Novel integrable higher-dimensional nonlinear Schrödinger equation: properties, solutions, applications

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May 20, 2013

Abstract

An integrable extension of the well known nonlinear Schrödinger (NLS) equation to a higher space-dimension, recently proposed by us, is investigated, exploring its various important aspects. Focusing on the idea of construction its connection with other known models like the Zakharov equation and the Strachan construction is shown. The underlying integrable structures like the Lax pair, the infinite conserved charges and the higher soliton solutions are presented in the explicit form. The related 2D rogue wave model and other applications are focused on.

02.30.lk, 05.45.Yv, 02.30.jr, 92.10.H+, 42.65.-k,

keywords: Integrable 2D NLS equation, Lax pair, exact solitons, 2D ocean rogue wave

Short title: Integrable 2D NLS

1 Introduction:

In the development of nonlinear integrable systems, the (1 + 1)-dimensional NLS equation, though being a rather late entrant solit, gained popularity very quickly due to its enormously rich theoretical and practical importance. Over the years, the 1D NLS model has penetrated into different subjects with its soliton solutions finding important applications in diverse fields, starting from nonlinear plasma, optical communication to nonlinear phenomena in deep sea waves. However, unfortunately since this equation is defined in (1 + 1) space-time dimensions, the nonlinear evolution along a line only is possible to describe by this model. Therefore there remains a persistent need for the extension of the NLS equation to higher dimensions preserving its integrable structures, for modelling more realistic nonlinear phenomena. An important example of such an event is the ocean rogue wave, one of the mysteries of nature, which is a genuine 2D surface wave appearing in the deep sea.

Unfortunately, until very recently, neither a satisfactory (2 + 1)-dimensional integrable NLS model with local and regular nonlinear interaction of the basic field nor an analytic model for the 2D ocean rogue wave was available. A straightforward generalization of the 1D NLS to 2D keeping its local cubic nonlinearity does not serve the required purpose, since such a higher dimensional generalization turns out to be a nonintegrable system with no stable soliton solution. Zakharov on the other hand has
proposed long ago an integrable (2 + 1)-dimensional variant of the NLS equation by constructing certain form of Lax operators by an innovative trick. However, unfortunately such an equation is coupled to another potential field, expressed through the basic fields only in a nonlocal way zakhar'. Similar situation arises also in the Davey-Stewartson equation, where again additional fields are coupled in the interacting term, which could be related to the basic fields only through nonlocal transformations DSE. Strachan had proposed another way of constructing higher-dimensional equations as a reduction of the self-dual Yang-Mills equation and reproduced the 2D coupled NLS equation of Zakharov and its integrable hierarchies Strach.

The nonavailability of a higher-dimensional integrable NLS equation seems to make the situation rather desperate, since the deep-sea rogue wave, clearly a 2D surface ocean wave, is described popularly by the Peregrine breather of the 1D NLS equation Peregrine,rog1D, which is a rational solution describing wave formation along an one dimensional line only. In the background of this situation, we have proposed recently a novel (2 + 1)-dimensional NLS equation with a local nonlinear interaction in the basic field. The integrability of this equation is achieved when the traditional cubic amplitude-type nonlinearity is replaced by a current-like nonlinear term 2DNLSarxiv12, producing a realistic and exact 2D model for the ocean rogue wave rog12. However, many important properties of this recent (2 + 1)-dimensional integrable system and its related structures remain unexplored. Therefore, the investigation of this (2 + 1)-dimensional NLS equation regarding the aspects like, insight into its associated Lax operators, related infinite set of conserved quantities, integrable hierarchy, explicit construction of higher soliton solutions and its possible connection with other integrable models etc., is our main goal here.

The arrangement of the paper is as follows. In sec. 2 we briefly list the basic facts about the 1D NLS equation. Sec. 3 presents the integrable 2D NLS model together with the related higher conserved quantities and investigates the associated Lax operators, exploring its relation with other known models. Sec. 4 derives the soliton and its higher forms through Hirota’s bilinearization. The next section followed by concluding remarks and bibliography, gives some account of the potential physical applications of this model.

2 Brief account on integrable 1D NLS equation

The well known integrable 1D NLS equation expressed through a complex field \( q(x,t) \):

\[
    iq_t = q_{xx} + 2|q|^2q, \tag{1}
\]

together with its complex conjugate equation, occupies an important position among the integrable models due to its natural appearance in various applicable fields and its rich integrable structure. This nonlinear integrable system is associated with a matrix form of the linear Lax equations solit:

\[
    \Phi_x = U_1(\lambda)\Phi, \quad \Phi_t = V_2(\lambda)\Phi, \tag{2}
\]

with the Lax pair

\[
    U_1(\lambda) = i[\lambda\sigma_z + q\sigma_+ + q^*\sigma_-] \tag{3}
\]

and

\[
    V_2(\lambda) = i[V_{11}^{(2)}\sigma_z + V_{12}^{(2)}\sigma_+ + V_{21}^{(2)}\sigma_-] \tag{4}
\]

where \( V_{11}^{(2)} = 2\lambda^2 - |q|^2, V_{12}^{(2)} = (V_{21}^{(2)})^* = 2\lambda q - iq_x \).

