SECONDARY STIEFEL-WHITNEY CLASS
OF SYMPLECTOMORPHISMS
OF RULED SYMPLECTIC 4-MANIFOLDS

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ABSTRACT. We prove the vanishing of the secondary Stiefel-Whitney class \( \tilde{w}_2 \) of homotopically trivial symplectomorphisms of an irrational ruled symplectic 4-manifold \( X \), either minimal or blown-up.

We also show that any value of \( \tilde{w}_2 \) can be realised by an appropriate diffeomorphism. This proves the existence of at least \( 2^{2g} \) different deformation classes of symplectic structures on \( X \) where \( 2g := \text{rank} H_1(X) \).

0. INTRODUCTION

One of problems of the symplectic geometry is to classify or at least distinguish possible symplectic structures on a given manifold \( X \). Except the easy case of surfaces, the only class of manifolds for which a complete classification is known consists of rational and ruled 4-manifolds and their symplectic blow-ups, see §1.2 for more details. An example of a ruled manifolds is the product \( X_0 = Y \times S^2 \) with a surface of genus \( g(Y) \). In general, for any \( X \) as above there exists a surface such that \( \pi_1(X) \cong \pi_1(Y) \) and we call \( g(Y) \) genus of \( X \).

The above classification of symplectic structures is made up to diffeomorphisms which could be topologically non-trivial. In this paper we show that there exists an invariant of homotopically trivial diffeomorphisms \( F : X \rightarrow X \), called secondary Stiefel-Whitney class \( \tilde{w}_2(F) \), which vanishes on symplectomorphisms of ruled 4-manifolds, and therefore allows to distinguish deformation classes of symplectic structures. In general, the class \( \tilde{w}_2(F) \) is defined for every diffeomorphism \( F : Z \rightarrow Z \) of arbitrary smooth manifold \( Z \) such that \( F \) acts trivially on all \( \mathbb{Z}_2 \)-homology groups \( H_\bullet(Z,\mathbb{Z}_2) \), and take values in the group \( H_1(Z,\mathbb{Z}_2) \).

The main result of the paper is as follows.

**Theorem 1.** Let \( (X,\omega) \) be a ruled symplectic manifold of genus \( g \geq 1 \).

i) \( \tilde{w}_2(F) = 0 \) for any symplectomorphism \( F : (X,\omega) \rightarrow (X,\omega) \) with trivial action in \( H_2(X,\mathbb{Z}) \) and \( \pi_1(X) \).

ii) For every \( w \in H_1(X,\mathbb{Z}_2) \) there exists a diffeomorphism \( F : X \rightarrow X \) with \( \tilde{w}_2(F) = w \)
and such that \( F \) acts trivially on \( \pi_1(X) \) and \( H_2(X,\mathbb{Z}) \).

**Corollary 2.** A ruled symplectic manifold \( (X,\omega) \) of genus \( g \geq 1 \) has at least \( 2^{2g} \) different deformation classes of symplectic structures.

The corollary follows immediately from **Theorem 3** below giving the description of the structure of the diffeomorphism group of ruled symplectic 4-manifolds. Let us introduce other relevant groups.

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Definition 0.1. Denote by $\Diff(X)$ the group of diffeomorphisms and by $\Diff_0(X)$ its component of the identity.

In the case of ruled 4-manifold $(X,\omega)$, $\Diff_+(X) < \Diff(X)$ is the subgroup which preserves the orientation of $X$ and the positive cone in $\mathbb{H}_2(X,\mathbb{R})$. Recall that the intersection form of $\mathbb{H}_2(X,\mathbb{R})$ has Lorentz signature. $\Diff_+(X)$ consists of diffeomorphisms $f \in \Diff(X)$ which are homotopically trivial. The latter means the following 2 conditions:

(i) $f_\ast : \mathbb{H}_2(X,\mathbb{Z}) \to \mathbb{H}_2(X,\mathbb{Z})$ is the identical homomorphisms;
(ii) $f$ is isotopic to a map $f'$ which preserves some base point $x_0$ and such that $f'_\ast : \pi_1(X,x_0) \to \pi_1(X,x_0)$ is the identical homomorphism.

Recall that $\pi_1(X) \cong \pi_1(Y)$ for some surface $Y$. Denote $\text{Aut}_+(\pi_1(Y))$ the group of automorphisms of $\pi_1(Y,\gamma)$ preserving some fixed orientation of $Y$. $\text{Out}_+(\pi_1(X))$ is the group of outer automorphisms of $\pi_1(X)$, defined as the quotient $\text{Aut}_+(\pi_1(X))/\text{Conj}(\pi_1(X))$. It is isomorphic to the mapping class group $\text{Map}(Y)$ of the surface $Y$, see [Ivn, F-M, Co-Zi].

Set $\Gamma_\ast := \Diff_+(X)/\Diff_0(X)$. Let $\Gamma_W$ be the image of $\Diff_+(X)$ in $\text{Aut}(\mathbb{H}_2(X,\mathbb{Z}))$ and $\Gamma_H$ the image of $\Diff_+(X)$ in the product $\text{Aut}(\mathbb{H}_2(X,\mathbb{Z})) \times \text{Out}_+(\pi_1(X))$.

Theorem 3. Let $(X,\omega)$ be a ruled symplectic manifold of genus $g \geq 1$ and $\Omega(X,\omega)$ the space of symplectic form cohomologous to $\omega$.

i) The group $\Diff_+(X)$ acts transitively on $\Omega(X,\omega)$. The group $\Gamma_\ast$ acts simply transitively on the set $\pi_0\Omega(X,\omega)$ of connected components of $\Omega(X,\omega)$. Moreover, the group $\Gamma_\ast$ remains unchanged after symplectic blow-ups of $X$.

ii) The group $\Gamma_H$ is isomorphic to the product $\Gamma_W \times \text{Map}(Y)$.

iii) The group $\Gamma_\ast$ is a semidirect product $\Gamma_\ast \rtimes \Gamma_H$.

iv) The secondary Stiefel-Whitney class induces a surjective homomorphism $\tilde{w}_2 : \Gamma_\ast \to H^1(X,\mathbb{Z}_2) \cong \mathbb{Z}_2^{2g}$.

A description of the group $\Gamma_W$ is given in Proposition 2.4. Here we only mention that this is a Coxeter-Weyl group of type $D_f$.

In view of Theorem 3 the complete classification of deformation classes of symplectic reduces to an intriguing question about the structure of the the kernel $\ker(\tilde{w}_2 : \Gamma_\ast \to \mathbb{Z}_2^{2g})$ which we denoted by $\Gamma_\bullet$. In this connection we notice that in the case of homeomorphisms of $S^2 \times S^2$ the corresponding group is trivial, see [Qui] and [Per].
2.4. Structure of the diffeotopy group $\Gamma_+(X)$.

References
$J \in \mathcal{J}_0$ is generic, then every singular fibre of $\text{pr} : X \to Y$ consists of two exceptional spheres meeting transversally in a single point.

1.2. Ruled symplectic 4-manifold. For the purpose of this paper we need a generalisation the standard definition of rulings on 4-manifolds admitting singular fibres.

**Definition 1.1.** A **holomorphic ruling** on a complex surface $X$ with the complex structure $J$ is a proper holomorphic projection $\text{pr} : X \to Y$ on a complex curve $Y$ for which there exists a holomorphic bundle $\overline{\text{pr}} : \overline{X} \to Y$ with the fibre $\mathbb{C}\mathbb{P}^1$ such that $X$ is obtained by a multiple blow-up and the projection $\text{pr} : X \to Y$ is the composition of the projection $\overline{\text{pr}} : \overline{X} \to Y$ with the natural map $p : X \to \overline{X}$.

A **ruling** on symplectic 4-manifold $(X,\omega)$ is given by a proper projection $\text{pr} : X \to Y$ on an oriented surface $Y$ such that

- there exist finitely many **singular values** $y_1^*,\ldots,y_n^*$ such that $\text{pr} : X \to Y$ is a spherical fibre bundle over $Y^0 := Y \setminus \{y_1^*,\ldots,y_n^*\}$;
- for every singular value $y_i^*$ there exists a disc neighbourhood $U_i \subset Y$ of $y_i^*$ and an integrable complex structures $J$ in $\text{pr}^{-1}(U_i)$ and $J_Y$ in $U_i$ such that the restricted projection $\text{pr} : \text{pr}^{-1}(U_i) \to U_i$ is a holomorphic ruling;
- every fibre $X_y := \text{pr}^{-1}(y)$ is a union $\bigcup_j S_j$ of spheres, called **components** of the fibre, and $\omega$ is positive on each component $S_j$ of a fibre.

A fibre $X_y = \text{pr}^{-1}(y)$ is **regular** if it has a single component and **ordinary singular** if it consists of two exceptional spheres.

The usual definition of the ruling does not allow singular fibres. Ruling with only ordinary singular fibres are a special case of **Lefschetz pencils**.

