A lower bound for the modulus of the Dirichlet eta function on a partition $\mathcal{P}$ from 2-D principal component analysis

Yuri Heymann

Abstract The aim of the present manuscript is to derive an expression for the lower bound of the modulus of the Dirichlet eta function on vertical lines $\Re(s) = \alpha$. The approach is based on a two-dimensional principal component analysis built on a parametric ellipse, to match the dimensionality of the complex plane. The lower bound $\forall s \in \mathbb{C}$ s.t. $\Re(s) \in \mathcal{P}$, $|\eta(s)| \geq 1 - \frac{\sqrt{2}}{s}$, where $\eta$ is the Dirichlet eta function is related with the Riemann hypothesis as $|\eta(s)| > 0$ for any $s \in \mathbb{C}$ s.t. $\Re(s) \in \mathcal{P}$, where $\mathcal{P}$ is a partition spanning one half of the critical strip on either side of the critical line $\Re(s) = 1/2$ depending upon a variable. We propose the composite lower bound $\forall s \in \mathbb{C}$ s.t. $\Re(s) \in ]1/2, 1[$, $|\eta(s)| \geq \min \left(1 - \frac{\sqrt{2}}{s}, \sqrt{\frac{3}{2}} - \frac{\sqrt{2}}{s} \right)$, resulting from transitivity of $\eta : \mathbb{C} \to \mathbb{C}$ in $\eta(s) = (1 - \frac{1}{s}) \zeta(s)$. The solution space of the set of solutions referring to such $L^2$-problem is a representation of the space spanned by explanatory variables satisfying its algebraic form, as a foundation of a Harrison Ford proof (see the little proof).

Keywords Dirichlet eta function, PCA, Analytic continuation

1 Introduction

The Dirichlet eta function is an alternating series related to the Riemann zeta function of interest in the field of number theory for the study of the distribution of primes [14]. Both series are tied together on a two-by-two relationship expressed as $\eta(s) = (1 - 2^{1-s}) \zeta(s)$ where $s$ is a complex number. The location of the non-trivial zeros of the Riemann zeta function in the critical strip $\Re(s) \in ]0, 1[$ is key in the prime-number theory. For example, the Riemann-von Mangoldt explicit formula, as an asymptotic expansion of the prime-counting function, involves a sum over the non-trivial zeros of the Riemann zeta function [5]. The Riemann hypothesis, which scope is the domain of existence of the zeros in the critical strip, has implications for the accurate estimate of the error involved in the prime-number theorem and a variety of conjectures such as the Lindelöf hypothesis [5], conjectures about short intervals containing primes [5], Montgomery’s pair correlation conjecture [10], the inverse spectral problem for fractal strings [12], etc. Moreover, variants of the Riemann hypothesis falling under the generalized Riemann hypothesis in the study of modular L-functions [10] are core for many fundamental results in number theory and related fields such as the theory of computational complexity. For instance, the asymptotic behavior of the number of primes less than $x$ described in the prime-number theorem, $\pi(x) \sim \frac{x}{\ln(x)}$, provides a smooth transition of time complexity as $x$ approaches infinity. As such, the time complexity of the prime-counting function

Yuri Heymann
E-mail: y.heymann@yahoo.com
Address in Switzerland: 9 rue Chantepoulet, 1201 Geneva.
using the \( x/(\ln x) \) approximation is of order \( O(M(n) \log n) \), where \( n \) is the number of digits of \( x \) and \( M(n) \) is the time complexity for multiplying two \( n \)-digit numbers. This figure is based on the time complexity to compute the natural logarithm with the arithmetic-geometric mean approach where \( n \) represents the number of digits of precision.

The below definitions are provided on an informal basis as a supplement to standard definitions when referring to reals, complex numbers and holomorphic functions. A complex number is the composite of a real and imaginary number, forming a 2-D surface, where the pure imaginary axis is represented by the letter \( i \) such that \( i^2 = -1 \). The neutral element and index \( i \) form a basis spanning some sort of vector space. The Dirichlet eta function is a holomorphic function having for domain a subset of the complex plane where reals are positive denoted \( \mathbb{C}^+ \), which arguments are sent to codomain in \( \mathbb{C} \). As an holomorphic function it is characterized by its modulus, a variable having for support an axial vertex orthonormal to the complex plane. The conformal way to describe space in geometrical terms is the orthogonal system, consisting of eight windows delimited by the axes of the Cartesian coordinates, resulting from the union of the three even surfaces of Euclidean space.

A Hilbert space as an extension of the former, is a multidimensional space which in current context of some functions in \( L^2 \) space referring to squared-integrable functions, is comprised of wave functions expressing components or basis elements of space, further equipped of an inner product defining a norm and angles between these functions. The well-known Riemann hypothesis as part of the eighth problem David Hilbert presented at the International Congress of Mathematicians in Paris in 1900 [5], has strong relation with such Hilbertian spaces, as unveiled in the remaining of the manuscript.

As a reminder, the definition of the Riemann zeta function and its analytic continuation to the critical strip are displayed below. The Riemann zeta function is commonly expressed as follows:

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

where \( s \) is a complex number and \( \Re(s) > 1 \) by convergence in the above expression.

The standard approach for the analytic continuation of the Riemann zeta function to the critical strip \( \Re(s) \in ]0, 1[ \) is performed with the multiplication of \( \zeta(s) \) with the function \( \left(1 - \frac{2}{s} \right) \), leading to the Dirichlet eta function. By definition, we have:

\[
\eta(s) = \left(1 - \frac{2}{2^{s}}\right) \zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},
\]

where \( \Re(s) > 0 \) and \( \pi \) is the Dirichlet eta function. By continuity as \( s \) approaches one, \( \eta(1) = \ln(2) \).