considering \( \lambda \) to be real and \( \sigma_\pm, \sigma_z \) are Pauli matrices

\[
    \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
The subscripts in $x$ and $t$ denote partial derivatives in the space and time variables, respectively, with the space Lax equation in (2) representing a scattering problem with $\lambda$ as the spectral parameter and the field $q(x, t)$ as the scattering potential. The subscripts is the Lax pair $U_1, V_2$ on the other hand designate the maximum power of the spectral parameter $\lambda$ contained in it, as may be checked from its explicit form (3) [4]. The compatibility of the Lax equations (2): $\Phi_{t\lambda} = \Phi_{\lambda t}$ leads to the flatness condition of the Lax pair:

$$U_{1t} - V_{2x} + [U_1, V_2] = 0,$$

which yields the 1D NLS equation (1) at the space Lax equation in (2) representing a scattering problem with the maximum power of the spectral parameter $\lambda$ contained in it, as may be checked from its explicit form (3) [4]. The compatibility of the Lax equations (2): $\Phi_{t\lambda} = \Phi_{\lambda t}$ leads to the flatness condition of the Lax pair:

$$U_{1t} - V_{2x} + [U_1, V_2] = 0,$$

which yields the 1D NLS equation (1) at the $\lambda^0$ power, while other relations at $\lambda^n$, $n = 1, 2, 3$ are trivially satisfied. The association of the 1D NLS equation with a Lax pair and its equivalence to the flatness condition of the Lax operators may be considered as the criteria of its integrability. In fact the space Lax equation $\Phi_x = U_1(\lambda) \Phi$ can be used to find analytic solutions of the 1D NLS equation through inverse scattering method, yielding also the N-soliton solution as reflection-less potentials soliton. In addition the same equation can be used through the Riccati equation for constructing the infinite set of its conserved quantities $c_j$, $j = 1, 2, 3, \ldots$ given in the explicit form for $x \in [-\infty, +\infty]$ as

$$c_1 = \int dx |q|^2, \quad c_2 = i \int dy (q^*_x q - q^* q_x)$$
$$c_3 = \int dx (q^*_x q_x + |q|^4)$$

and so on. If we consider $c_3 \equiv H$ as the Hamiltonian of the system, one can derive the NLS equation (1), which on the other hand can also be obtained from the flatness condition (5) of the Lax pair $(U_1, V_2)$, as mentioned above. If on the other hand we take the next higher conserved quantity $c_4 \equiv H$ as the Hamiltonian we get the higher order NLS equation

$$q_t + q_{xxx} - 6|q|^2 q_x = 0.$$

Interestingly, the same nonlinear equation can be obtained as the flatness condition of the Lax pair $(U_1(\lambda), V_3(\lambda))$ where $U_1(\lambda)$ is the same space-Lax operator (3), while the time-Lax operator $V_3(\lambda)$ may be expressed in an explicit form as

$$V_3 = i[V_1^{(3)} \sigma_z + V_{12}^{(3)} \sigma_+ + V_{21}^{(3)} \sigma_-]$$

Where

$$V_{11}^{(3)} = [-4\lambda^3 + 2\lambda |q|^2 + i(qq_x^* - q^* q_x)],$$
$$V_{12}^{(3)} = (V_{21}^{(3)})^* = [-4\lambda^2 q + 2i\lambda q_x - 2|q|^2 q + q_{xx}].$$

Similarly the infinite set of conserved quantities $c_j$, $j = 5, 6, \ldots$ can generate infinite number of integrable higher order 1D NLS equations forming the integrable NLS hierarchy. These equations can be shown to be equivalent to the flatness condition of the Lax pair $(U_1(\lambda), V_j(\lambda))$ where $U_1(\lambda)$ is the same space-Lax operator (3), while higher time-Lax operators $V_j(\lambda)$ are the $j$-th order polynomials in parameter $\lambda$ with matrix coefficients containing higher order derivatives and nonlinearity in the basic field (with the scaling dimension of each term being $j$ nonlin). Note however that in spite of the involved structure of the higher NLS equations their soliton solutions have very similar form with the simplest 1-soliton for all the equations given in the form

$$q = \text{sech} \eta(x - vt)e^{i(kx + \omega t)}$$
with only the constant soliton velocity $v$ and the modulation frequency $\omega$ changing for the hierarchal equations in a particular way.

Among many applications of the soliton solutions of the 1D NLS equations the solitonic communication in fiber optics, localized structures in nonlinear plasma, deep water waves are worth mentioning. A rational solution of the 1D NLS, known as the Peregrine breather is often used for modelling the rogue waves rog1D. However Ocean rogue wave being a 2D surface wave, modelling it using a 1D Peregrine breather is indeed unsatisfactory. Consequently, there is a strong need for a 2D analytic rogue wave solution having tunable parameters for amplitude, steepness, velocity etc. We present below the 2D NLS equation recently proposed by us 2DNLSarxiv12,rog12, with a 2D solution as model for the ocean rogue wave, where the above realistic requirements are mostly taken care of.

3 Integrable 2D NLS equation

The need for constructing a 2D generalization of the NLS equation boosted many attempts in this direction. In some studies a straightforward 2D extension of the NLS equation is used in the form 2dNLS

$$iu_t = u_{xx} - du_{yy} + 2|u|^2u$$

(11)

Where $d$ is a constant. This equation however turns out to be a nonintegrable system, with unstable soliton solutions and with solutions obtained only numerically.

