Classical Limits and Contextuality in a Scenario of Multiple Observers

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(Dated: April 10, 2019)

One of the most surprising features of Quantum Theory is contextuality, which defies the intuition behind Classical Theories and provides a resource for quantum computation. However, Classical Theories explain very well our everyday experience, reinforcing one’s belief in a non-contextual explanation of nature. This naturally raises the question: is it possible to see the emergence of non-contextuality under a suitable limit of Quantum Theory? Here we develop a game of multiple observers inspired by Quantum Darwinism, that allows for non-contextuality in N-cycle scenarios when redundancy among players spread, suggesting that, despite its non-classical features, Quantum Theory can also explain our daily non-contextual experience.

In Classical Theories (CTs), non-contextuality is a natural assumption [1, 2] and these theories are very successful on explaining our everyday experience. Hence, any contextual theory is both surprising and manifestly non-classical; in particular, Quantum Theory (QT) was proved incompatible with non-contextuality [3]. This incompatibility brings the question: in a suitable classical limit, does Quantum Theory give rise to non-contextual correlations as our classical intuition would expect? This is the central question we address in this work.

The incompatibility between non-contextuality and QT was first shown as a logical impossibility [3]. By reformulating the problem it is possible to construct non-contextuality inequalities [4, 5], which are obeyed by any non-contextual theory, but possibly violated by contextual theories, such as QT – and these quantum violations can be seen in the laboratory [6–10].

As one can focus on how QT is different from CTs, it is also possible to take the opposite path in order to conciliate these theories. In classical limits, one is usually trying to recover classicality inside QT [11]. Historically, the decoherence paradigm was a major step in this direction, accomplished by recognizing that a quantum system is rarely isolated. It led to several models where quantum superpositions were suppressed [12–14].

Another important breakthrough in classical limits of QT is Quantum Darwinism (QD) [15, 16]. From the idea of fractioning the environment in small pieces, QD singles out redundancy of the information gained by these environment subsystems (about the central one) as the main feature for emergence of objectivity in QT. Mutual information between the central system and fractions of the environment have been analyzed in specific models, showing the emergence of a classical plateau [16–18]. There is some debate whether mutual information is the most suitable tool to analyze the emergence of objectivity [11, 19] and the generality of the approach [13]. However, Brandão et al. show that any map given by an interaction of several systems with a central one and restricted to a small fraction of such environment leads generically to some broad notion of objectivity [20]. Then, we can say that the QD scenario of analyzing information stored in small fractions of a big environment is essential for emergence of objectivity inside QD program.

In this Letter, we focus on this important trait of QD developing a game where players act as fractions of an environment seeking for violation of N-cycle non-contextuality inequalities. Within this game approach we show that quantum contextuality can be killed for observers independently monitoring the system. Despite the differences between this game and the full process of QD we obtain results that have similar interpretations, showing that quantum contextuality vanishes in a QD-like classical limit in this implementation.

Background - Kochen-Specker contextuality scenarios are constituted by a set of available observables (or measurements), the compatibility restrictions between them and the set of outcomes [21, 22]. The compatibility restrictions can be depicted in a compatibility graph, where each vertex represents an observable and two vertices are connected if and only if they represent compatible observables. In the N-cycle scenarios, we have N observables and the compatibility graph is a cycle of length N [23]. In other words, denoting Ai the observables, the maximal contexts are the elements in the set \{\{A0, A1\}, ..., \{AN−1, A0\}\}. The observables are dichotomic with outcomes o_e \in \{-1, 1\} and subscript i ∈ \{0, 1, ..., N − 1\} denoting each measurement. The non-trivial inequalities defining the classical polytope for odd N can be written as

\[ \sum_{i=0}^{N-1} \langle A_i A_{i+1} \rangle_{\text{NC}} \geq 2 - N, \]  \hspace{1cm} (1)

with sums made modulo N. Considering our choice of outcomes, \( \langle A_i A_{i+1} \rangle = p(o_i = o_{i+1}|A_i A_{i+1}) - p(o_i \neq o_{i+1}|A_i A_{i+1}) \). The superscript “NC” remembers that the inequality was derived assuming non-contextuality. This means that the outcome assigned to a measurement

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does not depend on which context it was measured, e.g., if one measures \( \{A_0, A_1\} \) or \( \{A_1, A_2\} \), a non-contextual theory must assign the same \( \alpha_i \) irrespective of the context and so on. By using \( p(\alpha_i = \alpha_{i+1} | A_i A_{i+1}) + p(\alpha_i \neq \alpha_{i+1} | A_i A_{i+1}) = 1 \), one can arrive at other two equivalent inequalities:

\[
\sum_{i=0}^{N-1} p(\alpha_i \neq \alpha_{i+1} | A_i A_{i+1}) \leq N - 1, \quad (2a)
\]

\[
\sum_{i=0}^{N-1} p(\alpha_i = \alpha_{i+1} | A_i A_{i+1}) \geq 1. \quad (2b)
\]

