GLOBAL EXISTENCE AND BLOW-UP RESULTS FOR A NONLINEAR MODEL FOR A DYNAMIC SUSPENSION BRIDGE

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Abstract. The paper deals with global existence and blow-up results for a class of fourth-order wave equations with nonlinear damping term and superlinear source term with the coefficient depends on space and time variable. In the case the weak solution is global, we give information on the decay rate of the solution. In the case the weak solution blows up in finite time, estimate the lower bound and upper bound of the lifespan of the blow-up solution, and also estimate the blow-up rate. Finally, if our problem contains an external vertical load term, a sufficient condition is also established to obtain the global existence and general decay rate of weak solutions.

1. Introduction. In this paper, we study the initial value problem of following fourth-order suspension bridge equation with nonlinear damping and source terms

\[ u_{tt} - \Delta^2 u + au + \mu |u|^{q-2} u = K(x,y,t) |u|^{p-2} u, \quad (x,y,t) \in \Omega \times (0,\infty), \]

\[ u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), \quad (x,y) \in \Omega, \]

with boundary conditions

\[
\begin{align*}
    u(0,y,t) &= u_{xx}(0,y,t) = u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, \quad y \in (-l,l), \\
    u_{yy}(x,\pm l,t) + \sigma u_{xx}(x,\pm l,t) &= 0, \quad x \in (0,\pi), \\
    u_{yyy}(x,\pm l,t) + (2-\sigma) u_{xxy}(x,\pm l,t) &= 0, \quad x \in (0,\pi),
\end{align*}
\]

where \( \Omega = (0,\pi) \times (-l,l) \subset \mathbb{R}^2, \) \( q \geq 2, \) \( \mu > 0, \) \( p \geq 2, \) \( \sigma \in (0,\frac{1}{2}), \) \( a = a(x,y) \) is a sign-changing and bounded measurable function, \( K(x,y,t) \) is given function satisfying conditions specified later. The initial data \( u_0 \) and \( u_1 \) belong to suitable spaces.

Problem (1)-(3) arises from the physical model for the nonlinear dynamic suspension bridge. Suspension bridges are long, slender, and flexible structures that are very sensitive to dynamic actions like wind, vehicles, etc, which potentially cause a variety of instability phenomena. After the Tacoma Narrows Bridge failure in 1940,
major efforts were made to understand the dynamic of suspension bridges, including the effects of the stiffening girder. Hence reliable mathematical models appear necessary for a precise description of the instability and of the structural behavior of suspension bridges under the action of external force including gravity. On one hand, realistic models appear too complicated to give helpful hints when making plans. On the other hand, simplified models do not describe with sufficient accuracy the complex behavior of actual bridges. Based on this, Lazer and McKenna [11] proposed to consider the following equation with multiple degrees of freedom

$$\Delta^2 u + c^2 \Delta u + h(u) = 0, \ x \in \mathbb{R}^n,$$

where $h(u) \sim [u + 1]^+ - 1$ describes restoring force due to the hangers and external forces including gravity and $[u]^+$ stands for its positive part.

In other papers [5, 6, 19, 20], this kind of problem was also investigated under the Navier boundary condition. More recently, Ferrero and Gazzola [8] considered the boundaries of a plate $\Omega = (0, \pi) \times (-l, l)$ which represents the roadway of a suspension bridge. Because the edges $x = 0$ and $x = \pi$ connect with the ground, they are assumed to be hinged and then

$$u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0, \ \forall y \in (-l, l),$$

while the edges $y = \pm l$ are free and the boundary conditions there become

$$u_{yy}(x, \pm l) + \sigma u_{xx}(x, \pm l) = u_{yyy}(x, \pm l) + (2 - \sigma) u_{xxy}(x, \pm l) = 0, \ \forall x \in (0, \pi).$$

In order to consider the effect of internal friction on a system, Ferrero and Gazzola [8] studied the following linear damped wave equation

$$u_{tt} + \Delta^2 u + h(x, y) = f(x, y, t), \ (x, y, t) \in \Omega \times (0, T),$$

with the initial boundary data conditions (2) and (3), where $h(\cdot, \cdot, s)$ is a Lipschitz function and increasing with respect to $s \in \mathbb{R}$. They proved the existence and uniqueness of global solutions, but did not give any non-global results due to the linear source term. It is worth mentioning the work on an extensible beam model similar to the model equation in this paper, the so-called polyharmonic Kirchhoff equation was considered in [1], to show the global nonexistence and a priori estimates for the life span of solution.

In order to deal with the above problem, considering the more complex effect of internal friction on the system, Xu et. al. studied the problem (5) in another way. The authors considered the problem

$$u_{tt} - \Delta^2 u + a u + \mu |u_t|^{q-2} u_t = |u|^{p-2} u, \ (x, t) \in \Omega \times (0, \infty),$$

with boundary conditions (2) and initial condition (3). By using potential well method, the authors established some threshold results for the existence of global and blow-up solutions of problem (6). At the arbitrarily high initial energy level, the authors gave explicit conditions for blow-up result with linear weak damping. At the arbitrarily high initial energy level, the authors gave explicit conditions for blow-up result with linear weak damping, and leave some problems unsolved in [30].

We list here some interesting problems at high initial energy level

1. For $q = 2$ and some suitable conditions, we know that the weak solution blows up in finite time. But we are interested in how to estimate the lifespan and blow-up rate of solution.

2. For $q > 2$, does blow-up occur with some suitable condition?
Our paper is an extension of [30] in another way. We replace the nonlinear term \(|u|^{p-2}u\) by \(K(x,y,t)|u|^{p-2}u\). Let us explain in some detail which are our main results. We first state a local existence theorem (Theorem 2.4) that can be established by using standard Faedo-Galerkin method. We omit the proof of local existence. For an evolution equation, like our problem, we have the following questions. With respect to a functional framework and in terms on the initial data, study:

1. Blow-up in finite time: \(T_\infty < \infty\) where \(T_\infty\) is maximal existence time.
2. Global existence: \(T_\infty = \infty\).

In the blow-up case, we need to find or at least is give the estimates for lifespan and blow-up rate. In the latter case, the behavior of the solution as time approaches infinity. Since the potential well was introduced by Sattinger [23] in 1968 the potential well method has become one of the most important methods for studying nonlinear evolution equations and there has been a lot of results [10, 9, 7, 20, 17, 28, 29, 30], where Payne and Sattinger’s work [22] is the most important and typical work on the potential well. But in our model, when the nonlinear term depends on time variable, we think potential well method is not suitable. If we use potential well method, then we need to consider the family of potential well depend on time variable. Therefore, we need more complicated calculations. So we will use another method was first introduce by E. Vitillaro [24]. With the help of this method, under the some suitable conditions, we can prove some priori estimates for the weak solution. Therefore, by using standard continue argument or some differential inequality, we can establish (uniform) bounds, global existence, large-time behavior, decay rate, blow-up rate, lifespan. If we add external vertical load term \(f(x,y,t)\) in RHS of our problem i.e. we consider the problem

\[
\begin{align*}
  u_{tt} - \Delta u + au + \mu u_t &= K(x,y,t)|u|^{p-2}u + f(x,y,t), \quad (x,y,t) \in \Omega \times (0,\infty).
\end{align*}
\]

(7)

We emphasize that the previous methods like potential well method or energy method introduced by E. Vitillaro can not apply in this case (we will explain the reason in Sec. 7). Therefore, that is a challenging problem for us. We will introduce a new method to deal with this kind of problem.

This paper consists of seven sections. Section 2, we give some notations, functional setting and primary results. In Section 3, under some suitable conditions and some restrictions on the initial data, we prove that the solution of (1)-(3) globally and decay to zero when \(t\) tends to infinity. In Section 4, we prove that the weak solution blows up in finite time if initial datas large enough. We also estimate the blow-up rate and lifespan of solution. In Section 5, we focus on the case \(E(0) = E_1\). In Section 6, we prove the blow-up in finite time and global existence of solution with the arbitrarily high initial energy level. Finally, in Section 7, we will add external vertical load term \(f(x,y,t)\) in right-hand side of our problem. After, we give a sufficient condition, in which the initial energy is positive and small, to guarantee the global existence and general decay of weak solutions. In the proof, a suitable Lyapunov functional is constructed.

2. Preliminaries. We begin with some notations on the functional space. We denote the standard \(L^q(\Omega)\) norm by \(|\cdot|_q\) for \(q \in [1,\infty)\), where \(|\cdot|\) denotes the \(L^2(\Omega)\) norm, and the \(H^2(\Omega)\) norm by \(|\cdot|_{H^2} = \sqrt{|\cdot|^2 + |D^2\cdot|^2}\).
In addition, we introduce another function space defined in [8],

\[ H_2^* = H_2^*(\Omega) = \{ u \in H^2(\Omega) : u = 0 \text{ on } \{0, \pi\} \times (-l, l) \}, \]

the dual space of which is defined by \( \mathcal{H}(\Omega) \) and the corresponding duality between them is denoted by \( \langle \cdot, \cdot \rangle \). Clearly, \( H_2^* \) satisfies \( H_2^* \subset H^2(\Omega) \) and is a Hilbert space when endowed with the inner product

\[ \langle u, v \rangle_{H_2^*} = \int_\Omega \Delta u \Delta v + (1 - \sigma) \int_\Omega 2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx}, \forall u, v \in H_2^*. \]

This inner product induces a norm

\[ \| u \|_{H_2^*} = \sqrt{\int_\Omega |\Delta u|^2 + 2(1 - \sigma) \int_\Omega u_{xy}^2 - u_{xx}u_{yy}}, \]

for any \( u \in H_2^* \) which is equivalent to \( \| u \|_{H^2} \) for \( \sigma \in (0, \frac{1}{2}) \) (See [8, Lemma 7]). Moreover, we have a Sobolev embedding inequality for this case.

**Lemma 2.1.** Assume that \( q \in [1, \infty) \). Then for any \( u \in H_2^* \) the inequality

\[ \| u \|_q \leq S_q \| u \|_{H_2^*}, \]

holds, where \( S_q = \left( \frac{q}{2} + \frac{\sqrt{q^2 + 4}}{2} \right) \left( 2\pi I \right)^{\frac{q-2}{2}} \left( \frac{1}{1-\sigma} \right)^{\frac{q}{2}} \).

Let \( \{ \Lambda_i \}_{i=1}^\infty \) be the eigenvalue sequence to the eigenvalue problem

\[
\begin{cases}
\Delta^2 u = \Lambda_i u, \quad (x, y) \in \Omega, \\
u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0, \quad y \in (-l, l), \\
u_{yy}(x, \pm l) + \sigma u_{xx}(x, \pm l) = u_{yy}(x, \pm l) + (2 - \sigma) u_{xx}(x, \pm l) = 0, \quad x \in (0, \pi),
\end{cases}
\]

which has been solved in [8]. Particularly, \( \Lambda_1 < 1 \) and we have an elementary result as follows.

**Lemma 2.2.** Assume \(-\Lambda_1 < a_1 \leq a \leq a_2 \). Then for any \( u \in H_2^* \) there holds

\[ A_1 \| u \|_{H_2^*}^2 \leq \| u \|_{H_2^*}^2 + (au, u)_2 \leq A_2 \| u \|_{H_2^*}^2 \]

where \( (\cdot, \cdot)_2 \) is the \( L^2 \) inner product and \( A_1, A_2 \) are given by

\[ A_1 = \begin{cases} 1 + \frac{a_1}{\Lambda_1}, & \text{if } a_1 < 0, \\
1, & \text{if } a_1 \geq 0 
\end{cases} \quad \text{and} \quad A_2 = \begin{cases} 1, & \text{if } a_2 < 0, \\
1 + \frac{a_2}{\Lambda_2}, & \text{if } a_2 \geq 0. \end{cases} \]

**Remark 1.** From Lemma 2.2, we can define new inner product in \( H_2^*(\Omega) \) as follows

\[ (u, v)_* = (u, v)_{H_2^*} + (au, v), \forall u, v \in H_2^*(\Omega). \]

On \( H_2^*(\Omega) \), two norms \( v \mapsto \| v \|_{H_2^*} \) and \( v \mapsto \| v \|_* = \sqrt{(v, v)_*} \) are equivalent.

Before going further, let us give the definition of weak solutions.