Davey-Stewartson equation (DSE) however is a successful attempt for constructing a genuine $(2 + 1)$-dimensional integrable system, one of its several forms may be given by the coupled equations

$$iq_t + c_0q_{xx} + q_{yy} = c_1|q|^2q + c_2q\phi_x,$$

$$\phi_{xx} + c_3\phi_{yy} = (|q|^2)_x.$$  

(12)

with a Lax representation DSE. We may note however that using the first DSE and its conjugate we can express the external potential $\phi_x$ through the basic field $q$ and its derivatives, in a complicated but local way. This could be achieved for example, by multiplying the first equation by $q^*$ and its conjugate by $q$ and then adding them. Thus we could remove the external field from both the equations of the DSE, giving an equation for $q$ in $(2 + 1)$ dimensions together with an additional constraint equation. This feature of higher dimensional integrable equations to have an additional differential constraint seems to be rather universal, as supported by the available examples and also appears in constructing our 2D integrable NLS equation. In some cases additional potential fields can be expressed more simply through the basic fields, though with nonlocal interactions and with dromion fields sitting at the space boundaries DSE.

Another interesting 2D generalization of the NLS equation was proposed by Zakharov zakh:\n
$$iq_t - q_{xy} + Vq = 0, \ V_x = 2(|q|^2)_y$$

(13)

derived as a zero curvature condition from an innovatively constructed Lax pair zakh. As we see from 13 this coupled NLS type equation involves again an additional potential field, which could be related non -locally to the basic NLS field. At the same time the potential field $V$ can also be expressed in local terms through the basic field and thus can be removed from both the equations using the conjugate field, similar to the DSE as explained above. This would result to an equation together with an differential constraint involving all variables $x, y, t$, without any external potential. We will see similar situation for our proposed $(2 + 1)$-dimensional NLS equation.
Through the reduction of self-dual Yang-Mills equation Strachan has introduced a general scheme for constructing higher space-dimensional models and re-derived the Zakharov equation giving a geometrical meaning to it, together with its integrable hierarchies, involving more and more constraint equations.

Recently, a \((2 + 1)\)-dimensional integrable NLS equation was introduced, with local nonlinear interaction and without any external potential field \(2\text{DNLSarxiv12} \). In this 2D NLS equation the traditional cubic \(\text{amplitude-type nonlinearity}\) in \((11)\) is replaced by a \(\text{current-like nonlinearity}\):

\[
iq_t = q_{xx} - dq_{yy} + 2i\bar{q}(j_x - \sqrt{d}j_y)
\]

with the current terms defined as \(j_a = qq^*_a - q^*q_a, \ a \equiv x, y\). Interestingly, the replacement of such local nonlinearity turns the nonintegrable equation \((11)\) to an integrable system \((14)\) with all its characteristic properties like the Lax pair, infinite set of conserved quantities, higher soliton solutions etc., which we will present below sequentially.

Before proceeding further we rewrite equation \((14)\) in a more compact form by rotating the coordinate frame on the plane by an angle \(\frac{\pi}{4}\) together with a scale transformation: \((x, y) \rightarrow (\bar{x}, \bar{y})\) with \(\bar{x} = \frac{1}{\sqrt{2}}(-x + \frac{1}{\sqrt{d}}y), \ \bar{y} = \frac{1}{\sqrt{2}}(x + \frac{1}{\sqrt{d}}y)\) and \(\bar{t} = 2t\) and a scaling of the field, yielding

\[
iq_t + q_{xy} + 2iq(qq^*_x - q^*q_x) = 0
\]

where the \(\text{bar}\) over the coordinates is omitted for the sake of notational simplicity.

### 3.1 Lax operators for the 2D NLS equation

We introduce here the Lax pair associated with the integrable 2D NLS equation \((15)\). Since the idea is to generalize the NLS equation to \(x, y, t\) coordinate variables, preserving the integrable structure of the system, we start by generalizing the pair of Lax equations \((2)\) to

\[
\Phi_y = U_2(\lambda)\Phi, \quad \Phi_t = V_3(\lambda)\Phi,
\]

where we will use the pair \(U_2(\lambda), V_3(\lambda)\) for constructing the 2D generalization of the NLS, by taking them in the same form as in \((13)\), considering \(U_2(\lambda) \equiv V_2(\lambda)\). It is interesting to observed that the Lax operators taken usually in a rather complicated element-wise form as presented above, can be expressed in more compact and interrelated matrix form as

\[
U_2(\lambda) = 2\lambda U_1(\lambda) + U_2^{(0)}, \quad U_1(\lambda) = i(\lambda\sigma^3 + U^{(0)}), \quad V_3(\lambda) = 2\lambda U_2(\lambda) + V_3^{(0)}.
\]

We notice that the higher order Lax operators, finding of which is known to be a difficult task in general, could be simplified by expressing them through those of the lower orders, except the \(\lambda\)-independent terms \(U^{(0)}, U_2^{(0)}, V_3^{(0)}\). We choose these crucial terms as

\[
U_2^{(0)} = \sigma^3(U_2^{(0)} - iU^{(0)})^2, \quad V_3^{(0)} = D(U^{(0)}) - [U^{(0)}, U_2^{(0)}], \quad U^{(0)} = \begin{pmatrix} 0 & q \\ q^* & 0 \end{pmatrix},
\]

where for the standard choice:

\[
D(U^{(0)}) = iU_x^{(0)} + 2iU^{(0)}U_x^{(0)}
\]

we can recover the Lax operator forms including \((8, 9)\), well known for the NLS hierarchy.

