Models for operads

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Abstract. We study properties of differential graded (dg) operads modulo weak equivalences, that is, modulo the relation given by the existence of a chain of dg operad maps inducing a homology isomorphism. This approach, naturally arising in string theory, leads us to consider various versions of models. Some applications in topology (homotopy-everything spaces), algebra (cotangent cohomology) and mathematical physics (closed string-field theory) – are also given.

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Introduction. Our aim is to show that some basic constructions of rational homotopy theory (such as minimal models, bigraded and filtered models, etc.) are available also for the category of operads and to indicate also some applications of this fact in topology, homological algebra and mathematical physics. At the very end of the paper we also try to convince the reader that the “homotopy theory of operads” naturally arises in some situations of string theory. We also hope that the paper will help to understand better the (co)bar construction for operads and the related property of Koszulness introduced in [11].

Our constructions are related with a closed model category structure on the category of operads whose existence is more or less known. A stunning observation is that our constructions are even easier than the corresponding constructions in rational homotopy theory. Our explanation is that, while in rational homotopy theory we work basically with graded objects (algebras of various types), the nature of operads resembles more bigraded algebras, one grading (inner) being given by the grading of the underlying vector space, the other given by the “number of inputs”. Experience then says “the more independent gradings, the better”, which would certainly resonate for anyone familiar with the construction of a trigraded model of [4].

The concept presented here was originated in [21] where we tried to do “homological algebra for PROPs” (operad is an algebraic PROP). It was then the illuminating paper [11] which made us believe that the constructions presented here were possible.

As far as our constructions are concerned, we almost completely neglect the uniqueness problem, which would require a notion of homotopy for the category of operads and we are afraid that this

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would stretch the length of the paper beyond any reasonable limit. Fortunately, our applications are mostly independent on this assumption.

All our constructions are primarily made for (unital) operads $S$ with $S(1) = $ the ground field, which is always supposed to be of characteristic zero. This assumption plays here the rôle of simple connectivity in rational homotopy theory. We will see that for any (unital) augmented operad $S$ there exists an operad $\tilde{S}$ satisfying this assumption, constructed from $S$ in a canonical way, which may be understood as an analog of the universal covering of a topological space, and we may apply our constructions to $\tilde{S}$ instead of $S$. Even this could give interesting results.

Central for the applications of our theory is Proposition 4.4 which gives an easy criterion for intrinsic formality of an operad. Note that there is no analog of this result in rational homotopy theory. From this statement we deduce, for example, that a closed string-field theory induces a homotopy Lie algebra structure on the space of relative states (Example 5.13, Proposition 5.14), which is one of main results of [13]. For more applications, see par. 5.

Another by-product of our theory is a new, very general definition of homotopy versions of algebraic objects – homotopy $S$-algebra is an algebra over the minimal model of $S$, see par. 5. For Koszul operads our definition coincides with the definition of [1] (Example 5.1), but it is not restricted to the Koszul case, see Example 3.7.

The paper is not the only example of possible applications of the “rational way of thinking” in other branches of mathematics; let us mention at least the applications in local algebra, see [1].

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Plan of the paper:
1. Preliminaries.
2. Self-dual nature of the category of operads.
3. Models for operads.
4. Cotangent cohomology.
5. Homotopy $S$-algebras, homotopy everything spaces.

1. Preliminaries.

All algebraic objects in the paper will be considered over a fixed field $k$ of characteristic zero. For a graded vector space $V = \bigoplus_p V_p$ let $\uparrow V$ (resp. $\downarrow V$) be the suspension (resp. the desuspension) of $V$, i.e. the graded vector space defined by $(\uparrow V)_p = V_{p-1}$ (resp. $(\downarrow V)_p = V_{p+1}$). We have the obvious natural maps $\uparrow: V \to \uparrow V$ and $\downarrow: V \to \downarrow V$. By $\#V$ we denote the dual of $V$, i.e. the graded vector space with $(\#V)_p :=$ the space of linear maps $\phi : V_p \to k$. If, moreover, an action of the symmetric group $\Sigma_n$ on $V$ is given, we will consider the dual $\#V$ with the action given by $(\sigma \phi)(v) := (-1)^{\text{sgn}(\sigma)} \phi(\sigma^{-1} v)$, for $\sigma \in \Sigma$, $\phi \in \#V$ and $v \in V$. 
Let $\mathcal{P}$ be a (graded) operad in the usual sense (= an operad in the monoidal category of graded vector spaces, see, for example, \cite{10, 11}), i.e. a sequence $\mathcal{P} = \{\mathcal{P}(n); \ n \geq 1\}$ of graded vector spaces together with degree zero linear maps

$$
\gamma = \gamma_{m_1, \ldots, m_l} : \mathcal{P}(l) \otimes \mathcal{P}(m_1) \otimes \cdots \otimes \mathcal{P}(m_l) \rightarrow \mathcal{P}(m_1 + \cdots + m_l)
$$

given for any $m_1, \ldots, m_l \geq 1$. The usual axioms are assumed, including the existence of an identity $1 \in \mathcal{P}(1)$. To stress the existence of the unit $1$ we say that such an operad is unital.

**Definition 1.1.** A nonunital operad is a sequence $S = \{S(n); \ n \geq 1\}$ of graded vector spaces such that

(i) Each $S(n)$ is equipped with a $k$-linear (left) action of the group $\Sigma_n$, where $\Sigma_n$ is the symmetric group on $n$ elements, $n \geq 1$.

(ii) We have, for any $m, n$ and $1 \leq i \leq n$, degree zero linear maps

$$
\circ_i : S(m) \otimes S(n) \rightarrow S(m + n - 1)
$$

such that, for $f \in S(a)$, $g \in S(b)$ and $h \in S(c),$

$$
(1) \quad f \circ_i (g \circ_j h) = \begin{cases} 
(-1)^{|i-j|} g \circ_{j+a-1} (f \circ_i h), & 1 \leq i \leq j - 1, \\
(f \circ_{i-j+1} g) \circ_{j} h, & j \leq i \leq b + j - 1, \text{ and} \\
(-1)^{|i-j|} g \circ_{j} (f \circ_{i-j+1} h), & i \geq j + b.
\end{cases}
$$

(iii) Let $\sigma \in \Sigma_m$, $\tau \in \Sigma_n$ and let $\sigma \circ_i \tau \in \Sigma_{m+n-1}$ be given by inserting the permutation $\sigma$ at the $i$th place in $\tau$. The operations $\circ_i$ are equivariant in the sense that

$$
(2) \quad (\sigma f) \circ_i (\tau g) = \sigma \circ_i \tau (f \circ_i g),
$$

for any $f \in S(m)$ and $g \in S(n)$.

Notice the resemblance of (1) with the definition of a comp algebra of \cite{9}. We say that a unital operad $\mathcal{P}$ is augmented if there exists a homomorphism of unital operads $\epsilon : \mathcal{P} \rightarrow k$ which splits the identity homomorphism $\eta : k \rightarrow \mathcal{P}$ ($k$ denotes both the ground field and the trivial unital operad).

Let $\mathcal{P}$ be an augmented operad and define

$$
\overline{\mathcal{P}}(n) := \begin{cases} 
\mathcal{P}(n), & \text{for } n \geq 2, \text{ and} \\
\ker(\epsilon(1) : \mathcal{P}(1) \rightarrow k(1) = k), & \text{for } n = 1.
\end{cases}
$$

For $f \in \overline{\mathcal{P}}(m)$ and $g \in \overline{\mathcal{P}}(n)$ let

$$
f \circ_i g := \gamma_{1, \ldots, m, \ldots, 1}(g \otimes 1 \cdots \otimes f \otimes \cdots \otimes 1) \in \overline{\mathcal{P}}(m + n - 1) \ (f \text{ at the } i\text{th place}),
$$

$\gamma$ being the structure map of $\mathcal{P}$. It is easy to see that this gives on $\overline{\mathcal{P}}$ the structure of a nonunital operad.
On the other hand, for a nonunital operad $S$ set

$$\hat{S}(n) := \begin{cases} S(n), & \text{for } n \geq 2, \\ S(1) \oplus \mathbb{k}, & \text{for } n = 1. \end{cases}$$

Extend the structure maps $\circ_i : S(m) \otimes S(n) \to S(m+n-1)$ to $\hat{\circ}_i : \hat{S}(m) \otimes \hat{S}(n) \to \hat{S}(m+n-1)$ by

$$f \hat{\circ}_1 1 := f \text{ and } 1 \hat{\circ}_i g := g$$

for $f \in \hat{S}(m), g \in \hat{S}(n), m, n \geq 2$ and $1 \leq i \leq n$. Then the formula

$$\gamma_{m_1,\ldots,m_l}(\mu \otimes \nu_1 \otimes \cdots \otimes \nu_l) := \nu_1 \hat{\circ}_1 (\nu_2 \hat{\circ}_2 (\cdots \hat{\circ}_{n-1} (\nu_n \hat{\circ}_n \mu) \cdots))$$

defines on $S$ the structure of an unital operad. Notice that an unital operad $P$ with $P(1) = \mathbb{k}$ is always canonically augmented, by defining $\epsilon(1) := \mathbb{1}$, the identity map. We may formulate the following observation.

**Observation 1.2.** The correspondence above defines an equivalence between the category of unital augmented operads and the category of nonunital operads. This equivalence restricts to an equivalence of the category of unital operads $P$ with $P(1) = \mathbb{k}$ and the category of nonunital operads $S$ with $S(1) = 0$.

**Warning.** Notice that nonunital operads are not the same as the objects whose axioms are obtained from the axioms for unital operads by forgetting everything related with the unit! If $P$ is such an object, there is no way of defining the operations $\circ_i$ from the structure maps of $P$.

In what follows we will be primarily concerned with nonunital operads. The advantage of working with nonunital operads is that their axioms are quadratic (hence homogeneous) which will give a nice natural filtration on the corresponding free objects, see below. On the other hand, as we have seen in Observation 1.2, nonunital operads are the same as augmented unital operads, therefore the results and constructions of 11, 10 are in fact available also for nonunital operads.

The main results of the paper will be, unless otherwise stated, formulated for nonunital operads $S$ with $S(1) = 0$ (which are the same as unital operads $P$ with $P(1) = \mathbb{k}$), with one very important exception – topological and related singular chain operads of par. 5. We also assume that all operads in the paper are of finite type, meaning that $S(n)$ is a finite dimensional vector space or, if $S$ is a differential operad (see below for the definition), $S = (S, d)$, that the cohomology $H(S(n), d(n))$ is a finite dimensional vector space, for any $n \geq 2$. The last assumption is not really necessary everywhere, but it will simplify the exposition.

The following two constructions will be useful in the sequel. For an operad $S$ define its suspension $sS$ to be the operad defined by

$$(sS)(n) = \uparrow^{n-1} S(n),$$

the $(n-1)$-fold suspension of the graded vector space $S(n)$,
with the structure maps and the symmetric group action defined in the obvious way. For a nonunital operad \( T \), not necessarily with \( T(1) = 0 \), define its universal covering \( \tilde{T} \) as the operad with

\[
\tilde{T}(n) := \begin{cases} 
T(n), & \text{for } n \geq 2, \\
0, & \text{for } n = 1,
\end{cases}
\]

and the operad structure defined in the most obvious way. Using the terminology introduced later, we may say alternatively that \( \tilde{T} \) is the ideal in \( T \) generated by \( \{T(n); \ n \geq 2\} \).

By a collection we mean a sequence \( E = \{E(n); \ n \geq 2\} \) of (graded) vector spaces such that each \( E(n) \) is equipped with an action of the symmetric group \( \Sigma_n \). If \( F = \{F(n); \ n \geq 2\} \) is another collection, then by a map \( \alpha : E \to F \) of collections we mean a sequence \( \alpha = \{\alpha(n); \ n \geq 2\} \) of \( \Sigma_n \)-equivariant maps \( \alpha(n) : E(n) \to F(n) \). Instead of saying “\( \alpha \) is a map of collections” we sometimes say simply “\( \alpha \) is an equivariant map” hoping it will not confuse the reader.

Denote by \( \Box : \text{Oper} \to \text{Coll} \) the obvious forgetful functor from the category of (nonunital) operads to the category of collections. It has a left adjoint \( F : \text{Coll} \to \text{Oper} \) (see \([10, 11]\)) and it is natural to call \( F(E) \) the free operad on the collection \( E \). It follows easily from the homogeneity of the axioms of an (nonunital) operad that each \( F(E)(n) \) is naturally graded, \( F(E)(n) = \bigoplus_{l \geq 1} F^l(E)(n) \) and that the grading has the following properties.

(i) Each \( F^l(E)(n) \) is a \( \Sigma_n \)-invariant subspace of \( F(E)(n) \).
(ii) \( F^1(E)(n) = E(n) \) and \( F^{\geq n}(E)(n) = 0 \).
(iii) If \( f \in F^l(E)(m) \) and \( g \in F^k(E)(n) \) then \( f \circ_i g \in F^{k+l}(E)(m+n-1) \).

In the tree language of \([11]\), \( F^l(E)(n) \) is represented by trees having \((l-1)\) inner edges.

Let \( S \) be an operad. Define the collection \( Q = Q(S) \) (the indecomposables of \( S \)) by \( Q(n) := S(n)/D(n) \), where the collection \( D = D(S) \) (the decomposables of \( S \)) is defined as the subcollection of \( S \) spanned by the elements of the form \( \sigma(f \circ_i g) \) with \( \sigma \in \Sigma_m \Sigma_{n-1} \), \( m, n \geq 1 \), \( f \in S(m), \ g \in S(n) \) and \( 1 \leq i \leq n \). We have the natural projection \( \pi : S \to Q \) of collections. Notice that there always exists an equivariant splitting (i.e. a map of collections) \( s : Q \to S \) of \( \pi \). To see this, choose, for \( n \geq 2 \), a \( k \)-linear splitting \( s'(n) : Q(n) \to S(n) \) of \( \pi(n) \) and let \( s(n) \) be the symmetrization

\[
(3) \quad s(n) := \frac{1}{n!} \sum_{g \in \Sigma_n} gs'(n)g^{-1}.
\]

Then \( s(n) \) is obviously an \( \Sigma_n \)-equivariant splitting of \( \pi(n) \) and \( s := \{s(n); n \geq 2\} \) is the required section. We will use this trick in many places of the paper.

