Abstract

Jamming is a phenomenon occurring in systems as diverse as traffic, colloidal suspensions and granular materials. A theory on the reversible elastic deformation of jammed states is presented. First, an explicit granular stress-strain relation is derived that captures many relevant features of sand, including especially the Coulomb yield surface and a third-order jamming transition. Then this approach is generalized, and employed to consider jammed magneto- and electro-rheological fluids, again producing results that compare well to experiments and simulations.
We start our study of jamming in granular systems, by deriving an appropriate stress-strain relation from a simple, postulated elastic energy. It accounts for the reversible elastic deformation of granular systems, up to the point of yield, and reproduces many relevant results from granular physics and soil mechanics, including the compliance tensor, Rankine states, and shear dilatancy. Moreover, the elastic energy is convex only below the Coulomb yield condition and becomes unstable there. As a result, the system escapes from the strained state and loses shape-rigidity, providing an explanation why sand unjams. Next, the granular elastic energy is shown to be a special case of a more generally valid energy expansion, with respect to the shear strain. Realizing that this expansion may serve as the starting point to account for other jammed systems, we use it to consider colloidal suspensions, specifically magneto- and electro-rheological fluids, which solidify at fields strong enough. Again, an energy expression is proposed, from which the magnetic, dielectric and elastic behavior is deduced, especially the solid-fluid phase diagram.

Our basic understanding of sand is due to Coulomb, who noted that its most conspicuous property is yield: A pile of dry sand possesses a critical slope that it will not exceed. His insightful conclusion is that the quotient of shear stress over pressure must not exceed a certain value, \( \frac{\sigma_s}{P} \leq \mu_f \). Wet sand can sustain a small shear stress \( \sigma_c \) even at vanishing pressure. It satisfies the Mohr-Coulomb condition, \( |\sigma_s| \leq \mu_f P + \sigma_c \), see [7].

It is standard praxis in soil mechanics to calculate the stress distribution by taking the stress \( \sigma_{ij} \) as some function of the strain \( u_{ij} \). Unfortunately, the calculated stress distribution routinely contradicts the Coulomb condition, and yield must be postulated, ex post facto, where it is not satisfied. An improvement of this somewhat brute method is given by the Rankine states, \( \sigma_s = \pm P \mu_f \), which should hold close to yield. The ameliorated calculation is given by accepting the result of elasticity away from the region of failure, postulating a Rankine state close to it, and connecting both smoothly. Clearly, in spite of ingenious ways to circumvent it, the basic problem is the lack of a stress-strain relation \( u_{ij}(\sigma_{kl}) \), with which a realistic stress distribution can be calculated.

If we had \( u_{ij}(\sigma_{kl}) \), the incremental relation, \( \delta u_{ij} = (\partial u_{ij}/\partial \sigma_{kl})\delta \sigma_{kl} \equiv \lambda_{ijkl}\delta \sigma_{kl} \), is easily derived. The elements of the compliance tensor \( \lambda_{ijkl} \) can also be obtained from experiments, in which \( \delta u_{ij} \), the strain response to a stress change \( \delta \sigma_{ij} \), is measured. Although integrating the measured \( \lambda_{ijkl} \) should in principle lead to \( u_{ij}(\sigma_{kl}) \), this is a hard, backward operation – made more difficult by the typical scatter of data, partly from irreversible plastic deforma-
tions. This circumstance has led many to espouse the view that $\lambda_{ijkl}$ is history-dependent, that an explicit $u_{ij}(\sigma_{kl})$ (from which to deduce $\lambda_{ijkl}$) does not exist. Different elasto-plastic theories, some exceedingly complex, have been constructed to account for $\lambda_{ijkl}$, including both elastic and plastic deformations, though a universally accepted model is missing \[3\].

Confining our study to reversible elastic deformations, we derive a stress-strain relation to account for the listed granular behavior. We start from the elastic energy

$$w = \frac{1}{2} \delta^{0.5}(B\delta^2 + Au_s^2), \quad (1)$$

where $\delta \equiv -u_{\ell\ell}$ is the compression, $u_s^2 \equiv u_{ij}^0 u_{ij}^0$ is shear strain squared. ($u_{\ell\ell}$ denotes the trace of the strain and $u_{ij}^0$ its traceless part. $\delta, u_s = 0$ imply the grains are in contact but not compressed or sheared.) $A, B > 0$ are functions of the void ratio $e$, an independent variable. We adopt the same empirical expression for both, $A, B \sim (2.17 - e)^2/(1 + e)$, see \[8\]. Eq (1) is clearly evocative of the Hertz contact: The energy of compressing two elastic spheres scales with $(\Delta h)^{2.5}$, where $\Delta h$ is the change in height \[9\]. Writing the energy as $\frac{1}{2}E(\Delta h)^2$, the effective Young modulus $E \sim (\Delta h)^{0.5}$ vanishes with $\Delta h$. The physics for the shear modulus is assumed to be similar.

