ON THE WRONSKIAN COMBINANTS OF BINARY FORMS

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Abstract. For generic binary forms $A_1, \ldots, A_r$ of order $d$ we construct a class of combinants $C = \{C_q : 0 \leq q \leq r, q \neq 1\}$, to be called the Wronskian combinants of the $A_i$. We show that the collection $C$ gives a projective imbedding of the Grassmannian $G(r, S_d)$, and as a corollary, any other combinant admits a formula as an iterated transvectant in the $C$. Our second main result characterizes those collections of binary forms which can arise as Wronskian combinants. These collections are the ones such that an associated algebraic differential equation has the maximal number of linearly independent polynomial solutions. Along the way we deduce some identities which connect Wronskians with transvectants.

1. Introduction

This article extends some of the investigations in [2] to the case of several binary forms. We begin by recalling the classical notion of a combinant of binary forms (see [7, §250]). A summary of our results will appear in §1.6 below after the required notation is available.

1.1. Let $A_1, \ldots, A_r$ denote generic forms of order $d$ in the variables $x = \{x_1, x_2\}$ (assume $r \leq d$). Write

$$A_i = \sum_{j=0}^{d} \binom{d}{j} a_{ij} x_1^{d-j} x_2^j, \quad (1 \leq i \leq r),$$

where the $a_{ij}$ are independent indeterminates. Given a matrix $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that $\alpha \delta - \beta \gamma = 1$, make substitutions

$$x_1 = \alpha x_1' + \beta x_2', \quad x_2 = \gamma x_1' + \delta x_2';$$
and now define $a'_{ij}$ by forcing the equalities
\[ \sum_{j=0}^{d} \binom{d}{j} a_{ij} x_1^{d-j} x_2^j = \sum_{j=0}^{d} \binom{d}{j} a'_{ij} x'_1^{d-j} x'_2^j. \]

A polynomial function $Q(\{a_{ij}\}; x_1, x_2)$ is called a joint covariant of the the $\{A_i\}$ if
\[ Q(\{a_{ij}\}; x_1, x_2) = Q(\{a'_{ij}\}; x'_1, x'_2), \]
for every $g$. It is called a combinant if the following additional condition is satisfied: given a matrix $M \in SL_r$, define new constants $b_{ij}$ via the matrix equality $B = M A$, where $B = [b_{ij}], A = [a_{ij}]$. Then we should have an equality
\[ Q(\{a_{ij}\}; x_1, x_2) = Q(\{b_{ij}\}; x_1, x_2), \]
for every $M$. We say that $Q$ is of degree $m$ and order $n$, if it has total degree $m$ in the coefficients of each $A_i$ and total degree $n$ in $x$. By the first fundamental theorem, the coefficients of $Q$ can be written as degree $m$ forms in the $r \times r$ minors of the matrix $[a_{ij}]$.

For instance, for $r = 2$ the resultant $R(A_1, A_2)$ is a combinant of degree $d$ and order zero. The Jacobian $\begin{vmatrix} \frac{\partial A_1}{\partial x_1} & \frac{\partial A_1}{\partial x_2} \\ \frac{\partial A_2}{\partial x_1} & \frac{\partial A_2}{\partial x_2} \end{vmatrix}$ is a combinant of degree one and order $2d - 2$.

For fixed $(r, d)$, the combinants define a ring $R$ bigraded by $m$ and $n$. The structure of this ring can be very involved, and it is concretely known only for a few small values of $r$ and $d$ (see e.g. [6, 11]). Our objective, roughly speaking, is to construct distinguished elements of this ring $C_0, C_2, \ldots, C_r$ (sic) which generate it (in a slightly extended sense, to be made precise later).

1.2. Preliminaries. Throughout, the base field will be $\mathbb{C}$. We will write $S_d$ for the space of order $d$ forms in $x$, which is naturally a representation of $SL_2$. See [3, Ch. 11] or [14, Ch. 4] for standard facts about $SL_2$-representations. All of our constructions (and morphisms) will be $SL_2$-equivariant. Each finite-dimensional $SL_2$-module is canonically self-dual, we will use this identification without further comment.

Our basic reference for invariant theory is [7]; much of the same material is covered in [11]. We will freely use the classical symbolic calculus interpreted according to [11, §2]. Other treatments of this calculus, which we will not use, can be found in [10, 12].
If $E, F$ are two binary forms of orders $e, f$, their $k$-th transvectant is defined as

\[(E, F)_k = \frac{(e-k)!(f-k)!}{e!f!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{\partial^k E}{\partial x_1^{k-i} \partial x_2^i} \frac{\partial^k F}{\partial x_1^i \partial x_2^{k-i}}.\] (2)

It is identically zero outside the range $0 \leq k \leq \min\{e, f\}$.

