A NOTE ON RELATIVE HOMOLOGICAL EPIMORPHISMS OF TOPOLOGICAL ALGEBRAS

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Abstract. We give several characterizations of relative homological epimorphisms in the setting of locally convex topological algebras, thereby correcting a gap in our earlier paper [Trans. Moscow Math. Soc. 2008, 27–104].

1. Introduction

Homological epimorphisms of topological algebras were introduced by J. L. Taylor [24] under the name of absolute localizations. Since then, they were rediscovered several times under different names (see [2, 3, 4, 5, 6]), both in the purely algebraic and in the functional analytic contexts. A more detailed historical survey is given in [1, Remark 3.16].

In [20], we introduced a relative version of this notion, which roughly corresponds to the situation where homological properties of topological \(A\)-modules are considered relative to a subalgebra \(R\) of \(A\). In other words, we worked in the functional analytic version of G. Hochschild’s relative homological algebra [1]. Our main motivation was that, in order to prove that an algebra homomorphism \(\varphi: A \to B\) is a homological epimorphism, it is often convenient (and it often suffices, provided that some natural conditions are satisfied) to construct finite chains \(R_0 \subseteq R_1 \subseteq \cdots \subseteq R_n = A\) and \(S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n = B\) of subalgebras such that \(\varphi(R_i) \subseteq S_i\) for all \(i\), and such that \(\varphi|_{R_i}: R_i \to S_i\) is a homological epimorphism relative to \(R_{i-1} \subseteq R_i\) and \(S_{i-1} \subseteq S_i\), respectively. This method applies, in particular, to iterated Ore extensions and to their functional analytic counterparts (see [20, Section 9]).

The goal of this note is to correct a gap in [20, Theorem 6.3], where several characterizations of relative homological epimorphisms were given. For the non-relative case, such characterizations are known in the purely algebraic setting [3, 4, 5, 6] and in the setting of bornological algebras [11]. However, the setting of locally convex topological algebras requires some different tools, mostly because there is no internal Hom functor in the category of complete locally convex spaces, and so the adjunctions between Hom and \(\otimes\) are not available in the context of complete locally convex modules.

2. Preliminaries

Throughout, all vector spaces and algebras are assumed to be over the field \(\mathbb{C}\) of complex numbers. All algebras are assumed to be associative and unital. By a \(\hat{\otimes}\)-algebra we mean an algebra \(A\) endowed with a complete locally convex topology in such a way that the multiplication \(A \times A \to A\) is jointly continuous. Note that the multiplication uniquely extends to a continuous linear map \(A \hat{\otimes} A \to A\), \(a \otimes b \mapsto ab\), where the symbol \(\hat{\otimes}\) stands for the completed projective tensor product (whence the name “\(\hat{\otimes}\)-algebra”). If \(A\) is a \(\hat{\otimes}\)-algebra, then a left \(A\)-\(\hat{\otimes}\)-module is a left \(A\)-module \(X\) endowed with a complete locally

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convex topology in such a way that the action $A \times X \rightarrow X$ is jointly continuous. We always assume that $1_A \cdot x = x$ for all $x \in X$, where $1_A$ is the identity of $A$. Left $A$-$\hat{\otimes}$-modules and their continuous morphisms form a category denoted by $A$-$\text{mod}$. The categories $\text{mod}$-$A$ and $A$-$\text{mod}$-$B$ of right $A$-$\hat{\otimes}$-modules and of $A$-$B$-$\hat{\otimes}$-bimodules are defined similarly. Given left $A$-$\hat{\otimes}$-modules $X$ and $Y$, we denote by $\text{Hom}_A(X, Y)$ the space of morphisms from $X$ to $Y$. If $X$ is a right $A$-$\hat{\otimes}$-module and $Y$ is a left $A$-$\hat{\otimes}$-module, then their $A$-module tensor product $X \hat{\otimes}_A Y$ is defined to be the completion of the quotient $(X \hat{\otimes} Y)/L$, where $L \subset X \hat{\otimes} Y$ is the closed linear span of all elements of the form $x \cdot a \otimes y - x \cdot a \cdot y$ ($x \in X$, $y \in Y, a \in A$). As in pure algebra, the $A$-module tensor product can be characterized by the universal property that, for each complete locally convex space $E$, there is a natural bijection between the set of all continuous $A$-balanced bilinear maps from $X \times Y$ to $E$ and the set of all continuous linear maps from $X \hat{\otimes}_A Y$ to $E$.

