Birationally rigid complete intersections of high codimension

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Abstract. We prove that a Fano complete intersection of codimension $k$ and index 1 in the complex projective space $\mathbb{P}^{M+k}$ for $k \geq 20$ and $M \geq 8k \log k$ with at most multi-quadratic singularities is birationally superrigid. The codimension of the complement of the set of birationally superrigid complete intersections in the natural moduli space is shown to be at least $(M - 5k)(M - 6k)/2$. The proof is based on the technique of hypertangent divisors combined with the recently discovered $4n^2$-inequality for complete intersection singularities.

Keywords: birational rigidity, maximal singularity, multiplicity, hypertangent divisor, complete intersection singularity.

Introduction

0.1. Complete intersections of index 1. Let $k \geq 20$ be a fixed integer. For every $k$-tuple $d = (d_1, \ldots, d_k)$ of integers with $2 \leq d_1 \leq \cdots \leq d_k$ we put $M = |d| - k$, where $|d| = d_1 + \cdots + d_k$. Let

$$\mathcal{P}(d) = \prod_{i=1}^{k} \mathcal{P}_{d_i, M+k+1}$$

be the space of ordered $k$-tuples of homogeneous polynomials of degrees $d_1, \ldots, d_k$ respectively on the complex projective space $\mathbb{P} = \mathbb{P}^{M+k}$. Here the symbol $\mathcal{P}_{a, N}$ stands for the linear space of homogeneous polynomials of degree $a$ in $N$ variables, which may naturally be regarded as polynomials on $\mathbb{P}^{N-1}$. We write the elements of $\mathcal{P}(d)$ in the form $\underline{f} = (f_1, \ldots, f_k) \in \mathcal{P}(d)$. Moreover, let

$$\mathcal{P}_{\text{fact}}(d) \subset \mathcal{P}(d)$$

be the set of $k$-tuples $\underline{f} = (f_1, \ldots, f_k)$ whose common zero set

$$V(\underline{f}) = \{ f_1 = \cdots = f_k = 0 \} \subset \mathbb{P}$$

is an irreducible, reduced and factorial complete intersection of codimension $k$. Note that for every $\underline{f} \in \mathcal{P}_{\text{fact}}(d)$ the projective variety $V(\underline{f})$ is a primitive Fano variety of index 1, that is,

$$\text{Cl} V(\underline{f}) = \text{Pic} V(\underline{f}) = \mathbb{Z} H.$$
where \( H \) is the class of a hyperplane section (this follows from the Lefschetz theorem) and \( K_{V(f)} = -H \). Therefore we may ask whether the variety \( V = V(f) \) is birationally rigid or superrigid (see [1], Ch. 2, for definitions).

**Theorem 0.1.** Assume that \( M > 8k \log k \). Then there is a non-empty Zariski-open subset \( \mathcal{P}_{\text{reg}}(d) \subset \mathcal{P}_{\text{fact}}(d) \) such that

(i) the variety \( V = V(f) \) is birationally superrigid for every \( f \in \mathcal{P}_{\text{reg}}(d) \);

(ii) we have

\[
\text{codim}((\mathcal{P}(d) \setminus \mathcal{P}_{\text{reg}}(d)) \subset \mathcal{P}(d)) \geq \frac{(M - 5k)(M - 6k)}{2}.
\]

The birational superrigidity of generic complete intersections of index 1 (for \( M \geq k + 7 \)) was shown in [2], [3], but only non-singular complete intersections were considered there, so that the complement of the set of birationally superrigid varieties could be a divisor. In the present paper we consider complete intersections with multi-quadratic singularities. As a result, we get a much better estimate for the codimension of the complement: when \( k \) is fixed and \( M \) grows, the codimension is of order \( M^2/2 \), which is quite high.

We now proceed to give explicit definitions of some Zariski-open subsets in \( \mathcal{P}(d) \).

