SERRE FINITENESS AND SERRE VANISHING FOR NON-COMMUTATIVE $\mathbb{P}^1$-BUNDLES

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Abstract. Suppose $X$ is a smooth projective scheme of finite type over a field $K$, $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2, $\mathcal{A}$ is the non-commutative symmetric algebra generated by $\mathcal{E}$ and $\text{Proj}\mathcal{A}$ is the corresponding non-commutative $\mathbb{P}^1$-bundle. We use the properties of the internal Hom functor $\text{Hom}_{\mathcal{A}}(-,-)$ to prove versions of Serre finiteness and Serre vanishing for $\text{Proj}\mathcal{A}$. As a corollary to Serre finiteness, we prove that $\text{Proj}\mathcal{A}$ is Ext-finite. This fact is used in [3] to prove that if $X$ is a smooth curve over $\text{Spec} K$, $\text{Proj}\mathcal{A}$ has a Riemann-Roch theorem and an adjunction formula.

Keywords: non-commutative geometry, Serre finiteness, non-commutative projective bundle.

1. INTRODUCTION

Non-commutative $\mathbb{P}^1$-bundles over curves play a prominent role in the theory of non-commutative surfaces. For example, certain non-commutative quadrics are isomorphic to non-commutative $\mathbb{P}^1$-bundles over curves [10]. In addition, every non-commutative deformation of a Hirzebruch surface is given by a non-commutative $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ [9, Theorem 7.4.1, p. 29].

The purpose of this paper is to prove versions of Serre finiteness and Serre vanishing (Theorem 3.5 (1) and (2), respectively) for non-commutative $\mathbb{P}^1$-bundles over smooth projective schemes of finite type over a field $K$. As a corollary to the first of these results, we prove that such non-commutative $\mathbb{P}^1$-bundles are Ext-finite. This fact is used to prove that non-commutative $\mathbb{P}^1$-bundles over smooth curves have a Riemann-Roch theorem and an adjunction formula [3].

We now review some important notions from non-commutative algebraic geometry in order to recall the definition of non-commutative $\mathbb{P}^1$-bundle. We conclude the introduction by relating the results of this paper to Mori’s intersection theory.

If $X$ is a quasi-compact and quasi-separated scheme, then $\text{Mod}X$, the category of quasi-coherent sheaves on $X$, is a Grothendieck category. This leads to the following generalization of the notion of scheme, introduced by Van den Bergh in order to define a notion of blowing-up in the non-commutative setting.

Definition 1.1. [8] A quasi-scheme is a Grothendieck category $\text{Mod}X$, which we denote by $X$. $X$ is called a noetherian quasi-scheme if the category $\text{Mod}X$ is locally noetherian. $X$ is called a quasi-scheme over $K$ if the category $\text{Mod}X$ is $K$-linear.

If $R$ is a ring and $\text{Mod}R$ is the category of right $R$-modules, $\text{Mod}R$ is a quasi-scheme, called the non-commutative affine scheme associated to $R$. If $A$ is a graded
ring, \( \text{Gr} A \) is the category of graded right \( A \)-modules, \( \text{Tors} A \) is the full subcategory of \( \text{Gr} A \) consisting of direct limits of right bounded modules, and \( \text{Proj} A \) is the quotient category \( \text{Gr} A / \text{Tors} A \), then \( \text{Proj} A \) is a quasi-scheme called the non-commutative projective scheme associated to \( A \). If \( A \) is an Artin-Schelter regular algebra of dimension 3 with the same Hilbert series as a polynomial ring in 3 variables, \( \text{Proj} A \) is called a non-commutative \( \mathbb{P}^2 \).

The notion of non-commutative \( \mathbb{P}^1 \)-bundle over a smooth scheme \( X \) generalizes that of commutative \( \mathbb{P}^1 \)-bundle over \( X \). In order to recall the definition of non-commutative \( \mathbb{P}^1 \)-bundle, we review some preliminary notions. Let \( S \) be a scheme of finite type over \( \text{Spec} K \) and let \( X \) be an \( S \)-scheme. For \( i = 1, 2 \), let \( \text{pr}_i : X \times_S X \to X \) denote the standard projections, let \( \delta : X \to X \times_S X \) denote the diagonal morphism, and let \( \Delta \) denote the image of \( \delta \).

**Definition 1.2.** A coherent \( \mathcal{O}_X \)-bimodule, \( \mathcal{E} \), is a coherent \( \mathcal{O}_{X \times_S X} \)-module such that \( \text{pr}_i \mid \text{Supp} \mathcal{E} \) is finite for \( i = 1, 2 \). A coherent \( \mathcal{O}_X \)-bimodule \( \mathcal{E} \) is locally free of rank \( n \) if \( \text{pr}_i \mathcal{E} \) is locally free of rank \( n \) for \( i = 1, 2 \).

