TRANSFERENCE OF SCALE-INVARIANT ESTIMATES FROM LIPSCHITZ TO NON-TANGENTIALLY ACCESSIBLE TO UNIFORMLY RECTIFIABLE DOMAINS

STEVE HOFMANN, JOSÉ MARÍA MARTELL, AND SVITLANA MAYBORODA

Abstract. In relatively nice geometric settings, in particular, on Lipschitz domains, absolute continuity of elliptic measure with respect to the Lebesgue measure is equivalent to Carleson measure estimates for solutions, to square function estimates, to $\varepsilon$-approximability, for any second order elliptic PDE. In more general situations, notably, in a domain with a uniformly rectifiable boundary, absolute continuity of elliptic measure with respect to the Lebesgue measure may fail, already for the Laplacian. In the present paper the authors demonstrate that nonetheless, Carleson measure estimates for solutions, square function estimates, and $\varepsilon$-approximability remain valid. Moreover, the paper offers a general real-variable transference principle of certain scale-invariant estimates from Lipschitz to NTA to uniformly rectifiable domains, not restricted to harmonic functions or even to solutions of elliptic equations. In particular, this allows one to deduce the first bounds for higher order systems on uniformly rectifiable domains, in the setting where the elliptic measure does not exist, and to treat subharmonic functions.

Contents

1. Introduction 2
2. Preliminaries and relevant results from [HMM] 6
   Case ADR 11
   Case UR 11
   Case NTA 13
3. Transference of Carleson measure estimates from NTA to Uniformly Rectifiable domains 14
4. John-Nirenberg inequality and transference of Carleson measure estimates from Lipschitz to Uniformly Rectifiable domains 15
5. $A < N$ bounds: good-$\lambda$ arguments 20
6. $N < S$ bounds: from Lipschitz to NTA domains 27
7. From $N < S$ bounds on NTA domains to $\varepsilon$-approximability in a complement of a UR set 36
8. Applications: solutions of divergence form elliptic equations with bounded measurable coefficients 37

Date: April 30, 2019.

2010 Mathematics Subject Classification. 28A75, 28A78, 31B05, 42B20, 42B25, 42B37.

Key words and phrases. Carleson measures, square functions, non-tangential maximal functions, $\varepsilon$-approximability, uniform rectifiability, harmonic functions.

The first author was supported by NSF grant DMS-1664047. The second author acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the “Severo Ochoa Programme for Centres of Excellence in R&D” (SEV-2015-0554). He also acknowledges that the research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013)/ERC agreement no. 615112 HAPDEGMT. The third author was supported in part by the NSF INSPIRE Award DMS-1344235, NSF CAREER Award DMS-1220089, NSF-DMS 1839077, Simons Fellowship, and the Simons Foundation grant 563916, SM.
8.1. Second order divergence form elliptic operators with coefficients satisfying a Carleson measure condition
8.2. Higher order elliptic equations and systems with constant coefficients
Appendix A. Sawtooths have UR boundaries

References

1. Introduction

In the setting of a Lipschitz domain $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$, for any divergence form elliptic operator $L = -\text{div} A \nabla$ with bounded measurable coefficients, the following are equivalent:

(i) Every bounded solution $u$, of the equation $Lu = 0$ in $\Omega$, satisfies the Carleson measure estimate (see Definition 1.8 with $F = |\nabla u|/\|u\|_{L^\infty(\Omega)}$).

(ii) Every bounded solution $u$, of the equation $Lu = 0$ in $\Omega$, is $\varepsilon$-approximable, for every $\varepsilon > 0$ (see Definition 1.11).

(iii) The elliptic measure associated to $L$, $\omega_L$, is (quantitatively) absolutely continuous with respect to the Lebesgue measure, $\omega_L \in A_\infty(\sigma)$ on $\partial \Omega$.

(iv) Uniform Square function/Non-tangential maximal function ("$S/N$") estimates hold locally in "sawtooth" subdomains of $\Omega$ (see Definition 1.14 and the discussion following the Definition).

Historically, Dahlberg [Da3] obtained an extension Garnett’s $\varepsilon$-approximability result, observing that (iv) implies (ii). The explicit connection of $\varepsilon$-approximability with the $A_\infty$ property of harmonic measure, i.e., that (ii) $\Rightarrow$ (iii), appears in [KKoPT] (where this implication is established not only for the Laplacian, but for general divergence form elliptic operators). That (iii) implies (iv) is proved for harmonic functions in [Da2], and, for null solutions of general divergence form elliptic operators, in [DJK]. Finally, Kenig, Kirchheim, Pipher and Toro [KKiPT] have recently shown that (i) implies (iii), whereas, on the other hand, (i) may be seen, via good-lambda and John-Nirenberg arguments, to be equivalent to the local version of one direction of (iv) (the "$S < N$" direction).

The main goal of the present paper is to show that while (iii) may fail on general uniformly rectifiable domains even for harmonic functions [BJ] or might be not applicable in the absence of a suitable concept of elliptic measure (e.g., for systems), (i), (ii) and (iv) carry over from Lipschitz domains to uniformly rectifiable sets by a purely real variable mechanism. But let us start with more historical context.

Past several decades brought to the center of attention uniformly rectifiable sets as the most general geometric setting in which meaningful analytic properties continue to hold. It was shown in the beginning of 90’s that uniform rectifiability of a set $E$ is equivalent to boundedness of all singular integral operators with odd kernels in $L^2(E)$ [DS1], and, much more recently, that uniform rectifiability is equivalent to boundedness of the Riesz transform in $L^2(E)$ (see [MMV] for the case $n = 1$, and [NToV] in general).

However, it seemed to be vital for the estimates on solutions of elliptic PDEs in a domain $\Omega$ that, in addition to uniform rectifiability of $E = \partial \Omega$, $\Omega$ possesses some additional topological features, ensuring a reasonably nice approach to the boundary. In particular, it has been known

---

1 This implication holds more generally for null solutions of divergence form elliptic equations, see [KoPT] and [HKMP].
2 And thus all three properties hold for harmonic functions in Lipschitz domains, by the result of [Da1].
3 We will prove this fact in much bigger generality in this paper.
that \((i), (iii), (iv)\) hold for harmonic functions on non-tangentially accessible domains which satisfy an interior and exterior corkscrew condition (quantitative openness) and Harnack chains condition (quantitative connectedness) – see [JK], [DJK]. At the same time, the counterexample of Bishop and Jones [BJ] showed that absolute continuity of harmonic measure with respect to the Lebesgue measure \((iii)\) may fail on a general set with a uniformly rectifiable boundary: they construct a one dimensional \((\text{uniformly})\) rectifiable set \(E\) in the complex plane, for which harmonic measure with respect to \(\Omega = \mathbb{C} \setminus E\), is singular with respect to Hausdorff \(H^1\) measure on \(E\). In [HMM] the authors proved that, in spite of Bishop-Jones counterexamples, Carleson measure estimates \((i)\) and \(\varepsilon\)-approximability \((ii)\) for harmonic functions remain valid on all domains with a uniformly rectifiable boundary and shortly thereafter it was shown that, at least in the presence of interior corkscrew points, each of these properties is \textit{necessary and sufficient} for uniform rectifiability [GMT]. For the sake of completeness, we also want to point out that, in the absence of any additional topological assumptions, absolute continuity of the harmonic measure \(\omega\) with respect to the Hausdorff measure on \(E \subset \partial \Omega\) implies that \(\omega|_E\) is rectifiable [AHM3TV].

The present paper introduces a new transference mechanism, which shows that a passage from scale-invariant estimates, such as a Carleson measure bound or square function estimates/non-tangential maximal function estimates, on Lipschitz domains to analogous results on non-tangentially accessible domains and further to the same bounds on all sets with uniformly rectifiable boundaries is, in fact, a real variable phenomenon. That is, whenever one has suitable bounds for a given function on Lipschitz domains, they automatically carry over to uniformly rectifiable sets. This immediately entails a series of new results in very general PDE settings (for solutions of second order elliptic PDEs with coefficients satisfying a Carleson measure condition, for solutions of higher order systems, for subharmonic functions), but clearly the power of having a general, purely real-variable scheme, goes beyond these applications. Let us now discuss the details.

\textbf{Definition 1.1.} (ADR) \textit{(aka Ahlfors-David regular).} We say that a set \(E \subset \mathbb{R}^{n+1}\) is \(n\)-dimensional ADR \textit{(or simply ADR)} if it is closed, and if there is some uniform constant \(C \geq 1\) such that

\[
C^{-1}r^n \leq \sigma(\Delta(x, r)) \leq C r^n, \quad \forall r \in (0, \text{diam}(E)), \; x \in E,
\]

where \(\text{diam}(E)\) may be infinite. Here, \(\Delta(x, r) := E \cap B(x, r)\) is the “surface ball” of radius \(r\), and \(\sigma := H^n|_E\) is the “surface measure” on \(E\), where \(H^n\) denotes \(n\)-dimensional Hausdorff measure.

\textbf{Definition 1.3.} (UR) \textit{(aka uniformly rectifiable).} An \(n\)-dimensional ADR \textit{(hence closed)} set \(E \subset \mathbb{R}^{n+1}\) is \(n\)-dimensional UR \textit{(or simply UR)} if and only if it contains “Big Pieces of Lipschitz Images” of \(\mathbb{R}^n\) (“BPLI”). This means that there are positive constants \(\theta\) and \(M_0\), such that for each \(x \in E\) and each \(r \in (0, \text{diam}(E))\), there is a Lipschitz mapping \(\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}\), with Lipschitz constant no larger than \(M_0\), such that

\[
H^n\left( E \cap B(x, r) \cap \rho \left( \{z \in \mathbb{R}^n : |z| < r\} \right) \right) \geq \theta r^n.
\]

Note that, in particular, a UR set is closed by definition, so that \(\Omega := \mathbb{R}^{n+1} \setminus E\) is open, but need not be connected.

We recall that \(n\)-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of \(H^n\) measure 0, by a countable union of Lipschitz images of \(\mathbb{R}^n\); we observe that BPLI is a quantitative version of this fact.

\textbf{Definition 1.4.} (“UR character”). Given a UR set \(E \subset \mathbb{R}^{n+1}\), its “UR character” is just the pair of constants \((\theta, M_0)\) involved in the definition of uniform rectifiability, along with the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.
It is worth mentioning that there exist sets that are ADR (and that even form the boundary of an open set satisfying interior Corkscrew and Harnack Chain conditions), but that are totally non-rectifiable (e.g., see the construction of Garnett’s “4-corners Cantor set” in [DS2, Chapter1]).

**Definition 1.5. (Corkscrew condition).** Following [JK], we say that an open set \( \Omega \subset \mathbb{R}^{n+1} \) satisfies the “Corkscrew condition” if for some uniform constant \( c > 0 \) and for every surface ball \( \Delta := \Delta(x, r) = B(x, r) \cap \partial \Omega \), with \( x \in \partial \Omega \) and \( 0 < r < \text{diam}(\partial \Omega) \), there is a ball \( B(\Delta, cr) \subset B(x, r) \cap \Omega \). The point \( X_\Delta \subset \Omega \) is called a “Corkscrew point” relative to \( \Delta \). We note that we may allow \( r < C \text{diam}(\partial \Omega) \) for any fixed \( C \), simply by adjusting the constant \( c \).

**Definition 1.6. (Harnack Chain condition).** Again following [JK], we say that \( \Omega \) satisfies the Harnack Chain condition if there is a uniform constant \( C \geq 1 \), and every pair of points \( X, X' \in \Omega \) with \( \text{dist}(X, \partial \Omega) \geq \rho \), \( \text{dist}(X', \partial \Omega) \geq \rho \) and \( |X - X'| < \Lambda \rho \), there is a chain of open balls \( B_1, \ldots, B_N \subset \Omega \), \( N \leq C(\Lambda) \), with \( X \in B_1, X' \in B_N, B_k \cap B_{k+1} \neq \emptyset \) and \( C^{-1} \text{diam}(B_k) \leq \text{dist}(B_k, \partial \Omega) \leq C \text{diam}(B_k) \). The chain of balls is called a “Harnack Chain”.

**Definition 1.7. (NTA).** Again following [JK], we say that an open set \( \Omega \subset \mathbb{R}^{n+1} \) is NTA (“Non-tangentially accessible”) if it satisfies the Harnack Chain condition, and if both \( \Omega \) and \( \Omega_{\text{ext}} := \mathbb{R}^{n+1} \setminus \overline{\Omega} \) satisfy the Corkscrew condition.

As we pointed out above and as can be seen from the definitions, non-tangentially accessible domains possess certain quantitative topological features. One can characterize an NTA domain with an ADR boundary in terms close to (1.3), but ensuring Big Pieces of Lipschitz Subdomains, rather than Big Pieces of Lipschitz Images (see Proposition 4.8), the crucial difference being that in some sense, a nice access to the boundary of a Lipschitz domain is partially retained, contrary to the general UR case.

Finally, let us define the scale-invariant estimates at the center of this paper.

**Definition 1.8.** Let \( E \subset \mathbb{R}^{n+1} \) be an \( n \)-dimensional ADR set and let \( F \in L^2_{\text{loc}}(\mathbb{R}^{n+1} \setminus E) \). We say that \( F \) satisfies the Carleson measure estimate (CME) on \( \mathbb{R}^{n+1} \setminus E \) if there exists a constant \( C > 0 \) such that

\[
(1.9) \quad \sup_{x \in E, 0 < r < \infty} \frac{1}{r^n} \int_{B(x, r)} |F(Y)|^2 \delta(Y) \, dY \leq C.
\]

Similarly, we say that \( F \in L^2_{\text{loc}}(D) \) satisfies the Carleson measure estimate in some open set \( D \subset \mathbb{R}^{n+1} \) with \( \partial D \) being \( n \)-dimensional ADR if there exists a constant \( C > 0 \) such that

\[
(1.10) \quad \sup_{x \in \partial D, 0 < r < \infty} \frac{1}{r^n} \int_{B(x, r) \cap D} |F(Y)|^2 \delta(Y) \, dY \leq C,
\]

where \( \delta(Y) = \text{dist}(Y, \partial D) \).

More generally, if \( E \) is the boundary of some open set \( D \subset \mathbb{R}^{n+1} \), we say that a given property stated on \( \mathbb{R}^{n+1} \setminus E \) is satisfied on \( D \) if the function in question is supported on \( D \).

**Definition 1.11.** Let \( \Omega := \mathbb{R}^{n+1} \setminus E \), where \( E \subset \mathbb{R}^{n+1} \) is an \( n \)-dimensional ADR set (hence closed); thus \( \Omega \) is open, but need not be a connected domain. Let \( u \in L^{\infty}(\Omega) \), with \( ||u||_{\infty} \leq 1 \), and let \( \varepsilon \in (0, 1) \). We say that \( u \) is \( \varepsilon \)-approximable, if there is a constant \( C_{\varepsilon} \), and a function \( \varphi = \varphi^\varepsilon \in W^{1,1}_{\text{loc}}(\Omega) \) satisfying

\[
(1.12) \quad ||u - \varphi||_{L^{\infty}(\Omega)} < \varepsilon,
\]

and

\[
(1.13) \quad \sup_{x \in E, 0 < r < \infty} \frac{1}{r^n} \int_{B(x, r)} |\nabla \varphi(Y)| \, dY \leq C_{\varepsilon}.
\]
Theorem 1.17. Let $E$ be a $n$-dimensional UR set, and let $\imath$ imply the Carleson measure estimates on the sets with UR boundaries, via the following formalism.

\begin{equation}
\|A\|_{\ell_2(\mathbb{R}^{n+1})} \leq C \|N_h\|_{L^\infty(\partial D)} \quad \text{for all } D \in \Sigma,
\end{equation}

for some $\kappa > 0$ (see Remark 2.35).

Let us now list some highlights of main results of this paper. First, the Carleson measure estimates on Lipschitz domains imply the Carleson measure estimates in NTA domains, which, in turn, imply the Carleson measure estimates on the sets with UR boundaries, via the following formalism.

Theorem 1.18. Let $E$ be a $n$-dimensional UR set, and let $\Omega := \mathbb{R}^{n+1} \setminus E$.

If for some $F \in L^2_{\text{loc}}(\Omega)$ for every bounded NTA subdomain $D \subset \Omega$ with an ADR boundary the Carleson measure estimate (1.10) is satisfied with a constant depending on $n$ and the NTA/ADR constants of $D$ only, then the Carleson measure estimate holds on $\Omega$ as well, i.e., (1.9) is satisfied, with the constant depending on $n$ and the UR character of $E$ only.

Furthermore, given an NTA domain $D \subset \mathbb{R}^{n+1}$ with an ADR boundary $E = \partial D$ and $F \in L^2_{\text{loc}}(D)$ which satisfies (3.2), if $F$ satisfies the Carleson measure estimate (1.10) on all bounded Lipschitz subdomains of $D$ with the constant depending on the Lipschitz constants of the underlying domains only, then $F$ satisfies the Carleson measure estimate (1.10) in $D$ as well, with the bound depending on the constant in (3.2), the NTA constants of $D$ and the ADR constants of $\partial D$ only.

We do not explain in details condition (3.2) now, but let us mention that generally it is a harmless bound on interior cubes, which, in the context of solutions, is a simple consequence of Caccioppoli’s inequality.

Proof. The Theorem is a combination of (reduced versions of) Theorem 3.3 and Theorem 4.10 proved in the body of the paper.

Secondly, in the class of Lipschitz domains, or in the class of NTA domains with ADR boundaries, or in the class of sets with UR boundaries, the Carleson measure estimates are equivalent to local and global area integral bounds (AKA square function estimates).

Theorem 1.19. Let $\Sigma$ be a subclass of ADR domains in $\mathbb{R}^{n+1}$ with the property that if $D$ belongs to $\Sigma$ then all its local sawtooth subdomains belong to $\Sigma$ (for example, a class of ADR subdomains of a certain set, or a class of bounded NTA subdomains with ADR boundaries, or a class of sets with UR boundaries, with uniform relevant geometric constants). Let $G \in L^2_{\text{loc}}(D)$ and $H \in C(D) \cap L^\infty(D)$ for all $D \in \Sigma$.

Then the Carleson measure estimate (1.10) is satisfied for $F = G/\|H\|_{L^\infty(D)}$ for all $D \in \Sigma$ if and only if

\begin{equation}
\|A\|_{\ell_2(\partial D)} \leq C \|N_h\|_{L^\infty(\partial D)}, \quad \text{for all } D \in \Sigma,
\end{equation}

for some $0 < q < \infty$ if and only if (1.19) holds for all $0 < q < \infty$. 

Definition 1.14. (Area integral and non-tangential maximal function). Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional ADR set, as in (1.1). For a continuous function $H$ in $\mathbb{R}^{n+1} \setminus E$, we define the “non-tangential maximal function” as

\[ N_h(x) := \sup_{y \in \Gamma(x)} |H(Y)|, \quad x \in E \]

and for $G \in L^2_{\text{loc}}(\mathbb{R}^{n+1} \setminus E)$, we define the area integral $\mathcal{A}(G)$, as follows:

\begin{equation}
\mathcal{A}(G) := \left( \int_{\Gamma(x)} |G(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2}, \quad x \in E,
\end{equation}

with $\delta(Y) = \text{dist}(Y,E)$, as before. In the body of the paper we will always use these definitions with dyadic cones $\Gamma(x)$, $x \in \partial \Omega$, which will be introduced later – see (2.22). Equivalently though, the reader can think instead of the traditional cones

\begin{equation}
\Gamma_\Omega(x) := \{ Y \in \Omega : |Y - x| \leq (1 + \kappa) \text{dist}(Y,\partial \Omega) \}, \quad x \in \partial \Omega,
\end{equation}

for some $\kappa > 0$ (see Remark 2.35).
Theorem 1.20. Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional UR set, $\Omega = \mathbb{R}^{n+1} \setminus E$, and suppose that $u \in W^{1,2}_{\text{loc}}(\Omega) \cap C(\Omega)$ satisfies (7.2). Assume, in addition, that $F = |\nabla u|/|u|_{L^\infty(\Omega)}$ satisfies the Carleson measure estimate (1.9). If for every bounded NTA subdomain $\Omega' \subset \Omega$ with an ADR boundary
\begin{equation}
\|N_\epsilon(u - u(X^+_{\Omega'}))\|_{L^2(\Omega')} \leq C \|\mathcal{A}(\nabla u)\|_{L^2(\partial\Omega')},
\end{equation}
holds with a constant depending on $n$, the NTA constants of $\Omega'$ and the ADR constants of $\partial\Omega'$ only, then $u$ is $\epsilon$-approximable on $\Omega$, with the implicit constants depending on $n$ and the UR character of $E$ only. Here, $X^+_{\Omega'}$ is any interior corkscrew point of $\Omega'$ at the scale of $\text{diam}(\Omega')$.

The Theorem is a combination of (the reduced versions of) Theorem 7.1 and 6.2. The interior bound (7.2) is, again, a fairly harmless prerequisite which follows from known interior estimates in the context of solutions of elliptic PDEs. We remark that the estimate (1.21) itself would not make much sense for general UR sets, because of topological obstructions (there is no preferred component for a corkscrew point in such a general context), and for that reason we pass directly to $\epsilon$-approximability.

2. Preliminaries and relevant results from [HMM]

We start with some further notation and definitions.

- We use the letters $c, C$ to denote harmless positive constants, not necessarily the same at each occurrence, which depend only on dimension and the constants appearing in the hypotheses of the theorems (which we refer to as the “allowable parameters”). We shall also sometimes write $a \leq b$ and $a \approx b$ to mean, respectively, that $a \leq Cb$ and $0 < c \leq a/b \leq C$, where the constants $c$ and $C$ are as above, unless explicitly noted to the contrary. At times, we shall designate by $M$ a particular constant whose value will remain unchanged throughout the proof of a given lemma or proposition, but which may have a different value during the proof of a different lemma or proposition.

- Given a closed set $E \subset \mathbb{R}^{n+1}$, we shall use lower case letters $x, y, z$, etc., to denote points on $E$, and capital letters $X, Y, Z$, etc., to denote generic points in $\mathbb{R}^{n+1}$ (especially those in $\mathbb{R}^{n+1} \setminus E$).

- The open $(n + 1)$-dimensional Euclidean ball of radius $r$ will be denoted $B(x, r)$ when the center $x$ lies on $E$, or $B(X, r)$ when the center $X \in \mathbb{R}^{n+1} \setminus E$. A “surface ball” is denoted $\Delta(x, r) := B(x, r) \cap E$ where unless otherwise specified we implicitly assume that $x \in E$.

- Given a Euclidean ball $B$ or surface ball $\Delta$, its radius will be denoted $r_B$ or $r_\Delta$, respectively.

- Given a Euclidean or surface ball $B = B(X, r)$ or $\Delta = \Delta(x, r)$, its concentric dilate by a factor of $\kappa > 0$ will be denoted $\kappa B := B(X, kr)$ or $\kappa \Delta := \Delta(x, kr)$.

- Given a (fixed) closed set $E \subset \mathbb{R}^{n+1}$, for $X \in \mathbb{R}^{n+1}$, we set $\delta(X) := \text{dist}(X, E)$.

- We let $H^n$ denote $n$-dimensional Hausdorff measure, and let $\sigma := H^n|_E$ denote the “surface measure” on $E$.

- We will also work with open sets $\Omega \subset \mathbb{R}^{n+1}$ in which case the previous notations and definitions easily adapt by letting $E := \partial\Omega$. 
For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $1_A$ denote the usual indicator function of $A$, i.e. $1_A(x) = 1$ if $x \in A$, and $1_A(x) = 0$ if $x \notin A$.

- For a Borel set $A \subset \mathbb{R}^{n+1}$, we let $\text{int}(A)$ denote the interior of $A$.

- Given a Borel measure $\mu$, and a Borel set $A$, with positive and finite $\mu$ measure, we set $\int_A f \, d\mu := \mu(A)^{-1} \int_A f \, d\mu$.

- We shall use the letter $I$ (and sometimes $J$) to denote a closed $(n+1)$-dimensional Euclidean dyadic cube with sides parallel to the co-ordinate axes, and we let $\ell(I)$ denote the side length of $I$. If $\ell(I) = 2^{-k}$, then we set $k_I := k$. Given an ADR set $E \subset \mathbb{R}^{n+1}$, we use $Q$ to denote a dyadic “cube” on $E$. The latter exist (cf. [DS1], [Chr]), and enjoy certain properties which we enumerate in Lemma 2.1 below.

**Lemma 2.1. (Existence and properties of the “dyadic grid”)** [DS1, DS2], [Chr]. Suppose that $E \subset \mathbb{R}^{n+1}$ is an $n$-dimensional ADR set. Then there exist constants $a_0 > 0$, $\gamma > 0$ and $C_1 < \infty$, depending only on dimension and the ADR constant, such that for each $k \in \mathbb{Z}$, there is a collection of Borel sets (“cubes”)

$$D_k := \{Q^k_j \subset E : j \in \mathcal{J}_k\},$$

where $\mathcal{J}_k$ denotes some (possibly finite) index set depending on $k$, satisfying

(i) $E = \bigcup_j Q^k_j$ for each $k \in \mathbb{Z}$.

(ii) If $m \geq k$ then either $Q^m_i \subset Q^k_j$ or $Q^m_i \cap Q^k_j = \emptyset$.

(iii) For each $(j,k)$ and each $m < k$, there is a unique $i$ such that $Q^m_i \subset Q^k_j$.

(iv) $\text{diam} (Q^k_j) \leq C_1 2^{-k}$.

(v) Each $Q^k_j$ contains some “surface ball” $\Delta(x^k_j, a_0 2^{-k}) := B(x^k_j, a_0 2^{-k}) \cap E$.

(vi) $H^n(\{x \in Q^k_j : \text{dist}(x, E \setminus Q^k_j) \leq \varrho 2^{-k}\}) \leq C_1 \varrho^\gamma H^n(Q^k_j)$, for all $k, j$ and for all $\varrho \in (0, a_0)$.

A few remarks are in order concerning this lemma.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [Chr], with the dyadic parameter $1/2$ replaced by some constant $\delta \in (0, 1)$. In fact, one may always take $\delta = 1/2$ (cf. [HMMM, Proof of Proposition 2.12]). In the presence of the Ahlfors-David property (1.2), the result already appears in [DS1, DS2].

- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \text{diam}(E)$, in the case that the latter is finite.

- We shall denote by $\mathcal{D} = \mathcal{D}(E)$ the collection of all relevant $Q^k_j$, i.e.,

$$\mathcal{D} := \bigcup_k \mathcal{D}_k,$$

where, if $\text{diam}(E)$ is finite, the union runs over those $k$ such that $2^{-k} \lesssim \text{diam}(E)$.

- For a dyadic cube $Q \in \mathcal{D}_k$, we shall set $\ell(Q) = 2^{-k}$, and we shall refer to this quantity as the “length” of $Q$. Evidently, $\ell(Q) \approx \text{diam}(Q)$.

- For a dyadic cube $Q \in \mathcal{D}$, we let $k(Q)$ denote the “dyadic generation” to which $Q$ belongs, i.e., we set $k = k(Q)$ if $Q \in \mathcal{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.

- Properties (iv) and (v) imply that for each cube $Q \in \mathcal{D}$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r)$ and a surface ball $\Delta(x_Q, r) := B(x_Q, r) \cap E$ such that $c \ell(Q) \leq r \leq \ell(Q)$ for some uniform constant $0 < c < 1$ and

$$\Delta(x_Q, r) \subset Q \subset \Delta(x_Q, C r).$$
for some uniform constant $C$. We shall denote this ball and surface ball by

\begin{equation}
B_Q := B(x_Q, r), \quad \Delta_Q := \Delta(x_Q, r),
\end{equation}

and we shall refer to the point $x_Q$ as the “center” of $Q$.

At this stage we would like to recall some results from [HMM]. Many of them have been stated for harmonic functions, but here we would like to highlight a more general point of view. We first give a definition to then continue with some key geometric lemmas from [HMM].

