SOME REMARKS ON THE SYMPLECTIC AND KÄHLER GEOMETRY OF TORIC VARIETIES

CLAUDIO AREZZO, ANDREA LOI, AND FABIO ZUDDAS

ABSTRACT. Let $M$ be a projective toric manifold. We prove two results concerning respectively Kähler-Einstein submanifolds of $M$ and symplectic embeddings of the standard euclidean ball in $M$. Both results use the well-known fact that $M$ contains an open dense subset biholomorphic to $\mathbb{C}^n$.

1. INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

In this paper we use the well-known fact that toric manifolds are compactifications of $\mathbb{C}^n$ in order to prove two results, of Riemannian and symplectic nature, given by the following two theorems.

**Theorem 1.1.** Let $N$ be a projective toric manifold equipped with a toric Kähler metric $G$ and $(M, g) \xrightarrow{\varphi} (N, G)$ be an isometric embedding of a Kähler-Einstein manifold such that $\varphi(M)$ contains a point of $N$ fixed by the torus action. Then $(M, g)$ has positive scalar curvature.

**Theorem 1.2.** Let $(M, \omega)$ a toric manifold endowed with an integral toric Kähler form and let $\Delta \subseteq \mathbb{R}^n$ be the image of the moment map for the torus action. Then, there exists a number $c(\Delta)$ (explicitly computable from the polytope, see Corollary 3.5) such that any ball of radius $r > c(\Delta)$, symplectically embedded into $(M, \omega)$, must intersect the divisor $M \setminus \mathbb{C}^n$.

These two results are proved and discussed respectively in Section 2 (Theorem 2.6) and Section 3 (Corollary 3.5).

The paper ends with an Appendix where, for the reader’s convenience, we give an exposition (as self-contained as possible) of the classical facts about toric manifolds we need in Sections 2 and 3.

2. KÄHLER–EINSTEIN SUBMANIFOLDS OF TORIC MANIFOLDS

Let us briefly recall Calabi’s work on Kähler immersions and diastasis function [7].
Given a complex manifold $N$ endowed with a real analytic Kähler metric $G$, the ingenious idea of Calabi was the introduction, in a neighborhood of a point $p \in N$, of a very special Kähler potential $D_p$ for the metric $G$, which he christened diastasis. Recall that a Kähler potential is an analytic function $\Phi$ defined in a neighborhood of a point $p$ such that $\Omega = \frac{i}{2} \partial \bar{\partial} \Phi$, where $\Omega$ is the Kähler form associated to $G$. In a complex coordinate system $(Z)$ around $p$

$$G_{\alpha\beta} = 2G\left(\frac{\partial}{\partial Z_\alpha}, \frac{\partial}{\partial \bar{Z}_\beta}\right) = \frac{\partial^2 \Phi}{\partial Z_\alpha \partial \bar{Z}_\beta}.$$

A Kähler potential is not unique: it is defined up to the sum with the real part of a holomorphic function. By duplicating the variables $Z$ and $\bar{Z}$ a potential $\Phi$ can be complex analytically continued to a function $\tilde{\Phi}$ defined in a neighborhood $U$ of the diagonal containing $(p, \bar{p}) \in N \times \bar{N}$ (here $\bar{N}$ denotes the manifold conjugated of $N$). The diastasis function is the Kähler potential $D_p$ around $p$ defined by

$$D_p(q) = \tilde{\Phi}(q, \bar{q}) + \tilde{\Phi}(p, \bar{p}) - \tilde{\Phi}(p, \bar{q}) - \tilde{\Phi}(q, \bar{p}).$$

Among all the potentials the diastasis is characterized by the fact that in every coordinates system $(Z)$ centered in $p$

$$D_p(Z, \bar{Z}) = \sum_{|j|, |k| \geq 2} a_{jk} Z^j \bar{Z}^k,$$

with $a_{j0} = a_{0j} = 0$ for all multi-indices $j$. The following proposition shows the importance of the diastasis in the context of holomorphic maps between Kähler manifolds.

**Proposition 2.1. (Calabi)** Let $\varphi : (M, g) \rightarrow (N, G)$ be a holomorphic and isometric embedding between Kähler manifolds and suppose that $G$ is real analytic. Then $g$ is real analytic and for every point $p \in M$

$$\varphi(D_p) = D_{\varphi(p)},$$

where $D_p$ (resp. $D_{\varphi(p)}$) is the diastasis of $g$ relative to $p$ (resp. of $G$ relative to $\varphi(p)$).

In Proposition 2.3 below, we are going to require that $N$ is a compactification of $\mathbb{C}^n$, or more precisely that $N$ contains an analytic subvariety $Y$ such that $X = N \setminus Y$ is biholomorphic to $\mathbb{C}^n$; as far as the Kähler metric $G$ on $N$ is concerned, in addition to the requirement that $G$ is real analytic, we impose two other conditions. The first one is

**Condition (A):** there exists a point $p_0 \in X = N \setminus Y$ such that the diastasis $D_{p_0}$ is globally defined and non-negative on $X$.

In order to describe the second condition we need to introduce the concept of Bochner’s coordinates (cfr. [5], [7], [15], [16]). Given a real analytic Kähler metric $G$ on $N$ and a point $p \in N$, one can always find local (complex) coordinates in a neighborhood of $p$ such that

$$D_p(Z, \bar{Z}) = |Z|^2 + \sum_{|j|, |k| \geq 2} b_{jk} Z^j \bar{Z}^k,$$
where $D_p$ is the diastasis relative to $p$. These coordinates, uniquely defined up to a unitary transformation, are called the Bochner’s coordinates with respect to the point $p$.

One important feature of these coordinates which we are going to use in the proof of our main theorem is the following:

**Theorem 2.2. (Calabi)** Let $\varphi: (M, g) \rightarrow (N, G)$ be a holomorphic and isometric embedding between Kähler manifolds and suppose that $G$ is real analytic. If $(z_1, \ldots, z_m)$ is a system of Bochner’s coordinates in a neighborhood $U$ of $p \in M$ then there exists a system of Bochner’s coordinates $(Z_1, \ldots, Z_n)$ with respect to $\varphi(p)$ such that

$$Z_1|_{\varphi(U)} = z_1, \ldots, Z_m|_{\varphi(U)} = z_m.$$  \hspace{1cm} (1)

We can then state the following condition:

**Condition (B):** the Bochner’s coordinates with respect to the point $p_* \in X$, given by the previous condition (A), are globally defined on $X$.

Our first result is then the following:

**Proposition 2.3.** Let $N$ be a smooth projective compactification of $X$ such that $X$ is algebraically biholomorphic to $\mathbb{C}^n$ and let $G$ be a real analytic Kähler metric on $N$ such that the following two conditions are satisfied:

(A) there exists a point $p_* \in X$ such that the diastasis $D_{p_*}$ is globally defined and non-negative on $X$;

(B) the Bochner’s coordinates with respect to $p_*$ are globally defined on $X$.

Then any K–E submanifold $(M, g) \xrightarrow{\phi} (N, G)$ such that $p_* \in \phi(M)$ has positive scalar curvature.

**Remark 2.4.** The easiest example of compactification of $\mathbb{C}^n$ which satisfies condition (A) is given by $\mathbb{C}P^n = \mathbb{C}^n \cup Y$ endowed with the Fubini–Study metric $g_{FS}$, namely the metric whose associated Kähler form is given by

$$\omega_{FS} = \frac{i}{2} \partial \bar{\partial} \log \sum_{j=0}^n |Z_j|^2,$$  \hspace{1cm} (2)

and $Y = \mathbb{C}P^{n-1}$ is the hyperplane $Z_0 = 0$. Indeed the diastasis with respect to $p_* = [1, 0, \ldots, 0]$ is given by:

$$D_{p_*}(u, \bar{u}) = \log(1 + \sum_{j=1}^n |u_j|^2).$$

where $(u_1, \ldots, u_n)$ are the affine coordinates, namely $u_j = \frac{Z_j}{Z_0}, j = 1, \ldots, n$. Proposition 2.3 can be then considered as an extension of a theorem of Hulin [16] which asserts that a compact Kähler–Einstein submanifold of $\mathbb{C}P^n$ is Fano (see also [18]).
Other examples of compactifications of $\mathbb{C}^n$ satisfying conditions (A) and (B) are given by the compact homogeneous Hodge manifolds. These are not interesting since all compact homogeneous Hodge manifolds can be Kähler embedded into a complex projective space \(^{(19)}\) and so we are reduced to study the Hulin’s problem. We also remark that, by Proposition 2.1, condition (A) is satisfied also by all the Kähler submanifolds of the previous examples.

**Proof of Proposition 2.3.** Let $p$ be a point in $M$ such that $\varphi(p) = p_*$, where $p_*$ is the point in $N$ given by condition (A). Take Bochner’s coordinates $$(z_1, \ldots, z_m)$$ in a neighborhood $U$ of $p$ which we take small enough to be contractible. Since the Kähler metric $g$ is Einstein with (constant) scalar curvature $s$ then: $\rho_\omega = \lambda \omega$ where $\lambda$ is the Einstein constant, i.e. $\lambda = \frac{s}{2m}$, and $\rho_\omega$ is the Ricci form. If $\omega = \frac{1}{4} \sum_{j=1}^m g_{jk} dz_j \wedge d\bar{z}_k$ then $\rho_\omega = -i \partial \bar{\partial} \log \det g_{jk}$ is the local expression of its Ricci form.

