Lévy distributions for one-dimensional analysis of the Bose–Einstein correlations

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A general study of relations between the parameters of two centrally-symmetric Lévy distributions, often used for one-dimensional investigation of Bose–Einstein correlations, is given for the first time. These relations of the strength of correlations and of the radius of the emission region take into account possible various finite ranges of the Lorentz invariant four-momentum difference for two centrally-symmetric Lévy distributions. In particular, special cases of the relations are investigated for Cauchy and normal (Gaussian) distributions. The mathematical formalism is verified using the recent measurements given a generalized centrally-symmetric Lévy distribution is used. The reasonable agreement is observed between estimations and experimental results for all available types of strong interaction processes and collision energies.

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I. INTRODUCTION

Correlations between two identical bosons, called Bose–Einstein correlations (BEC), are a well-known phenomenon in high-energy and nuclear physics. These correlations play an important role in the studies of multiparticle production and soft physics. Constructive interference affects the joint probability for the emission of a pair of identical bosons with four-momenta $p_1$ and $p_2$. Experimentally, the one-dimensional BEC effect is observed as an enhancement at low values of the Lorentz invariant quantity $q = \sqrt{-(p_1 - p_2)^2} \geq 0$ in the two-particle correlation function (CF),

$$C_2(q) = \rho(q)/\rho_{\text{ref}}(q).$$

Here the $\rho$ is the two-particle density function, $\rho_{\text{ref}}$ is a reference two-particle density function that by construction is expected to include no BEC. The detailed shape analysis of the peak of CF is an important topic on theoretical and experimental points of view because this shape carries information about the possible features of space-time structure of particle source [1, 2]. For instance, the detail investigations have to do for shape of correlation peak in modern experiments with high statistics for verification of hypothesis of possible self-affine fractal-like geometry of emission region [3, 4]. The BEC effect in one dimension is usually described by a few-parameter function for which several different functional forms have been proposed. The power-law parametrization $C_2(q) \sim q^{-\beta}$ is the important signature for fractal-like source extending over a large volume [5, 6]. The quite reasonable fit is achieved with this parametrization of two-pion CF in various types of multiparticle production processes [1]. But unfortunately power-law fits are absent for high-statistics modern experimental data so far [7]. On the other hand the stable (on Lévy) distributions [8] are one of the most promising tools for studies of fractal-like space-time extent of emission region. These distributions are a rich class of probability distributions that allow skewness and heavy tails and have many important physical applications. As shown in [3, 4] the subclass of non-isotropic centrally-symmetric Lévy distributions is most useful for studies of Bose–Einstein CF. Therefore this subclass of centrally-symmetric Lévy distributions is considered regarding of BEC measurements in the present paper.

For low-dimensional (1D) analysis the centrally-symmetric Lévy distribution results in the most general parametrization of the experimental Bose–Einstein CF

$$C_2(q) \propto 1 + \Omega(\alpha, \lambda, z), \quad \Omega(\alpha, \lambda, z) \equiv \lambda \exp(-|z|^\alpha), \quad z \equiv qR. \quad (1)$$

Here $\lambda$ is the strength of correlations called also chaoticity, $R$ is the 1D BEC radius, $0 < \alpha \leq 2$ is the Lévy index called also index of stability. As known for a static source with no final state interactions [11, 12], there is the relation $C_2(q) \propto |\tilde{f}(q)|^2$, $\tilde{f}(q) = \int dx \exp(iqx)f(x)$, i.e. Bose–Einstein CF $C_2(q)$ measures the absolute value

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squared Fourier transformed source density in the coordinate space $f(x)$, $\int dx f(x) = 1$ called also coordinate-space distribution function of the particle emission points. The various experiments use the different forms of the $\Omega$ which correspond to the various hypotheses with regard of $f(x)$. For example, most of the earliest experiments with particle beams used the specific case of the $\Omega$ at $\alpha = 2$ – the Gaussian parametrization corresponded to the normal (Gaussian) distribution function $f_G(x) = (2\pi R^2_G)^{-1/2} \exp\left[-(x-x_0)^2/2R^2_G\right]$, where the Gaussian scale parameter is $R^2_G = \langle x^2 \rangle - x_0^2$, the standard deviation; then another specific case of the $\Omega$ at $\alpha = 1$ is used widely, especially, for particle (not nuclear) collisions. The equation $\Omega$ at $\alpha = 1$ is called exponential parametrization for Bose–Einstein CF $C_2(q)$ and it corresponds to the Cauchy (Lorentzian) distribution function $f_C(x) = \pi^{-1} R_C/[R^2_C + (x-x_0)^2]$ with scale parameter $R_C$. Furthermore the recent studies at the LHC demonstrate that general view of the $\Omega$ allows the reasonable description of experimental CF, particular for proton-proton $(p+p)$ collisions but for centrally-symmetric Lévy distribution with $\alpha \in (0;2)$, $\alpha \neq 1$ the corresponding source density in coordinate space $f(x)$ can be written analytically for $\alpha = 3/2,2/3,1/2,1/3$ only. It is often difficult to compare results from different experiments because of the many different data analysis methods, in particular due to various parameterizations for 1D Bose–Einstein CF $C_2(q)$. Therefore the derivation of the relations between the sets of BEC parameters for two centrally-symmetric Lévy distributions is the important task for correct comparison of the results from different experiments, creation of the global kinematic (energy, pair transverse momentum etc.) dependencies of BEC parameters and so on. Such studies are important for investigations of common features of soft-stage dynamics in various multiparticle production processes as well as for equation of state (EoS) of strongly interacting matter, in particular, search for phase transition to the quark-gluon deconfined matter. It would be noted the study of energy dependence of pion BEC parameters in heavy ion collisions was one of the main causes and drivers for hypothesis of cross-over transition from strongly-coupled quark-gluon phase to hadronic one at Relativistic Heavy Ion Collider (RHIC) energies $\sqrt{s_{NN}} \sim 100$ GeV. Furthermore some results for deconfinement in small system indicate remarkable similarity of both the bulk and the thermodynamic properties of strongly interacting matter created in high energy $p+p/\bar{p}+p$ and $A+A$ collisions. The BEC can provide new knowledge about collectivity and possible creation of droplets of quark-gluon matter in small system collisions. For these studies the correct comparison can be crucially important for BEC parameters in various multiparticle processes for wide energy range. But as mentioned above Bose–Einstein CF $C_2(q)$ is often described by different view of the $\Omega$ depending on type of reaction, collision energy and features of experimental analysis. Therefore the study of centrally-symmetric Lévy distributions and search for relations between parameters for corresponding CF has scientific interest for physics of strong interactions.

The paper is organized as follows. In Sec. III mathematical formalism is described for case of two general view centrally-symmetric Lévy distributions. Dependencies of desired 1D BEC observables on $q$ and $\alpha$ are studied for a priori known parameters for second centrally-symmetric Lévy distributions. Section IV is devoted to the detail discussion of specific case of these distributions, namely, Cauchy and Gaussian ones most used in experimental investigations of 1D CF $C_2(q)$. Database of experimental results for set of 1D BEC parameters $\{\Lambda, R\}$ for charged pion source in strong interaction processes is created within the framework of the paper in order to verify the mathematical formalism. Sec. V demonstrates the comparison between the estimations calculated for 1D BEC parameters with help of mathematical formalism under discussion and available experimental results for various reactions and in wide energy range. In Sec. VI some final remarks are presented. The experimental database is shown in the Appendix A for 1D BEC parameters.

II. RELATIONS BETWEEN BEC PARAMETERS IN GENERAL CASE

Let some experimental CF $C_2(q)$ is described by two parameterizations $\Omega$ with $\Omega_1 \equiv \Omega(\alpha_1, \lambda_1, z_1)$ and $\Omega_2 \equiv \Omega(\alpha_2, \lambda_2, z_2)$. Then relations between parameters of $\Omega_1$ and $\Omega_2$ can be deducted on the basis that both pa-

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1 In should be noted that in Sec. III and IV the 1D BEC parameters are supplied with the subindexes according with the names of corresponding source distribution function, namely, “$L$” is for the general view of centrally-symmetric Lévy distribution, “$G$” is for the Cauchy source distribution function and “$C$” – for Gaussian one. Otherwise the notations $\{\lambda_L, R_L\}$ are often used in papers for second case due to relation between Cauchy distribution for $f(x)$ and exponential parametrization for Bose–Einstein CF $C_2(q)$ discussed above. As consequence the mathematically rigorous terminology is used over full manuscript: the term “Cauchy distribution” corresponds to the source function in coordinate space $f_C(x)$ and the term “exponential function / parametrization” is used for the related parametrization of correlation function $C_{2,exp}(q)$; for the case of arbitrary $0 < \alpha < 2$, $\alpha \neq 1$ the term “centrally-symmetric Lévy” is suitable for both the source function in coordinate space $f_L(x)$ and the parametrization of correlation function $C_{2,L}(q)$; the similar situation is for $\alpha = 2$: the term “Gaussian” is applicable for both the source function in coordinate space $f_G(x)$ and the corresponding parametrization of correlation function $C_{2,G}(q)$.
rameterizations describe one experimental CF $C_2(q)$, i.e. one sample of experimental points\(^1\). Thus one can assume that the areas under fit curves for two parameterizations \(^1\) with $\Omega_1$ and $\Omega_2$ are approximately equal to each other as well as the first moments of the corresponding centrally-symmetric Lévy distributions.

