Random walks maximizing the probability to visit an interval

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Abstract

We consider random walks, say $W_n = (M_0, M_1, \ldots, M_n)$, of length $n$ starting at 0 and based on the martingale sequence $M_k$ with differences $X_m = M_m - M_{m-1}$. Assuming that the differences are bounded, $|X_m| \leq 1$, we solve the problem

$$D_n(x) \overset{\text{def}}{=} \sup \mathbb{P}\{W_n \text{ visits an interval } [x, \infty)\}, \quad x \in \mathbb{R}, \quad (1)$$

where sup is taken over all possible $W_n$. In particular, we describe random walks which maximize the probability in (1). We also extend the result to super-martingales.

1 Introduction and results

We consider random walks, say $W_n = (M_0, M_1, \ldots, M_n)$ of length $n$ starting at 0 and based on a martingale sequence $M_k = X_1 + \cdots + X_k$ (assume $M_0 = 0$)

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with differences \( X_m = M_m - M_{m-1} \). Let \( \mathcal{M} \) be the class of martingales with bounded differences such that \( |X_m| \leq 1 \) and \( \mathbb{E}(X_m | \mathcal{F}_{m-1}) = 0 \) with respect to some increasing sequence of algebras \( \emptyset \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n \). If a random walk \( W_n \) is based on a martingale sequence of the class \( \mathcal{M} \) then we write symbolically \( W_n \in \mathcal{M} \). Extensions to super-martingales are provided at the end of the Introduction.

In this paper we provide a solution of the problem

\[
D_n(x) \overset{\text{def}}{=} \sup_{W_n \in \mathcal{M}} \mathbb{P}\{W_n \text{ visits an interval } [x, \infty)\}, \quad x \in \mathbb{R}.
\] (2)

In particular, we describe random walks which maximize the probability in (2) and give an explicit expression of the upper bound \( D_n(x) \). It turns out that the random walk maximizing the probability in (2) is an inhomogeneous Markov chain, i.e., given \( x \) and \( n \), the distribution of \( k \)th step depends only on \( M_{k-1} \) and \( k \). For integer \( x \in \mathbb{Z} \) the maximizing random walk is a simple symmetric random walk (that is, a symmetric random walk with independent steps of length 1) stopped at \( x \). For non-integer \( x \), the maximizing random walk makes some steps of smaller sizes. Smaller steps are needed to make a jump so that the remaining distance becomes integer and then continue as a simple random walk. The average total number of the smaller steps is bounded by 2. For martingales our result can be interpreted as a maximal inequality

\[
\mathbb{P}\left\{ \max_{1 \leq k \leq n} M_k \geq x \right\} \leq D_n(x).
\]

The maximal inequality is optimal since the equality is achieved by martingales related to the maximizing random walks, that is,

\[
\sup_{W_1, \ldots, W_n \in \mathcal{M}} \mathbb{P}\left\{ \max_{1 \leq k \leq n} M_k \geq x \right\} = D_n(x),
\] (3)

where we denote by \( W_k \) a random walk \((M_0, M_1, \ldots, M_k, M_k, \ldots, M_k) \in \mathcal{M}\).

To prove the result we introduce a general principle for maximal inequalities for (natural classes of) martingales which reads as

\[
\sup_{W_1, \ldots, W_n \in \mathcal{M}} \mathbb{P}\left\{ \max_{1 \leq k \leq n} M_k \geq x \right\} = \sup_{M_n \in \mathcal{M}} \mathbb{P}\{M_n \geq x\}
\] (4)

in our case. It means that for martingales, the solutions of problems of type (2) are inhomogeneous Markov chains, i.e., the problem of type (2) can
be always reduced to finding a solution of (2) in a class of inhomogeneous Markov chains.

