Graphs, flags and partitions

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Abstract

This paper defines, for each graph $G$, a flag vector $f_G$. The flag vectors of the graphs on $n$ vertices span a space whose dimension is $p(n)$, the number of partitions on $n$. The analogy with convex polytopes indicates that the linear inequalities satisfied by $f_G$ may be both interesting and accessible. Such would provide inequalities both sharp and subtle on the combinatorial structure of $G$. These may be related to Ramsey theory.

1 Introduction

The formulation and proof of subtle combinatorial inequalities on the number of vertices, edges and so forth on a graph is the goal towards which this paper is intended to be the first step. This paper puts forward a precise definition of what is meant by ‘and so forth’. More exactly it assigns to each graph $G$ a flag vector $f_G$. The span of $f_G$, as $G$ ranges over all graphs on $n$ vertices, has dimension equal to $p(n)$, the number of partitions on $n$. The linear inequalities satisfied by $f_G$, or what comes to the same thing the convex hull $\Delta(n)$ of $f_G$ as $G$ ranges over all $n$-vertex graphs, seems not to be arbitrary, at least for small values of $n$.

Ramsey theory can be expressed in terms of combinatorial inequalities. For example, that any graph $G$ on six vertices contains a 3-element subset $S$ of its vertex set, such that the restriction $G|_S$ of $G$ to $S$ is either the complete graph or the empty graph: this statement is equivalent to the following. For each 3-element subset $S$ of the vertex set, temporarily let $\lambda(G,S)$ be 1 if $G|_S$ is the complete graph, and zero otherwise. Now let $\bar{\lambda}(G)$ be the sum of $\lambda(G,S)$ over all $S$. Define $\bar{\lambda}(G)$ to be as $\lambda(G)$, but this time counting empty graphs. The Ramsey statement is equivalent to $\lambda(G) + \bar{\lambda}(G) \geq 1$. Although this equivalence is trivial, it does indicate the sort of relationship we are looking for. Whether or not Ramsey-type statements can be formulated in terms of the flag vector $f_G$ is not yet clear. If so, they may very well involve non-linear functions of $f_G$.

The flag vector can be defined in several different ways, that are essentially equivalent in the sense that each is a linear function of any of the others. The easiest, although lacking in motivation, is via subgraphs. Here is its definition.

\textbf{Definition 1} Suppose $G$ is a graph on $n$ vertices, and that $H$ is a subgraph of $G$, also on $n$ vertices. Each such subgraph will contribute $\lambda^s(H)$, a quantity defined below. In other words

$$f^s G = \sum_H \lambda^s(H) \quad H \text{ a subgraph of } G$$

defines the subgraph form $f^s G$ of the flag vector.

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Definition 2 The subgraph contribution λ*(G) is defined as follows. Each graph G, via connectivity, determines a partition of its vertex set, which can then be thought of as a partition π = π(G) of the order n of the graph. The quantity λ*(G) will be a formal sum of partitions, and the contribution due to G will be s0(G)π(G), where s0(G) is next to be defined.

Definition 3 Let G be a graph on n vertices. A shelling σ of G is a sequence G = G0, G1, . . . , Gn of graphs obtained from G by removing the vertices from G one at a time. As a vertex is removed, so all edges through that vertex are also removed. An acyclic shelling is a shelling where at most one edge is removed at each stage. The acyclic shelling number s0(G) of G is the number of acyclic shellings that G possesses. For example, if G has a cycle then s0(G) is zero.

The flag vector is an example of a rule v that assigns to each graph G a vector v(G) lying in some vector space V. For such to be useful it should retain some, but perhaps not all, of the geometric information in G. For example, say that v distinguishes graphs if v(G) = v(G′) implies that G and G′ are isomorphic as graphs. The location of G in an enumeration of all isomorphism classes of graphs is an example of such a rule. The essentially random ordering ordering of graphs that such provides is unlikely to be useful. Similarly, say that v embeds graphs if the vectors v(G), as G ranges over all isomorphism types of graphs, are linearly independent. This is even worse. Enumeration gives only random relations between the vectors v(G). Embedding gives none at all.

Much of the motivation for this paper has come from the theory of convex polytopes. In this case the definition of the flag vector is straightforward, and many subtle combinatorial inequalities are either known or conjectured. When Δ is a simple n-polytope (exactly n facets at each vertex) McMullen’s conjectured conditions on the face vector [1] were proved sufficient (by Stanley [2]) and necessary (by Billera and Lee [3]). In the general case the ‘mph h-vector’ is known to be unimodal when Δ has rational vertices [4, 5, 13]. Latent in the author’s local-global variant of intersection homology [6, 7, 10] and his analogue of ring structure [9] are various conjectures on polytope flag vectors.