We notice however that The \(\lambda\)-independent parts of the Lax operators should be obtained additionally and could be constructed guided by the principle of maintaining the scaling dimensions nonlin, which
therefore seems to give some flexibility in their choice. We will exploit this freedom for constructing a
different Lax operator. In particular by choosing the matrix $D(U^{(0)})$ in $V_3^{(0)}$ differently, we can generate
different constraint equations.

If we start from the linear system (16), considering the Lax operators (18) with the standard choice
(19), the compatibility of the system leading to the flatness condition on the Lax pair: $(U_2, V_3)$ would
produce different equations at different powers of the spectral parameter $\lambda$. At $\lambda^2$ we get a NLS like
equation, involving only variables $x, y$:

$$iq_y = q_{xx} + 2|q|^2q,$$

which being not an evolution equation may be considered as a nonholonomic differential constraint on
the field. At $\lambda$ we obtain the integrable 2D NLS equation (15) together with its complex conjugate and
finally at $\lambda^0$ we get yet another nonlinear equation

$$q_{xt} + q_{yy} + 2i|q|^2q_y + 2q_x(qq_x^* - q^*q_x) = 0,$$

along with

$$i(|q|^2)t + (q^*q_{xy} - q_yq_x) = 0.$$

Note that we may consider the 2D NLS equation (15) as our main equation, together with the differential
constraints (20) and (21), since the additional equation (22) is derivable from equation (15) and its
conjugate and hence not an independent equation. We observe therefore that though we can derive the
integrable 2D NLS equation (15) from the Lax pair $(U_2, V_3)$ it give also two constraint equations (21)
and (20). However since too many constraints for a single equation is an undesirable situation, we wish
to seek now a way for the reduction of such constraints.

Intriguing, if we choose the matrix $D(U^{(0)})$ occurring in (18) different from the standard one (19) by
taking simply

$$D(U^{(0)}) = -U_y^{(0)},$$

preserving the scaling dimension, the constraint (20) disappears miraculously as we wanted and the Lax
pair $U_2, V_3$ with the important modification (23) would yield the 2D NLS equation (15):

$$iq_t + q_{xy} + 2iq(qq_x^* - q^*q_x) = 0$$

together with a single higher order constraint equation (21). Note that using further the evolution
equation (15) this constraint can be reduced to a nonevolutionary type constraint equation involving
only derivatives in $x$ and $y$:

$$iq_{xxy} + q_{yy} + 2i|q|^2q_y - 2q(qq_{xx}^* - q^*q_{xx}) = 0$$

Therefore, we may conclude, that the 2-dimensional integrable NLS equation (15) with local nonlinear
interaction and without introducing any external potentials is possible to construct from an associated
Lax pair as explained above. However this equation has a single differential constraint (24), common also
for the Zakharov equation and the DSE. We will be concerned here mainly with the integrable 2D NLS
equation (15) treating it as an independent equation, since unlike the DSE (12), the Zakharov equation
(13) or the Strachan’s construction Strach, it does not have any potential field linked to the constraints.
3.2 Relation with Zakharov and Strachan construction

Using the twistor space technique Strachan has generated a series of (2+1)-dimensional integrable models belonging to the NLS family, e.g. Zakharov equation having an external potential with one constraint, its higher order flow involving two additional potentials and two constraints etc. as well as a similar 2-dimensional generalization of the derivative NLS equation involving also external potential fields Strach.

In the scheme proposed by Strachan for generating (2+1)-dimensional integrable equations, the set of linear systems for some particular choice of parameters is given by

\[ \Phi_{x_1} = (-\lambda A_0 + D_1)\Phi, \quad \Phi_{x_2} = \lambda \Phi_{x_1} + D_2\Phi, \]
\[ \Phi_{x_3} = \lambda \Phi_{x_2} + D_3\Phi, \quad \Phi_{x_4} = \lambda \Phi_{x_3} + D_4\Phi, \]  

(25)

type etc. For establishing connection of the linear system (25) with our Lax equation (16) given through the Lax operators (17,18), let us take the first three equations in (25) and denote \( V \) equation together with the form in the Strachan series (25) our time-Lax equation \( \Phi_t \)

In the family of integrable equations in (2+1)-dimensions together with their hierarchies constructed by Strachan including the Zakharov equation (13) there involves an external potential field

\[ V = 2\lambda U_1(\lambda) + U_2^{(0)}, \]  
as in (17). In a similar way by replacing \( D_3 = V_3^{(0)} \) and using the second Lax equation together with the form \( V_3(\lambda) = 2\lambda U_2(\lambda) + V_3^{(0)} \) as in (17), one obtains from the third equation in the Strachan series (25) our time-Lax equation \( \Phi_t = V_3(\lambda)\Phi \). This shows an intimate relation of our model with the Strachan construction revealing a geometrical meaning through twistor space formalism and self-duality construction of the (2+1)-dimensional integrable NLS equation (15) we are investigating here. However a note of caution is that the explicit form for our matrices \( V_2^{(0)}, V_3^{(0)} \) containing only the basic fields are different from the Strachan construction \( D_2, D_3 \) involving external potentials.