We postulate Eq (1) to consider its ramifications – noting that it should be possible to derive it employing micro-mechanics \[10\]: Although an intricate task, it is not as difficult as calculating the stress $\sigma_{ij}$ or the compliance tensor $\lambda_{ijkl}$ directly. Remarkably, assuming that both moduli vanish with $\delta^{0.5}$, we take sand to be arbitrarily pliable, not at all “fragile” \[11\]. Differentiating the energy $w$ with respect to $\delta$, $u_s$ yields the pressure $P$ and shear $\sigma_s$, two scaler quantities; differentiating it with respect to $u_{ij}$ yields the complete stress tensor $\sigma_{ij}$,

$$P \equiv \partial w/\partial \delta = \frac{5}{4}B\delta^{1.5} + \frac{1}{4}A u_s^2/\delta^{0.5}, \quad (2)$$

$$\sigma_s \equiv \partial w/\partial u_s = A \delta^{0.5} u_s. \quad (3)$$

$$\sigma_{ij} \equiv \partial w/\partial u_{ij} = -P\delta_{ij} + A\delta^{0.5} u_{ij}^0. \quad (4)$$

This is the announced static stress-strain relation. The first term in $P$ is well-known and considered characteristic of Hertz contacts. The second term, accounting both for shear dilatancy and yield, is new. Dilatancy: Holding $P$ constant, $\delta$ decreases (and the volume expands) with growing $u_s$. Yield: For given $u_s$, the compressibility $(\partial P/\partial \delta)^{-1}$ is negative if $\delta$ is sufficiently small. This implies lack of local stability, and the system will not remain in the strained state. It is then, without the capability to sustain static shear, in
a fundamental sense “fluid” – though by no means necessarily Newtonian. In fact, the energy loses stability even before \( \partial P / \partial \delta \) turns negative, as the cross convexity condition 
\[
(\partial^2 w / \partial \delta^2)(\partial^2 w / \partial u_s^2) \geq (\partial^2 w / \partial \delta \partial u_s)^2,
\]
or \( u_s^2 / \delta^2 \leq 5B/2A \), also needs to be met. We saw the significance of instability in a previous work [12], but did not realize the following remarkable point and its consequences: Rewriting the cross convexity condition by replacing \( \delta, u_s \) with \( P, \sigma_s \) leads directly to (the Drucker-Prager variant [7] of) the Coulomb yield condition,
\[
|\sigma_s|/P \leq \sqrt{4A/5B}.
\]
(5)

To account for wet sand, the term \(-P_c \delta\) (with \( P_c > 0 \)) is added to the energy \( w \). This implies a force (typically supplied by the water’s surface tension) that compresses the grains even without an applied pressure. The additional term does not change the convexity condition, only substitutes \( P + P_c \) for \( P \) in Eq (2). As a result, Eq (5) assumes the Mohr-Coulomb form,
\[
|\sigma_s| \leq (P + P_c)\sqrt{4A/5B}.
\]

As any other elasticity theory, the stress-strain relation of Eqs (2,3,4) may be directly solved with appropriate boundary conditions to obtain a complete stress distribution. Because it includes yield as given by Eq (5), the Rankine states are automatically predicted where instability is close. And the compliance tensor \( \lambda_{ijkl} \) is obtained by simple differentiation. Writing \( \delta u_{ij} = \lambda_{ijkl} \delta \sigma_{kl} \) as a vector equation, \( \delta \sigma = \hat{M} \delta u \), with \( \hat{M} \) a 6 \times 6 matrix, we see yield is signified if an Eigenvalue \( m_1 \) of \( \hat{M} \) vanishes, with the Eigenvector \( \delta u_1 \) indicating the direction of instability. Explicit calculation shows \( \delta u_1 \parallel (\partial m_1 / \partial \sigma) \), implying \( \delta u_1 \) is perpendicular to the yield surface, \( m_1(\sigma) \sim |\sigma_s| - P\sqrt{4A/5B} = 0 \). If there is no plastic contribution, this implies flows perpendicular to the yield surface, a circumstance referred to as the “associated flow rule” [7].

