Effects of Turbulent Viscosity on A Rotating Gas Ring Around A Black Hole: The Density Profile of Numerical Simulation

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Abstract

In this paper, we present the time evolution of a rotationally axisymmetric gas ring around a non-rotating black hole using two dimensional grid-based hydrodynamic simulation. We show the way in which angular momentum transport is included in simulations of non-self-gravitating accretion of matter towards a black hole. We use the Shakura-Sunyaev \(\alpha\) viscosity prescription to estimate the turbulent viscosity. We investigate how a gas ring which is initially assumed to rotate with Keplerian angular velocity is accreted on to a back hole and hence forms accretion disc in the presence of turbulent viscosity. Furthermore, we also show that increase of the \(\alpha\) coefficient increases the rate of advection of matter towards the black hole. The density profile we obtain is in good quantitative agreement with that obtained from the analytical results. The dynamics of resulting angular momentum depends strongly on \(\alpha\).

Keywords: viscosity; advection; angular momentum; black hole.

1 Introduction

The main source of energy in many astrophysical objects is accretion which includes different types of binary stars, binary X-ray sources, most probably quasars and active galactic nuclei (AGN). Intensive development of the accretion disk theory begins after the first discovery of the extra solar X-ray sources by Giacconi and his team (Giacconi et al. 1967) in which accretion was the only possible way. There exists a few dominating factors that determine the morphology of an accretion flow. In general, some matter is assumed to rotate in Keplerian orbits inside an accretion disc. But it is only possible when it is in equilibrium, i.e., gravity is balanced by the centrifugal force and gas is sufficiently cold and has only rotational motion. If this is the case, then nothing would have ever happened inside the accretion disc. In this situation, matter would just go on revolving around the compact object forever. However, this is not what happens because of viscosity which transports momentum, and therefore angular momentum. Viscous processes are of great importance in the theory of accretion discs. In order to move radially inwards, the disc material has to lose its angular momentum. Therefore a steady flow of angular momentum radially outwards and of mass radially inwards takes place in the disc. But, the main source of this large amount of viscosity is still a mystery in this subject. The kinematic viscosity \(\eta\) has a dimension of \(L^2/T\), where \(L\) is length and \(T\) is time. Diffusion at the molecular level generates viscosity, and is called the molecular viscosity for which the length \(L\) is comparatively very small. Evidently, the molecular viscosity is microscopic in nature and caused by frictious and dragging interactions of individual neighbouring particles. So, the molecular viscosity fails to explain the cause of advection of matter towards the compact object. We therefore need to identify possible instabilities that can cause a turbulence which generate huge viscosity larger than that caused by the...
molecular viscosity. Hence, it is clear that some kind of macroscopic turbulent viscosity must be present. A significant factor is how fast the angular momentum of accreting matter can be eliminated. However, a very successful idea of parametrizing the viscosity without identifying its source was given in 1973 by Shakura & Sunyaev (hereafter SS73) who first proposed the so-called $\alpha$- parameter to measure the turbulent viscosity which is used to determine the efficiency of angular momentum transport. Later, analytically, it has been shown that the turbulent viscosity is an important parameter that controls the ability of the accreting disc to produce the inward transport of mass and outward transport of angular momentum (Pringle 1981, Frank et al. 2002).