The function \( \left(1 - \frac{2}{s} \right) \) has an infinity of zeros on the line \( \Re(s) = 1 \) given by \( s_k = 1 + \frac{2k \pi i}{\ln 2} \) where \( k \in \mathbb{Z}^\ast \). As \( \left(1 - \frac{2}{2^{s}}\right) = 2 \times \left(2^{s-1} e^{i \beta \ln 2} - 1 \right) / 2^s e^{i \beta \ln 2} \), the factor \( \left(1 - \frac{2}{s} \right) \) has no poles nor zeros in the critical strip \( \Re(s) \in ]0, 1[ \). As such, the Dirichlet eta function can be used as a proxy of the Riemann zeta function for zero finding in the critical strip \( \Re(s) \in ]0, 1[ \).

From the above, the Dirichlet eta function is expressed as:

\[
\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} e^{-i \beta \ln(n)}}{n^s},
\]

where \( s = \alpha + i \beta \) is a complex number, \( \alpha \) and \( \beta \) are real numbers.
2 Mathematical development

2.1 Elementary propositions no. 1 ~ 4

**Proposition 1** Given $z_1$ and $z_2$ two complex numbers, we have:

$$|z_1 + z_2| \geq |z_1| - |z_2|,$$

where $|z|$ denotes the modulus of the complex number $z$. This is the well-known reverse triangle inequality, which is valid for any normed vector space (including complex numbers), where the norm is subadditive over its domain of definition, see [27][11].

**Proposition 2** Let us consider an ellipse $(x_t, y_t) = [a \cos(t), b \sin(t)]$ where $a$ and $b$ are two positive reals corresponding to the lengths of the semi-major and semi-minor axes of the ellipse $(a \geq b)$ and $t \in [0, 2\pi]$ is a variable having a correspondance with the angle between the x-axis and the vector $(x_t, y_t)$.

Let us set $t$ such that the semi-major axis of the ellipse is aligned with the x-axis, which is the angle maximizing the objective function defined as the modulus of $(x_t, y_t)$. When $|(x_t, y_t)|$ is maximized, we have:

$$|(x_t, y_t)| = x_t + y_t = a.$$  

(5)

Note that by maximizing $x_t + y_t$, we would get $|(x_t, y_t)| < x_t + y_t$, as the expression $x_t + y_t$ is maximized when $t = \arctan(b/a)$, leading to $\max(x_t + y_t) = \sqrt{a^2 + b^2}$.

**Proof** (Background and construction) Principal Component Analysis (PCA), is a statistical concept for reducing the dimensionality of a variable space by representing it with a few orthogonal variables capturing most of the variability of an observable. An ellipse centered on the origin of the coordinate system can be parametrised as follows: $(x_t, y_t) = [a \cos(t), b \sin(t)]$ where $a$ and $b$ are positive real numbers corresponding to the lengths of the semi-major and semi-minor axes of the ellipse $(a \geq b)$ and $t \in \mathbb{R}$ is a variable having correspondance with the angle between the x-axis and the vector $(x_t, y_t)$. The objective function $|(x_t, y_t)|$ is maximized with respect to $t$ when the major axis is aligned with the x-axis. When $|(x_t, y_t)|$ is at its maximum value, we have $|(x_t, y_t)| = x_t + y_t = a$ where $a$ is the length of the semi-major axis.

The modulus of $(x_t, y_t)$ is as follows: 

$$|(x_t, y_t)| = \sqrt{a^2 \cos^2(t) + b^2 \sin^2(t)} \leq a, \forall t \in \mathbb{R}.$$ 

In the principal component analysis representation, $x_t$ represents the first principal component and $y_t$ the second principal component. Say $x_t$ and $y_t$ were not orthogonal, then there would be a non-zero phase shift $\varphi$ between the components, i.e. $x_t = a \cos(t)$ and $y_t = b \sin(t + \varphi)$.

**Proposition 3** Given a vector $V = [u(E), v(E)]$ defined in a bidimensional vector space, where $u(E)$ and $v(E)$ are two real-valued functions say on $\mathbb{R}^\nu$ $\rightarrow \mathbb{R}$, where $\nu$ represents the degrees of freedom of the system. The reference of a point in such system, is described by the set $E = \{e_1, e_2, ..., e_\nu\}$ representing a multidimensional coordinate system. Thus, we have:
\[ |V_2| = u(\mathcal{E}) + v(\mathcal{E}), \]  

if only \( u(\mathcal{E}) + v(\mathcal{E}) = 0 \) and \( u(\mathcal{E}) + v(\mathcal{E}) \geq 0 \).

Given a basis set \( \{e_1, e_2\} \) of the above-mentioned vector space, where \( |e_i| = 1 \) for \( i = 1, 2 \), proposition 3 is true if only the inner product across basis elements is equal to zero, i.e. \( e_1 \cdot e_2 = 0 \), meaning that the basis elements are disentangled from each other. We say that \( \{e_1, e_2\} \) is an orthonormal basis. This condition is also necessary for propositions 2 and 4 to be true, in 2-D Cartesian frame.

**Proof** By the square rule, we have \( (u + v)^2 = u^2 + v^2 + 2uv \). The modulus of a vector \( V \) as defined in such two-dimensional frame is \( |V| = \sqrt{u^2 + v^2} \), leading to \( |V| = |u + v| \) if only \( uv = 0 \), which is provided as a complement to proposition 2. The above as a support of pre-Hilbertian spaces by the scalar product \( uv \), is a prerequisite for Hilbertian spaces of squared-integrable functions referring to such \( \mathcal{L}^2 \) spaces equipped of an inner product.

