Mathematical Tools for Calculation of the Effective Action in Quantum Gravity

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We review the status of covariant methods in quantum field theory and quantum gravity, in particular, some recent progress in the calculation of the effective action via the heat kernel method. We study the heat kernel associated with an elliptic second-order partial differential operator of Laplace type acting on smooth sections of a vector bundle over a Riemannian manifold without boundary. We develop a manifestly covariant method for computation of the heat kernel asymptotic expansion as well as new algebraic methods for calculation of the heat kernel for covariantly constant background, in particular, on homogeneous bundles over symmetric spaces, which enables one to compute the low-energy non-perturbative effective action.
1 Introduction

One of the most important problems of modern fundamental physics is the problem of reconciling classical general relativity, the theory of macroscopic gravitational phenomena, with quantum theory, so-called Quantum Gravity problem. This is a really difficult task since one has to answer the very basic questions concerning the local and the global structure of the spacetime itself as well as deep questions about the nature of quantum mechanics.

Although, over the last several decades many competing approaches (Euclidean path integrals, string theory, loop gravity, non-commutative geometry, asymptotic safety, various lattice approaches and others) has been put forward and despite some real progress in some of these approaches in the last two decades, we still do not have a complete consistent theory of quantum gravitational phenomena. It looks like we are missing an important piece of the puzzle which prevents us to find the solution.

In this situation it seems to be wise to go back and to recall some pioneering works in quantum gravity. This review will concentrate on so-called covariant methods in quantum gravity. Some other approaches are reviewed by other lecturers of this school. The basis of the covariant methods in quantum gravity is the background field method. This method was developed mainly by De Witt in his classical papers [20, 21] and reviews [22, 23] (for the latest update see the book [24]). It is a generalization of the method of generating functionals in quantum field theory developed and successfully used by Schwinger [29, 30]. For a detailed review see, for example, [18, 16, 26].

The basic object in the background field method is the effective action. The effective action is a functional of the background fields that encodes, in principle, all the information of quantum field theory. It determines the full one-point propagator and the full vertex functions and, hence, the whole S-matrix. Moreover, the variation of the effective action gives the effective equations for the background fields, which makes it possible to study the back-reaction of quantum processes on the classical background. In particular, the low energy effective action (called the effective potential) is the most appropriate tool for investigating the structure of the physical vacuum in quantum field theory.

The only practical method for the calculation of the effective action is the semi-classical perturbative expansion of the path integral in the number of loops. All fields are split in background classical parts and quantum perturbations propagating on this background and the classical action is expanded in quantum fields. Then the quadratic part determines the propagators of the quantum fields and the
higher-order terms reproduce the vertex functions of the perturbation theory.

In the perturbation theory the effective action is expressed in terms of the propagators and the vertex functions. One of the most powerful methods to study the propagators is the proper time method (also called the heat kernel method, in particular, by mathematicians), which was originally proposed by Fock \[25\] and later generalized by Schwinger \[29, 30\] who also applied it to the calculation of the one-loop effective action in quantum electrodynamics. It was De Witt \[20, 22, 23\] who perfected the proper time method; he reformulated it in the geometrical language and applied it to the case of gravitational field.

At one-loop level, the contribution of the gravitational loop is of the same order as the contributions of matter fields. At low energies (lower than the Planckian energy, $\hbar c^5/G$) the contribution of higher gravitational loops should be highly suppressed. Therefore, a semi-classical concept applies when the quantum matter fields together with the linearized perturbations of the gravitational field interact with the background gravitational field (and, probably, with the background matter fields). This is what is usually called the one-loop quantum gravity. The main difficulty of quantum gravity is the fact that there is no consistent way to eliminate the ultraviolet divergences arising in perturbation theory, even at one loop level.

The present review is devoted to the development of the covariant methods for calculation of the effective action in quantum field theory and quantum gravity. In Sect. 2 we review the formal structure of quantum gauge field theory and quantum gravity and the construction of the effective action following \[10, 11\]. In Sect. 3 we describe the heat kernel method and develop the asymptotic expansion of the heat kernel following \[2, 3, 9, 10, 11\]. In Sect. 4 we describe the local structure of the Green function following \[8\]. In Sect. 5 we develop a method for the calculation of the heat kernel coefficients and describe their general structure following \[3, 9, 10, 11\]. In Sect. 6 we compute the heat trace in the high-energy approximation following \[1, 3, 10\]. In Sect. 7 we describe our results for the calculation of the low-energy heat trace following our recent work \[4, 5, 6, 7, 12, 13\]. In Sect. 8 we apply the obtained results to compute the low-energy one-loop effective action in quantum gravity.
2 Effective Action in Quantum Field Theory and Quantum Gravity

In this section we briefly describe the standard formal construction of the generating functional and the effective action in gauge theories. The basic object of any physical theory is the spacetime $M$, which we will assume to be a $n$-dimensional manifold with the topological structure of a cylinder

$$M = I \times \Sigma,$$

where $I$ is an open interval of the real line (or the whole real line) and $\Sigma$ is some $(n-1)$-dimensional manifold. The spacetime manifold is here assumed to be globally hyperbolic and equipped with a (pseudo)-Riemannian metric $g$ of signature $(- + \cdots +)$; thus, a foliation of spacetime exists into spacelike sections identical to $\Sigma$. Usually one also assumes the existence of a spin structure on $M$. A point $x = (x^\mu)$ in the spacetime is described locally by the time $x^0$ and the space coordinates $(x^1,\ldots,x^{n-1})$. We label the spacetime coordinates by Greek indices, which run from 0 to $(n-1)$, and sum over repeated indices.

Let us consider a vector bundle $\mathcal{V}$ over the spacetime $M$ each fiber of which is isomorphic to a vector space, $V$, on which the spin group $\text{Spin}(1,n-1)$, i.e. the covering group of Lorentz group, acts. The vector bundle $\mathcal{V}$ can also have an additional structure on which a gauge group acts. The sections of the vector bundle $\mathcal{V}$ are called fields. The tensor fields describe the particles with integer spin (bosons) while the spin-tensor fields describe particles with half-integer spin (fermions). Although the whole scheme can be developed for superfields (a combination of boson and fermion fields), we restrict ourselves in the present lecture to boson fields (which, without loss of generality can be considered real). A field $\varphi$ is represented locally by a set of real-valued functions $\varphi = (\varphi^A(x))$, where $A = 1,\ldots,\dim V$. Capital Latin indices will be used to label the local components of the fields. To construct invariant functionals we need to introduce an invariant fiber inner product and an $L^2$ inner product

$$\langle \psi, \varphi \rangle = \int_M d\text{vol}(x) \psi^A(x) E_{AB}(x) \varphi^B(x).$$

where $d\text{vol}(x) = dx g^{1/2}$, $g = |\det g_{\mu\nu}|$, is the natural Riemannian volume element defined by some background metric $g$, and $E^{AB}$ is a non-degenerate symmetric matrix (a fiber metric). As usual, we assume that a summation over repeated
indices is performed. This metric (and its inverse $E^{-1AB}$) can be used to naturally identify the bundle $\mathcal{V}$ with its dual $\mathcal{V}^*$ (that is to raise and lower the field indices). The sections of the dual bundle are called currents and are represented locally by a set of functions, e.g.

$$J_A = E_{AB}\varphi^B.$$  \hfill (2.3)

We will also use the condensed De Witt notation, where the discrete index $A$ and the spacetime point $x$ are combined in one lower case Latin index $i \equiv (A, x)$. Then the components of a field $\varphi$ are $(\varphi^i) \equiv (\varphi^A(x))$. There is a natural pairing between the bundles $\mathcal{V}$ and $\mathcal{V}^*$ defined by

$$\langle J, \varphi \rangle \equiv J_i \varphi^i \equiv \int_M d\text{vol}(x) J_A(x) \varphi^A(x).$$  \hfill (2.4)

It is assumed that a summation over repeated lower case Latin indices, i.e. a combined summation-integration, is performed.

The set of all sections of the vector bundle $\mathcal{V}$ is called the configuration space, which one assumes to be an infinite-dimensional manifold $M$. The fields $\varphi^i$ are the coordinates on this manifold, the variational derivative $\delta/\delta \varphi$ is a tangent vector, a small disturbance $\delta \varphi$ is a one-form and so on. If $S(\varphi)$ is a scalar field on the configuration space, then its variational derivative $\delta S/\delta \varphi$ is a one-form on $M$ defined by

$$\frac{d}{d\varepsilon} S(\varphi + \varepsilon h) \bigg|_{\varepsilon=0} = \left( \frac{\delta S}{\delta \varphi} , h \right) = \frac{\delta S}{\delta \varphi^i} h^i.$$  \hfill (2.5)

By using the functional differentiation one can define formally the concept of tangent space, the tangent vectors, Lie derivative, one-forms, metric, connection, geodesics and so on (for more details, see [24]).

2.1 Non-Gauge Field Theories

The dynamics of quantum field theory is determined by an action functional $S(\varphi)$, which is a differentiable real-valued scalar field on the configuration space. The dynamical field configurations are defined as the field configurations satisfying the stationary action principle, i.e. they must satisfy the dynamical equations of motion

$$\frac{\delta S}{\delta \varphi} = 0$$  \hfill (2.6)

with given boundary (and initial) conditions. The set of all dynamical field configurations, i.e. those that satisfy the dynamical equations of motion, $M_0$, is a
subspace of the configuration space called the dynamical subspace (or the mass-shell in the high-energy physics jargon).

Quantum field theory is basically a theory of small disturbances on the dynamical subspace. Most of the problems of standard quantum field theory deal with scattering processes, which are described by the transition amplitudes between some well defined initial and final states in the remote past and the remote future. The collection of all these amplitudes is called the scattering matrix, or shortly $S$-matrix.

Let us single out in the space-time two causally connected in– and out– regions, that lie in the past and in the future respectively relative to the region $\Omega$, which is of interest from the dynamical standpoint. Let $|\text{in}\rangle$ and $|\text{out}\rangle$ be some initial and final states of the quantum field system in these regions. Let us consider the transition amplitude $\langle\text{out}|\text{in}\rangle$ and ask the question: how does this amplitude change under a variation of the interaction with a compact support in the region $\Omega$. The answer to this question gives the Schwinger variational principle which states that

$$\delta \langle\text{out}|\text{in}\rangle = \frac{i}{\hbar} \langle\text{out}|\delta S |\text{in}\rangle,$$  \hspace{1cm} (2.7)

where $\delta S$ is the corresponding change of the action. This principle gives a very powerful tool to study the transition amplitudes. The Schwinger variational principle can be called the quantization postulate, because all the information about quantum fields can be derived from it.

Let us change the external conditions by adding a linear interaction with some external classical sources $J$ in the dynamical region $\Omega$, i.e.

$$\delta S = \langle J, \varphi \rangle.$$  \hspace{1cm} (2.8)

The amplitude $\langle\text{out}|\text{in}\rangle$ becomes a functional of the sources that we denote by $Z(J)$. The primary objects of interest in quantum field theory are the chronological mean values

$$q_{i_1\ldots i_n} = \frac{\langle\text{out}|T(\varphi^{i_n}\ldots\varphi^{i_1})|\text{in}\rangle}{\langle\text{out}|\text{in}\rangle}$$  \hspace{1cm} (2.9)

where $T$ denotes the operator of chronological ordering that orders the (non-commuting) operators in order of their time variables from right to left. Of course, in the presence of the sources they become functionals of $J$. By using the Schwinger variational principle one can obtain the chronological mean values in terms of the
functional derivatives of the functional $Z(J)$, that is,

$$Z(J + \eta) = Z(J) \left\{ 1 + \left( \frac{i}{\hbar} \right) \langle \eta, \Psi_1(J) \rangle + \frac{1}{2} \left( \frac{i}{\hbar} \right)^2 \langle \eta, \Psi_2(J) \eta \rangle + \sum_{n=3}^{\infty} \frac{1}{n!} \left( \frac{i}{\hbar} \right)^n \Psi_{1\ldots n}^{i_1\ldots i_n}(J) \eta_{i_1} \cdots \eta_{i_n} \right\}.$$  \hfill (2.10)

In other words, the functional $Z(J)$ is the generating functional for the chronological amplitudes $\Psi_n$.

Let us now define another functional $W(J)$ by

$$Z = \exp \left( \frac{i}{\hbar} W \right).$$  \hfill (2.11)

Its functional derivatives define so called full connected Green functions, $\mathcal{G}_{n}^{i_1\ldots i_n}$, (or the correlation functions) by

$$W(J + \eta) = W(J) + \langle \eta, \mathcal{G}_1(J) \rangle + \frac{1}{2} \langle \eta, \mathcal{G}_2(J) \eta \rangle + \sum_{n=3}^{\infty} \frac{1}{n!} \mathcal{G}_{n}^{i_1\ldots i_n}(J) \eta_{i_1} \cdots \eta_{i_n}.$$  \hfill (2.12)

The functional $\phi = \mathcal{G}_1$ is called the background (or the mean) field, the operator $\mathcal{G} = \mathcal{G}_2$ is called the full propagator. Then, it is easy to see that all chronological mean amplitudes can be expressed in terms of connected Green functions. In particular, we have

$$\Psi_1 = \phi,$$  \hfill (2.13)

$$\Psi_2^{jk} = \phi^j \phi^k + \frac{\hbar}{i} \mathcal{G}^{jk}.$$  \hfill (2.14)

Thus, whilst $Z(J)$ is the generating functional for chronological amplitudes, the functional $W(J)$ is the generating functional for the connected Green functions. The Green functions satisfy the boundary conditions which are determined by the states $|\text{in}\rangle$ and $|\text{out}\rangle$.

The mean field itself is a functional of the sources, $\phi = \phi(J)$. It is easy to see that the functional derivative of the mean field is equal to the full propagator, that is,

$$\frac{d}{d\epsilon} \phi(J + \epsilon \eta) \bigg|_{\epsilon=0} = \mathcal{G} \eta.$$  \hfill (2.15)

In the non-gauge theories the full propagator $\mathcal{G}$, which plays the role of the (infinite-dimensional) Jacobian, is non-degenerate. Therefore, one can change variables
and consider $\phi$ as independent variable and $J = J(\phi)$ (as well as all other functionals) as a functional of $\phi$.

There are many different ways to show that there is a functional $\Gamma(\phi)$ such that

$$
\left\langle \frac{\delta S(\varphi)}{\delta \varphi} \right\rangle = \frac{\delta \Gamma(\phi)}{\delta \phi}.
$$

(2.16)

This functional is defined by

$$
\langle \text{out}|\text{in} \rangle = \exp \left\{ \frac{i}{\hbar} \left[ \Gamma + \langle J, \phi \rangle \right] \right\},
$$

(2.17)

or by the functional Legendre transform

$$
\Gamma(\phi) = W(J(\phi)) - \langle J(\phi), \phi \rangle.
$$

(2.18)

This is the most important object in quantum field theory. It contains all the information about quantized fields. The functional expansion of this functional reads

$$
\Gamma(\phi + h) = \Gamma(\phi) - \langle J(\phi), h \rangle - \frac{1}{2} \langle h, G(J(\phi))h \rangle + \sum_{n=3}^{\infty} \frac{1}{n!} \Gamma_{i_1 \ldots i_n}(\phi)h^{i_1} \ldots h^{i_n}.
$$

(2.19)

Therefore, the first variation of $\Gamma$ gives the effective equations for the background fields

$$
\frac{\delta \Gamma}{\delta \phi} = -J.
$$

(2.20)

These equations replace the classical equations of motion and describe the effective dynamics of the background field with regard to all quantum corrections. That is why $\Gamma$ is called the effective action.

Furthermore, the second derivative of $\Gamma(\phi)$ determines the full propagator

$$
\mathcal{G} = \left( \frac{\delta^2 \Gamma}{\delta \phi^2} \right)^{-1}.
$$

The higher derivatives, $\Gamma_{i_1 \ldots i_k}$, determine the so-called full vertex functions (also called strongly connected, or one-particle irreducible, functions). In other words, $\Gamma(\phi)$ is the generating functional for the full vertex functions. The full vertex functions together with the full propagator determine the full connected Green functions and, therefore, all chronological amplitudes and, hence, the $S$-matrix.
Thus, the entire quantum field theory is summed up in the functional structure of the effective action.

One can obtain a very useful formal representation for the effective action in terms of functional integrals (called also path integrals, or Feynman integrals). A functional integral is an integral over the (infinite-dimensional) configuration space $\mathcal{M}$. Although a rigorous mathematical definition of functional integrals is absent, they can be used in perturbation theory of quantum field theory as an effective tool, especially in gauge theories, for manipulating the whole series of perturbation theory. The point is that in perturbation theory one encounters only functional integrals of Gaussian type, which can be well defined effectively in terms of classical propagators and vertex functions. The Gaussian integrals do not depend much on the dimension and, therefore, (after a proper normalization) all formulas from the finite-dimensional case, like Fourier transform, integration by parts, delta-function, change of variables etc., are valid in the infinite-dimensional case as well. One has to note that functional integrals are formally divergent — if one tries to evaluate the integrals, one encounters meaningless divergent expressions. This difficulty can be overcome in the framework of the renormalization theory (in so-called renormalizable field theories). In non-renormalizable theories (like quantum general relativity) this issue becomes the main difficulty of the theory.

