SPECTRA OF SCHREIER GRAPHS OF GRIGORCHUK’S GROUP AND SCHROEDINGER OPERATORS WITH APERIODIC ORDER

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Abstract. We study spectral properties of the Laplacians on Schreier graphs arising from Grigorchuk’s group acting on the boundary of the infinite binary tree. We establish a connection between the underlying dynamical system and a subshift associated to a non-primitive substitution and relate the Laplacians on the Schreier graphs to discrete Schroedinger operators with aperiodic order. We use this relation to prove that the spectrum of the anisotropic Laplacians is a Cantor set of Lebesgue measure zero. We also show absence of eigenvalues and establish equality of the integrated density of states and the von-Neumann-Kesten-Serre trace. Along the way we give a careful study of combinatorial and dynamical features of the subshift associated to the non-primitive substitution. In particular, we show that this subshift is an almost everywhere one-to-one extension of its maximal equicontinuous factor.

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INTRODUCTION

In this article we relate two previously unconnected areas. These are Schreier graphs of self-similar groups and Schrödinger operators associated to aperiodic order. This allows us to solve a problem, open for almost fifteen years, about the spectra of the Laplacians on Schreier graphs of the first group of intermediate growth, for all possible weights attached to the generators.

Schreier graphs are generalizations of Cayley graphs of finitely generated groups. A Schreier graph corresponds to a transitive action of a finitely generated group \(G\) with a finite symmetric generating system \(A\) on a set \(S\). Its vertex set is \(S\) and its edges describe the action of generators and are oriented and labeled accordingly. It can be shown (see [41, 52]) that every regular graph of even degree, finite or infinite, can be oriented and labeled so as to be a Schreier graph.

Each graph gives rise to a number of related operators on the \(l^2\)-space on its vertex set: the adjacency operator, the Laplace operator, the Markov operator. The spectral theory of graphs deals with the spectral theory of these operators. It has a long history with relations and applications to many areas of mathematics, see e.g. the monographs [19, 20, 21]. If the graph is regular the difference between the three operators is irrelevant for spectral considerations. For this reason we refer to the operators of interest in our article, denoted by \(M\), as Laplacians, even though, in a strict sense, they are rather given by the adjacency matrix.

An interesting class of regular graphs emerged in recent years – the Schreier graphs arising from actions of automorphism groups of infinite regular rooted trees [45, 69]. Of particular interest are the so-called self-similar groups whose action reflect the self-similar structure of the tree. Their Schreier graphs also have self-similarity features. Some are closely related to Julia sets [9, 69, 20], others are fractal sets close to e.g. the Sierpinski gasket or the Apollonian gasket [8, 46, 47].

The spectra of the Laplacians on such graphs have been described in some cases [8, 46, 47, 44, 49]. The spectrum can be a union of intervals [44], a Cantor set [8], or a union of a Cantor set together with an infinite set of isolated points that accumulate to it, as in the case of the so-called Hanoi tower group [47].

These investigations all use the method introduced in [8]. Roughly speaking, it consists of including the operator \(M\) in a multi-parameter family of operators such that the simultaneous spectrum of this family becomes invariant under a certain rational mapping. Implementing the method is far from obvious. Even if the corresponding rational mapping \(F\) can be identified,
the method further asks to identify a certain $F$-invariant set which, as computer experiments show, often has the shape of a 'strange attractor' and can be quite complicated (see e.g. [50, 48, 44]).

In [8], the method was applied to the group of intermediate word growth introduced by the first author in [39, 40]. By now it is generally known as Grigorchuk’s group $G$ and this is how we will refer to it. It is generated by four involutions $a, b, c, d$. The group $G$ can be viewed as a group acting by automorphisms on the full infinite binary tree $T$. The action extends by continuity to an action by homeomorphisms on the boundary $\partial T$. The action of $G$ gives rise to Schreier graphs: the finite graphs $\Gamma_n$ arise from the (transitive) action on the $n$-th level of the tree for $n \in \mathbb{N}$; and the infinite graphs $\Gamma_\xi$, $\xi \in \partial T$, arise from the orbits of the action on the boundary. As Schreier graphs, these graphs have edges labeled by $a, b, c, d$ in such a way that each vertex has exactly one edge of each color incident with it. As all generators are involutions, we can disregard the orientation.

Thinking of the labels $a, b, c, d$ as encoding some weights $t, u, v, w \in \mathbb{R}$ on the edges of the graphs, we obtain the Laplacians $M_n(t, u, v, w)$ and $M_\xi(t, u, v, w)$. In the case when $t, u, v, w > 0$ with $t + u + v + w = 1$ these operators are the Markov operators of the random walk on these graphs with transition probabilities $t, u, v, w$.

It is shown in [8] that

- the spectrum of $M_\xi(t, u, v, w)$ does not depend on $\xi \in \partial T$ (for given $(t, u, v, w)$);
- this spectrum coincides with the spectrum of the operator $M_\pi(t, u, v, w)$ arising from the unitary representation $\pi$ induced by the action of $G$ on the $L^2$-space of boundary $\partial T$ (the Koopman representation);
- for $n \to \infty$, the spectra of $M_n(t, u, v, w)$ converge towards the spectrum of $M_\pi(t, u, v, w)$.

These results remain true also more generally, as long as the orbital Schreier graphs $\Gamma_\xi$ are amenable. (The group $G$ being of subexponential growth implies that it is amenable, and hence so are all its Schreier graphs.) However, in the actual description of the spectrum, the 'shape' of the underlying graphs play a key role. It turns out that this shape is rather simple for the group $G$.

![Figure 1. Extract of a Schreier graph of $G$: labeled and unlabeled version.](image-url)
In fact, the graphs have a linear structure with consecutive vertices connected alternately by one edge of label $a$ or two edges none of which carries the label $a$. There is also one additional loop on each vertex, for the remaining generator (see Figure 1).

While the labels $b, c, d$ are mixed in a complicated way, the rule for the label $a$ is quite straightforward. Thus, if one disregards the labeling with $b, c, d$ or, equivalently, puts the same weight $u = v = w$ to the labels $b, c, d$, then the Laplacians have a rather simple 2-periodic structure which makes the spectral theory of these operators accessible. In fact, the case $u = v = w$ is the case of periodic Shrödinger type operators and can be easily be treated by classical means (Floquet decomposition). In [8], these spectra were studied by the method described above and were shown to be one or two intervals or just two points depending on the exact values of $t$ and $u = v = w$.

However, the case of arbitrary weights, corresponding to the general case of an anisotropic random walk, could not be solved by the method of [8] and remained open. As indicated above, the basic issue is the intricate rule of labeling of the double edges (equivalently, of loops) in the infinite Schreier graphs.

In the present article we solve the spectral problem of the operators $M_\xi(t, u, v, w)$ for arbitrary values of the parameters by a new and completely different approach. It relies on the construction of a certain subshift $(\Omega_\tau, T)$ associated with a non-primitive substitution $\tau$ acting on the set of words over the alphabet $\{a, x, y, z\}$ via

$$a \mapsto axa, x \mapsto y, y \mapsto z, z \mapsto x,$$

see Section 4 for more details. This substitution has already appeared in the context of the group $G$ (up to renaming some letters). In fact, Lysenok [67] showed that $G$ has the following presentation by generators and relations

$$G = \langle a, b, c, d \mid 1 = a^2 = b^2 = c^2 = d^2 = \kappa^k((ad)^4) = \kappa^k((adacac)^4), k = 0, 1, 2, \ldots \rangle,$$

where $\kappa$ is the substitution on $\{a, b, c, d\}$ obtained from $\tau$ by replacing $x$ by $c$, $y$ by $b$ and $z$ by $d$.

We show that $G$ acts naturally on $\Omega_\tau$. One of the main results of the paper (compare to Theorem 6.5) then establishes a connection between this action and the natural action of the group $G$ on the space of its Schreier graphs that we denote by $X$, see Section 2.2 for a precise definition and a discussion. In particular, we prove

Result. (Compare Theorem 6.5) There is an action of $G$ on $\Omega_\tau$ such that

- the system $(X, G)$ is a factor (i.e. an equivariant surjection) of the system $(\Omega_\tau, G)$;
- the orbits of the $G$-action and of the shift $T$ on $\Omega_\tau$ coincide.

This connection allows us to transfer the problem of spectral theory of the operators $M_\xi(t, u, v, w)$, $\xi \in \partial T$, to the field of the spectral theory of discrete Shrödinger operators with aperiodic order, $(H_\omega)_{\omega \in \Omega_\tau}$. We hope that this connection will be also useful in other contexts. Our construction has points of contact with the work of Vorobets [78, 79] and can, in fact, be seen as linking [78] with [79].

Aperiodic order denotes an intermediate regime of long range order between periodicity and randomness. In one dimension it is commonly modeled by subshifts of low complexity. In higher dimensions it is modeled by dynamical systems consisting of point sets with suitable regularity features (Delone dynamical systems). It has received a lot of attention over the last thirty years or so, see e.g. the article collections and monographs [3, 6, 54, 68, 72]. This
interest is due to the many remarkable and previously unknown features and phenomena arising from aperiodic order in various branches of mathematics. It is also due to the physical and chemical relevance of aperiodic order which provides a mathematical foundation of quasicrystals. Typical mathematical models include Penrose tiling or Sturmian subshifts, see [3] for detailed discussion.

Schroedinger operators occupy a prominent position in the theory of aperiodic order. Indeed, they arise in the quantum mechanical description of conductance properties of quasicrystals and exhibit quite interesting mathematical properties. In fact, already the first two papers on them written by physicists suggest that the corresponding spectral measures are purely singular continuous and the spectrum is a Cantor set of Lebesgue measure zero [55, 70]. By now these features as well as other conductance-related properties known as anomalous transport have been thoroughly studied in a variety of models by various authors, see the survey articles [16, 22, 23]. The phenomenon that the underlying spectrum is a Cantor set of Lebesgue measure zero is usually referred to as Cantor spectrum of Lebesgue measure zero and this is how we will refer to it subsequently.

Starting with [18, 75], Cantor spectrum of Lebesgue measure zero was established for various systems using so-called trace map dynamical systems. Later, a different method was devised by the second author. In particular, the article [59] shows that Cantor spectrum of Lebesgue measure zero holds whenever the underlying subshift is aperiodic and linearly repetitive (see Section 1.2 for the definition of linear repetitivity). It deals with discrete Schroedinger operators (whose first off diagonal entries are constantly equal to one). However, the corresponding considerations of [59, 27] can be extended [11] to Jacobi operators (whose off-diagonal entries are given by non-vanishing continuous functions on the underlying subshift).

In the present article we are led to consider Schroedinger operators, or rather Jacobi operators, associated to the subshift \((\Omega, T)\) of the substitution \(\tau\) defined above. The subshift \((\Omega, T)\) is studied in Section 4. As \(\tau\) is non-primitive, standard methods are not applicable. However, using results of [28] we are able to establish linear repetitivity of the subshift, and this is a key element in our proof of Cantor spectrum of Lebesgue measure zero, by the results discussed above.

We then go further with a detailed study of the subshift and provide a rather complete account of its basic combinatorial and dynamical features. To the best of our knowledge, this is the first instance of a detailed treatment of the subshift associated with a non-primitive substitution (see, however, [13, 14] for recent works concerning general and in particular non-minimal substitutions). We show in particular that \((\Omega, T)\) is almost automorphic. In fact, the map to its maximal equicontinuous factor is one-to-one in all points except the orbits of three specific elements (Theorem 4.15). For the one-sided subshift the corresponding result has already been obtained by Vorobets [78]. Our approach is different. It relies on finding suitable partitions of the sequences in the subshift. This is close in spirit to the partition based approach to Sturmian dynamical systems which was developed in [21] and then used in various subsequent works dealing with the spectral theory (see the survey [22]) and with some combinatorial questions [25, 26].

Having detailed information on \((\Omega, T)\) at our disposal, we can then use the general theory of Schroedinger operators (see Section 3) to study in detail the Schroedinger operators \((H_\omega)_{\omega \in \Omega}\) associated to the subshift. Our main results in this context are Theorem 5.3 and Theorem 5.5. Theorem 5.3 gives Cantor spectrum of Lebesgue measure zero for the operators in question. Theorem 5.5 asserts absence of eigenvalues resulting in purely singular continuous
spectrum for most values of \( \omega \in \Omega \) as well as for some specific values. These results provide a case study of Schroedinger operators with aperiodic order arising from a non-primitive substitution. This, again, is worth mentioning as - with the notable exception of [33, 32, 28] - all corresponding works dealing with Schroedinger operators so far assumed primitivity of the substitution.

When translated into the language of Schreier graphs via the connection established in Theorem 6.5 mentioned above, Theorem 5.3 gives the main result of the paper. Recall that by [8] the spectrum of \( M_{\xi}(t,u,v,w) \) is independent of \( \xi \in \partial T \).

**Result.** (Compare Theorem 7.3)

Let \( t,u,v,w \in \mathbb{R} \) be such that \( t \neq 0, u + v \neq 0, u + w \neq 0, v + w \neq 0 \). If \( u = v = w \) does not hold then the spectrum of \( M_{\xi}(t,u,v,w) \) is a Cantor set of Lebesgue measure zero.

This result completes the study initiated in [8], and provides the full description of the spectrum for the operators \( M_{\xi}(t,u,v,w), \xi \in \partial T \). Moreover, we also show absence of eigenvalues for the operators \( M_{\xi}(t,u,v,w) \) for almost all \( \xi \in \partial T \) with respect to the uniform measure on \( \partial T \) (Theorem 7.5) as well as for some specific most relevant \( \xi \).

As a by-product of our investigation we can also show that the Kesten-von-Neumann-Serre spectral measure, introduced in [8] (the name was given in [51]) whose existence was only known in the case \( u = v = w \) exists in the general case and agrees with the integrated density of states (Theorem 8.1). This also allows us to reformulate the statements on convergence of the spectra of the \( M_n(t,u,v,w) \) for \( n \to \infty \) in terms of the integrated density of states.

The article is organized as follows: In Section 1 we introduce some standard concepts from the theory of words, subshifts and dynamical systems. We also specify a topology on the set of all (isomorphism classes of) rooted labelled graphs, and in particular on the set of all (finite, one-sided infinite and two-sided infinite) words over a finite alphabet, that we will be using throughout the paper. In Section 2 we introduce Grigorchuk’s group \( G \) and its Schreier graphs and in Section 2.3 we present the associated operators \( M_{\xi}(t,v,u,w) \). As explained above, it is the spectral theory of these operators that we study in this article. In order to do so we will need the - at first seemingly completely unrelated - theory of Schroedinger operators associated to low complexity subshifts. The corresponding general theory is presented in Section 3. The specific subshift \((\Omega, T)\) we are interested in is studied in detail in Section 4. The results of these two sections are combined in Section 5 to provide informations about the spectra of the operators associated to \((\Omega, T)\). The connection between the operators associated to the Schreier graphs of \( G \) and the Schroedinger operators associated to the subshift \((\Omega, T)\) is established in Section 6. It allows us to complete the proofs of our main results in Section 7. The equality of the integrated density of states and the Kesten-von-Neumann-Serre spectral measure is shown in Section 8. The paper concludes with an outlook pointing towards some further questions and directions, in Section 9.

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1. General background: Dynamical systems, words and graphs

In this section we shortly discuss some notions and concepts from symbolic dynamics. In particular, we discuss a topology on the set of all (finite and infinite) words over a finite alphabet. The material of this section will be used throughout the paper.

1.1. Dynamical systems. A topological dynamical system consists of a compact space $Y$ and an action $\alpha$ of a group $G$ on $Y$. We will write this as $(Y, G, \alpha)$. We will often suppress the $\alpha$ in the notation and just write $(Y, G)$ instead of $(Y, G, \alpha)$. We will then also write $gy$ instead of $\alpha_g(y)$ for $y \in Y$ and $g \in G$. In the case when $G$ is the infinite cyclic group (i.e. $G \simeq \mathbb{Z}$) the action is completely determined by $T := \alpha(1)$ and we write $(Y, T)$ instead of $(Y, \mathbb{Z})$.

A topological dynamical system $(Y, G)$ is called minimal if the orbit $G \cdot y := \{\alpha(g)(y) : g \in G\}$ of $y$ is dense in $Y$ for every $y \in Y$. The system $(Y, G)$ is called uniquely ergodic if there exists exactly one $G$-invariant probability measure on $Y$. By Bogolubov’s theorem $(Y, G)$ has at least one invariant probability measure if $G$ is amenable, i.e. if $G$ admits a left invariant mean. This applies in particular if $G$ has subexponential growth, for example, when $G$ is $\mathbb{Z}$.

A dynamical system $(Y', G)$ is said to be a factor of the dynamical system $(Y, G)$ if there exists a continuous surjective map $\pi : Y \to Y'$ with $\pi(gy) = g\pi(y)$ for all $g \in G$ and $y \in Y$.

1.2. Subshifts. We will be interested in special dynamical systems encoded by subshifts over a finite alphabet $\mathcal{A}$. We will consider the set $\mathcal{A}^*$ of finite words (including the empty word) over the alphabet $\mathcal{A}$, as well as the set $\mathcal{A}^\mathbb{Z}$ of bi-infinite words over the alphabet $\mathcal{A}$ and the set $\mathcal{A}^\mathbb{N}$ of one-sided infinite words over $\mathcal{A}$.

If $v, w$ are finite words and $\omega \in \mathcal{A}^\mathbb{Z}$ satisfies
\[
\omega_1 \ldots \omega_{|v|} = v \quad \text{and} \quad \omega_{-|w|+1} \ldots \omega_0 = w
\]
we write
\[
\omega = \ldots w|v|
\]
and say that $|$ denotes the position of the origin.

We equip $\mathcal{A}$ with discrete topology and $\mathcal{A}^\mathbb{Z}$ with product topology. By Tychonoff theorem, $\mathcal{A}^\mathbb{Z}$ is then compact. In fact, it is homeomorphic to the Cantor set. A pair $(\Omega, T)$ is then called a subshift over $\mathcal{A}$ if $\Omega$ is a closed subset of $\mathcal{A}^\mathbb{Z}$ which is invariant under the shift transformation
\[
T : \mathcal{A}^\mathbb{Z} \to \mathcal{A}^\mathbb{Z}, \quad (Tu)(n) = u(n+1).
\]

If there exists a natural number $N$ with $T^N \omega = \omega$ for all $\omega \in \Omega$ then $(\Omega, T)$ is called periodic otherwise it is called aperiodic.