**Definition 2.4.** [DS2]. Let $S \subset \mathbb{D}(E)$. We say that $S$ is “coherent” if the following conditions hold:

(a) $S$ contains a unique maximal element denoted by $Q(S)$ which contains all other elements of $S$ as subsets.

(b) If $Q$ belongs to $S$, and if $Q \subset \tilde{Q} \subset Q(S)$, then $\tilde{Q} \in S$.

(c) Given a cube $Q \in S$, either all of its children belong to $S$, or none of them do.

We say that $S$ is “semi-coherent” if only conditions $(a)$ and $(b)$ hold.

**Lemma 2.5** (The bilateral “corona decomposition”, [HMM]). Suppose that $E \subset \mathbb{R}^{n+1}$ is $n$-dimensional UR. Then given any positive constants $\eta \ll 1$ and $K \gg 1$, there is a disjoint decomposition $\mathbb{D}(E) = \mathcal{G} \cup \mathcal{B}$, satisfying the following properties.

1. The “Good” collection $\mathcal{G}$ is further subdivided into disjoint stopping time regimes, such that each such regime $S$ is coherent (cf. Definition 2.4).

2. The “Bad” cubes, as well as the maximal cubes $Q(S)$ satisfy a Carleson packing condition:

\begin{equation}
\sum_{Q' \subset Q, Q' \in \mathcal{B}} \sigma(Q') + \sum_{S(Q(S) \subset Q)} \sigma(Q) \leq C_{\eta,K} \sigma(Q), \quad \forall Q \in \mathbb{D}(E).
\end{equation}

3. For each $S$, there is a Lipschitz graph $\Gamma_S$, with Lipschitz constant at most $\eta$, such that, for every $Q \in S$,

\begin{equation}
\sup_{x \in \Delta_Q} \text{dist}(x, \Gamma_S) + \sup_{y \in B'^*_Q \cap \Gamma_S} \text{dist}(y, E) < \eta \ell(Q),
\end{equation}

where $B'^*_Q := B(x, K\ell(Q))$ and $\Delta'^*_Q := B'^*_Q \cap E$.

Now we construct, for each stopping time regime $S$ in Lemma 2.5, a pair of NTA domains $\Omega^S_\pm$, with ADR boundaries, which provide a good approximation to $E$, at the scales within $S$, in some appropriate sense. To be a bit more precise, $\Omega^S := \Omega^S_+ \cup \Omega^S_-$ will be constructed as a sawtooth region relative to some family of dyadic cubes, and the nature of this construction will be essential to the dyadic analysis that we will use below. We first discuss some preliminary matters.

Let $W = \mathcal{W}(\mathbb{R}^{n+1} \setminus E)$ denote a collection of (closed) dyadic Whitney cubes of $\mathbb{R}^{n+1} \setminus E$, so that the cubes in $W$ form a pairwise non-overlapping covering of $\mathbb{R}^{n+1} \setminus E$, which satisfy

\begin{equation}
4 \text{diam}(I) \leq \text{dist}(4I, E) \leq \text{dist}(I, E) \leq 40 \text{diam}(I), \quad \forall I \in W
\end{equation}

(just dyadically divide the standard Whitney cubes, as constructed in [Ste, Chapter VI], into cubes with side length 1/8 as large) and also

\begin{equation}
(1/4) \text{diam}(I_1) \leq \text{diam}(I_2) \leq 4 \text{diam}(I_1),
\end{equation}

whenever $I_1$ and $I_2$ touch.

Let $E$ be an $n$-dimensional ADR set and pick two parameters $\eta \ll 1$ and $K \gg 1$. Define

\begin{equation}
\mathcal{W}^\rho_Q := \{ I \in W : \eta^{1/4} \ell(Q) \leq \ell(I) \leq K^{1/2} \ell(Q), \text{ dist}(I, Q) \leq K^{1/2} \ell(Q) \}.
\end{equation}
\textbf{Remark 2.9.} We note that $\mathcal{W}_Q^0$ is non-empty, provided that we choose $\eta$ small enough, and $K$ large enough, depending only on dimension and the ADR constant of $E$. Indeed, given an $n$-dimensional ADR set $E$, and given $Q \in \mathbb{D}(E)$, consider the ball $B_Q = B(x_Q, r)$, as defined in (2.2)-(2.3), with $r \approx \ell(Q)$, so that $\Delta_Q = B_Q \cap E \subset Q$. By [HM, Lemma 5.3], we have that for some $C = C(n, \text{ADR})$, 
\[ \left| \{ Y \in \mathbb{R}^{n+1} : \delta(Y) < \epsilon \} \cap B_Q \right| \leq C \epsilon r^{n+1}, \]
for every $0 < \epsilon < 1$. Consequently, fixing $0 < \epsilon_0 < 1$ small enough, there exists $X_Q \in \frac{1}{2} B_Q$, with $\delta(X_Q) \geq \epsilon_0 r$. Thus, $B(X_Q, \epsilon_0 r/2) \subset B_Q \setminus E$. We shall refer to this point $X_Q$ as a “Corkscrew point” relative to $Q$. Now observe that $X_Q$ belongs to some Whitney cube $I \in \mathcal{W}$, which will belong to $\mathcal{W}_Q^0$, for $\eta$ small enough and $K$ large enough.

Next, we choose a small parameter $\tau_0 > 0$, so that for any $I \in \mathcal{W}$, and any $\tau \in (0, \tau_0]$, the concentric dilate $I^*(\tau) := (1 + \tau)I$ still satisfies the Whitney property
\[ \text{diam } I \approx \text{diam } I^*(\tau) \approx \text{dist } (I^*(\tau), E) \approx \text{dist } (I, E), \quad 0 < \tau \leq \tau_0. \]
Moreover, for $\tau \leq \tau_0$ small enough, and for any $I, J \in \mathcal{W}$, we have that $I^*(\tau)$ meets $J^*(\tau)$ if and only if $I$ and $J$ have a boundary point in common, and that, if $I \neq J$, then $I^*(\tau)$ misses $(3/4)J$.

At this point the discussion splits into a few special cases depending whether we have some extra information about $E$. The main idea consists in constructing some kind of “Whitney regions” which will allow us to introduce some “Carleson boxes” and “sawtooth subdomains”. The construction of the Whitney regions depends very much on the background assumptions, having extra information about $E$ will allow us to augment the collections $\mathcal{W}_Q^0$ so that we gain some connectivity on the corresponding Whitney regions and hence the resulting subdomains would have better properties. We consider three cases. In the first one we assume only that $E$ is ADR (but is not UR) and we set $\mathcal{W}_Q = \mathcal{W}_Q^0$ (here we do not gain any connectivity). The second case deals with $E$ being UR, in which case we can invoke Lemma 2.5 and use the Lipschitz graphs associated to the good regimes so that the augmented collection $\mathcal{W}_Q$ creates two nice Whitney regions one lying respectively above and below the Lipschitz graph. Finally, when $E$ is the boundary of a bounded NTA domain with ADR boundary (hence $E$ is UR) we are just interested on keeping the Whitney regions inside $D$ and we can augment $\mathcal{W}_Q^0$ using that $D$ is Harnack chain connected so that the resulting collections $\mathcal{W}_Q$ give some Whitney regions which are indeed bounded NTA domains with ADR boundary.

In order to keep a unified presentation let us assume that for every $Q \in \mathbb{D}$ we are given $\mathcal{W}_Q \supseteq \mathcal{W}_Q^0$ (below we will give the specific definition in each different case) and a constant $C \geq 1$ so that the following hold:
\[ C^{-1} \eta^{1/2} \ell(Q) \leq \ell(I) \leq CK^{1/2} \ell(Q), \quad \forall I \in \mathcal{W}_Q, \]
\[ \text{dist } (I, Q) \leq CK^{1/2} \ell(Q), \quad \forall I \in \mathcal{W}_Q. \]

Fix $0 < \tau \leq \tau_0/4$ as above. Given an arbitrary $Q \in \mathbb{D}(E)$, we may define an associated \textbf{Whitney region} $U_Q$ (not necessarily connected), as follows:
\[ U_Q = U_{Q, \tau} := \bigcup_{I \in \mathcal{W}_Q} I^*(\tau) \]
For later use, it is also convenient to introduce some fattened version of $U_Q$
\[ \hat{U}_Q = \hat{U}_{Q, 2\tau} := \bigcup_{I \in \mathcal{W}_Q} I^*(2\tau). \]
When the particular choice of $\tau \in (0, \tau_0]$ is not important, for the sake of notational convenience, we may simply write $I^*$ and $U_Q$ in place of $I^*(\tau)$ and $U_{Q, \tau}$.
We may also define the Carleson box relative to \( Q \in D(E) \), by
\[
T_Q = T_{Q,\tau} := \text{int} \left( \bigcup_{Q' \in D_Q} U_{Q',\tau} \right),
\]
where
\[
D_Q := \{ Q' \in D(E) : Q' \subset Q \}.
\]
Let us note that we may choose \( K \) large enough so that, for every \( Q \),
\[
T_{Q,\tau} \subset T_{Q,\tau_0} \subset B^*_{Q} := B(x_Q, K\ell(Q)).
\]
For future reference, we also introduce dyadic sawtooth regions as follows. Given a family \( \mathcal{F} \) of disjoint cubes \( \{ Q_j \} \subset D \), we define the global discretized sawtooth relative to \( \mathcal{F} \) by
\[
D_{\mathcal{F}} := D \setminus \bigcup_{Q_j \in \mathcal{F}} D_{Q_j},
\]
i.e., \( D_{\mathcal{F}} \) is the collection of all \( Q \in D \) that are not contained in any \( Q_j \in \mathcal{F} \). Given some fixed cube \( Q \), the local discretized sawtooth relative to \( \mathcal{F} \) by
\[
D_{\mathcal{F},Q} := D \setminus \bigcup_{Q_j \in \mathcal{F}} D_{Q_j} = D_{\mathcal{F}} \cap D_{Q}.
\]
Note that we can also allow \( \mathcal{F} \) to be empty in which case \( D_{\emptyset} = D \) and \( D_{\emptyset, Q} = D_{Q} \).

Similarly, we may define geometric sawtooth regions as follows. Given a family \( \mathcal{F} \subset D \) of disjoint cubes as before we define the global sawtooth and the local sawtooth relative to \( \mathcal{F} \) by respectively
\[
\Omega_{\mathcal{F}} := \text{int} \left( \bigcup_{Q' \in D_{\mathcal{F}}} U_{Q'} \right), \quad \Omega_{\mathcal{F},Q} := \text{int} \left( \bigcup_{Q' \in D_{\mathcal{F},Q}} U_{Q'} \right).
\]
Notice that \( \Omega_{\emptyset, Q} = T_Q \). For the sake of notational convenience, we set
\[
W_{\mathcal{F}} := \bigcup_{Q' \in D_{\mathcal{F}}} W_{Q'}, \quad W_{\mathcal{F},Q} := \bigcup_{Q' \in D_{\mathcal{F},Q}} W_{Q'},
\]
so that in particular, we may write
\[
\Omega_{\mathcal{F},Q} = \text{int} \left( \bigcup_{I \in W_{\mathcal{F},Q}} I \right).
\]
Finally, for every \( x \in E \), we define non-tangential approach regions, cones, as
\[
\Gamma(x) = \bigcup_{Q \in D(E) : Q \ni x} U_Q.
\]
Their local versions are given by
\[
\Gamma^0(x) = \bigcup_{Q' \in D_Q : Q' \ni x} U_{Q'}, \quad x \in Q.
\]
We shall often change the “aperture” of cones, Carleson boxes, sawtooth regions, by either using \( \hat{U}_Q = U_{Q,2\tau} \) (cf. (2.13)) in place of \( U_Q \) in the definitions or by changing \( \eta \) and \( K \). The corresponding larger sets will be always distinguished by a “widehat”, and within the same notation, the aperture can become larger from line to line as long as \( \tau, \eta \) and \( K \) (or \( \kappa \)) depend only on allowable geometric characteristics, that is, ADR, UR, NTA constants (depending on the case). Standard real variable arguments show that the \( L^p \) norms of non-tangential maximal functions defined with different apertures are equivalent, and the same applies to area integrals and square functions.
**Case ADR.** Here we assume that the set $E$ under consideration is merely ADR, but possibly not UR. Let us set $\mathcal{W}_Q = \mathcal{W}_Q^0$ as defined in (2.8). By definition (cf. (2.8)) we clearly have (2.11) with $C = 1$ and all the previous are therefore at our disposal. In [HMM] it was shown that the ADR property is inherited by all dyadic local sawtooths and all Carleson boxes:

**Proposition 2.24** ([HMM, Proposition A.2]). Let $E \subset \mathbb{R}^{n+1}$ be a closed $n$-dimensional ADR set (which may be UR, or not; if so, the sawtooth regions and Carleson boxes are built using the augmented collections $\mathcal{W}_Q$ to be constructed momentarily). Then all dyadic local sawtooths $\Omega_{F, Q}$ and all Carleson boxes $T_Q$ have $n$-dimensional ADR boundaries. In all cases, the implicit constants are uniform and depend only on dimension, the ADR constant of $E$ and the parameters $\eta, K$, and $\tau$.

**Case UR.** Here we further assume that $E$ is UR and make the corresponding bilateral corona decomposition of Lemma 2.5 with $\eta \ll 1$ and $K \gg 1$. Given $Q \in \mathcal{D}(E)$, for this choice of $\eta$ and $K$, we set as above $B^*_Q := B(x_Q, K\ell(Q))$, where we recall that $x_Q$ is the center of $Q$ (see (2.2)-(2.3)). For a fixed stopping time regime $S$, we choose a co-ordinate system so that $\Gamma_S = \{(z, \varphi_S(z)) : z \in \mathbb{R}^n\}$, where $\varphi_S : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function with $\|\varphi\|_{\text{Lip}} \leq \eta$.

**Claim 2.25** ([HMM, Claim 3.4]). If $Q \in S$, and $I \in \mathcal{W}_Q^0$, then $I$ lies either above or below $\Gamma_S$. Moreover, $\text{dist}(I, \Gamma_S) \geq \eta^{1/2}\ell(Q)$ (and therefore, by (2.6), $\text{dist}(I, \Gamma_S) \approx \text{dist}(I, E)$, with implicit constants that may depend on $\eta$ and $K$).

Next, given $Q \in S$, we augment $\mathcal{W}_Q^0$. We split $\mathcal{W}_Q^0 = \mathcal{W}_Q^{0+} \cup \mathcal{W}_Q^{0-}$, where $I \in \mathcal{W}_Q^{0+}$ if $I$ lies above $\Gamma_S$, and $I \in \mathcal{W}_Q^{0-}$ if $I$ lies below $\Gamma_S$. Choosing $K$ large and $\eta$ small enough, by (2.6), we may assume that both $\mathcal{W}_Q^{0+},$ are non-empty. We focus on $\mathcal{W}_Q^{0+}$, as the construction for $\mathcal{W}_Q^{0-}$ is the same. For each $I \in \mathcal{W}_Q^{0+}$, let $X_I$ denote the center of $I$. Fix one particular $I_0 \in \mathcal{W}_Q^{0+}$, with center $X_{I_0}^* := X_{I_0}$. Let $\tilde{Q}$ denote the dyadic parent of $Q$, unless $Q = Q(S)$; in the latter case we simply set $\tilde{Q} = Q$. Note that $\tilde{Q} \in S$, by the coherency of $S$. By Claim 2.25, for each $I$ in $\mathcal{W}_Q^{0+}$, or in $\mathcal{W}_Q^{0+}$, we have

$$\text{dist}(I, E) \approx \text{dist}(I, Q) \approx \text{dist}(I, \Gamma_S),$$

where the implicit constants may depend on $\eta$ and $K$. Thus, for each such $I$, we may fix a Harnack chain, call it $\mathcal{H}_I$, relative to the Lipschitz domain

$$\Omega_S^+ := \{(x, t) \in \mathbb{R}^{n+1} : t > \varphi_S(x)\},$$

connecting $X_I$ to $X_{I_0}^*$. By the bilateral approximation condition (2.6), the definition of $\mathcal{W}_Q^0$, and the fact that $K^{1/2} \ll K$, we may construct this Harnack Chain so that it consists of a bounded number of balls (depending on $\eta$ and $K$), and stays a distance at least $c\eta^{1/2}\ell(Q)$ away from $\Gamma_S$ and from $E$. We let $\mathcal{W}_Q^{0+}$ denote the set of all $J \in \mathcal{W}$ which meet at least one of the Harnack chains $\mathcal{H}_I$, with $I \in \mathcal{W}_Q^{0+} \cup \mathcal{W}_Q^{0-}$ (or simply $I \in \mathcal{W}_Q^{0+}$, if $Q = Q(S)$), i.e.,

$$\mathcal{W}_Q^{0+} := \left\{ I \in \mathcal{W} : \exists J \in \mathcal{W}_Q^{0+} \cup \mathcal{W}_Q^{0-} \text{ for which } \mathcal{H}_J \cap J \neq \emptyset \right\},$$

where as above, $\tilde{Q}$ is the dyadic parent of $Q$, unless $Q = Q(S)$, in which case we simply set $\tilde{Q} = Q$ (so the union is redundant). We observe that, in particular, each $I \in \mathcal{W}_Q^{0+} \cup \mathcal{W}_Q^{0-}$ meets $\mathcal{H}_I$, by definition, and therefore

$$\mathcal{W}_Q^{0+} \cup \mathcal{W}_Q^{0+} \subset \mathcal{W}_Q^{0+}. (2.26)$$

Of course, we may construct $\mathcal{W}_Q^{0-}$ analogously. We then set

$$\mathcal{W}_Q := \mathcal{W}_Q^{0+} \cup \mathcal{W}_Q^{0-}.$$
It follows from the construction of the augmented collections $\mathcal{W}_{Q}^{*,\pm}$ that there are uniform constants $c$ and $C$ such that

\begin{equation}
\eta^{1/2}\ell(I) \leq \ell(I) \leq CK^{1/2}\ell(I), \quad \forall I \in \mathcal{W}_{Q},
\end{equation}

\begin{equation}
\text{dist}(I, Q) \leq CK^{1/2}\ell(I), \quad \forall I \in \mathcal{W}_{Q}'.
\end{equation}

It is convenient at this point to introduce some additional terminology.

**Definition 2.28.** Given $Q \in \mathcal{G}$, and hence in some $\mathcal{S}$, we shall refer to the point $X_{Q}$ specified above, as the “center” of $U_{Q}^{+}$ (similarly, the analogous point $X_{Q}^{-}$, lying below $\Gamma_{S}$, is the “center” of $U_{Q}^{-}$).

We also set $Y_{Q}:=X_{Q}$ and we call this point the “modified center” of $U_{Q}$, where as above $\tilde{Q}$ is the dyadic parent of $Q$, unless $Q = Q(\mathcal{S})$, in which case $Q = \tilde{Q}$, and $Y_{\tilde{Q}} = X_{\tilde{Q}}$.

Observe that $\mathcal{W}_{Q}^{*,\pm}$ and hence also $\mathcal{W}_{Q}$ have been defined for any $Q$ that belongs to some stopping time regime $\mathcal{S}$, that is, for any $Q$ belonging to the “good” collection $\mathcal{G}$ of Lemma 2.5. On the other hand, we have defined $\mathcal{W}_{Q}^{\ell}$ for arbitrary $Q \in \mathcal{D}(E)$. We now set

\begin{equation}
\mathcal{W}_{Q} := \left\{ \begin{array}{ll}
\mathcal{W}_{Q}^{*} & Q \in \mathcal{G}, \\
\mathcal{W}_{Q}^{\ell} & Q \in \mathcal{B},
\end{array} \right.
\end{equation}

and for $Q \in \mathcal{G}$ we shall henceforth simply write $\mathcal{W}_{Q}^{\pm}$ in place of $\mathcal{W}_{Q}^{*,\pm}$. Notice that by (2.8) when $Q \in \mathcal{B}$ and by (2.27) when $Q \in \mathcal{G}$ we clearly obtain (2.11) with $C$ depending on $\mathcal{U}$ character of $E$.

Given an arbitrary $Q \in \mathcal{D}(E)$ and $0 < \tau \leq \tau_{0}/4$, we may define an associated Whitney region $U_{Q}$ (not necessarily connected) as in (2.12) or the fattened version of $\tilde{U}_{Q}$ as in (2.13). In the present situation, if $Q \in \mathcal{G}$, then $U_{Q}$ splits into exactly two connected components

\begin{equation}
U_{Q}^{\pm} := \bigcup_{I \in \mathcal{W}_{Q}} I^\tau \cdot \mathcal{O}_{F},
\end{equation}

We note that for $Q \in \mathcal{G}$, each $U_{Q}^{\pm}$ is Harnack chain connected, by construction (with constants depending on the implicit parameters $\tau, \eta$ and $K$); moreover, for a fixed stopping time regime $\mathcal{S}$, if $Q'$ is a child of $Q$, with both $Q', Q \in \mathcal{S}$, then $U_{Q'}^{\pm} \cup U_{Q}^{\pm}$ is Harnack Chain connected, and similarly for $U_{Q'}^{-} \cup U_{Q}^{-}$.

We may also define the Carleson boxes $T_{Q}$, global and local sawtooth regions $\Omega_{F}, \Omega_{F,Q}$, cones $\Gamma$, and local cones $\Gamma_{Q}$ as in (2.14) (2.19), (2.22), and (2.23).

**Remark 2.31.** We recall that, by construction (cf. (2.26), (2.29)), given $Q \in \mathcal{G}$ $\mathcal{W}_{Q}^{\ell,\pm} \subset \mathcal{W}_{Q}$, and therefore $Y_{Q}^{\pm} \in U_{Q}^{\pm} \cap U_{Q}^{\pm}$. Moreover, since $Y_{Q}^{\pm}$ is the center of some $I \in \mathcal{W}_{Q}^{\ell,\pm}$, we have that dist $(Y_{Q}^{\pm}, \partial U_{Q}^{\pm}) \approx \ell(Q)$ (with implicit constants possibly depending on $\eta$ and/or $K$).

**Remark 2.32.** Given a stopping time regime $\mathcal{S}$ as in Lemma 2.5, for any semi-coherent subregime (cf. Definition 2.4) $\mathcal{S}' \subset \mathcal{S}$ (including, of course, $\mathcal{S}$ itself), we now set

\begin{equation}
\Omega_{\mathcal{S}'}^{\pm} = \text{int} \left( \bigcup_{Q \in \mathcal{S}'} U_{Q}^{\pm} \right),
\end{equation}

and let $\Omega_{\mathcal{S}'} := \Omega_{\mathcal{S}'}^{+} \cup \Omega_{\mathcal{S}'}^{-}$. Note that implicitly, $\Omega_{\mathcal{S}'}$ depends upon $\tau$ (since $U_{Q}^{\pm}$ has such dependence). When it is necessary to consider the value of $\tau$ explicitly, we shall write $\Omega_{\mathcal{S}'}(\tau)$.

The main geometric lemma for the previous sawtooth regions is the following.
Lemma 2.34 ([HMM, Lemma 3.24]). Let \( S \) be a given stopping time regime as in Lemma 2.5, and let \( S' \) be any nonempty, semi-coherent subregime of \( S \). Then for \( 0 < \tau \leq \tau_0 \), with \( \tau_0 \) small enough, each of \( \Omega_{S'}^\tau \) is an NTA domain with ADR boundary with character depending only on \( n, \tau, \eta, K \), and the ADR/UR constants for \( E \).

Case NTA. Here we assume that \( E \) is ADR and is the boundary of \( D \), a bounded NTA. This is, strictly speaking, a sub-case of the Case UR above, but the extra assumption that \( E \) is a boundary of some bounded NTA makes the construction simpler. In this case, we are basically in the situation which is equivalent to being within one regime \( S \), at least as far as the construction of \( \mathcal{W}_Q \) is concerned.

Let \( D \) be a bounded NTA with ADR boundary and write \( E = \partial D \). Define \( \mathcal{W} \) as above, but in this case we only keep those Whitney cubes contained in \( D \) (that is we are doing a Whitney decomposition of \( D \) rather than that of \( \mathbb{R}^{n+1} \setminus E \)). Let \( \mathcal{W}_Q^0 \) be as defined in (2.8) (once again, considering only the Whitney cubes in \( D \)). Next, given any \( Q \in \mathcal{D}(E) \), augment \( \mathcal{W}_Q^0 \) to \( \mathcal{W}_Q^* \) as done in [HM, Section 3] using the fact that one can construct a Harnack chain to connect \( X_Q \) (a corkscrew point relative to \( Q \) with any of the centers of the Whitney cubes in \( \mathcal{W}_Q^0 \). Notice that in the case when \( E \) is UR and \( Q \in S \) we have used a similar idea, the main difference is that the Harnack chain in that case comes from the fact that \( \Omega_{S'}^\tau \) is a Lipschitz domain, whereas here such property comes from the assumption that \( D \) is NTA and hence the Harnack chain condition holds.

With the appropriate choice of a sufficiently small \( \eta \) and a sufficiently large \( K \) depending on \( n \), the NTA constants of \( D \) only, we can guarantee the same key properties for the resulting augmented \( \mathcal{W}_Q^* \). In particular, (2.11) holds, the cubes in \( \mathcal{W}_Q^0 \cup \mathcal{W}_Q^* \) are contained in \( \mathcal{W}_Q^0 \) together with the associated Harnack chains, the corkscrew points \( X_Q \) and \( X_Q^\star \) are contained in \( \mathcal{W}_Q^* \), and others. Then one set \( \mathcal{W}_Q = \mathcal{W}_Q^* \) and uses (2.12)–(2.23) to define Whitney regions, Carleson boxes, sawtooth regions, cones, in the very same way as in the UR case and, respectively, satisfying the same properties.

We observe that from [HM, Lemma 3.61] it follows that all Carleson boxes, all sawtooth regions and local sawtooth regions have ADR boundary and satisfy the (interior) Harnack chain and Corkscrew condition. We claim that the exterior Corkscrew condition holds as well. Let \( D_\star \) be one of these subdomains and take \( x_\star \in \partial D_\star \) and \( 0 < r < \text{diam}(\partial D_\star) \). By construction \( \partial D_\star \subset \mathcal{D} \) and we consider two cases \( 0 \leq \delta(x_\star) \leq r/2 \) and \( \delta(x_\star) > r/2 \). In the first scenario we pick \( x \in \partial D \) so that \( |x - x_\star| = \delta(x_\star) \leq r/2 \) (notice that \( x = x_\star \) if \( x_\star \in \partial D \cap \partial D_\star \)). Since \( D \) is an NTA domain it satisfies the exterior Corkscrew condition, hence we can find \( X \in D_{\text{ext}} = \mathbb{R}^{n+1} \setminus \overline{D} \) so that \( B(X, c_0r') \subset B(x, r/2) \cap D_\star \) where \( c_0 \) is the exterior corkscrew constant. Note that \( D_\star \subset D \), hence \( B(X, c_0r/2) \subset (D_\star)_{\text{ext}} \). Also, \( B(X, c_0r/2) \subset B(x, r/2) \subset B(x_\star, r) \). This shows that \( X \) is an exterior corkscrew point relative to the surface ball \( B(x_\star, r) \cap \partial D_\star \) for the domain \( D_\star \) with constant \( c_0/2 \). Consider next the case on which \( \delta(x_\star) > r/2 \). Note that in particular \( x_\star \in \Omega \) and therefore we can find two Whitney cubes \( I, J \in \mathcal{W} \) so that \( x \in \partial I \cap J, \partial I \cap \partial J \neq \emptyset, \text{int}(I^\star) \subset D_\star \) and \( J \) is a Whitney cube which does not belong to any of the \( \mathcal{W}_Q \) that define \( D_\star \). Note that \( \ell(J) \geq \delta(x_\star)/C > r/(2C) \) for some uniform constant \( C \geq 1 \), that \( I^\star \) misses \( \frac{3}{2}J \) as observed before and that the center of \( J \) satisfies \( X(J) \in (D_\star)_{\text{ext}} \). It is then clear that the open segment joining \( x_\star \) with \( X(J) \) is contained in \( (D_\star)_{\text{ext}} \) and we pick \( X \) in that segment so that \( |X - x_\star| = r/(8C) \) and hence \( B(X, r/(16C)) \subset B(x_\star, r) \cap D_\star \). This shows that \( X \) is an exterior corkscrew point relative to the surface ball \( B(x_\star, r) \cap \partial D_\star \) for the domain \( D_\star \) with constant \( 1/(16C) \). Therefore, we have shown that \( D \) satisfies the exterior Corkscrew condition.