Integrating the Schwinger variational principle one can obtain the following functional integral:

$$\langle \text{out}|\text{in} \rangle = \int_{\mathcal{M}} D\phi \exp \left\{ \frac{i}{\hbar} \left[ S(\phi) + \langle J,\phi \rangle \right] \right\}. \quad (2.21)$$

Here $D\phi$ represents the functional measure; however, it should not be taken too seriously—it will just provide a formal device for manipulations of Gaussian integrals. Correspondingly, for the effective action one obtains the functional equation

$$\exp \left\{ \frac{i}{\hbar} \Gamma(\phi) \right\} = \int_{\mathcal{M}} D\phi \exp \left\{ \frac{i}{\hbar} \left[ S(\phi) - \left( \frac{\delta \Gamma(\phi)}{\delta \phi} \right)(\phi - \langle \phi \rangle) \right] \right\}. \quad (2.22)$$

The only way to get numbers from this formal expression is to take advantage of the semi-classical approximation within a formal (asymptotic) expansion in powers of Planck constant $\hbar$:

$$\Gamma \sim S + \sum_{k=1}^{\infty} \hbar^k \Gamma_{(k)}. \quad (2.23)$$
Next, we substitute this expansion in the functional equation for the effective action, shift the integration variable in the functional integral
\[ \varphi = \phi + \sqrt{\hbar} h, \] (2.24)
and expand the action \( S(\varphi) \) in functional Taylor series in quantum fields \( h \)
\[
S(\phi + \sqrt{\hbar} h) = S(\phi) + \hbar^{1/2} \left( \frac{\delta S(\phi)}{\delta \phi}, h \right) - \hbar \frac{1}{2} \left( h, L(\phi) h \right) + \sum_{n=3}^{\infty} \frac{1}{n!} \hbar^{n/2} S_{i_1...i_n}(\phi) h^{i_1} \cdots h^{i_n},
\] (2.25)
where \( L \) is a (usually, partial differential) operator defined by the second variation of the action
\[ L = -\frac{\delta^2 S}{\delta \varphi^2}. \] (2.26)
Notice that the operator \( L \) maps sections of the vector bundle \( \mathcal{V} \) to sections of the dual bundle \( \mathcal{V}^* \), that is,
\[ L : C^\infty(\mathcal{V}) \rightarrow C^\infty(\mathcal{V}^*). \] (2.27)
In order to have a well defined operator which is self-adjoint with respect to the \( L^2 \) inner product on the bundle \( \mathcal{V} \) we define another operator
\[ \hat{L} : C^\infty(\mathcal{V}) \rightarrow C^\infty(\mathcal{V}), \] (2.28)
such that
\[ (\varphi, \hat{L} h) = \int_M d\text{vol} \quad \varphi^A E_{AB} \hat{L}^B C h^C = \int_M d\text{vol} \quad \varphi^A L_{AC} h^C = \langle \varphi, L h \rangle, \] (2.29)
that is,
\[ E_{AB} \hat{L}^B C = L_{AC} h^C. \] (2.30)
Now, by expanding both sides of the functional equation for the effective action in powers on \( \hbar \) and equating the coefficients of equal powers of \( \hbar \), we get the recurrence relations that uniquely define all coefficients \( \Gamma_{(k)} \). The measure formally transforms as \( D\varphi = Dh \). All functional integrals appearing in this expansion are Gaussian and can be calculated in terms of the functional determinant, \( \text{Det} \hat{L} \), of the operator \( \hat{L} \) and the bare propagator \( G = L^{-1} \), i.e. the Green function of the
operator $L$ with Feynman boundary conditions. More precisely, with the proper normalization of the measure one can define
\[
\int_M \mathcal{D}h \exp\left(-\frac{i}{2}(h, \hat{L}h)\right) = (\text{Det} \hat{L})^{-1/2},
\]
(2.31)
\[
\int_M \mathcal{D}h \exp\left(-\frac{i}{2}(h, \hat{L}h)\right)h^{i_1} \cdots h^{i_{2m+1}} = 0,
\]
(2.32)
\[
\int_M \mathcal{D}h \exp\left(-\frac{i}{2}(h, \hat{L}h)\right)h^{i_1} \cdots h^{i_{2m}} = \frac{(2m)!}{2^m m! m!} (\text{Det} \hat{L})^{-1/2} G^{i_1 i_2} \cdots G^{i_{2m-1} i_{2m}},
\]
(2.33)
where parenthesis denote the complete symmetrization over all indices included. Of course, the Green functions of the operators $L$ and $\hat{L}$ are related by
\[
\hat{G}^A_{\beta}(x,y) = G^{AC}(x,y)E_{CB}(y).
\]
(2.34)
In particular, the one-loop effective action is determined by the functional determinant of the operator $L$
\[
\Gamma^{(1)} = -\frac{1}{2i} \log \text{Det} \hat{L},
\]
(2.35)
and the two-loop effective action is given by
\[
\Gamma^{(2)} = -\frac{1}{8} S_{,ijkl} G^{ij} G^{kl} - \frac{1}{12} S_{,ijkl} G^{im} G^{kn} S_{,lmn}.
\]
(2.36)
Strictly speaking, the Gaussian integrals are well defined for elliptic partial differential operators in terms of the functional determinants and their Green functions. Although the Gaussian integrals of quantum field theory are determined by hyperbolic partial differential operators with Feynman boundary conditions they can be well defined by means of the analytic continuation from the Euclidean sector of the theory where the operators become elliptic. This is done by so-called Wick rotation—one replaces the real time coordinate by a purely imaginary one $x^0 \rightarrow i\tau$ and singles out the imaginary factor also from the action $S \rightarrow iS$ and the effective action $\Gamma \rightarrow i\Gamma$. Then the metric of the spacetime manifold becomes positive definite and the classical action in all ‘nice’ field theories becomes a positive-definite functional. Then the fast oscillating Gaussian functional integrals become exponentially decreasing and can be given a rigorous mathematical meaning.
2.2 Gauge Field Theories

Let us try to apply the formalism described above to a gauge field theory. A characteristic feature of a gauge field theory is the fact that the dynamical equations

\[ \frac{\delta S}{\delta \varphi} = 0 \] (2.37)

are not independent — there are certain identities, called Nöther identities, between them. This means that there are some nowhere vanishing vector fields

\[ R_{\alpha} = R_{\alpha}^{i} \frac{\delta}{\delta \varphi^{i}} \] (2.38)

on the configuration space \( M \) that annihilate the action,

\[ R_{\alpha} S = 0, \] (2.39)

and, hence, define invariance flows on \( M \). The transformations of the fields

\[ \delta \xi \varphi^{i} = R_{\alpha}^{i} \xi^{\alpha} \] (2.40)

are called the invariance transformations and \( R_{\alpha} \) are called the generators of invariance transformations. The infinitesimal parameters of these transformations \( \xi \) are sections of another vector bundle (usually the tangent bundle \( TG \) of a compact Lie group \( G \)) that are represented locally by a set of functions \( (\xi^{a}) = (\xi^{a}(x)), \ a = 1, \ldots, \dim G \), over spacetime with compact support. To distinguish between the components of the gauge fields and the components of the gauge parameters we introduce lower case Latin indices from the beginning of the alphabet; the Greek indices from the beginning of the alphabet are used as condensed labels \( \alpha = (a, x) \) that include the spacetime point.

We assume that the vector fields \( R_{\alpha} \) are linearly independent and complete, which means that they form a complete basis in the tangent space of the invariant subspace of configuration space. The vector fields \( R_{\alpha} \) form the gauge algebra. We restrict ourselves to the simplest case when the gauge algebra is the Lie algebra of an infinite-dimensional gauge Lie group \( \mathcal{G} \). This is the case in Yang-Mills theory and gravity. Then the flow vectors \( R_{\alpha} \) decompose the configuration space into the invariants subspaces of \( M \) (called the orbits) consisting of the points connected by the gauge transformations. The space of orbits is then \( M/G \). The linear independence of the vectors \( R_{\alpha} \) at each point implies that each orbit is a copy of the group manifold. One can show that the vector fields \( R_{\alpha} \) are tangent to the dynamical
subspace \( M_0 \), which means that the orbits do not intersect \( M_0 \) and the invariance flow maps the dynamical subspace \( M_0 \) into itself. Since all field configurations connected by a gauge transformation, i.e., the points on an orbit, are physically equivalent, the physical dynamical variables are the classes of gauge equivalent field configurations, i.e., the orbits. The physical configuration space is, hence, the space of orbits \( M/G \). In other words the physical observables must be the invariants of the gauge group.

To quantize a gauge theory by means of the functional integral, we consider the in– and out– regions, define some \( |\text{in}\rangle \) and \( |\text{out}\rangle \) states in these regions and study the amplitude \( \langle \text{out}|\text{in}\rangle \). Since all field configurations along an orbit are physically equivalent we have to integrate over the orbit space \( M/G \). To deal with such situations one has to choose a representative field in each orbit. This can be done by choosing special coordinates \(( I^A(\varphi), \chi^\alpha(\varphi)) \) on the configuration space \( M \), where \( I^A \) label the orbits and \( \chi^\alpha \) the points in the orbit. Computing the Jacobian of the field transformation and introducing a delta functional \( \delta(\chi - \zeta) \) we can fix the coordinates on the orbits and obtain the measure on the orbit space \( M/G \)

\[
\mathcal{D}I = \mathcal{D}\varphi \text{Det} F(\varphi)\delta(\chi(\varphi) - \zeta),
\]

where

\[
F_{\alpha}\beta \ = \ R_{\alpha}\beta
\]

is a non-degenerate operator. Thus we obtain a functional integral for the transition amplitude

\[
\langle \text{out}|\text{in}\rangle = \int_M \mathcal{D}\varphi \text{Det} F(\varphi)\delta(\chi(\varphi) - \zeta) \exp\left\{ i\frac{\bar{h}}{\hbar} S(\varphi) \right\}.
\]

Now one can go further and integrate this equation over parameters \( \zeta \) with a Gaussian measure determined by a symmetric nondegenerate matrix \( \gamma = (\gamma_{\alpha\beta}) \), which most naturally can be chosen as the metric on the orbit (gauge group metric). As a result we get

\[
\langle \text{out}|\text{in}\rangle = \int_M \mathcal{D}\varphi (\text{Det} \gamma)^{1/2} \text{Det} F(\varphi) \exp\left\{ i\frac{\bar{h}}{\hbar} \left[ S(\varphi) + \frac{1}{2} \langle \chi(\varphi), \gamma \chi(\varphi) \rangle \right] \right\}.
\]

The functional equation for the effective action takes the form

\[
\exp\left\{ i\frac{\bar{h}}{\hbar} \Gamma(\phi) \right\} = \int_M \mathcal{D}\varphi (\text{Det} \gamma(\phi))^{1/2} \text{Det} F(\varphi)
\]
\[ \times \exp \left\{ \frac{i}{\hbar} \left[ S(\varphi) + \frac{1}{2} \langle \chi(\varphi), \gamma(\varphi) \rangle \chi(\varphi) - \left\langle \delta \Gamma(\phi), \left( \varphi - \phi \right) \right\rangle \right] \right\}. \]

The determinants of the operators $F$ and $\gamma$ are usually represented as a result of the integration over some auxiliary Grassmanian variables, so called ghost fields.

This equation can be used to construct the semi-classical perturbation theory in powers of the Planck constant (loop expansion), which gives the effective action in terms of the bare propagators and the vertex functions. In particular, one finds the one-loop effective action

\[ \Gamma_{(1)} = -\frac{1}{2l} \log \det \hat{L} + \frac{1}{l} \log \det F + \frac{1}{2l} \log \det \gamma, \]  

(2.46)

where $\hat{L}$ is an operator defined by

\[ \frac{d^2}{d\epsilon^2} \left\{ S(\varphi + \epsilon h) + \frac{1}{2} \langle \chi(\varphi + \epsilon h), \gamma \chi(\varphi + \epsilon h) \rangle \right\} \bigg|_{\epsilon = 0} = -(h, \hat{L} h). \]  

(2.47)

In De Witt notation it reads

\[ \hat{L}^k_j = E^{-ki} L_{ij}, \]  

(2.48)

where

\[ L_{ij} = -\frac{\delta^2 S}{\delta \phi^i \delta \phi^j} - \frac{\delta \chi^\alpha}{\delta \phi^i} \gamma_{\alpha \beta} \frac{\delta \chi^\beta}{\delta \phi^j}. \]  

(2.49)

### 2.3 Quantum General Relativity

Einstein’s theory of general relativity is an example of a gauge theory with the gauge group $G$ being the group of all diffeomorphisms of the spacetime manifold $M$ and the configuration space $\mathcal{M}$ being the space of all pseudo-Riemannian metrics on $M$. The physical configuration space $\mathcal{M}/G$ of all orbits of the gauge group is then the space of all geometries on the spacetime.

The gravitational field can be parametrized by the metric tensor of the spacetime

\[ g^i \equiv g_{\mu \nu}(x), \quad i \equiv (\mu \nu, x). \]  

(2.50)

An invariant fiber metric is defined by

\[ E^{\mu \nu \alpha \beta} = g^{(\alpha} g^{\beta)} - \kappa g^{\mu \nu} g^{\alpha \beta}, \]  

(2.51)

where $\kappa \neq 1/n$ is a real parameter. The inverse metric is then

\[ E^{-1}_{\mu \nu \alpha \beta} = g_{(\alpha \beta} g^{\nu)} - \kappa \frac{1}{n \kappa - 1} g_{\mu \nu} g_{\alpha \beta}. \]  

(2.52)
The parameters of gauge transformations are the components of the vector of the infinitesimal diffeomorphism,

$$\xi^\mu = \xi^\mu(x), \quad \mu \equiv (\mu, x). \quad (2.53)$$

An invariant metric in the gauge group can be chosen to be just a background metric $g_{\mu\nu}$.

The local generators of the gauge transformations in this parametrization are defined by their action as follows

$$R^i_{\alpha} \xi^\alpha = 2\nabla_{(\mu} \xi_{\nu)}, \quad i \equiv (\mu\nu, x), \quad (2.54)$$

$$J^i_{\alpha} = -2\nabla_\mu J^\mu_{\alpha}, \quad \alpha \equiv (\alpha, x). \quad (2.55)$$

The Hilbert-Einstein action of general relativity has the form

$$S = \frac{1}{k^2} \int_M dx \, g^{1/2} (R - 2\Lambda), \quad (2.56)$$

where $R$ is the scalar curvature, $k^2 = 16\pi G$ is the Einstein coupling constant, $G$ is the Newtonian gravitational constant and $\Lambda$ is the cosmological constant. Here we neglect the boundary term for simplicity; it will not affect our calculations.

The first variation of the action gives the classical equations of motion

$$g^{-1/2} \frac{\delta S}{\delta g_{\mu\nu}} = -\frac{1}{k^2} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right), \quad (2.57)$$

which satisfy, of course, the Noether identities

$$\nabla_\mu \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} \right) = 0. \quad (2.58)$$

Here $R_{\mu\nu}$ is the Ricci tensor defined in terms of the Riemann tensor by $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$.

The second variation of the action defines a second-order partial differential operator by

$$g^{-1/2} \frac{\delta^2 S}{\delta g_{\mu\nu} \delta g_{\alpha\beta}} h_{\alpha\beta} = P^\mu_{\nu\alpha\beta} h_{\alpha\beta}, \quad (2.59)$$

where

$$P^\mu_{\nu\alpha\beta} = -\frac{1}{2k^2} \left( g^{\alpha(\mu} g^{\nu)\beta} - g^{\alpha\beta} g^{\mu\nu} \right) \Delta$$
Here, of course, $\Delta = g^{\mu\nu} \nabla_\mu \nabla_\nu$ denotes the Laplacian.

Next, we choose the De Witt gauge condition

$$
\chi^\alpha = -E^{\alpha\beta\mu\nu} \nabla_\beta h_{\mu\nu} = -\left( g^{\alpha(\nu} \nabla^{\mu)} - \kappa g^{\mu\nu} \nabla_\alpha \right) h_{\mu\nu} .
$$

(2.61)

The ghost operator in this gauge is a second-order differential operator defined by

$$
F_{\mu\nu} = -2E^{\mu\alpha\beta\nu} \nabla_\alpha \nabla_\beta = -\delta_{\nu}^\mu \Delta + (2\kappa - 1) \nabla_\mu \nabla_\nu - R_{\nu}^\mu .
$$

(2.62)

For this operator to be non-singular, the gauge parameter should satisfy the condition $\kappa \neq 1$.

For the graviton operator $L$ to be non-degenerate it is necessary to choose the operator $\gamma$ as a zero order differential operator defined by

$$
\gamma_{\mu\nu} = \frac{\alpha}{k^2} g_{\mu\nu} ,
$$

(2.63)

where $\alpha \neq 0$ is a real parameter. Thus we obtain a two-parameter class of gauges involving two arbitrary parameters, $\kappa$ and $\alpha$.

The graviton operator $L$ now reads

$$
L^{\mu\nu,\alpha\beta} = \frac{1}{2k^2} \left\{ -(g^{\alpha(\mu} g^{\beta)} - (1 + 2\alpha\kappa^2)g^{\alpha\beta} g^{\mu\nu}) \Delta 
- (1 + 2\alpha\kappa) g^{\mu\nu} \nabla^{(\alpha} \nabla^{\beta)} - (1 + 2\alpha\kappa) g^{\alpha\beta} \nabla^{(\mu} \nabla^{\nu)} + 2(1 + \alpha) \nabla^{(\mu} g^{\nu)(\alpha} \nabla^{\beta)} 
- 2R^{(\mu\nu|\alpha\beta)} - g^{\alpha(\mu} R^{\nu)\beta} - g^{\beta(\mu} R^{\nu)\alpha} + R^{\mu\nu} g^{\alpha\beta} + g^{\mu\nu} R_{\alpha\beta} 
+ \left( g^{\mu(\alpha} g^{\beta)\nu} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \right) (R - 2\Lambda) \right\}
$$

(2.64)

The most convenient choice is the so-called minimal gauge

$$
\kappa = \frac{1}{2}, \quad \alpha = -1 .
$$

(2.65)
In this gauge the non-diagonal derivatives in both the graviton operator and the ghost operator vanish

\[ L_{\mu\nu,\alpha\beta} = \frac{1}{2k^2} \left( \left( g^{\alpha(\mu} g^{\nu)\beta} - \frac{1}{2} g^{\alpha\beta} g^{\mu\nu} \right) (-\Delta + R - 2\Lambda) \right) \]

\[ -2R^{(\mu(\alpha|\nu)\beta} - g^{\alpha(\mu} R_{\beta)\nu} - g^{\beta(\mu} R^{\alpha)\nu} + R^{\mu\nu} g^{\alpha\beta} + g^{\mu\nu} R^{\alpha\beta} \right) , \tag{2.66} \]

\[ F^\mu_{\nu} = -\delta^\mu_{\nu} \Delta - R^\mu_{\nu} . \tag{2.67} \]

Finally, we define the graviton operator in the canonical Laplace-type form, \( \hat{L} \), by factoring out the configuration space metric (in the minimal gauge \( \kappa = 1/2 \))

\[ \hat{L}_{\mu\nu,\alpha\beta} = 2k^2 E_{\mu\nu,\alpha\beta} . \tag{2.68} \]

We obtain

\[ \hat{L}_{\mu\nu}^{\alpha\beta} = -\delta_{(\alpha}^\mu \delta_{\beta)}^\nu \Delta + Q_{\mu\nu}^{\alpha\beta} , \tag{2.69} \]

where

\[ Q_{\mu\nu}^{\alpha\beta} = -2R^{(\alpha(\mu}_{\nu)\beta)} - 2\delta^{(\alpha}_{(\mu} R^{\beta)\nu)} + R_{\mu\nu} g^{\alpha\beta} + \frac{2}{n-2} g_{\mu\nu} R^{\alpha\beta} \]

\[ + \left( \delta^{(\alpha}_{(\mu} \delta^{\beta)}_{\nu)} - \frac{1}{n-2} g_{\mu\nu} g^{\alpha\beta} \right) R - 2\Lambda \delta^{(\alpha}_{(\mu} \delta^{\beta)}_{\nu)} . \tag{2.70} \]

One can show that the contribution of the determinant of the operator \( \gamma \) can be neglected (more precisely, it can be absorbed in the measure of the path integral) since it is of zero order. Thus, with this choice of gauge parameters the one-loop effective action of quantum general relativity is given by

\[ \Gamma^{(1)} = -\frac{1}{2i} \log \operatorname{Det} \hat{L} + \frac{1}{i} \log \operatorname{Det} F . \tag{2.71} \]

Therefore, in order to compute the effective action we need to compute the determinants of Laplace type partial differential operators acting on symmetric two-tensors and vectors.

### 3 Heat Kernel Method

As we described in the previous section the effective action in quantum field theory can be computed within the semi-classical perturbation theory. It is determined
by the functional determinants of second-order hyperbolic partial differential operators with Feynman boundary conditions and the higher-loop approximations are determined in terms of the Feynman propagators and the classical vertex functions. As we noted above these expressions are purely formal and need to be regularized and renormalized, which can be done in a consistent way in renormalizable field theories. One should stress, of course, that many physically interesting theories (including Einstein’s general relativity) are perturbatively non-renormalizable. Since we only need Feynman propagators we can do the Wick rotation and consider instead of hyperbolic operators the elliptic ones. The Green functions of elliptic operators and their functional determinants can be expressed in terms of the heat kernel. That is why we concentrate below on the calculation of the heat kernel.

The heat kernel is one of the most powerful tools in mathematical physics and geometric analysis (see, for example, the books [27, 15, 10, 28] and reviews [19, 9, 11, 32, 14]). The short-time asymptotic expansion of the trace of the heat kernel determines the spectral asymptotics of the differential operator. The coefficients of this asymptotic expansion, called the heat invariants, are extensively used in geometric analysis, in particular, in spectral geometry and index theorems proofs [27, 15].

The gauge invariance (or covariance) in quantum gauge field theory and quantum gravity is of fundamental importance. That is why, manifestly covariant methods present inestimable advantage. A manifestly covariant calculus is such that every step is expressed in terms of geometric objects; it does not have some intermediate non-covariant steps that lead to an invariant result. Below we describe a manifestly covariant method for calculation of the heat kernel following mainly our papers [2, 3, 9, 10, 11].