To provide some background, we here recall the basic definition and main result of [1]. (For the subtle inequalities, see the works just cited.)

Definition 4 Suppose Δ is a simple n-dimensional polytope. In that case the face vector fΔ of Δ is the sequence fΔ = (f0, . . . , fn−1), where fi is the number of i-dimensional faces on Δ. Now suppose that Δ is a general n-dimensional polytope. A flag δ of Δ is a sequence

δ = (δ1 ⊂ δ2 ⊂ . . . ⊂ δr ⊂ Δ)

of faces, each strictly contained in the next. Its dimension sequence, or dimension for short, is the sequence

d = (d1 < d2 < . . . < dr < n)

of integers, where di is the dimension of δi. The d-th component fdΔ of the flag vector fΔ is the number of d-dimensional flags on Δ. One can think of d as a subset I of {0, 1, . . . , n − 1}, and so fΔ has 2n components, indexed by such subsets. (If Δ is known to be simple, then there is a linear function that computes the flag vector from the face vector. Thus, the overloaded use of fΔ will not in this situation do any harm.)

Theorem 5 (Generalised Dehn-Sommerville equations) The span of the flag vectors of n-dimensional polytopes has dimension the (n + 1)st Fibonacci number.
2 The verbose flag vector $f^vG$

This definition takes its motivation from convex polytopes. Suppose $\Delta$ is an $n$-dimensional convex polytope. To define the flag vector of $\Delta$ it is enough to know what it means, for $\delta$ to be a facet of $\Delta$. For once this is known, the faces are facets of facets of facets etc., and the flag vector follows from the counting of chains.

What then is a ‘facet’ of a graph $G$ on $n$ vertices? Clearly, it should be a graph $H$ on $n - 1$ vertices. Such can be obtained by removing a single vertex $v$ from $G$, and when this is done some edges may have to be removed as well. The flag vector will record such information.

**Definition 6** Suppose $v$ is a vertex of a graph $G$. As usual, the order $m_v$ of $v$ (on $G$) is the number of edges of $G$ that pass through $v$. Define the $v$-facet $G_v$ of $G$ to be the result of removing $v$ (and also the $m_v$ edges) from $G$.

**Definition 7** The verbose flag vector of the unique graph on zero vertices is the number 1. For other graphs the verbose flag vector $f^vG$ is a formal sum of words in $a$ and $b$ obtained by the following recursion:

$$f^vG = \sum (a + m_v b)f^vG_v$$

where the sum is over all vertices $v$ (or facets $G_v$) of $G$. If $G$ is of order $n$ then $f^vG$ is a homogeneous polynomial of degree $n$ in the non-commuting variables $a$ and $b$. There are $2^n$ such words $w$ in $a$ and $b$. Each can be thought of as a subset $I$ of $\{1, \ldots, n\}$.

By way of example, let $3_0$, $3_1$, $3_2$ and $3_3$ denote the graphs on 3 vertices with 0, 1, 2 and 3 edges respectively. The reader is asked to verify the following results. Later, we will give Proposition 9 a geometric meaning.

**Proposition 8** The verbose flag vectors of the graphs on 3 vertices are as follows:

- $f^v(3_0) = 6aaa$
- $f^v(3_1) = 6aaa + 2aba + 4baa$
- $f^v(3_2) = 6aaa + 4aba + 8baa + 4bba$
- $f^v(3_3) = 6aaa + 6aba + 12baa + 12bba$

**Proposition 9** There is a single linear relation among these flag vectors, namely

$$f(3_0) - 3f(3_1) + 3f(3_2) - f(3_3) = 0.$$ 

Each form of the flag vector has its own advantages. Here is a result that is most easily proved by using the verbose form. But first a definition.

**Definition 10** As usual, if $G$ is a graph on $n$ vertices, let the complementary graph $\overline{G}$ be the graph on the same $n$ vertices, whose edges are exactly the non-edges of $G$. Let $f^vG$ denote the flag vector (in whatever form) of the complement $\overline{G}$ of $G$.

**Theorem 11** The verbose flag vector of $\overline{f^vG}$ of the complement $\overline{G}$ of $G$ is a linear function of $f^vG$. (This statement means that there is a linear function $L$ on degree $n$ words in $a$ and $b$, such that the equation $L(f^vG) = \overline{f^vG}$ holds true, for every $n$-vertex graph $G$.)
Proof. If $G$ has no vertices the result is trivial. We will use induction. The $v$-contribution to $\mathcal{F}G$ is $(a + (n - 1 - m_v)b)\mathcal{F}G_v$, where $m_v$ is the multiplicity of $v$ on $G$. Thus, the transformation

$$a \mapsto a + (n - 1)b \quad b \mapsto -b$$

applied to the first letter of each word in $fG$, together with the inductive formula applied to the remainder of the word, will define the order $n$ rule that takes $f^nG$ to $\mathcal{F}G$. □

In fact the induction in this proof is spurious. The transformation

$$a \mapsto a + (i - 1)b \quad b \mapsto -b$$

applied to each of the letters $w_i$ of a word $w = w_n w_{n-1} \ldots w_1$ will transform $fG$ to $\mathcal{F}G$. It is easily seen to be an involution, as it should be.