1.3. Let $G = G(r, S_d)$ denote the Grassmann variety of $r$-dimensional subspaces of $S_d$. Let $\lambda_m$ denote the partition $(m, \ldots, m)^{r\text{ times}}$, and $S_{\lambda_m}$ the associated Schur functor (see [3, Ch. 6]). By the Borel-Weil-Bott theorem (see [13, p. 687]) we have an isomorphism of $SL_2$-representations

\[H^0(G, \mathcal{O}_G(m)) \simeq S_{\lambda_m}(S_d).\]

An element of $S_{\lambda_m}(S_d)$ can be seen as a degree $m$ function in the Plücker coordinates via the imbedding

\[S_{\lambda_m}(S_d) \subseteq S_m(\wedge^r S_d).\]

Then, a combinant $Q$ of degree $m$ and order $n$ can be identified (up to a scalar) with a morphism

\[\mu : S_n \rightarrow H^0(G, \mathcal{O}_G(m))\]

which sends $F \in S_n$ to the transvectant $(F, Q)_n$. In the reverse direction, $\mu$ gives rise to a morphism

\[\mu' : C \rightarrow H^0(G, \mathcal{O}_G(m)) \otimes S_n,\]

and then $Q$ can be recovered (up to scalar) as the element $\mu'(1)$.

Hence combinants of degree-order $(m, n)$ (up to scalars) are in bijection with nonzero (and hence necessarily injective) morphisms $S_n \rightarrow S_{\lambda_m}(S_d)$. In particular all the linear (i.e., degree one) combinants correspond to the irreducible summands of

\[\wedge^r S_d \simeq S_r(S_{d-r+1}).\]

**Example 1.1.** For $r = 2$ and $m = 1$, we have

\[S_2(S_{d-1}) \simeq \bigoplus_{i=1}^{\lfloor \frac{d+1}{2} \rfloor} S_{2d-2(2i-1)}.\]

The $i$-th summand corresponds to the combinant $(A_1, A_2)_{2i-1}$. 
Example 1.2. Assume \((r, d) = (2, 5), m = 2\). We have a plethysm decomposition

\[
S_{\lambda_2}(S_5) = S_{(2,2)}(S_5) = S_{16} \oplus S_{12}^2 \oplus S_{10} \oplus S_8^3 \oplus S_6 \oplus S_4^3 \oplus S_0^2.
\]

(This was calculated using the Maple package ‘SF’.) This implies for instance, that two binary quintics \(A_1, A_2\) have a two dimensional space of combinants of degree 2 and order 12. Writing \(t_i = (A_1, A_2)_i\), a basis for this space is given by \((t_1, t_1)^2\) and \(t_1 t_3\).

Since the algebra \(\bigoplus_m S_{\lambda_m}(S_d)\) is generated in degree one, every combininant can be written as an iterated transvectant expression in linear combinants.

1.4. Wronskians. Given binary forms \(F_1, \ldots, F_s\) of order \(n\), we define their Wronskian

\[
W(F_1, \ldots, F_s) = \left(\frac{(n - s + 1)!}{n!}\right)^s \times \det \left(\frac{\partial^{s-1} F_i}{\partial x_1^{s-j}\partial x_2^{j-1}} \right)_{1 \leq i,j \leq s}.
\]

It is zero iff the \(F_i\) are linearly dependent over \(\mathbb{C}\). Using the classical symbolic calculus according to [1, §2], if \(F_i = f^{(i)}_x\) then

\[
W = \prod_{1 \leq i<j \leq s} (f^{(i)} f^{(j)}) \prod_{1 \leq i \leq s} f^{(i)(n-s+1)}_x.
\]

(The proof is easy: differentiate the symbolic expressions, and calculate the Vandermonde determinant.)

