Hidden Chaos

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When a medium composed of microscopic elements is subjected to a high intensity field, the individual behaviors of microscopic elements can become chaotic. In such cases it is important to consider the effects of this irregularity at a macroscopic level onto the macroscopic behavior of the medium. We show that the macroscopic field produced by a large group of chaotic scatterers can remain regular, due to the partial or complete phase coherence of the scattering elements and the incoherence of the chaotic components of their responses. Thus when only macroscopic fields are observed, one may be unaware of chaotic microscopical motion, as it appears to be hidden from the observer. The coupling among the elements may lead to partial chaos synchronization, which exposes the chaotic nature of the system making the oscillations of macroscopic fields more irregular.

The problem of wave scattering and dispersion is one of the basic problems in many branches of physics. In most physical problems scattering and dispersion are explained by the interaction of the wave field with particles that respond linearly or weakly nonlinearly to the external forcing. However over the last few years problems appeared where external fields are so powerful that they drive the individual scatterers into the chaotic regime. Examples of such problems are the chaotic excitation of atoms and molecules by powerful electro-magnetic fields and chaotic pulsations of cavitating bubbles in powerful ultrasonic fields. In such cases the problem of scattering and dispersion becomes more complicated. Even if the behavior of a microscopic element is chaotic, it is not obvious what manifestation this would have at the macroscopic level. The study of this issue is important from both theoretical and experimental prospective.

Scattering elements can typically be considered as passive damped oscillators. When the amplitude of the scattered field is small compared to the external field, the effect of the scattered field upon an individual element is negligible and the whole system becomes an ensemble of uncoupled damped oscillators driven by the same external force. Beyond some amplitude level the nonlinearity of the scatterers becomes essential and can lead to chaoticization of their oscillations. In many cases chaotic oscillations occurring in a single damped oscillator under the action of a periodic external force turn out to be in some sense phase locked to the driving force. As the result, chaotic oscillations in such systems contain components that are coherent with the external driving. If the macroscopic field is produced by a large group of such microscopic elements, these components add coherently, while the chaotic components of signals add non-coherently. The resulting macroscopic field becomes nearly periodic.

When the coupling through the scattered field is taken into account, an unusual situation occurs. Sufficiently strong coupling can lead to partial or complete synchronization of chaotic oscillations in individual elements. In such case chaotic components of these oscillations begin to add coherently and the macroscopic field becomes more chaotic. Thus the synchronization phenomenon, which is normally associated with the onset of a more regular behavior in the system, has the opposite effect on the observed macroscopic quantities, making their oscillations less regular.

To better understand this phenomenon, let us consider as an example the following system:

$$\frac{d^2 x_j}{dt^2} + \nu \frac{dx_j}{dt} - \frac{\alpha}{x_j^3} + \frac{1}{x_j^2} = A \sin(\Omega t) + \sum_{i=1}^{N} \frac{\kappa_{i,j}}{N} \frac{dx_i}{dt}. \quad (1)$$

These equations describe damped motion of $N$ particles in the potential $U(x) = \alpha/(2x^2) - 1/x$ and subject to the periodic external force. The last term is to account for possible synchronization effects due to the mutual coupling through the macroscopic field. Here we assume for simplicity that the scattering region is much smaller than the wavelength of the external field and neglect the delays in the coupling term. The values of the coupling coefficients depend on the geometry of the problem. We consider the case where the scattered field is proportional to the derivative of the state of a scattering element, $y_j(t) = dx_j(t)/dt$. In the following discussion we shall keep $\nu = 0.4$, $\alpha = 0.75$, $A = 0.45$, and $\Omega = 1.3$.

![Fig. 1. The chaotic attractor in one oscillator without coupling.](image)

With these values of parameters and with $\kappa_{i,j} = 0$ for all $i$ and $j$, system (1) oscillates in a chaotic regime, as illustrated in Fig. 1.

The response of the ensemble, $Y(t)$, observed far from
the scattering region is proportional to the sum of fields from individual scatterers:

\[ Y(t) \sim \sum_{j=1}^{N} y_j(t). \]

Assuming stationarity of \( y_j(t) \), the mean temporal power spectral density \( P(\omega) \) of the scattered field can be calculated using:

\[ P(\omega) = \frac{2}{\pi} \int_{0}^{\infty} C(\tau) \cos \omega \tau d\tau, \]

where \( C(\tau) \) is the auto-correlation function for \( Y(t) \):

\[ C(\tau) = \langle Y(t)Y(t+\tau) \rangle = \sum_{j,k=1}^{N} \langle y_j(t)y_k(t+\tau) \rangle. \]

The last sum contains auto-correlation terms \( j = k \) and cross-correlation terms \( j \neq k \).

Let us first consider the case when \( \kappa_{i,j} = 0 \) for all \( i \) and \( j \). Then all oscillators evolve on the same attractor so that \( \langle y_j(t)y_j(t+\tau) \rangle = C_0(\tau) \), and \( \langle y_j(t)y_k(t+\tau) \rangle = C_X(\tau) \) for \( i \neq j \), independent of \( j \) and \( k \). Thus the autocorrelation of the macroscopic field can be written as

\[ C(\tau) = N (C_0(\tau) - C_X(\tau)) + N^2 C_X(\tau), \]

and the power spectral density can be written as

\[ P(\omega) = N (P_1(\omega) - P_X(\omega)) + N^2 P_X(\omega), \tag{2} \]

where \( P_1(\omega) \) is the power spectral density of a single response, and \( P_X(\omega) \) is the cross spectral density.