A few works have been done on the time evolution of a rotating viscous gas ring which is used as a testing model for numerical methods developed to simulate accretion discs (Speith & Riffert 1999, Kley 1999). The dynamics of a rotationally symmetric viscous gas ring around a Kerr black hole is calculated in the thin-disk approximation (Riffert 1999). However, because their main objective was much wider, they did not elaborate on the estimations and effects of turbulent viscosity (SS73) on a rotating ring. It is noted that MHD turbulence considered in MRI is obviously a phenomenon that we cannot hope to capture in a hydro simulation, however we can simply introduce a shear viscosity term (SS73) into our numerical scheme to simulate its effect in angular momentum transportation. On the theoretical side, since the pioneering work by SS73 thin disc models, parametrized by the so-called turbulent or $\alpha$ viscosity, in which the gas rotates with Keplerian or sub-Keplerian angular momentum which is transported radially by viscous stresses, have been applied successfully to many numerical works in accretion (Robertson & Frank 1986, Chakrabarti 1990, Chakrabarti & Molteni 1995, Igumenshchev et al. 2000, McKinney & Narayan 2007, Yuan et. al 2012, Giri & Chakrabarti 2012, hereafter GC12). In GC12, we study the $\alpha$ viscosity effects on inflowing sub-Keplerian matter towards a black hole. In that paper we mainly concentrated on the time evolution of matter which was initially assumed to rotate with sub-Keplerian velocity at the outer boundary of our simulation. But, till now, however, we did not study on the time evolution of a ring of matter which is initially assumed to rotate around a black hole with Keplerian velocity. In the present paper, we have shown how the turbulent viscosity plays a key role in advection of the matter towards a black hole from an initial gas ring which is assumed to rotate in its Keplerian velocity. The plan of this paper is in next paragraph.

In the next section, we show how we introduce the governing equations that we have solved by a grid-based finite difference method. In sections 3 and 4, we present the methodology and the results of our simulation, respectively. Finally, in Section 5, we draw concluding remarks.

## 2 Governing Equations

A continuum physical system is described by the laws of conservation of mass, momentum and energy. The conservation of mass of the flow is described by the continuity equation for the density $\rho$ and flow velocity $\mathbf{v}$ which is given by (Landau & Lifshitz 1959)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{1}$$

The momentum conservation are given by the *Navier-Stokes equation*:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P + \mathbf{f}_{\text{external}}, \tag{2}$$

where, $P$ is the gas pressure at each point arising because of the thermal motion of the gas particles and $\mathbf{f}_{\text{external}}$ denotes the external forces like gravity, viscosity, body forces etc. The energy equation for the gas element is given by,

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho \mathbf{v}^2 + \rho E \right) + \nabla \cdot \left[ \frac{1}{2} \rho \mathbf{v}^2 + \rho E + P \right] \mathbf{v} = \mathbf{f} \cdot \mathbf{v} - \nabla \cdot \mathbf{F}_{\text{rad}} - \nabla \cdot \mathbf{q}, \tag{3}$$

where, the terms $\rho \mathbf{v}^2$ and $\rho E$ measure the kinetic energy density and internal energy density respectively.

The mass, momentum and energy conservation equations to describe a 2D axisymmetric inviscid flow around a Schwarzschild black hole in a compact form using non-dimensional units are already presented in
Giri et al. (2010, hereafter G10). The self-gravity of the accreting matter is ignored. Cylindrical coordinate \((x, \phi, z)\) is adopted with the \(z\) axis being the rotation axis of the disc. Instead of using general relativity, we use the well known pseudo-Newtonian potential prescribed by Paczyński & Wiita (1980, hereafter PW80). We use the mass of the black hole \(M_{bh}\), the velocity of light \(c\) and the Schwarzschild radius \(r_g = 2GM/c^2\) as the units of the mass, velocity and distance respectively. From Eqn. 2, in an inertial frame of reference, the modified general form of the equations of the viscous flow (Batchelor 1967) is

\[
\rho \left[ \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right] = -\nabla P + \mathbf{F}_b + \nabla \cdot \tau,
\]

where, \(\mathbf{v}\) is the flow velocity, \(\rho\) is the fluid density, \(P\) is the pressure, \(\tau\) is the stress tensor, and \(\mathbf{F}_b\) represents body forces (per unit volume) acting on the fluid and \(\nabla\) is the Del operator. Typically body forces consist of only gravity forces, but may include other types. Here \(\tau\) is the viscous stress having six mutually independent components. In cylindrical coordinates the components of the velocity vector given by \(\mathbf{v} = (v_x, v_\phi, v_z)\). The six independent components of the viscous stress tensor (LL59) are listed here in cylindrical coordinates, \(\tau_{xx}, \tau_{x\phi}, \tau_{xz}, \tau_{\phi\phi}, \tau_{\phi z} \, & \tau_{zz}\). In general, the component \(\tau_{x\phi}\) is the dominant contributor. So, in this work, we have neglected rest five components. The dimensionless equations governing the viscous flow in our system have been presented in GC12 in great detail and we do not repeat them here. The SSo-model (SS73) introduced a phenomenological shear stress into the equations of motion to model the effects of this turbulence. This shear stress is proportional to \(p\), the total pressure. This shear stress permits an exchange of angular momentum between neighbouring layers. Using SSo prescription, we can assume

