Abstract

An observation of autocorrelation of Wilson loops on lattice is presented, especially for the small separation in Markov chain. We give a possible explanation for such behavior. We also present the dependence of autocorrelation behavior on the chosen operators in this paper.
1 Introduction

In order to extract a meaningful physical quantity, e.g. glueball mass, from MC simulation of lattice gauge theory, we have to construct the correlation function \( G(t) \) for chosen operator \( o(\vec{x}, t) \):

\[
G(t) = \Sigma_{\vec{x}} \langle o(\vec{x}, t) o(\vec{0}, 0) \rangle,
\]

to extract the glueball mass \( m \) when \( t \to \infty \),

\[
G(t) \propto e^{-mt}.
\]

It is the average over the independent configurations. Meanwhile, we also have to make correctly error estimate for the measurement. We should generate a sequence of configurations and then choose samples separated by enough sweeps to ensure their independence. Therefore it is needed to investigate the correction from autocorrelation between configurations in Markov chain. Denoting sequence in Markov chain by \( T_{MC} \), some authors\[^1\] have studied autocorrelation with different algorithm and regard normalized autocorrelation asymptotic behavior with separation \( \Delta T_{MC} \) in Markov chain as \( e^{-\Delta T_{MC}/\tau_{exp}} \). In order to improve the fitting, ref. \[^2\] redefined \( \tau_{exp} \) via

\[
\tau_{exp} = \limsup_{\Delta T_{MC} \to \infty} \frac{\Delta T_{MC}}{-\log(\rho(\Delta T_{MC}))},
\]

where

\[
\rho(\Delta T_{MC}) = \frac{\langle v_i v_{i+\Delta T_{MC}} \rangle - \langle v_i \rangle \langle v_{i+\Delta T_{MC}} \rangle}{\langle v_i v_i \rangle - \langle v_i \rangle \langle v_i \rangle}
\]

is normalized autocorrelation and \( v_i \) is the measured value of certain operator in the \( i^{th} \) configuration. But this formula always works at large \( \Delta T_{MC} \) region\[^3\].

Since autocorrelation contribution to error is mainly from small \( \Delta T_{MC} \) region, we investigates autocorrelation at small \( \Delta T_{MC} \) using MC simulation in this paper. From the simulation, we find that autocorrelation behavior is like \( \rho \propto e^{-c\sqrt{\Delta T_{MC}}} \) rather than \( e^{-c\Delta T_{MC}} \), especially at small \( \Delta T_{MC} \) region. This discrepancy is important since the main autocorrelation contribution to error comes from small \( \Delta T_{MC} \) region. In section 2, we give a possible explanation for such behavior.

We also investigate the dependence of autocorrelation with the operators chosen for extracting the glueball mass. It is found that autocorrelation depends on the chosen operators and we also give a possible explanation of this dependence in section 2. We present our simulation results in section 3 and the conclusion in section 4.

We adopt improved action on anisotropic lattice ( one can find the notations and details in ref. \[^3\]):

\[
S_{II} = \beta \{ \frac{5\Omega_{sp}}{3\xi u_s} + \frac{4\xi\Omega_{tq}}{3u_s^2} - \frac{\Omega_{sr}}{12u_s^6} - \frac{\xi\Omega_{str}}{12u_s^6} \}.
\]

To simplify, we restrict ourselves to make the simulation in SU(2) pure gauge fields.
2 An Explanation for the Autocorrelation Behavior at Small $\Delta T_{MC}$ Region

A configuration can be described by $4N^4$ SU(2) matrices, where $N^4$ is lattice site number. This means we can depict configuration by $12N^4 \triangleq M$ variables, a factor of 3 multiplied here is due to 3-dimension Lie algebra for each link (We regard it as $E^3$ space for simplification). Therefore, configuration can be regarded as one point $P$ in M-dimensional Euclidean space $S$. To simplify, we assume the spacing of time-like and space-like links are of the same. Then all the $M$ parameters, or coordinates, are of equivalence due to the periodic boundary condition.