Now assume \( X \) is smooth. If \( \mathcal{E} \) is a locally free \( \mathcal{O}_X \)-bimodule, then let \( \mathcal{E}^* \) denote the dual of \( \mathcal{E} \) [9, p. 6], and let \( \mathcal{E}^{*2} \) denote the dual of \( \mathcal{E}^{*1} \). Finally, let \( \eta : \mathcal{O}_\Delta \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^* \) denote the counit from \( \mathcal{O}_\Delta \) to the bimodule tensor product of \( \mathcal{E} \) and \( \mathcal{E}^* \) [9, p. 7].

**Definition 1.3.** [9, Section 4.1] Let \( \mathcal{E} \) be a locally free \( \mathcal{O}_X \)-bimodule. The non-commutative symmetric algebra generated by \( \mathcal{E} \), \( \mathcal{A} \), is the sheaf-\( \mathbb{Z} \)-algebra generated by the \( \mathcal{E}^{*2} \) subject to the relations \( \eta(\mathcal{O}_\Delta) \).

A more explicit definition of non-commutative symmetric algebra is given in Section 2. We now recall the definition of non-commutative \( \mathbb{P}^1 \)-bundle.

**Definition 1.4.** [9] Suppose \( X \) is a smooth scheme of finite type over \( K \), \( \mathcal{E} \) is a locally free \( \mathcal{O}_X \)-bimodule of rank 2 and \( \mathcal{A} \) is the non-commutative symmetric algebra generated by \( \mathcal{E} \). Let \( \text{Gr} A \) denote the category of graded right \( A \)-modules, let \( \text{Tors} A \) denote the full subcategory of \( \text{Gr} A \) consisting of direct limits of right-bounded modules, and let \( \text{Proj} A \) denote the quotient of \( \text{Gr} A \) by \( \text{Tors} A \). The category \( \text{Proj} A \) is a non-commutative \( \mathbb{P}^1 \)-bundle over \( X \).

This notion generalizes that of a commutative \( \mathbb{P}^1 \)-bundle over \( X \) as follows. Let \( \mathcal{E} \) be an \( \mathcal{O}_X \)-bimodule on which \( \mathcal{A} \) acts centrally. Then \( \mathcal{E} \) can be identified with the direct image \( \text{pr}_i \mathcal{E} \) for \( i = 1, 2 \). If, furthermore, \( \mathcal{E} \) is locally free of rank 2 and \( \mathcal{A} \) is the non-commutative symmetric algebra generated by \( \mathcal{E} \), Van den Bergh proves [9, Lemma 4.2.1] that the category \( \text{Proj} A \) is equivalent to the category \( \text{Mod} \mathbb{P}_X (\text{pr}_i \mathcal{E}) \), where \( \mathbb{P}_X (\_2) \) is the usual (commutative) projectivization.

One of the major problems in non-commutative algebraic geometry is to classify non-commutative surfaces. Since intersection theory on commutative surfaces facilitates the classification of commutative surfaces, one expects intersection theory to be an important tool in non-commutative algebraic geometry. Mori shows [3, Theorem 3.11] that if \( Y \) is a noetherian quasi-scheme over a field \( K \) such that

(1) \( Y \) is Ext-finite,

(2) the cohomological dimension of \( Y \) is 2, and

(3) \( Y \) satisfies Serre duality

then versions of the Riemann-Roch theorem and the adjunction formula hold for \( Y \). Let \( X \) be a smooth curve over \( \text{Spec} K \). In [6], we prove that a non-commutative
P\textsuperscript{1}-bundle over X satisfies (2) and (3) above (see Section 4 for a precise statement of these results). In this paper we prove that a non-commutative P\textsuperscript{1}-bundle over a projective scheme of finite type satisfies (1) (Corollary 3.6). We conclude the paper by stating the versions of the Riemann-Roch theorem and the adjunction formula which hold for non-commutative P\textsuperscript{1}-bundles.

In what follows, K is a field, X is a smooth, projective scheme of finite type over Spec K, Mod\textsubscript{X} denotes the category of quasi-coherent O\textsubscript{X}-modules, and we abuse notation by calling objects in this category O\textsubscript{X}-modules.

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2. Preliminaries

Before we prove Serre finiteness and Serre vanishing, we review the definition of non-commutative symmetric algebra and the definition and basic properties of the internal Hom functor \textit{Hom}_{Gr\mathcal{A}}(-,-) on Gr\mathcal{A}.