**Definition 2.3.** A function \( u \) called a weak solution of the Problem (1)-(3) on \( (0, T) \) if \( u \in C([0, T]; H_2^*(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap C^2([0, T]; H(\Omega)) \) with \( u_t \in C((0,T]; L^q(\Omega)) \) satisfies (1) in the distribution sense, i.e.,

\[ \langle u''(t), v \rangle + (u(t), v)_* + \mu \langle |u'(t)|^{q-2}u'(t), v \rangle_2 = \langle K(t) |u(t)|^{p-2}u(t), v \rangle_2, \]
for all test function \( v \in H^2_s(\Omega) \) and for almost every \( t \in (0, T) \), together with the following initial conditions

\[
u(0) = u_0, \quad u'(0) = u_1.\]

Next, we recall the local existence and uniqueness result for the solution of the Problem (1)-(3). For the proof, we refer the reader to [30, Theorem 2.5].

**Theorem 2.4.** Suppose that \((u_0, u_1) \in H^2_s(\Omega) \times L^2(\Omega), \ K \in C^1(\overline{\Omega} \times [0, \infty))\), and \( \mu > 0, -\Lambda_1 < a_1 \leq a \leq a_2 \), then there exist \( T > 0 \) and a unique solution \( u \) of the Problem (1)-(3) over \([0, T]\). Moreover, if

\[
T_\infty = \sup \{T > 0 : u = u(t) \text{ exists on } [0, T]\} < \infty,
\]

then

\[
\lim_{t \nearrow T_\infty} \left( \|u(t)\|_s^2 + \|u'(t)\|^2 \right) = \infty.
\]

In the rest paper, we always assume that \((u_0, u_1) \in H^2_s(\Omega) \times L^2(\Omega), \ K \in C^1(\overline{\Omega} \times [0, \infty))\), and \( \mu > 0, -\Lambda_1 < a_1 \leq a \leq a_2 \). We also define the total energy functional

\[
E(t) = \frac{1}{2} \|u'(t)\|^2 + J(t),
\]

where

\[
J(t) = \frac{1}{2} \|u(t)\|_s^2 - \frac{1}{p} \int_{\Omega} K(x, y, t)|u(x, y, t)|^p \, dx \, dy,
\]

for \( t \in [0, T_\infty) \) and \( T_\infty \) is maximal existence time of solution.

We also define the functional

\[
I(t) = \|u(t)\|_s^2 - \int_{\Omega} K(x, y, t)|u(x, y, t)|^p \, dx \, dy, \quad \forall t \in [0, T_\infty).
\]

Therefore, from definition of the functional \( E, J \) and \( I \), we have

\[
E(t) = \frac{1}{2} \|u'(t)\|^2 + J(t) = \frac{1}{2} \|u'(t)\|^2 + \frac{p-2}{2p} \|u(t)\|_s^2 + \frac{I(t)}{p}, \quad \forall t \in [0, T_\infty).
\]

3. **Global existence and decay solutions at low energy level.** In this section, we make the following assumptions:

\([K] \quad K, K_s \in C(\overline{\Omega} \times [0, \infty))\) such that

i. \( K_1 \leq K(x, y, t) \leq K_2 \) for all \((x, y, t) \in \overline{\Omega} \times [0, \infty)\) with \( K_1 \) and \( K_2 \) are positive constants;

ii. \( K(x, y, t) \geq 0 \) for all \((x, y, t) \in \overline{\Omega} \times [0, \infty)\).

Firstly, we have the following lemma.

**Lemma 3.1.** Assume that \([K]\) holds. If \( u \) is the uniqueness local solution of Problem (1)-(3), then we have the following identity

\[
E'(t) = -\mu \|u'(t)\|_q^q - \frac{1}{p} \int_{\Omega} K_1(x, y, t)|u(x, y, t)|^p \, dx \, dy
\]

\[
\leq -\mu \|u'(t)\|_q^q,
\]

for all \( t \in [0, T_\infty) \).

**Proof of Lemma 3.1.** By testing (1) with \( u_t \), we have

\[
E'(t) = -\mu \|u'(t)\|_q^q - \frac{1}{p} \int_{\Omega} K_1(x, y, t)|u(x, y, t)|^p \, dx \, dy \leq -\mu \|u'(t)\|_q^q.
\]

Lemma 3.1 is proved.
Remark 2. From Lemma 3.1, we deduce that $E(t) \leq E(0)$ for all $t \in [0, T_\infty)$. This fact implies
\[
\frac{1}{2} \| u'(t) \|^2 + \frac{1}{2} \| u(t) \|^2 \leq E(0) + \frac{1}{p} \int_\Omega K(x, y, t) |u(x, y, t)|^p \, dx \, dy \lesssim \| u(t) \|^p_p.
\] (13)
So if the weak solution blows up in finite time, then $\lim_{t \nearrow T_\infty} \| u(t) \|_p = \infty$. Combine with Lemma 2.1, we know that $\lim_{t \nearrow T_\infty} \| u(t) \|_* = \infty$.

On the other hand, from definition of the functional $E$, we have
\[
E(t) = \frac{1}{2} \| u'(t) \|^2 + \frac{1}{2} \| u(t) \|^2 - \frac{1}{p} \int_\Omega K(x, y, t) |u(x, y, t)|^p \, dx \, dy
\]
\[
\geq \frac{1}{2} \| u(t) \|^2 + \frac{K_2}{p} \| u(t) \|^p_p
\]
\[
\geq \frac{1}{2} \| u(t) \|^2 - \frac{K_2 S_p^p}{p} \| u(t) \|^p_p
\]
\[
= H (\| u(t) \|_*) ,
\] (14)
where $S_p = \sup_{u \in H^2(\Omega) \setminus \{0\}} \frac{\| u \|_p}{\| u \|_{*p}} > 0$ and $H : [0, \infty) \rightarrow \mathbb{R}$ is the function defined by
\[
H (\lambda) = \frac{\lambda^2}{2} - \frac{K_2 S_p^p}{p} \lambda^p , \forall \lambda \in [0, \infty) .
\] (15)
Before stating our main results, we give a following useful lemma. The proof of this lemma is not difficult, so we omit it.

Lemma 3.2. Let
\[
H (\lambda) = \frac{\lambda^2}{2} - \frac{K_2 S_p^p}{p} \lambda^p , \forall \lambda \in [0, \infty) ,
\]
then we have following properties:
1. $H(0) = 0$ and $\lim_{\lambda \rightarrow \infty} H(\lambda) = -\infty$.
2. The equation $H'(\lambda) = 0$ has a unique positive solution $\lambda = \lambda_0 = K_2^{-\frac{1}{p-2}} S_p^{-\frac{p}{p-2}}$.
3. $H'(\lambda) > 0$ if and only if $\lambda \in (0, \lambda_0)$, $H'(\lambda) < 0$ if and only if $\lambda \in (\lambda_0, \infty)$ and
\[
H(\lambda_0) = \frac{p-2}{2p} K_2^{-\frac{2}{p-2}} S_p^{-\frac{2p}{p-2}} = E_1 .
\]

The following lemma will play an essential role in this paper, and it is similar to the lemma used firstly by E. Vitillaro in [24].

Lemma 3.3. Let $[K]$ holds. Moreover, we also assume that the initial data satisfies $0 \leq E(0) < E_1$. Then, for each weak solution $u$ of Problem (1)-(3), if $\| u_0 \|_* < \lambda_0$ then there exists $\lambda_1 \in (0, \lambda_0)$ such that $\| u(t) \|_* \leq \lambda_1$ for all $t \in [0, T_\infty)$.

Proof of Lemma 3.3. Since $0 \leq E(0) < E_1$, there are constants $\lambda_1 \in (0, \lambda_0)$ and $\lambda_2 \in (\lambda_0, \infty)$ such that
\[
E(0) = H(\lambda_1) = H(\lambda_2) .
\]
From the condition $\| u_0 \|_* < \lambda_0$ and Lemma 3.2, we obtain
\[
H(\lambda_1) = E(0) \geq H(\| u(0) \|_*) = H(\| u_0 \|_*) .
\] (16)
By Lemma 3.2 again, (16) leads to $\|u_0\|_\ast \leq \lambda_1$. We claim that $\|u(t)\|_\ast \leq \lambda_1$ for all $t \in [0, T_\infty)$. Indeed, suppose by contradiction, assume there is $t_\ast \in (0, T_\infty)$ such that $\|u(t_\ast)\|_\ast > \lambda_1$. By the continuity of the function $t \mapsto \|u(t)\|_\ast$, without loss of generality, we may assume that $\|u(t_\ast)\|_\ast = (\lambda_1, \lambda_0)$. By Lemma 3.1 and 3.2, we get

$$E(t_\ast) > H(\|u(t_\ast)\|_\ast) > H(\lambda_1) = E(0),$$

hence, this is a contradiction, because of (3.4). This completes the proof of Lemma 3.3.

Remark 3. In Lemma 3.3, we not need the assumption $E(0) \geq 0$. This condition can be implied by using the assumption $\|u_0\|_\ast < \lambda_0$ and (14).

Let us turn to the global existence of solutions starting with suitable initial data.

**Theorem 3.4.** Assume $[K]$ holds. For any $(u_0, u_1) \in H^2_\ast (\Omega) \times L^2(\Omega)$ satisfies conditions $\|u_0\|_\ast < \lambda_0$ and $0 \leq E(0) < E_1$, the weak solution of Problem (1)-(3) is global.

Proof of Theorem 3.4. By using Lemma 3.3, we have $\|u(t)\|_\ast \leq \lambda_1 < \lambda_0$ for all $t \in [0, T_\infty)$. This fact implies $H(\|u(t)\|_\ast) \geq 0$ for all $t \in [0, T_\infty)$ provided by Lemma 3.2. Furthermore, from Lemma 3.1 and (14), we deduce that

$$E_1 > E(0) \geq E(t) \geq \frac{1}{2}\|u'(t)\|^2 + H(\|u(t)\|_\ast) \geq \frac{1}{2}\|u'(t)\|^2, \forall t \in [0, T_\infty).$$

Hence,

$$\|u'(t)\|^2 + \|u(t)\|_\ast^2 \leq 1, \forall t \in [0, T_\infty),$$

which implies that $T_\infty = \infty$ by the continue principle. This completes our proof.

Remark 4. From Theorem (3.4), we know that

$$u \in C([0, \infty); H^2_\ast (\Omega)) \cap C^1 ([0, \infty); L^2 (\Omega)) \cap L_\infty (0, \infty; H^2_\ast (\Omega)),$$

and $u_t \in C ([0, \infty); L^2 (\Omega)) \cap L^\infty (0, \infty; L^2 (\Omega))$. Indeed, from (12), we get

$$\mu \int_0^t \|u'(s)\|_q^q \,ds \leq E(0) - E(t) \leq E(0), \forall t \in [0, \infty).$$

Therefore, we have $u_t \in L^q (0, \infty; L^q) \cap C ([0, \infty); L^q)$.

Next, we give one useful lemma to estimate the behavior of weak solution. For the proof, we refer the reader to [21].

**Lemma 3.5** (The Nakao inequality). Let $\varphi : [0, \infty) \rightarrow [0, \infty)$ be a bounded function for which there exist constant $\gamma \geq 0$ such that

$$\sup_{t \leq s \leq t+1} \varphi^{1+\gamma} (s) \leq \varphi (t) - \varphi (t + 1), \forall t \in [0, \infty).$$

Then

1. If $\gamma = 0$ then there exist positive constant $\theta > 0$ such that $\varphi (t) \leq \exp(-\theta t)$ for all $t \in [0, \infty)$.

2. If $\gamma > 0$ then $\varphi (t) \leq (1 + t)^{-\gamma}$ for all $t \in [0, \infty)$.

**Theorem 3.6.** Under the same assumptions as in Theorem 3.4, we have the following estimates:
1. If $q = 2$ then

\[ \|u'(t)\|^2 + \|u(t)\|^2 \lesssim E(t) \lesssim \exp(-\gamma t), \quad \forall t \in [0, \infty), \]  

where $\gamma > 0$.