1.5. Polarization. Introduce new letters \(y = (y_1, y_2)\). If \(E\) is a form of order \(n\) in \(x\), then define its \(k\)-th polarization

\[
E^{(k)} = \frac{(n-k)!}{n!} (y_1 \frac{\partial}{\partial x_1} + y_2 \frac{\partial}{\partial x_2})^k E.
\]

1.6. A summary of results. In this paper we will construct a set of linear combinants \(C = \{C_0, C_2, \ldots, C_r\}\) associated with a set of \(r\) binary \(d\)-ics \(A_1, \ldots, A_r\). In fact \(C_0\) is the Wronskian of the \(\{A_i\}\); the others are defined as transvectants of certain symbolic products derived from \(\{A_i\}\). By construction \(C_q\) is of order \(r(d - r + 1) - 2q\). We will show that the \(C\) enjoy a cluster of special properties:

- The subspace spanned by the \(\{A_i\}\) can be recovered from \(C\) as the solution space of the differential equation \(\sum_q (C_q, F)_{r-q} = 0\).
○ The assignment span \( \{A_i\} \rightarrow [C_0, \ldots, C_r] \) gives a projective imbedding of the Grassmannian \( G(r, S_d) \).

○ Every combinant \( Q \) admits a formula of the type

\[
Q = \frac{1}{C_0^N} \times (A \text{ compound transvectant expression in the } C),
\]

for some nonnegative integer \( N \).

In Theorem 4.1 we characterize all possible values of \( C \). Specifically we prove that a sequence of binary forms \( E_0, E_2, \ldots, E_r \) (of the correct orders) can arise as the Wronskian combinants of an \( r \)-dimensional subspace iff the differential equation

\[
\sum_q (E_q, F)_{r-q} = 0 \quad (3)
\]

admits \( r \) linearly independent polynomial solutions. It follows that the image of the imbedding above is a determinantal variety, defined by equations of degree \( d-r+2 \). The proof proceeds in two steps: firstly we establish some identities which connect Wronskians with transvectants. Then we use these identities to ‘peel off’ one summand at a time from equation (3).

2. The Wronskian combinants

Let us write \( \mathcal{N}_r = \{q : 0 \leq q \leq r, q \neq 1\} \). In this section we will construct the Wronskian combinants \( \mathcal{C} = \{C_q : q \in \mathcal{N}_r\} \). It is a special instance of the method used by Gordan in order to show that any covariant of a binary form is a linear combination of iterated transvectants (see [5, §2]).

2.1. Let

\[
A_i = \alpha_x^{(i)d}, \quad 1 \leq i \leq r,
\]

denote \( r \) binary \( d \)-ics, and \( F = f_x^d \) another binary \( d \)-ic. Then

\[
W = W(A_1, \ldots, A_r, F) = f_x^{d-r} \prod_{i=1}^r \alpha_x^{(i)d-r} \prod_{1 \leq i < j \leq r} (\alpha^{(i)} \alpha^{(j)}) \prod_{i=1}^r (\alpha^{(i)} f).
\]

We will rewrite this expression as follows. Define

\[
\Phi = \prod_{i=1}^r \alpha_x^{(i)d-r} \prod_{1 \leq i < j \leq r} (\alpha^{(i)} \alpha^{(j)}) \prod_{i=1}^r \alpha_y^{(i)}.
\]
Then
\[ W = (\Phi, f^d_y)_{y:=x}, \] (4)
which is to say, take the \( r \)-th transvectant of the pair \( \Phi, f^d_y \) as forms in \( y \) (treating the \( x \) as constants), and then substitute \( x \) for \( y \).

If we write \( G_x = \prod_{i=1}^r \alpha^{(i)}_x \), then
\[ \Phi = \left\{ \prod_{1 \leq i < j \leq r} (\alpha^{(i)} \alpha^{(j)}) \right\} G_x^{d-r} G_y. \]

Now the Gordan series (see [7, §52]) gives an identity
\[ G_x^{d-r} G_y = \sum_{q=0}^r \frac{(rd-r^2)^r}{(rd-r^2+1)(q+r)} [(G_x^{d-r}, G_x)_q]^{(r-q)} (x y)^q. \]

Hence let us define
\[ C_q = (-1)^q \frac{(rd-r^2)^r}{(rd-r^2+1)(q+r)} \left\{ \prod_{1 \leq i < j \leq r} (\alpha^{(i)} \alpha^{(j)}) \right\} (G_x^{d-r}, G_x)_q. \] (5)

By construction this is a combinant which is linear in the coefficients of each \( A_i \), and of order \( r(d-r+1) - 2q \). Notice that \( C_1 \) is identically zero, because it is the Jacobian of two functionally dependent forms (and hence it will not be mentioned any further). Since \( G_x \) is of order \( r \), the index \( q \) ranges over \( \mathcal{N}_r \).