Our methods are similar in spirit to a method used in [Ben01], where a solution of a problem (2) was provided for integer \( x \in \mathbb{Z} \). Namely, he showed that if \( R_n = \varepsilon_1 + \cdots + \varepsilon_n \) is a sum of Rademacher random variables such that \( \mathbb{P}\{\varepsilon_i = -1\} = \mathbb{P}\{\varepsilon_i = -1\} = 1/2 \) and \( B(n, k) \) is a normalized sum of \( n - k + 1 \) smallest binomial coefficients, i.e.,

\[
B(n, k) = 2^{-n} \sum_{i=0}^{n-k} \binom{\left\lceil \frac{i}{2} \right\rceil}{\frac{i}{2}}
\]

where \( \left\lfloor x \right\rfloor \) denotes an integer part of \( x \), then for all \( k \in \mathbb{Z} \)

\[
D_n(k) = B(n, k) = \begin{cases} 
2 \mathbb{P}\{R_n \geq k + 1\} + \mathbb{P}\{R_n = k\} & \text{if } n + k \in 2\mathbb{Z}, \\
2 \mathbb{P}\{R_n \geq k + 1\} & \text{if } n + k \in 2\mathbb{Z} + 1.
\end{cases}
\]

Recently Dzindzalieta, Juškevičius and Šileikis [DJS12] solved the problem (2) in the case of sums of bounded independent symmetric random variables. They showed that if \( S_n = X_1 + \cdots + X_n \) is a sum of independent symmetric random variables such that \( |X_i| \leq 1 \) then

\[
\mathbb{P}\{S_n \geq x\} \leq \begin{cases} 
\mathbb{P}\{R_n \geq x\} & \text{if } n + \left\lceil x \right\rceil \in 2\mathbb{Z}, \\
\mathbb{P}\{R_{n-1} \geq x\} & \text{if } n + \left\lceil x \right\rceil \in 2\mathbb{Z} + 1,
\end{cases}
\]

where \( \left\lceil x \right\rceil \) denotes the smallest integer number greater or equal to \( x \). We note that for integer \( x \) the random walk based on the sequence \( R_k \) stopped at a level \( x \) is a solution of (2).

As far as we are aware, the paper presents the first result where problems for martingales of type (2) and (3) are solved for all \( x \in \mathbb{R} \).

Let us turn to more detailed formulations of our results. For a martingale \( M_n \in \mathcal{M} \) and \( x \in \mathbb{R} \), we introduce the stopping time

\[
\tau_x = \min\{k : M_k \geq x\}.
\]

The stopping time \( \tau_x \) is a non-negative integer valued random variable possibly taking the value \( +\infty \) in cases where \( M_k < x \) for all \( k = 0, 1, \ldots \). For a martingale \( M_n \in \mathcal{M} \), define its version stopped at level \( x \) as

\[
M_{n,x} = M_{\tau_x \wedge n}, \quad a \wedge b = \min\{a, b\}.
\]
Given a random walk \( W_n = \{0, M_1, \ldots, M_n\} \) its stopped version is denoted as \( W_{n,x} = \{0, M_{1,x}, \ldots, M_{n,x}\} \).

Fix \( n \) and \( x > 0 \). The maximizing random walk \( RW_n = \{0, M^{x}_1, \ldots, M^{x}_n\} \) is defined as follows. We start at 0. Suppose that after \( k \) steps the remaining distance to the target \([x, \infty)\) is \( \rho_k \). The distribution of the next step is a Bernoulli random variable (which takes only two values), say \( X^{*} = X^{*}(k, \rho_k, n) \), such that

\[
\sup \mathbb{E} D_{n-k}(\rho_k - X) = \mathbb{E} D_{n-k}(\rho_k - X^{*})
\]

where \( \sup \) is taken over all random variables \( X \) such that \(|X| \leq 1 \) and \( \mathbb{E} X = 0 \).

The distribution of the next step \( X^{*} \) depends on four possible situations.

1) \( \rho_k \) is integer;
2) \( n - k \) is odd and \( 0 < \rho_k < 1 \);
3) the integer part of \( \rho_k + n - k \) is even;
4) the integer part of \( \rho_k + n - k \) is odd and \( \rho_k > 1 \).

