TWO-SIDED BOUNDS FOR THE VOLUME OF RIGHT-ANGLED HYPERBOLIC POLYHEDRA

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Abstract. For a compact right-angled polyhedron \( R \) in \( \mathbb{H}^3 \) denote by \( \text{vol}(R) \) the volume and by \( \text{vert}(R) \) the number of vertices. Upper and lower bounds for \( \text{vol}(R) \) in terms of \( \text{vert}(R) \) were obtained in \cite{Atkinson2011}. Constructing a 2-parameter family of polyhedra, we show that the asymptotic upper bound \( \frac{5v_3}{8} \), where \( v_3 \) is the volume of the ideal regular tetrahedron in \( \mathbb{H}^3 \), is a double limit point for ratios \( \text{vol}(R) / \text{vert}(R) \). Moreover, we improve the lower bound in the case \( \text{vert}(R) \leq 56 \).

1. Right-angled polyhedra in \( \mathbb{H}^3 \).

In any space, right-angled polyhedra are very convenient to serve as "building blocks" for various geometric constructions. In particular, they have several interesting properties in hyperbolic 3-space \( \mathbb{H}^3 \). One can try to obtain a hyperbolic 3-manifold using a right-angled polyhedron as its fundamental polyhedron. Or, one can construct a hyperbolic 3-manifold in such a way that its fundamental group is a torsion-free subgroup of the Coxeter group, generated by reflections across the faces of a right-angled polyhedron \cite{Lobell1989}. Below we consider only compact polyhedra, which do not admit ideal vertices.

We start by recalling two nice recent results. Inoue \cite{Inoue2011} introduced two operations on right-angled polyhedra called decomposition and edge surgery, and proved that Löbell polyhedra (which will be a subject of discussion below) are universal in the following sense:

**Theorem 1.1.** \cite[Theorem 9.1]{Inoue2011} Let \( P_0 \) be a right-angled hyperbolic polyhedron. Then there exists a sequence of disjoint unions of right-angled hyperbolic polyhedra \( P_1, \ldots, P_k \) such that for \( i = 1, \ldots, k \), \( P_i \) is obtained from \( P_{i-1} \) by either a decomposition or an edge surgery, and \( P_k \) is a set of Löbell polyhedra. Furthermore,

\[
\text{vol}(P_0) \geq \text{vol}(P_1) \geq \text{vol}(P_2) \geq \ldots \geq \text{vol}(P_k).
\]

Atkinson \cite{Atkinson2011} estimated the volume of a right-angled polyhedron in terms of the number of its vertices as follows:

**Theorem 1.2.** \cite[Theorem 2.3]{Atkinson2011} If \( P \) is a compact right-angled hyperbolic polyhedron with \( V \) vertices, then

\[
(V - 2) \cdot \frac{v_8}{32} \leq \text{vol}(P) < (V - 10) \cdot \frac{5v_3}{8},
\]
where \( v_8 \) is the volume of a regular ideal octahedron, and \( v_3 \) is the volume of a regular ideal tetrahedron. There is a sequence of compact polyhedra \( P_i \), with \( V_i \) vertices such that \( \text{vol}(P_i)/V_i \) approaches \( 5v_3/8 \) as \( i \) goes to infinity.

A family of polyhedra \( P_i \) suggested by Atkinson is described in the proof of [3, Prop. 6.4].

In this note we will demonstrate that Löbell polyhedra can serve as a suitable family realizing the upper bound. Thus these polyhedra play an important role not only in Theorem 1.1 but also in Theorem 1.2.

Let us denote by \( \text{vert}(R) \) the number of vertices of a right-angled polyhedron \( R \). In this note we prove that \( 5v_3/8 \) is a double limit point in the sense that it is the limit point of limit points for ratios \( \text{vol}(R)/\text{vert}(R) \).

**Theorem 1.3.** For any integer \( k \geq 1 \) there exists a series of compact right-angled polyhedra \( R_k(n) \) in \( \mathbb{H}^3 \) such that

\[
\lim_{n \to \infty} \frac{\text{vol}(R_k(n))}{\text{vert}(R_k(n))} = \frac{k}{k+1} \cdot \frac{5v_3}{8}.
\]

As one will see from the proof, \( R_1(n) \) are Löbell polyhedra and \( R_k(n) \) for \( k > 1 \) are towers of them.

Moreover, in Corollary 4.3 we improve the lower estimate from Theorem 1.2 in the case \( \text{vert}(R) \leq 56 \).

2. Löbell polyhedra and manifolds.

We introduced Löbell polyhedra in [10] as a generalization of a right-angled 14-hedron used in [5].

Recall that in order to give a positive answer to the question of the existence of “Clifford-Klein space forms” (that is, closed manifolds) of constant negative curvature, Löbell [5] constructed in 1931 the first example of a closed orientable hyperbolic 3-manifold. This manifold was obtained by gluing together eight copies of the right-angled 14-faced polytope (denoted below by \( R(6) \) and shown in Fig. 1) with an upper and a lower basis both being regular hexagons, and a lateral surface given by 12 pentagons, arranged similarly as in the dodecahedron. Obviously, \( R(6) \) can be considered as a generalization of a right-angled dodecahedron in the way of replacing basis pentagons to hexagons.