**0.2. Complete intersections with multi-quadratic singularities.** Let us describe conditions for the singularities of a complete intersection that guarantee its factoriality. Take an arbitrary \( k \)-tuple \( f \in \mathcal{P}(d) \) whose common zero set \( V = V(f) \) is an irreducible reduced complete intersection of codimension \( k \). Let \( o \in V \) be a point. We fix a system of affine coordinates \((z_1, \ldots, z_{M+k})\) on an affine chart \( \mathbb{C}^{M+k} \subset \mathbb{P} \) with origin at \( o \). Write the corresponding dehomogenized polynomials (denoted by the same symbols) in the form

\[
\begin{align*}
  f_1 &= q_{1,1} + q_{1,2} + \cdots + q_{1,d_1}, \\
  \quad&\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\qu
Note that, by Definition 0.1, the codimension of the singular set of $V$ near a correct multi-quadratic singularity is at least $2k + 2$.

We now discuss the conditions in Definition 0.1 in more detail. There is a subset $I \subset \{1, \ldots, k\}$ such that $|I| = k - l$ and the linear forms $q_{i,1}$, $i \in I$, are linearly independent:

$$\langle q_{1,1}, \ldots, q_{k,1} \rangle = \langle q_{i,1} | i \in I \rangle.$$  

Since $P$ is generic, the restrictions $q_{i,1}|_P$, $i \in I$, remain linearly independent. Hence the zero set

$$V_{P,I} = \{ f_i |_P = 0 | i \in I \}$$

is a non-singular complete intersection of codimension $k - l$ near $o$. Let

$$\varphi_{P,I} : V_{P,I}^+ \to V_{P,I}$$

be the blow-up of the point $o \in V_{P,I}$ whose exceptional divisor $E_{P,I} = \varphi_{P,I}^{-1}(o)$ is a projective space of dimension $\max\{k + l + 1, 4l + 2\}$. We may now regard the blow-up $\varphi_P$ as the restriction of the blow-up $\varphi_{P,I}$ to $V_P$, that is, $V_{P}^+$ is the strict transform of $V_P$ on $V_{P,I}^+$. In terms of this representation, the exceptional divisor $Q_P \subset E_{P,I}$ is given by the following system of $l$ equations:

$$q_{i,2}|_{E_{P,I}} = 0, \quad i \in \{1, \ldots, k\} \setminus I.$$  

Definition 0.1 requires $Q_P$ to be a non-singular complete intersection of type $2^l$ in the projective space $E_{P,I}$.

**Definition 0.2.** We say that an irreducible complete intersection $V = V(f)$ has at most correct multi-quadratic singularities if every point $o \in V$ is either a non-singular point or a correct multi-quadratic singularity of type $2^l$ for some $l \in \{1, \ldots, k\}$.

We write $P_{mq}(d)$ for the set of $k$-tuples $f \in P(d)$ such that $V(f)$ satisfies Definition 0.2. The subset $P_{mq}(d) \subset P(d)$ is clearly Zariski open. Since for $\bar{f} \in P_{mq}(d)$ we have

$$\text{codim}(\text{Sing} V(\bar{f}) \subset V(\bar{f})) \geq 2k + 2,$$

the complete intersection $V(f)$ is a factorial variety by Grothendieck's theorem on the parafactoriality of local rings [4]. Therefore, $P_{mq}(d) \subset P_{\text{fact}}(d)$.

**Theorem 0.2.** The following estimate holds:

$$\text{codim}( (P(d) \setminus P_{mq}(d)) \subset P(d)) \geq \frac{(M - 4k + 1)(M - 4k + 2)}{2} - (k - 1). \quad (2)$$