Whenever $\omega$ is a word over $\mathcal{A}$ (finite or infinite, indexed by $\mathbb{N}$ or by $\mathbb{Z}$) we define
\[
Sub(\omega) := \text{Finite subwords of } \omega.
\]

By convention, the set of finite subwords includes the empty word. Any subshift $(\Omega, T)$ comes naturally with the set $\mathcal{W}(\Omega)$ of associated finite words given by
\[
\mathcal{W}(\Omega) = \bigcup_{\omega \in \Omega} Sub(\omega).
\]
A word \( v \in W(\Omega) \) is said to occur with bounded gaps if there exists an \( L_v > 0 \) such that every \( w \in W(\Omega) \) with \( |w| \geq L_v \) contains a copy of \( v \). As is well known (and not hard to see) \((\Omega,T)\) is minimal if and only if every \( v \in W(\Omega) \) occurs with bounded gaps. For proofs and further discussion we refer to standard textbooks such as [65, 80]. We will be concerned here with the following strengthening of the bounded gaps condition.

**Definition 1.1** (Linearly repetitive). A subshift \((\Omega,T)\) is called linearly repetitive \((LR)\), if there exists a constant \( C > 0 \) such that any word \( v \in W(\Omega) \) occurs in any word \( w \in W(\Omega) \) of length at least \( C|v| \).

**Remark 1.2.** This notion has been discussed under various names by various people. In particular it was studied by Durand, Host and Skau [36] in the setting of subshifts (under the name 'linearly recurrent'). For Delone dynamical systems it was brought forward at about the same time by Lagarias and Pleasants [57] under the name 'linearly repetitive'. It has also featured in the work of Boris Solomyak [74] (under the name 'uniformly repetitive'). It was already discussed in an unpublished work of Boshernitzan in the 90s. That work also contains a characterization in terms of positivity of weights. A corresponding result for Delone systems was recently given in [12].

Durand [35] gives a characterization of such subshifts in terms of primitive \( S \)-adic systems and shows the following (which was already known to Boshernitzan).

**Theorem 1.3.** Let \((\Omega,T)\) be a linearly repetitive subshift. Then, the subshift is uniquely ergodic.

**Remark 1.4.** In fact, linear repetitivity implies a strong form of subadditive ergodic theorem [61]. Validity of such a result together with the fundamental work of Kotani [56] is at the heart of the approach to Cantor spectrum of Lebesgue measure zero developed in [59]. Our considerations below rely on an extension of that approach worked out in [11].

1.3. **The space of rooted labeled graphs.** Here we recall some terminology from the theory of graphs and introduce the topological space of (isomorphism classes of) rooted labeled graphs that will play an important role in the paper.

Let \( B \) be a finite non-empty set. A graph with edges labeled by \( B \) is a pair \((V,E)\) consisting of a set \( V \) and a set \( E \subset V \times V \times B \) that is symmetric in the sense that \((v,w,b)\) belongs to \( E \) if and only if \((w,v,b)\) belongs to \( E \) (for \( v,w \in V \) and \( b \in B \)). An edge of the form \((v,v,b)\) is called a loop at \( v \) (with label \( b \)).

If \( B \) has only one element we omit it from notation and speak about the associated graphs as non-labeled.

We will need the combinatorial distance on a graph given as follows. Each vertex has distance 0 to itself. The distance between different vertices \( v \) and \( w \) is one if and only if there exists a label \( b \) such that \((v,w,b)\) belongs to \( E \). More generally the distance between different vertices \( v \) and \( w \) is then defined inductively as the smallest natural number \( n \) such that there exists a vertex \( v' \) with distance \( n-1 \) to \( v \) and distance one to \( w \). If no such \( n \) exists the combinatorial distance is defined to be \( \infty \). The graph is called connected if the distance between any two if its vertices is finite. Likewise the connected component of a vertex is the set of all vertices with finite distance to it.
A ray in an infinite graph is an infinite sequence \( v_0, v_1, \ldots \) of pairwise different vertices with distance one between consecutive vertices. Two rays are equivalent if there exists a third ray containing infinitely many vertices of each of the rays. An equivalence class of rays is called an end of the graph.

A rooted graph is a pair consisting of a graph and a vertex belonging to the vertex set of the graph. This vertex is then called the root.

Two rooted graphs \((G_1, v_1)\) and \((G_2, v_2)\) labeled by the same set \(\mathcal{B}\) are called isomorphic if there exists a bijective map \(\beta\) from the vertices of \(G_1\) to the vertices of \(G_2\) taking \(v_1\) to \(v_2\) such that the vertices \(x\) and \(y\) in \(G_1\) are connected by an edge of color \(b\) if and only if their images in \(V(G_2)\) are connected by an edge of color \(b\). In this case we write \((G_1, v_1) \cong (G_2, v_2)\).

Let us now consider the set \(\mathcal{G}_* = \mathcal{G}_*(\mathcal{B})\) of isomorphism classes of connected rooted graphs labeled with elements from \(\mathcal{B}\) that we endow with the following natural metric. The distance between two rooted graphs \((Y_1, v_1)\) and \((Y_2, v_2)\) is then defined as

\[
\text{dist}((Y_1, v_1), (Y_2, v_2)) := \inf \left\{ \frac{1}{r+1} : B_1(v_1, r) \cong B_2(Y_1, Y_2) \right\}
\]

where \(B_Y(v, r)\) is the (labeled) ball of radius \(r\) centered in \(v\) in the combinatorial metric on \(Y\). If we only consider graphs of uniformly bounded degree (as we will in this paper), the space \(\mathcal{G}_*\) is compact.

1.4. The set of all words as a topological space. Let \(\mathcal{A}\) be a finite set. The three sets \(\mathcal{A}^*, \mathcal{A}^\mathbb{N}\) and \(\mathcal{A}^\mathbb{Z}\) can be viewed as linear graphs (respectively segments, rays and lines) labeled by \(\mathcal{A}\), and therefore the topology defined in the previous subsection can be considered on the set of all (finite and infinite) words. However, it will be convenient to us to describe the topology on the set of words explicitly.

To do so we will consider a bigger set containing all the three sets above. Elements of this bigger set will be called words associated to \(\mathcal{A}\). With the topology at hand the set of these words will be compact and even a dynamical system in a natural way.

The basic idea is to extend elements of \(\mathcal{A}^*\) and \(\mathcal{A}^\mathbb{N}\) to functions on \(\mathbb{Z}\) which take an additional value \(*\) at those places where they are not originally defined. More precisely, choose an element \(*\) which does not belong to \(\mathcal{A}\) and consider the new alphabet \(\mathcal{A} \cup \{\ast\}\) and equip it with the discrete topology. Then, \((\mathcal{A} \cup \{\ast\})^\mathbb{Z}\) is a compact set in the product topology. We will consider suitable elements of \(\omega \in (\mathcal{A} \cup \{\ast\})^\mathbb{Z}\). For any \(\omega : \mathbb{Z} \rightarrow \mathcal{A} \cup \{\ast\}\) we define its support, \(\text{supp}(\omega)\), via

\[
\text{supp}(\omega) := \omega^{-1}(\mathcal{A}).
\]

We then call the elements of

\[
\mathcal{W}(\mathcal{A}) := \{ \omega \in (\mathcal{A} \cup \{\ast\})^\mathbb{Z} : \text{supp}(\omega) \text{ is an interval} \}
\]

the words associated to \(\mathcal{A}\). Here, a subset \(I\) of \(\mathbb{Z}\) is called an interval if with \(a, b \in I\) also all \(c \in \mathbb{Z}\) with \(a \leq c \leq b\) belong to \(I\).

Clearly \(\mathcal{W}(\mathcal{A})\) is a closed and \(T\)-invariant subset of \((\mathcal{A} \cup \{\ast\})^\mathbb{Z}\). Hence, \(\mathcal{W}(\mathcal{A})\) is compact and \((\mathcal{W}(\mathcal{A}), T)\) is a subshift (over the alphabet \(\mathcal{A} \cup \{\ast\}\)).

The elements of \(\mathcal{A}^*\) and \(\mathcal{A}^\mathbb{N}\) can be canonically identified with elements of \(\mathcal{W}(\mathcal{A})\) by extension by \(*\). More specifically, we will identify \(w \in \mathcal{A}^*\) with the function \(\omega_w : \mathbb{Z} \rightarrow (\mathcal{A} \cup \{\ast\})\) defined by \(\omega(w)(n) = w_n\) for \(n \in \{1, \ldots, |w|\}\) and \(\omega(w)(n) = \ast\) otherwise. Similarly, we will identify
\( \xi \in A^\mathbb{N} \) with the function \( \omega_\xi : \mathbb{Z} \rightarrow (A \cup \{\ast\}) \) defined by \( \omega_\xi(n) = \xi(n) \) for \( n \in \mathbb{N} \) and \( \omega_\xi(n) = \ast \) otherwise. These identifications will be tacitly assumed in the sequel.

2. Grigorchuk’s group \( G \), its Schreier graphs and the associated Laplacians

In this section we introduce the main object of our interest: Grigorchuk’s group \( G \) and the associated Laplacians.

2.1. Grigorchuk’s group \( G \). Let us denote by \( T_q, q \in \mathbb{N} \) with \( q \geq 2 \), the rooted regular tree of degree \( q \). The vertex set of \( T_q \) is given by \( \{0, \ldots, q-1\}^* \), i.e. the set of all words over the alphabet \( \{0, \ldots, q-1\} \). The root of \( T_q \) is the empty word. There is an edge between \( v \) and \( w \) whenever \( w = vk \) or \( v = wk \) holds for some \( k \in \{0, \ldots q-1\} \). The words \( w \in \{0, \ldots, q-1\}^n \) constitute the \( n \)-the level of the tree. (In the tree, they are at combinatorial distance exactly \( n \) from the root.)

The set \( \{0,1,\ldots,q-1\}^\mathbb{N} \) of one-sided infinite words can be identified with the boundary \( \partial T_q \) of \( T_q \) consisting of infinite geodesic rays in \( T_q \) emanating from the root (i.e. infinite paths starting in the root all of whose edges are pairwise different). As mentioned above, the set \( \{0,1,\ldots,q-1\}^\mathbb{N} \) is equipped with the product topology and is thus a compact space homeomorphic to the Cantor set.

Any automorphism of \( T_q \) necessarily preserves the root (which is the only vertex with degree \( q \)) and maps paths starting in the root to paths starting in the root. This readily implies that any automorphisms group action on \( T_q \) is level preserving, i.e. maps words of length \( n \) to words of length \( n \). Any such action then extends to an action of the same group by homeomorphisms on the boundary \( \partial T_q \).

A regular rooted tree is a self-similar object. Indeed, the subtree rooted at an arbitrary vertex of the tree is isomorphic to the whole tree \( T_q \). The full group of automorphisms inherits this self-similarity property in the following sense: any automorphism of \( T_q \) is completely determined by the permutation it induces on the \( q \) branches growing from the root (an element of \( Sym(q) \)) and the collection of \( q \) automorphisms \( (g_0, \ldots, g_{q-1}) \) which coincide with the restrictions of \( g \) on the corresponding branches.

However, if one is interested in a subgroup \( H < \text{Aut}(T_q) \) and wants it to be self-similar, one has to impose the condition that all the restrictions \( (g_0, \ldots, g_{q-1}) \) are again elements of the same group \( H \), so that every \( g \in H \) can be represented as

\[
g = \alpha(g_0, \ldots, g_{q-1}),
\]

where \( \alpha \) belongs to \( Sym(q) \) and describes the action of \( g \) on the first level of \( T_q \) and \( g_i \in G, i = 0, \ldots, q-1 \) is the restriction of \( g \) on the full subtree of \( T_q \) rooted at the vertex \( i \) of the first level of \( T_q \). This leads to the following definition.

**Definition 2.1.** A group \( H \) of automorphisms of \( T_q \) is **self-similar** if, for all \( g \in H, x \in \{0, \ldots, q-1\} \), there exist \( h \in H, y \in \{0, \ldots, q-1\} \) such that

\[
g(xw) = yh(w),
\]

for all finite words \( w \) over the alphabet \( \{0, \ldots, q-1\} \).

We refer the interested reader to [69, 45] for more information about self-similar groups.
We now turn our attention to one particular example of a self-similar group that will be the central object of our study, the Grigorchuk group $G$. It is generated by four automorphisms $a, b, c, d$ of the rooted binary tree $T = T_2$ as follows:

\[
\begin{align*}
a(0w) &= 1w, \quad a(1w) = 0w; \\
b(0w) &= 0a(w), \quad b(1w) = 1c(w); \\
c(0w) &= 0a(w), \quad c(1w) = 1d(w); \\
d(0w) &= 0w, \quad d(1w) = 1b(w),
\end{align*}
\]

for an arbitrary word $w$ over $\{0, 1\}$. These automorphisms can also be expressed in the self-similar form, as above:

\[
a = \epsilon(id, id), \quad b = \epsilon(a, c), \quad c = \epsilon(a, d), \quad d = \epsilon(id, b),
\]

where $\epsilon$ and $\epsilon$ are, respectively, the trivial and the non-trivial permutations in the group $\text{Sym}(2)$.

Observe that all the generators are involutions and that $\{1, b, c, d\}$ commute and constitute a group isomorphic to the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Let us also mention that there are many more relations and the group is not finitely presented.

For our subsequent discussion it will be important that $G$ acts transitively on each level, i.e. for arbitrary words $w, u$ over $\{0, 1\}$ with the same length there exists a $g \in G$ with $gw = w$.

2.2. The Schreier graphs of $G$ and the dynamical system $(X, G)$. The action of $G$ on the set $V(T) = \{0, 1\}^*$ of vertices of the rooted binary tree and on its boundary $\partial T = \{0, 1\}^\mathbb{N}$ induces on this set the structure of Schreier graphs, with respect to the generating set $\{a, b, c, d\} \subset G$, as defined in the Introduction (see Figure 2). It consists of connected components corresponding to the orbits of the action. The ones that correspond to the levels of the tree are finite (recall that the action of $G$ on the levels of the tree is transitive), and the ones that correspond to the orbits of the action on the boundary and infinite and will referred to as orbital Schreier graphs. Specifically, the vertex set of this graph is given by $\{0, 1\}^* \cup \{0, 1\}^\mathbb{N}$ and there is an edge with label $s \in \{a, b, c, d\}$ and origin $v$ and terminal vertex $w$ if and only if $sv = w$ holds. Note that the set of arising edges has indeed the required symmetry property as any $s \in \{a, b, c, d\}$ is an involution. For the first three levels of the tree the resulting graphs are shown in Figure 2.

We will view the Schreier graphs as rooted and introduce therefore the map

\[
\mathcal{F} : V(T) \cup \partial T \longrightarrow \mathcal{G}_s(\{a, b, c, d\}), \quad \mathcal{F}(v) := [(\Gamma_v, v)],
\]

where $\Gamma_v$ is the connected component of $v$. As is well-known (see e.g. Section 2.2 of [30]) finite Schreier graphs converge to infinite orbital Schreier graphs, as follows.

**Lemma 2.2.** For any $\xi \in \{0, 1\}^\mathbb{N}$, the sequence $(\mathcal{F}(\xi_1 \ldots \xi_n))$ converges to $\mathcal{F}(\xi)$ in the space of isomorphism classes of rooted graphs labeled with $\{a, b, c, d\}$ (equipped with the topology discussed in Section 1.3).

The graphs $\Gamma_w$ and $\Gamma_v$ coincide (as non-rooted graphs) whenever $v$ and $w$ are in the same orbit of the action of $G$. In particular, as $G$ acts transitively on each level of the tree, For $n \in \mathbb{N}$ we can therefore define

\[
\Gamma_n := \Gamma_1^n
\]

which coincides with $\Gamma_w$ for all $w \in V(T)$ with $|w| = n$. In general, the Schreier graph $\Gamma_n$ has a linear shape; it has $2^{n-1}$ simple edges, all labeled by $a$, and $2^{n-1} - 1$ cycles of length 2.
whose edges are labeled by $b, c, d$. It is regular of degree 4, with one loop at each edge. The loop contributes 1 to the degree of the vertex because all generators are elements of order 2,
and the labeling of the loop is uniquely determined by the labeling of the other edges around
the vertex, as edges around one vertex are labeled by \{a, b, c, d\}.

The orbital Schreier graphs corresponding to the action on the boundary are infinite and
have either two ends or one end. The graph \( \Gamma_1^\infty \) corresponding to the orbit of the rightmost
infinite ray, is one-ended (see Figure 3), (and so, by the remark above are all graphs in the
same orbit).

All the other orbital Schreier graphs \( \Gamma_\xi, \xi \notin G \cdot 1^\infty \), are two-ended. They are all isomorphic
as unlabeled graphs.

In [?9], Vorobets studied the closure \( F(\partial T) \) of the space of Schreier graphs in the space of
isomorphism classes of rooted labeled graphs. We recall some of his results next. He showed
that the one-ended graphs are exactly the isolated points of this closure \( F(\partial T) \), and that the
other points in \( F(\partial T) \) are two ended graphs. This suggests to consider the compact set
\[ X := F(\partial T) \setminus \{\text{isolated points}\}. \]

Then, the group \( G \) acts on \( X \) by changing the root of the graph and this action is minimal
and uniquely ergodic with invariant probability measure \( \nu \).

A precise description of \( X \) can be given as follows. The space \( X \) consists of all isomorphism
classes of two-ended rooted Schreier graphs \( \{(\Gamma_\xi) : \xi \in \partial T \setminus G \cdot 1^\infty \} \) and of three additional
countable families of isomorphism classes of two-ended graphs. These families are obtained
by gluing two copies of the one-ended graph \( \Gamma_\xi, \xi \in G \cdot 1^\infty \), at the root in three possible
ways corresponding to choosing a pair \((b, c), (b, d)\) or \((c, d)\) and then choosing an arbitrary
vertex of the arising graph as the root. One of these three possibilities is shown in Figure 4.
There, the chosen pair is \((c, d)\) and to avoid confusion with other edges with the same labels,
the labels at the gluing point are denoted with a prime. These three new graphs are again
Schreier graphs of \( G \).

The decomposition of \( X \) into isomorphism classes of the \((\Gamma_\xi)\) and the three families
mentioned above gives immediately rise to a factor map
\[ \phi : X \to \partial T, \]
which is one-to-one except in a countable set of points, where it is three-to-one. In fact, the 
inverse map \( \phi^{-1} \) exists on the complement \( \partial T \setminus G \cdot 1^\infty \) of the orbit of the point \( 1^\infty \) and agrees 
there with \( F \).

