THE $p$-CURVATURE CONJECTURE IN FAMILIES

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Abstract. We prove two results on the $p$-curvature conjecture for families of algebraic varieties. First, we prove the conjecture for rank two vector bundles with connection on generic curves of positive genus. Second, we show that the $p$-curvature conjecture holds for the generic fiber of a strongly topologically constant family of varieties provided it holds on some special fiber.

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1. Introduction

1.1. This is a two-part study of the Grothendieck–Katz $p$-curvature conjecture in families of algebraic varieties. In the first part, we prove the $p$-curvature conjecture for rank two vector bundles with connection on generic curves of positive genus. In the second part, given a strongly topologically constant family of algebraic varieties, we show that the $p$-curvature conjecture holds for the generic fiber of the family if it holds for some special fiber; this in particular allows us to deduce new cases of the conjecture by deformation of known examples.

Formulated as follows, the $p$-curvature conjecture gives an arithmetic criterion for certain differential equations on algebraic varieties to admit algebraic solutions. Let $(V, \nabla)$ be a vector bundle with integrable connection on a smooth variety $X$ over a field $K$ of characteristic zero. Let $R$ be a finitely generated subring of $K$ to which $X$ and $(V, \nabla)$ descend. For each maximal ideal $\mathfrak{p}$ of $R$, one can consider the reduction $(V_{\mathfrak{p}}, \nabla_{\mathfrak{p}})$ of the connection modulo $\mathfrak{p}$ and the associated $p$-curvature operator $\psi_{\mathfrak{p}}$, whose vanishing is equivalent to the vector bundle $V_{\mathfrak{p}}$ being spanned by its sheaf of parallel sections.

Conjecture 1.1 (The $p$-curvature conjecture). If $(V, \nabla)$ has vanishing $p$-curvature for almost all primes $\mathfrak{p}$, then it has a full set of algebraic solutions, i.e. it trivializes on a finite étale cover of $X$.

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In this paper, we shall work in the scenario where $X/K$ is the generic fiber of a smooth family of varieties $X \to B$, where the family is defined over a number field $k$. In this setting, if the $p$-curvature conjecture holds for such $X/K$, then it holds for the fibers $X_b$ of the family over a Baire-generic subset of complex points $b \in B(\mathbb{C})$ in the base variety. The main theme of this paper is to make, conversely, progress towards the $p$-curvature conjecture for $X/K$ by analyzing degenerations and special fibers of the family $X \to B$ when $B$ has positive dimension.

1.2. Generic curves. In Section [2] which constitutes the first part of our study, we work with a smooth curve $C \to B$ of genus $g$ with $n$ punctures over a base scheme $B$ whose associated morphism $B \to \mathcal{M}_{g,n}$ to the moduli stack of curves is dominant (we shall call such $C/B$ generic). In [17], the second author used nodal degenerations of curves to prove that, over a generic curve $C/B$, every vector bundle with connection $(V, \nabla)$ with almost all $p$-curvatures zero must have finite monodromy along every simple loop on $C$. Building on this work, we prove the following new case of the $p$-curvature conjecture.

**Theorem 1.1.** The $p$-curvature conjecture is true for rank 2 vector bundles with connection on a generic curve of genus $g \geq 1$ with $n \geq 0$ punctures.

Previously known cases of the $p$-curvature conjecture include Gauss-Manin connections associated to families of varieties by Katz [13], connections on certain locally symmetric varieties by Farb–Kisin [6], and the case of solvable monodromy by Andrè [1], Bost [2], and D. Chudnovsky–G. Chudnovsky [3]. We shall obtain Theorem 1.1 by combining the results of [17] with the following topological result, which may be of independent interest.

**Theorem 1.2.** Let $\Sigma$ be a topological surface of genus $g \geq 1$ with $n \geq 0$ punctures. If a semisimple representation $\rho : \pi_1 \Sigma \to \text{GL}_2(\mathbb{C})$ of its fundamental group has finite monodromy along every simple loop on $\Sigma$, then the image of $\rho$ is finite.

The hypothesis $g \geq 1$ is essential in Theorem 1.2 as Fuchsian triangle groups furnish counterexamples to the analogous statement for $g = 0$. We also remark that a naïve generalization of Theorem 1.2 to all $\text{GL}_r(\mathbb{C})$-representations beyond $r = 2$ fails, as shown by counterexamples of Koberda–Santharoubane [15]. Determining the precise range of ranks $r$, for each fixed genus $g$, to which our approach generalizes is an interesting problem which will be pursued in future work.

Our proof of Theorem 1.2 is summarized as follows. Upon reducing to the case of representations into $\text{SL}_2(\mathbb{C})$, we shall show that, if a representation $\rho : \pi_1 \Sigma \to \text{SL}_2(\mathbb{C})$ satisfies the hypotheses of Theorem 1.2 then

1. (nonarchimedean bound) $\text{tr} \rho(a)$ is an algebraic integer for all $a \in \pi_1 \Sigma$; and
2. (archimedean bound) the image of $\rho$ is conjugate to a subgroup of $\text{SU}(2)$.

By (1), we can assume without loss of generality that $\rho$ has image in $\text{SL}_2(\mathbb{Q})$. Since the set of roots of unity is preserved by absolute Galois group $\text{Gal}(\mathbb{Q}/\mathbb{Q})$ of $\mathbb{Q}$, any conjugate of $\rho$ by $\sigma \in \text{Gal}(\mathbb{Q}/\mathbb{Q})$ also satisfies the hypotheses of Theorem 1.2. Thus, the eigenvalues of $\rho(a)$ are algebraic integers whose Galois conjugates all have absolute value 1 in $\mathbb{C}$, by (1) and (2). By Kronecker’s theorem, these eigenvalues must be roots of unity, and $\rho(a)$ has finite order for every $a \in \pi_1 \Sigma$ (note that $\rho(a)$ is semisimple by (2)). This allows us to conclude, by Selberg’s lemma, that the image of $\rho$ is finite.
1.3. Deformations. For the remaining results, we consider the following general setting. Let $B$ denote a smooth irreducible variety over a number field $k$, and $X \to B$ be a smooth map, and let $(V, \nabla)$ be a vector bundle on $X \times_B B^o$ with connection relative to $B^o$, where $B^o \subset B$ is some nonempty open subvariety. Our second result pertains to extending $(V, \nabla)$ over the fibers of $X$ at points $b \in B \setminus B^o$:

**Theorem 1.3 (Good reduction).** Suppose that notation is as above, and that $X \to B$ is strongly topologically constant (see Definition [3]). Let $b \in B$ denote a codimension 1 point of $B \setminus B^o$. If the $p$-curvatures of $(V, \nabla)$ vanish for almost all primes $p$, then there exists an étale neighbourhood $U$ of $b$ such that the following results hold.

1. Suppose that $X \to B$ is a family of curves. The pair $(V, \nabla)$ extends to an algebraic vector bundle on $X^o \times_B U$ with connection relative to $U$. Here $X^o \subset X$ is open, and the map $X^o \to U$ has non-trivial fibers.
2. The pair $(V, \nabla)$ extends to an analytic vector bundle on $X \times_B U$ with connection relative to $U$.

We prove the first part of Theorem 1.3 by showing that the vanishing of $p$-curvatures implies that the connection cannot have poles along the boundary divisor $b \mod p$ for all but finitely many primes $p$. The regularity of the connection along the boundary in characteristic zero follows from this. The second part of Theorem 1.3 follows from the case of curves by passing to a suitable curvilinear slice of the family $X$, and then applying the Lefschetz hyperplane theorem for fundamental groups of quasi-projective varieties.

Our subsequent results concern isomonodromy of the connection $\nabla$. In general it is expected (and indeed, would follow from Conjecture 1.1) that, given a family of vector bundles with flat connection, the vanishing of $p$-curvatures for almost all $p$ implies constancy of the monodromy representation. Let us consider the following conjecture, which is implied by the $p$-curvature conjecture:

**Conjecture 1.2.** Let $(V, \nabla)$ be a vector bundle with connection on $X$ so that almost all $p$-curvatures vanish. Then, $(V, \nabla)$ has semisimple monodromy.

Proving Conjecture 1.1 in the case of semisimple monodromy would imply Conjecture 1.1 in full generality by the results concerning solvable monodromy (see [1, Theorem 0.5.1], [2, Theorem 2.9] and [3]). We prove the following result in §4, conditional on Conjecture 1.2.

**Theorem 1.4.** Let $X \to B$ with $B$ irreducible denote a family of smooth strongly topologically constant varieties, and let $(V, \nabla)$ denote a vector bundle on $X$ with flat connection relative to $B$. If the $p$-curvatures of $(V, \nabla)$ vanish for almost all primes $p$, then Conjecture B implies that the monodromy of $(V, \nabla)$ pulled back to $X_b$ does not depend on $b \in B$.

The proof of this result relies on relating the infinitesimal deformations of $(V, \nabla)_b$ to self-extensions of $(V, \nabla)_b$. In the case when the monodromy of some single fiber is finite, then it is easy to show that such non-trivial self-extensions cannot exist. Indeed, the connected component of the Zariski-closure of monodromy would then be non-trivial and unipotent, and the work of [1, 2, 3] implies that this is not possible. Therefore, without assuming Conjecture 1.2 we get:

**Theorem 1.5.** Maintain the setting of $X \to B$, $(V, \nabla)$ above. If $(V, \nabla)_{b_0}$ has finite monodromy for some $b_0 \in B$, then $(V, \nabla)_b$ has finite monodromy for every $b \in B$. 
Theorems 1.3 and 1.5 can be used in conjunction to deduce new cases of the $p$-curvature conjecture.