In the family of integrable equations in (2+1)-dimensions together with their hierarchies constructed by Strachan including the Zakharov equation (13) there involves an external potential field \( V(x,y,t) \), expressed nonlocally through the basic NLS field as \( V(x,y,t) = 2\int_{-\infty}^{x} dy (|q|^2 y + f(y,t)) \), with arbitrary function \( f(y,t) \) defined at \( x \to \) -\( \infty \)-boundary. We can find therefore a link of the Zakharov equation to the 2D NLS (15), if we additionally consider the NLS-like constraint (20) together with its conjugate deriving the relation \( |q|^2 y = i(q q_x^* - q^* q_x) \).

3.3 Infinite set of conserved quantities

Systems with infinite degrees of freedom like the 2D-NLS equation (15), when integrable, should have infinite set of independent conserved quantities. We generate here the related infinite set of conserved charges \( C_n, n = 1, 2, \ldots \) in the explicit form, demonstrating again an important feature of the 2D NLS equation linked to its integrability. In analogy with the 1D NLS equation we start from the linear system (16), but use now the Lax equation along the \( y \)-direction: \( \Phi_y = U_2(\lambda)\Phi \). Note, that for the wave function \( \Phi(\lambda, y) = (\phi, \tilde{\phi}) \), the component

\[ \phi(y, \lambda) = e^{\int_{-\infty}^{y} \rho(\lambda, y') dy'}, \]

with \( \int_{-\infty}^{\infty} dy' \rho(\lambda, y') = \sum_{n=1}^{\infty} C_n \lambda^{-n} \) acts as a generator of the conserved quantities, yielding

\[ \ln \phi(y = \infty, \lambda) = \sum_{n=1}^{\infty} C_n \lambda^{-n}. \]
Therefore using $U_2(\lambda)$ as in (17) or in more explicit form (4) in the Lax equation $\Phi_y = U_2(\lambda)\Phi$, we can build systematically the infinite set of conserved charges: $C_n, \ n = 1, 2, \ldots$ through a recurrence relation giving

$$
C_1 = i \int dq^*q_x - q^*_xq, \ C_2 = \int dq^*(q^*q - q^*_xq_x + |q|^4), \ C_3 = \int dq^*(q^*_xq_x + q^*_yq_y),
$$

and $C_4 = \int dq^* x^2 + y^2 - i|q|^2(q^*q_y - q^*_xq) - 2|q|^2 q^*_xq + (q^*q_x + q^*_xq^2))$, (26)

and so on. We note the involvement of both the space-variables $x, y$ in this series of independent conserved quantities, which also gives another strong argument in favor of the integrability of the 2D nonlinear equation (15). Taking these conserved quantities as Hamiltonians $H \equiv C_n$ we can generate the integrable hierarchy for this 2D NLS equation.

4 Soliton solutions

Recall that the integrable nonlinear equations allowing linear spectral problem can be solved for the general initial value problem by the inverse scattering method, to obtain in particular the exact soliton solutions solit. However this method has been developed mostly for systems like the NLS, KdV, mKdv, sine-Gordon equations etc., belonging to the AKNS spectral problem or like the derivative NLS equation. For higher dimensional equations like DS equation the associated Lax operator on the other hand usually contains no spectral parameter and needs therefore a different treatment for their solution DSÈ. Noticing that the associated spectral problem given through the Lax operator $U_2(\lambda)$ (4) for our 2D NLS equation, does not fall into any of these well known problems, we resort to a direct method through Hirota’s bilinearization for extracting exact solutions to the $(2 + 1)$-dimensional nonlinear equation (15). As it is known, successful application of the Hirota’s bilinearization method should yield an exact 1-soliton and subsequently, through a recursive method the higher soliton solutions can also be obtained. For expressing the 2D NLS equation in the bilinear form we use the standard transformation

$$
q(x, y, t) = \frac{G(x, y, t)}{F(x, y, t)},
$$

where $G(x, y, t)$ and $F(x, y, t)$ are complex and real functions, respectively. Inserting (27) in (15) one derives the pair of bilinear equations:

$$
i(FG_t - GF_t) + (FG_{xy} + GF_{xy} - G_xF_y - G_yF_x) = 0, \quad (28)
$$

$$
2i(GG_x^* - G^*G_x) + 2(F_xF_y - FF_{xy}) = 0. \quad (29)
$$

Following the standard prescription of a formal series expansion:

$$
F = 1 + \epsilon^2 F_2 + \epsilon^4 F_4 + \cdots \quad (30)
$$

$$
G = \epsilon G_1 + \epsilon^3 G_3 + \cdots, \quad (31)
$$

where $\epsilon$ need not be small, we obtain the following equations corresponding to different orders in $\epsilon$.

$$
O(\epsilon) : \quad iG_{1t} + G_{1xy} = 0 \quad (32)
$$

$$
O(\epsilon^2) : \quad 2F_{2xy} = 2i[G_1G_{1x}^* - G_{1x}^*G_{1x}] \quad (33)
$$

$$
O(\epsilon^3) : \quad iG_{3t} + G_{3xy} = i[G_1F_2 - F_2G_1] - [F_2G_{1xy} + G_1F_{2xy} - G_{1x}F_2y - G_{1y}F_{2x}] = 0 \quad (34)
$$

$$
O(\epsilon^4) : \quad 2F_{4xy} = 2i[G_3G_{1x}^* + G_{1x}G_{3x}^* - G_3^*G_{1x} - G_{1x}^*G_{3x}] + 2[F_2F_2y - F_2F_{2xy}] \quad (35)
$$

and similarly higher order equations.
4.1 1-soliton

To construct 1-soliton solution for (15) we assume the ansatz

\[ G_1 = e^{\eta_1}, \eta_1 = k_1 x + p_1 y - w_1 t + \eta_0^1 \]

where \( k_1, p_1, w_1, \eta_0^1 \) are complex constants. From equation (32) therefore one obtains the associated dispersion relation \( w_1 = -ik_1 p_1 \), using which the equation (33) is solved easily to yield

\[ F_2 = i (k_1^* - k_1) \frac{e^{(\eta_1 + \eta_0^1)}}{(p_1 + p_1^*)(k_1 + k_1^*)} . \]  (37)