\[
\tau_{x\phi} = -\alpha p,
\]

where, \(\alpha\) is a proportionality factor which is assumed not be constant throughout the flow. In this work, we have used this method to quantify turbulent viscosity.

### 3 Methodology and Simulation Setup

The setup of our simulation is the same as presented in G10 and GC12. However, we briefly describe it again for completeness. To model a gaseous ring problem, we consider an axisymmetric rotating ring of flow of gas in the pseudo-Newtonian gravitational field of a black hole of mass \(M_{bh}\) located at the centre in the cylindrical coordinates \([x, \theta, z]\). We assume that at infinity, the gas pressure is negligible and the energy per unit mass vanishes. To mimic the general relativistic effects near the black hole, we assume the gravitational field which is described by PW80,

\[
\phi(r) = -\frac{GM_{bh}}{(r - r_g)},
\]

where, \(r = \sqrt{x^2 + z^2}\), and the Schwarzschild radius is given by,

\[
r_g = \frac{2GM_{bh}}{c^2}.
\]

We also assume a polytropic equation of state for the accreting (or, outflowing) matter, \(P = K\rho^\gamma\), where, \(P\) and \(\rho\) are the isotropic pressure and the matter density respectively, \(\gamma\) is the adiabatic index (assumed to be constant throughout the flow, and is related to the polytropic index \(n\) by \(\gamma = 1 + 1/n\)) and \(K\) is related to the specific entropy of the flow \(s\).

Our computational box occupies one quadrant of the \(x - z\) plane with \(0 \leq x \leq 200\) and \(0 \leq z \leq 200\). We have considered a circular ring (an annulus in 2D) at equatorial plane, i.e. \(xy\) plane. The centre of the ring is taken at \(r_c = 100r_g\). Without any loss of generality, the radius of the ring is taken as \(20r_g\). We have chosen the initial density of the gas inside the ring is \(\rho_{ring} = 1\) for convenience since, in the absence of self-gravity and cooling, the density is scaled out, rendering the simulation results valid for any amount of matter. In the beginning of the simulation, we considered that the matter inside the ring is rotating with it’s Keplerian velocity \((\Omega_k)\) is given by

\[
\Omega_k = \left[\frac{1}{r} \frac{\partial \Phi}{\partial r}\right]^{\frac{1}{2}},
\]
where, $\Phi$ is the gravitational potential. Initially, the radial velocity of the gas particle inside the ring is assumed to be zero, i.e. we assume the ring is rotating it’s Kepelrian orbit. We need the sound speed $a$ (i.e., temperature) of the matter inside the ring. We assumed the sound speed $a$ is low. In order to mimic the horizon of the black hole at the Schwarzschild radius, we placed an absorbing inner boundary at $r = 2.5r_g$, inside which all material is completely absorbed into the black hole. For the background matter (required to avoid division by zero) we used a stationary gas with density $\rho_{bg} = 10^{-6}$ and sound speed (or temperature) the same as that of the gas ring. Hence the incoming matter has a pressure $10^6$ times larger than that of the background matter. All the calculations were performed with $512 \times 512$ cells, so each grid has a size of approximately 0.40 in units of the Schwarzschild radius. All the simulations are carried out assuming a stellar mass black hole ($M = 10M_\odot$). The procedures remain equally valid for massive/super-massive black holes. We carry out the simulations till several thousands of dynamical time-scales are passed. In reality, this corresponds to a few seconds in physical units.