The following proposition shows that, as far as the presentation is concerned, operads behave as connected graded algebras (compare \([22]\)).

**Proposition 1.3.** In the situation above, define \( \alpha : F(Q) \to S \) by \( \alpha|_Q = s \). Then:

(i) The map \( \alpha \) is an epimorphism (meaning that each \( \alpha(n) \) is epi).
(ii) Ker(α) consists of reducible elements, Ker(α) ⊂ \( \mathcal{F}^{\geq 2}(Q) = D(\mathcal{F}(Q)) \).

(iii) The presentation \( \alpha : \mathcal{F}(Q) \to \mathcal{S} \) is minimal in the sense that for any collection \( Q' \) and for any epimorphism \( \alpha' : \mathcal{F}(Q') \to \mathcal{S} \) there exists a monomorphism \( \beta : \mathcal{F}(Q) \to \mathcal{F}(Q') \) such that the diagram

\[
\begin{array}{ccc}
\mathcal{F}(Q) & \xrightarrow{\alpha} & \mathcal{S} \\
\beta \downarrow & & \downarrow \alpha' \\
\mathcal{F}(Q') & \xrightarrow{\alpha'} & \mathcal{S}
\end{array}
\]

commutes.

**Proof.** To prove that \( \alpha \) is an epimorphism, observe first that \( Q(2) = \mathcal{S}(2) \), hence \( \alpha(2) = \text{Id} \).

Suppose we have proved that \( \alpha(k) \) is epi for any \( 2 \leq k < n \) and try to prove that \( \alpha(n) \) is an epimorphism too. Any \( f \in \mathcal{S}(n) \) can be decomposed uniquely as \( f = s(n)(x) + y \) for some \( x \in Q(n) \) and \( y \in \mathcal{S}(n) \) with \( \pi(n)(y) = 0 \). We have, by definition, \( s(n)(x) = \alpha(n)(x) \), so it is enough to prove that \( y \in \text{Im}(\alpha(n)) \). But \( \pi(n)(y) = 0 \) means that \( y \in D(n) \), i.e.

\[
y = \sum_t \sigma_t(f_t \circ_i g_t)
\]

for some \( \sigma_t \in \Sigma_n \), \( f_t \in \mathcal{S}(m_t) \), \( g_t \in \mathcal{S}(n_t) \) and \( 1 \leq i \leq n_t \). We know by induction (since \( m_t, n_t < n \)) that \( f_t = \alpha(m_t)(\phi_t) \) and \( g_t = \alpha(n_t)(\psi_t) \) for some \( \phi_t \in \mathcal{F}(Q)(m_t) \) and \( \psi_t \in \mathcal{F}(Q)(n_t) \).

As \( \alpha \) is a homomorphism of operads we have

\[
y = \alpha(m + n - 1)(\sum_t \sigma_t(\phi_t \circ_i \psi_t)),
\]

hence \( y \in \text{Im}(\alpha(n)) \) and (i) is proven by induction.

To prove (ii), observe first that \( \alpha \) maps \( \mathcal{F}^{\geq 2}(Q) \) into the decomposables \( D \) of \( \mathcal{S} \). Let \( f \in \text{Ker}(\alpha(n)) \subset \mathcal{F}(Q)(n) \) for some \( n \geq 2 \). We may decompose \( f = f_1 + f_{\geq 2} \) with \( f \in \mathcal{F}^1(Q)(n) = Q(n) \) and \( f \in \mathcal{F}^{\geq 2}(Q)(n) \). Then we have

\[
0 = \pi(n)\alpha(n)(f) = \pi(n)s(n)(f_1) + \pi(n)\alpha(n)(f_{\geq 2}) = f_1,
\]

hence \( f \in \mathcal{F}^{\geq 2}(Q)(n) \) as claimed.

We leave (iii) as an exercise for the reader.

The notions as (co)kernels, ideals, quotients, etc., translate in the obvious way into the category \( \text{Oper} \). For example, an ideal in \( \mathcal{S} \) is a suboperad \( \mathcal{I} = \{\mathcal{I}(n) \subset \mathcal{S}(n); \ n \geq 2\} \) such that \( \phi_i g \in I(m + n - 1) \) whenever \( f \in \mathcal{I}(m) \) or \( g \in \mathcal{I}(n) \), \( m, n \geq 1 \), \( 1 \leq i \leq n \). If \( \mathcal{I} \) is an ideal in \( \mathcal{S} \) then the quotient \( \mathcal{S}/\mathcal{I} := \{\mathcal{S}(n)/\mathcal{I}(n); \ n \geq 2\} \) has a natural structure of an operad, etc.

Let \( \mathcal{S} \) be an operad. As usual, we have several equivalent definitions of a module over \( \mathcal{S} \); for example, we can simply say that a module is an abelian group object in the slice category
Oper/S of operads over S. But we prefer to give the following more explicit definition, compare the notion of a module over a prop introduced in [21].

**Definition 1.4.** A module over an operad S is a collection M = {M(n); n ≥ 2} together with degree zero linear maps

\[ \circ_i = \circ_i^L : S(m) \otimes M(n) \to M(m + n - 1) \] (the left composition), and

\[ \circ_i = \circ_i^R : M(m) \otimes S(n) \to M(m + n - 1) \] (the right composition),

given for any m, n ≥ 2 and 1 ≤ i ≤ n such that the condition (7) of Definition 1.1 is satisfied for any \( f \otimes g \otimes h \in [M(a) \otimes S(b) \otimes S(c)] \) or \([S(a) \otimes M(b) \otimes S(c)]\) or \([S(a) \otimes S(b) \otimes M(c)]\) and that the condition (2) of Definition 1.1 is satisfied for any \( f \otimes g \in [M(m) \otimes S(n)] \) or \([S(m) \otimes M(n)]\).

**Example 1.5.** There are two important types of examples of S-modules. If \( \phi : S \to T \) is a homomorphism of operads, then \( \phi \) induces on T the natural structure of an operad. In particular (taking \( \phi = 1 \)), S is an S-module over itself.

Recall that an algebra over S (or an S-algebra) is a homomorphism \( A : S \to \text{End}(V) \), where \( \text{End}(V) \) denotes the operad of endomorphisms of a graded space V (see [11], compare also the differential case discussed later). Therefore A induces on \( \text{End}(V) \) the structure of an S-module which we denote (a bit ambiguously) by \( k_V \), because this module plays the rôle of the residue ring in our theory, see also the discussion in [21].

A second example is provided by an ideal \( K \subset S \), typically given as the kernel of a homomorphism. The operad structure of S obviously induces on K the natural structure of an S-module.

The forgetful functor \( \square : \text{S-Mod} \to \text{Coll} \) from the category of S-modules to the category of collections has a left adjoint (compare [21]), \( S(-) : \text{Coll} \to \text{S-Mod} \), and we call the S-module \( S\langle E \rangle \) the free S-module on the collection \( E \). The free module \( S\langle E \rangle \) is, like the free operad, naturally graded, \( S\langle E \rangle = \bigoplus_{l \geq 1} S\langle E \rangle^l \), by the “length of words”.

For an S-module M define the collection \( Q_S = Q_S(M) \) (the indecomposables of M) by \( Q_S(n) := M(n)/D_S(n) \), where the collection \( D_S = D_S(M) \) (the decomposables of M) is defined as the subcollection of M spanned by the elements of the form \( \sigma(f \circ_i g) \) with \( \sigma \in \Sigma_{m+n-1} \), \( m, n \geq 1 \), \( f \in M(m) \), \( f \otimes g \in M(m) \otimes S(n) \otimes S(m) \otimes M(n) \) and \( 1 \leq i \leq n \). We have the following analog of Proposition 1.3.

**Proposition 1.6.** Let M be an S-module and let \( t : Q_S \to M \) be a section of the natural projection \( p : M \to Q_S \). Define \( \gamma : S\langle Q_S \rangle \to M \) by \( \gamma|_{Q_S} = t \). Then the following holds.

(i) The map \( \gamma \) is an epimorphism.

(ii) \( \text{Ker}(\gamma) \) consists of decomposable elements, \( \text{Ker}(\gamma) \subset S\langle Q_S \rangle^{\geq 2} = D_S(S\langle Q_S \rangle) \).
(iii) The presentation $\gamma : S\langle Q \rangle \to M$ is minimal in the sense that for any collection $Q$ and for any epimorphism $\gamma' : S\langle Q \rangle \to M$ there exists a monomorphism $\delta : S\langle Q \rangle \to S\langle Q \rangle$ such that the diagram

```
\begin{array}{c}
S\langle Q \rangle \\
\downarrow \delta \\
S\langle Q \rangle \\
\end{array} \quad \begin{array}{c}
\gamma \\
\gamma' \\
\end{array}
```

commutes.

**Proof.** It is an obvious analog of the proof of Proposition 1.3 and we can leave it to the reader. □

By a degree $p$ derivation on an operad $S$ we mean a sequence $\theta = \{\theta(n) : S(n) \to S(n); n \geq 2\}$ of equivariant degree $p$ maps such that

$$\theta(m + n - 1)(f \circ_i g) = \theta(m)(f) \circ_i g + (-1)^{p|f|} \cdot f \circ_i \theta(n)(g),$$

for any $f \in S(m)$, $g \in S(n)$, $m, n \geq 1$ and $1 \leq i \leq n$. We denote by $\text{Der}_p(S)$ the vector space of all degree $p$ derivations of $S$.

**Lemma 1.7.** Let $\phi \in \text{Der}_p(S)$ and $\psi \in \text{Der}_q(S)$. Then the formula

$$[\phi, \psi](n) := \phi(n)\psi(n) - (-1)^{pq}\psi(n)\phi(n), \quad n \geq 2.$$ 

defines on $\text{Der}_*(S)$ the structure of a graded Lie algebra.

**Proof.** An easy exercise. □

By a differential operad we mean a couple $(S, d)$ where $S$ is an operad and $d$ is a degree $-1$ derivation of $S$, $d \in \text{Der}_{-1}(S)$, with $d^2 = 0$. It is clear that for such a differential operad the collection $H = H(S, d) := \{H(S(n), d(n)); n \geq 2\}$, where $H(S(n), d(n))$ denotes the homology of the differential space $(S(n), d(n))$, has a natural structure of an operad, called the homology (operad) of $(S, d)$. Observe that what we call a differential operad here is exactly an operad in the monoidal category of differential spaces.

Of course, any nondifferential operad $S$ can be considered as a differential operad with the trivial differential. If we wish to stress that we consider $S$ in this way, we write $(S, 0)$ instead of $S$.

**Example 1.8.** Any graded differential space $(V, d_V)$ has its (nonunital) endomorphism operad $(\text{End}(V), d_{\text{End}})$ defined by

$$\text{End}(V)(n) := \begin{cases} 
\text{Hom}_*(V^\otimes n, V), & \text{for } n \geq 2, \text{ and} \\
0, & \text{for } n = 1,
\end{cases}$$
where $\text{Hom}_p(V^{\otimes n}, V)$ denotes the vector space of homogeneous degree $p$ linear maps $f : V^{\otimes n} \to V$. The composition maps are defined in the obvious way and the differential $d_{\text{End}}$ is, for $f \in \text{Hom}_p(V^{\otimes n}, V)$ given by

$$d_{\text{End}}(f) := d \circ f - (-1)^p \cdot f \circ d^{\otimes n},$$

where $d^{\otimes n}$ is the usual differential induced by $d$ on $V^{\otimes n}$. An algebra over a differential operad $(S, d_S)$ (or an $(S, d_S)$-algebra) is then a differential operad homomorphism $A : (S, d_S) \to (\text{End}(V), d_{\text{End}})$.

For any two collections $X$ and $Y$, let $\text{Coll}_p(X, Y)$ denotes the space of all sequences $\phi = \{\phi(n); \ n \geq 2\}$ of $\Sigma_n$-equivariant degree $p$ linear maps $\phi(n) : X(n) \to Y(n)$. Using the universal property of the free operad, we may easily deduce the following lemma.

**Lemma 1.9.** For any collection $E$, the restriction defines an isomorphism

$$\text{Der}_p(\mathcal{F}(E)) \cong \text{Coll}_p(E, \mathcal{F}(E)).$$

Let us state also the following technical lemma.

**Lemma 1.10.** The correspondence $\mathcal{H} : (S, d_S) \mapsto \mathcal{H}(S, d_S)$ induces a functor from the category of differential operads to the category of nondifferential operads.

If $(V, d_V)$ is a differential vector space and $(\text{End}(V), d_{\text{End}})$ the endomorphism operad from Example 1.3, then there exists a noncanonical map $\Phi : (\text{End}(H(V, d_V), 0) \to (\text{End}(V), d_{\text{End}})$ of differential operads inducing the canonical isomorphism

$$(\text{End}(H(V, d_V), 0) \cong \mathcal{H}(\text{End}(V), d_{\text{End}})$$

given by the Künneth formula.

**Proof.** The first part of the lemma is an easy exercise. To prove the second part, choose a decomposition $V = H \oplus B \oplus C$ with $H \oplus B = \text{Ker}(d_V)$ and $B = \text{Im}(d_V)$. Let $\iota : H \to V$ be the corresponding inclusion and $\pi : V \to H$ be the corresponding projection. For an element $f \in \text{End}(H)(n)$, $f : H^{\otimes n} \to H$, let $\Phi(f) \in \text{End}(V)$ be defined as $\Phi(f) := \iota \circ f(\pi^{\otimes n})$. Let us verify that $\Phi : (\text{End}(H(V, d_V), 0) \to (\text{End}(V), d_{\text{End}})$ thus defined is a homomorphism of differential operads.