2. If $q > 2$ then

\[ \|u'(t)\|^2 + \|u(t)\|^2 \lesssim E(t) \lesssim (1 + t) \frac{2}{q-2}, \quad \forall t \in [0, \infty). \]  

**Proof of Theorem 3.6.** First, by using Lemma 3.3, we obtain

\[ I(t) = \|u(t)\|^2 - \int_{\Omega} K(x, y, t) |u(x, y, t)|^p \, dx \, dy \]

\[ \geq \|u(t)\|^2 - K_2 S^p \|u(t)\|^2 \]

\[ = \left( 1 - K_2 S^p \|u(t)\|^p - 2 \right) \|u(t)\|^2 \]

\[ \geq \left( 1 - K_2 S^p \lambda^{-2} \right) \|u(t)\|^2 \]

\[ = \omega \|u(t)\|^2, \]  

where $\omega = 1 - K_2 S^p \lambda^{-2} > 0$. Moreover, from (11), we also have

\[ \|u(t)\|^2 \leq \left( \frac{2p}{p-2} E(t) \leq \frac{2pE(0)}{p-2} \right), \quad \forall t \in [0, \infty). \]  

Next, by integration over $(t, t + 1)$ both side of (12), we get

\[ \mu \int_{t}^{t+1} \|u'(s)\|^q \, ds \leq E(t) - E(t + 1) = D(t). \]  

Since $u_t \in C([0, \infty) : L^q)$, there exist $t_1 \in [t, t + \frac{1}{2}]$ and $t_2 \in [t + \frac{3}{4}, t + 1]$ such that

\[ \|u'(t_i)\|^q \lesssim E(t) - E(t + 1) = D(t), \quad \forall i \in \{1, 2\}. \]  

On the other hand, multiplying (1) by $u$ and integrating over $\Omega \times (t_1, t_2)$, we obtain

\[ \int_{t_1}^{t_2} I(u(t)) \, dt = \int_{t_1}^{t_2} \|u'(t)\|^2 \, dt + (u'(t_1), u(t_1))_2 - (u'(t_2), u(t_2))_2 \]

\[ - \mu \int_{t_1}^{t_2} \left( |u'(t)|^{q-2} u'(t), u(t) \right)_2 \, dt. \]  

We estimate the terms on right-hand side of as follows.

Estimate $I_1 = \int_{t_1}^{t_2} \|u'(t)\|^2 \, dt$

By using Hölder’s inequality and (21), we get

\[ I_1 = \int_{t_1}^{t_2} \|u'(t)\|^2 \, dt \lesssim \int_{t_1}^{t_2} \|u'(t)\|^q \, dt \lesssim \left( \int_{t_1}^{t_2} \|u'(t)\|^q \, dt \right)^{\frac{2}{q}} \]

\[ \lesssim D^\frac{2}{q}(t). \]  

Estimate $I_2 = (u'(t_1), u(t_1))_2 - (u'(t_2), u(t_2))_2$

From Lemma 2.1 and (20), we have

\[ \|u(t)\|^2 \lesssim \|u(t)\|_q^2 \lesssim \sqrt{E(t)}, \quad \forall t \in [0, \infty). \]  

Then, we can estimate the remaining terms similarly.
From the Cauchy-Schwarz inequality, Lemma 2.1, (22) and (25), for any $\epsilon > 0$ and $i \in \{1, 2\}$ we get

$$
\begin{align*}
\|u' (t_i), u (t_i)\|_2 &\leq \|u' (t_i)\|_2 \|u (t_i)\|_2 \\
&\lesssim \|u' (t_i)\|_q \|u (t_i)\|_q \\
&\lesssim \frac{\epsilon^2}{2} E (t) + C (\epsilon) D^\frac{2}{q} (t).
\end{align*}
$$

Therefore, we have

$$
I_2 \lesssim \epsilon^2 E (t) + C (\epsilon) D^\frac{2}{q} (t). \tag{26}
$$

Estimate $I_3 = -\mu \int_{t_1}^{t_2} \left( |u' (t)|^{q-2} u' (t), u (t) \right) dt$

From the Hölder’s inequality, Young’s inequality and Lemma 2.1, we get

$$
I_3 \lesssim \int_{t_1}^{t_2} \left( |u' (t)|^{q-2} u' (t), u (t) \right)_2 dt \lesssim \int_{t_1}^{t_2} \|u' (t)\|_q^{q-1} \|u (t)\|_q dt
$$

$$
\lesssim \left( \int_{t_1}^{t_2} \|u' (t)\|_q^q dt \right)^{\frac{q-1}{q}} \left( \int_{t_1}^{t_2} \|u (t)\|_q^q dt \right)^{\frac{1}{q}}
$$

$$
\lesssim \epsilon^2 \int_{t_1}^{t_2} \|u (t)\|_q^q dt + C (\epsilon) \int_{t_1}^{t_2} \|u' (t)\|_q^q dt
$$

$$
\lesssim \epsilon^2 \int_{t_1}^{t_2} E (t) dt + C (\epsilon) D (t). \tag{27}
$$

On other hand, by using (11) and (19), we have

$$
E (t) = \frac{1}{2} \|u' (t)\|^2 + \frac{p - 2}{2p} \|u (t)\|^p + \frac{I (t)}{p} \leq \frac{1}{2} \|u' (t)\|^2 + \left( \frac{p - 2}{2p} + \frac{1}{p} \right) I (t). \tag{28}
$$

Therefore, with $\epsilon > 0$ small enough, from (23)-(28), we achieve that

$$
\int_{t_1}^{t_2} E (t) dt \lesssim \epsilon^2 E (t) + C (\epsilon) D (t) + C (\epsilon) D^\frac{2}{q} (t)
$$

$$
\lesssim \epsilon^2 E (t) + C (\epsilon) D^\frac{2}{q} (t). \tag{29}
$$

Furthermore, by using mean value theorem for integrals and Lemma 3.1, we deduce that

$$
E (t) \lesssim \int_{t_1}^{t_2} E (t) dt + D (t). \tag{30}
$$

Combine with (29), (30) leads to

$$
\sup_{t \in [t, t+1]} E^\frac{2}{q} (t) = E^\frac{2}{q} (t) \lesssim D (t) = E (t) - E (t + 1), \forall t \in [0, \infty). \tag{31}
$$

Thus, applying Lemma 3.5 to (31), we get (17) if $q = 2$ or (18) if $q > 2$. This completes the proof of Theorem 3.6.

Remark 5. In [30], the authors required the condition $q < p$ to estimated the term $\int_{t_1}^{t_2} \|u (t)\|_q^q dt$. We not really need this condition. So in our theorem, we eliminate this condition.
4. **Blow-up in finite time results at low energy level.** In principle, the blow-up phenomenon of a solution to a time dependent equation is devoted to the study of maximal time domain for which it is defined by a finite length. At the endpoint of that interval, the solution behaves in such a way that either it goes to infinity in some specific senses, or it stops being smooth, and so forth. Our main objective here is to show that with some suitable conditions, every weak solution of Problem (1)-(3) blow up in finite time.

First, we give two useful lemmas to estimate blow-up results.

**Lemma 4.1.** Let $\beta \in [2, p]$, We have following estimate

$$\|u\|_p^\beta \leq \|u\|_*^2 + \|u\|_p^p, \forall u \in H_*^2(\Omega).$$

**Lemma 4.2.** Assume that $[K]$ holds as well as at least one of the following conditions

1. $0 \leq E(0) < E_1$ and $\|u_0\|_* > \lambda_0$,
2. $E(0) < 0$,

then there exists $\lambda_2 \in (\lambda_0, \infty)$ such that $\|u(t)\|_* \geq \lambda_2$ for all $t \in [0, T_\infty)$.

Furthermore, there exists $\omega > 1$ such that

$$\int_\Omega K(x, y, t) |u(x, y, t)|^p \, dx \, dy \geq K_2 S_p^p \lambda_2^p > \frac{2p \omega}{p - 2} E_1, \forall t \in [0, T_\infty).$$

**Proof of Lemma 4.2.** With the same method as in Theorem 3.4, if the first condition holds, we can easily obtain the first result. So we will focus on the second case. From the condition $E(0) < 0$, there exists $\lambda_2 > \left(\frac{p}{2}\right)^{\frac{1}{p-2}} K_2^{-\frac{1}{p-2}} S_p^{\frac{1}{p-2}} = \lambda_0$ such that $H(\lambda_2) = E(0)$. From

$$0 > E(0) = H(\lambda_2) \geq H(\|u(0)\|_*) = H(\|u_0\|_*),$$

and Lemma 3.2, we have $\|u_0\|_* \geq \lambda_2$. Now, in order to prove $\|u(t)\|_* \geq \lambda_2$ for all $t \in (0, T_\infty)$, we assume by contradiction that there is $t_* \in (0, T_\infty)$ such that $\|u(t)\|_* < \lambda_2$. However, by the continuity, we can also assume that $\|u(t)\|_* \in (\lambda_0, \lambda_2)$. By using Lemma 3.1 and 3.2, we have

$$H(\lambda_2) = E(0) \geq E(t_*) \geq H(\|u(t_*)\|) > H(\lambda_2).$$

But this is impossible. Therefore, $\|u(t)\|_* \geq \lambda_2$ for all $t \in [0, T_\infty)$.

Furthermore, we have

$$\frac{1}{p} \int_\Omega K(x, y, t) |u(x, y, t)|^p \, dx \, dy \geq \frac{1}{2} \|u(t)\|_*^2 - E(0) \geq \frac{1}{2} \lambda_2^2 - H(\lambda_2)$$

$$= \frac{K_2 S_p^p \lambda_2^p}{p}.$$ (33)

On other hand, from $\lambda_2 > \lambda_0$, there exists $\omega > 1$ such that $\lambda_2^2 > \omega \lambda_0^p$. Then (33) gives us

$$\int_\Omega K(x, y, t) |u(x, y, t)|^p \, dx \, dy \geq K_2 S_p^p \lambda_2^p > \omega \omega K_2 S_p^p \lambda_0^p$$

$$= \omega K_2 S_p^p K_2^{-\frac{1}{p-2}} S_p^{\frac{1}{p-2}} = \omega K_2^{-\frac{1}{p-2}} S_p^{\frac{2}{p-2}}$$

$$= \frac{2p \omega}{p - 2} E_1.$$

Lemma 4.2 is proved. \(\square\)
Next, we show that the solution of Problem (1)-(3) blows up in finite time with some suitable conditions.

**Theorem 4.3.** Let \((u_0, u_1) \in H^2_0(\Omega) \times L^2(\Omega)\) and \(p > q\). Assume that \([K]\) holds as well as at least one of the following conditions

1. \(0 \leq E(0) < E_1\) and \(\|u_0\|_\sigma > \lambda_0\),
2. \(E(0) < 0\),

then the weak solution of Problem (1)-(3) blows up in finite time.

**Proof of Theorem 4.3.** We suppose that the solution exists for all time, and we reach to a contradiction. Set \(G\) and obtain

\[
G(t) = E_1 - E(t) \quad \forall t \in [0, \infty).
\]

Moreover, in this case, due to Lemma 4.2, \(G\) satisfies

\[
0 < G(t) = E_1 - \frac{1}{2} \|u'(t)\|^2 - \frac{1}{2} \|u(t)\|^2 + \frac{1}{p} \int_\Omega K(x,y,t) |u(x,y,t)|^p dx dy \leq E_1 - \frac{1}{2} \lambda_2^2 + \frac{1}{p} \int_\Omega K(x,y,t) |u(x,y,t)|^p dx dy
\]

\[\leq \frac{1}{p} \int_\Omega K(x,y,t) |u(x,y,t)|^p dx dy.
\]

Set a function \(M\) as follows

\[
M(t) = G^{1-\sigma}(t) + \epsilon(u'(t), u(t))_2,
\]

for \(\epsilon > 0\) small enough, to be determined later, and

\[
0 < \sigma < \min \left\{ \frac{p-2}{2p}, \frac{p-q}{p(q-1)} \right\}.
\]

Our goal is to show that \(M\) satisfies a differential inequality of the form

\[
M'(t) > M^\alpha(t), \quad \forall t \in [0, \infty).
\]

where \(\alpha > 1\). This inequality, of course, will leads to a blow up in finite time.

