Mathematical issues in eternal inflation

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Abstract

In this paper, we consider the problem of the existence and uniqueness of solutions to the Einstein field equations for a spatially flat Friedmann–Lemaître–Robertson–Walker universe in the context of stochastic eternal inflation, where the stochastic mechanism is modelled by adding a stochastic forcing term representing Gaussian white noise to the Klein–Gordon equation. We show that under these considerations, the Klein–Gordon equation actually becomes a stochastic differential equation. Therefore, the existence and uniqueness of solutions to Einstein’s equations depend on whether the coefficients of this stochastic differential equation obey Lipschitz continuity conditions. We show that for any choice of $V(\phi)$, the Einstein field equations are not globally well-posed, hence, any solution found to these equations is not guaranteed to be unique. Instead, the coefficients are at best locally Lipschitz continuous in the physical state space of the dynamical variables, which only exist up to a finite explosion time. We further perform Feller’s explosion test for an arbitrary power-law inflaton potential and prove that all solutions to the Einstein field equations explode in a finite time with probability one. This implies that the mechanism of stochastic inflation thus considered cannot be described to be eternal, since the very concept of eternal inflation implies that the process continues indefinitely. We therefore argue that stochastic inflation based on a stochastic forcing term would not produce an infinite number of universes in some multiverse ensemble. In general, since the Einstein field equations in both situations are not well-posed, we further conclude that the existence of a multiverse via the stochastic eternal inflation mechanism considered in this paper is still very much an open question that will require much deeper investigation.

Keywords: general relativity, stochastic eternal inflation, cosmology

(Some figures may appear in colour only in the online journal)
1. Introduction

In this paper, we consider the problem of the existence and uniqueness of solutions to the Einstein field equations for a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) universe in the context of stochastic eternal inflation. Stochastic eternal inflation has received a considerable amount of attention over the past few years as it is one of main motivations behind the multiverse [EMM12]. In particular, Mijić [Mij90] analysed the boundary conditions of the dynamics of inflation as a relaxation random process and gave a simple proof for the existence of eternal inflation. Salopek and Bond [SB91] studied nonlinear effects of the metric and scalar fields in the context of stochastic inflation. Linde et al [LLM94] considered chaotic inflation in theories with effective potentials that behave either as $\phi^n$ or as $e^{\alpha \phi}$. They also performed computer simulations of stochastic processes in the inflationary universe. Linde and Linde [LL94] investigated the global structure of an inflationary universe both by analytical methods and by computer simulations of stochastic processes in the early universe. Susperregi and Mazumdar [SM98] considered an exponential inflation potential and showed that this theory predicts a uniform distribution for the Planck mass at the end of inflation, for the entire ensemble of universes that undergo stochastic inflation. Vanchurin et al [VVW00] investigated methods of inflationary cosmology based on the Fokker–Planck equation of stochastic inflation and direct simulation of inflationary spacetime. Winitzki [Win02] explored the fractal geometry of spacetime that results from stochastic eternal inflation. Kunze [Kun04] considered chaotic inflation on the brane in the context of stochastic inflation. Winitzki [Win05] described some issues regarding the time-parameterization dependence in stochastic descriptions of eternal inflation. Gratton and Turok [GT05] investigated a simple model of $\lambda \phi^4$ inflation with the goal of analysing the continuous revitalization of the inflationary process in some regions. Li and Wang [LW07] used a stochastic approach to investigate a measure for slow-roll eternal inflation. Gálvez Gherzi et al [Tom11] used stochastic eternal inflation to analyse the connection between the Einstein equations and the thermodynamic equations. Qiu et al [QS12] used stochastic quantum fluctuations through a phenomenological Langevin analysis studying whether they can affect entropic inflation eternality. Harlow et al [HSSS12] described a discrete stochastic model of eternal inflation that shares many of the most important features of the corresponding continuum theory. Vanchurin [Van12] developed a dynamical systems approach to model inflation dynamics. Feng et al [FLS10] investigated conditions under which phantom inflation is prevented from being eternal.