3.1 Laplace Type Operators

Let \((M, g)\) be a smooth compact Riemannian manifold of dimension \(n\) without boundary, equipped with a positive definite Riemannian metric \(g\). We assume that it is complete simply connected orientable and spin. We denote the local coordinates on \(M\) by \(x^\mu\), with Greek indices running over \(1, \ldots, n\). Let \(e_\alpha^\mu\) be a local orthonormal frame defining a basis for the tangent space \(T_xM\). We denote the frame indices by low case Latin indices from the beginning of the alphabet, which also run over \(1, \ldots, n\). The frame indices are raised and lowered by the metric \(\delta_{ab}\). Let \(e^a_\mu\) be the matrix inverse to \(e_\alpha^\mu\), defining the dual basis in the cotangent space \(T_x^*M\). As usual, the orthonormal frame, \(e^a_\mu\) and \(e_\alpha^\mu\), will be
used to transform the coordinate (Greek) indices to the orthonormal (Latin) indices. The Riemannian volume element is defined as usual by $d\text{vol} = dx g^{1/2}$, where $g = \det g_{\mu\nu} = (\det e_a^{\mu})^2$. The spin connection $\omega^{ab}_\mu$ is defined in terms of the covariant derivatives of the orthonormal frame with the Levi-Civita connection. The curvature of the spin connection defines the Riemann tensor, $R^{\rho}_{\mu\rho\nu}$, the Ricci tensor, $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$, and the scalar curvature, $R = R^{\mu}_{\mu}$, as usual.

Let $T$ be a spin-tensor bundle realizing a representation $\Sigma$ of the spin group $\text{Spin}(n)$, the double covering of the group $SO(n)$, with the fiber $\Lambda$. Let $\Sigma_{ab}$ be the generators of the orthogonal algebra $SO(n)$, the Lie algebra of the orthogonal group $SO(n)$. The spin connection induces a connection on the bundle $T$ defining the covariant derivative of smooth sections $\varphi$ of the bundle $T$ by

$$\nabla^\text{spin}_\mu \varphi = \left( \partial_\mu + \frac{1}{2} \omega^{ab}_\mu \Sigma_{ab} \right) \varphi. \quad (3.1)$$

The commutator of covariant derivatives defines the curvature of this connection via

$$[\nabla^\text{spin}_\mu, \nabla^\text{spin}_\nu] \varphi = \frac{1}{2} R^{ab}_{\mu\nu} \Sigma_{ab} \varphi. \quad (3.2)$$

The covariant derivative along the frame vectors is defined by $\nabla_a = e_d^{\mu} \nabla_\mu$. For example, with our notation, $\nabla_a \nabla_b T_{cd} = e_d^{\mu} e_b^{\nu} e_c^{\alpha} e_d^{\beta} \nabla_\mu \nabla_\nu T_{ab}$. The metric $\delta_{ab}$ induces a positive definite fiber metric on tensor bundles.

Let $G_{YM}$ be a compact Lie group (called a gauge group). It naturally defines the principal fiber bundle over the manifold $M$ with the structure group $G_{YM}$. We consider a representation of the structure group $G_{YM}$ and the associated vector bundle through this representation with the same structure group $G_{YM}$ whose typical fiber is a $k$-dimensional vector space $W$. Then for any spin-tensor bundle $T$ we define the twisted spin-tensor bundle $V$ via the twisted product of the bundles $W$ and $T$. The fiber of the bundle $V$ is $V = \Lambda \otimes W$ so that the sections of the bundle $V$ are represented locally by $k$-tuples of spin-tensors.

Let $A_{YM}$ be a connection one form on the bundle $W$ (called Yang-Mills or gauge connection) taking values in the Lie algebra $g_{YM}$ of the gauge group $G_{YM}$. Then the total connection on the bundle $V$ is defined by

$$\nabla_\mu \varphi = \left( \partial_\mu + A_\mu \right) \varphi, \quad (3.3)$$

where

$$A_\mu = \frac{1}{2} \omega^{ab}_\mu \Sigma_{ab} \otimes I_W + I_\Lambda \otimes A^{YM}_\mu, \quad (3.4)$$
and the total curvature $\mathcal{R}$ of the bundle $\mathcal{V}$ is defined by

$$[\nabla_\mu, \nabla_\nu] \varphi = \mathcal{R}_{\mu\nu} \varphi,$$

(3.5)

where

$$\mathcal{R}_{\mu\nu} = \frac{1}{2} R^{ab}_{\underline{\mu\nu}} \Sigma_{ab} + R^{YM}_{\mu\nu},$$

(3.6)

and

$$R^{YM}_{\mu\nu} = \partial_\mu A^{YM}_\nu - \partial_\nu A^{YM}_\mu + [A^{YM}_\mu, A^{YM}_\nu]$$

(3.7)

is the curvature of the Yang-Mills connection.

We also consider the bundle $\text{End}(\mathcal{V})$ of endomorphisms of the bundle $\mathcal{V}$. The covariant derivative of sections of this bundle is defined by

$$\nabla_\mu Q = \partial_\mu Q + [A_\mu, Q],$$

(3.8)

and the commutator of covariant derivatives is equal to

$$[\nabla_\mu, \nabla_\nu] Q = [\mathcal{R}_{\mu\nu}, Q].$$

(3.9)

We assume that the vector bundle $\mathcal{V}$ is equipped with a Hermitian metric. This naturally identifies the dual vector bundle $\mathcal{V}^*$ with $\mathcal{V}$. We assume that the connection $\nabla$ is compatible with the Hermitian metric on the vector bundle $\mathcal{V}$. The connection is given its unique natural extension to bundles in the tensor algebra over $\mathcal{V}$ and $\mathcal{V}^*$. In fact, using the Levi-Civita connection of the metric $g$ together with the connection on the bundle $\mathcal{V}$, we naturally obtain connections on all bundles in the tensor algebra over $\mathcal{V}$, $\mathcal{V}^*$, $T M$ and $T^* M$; the resulting connection will usually be denoted just by $\nabla$. It is usually clear which bundle’s connection is being referred to, from the nature of the section being acted upon.

We denote by $C^\infty(\mathcal{V})$ the space of smooth sections of the bundle $\mathcal{V}$. The fiber inner product on the bundle $\mathcal{V}$ defines a natural $L^2$ inner product and the $L^2$-trace $\text{Tr}$ using the invariant Riemannian measure on the manifold $M$. The completion of $C^\infty(\mathcal{V})$ in this norm defines the Hilbert space $L^2(\mathcal{V})$ of square integrable sections. Let $\nabla^*$ be the formal adjoint to $\nabla$ defined using the Riemannian metric and the Hermitian structure on $\mathcal{V}$ and let $Q$ be a smooth Hermitian section of the endomorphism bundle $\text{End}(\mathcal{V})$.

A Laplace type operator $L: C^\infty(V) \rightarrow C^\infty(V)$ is a partial differential operator of the form

$$L = \nabla^* \nabla + Q = -\Delta + Q.$$  

(3.10)
In local coordinates the Laplacian is defined by
\[ \Delta = g^{\mu \nu} \nabla_\mu \nabla_\nu = g^{-1/2}(\partial_\mu + \mathcal{A}_\mu)g^{1/2}g^{\mu \nu}(\partial_\nu + \mathcal{A}_\nu). \] (3.11)
and, therefore,
\[ L = -g^{-1/2}(\partial_\mu + \mathcal{A}_\mu)g^{1/2}g^{\mu \nu}(\partial_\nu + \mathcal{A}_\nu) + Q \]
\[ = -g^{\mu \nu} \partial_\mu \partial_\nu - 2a^\mu \partial_\mu + q, \] (3.12)
where
\[ a^\mu = g^{\mu \nu} \mathcal{A}_\nu + \frac{1}{2}g^{-1/2}\partial_\nu (g^{1/2}g^{\mu \nu}) \] (3.13)
\[ q = Q - g^{\mu \nu} \mathcal{A}_\mu \mathcal{A}_\nu - g^{-1/2}\partial_\mu (g^{1/2}g^{\mu \nu} \mathcal{A}_\nu). \] (3.14)

Thus, a Laplace type operator is constructed from the following three pieces of geometric data: i) a Riemannian metric \( g \) on \( M \), which determines the second-order part, ii) a connection 1-form \( \mathcal{A} \) on the vector bundle \( \mathcal{V} \), which determines the first-order part, iii) an endomorphism \( Q \) of the vector bundle \( \mathcal{V} \), which determines the zeroth order part. It is worth noting that every second-order differential operator with a scalar leading symbol given by the metric tensor is of Laplace type and can be put in this form by choosing the appropriate connection and the endomorphism \( Q \).

It is easy to show that the Laplacian, \( \Delta \), and, therefore, the operator \( L \), is an elliptic symmetric partial differential operator satisfying
\[ (L \varphi, \psi) = (\varphi, L \psi), \] (3.15)
with a positive principal symbol. Moreover, the operator \( L \) is essentially self-adjoint, i.e., it has a unique self-adjoint extension. We will not be very careful about distinguishing between the operator \( L \) and its closure, and will simply say that the operator \( L \) is elliptic and self-adjoint.

It is well known \[27\] that:

i) the operator \( L \) has a discrete real spectrum, \( \{\lambda_n\}_{n=1}^\infty \), bounded from below:
\[ \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \] (3.16)
with some real constant \( \lambda_0 \),

ii) the eigenvalues grow as \( k \to \infty \) as \( \lambda_k \sim Ck^{2/n} \), where \( n = \text{dim} M \),

iii) all eigenspaces of the operator \( L \) are finite-dimensional, and

iv) the eigenvectors, \( \{\varphi_n\}_{n=1}^\infty \), of the operator \( L \), are smooth sections of the vector bundle \( \mathcal{V} \) that form a complete orthonormal basis in \( L^2(\mathcal{V}) \).
3.2 Spectral Functions

The spectrum of the operator $L$ can be described by certain spectral invariants, called spectral functions. First of all, we define the heat trace:

$$\Theta(t) = \sum_{n=1}^{\infty} e^{-t\lambda_n},$$  \hspace{1cm} (3.17)

where each eigenvalue is counted with multiplicities. The heat trace is well defined for real positive $t$. Notice that it can be analytically continued to an analytic function of $t$ in the right half-plane (for $\text{Re} \, t > 0$).

The heat trace determines other spectral functions by integral transforms: the distribution function (also called counting function), defined as the number of eigenvalues below the level $\lambda$,

$$N(\lambda) = \sum_{n=1}^{\infty} \theta(\lambda - \lambda_n) = \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} \frac{e^{\lambda t}}{t} \Theta(t),$$  \hspace{1cm} (3.18)

where $\varepsilon$ is a positive constant, the density function,

$$\rho(\lambda) = \sum_{n=1}^{\infty} \delta(\lambda - \lambda_n) = \frac{1}{2\pi i} \int_{\varepsilon-i\infty}^{\varepsilon+i\infty} dt \, e^{\lambda t} \Theta(t),$$  \hspace{1cm} (3.19)

and the zeta-function,

$$\zeta(s, \lambda) = \sum_{n=1}^{\infty} \frac{1}{(\lambda_n - \lambda)^s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \, t^{s-1} e^{t\lambda} \Theta(t),$$  \hspace{1cm} (3.20)

where $\lambda$ is a large negative constant such that $\text{Re} \, \lambda < \lambda_0$ and $s$ is a complex parameter with $\text{Re} \, s > n/2$.

In principle, if known exactly, they determine the spectrum. Of course, this is not valid for asymptotic expansions of the spectral functions. There are examples of operators that have the same asymptotic series of the spectral functions but different spectrum.

The zeta function enables one to define, in particular, the zeta-regularized determinant of the operator $(L - \lambda)$,

$$\zeta'(0, \lambda) \equiv \frac{\partial}{\partial s} \zeta(s, \lambda) \bigg|_{s=0} = -\log \text{Det}(L - \lambda),$$  \hspace{1cm} (3.21)

which determines the one-loop effective action in quantum field theory.
3.3 Heat Kernel

For \( t > 0 \) the operators

\[ U(t) = \exp(-tL) \]  

(3.22)
form a semi-group of bounded operators on \( L^2(V) \), so called heat semi-group. The kernel of this operator is defined by

\[ U(t|x, x') = \sum_{n=1}^{\infty} e^{-t\lambda_n} \varphi_n(x) \otimes \varphi_n^*(x'), \]  

(3.23)

where each eigenvalue is counted with multiplicities. It is a section of the external tensor product of vector bundles \( V \otimes V^* \) over \( M \times M \), which can also be regarded as an endomorphism from the fiber of \( V \) over \( x' \) to the fiber of \( V \) over \( x \). This kernel satisfies the heat equation

\[ (\partial_t + L) U(t) = 0 \]  

(3.24)
with the initial condition

\[ U(0^+|x, x') = \delta(x, x') \]  

(3.25)
and is called the heat kernel.

Moreover, the heat semigroup \( U(t) \) is a trace-class operator with a well defined \( L^2 \)-trace,

\[ \text{Tr} \exp(-tL) = \int_M d\text{vol} \text{ tr}_V U^{\text{diag}}(t). \]  

(3.26)
Hereafter \( \text{tr}_V \) denotes the fiber trace and the label ‘diag’ means the diagonal value of a two-point quantity, e.g.

\[ U^{\text{diag}}(t|x) = U(t|x, x') \bigg|_{x = x'}. \]  

(3.27)
It is easy to see that the heat trace defined above is equal to the trace of the heat semigroup, that is,

\[ \Theta(t) = \text{Tr} \exp(-tL). \]  

(3.28)

3.4 Asymptotic Expansion of the Heat Kernel

In the following we are going to study the heat kernel only locally, i.e. in the neighbourhood of the diagonal of \( M \times M \), when the points \( x \) and \( x' \) are close to
each other. The exposition will follow mainly our papers [3, 10, 9, 11]. We will keep a point $x'$ of the manifold fixed and consider a small geodesic ball, i.e. a small neighbourhood of the point $x'$: $B_{\varepsilon}(x') = \{ x \in M | r(x, x') < \varepsilon \}$, $r(x, x')$ being the geodesic distance between the points $x$ and $x'$. We will take the radius of the ball sufficiently small, so that each point $x$ of the ball of this neighbourhood can be connected by a unique geodesic with the point $x'$. This can be always done if the size of the ball is smaller than the injectivity radius of the manifold, $\varepsilon < r_{\text{inj}}$.

Let $\sigma(x, x')$ be the geodetic interval, also called world function, defined as one half the square of the length of the geodesic connecting the points $x$ and $x'$

$$\sigma(x, x') = \frac{1}{2} r^2(x, x').$$  \hfill (3.29)

The first derivatives of this function with respect to $x$ and $x'$ define tangent vector fields to the geodesic at the points $x$ and $x'$ respectively pointing in opposite directions

$$u^\mu = g^{\mu \nu} \nabla_\nu \sigma, \quad (3.30)$$
$$u'^\nu = g^{\mu ' \nu '} \nabla'_{\nu '} \sigma, \quad (3.31)$$

and the determinant of the mixed second derivatives defines a so called Van Vleck-Morette determinant

$$\Delta(x, x') = g^{-1/2}(x) \det \left[ -\nabla_\mu \nabla'_{\nu '} \sigma(x, x') \right] g^{-1/2}(x').$$  \hfill (3.32)

This object should not be confused with the Laplacian, which is also denoted by $\Delta$.

Let, finally, $\mathcal{P}(x, x')$ denote the parallel transport operator of sections of the vector bundle $\mathcal{V}$ along the geodesic from the point $x'$ to the point $x$. It is a section of the external tensor product of the vector bundle $\mathcal{V} \boxtimes \mathcal{V}^*$ over $M \times M$, or, in other words, it is an endomorphism from the fiber of $\mathcal{V}$ over $x'$ to the fiber of $\mathcal{V}$ over $x$. Here and everywhere below the coordinate indices of the tangent space at the point $x'$ are denoted by primed Greek letters. They are raised and lowered by the metric tensor $g_{\mu ' \nu '}(x')$ at the point $x'$. The derivatives with respect to $x'$ will be denoted by primed Greek indices as well.

We extend the local orthonormal frame $e_a^\mu(x')$ at the point $x'$ to a local orthonormal frame $e_a^\mu(x)$ at the point $x$ by parallel transport. The parameters of the geodesic connecting the points $x$ and $x'$, namely the unit tangent vector at the point $x'$ and the length of the geodesic, (or, equivalently, the tangent vector at the point...
\( x' \) with the norm equal to the length of the geodesic, provide normal coordinate system for \( B_x(x') \). Now, let us define the following geometric parameters

\[ y^a = e^a_{\mu} u^\mu = -e^a_{\mu'} u'^{\mu'}, \tag{3.33} \]

so that

\[ u^\mu = e^a_{\mu} y^a \quad \text{and} \quad u'^{\mu} = -e^a_{\mu'} y^a. \tag{3.34} \]

Notice that \( y^a = 0 \) at \( x = x' \). The geometric parameters \( y^a \) are nothing but the normal coordinates.

Near the diagonal of \( M \times M \) all these two-point functions are smooth single-valued functions of the coordinates of the points \( x \) and \( x' \). Let us note from the beginning that we will construct the heat kernel in form of covariant Taylor series in coordinates. In the smooth case these series do not necessarily converge. However, if one assumes additionally that the two-point functions are analytic, then the Taylor series converge in a sufficiently small neighborhood of the diagonal.

Further, one can easily prove that the function

\[ U_0(t|x,x') = (4\pi t)^{-n/2} \Delta^{1/2}(x,x') \exp \left( -\frac{1}{2t} \sigma(x,x') \right) \mathcal{P}(x,x') \tag{3.35} \]

satisfies the initial condition

\[ U_0(0^+|x,x') = \delta(x,x'). \tag{3.36} \]

Moreover, locally it also satisfies the heat equation in the free case, when the Riemannian curvature of the manifold, \( \text{Riem} \), the curvature of the bundle connection, \( \mathcal{R} \), and the endomorphism \( Q \) vanish:

\[ \text{Riem} = \mathcal{R} = Q = 0. \tag{3.37} \]

Therefore, \( U_0(t|x,x') \) is the exact heat kernel for a pure Laplacian in flat Euclidean space with a flat trivial bundle connection and without the endomorphism \( Q \).

### 3.4.1 Transport Function

This function gives a good framework for the approximate solution in the general case. Namely, by factorizing out this free factor we get an ansatz

\[ U(t|x,x') = (4\pi t)^{-n/2} \Delta^{1/2}(x,x') \exp \left( -\frac{1}{2t} \sigma(x,x') \right) \mathcal{P}(x,x') \Omega(t|x,x'). \tag{3.38} \]
The function $\Omega(t|x,x')$, called the transport function, is a section of the endomorphism vector bundle $\text{End}(V)$ over the point $x'$. Using the definition of the functions $\sigma(x,x'), \Delta(x,x')$ and $\mathcal{P}(x,x')$ it is not difficult to find that the transport function satisfies a transport equation

$$\left(\partial_t + \frac{1}{t}D + \tilde{L}\right)\Omega(t) = 0,$$

(3.39)

where $D$ is the radial vector field, i.e. operator of differentiation along the geodesic, defined by

$$D = u^\mu \nabla_\mu,$$

(3.40)

and $\tilde{L}$ is a second-order differential operator defined by

$$\tilde{L} = \mathcal{P}^{-1} \Delta^{-1/2} L \Delta^{1/2} \mathcal{P}.$$

(3.41)

The initial condition for the transport function is obviously

$$\Omega(t|x,x') = \mathbb{I}_V,$$

(3.42)

where $\mathbb{I}_V$ is the identity endomorphism of the vector bundle $\mathcal{V}$ over $x'$. It is obvious that if we replace the operator $L$ by $(L - \lambda)$, with $\text{Re} \lambda < \lambda_0$, then the heat kernel and the transport function are simply multiplied by $e^{t\lambda}$, i.e. the transport function for the operator $(L - \lambda)$ is $e^{t\lambda}\Omega(t)$. Further, for $\lambda < \lambda_0$ the operator $(L - \lambda)$ becomes a positive operator. Therefore, the function $e^{t\lambda}\Omega(t)$ satisfies the following asymptotic conditions

$$\lim_{t \to \infty, 0} t^\alpha \partial^N \left[ e^{t\lambda}\Omega(t) \right] = 0 \quad \text{for} \quad \lambda < \lambda_1, \quad \alpha > 0, \quad N \geq 0. \quad (3.43)$$

In other words, as $t \to \infty$ the function $e^{t\lambda}\Omega(t)$ and all its derivatives decreases faster than any power of $t$, actually it decreases exponentially, and as $t \to 0$ the product of $e^{t\lambda}\Omega(t)$ with any positive power of $t$ vanishes.

