Polynomial-time algorithms for minimum weighted colorings of $(P_5, \overline{P}_5)$-free graphs and related graph classes

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Abstract

We design an $O(n^3)$ algorithm to find a minimum weighted coloring of a $(P_5, \overline{P}_5)$-free graph. Furthermore, the same technique can be used to solve the same problem for several classes of graphs, defined by forbidden induced subgraphs, such as (diamond, co-diamond)-free graphs.

Keywords: Graph coloring, $P_5$-free graphs

1 Introduction

Graph coloring is a classical problem in computer science and discrete mathematics. The chromatic number $\chi(G)$ of a graph $G$ is the smallest number of colors needed to color the vertices of $G$ in such a way that no two adjacent vertices receive the same color. Determining the chromatic number of a graph is a NP-hard problem. But for many classes of graphs, such as perfect graphs, the problem can be solved in polynomial time.

Recently, much research have been done on coloring $P_5$-free graphs. Finding the chromatic number of a $P_5$-free graphs is NP-hard [17], but for every fixed $k$, the problem of coloring a graph with $k$ colors admits a polynomial-time algorithm [14, 15]. Research has also been done on $(P_5, \overline{P}_5)$-free graphs (graphs without $P_5$ and its complement $\overline{P}_5$). In [10], a polynomial-time algorithm is found for finding an approximate weighted coloring of a $(P_5, \overline{P}_5)$-free graph. Weighted colorings generalize vertex colorings. Given a graph $G$ with a nonnegative integral weight $w_G(v)$ on each vertex $v$ of $G$, the minimum weighted coloring problem (MWC) is to find stable sets $S_1, S_2, \ldots, S_t$ of $G$ and nonnegative integers $I(S_1), I(S_2), \ldots, I(S_t)$ such that for each vertex $v$, $\sum_{s\in S_i} I(S_i) \geq w_G(v)$ and that $\chi_w(G) = \sum_{i=1}^t I(S_i)$ is as small as possible; $\chi_w(G)$ is called the weighted chromatic number of $G$; the stable sets $S_i$ together with the weights $I(S_i)$ are called a weighted coloring of $G$.

The motivation of our paper is to find a polynomial-time algorithm for MWC for $(P_5, \overline{P}_5)$-free graphs. In the process of doing this, we actually solve a more general problem. We prove that for a hereditary class $C$ of graphs, if the minimum weighted coloring problem
can be solved for every prime graph of $C$ in polynomial time, then so can the problem for every graph in $C$ (definitions not given here will be given later). As a corollary, we obtain a polynomial-time algorithm to find a minimum weighted coloring of a $(P_5, \overline{P}_5)$-free graph. This algorithm runs in $O(n^3)$ time. Furthermore, the same technique can be used to solve the same problem for several classes of graphs, defined by forbidden induced subgraphs, such as (diamond, co-diamond)-free graphs. We will remark on this point in section 4. In section 2, we give definitions and discuss the background to our problem. In section 3, we establish the above theorem and give our algorithm for MWC for $(P_5, \overline{P}_5)$-free graphs.

2 Definitions and background

Let $G$ be a graph. A set $H$ of vertices of $G$ is a module if every vertex in $G - H$ is adjacent to either all vertices of $H$, or no vertices of $H$; if $|H| = 1$ or $|H| = |V(G)|$ then $H$ is a trivial module. A graph is prime if it does not contain a non-trivial module. For the rest of the paper, modules are non-trivial unless otherwise noted. A module $H$ is strong if for any module $A$, either $H \cap A = \emptyset$, or $H$ is contained in $A$ or vice versa. It is well known (for example, see [18]) that the vertex set of a graph can be partitioned into unique maximal strong modules in linear time.