Under the additional assumption \( p(-1, -1 | A_i A_{i+1}) = 0 \), one can simplify further these expressions. This assumption can be related to two different features of the theory under consideration: as a consequence of assuming completeness of a sharp measurement with at most 3 outcomes (as would occur in QT with a 3-dimensional system) – which obligates to exclude one outcome from the initially four possible (the choice of \((-1, -1)\) is arbitrary); or as an additional assumption in any finite dimension, called the exclusiveness assumption. In either case, since we want to study the limits of the best known quantum realizations for such scenarios and these realizations obey such assumption, to assume \( p(-1, -1 | A_i A_{i+1}) = 0 \) is well motivated – and have been used in several tests of contextuality [6–8]. The (now three) possible outcomes and effects for a measurement of a context \( \{A_i, A_{i+1}\} \) will be labeled \( a_i, a_{i+1} \) and \( b_i \), meaning respectively \( a_i = -1 \), \( a_{i+1} = -1 \), \( b_i = 1 \). Here, we are also assuming, in the Kochen-Specker way, that the effects are the same independently of which complete measurement they take part – called ‘Gleason property’ [24]. Under these considerations on the general theories framework, one can arrive at non-contextual bounds to the number of outcomes \( a_i = 1 \) or \( b_i = 1 \) (‘yes’) to the effects \( \{a_i\} \) and \( \{b_i\} \) (see Fig. 1):

\[
\alpha = \sum_{i=0}^{N-1} p(a_i = 1) \leq \frac{N - 1}{2}, \quad (3a)
\]

\[
\beta = \sum_{i=0}^{N-1} p(b_i = 1) \geq 1. \quad (3b)
\]

These are the Inequalities that will be used from now on (inequality (3a) for \( N = 5 \) is the KCBS inequality [4]). It is important noticing that violation of one inequality of (3) leads to violation of the other, and so, they are equivalent in respect to witnessing contextuality.

Despite this equivalence, Inequalities (3) suggest different measurement protocols to be evaluated, i.e., different collections of measurements. At least three natural protocols can be considered: one common to both inequalities, while the other two are straightforwardly suggested by the form of \( \beta \) and \( \alpha \). These are represented by the following sets of measurements (one for each \( i \in \{0, ..., N-1\} \)):

1. \( M_i = \{a_i, b_i, a_{i+1}\} \); suitable for both Inequalities;
2. \( M_i^q = \{a_i, -a_i\} \); suggested by Ineq. (3a);
3. \( M_i^b = \{b_i, -b_i\} \); suggested by Ineq. (3b).

Protocol 1 is motivated by the usual contextuality scenario, where joint or sequential measurements of the context allows to recover separately the results for both observables in that context. However, as Inequalities (3) show, this can carry excess of information: under the assumption \( p(-1, -1 | A_i A_{i+1}) = 0 \), one (surprisingly) might know only the occurrences of \( a_i = 1 \) or \( b_i = 1 \), which are less invasive measurements (used in several experimental tests of similar inequalities [6–8]). In a typical scenario with one observer this makes no difference (since after measuring one context the system is prepared again for another round). However, they might present different results for sequential observers as for the QD-game we will construct.

Since our motivation is to consider a game inspired by classical limits of QT, it is important to look closely to quantum realizations of the odd \( N \) scenarios. The maximum quantum violations for all odd \( N \geq 5 \) can be realized in a Hilbert space of dimension \( d = 3 \), with measurements defined by \( A_i = I - 2 |a_i\rangle \langle a_i| \), where \( I \) is the identity and, with an appropriate basis choice, the vectors \( |a_i\rangle \) are

\[
|a_i\rangle = K \left( \cos \left( \frac{i\pi(N-1)}{N} \right), \sin \left( \frac{i\pi(N-1)}{N} \right), \sqrt{\cos \left( \frac{\pi}{N} \right)} \right)^t,
\]

with \( t \) meaning transposition, and \( K = 1/\sqrt{(1 + \cos(\pi/N))} \) for normalization [23]. The vectors \( |b_i\rangle \) are orthogonal to \( \{a_i\}, |a_{i+1}\rangle \). The important point here is the symmetry obeyed by the vectors \( \{a_i\} \): they

![FIG. 1. Hypergraph representing a 5-cycle realization in a theory obeying \( p(-1, -1 | A_i A_{i+1}) = 0 \). Hyperedges represent each context \( \{A_i, A_{i+1}\} \). The \( a_i \) vertices denote the outcomes 'yes' to the related effects, important for expression \( \alpha \) (hence the continuous line) and the \( b_i \) vertices are 'yes' answers to the related effects, where the dashed line reinforce its exclusiveness only to \( \{a_i, a_{i+1}\} \). The Gleason property considers \( a_i \) 'the same' vertex for both hyperedges it belongs to. Non-contextuality implies Inequalities (3).](image-url)
form a regular polygon in the plane orthogonal to the
axis $(0,0,1)^t$. The state that reaches the maximum vi-
olations is $|\psi_{\text{handle}}\rangle = (0,0,1)^t$, symmetric with respect
to the $\{|a_1\rangle\}$. The form above for the vectors implies
$(a_i | a_{i\pm 1}) = 0$, obeying the compatibility constraints
required by the scenario and exclusiveness. This singles
out Inequalities (3) as good witnesses for emergence
of non-contextuality in theories reproducing quantum
predictions.

**QD-Inspired Game** - An important improvement given
by QD is that it treats a system’s environment as made of
several individual pieces, which interact with the central
system of interest, $S$. The main object to analyze is the
information about $S$ that ended up stored in independent
fractions of the environment [15–18]. It was shown that
this *scenario* where observers have access to sufficiently
small fractions of a big environment is essential to obtain,
generically, some notion of objectivity [20, 25]. Here,
we do not ask if the information in these independent
fractions is the same, but whether it allows the witnessing
of contextuality.