Definition 2.1. Let \mathcal{E} be a locally free \mathcal{O}_X-bimodule. The non-commutative symmetric algebra generated by \mathcal{E} is the sheaf-\mathbb{Z}-algebra \mathcal{A} = \bigoplus_{i,j \in \mathbb{Z}} A_{ij} with components

\begin{itemize}
  \item A_{ii} = \mathcal{O}_\Delta,
  \item A_{i,i+1} = \mathcal{E}^{i*},
  \item A_{ij} = A_{i,i+1} \otimes \cdots \otimes A_{j-1,j} / R_{ij} for j > i + 1, where R_{ij} \subset A_{i,i+1} \otimes \cdots \otimes A_{j-1,j} is the \mathcal{O}_X-bimodule

  \[\sum_{k=i}^{j-2} A_{i,i+1} \otimes \cdots \otimes A_{k-1,k} \otimes \mathcal{Q}_k \otimes A_{k+2,k+3} \otimes \cdots \otimes A_{j-1,j},\]

  and \mathcal{Q}_k is the image of the unit map \mathcal{O}_\Delta \rightarrow A_{i,i+1} \otimes A_{i+1,i+2}, and
  \item A_{ij} = 0 if i > j
\end{itemize}

and with multiplication, \mu, defined as follows: for i < j < k,

\[A_{ij} \otimes A_{jk} = \frac{A_{i,i+1} \otimes \cdots \otimes A_{j-1,j} \otimes A_{j,j+1} \otimes \cdots \otimes A_{k-1,k}}{R_{ij}} \approx \frac{A_{i,i+1} \otimes \cdots \otimes A_{k-1,k}}{R_{ij} \otimes A_{j,j+1} \otimes \cdots \otimes A_{k-1,k} + A_{i,i+1} \otimes \cdots \otimes A_{j-1,j} \otimes R_{jk}}\]

by [5 Corollary 3.18]. On the other hand,

\[R_{ik} \cong R_{ij} \otimes A_{j,j+1} \otimes \cdots \otimes A_{k-1,k} + A_{i,i+1} \otimes \cdots \otimes A_{j-1,j} \otimes R_{jk} + A_{i,i+1} \otimes \cdots \otimes A_{j-2,j-1} \otimes \mathcal{Q}_{j-1} \otimes A_{j+1,j+2} \otimes \cdots \otimes A_{k-1,k}.\]

Thus there is an epi \mu_{ijk} : A_{ij} \otimes A_{jk} \rightarrow A_{ik}.

If i = j, let \mu_{ijk} : A_{ij} \otimes A_{ik} \rightarrow A_{ik} be the scalar multiplication map \mathcal{O}_\mu : \mathcal{O}_\Delta \otimes A_{ik} \rightarrow A_{ik}. Similarly, if j = k, let \mu_{ijk} : A_{ij} \otimes A_{jj} \rightarrow A_{ij} be the scalar multiplication map \mu_{ij}. Using the fact that the tensor product of bimodules is associative, one can check that multiplication is associative.

Definition 2.2. Let Bimod\mathcal{A} - \mathcal{A} denote the category of \mathcal{A} - \mathcal{A}-bimodules. Specifically:
• an object of $\text{Bimod}\mathcal{A} - \mathcal{A}$ is a triple

$$(C = \{C_{ij}\}_{i,j \in \mathbb{Z}}, \{\mu_{ijk}\}_{i,j,k \in \mathbb{Z}}, \{\psi_{ijk}\}_{i,j,k \in \mathbb{Z}})$$

where $C_{ij}$ is an $\mathcal{O}_X$-bimodule and $\mu_{ijk} : C_{ij} \otimes A_{jk} \to C_{ik}$ and $\psi_{ijk} : A_{ij} \otimes C_{jk} \to C_{ik}$ are morphisms of $\mathcal{O}_{X_2}$-modules making $C$ an $\mathcal{A}$-$\mathcal{A}$ bimodule.

• A morphism $\phi : C \to D$ between objects in $\text{Bimod}\mathcal{A} - \mathcal{A}$ is a collection $\phi = \{\phi_{ij}\}_{i,j \in \mathbb{Z}}$ such that $\phi_{ij} : C_{ij} \to D_{ij}$ is a morphism of $\mathcal{O}_{X_2}$-modules, and such that $\phi$ respects the $\mathcal{A}$-$\mathcal{A}$-bimodule structure on $C$ and $D$.

