Quark matter under strong magnetic fields in the su(3) Nambu–Jona-Lasinio Model

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In the present work we use the mean field approximation to investigate quark matter described by the su(3) Nambu–Jona-Lasinio model subject to a strong magnetic field. We consider two cases: pure quark matter and quark matter in β-equilibrium possibly present in magnetars. The results are compared with the ones obtained with the su(2) version of the model. The energy per baryon of magnetized quark matter becomes more bound than nuclear matter made of iron nuclei, for B around 2 × 10¹⁵ G. When the su(3) NJL model is applied to stellar matter, the maximum mass configurations are always above 1.45 M☉ and may be as high as 1.86 M☉ for a central magnetic field of 5 × 10¹⁸ G. These numbers are within the masses of observed neutron stars.

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I. INTRODUCTION

In non-central heavy ion collisions such as the ones performed at RHIC and LHC-CERN, physicists have been looking for a possible signature of the presence of CP-odd domains in the presumably formed quark-gluon plasma phase [1]. The study of deconfinement quark matter subject to strong external magnetic fields is then mandatory if one intends to understand the physics taking place in such colliders.

Neutron stars with very strong magnetic fields of the order of 10¹⁴ − 10¹⁵ G are known as magnetars and they are believed to be the sources of the intense gamma and X rays detected in 1979 [2, 3]. The hypothesis that some neutron stars are constituted by unbound quark matter cannot be completely ruled out [4] since the Bodmer-Witten conjecture [5] cannot be tested on earthly experiments. This conjecture implies that the true ground state of all matter is (unbound) quark matter because theoretical predictions show that its energy per baryon at zero pressure is lower than ⁵⁶Fe binding energy.

In the present work our aim is to investigate quark matter described by the su(3) version of the Nambu–Jona-Lasinio [6] model exposed to strong magnetic fields. In the case of pure quark matter, as predicted by the QCD phase transition possibly taking place in heavy ion collisions, the magnetic field is certainly external. In the case of neutron stars, the magnetic field can be generated by the alignment of charged particles that are spinning very rapidly. We next use an external field to mimic the real situation, which we do not know how to determine. Albeit in an approximate way, the effect of the magnetic field on the macroscopic quantities as radius and masses can be obtained.

Recently the su(2) version of the NJL model was used to treat both situations described above [6]. We have shown that, for pure quark matter, the energy per baryon for magnetized quark matter has a minimum which is lower than the one determined for magnetic free quark matter. We have also obtained that a magnetic field of the order of 2 × 10¹⁵ G barely affects the effective mass as compared with the results for matter not subject to the magnetic field. For B = 5 × 10¹⁵ G matter is totally polarized for chemical potentials below 490 MeV. For small values of the magnetic fields the number of filled Landau levels (LL) is large and the quantisation effects are washed out, while for large magnetic fields the chiral symmetry restoration occurs for smaller values of the chemical potentials. When β-equilibrium is enforced, the numerical results show that, for the the su(2) case, only very high magnetic fields (B ≥ 10¹⁸ G) affect the equation of state (EOS) in a noticeable way.

The inclusion of the s-quarks, necessary in the su(3) NJL model, poses some new numerical difficulties and some questions that need to be addressed. Those problems are tackled through out the paper. One of the questions was raised in [8, 9] and refers to the stability of quark matter described by the NJL model. The authors show that it is not absolute stable. As already mentioned, in [8] we have seen that the inclusion of the magnetic field increases stability in the su(2) version and the same behavior is expected in the su(3) NJL, which is shown next.

The paper is organized is such a way that all calculations already shown explicitly in [6] are not repeated but all important differences are outlined. In sections II and III the formalism (mean field theory) and the equations of state are shown and in section IV the final results are displayed and the conclusions are drawn.

