TIME-DEPENDENT LAGRANGIANS INvariant BY A VECTOR FIELD

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Abstract

We study the reduction of non-autonomous regular Lagrangian systems by symmetries, which are generated by vector fields associated with connections in the configuration bundle of the system \( Q \times \mathbb{R} \to \mathbb{R} \). These kind of symmetries generalize the idea of “time-invariance” (which corresponds to taking the trivial connection in the above trivial bundle).

Key words: Connections, vector fields, symmetries, dynamical systems, Lagrangian formalism.

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1 Introduction

When dealing with mechanical systems coming from a regular Lagrangian, different formulations exist: If the Lagrangian is autonomous, then the symplectic formalism is the appropriate one, but if the Lagrangian depends on time, then the contact formulation is more suitable. Contact structures are built on odd dimensional manifolds, and symplectic ones on even dimensional manifolds, which are the phase space of coordinate-velocities.

But these are not the only differences between both formulations. In the symplectic case, the geometric structure is given by a non-degenerate differential 2-form, whereas in the case of the contact, the structural 2-form is degenerate and its kernel has dimension one. This kernel gives at every point the direction of the dynamical vector field.

Obviously, one may use the contact formulation for an autonomous Lagrangian. Then the symmetry given by this time independence allows us to reduce the system, making quotient by the action of time,
reducing the dimension of the manifold where the dynamical equations are written, and obtaining the symplectic formulation.

In this work we study the reduction of the contact formulation for a time dependent Lagrangian if there exists an infinitesimal symmetry of the Lagrangian.

The ideas come from [4, 12, 11, 13]. There is a description of the changes in the classical formulation of Lagrangian systems when the natural connection on the bundle $\pi : Q \times \mathbb{R} \to \mathbb{R}$, given by the vector field $\partial/\partial t$, is replaced by any other connection, given by a vector field $\partial/\partial t + Y$, where $Y$ is a vector field on $Q$ along the projection $\pi_2 : Q \times \mathbb{R} \to Q$. The natural notion of energy associated to any connection of that kind and its variation along the integral curves of the dynamical vector field of the system is also studied, in particular its invariance, if the Lagrangian is invariant by the action of the vector field associated with the connection.

In this last situation, where the vector field associated with the connection is a symmetry of the Lagrangian, we can mimic the full process of the autonomous case in the contact formulation and obtain a symplectic formulation in an appropriate manifold, given by the quotient of $TQ \times \mathbb{R}$ by the action of the vector field of the connection. So, every time we have a time dependent regular Lagrangian with a suitable symmetry, the system can be reduced and the symplectic formulation used.

The organization of the paper is as follows: In paragraph 2, we summarize the geometric foundations used in the sequel for non autonomous Lagrangian regular systems. We introduce a non standard connection in the configuration bundle and consider the consequences of this choice. Paragraph 3 is devoted to a study of the relation between symmetries and connections. In paragraph 4 we study the reduction of the system, in particular the projectability of the geometrical structures in order to construct a dynamical system on the quotient manifold. Paragraph 5 is devoted to studying the properties of the reduced system, which is a Hamiltonian but not a Lagrangian system, although the phase space is canonically diffeomorphic to a tangent bundle. The paper ends with some examples and conclusions.

Throughout the paper, every manifold, function and mapping is assumed to be smooth. Manifolds are assumed to be connected and second countable. Summation over crossed repeated indices is understood.

2 Time-dependent Lagrangian systems

2.1 The 1-jet bundle of $\pi: Q \times \mathbb{R} \to \mathbb{R}$. Geometric structures and connections

The ideas in this section are known. We merely emphasize the differences between the general situation and this particular one in order to make the paper more readable and self-contained. See [8] and [14] as general references.

Consider the bundle $\pi: Q \times \mathbb{R} \to \mathbb{R}$, where $Q$ is an $n$-dimensional differentiable manifold (the configuration space of a physical system). The 1-jet bundle of sections of $\pi$ is $\pi_1: TQ \times \mathbb{R} \to Q \times \mathbb{R}$.

Proposition 1 The following elements on $\pi: Q \times \mathbb{R} \to \mathbb{R}$ can be canonically constructed one from the other:

1. A section of $\pi_1: TQ \times \mathbb{R} \to Q \times \mathbb{R}$, that is a mapping $\nabla: Q \times \mathbb{R} \to TQ \times \mathbb{R}$ such that $\pi_1 \circ \nabla = \text{Id}_{Q \times \mathbb{R}}$.

2. A subbundle $H(\nabla)$ of $T(Q \times \mathbb{R})$ such that

$$T(Q \times \mathbb{R}) = V(\pi) \oplus H(\nabla) \tag{1}$$

3. A semibasic 1-form $\tilde{\nabla}$ on $Q \times \mathbb{R}$ with values in $T(Q \times \mathbb{R})$ (that is, an element of $\Gamma(Q \times \mathbb{R}, \pi^* T^* Q) \otimes \mathfrak{X}(Q \times \mathbb{R})$), such that $\alpha \circ \tilde{\nabla} = \alpha$, for every semibasic form $\alpha \in \Omega^1(Q \times \mathbb{R})$.

(We denote by $\Gamma(A, B)$ for the set of sections of the bundle $B \to A$).
Definition 1 A connection in the bundle \( \pi: Q \times \mathbb{R} \to \mathbb{R} \) is one of the above mentioned equivalent elements. \( \nabla \) is usually called a jet field. \( H(\nabla) \) is called the horizontal subbundle of \( T(Q \times \mathbb{R}) \) associated with \( \nabla \) and its sections horizontal vector fields. Finally \( \nabla \) is called the Ehresmann connection form.