A. Mathematical formalism

The relations between two sets of parameters $\{\lambda_1, R_1\}$ and $\{\lambda_2, R_2\}$ of the particle source can be derived from the following system of equations:

\[
S_i = S_2, \quad \forall i = 1, 2 : S_i = \int_{J_i} \Omega(\alpha_i, \lambda_i, z_i) dq;
\]

\[
\langle q \rangle_1 = \langle q \rangle_2, \quad \forall i = 1, 2 : \langle q \rangle_i = S_i^{-1} \int_{J_i} q \Omega(\alpha_i, \lambda_i, z_i) dq.
\]

The first equation (2a) corresponds to the equality of the areas under fit curves and the (2b) is the equality of the first moments of resonance contributions etc.), i.e. $\forall$ the fit range can be the set of subranges due to possible exception of some intervals of the relative 4-momentum (regions of resonance contributions etc.), i.e. $\forall i = 1, 2 : J_i = \bigcup_{k=1}^{N_i} j^k = \bigcup_{k=1}^{N_i} [q_{k, \text{min}}, q_{k, \text{max}}]$ and consequently for all types of integrals and $\forall i = 1, 2$ in the system (2): $\int_{J_i} \rightarrow \sum_{k=1}^{N_i} \int_{j^k}$. But usually the fit ranges $J_i$ are identical for both $\Omega_i$, $i = 1, 2$ in experimental studies (see, for example, \cite{15}). In general case of the centrally-symmetric Lévy distributions and finite fit ranges the system equations under consideration can not be solved analytically. The numerical procedure should be used in order to get the relations between two sets of parameters $\{\lambda_1, R_1\}$ and $\{\lambda_2, R_2\}$ of the particle source in this case. Without loss of generality the $\Omega_1$ and $\Omega_2$ are considered as known and values of BEC parameters $\{\lambda_2, R_2\}$ are supposed as desired below. Then for specific case of semi-infinite ranges for integration $\forall i = 1, 2 : J_i = [0; \infty)$ the system (2) can be solved analytically and one can derive the following ultimate relations between two sets $\{\lambda_1, R_1\}$ and $\{\lambda_2, R_2\}$ of BEC parameters for corresponding centrally-symmetric Lévy parameterizations with $\Omega_i$, $i = 1, 2$

\[
\lambda_1^u = \lambda_2 \left[ \alpha_1 \Gamma(2\alpha_1^{-1}) \Gamma^2(\alpha_2^{-1}) \right] \left[ \alpha_2 \Gamma^2(\alpha_1^{-1}) \Gamma(2\alpha_2^{-1}) \right]^{-1};
\]

\[
R_1^u = R_2 \left[ \Gamma(2\alpha_1^{-1}) \Gamma(\alpha_2^{-1}) \right] \left[ \Gamma(\alpha_1^{-1}) \Gamma(2\alpha_2^{-1}) \right]^{-1}
\]

and vice versa. Here $\Gamma(x) = \int_0^\infty t^{x-1} \exp(-t) dt$, $\Re x > 0$ is the gamma function.

In the point of view of data analysis the absence of the general analytic relations between $\{\lambda_1, R_1\}$ and $\{\lambda_2, R_2\}$ leads to the following approach for estimations of the errors of the unknown parameters. Let without the loss of generality suppose that the values are known for set of parameters $\{\alpha_1, \lambda_1, R_1\}$ with its errors $\Delta^+\alpha_1, \Delta^+\lambda_1, \Delta^+R_1$ for centrally-symmetric Lévy parametrisation with $\Omega_1$ as well as for $\alpha_2$ with $\Delta^+\alpha_2$ for parametrization with $\Omega_2$. The two sets of values for unknown BEC parameters $\{\lambda_2, R_2\}$ can be obtained with the help of suitable system of equations: the input values $\{\alpha_1 + \Delta^+\alpha_1, \lambda_1 + \Delta^+\lambda_1, R_1 + \Delta^+R_1\}$ and $\alpha_2 + \Delta^+\alpha_2$ produce the output set $\{\lambda_2^+, R_2^+\}$ and $\{\alpha_1 - \Delta^-\alpha_1, \lambda_1 - \Delta^-\lambda_1, R_1 - \Delta^-R_1, \alpha_2 - \Delta^-\alpha_2\} \rightarrow \{\lambda_2^-, R_2^-\}$. Then the error estimations for set $\{\lambda_2, R_2\}$ of BEC parameters for parametrization with $\Omega_2$ can be calculated as follows:

\[
\Delta Y_2 = |Y_2^+ - Y_2|, Y_2 \equiv \lambda_2, R_2.
\]

One can use the errors \(^1\) which are asymmetric in general case or make the averaging of up and low uncertainties and then to use the symmetric errors $\Delta Y_2 = (\Delta^+ Y_2 + \Delta^- Y_2)/2$.

\(^1\) In general the approximations of $C_2(q)$ are characterized by different qualities for various parameterizations \(^1\) with $\Omega_1$ and $\Omega_2$. The influence of this difference is not studied in present work and can be considered as separate task.

\(^2\) As discussed above the approximate equalities are expected for areas and first moments in general case. This softer condition is enough for applicability of the formalism suggested in the paper. But the exactly equal signs are used in the \cite{2} as well as in the text below in order to get the mathematically correct forms for the systems of equations.
B. Dependencies on $q$ and $\alpha$ variables

Fig. 1 shows dependence of 1D BEC radius $(a,b)$ and strength of correlations $(c,d)$ for centrally-symmetric Lévy parametrization $\Omega_1$ with known $\alpha_1 = 1.5$ on low $q_1$ $(a,c)$ and high $q_2$ $(b,d)$ limits of integration in the system (2) for set $\{\alpha_2, \lambda_2, R_2\} = \{0.5, 0.5, 1.5 \text{ fm}\}$ for $\Omega_2$. The solid lines correspond to the indicated values of the $q_2$ in GeV/c for $q_1$-dependence $(a,c)$ and to shown values of the $q_1$ in GeV/c for $q_2$-dependence $(b,d)$. Values of BEC parameters $\lambda_1$ and $R_1$ depend strongly on the fixed second limit of integration $(q_2)$ for both the results from the system (2) and the estimations for $R$ of small values of Lévy indexes. The region $\forall (\lambda, R)$ of small values of Lévy indexes is calculated with help of (3b) and presented by dotted line for any $\alpha_1$-dependence $(Fig. 1c,d)$. The dashed lines correspond to the results from the system (2) with $q_2 \to \infty$ for $q_1$-dependence $(Fig. 1c,d)$ and with $q_1 = 0$ for $q_2$-dependence $(Fig. 1b,d)$. As seen the curves for general case of (2) coincide with dashed lines at $q_2 = 10$ GeV/c for $q_1$-dependence $(Fig. 1c,d)$ and at $q_1 = 10^{-3}$ GeV/c for $q_2$-dependence $(Fig. 1b,d)$. These values for $q_1$ and especially for $q_2$ are far from the corresponding limit in modern experimental CF. Therefore one should use the system (2) for finite limits in an integrations in the case of experimentally available $q$-ranges for two parameterizations with $\Omega_1$ and $\Omega_2$. The thin dotted lines demonstrate the ultimate levels for $R_1$ $(Fig. 1b,d)$ and $\lambda_1$ $(Fig. 1d)$ calculated with $\Omega_2$ for given values of the $\alpha_1$ and the set of parameters $\{\alpha_2, \lambda_2, R_2\}$ for second Lévy parametrization $\Omega_2$. One can use the simple relations (3) for calculation $\{\lambda_1, R_1\}$ at $q_1 \lesssim 10^{-2}$ GeV/c and $q_2 \gtrsim 10$ GeV/c $(Fig. 1b,d)$ but as expected the values of BEC parameters $\{\lambda_1, R_1\}$ are far from the ultimate levels at any $q_1$ for $q_2$-dependence in the range $q_2 \leq 2$ GeV/c is considered in $Fig. 1b,d$.

In Fig. 2 dependence of 1D BEC radius $(a,b)$ and strength of correlations $(c,d)$ is demonstrated for centrally-symmetric Lévy parametrization with $\Omega_1$ on $\alpha_1$ at fixed values of $\alpha_2$ $(a,c)$ and on $\alpha_2$ at fixed values of $\alpha_1$ $(b,d)$ for given limits of integration in the system of equations (2) $q_1 = 0.02$ GeV/c, $q_2 = 2.0$ GeV/c and for certain values of the BEC parameters for second centrally-symmetric Lévy parametrization with $\Omega_2$: $\lambda_2 = 0.5$ and $R_2 = 1.5$ fm. The solid lines correspond to the indicated values of the $\alpha_2$ for $\alpha_1$-dependence $(a,c)$ and to shown values of the $\alpha_1$ for $\alpha_2$-dependence $(b,d)$. The values for limits of integration $q_1$ and $q_2$ in the system (2) are similar to those used in modern experiments. As seen dependences of both BEC parameters on $\alpha_i$, $i = 1, 2$ at fixed another Lévy index $\alpha_j$, $j \neq i$, $j = 1, 2$ change very fast at small value $\alpha_j = 0.2$ in narrow range $\alpha_1 \approx \alpha_j$. Such behavior is observed for both the results from the system (2) and the estimations for $R_1$ $(Fig. 2b,d)$ and $\lambda_1$ $(Fig. 2d)$ calculated with (3) for semi-infinite ranges for integration and shown by dotted lines. The dependence $R_1(\alpha_1)$ shown in $Fig. 2$ closes to the analytic one calculated with help of (3b) and presented by dotted line for any $\alpha_2$ under study with exception of the small value $\alpha_2 = 0.2$. For last case the agreement is obtained in very narrow range $\alpha_1 \approx 0.2$ between results from the system (2) and equation (3b). This feature maps clearly in corresponding dependences $R_2(\alpha_2)$ shown in $Fig. 2$, for $\alpha_1 = 0.2$. For large $\alpha_1 > 1.0$ solid and dotted lines close to each other in the range $\alpha_2 > 0.5$ but agreement is poor significantly between results from the system (2) and equation (3b) for $\alpha_1 = 0.6$ especially in domain $\alpha_2 < 0.5$ $(Fig. 2)$ taking into account the sharp behavior for corresponding dependence $R_1(\alpha_2)$. In general the behavior of dependencies of $\lambda_1$ on $\alpha_1$ $(Fig. 2c)$ and $\alpha_2$ $(Fig. 2d)$ is similar to the corresponding dependences of 1D BEC radius $R_1$. But the agreement is poor usually between results from the system (2) and shown by the solid lines and the estimations calculated based on the (3b) and presented by the dotted lines. Therefore for values of limits of integration $q_1, q_2$ under consideration the approximate relations (3) should be used carefully for experimental analysis of dependencies on Lévy indexes and last equations can produce the reasonable estimations for BEC parameters for ranges $\forall i = 1, 2: \alpha_i \gtrsim 1$ only.