Inspired by this QD paradigm of several independent
fragments monitoring the system and analyzing only its
partial information, the game to analyze the resistance of
quantum contextuality under dynamics is formulated as
follows (see Fig. 2). Fix a $N$-cycle scenario, an Inequality
of (3) (which all players will evaluate), and a related
protocol. Then, (i) an initial state of dimension $d = 3$
is prepared; (ii) an order of access of each player to $S$
is followed; (iii) each player chooses one of the possible
quantum projective measurements and executes it on $S$.
Steps (i) to (iii) (called henceforth a run) are repeated
for players to collect their individual data to estimate
the expression $\alpha^k_Q$ or $\beta^k_Q$, where superscript $k$
labels each player and $Q$ reminds us that we are looking at quantum
realizations. Player $k$ wins the game if capable of violate-
ing the chosen inequality. Players cannot communicate,
being ignorant with respect to others’ measurements and
outcomes.

The game is represented in figure 2 with the following
notation: for all protocols we denote the choice of mea-
surement by the $k$-th observer as $x_k \in \{0, \ldots, N - 1\}$
and the outcome obtained as $a_k|x_k$. For protocol 1,
$\alpha_k \in \{a_{x_k}, b_{x_k}, a_{x_k + 1}\}$; for protocol 2 $\alpha_k \in \{a_{x_k}, -a_{x_k}\}$;
while $\alpha_k \in \{b_{x_k}, -b_{x_k}\}$ for protocol 3. In the following,
we analyze the behavior of the observables involved in
the inequalities in each of the protocols.

For protocol 1, it is useful to define the probability
vector $\vec{P}_{x_k}$: each entry tells the probability of obtaining
each $a_k|x_k$ outcome. With $(\vec{P}_{x_k})_{x_k}$, each observer
can calculate either $\alpha_Q$ or $\beta_Q$ by choosing the relevant entries,
so, using the relation
$$I^k_Q = \sum_{x_k} \langle v_1, \vec{P}_{x_k} \rangle = \langle v_1, \sum_{x_k} \vec{P}_{x_k} \rangle,$$
where $\langle , \rangle$ is the usual Euclidean scalar product, $I = $

![FIG. 2. Scheme for the QD-inspired game. a) First player makes a measurement $x_1$ on state $\rho$, obtaining his outcome $\alpha_{x_1}|x_1$. All other players ignores $x_1$ and $\alpha_{x_1}$. b) Second player makes a measurement on state $\sigma$ which depends on $\rho$, $x_1$ and $\alpha_{x_1}$, and so on.](image)

Analyzing $I^k_Q$ for each $k$ via Eq. (4), we see that there is
no violation already for the second observer, for all $N$ (see
Table I and Fig. 3). This is a rather extreme behavior,
leading to emergence of non-contextuality after measure-
ment on $S$ by just one player. This can be considered
as a consequence of the high level of disturbance of these
measurements, since they completely destroys coherences
in the measured basis. It is natural to ask if this radical
emergence of no-violation also happens for the other
protocols, which preserve coherence in a 2-dimensional
subspace of the Hilbert space.

For protocols 2 and 3, the post-measurement state can
not be completely determined by a previous measure-
Remarkably, using the symmetries of the odd $N$-cycle quantum realizations (defined by the $C_{Nv}$ point group [28]), it is possible to obtain a relation between $I^k$ and $I^{k-1}$ (see [26]). Explicitly:

$$\beta^k_Q = B_N \beta^{k-1}_Q + b_N,$$  \hspace{1cm} (7a)$$
$$\alpha^k_Q = C_N \alpha^{k-1}_Q + c_N,$$  \hspace{1cm} (7b)$$

with $B_N, b_N, C_N, c_N$ fixed for each $N$. With Eqs. (7), it is possible to see again that the initial state allowing for maximum violations for $k = 1$ reaches the highest violations for any $k$, i.e., $|\psi_{\text{handle}}\rangle$. With the form of Eqs. (7) it is possible to calculate the quantities for each player and to obtain the asymptotic limit analytically, using the symmetries and dependencies of $B_N (C_N)$ and $b_N (c_N)$ with $N$ (see [26]). This gives

$$\lim_{k \to \infty} \beta^k_Q = \lim_{k \to \infty} \alpha^k_Q = \frac{N}{3},$$

which again means no violation of any of the inequalities and show that dynamics imposed by the game leads to a limit considerably above (below) the non-contextual bound (equal to the asymptotic limit for protocol 1). Results presented above are depicted in Fig. 3 for the case $N = 9$. Other cases are presented at Table I.

![Diagram](image)

FIG. 3. a) minimum value of $\beta^k_Q$ for the $k$-th player, for protocols 1 (black) and 3 (red). b) maximum value of $\alpha^k_Q$ for the $k$-th player for measurement protocols 1 (black) and 2 (red). Initial state is $|\psi_{\text{handle}}\rangle$ and $N = 9$. Blue line sets the quantum highest violation and Green line sets the non-contextual bounds.

Now, let us consider a slightly different version of the game, regarding step (ii): the order of the players is not fixed, but sorted in each run. This case is closer to the usual QD paradigm since the notion of order of access is relaxed. Calculations are straightforward convex combinations of the previous quantities. Results for the particular – but most interesting in this case – uniform distribution of $K$ observers are shown at Table I: the maximum value of $K$ which still let players to witness contextuality for some values of $N$. In this version, conclusions of non-violation emerge as a collective property, since the order of access is completely random.