Shellings can be used to compute the verbose flag vector. The next theorem follows immediately.

**Definition 12** Let $\sigma$ be a shelling of a graph $G$. Define the verbose shelling contribution $\sigma^vG$ to $f^vG$ to be the product $w_1 w_2 \ldots w_n$, where $w_i$ is $(a + m_i b)$, and where $m_i$ is the number of edges of $G$ of the form $v_i v_j$ for $i < j$, and where $v_i$ and $v_j$ are the $i$-th and $j$-th vertices in the shelling $\sigma$.

**Theorem 13** The formula $f^vG = \sum_\sigma \sigma^vG$ computes the verbose flag vector.

This gives another way of looking at $f^vG$. First fix a word $w$ in $a$ and $b$. Each shelling $\sigma$ will contribute to the $f_w$ component of $f^vG$ as follows. Whenever $w_i$ in $w = w_1 \ldots w_n$ is $b$, compute $m_i$ as above. The product of these $m_i$ is the $\sigma$ contribution to the coefficient of $w$ in $f^vG$, or in other words to the $f_w$ component.

In Ramsey theory, non-constructive proofs of the following type are common. First produce an overestimate of the number of graphs that do not have some property $P$. Next, count the number of graphs. If, for some number $n$ of vertices the first number is less than the second, then we have a non-constructive proof that there is a graph that has the property $P$. This is an example of the ‘random method’ that pervades graph theory and combinatorics. It is similar to the application of the pigeon-hole principle.

Here the property $P$ is a function $P(G)$ that takes the values zero and one on graphs. Now suppose that $l(G)$ is a non-negative linear function of the flag vector $fG$. (To produce such is to know something of the linear inequalities on the flag vector.) If we can compute the average value of $l$ over all graphs (on $n$ vertices), and this value is non-zero, then similarly we have a non-constructive proof that there is a $G$ for which $l(G)$ is non-zero.

The earlier presentation of the verbose flag vector as a sum over all shellings of a graph leads to the next result, whose proof is left to the reader.

**Definition 14** The total flag vector $f(n)$ of all graphs on $n$ vertices is the sum of $fG$, as $G$ ranges over all graphs on $n$ vertices. Here, such a graph is simply a subset of the $N = (\begin{array}{c} n \\ 2 \end{array})$ possible edges on an $n$-element vertex set. Thus, there are $2^N$ such graphs.

**Theorem 15** For each word $w = w_n \ldots w_1$, the $w$ component $f_w(n)$ is the product

$$n! \lambda_n \ldots \lambda_1$$

where $\lambda_i$ if $2^{i-1}$ is $w_i$ is $a$, while $\lambda_i$ is $(i - 1)2^{i-2}$ if $w_i$ is $b$. In particular, if $l$ is a linear function of $f$, then the average value of $l$ over all graphs can readily be computed.
Given a word w, certain pairs of shellings σ and σ′ will make equal contributions to \( f_w \), for any choice of an edge set for \( G \). Here is an example. Let \( w = aabbaaa \), and suppose that \( σ \) and \( σ' \) differ, if at all, only in the order in which the first two vertices are presented, and in the order of the last three. In other words, we may permute \( \{v_1, v_2\} \) and also \( \{v_5, v_6, v_7\} \) to obtain \( σ' \) from \( σ \). Clearly there are 12 such possibilities and as such changes do not affect \( m_3 \) and \( m_4 \), all these \( σ' \) make the same contribution to \( f_w \).

From this it follows that for \( w = aabbaaa \), the \( f_w \) component of \( f^v G \) is always divisible by 12. (Proposition 8 provides similar examples.) This is a redundancy of sorts, which can be reduced by using ‘partial shellings’ that ‘respect’ the word \( w \). There is another redundancy in \( f^v G \), which is that although it lies in a vector space of dimension \( 2^n \), it does not span this space. The next section will define a flag vector that provides an integral spanning set for the space in which it lies.