For $f \in \text{End}(H)(m)$, $g \in \text{End}(H)(n)$ and $1 \leq i \leq n$, we have

$$\Phi(f \circ_i g) = \iota \circ g(\mathbb{1}^{\otimes (i-1)} \otimes f \otimes \mathbb{1}^{\otimes (n-i)}) \circ (\pi^{\otimes (m+n-1)}) = \iota \circ g(\pi^{\otimes (i-1)} \otimes f \pi^{\otimes m} \otimes \pi^{\otimes (n-i)})$$

$$= \iota \circ g(\pi^{\otimes (i-1)} \otimes \pi f \pi^{\otimes m} \otimes \pi^{\otimes (n-i)}) = \Phi(f) \circ_i \Phi(g),$$

because $\pi \iota = \mathbb{1}$. We also easily have $d_{\text{End}}(\Phi(f)) = df \pi^{\otimes m} - (-1)^p tf(\pi d)^{\otimes n} = 0$, because $d\iota = \pi d = 0$, thus $\Phi$ is indeed a map of differential operads. The rest of the statement follows.
from the Künneth formula.

Let \( A : (\mathcal{S}, d_\mathcal{S}) \to (\text{End}(V), d_{\text{End}}) \) be an \((\mathcal{S}, d_\mathcal{S})\)-algebra structure on \((V, d_V)\). By the above lemma, \( A \) induces an \( \mathcal{H}(\mathcal{S}, d_\mathcal{S}) \)-algebra structure on \( \mathcal{H}(V, d_V) \),

\[
\mathcal{H}(A) : \mathcal{H}(\mathcal{S}, d_\mathcal{S}) \to \text{End}(\mathcal{H}(V, d_V))
\]

which we call the \textit{homology algebra structure} induced by \( A \).

All the objects introduced above and the majority of objects introduced below have an obvious and even easier \textit{nonsymmetric} form which means that we simply forget everything related to the action of the symmetric group, compare the similar situation in homotopy theory [23].

2. Self-dual nature of the category of operads.

This section has an auxiliary character though we think that the results may be of some interest in themselves. We aim to discuss here the following statement which is implicit in [11, 10].

**Theorem 2.1.** Let \( E \) be a (finite-type) collection. Then there exists a one-to-one correspondence between (nonunital) operad structures on \( E \) and quadratic differentials on \( \mathcal{F}(\# \downarrow E) \) (i.e. differentials \( d \) such that \( d(\# \downarrow E) \subset \mathcal{F}^2(\# \downarrow E) \)),

\[
\Omega : \{ \text{operad structures on } E \} \longleftrightarrow \{ \text{quadratic differentials on } \mathcal{F}(\# \downarrow E) \} : \Omega^{-1}
\]

where \( (\# \downarrow E) \) is the collection given by \( (\# \downarrow E)(n) = \# \downarrow E(n), n \geq 2 \). The functor \( \Omega \) (the dual bar construction) was explicitly constructed in [11].

Let \( \mathcal{R} \) be a \( k \)-linear graded rigid monoidal category, i.e. a category where both the objects and the hom-sets are given a structure of \( k \)-linear graded vector space, a natural transformation \( \odot : \mathcal{R} \times \mathcal{R} \to \mathcal{R} \) (the “tensor product”) is given, and every object \( V \in \mathcal{R} \) has a dual \( \#V \in \mathcal{R} \). We do not aim to give a full set of axioms here; we just mention two major examples instead: the category of graded vector spaces with the usual tensor product and dual, and the category \( \text{Coll} \) of collections with the rigid monoidal structure defined below.

For a category \( \mathcal{R} \) as above and \( V \in \mathcal{R} \), let \( \text{End}_\mathcal{R}(V) \) be a generalization of the nonsymmetric operad \( \text{End}(V) \) from Example 1.8 with \( \mathcal{R} \) in place of \( V \) defined by

\[
\text{End}_\mathcal{R}(V)(n) := \mathcal{R}(V^\odot n, V), \ n \geq 2,
\]

with the structure maps \( \circ_i \) given as the obvious compositions. If \( \mathcal{S} \) is an operad, then an \( \mathcal{S} \)-\textit{algebra} in \( \mathcal{R} \) is a homomorphism \( A : \mathcal{S} \to \text{End}_\mathcal{R}(V) \) of operads.

The following proposition is a generalized dual form of a theorem of [3].
Proposition 2.2. Let $\mathcal{R}$ be a $k$-linear graded rigid monoidal category and let $V \in \mathcal{R}$. Let $\mathcal{S}$ be a quadratic operad and let $\mathcal{S}'$ be its Koszul dual (see [11] for definitions). Then there is a natural one-to-one correspondence between $\mathcal{S}$-algebra structures on a given object $V \in \mathcal{R}$ and quadratic differentials on $F_{\mathcal{S}'}(\# V)$, where $F_{\mathcal{S}'}(\# V)$ denotes the free $\mathcal{S}'$-algebra on $V$.

Let us introduce now a rigid monoidal structure on the category of collections $\text{Coll}$. For two collections $E, F \in \text{Coll}$, let $E \odot F$ be the collection defined as

$$(E \odot F)(n) := \bigoplus_{k=l=n+1} k[\Sigma_n] \langle \bigoplus_{1 \leq i \leq l} (E(k) \otimes_i F(l)) \rangle / (\sim),$$

where $k[\Sigma_n] \langle \bigoplus_{1 \leq i \leq l} (E(k) \otimes_i F(l)) \rangle$ is the free $\Sigma_n$-space on $l$ copies of $(E(k) \otimes_i F(l)) := (E(k) \otimes F(l))$ and the equivalence relation $\sim$ is generated by

$$(\sigma \circ_i \tau)(e \otimes_i f) \sim \sigma e \otimes_i \tau f,$$

the meaning of $\sigma \circ_i \tau \in \Sigma_n$ being the same as in (iii) of Definition 1.1. The definition of the dual collection $\# E$ is the obvious one. The following lemma is a nice exercise left for the reader (compare [24]).

Lemma 2.3. Nonunital operads are associative algebras in the category $\text{Coll}$ with the monoidal structure introduced above.

The philosophy behind Theorem 2.1 becomes obvious now. An associative algebra is an algebra over the associative algebra operad $\text{Ass}$ which is, by [11], Koszul self-dual, $\text{Ass}' = \text{Ass}$, and we see that Theorem 2.1 is a consequence of Proposition 2.2.

3. Models for operads.

In the following theorem, we show that there exists an analog of a minimal model for differential operads.

Theorem 3.1. Let $(\mathcal{S}, d_{\mathcal{S}})$ be a differential graded operad (with $\mathcal{S}(1) = 0$ as usual). Then there exists a collection $M = \{M(n); n \geq 2\}$, a differential $d$ on $\mathcal{F}(M)$ and a homomorphism $\nu : (\mathcal{F}(M), d) \rightarrow (\mathcal{S}, d_{\mathcal{S}})$ such that

(i) the differential $d$ is minimal, $d(M) \subset \mathcal{F}^{\geq 2}(M)$, and

(ii) the map $\nu$ induces an isomorphism in homology.

The object $\nu : (\mathcal{F}(M), d) \rightarrow (\mathcal{S}, d_{\mathcal{S}})$ is called the minimal model of the differential operad $(\mathcal{S}, d_{\mathcal{S}})$. It is unique in the sense that if $\nu' : (\mathcal{F}(M'), d) \rightarrow (\mathcal{S}, d_{\mathcal{S}})$ is another minimal model of $(\mathcal{S}, d_{\mathcal{S}})$, then the differential operads $(\mathcal{F}(M), d)$ and $(\mathcal{F}(M'), d)$ are isomorphic.
Proof. Let $X(2) := \mathcal{H}(S, d_S)(2)$ and let $s(2) : X(2) \to Z(S, d_S)(2) \subset S(2)$ be an equivariant splitting of the projection $cl(2) : Z(S, d_S)(2) \to \mathcal{H}(S, d_S)(2)$ (here and later, $cl$ will denote the projection of a chain onto its homology class). Define a differential $d_2$ on $\mathcal{F}(X(2))$ and a map $\nu_2 : (\mathcal{F}(X(2), d_2) \to (S, d_S)$ by

$$d_2 = 0 \text{ and } \nu_2|_{X(2)} := s(2).$$

We finish the construction by induction. Suppose we have already constructed a collection $X(<n) = \{X(k); 2 \leq k < n\}$, a differential $d_{n-1}$ on $\mathcal{F}(X(<n))$ and a map $\nu_{n-1} : (\mathcal{F}(X(<n)), d_{n-1}) \to (S, d_S)$ such that

(i) the differential $d_{n-1}$ is minimal, $d_{n-1}(X(<n)) \subset \mathcal{F}^{22}(X(<n))$ and

(ii) the map $\mathcal{H}(\nu_{n-1})(k) : \mathcal{H}(\mathcal{F}(X(<n)), d_{n-1})(k) \to (S, d_S)(k)$ is an isomorphism for any $k \leq n - 1$.

Let

$$A(n) := \mathcal{H}(S, d_S)(n)/\text{Im}(\mathcal{H}(\nu_{n-1}))(n), \quad B(n) := \text{Ker}(\mathcal{H}(\nu_{n-1}))(n) \text{ and } B(n) := \uparrow B(n).$$

Let $s(n) : A(n) \to Z(S, d_S)(n)$ be an equivariant section of the composition $Z(S, d_S)(n) \xrightarrow{\text{cl}} A(n)$ and let $r'(n) : \mathcal{H}(\mathcal{F}(X(<n)), d_{n-1})(n) \to Z(\mathcal{F}(X(<n)), d_{n-1})(n)$ be an equivariant section of the projection $\text{cl}_n : Z(\mathcal{F}(X(<n)), d_{n-1})(n) \to \mathcal{H}(\mathcal{F}(X(<n)), d_{n-1})(n)$ and let $r(n) : B(n) \to Z(\mathcal{F}(X(<n)), d_{n-1})(n)$ be the composition of the inclusion $\uparrow B(n) \hookrightarrow \mathcal{H}(\mathcal{F}(X(<n)), d_{n-1})(n)$ and $r'(n)$. Define

$$X(n) := A(n) \oplus B(n), \quad X(\leq n) := X(<n) \oplus X(n),$$

$$d_n|_{X(<n)} := d_{n-1}|_{X(<n)}, \quad d_n|_{A(n)} := 0, \quad d_n|_{B(n)} := r(n) \circ \downarrow ,$$

$$\nu_n|_{X(<n)} := \nu_{n-1}|_{X(<n)}, \quad \nu_n|_{A(n)} := s(n), \quad \text{and } \nu_n|_{B(n)} := 0.$$
Let $Y$ be a collection and $\delta$ a (not necessary minimal) differential on $F(Y)$. Let $p^i : F(Y) \to F^i(Y)$ be the natural projection and let $\delta_i$ be the derivation on $F(Y)$ defined by $\delta_i|_Y := p^i \circ \delta$. Notice that

$$\delta_1^2 = 0 \text{ and } \delta_1 \circ \delta_2 + \delta_2 \circ \delta_1 = 0.$$  

This means that it makes sense to consider the collection $H(Y, \delta_1)$ and that $\delta_2$ induces on $F(H(Y, \delta_1))$ a quadratic differential $\overline{\delta}_2$. There is an obvious important special case when $\delta$ is minimal, which is the same as $\delta_1 = 0$. Then $\overline{\delta}_2 = \delta^2$ is a quadratic differential on $F(Y)$.

The following definition is motivated by the notion of the homotopy Lie algebra in rational homotopy theory.

**Definition 3.2.** Let $(S, d_S)$ be a differential operad and let $\nu : (F(M), d) \to (S, d_S)$ be its minimal model as in Theorem 3.1. The (nondifferential) operad $\Omega^{-1}(F(M), d_2)$ (see Theorem 2.1 for the notation) is called the homotopy operad of $(S, d_S)$ and denoted $\pi(S, d_S)$.

We state without proof, which is analogous to the proof of [19, Theorem V.7], the following statement.

**Proposition 3.3.** Let $(F(Y), \delta)$ be as above and let $\nu : (F(M), d) \to (F(Y), \delta)$ be its minimal model. Then the differential operads $(F(M), d_2)$ and $(F(H(Y, \delta_1)), \overline{\delta}_2)$ are isomorphic,

$$(F(M), d_2) \cong (F(H(Y, \delta_1)), \overline{\delta}_2).$$

Let us make some more comments on the dual bar construction $\Omega$ of [11]. It is defined for any differential operad $(S, d_S)$ and it is of the form $(F(Y), d_0)$, with $Y := \# \downarrow S$ and $d_0 := d_I + d_E$, where $d_I$ (the internal differential) is the dual of $d_S$, and $d_E$ (the external differential) is obtained by dualizing the structure maps of $S$.