A **section** of a ruling $\text{pr} : X \to Y$ is a surface $S \subset X$ such that the restriction $\text{pr}|_S : S \to Y$ is a diffeomorphism, or a map $\sigma : Y \to X$ such that $\text{pr} \circ \sigma = \id_Y$, with the obvious bijective correspondence between these two notions.

We shall also consider the following more general **singular symplectic rulings**: the map $\text{pr} : X \to Y$ is merely continuous, there are finitely many points $y_i^*$ on $Y$ such that $\text{pr}$ is smooth outside the fibres $F_i^* := \text{pr}^{-1}(y_i^*)$, and such that there exists a homeomorphism $\Phi : X \to X$ such that

- $\Phi : X \to X$ maps every fibre (singular or not) onto itself, $\Phi$ is smooth in outside the singular fibres $F_i^*$ and on each component of those, and such that $\text{pr} \circ \Phi$ is a smooth symplectic ruling in the above sense.

In particular, every regular fibre $\text{pr}^{-1}(y)$ and every component of a singular fibre is $\omega$-symplectic.

A symplectic 4-manifold $(X,\omega)$ is **ruled** if it admits a symplectic ruling, and **rational** if $X$ is either $\mathbb{C}\mathbb{P}^2$ or a ruled 4-manifold such that the base $Y$ is the sphere $S^2$. Non-rational ruled 4-manifolds are called **irrational**. A ruled 4-manifold $(X,\omega)$ is **minimal** if it admits a ruling without singular fibres. Notice that $\mathbb{C}\mathbb{P}^2$ is the only rational non-ruled 4-manifold.

The structure of (closed) ruled symplectic 4-manifold is well understood, see [La, La-McD, McD-2, McD-1, McD-Sa-2]. In particular, there exists a topological characterisation of ruled symplectic 4-manifold.
Proposition 1.3. Let \((X, \omega)\) be a ruled symplectic manifold.

i) If \(\omega'\) is another symplectic form, then there exists a ruling \(pr' : X \to Y\) with \(\omega'\)-symplectic fibres.

ii) If moreover \(\omega'\) is cohomologous to \(\omega\), then there exists a diffeomorphism \(F : X \to X\) with \(F_*\omega = \omega'\).

The crucial technique used in the proof is Seiberg-Witten theory. It provides the existence of symplectic sphere \(F \subset (X, \omega')\) with \([F]^2 = 0\). The rest follows by “usual” Gromov’s theory: \(\text{Corollary 1.2}\) show that for generic \(\omega'\)-tamed almost complex structure \(J\) there exists a unique singular ruling \(pr_J : X \to Y\) with \(J\)-holomorphic fibres homologous to \(F\) such that all singular fibres are ordinary.

Unfortunately, such a ruling \(pr_J : X \to Y\) in never smooth in a neighbourhood of singular fibres. However, it can be regularised by means of the following construction: Take a \(C^0\)-small perturbation \(\tilde{J}\) of \(J\) which is holomorphic in a neighbourhood of singular fibres and such that singular fibres remain \(\tilde{J}\)-holomorphic. Then the Gromov compactness theorem for \(C^0\)-continuous almost complex structures (see [Iv-Sh-4]) imply that the new ruling \(pr_J : X \to Y\) will be \(C^0\)-close to the old one. This implies the existence of the homeomorphism \(\Phi : X \to X\) in the definition above transforming the old ruling in the new one. Now, since \(\tilde{J}\) is holomorphic in a neighbourhood of the “problem locus” (singular fibres), the ruling \(pr_J : X \to Y\) is smooth.

Definition 1.2. We call the above construction \text{regularisation of the ruling}.

Notice also \(\tilde{J}\) remains generic enough. Namely, with exception of fibres, every compact \(\tilde{J}\)-holomorphic curve escapes somewhere from the set \(U\) where \(\tilde{J}\) is holomorphic. Since we can deform \(\tilde{J}\) in \(X \setminus U\) arbitrarily, the transversality/genericity property of \(\tilde{J}\) is remained. In particular, the results of \(\text{Corollary 1.2}\) are valid.

Finally, let us notice that every non-minimal ruled symplectic 4-manifolds is a multiple symplectic blow-up of some \textit{minimal} ruled complex surface \(X_0\). Depending on the parity \(w_2(X_0)\), \(X_0\) is topologically either a product \(S^2 \times Y\), or a twisted product \(Y \times S^2\) which is a topologically non-trivial \(S^2\)-bundle over \(Y\).

1.3. \textbf{Inflation and symplectic blow-up.} The standard symplectic form in \(\mathbb{R}^4\) is \(\omega_{st} := dx_1 \wedge dy_1 + dx_2 \wedge dy_2\). We shall use the following results about symplectic blow-ups proved \(\text{[McD-1][McD-2]}\): 

\begin{lemma}

i) Let \((X, \omega)\) be a symplectic 4-manifold and \(B \subset (X, \omega)\) a symplectic embedding of the closed ball \(B(R) \subset (\mathbb{R}^4, \omega_{st})\) of radius \(R\). Then there exists a symplectic manifold \((X', \omega')\) and a symplectic exceptional sphere \(\Sigma \subset (X', \omega')\) with the following properties:

\begin{itemize}
  \item \((X' \setminus \Sigma, \omega')\) and \((X \setminus B, \omega)\) are naturally symplectomorphic;
  \item The area of \(\Sigma\) is \(\int_{\Sigma} \omega' = \pi R^2\), the new volume is \(\int_{X'} \omega' = \int_X \omega^2 - (\pi R^2)^2\).
\end{itemize}

ii) Let \((X, \omega_\mu)\) be a symplectic 4-manifold and \(\Sigma \subset (X, \omega_\mu)\) be an exceptional symplectic sphere with \(\int_{\Sigma} \omega_\mu = \mu\). Then there exists a symplectic manifolds \((X_0, \omega_0)\) and a symplectic embedding \(B(R) \subset (X_0, \omega_0)\) of the radius \(R = \sqrt{\mu/\pi}\) with the following properties:

\begin{itemize}
  \item \(X \setminus \Sigma\) is naturally symplectomorphic to \(X_0 \setminus B(R)\);
\end{itemize}

\end{lemma}
For any $0 < \tau < \mu$, $X$ admits a symplectic form $\omega_\tau$ which coincides with $\omega_\mu$ outside a given neighbourhood of $\Sigma$ and such that $\int_\Sigma \omega_\tau = \tau$. Moreover $(X \setminus \Sigma, \omega_\tau)$ is naturally symplectomorphic to $X_0 \setminus B(r)$ with $r := \sqrt{\tau/\pi}$.

**Definition 1.3.** We call the deformation $(X, \omega_\mu) \Rightarrow (X, \omega_\tau)$ the *defl ation* of $\Sigma$, and the symplectic surgery $(X, \omega_\mu) \Rightarrow (X_0, \omega_0)$ the *symplectic contraction* or blow-down of $\Sigma$. The inverse procedures $(X, \omega_{\tau}) \Rightarrow (X, \omega_\mu)$ and $(X, \omega, B) \Rightarrow (X', \omega', \Sigma)$ are called inflation and *symplectic blow-up*.

Notice that our notion of inflation(deflation) resembles but does not coincide with that used in [La-McD]. In both cases one changes the cohomology class $[\omega]$ by an explicit construction which increases the volume, but resulting deformations of the class $[\omega]$ are different.

iii) For any ruled symplectic 4-manifold $(X, \omega)$ there exist a collection $\{E_i\}$ of disjoint exceptional symplectic spheres whose contraction yields a minimal ruled symplectic 4-manifold $(X_0, \omega_0)$.

**1.4. Deformation classes of symplectic structures.** Recall that $\Omega(X, \omega)$ denotes the space of symplectic forms $\omega'$ representing the class $[\omega]$. Let $\Omega_0(X, \omega)$ be the component containing the form $\omega$. We start with the following

**Lemma 1.5.** i) The group $\text{Diff}_0$ acts transitively on $\Omega_0(X, \omega)$.

ii) Let $\omega'_t$ and $\omega''_t$ be two families of symplectic forms on $X$ depending smoothly on $t \in [0, 1]$ such that $\omega'_0 = \omega''_0$ and such that $[\omega'_t] = [\omega''_t] \in H^2(X, \mathbb{R})$. Then there exists a family of diffeomorphisms $F_t$ depending smoothly on $t \in [0, 1]$ such that $F_0 = \text{id}_X$ and such that $\omega''_t = F_t^* \omega'_t$ for each $t$. In particular, the forms $\omega'_t, \omega''_t$ lie in the same component of $\Omega(X, \omega'_t)$.

iii) The set $\pi_0(X, \omega)$ parametrises the deformation classes of symplectic structures on $X$. Moreover, if $(X', \omega')$ is obtained from $(X, \omega)$ by symplectic blow-up or contraction of exceptional symplectic spheres, then $\pi_0(X, \omega)$ and $\pi_0(X', \omega')$ are naturally identified.

iv) Let $D_\omega(X) \subset H^2(X, \mathbb{R})$ be the set of the 2-cohomology classes represented by symplectic forms defining a given orientation and a given positive cone in $H^2(X, \mathbb{R})$. Then $D_\omega(X)$ is contractible.

