Two-fold mechanical squeezing in a cavity optomechanical system

Chang-Sheng Hu,1 Zhen-Biao Yang,1 Huaizhi Wu,1 Yong Li,2 and Shi-Biao Zheng1
1Fujian Key Laboratory of Quantum Information and Quantum Optics and Department of Physics, Fuzhou University, Fuzhou 350116, People’s Republic of China
2Beijing Computational Science Research Center, Beijing 100193, People’s Republic of China

We investigate the dynamics of an optomechanical system where a cavity with a movable mirror involves a degenerate optical parametric amplifier and is driven by a periodically modulated laser field. Our results show that the cooperation between the parametric driving and periodically modulated cavity driving results in a two-fold squeezing on the movable cavity mirror that acts as a mechanical oscillator. This allows the fluctuation of the mechanical oscillator in one quadrature (momentum or position) to be reduced to a level that cannot be reached by solely applying either of these two drivings. In addition to the fundamental interests, e.g., study of quantum effects at the macroscopic level and exploration of the quantum-to-classical transition, our results have potential applications in ultrasensitive sensing of force and motion.

I. INTRODUCTION

Squeezing associated with the mechanical motion of a massive object [1–9] refers to the reduction of the quantum fluctuation in its position or momentum below the vacuum level, which is not only important for fundamental test of quantum theory [10], such as exploration of the quantum-classical boundary [11], but also have potential applications in high-precision measurement [12, 13]. In analogy to the standard parametric techniques applied for squeezing of optical fields, the thermal noise of a mechanical oscillator can be reduced directly via parametrical modulation of the mechanical spring constant [14]. However, even though the mechanical oscillator is initially prepared in its quantum ground state, the parametric approach failed to generate a steady-state squeezing of mechanical motion below one half of the zero-point level (i.e. the well-known 3-dB limit) due to the onset of instability.

In cavity optomechanical systems [15–21], theoretical schemes for surpassing the 3-dB limit to realize mechanical squeezing have been proposed, e.g. by injecting a broad band squeezed light into the cavity to transfer optical squeezing into mechanical mode [22, 23] or by driving the optical cavity with two-tone control lasers of different amplitudes combined with a reservoir engineering technique [24, 25], based on which the experimental demonstration of stationary squeezing beyond the 3-dB limit was recently achieved [26]. Additionally, it was shown that mechanical squeezing can also be generated simply by using a periodically amplitude-modulated driving laser [27] or by directly coupling an optical parametric amplifier (OPA) to the optical cavity [28], without the requirement of classical feedback and of the input of squeezed light [29]. Despite the advantages of each scheme on certain conditions, it is still highly desirable to further strengthen the mechanical squeezing, and then the following important problems remain open: Does there exist a cooperative effect when the physical processes used for different methods are applied at the same time? If yes, to what extent can the mechanical squeezing be enhanced by this cooperative effect?

In this paper, we study the quantum dynamics of an optical cavity that has a movable mirror and contains a degenerate OPA, and which is driven by a laser field with periodically modulated amplitude, as shown in Fig. 1(a). Our results reveal a cooperation-based enhancement of the squeezing in the fluctuation of the momentum or position of the cavity mirror. Both the parametric pump driving and periodically modulated cavity driving contribute to the reduction of the mechanical fluctuation. The resulting two-fold squeezing exceeds the squeezing that can be achieved solely by either of these two processes [see Fig. 1(b)]. The idea may be generalized to realize cooperation-based...
enhancement of other quantum effects in complex optomechanical systems, e.g., entanglement between two mechanical oscillators or entanglement between a light field and a mechanical oscillator [30, 31].

II. THEORETICAL MODEL

We consider an optomechanical system where a degenerate OPA placed in a Fabry-Perot cavity of length $L$ and finesse $F$, with one fixed and partially transmitting mirror, and one movable and totally reflecting mirror, satisfies the standard canonical commutation relation

$$[a(t), a^\dagger(t')] = (n_a + 1)\delta(t - t')$$

with $n_a = [\exp(h\omega_c/k_BT) - 1]^{-1}$ being the thermal photon number and that of $\xi(t)$ is given by $\langle \xi(t)\xi(t') \rangle = \gamma_m c^{-4}\delta(t - t')$ [32, 37]. For the specific case where the mechanical oscillator has a good quality factor $Q \equiv \omega_m/\gamma_m \gg 1$, $\xi(t)$ becomes delta-correlated $\langle \xi(t)\xi(t') \rangle = \gamma_m(2n_m + 1)\delta(t - t')$ [38, 39], which corresponds to the Markovian process with $n_m = [\exp(h\omega_m/k_BT) - 1]^{-1}$ being the mean thermal excitation number in the mechanical mode.