Given the subbundle \( H(\nabla) \) and the splitting (\( L \)), we have the maps

\[
h_\nabla: T(Q \times \mathbb{R}) \to H(\nabla) \quad v_\nabla: T(Q \times \mathbb{R}) \to V(\pi)
\]
called the horizontal and vertical projections (we will use the same symbols \( h_\nabla \) and \( v_\nabla \) for the natural extensions of these maps to vector fields).

In a local chart \( (q^\mu, t, v^\mu) \) the expressions of all these elements are

\[
\nabla(q, s) = (q, s, \Gamma^\mu(q, s)) \quad \nabla = dt \otimes \left( \frac{\partial}{\partial t} + \Gamma^\mu \frac{\partial}{\partial q^\mu} \right)\quad H(\nabla) = \text{span} \left\{ \frac{\partial}{\partial t} + \Gamma^\mu \frac{\partial}{\partial q^\mu} \right\}
\]
(for every \( (q, s) \in Q \times \mathbb{R} \)).

Proposition 2 A connection in the bundle \( \pi: Q \times \mathbb{R} \to \mathbb{R} \) is equivalent to a time-dependent vector field in \( Q \), that is a vector field along \( \pi_Q: Q \times \mathbb{R} \to Q \).

Given a connection \( \nabla \), let \( Y \) be its associated vector field. The suspension of \( Y \) is the vector field \( \tilde{Y} := \frac{\partial}{\partial t} + Y \).

In the Lagrangian formalism, the dynamics takes place in the manifold \( TQ \times \mathbb{R} \). Then, in order to set the dynamics, we need to introduce some geometrical elements in the bundle \( \pi_1: TQ \times \mathbb{R} \to Q \times \mathbb{R} \) (see (\ref{2}), (\ref{3}), (\ref{4}), and (\ref{5}) for details). Every time it is required, we will use a local system given by \( (q^\mu, t, v^\mu) \).

We can define a 1-form \( \theta \) in \( TQ \times \mathbb{R} \), with values in \( \pi_1^*V(\pi) \), in the following way:

\[
\theta(((q, s), u), X) := (T_{((q, s), u)}\pi_1 - T_{((q, s), u)}(\phi \circ \pi \circ \pi_1))(X_{(q, s)})
\]

where \( \phi \) is a representative of \( ((q, s), u) \in TQ \times \mathbb{R} \). \( \theta \) is called the structure canonical form of \( TQ \times \mathbb{R} \).

Its local expression is \( \theta = (dq^\mu - v^\mu dt) \otimes \frac{\partial}{\partial q^\mu} \).

Taking into account that \( \pi_1: TQ \times \mathbb{R} \to Q \times \mathbb{R} \) is a vector bundle, and the fiber on \( (q, s) \in Q \times \mathbb{R} \) is \( T_qQ \times \{s\} \), there exists a canonical diffeomorphism between the \( \pi_1 \)-vertical subbundle and \( \pi_1^*(TQ \times \mathbb{R}) \), that is,

\[
V(\pi_1) \simeq \pi_1^*(TQ \times \mathbb{R}) \simeq \pi_1^*\pi_Q^*TQ \simeq \pi_1^*V(\pi)
\]

We denote by \( S: \pi_1^*V(\pi) \to V(\pi) \) the realization of this isomorphism, and we will use the same notation \( S \) for its action on the modules of sections of these bundles. \( S \) is an element of \( \Gamma(TQ \times \mathbb{R}, \pi_1^*V^*(\pi)) \otimes \Gamma(TQ \times \mathbb{R}, V(\pi_1)) \). Taking into account that the structure form \( \theta \) is an element of \( \Omega^1(TQ \times \mathbb{R}, \pi_1^*V(\pi_1)) = \Omega^1(TQ \times \mathbb{R}) \otimes \Gamma(TQ \times \mathbb{R}, \pi_1^*V(\pi)) \), using the natural duality, by contracting \( S \) with \( \theta \), we obtain an element

\[
\mathcal{V} := i(S)\theta \in \Omega^1(TQ \times \mathbb{R}) \otimes \Gamma(TQ \times \mathbb{R}, V(\pi_1))
\]
whose local expression is \( \mathcal{V} = (dq^\mu - v^\mu dt) \otimes \frac{\partial}{\partial v^\mu} \). Notice that \( \mathcal{V} \) can be thought of as a \( \mathcal{C}^\infty(TQ \times \mathbb{R}) \)-module morphism \( \mathcal{V}: \mathfrak{X}(TQ \times \mathbb{R}) \to \mathfrak{X}(TQ \times \mathbb{R}) \) with image on the \( \pi_1 \)-vertical vector fields.

\( S \) and \( \mathcal{V} \) are called the vertical endomorphisms of \( TQ \times \mathbb{R} \).
2.2 Lagrangian formalism. Connections and Lagrangian energy functions

A time-dependent Lagrangian function is a function $L \in C^\infty(TQ \times \mathbb{R})$.

**Definition 2** The Poincaré-Cartan 1 and 2-forms associated with the Lagrangian function $L$ are the forms in $TQ \times \mathbb{R}$ defined by

$$
\Theta_L := dL \circ \nabla + Ldt, \quad \Omega_L := -d\Theta_L
$$

The coordinate expressions of the Poincaré-Cartan forms are

$$
\Theta_L = \frac{\partial L}{\partial v^\mu} (dq^\mu - v^\mu dt) + L dt = \left( L - v^\mu \frac{\partial L}{\partial v^\mu} \right) dt + \frac{\partial L}{\partial v^\mu} dq^\mu
$$

$$
\Omega_L = -d \left( \frac{\partial L}{\partial v^\mu} \right) \wedge dq^\mu + d \left( \frac{\partial L}{\partial v^\mu} v^\mu - L \right) \wedge dt
$$

Observe that these elements do not depend on the connection.