**Remark 0.1.** In what follows we construct the subset $P_{\text{reg}}(d) \subset P_{mq}(d)$ by removing some additional closed subsets from $P_{mq}(d)$.
0.3. Regular complete intersections. We keep the coordinate notation in § 0.2 at a point \( o \in V \). For brevity and uniformity, we regard non-singular points \( o \notin \text{Sing} V \) as multi-quadratic points of type \( 2^l \) with \( l = 0 \). We list the homogeneous polynomials

\[
q_{i,1}, \quad i \in I, \quad q_{i,j}, \quad j \geq 2,
\]

in the standard order corresponding to the lexicographic order of the pairs \((i, j)\): \((i_1, j_1)\) precedes \((i_2, j_2)\) if \( j_1 < j_2 \) or \( j_1 = j_2 \) but \( i_1 < i_2 \). Thus we obtain a sequence

\[
h_1, h_2, \ldots, h_{M+k-l}
\]

(3)
of \( M+k-l \) homogeneous polynomials in \( z_\ast \) of non-decreasing degrees: \( \deg h_{e+1} \geq \deg h_e \).

Definition 0.3. A point \( o \in V \) is regular if the sequence of polynomials obtained from (3) by removing the last \( \lfloor 2 \log k \rfloor - l \) terms is regular in \( \mathcal{O}_o, \mathbb{P} \). (Here \( \lfloor \cdot \rfloor \) means the integer part of a non-negative real number. When \( l > \lfloor 2 \log k \rfloor \), we remove no terms of the sequence (3).)

In plain words, Definition 0.3 requires that the set of common zeros of the polynomials \( h_e(z) \) in the sequence obtained from (3) by removing the last \( \lfloor 2 \log k \rfloor - l \) terms is of the correct codimension. Since the polynomials \( h_\ast \) are homogeneous, we may regard them as polynomials in the homogeneous coordinates \((z_1: \cdots: z_{M+k})\) on the projective space \( \mathbb{P}^{M+k-1} \) and understand regularity in the projective sense.

Definition 0.4. A complete intersection \( V = V(f) \) with \( f \in \mathcal{P}_{mq}(d) \) is regular if it is regular at every point \( o \in V \), singular or non-singular. In this case we write \( f \in \mathcal{P}_{reg}(d) \).

Theorem 0.3. Assume that \( f \in \mathcal{P}_{reg}(d) \). Then \( V = V(f) \) is birationally superrigid.

Theorem 0.4. The following estimate holds:

\[
\text{codim}((\mathcal{P}_{mq}(d) \setminus \mathcal{P}_{reg}(d)) \subset \mathcal{P}(d)) \geq \frac{(M-5k)(M-6k)}{2}.
\]

Proof of Theorem 0.1. Since the right-hand side of (4) is obviously larger than that of (2), Theorem 0.1 follows immediately from Theorems 0.2, 0.3 and 0.4. \( \square \)

0.4. Structure of the paper. Our paper is organized in the following way. In § 1 we prove Theorem 0.3. This is done using the technique of hypertangent divisors (the constructions can be found in [2], in [1], Ch. 3, or in [5]) and the recently discovered \( 4n^2 \)-inequality for complete intersection singularities [6]. We need to take into account that the regularity condition holds, generally speaking, not for the whole sequence (3) but for a shorter one, whence the resulting estimates are weaker than those in [2]. However, we shall show that they are still sufficient for the proof of birational superrigidity. Incidentally, the biggest deviation from the computations in [2] occurs for non-singular points.

In § 2 we prove Theorem 0.2. This is rather straightforward. The proof is by induction on the codimension \( k \) of the complete intersection. (Here we need not assume that \( k \geq 20 \). The case \( k = 2 \) was proved in [7], \( k = 1 \) in [8].)
In §3 we prove Theorem 0.4. The computations needed for the proof are really hard. We did our best to make them as clear and compact as possible. The estimates for the codimension are obtained by the ‘projection’ technique suggested in [9] and also used in [7].

0.5. Historical remarks. The first complete intersection of codimension at least 2 which was shown to be birationally rigid was the complete intersection $V_{2,3} \subset \mathbb{P}^5$ of a quadric and a cubic; see [10] or the modernized exposition in [1], Ch. 2. The variety $V_{2,3}$ was assumed to be general in the sense that it contains no lines with ‘incorrect’ normal sheaf. Singular complete intersections $V_{2,3} \subset \mathbb{P}^5$ were later studied in [11].