III. DYNAMICS OF THE FIRST MOMENTS OF THE OPTICAL AND MECHANICAL MODES

Suppose that the external drivings are strong enough such that the intracavity photon number is much larger than 1, we can rewrite each Heisenberg operator as $O = \langle O(t) \rangle + \delta O (O = q, p, a)$, where $\delta O$ are quantum fluctuation operators with zero-mean values; and justify that $\langle a^\dagger(t)q(t) \rangle \approx (\langle a(t) \rangle)^2$ and $\langle a(t)q(t) \rangle \approx (\langle a(t) \rangle)^2$ are valid approximations. Applying the standard linearization techniques to the QLEs [2] and setting $\Delta_0 = \Omega/2$ for the consideration of mechanical squeezing, we thus obtain the equations for the first moments of the optical and mechanical modes

$$\langle \dot{q}(t) \rangle = \omega_m (p(t)) \rangle,$$

$$\langle \dot{p}(t) \rangle = -\omega_m \langle q(t) \rangle - \gamma_m \langle p(t) \rangle + g [\langle a(t) \rangle]^2,$$

$$\dot{a} = -(\kappa + i\Delta_0)\langle a(t) \rangle + ig\langle a(t) \rangle \langle q(t) \rangle + E(t) + 2\Lambda e^{i\phi} \langle a(t) \rangle e^{-i\Omega t},$$

and the linearized QLEs for the quantum fluctuations

$$\dot{\delta q} = \omega_m \delta p,$$

$$\dot{\delta p} = -\omega_m \delta q - \gamma_m \delta p + g [\langle a(t) \rangle] \delta a^\dagger + (\langle a(t) \rangle)^* \delta a + \xi(t),$$

$$\dot{\delta a} = -(\kappa + i\Delta_0)\delta a + ig \langle a(t) \rangle \delta q + 2\Lambda e^{i\phi} \delta a e^{-i\Omega t} + \sqrt{2\kappa a_n(t)},$$

where $\Delta(t) = \Delta_0 - g \langle q(t) \rangle$ is slightly modulated by the mechanical motion.

The phase space trajectories of the first moments $\langle O(t) \rangle$ can be found by simulating Eq. [3] for a set of typical parameters [see Figs. 1(c)-(d)] [10]. When the system is far away from the optomechanical instabilities and multistabilities [11], the semiclassical dynamics in the steady state will evolve toward a fixed orbit with a period being equal to the modulation period of the cavity driving $\tau$. Moreover, since the two non-linear terms in Eq. [3] are both proportional to the coupling strength $g$, the asymptotic solutions of $\langle O(t) \rangle$ can then be expanded perturbatively in the powers of $g$ and in terms of the Fourier components for $g \ll \omega_m$

$$\langle O(t) \rangle = \sum_{j=0}^{\infty} \sum_{n=-\infty}^{\infty} O_{n,j} e^{i\Omega t} g^j.$$  

Substituting Eq. [5] into Eq. [3], we can then obtain the recursive formulas for the time-independent coefficients $O_{n,j}$ (see Appendix A). By truncating the series to the first terms with indexes $j = 0, 1, ..., 6$ and...
where \( g_m = |g_m|e^{i\phi_m} = \frac{1}{\sqrt{2}} \sum_{j=0}^{\infty} a_{-n,j}g^{j+1} \) with \( n = -1, 0, 1 \).

IV. QUANTUM FLUCTUATIONS AND TWO-FOLD MECHANICAL SQUEEZING

To examine the effect of the modulation sidebands \((\sim e^{\pm i\Omega t})\), we introduce the mechanical annihilation and creation operators \( \delta b = (\delta g + i\delta p)/\sqrt{2} \), \( \delta b^\dagger = (\delta g - i\delta p)/\sqrt{2} \). Then, the QLEs for \( \delta a \) and \( \delta b \) are

\[
\delta \dot{a} = -i\Delta \delta a + iG(t)(\delta b^\dagger + \delta b) + 2\Lambda e^{i\theta} \delta a e^{-i\Omega t} - \kappa \delta a + \sqrt{2}a_{-m}(t),
\]

\[
\delta \dot{b} = -i\omega_m \delta b - \frac{\gamma}{2} (\delta b - \delta b^\dagger) + i\Gamma(t)\delta a^\dagger + G^*(t)\delta a + \sqrt{2}b_{-m}(t),
\]

(7)

with the mechanical noise operator \( b_{-m}(t) \) satisfying \( \langle b_{-m}(t) \rangle = 0 \), \( \langle b_{-m}(t)b_{-m}(t') \rangle = n_m\delta(t-t') \), and \( \langle b_{-m}(t)b_{m}(t') \rangle = (n_m + 1)\delta(t-t') \). We assume that the modulation frequency satisfies \( \Omega = 2\omega_m \) and the carrier frequency of the laser field driving the cavity is close to the anti-Stokes sideband, which leads to \( \Delta = \Delta_0 + g(q(t)) \approx \omega_m \) for weak optomechanical single-photon coupling. We further assume that the system is working in the resolved sideband regime: \( \omega_m \gg \kappa \), and the driving fields are weak: \( \omega_m \gg |g_0|, |g_{-1}|, |g_1| \). Under these conditions, if we substitute the slow varying fluctuation operators \( \delta a(t) = \delta a(t)e^{-i\Delta t} \), \( \delta b(t) = \delta b(t)e^{-i\omega_m t} \) and \( a_{-m}(t) = a_{-m}(t)e^{i\Delta t} \), and \( b_{-m}(t) = b_{-m}(t)e^{-i\omega_m t} \) into Eq. (7), the terms rotating at \( \pm 2\omega_m \) and \( \pm 4\omega_m \) can be ignored in the rotating wave approximation (RWA), which leads to