A Lagrangian function $L$ is regular if its associated form $\Omega_L$ has maximal rank, which is equivalent to demanding that $\det \left( \frac{\partial^2 L}{\partial v^\mu \partial v^\nu} \right)$ is different from zero at every point.

On the other hand, if we have a connection $\nabla$ on $\pi : Q \times \mathbb{R} \rightarrow \mathbb{R}$, we can split $\Theta_L$ and $\Omega_L$ into a sum of vertical and horizontal forms. The vertical form (resp. horizontal) is characterized by the fact that it vanishes under the action of the horizontal (resp. vertical) vector fields associated with the connection.

Thus $\Omega_L = \Omega_L^H + \Omega_L^V$ where $\Omega_L^H = dt \wedge i(\nabla^1 \tilde{Y}) \Omega_L$ and $\Omega_L^V = \Omega_L - \Omega_L^H$. The splitting for $\Theta_L$ follows in the same way as above, $\Theta_L^H = dt \wedge i(\nabla^1 \tilde{Y}) \Theta_L$, and $\Theta_L^V = \Theta_L - \Theta_L^H$.

It is easy to see that this construction can be generalized to every $\alpha \in \mathcal{A}(F)$ where $F$ is fibred manifold on the basis $B$; see [9] for more details.

Assuming the regularity of $L$, the dynamics of the system is described by a vector field $X_L \in \mathfrak{X}(TQ \times \mathbb{R})$, which is a Second Order Differential Equation (SODE), such that:

$$
i(X_L) \Omega_L = 0, \quad i(X_L) dt = 1
$$

As a consequence, the integral curves of $X_L$ verify the Euler-Lagrange equations.

A very different picture arises when we try to define intrinsically the Lagrangian energy function.

**Definition 3** Let $\nabla$ be a connection in $\pi : Q \times \mathbb{R} \rightarrow \mathbb{R}$, $\tilde{Y} \in \mathfrak{X}(Q \times \mathbb{R})$ the suspension of the vector field associated with it, and $j^1\tilde{Y} \in \mathfrak{X}(TQ \times \mathbb{R})$ its canonical lifting. The Lagrangian energy function associated with the Lagrangian function $L$ and the connection $\nabla$ is $E_L^\nabla = -i(j^1\tilde{Y}) \Theta_L$

In a local chart, if $\tilde{Y} = \frac{\partial}{\partial t} + \Gamma^\mu \frac{\partial}{\partial q^\mu}$, we have $j^1\tilde{Y} = \frac{\partial}{\partial t} + \Gamma^\mu \frac{\partial}{\partial q^\mu} + \left( \frac{\partial \Gamma^\mu}{\partial t} + v^\nu \frac{\partial \Gamma^\mu}{\partial q^\nu} \right) \frac{\partial}{\partial v^\mu}$ and $E_L^\nabla = \frac{\partial L}{\partial v^\mu} (v^\mu - \Gamma^\mu) - L$

It is obvious from this expression that the Lagrangian energy is connection-depending. If $\Gamma^\mu = 0$, the corresponding connection is the standard one. The use of this connection “hides” the explicit dependence on the connection of the energy in classical mechanics. The proof of the following propositions can be found in [4].

**Proposition 3** Let $X_L \in \mathfrak{X}(TQ \times \mathbb{R})$ be the dynamical vector field (solution of the equations (1)). Then

$$X_L(E_L^\nabla) = -(j^1\tilde{Y})L$$

**Proposition 4** If $L$ is a Lagrangian function such that its associated Legendre transformation is different from zero at every point, then every first integral of the dynamical vector field $X_L$ is the energy associated with some connection.
3 Symmetries of Lagrangian time-dependent systems. Infinitesimal symmetries and connections

Let $L \in C^\infty(TQ \times \mathbb{R})$ be a time-dependent regular Lagrangian, and $\Phi$ a bundle diffeomorphism of $Q \times \mathbb{R}$ such that its restriction $\Phi_R : \mathbb{R} \rightarrow \mathbb{R}$ verifies $\Phi_R(t) = t + c$. $\Phi$ is a symmetry if $(j^1 \Phi)^* L = L$, that is $L$ is invariant by $j^1 \Phi$. Let us recall that in this situation the canonical forms $\Theta_L$ and $\Omega_L$ are also invariant by $j^1 \Phi$, see [1].

**Proposition 5** If $L$ is invariant by $j^1 \Phi$, and $X_L$ is a solution of the equations (3), then $(j^1 \Phi)^* X_L$ is also a solution.

**Proof:** Since $(j^1 \Phi)^* L = L$ and $(j^1 \Phi)^* dt = d((j^1 \Phi)^* t) = d(t + c) = dt$, we have that

$$0 = j^1 \Phi^* (i(X_L) \Omega_L) = i(j^1 \Phi^*(X_L)) j^1 \Phi^* (\Omega_L) = i(j^1 \Phi^*(X_L)) \Omega_L$$

and in a similar way

$$1 = j^1 \Phi^* (i(X_L)dt) = i(j^1 \Phi^*(X_L))dt.$$

\[\blacksquare\]

**Remark:** As $L$ is regular, we have that $(j^1 \Phi)^* X_L = X_L$, hence if $\sigma$ is an integral curve of $X_L$, then $\Phi^* \sigma$ is too.