**Theorem 3.4.** Let $(S, d_S)$ be a differential operad. Then

$$\pi(S, d_S) \cong \mathcal{H}(\Omega(S, d_S)).$$

**Proof.** By a theorem of [11] there exists a natural map $\phi : \Omega(\Omega(S, d_S)) \to (S, d_S)$ such that $\mathcal{H}(\phi)$ is an isomorphism. This means that, if $(F(M), d)$ is a minimal model for the double dual bar construction $\Omega(\Omega(S, d_S))$, then it is also a minimal model for $(S, d_S)$. We have

$$\Omega(\Omega(S, d_S)) = (F(\# \downarrow \Omega(S, d_S)), d_\Omega),$$

where $(d_\Omega)_1 = d_I$ is the dual of the differential on $\Omega(S, d_S)$, therefore

$$F(H(\# \downarrow \Omega(S, d_S), (d_\Omega)_1), \overline{\delta}_2) = F(\# \downarrow \mathcal{H}(\Omega(S, d_S)), \overline{\delta}_2) = \Omega(\mathcal{H}(\Omega(S, d_S))),$$

and Proposition 3.3 gives the desired result.
Suppose that we have a collection $Z$ which decomposes as $Z = Z^0 \oplus Z^1 \oplus \cdots$ (meaning, of course, that for each $n \geq 2$ we have a $\Sigma_n$-invariant decomposition $Z(n) = Z^0(n) \oplus Z^1(n) \oplus \cdots$). This induces on $\mathcal{F}(Z)$ still another grading (because of the homogeneity of the axioms of a nonunital operad!), $\mathcal{F}(Z) = \bigoplus_{k \geq 1} \mathcal{F}(Z)^k$. We will call this grading the TJ-grading (from Tate-Jozefiak) here; the reason will become obvious below.

In this situation the free operad $\mathcal{F}(Z)$ is naturally trigraded, $\mathcal{F}(Z) = \bigoplus \mathcal{F}^l(Z)^k_j$, where the $k$ refers to the TJ-grading introduced above, the $l$ refers to the “length” grading given by the length of words in the free operad, and the $j$ indicates the “inner” grading given by the grading of the underlying vector space.

Suppose that $d$ is an (inner) degree $-1$ differential on $\mathcal{F}(Z)$ such that

\begin{equation}
\label{eq:4}
d(Z^k) \subset \mathcal{F}(Z)^{k-1}, \text{ for all } k \geq 1
\end{equation}

(meaning, of course, that $d(Z^k)(n) \subset \mathcal{F}(Z)^{k-1}(n)$ for all $n \geq 2$), i.e. that $d$ is homogeneous degree $-1$ with respect to the TJ-grading. Then the homology operad $\mathcal{H}(\mathcal{F}(Z), d)$ is naturally bigraded,

\[ \mathcal{H}(\mathcal{F}(Z), d) = \bigoplus \mathcal{H}^k_j(\mathcal{F}(Z), d), \]

the upper grading being induced by the TJ-grading and the lower one by the inner grading. We have the following analog of the bigraded model of a commutative graded algebra constructed in [14, par. 3].

**Theorem 3.5.** Let $\mathcal{H}$ be a (nonunital with $\mathcal{H}(1) = 0$ as usual) operad. Then there exists a collection $Z = Z^0 \oplus Z^1 \oplus \cdots$, a differential $d$ on $\mathcal{F}(Z)$ satisfying (4) and a map $\rho : (\mathcal{F}(Z), d) \to (\mathcal{H}, 0)$ of differential operads such that the following conditions are satisfied:

(i) $d$ is minimal in the sense that $d(Z) \subset \mathcal{F}^\geq 2(Z)$,

(ii) $\rho|_{Z^1} = 0$ and $\rho$ induces an isomorphism $\mathcal{H}^0(\rho) : \mathcal{H}^0(\mathcal{F}(Z), d) \cong \mathcal{H}$, and

(iii) $\mathcal{H}^\geq 1(\mathcal{F}(Z), d) = 0$.

We call $\rho : (\mathcal{F}(Z), d) \to (\mathcal{H}, 0)$ the TJ-model (or the bigraded model) of $\mathcal{H}$.

**Proof.** Let $Z^0 := Q(\mathcal{H})$ and let $s^0 : Z^0 \to \mathcal{H}$ be an (equivariant) section of the projection $\mathcal{H} \to Z^0$. Define $\overline{\rho} : \mathcal{F}(Z^0) \to \mathcal{H}$ by $\overline{\rho}|_{Z^0} = s$ and let $d$ be trivial on $\mathcal{F}(Z^0)$. Then $\mathcal{H}^0(\overline{\rho}) = \overline{\rho} : \mathcal{F}(Z^0) \to \mathcal{H}$ is an epimorphism by Proposition [1.3].

Let $K := \operatorname{Ker}(\overline{\rho})$ and consider the collection $K$ with the natural structure of an $\mathcal{F}(Z^0)$-module as in Example [1.5]. Define $\underline{Z}^1 := Q_{\mathcal{F}(Z^0)}(K)$ and $Z^1 \leftarrow \underline{Z}^1$.

Let $s^1 : \underline{Z}^1 \to K$ be a splitting of the projection $K \to \underline{Z}^1$ and extend $d$ by $d|_{Z^1} := s^1 \circ \downarrow$. By Proposition [1.6], $K \subset \mathcal{F}^{\geq 2}(Z^0)$, therefore $d(Z^1) \subset \mathcal{F}^{\geq 2}(Z^0)$. We have obviously

\[ \mathcal{H}^0(\mathcal{F}(Z^\leq 1), d) = \mathcal{F}(Z^0)/K \cong \mathcal{H}. \]
To finish the proof, we construct inductively, for any \( n \geq 2 \), a collection \( Z^{<n} = Z^0 \oplus Z^1 \oplus \cdots \oplus Z^{n-1} \) (where \( Z^0 \) and \( Z^1 \) are those already constructed above), a differential \( d \) on \( F(Z^{<n}) \) (which extends that already constructed above) such that, if \( \rho : F(Z^{<n}) \to H \) is given by \( \rho|Z^0 := \overline{\rho} \) and \( \rho|Z^k := 0 \) for \( 1 \leq k < n \), the following conditions are satisfied:

(i) \( d(Z^{<n}) \subset F^{\geq 2}(Z^{<n}) \),

(ii) The map \( \rho \) induces an isomorphism \( H^0(\rho): H^0(F(Z^{<n}), d) \cong H \), and

(iii) \( H^k(F(Z^{<n}), d) = 0 \) for any \( 1 \leq k \leq n - 2 \).

For \( n = 2 \) these data have already been constructed above. Suppose we have constructed them for some \( n \geq 2 \). The inclusion \( F(Z^0) \hookrightarrow F(Z^{<n}) \) induces on \( F(Z^{<n}) \) the structure of an \( F(Z^0) \)-module which in turn induces on \( H(F(Z^{<n}), d) \) the structure of an \( H \)-module (since \( H^0(F(Z^{<n}), d) = H \)). Let

\[ Z^n := Q_H(H^{n-1}(F(Z^{<n}), d)) \quad \text{and} \quad Z^n := Z^n. \]

Let \( s^n: Z^n \to Z^{n-1}(F(Z^{<n}), d) \subset F(Z^{<n}) \) be an equivariant section of the composition of the projections

\[ Z^{n-1}(F(Z^{<n}), d) \xrightarrow{cl} H^{n-1}(F(Z^{<n}), d) \to Q_H(H^{n-1}(F(Z^{<n}), d)). \]

Extend the differential \( d \) by \( d|_{Z^n} := s^n \circ \downarrow \) and prove that the data thus constructed satisfy the conditions (i)\(_n\)–(iii)\(_n\). We show first that \( d^2 = 0 \). By Lemma 1.9 it is enough to verify that \( d^2 = 0 \) on generators. We have \( d^2|_{Z^{<n}} = 0 \) by the induction and \( d^2|_{Z^n} = 0 \) because \( d|_{Z^n} = s^n \circ \downarrow \) by definition and \( \text{Im}(s^n) \subset Z^{n-1}(F(Z^{<n}), d) \).

To prove the minimality (i)\(_n\) we must show that \( Z^{n-1}(F(Z^{<n}), d) \subset F^{\geq 2}(Z^{<n})^{n-1} \). Any \( u \in Z^{n-1}(F(Z^{<n}), d) \) obviously decomposes as \( u = z + w \) with \( z \in F(\mathbb{Z}^0 \oplus \mathbb{Z}^{n-1}) \) and \( w \in F(Z^{<n-1}) \). Further analysis on the TJ-grading shows that \( z = z' + z'' \), with \( z' \in Z^{n-1} \) and \( z'' \in D_{F(\mathbb{Z}^0})(F(\mathbb{Z}^0 \oplus \mathbb{Z}^{n-1})) \) while in fact \( w \in F^{\geq 2}(Z^{<n-1}) \). We infer from \( 0 = dz = dz' + dz'' + dw \) that \( dz' \) represents an element of \( Q_H(H^{n-2}(F(Z^{<n-1}), d))) \), therefore \( z' = 0 \) by the definition of \( dz' \) as \( s^{n-1} \circ \downarrow z' \), which means that \( u \) is decomposable.

The condition (ii)\(_n\) is obviously satisfied because our construction does not change \( H^0 \) while (iii)\(_n\) is satisfied by the construction of \( d \).}

Let \( T \) be a (nondifferential) operad. Its dual bar construction \( \Omega(T) \) is of the form \( (F(Z), d_Q) \) with \( Z := \# \downarrow T \) and \( d_Q = d_E \), the internal part \( d_I \) of the differential \( d_Q \) being zero as \( T \) has trivial differential. The collection \( Z \) decomposes as \( Z = Z^0 \oplus Z^1 \oplus \cdots \), where the collections \( Z^k \) are, for \( k \geq 0 \), defined by

\[
Z^k(n) := \begin{cases} Z(n), & \text{for } n = k + 2, \text{ and} \\ Z(n) = 0, & \text{otherwise.} \end{cases}
\]
Then clearly $d_{\Omega}(Z^k) \subset F^{k-1}(Z)$. Let $S$ be the operad defined by

$$S := \mathcal{F}(Z^0)/\text{Im}(d_{\Omega} : \mathcal{F}^1(Z) \to \mathcal{F}(Z^0))$$

and let $\rho : \Omega(\mathcal{T}) \to S$ be the obvious map. The following proposition is more or less the definition of the Koszulness as it is given in [11].

**Proposition 3.6.** The object $\rho : \Omega(\mathcal{T}) \to S$ constructed above is a bigraded model for $S$ if and only if the operad $S$ is Koszul. In this case $\pi(S) = \mathcal{T} = sS^!$, where $sS^!$ denotes the suspension of the Koszul dual of the operad $S$ (see [11] for the definition of the Koszul dual).

The proposition above explicitly describes the structure of bigraded models for Koszul operads. A nice thing is that the most important examples of operads encountered in life are Koszul, in contrast with topology, where the “Koszul spaces” (i.e. the spaces such that the universal enveloping algebra of the homotopy Lie algebra is Koszul) are not of much interest. The following, though a bit artificial, example shows that some explicit computation is possible also for non-Koszul operads.

**Example 3.7.** All the objects considered in this example are nonsymmetric. Let $S$ be the operad defined as $S := \mathcal{F}(E)/(I)$ with $E = E(3) = \text{Span}(y)$ and the ideal $I$ generated by $y \circ_1 y \in \mathcal{F}(E)(5)$. Define the collection $Z = Z^0 \oplus Z^1 \oplus \cdots$ by

$$Z^k(n) := \begin{cases} \text{Span}(\eta_k), & \text{for } n = 2k + 3, \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

where $\eta_k$ are generators of inner degree $k$, $k \geq 0$, let the differential $d$ on $\mathcal{F}(Z)$ be given by

$$d(\eta_k) := \sum_{a+b=k-1} (-1)^a \eta_a \circ_1 \eta_b,$$

and the map $\rho : (\mathcal{F}(Z), d) \to S$ by $\rho(\eta_0) := y$ and $\rho(\eta_k) := 0$ for $k \geq 1$. A direct computation shows that $\rho : (\mathcal{F}(Z), d) \to S$ is a bigraded model of $S$.

The operad $S$ is manifestly not Koszul, since the definition of the Koszulness as it is given in [11] requires the quadraticity, which is not the case of $S$.

The associated homotopy operad $\mathcal{T}$ can be described as follows. Let

$$\mathcal{T}(n) := \begin{cases} \text{Span}(\xi_k), & \text{for } n = 2k + 3, \text{ and} \\ 0, & \text{otherwise,} \end{cases}$$

where $\xi_k$ are generators of inner degree $k + 1$, $k \geq 0$, and define the structure maps by

$$\xi_k \circ_i \xi_l := \begin{cases} \xi_{(k+l)}, & \text{for } i = 1, \text{ and} \\ 0, & \text{otherwise} \end{cases}$$

The operad $\mathcal{T}$ can be presented also as $\mathcal{T} := \mathcal{F}(F)/(J)$, where $F = F(3) = \text{Span}(z)$ is spanned by a generator $z$ of inner degree 1 and the ideal $J$ is generated by $z \circ_2 z, z \circ_3 z \in \mathcal{F}(F)(5)$.  


Let us prove the following technical lemma (an analog of [14, Lemma 4.5]).

**Lemma 3.8.** Let \( \rho : (\mathcal{F}(Z), d) \to (\mathcal{H}, 0) \) be a TJ model of an operad \( \mathcal{H} \) as in Theorem 3.3. Suppose that \( \eta : \mathcal{H} \to \mathcal{F}(Z^0) \) is such that \( \rho \eta = 1 \). Suppose that \( D \) is a differential on \( \mathcal{F}(Z) \) such that \( (D - d) : Z^l \to \mathcal{F}(Z) \leq l - 2 \) for each \( 0 \leq l \leq n \). Then there exist equivariant maps \( v_n : Z(\mathcal{F}(Z) \leq n - 1, D) \to \mathcal{F}(Z) \leq n \) and \( a_n : Z(\mathcal{F}(Z) \leq n - 1, D) \to \mathcal{H} \) such that

\[
u = Dv_n(u) + \eta(a_n(u)), \text{ for any } u \in Z(\mathcal{F}(Z) \leq n - 1, D).
\]

**Proof.** For any \( u \in \mathcal{F}(Z^0) = Z(\mathcal{F}(Z) \leq 0, D) \) define \( a_1(u) := \text{cl}(u) \). Because \( \text{cl}(u) = \text{cl}(\eta a_1(u)) \), there obviously exists a map \( v_1 : \mathcal{F}(Z^0) \to \mathcal{F}(Z) \leq 1 \) such that

\[
u - \eta(a_1(u)) = dv_1(u) = Dv_1(u)
\]

and we may suppose, using the same trick of the symmetrization as in (3), that the map \( v_1 \) is equivariant.