After \( k \) steps we make a step of length \( s_l \) or \( s_r \) to the left or right with probabilities \( p_i = \frac{s_r}{s_r + s_l} \) and \( q_i = \frac{s_l}{s_r + s_l} \) respectively. Let \( \{x\} \) denotes the

Depending on (i)-(iv) we have.

i) \( s_l = s_r = 1 \) with equal probabilities \( p_1 = q_1 = \frac{1}{2} \), i.e., we continue as a simple random walk;
ii) \( s_l = \rho_k \) and \( s_r = 1 - \rho_k \) with \( p_2 = 1 - \{\rho_k\} \) and \( q_2 = \{\rho_k\} \), i.e., we make a step so that the remaining distance \( \rho_{k+1} \) becomes equal either to 0 or 1;
(iii) \( s_l = \{\rho_k\} \) and \( s_r = 1 \) with \( p_3 = \frac{1}{1 + \{\rho_k\}} \) and \( q_3 = \frac{\{\rho_k\}}{1 + \{\rho_k\}} \), i.e., we make a step to the left so that \( \rho_{k+1} \) is of the same parity as \( n - k - 1 \) or to the right side as far as possible;
(iv) \( s_l = 1 \) and \( s_r = 1 - \{\rho_k\} \) with \( p_4 = \frac{1 - \{\rho_k\}}{2 - \{\rho_k\}} \) and \( q_4 = \frac{1}{2 - \{\rho_k\}} \), i.e., we make a step to the left so that \( \rho_{k+1} \) is of the same parity as \( n - k - 1 \) or to the right side as far as possible.

In other words if \( \rho_k \) is non-integer then the maximizing random walk jumps so that \( \rho_{k+1} \) becomes of the same parity as the remaining number of steps \( n - k - 1 \) or the step of length \( \min\{x, 1\} \) to the other side. If the remaining distance \( \rho_k \) is integer, then it continues as a simple random walk.

The main result of the paper is the following theorem.

**Theorem 1.** The random walk \( RW_n \) stopped at \( x \) maximizes the probability

\[
\mathbb{P}(\text{random walk stopped at } x) = \sup \mathbb{P}(\text{random walk stopped at } x) = \mathbb{P}(\text{random walk stopped at } x^{*})
\]
to visit an interval $[x, \infty)$ in first $n$ steps, i.e., the following equalities hold

$$D_n(x) = \mathbb{P}\{RW_{n,x} \text{ visits an interval } [x, \infty)\} = \mathbb{P}\{M_{n,x}^x \geq x\},$$

(9)

for all $x \in \mathbb{R}$ and $n = 0, 1, 2, \ldots$.

An explicit definition of $D_n(x)$ depends on the parity of $n$. Namely, let $x = m + \alpha$ with $m \in \mathbb{Z}$ and $0 \leq \alpha < 1$.

If $m + n$ is odd then

$$D_n(x) = \sum_{i=0}^{h} a_i B(n - i - 1, m + i), \quad a_i = \frac{\alpha^i}{(1 + \alpha)^{i+1}},$$

(10)

where $h = (n - m - 1)/2$.

If $m + n$ is even then

$$D_n(x) = \sum_{i=0}^{m+1} b_i B(n - i - 1, m - i + 1),$$

(11)

where $b_i = \frac{(1-\alpha)^i}{(2-\alpha)^{i+1}}$, for $i < m$, $b_m = \alpha \left(\frac{1-\alpha}{2-\alpha}\right)^m$ and $b_{m+1} = (1 - \alpha) \left(\frac{1-\alpha}{2-\alpha}\right)^m$.

It is easy to see from (10) and (11) that $D_n$ is decreasing and continuous for all $x \in \mathbb{R}$ except at $x = n$ it has a jump. In particular we have that $D_n(x) = 1$ for $x \leq 0$ and $D_n(x) = 0$ for $x > n$. In Section 3 we prove that the function $D_n$ is piecewise convex and piecewise continuously differentiable. We also give the recursive definition of the function $D_n$.

