A Linear Quantum Dynamic Theory for Coherent Output of Bose-Einstain Condensation

C. P. Sun\textsuperscript{a,c,} J. M. Li\textsuperscript{b,} H. Zhan\textsuperscript{c,} Y. X. Miao\textsuperscript{c,} S. R. Zhao\textsuperscript{d} and G. Xu\textsuperscript{a}

\textsuperscript{a}Institute of Theoretical Physics, Academia Sinica, Beijing 100080, China
\textsuperscript{b}Department of Physics, Peking University, Beijing 100871, China
\textsuperscript{c}Institute of Theoretical Physics, Northeast Normal University, Changchun 130024, China
\textsuperscript{d}Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada, N6A, 5B7
\textsuperscript{e}Department of Physics, the Chinese University of Hong Kong, Shatin, NT, Hong Kong

ABSTRACT

A model for the coherent output coupler of the Bose-Einstein condensed atoms from a trap in the recent MIT experiment (Phys. Rev. Lett., 78(1997)582) is established with a simple many-boson system of two states with linear coupling. Its exact solution for the many-body problem shows a factorization of dynamical evolution process, i.e., the wave function initially prepared in a direct product of a vacuum state and a coherent state remains in a direct product of two coherent states at any instance in the evolution of the total system. This conclusion always holds even for a system with a finite average particle number in the initial state. Its thermodynamical limit can be directly dealt with in the Bogoliubov approximation and manifests that an ideal condensate in the trap will remain in a coherent state after the r.f. interaction while the output-coupler pulse of atoms is also in a coherent state, which means a coherent output of atomic beam to form a macroscopic quantum state in a propagating mode.

PACS numbers:03.65,03.75,05.30
1. Introduction

According to de Broglie matter wave theory, all matter in general can behave itself like a light that spreads out in space and combines with another one to form interference pattern. In principle, it is logical to image that a “matter wave laser” can be realized, in analog to a light laser, to produce the output coherent matter wave. Based on such fundamental consideration and stimulated by the experiments observing atomic Boson-Einstein condensate (BEC) in last two years [1-3], a number of efforts coming from different research groups have been placed on both its experimental setup and theoretical possibilities [4-7]. Until the end of last year, a milestone experiment for the matter wave laser of bosonic atoms (also called atom laser) was accomplished by Mewes et al in MIT [8].

In their experiment, they coupled the toms trapped in a state $|1\rangle$ with that untrapped in another state $|2\rangle$ by a sweeping radiation frequency (r.f.) field. The r.f. field coherently turns the BEC atoms constrained in $|1\rangle$ into a propagating mode of atoms in $|2\rangle$. The experiment shows that “the condensate which is initially in a coherent state remains in such a state after the r.f. interaction while the output-coupler pulse of atom is also in a coherent state” [8]. In fact, a laser of photons above threshold can be understood in terms of the coherent state for its well-defined classical analogy with minimum quantum fluctuation in amplitude [9]. Similarly, it is reasonable to believe that such a output of atoms in a single coherent state is something like the photon laser. Roughly, we can regard the dynamic process of wave function evolution driven by the r.f. field as a “stimulated” process of (massive) matter wave.

In this paper, we first use the Bogoliubov approximation [10,11] in thermodynamic limit to give a theoretical analysis for the dynamic lasering process of BEC atoms in the MIT experiment. We will show that such an approximation is not actually needed to reach the conclusion about coherent output in the requirement of very large $N_c$. Namely, even for the case that the number $N_c$ of atoms condensed in the ground state is not very large, thereby the Bogoliubov approximation is violated, the output of finite atoms is still coherent so long as the system is interatomic interaction free and the atoms in the trap is initially in a coherent state. Theoretically, this is the factorization structure for the evolution of the many-boson model of two state with linear coupling, i.e., the wave function at any instance remains in a direct product of two coherent states if the system is initially prepared in a direct product of a vacuum state and a coherent state. This fact does not depend on the average number of condensed atoms in the initial state and it implies that the experiment result by Mewes et al is quite steady even for rather finite $N_c$.