First, we note that \(\sigma > 1\) gives us \(\frac{2}{p} < \frac{2\sigma}{p - 2 + 2\sigma}\). Fixed \(\epsilon_1 \in \left(\frac{2}{p}, \frac{2\sigma}{p - 2 + 2\sigma}\right)\), we have \(\frac{1}{\epsilon_1} - 1 > \frac{p-2}{2\sigma}\) and \(\frac{\epsilon_1}{\epsilon_2} > 1\). Next, from the limit

\[
\lim_{\epsilon_2 \to 0} \left( \frac{1}{\epsilon_1} - 1 - \frac{\epsilon_2}{\epsilon_1} \right) = \frac{1}{\epsilon_1} - 1 > \frac{p-2}{2\sigma},
\]

we can choose \(\epsilon_2 \in (0,1)\) such that \(\frac{1}{\epsilon_1} - 1 - \frac{\epsilon_2}{\epsilon_1} > \frac{p-2}{2\sigma}\). Using (32), we obtain

\[
\left( \frac{1}{\epsilon_1} - 1 - \frac{\epsilon_2}{\epsilon_1} \right) \int_\Omega K(x,y,t) |u(x,y,t)|^p dx dy > \frac{p-2}{2\sigma} \frac{2\sigma}{p-2} E_1 = pE_1,
\]

or equivalently

\[
(1 - \epsilon_1 - \epsilon_2) \int_\Omega K(x,y,t) |u(x,y,t)|^p dx dy > p\epsilon_1 E_1.
\]
Next, by direct calculation and using (38), we obtain
\[
M'(t) = (1 - \sigma) G^{-\sigma}(t) G'(t) + \epsilon \|u'(t)\|^2 + \epsilon (u''(t), u(t))_2 \\
= (1 - \sigma) G^{-\sigma}(t) G'(t) + \epsilon \|u'(t)\|^2 - \epsilon \|u(t)\|_s^2 \\
+ \epsilon \int_\Omega K(x, y, t)|u(x, y, t)|^p dx dy - \epsilon \mu \left( |u'(t)|^{q-2}u'(t), u(t) \right)_2 \\
= (1 - \sigma) G^{-\sigma}(t) G'(t) + \epsilon \|u'(t)\|^2 - \epsilon \|u(t)\|_s^2 \\
+ \epsilon_1 \int_\Omega K(x, y, t)|u(x, y, t)|^p dx dy + \epsilon_2 \int_\Omega K(x, y, t)|u(x, y, t)|^p dx dy \\
+ \epsilon (1 - \epsilon_1 - \epsilon_2) \int_\Omega K(x, y, t)|u(x, y, t)|^p dx dy \\
- \epsilon \mu \left( |u'(t)|^{q-2}u'(t), u(t) \right)_2 \\
= (1 - \sigma) G^{-\sigma}(t) G'(t) + \epsilon \left( \frac{\epsilon_1}{2} + 1 \right) \|u'(t)\|^2 + \epsilon \left( \frac{\epsilon_1}{2} - 1 \right) \|u(t)\|_s^2 \\
+ \epsilon_1 \int_\Omega K(x, y, t)|u(x, y, t)|^p dx dy - \epsilon \mu \left( |u'(t)|^{q-2}u'(t), u(t) \right)_2 \\
+ \epsilon_2 \int_\Omega K(x, y, t)|u(x, y, t)|^p dx dy \\
- \epsilon \mu \left( |u'(t)|^{q-2}u'(t), u(t) \right)_2 \\
\geq (1 - \sigma) G^{-\sigma}(t) G'(t) + \epsilon \left( \frac{\epsilon_1}{2} + 1 \right) \|u'(t)\|^2 + \epsilon \left( \frac{\epsilon_1}{2} - 1 \right) \|u(t)\|_s^2 \\
+ \epsilon_1 \int_\Omega K(x, y, t)|u(x, y, t)|^p dx dy \\
- \epsilon \mu \left( |u'(t)|^{q-2}u'(t), u(t) \right)_2 \\
\geq (1 - \sigma) G^{-\sigma}(t) G'(t) + \epsilon \left( \frac{\epsilon_1}{2} + 1 \right) \|u'(t)\|^2 + \epsilon \left( \frac{\epsilon_1}{2} - 1 \right) \|u(t)\|_s^2 \\
+ \epsilon_1 \int_\Omega K(x, y, t)|u(x, y, t)|^p dx dy \\
- \epsilon \mu \left( |u'(t)|^{q-2}u'(t), u(t) \right)_2 \quad (39)
\]
By using Young’s inequality and (34), for any \(\epsilon_3 > 0\), we have following estimate
\[
\left( |u'(t)|^{q-2}u'(t), u(t) \right)_2 \leq \epsilon_3 \frac{q}{q'} \left( u(t) \right)_{q'} + \epsilon_3^{q'} \left( u'(t) \right)^q \leq \epsilon_3 \frac{q}{q'} \left( u(t) \right)_{q'} + \epsilon_3^{q'} \mu q' G'(t),
\]
where \(q'\) is the exponent conjugate to \(q\).

Furthermore, from (39) and (40), with \(\epsilon_3 q' = \kappa G^{-\sigma}(t)\) where \(\kappa > 0\) is specified later, we obtain
\[
M'(t) \geq \left( 1 - \sigma - \frac{\epsilon_3 \kappa}{q'} \right) G^{-\sigma}(t) G'(t) + \epsilon \left( \frac{\epsilon_1}{2} + 1 \right) \|u'(t)\|^2 + \epsilon \left( \frac{\epsilon_1}{2} - 1 \right) \|u(t)\|_s^2.
\]
\[ + \epsilon \epsilon _1 p G (t) + \epsilon \epsilon _2 \int _\Omega K (x, y, t) |u (x, y, t)|^p \, dx \, dy \]
\[ - \frac{\epsilon \mu}{q \kappa ^{q-1}} G^{\sigma (q-1)} (t) \|u(t)\|_q^q. \]  

(41)

By using (35), we have the following estimate
\[ \|u(t)\|_p^p \lesssim G(t) + \|u'(t)\|_2^2 + \|u(t)\|_\infty^2, \forall t \in [0, \infty). \]  

(42)

Therefore, with \( \sigma \) satisfies (37), we have \( \sigma p (q - 1) + q < p \). By using (35), Lemma 4.1 and (42), we can easily show that
\[ G^{\sigma (q-1)} (t) \|u(t)\|_q^q \lesssim \|u(t)\|_p^{\sigma p (q-1) + q} \lesssim \|u(t)\|_2^2 + \|u(t)\|_p^2 \]
\[ \lesssim G(t) + \|u'(t)\|_2^2 + \|u(t)\|_\infty^2. \]  

(43)

From (41) and (43), there exists \( C_\ast > 0 \) such that
\[ M'(t) \geq \left(1 - \sigma - \frac{\epsilon K}{q'}\right) G^{-\sigma} (t) G^{\sigma} (t) + \epsilon \left(\frac{\epsilon _1 p}{2} + 1 - \frac{\epsilon C_\ast}{\kappa ^{q-1}}\right) \|u'(t)\|^2 \]
\[ + \epsilon \left(\frac{\epsilon _1 p}{2} - 1 - \frac{\epsilon C_\ast}{\kappa ^{q-1}}\right) \|u(t)\|_2^2 + \epsilon \left(\epsilon _1 p - \frac{\epsilon C_\ast}{\kappa ^{q-1}}\right) G(t) \]
\[ + \epsilon \epsilon _2 \int _\Omega K (x, y, t) |u (x, y, t)|^p \, dx \, dy. \]  

(44)

At this point, and for large value of \( \kappa > 0 \) such that
\[ \frac{\epsilon _1 p}{2} - 1 - \frac{\epsilon C_\ast}{\kappa ^{q-1}} > 0, \]
we choose \( \epsilon > 0 \) such that
\[ 1 - \sigma - \frac{\epsilon K}{q'} > 0, M (0) = G^{\sigma} (0) + \epsilon (u_0, u_1) > 0, \]
then (44) gives us
\[ M'(t) \gtrsim \|u'(t)\|^2 + \|u(t)\|_\infty^2 + G(t) + \int _\Omega K (x, y, t) |u (x, y, t)|^p \, dx \, dy. \]  

(45)

On the other hand, applying Hölder’s inequality and Lemma 4.1, we obtain
\[ (u'(t), u(t))^\frac{1}{2} \lesssim \|u'(t)\|_p^\frac{1}{p} \|u(t)\|_p^\frac{1}{p} \lesssim \|u'(t)\|_p \|u(t)\|_p \]
\[ \lesssim \|u'(t)\|^2 + \|u(t)\|_p^2 \lesssim \|u'(t)\|^2 + \|u(t)\|_\infty^2 + \|u(t)\|_p^2. \]  

(46)

Therefore, from definition of the functional \( M \) and (46), we have
\[ M^{\frac{1}{1-s}} (t) \lesssim G(t) + \|u'(t)\|^2 + \|u(t)\|_\infty^2 + \|u(t)\|_p^2 \]
\[ \lesssim G(t) + \|u'(t)\|^2 + \|u(t)\|_\infty^2 + \int _\Omega K (x, y, t) |u (x, y, t)|^p \, dx \, dy. \]  

(47)

Combine (45) and (47), we have
\[ M'(t) \gtrsim M^{\frac{1}{1-s}} (t), \forall t \in [0, \infty), \]
and \( M(0) > 0 \). Therefore, the weak solution of Problem (1)-(3) blows up in finite time. This completes the proof of Theorem 4.3. \( \square \)
Remark 6. In [30], the authors used another method to obtain blow-up result when \( E(0) < d \) when \( u_0 \) belongs to unstable set. In fact, in the proof of [30, Theorem 3.10], the authors omitted one detail to keep the work within a reasonable range of length. The authors assumed that the reader know that if there exists \( t_* \in (0, T_\infty) \) such that \( E(t_*) < 0 \) then the weak solution blows up in finite time. Therefore, with the assumption the weak solution is global, the authors just considered the case \( E(t) \geq 0 \) for all \( t \in [0, T_\infty) \). With our method as in Theorem 4.3, we will not assume that \( E(t) \geq 0 \) for all \( t \in [0, T_\infty) \).

Next, we give the estimates for the lower bound of lifespan and upper bound for blow-up rate.

**Theorem 4.4.** Under the same assumptions as in Theorem 4.3, if \( K_t(x, y, t) \leq \overline{K} \) for all \((x, y, t) \in \overline{\Omega} \times [0, \infty)\) then we have the following estimate

\[
T_\infty \geq \int_{K(0)}^\infty \frac{dz}{\frac{\overline{K}}{K_t} z + 2p^{-2} K_2 S_2^{p-1}(E(0) + p z)^{p-1} + K_2 (E(0) + p z)},
\]

where \( K(0) = \int_{\Omega} K(x, y, 0) |u_0(x, y)|^p \, dx \, dy \).

Furthermore, we also have following estimate

\[
\|u(t)\|_p \geq \|u(t)\|_p \geq \mathcal{X}^{-1} (T_\infty - t), \quad \forall t \in [0, T_\infty),
\]

where \( \mathcal{X}^{-1} \) is an inverse function of the function

\[
\mathcal{X}(s) = \int_s^\infty \frac{dz}{\frac{\overline{K}}{K_t} z + 2p^{-2} K_2 S_2^{p-1}(E(0) + p z)^{p-1} + K_2 (E(0) + p z)},
\]

for all \( s \in (0, \infty) \).

**Proof of Theorem 4.4.** First, we put

\[
K(t) = \int_{\Omega} K(x, y, t) |u(x, y, t)|^p \, dx \, dy, \quad \forall t \in [0, T_\infty).
\]

By direct calculation and Cauchy inequality, we obtain

\[
K'(t) = \int_{\Omega} K_t(x, y, t) |u(x, y, t)|^p \, dx \, dy
\]

\[
+ \int_{\Omega} K(x, y, t) |u(x, y, t)|^{p-2} u(x, y, t) u_t(x, y, t) \, dx \, dy
\]

\[
\leq \frac{\overline{K}}{K_1} K(t) + K_2 \left( \frac{1}{2} \|u(t)\|_2^{2(p-1)} + \frac{1}{2} \|u'(t)\|^2 \right).
\]

On other hand, by using (13), we have

\[
\frac{1}{2} \|u(t)\|_2^{2(p-1)} \leq \frac{S_2^{2(p-1)}}{2} \|u(t)\|_2^{2(p-1)} \leq 2p^{-2} S_2^{2(p-1)}(E(0) + pK(t))^{p-1},
\]

and

\[
\frac{1}{2} \|u'(t)\|^2 \leq E(0) + pK(t).
\]

Combine (52)-(54), we obtain

\[
K'(t) \leq \frac{\overline{K}}{K_1} K(t) + 2p^{-2} K_2 S_2^{2(p-1)}(E(0) + pK(t))^{p-1}
\]

\[
+ K_2 (E(0) + pK(t)).
\]
By direct calculation, (55) implies

$$t_2 - t_1 \geq \int_{K(t_1)}^{K(t_2)} \frac{dz}{K_1 z + 2p^{-2}K_2s^{2(p-1)}_2(E(0) + pz)^{p-1} + K_2(E(0) + pz)},$$

(56)

for all $t_1, t_2 \in [0, T_\infty)$ and $t_1 < t_2$.

