The star formation history of the solar neighbourhood from the white dwarf luminosity function

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Accepted 2013 June 18. Received 2013 June 18; in original form 2013 January 8

ABSTRACT

The termination in the white dwarf luminosity function is a standard diagnostic tool for measuring the total age of nearby stellar populations. In this paper, an algorithm is presented for inverting the full white dwarf luminosity function to obtain a maximum likelihood estimate of the time-varying star formation rate of the host stellar population. Tests with synthetic data demonstrate that the algorithm converges over a wide class of underlying star formation rate forms. The algorithm successfully estimates the moving average star formation rate as a function of lookback time in the presence of realistic measurement noise, though suffers from degeneracies around discontinuities in the underlying star formation rate. The inversion results are most sensitive to the choice of white dwarf cooling models, with the models produced by different groups giving quite different results. The results are relatively insensitive to the progenitor metallicity, initial mass function, initial–final mass relation and ratio of H/He atmosphere white dwarfs. Application to two independent determinations of the solar neighbourhood white dwarf luminosity function gives similar results. The star formation rate has a bimodal form, with broad peaks at 2–3 and 7–9 Gyr in the past, separated by a significant lull of magnitude 30–90 per cent depending on the choice of cooling models. The onset of star formation occurs around 8–10 Gyr ago. The total integrated star formation rate is $\sim 0.014$ stars pc$^{-3}$ in the solar neighbourhood, for stars more massive than 0.6 $M_\odot$.

Key words: white dwarfs – Galaxy: formation – solar neighbourhood.

1 INTRODUCTION

The white dwarf luminosity function (WDLF) is a useful tool for determining the age of a population of stars. The magnitude at which the function terminates is time dependent, and by fitting the faint end with theoretical WDLF models of different ages one can obtain a statistical estimate of the age of the population without having to determine the total age of any individual white dwarf (WD), which is considerably more difficult. This technique has been applied successfully to single-burst populations such as open and globular clusters (e.g. NGC 2158, Bedin et al. 2010; M4, Bedin et al. 2009), where the comparison of WD and main-sequence (MS) ages has proved to be very fruitful. NGC 6791 in particular has provided an important benchmark for understanding WD cooling processes at faint magnitudes (García-Berro et al. 2010) and the binary fractions of old, metal-rich clusters (Bedin et al. 2008).

The Galactic disc WDLF has been examined many times over the years (Winget et al. 1987; Oswalt et al. 1996; Leggett, Ruiz & Bergeron 1998), with studies finding ages in the range 8–10 Gyr depending on the WD evolutionary models adopted. For continuous populations such as the disc, the faintest WDs are the descendants of high-mass MS stars that formed at very early times, and their lifetimes are completely dominated by the WD cooling phase, leading to age estimates largely independent of uncertainties associated with MS lifetimes. The picture is considerably more complicated at brighter magnitudes, because the WDs are a mixture of ages: both young, high-mass WDs that are produced by recently formed MS progenitors and old, low-mass WDs produced by low-mass MS stars that formed at early times. It is for this reason that nearly all studies have focused exclusively on the faint turnover of the disc WDLF in an attempt to constrain the total disc age.

The groundbreaking study of Noh & Scalo (1990) revealed that the detailed shape of the WDLF at magnitudes brighter than the peak is far more sensitive to the time-varying star formation rate (SFR) than to variations in the initial mass function (IMF). By forward modelling methods, they demonstrated that a marginal feature in the WDLF at $M_{bol} \approx 10$ could be interpreted as evidence for a burst of star formation 0.3 Gyr ago. According to both Iben & Laughlin (1989) and Noh & Scalo (1990), the shape of the WDLF at intermediate magnitudes is also affected by the cooling rates of WDs: magnitudes at which the cooling is faster tend to have lower WD numbers, due to WDs transiting quickly to fainter magnitudes. Other authors have used this fact to examine the WDLF...
for evidence of additional WD cooling mechanisms, beyond those currently included in WD cooling theory. One possibility is cooling by emission of ‘axions’ (e.g. Isern et al. 2008; Melendez, Miller Bertolami & Althaus 2012), a light pseudo-scalar particle postulated by the Peccei–Quinn theory. Isern et al. (2008) argue that the rising slope of the WDLF at $M_{\text{bol}} < 13$ is independent of age, with the shape determined solely by WD cooling rates. By forward modelling the WDLF assuming a constant SFR, and with WD cooling models that include axion emission, they find a best-fitting axion mass of 5 meV.

1.1 Forward modelling the WDLF

The standard equation for modelling the WDLF for a given star formation history is (e.g. Iben & Laughlin 1989; Fontaine, Brassard & Bergeron 2001)

$$\Phi(M_{\text{bol}}) = \int_{M_{\text{bol}}}^{M_{\odot}} \frac{dM_{\text{cool}}}{dM_{\text{bol}}} \psi(T_0 - t_{\text{cool}} - t_{\text{MS}}) \phi(M) dM,$$

(1)

where $\Phi(M_{\text{bol}})$ is the number density of WDs at magnitude $M_{\text{bol}}$. The derivative inside the integral is the characteristic cooling time for WDs, $\psi(t)$ is the SFR at time $t$ and $\phi$ is the IMF. The integral also depends on the lifetimes of MS progenitors as a function of mass and metallicity $t_{\text{MS}}$, the WD cooling times as a function of mass and luminosity $t_{\text{cool}}$, and the total time since the onset of star formation $T_0$. The integral is over all MS masses that have had time to produce WDs at the present day, with the magnitude-dependent lower limit corresponding to the solution of

$$T_0 = t_{\text{cool}}(M_{\text{bol}}, \xi(M)) + t_{\text{MS}}(M, Z)$$

(2)

and the upper limit for WD production $M_{\odot} \approx 7 M_{\odot}$, $\xi$ is the initial–final mass relation (IFMR) that relates the mass of an MS star to the mass of the WD that it forms. We note in passing that for modelling single-burst populations, inserting a delta function for the SFR simplifies equation (1) to

$$\Phi(M_{\text{bol}}) = \phi(M) \frac{dM}{dM_{\text{bol}}},$$

(3)

This can also be derived by considering the conservation of stars between corresponding progenitor mass and WD bolometric magnitude intervals, $\Phi(M_{\text{bol}}) dM_{\text{bol}} = \phi(M) dM$. For such populations, there is a one-to-one correspondence between the WD bolometric magnitude and MS progenitor mass, given by the solution (if any) to equation (2).

1.2 On the invertibility of the WDLF

Although there have been several major studies to develop forward modelling approaches to estimating the Galactic age from the WDLF (e.g. Winget et al. 1987; Oswalt et al. 1996; Leggett et al. 1998), it appears that not much work has been done on the possibility of inverting the WDLF to obtain a direct estimate of the SFR. This is in stark contrast to comparable studies using MS stars, for which very mature Bayesian methods have been developed for inverting colour–magnitude diagrams (CMDs). Hernandez, VallsGabaud & Gilmore (2000) and Vergely et al. (2002) develop largely independent Bayesian techniques for inverting the Hipparcos CMD to measure the solar neighbourhood SFR. The non-parametric nature of these approaches is a great benefit of the inversion method: forward modelling techniques normally involve selecting a parametrization for the SFR and then optimizing the parameters using some choice of cost function to compare models to the data (see, for example, Bertelli & Nasi 2001). The solution is therefore only optimal in the context of the adopted model, which might not correspond to reality. Non-parametric methods allow the full form of the SFR solution to be determined from the data alone, rather than forcing it to conform to some imposed function.

It has generally been thought that the shape of the WDLF is almost independent of the SFR at all but the faintest magnitudes, where it is governed mainly by the total population age. Isern et al. (2008) present the following argument for the bright portion of the WDLF being independent of the SFR: the characteristic cooling time of a WD is not very sensitive to mass so the derivative in equation (1) can be taken out of the integral and replaced with an average over all WD masses. Because the cooling rates of WDs are highly non-linear, at bright magnitudes the cooling time is relatively small (of the order of 200 Myr at $M_{\text{bol}} = 10$) and the lower limit of the integral (determined from equation 2) is satisfied by low-mass stars and is almost constant. Therefore, as long as the SFR is a well-behaved function (no large bursts or lulls within the last ~200 Myr) and $T_0$ is large enough, the integral is not sensitive to the WD luminosity and the WDLF is determined almost entirely by the average cooling rates of WDs.