Suppose we have already proved our theorem for \( n - 1, n \geq 2 \). Any \( D \)-cycle \( u \in Z(\mathcal{F}(Z) \leq n - 1, D) \) decomposes as \( u = \sum_{j=0}^{n-1} u_j, u_j \in \mathcal{F}(Z)^j \). Then obviously \( du_{n-1} = 0 \) and we may find some \( w_n \in \mathcal{F}(Z)^n \) with \( u_{n-1} = dw_n \). Now apply the induction assumption on the \( D \)-cycle \( (u - Dw_n) \in \mathcal{F}(Z) \leq n - 2 \) to see that

\[
u - Dw_n = Dv_{n-1}(u - Dw_n) + \eta(a_{n-1}(u - Dw_n)).
\]

We see (using the same trick as in (3)) that \( w_n \) can be chosen to depend equivariantly on \( u \), and we may define \( v_n(u) := w_n(u) + v_{n-1}(u - Dw_n(u)) \) and \( a_n := a_{n-1}(u - Dw_n(u)) \).

In the following theorem we show that for a differential operad there exists an analog of the filtered model of [14, par. 4].

**Theorem 3.9.** Let \( (\mathcal{S}, d_{\mathcal{S}}) \) be a differential operad (with \( \mathcal{S}(1) = 0 \)) and let \( \mathcal{H} := \mathcal{H}(\mathcal{S}, d_{\mathcal{S}}) \) be its homology. Let \( \rho : (\mathcal{F}(Z), d) \to (\mathcal{H}, 0) \) be the TJ-model of \( \mathcal{H} \) as in Theorem 3.3. Then there exists a differential \( D \) on \( \mathcal{F}(Z) \) and a homomorphism \( \alpha : (\mathcal{F}(Z), D) \to (\mathcal{S}, d_{\mathcal{S}}) \) such that

(i) the differential \( D \) is a perturbation of the differential \( d \) in the sense that \( (D - d)(Z^k) \subseteq \mathcal{F}(Z) \leq k - 2 \) for any \( k \geq 0 \), and

(ii) the map \( \alpha \) induces an isomorphism \( \mathcal{H}(\alpha) : \mathcal{H}(\mathcal{F}(Z), D) \to \mathcal{H}(\mathcal{S}, d_{\mathcal{S}}) \).

An object \( \alpha : (\mathcal{F}(Z), D) \to (\mathcal{S}, d_{\mathcal{S}}) \) as above is called the filtered model of the differential operad \( (\mathcal{S}, d) \).
Proof. Fix an equivariant map $\eta: \mathcal{H} \to \mathcal{F}(Z^0)$ with $\rho \eta = \mathbb{I}$. Let $D := 0$ on $\mathcal{F}(Z^0)$ and let $\alpha|_{Z^0}$ be defined by the commutativity of the diagram

$$
\begin{array}{ccc}
Z^0 & \xrightarrow{\alpha|_{Z^0}} & Z(S, d_S) \\
\downarrow{\rho} & & \downarrow{\text{cl}} \\
\mathcal{H} & & \\
\end{array}
$$

Put $D := d$ on $Z^1$. Then $\text{cl}(\alpha D) = \text{cl}(\alpha d) = 0$ on $Z^1$ and we can define $\alpha|_{Z^1}$ by the commutativity of the diagram

$$
\begin{array}{ccc}
Z^1 & \xrightarrow{\alpha|_{Z^1}} & S \\
\downarrow{D} & & \downarrow{d_S} \\
\mathcal{F}(Z^0) & \xrightarrow{\alpha} & S \\
\end{array}
$$

Suppose that $z \in Z^2$, then $dz \in \mathcal{F}^1(Z)$, hence $Ddz = d^2z = 0$ and $d_S(\alpha dz) = 0$. Extend $D$ by $D(z) := d(z) - \eta(\text{cl}(\alpha dz))$, for $z \in Z^2$. Since, for any $a \in \mathcal{H}$, $\text{cl}(\alpha(\eta(a))) = \rho \eta(a) = a$,

$$
\text{cl}(\alpha Dz) = \text{cl}(\alpha dz) - \text{cl}(\alpha \eta(\text{cl}(\alpha dz))) = 0,
$$

for all $z \in Z^2$. This means that we can extend $\alpha$ by the commutativity of the diagram

$$
\begin{array}{ccc}
Z^2 & \xrightarrow{\alpha|_{Z^2}} & S \\
\downarrow{D} & & \downarrow{d_S} \\
\mathcal{F}(Z)^{<2} & \xrightarrow{\alpha} & S \\
\end{array}
$$

Suppose we have already extended $D$ and $\alpha$ to $\mathcal{F}(Z)^{\leq n}$ for some $n \geq 2$. For any $u \in Z^{n+1}$, $D(du)$ is a $D$-cycle in $\mathcal{F}(Z)^{\leq n-1}$ and Lemma 3.8 gives

$$
D(du) = Du_n(u) + \eta(a_n(u)).
$$

Applying $\alpha$ to the above equation we get

$$
d_S \alpha(du) = d_S \alpha(v_n(u)) + \alpha \eta(a_n(u)),
$$

which implies that $\text{cl}(\alpha \eta(a_n(u))) + a_n(u) = 0$, therefore $D(du - v_n(u)) = 0$. Extend $D$ on $Z^{n+1}$ by

$$
Du := du - v_n(u) - \eta(\text{cl}(\alpha(du - v_n(u))).
$$

then obviously $\text{cl}(\alpha Du) = 0$ and we can extend $\alpha$ to $Z^{n+1}$ so that the diagram

$$
\begin{array}{ccc}
Z^{n+1} & \xrightarrow{\alpha|_{Z^{n+1}}} & S \\
\downarrow{D} & & \downarrow{d_S} \\
\mathcal{F}(Z)^{<n+1} & \xrightarrow{\alpha} & S \\
\end{array}
$$
commutes.

To prove that \( H(\alpha) : H(\mathcal{F}(Z), D) \rightarrow H(S, d_S) \) is an isomorphism, observe that \( H(\eta) : H(S, d_S) \rightarrow H(\mathcal{F}(Z), D) \) is an epimorphism by Lemma 3.8 while obviously \( H(\alpha)H(\eta) = 1 \) which implies that \( H(\eta) \) is an isomorphism and \( H(\alpha) \) its inverse.

Let \( S \) be a nonsymmetric operad (possibly differential) and let \( \mathcal{M} \) be some of the models of \( S \) constructed above made in the category of nonsymmetric operads. We can consider \( S \) equally well as a symmetric operad \( S_\Sigma \) with the trivial action of the symmetric group. Let \( \mathcal{M}_\Sigma \) be the corresponding model for \( S_\Sigma \) in the category of symmetric operads. We may then formulate the following principle.

3.10. Imbedding Principle. The models \( \mathcal{M} \) and \( \mathcal{M}_\Sigma \) are isomorphic, \( \mathcal{M} \cong \mathcal{M}_\Sigma \).

Loosely speaking, the above principle says that if \( S \) is an operad whose axioms can be formulated without making use of the symmetric group action (for example the operad \( \text{Ass} \) for associative algebras or the operads from Example 3.7), then all the “higher syzygies” of \( S \) do not contain the action of the symmetric group as well.

4. Cotangent cohomology.

In [21] we constructed, for an arbitrary equationally given category, a cohomology theory (called the cotangent cohomology to stress the analogy with the cotangent cohomology of a commutative algebra) as “the best possible” cohomology theory controlling the deformations of the objects of this category. Our cohomology was defined only in small, relevant degrees, and we raised the question when this cohomology can be naturally extended in all degrees. In this paragraph we show that such an extension exists for an algebraic equationally given category, i.e. for a category of algebras over some operad. We indicate also some applications and connections with [5, 11].

Let \( S \) be an operad and \( M \) an \( S \)-module (see Definition 1.4). By a degree \( p \) derivation of the operad \( S \) in \( M \) we mean a sequence \( \theta = \{ \theta(n) : S(n) \rightarrow M(n); \ n \geq 2 \} \) of equivariant degree \( p \) linear maps such that

\[
\theta(m + n - 1)(f \circ_i g) = \theta(m)(f) \circ_i^R g + (-1)^{|f||f|} f \circ_i^L \theta(n)(g),
\]

for any \( f \in S(m), g \in S(n), m, n \geq 1 \) and \( 1 \leq i \leq n \). We denote by \( \text{Der}_p(S, M) \) the vector space of all degree \( p \) derivations of \( S \) in \( M \).

Let \( \rho : (\mathcal{F}(Z), d) \rightarrow (S, 0) \) be a bigraded model of \( S \) as in Theorem 3.3. If \( M \) is an \( S \)-module, then the homomorphism \( \rho \) induces on \( M \) a structure of an \( \mathcal{F}(Z) \)-module and we may consider the space \( C^*(S; M) := \text{Der}_*(\mathcal{F}(Z), M) \) of derivations of \( \mathcal{F}(Z) \) in \( M \).
**Proposition 4.1.** The restriction induces an isomorphism $C^*(S;M) \cong \text{Coll}_*(Z,M)$. If we denote

$$C^{p,*}(S;M) := \{ \theta \in C^*(S;M); \theta|_Z = 0 \text{ for } k \neq p \},$$

then the formula $\nabla(\theta) := \theta \circ d$ gives a well-defined endomorphism $\nabla$ of $C^*(S,M)$ with

$$\nabla(C^{p,q}(S;M)) \subset C^{p+1,q-1}(S;M),$$

and, moreover, $\nabla$ is a differential, $\nabla^2 = 0$.

**Proof.** The isomorphism is a consequence of the universal property of $F(Z)$. The only thing which has to be checked carefully is that $\nabla(\theta)$ is a derivation. Let us verify this. Let $\theta \in C^{p,q}(S;M), f \in F(Z)(m), g \in F(Z)(n)$ and $1 \leq i \leq n$. Then

$$\nabla(\theta)(f \circ_i g) = \theta d(f \circ_i g) = \theta(df \circ_i g) + (-1)^{|f|} \cdot \theta(f \circ_i dg) = \theta(df) \circ_i^R \rho(g) + (-1)^{|f|} \cdot \theta(f) \circ_i^L \rho(dg) + (1)^{|f||q+1|} \cdot \rho(f) \circ_i^L \theta(dg)$$

$$= \theta(df) \circ_i^R \rho(g) + (-1)^{f(|q+1|)} \cdot \rho(f) \circ_i^L \theta(dg)$$

$$= \nabla(\theta)(f) \circ_i^R \rho(g) + (-1)^{|f||q+1|} \cdot \rho(f) \circ_i^L \nabla(\theta)(g),$$

because $\rho \circ d = 0$. The condition $\nabla^2 = 0$ is easy to verify, $\nabla^2(\theta) = \nabla(\theta)(d) = \theta(d^2) = 0$. 

**Definition 4.2.** Let $S$ be an operad and $M$ an $S$-module. Then the cotangent cohomology of $S$ with coefficients in $M$, $T^{*,*}(S;M)$, is defined as

$$T^{*,*}(S;M) := H^{*,*}(C^{*,*}(S;M), \nabla).$$

The following two definitions are motivated by one of the most important concepts of rational homotopy theory, by the notion of the formality. We say that a differential operad $(T,d_T)$ is *formal* if it has the same minimal model as its homology operad $H(T,d_T)$. A (nondifferential) operad $H$ is said to be *intrinsically formal* if any differential operad $(T,d_T)$ with $H(T,d_T) \cong H$ is formal. The following proposition shows that intrinsic formality is obstructed by the cotangent cohomology constructed above, compare the corresponding statement of [14].

**Theorem 4.3.** Any (nondifferential) operad $H$ with $T^{\geq 2,-1}(H,H) = 0$ is intrinsically formal.

**Proof.** Let $\rho : (F(Z),d) \to (H,0)$ be a bigraded model of $H$. Consider the space $\text{Der}_*(F(Z))$ of derivations of $F(Z)$ with the differential $\Delta$ defined by $\Delta(\phi) := d \circ \phi - (-1)^{|\phi|}\phi \circ d$. We have on $\text{Der}_*F(Z)$ a second grading given by

$$\text{Der}_q^q(F(Z)) := \{ \phi \in \text{Der}_*(F(Z)); \phi(Z^k) \subset F^{k-q}(Z), k \geq 0 \}$$
and $\Delta(D_{p}^{\rho}(\mathcal{F}(Z)) \subset D_{p+1}^{\rho}(\mathcal{F}(Z))$. Define $\chi : \text{Der}^{*}_{\rho}(\mathcal{F}(Z)) \to C^{*,*}(\mathcal{H}, \mathcal{H})$ by $\chi(\phi) := (-1)^{|\phi|} \rho \circ \phi$. We have $\chi(\Delta(\phi)) = \chi(d\phi - (-1)^{|\phi|} \phi d) = \chi(\phi) \circ d = \nabla(\chi(\phi))$, so $\chi$ is a map of differential spaces, $\chi : (\text{Der}^{*}_{\rho}(\mathcal{F}(Z)), \Delta) \to (C^{*,*}(\mathcal{H}, \mathcal{H}), \nabla)$ and a spectral sequence argument based on the filtration induced by the TJ-grading shows that $\chi$ is a homology isomorphism. We may now reformulate the hypothesis of the theorem as

$$H^{>2}_{n-1}(\text{Der}^{*}_{\rho}(\mathcal{F}(Z)), \Delta) = 0.$$  

Let $(\mathcal{T}, d_{\mathcal{T}})$ be an operad such that $\mathcal{H} \cong \mathcal{H}(\mathcal{T}, d_{\mathcal{T}})$. By Theorem 3.3, the filtered model of $(\mathcal{T}, d_{\mathcal{T}})$ is of the form $(\mathcal{F}(Z), D)$, where $D$ is a perturbation of $d$ in the sense that $D = d + d_{2} + d_{3} + \cdots$, $d_{k} \in \text{Der}^{*}_{k}(\mathcal{F}(Z))$. Similarly as in [14] we may see that the triviality of such a perturbation is obstructed by (part of) the cohomology of $(\mathcal{T}, d_{\mathcal{T}})$, therefore the filtered model of $(\mathcal{T}, d_{\mathcal{T}})$ is isomorphic with the bigraded model of $\mathcal{H}$ which is minimal, by definition.

