A vertex model for LLT polynomials

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Overview

Plan:

1. Review a vertex model for Schur polynomials.
2. Introduce our vertex model for coinversion LLT polynomials.
3. Sketch the proofs of some results about the LLT polynomials in the vertex model framework.
4. Discuss possible future work involving super-symmetric LLT polynomials.
Part 1: A vertex model for Schur polynomials
There is a well-known bijection between semistandard skew Young tableaux and nonintersecting up-right lattice paths.

\[ \lambda/\mu = (6, 3, 3, 0)/(2, 1, 0, 0), \ n = 5 \]

In general,

1. The partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \) gives the top boundary condition. The \( i \)-th path exits in column \( \lambda_i + m - i \).
2. The partition \( \mu = (\mu_1, \ldots, \mu_m) \) gives the bottom boundary condition. The \( i \)-th path enters in column \( \mu_i + m - i \).
3. The number of rows is \( n \).
Local weights

We can consider this as a vertex model with local weights given by

\[
\begin{array}{ccccc}
\text{1} & \text{x}_i & \text{x}_i & \text{1} & \text{1}
\end{array}
\]

where row \(i\) has the variable \(x_i\). The weight of the configuration is the product of the weights of the vertices.

\[
\begin{array}{c}
\begin{array}{ccccc}
\text{1} & \text{x}_i & \text{x}_i & \text{1} & \text{1}
\end{array}
\end{array}
\]

The partition function of the vertex model is

\[
Z_{\lambda/\mu}(x_1, \ldots, x_n) := \sum_{\text{config. } C} w(C) = s_{\lambda/\mu}(x_1, \ldots, x_n).
\]
Part 2: A vertex model for LLT polynomials
A brief history of LLT polynomials

1. LLT polynomials were first defined by Lascoux, Leclerc, and Thibon (1997). They serve as a generating function for semistandard ribbon tableaux of shape $\lambda$, $q$-counting a spin statistic.

2. Haglund, Haiman, and Loehr (2005) gave an alternate description of the LLT polynomials as a generating function for tuples of semistandard Young tableaux, $q$-counting an inversion statistic.

3. Blasiak, Haiman, Morse, Pun, and Seelinger (2021) gave a formulation in terms of coinversions.

4. We formulate them as collection of up-right lattice paths in bijection with the coinversion LLT polynomials.
Let $\lambda/\mu = (\lambda^{(1)}/\mu^{(1)}, \ldots, \lambda^{(k)}/\mu^{(k)})$ be a tuple of skew partitions. Define the coinversion LLT polynomial

$$L_{\lambda/\mu}(X; t) = \sum_{T \in \text{SSYT}(\lambda/\mu)} t^{\text{coinv}(T)} X^T$$

where coinversion triples are given by

$$\text{triples } a \leq b \leq c$$

$$\begin{array}{c}
\text{a} \\
\text{b} \\
\text{c}
\end{array}$$

where $a = 0$ and $c = \infty$ if they are not in the diagram.

For example:

$$\begin{array}{ccc}
-2 & -1 & 0 \\
1 & 2 & \\
3 & 3 & \\
1 & \\
\end{array}$$

$T^{(1)}$

$T^{(2)}$
The inversion LLT polynomial is defined by

\[
G_{\lambda/\mu}(X; t) = \sum_{T \in \text{SSYT}(\lambda/\mu)} t^{\text{inv}(T)} X^T
\]

where \text{inversion triples} are triples that are not coinversion triples. Note that

\[
L_{\lambda/\mu}(X; t) = t^m G_{\lambda/\mu}(X; t^{-1})
\]

where \( m \) is the total number of triples in the diagram.
Other definitions of/approaches to LLT polynomials

- Our definition of LLT polynomials is in terms of tuples of SSYT, but the original definition of LLT polynomials by Lascoux, Leclerc, and Thibon (1997) was in terms of semistandard ribbon tableaux.