As seen the mathematical formalism described above as well as the results in Figs. 1, 2 are quantitative basis for choice of the applying of general equations (2) or ultimate relations (3) in data analysis for given experiment. Thus the method suggested in the paper is helpful for experimental and phenomenological studies of BEC in various processes at different parameterizations of CF $C_2(q)$ corresponded to the centrally-symmetric Lévy source distributions.

III. RELATIONS BETWEEN BEC PARAMETERS IN SPECIFIC CASES

As seen in Fig. 2 both dependencies of the 1D BEC radius $R_1$ $(a,b)$ and the strength of correlations $\lambda_1$ $(c,d)$ on Lévy indexes $\alpha_i$, $i = 1, 2$ show the weaker changing in the domain $\forall i = 1, 2: \alpha_i \gtrsim 1$ in comparison with the range of small values of Lévy indexes. The region $\forall i = 1, 2: \alpha_i \gtrsim 1$ includes in particular the specific cases of Cauchy and Gaussian distributions for which corresponding parameterizations of Bose–Einstein CF $C_2(q)$ with $\alpha = 1$ and $\alpha = 2$ are used mostly for experimental studies. Therefore these certain views of $\Omega_2$ are studied in detail below. Let $\Omega_1 = \Omega_G = \Omega(2, \lambda_G, R_G)$ for Gaussian parametrization (1) and $\Omega_2 = \Omega_C = \Omega(1, \lambda_C, R_C)$ for 1D approximation of experimental CF $C_2(q)$ by exponential function.
A. Mathematical formalism

The relations (3) are valid at any values of indexes of stability $0 < \alpha_i \leq 2$ in two centrally-symmetric Lévy parameterizations with $\Omega_i$, $i = 1, 2$. If without loss of generality the $\{\lambda_C, R_C\}$ are considered as a priori known and values of Gaussian BEC parameters $\{\lambda_G, R_G\}$ are supposed as desired then as expected the equations (3) result in the ultimate relations:

$$\lambda_G^u = 2\lambda_C / \pi; \quad (5a)$$

$$R_G^u = R_C / \sqrt{\pi}; \quad (5b)$$

and vice versa. The relation (5a) is derived in [22] for the first time while the formula (5b) for 1D BEC radii is well-known.

The following relations can be obtained from general system (2) for finite ranges of integrations and specific values $\alpha_1 = 2$ and $\alpha_2 = 1$:

$$\frac{\lambda_G \sqrt{\pi}}{2R_G} \sum_{j=1}^{N_G} [\text{erf}(z_{2,j,G}) - \text{erf}(z_{1,j,G})] = \lambda_C R_C \sum_{i=1}^{N_C} [\exp(-z_{1,i,C}) - \exp(-z_{2,i,C})]; \quad (6a)$$

$$\frac{1}{R_G \sqrt{\pi}} \frac{1}{\sum_{j=1}^{N_G} [\text{erf}(z_{2,j,G}) - \text{erf}(z_{1,j,G})]} \sum_{i=1}^{N_C} [\exp(-z_{1,i,C})(1 + z_{1,i,C}) - \exp(-z_{2,i,C})(1 + z_{2,i,C})] = \frac{1}{R_C} \frac{\sum_{i=1}^{N_C} [\exp(-z_{1,i,C}) - \exp(-z_{2,i,C})]}{\sum_{i=1}^{N_C} [\exp(-z_{1,i,C}) - \exp(-z_{2,i,C})]}. \quad (6b)$$

Here $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)dt$ is the error integral, $z_{1(2),i,C(G)} \equiv q_{1(2),i,C(G)} R_C(G)$ are the limits for integration over corresponding subranges for Cauchy (Gaussian) distribution. The detailed study of all available experimental results in strong interaction processes for 1D parametrization (1) with $\alpha_1 = 2$ and $\alpha_2 = 1$ for experimental CF $C_2(q)$ shows that (i) the ranges of integration for both the exponential and the Gaussian functions are equal; (ii) usually, the range of integration is not divided into subranges, in any case, such division is identical for both functions under consideration and maximum value of the $N_{C(G)}$ is equal 2 for experimental analyses. Thus the general statement with regard of identity of integration ranges for $\Omega_1$ and $\Omega_2$ is quite confirmed for case of Cauchy and Gaussian distributions and $N_{C(G)} \equiv N$ in the system (6).

Further simplification for the system of equations (6) depends on features of certain experiment; direction of calculations $\Omega_C \equiv \Omega_G$, i.e. what kind of a set of BEC parameters of the two, $\{\lambda_C, R_C\}$ and $\{\lambda_G, R_G\}$, it is regarded as a priori known, and which set is supposed as desired; and requirement on the accuracy level. For available experimental data for BEC of charged pion pairs produced in strong interaction processes (i) the accuracy for 1D BEC radius is better usually than that for $\lambda$ parameter and (ii) the accuracy for 1D BEC parameters in modern experiments is not better than $\sim 10^{-3}$ so far. Thus one can assume the conservative accuracy level $\varepsilon = 5 \times 10^{-4}$. At present the most complex case with $N = 2$ is for analyses of proton-proton collisions at some LHC energies only [13] [22]. For this case all contributions are negligible from the subrange of $q$ values larger than the region of the influence of meson resonances excluded from the experimental fits, i.e. all terms for $i = j = 2$ can be omitted at given $\varepsilon$ and direction of calculation from a priori known Cauchy parameters to desired Gaussian parameters $\{\lambda_C, R_C\} \rightarrow \{\lambda_G, R_G\}$. But the statement is wrong for opposite direction of calculation from a priori known Gaussian parameters to desired Cauchy parameters $\{\lambda_G, R_G\} \rightarrow \{\lambda_C, R_C\}$ at $\varepsilon = 5 \times 10^{-4}$. Therefore the sum can be omitted in the system (6) and equations can be re-written as follow:

$$\frac{\lambda_G}{R_G \sqrt{\pi}} [\text{erf}(z_{2,G}) - \text{erf}(z_{1,G})] = \frac{2\lambda_C}{\pi R_C} [\exp(-z_{1,C}) - \exp(-z_{2,C})]; \quad (7a)$$

$$\frac{1}{R_G \sqrt{\pi}} \frac{\exp(-z_{2,G}^2) - \exp(-z_{2,G}^2)}{\text{erf}(z_{2,G}) - \text{erf}(z_{1,G})} = \frac{1}{R_C} \frac{\exp(-z_{1,C})(1 + z_{1,C}) - \exp(-z_{2,C})(1 + z_{2,C})}{\exp(-z_{1,C}) - \exp(-z_{2,C})}. \quad (7b)$$

for all experiments in the case of estimation of unknown BEC parameters for Gaussian function based on the a priori known set of corresponding parameters for exponential function and for all experiments with exception of $p + p$ collisions at $\sqrt{s_{NN}} = 0.9, 2.36$ and 7 TeV [14] [22] in the case of inverse problem. Account the relation $R_G \leq R_C / \sqrt{\pi}$ and properties of the functions $\exp(-x^2), \text{erf}(x)$ allows us to simplify Eqs. (7) to the system

$$\frac{\lambda_G}{R_G \sqrt{\pi}} \text{erfc}(z_{1,G}) = \frac{2\lambda_C}{\pi R_C} [\exp(-z_{1,C}) - \exp(-z_{2,C})]; \quad (8a)$$
As expected one can get the ultimate relations (5) from the any systems of equations (9) or (10) at where \( \text{erfc}(z) \) corresponds to the range of integration \([z_{1,(c,G)}, \infty)\) with exception of the collision energy \( \varepsilon \) a priori C Plear data [25] for direction of calculation from a priori known Gaussian parameters to desired Cauchy parameters \( \{\lambda_G, R_G\} \rightarrow \{\lambda_C, R_C\} \) at \( \varepsilon = 5 \times 10^{-4} \). The simplest view of the system of equations (5)

\[
\frac{\lambda_G}{R_G\sqrt{\pi}} \text{erfc}(z_{1,G}) = \frac{1}{R_C} \frac{\exp(-z_{1,G}^2)}{\text{erfc}(z_{1,G})} = \frac{1 - \exp(-z_{1,C})(1 + z_{1,C}) - \exp(-z_{2,C})(1 + z_{2,C})}{\exp(-z_{1,C}) - \exp(-z_{2,C})},
\]

where \( \text{erfc}(x) = 1 - \text{erf}(x) \). The last system of equations is valid for remain set of experimental results with exception of the WA98 data [24] at given \( \varepsilon \). Also the transition from system (7) to simpler equations (8) is not valid for C Plear data [25] for direction of calculation from a priori known Gaussian parameters to desired Cauchy parameters \( \{\lambda_G, R_G\} \rightarrow \{\lambda_C, R_C\} \) at \( \varepsilon = 5 \times 10^{-4} \). The simplest view of the system of equations (5)

\[
\frac{\lambda_G}{R_G\sqrt{\pi}} \text{erfc}(z_{1,G}) = \frac{2\lambda_C}{\pi R_C} \exp(-z_{1,C});
\]

\[
\frac{1}{R_G\sqrt{\pi}} \frac{1 - \exp(-z_{2,G}^2)}{\text{erfc}(z_{2,G})} = \frac{1}{R_C} \frac{1 - \exp(-z_{2,C})(1 + z_{2,C})}{1 - \exp(-z_{2,C})}
\]

(9b)