The birational superrigidity of general complete intersections $V \subset \mathbb{P}^{M+k}$ of type $\underline{d}$ with $|\underline{d}| = M + k$ and $M \geq 2k + 1$ was proved in [2]. It was extended in [3] to families with $M \geq k + 3$, $M \geq 7$ and $d_k = \max\{d_i\} \geq 4$, and in [12] to complete intersections of $k_2$ quadrics and $k_3$ cubics with $M \geq 12$ and $k_3 \geq 2$. The birational superrigidity currently remains an open problem only for three infinite series: the complete intersections of type $\underline{d}$, where $\underline{d}$ is

$$(2, \ldots, 2), \quad (2, \ldots, 2, 3), \quad \text{or} \quad (2, \ldots, 2, 4),$$

and finitely many families with $M \leq 11$.

A bound for the codimension of the locus of non-superrigid hypersurfaces of index 1 was given in [8]. Such bounds are important for investigations of the birational geometry of Fano fibre spaces with higher-dimensional base; see [13], [14]. Similar bounds were obtained for complete intersections of codimension $k = 2$ in [7], and for double quadrics and cubics (which may be regarded as complete intersections of codimension 2 in a weighted projective space) in [15].

An alternative approach to proving the birational superrigidity of Fano complete intersections in the projective space can be found in [16]. Other papers on the birational geometry of Fano complete intersections and their generalizations are [17]–[22].

§1. Proof of birational rigidity

In this section we prove Theorem 0.3. We first recall the definition of a maximal singularity and prove that the centre of a maximal singularity is of codimension at least 3 (§ 1.1). Then we construct hypertangent divisors (§ 1.2). The construction is standard, but singular points need special attention. In § 1.3 we exclude the case when the centre of the maximal singularity is not contained in the singular locus of $V$. In § 1.4 we exclude the case when the centre of the maximal singularity is contained in the locus of multi-quadratic points of type $2^l$. Since it follows that a mobile linear system can not have a maximal singularity, this proves that the variety $V$ is birationally superrigid.

1.1. Maximal singularities. As usual, we prove the birational superrigidity of the variety $V = V(\underline{f})$, where $\underline{f} \in \mathcal{P}_{\text{reg}}(\underline{d})$, by assuming the opposite and obtaining a contradiction. So we fix a tuple $\underline{f} \in \mathcal{P}_{\text{reg}}(\underline{d})$ and the corresponding complete intersection $V = V(\underline{f})$ and assume that $V$ is not birationally superrigid. This
immediately implies that the Noether–Fano inequality holds for some mobile linear system $\Sigma \subset |nH|$ and some exceptional divisor $E$ over $V$:

$$\operatorname{ord}_E \Sigma > n \cdot a(E),$$

where $a(E)$ is the discrepancy of $E$ with respect to $V$. In other words, $E$ is a maximal singularity of $\Sigma$; see, for example, [1], Ch. 2. Let $B \subset V$ be the centre of $E$ on $V$. It is an irreducible subvariety of codimension $\geq 2$.

**Lemma 1.1.** $\operatorname{codim}(B \subset V) \geq 3$.

**Proof.** Assume the opposite: $\operatorname{codim}(B \subset V) = 2$. Then $B \not\subset \operatorname{Sing} V$, whence we have

$$\operatorname{mult}_B \Sigma > n.$$

Consider the self-intersection $Z = (D_1 \circ D_2)$ of the system $\Sigma$, where $D_1, D_2 \in \Sigma$ are general divisors. Clearly, $Z = \beta B + Z_1$, where $\beta > n^2$ and the effective cycle $Z_1$ of codimension 2 does not contain $B$ as a component.