When updating configurations, point $P$ forms a trajectory in $S$ space. Define distance between two points $x = (x_1, x_2, \cdots, x_M), y = (y_1, y_2, \cdots, y_M) \in S$ by

$$r(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_M - y_M)^2}.$$ 

In one sample, we regard the same-time-slice sub-configuration as a point $P_1$ in $9N^3 (\triangleq M_1)$-dimensional space $S_1$. As expected, the series of $N$ points in one sample can be regarded as the evolution of the sub-configuration with Euclidean time $t$. All the links in these sub-configurations are space-like, and we can define distance $r'$ of two points in $S_1$ in the similar way. After doing these, we find that point $P_1$ in $S_1$ has two independent evolution behaviors: the first one is the evolution with $T_{MC}$ at fixed $t$ and the second one is the evolution with $t$ at fixed $T_{MC}$ (in the same sample).

Suppose configurations have approached to equilibrium after enough pre-updatings. Updating configuration $n$ times, we get representation points $\{x^{(i)}\} (i = 1, 2, \cdots, n)$ in $S$ space. We denote the displacement of two adjoining points by $\vec{r}^{(i)} = x^{(i+1)} - x^{(i)}$. ($x^{(i)}$ is also a vector.) When $n \to \infty$, we get a sequence of $\vec{r}$ with certain distribution. Due to ergodicity and periodic condition, it can be expected that the distribution of $\vec{r}$ only depends on its modular value $r$. Since the procedure of updating configuration, or Markov chain, forms a trajectory in space $S$, and the procedure is of random, we may regard the procedure as random walking in space $S$. Then mean distance of two points separated by $\Delta T_{MC}$ in Markov chain is given by $< r > = \sqrt{2D_0 \Delta T_{MC}}$, where $D_0$ is diffusion coefficient which only depends on the choice of algorithm.

Now let’s consider sub-configuration evolves with $T_{MC}$ at fixed $t_1$ (the sub-configuration consists of space-like links in time-$t_1$-slice). We denote corresponding points and displacements in $S_1$ with $\{x^{(i)}\}$ and $\vec{r}^{(i)}$. When $n$ is large enough, we expect that mean distance of two sub-configurations adjoining in Markov chain is $\sqrt{M_1/M r_0}$ where $r_0$ is mean distance of two representation points in $S$ space adjoining in Markov chain. So, at certain $t_1$, mean distance of two sub-configurations separated by $\Delta T_{MC}$ in Markov chain is

$$< r' > = \text{const} \cdot \sqrt{\Delta T_{MC}}. \quad (1)$$
The distribution of $\mathbf{r}'$ has no dominant direction.

Then we consider sub-configuration evolves with time $t$ in one configuration. When configuration approaches to the equilibrium, the most possible contribution to configuration is from the vicinity near the solution of classical equation of motion, and we expect the distance of two sub-configurations separated by time $\Delta t$ is a well-defined function. Up to the first order, mean distance of two sub-configurations separated by $\Delta t$ in the same sample in $S_1$ space can be expanded as

$$< r' > \simeq c'_1 + c'_2 \Delta t,$$

where $c'_2 > 0$. The distribution of $\mathbf{r}'$ has no dominant direction, too. Eq.(2) can also be written as

$$\Delta t = c_1 + c_2 < r' > .$$

Since we are interested in operators for extracting glueball mass, we will only consider the operator with certain quantum number $J^{PC}$. Suppose $o$ is such kind of the operator and its measurement $o(t)$ in sub-configuration is a function of point $P_1$ in $S_1$. Autocorrelation of $o(t)$ is actually the correlation function $G(t)$. One can extract the glueball mass by its asymptotic behaviour

$$\rho(\Delta t) \propto e^{-mc_2 \sqrt{\Delta T_{MC}}}.$$ (4)

On the other hand, due to Eq.(3), it is

$$\rho(\Delta t) \propto e^{-c_2 m< r' >}.$$ (5)

When we calculate correlation function with certain $\Delta T_{MC}$ or $\Delta t$, we should average it in many samples, or average it in subspace $S_1$ due to ergodicity. Therefore, $\rho$ is function of $\mathbf{r}'$. Meanwhile, the distribution of $\mathbf{r}'$ has no dominant direction with certain $\Delta T_{MC}$ and $\Delta t$, so $\rho$ is only the function of $\mathbf{r}'$.