Given $Y \in \mathfrak{x}(Q, \tau_Q)$ a vector field along the projection $\tau_Q$, consider the vector field $j^1 \tilde{Y} \in \mathfrak{x}(TQ \times \mathbb{R})$ where $\tilde{Y} = \frac{\partial}{\partial t} + Y$. We say that $j^1 \tilde{Y}$ is an infinitesimal symmetry of the system if $\mathcal{L}(j^1 \tilde{Y})L = 0$, where $\mathcal{L}(X)$ is the Lie derivative operator with respect to $X$.

As a consequence of the above proposition, we have:

**Corollary 1** If $\mathcal{L}(j^1 \tilde{Y})L = 0$, then $\mathcal{L}(j^1 \tilde{Y})\Omega_L = 0$, $\mathcal{L}(j^1 \tilde{Y})dt = 0$, $\mathcal{L}(j^1 \tilde{Y})X_L = 0$, and $\mathcal{L}(j^1 \tilde{Y})E_L^Y = 0$.

\[\blacksquare\]

As a particular case, we can consider the invariance under a connection. Let $\nabla$ be a connection with $j^1 \tilde{Y} \in \mathfrak{x}(TQ \times \mathbb{R})$ the 1-jet prolongation of the field $\tilde{Y}$ associated with $\nabla$, which we will suppose to be complete. The Lagrangian system is called invariant under the connection $\nabla$ if $\mathcal{L}(j^1 \tilde{Y})L = 0$.

With these conditions, and taking into account that $j^1 \tilde{Y}$ is complete, we can define the next family of diffeomorphisms:

$$\Phi_s: TQ \times \mathbb{R} \rightarrow TQ \times \mathbb{R}$$

$$(v_q, t) \rightarrow j^1 \varphi_s(v_q, t)$$

where $j^1 \varphi_s$ is the uniparametric flow of $j^1 \tilde{Y}$. Observe that if $\mathcal{L}(j^1 \tilde{Y})L = 0$, then $\Phi_s$ is a symmetry of $L$ for every $s \in \mathbb{R}$. Therefore:

**Proposition 6** If $\mathcal{L}(j^1 \tilde{Y})L = 0$ then $d\Theta_L^Y = -\Omega_L^Y$.

**Proof:**

$$d\Theta_L^Y = d(\Theta_L - i(j^1 \tilde{Y})\Theta_L dt) = -\Omega_L - d(i(j^1 \tilde{Y})\Theta_L) \wedge dt$$

$$= -\Omega_L - i(j^1 \tilde{Y})\Omega_L \wedge dt + \mathcal{L}(j^1 \tilde{Y})\Omega_L = -\Omega_L + dt \wedge i(j^1 \tilde{Y})\Omega_L = -\Omega_L^Y$$

\[\blacksquare\]

**Assumption:** From now on, we will suppose that $j^1 \tilde{Y}$ is an infinitesimal symmetry of the system $(TQ \times \mathbb{R}, L)$.
4 Reduction of the time-dependent Lagrangian systems invariant under $\nabla$

Consider the action of $j^1\tilde{Y}$ on $TQ \times \mathbb{R}$, and let $[TQ \times \mathbb{R}] \equiv (TQ \times \mathbb{R})/j^1\tilde{Y}$ be the set of equivalence classes by this action. Let $\pi^1 : TQ \times \mathbb{R} \rightarrow [TQ \times \mathbb{R}]$ the natural projection. We will assume that $[TQ \times \mathbb{R}]$ is a manifold and $\pi^1$ a submersion.

Remember that given a projection of manifolds $\pi : M \rightarrow M/\sim$, and $\omega \in \Lambda^j(M)$, we say that $\omega$ is $\pi-$projectable if there exists $\tilde{\omega}$ such that $\pi^*(\tilde{\omega}) = \omega$. A necessary and sufficient condition to assure the projectability is that $i(X)\omega = 0$ and $L(X)\omega = 0$, for every $X \in \mathfrak{x}^j(\pi)$ or equivalently $i(X)\omega = 0$, $i(X)d\omega = 0$, for $X \in \mathfrak{x}^j(\pi)$ (see [10] for more details). Observe that in our situation $\mathfrak{x}^j(\pi^1)$ is spanned by $j^1\tilde{Y}$.

Let $\Omega^H_L = dt \wedge (j^1\tilde{Y})\Omega_L$, $\Omega^V_L = \Omega_L - \Omega^H_L$, and $\Theta^H_L = (i(j^1\tilde{Y})\Theta_L)dt$, $\Theta^V_L = \Theta_L - \Theta^H_L$ be the decompositions induced by $\nabla$, then we have:

**Proposition 7** $\Theta^V_L$, $\Omega^V_L$ and $E^V_L$ are $\pi^1-$projectable.

**Proof:** Taking into account that $i(j^1\tilde{Y})dt = 1$, and $i(j^1\tilde{Y})i(j^1\tilde{Y})\Omega_L = 0$, because $\Omega_L$ is skew-symmetric, we have:

\[
\begin{align*}
i(j^1\tilde{Y})\Omega^V_L &= i(j^1\tilde{Y})\Omega_L - (i(j^1\tilde{Y})dt)i(j^1\tilde{Y})\Omega_L + (i(j^1\tilde{Y})i(j^1\tilde{Y})\Omega_L)dt = 0 \\
i(j^1\tilde{Y})d\Omega^V_L &= i(j^1\tilde{Y})d\Omega_L - i(j^1\tilde{Y})(d(dt) \wedge i(j^1\tilde{Y})\Omega_L) + i(j^1\tilde{Y})(dt \wedge d(i(j^1\tilde{Y})\Omega_L)) \\
&= i(j^1\tilde{Y})(dt \wedge d(i(j^1\tilde{Y})\Omega_L))
\end{align*}
\]

where the last expression is equal to zero since $d(i(j^1\tilde{Y})\Omega_L) = L(j^1\tilde{Y})\Omega_L - i(j^1\tilde{Y})d\Omega_L$. In the same way we can prove the assertion for $\Theta^V_L$.