Let $P \subset \mathbb{P}$ be a general $(2k+1)$-dimensional subspace. Since $\operatorname{codim}(\operatorname{Sing} V \subset V) \geq 2k+2$, the intersection $V_P = V \cap P$ is non-singular. By the Lefschetz theorem, the numerical Chow group $A^2 V_P$ of cycles of codimension 2 on $V_P$ is $\mathbb{Z}H_P$, where $H_P$ is the class of a hyperplane section of $V_P$. Putting $Z_P = Z|_P$ and $B_P = B|_P$, we obtain the inequality

$$\deg(Z_P - \beta B_P) \geq 0.$$

Since $B_P \sim mH_P^2$ for some $m \geq 1$, this inequality implies that

$$\deg V \cdot (n^2 - m\beta) \geq 0,$$

which is impossible. $\square$

Note that if $\operatorname{codim}(B \subset V) \leq 2k+1$, then $B$ is not contained in the singular locus $\operatorname{Sing} V$ of the complete intersection $V$.

**1.2. Hypertangent divisors.** To exclude the maximal singularity $E$, we need the construction of hypertangent linear systems. This is well known and has been published many times; see [2] or [1], Ch. 3, or the most recent application in [5]. However, some minor modifications are needed in the multi-quadratic case. Therefore we sketch this construction here. We fix a point $o$ and use the notation in § 0.2, working in the affine chart $\mathbb{C}^{M+k}$ of the space $\mathbb{P}$ with coordinates $z_1, \ldots, z_{M+k}$ where the point $o \in V$ is the origin. Let $j \geq 2$ be an integer. We recall that one can find an $l \in \{0, 1, \ldots, k\}$ and a subset $I \subset \{1, \ldots, k\}$ such that $|I| = k - l$, the linear forms $q_{i,1}, i \in I$, are linearly independent and the other forms $q_{i,1}, i \notin I$, are linear combinations of them. We write

$$f_{i,\alpha} = q_{i,1} + \cdots + q_{i,\alpha}$$

for the truncated $i$th equation in the tuple $f$. 
Definition 1.1. The linear system
\[ \Lambda(j) = \left\{ \left( \sum_{i \in I} q_{i,1}s_{i,j-1} + \sum_{i=1}^{k} \sum_{\alpha=2}^{d_i-1} f_{i,\alpha}s_{i,j-\alpha} \right) \bigg|_V = 0 \right\}, \]

where the \( s_{i,j-\alpha} \) independently range over the set of homogeneous polynomials of degree \( j - \alpha \) in the variables \( z_* \) (if \( j - \alpha < 0 \), then \( s_{j-\alpha} = 0 \)), is called the \( j \)th hypertangent system at the point \( o \).

To keep the notation uniform, we write \( \Lambda(1) \) for the tangent linear system:
\[ \Lambda(1) = \left\{ \left( \sum_{i \in I} q_{i,1}s_{i,0} \right) \bigg|_V = 0 \right\}. \]
The Zariski tangent space \( \{ q_{i,1} = 0 \mid i \in I \} \) is denoted by \( T \). Put \( c(1) = k - l \) and \( c(j) = k - l + \sharp \{(i, \alpha) \mid i = 1, \ldots, k, 1 \leq \alpha \leq \min\{j, d_i - 1\}\} \) for \( j \geq 2 \). We also put \( m(j) = c(j) - c(j-1) \), where \( c(0) = 0 \) and take \( m(j) \) general divisors
\[ D_{j,1}, \ldots, D_{j,m(j)} \]
in the linear system \( \Lambda(j) \) for \( j = 1, \ldots, d_k - 1 \). Ordering them in the standard way, which corresponds to the lexicographic order of the pairs \( (j, \alpha) \) (see § 0.3 for a similar procedure), we obtain a sequence
\[ R_1, \ldots, R_{M-l} \]
of effective divisors \( V \). Put \( N_l = M - l \) if \( l > [2 \log k] \), and \( N_l = M - [2 \log k] \) otherwise. In what follows we use only \( R_1, \ldots, R_{N_l} \), but it is convenient to keep the entire sequence in mind.