Under this factor map \( \phi \) the unique \( G \)-invariant probability measure \( \nu \) on \( X \) is mapped to the 
\( \{1/2,1/2\} \) uniform Bernoulli measure \( \mu \) on \( \partial T \).

2.3. Laplacians associated to the Schreier graphs of \( G \). Whenever a group \( H \) acts 
on a measure space \( (Y,m) \) by measure preserving transformations, one obtains the so-called 
Koopman representation \( \hat{\rho} \) of \( G \) on \( L^2(Y,m) \) via 
\[
\hat{\rho}(g) : L^2(Y,m) \rightarrow L^2(Y,m), (\hat{\rho}(g)f)(y) = f(g^{-1}y),
\]
(for \( g \in G \)). Any \( \hat{\rho}(g) \) is then a unitary operator (as the action is measure preserving).

In our situation when \( H = G \), we have moreover that for \( s \in \{a,b,c,d\} \) the unitary operator 
\( \hat{\rho}(s) \) is its own inverse (as \( s \) is an involution) and hence must be selfadjoint. In particular, for 
any set of parameters \( t,u,v,w \in \mathbb{R} \) we obtain a selfadjoint operator 
\[
\hat{M}_\rho(t,u,v,w) := t\hat{\rho}(a) + u\hat{\rho}(b) + v\hat{\rho}(c) + w\hat{\rho}(d).
\]

Consider first an arbitrary \( \xi \in \partial T \). Then, there is an action of \( G \) on the (countable) vertex 
set of \( V(I_\xi) \) of \( I_\xi \). Specifically, \( s \in \{a,b,c,d\} \) maps the vertex \( x \in V(I_\xi) \) to the vertex 
\( sx \), which is the unique vertex of \( V(I_\xi) \) connected to \( v \) by an edge of label \( s \). Clearly, this 
action preserves the counting measure on \( V(I_\xi) \). Thus, we obtain a representation \( \hat{\rho}_\xi \) of \( G \) 
on \( \ell^2(V(I_\xi)) \).

**Definition 2.3** (Laplacian of the Schreier graph). An operator \( M_\xi(t,u,v,w) \) defined by 
\[
M_\xi(t,u,v,w) := \hat{M}_\rho(t,u,v,w) = t\hat{\rho}_\xi(a) + u\hat{\rho}_\xi(b) + v\hat{\rho}_\xi(c) + w\hat{\rho}_\xi(d)
\]
with \( \xi \in \partial T \) and \( t,u,v,w \in \mathbb{R} \) will be called (weighted) Laplacian of the 
Schreier graph \( I_\xi \).

**Remark 2.4.**
- It is possible to understand \( V(I_\xi) \) as \( G/G_\xi \), where \( G_\xi \) is the stabilizer 
of \( \xi \) in the action of \( G \) on \( \partial T \), and then \( \hat{\rho}_\xi \) is the quasi-regular representation \( \hat{\rho}_{G/G_\xi} \) 
associated to \( G/G_\xi \).
- If \( t,u,v,w \) are positive with \( 1 = t + u + v + w \) then it is possible to interpret the 
operators \( M_\xi \) as the Markov operators of a random walk on the graph \( I_\xi \). In the 
general case, the operator \( M_\xi(t,u,v,w) \) can still be seen as the natural weighted 
‘adjacency matrix’ or ‘Laplacian’ associated to the graph \( I_\xi \).

We can also equip \( \partial T = \{0,1\}^\mathbb{N} \) with the uniform \( \{1/2,1/2\} \) Bernoulli measure \( \mu \) and 
consider the Koopman representation \( \pi \) of \( G \) on \( L^2(\partial T,\mu) \) given via 
\[
\pi(g) : L^2(\partial T,\mu) \rightarrow L^2(\partial T,\mu), \pi(g)f(x) = f(g^{-1}x).
\]
This is a unitary representation of \( G \) and any \( \pi(s), s \in \{a,b,c,c\} \), is a unitary selfadjoint 
involution. For \( t,u,v,w \in \mathbb{R} \) we then obtain the operator \( M_\pi(t,u,v,w) \) via 
\[
M_\pi(t,u,v,w) = t\pi(a) + u\pi(b) + v\pi(c) + w\pi(d).
\]

The following is a crucial result on the spectral theory of the above operators.

**Theorem 2.5** (Independence of spectrum (Bartholdi / Grigorchuk [8])). For any given set 
of parameters \( t,u,v,w \in \mathbb{R} \) the spectrum of \( M_\xi(t,u,v,w) \) does not depend on \( \xi \in \partial T \) and 
coincides with the spectrum of \( M_\pi(t,u,v,w) \).
Of course, the theorem immediately raises the question what the spectrum is in terms of the parameters \( t, u, v, w \). In this context, a complete answer was given in [8] in the case \( u = v = w \). The spectrum then consists of two points or one or two intervals, and an explicit description of the spectrum can be given in terms of the parameter \( u = v = w \). In fact, the case \( u = v = w \) is the case of periodic Schrödinger type operators and can easily be treated by classical means (Floquet decomposition). It can also be treated by the method suggested in [8]. In the present paper we will be interested in the case where \( u = v = w \) does not hold.

We will show that in this case, the spectrum is a Cantor set of Lebesgue measure zero.

There is one more representation studied in the context of spectral approximation. This representation comes from the action of \( G \) on the \( n \)-th level of the tree. This action clearly preserves the counting measure on the finite set of vertices of the \( n \)-th level. It hence gives rise to a representation \( \varrho_n \) of \( G \) on the finite dimensional \( \ell^2(V(\Gamma_n)) \). For \( t, u, v, w \in \mathbb{R} \) we then set

\[
M_n(t, u, v, w) := M_{\varrho_n}(t, u, v, w) = t\varrho_n(a) + u\varrho_n(b) + v\varrho_n(c) + w\varrho_n(d).
\]

To each such \( M_n(t, u, v, w) \) we associate the spectral distribution which is the measure \( \mu_n(t, u, v, w) \) on \( \mathbb{R} \) given by

\[
\mu_n(t, u, v, w) := \frac{1}{|V(\Gamma_n)|} \sum_E \delta_E,
\]

where the sum runs over eigenvalues \( E \) of \( M_n(t, u, v, w) \) counted with multiplicities. In the case \( u = v = w \) it is shown in [51] that the measures \( \mu_n, n \in \mathbb{N} \), converge weakly and the limiting measure

\[
\mu_\infty(t, u, v, w) = \lim_{n \to \infty} \mu_n(t, u, v, w)
\]

is called the Kesten-von-Neumann-Serre spectral measure there. Below we will show that the limiting measure exists for any values of the parameters \( t, u, v, w \) and coincides with the so-called integrated density of states of the associated Schrödinger operators.

3. Schrödinger operators on (low complexity) subshifts

In this section, we present (parts of) the spectral theory of discrete Schrödinger operators associated to subshifts. We will later see that the operators on Schreier graphs introduced in the previous section are in fact unitarily equivalent to discrete Schrödinger operators associated to a certain subshift.

3.1. Constancy of the spectrum and the integrated density of states. In this section we review some basic theory of discrete Schrödinger operators associated to minimal topological dynamical systems. This includes constancy of the spectrum and uniform convergence of the so-called integrated density of states. All results of this section are well known.

Let \( (\Omega, T) \) be a topological dynamical system and let \( f, g : \Omega \to \mathbb{R} \) be continuous functions. Then, we associate a family of discrete Schrödinger operators \( (H_\omega)_{\omega \in \Omega} \). The spectral theory of these operators is our basic concern in this section.

Specifically, \( H_\omega \) is a bounded selfadjoint operator from \( \ell^2(\mathbb{Z}) \) to \( \ell^2(\mathbb{Z}) \) acting via

\[
(H_\omega \varphi)(n) = f(T^n \omega)\varphi(n - 1) + f(T^{n+1} \omega)\varphi(n + 1) + g(T^n \omega)\varphi(n)
\]
for \( \varphi \in \ell^2(\mathbb{Z}) \). As the operator \( H_\omega \) is selfadjoint, the operator \( H_\omega - z \) is bijective with continuous inverse \( (H_\omega - z)^{-1} \) for any \( z \in \mathbb{C} \setminus \mathbb{R} \). Moreover, for any \( \varphi \in \ell^2(\mathbb{Z}) \) there exists a unique positive Borel measure \( \mu_\omega^\varphi \) on \( \mathbb{R} \) with
\[
\int_{\mathbb{R}} \frac{1}{t - z} d\mu_\omega^\varphi(t) = \langle \varphi, (H_\omega - z)^{-1} \varphi \rangle
\]
for any \( z \in \mathbb{C} \setminus \mathbb{R} \). This measure if finite and assigns the value \( \|f\|^2 \) to the set \( \mathbb{R} \).

For fixed \( \omega \) the measures \( \mu_\omega^\varphi, \varphi \in \ell^2(\mathbb{Z}) \), are called the spectral measures of \( H_\omega \). The smallest set containing the support of any \( \mu_\omega^\varphi \) is the spectrum of \( H_\omega \) and denoted by \( \sigma(H_\omega) \). The spectrum is said to be purely absolutely continuous if all spectral measures are absolutely continuous with respect to the Lebesgue measure. The spectrum is said to be purely singular continuous if all spectral measures are both continuous (i.e. do not have discrete parts) and singular with respect to Lebesgue measure.

**Remark 3.1.** (a) If one thinks of a selfadjoint operator on \( \ell^2(\mathbb{Z}) \) as a two-sided infinite symmetric matrix, then the operator \( H_\omega \) has diagonal entries given by \( g(T^n_\omega) \) for \( n \in \mathbb{Z} \) and entries given by \( f(T^n_\omega) \) for \( n \in \mathbb{Z} \) on the two first off-diagonals (and all other entries vanish).

(b) Later we will require that \( f \) does not vanish anywhere. Note that this implies that the matrix given by the operator \( H_\omega \) can not be 'decomposed into blocks'.

(c) Sometimes the name 'discrete Schroedinger operator' is reserved for the situation \( f \equiv 1 \). The operators with general \( f \) are then called Jacobi operators.

(d) Subsequently we will mostly write 'Schroedinger operator' instead of 'discrete Schroedinger operator' (as all considered operators are discrete i.e. defined on \( \ell^2(\mathbb{Z}) \)).

The following result is well-known. It can be found in various places, see e.g. [60].

**Theorem 3.2 (Constancy of the spectrum).** Let \((\Omega, T)\) be minimal and \( f, g : \Omega \to \mathbb{R} \) continuous. Then, there exists a closed subset \( \Sigma \subset \mathbb{R} \) such that the spectrum \( \sigma(H_\omega) \) of \( H_\omega \) equals \( \Sigma \) for any \( \omega \in \Omega \).

We will refer to the set \( \Sigma \) in the previous theorem as the spectrum of the Schroedinger operator associated to \((\Omega, T) \) (and \( (f, g) \)). The spectrum \( \Sigma \) is a main quantity of interest in our study.

Before turning to a finer analysis of the spectrum we will introduce a further quantity of interest the so called integrated density of states. In order to do so, we will assume that the underlying dynamical system \((\Omega, T)\) is not only minimal but also uniquely ergodic with unique invariant probability measure \( \lambda \). Then, we can associate to the family \((H_\omega)\) the positive measure \( k \) on \( \mathbb{R} \) defined via
\[
\int_{\mathbb{R}} F(x) dk(x) := \int_{\Omega} \langle f(H_\omega) \delta_0, \delta_0 \rangle d\lambda(\omega)
\]
(for \( F \) any continuous function on \( \mathbb{R} \) with compact support). Here, \( \delta_0 \in \ell^2(\mathbb{Z}) \) is just the characteristic function of \( 0 \in \mathbb{Z} \). This measure \( k \) is called the integrated density of states. Let
\[
N : \mathbb{R} \to [0, 1], N(E) := \int_{(-\infty, E]} dk,
\]
be the distribution function of \( k \).
As is well-known, there is a direct relation between the measure $k$ and the spectrum of the $H_\omega$.

**Theorem 3.3.** The set $\Sigma$ is the support of the measure $k$. If the function $f$ does not vanish anywhere then $k$ does not have atoms (i.e. it assigns the value zero to any set containing only one element).

**Remark 3.4.** This is rather standard theory of random operators. Specific variants of it can be found in many places. In particular, the first statement can be found in [60]. In the case of $f$ which do not vanish anywhere the statements of the theorem are contained in Section 5 of [77]. Given the constancy of the spectrum, Theorem 3.2 the statements are also very special cases of the results of Section 5 of [63]. The key ingredient for the absence of atoms is amenability of the underlying group $\mathbb{Z}$. The statement on the support does not even need this property.

As is well known, it is possible to calculate $k$ via an approximation procedure. This will be discussed next. For $n, m \in \mathbb{Z}$ with $n \leq m$ let

\[ j_{n,m} : \ell^2([n, \ldots, m]) \longrightarrow \ell^2(\mathbb{Z}) \]

be the canonical inclusion and let $p_{n,m}$ be the adjoint of $j_{n,m}$. Thus,

\[ p_{n,m} : \ell^2(\mathbb{Z}) \longrightarrow \ell^2([n, \ldots, m]) \]

is the canonical projection. Define for $\omega \in \Omega$ then

\[ H_{n,m}^\omega := p_{n,m} H_\omega j_{n,m}. \]

We will be concerned with the spectral theory of these operators. Let the measure $k_{n,m}^\omega$ on $\mathbb{R}$ be defined as

\[ \int_\mathbb{R} F(x) dk_{n,m}^\omega(x) := \frac{1}{m-n+1} \sum_{k=n}^{m} \langle F(H_{n,m}^\omega) \delta_k, \delta_k \rangle \]

(for any continuous $F$ on $\mathbb{R}$ with compact support) and let

\[ N_{n,m}^\omega : \mathbb{R} \longrightarrow [0,1], N_{n,m}^\omega(E) := \int_{(\infty, E]} dk_{n,m}^\omega(x), \]

be its distribution function. Let $E_1, \ldots, E_{m-n+1}$ be the eigenvalues of $H_{n,m}^\omega$ counted with multiplicity. Then, straightforward linear algebra (diagonalization of $H_{n,m}^\omega$ and independence of the trace of the chosen orthonormal basis) shows that

\[ \int_\mathbb{R} F(x) dk_{n,m}^\omega(x) = \frac{1}{m-n+1} \sum_j F(E_j) \]

holds for any continuous $F$ on $\mathbb{R}$ with compact support and that the distribution functions are given by

\[ N_{n,m}^\omega(E) = \frac{\sharp\{\text{Eigenvalues of } H_{n,m}^\omega \text{ not exceeding } E\}}{m-n+1}, \]

where $\sharp$ denotes the cardinality of a set. In this sense, $k_{n,m}^\omega$ is just an averaged eigenvalue counting.
Theorem 3.5 (Convergence of the integrated density of states). For any continuous $F$ on $\mathbb{R}$ with compact support and any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ with
\[
\left| \int_{\mathbb{R}} F(x) dk(x) - \int_{\mathbb{R}} F(x) k_{\omega}^{n,m} \right| \leq \varepsilon
\]
for all $\omega \in \Omega$ and all $n, m \in \mathbb{Z}$ with $m - n \geq N$.

Proof. It is well-known that the measures $(k_{\omega}^{m})_{m}$ converge weakly toward $k$ for $m \to \infty$ for almost every $\omega \in \Omega$. This can be found in many places, see e.g. Lemma 5.12 in [77]. (That lemma assumes that $f$ does not vanish anywhere but its proof does not use this assumption.) A key step in the proof is the use of the Birkhoff ergodic theorem. The desired statement now follows by replacing the Birkhoff ergodic theorem with the uniform ergodic theorem (Oxtoby Theorem) valid for uniquely ergodic systems [80]. □

The operators $H_{\omega}^{n,m}$ are sometimes thought of as arising out of the $H_{\omega}$ by some form of 'Dirichlet boundary condition'. The previous result is stable under taking different 'boundary conditions'. In fact, even more is true as we will discuss next (and this more general statement will even save us from saying what we mean by boundary condition) Let for any $n, m \in \mathbb{Z}$ with $n \leq m$ and $\omega \in \Omega$ be a selfadjoint operator $C_{\omega}^{n,m}$ on $\ell^2\{\{n, \ldots, m\}\}$ be given. Then, the statement of the theorem essentially continues to hold if the operators $H_{\omega}^{n,m}$ are replaced by the operators
\[
\tilde{H}_{\omega}^{n,m} := H_{\omega}^{n,m} + C_{\omega}^{n,m}
\]
provided the rank of the $C_{\omega}$'s is not too big. Here, the rank of an operator $C$ on a finite dimensional space, denoted by $\text{rank}(C)$, is just the dimension of the range of $C$.

In order to be more specific, we introduce the measure $\tilde{k}_{\omega}^{n,m}$ on $\mathbb{R}$ defined as
\[
\int_{\mathbb{R}} F(x) d\tilde{k}_{\omega}^{n,m}(x) := \frac{1}{m-n+1} \sum_{k=n}^{m} \langle F(\tilde{H}_{\omega}^{n,m}) \delta_k, \delta_k \rangle
\]
(for any continuous $F$ on $\mathbb{R}$ with compact support) and its distribution function given by
\[
\tilde{N}_{\omega}^{n,m} : \mathbb{R} \to [0, 1], \quad \tilde{N}_{\omega}^{n,m}(E) := \frac{1}{m-n+1} \sharp\{\text{Eigenvalues of } \tilde{H}_{\omega}^{n,m} \text{ not exceeding } E\}.
\]

Corollary 3.6. Consider the situation just described. Let $\omega \in \Omega_{\tau}$ be given with
\[
\frac{1}{m-n+1} \text{rank}(C_{\omega}^{n,m}) \to 0, m-n \to \infty.
\]
Then, for any continuous $F$ on $\mathbb{R}$ with compact support and any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ with
\[
\left| \int_{\mathbb{R}} F(x) dk(x) - \int_{\mathbb{R}} F(x) d\tilde{k}_{\omega}^{n,m} \right| \leq \varepsilon
\]
for all $n, m \in \mathbb{Z}$ with $m - n \geq N$.