As shown in [10], the dodecahedron and \( R(6) \) are part of a larger family of polyhedra. For each \( n \geq 5 \) we consider the right-angled polyhedron \( R(n) \) in \( \mathbb{H}^3 \) with \( 2(n+2) \) faces, two of which (viewed as the upper and lower bases) are regular \( n \)-gons, while the lateral surface is given by \( 2n \) pentagons, arranged as one can easily imagine. Note that \( R(5) \) is the right-angled dodecahedron (see Fig. 1). Existence of polyhedra \( R(n) \) in \( \mathbb{H}^3 \) can be easy checked by involving Andreev’s theorem [1].

An algebraic approach suggested in [10] admits a construction of both orientable and non-orientable closed hyperbolic 3-manifolds from eight copies of any bounded right-angled hyperbolic polyhedron. More exactly, any coloring of the faces of a right-angled polyhedron by four colors so that no two faces of the same color share an edge encodes a torsion-free subgroup of orientation preserving isometries which is a subgroup of the polyhedral Coxeter group of index eight. Thus, any four-coloring encodes an orientable hyperbolic 3-manifold obtained from eight
copies of a right-angled polyhedron. This approach also allows one to construct non-orientable hyperbolic 3-manifolds, but in this case five to seven colors are needed.

It was mentioned in [10] that the manifold constructed by Löbell can be encoded by some four-coloring of $R(6)$, and it was shown how to construct concrete orientable and non-orientable manifolds using eight copies of $R(n)$ for any $n \geq 5$. Closed orientable hyperbolic 3-manifolds encoded by four-colorings of $R(n)$, $n \geq 5$, were called Löbell manifolds. (Observe that for each $n$ number of such manifolds do not need to be unique.) Polyhedra $R(n)$ can be naturally referred as Löbell polyhedra.

Various properties of Löbell manifolds were intensively studied: the volume formulae were obtained in [9] and [11], invariant trace fields for fundamental groups and their arithmeticity were numerically calculated in [2], many of Löbell manifolds were obtained in [8] as two-fold branched coverings of the 3-sphere, and two-sided bounds for complexity of Löbell manifolds were done in [7].

Since Lobachevsky's 1832 paper, the following Lobachevsky function has traditionally been used in volume formulae for hyperbolic polyhedra

$$\Lambda(x) = -\int_0^x \log |2 \sin(t)| \, dt.$$ 

The volume formula for Löbell manifolds established in [11] implies the following formula for $\text{vol}(R(n))$, since any Löbell manifolds indexed by $n$ is glued by isometries from eight copies of $R(n)$:

**Theorem 2.1.** For all $n \geq 5$ we have

$$\text{vol}(R(n)) = \frac{n}{2} \left(2\Lambda(\theta_n) + \Lambda \left(\theta_n + \frac{\pi}{n}\right) + \Lambda \left(\theta_n - \frac{\pi}{n}\right) + \Lambda \left(\frac{\pi}{2} - 2\theta_n\right)\right),$$

where

$$\theta_n = \frac{\pi}{2} - \arccos \left(\frac{1}{2 \cos(\pi/n)}\right).$$

It is easy to check that $\theta_n \to \pi/6$ and $\frac{\text{vol}(R(n))}{n} \to \frac{5\pi}{3}$ as $n \to \infty$. Here we use that $v_3 = 3\Lambda(\pi/3) = 2\Lambda(\pi/6)$. Moreover, the asymptotic behavior of volumes of Löbell manifolds was established in [7, Prop. 2.10]. This implies trivially the description of the asymptotic behavior of $\text{vol}(R(n))$ as $n$ tends to infinity.
Proposition 2.1. The following inequalities hold for sufficiently large $n$:
\[
\frac{5v_3}{4} \cdot n - \frac{17v_3}{2n} < \text{vol}(R(n)) < \frac{5v_3}{4} \cdot n.
\]
Since $\text{vert}(R(n)) = 4n$, we get

Corollary 2.1. The following inequalities hold for sufficiently large $n$:
\[
\frac{5v_3}{16} - \frac{17v_3}{8n^2} < \frac{\text{vol}(R(n))}{\text{vert}(R(n))} < \frac{5v_3}{16}.
\]

3. Proof of Theorem 1.3

We will use Löbell polyhedra $R(n)$ as building blocks to construct right-angled polyhedra with necessary properties. Let us present polyhedra $R(n)$ by their lateral surfaces as it is done in Fig. 2 for polyhedra $R(6)$ and $R(5)$, keeping in mind that left and right sides are glued together.

\[\text{Figure 2. Polyhedra } R(6) \text{ and } R(5).\]

For integer $k \geq 1$ denote by $R_k(n)$ the polyhedron constructed from $k$ copies of $R(n)$ gluing them along $n$-gonal faces similar to a tower. In particular, $R_1(n) = R(n)$. The polyhedron $R_3(6)$ is presented in Fig. 3.

\[\text{Figure 3. Polyhedron } R_3(6).\]

Obviously, $R_k(n)$ is a right-angled polyhedron with $n$-gonal top and bottom and the lateral surface formed by $2n$ pentagons and $(k - 1)n$ hexagons.