If $R$ is a $\hat{\otimes}$-algebra, then an $R$-$\hat{\otimes}$-algebra is a pair $(A, \eta_A)$ consisting of a $\hat{\otimes}$-algebra $A$ and a continuous algebra homomorphism $\eta_A: R \rightarrow A$. When speaking of $R$-$\hat{\otimes}$-algebras we often abuse the language by saying that $A$ is an $R$-$\hat{\otimes}$-algebra (without explicitly mentioning $\eta_A$); this should not lead to confusion. If $A$ is an $R$-$\hat{\otimes}$-algebra and $B$ is an $S$-$\hat{\otimes}$-algebra, then an $R$-$S$-homomorphism from $A$ to $B$ is a pair $(f, g)$, where $f: A \rightarrow B$ and $g: R \rightarrow S$ are continuous algebra homomorphisms such that $f \eta_A = \eta_B g$. Again, we often abuse the language by saying that $f: A \rightarrow B$ is an $R$-$S$-homomorphism (without explicitly mentioning $g$).

Let $A$ be an $R$-$\hat{\otimes}$-algebra. We say that a short sequence of left $A$-$\hat{\otimes}$-modules is $R$-admissible if it is split exact in $R$-$\text{mod}$. In the special case where $R = \mathbb{C}$, we recover the standard definition of an admissible (or $\mathbb{C}$-split) sequence used in the homology theory of topological algebras (see the original papers \[3, 13, 23\] and the monographs \[7, 10\]). The category $A$-$\text{mod}$ together with the class of all short $R$-admissible sequences is an exact category (in Quillen’s sense); we denote this exact category by $(A, R)$-$\text{mod}$. Therefore the derived categories $D^*((A, R)$-$\text{mod})$ are defined, where $* \in \{+, -, b\}$; see \[2, 22\] for details. The exact categories $\text{mod}$-$(A, R)$, $(A, R)$-$\text{mod}$-$(B, S)$, and their derived categories are defined similarly.

As was observed in \[20\], Section 2], $(A, R)$-$\text{mod}$ has enough projectives. Hence each additive covariant (respectively, contravariant) functor from $(A, R)$-$\text{mod}$ to any exact category has a total left (respectively, right) derived functor (see, e.g., \[5, 12\]). In particular, let $\text{Vect}$ denote the abelian category of vector spaces, and let $\text{LCS}$ denote the quasi-abelian category of locally convex spaces (see \[21, 22\] for details about the latter). Then for each $Y \in \text{Ob}((A, R)$-$\text{mod})$ we have the right derived functor

$$\text{RHom}_A(-, Y): D^-(((A, R)$-$\text{mod})) \rightarrow D^+(\text{Vect}),$$

and for each $X \in \text{Ob}(\text{mod}-(A, R))$ we have the left derived functor

$$X \hat{\otimes}^L_A(-): D^-((A, R)$-$\text{mod}) \rightarrow D^-(\text{LCS}).$$

There are some obvious variations of $\text{RHom}_A$ and $\hat{\otimes}^L_A$. For example, if $B$ is an $S$-$\hat{\otimes}$-algebra and $X \in \text{Ob}((B, S)$-$\text{mod}-(A, R))$, then $X \hat{\otimes}^L_A(-)$ may be viewed as a functor with values in $D^-((B, S)$-$\text{mod})$.