**Proposition 4** Given a circle of radius \( r \in \mathbb{R}^+ \) parametrized as follows: \( (x_t, y_t) = [r \cos(t), r \sin(t)] \) where \( t \) is a real variable in \( [0, 2\pi] \), we construct a function \( f(t) = a \cos(t) + b \sin(t + \varphi) \) where \( a \) and \( b \) are two positive reals and \( \varphi \) a real variable which can be positive or negative such that:

\[ r \cos(t) + r \sin(t) = a \cos(t) + b \sin(t + \varphi), \]

\( \forall t \in \mathbb{R} \) and where \( \varphi \) is a real variable of \( t \) (when \( a \) and \( b \) are scalars).

As an excerpt of below proof elements, \( \forall t \in [0, 2\pi] \) and \( \forall \delta \in [-r, r] \) we have:

\[ r \cos(t) + r \sin(t) = (r + \delta) \cos(t) + \sqrt{r^2 + \delta^2} \sin(t + \varphi), \]

where \( \varphi = -\arctan \delta/r \), making \( \varphi \) independent of \( t \) by some correspondence between \( a \), \( b \) and \( \varphi \) represented as parametric functions of \( \delta \) and where \( \delta/r \in [-1, 1] \).

As such \( a = r + \delta \) and \( b = \sqrt{r^2 + \delta^2} \) where \( \delta = -r \tan(\varphi) \), yielding \( a/r = 1 - \tan \varphi \) and \( b/r = \sqrt{1 + \tan^2 \varphi} \) where \( \varphi \in ]-\pi/4, \pi/4[ \).

Say \( u_t = a \cos(t) \) and \( v_t = b \sin(t + \varphi) \).

When \( a \geq b \), the first component \( u_t \) is the one carrying most of the variance\(^1\) of expression \( f(t) \), meaning it is leading component. Thus, we have:

\[ |(x_t, y_t)| \leq \max(v_t) \leq \max(u_t), \]

\( \forall t \in [0, 2\pi] \), where \( \max(u_t) \) is the maximum value of \( u_t \) and \( \max(v_t) \) the maximum value of \( v_t \) over the interval \([0, 2\pi]\).

When \( a \leq b \), the component \( v_t \) carries most of the variance of \( f(t) \) meaning it is leading component, and we have:

\[ \max(u_t) \leq |(x_t, y_t)| \leq \max(v_t), \]

\( \forall t \in [0, 2\pi] \), where \( \max(u_t) \) and \( \max(v_t) \) as above.

When the functions \( u_t \) and \( v_t \) are orthogonal i.e. \( \varphi = 0 \), we have \( r = a = b \).

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\(^1\) The variance of a \( \mathbb{R} \times \mathbb{R} \) univariate function \( f : t \rightarrow f(t) \) represented as \( f_t = f(t) \) is measured by its variance over time \( t \), e.g. \( \text{Var}(f(t)) = \frac{1}{|t_a - t_b|} \int_{t_a}^{t_b} f^2(t) \, dt \) over interval \([a, b] \). The variance of a component of \( f(t) \) is defined by its attribution of variance with respect to total variance.
In standard notations, the point $s$ is the lower bound of the Dirichlet eta modulus as a floor function as per the one-sided RH test. A little proof of the Riemann hypothesis is elaborated for the RH to be true. When in the critical strip, then $\Re \eta$ of the function.

According to the RH (Riemann hypothesis) all non-trivial zeros lie on the critical line $\Re(s) = 1/2$, meaning that RH is true if only $|\eta(s)| > 0$ for any $\alpha \in [0, 1/2]$ and $\Re(s) = 1/2, 1]$. The one-sided RH test is a consequence of the Riemann zeta functional, leading to prop 7 in [6], i.e. Given $s$ a complex number and $\bar{s}$ its conjugate, if $s$ is a zero of the Riemann zeta function in the strip $\Re(s) \in ]0, 1]$, then $1 - \bar{s}$ is also a zero of the function.

The converse is also true, meaning if $s$ is not a zero of the Riemann zeta function in the critical strip, then $1 - \bar{s}$ is not a zero as well. The one-sided test is enough for the RH to be true. When $\alpha = 1/2$, $|\eta(s)| > 0$ means the Dirichlet eta function can have some zeros on the critical line $\Re(s) = 1/2$, which is known to be true [15], p. 256. As the Dirichlet eta function and the Riemann zeta function share the same zeros in the critical strip, we have to show that $|\eta(s)| > 0$ for any $s$ on the critical strip not on the critical line to say the Riemann hypothesis is true.

In the present study, we propose several lower bounds for the modulus of the Dirichlet eta function that are one-sided lower bounds (i.e. apply on one side of the critical strip). These are eqns. (17), (18) and (20). Note the zeros of the Dirichlet eta function on the line $\Re(s) = 1$, produce violations of (17) in the neighbourhood of such zeros. Lower bounds (18) and (20) suggest that $|\eta(s)| > 0$ when $\alpha \in [1/2, 1]$ as per the one-sided RH test. A little proof of the Riemann hypothesis is elaborated in section 4, by analysing the solution set of a pair of Taylor polynomials.

3 The lower bound of the Dirichlet eta modulus as a floor function

In standard notations, the point $s$ is expressed as $s = \alpha + i \beta$ where $\alpha$ and $\beta$ are reals in their corresponding basis belonging to $\mathbb{C}$.

Note the zeros of the Dirichlet eta function and its complex conjugate are the same. For convenience, we introduce the conjugate of the Dirichlet eta function, expressed as follows:

$$\bar{\eta}(s) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{e^{i \beta \ln n}}{n^\alpha},$$

where $\Re(s) > 0$. By applying the reverse triangle inequality to (10) (see proposition 1), we get:

$$|\bar{\eta}(s)| \geqslant 1 - \left| \sum_{n=2}^{\infty} (-1)^{n+1} \frac{e^{i \beta \ln n}}{n^\alpha} \right|,$$

where $|z|$ denotes the norm of the complex number $z$.