We can verify using (36) and (37), that all higher order terms in \( \epsilon \) like (34, 35) etc., beyond \( G_1 \) and \( F_2 \) trivially vanish. Absorbing \( \epsilon \) in arbitrary constant \( \eta_0^1 \), we construct from (27) using (36) and (37) the 1 soliton solution in the form

\[ q(x, y, t) = G_1 + F_2 = e^{\eta_1} + \alpha e^{(\eta_1 + \eta_0^1)} \]  (38)

where \( \alpha \) depends on the parameter \( k_1, p_1 \). If additionally we use the constraint equation (24), through its dispersion relation together with that of the main equation we can link the independent parameters as \( p_1 = -ik_1^2 \), simplifying the soliton solution (38) to yield the conventional form

\[ q(x, y, t) = \text{sech} \xi \ e^{i \theta}, \text{ with } \xi = \eta(x + vy + vt), \theta = (k_x x + k_y y + \omega t). \]  (39)

where all the soliton parameters \( \eta, v, \nu, k_x, k_y, \omega \) can be expressed through two independent real spectral parameters \( \lambda = k_x + i \eta \). Note, that the wave front of the 2D soliton along a line travels in time as an exact solution of the \( (2+1) \)-dimensional NLS equation (15). Such solitons are often called a line-soliton. We unfortunately could not find a dromion-like exponentially decaying moving lump-soliton as found in DSE DSE. Note that the Zakharov equation also exhibit no dromion solution as reported in rasha. A frozen picture of the modulus of our travelling soliton solution (39) at time \( t = 0 \) is shown in Fig. 1.

4.2 2-Soliton

For obtaining 2-soliton solution we start with the standard procedure assuming

\[ G_1 = e^{\eta_1} + e^{\eta_2}, \text{ with } \]

\[ \eta_1 = k_1 x + p_1 y - w_1 t + \eta_0^1, \eta_2 = k_2 x + p_2 y - w_2 t + \eta_0^2, \]  (40)
Figure 2: Modulus of 2 soliton with \( k_{1r} = 1, k_{1i} = 1, k_{2r} = 2, k_{2i} = -1, \eta_{1r}^0 = 1, \eta_{1i}^0 = 1, \eta_{2r}^0 = 1, \eta_{2i}^0 = 1 \) and at \( t=2 \) where the parameters involved are complex numbers. Applying similar dispersion relations as earlier we get \( w_1 = -ik_1p_1, w_2 = -ik_2p_2 \) and obtain from (33)

\[
F_2 = [e^{(\eta_1 + \eta_1^* + R_1)} + e^{(\eta_2 + \eta_2^* + \delta_0)} + e^{(\eta_1 + \eta_2^* + \delta_0)}],
\]

(41)

where all the constant parameters can be worked out explicitly (see Appendix I). Similarly equation (34) at higher order expansion gives

\[
G_3 = e^{(\eta_1 + \eta_1^* + \eta_2 + \delta_1)} + e^{(\eta_1 + \eta_2^* + \eta_2 + \delta_2)},
\]

(42)

where the relevant parameter details are given in Appendix II. Using further equation (35) one obtains

\[
F_4 = e^{(\eta_1 + \eta_1^* + \eta_2 + \delta_2)},
\]

(43)

with the relevant parameters presented in Appendix III. Note that in all the expressions of \( F_2, G_3 \) and \( F_4 \) the two-soliton interaction is explicit. For simplifying the expressions, as mentioned earlier, we can use the constraint equation (24), imposing the relations between \( k_1, p_1 \) and \( k_2, p_2 \) as \( p_1 = -ik_2^2, p_2 = -ik_2^2 \) (see Appendix IV). Here we find again, that the higher order terms in \( \epsilon \) beyond \( G_3 \) and \( F_4 \) trivially vanish, leaving the exact 2-soliton solution in the form

\[
q(x, y, t) = \frac{G_1 + G_3}{1 + F_2 + F_4}
\]

(44)

A graphical plot of the modulus of this solution in \((2 + 1)\)-dimensions, frozen at time \( t = 2 \), is shown in Fig. 2, where the 2-soliton as two interacting 1-solitons is clearly seen on a 2D \((x, y)\)-plane.