2.2. Thus we have
\[ \Phi = \sum_{q \in \mathcal{N}_r} (-1)^q [C_q]^{(r-q)} (x y)^q. \]

We will calculate \( W \) by substituting this expression into (4). Write \( C_q = c_x^{r(d-r+1)-2q} \). Then
\[ [C_q]^{(r-q)} (x y)^q = c_x^{r(d-r)-q} c_y^{-q} (x y)^q, \]
and
\[ ([C_q]^{(r-q)} (x y)^q, f^d_y)_r = (-1)^q c_x^{r(d-r)-q} (c f)^{r-q} f^q_x f^{d-r}_y. \]

Letting \( y := x \), this reduces to
\[ (-1)^q c_x^{r(d-r)-q} (c f)^{r-q} f^{d-r+q}_x = (-1)^q (C_q, F)_{r-q}. \]

In sum,
\[ W = \sum_{q \in \mathcal{N}_r} (C_q, F)_{r-q}. \] (6)
2.3. Given an arbitrary collection of forms $E = \{E_q : q \in \mathbb{N}_r\}$ of orders $r(d - r + 1) - 2q$, we define
\[
\psi_E(F) = \sum_{q \in \mathbb{N}_r} (E_q, F)_{r-q}.
\]
Then $\psi_E(F) = 0$ is an algebraic differential equation of order $r$ dependent on the parameters $E$.

Let $C = \{C_q\}$ be the combinants constructed above associated with $\{A_1, \ldots, A_r\}$. If the $\{A_i\}$ are linearly dependent, then (and only then) all $C_q$ are zero. If they are independent, then a binary $d$-ic $F$ belongs to their linear span iff $W(A_1, \ldots, A_r, F) = 0$, i.e., iff $\psi_C(F) = 0$. Hence the $C$ completely characterize the subspace spanned by the $A$.

**Example 2.1.** Assume $r = 2$, then $C_0 = (A_1, A_2)_1$. Since $G_x$ is a quadratic,
\[
(G_x^{d-2}, G_x)_2 = \frac{2 - d}{2(2d - 5)} (G_x, G_x)_2 G_x^{d-3}.
\]
(This can be checked by a direct symbolic calculation.) Hence
\[
C_2 = \frac{\binom{2d-4}{2}}{\binom{2d-3}{2}} \frac{2 - d}{2(2d - 5)} (\alpha_1^{(1)} \alpha_2^{(2)})^3 \alpha_1^{(1)d-3} \alpha_2^{(2)d-3} = \frac{2 - d}{4d - 6} (A_1, A_2)_3.
\]

3. **The incomplete Plücker imbedding**

Let
\[
U = \bigoplus_{q \in \mathbb{N}_r} S_{r(d-r+1)-2q},
\]
and consider the morphism
\[
\pi : G(r, S_d) \longrightarrow \mathbb{P} U
\]
which sends the subspace $\Lambda = \text{span}\{A_1, \ldots, A_r\}$ to $[C_0, C_2, \ldots, C_r]$.

**Theorem 3.1.** The morphism $\pi$ is an imbedding.

**Proof.** Since $\Lambda$ can be recovered from $C$, we deduce that $\pi$ is set-theoretically injective. It remains to show that $\pi$ is injective on tangent spaces ([8, Ch. II, Prop. 7.3]).

The tangent space $T_{G,\Lambda}$ is canonically isomorphic to $\text{Hom}(\Lambda, S_d/\Lambda)$. Assume that $v \in T_{G,\Lambda}$ sends $A_i$ to $B_i + \Lambda$. 

The tangent space to $\mathbb{P}U$ at $\pi(v)$ is isomorphic to $U/[C_0, \ldots, C_q]$. To calculate the image vector $d\pi(v)$, define

$$D_q = \lim_{\epsilon \to 0} \frac{C_q(A_1 + \epsilon B_1, \ldots, A_r + \epsilon B_r) - C_q(A_1, \ldots, A_r)}{\epsilon}.$$ 

Then $d\pi(v) = [D_0, D_2, \ldots, D_q]$, considered modulo $[C_0, \ldots, C_q]$. The Wronskian combinants are multilinear in each argument, hence

$$D_q = \sum_{i=1}^r C_q(A_1, \ldots, \widehat{A_i} | B_i, \ldots, A_r),$$

where the last expression means that $A_i$ is to be replaced by $B_i$. Assume that $d\pi(v) = 0$, so there exists a constant $\alpha$ such that $D_q = \alpha C_q$ for all $q$. But then $\sum_q (D_q, A_1)_r_q = 0$, i.e.,

$$\sum_{i=1}^r \left\{ \sum_q (C_q(A_1, \ldots, \widehat{A_i} | B_i, \ldots, A_r), A_1)_r_q \right\} = 0.$$ 