With respect to expression $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{e^{i \beta \ln n}}{n^\alpha}$, its decomposition into sub-components $u_n = \frac{(-1)^{n+1}}{n^\alpha} e^{i \beta \ln n}$ is a vector representation where $\beta \ln n + (n + 1)\pi$ is the
angle between the real axis and the orientation of the vector itself, and where $\frac{1}{e}$ is its modulus. The idea is to apply a rotation by an angle $\theta$ to all component vectors simultaneously, resulting in a rotation of the vector of their sum. The resulting vector after rotation $\theta$ expressed in Euler’s notation is $v_{\theta, \beta} = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^{\alpha}} e^{i(\beta \ln n + \theta)}$, where $\theta, \alpha$ and $\beta$ are real numbers.

Let us introduce the objective function $w$, defined as the sum of the real and imaginary parts of $v_{\theta, \beta}$, i.e. $w = v_x + v_y$ where $v_x = \Re(v_{\theta, \beta})$ and $v_y = \Im(v_{\theta, \beta})$. We get:

$$w = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^{\alpha}} \left( \cos(\beta \ln n + \theta) + \sin(\beta \ln n + \theta) \right)$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^{n+1} \sqrt{2}}{n^{\alpha}} \cos \left( \beta \ln n + \theta - \frac{\pi}{4} \right).$$

The trigonometric identity $\cos(x) + \sin(x) = \sqrt{2} \cos \left( x - \frac{\pi}{4} \right)$ which follows from $\cos(a) \cos(b) + \sin(a) \sin(b) = \cos(a - b)$ with $b = \frac{\pi}{4}$ is invoked in (12), see [1] formulas 4.3.31 and 4.3.32, p. 72. The finite sum of a subset of the elements of the second line of (12) from 2 to $n \in \mathbb{N}$ is further referred to as the $w$-series.

While orthogonality between vectors is defined in terms of the scalar product between such pairs, for real functions on $\mathbb{R}$ to $\mathbb{R}$ we usually define an integration product forming an $L^2$ space. Let us say we have two real-valued functions $f$ and $g$, which are squared-Lebesgue integrable on a segment $[a, b]$ and where the inner product between $f$ and $g$ is given by:

$$\langle f, g \rangle = \int_{a}^{b} f(x) g(x) dx.$$  
(13)

The functions $f$ and $g$ are squared-Lebesgue integrable, meaning such functions can be normalized i.e. the squared norm as defined by $\langle f, f \rangle$ is finite. For sinusoidal functions such as sine and cosine, it is common to say $[a, b] = [0, 2\pi]$, which interval corresponds to one period. The condition for functions $f$ and $g$ to be orthogonal is that the inner product as defined in (13) is equal to zero.

We proceed with the decomposition of the objective function $w$ into dual components $w_1$ and $w_2$, expressed as follows:

$$w_1 = -\frac{\sqrt{2}}{2^\alpha} \cos \left( \beta \ln(2) + \theta - \frac{\pi}{4} \right),$$

and

$$w_2 = \sum_{n=3}^{\infty} \frac{(-1)^{n+1} \sqrt{2}}{n^{\alpha}} \cos \left( \beta \ln n + \theta - \frac{\pi}{4} \right),$$

where most of the variance of the $w$-series comes from the leading component in a direction of $L^2$ space.

By construction $w$ is the sum of the real and imaginary parts of $v_{\theta, \beta}$ which are orthogonal functions. Let us say $v_{\theta}$ is the parametric notation of $v_{\theta, \beta}$ for a given $\beta$ value, and $\alpha$ implicitly. We note that for any given $\beta$ value, the complex number $v_{\theta}$ describes a circle in the complex plane, centered on the origin. Hence, in symbolic notations, $w$ can be written as $w = r \cos(t) + r \sin(t)$. The components $w_1$ and $w_2$ can be expressed as $w_1 = a \cos(t)$ and $w_2 = b \sin(t + \varphi)$ where $t$ is a variable in $[0, 2\pi]$ and $\varphi$ some variable in $\mathbb{R}$. As we suppose that $w_1$ carries most of the variance of $w$ (i.e. $a \gg b$), the modulus $|v_{\theta}|$ is smaller or equal to the maximum value of $|w_1|$, by proposition 4. Yet, $|v_{\theta}|$ is equal to max{|$w_1$|, if $w_1$ and $w_2$ are orthogonal and $t = 0$. By proposition 2, at its maximum value $|\langle w_1, w_2 \rangle| = w_1 + w_2$ when $\varphi = 0$ and $t = 0$, which in light of the above, is also equal to the maximum value of
As the inner product between $w_1$ and $w_2$ does not depend on $\theta$, orthogonality between $w_1$ and $w_2$ is determined by $\beta$ values. We then apply a rotation by an angle $\theta$ to maximize the objective function $[(w_1, w_2)]$. We consider two complementary scenarios respectively, depending whether $w_1$ is the leading component of the $u$-series or some other function i.e. $w_2$ as the alternative, or in a direction of $L^2$ space.

**When $w_1$ is leading component:**

Say $w_1$ is leading component in some regions of the critical strip denoted $\mathcal{A}$. In this scenario, as we suppose $w_1 \text{ and } w_2$ are orthogonal at the maximum value of $||(w_1, w_2)||$, i.e. $w_1 = \frac{\sqrt{2}}{2}$ and $w_2 = 0$, we get $\max||(w_1, w_2)|| = \frac{\sqrt{2}}{2}$, also equal to max $|v_{\theta, \beta}|$ by **proposition 2**.