Let $\mathbb{B}$ denote the full subcategory of $\text{Bimod}\mathcal{A} - \mathcal{A}$ whose objects $C = \{C_{ij}\}_{i,j \in \mathbb{Z}}$ have the property that $C_{ij}$ is coherent and locally free for all $i, j \in \mathbb{Z}$.

Let $\text{Gr}\mathcal{A}$ denote the full subcategory of $\mathbb{B}$ consisting of objects $C$ such that for some $n \in \mathbb{Z}$, $C_{ij} = 0$ for $i \neq n$ (we say $C$ is left-concentrated in degree $n$).

**Definition 2.3.** [6 Definition 3.7] Let $C$ be an object in $\mathbb{B}$ and let $\mathcal{M}$ be a graded right $\mathcal{A}$-module. We define $\text{Hom}_{\text{Gr}\mathcal{A}}(C, \mathcal{M})$ to be the $\mathbb{Z}$-graded $\mathcal{O}_X$-module whose $k$th component is the equalizer of the diagram

$$
\begin{array}{ccc}
\Pi(M_i \otimes C_{ki}^*) & \xrightarrow{\alpha} & \Pi(M_j \otimes C_{kj}^*) \\
\beta \downarrow & & \gamma \\
\Pi(\Pi(M_j \otimes A_{ij}^*) \otimes C_{ki}) & \xrightarrow{\delta} & \Pi(\Pi(M_j \otimes (C_{ki} \otimes A_{ij})^*)
\end{array}
$$

where $\alpha$ is the identity map $\beta$ is induced by the composition

$$
\mathcal{M}_i \xrightarrow{\eta} \mathcal{M}_i \otimes A_{ij} \otimes A_{ij}^* \xrightarrow{\mu} \mathcal{M}_j \otimes A_{ij}^* ,
$$

$\gamma$ is induced by the dual of

$$
\mathcal{C}_{ki} \otimes A_{ij} \xrightarrow{\mu} \mathcal{C}_{kj} ,
$$

and $\delta$ is induced by the composition

$$(\mathcal{M}_j \otimes A_{ij}^*) \otimes C_{ki}^* \to \mathcal{M}_j \otimes (A_{ij}^* \otimes C_{ki}^*) \to \mathcal{M}_j \otimes (C_{ki} \otimes A_{ij})^*$$

whose left arrow is the associativity isomorphism and whose right arrow is induced by the canonical map [6 Section 2.1]. If $C$ is an object of $\text{Gr}\mathcal{A}$ left-concentrated in degree $k$, we define $\text{Hom}_{\text{Gr}\mathcal{A}}(C, \mathcal{M})$ to be the equalizer of $\Pi$.

Let $\tau : \text{Gr}\mathcal{A} \to \text{Tors}\mathcal{A}$ denote the torsion functor, let $\pi : \text{Gr}\mathcal{A} \to \text{Proj}\mathcal{A}$ denote the quotient functor, and let $\omega : \text{Proj}\mathcal{A} \to \text{Gr}\mathcal{A}$ denote the right adjoint to $\pi$. For any $k \in \mathbb{Z}$, let $e_k\mathcal{A}$ denote the right $\mathcal{A}$-module $\bigoplus_{i \geq k} \mathcal{A}_{i}$. We define $e_k\mathcal{A}_{\geq k+n}$ to be the sum $\bigoplus_{i \geq 0} e_k\mathcal{A}_{i+n+k}$ and we let $A_{\geq n} = \bigoplus_{k} e_k\mathcal{A}_{\geq k+n}$.

**Theorem 2.4.** If $\mathcal{M}$ is an object in $\text{Gr}\mathcal{A}$ and $C$ is an object in $\mathbb{B}$, $\text{Hom}_{\text{Gr}\mathcal{A}}(C, \mathcal{M})$ inherits a graded right $\mathcal{A}$-module structure from the left $\mathcal{A}$-module structure of $C$, making $\text{Hom}_{\text{Gr}\mathcal{A}}(-, -) : \mathbb{B}^{\text{op}} \times \text{Gr}\mathcal{A} \to \text{Gr}\mathcal{A}$ a bifunctor.