Proposition 1.1. We have
\[ \operatorname{codim}_o \left( \left( \bigcap_{j=1}^{N_l} |R_j| \right) \subset V \right) = N_l, \]
where \( |R_j| \) stands for the support of \( R_j \).

Proof. Since
\[ f_{i,\alpha}|_V = (-q_{i,\alpha+1} + \cdots)|_V \quad (5) \]
for \( 1 \leq \alpha \leq d_i - 1 \), where the dots stand for higher order terms in \( z_* \), the codimension of the base locus of the tangent linear system \( \Lambda(1) \) near \( o \) is equal to \((k - l)\), and that of the hypertangent linear system \( \Lambda(j), j \geq 2 \), is equal to
\[ (k - l) + \operatorname{codim}(\{ q_{i,\alpha}|_T = 0 \mid 1 \leq i \leq k, 1 \leq \alpha \leq 1 + \min\{j, d_i - 1\}\} \subset T). \]
Therefore, for a general choice of hypertangent divisors \( R_* \), the equality
\[ \operatorname{codim}_o \left( \left( \bigcap_{j=1}^{i} |R_j| \right) \subset V \right) = i \]
follows from the regularity of the subsequence

$$h_1, \ldots, h_i$$

of the sequence (3). The assertion now follows immediately from the regularity condition; see Definition 0.3. □

For a hypertangent divisor $R_i = D_{j,\alpha}$, where $j \in \{1, \ldots, d_k - 1\}$ and $\alpha \in \{1, \ldots, m(j)\}$, the number

$$\beta_{i,i} = \beta(R_i) = \frac{j + 1}{j}$$

is its slope.

Let $\varphi: V^+ \to V$ be the blow-up of the point $o$ with exceptional divisor $Q = \varphi^{-1}(o)$. The symbol $R_i^+$ means the strict transform of $R_i$ on $V^+$.

**Proposition 1.2.** (i) $R_i^+ \sim j\varphi^*H - \gamma_i Q$, where $\gamma_i \geq j + 1$.

(ii) For every irreducible subvariety $Y \subset V$ of codimension $\geq 2$ such that $Y \not\subset |R_i|$, the algebraic cycle $(Y \circ R_i)$ of the scheme-theoretic intersection satisfies the inequality

$$\frac{\text{mult}_o(Y \circ R_i)}{\deg(Y \circ R_i)} \geq \frac{\beta_{i,i}}{\deg Y}.$$

Here, as usual, the symbol $\text{mult}_o / \deg$ means the ratio of the multiplicity at $o$ and the degree in $\mathbb{P}$.

**Proof.** Part (i) follows from (5). Part (ii) follows from part (i). □

1.3. The non-singular case. In the notation of § 1.1, assume that $B \not\subset \text{Sing} V$. We want to show by contradiction that this case is impossible. To simplify the notation, we write $N$ instead of $N_0$ and $\beta_i$ instead of $\beta_{0,i}$.

By [1], Ch. 2, § 2, we have the $4n^2$-inequality

$$\text{mult}_B Z > 4n^2,$$

where $Z$ is the self-intersection of the mobile system $\Sigma \subset |nH|$. Take a point $o \in B$ in general position, $o \not\in \text{Sing} V$, and let $Y_2$ be an irreducible component of $Z$ with the maximal value of the ratio $\text{mult}_o / \deg$. Then

$$\frac{\text{mult}_o Y_2}{\deg Y_2} > \frac{4}{d}.$$

Take general hypertangent divisors $R_1, \ldots, R_M$ as described in § 1.2. The first $k$ of them, $R_1, \ldots, R_k$, are actually tangent divisors and we know that