Corresponds to the range of integration \([z_{1,(c,G)}, \infty)\) and can be used for experimental results from ALICE [26], CMS [13, 17, 27] with exception of the collision energy \( \sqrt{s_{NN}} = 2.36 \text{ TeV} \) [13] in the case of proton-proton collisions and WA80 [28] for asymmetric nucleus-nucleus collisions \( O + C, O + Cu \). On the other hand, the using of the range of integration \([0,0, z_{2,(c,G)}]\) allows the derivation from the Eqs. (5) the following system:

\[
\frac{\lambda_G}{R_G\sqrt{\pi}} \text{erfc}(z_{1,G}) = \frac{2\lambda_C}{\pi R_C} (1 - \exp(-z_{2,C}));
\]

(10a)

\[
\frac{1}{R_G\sqrt{\pi}} \frac{1 - \exp(-z_{2,G}^2)}{\text{erfc}(z_{2,G})} = \frac{1}{R_C} \frac{1 - \exp(-z_{2,C})(1 + z_{2,C})}{1 - \exp(-z_{2,C})}
\]

(10b)

As expected one can get the ultimate relations (10) from the any systems of equations (9) or (10) at \( q_1 \rightarrow 0 \) or \( q_2 \rightarrow \infty \) respectively. Therefore the system (9) can be replaced by ultimate system of equations (10) with some accuracy \( \varepsilon' \) for finite range of \( q \) if \( q_1 \leq q_1^b \) and \( q_2 \) value is large enough to consider this value as \( q_2 \rightarrow \infty \). Similarly, the system (10) can be replaced by ultimate system of equations (10) with some accuracy \( \varepsilon' \) for finite range of \( q \) if \( q_2 \geq q_2^b \) and \( q_1 \) is small enough to consider it as \( q_1 \rightarrow 0 \) for Eqs. (10). The high / low boundary values \( q_1^b / q_2^b \) for variables \( q_1 / q_2 \) are dominated by assigned value of accuracy. For instance, at \( \varepsilon' = 10^{-2} \) the ultimate system of equations (10) is valid for \( q_1 \lesssim q_1^b = 2 \times 10^{-3} R_{C(c,G)} \) or \( q_2 \gtrsim 1.3 R_{C(c,G)} \), i.e. \( q_1 \lesssim q_1^b = 2 - 4 \text{ MeV/c} \) or \( q_2 \gtrsim q_2^b = 1.3 - 2.6 \text{ GeV/c} \) for proton-proton collisions. The derived estimations are close to the values of \( q \) variable which used in present experimental analyses of BEC correlations.

These qualitative estimations are confirmed by quantitative analysis below for the \( q_1 \)-, \( q_2 \)-dependencies of the Gaussian parameters \( \lambda_G \) and \( R_G \) derived for some assigned values of the corresponding BEC parameters for exponential function \( \lambda_C, R_C \) and vice versa.

### B. Dependence on \( q \) for desired Cauchy / Gaussian parameters

For Fig. 3 the \( \Omega_G \) is considered as a priori known and set of BEC parameters \( \{\lambda_G, R_G\} \) are studied for Gaussian parametrization (1). The Fig. 3 shows the \( q_1 \)- and \( q_2 \)-dependence of 1D BEC radius (Fig. 3a,b) and strength of correlations (Fig. 3c,d) for parametrization (1) with Gaussian function \( \Omega_G \) at fixed values \( \lambda_C = \pi/2 \) and \( R_C \) as \( \sqrt{\pi} \). As seen the both Gaussian parameters show the similar behavior with changing the integration limits, namely \( \lambda_G \) and \( R_G \) growth with decreasing of the \( q_1, q_2 \) at fixed another limit of integration. The curves \( \lambda_G(q_1), R_G(q_1) \) approach to the asymptotic dashed lines calculated with help of the system (10) with increasing of the \( q_2 \). The similar situation is observed in Fig. 3c,d for curves \( R_G(q_2), \lambda_G(q_2) \) and asymptotic dashed lines calculated with help of the system (10) with decreasing of the \( q_1 \). As seen the asymptotic lines are achieved at \( q_1 \lesssim 1 \text{ MeV/c} \) (Fig. 3a,b,d) and \( q_2 \gtrsim 0.8 \text{ GeV/c} \) (Fig. 3c). Furthermore the ultimate values of the Gaussian BEC parameters \( \lambda_G^b \) and \( R_G^b \) are valid with good accuracy for \( q_1 < 2 \text{ MeV/c} \) and \( q_2 > 1.0 \text{ GeV/c} \). The last ranges are in the good agreement with qualitative estimations for proton-proton collisions obtained above. It should be emphasized that for specific case of exponential \( (\Omega_C) \) and Gaussian \( (\Omega_G) \) functions the asymptotic \( q_1 \)-dependence is achieved for both the 1D BEC radius (Fig. 3c) and the strength of correlations (Fig. 3a) at \( q_2 \) which is much smaller than that for case of two some another centrally-symmetric Lévy parameterizations (Fig. 3c). This \( q_1 \) value for case of \( \Omega_C \) and \( \Omega_G \) is similar to those used in analyses of experimental CF \( C_2(q) \). In Fig. 4 the dependencies of relative BEC parameters, namely \( R_C/R_G \) (a,b) and \( \lambda_C/\lambda_G \) (c,d), on \( q_1 \) (a,c) and \( q_2 \) (b,d) are presented for various assigned values of parameters for Cauchy distribution. The curves are calculated with the simpler system of equations (9) for \( q_1 \)-dependence (Figs. 4a,c) and system (10) for...
$q_2$-dependence (Fig. 3a,d) respectively. As seen the larger values of Cauchy parameters lead to the larger values of relative BEC parameters. The $q_1$-dependence of relative BEC parameters growth faster with increasing of the input values of Cauchy parameters (Fig. 3a,c). On the contrary the decrease of the $q_2$-dependence of $R_C/R_G$ (Fig. 3a) and $\lambda_C/\lambda_G$ (Fig. 3b) is slower with increasing of the input values of the $\{\lambda_C, R_C\}$. As expected the ultimate levels $R_C/R_G = \sqrt{7}$ (Fig. 3a,b) and $\lambda_C/\lambda_G = \pi/2$ (Fig. 3a,d) shown by thin dotted lines are valid for the same ranges of $q_1$ and $q_2$ as estimated above for Fig. 3.

Fig. 4 shows results for opposite direction of calculations, i.e ΩG is supposed a priori known and BEC parameters $\{\lambda_C,R_C\}$ for exponential parametrization of CF $C_2(q)$ are derived. Fig. 5 shows the $q_1$- and $q_2$-dependence of 1D BEC radius $\Omega_G$ (Fig. 5a,b) and strength of correlations (Fig. 5c,d) for exponential function $\Omega_G$ at fixed values $\lambda_G = 2/\pi$ and $R_G = 1/\sqrt{\pi}$. The $\forall i = 1, 2 : q_i$-dependencies show the opposite behavior for desired parameters of Cauchy source function $R_C$ (Fig. 5a,b) and $\lambda_C$ (Fig. 5c,d) with respect to the corresponding dependencies presented in Fig. 3 above for another direction of calculation $\{\lambda_C,R_C\} \rightarrow \{\lambda_G,R_G\}$. These differences are seen in domain of relatively large $q_1 \gtrsim 10^{-2}$ GeV/c for $q_1$-dependence and at relatively small $q_2 \lesssim 0.7$ GeV/c for $q_2$-dependence of BEC parameters. Furthermore the $q_1$-dependence for parameters from set for $\Omega_G$ (Fig. 5a,c) approaches to the constant at $q_1 \rightarrow 0$ faster noticeably than that for $R_G$ (Fig. 5a) and $\lambda_C$ (Fig. 5c). The opposite situation is observed for achievement of constants by $q_2$-dependence at $q_2 \rightarrow \infty$. It should be noted that dependencies $R_C(q_1)$ and $\lambda_C(q_1)$ approach to its asymptotic curves calculated with help of the system (9) and shown by dashed lines in Fig. 4c slower than corresponding dependencies for desired Gaussian parameters in Fig. 3a,c. As consequence the $R_C(q_1)$ and $\lambda_C(q_1)$ approach to its asymptotic curves at higher $q_2$ than that for Fig. 3a,c. The asymptotic value of $q_1 \simeq 10^{-3}$ GeV/c is the same for $q_2$-dependence for both directions of calculations $\{\lambda_C,R_C\} \rightleftharpoons \{\lambda_G,R_G\}$.

Fig. 6 demonstrates the dependence of relative BEC parameters, namely $R_C/R_G$ (a,b) and $\lambda_C/\lambda_G$ (c,d), on $q_1$ (a,c) and $q_2$ (b,d) for various assigned values of parameters for Gaussian parametrization. The smaller system of equations (9) is used for calculation of $q_1$-dependencies in Figs. 6a,c and curves on $q_2$ (Fig. 6b,d) are derived with help of the system (10). In general $\forall i = 1, 2 : q_i$-dependencies show similar behavior for corresponding relative 1D BEC parameters in both cases the Fig. 6 and the Fig. 4 with some faster changing of $q_i$-dependencies in the second case than that for the first one in domain of relatively large $q_1 \gtrsim 10^{-2}$ GeV/c for $q_1$-dependence and at relatively small $q_2 \lesssim 0.7$ GeV/c for $q_2$-dependence of $R_C/R_G$ and $\lambda_C/\lambda_G$.

Simultaneous consideration of available 1D BEC data analyses for strong interaction processes and Fig. 3 allows the assertion that ultimate relations (9) are not acceptable with reasonable accuracy for most of experimental results with exponential / Gaussian parametrization (11) of 1D CF $C_2(q)$. As seen from Fig. 3 the even the asymptotic values of relative 1D BEC parameters $\{\lambda_C/\lambda_G, R_C/R_G\}$ can differ up to several times from ultimate values calculated with help of the system of equations (9) in some domains of $q_1$ and $q_2$ variables. Therefore Figs. 3-6 confirm the conclusion formulated above for case of two general view centrally-symmetric Lévy parameterizations, namely, for desired 1D BEC parameters the finite values for limits of integrations can lead to the significant difference between values of BEC observables calculated on exact equations and asymptotic / ultimate values calculated on simpler relations.