In (56), by letting $t_2 \nearrow T_\infty$ and choosing $t_1 = 0$, we have the lower bound for $T_\infty$ as follow

$$T_\infty \geq \int_{K(0)}^{\infty} \frac{dz}{K_1 z + 2p^{-2}K_2s^{2(p-1)}_2(E(0) + pz)^{p-1} + K_2(E(0) + pz)}.$$

Finally, we prove (50) holds. By letting $t_2 \nearrow T_\infty$ and choosing $t_1 = t$ in (56), we deduce that

$$T_\infty - t \geq \mathcal{X}(K(t)), \forall t \in [0, T_\infty),$$

(57)

where

$$\mathcal{X}(s) = \int_s^{\infty} \frac{dz}{K_1 z + 2p^{-2}K_2s^{2(p-1)}_2(E(0) + pz)^{p-1} + K_2(E(0) + pz)}, \forall s \in (0, \infty).$$

We note that the function

$$\mathcal{X}(s) = \int_s^{\infty} \frac{dz}{K_1 z + 2p^{-2}K_2s^{2(p-1)}_2(E(0) + pz)^{p-1} + K_2(E(0) + pz)},$$

is continuous and strictly decreasing on $(0, \infty)$. Therefore the inverse function $\mathcal{X}^{-1} : \mathcal{X}(0, \infty) \rightarrow (0, \infty)$ is also continuous and strictly decreasing. Then (57) leads to

$$\|u(t)\|_p^p \geq \|u(t)\|_p^p \geq \mathcal{K}(t) \geq \mathcal{X}^{-1}(T_\infty - t), \forall t \in [0, T_\infty).$$

This completes the proof of Theorem 4.4. $\square$

We give another way to obtain lower bound of lifespan of solution.

**Theorem 4.5.** Under the same assumptions as in Theorem 4.3, we have the following estimate

$$T_\infty \geq \frac{1}{K_2p} \ln \left(1 + 2^{-p}S_2^{2(1-p)}P_2^{-p}(0)\right),$$

(58)

where $P(0) = \frac{1}{2} \|u_0\|_p^2 + \frac{1}{2} \|u_0\|_p^2$.

**Proof of Theorem 4.5.** First, we put $P(t) = \frac{1}{2} \|u(t)\|_p^2 + \frac{1}{2} \|u(t)\|_p^2$ for all $t \in [0, T_\infty)$. By Lemma 4.2, it is clear that $P(t) > 0$ for all $t \in [0, T_\infty)$. By direct calculation, we have

$$P'(t) = (u''(t), u'(t))_2 + (u(t), u'(t))_s$$

$$= -\mu \|u(t)\|_q^q + \int_\Omega K(x, y, t) |u(x, y, t) - u'(x, y, t)|^{p-2} u(x, y, t) u'(x, y, t) dx dy$$

$$\leq K_2 \left(\frac{1}{2} \|u(t)\|_2^{2(p-1)} + \frac{1}{2} \|u'(t)\|_2^2\right)$$

$$\leq K_2 P(t) + \frac{K_2S_2^{2(p-1)}}{2} \|u(t)\|_p^{2(p-1)}$$

$$\leq K_2 P(t) + K_2S_2^{2(p-1)} \|u(t)\|_p^{2(p-1)}.$$
Therefore, by direct calculation, for any \( t \in [0, T_\infty) \), we obtain

\[
t \geq \frac{1}{K_{2p}} \ln \left( \frac{S_{2(p-1)}2^{p-2} \mathcal{P}^2 - p(0)}{S_{2(p-1)}2^{p-2} \mathcal{P}^2 - p(t)} \right).
\]

(60)

By letting \( t \nearrow T_\infty \) in (60), we have the estimate (58) holds. Theorem 4.5 is proven completely.

\[\square\]

**Remark 7.** In fact, in Theorem 4.4 and 4.5, if we just require the weak solution blows up in finite time then these results also hold. Therefore, with any \( E(0) \), if the weak solution of Problem (1)-(3) blows up in finite time then (58) holds. Furthermore, if \( K_t(x, y, t) \leq K \) for all \((x, y, t) \in \Omega \times [0, \infty)\) then (48) holds.

Next, we give one upper bound for lifespan of solution when \( q = 2 \).

**Theorem 4.6.** Under the same assumptions as in Theorem 4.3, if \( q = 2 \) then we have the following estimate

\[
T_\infty \leq \frac{4}{p-2} \frac{\zeta + \sqrt{\zeta^2 + 2 \beta_* \|u_0\|^2}}{\beta_*} = T_\infty^{\text{max}},
\]

(61)

where

\[
\beta_* = \frac{(p-2) \lambda_2^2 - 2pE(0)}{2p}, \quad \zeta = \frac{2\mu}{p-2} \|u_0\|^2 - (u_0, u_1)_2.
\]

(62)

**Proof of Theorem 4.6.** By last statement in Theorem 2.4, it is enough to prove that no global solution in \([0, \infty)\) can exists. Then, we will assume, by contradiction, that weak solutions exist in the whole interval \([0, \infty)\). The main tool in proving the blow-up result is the concavity method (introduced by Levine [13, 12]) where the basis idea of the method is to construct a positive defined functional \( M \) of the solution by the energy inequality and show that \( M - \alpha \) is concave function of time variable. For this purpose, with \( T_0 > 0, \beta > 0 \) and \( \tau > 0 \) specified later, we define the auxiliary functional \( M : [0, T_0] \rightarrow \mathbb{R} \) by

\[
M(t) = \|u(t)\|^2 + \mu \int_0^t \|u(s)\|^2 \, ds + \mu (T_0 - t) \|u_0\|^2 + \beta(t + \tau)^2.
\]

(63)

By direct calculation, we obtain

\[
M'(t) = 2(u'(t), u(t))_2 + \mu \|u(t)\|^2 - \mu \|u_0\|^2 + 2\beta(t + \tau)
\]

\[
= 2(u'(t), u(t))_2 + 2\mu \int_0^t (u'(s), u(s))_2 \, ds + 2\beta(t + \tau),
\]

(64)

and

\[
M''(t) = 2(u''(t), u(t))_2 + 2\|u'(t)\|^2 + 2\mu (u'(t), u(t))_2 + 2\beta
\]

\[
= 2\|u'(t)\|^2 + 2\beta - 2I(t).
\]

(65)

From (63) and (64), we have \( M(t) \geq \beta \tau^2 > 0 \) for all \( t \in [0, T_0] \) and \( M'(0) = 2(u_0, u_1)_2 + 2\beta \tau \) for \( \beta \tau \) large enough.
By using Cauchy–Schwarz inequality, we can easily obtain
\[
\frac{(M'(t))^2}{4} = \left( (u'(t), u(t))_2 + \mu \int_0^t (u'(s), u(s))_2 ds + \beta (t + \tau) \right)^2
\]
\[
\leq \left( \|u'(t)\|^2 + \mu \int_0^t \|u'(s)\|^2 ds + \beta \right)
\times \left( \|u(t)\|^2 + \mu \int_0^t \|u(s)\|^2 ds + \beta (t + \tau)^2 \right)
\]
\[
\leq \left( \|u'(t)\|^2 + \mu \int_0^t \|u'(s)\|^2 ds + \beta \right) M(t).
\]
(66)

Combine with (63)-(65), (66) give us the following estimate
\[
M''(t) M(t) - \frac{p + 2}{4} (M'(t))^2 \geq M(t) \xi(t), \ \forall t \in [0, T_0],
\]
where
\[
\xi(t) = -p \|u'(t)\|^2 - \mu (p + 2) \int_0^t \|u'(s)\|^2 ds - 2I(t) - p\beta.
\]
(67)

By using (11) and (12), we have
\[
-p \|u'(t)\|^2 - 2I(t) \geq (p - 2) \|u(t)\|^2 - 2pE(0) + 2p\mu \int_0^t \|u'(s)\|^2 ds.
\]

Therefore, we have
\[
\xi(t) \geq (p - 2) \|u(t)\|^2 - 2pE(0) + \mu (p - 2) \int_0^t \|u'(s)\|^2 ds - p\beta
\]
\[
\geq (p - 2) \lambda^2_2 - 2pE(0) - p\beta.
\]

We know that
\[
(p - 2) \lambda^2_2 - 2pE(0) = 2p \left( \frac{p - 2}{2p} \lambda^2_2 - E(0) \right) = 2p \left( \frac{p - 2}{2p} \lambda^2_2 - H(\lambda_2) \right)
\]
\[
= -2H'(\lambda_2) > 0,
\]

then we can choose \( \beta \in (0, \beta_\ast) \) where \( \beta_\ast = \left( \frac{p - 2}{2p} \lambda^2_2 - 2pE(0) \right)^{-1} \) such that
\[
\xi(t) \geq (p - 2) \lambda^2_2 - 2pE(0) - p\beta \geq 0, \ \forall t \in [0, T_0].
\]
(69)

Therefore, (67) leads to
\[
M(t) \geq M(0) \left[ 1 - \frac{(p - 2) M'(0) t}{4M(0)} \right]^{-\frac{1}{1-p}}, \ \forall t \in [0, T_0].
\]
(70)

Choose \( \tau \in (\tau_\ast, \infty) \) where
\[
\tau_\ast = \begin{cases} 
0 & \text{if } \zeta = \frac{2\mu}{p-2} \|u_0\|^2 - (u_1, u_0)_2 \leq 0, \\
\frac{\zeta}{\beta} & \text{if } \zeta > 0,
\end{cases}
\]

and \( T_0 \in \left[ \frac{2}{p-2} \frac{\beta \tau^2 + \|u_0\|^2}{\beta \tau - \zeta}, \infty \right) \), then we have
\[
T_\ast = \frac{4M(0)}{(p - 2) M'(0)} = \frac{\|u_0\|^2 + \mu T_0 \|u_0\|^2 + \beta \tau^2}{\frac{4}{p-2} ((u_0, u_1)_2 + \beta \tau)} \in [0, T_0].
\]

Therefore, (70) gives us \( \lim_{t \to T^*} M(t) = \infty \). This is a contradiction with the fact that the solution is global and it shows that the solution blows up at finite time.

To derive the upper bound for \( T_\infty \), we know that
\[
T_\infty \leq \frac{2}{p-2} \frac{\beta \tau^2 + \|u_0\|^2}{\beta \tau - \zeta} = \frac{2}{p-2} f(\beta, \tau), \quad \forall (\beta, \tau) \in (0, \beta^*) \times (\tau_*, \infty).
\]
By direct calculation, we have
\[
f(\beta, \tau) = \frac{\beta (\beta \tau^2 - 2\zeta \tau - \|u_0\|^2)}{(\beta \tau - \zeta)^2} = 0 \iff \tau = \tau_\pm = \frac{\zeta \pm \sqrt{\zeta^2 + \beta \|u_0\|^2}}{\beta}.
\]
Therefore, for any \((\beta, \tau) \in (0, \beta^*) \times (\tau_*, \infty)\), we have
\[
f(\beta, \tau) \geq f(\beta, \tau_+) = 2\tau_+ = 2 \frac{\zeta + \sqrt{\zeta^2 + \beta \|u_0\|^2}}{\beta} = \varphi(\beta).
\]
Moreover, we also have
\[
\varphi_\beta(\beta) = -\frac{\zeta + \sqrt{\zeta^2 + \beta \|u_0\|^2}}{\beta^2 \sqrt{\zeta^2 + \beta \|u_0\|^2}} \leq 0, \quad \forall \beta \in (0, \beta^*].
\]
Therefore, we achieve
\[
T_\infty \leq \frac{4}{p-2} \frac{\zeta + \sqrt{\zeta^2 + \beta^* \|u_0\|^2}}{\beta^*}.
\]

Theorem 4.6 is proved.

**Remark 8.** In Theorem 4.4, if \( q = 2 \), we can eliminate the condition \( K_t(x, y, t) \leq K \) for all \((x, y, t) \in \Omega \times [0, \infty)\) and replace the constant \( K \) by \( \max_{(x, y) \in \Omega} K(x, y, T_{\max}) \).