Let $G$ be a graph with a maximal strong module $H$. The graph $G$ can be decomposed into two graphs: one is $H$ and the other is the graph $g(G, H, h)$ obtained from $G$ by substituting the vertex $h$ for $H$, i.e. removing $H$ from $G$, adding $h$ and the edge $hv$ for every vertex $v \in G - H$ with $vu \in E(G)$ for some $u \in H$ ($v$ has some neighbor in $H$). If $H$ or $g(G, H, h)$ is not prime, then we can recursively decompose the graph in the same way. We can associate this recursive decomposition of $G$ with a binary tree $T(G)$, where each node $X$ of $T(G)$ represents an induced subgraph $r(X)$ of $G$, as follows. The root $T$ of $T(G)$ represents $G$ (i.e., $r(T) = G$), $T$ has two children $L, R$ where node $L$ (left child) represents a maximal strong module $H$ and node $R$ (right child) represents the graph $g(G, H, h)$. If their representative graphs are not prime, then $L$ and $R$ in turn have children defined by some maximal strong modules. Thus, the leaves of $T(G)$ represent prime induced subgraphs of $G$. Figure 1 shows a graph $G$, Figure 2 shows $T(G)$ together with the representative graphs of the nodes of $T(G)$. A well known and easy proof by induction shows that the number of internal nodes of $T(G)$ is at most $2|V(G)|$ and the total number of edges in all prime graphs (produced by the decomposition) is at most $|E(G)|$. There are well known linear time algorithms to construct $T(G)$ and the associative graphs of its internal nodes from $G$ ([8], [18], see also the survey paper [12]).

Let $P_k$ (resp., $C_k$) denote the chordless path (resp., cycle) on $k$ vertices. If $F$ is a set of graphs, then we say a graph $G$ is $F$-free if $G$ does not contain an induced subgraph isomorphic to any of the graphs in $F$. A buoy is the graph whose vertex set can be partitioned into non-empty sets $S_1, S_2, S_3, S_4, S_5$ such that there are all edges between $S_i$ and $S_{i+1}$ and no edges between $S_i$ and $S_{i+2}$ with the subscript taken modulo 5. A buoy is complete if every $S_i$ is a complete graph.

Given an ordered graph $(G, <)$, the ordering $<$ is called perfect if for each induced ordered subgraph $(H, <)$ the greedy algorithm produces an optimal coloring of $H$. The graphs admitting a perfect order are called perfectly orderable. A stable set of a graph $G$ is strong if it meets all maximal cliques of $G$. (Here, as usual, “Maximal” is meant with respect
Figure 1: The graph $G$

Figure 2: The decomposition tree $T(G)$
to set-inclusion, and not size. In particular, a maximal clique may not be a largest clique.)
A graph is strongly perfect if each of its induced subgraphs contains a strong stable set. In [6], it is proved that perfectly orderable graphs contain strong stable sets and therefore are strongly perfect.

When $G$ is an input graph to some algorithm, $n(G)$ (resp., $m(G)$) denotes the number of vertices (resp., edges) of $G$. When the context is obvious, we will write $n = n(G)$ and $m = m(G)$.

**Theorem 1** [13] If there is a polynomial time algorithm $A$ to find a strong stable set of a strongly perfect graph then there is a polynomial time algorithm $B$ to find a minimum weighted coloring and maximum weighted clique of a strongly perfect graph. If algorithm $A$ runs in time $O(f(n))$ then algorithm $B$ runs in time $O(nf(n))$. 

In [7], it is proved that $(P_5, P_5, C_5)$-free graphs are perfectly orderable and that a strong stable set of a $(P_5, P_5, C_5)$-free graph can be found in $O(n + m)$ time. So the following result follows from Theorem 1.

**Corollary 1** MWC can be solved for $(P_5, P_5, C_5)$-free graphs in $O(n(n + m))$ time. 

In [9], the following result is obtained on the structure of $(P_5, P_5)$-free graph with a $C_5$.

**Theorem 2** [9] Let $G$ be a connected $(P_5, P_5)$-free graph having at least five vertices. If $G$ contains an induced $C_5$ then every $C_5$ is contained in a buoy and this buoy is either equal to $G$ or is a non-trivial module of $G$.

**Corollary 2** A prime $(P_5, P_5)$-free graph is either $C_5$-free or is the $C_5$.

In section 4 we will remark on several classes of graphs and so we need to introduce more definitions now.

- A graph $G$ is **chordal** if it does not contain as induced subgraphs the chordless cycle $C_k$ for $k \geq 4$.

- A graph $G$ is a **thin spider** if its vertex set can be partitioned into a clique $C$ and a stable set $S$ with $|C| = |S|$ or $|C| = |S| + 1$ such that the edges between $C$ and $S$ are a matching and at most one vertex is not covered by the matching.

- A graph is a **thick spider** if it is the complement of a thin spider.

- A graph $G$ is **matched co-bipartite** if its vertex set can be partitioned into two cliques $C_1, C_2$ with $|C_1| = |C_2|$ or $|C_1| = |C_2| + 1$ such that the edges between $C_1$ and $C_2$ are a matching and at most one vertex is not covered by the matching.

