QCD challenges in radiative $B$ decays

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Abstract. Radiative decays of the $B$ meson are known to provide important constraints on the MSSM and many other realistic new physics models in the sub-TeV range. The inclusive branching ratio $\mathcal{B}(\bar{B} \rightarrow X_s \gamma)$ being the key observable is currently measured to about $\pm 7\%$ accuracy. Reaching a better precision on the theory side is a challenge both for the perturbative QCD calculations and for analyses of non-perturbative hadronic effects. The current situation is briefly summarized here.\textsuperscript{1}

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\section*{INTRODUCTION}

The loop-generated effective $bs\gamma$ coupling is sensitive to new physics at scales (a few)$\times \mathcal{O}(100\mathrm{GeV})$ even in the simplest theories with Minimal Flavor Violation, like the Two-Higgs-Doublet Model or the Minimal Supersymmetric Standard Model. Measuring the inclusive $\mathcal{B}(\bar{B} \rightarrow X_s \gamma)$ is the most efficient way to constrain this coupling. The Standard Model (SM) contribution to this branching ratio forms a background to effects we would like to put bounds on. Its precise QCD calculation has been a challenge for a long time. A satisfactory situation has not yet been reached even for the perturbative contributions.

The current experimental world averages (for $E_\gamma > 1.6\GeV$ in the decaying meson rest frame) read:

$$\mathcal{B}(\bar{B} \rightarrow X_s \gamma) = \begin{cases} (3.55 \pm 0.24_{\text{exp}} \pm 0.09_{\text{model}}) \times 10^{-4} & [1], \\ (3.50 \pm 0.14_{\text{exp}} \pm 0.10_{\text{model}}) \times 10^{-4} & [2]. \end{cases}$$

(1)

They have been obtained by combining the measurements of CLEO [3], BABAR [4]–[6] and BELLE [7, 8] with different lower cuts $E_0$ on the photon energy, ranging from 1.7 to 2.0GeV. An extrapolation in $E_0$ down to 1.6GeV has been performed simultaneously. Uncertainties due to modeling the photon energy spectrum that matter both for the averaging and extrapolation have been singled out in Eq. (1). Ref. [1] gives a larger error than [2] because it uses results at $E_0 \geq 1.8\GeV$ from the older measurements [3]–[7] only, not taking into account the most precise ones from Ref. [8].

Calculations including $\mathcal{O}(\alpha_s^2)$ and $\mathcal{O}(\alpha_{\text{em}})$ effects in the SM give [9, 10]

$$\mathcal{B}(\bar{B} \rightarrow X_s \gamma) = (3.15 \pm 0.23) \times 10^{-4},$$

(2)

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where the error is found by adding in quadrature the non-perturbative (5%), perturbative (3% + 3%) and parametric (3%) uncertainties. The result in Eq. (2) is consistent with the averages (1) at the 1.2σ level. Its evaluation is based on an approximate equality of the hadronic and perturbatively calculable partonic decay widths

\[ \Gamma(\bar{B} \to X_s \gamma)_{E_\gamma > E_0} \simeq \Gamma(b \to X_s^p \gamma)_{E_\gamma > E_0}, \]  

where \( X_s^p \) stands for \( s, sg, sgg, sq\bar{q}, \) etc. This approximation works well only in a certain range of \( E_0 \), namely when \( E_0 \) is large \( (E_0 \sim m_b/2) \) but not too close to the endpoint \( (m_b - 2E_0 \gg \Lambda_{\text{QCD}}) \). Corrections to Eq. (3) of various origin have been widely discussed in the literature, most recently in Ref. [11].

\section*{PERTURBATIVE CALCULATIONS}

Radiative \( B \) decays are most conveniently analyzed in the framework of an effective theory that is obtained from the SM by decoupling of the \( W \) boson and all the heavier particles. Flavor-changing weak interactions are then given by

\[ \mathcal{L}_{\text{weak}} \sim \sum_i C_i(\mu)Q_i, \]  

where the operators \( Q_i \) are built of the light fields only, while \( C_i(\mu) \) are the Wilson coefficients. Once the higher-order electroweak and/or CKM-suppressed effects are neglected, eight operators matter for \( \bar{B} \to X_s \gamma \). They are displayed in Fig. 1. Their Wilson coefficients at the scale \( \mu_b \sim m_b/2 \) are presently known up to the Next-to-Next-to-Leading Order (NNLO) in QCD, i.e. including corrections up to \( \mathcal{O}(\alpha_s^2(\alpha_s \ln \frac{M_W}{m_b}))^n \) for \( n = 0, 1, 2, 3, \ldots \). The necessary matching [12, 13] and anomalous dimension [14, 15, 16] calculations involved Feynman diagrams up to three and four loops, respectively.