\[
\delta \dot{a} = ig_0\delta b + ig_1\delta b^\dagger + 2\Lambda e^{i\theta} \delta a^\dagger - \kappa \delta a + \sqrt{2}a_{-m}(t),
\]

\[
\delta \dot{b} = ig_0^* \delta a^\dagger + ig_1 \delta a - \frac{\gamma}{2} \delta b + \sqrt{2}b_{-m}(t).
\]

(8)

Note that \( a_{-m}(t) \) has the same correlation function as \( a_m(t) \). We then introduce the optical and mechanical quadratures with the tilded operators \( \tilde{\delta} x = (\delta a + \delta a^\dagger)/\sqrt{2} \), \( \tilde{\delta} \phi = (\delta a - \delta a^\dagger)/i\sqrt{2} \), \( \tilde{\delta} \phi = (\delta b + \delta b^\dagger)/\sqrt{2} \), \( \tilde{\delta} \phi = (\delta b - \delta b^\dagger)/iv2 \), and the corresponding noise operators \( \tilde{a}_{-m} = (a_{-m} + a_{-m}^\dagger)/\sqrt{2} \), \( \tilde{b}_{-m} = (b_{-m} + b_{-m}^\dagger)/\sqrt{2} \). The QLEs (8) can be rewritten as

\[
\tilde{U}(t) = \tilde{M}(t)N(t),
\]

(9)

where

\[
\tilde{U}(t) = \begin{bmatrix}
-2m & 0 & \text{Im}g_+ & -\text{Re}g_+
0 & 0 & \text{Re}g_+ & \text{Im}g_+
-\text{Im}g_+ & -\text{Re}g_+ & -\kappa + 2\Lambda \cos \theta & 2\Lambda \sin \theta
\text{Re}g_+ & \text{Re}g_+ & -\kappa - 2\Lambda \cos \theta & -2\Lambda \sin \theta
\end{bmatrix}
\]

(10)

with \( g_\pm = g_0 \pm g_1 \). Note that the stability conditions derived from the Routh-Hurwitz criterion require the parametric gain to fulfill \( \Lambda = 2\kappa/\kappa < 1 \), the calculation of which is fuzzy and will not be shown here.

The mechanical squeezing can be measured by the variance of the tilded fluctuations \( \langle \tilde{\delta} \phi^2 \rangle \) and \( \langle \tilde{\delta} \phi^2 \rangle \), which are just the first two diagonal elements of the tilded covariance matrix \( \tilde{V}(t) = [(U(t)U(t)\dagger) + \langle U(t)U(t)\dagger \rangle]/2 \). Using Eqs. (8)-(9), \( \tilde{V}(t) \) in the steady state is dominated by the Lyapunov equation (see Appendix A)

\[
\tilde{M}(t) = -D \]

(11)

with \( D = diag\{0, \gamma_0(2n_0 + 1), \kappa(2n_a + 1), \kappa(2n_a + 1)\} \). Eq. (11) can be analytically solved in the parameter regime with negligible mechanical damping \( \gamma_0 \approx 0 \) and null thermal photon number \( n_0 = 0 \), leading to

\[
\langle \tilde{\delta} \phi^2 \rangle = S_\phi^2 - \Lambda S_\phi^\Lambda, \langle \tilde{\delta} \phi^2 \rangle = S_\phi^2 + \Lambda S_\phi^\Lambda,
\]

(12)

where \( S_\phi^2 = (|g_0|^2 + |g_1|^2 + 2|g_0||g_1|\cos \phi_0)\mu^{-1} \), \( S_\phi^\Lambda = (|g_0|^2 \cos \phi_1 \cos \phi_0 + |g_1|^2 \cos \phi_2 \cos \phi_0 + 2|g_0||g_1|\cos \phi_0)\mu^{-1} \), \( \phi_0 \approx \phi_1 \approx \phi_0 \approx \phi_2 \approx \theta - 2\phi_0 \), and \( \phi_0, \phi_1 \approx \theta - 2\phi_0 \). Eq. (12) shows that, under the interplay between the periodic cavity driving and the parametric interaction, the fluctuations of the position and momentum of the mechanical oscillator strongly depend on the phase matching condition.