All the summands except $i = 1$ vanish for obvious reasons, hence so does the remaining one. This implies that $A_1 \in \text{Span} \{B_1, A_2, \ldots, A_r\}$, which forces $B_1 \in \text{Span} \{A_1, \ldots, A_r\}$. Similarly each $B_i \in \Lambda$, hence $v$ must be the zero vector. This shows that $\pi$ was injective on tangent spaces. The theorem is proved.

3.1. Let $Q$ be an arbitrary combinant of $r$ binary $d$-ics. We will show that $Q$ admits a ‘formula’ as mentioned in the introduction. In order to make this precise, assume $A_i$ to be as in equation (1). Let $T$ denote the smallest $\mathbb{C}$-subalgebra of the polynomial algebra $\mathbb{C}[\{a_{ij}\}_{i,j}, x_1, x_2]$ such that

1. $C_0, \ldots, C_q \in T$, and
2. if $e_1, e_2 \in T$, then $(e_1, e_2)_k \in T$ for all $k \geq 0$.

Each element of $T$ is a combinant, and there is a natural bigraded decomposition of $T$ induced by the degree $m$ and order $n$. For instance, the element

$$(C_0, C_3)_3, C_2)_2 + 5 (C_0^2, C_4)_6$$

is bihomogeneous of degree 3 and order $3r(d - r + 1) - 20$.

**Theorem 3.2.** Let $Q$ be a combinant of the $\{A_i\}$. Then there exists an integer $N \geq 0$ such that $C_N^0 Q \in T$. 

Since $C_0$ is always nonzero on linearly independent forms, this shows the existence of a formula for $Q$.

**Proof.** Given the imbedding $G \subseteq \mathbb{P} U$, for every integer $m \geq 0$ we have the restriction morphism

$$f_m : H^0(\mathbb{P} U, O_{\mathbb{P}}(m)) \rightarrow H^0(G, O_G(m)).$$

Let $m = N + \deg Q$. The combinant $C_0^N Q$ will lie in $T$ iff the image of the corresponding morphism (see §1.3)

$$S_{\text{ord}(C_0^N Q)} \rightarrow H^0(G, O_G(m))$$

is contained in the image of $f_m$. But this can always be arranged by choosing $N >> 0$, since $f_m$ is surjective for $m >> 0$. □

**Remark 3.3.** If $\rho$ is the Castelnuovo regularity of the ideal sheaf $I_G$, then $H^1(\mathbb{P} U, I_G(\rho - 1)) = 0$ implying that $f_{\rho - 1}$ is surjective. Hence $N$ can be chosen to be max \{0, $\rho - \deg Q - 1$\}. It is possible (but rather tedious) to calculate an explicit upper bound for $\rho$ from the Hilbert polynomial of $G$ (see [9]), but we have not attempted this.

**Example 3.4.** Assume $(r, d) = (2, 5)$, and write $t_i = (A_1, A_2)_i$. Then the element $C_0 t_5$ lies in $T$, in fact there is an identity

$$t_5 = \frac{1}{C_0} [50 C_2^2 - 15 (C_0, C_0)_4 - 40 (C_0, C_2)_2].$$

**Proof sketch:** $C_0 t_5$ is of degree-order $(2, 8)$. The plethysm $S_{(2, 2)}(S_5)$ contains 3 copies of $S_8$, hence there is a 3 dimensional space of such combinants. By specialising $A_1, A_2$ we can show that $C_2^2, (C_0, C_0)_4, (C_0, C_2)_2$ are linearly independent, hence they form a basis of this space. Thus $C_0 t_5$ must be expressible as their linear combination. To find the actual coefficients we only need to solve a system of linear equations.

### 4. Wronskians and Transvectants

We now come to our second main theorem which characterizes all possible values of $C$.

4.1. Let $E = \{ E_q : q \in \mathcal{N}_r \}$ be an arbitrary collection of binary forms of orders $r(d - r + 1) - 2q$, such that $E_1 \neq 0$. In general the $r$-th order differential equation

$$\psi_E(F) = 0$$

(7)
does not admit \( r \) linearly independent polynomial solutions. Our result says that if indeed it does, then the \( E \) must be values of Wronskian combinants.