Isern, Artigas & García-Berro (2012) take this further and assert that the WDLF is in fact non-invertible, due to equation (1) failing to satisfy the Picard–Lindelöf theorem for the inversion of integral equations, meaning that the solution is sensitive to the trial function used and its uniqueness is therefore not guaranteed, ultimately due to the smoothing effect of the integral washing out sensitivity to high-frequency components of the SFR. Some justification for this is provided by Fig. 1 which shows the similarity between WDLFs computed using quite different assumptions for the SFR (reproduction of fig. 5 of Isern et al. 2012). Brighter than $M_{\text{bol}} \sim 13$, these are almost totally degenerate. In the lower panel, the mean age of WDs is roughly constant at these magnitudes, but fainter than $M_{\text{bol}} \sim 12–13$; there is a correlation between magnitude and age. It is at these magnitudes that we expect the shape of the WDLF to contain information on time variations in the SFR. It is important to point out that although the star formation histories included in this figure cover a broad range of scenarios, they are all very smooth.

Figure 1. Upper panel: simulated WDLFs for a variety of different star formation histories and ages. The solid and dot-dashed lines are for a constant SFR with $T_0 = 10, 13$ Gyr, respectively; the dashed line is for $\psi \propto \exp(-t/\tau)$ with $\tau = 3$ Gyr and $T_0 = 13$ Gyr; the dotted line is for $\psi \propto (1 + \exp((t - t')/\tau))^{-1}$ with $t' = 10$ Gyr, $\tau = 3$ Gyr and $T_0 = 13$ Gyr. Lower panel: mean total stellar age as a function of WD bolometric magnitude, for the same star formation histories as in the upper panel.
functions with few degrees of freedom and little information content. We will demonstrate in this paper that for such smooth star formation histories, the few WDLF points around the peak are indeed sufficient to recover the overall form of the time-varying SFR by inversion methods, and that the non-uniqueness of the solution is limited to a loss of resolution around discontinuities. In addition, a simple trial function (in this case, a constant SFR) can recover a wide range of different underlying SFR forms with a good degree of confidence.

1.3 This paper

This paper presents the results of work to develop an inversion algorithm suitable for application to the WDLF, in order to measure the time-varying SFR. To a first approximation, the two parameters that determine the total age of a WD are the present-day bolometric magnitude and the mass. These can be used to determine both the total WD cooling time and the time spent on the MS. The approach to inverting the WDLF developed in this paper is based on the observation that if the distribution of WD mass was known at all magnitudes, then the WDLF could be directly transformed to the SFR, due to the correspondence between points in the progenitor mass/formation time plane and the WD mass/luminosity plane. As this quantity is generally not known observationally, this direct approach cannot be used. Instead, we use an inversion technique based on the expectation–maximization method (Dempster, Laird & Rubin 1977; Do & Batzoglou 2008), which is used to obtain maximum likelihood estimates of the solution to inverse problems in the presence of missing data. This technique is used widely in image restoration, where it is called Richardson–Lucy deconvolution (see also Binney & Merrifield 1998, appendix C). Applied to the WDLF, this iterative technique involves using an initial guess of the SFR to derive the present-day WD mass/luminosity distribution, which is then normalized to the observed luminosity function (LF), before transforming back to obtain an improved estimate of the SFR.

Note that although all the information pertaining to the time-varying SFR lies in the region fainter than $M \sim 0 \sim 12-13$, in the inversions presented in this paper we will include the hot branch of the WDLF: while this does not help to constrain the time-varying SFR, it improves overall constraint and reduces global errors on the SFR. Also, valid solutions should at least be consistent with this region of the WDLF so it provides an additional sanity check on the results.

This work is motivated by two related questions: given current WD cooling models, what constraint can features in the WDLF at all magnitudes place on the time-varying SFR? And as a corollary to this: can features in the WDLF be explained exclusively by time variations in the SFR, or are additional cooling mechanisms required?

2 STATISTICAL FRAMEWORK

The WDLF inversion algorithm involves iteratively refining an initial guess of the SFR. The general procedure for each iteration is as follows. The starting point is an initial guess of the SFR $\psi_0$,

$$\psi_0 \equiv \psi_0(t)$$

(4)

with $t$ the lookback time and $\psi$ measured in units of stars per year. In this work, $\psi_0$ is flat, i.e. a constant SFR. It will be demonstrated later that this is sufficient to recover a wide range of different underlying SFR forms. This is combined with the IMF $\phi$ to get the joint mass and formation time distribution of MS progenitors $P_{MS}$, where

$$P_{MS}(M_{MS}, t) = \phi(M_{MS}) \psi(t).$$

(5)

$P_{MS}$ is thus separable at this stage, assuming that the IMF $\phi$ is independent of time. $P_{MS}$ is the distribution of WD progenitors, i.e. the subset of MS stars that form WDs at the present day. The region of the $[M_{MS}, t]$ plane that these stars inhabit is bounded by the functions

$$M_{MS}^{\text{upper}} = M_{MS}^{\text{max}},$$

$$M_{MS}^{\text{lower}} = M_{MS}^{\text{lifetime}}(t),$$

$$t_{\text{upper}} = T_0,$$

(6)

where $M_{MS}^{\text{max}} = 7.0 M_\odot$ is the maximum progenitor mass for WD formation. The function $M_{MS}^{\text{lifetime}}(t)$ is the mass of the MS star with lifetime $t$; MS stars with lifetimes longer than the lookback time do not have time to form WDs at the present day. $T_0$ is the maximum lookback time and is a parameter of the algorithm. Note that $T_0$ does not enforce a fixed total age on the stellar population, because for populations younger than $T_0$, the SFR will be driven towards zero at lookback times less than $T_0$. Its main purpose is to exclude unphysical solutions. No a priori knowledge of the total time since the onset of star formation is required. Fig. 2 shows an example $P_{MS}$ distribution generated during testing.

Using standard rules of probability density functions, $P_{MS}$ is transformed to the joint mass and bolometric magnitude distribution of white dwarfs $P_{WD}(M_{WD}, M_{bol})$ at the present day:

$$P_{WD}(M_{WD}, M_{bol}) = P_{MS}(M_{MS}, t) \cdot \frac{\partial(M_{MS}, t)}{\partial(M_{WD}, M_{bol})},$$

(7)

where the Jacobian expresses the transformation of the $dM_{MS} \, d\text{are}$ element in order to conserve the volumetric probability between

Figure 2. Example plot of $P_{MS}$. In the top panel, the upper grey region of the plane corresponds to MS stars that do not produce WDs; the lower grey region corresponds to MS stars that have not had time to produce WDs at the present day. The plane is bounded on the left by the maximum lookback time, set to 13 Gyr here. The contours lie at intervals of $0.15 \times 10^{-12} \text{ yr}^{-1} M_\odot^{-1}$; for clarity only the first six are plotted. The solid black lines mark the region inhabited by progenitors that form $13 < M_{bol} < 13.5 \text{ WDs (H atmosphere)}$ at the present day. The lower panel shows the SFR after integrating over the progenitor mass.
corresponding intervals in $P_{\text{MS}}$ and $P_{\text{WD}}$. The $(M_{\text{WD}}, M_{\text{bol}})$ plane is not fully populated; it is bounded at high mass by the line

$$M_{\text{WD}}^{\text{upper}} = \xi(M_{\text{MS}}^{\text{max}})$$

and at bright and faint magnitudes by the functions

$$M_{\text{WD}}^{\text{upper}} = M_{\text{bol}}(M_{\text{WD}}, t_{\text{cool}} = T_0 - t_{MS}(\xi^{-1}(M_{\text{WD}}, Z)))$$

$$M_{\text{WD}}^{\text{lower}} = M_{\text{bol}}(M_{\text{WD}}, t_{\text{cool}} = 0).$$

The WD bolometric magnitude at given mass and cooling time $M_{\text{bol}}(M_{\text{WD}}, t_{\text{cool}})$ is obtained from theory. Fig. 3 shows an example $P_{\text{WD}}$ distribution generated during testing.