For the Lagrangian energy function holds:

\[
\begin{align*}
L(j^1\tilde{Y})E^V_L &= i(j^1\tilde{Y})E^V_L = (dE^V_L)(j^1\tilde{Y}) = -d(i(j^1\tilde{Y})\Theta_L)(j^1\tilde{Y}) \\
&= -i(j^1\tilde{Y})\Omega_L)(j^1\tilde{Y}) = -i(j^1\tilde{Y})i(j^1\tilde{Y})\Omega_L = 0.
\end{align*}
\]

**Remark:** As a consequence, there exist differential forms $\omega$ and $\theta$ in $[TQ \times \mathbb{R}]$, such that $\pi^1*(\omega) = \Omega^V_L$ and $\pi^1*(\theta) = \Theta^V_L$, and a function $E \in C^\infty([TQ \times \mathbb{R}])$ verifying that $\pi^1*(E) = E^V_L$.

**Proposition 8** $([TQ \times \mathbb{R}], \omega)$ is a symplectic manifold and $([TQ \times \mathbb{R}], \omega, E)$ is a Hamiltonian system.

**Proof:** We must show that $\omega$ is closed and non-degenerated.

- $\pi^1*(d\omega) = d(\pi^1*\omega) = d\Omega^V_L = d\Omega_L - d\Omega^H_L = -d(dt \wedge i(j^1\tilde{Y})\Omega_L) = dt \wedge d(i(j^1\tilde{Y})\Omega_L)) = 0$.

  Since $\pi^1$ is a submersion, the fact that $\pi^1*(d\omega) = 0$, implies that $d\omega = 0$.

- Suppose that $i(Z)\omega = 0$, then if $X \in \pi^1*(Z)$, then $0 = i(X)\Omega^V_L = i(X^V + X^H)\Omega^V_L = i(X^V)\Omega^V_L = i(X^V)\Omega_L$.

  Since, $L$ is regular, then $X^V = 0$, which implies that $Z = 0$.

  Observe that $\pi^1*(Z)$ is well-defined because $\pi^1$ is an exhaustive submersion.
Considering the dynamical equations in $TQ \times \mathbb{R}$, we write them in a suitable way to obtain the dynamics in the quotient.

$$0 = i(X_L)\Omega_L = i(X_L)(\Omega_L^H + \Omega_L^V) = i(j^1Y)\Omega_L - dt(i(X_L)i(j^1Y)\Omega_L) + i(X_L)\Omega_L^V$$

$$= i(j^1Y)\Omega_L + i(X_L)\Omega_L^V = -dE_L^V + i(X_L)\Omega_L^V.$$  

where we have taken into account that $i(X_L)i(j^1Y)\Omega_L = 0$, since $X_L$ is a solution of (2), and $i(j^1Y)\Omega_L = -L(j^1Y)\Theta_L + d(i(j^1Y)\Theta_L) = -dE_L^V$.

As the connection $\nabla$ associated with $j^1Y$ allows us to split $T(TQ \times \mathbb{R}) = H(\nabla) \oplus V(\pi \circ \pi^1)$, then $X_L = X_L^H + X_L^V$, and

$$0 = -dE_L^V + i(X_L^H + X_L^V)\Omega_L^V = -dE_L^V + i(X_L^H)\Omega_L^V + i(X_L^V)\Omega_L^V = -dE_L^V + i(X_L^V)\Omega_L^V.$$  

since $i(fj^1Y)\Omega_L^V = f(i(j^1Y)\Omega_L^V = 0$. Observe that the second equation does not give us any information about $X_L^V$, because $i(X_L^V)dt = 0$ holds. Then we have $i(X_L^V)\Omega_L^V = dE_L^V$.

As a consequence of the previous propositions

$$0 = -d(\pi^1*(E)) + i(X_L^V)\pi^1*\omega = \pi^1*(-dE) + \pi^1*(i(\chi)\omega) = \pi^1*(-dE + i(\chi)\omega).$$

This implies that $0 = -dE + i(\chi)\omega$, because $\pi^1$ is a submersion. So the dynamical equation in $[TQ \times \mathbb{R}]$ is $i(\chi)\omega = dE$.

**Proposition 9** $E$ is a first integral of the system $([TQ \times \mathbb{R}], \omega, E)$.

**Proof:** Since $(j^1Y)L = 0$ the energy function $E_L^V$ is constant along the trajectories of (2). Otherwise $X_L^H(E_L^V) = 0$ because $(dE_L^V)X_L^H = -i(fj^1Y)i(j^1Y)\Omega_L = 0$. As a consequence of this $0 = X_L(E_L^V) = X_L^H(E_L^V) + X_L^V(E_L^V) = X_L^V(E_L^V)$. Then we have that $0 = X_L^V(E_L^V) = \pi^1(\chi(E))$ and $\chi(E) = 0$.

**Remark:** The restriction of the Hamiltonian system $([TQ \times \mathbb{R}], \omega, E)$ to a hypersurface defined by $E = \text{ctn}$, gives the same result as if we apply the presymplectic reduction procedure, studied in [3], to the initial Lagrangian system $(TQ \times \mathbb{R}, L)$ under the action of $j^1Y$.