II. GENERAL FORMALISM

In order to consider (three flavor) quark stars in β equilibrium with strong magnetic fields one may define the following lagrangian density

\[
\mathcal{L} = \mathcal{L}_f + \mathcal{L}_l - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1}
\]

where the quark sector is described by the su(3) version of the Nambu–Jona-Lasinio model

\[
\mathcal{L}_f = \bar{\psi}_f [\gamma_\mu (i\partial^\mu - q_f A^\mu) - \hat{m}_c] \psi_f + \mathcal{L}_{sym} + \mathcal{L}_{det} \tag{2}
\]
where \( L_{\text{sym}} \) and \( L_{\text{det}} \) are given by:

\[
L_{\text{sym}} = G \sum_{a=0}^{8} \left( (\bar{\psi} f \lambda_a \psi f)^2 + (\bar{\psi} i \gamma_5 \lambda_a \psi f)^2 \right),
\]

\[
L_{\text{det}} = -K \left\{ \det f \left[ \bar{\psi} f (1 + \gamma_5) \psi f \right] + \det f \left[ \bar{\psi} f (1 - \gamma_5) \psi f \right] \right\},
\]

where \( \psi_f = (u, d, s)^T \) represents a quark field with three flavors, \( \tilde{m}_c = \text{diag}(m_u, m_d, m_s) \) is the corresponding (current) mass matrix while \( q_f \) represents the quark electric charge and \( \lambda_a \) denotes the Gell-Mann matrices. Here, we consider \( m_u = m_d \neq m_s \). The \( L_{\text{det}} \) term is the t’Hooft interaction which represents a determinant in flavor space which, for three flavors, gives a six-point interaction \[1]\]

\[
\det f (\bar{\psi} f \Omega \psi f) := \sum_{\epsilon_{ijk}} \epsilon_{ijk} (\bar{u} \bar{O} \psi_i) (\bar{d} \bar{O} \psi_j) (\bar{s} \bar{O} \psi_k),
\]

where \( \epsilon_{ijk} \) is the usual three-dimensional Levi-Civita symbol. The lagrangian also contains the \( L_{\text{sym}} \) term which is symmetric under global \( U(N_f)_L \times U(N_f)_R \) transformations and corresponds to a 4-point interaction in flavor space. In the appendix we discuss the steps to obtain \( L_f \) in the mean-field approximation (MFA).

The leptonic sector is given by

\[
L_l = \bar{\psi}_l \left[ \gamma_\mu \left( i \partial_\mu - q_i A_\mu \right) - m_l \right] \psi_l,
\]

where \( l = e, \mu \). One recognizes this sector as being represented by the usual QED type of lagrangian density. As usual, \( A_\mu \) and \( F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) are used to account for the external magnetic field. Then, since we are interested in a static and constant magnetic field in the \( z \) direction, \( A_\mu = \delta_\mu \sigma_1 B \).

### III. THE EOS

We need to evaluate the thermodynamical potential for the three flavor quark sector, \( \Omega_f \), which as usual can be written as \( \Omega_f = -P_f = \mathcal{E}_f - \sum \mu_f \rho_f \) where \( P_f \) represents the pressure, \( \mathcal{E}_f \) the energy density, \( T \) the temperature, \( S \) the entropy density, and \( \mu_f \) the chemical potential.

For the present study, just the zero temperature case is important and, as a consequence, the term with the entropy vanishes. The total pressure for three flavor in \( \beta \) equilibrium is given by

\[
P(\mu_f, \mu_l, B) = P_f^N |_{M_f} + P_f^N |_{M_l} + \frac{B^2}{2},
\]

where our notation means that \( P_f^N \) is evaluated in terms of the quark effective mass, \( M_f \), which is determined in a (nonperturbative) self consistent way while \( P_f^N \) is evaluated at the leptonic bare mass, \( m_l \). The term \( B^2/2 \) arises due to the electromagnetic term \( F_{\mu \nu} F^{\mu \nu}/4 \) in the original lagrangian density. The subscript \( N \) indicates normalized pressures. Here, our normalization choice is such that \( P_f^N = 0 \) at \( \mu_f = 0 \) (\( f = u, d, s \)) and \( P_f^N = 0 \) at \( \mu_l = 0 \) (\( l = e, \mu \)) implying that \( P(0, 0, B) = B^2/2 \).