- A graph $G$ is **co-matched bipartite** if $G$ is the complement of a matched co-bipartite graph.

- A bipartite graph $B = (X, Y, E)$ is a **bipartite chain graph** if there is an ordering $x_1, x_2, \ldots, x_k$ of all vertices in $X$ such that $N(x_i) \subseteq N(x_j)$ for all $1 \leq i < j \leq k$. (Note that then also the neighborhoods of the vertices from $Y$ are linearly ordered by set inclusion.) If, moreover, $|X| = |Y| = k$ and $N(x_i) = \{y_1, \ldots, y_i\}$ for all $1 \leq i \leq k$, then $B$ is prime.
• G is a co-bipartite chain graph if it is the complement of a bipartite chain graph.

• G is an enhanced co-bipartite chain graph if it can be partitioned into a co-bipartite chain graph with cliques $C_1, C_2$ and three additional vertices $a, b, c$ ($a$ and $c$ are optional) such that $N(a) = C_1 \cup C_2$, $N(b) = C_1$ and $N(c) = C_2$, and there are no other edges in G.

• G is an enhanced bipartite chain graph if it is the complement of an enhanced co-bipartite chain graph.

3 MWC algorithm for ($P_5, P_5$)-free graphs

Consider a weighted graph $G$ where each vertex $x$ has a weight $w_G(x)$. Let $H$ be a non-trivial module of $G$. By $f(G, H, h)$, we denote the weighted graph obtained from $G$ by substituting a vertex $h$ for $H$ where the weight function $w$ for $f_w(G, H, h)$ is defined as follows. With $F = f_w(G, H, h)$, for the vertex $h$, we let $w_F(h) = \chi_w(H)$ and $w_F(x) = w_G(x)$ for all $x \in G - H$.

**Theorem 3** For a weighted graph $G$, we have $\chi_w(f(G, H, h)) = \chi_w(G)$. Furthermore, given weighted coloring of $f(G, H, h)$ and $H$ with, respectively, $a$ and $b$ stable sets, a minimum weighted coloring of $G$ can be constructed in $O(n(a + b))$ time.

**Proof of Theorem 3** Write $F = f(G, H, h)$. We will first prove $\chi_w(F) \leq \chi_w(G)$. Consider a minimum weighted coloring of $G$ with stable sets $S_1, S_2, \ldots, S_t$ with each $S_i$ having weight $I(S_i)$. Let $\mathcal{X}$ be the stable sets $S_i$ with $S_i \cap H = \emptyset$. Write $W = \sum_{S_i \in \mathcal{X}} I(S_i)$. Since the restriction of the stable sets of $\mathcal{X}$ to $H$ is a weighted coloring of $H$, we have $W \geq \chi_w(H)$. Construct a weighted coloring $Y_1, Y_2, \ldots, F$ of the stable sets $S_1, S_2, \ldots$ as follows. For each $S_i$, if $S_i \cap H = \emptyset$ then $Y_i = S_i$; otherwise $Y_i = (S_i - H) \cup \{h\}$. Then let $I(Y_i) = I(S_i)$. To verify that the stable sets $Y_i$ is a weighted coloring of $F$, we only need see that $w(h) = \chi_w(H) \leq W = \sum_{y \in Y_i} I(Y_i)$. Thus, we have $\chi_w(F) \leq \sum_{i=1}^{\mathcal{X}} I(Y_i) = \sum_{i=1}^{\mathcal{X}} I(X_i) = \chi_w(G)$.