A great number of papers is devoted to construction of upper bounds for tail probabilities of sums of random variables. The reader can find classical results in books [PB75, SW09]. One of the first and probably the most known non-asymptotic bound for $D_n(x)$ was given by Hoeffding in 1963 [Hoe63]. He proved that for all $x$ the function $D_n(x)$ is bounded by $\exp\{-x^2/2n\}$. Hoeffding’s inequalities remained unimproved until 1995 when Talagrand [Tal95] inserted certain missing factors. Bentkus 1986–2007 [Ben87, Ben01, Ben04, BKZ06] developed induction based methods. If it is possible to overcome related technical difficulties, these methods lead to the best known upper bounds for the tail probabilities (see [BD10, DJS12] for examples of tight bounds received using these methods). In [Ben01] first tight bounds for $D_n(x)$ for integer $x$ was received. To overcome technical difficulties for non-integer $x$ in [Ben01] the linear interpolation between integer points was used, thus losing precision for non-integer $x$. Our method is similar in spirit to [Ben01].
1.1 An extension to super-martingales

Let $\mathcal{SM}$ be the class of super-martingales with bounded differences such that $|X_m| \leq 1$ and $\mathbb{E}(X_k|\mathcal{F}_{k-1}) \leq 0$ with respect to some increasing sequence of algebras $\emptyset \subset \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_n$. We show that

**Theorem 2.** For all $x \in \mathbb{R}$ we have

$$\sup_{SW_n \in \mathcal{SM}} \mathbb{P}\{SW_n \text{ visits an interval } [x, \infty)\} = D_n(x). \quad (12)$$

For super-martingales Theorem 2 can also be interpreted as the maximal inequality

$$\mathbb{P}\left\{ \max_{1 \leq k \leq n} M_k \geq x \right\} \leq D_n(x),$$

where $M_k \in \mathcal{SM}$, and furthermore, the sup over the class of super-martingales is achieved on a martingale class.

**Proof of Theorem 2.** Suppose that sup in (12) is achieved with some super-martingale $SM_n = X_1 + \cdots + X_n$. Let $M_n = Y_1 + \cdots + Y_n$ be a sum of random variables, such that

$$(Y_k|\mathcal{F}_{k-1}) = (\mathbb{E}(X_k|\mathcal{F}_{k-1}) - 1) \mathbb{E}(X_k|\mathcal{F}_{k-1}).$$

It is easy to see that $Y_k \geq 0$, $|X_k + Y_k| \leq 1$ and $\mathbb{E}(X_k + Y_k|\mathcal{F}_{k-1}) = 0$, so $SM_n + M_n \in \mathcal{M}$. Since $Y_k \geq 0$ we have that $M_n \geq 0$, so $\mathbb{P}\{SM_n + M_n \geq x\}$ is greater or equal to $\mathbb{P}\{SM_n \geq x\}$. This proves the theorem. $\square$

2 Maximal inequalities for martingales are equivalent to inequalities for tail probabilities

Let $\mathcal{M}$ be a class of martingales. Introduce the upper bounds for tail probabilities and in the maximal inequalities as

$$B_n(x) \overset{\text{def}}{=} \sup_{M_n \in \mathcal{M}} \mathbb{P}\{M_n \geq x\}, \quad B^*_n(x) \overset{\text{def}}{=} \sup_{M_n \in \mathcal{M}} \mathbb{P}\left\{ \max_{0 \leq k \leq n} M_k \geq x \right\}$$
for $x \in \mathbb{R}$ (we define $M_0 = 0$).

Let as before $\tau_x$ be a stopping time defined by

$$\tau_x = \min\{k : M_k \geq x\}. \quad (13)$$

**Theorem 3.** If a class $\mathcal{M}$ of martingales is closed under stopping at level $x$, then

$$B_n(x) \equiv B_n^*(x).$$

We can interpret Theorem 3 by saying that inequalities for tail probabilities for natural classes of martingales imply (seemingly stronger) maximal inequalities. This means that maximizing martingales are inhomogeneous Markov chains. Assume that for all $M_n \in \mathcal{M}$ we have

$$\mathbb{P}\{M_n \geq x\} \leq g_n(x)$$

with some function $g$ which depends only on $n$ and the class $\mathcal{M}$. Then it follows that

$$\mathbb{P}\left\{\max_{0 \leq k \leq n} M_k \geq x\right\} \leq g_n(x).$$

In particular, equalities (2)–(4) are equivalent.