**Remark.** In [McD-2] McDuff constructed a manifold admitting pairs of symplectic forms $\omega', \omega''$ which are cohomologous and deformationally equivalent but nevertheless not isomorphic.

**Proof.** i)--ii). These facts follow easily by Moser’s technique.

iii) The stability of $\pi_0(X, \omega)$ under small deformations of $\omega$ follows from the previous lemma. **Proposition 1.3** ii), ensures a global stability in the “monodromic” form: For any deformation path $\omega_t$ of symplectic structures no components of $\Omega(X, \omega_t)$ appear or disappear. Hence the sets $\pi_0(X, \omega_t)$ form a non-ramified covering of the parameter path $t \in I = [0, 1]$. The triviality of the monodromy along closed paths follows from iv).

Observe any two symplectic balls in $(X, \omega)$ of the same sufficiently small radius are symplectically isotopic. It follows that that two isotopic symplectic forms remain isotopic after a blow-up, provided the involved symplectic balls have the same small enough radius. The uniqueness of symplectic blow-down proved in [McD-2], **Theorem 1.1**, shows that
isotopic symplectic structure remain isotopic after contraction homologous exceptional spheres.

iv) Let $\omega, \omega'$ be two cohomologous symplectic forms on $X$. Fix a basis $\{E_i\}$ of integer homology group $H_2(X, \mathbb{Z})$. Perturbing the forms slightly we may assume that their periods $\int_{E_i} \omega = \int_{E_i} \omega'$ are rationally independent. By [La-McD, La] there exists a diffeomorphism $F : X \to X$ with $F_* \omega = \omega'$. The condition on periods ensure that $F : H_2(X, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ is trivial. It flows that if the class $[E] \in H_2(X, \mathbb{Z})$ is represented by an $\omega$-symplectic exceptional sphere, then it contains an $\omega'$-symplectic exceptional sphere.

Now let $\eta_\alpha$ be a family of cohomological classes parametrised by a sphere $s \in S^k$, such that each class is represented by a symplectic form. Find a sufficiently fine covering $U_\alpha$ of $S^k$ such that there exist families of symplectic forms $\omega_{\alpha,s}$, each parametrised by $s \in U_\alpha$ and representing the class $\eta_\alpha$. For a fixed value $s_0 \in S^k$, find a maximal collection of $\omega_{\alpha,s_0}$-symplectic exceptional spheres $E_1(s_0), \ldots, E_\ell(s_0)$. Refining the covering if needed, we may assume that there exists a families $J_{\alpha,s}$ of $\omega_{\alpha,s}$-tamed almost complex structures with the following property: For each $s \in U_\alpha$ there exists an exceptional $J_{\alpha,s}$-holomorphic sphere $E_{i,\alpha,s}$ homologous to $E_i$. Now deflate each sphere $E_{i,\alpha,s}$ forming a collection of families of symplectic forms $\omega_{\alpha,s,t}$, such that the areas $\langle \omega_{\alpha,s,t}, [E_i] \rangle$ remain dependent of $\alpha$. This is possible in view of Lemma 1.4. The above construction shows that each element $\psi \in \pi_k(D_\omega(X))$ is represented by a family $\eta_\alpha$ such that all periods $\langle \eta_\alpha, [E_i] \rangle$ are sufficiently small. It follows that the groups $\pi_k(D_\omega(X))$ are isomorphic to the group $\pi_k(D_\omega(X_0))$ where $X_0$ is the minimal ruled symplectic manifolds $(X_0 = \mathbb{CP}^2$ in the rational case) obtained by contraction all symplectic spheres $E_i$ above.

The set $D_\omega(\mathbb{CP}^2)$ is the positive ray. In the irrational case the set $D_\omega(X_0)$ was described in [McD-2, La], it is also contractible. This implies the assertion iv).

1.5. Symplectic Dehn twists in dimension 4. We refer to [Sei] for a detailed description of symplectic Dehn twists. In this subsection we simply give a definition and list some properties.

Let $\Sigma$ be a Lagrangian sphere in a symplectic 4-manifold $(X, \omega)$. Then there exists a symplectomorphism $T_\Sigma : X \to X$, called symplectic Dehn twist along $\Sigma$ which has the following properties:

- $T_\Sigma$ is supported in a given neighbourhood of $\Sigma$;
- $\Sigma$ is invariant with respect to $T_\Sigma$ and $T_\Sigma$ acts on $\Sigma$ as the antipodal map. In particular, $(T_\Sigma)_* [\Sigma] = -[\Sigma]$ and the action of $T_\Sigma$ on the homology groups $H_2(X, \mathbb{Z}), H_2(X, \mathbb{R})$ is the reflection with respect to the hyperplane in $H_2(X, \mathbb{R})$ orthogonal to $[\Sigma]$

\begin{equation}
[A] \in H_2(X, \mathbb{R}) \to [A] + ([A] \cdot [\Sigma]) [\Sigma].
\end{equation}

- The symplectic isotopy class of $T_\Sigma$ is defined uniquely.
- $T_\Sigma^2$ is smoothly isotopic to the identity.
- Assume that $\Sigma' \subset (X, \omega)$ is a symplectic $(-2)$-sphere. Then there exists a deformation $\omega'$ of $\omega$ supported in a given neighbourhood of $\Sigma'$ such that $\Sigma'$ is $\omega'$-Lagrangian. In particular, $T_{\Sigma'}$ is a well-defined diffeomorphism.

**Lemma 1.6.** Let $(X, \omega)$ be a symplectic 4-manifold and $E_1, E_2$ two disjoint symplectic exceptional spheres. Then there exists a deformation $\omega'$ of $\omega$ and an $\omega'$ symplectic sphere $\Sigma$ representing the homology class $[E_1] - [E_2]$. 
Remark. Observe that the Dehn twist along $\Sigma$ interchanges the classes $[E_1],[E_2]$.

Proof. Blow-down the sphere $E_2$. Let $(X^*,\omega^*)$ be the arising manifold, $B_2$ the arising symplectic ball, and $E^*_1$ the image of $E_1$ in $X^*$. Move the centre of $B_2$ into a point $p^*_2$ lying on $E^*_1$ and then blow-up back using a small symplectic ball centred at $p^*_2$. The obtained manifold $(X',\omega')$ is naturally diffeomorphic to $X$ and the form $\omega'$ is isotopic to the form obtained from $\omega$ by an appropriate deflation of $E_2$. The proper preimage of $E^*_1$ in $(X',\omega')$ is the desired $(-2)$-sphere $\Sigma$. \hfill $\square$

Starting from this point we use the language and certain well-known facts from complex geometry.

Consider the family of quadrics $Z_{\lambda} = \{z = (z_1,z_2,z_3) : z_1^2 + z_2^2 + z_3^2 = \lambda\}$ in $\mathbb{C}^3$, parametrised by the unit disc $\Delta := \{|\lambda| < 1\}$. Then $Z_{\lambda}$ are smooth for $\lambda \neq 0$ and $Z_0$ is the standard quadratic cone in $\mathbb{C}^3$. The vertex 0 of $Z_0$ is a singular point of type $A_1$ and its desingularisation by means of (holomorphic) blow-up yields the holomorphic cotangent bundle $T^*\mathbb{C}P^1$. Moreover, if $C$ is a holomorphic rational $(-2)$-curve on a smooth complex surface $X$, and $U$ its neighbourhood, then one can contract $C$ into a singular point such that the obtained complex space $U'$ is isomorphic a neighbourhood of the vertex 0 of $Z_0$.

Further, for $\lambda = \rho e^{i\theta}$ with $\rho > 0$ the quadric $Z_{\lambda}$ contains the Lagrangian sphere $\Sigma_{\lambda}$ given by $\Sigma_{\lambda} := \{e^{i\theta}/2(x_1,x_2,x_3) : x_1^2 + x_2^2 + x_3^2 = \rho\}$ where $(x_1,x_2,x_3) \in \mathbb{R}^3$. It is not difficult to show that the Lefschetz monodromy of the family $Z_{\lambda}$ around the origin $0 \in \Delta$ is exactly the Dehn twist along the Lagrangian sphere $\Sigma_{\lambda}$.