In this paper, we consider the effects of adding a stochastic forcing term in the form of Gaussian white noise to the right-hand side of the Klein–Gordon equation as is done by Gratton and Turok [GT05] in their study of employing the Langevin formalism to eternal inflation. We show that because this stochastic term is not differentiable at any point, the existence and uniqueness of Einstein’s equations depends on the Lipschitz continuity of the coefficients of the corresponding stochastic differential equation. Indeed, in [GT05], the authors also conclude, although through different means, namely that the Langevin approach leads to a monotonically decreasing Hubble constant, which implies that one cannot have eternal inflation in this scenario. Instead, as an alternative to the eternal inflation picture, they propose a model with a constraint on the initial and final states that lead to a picture similar to what is typically associated with eternal inflation.

Throughout, we use geometrized units, such that $G = c = 1$, and as a result of this, all physical quantities have dimensions of powers of length [Wal84].
2. Description of the model

We consider a spatially homogeneous and isotropic universe described by the FLRW metric [EMM12] as
\[ ds^2 = -dt^2 + a^2(t) \left[ dr^2 + f^2(r) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right], \quad u^\mu = \delta^\mu_0, \]
where
\[ f(r) = (\sinh r, r, \sin r) \quad \text{for} \quad K = (-1, 0, +1), \]
where \( K \) denotes the sign of the curvature of the particular FLRW model under consideration. Namely, \( K = -1 \) refers to hyperbolic FLRW models, \( K = 0 \) refers to flat FLRW models, and \( K = +1 \) refers to positively curved FLRW models.

We assume that energy–momentum tensor is that of a scalar field and has the form [EMM12]
\[ T^{ab}_\phi = \nabla^a \phi \nabla^b \phi - \left[ \frac{1}{2} \nabla^c \phi \nabla^c \phi + V(\phi) \right] g^{ab}. \]

The Einstein field equations that also describe the dynamics of the model are given by the Raychaudhuri and Friedmann equations [EMM12]
\[ \dot{H} = -H^2 + \frac{1}{3} \left[ V(\phi) - \dot{\phi}^2 \right], \]
\[ 3H^2 = V(\phi) + \frac{\dot{\phi}^2}{2} + \frac{1}{2} R, \]
where \( H \) is the Hubble parameter, and \( R \) is the three-dimensional Ricci scalar, which is constant for the FLRW models.

The contracted Bianchi identities give an evolution equation for the scalar field, \( \phi \), which in this case is precisely the Klein–Gordon equation [EMM12]
\[ \ddot{\phi} + 3H \dot{\phi} + V'(\phi) = 0. \]
Together equations (4)–(6) fully describe the dynamics of the cosmological model described by equations (1) and (3).

3. A model of stochastic eternal inflation

Following [EMM12], we note that the classical dynamics of the inflaton dictates that the inflaton always rolls down its potential. However, quantum fluctuations can also drive the inflaton uphill, which causes inflation to last longer in some regions, which, in turn, enlarges the volume of the region. In some regions, the inflaton will remain high enough up the potential hill to maintain acceleration. This is a stochastic scenario that is typically described as eternal inflation, which is one of the main motivating ideas behind the multiverse concept.

We will model this stochastic behaviour following [GT05] and [LLM94] where a stochastic forcing term representing Gaussian white noise is added to the right-hand side of equation (6).

As discussed in [GT05], the purpose of this stochastic term is to describe the scalar field fluctuations. Further, this stochastic term arises because of initially small-scale quantum fluctuations across the de Sitter horizon. It is typically claimed in inflationary studies that the physical size of the region that is known to be inflating increases quite rapidly. However, this statement only applies when the initial region is so large that causal influences coming from
outside the initial region cannot propagate far enough into it to stop the inflationary process. Therefore, the stochastic model under consideration only applies to a region that is small enough to lie within the past light cone of a future observer, and large enough to remain in an inflating state well into the future. The reader is encouraged to see figure 1 in [GT05].