**Proof of Theorem 3.** It is clear that $B_n \leq B_n^*$ since $M_n \leq \max_{0 \leq k \leq n} M_k$. Therefore it suffices to check the opposite inequality $B_n \geq B_n^*$. Let $M_n \in \mathcal{M}$. Using the fact that $M_{\tau_x \wedge n} \in \mathcal{M}$, we have

$$\mathbb{P}\left\{\max_{0 \leq k \leq n} M_k \geq x\right\} = \mathbb{P}\{M_{\tau_x \wedge n} \geq x\} \leq B_n(x). \quad (14)$$

Taking in (14) sup over $M_n \in \mathcal{M}$, we derive $B_n^* \geq B_n$. \qed

In general conditions of Theorem 3 are fulfilled under usual moment and range conditions. That is, conditions of type

$$\mathbb{E}(|X_k|^{\alpha_k} \mid \mathcal{F}_{k-1}) \leq g_k, \quad (X_k \mid \mathcal{F}_{k-1}) \in I_k,$$

with some $\mathcal{F}_{k-1}$-measurable $\alpha_k \geq 0$, $g_k \geq 0$, and intervals $I_k$ with $\mathcal{F}_{k-1}$-measurable endpoints. One can use as well assumptions like symmetry, unimodality, etc.
3 Proofs

In order to prove Theorem 1 we need some additional lemmas.

**Lemma 4.** Suppose $f \in C^1(0,2)$ is a continuously differentiable, non-increasing, convex function on $(0,2)$. Suppose that $f$ is also two times differentiable on intervals $(0,1)$ and $(1,2)$. The function $F : (0,2) \rightarrow \mathbb{R}$ defined as

\[
F(x) = \begin{cases} 
\frac{1}{x+1} f(0) + \frac{x}{x+1} f(x+1) & \text{for } x \in (0,1]; \\
\frac{2-x}{3-x} f(x-1) + \frac{1}{3-x} f(2) & \text{for } x \in (1,2) 
\end{cases}
\]

is convex on intervals $(0,1)$ and $(1,2)$.

**Proof** Since the function $f$ is decreasing and convex, we have that

\[
f'(x+1) \geq \frac{f(x+1) - f(0)}{x+1} \quad \text{for } x \in (0,1); \quad \text{(15)}
\]

\[
f'(x-1) \leq \frac{f(2) - f(x-1)}{3-x} \quad \text{for } x \in (1,2). \quad \text{(16)}
\]

For $x \in (0,1)$ simple algebraic manipulations gives

\[
F''(x) = \frac{x}{x+1} f''(x+1) + \frac{2}{(x+1)^2} \left( f'(x+1) - \frac{f(x+1) - f(0)}{x+1} \right). \quad \text{(17)}
\]

By (15) the second term in right hand side of (17) is non-negative. Thus $F''(x) \geq 0$ for all $x \in (0,1)$.

For $x \in (1,2)$ similar algebraic manipulation gives

\[
F''(x) = \frac{2-x}{3-x} f''(x-1) - \frac{2}{(3-x)^2} \left( f'(x-1) - \frac{f(2) - f(x-1)}{3-x} \right). \quad \text{(18)}
\]

By (16) the second term in right hand side of (18) is non-negative. Thus $F''(x) \geq 0$ for all $x \in (1,2)$.

We use Lemma 4 to prove that the function $x \rightarrow D_n(x)$ satisfies the following analytic properties.

**Lemma 5.** The function $D_n$ is convex and continuously differentiable on intervals $(n-2,n), (n-4,n-2), \ldots, (0,2\{n/2\})$. 

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Proof In order to prove this lemma it is very convenient to use a recursive
definition of the function $D_n(x)$ which easily follows from the the description
of the maximizing random walk $RW_{n,x}$. We have $D_0(x) = \mathbb{I}\{x \leq 0\}$ and

$$D_{n+1}(x) = \begin{cases} 
1 & \text{if } x \leq 0, \\
p_1 D_n(x-1) + q_1 D_n(x+1) & \text{if } x \in \mathbb{Z} \text{ and } x > 0, \\
p_2 D_n(0) + q_2 D_n(1) & \text{if } n \in 2\mathbb{Z} + 1 \text{ and } x < 1, \\
p_3 D_n(\lfloor x \rfloor) + q_3 D_n(x+1) & \text{if } \lfloor x \rfloor + n \in 2\mathbb{Z} \text{ and } x > 0, \\
p_4 D_n(x-1) + q_4 D_n(\lceil x \rceil) & \text{if } \lceil x \rceil + n \in 2\mathbb{Z} \text{ and } x > 1, \\
0 & \text{if } x > n.
\end{cases} \tag{19}$$

where $p_i + q_i = 1$ with $p_1 = 1/2$, $p_2 = 1 - \{x\}$, $p_3 = \frac{1}{1+(x)}$ and $p_4 = \frac{1-(x)}{2-(x)}$.