If we suppose that $w_1$ and $w_2$ are not orthogonal and form an accute angle, by **proposition 4** eq. (8) we would get $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\epsilon^{i \beta \ln n}}{\ln n} < \frac{\sqrt{2}}{2}$, and $|\eta(s)|$ would be strictly larger than zero when $\alpha = 1/2$ in (11). This would imply that the Dirichlet eta function does not have zeros on the critical line $\Re(s) = 1/2$, which is known to be false. Hence, we can say that when $w_1$ is the leading component, the functions $w_1$ and $w_2$ are orthogonal at the maximum value of $|v_{\theta}|$, i.e. $w_1$ acts as first principal component. As we suppose that:

$$\forall s \in \mathbb{C}^+ \text{ s.t. } s \in \mathcal{A}, \quad \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\epsilon^{i \beta \ln n}}{\ln n} \leq \frac{\sqrt{2}}{2^\alpha},$$

(16)

where $\mathbb{C}^+$ is the subset of $\mathbb{C}$ s.t. $\Re(s) > 0$ (acronym s.t. standing for "such that").

Say for $\Re(s) = \alpha \geq \frac{1}{2}$, (11) and (16) imply that:

$$|\eta(s)| \geq \left| 1 - \frac{\sqrt{2}}{2^\alpha} \right|,$$

(17)

for any $s \in \mathcal{A}$ s.t. $\Re(s) = \alpha \in [1/2, \infty[$, where $\mathcal{A} \subseteq \mathbb{C}^+$. This is a one-sided lower bound function $L_\alpha$ which is always true when (16) applies.

By the conjugate, we have:

$$|\eta(1-s)| \geq \left| 1 - \frac{\sqrt{2}}{2^{1-\alpha}} \right|,$$

(18)

which is proposed as an alternative one-sided lower bound applicable when (17) fails, i.e. as the left-side of the critical strip for $\Re(s) \leq 1/2$ is not affected by the zeros of the Dirichlet eta function on the line $\Re(s) = 1$.

As such the lower bound $\forall s \in \mathbb{C}$ s.t. $\Re(s) \in \mathcal{P}$, $|\eta(s)| \geq \left| 1 - \frac{\sqrt{2}}{2^\alpha} \right|$, where $\eta$ is the Dirichlet eta function is related with the Riemann hypothesis as $|\eta(s)| > 0$ for any $s \in \mathbb{C}$ s.t. $\Re(s) \in \mathcal{P}$, where $\mathcal{P}$ is a partition spanning one half of the critical strip on either side of the critical line $\Re(s) = 1/2$ depending upon a variable.

**When $w_1$ is not the leading component, i.e. $w_2$ as the alternative, or in a direction of $L^2$ space:**

As a complementary of the former, say $\mathcal{A}^c$ is some regions of the critical strip where $w_1$ is not leading. In this scenario, we fall on eq. (9) of **proposition 4**, yielding:

$$\forall s \in \mathbb{C}^+ \text{ s.t. } s \in \mathcal{A}^c, \quad \sum_{n=2}^{\infty} (-1)^{n+1} \frac{\epsilon^{i \beta \ln n}}{\ln n} \geq \frac{\sqrt{2}}{2^\alpha},$$

(19)

where $\mathbb{C}^+$ is defined as above, hence (17) is no longer guaranteed to work and we need some additional thinking.

The function $(1 - \frac{\sqrt{2}}{2^\alpha})$ has an infinity of zeros on the line $\Re(s) = 1$ given by $s_k = 1 + \frac{2k\pi i}{\ln 2}$ where $k \in \mathbb{Z}_*$. The Dirichlet eta function as a product of the former and
the Riemann zeta function has some zeros on the line $\Re(s) = 1$. A violation of (17) by a tiny epsilon was reported by Vincent Granville in the neighbourhood of a point given by $s = 0.75 + 580.13i$. This point is located near a zero of the Dirichlet eta function on the line $\Re(s) = 1$, for $k = 64$. From the pendulum model (see Fig. 1), cases when $\alpha$ approaches 1 producing a violation of (17) in the neighbourhood of zeros on the line $\Re(s) = 1$ are taken care of by replacing the wider gap $1 - \frac{\sqrt{3}}{2}$ by the more narrow gap $\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}$ in 3-d space. We propose the composite lower bound

$$|\eta(s)| \geq \text{Min} \left(1 - \frac{\sqrt{3}}{2\alpha}, \frac{\sqrt{3}}{2\alpha} - \frac{\sqrt{2}}{2}\right), \tag{20}$$

where $\alpha \in ]1/2, 1[$, resulting from transitivity in $\eta(s) = (1 - \frac{2}{\pi}) \zeta(s)$. Case when $\alpha = 1$ show no positive gap (i.e. $|\eta(s)| \geq 0$) due to the zeros on the line $\Re(s) = 1$.

The narrower gap function in the pendulum model is an application of the reverse triangle inequality to function $\frac{1}{\sqrt{2}} \times (1 - \frac{2}{\pi})$. Validity of this gap function follows from observation that $\forall k \in \mathbb{Z}^+$, when $(\beta \ln 2) \in [2k\pi, 2k\pi + 3\pi]$, $3 \delta > 0$ such that $\forall \alpha \in ]1 - \delta, 1]$, $|\zeta(s)| \geq \sqrt{(1 - \frac{2}{\pi}) \cos(\beta \ln 2)} + (\frac{2}{\pi}) \sin(\beta \ln 2))^2 \geq \frac{1}{\sqrt{2}} |1 - \frac{2}{\pi}|$, where $|\zeta(s)| > 0$ on the line $\Re(s) = 1$. The narrow gap function is a specificity of the Dirichlet composite functional i.e. a one-sided leg carrying zeros.

By 2-D PCA, near zeros of $(1 - \frac{2}{\pi})$ at points $s_k = 1 + \frac{2k\pi}{\ln 2} i$ where $k \in \mathbb{Z}$, we have $|\zeta(s_k)| \geq |1 + \frac{1}{2} - \frac{\sqrt{2}}{\alpha}|$, meaning $\forall k \in \mathbb{Z}, |\zeta(s_k)| > \frac{1}{\sqrt{2}}$, as a basis motivating the application of the reverse triangle inequality to $\frac{1}{\sqrt{2}} \times (1 - \frac{2}{\pi})$.