Thus the volume form of $(M, g)$ reads on $U$ as:

$$\omega^m \over m! = \frac{i^m}{2m} e^{-\frac{s}{2} D_p + \bar{F}} dz_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m,$$

where $F$ is a holomorphic function on $U$ and $D_p = \varphi^{-1}(D_{p*})$ is the diastasis on $p$ (cfr. Proposition 2.1).

We claim that $F + \bar{F} = 0$. Indeed, observe that

$$\omega^m \over m! = \frac{i^m}{2m} \det \left( \frac{\partial^2 D_p}{\partial z_\alpha \partial \bar{z}_\beta} \right) dz_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m.$$

By the very definition of Bochner’s coordinates it is easy to check that the expansion of $\log \det \left( \frac{\partial^2 D_p}{\partial z_\alpha \partial \bar{z}_\beta} \right)$ in the $(z, \bar{z})$-coordinates contains only mixed terms (i.e. of the form $z^j \bar{z}^k$, $j \neq 0, k \neq 0$). On the other hand by formula (4)

$$-\frac{\lambda}{2} D_p + F + \bar{F} = \log \det \left( \frac{\partial^2 D_p}{\partial z_\alpha \partial \bar{z}_\beta} \right).$$

Again by the definition of the Bochner’s coordinates this forces $F + \bar{F}$ to be zero, proving our claim. By Theorem 2.2 there exist Bochner’s coordinates $(Z_1, \ldots, Z_n)$ in a neighborhood of $p_*$ satisfying (11). Moreover, by condition (B) this coordinates are globally defined on $X$. Hence, by formula (3) (with $F + \bar{F} = 0$), the $m$-forms $\Omega^m \over m!$ and $e^{-\frac{s}{2} D_p} dZ_1 \wedge \cdots \wedge dZ_m$ globally defined on $X$ agree on the open set $\varphi(U)$. Since they are real analytic they must agree on the connected open set $\tilde{M} = \varphi(M) \cap X$, i.e.

$$\Omega^m \over m! = \frac{i^m}{2m} e^{-\frac{s}{2} D_p} dZ_1 \wedge \cdots \wedge dZ_m \wedge d\bar{Z}_m.$$

Since $\Omega^m \over m!$ is a volume form on $\tilde{M}$ we deduce that the restriction of the projection map

$$\pi : X \cong \mathbb{C}^n \to \mathbb{C}^m : (Z_1, \ldots, Z_n) \mapsto (Z_1, \ldots, Z_m)$$

to $\tilde{M}$ is open. Since it is also algebraic (because the biholomorphism between $X$ and $\mathbb{C}^n$ is algebraic), its image contains a Zariski open subset of $\mathbb{C}^m$ (see Theorem 13.2 in [6]), hence its euclidean volume,
vol_{euc}(\pi(\hat{M})), has to be infinite. Suppose now that the scalar curvature of \( g \) is non-positive. By formula (4) and by the fact that \( D_p \) is non-negative, we get
\[
\text{vol}(\hat{M}, g) \geq \text{vol}_{euc}(\pi(\hat{M}))
\]
which is the desired contradiction, being the volume of \( M \) (and hence that of \( \hat{M} \)) finite. \( \square \)

Now, we are going to apply Proposition 2.3 to toric manifolds endowed with toric Kähler metrics.

Recall that a toric manifold \( M \) is a complex manifold which contains an open dense subset biholomorphic to \((\mathbb{C}^*)^n\) and such that the canonical action of \((\mathbb{C}^*)^n\) on itself by \((\alpha_1, \ldots, \alpha_n)(\beta_1, \ldots, \beta_n) = (\alpha_1\beta_1, \ldots, \alpha_n\beta_n)\) extends to a holomorphic action on the whole \( M \) (see the Appendix for more details). A toric Kähler metric \( \omega \) on \( M \) is a Kähler metric which is invariant for the action of the real torus \( T^n = \{ (e^{i\theta_1}, \ldots, e^{i\theta_n}) | \theta_i \in \mathbb{R} \} \) contained in the dense, complex torus \((\mathbb{C}^*)^n\), that is for every fixed \( \theta \in T^n \) the diffeomorphism \( f_\theta : M \to M \) given by the action of \((e^{i\theta_1}, \ldots, e^{i\theta_n})\) is an isometry.

We have the following, well-known fact (compare, for example, with Section 2.2.1 in [10] or Proposition 2.18 in [3]).

**Proposition 2.5.** If \( M \) is a projective, compact toric manifold then there exists an open dense subset \( X \subseteq M \) which is algebraically biholomorphic to \( \mathbb{C}^n \). More precisely, for every point \( p \in M \) fixed by the torus action there are an open dense neighborhood \( X_p \) of \( p \) and a biholomorphism \( \phi_p : X_p \to \mathbb{C}^n \) such that \( p \) is sent to the origin and the restriction of the torus action to \( X_p \) corresponds via \( \phi \) to the canonical action of \((\mathbb{C}^*)^n\) on \( \mathbb{C}^n \).

A self-contained proof of this proposition is given in Section A.1 of the Appendix. Now we are ready to prove the main result of this section.

**Theorem 2.6.** Let \( N \) be a projective toric manifold equipped with a toric Kähler metric \( G \). Then any K-E submanifold \((M, g) \overset{\phi}{\to} (N, G)\) such that \( \phi(M) \) contains a point of \( N \) fixed by the torus action has positive scalar curvature.

**Proof.** As we have just recalled, \( N \) is a smooth projective compactification of an open subset algebraically biholomorphic to \( \mathbb{C}^n \). So, the Theorem will follow from Proposition 2.3 once we have shown that, for \( p_\ast \) equal to a point \( N \) fixed by the torus action, then the conditions (A) and (B) of the statement of Proposition 2.3 are satisfied.

Let then \( p_\ast \in N \) be such a point, and let \( \xi = (\xi_1, \ldots, \xi_n) \) be the system of coordinates given by the biholomorphism \( \phi_{p_\ast} : X_{p_\ast} \to \mathbb{C}^n \) given in Proposition 2.5 above.

Let \( \Omega \) be the Kähler form associated to \( G \) and let \( \Phi \) be a local potential for \( \Omega \) around the origin in the coordinates \( \xi = (\xi_1, \ldots, \xi_n) \). Since \( X = X_{p_\ast} \) is contractible, \( \Phi \) can be extended to all \( X \) (see, for example, Remark 2.6.2 in [12]) and
\[
D(\xi, \bar{\xi}) = \Phi(\xi, \bar{\xi}) + \Phi(0, 0) - \Phi(0, \bar{\xi}) - \Phi(\xi, 0)
\]
is a diastasis function on all \( X \) in the coordinates \( \xi_1, \ldots, \xi_n \).
For every $\theta \in T^n$ and $\xi \in \mathbb{C}^n$, let us denote

$$e^{i\theta} \xi := (e^{i\theta_1}, \ldots, e^{i\theta_n})(\xi_1, \ldots, \xi_n) = (e^{i\theta_1} \xi_1, \ldots, e^{i\theta_n} \xi_n)$$

and $D_\theta(\xi, \bar{\xi}) := D(e^{i\theta} \xi, e^{-i\theta} \bar{\xi})$. Then

$$i\partial \bar{\partial} D_\theta = i \frac{\partial^2 D_\theta}{\partial \xi_k \partial \bar{\xi}_l}(\xi, \bar{\xi}) d\xi_k \wedge d\bar{\xi}_l = ie^{i(\theta_k - \theta_l)} \frac{\partial^2 D}{\partial \xi_k \partial \bar{\xi}_l}(e^{i\theta} \xi, e^{-i\theta} \bar{\xi}) d\xi_k \wedge d\bar{\xi}_l =$$

where the last equality follows by the invariance of $\Omega$ for the action of $T^n$. Then, for every $\theta \in T^n$, the function $D_\theta$ is a potential for $\Omega$ on $X$; moreover, it clearly satisfies the characterization for the diastasis. By the uniqueness of the diastasis around the origin, we then have $D = D_\theta$, that is

$$D(\xi, \bar{\xi}) = D(e^{i\theta_1} \xi_1, \ldots, e^{i\theta_n} \xi_n, e^{-i\theta_1} \bar{\xi}_1, \ldots, e^{-i\theta_n} \bar{\xi}_n).$$

This last equality means that $D$ depends on the norms $|\xi_1|^2, \ldots, |\xi_n|^2$ (i.e. $D$ is a rotation invariant function), and in particular it is immediately seen to satisfy the condition for $\xi_1, \ldots, \xi_n$ to be Bochner coordinates.