### A. Quark Contribution to the EOS

In the mean field approximation the pressure can be written as

\[
P_f = \theta_u + \theta_d + \theta_s - 2G(\phi_u^2 + \phi_d^2 + \phi_s^2) + 4K \phi_u \phi_d \phi_s ,
\]

where an irrelevant term has been discarded. The pressure due to the three quarks is diagrammatically represented in figure 1a.

![Diagram](https://via.placeholder.com/150)

**FIG. 1**: a) Feynman diagrams contributing to the quark pressure in the MFA. The lines represent the three dressed quark propagators for the different flavors: \( u \) (continuous line), \( d \) (dashed line) and \( s \) (dotted line). The black dot represents \( G \) and the black hexagon represents \( K \). b) Diagrammatic representation of the effective mass for flavor \( u \). The diagrams contributing to the other two flavor display the same topology.

For a given flavor, the \( \theta_f \) term is given by

\[
\theta_f = -\frac{i}{2} \text{tr} \int \frac{d^4 p}{(2\pi)^4} \ln \left( -p^2 + M_f^2 \right)
\]

and the condensates, \( \phi_f \) are given by

\[
\phi_f = \langle \bar{\psi} f \psi f \rangle = -i \int \frac{d^4 p}{(2\pi)^4} \text{tr} \frac{1}{(\not{p} - M_f + i\epsilon)}
\]

where all the traces are to be taken over color (\( N_c = 3 \)) and Dirac space, but not flavor. In order to obtain results valid at finite \( T \) and \( \mu \) in the presence of an external magnetic field \( B \) one can use the following replacements

\[
p_0 \rightarrow i(\omega_\nu - i\mu_f),
\]

\[
p^2 \rightarrow p^2 + (2n + 1 - s),
\]

with \( s = \pm 1 \) , \( n = 0, 1, \ldots \)

\[
\int \frac{d^4 p}{(2\pi)^4} \rightarrow \frac{|T| q_f |B|}{2\pi} \sum_{\nu = -\infty}^{\infty} \sum_{n=0}^{\infty} \int \frac{dp_z}{(2\pi)^4}.
\]

In the above relations, \( \omega_\nu = (2\nu + 1)\pi T \), with \( \nu = 0, \pm 1, \pm 2, \ldots \) representing the Matsubara frequencies for
fermions while \(n\) represents the Landau levels (LL) and \(s\) represents the spin states which, at \(B \neq 0\), must be treated separately. The case \(T = 0\) in which we are interested can be easily obtained after the above substitutions (see Ref. [7]).

The effective quark masses can be obtained self-consistently from (see figure 1b)

\[
M_i = m_i - 4G \phi_i + 2K \phi_j \phi_k, \tag{11}
\]

with \((i, j, k)\) being any permutation of \((u, d, s)\). So, to determine the EOS for the \(su(3)\) NJL at finite density and in the presence of a magnetic field we need to know the condensates, \(\phi_f\), as well as the contribution from the gas of quasiparticles, \(\phi_f\). Both quantities, which are related by \(\phi_f \sim d\theta_f/dM_f\), have been evaluated with great detail in Ref. [7]. Here, we just quote the results

\[
P_f = \left( \phi_f^{vac} + \phi_f^{mag} + \phi_f^{med} \right)_{M_f}, \tag{12}
\]

where the vacuum contribution reads

\[
\phi_f^{vac} = -\frac{N_c}{8\pi^2} \left\{ M_f^2 \ln \left[ \frac{(\Lambda + \epsilon_\Lambda)}{M_f} \right] - \epsilon_\Lambda \left( \Lambda^2 + \epsilon_\Lambda^2 \right) \right\}, \tag{13}
\]

where we have defined \(\epsilon_\Lambda = \sqrt{\Lambda^2 + M_f^2}\) with \(\Lambda\) representing a non-covariant ultra violet cut off. The evaluations performed in Ref. [7] also give the following finite magnetic contribution

\[
P_f^{mag} = \frac{N_c |q_f| B^2}{2\pi^2} \left[ \zeta'(1, x_f) - \frac{1}{2}(x_f^2 - x_f) \ln x_f + \frac{x_f^2}{4} \right], \tag{14}
\]