Given $X, Y \in \text{Ob}((A, R)$-$\text{mod})$ and $n \in \mathbb{Z}_+$, we let $\text{Ext}^n_{A,R}(X, Y) = H^n(\text{RHom}_A(X, Y))$, where $\text{RHom}_A$ is given by \[2.1\] and $H^n$ is the $n$th cohomology functor. Similarly, given $X \in \text{Ob}(\text{mod}-(A, R))$ and $Y \in \text{Ob}((A, R)$-$\text{mod})$, we let $\text{Tor}^n_{A,R}(X, Y) = H^{-n}(X \hat{\otimes}^L_A Y)$, where $\hat{\otimes}^L_A$ is given by \[2.2\]. Thus $\text{Ext}^n_{A,R}(X, Y)$ is a vector space, while $\text{Tor}^n_{A,R}(X, Y)$ is
a (not necessarily Hausdorff) locally convex space. The completion of $\text{Tor}_0^{A,R}(X,Y)$ is naturally isomorphic to $X \hat{\otimes}_A Y$ [20, Section 2].

Suppose that $A$ is an $R$-$\otimes$-algebra, $B$ is an $S$-$\otimes$-algebra, and $f: A \to B$ is an $R$-$S$-homomorphism. The “restriction of scalars” functor $f^*: (B,S)\text{-mod} \to (A,R)\text{-mod}$ is obviously exact (i.e., it takes admissible sequences to admissible sequences). Hence $f^*$ extends to a triangle functor $Df^*: D^*((B,S)\text{-mod}) \to D^*((A,R)\text{-mod})$.

We will also need the notion of a homotopy limit (see, e.g., [3,16]). Suppose that $\mathcal{T}$ is a triangulated category with countable products, and let $(X_n: f_{n+1}: X_{n+1} \to X_n)_{n \in \mathbb{N}}$ be an inverse system in $\mathcal{T}$. Define

$$s: \prod_{n \in \mathbb{N}} X_n \to \prod_{n \in \mathbb{N}} X_n, \quad s(x_1, x_2, \ldots) = (f_2(x_2), f_3(x_3), \ldots).$$

An object $X$ of $\mathcal{T}$ is a homotopy limit of $(X_n, f_n)$ if there exists a distinguished triangle

$$X \to \prod_{n \in \mathbb{N}} X_n \xrightarrow{1-s} \prod_{n \in \mathbb{N}} X_n \to X[1].$$

In this case, we write $X = \text{holim}_n X_n$. The homotopy limit exists and is unique up to a noncanonical isomorphism.

Finally, recall that, if $\mathcal{A}$ and $\mathcal{B}$ are exact categories and $F: \mathcal{A} \to \mathcal{B}$ is a covariant additive functor, then an object $X$ of $\mathcal{A}$ is (left) $F$-acyclic [13, Section 15] if the left derived functor $LF$ is defined at $X$ and if the canonical morphism $LF(X) \to F(X)$ is an isomorphism in $D(\mathcal{B})$.

3. THE RESULTS

Lemma 3.1. Let $\mathcal{A}$ be an additive category with countable products, and let $f: X \to Y$ be a morphism in $\mathcal{A}$. Define $i: X \to Y \times X^\mathbb{N}$ and $t: Y \times X^\mathbb{N} \to Y \times X^\mathbb{N}$ by

$$i(x) = (f(x), x, x, \ldots),$$

$$t(y, x_1, x_2, \ldots) = (y - f(x_1), x_1 - x_2, x_2 - x_3, \ldots).$$

Then the sequence

$$0 \to X \xrightarrow{i} Y \times X^\mathbb{N} \xrightarrow{t} Y \times X^\mathbb{N} \to 0$$

is split exact in $\mathcal{A}$.