In order to show that $D$ is non-negative, recall that, since $i\partial \bar{\partial} D$ is a Kähler form, $D$ must be a plurisubharmonic function, which means that, for any $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{C}^n$, the function of one complex variable $f(\xi) = D(a\xi + b) = D(a_1 \xi_1 + b_1, \ldots, a_n \xi_n + b_n)$ is a subharmonic function, i.e. $\frac{\partial^2 f}{\partial \xi_k \partial \bar{\xi}_l} \geq 0$. To prove the claim it will be enough to show that, for any $a \in \mathbb{C}^n$, the rotation invariant subharmonic function $f_a(\xi) = D(a\xi)$ is non-negative. Now, we have

$$0 \leq \frac{\partial^2 f_a}{\partial \xi_k \partial \bar{\xi}_l} = t \cdot \frac{d^2 f_a}{dt^2} + \frac{df_a}{dt} = \frac{d}{dt}(tf_a(t))$$

where we are denoting $t = |\xi|^2$.

It follows that $g(t) = tf_a(t)$ is a non-decreasing function, and since $g(0) = 0$ we have $g(t) = tf_a(t) \geq 0$, that is $f_a(t) \geq 0$, as required.

\[\square\]

**Remark 2.7.** If $\phi(M)$ does not contain any point of $N$ fixed by the torus action, then for any $f \in Aut(N) \cap Isom(N, G)$ one could be tempted to replace $\phi$ by $f \circ \phi$ (which is clearly again a Kähler embedding) so to have that $f(\phi(M))$ contains a fixed point.

Anyway, while the automorphisms group of a toric manifold can be explicitly described, in general we do not have control on $Isom(N, G)$, and in general this group can be too small. For example, for the Kähler-Einstein metric $G$ on $N = \mathbb{CP}^2 \times \overline{\mathbb{CP}^2}$ (24, 25), one has that $Isom(N, G)$ is the real part of $Aut(N)$, whose component of the identity $Aut^0(N)$ contains only the automorphisms given
by the action of the complex torus $(\mathbb{C}^*)^n$ (indeed, one easily sees that the set of the Demazure roots is empty in this case, see for example Section 3.4 in [22]), so $\text{Isom}(N,G) \simeq T^n$ and the isometries do not move the fixed points. By contrast, if $N$ is the complex projective space endowed with the Fubini-Study metric $G$, then $\text{Isom}(N,G)$ acts transitively and we can always guarantee the validity of the assumption of Theorem 2.6, so that we recover Hulin’s theorem (Remark 2.4).

Notice that if $f \in \text{Aut}(N) \setminus \text{Isom}(N,G)$ then, in order to guarantee that $f \circ \phi$ is a Kähler embedding one has to replace $G$ by $(f^{-1})^*(G)$, and consequently the torus action, say $\rho = f \circ \rho \circ f^{-1}$. Then any new fixed point is of the form $f(p)$, where $p$ is a point fixed by the action $\rho$. This implies that if $\phi(M)$ does not contain any point fixed by $\rho$, then $f(\phi(M))$ does not contain any point fixed by $\tilde{\rho}$.

3. Gromov width of toric varieties

Let us recall that the Gromov width (introduced in [13]) of a $2n$-dimensional symplectic manifold $(M,\omega)$ is defined as

$$c_G(M,\omega) = \sup \{ \pi r^2 \mid (B^{2n}(r),\omega_{can}) \text{ symplectically embeds into } (M,\omega) \}$$

where $\omega_{can} = \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$ is the canonical symplectic form in $\mathbb{C}^n$.

By Darboux’s theorem $c_G(M,\omega)$ is a positive number. Computations and estimates of the Gromov width for various examples can be found in [4], [8], [17] and in particular for toric manifolds in [21].

In what follows, we are going to make some remarks about the Gromov width of toric manifolds. More precisely, let $(M,\omega)$ be a toric manifold endowed with an integral toric Kähler form $\omega$. As it is known ([6], [14]), the image of the moment map $\mu : M \rightarrow \mathbb{R}^n$ for the isometric action of the real torus $T^n$ on $M$ is a convex Delzant polytope $\Delta = \{ x \in \mathbb{R}^n \mid \langle x, u_i \rangle \geq \lambda_i, \ i = 1, \ldots, d \} \subseteq \mathbb{R}^n$, i.e. such that the normal vectors $u_i$ to the faces meeting in a given vertex form a $\mathbb{Z}$-basis of $\mathbb{Z}^n$. The vertices of $\Delta$ (which, by the integrality of $\omega$, belong to $\mathbb{Z}^n$) are the images by $\mu$ of the fixed points for the action of $T^n$ on $M$.

As recalled in Section A.2 of the Appendix, such a polytope $\Delta$ represents a very ample line bundle on the toric manifold $X_\Sigma$ associated to the fan $\Sigma$ which has the $u_i$’s as generators. Then, by the Kodaira embedding $i_\Delta$ we can embed $X_\Sigma$ into a complex projective space $\mathbb{CP}^{N-1}$ and endow $X_\Sigma$ with the pull-back $i_\Delta^*(\omega_{FS})$ of the Fubini-Study form $\omega_{FS} = i \log(\sum_{j=1}^{N} |z_j|^2)$ of $\mathbb{CP}^{N-1}$.

We have the following, important result.

**Theorem 3.1.** (see, for example, [1], page 3 or [14], Section A2.1) The manifolds $(X_\Sigma, i_\Delta^*(\omega_{FS}))$ and $(M,\omega)$ are equivariantly symplectomorphic.

Now, by the following well-known result we can write the Kodaira embedding explicitly.
Proposition 3.2. Let $p \in M$ be a fixed point for the torus action and $X_p, \phi_p : X_p \to \mathbb{C}^n$ be as in Proposition 2.5. The restriction to $X_p$ of the Kodaira embedding $i_\Delta : M \to \mathbb{CP}^{N-1}$ writes, in the coordinates given by $\phi_p$, as

$$i_g|_{X_p} \circ \phi_p^{-1} : \mathbb{C}^n \to \mathbb{CP}^{N-1}, \ \xi \mapsto [\ldots, \xi_1^{x_1}, \ldots, \xi_n^{x_n}, \ldots]$$

where $(x_1, \ldots, x_n)$ runs over all the points with integral coordinates in the polytope $\Delta'$ of $\mathbb{R}^n$ obtained by $\Delta$ via the transformation in $\text{SL}_n(\mathbb{Z})$ and the translation which send the vertex of $\Delta$ corresponding to $p$ to the origin and the corresponding edge to the edge generated by the vectors $e_1, \ldots, e_n$ of the canonical basis of $\mathbb{R}^n$.

Notice that the existence of the transformation in $\text{SL}_n(\mathbb{Z})$ invoked in the statement follows from the fact that the normal vectors to the faces meeting in any vertex of the polytope form a $\mathbb{Z}$-basis of $\mathbb{R}^n$.

We will give a detailed proof of Proposition 3.2 in the Appendix (Proposition A.6).

It follows by Proposition 3.2 that the restriction $\omega_\Delta$ of the pull-back metric $i_\Delta^*(\omega_{FS})$ to the open subset $X_p$ is given in the coordinates $\xi_1, \ldots, \xi_n$ by $i \log(\sum_{j=1}^N |\xi_j^{2j}|)$, where $\{J_k\}_{k=1, \ldots, N} = \Delta' \cap \mathbb{Z}^n$ and for any $J = (J_1, \ldots, J_n) \in \mathbb{Z}^n$ we are denoting $|\xi|^{2J} := |\xi_1|^{2J_1} \cdots |\xi_n|^{2J_n}$.

Then, by combining Theorem 3.4 and Proposition 3.2 we conclude that the manifold $(M, \omega)$ has an open dense subset, say $A$, symplectomorphic to $(\mathbb{C}^n, \omega_\Delta := i \log(\sum_{j=1}^N |\xi_j^{2j}|))$. We now estimate from above the Gromov width of $(\mathbb{C}^n, \omega_\Delta)$.

Lemma 3.3. Let $A$ be an open, connected subset of $\mathbb{C}^n$ such that $A \cap \{z_j = 0\} \neq \emptyset$, $j = 1, \ldots, n$, endowed with a Kähler form $\omega = \frac{i}{2} \partial \bar{\partial} \Phi$, where $\Phi(\xi_1, \ldots, \xi_n) = \tilde{\Phi}(|\xi_1|^2, \ldots, |\xi_n|^2)$ for some smooth function $\tilde{\Phi} : \tilde{A} \to \mathbb{R}$, $\tilde{A} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = |\xi_i|^2, (\xi_1, \ldots, \xi_n) \in A\}$ (we say that $\omega$ is a rotation invariant form). Assume $\frac{\partial \tilde{\Phi}}{\partial x_k} > 0$ for every $k = 1, \ldots, n$. Then the map

$$\Psi : (A, \omega) \to (\mathbb{C}^n, \omega_0), \quad (\xi_1, \ldots, \xi_n) \mapsto \left(\sqrt{\frac{\partial \tilde{\Phi}}{\partial x_1}} \xi_1, \ldots, \sqrt{\frac{\partial \tilde{\Phi}}{\partial x_n}} \xi_n\right) \quad (5)$$

is a symplectic embedding (where $\omega_0 = \frac{i}{2} \partial \bar{\partial} \sum_{k=1}^n |z_k|^2$).