Because both MS lifetimes and WD cooling rates are mass dependent, $P_{\text{WD}}$ is not separable; the variables $(M_{\text{WD}}, M_{\text{bol}})$ are reasonably correlated, with WDs of all masses existing at high magnitudes, and faint magnitudes inhabited exclusively by high-mass WDs. $P_{\text{WD}}$ can, however, be separated into a part of the marginal luminosity distribution $\Phi_{\text{sim}}(M_{\text{bol}})$ and the conditional probability of $M_{\text{WD}}$ given $M_{\text{bol}}$

$$P_{\text{WD}}(M_{\text{WD}}, M_{\text{bol}}) = \Phi_{\text{sim}}(M_{\text{bol}})P_{\text{WD}}(M_{\text{WD}}|M_{\text{bol}}).$$

The quantity $\Phi_{\text{sim}}(M_{\text{bol}})$ is just the WDLF for the initial guess SFR model, up to a normalization factor. The next crucial step is to replace the simulated WDLF in equation (10) with the observed WDLF $\Phi_{\text{obs}}(M_{\text{bol}})$, in order to obtain the updated joint distribution $P_{\text{WD}}'$, where

$$P_{\text{WD}}'(M_{\text{WD}}, M_{\text{bol}}) = \Phi_{\text{obs}}(M_{\text{bol}})P_{\text{WD}}(M_{\text{WD}}|M_{\text{bol}}).$$

This updated WD distribution has the same marginal luminosity distribution as the observed WDLF and the magnitude-dependent mass distribution derived from the initial guess SFR model. We can now transform this distribution to obtain the updated distribution for MS stars $P_{\text{MS}}'$ again using standard transformation rules:

$$P_{\text{MS}}'(M_{\text{MS}}, t) = P_{\text{WD}}'(M_{\text{WD}}, M_{\text{bol}}) \frac{\partial (M_{\text{WD}}, M_{\text{bol}})}{\partial (M_{\text{MS}}, t)}.$$ 

In general, $P_{\text{MS}}'(M_{\text{MS}}, t)$ is not separable; the correction to $P_{\text{MS}}$ produces a distribution $P_{\text{MS}}'$ for which the marginal MS mass distribution varies over time. It is implicit in the algorithm that the IMF is independent of time, and that valid solutions should have this property. It will be demonstrated empirically that as the algorithm proceeds, it converges towards an SFR that produces a present-day WDLC that is an increasingly better match to the observed WDLF. Close to convergence, the correction to $P_{\text{MS}}$ is very small, and $P_{\text{MS}}'$ becomes separable as required.

The final step is to marginalize $P_{\text{MS}}'$ over the MS mass, to obtain the updated SFR model $\psi_1$:

$$\psi_1(t) = \frac{1 - A(t)}{\int_{0.6M_\odot}^{M_{\text{MS,obs}}}} P_{\text{MS}}'(M_{\text{MS}}, t) \, dM_{\text{MS}}.$$ 

The integral is over all MS stars that produce WDs at the present day. The factor $\frac{1 - A(t)}{\int_{0.6M_\odot}^{M_{\text{MS,obs}}}}$ corrects for low-mass MS stars that have not had time to form WDs at the present day, and is calculated by

$$A(t) = \int_{0.6M_\odot}^{M_{\text{MS,obs}}(t)} \Phi(M_{\text{MS}}) \, dM_{\text{MS}}.$$ 

The lower mass limit here is chosen as $0.6 M_\odot$; the IMF is still well constrained at this mass, and no star less massive than this forms a WD in any realistic lookback time, so using this as the normalization point avoids overcorrection. The SFR that we measure therefore only accounts for stars more massive than $0.6 M_\odot$.

### 2.1 Additional considerations

#### 2.1.1 WD atmosphere types

Along with the mass and present luminosity, the H/He atmosphere type is a third parameter affecting the cooling time of a WD. This has a significant effect at larger cooling times ($\gtrsim 6$ Gyr depending on the choice of models), with hydrogen atmosphere WDs being brighter at a given age. In this paper, we use the cooling sequences of Fontaine et al. (2001) and Salaris et al. (2010) (see Section 3.1.4), each of which provides cooling times for both H and He atmosphere WDs.

The two atmosphere types can be included in the algorithm in a relatively straightforward manner. We compute the joint mass and bolometric magnitude distribution separately for each type to obtain $P_{\text{WD}}^\text{H}$ and $P_{\text{WD}}^\text{He}$. Under the assumption that the atmosphere type is fixed at birth for WDs, these functions can be superposed to obtain the full distribution

$$P_{\text{WD}} = \alpha P_{\text{WD}}^\text{H} + (1 - \alpha) P_{\text{WD}}^\text{He},$$

where $\alpha = \frac{M_\odot}{M_{\text{MS,obs}}}$ lies in the range 0–1. Observational determinations of $\alpha$ are complicated by the fact that the spectral energy distribution of cooling WDs appears to evolve over time, presumably due to convective mixing of stratified layers. Tremblay & Bergeron (2008) find a value of $\frac{M}{M_\odot}$ to be 0.25 to be appropriate at high temperatures ($T_{\text{eff}} \gtrsim 12$ 000 K), before the effect of convective mixing has set in; this corresponds to $\alpha = 0.8$, which is the value we adopt in this work.
2.1.2 Undetected WDs and algorithm convergence

Observational determinations of the WDLF generally do not cover the full range of bolometric magnitude. In general, due to the intrinsic rarity of very bright WDs, and the difficulty in detecting very faint WDs, there are ranges over which the number density of WDs is unconstrained observationally. However, it is still possible to constrain the SFR over the full history, because at any given formation time stars produce WDs over a relatively wide range of \( M_{\text{bol}} \), and as long as some of these are observed, the SFR can in principle be constrained.

The implication of this with respect to the inversion algorithm is that at any given formation time, some fraction of the WD progenitors may produce WDs that are undetected at the present day, i.e. lie outside the range of the observed WDLF used to constrain the algorithm. These ‘unobserved’ stars will lie in one or more non-contiguous ranges of progenitor mass, corresponding to intervals in bolometric magnitude where the WD density is unconstrained observationally. The probability density of these objects remains at the initial value on applying equation (11), and they provide no constraint on the SFR. This can cause the algorithm to take a long time to converge, especially at times where the true SFR is very low relative to the initial guess. The lower constraint also leads to a greater risk of systematic error on the recovered SFR.

A better way to handle unobserved stars is to exclude them from the integral when calculating the updated SFR (equation 13), and then to multiply \( \psi_1 \) by a suitable factor to correct for the fraction of missing stars. The correction factor is determined from the IMF, in the same way as for low-mass stars. This allows the algorithm to converge much faster, although the risk of systematic error remains. For this reason, we record the fraction of unobserved stars as a function of the formation time as a diagnostic for the systematic error on the recovered SFR. In practice however, the observed WDLF – at least in the case of the Galactic disc – covers almost the entire bolometric magnitude range, and only a few per cent of WDs at the faintest magnitudes are missing.

2.1.3 Lookback time resolution

The lower limit on the lookback time is set by the lifetime of the most massive WD progenitor, which in practice is \( \sim 10^7 - 10^8 \) yr depending on metallicity (Bertelli et al. 2008, 2009). The algorithm is completely insensitive to variations in the SFR that occurred more recently than this. There is no theoretical upper limit on the lookback time, although most IFMRs break down at low mass (e.g. 0.48 M_\odot for Kalirai et al. 2008; 0.47 M_\odot for Catalán et al. 2008), producing WDs more massive than their MS progenitors. As these live for much longer than a Hubble time, this imposes no constraint on realistic models. Between these extremes, the finite magnitude resolution of the observational WDLF leads to a lower limit on the SFR time resolution. At any age, there is a frequency above which variations in the SFR produce no discernible change in the observed WDLF. Generally the resolution is poorer at older times, due to the cooling rates of WDs slowing with age.

Considering this, the set of lookback time bins used to represent the initial guess SFR needs to be selected with some care. Attempting to match bin sizes to cooling times of WDs over constant intervals in magnitude (so that, for example, more recent bins are narrower where cooling is faster) is attractive from an observational point of view, but suffers from highly underpopulated bins at recent times where cooling is very rapid and only high-mass stars contribute.

A scheme using lookback time intervals corresponding to a constant number of WDs at the present day (so that, for example, older time bins are narrower where the density of WD progenitors is greater) is attractive due to the statistical noise being roughly uniform in each time bin, but requires very narrow magnitude bins around the peak of the WDLF which is not justified observationally because magnitude errors are too large.

A simple compromise between these two is to use lookback time bins of a constant width. This is the approach taken in this work. However, it should be remembered that high-frequency components of the underlying SFR are likely to be lost at older times. Experiments with synthetic data presented in the following sections will attempt to quantify this.

2.1.4 Convergence criteria

The convergence of the inversion algorithm is assessed by checking the goodness of fit between the simulated and observed WDLFs. Once an SFR has been arrived at that results in a WDLF that closely matches the observed present-day WDLF, no further improvement can be made and the algorithm must be halted to prevent further iterations from overfitting the noise in the data, causing unrealistic spikes to develop in the SFR.