Proof. Define $p: Y \times X^\mathbb{N} \to X$ and $q: Y \times X^\mathbb{N} \to Y \times X^\mathbb{N}$ by

$$p(y, x_1, x_2, \ldots) = x_1,$$

$$q(y, x_1, x_2, \ldots) = \left(y, 0, -x_1, -(x_1 + x_2), \ldots, -\sum_{i=1}^{k-1} x_i, \ldots \right).$$

An elementary calculation shows that $pi = 1$, $qt = 1$, and $ip + qt = 1$. \hfill \Box

Let $\mathcal{A}$ be an exact category with cokernels, and let $X = (X^i, d^i: X^i \to X^{i+1})$ be a cochain complex in $\mathcal{A}$. For each $n \in \mathbb{N}$ let $\beta_n X$ denote the bounded complex

$$0 \to \text{Coker} d^{-n-1} \to X^{-n+1} \to \ldots \to X^{n-1} \to X^n \to 0$$

concentrated in degrees from $-n$ to $n$. We have the obvious morphisms $\beta_{n+1} X \to \beta_n X$, so $(\beta_n X)$ is an inverse system in $C(\mathcal{A})$ and hence in $D(\mathcal{A})$. We clearly have $X = \varprojlim \beta_n X$ in $C(\mathcal{A})$. 

Given $\mathcal{A}$ as above, we say that $\mathcal{A}$ has exact countable products if countable products exist in $\mathcal{A}$, and if the product of every countable family of short admissible sequences in $\mathcal{A}$ is admissible.

The following result is essentially contained in [15, Prop. 6].

**Lemma 3.2.** If $\mathcal{A}$ has exact countable products, then $X = \varprojlim \beta_nX$ in $D(\mathcal{A})$.

**Proof.** Since countable products are exact in $\mathcal{A}$, it follows that the functor $C(\mathcal{A}) \to D(\mathcal{A})$ preserves countable products [3, Lemma 1.5]. In particular, the product complex $\prod_n \beta_nX$ taken in $C(\mathcal{A})$ (i.e., componentwise) is also a product of the objects $(\beta_nX)$ in $D(\mathcal{A})$. To construct a triangle

$$X \to \prod_n \beta_nX \xrightarrow{1-s} \prod_n \beta_nX \to X[1]$$

(3.2)

in $D(\mathcal{A})$, it suffices to find a sequence $X \xrightarrow{i} \prod_n \beta_nX \xrightarrow{1-s} \prod_n \beta_nX$ in $C(\mathcal{A})$ such that, for each $k \in \mathbb{Z}$, the sequence

$$0 \to X^k \xrightarrow{i} \prod_n (\beta_nX)^k \xrightarrow{(1-s)^k} \prod_n (\beta_nX)^k \to 0$$

(3.3)

is admissible [12, Section 11]. The obvious maps $X \to \beta_nX$ yield $i: X \to \prod_n \beta_nX$. Now observe that (3.3) is a special case of (3.1) (for $k \geq 0$ we let $Y = 0$, while for $k < 0$ we let $Y = \text{Coker } d^{k-1}$ and $f: X^k \to \text{Coker } d^{k-1}$ be the canonical map). Applying Lemma 3.1, we see that (3.3) is split exact and is a fortiori admissible in $\mathcal{A}$.

**Remark 3.3.** Since (3.3) is not only admissible, but is also split exact in $\mathcal{A}$, it follows from [12, Section 6] that (3.2) is a triangle not only in $D(\mathcal{A})$, but also in the homotopy category $H(\mathcal{A})$. Thus $X = \varprojlim \beta_nX$ in $H(\mathcal{A})$ as well (cf. [15, Prop. 6]).

**Lemma 3.4.** Let $\mathcal{A}$ be an exact category with cokernels and with exact countable products, and let $X \to Y$ be a morphism in $D(\mathcal{A})$. Suppose that for each $Z \in \text{Ob}(D^b(\mathcal{A}))$ the induced map $\text{Hom}_{D(\mathcal{A})}(Y, Z) \to \text{Hom}_{D(\mathcal{A})}(X, Z)$ is a bijection. Then $X \to Y$ is an isomorphism.