- The Littlewood quotient map gives a bijective correspondence

\[
\{\text{semistandard } k\text{-ribbon tableaux of shape } \lambda \} \leftrightarrow \{k\text{-tuples of SSYT of shapes } (\lambda^{(1)}, \ldots, \lambda^{(k)})\}
\]

where \((\lambda^{(1)}, \ldots, \lambda^{(k)})\) is the \(k\)-quotient of \(\lambda\).
The $k$-quotient map

the $k$-quotient $(\lambda^{(0)}, \ldots, \lambda^{(k-1)})$ of a partition $\lambda$: label the Maya diagram on the boundary of $\lambda$ with $0, \ldots, k-1$

picture adapted from https://www2.math.upenn.edu/~peal/polynomials/borderStripTableaux.htm
The Littlewood quotient map

the Littlewood quotient map:
look at the minimum content of each ribbon modulo $k$

picture adapted from https://www2.math.upenn.edu/~peal/polynomials/borderStripTableaux.htm
Other definitions of/approaches to LLT polynomials

- Lascoux, Leclerc, and Thibon (1997) define LLT polynomials using semistandard ribbon tableaux and the spin statistic.
- Given \( T \in \text{SSRT}_k(\lambda/\mu) \), we define

\[
\text{spin}(T) = \sum_{\text{ribbons } r \text{ in } T} (\text{height}(r) - 1).
\]

As an example:

\[
\begin{array}{cccc}
 & 1 & 1 & 2 \\
2 & 0 & 2 & 1 \\
 & 0 & 1 & 1 \\
 & 1 & & \\
\end{array}
\]

\[
\text{spin}(T) = 14
\]

picture adapted from https://www2.math.upenn.edu/~peal/polynomials/borderStripTableaux.htm
Other definitions of/approaches to LLT polynomials

- Curran, Yost-Wolff, Zhang, and Zhang (2019) define LLT polynomials by
  \[ \tilde{G}_{\lambda/\mu}^{(k)}(X; t) = \sum_{T \in \text{SSRT}_k(\lambda/\mu)} t^{\text{spin}(T)} X^T. \]

- \( \mathcal{L}_{\lambda/\mu}(X; t) = t^e \tilde{G}_{\lambda/\mu}^{(k)}(X; t^{1/2}) \) for some integer \( e \), where \( \lambda/\mu \leftrightarrow \lambda/\mu \) via the Littlewood quotient map

- Starting with this definition, they construct a vertex model of a similar flavor as ours.
Let $\lambda/\mu = (\lambda^{(1)}/\mu^{(1)}, \ldots, \lambda^{(k)}/\mu^{(k)})$. It is known that when $t = 1$ the LLT polynomial is a product of Schur polynomials

$$\mathcal{L}_{\lambda/\mu}(X; 1) = s_{\lambda^{(1)}/\mu^{(1)}}(X) \ldots s_{\lambda^{(k)}/\mu^{(k)}}(X).$$

We can superimpose the paths:
Local weights

Let $i, j, k, l \in \{0, 1\}$. Recall the weights for one color:

\[
L_x^{(1)}(i, j, k, l) = \begin{cases} 
1 & \text{if } x = 0 \\
1 & \text{if } x = 1
\end{cases}
\]

Let $I, J, K, L \in \{0, 1\}^k$, $A = (A_1, \ldots, A_k)$ for $A \in \{I, J, K, L\}$. We introduce $t$ into the weights for $k$ colors

\[
L_x^{(k)}(I, J, K, L) = \prod_{i=1}^{k} L_x^{(1)}(i, j, k, l) = \prod_{i=1}^{k} L_x^{(1)}(i, j, k, l)
\]

where $\delta_i = \# \text{ colors greater than } i \text{ that appear at the vertex}$. In other words, we pick up a factor of $t$ when a color exits right and a larger color is present.
LLT polynomials as multicolored collection of paths

With **blue** as color 1 and **red** as color 2, the faces that contribute a factor of $t$ are

As an example:

$w = x_1^3 x_2^2 x_3^2 t^3$
Coinversion LLT polynomials

The coinversion triples map to faces of the vertex model that contribute a $t$. For instance, when $0 < a < b < c < \infty$,

$$
T^{(2)}(2) \leftrightarrow T^{(1)}(3)
$$

**Theorem**

Let $\lambda/\mu$ be a tuple of skew partitions. Let $Z_{\lambda/\mu}$ be the partition function of our vertex model. Then

$$
Z_{\lambda/\mu}(x_1, \ldots, x_n; t) = \mathcal{L}_{\lambda/\mu}(x_1, \ldots, x_n; t).
$$
Part 3: Yang-Baxter integrability + other results
Yang-Baxter equation