It should be emphasized the results of the present paper shown in Figs. 3-6 are useful for experimental data analysis as well as for phenomenological studies because it allow, in particular, the quantitative choice between systems of equations (9) – (10) for estimations of 1D BEC parameters for specific cases of centrally-symmetric Lévy parametrization (11) at $\alpha = 1, 2$ depending on some features in given experiment.

IV. COMPARISON WITH EXPERIMENTAL RESULTS

Database is created for 1D BEC results for identical charged pions produced in strong interaction processes in order to verify the mathematical formalisms suggested above. This database is shown in the Appendix A and it is used as input for calculations below. Experimental results for strength of correlations and 1D source radius are considered for all types of the processes, centrally-symmetric Lévy parameterizations (11) and for total available energy range in the paper. The results for most central nucleus-nucleus collisions are usually included in the database because these collisions are used for studying of new features of final-state matter [29]. The dependence of 1D BEC parameters on the outgoing charged particle multiplicity, $N_{ch}$, is wide studied for $p+p$ and $\bar{p}+p$ collisions at least. Therefore the additional separation is made on experimental 1D BEC values deduced for minimum bias and for high multiplicity event classes sometimes1. This consideration seems important for both the additional verification of mathematics above and the more careful comparison with nucleus-nucleus results. As seen the additional information is required

1 This separation will be stipulated additionally if experimental 1D BEC results are available for various multiplicity event classes in $p+p$, $\bar{p}+p$ collisions.
about experimental $q$ ranges for systems (2), (3) in comparison with the ultimate relations (3) and (4). Therefore experimental $q$ ranges are estimated based on the available published data. In this Section in Tables VI the statistical errors are shown first, available systematic uncertainties – second, unless otherwise specifically indicated; the types of uncertainties (statistical / total, symmetric / asymmetric) is chosen just the same as well as input parameters for the sake of simplicity.

A. Relations between parameters for Cauchy / Gaussian distribution and Lévy one

The general system of equations (2) allow us to estimate the 1D BEC parameters for exponential / Gaussian function $\Omega_{C(G)}$ based on the a priori known parameter values for $\Omega_{L} \equiv \Omega(\alpha_{L}, \lambda_{L}, R_{L})$ corresponded to general view of centrally-symmetric Lévy distribution and vice versa.

1. Direction of calculations $\Omega_{L} \rightarrow \Omega_{C(G)}$. The sets $\{\lambda_{C}, R_{C}\}$ and $\{\lambda_{G}, R_{G}\}$ are estimated for experimentally known $\Omega_{L}$ and finite $q$-ranges with help of (2). The estimations are shown in Table I together with the available experimental results and the data for Cauchy distribution are shown on the first line, for Gaussian parametrization – on the second line for certain experiment at given energy. As seen estimations for strength of correlations and 1D radius calculated with help of (2) agree with experimental values within errors for both the Cauchy and the Gaussian distributions for particle emission points at all energies under study. One can note that estimation for $\lambda_{G}$ coincides with experimental values within total errors only at $\sqrt{s_{NN}} = 7$ TeV. Nevertheless the general system of equations (2) provides rather well estimations of 1D BEC parameters for both the Cauchy and the Gaussian distributions. The possibility is considered for application of ultimate relations (3) for the experimental data under study. All estimations from (3) coincide with results from general system (2) within errors with exception of the $\lambda_{G}$ for CMS at $\sqrt{s_{NN}} = 7$ TeV. For last case the estimations from (2) and (3) coincide with each other within 2σ. Thus the ultimate relations (3) provide reasonable estimations for 1D BEC parameters in both cases of the Cauchy and the Gaussian distributions within features of modern experiments under consideration, i.e. at $q_{1} \sim 10^{-2}$ GeV/c, $q_{2} \sim 2$ GeV/c and $\alpha_{L} \sim 0.8$ which is close to the region of Lévy index values with weaker changing of 1D BEC given a priori.

2. Direction of calculations $\Omega_{C(G)} \rightarrow \Omega_{L}$. Here the set of 1D BEC parameters $\{\lambda_{L}, R_{L}\}$ is estimated at a priori given $\alpha_{L}$ with help of the system of equations (2) for experimentally known $\Omega_{C}$ and $\Omega_{G}$ and finite $q$-ranges. The estimations for parameters of $\Omega_{L}$ are shown in Table I together with the available experimental results and the values of $\lambda_{L}$, $R_{L}$ derived from experimental analysis with Cauchy distribution are shown on the first line, the second one corresponds to the calculations with data for Gaussian distribution for certain experiment at given energy. The relatively large systematic uncertainties for ATLAS are driven by corresponding error for Lévy index (4). There is a remarkable agreement between results of calculations and experimental analyses (Table I): estimations for all 1D BEC parameters coincide with corresponding experimental values within statistical errors with exception of the $\lambda_{L}$ at $\sqrt{s_{NN}} = 7$ TeV for ATLAS data. For last case the coincidence between estimations from (2) experiment is achieved within total errors. This conclusion is for both $\Omega_{C} \rightarrow \Omega_{L}$ and $\Omega_{G} \rightarrow \Omega_{L}$ schemes of calculations. Thus the system of general equations (2) provides the high-quality estimations of 1D BEC parameters for general view centrally-symmetric Lévy parametrization $\Omega_{L}$ based on the a priori known $\alpha_{L}$ and results for exponential / Gaussian function. One can note the ultimate relations (3) for semi-infinite range of Lorentz invariant quantity $q$ result in reasonable estimations for the set of 1D BEC parameters $\{\lambda_{L}, R_{L}\}$ with help of results for exponential function $\Omega_{C}$ as well as for Gaussian one $\Omega_{G}$. Nevertheless the general system (2) allows the noticeable improvement of the results with respect of the (3) for chaoticity $\lambda_{L}$ derived from results for Gaussian distribution. This feature can be expected from Fig 2 because curve calculated with (2) differs from the corresponding ultimate $\alpha_{2}$-dependence at values $\alpha_{1} \lesssim 0.8$ which are close to the experimental data (Table VI).

| Collision $\sqrt{s_{NN}}$, GeV | Experiment | Estimation based on the (2) | Experimental values |
|-------------------------------|------------|-----------------------------|---------------------|
| $p + p$ 900 CMS | $\lambda = 0.62 \pm 0.08$ | $R = 1.47 \pm 0.24$ | $\lambda = 0.616 \pm 0.011 \pm 0.029$ | $R = 1.56 \pm 0.02 \pm 0.15$ |
| $7000$ ATLAS | $\lambda = 0.35 \pm 0.06$ | $R = 0.81 \pm 0.22$ | $\lambda = 0.32 \pm 0.01$ | $R = 0.98 \pm 0.03$ |
| CMS | $\lambda = 0.39 \pm 0.01 \pm 0.02$ | $R = 1.07 \pm 0.04 \pm 0.82$ | $\lambda = 0.302 \pm 0.002 \pm 0.019$ | $R = 1.046 \pm 0.005 \pm 0.114$ |

TABLE I: Parameter values for exponential and Gaussian parameterizations $\Omega_{C(G)}$. 
TABLE II: Parameter values for general Lévy parametrization $\Omega_L$ at given $\alpha_L$

| Collision $\sqrt{s_{NN}}$, Experiment | Estimation based on the [9] | Experimental values |
|---------------------------------------|-----------------------------|---------------------|
| GeV | $\lambda$ | $R$, fm | $\lambda$ | $R$, fm | Ref. |
| $p+p$ 900 CMS | 0.85 ± 0.04 ± 0.05 | 2.33 ± 0.18 ± 0.23 | 0.85 ± 0.06 | 2.20 ± 0.17 | [16] |
| 7000 ATLAS | 0.973 ± 0.012 ± 0.332 | 2.97 ± 0.06 ± 1.27 | 1.02 ± 0.03 ± 0.41 | 2.96 ± 0.09 ± 1.31 | [18] |
| CMS | 0.89 ± 0.04 ± 0.07 | 2.96 ± 0.17 ± 0.34 | 0.90 ± 0.05 | 2.83 ± 0.18 | [16] |

B. Relations between parameters for Cauchy and Gaussian distributions

The system of equations (9) derived above is used for estimation of the 1D BEC parameter values for Gaussian parametrization based on the $a$ priori known values for set $\{\lambda_C, R_C\}$ of BEC parameters for exponential parametrization, experimental ranges on $q$ and vice versa. In the subsection the separation is used on various multiplicity event classes in $p+p$, $\bar{p}+p$ collisions for 1D BEC results in some experiments. The results for minimum bias events are shown on the first line, for high multiplicity events – on the second line for certain experiment at given energy in Tables III and IV.

1. Direction of calculations $\Omega_C \to \Omega_G$. Parameters for Gaussian function are calculated with help of system (6) and $a$ priori known set $\{\lambda_C, R_C\}$. The results are shown in the Table IV together with available published experimental results for the Gaussian set $\{\lambda_G, R_G\}$. As seen from the Table IV the estimations for the set of Gaussian parameters are equal for published results within (total) errors for proton-proton collisions with exception of the value of strength of correlations $\lambda_G$ in ATLAS minimum bias events at $\sqrt{s_{NN}} = 7$ TeV and CMS results at collision energy $\sqrt{s_{NN}} = 2.36$ TeV. In the two last cases the agreement is observed within $2\sigma$. The similar situation is for symmetric nucleus-nucleus collisions, i.e. the estimations within the present paper for set of Gaussian parameters $\{\lambda_G, R_G\}$ agree with the results of the WA98 experiment [24] within $2\sigma$. But there is qualitative agreement only between results of calculations with help of (7) and experimental data for $\bar{p}+p$ collisions [25]. Perhaps, this discrepancy is dominated by some features of experiment provided unusually large values of chaoticity for both the exponential and the Gaussian parameterizations of 1D CF $C_2(q)$. For asymmetric nuclear interactions the agreement between results of calculations in the present paper and available experimental data is achieved mostly within errors for both the $\lambda_G$ and the 1D BEC radius. Only estimations for Gaussian 1D radius $R_G$ in O + Ag and for $\lambda_G$ in O + Au coincide with corresponding results of the WA80 experiment [28] within $2\sigma$. It should be emphasized that approximate calculations demonstrate the same results as in Table III within errors for all consecutive simplifications [7] – [11] which are valid and can be applied for certain experiment. One can note in particular that as expected the ultimate relations [5] work rather well for the CMS results at $\sqrt{s_{NN}} = 2.76$ TeV with low enough $q_1 \approx 0.6$ MeV/c and high enough $q_2 \approx 2.0$ GeV/c. Thus detail calculations for case $\Omega_C \to \Omega_G$ confirm both the correctness of suggestions made above for certain experiments and the validity of corresponding systems of equations [3] – [10].