To prove Lemma 4 we use induction on $n$. If $n = 0$ then $D_n(x) = \mathbb{I}\{x \leq 0\}$
clearly satisfies Lemma 5. Suppose that Lemma 4 holds for $n = k - 1 \geq 0$. Assume $n = k$.

First we prove that $D_k$ is convex and continuously differentiable on intervals $(0,1), (1,2), \ldots, (k-1,k)$. Since $D_k$ is rational and do not have discontinuities between integer points, it is clearly continuously differentiable on intervals $(0,1), (1,2), \ldots, (k-1,k)$. If $x \in (k-1,k]$ then by (10) we have that $D_k(x) = 2^{-k+1}/(x-k+1)$. Thus the function $D_k$ is clearly convex on interval $(k-1,k)$. The convexity of $D_k$ on intervals $(0,1), (1,2), \ldots, (k-2,k-1)$ follows directly from Lemma 4 and recursive
definition (19). To prove that the function $D_k$ is also continuously differentiable on intervals $(k-2,k), (k-4,k-2), \ldots, (0,2\{k/2\})$ it is enough to show that $D_k'(m-0) = D_k'(m+0)$ for all $m \in \mathbb{N}$ such that $k+m \in 2\mathbb{Z} + 1$. If $x = m - 0$ (we consider only the case $m > 0$, since for $m = 0$ the function $D_k(x)$ is linear), then by (19) we have

$$D_k(x) = p_4 D_{k-1}(x-1) + q_4 D_{k-1}(m) \tag{20}$$

and since $D_{k-1}$ is continuously differentiable at $x - 1$ we have

$$D_k'(x) = q_4^2 D_{k-1}(x-1) + p_4 D_{k-1}'(x-1) - q_4^2 D_{k-1}(m).$$

Since $x = m - 0$ we get that $D_k'(x) = D_{k-1}(x-1) - D_{k-1}(m)$. Similarly we have that if $x = m + 0$ then $D_k'(x) = D_{k-1}(m) - D_{k-1}(x-1)$. Since $D_{k-1}(m - 1) - D_{k-1}(m) = D_{k-1}(m) - D_{k-1}(m - 1)$ we get that
\[ D_k'(m - 0) = D_k'(m + 0). \] Since \( D_k \) is continuously differentiable on intervals \((k - 2, k), (k - 4, k - 2), \ldots, (0, 2\{k/2\})\) and \( D_k \) is convex on intervals \((0, 1), (1, 2), \ldots, (k - 1, k)\) we have that \( D_k(x) \geq 0 \) is convex on \( x = m \) for all \( m \in \mathbb{N} \) such that \( k + m \in 2\mathbb{Z} + 1 \). This ends the proof of Lemma 4.

We also need the following lemma, which is used to find the minimal dominating linear function in a proof of Theorem 1.

**Lemma 6.** The function \( D_n \) satisfies the following inequalities.

a) If \( n \in 2\mathbb{Z} + 1 \) and \( 0 < x < 1 \) then
\[
p_2 D_n(0) + q_2 D_n(1) - p_3 D_n(0) - q_3 D_n(x + 1) \geq 0. \tag{21}
\]
b) If \( \lfloor n + x \rfloor \in 2\mathbb{Z} \) then
\[
p_3 D_n(\lfloor x \rfloor) + q_3 D_n(x + 1) - p_1 D_n(x - 1) - q_1 D_n(x + 1) \geq 0. \tag{22}
\]
c) If \( \lfloor n + x \rfloor \in 2\mathbb{Z} + 1 \) and \( x > 1 \)
\[
p_4 D_n(x - 1) + q_4 D_n(\lfloor x \rfloor + 1) - p_1 D_n(x - 1) - q_1 D_n(x + 1) \geq 0. \tag{23}
\]

Here \( p_i \) and \( q_i \) are the same as in Lemma 4.

