Thermal relics as hot, warm and cold dark matter in power-law $f(R)$ gravity

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Abstract

We investigate the thermal relics as hot, warm and cold dark matter in $\mathcal{L} = \varepsilon^{2-2\beta} R^\beta + 16\pi m_\text{Pl}^{-2} \mathcal{L}_m$ gravity, where $\varepsilon$ is a constant balancing the dimension of the field equation, and $1 < \beta < (4 + \sqrt{6})/5$ for the positivity of energy density and temperature. If light neutrinos serve as hot/warm relics, the entropic number of statistical degrees of freedom $g_\ast s$ at freeze-out and thus the predicted fractional energy density $\Omega_\psi h^2$ are $\beta$-dependent, which relaxes the standard mass bound $\Sigma m_\nu$. For cold relics, by exactly solve the simplified Boltzmann equation in both relativistic and nonrelativistic regimes, we show that the Lee-Weinberg bound for the mass of heavy neutrinos can be considerably relaxed, and the “WIMP miracle” for weakly interacting massive particles (WIMPs) gradually invalidates as $\beta$ deviates from $\beta = 1^+$. The whole framework reduces to become that of GR in the limit $\beta \to 1^+$.

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1. Introduction

With the development of observational astrophysics and cosmology, the investigations of galaxy rotation curves, gravitational lensing and large scale structures have provided strong evidences for the existence and importance of dark matter. The abundance of dark matter has been measured with increasingly high precision, such as $\Omega_\text{dm} h^2 = 0.1198 \pm 0.0026$ by the latest Planck data [1]; however, since our knowledge of dark matter exclusively comes from the gravitational effects, the physical nature of dark-matter particles remain mysterious.

Nowadays it becomes a common view that to account for the observed dark matter, one needs to go beyond the SU(3)$_c \times$SU(2)$_W \times$U(1)$_Y$ minimal standard model. There are mainly two leading classes of dark-matter candidates: axions that are non-thermally produced via quantum phase transitions in the early universe, and generic weakly interacting massive particles (WIMPs) [2] that freeze out of thermal equilibrium from the very early cosmic plasma and leave a relic density matching the present-day Universe. In this paper, we are interested in the latter class, i.e. dark matter created as thermal relics. We aim to correct and complete the pioneering investigations in Ref.[3] for cold relics in $\mathcal{L} = m_\text{Pl}^{-2} R^\beta + 16\pi m_\text{Pl}^{-2} \mathcal{L}_m$ gravity, and provide a comprehensive investigation of thermal relics as hot, warm and cold dark matter in $\mathcal{L} = \varepsilon^{2-2\beta} R^\beta + 16\pi m_\text{Pl}^{-2} \mathcal{L}_m$ gravity.

This paper is organized as follows. Sec. 2 sets up the gravitational framework of $\mathcal{L} = \varepsilon^{2-2\beta} R^\beta + 16\pi m_\text{Pl}^{-2} \mathcal{L}_m$ gravity, while Sec. 3 generalizes the time-temperature relation for cosmic expansion and derives the simplified Boltzmann equation. Sec. 4 studies hot/warm thermal relics, and shows the influences of $\beta$ and $\varepsilon$ to the bound of light neutrino mass. Sec. 5 investigates cold thermal relics by solving...
the simplified Boltzmann equation, while Sec. 6 rederives the Lee-Weinberg bound on fourth-generation massive neutrinos, and examines the departure from electroweak energy scale. Finally, the GR limit of the whole theory is studied in Sec. 7.

Throughout this paper, for the physical quantities involved in the calculations of thermal relics, we use the natural unit system of particle physics which sets $\hbar = k_B = 1$ and is related to le système international d’unités by $1\text{ MeV} = 1.16 \times 10^{10}\text{ kelvin} = 1.78 \times 10^{-30}\text{ kg} = (1.97 \times 10^{-13}\text{ meters})^{-1} = (6.58 \times 10^{-22}\text{ seconds})^{-1}$. On the other hand, for the spacetime geometry, we adopt the conventions $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma}$, $R^\alpha_{\beta\gamma\delta} = \partial_\gamma \Gamma^\alpha_{\delta\beta} - \partial_\delta \Gamma^\alpha_{\gamma\beta} + \Gamma^\alpha_{\gamma\epsilon} \Gamma^\epsilon_{\delta\beta} - \Gamma^\alpha_{\delta\epsilon} \Gamma^\epsilon_{\gamma\beta}$ and $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$ with the metric signature $(-, +, +, +)$.

2. Gravitational framework of power-law $f(R)$ gravity

$f(R)$ gravity is a direct generalization of GR and extends the Hilbert-Einstein action $I_{\text{HE}} = \int \sqrt{-g} \, d^4x \left( R + 16\pi m_{\text{Pl}}^{-2} \mathcal{L}_m \right)$ into

$$I = \int d^4x \sqrt{-g} \left[ f(R, \varepsilon) + 16\pi m_{\text{Pl}}^{-2} \mathcal{L}_m \right],$$

where $R$ denotes the Ricci scalar of the spacetime, $\varepsilon$ is some constant balancing the dimensions of the field equation, and $\mathcal{L}_m$ is the matter Lagrangian density. This paper considers the spatially flat, homogeneous and isotropic Universe, which, in the $(t, r, \theta, \varphi)$ comoving coordinates along the cosmic Hubble flow, is depicted by the Friedmann-Robertson-Walker (FRW) line element

$$ds^2 = -dt^2 + a(t)^2 dr^2 + a(t)^2 r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),$$

where $a(t)$ denotes the cosmic scale factor. Assume a perfect-fluid material content $\mathcal{T}^{(m)}_{\mu\nu} = \text{diag}[-\rho, P, P, P]$, with $\rho$ and $P$ being the energy density and pressure, respectively. Then Eq.(2) under the flat FRW metric yields the generalized Friedmann equations