Proof. A consequence of the minimax principle, see e.g. Theorem 4.3.6 in [53] shows
\[
|\tilde{N}_{\omega}^{n,m}(E) - N_{\omega}^{n,m}(E)| \leq \frac{1}{m-n+1} \text{rank}(C_{\omega}^{n,m})
\]
independent of $E \in \mathbb{R}$. This directly gives the desired statement. □
3.2. Subshifts or when \((f, g)\) takes only finitely many values. As discussed in the previous section we are interested in Schroedinger operators arising from continuous functions \(f, g : \Omega \to \mathbb{R}\) on minimal dynamical systems \((\Omega, T)\). For our further analysis we will restrict our attention to the situation that

- the function \((f, g) : \Omega \to \mathbb{R}^2\) takes only finitely many values and
- the function \(f\) does not vanish anywhere.

In this context we will have to distinguish two cases: the case that \((f, g)\) is periodic and the case that it is not periodic. Here, a function \(h\) on \(\Omega\) (with values in an arbitrary set) is called periodic if there exists an natural number \(N\) such that \(h(T^n \omega) = h(\omega)\) for all \(\omega\) in \(\Omega\). The smallest positive such \(N\) is then called the period of \(h\).

The restriction to functions \((f, g)\) taking only finitely many values amounts in essence to the consideration of locally constant functions on subshifts as will be discussed next. Here, a function \(h\) on a subshift \(\Omega\) is called locally constant if there exists a natural number \(N\) such the value \(h(\omega)\) depends only on the finite word \(\omega_{-N} \ldots \omega_N\). Local constancy is a strong form of continuity. Now, obviously, any locally constant function on a subshift takes only finitely many values (as there are only finitely many words of length \(2N + 1\)). Conversely, whenever \(h\) is a continuous function on an arbitrary dynamical system \((\Omega, T)\) taking only finitely many values we can associate a subshift \(\Omega_h\) over the alphabet \(A\) of values of \(h\) given by all sequences

\[ \mathbb{Z} \ni n \mapsto h(T^n \omega) \in A \]

for \(\omega \in \Omega\). Indeed, it is not hard to see that the arising set of sequences forms a subshift and the map

\[ \Phi : \Omega \to \Omega_h, \omega \mapsto (n \mapsto h(T^n \omega)), \]

is continuous and onto and satisfies

\[ \Phi \circ T = T_h \circ \Phi, \]

where \(T_h\) denotes the canonical subshift action on \(\Omega_h\). This means that \((\Omega_h, T_h)\) is a factor of \((\Omega, T)\) with factor map \(\Phi\). Let

\[ q : \Omega_h \to A, \omega' \mapsto \omega'(0), \]

be the evaluation at the origin. Then, \(q\) is obviously locally constant with

\[ h = q \circ \Phi. \]

In this sense, the continuous function \(h\) with finitely many values on \((\Omega, T)\) is recoded by the locally constant function \(q\) on the subshift \((\Omega_h, T_h)\).

We finish this section by noting that \(\Omega_h\) will be a Cantor set if \((f, g)\) is not periodic.

3.3. The spectrum as a set and the absolute continuity of spectral measures. In this section we consider Schroedinger operators over minimal dynamical systems with the associated functions \((f, g)\) taking only finitely many values. As discussed in the previous section, we can then restrict attention to locally constant functions on subshifts. As mentioned already the key distinction will be whether \((f, g)\) is periodic or not.

The overall structure of the spectrum in the periodic case is well-known. This can be found in many references, see e.g. the monograph [77].
Theorem 3.7 (Periodic case). Let \((\Omega,T)\) be a minimal subshift and \(f,g: \Omega \to \mathbb{R}\) locally constant with \(f(\omega) \neq 0\) for all \(\omega \in \Omega\). If \((f,g)\) is periodic (with period \(N\)), then the spectrum \(\Sigma\) of the associated Schrödinger operator consists of finitely many (and not more than \(N\)) closed intervals of positive length and all spectral measures are absolutely continuous with respect to Lebesgue measure.

Remark 3.8. Note that periodicity of \((f,g)\) may have its origin in both properties of \((\Omega,T)\) and properties of \((f,g)\). For example periodicity always occurs if \(f = g = 1\) irrespective of the nature of \((\Omega,T)\). Also, periodicity occurs for arbitrary \(f,g\) if \((\Omega,T)\) is periodic (i.e. there exists a natural number \(N\) with \(T^N\omega = \omega\) for all \(\omega \in \Omega\)). However, if \((\Omega,T)\) is not periodic then \((f,g)\) will be non-periodic if it is not 'degenerate'.

The previous theorem gives rather complete information on \(\Sigma\) in the periodic case. In order to deal with the non-periodic case, we will need a further assumption on \((\Omega,T)\). This condition is linear repetitivity.

Theorem 3.9 (Aperiodic case [11]). Let \((\Omega,T)\) be a linearly repetitive subshift and \(f,g: \Omega \to \mathbb{R}\) locally constant with \(f(\omega) \neq 0\) for all \(\omega \in \Omega\). If \((f,g)\) is non-periodic, then there exists a Cantor set \(\Sigma\) of Lebesgue measure zero in \(\mathbb{R}\) such that \(\sigma(H_\omega) = \Sigma\) for all \(\omega \in \Omega\).

Remark 3.10. The above theorem was first proven in [59] in the case \(f \equiv 1\). This result was then extended in [27] from linearly repetitive subshifts to arbitrary subshifts satisfying a certain condition known as Boshernitzan condition (B) (again for the case \(f \equiv 1\)). In the form stated above it can be inferred from the recent work [11], Corollary 4. This corollary treats the even more general situation, where condition (B) is satisfied. Condition (B) was introduced by Boshernitzan as a sufficient condition for unique ergodicity [17] (see [27] for an alternative approach as well). In our context, we do not actually need its definition here. It suffices to know that linear repetitivity implies (B) (see e.g. [27]).

The previous result deals with the appearance of \(\Sigma\) as a set. It also gives some information on the spectral type.

Corollary 3.11. Assume the situation of the previous theorem. Then, no spectral measure, \(\mu_\omega\), \(\omega \in \Omega\), \(\varphi \in \ell^2(\mathbb{Z})\), can be absolutely continuous with respect to the Lebesgue measure.

Proof. This is clear as the support of any spectral measure is contained in \(\Sigma\). Now, by the previous theorem \(\Sigma\) has Lebesgue measure zero and the corollary follows. \(\square\)

3.4. Absence of eigenvalues and purely singular continuous spectrum. In the preceding considerations we have discussed the spectrum as a set as well as (absence of) absolute continuity of spectral measures. In some situations it is possible to infer even further spectral properties such as absence of eigenvalues.

A key tool in the exclusion of eigenvalues is the following well-known result. It is named after the paper [37] of Gordon which deals with a slightly different context. For a survey on applications in a similar spirit to our work we refer to [22].
Lemma 3.12 (Gordon Lemma). Let bounded sequences \((f_n)\) and \((g_n)\) in \(\mathbb{R}\) be given. Assume \(f_n \neq 0\) for all \(n \in \mathbb{Z}\). Consider the self adjoint operator \(H\) acting on \(\ell^2(\mathbb{Z})\) via
\[H\varphi(n) = f_n\varphi(n + 1) + f_{n-1}\varphi(n - 1) + g_n\varphi(n),\]
If there exists a sequence of natural numbers \((L_k)\) with \(L_k \to \infty\) and
\[(f_n, g_n) = (f_{n-L_k}, g_{n-L_k}) = (f_{n-2L_k}, g_{n-2L_k})\]
for \(n = 1, \ldots, L_k\), then the operator \(H\) does not have any eigenvalues.

**Proof.** In the case \(f \equiv 1\) this is well-known (see e.g. the review [22]). The corresponding proof can easily be adapted to give the above lemma. \(\square\)

The previous lemma has consequences for subshifts possessing infinite words containing many cubes (i.e. words of the form \(www\)).

**Corollary 3.13.** Let \((\Omega, T)\) be a subshift over a finite alphabet. Let \(f, g : \Omega \to \mathbb{R}\) be locally constant such that \(f\) is not zero anywhere. Let \(\omega \in \Omega\) be given. Assume that there exists a sequence of finite words \(w_n\) with \(|w_n| \to \infty\) as well as non-empty prefixes \(v_n\) of \(w_n\) with \(|v_n| \to \infty\) such that
\[\omega = \ldots w_n w_n |w_n v_n| \ldots,\]
where \(|\cdot|\) denotes the position of the origin. Then, \(H_\omega\) does not have any eigenvalues.

**Proof.** By assumption \(f, g\) are locally constant. After a suitable shift we can then assume without loss of generality that there exists an \(N \in \mathbb{N}\) such that the values of \(f\) and \(g\) only depend on the positions \(0, \ldots, N\). Now, for sufficiently large values of \(n\) we will have \(|v_n| \geq N\). Thus, the sequences \(k \mapsto f(T^k \omega)\) and \(k \mapsto g(T^k \omega)\) will satisfy the assumptions of the previous lemma. This lemma then gives the desired statement. \(\square\)

**4. The substitution \(\tau\) and its subshift \((\Omega_\tau, T)\)**

In this section we study the two-sided subshift induced by a particular substitution. The one-sided subshift induced by this substitution had already been studied by Vorobets [78]. Some of our results can be seen as providing the two-sided counterparts to his investigations. These investigations rely on a connection to Toeplitz sequences. Here, we develop a new approach based on what we call \(n\)-decomposition and \(n\)-partition respectively. The subshift \((\Omega_\tau, T)\) will be of crucial importance for us as it will turn out that the Schroedinger operators associated to it are unitarily equivalent to the Laplacians on the Schreier graphs of the Grigorchuk group \(G\). The substitution in question is (a version of) the substitution used by Lysenok [67] for getting a presentation of \(G\), as was already discussed in the introduction.

**4.1. Basic features of the substitution \(\tau\).** Let the alphabet \(\mathcal{A} = \{a, x, y, z\}\) be given and let \(\tau\) be the substitution mapping \(a \mapsto axa, x \mapsto y, y \mapsto z, z \mapsto x\). Let \(\mathcal{W}_\tau\) be the associated set of words given by
\[\mathcal{W}_\tau = \bigcup_{w \in \mathcal{A}, n \in \mathbb{N} \cup \{0\}} \text{Sub}(\tau^n(w)).\]
Then, the following three properties obviously hold true:

- The letter \(a\) is a prefix of \(\tau^n(a)\) for any \(n \in \mathbb{N} \cup \{0\}\).
- The lengths \(|\tau^n(a)|\) converge to \(\infty\) for \(n \to \infty\).
- Any letter of \(\mathcal{A}\) occurs in \(\tau^n(a)\) for some \(n\).
By the first two properties $\tau^n(a)$ is a prefix of $\tau^{n+1}(a)$ for any $n \in \mathbb{N} \cup \{0\}$. Thus, there exists a unique one-sided infinite word $\eta$ such that $\tau^n(a)$ is a prefix of $\eta$ for any $n \in \mathbb{N} \cup \{0\}$. This $\eta$ is then a fixed point of $\tau$ i.e. $\tau(\eta) = \eta$. We will refer to it as the fixed point of the substitution $\tau$. By the third property we then have

$$W_\tau = \text{Sub}(\eta).$$

We can now associate to $\tau$ the subshift

$$\Omega_\tau := \{ \omega \in A^\mathbb{Z} : \text{Sub}(\omega) \subset W_\tau \}.$$ 

Note that every other letter of $\eta$ is an $a$ (as can easily be seen). Thus, $a$ occurs in $\eta$ with bounded gaps. This implies that any word of $W_\tau$ occurs with bounded gaps (as the word is a subword of $\tau^n(a)$ and $\eta$ is a fixed point of $\tau$). For this reason $(\Omega_\tau, T)$ is minimal and $\text{Sub}(\omega) = W$ holds for any $\omega \in \Omega_\tau$. We can then apply Theorem 1 of [28] to obtain the following.

**Theorem 4.1.** The subshift $(\Omega_\tau, T)$ is linearly repetitive. In particular, $(\Omega_\tau, T)$ is uniquely ergodic and minimal.

**Remark 4.2.** It is well-known that subshifts associated to primitive substitutions are linearly repetitive (see e.g. [30, 29]). Theorem 1 of [28] shows that linearity holds for subshifts associated to any substitution provided minimality holds. Unique ergodicity is then a direct consequence of linear repetitivity due to Theorem 1.3.

Our further considerations will be based on a more careful study of the $\tau^n(a)$. We set $p^{(0)} := a$ and $p^{(n)} := \tau^n(a)$ for $n \in \mathbb{N}$.

A direct calculation gives

$$p^{(n+1)} = \tau^{n+1}(a) = \tau^n(axa) = \tau^n(a)\tau^n(x)\tau^n(a),$$

i.e.

$$(RF) \quad p^{(n+1)} = p^{(n)}s_np^{(n)}$$

with

$$s_n = \tau^n(x) = \begin{cases} 
  x & n = 3k, k \in \mathbb{N} \cup \{0\} \\
  y & n = 3k + 1, k \in \mathbb{N} \cup \{0\} \\
  z & n = 3k + 2, k \in \mathbb{N} \cup \{0\}
\end{cases}$$

We will refer to $(RF)$ as the recursion formula for the $p^{(n)}$.

For later use we note the following rather direct consequences of the definitions.

**Lemma 4.3** (The $n$-decomposition of the fixed point $\eta$). For any $n \in \mathbb{N} \cup \{0\}$ the word $\eta$ has a (unique) decomposition as

$$\eta = p^{(n)}r_1^{(n)}p^{(n)}r_2^{(n)}...$$

with $r_j^{(n)} \in \{x, y, z\}$. Moreover, setting $r_j := r_j^{(0)}$ we have $r_j^{(n)} = \tau^n(r_j)$ for any $j \in \mathbb{N}$.

**Proof.** Uniqueness of such a decomposition is clear. It remains to show existence. Using $(RF)$ it is not hard to see that any other letter of $\eta$ is an $a$ and the letters between two such $a$’s belong to $\{x, y, z\}$. Thus,

$$\eta = ar_1ar_2a...$$
with a sequence \( r_1r_2\ldots \in \{x,y,z\}^\mathbb{N} \). This gives the desired statement for \( n = 0 \). Now, the remaining claims follow as
\[
\eta = \tau^n(\eta) = \tau^n(\eta_1)\tau^n(\eta_2)\ldots
\]
for any \( n \in \mathbb{N} \) due to the fixed point property of \( \eta \).

The way of writing \( \eta \) given in the previous lemma will be called the \( n \)-decomposition of \( \eta \). We will have more to say about it in the next sections.

Recall that a non-empty word \( w = w_1\ldots w_n \in A^* \) with \( w_j \in A \) is called a palindrome if \( w = w_n\ldots w_1 \).

**Proposition 4.4 (Properties of the \( p^{(n)} \)).** For any \( n \in \mathbb{N} \cup \{0\} \) the word \( p^{(n)} \) is a palindrome of length \( 2^n - 1 \). It starts and ends with \( p^{(k)} \) for any \( k \in \mathbb{N} \cup \{0\} \) with \( k \leq n \).

**Proof.** This follows directly by induction and the recursion formula \((RF)\). \(\square\)

We are now heading towards introducing some special elements \( \omega^{(x)}, \omega^{(y)}, \omega^{(z)} \in \Omega_\tau \). These will play an important role in our subsequent analysis. Indeed, they will be shown later to be exactly those elements of \( \Omega_\tau \) which agree with \( \eta \) on \( \mathbb{N} \).

**Lemma 4.5 (The special words \( \omega^{(x)}, \omega^{(y)}, \omega^{(z)} \)).** For any \( n \in \mathbb{N} \cup \{0\} \) and any single letter \( s \in \{x,y,z\} \) the word \( p^{(n)}sp^{(n)} \) occurs in \( \eta \). In particular, for any \( s \in \{x,y,z\} \) there exists a unique element \( \omega^{(s)} \in \Omega_\tau \) such that
\[
\omega^{(s)} = \ldots p^{(n)}sp^{(n)}\ldots
\]
holds for all natural numbers \( n \), where the \(|\) denotes the position of the origin.

**Proof.** Note that \( \tau^3(a) \) contains \( axa, aya \) and \( aza \). As \( \eta \) is a fixed point of \( \tau \) and \( \tau^n \) is injective on \( \{x,y,z\} \) the first statement follows. Now, we turn to the second statement. Existence follows directly from the first statement and the previous proposition. Uniqueness is clear as the lengths of the \( p^{(n)} \) tend to infinity \((n \to \infty)\) and hence the \( p^{(n)}sp^{(n)} \) determine arbitrary long stretches around the origin. \(\square\)

As mentioned already, the infinite sequences \( \omega^{(x)}, \omega^{(y)}, \omega^{(z)} \) will play a special role in our subsequent considerations. Here, we already note that these three sequences are different and agree on \( \mathbb{N} \). Thus, \( \Omega_\tau \) contains different sequences which agree on \( \mathbb{N} \) and hence is not periodic. In fact, even a renaming of the letters will not destroy aperiodicity provided not all letters \( x, y, z \) are given the same name. This is the content of the next proposition.

**Proposition 4.6.** Let \( B \) be a finite set and \( C : A \to C \) a map such that \( C(x) = C(y) = C(z) \) does not hold. Then, the subshift
\[
\Omega^{(C)}_\tau := \{ C \circ \omega : \omega \in \Omega_\tau \}
\]
is not periodic.

**Proof.** Obviously, the map \( \Omega_\tau \to \Omega^{(C)}_\tau, \omega \to C \circ \omega \), is continuous and onto. In particular, \((\Omega^{(C)}_\tau, T)\) is a minimal subshift. Moreover, the two - sided infinite words \( C\omega^{(x)}, C\omega^{(y)}, C\omega^{(z)} \) all agree on \( \mathbb{N} \) but are not all equal (due to the assumption on \( C \)). This can easily be seen to imply that the subshift \( \Omega^{(C)}_\tau \) is not periodic. \(\square\)
Remark 4.7.  
- As the $p^{(n)}$ are palindromes, the words $\omega^{(s)}$ are symmetric around the $s$ at the origin for $s \in \{x, y, z\}$. In fact, the words $\omega^{(s)}$ are characterized within $\Omega_\tau$ by this property, see Theorem 4.26 and its Corollary 4.27.
- Later we will associate graphs to the sequences of $\Omega_\tau$ (see Section 6). Then, the three graphs corresponding to $\omega^{(s)}$, $s \in \{x, y, z\}$, are exactly the graphs $\Delta_0, \Delta_1, \Delta_2$ from [49].