2. A Dynamic Model of Output Coupler via Bogoliubov Approximation

The second quantization of model Hamiltonian for the MIT experiment can be written as

$$H = \hbar \omega_a b_1^{\dagger} b_2 + \hbar \omega_R \left[ b_2^{\dagger} b_1 \exp \left( -i \int_0^t \omega(\tau) d\tau \right) + h.c \right]$$

(1)

in terms of the creation and annihilation operators, $b_1^{\dagger}$, $b_2^{\dagger}$, $b_1$ and $b_2$, of bosonic atoms for the magnetically trapped state $|1\rangle$ and the untrapped state $|2\rangle$ with level difference $\hbar \omega_a$. The r.f. pulse of varying frequency $\omega(t)$ is a sweeping classical electromagnetic field, which couples $|1\rangle$ and $|2\rangle$ through the dipole matrix element $\hbar \omega_R = \sqrt{\hbar \omega/2 \epsilon_0 V} \equiv \hbar g/\sqrt{V}$. $V$ is the effective mode
volume and $\varepsilon_0$ is the vacuum permittivity. This many atom system without interatomic interaction can be modeled as a linear coupling system of two-oscillators. The main simplification is to ignore the quantized motion of atomic center of mass in the trapped state by an inhomogeneous magnetic field. As pointed out by Mewes et al, this inhomogeneous feature can be neglected for a sufficiently short r.f. pulse.

The MIT experiment is theoretically described as an initial value problem for Schrödinger equation governed by $H$ with the initial state $|\psi(0)\rangle = |\alpha = \sqrt{N_c}\rangle_1 \otimes |0\rangle_2$. Here, $|\alpha = \sqrt{N_c}\rangle_1$ is a Glauber coherent state of the operator $b_1$ characterizing $N_c$ atoms condensed in the trapped state $|1\rangle$. No atoms occupy the untrapped state $|2\rangle$ initially. Why does the coherent state $|\alpha\rangle$ represent the BEC is still an open question, but there are some reasons enable one to believe it is right, such as the phase locking implied by BEC [12] and the correct average atomic number $N_c$. The additional reason we believe is due to the connection with the Bogoliubov approximation [10,11], which replaces operators $b_1$ and $b_1^{\dagger}$ with a $c$-number $\sqrt{N_c}$ when much many atoms condensate in the trapped state. A factorizable evolution structure in the MIT experiment is just rooted in this connection and will be discussed as following. There is a close relation between the selection of initial coherent state $|\alpha\rangle$ and the Bogoliubov approximation. In fact, a unitary transformation $|\phi(t)\rangle = D^{-1}(\alpha)|\psi(t)\rangle$, in terms of the generator $D(\alpha) = \exp[\alpha b_1^{\dagger} - \alpha b_1]$ of coherent state, defines an equivalent initial value problem of Schrödinger equation for $|\phi(0)\rangle = |0\rangle_1 \otimes |0\rangle_2$ and the equivalent Hamiltonian

$$D(\alpha)^{-1}H D(\alpha) = H + \hbar \omega R \sqrt{N_c} \left[ b_2^{\dagger} \exp \left( -i \int_0^t \omega(\tau) d\tau \right) + h.c \right].$$

For the case of very large $N_c$ in BEC, the coupling term is very small in comparison with the term of $\hbar \omega R \sqrt{N_c}$ and can be neglected to obtain an effective Hamiltonian

$$\mathcal{H}_B = \hbar \omega \sqrt{N_c} \left[ b_2^{\dagger} \exp \left( -i \int_0^t \omega(\tau) d\tau \right) + h.c \right].$$

(2)

This is just the Bogoliubov approximation of the original Hamiltonian $H$.

Now let’s consider the thermodynamical limit of an infinite number of atoms in an infinite volume but with the density fixed, i.e., $n_c = \lim_{N,V \to \infty} (N_c/V) \to constant$, the coupling term $\hbar \omega R \propto 1/\sqrt{V} \to 0$, but $\hbar \omega R \sqrt{N_c} = \hbar g \sqrt{n_c}$ is finite. Therefore, in the point of view of Schrödinger evolution, the Bogoliubov approximation for our problem just describes an excitation motion of system in a large background provided by the BEC initial state. Namely, for a very large $N_c$, the system initially in a coherent state remains in such a state while another component $|\Phi(t)\rangle$ is governed by the Bogoliubov approximate Hamiltonian. Mathematically, the evolution wave function enjoys a factorizable structure

$$|\psi(t)\rangle = |\alpha\rangle \otimes |\Phi(t)\rangle.$$  

(3)

The similar factorization structure has been shown in the analysis about the transition from quantum domain to macroscopic, classical domain concerning the wave function collapse in quantum measurement [13] and the macroscopic damping in quantum dissipation [14,15].
method calculating wave function developed in [14,15] will be used in this paper to obtain the exact solution for factorized wave function in section 4.