$$\frac{3\ddot{a}}{a} f_R - \frac{1}{2} f - 3\frac{\dot{a}}{a} f_{RRR} \dot{R} = -8\pi m_{\text{Pl}}^{-2} \rho,$$

$$\left( \frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \right) f_R - \frac{1}{2} f - f_{RRR}(R^2) - 3\frac{\dot{a}}{a} f_{RR} \dot{R} = 8\pi m_{\text{Pl}}^{-2} P,$$

where overdot denotes the derivative with respect to the comoving time, $f_{RR} \equiv d^2 f(R, \varepsilon)/dR^2$, and $f_{RRR} \equiv d^3 f(R, \varepsilon)/dR^3$. In addition, the equation of local energy-momentum conservation gives rise to the continuity equation,

$$\nabla^\mu \mathcal{T}^{(m)}_{\mu\nu} = 0 \quad \Rightarrow \quad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0;$$

for the very early Universe that is radiation-dominated, integration of Eq.(6) with the equation of state...
\( \rho = 3P \) yields that the radiation density is related to the cosmic scale factor by

\[ \rho = \rho_0 a^{-4} \propto a^{-4}. \]  

(7)

In this paper, we will work with the specific power-law \( f(R) \) gravity

\[ I = \int d^4x \sqrt{-g} \left( e^{2-2\beta} R^\beta + 16\pi m_{\text{Pl}}^{-2} \mathcal{L}_m \right), \]

(8)

where \( \beta = \text{constant} > 0 \). With \( \rho = \rho_0 a^{-4} \) and \( f(R) = e^{2-2\beta} R^\beta \), the generalized first Friedmann equation (4) yields

\[ a = a_0^{\beta/2} \propto t^{\beta/2}, \]

(9)

and

\[ \left[ \frac{12(\beta - 1)}{\beta} \right]^{\beta/2} \left( \frac{-5\beta^2 + 8\beta - 2}{\beta - 1} \right) = 32\pi e^{2\beta - 2} m_{\text{Pl}}^{-2} \rho, \]

(10)

where \( H \) refers to the cosmic Hubble parameter. Moreover, the weak, strong and dominant energy conditions for classical matter fields require the energy density \( \rho \) to be positive definite, and as a consequence, the positivity of the left hand side of Eq. (10) limits \( \beta \) to the domain

\[ 1 < \beta < \frac{4 + \sqrt{6}}{5} \approx 1.2899. \]  

(11)

Note that the Ricci scalar for the flat FRW metric with \( a = a_0^{\beta/2} \) reads

\[ R = 6 \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) = \frac{3\beta(\beta - 1)}{t^2}, \]

(12)

so \( R > 0 \) and \( R^\beta \) is always well defined in this domain. Moreover, we will utilize two choices of \( \varepsilon \) to balance the dimensions in \( \mathcal{L} = e^{2-2\beta} R^\beta + 16\pi m_{\text{Pl}}^{-2} \mathcal{L}_m \) gravity:

(i) \( \varepsilon = 1 \) [sec\(^{-1}\)]. This choice can best respect and preserve existent investigations in mathematical relativity for the \( f(R) \) class of modified gravity, which have been analyzed for \( \mathcal{L} = f(R) + 16\pi m_{\text{Pl}}^{-2} \mathcal{L}_m \) without caring the physical dimensions.

(ii) \( \varepsilon = m_{\text{Pl}} \approx 0.1854 \times 10^{44} \) [1/s], or \( 1/\ell_{\text{Pl}} \) where \( \ell_{\text{Pl}} = \sqrt{G} \) refers to Planck length. The advantage of this choice is there is no need to employ extra parameters outside the mathematical expression \( \mathcal{L} = f(R) + 16\pi m_{\text{Pl}}^{-2} \mathcal{L}_m \).

3. Thermal relics

3.1. Time-temperature relation of cosmic expansion

For the very early Universe, the radiation energy density \( \rho \) attributes to all relativistic species, which are exponentially greater than those of the nonrelativistic particles, and therefore \( \rho = \sum \rho_i \) (boson) + \( \sum \rho_f \) (fermion) = \( \sum \frac{\pi^2}{30} g_i^{(b)} T_i^4 \) (boson) + \( \sum \frac{\pi^2}{30} g_j^{(f)} T_j^4 \) (fermion), where \( \{g_i^{(b)}, g_j^{(f)}\} \) are the numbers of statistical degrees of freedom for relativistic bosons and fermions, respectively. More concisely, normalizing the temperatures of all relativistic species with respect to photons’ temperature \( T_\gamma \equiv T \), one has the
generalized Stefan-Boltzmann law
\[ \rho = \frac{\pi^2}{30} g_* T^4 \quad \text{with} \quad g_* := \sum_{\text{boson}} g_i^{(b)} \left( \frac{T_i}{T} \right)^4 + \frac{7}{8} \sum_{\text{fermion}} g_j^{(f)} \left( \frac{T_j}{T} \right)^4, \]
where, in thermodynamic equilibrium, \( T \) is the common temperature of all relativistic particles. To facilitate the discussion of thermal relics, introduce a dimensionless variable
\[ x := \frac{m_\psi}{T} \]
(14)
to relabel the time scale, where \( m_\psi \) denotes the mass of dark-matter particles. \( x \) is a well defined variable since the temperature monotonically decreases after the Big Bang: Reheatings due to pair annihilations at \( T \gtrsim 0.5486 \text{ MeV} = m(e^\pm) \) only slow down the decrement of \( T \) rather than increase \( T \) [5].