Hereafter we fix $\lambda < \lambda_0$, so that $(L - \lambda)$ is a positive operator. Now, let us consider a slightly modified version of the Mellin transform of the function $e^{t\lambda}\Omega(t)$ introduced in [3]

$$b_q(\lambda) = \frac{1}{\Gamma(-q)} \int_0^\infty dt \ t^{-q-1} e^{t\lambda}\Omega(t). \quad (3.44)$$

Note that for fixed $\lambda$ this is a Mellin transform of $e^{t\lambda}\Omega(t)$ and for a fixed $q$ this is a Laplace transform of the function $t^{-q-1}\Omega(t)$. The integral (3.44) converges
for $\text{Re}q < 0$. By integrating by parts $N$ times and using the asymptotic conditions (3.43) we also get

$$b_q(\lambda) = \frac{1}{\Gamma(-q + N)} \int_0^\infty dt \ t^{-q-1+N} (-\partial_t)^N \left[ e^{f' \Omega(t)} \right]. \quad (3.45)$$

This integral converges for $\text{Re}q < N - 1$. Using this representation one can prove that the function $b_q(\lambda)$ is an entire function of $q$ (analytic everywhere) satisfying the asymptotic condition

$$\lim_{|q| \to \infty, \text{Re}q < N} \Gamma(-q + N) b_q(\lambda) = 0, \quad \text{for any } N > 0. \quad (3.46)$$

Moreover, the values of the function $b_q(\lambda)$ at the integer positive points $q = k$ are given by

$$b_k(\lambda) = (-\partial_t)^k \left[ e^{f' \Omega(t)} \right] \bigg|_{t=0} = \sum_{n=0}^{k} \binom{k}{n} a_n, \quad (3.47)$$

where

$$a_k = (-\partial_t)^k \left[ \Omega(t) \right] \bigg|_{t=0}. \quad (3.48)$$

By inverting the Mellin transform we obtain a new ansatz for the transport function and, hence, for the heat kernel

$$\Omega(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \ e^{-t^{q+1} \Gamma(-q)} \Gamma(-q) b_q(\lambda) \quad (3.49)$$

where $c < 0$ and $\text{Re} \lambda < \lambda_0$. Clearly, since the left-hand side of this equation does not depend on $\lambda$, neither does the right hand side. Thus, $\lambda$ serves as an auxiliary parameter that regularizes the behavior at $t \to \infty$. If we invert instead the Laplace transform, we obtain another representation

$$\Omega(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} d\lambda \ e^{-t^{q+1} \Gamma(-q)} \Gamma(-q) b_q(\lambda) \quad (3.50)$$

where $\gamma < \lambda_0$ and $\text{Re} q < 0$.

Substituting this ansatz into the transport equation we get a functional equation for the function $b_q$

$$\left( 1 + \frac{1}{q} D \right) b_q(\lambda) = (\tilde{L} - \lambda) b_{q-1}(\lambda). \quad (3.51)$$
The initial condition for the transport function is translated into

\[ b_0(\lambda) = \mathbb{I}_V. \] (3.52)

Thus, we have reduced the problem of solving the heat equation to the following problem: one has to find an entire function of \( q, b_q(\lambda | x, x') \), that satisfies the functional equation (3.51) with the initial condition (3.52) and the asymptotic condition (3.46).

Although the variables \( q \) and \( \lambda \) seem to be independent they are very closely related to each other. In particular, by differentiating with respect to \( \lambda \) we obtain an important result

\[ \frac{\partial}{\partial \lambda} b_q(\lambda) = -qb_{q-1}(\lambda). \] (3.53)

Also, by differentiating the eq. (3.51) with respect to \( q \) one obtains another recursion

\[ \left( 1 + \frac{1}{q} D \right) b'_q(\lambda) = \tilde{L} b'_{q-1}(\lambda) + \frac{1}{q} Db_q(\lambda), \] (3.54)

where

\[ b'_q(\lambda) = \frac{\partial}{\partial q} b_q(\lambda), \] (3.55)

which enables one to compute the derivatives of the function \( b_q(\lambda) \) at positive integer points if one fixes its value \( b'_0(\lambda) \). This turns out to be useful when computing the determinant of the operator \((L - \lambda)\).

Moreover, one can actually manifest the dependence of \( b_q(\lambda) \) on \( \lambda \). It is not difficult to prove that [3] the integral

\[ b_q(\lambda) = \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} dp \frac{\Gamma(-p)\Gamma(p-q)}{\Gamma(-q)} (-\lambda)^{q-p} a_p, \] (3.56)

with \( \text{Re} q < c_1 < 0 \), satisfies the equation (3.51) if \( a_p \) satisfies this equation for \( \lambda = 0 \), i.e.

\[ \left( 1 + \frac{1}{q} D \right) a_q = \tilde{L} a_{q-1}. \] (3.57)

with the initial condition

\[ a_0 = \mathbb{I}_V. \] (3.58)

For integer \( q = k = 1, 2, \ldots \) the functional equation (3.57) becomes a recursion system that, together with the initial condition (3.58), determines all coefficients \( a_k \).
Now, from eq. (3.56) we also obtain the asymptotic expansion of $b_q(\lambda)$ as $\lambda \to -\infty$,

$$b_q(\lambda) \sim \sum_{n=0}^{\infty} \frac{\Gamma(q+1)}{n!\Gamma(q-n+1)}(-\lambda)^{q-n} a_n.$$  \hspace{2cm} (3.59)

For integer $q$ this coincides with (3.47).

The function $b_q(\lambda)$ turns out to be extremely useful in computing the heat kernel, the resolvent kernel, the zeta-function and the determinant of the operator $L$. It contains the same information about the operator $L$ as the heat kernel. In some cases the function $b_q(\lambda)$ can be constructed just by analytical continuation from the integer positive values $b_k$ [3].

### 3.4.2 Asymptotic Expansion of the Transport Function

Now we are going to do the usual trick, namely, to move the contour of integration over $q$ to the right. Due to the presence of the gamma function $\Gamma(-q)$ the integrand has simple poles at the non-negative integer points $q = 0, 1, 2 \ldots$, which contribute to the integral while moving the contour. So, we get

$$\Omega(t) = e^{-t\lambda} \left\{ \sum_{k=0}^{N-1} \frac{(-t)^k}{k!} b_k(\lambda) + R_N(t) \right\},$$  \hspace{2cm} (3.60)

where

$$R_N(t) = \frac{1}{2\pi i} \int_{c_N^{-i\infty}}^{c_N+i\infty} dq \Gamma(-q) b_q(\lambda)$$  \hspace{2cm} (3.61)

with $c_N$ is a constant satisfying the condition $N - 1 < c_N < N$. As $t \to 0$ the rest term $R_N(t)$ behaves like $O(t^N)$, so we obtain an asymptotic expansion as $t \to 0$

$$\Omega(t) \sim e^{-t\lambda} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} b_k(\lambda) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} a_k.$$  \hspace{2cm} (3.62)

Using our ansatz (3.38) we find immediately the heat trace

$$\Theta(t) = (4\pi t)^{-n/2} e^{-t\lambda} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \Gamma(-q) b_q(\lambda),$$  \hspace{2cm} (3.63)
where

$$B_q(\lambda) = \int_M d\text{vol} \; tr \, b_q^{\text{diag}}(\lambda). \quad (3.64)$$

The heat trace has an analogous asymptotic expansion as $t \to 0$

$$\Theta(t) \sim (4\pi t)^{-n/2} e^{-t\lambda} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} B_k(\lambda) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A_k, \quad (3.65)$$

where

$$A_k = \int_M d\text{vol} \; tr \, a_k^{\text{diag}}. \quad (3.66)$$

This is the famous Minakshisundaram-Pleijel asymptotic expansion. The physicists call it the Schwinger-De Witt expansion [14]. Its coefficients $A_k$ are also called sometimes Hadamard-Minakshisundaram-De Witt-Seeley (HMDS) coefficients. This expansion is of great importance in differential geometry, spectral geometry, quantum field theory and other areas of mathematical physics, such as theory of Huygens’ principle, heat kernel proofs of the index theorems, Korteweg-De Vries hierarchy, Brownian motion etc.

One should stress, however, that this series does not converge, in general. In that sense our ansatz (3.49) or (3.60) in form of a Mellin transform of an entire function is much better since it is exact and gives an explicit formula for the rest term.

### 3.5 Zeta Function and Determinant

Let us apply our ansatz for computation of the complex power of the operator $(L - \lambda)$ (with $\lambda < \lambda_0$ so that the operator $(L - \lambda)$ is positive) defined by

$$G_s(\lambda) = (L - \lambda)^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} e^{t \lambda} U(t). \quad (3.67)$$

Using our ansatz for the heat kernel one can obtain [3]

$$G_s(\lambda) = (4\pi)^{-n/2} \Delta^{1/2} P \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dq \frac{\Gamma(-q)\Gamma(-q-s+n/2)}{\Gamma(s)} \left(\frac{\sigma}{2}\right)^{q+s-n/2} b_q(\lambda), \quad (3.68)$$
where \( c < - \Re \rho + n/2 \).

Outside the diagonal, i.e. for \( \sigma \neq 0 \), this integral converges for any \( s \) and defines an entire function of \( s \). The integrand in this formula is a meromorphic function of \( q \) with some simple and maybe some double poles. If we move the contour of integration to the right, we get contributions from the simple poles in form of powers of \( \sigma \) and a logarithmic part due to the double poles (if any). This gives the complete structure of diagonal singularities of \( G_s(x, x') \). Thus the function \( b_q(\lambda) \) turns out to be very useful to study the diagonal singularities.

Now, let us consider the diagonal limit of \( G_s \). By taking the limit \( \sigma \to 0 \) we obtain a very simple formula in terms of the function \( b_q \)

\[
G_s^{\text{diag}}(\lambda) = (4\pi)^{-n/2} \frac{\Gamma(s-n/2)}{\Gamma(s)} b_n^{\text{diag}}(\lambda). \tag{3.69}
\]

This gives automatically the zeta-function

\[
\zeta(s, \lambda) = (4\pi)^{-n/2} \frac{\Gamma(s-n/2)}{\Gamma(s)} B_{n/2-s}(\lambda). \tag{3.70}
\]

Herefrom we see that both \( G_s^{\text{diag}}(\lambda) \) and \( \zeta(s, \lambda) \) are meromorphic functions of \( s \) with simple poles at the points \( s = [n/2] + 1/2 - k, \) (\( k = 0, 1, 2, \ldots \)) and \( s = 1, 2, \ldots, [n/2] \). In particular, the zeta-function is analytic at the origin. Its value at the origin is given by

\[
\zeta(0, \lambda) = \begin{cases} 
0 & \text{for odd } n, \\
(4\pi)^{-n/2} \frac{(-1)^{n/2}}{\Gamma(n/2+1)} B_{n/2}(\lambda) & \text{for even } n.
\end{cases} \tag{3.71}
\]

This gives the regularized number of all modes of the operator \( L \).

Moreover, the derivative of the zeta-function at the origin is also well defined. As we already mentioned above it determines the regularized determinant of the operator \( (L - \lambda) \)

\[
\log \Det (L - \lambda) = -(4\pi)^{-n/2} \frac{\pi(-1)^{(n+1)/2}}{\Gamma(n/2+1)} B_{n/2}(\lambda) \tag{3.72}
\]

for odd \( n \), and

\[
\log \Det (L - \lambda) = (4\pi)^{-n/2} \frac{(-1)^{n/2}}{\Gamma(n/2+1)} \left( B'_{n/2}(\lambda) - [\Psi(n/2 + 1) + C]B_{n/2}(\lambda) \right) \tag{3.73}
\]
for even $n$. Here $\Psi(z) = (d/dz)\log \Gamma(z)$ is the psi-function, $C = -\Psi(1) = 0.577\ldots$ is the Euler constant, and

$$B'_{n/2}(\lambda) = \frac{\partial}{\partial q} B_q(\lambda) \bigg|_{q=n/2}. \quad (3.74)$$

## 4 Green Function

In this section we closely follow our paper [8]. Let $\lambda$ be a sufficiently large negative parameter, such that $\lambda < \lambda_0$ and, therefore, $(L - \lambda)$ be a positive operator. The Green function of the operator $(L - \lambda)$ reads

$$G(\lambda|x,x') = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \varphi_n(x) \otimes \varphi_n^*(x'). \quad (4.1)$$

It is not difficult to see that the Green function can be represented as the Laplace transform of the heat kernel

$$G(\lambda) = \int_0^\infty dt \ e^{t\lambda} U(t). \quad (4.2)$$

Using our ansatz for the heat kernel we obtain

$$G(\lambda) = (4\pi)^{-n/2} \Delta^{1/2} \int_{c-i\infty}^{c+i\infty} dq \Gamma(-q)\Gamma(-q-1+n/2) \left(\frac{\sigma}{2}\right)^{q+1-n/2} b_q(\lambda) \quad (4.3)$$

where $c < n/2 - 1$.

This ansatz is especially useful for studying the singularities of the Green function, or more general, for constructing the Green function as a power series in $\sigma$. The integrand in (4.3) is a meromorphic function with poles at the points $q = k$ and $q = k - 1 + n/2$, where $(k = 0, 1, 2, \ldots)$. Here one has to distinguish between odd and even dimensions. In odd dimensions, the poles are at the points $q = k$ and $q = k + [n/2] - 1/2$ and are simple, whereas in even dimension there are simple poles at $q = 0, 1, 2, \ldots, n/2 - 2$ and double poles at the points $q = k + n/2 - 1$.

Moving the contour of integration in (4.3) to the right one can obtain an expansion of the Green function in powers of $\sigma$ (Hadamard series). Generally, we obtain

$$G(\lambda) = G^{\text{sing}}(\lambda) + G^{\text{non-anal}}(\lambda) + G^{\text{reg}}(\lambda). \quad (4.4)$$
Here $G^{\text{sing}}(\lambda)$ is the singular part which is polynomial in the inverse powers of $\sqrt{\sigma}$

$$G^{\text{sing}}(\lambda) = (4\pi)^{-n/2} \Delta^{1/2} \mathcal{P} \sum_{k=0}^{[n(n+1)/2]-1} \frac{(-1)^k}{k!} \pi \Gamma(n/2-k-1) \left(\frac{\sigma}{2}\right)^{n/2-k-1} b_k(\lambda), \quad (4.5)$$

Let us fix an integer $N$ such that $N > (n-1)/2$.

For the rest we get in odd dimensions

$$G^{\text{non-anal}}(\lambda) + G^{\text{reg}}(\lambda)$$

$$= (-1)^{(n-1)/2}(4\pi)^{-n/2} \Delta^{1/2} \mathcal{P} \sum_{k=0}^{N-(n+1)/2} \frac{\pi}{\Gamma(k+n+1/2)\Gamma(k+3/2)} \left(\frac{\sigma}{2}\right)^{k+1/2} b_k(\lambda)$$

$$+ (-1)^{(n+1)/2} \int_{c_N-i\infty}^{c_N+i\infty} dq \left(\frac{\sigma}{2}\right)^{q+1-n/2} \Gamma(-q) \Gamma(-q-1+n/2) b_q(\lambda), \quad (4.6)$$

where $N-1 < c_N < N-1/2$. Thus, by putting $N \to \infty$ we recover the Hadamard power series in $\sigma$ for odd dimension $n$

$$G^{\text{non-anal}}(\lambda) \sim (-1)^{(n+1)/2}(4\pi)^{-n/2} \Delta^{1/2} \mathcal{P} \sum_{k=0}^{\infty} \frac{\pi}{\Gamma(k+n+1/2)\Gamma(k+3/2)} \left(\frac{\sigma}{2}\right)^{k+1/2} b_k(\lambda) \quad (4.7)$$

$$G^{\text{reg}}(\lambda) \sim (-1)^{(n-1)/2}(4\pi)^{-n/2} \Delta^{1/2} \mathcal{P} \sum_{k=0}^{\infty} \frac{\pi}{k!\Gamma(k+n/2)} \left(\frac{\sigma}{2}\right)^k b_{k-1+n/2}(\lambda). \quad (4.8)$$

In even dimensions, the point is more subtle due to the presence of double poles. Moving the contour in (4.3) to the right and calculating the contribution of the residues at the simple and double poles we obtain

$$G^{\text{non-anal}}(\lambda) + G^{\text{reg}}(\lambda)$$

$$= (-1)^{n/2-1}(4\pi)^{-n/2} \Delta^{1/2} \mathcal{P} \log \left(\frac{\mu^2\sigma}{2}\right) \sum_{k=0}^{N-1} \frac{1}{k!\Gamma(k+n/2)} \left(\frac{\sigma}{2}\right)^k b_{k-1+n/2}(\lambda)$$
\[ +(-1)^{n/2-1}(4\pi)^{-n/2}\Delta^{1/2}(k+n/2) \sum_{k=0}^{N-1} \frac{1}{k!\Gamma(k+n/2)} \left( \frac{\sigma}{2} \right)^k \]

\[ \times \left\{ b'_{k-1+n/2}(\lambda) - \left[ \log \mu^2 + \Psi(k+1) + \Psi(k+n/2) \right] b_{k-1+n/2}(\lambda) \right\} \]

\[ + (4\pi)^{-n/2}\Delta^{1/2}(k-n/2) \sum_{k=0}^{N-1} \frac{1}{k!\Gamma(k+n/2)} \left( \frac{\sigma}{2} \right)^k b_{k-1+n/2}(\lambda) \]

(4.9)

where \( \mu \) is an arbitrary mass parameter introduced to preserve dimensions, \( N < c_N < N \) and \( \Psi(z) = (d/dz) \log \Gamma(z) \). If we let \( N \to \infty \) we obtain the Hadamard expansion of the Green function for even dimension \( n \geq 2 \).

\[ G^\text{non-anal}(\lambda) \sim (-1)^{n/2-1}(4\pi)^{-n/2}\Delta^{1/2}(k+n/2) \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+n/2)} \left( \frac{\sigma}{2} \right)^k b_{k-1+n/2}(\lambda) \]

(4.10)

\[ G^\text{reg}(\lambda) \sim (-1)^{n/2-1}(4\pi)^{-n/2}\Delta^{1/2}(k+n/2) \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(k+n/2)} \left( \frac{\sigma}{2} \right)^k \]

\[ \times \left\{ b'_{k-1+n/2}(\lambda) - \left[ \log \mu^2 + \Psi(k+1) + \Psi(k+n/2) \right] b_{k-1+n/2}(\lambda) \right\} \]

Notice that the singular part (which is a polynomial in inverse powers of \( \sqrt{\sigma} \)) and the non-analytical parts (proportional to \( \sqrt{\sigma} \) and \( \log \sigma \)) are expressed in terms of the the values of the function \( b_q(\lambda) \) at the integer points \( q \), which are uniquely locally computable from the recursion relation, whereas the regular analytical part contains the values of the function \( b_q(\lambda) \) at half-integer positive points \( q \) and the derivatives of the function \( b_q(\lambda) \) with respect to \( q \) at integer positive points \( q \), which are not expressible in terms of the local information. These objects are global and cannot be expressed further in terms of the local heat kernel coefficients. However, they can be computed from the eqs. (3.51) and (3.54) in terms of the value of the function \( b(q) \) at some fixed point \( q_0 \) (see [3]).

