, and δc on the local Nusselt number. An increase in the values of N, Pr, and δt increases the values of the Nusselt number. However, an increase in the values of Nt, Nb, Pr, and δc results in a decrease in the values of the Nusselt number and the effect of N, Nt, Nb, Pr, δt, and δc on the Sherwood number. An increase in the values of Nb and δt increases the values of the Sherwood number. However, an increase in Nt and Pr results in a decrease in the value of the local Sherwood number.

V. CONCLUSIONS

This study considers the influence of the Cattaneo-Christov heat and mass flux model with third order slip and convective boundary conditions on three-dimensional mixed convection flow of the Oldroyd-B nanofluid over a stretching surface, and the main finding of this study was summarized and concluded as follows:

1. The effects of the Deborah numbers β1 and β2 on the temperature and concentration profiles were similar.
2. An increase in λ results in an increase in velocity along the x-axis but a decrease in temperature and concentration.
3. An increase in Prandtl number Pr and thermal relaxation parameter δt decreases the temperature and thermal layer thickness.
4. Temperature profile θ(y) and thermal layer thickness have enhanced behavior with an increase in the values of Brownian motion parameter Nb and thermophoresis parameter Nt.
5. The concentration profile reduced with the increase in the Brownian motion parameter Nb, and the opposite behavior was noted with an increase in thermophoresis parameter Nt.
6. The magnitude of the local Nusselt number ~θ(0) and Sherwood number ~ϕ(0) enhanced with an increase in Brownian motion parameter Nb.

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 NOMENCLATURE

| Symbol | Description |
|--------|-------------|
| A, B, C, D, E, F | constants |
| A1 | tensor |
| Bi | Biot number |
| Cw | concentration at the surface |
| C∞ | ambient concentration |
| DB | Brownian diffusion coefficient |
| DT | thermophoresis diffusion coefficient |
| f′, g′ | dimensionless velocity functions along the x- and y-direction |
| I | identity tensor |
| J | mass flux |
| k | thermal conductivity |
| L | velocity gradient |
| M | Cauchy stress tensor |
| N | buoyancy ratio variable |
| Nb | Brownian motion parameter |
| Nt | thermophoresis parameter |
| Nu | local Nusselt number |
| P | fluid pressure |
| q | heat flux |
| Re, Rey | local Reynolds numbers |
| S, G | second and third order slip parameters on velocity along the y-direction |
| Sc | Schmidt number |
| Sh | local Sherwood number |
| T | temperature of the fluid |
| T∞ | ambient temperature |
| u, v, w | velocity component along the x-, y-, and z-directions |
| ν | kinematic viscosity of the fluid |
| μ | dynamic viscosity of the fluid |
| λ | mixed convection parameter |
| λ1 | the thermal relaxation time |
| λ2 | retardation time |
| δt | thermal relaxation parameter |
| δc | concentration relaxation parameter |
| α | second order slip parameter in the y-direction |
| β1, β2 | Deborah numbers |
| T | the extra stress tensor |
| δt | thermal relaxation time |
| δc | concentration relaxation time |

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