The goodness of fit is measured using the \( \chi^2 \) statistic between the simulated and observed WDLFs

\[
\chi^2 = \sum_k \frac{(\Phi^\text{sim}_k - \Phi^\text{obs}_k)^2}{\sigma^2_{\Phi^\text{obs}_k}},
\]

where the sum is over all bolometric magnitude bins in the observed WDLF.

Tests with synthetic data suggest that convergence is reached when the relative change in \( \chi^2 \) from one iteration to the next falls below a threshold of approximately 1 per cent. This level allows a good fit to be reached, while avoiding overfitting of the data. In order to prevent statistical noise in \( \chi^2 \) from affecting the convergence, we use a sliding linear fit to the most recent five \( \chi^2 \) values and calculate the relative change at the latest iteration from this. The first iteration is always omitted from the fit, because if the initial guess is significantly far from the truth, then the first \( \chi^2 \) is an outlier.

3 Monte Carlo modelling procedure

In this work, the inversion algorithm is implemented using a Monte Carlo method, which is described now. The initial guess SFR (in units of stars per year) is first broken into a fixed number of discrete lookback time bins of width \( \delta t \)

\[
\psi_0(t) \rightarrow \psi^i_0
\]

over which the SFR is assumed to be constant. A finite number \( N \) of simulated stars are generated in each bin with formation times drawn uniformly within the range of the bin and masses drawn from the IMF. We also randomly assign a fixed H/He WD atmosphere type to each star at this point: stars are assigned an H atmosphere with probability \( \alpha \). Initially, each simulated star represents \( n = 1 \pm 1 \) real stars, assuming Poisson statistics. Within each bin, the number of real stars that each simulation star represents is then scaled to

\[
n = \frac{\psi^i_0 \delta t}{N}
\]
and the variance propagated to obtain

$$\sigma_{n_i}^2 = \left( \frac{\psi_i \delta t}{N} \right)^2. \tag{17}$$

The simulated stars are then evolved to the present day and binned according to their WD bolometric magnitude at a resolution that matches the observed WDLF used as input. Unobserved stars that fall outside the range of the WDLF are identified at this point, but are not purged from the simulated population. Prior to binning, we add a bolometric magnitude error to each star drawn from a Gaussian distribution with width $\sigma_{\delta M}$, which is designed to simulate photometric parallax errors. The value of $\sigma_{\delta M}$ should be close to the approximate size of bolometric magnitude errors on the observed WDLF. The LF (in units of stars per magnitude) in a given bolometric magnitude bin $k$ is obtained for the simulated population by

$$\Phi_{\text{sim}}^k = \frac{\sum_{i=1}^{N_i} n_i^j}{\delta M_{\text{bol}}} \tag{18}$$

with the associated statistical uncertainty

$$\sigma_{\Phi_{\text{sim}}}^2 = \frac{\sum_{i=1}^{N_i} \sigma_{n_i}^2}{\left( \delta M_{\text{bol}} \right)^2}, \tag{19}$$

where $n_i^j$ is the number of real stars represented by the $i$th simulated star in bolometric magnitude bin $k$. At this point, we measure the goodness of fit between the simulated and observed WDLFs using equation (16), and check for convergence of the algorithm. If convergence has been reached, then $\psi_i$ is the solution for the SFR.

Next, the number of real stars that each simulated star represents is scaled so that $\Phi_{\text{sim}} = \Phi_{\text{obs}}$, i.e. for star $i$ in bolometric magnitude bin $k$

$$n_i \rightarrow n_i^j = n_i \times \frac{\Phi_{\text{obs}}^k}{\Phi_{\text{sim}}^k}$$

and

$$\sigma_{n_i}^2 \rightarrow \sigma_{n_i}^2 = n_i^2 \times \sigma_{\Phi_{\text{obs}}}^2 \frac{\Phi_{\text{obs}}^k}{\Phi_{\text{sim}}^k} \times \sigma_{\Phi_{\text{sim}}}^2.$$

This results in a simulated population of WD progenitors for which the present-day WDLF obtained using equation (17) exactly equals the observed WDLF, and for which the uncertainty is purely statistical arising from the finite number of simulated stars. This is not strictly appropriate, as the error on the simulated WDLF and recovered SFR could be driven arbitrarily low by using a large enough number of simulation stars. In fact, we wish to assign uncertainties to our simulated stars in such a way that the uncertainty on the simulated WDLF matches that on the observed WDLF, in the limit of a large number of simulation stars. This is the best that could be achieved, given the errors on the data. Any finite number of simulation stars will result in increased uncertainty, due to the additional contribution from statistical errors. This is achieved in a given bolometric magnitude bin $k$ by adding the term

$$\sigma_{n_i}^2 \rightarrow \sigma_{n_i}^2 = \sigma_{n_i}^2 + \frac{\left( \delta M_{\text{bol}} \right)^2}{n_i} \sigma_{\Phi_{\text{obs}}}^2,$$

where $N_i$ is the number of simulated stars in bin $k$ and $\langle n_i \rangle$ is the mean number of real stars that each simulated star represents.

Inserting this term alone into the equation for the error on the simulated WDLF (equation 18) gives the desired result $\sigma_{\Phi_{\text{sim}}}^2 = \sigma_{\Phi_{\text{obs}}}^2$ in the limit of zero statistical error.

The updated SFR $\psi_i$ and formal error $\sigma_{\psi_i}$ can now be obtained from the simulated star population using the equations

$$\psi_i^j = \frac{N}{\sum_{i=1}^{N_{\text{obs}}} \left( \frac{N}{N_{\text{obs}}^j} \right) \left( \frac{1}{1 - A_j} \right)} \tag{20}$$

$$\sigma_{\psi_i}^2 = \frac{\sum_{i=1}^{N} \sigma_{n_i}^2}{\left( \delta t \right)^2} \left( \frac{N}{N_{\text{obs}}^j} \right)^2 \frac{\sigma_{\Phi_{\text{obs}}}^2}{\Phi_{\text{sim}}} \frac{1}{\left( 1 - A_j \right)^2}, \tag{21}$$

where $n_i^j$ is the number of real stars represented by the $i$th simulated star in formation time bin $j$. The sum includes only observed stars, i.e. stars that form WDs that lie within the range of the observed WDLF at the present day. The factor $\frac{N}{N_{\text{obs}}^j}$ corrects the rate for the fraction of missing stars, where $N$ is the number of simulation stars in each formation time bin and $N_{\text{obs}}^j$ is the number of observed simulation stars in bin $j$. The factor $\frac{\sigma_{\Phi_{\text{obs}}}^2}{\Phi_{\text{sim}}}$ accounts for low-mass stars that form in bin $j$ that do not produce WDs at the present day, and is calculated by

$$A_j = \frac{1}{\delta t} \int_{M_{\text{MS}}}^{M_{\text{final}}(Y)} \phi(M) \, dM \, d \tau. \tag{22}$$

This is analogous to equation (14) in the continuous case.

### 3.1 Input parameters

#### 3.1.1 The initial mass function

In this study, the IMF is a simple power law with exponent $–2.3$. This is appropriate for stars more massive than $0.5 \, M_\odot$ (Kroupa 2001), which encompasses the entire range of interest.

#### 3.1.2 MS lifetimes

In order to provide an estimate of the MS lifetime as a function of progenitor mass, we adopt the stellar evolutionary model grid of the Padova group (Bertelli et al. 2008, 2009). These cover the mass range $0.15–20 \, M_\odot$ for 39 different metal and helium abundances. We consider the pre-WD phase to last from the zero-age MS to the first thermal pulse, and for low-mass stars that do not produce WDs at the present day, and is calculated by

$$A_j = \frac{1}{\delta t} \int_{M_{\text{MS}}}^{M_{\text{final}}(Y)} \phi(M) \, dM \, d \tau. \tag{22}$$

This is analogous to equation (14) in the continuous case.

#### 3.1.3 The initial–final mass relation

The IFMR $\xi$ determines the mass of the white dwarf $M_{\text{WD}}$ that forms from a progenitor of mass $M_{\text{MS}}$. In this work, we consider
a range of IFMRs and adopt the empirical linear model of Kalirai et al. (2008) as a baseline. This relation has the form

$$M_{\text{WD}} = 0.109M_{\text{MS}} + 0.428. \quad (23)$$

In conjunction with the upper mass limit for WD formation, this relation fixes the mass of the heaviest WD at 1.19 $M_\odot$. At low masses, this relation breaks down at $M_{\text{WD}} = M_{\text{MS}} \sim 0.48 M_\odot$. However, stars of this mass have ages well in excess of the age of the Universe and do not contribute to our models.