The striking difference between protocols 2 and 3 shows that coherence is not the only ingredient necessary to extend survival of contextuality. This difference must be a consequence of orthogonality in the $\{a_i\}$ set: if a player obtains outcome $a_i$, it is automatically forbidden for the next one to obtain $a_{i+1}$ or $a_{i-1}$, while the $|b\rangle$ vectors are less restrictive, since obtaining an outcome $b_i$ implies (with high probability) less disturbance on the incoming state. Finally, all limits are given by $N/3$ because the average asymptotic state is the maximally mixed [26].

It is noteworthy that independence of players is crucial since collective strategies can guarantee that all players win. For instance, if all players combine which measurements to make or post-select the data to keep only those not perturbed by precedent players, (maximal) violation is always reachable.

**Relation to QD-terms** - In usual QD there is a process of einselection and dynamical transfer of information about pointer states trough interaction. By substituting interaction for measurements, this is no longer the case. Nonetheless, as results of Brandão et al. show [20], the fragmented environment of QD and restricting analysis to a fraction of the environment is its essence. In this sense there are strong similarities between the game and the QD paradigm: players assume the role of the fragments, independently obtaining information from a central system; only partial information is available to each player and one looks for what this partial information tells about (contextuality of) $S$. Not only this picture is borrowed, but for both versions of the game we can relate results to the usual way of seeing emergence of classicality in QD: the redundant information that leaks to players points to no violation. For the randomized access version, it is possible to say even more: non-contextuality is an objective feature, as all observers would agree on the fact that a non-contextual model could explain the obtained results [11]. These results, however, only appear in right conditions: a big enough environment, no collective strategies preceding measurements or post-selection. This is similar to the fact that, in QD, correlations between the fractions or access to almost the whole environment can hinder the emergence of classicality [15, 20].

**Concluding remarks** - Since contextuality is a striking non-classical feature, it is important to understand classical limits in which non-contextuality emerges. By developing a QD-inspired game language we obtained...
N-cycle non-contextuality as an emergent property in conditions similar to those where QD holds. In other words, it emerges for most players when many independent observers, with no collective strategies, play the game. Redundancy, as in QD models, leads to emergence of non-contextuality. Interestingly, measurement protocols which are equivalent in the usual scenario achieve different results in this QD-game.

This work also opens questions and directions for future research. First, it is valuable to study variations of this game that can be even more related to usual classical limits. For example, considering interaction between players and $S$ in place of measurements, leading to an approach that is an interplay between collisional models [29–32] and QD. It is essential to study other forms of contextuality, specially state-independent contextuality since our approach is manifestly state-dependent.

This work demonstrates that QT cannot always exhibit contextuality if a system is monitored by several independent observers. This is a first step to a conciliation between classical limits of QT and non-contextuality.

RDB acknowledges funding by São Paulo Research Foundation - FAPESP, grant no. 2016/24162-8. Authors thank the Brazilian agencies CNPq and CAPES for financial support. We thank Felippe Barbosa, János Bergou, Raphael Drumond and Rafael Rabelo for fruitful discussions. Important clarifications were made thanks to an anonymous Referee.

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Supplemental Material for “Classical Limits and Contextuality in a Scenario of Multiple Observers”

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(Dated: April 10, 2019)

In this Supplementary Material we present the proofs for the results shown in the Letter “Classical Limits and Contextuality in a Scenario of Multiple Observers”. In particular, the relations between the expressions for the $k$-th observer in terms of precedent players, Markovianity of the process of the measurement protocol number 1 as well as the asymptotic limits of many observers and average state in these limits.

I. PROTOCOL NUMBER 1

In protocol number 1, the measurements are given by the set of complete projective measurements $\{|a_i\rangle\langle a_i|, |b_i\rangle\langle b_i|, |a_{i+1}\rangle\langle a_{i+1}|\rangle$, where $i \in \{0,...,N-1\}$ denotes the $N$ possible choices and the sets of outcomes are $\{a_i, b_i, a_{i+1}\}$. We can define the vector $\vec{P}_{x_k}$ for the $k$-th observer ($x_k$ labeling the choice of measurement for this player) as:

$$\vec{P}_{x_k} = \begin{pmatrix} p(a_{x_k} | x_k) \\ p(b_{x_k} | x_k) \\ p(a_{x_k+1} | x_k) \end{pmatrix},$$  \hspace{1cm} (1)$$

where the entries define probabilities for the different outcomes of the measurement. Using the definitions for $\alpha^k$ and $\beta^k$, given by Ineq. (3) of the Letter, we can write the sums involved in terms of this vector, obtaining the following:

$$\beta^k = \sum_{x_k} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \vec{P}_{x_k} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \sum_{x_k} \vec{P}_{x_k}$$ \hspace{1cm} (2a)$$

$$\alpha^k = \sum_{x_k} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \vec{P}_{x_k} = \frac{1}{2} \sum_{x_k} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \vec{P}_{x_k} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \sum_{x_k} \vec{P}_{x_k}. \hspace{1cm} (2b)$$

Some comments on (2b) are important: the original expression for $\alpha$, Eq. (3a) of the Letter, suggests the inner product of each $\vec{P}_{x_k}$ with $(1,0,0)^T$ which picks out the value $p(a_{x_k} | x_k) = \langle a_i \rangle |a_i\rangle$. However, the last expression is also correct under the non-disturbance condition [22], which is valid for quantum theory and can be stated in this case as:

$$p(a_{x_k+1} | x_k) = p(a_{x_k+1} | x_k + 1), \forall x_k, x_{k+1} \hspace{1cm} (3)$$

this means, in vector form:

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \vec{P}_{x_k} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \vec{P}_{x_{k+1}}, \hspace{1cm} (4)$$

which in turn implies that any convex combination of these entries is valid; in particular, the right hand side of expression (2b). We chose this particular convex combination because of its simplicity and symmetry. These forms for $\alpha^k$ and $\beta^k$ can be summarized by the expression of Eq. (4) of the refereed Letter, reproduced below:

$$I^k = \sum_{x_k} \langle v_I, \vec{P}_{x_k} \rangle, \hspace{1cm} (5)$$

where $I$ represent either $\alpha$ or $\beta$ and $v_I$ is respectively $(1/2)(1 0 1)$ or $(1 0 0)$. We can understand Expression (5) as follows: for every observer $k$, the collection $(\vec{P}_{x_k})_{x_k}$ that maximizes $\alpha^k$ is the one that maximizes the extreme entries of $\sum \vec{P}_{x_k}$ (and with the similar understanding for minimizing $\beta^k$).
Given Expressions (2) for $\beta_k^k$ and $\alpha_k$ in terms of the vectors $\vec{P}_{xk}$ we can state the problem as a Markovian process, as follows. Denoting as $\rho^k$ the state that is available to the $k$-th player after the measurements of the precedent ones denoting the outcomes as $o_{x_k} | x_k \in \{a_i, b_i, a + 1\}$ we can write
\[
p(o_{x_k} | x_k) = \langle o_{x_k} | \rho^k | o_{x_k} \rangle = \text{Tr} \left[ | o_{x_k} \rangle \langle o_{x_k} | \rho^k | o_{x_k} \rangle \langle o_{x_k} | \right]
\]
\[
= \frac{1}{N} \sum_{x_{k-1}, o_{x_k-1} | x_{k-1}} \text{Tr} \left[ | o_{x_k} \rangle \langle o_{x_k} | o_{x_k-1} \rangle \langle o_{x_k-1} | \rho_{k-1} | o_{x_k-1} \rangle \langle o_{x_k-1} | o_{x_k} \rangle \langle o_{x_k} | \right] = \frac{1}{N} \sum_{x_{k-1}, o_{x_k-1} | x_{k-1}} |o_{x_k} \rangle \langle o_{x_k-1} | p(o_{x_k-1} | x_{k-1}).
\]

We used the fact that the players choose the measurement independently and from a uniform distribution under the $N$ possible choices. We can rewrite the relation above in matrix form as
\[
\vec{P}_{x_k} = \frac{1}{N} \sum_{x_{k-1}=0}^{N-1} \left( |a_{x_k-1} | a_{x_k} \rangle \langle a_{x_k} | + |b_{x_k-1} | b_{x_k} \rangle \langle b_{x_k} | + |a_{x_k-1+1} | a_{x_k+1} \rangle \langle a_{x_k+1} | + |b_{x_k-1+1} | b_{x_k+1} \rangle \langle b_{x_k+1} | \right) \vec{P}_{x_{k-1}} = \frac{1}{N} \sum_{x_{k-1}=0}^{N-1} \left[ M_{x_k,x_{k-1}} \right] \vec{P}_{x_{k-1}},
\]
which defines the matrix $M_{x_k,x_{k-1}}$: each column is labeled by a fixed state representing an outcome $o_{x_{k-1}} | x_{k-1}$ of the $k - 1$-th observer, while the rows are labeled by states representing the outcomes $o_{x_k} | x_k$ of the $k$-th observer. It is important to note that, once $N$ is fixed, this matrix depends only on the difference $x_k - x_{k-1}$ (modulo $N$) and is bistochastic – rows and columns sum to one as they are the norm of normalized states– denoting the Markovianity of this process. If the choice $x_{k-1}$ and $\vec{P}_{x_{k-1}}$ are known, then the whole vector $\vec{P}_{x_k}$ can be known.

Now we are going to investigate the consequences of such Markovianity, leading to the results for this protocol. Using (7) in Eq. (5) we can obtain the expression for $I^k$ in terms of the matrices $\{M_{x_k,x_{k-1}}\}$:
\[
I^k = \sum_{x_k} \langle v_I | \vec{P}_{x_k} \rangle = \sum_{x_k} \left( \frac{1}{N} \sum_{x_{k-1}} M_{x_k,x_{k-1}} \vec{P}_{x_{k-1}} \right) = \langle v_I, \frac{1}{N} \sum_{x_{k-1}} \left( \frac{1}{N} \sum_{x_k} M_{x_k,x_{k-1}} \right) \vec{P}_{x_{k-1}} \rangle.
\]
The term in parenthesis suggests the definition of the matrix
\[
M_N = \frac{1}{N} \sum_{x_k} M_{x_k,x_{k-1}},
\]
which does not depend on $x_{k-1}$, since each $M_{x_k,x_{k-1}}$ depend only on the distance $x_k - x_{k-1}$ and, by summing $M_{x_k,x_{k-1}}$ over $x_k$, all differences $x_k - x_{k-1}$ are considered. We can see from Eq. (9) that $M_N$ is also bistochastic. The symmetry of the $N$-cycles and the bistochastic feature reflect into $M_N$ not only by making it independent of $x_{k-1}$ but also imply that $(M^1)_{11} = (M^3)_{13} = (M^3)_{31} = (M^3)_{33} \equiv t_N$ and $(M^3)_{12} = (M^3)_{21} = (M^3)_{23} = (M^3)_{32} = 1 - 2t_N$. So, we can define $M_N$ only in terms of $t_N$ as:
\[
M_N = \begin{pmatrix}
t_N & 1 - 2t_N & t_N \\
1 - 2t_N & 4t_N - 1 & 1 - 2t_N \\
t_N & 1 - 2t_N & t_N
\end{pmatrix}.
\]

With $M_N$ in this form we can get bounds on $t_N$, since all elements being positive implies $1 \leq 4t_N \leq 2$ for all odd $N$.