Substitute Eq.(13) into Eq.(10), and it follows that the cosmic expansion rate is related to the radiation temperature by
\[ H = \sqrt{\frac{\beta}{12(\beta - 1)}} \left( \frac{(\beta - 1) g_*}{-5\beta^2 + 8\beta - 2} \right)^{1/\beta} \left( \frac{32\pi^2}{30} \frac{T^2}{m_{\text{Pl}}} \right)^{1/\beta} e^{1-1/\beta} \]
\[ = \sqrt{\frac{\beta}{12(\beta - 1)}} \left( \frac{(\beta - 1) g_*}{-5\beta^2 + 8\beta - 2} \right)^{1/\beta} \left( \frac{32\pi^2}{30} \frac{m^2}{m_{\text{Pl}}^2} \right)^{1/\beta} e^{1-1/\beta} x^{-2/\beta}, \]
(15)
which can be compactified into
\[ H = H(m)x^{-2/\beta} \quad \text{with} \quad H(m) := H(T = m_\psi). \]
(16)

As time elapses after the Big Bang, the space expands and the Universe cools. Eq.(15) along with \( H = \beta/(2t) \) leads to \( t = \frac{\beta}{2 H} = \frac{\beta e^{2+2\beta}}{H(m)} \) and the time-temperature relation
\[ t = \sqrt{3\beta(\beta - 1)} \left( \frac{-5\beta^2 + 8\beta - 2}{(\beta - 1) g_*} \right)^{1/\beta} \left( \frac{30 m_{\text{Pl}}}{32\pi^3 T^2} \right)^{1/\beta} e^{1/\beta - 1} \]
\[ = \sqrt{3\beta(\beta - 1)} \left( \frac{-5\beta^2 + 8\beta - 2}{(\beta - 1) g_*} \right)^{1/\beta} \left( \frac{30 m_{\text{Pl}}}{32\pi^3 m_\psi^2} \right)^{1/\beta} e^{1/\beta - 1} x^{2/\beta}. \]
(17)

3.2. Boltzmann equation

For dark-matter particles \( \psi \) in the very early Universe (typically before the era of primordial nucleosynthesis), there are various types of interactions determining the \( \psi \) thermal relics, such as elastic scattering between \( \psi \) and standard-model particles, and self-annihilation \( \psi + \psi \rightleftharpoons \psi + \psi + \cdots \). In this paper, we are interested in \( \psi \) initially in thermal equilibrium via the pair annihilation into (and creation from) standard-model particles \( \ell = \gamma, e^\pm, \mu^\pm, \tau^\pm, \cdots \).
\[ \psi + \bar{\psi} \rightleftharpoons \ell + \bar{\ell}. \]
(18)

As the mean free path of \( \psi \) increases along the cosmic expansion, the interaction rate \( \Gamma_{\psi\psi} \) of Eq.(18) gradually falls below the Hubble expansion rate \( H \), and the abundance of \( \psi \) freezes out. The number
Substitute the simplified Boltzmann equation (19) into Eq.(24), and one obtains

\[ \dot{n}_\psi + 3Hn_\psi = -\langle\sigma v\rangle_n^2 \left( n_\psi^2 - n_\psi^\text{eq}^2 \right), \]  

(19)

where \( \langle\sigma v\rangle \) is the thermally averaged cross-section. Employ the following quantity to describe the evolution of \( \psi \) at different temperature scales:

\[ Y := \frac{n_\psi}{s} \propto \frac{n_\psi}{g_sT^3}, \]  

(20)

where \( s \) is the comoving entropy density \( s := S/V \),

\[ s = \sum_i \rho_i + P_i - \mu_i n_i = \frac{2\pi^2}{45} g_s T^3 \quad \text{with} \quad g_s := g_{s(b)}^i \left( \frac{T_i}{T} \right)^3 + g_{s(f)}^i \left( \frac{T_i}{T} \right)^3. \]  

(21)

Here we have applied \( P_i = \rho_i/3 \) and \( \mu_i \ll T_i \) in \( s \) for relativistic matter, and \( g_s \) denotes the entropic number of statistic degrees of freedom. According to the continuity equation Eq.(6) and the thermodynamic identities

\[ \frac{\partial P}{\partial T}\bigg|_\mu = s, \quad \frac{\partial P}{\partial \mu}\bigg|_T = n, \]  

(22)

one has

\[ \frac{d(sa^3)}{dt} = -\frac{\mu}{T} \frac{d(na^3)}{dt}, \]  

(23)

so the comoving entropy density \( sa^3 \) of a particle species is conserved when the comoving particle number density \( n_\psi a^3 \) is conserved or the chemical potential \( \mu \) is far smaller than the temperature. Thus, \( d(sa^3)/dt = 0 = a^3(\dot{s} + 3Hs) \), \( \dot{s}/s = -3H \), and the time derivative of \( Y \) becomes

\[ \frac{dY}{dt} = \frac{\dot{n}_\psi}{s} - \frac{s}{\dot{s}} Y = \frac{\dot{n}_\psi}{s} + 3HY = s^{-1} \left( \dot{n}_\psi + 3Hn_\psi \right). \]  

(24)

Substitute the simplified Boltzmann equation (19) into Eq.(24), and one obtains

\[ \frac{dY}{dt} = -s \langle\sigma v\rangle \left( Y^2 - Y_\text{eq}^2 \right). \]  

(25)

Now rewrite \( dY/dt \) into \( dY/dx \). Since

\[ T = \left( \frac{30}{\pi^2 g_s} \right)^{1/4} = \left( \frac{30e^2 - 2\beta \mu^2}{8\beta - 2} \right)^{1/4} \left( \frac{\beta \mu - 1}{2\beta - 1} \right)^{1/4} \propto t^{-\beta/2}, \]  