2. Direction of calculations $\Omega_G \to \Omega_C$. Values of 1D BEC parameters for exponential function are estimated with help of system (6) and $a$ priori known values for Gaussian BEC quantities $\{\lambda_G, R_G\}$. The results are presented in Table IV together with available published experimental results for the set of BEC parameters $\{\lambda_C, R_C\}$ corresponded to the Cauchy distribution function in coordinate space for particle emission points. For $p+p$ collisions there is agreement between estimations for parameters for exponential parametrization of the CF $C_2(q)$ calculated with (6) and experimental results within (total) errors with exception of the 1D BEC radius $R_C$ in CMS at $\sqrt{s_{NN}} = 0.9$ TeV. In the last case results from calculation and experiment coincide within $2\sigma$. The similar situation is observed for nucleus-nucleus collisions: estimations for parameters of exponential function $\Omega_C$ derived with (6) agree with corresponding experimental results mostly within $1\sigma$, but the coincidence is achieved within $2\sigma$ for $R_C$ in both the Pb + Pb collisions at $\sqrt{s_{NN}} = 17.3$ GeV [22] and the O + Ag reactions at $\sqrt{s_{NN}} = 19.4$ GeV [28]. The estimations of 1D BEC parameters $\lambda_C, R_C$ obtained for $\bar{p}+p$ with help of (7) are in qualitative agreement with corresponding experimental results [25] even for the case of unusually large chaoticity. Thus the system of equations (6) provides quite reasonable estimations for parameters for exponential function based on the $a$ priori known values of 1D BEC observables for Gaussian function in various strong interaction processes at all available experimental energies.

In summary of the section, the systems of equations (2), (6) provide correct estimations for both desired BEC parameters, namely, the strength of correlations and the 1D radius in the case of centrally-symmetric Lévy distribution $\Omega_L$ as well as for specific Cauchy and Gaussian ones. In general the estimations show remarkable agreement with
available experimental data. Thus the systems of equations suggested in Sec. III can be useful in experimental data analysis as well as in phenomenological study for estimation of unknown values of BEC parameters for some parametrization of the 1D CF $C_{\lambda}(q)$ based on the available values of $\lambda$ and $R$ for another centrally-symmetric Lévy distribution. As seen from Tables I – IV the new estimations are obtained for 1D BEC parameters in $\Omega_C$, $\Omega_G$ in many cases for which the corresponding experimental results are absent. Thus the systems of equations derived within the framework of this paper allow the expansion of the available ensemble of values for 1D BEC parameters $\lambda$ and $R$ which it is useful for future investigations.

| Collision $\sqrt{s_{NN}}$, Experiment GeV | Estimation based on the (6) | Experimental values | Ref. |
|------------------------------------------|-----------------------------|---------------------|-----|
| $p + p$ 63.0 AFS                       | $0.47 \pm 0.04$             | $0.73 \pm 0.07$     | $0.40 \pm 0.03$ | $0.82 \pm 0.05$ [30] |
| 900 ALICE                               | $0.305 \pm 0.024$           | $1.00 \pm 0.09^{+0.06}_{-0.18}$ | $0.35 \pm 0.03$ | $1.00 \pm 0.06^{+0.10}_{-0.20}$ [26] |
|                                          | $0.357 \pm 0.025$           | $0.89 \pm 0.06^{+0.18}_{-0.08}$ | $0.310 \pm 0.026$ | $1.18 \pm 0.09^{+0.07}_{-0.17}$ [26] |
| ATLAS                                    | $0.41 \pm 0.01 \pm 0.03$    | $1.00 \pm 0.03 \pm 0.08$ | $0.34 \pm 0.01 \pm 0.03$ | $1.00 \pm 0.03 \pm 0.08$ [31] |
| CMS                                      | $0.351 \pm 0.006 \pm 0.013$ | $0.83 \pm 0.01 \pm 0.08$ | $0.32 \pm 0.01$ | $0.98 \pm 0.03$ [15] |
| 2360                                     | $0.348 \pm 0.030 \pm 0.013$ | $1.04 \pm 0.08 \pm 0.10$ | $0.32 \pm 0.01$ | $0.98 \pm 0.03$ [15] |
| 2760                                     | $0.366 \pm 0.005 \pm 0.025$ | $0.915 \pm 0.007 \pm 0.116$ | – | – |
| 7000 ALICE                               | $0.719 \pm 0.002 \pm 0.047$ | $1.148 \pm 0.007^{+0.04}_{-0.02} \pm 0.047$ | $0.645 \pm 0.003 \pm 0.047$ | $1.430 \pm 0.005^{+0.16}_{-0.13} \pm 0.02$ [32] |
| ATLAS                                    | $0.381 \pm 0.003 \pm 0.022$ | $1.092 \pm 0.005 \pm 0.074$ | $0.327 \pm 0.002 \pm 0.020$ | $1.130 \pm 0.005 \pm 0.086$ [31] |
|                                          | $0.266 \pm 0.009 \pm 0.015$ | $1.25 \pm 0.03 \pm 0.09$ | $0.251 \pm 0.010 \pm 0.018$ | $1.38 \pm 0.04 \pm 0.12$ |
| CMS                                      | $0.344 \pm 0.005 \pm 0.018$ | $1.00 \pm 0.01 \pm 0.10$ | – | – |
| $\bar{p} + p$ 1.89 CPLEAR               | $2.332 \pm 0.025$           | $0.972 \pm 0.014$     | $1.96 \pm 0.03$ | $1.04 \pm 0.01$ [25] |
| $p + Pb$ 5020                            | $0.358 \pm 0.007 \pm 0.021$ | $1.70 \pm 0.02 \pm 0.12$ | – | – |
| Pb + Pb 17.3 WA98                        | $0.327 \pm 0.008$           | $6.51 \pm 0.10$      | $0.307 \pm 0.008$ | $6.83 \pm 0.10$ [24] |
| O + C 19.4 WA80                          | $0.44 \pm 0.05$             | $2.8 \pm 0.3$        | $0.40 \pm 0.03$ | $2.90 \pm 0.21$ [28] |
| O + Cu                                   | $0.24 \pm 0.07$             | $2.53 \pm 0.11$      | $0.17 \pm 0.03$ | $2.35 \pm 0.11$ |
| O + Ag                                   | $0.28 \pm 0.10$             | $2.71 \pm 0.11$      | $0.17 \pm 0.04$ | $2.44 \pm 0.11$ |
| O + Au                                   | $0.110 \pm 0.015$           | $1.63 \pm 0.05$      | $0.085 \pm 0.007$ | $1.68 \pm 0.06$ |

V. SUMMARY

The case is investigated for smooth approximation of the one experimental 1D Bose–Einstein correlation function by two various centrally-symmetric Lévy parameterizations. It is suggested that lowest moments of corresponding distributions are equal approximately. Then the relations are derived between sets of 1D BEC observables, namely, strength of correlations and source radius, for two general view centrally-symmetric Lévy parameterizations under consideration for the first time. The relations obtained in the paper take into account the finiteness of range of Lorentz invariant four-momentum difference in experimental studies. Detailed analysis results in the systems of transcendental equations for various finite ranges of the Lorentz invariant four-momentum difference in the specific case of the exponential and Gaussian parameterizations for correlation function. It is shown that finite range of $q$ should be taken into account and corresponding systems of equations should be used for derivation of set $\{\lambda, R\}$ based on the $a priori$ known values of corresponding parameters for both cases the two general view centrally-symmetric Lévy parameterizations and the two specific functions (exponential and Gaussian) most used in experimental studies. The ultimate relations derived for semi-infinite range of $q$ can be utilized carefully for experimental analysis and these equations can produce the reasonable estimations for 1D BEC parameters for ranges of Lévy indexes $\forall i = 1, 2 : \alpha_i > 1$ only. Furthermore it is demonstrated that the corresponding ultimate relations for specific case of Cauchy and Gaussian distributions for source in coordinate space produce the reasonable estimations of 1D BEC parameters for few modern experimental analyses only. The mathematical formalism suggested within the framework of the present paper is verified with help of experimental results obtained for wide set of strong interaction processes in all available energy range. The two pairs of distributions are considered: general view centrally-symmetric Lévy one with specific case (Cauchy / Gaussian); two specific Cauchy and Gaussian distributions. For both cases verifications are made for both directions of calculations. Namely, the calculations have been made for estimation of 1D BEC observables for Cauchy / Gaussian function based on the $a priori$ known values of parameters of general Lévy parametrization
TABLE IV: Parameter values for exponential function $\Omega_C$ in [1]