**Theorem 1.6.** Let $X_0$ denote a variety over some field which is finitely generated over $\mathbb{Q}$ for which the $p$-curvature conjecture is known. Then, the $p$-curvature conjecture is true for the generic fiber of any strongly topologically constant family of varieties which contains $X_0$ as a fiber.

Farb-Kisin [6] proved Conjecture 1.1 for certain Shimura varieties (see [6] to see the exact list which they treat). When the Shimura variety is compact, has dimension $\geq 3$ and is of Hodge type, their proof goes through verbatim for hyperplane slices. Theorem 1.6 would then apply to the generic fiber of any smooth family of varieties containing one such hyperplane slice as a fiber.

We end the paper with the following application of (variants of) Theorems 1.3 and 1.5.

**Theorem 1.7.** Suppose $C$ is a genus 0 curve with $d$ generic punctures, and let $(V, \nabla)$ be a rank 2 vector bundle with connection on $C$ with almost all $p$-curvatures vanishing. If $P \subset C(\mathbb{C}^n)$ is any pair of pants, then $(V, \nabla)^{an}$ restricted to $P$ has finite monodromy.

The idea of proof is to use nodal degenerations of the generic curve to reducible curves containing $\mathbb{P}^1 \setminus 0, 1, \infty$ as irreducible components, and the fact that the $p$-curvature conjecture is known (due to work of Katz) for rank two bundles on $\mathbb{P}^1 \setminus 0, 1, \infty$.

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2. RANK TWO LOCAL SYSTEMS ON SURFACES

The purpose of this section is to prove Theorems 1.1 and 1.2. Given a surface $\Sigma$ of positive genus $g \geq 1$ with $n \geq 0$ punctures and a semisimple representation $\rho : \pi_1 \Sigma \to \text{SL}_2(C)$ having finite monodromy along all simple loops of $\Sigma$, we shall prove its nonarchimedean (resp. archimedean) boundedness property in Section 2.1 (resp. Section 2.2). In Section 2.3 we shall then use these results to prove Theorem 1.1 and deduce from it Theorem 1.2 as outlined in Section 1.2.

2.1. Nonarchimedean bound. The following observation is elementary.

**Lemma 2.1.** Let $\Sigma$ be a topological surface of genus $g \geq 0$ with $n \geq 0$ punctures. If $\rho : \pi_1 \Sigma \to \text{SL}_2(C)$ is a semisimple representation with finite monodromy along every simple loop on $\Sigma$, then $\text{tr} \rho(a)$ is an algebraic integer for every $a \in \pi_1 \Sigma$.

**Proof.** Fix a base point on $\Sigma$, and let $a : S^1 \to \Sigma$ be a based loop. Up to homotopy, we may assume $a$ is an immersion with a minimum possible number (denoted $m(a)$) of self-intersection points in its image, each intersection point being required to be a simple normal crossing. If $m(a) = 0$, i.e. $a$ is a simple loop, then $\text{tr} \rho(a) = \zeta + \zeta^{-1}$ for some root of unity $\zeta$ by our hypothesis on $\rho$, so $\rho(a)$ is an algebraic integer.

So suppose $m(a) \geq 1$. We shall proceed by induction on $m(a)$ as follows. Up to homotopy, we may assume that one of the $m(a)$ self-intersection points of $a$ is the
base point. The loop $a$ is thus a concatenation $a = bc$ of two uniquely determined nontrivial based loops $b$ and $c$ (namely, travelling along $a$, we define $b$ as the loop of first return to the base point, and $c$ is the remainder). We observe that $m(b)$, $m(c)$, and $m(bc^{-1})$ are all strictly less than $m(a)$. Thus, $\text{tr}\rho(b)$, $\text{tr}\rho(c)$, and $\text{tr}\rho(bc^{-1})$ are all algebraic integers by our inductive hypothesis. But

$$\text{tr}\rho(a) = \text{tr}\rho(bc) = \text{tr}\rho(b) \cdot \text{tr}\rho(c) - \text{tr}\rho(bc^{-1})$$

since $\text{tr}(xy) + \text{tr}(yx^{-1}) = \text{tr}(x)\text{tr}(y)$ for every $x, y \in \text{SL}_2(\mathbb{C})$. This shows that $\text{tr}\rho(a)$ is also an algebraic integer, completing the inductive step and the proof. □

2.2. Archimedean bound. Let $\Sigma$ be a surface of genus $g$ with $n$ punctures. We fix a base point in $\Sigma$. For convenience of exposition, we shall say that a sequence $\ell = (\ell_1, \cdots, \ell_m)$ of based loops on $\Sigma$ is clean if each loop is simple and the loops pairwise intersect only at the base point. Recall the standard presentation of the fundamental group

$$\pi_1\Sigma = \langle a_1, b_1', \cdots, a_g, b_g', c_1, \cdots, c_n | [a_1, b_1'] \cdots [a_g, b_g'] c_1 \cdots c_n \rangle.$$

We can choose (the based loops representing) the generators so that the sequence of loops $\langle a_1, b_1', \cdots, a_g, b_g', c_1, \cdots, c_n \rangle$ is clean. For $i = 1, \cdots, g$, let $b_i$ be the based simple loop parametrizing the curve underlying $b_i'$ with the opposite orientation. Note that $\langle a_1, b_1', \cdots, a_g, b_g', c_1, \cdots, c_n \rangle$ is a clean sequence with the property that any product of distinct elements preserving the cyclic ordering on the sequence, such as $a_1 b_g$ or $a_1 a_2 b_2 b_g$ or $b_g c_n a_1$, is homotopic to a simple loop in $\Sigma$. We shall refer to $\langle a_1, b_1', \cdots, a_g, b_g', c_1, \cdots, c_n \rangle$ as an optimal sequence of generators of $\pi_1\Sigma$. See Figure 1 for an illustration of the optimal generators for $(g, n) = (2, 1)$.

We shall say that a pair of loops $(\ell_1, \ell_2)$ on $\Sigma$ is in $(1, 1)$-position if it is homotopic to a clean pair $(\ell_1', \ell_2')$ such that a closed tubular neighborhood $\Sigma' \subset \Sigma$ of the union of images of $\ell_1'$ and $\ell_2'$ in $\Sigma$ is a subsurface of genus one with one boundary curve, and $(\ell_1', \ell_2')$ completes to an optimal sequence of generators for $\pi_1\Sigma'$. For instance, if $\langle a_1, b_1', \cdots, a_g, b_g', c_1, \cdots, c_n \rangle$ is an optimal sequence of generators for $\pi_1\Sigma$, then each of the pairs $(a_i, b_i)$ is in $(1, 1)$-position.

![Figure 1. Optimal generators for $(g, n) = (2, 1)$](image-url)

Recall that the group $\text{SL}_2(\mathbb{R})$ acts transitively on the Poincaré upper half plane $\mathbb{H}^2 = \{ z \in \mathbb{C} : \text{Im} z > 0 \}$ by Möbius transformations. The stabilizer of $i \in \mathbb{H}^2$ is the special orthogonal group $\text{SO}(2)$. Recall that an element $g \in \text{SL}_2(\mathbb{C})$ is said to be elliptic if $\text{tr}(g) \in (-2, 2)$, and central if $g = \pm 1$. A non-central element $g \in \text{SL}_2(\mathbb{R})$ is elliptic precisely when it is conjugate in $\text{SL}_2(\mathbb{C})$ to an element of $\text{SO}(2)$, and precisely when it has a unique fixed point in $\mathbb{H}^2$.

**Lemma 2.2.** Let $\Sigma$ be a surface of positive genus $g \geq 1$ with $n \geq 0$ punctures. If $\rho : \pi_1\Sigma \to \text{SL}_2(\mathbb{C})$ is a semisimple representation with elliptic or central monodromy
along every simple loop of $\Sigma$, then the image of $\rho$ is conjugate to a subgroup of $\text{SU}(2)$. In particular, the eigenvalues of $\rho(a)$ have absolute value 1 for every $a \in \pi_1 \Sigma$.

**Proof.** In the case $(g, n) = (1, 1)$, this result can be seen from the characterization of the locus of unitary characters in the $\text{SL}_2(\mathbb{C})$-character variety of a once-punctured torus (see e.g. [8, Section 5]). More precisely, if $(a_1, b_1, c_1)$ is an optimal sequence of generators for $\pi_1 \Sigma$, and $\rho : \pi_1 \Sigma \to \text{SL}_2(\mathbb{C})$ is a semisimple representation with

$$\text{tr} \rho(a_1), \text{tr} \rho(b_1), \text{tr} \rho(a_1 b_1), \text{tr} \rho(c_1) \in [-2, 2],$$

then the character of $\rho$ is the character of a representation $\pi_1 \Sigma \to \text{SU}(2)$, and the claim follows. We also remark that, if $\Sigma$ is a once-punctured torus, then any representation $\rho : \pi_1 \Sigma \to \text{SL}_2(\mathbb{C})$ with elliptic or central monodromy along every simple loop must automatically be semisimple.