To clarify the underlying physics clearly, we assume \( \phi_0 = \pi, \phi_0, \phi_1 \approx \pi, \phi_1, \phi_1 \approx \pi \), then the variance of the position and momentum fluctuations reduce to

\[
\langle \tilde{\delta} \phi^2 \rangle = \frac{1}{2} \left[ 1 + \frac{|g_0|^2}{|g_1|^2} \right] (1 - \Lambda)^{-1},
\]

(13)

\[
\langle \tilde{\delta} \phi^2 \rangle = \frac{1}{2} \left[ 1 + \frac{|g_0|^2}{|g_1|^2} \right] (1 + \Lambda)^{-1},
\]

(14)

which reveal that the mechanical mode is squeezed in momentum (i.e. \( \langle \tilde{\delta} \phi^2 \rangle < 0.5 \)). Alternatively, the position squeezing can be achieved by setting \( \phi_0 = 0, \phi_0, \phi_1 = \pi \) and \( \phi_0, \phi_0, \phi_0 \approx 0 \). More importantly, Eq. (14) shows that the cooperation between the two driving fields results in a two-fold squeezing: The coefficient \( (1 - |\frac{|g_0|^2}{|g_1|^2}|)/(1 + |\frac{|g_0|^2}{|g_1|^2}|) \) describes the squeezing effect produced by the periodically modulated cavity driving, while \( (1 + \Lambda)^{-1} \) corresponds to the effect associated with the parametric driving.

The two-fold mechanical squeezing can be further understood by introducing the Bogoliubov mode defined as \( \delta B \equiv \delta b \cos \theta + e^{i\phi_0} \delta b^\dagger \sin \theta \) with \( \tan \theta = |g_1|/|g_0| \), which evolves according to the QLEs

\[
\delta \dot{B} = ig_0^* \delta a^\dagger + \frac{\gamma}{2} \delta B + \sqrt{\gamma_m} B_{-m},
\]

\[
\delta \dot{a} = -\kappa \delta a + ig_0 \delta B + 2\Lambda e^{i\theta} \delta a^\dagger + \sqrt{2}a_{-m}(t).
\]
duces a squeezing effect on the momentum fluctuation with zero mean and the nonzero correlation functions \( \kappa \). For \( \delta p \) versus \( \Lambda \) and \( \tanh r \) for \( C = 1 \times 10^4 \), the black region indicates that the mechanical oscillator is not squeezed. The white line denotes the optimal parametric gain \( \Lambda \) with which the momentum squeezing reaches its maximum for a given \( r \). (c) \( \delta p \) versus \( \Lambda \) for different cooperativity parameters \( C \) with \( r = 0.6 \). The black arrows indicate the optimal squeezing. (d) \( \delta p \) versus \( \Lambda \) for different modulations of the cavity driving with \( C = 5 \times 10^7 \). The makers in (c)-(d) indicate the numerical counterpart via the Fourier transformation, see Appendix C. In all figures other parameters are \( \kappa/\omega_m = 0.1 \), \( n_a = 0 \), \( n_m = 100 \), \( \gamma_m/\omega_m = 10^{-6} \).

The white line denotes the optimal parametric gain \( \Lambda \) which agrees well with its numerical counterpart obtained by simulation of Eq. (7), as shown in Fig. 2(a). For \( |g| \approx |g'| \) (corresponding to \( r \to 1 \)), the effective coupling between the Bogoliubov mode and the cavity mode becomes negligible, and \( \langle \delta p^2 \rangle \approx n_a + \frac{1}{4} \) is mainly determined by the thermal occupation of the mechanical mode with \( \gamma_m \ll \kappa \), implying that the mechanical mode is not squeezed [see Fig. 2(b)] [25]. In this case, the self-cooling of the mechanical oscillator through the photon-phonon sideband coupling is suppressed [43,45], therefore, the mechanical oscillator may stay far away from the ground state [24,66]. Generally, there exists an optimal squeezing for \( \langle \delta p^2 \rangle \) corresponding to the best efficiency of the cooperation between the two driving fields, which can be readily found by setting \( \mathrm{d} \langle \delta p^2 \rangle / \mathrm{d}\Lambda = 0 \) for an appropriate amplitude modulation \( |g_1|/|g'| \) (i.e. a given \( r \)). For \( C \gg 1 \), \( \langle \delta p^2 \rangle \) reaches its minimum when the optimal parametric gain satisfies

\[
\bar{\Lambda}_{\text{opt}} = \frac{1}{2} \left( 1 + \sqrt{1 + \frac{C}{\kappa}} \right) - 1
\]

\[
= \frac{1}{2} \left( \cosh r - \sinh r \right)^2 \langle \delta p^2 \rangle.
\]

Considering the effect of the mechanical damping \( \gamma_m / \omega_m \), the variances of the fluctuations \( \langle \delta q^2 \rangle \) and \( \langle \delta p^2 \rangle \) can again be calculated by the Lyapunov equation (11). As an example, when the phases \( \phi_0 = 0 \), \( \phi_r = \pi \) and \( \theta = \pi \) are set and the cooperativity parameter \( C = 4|g|^2/(\kappa r) \) is large so that \( C \approx C(1 - \tanh^2 r) \gg 2(1 + \bar{\Lambda}) \), the variance of the momentum is approximately given by

\[
\langle \delta p^2 \rangle \approx \frac{(\cosh r - \sinh r)^2}{2(1 + \bar{\Lambda})} + (2n_a + 1) \left[ \frac{1 + \bar{\Lambda}}{C} + \frac{\gamma_m}{4\kappa(1 + \bar{\Lambda})} \right].
\]