Remark 2.35. When \( \Omega \) is an NTA domain with ADR boundary, or more generally, an open set \( \Omega \) with a UR or even ADR boundary \( E = \partial \Omega \), we will also use a non-dyadic definition of cones (1.16). It is straightforward to see that given \( \eta \) and \( K \) as above there exists \( \kappa \) such that dyadic cones \( \Gamma(x) \) are contained in \( \Gamma_\Omega(x) \) for all \( x \in \partial \Omega \). Vice versa, given \( \kappa > 0 \), there exist \( \eta \) and \( K \) such that \( \Gamma_\Omega(x) \) are contained in \( \Gamma(x) \) for all \( x \in \partial \Omega \).
3. Transference of Carleson measure estimates from NTA to Uniformly Rectifiable domains

Let us now discuss the “transference” mechanism allowing one to pass from the Carleson measure estimates on NTA domains to those for domains with UR boundaries. They are due to [HMM], although there the discussion is formally confined to the case of harmonic functions.

When \( u \) is a bounded solution of a second order elliptic PDE, e.g., a harmonic function in \( \mathbb{R}^{n+1} \setminus E \) or in a domain \( \Omega \), for reasonably nice \( E \) and \( \Omega \), one expects (1.9)–(1.10) with \( F = |\nabla u|/\|u\|_{L^\infty(\Omega)} \), and for a solution of a \( 2m \)-th order elliptic PDE, \( m \in \mathbb{N} \), we will be aiming at \( F = |\nabla^m u|/\|\nabla^{m-1} u\|_{L^\infty(\Omega)} \).

We shall come back to this point with more details in Section 8 and for now try to keep the discussion general for as long as possible.

Remark 3.1. There is a slightly glitchy notation point here. For homogeneity reasons one could prefer to normalize so that \( F = \delta|\nabla u|/\|u\|_{L^\infty(\Omega)} \). However, making the function \( F \) and later on \( G \) and \( H \) in Section 5 depend on the domain (via distance) has its own dangers and kills the beauty of the generality here.

Recall now the dyadic grid in Lemma 2.1 and the Whitney regions \( U_\delta \) from (2.12). Since every Whitney region is contained in a ball (of a possibly larger but proportional to the scale radius) by (2.16), a necessary condition for (1.9) is that

\[
(3.2) \sup_{Q \in \mathcal{D}(E)} \frac{1}{\sigma(Q)} \int_{\partial Q} |F(Y)|^2 \delta(Y) \, dY \leq C.
\]

This will be a starting assumption in most of our statements, which however in all applications to the CME for solutions of elliptic PDEs will be automatically fulfilled by Caccioppoli’s inequality. We shall discuss this in more details together with the corresponding applications.

While stated exclusively for harmonic functions, the main result in [HMM] can be reformulated as follows.

Theorem 3.3. Let \( E \subset \mathbb{R}^{n+1} \) be an \( n \)-dimensional UR set, and suppose that \( F \in L^2_{\text{loc}}(\mathbb{R}^{n+1} \setminus E) \) satisfies (3.2).

If for every \( \Omega^\pm_k \) defined by (2.33) (with \( S' = S \)) we have

\[
(3.4) \sup_{x \in \partial \Omega^\pm_k, 0 < r < \infty} \frac{1}{r^n} \int_{B(x,r) \cap \Omega^\pm_k} |F(Y)|^2 \text{dist}(Y, \partial \Omega^\pm_k) \, dY \leq C_0,
\]

for some \( C_0 > 0 \), then

\[
(3.5) \sup_{x \in \partial \Omega, 0 < r < \infty} \frac{1}{r^n} \int_{B(x,r) \cap \Omega} |F(Y)|^2 \delta(Y) \, dY \leq C,
\]

with \( C > 0 \) depending on \( C_0 \), the constant in (3.2), \( n \), the ADR/UR constants of \( E \), and the choice of \( \eta, K, \tau \) only.

In particular, if for some \( F \in L^2_{\text{loc}}(\mathbb{R}^{n+1} \setminus E) \) for every bounded NTA subdomain \( \Omega \subset \mathbb{R}^{n+1} \setminus E \) with an ADR boundary

\[
(3.6) \sup_{x \in \partial \Omega, 0 < r < \infty} \frac{1}{r^n} \int_{B(x,r) \cap \Omega} |F(Y)|^2 \text{dist}(Y, \partial \Omega) \, dY \leq C_0,
\]

with a constant \( C_0 > 0 \) depending on \( n \), and the NTA/ADR constants of \( \Omega \) only, then (3.5) holds, and \( C > 0 \) depends on \( n, \eta, \tau, K \), and the ADR/UR constants of \( E \) only.

Note that under the assumption (3.6) pertaining to all bounded NTA subdomains of \( \mathbb{R}^{n+1} \setminus E \), the condition (3.2) is automatically satisfied.
4. John-Nirenberg Inequality and Transference of Carleson Measure Estimates from Lipschitz to Uniformly Rectifiable Domains

We start with a following version of the John-Nirenberg inequality. It is a suitable modification of Lemma 10.1 in [HMa] which, in turn, was inspired by Lemma 2.14 in [AHLT].

Lemma 4.1. Let \( E \subset \mathbb{R}^{n+1} \) be an \( n \)-dimensional ADR set. Suppose there exist numbers \( 0 < \alpha < 1 \) and \( 0 < N < \infty \) such that for some function \( F \in L^2_{\text{loc}}(\mathbb{R}^{n+1} \setminus E) \) and every \( Q \subset \mathcal{D}(E) \)

\[
\sigma \left\{ x \in Q : \left( \int_{\Gamma Q(x)} |F(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2} > N \right\} \leq \alpha \sigma(Q).
\]

Then there exists \( C > 0 \) such that

\[
\sup_{Q \in \mathcal{D}(E)} \frac{1}{\sigma(Q)} \int_Q \left( \int_{\Gamma Q(x)} |F(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{p/2} \, d\sigma(x) \leq C,
\]

for all \( p \in (0, \infty) \).

Proof. Fix \( Q \in \mathcal{D}(E) \) and denote the set on the left-hand side of (4.2) by \( E_{N,Q} \), so that \( \sigma(E_{N,Q}) \leq \alpha \sigma(Q) \). We can assume that \( \sigma(E_{N,Q}) \neq 0 \), for, otherwise, the contribution of \( Q \) into (4.3) is \( N^p \) (which can be absorbed in \( C \)). Clearly, also \( E_{N,Q} \neq Q \) since \( \alpha < 1 \). Moreover, by outer regularity of the measure, we can find a set \( \tilde{E}_{N,Q} \) such that \( E_{N,Q} \subset \tilde{E}_{N,Q} \subset Q \), \( \tilde{E}_{N,Q} \) is relatively open in \( Q \), and

\[
\sigma(\tilde{E}_{N,Q}) \leq \frac{1 + \alpha}{2} \sigma(Q).
\]

Thus, one can build a collection of (pairwise disjoint) maximal dyadic cubes \( \{ Q_j \} \subset \mathcal{D}_Q \setminus Q \) with \( \bigcup_j Q_j = \tilde{E}_{N,Q} \).

Fix one of the maximal cubes \( Q_j \). By maximality, for every \( P \supset Q_j, P \neq Q_j, P \in \mathcal{D}_Q \), (and since \( \{ Q_j \} \subset \mathcal{D}_Q \setminus Q \), at least one such \( P \) always exists) there exists \( x' \in P \) such that

\[
\left( \int_{\Gamma x'} |F(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2} \leq N.
\]

Hence, in particular, for every \( P \in \mathcal{D}_Q \setminus \mathcal{D}_{Q_j}, P \supset Q_j, P \neq Q_j \) (and again, at least one such \( P \) always exists) we have

\[
\left( \int_{U_P} |F(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2} \leq N.
\]

Now let

\[
M_Q(k) := \sup_{Q' \in \mathcal{D}_Q} \frac{1}{\sigma(Q')} \int_{Q'} \left( \int_{\Gamma Q'(x)} |F(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{p/2} \, d\sigma(x),
\]

where, much as before,

\[
\Gamma^{Q,k}(x) = \bigcup_{P \in \mathcal{D}_Q : P \ni x} U_P, \quad x \in Q,
\]

and \( M_Q(k) = 0 \) whenever \( \Gamma^{Q,k}(x) = \emptyset \) for all \( x \in Q', Q' \in \mathcal{D}_Q \). Then

\[
\int_Q \left( \int_{\Gamma Q(x)} |F(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{p/2} \, d\sigma(x)
\]
Note that, by definition, \(\text{(ii)}\) of Lemma 2.1, if there is a point \(x\) and observe that the conditions

\[
\leq \int_{Q \cap E_{x,Q}} \left( \iint_{\Gamma^{Q,j}(x)} |F(Y)|^2 \, dY \right)^{p/2} \, d\sigma(x)
\]

\[
+ \sum_j \int_{Q_j} \left( \iint_{\Gamma^{Q,j}(x)} |F(Y)|^2 \, dY \right)^{p/2} \, d\sigma(x)
\]

\[
\leq N^p \sigma(Q) + \sum_j \int_{Q_j} \left( \iint_{\Gamma^{Q,j}(x)} |F(Y)|^2 \, dY \right)^{p/2} \, d\sigma(x)
\]

\[
+ \sum_j \int_{Q_j} \left( \iint_{\Gamma^{Q,j}(x)} |F(Y)|^2 \, dY \right)^{p/2} \, d\sigma(x)
\]

\[
\leq N^p \sigma(Q) + \sum_j \sigma(Q_j) M_Q(k)
\]

\[
+ \sum_j \int_{Q_j} \left( \iint_{\Gamma^{Q,j}(x)} |F(Y)|^2 \, dY \right)^{p/2} \, d\sigma(x)
\]

\[
\leq N^p \sigma(Q) + \frac{1 + \alpha}{2} \sigma(Q) M_Q(k)
\]

\[
+ \sum_j \int_{Q_j} \left( \iint_{\Gamma^{Q,j}(x)} |F(Y)|^2 \, dY \right)^{p/2} \, d\sigma(x).
\]

Note that, by definition,

\[
\Gamma^{Q,j}(x) \setminus \Gamma^{Q,j}(x) = \bigcup_{P \in \mathcal{D}_Q \setminus \mathcal{D}_{Q_j}, P_x} \ U_P, \quad x \in Q_j,
\]

and observe that the conditions \(x \in Q_j\) and \(x \in P, P \in \mathcal{D}_Q \setminus \mathcal{D}_{Q_j}\), guarantee that \(P \supseteq Q_j\). Indeed, by (ii) of Lemma 2.1, if there is a point \(x \in Q_j\), then either \(P \supseteq Q_j\) or \(P \subset Q_j\), and the latter is not possible since \(P \in \mathcal{D}_Q \setminus \mathcal{D}_{Q_j}\). Thus, (4.4) applies, and the last term on the right-hand side of (4.6) is bounded by \(\frac{1 + \alpha}{2} \sigma(Q) N^p\). All in all, (4.6) demonstrates that

\[
M_Q(k) \leq C N^p, \quad \text{for all } k \in \mathbb{N},
\]

and thus, letting \(k \to \infty\), we arrive at

\[
\frac{1}{\sigma(Q')} \int_{Q'} \left( \iint_{\Gamma^{Q'}(x)} |F(Y)|^2 \, dY \right)^{p/2} \, d\sigma(x) \leq C N^p,
\]

for all \(Q \in \mathcal{D}(E)\) which finishes the proof of the Lemma. \(\square\)

At this point we are ready to address the transference of the Carleson measure condition from Lipschitz to NTA domains. We shall use the fact that NTA domains with ADR boundaries contain big pieces of Lipschitz subdomains due to [DJ]. To be more precise, the following holds.

**Definition 4.7.** We say that the domain \(\Omega \subset \mathbb{R}^{n+1}\) is a Lipschitz graph domain if there is some Lipschitz function \(\psi : \mathbb{R}^n \to \mathbb{R}\) and some coordinate system such that

\[
\Omega = \{(x', t) : x' \in \mathbb{R}^n, t > \psi(x')\}.
\]

We refer to \(M = \|\nabla \psi\|_{L^\infty(\mathbb{R}^n)}\) as the Lipschitz constant of \(\Omega\).
The open connected set $\Omega$ is said to be a bounded Lipschitz domain if there is some positive scale $r = r_\Omega$, some constants $M > 0$ and $c_0 \geq 1$, and some finite set $\{x_j\}_{j=1}^m$ of points with $x_j \in \partial \Omega$, such that the following conditions hold. First,

$$\partial \Omega \subset \bigcup_{j=1}^m B(x_j, r_j)$$

for some $r_j$ with $\frac{1}{c_0} r_\Omega < r_j < c_0 r_\Omega$.

Second, for each $x_j$ there is some Lipschitz graph domain $V_j$, with $x_j \in \partial V_j$ and with Lipschitz constant at most $M$, such that

$$Z_j \cap \Omega = Z_j \cap V_j$$

where $Z_j$ is a cylinder of height $(8 + 8M)r_j$, radius $2r_j$, and with axis parallel to the $t$-axis (in the coordinates associated with $V_j$).

We refer to the triple $(M, m, c_0)$ as the Lipschitz character of $\Omega$.

**Proposition 4.8** ([DJ]). Given $D \subset \mathbb{R}^{n+1}$, a bounded NTA with ADR boundary, there exist constants $C \geq 2$ and $0 < \theta < 1$ such that for every surface ball $\Delta(x, r) = B(x, r) \cap \partial D, x \in \partial D, r < \text{diam}(\partial D)$, there exists a bounded Lipschitz domain $\Omega'$ for which we have the following conditions:

1. $H^n(\partial \Omega \cap \partial \Omega' \cap B(x, r)) \geq \theta H^n(\Delta(x, r)) \approx \theta r^n$.
2. There exists $\Lambda_\delta$ so that $B(\theta \Lambda_\delta, r/C) \subset B(x, r) \cap D \cap \Omega'$.
3. $\Omega' \subset \Omega \cap B(x, r)$.

The Lipschitz character of $\Omega'$ as well as $0 < \theta < 1$ and $C \geq 2$ depend on $n$, the NTA constants of $D$, and the ADR constant of $\partial D$ only (and are independent of $x, r$).

We remark that in [DJ], Proposition 4.8 is proved under a weaker assumption than that of NTA, namely, only an interior corkscrew condition, and a “weak exterior corkscrew condition” which entails exterior disks rather than exterior balls, and with no hypothesis of Harnack chains. Later on, in [Bad], existence of big pieces of Lipschitz subdomains was also proved for usual NTA domains, with no upper ADR assumption on $\partial \Omega$ (the lower ADR bound holds automatically in the presence of a two-sided corkscrew condition, by virtue of the relative isoperimetric inequality). For the applications that we have in mind here, neither amelioration is significant, and we will simply work with the NTA domains in the sense of Definition 1.7 with ADR boundaries.

For future reference we also would like to provide the following corollary.

**Corollary 4.9.** Given a bounded NTA domain $D \subset \mathbb{R}^{n+1}$, with an ADR boundary $E = \partial D$, there exist constants $C > 0$ and $0 < \theta < 1$ such that for every $Q \in \mathbb{D}(E)$ there exists a bounded Lipschitz domain $\Omega_Q$ for which we have the following:

1. $\sigma(\partial \Omega_Q \cap Q) \geq \theta \sigma(Q) \approx \theta \sigma(Q)^n$.
2. For every $Q' \in \mathbb{D}(Q)$ such that there exists a point $y_{Q'} \in Q' \cap \partial \Omega_Q$ it follows that the domain $\Omega_Q$ contains a corkscrew point $Y_{Q'}$ relative to $B(y_{Q'}, r) \cap \partial \Omega_Q, r \approx \ell(Q')$, and the domain $\Omega_Q$, and furthermore, with the appropriate choice of $\eta$ and $K$ in (2.8), we have $B(Y_{Q'}, C\ell(Q')) \subset U_Q$.
3. $\Omega_Q \subset \Omega \cap B(x, C\ell(Q))$.

The Lipschitz character of $\Omega_Q$ as well as $0 < \theta < 1, c, C > 0$, and the constants implicitly used in the statement that “$\Omega_Q$ contains a corkscrew point $Y_{Q'}$ relative to $B(y_{Q'}, r) \cap \partial \Omega_Q, r \approx \ell(Q')$” depend on $n$, the NTA constants of $D$ and the ADR constants of $E = \partial D$ only (uniformly in $Q, Q'$).

**Proof.** The corollary follows directly from Proposition 4.8. Indeed, for any $Q \in \mathbb{D}(E)$ there exists $\Delta(x, r) \subset Q, x \in Q, r \approx \ell(Q)$. One can build a Lipschitz domain from Proposition 4.8 corresponding to this $\Delta(x, r)$, and then the conditions (1), (3) in Proposition 4.8 entail (1) and (3) in Corollary 4.9, respectively. The condition (2) in Corollary 4.9 follows from the fact that a Lipschitz domain $\Omega_Q$ is, in particular, an NTA domain, and hence, it has a corkscrew point corresponding to $B(y_{Q'}, r) \cap \partial \Omega_Q$.
as long as \( r < \text{diam}(\partial \Omega_Q) \). Using the fact that \( \Omega_Q \subset D \), one can easily see that \( Y_Q \) is also a corkscrew point of \( D \), relative to \( B(y_Q, r) \cap E, r \approx \ell(Q') \). It remains to observe that a suitable choice of \( \eta \) and \( K \) (uniform in \( Q' \)) ensures that such a corkscrew point always belongs to \( U_Q \) and moreover, \( B(y_Q, c(\ell(Q')) \subset U_Q \), for some uniform constant \( c \) depending on \( n \), the NTA constants of \( D \) and the ADR constants of \( E = \partial D \) only.

**Theorem 4.10.** Given an NTA domain \( D \subset \mathbb{R}^{n+1} \) with an ADR boundary \( E = \partial D \) and \( F \in L^2_{\text{loc}}(D) \) which satisfies (3.2), the following holds. If \( F \) satisfies the Carleson measure estimate (1.10) on all bounded Lipschitz subdomains of \( D \) with the constant \( C = C_0 \) depending on the Lipschitz constants of the underlying domains only, then \( F \) satisfies the Carleson measure estimate (1.10) in \( D \) as well, with the bound depending on \( C_0, \) the constant in (3.2), the NTA constants of \( D \) and the ADR constants of \( \partial D \) only.

Let us remark that in the course of the proof we ensure a suitable choice of a (sufficiently small) \( \eta \) and a (sufficiently large) \( K \) is (2.8) which strictly speaking affect the constant in (3.2). However, as all choices depend on the NTA constants of \( E \) and ADR constants of \( E = \partial D \) only, this does not affect the result as stated above.

**Proof.** First of all, given that \( F \) satisfies (3.2), the same argument as in [HMM] allows one to reduce matters to proving that

\[
\sup_{Q \in \mathcal{D}(E)} \frac{1}{\sigma(Q)} \int_{T_Q} |F(X)|^2 \delta(X) \, dX \leq C. \tag{4.11}
\]

Note that here and below, \( \delta = \delta_E \) denotes distance to \( E = \partial D \); the distance to the subdomains will be distinguished by the corresponding subscript. Furthermore, due to John-Nirenberg Lemma 4.1, it is in fact sufficient to show that there exist numbers \( 0 < \alpha < 1 \) and \( 0 < N < \infty \) such that for every \( Q \subset \mathcal{D}(E) \)

\[
\sigma \left\{ x \in Q : \left( \int_{T_{Q'}(x)} |F(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2} > N \right\} \leq \alpha \sigma(Q). \tag{4.12}
\]

Fix some \( Q \subset \mathcal{D}(E) \). According to Corollary 4.9 (along with the inner regularity property of the measure) there exists a bounded Lipschitz domain \( \Omega_Q \) and a closed set \( F_Q \subset \partial \Omega_Q \cap Q \) such that \( \sigma(F_Q) \geq \theta \sigma(Q) \), and the Lipschitz character of \( \Omega_Q \) as well as \( 0 < \theta < 1 \) depend on \( n \), the NTA constants of \( D \) and the ADR constants of \( E \) only (uniformly in \( Q \)). The domain \( \Omega_Q \) further satisfies properties (1)–(3) in Corollary 4.9.

Now let us take a relatively open set \( Q \setminus F_Q \), single out the collection of maximal disjoint cubes \( \mathcal{T} = \{Q_j\} \) such that \( Q \setminus F_Q = \bigcup_j Q_j \), and build the corresponding sawtooth region \( \Omega_{\mathcal{T}, Q} \).

By Tchebyshev inequality, it is sufficient to prove that

\[
\frac{1}{\sigma(Q)} \int_{\Omega_{\mathcal{T}, Q}} |F(X)|^2 \delta(X) \, dX \leq C, \tag{4.13}
\]

in order to conclude (4.12). Indeed,

\[
\sigma \left\{ x \in F_Q : \left( \int_{T_{Q'}(x)} |F(Y)|^2 \delta(Y)^{1-n} \, dY \right)^{1/2} > N \right\} \leq \frac{1}{N^2} \int_{F_Q} \int_{T_{Q'}(x)} |F(Y)|^2 \delta(Y)^{1-n} \, dY
\]

\[
\approx \frac{1}{N^2} \int_{\bigcup_{Q' \sim Q} U_{Q'}} |F(Y)|^2 \delta(Y) \, dY.
\]


Now recall that by construction dyadic “cubes” are either contained in each other or do not intersect. Hence, if $Q' \in D_Q$ is such that $Q' \cap F_Q \neq \emptyset$, we have $Q' \in D_Q \setminus \bigcup F D_Q$, for, otherwise, $Q' \in \bigcup F D_Q$ and hence $Q' \subset Q \setminus F_Q$. Thus, the right-hand side of (4.14) is bounded by

$$\frac{C}{N^2} \int_{D_{f,Q}} |F(Y)|^2 \delta(Y) dY,$$

so that (4.13), in conjunction with $\sigma(Q \setminus F_Q) \leq (1 - \theta) \sigma(Q)$ yield the desired result (4.12).

Hence, it remains to show (4.13) with a constant $C$ depending on $C_0$, the constant in (3.2), the NTA constants of $D$ and the ADR constants of $\partial D$ only.

To this end, let us write

$$(4.15) \quad \int_{D_{f,Q}} |F(X)|^2 \delta(X) dX = \sum_{Q' \in D_{f,Q}} \int_{U_{Q'}} |F(Y)|^2 \delta(Y) dY$$

and split this sum according to whether $\text{dist}(U_{Q'}, E) \leq \frac{1}{\ell} \text{dist}(U_{Q'}, \partial \Omega_Q)$ or else $\text{dist}(U_{Q'}, E) > \frac{1}{\ell} \text{dist}(U_{Q'}, \partial \Omega_Q)$, for some small $\varepsilon > 0$ to be defined later. Note that, in principle, $U_{Q'}$ can intersect $\partial \Omega_Q$. We also record that $\ell(Q') \approx \text{dist}(U_{Q'}, E)$ by definitions (see (2.10), (2.11), and (2.8)).

Let us start with

**Case I:**

$$(4.16) \quad Q' \in D_{f,Q} : \text{dist}(U_{Q'}, E) \leq \frac{1}{\ell} \text{dist}(U_{Q'}, \partial \Omega_Q).$$

We claim that in this scenario

$$(4.17) \quad \ell(Q') \approx \text{dist}(U_{Q'}, E) \approx \text{dist}(U_{Q'}, F_Q) \approx \text{dist}(U_{Q'}, \partial \Omega_Q).$$

As discussed above, the first equivalence follows from definitions. Now,

$$(4.18) \quad \ell(Q') \approx \text{dist}(U_{Q'}, E) \geq \text{dist}(U_{Q'}, Q').$$

This is because for any $I \in W_{Q'}$ we have $\ell(I', Q') \geq \text{dist}(I, Q')$ by (2.11) and (2.8) and hence, $\ell(Q') \geq \text{dist}(I'(\tau), Q')$ as well. Next, note that there exists $y \in Q'$ such that $y \in F_Q$. Indeed, if not, $Q' \subset Q \setminus F_Q$ and hence, $Q' \subset Q_j$ for some $j$ which is a contradiction with $Q' \in D_Q \setminus \bigcup F D_Q$.

Hence,

$$(4.19) \quad \text{dist}(U_{Q'}, F_Q) \leq \text{dist}(U_{Q'}, y) \leq \text{dist}(U_{Q'}, Q') + \ell(Q') \leq \ell(Q').$$

In addition,

$$(4.20) \quad \text{dist}(U_{Q'}, F_Q) \geq \text{dist}(U_{Q'}, E)$$

for trivial reasons ($F_Q \subset E$). Thus, combining (4.18)–(4.20), we have proved the second equivalence in (4.17).

As for the third one, we have

$$(4.21) \quad \text{dist}(U_{Q'}, F_Q) \geq \text{dist}(U_{Q'}, \partial \Omega_Q)$$

once again due to the fact that $F_Q \subset \partial \Omega_Q$. This, in combination with the current assumption (4.16) finally finishes the proof of (4.17).

In particular, we conclude that for every $Y \in U_{Q'}$ with $U_{Q'}$ satisfying (4.16) we have

$$\delta(Y) = \text{dist}(Y, E) \leq \ell(Q') + \text{dist}(U_{Q'}, \partial \Omega_Q) \leq \text{dist}(U_{Q'}, \partial \Omega_Q) \leq \text{dist}(Y, \partial \Omega_Q).$$

Note also that the condition $Q' \cap F_Q \neq \emptyset$ proved above implies that, according to Corollary 4.9, $\Omega_Q$ contains a corresponding corkscrew point, which is in turn contained in $U_{Q'}$. Hence, $\Omega_Q \cap U_{Q'} \neq \emptyset$, and then due to (4.17), $U_{Q'} \subset \Omega_Q$ in this case.

With this at hand, the part of the sum on the right-hand side of (4.15) corresponding to Case I can be bounded as follows:
We shall demonstrate a packing condition on the cubes satisfying (4.22). Indeed, recall from above some uniform constant $C$ (4.25) simply invoking (3.2). Then, combining (4.24)–(4.25) we finish the argument for Case II.

It follows that for a suitably small $\varepsilon$ that CME holds on all Lipschitz subdomains of $D$ in the last one. Note that $\Omega_Q \subset B(x_Q, C\ell(Q))$ for some uniform constant $C$, which justifies the bound by $\sigma(Q)$.