As we know the important features of the matrix $M_N$ we can turn back to its effects on the inequalities for the $k$-th player. From Eq. (8) and the symmetries of $M_N$, we can write:
\[
I^k = \langle v_I, \sum_{x_{k-1}} M_N \vec{P}_{x_{k-1}} \rangle = \langle v_I, M_N \sum_{x_{k-1}} \vec{P}_{x_{k-1}} \rangle.
\]

Equation (11) highlights the Markovianity of the game under protocol number 1. We can see that the relevant vector for the $k$-th player to calculate $I_k$ – i.e., $\sum \vec{P}_{x_k}$ – can be obtained only from $M_N$ and the relevant vector of the previous player, $\sum \vec{P}_{x_{k-1}}$. Using the same reasoning as in Eq. (8) for $\vec{P}_{x_{k-1}}$ in terms of $\vec{P}_{x_{k-2}}$ and so on, we arrive at
\[
I^k = \langle v_I, M_N \sum_{x_{k-1}} \vec{P}_{x_{k-1}} \rangle = \langle v_I, (M_N)^{k-1} \sum_{x_1} \vec{P}_{x_1} \rangle.
\]
Since the matrix $M_N$ is fixed for each $N$, Eq. (12) already gives all that is necessary to calculate $\alpha^k$ and $\beta^k$ for any $k$ and initial state in terms of $(P_{x_i})_{x_i}$.

With Eqs. (11) and (10) it is possible to prove that the initial state that gives maximal violation for the first player also allows for the best attempt for the $k$-th player to win. The proof goes as follows. Writing $\sum_{x_{k-1}} P_{x_{k-1}} = (c \ d \ e)^t$ for the $(k-1)$-th player, $\alpha^{k-1}$ is maximized ($\beta^{k-1}$ is minimized) by the vector with maximum $(c + e)$, as pointed above [? ]. Now, if this vector is multiplied by $M_N$, from Eq. (10) we have

$$\sum_{x_k} P_{x_k} = M_N \sum_{x_{k-1}} P_{x_{k-1}} = (c + e) \left( \begin{array}{c} t_N \\ t_N \end{array} \right) + d \left( \frac{1 - 2t_N}{1 - 2t_N} \right).$$

(13)

Now, $\alpha^k$ can be written as

$$\alpha^k = (c + e)t_N + d(1 - 2t_N) = (c + e)(3t_N - 1) + N(1 - 2t_N),$$

(14)

where we used that $c + d + e = N$. We know from the positivity of every element of $M_N$ that $t_N < 1/2$, ensuring positivity of the last term, $(1 - 2t_N)$. Now, we see that the maximum value of $\alpha^k$ will be given by the maximum value of $(c + e)$ only if $t_N > 1/3$. For those cases, we see that the vector $\sum_{x_{k-1}} P_{x_{k-1}}$ that maximizes $\alpha^{k-1}$ (the one with maximum $(c + e)$) leads to the vector $\sum_{x_k} P_{x_k}$ that maximizes $\alpha^k$, i.e., leads to the vector $\sum_{x_k} P_{x_k}$ with maximized extreme entries. Then, continuing the argument for the previous players, the vector $\sum_{x_k} P_{x_k}$ that leads to the best value of $P^k$ is the one that leads to the best value for $P^t$. Since we know that the quantum state $|\Psi_{\text{handle}}\rangle$ leads to maximum violation for the first player, we know by the previous argument that this initial state also leads to the best hope for the $k$-th player to win, for all $k$. It is left to prove now that $t_N > 1/3$ is not a restriction in the scenarios we are considering. In fact, it follows from (i) $t_N = (1/N) \sum |\langle a_0 | a_i \rangle|^2$ increases with $N$ and (ii) $t_{N=3} = 1/3$, implying that $t_N > 1/3$ $\forall N \geq 5$ [? ].

The calculation given by Eq. (12) can be used for every $k$ and leads naturally to the evaluation of the limit $k \to \infty$. The asymptotic limit can be calculated exactly by noting that the matrix $M_N$ is regular (which implies irreducibility). A stochastic matrix $M$ is said to be regular if there is a natural $r$ such that all entries of the $r$-th power of $M$ are positive:

$$(M^r)_{ij} > 0, \forall i, j.$$  

(15)