which agrees well with its numerical counterpart obtained by simulation of Eq. (7), as shown in Fig. 2(a). For \( |g| \approx |g'| \) (corresponding to \( r \to 1 \)), the effect of the parameter \( C \) on the variance of the momentum fluctuations is illustrated in Fig. 2(b). Note that an effective cooperation parameter \( C_{\text{thr}} = 4(\eta^{-1} - 1) \) on \( \bar{\Lambda} \), namely \( \bar{\Lambda} > C_{\text{thr}} \). In addition, the stability condition \( \Lambda_{\text{opt}} < 1 \) requires \( \bar{\Lambda} < C_{\text{ins}} \) with \( C_{\text{ins}} = 8(2(1 + \bar{\Lambda})^{-1} - 1) \). This result can be roughly explained as follows: The periodically modulated cavity driving produces a squeezing effect on the momentum fluctuation of the mechanical mode, which mathematically corresponds to converting the normal mechanical mode into the Bogoliubov mode through a unitary transformation equivalent to a squeezed operator. As a consequence, the “momentum” fluctuation of the Bogoliubov mode at the “vacuum” level corresponds to the normal momentum fluctuation below the vacuum level \( \langle \delta p^2 \rangle < 0.5 \). The parametric driving further reduces the “momentum” fluctuation of the Bogoliubov mode below the “vacuum” level, resulting in a second squeezing effect.

V. THE EFFECT OF MECHANICAL DAMPING AND EXPERIMENTAL FEASIBILITY
VI. CONCLUSION

In summary, we have shown that the parametric driving and the periodically modulated cavity driving, simultaneously applied to a cavity optomechanical system, can result in a two-fold squeezing effects on the mechanical oscillator. This enables implementation of strong squeezing for a macroscopic oscillator, which exceeds the result that is solely produced by either of these two drivings. Our results show that different physical processes, each producing a weak quantum effect, can cooperate to enhance the quantum effect. Our idea can be generalized to more complex optomechanical systems to realize two-fold two-mode squeezing, offering a possibility to produce strong mechanical-mechanical or optomechanical entanglement that can exceed the bound imposed by present methods.

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Appendix A: Periodic motion and quantum dynamics of the mechanical oscillator in the steady state

The time-independent coefficients $a_{n,j}$ in the Fourier expansion of $\langle O(t) \rangle$ ($O = q, p, a$) given by Eq. 5 can be found by substituting Eq. 5 into Eq. 3, leading to the following recursive formulas

$$p_{n,0} = 0, \quad q_{n,0} = 0, \quad a_{n,0} = \frac{[\kappa - i(\Delta_0 - (n + 1)\Omega)]E_{-n} + 2a e^{i\theta} E_{n+1}}{[\kappa - i(\Delta_0 - (n + 1)\Omega)]\kappa + i(\Delta_0 + n\Omega) - 4\Lambda^2},$$

$$q_{n,j} = \omega_m \sum_{k=0}^{j-2} \sum_{m=-\infty}^{\infty} \omega_m^2 - (n\Omega)^2 + i\gamma_m n\Omega,$$

$$a_{n,j} = \frac{2a e^{i\theta} a_{n-1,j}}{\kappa + i(\Delta_0 + n\Omega)} + i \sum_{k=0}^{j-2} \sum_{m=-\infty}^{\infty} a_{m,k} a_{n-m,j-k-1},$$

Using Eq. (A1) and (A2), the quantum dynamics of the mechanical oscillator can be studied through the linearized QLEs.

We introduce the amplitude and phase quadratures of the cavity mode as $\delta x = (\delta a + \delta a^*)/\sqrt{2}$, $\delta y = (\delta a - \delta a^*)/i\sqrt{2}$ and the analogous input quantum noise quadratures as $\delta x_{in} = (\delta a_{in} + \delta a_{in}^*)/\sqrt{2}$, $\delta y_{in} = (\delta a_{in} - \delta a_{in}^*)/i\sqrt{2}$ for convenience. Then the time-dependent equations of motion for the quantum fluctuations $u(t) = [\delta q, \delta p, \delta x, \delta y]^T$ arise as

$$\dot{u}(t) = M(t)u(t) + n(t),$$

with the drift matrix

$$M(t) = \begin{pmatrix} 0 & \omega_m & 0 & 0 \\ -\omega_m & -\gamma_m & 2G_x(t) & 2G_y(t) \\ 2G_y(t) & 0 & -\kappa + 2a \cos \theta & -2\Lambda \sin \theta \\ 2G_x(t) & 0 & -\Delta - 2a \sin \theta & -\kappa - 2\Lambda \cos \theta \end{pmatrix},$$