Since $\text{vol}(R_k(n)) = k \cdot \text{vol}(R(n))$, Proposition 2.1 implies that for sufficiently large $n$
\[
k \cdot \frac{5v_3}{4} \cdot n - k \cdot \frac{17v_3}{2n} < \text{vol}(R_k(n)) < k \cdot \frac{5v_3}{4} \cdot n.
\]
Since $\text{vert}(R_k(n)) = (2k + 2)n$, we obtain
\[
k \cdot \frac{5v_3}{8} - k \cdot \frac{17v_3}{4n^2} < \frac{\text{vol}(R_k(n))}{\text{vert}(R_k(n))} < \frac{k}{k + 1} \cdot \frac{5v_3}{8}.
\]
Thus family of right-angled polyhedra $R_k(n)$ is such that for any integer $k \geq 1$
\begin{equation}
\lim_{n \to \infty} \frac{\text{vol}(R_k(n))}{\text{vert}(R_k(n))} = \frac{k}{k+1} \cdot \frac{5v_3}{8},
\end{equation}
and the upper bound $5v_3/8$ is a double limit point in the sense that it is the limit
of above limit points as $k \to \infty$:
\begin{equation}
\lim_{k,n \to \infty} \frac{\text{vol}(R_k(n))}{\text{vert}(R_k(n))} = \frac{5v_3}{8}.
\end{equation}
Thus, the theorem is proved. \qed

4. Other volume estimates.

Since 1-skeleton of a right-angled compact hyperbolic polyhedron $P$ is a trival-
ent plane graph, one can easy see that Euler formula for a polyhedron implies
\begin{equation}
V = 2F - 4,
\end{equation}
where $V$ is number of vertices of $P$ and $F$ is number of its faces. Moreover, Euler
formula implies also that $P$ has at least 12 faces (this smallest number of faces
corresponds to a dodecahedron). Thus, Theorem 1.2 implies the following result.

**Corollary 4.1.** If $P$ is a compact right-angled hyperbolic polyhedron with $F$ faces,
then
\begin{equation}
(F - 3) \cdot \frac{v_8}{16} \leq \text{vol}(P) < (F - 7) \cdot \frac{5v_4}{4}.
\end{equation}

We recall that constants $v_3$ and $v_8$ are
\begin{align}
v_3 &= 3 \Lambda(\pi/3) = 1.0149416064096535 \ldots,
\end{align}
and
\begin{align}
v_8 &= 8 \Lambda(\pi/4) = 3.663862376708876 \ldots.
\end{align}

Since a right-angled hyperbolic $n$-gon has area $\pi/2 \cdot (n - 4)$, the lateral surface
area of a compact hyperbolic right-angled polyhedron $P$ with $F$ faces is equal to
$\pi \cdot (F - 6)$. Thus, Corollary 4.1 implies the following result.

**Corollary 4.2.** If $P$ is a compact right-angled hyperbolic polyhedron with lateral
surface area $S$, then
\begin{equation}
(S/\pi + 3) \cdot \frac{v_8}{16} \leq \text{vol}(P) < (S/\pi - 1) \cdot \frac{5v_3}{4}.
\end{equation}

Observe, that Theorem 2.1 can be used to show that the volume function
$\text{vol} R(n)$ is a monotonic increasing function of $n$ (see [4] and [7] for proofs), and
to calculate volumes of L"obell polyedra. In particular,
\begin{equation}
\text{vol} R(5) = 4.306 \ldots, \quad \text{vol} R(6) = 6.023 \ldots, \quad \text{vol} R(7) = 7.563 \ldots.
\end{equation}
Together with Theorem 1.1 it gives that the right-angled hyperbolic polyhedron of
smallest volume is $R(5)$ (a dodecahedron) and the second smallest is $R(6)$. Thus,
if a compact right-angled hyperbolic polyhedron $P$ is differ of a dodecahedron,
then
\begin{equation}
\text{vol}(P) \geq 6.023 \ldots.
\end{equation}
Thus, we get the following
Corollary 4.3. If $P$ is a compact right-angled hyperbolic polyhedron different than a dodecahedron, having $V$ vertices and $F$ faces. Then

$$\text{vol}(P) \geq \max\{(V - 2) \cdot \frac{v_8}{32}, 6.023 \ldots\}$$

and

$$\text{vol}(P) \geq \max\{(F - 3) \cdot \frac{v_8}{16}, 6.023 \ldots\}.\text{vol}(P)$$

The estimates from Corollary 4.3 improve the lower estimate from Theorem 1.2 for $V \leq 54$ and the lower estimate from Corollary 4.1 for $F \leq 29$.

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