3. Coherent Output via Factorization

The next step is to prove $|\Phi(t)\rangle$ is still a coherent state evolving from its vacuum state $|0\rangle_2$. It is quite clear that the Bogoliubov approximation Hamiltonian (2) is just a forced harmonic oscillator (FHO). It is well-known that the displacement property of a quantized FHO results dynamically in a coherent state for the initial vacuum state. By considering the case of $\omega(\tau)$ independent of time, i.e., $\omega(\tau) = \omega$, the solution of the Hamiltonian (2) can be found in some standard textbook [16] $|\Phi(t)\rangle = |\tilde{\alpha}(t)\rangle$ with

$$\tilde{\alpha}(t) = \frac{\omega_R \sqrt{N_c}}{\Delta} (e^{i\Delta t} - 1) e^{-i\omega_a t}, \quad (4)$$

Namely, the total wave function in the BEC with large $N_c$ can be factorized in the direct product of two coherent states $|\psi(t)\rangle = |\alpha\rangle \otimes |\tilde{\alpha}(t)\rangle$. The second components implies the output-couple pulse of the atomic beam is in the coherent state, which “can be regarded as a pulsed atomic laser”.

Notice again that the above theoretical discussion about the output-coupler for atomic laser is only with the help of a highly-simplified many-boson model of two states with linear coupling rather than the complicated nonlinear one, such as the system of Gross-Petavskii equation. Because the atomic number of the coherent output in the untrapped state $\langle \psi(t)|b_{2}^\dagger b_{2}|\psi(t)\rangle = |\tilde{\alpha}|^2 = 2 \frac{\omega_a^2 N_c}{\Delta^2} (1 - \cos \Delta t)$ (5) is an oscillating function of $t$, the sweeping mechanism [17, 8] of the r.f. pulse must be invoked here to localized the population in the untrapped state at $t = \infty$. This will be in the aid of Landau-Zener method [17] to control the populations in trapped and untrapped states by adjusting the time-dependent frequency $\omega(t)$ of the r.f. pulse from diabetic to adiabatic point in the avoid level crossing. A direct treatment for the r.f. sweeping is changing the factors $\exp(\pm i\omega t)$ into $\exp\left(\pm i \int_0^t \omega(\tau) d\tau\right)$ of $H$ in the above approach. Near the resonance $\omega(t_0) = \omega_a$, $\Delta \approx \dot{\omega}(t_0)(t - t_0)$,

$$\tilde{\alpha}(\infty) = ig \sqrt{n_c} \int_0^\infty \exp\left(\frac{i}{2} \dot{\omega}(t_0)(t - t_0)^2\right) dt = ig \left[ \frac{\pi n_c}{4\omega(t_0)} \right]. \quad (6)$$

This leads to a fixed coherent population in the untrapped state, which can be controlled by adjustment of the changing rate $\dot{\omega}(t_0)$ of the frequency of the r.f. pulse at resonance.

4. Coherence From Finite Condensed Atoms

The above discussion shows that the factorized structure of wave function crucially result in the output of coherent atomic beam as an atomic laser when the initial state of system is prepared on the BEC in a trap for which the Bogoliubov approximation is hold. This fact seems that the factorizable structure of evolution depends on the Bogoliubov approximation, namely, it is decided by whether or not there are very many atoms in the BEC trapped state. In the following, it will be demonstrated by the exact solution of the Hamiltonian (1) that the requirement of
very large \( N_c \) is not necessary so long as the atoms are prepared in a coherent state \( |\alpha\rangle \) even with rather finite average atom number \( N_c = \langle \alpha | b_1^\dagger b_2 | \alpha \rangle \). The coherent state is only a special Poisson superposition of infinite number states with matched phases \( \Theta_n = n\theta, n = 1, 2, \ldots \).