**Theorem**

*This vertex model is Yang-Baxter integrable.*

The Yang-Baxter equation is given by

\[
\sum_{\text{interior paths}} w \begin{pmatrix}
  l_1 & y & J_3 \\
  l_2 & x & J_2 \\
  l_3 & & J_1
\end{pmatrix} = \sum_{\text{interior paths}} w \begin{pmatrix}
  l_1 & x & J_3 \\
  l_2 & y & J_2 \\
  l_3 & & J_1
\end{pmatrix}
\]

for any choice of boundary condition \( l_1, l_2, l_3 \) and \( J_1, J_2, J_3 \). The weights

\[
\begin{array}{c}
  \uparrow \\
  J \\
  \downarrow \\
  I
\end{array} \quad \begin{array}{c}
  \uparrow \\
  K \\
  \downarrow \\
  L
\end{array}
\]

are the \( R \)-matrix for the model.
The weights of the one-color $R$-matrix are

\[
R_{y/x}^{(1)}(i,j,k,l): \begin{cases} 
1 - y/x & \text{Type 1} \\
y/x & \text{Type 2} \\
1 & \text{Type 3} \\
y/x & \text{Type 4} \\
y/x & \text{Type 5}
\end{cases}
\]

The weights of the $k$-color $R$-matrix are given by

\[
R_{y/x}^{(k)}(I,J,K,L) = \prod_{i=1}^{k} R_{y/(xt^{\epsilon_i})}^{(1)}(l_i, j_i, K_i, L_i)
\]

where $\epsilon_i = \#$ colors greater than $i$ of type 1.
With blue as color 1 and red as color 2

\[
\begin{align*}
\begin{pmatrix}
\text{blue} & y \\
x & \text{red}
\end{pmatrix}
&= \begin{pmatrix}
\text{red} & x \\
y & \text{blue}
\end{pmatrix} \\
\begin{pmatrix}
x^2t(1 - y/x)
\end{pmatrix}
&= \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
x \\
y
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix}
\end{align*}
\]

The proof of the YBE follows by a (slightly complicated) induction argument on the number of colors.
Symmetry

The YBE implies that the polynomials are symmetric. Let 0 indicate “no paths.” Then using the YBE repeatedly gives

\[
\begin{array}{c|c|c}
0 & x_j & \cdots & x_j \\
\hline
0 & x_i & \cdots & x_i \\
0 & \cdots & x_j & \cdots \\
\end{array}
\]

\[=\]

\[
\begin{array}{c|c|c}
0 & x_i & x_j \\
\hline
0 & x_j & x_i \\
0 & \cdots & \cdots \\
\end{array}
\]

\[=\]

\[
\begin{array}{c|c|c}
x_i & \cdots & x_j \\
\hline
x_j & \cdots & x_i \\
0 & \cdots & \cdots \\
\end{array}
\]

\[=\]

\[
\begin{array}{c|c|c}
0 & x_i & x_j \\
\hline
0 & x_j & x_i \\
0 & \cdots & \cdots \\
\end{array}
\]
With this formulation of the vertex model, we can give a combinatorial proof of a Cauchy identity

\[
\sum_{\lambda} t^{d(\lambda)} \mathcal{L}_\lambda(X_n; t)\mathcal{L}_\lambda(Y_n; t) = \prod_{i,j=1}^{n} \prod_{m=0}^{k-1} (1 - x_i y_j t^m)^{-1}.
\]
Before proving the Cauchy identity, we introduce the $L^*$-matrix, with weights given by

$$L^*_x(I, J; K, L) = x^k t^{(k)} L_x(I, J; K, L)$$

where $\bar{x} = \frac{1}{xt^{k-1}}$.

We represent the $L^*$-matrix with a gray box.
The $L^*$-matrix

Here’s one way to think about the weights of the $L^*$-matrix. Similarly as before, we can think of each color as contributing a multiplicative factor to the overall weight. The contribution of color $i$ is given by

\[
\begin{array}{cccc}
\text{Type 1} & \text{Type 2} & \text{Type 3} & \text{Type 4} \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
xt^{\alpha_i} & xt^{\alpha_i} & xt^{\alpha_i} & t^{\beta_i} \\
\end{array}
\]

where

\[
\alpha_i = \# \text{ colors greater than } i \text{ of Type 1, 4, or 5},
\]

\[
\beta_i = \# \text{ colors greater than } i \text{ of Type 4 or 5}.
\]
Cauchy identity