As a consequence of our computations we get the following stunning result.

**Proposition 4.4.** Let $\mathcal{H}$ be a (nondifferential) operad such that $\mathcal{H}(n)$ is concentrated in degree $N(n - 1)$, for some $N$ and any $n \geq 2$. Then $\mathcal{H}$ is intrinsically formal.

**Proof.** Let $\rho : (\mathcal{F}(Z), d) \to \mathcal{H}$ be the bigraded model of $\mathcal{H}$. We show that the collection $Z^{k}$ has the property that $Z^{k}(n)$ is concentrated in degree $N(n - 1) + k$, for any $n \geq 2$ and $k \geq 0$. Let us prove this statement inductively.

For $k = 0$, the collection $Z^{0}$ consists of generators of $\mathcal{H}$, therefore $Z^{0}(n)$ is concentrated in degree $N(n - 1)$ by the assumption. Suppose we have proved the statement for any $k < l$, for some $l \geq 1$. We prove that $\mathcal{F}(Z^{<l})^{l-1}(n)$ is concentrated in degree $N(n - 1) + (l - 1)$ which would give the induction step, since $Z^{l}$ was constructed as the suspension of the space of generators of $\mathcal{H}^{l-1}(\mathcal{F}(Z^{<l}), d)$, see the proof of Theorem 3.3.

Any element $w \in \mathcal{F}(Z^{<l})^{l-1}(n)$ is a sum of elements of the form

$$w_{1} \circ_{j_{1}}(w_{2} \circ_{j_{2}}(\cdots \circ_{j_{p-1}}w_{p}) \cdots),$$

for some $p \geq 2$, $w_{i} \in Z^{s_{i}}(n_{i})$, with $s_{1} + \cdots + s_{p} = l - 1$ and $n_{1} + \cdots + n_{p} = n + p - 1$. By the induction assumption, $\deg(w_{i}) = N(n_{i} - 1) + s_{i}, 1 \leq i \leq p$, therefore

$$\deg(w) = \sum_{i=1}^{p}(N(n_{i} - 1) + s_{i}) = N(n - 1) + (l - 1)$$

as claimed.
Suppose $\theta \in C^{2,-1}(\mathcal{H}, \mathcal{H})$. We show that $\theta = 0$. It is enough to verify it on generators, so let us pick some $z \in Z^k(n)$; we already know that $\deg(z) = N(n - 1) + k$. Since $\theta(z) \in \mathcal{F}^{k-2}(n)$, $\theta(z)$ must be a sum of elements of the form

$$w_1 \circ_{j_1} (w_2 \circ_{j_2} (\cdots \circ_{j_{p-1}} w_p) \cdots),$$

for some $p \geq 2$, $w_i \in Z^{s_i}(n_i)$, with $s_1 + \cdots + s_p \leq k - 2$ and $n_1 + \cdots + n_p = n - p + 1$. The similar degree argument as above gives that $\deg(\theta(z)) \leq N(n - 1) + k - 2 = \deg(z) - 2$, which is impossible, since we should have $\deg(\theta(z)) = \deg(z) - 1$.

We proved that $C^{2,-1}(\mathcal{H}, \mathcal{H}) = 0$, therefore the relevant part of the cohomology of Theorem 4.3 vanishes.

Notice that an equivalent way to formulate the hypothesis of Proposition 4.4 is to say that the operad $\mathcal{H}$ is an $N$-fold suspension $s^N \mathcal{R}$ of a nongraded (= concentrated in degree zero) operad $\mathcal{R}$.

**Example 4.5.** In this example we construct a differential operad which is not formal. Let $E = E(2) \oplus E(4)$ be the collection given by $E(2) := \text{Span}(f)$, $E(4) := \text{Span}(g)$, where $f$ (resp. $g$) is a generator of inner degree 1 (resp. 4). Let $T := F(E)$ and define a differential $d_T$ on $T$ by

$$d_T(f) := 0 \text{ and } d_T(g) := f \circ_1 f \circ_1 f.$$

Then the differential operad $\Omega(T, d_T)$ is not formal.

To see this, recall that $\Omega(T, d_T)$ is of the form $(F(Z), d_\Omega)$ with $Z = \# \downarrow F(E)$. Let $Z^k := \# \downarrow F^{k+1}(E) \subset \# \downarrow F(E) = Z$. Then $d_E(Z^k) \subset F(Z)^{k-1}$ (where $d_E$ is the external part of the differential $d_\Omega$) and it is not difficult to see that $(F(Z), d_E)$ is the bigraded model of the operad $\mathcal{Y} := F(Z)/F^{\geq 1}(Z)$, which is nothing else but the collection $\downarrow \#E$ with trivial structure maps. The differential $d_\Omega = d_E + d_I$ is then a nontrivial perturbation of the “formal” differential $d_E$.

Therefore $\Omega(T, d_T)$ is a differential operad which is not formal, its homology operad is isomorphic to the operad $\mathcal{Y}$ described above and its homotopy operad $\pi \Omega(T, d_T)$ is the homology operad $\mathcal{H}(T, d_T)$.

A topological analog of this types of examples would be a nonformal space obtained from a wedge of spheres by attaching a cell by a suitable map.

Let $A$ be an $\mathcal{S}$-algebra, i.e. a vector space $V$ together with a homomorphism $A : \mathcal{S} \to \text{End}(V)$ and let $k_V$ be the operad $\text{End}(V)$ together with the $\mathcal{S}$-module structure induced by $A$ (see Example 1.5).

**Definition 4.6.** Define the $\mathcal{S}$-cohomology of $A$ (with coefficients in itself) as

$$H^{*,*}_{\mathcal{S}}(A) := T^{*,*}(\mathcal{S}; k_V).$$
It is possible to show that, for a Koszul operad $S$, the cochain complex $(C^{\ast,\ast}(S), \nabla)$ used in the definition of the above cohomology coincides with the cochain complex used in the definition of the cohomology introduced in [11] (or, more precisely, with the dual form of this complex introduced in [3]). Thus the above cohomology is a natural extension of the construction of [11] for algebras over arbitrary, not necessary Koszul, operads, hence an extension of the “classical cohomology” as well.

Using a trick similar to the one in [5] we may extend the cohomology of Definition 4.6 for an arbitrary coefficient module $N$ (denote this extension by $H_{S}^{\ast,\ast}(A; N)$) and we may show that $H_{S}^{\ast,\ast}(A; N)$ is trivial whenever $A$ is a free $S$-algebra. A theorem of [2] then says that $H_{S}^{\ast,\ast}(A; N)$ coincides with the Barr-Becker cohomology of $A$ with coefficients in $N$ (notice that the category of $S$-algebras is obviously tripleable).

The above observation may seem to be an extremely strong result. For example, let $Comm$ be the operad for the category of commutative algebras. This operad is Koszul (see [11]), hence the cohomology of Definition 4.6 coincides with the cohomology of [11] which is in this case manifestly isomorphic to the Harrison cohomology. Our claim about the vanishing of $H_{S}^{\ast,\ast}(A; N)$ implies that the Harrison cohomology of the polynomial algebra over a field of characteristic zero is trivial, for arbitrary coefficients. This is a very difficult result and we do not know any reasonably simple proof of this statement, which was proved by M. Barr using the Hodge decomposition trick. The point is that the Koszulness of $Comm$ is in fact equivalent with the vanishing property for the Harrison cohomology, as it was pointed out in [11].

5. Homotopy $S$-algebras, homotopy everything spaces.

By a homotopy algebra we mean an algebra over a minimal (in the sense of Theorem 3.1) differential operad. If $A : (\mathcal{F}(M), d) \to (\text{End}(V), d_{\text{End}})$ is a homotopy algebra structure on $(V, d_{V})$ and $(\mathcal{F}(M), d)$ is a minimal model of some $(S, d_{S})$ we sometimes say also that $A$ is a homotopy $(S, d_{S})$-algebra structure on $(V, d_{V})$.

Example 5.1. If the operad $S$ is Koszul, then $(\mathcal{F}(Z), d) = \Omega(sS)$ by Proposition 3.6 and our definition coincides with that of [11]. Especially, homotopy associative algebras are $A(\infty)$-algebras of [25], homotopy commutative associative algebras are the balanced $A(\infty)$-algebras of [13, 20] while homotopy Lie algebras are strong Lie homotopy algebras of [12, 18, 17].

Example 5.2. Let us consider the nonsymmetric operad $S$ of Example 3.7. An $S$-algebra on a differential graded vector space $(V, d_{V})$ is given by a degree zero map $\mu : V^{|S|} \to V$ which commutes with the differential and satisfy $\mu(\mu \otimes 1^{2}) = 0$.

A homotopy $S$-algebra is then a differential space $(V, d_{V})$ together with degree $k$ linear maps
\[ h_k : V^{\otimes(2k+3)} \to V, \ k \geq 0, \text{ satisfying} \]
\[
0 = dh_0 + h_0(d \otimes 1^2 + 1 \otimes d \otimes 1 + 1^2 \otimes d),
\]
\[
h_0(h_0 \otimes 1^2) = dh_1 - h_1(d \otimes 1^4 + \cdots + 1^4 \otimes d),
\]
\[
-h_1(h_0 \otimes 1^4) + h_0(h_1 \otimes 1^2) = dh_2 + h_2(d \otimes 1^6 + \cdots + 1^6 \otimes d),
\]
\[
\sum_{i+j=k} (-1)^j \cdot h_i(h_j \otimes 1^{(2i+2)}) = dh_k + (-1)^k \cdot h_k(d \otimes 1^{(2k+2)} + \cdots + 1^{(2k+2)} \otimes d).
\]

**Definition 5.3.** Let \((\mathcal{S}, d_{\mathcal{S}})\) be a differential operad, \(A : (\mathcal{S}, d_{\mathcal{S}}) \to (\text{End}(V), d_{\text{End}})\) and \((\mathcal{S}, d_{\mathcal{S}})\)-algebra structure on a differential space \((V, d_V)\) and \(\nu : (\mathcal{F}(M), d) \to (\mathcal{S}, d_{\mathcal{S}})\) the minimal model of \((\mathcal{S}, d_{\mathcal{S}})\). We then call the structure \(A \circ \nu : (\mathcal{F}(M), d) \to (\text{End}(V), d_{\text{End}})\) the homotopy structure associated to \(A\).

Let us discuss the possibility of an intrinsic characterization of the associated homotopy structure defined above. We first state the following very natural definition.

Let \(\mathcal{T}\) and \(\mathcal{U}\) be two (nondifferential) operads, let \(H\) be a graded vector space and let \(a : \mathcal{T} \to \text{End}(H)\) and \(b : \mathcal{U} \to \text{End}(H)\) be two structures on \(H\). We say that the structures \(a\) and \(b\) are equivalent (= are the same) if there exists an operad isomorphism \(\varphi : \mathcal{T} \to \mathcal{U}\) such that the diagram

\[
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{a} & \text{End}(H) \\
\varphi \downarrow & & \downarrow b \\
\mathcal{U} & \end{array}
\]

commutes. The first important property of the associated homotopy structure is that it induces on \(H = H(V, d_V)\) the same homology structure as the operad \((\mathcal{S}, d_{\mathcal{S}})\). This is immediately obvious from the diagram

\[
\begin{array}{ccc}
\mathcal{H}(F(M), d) & \xrightarrow{\mathcal{H}(A \circ \nu)} & \mathcal{H}(A) \\
\mathcal{H}(\nu) \downarrow & & \downarrow \mathcal{H}(\text{End}(V), d_{\text{End}}) \cong (\text{End}(H), 0) \\
\mathcal{H}(\mathcal{S}, d_{\mathcal{S}}) & \xrightarrow{\mathcal{H}(A)} & \end{array}
\]

In this context, it is interesting to formulate the following observation.

**Observation 5.4.** Let \(A : (\mathcal{S}, d_{\mathcal{S}}) \to (\text{End}(V), d_{\text{End}})\) be an \((\mathcal{S}, d_{\mathcal{S}})\)-algebra structure on \((V, d_V)\) and denote \(\mathcal{H} := \mathcal{H}(\mathcal{S}, d_{\mathcal{S}})\). Then there exists on \((V, d_V)\) an \(\mathcal{H}\)-algebra structure inducing on \((V, d_V)\) the same homology structure as \(A\).
To see this, let $\Phi : (\text{End}(H), 0) \to (\text{End}(V), d_{\text{End}})$ be the (noncanonical) homomorphism constructed in Lemma 1.10. Then $\Phi \circ \mathcal{H}(A)$ is the requisite $\mathcal{H}$-algebra structure.

The second important property of the associated homotopy algebra structure is that it factors through $A$. This means, loosely speaking, that both the operations and the axioms of the associated homotopy structure are expressible in terms of compositions of the operations of the $(S, d_S)$-structure. Using the terminology borrowed from the universal algebra we may say that the associated homotopy algebra structure is a specialization of $A$.