In general, let $\rho : \pi_1 \Sigma \to \text{SL}_2(\mathbb{C})$ be given as above. Since $\text{tr} \rho(a) \in [-2, 2] \subset \mathbb{R}$ for each simple loop $a$ on $\Sigma$, it follows by arguing as in the proof of Lemma 2.1 that $\text{tr} \rho(a) \in \mathbb{R}$ for every $a \in \pi_1 \Sigma$. Since $\rho$ is semisimple, this implies that the image of $\rho$ is conjugate to a subgroup of $\text{SU}(2)$ or to a subgroup of $\text{SL}_2(\mathbb{R})$ (see e.g. [10, Proposition III.1.1.]). If the former occurs, then we are done. So let us assume that $\rho$ has image in $\text{SL}_2(\mathbb{R})$.

The representation $\rho$ induces an action of $\pi_1 \Sigma$ on $\mathbb{H}^2$, and it suffices for us to show that this action has a fixed point. By case $(g, n) = (1, 1)$, we know that any pair of loops $(\ell_1, \ell_2)$ in $(1, 1)$-position on $\Sigma$ must have a common fixed point on $\mathbb{H}^2$. Now, we may assume that there is a simple loop $a_1$ on $\Sigma$ such that the image of $a_1$ has connected complement in $\Sigma$, and such that $\rho(a_1)$ is elliptic. Indeed, if such $a_1$ does not exist, then the image of $\rho$ can be easily shown to be contained in $\{\pm 1\}$, and we are done. Given such $a_1$, let us complete it to an optimal sequence $S = (a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n)$ of generators for $\pi_1 \Sigma$. Let $q \in \mathbb{H}^2$ denote the unique fixed point of $a_1$. Since $(a_1, b_1)$ is in $(1, 1)$-position, $a_1$ and $b_1$ have a common fixed point in $\mathbb{H}^2$, namely $q$. For any other loop $\ell \neq a_1, b_1$ in $S$, the pair $(a_1, b_1 \ell)$ is also in $(1, 1)$-position, so has common fixed point $q$; thus, $\ell$ also fixes $q$. Since $S$ generates $\pi_1 \Sigma$, it follows that $\pi_1 \Sigma$ fixes $q$, as desired.

2.3. **Proofs of Theorems 1.1 and 1.2.** With the results of Sections 2.1 and 2.2 in hand, we now prove Theorem 1.2 and deduce from it Theorem 1.1.

**Theorem 1.2.** Let $\Sigma$ be a topological surface of genus $g \geq 1$ with $n \geq 0$ punctures. If a semisimple representation $\rho : \pi_1 \Sigma \to \text{GL}_2(\mathbb{C})$ of its fundamental group has finite monodromy along every simple loop on $\Sigma$, then the image of $\rho$ is finite.

**Proof.** Let $\rho : \pi_1 \Sigma \to \text{GL}_2(\mathbb{C})$ be given as above. Since the morphism

$$(\text{pr}, \text{det}) : \text{GL}_2(\mathbb{C}) \to \text{PGL}_2(\mathbb{C}) \times \text{GL}_1(\mathbb{C})$$

has finite kernel, it suffices to show that the compositions $\text{pr} \circ \rho : \pi_1 \Sigma \to \text{PGL}_2(\mathbb{C})$ and $\text{det} \circ \rho : \pi_1 \Sigma \to \text{GL}_1(\mathbb{C})$ each have finite image. The fact that $\text{det} \circ \rho$ has finite image is clear. To show that $\text{pr} \circ \rho$ has finite image, let us first introduce a punctured surface $\Sigma' = \Sigma \setminus p$ for some $p \in \Sigma$. Since the morphism $i : \pi_1 \Sigma' \to \pi_1 \Sigma$ is surjective, it suffices to show that $\text{pr} \circ \rho \circ i$ has finite image; choosing a semisimple lift $\tilde{\rho} : \pi_1 \Sigma' \to \text{SL}_2(\mathbb{C})$ of $\text{pr} \circ \rho \circ i$ (which exists since $\pi_1 \Sigma'$ is free), it suffices to show that $\tilde{\rho}$ has finite image. Note that $\tilde{\rho}$ has finite monodromy along every simple loop on $\Sigma'$.
Thus, it only remains to prove the theorem in the case where the image of $\rho$ lies in $\text{SL}_2(\mathbb{C})$. We proceed as outlined in Section 1.2. By Lemma 2.1, the character $\text{tr}\rho$ of $\rho$ takes values in the ring of algebraic integers (and a fortiori in $\bar{\mathbb{Q}}$). Since $\rho$ is semisimple, up to $\text{SL}_2(\mathbb{C})$-conjugation we may thus assume that the image of $\rho$ lies in $\text{SL}_2(\bar{\mathbb{Q}})$. Since the set of roots of unity is preserved by the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ of $\mathbb{Q}$, any conjugate $\rho_\sigma$ of $\rho$ by $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, given as $\rho_\sigma(a) = (\rho(a))^{\sigma}$ for all $a \in \pi_1^0\Sigma$, also has finite monodromy along simple loops on $\Sigma$. It follows by Lemmas 2.1 and 2.2 that the eigenvalues of $\rho(a)$ are algebraic integers (being roots of the characteristic polynomial $X^2 - (\text{tr}\rho(a))X + 1 = 0$) whose Galois conjugates all have absolute value 1 in $\mathbb{C}$, and by Kronecker’s theorem they must be roots of unity, i.e. $\rho(a)$ has finite order (note that $\rho(a)$ is semisimple by Lemma 2.2). Finally, Selberg’s lemma states that any finitely generated subgroup of $\text{GL}_r(L)$ for a field $L$ of characteristic zero has a torsion-free finite-index subgroup. Applied to the image of $\rho$, this shows that the image of $\rho$ is therefore finite. □

Theorem 1.1. The $p$-curvature conjecture is true for rank 2 vector bundles with connection on a generic curve of genus $g \geq 1$ with $n \geq 0$ punctures.

Proof. Let $(V, \nabla)$ be a rank two vector bundle with connection on a generic curve of genus $g \geq 1$ with $n \geq 0$ punctures, such that almost all $p$-curvatures of $(V, \nabla)$ vanish. By [17, Theorem 1.3], every simple loop has finite monodromy.

Choose a specialization of the generic curve and $(V, \nabla)$ to a curve over $\mathbb{C}$ with underlying topological surface $\Sigma$, and consider the associated monodromy representation $\rho : \pi_1^0\Sigma \to \text{GL}_2(\mathbb{C})$. Since the $p$-curvature conjecture is known to be true in the case of solvable monodromy by the works of Andre [1], Bost [2], and D. Chudnovsky–G. Chudnovsky [3], we may assume that $\rho$ is irreducible (and in particular hence semisimple). It follows by Theorem 1.2 that the image of $\rho$ is finite, as desired. □

3. Good reduction at the special fiber

The first step in the proof of Theorem 1.3 is to analyze the case of affine curves satisfying certain conditions. We will then deduce the higher dimensional case from the case of curves.

3.1. Affine curves. Let $k$ denote a number field, and let $B$ denote a smooth variety over $k$, and suppose that $B^o \subset B$ is a non-empty open subvariety also defined over $k$. Let $C \to B$ denote a smooth family of curves. Let $(V, \nabla)$ denote a vector bundle on $C \times_B B^o$ with connection relative to $B_0$. Let $E \subset B$ denote a smooth divisor, such that $E$ is not contained in $B^o$. There exists an integer $N$ such that all this data can be spread out smoothly to $\mathcal{O}_k[1/N]$. We will therefore assume that all objects are schemes defined over $\mathcal{O}_k[1/N]$. By choosing a small enough open subscheme of $B$ and a suitable opensubscheme of $C$, and up to replacing $N$ by some larger integer, we may assume the following:

1. $B = \text{Spec} R$, $E \subset B$ is defined by the vanishing of a single equation $q = 0$.
2. $C$ is affine, $\Omega_{C/R}$ is the trivial line bundle, and there exists $D \in \text{Der}(\mathcal{O}_C/R)$ such that $D^p \equiv D \mod p$, where $p$ is a maximal ideal of $\mathcal{O}_k[1/N]$ with residue characteristic $p$. 
(3) The vector bundle \( V \) is trivial, and there exists a cyclic basis with respect to the endomorphism \( \nabla(D) \) ([14] Theorem 4.4.2]).

**Definition 1.** Let \( S \) denote a domain which is an \( R \)-algebra. Let \( F(S) \) denote the fraction field of \( S \), and let \( D \in \text{Der}_R(F(S)) \). Let \( \nu \) denote a discrete valuation on \( F(S) \). We say that the derivation \( D \) is \( \nu \)-**integral** if \( \nu(D(\alpha)) \geq \nu(\alpha) \) for all \( \alpha \in F(S) \).

The fact that \( D \in \text{Der}(\mathcal{O}_C/R) \) implies that \( D \) is \( q \)-integral (here, we abuse notation to allow \( q \) to denote the \( q \)-adic valuation on \( \mathcal{O}_C \)).

**Theorem 3.1.** Let \( C \to \text{Spec} \, R \) be the affine curve satisfying the conditions above, and let \( (V, \nabla) \) denote a vector bundle on \( C \) with connection relative to \( R[1/q] \) satisfying the above assumptions. If the \( p \)-curvatures of \( (V, \nabla) \) vanish for almost all primes \( p \), then \( \nabla \) extends to a connection over \( C \to \text{Spec} \, R \).