**Theorem 4.1.** Assume that there exist \( r \) linearly independent \( d \)-ics \( A_1, \ldots, A_r \) such that \( \psi_E(A_i) = 0 \). Then there exists a nonzero constant \( k \) such that
\[
E_q = k C_q(A_1, \ldots, A_r)
\]
for all \( q \in \mathbb{N}_r \).

Then, of course, we can arrange that \( E_q = C_q \) by replacing \( A_1 \) with \( k A_1 \).

4.2. The proof hinges upon certain identities involving transvectants and Wronskians. Let \( B \) denote a form of order \( n \). For \( 0 \leq p \leq \min \{ d, n \} \), define
\[
\Gamma_p(B; A_1, \ldots, A_r) = \sum_{i=1}^{r} (-1)^{i+1} (B, A_i)_p W(A_1, \ldots, \widehat{A}_i, \ldots, A_r).
\]
We will tentatively abbreviate this to \( \Gamma_p \). Now the key result is the following.

**Proposition 4.2.** We have identities
\[
\Gamma_p = \begin{cases} 
0 & \text{for } 0 \leq p \leq r - 2, \\
(-1)^{r-1} B W(A_1, \ldots, A_r) & \text{for } p = r - 1, \\
(-1)^{r-1} r (B, W(A_1, \ldots, A_r))_1 & \text{for } p = r.
\end{cases}
\]

We have found no such simple identities for \( p > r \). The proposition will be proved in \( \S 5 \) meanwhile let us use it to prove the theorem.

**Proof of Theorem 4.1.** By hypothesis
\[
\sum_{q \in \mathbb{N}_r} (E_q, A_i)_{r-q} = 0. \tag{8}
\]
Multiply this equation by \( (-1)^{i+1} W(A_1, \ldots, \widehat{A}_i, \ldots, A_r) \) and sum over \( 1 \leq i \leq r \). This gives
\[
\sum_{q \in \mathbb{N}_r} \Gamma_{r-q}(E_q; A_1, \ldots, A_r) = 0.
\]
By the proposition, we have $\Gamma_{r-q} = 0$ for $r - q \leq r - 2$, i.e., for $q \geq 2$. Fortunately there is no $q = 1$ term, hence

$$\Gamma_r(E_0; A_1, \ldots, A_r) = (E_0, W(A_1, \ldots, A_r))_1 = 0.$$  

In general, if $M, N$ are forms of the same order, then $(M, N)_1 = W(M, N)$; which can be zero only if $M, N$ are multiples of each other. Hence there exists a constant $k$ such that

$$E_0 = k W(A_1, \ldots, A_r) = k C_0.$$  

Now write $\tilde{E}_q = E_q - k C_q$. Subtract the equation $k \psi C(A_i) = 0$ from (8), this gives

$$\sum_{q=2}^{r} (\tilde{E}_q, A_i)_{r-q} = 0.$$  

Multiply (9) by $(-1)^{i+1} W(A_1, \ldots, A_i, \ldots, A_{r-1})$, (note that $A_r$ is missing), and sum over $1 \leq i \leq r - 1$. Then we have

$$\sum_{q=2}^{r} \Gamma_{r-q}(\tilde{E}_q; A_1, \ldots, A_{r-1}) = 0.$$  

By the proposition, all the summands for $r - q \leq r - 3$, (i.e., $q \geq 3$) are zero. Hence

$$\Gamma_{r-2}(\tilde{E}_2; A_1, \ldots, A_{r-1}) = (-1)^{r-2} \tilde{E}_2 W(A_1, \ldots, A_{r-1}) = 0.$$  

Since the $A_i$ are linearly independent, the Wronskian on the right is nonzero, hence $\tilde{E}_2 = 0$. We can now repeat this procedure by dropping $A_{r-1}, A_{r-2}$ etc.; this will successively force $\tilde{E}_3, \tilde{E}_4$ etc. to be zero. \hfill \Box

4.3. The assignment $F \rightarrow \psi_E(F)$ gives a morphism

$$\psi : S_d \otimes \mathcal{O}_{PU}(-1) \rightarrow S_{d+r(r-d-1)} \otimes \mathcal{O}_{PU}.$$  

The theorem implies that $\pi(G)$ is equal to the locus \{rank $\psi \leq d-r+1$\}. Consequently $\pi(G)$ is set-theoretically defined by equations of degree $d - r + 2$.