Furthermore

1. $\tau(-) \cong \lim_{n \to \infty} \text{Hom}_{\text{Gr}\mathcal{A}}(A/A_{\geq n}, -)$,
2. If $\mathcal{F}$ is a coherent, locally free $\mathcal{O}_X$-bimodule,

$$
\text{Hom}_{\text{Gr}\mathcal{A}}(\mathcal{F} \otimes e_k\mathcal{A}, -) \cong (-)_k \otimes \mathcal{F}^*
$$

and
(3) If $L$ is an $O_X$-module and $M$ is an object of $\text{Gr} \mathcal{A}$,

$$\text{Hom}_{O_X}(L, \text{Hom}_{\text{Gr} \mathcal{A}}(e_k \mathcal{A}, M)) \cong \text{Hom}_{\text{Gr} \mathcal{A}}(L \otimes e_k \mathcal{A}, M).$$

Proof. The first statement is [6, Proposition 3.11], (1) is [6, Proposition 3.19], (2) is [6, Theorem 3.16(4)] and (3) is a consequence of [6, Proposition 3.10]. □

By Theorem 2.4 (2), $\text{Hom}_{\text{Gr} \mathcal{A}}(-, M)$ is $F \otimes e_k \mathcal{A}$-acyclic when $F$ is a coherent, locally free $O_X$-bimodule. Thus, one may use the resolution [9, Theorem 7.1.2] to compute the derived functors of $\text{Hom}_{\text{Gr} \mathcal{A}}(A/A_{\geq 1}, -)$. By Theorem 2.4(1), we may thus compute the derived functors of $\tau$:

**Theorem 2.5.** The cohomological dimension of $\tau$ is 2. For $i < 2$ and $L$ a coherent, locally free $O_X$-module,

$$R^i \tau(L \otimes e_k \mathcal{A}) = 0$$

and

$$(R^2 \tau(L \otimes e_k \mathcal{A}))_{l-2-i} \cong \begin{cases} 
L \otimes Q_{l-2} \otimes A_{l-2-i} & \text{if } i \geq 0, \\
0 & \text{otherwise}.
\end{cases}$$

Proof. The first result is [6, Corollary 4.10], while the remainder is [6, Lemma 4.9]. □

3. SERRE FINITENESS AND SERRE VANISHING

In this section let $I$ denote a finite subset of $\mathbb{Z} \times \mathbb{Z}$. The proof of the following lemma is straightforward, so we omit it.

**Lemma 3.1.** If $M$ is a noetherian object in $\text{Gr} \mathcal{A}$, $\pi M$ is a noetherian object in $\text{Proj} \mathcal{A}$ and $M$ is locally coherent.

**Lemma 3.2.** If $M$ is a noetherian object in $\text{Gr} \mathcal{A}$, $R^i \tau M$ is locally coherent for all $i \geq 0$.

Proof. The module $O_X(j) \otimes e_k \mathcal{A}$ is noetherian by [6, Lemma 2.17] and the lemma holds with $M = \bigoplus_{(j,k) \in I} O_X(j) \otimes e_k \mathcal{A}$ by Theorem 2.3.

To prove the result for arbitrary noetherian $M$, we use descending induction on $i$. For $i > 2$, $R^i \tau M = 0$ by Theorem 2.5 so the result is trivial in this case. Since $M$ is noetherian, there is a finite subset $I \subset \mathbb{Z} \times \mathbb{Z}$ and a short exact sequence

$$0 \rightarrow R \rightarrow \bigoplus_{(j,k) \in I} O_X(j) \otimes e_k \mathcal{A} \rightarrow M \rightarrow 0$$

by [6, Lemma 2.17]. This induces an exact sequence of $\mathcal{A}$-modules

$$\ldots \rightarrow (R^i \tau(\bigoplus_{(j,k) \in I} O_X(j) \otimes e_k \mathcal{A}))_l \rightarrow (R^i \tau M)_l \rightarrow (R^{i+1} \tau R)_l \rightarrow \ldots$$

The left module is coherent by the first part of the proof, while the right module is coherent by the induction hypothesis. Hence the middle module is coherent since $X$ is noetherian.

**Corollary 3.3.** If $M$ is a noetherian object in $\text{Gr} \mathcal{A}$, $R^i(\omega(-)_{k})(\pi M)$ is coherent for all $i \geq 0$ and all $k \in \mathbb{Z}$. 

Proof. Since \((-)_k : \text{Gr}A \rightarrow \text{Mod}X\) is an exact functor, \(R^i(\omega(-)_k)(\pi M) \cong R^i\omega(\pi M)_k\).

Now, to prove \(\omega(\pi M)_k\) is coherent, we note that there is an exact sequence in \(\text{Mod}X\)

\[
0 \rightarrow \tau M_k \rightarrow M_k \rightarrow \omega(\pi M)_k \rightarrow (R^1\tau M)_k \rightarrow 0
\]

by [6] Theorem 4.11. Since \(M_k\) and \((R^1\tau M)_k\) are coherent by Lemma 3.1 and Lemma 3.2 respectively, \(\omega(\pi M)_k\) is coherent since \(X\) is noetherian.