**Proof.** The proof goes along the lines of the arguments used in [17], Lemma 3.3, Proposition 3.4]. Let \( n \) be the dimension of \( V \). The connection matrix \( A \) (of \( \nabla(D) \)) with respect to the cyclic basis \( e \) has the form

\[
A = \begin{pmatrix}
1 & f_0 \\
& f_1 \\
& \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & 1 & f_{n-1}
\end{pmatrix}
\]

It suffices to prove that the \( f_i \) all have non-negative \( q \)-adic valuation. Let \( p \) (with associated rational prime \( p \)) be such that \( p > n \), the \( q \)-adic valuations of the \( f_i \) stay the same modulo \( p \), and the \( p \)-curvature vanishes modulo \( p \). It suffices to prove that the reductions of the \( f_i \) modulo \( p \) have non-negative \( q \)-adic valuation.

To that end, we work modulo \( p \) for the rest of the proof, and we assume that some \( f_i \) has negative \( q \)-adic valuation. Let \( C = \text{Spec} \, S \). We also localize \( R \) and \( S \) at \( (q) \), and replace the rings by their \( q \)-adic completions. Let \( S' \) be the smallest \( S \)-algebra over which all the eigenvalues of \( A \) are defined. We choose an extension \( \nu \) of the \( q \)-adic valuation on \( S \) to \( S' \). By Lemma 3.2 below, it follows that the unique extension \( D' \) of \( D \) to \( S' \) is \( \nu \)-integral.

Let \( \lambda \) be an eigenvalue of \( A \) with the largest \( \nu \)-adic size (equivalently, smallest \( \nu \)-adic valuation). Let \( \ell = |\lambda|_\nu \). By assumption \( \ell > 1 \). Further, \( |f_i|_\nu \leq \ell^{n-1} \) with equality holding for some \( i \). Let \( w_\lambda \) be an eigenvector of \( A \) with eigenvector \( \lambda \). We will prove that \( \nabla(D^p)w_\lambda \neq \nabla(D^p)w_\lambda \), from which the result would follow. We now introduce some notation. For any vector \( v \), denote by \( v[m] \) its \((m+1)\)th entry. We have

\[
\nabla(D^p)w_\lambda = \sum_{W \in I} Ww_\lambda,
\]

where \( I \) is the set of all length \( p \) words in the letters \( A \) and \( D \). Here, \( A \) acts on any vector by left-multiplication, and \( D \) acts on each coordinate in the natural way. We will need the following claim to conclude the proof of this result:

**Claim 3.1.** Suppose that a vector \( w \) has the property that for all \( m \), \( |w[m]| \leq (\text{resp. } <) \ell^{n-m-1}|w_\lambda[n-1]| \). Then:

1. \( |Dw[m]| \leq \ell^{n-m-1}(|\lambda w_\lambda[n-1]| \), i.e. the coordinates of \( Dw \) and \( \lambda w_\lambda \) will satisfy the strict inequalities.
(2) The coordinates of $Aw$ and $\lambda w_\lambda$ satisfy the same inequalities as those satisfied by the coordinates of $w$ and $w_\lambda$.

Proof of claim. The first part of the claim follows from the $\nu$-integrality of $D$. It remains to prove the second part.

We must show that $|Aw[m]| \leq (\text{resp. } <) \ell^{n-m} |w_\lambda[n-1]|$. This is clear for $Aw[0]$. For $m \geq 1$, we have $Aw[m] = f_m w[n-1] + w[m-1]$. As $\lambda$ is the eigenvalue with greatest $\nu$-adic norm, it follows that $|f_m| \leq \ell^{n-m}$. Therefore, $|f_m w[n-1]| \leq (\text{resp. } <) \ell^{n-m} |w_\lambda[n-1]|$. Therefore it suffices to show that $w[m-1]| \leq (\text{resp. } <) \ell^{n-m} |w_\lambda[n-1]|$. This follows, because $w[m-1]$ satisfies the required inequality by hypothesis. 

As in [17], we will prove that the word $W_0 = AA \ldots A$ $(p$ times) has the property that $|(W_0 w_\lambda)[n-1]|_\nu$ is strictly larger than $|(W_0 w_\lambda)[n-1]|_\nu$ for every $W_0 \neq W \in I$. The result follows from Claim 3.1 as follows:

Let $w^j$ and $w^j_0$ be the vectors obtained by applying the first $j$ letters of $W$ and $W_0$ respectively on $w_\lambda$. Claim 3.1 implies $|w^j[m]| \leq \ell^{n-m-1} |w^j_0[m]|$ for any $0 \leq m \leq n-1$. However, $W$ differing from $W_0$, must contain the letter $D$. By Claim 3.1 a letter would render the inequality strict, i.e. if such a letter first occurred at the $j_0^\text{th}$ stage, then $|w^{j_0}[m]| < \ell^{n-m-1} |w^{j_0}_0[n-1]|$. According to Claim 3.1 this said strictness would persist through the application of the rest of the word, i.e. $|w^j[m]| < \ell^{n-m-1} |w^j_0[n-1]|$, for $j \geq j_0$. Therefore, $|w_{\nu p}(n-1)| = |(Aw_\lambda)[n-1]| = \ell^n |w_\lambda[n-1]|$. All the coordinates of $(D)w_\lambda$ are bounded by $\ell^n |w_\lambda[n-1]|$ (as the sizes of all the coordinates of $w_\lambda$ are bounded above by $\ell^n |w_\lambda[n-1]|$ and the entries of $B$ are bounded by $\ell^n$), which establishes the result. 

Lemma 3.2. The derivation $D$ can be $\nu$-integrally extended to $S'$.

Proof. There is a unique extension of $D$ to $F(S')$ (which we will also denote by $D$), and it remains to prove the $\nu$-integrality of $D$.

Note that $S = \kappa[[q]]$, where $\kappa$ is some characteristic $p$ field. Any unramified extension of its quotient field will be of the form $\kappa'((q))$, where $\kappa'/\kappa$ is a finite separable extension. The extension $S'/S$ is at worst tamely ramified over $q$ (as the degree of $S'$ over $S$ is less than $p$). Therefore, $S'$ is contained in $T[q^{1/m}]$, where $T$ is unramified over $S$ and $m$ is relatively prime to $p$. Therefore, $T = \kappa'[[q]]$, with $\kappa'/\kappa$ as above.

We claim that extension of $D$ to $T$ is $q$-integral because it is unramified over $q$: indeed, let $T = S[\alpha]$, where $\alpha \in k'$. Let $g \in S[x]$ denote the monic irreducible polynomial which $\alpha$ satisfies. Denote by $D(g)$ the polynomial obtained by applying $D$ to the coefficients of $g$. Then, $D(\alpha)$ is easily seen to equal $\frac{D(g)(\alpha)}{g'(\alpha)}$. As $T/S$ is unramified, it follows that $g'(\alpha)$ is a $q$-adic unit.

We now show that $D$ can be $q$-integral extended from $T$ to $S'$. Let $s = \sum_{i=0}^{m-1} t_i q^{i/m}$ be an element of $S'$, where $t_i \in T$. The unique extension of $D$ to $S$ satisfies $D(q^{i/m}) = 0$ for every $i$, therefore $D(s) = \sum_i D(t_i) q^{i/m}$. The $q$-adic valuations of $t_i q^{i/m}$ is different from the valuation of $t_j q^{j/m}$ for $i \neq j$. Therefore, the $q$-adic valuation of $s$ equals the $q$-adic valuation of $t_i q^{i/m}$ for some $i$. Because of the $q$-integrality of $D$ on $T$, it follows that $D(t_i) q^{i/m}$ has $q$-adic valuation larger
than that of \( s \). The result follows from the observation that the \( q \)-adic valuation of \( D(s) \) is at least that of \( D(t_i)q^{i/m} \).

\[\square\]

3.2. Families of holomorphic vector bundles with connection. Here we spell out for the reader standard constructions and results involving families of vector bundles with connection.

Our common setting is: \( f: X \to B \) is a smooth morphism between smooth complex manifolds with \( d \)-dimensional fibers, and \( p: B \to X \) is a section. Further, \( (V, \nabla) \) is a rank \( n \) holomorphic vector bundle with integrable connection relative to \( f \) on \( X \).

When the family is homeomorphic to a product, we review the construction of the corresponding holomorphic family of monodromy representations attached to \( (V, \nabla) \). We assume that the reader is familiar with the basic correspondence between locally constant sheaves on a manifold and representations of its fundamental group.

The essential ingredient is:

**Lemma 3.2** (Cauchy-Kowalewski Theorem). The sheaf of solutions \( \ker \nabla \) is locally (in the Euclidean topology on \( X \)) isomorphic to the inverse-image sheaf \( f^{-1}\mathcal{O}_B^\times \).

**Proof.** This statement is contained in the proof of [4, Theorem 2.23]. It is known as the Cauchy-Kowalewski theorem on existence and uniqueness of solutions to certain types of differential equations. \( \square \)

**Corollary 3.3.** Let \( i: X_b \to X \) denote the inclusion of the fiber over \( b \in B \). Then the sheaf \( i^{-1}\ker \nabla \) is a locally constant sheaf of free rank \( n \) \( \mathcal{O}_{B,b} \)-modules on \( X_b \) (in the Euclidean topology).