For the matrices $M_N$ it is possible to see that $r = 1$. This is so because there are null entries in $M_{x_k,x_{k-1}}$ if, and only if, $|x_k - x_{k-1}| \leq 1$. In all other cases, the definition of the vectors $|o_{x_k}\rangle$ imply non-null entries for $M_{x_k,x_{k-1}}$. As $M_N$ is the sum of all elements of $\{M_{x_k,x_{k-1}}\}$, $M_N$ has all entries strictly higher than zero. Now, the Perron-Fr"obenius theorem guarantees the existence of a unique stationary distribution eigenvector (i.e., eigenvector with eigenvalue 1) for any regular stochastic matrix, for which the system will tend by successive applications of this matrix [27]. This in turn implies that $M_N$ has such an unique stationary distribution; more than that, as $M_N$ is not only stochastic but bistochastic, it is a fact that the uniform distribution $\tilde{P}^* = (1/3)(1,1,1)^T$ is an eigenvector with eigenvalue 1. As the Perron-Fr"obenius theorem guarantees the stationary vector is unique, this is the only stationary distribution. In other words:

$$\lim_{n \to \infty} (M_N)^n \tilde{P} = \frac{1}{3} \left( \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right), \forall \tilde{P} \text{ s. t. } \sum_i P(i) = 1.$$  

(16)

This implies the asymptotic limit presented in the Letter.

II. PROTOCOLS NUMBER 2 AND 3

For protocols 2 and 3, the measurements are $\{|a_{x_k}\rangle \langle a_{x_k}|, \mathcal{I} - |a_{x_k}\rangle \langle a_{x_k}|\}_{x_k}$ with outcomes $o_{x_k} = x_k \in \{a_{x_k}, -a_{x_k}\}$ and $\{|b_{x_k}\rangle \langle b_{x_k}|, \mathcal{I} - |b_{x_k}\rangle \langle b_{x_k}|\}_{x_k}$ with outcomes $o_{x_k} = x_k \in \{b_{x_k}, -b_{x_k}\}$, respectively. Here we prove Equations (7) of the Letter, relating $\beta^k$ to $\beta^{k-1}$ and $\alpha^k$ to $\alpha^{k-1}$, given by

$$\beta^k = B_N \beta^{k-1} + b_N,$$

(17a)

$$\alpha^k = C_N \alpha^{k-1} + c_N.$$  

(17b)

We will present the calculations for $\beta^k$ and the results for $\alpha^k$ follows an analogous path.
Expression for $\beta^k$ can be written in the form

$$\beta^k = \sum_{x_k} \text{Tr} \left\{ |b_{x_k}\rangle\langle b_{x_k}| \rho^k |b_{x_k}\rangle\langle b_{x_k}| \right\} = \text{Tr} \left\{ \left( \sum_{x_k} |b_{x_k}\rangle\langle b_{x_k}| \right) \rho^k \right\} = \text{Tr} \left\{ B_N \rho^k \right\}$$

(18)

with $B_N = \sum_{x_k} |b_{x_k}\rangle\langle b_{x_k}|$, and $\rho^k$ the state available for the $k$-th player. Since this state is given by the initial state $\rho^1$ measured by the $k - 1$ previous players without registering of which measurement and which result occurred, $\rho^k$ is given by:

$$\rho^k = \left( \frac{1}{N} \right)^{k-1} \sum_{\vec{x}, \vec{x}'} \Pi_{x_{k-1}}^{\vec{x}'} \ldots \Pi_{x_1}^{\vec{x}'} \rho \Pi_{x_1}^{\vec{x}} \ldots \Pi_{x_{k-1}}^{\vec{x}}$$

(19)

where we denote the projective elements of the POVM $\{ |b_{x_k}\rangle\langle b_{x_k}|, I - |b_{x_k}\rangle\langle b_{x_k}| \}$ as $\{ \Pi_{x_k}^{b_{x_k}} \}_{b_{x_k}}$ with $o_{x_k} \in \{ b_{x_k}, -b_{x_k} \}$ and $\vec{a}$ stands for $a_1 |x_1, \ldots, a_{x_{k-1}} |x_{k-1}$ while $\vec{x} = x_1, \ldots, x_{k-1}$. As one can see, the operator $B_N$ defines the expression $\beta^k$ through its expected value on a given state $\rho^k$. Now, some symmetry aspects of such operator (and the analogous for the operator for $\alpha$) will be discussed.

A. Symmetries of the odd N-cycle

The symmetry of the vectors defining the quantum realization leads to some important relations. First, the action of a rotation of $2\pi/N$ by an axis defined by the $|\psi_{\text{handle}}\rangle$ or the reflection by a plane that contains both the handle and one of the vectors let the set of operators $\{ |b_{x_k}\rangle\langle b_{x_k}| \}_{x_k}$ unchanged, since it only reorganize its elements [25]. This means that the operator $B_N$ is the same after action of any element of such symmetry group (denoted $C_{Nv}$ point group in Ref. [28]). In other words, for any unitary $U_N$ representing any of the elements of the group, the following relation is valid:

$$U_N B_N U_N^{-1} = \sum_i U_N | b_i \rangle \langle b_i | U_N^{-1} = B_N \Rightarrow [B_N, U_N] = 0 \forall U_N \in C_N.$$  

(20)

where we used subscript $i$ instead of $x_k$ for sake of an enlightened notation. If the representations given by $\{ U_N \}$ were irreducible, Schur’s lemma would imply that the operator $B_N$ is a multiple of identity. However, this is not the case, as none of the groups $C_{Nv}$ with odd $N$ has irreducible representations of dimension 3 [28]. This implies, by Schur’s Lemma, that this matrix takes the form:

$$B_N = \sum_i | b_i \rangle \langle b_i | = \lambda_0^N I_{2 \times 2} \oplus \lambda_1^N I_{1 \times 1}.$$  

(21)