(26)

thus \( \dot{T}/T = -\beta/(2t) = -H(t) = -H(x) = -H(m)x^{-2/\beta} \), and \( \frac{dY}{dx} \frac{dx}{dt} = \frac{dY}{dt} \frac{dT}{dx} = \frac{dY}{dx} \left( \frac{-x}{H(m)} \right) H(m)x^{-2/\beta} \), which recast Eq.(25) into

\[ \frac{dY}{dx} = -\frac{\chi^{\beta-1}}{H(m)} \langle\sigma v\rangle_s (Y^2 - Y_\text{eq}^2) = -\frac{\langle\sigma v\rangle}{H x} \frac{s}{s} \left( Y^2 - Y_\text{eq}^2 \right). \]  

(27)

Defining the annihilation rate of \( \psi \) as \( \Gamma_\psi := n_\text{eq} \langle\sigma v\rangle \), then Eq.(27) can be rewritten into the form

\[ x \frac{dY}{Y_\text{eq}} \frac{dx}{dx} = -\frac{n_\text{eq}}{H} \langle\sigma v\rangle \left[ \left( \frac{Y}{Y_\text{eq}} \right)^2 - 1 \right] = -\frac{\Gamma_\psi}{H} \left[ \left( \frac{Y}{Y_\text{eq}} \right)^2 - 1 \right], \]  

(28)
which will be very useful in calculating the freeze-out temperature of cold relics in Sec. 5.

4. Hot/warm relic dark matter and light neutrinos

4.1. Generic bounds on $\psi$ mass

Having set up the modified cosmological dynamics and Boltzmann equations in $\mathcal{L} = e^{-2\phi} R^2 + 16\pi m_{\psi}^2 \mathcal{L}_m$ gravity, we will continue to investigate hot dark matter which is relativistic for the entire history of the Universe until now, and warm dark matter which is relativistic at the time of decoupling but become nonrelativistic nowadays.

In the relativistic regime $T \gg 3m_\psi$ or equivalently $0 < x \ll 3$, the abundance of $m_\psi$ is given by

$$Y_{\text{eq}} = Y^{(R)}_{\text{eq}} = \frac{45 \zeta(3)}{2\pi^4} \frac{b_\psi g_\psi}{g_{*s}} \approx 0.2777 \frac{b_\psi g_\psi}{g_{*s}} ,$$

(29)

where $\zeta(3) = 1.20206$, $b_\psi = 1$ for bosons and $b_\psi = 3/4$ for fermions. $Y_{\text{eq}}$ only implicitly depends on $x$ through the evolution of $g_{*s}$ along the temperature scale. Then, the relic abundance is still given by $Y_{\text{eq}}$ at the time of freeze-out $x_f$:

$$Y_{\infty} := Y(x \to \infty) = Y^{(R)}_{\text{eq}}(x_f) = 0.2777 \times \frac{b_\psi g_\psi}{g_{*s}(x_f)} .$$

(30)

At the present time with $T_{\text{cmb}} = 2.7255 \, K$ [8], the entropy density is

$$s_0 = \frac{2\pi^2}{45} g_{*s0} T_{\text{cmb}}^3 = 2891.2 \, \text{cm}^{-3} ,$$

(31)

where in the minimal standard model with three generations of light neutrinos ($N_\nu = 3$),

$$g_{*s0} = 2 + \frac{7}{8} \times 2 \times N_\nu \times \left( \frac{T_\nu}{T_{\text{cmb}}} \right)^3 = 3.9091 .$$

(32)

Thus, the present-day number density and energy density of hot/warm relic $\psi$ can be found by

$$n_{\psi0} = s_0 Y_{\infty} = 802.8862 \times \frac{b_\psi g_\psi}{g_{*s}(x_f)} \, \text{cm}^{-3} ,$$

(33)

$$\rho_{\psi0} = m_\psi n_{\psi0} = 802.8862 \times \frac{b_\psi g_\psi}{g_{*s}(x_f)} \left( \frac{m_\psi}{eV} \right) \, eV \, \text{cm}^{-3} ,$$

(34)

which, for $\rho_{\text{crit}} = 1.05375 \times 10^4 h^2 \, eV/cm^3$, correspond to the fractional energy density

$$\Omega_\psi h^2 = \frac{\rho_{\psi0}}{\rho_{\text{crit}}} h^2 \times \frac{b_\psi g_\psi}{g_{*s}(x_f)} \left( \frac{m_\psi}{eV} \right) = 0.0762 \times \frac{b_\psi g_\psi}{g_{*s}(x_f)} \left( \frac{m_\psi}{eV} \right) .$$

(35)

This actually stands for an attractive feature of the paradigm of thermal relics: the current abundance $\Omega_\psi h^2$ of relic dark matter (hot, warm, or cold) can be predicted by $\psi$’s microscopic properties like mass, annihilation cross-section, and statistical degrees of freedom.