To complete the theorem, we will now prove $\chi_w(F) \geq \chi_w(G)$. Let $\mathcal{X}$ (resp., $Y$) be the collection of stable sets $X_1, X_2, \ldots, X_a$ (resp., $Y_1, Y_2, \ldots, Y_b$) with weights $I(X_i)$ (resp., $I(Y_i)$) be a minimum weighted coloring of $H$ (resp., $F = f(G, H, h)$). We can rearrange the stable sets $Y_i$’s such that there is an integer $c$ such that $h \in Y_i$ for $i \leq c$, and $h \notin Y_i$ for $i > c$. We will describe an algorithm that produces a (minimum) weighted coloring of $G$ with a collection $Z$ of stable sets $Z_i$ and integers $I(Z_i)$ with $\sum_{Z_i \in Z} I(Z_i) = \sum_{Y_i \in Y} I(Y_i) = \chi_w(F)$ (the detail is spelled out in Algorithm 2 of the Appendix). The algorithm takes as input the list $L_1$ of stable sets $X_1, X_2, \ldots, X_a$ of $H$, and the list $L_2$ of stable sets $Y_1, Y_2, \ldots, Y_b$ of $F$, and produces the desired sets $Z$. We scan sequentially the stable sets $X_1, X_2, \ldots, X_a$ of $L_1$ and in parallel the stable sets $Y_1, \ldots, Y_c$ of $L_2$ and merge them into stable sets of $Z$. Suppose $X_i$ and $Y_j$ are being scanned. We merge them into a stable set of $Z$ by introduce a stable set $Z_k = X_i \cup Y_j - h$. If $I(X_i) \leq I(Y_j)$, then we give $Z_k$ the weight of $X_i$, i.e. $I(Z_k) = I(X_i)$, and reduce the weight of $Y_j$ appropriately, i.e. $I(Y_j) = I(Y_j) - I(X_i)$. Now, $X_i$ can be eliminated from the first list ($Y_j$ remains in the second list if its weight is not zero). Similarly, if $I(X_i) > I(Y_j)$, then we give $Z_k$ the weight of $Y_j$, i.e. $I(Z_k) = I(Y_j)$, and reduce the weight of $X_i$ appropriately; now $Y_j$ can be eliminated from the second list. Since $\sum_{i=1}^{\mathcal{X}} I(Y_i) \geq \sum_{i=1}^{\mathcal{X}} I(X_i)$, after $Y_c$ is processed, all the stable sets in the first list will be
eliminated. Now, the stable sets $Y_{c+1}, \ldots, Y_b$ in the second list are made to be stable sets of $Z$; and we have $\sum_{Z_i \in Z} I(Z_i) = \sum_{Y_i \in Y} I(Y_i) = \chi_w(F)$. It is easy to verify that the stable sets $Z_i$ form a weighted coloring of $G$. The algorithm produces at most $a + b$ stable sets, and each stable set has size at most $n$. This establishes the claimed time bound. $\square$

**Theorem 4** Let $C$ be a hereditary class of graphs. If there is an $O(f(n))$ MWC algorithm for every prime graph in $C$, then there is an $O(n^2 f(n))$ MWC algorithm for every graph in $C$. $\square$

**Proof of Theorem 4** As remarked in section 2, the modular decomposition produces $O(n)$ prime graphs. The result then follows from Theorem 3. $\square$

Now, we turn our attention to solving MWC for weighted $(P_5, \overline{P_5})$-free graphs.

**Theorem 5** There is an $O(n^3)$ algorithm to solve MWC for $(P_5, \overline{P_5})$-free graph.

**Proof of Theorem 5** Let $G$ be a $(P_5, \overline{P_5})$-free graph. Use the modular decomposition algorithms of [18] or [8] to construct the decomposition tree $T(G)$ with root $S$. If $G$ is a prime $(P_5, \overline{P_5})$-free graphs, then $G$ is the $C_5$ or $(P_5, \overline{P_5}, C_5)$-free and we are done by Corollary 4. Otherwise, consider the left child $L$ and the right child $R$ of $S$ in $T(G)$. Let $H$ be the representative graph of $L$, that is, $r(L) = H$. We know $H$ is a non-trivial module of $G$. We now recursively solve MWC on $H$ and $f(G, H, h)$, the latter being the representative graph of $R$. Given minimum weighted colorings of $H$ and $f(G, H, h)$, we apply the stable sets merging algorithm of Theorem 3 to construct a minimum weighted coloring of $G$. The detail is spelled out in Algorithms 1 and 2 in the Appendix. We start the algorithm by calling COLOR($S$) on the root $S$ of $T(G)$. We may assume the total time used by COLOR-PRIME() on all graphs produced by the algorithm is $O(n(n + m))$ since the total number of edges in all prime graphs is bounded by $m$. Assume without loss of generality COLOR-PRIME($G$) returns a minimum weighted coloring of a prime $(P_5, \overline{P_5})$-graph $G$. An easy proof by induction shows that the number of stable sets in the minimum weighted coloring produced by the call COLOR($S$) is at most $2n - 1$. Each call to MERGE-COLOR can be implemented in $O(n^3)$ time. Since the number of internal nodes of $T(G)$ is $O(n)$ (see [8]), the number of calls to MERGE-COLOR is $O(n)$. It follows our algorithm runs in $O(n^3)$ time. $\square$