Of course the same happens with operator $A_N = \sum_i | a_i \rangle \langle a_i |$, which acquires the form

$$A_N = \sum_i | a_i \rangle \langle a_i | = \mu_0^N I_{2 \times 2} \oplus \mu_1^N I_{1 \times 1}.$$  

(22)

B. Proof for protocols 2 and 3

The symmetry relation (21) implies the operator identity

$$\sum_i \Pi_{b_i}^{h_i} B_N \Pi_{b_i}^{h_i} = z_N B_N;$$  

(23)

which says that when $\Pi_{b_i}^{h_i}$ acts on $B_N$ we get a multiple of the same $O_N$, where the constant of proportionality is given by $z_N = (1/N)[2(\lambda_0^N)^2 + (\lambda_1^N)^2]$ and is dependent only on the scenario, i.e., on the odd $N$. It is also possible to analyze the action of $\Pi_{b_i}^{h_i}$ on the same operator, and one gets:

$$\sum_i \Pi_{a_i}^{h_i} B_N \Pi_{a_i}^{h_i} = u_N B_N + 2\lambda_0^N \lambda_1^N I_{3 \times 3};$$  

(24)
where \( u_N = N + z_N - 2(\lambda_0^N + \lambda_1^N) \), \( \lambda_0^N \) and \( \lambda_1^N \) are the eigenvalues of \( B_N \) and so \( u_N \) is also only dependent on \( N \). Now, it is good to rewrite the expressions for \( \beta^k \) making explicit the dependence on \( \rho^{k-1} \):

\[
\beta^k = \sum_{x_k} p(o_{x_k} = b|x_k) = \text{Tr} \left\{ B_N \rho^k \right\} = \left( \frac{1}{N} \right) \sum_{x_{k-1},o_{x_{k-1}}} \text{Tr} \left\{ B_N \left( \Pi_{x_{k-1}}^o \rho^{k-1} \Pi_{x_{k-1}}^o \right) \right\}. \tag{25}
\]

Using in Eq. (25) the relations (23) and (24) together with the cyclic property of the trace operation, we get:

\[
\beta^k = \frac{z_N + u_N}{N} \beta^{k-1} + 2 \frac{\lambda_0^N \lambda_1^N}{N} = B_N \beta^{k-1} + b_N, \tag{26}
\]

which is the relation that we aimed to prove. Now, for the asymptotic limit, we use the explicit form for \( u_N \) and \( z_N \), to get

\[
B_N = 1 - \frac{3b_N}{N}. \tag{27}
\]

Now, we just have to calculate the limit

\[
\lim_{k \to \infty} \beta^k = \lim_{k \to \infty} \left( \sum_{n=0}^{k} [B_N]^n \right) b_N = \frac{1}{1 - B_N} b_N = \frac{N}{3}, \tag{28}
\]

where the fact that \( |B_N| < 1 \) for all odd \( N \) was used. This finishes the proof for protocol 2, and for protocol 3 the proof is completely analogous. Now it is only left to prove that the average asymptotic state is the the maximally mixed, for all the protocols.

### III. Average State in the Asymptotic Limit

To see that the average state is indeed the maximally mixed, we first note that for this state \( p(o = b_i|x_i) = (1/3) \) for all choices of \( x_i \). This implies that \( \bar{\rho}_\infty \) can be written as

\[
\bar{\rho}_\infty = \frac{1}{3} |b_i\rangle \langle b_i| + \frac{2}{3} \Pi_i^{-b_i} R_i \Pi_i^{-b_i}, \quad \forall i \in \{0,...N-1\} \tag{29}
\]

with \( R_i \) being a positive semidefinite matrix with unit trace. Now,

\[
\bar{\rho}_\infty = \frac{N \bar{\rho}_\infty}{N} = \frac{1}{N} \sum_i \left( \frac{1}{3} |b_i\rangle \langle b_i| + \frac{2}{3} \Pi_i^{-b_i} R_i \Pi_i^{-b_i} \right) = \frac{1}{N} \left( \frac{1}{3} B_N + \frac{2}{3} \sum_i \Pi_i^{-b_i} R_i \Pi_i^{-b_i} \right). \tag{30}
\]

The matrix \( \sum_i \Pi_i^{-b_i} R_i \Pi_i^{-b_i} \) commutes with every operator representing the group \( C_N \). Then, by Schur’s lemma this means

\[
\sum_i \Pi_i^{-b_i} R_i \Pi_i^{-b_i} = (r_1 I_{2 \times 2}) \oplus r_0 I_{1 \times 1}. \tag{31}
\]

We also showed, in Eq. (21) that \( B_N \) has an analogous form. This means that the average state also has this form and we can write

\[
\bar{\rho}_\infty = (r'_1 I_{2 \times 2}) \oplus r'_0 I_{1 \times 1}, \tag{32}
\]

with \( r'_j \) related to the eigenvalues of \( B_N \) and to \( r_j \). However, it is not necessary to enter into these details. By (29), (32) and the orthogonality of \( \Pi_i^{-b_i} R_i \Pi_i^{-b_i} \) with \( |b_i\rangle \langle b_i| \), the average state is a diagonal matrix, with one eigenvalue equal to \( r'_0 = 1/3 \) and two other eigenvalues equal to \( r'_1 \). By the condition of unit trace for \( \bar{\rho}_\infty \) this means that \( r'_1 = 1/3 \) as well. This implies \( \rho_\infty = I/3 \).

[1] R. W. Spekkens, Phys. Rev. A 71, 052108 (2005).