We will denote the stochastic forcing term by $\eta(t)$ [GT05], and we find that the dynamical equations (4) and (6) can be written as coupled first-order ordinary differential equations in the form:

\[
\frac{d\phi}{dt} = f, \tag{7}
\]

\[
\frac{dH}{dt} = -H^2 + \frac{1}{3} \left[ V(\phi) - f^2 \right], \tag{8}
\]

\[
\frac{df}{dt} = H^{3/2} \eta(t) - 3Hf - V'(\phi), \tag{9}
\]

\[
3H^2 = V(\phi) + \frac{f^2}{2} + \frac{1}{2} R, \tag{10}
\]

where equation (10) is the generalized Friedmann equation and acts as a constraint on the initial conditions. Analysing the system (7)–(9), we see that the right-hand side of equation (9) is not $C^1$ because the stochastic function $\eta(t)$ is not anywhere differentiable. To make this point more precise, we describe some properties of the stochastic function $\eta(t)$. Following [Lon10], we note that $\eta(t)$ is defined as the time derivative of the Wiener process, which we denote by $W$, that is

\[
\frac{dW}{dr} = \eta(t), \tag{11}
\]

where $W(0) = 0$ by definition. As noted in [GT05], $\eta(t)$ also satisfies the autocorrelation relation

\[
\langle \eta(t)\eta(t') \rangle = \delta(t - t'). \tag{12}
\]

One has to be careful with the definition of $\eta(t)$ as given in equation (11) since $W$ is nowhere differentiable. In fact, for completeness, following [Wie03] we now state some properties of $W$ based on the so-called Lévy characterization:

1. the path of $W$ is continuous and starts at 0,
2. $W$ is a martingale and $\{dW(t)^2\} = dt$,
3. the increment of $W$ over time period $[s, t]$ is normally distributed, with mean 0 and variance $(t - s)$,
4. the increments of $W$ over non-overlapping time periods are independent.

Note that we will not go into extensive detail about martingales. The interested reader is asked to consult [Wie03] for more details on martingales and their properties. Based on the arguments provided in [Wie03], we will now show that $W$ is not anywhere differentiable, that is, not $C^1$ anywhere. Consider a time interval of length $\Delta t = 1/n$ starting at $t$. We will define the rate of change over an infinitesimal time interval $[t, t + \Delta t]$ is
\( X_n \equiv \frac{W(t + \Delta t) - W(t)}{\Delta t} = \frac{W(\frac{t + \frac{1}{n} - W(t)}{\frac{1}{n}}) = n \left[ W\left(t + \frac{1}{n}\right) - W(t)\right]. \)  

(13)

Therefore, \( X_n \) is normally distributed with expectation value 0, variance \( n \), and of course, standard deviation, \( \sqrt{n} \). It is clear then that \( X_n \) has the same probability distribution as \( \sqrt{n}Z \), where \( Z \) is the standard normal distribution. To analyse the \( C^1 \) properties, we must see what happens to \( X_n \) as \( \Delta t \to 0 \), in other words as \( n \to \infty \). For any \( k > 0 \), let \( X_n = \sqrt{n}Z \), then

\[
P\left[ \|X_n\| > k \right] = P\left[ \|Z\| > \frac{k}{\sqrt{n}} \right].
\]

(14)

Clearly, as \( n \to \infty \), \( k/\sqrt{n} \to 0 \), so we have that

\[
P\left[ \|Z\| > \frac{k}{\sqrt{n}} \right] \to P[\|Z\| > 0] = 1.
\]

(15)

The point is that we can choose \( k \) to be arbitrarily large, so that the rate of change at time \( t \) is not finite, and therefore, \( W \) is not differentiable at \( t \). Since \( t \) is arbitrary, one concludes that \( W \) is nowhere differentiable.

4. Dynamical equations

One notices that the Einstein field equations in the present context yield a coupled system of nonlinear ordinary differential equations (7)–(9). As stated from the onset, our goal in this paper is to analyse the notion of existence and uniqueness of solutions to these equations and to determine the existence of the possibility of exploding solutions within the context of the powerful theorems related to stochastic differential equations. The issue, however, is that all of the theorems concerning the aforementioned properties of such equations that exist in the present literature only deal with one-dimensional stochastic processes. Clearly, equation (9) is not a one-dimensional process, as its drift term is coupled to the other differential equations of the system. The interested reader is encouraged to refer to [KS91] and references therein for a careful review of these ideas. We therefore will employ the powerful technique of re-writing our system of equations in terms of expansion-normalized variables [WE97] that will effectively lead to us obtaining a single stochastic differential equation that describes the dynamics of the system.