**Case II:**

\begin{equation}
Q' \in \mathcal{D}_{F,Q} : \text{dist}(U_{Q'}, E) > \frac{1}{\varepsilon} \text{dist}(U_{Q'}, \partial \Omega_Q).
\end{equation}

We shall demonstrate a packing condition on the cubes satisfying (4.22). Indeed, recall from above that $\ell(Q') \approx \text{dist}(U_{Q'}, E)$, so that in particular,

\begin{equation}
\ell(Q') \geq \frac{1}{\varepsilon} \text{dist}(U_{Q'}, \partial \Omega_Q).
\end{equation}

It follows that for a suitably small $\varepsilon$ depending on the implicit constant in (4.23) and $\tau$, we can ensure that fattened regions $\tilde{U}_{Q'}$ corresponding to $U_{Q'}$ from (4.22) necessarily intersect $\partial \Omega_Q$ and, moreover, $\sigma(\tilde{U}_{Q'} \cap \partial \Omega_Q) \approx \ell(Q')$, while $\tilde{U}_{Q'}$’s still have finite overlap. Since the Lipschitz character of $\partial \Omega_Q$ is controlled, $\sigma(\partial \Omega_Q) \approx \sigma(Q)$ with some uniform in $Q$ constants. Thus, all in all,

\begin{equation}
\sum_{Q' \in \mathcal{D}_{F,Q} : \text{dist}(U_{Q'}, E) > \frac{1}{\varepsilon} \text{dist}(U_{Q'}, \partial \Omega_Q)} \sigma(Q') \lesssim \sigma(Q).
\end{equation}

However, then

\begin{equation}
\sum_{Q' \in \mathcal{D}_{F,Q} : \text{dist}(U_{Q'}, E) > \frac{1}{\varepsilon} \text{dist}(U_{Q'}, \partial \Omega_Q)} \int_{U_{Q'}} |F(Y)|^2 \delta(Y) \, dY \lesssim \sum_{Q' \in \mathcal{D}_{F,Q} : \text{dist}(U_{Q'}, E) > \frac{1}{\varepsilon} \text{dist}(U_{Q'}, \partial \Omega_Q)} \sigma(Q'),
\end{equation}

simply invoking (3.2). Then, combining (4.24)–(4.25) we finish the argument for Case II. \hfill \Box

5. $\mathcal{A} < N$ bounds: good-$\lambda$ arguments

Recall now definitions of the area integral, square function, and non-tangential maximal function from Definition 1.14. We point out that we work with the dyadic cones which were not defined in the introduction but rather in Section 4.

When $u$ is a solution of a second order elliptic PDE, e.g., a harmonic function in $\mathbb{R}^{n+1} \setminus E$, one normally works with the square function

\begin{equation}
Su(x) := \mathcal{A}(|\nabla u|), \quad x \in E,
\end{equation}

and for a solution of a $2m$-th order elliptic PDE, $m \in \mathbb{N}$, we will be interested in

\begin{equation}
S_m u(x) := \mathcal{A}(|\nabla^m u|), \quad x \in E.
\end{equation}

We shall come back to this point with more details in Section 8 and for now try to keep the discussion general for as long as possible. This is the same normalization as in the previous sections, and thus is also susceptible to the issue raised in Remark 3.1.
By $\hat{\mathcal{A}}, \hat{S}, \hat{N}$, we denote the area integral, square function, and the non-tangential maximal function defined using the family $\hat{F}$ in place of $\Gamma$. Note that according to these definitions, the cones are unbounded when $E$ is unbounded. On the other hand, when $E$ is bounded, so are the cones, all being contained in a $C$-dimensional $(E)$-neighborhood of $E$. We note also that when $E$ is bounded, there exists a cube $Q_0 \in \mathbb{D}(E)$ such that $Q_0 = E$ and for any $Q \in \mathbb{D}(E)$ we have $Q \in \mathbb{D}_{Q_0}$. It is, however, particularly useful to work with local versions. To this end, by $\mathcal{A}^Q, S^Q, N^Q$ we denote the area integral, square function, and the non-tangential maximal function defined using the family $\Gamma^Q$ in place of $\Gamma$, and similarly for $\hat{\mathcal{A}}^Q, \hat{S}^Q, \hat{N}^Q$.

**Definition 5.3.** Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional ADR set, as in (1.1). By $M^D = M^D_E$ we denote the dyadic Hardy-Littlewood maximal function on $E$, that is, for $f \in L^1_{loc}(E)$

$$M^D f(x) = \sup_{Q \in \mathbb{D}(E); x \in Q} \int_Q |f(y)| \, d\sigma(y),$$

and we also write $M^D f = M^D(|f|^p)^{1/p}$.

**Definition 5.4.** ("$\mathcal{A}/N$" estimates). Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional ADR set, $G \in L^2_{loc}(\mathbb{R}^{n+1} \setminus E)$, $H \in C(\mathbb{R}^{n+1} \setminus E)$. We say that "$\mathcal{A} < N$" estimates hold for $G, H$ in $L^q(E)$ if

$$(5.5) \quad \|\mathcal{A}G\|_{L^q(E)} \leq C\|\hat{N}_s H\|_{L^q(E)},$$

for some $q \in (1, \infty)$, and some uniform constant $C$. Same conventions apply if $G$ and $H$ are supported on a subset $D \subset \Omega$ with an ADR boundary $E = \partial D$. The $L^p$ norms are taken with respect to "surface measure" $\sigma := H^{n-1}E$.

Similarly, we will say that "$\mathcal{A}^Q < N^Q$" estimates hold for $G, H$ in $L^q(E)$ if

$$(5.6) \quad \|\mathcal{A}^Q G\|_{L^q(Q)} \leq C\|\hat{N}^Q_s H\|_{L^q(Q)},$$

for all $Q \in \mathbb{D}(E)$, for some $q \in (1, \infty)$.

**Theorem 5.7.** Let $E$ be an $n$-dimensional ADR set in $\mathbb{R}^{n+1}$, $\Omega = \mathbb{R}^{n+1} \setminus E$, $G \in L^2_{loc}(\Omega)$, $H \in C(\Omega) \cap L^\infty(\Omega)$. Given $0 < q < \infty$, consider the following statements:

(A) Carleson measure estimate (1.10) holds for $F = G/\|H\|_{L^q(\Omega)}$ in $\Omega$;

(A$_{loc}$) Carleson measure estimate (1.10) holds on every (bounded) local sawtooth subdomain $\hat{\mathcal{F}}_{F,Q}$ for $F = G/\|H\|_{L^q(\hat{\mathcal{G}}_F,Q)}$, for any $Q \in \mathbb{D}(E)$ and any pairwise disjoint family of cubes $\mathcal{F}$ in $\mathbb{D}(E)$;

(B)$_q$ $\mathcal{A} < N$ on $L^q(E)$ holds for $G$ and $H$, in the sense of Definition 5.4, i.e., (5.5) is valid;

(B$_{loc}$)$_q$ $\mathcal{A}^Q < N^Q$ on $L^q(E)$ holds for $G$ and $H$, in the sense of Definition 5.4, i.e., (5.5) is valid;

(GA)$_q$ for every $\varepsilon, \gamma > 0$ and for all $\alpha > 0$

$$(5.8) \quad \sigma(x \in E : \mathcal{A}G(x) > (1 + \varepsilon) \alpha, \hat{N}_s H(x) \leq \gamma \alpha) \leq C (\gamma/\varepsilon)^2 \sigma(x \in E : M^D_q(\mathcal{A}G)(x) > \alpha);$$

(GA)$_{loc}$ for every $\varepsilon, \gamma > 0$ and for all $\alpha > 0$

$$(5.9) \quad \sigma(x \in Q : \mathcal{A}^Q G(x) > (1 + \varepsilon) \alpha, \hat{N}^Q_s H(x) \leq \gamma \alpha) \leq C (\gamma/\varepsilon)^2 \sigma(x \in Q : M^D_q(\mathcal{A}^Q G)(x) > \alpha), \quad \text{for any } Q \in \mathbb{D}(E).$$

Consider, in addition, a condition

$$(5.10) \quad \frac{1}{\sigma(Q)} \left( \int_{U^0} |G(Y)|^2 \delta(Y) \, dY \right)^{1/2} \leq C||H||_{L^\infty(U^0)}, \quad \text{for all } Q \in \mathbb{D}(E).$$
Then
\[
(A)_{\text{loc}} \implies (G\lambda)_{q}, \quad \text{for all } 0 < q < \infty,
\]
\[
([A]_{\text{loc}} \& (5.10)) \implies (B)_{q}, \quad \text{for all } 0 < q < \infty,
\]
\[
(A)_{\text{loc}} \implies (G\lambda)_{q}, \quad \text{for all } 0 < q < \infty,
\]
\[
([A]_{\text{loc}} \& (5.10)) \implies (B)_{q}, \quad \text{for all } 0 < q < \infty,
\]
\[
(B)_{q} \implies (B)_{q}, \quad \text{for all } 0 < q < \infty,
\]
\[
((B)_{q} \text{ for some } 0 < q < \infty) \implies (A).
\]

In particular, if there exists a subclass of ADR domains, \(\Sigma\), such that \(\Omega \in \Sigma\); all (bounded) local sawtooth subdomains \(\hat{\Omega}_{F,Q}\) are in \(\Sigma\), for any \(Q \in \mathbb{D}(E)\) and any pairwise disjoint family of cubes \(\mathcal{F} \in \mathbb{D}(E)\); all local sawtooth subdomains of each of \(\hat{\Omega}_{F,Q}\) are also in \(\Sigma\), etc., and \(G \in L^{2}_{\text{loc}}(\Omega), H \in C(\Omega) \cap L^{\infty}(\Omega)\), then \((5.10)\) holds for any domain in \(\Sigma\) then
\[
(A) \text{ on every } D \in \Sigma
\]
\[
\iff (B)_{q} \text{ on the boundary of every } D \in \Sigma \text{ for some } 0 < q < \infty
\]
\[
\iff (B)_{q} \text{ on the boundary of every } D \in \Sigma \text{ for all } 0 < q < \infty
\]
\[
\iff (B)_{q} \text{ on the boundary of every } D \in \Sigma \text{ for some } 0 < q < \infty
\]
\[
\iff (B)_{q} \text{ on the boundary of every } D \in \Sigma \text{ for all } 0 < q < \infty,
\]
with the understanding that all implicit constants in the statements above are uniform within \(\Sigma\).

An example of a class \(\Sigma\) as above (which is used in the present paper) is the class of uniformly rectifiable domains with uniformly controlled ADR and UR constants. One has to point out that the definition of the sawtooth regions and with it, the meaning of the statements above, is slightly different depending on whether the involved domains are just ADR or UR as well (or even NTA). For that reason, the reader will see statements like “the constant depends on the ADR constants of \(E\) (or the UR character if \(E\) is UR)”. We consider, however, the resulting dependence of either ADR constants or UR (or NTA) character harmless.

We remark that the assumption \((5.10)\) is only needed to justify finiteness of some integrals in \(A < N\) arguments. If, e.g., it is known a priori that the \(L^{q}\) norm of \(A\) is finite (or even that the \(L^{q}\) norm of a certain truncated from above and below version of \(A\) is finite), then the result of Theorem 5.7 carries over without \((5.10)\). However, in all practical applications to solutions of elliptic PDEs \((5.10)\) is easily justified by Caccioppoli’s inequality. Remark that it is an analogue of the assumption that \(F = G/\|H\|_{L^{\infty}(\Omega)}\) satisfies \((3.2)\), which is used in Theorem 3.3. In fact, the latter follows from \((5.10)\).

Finally, a combination of \((5.14)\) and \((5.15)\) of course absorbs \((5.12)\), but we will prove the latter earlier, based on \((5.11)\), and then use an analogous argument towards \((5.14)\).

\textbf{Proof of Theorem 5.7. Step I:} \((A)_{\text{loc}} \implies (G\lambda)_{p}\), for all \(0 < p < \infty\). We start by proving that in the assumptions of the Theorem (with or without \((5.10)\) at this stage) the statement \((A)_{\text{loc}}\) implies that for any \(0 < p < \infty\), for every \(\epsilon, \gamma > 0\) and for all \(\alpha > 0\)
\[
(1 + \epsilon)\alpha, \quad \tilde{N}_{e}H(x) \leq \gamma a
\]
\[
\leq C(\gamma/\epsilon)^{2} \sigma\{x \in E : \bar{A}G(x) > (1 + \epsilon)\alpha, \tilde{N}_{e}H(x) \leq \gamma a\}
\]

with the constant \(C\) depending on the ADR constants of \(E\) (or the UR character if \(E\) is UR) and the constant in \((A)_{\text{loc}}\) only.
We can assume that the set on the right-hand side of (5.8) is not empty (otherwise \( \mathcal{AG}(x) \leq \alpha \) for a.e. \( x \in E \) and the left-hand side of (5.8) has measure zero, as desired). We can also assume that it is finite, even if \( E \) is unbounded (for, otherwise, there is nothing to prove).

Thus, one can then build a collection of maximal dyadic cubes comprising the set \( \{ x \in E : M^D_p(\mathcal{AG})(x) > \alpha \} \), following the Calderón-Zygmund decomposition argument and extracting the cubes maximal with respect to the property \( \left( \int_{Q} |\mathcal{AG}|^p \ d\sigma \right)^{1/p} > \alpha \). One can check that the union of such cubes is equal to \( \{ x \in E : M^D_p(\mathcal{AG})(x) > \alpha \} \). In our assumptions, a maximal cube always exists.

Let us denote by \( Q \) one of these maximal cubes. We will prove that for every such \( Q \) we have
\[
\sigma(x \in Q : \mathcal{AG}(x) > (1 + \varepsilon)\alpha, \hat{\mathcal{N}}_* H(x) \leq \gamma \alpha) \leq C(\gamma/\varepsilon)^2 \sigma(Q),
\]
with the constant \( C \) depending on the ADR constants of \( E \) (or UR character if \( E \) is UR) and the constants in \( A_{\text{loc}} \) only.

Let us temporarily separate the cases. We note that if \( E \) is bounded, then \( E \) itself is the largest cube in \( \mathbb{D}(E) \), and in this case we set \( E = Q_0 \).

**Case I of Step I:** \( E \) is unbounded, or \( E \) is bounded and \( \left( \int_{Q_0} |\mathcal{AG}|^p \ d\sigma \right)^{1/p} \leq \alpha \).

In this case the considered collection of maximal cubes does not cover the entire \( E \), so that \( Q \) has a parent \( \tilde{Q} \in \mathbb{D}(E) \), \( \tilde{Q} \neq Q \). Indeed, when \( E \) is bounded, such a property is guaranteed by the condition \( \left( \int_{Q_0} |\mathcal{AG}|^p \ d\sigma \right)^{1/p} \leq \alpha \). When \( E \) is unbounded, the existence of a parent \( \tilde{Q} \in \mathbb{D}(E) \), \( \tilde{Q} \neq Q \), is clear from the implicit assumption that \( \sigma(x \in E : M^D_p(\mathcal{AG})(x) > \alpha) < \infty \) (for, otherwise there would be nothing to prove).

Therefore, since \( \tilde{Q} \) is maximal, there exists an \( \tilde{x} \) belonging to \( \tilde{Q} \), such that \( M^D_p(\mathcal{AG})(\tilde{x}) \leq \alpha \) and hence, \( \left( \int_{\tilde{Q}} |\mathcal{AG}|^p \ d\sigma \right)^{1/p} \leq \alpha \). Then there exists \( z \in \tilde{Q} \) such that \( \mathcal{AG}(z) \leq \alpha \).

Thus, if we denote (cf. (2.23))
\[
\Gamma_1(x) = \Gamma_0(x) = \bigcup_{Q' \in \mathbb{D}(E)} U_{Q'}, \quad \Gamma_2(x) = \bigcup_{Q' \in \mathbb{D}(E)} U_{Q'}, \quad x \in Q,
\]
and by \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) the corresponding portions of the square function, then
\[
(5.20) \quad \mathcal{A}_2 G(x) \leq \alpha, \quad \text{for every } x \in Q,
\]
since for every \( x \in Q \) we have \( \mathcal{A}_2 G(x) \leq \mathcal{AG}(z) \). This follows from the properties (ii) and (iii) of the dyadic decomposition, see Lemma 2.1. Thus, for every \( x \) belonging to the set on the left-hand side of (5.19) we have \( \mathcal{A}_1 G(x) > \varepsilon \alpha \) and in particular, it is sufficient to prove that
\[
(5.21) \quad \sigma(x \in Q : \mathcal{A}_1 G(x) > \varepsilon \alpha, \hat{\mathcal{N}}_* H(x) \leq \gamma \alpha) \leq C(\gamma/\varepsilon)^2 \sigma(Q).
\]

**Case II of Step I:** \( E \) is bounded (so that \( E = Q_0 \)) and \( \left( \int_{Q_0} |\mathcal{AG}|^p \ d\sigma \right)^{1/p} > \alpha \).

In this case, by definition, \( \{ x \in E : M^D_p(\mathcal{AG})(x) > \alpha \} = E = Q_0 \), and therefore, again by definitions, \( \mathcal{A}_2 = 0 \) and \( \mathcal{A}_1 = \mathcal{A} \). Hence, in this case (5.19) reduces to (5.21) trivially.

Thus, the two cases are now merged and we concentrate on proving (5.21). To this end, we denote by \( F \) the set \( \{ x \in Q : \hat{\mathcal{N}}_* H(x) \leq \gamma \alpha \} \). If \( \sigma(F) = 0 \), there is nothing to prove.

Otherwise, if \( \sigma(F) > 0 \), we subdivide \( Q \) dyadically and stop the first time that \( Q' \cap F = \emptyset \). If one never stops, we set \( F = \emptyset \), otherwise we let \( F = \mathbb{D}_Q \setminus \{ Q \} \) be the family of stopping cubes which is maximal by construction.

The sawtooth regions \( \Omega_F \) and \( \Omega_{F,Q} \) retain the same significance as in (2.19) and by \( \Omega_F \) and \( \Omega_{F,Q} \) we denote analogous sawtooth regions defined with \( \hat{U}_Q \) in place of \( U_{Q'} \). Of course, in the
case that $\mathcal{F}$ is empty, then $\Omega_{F,Q} = T_Q$ and $\hat{\Omega}_{F,Q} = \hat{T}_Q$ are just the corresponding Carleson boxes associated to $Q$.

Observe that by construction $|H(X)| \leq \gamma \alpha$ for every $X \in \hat{\Omega}_{F,Q}$. Indeed, if $X \in \hat{\Omega}_{F,Q}$ then $X \in \hat{U}_{Q'}$ for some $Q' \in \mathbb{D}_{F,Q}$, where we recall that by definition, $\mathbb{D}_{F,Q}$ is comprised of those dyadic sub-cubes of $Q$ that are not contained in any $Q_j \in \mathcal{F}$. Thus, such a $Q'$ necessarily contains a point from $F$. Now, let $z \in Q' \cap F$. By definition, $\hat{N}_s H(z) \leq \gamma \alpha$, and therefore, $|H(X)| \leq \gamma \alpha$ for every $X \in \hat{U}_{Q'}$, as desired.

Next, $\sigma[x \in F : \mathcal{A}_1 G(x) > \varepsilon \alpha] \leq (\varepsilon \alpha)^{-2} \int_F (\mathcal{A}_1 G(x))^2 \, d\sigma(x)$

$$= (\varepsilon \alpha)^{-2} \int_{\Omega} \int_{\partial \Omega} \frac{|G(Y)|^2}{\delta(Y)^{1-n}} \, dY \, d\sigma(x)$$

$$\approx (\varepsilon \alpha)^{-2} \int_{Q' \cap \mathbb{D}_F} \sum_{Q \in \mathbb{D}_Q \cap \mathbb{D}_F} \int_{U_{Q'}} |G(Y)|^2 \, \delta(Y)^{-1-n} \, dY \, d\sigma(x).$$

Any $Q' \in \mathbb{D}_Q$ which contains points of $F$ must be an element of $\mathbb{D}_Q \cap \mathbb{D}_F$. Hence, the expression above is bounded modulo a multiplicative constant by

$$\sigma[x \in F : \mathcal{A}_1 G(x) > \varepsilon \alpha] \leq (\varepsilon \alpha)^{-2} \sum_{Q' \in \mathbb{D}_Q \cap \mathbb{D}_F} \int_{Q'} \int_{U_{Q'}} |G(Y)|^2 \, \delta(Y)^{-1-n} \, dY \, d\sigma(x).$$

Observe, however, that for every $Y \in U_{Q'}$ as above $\delta(Y) = \text{dist}(Y,E) \approx \ell(Q') \approx \text{dist}(Y,\hat{\Omega}_{F,Q})$ since, as explained above, $\hat{\Omega}_{F,Q}$ is comprised of fattened Whitney regions $\hat{U}_{Q'}$. Using the bounded overlap of the Whitney regions, we have

$$\sigma[x \in F : \mathcal{A}_1 G(x) > \varepsilon \alpha] \leq (\varepsilon \alpha)^{-2} \int_{\mathbb{D}_Q \cap \mathbb{D}_F} \int_{U_{Q'}} |G(Y)|^2 \, \text{dist}(Y,\hat{\Omega}_{F,Q}) \, dY$$

$$\leq (\varepsilon \alpha)^{-2} \int_{\hat{\Omega}_{F,Q}} |G(Y)|^2 \, \text{dist}(Y,\hat{\Omega}_{F,Q}) \, dY.$$

At this stage, we recall two facts. First, $|H(X)| \leq \gamma \alpha$ for every $X \in \hat{\Omega}_{F,Q}$ and hence, $\|H\|_{L^q(\hat{\Omega}_{F,Q})} \leq \gamma \alpha$. Secondly, by definition $\hat{\Omega}_{F,Q}$ is a subset of $B(x, C \ell(Q))$ for some $x \in \partial \hat{\Omega}_{F,Q}$ and $C$ depending on the ADR constants (or UR character if $E$ is UR) only. It follows then from $(A)_{\text{loc}}$ that we can bound the right-hand side of (5.24) by $C(\varepsilon \alpha)^{-2}(\gamma \alpha)^2 \sigma(Q)$, as desired.

**Step II:** $[(A)_{\text{loc}} \& (5.10)] \implies (B)_q$, for all $0 < q < \infty$. Due to Step I, we have at hand good-$\lambda$ inequalities $(G)_{\alpha}$, for all $0 < p < \infty$. Assuming that the left-hand side of (5.5) is finite, the proof of (5.5) would be a standard argument using the $L^q$ boundedness of the Hardy-Littlewood maximal operator $M^p$, $p < q$. Let us recall it for future reference. We have

$$\|\mathcal{A}G\|_{L^q(E)}^q = (1 + \varepsilon)^q \int_0^\infty q \alpha^q \sigma[x \in E : \mathcal{A}G(x) > (1 + \varepsilon) \alpha] \, d\alpha$$

$$\leq (1 + \varepsilon)^q \int_0^\infty q \alpha^q \sigma[x \in E : \mathcal{A}G(x) > (1 + \varepsilon) \alpha, \hat{N}_s H(x) \leq \gamma \alpha] \, d\alpha$$
Note that we used Step I in the second inequality above. Then, assuming that \( \|A\|_{L^p(E)} \) is finite (albeit with the norm depending on \( k \)), we remove the restriction \( 2^k \leq \ell(Q^*) \leq 2^k \). Here, \( \gamma \) is chosen so that the area integral defined with \( \gamma \) is bounded, the truncation of cones from above is invisible when \( 2^k > \ell(Q_0) \approx \text{diam}(E) \). We have the following analogue of (4.5) for \( \mathcal{A}_k \). Take any \( 0 < p < \infty \). Then \( (A)_{\text{loc}} \) implies that for every \( \varepsilon, \gamma > 0 \), and for all \( \alpha > 0 \)

\[
(\mathcal{A}_k) E \quad \sigma(x \in E : \mathcal{A}_k G(x) > (1 + \varepsilon)\alpha, \mathcal{N}_e H(x) \leq \gamma \alpha) \leq C (\gamma/\varepsilon)^2 \sigma(x \in E : M^2_p(\mathcal{A}_k G)(x) > \alpha),
\]

where \( C \) is independent of \( k \in \mathbb{N} \).

This is proved following line-by-line the argument of Step I and systematically changing \( \mathcal{A} \) to \( \mathcal{A}_k \) etc. We only mention that at the final step, the appropriate analogue of (5.22) becomes

\[
(\mathcal{A}_k) E \quad \sigma(x \in F : \mathcal{A}_k^\gamma G(x) > \varepsilon \alpha) \leq (\varepsilon \alpha)^{-2} \int_{F} \sum_{Q^* \subset D_Y : x \in Q^*, 2^{-k} \leq \ell(Q^*) \leq 2^k} \int_{U_{Q^*}} |G(Y)|^2 \delta(Y)^{1-n} dY d\sigma(x).
\]

Here, \( \mathcal{A}_k^\gamma \) corresponds to the integration over \( \Gamma_k^{\gamma}(x) := \Gamma^{\gamma}(x) \) (see (4.5)). At this point we can remove the restriction \( 2^{-k} \leq \ell(Q^*) \leq 2^k \), dominating the right-hand side of (5.27) by the right-hand side of (5.22) and finish the argument as in Step I.

Now, if \( \|\mathcal{N}_e H\|_{L^p(E)} < \infty \) (and otherwise there is nothing to prove) then \( \|S^k u\|_{L^p(E)} \) is qualitatively finite (albeit with the norm depending on \( k \)) by (5.10). Therefore, we can apply the argument in (5.25) to conclude that (5.26) implies

\[
(\mathcal{A}_k) E \quad \|\mathcal{A}_k G\|_{L^p(E)} \leq C \|N^e H\|_{L^p(E)}.
\]
where $C$ is independent of $k$ (since $C$ in the good-$A$ estimate (5.26) was independent of $k$ and we used finiteness of $\|\mathcal{A}^k\mathcal{G}\|_{L^1(E)}$ only qualitatively). Now one can pass to the limit as $k \to \infty$ and conclude $(B)_q$, as desired.

**Step III:** $(A)_{\text{loc}}$ implies $(G)_{\text{loc}}$, and $[(A_{\text{loc}})\&(5.10)]$ imply $(B)_{\text{loc}}$ for all $0 < p, q < \infty$.

The argument follows line-by-line Steps I and II for the case of a bounded $E$. Further details are left to the interested reader.

**Step IV:** $(B)_{\text{loc}}$ implies $(B)_q$, for all $0 < q < \infty$.

Recall that when $E$ is bounded, we assign $E = Q_0$ and hence, $(B)_q$ is a particular case of $(B)_{\text{loc}}$.

When $E$ is unbounded, we proceed as follows. Given $k \gg 1$ we define $\mathcal{A}_k$ a truncated square functions where the dyadic cones incorporate the restriction that the cubes involved satisfies $\ell(Q) \leq 2^k$. Clearly $\mathcal{A}_k \mathcal{G} \not\supset \mathcal{G}$. Clearly, $(\mathcal{A}_k \mathcal{G})_1 Q = \mathcal{A}^k \mathcal{G}$ for every $Q \in \mathcal{D}_k$. In each such $Q$, the estimate (5.6) holds uniformly on $Q$. With this in hand we can sum over those cubes and conclude $\mathcal{A}_k \mathcal{G}$ is uniformly controlled by $H$. This an the monotone convergence theorem gives the desired estimate.

**Step V:** validity of $(B)_{\text{loc}}$ for some $0 < q < \infty$ implies $(A)$. Assume that $(B)_{\text{loc}}$, for some $0 < q < \infty$ holds. Fix the corresponding constant $C$ from (5.6). Upon renormalization $\tilde{G} := G \left( C \|H\|_{L^{\infty}(\Omega)} \right)^{-1}$ (with $C$ coming from (5.6)), we have

$$\int_Q \left( \mathcal{A}^k \tilde{G}(x) \right)^q \, d\sigma(x) \leq 1, \quad \text{for all } Q \in \mathcal{D}(E),$$

in particular, (4.2) is verified for $F = \tilde{G}$ uniformly on all $Q \in \mathcal{D}(E)$. It follows that (4.3) holds with the same $F$ for $0 < p < \infty$. The case $p = 2$ furnishes the desired Carleson measure estimate (4.11) and hence, as argued right after (4.11), the estimate (1.9) as well.

Let us mention that the case $q = 2$ is, in fact, due to a trivial observation that

$$\frac{1}{\sigma(Q)} \int_{T_0} |\tilde{G}(Y)|^2 \delta(Y) \, dY \approx \int_Q \left( \mathcal{A}^k \tilde{G}(x) \right)^2 \, d\sigma(x),$$

and then the statement for any other $q \geq 2$ follows from Hölder inequality. The above argument was designed to treat $0 < q \leq 2$.