The third important property of the associated homotopy algebra structure is the following universality. Let $(U, d_U)$ is another differential operad which satisfies the above conditions, then there exists a homomorphism $\psi : (F(M), d) \to (U, d_U)$ inducing the isomorphism in homology such that the diagram

$$
\begin{array}{ccc}
(F(M), d) & \xrightarrow{\nu} & (S, d_S) \\
\psi \downarrow & & \downarrow \\
(U, d_U) & \xrightarrow{\iota} & (S, d_S)
\end{array}
$$

commutes up to homotopy. Of course, strictly speaking, we have no right to formulate this statement as we never have introduced the notion of a homotopy for maps of operads (see also the comments in the introduction), but we may at least claim that the three properties listed above determines uniquely the associated homotopy algebra structure up to an acyclic factor.

The following proposition shows the relevance of the notion of the (intrinsic) formality introduced in the previous paragraph.

**Proposition 5.5.** Suppose that the differential operad $(S, d_S)$ is formal and let $\mathcal{H} = \mathcal{H}(S, d_S)$ be its homology operad. Then every $(S, d_S)$-algebra admits an associated homotopy $\mathcal{H}$-algebra structure.

The above statement is in particular true for any operad $(S, d_S)$ whose homology operad $\mathcal{H} = \mathcal{H}(S, d_S)$ is concentrated in degree zero or which is an (iterated) suspension of such an operad.

**Proof.** Let $A : (S, d_S) \to (\text{End}(V), d_{\text{End}})$ be an $(S, d_S)$ algebra structure on $(V, d_V)$. By the definition of formality, the minimal model of $(S, d_S)$ is the same as the minimal model (= the bigraded model) $\nu : (F(M), d) \to (\mathcal{H}, 0)$ of $\mathcal{H}$. Let $\alpha : (F(M), d) \to (S, d_S)$ be the corresponding minimal model map. The composition $A \circ \alpha : (F(M), d) \to (\text{End}(V), d_{\text{End}})$ is then the desired associated $\mathcal{H}$-homotopy algebra structure. The second part follows from Proposition 4.4.

Having in mind future applications we generalize our definitions a bit. Let $(S, d_S)$ be a differential operad with the condition $S(1) = 0$ replaced by $H(S(1), d_S(1)) = 0$. Let $(\tilde{S}, d_{\tilde{S}})$ be its universal covering constructed in section 1. The inclusion $\iota : (\tilde{S}, d_{\tilde{S}}) \hookrightarrow (S, d_S)$ is a differential
operad map inducing an isomorphism in homology and for any \((S, d_S)\)-algebra structure there exists a natural \((\tilde{S}, d_{\tilde{S}})\)-algebra structure induced from the former one by \(\iota\). In the rest of the paper, by an associated homotopy algebra structure we mean a homotopy structure associated to this induced \((\tilde{S}, d_{\tilde{S}})\)-algebra structure.

**Example 5.6.** Let \((O, d_O)\) be an operad such that \(H(O, d_O)\) is isomorphic to the operad \(Comm\) for commutative algebras. Then any \((O, d_O)\)-algebra structure admits an associated balanced \(A(\infty)\)-algebra structure, see \([20]\) for the definition. This is in particular true for so-called May algebras (see \([13]\)) which are, by definition, acyclic operads augmented over the operad \(Comm\). Our statement follows from the fact that the operad \(Comm\) is concentrated in degree zero (notice that \(Comm\) is, in a sense, a trivial symmetric operad, \(Comm(n) = k\) for any \(n \geq 2\)) and from the fact that homotopy \(Comm\)-algebras are balanced \(A(\infty)\)-algebras, see the discussion in Example 5.1. A nonsymmetric analog of the above claim is the following.

Let \((U, d_U)\) be a nonsymmetric operad such that \(H(U, d_U) \cong Ass\), where \(Ass\) denotes the nonsymmetric operad for the category of associative algebras. As above, the operad \(Ass\) is, in a sense, the trivial nonsymmetric operad, \(Ass(n) = k\) for any \(n \geq 2\). Any \((U, d_U)\)-algebra then admits an associated \(A(\infty)\)-algebra structure. This again follows from the fact that the operad \(Ass\) is nongraded and from the remarks of Example 5.1.

**Example 5.7.** In this example we show that both the bigraded model of an operad and the associated homotopy algebra structure is implicitly hidden in the 30 years old papers \([25]\). Recall that the associahedron \(K_n\) is, for \(n \geq 2\), an \((n - 2)\)-dimensional polyhedron whose \(i\)-dimensional cells are, for \(0 \leq i \leq n - 2\), indexed by all (meaningful) insertions of \((n - i - 2)\) pairs of brackets between \(n\) independent indeterminates, with suitably defined incidence maps. There exists a natural structure of a nonsymmetric topological operad on the nonsymmetric collection \(K := \{K_n; n \geq 2\}\). A topological space \(X\) is called an \(A(\infty)\)-space if it admits an action of \(K\), see \([25]\) for details.

On the other hand, let \(Ass\) be the nonsymmetric operad for associative algebras, i.e. the operad with \(Ass(n) = k\) for any \(n \geq 2\) and all the structure operations equal to the identity, \(\circ_i = 1 : k \otimes k \cong k \to k\). Let \((\mathcal{F}(Z), d) \to (Ass, 0)\) be its bigraded model. We know from Proposition 3.6 and from the fact that the operad \(Ass\) is Koszul with \(Ass^1 = Ass\) (see \([11]\)) that

\[(\mathcal{F}(Z), d) \cong (\Omega(sAss), d_\Omega),\]

and we can obtain from this the following very explicit description of the bigraded model \((\mathcal{F}(Z), d)\): there are some \(\xi_k \in Z(k + 2)\) of inner degree \(k\) such that \(Z^k = \text{Span}(\xi_k), k \geq 0\). The differential \(d\) is given by the formula

\[d(\xi_k) = \sum (-1)^{(a+1)(i+1)+a} \cdot \xi_a \circ_i \xi_b,\]
where the summation is taken over all \(a, b \geq 0\) with \(a + b = k - 1\) and \(1 \leq i \leq b + 2\). Due to the integrality of the coefficients at the right-hand side of the above equation, we can consider an obvious integral variant of the above construction, \((\mathcal{F}_Z(Z), d)\).

Let \((\CC(K), d_C)\) be the nonsymmetric operad of cellular chains on \(K\), i.e. the operad given by \((\CC(K), d_C)(n) := (CC(K_n), d_C)\) and the structure maps induced by those of \(K\); here \((CC(-), d_C)\) denotes the cellular chain complex functor. According to the definition of the associahedra recalled above, there is, for any \(n \geq 2\), exactly one top-dimensional cell \(u_n\) in \(K_n\) (given by the insertion of no pair of brackets between \(n\)-indeterminates and having dimension \(= (n - 2)\)) and it is a very stimulating exercise to prove that the map \(J : (\mathcal{F}_Z(Z), d) \rightarrow (\CC(K), d_C)\) given by

\[\mathcal{F}_Z(Z) \ni \xi_n \longmapsto 1 \cdot u_n \in CC(K)\]

is in fact an isomorphism of differential operads.

Suppose that a topological space \(X\) admits an action of \(K\). Then there exists an induced action of the singular chain operad \((CS(K), d_S)\) on the singular chain complex \((CS(X), d_S)\) of \(X\) and the inclusion \((\CC(K), d_C) \hookrightarrow (CS(K), d_S)\) induces on \((CS(X), d_S)\) a structure of an \((\CC(K), d_C)\)-algebra. This is exactly the \(A(\infty)\)-structure whose existence is proved in \([25]\). Summing up the above remarks we see that this structure is in fact an integral variant of the associated homotopy algebra structure discussed in this paper.

We may say also that the existence of the associated \(A(\infty)\)-structure was in \([25]\) proven as a consequence of a very rich geometrical structure on the corresponding topological operad. Proposition \([5.4]\) says that structures of this type exist, under fairly general assumptions, regardless of the concrete geometric structure of the corresponding topological operad.

The above example also explains a close relationship between the coherence implying the asphericity of the associahedra on one hand, and the Koszulness of the operad \(Ass\) on the other hand, the fact that puzzled us a lot in \([21]\).

**Example 5.8.** Let \(N\) be the symmetric discrete topological operad with \(N(n) = P\) for \(n \geq 2\), where \(P\) is the one-point discrete topological space. Any topological operad \(\mathcal{E}\) such that any \(\mathcal{E}(n)\) is connected, \(n \geq 2\), is obviously augmented over \(N\).

Suppose moreover that \(\mathcal{E}\) is acyclic (meaning that \(\mathcal{E}(n)\) is acyclic for any \(n \geq 2\)). Then the singular chain operad \((CS(\mathcal{E}), d_S)\) is obviously augmented over \((CS(N), d_S) = (Comm, 0)\) and the augmentation induces a homology isomorphism. If \(X\) is a topological space on which such an operad acts, then there exists an associated balanced \(A(\infty)\)-algebra structure on the singular chain complex \((CS(X), d_S)\) of \(X\). This is in particular true for homotopy-everything spaces introduced in \([23]\).

There is an obvious nonsymmetric analog of the previous example. Let \(M\) be the nonsymmetric discrete topological operad with \(M(n) = P\) for \(n \geq 2\). Any connected nonsymmetric topological
operad is clearly augmented over \( \mathcal{M} \).

If \( \mathcal{A} \) is a nonsymmetric aspheric topological operad, then the singular nonsymmetric operad \( (CS(\mathcal{A}), d_S) \) is obviously augmented over \( (CS(\mathcal{M}), d_S) = (\text{Ass}, 0) \) and the augmentation induces an isomorphism in homology. If \( Y \) is a topological space endowed with an action of \( \mathcal{A} \), then the singular chain complex \( (CS(Y), d_S) \) of \( Y \) has an associated \( A(\infty) \)-algebra structure.

**Example 5.9.** By an \((m, n)\)-algebra we mean a (graded) vector space \( V \) together with two bilinear maps, \( - \cup - : V \otimes V \to V \) of degree \( m \), and \([-,-] : V \otimes V \to V \) of degree \( n \) (\( m \) and \( n \) are natural numbers), such that, for any homogeneous \( a, b, c \in V \),

(i) \( a \cup b = (-1)^{|a||b|+m} \cdot b \cup a \),

(ii) \([a, b] = -(-1)^{|a||b|+n} \cdot [b, a] \),

(iii) \(- \cup - \) is associative in the sense that

\[
a \cup (b \cup c) = (-1)^{m\cdot(|a|+1)} \cdot (a \cup b) \cup c,
\]

(iv) \([-,-]\) satisfies the following form of the Jacobi identity:

\[
(-1)^{|a|\cdot(|c|+n)} \cdot [a, [b, c]] + (-1)^{|b|\cdot(|a|+n)} \cdot [b, [c, a]] + (-1)^{|c|\cdot(|b|+n)} \cdot [c, [a, b]] = 0,
\]

(v) the operations \(- \cup -\) and \([-,-]\) are compatible in the sense that

\[
(-1)^{m\cdot|a|} [a, b \cup c] = [a, b] \cup c + (-1)^{(|b|\cdot|c|+m)} [a, c] \cup b.
\]

Obviously \((0,0)\)-algebras are exactly (graded) Poisson algebras, \((0,1)\)-algebras are Gerstenhaber algebras introduced in [7] while \((0, n - 1)\)-algebras are the \( n \)-algebras of [10]. We may think of an \((m, n)\)-structure on \( V \) as of a Lie algebra structure on \( \uparrow^n V \) together with an associative commutative algebra structure on \( \uparrow^m V \) such that both structures are related via the compatibility axiom (v).

Denote by \( \mathcal{P}(m, n) \) the operad for \((m, n)\)-algebras, see [3] for a very explicit description of this operad. In [3] we also computed the Koszul dual as \( \mathcal{P}(m, n) ! = \mathcal{P}(n, m) \), it can also be proven, using the fact that \( \mathcal{P}(m, n) \) is an operad with a distributive law and a suitable spectral sequence argument, that \( \mathcal{P}(m, n) \) is Koszul (for the special case of \( n \)-algebras it was done in [10]). Let \( \mathcal{C}_N \) be, for \( N \geq 2 \), the “\( N \)-little cubes operad” of [3] and let \( (CS(\mathcal{C}_N), d_S) \) be the corresponding differential operad of singular chains. It is known [10] that \( \mathcal{H}(CS(\mathcal{C}_N), d_S) = \mathcal{P}(0, N - 1) \).

We do not know whether the operad \((CS(\mathcal{C}_N), d_S) \) is formal (this would imply that the singular chain complex \( (CS(X), d_S) \) would have an associated homotopy \( \mathcal{P}(0, N - 1) \)-structure for any topological space \( X \) on which the little \( N \)-cubes operad \( \mathcal{C}_N \) acts). We show at least that \((CS(\mathcal{C}_N), d_S) \) contains a formal suboperad with the homology operad isomorphic to \( s^N \text{Lie} \), the \( N \)-fold suspension of the operad \( \text{Lie} \) for Lie algebras.
Let \((\mathcal{R}, d_\mathcal{R})\) be the differential suboperad of \((\text{CS}(\mathcal{C}_N), d_\mathcal{S})\) defined by

\[
\mathcal{R}_i(n) := \begin{cases} 
0, & \text{for } i < N(n-1), \\
\text{Ker}(d_\mathcal{S} : \text{CS}_{N(n-1)}(\mathcal{C}_N(n)) \rightarrow \text{CS}_{N(n-1)-1}(\mathcal{C}_N(n))), & \text{for } i = N(n-1), \text{ and} \\
\mathcal{S}_i(\mathcal{C}_N(n)), & \text{for } i > N(n-1).
\end{cases}
\]

It is easy to see that \(\mathcal{H}(\mathcal{R}, d_\mathcal{R}) = s^N\text{Lie} \) and we may apply Proposition 4.4 to infer that the operad \((\mathcal{R}, d_\mathcal{R})\) is formal. This means, by definition, that the minimal model of \((\mathcal{R}, d_\mathcal{R})\) is the same as the minimal model of \((s^N\text{Lie}, 0)\) (which is \(\Omega(s^N\text{Comm})\)).