### 3.1.4 WD cooling times

WD luminosities at a given cooling time, mass and atmosphere composition are obtained by interpolating grids of model cooling sequences. In order to check the sensitivity of the results to the WD models, we use two independent sets.

The first set of models is that of the Montreal group. We use the latest set of cooling sequences available online\(^1\) at the time of writing: these are described in Fontaine et al. (2001) and Bergeron, Leggett & Ruiz (2001), and updated in Tremblay, Bergeron & Gianninas (2011), Bergeron et al. (2011) and references therein. Note that we only require bolometric magnitude as a function of the cooling time: colours in specific filter systems and synthetic spectra are not used. These cooling sequences will be referred to as the F01 models from now on. The F01 models are computed using pure carbon cores at high temperatures ($T_{\text{eff}} > 30,000$ K) and uniformly mixed carbon/oxygen cores of equal mass fraction at lower temperatures. The core composition is important because the rate of cooling is determined, among other things, by the ionic specific heat. In these models, the additional energy source at low temperatures associated with the sedimentation of carbon and oxygen upon crystallization of the core is not included. The hydrogen atmosphere models have standard ‘thick’ envelopes, consisting of an outer H layer of mass fraction $q_H = 10^{-4}$ on top of an He layer of mass fraction $q_{\text{He}} = 10^{-2}$. The helium atmosphere models are similar, but with $q_{\text{He}} = 10^{-10}$. For both atmosphere types, models are computed at constant mass over the range 0.2–1.2 $M_\odot$ in steps of 0.1 $M_\odot$. For each mass, the cooling time varies from up to several Myr to 15 (8) Gyr for H (He) models, over which time the bolometric magnitude varies approximately in the range 0(6)–20(17). For the He models, a discontinuity exists between around 25 000 and 35 000 K which we interpolate over. This is due to the use of two different sets of evolutionary models at high and low temperatures, which match well on stellar radii but have small discontinuities in the cooling times at the point where they are joined (Bergeron, private communication). No such discontinuity is present in the H models.

The second set of models is taken from the BaSTI data base\(^2\) and is described in Salaris et al. (2010). For these models, the core is composed of a carbon and oxygen mixture with relative abundance and distribution that varies as a function of WD mass (see Salaris et al Fig. 2). For each WD mass, the C/O stratification is taken from a connecting MS evolutionary model at the first thermal pulse (obtained from the BaSTI data base for solar metallicity). Cooling sequences are computed with and without the effects of C/O phase separation and sedimentation on crystallization, which slows the cooling of stars at low luminosities. These models will be referred to as S10 and S10p (including C/O phase separation effects) from now on. For the hydrogen atmosphere models, the envelope consists of a surface H layer with mass fraction $q_H = 10^{-4}$ on top of a $q_{\text{He}} = 10^{-2}$ He layer (the same as for F01), and the helium atmosphere models have an envelope consisting of a single $q_{\text{He}} = 10^{-3.5}$ He layer. Nine discrete masses are computed in the range $0.54$–$1.2 M_\odot$, in intervals varying from 0.01 to 1.0 $M_\odot$. Models are denser in the $0.5$–$0.8 M_\odot$ range, to reflect the general higher abundance of these stars. For each mass, the cooling time varies from around several Myr to 15 (8) Gyr for H (He) models, and the bolometric magnitude varies approximately in the range 3–16 for both types. Two representative cooling tracks are shown in Fig. 4 comparing the F01, S10 and S10p models.

We experimented with a variety of interpolating functions and found a bilinear scheme to be the most appropriate. It is sometimes necessary to extrapolate bolometric magnitudes at points outside the model grid, either at very early or late cooling times, or at very low masses in the case of the S10 models. We use the same bilinear method for this, and are careful to check that results do not rely too heavily on extrapolated points far outside the model grid.

### 4 VALIDATION WITH SYNTHETIC DATA

#### 4.1 Synthetic data generation

In order to test the performance of the inversion algorithm, we run it on a set of synthetic WDLFs derived from known input star formation histories. Synthetic WDLFs for test purposes can be calculated in the same way as is done during the inversion algorithm, i.e. generating a synthetic population of WDS and using equations (17) and (18). A WDLF generated by following the algorithm up to that point is strictly appropriate for a volume-limited sample of WDS; each WD that is present is used to determine the WDLF without selection according to magnitude. This is the best method for the inversion algorithm, when good constraint over the entire WDLF range is required. However, this is not realistic for modelling observational WDLFs: these are typically derived from magnitude-limited catalogues and have quite different noise profiles. The main difference is that the statistical noise at the faint end is much greater and the WDLF generally does not extend as far, due to intrinsically faint stars being preferentially lost. Fig. 5 shows two synthetic WDLFs derived from volume- and magnitude-limited WD populations. It is
important that our sensitivity and validation tests consider realistic noise models in order to get a true estimate of the performance of the algorithm on real data.

We simulate a magnitude-limited WDLF in the following manner. First, we select an appropriate apparent magnitude limit and use the minimum absolute magnitude of any simulated WD to determine the theoretical survey edge. In conjunction with a model of the density profile, this defines the total volume of the survey region $V_{\text{tot}}$. Then, for each simulated WD, we calculate the maximum observable distance and use this to determine the accessible survey volume $V_{\text{max}}$ in which the star is observable. This is used to assign an observation probability

$$p_{\text{obs}} = \frac{V_{\text{max}}}{V_{\text{tot}}},$$

assuming that stars are distributed uniformly within the survey volume. We use $p_{\text{obs}}$ to randomly select or reject each simulated WD; those that are selected have their density contribution scaled according to

$$n_i \rightarrow n'_i = n_i \times p_{\text{obs}}^{-1}$$

and

$$\sigma^2_{n_i} \rightarrow \sigma^2_{n'_i} = \sigma^2_{n_i} \times p_{\text{obs}}^{-2}$$

to correct for the missing stars. This is a variation on the standard $V_{\text{max}}$ technique that ensures that the WDLF spatial density is dimensionless, as required.

### 4.2 Convergence tests

The first round of tests is designed to check the convergence of the algorithm under a range of different SFR scenarios. The particular set of scenarios chosen is summarized in Table 1; these are intended to cover a wide variety of stellar population types. In all cases, the initial guess SFR model is a flat rate with a maximum lookback time of 13 Gyr, divided into 100 bins of approximately 130 Myr width. These tests use synthetic WDLFs generated from a large (2 x $10^4$) WD sample with no bolometric magnitude error. WDLFs are computed at a resolution and magnitude range that matches recent determinations of the WDLF in the solar neighbourhood ($M_{\text{bol}}$ = 1–18, $\Delta M_{\text{bol}}$ = 0.5; e.g. Rowell & Hambly 2011), but are otherwise noise-free, and the inversion algorithm uses the same set of input parameters as those used to generate the WDLFs. These are listed in Table 2.

The results of the inversion are shown in Fig. 6. In each case, the first three iterations are plotted along with the final converged fit, for both the SFR model and the WDLF. The algorithm achieves a reasonably good approximation to the underlying SFR in all cases, though with varying success. In all cases, the recent ($t < 5$ Gyr) SFR is accurately recovered. The constant SFR model (Fig. 6a) shows a significant deviation at earlier times, and the onset of star formation is not well resolved. However, the total integrated SFR of 8.97 x $10^{-3}$ is within 0.4 per cent of the true value. The exponentially decaying model is similar, and has an integrated SFR (9.89 x $10^{-3}$) that is within 0.1 per cent of the true value. In the case of the single-burst model, the shape of the burst is poorly resolved but the location of the peak is well recovered, and the integrated rate (1.53 x $10^{-3}$) is within 2 per cent of the true value. The fractal SFR fit reveals some important behaviour of the algorithm. High-frequency components and discontinuities in the SFR are only well resolved at recent times; after around 4 Gyr any features narrower than several Gyr are lost, and the converged fit approximates a moving average of the time-varying SFR. The integrated SFR of 5.007 x $10^{-2}$ is within 0.7 per cent of the true value. In all cases, the onset of star formation is resolved to around ~1 Gyr.