In view of these results, there can be little doubt that Eq (11) indeed captures the essence of granular elasticity. And the remaining question is: To which extent is it also a quantitative rendition. To test this, we compare the calculated \( \lambda_{ijkl} \) to the data gathered recently [8], over a wide range of pressure, shear stress and void ratio. (Specifying these three variables, the reversible granular response is unique, showing no history-dependence.) Fig. 1 is a typical plot, with an overall agreement that further confirms Eq (11). (The expression for \( \lambda_{ijkl} \) is too cumbersome to be displayed here. It will be given in a forthcoming single-issue paper containing extensive comparison.) Note the ratio \( A/B \) is fixed by the Coulomb friction coefficient \( \mu_f \), so the theory has only one overall scale factor, and no actual adjustable
FIG. 1: The Poisson ratios $\nu_{zx}, \nu_{xy}$, the Young moduli $E_z, E_x = E_y$, and the shear moduli $G_{zx}, G_{xy}$, measured \(^8\) with Ham River sand at $\sigma_{xx}/\sigma_{zz} = 0.45$ and a void ratio of 0.66, compared to the calculated curves assuming $B = \frac{2}{3}A = 6800$MPa, with $E_i \equiv \lambda_{iii}^{-1}, G_{ij} \equiv \frac{1}{2}\lambda_{ijij}^{-1}, \nu_{ij} \equiv -\lambda_{ijjj}/\lambda_{iii}$. ($x, y$ are horizontal directions, $z$ the vertical one.) Note these coefficients are pairwise equal for linear elasticity, but deviate from each other nonlinearly; theory and experiment especially agree with respect to the direction of deviations, i.e., the fact that $\nu_{zx} > \nu_{xy}, E_z > E_x = E_y, G_{zx} = G_{xy}$. parameter. (The most important effect missing in Eq (1) is probably “fabric-anisotropy” \(^2\).)

Switching now to a broader context, we proceed to discriminate between the general feature of the above theory and those aspects specific to granular elasticity. This should give us a better appreciation why Eq (1) is as successful, and also help to apply the same approach to other jammed systems. Generally speaking, the energy should be a function of at least two variables, $u_s$ and $f$, with $f$ being the one driving the transition, taking place at $f_c$. In sand, suspensions, and electro-rheological fluids, $f$ is respectively given by the compression $\delta$, concentration, and the electric field. Expanding the energy in $u_s$,

$$w = w_0(f) + \frac{1}{2}Ku_s^2,$$

the shear modulus $K$ is a function of $f$, typically $K \sim (f - f_c)^a$ with $a > 0$ in the solid phase ($f > f_c$), and $K \equiv 0$ in the liquid one ($f < f_c$). This dependence is observed in suspensions \(^4\), simulations \(^14\) and, with $a \approx \frac{1}{2}$, works well for sand. We take it as an input. Local stability requires $K > 0$ and

$$w''_0 > [(K')^2/K - \frac{1}{2}K'']u_s^2 \equiv \kappa u_s^2,$$
ensuring $w$ is convex in $f, u_s$. Because $\kappa \sim a(a + 1) \times (f_c - f)^{a-2}$ is positive, the inequality is always violated when $u_s$ becomes sufficiently large, rendering instability, and hence the unjamming transition, a generic feature. If $a < 2$, $\kappa$ diverges for $f \to f_c$, and unjamming occurs at vanishing values of $u_s$ (assuming $w_0''$ remains finite). This ensures the validity of the expansion of Eq (6).

Considering the jamming transition in the shear-free limit $u_s \to 0$, we identify it – by analogy to conventional phase transitions – as of $n$th order, if $\partial^i w_0 / \partial f^i$ is continuous for $i < n$, but not for $i = n$. With $w_0 \sim \delta^{2.5}$, sand displays a third-order jamming transition.

Yield at finite shear, as a result of the energetic instability, Eq (7), is not an equilibrium transition, because the liquid phase moves and dissipates. This may well be compared to raising the temperature $T$ in a current-carrying superconductor, such that the metal is pushed into its normal state carrying a dissipative, ohmic current. In fact, if one identifies $f$ as $T$, replaces $u_s$ with the superfluid velocity $v_s$ (and hence $\sigma_s$ with the current, $j_s = \rho_s v_s$), Eq (6) is valid for superconductors, and superfluid helium, $\frac{1}{2}Ku_s^2 \to \frac{1}{2}\rho_s v_s^2$, respectively with $\rho_s \sim T_c - T$ and $\rho_s \sim (T_c - T)^{2/3}$ [15]. Macroscopically, jamming and phase transition are clearly hard to tell apart, and their conceptual difference must be subtle.