**Step VI:** the proof of (5.17). Fix any domain $D_0 \in \Sigma$. The fact that the first line (5.17) for all $D \in \Sigma$, and, in particular, for all sawtooth subdomains of $D_0$, implies any other line, is a consequence of (5.12) and (5.14) applied to $G_{\chi_{D_0}}$ and $H_{\chi_{D_0}}$ in place of $G$ and $H$. The third line implies the second on any $D_0$ and similarly the fifth line implies the fourth for trivial reasons. The second one implies first on any $D_0 \in \Sigma$ by (5.16).

It remains to show that the validity of $(B)_q$ on the boundary of every $D \in \Sigma$ implies the validity of $(A)$ on $\Omega$ (and similarly for any subdomain of $\Omega$ in $\Sigma$). To this end, recall that $\tilde{Q}_{D_0} = \tilde{T}_0 \in \Sigma$ by our assumptions. And thus, with the same $\tilde{G}$ as in (5.29), $(B)_q$ on $\tilde{T}_0$’s implies that $\|\mathcal{A}^k \tilde{G}\|_{L^1(\tilde{T}_0)} \leq \sigma(\partial\tilde{T}_0) \leq C\sigma(Q)$, where the cones and $\mathcal{A}^k$ are built from the dyadic decomposition of $\partial\tilde{T}_0$. What we want though is (5.29), with the cones and $\mathcal{A}^k$ built from the dyadic decomposition of $E$. To this end, it remains to pass from $\mathcal{A}^k$ in the latter statement (built with cones associated to Whitney decomposition of $\tilde{T}_0$) to $\mathcal{A}^k$ from (5.29) (built with truncated cones associated to Whitney decomposition of $\Omega$). This is a fairly straightforward step, using, in particular, (5.10) to handle the Whitney cubes at distance roughly $\ell(Q)$ from $Q$. We leave the details to the interested reader. Thus, with possibly another renormalization, we are getting (5.29) and proceed to $(A)$ on $\Omega$ as before. $\square$
6. $N < S$ bounds: from Lipschitz to NTA domains

Before the start, let us observe that for any bounded NTA domain $D \subset \mathbb{R}^{n+1}$, with an ADR boundary $E = \partial D$, there exists a cube $Q_0 \in \mathcal{D}(E)$ such that $Q_0 = E$ and for any $Q \in \mathcal{D}(E)$ we have $Q \in \mathcal{D}_{Q_0}$, and since the domain is NTA, there exists at least one interior corkscREW point corresponding to $Q_0$ (or rather to a surface ball containing $Q_0$ with the radius proportional to $\ell(Q_0)$ – see (2.2)). We shall refer to this point as $X_D^+$. Also, recall the dyadic Hardy-Littlewood maximal function from Definition 5.3. In addition, we will be using its continuous analogue.

**Definition 6.1.** Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional ADR set, as in (1.1). By $M = M_E$ we denote the continuous (non-centered) Hardy-Littlewood maximal function on $E$, that is, for $f \in L^1_{\text{loc}}(E)$

$$Mf(x) = \sup_{\Delta \ni x} \frac{1}{|\Delta|} \int_{\Delta} |f(y)| \, d\sigma(y),$$

and we also write $M_p f = M(|f|^p)^{\frac{1}{p}}$. Here, the supremum is taken over all $\Delta$, surface balls on $E$ containing $x$.

It is clear from (2.2) that $M^D f(x) \leq M f(x)$ for every $x \in E$. The converse might fail pointwise, but both maximal functions are bounded in $L^p(E)$, $p > 1$.

With this notation in mind, we have the following result.

**Theorem 6.2.** Given a bounded NTA domain $D \subset \mathbb{R}^{n+1}$ with an ADR boundary $E = \partial D$ let $u \in W^{1,2}_{\text{loc}}(D) \cap C(D)$ be such that for any $c \in \mathbb{R}$, for any $Q \in \mathcal{D}(E)$,

$$(6.3) \quad \sup_{x \in \partial Q} |u(X) - c| \leq C \left( \frac{\ell(Q)^{-n-1}}{\partial Q} \right)^{1/2} \int_{\partial Q} |u - c|^2 \, dX.$$  

Then the following holds.

Suppose that the “$N < S$” estimates are valid on all bounded Lipschitz subdomains $\Omega' \subset D$. That is, for any $\Omega' \subset D$

$$(6.4) \quad \left\| N_{\star, \Omega'} (u - u(X_{\Omega'}^+)) \right\|_{L^2(\Omega')} \leq C_0 \| S_{\partial \Omega'} u \|_{L^2(\partial \Omega')}.$$  

Here $X_{\Omega'}^+$ is any interior corkscREW point of $\partial \Omega'$ at the scale of diam($\Omega'$) (see the discussion above the statement of the Theorem) and $N_{\star, \Omega'}$ and $S_{\partial \Omega'}$ are defined on the boundaries of bounded Lipschitz domains using the traditional non-tangential cones (1.16) on $\partial \Omega'$ for $\kappa > 0$. The constant $C_0$ in (6.4) depends on the Lipschitz character of $\partial \Omega'$, the dimension $n$, and the choice of $\kappa$ only.

Then there exists $0 < c_0 << 1$ depending on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only such that for every $\varepsilon > 0$, $0 < \gamma < c_0 \varepsilon$ and for all $\alpha > 0$

$$(6.5) \quad \sigma \{ x \in E : N_{\star}(u - u(X_D^+))(x) > (1 + \varepsilon) \alpha, \ M_2(\hat{S} u)(x) \leq \gamma \alpha \} \leq C_{\gamma, \varepsilon} \sigma \{ x \in E : N_{\star}(u - u(X_D^+))(x) > \alpha \}.$$  

with the constant $C_{\gamma, \varepsilon} < 1$. To be more precise, $C_{\gamma, \varepsilon} = 1 - \theta + C_1 \left( \frac{\gamma}{\varepsilon} \right)^2$ where $C_1 > 0$ is a constant depending on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only and $0 < \theta < 1$, depending on the same parameters, is from Corollary 4.9.

In particular, for every $u \in L^\infty_{\text{loc}}(D)$ satisfying the assumptions of the Theorem we have

$$(6.6) \quad \| N_{\star}(u - u(X_D^+)) \|_{L^p(E)} \leq C \| \hat{S} u \|_{L^p(E)}, \quad \text{for all } \ p > 2.$$  

We remark that contrary to the previous sections, we do not consider general $\mathcal{A}G$ and $N, H$ any more. This is a necessity as the argument of the area integral has to be the gradient of the argument of the non-tangential maximal function in the course of this proof. Thus, we might as well work
directly with $S$ rather than $A$ (cf. (5.1)). The assumption (6.3) is a standard interior regularity estimate for solutions of elliptic equations (also known as Moser estimate). In principle, we need a slightly weaker version,

$$\|u(Y_Q) - c\| \leq C \left( \ell(Q)^{-n-1} \iint_{U_Q} |u - c|^2 \, dX \right)^{1/2},$$

where $Y_Q$ is any point lying in $U_Q$ together with a ball centered at $Y_Q$ of radius proportional to $\ell(Q)$. Using (6.7) directly would permit us not to enlarge the “aperture of cones”, that is, in this context, not to pass from $U_Q$ to $\tilde{U}_Q$, but that is minor and (6.3) looks a bit more familiar and more in line with (7.2).

We also remark that we could be more careful, as in Theorem 5.7, to try to avoid the assumption $u \in L^\infty(D)$ in (6.6), but in practice we will always work with bounded solutions.

**Proof.** For brevity, we shall write $u_D := u(X_D^+)$. We can assume that the set on the right-hand side of (6.5) is not empty (otherwise $N_\gamma(u - u_D)(x) \leq \alpha$ for a.e. $x \in E$ and the left-hand side of (6.5) has measure zero, as desired). It is also finite as $E$ is bounded by our assumptions.

We also assume for the time being that the set on the right-hand side of (6.5) is not the entire $E = Q_0$. This case will be addressed in the end of the proof.

Let $(Q_j)_j \subset D(E)$ be a (disjoint) collection of maximal cubes such that

$$\bigcup_j Q_j = \{x \in E : (N_\gamma(u - u_D))(x) > \alpha\}.$$

Indeed, one can subdivide $Q_0 = E$ into dyadic cubes stopping whenever for some $Q' \subset Q$ there exists $Y \in U_Q$ such that $u(Y) - u_D > \alpha$. Since we assume for now that the right-hand side of (6.5) is not the entire $E = Q_0$, it follows that $Q_0$ is not the stopping cube. The resulting collection of stopping time cubes $Q'$ is automatically maximal (for, the parent $\tilde{Q}'$ does not belong to this collection by construction) and disjoint (again, by construction). We will denote it by $\bigcup_j Q_j$. The fact that the union of stopping time cubes coincides with the desired set, that is, (6.8) holds, can be seen as follows. Since there exists $Y \in U_Q$ such that $u(Y) - u_D > \alpha$, by definition $(N_\gamma(u - u_D))(x) > \alpha$ for every $x \in Q_j$. Thus, $Q_j \subset \{x \in E : (N_\gamma(u - u_D))(x) > \alpha\}$ for every $j$. Conversely, if $x$ is such that $(N_\gamma(u - u_D))(x) > \alpha$ then there exists $Q' \subset Q$ containing $x$ such that for some $Y \in U_{Q'}$ we have $u(Y) - u_D > \alpha$. However, in that case either $Q'$ or one of $Q \in D(E)$ with $Q \supset Q'$ must be the stopping time cube. Hence, $Q' \subset \bigcup_j Q_j$. This finishes the proof of (6.8).

Let us denote by $Q$ one of the maximal cubes from the collection $(Q_j)_j$ constructed above. We will prove that for every such $Q$ we have

$$\sigma\{x \in Q : N_\gamma(u - u_D)(x) > (1 + \varepsilon)\alpha, M_2(\tilde{u})(x) \leq \gamma\alpha\} \leq (1 - \theta + C(\gamma, \varepsilon)) \sigma(Q),$$

with $0 < \theta < 1$ from the “Interior Big Pieces of Lipschitz Graph” condition (cf. Corollary 4.9) and the constant $C(\gamma, \varepsilon) \approx C_0 \left( \frac{\gamma}{\varepsilon} \right)^2$ for all $\gamma < c_0 \varepsilon$ with a suitably small $c_0$. Here, $C_0$ will be a constant form (6.4) for a collection of bounded Lipschitz subdomains of $D$ as in Corollary 4.9. Hence, $C_0$ will depend on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only. We write $C(\gamma, \varepsilon)$ as $C_1 \left( \frac{\gamma}{\varepsilon} \right)^2$ in the statement of the theorem as both $C_0$ and the implicit constant in the equivalence $C(\gamma, \varepsilon) \approx C_0 \left( \frac{\gamma}{\varepsilon} \right)^2$ depend on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only. In particular, given any $\varepsilon > 0$ one can choose $\gamma$ small enough, depending on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only, so that $C_{\gamma, \varepsilon} := 1 - \theta + C(\gamma, \varepsilon) < 1$. Then, possibly further shrinking $c_0$, we have $C_{\gamma, \varepsilon} < 1$ for all $\gamma < c_0 \varepsilon$, as desired.

Let us now turn to (6.9). First, we claim that we can reduce matters to proving
where as before, $N^Q_F$ is defined by taking the supremum within a truncated cone $\Gamma^Q$ (see (2.23)). Indeed, by maximality, for any $P \in \mathbb{E}(E)$, $P \supset Q$, we have $N_e(u-u_D) \leq \alpha$ for some $x \in P$ and hence $u-u_D \leq \alpha$ on the entire $U_P$ for every such $P$. Hence, if $N_e(u-u_D)(x) > (1+\epsilon)\alpha$ for some $x \in Q$, then necessarily there is a point $Y \in \bigcup_{Q \subseteq Q' \supset \Theta} U^Q$ such that $u(Y) - u_D > \alpha$. And hence, the set on the left-hand side of (6.10) contains the set on the left-hand side of (6.9), as desired.

Next we invoke Corollary 4.9 and take a bounded Lipschitz domain $\Omega_Q \subset D$ satisfying properties (1)-(3) in the statement of the Corollary. In particular, we set $F_Q := \partial \Omega_Q \cap Q \subset Q$ such that $\sigma(F_Q) \geq \theta \sigma(Q)$. Since $\sigma(Q \setminus F_Q) \leq (1-\theta)\sigma(Q)$, it remains to prove that

$$\sigma(x \in F_Q : N^Q_F(u-u_D)(x) > (1+\epsilon)\alpha, M_2(\hat{S}u)(x) \leq \gamma \alpha) \leq C(\gamma, \epsilon)\sigma(Q).$$

Going further, let us denote by $\tilde{Y}_Q \in U_Q$ the corkscrew point of $\Omega_Q$ relative to $Q$ (cf. property (2) in Corollary 4.9). Denoting by $\tilde{Q}$ the parent of $Q$, we observe that by maximality $u-u_D \leq \alpha$ in $U_{\tilde{Q}}$ and, in particular,

$$u(\tilde{Y}_Q) - u_D \leq \alpha$$

(here $\tilde{Y}_Q$ can retain the same significance as in Corollary 4.9 or just be any point lying in $U_{\tilde{Q}}$ together with its corkscrew ball). On the other hand,

$$|u(\tilde{Y}_Q) - u(Y_Q)| \leq \gamma \alpha.$$

Indeed, since $M_2^Q(\hat{S}u)(x) \leq M_2(\hat{S}u)(x) \leq \gamma \alpha$, we have, in particular,

$$\int_{\tilde{Q}} |\nabla u(Y)|^2 \delta(Y) \ dY \leq (\gamma \alpha)^2 \sigma(\tilde{Q}) \approx (\gamma \alpha)^2 \sigma(Q),$$

and hence,

$$\int_{\tilde{Q}_\tilde{Q}} |\nabla u(Y)|^2 \ dY \leq (\gamma \alpha)^2 \ell(Q)^{n-1}.$$

Then, denoting temporarily

$$c_Q := \frac{1}{|U_{\tilde{Q}} \cup U_Q|} \int_{U_{\tilde{Q}} \cup U_Q} u \ dX,$$

we have

$$|u(\tilde{Y}_Q) - u(Y_Q)| \leq |u(\tilde{Y}_Q) - c_Q| + |u(Y_Q) - c_Q| \leq \left( \ell(Q)^{-n-1} \int_{U_{\tilde{Q}} \cup U_Q} |u - c_Q|^2 \ dX \right)^{1/2} \leq \left( \ell(Q)^{-n+1} \int_{U_{\tilde{Q}} \cup U_Q} |\nabla u|^2 \ dX \right)^{1/2},$$

by (6.3). Now we can invoke standard Poincaré inequality considerations to show (6.13) and then combine this with (6.12) to reduce (6.11) to

$$\sigma(x \in F_Q : N^Q_F(u-u(Y_Q))(x) > \epsilon \alpha/2, M_2(\hat{S}u)(x) \leq \gamma \alpha) \leq C(\gamma, \epsilon)\sigma(Q).$$
assuming that $\gamma < c_0 \varepsilon$ with a suitably small $c_0$ depending on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only.

At this stage let us recall once again condition (2) of Corollary 4.9. By definition of a corkscrew point and given the fact that all implicit constants depend on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only, we can assure that for a suitable $\kappa$, once again depending on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only, the non-tangential approach regions of $\Omega_Q$ defined by

$$
\Gamma_{\Omega_Q}(x) := \{ Y \in \Omega_Q : |Y - x| \leq (1 + \kappa) \text{dist}(Y, \partial \Omega_Q) \}, \quad x \in \partial \Omega_Q,
$$

contain arising corkscrew points. That is, in the notation of Corollary 4.9, $Y_{Q'} \in \Gamma_{\Omega_Q}(y_Q')$ for all $Q' \in \mathcal{D}(Q)$.

Now, if $x \in F_Q$ is such that $N_s^Q(u - u(Y_Q))(x) > \varepsilon \alpha / 2$, it follows that there exists $Q' \ni x$ and there exists $X \in U_Q$ such that $u(X) - u(Y_Q) > \varepsilon \alpha / 2$. Much as above, using the fact that $M^2_\Omega(\tilde{S}u)(x) \lesssim \gamma \alpha$, and Poincaré inequality considerations, we deduce that $u(Y_{Q'}) - u(Y_Q) > \varepsilon \alpha / 4$ provided that $\gamma < c_0 \varepsilon$ with $c_0$ small enough depending on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only. Here $Y_{Q'} \in U_Q$ is a special point from the condition (2) of Corollary 4.9 corresponding to $y_{Q'} = x$. Since by construction $Y_{Q'} \in \Gamma_{\Omega_Q}(x)$, we have

$$
N_s^Q(u - u(Y_Q))(x) := \sup_{\Gamma_{\Omega_Q}(x)} (u - u(Y_Q)) > \varepsilon \alpha / 4,
$$

that is, we can change the cones in the definition of the non-tangential maximal function from those with respect to $D$ to those with respect to $\partial \Omega_Q$. Using (a simplified version of) the same argument, we also can switch from $Y_Q$, which is a corkscrew point of $\partial \Omega_Q$ relative to some surface ball of the radius $r \approx \ell(Q)$ in $\Omega_Q^+$ in (6.4), and obtain

$$
N_s^Q(u - u(X_{\Omega_Q}^+))(x) := \sup_{\Gamma_{\Omega_Q}(x)} (u - u(Y_Q)) > \varepsilon \alpha / 8.
$$

Now (6.17) further reduces to showing that

$$
\sigma\{x \in F_Q : N_s^Q(u - u(X_{\Omega_Q}^+))(x) > \varepsilon \alpha / 8, \ M_2(\tilde{S}u)(x) \leq \gamma \alpha\} \leq C(\gamma, \varepsilon) \sigma(Q).
$$

At this stage, using the Tchebyshev inequality and the assumption (6.4), we can write

$$
\sigma\{x \in F_Q : N_s^Q(u - u(X_{\Omega_Q}^+))(x) > \varepsilon \alpha / 8, \ M_2(\tilde{S}u)(x) \leq \gamma \alpha\} \leq \sigma\{x \in F_Q : N_s^Q(u - u(X_{\Omega_Q}^+))(x) > \varepsilon \alpha / 8\}
$$

$$
\leq \left(\frac{8}{\varepsilon \alpha}\right)^2 \int_{\partial \Omega_Q} \left( N_s^Q(u - u(X_{\Omega_Q}^+))(x) \right)^2 \ d\sigma
$$

$$
\leq C_0 \left( \frac{1}{\varepsilon \alpha} \right)^2 \int_{\partial \Omega_Q} (S_{\Omega_Q^+}u)(x)^2 \ d\sigma \approx C_0 \left( \frac{1}{\varepsilon \alpha} \right)^2 \int \int_{\Omega_Q} |\nabla u(Y)|^2 \text{dist}(Y, \partial \Omega_Q) \ dY
$$

$$
\leq C_0 \left( \frac{1}{\varepsilon \alpha} \right)^2 \int \int_{D \cap B(x_Q, \ell(Q))} |\nabla u(Y)|^2 \text{dist}(Y, E) \ dY.
$$

The last inequality is due to the fact that $\Omega_Q \subset D \cap B(x_Q, \ell(Q))$ (see (3) in Corollary 4.9) and, in particular, $\text{dist}(Y, \partial \Omega_Q) \leq \text{dist}(Y, E)$ for every $Y \in \Omega_Q$. Note that all our choices depended on $n$, the NTA constants of $D$ and the ADR constants of $\partial D$ only, and hence, so does $C_0$.

Now, let us return to the set on the left-hand side of (6.19). At this stage, we have dropped the condition $M_2(\tilde{S}u)(x) \leq \gamma \alpha$. However, if for every $x \in F_Q$ we have $M_2(\tilde{S}u)(x) > \gamma \alpha$, then the set on the left-hand side of (6.19) has measure zero and there is nothing to prove. Hence, we can proceed...
assuming that there is a point \( x_0 \in F_Q \) such that \( M_2(\widehat{S}u)(x_0) \leq \gamma \alpha \). Therefore, for all surface balls \( \Delta \ni x_0 \) we have \( \int_{\Delta} (|\widehat{S}u(y)|^2 \, dy \leq (\gamma \alpha)^2 \). This gives an upper bound for the right-hand side of (6.19). Indeed, recall, e.g., from the argument in [HMM] passing from the Carleson measure on tent regions (4.13), [HMM], loc.cit., to Carleson measure on balls (4.12), [HMM], loc.cit., that for any \( B(x_0, \mathcal{C}(Q)) \cap D \) on the right-hand side of (6.19) there exists a collection of dyadic cubes \( \{P_j\}_{j=1}^J \subset \mathcal{D}(E) \) of uniformly controlled cardinality \( M \) with \( \ell(P_j) \approx \ell(Q) \) such that \( \bigcup_j P_j \) covers \( B(x_0, \mathcal{C}(Q)) \cap E \) and \( \bigcup_j T_{P_j} \) covers \( B(x_0, \mathcal{C}(Q)) \cap D \). Now if we take \( \Delta \) to be a surface ball on \( E \) of radius \( r \approx \ell(Q) \) containing \( \bigcup_j P_j \) and \( x_0 \in F_Q \), then

\[
\int_{D \cap B(x_0, \mathcal{C}(Q))} |\nabla u(Y)|^2 \, \text{dist}(Y, E) \, dY
\leq \sum_{j=1}^M \int_{T_{P_j}} |\nabla u(Y)|^2 \, \text{dist}(Y, E) \, dY \leq \int_{\Delta} (|\widehat{S}u(y)|^2 \, dy \leq (\gamma \alpha)^2 \sigma(\Delta) \leq (\gamma \alpha)^2 \sigma(Q).
\]

Combining this with (6.19), we conclude that

\[
(6.20) \quad \sigma\{x \in F_Q : N_s \Omega_0 (u - u(X^+_D)) (x) > \varepsilon \alpha / 4, \ M_2(\widehat{S}u)(x) \leq \gamma \alpha \} \\ \leq C_0 \left( \frac{\ell}{\varepsilon} \right)^2 \sigma(Q) =: C(\gamma, \varepsilon) \sigma(Q).
\]

Now it only remains to treat a special case when \( E = Q_0 = Q \) is itself the first stopping cube. In this case, \( N_s = N_s^{Q_0} \) by definition and hence, the reduction to (6.11) is automatic. Using (6.13) we can swipe \( u_D \) for \( u(Y_{Q_0}) \) (as they are both corkscrew points of \( E \) at the scale \( \ell(Q_0) = \text{diam}(E) \)) and further reduce to (6.17). From that point on, the argument is the same as before.

Finally, having at hand (6.5), an argument analogous to (5.25) yields (6.6). To be specific, we show that taking \( \varepsilon > 0 \) in (6.5) small enough depending on the NTA constants of \( D \) and ADR constants of \( E \) and then taking \( \gamma > 0 \) small enough depending on the same parameters and \( \varepsilon \), the estimate (6.5) yields (6.6). It is here that we use a possibility to pick \( \varepsilon > 0 \) sufficiently small.

Indeed, fix any \( q > 2 \). The assumption that \( u \in L^\infty(D) \) and boundedness of \( D \) guarantee that \( \|N_s(u - u(X^+_D))\|_{L^q(E)} \) is a priori finite. Then much as in (5.25),

\[
(6.21) \quad \|N_s(u - u(X^+_D))\|_{L^q(E)}^q = (1 + \varepsilon)^q \int_0^\infty q a^q \sigma\{x \in E : N_s(u - u(X^+_D))(x) > (1 + \varepsilon) a\} \, \frac{da}{a} \\ \leq (1 + \varepsilon)^q \int_0^\infty q a^q \sigma\{x \in E : N_s(u - u(X^+_D))(x) > (1 + \varepsilon) a\} \, \frac{da}{a} + \left( \frac{1 + \varepsilon}{\gamma} \right)^q \|M_2(\widehat{S}u)\|_{L^q(E)}^{q} \\ \leq C_{\gamma, \varepsilon} (1 + \varepsilon)^q \int_0^\infty q a^q \sigma\{x \in E : N_s(u - u(X^+_D))(x) > a\} \, \frac{da}{a} + \left( \frac{1 + \varepsilon}{\gamma} \right)^q \|M_2(\widehat{S}u)\|_{L^q(E)}^{q}.
\]

At this point we note that we can choose \( \varepsilon > 0 \) and \( 0 < \gamma < c_0 \varepsilon \) depending on the NTA constants of \( D \) and ADR constants of \( E \) such that \( C_{\gamma, \varepsilon} (1 + \varepsilon)^q = \left( 1 - \theta + C_1 \left( \frac{\varepsilon}{2} \right)^{\gamma} \right) (1 + \varepsilon)^q < \frac{1}{2} \). Then the right-hand side of (6.21) is bounded by

\[
(6.22) \quad \frac{1}{2} \|N_s(u - u(X^+_D))\|_{L^q(E)}^q + C \|M_2(\widehat{S}u)\|_{L^q(E)}^{q}.
\]
and a priori finiteness of \( \|N_e(u - u(X_D^*))\|_{L^p(E)} \) allows us to hide the corresponding term on the left-hand side.

\[ \leq \frac{1}{n} \|N_e(u - u(X_D^*))\|_{L^p(E)}^n + C \|\hat{S}u\|_{L^p(E)}^n, \]

for some \( C > 0 \) depending on the ADR and NTA constants on \( E \) only. Here, as before, \( Y_Q \) is any corkscREW point of \( D \) relative to \( Q \).

**Theorem 6.25.** Given a bounded NTA domain \( D \subset \mathbb{R}^{n+1} \) with an ADR boundary \( E = \partial D \), let \( u \in W^{1,2}_{\text{loc}}(D) \), continuous and bounded on \( D \), be such that for every \( \alpha \in \mathbb{R} \), for any \( Q \in \mathbb{D}(E) \) (6.3) is valid and

\[(6.26) \quad \left( \ell(Q)^{n-1} \int_{U_Q} |\nabla u|^p \, dX \right)^{1/p} \lesssim \left( \ell(Q)^{n-1} \int_{U_Q} |\nabla u|^2 \, dX \right)^{1/2}, \]

for some \( p > 2 \). Then the following holds.

If the local “\( N < S \)” estimates are valid on all bounded NTA subdomains \( \Omega' \subset D \) for the same \( p > 2 \) as above, that is, for every \( Q \in \mathbb{D}(\partial \Omega') \) and any \( Y_Q \), a corkscREW point of \( \Omega' \) relative to \( Q \),

\[(6.27) \quad \|N_e^Q(u - u(Y_Q))\|_{L^q(Q)} \leq C \|\hat{S}^Q(u\|_{L^q(Q)}, \]

then (6.6) holds in \( D \) for all \( 0 < q < \infty \).

**Proof.** Much as in the proof of Theorem 6.2, matters can be reduced to showing that there exists \( 0 < c_0 \ll 1 \) depending on \( n \), the NTA constants of \( D \) and the ADR constants of \( \partial D \) only such that for every \( \varepsilon > 0 \), \( 0 < \gamma < c_0 \varepsilon \) and for all \( \alpha > 0 \)

\[(6.28) \quad \sigma \{ x \in E : N_e(u - u(X_D^*))(x) > (1 + \varepsilon) \alpha \}, \hat{S}u(x) \leq \gamma \alpha \}

\[ \leq C^\alpha_{\gamma, \varepsilon} \sigma \{ x \in E : N_e(u - u(X_D^*))(x) > \alpha \}. \]

with the constant \( C^\alpha_{\gamma, \varepsilon} < 1 \). We will show that in this case \( C^\alpha_{\gamma, \varepsilon} \to 0 \) as \( \gamma \to 0 \) for a fixed \( \varepsilon \). In fact, at this stage we do not have to be as careful keeping \( \varepsilon > 0 \) arbitrary and could just work with \( \varepsilon = 1 \), but it is convenient to keep the notation in line with that in Theorem 6.2.