4.2. The $n$-partition. By Lemma 4.3 the fixed point $\eta$ of $\tau$ has a (unique) decomposition as

$$\eta = p^{(n)}r_1^{(n)}p^{(n)}r_2^{(n)} \ldots$$

with $r_j^{(n)} \in \{x, y, z\}$ for any $n \in \mathbb{N} \cup \{0\}$. This was called the $n$-decomposition of $\eta$. It turns out that an analogue decomposition can actually be given for any element $\omega \in \Omega_\tau$. Based on this one can then study the dynamical system $(\Omega_\tau, T)$. This is discussed in this section.

Our aim is to show that each $\omega \in \Omega_\tau$ admits for each $n \in \mathbb{N} \cup \{0\}$ a unique decomposition of the form

$$\omega = \ldots p^{(n)}s_0p^{(n)}s_1p^{(n)}s_2\ldots$$

with

- $s_k \in \{x, y, z\}$ for all $k \in \mathbb{Z}$,
- the origin $\omega_0$ belongs to $s_0p^{(n)}$.

Such a decomposition will be referred to as an $n$-decomposition of $\omega$. A short moment’s thought reveals that if such a decomposition exists at all, then it is uniquely determined by the position of any of the $s_j$’s in $\omega$. Moreover, the positions of the $s_j$’s are given by $P + 2^n\mathbb{Z}$ with $P \in \{0, \ldots, 2^n - 1\}$. Thus, the positions are given by an element of $\mathbb{Z}/2^n\mathbb{Z}$. This suggests the following definition as an alternative but completely equivalent way of thinking about $n$-decompositions.

Definition 4.8 ($n$-partition). For $n \in \mathbb{N} \cup \{0\}$ we call an element $P \in \mathbb{Z}/2^n\mathbb{Z}$ an $n$-partition of $\omega \in \Omega_\tau$ if for any $q \in P$ both

- $\omega_q \in \{x, y, z\}$ and
- $\omega_{q+1} \ldots \omega_{q+2^n-1} = p^{(n)}$

hold.

It is not apparent that such an $n$-partition exists at all. Here is our corresponding result.

Theorem 4.9 (Existence and Uniqueness of $n$-partitions). Let $n \in \mathbb{N} \cup \{0\}$ be given. Then any $\omega \in \Omega_\tau$ admits a unique $n$-partition $P^{(n)}(\omega)$ and the map

$$P^{(n)} : \Omega_\tau \rightarrow \mathbb{Z}/2^n\mathbb{Z}, \ \omega \mapsto P^{(n)}(\omega),$$

is continuous and equivariant (i.e. $P^{(n)}(T\omega) = P^{(n)}(\omega) + 1$).
Proof. The last statement is an immediate consequence of the previous ones. Thus, it suffices to show those.

Existence of \( P^{(n)}(\omega) \). Recall that \( \omega^{(x)} \in \Omega_r \) is the unique word with \( \omega^{(x)} = \ldots p^{(n)}x[p^{(n)}] \ldots \), where \( | \) denotes the position of the origin. Now, obviously, \( \omega^{(x)} \) admits an \( n \)-partition (by its very definition and (RF)). Moreover, the subshift is minimal. Hence, any \( \omega \in \Omega_r \) can be approximated by a sequence of translates of \( \omega^{(x)} \). These all carry natural \( n \)-partitions coming from the \( n \)-partition of \( \omega^{(x)} \). As the values of these \( n \)-partitions all lie within the finite set \( \mathbb{Z}/2^n\mathbb{Z} \) we can assume (after restricting attention to a subsequence) without loss of generality that all these values are all equal. This easily gives existence of an \( n \)-partition for \( \omega \).

Uniqueness. As discussed at the beginning of this section the concepts of \( n \)-decomposition and \( n \)-partition are equivalent in the sense that existence (uniqueness) of an \( n \)-partition implies existence (uniqueness) of an \( n \)-decomposition and vice versa. This will be used in order to obtain uniqueness. Our proof proceeds by induction. The case \( n = 0 \) is clear. (In this case \( p^{(0)} = a \).) Let us now show how to proceed from \( n \) to \( n + 1 \). Consider an \( (n + 1) \)-decomposition of \( \omega \). Such a decomposition exists by the already shown part. Chose \( s \in \{x, y, z\} \) with \( p^{(n+1)} = p^{(n)}sp^{(n)} \). Then, out of the \( n + 1 \) decomposition of \( \omega \) we obtain an \( n \)-decomposition by just replacing \( p^{(n+1)} \) by \( p^{(n)}sp^{(n)} \) in the corresponding decomposition of \( \omega \). This \( n \)-decomposition is unique by our induction assumption. Now, it is not hard to see that non-uniqueness of the \( (n + 1) \)-decomposition can only occur if \( \omega \) is periodic. However, as discussed above in Proposition 4.6 there is no periodic sequence in \( \Omega \).

Continuity. This is a rather direct consequence of uniqueness: Let \( \omega^{(k)} \) be a sequence converging to \( \omega \) and let \( P_k \) and \( P \) be the respective \( n \)-partitions. We have to show \( P_k \to P \). As the space \( \mathbb{Z}/2^n\mathbb{Z} \) is finite (hence compact), it suffices to show that any converging subsequence of \( (P_k) \) converges to \( P \). Now, it is clear that whenever a subsequence of \( (P_k) \) converges to some \( P' \) then \( P' \) is an \( n \)-partition of \( \omega \). By uniqueness we infer \( P' = P \) and this gives the desired statement. \( \square \)

For any \( n \in \mathbb{N} \cup \{0\} \) we can consider \( \mathbb{T}^{(n)} := \mathbb{Z}/2^n\mathbb{Z} \) with the addition map \( A^{(n)} \) mapping \( m + 2^n\mathbb{Z} \) to \( m + 1 + 2^n\mathbb{Z} \). Then, \( (\mathbb{T}^{(n)}, A^{(n)}) \) is a periodic minimal dynamical system. In this context, the previous theorem has the following direct consequence.

Corollary 4.10. The dynamical system \( (\mathbb{T}^{(n)}, A^{(n)}) \) is a factor of \( (\Omega_r, T) \) via the factor map \( P^{(n)} \).

4.3. The dynamical system \( (\Omega_r, T) \) associated to \( \tau \). By Corollary 4.10 the dynamical system \( (\mathbb{T}^{(n)}, A^{(n)}) \) is a factor of \( (\Omega_r, T) \) via the continuous map

\[
\Omega_r \longrightarrow \mathbb{T}^{(n)}, \ \omega \mapsto P^{(n)}(\omega).
\]

We will use this to provide a further analysis of properties of the dynamical system \( (\Omega_r, T) \).

We first turn to a study of continuous eigenvalues. Let \( (Y, S) \) be a dynamical system (i.e. \( Y \) is a compact space and \( S \) is a homeomorphism). Denote the unit circle in \( \mathbb{C} \) by \( S^1 \). Then, \( k \in S^1 \) is called a continuous eigenvalue of the dynamical system \( (Y, S) \) if there exists a continuous function \( f \neq 0 \) with values in \( \mathbb{C} \) on \( Y \) with

\[
f(S(y)) = kf(y)
\]

for all \( y \in Y \). Such a function is then called a continuous eigenfunction. By standard reasoning the continuous eigenvalues form a group whenever the underlying dynamical system
is minimal. Indeed, by minimality any continuous eigenfunction has constant (non-vanishing) modulus. Then, the product of two eigenfunctions is an eigenfunction to the product of the corresponding eigenvalues. The complex conjugate of an eigenfunction is an eigenfunction to the inverse of the corresponding eigenvalue and the constant function is an eigenfunction to the eigenvalue 1. Moreover, it is not hard to see that minimality implies that the multiplicity of each continuous eigenvalue is one (i.e. for any two continuous eigenfunctions \( f, g \) to the same eigenvalue there exists a complex number \( c \) with \( f = cg \)).

Let now \( \mathcal{E}_n \) be the group of continuous eigenvalues of \( T^{(n)} \). This is just the subgroup of \( S^1 \) given by \( \{e^{2\pi i \frac{k}{2^n}} : 0 \leq k \leq 2^n - 1 \} \). Thus, obviously, \( \mathcal{E}_n \) is a subgroup of \( \mathcal{E}_{n+1} \). Let \( \mathcal{E} \) be the group arising as the union of the \( \mathcal{E}_n \). Equip it with the discrete topology and denote its Pontryagin dual by \( \mathbb{T} \). As any eigenvalue belongs to \( S^1 \), there is a canonical embedding of groups \( \mathcal{E} \to S^1 \). By duality, this gives rise to a group homomorphism \( j : \mathbb{Z} \to \mathbb{T} \) with dense range. This homomorphism induces then an action \( A \) of \( \mathbb{Z} \) on \( \mathbb{T} \) via

\[
A : \mathbb{T} \to \mathbb{T}, A\gamma := j(1)\gamma.
\]

Disentangling definitions, we infer that the action is given by

\[
(A\gamma)(k) = k\gamma(k)
\]

(for \( \gamma \in \mathbb{T} \) and \( k \in \mathcal{E} \)). We denote the arising dynamical system as \((\mathbb{T}, A)\). It is known as binary odometer. By construction it is a rotation on a compact abelian group. Thus, its group of continuous eigenvalues is exactly given by \( \mathcal{E} \).

Now, obviously any eigenfunction of \( T^{(n)} \) gives immediately rise to an eigenfunction of \( \Omega_\tau \) for the same eigenvalue (by composing with the factor map). As the factor map is continuous, we obtain in this way continuous eigenfunctions to the eigenvalues from \( \mathcal{E} \). Minimality easily shows that (up to an overall scaling) each of these eigenfunctions is unique. Thus, we obtain a family of continuous eigenfunctions. At this point it is not clear that all eigenvalues of \((\Omega_\tau, T)\) belongs to \( \mathcal{E} \). However, this will be shown later.

We can use the preceding considerations to introduce an equivalence relation \( \approx \) on \( \Omega_\tau \) via

\[
\omega \approx \omega' \iff f(\omega) = f(\omega') \quad \text{for all eigenfunctions to eigenvalues from } \mathcal{E}.
\]

Clearly, \( \omega \approx \omega' \) if and only if \( T\omega \approx T\omega' \). Thus, the relation \( \approx \) is compatible with the shift operation. Hence, the quotient \( \Omega_\tau/\approx \) becomes a dynamical system with the operation \( T\approx \) induced by the shift.

Fix now an \( \omega_0 \in \Omega_\tau \). As discussed above continuous eigenfunctions do not vanish anywhere and the multiplicity of each continuous eigenvalue is one. Thus, for each \( k \in \mathcal{E} \) there exists a unique eigenfunction \( f_k \) to \( k \) on \( \Omega_\tau \) with \( f_k(\omega_0) = 1 \). Then, the arising system of eigenfunctions will have the property that

\[
f_{k_1}f_{k_2} = f_{k_1+k_2}, \quad f_{-k} = \overline{f_k}
\]

for all \( k, k_1, k_2 \in \mathcal{E} \). Then, the following result holds. It is well-known and can be found in various places in the literature. Recent discussions are given in [7, 4, 2].

**Lemma 4.11.** The dynamical systems \((\Omega_\tau/\approx, T\approx)\) and \((\mathbb{T}, A)\) are conjugate via the map

\[
[\omega] \mapsto (k \mapsto f_k(\omega)).
\]

In particular, the eigenvalues of \((\Omega_\tau/\approx, T\approx)\) are exactly given by the elements of \( \mathcal{E} \).

We now further investigate \( \approx \) and provide a characterization of \( \omega \approx \omega' \). Here, we will again meet the special words \( \omega^{(x)}, \omega^{(y)}, \omega^{(z)} \) introduced above in Lemma 4.5.
Proposition 4.12 (Characterization $\approx$). For $\omega, \omega' \in \Omega_T$ the relation $\omega \approx \omega'$ holds if and only if one of the following two properties hold:

- $\omega = \omega'$.
- There exist $s, s' \in \{x, y, z\}$ with $s \neq s'$ and $N \in \mathbb{Z}$ with $\omega = T^N \omega^{(s)}$ and $\omega' = T^N \omega^{(s')}$. 

Proof. Let $\omega, \omega'$ with $\omega \neq \omega'$ and $\omega \approx \omega'$ be given. By definition of $\approx$ and the construction of the eigenfunction, we then have that

$$P^{(n)}(\omega) = P^{(n)}(\omega')$$

for all $n \in \mathbb{N} \cup \{0\}$. Call this quantity $P^{(n)}$. In the remaining part of the proof we will identify such a $P^{(n)}$ with its unique representative in $\{0, \ldots, 2^n - 1\}$.

As $\omega \neq \omega'$ we infer that one of the sequences $(P^{(n)})_{n \in \mathbb{N} \cup \{0\}}$ or $(2^n - P^{(n)})_{n \in \mathbb{N} \cup \{0\}}$ must be bounded. (Otherwise, $\omega$ and $\omega'$ would agree on larger and larger pieces around the origin and then had to be equal.) Assume without loss of generality that $P^{(n)} = P$ for all $n$. After shifting the sequences by $P$ to the left we can then assume without loss of generality that $P^{(n)} = 0$ for all $n$. By definition of $P$, there exist then letters $s, s' \in \{x, y, z\}$ with

$$\omega = \ldots s | p^{(n)} \ldots \quad \text{and} \quad \omega' = \ldots s' | p^{(n)} \ldots$$

for all $n \geq 0$, where $|$ denotes the position of the origin. This gives, by definition of the $n$-partition that in fact

$$\omega = \ldots p^{(n)} s \ | \ p^{(n)} \quad \text{and} \quad \omega' = \ldots p^{(n)} s' \ | \ p^{(n)}$$

for all $n \geq 0$. Thus, we obtain $\omega = \omega^{(s)}$ and $\omega' = \omega^{(s')}$. As $\omega \neq \omega'$ we infer that $s \neq s'$. This finishes the proof. \[\square\]

The previous result shows that the factor map from $\Omega_T$ to $\Omega_T/\sim$ is one-to-one except on three orbits. This has strong consequences as will be discussed next.

As $(\Omega, T)$ is uniquely ergodic, there exists a unique $T$-invariant probability measure $\lambda$ on $\Omega_T$. The operation $T$ then induces a unitary operator $U_T$ on the associated $L^2$-space via

$$U_T : L^2(\Omega, \lambda) \rightarrow L^2(\Omega, \lambda), U_T f = f \circ T.$$ 

An element $f \in L^2(\Omega, \lambda)$ (with $f \neq 0$) is called a measurable eigenfunction to $k \in S^1$ if $U_T f = kf$. The subshift is said to have pure point spectrum if there exists an orthonormal basis of measurable eigenfunctions. From the previous result we immediately infer the following.

Theorem 4.13. The dynamical system $(\Omega_T, T)$ has pure point spectrum and any measurable eigenvalue is a continuous eigenvalue and belongs to $\mathcal{E}$.

Proof. The dynamical system $(\mathbb{T}, A)$ has pure point spectrum with all eigenvalues being continuous and belonging to $\mathcal{E}$ as it is a shift on the compact abelian group $\mathbb{T}$ which is the Pontryagin dual of $\mathcal{E}$. As the dynamical system $\Omega_T / \sim$ is conjugate to $(\mathbb{T}, A)$ due to Lemma 4.11 it has also pure point spectrum with all eigenvalues being continuous and belonging to $\mathcal{E}$.

Now, the previous result shows that factor map from $\Omega_T$ to $\Omega_T / \sim$ is one-to-one except on three countable orbits. This implies that in terms of measures the associated $L^2$-spaces are isomorphic. This easily gives the desired result. \[\square\]
Remark 4.14. The occurrence of pure point dynamical spectrum is a key feature in the investigation of aperiodic order. In fact, while there is no axiomatic framework for aperiodic order a distinctive feature is (pure) point diffraction. Now, pure point diffraction has been shown to be equivalent to pure point dynamical spectrum. For the case of subshifts at hand this can be inferred (after some work) from [71]. A more general result (dealing with uniquely ergodic Delone systems) was then given in [58]. The result can even further be generalized to measure dynamical systems and even processes [5, 64, 62].

The previous theorem implies that $(\mathbb{T}, A)$ is exactly the maximal equicontinuous factor of $(\Omega_\tau, T)$ (see e.g. [1] for definition). Indeed, one of the many equivalent ways to describe this factor is as the dual group of the group of continuous eigenvalues. A recent discussion of this and various related facts can be found in [2]. Henceforth we will denote the maximal equicontinuous factor of $(\Omega_\tau, T)$ by $(\Omega_{\tau}^{\text{max}}, T_{\text{max}})$ and the corresponding factor map by $\pi_{\text{max}}$. Then, our findings so far provide the following theorem.

**Theorem 4.15 (Factor map onto $\Omega_{\tau}^{\text{max}}$).** The three systems $(\Omega_\tau / \approx, T^\infty)$, $(\mathbb{T}, A)$ and $(\Omega_{\tau}^{\text{max}}, T_{\text{max}})$ are conjugate. The factor map $\pi_{\text{max}} : \Omega_\tau \to \Omega_{\tau}^{\text{max}}$ is one-to-one in all points except on the images of the points of the orbits of $\omega(x), \omega(y), \omega(z)$. In these points it is three-to-one.

**Remark 4.16.**

- Minimal systems with the property that their factor map to the maximal equicontinuous factor is one-to-one in at least one point are known as *almost automorphic systems* (see e.g. [2] for further details). Their study has attracted a lot of attention. As the previous result shows, $(\Omega_\tau, T)$ is an almost automorphic system. In fact, as $\Omega_\tau$ is uncountable, the factor map is one-to-one in almost every point with respect to the unique invariant probability measure $\lambda$ on $\Omega_\tau$. This is sometimes expressed as *the factor map from $(\Omega_\tau, T)$ to its maximal equicontinuous factor is almost-everywhere one-to-one*. This feature is remarkable as the underlying substitution is not primitive.