The regular part of the Green function has a well defined diagonal value and the functional trace. It reads in odd dimensions \( n \):

\[ \text{Tr} \ G^\text{reg}(\lambda) = (-1)^{(n+1)/2}(4\pi)^{-n/2} \frac{\pi}{\Gamma(n/2)} B_{n/2-1}(\lambda) \]
\[ \lambda \to -\infty \sim (1 + (4\pi)^{-n/2})^n \sum_{k=0}^{\infty} \frac{(-\lambda)^{n/2 - 1 - k}}{k! \Gamma(n/2 - k)} A_k \]  \tag{4.12}

and in even dimensions \( n \)

\[
\text{Tr} \ G^{\text{reg}}(\lambda) = (-1)^{n/2 - 1} \left( \frac{4\pi}{\Gamma(n/2)} \right)^{-n/2} \left\{ B'_{n/2 - 1}(\lambda) - \left[ \log \mu^2 + \Psi(n/2) - C \right] B_{n/2 - 1}(\lambda) \right\}
\]

\[ \lambda \to -\infty \sim (1 + (4\pi)^{-n/2})^n \sum_{k=0}^{\infty} \frac{(-\lambda)^{n/2 - 1 - k}}{k! \Gamma(n/2 - k)} \left[ C - \Psi(n/2 - k) + \log \left( \frac{-\lambda}{\mu^2} \right) \right] A_k 
\]

\[ + \sum_{k=n/2}^{\infty} \frac{(-1)^{k-n/2}}{k!} (-\lambda)^{n/2 - 1 - k} \Gamma(k + 1 - n/2) A_k \]  \tag{4.13}

This trace determines the regularized vacuum expectation values like \( \langle \varphi^2 \rangle \) in quantum field theory.

Thus, we see that

i) all the singularities of the Green function and the non-analytical parts thereof
(proportional to \( \sqrt{\sigma} \) in odd dimensions and to \( \log \sigma \) in even dimensions) are determined by the values of the function \( b_q(\lambda) \) at integer points \( q \), which are determined, in turn, by the heat kernel coefficients \( a_k \);

ii) there are no power singularities, i.e. \( G^{\text{sing}}(\lambda) = 0 \), in lower dimensions \( n = 1, 2 \);

iii) there is no logarithmic singularity (more generally, no logarithmic part at all) in odd dimensions;

iv) the regular part depends on the values of the function \( b_q(\lambda) \) at half-integer points \( q \) and its derivative \( b'_q(\lambda) \) at integer points \( q \) and is a global object that cannot be reduced to purely local information like the heat kernel coefficients \( a_k \).

The logarithmic part of the Green function is very important. On the one hand it determines, as usual, the renormalization properties of the regular part of the Green function, i.e. the derivative \( \mu(\partial/\partial \mu)G^{\text{reg}}(\lambda) \). In particular,

\[
\mu \frac{\partial}{\partial \mu} \text{Tr} \ G^{\text{reg}}(\lambda) = \begin{cases} 0 & \text{for odd } n \\ (4\pi)^{-n/2} \frac{B_{n/2 - 1}(\lambda)}{\Gamma(n/2)} & \text{for even } n \end{cases} \]  \tag{4.14}
On the other hand, it is of crucial importance in studying the Huygens principle. Namely, the absence of the logarithmic part of the Green function is a necessary and sufficient condition for the validity of the Huygens principle for hyperbolic operators. The heat kernel coefficients and, therefore, the logarithmic part of the Green function are defined for the hyperbolic operators just by analytic continuation from the elliptic case. Thus, the condition of the validity of Huygens principle reads

$$\sum_{k=0}^{\infty} \frac{\Gamma(n/2)}{k!\Gamma(k+n/2)} \left( \frac{\sigma}{2} \right)^k b_{k-1+n/2}(\lambda) = 0,$$

or, by using (3.47),

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k-1+n/2} \frac{\Gamma(n/2)}{k!j!\Gamma(k-j+n/2)} \left( \frac{\sigma}{2} \right)^k (-\lambda)^{k-j} a_j = 0.$$  (4.16)

By expanding this equation in covariant Taylor series using the methods of [3] one can obtain an infinite set of local conditions for validity of the Huygens principle, see [8]. In particular,

$$[b_{n/2-1}(\lambda)]^{\text{diag}} = 0,$$

$$[\nabla_\mu b_{n/2-1}(\lambda)]^{\text{diag}} = 0,$$  (4.18)

$$[\nabla_\mu \nabla_\nu b_{n/2-1}(\lambda)]^{\text{diag}} + \frac{1}{2n} g_{\mu\nu} [b_{n/2}(\lambda)]^{\text{diag}} = 0.$$  (4.19)

### 5 Heat Kernel Coefficients

As we have shown above the calculation of the effective action and the Green function reduces to the calculation of the heat kernel. An important part of that calculation is the calculation of the coefficients of the asymptotic expansion of the heat kernel. They are determined by a recursion system which is obtained simply by restricting the complex variable $q$ in the eq. (3.57) to positive integer values $q = 1, 2, \ldots$. 
5.1 Non-recursive Solution of the Recursion Relations

This problem was solved in [2, 3, 10] where a systematic technique for calculation of \(a_k\) was developed. The formal solution of this recursion system is

\[
a_k = \left(1 + \frac{1}{k}D\right)^{-1} \bar{L} \left(1 + \frac{1}{k-1}D\right)^{-1} \bar{L} \cdots \left(1 + \frac{1}{1}D\right)^{-1} \bar{L} \cdot I.
\]  

(5.1)

Now, the problem is to give a precise practical meaning to this formal operator solution. To do this one has, first of all, to define the inverse operator \((1 + D/k)^{-1}\). This can be done by constructing the complete set of eigenvectors of the operator \(D\). However, first we introduce some auxiliary notions from the theory of symmetric tensors.

Let \(S^n_m\) be the bundle of symmetric tensors of type \((m,n)\). First of all, we define the exterior symmetric tensor product \(\lor: S^n_m \times S^i_j \to S^{n+i}_{m+j}\) of symmetric tensors by

\[
(A \lor B)^{\beta_1 \ldots \beta_{n+i}}_{\alpha_1 \ldots \alpha_{m+j}} = A^{\alpha_1 \ldots \alpha_m}_{\alpha_1 \ldots \alpha_m} B^{\beta_{n+1} \ldots \beta_{n+i}}_{\alpha_{m+1} \ldots \alpha_{m+j}}.
\]  

(5.2)

Next, we define the inner product \(\star: S^n_m \times S^i_j \to S^i_m\) by

\[
(A \star B)^{\alpha_1 \ldots \alpha_m}_{\beta_1 \ldots \beta_i} = A^{\gamma_1 \ldots \gamma_n}_{\gamma_1 \ldots \gamma_n} B^{\alpha_1 \ldots \alpha_m}_{\beta_{n+1} \ldots \beta_i}.
\]  

(5.3)

Finally, let \(\mathbb{I}_{(n)}\) be the identity endomorphism on the space of symmetric tensors of type \((n,0)\); it is a section of the bundle \(S^n_n\), that is,

\[
\mathbb{I}_{(n)}^{\mu_1 \ldots \mu_n}_{\nu_1 \ldots \nu_n} = \delta^{(\mu_1} (\nu_1 \cdots \delta^{\mu_n)} (\nu_n).
\]  

(5.6)

We also define the exterior symmetric covariant derivative \(\nabla^S: S^n_m \to S^{n+1}_m\) by

\[
(\nabla^S A)^{\beta_1 \ldots \beta_m}_{\alpha_1 \ldots \alpha_{n+1}} = \nabla^{(\alpha_1} A^{\beta_1 \ldots \beta_m}_{\alpha_2 \ldots \alpha_{n+1})}.
\]  

(5.8)

These definitions are naturally extended to \(\text{End}(\mathcal{V})\)-valued symmetric tensors, i.e. to the sections of the bundle \(S^n_m \otimes \text{End}(\mathcal{V})\).
5.2 Covariant Taylor Basis

Let us consider the space of smooth two-point functions in a small neighborhood of the diagonal $x = x'$; we will denote such functions by $|f\rangle$. Let us define a special set of such functions $\{|n\rangle\}_{n=0}^{\infty}$, labeled by a non-negative integer $n$, by

$$|0\rangle = 1,$$
$$|n\rangle = \frac{1}{n!} y^{a_1} \ldots y^{a_n},$$

where $y^a$ are the geometric parameters (normal coordinates) defined by (3.33). These functions are scalars at the point $x$ and symmetric tensors of type $(0, n)$ at the point $x'$. It is easy to show that these functions satisfy the equation

$$D|n\rangle = n|n\rangle,$$

and, hence, are the eigenfunctions of the operator $D$ with positive integer eigenvalues.

Let $\langle n|$ denote the dual linear functionals defined by

$$\langle n| f \rangle = (\nabla^S)^a f \bigg|_{x=x'},$$

so that

$$\langle n|m \rangle = \delta_{mn} \beta(n),$$

Using this notation the covariant Taylor series for an analytic function $|f\rangle$ can be written in the form

$$|f\rangle = \sum_{n=0}^{\infty} |n\rangle \star \langle n| f \rangle,$$

For smooth functions the Taylor series is only an asymptotic series, which does not necessarily converge. For analytic functions, however, the Taylor series converges in a sufficiently small neighborhood of the fixed point $x'$. Therefore, the functions $|n\rangle$ form a complete orthonormal basis in the subspace of analytic functions. This is a reflection of the fact that an analytic function that is orthogonal to all functions $|n\rangle$, that is, whose all symmetrized derivatives vanish at the point $x'$, is, in fact, identically equal to zero in this neighborhood. Note, however, that the space of functions we are talking about is not a Hilbert space since there are many analytic functions $|f\rangle$ such that the norm $\langle f|f \rangle$ defined above diverges. If we restrict ourselves to polynomials of some order, then this problem does not appear, and, hence, the space of polynomials is a Hilbert space with the inner product defined above.
5.3 Matrix Algorithm

The complete set of eigenfunctions $|n\rangle$ can be employed to present the action of the operator $\tilde{L}$ on a function $|f\rangle$ in the form

$$\tilde{L}|f\rangle = \sum_{m,n \geq 0} |m\rangle \star \langle m|\tilde{L}|n\rangle \star \langle n|f\rangle,$$

(5.15)

where $\langle m|\tilde{L}|n\rangle$ are the ‘matrix elements’ of the operator $\tilde{L}$ that are just $\text{End}(\mathcal{V})$-valued symmetric tensors, i.e. sections of the vector bundle $S_{m,n} \otimes \text{End}(\mathcal{V})$. When acting on an analytic function this series is nothing but the Taylor series of the result and converges in a sufficiently small neighborhood of the point $x'$; for a smooth functions it gives an asymptotic expansion.

Now it should be clear that the inverse of the operator $(1 + \frac{1}{k}D)^{-1}$ can be defined by

$$\left(1 + \frac{1}{k}D\right)^{-1}|f\rangle = \sum_{n=0}^{\infty} \frac{k}{k+n} |n\rangle \star \langle n|f\rangle.$$

(5.16)

Using such representations for the operators $(1 + \frac{1}{k}D)^{-1}$ and $\tilde{L}$ we obtain a covariant Taylor series for the coefficients $a_k$

$$a_k = \sum_{n=0}^{\infty} |n\rangle \star \langle n|a_k\rangle$$

(5.17)

where

$$\langle n|a_k\rangle = \sum_{n_1,\ldots,n_{k-1} \geq 0} \frac{k}{k+n} \frac{k-1}{k-1+n_{k-1}} \cdots \frac{2}{2+n_2} \frac{1}{1+n_1} \times \langle n|\tilde{L}|n_{k-1}\rangle \star \langle n_{k-1}|\tilde{L}|n_{k-2}\rangle \star \cdots \star \langle n_1|\tilde{L}|0\rangle,$$

(5.18)

where the summation is over all non-negative integers $n_1,\ldots,n_{k-1}$. It is not difficult to show that for a differential operator of second order the matrix elements $\langle m|\tilde{L}|n\rangle$ do not vanish only for $n \leq m + 2$. Therefore, the summation over $n_i$ here is limited from above by

$$0 \leq n_1, \quad n_i \leq n_{i+1} + 2, \quad (i = 1, 2, \ldots, k-1),$$

(5.19)

where $n_k \equiv n$. Thus, the sum (5.18) contains only a finite number of terms.
Thus, we have reduced the problem of computation of the heat kernel coefficients $a_k$ to the evaluation of the matrix elements $\langle m|\bar{L}|n \rangle$ of the operator $\bar{L}$. For a differential operator $\bar{L}$ of second order, the matrix elements $\langle m|\bar{L}|n \rangle$ vanish for $n > m + 2$. Therefore, the summation over $n_i$ in (5.18) is limited from above: $n_1 \geq 0$, and $n_i \leq n_{i+1} + 2$, for $i = 1, 2, \ldots, k - 1$, and, hence, the sum (5.18) always contains only a finite number of terms.

The matrix elements $\langle n|L|m \rangle$ of a Laplace type operator have been computed in our papers [3, 1]. They have the following general form

\begin{align}
\langle m|L|m + 2 \rangle &= -g^* \vee \mathbb{I}(m), \\
\langle m|L|m + 1 \rangle &= 0, \\
\langle m|L|n \rangle &= \binom{m}{n} \mathbb{I}(m-n) \vee Z(n-m) + \binom{m}{n-1} \mathbb{I}(n-1) \vee Y(n-m+1) + \binom{m}{n-2} \mathbb{I}(n-2) \vee X(n-m+2),
\end{align}

where $g^*$ is the metric on the cotangent bundle, $Z(n)$ is a section of the vector bundle $S_n \otimes \text{End}(\mathcal{V})$, $Y(n)$ is a section of the vector bundle $S_{n+1} \otimes \text{End}(\mathcal{V})$ and $X(n)$ is a section of the vector bundle $S_{n+2}$ (a symmetric tensor of type $(2,n)$). Here it is also meant that the binomial coefficient $\binom{n}{k}$ is equal to zero if $k < 0$ or $n < k$.

We will not present here explicit formulas, (they have been computed explicitly for arbitrary $m, n$ in [3, 10]), but note that all these quantities are expressed polynomially in terms of three sorts of geometric data:

i) symmetric tensors of type $(2,n)$, i.e. sections of the bundle $S_n^2$ obtained by symmetric derivatives

$$K(n) = (\nabla^S)^{n-2} \text{Riem}$$

of the symmetrized Riemann tensor $\text{Riem}$ taken as a section of the bundle $S_2^2$,

ii) sections

$$\mathcal{R}(n) = (\nabla^S)^{n-1} \mathcal{R}$$

of the vector bundle $S_n^1 \otimes \text{End}(\mathcal{V})$ obtained by symmetrized derivatives of the curvature $\mathcal{R}$ of the connection $\nabla^\mathcal{V}$ taken as a section of the bundle $S_1^1 \otimes \text{End}(\mathcal{V})$. 
iii) End(\mathcal{V})-valued symmetric forms, i.e. sections of the vector bundle $S_n^0 \otimes \text{End}(\mathcal{V})$, constructed from the symmetrized covariant derivatives

$$Q_{(n)} = (\nabla^S)^n Q$$

of the endomorphism $Q$.

From dimensional arguments it is obvious that the matrix elements $\langle n|L|n \rangle$ are expressed in terms of the Riemann curvature tensor, Riem, the bundle curvature, $\mathcal{R}$, and the endomorphism $Q$: the matrix elements $\langle n + 1|L|n \rangle$ — in terms of the quantities $\nabla \text{Riem}$, $\nabla \mathcal{R}$ and $\nabla Q$; the elements $\langle n + 2|L|n \rangle$ — in terms of the quantities of the form $\nabla \nabla \text{Riem}$, $\text{Riem} \cdot \text{Riem}$, etc.

### 5.4 Diagramatic Technique

In the computation of the heat kernel coefficients by means of the matrix algorithm a “diagrammatic” technique, i.e., a graphic method for enumerating the different terms of the sum (5.18), turns out to be very convenient and pictorial [3, 10].

The matrix elements $\langle m|L|n \rangle$ are presented by some blocks with $m$ lines coming in from the left and $n$ lines going out to the right (Fig. 1),

![Fig. 1](image)

and the product of the matrix elements $\langle m|L|k \rangle \star \langle k|L|n \rangle$ — by two blocks connected by $k$ intermediate lines (Fig. 2),

![Fig. 2](image)

that represents the contractions of the corresponding tensor indices (the inner product).

To obtain the coefficient $\langle n|a_k \rangle$ one should draw, first, all possible diagrams which have $n$ lines incoming from the left and which are constructed from $k$ blocks
connected in all possible ways by any number of intermediate lines. When doing this, one should keep in mind that the number of the lines, going out of any block, cannot be greater than the number of the lines, coming in, by more than two and by exactly one. Then one should sum up all diagrams with the weight determined for each diagram by the number of intermediate lines from the analytical formula (5.18). Drawing of such diagrams is of no difficulties. This helps to keep under control the whole variety of different terms. Therefore, the main problem is reduced to the computation of some standard blocks, which can be computed once and for all.

For example, the diagrams for the diagonal values of the HMDS-coefficients $a^{\text{diag}}_k = \langle 0 | a_k \rangle$ have the form,

$$a^{\text{diag}}_1 = \phantom{\bigg|}$$

$$a^{\text{diag}}_2 = \phantom{\bigg|} + \frac{1}{3} \phantom{\bigg|}$$

$$a^{\text{diag}}_3 = \phantom{\bigg|} + \frac{1}{3} \phantom{\bigg|} + \frac{2}{4} \phantom{\bigg|}$$

$$+ \frac{2}{4} \cdot \frac{1}{2} \phantom{\bigg|} + \frac{2}{4} \cdot \frac{1}{3} \phantom{\bigg|} + \frac{2}{4} \cdot \frac{1}{5} \phantom{\bigg|} .$$

As an illustration let us compute the coefficients $a^{\text{diag}}_1$ and $a^{\text{diag}}_2$. We have [3]

$$\phantom{\bigg|} = \langle 0 | L | 0 \rangle = Z_{(0)}$$

$$\phantom{\bigg|} = \langle 0 | L | 2 \rangle = -g^{ab}$$

$$\phantom{\bigg|} = \langle 2 | L | 0 \rangle = Z_{(2)ab}$$

$$\phantom{\bigg|} = \langle 0 | L | 2 \rangle \star \langle 2 | L | 0 \rangle = -g^{ab} Z_{(2)ab} ,$$

where

$$Z_{(0)} = Q - \frac{1}{6} R \mathbb{I}_V ,$$

$$Z_{(2)ab} = -g^{ab} Z_{(2)ab} ,$$

$Q$ and $R$ are matrices representing the operators. The index $V$ denotes the vector space on which these operators act.
\[ Z_{(2)ab} = \nabla_{(a} \nabla_{b)} Q - \frac{1}{2} R_{c(a} R^{c}_{b)} + \frac{1}{2} \nabla_{(a} \nabla_{c)} R^{c}_{b)} \]  

\[ + \mathbb{I}_V \left( - \frac{3}{20} \nabla_{a} \nabla_{b} R - \frac{1}{20} \Delta R_{ab} + \frac{1}{15} R_{ac} R^{c}_{b} - \frac{1}{30} R_{acde} R^{cde}_{b} - \frac{1}{30} R_{cd} R^{c}_{a} R^{d}_{b} \right). \]  

Here, as usual, the parenthesis denote the complete symmetrization over all indices included and the vertical lines indicate the indices excluded from the symmetrization. Hence, we immediately get

\[ a_1^{\text{diag}} = Q - \frac{1}{6} \mathbb{I}_V, \]  

and, by taking the trace of \( Z_{(2)} \) and using the identity \( \nabla_a \nabla_b R^{ab} = 0 \), we obtain the well known result \([3]\)

\[ a_2^{\text{diag}} = \left( Q - \frac{1}{6} \mathbb{I}_V \right)^2 - \frac{1}{3} \Delta Q + \frac{1}{6} R_{ab} R^{ab} + \mathbb{I}_V \left( \frac{1}{15} \Delta R - \frac{1}{90} R_{ab} R^{ab} + \frac{1}{90} R_{abcd} R^{abcd} \right). \]

The technique described above is manifestly covariant and is applicable for any Riemannian (or pseudo-Riemannian) manifold \( M \) and for any vector bundle \( V \). It is also valid for local analysis on noncompact manifolds and manifolds with boundary (on a finite distance from the boundary). This method gives not only the diagonal values of the heat kernel coefficients but also the diagonal values of all their derivatives, that is, it gives also the off-diagonal coefficients in form of a covariant Taylor series. Due to the use of symmetric forms and symmetric covariant derivatives the famous ‘combinatorial explosion’ in the complexity of the heat kernel coefficients is avoided. This technique is very algorithmic and well suited to automated computation. The developed method is very powerful; it enabled us to compute for the first time the diagonal value of the fourth HMDS-coefficient \( a_4^{\text{diag}} \) \([2, 3]\). It was used in \([33, 31]\) to compute the coefficient \( a_5^{\text{diag}} \). Lastly, this technique enables one not only to carry out explicit computations, but also to analyse the general structure of the heat kernel coefficients for all orders \( k \).