Let us, now, consider the evolution of sub-configuration with $\Delta T_{MC}$ again. Due to Eqs. (1), (5), we get

$$\rho(\Delta T_{MC}) \propto e^{-mc_2 \sqrt{\Delta T_{MC}}}. $$ (6)

In our simulation, we find it is better to express the autocorrelation $\rho$ by

$$\rho(\Delta T_{MC}) \propto e^{-(c_0 + c_2 m) \sqrt{\Delta T_{MC}}}. $$ (7)

Presumably, it is due to finite size effect in lattice simulation. One can find that autocorrelation gets weaker with the increasing of the mass extracted from the correlation function of the operator $G(t)(c_2 > 0)$.
3 Simulation Results

As we know, one can construct certain combination of Wilson loops to extract the mass of $J^{PC}$ glueball states. These special combination of Wilson loops transforms according to certain representation of cubic group($A_1$, $A_2$, $E$, $T_1$, $T_2$). We choose some of these operators to observe the autocorrelations. These operators are the combination of the certain prototype Wilson loop which are shown in Fig. 1. Operator a) belongs to representation $A_1^+(J^P = 0^+)$; b) belongs to representation $E^+(J^P = 2^+)$; c) belongs to representation $T_2^+(J^P = 2^+)$ and d) belongs to representation $A_1^-(J^P = 0^-)$. The prototype of operators a) and b) are plequettes. We choose different combination coefficient and different orientation to make operator a) belongs to representation $A_1^+$ and operator b) belongs to representation $E^+$($J^P = 2^+$). Therefore, one can extract different mass states from them. Operator b) and c) belong to different representations of cubic group but correspond to the same continuum $J^P$ states (Masses extracted from operator b) and c) approach to the same value in the continuum limit).

Our simulation is performed on a $8^3 \times 24$ anisotropic lattice with $\xi = \frac{a_s}{a_t} = 3.0$ and $\beta = 1.0$. The mass $m_a$ extract from $A_1^+$, $E^+$, $T_2^+$, $A_1^-$ operators are 0.74(2), 1.27(10), 1.21(7), and 1.59(7) respectively.

We present the operators autocorrelation behavior and their fitting curve in Fig. 2. Except at the large $\Delta T_{MC}$ region where the noise will overwhelm the signal, the formula (7) works very well.

When analyzing autocorrelation data for operator $P_{SS}$, spatial plaquette and for operator $R_{SS}$, spatial $2 \times 1$ rectangle from ref. [5], one also find our formula (7) does work well. Our argument is also supported by the data for different operators of ref. [5] (their Fig.2), although their data is to describe fermions.

When $\Delta T_{MC}$ is not small, the space $S$ can not be regarded as Euclidean one, so Eq.(1) is not valid. At the same time, the expansion in Eq.(2) is doubtful at large $\Delta t$. Therefore, the explanation in section 2 is not advisable at large $\Delta T_{MC}$ region or weak autocorrelation region. One may expect that when $\Delta T_{MC}$ is very large, the autocorrelation behavior will transform into the behavior $e^{-\Delta T_{MC}/\tau_{exp}}$. This can be seen obviously in Fig. 2 a). (To guide our eyes, we extract it in Fig.3 with $-\ln \rho$ verse $\Delta T_{MC}$). We find that the fitting curve is $-\ln \rho = -0.0786 + 0.675\sqrt{\Delta T_{MC}}$ (for small $\Delta T_{MC}$) and $-\ln \rho = -2.01 + 0.261\Delta T_{MC}$ (for large $\Delta T_{MC}$) respectively. But this transition does not happen suddenly. In fact, at intermediate $\Delta T_{MC}$ region, the formula $-\ln \rho = -1.0 + 0.104\Delta T_{MC}$ works as well as the first formula.