For a proof of this lemma, see Theorem 1.1 in [20]. Our result is

Theorem 3.4. Let $\omega_\Delta = i \partial \bar{\partial} \log(\sum_{j=1}^N |\xi_j^{2j}|)$. Then

$$c_G(\mathbb{C}^n, \omega_\Delta) \leq 2\pi \min_{j=1, \ldots, n} \left(\max_k \{J_k\}_{j=1, \ldots, n}\right) \quad (6)$$

Proof. Let us apply Lemma 3.3 to $A = \mathbb{C}^n$ endowed with the rotation invariant Kähler form $i \partial \bar{\partial} \log(\sum_{j=1}^N |\xi_j^{2j}|)$. In the notation of the statement of the lemma, we have then $\tilde{\Phi} = 2 \log \sum_{k=0}^N x_{J_k}$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and we are denoting $x^J = x_1^{J_1} \cdots x_n^{J_n}$, for $J = (j_1, \ldots, j_n)$. Since for
every \( k = 0, \ldots, N \) we have \( J_k = ((J_k)_1, \ldots, (J_k)_n) \in (\mathbb{Z}_{\geq 0})^n \), we have
\[
\frac{\partial \tilde{\Phi}}{\partial x_j} = \frac{2}{\pi} \frac{\sum_{k=0}^{N} (J_k)_j x^j_k}{\sum_{k=0}^{N} x^j_k} > 0
\]
and then we can embed symplectically \( \mathbb{C}^n \) (endowed with the toric form) into \( \mathbb{C}^n \) (endowed with the standard symplectic form) by \( (\xi_1, \ldots, \xi_n) \mapsto \left( \sqrt{\frac{\partial \Phi}{\partial x_1}} \xi_1, \ldots, \sqrt{\frac{\partial \Phi}{\partial x_n}} \xi_n \right) \) so that \( (\mathbb{C}^n, i\partial \bar{\partial} \log(\sum_{j=1}^{N} \xi_j^2)) \) is symplectomorphic to the domain \( D = \Psi(\mathbb{C}^n) \subseteq \mathbb{C}^n \) endowed with the canonical symplectic form \( \omega_{can} \). Now, let \( \pi_k : \mathbb{C}^n \to \mathbb{C}, \pi_k(z_1, \ldots, z_n) = z_k \) denote the projection onto the \( k \)-th coordinate. Then \( D \) is clearly contained in the cylinder \( \pi_k(D) \times \mathbb{C}^{n-1} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid \text{there exists } p \in D \text{ with } p_k = z_k \} \) over \( \pi_k(D) \), and then in the cylinder
\[
C_R = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_k|^2 < R^2 \},
\]
where \( R \) is the radius of any ball in the \( z_k \)-plane containing \( \pi_k(D) \). By the celebrated Gromov’s non-squeezing theorem, which states that the Gromov width of \( C_R \) endowed with the canonical symplectic form \( \omega_{can} \) is \( \pi R^2 \), we conclude that the Gromov width of \( D \) is less or equal to \( \pi R^2 \), where \( R \) is the radius of any euclidean ball of the \( z_k \)-plane containing \( \pi_k(D) \).

In order to calculate the best value of \( R \), notice that
\[
\pi_k(D) = \left\{ \sqrt{\frac{\partial \Phi}{\partial x_j}} \xi_j \mid (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n \right\} = \\
= \left\{ \sqrt{\frac{\partial \Phi}{\partial x_j}} x_j e^{i\theta_j} \mid (x_1, \ldots, x_n) \in (\mathbb{R}_{\geq 0})^n, \theta_j \in [0, 2\pi] \right\}
\]
(since \( x_j = |\xi_j|^2 \) and \( \xi_j = \sqrt{x_j e^{i\theta_j}} \)) that is the circle in \( \mathbb{R}^2 \) of radius
\[
\sup \left\{ \sqrt{\frac{\partial \Phi}{\partial x_j}} x_j \mid (x_1, \ldots, x_n) \in (\mathbb{R}_{\geq 0})^n \right\}.
\]
Now,
\[
\sqrt{\frac{\partial \Phi}{\partial x_j}} x_j = \sqrt{\frac{2 \sum_{k=0}^{N} (J_k)_j x^j_k}{\sum_{k=0}^{N} x^j_k}} \tag{7}
\]
where we are denoting \( J_k = ((J_k)_1, \ldots, (J_k)_n) \). Now, fix \( j = 1, \ldots, n \). On the one hand, we clearly have
\[
\sum_{k=0}^{N} (J_k)_j x^j_k \leq \sum_{k=0}^{N} \max_{k} ((J_k)_j) x^j_k = \max_{k} ((J_k)_j) \sum_{k=0}^{N} x^j_k
\]
so that
\[
\sup \sqrt{\frac{2 \sum_{k=0}^{N} (J_k)_j x^j_k}{\sum_{k=0}^{N} x^j_k}} \leq \sqrt{2 \max_{k} ((J_k)_j)}.
\]
On the other hand, we can show that the sup is actually equal to \( \sqrt{2 \max_k \{(J_k)_j\}} \) by setting \( x_i = t \) for \( i \neq j \) and \( x_j = t^s \), for an integer \( s \) large enough, and letting \( t \to +\infty \). Indeed, after substituting \( x_i = t \) for \( i \neq j \) and \( x_j = t^s \) we get the one variable function

\[
\sqrt{2 \sum_{k=0}^N (J_k)_j t^{(J_k)_j s + \sum_{i \neq j} (J_k)_i}},
\]

and, if we set \( f_k(s) = (J_k)_j s + \sum_{i \neq j} (J_k)_i \), it is clear that there is a value of \( s \) for which the largest \( f_k(s) \) is obtained for the value of \( k \) for which \((J_k)_j\) (i.e. the slope of the affine function \( f_k(s) \)) is maximum. This concludes the proof. \( \square \)

As an immediate corollary of Theorem 3.4, we get the following

**Corollary 3.5.** Let \((M,\omega)\) be a toric manifold endowed with an integral toric form, let \( \Delta \subseteq \mathbb{R}^n \) be the image of the moment map for the torus action (which, up to a translation and a transformation in \( SL_n(\mathbb{Z}) \), can be assumed to have the origin as vertex and the edge at the origin generated by the canonical basis of \( \mathbb{R}^n \)) and let \( \{J_k\}_{k=0,\ldots,N} = \Delta \cap \mathbb{Z}^n \). Let \( p \) be the point fixed by the torus action corresponding to the origin of \( \Delta \) and \( X_p \simeq \mathbb{C}^n \) be the open subset given by Proposition 2.5.

Then, any ball of radius \( r > \sqrt{2 \min_{j=1,\ldots,n} (\max_k \{(J_k)_j\})} \) symplectically embedded into \((M,\omega)\), must intersect the divisor \( M \setminus X_p \).

Let

\[
\Delta = \{x \in \mathbb{R}^n \mid \langle x, u_k \rangle \geq \lambda_k, \ k = 1, \ldots, d\}.
\]

Then Lu proves in Corollary 1.4 of [21] that the Gromov width of the corresponding toric manifold is bounded from above by

\[
\Lambda(\Delta) := 2\pi \max\{-\sum_{i=1}^d \lambda_i a_i \mid a_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^d a_i u_i = 0, \ 1 \leq \sum_{i=1}^d a_i \leq n + 1\}
\]

in general, and by

\[
\gamma(\Delta) := 2\pi \inf\{-\sum_{i=1}^d \lambda_i a_i \mid a_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^d a_i u_i = 0, \ \gamma \text{ if the polytope } \Delta \text{ is Fano}\}
\]

that is if there exist \( m \in \mathbb{R}^n \) and \( r > 0 \) such that

\[
r(\lambda_i + \langle m, u_i \rangle) = \pm 1, \ i = 1, \ldots, d, \quad \text{Int}(r \cdot (m + \Delta)) \cap \mathbb{Z}^n = \{0\}, \tag{8}
\]

**Example 3.6.** Take the polytope

\[\text{It is easy to see that this condition is equivalent to the fact that the Kähler form on the manifold represents the first Chern class of a multiple of the anticanonical bundle.}\]
\[ \Delta = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0, x_1 - x_2 \geq -1, x_2 - x_1 \geq -1, \\
\quad x_1 - 2x_2 \geq -3, x_2 \leq 3 \} \]

which represents a Kähler class \( \omega_\Delta \) on the Hirzebruch surface \( S_2 \) blown up at two points, denoted in the following by \( \tilde{S}_2 \).

Notice that \( \Delta \) is of the kind \( \{ x \in \mathbb{R}^n \mid (x, u_k) \geq \lambda_k, \ k = 1, \ldots, d \} \), where \( u_1 = (1, 0), u_2 = (0, 1), u_3 = (1, -1), u_4 = (-1, 1), u_5 = (1, -2), u_6 = (0, -1) \) and \( \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -1, \lambda_4 = -1, \lambda_5 = -3, \lambda_6 = -3 \). We first show that \( \Delta \) does not satisfy the above Fano condition (8). Indeed, these conditions read, for \( m = (m_1, m_2) \) and \( i = 1, 2, 3, 6, \)

\[
rm_1 = \pm 1, \quad rm_2 = \pm 1, \quad r(-1 + m_1 - m_2) = \pm 1, \quad r(-3 - m_2) = \pm 1.
\]

Combining the second and the last condition we get the four possibilities (the signs have to be taken independently) \(-3r - 1 = +1, -3r - 1 = -1, -3r + 1 = +1, -3r + 1 = -1\), that is \( r = -\frac{2}{3}, r = 0, r = \frac{2}{3} \). Since \( r > 0 \) the only possibility is \( r = \frac{2}{3} \), and \( m_2 = -\frac{2}{3} \). Replacing this in the third condition, and taking into account the first one, we have

\[
r(-1 + m_1 - m_2) = \frac{2}{3}(-1 + \frac{3}{2} \pm \frac{3}{2})
\]

which is either \( \frac{4}{3} \) or \( -\frac{4}{3} \), so different from \( \pm 1 \) for any choice of the signs. This proves the claim.