**Proof.** By Yoneda’s lemma, it suffices to show that $\text{Hom}_{D(\mathcal{A})}(Y, Z) \to \text{Hom}_{D(\mathcal{A})}(X, Z)$ is a bijection for each $Z \in \text{Ob}(D(\mathcal{A}))$. Applying Lemma 3.2, we see that $Z = \varprojlim \beta_nZ$ in $D(\mathcal{A})$. We have a commutative diagram of abelian groups

$$
\begin{array}{ccc}
0 & \xrightarrow{\iota} & \lim^1 \text{Hom}_{D(\mathcal{A})}(Y, \beta_nZ[-1]) \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\iota} & \lim^1 \text{Hom}_{D(\mathcal{A})}(X, \beta_nZ[-1])
\end{array}
$$

with exact rows (see [23, Lemma 15.85.3]). Since the left and the right vertical arrows are isomorphisms, so is the middle vertical arrow. This completes the proof.

Our next lemma is a variation of a well-known fact of category theory (see, e.g., [14, IV.3, Theorem 1] or [4, Prop. 3.4.1]).

**Lemma 3.5.** Let $\mathcal{A}$ and $\mathcal{B}$ be categories, and let $F: \mathcal{A} \to \mathcal{B}$ and $G: \mathcal{B} \to \mathcal{A}$ be an adjoint pair of functors, with $F$ left adjoint to $G$. Let $\mathcal{B}_0$ be a full subcategory of $\mathcal{B}$, and let $G_0: \mathcal{B}_0 \to \mathcal{A}$ denote the restriction of $G$ to $\mathcal{B}_0$. Suppose that $\mathcal{B}_0$ has the following property:
A morphism \( X \to Y \) in \( \mathcal{B} \) is an isomorphism if and only if the induced map 
\( \text{Hom}_\mathcal{B}(Y, Z) \to \text{Hom}_\mathcal{B}(X, Z) \) is bijective for all \( Z \in \text{Ob}(\mathcal{B}_0) \).

Then the following conditions are equivalent:

(i) \( G_0 \) is fully faithful;
(ii) for each \( X \in \text{Ob}(\mathcal{B}_0) \), the canonical morphism \( \varepsilon_X : FG(X) \to X \) is an isomorphism.

Proof. For each \( X, Y \in \text{Ob}(\mathcal{B}) \) we have the following commutative diagram:

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{B}(X, Y) & \xrightarrow{G_{X,Y}} & \text{Hom}_\mathcal{B}(GX, GY) \\
& \text{adj.} \sim & \text{Hom}_\mathcal{B}(FGX, FY) \times \varepsilon_{X,Y} \\
\end{array}
\]

(where “adj.” is the adjunction isomorphism). Hence (i) holds if and only if \( \text{Hom}(\varepsilon_X, Y) \) 
otherwiseform{is bijective for all \( X, Y \in \text{Ob}(\mathcal{B}_0) \). By (\ast), this is equivalent to (ii).}

We now come to the following corrected version of [20, Theorem 6.3].

Theorem 3.6. Let \( f : A \to B \) be an \( R-S \)-homomorphism from an \( R-\hat{\otimes} \)-algebra \( A \) to an \( S-\hat{\otimes} \)-algebra \( B \). The following conditions are equivalent:

(i) \( Df^\bullet : D^b((B, S)-\text{mod}) \to D^b((A, R)-\text{mod}) \) is fully faithful;
(ii) for every \( X, Y \in \text{Ob}((B, S)-\text{mod}) \) and every \( n \in \mathbb{Z}_+ \) the canonical morphism 
\( \text{Ext}^n_{B,S}(X, Y) \to \text{Ext}^n_{A,R}(X, Y) \)
is bijective;
(iii) for every \( X \in \text{Ob}(D^b((B, S)-\text{mod})) \) the canonical morphism 
\( B \hat{\otimes}_A^L Df^\bullet(X) \to X \)
is an isomorphism in \( D^-(B, S)-\text{mod}) \);
(iv) \( f \) is an epimorphism of \( \hat{\otimes} \)-algebras, and, moreover, every \( X \in \text{Ob}((B, S)-\text{mod}) \)
viewed as an object of \( (A, R)-\text{mod} \) is acyclic relative to the functor 
\( B \hat{\otimes}_A(-) : (A, R)-\text{mod} \to (B, S)-\text{mod} \).

