High-order numerical method for scattering data of the Korteweg–De Vries equation

A Gudko$^{1,2}$, A Gelash$^{3,4}$ and R Mullyadzhanov$^{1,2}$

$^1$Institute of Thermophysics SB RAS, Novosibirsk 630090, Russia
$^2$Novosibirsk State University, Novosibirsk 630090, Russia
$^3$Institute of Automation and Electrometry SB RAS, Novosibirsk 630090, Russia
$^4$Skolkovo Institute of Science and Technology, Moscow 121205, Russia

E-mail: a.gudko@g.nsu.ru

Abstract. Nonlinear wavefields governed by integrable models such as the Korteweg–De Vries (KdV) equation can be decomposed into the so-called scattering data playing the role of independent elementary harmonics evolving trivially in time. A typical scattering data portrait of a spatially localised wavefield represents nonlinear coherent wave structures (solitons) and incoherent radiation. In this work we present a fourth-order accurate algorithm to compute the scattering data within the KdV model. The method based on the Magnus expansion technique provides accurate information about soliton amplitudes, velocities and intensity of the radiation. Our tests performed using a box-shaped wavefield confirm that all components of the scattering data are computed correctly, while the test based on a single-soliton solution verifies the declared order of a numerical scheme.

1. Introduction

The Korteweg–De Vries (KdV) equation represents a fundamental model to describe the propagation of nonlinear waves with a weak dispersion and nonlinearity. This is a remarkable example of a nonlinear partial differential equation which can be completely integrated using the Inverse scattering transform (IST) method [1, 2]. In a standard non-dimensional form the KdV equation is written for a real-valued function $u(x, t)$ with $x$ and $t$ playing the role of space and time coordinates:

$$u_t + 6uu_x + u_{xxx} = 0.$$ (1)

This equation describes the evolution of surface waves on shallow water, long internal waves in density-stratified fluids or ion acoustic waves in a plasma [1, 2]. The KdV equation features a wide family of exact multi-soliton solutions describing stable propagation of solitary waves and their elastic collisions.

The IST method establishes a one-to-one correspondence between the wavefield $u(x, t_0)$ at a certain moment of time and the so-called scattering data which may be interpreted as a nonlinear Fourier transform spectrum. The scattering data provides information about the number of solitons in the wavefield and their parameters, i.e. their amplitudes, velocities and spatial positions as well as about incoherent nonlinear dispersive waves. Most importantly the IST theory establishes a trivial time evolution of the scattering data providing a powerful tool to describe nonlinear wave propagation [3, 4]. The scattering data is typically found using the...
Direct scattering transform (DST), which can be performed analytically only for a few cases, stimulating the development of advanced numerical tools for different integrable models and physical problems [3, 5, 6, 7, 8, 9, 10, 11, 12, 13]. The pioneering numerical DST algorithm for the KdV equation was suggested by Osborne [14], see also [15], followed by its further efficient improvements [4, 16]. Meanwhile, our recent studies [17, 18] reveal numerical difficulties in identification of soliton positions. Thus, the issue of developing efficient high-order accurate DST algorithms is of utmost importance for fundamental and practical needs.

In this work we present a fourth-order accurate algorithm to compute scattering data within the KdV model. Our approach is based on the application of the Magnus expansion [19], see also [13], to the basic DST equations similar to our recent developments for the nonlinear Schrödinger model. Our approach is based on the application of the Magnus expansion [19], see also [13], to the basic DST equations similar to our recent developments for the nonlinear Schrödinger equation [17]. We verify this algorithm using box-shaped and single-soliton wavefields.

2. Inverse scattering transform

Historically the IST method was discovered in seminal works [20, 21] using the KdV equation. They revealed the key role of the following linear problem for the operator $L$:

$$
L \phi = \lambda^2 \phi, \quad L = -\partial_{xx} - u(x,t),
$$

where $\phi(x, t)$ is the wave function and $\lambda = k + i \alpha$ is the time-independent spectral parameter. Eq. (2) is solved at a fixed moment of time $t_0$ assuming $t_0 = 0$ with $u(x) = u(x,t_0)$ playing the role of the potential. This is a well-known problem for stationary Schrödinger equation in quantum mechanics [22]. For spatially localized and rapidly decaying potentials $u(x)$ the eigenvalues of the operator $L$ represent a finite number of discrete points $\{\lambda_n = i \alpha_n\}, n = 1, ..., N$ and the real line $\lambda = k \in \mathbb{R}$. It is sufficient to consider only the upper half of the complex plane with $\lambda \geq 0$ [2]. We rewrite Eq. (2) in the form of a $2 \times 2$ system of the first order linear equations for the vector wave function $\Psi = (\phi, \phi')^T$:

$$
\Psi' = A \Psi, \quad A = \begin{pmatrix} 0 & 1 \\ -\lambda^2 - u(x) & 0 \end{pmatrix},
$$

where the prime denotes differentiation with respect to $x$ and the superscripts $T$ stands for the transposition. We impose the following boundary conditions for real values of the spectral parameter corresponding to the right scattering problem:

$$
\Psi|_{x \to -\infty} = \begin{pmatrix} e^{-ikx} \\ ike^{-ikx} \end{pmatrix}, \quad \Psi|_{x \to \infty} = \begin{pmatrix} a(k)e^{-ikx} + b(k)e^{ikx} \\ -ika(k)e^{-ikx} + ikb(k)e^{ikx} \end{pmatrix},
$$

where $a(k)$ and $b(k)$ are the scattering coefficients.

The first coefficient has an analytic continuation $a(\lambda)$ to the upper $\lambda$-plane with zeros at the discrete eigenvalues $\lambda_n$. The second coefficient is defined on the real axis and at the eigenvalue points $\lambda_n$ with $b(\lambda_n) = b_n$. Note that $b(k)$ can be analytically continued to the $\lambda$-plane only if the potential $u(x)$ has compact support, i.e. when $u = 0$ outside of a compact set in space [23]. The scattering coefficients are connected to the wavefield scattering data $\{\lambda_n, \rho_n; r\}$ as follows:

$$
a(\lambda_n) = 0, \quad \rho_n = (b_n/\partial_\lambda a)|_{\lambda = \lambda_n}; \quad r(k) = b(k)/a(k),
$$

where $\{\lambda_n, \rho_n\}$ represent the eigenvalues and associated norming constants (discrete spectrum), while $r(k)$ is the reflection coefficient defined on a real axis (continuous spectrum). The scattering data (5) is in one-to-one correspondence with the potential $u(x)$ and can be used to reconstruct it with the IST procedure, see [2]. Most importantly the time evolution of the scattering data is trivial:

$$
\lambda_n = i \alpha_n = \text{const}, \quad \rho_n(t) = \rho_n(0)e^{8\alpha_n^2 t}, \quad r(k,t) = r(k,0)e^{8k^2 t}.
$$
Within this theory the Cauchy initial-value problem for the KdV equation can be solved by identifying the scattering data with the DST for $u(x)$, finding its evolution in time and applying the IST recovering $u(x,t)$.

For further convenience we define the following extended $4 \times 4$ linear system for the vector wave function $\Phi = (\Psi, \partial_\lambda \Psi)^T$:

$$
\Phi' = \bar{A} \Phi, \quad \bar{A} = \left( \begin{array}{cc} A & 0 \\ \partial_\lambda A & A \end{array} \right),
$$

(7)

where $\partial_\lambda$ stands for the derivative with respect to the eigenvalue $\lambda$.

### 3. Scattering coefficients and $S$-matrix

We introduce a $4 \times 4$ $S$-matrix which transfers the wave function $\Phi$ from $-\infty$ to $\infty$ in the spatial domain, see also Eq. (7):

$$
\Phi|_{x \to \infty} = \left( \sum \frac{\partial_\lambda \Sigma}{\Sigma} \right) \Phi|_{x \to -\infty},
$$

(8)

where $\Sigma(\lambda)$ is a $2 \times 2$ matrix for $\Psi$, i.e. $\Psi|_{x \to \infty} = \Sigma \Psi|_{x \to -\infty}$. Further we consider a finite domain of the width $L$ and move the asymptotics for $\Phi(x)$ from infinity to $[-L/2, L/2]$. Thus, Eq. (8) can be presented as the following:

$$
\Phi(L/2) = \begin{pmatrix}
S_{11} & S_{12} & 0 & 0 \\
S_{21} & S_{22} & 0 & 0 \\
S_{31} & S_{32} & S_{11} & S_{12} \\
S_{41} & S_{42} & S_{21} & S_{22}
\end{pmatrix}
\Phi(-L/2),
$$