Now suppose that $\chi \in \mathcal{X}([TQ \times \mathbb{R}])$ is the solution of the dynamical equation $i(\chi)\omega = dE$. We want to recover the solution $X_L \in \mathcal{X}(TQ \times \mathbb{R})$ of (2) from $\chi$. Let $Z := j^1Y + v_\chi(\pi^1*(\chi)) \in \mathcal{X}(TQ \times \mathbb{R})$. It is well defined because every $X \in \mathcal{X}(TQ \times \mathbb{R})$ such that $\pi^1*(\chi) = X$ has the same vertical part. Then:

**Proposition 10** $Z \in \mathcal{X}(TQ \times \mathbb{R})$ is a solution of (2).

**Proof:** As $dt$ is a semibasic form we have:

$$i(Z)dt = i(j^1Y)dt + i(v_\chi(\pi^1*(\chi)))dt = 1 + i(v_\chi(\pi^1*(\chi)))dt = 1$$

$$i(Z)\Omega_L = i(j^1Y)\Omega_L + i(v_\chi(\pi^1*(\chi)))\Omega_L^V = i(j^1Y)\Omega_L + i(\pi^1*(\chi))\Omega_L^V$$

$$= -\pi^1*(dE) + i(\pi^1*(\chi))\pi^1*\omega = -\pi^1*(dE) + \pi^1*(i(\chi)\omega) = \pi^1*(-dE + i(\chi)\omega) = 0$$

\[\square\]
5 Properties of the system \([TQ \times \mathbb{R}], \omega, \mathcal{E}\)

Consider now the manifold \(\mathscr{Q} = (Q \times \mathbb{R})/Y\) and the submersion \(\pi : Q \times \mathbb{R} \to [Q \times \mathbb{R}]\). From every element \([(q, t)] \in [Q \times \mathbb{R}]\) we can choose a representative of the form \((\tilde{q}, 0)\): if \((q, t) \in [(q, t)]\) then, \((\tilde{q}, 0) = \varphi_{-t}(q, t)\) where \(\varphi_s(q, t) = (\varphi^Q_s(q, t), t + s)\) is the flow of \(Y\). This element is unique, since if there exists \((q_1, 0)\) and \((q_2, 0)\) in the same class \([(q, t)]\), then for some \(s \in \mathbb{R}\), we have \((q_2, 0) = (\varphi^Q_s(q_1, 0), s)\).

Hence we can conclude that \(s = 0\) and thus \(q_1 = q_2\).

From the above considerations, there exists a natural bijection \(\psi\) from \([Q \times \mathbb{R}]\) to \(Q\):

\[
\psi : [Q \times \mathbb{R}] \to Q \quad \psi^{-1} : Q \to [Q \times \mathbb{R}]
\]

\(\psi \circ \psi^{-1} = Id_{[Q \times \mathbb{R}]}\) and \(\psi^{-1} \circ \psi = Id_{Q}\).

**Proposition 11** \(\psi\) is a diffeomorphism.

**Proof:** Consider the diagram

\[
\begin{array}{ccc}
Q \times \mathbb{R} & \overset{\pi}{\longrightarrow} & [Q \times \mathbb{R}] \\
\downarrow{\phi} & & \downarrow{\psi} \\
Q & & \psi^{-1} \\
\end{array}
\]

where \(\phi(t, q) = \varphi^Q_t(t, q)\). \(\psi\) is smooth iff \(\phi\) is smooth, see [8], and \(\phi\) is smooth trivially.

On the other hand, we can see \(Q\) as a quotient manifold of \(Q \times \mathbb{R}\), then using the same pattern as above, \(\psi^{-1}\) is smooth because \(\pi\) is too. Therefore \(\psi\) is a diffeomorphism. \(\blacksquare\)

Observe that by means of \(\psi\) we can build natural local charts on \([Q \times \mathbb{R}]\). Let \(\{U_\alpha, \zeta_\alpha\}\) one atlas on \(Q\), then it is easy to see that \(\{\psi^{-1}(U_\alpha), \zeta_\alpha \circ \psi\}\) is an atlas on \([Q \times \mathbb{R}]\).

**Remark:** As a consequence of the above considerations, we can also construct a diffeomorphism \(\tilde{\psi} : TQ \times \mathbb{R} \to TQ\).

Now we can define the projection map \(\pi_1 : TQ \times \mathbb{R} \to [Q \times \mathbb{R}]\), because given an integral curve, \(\gamma\), of \(j^1Y\), \(\pi_1 \circ \gamma\) is an integral curve of \(\tilde{Y}\). Then we have:

\[
\pi_1 : TQ \times \mathbb{R} \to [Q \times \mathbb{R}] \\
[(v_q, t)] \to [(q, t)]
\]

**Proposition 12** \(\pi_1\) is a submersion.

**Proof:** In natural local charts, \(\pi_1(t, q, v) = (t, q)\), hence \(\pi_1\) is smooth and surjective. \(\blacksquare\)

From the previous considerations, the following diagram is commutative:

\[
\begin{array}{ccccccc}
TQ \times \mathbb{R} & \overset{\pi_1}{\longrightarrow} & [TQ \times \mathbb{R}] & \overset{\psi}{\cong} & TQ \\
\downarrow{\pi_1} & & \downarrow{\pi_1} & & \downarrow{\pi_Q} \\
Q \times \mathbb{R} & \overset{\pi}{\longrightarrow} & [Q \times \mathbb{R}] & \overset{\psi}{\cong} & Q
\end{array}
\]
Since $L \in C^\infty(TQ \times \mathbb{R})$ is constant along the integral curves of $j^1\tilde{Y}$, we can define a function $\mathcal{L}$ on $[TQ \times \mathbb{R}]$. Consider the Lagrangian forms $\theta_\Sigma$ and $\omega_\Sigma$ on $TQ$ associated with the Lagrangian function $\mathcal{L} = \tilde{\psi}_s(\mathcal{L})$. We wish to compare them with $\psi_s\omega$ and $\psi_s\theta$.