Now consider the following situation. Let $\Delta'$ be the unit disc with the coordinate $w'$, $X'$ the product $\Delta' \times \mathbb{C}P^1$, and $\varphi' : X' \to \Delta'$ the natural holomorphic ruling. Blow up a point on the central fibre $\{0\} \times \mathbb{C}P^1$. Denote the obtained manifold by $X''$ and the obvious induced ruling by $\varphi'' : X'' \to \Delta'$. Then the new central fibre is the union of two exceptional rational holomorphic curves, say $E_1$ and $E_2$. Further, in a neighbourhood of the nodal point of the central fibre there exist local complex coordinates $z_1,z_2$ in which $\varphi''$ is given $(z_1,z_2) \mapsto w' = z_1^2 + z_2^2$. Take another unit disc $\Delta$ with the coordinate $w$ and consider the covering $f : \Delta \to \Delta'$ given by $w \mapsto w = w^2$. Let $f^*X'' =: X'''$ be the pull-back of $X''$ and $\varphi''' : X''' \to \Delta$ the induced projection. The relations $w' = w^2$ and $w' = z_1^2 + z_2^2$ imply that $X'''$ contains a singular point of type $A_1$, say $p^*$, such that the functions $w,z_1,z_2$ generate the local ring $\mathcal{O}_{X''',p^*}$ and satisfy the relation $w^2 - z_1^2 - z_2^2 = 0$. In other words, $w^2 - z_1^2 - z_2^2 = 0$ is a local equation for $X'''$ at $p^*$. Furthermore, the obvious induced projection $\varphi''' : X''' \to \Delta$ is given locally by $(z_1,z_2) \mapsto \pm\sqrt{z_1^2 + z_2^2}$. Let $X_0$ be the desingularisation of $X'''$, $\Sigma$ the arising holomorphic $(-2)$-sphere, and $\varphi_0 : X_0 \to \Delta$ the induced ruling. It has a unique singular fibre which is a chain of curves $E_1,\Sigma,E_2$. On the other hand, let $X_{\lambda}$ be the deformations of $X'''$ given in a neighbourhood of the singular point $p^*$ by the equation $w^2 - z_1^2 - z_2^2 = \lambda$, $\lambda \in \Delta$. Then $X_{\lambda}$ admit rulings $\varphi_{\lambda} : X_{\lambda} \to \Delta$ locally given by $(w,z_1,z_2) \mapsto w$. In particular, for $\lambda \neq 0$ the ruling $\varphi_{\lambda}$ has two ordinary singular fibre over the singular values $w_{\pm}(\lambda) := \pm\sqrt{\lambda}$.

Braid groups. For the definition and basic properties of braid group $Br_d$, pure braid group $P_d$ we refer to [Bir]. For the definition of the usual 2-dimensional Dehn twist $T_\delta$ along an embedded circle $\delta$ on a surface $Y$ we refer to [FM-MJ [IY]].

Below we shall make use of the following facts: (i) In one of its geometric realisations, $Br_d$ is the fundamental group of the configuration space of $d$ pairwise distinct points on a disc $\Delta$ or in the complex plane $\mathbb{C}$. Let $w_1^*,\ldots,w_d^* \in \Delta$ be a basic configuration. (ii)
In this realisation, the standard generators of $\text{Br}_d$ are so called **half-twists** $\tau$ exchanging exactly two points $w_0^*$, $w_1^*$. (iii) The square $\tau^2$ of such a generator $\tau$ is a Dehn twist $T_\delta$ in $\Delta$ along an embedded closed curve $\delta$ surrounding the points $w_0^*$, $w_1^*$ and no other points. Such Dehn twists $\tau^2 = T_\delta$ form a system of generators of the pure braid group $\mathbb{P}$.

**Theorem 1.7.** i) The monodromy $\Phi$ of the above family $(X_\lambda, \text{pr}_\lambda)$ along the path $|\lambda| = \rho > 0$ act on $X_\rho$ as the Dehn twist along $\Sigma$ and on the critical values $w_\pm := \pm \sqrt{\rho}$ as a geometric half-twist in the braid group $\text{Br}_2(\Delta)$.

In particular, the monodromy interchanges the fibres over $+\sqrt{\rho}$ and $-\sqrt{\rho}$.

ii) Let $\delta \subset \Delta$ be a closed curve surrounding the points $w_\pm = \pm \sqrt{\rho}$, $A$ an annual neighbourhood of $\delta$ disjoint from $w_\pm$, and $f : \Delta \to \Delta$ its geometric realisation of the Dehn twist $T_\delta$ supported in $A$. Trivialise the set $\text{pr}_\rho^{-1}(A) \cong A \times \mathbb{C}\mathbb{P}^1$ and define the map $F : X_\rho \to X_\rho$ setting $F := f \times \text{id}_{\mathbb{C}\mathbb{P}^1}$ in $\text{pr}_\rho^{-1}(A)$ and $F := \text{id}$ outside. Then $F$ is smoothly isotopic to the square $\Phi^2$ of the monodromy $\Phi$ above and thus smoothly isotopic to the identity map $\text{id} : X_\rho \to X_\rho$.

**Proof.** i) follows directly from the construction of the family.

ii) Let us make a geometric analysis of the problem. As we show below, changing coordinates one can realise the Dehn twist $T_\delta$ as follows: Instead of points $w_\pm = \pm \sqrt{\rho}$ we take new $w_0 := 0$, the centre of $\Delta$, and $w_1 := \rho$, letting $w_1$ make one turn around $w_0$. The parameter expression for this turn is is given by $w_1(t) = \rho e^{2\pi it}$. The equation in coordinates is $(w - \lambda)^2 = \lambda^2$. Notice that the appearance of $\lambda^2$ instead of $\lambda$ expresses the fact that we take the full twist which is the square of the half-twist. Notice also that the square of the Dehn twist along $\Sigma$ is given by a local equation $(w - \lambda)^2 - z_1^2 - z_2^2 = \lambda^2$. However, this argument is not correct: The Dehn twist $T_\Sigma$ is supported in a neighbourhood of the curve $\Sigma$ which is 2-dimensional, whereas the map $F$ is supported in the set $A \times \mathbb{C}\mathbb{P}^1$ which is a neighbourhood of a 3-dimensional set $\delta \times \mathbb{C}\mathbb{P}^1$.

The idea of the proof is to reverse certain part of $F$ by means of an appropriate isotopy such that the obtained will have substantially smaller support. First, we rearrange our geometric objects. Recall that we changed our basic constellation to points $w_0 :=$ which will be constant and $w_1 := \rho$ which will move. As the new curve $\delta$ we take the circle $|w| = \rho + 2\varepsilon$ with sufficiently small $\varepsilon$. Choose a function $\chi_1(r)$ such that $0 \leq \chi_1(r) \leq 1$, $\chi_1(r) \equiv 1$ for $r \leq \rho + \varepsilon$, and $\chi_1(r) \equiv 0$ for $r \geq \rho + 4\varepsilon$. Then the Dehn twist $T_\delta$ can be realised by a map $f_1 : \Delta \to \Delta$ given in polar coordinates $(r, \theta)$ by the formula $f_1(r, \theta) = (r, \theta + 2\pi \chi_1(r))$. The geometric meaning of $f_1$ is clear: we make the full rotation on the disc $\{|w| \leq \rho + \varepsilon\}$ leaving fixed the annulus $\{\rho + 4\varepsilon \leq |w| \leq 1\}$. This description can be viewed via the family $f_{1,t}(r, \theta) := (r, \theta + t \cdot 2\pi \chi_1(r))$.

Now choose another function $0 \leq \chi_2(r) \leq 1$ such that $\chi_2(r) = \chi_1(r)$ for $\rho + \varepsilon \leq r \leq \rho + 3\varepsilon$ and $\chi_2(r) \equiv 0$ for $r \leq \rho - \varepsilon$. Then the map $f_2 : \Delta \to \Delta$ given by $f_2(r, \theta) = (r, \theta + 2\pi \chi_2(r))$ is isotopic to $f_1$ relative boundary $\partial \Delta$ and the points $w_0, w_1$. Thus $f_2$ is another geometric realisation of the Dehn twist $T_\delta$. The geometric meaning of $f_2$ is as follows: we make the full rotation of the circle $\{|w| = \rho\}$ leaving fixed the annulus $\{\rho + 4\varepsilon \leq |w| \leq 1\}$ and the disc $\{|w| \leq \rho - \varepsilon\}$, and as in the case of $f_1$, the family $f_{2,t}(r, \theta) := (r, \theta + t \cdot 2\pi \chi_2(r))$ illustrates the interpretation.

Next, we want to make a similar rearrangement in $\mathbb{C}\mathbb{P}^1$-direction. For this purpose let us realise $X_\rho$ as the blow-up of the product $\Delta \times \mathbb{C}\mathbb{P}^1$ in two points $p_0$ and $p_1$ lying on
the fibres $C_0 := \{w_0\} \times \mathbb{CP}^1$ and $C_1 := \{w_1\} \times \mathbb{CP}^1$. The exact position of points plays no role. Let $z$ be a complex projective coordinate on $\mathbb{CP}^1$ such that $z = 0$ at $p_1$. Fix a function $0 \leq \eta(z) \leq 1$ such that $\eta(z) \equiv 1$ in the disc $\{|z| \leq \varepsilon\}$ and $\eta(z) \equiv 0$ outside the disc $\{|z| \geq 2\varepsilon\}$ where as above $\varepsilon$ is sufficiently small. Define the family of maps $F_{2,t}: \Delta \times \mathbb{CP}^1 \to \Delta \times \mathbb{CP}^1$ setting

$$F_{2,t}(r,\theta;z) := (f_{2_t}(\eta(z) + 1 - t) \cdot 2\pi \chi_2(r); z).$$

Then $F_{2,t}$ is an isometry of diffeomorphisms relative points $p_0, p_1$, $F_{2,t}$ are identical in a neighbourhood of $p_0, p_1$, and for $t = 0$ we have $F_{2,0} = f_2 \times \text{id}_z$. The geometric structure of $F_{2,1}$ is as follows. Some small neighbourhood of $p_1$ turns around the fibre $C_0 = \{w_0\} \times \mathbb{CP}^1$ along the circle $\{|w| = \rho\} \times \{p_1\}$.