4 The roots of a biquadratic form and higher orders - the Riemann hypothesis little proof (a Harrison Ford proof)

The Riemann hypothesis little proof is elaborated further down.

Given a weight function $w_n \in [-1, 1]$ where $n = 1, ..., N$ and $N \in \mathbb{N}$, the question arises whether the number of non-trivial roots such that $\sum_{n=2}^{\infty} \frac{w_n}{n^{\alpha}} = \sum_{n=2}^{\infty} \frac{w_n}{n^{\alpha-1}} = \lambda_i$ is a countable set, where trivial roots lie on the critical line $\alpha = 1/2$.

By Taylor expansion of $\varphi_n(\alpha) = \frac{1}{\sqrt{n}}$ in $\alpha_0 = 1/2 + \Delta \alpha$, we have:

$$\varphi_n(\alpha) = \varphi_n(1/2) + \varphi_n'(1/2) \Delta \alpha + \frac{\varphi_n''(1/2)}{2!} \Delta \alpha^2 + .... \tag{21}$$
By Taylor expansion of $\phi_n(\alpha) = n^{\alpha-1}$ in $\alpha_0 = 1/2 + \Delta \alpha$, we have:

$$
\phi_n(\alpha) = \phi_n(1/2) + \phi_n'(1/2) \Delta \alpha + \frac{\phi_n''(1/2)}{2!} \Delta \alpha^2 + \ldots 
$$

$$
= \varphi_n(1/2) - \varphi_n'(1/2) \Delta \alpha + \frac{\varphi_n''(1/2)}{2!} \Delta \alpha^2 + \ldots . 
$$

From (21) and (22), $\sum_{n=2}^\infty \frac{\varphi_n'(1/2)}{n}$ is rewritten as follows:

$$
\sum_{n=2}^\infty w_n \varphi_n'(1/2) + \sum_{n=2}^\infty w_n \frac{\varphi_n''(1/2)}{3!} \Delta \alpha^2 + \sum_{n=2}^\infty w_n \frac{\varphi_n''(1/2)}{5!} \Delta \alpha^4 + \ldots = 0 .
$$

As a first guess we suppose the higher order terms of the Taylor expansion are negligible, meaning (23) is asymptotic to a biquadratic equation $Q(x) = ax^4 + bx^2 + c$. Let the auxiliary variable be $z = \Delta \alpha^2$. Then $Q(x)$ can be expressed in quadratic form $q(z) = a_4 z^2 + a_2 z + a_0$, which has at most two real roots given by $z = \frac{a_2 \pm \sqrt{\Delta}}{2a_4}$, with discriminant $\Delta = a_2^2 - 4a_4a_0$. Assuming that the roots of $q(z)$ are expressed as $z = a + bi$ where $a, b \in \mathbb{R}$, we can show that $Q(x) \in \mathbb{R}$ if and only if $b = 0$ or $a = 0$, meaning vertical axis crosses the real line at a point where discriminant is zero. Furthermore, there can be at most one real root of the biquadratic having a value equal to $\lambda = 1$, for any pair of distinct roots (see Fig. 2). As such (23) has a unique root given by $\Delta = 0$ in the biquadratic scenario (i.e. $\lambda$ countable). Unicity of the roots does not necessarily hold for higher orders.

![Fig. 2 Roots of the biquadratic equation Q(x)=0.](image)

Any point on the critical strip beside the trivial case $\alpha = 1/2$, is a zero of the Dirichlet eta function, provided the two conditions set forth below are satisfied simultaneously (see prop 7 in [6], from Riemann zeta functional):

$$
\sum_{n=2}^\infty (-1)^n \frac{\cos(\beta \ln n)}{n^{\alpha}} = \sum_{n=2}^\infty (-1)^n \frac{\cos(\beta \ln n)}{n^{1-\alpha}} = 1 ,
$$

and

$$
\sum_{n=2}^\infty (-1)^n \frac{\sin(\beta \ln n)}{n^{\alpha}} = \sum_{n=2}^\infty (-1)^n \frac{\sin(\beta \ln n)}{n^{1-\alpha}} = 0 .
$$
Stemming out of prop 7, these two equalities stand for the real and imaginary parts of the Dirichlet eta function to be equal to zero simultaneously, respectively on the right-hand side of the critical strip i.e. $\alpha \in ]1/2, 1[$ and conjugated form $1 - \alpha$ (on the left). This entails that both polynomials relative to (24) and (25) as obtained by Taylor expansion of $\varphi_n(\alpha) = \frac{1}{\alpha^n}$ and $\phi_n(\alpha) = n^{\alpha - 1}$ in $\alpha_0 = 1/2$ have to be satisfied simultaneously at a value of $\Delta \alpha$ for a given point to be a zero of the Dirichlet eta function, whereas both sets of weights in (24) and (25) are orthogonal with each other by the sine-cosine orthogonality.

The factors relative to both polynomials as seen in (23) are expressions of the form

$$\sum_{n=2}^{\infty} a_n \cos(\theta_n) + \sum_{n=2}^{\infty} a_n \sin(\theta_n)$$

respectively, where $a_n$ and $\theta_n$ are real series indexed by $n$ natural. Hence, we can say there exists reals $A$ and $B$, such that

$$\sum_{n=2}^{\infty} a_n \cos(\theta_n) = A \cos(\Omega) \quad \text{and} \quad \sum_{n=2}^{\infty} a_n \sin(\theta_n) = B \sin(\Omega).$$

The former system of biquadratic equations extended by Taylor expansions up to order $n$, leads to its canonical form expressed as follows:

$$A \cos(\Omega) + B \cos(\Phi) z + C \cos(\Psi) z^2 + \ldots + L_n \cos(\Upsilon_n) z^n = 0,$$

and

$$A \sin(\Omega) + B \sin(\Phi) z + C \sin(\Psi) z^2 + \ldots + L_n \sin(\Upsilon_n) z^n = 0,$$

where $A$, $B$, $C$, $\Omega$, $\Phi$, $\Psi$ are real variables and $z = \Delta \alpha^2$, with $L_n$ and $\Upsilon_n$ arbitrary letters representing real variables and $n \in \mathbb{N}$.