- There is an alternative description of the relation $\approx$ for almost automorphic systems. More specifically, define the *proximality relation* $\sim$ by

$\omega \sim \omega' \iff \inf_{n \in \mathbb{Z}} d(T^n \omega, T^n \omega') = 0,$

where $d$ is any metric on $\Omega_\tau$ inducing the topology. (Due to compactness of $\Omega_\tau$ the relation is indeed independent of the chosen metric.) Note that the proximality relation can be thought of as describing asymptotic agreement. Then, for almost automorphic systems the proximality relation $\sim$ and the relation $\approx$ agree. This can be found in the book of Auslander [1]. A recent discussion is given in [2]. In fact, this result is even more general in that one does not need almost automorphy but only a weaker condition called *coincidence rank one*. We refrain from further discussion of this concept and refer the reader to e.g. [7] for further investigation. We just note that in our situation equality of $\sim$ and $\approx$ means that sequences which are proximal (i.e. asymptotically equal) are in fact equal everywhere up to one position.

- In [78] Vorobets shows that the one-dimensional subshift associated to $\tau$ has the binary odometer $(\mathbb{T}, A)$ as a factor with the factor map being $1 : 1$ in all points except three orbits. He uses this to conclude pure point spectrum and unique ergodicity. Our corresponding results above for the two-sided subshift can therefore be seen as...
analogs to his results. However, our approach is different: It is based on \( n \)-partition whereas his approach is based on Toeplitz sequences.

4.4. **Powers and the index of set of words** \( W_\tau \) **associated to** \( \tau \). In this section we have a closer look at the structure of \( W_\tau \). The main focus will be on occurrences of three blocks and the index of words (to be explained below).

We start by investigating occurrences of almost four blocks.

**Lemma 4.17.** The word \( axaxaxa \) belongs to \( W_\tau \).

**Proof.** This follows by close inspection of \( \tau^4(a) \). □

Similar structures occur in any of the special sequences \( \omega(s) \) as shown in the next lemma. The lemma gives ‘almost’ the occurrence of a four block and, somewhat loosely, we will summarize this as occurrence of a three plus \( \varepsilon \) block for \( 0 < \varepsilon < 1 \).

**Lemma 4.18** (Three plus \( \varepsilon \) blocks with \( 0 < \varepsilon < 1 \) in \( \omega(s) \)). Let \( s \in \{ x, y, z \} \) be given and \( \omega(s) \) the unique word with \( \omega(s) = \ldots p^{(n)}s|p^{(n)}| \ldots \) for all \( n \) (where \( | \) denotes the position of the origin). Then,

\[
\omega(s) = \ldots p_{3n+k}s p_{3n+k} p_{3n+k} s p_{3n+k} \ldots
\]

for any \( n \in \mathbb{N} \), where \( k = 0 \) for \( s = x \), \( k = 1 \) for \( s = y \) and \( k = 2 \) for \( s = z \).

**Proof.** From the definition of \( \omega(s) \) we have

\[
\omega(s) = \ldots p_{3n+3}s p_{3n+3} p_{3n+3} s p_{3n+3} \ldots
\]

for any \( n \in \mathbb{N} \). Now, the lemma follows after we apply the recursion formula (RF) from page 22.

\[
p^{(0)} = a, \text{ and } p^{(n+1)} = p^{(n)} r p^{(n)}
\]

the corresponding number of times. □

In fact, similar structure occur in \( \lambda \)-almost every element of \( \Omega_\tau \), where \( \lambda \) is the unique \( T \)-invariant probability measure on \( \Omega_\tau \).

**Lemma 4.19** (Three plus \( \varepsilon \) blocks with \( 0 < \varepsilon < 1 \) in almost every \( \omega \)). For \( \lambda \) - almost every \( \omega \in \Omega_\tau \), there exist sequences of words \( w_n, v_n \) with \( v_n \) prefix of \( w_n \) and \( |w_n|, |v_n| \to \infty \) with \( \omega = \ldots w_n w_n v_n \ldots \).

**Proof.** This follows from Lemma 4.17 and Lemma 4.2 in [28]. The Lemma 4.2 of [28] only claims almost sure existence of \( w_n \) with the desired properties. However, close inspection of the proof shows existence of \( v_n \) as well. □

The above results show that there is quite a supply of three blocks at hand for elements of \( \Omega_\tau \). This will be rather useful for our investigation of the associated Schroedinger operators.

The next part of this section is devoted to showing that essentially there are no higher powers than \( 4 = 3 + 1 \) blocks appearing. While this is not used in our dealing with the Schroedinger operators it provides quite a bit of extra information.
Proposition 4.20. In the derived sequence $\tau$ the letters $y$ and $z$ always occur isolated preceded and followed by an $x$. The letter $x$ always occurs either isolated (i.e. preceded and followed by elements of \{y, z\}) or in the form $xxx$. The analogue statements hold for any natural number $n$ for the sequence $\tau^{(n)}$ (with $x, y, z$ replaced by $\tau^n(x), \tau^n(y)$ and $\tau^n(z)$).

Proof. As $\eta$ is a fixed point, we have

$$\eta = \tau(a)\tau(r_1)\tau(a)\tau(r_2)\ldots = p_1\tau(r_1)p_1\tau(r_2)\ldots$$

with $p_1 = \tau(a) = axa$ and $\tau(r_j) \in \{x, y, z\}$ for any $j \in \mathbb{N}$. Comparing with the definition of $r$ we find that any other letter of $r$ must be an $x$. This shows the claim on $y$ and $z$. It remains to show the statement on the occurrences of $x$. Assume that there is a block of the form $xxxx$ occurring in $\tau$. Then, there must $axazazazal$ occur in $\eta$ with $l \neq x$. In the 1-decomposition of $\eta$ this yields $p_1xp_1xp_1l$. This gives a contradiction when we consider the 2-decomposition as $p_2 \neq p_1xp_1$.

The last statement follows as $\tau^n$ is injective on \{x, y, z\}. \hfill \Box

Remark 4.21. It is not hard to establish an analogue of the previous proposition for arbitrary elements of $\Omega_r$.

The $n$-decomposition of $\eta$ gives a way of writing $\eta$ as a concatenation of the words $p^{(n)}$ and elements from \{x, y, z\}. Of course, this does not mean that any $p^{(n)}$ occurring somewhere in $\eta$ is in fact one of the words $p^{(n)}$ appearing in the $n$-decomposition of $\eta$. Still there is some form of alignment. This is the content of the next proposition.

Proposition 4.22. Consider $p^{(n+1)} = p^{(n)}sp^{(n)}$ and denote the first $p^{(n)}$ by $p^{(n),1}$ and the second $p^{(n)}$ by $p^{(n),2}$ and assume that $p^{(n+1)} = p^{(n),1}sp^{(n),2}$ occurs somewhere in $\eta$. Then, both $p^{(n),1}$ and $p^{(n),2}$ agree with blocks $p^{(n)}$ appearing in the $n$-decomposition $\eta = p^{(n)}r_1^{(n)}p^{(n)}r_2^{(n)}p^{(n)}\ldots$.

Proof. This follows by induction on $n$. The case $n = 0$ is clear. Let us assume that the statement holds for $n$ and let

$$p_{n+2} = p^{(n+1)}sp^{(n+1)} = p^{(n)}tp^{(n)}sp^{(n)}tp^{(n)}$$

with suitable letters $s$ and $t$ in \{x, y, z\}. By assumption the $p^{(n)}$ are well aligned with the $n$-decomposition.

Case 1: $s \neq t$. Then the situation is clear as both $p^{(n)}tp^{(n)}$ will become a $p^{(n+1)}$ in the $(n + 1)$-decomposition.

Case 2: $s = t$. Then the desired statement follows as there can be no more than three occurrences of $s = t$ in a row. \hfill \Box
If \( w \) is a finite word in \( W_\tau \) and \( v \) is a prefix of \( w \) and \( N \) is a natural number we define the index of the word \( w \) in \( w^Nv \) by \( \text{Ind}(w,w^Nv) = N + |v|/|w| \) and denote it by \( \text{Ind}(w,w^Nv) \). We then define the index of the word \( w \) by
\[
\text{Ind}(w) := \max\{\text{Ind}(w,w^Nv) : v \text{ prefix of } w, N \in \mathbb{N}, w^Nv \in W_\tau\}.
\]
As our subshift is minimal and aperiodic the index of every word can easily be seen to be finite.

**Theorem 4.23** (Index of \( \Omega_\tau \)). (a) For every \( w \in W_\tau \) the inequality \( \text{Ind}(w) < 4 \) holds.
(b) We have \( 4 = \sup\{\text{Ind}(w) : w \in W_\tau\} \).

**Proof.** (a) It suffices to consider occurrences of words in \( \eta \). The case of words of length 1 (i.e. single letters) is clear. Consider now a word \( w \) of length at least 2 and chose the natural number \( n \) with \( 2^n \leq |w| < 2^n+1 \). Then \( w \) must contain an occurrence of \( p_{n-1} \). This occurrence is well-aligned with the \((n-1)\) decomposition. Thus, the number of occurrences of complete \( w \) can be bound by considering the consecutive number of occurrences of \( \pi_{n-1}s \) with a suitable \( s \in \{x,y,z\} \). Now, the statement follows rather directly.

(b) By part (a) the supremum in question is clearly bounded above by 4. Thus, it remains to bound it below by 4. Now, we have already seen above that the word \( u = axaxaxa = w^3v \) with \( w = ax \) and \( v = a \) occurs in \( \eta \). Thus, all words of the form \( \tau^n(w^3v) \) will occur in \( \eta \) as well. This easily gives the desired statement. \( \square \)

**Remark 4.24.** In terms of information the derived sequence is as useful as the original sequence. The above considerations are based on the \( n \)-decompositions and a study of relatively simple properties of the derived sequence. It is clear that much more information can be obtained from a more detailed study of the derived sequences.

### 4.5. Palindromes and reflection symmetry in the set of words \( W_\tau \) associated to \( \tau \).

In this section we study palindromes and a corresponding reflection symmetry.

For a finite word \( w = w_1 \ldots w_n \) we define the reflected word \( w^R \) by \( w^R = w_n \ldots w_1 \). There are two different ways to extend this operation to double sided infinite words. One way is to associate to a double sided infinite sequence \( \omega \) the sequence \( \omega^R \) defined by \( \omega^R(n) := \omega(-n) \). Thus, the operation \( R \) on the double sided infinite words is just the reflection at the origin. The other way, which in some sense will be even more relevant for us, is to associate to a double sided infinite sequence \( \omega \) the sequence \( \tilde{\omega} \) given by \( \tilde{\omega}(n) := \omega(1-n) \). Thus, for \( \omega = \ldots \omega_{-3}\omega_0\omega_1\omega_2 \ldots \) we have
\[
\tilde{\omega} = \ldots \omega_2\omega_1\omega_0\omega_{-1} \ldots
\]
In this sense, \( \tilde{\omega} \) is just the reflection at \( 1 \). Of course, both reflections are related. In fact, the formula
\[
\tilde{\omega} = T^{-1}\omega^R
\]
holds. We now turn to studying how \( \Omega_\tau \) is compatible with the reflection operations \( R \) and \( \tilde{\omega} \).

**Proposition 4.25.** For any \( \omega \in \Omega_\tau \) the element \( \tilde{\omega} \) belongs to \( \Omega_\tau \) as well and the map
\[
\Omega_\tau \rightarrow \Omega_\tau, \ \omega \mapsto \tilde{\omega}
\]
is a homeomorphism without any fixed point.
Proof. As any $p^{(n)}$ is a palindrome, the set $\mathcal{W}_\tau$ is invariant under $R$, i.e. $R(\mathcal{W}_\tau) = \mathcal{W}_\tau$. This easily gives the first statement. As exactly one of the letters $\omega_0$ and $\omega_1$ is an $a$, the map $\tau$ cannot have a fixed point. 

By the previous proposition, the map $\tau$ does not have fixed points on $\Omega_\tau$. However, there are words which are fixed points of $R$. We say that $\omega$ is symmetric around 0. Indeed, the only symmetric words are the translates of those words as shown in the next theorem.

**Theorem 4.26.** Let $\omega \in \Omega_\tau$ be symmetric around $p \in \mathbb{Z}$. Then, there exists $s \in \{x,y,z\}$ with $\omega = T^p\omega^{(s)}$.

Proof. Without loss of generality we can assume $p = 0$. Consider the word $\omega_{-1}\omega_0\omega_1$ and note that $\omega_{-1} = \omega_1$ must hold by symmetry. Assume that $\omega_0 = a$ holds. As $a$’s occur isolated, we infer that $\omega_{-1} = \omega_1$ can then not be an $a$. As - after deletion of the $a$’s - the letters $y, z$ occur isolated and $x$ occurs either isolated or with power 3 we infer a contradiction to $\omega_0 = a$. Set $s := \omega_0 \in \{x,y,z\}$.

We now consider the $n$-decomposition of $\omega$. We say that the $n$-decomposition has a break point at the origin if the $n$-decomposition around the origin looks like $p^{(n)}|p^{(n)}$ with $|$ denoting the position of origin. By what we have shown above the $n$-decomposition has a break point at the origin for $n = 0$. Proposition 4.20 (and it subsequent remark) give that the occurrences of the $x,y,z$ in the $n$-decomposition of $\omega$ underly analogue restrictions as the occurrences of $x,y,z$ in the derived sequence of $n$. Thus, we can repeat the argument given in the case $n = 0$ for arbitrary $n$. An easy induction then shows that $\omega$ must have a break point at the origin for any $n$. This gives the desired statement.

**Corollary 4.27.** If $\omega \in \Omega$ is not equal to $T^p\omega^{(s)}$ for all $s \in \{x,y,z\}$ and $p \in \mathbb{Z}$ then the orbits $\{T^n\omega : n \in \mathbb{Z}\}$ and $\{T^n\omega : n \in \mathbb{Z}\}$ are disjoint.

Proof. It is not hard to see that the two orbits in question can only intersect if $\omega$ is symmetric. Thus, the statement of the corollary follows from the previous theorem.

5. **Spectral theory of Schroedinger operators associated to $\Omega_\tau$**

We consider the subshift $(\Omega_\tau, T)$. In order to define the Schroedinger operators we will define specific functions $f, g$ on $\Omega_\tau$ depending on four real parameters $t,u,v,w$. Given these parameters we set 

$$D := u + v + w$$

and define

$$f : \Omega_\tau \rightarrow \mathbb{R} \text{ by } f(\omega) := \begin{cases} t & : \omega_0 = a \\ D - w & : \omega_0 = x \\ D - v & : \omega_0 = y \\ D - u & : \omega_0 = z \end{cases}$$

and

$$g : \Omega_\tau \rightarrow \mathbb{R} \text{ by } g(\omega) := \begin{cases} w : \omega_0\omega_1 \in \{ax, xa\} \\ v : \omega_0\omega_1 \in \{ay, ya\} \\ u : \omega_0\omega_1 \in \{az, za\} \end{cases}$$
We will also need to consider the set 

\[ \mathcal{P} := \{(t, u, v, w) \in \mathbb{R}^4 : t \neq 0, u + v \neq 0, u + w \neq 0, v + w \neq 0\}. \]

**Proposition 5.1.** Let \((t, u, v, w) \in \mathbb{R}^4\) and \(f, g\) as above be given. Then, \((f, g): \Omega_T \to \mathbb{R}^2\) is locally constant. Moreover, the following holds:

(a) If \(u = v = w\), then \((f, g)\) is periodic (with period 1 if \(u = \frac{1}{2}t\) and with period 2 otherwise).

(b) If \(u = v = w\) does not hold, then \((f, g)\) is not periodic.

(c) The function \(f\) does not vanish anywhere if and only if \((t, u, v, w)\) belongs to \(\mathcal{P}\).

**Proof.** The first statement is clear from the definition. Similarly, (a) is immediate from the definition. To show the aperiodicity in (b) it suffices to show that \(f\) is not periodic. This is a rather direct consequence of Proposition 4.6 applied to 

\[ C : A \to \{t, u, v, w\} \]

\[ C(a) = t, C(x) = u, C(y) = v, C(z) = w. \] Finally, (c) is clear from the definition. \qed

**Remark 5.2.** It is not hard to see that the function \(g\) is also non-periodic if \(u = v = w\) does not hold. We leave the details to the reader.

We now come to the main result on the Schrödinger operators associated to \((\Omega_T, T)\).

**Theorem 5.3 (Intervals vs Cantor spectrum).** Let \((t, u, v, w) \in \mathcal{P}\) and \(f, g\) be as above and let \(\Sigma\) be the spectrum of the associated family of Schrödinger operators. Then, the following holds:

(a) If \(u = v = w\) holds then \(\Sigma\) consists of one or two closed non-trivial intervals and all spectral measures are absolutely continuous.

(b) If \(u = v = w\) does not hold then \(\Sigma\) is a Cantor set of Lebesgue measure zero and no spectral measure is absolutely continuous.

**Proof.** (a) This follows from the previous proposition and Theorem 3.7.

(b) By Theorem 4.4 and the previous proposition the assumptions of Theorem 3.9 are satisfied. That theorem and its Corollary 3.11 imply the statement. \qed

**Remark 5.4.** In a sense made precise in the next sections, the periodic case has already been treated in Bartholdi / Grigorchuk \[8\] and this work provides a concrete description of the spectrum.

For special \(\omega \in \Omega_T\) we can even infer more. Recall that \(\lambda\) is the unique \(T\)-invariant probability measure on \(\Omega_T\).

**Theorem 5.5 (Singular continuous spectrum).** Let \((t, u, v, w) \in \mathcal{P}\) and \(f, g\) be as above and assume that \(u = v = w\) does not hold. Let \((H_\omega)\) be the associated family of Schrödinger operators.

(a) For \(\omega = \omega^{(s)}\) with \(s \in \{x, y, z\}\) the operator \(H_\omega\) does not have eigenvalues. In particular, that operator has purely singular continuous spectrum supported on a Cantor set of Lebesgue measure zero.

(b) For \(\lambda\)-almost every \(\omega \in \Omega_T\) the operator \(H_\omega\) does not have eigenvalues. In particular, for almost-every \(\omega\) that operator has purely singular continuous spectrum supported on a Cantor set of Lebesgue measure zero.
Proof. Given the previous theorem, it suffices to show the statements on absence of eigenvalues. Now, (a) follows by combining Corollary 3.13 with Lemma 4.18 and (b) follows by combining Corollary 3.13 with Lemma 4.19.

□

Remark 5.6. The considerations of this section are concerned with the case \((t, u, v, w) \in \mathcal{P}\). So, one may wonder what happens for \((t, u, v, w) \notin \mathcal{P}\). Now, it is not hard to see that for such values of the parameters the operators \(H_\omega\) can be decomposed as a direct sum of finitely many different finite dimensional operators each appearing with infinite multiplicity. Thus, in this case the spectrum is pure point with finitely many eigenvalues each with infinite multiplicity.