5 Applicable aspects of integrable 2D NLS model

Similar to the well known integrable 1D NLS equation, we expect the NLS equation (15) in \((2 + 1)\) dimensions to have also diverse applications in different fields. Moreover, since the 2D NLS model can describe more realistic nonlinear phenomena in two-dimensional plane, its importance for physical application should be more significant. A key reason for the applicable success of the 1D NLS model is its link to and derivability from the basic physical equations like the Maxwell equation in electrodynamics and the Euler equation in hydrodynamics. It is therefore encouraging that the 2D NLS equation (15) has also found to be derivable under a certain space-asymmetry in 2D from the Euler equation rog12 as well as from the Maxwell equation with physical conditions relevant to the plasma physics plasma13. We briefly report below few important application of the model found recently by us rog12,light13.
5.1 Ocean rogue wave model based on 2D NLS equation

Ocean rogue waves are extremely high and steep surface waves, appearing suddenly and disappearing equally fast in a relatively calm sea. However due to the lack of satisfactory 2D models the real phenomena as well as experimental observations in various fields are usually attempted to be described by one dimensional models based on the well known 1D NLS equation and most popularly by its 1D Peregrine breather solution

\[ q_P(x, t) = e^{-2it}(u + iv), \quad u = G - 1, \quad v = -4tG, \]
\[ G = 1/F(x, t), \quad F(x, t) = x^2 + 4t^2 + \frac{1}{4}. \]  \hspace{1cm} (45)

The solution (45) represents a breather mode \( \cos 2t \) with unit intensity at both \( t \to \pm \infty \), while at \( t = 0 \) amplitude of the wave rises suddenly, attaining its maximum at \( x = 0 \) as shown in Fig. 3.

Notice however that, the NLS equation (1) together with its different generalizations are equations only in \((1 + 1)\)-dimensions and therefore all of their solutions, including the PB and its higher order generalizations, can describe the time evolution of a wave only along an one dimensional line (as in Fig. 3). Moreover, due to the absence of any free parameter in (45) and its higher order generalizations, the maximum amplitude and steepness of the rogue wave model as well as the duration and the speed of its appearance are all fixed. Therefore describing the actual ocean rogue waves, parameters of which may vary continuously from one event to another, becomes difficult.

Therefore there is an immediate need for a realistic rogue wave model. We have proposed recently a 2D ocean rogue wave model \( q_{2D} \) based on a modification of the integrable 2D NLS equation with the addition of an ocean current term, since the role of ocean currents in the formation of the rogue wave is found to be crucial. This modified 2D nonlinear equation with a specific form of the ocean current is found to yield an interesting 2D dynamical lump solution

\[ q_{P(2D)}(x, y, t) = e^{i4x}(-1 + (1 - i4x)\frac{1}{F(x, y, t)}), \]
\[ F(x, y, t) = 4x^2 + \alpha y^2 + \mu t^2 + c, \]  \hspace{1cm} (46)

which can serve as a satisfactory ocean rogue wave model. The solution (46) forms into a full grown lump rogue wave at \( t = 0 \) (see Fig. 4), while disappearing fast to the background plane wave at distant past and future \( (t \to \pm) \). Comparison with the Peregrine rogue wave (Fig. 3) shows vividly the 2D nature of our rogue model. The presence of arbitrary parameters \( \alpha, c, \mu \) in solution (46) plays crucial roles for
regulating the maximum amplitude, steepness, speed and duration of the rogue wave, which is difficult to achieve in the conventional Peregrine type model.

5.2 Application to Nonlinear Optics

1D NLS equation is a successful model in nonlinear optics. Apart from its soliton based optical communication hasegawa, possibility of achieving the bending of a light beam based on this model is also proposed recently NLtheor. Therefore, a natural expectation is, that the integrable 2D NLS equation admitting stable soliton and other localized solutions should be equally applicable to the nonlinear optics extending the NLS based models to 2D plane. Interestingly, the 2D NLS equation, like its well known 1D counterpart mainistov, is found to retain its integrability when coupled to the self-induced transparency (SIT) equations:

\[
\begin{align*}
    iE_z + E_{yt} + 2iE(E^*E_t - EE^*_t) &= 2p, \\
    ip_t &= 2(NE - \omega_0p), \\
    iN_t &= E^*p - p^*E,
\end{align*}
\]

(47)

opening up a novel possibility of more stable 2D nonlinear optical wave propagation through coherently excited resonant medium doped with Erbium atoms light13. Here \( p(t, y, z) \) is the polarization in the resonant medium induced by the electric field \( E(t, y, z) \) and \( -1 \leq N(t, y, z) \leq +1 \) is the population inversion profile of the dopant atoms. In the set of equations (47) the transverse dimension \( x \) is replaced by time variable \( t \), as customary in nonlinear optics models. The coupled set of 2DNLS-SIT equations (47) turns out to be an integrable system associated with a Lax pair, given by a deformation of the NLS Lax operator: \( V_3(\lambda) + \frac{1}{\lambda}V_{-1} \), where the additional matrix \( V_{-1}(N, p, p^*) \), contains the SIT fields: \( N, p, p^* \).

The set of the coupled integrable equations (47) for the electric field admits soliton solution again in the same form (39), though with a different expression for velocity \( v \) and modulation frequency \( \omega \). Similarly the other fields \( p(t, y, z) \) and \( N(t, y, z) \) also exhibit soliton solutions. Note that the stable loss free soliton solution of (47) for the electric field, valid under the conventional assumption of a constant initial condition for the population inversion: \( N(t \to -\infty, y, z) = N_0 = -1 \), would propagate with a constant velocity and with an invariant shape and is suitable for information transfer through nonlinear media and can have applications in 2D processes where transverse direction also plays a prominent role. However, intriguing in place of constant initial population inversion \( N_0 = -1 \), if we can maintain its initial setting at time \( t \to -\infty \) as an arbitrary function \( -1 \leq N_0(y, z) \leq +1 \), one can obtain an accelerating soliton with variable velocity \( v(y, z) \) and modular frequency \( \omega(y, z) \) light13.
Existence of such accelerating solitons in an integrable system, apparently violating the conventional belief, is explained by the fact that due to nontrivial initial condition, the energy is stored initially and acts without changing the total energy of the system. The application of such novel solitonic features in two-space dimensions could have innovative applications. One of such applications is a possibility for bending of the light beams in 1D and 2D by using integrable NLS-SIT equations have been found recently.