and the diffusion $n(t) = [0, \xi(t), \sqrt{2\kappa_0} \delta x_{in}, \sqrt{2\kappa_0} \delta y_{in}]^T$ being the noise sources. Here $\theta = \Omega t - \theta$, and $G_x$, $G_y$ are real part and imaginary part of the effective optomechanical coupling $G(t) \equiv g \langle a(t) / \sqrt{2} \rangle$. If all the eigenvalues of the matrix $M(t)$ have negative real parts at any time (i.e. the Routh-Hurwitz criterion [37]), the system will be in stable in the steady state. On the other hand, since the system in the steady state will evolve into an asymptotic Gaussian state for a Gaussian-type of noise [15], we can then characterize the second moments of the quadratures of the asymptotic state through the covariance matrix (CM) $V(t)$, with the matrix elements being

$$V_{k,l}(t) = \langle u_k(t)u_l^*(t) + u_l^*(t)u_k(t) \rangle / 2.$$

From Eqs. (A3) and (A4), we can easily derive a linear differential equation governing the evolution of the CM $V(t)$

$$\dot{V}(t) = M(t)V(t) + V(t)M^T(t) + D,$$

where $M(t)^T$ is the transpose matrix of $M(t)$, and

$$D = \text{diag}[0, \gamma_m(2n_m + 1), \kappa(2n_a + 1), \kappa(2n_a + 1)]$$

is a diagonal noise correlations matrix, defined by $\delta(t - t') D_{k,l} = \langle n_k(t)a_l^*(t') + n_l^*(t')a_k(t) \rangle / 2.$
The first two diagonal elements 
\[ V_{11}(t) = \langle \delta q(t)^2 \rangle, \]
\[ V_{22}(t) = \langle \delta p(t)^2 \rangle \] of \( \Gamma(t) \)
represent the variances of the 
fluctuations in the mechanical position and 
momentum, and the last two terms \( V_{33}(t) = \langle \delta x(t)^2 \rangle, \)
\[ V_{44}(t) = \langle \delta y(t)^2 \rangle \]
represent the variances of the 
fluctuations in the amplitude and phase of the 
cavity mode. The mechanical oscillator is position- 
or momentum-squeezed if either \( \langle \delta q(t)^2 \rangle < 1/2 \) or 
\( \langle \delta p(t)^2 \rangle < 1/2 \) in the steady state.
The degree of the squeezing can be expressed in the dB unit, 
which can be calculated by 
\[ -10 \log_{10} \left( \frac{\langle \delta q(t)^2 \rangle_{\text{vac}}}{\langle \delta q(t)^2 \rangle_{\text{vac}} / 2} \right) \] 
(or \( o = p, q \)), with \( \langle \delta q(t)^2 \rangle_{\text{vac}} = \langle \delta p(t)^2 \rangle_{\text{vac}} = 1/2 \) being 
the position and momentum variances of the vacuum state.

Recalling that the asymptotic behavior of the first 
moments of the mechanical mode and the cavity mode 
is \( t = 2\pi/\Omega \) periodic in the steady state, then we can find 
that the drift matrix \( \mathcal{M} \), which is related to 
\( \langle a(t) \rangle \) and \( \langle q(t) \rangle \), satisfies \( \mathcal{M}(t+\tau) = \mathcal{M}(t) \) and therefore 
\( V(t+\tau) = V(t) \) according to the Floquet theory [19]. By solving the evolitional equation \( \mathcal{A} \mathcal{V}(t) \) of 
the CM \( V(t) \), we have calculated the time-dependent 
variances of the mechanical moments \( \langle \delta p(t)^2 \rangle \) for 
the optomechanical system with (i) OPA, (ii) periodic 
driving, and (iii) both OPA and periodic driving, as shown in Fig. [3]. It has been realized that the 
cavity solely pumped by parametric interaction (with \( E_{\pm1} = 0 \)) [29] 
and solely modulated by periodic driving (\( \Lambda = 0 \)) [27] 
can both reach to mechanical squeezing, the degree of which (corresponding to the minimum of \( \langle \delta p(t)^2 \rangle \)) can reach 1.44 dB (\( \langle \delta p(t)^2 \rangle_{\text{min}} = 0.359 \)) for \( \Lambda/\kappa = 0.3 \), and 5.13 dB 
(\( \langle \delta p(t)^2 \rangle_{\text{min}} = 0.153 \)) for \( E_{\pm1}/\omega_m = 0.7 \times 10^4 \), respectively. However, we note that, by combining OPA 
and periodic driving simultaneously, the mechanical squeezing will be greatly enhanced, the degree of 
momentum squeezing can achieve as large as 6.31 dB 
(\( \langle \delta p(t)^2 \rangle_{\text{min}} = 0.117 \)), which is far beyond the 3 dB 
limit, required for ultrahigh-precision measurements.