$$\text{codim}_o((|R_1| \cap \cdots \cap |R_k|) \subset V) = k.$$

Proceeding as in § 1 of [2], we construct a sequence of irreducible subvarieties

$$Y_2, \ldots, Y_k$$

such that $\text{codim}(Y_i \subset V) = i$, $Y_2$ is an irreducible component of $Z$ with the maximal value of the ratio $\text{mult}_o / \deg$, and $Y_{i+1}$ is an irreducible component
of the scheme-theoretic intersection \((Y_i \circ R_{i-1})\) with the maximal value of the ratio \(\frac{\text{mult}_o}{\deg}\) for \(i = 2, \ldots, k - 1\). Thus, \(Y_k \subset V\) is an irreducible subvariety of codimension \(k\) satisfying the inequality
\[
\frac{\text{mult}_o}{\deg} Y_k > \frac{2^k}{d},
\]
where \(d = d_1 \cdots d_k = \deg V\).

**Lemma 1.2.** \(Y_k \not\subset |R_{k-1}|\).

**Proof.** Assume the opposite: \(Y_k \subset |R_{k-1}|\). Since the hypertangent divisors are general, this implies that
\[
Y_k \subset \{q_{1,1}|V = \cdots = q_{k,1}|V = 0\}.
\]
However, since \(\text{codim}(\text{Sing} V \subset V) \geq 2k + 2\), the section \(V_P\) of \(V\) by a generic linear subspace \(P \subset \mathbb{P}\) of dimension \(3k + 1\) is a \((2k + 1)\)-dimensional non-singular complete intersection in \(\mathbb{P}^{3k+1}\). By the Lefschetz theorem, the scheme-theoretic intersection
\[
(\{q_{1,1}|V_P = 0\} \circ \cdots \circ \{q_{k,1}|V_P = 0\})
\]
of codimension \(k\) on \(V_P\) must be irreducible and reduced. Hence the scheme-theoretic intersection
\[
(\{q_{1,1}|V = 0\} \circ \cdots \circ \{q_{k,1}|V = 0\})
\]
of codimension \(k\) on \(V\) is irreducible and reduced. By the regularity condition, the multiplicity of this irreducible subvariety at \(o\) is equal to \(2^k\) and its degree is equal to \(d\). Therefore, it cannot coincide with \(Y_k\). The resulting contradiction proves the lemma. \(\square\)

By Lemma 1.2 we can now proceed in exactly the same way as in [2], § 2: form the scheme-theoretic intersection \((Y_k \circ R_{k-1})\) and obtain an irreducible subvariety \(Y_{k+1} \subset V\) of codimension \(k + 1\) satisfying the inequality
\[
\frac{\text{mult}_o}{\deg} Y_{k+1} > \frac{2^{k+1}}{d}.
\]
After this, still following the arguments in [2], § 2, we use the hypertangent divisors \(R_{k+2}, \ldots, R_N\) to obtain a sequence of irreducible subvarieties \(Y_{k+2}, \ldots, Y_N\) of codimension \(\text{codim}(Y_i \subset V) = i\) such that \(Y_i\) is a component of the algebraic cycle \((Y_{i-1} \circ R_i)\) of the scheme-theoretic intersection of \(Y_{i-1}\) and \(R_i\) with the maximal value of the ratio \(\frac{\text{mult}_o}{\deg}\) (the regularity condition and the genericity of hypertangent divisors in their linear systems guarantee that \(Y_{i-1} \not\subset |R_i|\)). Therefore,
\[
\frac{\text{mult}_o}{\deg} Y_i \geq \beta_i \frac{\text{mult}_o}{\deg} Y_{i-1}.
\]
The last subvariety \(Y_N\) is positive-dimensional and satisfies the estimate
\[
\frac{\text{mult}_o}{\deg} Y_N > \gamma = \frac{2^{k+1}}{d} \prod_{i=k+2}^{N} \beta_i.
\]
Proposition 1.3. We have $\gamma \geq 1$.

We note that this proposition provides the desired contradiction and excludes the non-singular case.