The Cauchy identity

\[ \sum_{\lambda} t^{d(\lambda)} \mathcal{L}_\lambda(X_n; t) \mathcal{L}_\lambda(Y_n; t) = \prod_{i,j=1}^{n} \prod_{m=0}^{k-1} (1 - x_i y_j t^m)^{-1} \]

is equivalent to the following identity, which follows from repeated use of the YBE:

\[ \sum_{\text{interior paths}} w \left( \begin{array}{c}
\vdots \\
y_1 \\
\vdots \\
x_1 \\
\vdots
\end{array} \right) = \sum_{\text{interior paths}} w \left( \begin{array}{c}
\vdots \\
y_1 \\
\vdots \\
y_n \\
\vdots
\end{array} \right) \]

- \( d(\lambda) \) is an integer that depends only on \( \lambda \)
- the lattice has infinitely many columns
Cauchy identity

\[
\begin{align*}
\text{vertex model} & = 1 & \Rightarrow & \text{vertex model} & = 1 \\
\text{vertex model} & = x^k t^{\binom{k}{2}} & \Rightarrow & \text{vertex model} & = 0 \\
\text{vertex model} & = x^k t^{\binom{k}{2}} & \Rightarrow & \text{vertex model} & = 0 \\
\text{vertex model} & = 1 & \Rightarrow & \text{vertex model} & = 1
\end{align*}
\]
Cauchy identity

\[
\text{LHS} = w \left( \begin{array}{c}
  \vdots \\
  y_1 \\
  \vdots \\
  y_n \\
  \bar{x}_1 \\
  \vdots \\
  \bar{x}_n
\end{array} \right) = w \left( \begin{array}{c}
  \vdots \\
  \bar{y}_1 \\
  \vdots \\
  \bar{y}_n \\
  \bar{x}_1 \\
  \vdots \\
  \bar{x}_n
\end{array} \right) = (x^\rho)^k t^{n\choose 2}(k)
\]

where \( x^\rho = x_1^{n-1}x_2^{n-2} \cdots x_{n-1}^1 x_n^0 \)
Cauchy identity

\[
\text{RHS} = \sum_{\lambda} w \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
y_{\lambda 0} \\
\vdots \\
y_{1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
y_{n} \\
\lambda \\
\vdots \\
\vdots \\
\vdots \end{array} \right) w \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\bar{x}_{n} \\
\vdots \\
\bar{x}_{1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\bar{x}_{1} \\
\lambda \\
\vdots \\
\vdots \\
\vdots \end{array} \right) w \left( \begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
y_{\lambda 0} \\
\vdots \\
y_{1} \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\

\sum_{\lambda} \mathcal{L}_{\lambda}(Y_n; t) \cdot (x^\rho)^{k} t^{\binom{n}{2}}(\binom{2}{2}) t^{d(\lambda)} \mathcal{L}_{\lambda}(X_n; t) \cdot \prod_{i,j=1}^{n} \prod_{m=0}^{k-1} (1 - x_i y_j t^m)
\]
Part 4: Purple vertices + super-symmetric LLT
Generalizing the vertex model

- Aggarwal, Borodin, and Wheeler (2021) independently derived many of the results in this presentation. They used a more general vertex model, which has the $L$ and $R$ matrices as specializations.

- Their vertex model has other interesting specializations, including the $L'$ and $R'$ matrices (to be introduced shortly), which can be used to prove results about LLT polynomials and super-symmetric LLT polynomials.
The $L'$-matrix

We introduce the $L'$-matrix, which we represent with a purple box.

To get the $L'$-matrix from the $L$-matrix, we invert the vertical parts of the paths and then reflect the paths over the vertical axis.
To be more precise, the weights for one color are

|   |   | 1 | 0 | 0 | 0 |
|---|---|---|---|---|---|
| 1 | 0 |   |   |   |   |
| 1 | 0 | 1 |   | 1 | 0 |
| 0 | 0 |   | 0 | 0 | 1 |

and the weights for \( k \) colors are

\[
L_x^{(k)}(I, J, K, L) = \prod_{i=1}^{k} L_x^{(1)}(I_i, J_i, K_i, L_i) x t^{\delta_i'}
\]

where \( \delta_i' = \# \) colors greater than \( i \) that are vertical.
The partition function of the $L'$-matrix

Proposition (Aggarwal, Borodin, and Wheeler (2021))

Let $\lambda/\mu$ be a tuple of skew partitions. Let $Z'_{\lambda/\mu}$ be the partition function associated to the $L'$-matrix. Then

$$Z'_{\lambda/\mu}(X_n; t) = t^{-\frac{1}{2}(d(\lambda)+d(\lambda')-d(\mu)-d(\mu'))} L_{\lambda'/\mu'}(t^{\frac{k-1}{2}} X_n; t^{-1})$$

where $\lambda' = (\lambda^{(k)'}, \ldots, \lambda^{(1)'}$) (and similarly for $\mu'$).
Theorem (Aggarwal, Borodin, and Wheeler (2021))