As in (6.8), we decompose the set on the right-hand side of (6.28) into maximal cubes, let \( Q \) be one of such cubes, and reduce (6.28) to

\[(6.29) \quad \sigma \{ x \in Q : N_e^Q(u - u(Y_Q))(x) > \varepsilon \alpha / 2 \}, \hat{S}u(x) \leq \gamma \alpha \}

assuming that \( \gamma < c_0 \varepsilon \) with a suitably small \( c_0 \) depending on \( n \), the NTA constants of \( D \) and the ADR constants of \( \partial D \) only (and the constant \( C^\alpha_{\gamma, \varepsilon} \) satisfies the same constraints as before, although possibly the actual value is different). The notation here is the same as in (6.17) and the reduction argument is, in fact, even simpler, because one does need to pass to \( F_Q \) and because the use of \( \hat{S}u(x) \leq \gamma \alpha \) for some \( x \in Q \) directly yields (6.15) avoiding (6.14).

Now let us denote

\[(6.30) \quad E_Q := \{ x \in Q : N_e^Q(u - u(Y_Q))(x) > \varepsilon \alpha / 2, \hat{S}u(x) \leq \gamma \alpha \}. \]

By inner regularity, we can choose \( E_Q^\circ \), a closed subset of \( E_Q \), with the size arbitrarily close to that of \( E_Q \). Now we denote by \( \mathcal{F} \) the decomposition of an open set \( (E_Q^\circ)^\circ \) into dyadic maximal cubes from the family \( \mathbb{D}(E) \) and let \( \hat{\Omega}_F \) be a global sawtooth region corresponding to such a decomposition. The “fatness” of the Whitney regions defining \( \hat{\Omega}_F \) is bigger than that for \( N_e^Q \) and smaller that for \( \hat{S} \).
as will become clear soon. For now, just take them slightly larger than the Whitney regions of $N^Q$. We start with

$$
\sigma(E'_Q) \leq \left( \frac{2}{\varepsilon \alpha} \right)^p \int_{E'_Q} |N^Q(u - u(Y_Q))(x)|^p \, dx,
$$

and now change the cones from those used in $N^Q$ (dyadic, with respect to $D$) to the traditional ones (1.16) with respect to $\Omega_F$. This is possible because every dyadic cone with respect to $D$, $\Gamma(x)$, $x \in E'_Q$, is comprised of $U_Q$ with $Q \ni x$, and $\Omega_F$ contains all the corresponding $\hat{U}_Q$, so that for $Y \in U_Q$ we have $|Y - x| \approx \ell(Q)$ and $\text{dist}(Y, \partial \hat{\Omega}_F) \approx \ell(Q)$.

To lighten the notation, we will write $\Omega_F$ in place of $\hat{\Omega}_F$ from now on. The non-tangential maximal function defined with the traditional cones with respect to $\Omega_F$ will be denoted by $N_{*\Omega_F}$, and we have then the right-hand side of (6.31) bounded by

$$
\left( \frac{2}{\varepsilon \alpha} \right)^p \int_{E'_Q} |N^{C(Q)}_{*\Omega_F}(u - u(Y_Q))(x)|^p \, dx.
$$

We remark that by construction $Y_Q$, which is a corkscrew point for $Q$ with respect to $D$ is also a corkscrew point for some $B(x_Q, \hat{\ell}(Q)) \cap \partial \Omega_F \ni Q$ with respect to $\Omega_F$, since $\text{int}(\hat{U}_Q) \subset \Omega_F$. This is assuming that $Q \neq F$, but otherwise $E_Q = \emptyset$ and there is nothing to prove.

Now, with the constant $C$ changing value from line to line but still depending on the NTA/ADR constants of $D$ only, we have

$$
\left( \frac{2}{\varepsilon \alpha} \right)^p \int_{E'_Q} |N^{C(Q)}_{*\Omega_F}(u - u(Y_Q))(x)|^p \, dx
\begin{align*}
\leq \left( \frac{2}{\varepsilon \alpha} \right)^p \int_{\partial \Omega_F \cap B(x_Q, \hat{\ell}(Q))} |N^{C(Q)}_{*\Omega_F}(u - u(Y_Q))(x)|^p \, dx \\
\leq \left( \frac{2}{\varepsilon \alpha} \right)^p \int_{\partial \Omega_F \cap B(x_Q, \hat{\ell}(Q))} S^{C(Q)}_{\Omega_F} u(x)^p \, dx.
\end{align*}
$$

The first inequality here is just integration on a larger set. For the second one, we first recall that a sawtooth domain with respect to an NTA domain with an ADR boundary is itself an NTA domain with an ADR boundary (for the fact that the ADR property is preserved, see [HMM], and for NTA features see [HM]). With this at hand, we pass from traditional cones with respect to $\Omega_F$ to dyadic cones with respect to $\Omega_F$ by Remark 2.35, cover $\partial \Omega_F \cap B(x_Q, \hat{\ell}(Q))$ by dyadic cubes of $\partial \Omega_F$ at the scale $\hat{\ell}(Q)$ with a suitable $C$, use (6.27) on $\Omega_F$ and then pass back to the traditional cones with respect to $\Omega_F$, enlarging aperture and enlarging $C$ in (6.33) in the process, but still keeping dependence on the NTA/ADR constants only.

At this point, (6.31)–(6.33) can be summarized as

$$
\sigma(E'_Q) \leq \left( \frac{1}{\varepsilon \alpha} \right)^p \int_{\partial \Omega_F \cap B(x_Q, \hat{\ell}(Q))} S^{C(Q)}_{\Omega_F} u(x)^p \, dx
\begin{align*}
\leq \left( \frac{1}{\varepsilon \alpha} \right)^p \int_{E'_Q} S^{C(Q)}_{\Omega_F} u(x)^p \, dx + \left( \frac{1}{\varepsilon \alpha} \right)^p \int_{(\partial \Omega_F \cap B(x_Q, \hat{\ell}(Q))) \setminus E'_Q} S^{C(Q)}_{\Omega_F} u(x)^p \, dx \\
=: I + II.
\end{align*}
$$

The estimate on part $I$ is now quite straightforward. For points on the common boundary of $\partial D$ and $\partial \Omega_F$, that is, for $x \in E'_Q$, the traditional cones $\Gamma_{\partial D}(x)$ are trivially contained in traditional cones with respect to $\Omega$, $\Gamma_{\Omega}(x)$, which are in turn contained in dyadic cones with respect to $D$, $\Gamma(x)$, provided
that \( \eta \) and \( K \) in their definition are sufficiently small and large, respectively. This means that

\[
S_{C^2(Q)} \leq \tilde{S}_{C^2(Q)} \leq \gamma r, \quad \text{for } x \in E'_Q,
\]

provided that \( \tilde{S}_{C^2(Q)} \) in the definition of \( E_Q \) uses sufficiently wide “cones”. Hence,

\[
(6.35) \quad I \leq \left( \frac{Y}{\epsilon} \right)^p \sigma(Q).
\]

Turning to \( II \), we start with the following

**Claim 6.36.** For any \( x \in \partial \Omega_F \) there exists \( x_0 \in E'_Q \) such that

\[
\Gamma_{\Omega_F}(x) \subset \Gamma(x_0),
\]

where \( \Gamma(x_0), x_0 \in \partial D \), is a family of dyadic cones with respect to \( D \), with suitably large aperture (that is, suitable \( \eta \) and \( K \)).

**Proof.** We have already discussed that the Claim is straightforward with \( x = x_0 \) when \( x \in E'_Q \), so we concentrate on \( x \in \partial \Omega_F \setminus E'_Q \).

Since \( x \in \partial \Omega_F \setminus E'_Q \), we know that \( x \) belongs to the closure of some \( \tilde{U}_{Q_0}, Q_0 \in \mathcal{D}(E) \), such that \( Q_0 \cap E'_Q \neq \emptyset \). Indeed, if \( Q_0 \cap E'_Q = \emptyset \) then \( Q_0 \) is either \( Q_j \) or one of its subcubes, which contradicts the definition of the sawtooth region \( \Omega_F \). Therefore, there exists a point \( x_0 \in Q_0 \cap E'_Q \).

By Remark 2.35, it is sufficient to show that a dyadic cone with respect to \( \Omega_F \) with a vertex at \( x \), \( \Gamma_{\Omega_F}(x) \), is contained in a dyadic cone with respect to \( D \) with a vertex at \( x_0, \Gamma(x_0) \).

Now, if \( Y \in \Gamma_{\Omega_F}(x) \) then \( Y \in U_{p\Omega_F} \) for some \( P \in \mathcal{D}(\partial \Omega_F) \), \( P \ni x \), where \( U_{p\Omega_F} \) is a Whitney region of \( \Omega_F \) corresponding to \( P \in \mathcal{D}(\partial \Omega_F) \). If \( \ell(P) \leq c_0 \ell(Q_0) \), then \( |Y - x| \leq c_0^p \ell(Q_0) \) and hence, \( Y \in \tilde{U}_{Q_0} \), together with \( x \), provided that \( c_0 \) and hence \( c_0^p \) are sufficiently small depending on the usual geometric parameters only. Therefore, \( Y \in \Gamma(x_0) \) in this case.

If \( \ell(P) \geq c_0 \ell(Q_0) \) then let \( Q_P \in \mathcal{D}(\partial D) \) denote any cube containing \( Q_0 \) at the scale \( \frac{1}{c_0} \ell(P) \). We claim that \( Y \in U_{Q_P} \), provided that the parameters \( \eta \) and \( K \) in the definition of Whitney regions have been sufficiently adjusted. Indeed,

\[
\text{dist}(Y, Q_P) \leq \text{dist}(Y, P) + \text{dist}(P, Q_P) \leq \ell(P) + |x - x_0| \leq \ell(P) + \ell(Q_0) \leq \ell(P).
\]

In particular, \( \text{dist}(Y, \partial D) \leq \ell(P) \). On the other hand,

\[
\text{dist}(Y, \partial D) \geq \text{dist}(Y, \partial \Omega_F) \geq \ell(P)
\]

since \( Y \in U_P \). This is sufficient to show that \( Y \in U_{Q_P} \), with suitable \( \eta \) and \( K \), finishing the proof of Claim 6.36. \( \square \)

We observe that \( \Gamma(x_0) \) in the statement of Claim 6.36 can of course exceed the limits of \( \Omega_F \), which is not a problem.

Let us now get back to the proof of the Theorem, specifically, to the estimate for \( II \) in (6.34). To this end, we split it further into \( II_1 \) and \( II_2 \), as follows. The square function in the integrand of \( II \) is

\[
\tilde{S}_{C^2(Q)}(x) = \left( \int_{\Gamma_{C^2(Q)}(x)} |\nabla u(Y)|^2 \frac{dY}{\text{dist}(Y, \partial \Omega_F)^{n-1}} \right)^{1/2}.
\]

We divide the domain of integration into \( Y \in \Gamma_{C^2(Q)}(x) \) such that \( \text{dist}(Y, \partial \Omega_F) \leq c_0 \text{dist}(Y, \partial D) \) and \( Y \in \Gamma_{C^2(Q)}(x) \) such that \( \text{dist}(Y, \partial \Omega_F) \geq c_0 \text{dist}(Y, \partial D) \), with small \( c_0 \) to be determined below. The corresponding parts of \( II \) will be referred to as \( II_1 \) and \( II_2 \), respectively.

The estimate on \( II_2 \) is easier. Using Claim 6.36, we have for every \( \partial \Omega_F \cap B(x_0, C2(Q)) \setminus E'_Q \)
\[(6.37) \quad \left( \iint_{Y \in \Gamma^{C(Q)}(x)} |\nabla u(Y)|^2 \frac{dY}{\text{dist}(Y, \partial \Omega_F)^{n-1}} \right)^{1/2} \leq \left( \iint_{Y \in \Gamma^{C(Q)}(x); \text{dist}(Y, \partial \Omega_F) \geq c_0 \text{dist}(Y, \partial D)} |\nabla u(Y)|^2 \frac{dY}{\text{dist}(Y, \partial D)^{(n-1)}} \right)^{1/2} \leq \sup_{x_0 \in E_Q} \left( \iint_{Y \in \Gamma^{C(Q)}(x_0)} |\nabla u(Y)|^2 \frac{dY}{\text{dist}(Y, \partial \Omega_F)^{n-1}} \right)^{1/2} \leq \sup_{x_0 \in E_Q} \tilde{\gamma}^{C(Q)} u(x_0) \leq \gamma \alpha. \]

Thus,
\[(6.38) \quad H_2 \leq \left( \frac{2}{E} \right)^p \sigma((\partial \Omega_F \cap B(x_Q, C \ell(Q))) \setminus E'_Q) \leq \left( \frac{2}{E} \right)^p \sigma(Q). \]

It remains to handle \( H_1 \). To start,
\[(6.39) \quad \int_{(\partial \Omega_F \cap B(x_Q, C \ell(Q))) \setminus E'_Q} \left( \iint_{Y \in \Gamma^{C(Q)}(x); \text{dist}(Y, \partial \Omega_F) \leq c_0 \text{dist}(Y, \partial D)} |\nabla u(Y)|^2 \frac{dY}{\text{dist}(Y, \partial \Omega_F)^{n-1}} \right)^{p/2} dx \leq \iint_{(\partial \Omega_F \cap B(x_Q, C \ell(Q))) \setminus E'_Q} \left( \iint_{Y \in \Gamma^{C(Q)}(x); \text{dist}(Y, \partial \Omega_F) \leq c_0 \text{dist}(Y, \partial D)} |\nabla u(Y)|^p \frac{dY}{\text{dist}(Y, \partial \Omega_F)^{n-1}} \right)^{\frac{p}{2}} \times \left( \iint_{Y \in \Gamma^{C(Q)}(x); \text{dist}(Y, \partial \Omega_F) \leq c_0 \text{dist}(Y, \partial D)} \frac{dY}{\text{dist}(Y, \partial \Omega_F)^n} \right)^{\frac{p-1}{2}} dx. \]

We claim that
\[(6.40) \quad \iint_{Y \in \Gamma^{C(Q)}(x); \text{dist}(Y, \partial \Omega_F) \leq c_0 \text{dist}(Y, \partial D)} \frac{dY}{\text{dist}(Y, \partial \Omega_F)^n} \leq \text{dist}(x, E'_Q). \]

It is here that the smallness of \( c_0 \) is used. To show this, let us first observe that \( x \in (\partial \Omega_F \cap B(x_Q, C \ell(Q))) \setminus E'_Q \subset D \) belongs to some \( U_P \), a Whitney region of \( D \) corresponding to \( P \in \mathcal{D}(E) \), with \( \text{dist}(P, E'_Q) \approx \ell(P) \). Indeed, the sawtooth domain \( \Omega_F \) was formed based on cubes \( Q_j \) such that \( \ell(Q_j) \approx \text{dist}(Q_j, E'_Q) \) and their subcubes. If \( \ell(P) \geq C \text{dist}(P, E'_Q) \) for sufficiently large \( C \), this cube cannot belong to \( T \) and thus, int \( U_P \subset \Omega_F \). If \( \ell(P) \leq c \text{dist}(P, E'_Q) \) with sufficiently small \( c \), then first of all, \( P \subset \bigcup Q_j \) (since \( \text{dist}(P, E'_Q) > 0 \)) and secondly, it cannot contain any \( Q_j \) by maximality. Hence, \( P \notin T \) and \( x \notin \Omega_F \). Thus, indeed, \( \text{dist}(P, E'_Q) \approx \ell(P) \) if \( U_P \ni x \), or in other words, \( \partial \Omega_F \cap B(x_Q, C \ell(Q)) \setminus E'_Q \) is covered by such \( U_P \). Notice that \( \text{dist}(x, E'_Q) \approx \ell(P) \) in this notation:
\[ \ell(P) \approx \text{dist}(x, \partial D) \leq \text{dist}(x, E'_Q) \leq \text{dist}(x, P) + \text{dist}(P, E'_Q) \leq \ell(P). \]

If \( Y \in \Gamma^{C(Q)}(x) \) is such that \( \text{dist}(Y, \partial \Omega_F) \leq c_0 \text{dist}(Y, \partial D) \), then \( |x - Y| \leq (1 + \kappa) \text{dist}(Y, \partial \Omega_F) \leq c_0(1 + \kappa) \text{dist}(Y, \partial D) \). Hence, if \( c_0 \) is sufficiently small,
\[(6.41) \quad \text{dist}(Y, \partial \Omega_F) \leq \text{dist}(Y, \partial D) \approx \text{dist}(x, \partial D) \approx \ell(P) \approx \text{dist}(x, E'_Q). \]

In fact, and it will be useful soon, we can choose \( c_0 \) so small that \( Y \) belongs to \( \tilde{U}_P \) for the same \( U_P \) that contains \( x \). Having this at hand, \( 6.40 \) is established simply splitting the integral into the slices \( |Y - x| \leq (1 + \kappa) \text{dist}(Y, \partial \Omega_F) \approx 2^{-j}(1 + \kappa) \text{dist}(x, E'_Q), j \in \mathbb{N} \), and then summing up back.

Now, using \( 6.40 \) and \( 6.41 \) again, we have
\begin{equation}
\begin{split}
(\ell(1.9) \sup (\text{as above, square function in (6.30) yet again, we can show that}) \\
\text{Finally, we recall that } \text{dist}(\text{set } \Omega, \Omega^\sim) \text{ is such that for any } Q \in \mathcal{D}(E), \text{for every component } U_Q' \text{ of } U_Q \text{ we have}
\end{split}
\end{equation}

By Fubini’s theorem (keeping in mind our choice of small \( c_0 \)), the integral above is bounded by

\begin{equation}
\sum_{P \in \mathcal{D}(E): \Omega \cap (\partial \Omega \cap (\partial B(x_0, C) \cup (Q))) \neq \emptyset} \ell(P)^{-1} \int_{Y \in \hat{U}_P} \frac{|\nabla u(Y)|^p}{\text{dist}(Y, \partial \Omega)^{n-1}} d\sigma(x) dY
\end{equation}

Finally, we recall that \( \text{dist}(P, \Omega') \approx \ell(P) \) and hence, enlarging the Whitney regions that define the square function in (6.30) yet again, we can show that \( Y \in \Gamma(x_0) \) for some \( x_0 \in E_Q' \) and hence, for \( Y \) as above,

\begin{equation}
\left( \ell(P)^{-n-1} \int_{Y \in \hat{U}_P} |\nabla u(Y)|^2 dY \right)^{p/2} \leq \sup_{E_Q'} \hat{S} u(x_0)^p \leq (\gamma a)^p.
\end{equation}

Summing up \( \ell(P)^n \) over all \( P \) in (6.44) yields a multiple of \( \sigma(Q) \) and finishes the argument. \( \Box \)

7. From \( N < S \) bounds on NTA domains to \( \varepsilon \)-approximability in a complement of a UR set

Recall the definition of \( \varepsilon \)-approximability (Definition 1.11). The second main result in [HMM], stated there for harmonic functions but proved in full generality, can be formulated as follows.

**Theorem 7.1.** Let \( E \subset \mathbb{R}^{n+1} \) be an \( n \)-dimensional UR set, \( \Omega = \mathbb{R}^{n+1} \setminus E \), and suppose that \( u \in W^{1,2}_{\text{loc}}(\Omega) \cap C(\Omega) \) is such that for any \( Q \in \mathcal{D}(E) \), for every component \( U_Q' \) of \( U_Q \) we have

\begin{equation}
(7.2) \sup_{X \in U_Q'} |u(X) - u(Y)| \leq C \left( \ell(Q)^{-n-1} \int_{Y \in \hat{U}_P} |u|^2 dX \right)^{1/2}.
\end{equation}

Assume, in addition, that \( u \in L^\infty(\Omega) \) and \( F := \|\nabla u\|_{L^\infty(\Omega)} \) satisfies the Carleson measure estimate (1.9).

Finally, suppose that for every \( \Omega_{S'}^{\sim} \) defined by (2.33) (with \( S' = S \)) we have

\begin{equation}
(7.3) \left\| N_s(u - u(X_{S'})) \right\|_{L^2(\partial \Omega_{S'}^{\sim})} \leq C_0 \| u \|_{L^2(\partial \Omega_{S'}^{\sim})}.
\end{equation}

Here \( X_{\Omega_{S'}}^{\sim} \) is any interior corkscrew point of \( \Omega_{S'}^{\sim} \) at the scale of \( \text{diam}(\Omega_{S'}^{\sim}) \) (see the discussion above the statement of Theorem 6.2). Then \( u \) is \( \varepsilon \)-approximable on \( \Omega \), with the implicit constants depending only on \( n \), the ADR/UR constants of \( E \), and the choice of \( \eta, K, \tau, \kappa \) only.
In particular, if one substitutes \((7.3)\) by a more general assumption that the estimate
\[
(7.4) \quad \|N_s(u - u(X'))\|_{L^2(\partial\Omega')} \leq C_0 \|S u\|_{L^2(\partial\Omega')}
\]
on every bounded NTA domain \(\Omega' \subset \Omega\) with an ADR boundary, with the constant \(C_0\) depending on the NTA constants of \(\Omega'\) and ADR constants of \(\partial\Omega'\) only, then the same conclusion follows.

Strictly speaking, the Theorem above was proved in [HMM] departing from the bound \((7.3)\) for the non-tangential maximal function defined with traditional non-tangential cones (1.16) rather than the dyadic ones, but that is easy to change by Remark 2.35.

8. Applications: solutions of divergence form elliptic equations with bounded measurable coefficients

8.1. Second order divergence form elliptic operators with coefficients satisfying a Carleson measure condition. Let \(E \subset \mathbb{R}^{n+1}\) be an \(n\)-dimensional ADR set and let \(\Omega = \mathbb{R}^{n+1} \setminus E\). Consider a divergence form elliptic operator
\[
L := -\text{div} A(X) \nabla,
\]
defined in \(\Omega\), where \(A\) is an \((n + 1) \times (n + 1)\) matrix with real bounded measurable coefficients, possibly non-symmetric, satisfying the ellipticity condition
\[
(8.1) \quad A \xi^2 \leq \langle A(X) \xi, \xi \rangle := \sum_{i,j=1}^{n+1} A_{ij}(X) \xi_i \xi_j, \quad \|A\|_{L^\infty(\mathbb{R}^n)} \leq A^{-1},
\]
for some \(A > 0\), and for all \(\xi \in \mathbb{R}^{n+1}, X \in \Omega\). As usual, the divergence form equation is interpreted in the weak sense, i.e., we say that \(Lu = 0\) in a domain \(\Omega\) if \(u \in W^{1,2}_{\text{loc}}(\Omega)\) and
\[
\int_{\Omega} A(X) \nabla u(X) \cdot \nabla \Psi(X) \, dX = 0,
\]
for all \(\Psi \in C^\infty_0(\Omega)\).

Assume furthermore that the distributional derivatives of the coefficients of \(A\) satisfy the following Carleson measure condition:
\[
(8.3) \quad F(X) = \epsilon(X) := \sup_{i,j=1,...,n+1} \sup_{Z \in B(X, \delta(X)/2)} \sup_{i,j=1,...,n+1} \{ |\nabla A(Z)|, \quad Z \in B(X, \delta(X)/2) \} \quad \text{satisfies (1.9)}.
\]
It has been demonstrated in [KP] that the condition \((8.3)\) implies that solutions to the corresponding elliptic equation satisfy square function/non-tangential maximal function estimates on Lipschitz domains. The results of the present paper allow us to establish analogous facts in the full generality of uniformly rectifiable sets. To show this, we start with the following auxiliary fact (cf. Lemma 3.1 in [KP]).

**Lemma 8.4.** Let \(E \subset \mathbb{R}^{n+1}\) be an \(n\)-dimensional ADR set and let \(\Omega = \mathbb{R}^{n+1} \setminus E\), and let \(A\) be an \((n + 1) \times (n + 1)\) matrix defined on \(\Omega\) with real bounded measurable coefficients, possibly non-symmetric, satisfying the ellipticity condition \((8.1)\).

If the Carleson measure condition \((8.3)\) is satisfied on \(\Omega = \mathbb{R}^{n+1} \setminus E\) then it is also satisfied on any subset \(D \subset \Omega\) with an ADR boundary, that is,
\[
(8.5) \quad \epsilon_D(X) := \sup_{Z \in B(X, \text{dist}(X, \partial D)/2)} \{ |\nabla A(Z)|, \quad Z \in B(X, \text{dist}(X, \partial D)/2) \}
\]
satisfies
\[
(8.6) \quad \sup_{x \in \partial D, 0 < r < \infty} \frac{1}{r^n} \int_{B(x,r) \cap D} |\epsilon_D(X)|^2 \, \text{dist}(X, \partial D) \, dX \leq C,
\]
with the constant \(C\) depending on the constant in \((8.3)\) and ADR constants of \(E\) only.
\textbf{Proof.} Fix any $B(x,r)$ from (8.6). We shall consider two cases. First, if $\delta(x) \leq 2r$ then there exists $B(z,4r)$ such that $z \in E$ and $B(x,r) \subset B(z,4r)$. In addition, for every $X \in D$ we have $B(X, \text{dist}(X, \partial D)/2) \subset B(X, \delta(X)/2)$, so that $\epsilon_D \leq \epsilon$ on $D$. Hence, in this case,

\begin{equation}
\frac{1}{r^4} \int_{B(x,r) \cap D} |\nabla u|^2 \text{dist}(X, \partial D) \, dx \leq \frac{1}{r^4} \int_{B(z,4r)} |\nabla u|^2 \, dx,
\end{equation}

so that (8.3) gives the desired bound.

In the second case, $\delta(x) > 2r$, we have $\delta(x)/2 < \delta(Y) < 3\delta(x)/2$ for all $Y \in B(x,r) \cap D$.

Furthermore, by the ADR property of $E$, for any $B(y, r) \cap E, y \in E$, there exists a corkscrew point $Y \in E$ corresponding to $B(y, r) \cap E$ with the property that $B(Y, cr) \subset \Omega$ for some $c$ depending on the ADR constants of $E$ only. Evidently, $|\nabla A(Y)| \leq \epsilon(Z)$ for all $Z \in B(Y, cr)$. Now, for every $y \in E$, $r > 0$,

\begin{equation}
\frac{1}{r^4} \int_{B(y,r)} |\nabla u|^2 \, dx \geq \frac{1}{r^4} \int_{B(y,r)} |\nabla u|^2 \, dx \geq |\nabla A(Y)|^2 \delta(Y)^2.
\end{equation}

Hence, (8.3) implies that $|\nabla A(Y)| \delta(Y) \leq C$ for all $Y \in \Omega$, with $C$ depending on the constant in (8.3) and ADR constants of $E$ only.

Returning to $D$, and specifically, to $B(x,r)$, $x \in \partial D$, $\delta(x) > 2r$, we then have

\begin{equation}
\frac{1}{r^4} \int_{B(x,r) \cap D} |\nabla u|^2 \text{dist}(X, \partial D) \, dx \leq \frac{1}{r^4} \int_{B(x,r)} |\nabla u|^2 \, dx \leq \frac{r^2}{\delta(X)^2} \leq C,
\end{equation}
as desired. \hfill \Box

\textbf{Theorem 8.10.} Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional UR set and let $\Omega = \mathbb{R}^{n+1} \setminus E$. Let $A$ be an $(n+1) \times (n+1)$ matrix defined on $\Omega$ with real bounded measurable coefficients, possibly non-symmetric, satisfying the ellipticity condition (8.1) and the Carleson measure condition (8.3) on $\Omega = \mathbb{R}^{n+1} \setminus E$. Then any weak solution $u$ to $Lu = 0$ satisfies square function estimate

\begin{equation}
\|Su\|_{L^p(E)} \leq C \|\nabla u\|_{L^p(E)}, \quad 0 < p < \infty,
\end{equation}
as well as its local analogue

\begin{equation}
\| Su_{Q} \|_{L^p(Q)} \leq C \|\nabla u_{Q}\|_{L^p(Q)}, \quad Q \in \mathcal{D}(E), \quad 0 < p < \infty.
\end{equation}

\textbf{If} $u$ \textbf{is, in addition, bounded, then the Carleson measure estimate}

\begin{equation}
\sup_{x \in E, 0 < r < \infty} \frac{1}{r^4} \int_{B(x,r)} |\nabla u(Y)|^2 \delta(Y) \, dY \leq C \|u\|_{L^\infty(\Omega)}^2,
\end{equation}
holds and $u$ is $\epsilon$-approximable in the sense of Definition 1.11. All constants depend on the UR character of $E$ only.