We can easily show that the system of polynomials (26) and (27) admits for solution points on the critical line as given by $z = 0$ and $A = 0$ and a set of unfeasible solutions. By linear combination of (26) and (27), we get that $\forall \varphi \in \mathbb{R}$:

$$A \sin(\Omega + \varphi) + B \sin(\Phi + \varphi) z + C \sin(\Psi + \varphi) z^2 + \ldots + L_n \sin(\Upsilon_n + \varphi) z^n = 0.$$  \hfill (28)

**Proof** From the trigonometric identity $a \cos x + b \sin x = \sqrt{a^2 + b^2} \sin(x + \varphi)$ and the half-phase $\varphi = \arctan(b/a)$, the linear combination of a unit of (26) with $\lambda$ units of (27), leads to $a \cos x + \lambda a \sin x = \sqrt{1 + \lambda^2} a \sin(x + \varphi)$, where $\varphi = \arctan \lambda$. As such we can factor out $\sqrt{1 + \lambda^2}$ from the terms of the linear combination, yielding (28). By the definition of $\arctan : \mathbb{R} \to ]-\pi/2, \pi/2[$, (28) is defined for all $\varphi \in ]-\pi/2, \pi/2[$ and using a four-quadrant inverse tangent function as ATAN2 [13], (28) is true $\forall \varphi \in \mathbb{R}$.

By excluding the case $z = 0$ and $A = 0$ (zeros on the critical line), a trivial solution satisfying (28) is when all latin letters $A, B, C, \ldots$etc in (26, 27) are equal to zero simultaneously. As an example, the biquadratic scenario where letter $A = 0$ (expansion up to $z^2$), (26) and (27) as a pair carries two candidate solutions $z = 0$ or $z = -B/C$. We show that such solutions where $z \neq 0$ are not feasible. When $A = 0$, (28) is satisfied provided $z = 0$ (by linear combination of (26) and (27)). The case when $A \neq 0$, (28) cannot be satisfied for all $\varphi \in \mathbb{R}$ (it is only satisfied for distinct values of $\varphi$ and $z$). The case all latin letters $A, B, C, \ldots$ etc are equal to zero simultaneously, is a hypothetical scenario elaborated further down.

The question arises whether there exists a solution such that (24) or (25) is satisfied for all $\alpha$ in the critical strip (see scenario all latin letters $A, B, C, \ldots$ etc are equal to zero simultaneously). By Taylor expansion, the curves spanned by such polynomials are asymptotic to a symmetrical shape about axis $\Re(s) = 1/2$, see Fig. 3. This problem can be formulated as follows: Can a linear combination of exponentials $e^{-a_n x}$, where $a_n = \ln n$ is a real series indexed by natural $n$ lead to a symmetrical function about the vertical axis $x = 1/2$?

The asymptotic expression of a linear combination of $e^{-a_n x}$ forming a weighted sum, is defined by function $F : \mathbb{R} \to \mathbb{R}$ where $F(x) = \sum_{n=2}^{N} w_n e^{-a_n x}$, and $w_n$, $n = 2, \ldots, N \in \mathbb{N}$ are the weights. This function is symmetrical with respect to vertical axis $x = 1/2$ if only:
Fig. 3 Hypothetical scenario of a symmetrical curve about the vertical axis $\Re(s) = 1/2$. As the roots of such biquadratics (extendable to polynomials of arbitrary orders) are given by equidistant points to the axis $\Re(s) = 1/2$, the scenario of a symmetrical curve about $\Re(s) = 1/2$ yields an infinity of solutions, i.e. any real $\alpha$ satisfy the biquadratic polynomial.

$$\frac{\partial F(x)}{\partial x} = -\frac{\partial F(1-x)}{\partial x},$$

(29)

for all $x \geq 0$ (as Dirichlet proxy is defined for positive $\alpha$). By expansion into a weighted sum, the former gives $\forall x \in \mathbb{R}^+$:

$$\sum_{n=2}^{N} w_n \left( e^{-a_n x} - e^{-a_n(1-x)} \right) = 0,$$

(30)

which is only true provided all weights are equal to zero. Hence, the construction of a symmetrical function about the vertical axis $x = 1/2$ from a linear combination of exponentials is not feasible without having all weights equal to zero.

The zero-weighted sum scenario only occurs when $\beta = 0$, for which all weights in (25) are equal to zero. Still, $\sum_{n=2}^{\infty} \left( \frac{-1}{n} \right)^n < 1$ for $\forall \alpha > 0$, by upper trailing of the sum of a decreasing alternating series (see [13] p.74), meaning the real part of the Dirichlet eta function is strictly positive when $\Re(s) > 0$.

5 Discussion

The present work employs the Dirichlet eta function as a proxy of the Riemann zeta function for zero finding in the critical strip, and the interpretation about the lower bound of the modulus of the Dirichlet eta function as a floor function. The surface spanned by the modulus of the Dirichlet eta function is a continuum resulting from the application of a real-valued function over the dimensions of the complex plane, which is a planar representation where the reals form a line continuous to the right intersecting the imaginary axis, and where the square of imaginary numbers are subtracted from zero. As a design aspect, the modulus of the Dirichlet eta function is a holographic function sending a complex number into a real number, whereas the floor function is a projection of the former onto the real axis.