The $L$ and $L'$ matrices satisfy the Yang-Baxter equation

$$\sum_{\text{interior paths}} w \left( \begin{array}{ccc} J_1 & & J_3 \\ I_1 & J_2 & I_3 \\ I_2 & x & J_1 \end{array} \right) = \sum_{\text{interior paths}} w \left( \begin{array}{ccc} J_3 & & J_1 \\ I_1 & x & I_3 \\ I_2 & y & J_2 \end{array} \right)$$

for any choice of boundary condition $I_1, I_2, I_3$ and $J_1, J_2, J_3$. The weights

$$\begin{array}{c} J \\ J' \end{array} \quad \begin{array}{c} K \\ L \end{array}$$

are the $R'$-matrix for the model.
The weights of the one-color $R'$-matrix are

$$R'_{y/x}^{(1)}(i, j, k, l) = \begin{cases} 
\frac{1}{1+y/x} & \frac{y/x}{1+y/x} \\
\frac{1}{1+y/x} & \frac{y/x}{1+y/x} \\
1 & 
\end{cases}$$

The weights of the $k$-color $R'$-matrix are given by

$$R'_{y/x}^{(k)}(I, J, K, L) = \prod_{i=1}^{k} R'_{y/(xt^{\epsilon'_i})}^{(1)}(I_i, J_i, K_i, L_i)$$

where $\epsilon'_i = \#$ colors greater than $i$ that are present at the vertex.
A dual Cauchy identity

Aggarwal, Borodin, and Wheeler (2021) use this YBE to prove a dual Cauchy identity

\[ \sum_{\lambda} t^{-\frac{1}{2}(d(\lambda) - d(\lambda'))} \mathcal{L}_\lambda(X_n; t) \mathcal{L}_{\lambda'}(Y_n; t^{-1}) = \prod_{i,j=1}^{n} \prod_{m=0}^{k-1} \left(1 + x_i y_j t^{\frac{k-1}{2}} t^m\right). \]
A super ribbon tableaux is a ribbon tableaux with ribbons labelled by the alphabet $1 < ... < n < 1' < ... < m'$ such that

1. across each row, the primed ribbons strictly increase
2. down each column, the un-primed ribbons strictly increase.

The super-symmetric LLT polynomial is defined by

$$L_s^{S(k)}(X_n, Y_m; t) = \sum_{T \in SRT_k(\lambda/\mu)} t^{\text{spin}(T)} x^{\text{wt}(T)} y^{\text{wt}'(T)}.$$ 

We define

$$L_s^{S}(X_n, Y_m; t) = L_s^{S(k)}(X_n, Y_m; t)$$

when $\lambda/\mu \leftrightarrow \lambda/\mu$ via the Littlewood quotient map.
A vertex model for the super-symmetric LLT polynomials

The $L$ and $L'$ matrices form a vertex model for the super-symmetric LLT polynomials, up to a power of $t$.

**Theorem**

*For some integer $e$,*

$$\frac{1}{t^e} \mathcal{L}^S_{\lambda/\mu}(X_n, Y_m; t) = \sum_{\nu} \mathcal{L}_{\nu/\mu}(X_n; t) Z'_{\lambda/\nu}(Y_m; t)$$

$$= \sum_{\nu} w \left( \begin{array}{c}
\ldots \\
x_1 & \nu \\
\vdots \\
x_n \\
y_1 & \lambda \\
y_m \\
\ldots \\
\ldots \\
x_1
\end{array} \right) = \sum_{\text{interior paths}} w \left( \begin{array}{c}
\ldots \\
x_1 & \mu \\
\vdots \\
x_n \\
y_1 & \lambda \\
y_m \\
\ldots \\
\ldots \\
x_1
\end{array} \right).$$
The dual relationship between the purple and white vertices gives

$$\mathcal{L}^S_{\lambda/\mu}(X_n, Y_m; t) = t^\Delta \mathcal{L}^S_{\lambda'/\mu'}(Y_m, X_n; t^{-1})$$

for some integer $\Delta$. The idea is to invert vertical parts, reflect over the left edge of the lattice, and swap purple and white.
A vertex model for Schur polynomials
A vertex model for LLT polynomials
Yang-Baxter integrability + other results
Purple vertices + super-symmetric LLT

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Thank You!