Once the Wilson coefficients are at hand, the partonic decay rate is evaluated according to the formula

\[ \Gamma(b \to X_s^p \gamma)_{E_\gamma > E_0} = N \sum_{i,j=1}^{8} C_i(\mu_b)C_j(\mu_b)G_{ij}(E_0, \mu_b), \]  

where \( N = |V_{ts}^*V_{tb}|^2(G_F^2m_b^5\alpha_{\text{em}})/(32\pi^4) \). At the Leading Order (LO), we have \( G_{ij} = \delta_{i7}\delta_{j7} \) and the \( \mathcal{O}(\alpha_s) \) Next-to-Leading Order (NLO) contributions are known since a
FIGURE 2. Examples of Feynman diagrams that contribute to $G_{77}$, $G_{78}$ and $G_{27}$ at $\mathcal{O}(\alpha_s^2)$. Dashed vertical lines mark the unitarity cuts.

long time (see Ref. [17] for a description and references). At the NNLO, it is sufficient to restrict our attention to $i, j \in \{1, 2, 7, 8\}$ because the penguin operators have very small Wilson coefficients ($|C_{3,5,6}(\mu_b)| < |C_4(\mu_b)| \sim \alpha_s(\mu_b)/\pi$). In the following, we shall treat the two similar operators $Q_1$ and $Q_2$ as a single one (represented by $Q_2$), and consider six independent cases of the NNLO contributions to $G_{ij}$.

Three of those six cases ($G_{77}$, $G_{78}$ and $G_{27}$) involve the photonic dipole operator $Q_7$. Examples of the corresponding contributions to the decay rate are shown in the subsequent columns of Fig. 2 as propagator diagrams with unitarity cuts. Two-, three- and four-body final states appear in the first, second and third rows, respectively. The rows cannot be considered separately because cancellation of IR divergences takes place among them. While $G_{77}$ was found already several years ago [19]–[23], the complete calculation of $G_{78}$ has been finalized only very recently [24, 25]. Evaluation of $G_{27}$ is still in progress (see below).

The remaining three cases ($G_{22}$, $G_{28}$ and $G_{88}$) receive contributions from diagrams like those displayed in Fig. 3. Diagrams in the first row involve two-body final states and are IR-convergent. They are just products of the known NLO amplitudes. Three- and four-body final state contributions remain unknown at the NNLO beyond the BLM approximation [26]. The BLM calculation for them has been completed very recently [27, 28] providing new results for $G_{28}$ and $G_{88}$, and confirming the old ones [29] for $G_{22}$. The overall NLO + (BLM-NNLO) contribution to the decay rate from three- and four-body final states in $G_{22}$, $G_{28}$ and $G_{88}$ remains below 4% due to the phase-space suppression by the relatively high photon energy cut $E_0$. Thus, the unknown non-BLM effects here can hardly cause uncertainties that could be comparable to higher-order $\mathcal{O}(\alpha_s^3)$ uncertainties in the dominant terms ($G_{77}$ and $G_{27}$). One may conclude that the considered $G_{ij}$ are known sufficiently well.

It follows that the only contribution that is numerically relevant but yet unknown at

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2 These statements at the LO and NLO hold up to tiny but yet unknown contributions involving four-quark penguin operators and $s\gamma q\bar{q}$ final states. See Appendix E in Ref. [18].
the NNLO is $G_{27}$. So far, it has been evaluated for arbitrary $m_c$ in the BLM approximation [30, 29] supplemented by quark mass effects in loops on the gluon lines [31]. Non-BLM terms have been calculated only in the $m_c \gg m_b/2$ limit [10, 32], and then interpolated downwards in $m_c$ using BLM-based assumptions at $m_c = 0$. Such a procedure introduces a non-negligible additional uncertainty to the calculation, which has been estimated at the level $\pm 3\%$ in the decay rate.