Since hot/warm relics can at most reach the total dark matter density $\Omega_\psi h^2 = 0.1198 \pm 0.0026$ [8],
$\Omega_{\nu} h^2$ has to satisfy $\Omega_{\nu} h^2 \lesssim 0.1198$, and it follows from Eq. (35) that $m_\nu$ is limited by the upper bound

$$m_\nu \leq 1.5723 \times \frac{g_{*s}(x_f)}{b_q g_\nu} \text{ eV}. \quad (36)$$

Moreover, particles of warm dark matter become nonrelativistic at present time, which imposes a lower bound to $m_\nu$,

$$m_\nu \gtrsim T_{\nu 0} = T_{\nu 0} \frac{a_f}{a_0} = \left( \frac{g_{*0}}{g_{*s}(x_f)} \right)^{1/3} T_{\text{cmb}} = 2.3496 \times 10^{-4} \times \left( \frac{3.9091}{g_{*s}(x_f)} \right)^{1/3} \text{ eV}, \quad (37)$$

where we have applied $g_{*s}^{1/3} a T = \text{constant due to } sa^3 = \text{constant}$. Eqs. (36) and (37) lead to the mass bound for warm relics that

$$2.3496 \times 10^{-4} \times \left( \frac{3.9091}{g_{*s}(x_f)} \right)^{1/3} \lesssim \frac{m_\nu}{\text{eV}} \lesssim 1.5723 \times \frac{g_{*s}(x_f)}{b_q g_\nu}. \quad (38)$$

4.2. Example: light neutrinos as hot relics

Light neutrinos are the most popular example of hot/warm dark matter [10]. One needs to figure out the temperature $T_f^\nu$ and thus $g_{*s}(T = T_f^\nu)$ when neutrinos freeze out from the cosmic plasma. The decoupling occurs when the Hubble expansion rate $H$ balances neutrinos’ interaction rate $\Gamma_\nu$. For the cosmic expansion, it is convenient to write Eq. (15) into

$$H = 0.2887 \times \sqrt{\frac{\beta}{\beta - 1}} \times \left( \frac{(\beta - 1) g_*}{-5\beta^2 + 8\beta - 2} \right)^{1/\beta} \left( 0.7164 \cdot T_{\text{MeV}}^2 \right)^{1/\beta} e_s^{-1/\beta} \text{ [1/s]}, \quad (39)$$

where $T_{\text{MeV}}$ refers to the value of temperature in the unit of MeV, $T = T_{\text{MeV}} \times [1 \text{ MeV}]$, $e_s$ is the value of $\epsilon$ in the unit of [1/s], and numerically $T^2/m_{\text{Pl}} = T_{\text{MeV}}^2/8.0276 [1/\text{s}]$.

On the other hand, the event of neutrino decoupling actually indicates the beginning of primordial nucleosynthesis, when neutrinos are in chemical and kinetic equilibrium with photons, nucleons and electrons via weak interactions and elastic scattering. The interaction rate $\Gamma_\nu$ is [6]

$$\Gamma_\nu \approx 1.3 G_F^2 T^5 \approx 0.2688 T_{\text{MeV}}^5 [1/\text{s}], \quad (40)$$

where $G_F$ is Fermi’s constant in beta decay and generic weak interactions, and $G_F = 1.1664 \times 10^{-11} \text{ MeV}^{-2}$. Neutrinos decouple when $\Gamma_\nu = H$, and according to Eqs. (39) and (40), the weak freeze-out temperature $T_f^\nu$ is the solution to

$$T_{\text{MeV}}^{5-2/\beta} = 1.0741 \times \sqrt{\frac{\beta}{\beta - 1}} \times \left( 0.7164 \cdot \frac{(\beta - 1) g_*}{-5\beta^2 + 8\beta - 2} \right)^{1/\beta} e_s^{1-1/\beta}. \quad (41)$$

Figs. 1 and 2 have shown the dependence of $T_f^\nu$ on $\beta$ for $\epsilon = 1 \text{ sec}^{-1} = 6.58 \times 10^{-22} \text{ MeV}$ and $\epsilon = m_{\text{Pl}} = 1.2209 \times 10^{-22} \text{ MeV}$, respectively. Fig. 2 clearly illustrates that $T_f^\nu$ spreads from 1.3030 MeV to over 1000 MeV, which goes far beyond the scope of $1 \sim 10 \text{ MeV}$; thus, as shown in Table 1, $g_{*s}$ varies and the mass bound $\Sigma m_\nu$ in light of Eq. (38) is both $\beta$–dependent and $\epsilon$–dependent.
5. Cold relic dark matter

Now let’s consider cold dark matter which is already nonrelativistic at the time of decoupling. In the nonrelativistic regime $T \ll 3m_\psi$ or equivalently $x \gg 3$, the number density and entropy density are given by

$$n_\psi = g_\psi \left( \frac{m_\psi^2}{2\pi} \right)^{3/2} x^{-3/2} e^{-x}, \quad s = \frac{2\pi^2}{45} g_\ast m_\psi^3 x^{-3} = s(m) x^{-3}, \quad (42)$$
Table 1: $g_{ss}$ for the $T_s'$ in Fig. 2, based on the data of Particle Data Group. Note that between 100~200 MeV, $g_{ss}$ is also subject to the phase transition of quantum chromodynamics for strange quarks.

| Temperature | Temperature (in MeV) | $g_{ss}$ |
|-------------|----------------------|----------|
| $m_e < T < m_s$ (strange) | 0.5110 < $T$ < 95 | 43/4 |
| $m_s < T < m_{\mu}$ | 95 < $T$ < 105.6584 | 57/4 |
| $m_{\mu} < T < m_{\pi}$ | 105.6584 < $T$ < 134.9766 | 69/4 |
| $m_{\pi} < T < T_c$ | 134.9766 < $T$ < $T_c$ | 205/4 |
| $T_c < T < m_c$ (charm) | $T_c$ < $T$ < 1275 | 247/4 |
| $m_c < T < m_t$ | 1275 < $T$ < 1776.82 | 289/4 |
| $m_t < T < m_b$ (bottom) | 1776.82 < $T$ < 4180 | 303/4 |

so one obtains the equilibrium abundance of nonrelativistic $\psi$ particles

$$Y_{eq} = Y_{eq}^{(NR)} = \frac{45}{4\pi^4} \left( \frac{\pi}{2} \right)^{1/2} \frac{g_w}{g_{ss}} x^{3/2} e^{-x} \approx 0.1447 \times \frac{g_w}{g_{ss}} x^{3/2} e^{-x}.$$  