where \(x_f = M_f^2/(2|q_f| B)\) while \(\zeta'(1, x_f) = d\zeta(z, x_f)/dz\big|_{z=1}\) where \(\zeta(z, x_f)\) is the Riemann-Hurwitz zeta function [10]. Finally, after integration, the medium contribution can be written as

\[
P_f^{med} = \sum_{k=0}^{k_f, max} \alpha_k \frac{|q_f| B N_c}{4\pi^2} \left[ \mu_f \sqrt{\mu_f^2 - s_f(k, B)^2} \right] \tag{15}
\]

where \(s_f(k, B) = \sqrt{M_f^2 + 2|q_f| B k}\), \(\alpha_0 = 1\), \(\alpha_{k>0} = 2\). The upper Landau level (or the nearest integer) is defined by

\[
k_f, max = \frac{\mu_f^2 - M_f^2}{2|q_f| B} = \frac{p_{f, vac}^2}{2|q_f| B}. \tag{16}
\]

Finally, the condensates \(\phi_f\) entering the quark pressure at finite density and in the presence of an external magnetic field can also be written as

\[
\phi_f = (\phi_f^{vac} + \phi_f^{mag} + \phi_f^{med})_{M_f}, \tag{17}
\]

where

\[
\phi_f^{vac} = -\frac{M_f N_c}{2\pi^2} \left[ \Lambda \epsilon_\Lambda - M_f^2 \ln \left( \frac{(\Lambda + \epsilon_\Lambda)}{M_f} \right) \right], \tag{18}
\]

\[
\phi_f^{mag} = -\frac{M_f |q_f| B N_c}{2\pi^2} \ln \Gamma(x_f) - \frac{1}{2}\ln(2\pi) + x_f - \frac{1}{2}(2x_f - 1) \ln(x_f) \tag{19}
\]

and

\[
\phi_f^{med} = \sum_{k=0}^{k_f, max} \alpha_k \frac{M_f |q_f| B N_c}{2\pi^2} \left[ \mu_f + \sqrt{\mu_f^2 - s_f(k, B)^2} \right] \tag{20}
\]

From the pressure one can obtain the density, \(\rho_f\), corresponding to each different flavor, which is given by

\[
\rho_f = \sum_{k=0}^{k_f, max} \alpha_k \frac{|q_f| B N_c}{2\pi^2} \epsilon_{F, f} \tag{21}
\]

where \(\epsilon_{F, f} = \sqrt{\mu_f^2 - s_f(k, B)^2}\), since \(dP/d\phi_f = 0\).

The quark contribution to the energy density is

\[
\mathcal{E}_f(\mu_f, B) = -P_f^{N} + \sum_f \mu_f \rho_f, \tag{22}
\]

where \(P_f^{N} = P_f(\mu_f)|_{M_f(\mu_f)} - P_f(0)|_{M_f(0)}\).

Throughout this paper we consider the following set of parameters [11]: \(\Lambda = 631.4\text{ MeV}\), \(m_u = m_d = 5.5\text{ MeV}\), \(m_s = 105.66\text{ MeV}\), \(G^2 = 1.835\) and \(\Lambda^5 = 9.29\).

B. Lepton Contribution to the EOS

The leptonic contribution, \(P_l\) has also been evaluated in detail in Ref. [2] where the normalization requirement \(P_l^N = 0\) at \(\mu_l = 0\) has been adopted. The result shows that, at the one loop level, only the following (finite) medium contribution has to be considered

\[
P_f^{N} = \sum_{l=e}^{\mu} \sum_{k=0}^{k_{l, max}} \alpha_k \frac{|q_f| B}{4\pi^2} \left[ \mu_l \sqrt{\mu_l^2 - s_l(k, B)^2} - s_l(k, B)^2 \ln \left( \frac{\mu_l + \sqrt{\mu_l^2 - s_l(k, B)^2}}{s_l(k, B)} \right) \right]. \tag{23}
\]
Then, the leptonic density is also easily evaluated yielding
\[
\rho_l = \sum_{k=0}^{k_{l,\text{max}}} \alpha_k \frac{|q_l| B}{2\pi^2} k_{F,l}(k, s_l),
\]  
(24)
where \(k_{F,l}(k, s_l) = \sqrt{\mu_l^2 - s_l(k, B)^2}\). Finally, the leptonic energy density reads
\[
\mathcal{E}_l(\mu_l, B) = -P_{l}N + \sum_l \mu_l \rho_l.
\]  
(25)

The lepton masses are \(m_e = 0.511\) MeV and \(m_\mu = 105.66\) MeV.