### 5.5 General Structure of Heat Kernel Coefficients

Now we are going to investigate the general structure of the heat kernel coefficients. We will follow mainly our papers \([10, 9, 11]\).

Our analysis will be again purely local. Since locally one can always expand the metric, the connection and the endomorphism \( Q \) in the covariant Taylor series,
they are completely characterized by their Taylor coefficients, i.e. the covariant
derivatives of the curvatures, more precisely by the objects $K_{(n)}$, $R_{(n)}$ and $Q_{(n)}$ introduced above. We introduce the following notation for all of them

$$R_{(n)} = \{K_{(n+2)}, R_{(n+1)}, Q_{(n)}\}, \quad (n = 0, 1, 2, \ldots),$$

(5.37)

and call these objects covariant jets; $n$ will be called the order of a jet $R_{(n)}$. It is
worth noting that the jets are defined by symmetrized covariant derivatives. This
makes them well defined as the order of the derivatives becomes not important. It is
only the number of derivatives that plays a role.

The low-order coefficients $A_0$ and $A_1$ have been described above. As far as
the higher order coefficients $A_k$, ($k \geq 2$), are concerned they are integrals of local
invariants which are polynomial in the jets. One can classify all the terms in them according to the number of the jets and their order. The terms linear in the jets in higher order coefficients $A_k$, ($k \geq 2$), are given by integrals of total
derivatives, symbolically $\int_M j_{\omega}^V V^{A_{k-1}}$. They are calculated explicitly in [3, 10]. Since the total derivative do not contribute to an integral over a complete
compact manifold, it is clear that the linear terms vanish. Thus $A_k$, ($k = 2, 3, \ldots$),
begin with the terms quadratic in the jets. These terms contain the jets of highest
order (or the leading derivatives of the curvatures) and can be shown to be of the
form $\int_M j_{\omega}^V R^{A_{k-2}}$. Then it follows a class of terms cubic in the jets etc.
The last class of terms does not contain any covariant derivatives at all but only
the powers of the curvatures. In other words, the higher order HMDS-coefficients
have a general structure, which can be presented symbolically in the form

Thus, for $k \geq 2$ one can classify the terms in $A_{2k}$ according to the number of
the jets and their order

$$A_k = \sum_{j=2}^{k} A_{k,(j)},$$

(5.38)

where $A_{k,(j)}$ is the contribution of order $j$ in the jets; they can be presented sym-
bolically in the form

$$A_{k,(2)} = \int_M j_{\omega}^V \sum_{(0)} \sum_{(2k-4)} R_{(0)} R_{(2k-4)},$$

(5.39)

$$A_{k,(3)} = \int_M j_{\omega}^V \sum_{i=0}^{2k-6} \sum_{(0)} \sum_{(2k-6-i)} R_{(0)} R_{(i)} R_{(2k-6-i)},$$

(5.40)

\[ \ldots \]
\begin{align*}
A_{k,(k-1)} &= \int_M d\text{vol} \, \text{tr}_V \left[ \sum \mathcal{R}(2) (\mathcal{R}(0))^{k-2} + \sum (\mathcal{R}(1))^{2} (\mathcal{R}(0))^{k-3} \right], \\
A_{k,(k)} &= \int_M d\text{vol} \, \text{tr}_V \sum (\mathcal{R}(0))^k.
\end{align*}

More precisely, the functionals \( A_{k,(j)} \) transform under the rescaling of the jets \( \mathcal{R}(k) \mapsto \varepsilon^k \mathcal{R}(k) \) as follows

\begin{equation}
A_{k,(j)} \mapsto \varepsilon^j a^{2(k-j)} A_{k,(j)}.
\end{equation}

### 6 High Energy Approximation

One can show that all quadratic terms can be reduced to five independent invariants, viz. \([1, 3, 10]\)

\begin{align*}
A_{k,(2)} &= \frac{k!(k-2)!}{2(2k-3)!} \int_M d\text{vol} \, \text{tr}_V \left\{ f_k^{(1)} Q \Delta^{k-2} Q + 2 f_k^{(2)} R^{bc} \nabla_b \Delta^{k-3} \nabla_a R^a_{\sigma} \\
&\quad + f_k^{(3)} Q \Delta^{k-2} R + f_k^{(4)} R_{ab} \Delta^{k-2} R^{ab} + f_k^{(5)} R \Delta^{k-2} R \right\},
\end{align*}

where \( f_k^{(i)} \) are some numerical coefficients. These numerical coefficients can be computed by the technique developed in the previous section. From the formula (5.18) we have for the diagonal coefficients \( a_{k}^{\text{diag}} \) up to cubic terms in the jets

\begin{align*}
\langle 0|a_k \rangle &= (-1)^{k-1} \binom{2k-1}{k} \langle 0;k-1|L|0 \rangle \\
&+ (-1)^k \sum_{i=1}^{k-1} \sum_{n_i=0}^{2k-1-2i} \binom{2k-1}{i} \binom{2k-1-2i}{i} \langle 0;k-i-1|L|n_i \rangle \star \langle n_i;i-1|L|0 \rangle \\
&+ O(\mathcal{R}^3),
\end{align*}

where

\begin{equation}
\langle n;k|L|m \rangle = (\star^k g^+) \star \langle n|L|m \rangle
\end{equation}
and $O(\mathcal{R}^3)$ denote terms of third order in the jets.

By computing the matrix elements in the second order in the jets and integrating over $M$ one obtains \cite{1, 3}

$$f^{(1)}_k = 1,$$

$$f^{(2)}_k = \frac{1}{2(2k - 1)},$$

$$f^{(3)}_k = \frac{k - 1}{2(2k - 1)},$$

$$f^{(4)}_k = \frac{1}{2(4k^2 - 1)},$$

$$f^{(5)}_k = \frac{k^2 - k - 1}{4(4k^2 - 1)}.$$

(6.4)  

(6.5)  

(6.6)  

(6.7)  

(6.8)

One should note that the same results were obtained by a completely different method in \cite{17}.

Let us consider the situation when the curvatures are small but rapidly varying (high energy approximation in quantum field theory), i.e. the derivatives of the curvatures are more important than the powers of them. This corresponds to an asymptotic expansion in the deformation parameter $\varepsilon$ as $\varepsilon \to 0$. Then the leading derivative terms in the heat kernel are the largest ones. Thus the heat trace has the form

$$\Theta(t) \sim (4\pi t)^{-n/2} \left\{ A_0 - tA_1 + \frac{t^2}{2} H_2(t) \right\} + O(\mathcal{R}^3),$$

(6.9)

where $H_2(t)$ is some complicated nonlocal functional that has the following asymptotic expansion as $t \to 0$

$$H_2(t) \sim 2 \sum_{k=2}^{\infty} \frac{(-t)^{k-2}}{k!} A_{k,(2)}.$$

(6.10)

Using the results for $A_{k,(2)}$ one can easily construct such a functional $H_2$ just by a formal summation of the leading derivatives

$$H_2(t) = \int_{M} d\text{vol} \ tr_Y \left\{ Q \gamma^{(1)}(-t\Delta)Q + 2R^a_c \nabla_a \frac{1}{\Delta} \gamma^{(2)}(-t\Delta)\nabla_b R^{bc} \\
- 2Q \gamma^{(3)}(-t\Delta)R + R_{ab} \gamma^{(4)}(-t\Delta)R^{ab} + R \gamma^{(5)}(-t\Delta)R \right\},$$

(6.11)
where $\gamma^{(i)}(z)$ are entire functions defined by [13]:

$$\gamma^{(i)}(z) = \sum_{k=0}^{\infty} \frac{k!}{(2k+1)!} f_k^{(i)} z^k = \int_0^1 d\xi f^{(i)}(\xi) \exp \left( -\frac{1-\xi^2}{4} z \right). \quad (6.12)$$

where

$$f^{(1)}(\xi) = 1, \quad (6.13)$$
$$f^{(2)}(\xi) = \frac{1}{2} \xi^2, \quad (6.14)$$
$$f^{(3)}(\xi) = \frac{1}{4} (1 - \xi^2), \quad (6.15)$$
$$f^{(4)}(\xi) = \frac{1}{6} \xi^4, \quad (6.16)$$
$$f^{(5)}(\xi) = \frac{1}{48} (3 - 6\xi^2 - \xi^4). \quad (6.17)$$

Therefore, $H_2(t)$ can be regarded as generating functional for quadratic terms $A_{k,(2)}$ (leading derivative terms) in all coefficients $A_k$. It also plays a very important role in investigating the nonlocal structure of the effective action in quantum field theory in high-energy approximation [13].

## 7 Low Energy Approximation

Let us consider now the opposite case, when the curvatures are strong but slowly varying (low-energy approximation in quantum field theory), i.e. the powers of the curvatures are more important than the derivatives of them. This corresponds to the asymptotic expansion in the deformation parameter $\alpha$ as $\alpha \to 0$. The main terms in this approximation are the terms without any covariant derivatives of the curvatures, i.e. the lowest order jets. We will consider mostly the zeroth order of this approximation which corresponds simply to covariantly constant background curvatures

$$\nabla \text{Riem} = 0, \quad \nabla R = 0, \quad \nabla Q = 0. \quad (7.1)$$

The asymptotic expansion of the heat trace

$$\Theta(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} A_{k,(k)}. \quad (7.2)$$
determines then all the terms without covariant derivatives (highest order terms in the jets), $A_{k,(k)}$, in all heat kernel coefficients $A_k$. These terms do not contain any covariant derivatives and are just polynomials in the curvatures and the endomorphism $Q$. Thus the heat trace is a generating functional for all heat kernel coefficients for a covariantly constant background, in particular, for all symmetric spaces. Thus the problem is to calculate the heat trace for covariantly constant background.

### 7.1 Algebraic Approach

There exist a very elegant indirect way to construct the heat kernel without solving the heat equation but using only the commutation relations of some covariant first order differential operators. Below we will follow our papers [4, 5, 6, 7, 12, 13]. The main idea is to employ a generalization of the usual Fourier transform to the case of operators; it consists in the following. We are going to use the following representation of the heat trace

\[
\Theta(t) = \int_M d\operatorname{vol} \operatorname{tr}_V \left[ \exp(-tL)\delta(x,x') \right]^{\text{diag}}. \tag{7.3}
\]

Let us consider for a moment a trivial case, where the curvatures vanish but the potential term does not:

\[
\text{Riem} = 0, \quad \mathcal{R} = 0, \quad \nabla Q = 0. \tag{7.4}
\]

In this case the operators of covariant derivatives obviously commute and form together with the potential term an Abelian Lie algebra

\[
[\nabla_\mu, \nabla_\nu] = 0, \quad [\nabla_\mu, Q] = 0. \tag{7.5}
\]

It is easy to show that the heat semigroup can be presented in the form

\[
\exp(-tL) = (4\pi t)^{-n/2} \exp(-tQ) \int_{\mathbb{R}^n} dk \, g^{1/2} \exp\left(-\frac{1}{4t} \langle k, g k \rangle + k \cdot \nabla \right), \tag{7.6}
\]

where $\langle k, g k \rangle = k^\mu g_{\mu\nu} k^\nu$ and $k \cdot \nabla = k^\mu \nabla_\mu$. Here, of course, it is assumed that the covariant derivatives also commute with the metric

\[
[\nabla, g] = 0. \tag{7.7}
\]
Acting with this operator on the Dirac distribution and using the obvious relation

\[
[\exp(k \cdot \nabla) \delta(x, x')]^{\text{diag}} = \delta(k),
\]

one can integrate easily over \(k\) to obtain the heat trace

\[
\Theta(t) = (4\pi t)^{-n/2} \int_M \text{dvol} \text{ tr}_V \exp(-tQ).
\]

Of course, on curved manifolds the covariant differential operators \(\nabla\) do not commute—their commutators are determined by the curvatures \(\mathcal{R}\). The commutators of covariant derivatives \(\nabla\) with the curvatures \(\mathcal{R}\) give the first derivatives of the curvatures, i.e. the jets \(\mathcal{R}_{(1)}\), the commutators of covariant derivatives with \(\mathcal{R}_{(1)}\) give the second jets \(\mathcal{R}_{(2)}\), etc. Thus the operators \(\nabla\) together with the whole set of the jets \(\mathcal{J}\) form an infinite dimensional Lie algebra \(\mathcal{G} = \{\nabla, \mathcal{R}_{(i)}; (i = 1, 2, \ldots)\}\).

Now, let us remember that the heat trace is a functional of the jets, with the jets being defined by symmetrized covariant derivatives. This makes the order of a jet well defined. For example, the structures involving commutators of covariant derivatives, (like \([\nabla_a, \nabla_b]R^{c}_{\phantom{c}d}\), which involve 2-jets of the Riemann tensor on the left but, after using the Ricci identity, only 0-jets on the right) are not allowed. After symmetrizing over \(abcd\) this jet vanishes. So, if we express the final answer for the heat kernel diagonal or for the heat kernel coefficients in terms of the symmetrized jets, then there is a natural filtration with respect to the order of the jets involved. In other words, one can always say, what is the maximal order of symmetrized covariant derivative of the curvature involved in the result. This is especially true for the heat kernel coefficients \(A_k\) since they are polynomial in the jets.

If we identify a small deformation parameter \(\alpha\) with each derivative then a jet of order \(n\) is, actually, of order \(\alpha^n\). Thus, we get a perturbation theory in this small parameter. Since the derivatives are naturally identified with the momentum (or energy), the physicists call a situation when the derivatives are small the low-energy approximation. To evaluate the heat kernel in the low-energy approximation one can take into account only a finite number of low-order jets, i.e. the low-order covariant derivatives of the background fields, \(\{\mathcal{R}_{(i)}; (i \leq N)\}\) with some fixed \(N\), and neglect all higher order jets, i.e. the covariant derivatives of higher orders, i.e. put \(\mathcal{R}_{(i)} = 0\) for \(i > N\). Then one can show that there exist a set of covariant differential operators that together with the low-order jets generate a finite-dimensional Lie algebra \(\mathcal{G}_N = \{\nabla, \mathcal{R}_{(i)}; (i = 1, 2, \ldots, N)\}\). One should stress
here what problem one can solve this way. We try to answer the following con-
crete question: how do the heat kernel coefficients look if we throw away all the
(symmetrized) jets of order higher than \( N \)?

Thus one can try to generalize the above idea in such a way that (7.6) would
be the zeroth approximation in the commutators of the covariant derivatives, i.e.
in the curvatures. Roughly speaking, we would like to find a representation of the
heat semigroup in the form

\[
\exp(-tL) = (4\pi t)^{-D/2} \int_{\mathbb{R}^D} dk \Phi(t,k) \exp \left( -\frac{1}{4t} \langle k, \Psi(t)k \rangle + T(k) \right),
\]

(7.10)

where \( \langle k, \Psi(t)k \rangle = k^A \Psi_{AB}(t)k^B \), \( T(k) = k^A T_A \). \( T_A \) are some first
order differential operators and the functions \( \Psi(t) \) and \( \Phi(t,k) \) are expressed in
terms of commutators of these operators, i.e., in terms of the curvatures; that is,
these functions are analytic functions of \( t \). In general, the operators \( T_A \) do not
form a closed finite dimensional algebra because at each step, by taking more
commutators, there appear more and more derivatives of the curvatures. It is
the low-energy reduction \( \mathcal{G} \mapsto \mathcal{G}_N \), i.e. the restriction to the low-order jets, that
actually closes the algebra \( \mathcal{G} \) of the operators \( T_A \) and the background jets, i.e.
makes it finite dimensional.

Using this representation one can, as above, act with \( \exp[T(k)] \) on the Dirac
distribution to get the heat kernel. The main point of this idea is that it is much
easier to calculate the action of the exponential of the first order operator \( T(k) \)
on the Dirac distribution than that of the exponential of the second order operator \( L \).

7.2 Covariantly Constant Background in Flat Space

Let us consider now the more complicated case of nontrivial covariantly constant
curvature of the connection on the vector bundle \( V \) in flat space:

\[
\text{Riem} = 0, \quad \nabla \mathcal{R} = 0, \quad \nabla Q = 0.
\]

(7.11)

Using the condition of covariant constancy of the curvatures one can show that
in this case the covariant derivatives form a nilpotent Lie algebra \( [\nabla_\mu, \nabla_\nu] = \mathcal{R}_{\mu\nu}, \)

(7.12)

\[
[nabla_\mu, R_{\alpha\beta}] = [nabla_\mu, Q] = 0,
\]

(7.13)

\[
[R_{\mu\nu}, R_{\alpha\beta}] = [R_{\mu\nu}, Q] = 0.
\]

(7.14)
For this algebra one can prove a theorem expressing the heat semigroup operator in terms of an average over the corresponding Lie group [4]

\[
\exp(-tL) = (4\pi t)^{-n/2} \exp(-tQ) \left[ \det_{\mathcal{T}M} \left( \frac{tR}{\sinh(tR)} \right) \right]^{1/2} \times \int_{\mathbb{R}^n} dk \, g^{1/2} \exp \left( -\frac{1}{4t} \langle k, g tR \coth(tR) k \rangle + k \cdot \nabla \right),
\]

(7.16)

where \( k \cdot \nabla = k^\mu \nabla_\mu \). Here functions of the curvatures \( R \) are understood as functions of sections of the bundle \( \text{End}(TM) \otimes \text{End}(\mathcal{V}) \), and the determinant \( \det_{\mathcal{T}M} \) is taken with respect to the tangent space indices; the fiber indices of the bundle \( \mathcal{V} \) being intact.

It is not difficult to show that in this case we also have

\[
\left[ \exp(k \cdot \nabla) \delta(x, x') \right]^{\text{diag}} = \delta(k).
\]

(7.17)

Subsequently, the integral over \( k \) becomes trivial and we obtain immediately the trace of the heat kernel [4]

\[
\Theta(t) = (4\pi t)^{-n/2} \int_{\mathcal{M}} d\text{vol} \, \text{tr}_{\mathcal{V}} \exp(-tQ) \left[ \det_{\mathcal{T}M} \left( \frac{tR}{\sinh(tR)} \right) \right]^{1/2}.
\]

(7.18)

Expanding it in a power series in \( t \) one can find all covariantly constant terms in all heat kernel coefficients \( A_k \).

As we have seen the contribution of the curvature \( R_{\mu\nu} \) is not as trivial as that of the potential term. However, the algebraic approach does work in this case too. It is a good example how one can get the heat kernel without solving any differential equations but using only the algebraic properties of the covariant derivatives.

### 7.2.1 Quadratic Potential in Flat Space

In fact, in flat space it is possible to do a bit more, i.e. to calculate the contribution of the first and the second derivatives of the potential term \( Q [6] \). That is, we consider the case when the derivatives of the endomorphism \( Q \) vanish only starting from the third order, i.e.

\[
\text{Riem} = 0, \quad \nabla R = 0, \quad \nabla \nabla Q = 0.
\]

(7.19)
Besides we assume the background to be Abelian, i.e. all the nonvanishing background quantities, $R_{\alpha\beta}$, $Q$, $Q_{,\mu} \equiv \nabla_\mu Q$ and $Q_{,\nu\mu} \equiv \nabla_\mu \nabla_\nu Q$, commute with each other. Thus we have again a nilpotent Lie algebra

\begin{align}
[\nabla_\mu, \nabla_\nu] &= R_{\mu\nu}, \quad (7.20) \\
[\nabla_\mu, Q] &= Q_{,\mu}, \quad (7.21) \\
[\nabla_\mu, Q_{,\nu}] &= Q_{,\nu\mu}, \quad (7.22)
\end{align}

all other commutators being zero.