**Proposition 13** $\tilde{\psi}^1(\theta_\Sigma) = \Theta^V_L = \pi^1(\theta)$.

**Proof:** It is enough to show this proposition in local coordinates. Let $\phi := \tilde{\psi} \circ \pi^1$ be, as

$$
\phi(t, q^i, v^i) = \left( \varphi^i_t(q^i, t), \frac{\partial \varphi^i_s}{\partial t}(-t; (q^i, t)) + \frac{\partial \varphi^i_s}{\partial q^j}(-t; (q^i, t)) v^j \right) =: (\varphi_t^{i, \bar{v}}, \bar{v}^j)
$$

and $L(t, q^i, v^i) = \mathcal{L}(\phi(t, q^i, v^i))$, we have that

$$
\tilde{\psi}^1(\theta_\Sigma) = \left. \frac{\partial \mathcal{L}}{\partial \bar{v}^j} \right|_{(t, q^i, v^i)} \left( \left. \frac{\partial \varphi^i_t}{\partial q^j} \right|_{(t, q^i)} \, dq^j + \left. \frac{\partial \varphi^i_s}{\partial t} \right|_{(t, q^i)} \, dt \right) = \left. \frac{\partial \mathcal{L}}{\partial \bar{v}^j} \right|_{(t, q^i, v^i)} \left( \left. \frac{\partial \varphi^i_t}{\partial q^j} \right|_{(t, q^i)} \, dq^j + \left. \frac{\partial \varphi^i_s}{\partial t} \right|_{(t, q^i)} \, dt \right) + \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{v}^j} \left( -\Gamma^i(j, q^i, t) \left. \frac{\partial \varphi^i_t}{\partial q^j} \right|_{(t, q^i)} + \Gamma^j(q^i, t) \left. \frac{\partial \varphi^i_s}{\partial q^j} \right|_{(t, q^i)} \right) \, dt
$$

$$
= \left. \frac{\partial \mathcal{L}}{\partial \bar{v}^j} \right|_{(t, q^i, v^i)} \, dq^j - \Gamma^i(q^i, t) \left. \frac{\partial \mathcal{L}}{\partial \bar{v}^j} \right|_{(t, q^i, v^i)} \, dt = \Theta^V_L
$$

where we have taken into account that

$$
\left. \frac{\partial \mathcal{L}}{\partial \bar{v}^j} \right|_{(t, q^i, v^i)} = \left. \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{v}^j} \right|_{(t, q^i, v^i)} \left. \frac{\partial \varphi^i}{\partial \bar{v}^j} \right|_{(t, q^i, v^i)} = \left. \frac{\partial \tilde{\mathcal{L}}}{\partial \bar{v}^j} \right|_{(t, q^i, v^i)} \left. \frac{\partial \varphi^i}{\partial \bar{v}^j} \right|_{(t, q^i, v^i)}.
$$

Observe that $\left. \frac{\partial \varphi^i}{\partial t} \right|_{(t, q^i)} + \Gamma^i \left. \frac{\partial \varphi^i}{\partial q^j} \right|_{(t, q^i)} = 0$, since the tangent mapping of $(t, q^i) \mapsto (t, \varphi^i_t(q^i, t))$ transforms the vector field $\frac{\partial}{\partial t} + \Gamma^i(q^i, t) \frac{\partial}{\partial q^j}$ into $\frac{\partial}{\partial t}$.

**Remark:** The above result is due to the fact that the symmetry is natural, that is, a jet prolongation of a vector field on $Q \times \mathbb{R}$.

**Proposition 14** $\pi^1(\tilde{\psi}^1(\omega_\Sigma)) = \Omega^V_L$.

**Proof:** Since $\omega_\Sigma = -d\theta_\Sigma$, we have that $(\psi \circ \pi^1)^* (d\theta_\Sigma) = d((\psi \circ \pi^1)^* \theta_\Sigma) = d\Theta^V_L$. On the other hand, it verifies that $d\Theta^V_L = -\Omega^V_L$. Therefore $\pi^1(\tilde{\psi}^1(\omega_\Sigma)) = \Omega^V_L$.

**Corollary 2** $\theta = \tilde{\psi}^1(\theta_\Sigma)$ and $\omega = \tilde{\psi}^1(\omega_\Sigma)$. 
Proof: From the above proposition we have that \( \pi^\dagger x(\psi^*(\theta_\Sigma)) = \pi^\dagger x(\theta) \), then, since \( \pi^\dagger \) is a submersion, \( \theta = \psi^*(\theta_\Sigma) \). The second assertion follows in the same way.

A different situation arises when we try to do the same with the energy. Let \( E_\Sigma := \Delta \Sigma - \Sigma \) be the energy associated with the Lagrangian \( \Sigma \), then in general \( \pi^\dagger x(\psi^*(E_\Sigma)) \neq E^\nabla_L \). An example of this is given in the next section.

**Remark:** So the Hamiltonian system \( ([TQ \times \mathbb{R}], \omega, \mathcal{E}) \) is not, in general, a Lagrangian one.

### 6 Examples

#### 6.1 Autonomous dynamical systems

First we analyze the time-independent dynamical systems as a particular case of non-autonomous regular systems which are invariant under time translations.