Proof. (i) \( \iff \) (ii). This is a special case of [20, Lemma 6.2].

(i) \( \iff \) (iii). Let \( \mathcal{A} = D^-(A, R)-\text{mod} \), \( \mathcal{B} = D^-(B, S)-\text{mod} \), and \( \mathcal{B}_0 = D^b((B, S)-\text{mod}) \).

By Lemma 3.4, \( \mathcal{B} \) and \( \mathcal{B}_0 \) satisfy condition (\ast) of Lemma 3.3. Let now 
\( F = B \hat{\otimes}_A^L(-) : \mathcal{A} \to \mathcal{B} \), \( G = Df^\bullet : \mathcal{B} \to \mathcal{A} \).

By [12] Lemma 13.6], \( (F, G) \) is an adjoint pair, with \( F \) left adjoint to \( G \). Applying now Lemma 3.5, we see that (i) \( \iff \) (iii).

(iii) \( \implies \) (iv). By (iii), for every \( X \in \text{Ob}((B, S)-\text{mod}) \) the morphism \( B \hat{\otimes}_A^L X \to X \)
is an isomorphism in \( D^-(B, S)-\text{mod} \) and therefore in \( D^-(\text{LCS}) \). Applying the functor \( H_0^\bullet \), we see that the composition of the canonical morphisms 
\( \text{Tor}_0^{A,R}(B, X) \to B \hat{\otimes}_A X \to X \)
is an isomorphism in \( \text{LCS} \). Since the first morphism in this composition is the canonical 
map from a locally convex space to its completion (see [20, Section 2]), we conclude that 
both morphisms are isomorphisms in \( \text{LCS} \). Hence \( B \hat{\otimes}_A X \to X \) is an isomorphism in 
\( B-\text{mod} \). By composing the isomorphisms \( B \hat{\otimes}_A^L X \cong X \) and \( X \cong B \hat{\otimes}_A X \), we conclude
that $X$ is $B \hat{\otimes}_A (-)$-acyclic. Finally, the canonical isomorphism $B \hat{\otimes}_A B \cong B$ means precisely that $f$ is a $\hat{\otimes}$-algebra epimorphism \cite[Prop. 6.1]{BokstedtNeeman}.

(iv) $\implies$ (ii). Let $P_\bullet$ be a projective resolution of $X$ in $(A, R)$-mod. By (iv), $B \hat{\otimes}_A P_\bullet$ is a projective resolution of $X$ in $(B, S)$-mod. Therefore

$$\text{Ext}^n_{A,R}(X,Y) \cong H^n(\text{Hom}_A(P_\bullet,Y)) \cong H^n(\text{Hom}_B(B \hat{\otimes}_A P_\bullet,Y)) \cong \text{Ext}^n_{B,S}(X,Y).$$

\[ \square \]

**Definition 3.7.** If $f: A \to B$ satisfies the conditions of Theorem 3.6, then we say that $f$ is a left relative homological epimorphism. We say that $f$ is a right relative homological epimorphism if $f: A^{\text{op}} \to B^{\text{op}}$ is a left relative homological epimorphism.

**Remark 3.8.** In \cite[Theorem 6.3]{BokstedtNeeman}, which is the original version of Theorem 3.6, the morphism $B \hat{\otimes}_A^L Df^\bullet(X) \to X$ (see (iii)) is claimed to be a morphism in $D^b((B, S)$-mod). However, $B \hat{\otimes}_A^L (-)$ does not take $D^b((A, R)$-mod) to $D^b((B, S)$-mod) \emph{a priori} (unless we impose some homological finiteness conditions on $A$ or $B$). This resulted in a gap in the proof of equivalence (i) $\iff$ (iii), because the functors $B \hat{\otimes}_A^L (-)$ and $Df^\bullet$ cannot be viewed as adjoint functors between the respective bounded derived categories, and so \cite[IV.3, Theorem 1]{BenBassatKremnizer} does not apply. That is why we had to use Lemmas 3.4 and 3.5 instead.

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