5. Global existence and blow-up of weak solutions with critical initial data. In Sec. 3 and 4, we always assume that \( E(0) < E_1 \) to obtain the global existence, decay estimate or blow-up results. Therefore, a natural question arises: “What happen in the case \( E(0) = E_1 \)” To answer this question, in this section, we will consider our problem when \( E(0) = E_1 \). We will not use the “Approximate methods” was first introduced in [27]. We have following theorem.

**Theorem 5.1.** Let \((u_0, u_1) \in H^2_\infty(\Omega) \times L^2(\Omega)\) and \([K]\) hold. Assume that \( E(0) = E_1 \) and \( K_t(x, t, 0) > 0 \) for all \((x, t) \in \Omega\) if \( u_1 = 0 \).

1. If \( \|u_0\|_\infty < \lambda_0 \) then every local solution of Problem (1)-(3) are global solution. Furthermore, the global solution has the following decay property as in Theorem 3.6.

2. If \( \|u_0\|_\infty > \lambda_0 \) then the weak solution of Problem (1)-(3) blows up in finite time.

**Proof of Theorem 5.1.** First, we note that if \( E(0) = E_1 \) then \((u_0, u_1) \neq (0, 0)\). Therefore, we will consider two cases. The first case is \( u_0 = 0 \). This fact implies \( u_1 \neq 0 \). Recall (12), we know that there exists \( t_* > 0 \) such that
\[
E(t_0) \leq E(0) - \mu \int_0^{t_0} \|u'(s)\|^q_\infty ds < E(0) = E_1, \quad \forall t_0 \in (0, t_*].
\]
In the second case, if \( u_1 = 0 \) then \( u_0 \neq 0 \). One again, recall (12), we know that there exists \( t_* > 0 \) such that

\[
E(t_0) \leq E(0) - \frac{1}{p} \int_0^{t_0} \int_{\Omega} K_t(x, y, t)|u(x, y, t)|^p \, dx \, dy \leq E(0) = E_1, \forall t_0 \in (0, t_*].
\]

Roughly speaking, under our assumption as in Theorem 5.1, if \( E(0) = E_1 \) then there exists \( t_* > 0 \) such that \( E(t_0) < E_1 \) for all \( t_0 \in (0, t_*) \).

Next, we consider the case \( \|u_0\|_\ast < \lambda_0 \). By regularity property of \( u \), without loss of generality, we can assume that \( \|u(t_0)\|_\ast < \lambda_0 \) for some \( t_0 \in (0, t_*] \). By using Theorem 3.4 and 3.6, we can easily obtain the first result in our theorem. We also easily obtain the second result with the same method. So we omit this proof here. Theorem 5.1 is proved. \( \square \)

6. Blow-up and global existence of weak solution at arbitrary initial energy level. For convenience, we will split this section into two subsection.

6.1. Blow-up results. First, we consider the case \( q = 2 \). With the same method as in Theorem 4.6, we can easy to see that the difficult part of the proof is prove that

\[
(p - 2)\|u(t)\|_\ast^p - pE(0) \geq \gamma, \forall t \in [0, T_0],
\]

for some \( \gamma > 0 \). Lemma 4.2 gives us one method to prove that. Unfortunately, this method can not apply in this situation. Therefore, we will use another method obtain (71). In order to prove our main result, we will use the following lemmas which are similar to Lemma 8.1 and Lemma 8.2 in [9] with a slight modification.

**Lemma 6.1** (See [9]). Let \( \delta > 0, T > 0 \) and let \( h \) be a Lipschitzian function over \([0, T)\). Assume that \( h(0) \geq 0 \) and \( h'(t) + \delta h(t) > 0 \) for a.e. \( t \in (0, T) \). Then \( h(t) > 0 \) for all \( t \in (0, T) \).

**Lemma 6.2.** Assume that \( \mu > 0 \). And let \( (u_0, u_1) \in H^2_0(\Omega) \times L^2(\Omega) \) such that

\[
(u_0, u_1)_2 \geq 0.
\]

Let \( u(t) \) be the solution of Problem (1)-(3) with \( q = 2 \). Then the map \( t \mapsto \|u(t)\|^2 \) is strictly increasing as long as \( I(t) < 0 \).

**Proof of Lemma 6.2.** Let \( Y(t) = \|u(t)\|^2 \) and \( G(t) = Y'(t) = 2(u'(t), u(t))_2 \). A simple computation gives

\[
Y''(t) = 2\|u'(t)\|^2 + 2(u''(t), u(t))_2
= 2\|u'(t)\|^2 - \frac{\mu}{2} G(t) - 2I(t).
\]

Therefore, we have

\[
G'(t) + \frac{\mu}{2} G(t) = 2\|u'(t)\|^2 - 2I(t) > 0.
\]

By using Lemma 6.1, we have \( G(t) = Y'(t) > 0 \). Hence \( Y(t) \) is strictly increasing. Lemma 6.2 is proved. \( \square \)

**Lemma 6.3.** Suppose that \( (u_0, u_1) \in H^2_0(\Omega) \times L^2(\Omega), \mu > 0, q = 2 \) and \([K]\) hold. Assume that the initial data satisfy

\[
\|u_1\|^2 - 2(u_0, u_1)_2 + \alpha E(0) < 0,
\]

where \( \alpha = \frac{2pS^2}{(p - 2)^2} \). If \( E(0) > 0 \) and \( I(0) < 0 \), then \( I(t) < 0 \) for all \( t \in (0, T_\infty) \).
Proof of Lemma 6.3. Arguing by contradiction, by the continuity of $I$ in $t$, we suppose that there exists a first time $t_0 \in (0, T_\infty)$ such that $I(t_0) = 0$ and $I(t) < 0$ for $t \in [0, t_0)$. By the Cauchy-Schwarz inequality, we have
\[
(u_0, u_1)_2 \leq \|u_0\| \|u_1\| \leq \frac{1}{2} \left(\|u_0\|^2 + \|u_1\|^2\right).\] (74)

By Lemma 6.2 and (73) and (74), we infer that
\[
\|u(t)\|^2 > \|u_0\|^2 \geq 2(u_0, u_1)_2 - \|u_1\|^2 > \alpha E(0), \forall t \in (0, t_0).
\] (75)

Moreover, from the continuity of $u$ with respect to time variable, we have
\[
\|u(t_0)\|^2 > \alpha E(0).\] (76)

From Lemma 3.1, we obtain
\[
E(0) \geq E(t_0) = \frac{1}{2}\|u'(t_0)\|^2 + \frac{p - 2}{2p} \|u(t_0)\|_\mu^2 + \frac{I(t_0)}{p} \geq \frac{(p - 2)}{2pS_2^2} \|u(t_0)\|^2.
\]

Therefore, we have
\[
\|u(t_0)\|^2 \leq \frac{2pS_2^2}{(p - 2)} E(0) = \alpha E(0),
\]
which contradicts (76). Lemma 6.3 is proved. \(\square\)

We now state the main blow-up theorem for the weak solution with arbitrary positive initial energy and linear weak damping.

**Theorem 6.4.** Let $(u_0, u_1) \in H^2_0(\Omega) \times L^2(\Omega)$ be the given initial data, $q = 2$, $[K]$ and (73) hold. Assume that $E(0) > 0$ and $I(0) < 0$. Then the weak solution of the Problem (1)-(3) blows up in finite time. Furthermore, we have the following estimate for upper bound of lifespan:
\[
T_\infty \leq \frac{4}{p - 2} \frac{\zeta + \sqrt{\zeta^2 + \beta_* \|u_0\|^2}}{\beta_*},\] (77)
where
\[
\beta_* = \frac{p - 2}{pS_2^2} \left(\|u_0\|^2 - \frac{2pS_2^2}{p - 2} E(0)\right), \quad \zeta = \frac{2\mu}{p - 2} \|u_0\|^2 - (u_0, u_1)_2.
\] (78)

**Proof of Theorem 6.4.** With the same method as in Theorem 4.6, we need to prove (71) holds. Indeed, we have
\[
(p - 2) \|u(t)\|_{\mu}^2 - 2pE(0) \geq \frac{p - 2}{S_2^2} \|u(t)\|^2 - 2pE(0)
\]
\[
> \frac{p - 2}{S_2^2} \|u_0\|^2 - 2pE(0)
\]
\[
= \frac{p - 2}{S_2^2} \left(\|u_0\|^2 - \frac{2pS_2^2}{p - 2} E(0)\right)
\]
\[
> 0.
\]

If we choose $\beta \in (0, \beta_*)$ where $\beta_* = \frac{p - 2}{pS_2^2} \left(\|u_0\|^2 - \frac{2pS_2^2}{p - 2} E(0)\right)$, then we obtain
\[
\xi(t) \geq \frac{p - 2}{S_2^2} \left(\|u_0\|^2 - \frac{2pS_2^2}{p - 2} E(0)\right) - p\beta > 0, \forall t \in [0, T_0].\] (79)
The rest of the proof of Theorem 6.4 follow argued as in the proof of Theorem 4.6 so we omit the detail. Theorem 6.4 is proved. □

Next, we will focus on the case $q > 2$.

**Theorem 6.5.** Suppose that $2 < q < p$ and $[K]$ hold. Let $u$ is solution of Problem (1)-(3), satisfying

$$(u_1, u_0) > ME(0) > 0,$$  \hspace{1cm} (80)

then $u$ blow up in finite time, where

$$M = \frac{q - 1}{q} \left[ \frac{\mu (1 - \theta)}{\epsilon \alpha K_1} \right]^{\frac{1}{q}},$$  \hspace{1cm} (81)

with $\epsilon_0$ is a root of the equation

$$\frac{q - 1}{q} \left[ \frac{\mu (1 - \theta)}{\epsilon \alpha K_1} \right]^{\frac{1}{q}} = \frac{p (1 - \epsilon)}{\alpha (\epsilon)},$$

and

$$\alpha (\epsilon) = 2 \sqrt{\left[ \frac{p}{2} (1 - \epsilon) + 1 \right] \left[ \kappa (\epsilon) - \frac{\epsilon p \theta K_1}{2 (1 - \theta)} \right]}, \kappa (\epsilon) = \left[ \frac{p}{2} (1 - \epsilon) - 1 \right] S_2^{-2},$$

and $\theta = \frac{p - q}{p - 2}$.

**Proof of Theorem 6.5.** Assume $u$ is a global solution of Problem (1)-(3). Without loss of generality, we may assume that $E(t) > 0$ for all $t \in [0, \infty)$. By using $u$ to test equation (1), for any $\epsilon > 0$, we have

$$\frac{d}{dt} (u^0 (t), u(t)) = \|u^0 (t)\|^2 + (u^0 (t), u(t))_2$$

$$= \|u^0 (t)\|^2 - \|u(t)\|^2 + \int_{\Omega} K (x, y, t) |u(x, y, t)|^p dx dy$$

$$- \mu \left( |u^0 (t)|^{p - 2} u^0 (t), u(t) \right)_2$$

$$\geq \left[ \frac{p}{2} (1 - \epsilon) + 1 \right] u^0 (t) \|u^0 (t)\|^2 + \left[ \frac{p}{2} (1 - \epsilon) - 1 \right] S_2^{-2} u^0 (t) \|u^0 (t)\|^2$$

$$- p (1 - \epsilon) E(t) + \epsilon K_1 \|u(t)\|_p$$

$$- \mu \left( |u^0 (t)|^{p - 2} u^0 (t), u(t) \right)_2.$$  \hspace{1cm} (82)

Using Hölder inequality, Young inequality and Jensen inequality, for any $\epsilon_1 > 0$, we obtain

$$\left( |u^0 (t)|^{p - 2} u^0 (t), u(t) \right)_2 \leq \frac{\epsilon_1^q}{q} \|u(t)\|_q^q + \frac{\epsilon_1^{q'}}{q'} \|u^0 (t)\|_q^q$$

$$\leq \epsilon_1 \left[ \frac{\theta}{2} \|u(t)\|^2 + \frac{1 - \theta}{p} \|u(t)\|_p^p \right] + \frac{\epsilon_1^{q'}}{q'} \|u^0 (t)\|_q^q.$$  \hspace{1cm} (83)

where $\theta = \frac{p - q}{p - 2}$.