We investigated the effect on the SFR solution of using fainter ($M_{\text{bol}}$ = 24) and higher resolution ($\Delta M_{\text{bol}}$ = 0.25) WDLFs, and of setting the maximum lookback time equal to the time since the onset of star formation (for the case of measuring the time variation in star formation for a population of known total age). For both the faint and the high-resolution WDLFs, no significant difference is seen around the onset of star formation, although the fainter WDLF resolves the total age slightly better. The artefact in the constant and

### Table 1. SFR models considered.

| Type             | Integrated rate (stars) |
|------------------|-------------------------|
| Constant         | 9 x $10^{-3}$           |
| Exponential decay| 9.89 x $10^{-3}$        |
| Single burst     | 1.5 x $10^{-3}$         |
| Fractal          | 5.039 x $10^{-2}$       |

### Table 2. Input physics used in convergence tests.

| Parameter       | Value                           |
|-----------------|---------------------------------|
| IMF exponent    | -2.3                            |
| Metallicity     |                                 |
| Z               | 0.017                           |
| Y               | 0.30                            |
| IFMR            | $M_{\text{WD}} = 0.109 M_{\text{MS}} + 0.428$ (Kalirai et al. 2008) |
| $\sigma$        | 0.8                             |
| WD cooling sequences | F01                           |

Footnote: 1 The minimum lookback time of the initial guess rate is slightly greater than zero (~7–10 yr), due to the finite lifetime of the most massive WD progenitors, so bins are slightly narrower than 130 Myr.
Figure 6. Results of algorithm convergence tests on noise-free data. In each case, the panel on the left shows the recovered SFR model for the first three iterations (dashed grey) and the converged solution (dot–dashed), along with the ground truth (solid). The panel on the right shows the simulated WDLF derived for the same iterations, along with the observed WDLF used to constrain the algorithm. The inset panel shows the residuals for the final converged WDLF. (a) Constant SFR. (b) Exponentially decaying SFR. (c) Single-burst SFR. (d) Fractal SFR.
that the observed WDLF will be subject to both types to some degree.

We have repeated the tests of Section 4.2, using simulated WDLFs with realistic error models. These were calculated by reducing the number of simulated WDs to $10^4$, to give a statistical uncertainty on the number density that matches the WDLFs of Harris et al. (2006) and Rowell & Hambly (2011). Bolometric magnitude errors were added to each star by drawing from a Gaussian distribution of width 0.25. This is the approximate size of errors in the Rowell & Hambly (2011) WDLF; the Harris et al. (2006) errors are likely to be smaller given the better photometry of the Sloan Digital Sky Survey (SDSS) relative to SuperCOSMOS and the sharper features of their WDLF.

The effect of each type of error is shown in Fig. 7. The presence of realistic errors in the WDLF density (green lines) does not have a significant effect on the inversion results for both the exponentially decaying and single-burst SFR cases. The constant SFR case is quite noisy at recent times, and the fit is marginally improved in the fractal SFR case. In all cases, the integrated SFR is within 4 per cent of the true value.

When errors are introduced on the bolometric magnitude of stars in the observed WDLF (blue lines), the inversion results are not significantly affected. In the case of the fractal SFR, the additional smoothing of the observed WDLF that results causes a loss in resolution at recent times, and high-frequency components in the

4.3 Noise degradation

Inverse problems are notoriously susceptible to noise. In the case of the WDLF, this will manifest as errors on both the number density at a given magnitude and the bolometric magnitude of individual stars, resulting in an overall smoothing. It is important to estimate the effect that these errors have on the inversion procedure, given that the observed WDLF will be subject to both types to some degree.

We have repeated the tests of Section 4.2, using simulated WDLFs with realistic error models. These were calculated by reducing the number of simulated WDs to $10^4$, to give a statistical uncertainty on the number density that matches the WDLFs of Harris et al. (2006) and Rowell & Hambly (2011). Bolometric magnitude errors were added to each star by drawing from a Gaussian distribution of width 0.25. This is the approximate size of errors in the Rowell & Hambly (2011) WDLF; the Harris et al. (2006) errors are likely to be smaller given the better photometry of the Sloan Digital Sky Survey (SDSS) relative to SuperCOSMOS and the sharper features of their WDLF.

The effect of each type of error is shown in Fig. 7. The presence of realistic errors in the WDLF density (green lines) does not have a significant effect on the inversion results for both the exponentially decaying and single-burst SFR cases. The constant SFR case is quite noisy at recent times, and the fit is marginally improved in the fractal SFR case. In all cases, the integrated SFR is within 4 per cent of the true value.

When errors are introduced on the bolometric magnitude of stars in the observed WDLF (blue lines), the inversion results are not significantly affected. In the case of the fractal SFR, the additional smoothing of the observed WDLF that results causes a loss in resolution at recent times, and high-frequency components in the

![Figure 7](https://academic.oup.com/mnras/article-abstract/434/2/1549/1071622) Results of noise degradation tests on the inversion algorithm. In each case, the solid line indicates the true underlying SFR. The dashed line shows the results of the inversion algorithm when applied to a WDLF determined using $10^4$ simulated WDs with no bolometric magnitude errors. This results in statistical uncertainty on the WD density of the same order as that of recent solar neighbourhood WDLF measurements. The dotted line shows the results when a large number of simulation stars ($2 \times 10^6$) are used resulting in very low density uncertainty, and bolometric magnitude errors drawn from a $\sigma_M = 0.25$ Gaussian distribution. The dot–dashed line shows the results when both types of error are included. (a) Constant SFR. (b) Exponentially decaying SFR. (c) Single-burst SFR. (d) Fractal SFR.
SFR are not as well recovered. Again, the integrated SFR is within 3 per cent of the true value in all cases.

Errors in both the WDLF number density and bolometric magnitude of individual stars (red lines) cause additional degradation in all cases. This is particularly bad for the constant SFR model, for which the noise at recent times is quite severe. However, the overall form of the SFR is still reasonably well recovered, to the extent that significantly different SFR scenarios can still be distinguished, and it is encouraging that the fractal SFR is still well recovered, albeit with rather lower resolution on the SFR features. The integrated SFR is within 4 per cent of the true value in each case.

### 4.3.1 Error estimation

The inversion algorithm is capable of estimating the error on the converged SFR solution, by considering the propagation of number density errors from the observed WDLF to the corresponding SFR bins according to equation (19). Fig. 8 shows again the SFR solution from Fig. 7(d) for the case of both density and bolometric magnitude errors, and includes the estimated uncertainty on the solution drawn in light grey. We find that the errors are generally well estimated in regions free of discontinuities; however, in the vicinity of discontinuities significant departures are present, arising from degeneracies in the inversion solution that are not accounted for by the error estimation process.

### 4.4 Critical parameters

In order to fully characterize the inversion algorithm, and to properly interpret results derived from real data, it is important to estimate the sensitivity of the inversion process to variations in the input parameters. Some parameters, such as the slope of the IMF over the relevant mass range, are relatively well constrained; others, such as the fractions of hydrogen and helium atmosphere WDs and their cooling rates, are less well constrained, and could potentially lead to systematic errors in the SFR solution.

We have repeated the convergence and noise sensitivity tests of Sections 4.2 and 4.3, using the same set of synthetic WDLFs that were calculated using the parameters listed in Table 2. In the present tests, we vary the parameters that the inversion algorithm uses, and compare the results using the incorrect parameters with those obtained using the true values. This involves a large number of individual tests, from which we present a selection of results in Fig. 9. These were obtained for the fractal SFR model, using a synthetic WDLF that includes both density and bolometric magnitude errors. The results are similar for the remaining SFR models and for the noise-free WDLFs.

#### 4.4.1 Parameter \( \alpha \)

Fig. 9(a) shows the effect of varying the fraction of hydrogen atmosphere WDs by \( \pm 20\text{ per cent} \). There is very little difference within the last 3 Gyr, due to the cooling rates of the two types being very similar at intermediate temperatures. At older times, differential cooling starts to become significant, and the recovered SFRs show systematic deviations of up to around 25 per cent. However, the integrated SFR is mostly unaffected and only deviates by less than 1 per cent in these tests.

#### 4.4.2 Initial–final mass relation

Fig. 9(b) shows the effect of varying the IFMR. In these tests, the true IFMR is that of Kalirai et al. (2008), which is derived from two old open clusters to better constrain the low-mass end of the IFMR. The first alternative IFMR is the linear fit of Ferrario et al. (2005) that is derived from a selection of young open clusters in the solar neighbourhood, and uses the F01 models to obtain WD masses and cooling times. This function is shallower than the Kalirai et al. (2008) IFMR. The second alternative IFMR is that of Catalán et al. (2008), which is derived from a sample of local open clusters and common proper motion pairs that largely overlaps with the Ferrario et al. (2005) study. However, they use the WD cooling models of Salaris et al. (2000) (a predecessor of the S10 models), so the analysis is largely independent. Their piecewise linear fit is steeper than the Kalirai et al. (2008) IFMR for masses greater than 2.7 \( M_\odot \), and shallower at lower masses.