The autocorrelation, \( C_0(\tau) \), the cross-correlation \( C_X(\tau) \) and the difference between the two, \( C_C(\tau) = C_0(\tau) - C_X(\tau) \), are shown in Fig.2. We see that \( C_C(\tau) \) decays at large \( |\tau| \), which means that \( P_1(\omega) - P_X(\omega) \) is the continuous component of the power spectrum. More interestingly, we observe that \( C_X(\tau) \) is a purely periodic function, meaning \( P_X(\omega) \) is the discrete component of the spectrum.

Equation (2) indicates that as the number of scattering elements increases, the coherent component of the spectrum increases quadratically with the number of elements, while the continuous spectrum component increases linearly, as it typically happens with non-coherent signals. Therefore, with a large number of chaotic microscopic elements, the macroscopic response of the system is highly regular, nearly periodic. Thus the chaotic nature of the system is hidden from the macroscopic observer. This is illustrated in Fig.3 and Fig.4. The faster growth of the discrete spectral component, compared to the continuous chaotic component, is evident in Fig.4. A closer look at a proper projection of this figure reveals good agreement with the prediction of (2): the slope for the discrete part of the spectrum is twice larger than for the continuous part.

![FIG. 2. Autocorrelation of the chaotic response of oscillator (top) without coupling (middle); and the difference between these correlations (bottom).](image-url)
with the periodic driving and the fluctuations of the relative phase are less than half a period. Due to this, the cross-correlation of two oscillators driven by the same external field is purely periodic because the effect of chaos averages to zero, while the component coherent with the periodic driving does not. When \( \tau \) is large, due to the divergence of trajectories in the phase space of chaotic systems and the loss of information about the initial condition, the difference between computing autocorrelation for a single oscillator and computing cross-correlation for signals from two different oscillators disappears. This explains why \( C_0(\tau) \) becomes purely periodic, and \( C_C(\tau) \) decays to zero, see Fig. 2.

![Graph](image1)

**FIG. 3.** The mean field response of 4096 oscillators. The response of a single element is shown with a dotted line for comparison.

![Graph](image2)

**FIG. 4.** The mean power spectral density of the macroscopic response as a function of the number of oscillators (in logarithmic scale).

Let us now consider what happens when the coupling among the elements is taken into account. For simplicity we assume that \( \kappa_{i,j} = K \). Then the system has the solution where \( x_i(t) = X(t) \) for all \( i \), which corresponds to identically synchronized chaotic oscillations in the ensemble. If the coupling among the elements is sufficiently strong to stabilize this solution, clearly, the mean field response is proportional to that of a single element and is chaotic. Thus, although as synchronization sets in the dynamics of the system becomes less chaotic, with a lower dimension of the attractor, the mean field response becomes more irregular.

Not quite so obvious is the effect of weak synchronizing coupling. When the coupling is weak, the state of complete synchronization may not be achieved. To make the matter more complicated, the variations of coupling lead to changing dynamics of the entire systems. In addition to chaotic attractors for some values of coupling periodic or quasiperiodic stated can become stable. Nevertheless, the de-regularizing effects can be seen even for very small values of the coupling. This is illustrated in Fig. 6 which shows the ratio of the power in the continuous component of the spectrum to the total power. The solid line corresponds to increasing the coupling coefficient from zero, the dashed line, to decreasing it.

![Graph](image3)

**FIG. 5.** The detrended phase of the response of an individual oscillator.

![Graph](image4)

**FIG. 6.** The ratio of the power in the continuous component of the spectrum to the total power. The solid line corresponds to increasing the coupling coefficient from zero, the dashed line, to decreasing it.

![Graph](image5)

**FIG. 7.** The ratio of the power in the continuous component of the spectrum to the total power. The solid line corresponds to increasing the coupling coefficient from zero, the dashed line, to decreasing it.
of two oscillators looks almost identical to the autocorrelation of a single oscillator, and the oscillations become quite irregular.

In conclusion, we showed that when the only observed quantities are obtained by averaging the responses from many chaotic oscillators to external fields, these quantities can remain periodic. This presents a difficulty in observing chaos in such systems. Fig. illustrates the effect of averaging onto the bifurcation diagram of our example system. We see that the period doubling cascade and the chaotic regime, evident in the bifurcation diagram of a single element, are hardly visible in the bifurcation diagram of the mean field. Under experimental conditions the bifurcation sequence and the transition to chaos can easily be obscured by measurement noise.

The macroscopic chaotic oscillations may arise in such systems as a result of mutual coupling between the elements. When this coupling is strong chaotic oscillations in individual elements may become synchronized, which causes non-periodic components of individual responses add coherently. As the coupling strength varied, the transition between regimes characterized by different degrees of regularity of the mean field response can occur in a non-trivial way, for example, exhibit hysteresis.

Partial coherence and effects similar to those discussed in this communication can also occur in ensembles of mean field coupled chaotic generators. In such systems the external field may not be necessary to achieve phase coherence, which can arise spontaneously due to a global mean field coupling in the system.

We thank Lev Tsimring and Ulrich Parlitz for fruitful discussions. This research was supported in part by ARO, grant No.DAAG55-98-1-0269 and in part by the US DOE, grant No. DE-FG03-95ER14516.

FIG. 7. Bifurcation diagrams for a single oscillator and for the mean field of 4086 elements (bold line). The diagram was created using the Poincare map with the period of the driving.

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