We numerically solve the set of Hydrodynamic equations shown in above using a finite difference method based code which uses the principle of Total Variation Diminishing (TVD) which was originally developed by Harten (1983). TVD is a completely Eulerian numerical scheme, which uses an ensemble of grids to model a fluid. In the astrophysical context, Ryu et al. (1995) developed the TVD scheme to study astrophysical inviscid flows around black holes. We incorporate the viscosity in this TVD code.

4 Numerical Results

In this section we present the results of the simulations with various input viscosity parameters. First, we consider for non-viscous case. Afterwards, the cases for various viscous parameters are investigated. As
mentioned earlier, we chose the outer boundary of the simulation grid at $r = 200$. The specific angular momentum ($\lambda$) of the initial gas ring is chosen to be Keplerian. We stop the simulation at $t = 100$ s. This is more than several thousand times the dynamical time. Thus, the solution has most certainly come out of the transient regime and started exhibiting solution characteristics of its flow parameters. The simulation results will be discussed now.

In order to check the effects of turbulent viscosity, for an optimal case, we run our simulation with large amount of $\alpha = 0.15$. To show how do the matter gradually advect towards the black hole from a initial gas ring, in Fig.1, we show the time evolution of density distribution at equatorial plane for the case where, $\alpha = 0.15$. We have plotted the results at six different times: $t_0$ (0 s), $t_1$ (12.5 s), $t_2$ (25 s), $t_3$ (50 s), $t_4$ (75 s) and $t_5$ (100 s). It is evident that, the matter from the initial ring gradually advected towards the black hole as viscosity helps to transport the angular momentum of the flow outwards. It is interesting to show the redistributed angular momentum profile for the previous viscous flow case. Hence, in Fig.2, we plot the radial distributions of specific angular momentum at equatorial plane at the end of the simulation (i.e. at $t = 100$ s). The angular momentum distribution in our simulated result attains slightly less than it’s standard Keplerian distribution. We note that on the equatorial plane, the distribution become sub-Keplerian and the matter is advected inwards. These effects will be more clear when we run several cases with different $\alpha$.

Now, we run several cases for various viscous parameter $\alpha$ including a run where viscosity in our system is assumed to be zero, i.e $\alpha = 0$. To able to distinguish between numerical and viscous effects, we selected
Figure 3: Variations of the radial density distribution (in normalized units) at equatorial plane with the initial distribution (shown in rectangular shape) centered at $r = 100rg$ with width $20rg$. All plots are taken at $t = 100s$. As the viscosity parameter is increased, the angular momentum is transported outwards and hence matter is advected inwards. Density profiles are drawn for $\alpha = 0, 0.01, 0.05, 0.1&0.15$. 
Figure 4: Changes in the density distribution at $xy$ plane with the change of the viscous parameter $\alpha$ at $t = 100s$. Here, densities are in normalized unit, and radius and velocity are in Schwarzschild unit. All cases are started for the same initial condition (top-left). The $\alpha$ parameters are 0 (top-right), 0.01 (middle-left), 0.05 (middle-right), 0.1 (bottom-left) & 0.15 (bottom-right). The initial density distribution at $t = 0$ is shown in the top left plot.

Various values of $\alpha$ parameters for the simulations, and we inspected different types of result for a wide range of viscous parameter. In Fig.3, we compare the density distribution at the equatorial plane of the flow for various viscous parameter $\alpha$. Each distribution is the time evolution for an initial distribution centered around $r = 100rg$ with width $20rg$. To make the comparison meaningful, all the runs were carried out up to $t = 100s$. Each result is obtained starting with an same initial gas ring with Keplerian rotation as mentioned earlier. The values of $\alpha$ for which the curves are drawn are shown in the figure; $\alpha = 0$, $\alpha_1 = 0.01$, $\alpha_2 = 0.05$, $\alpha_3 = 0.1$ and $\alpha_4 = 0.15$. As the viscosity parameter is increased, the flow behaviour changes dramatically. The matter from the initial gas ring gradually accreting towards the black hole with enhancement of viscous parameter. It is noted that when no viscosity is there, with time evolution, the matter of the initial gas ring just go on revolving at approximately same orbit, although numerical viscosity which is negligible with respect to turbulent viscosity slightly changes the shape of the ring. As turbulent viscosity is enhanced, the angular momentum is transported outwards and this causes in most of the matter moving inwards and eventually accreting towards the black hole. It is also evident that angular momentum carried outwards by a small amount of material.