Finally, define the map $F_2 : X_\rho \to X_\rho$ as the blow-up of the map $F_{2,1}: \Delta \times \mathbb{CP}^1 \to \Delta \times \mathbb{CP}^1$ at points $p_0, p_1$. The geometric picture of $F_2$ is reformulation of that for the map $F_{2,1}$. It has also another description: It is the monodromy map of the family of blow-ups of $\Delta \times \mathbb{CP}^1$ at two points in which the point $p_0$ lies on the central fibre $C_0 = \{w_0\} \times \mathbb{CP}^1$ and $p_2$ turns around $C_0$. But this is exactly the family $X_\lambda$ above, so $F_2 : X_\rho \to X_\rho$ is smoothly isotopic to the squared symplectic Dehn twist $\Phi^2$.

2. Secondary Stiefel-Whitney class and symplectic structures.

2.1. Secondary Stiefel-Whitney class. Let $X$ be a (connected!) smooth manifold and $f : X \to X$ a diffeomorphism which acts trivially in $H^*(X,\mathbb{Z}_2)$. Denote $I := [0,1]$. Define the map torus $T(X,f)$ as the quotient of $X \times [0,1]$ under the identification $(x,1) \sim (f(x),0)$. In other words, $T(X,f)$ can be realised as a fibre bundle over $S^1$ with fibre $X$ and monodromy $f$. The condition on $H^*(X,\mathbb{Z}_2)$ implies the Künneth formula $H^*(T(X,f),\mathbb{Z}_2) \cong H^*(X,\mathbb{Z}_2) \otimes H^*(S^1,\mathbb{Z}_2)$. Define the class $[S^1] \in H_1(T(X,f),\mathbb{Z}_2)$ as the image of the fundamental class of the circle $[S^1]$ under the Künneth isomorphism.

**Definition 2.1.** In the above situation, the relative secondary Stiefel-Whitney class $\tilde{w}_2(f)$ is defined as the $/$-product $w_2(T(X,f))/[S^1]$. This is an element of $H^1(X,\mathbb{Z}_2) \otimes H^0(S^1,\mathbb{Z}_2) \cong H^1(X,\mathbb{Z}_2)$.

Let us consider some trivial properties of the class $\tilde{w}_2$.

**Lemma 2.1.** i) (Restriction formula) Let $Z \subset X$ be a connected cooriented hypersurface and $f : X \to X$ a diffeomorphism stabilising $Z$, preserving its coorientation, and acting trivially on $H^*(X,\mathbb{Z}_2)$ and $H^*(Z,\mathbb{Z}_2)$. Then $\tilde{w}_2(f|_Z) = \tilde{w}_2(f)|_Z$.

ii) Let $\gamma := S^1$ be a circle and $X_\gamma := \gamma \times S^2$ an $S^2$-bundle. Then there exists two isotopy classes of fibrewise trivialisations $\varphi : X_\gamma \xrightarrow{\cong} \gamma \times S^2$. Moreover, fibrewise map $F : X_\gamma \to X_\gamma$ identical on $\gamma$ and preserving the orientation of the fibre interchanges the isotopy classes of fibrewise trivialisations if and only if $\tilde{w}_2(F) \neq 0$.

**Remark.** Let us note in this connection that $X_\gamma$ admits exactly two non-isomorphic spin structures, and $F : X_\gamma \to X_\gamma$ as above exchanges spin structures if and only if $\tilde{w}_2(F) \cdot [\gamma] \neq 0$, i.e., when $F$ exchanges also the classes of trivialisations. The proof is omitted since we shall not use this fact.

**Proof.** i) follows directly from the definition.
The group $\mathcal{D}iff_+(S^2)$ is homotopy equivalent to $SO(3)$. Since $\pi_1(SO(3)) \cong \mathbb{Z}_2$ there exist two isotopy classes of trivialisations $\varphi : X_\gamma \cong \gamma \times S^2$, and any fibrewise diffeomorphism $F : X_\gamma \rightarrow X_\gamma$ as in the hypotheses either preserves or interchanges these classes of trivialisations $\varphi : X_\gamma \cong \gamma \times S^2$. In the first case $T(F,X_\gamma)$ is diffeomorphic to $T^2 \times S^2$ and $w_2(T^2 \times S^2) = 0$, in the second case $T(F,X_\gamma)$ is the non-trivial $S^2$-bundle $T^2 \times S^2$. This manifold is diffeomorphic to a complex minimal ruled surface $Z$ with a ruling $pr : Z \rightarrow E$ over a complex elliptic curve $E$ admitting disjoint holomorphic sections $S_1, S_{-1}$ of self-intersection +1 and respectively −1. The first Chern class of $Z$ is Poincaré dual to $[S_1] + [S_{-1}]$, and its reduction modulo 2 gives $w_2(Z) = w_2(T^2 \times S^2)$. In particular, the evaluation of $w_2(Z)$ on each section is always non-trivial. As a consequence, we conclude that $F : X_\gamma \rightarrow X_\gamma$ as above exchanges the classes of trivialisations if and only if $\tilde{w}_2(F) \cdot [\gamma] \neq 0$. Here $[\gamma] \in H_1(\gamma, \mathbb{Z}_2) \cong H_1(X_\gamma, \mathbb{Z}_2)$.

Now we prove the easier second assertion of Theorem 1.

**Proposition 2.2.** Let $(X, \omega)$ be an irrational ruled symplectic 4-manifold and $pr : X \rightarrow Y$ some singular symplectic ruling. Then there exist diffeomorphisms $F : X \rightarrow X$ with any prescribed value $\tilde{w}_2(F)$ which acts trivially on $\pi_1(X)$ and $H_2(X, \mathbb{Z})$.

**Proof.** First, we construct a map $F : X \rightarrow X$ with a given $\tilde{w}_2(F)$ for $X_0 = Y \times S^2$. Any class $\alpha \in H^1(Y, \mathbb{Z}_2) \cong H^1(X_0, \mathbb{Z}_2)$ is Poincaré dual to an embedded curve $\delta \subset Y$. Let $\varphi(t), t \in [1, 2]$ be a smooth loop in $SO(3)$ representing the non-trivial element in $\pi_1(SO(3))$ and such that $\varphi(1) = \varphi(2) = 1$. Find a neighbourhood $A$ of $\delta \subset Y$ isomorphic to an annulus. Let $\theta \in [0, 2\pi]$ and $r \in [1, 2]$ be polar coordinates in $A$. Define the map $F : X_0 \rightarrow X_0$ setting $F(y,s) := (y,s)$ if $x \notin A$ and $F(r, \theta; s) := (r, \theta; \varphi(r)(s))$ in the case $y = (r, \theta) \in A$. Then $\tilde{w}_2(F) \cdot [\gamma] \equiv [\delta] \cdot [\gamma] \mod 2$ every embedded curve $\gamma \subset Y$. Moreover, $F$ acts trivially on $\pi_1(X)$ and $H_2(X, \mathbb{Z})$.

The case $X_1 := Y \times S^2$ can be treated similarly. The general case follows from the blow-up construction and Proposition 1.3.

### 2.2. Proof of Theorem 1
In this paragraph we assume that the hypotheses of the theorem are fulfilled.

The group $SO(3)$, and hence the group $\mathcal{D}iff_+(S^2)$, are homotopy equivalent to the group $PGL(2, \mathbb{C})$. The group $PGL(2, \mathbb{C})$ acts simply transitively on the space $M_3(\mathbb{CP}^1)$ of triples $(z_1, z_2, z_3)$ of pairwise distinct points on $S^2 = \mathbb{CP}^1$, so that varieties $PGL(2, \mathbb{C})$ and $M_3(\mathbb{CP}^1)$ are isomorphic. Moreover, for any complex structure $J$ on $S^2$ and any triple $p_0, p_1, p_2$ there exists unique $J$-holomorphic isomorphism $\psi : (S^2, J) \rightarrow \mathbb{CP}^1$ with $\psi(p_0) = 0, \psi(p_1) = 1$ and $\psi(p_2) = \infty$. Consequently, if $J$ is an almost complex structure on $X$, $pr : X \rightarrow Y$ a $J$-holomorphic ruling, and $s_0, s_1, s_2 : Y \rightarrow X$ three sections of the ruling, then the fibre $pr^{-1}(y)$ is equipped a distinguished trivialisations provided $y$ is not a critical value of the ruling and the points $s_0(y), s_1(y), s_2(y) \in pr^{-1}(y) \cong S^2$ are distinct.

The main idea of the proof is to construct families $(J_t; pr_t; s_1(t), s_2(t), s_3(t))$ of such structures and then to transform $X$ removing all “bad” points.