3 The concise flag vector \( f^c G \)

This section introduces the concise flag vector \( f^c G \), which records the same information as the verbose flag vector \( f^v G \), but concisely. In particular, like the subgraph flag vector \( f^s G \), it takes values in a space whose dimension is \( p(n) \), the number of partitions of the number of vertices on \( G \), rather than the \( 2^n \) dimensional space used by \( f^v G \).

First we introduce a device that help us to understand Proposition 9. It allows us to produce formal sums of graphs, whose total flag vector is zero. If we think of a graph as a bunch of points joined by lines, then a graph with optional edges is the same, but some of the edges are represented by dashed lines. Between any two vertices there is at most one edge, either dashed or ordinary. Each dashed edge can be either completed or removed, and so from a graph with \( r \) dashed edges a total of \( 2^r \) graphs can be produced. The flag vector \( fG \) of a graph \( G \) with optional edges is the alternating sum of the \( 2^r \) graphs so obtained. A graph with no optional edges is, of course, just an ordinary graph. The following formal definition takes care of the details.

**Definition 16** A graph \( G \) with optional edges consists of a vertex set \( V \), together with disjoint sets \( E \) and \( C \) of unordered pairs of vertices. Here \( E \) is the regular edge set, while \( C \) (for choice) is the set of optional edges. Such is usually regarded as a formal sum

\[
(V, E, C) = \sum_{B \subseteq C} (-1)^{|C| - |B|} (V, E \cup B)
\]

of normal graphs. The quantity

\[
f(G) = \sum_{B \subseteq C} (-1)^{|C| - |B|} f(V, E \cup B)
\]

is its flag vector, of whatever type.

Each graph with only optional edges is of course an alternating sum of its subgraphs, thought of as ordinary graphs. Conversely, each ordinary graph can be written as a formal sum of graphs with only optional edges. We can think of an optional edge as the difference between an edge and a non-edge. Conversely, an edge can be thought of as the sum of an optional edge and a non-edge. The proof of the next result is left to the reader. It is at root nothing more than \( x = (x - y) + y \).

**Theorem 17** Let \( G \) be a graph \((V, E)\). For each subset \( C \) of \( E \), let \((V, \emptyset, C)\) be the graph with \( C \) as its optional edge set, and no actual edges. Then

\[
(V, E) = \sum_{C \subseteq E} (V, \emptyset, C)
\]

5
or in other words, $G$ is equivalent to a formal sum of graphs with only optional edges, namely those associated to all its subgraphs.

The advantage of graphs with only optional edges is that their flag vectors are much easier to calculate and otherwise deal with. Applied to the graph on 3 vertices, with 3 optional edges, the next result establishes Proposition 9.

**Theorem 18** Suppose $G$ is a graph with optional edges, and in those edges a cycle can be found. Then the verbose flag vector $f^vG$ of $G$ is zero.

**Proof.** Let $w$ be a word in $a$ and $b$, and let $\sigma$ be a shelling of $G$. Let $v_i$ be the first vertex removed by $\sigma$ that lies on the optional cycle. At this point $w_i$ is either $a$ or $b$. By assumption, at $v_i$ there are at least two optional edges of the form $v_iv_j$, for $j > i$ in the $\sigma$ ordering.

If $w_i$ is $a$ and there is just a single optional edge $v_iv_j$, then the $w$-\(\sigma\) contribution to $f^vG$ is zero. This is because changing the choice made for $v_iv_j$ changes the sign of the contributions made at this point. If there are several optional edges, choose just one of them and apply this argument to it.

If $w_i$ is a $b$, we will need two optional edges, say $v_iv_j$ and $v_kv_l$. This gives four $(2 \times 2)$ choices altogether, and when we determine their contributions we find that they are $+2$, $-1$ and $-1$, and $0$. (Both optional edges present, just one optional edge, and no optional edge.) The sum is zero and so the result follows. As before, if there are more than two optional edges, the result still holds, by applying the argument to just two of them.

The argument we just used can also be applied to compute the verbose flag vector $f^vG$ of a graph $G$ with only optional edges. To begin with, we will apply it just to a tree.

**Theorem 19** Suppose $T$ is a tree (connected acyclic graph) on $n$ vertices. Consider $T$ as a graph with only optional edges. Then $f^v(T) = s^a(T)b^{n-1}a$, where $s^a(T)$ is the acyclic shelling number of $T$.