It is evident from Fig. 9(b) that variations in the IFMR on this level do not have a large effect on the performance of the inversion algorithm. The shape of the SFR is preserved, and the integrals vary by less than 4 per cent.

#### 4.4.3 IMF exponent

Fig. 9(c) shows the effect of changes in the IMF power-law exponent of \( \pm 0.3 \). This leads to an over- and underestimate of the total integrated SFR of about 20 per cent, but does not significantly affect the overall shape of the recovered function. The explanation for this is that variations in the slope of the IMF change the fraction of high-mass stars that the algorithm forms. For a given SFR model, a flatter IMF (\( \Delta \phi = \pm 0.3 \)) will result in a greater number of WDs at all ages, due to the increased fraction of high-mass MS stars with short lifetimes. The magnitude of the SFR will then be reduced by the inversion algorithm in order to match the observed WDLF and compensate for the overproduction of WDs, leading to an overall reduction in the SFR that is roughly independent of age.
Figure 9. Selected results from critical parameters tests. In all cases, the solid line indicates the true underlying SFR, and the dashed line indicates the results of the inversion algorithm when the same set of parameters is used both to generate the synthetic WDLF and to invert it. The dotted and dot–dashed lines show the inversion results when perturbed parameter values are used. In these tests, a noisy synthetic WDLF is used. The algorithm is relatively insensitive to differences in the fraction of H atmosphere WDs within the last few Gyr (panel a), although artefacts appear at older times due to the different cooling rates of the two types. Variations in the IMF do not have a significant effect on the performance (panel b). Uncertainty in the IMF exponent has a significant effect on the normalization of the SFR solution but not on the functional form (panel c), and the effect of variations in the metallicity is very similar (panel d). In both cases, overproduction of massive stars in the inversion algorithm results in an overall reduction in the SFR in order to match the observed WDLF. By far the largest source of error in the recovered SFR arises from the choice of WD cooling models (panel e). (a) $\alpha \ [\text{ratio H/(H+He)}]$. (b) IFMR. (c) IMF exponent. (d) Metallicity $Z$. (e) WD cooling models.

4.4.4 Metallicity

The effect of variations in the progenitor metallicity is similar to that of variations in the IMF exponent. At constant stellar mass, lower metallicity results in shorter MS lifetimes, so a reduction in the metallicity parameter $Z$ results in an overproduction of WDs and an overall suppression in the recovered SFR in order to match the observed WDLF. In Fig. 9(d), the true value of $Z$ is 0.017. Reducing to $Z = 0.008$ causes the integrated SFR to be underestimated by around 10 per cent, and increasing to $Z = 0.040$ leads to an overestimation of around 20 per cent, while in both cases the overall shape of the SFR is preserved. This assumes of course that the metallicity is independent of time; the existence of an age–metallicity relation (AMR) would cause the shape of the SFR to be incorrectly estimated as well as the normalization.

4.4.5 WD cooling models

Fig. 9(e) demonstrates the effect of using different sets of WD cooling sequences. In these tests, the F01 models were used to generate the observed WDLF, which was then inverted using the S10 and S10p models. Clearly, the choice of WD cooling models is a significant source of error in the WDLF inversion algorithm, and for all SFR models this has the largest effect on the integrity of the solution. Although the integrated SFR is quite well preserved (within a few per cent in these tests), the shape of the SFR solution is quite severely compromised beyond $\sim 2$ Gyr in the past, with broad peaks in the SFR shifted significantly (the peak at $\sim 3$ Gyr) or lost entirely (the peak at $\sim 9$ Gyr). These effects can be explained as follows.

The loss of the peak at $\sim 9$ Gyr in the S10 and S10p solutions is due to the fact that these cooling sequences cool slower than the F01 sequences at faint magnitudes (equivalently, long cooling times). This is important because when WDs cool slower at certain magnitudes, they traverse the WDLF bins slower and tend to ‘pile up’, leading to larger number densities. Fig. 4 demonstrates this differential cooling for $0.6 M_\odot$ WDs; the effect is stronger for higher mass WDs, which is significant because these are over-represented at faint magnitudes. Because the F01 sequences were used to generate the synthetic WDLF in these tests, when the S10 and S10p sequences are used to invert it, the recovered SFR is suppressed at old times in order to match the observed WD density in the faint magnitude bins. We can therefore predict that the S10 and S10p sequences will always recover a lower SFR at old times ($\gtrsim 8$ Gyr) than the F01 sequences.
In addition, the inclusion of phase separation effects in the S10p cooling sequences, which slows the cooling of WDs relative to the S10 sequences at ages greater than ~2 Gyr, has the effect of shifting features in the S10p SFR solution to older times. This effect is analogous to that described above, and explains the shift in SFR features at more recent times.

5 THE SOLAR NEIGHBOURHOOD STAR FORMATION HISTORY

We are now ready to apply the inversion algorithm to a selection of recent determinations of the solar neighbourhood WDLF. In these tests, we have fixed the IMF exponent, IFMR, metallicity and α at the values listed in Table 2, which are reasonable for disc stars. The algorithm is not too sensitive to the chosen values for these parameters. However, we have repeated the inversion for each set of WD cooling models, as the solution is likely to be highly cooling model dependent.

Note that the SFR and WDLF models that are generated during the inversion process are dimensionless in spatial density (their units are just yr⁻¹ and M⊙ yr⁻¹). and will be automatically calibrated to whatever spatial density units the observed WDLF has (normally pc⁻³). In all cases, the inverted SFR is only for stars more massive than 0.6 M⊙.

5.1 The Sloan Digital Sky Survey

The SDSS has produced some of the deepest and cleanest WDLFs in recent years using a variety of different survey techniques. The WDLF of Harris et al. (2006, hereafter H06) is derived from a catalogue of ~6000 WDs obtained from Data Release 3 using the reduced proper motion technique, with proper motions obtained by combination with USNO-B data. This survey method does not work well for intrinsically bright stars (which have on average lower proper motions), and their LF covers the range 7 < Mbol < 16. At brighter magnitudes, selection of WDs on colour works well due to the UV excess shown by these objects, and the WDLF of Krzesinski et al. (2009, hereafter K09) covers the range 0 < Mbol < 7, which in conjunction with the H06 LF provides constraint on the WDLF over nearly the complete range of luminosity.

Fig. 10 shows the SFR solution obtained when the concatenated H06 and K09 WDLF is inverted. Although the results differ quite significantly depending on which set of WD cooling models is used, all show a certain bimodality in the SFR with broad peaks at ~2–3 and ~6–10 Gyr in the past. The shape of the SFR functions places the onset of star formation roughly 8–11 Gyr ago depending on the cooling models. The differences between the solutions are due to the differences in the predicted cooling rates of WDs (see Section 4.4.5); essentially, the SFR peak at ~6–10 Gyr is higher for the F01 cooling sequences due to the fact that these cool faster at faint magnitudes, and the SFR at early times is inflated in order to match the observed density of faint WDs at the present day. If the true WD cooling rates are closer to the S10/S10p sequences, then the size of the SFR peak is overestimated in the F01 solution. Alternatively, if the true cooling rates are closer to the F01 sequences, then the S10/S10p solutions underestimate the peak. The shift in features between the S10 and S10p solutions older than ~2 Gyr is due to C/O phase separation effects slowing the cooling of WDs, which pushes features to older times in the S10p solution. The integrated SFR for each of the models agrees very well: we find (13.9 ± 0.6) × 10⁻³ stars pc⁻³ for the F01 solution, (14.0 ± 0.6) × 10⁻³ stars pc⁻³ for the S10 solution and (13.8 ± 0.6) × 10⁻³ stars pc⁻³ for the S10p solution.

Fig. 11 shows the final converged WDLF models. These fit the observed WDLF very well, with χ² statistics of 26.4, 53.4 and 44.6 for the F01, S10 and S10p solutions, respectively. The residuals of the WDLF models to the data (Fig. 12) highlight any systematic errors in the fits. Generally, we find that the recovered SFR models produce present-day WDLFs that match the observed WDLF very closely.
Figure 12. Residuals for converged WD LFs. The S10 and S10p models have been offset horizontally by 0.1 and 0.2 mag for clarity. No significant departures from the observed WD LFs are seen.