Then Lu’s estimate by \( \gamma(\Delta) \) does not apply \(^2\). Since \( \sum_i a_iu_i = 0 \) reads \( a_1 = a_4 = a_3 - a_5, a_2 = a_3 - a_4 + 2a_5 + a_6 \) we have

\[
\Lambda(\Delta) = \max\{2\pi(a_3 + a_4 + 3a_5 + 3a_6) \mid a_i \in \mathbb{Z}_{\geq 0}, 1 \leq 2a_5 + 2a_6 + a_3 + a_4 \leq 3\}.
\]

It is easy to see that \( \Lambda(\Delta) = 8\pi \) (attained for \( a_2 = a_4 = a_5 = 1 \) and \( a_1 = a_3 = a_6 = 0 \)). We then get \( c_G(\tilde{S}_2, \omega_\Delta) \leq 8\pi \), while it is easy to see that our estimate \(^4\) yields \( c_G(\mathbb{C}^n, \omega_\Delta) \leq 6\pi \).

Then, Corollary \(^3\) in this case states that any ball of radius strictly larger than \( \sqrt{6} \), symplectically embedded into \( (\tilde{S}_2, \omega_\Delta) \), must intersect the divisor.

**Example 3.7.** Consider the family of polytopes

\[
\Delta(m) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq x_1 \leq 4, 0 \leq x_2 \leq 4, -2 \leq x_1 - x_2 \leq 2, \\
\quad 2x_1 - x_2 \geq -\frac{2m}{m+1} \}
\]

\(^2\)At page 169 of \([21]\), studying the case of the projective space blown up at one point, the author applies the same estimate valid for Fano polytopes also in the case when the polytope is not Fano: by looking at the proof, it turns out that this is possible because the projective space blown up at one point is Fano and any two Kähler forms on it are deformedly equivalent (see \([23]\), Example 3.7). This argument cannot be used here because \( \tilde{S}_2 \) is not Fano. As told to the second author by D. Salamon in a private communication, it is not known if the same result on the equivalence by deformation holds on the higher blowups of the projective spaces \( \mathbb{C}P^n \), with \( n > 2 \).
which, for every natural number \( m \geq 1 \), represents a Kähler class \( \omega_{A(m)} \) on the projective plane blown up at three points and blown up again (at one of the new fixed points by the toric action), which we denote from now on by \( M \).

Notice that \( A(m) \) is of the kind \( \{ x \in \mathbb{R}^n \mid \langle x, u_k \rangle \geq \lambda_k, \ k = 1, \ldots, d \} \), where \( u_1 = (1, 0), u_2 = (0, 1), u_3 = (-1, 1), u_4 = (-1, 0), u_5 = (0, -1), u_6 = (1, -1), u_7 = (2, -1) \) and \( \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = -2, \lambda_4 = -4, \lambda_5 = -4, \lambda_6 = -2, \lambda_7 = -\frac{2m}{m+1} \).

One easily sees by a straightforward calculation as in the previous example that \( A(m) \) does not satisfy the above Fano condition \( \mathbf{[5]} \). In fact, it is known (see for example Proposition 2.21 in \( \mathbf{[22]} \)) that \( M \) is not Fano, so Lu’s estimate by \( \gamma(A) \) does not apply (see also the footnote at page \( \mathbf{[11]} \)). Since \( \sum_i a_i u_i = 0 \) reads \( a_1 = a_3 + a_4 - a_6 - 2a_7, a_2 = a_5 + a_6 + a_7 - a_3 \) we have

\[
\Lambda(A) = \max\{2\pi(2a_3 + 4a_4 + 4a_5 + 2a_6 + \frac{2m}{m+1}a_7)\}
\]

over all the \( a_i \)'s in \( \mathbb{Z}_{\geq 0} \) such that \( 1 \leq a_3 + 2a_4 + 2a_5 + a_6 \leq 3 \).

It is easy to see that \( \Lambda(A) = 2\pi(6 + \frac{2m}{m+1}) \) (attained for \( a_3 = a_4 = a_7 = 1 \) and \( a_1 = a_2 = a_5 = a_6 = 0 \)). We have then \( c_G(M, \omega_{A(m)}) \leq 2\pi(6 + \frac{2m}{m+1}) \), while it is easy to see that our estimate \( \mathbf{[6]} \) yields \( c_G(C^n, \omega_{A(m)}) \leq 8\pi \) for every \( m \geq 1 \) (in fact, we need first to multiply \( A(m) \) by \( m+1 \) in order to get an integral polytope for which \( \min_j \{ (\langle J_k \rangle_j) \} = 4(m+1) \), and then we rescale by \( \frac{1}{m+1} \), and use the fact that \( c_G(M, \lambda \omega) = \lambda c_G(M, \omega) \)).

Then, Corollary \( \mathbf{[3.5]} \) in this case states that any ball of radius strictly larger than \( 2\sqrt{2} \), symplectically embedded into \((M, \omega_{A(m)})\), must intersect the divisor.

**Remark 3.8.** It is worth to notice that, for the complex projective space \( \mathbb{C}P^n \) endowed with the Fubini-Study form \( \omega_{FS} = i \log(\sum_i |Z_i|^2) \), the Gromov width is known to be equal to \( 2\pi \) and in fact it is attained by embedding symplectically an open ball of radius \( \sqrt{2} \) without intersecting the divisor (more precisely, one can see that the image of the symplectic embedding \((C^n, \omega_{FS}) \to (\mathbb{C}^n, \omega_0) \) given by \( \mathbf{[3]} \) is exactly a ball of radius \( \sqrt{2} \)).

### Appendix A. Toric Manifolds

#### A.1. Toric manifolds as compactifications of \( \mathbb{C}^n \)

Let us recall the following

**Definition A.1.** A toric variety is a complex variety \( M \) containing an open dense subset biholomorphic to \((\mathbb{C}^*)^n \) and such that the canonical action of \((\mathbb{C}^*)^n \) on itself by \((\alpha_1, \ldots, \alpha_n)\langle \beta_1, \ldots, \beta_n \rangle = (\alpha_1\beta_1, \ldots, \alpha_n\beta_n) \) extends to a holomorphic action on the whole \( M \).

A toric variety can be described combinatorially by means of fans of cones. In detail, by the cone \( \sigma = \sigma(u_1, \ldots, u_m) \) in \( \mathbb{R}^n \) generated by the vectors \( u_1, \ldots, u_m \in \mathbb{Z}^n \) we mean the set

\[
\{ x \in \mathbb{R}^n \mid x = \sum_{i=1}^m c_i u_i, \ c_i \geq 0 \}
\]
of linear combinations of \( u_1, \ldots, u_m \) with non-negative coefficients. The \( u_i \)'s are called the generators of the cone. The dimension of a cone \( \sigma = \sigma(u_1, \ldots, u_m) \) is the dimension of the linear subspace of \( \mathbb{R}^n \) spanned by \( \{u_1, \ldots, u_m\} \).

We will always assume that our cones are convex, i.e. that they do not contain any straight line passing through the origin, and that the generators of a cone are linearly independent.

The faces of a cone \( \sigma = \sigma(u_1, \ldots, u_m) \) are defined as the cones generated by the subsets of \( \{u_1, \ldots, u_m\} \). By definition, the cone generated by the empty set is the origin \( \{0\} \).

**Definition A.2.** A fan \( \Sigma \) of cones in \( \mathbb{R}^n \) is a set of cones such that

(i) for any \( \sigma \in \Sigma \) and any face \( \tau \) of \( \sigma \), we have \( \tau \in \Sigma \);

(ii) any two cones in \( \Sigma \) intersect along a common face.

Let us now recall how one can construct from a fan \( \Sigma \) a toric variety.

Let \( \{u_1, \ldots, u_d\} \), \( u_k = (u_{k1}, \ldots, u_{kn}) \in \mathbb{Z}^n \), be the union of all the generators of the cones in \( \Sigma \).