**Proof.** Let $\sigma$ be a shelling of $T$. If $\sigma$ is not acyclic then the previous argument, applied to the first vertex $v_i$ at which $\sigma$ fails to be acyclic, shows that the $\sigma$ contribution to $f^vT$ is zero. This holds whether $w_i$ in $w=w_1\ldots w_n$ is $a$ or $b$. Thus, only acyclic shellings contribute. Now suppose $w$ contains an $a$, and that $w_i$ is the first $a$ in $w$. Assume that $i < n$. Provided $\sigma$ is acyclic, when $w_i$ is removed an optional edge will be removed also, and the previous argument shows that this $w$-\(\sigma\) contribution is zero. Finally, if $w_n$ is $b$ then the $w$ contribution must be zero, for at the last vertex there is no edge left to remove. Thus, the only non-zero contributions are when $\sigma$ is an acyclic shelling, and then $b^{n-1}a$ is the contribution. By definition, $T$ has $s^a(T)$ such shellings.

Now suppose that $G = T_1 \sqcup T_2 \ldots \sqcup T_r$ is an acyclic graph, broken up as a disjoint union of trees. As before, we will treat $G$ as a graph with only optional edges. Suppose that an acyclic shelling $\sigma_i$ is given for each component $T_i$ of $G$. From this information many acyclic shellings $\sigma$ of $G$ can be constructed, such that when $\sigma$ is restricted to each $T_i$ (and also renumbered to account for absent vertices), the result is $\sigma_i$. Looked at abstractly, we have $r$ disjoint sets with an ordering $\sigma_i$ on each of them, and an ordering $\sigma$ on the union of these sets, that extends the $\sigma_i$. This is an example of what is known as the merging of ordered sets.

Now we will look at $f^vG$. The main idea is that it can be calculated from the $f^vT_i$. Considered in isolation each acyclic shelling $\sigma_i$ of each $T_i$ contributes $b^{n(i)−1}a$, where $n(i)$ is the number of vertices on $T_i$. If we have such a $\sigma_i$ for each component $T_i$, then the total contribution due to the
σ_i will be the sum of all the possible ‘mergings’ of the b^{n(i)-1}a. Here, each merging of the σ_i to σ will induce in a natural way a ‘merging’ of the b^{n(i)-1}a.

It should now be clear that f^vG is determined in a straightforward manner from the f^vT_i of its components. This follows from a technical lemma, whose proof is left to the reader.

**Lemma 20** Suppose G = G_1 △ G_2 is a disjoint union of two graphs with only optional edges. Then there is a universal bilinear function μ for computing f^vG. In other words, the equation

\[ f^vG = \mu(f^vG_1, f^vG_2) \]

holds. Moreover, the coefficients are all nonnegative integers. (By induction, there will also be a multilinear μ for when G is broken into more than two components G_i.)

We now come to the main definitions and theorem of this section.

**Definition 21** Let T be a tree on n ≥ 3 vertices. Then the shelling number s(T) of T is the number of ways of shelling T down to a 3-vertex tree. If T has 3 or fewer vertices, then s(T) is defined to be 1.

As a 3-vertex tree has precisely 4 acyclic shellings, s^a(T) = 4s(T) for n ≥ 3. For n = 2, s^a(T) = 2s(T). For n = 1, s^a(T) = s(T).

**Definition 22** The shelling number s(G) of an acyclic graph G is the product s(T_1)\ldots s(T_r) of the shelling numbers of its component trees T_i. If G contains a cycle, then s(G) is defined to be zero.

**Definition 23** Let G be a graph on n vertices. The concise flag vector f^cG is a formal sum of partitions of n. Each subgraph H contributes s(H)π(H) to f^cG, where s(H) is as above, and π(H) is the partition of n induced by H-connectivity on the vertex set.

The previous results, taken together, prove the following:

**Theorem 24** Let G be a graph. Then the verbose flag vector f^vG is a linear function of the concise flag vector f^cG. Further, the subgraph and concise flag vectors are linear functions of each other.

In the next section we will show that f^cG is a linear function of f^vG, and thus the concept of a linear function of the flag vector does not depend on the form of flag vector that is chosen. We will also show that the f^cG provide an integral spanning set for the space of formal sums of partitions.

4 Basis and dimension

In this section we will show that the concise flag vector is a linear function of the verbose flag vector. This will complete the proof of their equivalence. But first we will produce formal sums of graphs, whose flag vectors provide a basis for formal sums of partitions.

For this we will need some notation. We will let [1 + 1 + 2] denote a partition of 4, and [1 + 1 + 2] + [3] will denote a formal sum of two partitions. If we need to do arithmetic within a partition, we will enclose it by parentheses, like so [1 + (n - 1)]. We will need some graphs. We will use A_n and D_n to denote the corresponding Coxeter-Dynkin diagrams, considered as graphs. Thus, A_n is n vertices in a row, each joined by an edge to the next. The graph D_n has a central
vertex, from which three arms extend, two of length one, and one of length \( n - 3 \). We set \( D_3 \) equal to \( A_3 \).
Throughout this section \( A_n \) and \( D_n \) will denote graphs with only optional edges, considered as formal sums of actual graphs. Their concise flag vectors are as follows.