Let \(X\) be a topological space on which the little \(N\)-cubes operad \(\mathcal{C}_N\) acts. We have the induced action of \((\text{CS}(\mathcal{C}_N), d_\mathcal{S})\) on \((\text{CS}(X), d_\mathcal{S})\) and, via the inclusion \((\mathcal{R}, d_\mathcal{R}) \hookrightarrow (\text{CS}(\mathcal{C}_N), d_\mathcal{S}),\) also the action of \((\mathcal{R}, d_\mathcal{R})\) on \((\text{CS}(X), d_\mathcal{S})\). The above shows that there is an associated homotopy \(s^N\text{Lie}\)-action on the singular chain complex \((\text{CS}(X), d_\mathcal{S})\) of \(X\).

In the following proposition we generalize a bit the arguments used in the previous example.

**Proposition 5.10.** Let \((\mathcal{S}, d_\mathcal{S})\) be a differential operad and let \(\mathcal{H} = \mathcal{H}(\mathcal{S}, d_\mathcal{S})\) be its homology operad. Suppose that there exists a natural number \(N\) such that the graded vector space \(\mathcal{H}(n)\) is trivial in degrees \(> N(n-1)\). Let \(\mathcal{E}\) be the suboperad of the operad \(\mathcal{H}\) defined by \(\mathcal{E}_*(n) = \mathcal{E}_{N(n-1)}(n) := \mathcal{H}_{N(n-1)}(n), \) for \(n \geq 2\).

Then every \((\mathcal{S}, d_\mathcal{S})\)-algebra admits an associated homotopy \(\mathcal{E}\)-algebra structure.

**Proof.** Let \((\mathcal{R}, d_\mathcal{R})\) be the differential suboperad of \((\mathcal{S}, d_\mathcal{S})\) defined by

\[
\mathcal{R}_i(n) := \begin{cases} 
0, & \text{for } i < N(n-1), \\
\text{Ker}(d_\mathcal{S} : \mathcal{S}_{N(n-1)}(n) \rightarrow \mathcal{S}_{N(n-1)-1}(n)), & \text{for } i = N(n-1), \text{ and} \\
\mathcal{S}_i, & \text{for } i > N(n-1).
\end{cases}
\]

Then \(\mathcal{H}(\mathcal{R}, d_\mathcal{R}) = \mathcal{E}\) and Proposition 4.4 implies the formality of the operad \((\mathcal{R}, d_\mathcal{R})\). \(\square\)

In the last two examples we discuss some applications of our methods to string theory. The terminology and the notation (which may in some places collide with the notation we have used so far) was taken from [16]; we refer to this paper for more details.

**Example 5.11.** Let \(\mathcal{P} = \{\mathcal{P}(n); \ n \geq 1\}\) be the collection defined by \(\mathcal{P}(n) := \) the moduli space of nondegenerate Riemann spheres with \((n+1)\) punctures and holomorphic disks at each puncture. There exists an operad structure on \(\mathcal{P}\) induced by sewing Riemann spheres at punctures. Let us recall that a (tree level) **conformal field theory** (CFT) based on a state space \(\mathcal{H}\) is an action of the operad \(\mathcal{P}\) on the topological vector space \(\mathcal{H}\), see [10, 4.1] for details.

Let us recall also that a **string background** is a CFT based on a vector space \(\mathcal{H}\) with the following additional data.

(i) A grading \(\mathcal{H} = \bigoplus \mathcal{H}_i\) on the state space.
(ii) An action of the Clifford algebra $C(V \oplus V^*)$ on $\mathcal{H}$; here $V$ denotes the complexification of the Virasoro algebra.

(iii) A differential $Q : \mathcal{H} \to \mathcal{H}$ of degree 1.

These data are supposed to satisfy some axioms, see [16, 4.2]. The differential $Q$ is called a BRST operator and the complex $(\mathcal{H}, Q)$ is called the (absolute) BRST complex.

As it was shown in [16, 4.3], the string background defines a sequence of $\text{Hom}(\mathcal{H}^{\otimes n}, \mathcal{H})$-valued differential forms $\Omega_{n+1}$ on $\mathcal{P}(n)$ having the property that the integration map $\int : \text{CS}(\mathcal{P}(n)) \to \text{Hom}(\mathcal{H}^{\otimes n}, \mathcal{H})$, $\sigma \mapsto \int_{\sigma} \Omega_{n+1}$, makes the absolute BRST complex $(\mathcal{H}, Q)$ into an algebra over the operad $(\text{CS}(\mathcal{P}), d_S)$ of singular chains on $\mathcal{P}$.

Let us recall some more or less classical definitions. Let $\mathcal{D}$ be the unit disk in the complex plane and let $\mathcal{F}(n)$ be, for $n \geq 1$, the space of all maps $f$ from the disjoint union of $n$ disks such that $f$, when restricted to each disk, is the composition of the translation and the multiplication by a nonzero complex number, and the images of $f$ are disjoint. There is an obvious operad structure on the collection $\mathcal{F} = \{\mathcal{F}(n); n \geq 1\}$ and the resulting operad is called the framed little disks operad. The operad $\mathcal{F}$ has a suboperad $\mathcal{D}$ consisting of those maps $f$ which, when restricted to each disk, are the compositions of translations and multiplications by positive real numbers. The operad $\mathcal{D}$ is called the little disks operad. It is immediate to see that the homotopy equivalence $\mathcal{D}(n) \cong C_2(n)$, where $C_2$ is the little 2-cubes operad discussed in Example 5.3, given by the collapsing of disks (resp. cubes) into a point, induces an isomorphisms of homology operads. Therefore $\mathcal{H}(\text{CS}(\mathcal{D}), d_S) \cong \mathcal{P}(0, 1)$. We see that the algebras over the operad $\mathcal{H}(\text{CS}(\mathcal{D}), d_S)$ are exactly the Gerstenhaber algebras and all the tricks of Example 5.9 are available, namely every action of the operad $(\text{CS}(\mathcal{D}), d_S)$ on a chain complex induces an associate homotopy $s\text{Lie}$ structure. Let us remark, for the sake of completeness, that the algebras over the homology operad $\mathcal{H}(\text{CS}(\mathcal{F}), d_S)$ of the framed little disks operad are so-called Batalin-Vilkovisky algebras, see [8].

As it was observed in [16, 4.7], there exists a natural operad map $j : \mathcal{F} \hookrightarrow \mathcal{P}$. This map is even a homotopy equivalence, but we will not use this fact. The composition $\mathcal{D} \hookrightarrow \mathcal{F} \hookrightarrow \mathcal{P}$ combined with the string background data then induces an action of the operad $(\text{CS}(\mathcal{D}), d_S)$ on the absolute BRST complex $(\mathcal{H}, Q)$. Summing up the above remarks, we obtain the following proposition.

**Proposition 5.12.** String background induces a natural homotopy $s\text{Lie}$ action on the absolute BRST complex $(\mathcal{H}, Q)$.

**Example 5.13.** Let $\mathcal{N}(n)$ denote, for $n \geq 1$, the moduli space of stable $(n + 1)$-punctured complex curves of genus zero decorated with relative phase parameters at double points and phase parameters at punctures. The collection $\mathcal{N} := \{\mathcal{N}(n); n \geq 1\}$ has a natural operad
structure, see [16, 3.4]. A closed string field theory (CSFT) consists of a string background as in the previous example and of a choice of a smooth operad map $s : \mathcal{N} \to \mathcal{P}$, see [16, 4.4]. In the previous example we mentioned $\text{Hom}(\mathcal{H}^n, \mathcal{H})$-valued forms $\Omega_{n+1}$ on $\mathcal{P}(n)$; let us denote by $\Omega_{n+1}$ their restrictions $s^*(\Omega_{n+1})$ to $\mathcal{N}(n)$.

Let $\mathcal{M}(n)$ be the moduli space of $(n + 1)$-punctured complex projective lines [16, 3.2] and let $\mathcal{M}(n)$ be the Fulton-MacPherson real compactification of $\mathcal{M}(n)$. We stress that neither $\mathcal{M} := \{\mathcal{M}; n \geq 1\}$ nor $\mathcal{M} := \{\mathcal{M}; n \geq 1\}$ have an operad structure, but there still exists an operad structure on the suspension $(\mathcal{CS}(\mathcal{M}), \mathcal{d}_{S^1})$ of the singular chain complex collection [16, 3.3].

The BRST complex $(\mathcal{H}, Q)$ of the previous example has a naturally defined smaller subcomplex $(\mathcal{H}_{rel}, Q)$ called the relative BRST complex. The inclusion $\mathcal{H}_{rel} \hookrightarrow \mathcal{H}$ naturally splits, therefore the forms $\Omega_{n+1}$ induce $\text{Hom}(\mathcal{H}_{rel}^n, \mathcal{H}_{rel})$-valued forms $\Omega'^{n+1}$ on $\mathcal{N}(n)$, $n \geq 1$, see [16, 4.5]. There exists a natural projection $p : \mathcal{N}(n) \to \mathcal{M}(n)$ with the fiber $(S^1)^{n+1}$ and there are $\text{Hom}(\mathcal{H}_{rel}^n, \mathcal{H}_{rel})$-valued forms $\omega_{n+1}$ on $\mathcal{M}(n)$ such that $\Omega_{n+1} = p^*(\omega_{n+1})$ [16, Proposition 4.4]. The integration map $f : \mathcal{CS}(\mathcal{M}(n)) \to \text{Hom}(\mathcal{H}_{rel}^n, \mathcal{H}_{rel})$, $\sigma \mapsto \int_\sigma \omega_{n+1}$, defines on the relative BRST complex a structure of an algebra over the operad $(\mathcal{CS}(\mathcal{M}), \mathcal{d}_{S^1})$ see [16, Theorem 4.5].

Let us recall that algebras over the homology operad $\mathcal{H}(\mathcal{CS}(\mathcal{M}), \mathcal{d}_{S^1}) = \mathcal{H}(\mathcal{M})$ are called gravity algebras. Let us inspect more closely the homology of the space $\mathcal{M}(n)$. This space has the same homotopy type as $\mathcal{M}(n)$ and it is not difficult to see that there exists a natural free action of the circle $S^1$ on the space $\mathcal{D}(n)$ such that $\mathcal{M}(n)$ is homotopically equivalent to $\mathcal{D}(n)/S^1$. We conclude that $\mathcal{H}(\mathcal{M}(n)) = \mathcal{H}(\mathcal{D}(n)/S^1)$.

On the other hand, the Serre spectral sequence of the principal fibration $S^1 \hookrightarrow \mathcal{D}(n) \to \mathcal{D}(n)/S^1$ gives rise to a degree $+1$ operator $\Delta : H(\mathcal{D}(n)) \to H(\mathcal{D}(n))$ such that $\mathcal{H}(\mathcal{D}(n)/S^1) = \text{Ker}(\Delta)$, compare [3]. As we already know, the homology operad of $\mathcal{D}$ is the operad for Gerstenhaber algebras, thus $H_i(\mathcal{D}(n)) = 0$ for $i > n - 1$ while $H_{n-1}(\mathcal{D}(n)) = s\text{Lie}(n)$. As $\Delta$ has degree $+1$, $H_{n-1}(\mathcal{D}(n)) \subset \text{Ker}(\Delta)$ and we conclude that

\begin{equation}
\mathcal{H}(\mathcal{M}(n))_i = 0 \text{ for } i > n - 1 \text{ while } \mathcal{H}(\mathcal{M}(n))_{n-1} = s\text{Lie}(n).
\end{equation}

Moreover, the second equation of (6) is compatible with the operad structures and Proposition 5.10 gives the following statement.

**Proposition 5.14.** Every action of the operad $(\mathcal{CS}(\mathcal{M}), \mathcal{d}_{S^1})$ induces an associated $s\text{Lie}$ homotopy structure. Especially, there exists an associated homotopy $s\text{Lie}$-structure on the relative BRST complex.

The structure predicted by the second part of the proposition was constructed in a very explicit manner in [16, Corollary 4.7].
Final remark. Let us indulge at the very end of our paper in some far-stretched remarks. We may say that two differential operads \((\mathcal{S}, d\mathcal{S})\) and \((\mathcal{T}, d\mathcal{T})\) are homotopically or weakly equivalent if they have isomorphic minimal models. We claim that this notion could be a very useful tool for the study of some operads appearing in string theory and, moreover, that we are in fact almost forced to introduce this notion because we feel that when physicists speak about an operad, they in fact mean an equivalence class in the above sense. For example, we may read in [16, par. 3.3] the following sentence (\(\mathcal{M}_{m+1}\) denotes the real compactification of the moduli spaces of \(n\)-punctured complex projective lines – we already discussed this space in Example 5.13 where it was denoted \(\mathcal{M}(n)\) – and “for instance” was italicized by ourselves):

“Item (5) prevents \(\{\mathcal{M}_{n+1} \mid n \geq 2\}\) from being an operad. But some operad structure is naturally defined on the (singular, for instance) chain complexes \(C_\bullet(\mathcal{M}_{m+1})\) of \(\mathcal{M}_{m+1}\)’s with coefficients in the ground field.”

This suggests that physicists would like to have the chain functor from the category of topological (or geometrical) operads to the category of differential algebraic operads independent on the concrete chains used. The only consistent way to formalize this is to identify two operads \((\mathcal{S}, d\mathcal{S})\) and \((\mathcal{T}, d\mathcal{T})\) if there exists a differential operad homomorphism \(\phi : (\mathcal{S}, d\mathcal{S}) \rightarrow (\mathcal{T}, d\mathcal{T})\) inducing the isomorphism of homology. This is, however, not an equivalence relation, but we may take the smallest equivalence generated by this relation, and it can be shown that this equivalence coincides with the homotopy equivalence relation introduced above. As in rational homotopy theory, (minimal) models naturally appear when we begin to study operads modulo this relation.

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