Appendix B: The steady-state “momentum” fluctuation of the Bogoliubov mode

The QLEs [15] can be solved in the adiabatic approximation under the condition of \( \kappa \gg |g_B| \).
For this purpose, we rewrite the equations for \( \delta \hat{a} \) and \( \delta \hat{a}^\dagger \):

\[
\delta \hat{a} = -\kappa \delta \hat{a} + \frac{g_B}{\kappa(1 - \Lambda^2)} \delta B + B E \delta \theta + \sqrt{2\kappa} \delta a_n,
\]
\[
\delta \hat{a}^\dagger = -\kappa \delta \hat{a}^\dagger - \frac{g_B^*}{\kappa(1 - \Lambda^2)} \delta B^\dagger + B E^\dagger \delta \theta + \sqrt{2\kappa} \delta a_n^\dagger \, \text{(B1)}
\]

Setting \( \delta \hat{a} = \delta \hat{a}^\dagger = 0 \) and \( \theta - 2\phi_0 = 0 \), it is readily to find

\[
\delta \hat{B} = \frac{g_B}{\kappa(1 - \Lambda^2)} (\delta B + \Lambda \delta B^\dagger)
\]
\[
+ \sqrt{\frac{2\kappa}{\kappa(1 - \Lambda^2)}} (\delta a_n + \Lambda e^{\theta} \delta a_n^\dagger).
\, \text{(B2)}
\]

Inserting Eq. [B2] into \( \delta B = i g_B \delta \hat{a} - \frac{\gamma_m}{2} \delta B + \sqrt{\gamma_m} B_{in} \), we have

\[
\delta \hat{B} = \frac{-|g_B|^2}{\kappa(1 - \Lambda^2)} (\delta B + \Lambda \delta B^\dagger) + \sqrt{\gamma_m} B_{in}
\]
\[
+ \frac{i g_B \sqrt{2\kappa}}{\kappa(1 - \Lambda^2)} (\delta a_n + \Lambda e^{\theta} \delta a_n^\dagger). \, \text{(B3)}
\]

Here, the term \( \gamma_m \) in the coefficient of \( \delta B \) is safely ignored in our parameter regime.
Then the equation of motion for the “momentum” of the Bogoliubov mode \( \delta p_B \) is given by

\[
\delta \hat{p}_B = -\frac{|g_B|^2}{\kappa(1 + \Lambda)} \delta p_B + d(t) + v(t), \, \text{(B4)}
\]

where \( d(t) = \frac{1}{t} \sqrt{\frac{\gamma_m}{2}} (B_{in} - B_{in}^\dagger) \) and \( v(t) = \frac{g_B \sqrt{2\kappa} \delta a_n}{\kappa(1 + \Lambda)} \), whose correlation functions are 
\( \langle d(t)d(t') \rangle = \frac{\gamma_m}{2}(2\sinh^2 r + 1)(2n_m + 1) \delta(t - t') \) and 
\( \langle v(t)v(t') \rangle = \frac{|g_B|^2}{\kappa(1 + \Lambda)}(2n_m + 1) \delta(t - t') \), respectively.

Based on these equations, we obtain the equation for

\[
\frac{\partial \langle \delta p_B^2 \rangle}{\partial t} = -\frac{2|g_B|^2}{\kappa(1 + \Lambda)} \langle \delta p_B^2 \rangle + \frac{|g_B|^2}{\kappa(1 + \Lambda)}(2n_m + 1)
\]
\[
+ \frac{\gamma_m}{2}(2\sinh^2 r + 1)(2n_m + 1). \, \text{(B5)}
\]

As a result, the steady-state “momentum” fluctuation of the Bogoliubov mode can be solved by setting
\( \frac{\partial \langle \delta p_B^2 \rangle}{\partial t} = 0 \), giving rise to Eq. (16) in the main text.
Appendix C: Optomechanical squeezing in the rotating wave approximation

Taking the Fourier transform of Eq. (9) by using $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\omega)e^{-i\omega t}d\omega$ and $f'(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f'(\omega)e^{-i\omega t}d\omega$, we obtain the position and momentum fluctuations of the movable mirror in the frequency domain, i.e.

$$
\delta \tilde{q}(\omega) = A_1(\omega)\tilde{x}_m(\omega) + B_1(\omega)\tilde{y}_m(\omega) + \dot{E}_1(\omega)\tilde{\omega}_m + E_1(\omega),
$$

$$
\delta \tilde{p}(\omega) = A_2(\omega)\tilde{x}_m(\omega) + B_2(\omega)\tilde{y}_m(\omega) + \dot{E}_2(\omega)\tilde{\omega}_m + E_2(\omega),
$$

where

$$
A_1(\omega) = -\frac{\sqrt{\pi}}{\sqrt{\omega}} \left\{ \left[ \frac{\Lambda(\alpha_0 - \alpha_1)}{2} + i\omega(\omega_0 - g_1) \right]v(\omega) - i\left| g_0^2 \right| \right\},
$$

$$
B_1(\omega) = \frac{\sqrt{\pi}}{\sqrt{\omega}} \left\{ \left[ \frac{\Lambda(\beta_0 - \beta_1)}{2} - \imath(\omega_0 - g_1) \right] v(\omega) \right\},
$$

$$
E_1(\omega) = \frac{\sqrt{\pi}}{\sqrt{\omega}} \left\{ \left[ \left| g_0^2 \right| \imath(\omega_0 + g_1) \right] v(\omega) \right\},
$$

$$
F_1(\omega) = \frac{\sqrt{\pi}}{\sqrt{\omega}} \left\{ \left[ \imath(\omega_0 - g_1) \right] v(\omega) \right\},
$$