| Collision $\sqrt{s_{NN}}$, Experiment | Estimation based on the [6] | Experimental values |
|--------------------------------------|-----------------------------|---------------------|
|                                      | $\lambda_C$, $R_C$, fm      |                     |
|                                      | GeV                         |                     |
| $p + p$                              | $7.21$                      | $1.63 \pm 0.10$     | $2.29 \pm 0.08$ |
|                                      | $26.0$                      | $1.4 \pm 0.7$       | $2.6 \pm 0.7$  |
|                                      | $63.0$                      | $0.67 \pm 0.05$     | $1.50 \pm 0.10$|
|                                      | $200$                       | $0.588 \pm 0.010$   | $2.43 \pm 0.04$|
|                                      | $900$                       | $0.63 \pm 0.06$     | $1.89 \pm 0.12$|
|                                      | $2360$                      | $0.60 \pm 0.03$     | $1.86 \pm 0.07$|
|                                      | $7000$                      | $1.104 \pm 0.006$   | $2.627 \pm 0.010$|
|                                      |                             | $0.62 \pm 0.03$     | $1.84 \pm 0.07$|
|                                      | $\bar{p} + p$               | $4.21 \pm 0.09$     | $2.079 \pm 0.028$|
|                                      |                             | $0.86 \pm 0.08$     | $12.4 \pm 0.6$ |
|                                      |                             | $1.06 \pm 0.10$     | $15.1 \pm 0.9$ |
|                                      |                             | $0.69 \pm 0.02$     | $14.1 \pm 0.3$ |
|                                      |                             | $0.99 \pm 0.03$     | $13.30 \pm 0.3$|
|                                      |                             | $1.23 \pm 0.08$     | $8.4 \pm 0.3$  |
|                                      |                             | $0.95 \pm 0.05$     | $9.4 \pm 0.3$  |
|                                      |                             | $0.81 \pm 0.05$     | $7.8 \pm 0.6$  |
|                                      |                             | $0.85 \pm 0.08$     | $5.9 \pm 0.5$  |
|                                      |                             | $0.34 \pm 0.06$     | $4.7 \pm 0.2$  |
|                                      |                             | $0.34 \pm 0.09$     | $4.9 \pm 0.2$  |
|                                      |                             | $0.156 \pm 0.014$   | $3.19 \pm 0.13$|

and vice versa; for estimations of Gaussian parameters based on the a priori known values of observables for Cauchy distribution and vice versa. Comparison shows the quantitative agreement between estimations derived with help of mathematical formalism developed in the paper and most of available experimental results for both pairs consisting of the general view centrally-symmetric Lévy parametrization and specific (exponential / Gaussian) function and the two specific source distributions (Cauchy and Gaussian) most used in experimental studies for any direction of calculations.

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Appendix A: Data for 1D BEC parameters in strong interactions

In this Appendix experimental database is shown in Tables IV, V for 1D BEC parameters for identical charged pions produced in $p + p$, $\bar{p} + p$ and $A_1 + A_2$ interactions[1]. Some of the numerical values used in the Sec. IV The pion pairs with low average transverse momentum, $<k_T>$, are considered for all types of strong interaction processes. The results for most central nucleus-nucleus collisions are used and as consequence additional separation is made for $p + p$ collisions on minimum bias and high multiplicity events if corresponding experimental 1D BEC values are available. In the last case the results for minimum bias events are shown on the first line, for high multiplicity events – on the second line for certain experiment at given energy.

1 In Table IV total uncertainties are shown for exponential parametrization in CPLEAR [25] and for NA44 experiment [18].
### TABLE V: Experimental results for special cases of parametrization [1]

| Collision $\sqrt{s_{NN}}$, GeV | $\lambda_C$ | $R_C$, fm | $\lambda_G$ | $R_G$, fm | Exp. | Gaussian function $\Omega_C$ | Gaussian function $\Omega_G$ | Ref. |
|--------------------------------|-------------|------------|-------------|------------|------|----------------|----------------|------|
| $p + p$ 7.21                   | E766        | –          | –           | 0.466 ± 0.015 | 0.95 ± 0.03 | [33]          |                  |      |
| 26.0 NA23                      | –           | –          | 0.32 ± 0.08  | 1.02 ± 0.20  | [34]          |                  |                  |      |
| 27.4 NA27                      | –           | –          | –           | 1.20 ± 0.03  | [35]          |                  |                  |      |
| 31.0 ABCDHW                    | –           | –          | 0.35 ± 0.04  | 1.01 ± 0.08  | [36]          |                  |                  |      |
| 44.0                           | –           | –          | 0.42 ± 0.04  | 1.13 ± 0.07  | [37]          |                  |                  |      |
| 62.0                           | –           | –          | 0.42 ± 0.08  | 1.69 ± 0.25  | [38]          |                  |                  |      |
| 63.0 AFS                       | 0.77 ± 0.07 | 1.32 ± 0.13 | 0.40 ± 0.03  | 0.82 ± 0.05  | [39]          |                  |                  |      |
| 200 STAR                       | –           | –          | 0.345 ± 0.005| 1.32 ± 0.02 ± 0.13 | [40]          |                  |                  |      |
| 900 ALICE                      | 0.63 ± 0.05 | 1.87 ± 0.12^{+0.16}_{-0.18} | 0.35 ± 0.03 | 1.00 ± 0.06^{+0.10}_{-0.20} | [41]          |                  |                  |      |
| ATLAS                          | 0.74 ± 0.03 ± 0.09 | 1.83 ± 0.07 ± 0.20 | 0.34 ± 0.01 ± 0.03 | 1.00 ± 0.03 ± 0.08 | [42]          |                  |                  |      |
| CMS                            | 0.616 ± 0.011 ± 0.029 | 1.56 ± 0.02 ± 0.15 | 0.32 ± 0.01 | 0.98 ± 0.03 | [43]          |                  |                  |      |
| 2360                           | 0.66 ± 0.07 ± 0.05 | 1.99 ± 0.18 ± 0.24 | 0.32 ± 0.01 | 0.98 ± 0.03 | [44]          |                  |                  |      |
| 2760                           | 0.808 ± 0.017 ± 0.062 | 2.35 ± 0.07 ± 0.31 | –           | –          | [45]          |                  |                  |      |
| 7000 ALICE                     | 1.180 ± 0.005 ± 0.084 | 2.038 ± 0.014^{+0.083}_{-0.046} | 0.645 ± 0.003 ± 0.047 | 1.430 ± 0.005^{+0.158}_{-0.300} | [46]          |                  |                  |      |
| ATLAS                          | 0.718 ± 0.006 ± 0.062 | 2.067 ± 0.012 ± 0.182 | 0.327 ± 0.002 ± 0.020 | 1.130 ± 0.005 ± 0.086 | [47]          |                  |                  |      |
| CMS                            | 0.618 ± 0.009 ± 0.042 | 1.89 ± 0.02 ± 0.21 | –           | –          | [48]          |                  |                  |      |
| $\bar{p} + p$ 1.89             | CPLEAR      | 4.79 ± 10  | 1.89 ± 0.04  | 1.96 ± 0.03  | 1.04 ± 0.01 | [49]          |                  |      |
| 1800 E755                      | –           | –          | 0.24 ± 0.02  | 1.46 ± 0.10 ± 0.23 | [50]          |                  |                  |      |
| 1960 CDF                       | 0.89 ± 0.03  | 1.67 ± 0.05  | 0.50 ± 0.04  | 1.79 ± 0.08  | [51]          |                  |                  |      |
| $p + Pb$ 5020                  | ALICE       | 1.230 ± 0.016^{+0.088}_{-0.141} | 4.82 ± 0.05^{+0.25}_{-0.22} | 0.603 ± 0.006 ± 0.056 | 2.780 ± 0.018^{+0.418}_{-0.068} | [52]          |                  |      |
| ATLAS                          | –           | 5.32 ± 0.06^{+0.14}_{-0.11} | –           | –          | [53]          |                  |                  |      |
| CMS                            | 0.81 ± 0.02 ± 0.07 | 3.55 ± 0.05 ± 0.30 | –           | –          | [54]          |                  |                  |      |
| Au + Au 4.86                   | E802        | –           | –           | 0.44 ± 0.03  | 6.32 ± 0.29 | [55]          |                  |      |
| Au + Pb 17.3                   | NA49        | –           | –           | 0.560 ± 0.023 | –          | [56]          |                  |      |
| Pb + Pb                        | NA44        | –           | –           | 0.52 ± 0.04  | 7.6 ± 0.4   | [57]          |                  |      |
| WA98                           | 0.718 ± 0.023 | 13.34 ± 0.26 | 0.307 ± 0.008 | 6.83 ± 0.10 | [58]          |                  |                  |      |
| Au + Au 130                    | PHENIX      | –           | –           | –           | 6.0 ± 0.3   | [59]          |                  |      |
| STAR                           | –           | –           | 0.450 ± 0.009 ± 0.027 | 6.30 ± 0.12 ± 0.38 | [60]          |                  |                  |      |
| Pb + Pb 2760                   | ALICE       | 1.830 ± 0.003 ± 0.156 | 19.85 ± 0.002 ± 0.28 | 0.689 ± 0.001 ± 0.096 | 9.70 ± 0.06 ± 1.17 | [61]          |                  |      |
| Si + Al 5.41                   | E802        | –           | –           | 0.68 ± 0.04  | 4.42 ± 0.16 | [62]          |                  |      |
| Si + Au                        | –           | –           | 0.511 ± 0.026 | 4.91 ± 0.15 | [63]          |                  |                  |      |
| S + Pb 17.3                    | NA44        | –           | –           | 0.42 ± 0.02  | 4.00 ± 0.27 | [64]          |                  |      |
| O + C 19.4                     | WA80        | 0.92 ± 0.13 | 5.7 ± 0.7   | 0.40 ± 0.03  | 2.90 ± 0.21 | [65]          |                  |      |
| O + Cu                         | 0.49 ± 0.14 | 5.05 ± 0.25 | 0.17 ± 0.03  | 2.35 ± 0.11  | [66]          |                  |                  |      |
| O + Ag                         | 0.59 ± 0.21 | 5.46 ± 0.24 | 0.17 ± 0.04  | 2.44 ± 0.11  | [67]          |                  |                  |      |
| O + Au                         | 0.20 ± 0.03 | 3.07 ± 0.12 | 0.085 ± 0.007 | 1.68 ± 0.06  | [68]          |                  |                  |      |
| NA35                           | –           | –           | 0.29 ± 0.03  | 4.00 ± 0.20  | [69]          |                  |                  |      |