Next, we consider ER and MR (or electro- and magneto-rheological) fluids, employing them as further examples for the above notion of jamming. Although experimental data are as yet not confining enough for an unambiguous determination of their energy, plausibility may be drawn on to fill the gap. In ER fluids, the dielectric displacement $D$ assumes the role of the transition-driving variable $f$. Writing the shear-free part of the energy as $w_0 = w_1 + w_2$, we take $w_1 = \frac{1}{2}D^2 / \epsilon_1$, accounting for a linear dielectric relation, and $w_2 = -\frac{1}{2}\Delta(D - D_c)^2$, assuming that linearity prevails after the transition at $D_c$. This is a second-order transition, and the electric field $E \equiv \partial w / \partial D$ has a kink at $D_c$: We have $E = D / \epsilon_1$ for $D \leq D_c$, and $E - E_c = (D - D_c) / \epsilon_2$ for $D > D_c$, with $1/\epsilon_2 = 1/\epsilon_1 - \Delta$, $E_c \equiv D_c / \epsilon_1$. (A discontinuity in $E$, or a first-order transition, was to our knowledge never reported. Higher order transitions are possible, seem even likely, but they are not compatible with a linear constitutive relation after the transition.) $w_2$ is the condensation energy, so $\Delta$ must be positive for solidification to take place. (Taking $D \to B, E \to H$ yields the analogous formulas for MR fluids.) Given $w_0$ and $K = A(D - D_c)^a$, the energy $w$ of Eq (6) is specified. We calculate the dielectric relation $E \equiv \partial w / \partial D|_{u_s}$, elastic relation $\sigma_s \equiv \partial w / \partial u_s|_D$, and rewrite Eq (7) in terms of $E, \sigma_s$. 

FIG. 2: Elastic and dielectric properties, including the yield point, for electro- and magneto-rheological fluids: Shear stress $\sigma_s$ versus shear strain $u_s$ at fixed electric field $E$, and $E - E_c$ versus $D - D_c$ (or $H - H_c$ versus $B - B_c$) at fixed $\sigma_s$, for the exponents $a = 2, 1, 0.2$. Choosing the dimension of both curves such that the yield points are at (1,1) render the curves universal – removing the dependency on (i) all material parameters other than the exponent $a$, (ii) $E$ in the upper plot, and (iii) $\sigma_s$ in the lower one. The relation $\sigma_s(u_s)$ for $a = 1, 2$ agree with data from experiments and simulations [18].

to obtain the yield condition,

$$|\sigma_s| \leq (E - E_c)^{1+\frac{a}{2}} \sqrt{2A \left[ \frac{\epsilon_2(a+1)}{a(a+2)} \right]^{a+1}.} \tag{8}$$

The exponent $a = 1$, or a yield stress $|\sigma_s| \sim (H - H_c)^{3/2}$ is observed for most MR-fluids [16]. The same value is also appropriate for a few ER-fluids [17], though the yield stress is typically quadratic [3, 18], $|\sigma_s| \sim (E - E_c)^2$, indicating $a = 2$. An ER-fluid capable of sustaining an unusually high shear strength was reported [6] to display a nearly linear dependence of the yield stress, $|\sigma_s| \sim E - E_c$, or $a \ll 1$. For $a = 0$, the shear modulus $K$ is independent of the field, and there is no yield at all. This is the reason the square root in Eq (8) diverges for $a \to 0$, and possibly explains the observed high yield stress.

Finally, the above approach and results are critically appraised. (Granular vocabulary is employed for this purpose, though the statements are equally valid for ER and MR fluids.) In physics, every microscopic state has a unique energy. The same holds for macroscopic ones if we insist on a consistent description. The macroscopic energy always depends on entropy.
and conserved quantities, such as momentum and mass density. And if the considered system can sustain static shear stresses, the strain field $u_{ij}$ must also be included as an independent variable, where $u_{ij}$ is to be understood, in soil-mechanical parlance, as the reversible elastic portion of the strain field.

It is a plain fact that sand piles, if left alone under gravity, are stable – in spite of every kind of infinitesimal perturbations, which are always present. This demonstrates sand’s capability to sustain static shear and is the reason for including $u_{ij}$. Irrespective whether a unique displacement field exists, the elastic description employing $u_{ij}$ is robust enough to be valid. This is not different from superfluid helium with vortex lines, in which the description in terms of the velocity $v_s = \frac{\hbar}{m} \nabla \phi$ remains sound, although the phase field $\phi$ is multivalued.