(9)

Resolving the system of linear equations with respect to scattering coefficients, we obtain

$$
a = e^{i\lambda L} \frac{-i\lambda S_{12} - S_{21}/i\lambda + S_{11} + S_{22}}{2},
b = \frac{[S_{11} - S_{22} - i\lambda S_{12} + S_{21}/i\lambda]}{2},
\partial_\lambda a = e^{i\lambda L} \frac{-iS_{21} + \lambda(-LS_{21} + \lambda^2(LS_{12} - iS_{32}) + iS_{41} + \lambda(-iS_{12} + iLS_{11} + S_{22}) + S_{31} + S_{42})}{(2\lambda^2)},
\partial_\lambda b = \frac{-S_{21} + \lambda(S_{41} + \lambda(\lambda S_{32} + S_{12} + iS_{31} - iS_{42}))}{(2i\lambda^2)}.
$$

(10)

Note, that considering the finite domain we guarantee the presence of the compact support for $u(x)$. This features allows one to analytically continue both scattering coefficients to the upper half of the $\lambda$-plane and finally use arbitrary $\lambda$ in Eq. (10) and further, see also Sec. 2. Thus, the numerical algorithm is expected to express the elements of $S$-matrix in order to compute the whole set of scattering data, see Eq. (5).

### 4. Magnus expansion and numerical algorithm

We discretize the interval $[-L, L]$ into $M$ bins and denote the width of $m$th bin as $\Delta x_m$ with its center at $x_m$. To derive a general solution of Eq. (3) within the interval $[x_c, x_p]$ we use the Magnus expansion [19] leading to the following expression:

$$
\Psi(x_p) = U(x_m)\Psi(x_c), \quad U(x_m) = e^{\Omega(x_m)},
$$

(11)

where $\Omega$ represents an infinite series:

$$
\Omega(x_m) = \sum_{j=1}^{\infty} \Omega_j(x_m),
$$

(12)
with the following first two terms:

$$\Omega_1(x_m) = \int_{x_c}^{x_p} A(x) dx,$$

(13)

$$\Omega_2(x_m) = \frac{1}{2} \int_{x_c}^{x_p} dx_1 \int_{x_c}^{x_1} dx_2 [A_1, A_2],$$

(14)

where $x_c = x_m - \Delta x_m/2$ and $x_p = x_m + \Delta x_m/2$. $[A, B] = AB - BA$ is the matrix commutator, $A_i = A(x_i)$. The potential appearing in the matrix $A$ can be represented using the Taylor series within $m$th bin:

$$u(x) = u(x_m) + u'(x_m)(x - x_m) + u''(x_m)(x - x_m)^2/2 + \ldots$$

(15)

These derivations allow us to use the Taylor series for $\Omega(x_m)$ within the Magnus expansion in powers of $\Delta x_m$ assuming the interval width is small. The trace-vanishing feature of $A$ leads to the form of matrix $U$ as follows:

$$e^{\Omega(x_m)} = \begin{pmatrix}
\cosh \kappa_m + \zeta_m \sinh \kappa_m/\kappa_m \\
\beta_m \sinh \kappa_m/\kappa_m \\
\cosh \kappa_m - \zeta_m \sinh \kappa_m/\kappa_m
\end{pmatrix}$$

(16)

where the terms for second- and fourth-order schemes are retained:

$$\zeta_m = 0 + u'_m \Delta x^4/12,$$

$$\beta_m = - (\lambda^2 + u_m) \Delta x^2 - u''_m \Delta x^4/24,$$

$$\kappa_m^2 = \beta_m + \zeta_m^2 = - (\lambda^2 + u_m) \Delta x^2 + [u_m^2/144 - u''_m/24] \Delta x^4,$$

(17)

where $u_m = u(x_m)$, $u'_m = u'(x_m)$ and $u''_m = u''(x_m)$. The higher-order schemes can be constructed in a similar manner and will be presented elsewhere. Note that the second-order Osborne scheme is recovered keeping $\Omega_1$ in the Magnus expansion and $u(x_m)$ in the Taylor series. These expressions lead to $\Sigma$ multiplying the results within each bin:

$$\Sigma = \prod_{m=1}^{M} U(x_m) = \prod_{m=1}^{M} e^{\Omega(x_m)}.$$  

(18)