Therefore, with $\epsilon_1 = \left[ \frac{\epsilon p K_1}{\mu (1 - \theta)} \right]^{\frac{1}{q}}$, by using Lemma 3.1, (82) and (83) lead to

$$\mathcal{T}^0 (t) \geq \left[ \frac{p}{2} (1 - \epsilon) + 1 \right] \|u^0 (t)\|^2 + \left[ \kappa (\epsilon) - \frac{\epsilon p \theta K_1}{2 (1 - \theta)} \right] \|u(t)\|^2$$

$$- p (1 - \epsilon) E(t).$$  \hspace{1cm} (84)
where
\[
\kappa (\epsilon) = \left[ \frac{p}{2} - 1 \right] S^2 - \frac{\epsilon p K_1}{2 (1 - \theta)} ,
\]
and
\[
\Upsilon' (t) = (u' (t), u (t))_2 - \frac{\epsilon - \mu}{q} E (t)
\]
\[
= (u' (t), u (t))_2 - \frac{q - 1}{q} \left( \frac{\mu (1 - \theta)}{\epsilon p K_1} \right)^{\frac{1}{q}} E (t) .
\]

From Cauchy–Schwarz inequality, we have
\[
\kappa (0) > 0, \text{ we can take } \epsilon > 0 \text{ small enough such that } \kappa (\epsilon) - \frac{\epsilon p K_1}{2 (1 - \theta)} > 0 .
\]

Using Cauchy–Schwarz inequality, we have
\[
\left( \frac{\mu (1 - \theta)}{\epsilon p K_1} \right)^{\frac{1}{q}} E (t) .
\]

Therefore, from (84) and (87), we have
\[
\Upsilon' (t) \geq \alpha (\epsilon) \left[ (u' (t), u (t))_2 - \frac{p (1 - \epsilon)}{\alpha (\epsilon)} E (t) \right] ,
\]
where \( \alpha (\epsilon) = 2 \sqrt{\frac{p}{2} (1 - \epsilon) + 1} \left[ \kappa (\epsilon) - \frac{\epsilon p K_1}{2 (1 - \theta)} \right] . \) It is easy to see that
\[
\lim_{\epsilon \to 0} \kappa (\epsilon) - \frac{\epsilon p K_1}{2 (1 - \theta)} > 0, \lim_{\epsilon \to 1} \kappa (\epsilon) - \frac{\epsilon p K_1}{2 (1 - \theta)} < 0 .
\]

Hence, there exists \( \epsilon_* \in (0, 1) \) such that \( \alpha (\epsilon) > 0 \) for all \( \epsilon \in (0, \epsilon_*) \) and \( \alpha (\epsilon_*) = 0 . \)

Furthermore, we have
\[
\lim_{\epsilon \to 0} \left[ \frac{\mu (1 - \theta)}{\epsilon p K_1} \right]^{\frac{1}{q}} = \infty, \lim_{\epsilon \to \epsilon_*} \left[ \frac{\mu (1 - \theta)}{\epsilon p K_1} \right]^{\frac{1}{q}} = \left[ \frac{\mu (1 - \theta)}{\epsilon_* p K_1} \right]^{\frac{1}{q}} ,
\]
and
\[
\lim_{\epsilon \to 0} \frac{p (1 - \epsilon)}{\alpha (\epsilon)} > 0, \lim_{\epsilon \to \epsilon_*} \frac{p (1 - \epsilon)}{\alpha (\epsilon)} = \infty ,
\]
then by the continuity in \( \epsilon \) of \( \frac{q - 1}{q} \left[ \frac{\mu (1 - \theta)}{\epsilon p K_1} \right]^{\frac{1}{q}} \) and \( \frac{p (1 - \epsilon)}{\alpha (\epsilon)} , \) there exists \( \epsilon_0 \in (0, \epsilon_*) \subset (0, 1) \) such that
\[
\frac{q - 1}{q} \left[ \frac{\mu (1 - \theta)}{\epsilon_0 p K_1} \right]^{\frac{1}{q}} = \frac{p (1 - \epsilon_0)}{\alpha (\epsilon_0)} .
\]

Choosing \( \epsilon = \epsilon_0 , \) (88) leads to
\[
\Upsilon' (t) \geq \alpha (\epsilon_0) \Upsilon (t) , \forall t \in [0, \infty) .
\]

Therefore, we obtain
\[
(u' (t), u (t)) \geq \Upsilon (t) \geq \exp (\alpha (\epsilon_0) t) , \forall t \in [0, \infty) .
\]

By direct calculation, gives us
\[
\| u (t) \|^2 \geq \exp (\alpha (\epsilon_0) t) , \forall t \in [0, \infty) .
\]
Moreover, by using Hölder inequality, we have the following estimate
\[ \|u(t)\| \lesssim \|u_0\| + \int_0^t \|u'(s)\| \, ds \lesssim \|u_0\| + t^{\frac{1}{p}} \left( \int_0^t \|u'(s)\|_q^q \, ds \right)^{\frac{1}{q}} \]
\[ \lesssim \|u_0\| + t^{\frac{1}{p}} (E(0) - E(t))^{\frac{1}{q}} \lesssim \|u_0\| + t^{\frac{1}{p}} \]
which contradicts (90). Therefore, the weak solution blows up in finite time. Theorem 6.5 is proved. \( \square \)

6.2. Global existence results. In Theorem 4.3, 4.6, 6.4 and 6.5, we always assume that \( p > q \). In the physical viewpoint, that means if the restoring force (describe by term \( K(x, y, t)u|^{p-2}u \)) strictly stronger than the damping source (describe by term \( \mu|u|^{q-2}u_t \)) then blow-up in finite time phenomenon may be occur with some suitable conditions. Therefore, what happen when the restoring force weaker than the damping source i.e. \( q \geq p \). The next theorem will answer this question.

**Theorem 6.6.** Assume that \( q \geq p, \mu > 0, |K| \) and \( K_t(x, y, t) \leq \overline{K} \) for all \( (x, y, t) \in \Omega \times [0, \infty) \) hold. Then for any \( (u_0, u_1) \in H_2^2(\Omega) \times L^2(\Omega) \), Problem (1)-(3) admits global solution.

**Proof of Theorem 6.6.** We put
\[ \Sigma(t) = E(t) + \frac{2}{p} \int_\Omega K(x, y, t) |u(x, y, t)|^p \, dx \, dy. \]
By direct calculation, for \( \delta > 0 \) small enough, we have
\[ \Sigma'(t) = E'(t) + 2 \frac{2}{p} \int_\Omega K_t(x, y, t) |u(x, y, t)|^p \, dx \, dy \]
\[ + 2 \int_\Omega K(x, y, t) |u(x, y, t)|^{p-2}u(x, y, t) u_t(x, y, t) \, dx \, dy \]
\[ \leq - \mu \|u'(t)\|_q^q + \frac{2\overline{K}}{p} \|u(t)\|_p^p + 2K_2 \left( \frac{\delta^p}{p} \|u'(t)\|_p^p + \frac{\delta^{-p}}{p'} \|u(t)\|_p^p \right) \]
\[ \leq \left( -\mu \|u'(t)\|_q^{p-q} + C\delta^p \right) \|u'(t)\|_q^q + 2 \left( \frac{\overline{K}}{p} + \frac{K_2}{p'\delta^p} \right) \|u(t)\|_p^p \]
\[ \lesssim 1 + \|u(t)\|_p^p \]
\[ \lesssim 1 + \Sigma(t). \] (91)
By using Grönwall’s inequality, we get
\[ \|u'(t)\|_q^q + \|u(t)\|_p^p \lesssim \Sigma(t) \lesssim \exp(\gamma t), \forall t \in [0, \infty), \]
for some \( \gamma > 0 \). The last estimate together with the continuation principle completes our proof. \( \square \)

**Remark 9.** In Theorem 6.6, the condition \( \mu > 0 \) is essential. When the model not contain a damping term, even for the classical wave equation of form \( u_{tt} - \Delta u = |u|^{p-2}u \) or \( u_{tt} - \Delta u = |u|^{p-2}u \ln |u| \) for \( p \geq 2 \) (See [3, 4, 5, 6, 14, 15, 17, 19, 25, 31], due to the lack of the invariant sets, the global existence of solution when \( E(0) > 0 \) is still unsolved within potential well’s framework. Although we have full confidence and reason to believe that there exist such initial data satisfying \( E(0) > d \) and
leading to global existence, and we can even confirm this by knowledge of parabolic equation [2, 10, 16, 18, 26, 29, 28, 32, 33], we have not been able to find an effective way to prove it. Up to now, this question is still open.

7. Global existence and decay estimate with live load in right hand side.
In this section, we add live load term \( f(x, y, t) \) in RHS of our problem. Furthermore, we also assume that \( q = 2 \). That means we consider a new problem as follows

\[
u_{tt} - \Delta u + au + \mu u_t = K(x, y, t) |u|^{p-2} u + f(x, y, t), \quad (x, y, t) \in \Omega \times (0, \infty), \quad (92)
\]

with boundary conditions (2) and initial condition (3).

We will consider a live load \( f \) satisfying the following assumptions:

\[
f \in C([0, \infty) ; L^2(\Omega)) \cap L^2(0, \infty ; L^2(\Omega)).
\]

Even if \( f \in C([0, \infty) ; L^2(\Omega)) \cap L^2(0, \infty ; L^2(\Omega)) \) we still not have \( \lim_{t \to \infty} \| f(t) \| = 0 \). We just know that \( \liminf_{t \to \infty} \| f(t) \| = 0 \) and in some case we may be have \( \limsup_{t \to \infty} \| f(t) \| = \infty \). By using standard Galerkin method, we can easily obtain the following short time existence result.

**Theorem 7.1.** Suppose that \( f \in C([0, \infty) ; L^2(\Omega)) \cap L^2(0, \infty ; L^2(\Omega)) \), then there exist \( T > 0 \) such that Problem (92) with boundary conditions (2) and initial condition (3) admits unique weak solution

\[
u \in C([0, T] ; H^2(\Omega)) \cap C^1([0, T] ; L^2(\Omega)) \cap C^2([0, T] ; H(\Omega)),
\]

satisfies (92) in distribution sense, i.e.,

\[
\langle u''(t), v \rangle + \langle u(t), v \rangle_\infty + \mu \langle u'(t), v \rangle_2 = \left( K(t) |u(t)|^{p-2} u(t), v_2 \right) + \langle f(t), v \rangle_2,
\]

for all test function \( v \in H^2(\Omega) \) and for almost all \( t \in (0, T) \), together with the following initial conditions

\[
u(0) = u_0, \quad u'(0) = u_1.
\]

Moreover, if

\[
T_\infty = \sup \{ T > 0 : \nu = u(t) \text{ exists on } [0, T] \} < \infty,
\]

then

\[
\lim_{t \to T_\infty} \left( \| u(t) \|^2 + \| u'(t) \|^2 \right) = \infty.
\]

Before going further, let us explain some difficulties which we will face. First, by some direct calculations, we have

\[
E'(t) = -\mu \| u'(t) \|^2 - \int_\Omega K_t(x, y, t) |u(x, y, t)|^p dx dy + \langle f(t), u'(t) \rangle_2,
\]

for all \( t \in [0, T_\infty) \). Therefore, even if \([K]\) holds, we still not have \( E'(t) \leq 0 \). And this fact leads to we can not apply the previous technique like potential well method or method was first introduce by E. Vitillaro in [24]. That is the first difficult. Next, from Theorem 3.4, we know if \( f \equiv 0 \) and the weak solution is global then \( E'(t) \geq 0 \) for all \( t \in [0, \infty) \), if initial data large enough then the weak solution blows up in
finite time. These results will not hold if $f \neq 0$. Therefore, we need new technique to pass these difficulties. For this purpose, we introduce the functional

$$
\mathcal{E}(t) = \frac{1}{2} \|u'(t)\|^2 + \frac{1}{2} \|u(t)\|^2 + \frac{\epsilon}{2} (g \ast u')(t) - \frac{1}{p} \int_{\Omega} K(x,y,t) |u(x,y,t)|^p dxdy,
$$

(94)

where $\epsilon > 0$ will be specified later and convolution product $(g \ast u)$ defined by

$$
(g \ast u)(t) = \int_0^t g(t-s) \|u(s)\|^2 ds,
$$

(95)

with $g(t) = \exp(-t)$. We also put

$$
\mathcal{J}(t) = \frac{1}{2} \|u(t)\|^2 + \frac{\epsilon}{2} (g \ast u')(t) - \frac{1}{p} \int_{\Omega} K(x,y,t) |u(x,y,t)|^p dxdy,
$$

(96)

and

$$
\mathcal{I}(t) = \|u(t)\|^2 + \epsilon (g \ast u')(t) - \int_{\Omega} K(x,y,t) |u(x,y,t)|^p dxdy.
$$

(97)

From (95)-(97), we deduce that

$$
\mathcal{E}(t) = \frac{1}{2} \|u'(t)\|^2 + \mathcal{J}(t)
$$

$$
= \frac{1}{2} \|u'(t)\|^2 + \frac{p-2}{2p} \left( \|u(t)\|^2 + (g \ast u')(t) \right) + \frac{\mathcal{I}(t)}{p}.
$$

(98)

We need the following lemmas.