Assume that $X$ is not minimal and contains exceptional symplectic spheres. Let $g := g(X) = g(Y)$ be the genus of $X$. Equip $Y$ with some generic complex structure. Consider a generic holomorphic line bundle $L$ of degree $g$. Let $X_0$ be the fibrewise projective completion of the total space of $L$. Then $X_0$ is the projectivisation of the rank-2
holomorphic bundle $L \oplus \mathcal{O}_Y$. Let $pr : X_0 \to Y$ be the corresponding ruling, $Y_g$ the zero section of $pr : X_0 \to Y$, and $Y_{-g}$ the infinity section. Then $|Y_{\pm g}|^2 = \pm g$. Blowing up $X_0$ in $\ell$ generic points $x^*_i, \ldots, x^*_\ell$ we obtain a manifold diffeomorphic to $X$. This gives an integrable complex structure $J^*$ on $X$. It could occur that this structure is not compatible with any symplectic form $\omega'$ cohomologous to our form $\omega$. However, Lemma 1.5 ensures that if we deform $\omega$, the deformed form $\omega^*$ admits a symplectomorphism $F^*$ isotopic to $F$ and hence having the same class $\tilde{w}_2(F) = \tilde{w}_2(F^*)$. As such a deformation of $\omega$, we deflate almost completely the exceptional symplectic spheres $E_1, \ldots, E_\ell$ obtained above and increase the volume of the base $Y$, until we achieve the relation $\langle [\omega^*], [Y_{-g}] \rangle > 0$. Now the structure $J^*$ would be tamed by some symplectic form of the cohomology class $[\omega^*]$. Without loss of generality we may assume that $J^*$ is tamed by $\omega$.

By the choice of $L$, $H^0_{\mathbb{R}}(Y,L) \cong \mathbb{C}$ and $H^1_{\mathbb{R}}(Y,L) = 0$. Consequently, $X_0$ admits complex 1-dimensional family of holomorphic sections of the ruling $pr : X_0 \to Y$. Fix a generic point $y_0 \in Y$ and three generic points $x_0, x_1, x_2$ on the fibre $pr^{-1}(y_0)$. Now choose a generic almost complex structure $J_0$ sufficiently close to $J^*$ and set $J_1 := F^*_\ell(J_0)$. Then $J_1$ is also an $\omega$-tamed almost complex structure, and so there exists a path $J_t$ of $\omega$-tamed structures connecting $J_0$ and $J_1$. We choose the path $J_t$ to be generic enough. Let $pr_t : X \to Y$ be the induced family of $J_t$-holomorphic rulings. We also assume that the family of structures was regularised (see Definition 1.2) so that the maps $pr_t$ are smooth. Further, we also assume that the base spaces of the rulings $pr_t$, which are apriori different, is the same reference surface $Y$ such that the maps $pr_t : X \to \text{depend smoothly on } t$. We also assume that $pr_0 \circ F = f \circ pr_0$ for some diffeomorphism $f : Y \to Y$.

Let $y^*_i(t), \ldots, y^*_\ell(t)$ be the images of singular fibres of $pr_t : X \to Y$, and $y^*(t)$ the whole set. Further, choose a generic family $y_0(t)$ of points on $Y$, and three generic families $x_0(t), x_1(t), x_2(t)$ of points lying on the fibre $pr_t^{-1}(y_0(t))$. Consider the moduli space $\mathcal{M}$ of quadruples $(t; C_0, C_1, C_2)$ such that $t \in [0, 1]$ and $C_i$ are irreducible $J_t$-holomorphic curves in the homology class $[Y_g]$ each passing through the corresponding point $x_i(t)$. Then $\mathcal{M}$ is a smooth manifold of expected dimension $\dim_{\mathbb{R}} \mathcal{M} = 1$ equipped with the natural projection $\pi_\mathcal{M} : \mathcal{M} \to I := [0, 1]$. The choice of the involved data ensures that for a given $t$ close to 0 or to 1 there exists a unique triple $(C_0(t), C_1(t), C_2(t))$ in the fibre $\pi_\mathcal{M}^{-1}(t)$.

Now let us observe that $\mathcal{M}$ is compact and $\pi_\mathcal{M} : \mathcal{M} \to I$ is proper. Indeed, if one of the curves $C_i(t)$ breaks, then the expected dimension of the arising constellation would be negative. This eventuality is prohibited by the generic choice of the path $J_t$. It follows that $\mathcal{M}$ contains a unique component $\mathcal{M}^*$ which is an interval, whereas all remaining components are circles whose $\pi_\mathcal{M}$-projections lie in some proper subinterval $[\varepsilon, 1 - \varepsilon]$, $\varepsilon > 0$. Choose a parameter (coordinate) $\tau$ on $\mathcal{M}^*$ varying in the interval $I = [0, 1]$. Then the dependence of the old parameter $t$ on $\tau$ is established by the projection $\pi_\mathcal{M}$. We write this dependence in the form $J_\tau$ instead of formally correct $J_{\pi_\mathcal{M}(\tau)}$ and so on. This gives us the desired family of structures $(J_\tau, pr_\tau; C_0(\tau), C_1(\tau), C_2(\tau))$.

The genericity condition implies that for each $\tau$ every pair of the curves $C_i(\tau), C_j(\tau)$ has $g$ transversal intersection points which depend smoothly on $\tau$. In this way we obtain $3g$ families $x^*_j(\tau)$ of nodal points. We denote by $y^*_j(\tau) := pr_\tau(x^*_j(\tau))$ their projections on $Y$, they are pairwise distinct. Let $x^*(\tau)$ and $y^*(\tau)$ be the whole collections. Outside the points $y^*_j(\tau)$ and the images of singular fibres $y^*(\tau)$ we obtain the family of canonical trivialisations $\varphi_{\tau,y} : pr_\tau^{-1}(y) \to \mathbb{CP}^1 = S^2$. The relation between the points $y^*_j(\tau)$ and the canonical trivialisations $\varphi_{\tau,y} : pr_\tau^{-1}(y) \to S^2$ is as follows: Let $A \subset Y$ be an annulus
containing exactly one point $y_j^x(\tau)$, and no critical values $x^*(\tau)$ of the projection $\text{pt}_\tau : X \to Y$. Let $\gamma', \gamma''$ be the boundary circles of $A$. Then the classes of canonical trivialisations of $\text{pr}_\tau^{-1}(\gamma')$ and $\text{pr}_\tau^{-1}(\gamma'')$ given by $\varphi_{\tau,y}$ are opposite. Further, as $\tau$ varies, the points $y_j^x(\tau)$ move along $Y$. Those two phenomena do not allow to compare directly the homotopy classes of the canonical trivialisations for $\tau = 0$ and $\tau = 1$.

To manage this difficulty we apply an appropriate birational transformation of $X$ making certain blow-ups and blow-downs. The possibility to deform $\omega$ provided by Lemma 1.5 allows us to make arbitrarily small the area of given symplectic exceptional spheres. As the limit, we may make arbitrary symplectic contractions (blow-downs). Vice versa, we can make blow-ups at any given points provided the area of arising symplectic exceptional spheres remains small enough. In both constructions, the value $\tilde{u}_2(F)$ remains unchanged after such blow-ups or blow-downs.

Since the transformation are made along non-constant families of points/spheres, it is better to consider the total spaces of families of arising manifolds. We set $\mathcal{X} : = X \times I$ where $I = [0,1]$ is the parameter space for $\tau$ and let $\pi : \mathcal{X} \to I$ be the projection. The space $\mathcal{X}$ is equipped with the form $\omega \oplus 0$ for which we use the notation $\omega$. By the constructions, for every ordinary singular fibre $F^x_j(\tau) := \text{pr}_\tau^{-1}(y_j^x(\tau))$ all three sections $C_0(\tau), C_1(\tau), C_2(\tau)$ are disjoint from one of the component of $F^x_j(\tau)$, and we contract all these components. As the result, we obtain a new family $\mathcal{X}^\circ$ and a projection $\pi^\circ : \mathcal{X}^\circ \to I$ whose fibres $X^\circ_\tau := \pi^\circ^{-1}(\tau)$ are diffeomorphic to $X_0$. The fibres $X^\circ_\tau$ are equipped with symplectic forms $\omega^\circ_\tau$ isomorphic to the one on $X_0$. Doing $C^0$-small perturbation of the structures $J$, and making them integrable near contracted exceptional sphere, we obtain a family of $\omega^\circ_\tau$-tamed almost complex structures $J^\circ_\tau$ on $X^\circ_\tau$. This family induces the family of rulings $\text{pr}_\tau^\circ : X^\circ_\tau \to Y$. The families of $J^\circ_\tau$-holomorphic sections $C_i(\tau)$ survive without changes since they are disjoint from blow-down locus.