3.2.1. Relative local systems and monodromy representations. Throughout this section, let \( f: X \to B \) be a smooth morphism between complex manifolds, which is locally on \( B \) homeomorphic to a product. Furthermore, assume \( p: B \to X \) is a section, and suppose the fundamental groups of the fibers of the family are finitely generated. All sheaves considered here are in the analytic topology.

**Definition 2.** A \( f \)-local system of rank \( n \) on \( X \) is a sheaf of \( f^{-1}\mathcal{O}_B \)-modules on \( X \) which is locally (in the Euclidean topology) isomorphic to (\( f^{-1}\mathcal{O}_B \))^\( n \).

If \( M \) is an \( f \)-local system, we let \( p^*M \) denote the rank \( n \) locally free sheaf of \( \mathcal{O}_B \)-modules \( p^{-1}(M) \).

We define \( \pi_1(X/B) \) to be the locally constant sheaf on \( B \) whose stalk at a point \( b \in B \) is \( \pi_1(X_b, p(b)) \).

**Proposition 3.4.** A \( f \)-local system \( M \) on \( X \) naturally defines a homomorphism of sheaves of groups: \( \rho_M: \pi_1(X/B) \to \text{Aut}(p^*M) \).

**Proof.** If \( b \in B \) is any point and \( i_b: X_b \hookrightarrow X \) is the inclusion of the fiber, we obtain a locally constant sheaf \( i^{-1}_b(M) \) of \( \mathcal{O}_{B,b} \)-modules on \( X_b \). The correspondence between locally constant sheaves and representations of \( \pi_1(X_b, p(b)) \) on the stalk \( (p^*(M))_b \) provides a canonical representation

\[ \rho_b: \pi_1(X_b, p(b)) \to \text{Aut}(p^*M)_b \]
Since the above fundamental group is finitely generated, there exists an open neighborhood $U \subset B$ of $b$ and a representation $\rho_U : \pi_1(X_b, p(b)) \to \text{Aut}(p^*M)(U)$ whose germ is $\rho_b$.

The representations $\rho_U$ are clearly compatible on intersections, and collectively define $\rho_M$.

If $M$ is a local system relative to $f$, we call $\rho_M$ the monodromy representation of $M$.

**Proposition 3.5.** Let $W$ be a vector bundle on $B$, and $\rho : \pi_1(X/B) \to \text{Aut}(W)$ a homomorphism of sheaves of groups. Then there exists a unique $f$-local system $M$ such that $\rho_M = \rho$.

**Proof.** The $f$-local system $M$ is constructed as follows. Let $\{U_\alpha\}$ be a cover of $B$ consisting of contractible open sets and such that every pairwise and triple intersection are also contractible, and let $\pi_\alpha : \tilde{X}_\alpha \to X_\alpha \to U_\alpha$ denote the universal covers of $X_\alpha$ mapping down to $U_\alpha$. Then the sheaf $\pi_\alpha^{-1}(W)$ inherits, through $\rho_{U_\alpha}$, an action of $\pi_1(X_\alpha, p_\alpha)$ which is compatible with the action of $\pi_1(X_\alpha, p_\alpha)$ on the universal cover $\tilde{X}_\alpha$. Thus, the sheaf descends to a sheaf $M_\alpha$ on $X_\alpha$, which is locally on $X_\alpha$ isomorphic to $f^{-1}(W)$ by construction.

Next, over an intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we get natural sheaf isomorphisms $\phi_{\alpha\beta} : M_\alpha|_{X_{\alpha\beta}} \to M_\beta|_{X_{\alpha\beta}}$ by noticing that the identity map $X_{\alpha\beta} \times \pi_\alpha^{-1}(W|_{U_\alpha})_{|\alpha\beta} \to X_{\alpha\beta} \times \pi_\beta^{-1}(W|_{U_\beta})_{|\alpha\beta}$ is equivariant with respect to the action of $\pi_1(X_\alpha, p_\alpha)$ on the left and $\pi_1(X_\beta, p_\beta)$ on the right. (Note that the existence of the section $p$ provides canonical inclusions of the universal cover $\tilde{X}_{\alpha\beta}$ into the universal covers $\tilde{X}_\alpha$ and $\tilde{X}_\beta$.) The equivariance comes from the agreement of the restrictions of $\rho_{U_\alpha}$ and $\rho_{U_\beta}$ to $U_{\alpha\beta}$.

Similarly, on triple intersections we get that the cocycle condition $\phi_{\beta\gamma} \circ \phi_{\alpha\beta} = \phi_{\alpha\gamma}$ holds automatically.

**Proposition 3.6.** Let $i : C \hookrightarrow X$ be a closed complex submanifold containing the section $p$ which is topologically trivial over $B$, with the property that the induced map $\pi_1(C/B) \to \pi_1(X/B)$ is surjective.

If $L$ is a $f$-local system on $C$ which, over a dense open subset $U \subset B$ is the inverse image of an $f$-local system $M_U$ on $X_U$, then $M_U$ extends to an $f$-local system $M$ on $X$.

**Proof.** The $f$-local system $L$ gives, by [Theorem 3.4] a monodromy representation $\rho_L : \pi_1(C/B) \to \text{Aut}(p^*L)$ which factors through $\pi_1(X/B)$ over the dense open subset $U \subset B$. By continuity, $\rho_L$ must factor through $\pi_1(X/B)$ on all of $B$, and therefore defines by [Theorem 3.5] the required $f$-local system $M$.

We make the following remark for future use:

**Remark 1.** If $f : X \to B$ is a smooth morphism between complex varieties with section $p : B \to X$ and if $b \in B$ is any point, then the discussion in this section attaches to a vector bundle with integrable connection $(V, \nabla)$ on $X$ a monodromy representation

$$\rho : \pi_1(X_b, p(b)) \to \text{Aut} p^*(V)_b$$
This local monodromy representation exists irrespective of any hypotheses on the topological triviality of the family \( f \).

### 3.3. The general case.

#### 3.3.1. Topologically constant families, relative slice, and Lefschetz theorem.

**Definition 3.** Let \( f : X \to B \) be a morphism between two \( k \)-varieties, with \( k \subset \mathbb{C} \) a field. Then \( f \) is strongly topologically constant if

1. \( f \) is smooth;
2. \( X(\mathbb{C}) \to B(\mathbb{C}) \) is locally (in the Euclidean topology on \( B(\mathbb{C}) \)) a fibration;
3. There exists a projective \( B \)-scheme \( Y \), smooth over \( B \), and an open immersion \( i : X \hookrightarrow Y \) over \( B \) such that the complement \( D := Y \setminus i(X) \) is a union of smooth divisors \( D_1, \ldots, D_m \);
4. For each subset \( I \subset \{1, 2, \ldots, m\} \), the intersection \( \bigcap_{i \in I} D_i \) is \( B \)-smooth, and every geometric fiber of \( D \to B \) is reduced and normal crossings.

The following lemma is a trivial consequence of Bertini’s theorem and generic smoothness.

**Lemma 3.7.** Let \( f : X \to B \) be strongly topologically constant, \( b \in B(\mathbb{K}) \) a point, and suppose \( L \) is a line bundle on \( X \) which is very ample relative to \( f \).

Then there exists a Zariski open neighborhood \( U \subset B \) containing \( b \) and a non-zero element \( s \in H^0(X_U, L|_{f^{-1}(U)}) \) such that the zero scheme \( V(s) \subset X \) is strongly topologically constant over \( U \).

**Proof.** Omitted. \( \square \)

**Theorem 3.8.** (Lefschetz hyperplane theorem) Let \((Z, \Delta)\) be a smooth, irreducible, complex projective variety of dimension \( d \geq 2 \) with normal crossings divisor \( \Delta \) in projective space \( \mathbb{P}^N \) and let \( U \subset Z \) denote the complement of \( \Delta \). If \( H \subset \mathbb{P}^N \) is a hyperplane such that \( H \cap Z \) is smooth and \( H \cap S \) is smooth for every stratum \( S \) of \( \Delta \), then

\[ \pi_1(H \cap U) \to \pi_1(U) \]

is surjective.

**Proof.** This is a consequence of the main theorems in \[9\] – see especially page 130, where the set \( \Omega \) of admissible hyperplane sections is defined. \( \square \)

(Note that a choice of basepoint is not necessary when claiming surjectivity.)

**Corollary 3.9.** Let \( f : X \to B \) be strongly topologically constant, \( b \in B(\mathbb{K}) \) a chosen point, and suppose \( p : B \to X \) is a section of \( f \).

Then there exists a Zariski neighborhood \( U \subset B \) of \( b \) and a closed subscheme \( C \subset X_U \) containing the induced section \( p_U : U \to X_U \) such that:

1. \( C \to U \) is a strongly topologically constant family of curves, and
2. for every point \( u \in U(\mathbb{C}) \), the natural map
   \[ \pi_1(C_u, p(u)) \to \pi_1(X_u, p(u)) \]

is surjective.

**Proof.** This is obtained by a straightforward combination of a simple variant of \[Theorem 3.7\] (to include the condition of containing the section \( p \)) and \[Theorem 3.8\]. We omit the details. \( \square \)
3.4. Proof of part (2) of Theorem 1.3

Proof of Theorem 1.3. Let $X \to B$ be a strongly topologically constant family of quasi-projective $k$-varieties, and $D \subset B$ an irreducible divisor. Suppose further that $(V, \nabla)$ is an algebraic vector bundle with relative flat connection on $X|_{B \setminus D}$ whose $p$-curvatures vanish for almost all $p$.