We define expansion-normalized variables denoted by \( X, Y \) as

\[
X = \sqrt{\frac{V}{3}} \frac{1}{H^3}, \quad Y = \frac{\phi}{\sqrt{6}H^3}.
\]

(16)

As is done in [WE97], we additionally introduce a dimensionless time variable \( \tau \), such that

\[
\frac{dt}{d\tau} = \frac{1}{H}.
\]

(17)

Substituting equations (16) and (17) into equations (7)–(10), one obtains the following dynamical system

\[
X' = X (1 + q + Y),
\]

(18)

\[
Y' = N - 3Y - \frac{1}{3} \left( 1 - Y^2 \right) + Y (1 + q),
\]

(19)
subject to the constraint
\[ X^2 + Y^2 = 1. \] (20)

Note that \( q \) is the deceleration parameter found through Raychaudhuri’s equation and is in this case
\[ q = 2Y^2 - X^2. \] (21)

In addition, we have also defined an expansion-normalized Gaussian noise term
\[ \eta = \frac{\sqrt{6}}{\sqrt{H}} \mathcal{N}, \] (22)

where \( \mathcal{N} \) is the formal time derivative of the Wiener process \( W(\tau) \) as explained above. One sees that the motivation for this definition of \( \mathcal{N} \) is in fact purely physical. As discussed in [GT05], the dimensions of \( \eta(t) \) are length to the power one half (in our geometrized units). We scale \( \mathcal{N} \) according to this notion so that it is a dimensionless variable.

Following [WD11], we also have defined a model-dependent dimensionless parameter
\[ \lambda = \sqrt{\frac{3}{2}} \frac{V'}{V}. \] (23)

Note that in the definition of \( \lambda \) in equation (23), we have set the Planck mass to unity as per our choice of units.

Making use of equations (20) and (21) in equations (18) and (19), we see that the equations decouple
\[ X' = X \left( 3 - 3X^2 + \sqrt{1 - X^2} \lambda \right). \] (24)
\[ Y' = \left( Y^2 - 1 \right) \left( 3Y + \lambda \right) + \mathcal{N}(\tau). \] (25)

One sees then, that the stochastic dynamics are entirely represented by equation (25).

We will return to analysing equation (24) later in the paper. For now, we will focus our attention on equation (25). In doing so, it is important to note that the state space corresponding to equation (25) is not \( Y \in \mathbb{R} \). Notice that from equation (3), that the energy–momentum tensor is that of a perfect fluid with energy density
\[ \mu = \frac{1}{2} \dot{\phi}^2 + V(\phi), \] (26)

and pressure
\[ p = \frac{1}{2} \dot{\phi}^2 - V(\phi). \] (27)

Following [EMM12], we note that for a non-negative potential energy \( V \geq 0 \), one has the restriction that
\[ \frac{p}{\mu} \leq 1. \] (28)

Substituting equations (26), (27), (16), (20), and (21) into equation (28), we obtain that
\[ -1 \leq Y \leq 1, \] (29)

which defines the state space for equation (25).
5. Existence and uniqueness of solutions

In this section, we analyse in detail the solutions to the stochastic differential equation (25), with the state space defined in (29). Using equation (11), we write stochastic differential equation (25) in Itō form as

\[ dY = \left( (Y^2 - 1)(3Y + \lambda) \right) d\tau + dW, \]  

(30)

where \(-1 \leq Y \leq 1\).

We wish to analyse with respect to this equation whether there exist solutions in the domain (29), whether they are unique, and whether there exists some finite time where solutions explode, that is, rapidly converge to infinity with unitary probability. To answer these questions, we state some essential theorems and definitions from stochastic differential equations theory.