**Lemma 25** The concise flag vector \( f^c A_n \) is given by

\[
f^c(A_1) = [1], \quad f^c(A_2) = [2], \quad f^c(A_3) = [3], \quad f^c(A_n) = 2^{n-3}[n] \quad \text{for } n \geq 3
\]

while for \( D_n \) we have

\[
f^c(D_3) = [3], \quad f^c(D_4) = 3[4], \quad f^c(D_5) = (2 \times 2 + 3)[5] = 7[5]
\]

when \( n \) is small, and \( f^c(D_n) = (2^{n-2} - 1)[n] \) is the general rule.

**Proof.** The \( A_n \) part of this lemma is straightforward. We simply remove edges one at a time from either end, until only two edges are left. Each additional edge gives another two-fold choice to make. The \( D_n \) part is not difficult either. The first edge to be removed can be one of the two short arms, or can come from the end of the large arm. Removing a short arm from \( D_n \) turns it into \( A_{n-1} \), which we know how to do. Removing an edge from the end of the long arm turns \( D_n \) into \( D_{n-1} \), which we also know, this time by induction. \( \square \)

The following result is now obvious. It can also be seen by comparing the shellings of \( 2A_n \) and of \( D_n \).

**Proposition 26** \( f^c(2A_n - D_n) = [n] \) for \( n \geq 3 \).

Now suppose that \( \pi = [n(1) + \ldots + n(r)] \) is a partition of \( n \). It now follows that the formal expression

\[
(2A_{n(1)} - D_{n(1)}) \sqcup \ldots \sqcup (2A_{n(r)} - D_{n(r)})
\]

considered as a formal sum of graphs on \( n \) vertices has concise flag vector the given partition \( \pi \). This proves:

**Theorem 27** The concise flag vectors of graphs on \( n \) vertices are an integral spanning set for the formal sums of partitions of \( n \).

Recall that \( f^v G \) is a linear function of \( f^c G \). We have just shown that \( f^c G \) spans a space of dimension \( p(n) \), the number of partitions of \( n \). If we can show that \( f^v G \) also spans a \( p(n) \)-dimensional space, it will follow that \( f^c G \) is a linear function of \( f^v G \). It is to this task that we now turn.

We will produce \( p(n) \) graphs, or more exactly formal sums of graphs, whose verbose flag vectors can be seen to be linearly independent. Although not providing an integral basis, the simplest graphs for this purpose are disjoint unions of \( A_1 \), considered as always in this section as graphs with only optional edges.

Consider for example \( G = A_1 \sqcup A_2 \sqcup A_4 \). This is an acyclic graph with 3 components and 4 edges. From this it follows that for the \( w \)-component \( f^v w G \) to be non-zero, \( w \) must be a product of 3 \( a \)'s and 4 \( b \)'s. However, not for every such product will \( f^v w G \) be non-zero. For example, if \( w = aabbbba \) then \( f^v w G \) is zero. This is because of the leading pair of \( a \)'s. It is not possible to begin the shelling of this \( G \) by removing two vertices, without along the way removing an edge. (Acyclic shellings only contribute. If \( G \) were \( A_1 \sqcup A_1 \sqcup A_5 \) it could be done.)
Table 1: The concise flag vectors for graphs on 4 vertices

|        | [1 + 1 + 1 + 1] | [2 + 1 + 1] | [2 + 2] | [3 + 1] | [4] |
|--------|-----------------|-------------|---------|---------|-----|
| 4₀     | 1               | 0           | 0       | 0       | 0   |
| 4₁     | 1               | 1           | 0       | 0       | 0   |
| 2₁ ⊔ 2₁| 1               | 2           | 1       | 0       | 0   |
| 3₂ ⊔ 1₀| 1               | 2           | 0       | 1       | 0   |
| 4₃ = A₄| 1               | 3           | 1       | 2       | 2   |
| 3₃ ⊔ 1₀| 1               | 3           | 0       | 3       | 0   |
| 3₂ ⊔ 1₀ = D₄| 1 | 4 | 1 | 5 | 7 |
| 2₁ ⊔ 2₁ | 1 | 4 | 2 | 4 | 8 |
| 1₁ = 4₅ | 1 | 5 | 2 | 8 | 18 |
| 4₀ = 4₆ | 1 | 6 | 3 | 12 | 36 |

We can now formulate an upper diagonal argument that will prove that the vectors in our candidate basis are indeed linearly independent. Let \( \pi \) be a partition of \( n \), and let \( G_\pi \) be the disjoint union of \( A_i \) corresponding to \( \pi \), thought of as usual as a graph with only optional edges. Now let \( w(\pi) \) be the first word \( w \) in \( a \) and \( b \), under the lexicographic order, for which \( f^w_\pi G_\pi \) is nonzero. The argument of the previous paragraph shows the following lemma, from which the theorem follows easily.