The second reason to consider the exact solution concerns the analogy in the photon case. In the theory of light laser, a sufficient strong light field can be treated as a classical electromagnetic field and it can be regarded as a coherent state above the threshold [9]. In this sense, the quantum effect mainly results from the quantum fluctuation, which is demonstrated by a weak light field. Therefore, for the matter wave of massive particles, the infinite number case only corresponds to the classical analogy of photon field. Then, we need to study the case corresponding to the quantum fluctuation of the weak light field, for which the Bogoliubov approximation is broken down.

According to [14,15], the exact solution of the wave function of the linear system in the Schrödinger picture can be easily obtained from that of the canonical operators, such as \( b_1(t) \) and \( b_2(t) \), in Heisenberg picture. In the case of \( \omega(\tau) \) independent of time, by solving a system of one-order differential equations resulting from the Heisenberg equations of \( b_1(t) \) and \( b_2(t) \), the exact expressions of these solutions are

\[
\begin{align*}
\alpha_1(t) &= \frac{1}{\sqrt{\Delta^2 + 4\omega^2_R}} \left[ \omega_+ e^{i\omega_+ t} - \omega_- e^{i\omega_- t} \right] e^{-i\omega t}, \\
\alpha_2(t) &= \frac{-\omega_R}{\sqrt{\Delta^2 + 4\omega^2_R}} \left[ e^{i\omega_+ t} - e^{i\omega_- t} \right], \\
\beta_1(t) &= \frac{-\omega_R}{\sqrt{\Delta^2 + 4\omega^2_R}} \left[ e^{i\omega_+ t} - e^{i\omega_- t} \right] e^{-i\omega t}, \\
\beta_2(t) &= \frac{1}{\sqrt{\Delta^2 + 4\omega^2_R}} \left[ \omega_+ e^{i\omega_+ t} - \omega_- e^{i\omega_- t} \right] e^{-i\omega t}, \\
\omega_\pm &= \frac{1}{2} \left( \Delta \pm \sqrt{\Delta^2 + 4\omega^2_R} \right). 
\end{align*}
\]

From the property of the evolution operator, \( U(t) \), of the total system, i.e., \( b_\alpha(t) = U(t)^\dagger b_\alpha(0) U(t) \), \( (\alpha = 1, 2) \), the initial state \( |\psi(0)\rangle = |\alpha = \sqrt{N_c}\rangle \otimes |0\rangle \) will evolve into

\[
|\psi(t)\rangle = e^{-\frac{i}{2} |\alpha|^2} \sum_{n=0}^\infty \frac{\alpha^n}{n!} b_1^\dagger(-t)|0\rangle \otimes |0\rangle \\
= e^{-\frac{i}{2} |\alpha|^2} \sum_{n=0}^\infty \frac{\alpha^n}{n!} \left( \alpha_1(-t)b_1^\dagger(0) + \alpha_2(-t)b_2^\dagger(0) \right)^n |0\rangle \otimes |0\rangle. 
\]
That is just a direct product of two coherent states

\[
|\psi(t)\rangle = |\alpha_1(-t)\rangle \otimes |\alpha_2(-t)\rangle
\]

\[
= |\sqrt{N_c}e^{-\frac{i}{2}\Delta t} (\cos(\Omega t) + i \sin(\Omega t) \cos \theta)\rangle \otimes | -i\sqrt{N_c}e^{-\frac{i}{2}\Delta t} \sin \theta \sin(\Omega t)e^{-i\omega t}\rangle
\]

where \(tg\theta = 2\omega_R/\Delta\), where \(\Omega = \sqrt{\Delta^2/4 + \omega_R^2}\).

This result indeed leads to a coherent output of atoms in the untrapped state so long as the atoms are initially prepared as the BEC in trapped state even for the finite average number of atoms. In this output state with wave vector \(k\), the expectation value

\[
\langle \hat{\phi}(x) \rangle = \langle \psi(t)|\hat{\phi}(x)|\psi(t)\rangle = \sum_k \sqrt{\frac{2\hbar n_c}{\omega}} \sin \theta \sin(\Omega t) \sin \left(kx - \frac{1}{2}\Delta t - \omega t\right)
\]