The dependence of autocorrelation on the mass $m_a$ extracted from the correlation function $G(t)$ is very intricacy. Fitting autocorrelation by $\rho \propto e^{-c\sqrt{\Delta T_{MC}}}$, Fig.4 presents c verse $m_a$ by $c = -0.65 + 1.94 m_a$. Our data implies that with the increasing of the mass, the autocorrelation gets more weak, as expected in eq.(7).
4 Conclusion

From our simulation, one can find that autocorrelation behavior is like $e^{-c\sqrt{\Delta T_{MC}}}$ at small $\Delta T_{MC}$ region. As $\Delta T_{MC}$ increases, autocorrelation approaches to $e^{-c\Delta T_{MC}}$ gradually, as one expected. Therefore, we can get fair error estimation from the formula $e^{-c\sqrt{\Delta T_{MC}}}$. An explanation is also given in this paper for such behavior.

Our simulation also show, the autocorrelation gets more weak with the increasing of the extracted mass $ma_t$. It is consistent with the fact, as one usually expect, the autocorrelation is dependent on the operator chosen. Since $0^{++}$ glueball is the lightest state of glueballs, therefore, the correction from autocorrelation is larger in $0^{++}$ glueball case then the others.

References

[1] For instance, Montvay, Quantum Fields on a Lattice, Cambridge University Press. 1994,

[2] SESAM+TχL-Collaboration, Nucl. Phys. B(Proc. Suppl.)63(1998) 946,

[3] C. J. Morningstar and M. Peardon, Phys. Rev. D56(1997) 4043,

[4] J. B. Zhang, M. Jin and D. R. Ji, Chin. Phys. Lett. 15(1998) 865,

[5] T. Draper, C. Nenkov and M. Peardon, Nucl. Phys. B(Proc. Suppl.)53(1997) 997,

[6] B. Berg and A. Billoire, Nucl. Phys. B221(1983) 109,

[7] D. Q. Liu, J. M. Wu and Y. Chen, hep-lat/0011087
Figure Captions

- **Figure 1** Prototypes of the corresponding operators a) - d) in the paper. Each operator is the different combination of the prototype \([6][7]\) with directions in 3-dimensional space. a) and b) are both plaquette, but we choose different combination coefficients to make operator a) belong to representation \(A^+_1\) and operator b) belong to \(E^+_1\) in the cubic group. We also choose proper coefficients to make operator c) belong to \(T^+_2\) and operator d) belong to \(A^-_1\) in the cubic group.

- **Figure 2** autocorrelation behaviors \(\rho\) of operator a) - d) in Fig. 1 and their fitting curves. Fitting curves (dashing curve): a) is \(\rho = e^{0.0786 - 0.675\sqrt{\Delta T_{MC}}}\); b) is \(\rho = e^{0.92 - 2.02\sqrt{\Delta T_{MC}}}\); c) is \(\rho = e^{0.7 - 1.83\sqrt{\Delta T_{MC}}}\); and d) is \(\rho = e^{0.64 - 2.27\sqrt{\Delta T_{MC}}}\). The solid curves in a) are \(\rho = e^{2.01 - 0.261\Delta T_{MC}}\) (large \(\Delta T_{MC}\) region) and \(\rho = e^{-1.0 - 0.104\Delta T_{MC}}\) (intermediate \(\Delta T_{MC}\) region) respectively. The vertical coordinate is \(\rho\) and the horizontal one is \(\Delta T_{MC}\).

- **Figure 3** \(-ln\rho\) verse \(m_\alpha\). The dashing curve is \(-ln\rho = -0.0786 + 0.675\sqrt{\Delta T_{MC}}\). The solid line are \(-ln\rho = -2.01 + 0.261\Delta T_{MC}\) (large \(\Delta T_{MC}\) region) and \(-ln\rho = 1.0 + 0.104\Delta T_{MC}\) (intermediate \(\Delta T_{MC}\) region) respectively. The vertical coordinate is \(-ln\rho\) and the horizontal one is the mass extracted from the operator.

- **Figure 4** \(c\) verse \(m_\alpha\). The fitting line is \(c = -0.65 + 1.94m_\alpha\) (The dashing line is \(c = 1.487m_\alpha\)). The vertical coordinate is \(c\) and the horizontal one is the mass extracted from the operator.
plot3.m