\textbf{Proof.} First of all, all auxiliary estimates (3.2) for $F := |\nabla u|/\|u\|_{L^\infty(\Omega)}$, (7.2), (6.3), and (6.26) for $u$, (5.10) for $G = \nabla u$ and $H = u$, hold by the usual interior estimates for solutions of elliptic PDEs (see, e.g., [K]).

The square function bounds (8.11)–(8.12) and thus Carleson measure estimates (1.10) (as a particular case $p = 2$ of (8.12)) on bounded Lipschitz domains should be attributed to [KP]. Strictly speaking, the details of the proof are only provided there for $N \leq S$ direction (and only for $p > 2$), but all ingredients are laid out for a reader to reconstruct a complete proof. One can also consult [DFM] for complete details presented in this and more general, higher co-dimensional, case. With this at hand, using Theorem 4.10, one concludes that the Carleson measure estimates (1.10) hold in all NTA subdomains of $\Omega$ as well. Then, by Theorem 3.3, the CME estimates hold on $\Omega$ and any
UR subdomain of $\Omega$, with the appropriate control of constants. This proves (8.13). Recalling that all sawtooth subdomains of $\Omega$ are UR as well by Proposition A.10, we now use Theorem 5.7, to conclude (8.11)–(8.12).

Passing to the question of $\varepsilon$-approximability, we point out again that $N < S$ estimates (6.4) on all Lipschitz subdomains of $\Omega$ hold by [KP]. Once again, only the case $p > 2$ is explicitly proved there, but analogously one can obtain a local estimate for $p > 2$ and ultimately a global estimate for all $p$'s, in particular, (6.4), by Theorem 6.25. By Theorem 6.2 we now get (6.6) and even (6.24) for all bounded NTA subdomains of $\Omega$. This yields (6.6) for all $0 < p < \infty$ by Theorem 6.25 and puts us in the context of Theorem 7.1. Then $u$ is $\varepsilon$-approximable on $\Omega$ due to Theorem 7.1. □

8.2. Higher order elliptic equations and systems with constant coefficients. In [DKPV] the authors obtained square function/non-tangential maximal function estimates for higher order elliptic equations and systems on bounded Lipschitz domains. These results have never been extended, even to NTA domains, and here we present a generalization of Carleson measure estimates to the complements of UR sets. It is not clear if $\varepsilon$-approximability of (derivatives of) solutions to (higher order) systems has any consequences: indeed, the traditional connection with elliptic measure is not available in this context, and for that reason we do not pursue $N < S$ bounds and $\varepsilon$-approximability in this section.

Let $L^k_l = \sum_{|\alpha|=|\beta|=m} D^\alpha a_{\alpha \beta}^k D^\beta$, where $m, k, l \in \mathbb{N}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are multiindices, and $D^\alpha, D^\beta$ are the corresponding vectors of partial derivatives. The coefficients $a_{\alpha \beta}^k$ are real constants. We say that $Lu = 0$, $u = (u_1, \ldots, u_K)$, $K \in \mathbb{N}$, $u_i \in W^{m,2}_{\text{loc}}(\Omega)$, if

$$\sum_{k=1}^{K} L^k_l u^l = 0, \quad k = 1, \ldots, K,$$

as usual, in the weak sense, similarly to (8.2). Here, $W^{m,2}_{\text{loc}}(\Omega)$ is the space of functions with all derivatives of orders $0, \ldots, m$ in $L^2(\Omega)$ and $W^{m,2}_{\text{loc}}(\Omega)$ is the space of functions locally in $W^{m,2}(\Omega)$. We assume, in addition, that $L$ is symmetric: $L^k_l = L^l_k$ for $1 \leq k, l \leq K$, and that the Legendre-Hadamard ellipticity condition holds: there exists $\lambda > 0$ such that

$$\sum_{k,l=1}^{K} \sum_{|\alpha|=|\beta|=m} a_{\alpha \beta}^k \xi^\alpha \xi^\beta \xi^l \geq \lambda |\xi|^{2m} |\xi|^2, \quad \text{for all } \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{C}^n, \xi \in \mathbb{R}^n.$$

**Theorem 8.14.** Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional UR set and let $\Omega = \mathbb{R}^{n+1} \setminus E$. Let $L$ be a symmetric constant coefficient elliptic system on $\Omega$, satisfying the Legendre-Hadamard ellipticity condition, as above. Then any weak solution $u$ to $Lu = 0$ in $\Omega$ satisfies square function estimate

$$\|S(\nabla^{m-1} u)\|_{L^p(\Omega)} \leq C \|\nabla^{Q}(|\nabla^{m-1} u|)\|_{L^p(\Omega)}, \quad 0 < p < \infty,$$

as well as its local analogue

$$\|S(\nabla^{m-1} u)\|_{L^p(\Omega)} \leq C \|\nabla^{Q}(|\nabla^{m-1} u|)\|_{L^p(\Omega)}, \quad Q \in \mathcal{D}(E), \quad 0 < p < \infty.$$

If $u$ is, in addition, such that $\nabla^{m-1}u$ is bounded, then the Carleson measure estimate

$$\sup_{x \in E, 0 < r < \infty} \frac{1}{r^n} \int_{B(x,r)} |\nabla^{m} u(Y)|^2 \delta(Y) dY \leq C \|\nabla^{m-1} u\|_{L^p(\Omega)}^2,$$

holds. All constants depend on the UR character of $E$ only. Here $\nabla^k$, $k \in \mathbb{N}$, stands for the vector of all partial derivatives of order $k$.

**Proof.** As mentioned above, the square function/non-tangential maximal function estimates on bounded Lipschitz domains in the present context have been proved in [DKPV]. In particular, the $p = 2$ case which yields (1.10) for $F = \nabla^{m} u/|\nabla^{m-1} u|_{L^p}$ is Theorem 2, p. 1455, of [DKPV]. Much as before, using Theorem 4.10, one concludes that the Carleson measure estimates (1.10)
hold in all NTA subdomains of $\Omega$ as well. Then, by Theorem 3.3, the CME estimates hold on $\Omega$ and any UR subdomain of $\Omega$, with the appropriate control of constants. This proves (8.17). Recalling that all sawtooth subdomains of $\Omega$ are UR as well by Proposition A.10, we now use Theorem 5.7, to conclude (8.15)–(8.16). Throughout, $\nabla^{m-1}u$ is used in place of $u$ and auxiliary estimates (3.2) for $F := |\nabla^{m}u|/|\nabla^{m-1}u|_{L^{\infty}(\Omega)}$ and (5.10) for $G = \nabla^{m}u$ and $H = \nabla^{m-1}u$, hold by the usual interior estimates for solutions of elliptic PDEs – see, e.g., [PV]. □

Appendix A. Sawtooths have UR boundaries

To start, recall from [HMM] the fact that the sawtooth regions and Carleson boxes inherit the ADR property. In [HMM], we treated simultaneously the case that the set $E$ is ADR, but not necessarily UR, and also the case that $E$ is UR. The point was that the Whitney regions in the two cases (and thus also the corresponding sawtooth regions and Carleson boxes) were somewhat different. To make this more precise, we need to recall notational conventions set in Section 2.

If the set $E$ under consideration is merely ADR, but not UR, then we set $W_{Q} = W_{Q}^{0}$ as defined in (2.8) (see “Case ADR” in Section 2). If in addition, the set $E$ is UR, then we define $W_{Q}$ as in (2.29) (see “Case UR” in Section 2). In the first case, the constants involved in the construction of $W_{Q}$ depend only on the ADR constant $\eta$ and $K$, and in the UR case, on dimension and the ADR/UR constants (compare (2.8) and (2.11)). Therefore there are numbers $m_{0} \in \mathbb{Z}_{+}$, $C_{0} \in \mathbb{R}_{+}$, with the same dependence, such that

$$
2^{-m_{0}} \ell(Q) \leq \ell(I) \leq 2^{m_{0}} \ell(Q), \quad \forall I \in W_{Q}.
$$

(A.1)

This dichotomy in the choice of $W_{Q}$ is convenient for the results we have in mind, in the sense that the results that we quote from [HMM] may be applied in the purely ADR case, as well as in the UR case, under the conventions described above.

For any $I \in \mathcal{W}$ such that $\ell(I) < \text{diam}(E)$, we write $Q_{j}^{I}$ for the nearest dyadic cube to $I$ with $\ell(I) = \ell(Q_{j}^{I})$ so that $I \subseteq W_{Q_{j}^{I}}$. Notice that there can be more than one choice of $Q_{j}^{I}$, but at this point we fix one so that in what follows $Q_{j}^{I}$ is unambiguously defined.

Let us now recall some results from [HMM] that we shall use in the sequel.

**Proposition A.2.** [HMM, Proposition A.2] Let $E \subset \mathbb{R}^{n+1}$ be an $n$-dimensional ADR set. Then all dyadic local sawtooths $\Omega_{F_{0}}$ and all Carleson boxes $Q_{0}$ have $n$-dimensional ADR boundaries. In all cases, the implicit constants are uniform and depend only on dimension, the ADR constant of $E$ and the parameters $m_{0}$ and $C_{0}$.

Given a cube $Q_{0} \in \mathbb{D}$ and a family $\mathcal{F}$ of disjoint cubes $\mathcal{F} = \{Q\} \subset \mathbb{D}_{Q_{0}}$ (for the case $\mathcal{F} = \emptyset$ the changes are straightforward and we leave them to the reader, also the case $\mathcal{F} = \{Q_{0}\}$ is disregarded since in that case $\Omega_{F_{0},Q_{0}}$ is the null set). We write $\Omega_{\Sigma} = \Omega_{F_{0},Q_{0}}$ and $\Sigma = \partial \Omega_{\Sigma} \setminus E$. Given $Q \in \mathbb{D}$ we set

$$
\mathcal{R}_{Q} := \bigcup_{Q' \in \mathcal{D}_{Q}} W_{Q'}, \quad \text{and} \quad \Sigma_{Q} = \Sigma \bigcap \bigcup_{I \in \mathcal{R}_{Q}} I.
$$

Let $C_{1}$ be a sufficiently large constant, to be chosen below, depending on $n$, the ADR constant of $E$, $m_{0}$ and $C_{0}$. Let us introduce some new collections:

$$
\mathcal{F}_{\|} := \{ Q \in \mathbb{D} \setminus \{Q_{0}\} : \ell(Q) = \ell(Q_{0}), \text{dist}(Q, Q_{0}) \leq C_{1} \ell(Q_{0}) \}
$$

$$
\mathcal{F}_{\perp} := \{ Q' \in \mathbb{D} : \text{dist}(Q', Q_{0}) \leq C_{1} \ell(Q_{0}), \ell(Q_{0}) < \ell(Q') \leq C_{1} \ell(Q_{0}) \}
$$

$$
\mathcal{F}_{\|}' := \{ Q \in \mathcal{F}_{\|} : \Sigma_{Q} \neq \emptyset \} = \{ Q \in \mathcal{F}_{\|} : \exists I \in \mathcal{R}_{Q} \text{ such that } \Sigma \cap I \neq \emptyset \}
$$

$$
\mathcal{F}_{\perp}' := \{ Q \in \mathcal{F} : \Sigma_{Q} \neq \emptyset \} = \{ Q \in \mathcal{F} : \exists I \in \mathcal{R}_{Q} \text{ such that } \Sigma \cap I \neq \emptyset \}.
$$
We also set
\[ R_\perp = \bigcup_{Q \in F^*} R_Q, \quad R_\parallel = \bigcup_{Q \in F_1^*} R_Q, \quad R_T = \bigcup_{Q \in F_T} W_Q. \]

**Lemma A.3.** [HMM, Lemma A.3] Set \( W_\Sigma = \{ I \in W : I \cap \Sigma \neq \emptyset \} \) and define
\[ W^*_\Sigma = \bigcup_{Q \in F^*} W_{\Sigma, Q}, \quad W^0_\Sigma = \bigcup_{Q \in F_1^*} W_{\Sigma, Q}, \quad W^*_\Sigma = \{ I \in W_\Sigma : Q_I \in F_T \}. \]
where for every \( Q \in F^* \cup F_1^* \) we set
\[ W_{\Sigma, Q} = \{ I \in W_\Sigma : Q_I \in \mathbb{D}_Q \}; \]
and where we recall that \( Q_I \) is the nearest dyadic cube to \( I \) with \( \ell(I) = \ell(Q_I) \) as defined above. Then
\[ W_\Sigma = W^*_\Sigma \cup W^0_\Sigma \cup W^*_\Sigma, \]
where
\[ W^*_\Sigma \subset R_\perp, \quad W^0_\Sigma \subset R_\parallel, \quad W^*_\Sigma \subset R_T. \]
As a consequence,
\[ \Sigma = \Sigma_\perp \cup \Sigma_\parallel \cup \Sigma_T := \left( \sum_{I \in W^*_\Sigma} \Sigma \cap I \right) \bigcup \left( \sum_{I \in W^0_\Sigma} \Sigma \cap I \right) \bigcup \left( \sum_{I \in W^*_\Sigma} \Sigma \cap I \right). \]

**Lemma A.7.** [HMM, Lemma A.7] Given \( I \in W_\Sigma \), we can find \( Q_I \in \mathbb{D} \), with \( Q_I \subset Q_I' \), such that \( \ell(I) \approx \ell(Q_I) \), \( \text{dist}(Q_I, I) \approx \ell(I) \), and in addition,
\[ \sum_{I \in W_{\Sigma, Q}} 1_{Q_I} \leq 1_Q, \quad \text{for any } Q \in F^* \cup F_1^*, \]
and
\[ \sum_{I \in W^*_\Sigma} 1_{Q_I} \leq 1_{E_0 \cap E}, \]
where the implicit constants depend on \( n \), the ADR constant of \( E \), \( m_0 \) and \( C_0 \), and where \( B^*_Q = B(x_Q, C \ell(Q)) \) with \( C \) large enough depending on the same parameters.

With the preceding results in hand, we turn to the main purpose of this appendix: to prove that uniform rectifiability is also inherited by the sawtooth domains and Carleson boxes.

**Proposition A.10.** Let \( E \subset \mathbb{R}^{n+1} \) be an \( n \)-dimensional UR set. Then all dyadic local sawtooths \( \Omega_{F, Q} \) and all Carleson boxes \( T_Q \) have \( n \)-dimensional UR boundaries. In all cases, the implicit constants are uniform and depend only on dimension, the ADR and UR constants of \( E \) and the parameters \( m_0 \) and \( C_0 \).

The proof of this result follows the ideas from [HM, Appendix C] which in turn uses some ideas from Guy David, and uses the following singular integral characterization of UR sets, established in [DS1]. Suppose that \( E \subset \mathbb{R}^{n+1} \) is \( n \)-dimensional ADR. The singular integral operators that we shall consider are those of the form
\[ T_{E, f}(x) = T_E f(x) := \int_E K_E(x-y) f(y) \, dH^n(y), \]
where \( K_E(x) := K(x) \Phi(|x|/\varepsilon) \), with \( 0 \leq \Phi \leq 1 \), \( \Phi(\rho) \equiv 1 \) if \( \rho \geq 2 \), \( \Phi(\rho) \equiv 0 \) if \( \rho \leq 1 \), and \( \Phi \in C^\infty(\mathbb{R}) \), and where the singular kernel \( K \) is an odd function, smooth on \( \mathbb{R}^{n+1} \setminus \{0\} \), and satisfying
\[ |K(x)| \leq C_0 |x|^{-n} \]
and
\[ |\nabla^m K(x)| \leq C_m |x|^{-n-m}, \quad \forall m = 1, 2, 3, \ldots. \]
Then $E$ is UR if and only if for every such kernel $K$, we have that
\begin{equation}
\tag{A.13}
\sup_{\epsilon > 0} \int_E |T_{E, \epsilon} f|^2 \, dH^n \leq C_K \int_E |f|^2 \, dH^n.
\end{equation}
We refer the reader to [DS1] for the proof. For $K$ as above, set
\begin{equation}
\tag{A.14}
T_{E, \epsilon} f(X) := \int_E K(X - y) f(y) \, dH^n(y), \quad X \in \mathbb{R}^{n+1} \setminus E.
\end{equation}
We define (possibly disconnected) non-tangential approach regions $\mathcal{T}_\alpha(x)$ as follows. Set $\mathcal{W}_\alpha(x) := \{ I \in \mathcal{W} : \text{dist}(I, x) < \alpha(I) \}$. Then we define
\[ \mathcal{T}_\alpha(x) := \bigcup_{I \in \mathcal{W}_\alpha(x)} I^* \]
(thus, roughly speaking, $\alpha$ is the “aperture” of $\mathcal{T}_\alpha(x)$). For $F \in C(\mathbb{R}^{n+1} \setminus E)$ we may then also define the non-tangential maximal function
\[ N_{\alpha, \epsilon}(F)(x) := \sup_{Y \in \mathcal{T}_\alpha(x)} |F(Y)|. \]
We shall sometimes write simply $N_\alpha$ when there is no chance of confusion in leaving implicit the dependence on the aperture $\alpha$. The following lemma is a standard consequence of the usual Cotlar inequality for maximal singular integrals, and we omit the proof.

**Lemma A.15.** Suppose that $E \subset \mathbb{R}^{n+1}$ is $n$-dimensional UR, and let $T_E \subset \mathcal{D} \subset \mathcal{D}_0$ be defined as in (A.14). Then for each $1 < p < \infty$ and $\alpha \in (0, \infty)$, there is a constant $C_{p, \alpha, \epsilon}$ depending only on $p, n, \alpha, K$ and the UR constants such that
\begin{equation}
\tag{A.16}
\int_E \left( N_{\alpha, \epsilon}(T_E f) \right)^p \, dH^n \leq C_{\alpha, K} \int_E |f|^p \, dH^n.
\end{equation}

**Proof of Proposition A.10.** We now fix $Q_0 \in \mathbb{D}$ and a family $\mathcal{F}$ of disjoint cubes $\mathcal{F} = \{ Q_j \} \subset \mathbb{D}_{Q_0}$ (for the case $\mathcal{F} = \emptyset$ the changes are straightforward and we leave them to the reader, also the case $\mathcal{F} = \{ Q_0 \}$ is disregarded since $\Omega_{F, Q_0, e_0}$ is the null set). We write $\Omega_\bullet = \Omega_{F, Q_0}$ and also $E_\bullet = \partial \Omega_\bullet$. We fix $0 < \Phi \leq 1$, $\Phi(\rho) \equiv 1$ if $\rho \geq 2$, $\Phi(\rho) \equiv 0$ if $\rho \leq 1$, and $\Phi \in C^\infty(\mathbb{R})$. According to previous consideration we fix $e_0 > 0$ and our goal is to show that $T_{E_\bullet, e_0}$ is bounded in $L^2(E_\bullet, \omega)$ with bounds that are independent of $e_0$. To simplify the notation we write $K_0 = K_{e_0}$ and set for every $X \in \mathbb{R}^{n+1}$
\[ T_{E_\bullet, \epsilon, c}(X) = \int_{E_\bullet} K_0(x - y) f(y) \, d\sigma(y), \quad T_{E_\bullet, \epsilon, c}(X) = \int_{E_\bullet} K_0(x - y) g(y) \, d\sigma_\bullet(y). \]

We first observe that $K_0$ is not singular and therefore, for any $p, 1 < p < \infty$, and for every $f \in L^p(E)$, respectively $g \in L^p(E_\bullet)$, the previous operators are well-defined (by means of an absolutely convergent integral) for every $X \in \mathbb{R}^{n+1}$. Also for such functions it is easy to see that the dominated convergence theorem implies that $T_{E_\bullet, \epsilon, c} f, T_{E_\bullet, \epsilon, c} g \in C(\mathbb{R}^{n+1})$.

**Remark A.17.** We notice that $K_0$ is an odd smooth function which satisfies (A.11) and (A.12) with uniform constants (i.e. with no dependence on $e_0$) and the fact that $E$ is UR implies that (A.13) and (A.16) hold with constants that do not depend on $e_0$.

We are going to see that $T_{E_\bullet, \epsilon} : L^p(E) \rightarrow L^p(E_\bullet)$ for every $1 < p < \infty$. To do that we take $f \in L^p(E)$ and write
\begin{align*}
\int_{E_\bullet} |T_{E_\bullet, \epsilon, c}(x)|^p \, d\sigma_\bullet(x) \\
= \int_{E_\bullet \cap E} |T_{E_\bullet, \epsilon, c}(x)|^p \, d\sigma_\bullet(x) + \int_{E_\bullet \setminus E} |T_{E, \epsilon, c}(x)|^p \, d\sigma_\bullet(x) =: A + S.
\end{align*}
The estimate for $A$ follows from the fact that $E$ is UR

$$A \leq \int_E |T_{E,0}f(x)|^p \, d\sigma(x) = \int_E |T_{E,0}f(x)|^p \, d\sigma(x) \leq C_K \int_E |f(x)|^p \, d\sigma(x)$$

where we have used (A.13) and the standard Calderón-Zygmund theory (taking place in the ADR set $E$) and $C_K$ does not depend on $\epsilon_0$. For $S$ we use that $\Sigma = E_{\star} \setminus \{\partial \Omega_{\star} \setminus E$ and invoke Lemmas A.3 and A.7; let $Q_I$ be the cube constructed in the latter, so that

$$S = \sum_{I \in W_\Sigma} \int_{I \cap \Sigma} |T_{E,0}f(x)|^p \, d\sigma_{\star}(x) = \sum_{I \in W_\Sigma} \int_{Q_I \cap \Sigma} |T_{E,0}f(x)|^p \, d\sigma_{\star}(x) \, d\sigma(y).$$

Notice that if $y \in Q_I$ and $x \in \Sigma \cap I$ then dist$(I, y) \leq \ell(Q_I)$. Then taking $\alpha > 0$ large enough we obtain that $I \subset W_\alpha(y)$. Write $\tilde{F} = F^{\star} \cup F^\flat_{E_{\star}}$, and observe that by construction the cubes in $\tilde{F}$ are pairwise disjoint. Then by the ADR property of $E_{\star}$, along with Lemmas A.3 and A.7,

$$S \leq \sum_{I \in W_\Sigma} \sigma_{\star}(\Sigma \cap I) \int_{Q_I} |N_{\ast,\alpha}(T_{E,0}f)(y)|^p \, d\sigma(y)$$

$$\leq \sum_{Q \in \tilde{F}} \sum_{I \in W_\Sigma, Q} \int_{Q_I} |N_{\ast,\alpha}(T_{E,0}f)(y)|^p \, d\sigma(y) + \sum_{I \in W_\Sigma} \int_{Q_I} |N_{\ast,\alpha}(T_{E,0}f)(y)|^p \, d\sigma(y)$$

$$\leq \sum_{Q \in \tilde{F}} \int_{E} |N_{\ast,\alpha}(T_{E,0}f)(y)|^p \, d\sigma(y) + \int_{B_{\ell_0}} |N_{\ast,\alpha}(T_{E,0}f)(y)|^p \, d\sigma(y)$$

where in the last estimate we have employed Lemma A.15 and Remark A.17, and the implicit constants do not depend on $\epsilon_0$.

We have thus established that $T_{E,0} : L^p(E) \rightarrow L^p(E_{\star})$ for every $1 < p < \infty$. Since $K$ is odd, so is $K_0$, and by duality we therefore obtain that

$$(A.18) \quad T_{E_{\star},0} : L^p(E_{\star}) \rightarrow L^p(E), \quad 1 < p < \infty.$$  

Our goal is to show that $T_{E_{\star},0} : L^2(E_{\star}) \rightarrow L^2(E_{\star})$ with bounds that do not depend on $\epsilon_0$. Note that $T_{E_{\star},0}f$ is a continuous function for every $f \in L^2(E_{\star})$ and therefore $T_{E_{\star},0}f |_{E_{\star}} = T_{E_{\star},\epsilon_0}f$ everywhere on $E_{\star}$.

We take $f \in L^2(E_{\star})$ and write as before

$$(A.19) \quad \int_{E_{\star}} |T_{E_{\star},0}f(x)|^2 \, d\sigma_{\star}(x)$$

$$= \int_{E_{\star}\cap E} |T_{E_{\star},0}f(x)|^2 \, d\sigma_{\star}(x) + \sum_{I \in W_\Sigma} \int_{I \cap \Sigma} |T_{E_{\star},0}f(x)|^2 \, d\sigma_{\star}(x)$$

$$=: A + \sum_{I \in W_\Sigma} S_I = A + S.$$  

For $A$ we use (A.18) with $p = 2$ and conclude the desired estimate

$$(A.20) \quad A \leq \int_{E_{\star}\cap E} |T_{E_{\star},0}f(x)|^2 \, d\sigma_{\star}(x) \leq \int_E |T_{E_{\star},0}f(x)|^2 \, d\sigma(x) \leq \int_{E_{\star}} |f(x)|^2 \, d\sigma_{\star}(x).$$

We next fix $I \in W_\Sigma$ and estimate each $S_I$. Let $M > 1$ be large parameter to be chosen below and set $\zeta_I = \ell(I)/M$, $\xi_I = M \ell(I)$. Write
\[(A.21)\quad K_0(x) = K_0(x) \Phi \left( \frac{|x|}{\xi_1} \right) + K_0(x) \left( \Phi \left( \frac{|x|}{\xi_1} \right) - \Phi \left( \frac{|y|}{\xi_1} \right) \right) + K_0(x) \left( 1 - \Phi \left( \frac{|y|}{\xi_1} \right) \right)
\]
\[=: K_{0,\xi_1}(x) + K_{0,\xi_1,\xi_1}(x) + K_{0,\xi_1}^2(x).\]

Corresponding to any of these kernels we respectively set the operators \(T_{E_0,0,\xi_1,\xi_1}, T_{E_0,0,\xi_1,\xi_1}^*\) and \(T_{E_0,0,\xi_1}^*\).

We start with \(T_{E_0,0,\xi_1}\). Fix \(x \in \Sigma \cap I\). Write \(\Delta_{x,I} = B(x, \xi_1) \cap E_0\) and split \(f = f_1 + f_2 := f 1_{\Delta_{x,I}} + f 1_{E_0 \setminus \Delta_{x,I}}\). Then we use Remark A.17, the fact \(\text{supp} \Phi \subset [1, \infty)\) and that \(E_0\) is ADR to easily obtain that for every \(y \in Q_I\), with \(Q_I\) as in Lemma 7.17,
\[(A.22)\quad |T_{E_0,0,\xi_1} f_1(x)| + |T_{E_0,0,\xi_1} f_1(y)|
\leq \int_{\Delta_{x,I}} |K_0(x - z)| \Phi \left( \frac{|x - z|}{\xi_1} \right) + |K_0(y - z)| \Phi \left( \frac{|y - z|}{\xi_1} \right) |f(z)| \, d\sigma_*(z)
\leq \frac{1}{\xi_1} \int_{\Delta_{x,I}} |f(y)| \, d\sigma_*(z) \approx \int_{\Delta_{x,I}} |f(y)| \, d\sigma_*(z) \leq M_{E_0} f(x),
\]
where \(M_{E_0}\) is the Hardy-Littlewood maximal function on \(E_0\), and the constants are independent of \(\epsilon_0\) and \(I\).