**Lemma 28** If \( \pi = [n(1) + \ldots + n(r)] \) is a partition of \( n \), and the \( n_i \) are in non-decreasing order, then

\[
w(\pi) = b^{n(1) - 1}a \cdot b^{n(2) - 1}a \cdot \ldots \cdot b^{n(r) - 1}a
\]

and so the \( w(\pi) \) are distinct, as \( \pi \) ranges over all partitions of \( n \).

**Theorem 29** The \( p(n) \) vectors \( f^v G_\pi \), as \( \pi \) ranges over the partitions of \( n \), are linearly independent. Thus, the verbose and the concise flag vectors are linear functions of each other.

**Proof.** Let \( \pi \) and \( \pi' \) be two partitions of \( n \). We have shown that the matrix \( f^v_{w(\pi')} G_\pi \) is upper triangular, when partitions are ordered lexicographically via \( w(\pi) \) and \( w(\pi') \). The diagonal entries are clearly non-zero, and so the result follows. \( \square \)

5 Graphs on 4 vertices

It is possible to calculate by hand the flag vectors of all graphs on 4 vertices. With a certain amount of care the 3₀, 3₁, 3₂ and 3₃ notation used for the 3-vertex graphs can be extended to this case.

A graph on 4 vertices can have at most 6 edges. Let 4₀ and 4₁ denote the 4-vertex graphs with 0 and 1 edges respectively. There are two 4-vertex graphs with 2 edges. They can be denoted by \( 2₁ ⊔ 2₁ \) and \( 3₂ ⊔ 1₀ \) respectively. There are three 4-vertex graphs with 3 edges. One of them is \( 3₃ ⊔ 1₀ \), while another is its complement \( \overline{3₃ ⊔ 1₀} \), which is also known as \( D₄ \). The third we will denote by \( 4₃ \), which is also known as \( A₄ \). (Here \( A₄ \) and \( D₄ \) are graphs with ordinary, not optional, edges.) Finally, the graphs with 4, 5 and 6 edges are the complements of graphs with 0, 1 and 2 edges. Thus, we have a compact notation for every graph on four vertices.
Theorem 30  The concise flag vectors of the graphs on 4 vertices are as in Table 1. Each such flag vector is a distinct vertex of the convex hull $\Delta(4)$ of these vectors.

Proof. The first statement is left to the reader. The second is the result of applying the PORTA convex polytope software package to these values. (Alternatively, it could be computed by hand.)

6  Flags

Flags are important in the theory of convex polytopes not only numerically but also geometrically. Let $\delta$ be a flag $(\delta_1 \subset \ldots \subset \delta_r \subset \Delta)$ on a convex polytope $\Delta$. From $\delta$ a flag $(\langle \delta \rangle)$ of vector spaces can readily be constructed. (Each $\langle \delta \rangle_i$ will be the span of the vectors lying on $\delta_i$.) Using linear algebra all manner of vector spaces can be constructed from this flag. The standard formula \[3, 4, 13\] for the middle perversity intersection homology Betti numbers $h_i \Delta$ of $\Delta$ unwinds \[7\] to suggest a construction of a vector space $\Lambda(\delta, i)$ attached to each flag $\delta$, such that the dimension of $\Lambda(\delta, i)$ is precisely the contribution $\delta$ makes to $h_i \Delta$.

That the $h_i \Delta$ are non-negative, at least when $\Delta$ has rational vertices, is a subtle combinatorial inequality. When $\delta'$ is obtained from $\delta$ by removing any one of the $\delta_i$ from $\delta$, the space $\Lambda(\delta', i)$ is contained in $\Lambda(\delta, i)$. This allows the construction of a complex of vector spaces that is, the author conjectures, exact except at just one location. If this is true then the homology at this location will by construction have dimension $h_i \Delta$.

In much the same way, vector spaces can be attached to each flag on a graph. This is an important property, without which the flag vector concept would in some sense be inadequate.

Definition 31  A semi-concise flag $\delta$ of type $w$ on a graph $G$ on $n$ vertices consists of the following. Write $w$ as $a^{d(1)} b \cdot a^{d(2)} \ldots$. To each $d(i)$ associate a $d(i)$-element subset $S_i$ of the vertices, and to each $b$ associate a vertex $v_i$. The sets and vertices are to be disjoint. Call this a $w$-partition of the vertex set $V$. In addition, to each $b$ associate an edge from $v_i$ to some vertex that either lies in $S_j$ or is $v_j$, for some $j > i$. Such a configuration is a semi-concise flag.