### IV. RESULTS AND CONCLUSIONS

In the sequel we consider two different situations of quark matter under a strong magnetic field: a) pure quark matter with the same chemical potential for all quark flavors; b) \(\beta\)-equilibrium quark stellar matter.

We first discuss the properties of pure quark matter with equal chemical potentials for all flavors, namely the behavior of the dynamical quark masses, the chiral symmetry restoration with density and the energy per baryon. In Fig. 2 we display the masses of quarks \(u\) and \(d\) as function of the chemical potential for different values of the magnetic field and the two versions of the NJL model. For the magnetic field intensities used, one can clearly identify the filling of different Landau levels causing the usual kinks in the curves. For the three intensities considered the chiral symmetry is approximately restored for \(\mu = 400\) MeV.

It is interesting to see that although the general behavior is the same, the effect of the LL is more pronounced in the \(su(2)\) version.

In Fig. 3 the mass of the \(s\) quark is shown as a function of the chemical potential for different values of the magnetic field. One can see how drastically it falls around \(\mu = 450\) MeV. For magnetic free quark matter, this is the same behavior shown in Fig. 3 of [12]. One can observe that the curve is no longer smooth when \(B\) is turned on, but the values of the strange quark mass do not vary much. According to [12], the fact that the strange quark mass remains relatively high as compared with the masses of the other two quarks is the main reason why deconfined quark matter may not be likely to appear in the core of hybrid neutron stars. For a magnetic field larger than \(10^{19}\) the restoration of chiral symmetry for the \(s\)-quark occurs in steps and starts at a smaller chemical potential than the \(B=0\) case.

The phenomenon of magnetic catalysis, which enhances chiral symmetry breaking, has been well discussed within the \(su(2)\) version of the NJL model [14]. Here, for reference, we show the vacuum effective mass of the three quarks as a function of the magnetic field in Fig. 4. For \(B > 10^{19}\) G the vacuum masses increase dramatically with the magnetic field as expected. A similar increase of the vacuum mass was also obtained for the \(su(2)\) version of NJL in [14] and the effect is related to the fact that the B field facilitates the binding by antialigning the helicities of the quark and the antiquark, which are then bound by the NJL interaction. As shown in Fig. 4 an interesting result of the \(su(3)\) version is that, due to its larger electric charge, the \(u\) quark has an effective mass that becomes larger than that of the \(s\) quark for \(B > 5 \times 10^{20}\) G.

In Fig. 5 the baryonic density is shown as a function of the magnetic field for \(B = 0, 10^{19}\) and \(2 \times 10^{19}\) G within \(su(3)\) NJL.

![FIG. 2: Mass of the quarks (a) \(u\) and (b) \(d\) as a function of the chemical potential for \(B = 0, 10^{19}\) and \(2 \times 10^{19}\) G within the \(su(2)\) NJL and \(su(3)\) NJL.][2]

![FIG. 3: Mass of the \(s\) quarks as a function of the chemical potential for \(B = 0, 10^{19}\) and \(2 \times 10^{19}\) G within \(su(3)\) NJL.][3]
FIG. 4: Vacuum mass of the quarks as a function of the magnetic field B.

of the number of the filled LL and the amplitude of the oscillations is more clear in the graphics. For each value of the magnetic field, the kink appearing at the smallest chemical potential corresponds to the case when only the first LL has been occupied.

FIG. 5: Baryonic density as a function of the quark chemical potential for $B = 0$, $10^{19}$ and $2 \times 10^{19}$ G within both $su(2)$ and $su(3)$ NJL.

In Fig. 6 one can see that the inclusion of the magnetic field makes matter more and more bound in both versions of the model. For the present set of parameters, the energy per baryon $E/A$ of magnetized quark matter becomes more bound than nuclear matter made of iron nuclei, $E/A_{Fe} \sim 930$ MeV for $B$ around $2 \times 10^{19}$ G.