6. Connecting the dynamical system of Schreier graphs of \((X, G)\) with \((\Omega_\tau, T)\)

In this section we will link the Schreier graphs of Grigorchuk’s group \(G\) and the subshift \(\Omega_\tau\) in a precise way. More specifically, we will show that \(\Omega_\tau\) admits a continuous action of \(G\) and that the dynamical system of Schreier graphs \((X, G)\) introduced in Subsection 2.2 is a factor of \((\Omega_\tau, G)\) under an rather simple map. This seems an interesting result in itself. In our context, it will allow us to ‘translate’ the results on Schrödinger operators obtained in the last section into results on Laplacians on Schreier graphs.

Throughout this section we will use the alphabet \(A = \{a, x, y, z\}\) and the alphabet \(B = \{a, b, c, d\}\). Recall that \(\mathcal{G}_r(B)\) denotes the metric space of rooted graphs whose edges are labeled by elements from \(B\) (compare Subsection 1.3).

6.1. The substitution \(\Theta\). In this section we present the graph version of \(\tau\).

In Section 2.2 we have seen that the action of \(G\) on the \(n\)-th level of the binary tree gives rise to the \(n\)-th level Schreier graphs \(\Gamma_n\), \(n \in \mathbb{N}\). As discussed in [8, 41], the substitutional rules given in Figure 5 describe how to construct recursively the graph \(\Gamma_{n+1}\) from \(\Gamma_n\), starting from the Schreier graph of the first level \(\Gamma_1\). Specifically, the construction consists in replacing the labeled subgraphs of \(\Gamma_n\) on the top of Figure 5 by the new labeled graphs given on the bottom of Figure 5:

![Figure 5: The substitution \(\Theta\).](image)

To illustrate the procedure we show how the graphs \(\Gamma_1\), \(\Gamma_2\) and \(\Gamma_3\) look in Figure 6 (compare Figure 2).

The substitution rules and an easy induction directly give that for any natural number \(n\) the graph \(\Gamma_n\) has a linear structure with rightmost vertex given by \(1^n\) and this vertex ‘becomes’
SPECTRA OF SCHREIER GRAPHS OF GRIGORCHUK’S GROUP

Figure 6. The graphs $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$.

the rightmost vertex $1^{n+1}$ under the substitution. (The leftmost vertex is given by the vertex $1^k01$, $k = n - 1$, for $n \in \mathbb{N}$.)

The previous rules describing how to proceed from $\Gamma_n$ to $\Gamma_{n+1}$ suggests to study the substitution $\Theta$ acting on the set of rooted connected graph labeled with labels from $\mathcal{B}$ as follows (compare Figure 6 disregarding the notation under the vertices):

- It keeps the root.
- It replaces the edges labeled by $b$ with edges labeled by $d$, edges labeled by $c$ with edges labeled by $b$ and edges labeled by $d$ with edges labeled by $c$.
- It inserts between two vertices $v$ and $w$ connected by an edge of label $a$ two additional vertices $v_1, v_2$ as well as the following edges: edges with label $a$ from $v$ to $v_1$ and from $w$ to $v_2$; edges with label $b$ and $c$ respectively between $v_1$ and $v_2$; edges with label $d$ from $v_1$ to itself and from $v_2$ to itself.

By the very definition of this substitution and the preceding discussion we have the following result for the finite Schreier graphs. After passing to the limit in the space of rooted graphs, this result will enable us in the end to also deal with the infinite Schreier graphs $\Gamma_\xi$, $\xi \in \partial T$, in Theorem 6.5 and Corollary 6.7.

**Proposition 6.1.** For all $n \in \mathbb{N}$ the equality $\Theta((\Gamma_1^n, 1^n)) = (\Gamma_1^{n+1}, 1^{n+1})$ holds.

It is not hard to see that $\Theta$ is compatible with graph isomorphisms. Thus, $\Theta$ induces a substitution on the set $\mathcal{G}_*(\mathcal{B})$ of isomorphism classes of rooted connected graphs labeled by $\mathcal{B}$. We will denote the induced action also by $\Theta$.

6.2. The mapping $\text{Gr}$ from words to graphs. We need some notation. Let $\tau$ be the substitution considered above. Recall that we have associated to the alphabet $\mathcal{A}$ the compact space $\mathcal{W}(\mathcal{A})$ containing all finite and infinite words over $\mathcal{A}$ (see Section 1.2). We will be interested in a special subset of $\mathcal{W}(\mathcal{A})$. This is the subset $\mathcal{W}^*(\mathcal{A})$ consisting of all $\omega \in \mathcal{W}(\mathcal{A})$ satisfying the following two properties:

- 1 belongs to the support of $\omega$ (i.e. $\omega(1) \in \mathcal{A}$).
- Whenever $\omega_n\omega_{n+1}$ is not equal to $\ast\ast$ then, exactly one of the two letters $\omega_n$ and $\omega_{n+1}$ is an $a$. 


Note that the first condition is automatically satisfied for all elements of \( W(A) \) coming from \( A^*, A^N \) and \( A^Z \). The second condition means that any other letter is an \( a \). Moreover, it ensures that the words in question start and finish with the letter \( a \).

We will now provide a map from \( W'(A) \) to (isomorphism classes of) labeled rooted graphs with labels belonging to the alphabet \( B = \{a, b, c, d\} \). In order to get a better understanding of this map it will be useful to think of the letters \( x, y, z \) as encoding the pairs
\[
\begin{align*}
(b, c); &
(b, d); \\
(c, d)
\end{align*}
\]
respectively. Roughly speaking the map will replace letters in the word by graphs with two vertices connected by labeled edges according to the specifi c letter. In particular, in the case of finite words, the number of vertices of the graphs will exceed the number of letters of the word by one. Here are the details:

To \( \omega \in W'(A) \) we associate a labeled rooted graph \( \text{gr}(\omega) \) in the following way:

**Vertices:** The set of vertices is a subset of \( Z \) given by the support support(\( \omega \)) of \( \omega \) together with \( m + 1 \in Z \) if support(\( \omega \)) possesses a maximal element \( m \).

**Root:** By the very definition of \( W'(A) \) the number 1 is always a vertex and this vertex is chosen as the root.

**Edges:** There are edges between vertices \( n, k \) if an only if \( |n - k| \leq 1 \). Specifically, edges are assigned between \( n \) and \( n + 1 \) and from \( n \) to itself and from \( n + 1 \) to itself in the following way:

- If \( \omega(n) = a \), then there is an edge between \( n \) and \( n + 1 \) and this edge carries the label \( a \).
- If \( \omega(n) = x \), then there are two edges between \( n \) and \( n + 1 \); one carries the label \( b \) and the other carries the label \( c \). Moreover, there is an additional edge from \( n \) to itself labeled with \( d \) and an additional edge from \( n + 1 \) to itself labeled with \( d \).
- If \( \omega(n) = y \), then there are two edges between \( n \) and \( n + 1 \); one carries the label \( b \) and the other carries the label \( d \). Moreover, there is an additional edge from \( n \) to itself labeled with \( c \) and an additional edge from \( n + 1 \) to itself labeled with \( c \).
- If \( \omega(n) = z \), then there are two edges between \( n \) and \( n + 1 \); one carries the label \( c \) and the other carries the label \( d \). Moreover, there is an additional edge from \( n \) to itself labeled with \( b \) and an additional edge from \( n + 1 \) to itself labeled with \( b \).
- If \( n \) is the minimal element of the support of \( \omega \) then there are additional three edges labeled with \( b, c, d \) from \( n \) to itself. If \( n \) is the maximal element of the support of \( \omega \) then there are additional three edges labeled with \( b, c, d \) from \( n \) to itself.

We note that in this way any vertex has for each label \( \{a, b, c, d\} \) exactly one edge of this color emanating from it. We also note that the arising graphs have a 'linear structure' in a natural sense. In fact, the arising graphs look like the graphs given in Figure 1.

Let now \( G_\ast = G_\ast(B) \) be the metric space of isomorphism classes of connected rooted graphs labeled with elements from \( B \). Then, \( \text{gr} \) gives rise to a map \( \text{Gr} \) from words to \( G_\ast \) by taking isomorphism classes via

\[
\text{Gr} : W'(A) \longrightarrow G_\ast, \omega \mapsto [\text{gr}(\omega)],
\]

where \([\cdot]\) denotes the isomorphism class.

**Proposition 6.2.** The map \( \text{Gr} \) is continuous.
Proof. Obviously, only local information enters the definition of $gr$ i.e. two elements of $W'(A)$ which agree on a large interval $J \subset \mathbb{Z}$ around 1 will give rise to graphs which agree on a large neighborhood around the root. Now, agreement on large balls around the root is exactly how the topology on $G_*$ is defined and the continuity statement easily follows. □

6.3. The connection: $(X, G)$ as a factor of $(\Omega_\tau, G)$. In this section we are going to connect $\tau$, $\Theta$ and $Gr$. More specifically, we will show that $\Omega_\tau$ allows for an action of $G$ by homeomorphisms and $(X, G)$ is a factor of $(\Omega_\tau, G)$.

The following easy proposition provides a connection between the substitution $\tau$ and the substitution $\Theta$. It shows that the map $Gr$ intertwines the actions of $\tau$ and $\Theta$.

**Proposition 6.3** (Gr as intertwiner between $\tau$ and $\Theta$). The substitution $\tau$ maps $A^* \cap W'(A)$ into itself and on $A^* \cap W'(A)$ the equality $Gr \circ \tau = \Theta \circ Gr$ holds.

**Proof.** This is immediate from the definitions of $Gr$, $\tau$ and $\Theta$. □

After these preparations we can now state and prove the main lemma connecting the substitution $\tau$ and the (finite) Schreier graphs of Grigorchuk’s group $G$. Recall that $F$ maps a finite or infinite word $w$ to the isomorphism class of the rooted graph $(\Gamma_w, w)$.

**Lemma 6.4** (Connecting $\Gamma$ and $\tau$: finite situation). For any $n \in \mathbb{N}$ the equality $Gr(\tau^{n-1}(a)) = [(\Gamma_{1n}, 1^n)] = F(1^n)$ holds. In particular, $Gr(\eta) = [(\Gamma_{1\infty}, 1^\infty)] = F(1^\infty)$ holds.

**Proof.** The first statement follows by an easy induction (compare Figure 5):

- $n = 1$: We have $\tau^{n-1}(a) = a$. This translates into a graph $gr(a)$ with two vertices 1 and 2 and one edge between them with label $a$ (as well as loops on both vertices with labels $b, c, d$). This graph is clearly isomorphic to $\Gamma_1$.

- $n \Rightarrow n + 1$: We can calculate

\[
Gr(\tau^{n+1}(a)) = Gr(\tau^{n}(a)) = \Theta \circ Gr(\tau^{n}(a)) = \Theta(F(1^{n+1})) = F(1^{n+2}).
\]

This shows the first statement. Given the first statement, the ‘in particular’ statement is a rather direct consequence of continuity of $Gr$ shown in Lemma 6.2 and the continuity of $F$ shown in Lemma 2.2.

\[
Gr(\eta) = Gr(\lim_{n \to \infty} \tau^n(a)) = \lim_{n \to \infty} Gr(\tau^n(a)) = \lim_{n \to \infty} F(1^{n+1}) = F(1^\infty).
\]
This finishes the proof. □

Recall that $X$ denotes the closure of $\mathcal{F}(T)$ in $G_+(\mathcal{B})$ without its isolated points (see Section 2.2). Here comes our main result on the connection of the subshift $(\Omega_\tau, T)$ and the dynamical system $(X, G)$.

**Theorem 6.5** $(X, G)$ as a factor of $(\Omega_\tau, G)$. The range of the restriction of $\text{Gr}$ to $\Omega_\tau$ is given by $X$ and the map

$$
\psi : \Omega_\tau \rightarrow X, \omega \mapsto \text{Gr}(\omega),
$$

is two-to-one. Moreover, there exists a unique action $\alpha$ of $G$ on $\Omega_\tau$ with the following properties:

- $\alpha_a(\omega) = ...\omega_0\omega_1|\omega_2...$ if $\omega_1 = a$ and $\alpha_a(\omega) = ...\omega_{-1}|\omega_0\omega_1...$ if $\omega_0 = a$.
- $\alpha_0(\omega) = ...\omega_0|\omega_1|\omega_2...$ if $\omega_1 \in \{x, y\}$, $\alpha_b(\omega) = ...\omega_{-1}|\omega_0\omega_1...$ if $\omega_0 \in \{x, y\}$ and $\alpha_b(\omega) = \omega$ in all other cases.
- $\alpha_c(\omega) = ...\omega_0\omega_1|\omega_2...$ if $\omega_1 \in \{x, z\}$, $\alpha_c(\omega) = ...\omega_{-1}|\omega_0\omega_1...$ if $\omega_0 \in \{x, z\}$ and $\omega_c(\omega) = \omega$ in all other cases.
- $\alpha_d(\omega) = ...\omega_0\omega_1|\omega_2...$ if $\omega_1 \in \{y, z\}$, $\alpha_c(\omega) = ...\omega_{-1}|\omega_0\omega_1...$ if $\omega_0 \in \{y, z\}$ and $\omega_d(\omega) = \omega$ in all other cases.

This action makes $\psi$ into a factor map. For any $\omega \in \Omega_\tau$ the orbits $\{T^n\omega : n \in \mathbb{Z}\}$ and $\{\alpha_g(\omega) : g \in G\}$ agree. Finally, the unique $T$-invariant measure on $\Omega_\tau$ is $G$-invariant.

**Remark 6.6.** A recent article of Nicolás Matte Bonn [15] shows that the group $G$ (and other groups of intermediate growth introduced by the first author in [39]) embed into the topological full group $[[\phi]]$ of a minimal subshift $\phi$ over a finite alphabet. The fact that the group $G$ embeds into the topological full group of a minimal subshift over a finite alphabet also follows from our construction. In fact $G$ embeds into the topological full group of $(\Omega_\tau, \tau)$, as the action of generators $a, b, c, d$ on $\Omega_\tau$ can be represented as the action by $\tau^{\pm 1}$, so $G$ embeds into $[[\tau]]$.

**Proof.** We first show the statement on the range of the restriction of $\text{Gr}$ to $\Omega_\tau$. Recall that we consider finite words over $A$ as elements of the larger compact set $\mathcal{W}(A)$ and that this even allows us to also shift finite words by $T$ (compare Subsection 1.4). We define now

$$
\Omega_1 := \{T^k \tau^n(a) : n \in \mathbb{N} \cup \{0\}, 0 \leq k \leq 2^n - 1\} \quad \text{and} \quad \Omega_2 := \{T^n \eta : n \in \mathbb{N} \cup \{0\}\}.
$$

In the subsequent part of the proof we will make use of the reflection $\tilde{\tau}$ on the set $\Omega_\tau$, which was defined and thoroughly discussed in Section 1.5. It maps $\omega = ...\omega_{-1}|\omega_0|\omega_1|\omega_2...$ to $\tilde{\omega} = ...\omega_2|\omega_1|\omega_0|\omega_{-1}...$. It will be most relevant for us as the map $\text{Gr}$ is clearly invariant under it i.e. satisfies $\text{Gr}(\omega) = \text{Gr}(\tilde{\omega})$. In the context of the present proof it will then be useful for us to also have a reflected version of the fixed point $\eta$ at our disposal. To this end we define $\tilde{\eta} : \{..., -2, -1, 0, 1\} \rightarrow A, \tilde{\eta}(n) = \eta(2 - n)$. The basic idea behind this definition is that $\tilde{\eta}$ is a reflected version of $\eta$ which is additionally shifted so that 1 belongs to its support. This latter property is needed as we can only associate rooted graphs to words having 1 in their support.

**Claim 1.** The sets $\overline{\Omega_1}$ and $\overline{\Omega_2}$ are compact and the equalities

$$
\overline{\Omega_1} = \Omega_\tau \cup \{T^n \eta : n \geq 0\} \cup \{T^{-n} \tilde{\eta} : n \geq 0\} \cup \Omega_1 \quad \text{and} \quad \overline{\Omega_2} = \Omega_\tau \cup \Omega_2
$$

hold, where the unions are disjoint.
Proof of the claim. The sets in question are compact as they are closed subsets of the compact $W(A)$. It is clear that the unions are disjoint. The equalities follow easily from the repetitivity features of the $p_n$ and $\eta$.

Claim 2. The equalities

$$\text{Gr}(\Omega_1) = \text{Gr}(\Omega_\tau) \cup \text{Gr}(\Omega_1) \cup \text{Gr}(\Omega_2) \quad \text{and} \quad \text{Gr}(\Omega_2) = \text{Gr}(\Omega_\tau) \cup \text{Gr}(\Omega_2)$$

hold, where the unions are disjoint.

Proof of claim. We only show the statement for $\Omega_1$. The statement for $\Omega_2$ can be shown similarly (and even easier). As $\text{Gr}$ is continuous and $\Omega_1$ is compact, we have

$$\text{Gr}(\Omega_1) = \text{Gr}(\Omega_1).$$

Now, the desired equality follows from Claim 1 and the fact that the graphs associated to $\eta$ and to $\tilde{\eta}$ agree. Disjointness of the sets in question is clear.