On the other hand, very much as before we have that \(K_{0,\xi_1}\) is a Calderón-Zygmund kernel with constants that are uniform in \(\epsilon_0\) and \(\xi_1\). Also, if \(M\) is taken large enough we have that \(2|x - y| < M \ell(I) \leq |x - z|\) for every \(z \in E_0 \setminus \Delta_{x,I}, x \in \Sigma \cap I\) and \(y \in Q_I\). Therefore using standard Calderón-Zygmund estimates and the fact that \(E_0\) is ADR we obtain that for every \(y \in Q_I\)
\[(A.23)\quad |T_{E_0,0,\xi_1} f_2(x) - T_{E_0,0,\xi_1} f_2(y)|
\leq \int_{E_0 \setminus \Delta_{x,I}} |K_{0,\xi_1}(x - z) - K_{0,\xi_1}(y - z)| |f(z)| \, d\sigma_*(z)
\leq \int_{E_0 \setminus \Delta_{x,I}} \frac{|x - y|}{|x - z|^{p+1}} |f(z)| \, d\sigma_*(z) \leq CM_{E_0} f(x).
\]

We next use (A.22) and (A.23) to conclude that
\[
\left| T_{E_0,0,\xi_1} f(x) - \int_{Q_I} T_{E_0,0,\xi_1} f(y) \, d\sigma(y) \right|
\leq |T_{E_0,0,\xi_1} f_1(x)| + \int_{Q_I} |T_{E_0,0,\xi_1} f_1(y)| \, d\sigma(y)
+ \int_{Q_I} |T_{E_0,0,\xi_1} f_2(x) - T_{E_0,0,\xi_1} f_2(y)| \, d\sigma(y) \leq M_{E_0} f(x),
\]
which in turn yields
\[(A.24)\quad \int_{\Sigma \cap I} \left| T_{E_0,0,\xi_1} f(x) - \int_{Q_I} T_{E_0,0,\xi_1} f(y) \, d\sigma(y) \right|^2 \, d\sigma_*(x) \leq \int_{\Sigma \cap I} M_{E_0} f(x)^2 \, d\sigma_*(x).
\]

We next introduce another operator
\[
T_{E_0,0,\xi_1} f(y) = \int_{E_0 \setminus B(y - |y - z| \xi_1)} K_0(y - z) f(z) \, d\sigma_*(z), \quad y \in E.
\]
We fix \(x \in \Sigma \cap I\) and \(y \in Q_I\). We first observe that, for \(M\) large enough, Remark A.17 and the ADR property for \(E_0\) imply that
\[
\left| T_{E_0,0,\xi_1} f(y) - T_{E_0,0,\xi_1} f(y) \right|
\leq \int_{E_0} |K_0(y - z)| \, \Phi \left( \frac{|y - z|}{\xi_1} \right) \left. 1_{[1, \infty)} \left( \frac{|y - z|}{\xi_1} \right) \right| |f(z)| \, d\sigma_*(z)
\leq \int_{E_0} |K_0(y - z)| \Phi \left( \frac{|y - z|}{\xi_1} \right) \left. 1_{[1, \infty)} \left( \frac{|y - z|}{\xi_1} \right) \right| |f(z)| \, d\sigma_*(z).
\]
\[
\frac{1}{\xi I} \int_{\mathbb{R}^3} |f(z)| \, d\sigma_\ast(z) \\
\leq \frac{1}{\xi I} \int_{\mathbb{R}^3} |f(z)| \, d\sigma_\ast(z) \\
\leq \frac{1}{\xi I} \int_{\mathbb{R}^3} |f(z)| \, d\sigma_\ast(z) \leq M_{E_\ast} f(x).
\]

On the other hand, we can introduce another decomposition
\[
f = f_3 + f_4 := f 1_{B(y, \xi I) \cap E_\ast} + f 1_{E_\ast \setminus B(y, \xi I)},
\]
and then for every \(\tilde{y} \in Q_I\)
\[
(A.25) \quad |T_{E_\ast, 0, \xi I} f'(y)| \leq |T_{E_\ast, 0, f_4(y)}| \leq |T_{E_\ast, 0, f_4(y)} - T_{E_\ast, 0, f_3(\tilde{y})}| + |T_{E_\ast, 0, f_3(\tilde{y})}|
\]
We estimate each term in turn. We first observe that, for \(M\) large enough, for every \(z \in E_\ast \setminus B(y, \xi I)\) and \(\tilde{y} \in Q_I\). Therefore, using standard Calderón-Zygmund estimates and the fact that \(E_\ast\) is ADR, we obtain that for every \(y \in Q_I\)
\[
(A.26) \quad |T_{E_\ast, 0, f_4(y)} - T_{E_\ast, 0, f_3(\tilde{y})}| \leq \int_{E_\ast \setminus B(y, \xi I)} |K_0(y - \tilde{y}) - K_0(y - z)| |f(z)| \, d\sigma_\ast(z)
\]
\[
\leq \int_{E_\ast \setminus B(y, \xi I)} \frac{|y - \tilde{y}|}{|y - z|^{n+1}} |f(z)| \, d\sigma_\ast(z) \leq M_{E_\ast} f(x),
\]
where we have used that, for \(M\) large enough, \(x \in B(y, \xi I/2)\). Fix \(1 < p < 2\). We next average \((A.25)\) on \(\tilde{y} \in Q_I\) and use \((A.26)\) and \((A.18)\) to obtain
\[
(A.27) \quad |T_{E_\ast, 0, f_3(\tilde{y})}| \leq \int_{Q_I} \left( |T_{E_\ast, 0, f_3(\tilde{y})}| + |T_{E_\ast, 0, f_3(\tilde{y})} + |T_{E_\ast, 0, f_3(\tilde{y})}| \right) \, d\sigma(\tilde{y})
\]
\[
\leq M_{E_\ast} f(x) + M_E(T_{E_\ast, 0, f}(y)) + \sigma(Q_I) \frac{1}{\ell(I)} \|T_{E_\ast, 0, f_3}\|_{L^p(E)}
\]
\[
\leq M_{E_\ast} f(x) + M_E(T_{E_\ast, 0, f}(y)) + \sigma(Q_I) \frac{1}{\ell(I)} \|f_3\|_{L^p(E_\ast)}
\]
\[
\leq M_{E_\ast} f(x) + M_E(T_{E_\ast, 0, f}(y)) + \left( \frac{1}{\ell(I)} \int_{B(y, \xi I) \cap E_\ast} |f(z)|^p \, d\sigma_\ast(z) \right)^{\frac{1}{p}}
\]
\[
\leq M_{E_\ast, p} f(x) + M_E(T_{E_\ast, 0, f}(y)),
\]
where \(M_E\) is the Hardy-Littlewood maximal function on \(E\) and we also write \(M_{E_\ast, p} f = M_{E_\ast}(|f|^p)^{\frac{1}{p}}\). Note that this estimate holds for every \(x \in \Sigma \cap I\) and for every \(y \in Q_I\). Hence,
\[
(A.28) \quad \int_{\Sigma \cap I} \left( \int_{Q_I} |T_{E_\ast, 0, f_3(\tilde{y})}| \, d\sigma(\tilde{y}) \right)^2 \, d\sigma_\ast(x)
\]
\[
\leq \int_{\Sigma \cap I} M_{E_\ast, p} f(x)^2 \, d\sigma_\ast(x) + \int_{Q_I} M_E(T_{E_\ast, 0, f}(y))^2 \, d\sigma(y),
\]
where we have used that \(\sigma_\ast(\Sigma \cap I) \leq \ell(I)^p\). We now gather \((A.24)\) and \((A.28)\) to obtain that for every \(I \in \mathcal{W}_\Sigma\)
\[
(A.29) \quad \int_{\Sigma \cap I} \left( |T_{E_\ast, 0, f_3(\tilde{y})}| \right)^2 \, d\sigma_\ast(x)
\]
\[
\leq \int_{\Sigma \cap I} \left( |T_{E_\ast, 0, f_3(\tilde{y})} - \int_{Q_I} T_{E_\ast, 0, f_3(\tilde{y})} \, d\sigma(\tilde{y}) \right)^2 \, d\sigma_\ast(x)
\]
\[
+ \int_{\Sigma \cap I} \left( \int_{Q_I} T_{E_\ast, 0, f_3(\tilde{y})} \, d\sigma(\tilde{y}) \right)^2 \, d\sigma_\ast(x)
\]
\[\leq \int_{\Sigma \cap I} M_{E, \rho}f(x)^2 \, d\sigma_\star(x) + \int_{Q_I} M_E(T_{E, 0}f)(y)^2 \, d\sigma(y).\]

We next consider \(T_{E, 0, \xi, \delta}f\). Notice that for every \(x \in \Sigma \cap I\) and \(z \in E_\star\) we have

\[|K_{0, \xi, \delta}(z - x)| = |K_0(z - x)| \left| \Phi\left(\frac{|z - x|}{\xi I}\right) - \Phi\left(\frac{|z - x|}{\xi I}\right) \right| \leq \frac{1}{|z - x|^n} 1_{|z - x| \leq 2\xi I} \leq \frac{1}{\xi} 1_{|z - x| \leq 2\xi I},\]

and therefore

\[
(A.30) \quad \int_{\Sigma \cap I} |T_{E, 0, \xi, \delta}f(x)|^2 \, d\sigma_\star(x) \leq \int_{\Sigma \cap I} \left( \frac{1}{\xi^n} \right) \int_{B(x, 2\xi I) \cap E_\star} |f(z)| \, d\sigma_\star(z) \, d\sigma_\star(x) \leq C_M \int_{\Sigma \cap I} M_{E, \star}f(x)^2 \, d\sigma_\star(x).
\]

Let us finally address \(T^\xi_{E, \star, 0}\). Observe first that

\[K^\xi_0(\cdot) = K(\cdot) \Phi\left(\frac{|\cdot - 1|}{\xi_0}\right) \left(1 - \Phi\left(\frac{|\cdot - 1|}{\xi_I}\right)\right).\]

We consider different cases.

**Case 1:** \(\xi_I \leq \frac{\epsilon_0}{2}\). We have that \(K^\xi_0 \equiv 0\) and thus \(T^\xi_{E, \star, 0} \equiv 0\).

**Case 2:** \(\frac{\epsilon_0}{2} < \xi_I \leq 2 \epsilon_0\). In this case for every \(x \in \Sigma \cap I\) and \(z \in E_\star\)

\[|K^\xi_0(x - z)| \leq \frac{1}{|x - z|^n} 1_{|z - x| \leq 2\xi I} \leq \frac{1}{\epsilon_0} 1_{|z - x| \leq 4 \epsilon_0},\]

and therefore

\[
(A.31) \quad \int_{\Sigma \cap I} |T^\xi_{E, \star, 0}f(x)|^2 \, d\sigma_\star(x) \leq \int_{\Sigma \cap I} \left( \frac{1}{\epsilon_0^n} \right) \int_{B(x, 4 \epsilon_0)^\star \cap E_\star} |f(z)| \, d\sigma_\star(z) \, d\sigma_\star(x) \leq \int_{\Sigma \cap I} M_{E, \star}f(x)^2 \, d\sigma_\star(x)
\]

where the implicit constants are independent of \(\epsilon_0\) and \(\xi_I\).

**Case 3:** \(\xi_I > 2 \epsilon_0\). In this case \(T^\xi_{E, \star, 0}f\) is a double truncated integral whose smooth Calderón-Zygmund kernel \(K^\xi_0\) is odd, smooth in \(\mathbb{R}^{n+1}\) and satisfies the estimates \((A.11), (A.12)\) with uniform bounds (i.e., independent of \(\epsilon_0\) and \(\xi_I\)). Fix \(z_I \in \Sigma \cap I\) and notice that if \(x \in \Sigma \cap I\) and \(z \in B(x, 2 \xi_I) \cap E_\star\) then, taking \(M\) large enough, we have

\[|z - z_I| \leq |z - x| + |x - z_I| \leq 2 \xi_I + \text{diam}(I) = \frac{\ell(I)}{2M} + \text{diam}(I) < \frac{3}{2} \text{diam}(I)\]

and therefore the fact that \(\text{supp} \, K^\xi_0 \subset B(0, 2 \xi_I)\) immediately gives \(T^\xi_{E, \star, 0}f(x) = T^\xi_{E, \star, 0}(f 1_{\widetilde{\Delta}_\star}) (x)\) where \(\widetilde{\Delta}_\star := \widetilde{B}_\star \cap \Delta_\star := B(z_I, 2 \text{diam}(I)) \cap E_\star\). Notice that \((2.7)\) yields

\[4 \text{diam}(I) \leq \text{dist}(4 I, E) \leq \text{dist}(z_I, E) \leq \text{dist}(\widetilde{B}_\star, I, E) + 2 \text{diam}(I)\]

and therefore \(\text{dist}(\widetilde{B}_\star, I, E) \geq 2 \text{diam}(I)\). This implies that \(\frac{3}{2} \widetilde{B}_\star \subset \mathbb{R}^{n+1} \setminus E\). Also if \(J \in \mathcal{W}\) satisfies that \(J^* \cap \widetilde{B}_\star \neq \emptyset\) we can easily check that \(\ell(J) = \ell(J)\) and \(\text{dist}(I, J) \leq \ell(J)\). This implies that only
a bounded number of $J$'s have the property that $J^*$ intersects $\tilde{B}_{*,J}$. We recall that $\Sigma = E_* \setminus E$ is a union of portion of faces of fattened Whitney cubes $J^*$. Thus we have

$$\tilde{\Delta}_{*,J} \subset \bigcup_{m=1}^{M_0} F_{m,l},$$

where $M_0$ is a uniform constant and each $F_{m,l}$ is either a portion of a face of some $J^*$, or else $F_{m,l} = \emptyset$ (since $M_0$ is not necessarily equal to the number of faces, but is rather an upper bound for the number of faces.) Note also that $I \subset \tilde{B}_{*,J}$ and therefore we also have that

$$\Sigma \cap I \subset \bigcup_{m=1}^{M_0} F_{m,l}.$$

Thus

$$\int_{\Sigma \cap I} |T_{E*,0}^j f(x)|^2 \, d\sigma_\star(x) = \int_{\Sigma \cap I} |T_{E*,0}^j (f 1_{\tilde{\Delta}_{*,J}})(x)|^2 \, d\sigma_\star(x) \leq \sum_{1 \leq m,m' \leq M_0} \int_{F_{m,l}} |T_{E*,0}^j (f 1_{F_{m',l}})(x)|^2 \, d\sigma_\star(x).$$

In the case $m = m'$ we take the hyperplane $H_{m,l}$ with $F_{m,l} \subset H_{m,l}$ and then

$$\int_{F_{m,l}} |T_{E*,0}^j (f 1_{F_{m,l}})(x)|^2 \, d\sigma_\star(x) \leq \int_{H_{m,l}} |T_{E*,0}^j (f 1_{F_{m,l}})(x)|^2 \, dH^n(x) \leq \int_{F_{m,l}} |f(x)|^2 \, dH^n(x) = \int_{F_{m,l}} |f(x)|^2 \, d\sigma_\star(x),$$

where, after a rotation, we have used the $L^2$ bounds of Calderón-Zygmund operators with nice kernels on $\mathbb{R}^n$. For $m \neq m'$ we consider two cases: either $\text{dist}(F_{m,l}, F_{m',l}) \approx \ell(I)$ or $\text{dist}(F_{m,l}, F_{m',l}) \ll \ell(I)$. In the first scenario, using that $K_0^{G_0}$ satisfies (A.11) uniformly we obtain that

$$\int_{F_{m,l}} |T_{E*,0}^j (f 1_{F_{m',l}})(x)|^2 \, d\sigma_\star(x) \leq \int_{F_{m,l}} \left( \int_{F_{m',l}} \frac{1}{|x - z|^{d-2n}} |f(z)| \, d\sigma_\star(z) \right)^2 \, d\sigma_\star(x) \leq \int_{F_{m,l}} \frac{1}{\ell(I)^d} \int_{B(x,C \ell(I)) \cap E_*} |f(z)| \, d\sigma_\star(z)^2 \, d\sigma_\star(x) \leq \int_{F_{m,l}} M_{E_*} f(x)^2 \, d\sigma_\star(x).$$

Finally if $\text{dist}(F_{m,l}, F_{m',l}) \ll \ell(I)$, we have that $F_{m,l}$ and $F_{m',l}$ are contained in respective faces which either lie in the same hyperplane, or else meet at an angle of $\pi/2$. In the first case we may proceed as in the case $m = m'$. In the second case, after a possible rotation of co-ordinates, we may view $F_{m,l} \cup F_{m',l}$ as lying in a Lipschitz graph with Lipschitz constant 1, so that we may estimate $\mathcal{T}_{E*,0}^j$ using an extension of the Coifman-McIntosh-Meyer theorem:

$$\int_{F_{m,l}} |T_{E*,0}^j (f 1_{F_{m',l}})(x)|^2 \, d\sigma_\star(x) \leq \int_{F_{m,l}} |f(x)|^2 \, d\sigma_\star(x).$$

Gathering all the possible cases we may conclude that

\begin{equation}
\int_{\Sigma \cap I} |T_{E*,0}^j f(x)|^2 \, d\sigma_\star(x) \leq \sum_{1 \leq m \leq M_0} \int_{F_{m,l}} M_{E_*} f(x)^2 \, d\sigma_\star(x) \leq \sum_{l' \in W_2 : l' \cap \tilde{\Delta}_{*,J} \neq \emptyset} \int_{l' \cap \Sigma} M_{E_*} f(x)^2 \, d\sigma_\star(x).
\end{equation}
We now gather (A.29), (A.30) and (A.32) to get the following estimate for $S_I$ after using (A.21):

$$S_I = \int_{\Sigma_I} |T_{E_{i_0}} f(x)|^2 \, d\sigma_*(x)$$

$$\leq \int_{\Sigma_I} |T_{E_{i_0} \Delta_{i_0}} f(x)|^2 \, d\sigma_*(x) + \int_{\Sigma_I} |T_{E_{i_0} \Delta_{i_0}} f(x)|^2 \, d\sigma_*(x)$$

$$+ \int_{\Sigma_I} |T_{E_{i_0} \Delta_{i_0}} f(x)|^2 \, d\sigma_*(x)$$

$$\leq \int_{\Sigma_I} \sum_{E_{i_0} \Delta_{i_0} \subseteq \sigma} |E_{i_0} (T_{E_{i_0}} f(y))|^2 \, d\sigma(y) + \int_{Q_i} M_E (T_{E_{i_0}} f(y))^2 \, d\sigma(y)$$

$$+ \sum_{E_{i_0} \Delta_{i_0} \subseteq \sigma} \int_{Q_i} M_E (T_{E_{i_0}} f(y))^2 \, d\sigma(y).$$

Notice that since $1 < p < 2$ we have

$$\sum_{E_{i_0} \Delta_{i_0} \subseteq \sigma} \int_{Q_i} M_E (T_{E_{i_0}} f(y))^2 \, d\sigma(y) \leq \int_{E_*} M_E (T_{E_{i_0}} f(y))^2 \, d\sigma(y) \leq \int_{E_*} |f(y)|^2 \, d\sigma(y).$$

On the other hand, we set $\tilde{F} = F^+ \cup F^-$ and observe that, by construction, the cubes in $\tilde{F}$ are pairwise disjoint. Lemmas A.3 and A.7 then imply that

$$\sum_{E_{i_0} \Delta_{i_0} \subseteq \sigma} \int_{Q_i} M_E (T_{E_{i_0}} f(y))^2 \, d\sigma(y)$$

$$\leq \sum_{Q \in Q} \int_{Q} M_E (T_{E_{i_0}} f(y))^2 \, d\sigma(y) + \sum_{E_{i_0} \Delta_{i_0} \subseteq \sigma} \int_{Q_i} M_E (T_{E_{i_0}} f(y))^2 \, d\sigma(y)$$

$$\leq \int_{E} M_E (T_{E_{i_0}} f(y))^2 \, d\sigma(y)$$

$$\leq \int_{E} |T_{E_{i_0}} f(y)|^2 \, d\sigma(y)$$

$$\leq \int_{E_*} |f(y)|^2 \, d\sigma(y),$$

where in the last estimate we have used (A.18) with $p = 2$.

Finally, by the nature of the Whitney boxes (see (2.7)), we have that the family $\{ I \}_{I \in W}$ has the bounded overlap property and therefore

$$\sum_{I \in W} \sum_{I' \in W : I' \cap \Delta_{i_0} \neq \emptyset} 1_{\Sigma \cap I'} \leq \sup_{I \in W} \left\{ I \in \mathcal{W}_2 \cap I' \cap \Delta_{i_0} \neq \emptyset \right\}$$

which we claim that is uniformly bounded. Indeed, fix $I' \in \mathcal{W}_2$ and let $I_1, I_2 \in \mathcal{W}_2$ with $I' \cap \Delta_{i_0} \neq \emptyset$ and $I' \cap \Delta_{i_0} \neq \emptyset$. Recall that $\text{dist}(\tilde{B}_{i_0}, E) \geq 2 \text{diam}(I)$ with $\tilde{B}_{i_0} = B(z_1, 2 \text{diam}(I))$ and $z_1 \in I \cap \Sigma$. This implies that $\ell(I_1) \approx \ell(I_2)$ and also $\ell(I_1, I_2) \leq \ell(I_1)$. This easily gives our claim. Using this we conclude that

$$\sum_{I \in W} \sum_{I' \in W : I' \cap \Delta_{i_0} \neq \emptyset} \int_{I \cap \Sigma} M_E f(x)^2 \, d\sigma_*(x)$$

$$\leq \int_{E_*} M_E f(x)^2 \, d\sigma_*(x) \leq \int_{E_*} |f(x)|^2 \, d\sigma_*(x).$$
We now combine (A.33), (A.34), (A.35) and (A.36) to obtain that
\[ S = \sum_{b \in W \cap E} S_b \leq \int_{E_*} |f(x)|^2 \, d\sigma_*(x). \]
This, (A.20) and (A.19) give as desired that
\[ \int_{E_*} |T_{E_*,0}f(x)|^2 \, d\sigma_*(x) \leq \int_{E_*} |f(x)|^2 \, d\sigma_*(x), \]
and the implicit constant does not depend on \( \epsilon_0 \). Hence, \( T_{E_*,0} : L^2(E_*) \to L^2(E_*) \) with bounds that do not depend on \( \epsilon_0 \). Since \( T_{E_*,0}f \) is a continuous function for every \( f \in L^2(E_*) \), we have that \( T_{E_*,0}f \big|_{E_*} = T_{E_*,\epsilon_0}f \) everywhere on \( E_* \). Thus, all these show that \( T_{E_*,0} : L^2(E_*) \to L^2(E_*) \) uniformly in \( \epsilon \). This in turn gives, by the aforementioned result of [DS1], that \( E_* \) is UR as desired.

\[ \square \]

REFERENCES

[AHLT] P. Auscher, S. Hofmann, J.L. Lewis, and P. Tchamitchian, Extrapolation of Carleson measures and the analyticity of Kato’s square-root operators, Acta Math. 187 (2001), no. 2, 161–190. 15

[AHM3TV] J. Azzam, S. Hofmann, J. M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, and A. Volberg, Rectifiability of harmonic measure, Geom. Funct. Anal. 31, 33, 36, 37, 40, 41

[BJ] C. Bishop and P. Jones, Harmonic measure and arclength, Ann. of Math. (2)

[Bad] M. Badger, Null sets of harmonic measure on NTA domains: Lipschitz approximation revisited, Math. Z.

[Chr] M. Christ, A \( T(b) \) theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math., LX/LXI (1990), 601–628. 7

[Da1] B. Dahlberg, On estimates for harmonic measure, Arch. Rat. Mech. Analysis 65 (1977), 272–288. 2

[Da2] B. Dahlberg, Weighted norm inequalities for the Lusin area integral and the non-tangential maximal function for functions harmonic in a Lipschitz domain, Studia Math. 67 (1980), 297-314. 2

[Da3] B. Dahlberg, Approximation of harmonic functions Ann. Inst. Fourier (Grenoble) 30 (1980) 97-107. 2

[DJK] B.E. Dahlberg, D.S. Jerison, and C.E. Kenig, Area integral estimates for elliptic differential operators with nonsmooth coefficients, Ark. Mat. 22 (1984), no. 1, 97–108. 2, 3

[DKPV] B. E. J. Dahlberg, C. E. Kenig, J. Pipher, and G. Verchota, Area integral estimates for higher order elliptic equations and systems, Ann. Inst. Fourier (Grenoble) 47 (1997), no. 5, 1425–1461. 39

[DFM] G. David, J. Feneuil, and S. Mayboroda, Dahlberg’s theorem in higher co-dimension, arXiv:1704.00667.

[DJ] G. David and D. Jerison, Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals, Indiana Univ. Math. J. 39 (1990), no. 3, 831–845. 16, 17

[DS1] G. David and S. Semmes, Singular integrals and rectifiable sets in \( \mathbb{R}^n \): Beyond Lipschitz graphs, Astérisque 193 (1991). 2, 7, 41, 42, 49

[DS2] G. David and S. Semmes, Analysis of and on Uniformly Rectifiable Sets, Mathematical Monographs and Surveys 38, AMS 1993. 4, 7, 8

[GMT] J. Garnett, M. Mourgoglou, and X. Tolsa, Uniform rectifiability in terms of Carleson measure estimates and \( \epsilon \)-approximability of bounded harmonic functions, Duke Math. J. 167 (2018), no. 8, 1473–1524. 3

[HKMP] S. Hofmann, C. Kenig, S. Mayboroda, and J. Pipher, Square function/Non-tangential maximal function estimates and the Dirichlet problem for non-symmetric elliptic operators, J. Amer. Math. Soc. 28 (2015), 483–529. 2

[HM] S. Hofmann and J.M. Martell, Uniform rectifiability and harmonic measure I: Uniform rectifiability implies Poisson kernels in \( L^p \), Ann. Sci. École Norm. Sup. 47 (2014), no. 3, 577–654. 9, 13, 33, 41

[HMM] S. Hofmann, J.M. Martell, and S. Mayboroda, Uniform rectifiability, Carleson measure estimates, and approximation of harmonic functions, Duke Math. J. 165 (2016), no. 12, 2331–2389. 1, 3, 6, 8, 11, 13, 14, 18, 31, 33, 36, 37, 40, 41

[HMa] S. Hofmann, S. Mayboroda, Hardy and BMO spaces associated to divergence form elliptic operators, Math. Ann. 344 (2009), no. 1, 37–116. 15
S. Hofmann, D. Mitrea, M. Mitrea, and A.J. Morris. $L^p$-square function estimates on spaces of homogeneous type and on uniformly rectifiable sets. *Mem. Amer. Math. Soc.* **245** (2017), no. 1159.

D. Jerison and C. Kenig. Boundary behavior of harmonic functions in nontangentially accessible domains, *Adv. in Math.* **46** (1982), no. 1, 80–147.

C. Kenig, Harmonic analysis techniques for second order elliptic boundary value problems, *CBMS Regional Conference Series in Mathematics* **83.** Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1994.

C. Kenig, H. Koch, H. J. Pipher and, T. Toro. A new approach to absolute continuity of elliptic measure, with applications to non-symmetric equations. *Adv. Math.* **153** (2000), no. 2, 231–298.

C. Kenig, B. Kirchheim, J. Pipher, and T. Toro. Square functions and the $A^\infty$ property of elliptic measures, *J. Geom. Anal.* **26** (2016), no. 3, 2383–2410.

C. Kenig and J. Pipher. The Dirichlet problem for elliptic equations with drift terms, *Publ. Mat.* **45** (2001), no. 1, 199–217.

P. Mattila, M. Melnikov, and J. Verdera. The Cauchy integral, analytic capacity, and uniform rectifiability, *Ann. of Math. (2)* **144** (1996), no. 1, 127–136.

F. Nazarov, X. Tolsa and A. Volberg. On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1, *Acta Math.* **213** (2014), no. 2, 237–321.

J. Pipher, G. Verchota. Dilation invariant estimates and the boundary Gårding inequality for higher order elliptic operators. *Ann. of Math. (2)* **142** (1995), no. 1, 1–38.

E. M. Stein. *Singular Integrals and Differentiability Properties of Functions,* Princteon University Press, Princeton, NJ, 1970.

---

**Steve Hofmann, Department of Mathematics, University of Missouri, Columbia, MO 65211, USA**  
*Email address: hofmanns@missouri.edu*

**José María Martell, Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera, 13-15, E-28049 Madrid, Spain**  
*Email address: chema.martell@icmat.es*

**Svitlana Mayboroda, Department of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA**  
*Email address: svitlana@math.umn.edu*