of the bosonic field, \(\hat{\phi}(x) = \sum_k \sqrt{\hbar/2\omega V} \left(b_k e^{ikx} + h.c\right)\), looks just like a classical coherent field, where the variance of the field \(\Delta \phi = \sqrt{\langle \hat{\phi}(x)^2 \rangle - \langle \hat{\phi}(x) \rangle^2}\) do not depend on \(N_c\). This means the relative uncertainty of the amplitude, \(\Delta \phi/\langle \hat{\phi}(x) \rangle\), will approach zero only in the thermodynamics limit: \(N_c, V \rightarrow \infty\), but \(N_c/V \rightarrow constant\). Therefore, for finite \(N_c\), the quantum fluctuation \(\Delta \phi \propto \sqrt{\hbar}\) can be neglected for the dynamic problem. The Bogoliubov approximation in the last section is just the thermodynamics limit of the results in this section. Notice that, in the thermodynamics limit, we have \(\omega_R \rightarrow 0, \Omega \rightarrow \frac{\Delta}{2}\), but \(N_c \sin \theta \propto \sqrt{n_c}\) do not approach zero.

5. Discussions

In summary, for the MIT experiment on the output coupler of atomic laser, we have firstly shown that, if the atoms with an average number \(N_c\) are accumulated in a single quantum state \(|1\rangle\) to form a coherent state \(|\alpha = \sqrt{N_c}\rangle_1\) and no atoms occupy the untrapped state \(|2\rangle\), the spin-like dynamics of Rabi oscillation will create a factorized wave function \(|\psi(t)\rangle = |\alpha(t)\rangle_1 \otimes |\beta(t)\rangle_2\rangle\), which is a product of two coherent states. Even for the case without ideal Bose-Einstein condensation that \(N_c\) is not sufficiently large and thus \(|\alpha = \sqrt{N_c}\rangle_1\) is not a macroscopic quantum state, this wave-function of evolution remains in a factorizable structure so long as \(|\alpha = \sqrt{N_c}\rangle_1\) is coherent states. However, it seems very difficult to prepare finite numbers of atoms in a coherent state due to the massive feature of atoms. The spin-like dynamics of Rabi oscillation will only create an entanglement evolution state \(|\psi(t)\rangle = \sum_{ij} |\alpha_j(t)\rangle_1 \otimes |\beta_i(t)\rangle_2\rangle\) for an arbitrary initial components \(|\psi(0)\rangle_1\) other than a coherent state. Only in BEC case that the fixed phase and amplitude of macroscopic wave function are defined and thus to form a Glauber coherent state, this entanglement state can be completely factorized.

Such a factorized structure of wave function first pointed out by Mewes et al is very crucial to realize their experiment, but it dynamical origins mainly depend on the linearity of coupling. For the case with an interatomic interaction in the trapped or untrapped states, which roughly is of the form \(\hat{b}_i^\dagger \hat{b}_j^\dagger \hat{b}_i \hat{b}_j\), the corresponding Heisenberg equations are no longer linear and thus this factorized structure will be broken down. Therefore, the sufficiently weak interaction should be required for a coherent output in practical system. In fact, Ballagh et al [18] have modeled a theory for the evolution of two component BEC by generalizing the Gross-Pitaevskii equation.
Their theory introduces a non-linear element resulting from the interatomic interaction and shows a rich variety of phenomena of nonlinear effect such as the localization of atom in the untrapped state due to the nonlinear coherent coupling.

Notice that an interference experiment was also carried out by Andrews et al in the same research group at MIT [19] to examine the coherence of atoms in BEC by overlapping two expanding condensates flying from the double well magnetically-trapped potential when this trap is adiabatically switched off. The next step following this experiment was immediately accomplished to test the coherence of the output couplers rather than a pair of expanding atomic sources. They also have shown that a similar high-contrast regular pattern of interference. Based on an intuitional understanding of the MIT “atomic laser” in terms of coherent states described in this paper, a many-body quantum theory for the interference of output couplers can be present in a forecoming paper as a space-dependent generalization of present results. The non-vanishing order parameter of quantum system, concerning a symmetry broken involving the atomic number conservation, can be obtained for the description of regular pattern of stripes in these interference experiments.

One (CPS) of the authors wish to express his sincere thanks to K. Young for inviting him to visit the Chinese University of Hong Kong as a C. N. Yang’s Fellow. The work is supported in part by the NSF of China.
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