Next, we make small symplectic blow-up at each nodal point $x_j^x(\tau)$, such that the resulting exceptional symplectic spheres, denoted by $E^x_j(\tau)$, have the same sufficiently small area $\varepsilon > 0$. The technical details are the same as in the previous construction, and we simply describe the resulting objects and structures: We obtain a family of symplectic manifolds $(X'_\tau, \omega'_\tau)$ with the total space $\mathcal{X}'$ and the projection $\pi' : \mathcal{X}' \to I$ such that $\pi'^{-1}(\tau) = X'_\tau$. The forms $\omega'_\tau$ are restriction of a smooth 2-form $\omega'$ on $\mathcal{X}'$, possibly not closed. There exists a family $J'_\tau$ of $\omega'_\tau$-tamed almost complex structures on $X'_\tau$, and a family $\text{pr}_\tau' : X'_\tau \to Y$ of rulings with $J'_\tau$-holomorphic fibres. Also, there exist families $C_0'(\tau), C_1'(\tau), C_2'(\tau)$ of $J'_\tau$-holomorphic sections.

The singular fibres of $\text{pr}_\tau'$ are ordinary and lie over points $y_j^x(\tau)$. Moreover, one of two components of each singular fibre is the exceptional spheres $E^x_j(\tau)$. Let us denote by $E^{-x}_j(\tau)$ the other component. Then each $E^x_j(\tau)$ meets exactly two of three sections $C_0'(\tau), C_1'(\tau), C_2'(\tau)$ and $E^{-x}_j(\tau)$ the third one.

As the next step we contract all symplectic exceptional spheres $E^{-x}_j(\tau)$. Again, the technical details are the same and we simply describe the resulting spaces and structures. A new family $(X''_\tau, \omega''_\tau)$ with the total space $\mathcal{X}''$ and the projection $\pi'' : \mathcal{X}'' \to I$ with $\pi''^{-1}(\tau) = X''_\tau$ have similar properties as before. The same holds for a new family of $\omega''_\tau$-tamed structures $J''_\tau$, for the induced family of rulings $\text{pr}_\tau'' : X''_\tau \to Y$, and for new families of sections $C_0''(\tau), C_1''(\tau), C_2''(\tau)$. The main difference is that the rulings $\text{pr}_\tau'' : X''_\tau \to Y$
have no singular fibres and that the sections $C''_0(\tau), C''_1(\tau), C''_2(\tau)$ are disjoint and have vanishing self-intersection. Thus each $X''_y$ is diffeomorphic to $Y \times S^2$.

The key point in the proof is that for the boundary values $\tau = 0$ and $\tau = 1$ of the parameter all the constructions are made to be compatible with the symplectomorphism $F : X \to X$ and the arising diffeomorphisms $F' : X'_0 \to X'_1$, $F'' : X'_0 \to X''_y$. Indeed, since $F : X \to X$ transforms $(\omega, J_0, C_0(0))$ into $(\omega, J_1, C_1(1))$, one can make the blow-down construction compatible with $F$. This gives the symplectomorphism $F' : (X'_0, \omega'_0) \to (X'_1, \omega'_1)$. After adjustment of structures we achieve the properties $F'_* (J'_0) = J'_1$ and $F'(C'_0(0)) = C'_1(1)$. Doing the same at the next step, we obtain a symplectomorphism $F'' : (X''_0, \omega''_0) \to (X''_1, \omega''_1)$ satisfying $F''_* (J''_0) = J''_1$ and $F''(C''_0(0)) = C''_1(1)$.

Now we can enjoy the constructed objects and finish the proof. As we mentioned above, $\tilde{w}_2(F'') = \tilde{w}_2(F) \in H^1(Y, \mathbb{Z}_2)$. The families of structures $J''_y$, of the induced rulings $pr''_y : X''_y \to Y$, and of the sections $C''_y(\tau)$ define the family of canonical trivialisations $\varphi''_{\tau,y} : pr''_{\tau}^{-1}(y) \to S^2$, this time for each $y \in Y$. The compatibility of $F''$ with $pr''_y$ implies that $pr''_y \circ F'' = f \circ pr''_0$ for some diffeomorphism $f : Y \to Y$.

The homotopy triviality of $F : X \to X$ means that $F$ acts trivially on the fundamental group $\pi_1(X) \cong \pi_1(Y)$, or more precisely, that $F$ defines the identity element in $\text{Out}_+(\pi_1(Y))$. The same holds for $F''$, and consequently also for $f$. Since $\text{Out}_+(\pi_1(Y))$ is naturally isomorphic to $\text{Map}(Y)$, $f$ is isotopic to the identity. In particular, for any embedded curve $\gamma \subset Y$ its image $f(\gamma)$ is isotopic to $\gamma$. Finally, for $\gamma$ as above, the homotopy classes of canonical trivialisations of $X''_y := pr''_0^{-1}(y)$ and $F''(X''_y) = pr''_1^{-1}(f(\gamma))$ are equal. This proves the desired identity $\tilde{w}_2(F) = 0$. 

\section{Homotopically trivial symplectomorphisms.}

\textbf{Proposition 2.3.} Every homotopically trivial symplectomorphism $F : (X, \omega) \to (X, \omega)$ is smoothly homotopic to the identity.

\textbf{Proof.} \textit{Step 1. Deforming a symplectomorphism into a fibrewise map.} Let $F : (X, \omega) \to (X, \omega)$ be a homotopy trivial symplectomorphism. Fix a generic almost complex structure $J_0$ tamed by $\omega$ and denote $J_1 := F_* J_0$. Connect $J_0$ and $J_1$ by a path $J_t$ of generic structures tamed by $\omega$. Then there exists a family $pr_t : X \to Y$ of singular ruling which depends continuously on $t$ and such that the fibres of $pr_t$ are $J_t$-holomorphic. The singular fibres of $pr_t$ are all ordinary and depend smoothly on $t$. Correcting $J_t$ and $pr_t$ near singular fibres we may assume that all rulings $pr_t$ are smooth and depend smoothly on $t$. It follows that there exists families $F_t : X \to X$ and $f_t : Y \to Y$ of diffeomorphisms starting from $F =: F_1$ such that $pr_t \circ F_t = f_t \circ pr_0$. This construction gives us a diffeomorphism $F_0 : X \to X$ isotopic to $F$ and such that $pr_0 \circ F_0 = f_0 \circ pr_0$.

\textit{Step 2. Correction of the base map $f_0 : Y \to Y$.} Let $y_1, \ldots, y_k \in Y$ be the projections of singular fibre of $pr_0$. Since $F$ and $F_0$ act trivially on $H_2(X, \mathbb{Z})$, $F_0$ does not permute the singular fibres. This means that $y := \{y_1, \ldots, y_k\}$ are fixed points of $f_0$. In turn, this means that $f_0$ defines an element in the mapping class group $\text{Map}(Y, y)$ of isotopy classes of diffeomorphisms which fix the given points. Without loss of generality we may assume that the points $y = \{y_1, \ldots, y_k\}$ lie in a given disc $D \subset Y$. Fix a base point $y_0$ of $Y$ lying in $D$. Let $g_1, \ldots, g_k$ be a system of embedded mutually disjoint paths in $D$ connecting $y_0$ with the corresponding $y_i$. The condition of the triviality of the action of $F$ on $\pi_1(X, x_0)$ implies that each path $f_0(\tilde{g}_i)$ is homotopic relative endpoints $y_0, y_i$ to the path $\tilde{g}_i$. It
follows that \( f_0 \) is isotopic relative points \( y_0, y_1, \ldots, y_\ell \) to a diffeomorphism \( f' : Y \to Y \) which is identical outside the disc \( D \). In turn, this implies that \( F_0 \) is isotopic to a map \( F' : X \to X \) such that \( \text{pr}_0 \circ F' = f' \circ \text{pr}_0 \).

Step 3. Correction in the braid group. Since the map \( f' : Y \to Y \) identical outside the disc \( D \), it induces an element of the group \( \text{Map}(D,y) \). This group is the pure braid group \( P_\ell \) on \( \ell \) strands. It is generated by Dehn twists \( f_\delta : Y \to Y \) along closed embedded curves \( \delta \subset Y \setminus y \) surrounding exactly two points, say \( y_i, y_j \). Let \( V_\delta \) be a small bicollar neighbourhood of such a curve \( \delta \) such that \( f_\delta \) is supported in \( V_\delta \). Fix a trivialisation \( \text{pr}_0^{-1}(V_\delta) \cong V_\delta \times S^2 \). This allows us to lift \( f_\delta \) to a diffeomorphism \( F_\delta : X \to X \) which is identical outside \( \text{pr}_0^{-1}(V_\delta) \) and is of the product form \( F_\delta = f_\delta \times \text{id}_{S^2} \) inside \( \text{pr}_0^{-1}(V_\delta) \cong V_\delta \times S^2 \). By Theorem 1.7, page 9, every such diffeomorphism \( F_\delta \) is isotopic to the identity. It follows that \( F' \) is isotopic to a map \( F'' : X \to X \) such that \( \text{pr}_0 \circ F'' = f'' \circ \text{pr}_0 \) for an appropriate diffeomorphism \( f'' : Y \to Y \) which preserves the points \( y = \{y_1, \ldots, y_\ell\} \) and which defines trivial element \( [f'] = [\text{id}_Y] \) in group \( \text{Map}(Y,y) \). It should be noticed that that the isotopies between \( F_\delta \) and \( \text{id}_X \), and hence the isotopy between \( F' \) and \( F'' \), are not compatible with any ruling \( \text{pr} : X \to Y \).