Let \( (TQ \times \mathbb{R}, \Omega_L) \) the non-autonomous regular Lagrangian dynamical system. \( \nabla_0 \) is the standard connection, and as a consequence \( \nabla = \frac{\partial}{\partial t} \). Let \( L \in C^\infty(\mathbb{R} \times TQ) \) be the Lagrangian function, such that \( L(j^1\nabla) = L(\frac{\partial}{\partial t}) = 0 \), that is \( \frac{\partial L}{\partial \frac{\partial}{\partial t}} = 0 \). Then \( \varphi(t, q, v_q) = (q, v_q) =: (\bar{q}, \bar{v}_q) \), and \( \varphi(\bar{q}, \bar{v}_q) = L(0, q, v_q) \).

Therefore \( \theta = \theta_\Sigma, \omega = \omega_\Sigma \) and \( \mathcal{E} = E_\Sigma \).

Observe that in this particular case the system \( ([TQ \times \mathbb{R}], \omega, \mathcal{E}) \) is a Lagrangian one.

#### 6.2 Another example

Let \( (T\mathbb{R}^2 \times \mathbb{R}, \Omega_L) \) be the regular Lagrangian dynamical system associated with the Lagrangian function

\[
L(t, x, y, v_x, v_y) = \frac{1}{2}(v_x^2 + v_y^2) - V(y) + (t - x)v_x + (t - x)v_y.
\]

Consider the infinitesimal symmetry \( j^1\nabla = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \). Taking into account the coordinate expression of the flow, we have that \( \bar{\Sigma}(\bar{x}, \bar{y}, \bar{v}_x, \bar{v}_y) = \frac{1}{2}(\bar{v}_x^2 + \bar{v}_y^2) - V(\bar{y}) - \bar{x}\bar{v}_x - \bar{x}\bar{v}_y \). From the above considerations \( \theta_\Sigma = (\bar{v}_x - \bar{x})d\bar{x} + (\bar{v}_y - \bar{x})d\bar{y} \), and as we know

\[
\varphi^*(\theta_\Sigma) = (v_x - x + t)(dx - dt) + (v_y - x + t)dy = (x - v_x + t)dt + (v_y - x + t)dx + (v_y - x + t)dy = \Theta_L^V.
\]

On the other hand, since \( E^\nabla_L = \frac{1}{2}(\bar{v}_x^2 + \bar{v}_y^2) + V(\bar{y}) + x - \bar{v}_x \), \( \mathcal{E} = \frac{1}{2}(\bar{v}_x^2 + \bar{v}_y^2) + V(\bar{y}) \), which is different from \( E_\Sigma = \frac{1}{2}(\bar{v}_x^2 + \bar{v}_y^2) + V(\bar{y}) \).

Now we can solve the dynamical equation \( i(X_\Sigma)\omega_\Sigma = d\mathcal{E} \), obtaining that

\[
X_\Sigma = (\bar{v}_x - 1)\frac{\partial}{\partial \bar{x}} + \bar{v}_y\frac{\partial}{\partial \bar{y}} - (1 + \bar{v}_y)\frac{\partial}{\partial \bar{v}_x} + (v_x - 1 - \frac{\partial V}{\partial \bar{y}})\frac{\partial}{\partial \bar{v}_y}
\]

and therefore

\[
X_L = \frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} - (1 + v_y)\frac{\partial}{\partial v_x} + (v_x - 1 - \frac{\partial V}{\partial y})\frac{\partial}{\partial v_y}.
\]

### 7 Conclusions and outlook

Let \( (TQ \times \mathbb{R}, L) \) be a time-dependent regular Lagrangian system. Every connection \( \nabla \) in \( Q \times \mathbb{R} \to \mathbb{R} \) is associated with a vector field \( \nabla \in \mathfrak{X}(Q \times \mathbb{R}) \). Then, we study the reduction of the system when \( j^1\nabla \in \mathfrak{X}(TQ \times \mathbb{R}) \) is an infinitesimal symmetry of \( L \). First, the connection \( \nabla \) allows us to split the Lagrangian 2-form \( \Omega_L \) into the corresponding vertical and horizontal parts, \( \Omega_L^V \) and \( \Omega_L^H \), and then, introducing the Lagrangian energy function \( E^\nabla_L \) associated with \( \nabla \), and using \( \Omega_L^V \), we can write the
dynamical equations in a similar way to the time-independent case. Therefore, considering the reduced manifold \([TQ \times \mathbb{R}] \equiv TQ \times \mathbb{R}/j^1\bar{Y}\), and the corresponding natural projection \(TQ \times \mathbb{R} \to [TQ \times \mathbb{R}]\), we prove that both, \(\Omega^V_L\) and \(E^V\), project to \(\omega \in \Omega^2([TQ \times \mathbb{R}])\) and \(E \in C^\infty([TQ \times \mathbb{R}])\), in such a way that \(([TQ \times \mathbb{R}],\omega,E)\) is a regular (symplectic) Hamiltonian system, although it is not a Lagrangian system, in general, in spite of \([TQ \times \mathbb{R}]\) being canonically diffeomorphic to a tangent bundle.

The generalization of these results to other more general situations is a matter of future research. In particular to the following cases: time-dependent non regular Lagrangian systems, regular and non-regular time-dependent Lagrangian systems whose configuration space is a non-trivial fiber bundle \(E \to \mathbb{R}\) (and hence the phase space is a non-trivial jet bundle \(J^1E \to E \to \mathbb{R}\)), and regular and non-regular classical field theories.

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