5. Proof of proposition 4.2.

5.1. Let us write $B = \beta_x^n$. Then $\Gamma_p$ has the symbolic expression

$$\sum_{i=1}^{r} (-1)^{i+1} \{ (\beta \alpha^{(i)})^p \beta_x^{m-p} \alpha_x^{(i)d-p} \prod_{1 \leq j < k \leq r} (\alpha^{(j)} \alpha^{(k)}) \prod_{j \neq i} \alpha_x^{(j)d-r+2} \}.$$
Now dehomogenize using the following substitutions:

\[(\beta_1, \beta_2) = (b, 1), \quad (\alpha_1^{(i)}, \alpha_2^{(i)}) = (a_i, 1), \quad (x_1, x_2) = (1, -u).\]

Then we have

\[
\Gamma_p = \sum_{i=1}^{r} (-1)^{i+1} \left\{ (b - a_i)^p (b - u)^{n-p} (a_i - u)^{d-p} \times \prod_{1 \leq j < k \leq r, j, k \neq i} (a_j - a_k) \times \prod_{1 \leq j < r} (a_j - u)^{d-r+2} \right\}.
\]

This can be rewritten as \((b - u)^{n-p} \times \prod_{i=1}^{r} (a_i - u)^{d-r+2} \times \)

\[
\sum_{i=1}^{r} (-1)^{i+1} (b - a_i)^p (a_i - u)^{r-p-2} \times \begin{bmatrix}
\alpha_1^{r-2} & \cdots & a_1 & 1 \\
\vdots & \ddots & \vdots \\
a_i^{r-2} & \cdots & a_{i-1} & 1 \\
a_{i+1}^{r-2} & \cdots & a_{i+1} & 1 \\
\vdots & \ddots & \vdots \\
a_{r}^{r-2} & \cdots & a_r & 1
\end{bmatrix}.
\]

Hence

\[
\Gamma_p = (b - u)^{n-p} \times \prod_{i=1}^{r} (a_i - u)^{d-r+2} \times \begin{bmatrix}
Q(a_1) & a_1^{r-2} & \cdots & a_1 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
Q(a_r) & a_r^{r-2} & \cdots & a_r & 1
\end{bmatrix}, \quad (10)
\]

with \(Q(a) = (b - a)^p (a - u)^{r-p-2}.\) (To see this, expand the last determinant by its first column.)

If \(p \leq r - 2,\) then \(Q(a)\) is a polynomial in \(a\) of degree \(r - 2,\) hence the first column is a linear combination of the others. This forces \(\Gamma_p = 0,\)

which is the first part of the proposition. \(\square\)

5.2. Now let \(p = r - 1,\) so that

\[
Q(a) = \frac{(b - a)^{r-1}}{a - u} = (-1)^{r-1} \frac{[(a - u) + (u - b)]^{r-1}}{a - u} = (-1)^{r-1} \sum_{j=0}^{r-1} \binom{r-1}{j} (a - u)^{j-1} (u - b)^{r-1-j}
\]
By the previous argument on columns, we only need the $j = 0$ term to calculate the determinant. Hence $\Gamma_{r-1}$ is equal to

\[
(b - u)^{n-r+1}(u - b)^{r-1} \times \prod_{i=1}^{r} (a_i - u)^{d-r+2} \times \begin{vmatrix}
\frac{1}{a_1 - u} & a_1^{r-2} & \cdots & a_1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{a_r - u} & a_r^{r-2} & \cdots & a_r & 1
\end{vmatrix}
\]

\[
= -(b - u)^{n} \prod_{i=1}^{r} (a_i - u)^{d-r+2} \times \begin{vmatrix}
\frac{1}{u-a_1} & a_1^{r-2} & \cdots & a_1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{1}{u-a_r} & a_r^{r-2} & \cdots & a_r & 1
\end{vmatrix}
\]

The last determinant can be written as

\[
\sum_{s \ge 0} \frac{1}{u^{s+1}} \begin{vmatrix}
a_1^s & a_1^{r-2} & \cdots & a_1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_r^s & a_r^{r-2} & \cdots & a_r & 1
\end{vmatrix}
\]