We next consider stellar matter made out of quarks, electrons and muons in $\beta$-equilibrium, as possibly occurring in the interior of magnetars. It is worth mentioning that, in this case, the three different quarks bear different chemical potentials, determined by the chemical equilibrium conditions

$$\mu_d = \mu_s = \mu_u + \mu_e, \quad \mu_\mu = \mu_e.$$ 

We start by plotting the quark effective masses for different values of the magnetic field in Fig. 7. It is seen that the results for non-magnetized matter ($B = 0$) almost coincide with the ones obtained for $B = 10^{18}$G. A decrease of the $s$ quark mass starts only at $\rho \sim 0.8$ fm$^{-3}$. This behavior had already been discussed in [13]. If the magnetic field is strong enough the mass of quark $s$ occurs in finite jumps which may give rise to an increase of the strangeness fraction as shown in Fig. 8.

FIG. 6: Energy per nucleon as a function of density for $B = 0$, $10^{19}$ and $2 \times 10^{19}$ G within NJL $su(2)$ and NJL $su(3)$.

FIG. 7: The quark effective mass for $\beta$-equilibrium quark matter with a constant magnetic field within NJL $su(3)$.

The quark fractions $Y_i = \rho_i/\rho$, $i = u,d,s$ are shown in Fig. 8 Again the results for $B = 0$ are similar to the ones for $B = 10^{18}$G. For strong enough fields the quark $u$ fractions increase with a reduction of the quark $d$ fraction. The quark $s$ fraction has a sudden increase for $\rho \sim 0.7$ fm$^{-3}$ but above $\rho \sim 0.9$ fm$^{-3}$ remains below the $B = 0$ fraction.

FIG. 8: Fraction of quarks in $\beta$-equilibrium quark matter for a constant magnetic field within NJL $su(3)$.
In Fig. 9 the EOS for different values of the magnetic field is shown. For magnetic fields as large as $B = 10^{18} \text{G}$ the differences are very small as compared with non-magnetized matter. For larger fields there is an overall net softening of the EOS.

\[ B_i = B_{\text{surf}}^i + B_0 \left[ 1 - \exp \left\{ -\beta \left( \frac{\rho_i}{\rho_0} \right) ^\gamma \right\} \right], \quad (26) \]

where $B_{\text{surf}}^i = 10^{15} \text{G}$ is the magnetic field at the surface, $B_i$ is the magnetic field at the interior of the star for large densities and the parameters $\alpha = 5 \times 10^{-5}$ and $\gamma = 3$ were chosen in such a way that the field increases fast with density to its central value but still describes correctly the surface of the star where the pressure is zero. We show the equations of state for quark matter in $\beta$-equilibrium and a density dependent magnetic field within both versions of the NJL model in Fig. 10. As implicit in eq. (20), the field at the surface is $10^{15} \text{G}$. The magnetic field makes the EOS harder with consequences in the gravitational and baryonic masses of compact stars, whose properties are obtained from the integration of the Tolman-Oppenheimer-Volkoff equations, which use as input the EOS obtained with the density dependent magnetic field. The results are displayed both in Fig. 11 and in Table I from where it is seen that both the gravitational and the baryonic masses increase with the increase of the magnetic field for an intensity larger than $\sim 5 \times 10^{18} \text{G}$ for the $su(3)$ version and $10^{18} \text{G}$ for the $su(2)$ NJL. However, the increase of the gravitational mass is larger than the increase of the baryonic mass because the contribution of the magnetic field becomes more and more important as the field increases. This explains the decrease of the central energy/baryonic density for the stronger fields considered.

Another important effect of the field on the properties of the stars is the increase of the radius of the star with the largest radius, which may be as high as 9.5 Km for the $su(3)$ NJL. In general, the maximum mass star configurations for the $su(2)$ version of the NJL model are smaller with smaller radius, $\sim 7$ Km, in average 2 Km smaller than the corresponding stars in the $su(3)$ version of the NJL model.