The fact that \(R^i\omega(\pi M)_k\) is coherent for \(i > 0\) follows from Lemma 3.2 since, in this case,

\[
(2) \quad (R^i\omega(\pi M))_k \cong (R^{i+1}\tau M)_k
\]

by [6] Theorem 4.11. □

Lemma 3.4. For \(\mathcal{N}\) noetherian in \(\text{Gr}A\), \(R^1\omega(\pi \mathcal{N})_k = 0\) for \(k >> 0\).

Proof. When \(\mathcal{N} = \bigoplus_{(l,m) \in I} (\mathcal{O}_X(l) \otimes e_mA)\), the result follows from (2) and Theorem 2.5.

More generally, there is a short exact sequence

\[
0 \rightarrow \mathcal{R} \rightarrow \pi(\bigoplus_{(l,m) \in I} \mathcal{O}_X(l) \otimes e_mA) \rightarrow \pi \mathcal{N} \rightarrow 0
\]

which induces an exact sequence

\[
\cdots \rightarrow R^1\omega(\pi(\bigoplus_{(l,m) \in I} \mathcal{O}_X(l) \otimes e_mA)) \rightarrow R^1\omega(\pi \mathcal{N}) \rightarrow R^2\omega(\mathcal{R}) = 0.
\]

where the right equality is due to (2) and Theorem 2.5. Since the left module is 0 in high degree, so is \(R^1\omega(\pi \mathcal{N})\). □

Theorem 3.5. For any noetherian object \(\mathcal{N}\) in \(\text{Gr}A\),

1. \(\text{Ext}^i_{\text{proj}A}(\bigoplus_{(j,k) \in I} \pi(\mathcal{O}_X(j) \otimes e_kA), \pi \mathcal{N})\) is finite-dimensional over \(K\) for all \(i \geq 0\), and
2. for \(i > 0\), \(\text{Ext}^i_{\text{proj}A}(\bigoplus_{(j,k) \in I} \pi(\mathcal{O}_X(j) \otimes e_kA), \pi \mathcal{N}) = 0\) whenever \(j << 0\) and \(k >> 0\).

Proof. Let \(d\) denote the cohomological dimension of \(X\). Since \(\text{Ext}^i_{\text{proj}A}(-, \pi \mathcal{N})\) commutes with finite direct sums, it suffices to prove the theorem when \(I\) has only one element.

\[
\text{Hom}_{\text{proj}A}(\pi(\mathcal{O}_X(j) \otimes e_kA), \pi \mathcal{N}) \cong \text{Hom}_{\text{Gr}A}(\mathcal{O}_X(j) \otimes e_kA, \omega \pi \mathcal{N})
\]

\[
\cong \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(j), \mathcal{H}om_{\text{Gr}A}(e_kA, \omega \pi \mathcal{N}))
\]

\[
\cong \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(j), \omega(\pi \mathcal{N})_k)
\]

\[
\cong \Gamma(\mathcal{O}_X(-j) \otimes \omega(\pi \mathcal{N})_k)
\]

where the second isomorphism is from Theorem 2.4 (3), while the third isomorphism is from Theorem 2.4 (2). Thus,

\[
\text{Ext}^i_{\text{proj}A}(\pi(\mathcal{O}_X(j) \otimes e_kA), \pi \mathcal{N}) \cong R^i(\Gamma \circ (\mathcal{O}_X(-j) \otimes \omega(-)_k))\pi \mathcal{N}.
\]

If \(i = 0\), (1) follows from Corollary 3.3 and [1] III, Theorem 5.2a, p. 228. If \(0 < i < d + 1\), the Grothendieck spectral sequence gives us an exact sequence

\[
\cdots \rightarrow R^i(\Gamma \circ (\mathcal{O}_X(-j) \otimes \omega(\pi \mathcal{N})_k)) \rightarrow R^i(\Gamma \circ \mathcal{O}_X(-j) \otimes \omega(-)_k)\pi \mathcal{N} \rightarrow
\]
\[ R^{i-1} \Gamma R^1 (\mathcal{O}_X(-j) \otimes \omega(-) \pi N) \rightarrow \ldots \]

Since \( \omega(\pi N)_k \) and \( R^1 (\mathcal{O}_X(-j) \otimes \omega(-) \pi N) \cong \mathcal{O}_X(-j) \otimes R^1 (\omega(-) \pi N) \) are coherent by Corollary 3.3, the first and last terms of (4) are finite-dimensional by [1, III, Theorem 5.2a, p.228]. Thus, the middle term of (4) is finite-dimensional as well, which proves (1) in this case. To prove (2) in this case, we note that, since \( \omega(\pi N)_k \) is coherent, the first module of (4) is 0 for \( j \ll 0 \) by [1, III, Theorem 5.2b, p.228]. If \( i > 1 \), the last module of (4) is 0 for \( j \ll 0 \) for the same reason. Finally, if \( i = 1 \), the last module of (4) is 0 since \( R^1 \omega(\pi N)_k = 0 \) for \( k \gg 0 \) by Lemma 3.4.