Now, let us represent the endomorphism $Q$ in the form

$$Q = Q_0 - \alpha^{ik} N_i N_k, \quad (7.23)$$

where $(i = 1, \ldots, q; q \leq n)$, $\alpha^{ik}$ is some constant symmetric nondegenerate $q \times q$ matrix, $Q_0$ is a covariantly constant endomorphism and $N_i$ are some endomorphisms with vanishing second covariant derivative:

$$\nabla Q_0 = 0, \quad \nabla \nabla N_i = 0. \quad (7.24)$$

Next, let us introduce the operators $X_A = (\nabla_\mu, N_i), (A = 1, \ldots, n + q)$ and the matrix

$$(F_{AB}) = \begin{pmatrix} R_{\mu\nu} & N_{i,\mu} \\ -N_{i,\nu} & 0 \end{pmatrix}, \quad (7.25)$$

with $N_{i,\mu} \equiv \nabla_\mu N_i$.

The operator $L$ can now be written in the form

$$L = -G^{AB} X_A X_B + Q_0, \quad (7.26)$$

where

$$(G^{AB}) = \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & \alpha^{ik} \end{pmatrix}, \quad (7.27)$$

and the commutation relations (7.22) take a more compact form

$$[X_A, X_B] = F_{AB}, \quad (7.28)$$

all other commutators being zero.

This algebra is again a nilpotent Lie algebra. Thus one can apply the previous results in this case too to get [6]

$$\exp(-tL) = (4\pi t)^{-(n+q)/2} \exp(-tQ_0) \left[ \det \left( \frac{tF}{\sinh(tF)} \right) \right]^{1/2} \times \int_{\mathbb{R}^{n+q}} dk \ G^{1/2} \exp \left( -\frac{1}{4t} \langle k, GrF \coth(tF)k \rangle + X(k) \right), \quad (7.29)$$
where \( G = \det G_{AB} \) and \( X(k) = k^A X_A \).

Thus we have expressed the heat semigroup operator in terms of the operator \( \exp[X(k)] \). The integration over \( k \) is Gaussian except for the noncommutative part. Splitting the integration variables \( (k^A) = (q^\mu, \omega^i) \) and using the Campbell-Hausdorf formula we obtain \([6]\)

\[
[\exp[X(k)] \delta(x,x')]^{\text{diag}} = \exp[N(\omega)] \delta(q),
\]

where \( N(\omega) = \omega^j N_j \). Further, after taking off the trivial integration over \( q \) and a Gaussian integral over \( \omega \), we obtain the heat trace \([6]\).

To describe the result let us introduce a matrix determined by second derivatives of the potential term as follows

\[
P = (P^\mu_\nu), \quad P^\mu_\nu = \frac{1}{2} \nabla^\mu \nabla_\nu Q,
\]

and the matrices \( C(t) = (C^\mu_\nu(t)), K(t) = (K^\mu_\nu(t)) S(t) = (S^\mu_\nu(t)) \) and \( E(t) = (E^\mu_\nu(t)) \) by

\[
C(t) = \oint_C \frac{dz}{2\pi i} F(z) \frac{t}{z} \coth \left( \frac{t}{z} \right),
\]

\[
K(t) = \oint_C \frac{dz}{2\pi i} F(z) \frac{t}{z^2} \sinh \left( \frac{t}{z} \right),
\]

\[
S(t) = \oint_C \frac{dz}{2\pi i} F(z) \frac{t}{z} \sinh \left( \frac{t}{z} \right),
\]

\[
E(t) = \oint_C \frac{dz}{2\pi i} F(z) t \sinh \left( \frac{t}{z} \right),
\]

where

\[
F(z) = (1 - z R - z^2 P)^{-1}
\]

and the integral is taken along a sufficiently small closed contour \( C \) that encircles the origin counter-clockwise, so that \( F(z) \) is analytic inside this contour.

Then the heat trace has the form

\[
\Theta(t) = (4\pi t)^{-n/2} \int_M d\text{vol} \ tr_V [\Phi(t)]^{-1/2} \exp \left[ -t Q + \frac{1}{4} t^3 \langle \nabla Q, \Psi(t) \nabla Q \rangle \right],
\]
where \( \langle \nabla Q, \Psi(t) \nabla Q \rangle = \nabla_\mu Q \Psi^\mu \nabla_\nu Q \).

\[
\Phi(t) = \det_{TM} K(t) \det_{TM} \left[ 1 + t^2 C(t) P \right] \\
\times \det_{TM} \left[ 1 + t^2 [E(t) - S(t) K^{-1}(t) S(t)] P \right],
\]

\[
(7.38)
\]

\[
\Psi(t) = \left( \Psi^\mu_{\nu}(t) \right) = \left[ 1 + t^2 C(t) P \right]^{-1} C(t).
\]

\[
(7.39)
\]

The formula (7.37) exhibits the general structure of the heat trace. One sees immediately how the endomorphism \( Q \) and its first derivatives \( \nabla Q \) enter the result. The nontrivial information is contained only in a scalar, \( \Phi(t) \), and a tensor, \( \Psi_{\mu\nu}(t) \). These objects are constructed purely from the curvature \( R_{\mu\nu} \) and the second derivatives of the endomorphism \( Q, \nabla Q \). Thus, the heat kernel coefficients \( A_k \) are constructed from three different types of scalar (connected) blocks, \( Q, \Phi(t) \), and \( \Psi_{\mu\nu}(t) \). They are listed explicitly up to \( A_8 \) in [6].

### 7.3 Homogeneous Bundles over Symmetric Spaces

The exposition in this section closely follows our papers [5, 7, 12, 13]. Our goal is to compute the heat kernel of the Laplace type operator \( L = -\Delta + Q \) in the zero-order of the low-energy approximation. The difference with the previous sections is that now we are going to do it on most general covariantly constant background, that is, bundles with parallel curvature (that are called homogeneous bundles) on Riemannian manifolds with parallel curvature (that are called symmetric spaces).

It is well known that heat invariants are determined essentially by local geometry. They are polynomial invariants in the curvature with universal constants that do not depend on the global properties of the manifold [27]. It is this universal structure that we are interested in this paper. Our goal is to compute the heat kernel asymptotics of the Laplacian acting on homogeneous vector bundles over symmetric spaces.

In this section we will further assume that \( M \) is a locally symmetric space with a Riemannian metric with the parallel curvature

\[
\nabla_\mu R_{\alpha\beta\gamma\delta} = 0,
\]

\[
(7.40)
\]

which means, in particular, that the curvature satisfies the integrability constraints

\[
R_{\eta\lambda}^{\xi} R_{\beta\sigma}^{\mu} - R_{\eta\lambda}^{\xi} R_{\beta\sigma}^{\mu} + R_{\mu\lambda}^{\xi} R_{\beta\sigma}^{\eta} - R_{\mu\lambda}^{\xi} R_{\beta\sigma}^{\eta} = 0.
\]

\[
(7.41)
\]
In the following we will also consider \emph{homogeneous vector bundles} with parallel bundle curvature
\[ \nabla_\mu R_{\alpha\beta} = 0, \quad (7.42) \]
which means that the curvature satisfies the integrability constraints
\[ [R_{cd}, R_{ab}] - R^f_{\ acdf} R_{fb} - R^f_{\ bcdf} R_{af} = 0. \quad (7.43) \]
Finally, we consider a parallel section $Q$ of the endomorphism bundle $\text{End}(\mathcal{V})$, that is,
\[ \nabla_\mu Q = 0, \quad (7.44) \]
which means that
\[ [R_{cd}, Q] = 0. \quad (7.45) \]

We will use normal coordinates defined above. Note that for symmetric spaces normal coordinates cover the whole manifold except for a set of measure zero where they become singular \[19\]. This set is precisely the set of points conjugate to the fixed point $x'$ (where $\Delta^{-1}(x, x') = 0$) and of points that can be connected to the point $x'$ by multiple geodesics. In any case, this set is a set of measure zero and, as we will show below, it can be dealt with by some regularization technique. Thus, we will use the normal coordinates defined above for the whole manifold.

### 7.3.1 Curvature Group of a Symmetric Space

We assumed that the manifold $M$ is locally symmetric. Since we also assume that it is simply connected and complete, it is a globally symmetric space (or simply symmetric space). A symmetric space is said to be compact, non-compact or Euclidean if all sectional curvatures are positive, negative or zero. A generic symmetric space has the structure $M = M_0 \times M_s$, where $M_0 = \mathbb{R}^{n_0}$ and $M_s$ is a semi-simple symmetric space; it is a product of a compact symmetric space $M_+$ and a non-compact symmetric space $M_-, M_s = M_+ \times M_-$. Of course, the dimensions must satisfy the relation $n_0 + n_s = n$, where $n_s = \dim M_s$.

Let $\Lambda_2$ be the vector space of 2-forms on $M$ at a fixed point $x'$. It has the dimension $\dim \Lambda_2 = n(n - 1)/2$, and the inner product in $\Lambda_2$ is defined by
\[ \langle X, Y \rangle = \frac{1}{2} X_{ab} Y^{ab}. \quad (7.46) \]
The Riemann curvature tensor naturally defines the curvature operator
\[ \text{Riem} : \Lambda_2 \to \Lambda_2 \quad (7.47) \]
by

\[(\text{Riem} X)_{ab} = \frac{1}{2} R_{ab}^{\ \cd} X_{\cd} . \quad (7.48)\]

This operator is symmetric and has real eigenvalues which determine the principal sectional curvatures. Now, let \(\text{Ker}(\text{Riem})\) and \(\text{Im}(\text{Riem})\) be the kernel and the range of this operator and

\[p = \dim \text{Im}(\text{Riem}) = \frac{n(n - 1)}{2} - \dim \text{Ker}(\text{Riem}) . \quad (7.49)\]

Further, let \(\lambda_i, (i = 1, \ldots, p)\), be the non-zero eigenvalues, and \(E^i_{ab}\) be the corresponding orthonormal eigen-two-forms. Then the components of the curvature tensor can be presented in the form \([7]\)

\[R_{abcd} = \beta_{ik} E^i_{ab} E^k_{cd} , \quad (7.50)\]

where \(\beta_{ik}\) is a symmetric, in fact, diagonal, nondegenerate \(p \times p\) matrix. Of course, the zero eigenvalues of the curvature operator correspond to the flat subspace \(M_0\), the positive ones correspond to the compact submanifold \(M_+\) and the negative ones to the non-compact submanifold \(M_-\). Therefore, \(\text{Im}(\text{Riem}) = T_x M_s\).

In the following the Latin indices from the middle of the alphabet will be used to denote tensors in \(\text{Im}(\text{Riem})\); they should not be confused with the Latin indices from the beginning of the alphabet which denote tensors in \(M\). They will be raised and lowered with the matrix \(\beta_{ik}\) and its inverse \(\beta^{ik}\).

Next, we define the traceless \(n \times n\) matrices \(D_i = (D^a_{\ ib})\), where

\[D^a_{\ ib} = -\beta_{ik} E^k_{\ cb} \delta^{ca} . \quad (7.51)\]

The matrices \(D_i\) are known to be the generators of the holonomy algebra, \(\mathcal{H}\), i.e. the Lie algebra of the restricted holonomy group, \(H\),

\[[D_i, D_k] = F_j^{\ ik} D_j , \quad (7.52)\]

where \(F_j^{\ ik}\) are the structure constants of the holonomy group. The structure constants of the holonomy group define the \(p \times p\) matrices \(F_i\), by \((F_i)^j_{\ k} = F_j^{\ ik}\), which generate the adjoint representation of the holonomy algebra,

\[[F_i, F_k] = F_j^{\ ik} F_j . \quad (7.53)\]

For symmetric spaces the introduced quantities satisfy additional algebraic constraints. The most important consequence of the eq. \(7.41\) is the equation \([7]\)

\[E^i_{\ ac} D^j_{\ kb} - E^i_{\ bc} D^j_{\ ka} = F^j_{\ k} E^j_{\ ab} . \quad (7.54)\]
Now, we introduce a new type of indices, the capital Latin indices, $A, B, C, \ldots$, which split according to $A = (a, i)$ and run from 1 to $N = p + n$. We define new quantities $C^A_{BC}$ by

$$
C^i_{ab} = E^i_{ab}, \quad C^a_{ib} = -C^a_{bi} = D^a_{ib}, \quad C^i_{kl} = E^i_{kl},
$$

(7.55)

all other components being zero. Let us also introduce rectangular $p \times n$ matrices $T_a$ by $(T_a)^j_i = E^j_{ac}$ and the $n \times p$ matrices $\bar{T}_a$ by $(\bar{T}_a)^b_i = -D^b_{ia}$. Then we can define $N \times N$ matrices $C_A = (C_a, C_i)$

$$
C_a = \begin{pmatrix} 0 & \bar{T}_a \\ T_a & 0 \end{pmatrix}, \quad C_i = \begin{pmatrix} D_i & 0 \\ 0 & F_i \end{pmatrix},
$$

(7.56)

so that $(C_A)^B_C = C^B_{AC}$.

Then one can prove the following [7]:

**Theorem 1** The matrices $C_A$ generate the adjoint representation of a Lie algebra $\mathcal{G}$ with the structure constants $C^A_{BC}$, that is,

$$
[C_A, C_B] = C^C_{AB}C_C,
$$

(7.57)

For the lack of a better name we call the algebra $\mathcal{G}$ the curvature algebra. As it will be clear from the next section it is a subalgebra of the total isometry algebra of the symmetric space. It should be clear that the holonomy algebra $\mathcal{H}$ is the subalgebra of the curvature algebra $\mathcal{G}$. The curvature algebra $\mathcal{G}$ is compact; it is a direct sum of two ideals, $\mathcal{G} = \mathcal{G}_0 \oplus \mathcal{G}_s$, an Abelian center $\mathcal{G}_0$ of dimension $n_0$ and a semi-simple algebra $\mathcal{G}_s$ of dimension $p + n_s$.

Next, we define a symmetric nondegenerate $N \times N$ matrix

$$
(\gamma_{AB}) = \begin{pmatrix} \delta_{ab} & 0 \\ 0 & \beta_{ik} \end{pmatrix}.
$$

(7.58)

This matrix and its inverse $\gamma^{AB}$ will be used to lower and to raise the capital Latin indices.

### 7.3.2 Killing Vector Fields

We will use extensively the isometries of the symmetric space $M$. We follow the approach developed in [7, 8, 10, 13]. The generators of isometries are the Killing vector fields $\xi$. The set of all Killing vector fields forms a representation
of the isometry algebra, the Lie algebra of the isometry group of the manifold $M$. We define two subspaces of the isometry algebra. One subspace is formed by Killing vectors (called translations) satisfying the initial conditions $\nabla_\mu \xi^\nu |_{x=x'} = 0$, and another subspace is formed by Killing vectors (called rotations) satisfying the initial conditions $\xi^\nu |_{x=x'} = 0$.

One can easily show that a basis of translations can be chosen as

$$P_a = \left( \sqrt{K} \cot \sqrt{K} \right)^b_a \frac{\partial}{\partial y^b}. \tag{7.59}$$

where $K = (K^a_b)$ is a matrix defined by

$$K^a_b = R^a_{cbd} y^c y^d. \tag{7.60}$$

We can also show that the vector fields

$$L_i = -D^b_{ia}y^a \frac{\partial}{\partial y^b} \tag{7.61}$$

define $p$ linearly independent rotations. By adding the trivial Killing vectors for flat subspaces we find that the number of independent rotations is $p + n_0 n_s + n_0(n_0 - 1)/2$. We introduce the following notation $(\xi_A) = (P_a, L_i)$.

By using the explicit form of the Killing vector fields obtained above [7] one can prove the following theorem.

**Theorem 2** The Killing vector fields $\xi_A$ form a representation of the curvature algebra $G$

$$[\xi_A, \xi_B] = C^C_{AB} \xi_C. \tag{7.62}$$

Notice that they do not generate the complete isometry algebra of the symmetric space $M$. The curvature algebra $G$ introduced in the previous section is a subalgebra of the total isometry algebra. It is clear that the Killing vector fields $L_i$ form a representation of the holonomy algebra $H$, which is the isotropy algebra of the semi-simple submanifold $M_s$, and a subalgebra of the total isotropy algebra of the symmetric space $M$.

### 7.3.3 Homogeneous Vector Bundles

Let $h^a_b$ be the projection to the subspace $T_x M_s$ of the tangent space and

$$q^a_b = \delta^a_b - h^a_b \tag{7.63}$$
be the projection tensor to the flat subspace $\mathbb{R}^{r_0}$. Since the curvature exists only in the semi-simple submanifold $M_s$, the components of the curvature tensor $R_{abcd}$, as well as the tensors $E_{ab}^i$, are non-zero only in the semi-simple subspace $T_xM_s$. Then

$$R_{abcd}q^a_e = R_{ab}q^a_e = E_{ab}^i q^a_e = D_{i b}^a q^b_e = D_{i b}^a q^a_e = 0.$$  

Equation (7.43) imposes strong constraints on the curvature of the homogeneous bundle $W$. We define

$$\mathcal{B}_{ab} = R_{cd}^{YM} q^c_a q^d_b, \quad \mathcal{E}_{ab} = R_{cd}^{YM} h^c_b h^d_b,$$  

so that

$$\mathcal{R}_{ab}^{YM} = \mathcal{E}_{ab} + \mathcal{B}_{ab}.$$  

Then, from eq. (7.43) we obtain

$$[\mathcal{B}_{ab}, \mathcal{B}_{cd}] = [\mathcal{B}_{ab}, \mathcal{E}_{cd}] = 0,$$  

and

$$[\mathcal{E}_{cd}, \mathcal{E}_{ab}] - R^{f ac d} \mathcal{E}_{f b} - R^{f b c d} \mathcal{E}_{a f} = 0.$$  

This means that $\mathcal{B}_{ab}$ takes values in an Abelian ideal of the gauge algebra $G_{YM}$ and $\mathcal{E}_{ab}$ takes values in the holonomy algebra. More precisely, eq. (7.68) is only possible if the holonomy algebra $\mathcal{H}$ is an ideal of the gauge algebra $G_{YM}$. Thus, the gauge group $G_{YM}$ must have a subgroup $Z \times H$, where $Z$ is an Abelian group and $H$ is the holonomy group.

Let $X_{ab}$ be the generators of the orthogonal algebra $SO(n)$ is some representation $X$. Then the matrices $T_i = -\frac{1}{2} D_{i b}^a X_{b a}$ are the generators of the gauge algebra $G_{YM}$ realizing a representation $T$ of the holonomy algebra $\mathcal{H}$. Next, we can show that the curvature of the homogeneous bundle $W$ is given by

$$\mathcal{R}_{ab}^{YM} = -E_{ab}^i T_i + \mathcal{B}_{ab} = \frac{1}{2} R_{cd}^{YM X_{cd}} + \mathcal{B}_{ab}.$$  

Now, we consider the representation $\Sigma$ of the orthogonal algebra $SO(n)$ defining the spin-tensor bundle $T$ and define the matrices

$$G_{ab} = \Sigma_{ab} \otimes \mathbb{I}_X + \mathbb{I}_\Sigma \otimes X_{ab}.$$  

Obviously, these matrices are the generators of the orthogonal algebra $SO(n)$ in the product representation $\Sigma \otimes X$. Next, the matrices $Y_i = -\frac{1}{2} D_{i b}^a \Sigma^b_{a}$ form a representation $Y$ of the holonomy algebra $\mathcal{H}$ and the matrices

$$\mathcal{R}_i = -\frac{1}{2} D_{i b}^a G^b_{a}$$  

(7.71)
are the generators of the holonomy algebra in the product representation \( \mathcal{R} = Y \otimes T \).

Then the total curvature, that is, the commutator of covariant derivatives, of a twisted spin-tensor bundle \( \mathcal{V} \) is

\[
\mathcal{R}_{ab} = -E^i_{ab}\mathcal{R}_i + \mathcal{B}_{ab} = \frac{1}{2} R^{cd}_{ab} G_{cd} + \mathcal{B}_{ab}.
\]

(7.72)

### 7.3.4 Twisted Lie Derivatives

Let \( \varphi \) be a section of a twisted homogeneous spin-tensor bundle \( \mathcal{T} \). Let \( \xi_A \) be the basis of Killing vector fields. Then the covariant (or generalized, or twisted) Lie derivative of \( \varphi \) along \( \xi_A \) is defined by

\[
\mathcal{L}_A \varphi = \left( \xi_A^\mu \nabla_\mu + \frac{1}{2} \xi_A^a \mathcal{G}_{ab} \right) \varphi.
\]

(7.73)

One can prove the theorem \([12, 13]\).