Within the $su(3)$ NJL the maximum mass configurations are always above $1.45 M_\odot$ and may be as high as $1.86 M_\odot$ for a central magnetic field of $5 \times 10^{18} \text{G}$. These numbers are within the masses of observed neutron stars. On the other hand the $su(2)$ version of the NJL model foresees too small star masses except for very large magnetic fields.

The effects of the anomalous magnetic moments has been shown to be relevant [16, 17, 18] and we intend to take them into account in the next calculations. The color superconductivity (CS) [19], which allows the quarks near the Fermi surface to form Cooper pairs that condense and break the color gauge symmetry [20] is known to be present in the QCD phase diagram at sufficiently high densities. The effect of strong magnetic fields on the CS properties of quark matter, which can be drastic for sufficiently high fields, has already been studied by several authors [21]. It would be important to investigate how this SC phase could affect the properties of quark stars under strong magnetic fields. However, it could be that CS is only affected by magnetic fields.
stronger than the ones considered in the present paper, which, however, predicts already a very high maximum mass, \( M \sim 1.9 \) M\(_{\odot}\). The largest magnetic field we got in the center of a quark star is \( 5 \times 10^{18} \) G, while in \[21\] it is shown that a noticeable effect requires fields above \( \sim 10^{19} \) G.

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**APPENDIX A: THE SU(3) NJL MODEL IN THE MFA**

In this appendix the main steps in order to obtain the Nambu-Jona-Lasinio lagrangian given eq.\[2\] in the mean field approximation are explicitly shown. Firstly, we consider the \( \mathcal{L}_{\text{sym}} \) term given in eq.\[3\]. For later convenience, we define the matrix elements of \( \Phi \) and its adjoint \( \Phi^\dagger \) as \[22\],

\[
\Phi_{ij} = \bar{\psi}_j(1 - \gamma_5)\psi_i, \quad \Phi_{ij}^\dagger = \bar{\psi}_j(1 + \gamma_5)\psi_i,
\]

where \( i, j \) are flavor labels. From these definitions, one can easily show that:

\[
\bar{\psi}_f(1 - \gamma_5)\lambda_a\psi_f = tr(\lambda_a\Phi), \quad \bar{\psi}_f(1 + \gamma_5)\lambda_a\psi_f = tr(\lambda_a\Phi^\dagger),
\]

where \( tr \) is the trace operator in flavor space. So, adding and subtracting these expressions, we can rewrite the NJL symmetric four-point interaction term as:

\[
\mathcal{L}_{\text{sym}} = G \sum_{a=0}^8 \left[ (\bar{\psi}_f\lambda_a\psi_f)^2 + (\bar{\psi}_f i\gamma_5\lambda_a\psi_f)^2 \right]
\]

\[
= G \sum_{a=0}^8 tr(\lambda_a\Phi) tr(\lambda_a\Phi^\dagger) = 2G tr(\Phi\Phi^\dagger). \tag{A2}
\]

The summation involved in the latter equality can be performed noting that an arbitrary matrix \( A \) in the \( N_f=3 \) flavor space, can be expanded in terms of Gell-Mann matrices as follows:

\[
A = \sum_{a=0}^8 c_a\lambda_a , \text{ with } c_a = \frac{1}{2} tr(\lambda_a A). \tag{A3}
\]

The expansion coefficients \( c_a \) are obtained using the the Gell-Mann matrices property: \( tr(\lambda_a\lambda_b) = 2\delta_{ab} \). So, we can write:

\[
tr(\lambda_A\lambda_B) = tr(\sum_{a=0}^8 c_a\lambda_a \sum_{b=0}^8 c_b\lambda_b) = \sum_{a=0}^8 tr(\lambda_a A) tr(\lambda_a A^\dagger), \tag{A4}
\]

where in the latter term we have used that the Gell-Mann matrices are hermitian, i.e., \( \lambda_a = \lambda_a^\dagger \). We then evaluate \( \mathcal{L}_{\text{sym}} \) in the mean field approximation linearizing the interaction terms. We follow refs \[11, 22\] approximating the product of two operators \( \hat{O}_1 \) and \( \hat{O}_2 \) by:

\[
\hat{O}_1\hat{O}_2 \approx \hat{O}_1\langle\hat{O}_2\rangle + \langle\hat{O}_1\rangle\hat{O}_2 - \langle\hat{O}_1\rangle\langle\hat{O}_2\rangle. \tag{A5}
\]