Now suppose that $\delta$ is a semi-concise flag on $G$, whose word $w$ ends in $ba$. By permuting the last two vertices another semi-concise flag $\delta'$ can be obtained. But $\delta'$ gives no new combinatorial information, and so $\delta$ and $\delta'$ should be counted together. The same argument applies when $\delta$ is a semi-concise flag ending in $ba d^{(r)}$. In this case, permuting the ends of this last edge produces a $\delta'$, that should be counted along with $\delta$.

Now suppose that $\delta$ is a semi-concise flag whose word ends with $bba$. This means that we are given two edges that join the last three vertices. Now suppose we are given the last three vertices, in no particular order, and two edges linking them. For four out of the six possible permutations of the last three vertices, we will obtain the tail end of a flag whose word ends in $bba$. These four flags should be counted together, because each contains the same information as the others.

Definition 32  A concise flag $\delta$, or flag for short, is the same as a semi-concise flag except that one consolidates information as in the previous two paragraphs.

Now let $E$ be the $n$-dimensional vector space that has the vertices $v_i$ of $G$ as basis vectors. Each concise flag $\delta$ of $G$ will induce subspaces in $E$, whose dimensions and mutual dispositions are determined by the type $w$ of the flag $\delta$. As in convex polytopes, one can now seek to interpret linear functions of flag vectors in terms of linear algebra constructions on the induced subspaces of $E$. However, at present we do not have any idea as to which linear functions of the graph flag vector will play the rôle that the $h_i \Delta$ do for polytope flag vectors.
7 Summary and conclusions

We have defined for each graph \( G \) on \( n \) vertices a flag vector \( fG \), which lies in a space whose dimension is the number \( p(n) \) of partitions of \( n \). This flag vector can be given in verbose, subgraph and concise forms. Each is a linear function of the others. Here is another form for the flag vector.

**Definition 33** Let \( \mathcal{G}_n \) denote the space of all formal sums of graphs on \( n \) vertices, up to isomorphism. Now define the nullspace \( \mathcal{Z} = \mathcal{Z}(\mathcal{G}_n) \) to be all such formal sums \( G \), such that the flag vector \( fG \) is zero. The abstract flag vector \( f^aG \) of a graph \( G \) is the residue of \( G \), as a vector in \( \mathcal{G}_n \), in the quotient space \( \mathcal{G}_n/\mathcal{Z}_n \).

We already know that graphs with an optional cycle will generate elements of nullspace \( \mathcal{Z}_n \). It is not clear whether or not these elements will span the whole of the \( \mathcal{Z}_n \). If not, then finding a pleasant geometric characterisation of \( \mathcal{Z}_n \) is an open problem.

Next we turn to \( \Delta(n) \), or in other words the linear inequalities satisfied by the flag vector. For \( n \leq 4 \) each distinct graph gives a distinct vertex of \( \Delta(n) \), or in other words each graph is extremal for some linear function of the flag vector. On would like to know if the same holds for, say, \( n \leq 10 \). (This statement is much stronger than saying the flag vector distinguishes graphs, although of course it is still considerably weaker than the embedding of graphs in \( \mathcal{G}_n \).)

The theory of flag vectors for graphs can be generalised in two ways. The first is an extension to hypergraphs. A graph is a collection \( E \) of 2-element subsets of the vertex set \( V \). An \( i \)-graph is a collection of \( i \)-element subsets of \( V \). In [\ref{5}], and more concisely in [\ref{8}], a flag vector is defined for such hypergraphs, and various other objects besides.

The second generalisation is this. We have been thinking of a graph as a collection of vertices joined by edges, or in other words, the natural companions of a graph \( G \) on \( n \) vertices are the other graphs on \( n \) vertices. We can however think of a graph as a collection of edges joined at vertices. When this is done, the number of edges is paramount, and shelling consists of removing the edges one at a time.

The edge flag vector of \( G \) (this paper has hitherto studied the vertex flag vector) can be defined in the following way. Each edge has two ends, and at each end there is a multiplicity. When in a shelling an edge is removed, record (a) the fact that the edge is removed, (b) the smaller of the end multiplicities, less one for the edge itself, and (c) the difference between the larger and smaller of the end multiplicities. Use expressions such as \( a+4b+2c \) to record this data. (Perhaps whimsically, \( a \) stands for always, \( b \) for both, and \( c \) for choice.)

Each edge shelling will thus determine a sequence of linear expressions in \( a, b \) and \( c \). Form their product, and define the edge flag vector to be the formal sum of these products, over all edge shellings. This edge flag vector has not yet been investigated.

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