**Theorem 3** The operators \( \mathcal{L}_A \) satisfy the commutation relations

\[
[\mathcal{L}_A, \mathcal{L}_B] = C^{C AB} \mathcal{L}_C + \mathcal{B}_{AB},
\]

(7.74)

where

\[
\mathcal{B}_{AB} = \begin{pmatrix} \mathcal{B}_{ab} & 0 \\ 0 & 0 \end{pmatrix}.
\]

(7.75)

The operators \( \mathcal{L}_A \) form an algebra that is a direct sum of a nilpotent ideal and a semisimple algebra. For the lack of a better name we call this algebra *gauged curvature algebra* and denote it by \( \mathcal{G}_{\text{gauge}} \).

Now, let us define the operator

\[
\mathcal{L}^2 = \gamma^{AB} \mathcal{L}_A \mathcal{L}_B
\]

(7.76)

and the Casimir operator of the holonomy group

\[
\mathcal{R}^2 = \frac{1}{4} R_{abcd} G_{ab} G_{cd}.
\]

(7.77)

Then one can prove that \([13]\).

**Theorem 4** The Laplacian \( \Delta \) acting on sections of a twisted spin-tensor bundle \( \mathcal{V} \) over a symmetric space has the form

\[
\Delta = \mathcal{L}^2 - \mathcal{R}^2.
\]

(7.78)
7.3.5 Geometry of the Curvature Group

Let \( G_{\text{gauge}} \) be the gauged curvature group and \( H \) be its holonomy subgroup. Both these groups have compact algebras. However, while the holonomy group is always compact, the curvature group is, in general, a product of a nilpotent group, \( G_0 \), and a semi-simple group, \( G_s \). \( G_{\text{gauge}} = G_0 \times G_s \). The semi-simple group \( G_s \) is a product \( G_s = G_+ \times G_- \) of a compact \( G_+ \) and a non-compact \( G_- \) subgroups.

Let \( \xi_A \) be the basis Killing vectors, \( k^A \) be the canonical coordinates on the curvature group \( G \) and \( \xi(k) = k^A \xi_A \). The canonical coordinates are exactly the normal coordinates on the group defined above. Let \( C_A \) be the generators of the curvature group in adjoint representation and \( C(k) = k^A C_A \).

Let \( X = (X_A^M) \) be the matrix defined by
\[
X = \frac{C(k)}{1 - \exp[-C(k)]},
\] (7.79)
and \( X_A \) be the vector fields on the group \( G \) defined by
\[
X_A = X_A^M \frac{\partial}{\partial k^M}.
\] (7.80)

Then one can show that [13] the vector fields \( X_A \) form a representation of the curvature algebra \( G \)
\[
[X_A, X_B] = C_{ABC} X_C.
\] (7.81)
The vector fields \( X_A \) are nothing but the right-invariant vector fields.

Since we will actually be working with the gauged curvature group, we introduce now the operators (covariant right-invariant vector fields) \( J_A \) by
\[
J_A = X_A - \frac{1}{2} B_{AB} k^B.
\] (7.82)
Then we show [13] that the operators \( J_A \) form the following algebra
\[
[J_A, J_B] = C_{ABC} J_C + B_{AB}.
\] (7.83)
Thus, the operators \( J_A \) form a representation of the gauged curvature algebra \( G_{\text{gauge}} \).

Now, let \( L_A \) be the Lie derivatives and \( L(k) = k^A L_A \). Then we find [13]
\[
J_A \exp[L(k)] = \exp[L(k)] L_A.
\] (7.84)
Notice that \( J_A \) are first order differential operators with respect to \( k^A \), whereas \( L_A \) are first-order partial differential operators with respect to the coordinates \( x \) acting on sections of the bundle \( \mathcal{V} \).
7.3.6 Heat Kernel on the Curvature Group

Now, let us define the operator

\[ J^2 = \gamma^{AB} J_A J_B \]  

and the invariant (scalar curvature of the curvature group)

\[ R_G = -\frac{1}{4} \gamma^{AB} C^{AD} C^{BC}. \]

Then by using the properties of the right-invariant vector fields \( J_A \) one can find the heat kernel of the operator \( J^2 \) on the curvature group \( G \).

**Theorem 5** Let \( \Phi(t; k) \) be a function on the curvature group defined in canonical coordinates \( k^A \) by

\[
\Phi(t; k) = (4\pi t)^{-N/2} \left[ \det_{TM} \left( \frac{\sinh(tB)}{tB} \right) \right]^{-1/2} \left[ \det_G \left( \frac{\sinh[C(k)/2]}{C(k)/2} \right) \right]^{-1/2} \times \exp \left( -\frac{1}{4t} \langle k, \gamma tB \coth(tB) k \rangle + \frac{1}{6} R_G t \right),
\]

where \( \langle u, \gamma v \rangle = \gamma^{AB} u^A v^B \) is the inner product on the algebra \( G \). Then \( \Phi(t; k) \) satisfies the heat equation

\[ \partial_t \Phi = J^2 \Phi, \]

and the initial condition

\[ \Phi(0; k) = \gamma^{-1/2} \delta(k), \]

where \( \gamma = \det \gamma_{AB} \).

In the following we will complexify the gauged curvature group in the following sense. We extend the canonical coordinates \( (k^A) = (p^a, \omega^i) \) to the whole complex Euclidean space \( \mathbb{C}^N \). Then all group-theoretic functions introduced above become analytic functions of \( k^A \) possibly with some poles on the real section \( \mathbb{R}^N \) for compact groups. In fact, we replace the actual real slice \( \mathbb{R}^N \) of \( \mathbb{C}^N \) with an \( N \)-dimensional subspace \( \mathbb{R}^N_{\text{reg}} \) in \( \mathbb{C}^N \) obtained by rotating the real section \( \mathbb{R}^N \) counterclockwise in \( \mathbb{C}^N \) by \( \pi/4 \). That is, we replace each coordinate \( k^A \) by \( e^{\pi i/4} k^A \). In the complex domain the group becomes non-compact. We call this procedure the **decompactification**. If the group is compact, or has a compact subgroup, then this plane will cover the original group infinitely many times.
Since the metric \((\gamma_{AB}) = \text{diag}(\delta_{ab}, \beta_{ij})\) is not necessarily positive definite, (actually, only the metric of the holonomy group \(\beta_{ij}\) is non-definite) we analytically continue the function \(\Phi(t; k)\) in the complex plane of \(t\) with a cut along the negative imaginary axis so that \(-\pi/2 < \arg t < 3\pi/2\). Thus, the function \(\Phi(t; k)\) defines an analytic function of \(t\) and \(k^A\). For the purpose of the following exposition we shall consider \(t\) to be real negative, \(t < 0\). This is needed in order to make all integrals convergent and well defined and to be able to do the analytical continuation.

As we will show below, the singularities occur only in the holonomy group. This means that there is no need to complexify the coordinates \(p^a\). Thus, in the following we assume the coordinates \(p^a\) to be real and the coordinates \(\omega^j\) to be complex, more precisely, to take values in the \(p\)-dimensional subspace \(\mathbb{R}_{\text{reg}}^p\) of \(\mathbb{C}^p\) obtained by rotating \(\mathbb{R}^p\) counterclockwise by \(\pi/4\) in \(\mathbb{C}^p\). That is, we have \(\mathbb{R}^N_{\text{reg}} = \mathbb{R}^n \times \mathbb{R}_{\text{reg}}^p\).

This procedure (that we call a regularization) with the nonstandard contour of integration is necessary for the convergence of the integrals below since we are treating both the compact and the non-compact symmetric spaces simultaneously. Remember, that, in general, the nondegenerate diagonal matrix \(\beta_{ij}\) is not positive definite. The space \(\mathbb{R}^p_{\text{reg}}\) is chosen in such a way to make the Gaussian exponent purely imaginary. Then the indefiniteness of the matrix \(\beta\) does not cause any problems. Moreover, the integrand does not have any singularities on these contours. The convergence of the integral is guaranteed by the exponential growth of the sine for imaginary argument. These integrals can be computed then in the following way. The coordinates \(\omega^j\) corresponding to the compact directions are rotated further by another \(\pi/4\) to imaginary axis and the coordinates \(\omega^j\) corresponding to the non-compact directions are rotated back to the real axis. Then, for \(t < 0\) all the integrals below are well defined and convergent and define an analytic function of \(t\) in a complex plane with a cut along the negative imaginary axis.

### 7.3.7 Heat Trace

Now, by using the heat kernel \((7.87)\) of the operator \(J^2\) on the curvature group obtained above, the relation \((7.78)\) of the Laplacian and the operator \(L^2\), and the property \((7.84)\) one can find the following integral representation of the heat semi-group of the Laplace-type operator \([13]\).

**Theorem 6** Let \(L = -\Delta + Q\) be the Laplace type operator acting on sections of a homogeneous twisted spin-tensor vector bundle over a symmetric space. Then the
heat semigroup $\exp(-tL)$ can be represented in form of an integral

$$
\exp(-tL) = (4\pi t)^{-N/2} \left[ \det_T \left( \frac{\sinh(tB)}{tB} \right) \right]^{-1/2} \exp \left( -tQ - tR^2 + \frac{1}{6} R_G t \right) \\
\times \int_{\mathbb{R}^N_{\text{reg}}} dk \gamma^{1/2} \left[ \det_G \left( \frac{\sinh[C(k)/2]}{C(k)/2} \right) \right]^{1/2} \\
\times \exp \left\{ -\frac{1}{4t} \langle k, \gamma tB \coth(tB)k \rangle \right\} \exp[\mathcal{L}(k)]. \quad (7.90)
$$

The heat trace can be obtained by acting by the heat semigroup $\exp(-tL)$ on the delta-function. To be able to use this integral representation we need to compute the action of the isometries $\exp[\mathcal{L}(k)]$ on the delta-function.

Let $\omega^j$ be the canonical coordinates on the holonomy group $H$ and $(k^A) = (p^a, \omega^i)$ be the natural splitting of the canonical coordinates on the curvature group $G$. Then we can prove that \[13\]

$$
[\exp[\mathcal{L}(k)]\delta(x,x')]^{\text{diag}} = \left[ \det_T \left( \frac{\sinh[D(\omega))/2]}{D(\omega)/2} \right) \right]^{-1} \exp[\mathcal{R}(\omega)]\delta(p), \quad (7.91)
$$

where $D(\omega) = \omega^j D_j$ and $\mathcal{R}(\omega) = \omega^j \mathcal{R}_j$.

We implicitly assumed that there are no closed geodesics and that the equation of closed orbits of isometries has a unique solution. On compact symmetric spaces this is not true: there are infinitely many closed geodesics and infinitely many closed orbits of isometries. However, these global solutions, which reflect the global topological structure of the manifold, will not affect our local analysis. In particular, they do not affect the asymptotics of the heat kernel. That is why, we have neglected them here. This is reflected in the fact that the Jacobian in \[7.91\] can become singular when the coordinates of the holonomy group $\omega^j$ vary from $-\infty$ to $\infty$. Note that the exact results for compact symmetric spaces can be obtained by an analytic continuation from the dual noncompact case when such closed geodesics are absent \[19\]. That is why we proposed above to complexify our holonomy group. If the coordinates $\omega^j$ are complex taking values in the subspace $\mathbb{R}^{\text{reg}}_n$ defined above, then the equation of closed orbits should have a unique trivial solution and the Jacobian is an analytic function. It is worth stressing once again that the canonical coordinates cover the whole group except for a set of measure zero. Also a compact subgroup is covered infinitely many times.
Now by using the above lemmas and the theorem we can compute the heat trace. We define the invariant (scalar curvature of the holonomy group)

$$ R_H = -\frac{1}{4} \beta^{ij} F_i^k F_j^l. \quad (7.92) $$

**Theorem 7**  
The heat trace of the operator $L$ has the form

$$ \Theta(t) = (4\pi t)^{-n/2} \int_{M} dvol_{\mathcal{V}} \left[ \det_{TM} \left( \frac{\sinh(t\mathcal{B})}{t\mathcal{B}} \right) \right]^{-1/2} \exp \left\{ \left( \frac{1}{8} R + \frac{1}{6} R_H - R^2 - Q \right) t \right\} $$

$$ \times \int_{\mathbb{R}^{reg}} \frac{d\omega}{(4\pi t)^{p/2}} \beta^{1/2} \exp \left\{ -\frac{1}{4t} \langle \omega, \beta \omega \rangle \right\} \cosh \left[ R(\omega) \right] $$

$$ \times \left[ \det_{\mathcal{H}} \left( \sinh \left[ \frac{F(\omega)}{2} \right] / F(\omega)/2 \right) \right]^{1/2} \left[ \det_{TM} \left( \frac{\sinh \left[ D(\omega)/2 \right]}{D(\omega)/2} \right) \right]^{-1/2}, \quad (7.93) $$

where $\beta = \det \beta_{ij}$, $\langle \omega, \beta \omega \rangle = \beta_{ij} \omega^i \omega^j$ and $F(\omega) = \omega^j F_i$.

This equation can be used now to generate all heat kernel coefficients $A_k$ for any locally symmetric space simply by expanding it in a power series in $t$. By using the standard Gaussian averages one can obtain now all heat kernel coefficients in terms of traces of various contractions of the matrices $D^a_{ib}$ and $F^j_{ik}$ with the matrix $\beta^{ik}$. All these quantities are curvature invariants and can be expressed directly in terms of the Riemann tensor.

### 8 Low Energy Effective Action in Quantum General Relativity

We can apply now the obtained results for the heat trace to compute the low-energy one-loop effective action in quantum general relativity given by (2.71). In the Euclidean formulation we have

$$ \Gamma(1) = \frac{1}{2} \left( \log \det \hat{L} - 2 \log \det F \right), \quad (8.1) $$

which, in the zeta regularization takes the form

$$ \Gamma(1) = -\frac{1}{2} \left( \zeta_F'(0) - 2 \zeta_F(0) \right), \quad (8.2) $$
where \( \zeta_L(s) \) and \( \zeta_F(s) \) are the zeta functions of the graviton operator \( \hat{L} \) and the ghost operator \( F \). Now, let us define the total zeta function by

\[
\zeta_{GR}(s) = \zeta_L(s) - 2\zeta_F(s).
\] (8.3)

Then the effective action is

\[
\Gamma(1) = -\frac{1}{2} \zeta'_{GR}(0).
\] (8.4)

Next, by using the definition of the zeta function we obtain

\[
\zeta_{GR}(s) = \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt \ t^{s-1} e^{t\lambda} \Theta_{GR}(t),
\] (8.5)

where

\[
\Theta_{GR}(t) = \Theta_L(t) - 2\Theta_F(t),
\] (8.6)

\( \mu \) is a renormalization parameter introduced to preserve dimensions and \( \Theta_L(t) \) and \( \Theta_F(t) \) are the heat traces of the operators \( \hat{L} \) and \( F \). Here \( \lambda \) is a sufficiently large negative infrared cutoff parameter introduced to regularize any infrared divergences which are present if the operators \( \hat{L} \) and \( F \) have negative modes. The parameter \( \lambda \) should be set to zero at the end of the calculations.

Now, notice that both operators \( \hat{L} \) and \( F \) are of Laplace type, that is, \(-\Delta + Q\), acting on pure tensor bundles; so, there is no Yang-Mills group here, \( \tilde{R}_{ab} = E_{ab} = B_{ab} = 0 \). The operator \( \hat{L} \) acts on the bundle \( \mathcal{T}(2) = T^*M \otimes T^*M \) of symmetric two-tensors and the operator \( F \) acts on sections of the tangent bundle \( \mathcal{T}(1) = TM \). The potentials, \( Q \), for both operators are obviously read off from their definition

\[
(Q_{(1)})^{ab}_{\phantom{ab}c}_{d} = -R_{a \phantom{b}b}^{cd},
\] (8.7)

\[
(Q_{(2)})_{cd}^{ab} = -2R_{a \phantom{b}b}^{c \phantom{d}d} - 2\delta_{(a}^{(c} R_{b)_{d}} + R_{cd} g^{ab} + \frac{2}{n-2} g_{cd} R^{ab} - \frac{1}{(n-2)} g_{cd} g^{ab} R + \delta_{(a}^{a} \delta_{b)_{d}} (R - 2\Lambda).
\] (8.8)

The generators of the orthogonal group \( SO(n) \) in the vector representation are

\[
(\Sigma_{(1)ab})_{c}^{d} = 2\delta_{c [a} g_{b]d}.
\] (8.9)
The generators of the orthogonal group $SO(n)$ in the symmetric 2-tensor representation are
\[
(\Sigma_{(2)}^{(e)}\mathcal{S})_{cd}^{ef} = -4\delta^{(e)[ab]}(d\delta^{f)}c).
\] (8.10)

The generators of the holonomy group are
\[
R^{(1)}i = D_i,
\] (8.11)
and
\[
R^{(2)}i = -2D_i \lor I_{(1)},
\] (8.12)
which, in component language, reads
\[
(R^{(1)}i)_a^b = D^a_{ib},
\] (8.13)
and
\[
(R^{(2)}i)_{cd}^{ab} = -2D^{(a}_{(i}(d\delta^{b)}{c)}).
\] (8.14)

The Casimir operators are
\[
(R^2_{(1)})^a_b = -R^a_b,
\] (8.15)
and
\[
(R^2_{(2)})_{cd}^{ab} = 2R^{(a}_{d}(b)_{c} - 2\delta^{(a}_{(c}R^{b)}{d)}).
\] (8.16)

By using the results for the heat traces described above we obtain the total heat trace
\[
\Theta_{GR}(t) = (4\pi t)^{-n/2} \int_{M} d\text{vol} \exp\left\{\left(\frac{1}{8}R + \frac{1}{6}R_H\right)t\right\} \times \int_{\mathbb{R}^n_{\text{reg}}} \frac{d\omega}{(4\pi t)^{n/2}} \beta^{1/2} \exp\left\{-\frac{1}{4t} \langle \omega, \beta\omega \rangle\right\} \Psi(t; \omega) \times \left[\det_H\left(\frac{\sinh[F(\omega)/2]}{F(\omega)/2}\right)\right]^{1/2} \left[\det_{TM}\left(\frac{\sinh[D(\omega)/2]}{D(\omega)/2}\right)\right]^{-1/2}
\] (8.17)

where
\[
\Psi(t; \omega) = \exp[-t(R - 2\Lambda)] \text{tr}_T^{(2)} \exp(tV^{(2)}) \cosh[2D(\omega) \lor I_{(1)}] - 2\text{tr}_T^{(1)} \exp(tV^{(1)}) \cosh[D(\omega)],
\] (8.19)
and the matrices $V_{(1)}$ and $V_{(2)}$ are defined by

\[
\begin{align*}
(V_{(1)})^a_b &= 2R^a_b, \\
(V_{(2)})^{ab}_{cd} &= 4\delta^a_c (e^{R^b_d}) - R_{cd}g^{ab} - \frac{2}{n-2}g_{cd}R^{ab} + \frac{1}{(n-2)}g_{cd}g^{ab}R.
\end{align*}
\]

One can go further and compute the function $\Psi(t; \omega)$ by finding the eigenvalues of the endomorphisms $V_{(1)}$ and $V_{(2)}$. However, we will not do it here and leave the answer in the form (8.19). By using the obtained heat trace one can compute now the zeta function and then the effective action. We would like to stress two points here. First of all, quantum general relativity is a non-renormalizable theory. Therefore, even if one gets a final result via the zeta-regularization one should not take it too seriously. Secondly, our results for the heat kernel and, hence, for the effective action are essentially non-perturbative. They contain an infinite series of Feynmann diagrams and cannot be obtained in any perturbation theory. One could try now to use this result for the analysis of the ground state in quantum gravity. But this is a rather ambitious program for the future.

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