Thus, $n_\psi$ and $Y_{eq} = Y_{eq}^{(NR)}$ are exponentially suppressed when the temperature drops below $m_\psi$. Moreover, since cold relics are nonrelativistic when freezing out, one can expand the thermally averaged cross-section by $\langle \sigma v \rangle = c_0 + c_1 v^2 + c_2 v^4 + \cdots + c_q v^{2q} + \cdots$, where $c_0$ corresponds to the decay channel of $s$–wave, $c_1$ to $p$–wave, $c_2$ to $d$–wave, and so forth; recalling that $\langle \sigma v \rangle \sim \sqrt{T}$ in light of the Boltzmann velocity distribution, thus the annihilation cross-section can be expanded by the variable $x$ into

$$\langle \sigma v \rangle = \langle \sigma v \rangle_0 x^{-n} \quad \text{with} \quad n = q/2.$$  

Then the Boltzmann equation (27) becomes

$$\frac{dY}{dx} = -s(m) \frac{\langle \sigma v \rangle_0}{H(m)} x^{3/2 - 4-n} \left( Y^2 - Y_{eq}^2 \right) = -s(m) \frac{\langle \sigma v \rangle_0}{H(m)} x^{3/2 - 4-n} \left[ Y^2 - 0.0209 \left( \frac{g_w}{g_{ss}} \right)^2 x^3 e^{-2x} \right],$$  

where

$$s(m) \frac{\langle \sigma v \rangle_0}{H(m)} = \frac{1.519525}{(5.750944)^{1/\beta}} \left( \frac{\beta - 1}{\beta} \right)^{1/\beta} \left( \frac{-5\beta^2 + 8\beta - 2}{\beta - 1} \right)^{1/\beta} \frac{g_{ss}}{\sqrt{2\epsilon_0}} \frac{1}{\left( \sqrt{2\epsilon_0} \right)^{1/\beta}} \frac{1}{m_{Pl}^{3/2}} \langle \sigma v \rangle_0.$$  

Though initially in equilibrium $Y \approx Y_{eq} = Y_{eq}^{(NR)}$ (NR), the actual abundance $Y$ gradually departures from the equilibrium value $Y_{eq}^{(NR)}$ as the temperature decreases; $Y$ freezes out and escapes the exponential Boltzmann suppression when the interaction rate $\Gamma_\psi$ equates the cosmic expansion rate $H$. Transforming
Eq. (45) into the form

\[
\frac{x}{Y_{eq}} \frac{dY}{dx} = -s(m) \langle \sigma v \rangle_0 Y_{eq} \left[ \left( \frac{Y}{Y_{eq}} \right)^2 - 1 \right] x^{\frac{2}{3} - 3 - n} = -\frac{\Gamma_{\psi}}{H} \left[ \left( \frac{Y}{Y_{eq}} \right)^2 - 1 \right],
\]

(47)

and the coupling condition \( \Gamma_{\psi}(x_f) = H(x_f) \) at the freeze-out temperature \( T_f^\psi = m_{\psi}/x_f \) yields

\[
\frac{\Gamma_{\psi}}{H}(x_f) = 1 = \frac{s(m) \langle \sigma v \rangle_0}{H(m)} Y_{eq} x^{\frac{2}{3} - 3 - n} \approx 0.1447 \frac{s(m) \langle \sigma v \rangle_0}{H(m)} \frac{g_{\psi}}{g_*} x^{\frac{2}{3} - 3/2 - n} e^{-x}.
\]

(48)

Thus, it follows that

\[
e^{x_f} = \frac{0.1447 s(m) \langle \sigma v \rangle_0 g_{\psi}}{H(m)} \frac{g_{\psi}}{g_*} x^{\frac{2}{3} - 3/2 - n}
\]

\[= \frac{0.2199}{(5.7509)^{1/\beta}} \sqrt{\frac{\beta - 1}{\beta}} \left( \sqrt{\frac{-5\beta^2 + 8\beta - 2}{\beta - 1}} \right)^{1/\beta} \frac{g_{\psi}}{(\sqrt{g_*})^{1/\beta}} e^{\frac{1}{\beta} - 1} m^{3 - \frac{2}{\beta}} m_{Pl}^{1/\beta} \langle \sigma v \rangle_0 x^{\frac{2}{3} - 3/2 - n}.
\]

(49)

After taking the logarithm of both side, Eq. (49) can be iteratively solved to obtain

\[
x_f = \ln \left[ \frac{0.2199}{(5.7509)^{1/\beta}} \sqrt{\frac{\beta - 1}{\beta}} \left( \sqrt{\frac{-5\beta^2 + 8\beta - 2}{\beta - 1}} \right)^{1/\beta} \frac{g_{\psi}}{(\sqrt{g_*})^{1/\beta}} e^{\frac{1}{\beta} - 1} m^{3 - \frac{2}{\beta}} m_{Pl}^{1/\beta} \langle \sigma v \rangle_0 \right]
\]

(50)

\[+ \left( \frac{2}{\beta} - \frac{3}{2} - n \right) \ln \left[ \ln \left[ \frac{0.2199}{(5.7509)^{1/\beta}} \sqrt{\frac{\beta - 1}{\beta}} \left( \sqrt{\frac{-5\beta^2 + 8\beta - 2}{\beta - 1}} \right)^{1/\beta} \frac{g_{\psi}}{(\sqrt{g_*})^{1/\beta}} e^{\frac{1}{\beta} - 1} m^{3 - \frac{2}{\beta}} m_{Pl}^{1/\beta} \langle \sigma v \rangle_0 \right] \right],
\]

where \( g_* \) has been treated as a constant, as the time scale over which \( g_* \) evolves is much greater than the time interval near \( x_f \).