As a first attempt to improve the situation, a calculation of $G_{27}$ at $m_c = 0$ has been undertaken [33, 34]. Two- and three-particle cut contributions have been already found [35]. They contain an IR divergence which should be canceled by diagrams with four-particle cuts [34].

A recently started calculation [36] for arbitrary $m_c$ is supposed to cross-check the $m_c = 0$ result and, at the same time, make it redundant, because no interpolation in $m_c$ will be necessary any more. The method to be used is the same as in the BLM calculation of Ref. [31]. However, the number of master integrals to be considered is now much larger ($\mathcal{O}(500)$). A system of differential equations for them with respect to the variable $z = m_c^2/m_b^2$ needs to be solved (numerically) along an ellipse in the complex $z$-plane. The boundary conditions at $z \gg 1$ are going to be found with the help of asymptotic expansions. The most computer-power demanding part is the integration-by-parts reduction to master integrals that is currently being performed.

**NON-PERTURBATIVE CONTRIBUTIONS**

The question to what accuracy the approximate equality (3) holds has been subject of many investigations since early 1990’s. However, a quantitative analysis of all the dominant contributions to the resulting uncertainty in $\mathcal{B}(\bar{B} \to X_s \gamma)$ has been performed only very recently [11].

Obviously, corrections to Eq. (3) depend on $E_0$. As already mentioned in the introduction, they are minimized at a certain “optimal” value of $E_0$ that is high enough ($E_0 \sim m_b/2$) but not too close to the endpoint ($m_b - 2E_0 \gg \Lambda$). The value of $E_0 = 1.6\text{GeV} \simeq m_b/3$ that has been chosen in Ref. [18] as default seems to have a chance to

FIGURE 3. Examples of Feynman diagrams that contribute to $G_{22}$, $G_{28}$ and $G_{88}$ at $\mathcal{O}(\alpha_s^2)$. 
be in the vicinity of the optimal point. In the following, I will discuss non-perturbative effects at this value of the cutoff, leaving aside the problem of photon energy extrapolation in the experimental averages.\(^3\)

So long as only the photonic dipole operator \(Q_7\) is considered, non-perturbative corrections to Eq. (3) for \(m_b - 2E_0 \gg \Lambda\) can be described in terms of the so-called fixed-order approach that has been derived \[37\] using the optical theorem and the Operator Product Expansion. The corrections can then be written as a series in \((\Lambda/m_b)^n\alpha_s^k\) with \(n = 2, 3, 4, \ldots\) and \(k = 0, 1, 2, \ldots\), where perturbatively calculable coefficients multiply matrix elements of local operators between the \(B\)-meson states at rest. Such matrix elements (at least the leading ones) can be extracted from measurements of observables that are insensitive to new physics, like the semileptonic \(\bar{B} \to X_c e \nu\) decay spectra or mass differences between various \(b\)-flavored hadrons. Coefficients at the terms of order \(\Lambda^2/m_b^2\) and \(\Lambda^3/m_b^3\) have been evaluated in Refs. \[38, 39\] and \[40\], respectively. Very recently, a calculation at order \(\alpha_s\Lambda^2/m_b^2\) has been completed \[41\]. The result explodes near the endpoint \(E_0 \simeq m_b/2\) but remains perfectly consistent with the fixed-order approach at \(E_0 = 1.6\) GeV or even 1.7 GeV. Thus, non-perturbative corrections to the "77" interference term are well under control. In many phenomenological analyses, normalization to the semileptonic rate is applied in such a way that most of those corrections cancel, leaving out a sub-percent effect.

The most important non-perturbative uncertainty originates from the "27" interference term (that stands for "27" and "17"). In Ref. \[11\], photons that can be treated in analogy to the "77" term are called "direct", while all the other ones are called "resolved", i.e. produced far away from the \(b\)-quark annihilation vertex. Contributions from the resolved photons can still be written in terms of a series in powers of \((\Lambda/m_b)^n\alpha_s^k\), but this time the \((n = 1, k = 0)\) term is non-vanishing when \(m_c\) is treated as \(\mathcal{O}(\sqrt{\Lambda}m_b)\). Moreover, they are uncertain, as they depend on matrix elements of non-local operators that cannot be easily extracted from other measurements. Diagrams representing such terms are displayed in Fig. 4, where the external gluon is understood to be soft, while the other one (if present) is considered to be non-soft.