Then because $f$ is smooth, there exists a quasi-finite base change $U \to B$ whose image contains the generic point of $D$ and which is generically unramified over $D$, such that $X \times_B U \to U$ has a section $p : U \to X \times_B U$. We choose a divisor $D' \subset U$ lying over $D$, and we replace $B$ with $U$. $U$ will serve as the étale neighborhood of the generic point of $D$ in the statement of the theorem.

Next, we use Corollary 3.9 to obtain a relative strongly topologically constant curve $C \to U$ (defined over $k$) contained in $X \to U$ and containing the section $p$ such that the induced map

$$\pi_1(C/U) \to \pi_1(X/U)$$

is surjective.

The vector bundle with relative flat connection $(V, \nabla)$ on $X|_{U \setminus D'}$ restricts to $C|_{U \setminus D'}$, and after replacing $X$ and $C$ with a suitable Zariski open, we obtain the situation described in Theorem 3.1 (The Zariski open can be chosen to preserve strong topological constancy, and to contain the generic point of the section $p$ over $D'$.)

By Theorem 3.1, the vector bundle with relative connection $(V, \nabla)$ extends over $C$, and therefore the hypotheses in Proposition 3.6 are fulfilled. By that proposition, we conclude that the $f$-local system $\ker \nabla$ extends over $X$, and by the correspondence between $f$-local systems and vector bundles with relative flat connection, we produce the required analytic extension. This completes the proof.

□

4. Constancy of monodromy

We spend this section proving Theorems 1.4, 1.5 and 1.6. We first need the following result, which proves formal constancy of the monodromy representation in the setting of Theorems 1.4 and 1.5.

Proposition 4.1. Let $C \to k[[q]]$ denote a strongly topologically constant affine curve, and let $(V, \nabla)$ denote a vector bundle on $C$ with connection relative to $k[[q]]$. Suppose that the data of $C$, $(V, \nabla)$ arises from an algebraic family of curves and an algebraic vector bundle with connection, and that the $p$-curvatures vanish for almost all primes $p$. Then Conjecture 1.2 implies that for any positive integer $m$, there exists a basis with respect to which the connection is constant (in $q$) modulo $q^m$. Further, if there exists a full set of algebraic solutions modulo $q$, then the result holds without assuming Conjecture 1.2.

Proof. Without loss of generality, we assume the existence of a derivation $D \in \text{Der}(\mathcal{O}_C)$, such that $D^p \equiv D \mod p$, and that $V$ is the trivial bundle.

We proceed by induction on $m$, and will assume the existence of a basis such that the connection matrix $\nabla(D)$ is constant modulo $q^{m-1}$. As $C$ is affine and smooth, the infinitesimal lifting property states that $C \times_{k[[q]]} k[[q]]/q^m$ is isomorphic to the trivial deformation $C_0 \times_k k[[q]]/q^m$. Therefore, the connection matrix $\nabla(D)$ is of the form $A + q^{m-1}B$, with $A, B \in M_n(\mathcal{O}_{C_0})$. 
Consider the rank 2n trivial vector bundle with connection \( \nabla' \) on \( C_0 \), whose connection matrix with respect to \( D \) is

\[
M = \begin{bmatrix} A & B \\ 0 & A \end{bmatrix}.
\]

We will prove the following two statements:

**Claim 4.2.** If there exists a block upper-triangular change of coordinates with respect to which \( \nabla'(D) \) is block-diagonal, then the assertions of the proposition hold.

**Claim 4.3.** The \( p \)-curvatures of \( \nabla' \) vanish for almost all primes \( p \).

The Proposition follows immediately from these claims. Indeed, if \( \nabla \mod q \) had a full set of algebraic solutions, then the connected component of the monodromy representation associated to \( \nabla' \) would be solvable. The result then follows by applying Claim 4.3 and the \( p \)-curvature conjecture for solvable monodromy.

On the other hand, regardless of the monodromy of the special fiber, Conjecture 1.2 and Claim 4.3 imply that \( \nabla' \) has semisimple monodromy. Thus there exists a block upper-triangular change of coordinates with respect to which \( \nabla'(D) \) is block diagonal, and so the proposition would follow from the assertion of Claim 4.2. Therefore, it suffices to prove the two claims.

**Proof of Claim 4.2** Suppose that the block upper-triangular matrix is of the form

\[
G = \begin{bmatrix} X & Y \\ 0 & X' \end{bmatrix}.
\]

By again changing coordinates by the matrix

\[
\begin{bmatrix} X^{-1} \\ 0 \\ X'^{-1} \end{bmatrix},
\]

it follows that we may assume the initial block upper-triangular matrix has its block-diagonal entries equalling the identity.

Further, the connection matrix in the new coordinates equals \( G^{-1}MG+G^{-1}D(G) \). A short calculation shows that the top-left block (in terms of \( G \) and \( M \)) equals \( B + AY - YA + D(Y) \). Therefore, our assumptions imply that there exist a matrix \( Y \) such that \( B + AY - YA + D(Y) = 0 \).

We now shift our focus back to \( \nabla \). Consider the connection matrix in the basis given by the matrix \( I + q^{m-1}Y \). A short calculation shows that \( \nabla(D) \) in these new coordinates equals \( A - q^{m-1}(B + AY - YA + D(Y)) \), and this quantity equals \( A \) as our previous calculation yielded that \( B + AY - YA + D(Y) = 0 \). The claim follows.

**Proof of Claim 4.3** We work modulo \( p \) for the entirety of this proof. As \( D^p \equiv D \mod p \), it suffices to prove that \( \nabla'(D) \equiv \nabla'(D)^p \). Suppose that the matrix \( \nabla(D)^j = P_j + q^{m-1}Q_j \), for some positive integer \( j \). We will show by induction that the matrix \( \nabla'(D)^j \) equals

\[
\begin{bmatrix} P_j & Q_j \\ 0 & P_j \end{bmatrix}.
\]

By induction, we may assume that this holds for \( j - 1 \). Then, \( \nabla(D)^j = (A + q^{m-1}B)(P_{j-1} + q^{m-1}Q_{j-1}) + D(P_j - 1 + q^{m-1}Q_j - 1) = AP_{j-1} + D(P_{j-1}) + q^{m-1}(AQ_{j-1} + BP_{j-1} + D(Q_{j-1})) \). Therefore, \( P_j = AP_{j-1} + D(P_{j-1}) \) and \( Q_j = AQ_{j-1} + BP_{j-1} + D(Q_{j-1}) \).
A short calculation analogous to the one above shows that
\[ \nabla'(D)^j = \begin{bmatrix} AP_{j-1} + D(P_{j-1}) & AQ_{j-1} + BP_{j-1} + D(Q_{j-1}) \\ 0 & AP_{j-1} + D(P_{j-1}) \end{bmatrix}, \]
as required. The claim now follows from the vanishing of the \( p \)-curvatures of \( \nabla \).

The following result is the other ingredient needed to prove Theorems 1.4 and 1.5.

**Proposition 4.4.** Let \( C \to \mathbb{C}\{q\} \) denote a family of strongly topologically constant curves over the unit disc, and let \((V, \nabla)\) be a holomorphic family of vector bundles with connections. For each point \( z \) and integer \( n \), suppose that there exists a basis with respect to which \( \nabla \) is constant modulo \( q^n \). Then, the family of connections is isomonodromic.

**Proof.** The family of vector bundles with connection gives rise to a representation \( \rho : \pi_1 \to \text{GL}_n(\mathbb{C}\{q\}) \). Let \( R \) denote any \( \mathbb{C}\{q\}\)-algebra. We denote by \( \rho_R \) the representation of \( \pi_1 \) valued in \( \text{GL}_n(R) \). For any point \( z \), we let \( \rho_z : \pi_1 \to \text{GL}_n(\mathbb{C}) \) denote \( \rho \) specialized to \( q = z \).

Let \( L \) denote the field of fractions of \( \mathbb{C}\{q\} \). We will show that \( \rho_0 \) and \( \rho_L \) are isomorphic, when both basechanged to \( \overline{L} \). This suffices to prove the claim, because the same would hold for \( \rho_z \) and \( \rho_{L,z} \) for any point \( z \). It follows that \( \rho_0 \) and \( \rho_z \) are isomorphic when base-changed to \( \overline{L} \), and thus isomorphic when base-changed to any algebraically closed field.

Therefore, it suffices to prove that \( \rho_0 \) and \( \rho_{\overline{L}} \) are isomorphic. As \( \rho \) mod \( q^n \) is isomorphic to \( \rho_0 \), \( \rho_0 \) is isomorphic to \( \rho_{\mathbb{C}\{q\}} \). Indeed, given two representations valued in \( \mathbb{C}\{q\} \) that are identical modulo \( q^n \) and isomorphic modulo \( q^{n+1} \), it is easy to see that there exists an invertible matrix which is congruent to the identity modulo \( q^n \) so that conjugation by it renders the two representations equal modulo \( q^{n+1} \). The infinite product ranging over \( n \) clearly converges \( q \)-adically, whence it follows that \( \rho_0 \) and thus \( \rho_{\mathbb{C}\{q\}} \) are isomorphic. Therefore, \( \rho_0 \) is isomorphic to \( \rho_{\mathbb{C}\{q\}} \).