5.2 The SuperCOSMOS Sky Survey

The all-sky SuperCOSMOS Sky Survey is based on digitized photographic plates, and is accessible online via a Structured Query Language (SQL) interface. The merged source table contains multi-epoch photometry in three photographic bands BRI and proper motions for nearly two billion objects. This was used by Rowell & Hambly (2011, hereafter RH11) to measure the WD LFs for a sample of around 10 000 WDs, using the reduced proper motion technique to cover the range 1 < $M_{bol}$ < 18. The RH11 catalogue overlaps with that of H06, but covers a significantly larger area of sky. It is incomplete at around the 50 per cent level, but the incompleteness is independent of colour and does not bias the WD LFs. RH11 developed a new method of measuring the WD LFs that allowed the different kinematic populations to be resolved in a self-consistent way, allowing the thin- and thick-disc WD LFs to be measured separately for the first time. In this work, we use their WD LFs derived using the standard $V_{int}^{-1}$ technique, for a more direct comparison with the H06 and K09 WD LFs. In order to achieve a reasonable fit to the RH11 WD LFs and ensure smooth convergence of the algorithm, we had to remove both the faintest and brightest three points, restricting the WD LFs to the range 2.5 < $M_{bol}$ < 16.5. The faintest bins are sparsely populated and contain poorly characterized ultracool WDs, for which the photometric parallaxes are extremely uncertain, and at the bright end the bins are severely underpopulated leading to highly uncertain data points that cannot be fitted by any SFR.

Fig. 13 presents the SFR results obtained by inverting the RH11 WD LFs. The integrated SFR for each of the F01, S10 and S10p solutions is $(8.1 \pm 0.1) \times 10^{-3}$ stars pc$^{-3}$, $(8.1 \pm 0.1) \times 10^{-3}$ and $(8.0 \pm 0.1) \times 10^{-3}$ stars pc$^{-3}$. These values are around 60 per cent of those obtained from the H06+K09 WD LFs, due to the incompleteness present in the RH11 survey. Fig. 14 shows the final converged WD LFs. In each case, the fit is not as good as for the H06+K09 WD LFs, achieving $\chi^2$ of 75.9, 63.5 and 62.4 for the F01, S10 and S10p cooling models, respectively, though some of this increase is due to RH11 having two additional data points (28 compared to 26). In contrast to the H06+K09 WD LFs fits, the RH11 WD LFs shows some minor deviation in the residuals over the range $8 < M_{bol} < 11$ that none of the models have been able to fit (see Fig. 15, inset).

5.3 Comparison to other studies

As far as we are aware, this is the first study to use WDs to probe in detail the time-varying SFR of the Galactic disc, and the most directly comparable work has been done using MS stars, specifically, the CMD for MS stars in the Hipparcos catalogue.

\[\chi^2 \text{ for each of the F01, S10 and S10p models is 75.9, 63.5 and 62.4.}\]
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Figure 15. Residuals for converged WDLF models. The S10 and S10p models have been offset horizontally by 0.1 and 0.2 mag for clarity. A marginal excess is seen over the range $8 < M_{bol} < 11$, where none of the models have been able to adequately fit the data.

Hernandez et al. (2000) use a non-parametric Bayesian approach to derive the SFR by inverting the CMD. Their technique provides a high resolution of $\sim 50$ Myr on the recovered SFR, but requires a volume-complete sample of stars that restricts their results to the last 3 Gyr. Over this range, they derive a solar neighbourhood SFR with an oscillation of period $\sim 0.5$ Gyr, superposed on a weak constant rate. Our results hint at such an oscillation at $t < 1.5$ Gyr, although this may just be inversion noise, and the lack of resolution in our SFR prevents us from detecting any feature like this further in the past. Also, the method of Hernandez et al. (2000) enforced zero SFR at the present day so there may be a risk of artefacts at early times.

Vergely et al. (2002) use a similar Bayesian inversion method, but without the requirement of a volume-limited sample, allowing them to use a much larger number of stars and probe the full star formation history of the disc. They simultaneously fit the SFR and the AMR, obtaining a column-integrated SFR that shows a peak at around 2 Gyr followed by a smooth decline which levels off to a constant SFR older than $\sim 5$ Gyr. Their results are plotted in Fig. 16 over our own results obtained by inverting the H06 + K09 WDLF using the S10p cooling models. Both of the functions are normalized to unity; due to the different units we compare only the shape of the SFR.

Cignoni, Degl’Innocenti & Moroni (2006) use a different technique that combines Bayesian reconstruction of the noise-free Hipparcos CMD with a maximum likelihood fitting technique for their model CMDs. They again use a volume-complete sample, but with a fainter magnitude limit than that in Hernandez et al. (2000) allowing them to constrain the SFR over the last 12 Gyr with a time resolution varying from 0.5 to 2.0 Gyr. Their results are plotted in Fig. 17 along with our own; both functions are again normalized. They obtain a similar SFR to that of Vergely et al. (2002), with a peak at around 2–3 Gyr followed by a gradual decline to older times, but with a secondary peak at 10–12 Gyr.

The results of both Vergely et al. (2002) and Cignoni et al. (2006) agree with our own finding of a broad peak in star formation at $\sim 2–3$ Gyr ago. Neither predicts the strong recent bursts in star formation that we obtain from the H06 + K09 WDLF, though this is not seen in the RH11 WDLF results and may be due to noise. The secondary peak in star formation at ancient times that is seen in all of our results is not present in the Vergely et al. SFR. A similar feature is seen in the Cignoni et al. SFR, though the peak is shifted by around $\sim 2–3$ Gyr so a positive identification is difficult. We note that both studies use different values of the IMF slope ($\sim -3.0$ and $\sim -2.35$, respectively, compared to $\sim -2.3$ in this work), though this is not expected to change the shape of the recovered SFR much. Both also use an AMR, the existence of which is expected to change the shape of our own SFR. The total variation in metallicity in their AMRs is roughly the same as the range of metallicities considered during testing of our algorithm (though we only use constant-metallicity models), and the variation in SFR present in Fig. 9(d) therefore corresponds roughly to the additional uncertainty in the SFR solution were such an AMR true. This is not enough to explain the secondary peak seen in our results, so this would seem to be a real feature.

6 CONCLUSIONS

In this paper, we have presented an algorithm for use in inverting the WDLF to obtain the time-varying star formation history of the host stellar population. We have verified the performance and sensitivity to noise and various parameters by analysis with synthetic data, and applied the algorithm to two recent independent measurements of the solar neighbourhood WDLF. The SFR in the solar neighbourhood appears to be characterized by a bimodal distribution

Figure 16. The results of Vergely et al. (2002) compared to our own. No secondary peak in star formation is observed at older times.

Figure 17. The results of Cignoni et al. (2006) compared to our own. Both studies find a bimodal SFR with a secondary peak at ancient times.
with broad peaks at 2–3 and 7–9 Gyr in the past, separated by a significant lull. The onset of star formation occurs around 8–10 Gyr ago. These broad results are consistent across both data sets and are independent of the WD cooling models used. However, the finer details of the SFR, such as the relative size of the peaks and lull and the precise timing of the various features, are highly dependent on the choice of WD cooling sequences.

The model WDLFs that the algorithm computes match the observations very well, and we find no systematic deviation that might indicate additional sources of WD cooling that are unaccounted for in the models. The marginal feature seen at 8 < \(M_{\text{bol}}\) < 11 in the RH11 WDLF is not observed in the H06 WDLF; because the latter uses higher quality SDSS photometry, we conclude that this feature is likely an artefact in the RH11 data. However, we stress that this is not evidence against the existence of additional cooling processes, rather that differences between simple WDLF models and the observed WDLF cannot be reliably interpreted as evidence for additional cooling processes (at least at magnitudes fainter than \(M_{\text{bol}}\) ~ 10) without considering the full time-varying star formation history. Note that it may be possible to use a similar algorithm to measure directly the cooling rates of WDs, if the SFR can be constrained from other studies.

In principle, the algorithm can be applied to any population for which the WDLF can be measured, for example the spheroid and nearby clusters, although at present the algorithm does not consider binary stars or more exotic objects such as He or O/Ne core WDs. This may be a direction for future development work.

**ACKNOWLEDGEMENT**

The author wishes to thank an anonymous reviewer for helpful comments and suggestions.

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