Given an energy expression $w$, its derivative $\partial w / \partial u_{ij}$ yields the stress tensor $\sigma_{ij}$, and its second derivative $\partial^2 w / \partial u_{ij} \partial u_{kl}$ the inverse of the compliance tensor $\lambda_{ijkl}$. In soil mechanics, the usual approach consists of postulating the stress dependence of the 18 independent components of $\lambda_{ijkl}$ directly, while seeking the account for the plastic contribution at the same time, referring to the result as constitutive relations [3]. This is quite obviously a much harder task than finding the one appropriate scalar expression for the energy $w$ which, even if heavy-handedly simplified, preserves a large number of geometric correlation by the mere fact that $\lambda_{ijkl}$ is obtained via a double differentiation. We believe this to be the main reason why the calculated $\lambda_{ijkl}$ stood up so well when compared to the extensive data of [8].

The expression we proposed in Eq (1) is indeed the result of weighing simplicity versus accuracy while stressing the former, and hence is subject to further scrutiny. As discussed, it includes first of all an expansion in $u_s$: $w = \frac{1}{2} K u_s^2$ assuming $K \sim \delta^a$. Starting from $w = \frac{1}{2}(B\delta^{2+b} + A\delta^a u_s^2)$ in [12], we considered the experiments of inclined plane, simple shear and triaxial test to arrive at $a \approx 0.4, b \approx 0.5$ giving the best agreement [19]. On the other hand is the fact that the Coulomb yield condition, Eq (5), remains unchanged as long as $a = b$. And it becomes implicit if $a, b$ deviate from each other – though the numerical difference is at first modest. Our tentative choice is $a = b = \frac{1}{2}$.

[1] A. J. Liu, S. R. Nagel, Nature 396, 21 (1998).
[2] P. G. de Gennes, Rev. Mod. Phys 71, 374 (1999).
[3] D. Kolymbas in *Constitutive Modelling of Granular Materials* (ed. D. Kolymbas) 11 (Springer, New York, 2000), S. B. Savage, in *Powders & Grains 97* (eds. Behringer & Jenkins) 185 (Balkema, Rotterdam, 1997).

[4] V. Trappe, V. Prasad, L. Cipelletti, P.N. Segre, D.A. Weitz, Nature 411, 772 (2001).

[5] T. C. Halsey Electrorheological fluids. Science 258, 761 (1992).

[6] W. Wen, X. Huang, S. Yang, K. Lu, P. Sheng, Nature Materials, 2, 727 (2003) Letters.

[7] A. Schofield, P. Wroth, *Critical State Soil Mechanics* (Mcgraw-Hill, London, 1968); W. X. Huang (ed) *Engineering Properties of Soil* 1st edn (Hydroelectricity Publishing, Beijing, 1983) (in Chinese).

[8] R. Kuwano R.J. Jardine, Géotechnique 52, 727 (2002).

[9] L. D. Landau, E. M. Lifshitz, *Theory of Elasticity* 3rd edn (Pergamon Press, New York, 1986).

[10] C. Goldenberg, I. Goldhirsch, Phys. Rev Lett. 89, 84302 (2002). F. Alonso-Morroquin, H.J. Herrmann, Phys. Rev. E 66, 021301 (2002).

[11] The theory of “fragile-state” by M.E. Cates, J.P. Wittmer, J.P. Bouchaud, P. Claudin, Phys. Rev Lett, 81, 1841 (1998), plausibly approximates sand grains as infinitely rigid, seemingly rendering the strain field irrelevant, see also [1, 4].

[12] Y. Jiang, M. Liu, Phys. Rev Lett. 91, 144301(2003).

[13] J.D. Goddard, Proc. R. Soc. London A 430, 105 (1990).

[14] C.S. O’Hern, S.A. Langer, A.J. Liu, S.R. Nagel, Phys. Rev Lett. 88, 75507(2002).

[15] Eq (7) is not valid for superconductors, because $w(T)$ is a free energy and possesses different stability conditions.

[16] P. P. Phulé, J. M. Ginder, Int. J. Mod. Phys. B13, 2019 (1999).

[17] L. C. Davis J. M. Ginder, in *Progress in Electrorheology* (eds K.O. Havelka, F.E. Filisko) 107- (Pleumum, New York, 1995).

[18] H. Ma, W. Wen, W. Y. Tam, P. Sheng, Adv. Phys. 52, 343 (2003).

[19] Our results cannot be directly compared to that in [13], because the pressure contains shear contributions, see Eq (2). However, if we may neglect them, it is easy to see that we need to set $a = 1$ to obtain a Young modulus $\sim \sqrt{P}$. 