5.1. Abundance \( Y \) before freeze-out

To work out the actual abundance \( Y \) before the decoupling of \( \psi \), employ a new quantity \( \Delta := Y - Y_{eq} \), and then Eq. (45) can be recast into

\[
\frac{d\Delta}{dx} = -\frac{s(m) \langle \sigma v \rangle_0}{H(m)} x^{\frac{2}{3} - 4 - n} \Delta \left( \Delta + 2Y_{eq} \right) - \frac{dY_{eq}}{dx}.
\]

(51)

In the high-temperature regime \( x \ll x_f \) before \( \psi \) freezes out, \( Y \) is very close to \( Y_{eq} \), so that \( \Delta \ll Y_{eq} \) and \( d\Delta/dx \ll -dY_{eq}/dx \). With \( Y_{eq} = Y_{eq}^{NR} \) in Eq. (43), Eq. (51) can be algebraically solved to obtain

\[
\Delta = -\frac{dY_{eq}}{dx} \frac{H(m)}{s(m) \langle \sigma v \rangle_0} \frac{x^{n+4 - \frac{2}{\beta}}}{2Y_{eq} + \Delta} = \left( 1 - \frac{3}{2x} \right) \frac{H(m)}{s(m) \langle \sigma v \rangle_0} x^{n+4 - \frac{2}{\beta}} \approx \left( 1 - \frac{3}{2x} \right) \frac{H(m)}{2s(m) \langle \sigma v \rangle_0} x^{n+4 - \frac{2}{\beta}},
\]

(52)
and consequently

\[ Y = \Delta + Y_{eq} = \left(1 - \frac{3}{2x}\right) \frac{H(m)}{s(m) \langle \sigma v \rangle_0} x^{n+4-\frac{3}{\beta}} + \frac{45}{4\pi} \left(\frac{\pi}{2}\right)^{1/2} \frac{g_w}{g_{ss}} x^{3/2} e^{-x} \]

\[ \approx \left(1 - \frac{3}{2x}\right) \frac{H(m)}{2s(m) \langle \sigma v \rangle_0} x^{n+4-\frac{3}{\beta}} + 0.1447 \times \frac{g_w}{g_{ss}} x^{3/2} e^{-x}. \]  

(53)

5.2. Freeze-out abundance \( Y_{\infty} \)

After the decoupling of \( \psi \) particles, the actual number density \( n_{\psi} \) becomes much bigger than the ideal equilibrium value \( n_{\psi}^{eq} \). One has \( Y \gg Y_{eq} \), and the differential equations (45) or (51) leads to

\[ \frac{dY}{dx} = -\frac{s(m) \langle \sigma v \rangle_0}{H(m)} x^{5/2-n} Y^{2} \quad \text{or} \quad \frac{dY}{dx} = -\frac{s(m) \langle \sigma v \rangle_0}{H(m)} x^{5/2-n} Y^{2}, \]  

(54)

which integrates to yield the freeze-out abundance \( Y_{\infty} := Y(x = x_f) \approx Y(x \to \infty) \) that

\[ Y_{\infty} = \left(3 + n - \frac{2}{\beta}\right) \frac{H(m)}{s(m) \langle \sigma v \rangle_0} x^{3+n-\frac{3}{\beta}} \]

\[ = \frac{1.5195}{(5.7509)^{5/\beta}} \sqrt{\frac{\beta-1}{\beta}} \left(\frac{5\beta^2+8\beta-2}{\beta-1}\right)^{1/\beta} \frac{g_{ss}}{\langle \sigma v \rangle_0} e^{\frac{1}{\beta} - 1} m_{\psi}^{3/2} m_{Pl}^{1/\beta} \langle \sigma v \rangle_0. \]  

(55)

Following \( Y_{\infty} \), the number density and energy density of \( \psi \) are directly found to be

\[ n_{\psi 0} = s_0 Y_{\infty} = \frac{2891.2 \left(3 + n - \frac{2}{\beta}\right) x_f^{3+n-\frac{3}{\beta}}}{1.5195 \sqrt{\frac{\beta-1}{\beta}} \left(\frac{5\beta^2+8\beta-2}{\beta-1}\right)^{1/\beta} \frac{g_{ss}}{\langle \sigma v \rangle_0} e^{\frac{1}{\beta} - 1} m_{\psi}^{3/2} m_{Pl}^{1/\beta} \langle \sigma v \rangle_0} \]  

\[ \text{cm}^{-3}, \]  

(56)

\[ \rho_{\psi 0} = m_{\psi} n_{\psi 0} = \frac{2891.2 \left(3 + n - \frac{2}{\beta}\right) x_f^{3+n-\frac{3}{\beta}}}{1.5195 \sqrt{\frac{\beta-1}{\beta}} \left(\frac{5\beta^2+8\beta-2}{\beta-1}\right)^{1/\beta} \frac{g_{ss}}{\langle \sigma v \rangle_0} e^{\frac{1}{\beta} - 1} m_{\psi}^{2/2} m_{Pl}^{1/\beta} \langle \sigma v \rangle_0} \]  

\[ \text{eV cm}^{-3}, \]  

(57)

which gives rise to the fractional energy density

\[ \Omega_{\psi} h^2 = \frac{\rho_{\psi 0}}{\rho_{\text{crit}}} h^2 = \frac{2743.7248 \left(3 + n - \frac{2}{\beta}\right) x_f^{3+n-\frac{3}{\beta}}}{1.5195 \sqrt{\frac{\beta-1}{\beta}} \left(\frac{5\beta^2+8\beta-2}{\beta-1}\right)^{1/\beta} \frac{g_{ss}}{\langle \sigma v \rangle_0} e^{\frac{1}{\beta} - 1} m_{\psi}^{2/2} m_{Pl}^{1/\beta} \langle \sigma v \rangle_0}. \]  

(58)

Unlike Eq.(35) for hot/warm relics, the relic density \( \Omega_{\psi} h^2 \) for cold dark matter is not only much more sensitive to the temperature of cosmic plasma, but also relies on the annihilation cross-section.