We first note that the coefficient of \(d\tau\) in equation (30) is known as the drift coefficient, which we will denote by \(b(\tau, Y)\). Further, the coefficient of \(dW\) is known as the dispersion coefficient, which we will denote by \(\sigma(\tau, Y)\). It is clear from equation (30) that

\[ b(\tau, Y) = (Y^2 - 1)(3Y + \lambda), \quad \sigma(\tau, Y) = 1, \quad -1 \leq Y \leq 1. \]  

(31)

Following [MT03], we note that a function \(f\) is locally Lipschitz continuous if and only if for \(f : A \subset \mathbb{R}^n \rightarrow \mathbb{R}^n\) satisfies \(\|f(x) - f(y)\| \leq K \|x - y\|\) for all \(x, y \in A\) for \(K > 0\).

Similarly, a function \(f\) is globally Lipschitz continuous if one takes according to the previous definition \(A = \mathbb{R}^n\), that is, the domain is the whole space itself.

With these definitions in mind, we note that if the coefficients \(b\) and \(\sigma\) as defined in equation (31) are locally Lipschitz continuous then one can only guarantee existence up to an explosion time and uniqueness of solutions in a weak sense. Further, by Feller’s explosion test (discussed below) the solutions to a stochastic differential equation with only locally Lipschitz continuous functions as coefficients may explode in a finite time with probability one. We show below that indeed, by Feller’s explosion test, that solutions to the field equations within the domain (29) do indeed explode within a finite time with unitary probability.

Let us look at the function \(b(\tau, Y)\) from equation (31). Clearly from the above definition of Lipschitz continuity, one has that for any domain \(A \subset Y \in [-1, 1]\), there clearly exists a \(K > 0\) such that

\[ \|((x^2 - 1)(3x + \lambda) - (y^2 - 1)(3y + \lambda))\| \leq K \|x - y\|, \quad x, y \in A \subset Y \in [-1, 1]. \]  

(32)

In addition, note that indeed the interval \([-1, 1]\) is closed, so the derivative of \(b\) in equation (31), will be bounded on this interval as well, and by definition, is therefore, locally Lipschitz continuous.

However, it is quite clear, that if one extends the domain such that \(A = \mathbb{R}\), then there exists no \(K > 0\) such that

\[ \|((x^2 - 1)(3x + \lambda) - (y^2 - 1)(3y + \lambda))\| \leq K \|x - y\|, \quad x, y \in A = \mathbb{R}, \]  

(33)

since the polynomial function \(b\) is unbounded on \(\mathbb{R}\) and one can arbitrarily make the left-hand side of this inequality large.

Therefore, the coefficients of the stochastic differential equation (30) are not both globally Lipschitz continuous, and one therefore does not have global existence or uniqueness of solutions to the field equations. The coefficients are however, both locally Lipschitz.
continuous over the state space $Y \in [-1, 1]$ which means that one at most can have a solution up to an explosion time. In the section that follows, we will show using Feller’s explosion test that the solutions to the field equations in this context indeed explode in a finite time over the state space $Y \in [-1, 1]$ with unitary probability.

From [KS91], one defines the sequence

$$S_n = \inf \{ \tau \geq 0; \| Y_t \| \geq n \}. \tag{34}$$

The explosion time for $Y$ is then defined as

$$S = \lim_{n \to \infty} S_n. \tag{35}$$

We now define the functions $p(x), v(x)$ such that

$$p(x) = \int_{\zeta}^{x} \exp \left[ -2 \int_{\zeta}^{r} \frac{b(r) dr}{\sigma^2(r)} \right] ds, \tag{36}$$

$$v(x) = \int_{\zeta}^{x} p'(y) \int_{\zeta}^{y} \frac{2dz}{p'(z) \sigma^2(z)} dy, \tag{37}$$

where $\zeta \in (-1, 1)$, that is, $\zeta$ assumes a value over the physical state space of the problem as discussed earlier. Feller’s test [KS91, LPVM14] then says that suppose that for $b, \sigma : (-1, 1) \to \mathbb{R}$, are continuous functions and $\sigma^2 > 0 \in (-1, 1)$. The explosion time $\tau$ of the solution $Y$ of equation (30) is finite with probability 1 if and only if any one of the following conditions hold:

$$\lim_{x \to -1} v(x) < \infty, \quad \lim_{x \to 1} v(x) = -\infty, \tag{38}$$

$$\lim_{x \to -1} v(x) < \infty, \quad \lim_{x \to 1} p(x) = -\infty, \tag{39}$$

$$\lim_{x \to -1} v(x) < \infty, \quad \lim_{x \to 1} p(x) = \infty. \tag{40}$$

Per the above definition of the explosion time $S$, the condition for an explosion in finite time is defined probabilistically.