For any cone \( \sigma = \sigma(I) \in \Sigma \), \( I \subseteq \{1, \ldots, d\} \), let us denote

\[
C^d_\sigma = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_i = 0 \leftrightarrow i \in I\}.
\]

Notice that if \( \sigma = \sigma(\emptyset) \) is the cone consisting of the origin alone, then \( C^d_\emptyset = (\mathbb{C}^*)^d \). Now, let \( C^d_\Sigma = \bigcup_{\sigma \in \Sigma} C^d_\sigma \) and \( K_\Sigma \) be the kernel of the surjective homomorphism

\[
\pi : (\mathbb{C}^*)^d \to (\mathbb{C}^*)^n, \quad \pi(\alpha_1, \ldots, \alpha_d) = (\alpha_1^{u_{11}} \cdots \alpha_d^{u_{1n}}, \ldots, \alpha_1^{u_{dn}} \cdots \alpha_d^{u_{dn}}).
\]

**Definition A.3.** The toric variety \( X_\Sigma \) associated to \( \Sigma \) is defined to be the quotient \( X_\Sigma = \mathbb{C}^d_\Sigma / K_\Sigma \) of \( \mathbb{C}^d_\Sigma \) for the action of \( K_\Sigma \) given by the restriction of the canonical action \( (\alpha_1, \ldots, \alpha_d)(z_1, \ldots, z_d) = (\alpha_1z_1, \ldots, \alpha_dz_d) \) of \( (\mathbb{C}^*)^d \) on \( \mathbb{C}^d \).

The importance of this construction consists in the fact that any toric variety \( M \) of complex dimension \( n \) can be realized as \( M = X_\Sigma \) for some fan \( \Sigma \) in \( \mathbb{R}^n \) (see Section 1.4 in [11]).

Notice that, by definition of \( K_\Sigma \), we have \( (\mathbb{C}^*)^d / K_\Sigma \simeq (\mathbb{C}^*)^n \). So we have a natural action of this complex torus on \( X_\Sigma \) given by

\[
[(\alpha_1, \ldots, \alpha_d)](z_1, \ldots, z_d) = [\alpha_1z_1, \ldots, \alpha_dz_d].
\]

From now on, and throughout this section, we will assume that \( X_\Sigma \) is a compact, smooth manifold. From a combinatorial point of view, it is known ([11], Chapter 2) that:

(i) \( X_\Sigma \) is compact if and only if the support \( |\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma \) of \( \Sigma \) equals \( \mathbb{R}^n \).

(ii) \( X_\Sigma \) is a smooth complex manifold if and only if for each \( n \)-dimensional cone \( \sigma \) in \( \Sigma \) its generators form a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^n \).

Under these assumptions, we have the following well-known result.

**Proposition A.4.** Let \( X_\Sigma \) be a compact, smooth toric manifold of complex dimension \( n \). Then, for each \( p \in X_\Sigma \) fixed by the torus action there exists an open neighbourhood \( X_p \) of \( p \), dense in
\(X_\Sigma\), containing the complex torus \(\mathbb{C}^d/\mathbb{K}_\Sigma \simeq (\mathbb{C}^*)^n\) and a biholomorphism \(\phi_p : X_p \to \mathbb{C}^n\) such that \(\phi_p(p) = 0\) and that the restriction of the torus action \(\mathbb{C}^*\) to \(X_p\) coincides, via \(\phi_p\), with the canonical action of \((\mathbb{C}^*)^n\) on \(\mathbb{C}^n\) by componentwise multiplication. In particular, any compact, smooth toric manifold of complex dimension \(n\) is a compactification of \(\mathbb{C}^n\).

**Proof.** Let \(\sigma = \sigma(u_{j_1}, \ldots, u_{j_n})\) be an \(n\)-dimensional cone in \(\Sigma\), and let \(\{j_{n+1}, \ldots, j_d\} = \{1, \ldots, d\} \setminus \{j_1, \ldots, j_n\}\). Let us consider the open dense subset

\[
X_\sigma = \frac{\bigcup_{\tau \subseteq \sigma} \mathbb{C}_d^\tau}{\mathbb{K}_\Sigma} = \{([z_1, \ldots, z_d] \in X_\Sigma \mid z_{j_{n+1}}, \ldots, z_{j_d} \neq 0)\}.
\]

We are going to define a biholomorphism \(\phi_\sigma : X_\sigma \to \mathbb{C}^n\). Recall that, by the assumption of smoothness, \(u_{j_1}, \ldots, u_{j_n}\) form a \(\mathbb{Z}\)-basis of \(\mathbb{Z}^n\), or equivalently (with a further permutation if necessary) the matrix

\[
U = \begin{pmatrix}
u_{j_11} & \cdots & u_{j_1n} \\
\vdots & \ddots & \vdots \\
u_{j_n1} & \cdots & u_{j_nn}
\end{pmatrix}
\]

belongs to \(SL(n, \mathbb{Z})\). Let \(U^{-1} = \begin{pmatrix}w_{11} & \cdots & w_{n1} \\
\vdots & \ddots & \vdots \\
w_{1n} & \cdots & w_{nn}\end{pmatrix}\) and let

\[
\begin{pmatrix}v_{j_{n+1}1} & \cdots & v_{j_d1} \\
\vdots & \ddots & \vdots \\
v_{j_{n+1}n} & \cdots & v_{j_{d}n}\end{pmatrix}
\]

be the matrix in \(M_{n,d-n}(\mathbb{Z})\) obtained by deleting from

\[
\begin{pmatrix}w_{11} & \cdots & w_{n1} \\
\vdots & \ddots & \vdots \\
w_{1n} & \cdots & w_{nn}\end{pmatrix}
\]

the \(j\)-th column, for \(j = j_1, \ldots, j_n\).

We claim that

\[
\phi_\sigma([z_1, \ldots, z_d]) = (z_{j_1}^{-v_{j_{n+1}1}} \cdots z_{j_d}^{-v_{j_d1}}, \ldots, z_{j_1}^{-v_{j_{n+1}n}} \cdots z_{j_d}^{-v_{j_dn}})
\]

defines the required biholomorphism. In order to verify this, notice first that if \((\alpha_1, \ldots, \alpha_d) \in K_\Sigma\) then, by definition, for every \(k = 1, \ldots, n\), we have

\[
1 = (\alpha_1^{-u_{j_11}} \cdots \alpha_d^{-u_{j_d1}})^{w_{nk}} \cdot (\alpha_1^{-u_{j_1n}} \cdots \alpha_d^{-u_{j_dn}})^{w_{nk}} = \alpha_{j_k}^{-v_{j_{n+1}k}} \cdots \alpha_{j_d}^{-v_{j_{d}k}}
\]

so that

\[
\alpha_{j_k}^{-v_{j_{n+1}k}} \cdots \alpha_{j_d}^{-v_{j_{d}k}}, \quad k = 1, \ldots, n.
\]

(11)
For any \( \alpha_{j_1}, \ldots, \alpha_{j_d} \in \mathbb{C}^* \), these equations give a parametric representation of \( K_\Sigma \), using which it is easy to see that (10) is well defined. More in detail, if \([(z_1, \ldots, z_d)] = [(w_1, \ldots, w_d)]\) then there exist \( \alpha_{j_1}, \ldots, \alpha_{j_d} \in \mathbb{C}^* \) such that \( w_{j_1} = \alpha_{j_1} z_{j_1}^{v_{j_1}} \), \( w_{j_d} = \alpha_{j_d} z_{j_d}^{v_{j_d}} \) and
\[
\begin{align*}
    w_{j_1} &= \alpha_{j_1}^{-v_{j_1}+1} \cdots \alpha_{j_d}^{-v_{j_d}+1} z_1^{v_{j_1}} \cdots z_d^{v_{j_d}}, \\
    w_{j_d} &= \alpha_{j_1}^{-v_{j_1}} \cdots \alpha_{j_d}^{-v_{j_d}} z_1^{v_{j_1}} \cdots z_d^{v_{j_d}}.
\end{align*}
\]
from which it is immediate to see that \( \phi_\sigma([z_1, \ldots, z_d]) = \phi_\sigma([w_1, \ldots, w_d]) \).

Moreover, one sees that
\[
\psi_\sigma : \mathbb{C}^n \to X_\sigma, \quad \psi_\sigma(\xi_1, \ldots, \xi_n) = [(\psi_1, \ldots, \psi_d)]
\]
where
\[
\psi_{j_1} = \xi_1, \quad \psi_{j_n} = \xi_n, \quad \psi_{j_{n-1}} = \cdots = \psi_{j_d} = 1
\]
and is the inverse of \( \phi_\sigma \). Indeed, on the one hand it is clear that \( \phi_\sigma \circ \psi_\sigma = id_{\mathbb{C}^n} \). On the other hand, for every \([(z_1, \ldots, z_d)] \in X_\sigma \) we have \( (\psi_\sigma \circ \phi_\sigma)([z_1, \ldots, z_d]) = [(\psi_1, \ldots, \psi_d)] \) where
\[
\psi_{j_k} = z_{j_k}^{v_{j_k}+k} \cdots z_{j_d}^{v_{j_d}}, \quad k = 1, \ldots, n
\]
and \( \psi_{j_{n-1}} = \cdots = \psi_{j_d} = 1 \). But \([(\psi_1, \ldots, \psi_d)] = [(z_1, \ldots, z_d)] \) since \( (z_1, \ldots, z_d) = (\alpha_1, \ldots, \alpha_d)(\psi_1, \ldots, \psi_d) \) for the element \((\alpha_1, \ldots, \alpha_d) \in K_\Sigma \) given by \( \alpha_{j_1} = z_{j_1}^{v_{j_1}} \), \( \alpha_{j_d} = z_{j_d}^{v_{j_d}} \) (recall that, by definition of \( X_\sigma \), we have \( z_{j_{n-1}}, z_{j_d} = 0 \)) and
\[
\alpha_{j_k} = z_{j_k}^{v_{j_k}} \cdots z_{j_d}^{v_{j_d}}, \quad k = 1, \ldots, n.
\]
This proves the claim. Now, by the very definition of \( X_\sigma \) it is clear that it contains the complex torus \( \mathbb{C}^n \) and that \( X_\sigma \) is invariant by the action (9). In fact, one has \( \phi_\sigma \left( \frac{(\mathbb{C}^*)^n}{K_\Sigma} \right) = (\mathbb{C}^*)^n \) and, if \( \phi_\sigma(\alpha_1, \ldots, \alpha_d) = (a_1, \ldots, a_n) \), \( \phi_\sigma(z_1, \ldots, z_d) = (\xi_1, \ldots, \xi_n) \), then \( \phi_\sigma(\alpha_1 z_1, \ldots, \alpha_d z_d) = (a_1 \xi_1, \ldots, a_n \xi_n) \), which means that the action of \( \frac{(\mathbb{C}^*)^n}{K_\Sigma} \) on \( X_\sigma \) corresponds, via \( \phi_\sigma \), to the canonical action of \( (\mathbb{C}^*)^n \) on \( \mathbb{C}^n \).