To compute $\partial_\lambda \Sigma$ similar to Eq. (18) we also need the matrix $\partial_\lambda U$:

$$\partial_\lambda U_{11} = \partial_\lambda (\cosh \kappa_m + \zeta_m \sinh \kappa_m/\kappa_m),$$

(19)

$$\partial_\lambda U_{12} = \partial_\lambda (\sinh \kappa_m/\kappa_m),$$

(20)

$$\partial_\lambda U_{21} = \partial_\lambda (\beta_m \sinh \kappa_m/\kappa_m),$$

(21)

$$\partial_\lambda U_{22} = \partial_\lambda (\cosh \kappa_m - \zeta_m \sinh \kappa_m/\kappa_m).$$

(22)

Further we verify the algorithm using the box-shaped potential and one-soliton solution.

5. Box potential
The first test case corresponds to a box-shaped potential with $u(x) = q_0$ within $-L_0/2 \leq x \leq L_0/2$ and equal to zero elsewhere. One can derive the analytic solution of the Schrödinger equation for wave functions and scattering coefficients, see [22], which are as follows:

$$a = e^{i\lambda L_0} [\cosh(pL_0) - \frac{i(\lambda^2 - p^2)}{2\lambda p} \sinh(pL_0)], \quad b = \frac{iq_0}{\lambda p} \sinh(pL_0),$$

(23)
where \( p = \sqrt{-\lambda^2 - q_0} \). As mentioned above, the eigenvalues \( \{\lambda_n\} \) can be obtained using the condition \( a(\lambda_n) = 0 \), see Eq. (5), leading to a transcendental equation:

\[
\cosh(p_n L_0) = \frac{i(\lambda_n^2 - p_n^2)}{2\lambda_n p_n} \sinh(p_n L_0), \quad \text{where} \quad p_n = \sqrt{-\lambda_n^2 - q_0}.
\]

(24)

Employing this condition and Eq. (5), the expression for norming constants can be derived explicitly:

\[
\rho_n = \left( \frac{b}{\partial \lambda a} \right)|_{\lambda = \lambda_n} = e^{-i\lambda_n L_0} \frac{2i\lambda_n(\lambda_n^2 + q_0)}{q_0(2i + \lambda_n L_0)}.
\]

(25)

As a test we consider a box with \( q_0 = 4 \) and \( L_0 = 7 \) containing \( N = 5 \) solitons. We compute the discrete spectrum using explicit expressions (24) and (25) and numerical DST algorithm described in Sec. 4. Since the box function is described by a constant value of \( u(x) \), the second-order scheme turns out to be exact since all commutators in Eq. (12) go to zero. The exact values \( \{\lambda_n\} = \{0.653i, 1.272i, 1.623i, 1.840i, 1.961i\} \) and \( \{\rho_n\} = \{17.202i, 1022.991i, 7139.243i, 14913.258i, 8793.836i\} \) are fully recovered by a numerical solution.

The computational accuracy of the continuous spectrum is presented in Fig. 1 where both scattering coefficients and their derivatives are presented along the real axis. The next example of the one-soliton solution will help us to assess the efficiency of the high-order Magnus-based numerical scheme developed above.

6. One-soliton solution

The solution describing one soliton has the following form [2]:

\[
u(x) = \frac{2\kappa^2}{\cosh^2[\kappa(x - x_0)]}, \quad x_0 = \frac{1}{2\kappa} \log\left(\frac{\rho}{2i\kappa}\right),
\]

(26)

where \( x_0 \) represent the spatial position of a soliton. To test the numerical algorithm we set the parameters \( \lambda = i\kappa = i \), \( \rho = 60i \) and perform the numerical DST on the interval \( L = 32 \) varying the number of bins from \( M = 32 \) to 8192. Fig. 2 shows the errors for \( \lambda \) and \( \rho \) due to numerical discretization for both second- and forth-order Magnus-based schemes. The high-order scheme demonstrates significant improvements over a standard Osborne approach with a slight increase in computational time.