4 MWC algorithms for some related graph classes

In the previous section, we provide a polynomial time algorithm to find a minimum weighted coloring of a $(P_5, \overline{P_5})$-free graph. The insight of our result is that to solve MWC for a hereditary class of graphs, only prime graphs need to be considered. It turns out that this idea can be used to solve MWC for several graph classes that have been studied in the literature. These graph classes are defined by forbidden certain graphs defined in Figure 3 below. For these classes of graphs, it has been proved that the prime graphs in the classes have special structures (such as being perfect) and therefore it is easy to solve MWC for them. We will now elaborate on this point. Consider the following theorems.
Theorem 6 [4] Let $G$ be a prime graph.

(i) If $G$ is (diamond, co-diamond)-free then $G$ or $\overline{G}$ is a matched co-bipartite graph or $G$ has at most nine vertices.

(ii) If $G$ is (paw, co-paw)-free then $G$ is a $P_4$ or $C_5$.

Theorem 7 [1] Prime ($P_5$,diamond)-free graphs are either matched co-bipartite or a thin spider or an enhanced bipartite chain graph or have at most 9 vertices.

There are polynomial time MWC algorithms for all graphs described in Theorems 6 and 7 because bipartite graphs, co-bipartite graphs, matched co-bipartite graph, spiders, and enhanced bipartite chain graphs are perfect graphs; and there is a well known MWC algorithm for perfect graphs [11]. In some special cases, there are fast MWC algorithms. For example, spiders are chordal graphs and so the MWC problem can be solved in $O(n^2)$ time [13] on them.

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Algorithm 1 COLOR($X$)

**input:** Node $X$ in $T(G)$ with representative graph $r(X)$, $G$ being a weighted $(P_5, \overrightarrow{P_5})$-free graph.

**output:** A minimum weighted coloring of $r(X)$.

if $X$ is a leaf of $T(G)$ then
    return the output of COLOR-PRIME($r(X)$)
else
    Let $L$ and $R$ be the left and right children of $X$ in $T(G)$ where $r(L)$ is a maximal module of $r(X)$
    Call COLOR($L$) to get a minimum weighted coloring of $r(L)$
    Call COLOR($R$) to get a minimum weighted coloring of $r(R)$
    Call MERGE-COLOR($X, L, R$) and output a minimum weighted coloring of $r(X)$
end if
Algorithm 2 MERGE-COLOR($X, L, R$)

input:
$X, L, R$ are nodes of $T(G)$ with $L$ (resp., $R$) being the left (resp., right) child of $X$.
A minimum weighted coloring of $H = r(L)$ with stable sets $X_1, X_2, \ldots X_a$ with weights $I(X_i)$
A minimum weighted coloring of $f(r(X), H, h) = r(R)$ with stable sets $Y_1, Y_2, \ldots Y_b$ with weights $I(Y_i)$

output: A minimum weighted coloring of $r(X)$ with stable sets $Z_1, \ldots, Z_d$ with weights $I(Z_i)$ with $d \leq a + b$.

1. Enumerate the stable sets of $f(r(X), H, h)$ as $Y_1, \ldots, Y_c, Y_{c+1}, \ldots Y_b$ such that $h \in Y_i$ if $i \leq c$, and $h \notin Y_i$ otherwise
2. $i \leftarrow 1, j \leftarrow 1, k \leftarrow 1$
3. while $i \leq a$ do
   $Z_k \leftarrow X_i \cup Y_j - h$
   if $I(X_i) \leq I(Y_j)$ then
      $I(Z_k) \leftarrow I(X_i)$
      $i \leftarrow i + 1$
      $I(Y_j) \leftarrow I(Y_j) - I(X_i)$
      if $I(Y_j) = 0$ then
         $j \leftarrow j + 1$
      end if
   else
      $I(Z_k) \leftarrow I(Y_j)$
      $I(X_i) \leftarrow I(X_i) - I(Y_j)$
      $j \leftarrow j + 1$
   end if
   $k \leftarrow k + 1$
end while
4. for $r = j \rightarrow b$ do
   $Z_k \leftarrow Y_r$
   $k \leftarrow k + 1$
end for
Output the stable sets $Z_1, Z_2, \ldots$