So finally we conclude that our original symplectomorphism \( F : (X,\omega_0) \to (X,\omega_0) \) is smoothly isotopic to a diffeomorphism \( F''' : X \to X \) which preserves the projection \( \text{pr}_0 : X \to Y \), i.e., \( \text{pr}_0 \circ F''' = \text{pr}_0 \).

Step 4. Existence of fibrewise isotopies and \( \tilde{w}_2(F) \). Take any embedded curve \( \gamma \subset Y \) avoiding the points \( y = \{y_1, \ldots, y_\ell\} \) and denote \( X_\gamma := \text{pr}_0^{-1}(\gamma) \). Fix an orientation of \( \gamma \) and trivialisation \( \varphi : X_\gamma \cong \gamma \times S^2 \). Since \( F''' : X \to X \) acts fibrewisely, we obtain a family \( y \in \gamma \mapsto F'''|_{X_y} \) of diffeomorphisms of \( S^2 \). The group \( \text{Diff}_+(S^2) \) is homotopy equivalent to \( \text{SO}(3) \). Since \( \pi_1(\text{SO}(3)) \cong \mathbb{Z}_2 \) there exist two isotopy classes of trivialisations \( \varphi : X_\gamma \cong \gamma \times S^2 \). Moreover, the latter dichotomy is distinguished by the value \( \tilde{w}_2(F''') \cdot [\gamma] \) where the class \( [\gamma] \) is defined via the isomorphism \( H_1(X,\mathbb{Z}_2) = H_1(Y,\mathbb{Z}_2) \).

By Theorem 1 and Lemma 2.1.1 the map \( F''' : X \to X \) must preserve each class of trivialisations \( \varphi : X_\gamma \cong \gamma \times S^2 \). This implies that \( F''' \) is fibrewise isotopic to the identity map. \( \square \)

2.4. Structure of the diffeotopy group \( \Gamma_+(X) \). In this paragraph we give a proof of Theorem 3. We start with description of the group \( \Gamma_W \). Recall that this is the image of \( \text{Diff}_+(X) \) in \( \text{Aut}(H_2(X,\mathbb{Z})) \). We refer to [Bou, Hum] for information about Coxeter-Weyl groups.

**Proposition 2.4.** Let \( (X,\omega) \) be an irrational ruled symplectic 4-manifolds. Then the group \( \Gamma_W \) is isomorphic to the Coxeter-Weyl group of type \( D_\ell \). Moreover, \( \Gamma_W \) generated by Dehn twists along symplectic or Lagrangian \((-2)\)-spheres. In particular, there exists a subgroup \( G_W \) of \( \Gamma_H \) such that

(a) the homomorphism \( \Gamma_H \to \text{Aut}(H_2(X,\mathbb{Z})) \) induces an isomorphism between \( G_W \) and \( \Gamma_W \);

(b) \( G_W \) acts trivially on \( \pi_1(X) \).
Proof. Take a generic \( \omega \)-tamed almost complex structure \( J \) on \( X \). Then there exists a singular ruling \( pr : X \to Y \) with only ordinary singular fibres. Every such fibre consists of two exceptional \( J \)-holomorphic curves.

Let \( E \) be some \( \omega \)-symplectic exceptional sphere. Then by Corollary \( \ref{cor:lagrangian} \) \( E \) is isotopic to a \( J \)-holomorphic exceptional curve \( E' \). \( E' \) can not have positive intersection index with fibres of \( pr : X \to Y \) since otherwise we would have a map \( E' \to Y \) of positive degree on a surface of positive genus \( g(Y) > 0 \), which is impossible. Therefore \( E' \) must be a component of a fibre. Consequently, the components of all singular fibres of \( pr : X \to Y \) indexate all homology classes of symplectic exceptional spheres on \( X \). Notice that they appear in natural pairs. Let us denote these exceptional curves and their homology classes by \( E_1, E'_1; \ldots ; E_t, E'_t \).

Contracting curves \( E_1, \ldots , E_t \) we obtain a minimal ruled irrational symplectic manifold. It follows that any diffeomorphism \( F \in \text{Diff}_+(X) \) must permute the classes \( E_1, E'_1; \ldots ; E_t, E'_t \) preserving the pairs.

Deforming \( \omega \) if needed we can find a symplectic \((-2)\)-sphere \( \Sigma_{i,i+1} \) representing the class \( E_i - E_{i+1} \), see Lemma \( \ref{lem:unique_class} \). Then the Dehn twist along \( \Sigma_{i,i+1} \) interchanges the classes \( E_i, E_{i+1} \) and classes \( E'_i, E'_{i+1} \). Similarly, if \( \Sigma'_{i,i+1} \) is a symplectic \((-2)\)-sphere representing the class \( E_i - E'_{i+1} \), then the Dehn twist along \( \Sigma'_{i,i+1} \) interchanges the classes \( E_i, E'_{i+1} \) and classes \( E'_i, E_{i+1} \). The composition of these two Dehn twists interchanges the classes \( E_i, E'_i \) and classes \( E_{i+1}, E'_{i+1} \). This shows that the Dehn twists along spheres \( \Sigma_{i,i+1}, \Sigma'_{i,i+1} \) generate the Weyl group \( W(D_\ell) \) of type \( D_\ell \). Let us observe that there exists no diffeomorphism which interchanges the classes \( E_i, E'_i \) in exactly one pair. The reason is that the contractions of classes \( E_1, \ldots , E_t \) and respectively \( E'_1, E_2, \ldots , E_t \) yield minimal irrational symplectic manifolds of different parity, \( Y \times S^2 \) and \( Y \times S^2 \), which are not diffeomorphic.

Finally, we notice that for an appropriate form \( \omega \) the spheres \( \Sigma_{i,i+1}, \Sigma'_{i,i+1} \) can be made Lagrangian. Namely, one starts with the product form \( \omega_0 = \omega_Y \oplus \omega_{S^2} \) on \( X_0 := Y \times S^2 \) such that the volume of \( Y \) is much larger than that of sphere, and then make blow-ups and inflation of sphere \( E_1, E'_1; \ldots ; E_t, E'_t \) such that all of them have equal volume. In this case the whole group \( \Gamma_W \) is represented by symplectomorphisms. \( \square \)

Proof of Theorem \( \ref{thm:main} \).

i)–ii). By Proposition \( \ref{prop:transitivity} \) the group \( \text{Diff}_+(X) \) acts transitively on \( \Omega(X, \omega) \). Lemma \( \ref{lem:independence} \) ensures the independence of the set \( \pi_0 \Omega(X, \omega) \) of symplectic forms \( \omega \) in a given deformation class and its stability under blow-up. This allows us to use suitable symplectic forms. In particular, we may assume that \( \omega \) is generic. Then the periods of \( \omega \) are rationally independent. Consequently, any \( F \in \text{Diff}_+(X) \) preserving the cohomology class of \( \omega \) acts trivially in \( H_2(X, \mathbb{Z}) \). This means that the kernel \( \text{Ker}(\text{Diff}_+(X) \to \text{Aut}(H_2(X, \mathbb{Z}))) \) acts also transitively on \( \Omega(X, \omega) \).

Now let \( X_0 \) be a product \( Y \times S^2 \) equipped with a product symplectic form \( \omega = \omega_Y \oplus \omega_{S^2} \). Notice that any element \( \varphi \in \text{Map}(Y) \) is represented by a symplectomorphism \( f : (Y, \omega_Y) \to (Y, \omega_Y) \). Then the product \( f \times \text{id}_{S^2} \) is a symplectomorphism with the same action on \( \pi_1(X) = \pi_1(Y) \). The argument with deformations and symplectic blow-up implies that for any ruled symplectic 4-manifold \( X \) the group \( \Gamma_H \) contains a subgroup \( G \) with the following properties: (a) every element of \( G \) is represented by a symplectomorphism; (b) the homomorphism \( \Gamma_H \to \text{Out}(\pi_1(X)) \) induces an isomorphism between \( G \) and \( \text{Out}_+(\pi_1(X)) = \text{Out}_+(\pi_1(Y)) = \text{Map}(Y); \) (c) \( G \) acts trivially on \( H_2(X, \mathbb{Z}) \).
A similar result for $\Gamma_W$ was demonstrated above. This proves assertion $\text{ii).}$

$\text{iii)}$ As we have shown in the proof of Proposition 2.4, the group $\Gamma_W$ can be represented by symplectomorphisms of an appropriate symplectic form. The same property for the group $\text{Out}_+(\pi_1(X))$ was shown above. Moreover, we have also shown that these groups can be realised as subgroups of $\Gamma_+$. This implies $\text{iii).}$

$\text{iv)}$ was proved in Proposition 2.2.

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