If the charm quark was heavy enough \((m_c^2/m_b \gg \Lambda)\), its loop in the first diagram of Fig. 4 would become effectively local for soft gluons, and we would be back to the local operator description, as in the "77" term. This limit has been analyzed in Refs. \[42\]–\[46\].

\(^3\) Measurements at \(E_0 = 1.6\) GeV or 1.7 GeV will always be plagued with much larger background subtraction errors than the ones at 1.9 GeV or so. Thus, we will always need to search for a proper balance between those errors and uncertainties due to the photon spectrum modeling. Background subtraction requires modeling, too.
A series of the form

$$\sum_{n=0}^{\infty} b_n \mathcal{O} \left( \frac{\Lambda^2}{m_c^2} \left( \frac{m_b \Lambda}{m_c^2} \right)^n \right)$$

(6)

was found as a relative correction to Eq. (3). Explicit results for all the coefficients $b_n$ showed that they are small and quickly decreasing with $n$, which led to a conclusion that the first term in the series is a good approximation to the whole correction even in the $m_b \Lambda / m_c^2 \sim \mathcal{O}(1)$ case that we encounter in Nature. The leading term is proportional to a local operator matrix element that can be extracted from the measured $B$–$B^*$ mass difference. This way, a relative correction of around $+3\%$ to Eq. (3) has been found.

This conclusion has recently been questioned in Ref. [11] on the basis of realistic shape function models that allowed to vary $m_c$ in the physically interesting range, and test applicability of the expansion (6). It has been found that the first term of such an expansion in not really a good approximation if we allow for alternating-sign subleading shape functions (see Eq. (108) in that paper). Alternating signs were necessary to overpass normalization constraints, and make the shape function exponential tails wide enough for more energetic (though still soft) gluons. This is the main source of the overall $\pm 5\%$ non-perturbative uncertainty in the branching ratio that was estimated in Ref. [11]. So long as $m_c$ is treated as $\mathcal{O}(\sqrt{m_b})$, the considered correction is just $\mathcal{O}(\Lambda / m_b)$. Other (smaller) corrections studied in that paper were of order $\mathcal{O}(\alpha_s \Lambda / m_b)$, and did not originate from the “27” interference term.

Apparently, the $\mathcal{O}(\alpha_s \Lambda / m_b)$ corrections alone were the reason for assigning a $\pm 5\%$ non-perturbative uncertainty to the branching ratio in Refs. [9, 10]. Thus the two $\pm 5\%$ uncertainty estimates agree just by coincidence. The main worry in Refs. [9, 10] were the second and third diagrams in Fig. 4. Although they look like $\alpha_s$ corrections to the first one, they are not necessarily unimportant given that the smallness of the first one is partly accidental. It would be interesting to test their relevance using the shape function methods. This would make the analysis of $\mathcal{O}(\alpha_s \Lambda / m_b)$ uncertainties really complete.

In the end, let us recall that there exist non-perturbative corrections to Eq. (3) that are not suppressed by $\Lambda / m_b$ at all. Their intuitive description can be found in Ref. [47]. In particular, collinear photon emission effects belong to this class [48, 27]. Fortunately, they are numerically small due to interplay of several minor suppression factors.

**SUMMARY**

Given the present consistency of measurements and SM calculations, observing clean signals of new physics in $\bar{B} \rightarrow X_s \gamma$ is unlikely, even if the uncertainties are reduced by factors of 2 on both sides, which may be hoped for in the Super-$B$ era. However, achieving such a reduction is worth an effort, as it would lead to strengthening constraints on most popular beyond-SM theories. Several new perturbative NNLO results have been published this year, and new ones are expected in 2011. As far as the non-perturbative corrections are concerned, they are still dominated by unknown contributions, but at least their estimates are now based on calculations rather than order-of-magnitude considerations. Some room for improvement seems to remain both in the $\mathcal{O}(\Lambda / m_b)$ and $\mathcal{O}(\alpha_s \Lambda / m_b)$ cases.
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