To make a better representation, we now draw the two dimensional density maps in Fig. 4. In Fig.4, we
Figure 5: Changes in the density distribution at $xz$ plane with the change of the viscous parameter $\alpha$ at $t = 100s$. Here, densities are in normalized unit, and radius and velocity are in Schwarzschild unit. All cases are started for the same initial condition as in (a). The $\alpha$ parameters are (b) 0, (c) 0.01 (d) 0.05 (e) 0.1 & (f) 0.15.
show how the density of matter vary with viscosity through out the $xy$ plane. Though we performed our simulation in $xz$ plane, but, with the help of axisymmetric property which is assumed in our simulation, we present the results in $xy$ plane for better visualization. All cases are started for the same initial condition which is shown in (top-left) of Fig. 4. It is clear that for a large amount of viscosity, the density maximum of the gas ring moves toward the black hole, and at the same time some mass is moving outward, and subsequently the initial gas ring spreads toward larger radii. For the case where $\alpha = 0$, the gas ring (top right in Fig. 4) remain almost unaltered even at the end of the simulation. A very slight change in shape is caused due to the artificial numerical viscosity which is negligible related to the turbulent viscosity present in the numerical code. Though, this artificial viscosity is negligible in contrast of turbulent viscosity.

It is interesting to see the density distributions at $xz$ plane for the previous cases. In Figs. 5(a)-(f), we show the distributions of the density of the flow at the end of our simulation for different values of $\alpha$. For comparison, we keep the the initial distribution (i.e. at $t = 0$) at Fig. 5(a). Each of the results of Fig. 5(b-f) are also shown at $t = 100s$. The values of $\alpha$ are 0.0, 0.01, 0.05, 0.1 and 0.15 for Fig. (b), (c), (d), (e) and (f) respectively. This figure shows that the presence of large amount of viscosity is essential to advect the matter inwards.

5 Summary

Our investigation of the time evolution of the flow dynamics of a rotating gas ring (Keplerian rotation) around a black hole in the presence of viscosity is based on the different values of $\alpha$ of turbulent viscosity. We have shown that, in the absence of viscosity, an initially rotating gas ring revolves in same orbit around a black hole without being advected towards the black hole. We find, however, that in all viscous cases where turbulent viscosity is present, the rotating ring gradually advected towards the black hole and forms a circular disc. The results of our simulations show the good agreement with the standard viscous theory in accretion. We described that the simulations completed with zero viscosity or very low viscosity gave results that were qualitatively different from more viscous calculations. We find that the matter from the initial gas ring move inwards as the viscosity is enhanced and the whole region roughly attains a Keplerian disc. When the viscosity parameter is so high ($\sim 0.15$), the matter reaches up to marginally stable orbit ($\sim 3r_g$) which is close to the black hole and the whole disc becomes a Keplerian disc. So, the action of viscosity on the initial gas ring helps most of the mass moves inwards losing energy and angular momentum and a few percentage of matter moves outward to larger radii in order to take up the transported angular momentum. The radial profiles of density obtained by numerical simulations are in good agreement with standard results. We also found that the condition of standing gas ring, i.e. the matters in same circular orbit, satisfied only in a narrow range of the viscous parameter (0 to $\sim 0.01$). It is evident from our simulation that the momentum conservation is so important in accretion disc formation that may run for thousands of dynamical time scales. The two-dimensional simulation presented here can, of course, be extended to a three-dimensional simulation. In future, we have a plan to do it.

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