As a consequence, since the only fixed point for this canonical action is the origin, we have that the only point of \( X_\sigma \) fixed by the action of \( \frac{(\mathbb{C}^*)^n}{K_\Sigma} \) is the point \( p = [z_1, \ldots, z_d] \) having \( z_{j_1} = \cdots = z_{j_n} = 0 \).

So \( X_\sigma \) turns out to be a neighbourhood \( X_\rho \) of the fixed point \( p \) which satisfies all the requirements of the statement of the Proposition.

Since the \( X_\sigma \)'s, when \( \sigma \) runs over all the \( n \)-dimensional cones of \( \Sigma \), cover \( X_\Sigma \), we get in this way all the fixed points by the torus action, and this concludes the proof of the Proposition.

\[\square\]

A.2. Toric bundles and Kodaira embeddings. Let us recall how one constructs combinatorially the line bundles on a toric manifold \( X_\Sigma \).

**Definition A.5.** Let \( \Sigma \) be a fan of cones in \( \mathbb{R}^n \). A \( \Sigma \)-linear support function (or simply a support function when the context is clear) is a continuous function \( g : \mathbb{R}^n \to \mathbb{R} \) such that
(i) on every \( n \)-dimensional cone \( \sigma \in \Sigma \), \( g \) is the restriction of a linear function \( g_\sigma : \mathbb{R}^n \to \mathbb{R} \);

(ii) \( g \) has integer values on \( \mathbb{Z}^n \).

A support function is clearly determined by the values it has on the generators of the cones.

One associates to any such function \( g \) a line bundle, denoted \( X_{\Sigma g} \), on the manifold \( X_\Sigma \) and defined as \( X_{\Sigma g} = \mathbb{C}^d_\Sigma \times \mathbb{C} \) where \( \mathbb{C}^d_\Sigma \), \( K_\Sigma \) are as in Definition \( \ref{definition:toric-manifold} \) and the quotient comes from the action of \( K_\Sigma \) on \( \mathbb{C}^d_\Sigma \times \mathbb{C} \) given by

\[
(\alpha_1, \ldots, \alpha_d) \cdot (z_1, \ldots, z_d, z_{d+1}) = (\alpha_1 z_1, \ldots, \alpha_d z_d, \alpha_1^{-g(u_1)} \cdots \alpha_d^{-g(u_d)} z_{d+1}).
\]

The projection \( p : X_{\Sigma g} \to X_\Sigma \) is just given by \( p([z_1, \ldots, z_{d+1}]) = [z_1, \ldots, z_d] \), which is clearly well-defined by the very definition of the equivalence relations involved.

It is known that \( X_{\Sigma g} \) is very ample if and only if \( g \) is strictly convex, i.e. it fulfills the following requirements:

(a) for every \( v_1, v_2 \in \mathbb{R}^n \), \( t \in [0, 1] \), one has \( g(tv_1 + (1 - t)v_2) \geq tg(v_1) + (1 - t)g(v_2) \) (i.e. \( -g \) is convex);

(b) distinct \( n \)-dimensional cones \( \sigma \) give distinct functions \( g_\sigma \).

A nice representation of the very ample line bundle \( p : X_{\Sigma g} \to X_\Sigma \), encoding combinatorially both the structure of \( X_\Sigma \) and the function \( g \), is given by the convex polytope

\[
\Delta_g = \{ x \in \mathbb{R}^n \mid \langle x, u_i \rangle \geq g(u_i), \ i = 1, \ldots, d \}
\]

where \( u_1, \ldots, u_d \) are the generators of \( \Sigma \).

Every \( k \)-dimensional face of \( \Delta_g \) is given by the intersection of \( n - k \) hyperplanes \( \langle x, u_i \rangle = g(u_i) \), for \( i \in I \subseteq \{1, \ldots, d\} \) such that \( \{u_i\}_{i \in I} \) generates an \((n - k)\)-dimensional cone of \( \Sigma \). In particular, the vertices of \( \Delta_g \) correspond to the \( n \)-dimensional cones of \( \Sigma \) and then (see the proof of Proposition \( \ref{proposition:toric-manifold} \)) to the fixed points of the torus action.

Conversely, every convex polytope \( \Delta = \{ x \in \mathbb{R}^n \mid \langle x, u_i \rangle \geq \lambda_i, \ i = 1, \ldots, d \} \) with the property that the normal vectors \( u_i \) to the faces meeting in a given vertex form a \( \mathbb{Z} \)-basis of \( \mathbb{Z}^n \) determine a toric manifold together with a very ample line bundle.

We are now ready to prove the following

**Proposition A.6.** Let \( p \in X_\Sigma \) be a fixed point for the torus action and \( X_p, \phi_p : X_p \to \mathbb{C}^n \) be as in Proposition \( \ref{proposition:toric-manifold} \).

The restriction to \( X_p \) of the Kodaira embedding \( i_g : X_\Sigma \to \mathbb{CP}^{N-1} \) associated to \( X_{\Sigma g} \) writes, in the coordinates given by \( \phi_p \), as

\[
i_g|_{X_p} \circ \phi^{-1}_p : \mathbb{C}^n \to \mathbb{CP}^{N-1}, \ \xi \mapsto [\cdots, \xi_1^x \cdots \xi_n^x, \cdots]
\]

where \( (x_1, \ldots, x_n) \) runs over all the points with integral coordinates in \( \Delta \), being \( \Delta \) the polytope in \( \mathbb{R}^n \) obtained by \( \Delta_g \) via the transformation in \( SL_n(\mathbb{Z}) \) and the translation which send the vertex of
\( \Delta_g \) corresponding to \( p \) to the origin and the corresponding edge to the edge generated by the vectors \( e_1, \ldots, e_n \) of the canonical basis of \( \mathbb{R}^n \).

**Proof.** For the sake of simplicity and without loss of generality, we can assume that the fixed point \( p \) corresponds (in the sense of the proof of Proposition \[A.4\]) to the \( n \)-dimensional cone of \( \Sigma \) generated by \( u_1, \ldots, u_n \), so that \( X_p = \{ ([z_1, \ldots, z_d]) \in X_\Sigma \mid z_{n+1}, \ldots, z_d \neq 0 \} \). Given the line bundle \( p : X_\Sigma \to X_\Sigma \), we clearly have

\[
p^{-1}(X_p) = \{ ([z_1, \ldots, z_d, z_{d+1}]) \in X_\Sigma \mid z_{n+1}, \ldots, z_d \neq 0 \}.
\]

An explicit trivialization \( f : p^{-1}(X_p) \to X_p \times \mathbb{C} \) of \( X_\Sigma \) on \( X_p \) is given by

\[
f([z_1, \ldots, z_d, z_{d+1}]) = ([z_1, \ldots, z_d, z_{d+1}\cdot z_1^{n+1} \cdots z_d^{n+1}])
\]

where, for every \( j = n + 1, \ldots, d \),

\[
c_j = g(u_j) - \sum_{k=1}^n v_{jk} g(u_k)
\]

and the \( v_{jk} \)'s are defined in the proof of Proposition \[A.4\].

Indeed, \( f \) is well defined because \( z_{n+1}, \ldots, z_d \neq 0 \) and because, if \( ([z_1, \ldots, z_d, z_{d+1}]) = ([w_1, \ldots, w_d, w_{d+1}]) \) then, for some \( (\alpha_1, \ldots, \alpha_d) \in K_\Sigma \),

\[
w_{d+1}w_n^{\alpha_1} \cdots w_d^{\alpha_d} = (z_{d+1}\alpha_1^{-g(u_1)} \cdots \alpha_d^{-g(u_d)})z_1^{\alpha_1} \cdots z_d^{\alpha_d} =
\]

\[
= z_{d+1}z_n^{\alpha_1} \cdots z_d^{\alpha_d} \alpha_1^{-g(u_1)} \cdots \alpha_n^{-g(u_n)} = \sum_{k=1}^n v_{nk} g(u_k)
\]

by \([11]\) in the proof of Proposition \[A.4\].