**Lemma 7.2.** The perturbed energy functional $\mathcal{E}(t)$ satisfies

$$
\mathcal{E}'(t) \leq - \left( \mu - \frac{\epsilon}{2} - \frac{\epsilon_1}{2} \right) \|u'(t)\|^2 - \int_{\Omega} K_t(x,y,t) |u(x,y,t)|^p dxdy
$$

$$
- \frac{\epsilon}{2} (g \ast u')(t) + \frac{1}{2\epsilon_1} \|f(t)\|^2.
$$

(99)

for all $t \in [0, T_{\infty})$ and for any $\epsilon > 0$, $\epsilon_1 > 0$

**Proof of Lemma 7.2.** By testing (92) with $u_t$, we have

$$
\mathcal{E}'(t) = - \left( \mu - \frac{\epsilon}{2} \right) \|u'(t)\|^2 - \int_{\Omega} K_t(x,y,t) |u(x,y,t)|^p dxdy
$$

$$
- \frac{\epsilon}{2} (g \ast u')(t) + \left( f(t), u'(t) \right)_2.
$$

By using Cauchy-Schwarz inequality, It is not difficult to have the following estimate

$$
\mathcal{E}'(t) \leq - \left( \mu - \frac{\epsilon}{2} - \frac{\epsilon_1}{2} \right) \|u'(t)\|^2 - \int_{\Omega} K_t(x,y,t) |u(x,y,t)|^p dxdy
$$

$$
- \frac{\epsilon}{2} (g \ast u')(t) + \frac{1}{2\epsilon_1} \|f(t)\|^2.
$$

Lemma 7.2 proved. \( \square \)

**Remark 10.** If $[K]$ holds, then with $\epsilon = \frac{\mu}{2}$ and $\epsilon_1 = \frac{3\mu}{2}$, we obtain

$$
\mathcal{E}'(t) \leq \frac{1}{3\mu} \|f(t)\|^2, \forall t \in [0, T_{\infty}).
$$
Therefore, we deduce that
\[ E(t) \leq E(0) + \frac{1}{3\mu} \int_0^t \|f(s)\|^2 ds \leq E(0) + \frac{1}{3\mu} \|I_L^2(0, \infty; L^2(\Omega)) = R^2. \quad (100) \]

**Theorem 7.3.** Let \([K]\) holds. For any \((u_0, u_1) \in H^2(\Omega) \times L^2(\Omega)\) such that \(I(0) > 0\) and
\[ \eta_* = 1 - K_2S_p\left(\frac{2pR^2}{p - 2}\right) > 0, \quad (101) \]
then the solution \(u\) of Problem (92) with boundary-initial conditions (2) and (3) is global.

Furthermore, we have the following estimate
\[ I(t) \geq \eta_* \|u(t)\|_*^2 + \epsilon \|(g * u') (t)\|, \forall t \in [0, T\infty). \quad (102) \]

**Proof of Theorem 7.3.** First, we note that if \(I(0) > 0\) then \(R^2 > 0\). Furthermore, by the continuity of the functional \(I\) on time variable, there exists \(T > 0\) such that \(I(t) > 0\) for all \(t \in [0, T]\). We put \(T_* = \sup\{t > 0 : I(t) > 0, \forall t \in [0, T]\}\). It is clear that \(T_* \leq T\infty\). We claim that \(T_* = T\infty\). By contradiction, we assume that \(T_* < T\infty\). Then by continuity of the functional \(I\) and definition of \(T_*\), we have \(I(T_* > 0\). If \(I(T_*) > 0\) then by continuity, there exists \(T_\ast \in (T_*, T\infty)\) such that \(I(t) > 0\) for all \(t \in [0, T\ast]\). This contradicts the maximality of \(T_*\). This fact implies \(I(T_*) = 0\). From (98), we have
\[ E(t) \geq \frac{p - 2}{2p} \|u(t)\|^2, \forall t \in [0, T_*]. \]

Hence,
\[ \|u(t)\|_*^2 \leq \frac{2p}{p - 2} E(t) \leq \frac{2pR^2}{p - 2}, \forall t \in [0, T_*]. \]

On the other hand, for any \(t \in [0, T_*]\), we have
\[
I(t) = \|u(t)\|_*^2 + \epsilon \|(g * u') (t)\| + \int_{\Omega} K(x, y, t)|u(x, y, t)|^p dxdy \\
\geq \|u(t)\|_*^2 - K_2S_p\|u(t)\|_p^p + \epsilon \|(g * u') (t)\| \\
= \left(1 - K_2S_p\|u(t)\|_p^{p-2}\right)\|u(t)\|_*^2 + \epsilon \|(g * u') (t)\| \\
\geq \eta_* \|u(t)\|_*^2 + \epsilon \|(g * u') (t)\|. \quad (103)
\]

From \(I(T_*) = 0\) and (103), we have
\[
\begin{cases}
\int_0^{T_*} g(T_* - s)\|u'(s)\|^2 ds = 0, \\
u(T_*) = 0
\end{cases} \implies \begin{cases}
u(t) = u_0, \forall t \in [0, T_*], \\
u(T_*) = 0
\end{cases}.
\]

This fact implies \(u_0 = 0\). So we have \(I(0) = 0\). We get in contradiction with \(I(0) > 0\). Then we deduce that \(T_* = T\infty\). And this fact leads to \(I(t) > 0\) for all \(t \in [0, T\infty)\). Therefore, (98) leads to
\[ R^2 \geq E(t) \geq \frac{1}{2} \|u'(t)\|^2 + \frac{p - 2}{2p} \|u(t)\|_*^2, \forall t \in [0, T\infty). \]

The last estimate together with the continuation principle completes our proof. \(\square\)
Next, we prove a condition sufficient to obtain the general decay of weak solutions. For this purpose, we construct the Lyapunov functional

\[ \mathcal{L}(t) = \mathcal{E}(t) + \delta \psi(t), \quad \forall t \in [0, \infty), \]  

where \( \delta > 0 \) will be chosen later and

\[ \psi(t) = (u'(t), u(t))_2 + \mu \|u(t)\|^2. \]  

In order to get the decay result, we need some lemmas.

**Lemma 7.4.** Under the same assumption as in Theorem 7.3, then there exists positive constants \( \beta_1, \beta_2 \) such that

\[ \beta_1 \mathcal{E}_1(t) \leq \mathcal{L}(t) \leq \beta_2 \mathcal{E}_1(t), \quad \forall t \in [0, \infty), \]  

where

\[ \mathcal{E}_1(t) = \|u'(t)\|^2 + \|u(t)\|_s^2 + (g \ast u')(t) + I(t), \quad \forall t \in [0, \infty), \]  

for \( \delta \) is small enough.

**Proof of Lemma 7.4.** It is obviously that the energy functional \( \mathcal{E}(t) \) is equivalent to \( \mathcal{E}_1(t) \) in the sense that there exist two positive constants \( \bar{\alpha}_1 \) and \( \bar{\alpha}_2 \) such that

\[ \bar{\alpha}_1 \mathcal{E}_1(t) \leq \mathcal{E}(t) \leq \bar{\alpha}_2 \mathcal{E}_1(t), \quad \forall t \in [0, \infty). \]

Furthermore, by Cauchy-Schwarz’s inequality, we have

\[ \|u'(t), u(t)\|_2 + \frac{\mu}{2} \|u(t)\|^2 \lesssim \|u'(t)\|^2 + \|u(t)\|_s^2 \lesssim \mathcal{E}(t), \quad \forall t \in [0, \infty). \]

So with \( \delta \) small enough, we obtain (106). This completes the proof of Lemma 7.4. \( \square \)

**Theorem 7.5.** Under the same assumption as in Theorem 7.3, the weak solution of Problem (92) with boundary conditions (2) and initial condition (3) satisfies

\[ \lim_{t \to \infty} \left( \|u'(t)\|^2 + \|u(t)\|_s^2 \right) = 0. \]

Furthermore,

1. if \( \|f(t)\|^2 \lesssim \exp(-\alpha_0 t) \) for all \( t > 0 \), with \( \alpha_0 > 0 \) being constant, then for all \( t > 0 \), we have

\[ \|u'(t)\|^2 + \|u(t)\|_s^2 \lesssim \exp(-\alpha_1 t), \quad \forall t \in [0, \infty), \]

where \( \alpha_1 > 0 \).
2. if \( \|f(t)\|^2 \lesssim (1 + t)^{-p_*} \) for all \( t > 0 \), where \( p_* > 1 \) is given constant, then for all \( t > 0 \), we have

\[ \|u'(t)\|^2 + \|u(t)\|_s^2 \lesssim (1 + t)^{1-p_*}, \quad \forall t \in [0, \infty). \]

**Proof of Theorem 7.5.** By testing (92) with \( u \), we have

\[ \psi'(t) = \|u'(t)\|^2 + (u''(t), u(t))_2 + \mu(u'(t), u(t))_2 + (f(t), u(t))_2 \]
\[ = \|u'(t)\|^2 + \epsilon (g \ast u')(t) - I(t) + (f(t), u(t))_2 \]
\[ \leq \|u'(t)\|^2 + \epsilon (g \ast u')(t) - \epsilon_2 I(t) - (1 - \epsilon_2) I(t) \]
\[ + \frac{\varepsilon_3 S_2^2}{2} \|u(t)\|_*^2 + \frac{1}{2\varepsilon_3} \|f(t)\|^2 \]
\[ \leq \|u'(t)\|^2 + \varepsilon_2 \|g \ast u'(t)\| - \varepsilon_2 I(t) \]
\[ - \left[ \eta_* \left( 1 - \varepsilon_2 \right) - \frac{\varepsilon_3 S_2^2}{2} \right] \|u(t)\|_*^2 + \frac{1}{2\varepsilon_3} \|f(t)\|^2, \]
\[(108)\]
for any \( \varepsilon_2 > 0 \) and \( \varepsilon_3 > 0 \).

Combine (99) and (108), we obtain
\[
\mathcal{L}'(t) \leq - \left( \mu - \frac{\varepsilon}{2} - \frac{\varepsilon_1}{2} - \delta \right) \|u'(t)\|^2 - \varepsilon \left( \frac{1}{2} - \delta \right) (g \ast u')(t) - \varepsilon_2 \delta I(t) 
\]
\[ - \left[ \eta_* \left( 1 - \varepsilon_2 \right) - \frac{\varepsilon_3 S_2^2}{2} \right] \|u(t)\|_*^2 + \left( \frac{1}{2\varepsilon_1} + \frac{\delta}{2\varepsilon_3} \right) \|f(t)\|^2. \]
\[(109)\]
We choose \( \varepsilon > 0, \varepsilon_1 > 0 \) and \( \delta > 0 \) such that
\[
\mu - \frac{\varepsilon}{2} - \frac{\varepsilon_1}{2} - \delta > 0, \frac{1}{2} - \delta > 0, 1 - \varepsilon_2 > 0.
\]
After that, we choose \( \varepsilon_3 > 0 \) such that
\[
\eta_* \left( 1 - \varepsilon_2 \right) - \frac{\varepsilon_3 S_2^2}{2} > 0.
\]
By using Lemma 7.4, then we can find the positive constant \( C \) such that
\[
\mathcal{L}'(t) \leq - \mathcal{L}(t) + C \|f(t)\|^2, \forall t \in [0, \infty).
\[(110)\]
By direct computation, we achieve that \( \lim_{t \to \infty} \mathcal{L}(t) = 0 \). This fact implies
\[
\lim_{t \to \infty} \left( \|u'(t)\|^2 + \|u(t)\|_*^2 \right) = 0.
\]
We also obtain the decay results as in Theorem 7.5 by solving integral inequality (110). Theorem 7.5 is proved completely.

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