### TABLE VI: Experimental results for general centrally-symmetric Lévy parametrization [1]

| Collision $\sqrt{s_{NN}}$, GeV | Experiment | $\lambda_L$, fm | $R_L$, fm | $\alpha_L$ | Ref. |
|--------------------------------|------------|-----------------|-----------|------------|------|
| $p + p$ 900                    | CMS        | 0.93 ± 0.11     | 2.5 ± 0.4 | 0.76 ± 0.06 | [70] |
| 7000 ATLAS                     | 1.02 ± 0.03 ± 0.41 | 2.96 ± 0.09 ± 1.31 | 0.81 ± 0.01 ± 0.18 | 0.792 ± 0.024 | [71] |
| CMS                            | 0.90 ± 0.05 | 2.83 ± 0.18     | 0.792 ± 0.024 | [72]      |      |
[1] W. Kittel, Acta Phys. Polon. B32, 3927 (2001).
[2] T. Csomgo, Heavy Ion Phys. 15, 1 (2002).
[3] V.A. Okorokov, arXiv: 1312.4269 [nucl-ex], 2013.
[4] V.A. Okorokov, Adv. High Energy Phys. 2016, 5972709 (2016).
[5] A. Bialas, Acta Phys. Pol. B23, 561 (1992).
[6] A. Bialas, Nucl. Phys. A545, 285c (1992).
[7] V.A. Okorokov, in preparation.
[8] P. Lévy, Theorie de l'Addition des Variables Aleatoires, Gauthier–Villiers, Paris, 1937.
[9] V. V. Uchaikin, JEPT 97, 810 (2003).
[10] G. Goldhaber et al., Phys. Rev. 120, 300 (1960).
[11] D.H. Boal, C.-K. Gelbke, B.K. Jennings, Rev. Mod. Phys. 62, 553 (1990).
[12] T. Csorgo et al., Eur. Phys. J. C36, 67 (2004).
[13] C. Forbes et al., Statistical distributions, John Wiley & Sons Inc., New Jersey, 2011.
[14] V. Khachatryan et al. (CMS Collaboration), Phys. Rev. Lett. 105, 032001 (2010).
[15] V. Khachatryan et al. (CMS Collaboration), J. High Energy Phys. 0511, 029 (2011).
[16] S.S. Padula, Proceedings of the X International workshop on particle correlations and femtoscopy, eConf C140825.8 (2015).
[17] T. Csorgo et al., Eur. Phys. J. C36, 67 (2004).
[18] C. Forbes et al., Statistical distributions, John Wiley & Sons Inc., New Jersey, 2011.
[19] V. Khachatryan et al. (CMS Collaboration), Phys. Rev. Lett. 105, 032001 (2010).
[20] V. Khachatryan et al. (CMS Collaboration), J. High Energy Phys. 0511, 029 (2011).
[21] S.S. Padula, Proceedings of the X International workshop on particle correlations and femtoscopy, eConf C140825.8 (2015).
[22] T. Csorgo et al., Eur. Phys. J. C36, 67 (2004).
[23] C. Forbes et al., Statistical distributions, John Wiley & Sons Inc., New Jersey, 2011.
[24] V. Khachatryan et al. (CMS Collaboration), Phys. Rev. Lett. 105, 032001 (2010).
[25] V. Khachatryan et al. (CMS Collaboration), J. High Energy Phys. 0511, 029 (2011).
[26] S.S. Padula, Proceedings of the X International workshop on particle correlations and femtoscopy, eConf C140825.8 (2015).
[27] T. Csorgo et al., Eur. Phys. J. C36, 67 (2004).
[28] C. Forbes et al., Statistical distributions, John Wiley & Sons Inc., New Jersey, 2011.
FIG. 1: Dependence of 1D BEC radius (a,b) and strength of correlations (c,d) for centrally-symmetric Lévy parametrization with $\alpha_1 = 1.5$ on low $q_1$ (a,c) and high $q_2$ (b,d) limits of integration in the system of equations for fixed parameter values for second centrally-symmetric Lévy parametrization: $\alpha_2 = 0.5$, $\lambda_2 = 0.5$ and $R_2 = 1.5$ fm. The solid lines correspond to the indicated values of the $q_2$ for $q_1$-dependence (a,c) and to shown values of the $q_1$ for $q_2$-dependence (b,d). The dashed lines correspond to the calculations with $q_2 \to \infty$ for $q_1$-dependence (a,c) and with $q_1 = 0$ for $q_2$-dependence (b,d). The thin dotted lines are the ultimate levels for $R_1$ (a,b) and $\lambda_1$ (c,d) calculated with for given values of the $\alpha_1$ and the set of parameters ($\alpha_2$, $\lambda_2$, $R_2$) for second Lévy parametrization.
FIG. 2: Dependence of 1D BEC radius \((a,b)\) and strength of correlations \((c,d)\) for centrally-symmetric Lévy parametrization on \(\alpha_1\) at fixed values of \(\alpha_2\) \((a,c)\) and on \(\alpha_2\) at fixed values of \(\alpha_1\) \((b,d)\) for given limits of integration in the system of equations \([2]\) \(q_1 = 0.02 \text{ GeV}/c\), \(q_2 = 2.0 \text{ GeV}/c\) and for fixed values of the BEC parameters for second centrally-symmetric Lévy parametrization: \(\lambda_2 = 0.5\) and \(R_2 = 1.5 \text{ fm}\). The solid lines correspond to the indicated values of the \(\alpha_2\) for \(\alpha_1\)-dependence \((a,c)\) and to shown values of the \(\alpha_1\) for \(\alpha_2\)-dependence \((b,d)\). The dotted lines are the ultimate cases for \(R_1\) \((a,b)\) and \(\lambda_1\) \((c,d)\) calculated with \([3]\) for given values of the \(\alpha_1\) \((a,c)\) or \(\alpha_2\) \((b,d)\) and the set of BEC parameters \(\{\lambda_2, R_2\}\) for second Lévy parametrization.
FIG. 3: Dependence of 1D BEC radius \((a,b)\) and strength of correlations \((c,d)\) for Gaussian parametrization on low \(q_1\) \((a,c)\) and high \(q_2\) \((b,d)\) limits of integration in the system of equations \((7)\) for fixed values of the parameters for exponential parametrization: \(\lambda_C = \pi/2\) and \(R_C = \sqrt{\pi}\) fm. The solid lines correspond to the indicated values of the \(q_2\) for \(q_1\)-dependence \((a,c)\) and to shown values of the \(q_1\) for \(q_2\)-dependence \((b,d)\). The dashed lines correspond to the calculations based on the system \((9)\) for \(q_1\)-dependence \((a,c)\) and on the system \((10)\) for \(q_2\)-dependence \((b,d)\). The thin dotted lines are the ultimate levels \(R_G = 1.0\) fm \((a,b)\) and \(\lambda_G = 1.0\) \((c,d)\) calculated with help of \((5)\) for given values of the set of Cauchy parameters \(\{\lambda_C, R_C\}\).
FIG. 4: Dependence of relative 1D BEC radius (a,b) and strength of correlations (c,d) on $q_1$ (a,c) and $q_2$ (b,d) for various fixed values of the parameters for exponential parametrization. The calculations are made for simpler range of integration $[z_{1,C(G)}, \infty)$ with help of system (9) for $q_1$-dependence (a,c) and for $[0, z_{2,C(G)}]$ with system (10) for $q_2$-dependence (b,d) respectively. The dashed lines correspond to the $\lambda_C = 0.6\pi$, $R_C = 1.2\sqrt{\pi}$ fm; solid lines – $\lambda_C = 0.5\pi$, $R_C = \sqrt{\pi}$ fm; dotted lines – $\lambda_C = 0.4\pi$, $R_C = 0.8\sqrt{\pi}$ fm. The thin dotted lines are the ultimate levels $R_C/R_G = \sqrt{\pi}$ (a,b) and $\lambda_C/\lambda_G = \pi/2$ (c,d) corresponded to the system (1).
FIG. 5: Dependence of 1D BEC radius (a,b) and strength of correlations (c,d) for Cauchy distribution on low $q_1$ (a,c) and high $q_2$ (b,d) limits of integration in the system of equations (7) for fixed values of the parameters for Gaussian parametrization: $\lambda_G = 2/\pi$ and $R_G = 1/\sqrt{\pi}$ fm. The solid lines correspond to the indicated values of the $q_2$ for $q_1$-dependence (a,c) and to shown values of the $q_1$ for $q_2$-dependence (b,d). The dashed lines correspond to the calculations based on the system (9) for $q_1$-dependence (a,c) and on the system (10) for $q_2$-dependence (b,d). The thin dotted lines are the ultimate levels $R_C = 1.0$ fm (a,b) and $\lambda_C = 1.0$ (c,d) calculated with help of (5) for given values of the set of Gaussian parameters $\{\lambda_G, R_G\}$.
FIG. 6: Dependence of relative 1D BEC radius \((a,b)\) and strength of correlations \((c,d)\) on \(q_1\) \((a,c)\) and \(q_2\) \((b,d)\) for various fixed values of the parameters for Gaussian parametrization. The calculations are made for simpler range of integration \([z_1,C(G), \infty)\) with help of system \((9)\) for \(q_1\)-dependence \((a,c)\) and for \([0.0, z_2,C(G)]\) with system \((10)\) for \(q_2\)-dependence \((b,d)\) respectively. The dashed lines correspond to the \(\lambda_G = 1.2, R_G = 1.2\) fm; solid lines \(-\lambda_G = 1.0, R_G = 1.0\) fm; dotted lines \(-\lambda_G = 0.8, R_G = 0.8\) fm. The thin dotted lines are the ultimate levels \(R_C/R_G = \sqrt{\pi}\) \((a,b)\) and \(\lambda_C/\lambda_G = \pi/2\) \((c,d)\) corresponded to the system \((5)\).