6. Example: Fourth generation massive neutrinos and Lee-Weinberg bound

An example of cold relics can be the hypothetical fourth generation massive neutrinos [2, 9, 10]. For the Dirac-type neutrinos whose annihilations are dominated by s–wave (\( n = 0 \)), the interaction
cross-section reads
\[ \langle \sigma v \rangle_0 = G_F^2 m^2 = 1.3604 \times 10^{-10} \left( \frac{m_\nu}{\text{GeV}} \right)^2 \text{GeV}^{-2} \] (59)

where \( G_F \) is Fermi’s constant in beta decay and generic weak interactions, and \( G_F = 1.16637 \times 10^{-5} \text{GeV}^{-2} \). Then with \( g_\nu = 2 \) and \( g_\ast \sim 60 \), the neutrinos decouple at

\[
\bar{x}_f = \ln \left[ \frac{0.5983 \times 10^{-10}}{(44.5463)^{1/\beta}} \sqrt{\frac{\beta - 1}{\beta}} \left( \frac{-5\beta^2 + 8\beta - 2}{\beta - 1} \right)^{1/\beta} e^{\frac{1}{\beta} - 1} m^{5/2} m_{\text{Pl}}^{1/\beta} \right] \\
+ \left( \frac{2}{\beta} - \frac{3}{2} - n \right) \ln \left[ \ln \left( \frac{0.5983 \times 10^{-10}}{(44.5463)^{1/\beta}} \sqrt{\frac{\beta - 1}{\beta}} \left( \frac{-5\beta^2 + 8\beta - 2}{\beta - 1} \right)^{1/\beta} e^{\frac{1}{\beta} - 1} m^{5/2} m_{\text{Pl}}^{1/\beta} \right] \\
+ \left( \frac{2}{\beta} - \frac{3}{2} - n \right) \ln \left[ \ldots \right],
\] (60)

which, through Eq.(61), gives rise to the fractional energy density

\[ \Omega_{\nu} h^2 = \frac{\rho_\nu h^2}{\rho_{\text{crit}}} = \frac{2743.7248 \times 10^{10} \times (3 + n - \frac{2}{\beta}) \bar{x}_f^{3 + n - \frac{2}{\beta}}}{0.5983 \sqrt{\frac{\beta - 1}{\beta}} \left( \frac{-5\beta^2 + 8\beta - 2}{\beta - 1} \right)^{1/\beta} e^{\frac{1}{\beta} - 1} m^{5/2} m_{\text{Pl}}^{1/\beta}}. \] (61)

With the same amount of anti-particles, we finally have \( \Omega_{\nu\bar{\nu}} h^2 = 2\Omega_{\nu} h^2 \leq 0.1198 \). Thus the Lee-Weinberg bound [2, 9] for massive neutrinos are relaxed in \( \mathcal{L} = e^{2-2\beta} R^8 + 16\pi m_{\text{Pl}}^{-2} \mathcal{L}_m \) gravity.

7. Conclusions

In this paper, we have comprehensively investigated the thermal relics as hot, warm and cold dark matter in \( \mathcal{L} = e^{2-2\beta} R^8 + 16\pi m_{\text{Pl}}^{-2} \mathcal{L}_m \) gravity. When light neutrinos act as hot and warm neutrinos, the upper limit of neutrino mass \( \Sigma m_\nu \) relies on the value of \( \beta \) and the choice of \( \epsilon \). For cold relics, we have derived the freeze-out temperature \( T_f = m/x_f \) in Eq.(50), \( Y \) before the freeze-out in Eq.(53), the freeze-out value \( Y_{\ast} \) in Eq.(55), and the dark-matter fractional density \( \Omega_{\nu} h^2 \) in Eq.(61). Note that we focused on power-law \( f(R) \) gravity because unlike the approximated power-law ansatz \( a = a_0 e^\alpha (\alpha = \text{constant} > 0) \) for generic \( f(R) \) gravity, \( a = a_0 e^{\alpha/2} \) is an exact solution to \( \mathcal{L} = e^{2-2\beta} R^8 + 16\pi m_{\text{Pl}}^{-2} \mathcal{L}_m \) gravity for the radiation-dominated Universe; for GR with \( \beta \rightarrow 1^+ \), Eq.(9) reduces to recover the behavior \( a \propto t^{1/2} \) which respects \( 3a^2 / a^2 = -8\pi m_{\text{Pl}}^{-2} \rho_{\odot} a^{-4} \).

When light neutrinos serve as hot/warm relics, the entropic number of statistical degrees of freedom \( g_{\ast\ast} \) at freeze-out and thus the predicted fractional energy density \( \Omega_{\nu} h^2 \) are \( \beta \)-dependent, which relaxes the standard mass bound \( \Sigma m_\nu \). For cold relics, by exactly solve the simplified Boltzmann equation in both relativistic and nonrelativistic regimes, we show that the Lee-Weinberg bound for the mass of heavy neutrinos can be considerably relaxed, and the “WIMP miracle” for weakly interacting massive particles (WIMPs) gradually becomes invalid when \( \beta \) departs \( \beta = 1^+ \). The whole framework reduces to become that of GR in the limit \( \beta \rightarrow 1^+ \).

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