As \( L \to \mathbb{C}\{q\} \), it follows that \( \rho_0 \) and \( \rho_L \) are isomorphic when both basechanged to some large enough algebraically closed field. Therefore, they are isomorphic over \( \overline{L} \), as required.

**Proof of Theorems 1.4 and 1.5.** In order to prove these results it suffices to treat the case when \( B \) is a curve, so we make this assumption. Further, we may assume that \( X \to B \) is a family of strongly topologically constant curves by Corollary 3.9.

We first treat the case where there exists \( b \in B \) such that the monodromy of \( (V, \nabla) \) restricted to \( X_b \) is finite. Let \( q \) denote a uniformizing parameter for the point \( b \in B \). Proposition 1.1 implies the existence of a full set of algebraic solutions modulo \( q^n \). We can then apply Proposition 4.4 to deduce finiteness of monodromy of \( (V, \nabla) \) for \( b' \in B(\mathbb{C}) \) in some holomorphic neighbourhood of \( b \). It therefore follows that \( (V, \nabla) \) restricted to the generic fiber of \( X \to B \) has finite monodromy, thereby concluding the proof of Theorem 1.5.

We now prove Theorem 1.4. Let \( b \in B \) denote any point and let \( q \) again denote a uniformizing parameter at \( b \). By Proposition 1.1 we may assume that the connection matrix is constant modulo \( q^n \), and this is true for all \( n \) and \( b' \in B \). We
now choose a simply-connected neighbourhood $U$ of $b \in B$. By Proposition 4.4, it follows that the monodromy of $(V, \nabla)$ restricted to any $X_b'$ for $b' \in U$ is independent of $b'$. As $U$ was an arbitrary simply connected open subset of $B$, it follows that the monodromy of $(V, \nabla)$ restricted to any $X_b'$ for $b' \in B$ is independent of $b'$. The result follows.

\[ \Box \]

We are finally ready to deduce Theorem 1.6 from Theorems 1.3 and 1.5.

**Proof of Theorem 1.6.** Let $X \rightarrow B$ denote such a family, and suppose that $(V, \nabla)$ is a vector bundle on $X_{B^0} \rightarrow B^0$ with flat connection relative to $B^0$ with vanishing $p$-curvatures for almost all primes $p$. Let $b \in B$ denote a point such that $X_b = X_b^0$. The result follows directly if $b \in B^0$ so we assume this is not the case. By blowing up $B$ at $b$, we may assume that $b$ is a codimension-one point, and so part 2 of Theorem 1.3 implies that $(V, \nabla)$ extends (analytically) to $X_b$. By [4], $(V, \nabla)_b$, which is a-priori an analytic vector bundle with connection, has a canonical algebraic structure. We will first prove that the $p$-curvatures of $(V, \nabla)_b$ vanish for almost all primes. Note that part 1 of Theorem 1.3 implies that any subfamily of curves $C \rightarrow B \subset X \rightarrow B$ has the property that family $(V, \nabla)|_C$ is algebraic (where $C^0 \subset C$ is a suitable family of open subvarieties of $C$ with fiber over $b$ non-trivial), and hence the $p$-curvatures of $(V, \nabla)_b$ restricted to $C_b$ vanish. The following result shows the vanishing of $p$-curvatures of $(V, \nabla)_b$ restricted to $X_b$ for almost all primes $p$.

**Claim 4.5.** Suppose that $Y$ be a smooth quasi-projective variety over $\overline{\mathbb{F}}_p$. Let $(V, \nabla)$ denote a vector bundle with flat connection on $Y$, such that the $p$-curvature of $(V, \nabla)$ restricted to $C$ vanishes for every smooth plane-section $C$ of $Y$. Then the $p$-curvature of $(V, \nabla)$ vanishes on all of $Y$.

**Proof.** By induction on the dimension of $Y$, we may assume that the $p$-curvature of $(V, \nabla)$ pulled back to every smooth hyperplane section of $Y$ vanishes.

Suppose $Q \in Y$ is a point and $Z \subset Y$ is a smooth divisor containing $Q$. Then we get the conormal exact sequence:

\[ 0 \rightarrow \mathcal{I}_Z/\mathcal{I}_Z^2 \rightarrow \Omega_Y|_Z \rightarrow \Omega_Z \rightarrow 0 \quad (2) \]

By [12], the $p$-curvature $\Psi$ is a section of the coherent sheaf of $\mathcal{O}_Y$-modules $\text{Hom}(V, V) \otimes (\Omega_Y)^{(p)}$. Here $(p)$ denotes the Frobenius twist of $\Omega_Y$. Furthermore, its restriction $\Psi|_Z \in \text{Hom}(V, V)|_Z \otimes_{\mathcal{O}_Z} (\Omega_Y)^{(p)}|_Z$ agrees with the $p$-curvature of the restriction of the pair $(V, \nabla)$ to $Z$.

Now choose a point $Q \in Y$, and suppose $Z_1, ..., Z_n$ are smooth divisors such that the conormal vectors $\mathcal{I}_Z/\mathcal{I}_Z^2|_Q \in \Omega_Y|_Q$ form a basis. Then the natural map of vector spaces

\[ \Omega_Y|_Q \rightarrow \bigoplus_{i=1}^n \Omega_{Z_i}|_Q \quad (3) \]

is injective.

By Bertini’s theorem, at a general point $Q \in Y$, there exists such a collection of divisors $Z_i$. By induction, we may assume the $p$-curvatures of $(V|_{Z_i}, \nabla|_{Z_i})$ vanish, and hence by the injectivity of $\Psi$, we conclude that $\Psi$ also vanishes. This concludes the proof.

\[ \Box \]
As the $p$-curvature conjecture is known for $X_0$ (by hypothesis), it follows that $(V,\nabla)_b$ has finite monodromy. In order to deduce that $(V,\nabla)$ has finite monodromy, we use Corollary 3.9 to replace $B$ with an étale neighbourhood $U$ of $b$ and $X \to B$ with $C \to U$, such that the map $\pi_1(C_u) \to \pi_1(X_u)$ is surjective. It suffices to prove that $(V,\nabla)$ pulled back to $C$ has finite monodromy. We know that $(V,\nabla)$ pulled back to $C_b$ has finite monodromy (as we have proved that $(V,\nabla)_b$ has finite monodromy). Hence we may apply Theorem 1.5 to finish the proof of this theorem.

\[\square\]

5. AN APPLICATION TO GENERIC GENUS ZERO CURVES

We spend this section proving Theorem 1.7. To aid the reader, we give a sketch of the argument. There are three main inputs to our proof. We first use the genericity of the punctures to specialize to a nodal curve containing $\mathbb{P}^1 \setminus \{0,1,\infty\}$ as an irreducible component, and use Theorem 1.3 to prove that $(V,\nabla)$ extends to a vector bundle with connection on this irreducible component. By work of Katz [13], the $p$-curvature conjecture is known for rank two bundles\footnote{We thank Hélène Esnault for pointing out to us that every rank 2 vector bundle with connection on $\mathbb{P}^1 \setminus 0,1,\infty$ is Hypergeometric.} on $\mathbb{P}^1 \setminus 0,1,\infty$, and this is our second input. Finally, we use (a slight generalization of) the isomonodromy Theorem 1.5 to deduce our result.

We work over a number field $k$. Let $f: C \to B$ be a flat family of genus 0, $d$-punctured (marked) curves over a pointed curve $(B,0)$ with smooth generic fiber and with special fiber $C_0 = f^{-1}(0)$ reduced and nodal, corresponding to a map $B \to \mathcal{M}_{0,d}$, the moduli space of stable $d$-pointed genus 0 curves. Let $B^\circ$ denote the open set $B \setminus \{0\}$, assume that $C$ is smooth over $B^\circ$, and suppose that there exists an irreducible component $E$ of the special fiber $C_0$ which contains exactly three special points (either punctures/marked points or nodes). We let $C^0 \subset C$ denote the preimage of $B^\circ$.

Upon blowing down the components $C_0 \setminus E$, and after appropriately choosing three of the $d$-marked punctures to be at 0, 1, and $\infty$, we get an affine curve $A \subset (\mathbb{P}^1 \setminus \{0,1,\infty\}) \times B$ which is the complement of $d$ sections $\sigma_i : B \to \mathbb{P}^1$ satisfying $\sigma_i(0) \in \{0,1,\infty\} \subset \mathbb{P}^1$ for all $i$. (The first three sections are taken to be the constant sections 0, 1, $\infty$). By construction, the affine curve $A$ is isomorphic to an open subset of $C$ under the blowdown map.

This general setup will be referred to consistently throughout the remainder of the section.

**Lemma 5.1.** Maintain the setting of $f: C \to B$ above. Then there exists an affine open set $A \subset C$ containing the generic point of $E$ and an everywhere non-zero $B$-derivation $D$ on $A$ such that $D^p \equiv D \mod p$ for all primes $p$.

**Proof.** We take $A$ as in the discussion preceding the lemma. We then take, for instance, the derivation $D = x \frac{d}{dx}$ on $A \subset \mathbb{A}^1 \times B$, where $x$ denotes the coordinate on $\mathbb{A}^1$.