Therefore, calculating explicitly the trace involved in eq.\( A2 \) and taking into account the prescription above, \( \mathcal{L}_{\text{sym}} \), can be written in the MFA as:

\[
\mathcal{L}_{\text{sym}} = 4G \left[ \phi_u u^\dagger u + \phi_d d^\dagger d + \phi_s s^\dagger s - \frac{1}{2}(\phi_u^2 + \phi_d^2 + \phi_s^2) \right], \tag{A6}
\]

where we have used:

\[
\langle\bar{\psi}_i\psi_j\rangle = \delta_{ij}\phi_i \quad \text{and} \quad \langle\bar{\psi}_i\gamma_5\psi_j\rangle = 0. \tag{A7}
\]

The only 3 non-vanishing terms are the condensates which were defined in eq.\[11\]. Finally, we consider the 't Hooft term, eq.\[4\], which is a six-point interaction in the \( su(3) \) flavor space. Notice that term involves the product of three operators which we linearize analogously to eq.\[A5\]:

\[
\hat{O}_1\hat{O}_2\hat{O}_3 \approx \hat{O}_1\langle\hat{O}_2\rangle\langle\hat{O}_3\rangle + \langle\hat{O}_1\rangle\hat{O}_2\langle\hat{O}_3\rangle + \langle\hat{O}_1\rangle\langle\hat{O}_2\rangle\hat{O}_3 - 2\langle\hat{O}_1\rangle\langle\hat{O}_2\rangle\langle\hat{O}_3\rangle. \tag{A8}
\]

So, in the MFA the determinants which appear in the 't Hooft term can be written as:

**TABLE I: Maximum mass configurations for NJL \( su(3) \) and \( su(2) \), and several magnetic field intensities: the gravitational mass \( (M) \), baryonic mass \( (M_b) \), radius \( (R) \), central energy density \( (\epsilon_c) \), baryonic density \( (\rho_c) \) and magnetic field \( (B_c) \) are given**

| \( B_c \) (G) | \( M \) (M\(_{\odot}\)) | \( M_b \) (M\(_{\odot}\)) | \( R \) (Km) | \( \epsilon_c \) (fm\(^{-4}\)) | \( \rho_c \) (fm\(^{-3}\)) | \( B_c \) (G) |
|----------------|-----------------|-----------------|--------|-----------------|-----------------|--------|
| \( su(3) \)    | 0               | 1.46            | 1.53   | 8.93            | 7.49            | 1.19   | \( 10^{15} \) |
| \( su(2) \)    | 0               | 1.29            | 1.24   | 7.09            | 13.68           | 1.86   | \( 10^{15} \) |
| \( 10^{18} \)  | 1.46            | 1.54            | 8.88   | 7.94            | 1.24            | 8.8    | \( 10^{17} \) |
| \( 5 \times 10^{18} \) | 1.50           | 1.58            | 8.78   | 8.36            | 1.25            | 1.8    | \( 10^{18} \) |
| \( 1 \times 10^{19} \)  | 1.61            | 1.69            | 8.53   | 9.64            | 1.25            | 3.6    | \( 10^{18} \) |
| \( 2 \times 10^{19} \)  | 1.86            | 1.88            | 8.81   | 9.26            | 1.01            | 5.0    | \( 10^{18} \) |

\[5\times 10^{18}\] mass, \( R \sim 1.5 \times 10^{18} \) G, while in \[21\] it is shown that a noticeable effect requires fields above \( \sim 10^{19} \) G.
From eq. (2) and eqs. (A6, A8) the su(3) NJL lagrangian in the MFA is given by:

$$L_f^{MFA} = \tilde{\psi}_f \left( \gamma_\mu (i\partial^\mu - q_f A^\mu) - \hat{M} \right) \psi_f - 2G(\phi_u^2 + \phi_d^2 + \phi_s^2) + 4K\phi_u\phi_d\phi_s,$$

where $\hat{M}$ is a diagonal matrix with elements defined in eq. (11).

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