**Proof** We prove this lemma by induction on \( n \). If \( n = 0 \) then Lemma 6 is equivalent to the trivial inequality \( 1 - 1 \geq 0 \). Suppose that the properties (a)–(c) holds for \( n = k - 1 \geq 0 \). Assume \( n = k \).

**Proof of (a).** We use the following equalities directly following from the definition of the function \( D_k \). If \( k \in 2\mathbb{Z} + 1 \) and \( x \in (0, 1) \) then
\[
D_k(0) = D_{k-1}(0);
D_k(1) = p_1 D_{k-1}(0) + q_1 D_{k-1}(2);
D_k(x + 1) = p_4 D_{k-1}(2) + q_4 D_{k-1}(x);
D_{k-1}(x) = D_{k-1}(0) + x (D_{k-1}(1) - D_{k-1}(0));
\]
We substitute all these equalities to (21) we get that the left hand side of (21) is equal to
\[
q_2 p_3 p_4 x (p_1 D_{k-1}(0) + q_1 D_{k-1}(2) - D_{k-1}(1))
\]
The inequality (21) follows from the inequality \( D_{k-1}(1) \leq D_k(1) = p_1 D_{k-1}(0) + q_1 D_{k-1}(2) \).
Proof of (b). We rewrite every term in the inequality (22) using the definition of the function $D_k$ to get

$$p_1p_3(D_{k-1}([x] - 1) + D_{k-1}([x] + 1)) + q_3(p_3D_{k-1}([x] + 1) + q_3D_{k-1}(x + 2)) - p_1p_3(D_{k-1}([x] - 1) + D_{k-1}([x] + 1)) - p_1q_3(D_{k-1}(x) + D_{k-1}(x + 2)) \geq 0.$$ 

The inequality

$$p_3D_{k-1}([x] + 1) + q_3D_{k-1}(x + 2) \geq p_1D_{k-1}(x) + q_1D_{k-1}(x + 2)$$

follows from the inductive assumption (22) for $n = k - 1$.

Proof of (c). In this case we have to consider two separate cases.

Case $x > 2$. We again rewrite every term in the inequality (23) using the definition of the function $D_k$ to get

$$p_4(p_4D_{k-1}(x - 2) + q_4D_{k-1}([x])) + q_4p_1(D_{k-1}([x]) + D_{k-1}([x] + 2)) - q_4p_1(D_{k-1}([x]) + D_{k-1}([x] + 2)) - p_4p_1(D_{k-1}(x - 2) + D_{k-1}(x)) \geq 0.$$ 

The inequality

$$p_4D_{k-1}(x - 2) + q_4D_{k-1}([x]) \geq p_1D_{k-1}(x - 2) + q_1D_{k-1}(x)$$

follows from the inductive assumption (23) for $n = k - 1$.

Case $1 < x < 2$. Firstly let us again rewrite the inequality (23) using the recursive definition of $D_k$. After combining the terms we get that (23) is equivalent to

$$xD_{k-1}(1) + (1 - x)D_{k-1}(0) - D_{k-1}(x) \geq 0. \quad (24)$$

Now we use the inequality

$$D_{k-1}(x) \leq (2 - x)D_{k-1}(1) + (x - 1)D_{k-1}(2).$$

to get that

$$xD_{k-1}(1) + (1 - x)D_{k-1}(0) - D_{k-1}(x) \geq$$

$$2(x - 1)D_{k-1}(1) + (1 - x)D_{k-1}(0) - (x - 1)D_{k-1}(2) =$$

$$(x - 1)(2D_{k-1}(1) - D_{k-1}(0) - D_{k-1}(2)) = 0.$$ 

which proves the inequality (23).
Now we are ready to prove Theorem 1.

**Proof** For \( x \leq 0 \) to achieve sup in (9) take \( M_n \equiv 0 \). For \( x > n \) the sup in (9) is equal to zero since \( M_n \leq n \) for all \( n = 0, 1, \ldots \). To prove Theorem 1 for \( x \in (0, n] \) we use induction on \( n \).