Let us write $\Delta_r(a)$ for $\prod_{1 \le j < k \le r} (a_j - a_k)$. Using the Schur polynomials $S_{\lambda}(a)$ (see [3, Appendix 1]), we rewrite $\Gamma_{r-1}$ as

\[
- (b - u)^{n} \prod_{i=1}^{r} (a_i - u)^{d-r+2} \times \Delta_r(a) \sum_{s \ge r-1} \frac{1}{u^{s+1}} S_{(s-r+1)}(a_1, \ldots, a_r)
\]

\[
= -(b - u)^{n} \prod_{i=1}^{r} (a_i - u)^{d-r+2} \times \Delta_r(a) \times \frac{1}{u^r} \times \prod_{i=1}^{r} \frac{1}{1 - \frac{a_i}{u}}
\]

\[
= (-1)^{r-1} (b - u)^{n} \Delta_r(a) \prod_{i=1}^{r} (a_i - u)^{d-r+1}.
\]

Rehomogenizing this, we get

\[
\Gamma_{r-1} = (-1)^{r-1} \beta^n \times \prod_{i=1}^{r} \alpha_x^{(i)^{d-r+1}} \times \prod_{1 \le j < k \le r} (\alpha^{(j)} \alpha^{(k)}),
\]

which proves the second part.

5.3. Finally let $p = r$. Then $Q(a)$ is equal to

\[
= \frac{(b - a)^r}{(a - u)^2} = (-1)^r \frac{[(u - b) + (a - u)]^r}{(a - u)^2}
\]

\[
= (-1)^r \left[ \frac{(u - b)^r}{(a - u)^2} + \frac{r (u - b)^{r-1}}{a - u} + \text{irrelevant terms} \right]
\]
Now substitute this into (10). The positive powers of \((a-u)\) contribute nothing to the sum, hence

\[
\Gamma_r = (-1)^r (b-u)^{n-r} \left\{ \prod_{i=1}^{r} (a_i - u)^{d-r+2} \right\}
\times \left[ (u-b)^r D_2 + r (u-b)^{r-1} D_1 \right],
\]

where

\[
D_\nu = \left| \begin{array}{cccc}
\frac{1}{(a_1-u)^\nu} a_1^{r-2} & \cdots & a_1 & 1 \\
\vdots & \ddots & \vdots & \vdots \\
\frac{1}{(a_r-u)^\nu} a_r^{r-2} & \cdots & a_r & 1
\end{array} \right|.
\]

As in the previous case,

\[
D_1 = -\frac{\Delta_r(a)}{u^r} \times \prod_{i=1}^{r} \frac{1}{1 - \frac{a_i}{u}} = \frac{(-1)^{r-1} \Delta_r(a)}{\prod_{i=1}^{r} (a_i - u)},
\]

whereas

\[
D_2 = \frac{\partial D_1}{\partial u} = D_1 \times \sum_{i=1}^{r} \frac{1}{a_i - u}.
\]

Putting everything together, \(\Gamma_r\) equals

\[
(b-u)^n \left\{ \prod_{i=1}^{r} (a_i - u)^{d-r+2} \right\} \times \left[ D_2 + \frac{r D_1}{u-b} \right]
\]

\[
= (-1)^{r-1} (b-u)^n \Delta_r(a) \left\{ \prod_{i=1}^{r} (a_i - u)^{d-r+1} \right\} \times \sum_{i=1}^{r} \left\{ \frac{1}{a_i - u} + \frac{r}{u-b} \right\}
\]

\[
= (-1)^{r-1} (b-u)^{n-1} \Delta_r(a) \left\{ \prod_{i=1}^{r} (a_i - u)^{d-r+1} \right\} \times \sum_{i=1}^{r} \frac{b-a_i}{a_i - u}.
\]

Homogenizing,

\[
\Gamma_r = (-1)^{r-1} \beta_x^{n-1} \times \prod_{1 \leq j < k \leq r} (\alpha^{(j)}_x \alpha^{(k)})
\times \sum_{i=1}^{r} \left( (\beta \alpha^{(i)}) \times \alpha^{(i)}_x^{d-r} \times \prod_{1 \leq j \leq r \atop j \neq i} \alpha^{(j)}_x^{d-r+1} \right)
\]

\[
= (-1)^{r-1} \left( B, W(A_1, \ldots, A_r) \right)_1.
\]
In the last step we have used the general formula for transvectants of symbolic products (see \[4, \S3.2.5\]). The proposition is proved; this also completes the proof of Theorem 4.1. □

The following question arises naturally: given a reductive group \(G\) and a \(G\)-module \(V\), investigate how much of the theory carries over to the Grassmannian \(G(r, V)\).

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- The Göttinger DigitalisierungsZentrum (GDZ)
- Project Gutenberg (PG)
- The University of Michigan Historical Mathematics Collection (UM)

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