**Proposition 5.2.** Suppose $(V^\circ,\nabla^\circ)$ is an algebraic vector bundle with connection on $C^0$ relative to $B^\circ$ with the property that almost all $p$-curvatures vanish. Then $(V^\circ,\nabla^\circ)$ extends to an algebraic vector bundle with connection $(V,\nabla)$ on an open set $U \subset C$ containing the generic point of $E \subset C$.
Proof. The proposition is directly parallel to [3.1] so we omit the proof. \qed

We denote by \( \hat{E} \subset E \) the open set on which \( f \) is smooth, i.e. the complement of the nodes on \( E \). Note that \( \hat{E} \cong \mathbb{P}^1 \).

**Proposition 5.3.** Keep the notation from Proposition [5.2]. When restricted to \( U \cap E \), the extension \((V, \nabla)\) is the restriction of an algebraic vector bundle with connection \((\hat{V}, \hat{\nabla})\) on \( \hat{E} \). Furthermore, almost all \( p \)-curvatures of \((\hat{V}, \hat{\nabla})\) vanish.

Proof. Let \( p_1, \ldots, p_s \) denote the finitely many points in \( \hat{E} \) but not in \( U \), and let \( \hat{U} = U \cap E \). The algebraic vector bundle with connection on \( U \) has vanishing \( p \)-curvature (since the vanishing of \( p \)-curvature is a closed condition), and hence has regular singularities. Therefore, by the main result of [4], in order to prove this proposition it suffices to show that \((V, \nabla)|_{\hat{U}}\) on \( \hat{U} \) has trivial local monodromy around the points \( p_i \).

Let \( \Delta \subset B \) be a sufficiently small complex-analytic disk around \( 0 \in B \) with coordinate \( t \). Then for each \( p_i \) we choose a small loop \( \gamma_i \subset \hat{U} \) encircling \( p_i \). By the smoothness of \( f \), we may deform \( \gamma_i \) continuously with \( t \), obtaining a family of loops \( \gamma_i(t) \subset C_t \). The loops \( \gamma_i(t) \) are null-homotopic in \( C_t \). Since \((V, \nabla)\) is defined globally on \( C_t \) for all \( t \neq 0 \), we get that the monodromy of \( \gamma_i(t) \) is trivial for all \( t \neq 0 \). Therefore, by continuity, we conclude that the local monodromy of \((V, \nabla)|_{\hat{U}}\) around \( p_i \) is also trivial, as claimed. \qed

Suppose \( Y \to B \) is a smooth map of complex varieties, and suppose \( b \in B \) is a point sufficiently close to \( B \). Then there is a map \( \tau : \pi_1(Y_0) \to \pi_1(Y_b) \), well-defined up to choosing base-points, obtained by parallel transport of loops.

**Theorem 5.4** (Bootstrapping). Let \( A \subset C \) be an affine open subset of \( C \) such that the special fiber equals \( \hat{U} \) of \( \hat{E} \), and suppose \((V, \nabla)\) is a vector bundle with flat connection on \( A \) relative to \( B \) whose \( p \)-curvatures vanish for almost all \( p \).

If the restriction \((V, \nabla)|_{\hat{U}}\) has finite monodromy, then \( \tau(\pi_1(\hat{U})) \subset \pi_1(A_b) \) has finite image in \( \text{GL}_n(\mathbb{C}) \) under the monodromy representation of \((V, \nabla)|_{A_b} \) for \( b \in B \) sufficiently close to \( 0 \).

Proof. Let \( q \) denote a local equation on \( B \) cutting out \( 0 \). Also, let \( N \subset B(\mathbb{C}) \) denote a small analytic open neighbourhood of \( 0 \). Let \( \hat{U} \) and \( U \) be as in Figure 2. It suffices to prove that the monodromy of \((V, \nabla)\) restricted to any fiber of \( U \) is finite. The identical argument used in the proof of Proposition [4.1] (and also the fact that \( A \times_{\text{Spec } R} \text{Spec } R/q^n \) is the trivial deformation of \( A \times_{\text{Spec } R} \text{Spec } R/q \)) yields that \((V, \nabla)|_{A/q^n} \) is isomorphic to \((V, \nabla)_0 \times \text{Spec } R/q^n \). Note that \( U \) is an open subset of \( A^{\text{hol}} \). Therefore, \( U \subset A^{\text{hol}} \mod q^n \) for every integer \( n \), and so there exists a basis for \( V^{\text{hol}}|U \) with respect to which the connection is constant (in \( q \)) modulo \( q^n \). Now, consider the holomorphic family of monodromy representations: \( \rho : \pi_1(\hat{U}) \to \text{GL}_n(\mathcal{O}^{\text{hol}}(N)) \). As the connection is isomorphic to a constant connection mod \( q^n \), it follows that the kernel of \( \rho \mod q^n \) is independent of \( n \). We now claim that the kernel of \( \rho \) is the same as the kernel of \( \rho \mod q \). Let \( \alpha \in \pi_1(\hat{U}) \) be in the kernel of \( \rho \mod q \). Then, \( \alpha \) is in the kernel mod \( q^m \) for all \( m \), and so \( \rho(\alpha) \) is the identity element, as required. The result follows. \qed

We are now ready to put the above results together to prove Theorem 1.7.
Figure 2. Degeneration of smooth genus zero curves to a nodal curve

Proof of Theorem 1.7. Let $\mathcal{M}_{0,d}$ denote the (fine) moduli space of genus zero curves with $d$ marked points, with $d \geq 4$, and let $M$ denote some Zariski-open subset over which the moduli problem is fine. Let $\overline{\mathcal{M}}_{0,d}$ denote the Deligne-Mumford compactification of $\mathcal{M}_{0,d}$, which is a projective variety. There are three different families of “pairs of pants” inside $C(\mathbb{C})$, where $C$ a genus 0 curve with $d$ punctures:

1. Pick any simple closed loop $\gamma \subset C(\mathbb{C})$ as in Figure 2 such that the complement of $\gamma$ consists of 2 disks, the first containing two of the marked points and the second, containing the remaining $d - 2$ marked points. The disc containing two of the marked points is a pair of pants contained inside $C(\mathbb{C})$, and can be realised by approaching a divisorial boundary component of $\mathcal{M}_{0,d}$. We define this type of pairs of pants to be $P^2$. 

We thank Joe Harris and Ian Morrison for allowing us to use Figure 2, which can be found in their book *Moduli of Curves.*
(2) Fix one of the marked points, and partition the remaining \( d - 1 \) points into two non-empty sets containing \( a \) and \( b = d - 1 - a \) points respectively. Let \( \gamma_a \subset C(\mathbb{C}) \) denote a simple closed loop bounding the set of \( a \) marked points, and let \( \gamma_b \) denote the analogous simple closed loop. The complement of \( \gamma_a \) and \( \gamma_b \) consists of two discs (containing \( a \) and \( b \) marked points respectively), and a pair of pants. The pair of pants can be realised by approaching a suitable codimension-2 boundary component of \( \mathcal{M}_{0,d} \). Further, this codimension-2 boundary component is the intersection of two boundary divisors. We define this type of pairs of pants to be \( P_{a,b} \).

(3) Partition the \( d \) points into three nonempty sets containing \( a, b \) and \( c = d - a - b \) points respectively. Let \( \gamma_a \subset C(\mathbb{C}) \) denote a simple closed loop which bounds the first \( a \) points, and let \( \gamma_b, \gamma_c \) denote the analogous simple closed loops. The complement of these three loops equals the union of a pair of pants and three discs (with \( a, b \) and \( c \) punctures respectively). The pair of pants can be realised by approaching a suitable codimension-3 boundary component. Further, this boundary component is the intersection of three boundary divisors. We define this type of pairs of pants to be \( P_{a,b,c} \).

For more details about \( \mathcal{M}_{0,d} \), see [10, Chapter 3, Section G]. We now blow \( \mathcal{M}_{0,d} \) up at all the codimension 2 and codimension 3 boundary points considered just above. Let \( \overline{\mathcal{M}} \) denote this blown-up scheme; \( \overline{\mathcal{M}} \) is still projective, and we fix a projective embedding.

Let \( B \) denote a one-dimensional plane section. We may assume that \( B \) is irreducible, is defined over a number field, and also that the map \( \pi_1(B \cap M(\mathbb{C})) \to \pi_1(M(\mathbb{C})) \) is surjective (by the quasi-projective Lefschetz theorem). We also have that \( B \) intersects every divisorial boundary component of \( \overline{\mathcal{M}} \). Fix any type of pairs of pants \( T \), where \( T \) either equals \( P \), or \( P_{a,b} \) (for a fixed pair of integers \( a, b \) with \( a + b = d - 1 \), or \( P_{a,b,c} \) with \( a + b + c = d \). By [7, Page 37], the action of \( \pi_1(M(\mathbb{C})) \) (by parallel transport) on a fiber over \( M \) is transitive (up to isotopy) on pairs of pants of type \( T \), and thus the same is true about the action of \( \pi_1(B \cap M(\mathbb{C})) \). In sum, given any pair of pants \( P \) contained in a smooth fiber \( C_0 \), there exists a path contained in \( B(\mathbb{C}) \) connecting \( b \) to an appropriate boundary point \( 0 \) such that the pair of pants deforms to \( P_{0,1,\infty} \subset C_0 \). We now apply Theorem 5.4, noting that the \( p \)-curvature conjecture is known (by work of Katz) for rank 2 vector bundles on \( \mathbb{P}^1 \setminus 0, 1, \infty \). The theorem follows. □

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