In the common scenario when $w_1$ is the leading component, the floor function of the modulus of the Dirichlet eta function on vertical lines $\Re(s) = \alpha$ does not depend on $\beta$, which is reflected by the linear relationship between $\theta$ and $\beta$ in the cosine argument of the first principal component, as a single term of $w$-series. This is no longer the case, when adding together several terms of the $w$-series as first principal component. As a complementary of the former, scenarios when $w_1$ is not leading component i.e. $w_2$ is as the alternative or some direction of $L^2$ space, exist in various
parts of the domain. For such scenarios, though there is no straight-forward linear relationship between \( \theta \) and \( \beta \) of the arguments of component \( w_2 \) as the alternative, for \( L_\alpha = 1 - \sqrt{2}/2^\alpha \) to qualify as floor function in all the domain. \( L_\alpha \) is floor function of the modulus of the Dirichlet by mirror symmetry with respect to line \( \Re(s) = 1/2 \), c.f. (18) in complement of (17). By perfect matching principle under mirror symmetry, components \( w_1 \) and \( w_2 \) at zeros of the Dirichlet eta function resulting from the continuity to the right of the critical line as given by (18), are meant to match corresponding components at such zeros as given by (16) at the limit to the left of line \( \Re(s) = 1/2 \). An inversion of the roles of components \( w_1 \) and \( w_2 \) when approaching the critical line from both sides, suggests there is no such regions where \( w_1 \) is not leading which is contiguous with the critical line.

A special case of polynomial made up of a subset of the cosine terms of the \( w \)-series which does not depend on \( \beta \) occurs, if there exists such a polynomial which is equal to zero for any \( \beta \). For such a polynomial to be first principal component involves subsequent components are also equal to zero, leading to the degenerate case \( |v_\alpha| = 0 \). This occurs when \( \alpha \) tends to infinity, leading to \( w = 0 \) for \( \beta \) real, as a special case of the Dirichlet eta function converging towards unity.

The combination of multiple terms of the \( w \)-series as principal components involves such components are functions composed of terms of the form

\[
a_\alpha = \pm \frac{1}{\pi} \cos(\beta \ln(n) + \theta),
\]

where \( n \) is the index of the corresponding term in \( v_\alpha \). Due to the multiplicity of bivariate collinear arguments in the cosine functions, which comovements are not parallel across the index \( n \) (as a finite set), there is no straight-forward bijection between \( \theta \) and \( \beta \), i.e. a one-degree of freedom relationship, such that all cosine arguments \( \beta \ln n + \theta \) of the component are decoupled from \( \beta \). As aforementioned, the lower bound of the modulus of the Dirichlet eta function needs to be decoupled from \( \beta \), to be a floor function on vertical lines. Moreover, the principal components involved in modulus maximisation need to be disentangled for PCA to be applicable, which in current context means the maximum of \( |\{(w_1, w_2)\}| \) is reached at orthogonality between \( w_1 \) and \( w_2 \), as a prerequisite for \( L_\alpha \) to be a floor function of the modulus of the Dirichlet eta function. As a rule of thumb, one degree of freedom is needed for every additional principal component, when matching the dimensionality of the variable space in the parametric ellipsoidal model.

Lastly, a little proof of the Riemann is provided in section 4, by reformulating the problem of finding the zeros into a system of polynomials by a Taylor expansion. We show that this system admits for solution points on the critical line \( \Re(s) = 1/2 \) and a set of unfeasible solutions resulting from the orthogonality between the weight functions of the polynomials. A solution exists if we can construct a symmetrical function about the vertical axis \( x = 1/2 \) from a linear combination of exponentials (see hypothetical scenario all latin letters in (26, 27) are equal to zero simultaneously), which is prevented by the positivity of the curve, i.e. all weights are not equal to zero when \( \beta \neq 0 \). For the case \( \beta = 0 \), the real part of the Dirichlet eta function never reaches zero.

### 6 Conclusion

The lower bound of the modulus of the Dirichlet eta function in present manuscript as obtained from 2-D principal component analysis is expressed as \( \forall s \in \mathbb{C} \) s.t. \( \Re(s) \in [1/2, 1], |\eta(s)| \geq \text{Min} \left(1 - \sqrt{2}/2^\alpha, \frac{\sqrt{3}}{\pi} - \frac{\sqrt{2}}{\pi} \right) \), where \( \eta \) is the Dirichlet eta function. As a proxy of the Riemann zeta function for zero finding in the critical strip \( \Re(s) \in [0, 1] \), the above as a floor function of the modulus of the Dirichlet eta function has implication for the Riemann hypothesis (i.e. non-trivial zeros lie on the critical line \( \Re(s) = 1/2 \)). We finally develop a little proof of the Riemann hypothesis by linearization of the Dirichlet representation around \( \Re(s) = 1/2 \), i.e. analysing the solutions of a pair of polynomials obtained by a Taylor expansion of \( \varphi_\alpha(x) = \frac{1}{\alpha x} \) and \( \phi_\alpha(x) = \frac{1}{\pi x^2} \) in \( \alpha_0 = 1/2 + \Delta \alpha \). We show that this system admits for solution points on the critical line \( \Re(s) = 1/2 \), and a set of unfeasible solutions resulting
from orthogonality between the weight functions relative to both polynomials. We further verify that a symmetrical function about the vertical axis \( x = 1/2 \) cannot be constructed from a linear combination of exponentials (except when \( \beta = 0 \), all weights are zeros), preventing scenario all latin letters of the canonical form are equal to zero simultaneously.

**Colloquial language and non-standard notations:** Yuri Heymann has over thirteen years’ experience as a practitioner in financial economics. Although, present work covers research in number theory, some of the language in the manuscript comes from statistics as reference to principal component analysis, variable space, parametric models, degrees of freedom and dimensionalities. Some of the notations are non-standard as alpha-numerical letters from latin and non-latin alphabet.

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