![Figure 1](image_url)

Figure 1. The comparison of scattering coefficients and their derivatives on a real axis \( k \) obtained from analytical (continuous red lines) and numerical (dashed blue lines) solutions for a box potential with \( q_0 = 4 \) and \( L_0 = 7 \).
7. Conclusions
We presented a novel fourth-order accurate numerical scheme for scattering data of the KdV equation. The derived scheme based on the Magnus expansion was employed to compute the extended $4 \times 4$ scattering matrix providing accurate information about all components of the scattering data, i.e., soliton eigenvalues, norming constants as well as the reflection coefficient. Physically, within the KdV model this data describes soliton amplitudes, velocities and intensity of incoherent radiation. Our tests performed using a box-shaped wavefield confirmed that all components of the scattering data were computed correctly, while the test based on a single-soliton solution verified the declared order of a numerical scheme. The developed algorithm can be used for fast and efficient analysis of experimental and numerical data, see [3, 4], with particular focus on statistical problems such as soliton gas dynamics [24, 25]. Together with an appropriate IST algorithm, see e.g. [26], the Cauchy initial-value problem for the KdV equation can be completely solved. One of the key perspectives is to apply the developed algorithm to complex wavefields containing a large number of solitons when the advantages of the high-order approaches are especially pronounced, see [17]. Another perspective is to study the issue of the so-called anomalous numerical errors of the DST procedure which were recently addressed [18], see also [16].

Acknowledgments
This work is funded by the Russian Foundation for Basic Research grants No. 18-02-00042 and 19-31-60028, the development of the numerical code is conducted under state contract with IT SB RAS.

References
[1] Ablowitz M J and Segur H 1981 Solitons and the inverse scattering transform vol 4 (Siam)
[2] Novikov S, Manakov S, Pitaevskii L and Zakharov V 1984 Theory of solitons: the inverse scattering method (Springer Science & Business Media)
[3] Osborne A 2010 Nonlinear ocean waves (Academic Press) ISBN 0125286295
[4] Slunyaev A 2018 Radiophysics and Quantum Electronics 61 1–21
[5] Boffetta G and Osborne A R 1992 *Journal of Computational Physics* **102** 252–264
[6] Burtsev S, Camassa R and Timofeyev I 1998 *Journal of Computational Physics* **147** 166–186
[7] Frumin L L, Belai O V, Podivilov E V and Shapiro D A 2015 *Journal of the Optical Society of America B* **32** 290–296
[8] Vaibhav V 2018 *IEEE Photonics Technology Letters* **30** 700–703
[9] Vasilychenkova A, Prilepsky J E and Turitsyn S K 2018 *Optics Letters* **43** 3690–3693
[10] Vasilychenkova A, Prilepsky J, Shepelsky D and Chattopadhyay A 2019 *Communications in Nonlinear Science and Numerical Simulation* **68** 347–371
[11] García-Gómez F J and Aref V 2019 *Journal of Lightwave Technology* **37** 3563–3570
[12] Medvedev S, Vaseva I, Chekhovskoy I and Fedoruk M 2019 *Optics Letters* **44** 2264–2267
[13] Medvedev S, Vaseva I, Chekhovskoy I and Fedoruk M 2020 *Optics Express* **28** 20–39
[14] Osborne A 1991 *Journal of Computational Physics* **94** 284–313
[15] Provenzale A and Osborne A 1991 *Journal of Computational Physics* **94** 314–351
[16] Prins P J and Wahl S 2019 *IEEE Access* **7** 122914–122930
[17] Mullyadzhanov R and Gelash A 2019 *Optics Letters* **44** 5298
[18] Gelash A and Mullyadzhanov R 2020 *Physical Review E* **101**(5) 052206
[19] Blanes S, Casas F, Oteo J A and Ros J 2009 *Physics Reports* **470** 151–238
[20] Gardner C S, Greene J M, Kruskal M D and Miura R M 1967 *Physical Review Letters* **19**(19) 1095–1097
[21] Lax P D 1968 *Communications on pure and applied mathematics* **21** 467–490
[22] Landau L D and Lifshitz E M 1958 *Quantum Mechanics: Non-relativistic Theory. V. 3 of Course of Theoretical Physics* (Pergamon Press)
[23] Faddeev L D and Takhtajan L A 2007 *Hamiltonian methods in the theory of solitons* (Springer Science & Business Media, Berlin)
[24] Dutkyh D and Pelinovsky E 2014 *Physics Letters A* **378** 3102–3110
[25] Shurgalina E and Pelinovsky E 2016 *Physics Letters A* **380** 2049–2053
[26] Trogdon T, Olver S and Deconinck B 2012 *Physica D: Nonlinear Phenomena* **241** 1003–1025