If \( i = d + 1 \), the Grothendieck spectral sequence gives an isomorphism

\[ R^{d+1} (\Gamma \circ (\mathcal{O}_X(-j) \otimes \omega(-) \pi N) \cong R^d \Gamma R^1 (\mathcal{O}_X(-j) \otimes \omega(-) \pi N). \]

In this case, (1) again follows from Corollary 3.3 and [1, III, Theorem 5.2a, p.228], while (2) follows from Lemma 3.4.

**Corollary 3.6.** If \( M \) and \( N \) are noetherian objects in \( \text{Gr}A \), \( \text{Ext}^i_{\text{proj}A}(\pi M, \pi N) \) is finite-dimensional for \( i \geq 0 \).

**Proof.** Since \( M \) is noetherian, there is an exact sequence

\[ 0 \rightarrow R \rightarrow \pi( \bigoplus_{(j,k) \in I} \mathcal{O}_X(j) \otimes e_k A) \rightarrow \pi M \rightarrow 0. \]

Since the central term is noetherian by Lemma 3.4, so is the \( R \). Since \( \text{Hom}_{\text{proj}A}(-, \pi N) \) is left exact, there are exact sequences

(4) \[ 0 \rightarrow \text{Hom}_{\text{proj}A}(\pi M, \pi N) \rightarrow \text{Hom}_{\text{proj}A}(\pi( \bigoplus_{(j,k) \in I} \mathcal{O}_X(j) \otimes e_k A), \pi N) \rightarrow \]

and, for \( i \geq 1 \),

(5) \[ \rightarrow \text{Ext}^{i-1}_{\text{proj}A}(R, \pi N) \rightarrow \text{Ext}^i_{\text{proj}A}(\pi M, \pi N) \rightarrow \text{Ext}^i_{\text{proj}A}(\pi( \bigoplus_{(j,k) \in I} \mathcal{O}_X(j) \otimes e_k A), \pi N) \rightarrow \]

Since \( \pi \) commutes with direct sums, the right-hand terms of (4) and (5) are finite-dimensional by Theorem 3.3(1), while the left hand term of (5) is finite-dimensional by the induction hypothesis. \( \square \)

### 4. Riemann-Roch and Adjunction

Let \( X \) be a smooth projective curve, let \( A \) be the noncommutative symmetric algebra generated by a locally free \( \mathcal{O}_X \)-bimodule \( E \) of rank 2, and let \( Y = \text{Proj}A \). In this section, we state the Riemann-Roch theorem and adjunction formula for \( Y \). In order to state these results, we need to define an intersection multiplicity on \( Y \). This definition depends on the fact that \( Y \) has well behaved cohomology, so we begin this section by reviewing relevant facts regarding the cohomology of \( Y \).

Let \( \mathcal{O}_Y = \pi \text{pr}_2^* e_0 A \). By [1] Theorem 5.20, \( Y \) satisfies Serre duality, i.e., there exists an object \( \omega_Y \) in \( \text{Proj}A \), called the canonical sheaf on \( Y \), such that

\[ \text{Ext}^{2-i}_{Y}(\mathcal{O}_Y, -) \cong \text{Ext}^i_Y(-, \omega_Y) \]

for all \( 0 \leq i \leq 2 \). Furthermore, the canonical sheaf \( \omega_Y \) is noetherian [2].

By [1] Theorem 4.16, \( Y \) has cohomological dimension two, i.e.

(7) \[ 2 = \sup\{i | \text{Ext}^i_Y(\mathcal{O}_Y, M) \neq 0 \text{ for some noetherian object } M \text{ in } \text{Proj}A\}. \]

We write \( D: Y \rightarrow Y \) for an autoequivalence, \(-D: Y \rightarrow Y \) for the inverse of \( D \), and \( M(D) := D(M) \) for \( M \in Y \).
Definition 4.1. [3, Definition 2.3] A weak divisor on $Y$ is an element $\mathcal{O}_D \in K_0(Y)$ of the form $\mathcal{O}_D = [\mathcal{O}_Y] - [\mathcal{O}_Y(-D)]$ for some autoequivalence $D$ of $Y$.