**Figure 1.** Some results of the numerical integration of equation (37), which clearly shows that $\lim_{x \to -1} v(x) < \infty$. 

Class. Quantum Grav. 32 (2015) 075001 I S Kohli and M C Haslam
Substituting the expressions for $b$ and $\sigma$ from equation (31) into equations (36) and (37), one obtains

$$\int \lambda \xi \xi \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda \lambda 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We now analyse the solutions to the evolution equation for $X$, equation (24),

$$X' = X \left( 3 - 3X^2 + \sqrt{1 - X^2} \lambda \right),$$

where as mentioned above $\lambda \in \mathbb{R}$. We use a similar technique as above to determine the state space for this evolution equation. Namely, we obtain the condition that

$$1 - 2X^2 \leq 1 \Rightarrow X \in \mathbb{R}. \quad (45)$$

Now, it is clear that equation (24) is an ordinary differential equation that contains no stochastic terms. We recall the following theorem from ordinary differential equations theory [BD05]. Let the functions $f$ and $\frac{\partial f}{\partial x}$ be continuous in some rectangle $\alpha < \tau < \beta$, $\gamma < x < \delta$ containing the point $(\tau_0, x_0)$. Then, in some interval $\tau_0 - h < \tau < \tau_0 + h$ contained in $\alpha < \tau < \beta$, there exists a unique solution $x = \phi(\tau)$ of the initial value problem

$$x' = f(\tau, x), \quad x(\tau_0) = x_0. \quad (46)$$

As noted in [BD05], it is enough to guarantee existence, but not uniqueness of solutions on the basis of continuity of $f$ alone. Let us denote the right-hand side of equation (24) by $f(X)$, such that

$$f(X) = 3X - 3X^3 + X \sqrt{1 - X^2} \lambda, \quad (47)$$

where $\lambda \in \mathbb{R}$. This implies that

$$f'(X) = X^2 \left( -9 - \frac{\lambda}{\sqrt{1 - X^2}} \right) + \lambda \sqrt{1 - X^2} + 3. \quad (48)$$

Clearly, $f(X)$ is only valid for $-1 \leq X \leq 1$, while $f'(X)$ has singularities at $X = -1$ and $X = 1$. Therefore, by the above theorem, one can only guarantee existence and uniqueness of solutions in the open interval $X \in (-1, 1)$ up to a finite explosion time, which is a much smaller subset of the physical state space of $X$ is $X \in \mathbb{R}$ by the inflation conditions we stated previously.

6. Conclusions

In this paper, we considered the problem of the existence and uniqueness of solutions to the Einstein field equations for a spatially flat FLRW universe in the context of stochastic eternal inflation where the stochastic mechanism is modeled by adding a stochastic forcing term representing Gaussian white noise to the Klein–Gordon equation. We showed that under these considerations, the Klein–Gordon equation actually becomes a stochastic differential equation. We further demonstrated that the existence and uniqueness of solutions to Einstein’s equations depended on whether the coefficients of this stochastic differential equation obeyed global Lipschitz continuity and growth conditions. We showed that for any choice of $V(\phi)$, the Einstein field equations were not well-posed, hence, any solution found to these equations was not guaranteed to be unique. Instead, we showed that the coefficients were at best, locally Lipschitz continuous in the physical state space of the Einstein equations, which only existed up to a finite explosion time. We then performed Feller’s explosion test for an arbitrary power-law inflaton potential and proved that all solutions to the Einstein field equations explode in a finite time with probability one. This implies that the mechanism of stochastic inflation thus considered cannot be described to be eternal, since the very concept of eternal inflation implies that the process continues indefinitely. As mentioned in the introduction of
this paper, this was the same conclusion reached in [GT05], namely that eternal inflation described by such a stochastic force term does not lead to future eternal behaviour. Further, we believe we have extended the results of [GT05] to a more general case as the study of Gratton and Turok only considered a potential of the form $\lambda \phi^4$. Our work explicitly demonstrates that stochastic eternal inflation of the specific form considered here and in [GT05] for any power-law scalar field potential is not future eternal.