For \( n = 0 \) the statement is obvious since \( P\{M_0 \geq x\} = I\{x \leq 0\} = D_0(x) \).

Suppose that Theorem 1 holds for \( n = k > 0 \). Assume \( n = k + 1 \). In order to prove Theorem 1 it is enough to prove that \( D_{k+1} \) satisfies the recursive relations (19). We have

\[
P\{M_{k+1} \geq x\} = P\{X_2 + \cdots + X_{k+1} \geq x - X_1\}
\]

\[
= E P\{X_2 + \cdots + X_k \geq x - t | X_1 = t\}
\]

\[
\leq ED_k(x - X_1).
\]

Now for every \( x \) we find a linear function \( t \mapsto f(t) \) dominating the function \( t \mapsto D_k(x - t) \) on interval \([-1, 1] \) and touching it at two points, say \( x_1 \) and \( x_2 \), on different sides of zero. After this we consider a random variable, say \( X \in \{x_1, x_2\} \) with mean zero. It is clear that \( ED_k(x - X_1) \leq ED_k(x - X) \).

We show that the numbers \( x_1 \) and \( x_2 \) are so that (19) holds.

Since \( D_k \) is piecewise convex between integer points, the points where \( f(t) \) touches \( D_k(x - t) \) can be only the endpoints of an interval \([-1, 1] \) or the points where \( D_k(x - t) \) is not convex.

We consider four separate cases.

1. \( x \in \mathbb{Z} \);
2. \( k \in 2\mathbb{Z} + 1 \) and \( x < 1 \);
3. \( \lfloor x \rfloor + k \in 2\mathbb{Z} \);
4. \( \lfloor x \rfloor + k \in 2\mathbb{Z} \) and \( x > 1 \).

**Case (i).** Since \( x \in \mathbb{Z} \) the dominating linear function touches \( D_k(x - t) \) at integer points. So maximizing \( X_1 \in \{-1, 0, 1\} \).

If \( x+k \in 2\mathbb{Z} +1 \) then the function \( D_k(x-t) \) is convex on \((-1, 1) \) so maximizing \( X \) takes values 1 or \(-1 \) with equal probabilities 1/2.

If \( x + k \in 2\mathbb{Z} \), then

\[
D_k(x) = \frac{1}{2}(D_{k-1}(x-1) + D_{k-1}(x+1)) = \frac{1}{2}(D_k(x-1) + D_k(x+1)),
\]

so the dominating function touches \( D_k(x - t) \) at all three points \(-1, 0, 1 \). Taking \( X \in \{-1, 1\} \) we end the proof of the case (i).
The case (i) was firstly considered in [Ben01].

**Case (ii).** Since $D_k$ is convex on intervals $(0, 1)$ and $(1, 3)$ the dominating minimal function can touch $D_k(x-t)$ only at $x, x-1, -1$. But due to an inequality (21) the linear function $f(t)$ going through $(x, D_k(0))$ and $(x-1, D_k(1))$ is above the point $(-1, D_k(x+1))$.

**Case (iii).** Since the function $D_k$ is convex on intervals $(\lfloor x \rfloor - 1, \lfloor x \rfloor)$ and $(\lfloor x \rfloor, \lfloor x \rfloor + 2)$ the dominating minimal function can touch $D_k(x-t)$ only at $-1, \{x\}, 1$. But due to an inequality (22) the linear function $f(t)$ going through $(\{x\}, D_k(\lfloor x \rfloor))$ and $(-1, D_k(x+1))$ is above the point $(1, D_k(x-1))$.

**Case (iv).** Since the function $D_k$ is convex on intervals $(\lfloor x \rfloor - 1, \lfloor x \rfloor + 1)$ and $(\lfloor x \rfloor + 1, \lfloor x \rfloor + 3)$ the dominating minimal function can touch $D_k(x+t)$ only at $-1, \{x\} - 1, 1$. But due to an inequality (23) the linear function $f(t)$ going through $(1, D_k(x-1))$ and $(\{x\} - 1, D_k(\lfloor x \rfloor + 1))$ is above the point $(-1, D_k(x+1))$.

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