After these preparations we can now prove the desired statement on the range. By the continuity property of the map $\mathcal{F}$ given in Lemma 2.2 we clearly have

$$\mathcal{F}(\partial T) \subset \{\mathcal{F}(x) : x \in \{0, 1\}^*\}.$$

Moreover, the previous lemma implies that for any $x \in \{0, 1\}^*$ with $|x| \geq 1$ there exists a $k \in \{0, \ldots, 2^{|x|} - 1\}$ with

$$\mathcal{F}(x) = \text{Gr}(T^k(T^{1\cdot|x|}^{-1}(a)))$$

(as the graph underlying $\mathcal{F}(x)$ is exactly $\Gamma_1|_x$ and the only choice left is the root). Putting this together we infer

$$\mathcal{F}(\partial T) \subset \text{Gr}(\Omega_1).$$

By Claim 2, this implies

$$\mathcal{F}(\partial T) \subset \text{Gr}(\Omega_\tau) \cup \text{Gr}(\Omega_1) \cup \text{Gr}(\Omega_2).$$

Clearly, the elements of $\text{Gr}(\Omega_1) \cup \text{Gr}(\Omega_2)$ are isolated point of the right hand side. This easily implies

$$X \subset \text{Gr}(\Omega_\tau).$$

Conversely, we obviously have

$$\text{Gr}(\eta) = \mathcal{F}(1^\infty) \in \mathcal{F}(\partial T),$$

where we used the previous lemma to obtain the first equality. The $G$-invariance of the boundary of the tree, then gives

$$\text{Gr}(\Omega_2) = \text{Gr}(|T^n\eta : n \geq 0|) \subset \mathcal{F}(G \cdot 1^\infty) \subset \mathcal{F}(\partial T).$$

By Claim 2 this implies

$$\text{Gr}(\Omega_\tau) \subset \text{Gr}(\Omega_2) \subset \mathcal{F}(\partial T).$$

Now, clearly the points in $\text{Gr}(\Omega_\tau)$ are not isolated, as $(\Omega, T)$ is minimal (and hence any point is approximated arbitrarily well by the orbit of any other point). This implies

$$\text{Gr}(\Omega_\tau) \subset X.$$

Put together, these considerations give

$$X = \text{Gr}(\Omega_\tau).$$
We next show that the map \( \psi \) is two-to-one: Let \( x \in X \) be arbitrary. As we have already shown, \( X \) equals \( \text{Gr}(\Omega_\tau) \). Thus, there exists an \( \omega \in \Omega_\tau \)

\[
\omega = \ldots \omega_{-1} \omega_0 | \omega_1 \omega_2 \ldots
\]

with \( \text{Gr}(\omega) = x \). Then, \( \tilde{\omega} \in \{a, x, y, z\} \mathbb{Z} \) with

\[
\tilde{\omega} = \ldots \omega_2 \omega_1 | \omega_0 \omega_{-1} \ldots
\]

belongs to \( \Omega_\tau \) by Proposition \ref{prop:grOmega} and clearly satisfies \( \text{Gr}(\tilde{\omega}) = x \) as well by the definition of \( \text{Gr} \). As there is exactly one \( a \) among \( \omega_0 \omega_1 \) the two sequences \( \omega \) and \( \tilde{\omega} \) are different. This shows that any \( x \in X \) has at least two inverse images under \( \text{Gr} \). Conversely, the Schreier graph \( \text{Gr}(\omega) \) clearly determines the sequence \( \omega \) up to one overall reflection given by \( \tilde{\omega} \) and the statement follows.

We now turn to the statement on the action of \( G \): Uniqueness of a \( G \) action with the desired properties is clear as \( \{a, b, c, d\} \) generates \( G \). By construction, and as every other letter of any \( \omega \in \Omega_\tau \) is an \( a \), the map \( \alpha_a \) is an involution and

\[
(*) \quad \text{Gr}(\alpha_a \omega) = s \text{Gr}(\omega)
\]

holds for every \( s \in \{a, b, c, d\} \). Consider now the group \( H \) generated by \( \{\alpha_a, \alpha_b, \alpha_c, \alpha_d\} \). Then any element of \( H \) can be written in the form

\[
\alpha_v := \alpha_{v_1} \circ \ldots \circ \alpha_{v_n}
\]

with \( v_1 \ldots v_n \in \{a, b, c, d\}^n \) for some \( n \in \mathbb{N} \cup \{0\} \). Moreover, by \( (*) \)

\[
\text{Gr}(\alpha_v \omega) = v_1 \cdot \ldots \cdot v_n \text{Gr}(\omega)
\]

holds.

**Claim 3.** For any \( \omega \in \Omega_\tau \) with \( T^p \omega \neq \omega^{(s)} \) for all \( s \in \{x, y, z\} \) and \( p \in \mathbb{Z} \) we have

\[
\text{Gr}^{-1}(v \text{Gr}(\omega)) \cap \{T^n \omega : n \in \mathbb{Z}\} = \{\alpha_v \omega\}
\]

for any \( v = v_1 \ldots v_n \in \{a, b, c, d\}^n \).

**Proof of the claim.** We already know that the two inverse images of \( \text{Gr} \) of an element of \( x \) differ by a reflection \( \tilde{\omega} \). Now, the claim follows from Corollary \ref{cor:inverseImages}.

**Claim 4.** Let \( v = v_1 \ldots v_n \in \{a, b, c, d\}^n \) be given. Then, \( \alpha_v = id \iff v_1 \cdot \ldots \cdot v_n = e \in G \).

**Proof of the claim.** \( \implies \): Assume \( \alpha_v = id \). By \( \text{Gr}(\alpha_v \omega) = v \text{Gr}(\omega) \) and the assumption we infer

\[
\text{Gr}(\omega) = \text{Gr}(\alpha_v \omega) = v \text{Gr}(\omega)
\]

for any \( \omega \in \Omega_\tau \). This shows that \( v \) acts as the identity on \( X \). Invoking the factor map \( \phi : X \rightarrow \partial X \) we infer that the action of \( v \) on \( \partial X \) is the identity as well. As \( G \) acts faithfully on \( \partial X \), we conclude \( v = e \in G \).

\( \impliedby \): Assume \( v = v_1 \cdot \ldots \cdot v_n = e \in G \). Then, by Claim 3 we have

\[
\{\alpha_v \omega\} = \text{Gr}^{-1}(v \text{Gr}(\omega)) \cap \{T^n \omega : n \in \mathbb{Z}\} = \{\omega\}
\]

for all \( \omega \in \Omega_\tau \) with \( T^p \omega \neq \omega^{(s)} \) for all \( s \in \{x, y, z\} \) and \( p \in \mathbb{Z} \). This, shows \( \alpha_v = id \) on a dense set in \( \Omega_\tau \) and \( \alpha_v = id \) follows.

The equality \( G = H \) follows now easily from Claim 4.
We finally turn to the statement on the measure: This statement follows easily by considering cylinder sets around the origin i.e. sets of the form
\[ \{ \omega \in \Omega : \omega(m) \ldots \omega(m + |v| - 1) = v \} \]
for \( v \in \mathcal{W}_T \) and \( m \in \mathbb{Z} \) with \( m < 0 \) and \( m + |v| - 1 > 0 \), and noting that the generators of \( G \) act on such sets either as identity or as a \( T \) or as \( T^{-1} \).

\[ \square \]

As a consequence we obtain the following result.

**Corollary 6.7.** Let \( \xi \in \partial T \setminus G \cdot 1^\infty \) be arbitrary. Then, there exists an \( \omega \) in \( \Omega_T \) with \( \text{Gr}(\omega) = F(\xi) \).

**Proof.** This follows directly from the previous result and the results of Vorobets discussed in Section 2.2. \( \square \)

### 7. Spectral theory of the Laplacians associated to the Schreier graphs

In this section we will combine the results of the previous two sections in order to describe the spectral properties of the Laplacians \( M_\xi, \xi \in \partial T \), introduced in Section 2.3.

For a given \( (t,u,v,w) \in \mathbb{R}^4 \) we chose \( f,g \) as in Section 3 and let \( H_\omega, \omega \in \Omega_T \) be the associated operators.

**Proposition 7.1.** Let \( (t,u,v,w) \in \mathbb{R}^4 \) be given. Let \( \xi \in \partial T \setminus G \cdot 1^\infty \) be arbitrary. Then, there exists an \( \omega \) in \( \Omega_T \) such that \( H_\omega \) is unitarily equivalent to \( M_\xi(t,u,v,w) \).

**Proof.** By Corollary 6.7 there exists an \( \omega \in \Omega_T \) with \( \text{Gr}(\omega) = F(\xi) \). Now, consider the representative of \( \text{Gr}(\omega) \) given by \( \text{gr}(\omega) \) and the representative of \( F(\xi) \) given by \( (I_\xi, \xi) \). Note that \( \text{gr}(\omega) \) has vertex set given by \( \mathbb{Z} \). Choose a graph isomorphism \( \beta \) between \( \text{gr}(\omega) \) and \( (I_\xi, \xi) \). Then, this graph isomorphism induces a unitary map between \( \ell^2(\mathbb{Z}) \) and \( \ell^2(V(I_\xi)) \). A straightforward calculation shows that this unitary map establishes the desired unitary equivalence between \( H_\omega \) and \( M_\xi \).

\[ \square \]

**Remark 7.2.** The proposition deals with the Schreier graphs \( I_\xi, \xi \in \partial T \setminus G \cdot 1^\infty \). The remaining Schreier graphs in \( X \) belong to \( \phi^{-1}(G \cdot 1^\infty) \). A variant of the proof of the proposition shows that these Schreier graphs arise as images of \( \text{Gr}(T^n \omega(s)) \) for \( s \in \{x,y,z\} \). Thus, the corresponding Laplacians are also encoded by operators of the form \( H_\omega \).

As a consequence of the previous proposition we can translate the results of Section 5 as follows. Recall that the spectrum of the \( M_\xi(t,u,v,w) \) does not depend on \( \xi \in \partial T \) (due to Theorem 2.5). Recall also the definition of the set
\[ P := \{(t,u,v,w) \in \mathbb{R}^4 : t \neq 0, u + v \neq 0, u + w \neq 0, v + w \neq 0 \}. \]

**Theorem 7.3** (Intervals vs Cantor spectrum for the \( M_\xi \)). Let \( (t,u,v,w) \in P \) be given and let \( \Sigma = \Sigma(t,u,v,w) \) be the spectrum of the associated family of Laplacians \( M_\xi(t,u,v,w) \), \( \xi \in \partial T \setminus G \cdot 1^\infty \). Then, the following holds:

(a) If \( u = v = w \) holds then \( \Sigma \) consists of one or two closed non-trivial intervals and all spectral measures are absolutely continuous.

(b) If \( u = v = w \) does not hold then \( \Sigma \) is a Cantor set of Lebesgue measure zero and no spectral measure is absolutely continuous.
Proof. This follows from the previous proposition and Theorem 5.3.

Remark 7.4.
- The case \( u = v = w \) has already be treated in [8] and an explicit description of the spectrum (in terms of the value of \( u \)) has been given there.
- As noted just before the theorem, the spectrum of the \( M_\xi(t,u,v,w) \) does not depend on \( \xi \in \partial T \) (due to Theorem 2.5). Thus, the set \( \Sigma \) of the previous theorem is also the spectrum of the \( M_\xi(t,u,v,w) \) for \( \xi \in G \cdot 1^\infty \).
- The operators associated to the Schreier graphs in \( \phi^{-1}(G \cdot 1^\infty) \) can be easily be seen to have no point spectrum (by the remark after Proposition 7.1 and Theorem 5.5).

We can also translate the result on absence of eigenvalues for the \((H_\omega)\). Recall that \( \mu \) denotes the uniform \( \{1/2,1/2\} \) Bernoulli measure on \( \partial T = \{0,1\}^\mathbb{N} \).

Theorem 7.5 (Almost sure absence of eigenvalues). Let \((t,u,v,w) \in \mathcal{P}\) be given and assume that \( u = v = w \) does not hold. Then, for \( \mu \)-almost every \( \xi \in \partial T \) the operator \( M_\xi \) does not have eigenvalues.

Proof. By Theorem 5.5 together with Theorem 6.3 we can infer absence of eigenvalues for the Laplacian of almost every graph \( x \in X \). By the properties of the map \( \phi \) discussed in Section 2.2 we then obtain the desired statement.

Remark 7.6. The considerations of this section are concerned with the case \((t,u,v,w) \in \mathcal{P}\). For \((t,u,v,w) \notin \mathcal{P}\) the operators in question can be decomposed as a sum of finitely many different finite dimensional operators each appearing with infinite multiplicity. Thus, the spectrum is pure point with finitely many eigenvalues each with infinite multiplicity. (Compare remark at the end of Section 5 as well.)

8. Spectral convergence: IDS and Kesten-von-Neumann-Serre spectral measure

As a by-product of our approach we can show that the Kesten-von-Neumann-Serre spectral measure arising from the spectral distributions \( \mu_n(t,u,v,w) \) (see Section 2.3) actually agrees with the integrated density of states (see Section 3).

For given \((t,u,v,w) \in \mathbb{R}^4\) we chose the functions \( f, g \) as in the previous sections and let \( H_\omega, \omega \in \Omega_\tau \), be the associated Schroedinger operators.

Theorem 8.1. Let \((t,u,v,w) \in \mathbb{R}^4\) be given. For \( n \in \mathbb{N} \), let \( M_n(t,u,v,w) \) be the corresponding operator on \( \Gamma_n \) and \( \mu_n \) its spectral distribution. Then, the sequence of measures \( (\mu_n(t,u,v,w))_n \) converges weakly towards the integrated density of states \( \kappa \) of \((H_\omega)\). In particular, the integrated density of states of the \((H_\omega)\) agrees with the Kesten-von-Neumann-Serre spectral measure.

Proof. The graph arising from restricting \( gr(\omega^{(x)}) \) to the vertex set \([1,|\tau^n(a)|]\) and the graph \( \Gamma_n \) differ at most in 6 loops at the ends (as is clear from the definitions). Thus, a simple variant of the argument in Proposition 7.1 shows that the restriction of the operator \( H_{\omega^{(x)}} \) to \([1,|\tau^n(a)|]\) is a perturbation of \( M_n(t,u,v,w) \) of rank at most 6. Now, Corollary 3.6 gives the convergence of the \( \mu_n(t,u,v,w) \) towards the integrated density of states of \((H_\omega)\).
A few comments on this result are in order. Denote the $\xi$-independent spectrum of the operators $M_\xi(t, u, v, w)$ by $\Sigma(t, u, v, w)$ and the spectrum of $M_n(t, u, v, w)$ by $\Sigma_n(t, u, v, w)$. Clearly, the support of $\mu_n$ is given by $\Sigma_n(t, u, v, w)$. By Theorem 3.3 support of the integrated density of states $k$ is given by $\Sigma(t, u, v, w)$. Thus, the previous theorem gives in a certain and very weak sense the convergence of the $\Sigma_n(t, u, v, w)$ towards $\Sigma(t, u, v, w)$. In the case at hand, however, convergence of the $\Sigma_n(t, u, v, w)$ towards $\Sigma(t, u, v, w)$ holds in a much stronger sense. More precisely, the results of [8] give the following:

- $\Sigma_n(t, u, v, w) \subset \Sigma_{n+1}(t, u, v, w)$ for all $n \in \mathbb{N}$.
- $\Sigma(t, u, v, w) = \bigcup_n \Sigma_n(t, u, v, w)$.

These are rather remarkable features and not at all true for approximation of the integrated density of states of Schrödinger operators in other cases.

The feature presented in the preceding two points also explain that the spectrum of $M_\pi(t, u, v, w)$ agrees with $\Sigma(t, u, v, w)$. The reason is simply that the representation $\pi$ decomposes as a sum of the representations $\pi_n$ (as shown in [8]). Accordingly, $M_\pi(t, u, v, w)$ is a sum of the finite dimensional operators $M_n(t, u, v, w)$, $n \in \mathbb{N}$, and its spectrum is then given as the closure of the union of the spectra of the $M_n(t, u, v, w)$.

9. Outlook

The considerations above suggest various further alleys of research. Some of them will be discussed here.

Absence of eigenvalues in the remaining cases. Our results above show absence of eigenvalues for 'most' operators $M_\xi(t, u, v, w)$, $\xi \in \partial T$, and $(H_\omega), \omega \in \Omega_T$ as well as for a few particularly interesting special cases. It is an open question whether this absence of eigenvalues actually holds for all values of the parameters $\xi$ and $\omega$.

Other Grigorchuk’s groups. In [39] an uncountable family of 4-generated groups of automorphisms of the rooted binary tree was constructed, indexed by one-sided infinite sequences in the alphabet $\{0, 1, 2\}$, and it was shown that most of them have intermediate growth. The group $G$ is the first example in this family, corresponding to the sequence $(012)^\infty$. While the group $G$ is widely studied, less is known about other groups in the family. It would be interesting to study the dynamical properties of the action of these groups on the space of their Schreier graphs and to see whether considerations similar to the ones here can be carried out.

Other self-similar groups. The Grigorchuk group $G$ studied above is a member of the class of self-similar groups. In many cases, self-similarity of a group action on a regular rooted tree leads to a finite collection of rules that allow one to construct inductively the Schreier graphs $\Gamma_n$ for the action on the levels of the tree. Thus, it seems plausible that considerations similar to ours can be carried out for other self-similar groups. This is especially interesting in relation with the spectral problem of Schreier graphs of other 'important' self-similar groups like the Basilica group or the Hanoi Tower group where only partial results exist so far. In particular, nothing is known in the anisotropic case which hopefully could also be treated by reduction to Schroedinger type operators associated with aperiodic order.
Finer spectral properties. There is quite some machinery available nowadays to study finer Hausdorff properties of spectra of Schrödinger operators associated with aperiodic order. This machinery is (mostly) based on showing that the spectrum is dynamically generated by some maps. These maps arise via traces of periodic approximants and are called ‘trace maps’. It is tempting to think that this can be applied to our example. In fact, this may tie in well with the way how the Schreier graphs are generated via approximation of the \( \Gamma_n \).

Exploration of spectra via the map \( F \). Our approach to the spectral properties of the Laplacian is very different from the approach given in [8]. That approach works via a map \( F \) encoding a self-similarity structure of the spectrum, and the spectrum is then seen as a kind of fixed point of \( F \). Specifically, the operator \( M \) in extended to a multi-parameter family of operators \( \{ M_p \} \), with the parameter \( p \) taken from \( \mathbb{R}^d \), in such a way that there is a rational mapping

\[
F : \mathbb{R}^{d+1} \longrightarrow \mathbb{R}^{d+1},
\]

for which the simultaneous spectrum

\[
\Sigma = \{(p, E) : M_p - E \cdot I \text{ not invertible}\}
\]

is \( F \) invariant i.e. satisfies \( F^{-1}(\Sigma) = \Sigma \). (Here, \( I \) is the identity operator.) The desired spectrum, \( \sigma(M) \), of \( M \) is given by the intersection \( \ell \cap \Delta \), where \( \ell \in \mathbb{R}^{d+1} \) is a line determined by the \( p_0 \) with \( M_{p_0} = M \) and \( \Delta \) is a certain \( F \)-invariant set that has to be identified among the set family of all \( F \)-invariant subsets. It will be very interesting to see whether the knowledge of spectral properties gained in the present paper can help to further explore the approach to the spectrum via the map \( F \). Also, it seems not unreasonable that there is a connection between the map \( F \) and the approach to spectral properties via trace maps mentioned in the previous point.

We plan to work on these issues in the future.

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