$$
A_2(\omega) = \frac{\sqrt{\pi}}{\sqrt{\omega}} \left\{ \left[ \frac{\Lambda(\alpha_0 + \alpha_1)}{2} - \imath(\omega_0 + g_1) \right] v(\omega) \right\},
$$

$$
B_2(\omega) = \frac{\sqrt{\pi}}{\sqrt{\omega}} \left\{ \left[ \frac{\Lambda(\beta_0 + \beta_1)}{2} + i\omega(\omega_0 + g_1) \right] v(\omega) \right\},
$$

$$
E_2(\omega) = \frac{\sqrt{\pi}}{\sqrt{\omega}} \left\{ \left[ \left| g_0^2 \right| \imath(\omega_0 - g_1) \right] v(\omega) \right\},
$$

$$
F_2(\omega) = \frac{\sqrt{\pi}}{\sqrt{\omega}} \left\{ \left[ \left| g_0^2 \right| \imath(\omega_0 + g_1) \right] v(\omega) \right\},
$$

with $\alpha_0 = g_0 e^{-\imath \omega} - g_0 e^{\imath \omega}$, $\alpha_1 = g_1 e^{-\imath \omega} - g_1 e^{\imath \omega}$, $\beta_0 = g_0 e^{-\imath \omega} + g_0 e^{\imath \omega}$, $\beta_1 = g_1 e^{-\imath \omega} + g_1 e^{\imath \omega}$, $\Gamma_0 = g_0 e^{-\imath \omega} + g_0 e^{\imath \omega}$, $\Gamma_1 = g_1 e^{-\imath \omega} + g_1 e^{\imath \omega}$, and $\omega = \frac{2\pi}{\lambda} - \imath \omega$. The two terms $d(\omega)$ in Eq. (10) originate from the radiation pressure contribution, and the last two terms are from the thermal noise contribution.

Without optomechanical coupling ($g_0 = g_1 = 0$), the mechanical mode subjected to the purely thermal noise will make quantum Brownian motion leading to $\delta \tilde{p}(\omega) = \frac{\sqrt{\pi}}{\sqrt{\omega}} \tilde{\omega}_m$ and $\delta \tilde{q}(\omega) = \frac{\sqrt{\pi}}{\sqrt{\omega}} \tilde{\omega}_m$. The expressions of the spectra for the position and momentum fluctuations of the mechanical mode are ($Z = \tilde{q}, \tilde{p}$)

$$
S_Z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega' \epsilon^{-i(\omega' - \omega)t} \times \left[ (\delta \tilde{Z}(\omega)\delta \tilde{Z}(\omega')) + (\delta \tilde{Z}(\omega')\delta \tilde{Z}(\omega)) \right],
$$

which can be solved by using the correlation functions of the noise sources in the frequency domain.

$$
\langle \tilde{q}_m(\omega)\tilde{q}_m(\omega') \rangle = \langle \tilde{p}_m(\omega)\tilde{p}_m(\omega') \rangle = 0,
$$

$$
\langle \tilde{q}_m(\omega)\tilde{p}_m(\omega') \rangle = -\langle \tilde{p}_m(\omega)\tilde{q}_m(\omega') \rangle = i2\pi\delta(\omega - \omega'),
$$

$$
\langle \tilde{x}_m(\omega)\tilde{x}_m(\omega') \rangle = \langle \tilde{\omega}_m(\omega)\tilde{\omega}_m(\omega') \rangle = 0,
$$

$$
\langle \tilde{x}_m(\omega)\tilde{\omega}_m(\omega') \rangle = \langle \tilde{\omega}_m(\omega)\tilde{x}_m(\omega') \rangle = \frac{1}{2}\pi\delta(\omega - \omega'),
$$

and are given by

$$
S_q(\omega) = \left[ A_1(\omega)A_1(-\omega) + B_1(\omega)B_1(-\omega) \right](n_a + \frac{1}{2}) + \left[ E_1(\omega)E_1(-\omega) \right](n_m + \frac{1}{2}),
$$

$$
S_p(\omega) = \left[ A_2(\omega)A_2(-\omega) + B_2(\omega)B_2(-\omega) \right](n_a + \frac{1}{2}) + \left[ E_2(\omega)E_2(-\omega) \right](n_m + \frac{1}{2}),
$$

where the first term proportional to $(n_a + \frac{1}{2})$ and the second term proportional to $(n_m + \frac{1}{2})$ correspond to the radiation pressure contribution and thermal noise contribution, respectively. For $g_0 = g_1 = 0$, Eqs. (C6) and (C7) are simply Lorentzian lines with full width $\gamma_m$ at half maximum.

The variances in the position $\langle \delta \tilde{q}^2 \rangle$ and momentum $\langle \delta \tilde{p}^2 \rangle$ of the mechanical mode are finally obtained by

$$
\langle \delta Z^2 \rangle = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_Z(\omega)d\omega,
$$

giving rise to the numerical results in Fig. 2(c)-(d) (marker).

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