The inverse of \( f \) is clearly given by \( f^{-1} : X_p \times \mathbb{C} \to p^{-1}(X_p) \),

\[
f^{-1}([z_1, \ldots, z_d], z) = ([z_1, \ldots, z_d, z_1^{-z_1^{n+1} \cdots z_d^{n+1}}]),
\]

which is well defined by the same arguments as above.

A section of \( p : X_\Sigma \to X_\Sigma \) is determined by a function \( F = F(z_1, \ldots, z_d) \) which satisfies

\[
F(\alpha_1 z_1, \ldots, \alpha_d z_d) = \alpha_1^{-g(u_1)} \cdots \alpha_d^{-g(u_d)} F(z_1, \ldots, z_d).
\]

for every \( (\alpha_1, \ldots, \alpha_d) \in K_\Sigma \). Indeed, this is exactly the condition which assures that \( s : X_\Sigma \to X_\Sigma \),

\[
s([z_1, \ldots, z_d]) = ([z_1, \ldots, z_d, F(z_1, \ldots, z_d)]) \]

is well-defined.

By a straight calculation and by \([11]\), a basis for the space of global sections is given by the polynomials \( F(z_1, \ldots, z_d) = z_1^{x_1+1} \cdots z_d^{x_d+1}, x_i \geq 0 \), which satisfy

\[
x_j + g(u_j) = v_{j1}(x_1 + g(u_1)) + \cdots + v_{jn}(x_n + g(u_n)), \quad j = n + 1, \ldots, d
\]

(14)
where the $v_{jk}$’s are defined in the proof of Proposition A.4. We will refer to this basis as the monomial basis. Let $\{F_0, \ldots, F_N\}$ be the monomial basis. Then, by the celebrated theorem of Kodaira, the map

$$X_{\Sigma} \xrightarrow{i_{\Sigma}} \mathbb{C}P^N, \quad [(z_1, \ldots, z_d)] \mapsto [F_0(z_1, \ldots, z_d), \ldots, F_N(z_1, \ldots, z_d)].$$

(15)

yields an embedding of $X_{\Sigma}$ in the complex projective space. Restricting $i_\Sigma$ to $X_p$ and composing with $\phi_p^{-1} : \mathbb{C}^n \to X_p$ we get

$$(\xi_1, \ldots, \xi_n) \mapsto [F_0(\xi_1, \ldots, \xi_n, 1 \ldots, 1), \ldots, F_N(\xi_1, \ldots, \xi_n, 1 \ldots, 1)].$$

(16)

Now, since the $x_i$’s are all non-negative integers, conditions (14) are equivalent to

$$\langle x + g_u, v_j \rangle \geq g(u_j), \quad j = 1, \ldots, d,$$

(17)

where $x = (x_1, \ldots, x_n)$, $g_u = (g(u_1), \ldots, g(u_n))$ and for $j = 1, \ldots, n$ we are setting $v_j = e_j$ (the canonical basis of $\mathbb{R}^n$). Since $e_1, \ldots, e_n, v_{n+1}, \ldots, v_d$ are the images of $u_1, \ldots, u_n, u_{n+1}, \ldots, u_d$ via the map

$$A = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{1n} & \cdots & u_{nn} \end{pmatrix}^{-1} \in SL_n(\mathbb{Z}),$$

one easily sees that (17) are the defining equations of the polytope $\Delta = T A^{-1}(\Delta_g) - g_u$, obtained from $\Delta_g = \{x \in \mathbb{R}^n \mid \langle x, u_i \rangle \geq g(u_i)\}$ by the map in $SL_n(\mathbb{Z})$ and the translation which send the edge given by the faces having $u_1, \ldots, u_n$ as normals (i.e. the edge at the vertex corresponding to $p$) to the edge at the origin having the vectors of the canonical basis as edge. Then the embedding (16) turns out to be the map

$$\mathbb{C}^n \to \mathbb{C}P^N, \quad (\xi_1, \ldots, \xi_n) \mapsto [\ldots, \xi_1^{x_1} \cdots \xi_n^{x_n}, \ldots]$$

where $(x_1, \ldots, x_n)$ runs over all the points with integral coordinates in $\Delta$, as required.

\[\square\]

**Remark A.7.** Notice that the transformed polytope $\Delta$ represents, up to isomorphism, the same line bundle and the same toric manifold as $\Delta_g$, because we are always free to apply to the fan $\Sigma$ a transformation in $SL_n(\mathbb{Z})$ (see, for example, Proposition VII.1.16 in [2]) and we can always add to $g$ an integral linear function $\mathbb{Z}^n \to \mathbb{Z}$, which correspond to a translation of the polytope (this comes from the fact that two bundles $X_{\Sigma,g}$, $X_{\Sigma,g'}$ associated to $\Sigma$-linear support functions $g, g'$ on $\Sigma$ are isomorphic if and only if $g - g' : \mathbb{R}^n \to \mathbb{R}$ is a linear function).

In fact, it is well-known that if a toric manifold endowed with a very ample line bundle is represented by a polytope $\Delta$, then the monomials $a_1^{y_1} \cdots a_n^{y_n}$, for $(y_1, \ldots, y_n) \in \Delta \cap \mathbb{Z}^n$ and $(a_1, \ldots, a_n) \in (\mathbb{C}^*)^n$, give the restriction of a Kodaira embedding (associated to the given line bundle) to the complex torus $(\mathbb{C}^*)^n$ contained in $X_{\Sigma}$. What we have seen here in detail is exactly that this embedding can be extended to $\mathbb{C}^n$ if the polytope has the origin as vertex and the edge at the origin is generated by the canonical basis of $\mathbb{R}^n$, and coincides with (16) in this case.
**References**

[1] M. Abreu, *Kähler geometry of toric manifolds in symplectic coordinates*, Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), 124, Fields Inst. Commun., 35, Amer. Math. Soc., Providence, RI, 2003.

[2] M. Audin, *Torus actions on symplectic manifolds*, Progress in Mathematics, Vol. 93 (2004).

[3] V. Batyrev, *Quantum cohomology rings of toric manifolds*, Astérisque No. 218 (1993), 9-34.

[4] P. Biran, *A stability property of symplectic packing*, Invent. Math. 136 (1999) 123-155.

[5] S. Bochner, *Curvature in Hermitian metric*, Bull. Amer. Math. Soc. 53 (1947), 179-195.

[6] A. Borel, *Linear algebraic groups*, second ed., GTM n. 126 Springer–Verlag, New-York (1991).

[7] E. Calabi, *Isometric Imbeddings of Complex Manifolds*, Ann. of Math. 58 (1953), 1-23.

[8] A. C. Castro, *Upper bound for the Gromov width of coadjoint orbits of type A*, arXiv:1301.0158v1

[9] T. Delzant, *Hamiltoniens periodiques et image convexe de l’application moment*, Bull. Soc. Math. France, 116 (1988), 315-339.

[10] S. K. Donaldson, *Kähler geometry on toric manifolds, and some other manifolds with large symmetry*, Handbook of geometric analysis. No. 1, 2975, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.

[11] W. Fulton, *Introduction to toric varieties*, Princeton University press (1993).

[12] G. Giachetta, L. Mangiarotti, G. A. Sardanashvili, *Geometric and algebraic topological methods in quantum mechanics*, World Scientific (2005).

[13] M. Gromov, *Pseudoholomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985), no. 2, 307-347.

[14] V. Guillemin, *Moment maps and combinatorial invariants of hamiltonian Tⁿ-spaces*, Birkhauser 1994.

[15] D. Hulin, *Sous-varietes complexes d’Einstein de l’espace projectif*, Bull. Soc. math. France 124 (1996), 277-298.

[16] D. Hulin, *Kähler-Einstein metrics and projective embeddings*, J. Geom. Anal. 10 (2000), 525-528.

[17] Y. Karshon, S. Tolman, *The Gromov width of complex Grassmannians*, Algebr. Geom. Topol. 5 (2005), 911-922.

[18] A. Loi, M. Zedda, *Kähler-Einstein submanifolds of the infinite dimensional projective space*, Math. Ann. 350 (2011), 145-154.

[19] A. J. Di Scala, A. Loi and H. Hishi, *Kähler immersions of homogeneous Kähler manifolds into complex space forms*, Asian J. of Math.s Vol. 16 No. 3 (2012), 479-488.

[20] A. Loi, F. Zuddas, *Symplectic maps of complex domains into complex space forms*, Journal of Geometry and Physics 58 (2008), 888-899.

[21] G. Lu, *Symplectic capacities of toric manifolds and related results*, Nagoya Math. J. Vol. 181 (2006), 149-184.

[22] T. Oda, *Convex bodies and algebraic geometry*, Springer Verlag (1985).

[23] D. Salamon, *Uniqueness of symplectic structures*, Acta Math. Vietnam. 38 (2013), no. 1, 123-144

[24] Y. T. Siu, *The existence of Kähler-Einstein metrics on manifolds with positive anticanonical line bundle and a suitable finite symmetry group*, Ann. Math. 127 (1988), 585-627.

[25] G. Tian, *Kähler-Einstein metrics on complex surfaces with χ₁(M) positive*, Math. Ann. 274 (1986), 503-516.