6 Concluding Remarks

The well known NLS equation with wide applications and rich mathematical structure is an integrable system in (1 + 1)-dimensions. A (2 + 1)-dimensional integrable extension of this equation proposed by us recently is investigated here to show similar rich properties for this equation. Apart from analysing its underlying integrable structure associated with a Lax pair and exploring the infinite set of conserved quantities of this system in the explicit form, we have found also an exact 1-soliton solution along with its generalizations and shown the relation of this model with other known integrable models in higher dimensions. One of the common properties of these higher dimensional models, namely the occurrence of nonholonomic differential constraints, is found also to be present in our model.

To obtain exponentially localized dromion-like solutions for the present model, which we have failed to find, as well as to extend its practical applications at par with the well known NLS model, would be important future problems.

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7 Appendix: parametrs related to 2 soliton solution

I. Parameter details for $F_2$

$$e^{R_1} = i \frac{(k^r_1 - k_1)}{(p_1 + p^*_1)(k_1 + k^*_1)}, e^{R_2} = i \frac{(k^r_2 - k_2)}{(p_2 + p^*_2)(k_2 + k^*_2)},$$

$$e^{\delta_0} = i \frac{(k^r_1 - k_2)}{(p_2 + p^*_1)(k_2 + k^*_1)}, e^{\delta_0} = i \frac{(k^r_2 - k_1)}{(p_1 + p^*_2)(k_1 + k^*_2)}$$

II. Parameter details related to $G_3$

$$e^{\delta_1} = \frac{i}{[(k_2 + k^*_2)(p_1 + p^*_1) + (k_1 + k^*_1)(p_2 + p^*_2)]} \frac{(k^r_1 - k_1)(p_2 - p_1)}{(p_1 + p^*_1)} +$$

$$+ \frac{(k^r_1 - k_1)(k_2 - k_1)}{(k_1 + k^*_1)} + \frac{(k^r_2 - k_2)(p_1 - p_2)}{(p_2 + p^*_2)} + \frac{(k^r_2 - k_2)(k_1 - k_2)}{(k_2 + k^*_2)} \right) \] (48)$$

$$e^{\delta_2} = \frac{i}{[(k_1 + k^*_1)(p_2 + p^*_2) + (k_2 + k^*_2)(p_1 + p^*_2)]} \frac{(k^r_2 - k_2)(p_1 - p_2)}{(p_2 + p^*_2)} +$$

$$+ \frac{(k^r_2 - k_2)(k_1 - k_2)}{(k_2 + k^*_2)} + \frac{(k^r_1 - k_1)(p_2 - p_1)}{(p_1 + p^*_1)} + \frac{(k^r_1 - k_1)(k_2 - k_1)}{(k_1 + k^*_1)} \right) \] (49)$$

III. Parameter details for $F_4$

$$e^{R_3} = \frac{1}{(k_1 + k^*_1 + k_2 + k^*_2)(p_1 + p^*_1 + p_2 + p^*_2)} \left\{ (ie^{\delta_2}(k^r_1 - k_1 - k_2 - k^*_2)) +$$

$$+ ie^{\delta_1}(k^r_2 - k_1 - k_2 - k^*_1) + ie^{\delta_1}(k^r_1 + k^*_2 + k_1 - k_2) + ie^{\delta_2}(k^r_1 + k^*_1 + k_2 - k_1) \right\} e^{R_1 + R_2}$$

$$+ \left\{ (k_2 + k^*_2 - k_1 - k^*_1)(p_1 + p^*_1) + (k_1 + k^*_1 - k_2 - k^*_2)(p_2 + p^*_2) \right\} e^{R_1 + R_2}$$

$$+ \left\{ (k_2 + k^*_2 - k_1 - k^*_2)(p_1 + p^*_2) + (k_1 + k^*_2 - k_2 - k^*_1)(p_2 + p^*_1) \right\} e^{\delta_0 + \delta_0} \right\} e^{R_1 + R_2}$$

IV. Simplifying expressions for the parametrs $R_i, \ i = 1, 2, 3.$
For simplifying the expressions we can impose the relations between $k_1$, $p_1$ and $k_2$, $p_2$ as $p_1 = -i k_1^2$, $p_2 = -i k_2^2$, which would yield

$$e^{R_1} = \frac{1}{(k_1 + k_1^*)^2}, e^{R_2} = \frac{1}{(k_2 + k_2^*)^2}, e^{\delta_0} = \frac{1}{(k_1 + k_2^*)^2}, e^{\delta_0^*} = \frac{1}{(k_2 + k_1^*)^2},$$

$$e^{\delta_1} = \frac{(k_1 - k_2)^2}{(k_1 + k_1^*)^2(k_2 + k_1^*)^2}, e^{\delta_2} = \frac{(k_2 - k_1)^2}{(k_2 + k_2^*)^2(k_1 + k_2^*)^2},$$

$$e^{R_3} = \frac{|(k_1 - k_2)|^4}{(k_1 + k_1^*)^2(k_2 + k_2^*)^2(|(k_1 + k_2^*)|^4).}$$