We therefore conclude that stochastic inflation based on a stochastic forcing term would not produce an infinite number of universes in some multiverse ensemble. In general, since the Einstein field equations in both situations were not well-posed, we conclude that the existence of a multiverse via the stochastic eternal inflation mechanism considered in this paper is still very much an open question that will require much deeper investigation.

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References

[BD05] Boyce W E and DiPrima R C 2005 Elementary Differential Equations and Boundary Value Problems 8th edn (New York: Wiley)
[EMM12] Ellis G F R, Maartens R and MacCallum M A H 2012 Relativistic Cosmology 1st edn (Cambridge: Cambridge University Press)
[FLS10] Feng C-J, Li X-Z and Saridakis E N 2010 Preventing eternality in phantom inflation Phys. Rev. D 82 023526
[GT05] Gratton S and Turok N 2005 Langevin analysis of eternal inflation Phys. Rev. D 72 043507
[HSSS12] Harlow D, Shenker S H, Stanford D and Susskind L 2012 Tree-like structure of eternal inflation: a solvable model Phys. Rev. D 85 063516
[KS91] Karatzas I and Shreve S 1991 Brownian Motion and Stochastic Calculus (Graduate Texts in Mathematics vol 113) 2nd edn (Berlin: Springer)
[LL94] Linde A and Linde D 1994 Topological defects as seeds for eternal inflation Phys. Rev. D 50 2456–68
[LLM94] Linde A, Linde D and Mezhlumian A 1994 From the big bang theory to theory of a stationary universe Phys. Rev. D 49 1783–826
[Lon10] Longtin A 2010 Stochastic dynamical systems Scholarpedia 5 1619
[LPVM14] León J A, Peralta L and Villa-Morales J 2014 An Osgood’s criterion for a semilinear stochastic differential equation 1–21 (arXiv:1401.7905)
[LIW07] Li M and Wang Y 2007 A stochastic measure of eternal inflation J. Cosmol. Astropart. Phys. 08(2007)007
[Mij90] Mijic M 1990 Random walk after the big bang Phys. Rev. D 42 2469–82
[MT03] Marsden J E and Tromba A 2003 Vector Calculus 5th edn ed W H Freeman (San Francisco: Freeman)
[QS12] Qiu T and Saridakis E N 2012 Entropic force scenarios and eternal inflation Phys. Rev. D 85 043504
[SB91] Salopek D S and Bond J R 1991 Stochastic inflation and nonlinear gravity Phys. Rev. D 43 1005–31
[SM98] Susperregi M and Mazumdar A 1998 Textended inflation with an exponential potential Phys. Rev. D 58 083512
[Tom11] Gámez Ghersi J T, Geshnizjani G, Piazza F and Shandera S 2011 Eternal inflation and a thermodynamic treatment of Einstein’s equations J. Cosmol. Astropart. Phys. JCAP06 (2011)005

[Van12] Vanchurin V 2012 Dynamical systems of eternal inflation: a possible solution to the problems of entropy, measure, observables, and initial conditions Phys. Rev. D 86 043502

[VVVW00] Vanchurin V, Vilenkin A and Winitzki S 2000 Predictability crisis in inflationary cosmology and its resolution Phys. Rev. D 61 083507

[Wal84] Wald R M 1984 General Relativity 1984 edn (Chicago, IL: University of Chicago Press)

[WD11] Wagstaff J M and Dimopoulos K 2011 Particle production of vector fields: scale invariance is attractive Phys. Rev. D 83 023523

[WE97] Wainwright J and Ellis G F R 1997 Dynamical Systems in Cosmology 1st edn (Cambridge: Cambridge University Press)

[Wie03] Wiersma U F 2003 Brownian Motion Calculus 1st edn (New York: Wiley)

[Win02] Winitzki S 2002 Eternal fractal in the universe Phys. Rev. D 65 083506

[Win05] Winitzki S 2005 Time-reparametrization invariance in eternal inflation Phys. Rev. D 71 123507