We now define an intersection multiplicity on $Y$ following [3]. Let $\mathcal{M}$ be a noetherian object in $\text{Proj} \mathcal{A}$, and let $[\mathcal{M}]$ denote its class in $K_0(Y)$. We define, for $\mathcal{O}_D$ a weak divisor on $Y$, a map $\xi(\mathcal{O}_D, -) : K_0(Y) \to \mathbb{Z}$ by

$$\xi(\mathcal{O}_D, [\mathcal{M}]) = \sum_{i=0}^{\infty} (-1)^i (\dim K \text{Ext}^i_Y(\mathcal{O}_Y, \mathcal{M}) - \dim K \text{Ext}^i_Y(\mathcal{O}_Y(-D), \mathcal{M})).$$

This map is well defined by (7) and Corollary 3.6. We define the intersection multiplicity of $\mathcal{O}_D$ and $\mathcal{M}$ by

$$\mathcal{O}_D \cdot \mathcal{M} := -\xi(\mathcal{O}_D, [\mathcal{M}]).$$

Finally, we define a map $\chi(-) : K_0(Y) \to \mathbb{Z}$ by

$$\chi([\mathcal{M}]) := \sum_{i=0}^{\infty} (-1)^i \dim K \text{Ext}^i_Y(\mathcal{O}_Y, \mathcal{M}).$$

Corollary 4.2. Let $Y = \text{Proj} \mathcal{A}$, let $\omega_Y$ denote the canonical sheaf on $Y$, and suppose $\mathcal{O}_D$ is a weak divisor on $Y$. Then we have the following formulas:

1. (Riemann-Roch)

$$\chi(\mathcal{O}_Y(D)) = \frac{1}{2}(\mathcal{O}_D \cdot \mathcal{O}_D - \mathcal{O}_D \cdot \omega_Y + \mathcal{O}_D \cdot \mathcal{O}_Y) + 1 + p_a$$

where $p_a := \chi([\mathcal{O}_Y]) - 1$ is the arithmetic genus of $Y$.

2. (Adjunction)

$$2g - 2 = \mathcal{O}_D \cdot \mathcal{O}_D + \mathcal{O}_D \cdot \omega_Y - \mathcal{O}_D \cdot \mathcal{O}_Y$$

where $g := 1 - \chi(\mathcal{O}_D)$ is the genus of $\mathcal{O}_D$.

Proof. The quasi-scheme $Y$ is Ext-finite by Corollary 3.6 has cohomological dimension 2 by [6, Theorem 4.16], and satisfies Serre duality with $\omega_Y$ by [6, Theorem 5.20]. Thus, $Y$ is classical Cohen-Macaulay, and the result follows [3, Theorem 3.11].

In stating the Corollary, we defined the intersection multiplicity only for specific elements of $K_0(Y) \times K_0(Y)$. In order to define an intersection multiplicity on the entire set $K_0(Y) \times K_0(Y)$, one must first prove that $Y$ has finite homological dimension. In [4, Section 6], Mori and Smith study noncommutative $\mathbb{P}^1$-bundles $Y = \text{Proj} \mathcal{A}$ such that $\mathcal{A}$ is generated by a bimodule $\mathcal{E}$ with the property that $\mathcal{E} \otimes \mathcal{E}$ contains a nondegenerate invertible bimodule. In this case, they use the structure of $K_0(Y)$ to prove that $Y$ has finite homological dimension. They then compute various intersections on $Y$ without the use of either the Riemann-Roch theorem or the adjunction formula. In particular, they prove that distinct fibers on $Y$ do not meet, and that a fiber and a section on $Y$ meet exactly once.

References

[1] R. Hartshorne, Algebraic Geometry, Springer-Verlag, 1977.
[2] I. Mori, private communication.
[3] I. Mori, Riemann-Roch like theorems for triangulated categories, J. Pure Appl. Algebra, to appear.
[4] I. Mori and S. P. Smith, The grothendieck group of a quantum projective space bundle, preprint.

[5] A. Nyman, The geometry of points on quantum projectivizations, \textit{J. Algebra}, 246 (2001) 761-792.

[6] A. Nyman, Serre duality for non-commutative \( P^1 \)-bundles, \textit{Trans. Amer. Math. Soc.}, to appear.

[7] M. van den Bergh, A translation principle for the four-dimensional Sklyanin algebras, \textit{J. Algebra}, 184 (1996) 435-490.

[8] M. Van den Bergh, Blowing up of non-commutative smooth surfaces, \textit{Mem. Amer. Math. Soc.}, 154 (2001).

[9] M. Van den Bergh, Non-commutative \( P^1 \)-bundles over commutative schemes, to appear.

[10] M. Van den Bergh, Non-commutative quadrics, in preparation.

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