Proof. It is now convenient to use the whole set $R_1, \ldots, R_M$ of hypertangent divisors since we have an obvious identity

$$d = d_1 \cdots d_k = \prod_{j=1}^{k} \prod_{\alpha=1}^{d_j-1} \frac{\alpha + 1}{\alpha} = \prod_{i=1}^{M} \beta_i.$$  

We recall that $\beta_1 = \cdots = \beta_k = 2$ and $\beta_{k+1} = 3/2$. Hence the expression for $\gamma$ can be rewritten in the form

$$\gamma = \frac{4}{3d} \prod_{i=1}^{N} \beta_i = \frac{4}{3} \beta^{-1},$$  

where

$$\beta = \prod_{i=N+1}^{M} \beta_i,$$  

(6)

and our proposition can be deduced from the following lemma.

Lemma 1.3. We have $\beta < 4/3$.

Proof. First of all, we note that when $j \geq N + 1$ we have $\beta_j \leq 1 + 1/a$, where $a = [M/k]$. Indeed, assume the opposite: $\beta_{N+1} > 1 + 1/a$. This means that all the homogeneous polynomials $h_{k+1}, \ldots, h_{k+N}$ in the sequence (3) are some $q_{i,\alpha}$ with $\alpha < a$. Therefore,

$$N \leq \sharp \{ q_{i,\alpha} \mid 1 \leq i \leq k, \ 2 \leq \alpha \leq a - 1 \}.$$  

But the right-hand side does not exceed $k \cdot (a - 2) < M - k$. Hence we get

$$M - \lceil 2 \log k \rceil < M - k,$$  

which gives rise to a contradiction.

We have shown that

$$\beta \leq \left( 1 + \frac{1}{a} \right)^{\lceil 2 \log k \rceil} \leq \left( 1 + \frac{1}{a} \right)^{a/4}$$  

since $M \geq 8k \log k$ by assumption. Therefore, $\beta < e^{1/4} < 4/3$, as required. □

This completes the proof of Proposition 1.3. □

We have shown that the case $B \not\subset \text{Sing} V$ is impossible.
1.4. The multi-quadratic case. We now assume that $B$ is contained in the closure of the locus of multi-quadratic points of type $2^l$ but not in the closure of the locus of multi-quadratic points of type $2^j$ for $j \geq l+1$. In other words, a general point $o \in B$ is a singular multi-quadratic point of type $2^l$. We fix this point.

**Proposition 1.4.** The self-intersection $Z$ satisfies the inequality $\text{mult}_o Z > 2^{l+2}n^2$.

**Proof.** This is the $4n^2$-inequality for complete intersection singularities; see [6].

**Remark 1.1.** Requiring a point $o \in V$ to be a correct multi-quadratic singularity (see Definition 0.1) is actually much stronger than the conditions required in [6].

We now exclude the multi-quadratic case and thus complete the proof of Theorem 0.3.

We first assume that $1 \leq l \leq k - 2$. Let

$$R_1, \ldots, R_{k-l}$$

be general tangent divisors. By the regularity condition,

$$\text{codim}_o \left( \bigcap_{i=1}^{k-l} |R_i| \subset V \right) = k - l.$$ 

Therefore, we may argue as in the non-singular case and construct a sequence of irreducible subvarieties

$$Y_2, \ldots, Y_{k-l}$$

of codimension $\text{codim}(Y_i \subset V) = i$, where $Y_2$ is an irreducible component of $Z$ with the maximal value of $\text{mult}_o / \text{deg}$, and $Y_{i+1}$ is an irreducible component of $(Y_i \circ R_{i-1})$ with the same property. Clearly,

$$\frac{\text{mult}_o}{\text{deg}} Y_{k-l} > \frac{2^k}{d}.$$ 

By the Lefschetz theorem, the scheme-theoretic intersection

$$(R_1 \circ R_2 \circ \cdots \circ R_{k-l})$$

is irreducible and reduced. We can reach this conclusion by intersecting this cycle with the section $V_P$ of $V$ with a general linear subspace $P$ of dimension $3k + 1$, exactly as in the proof of Lemma 1.2 (in fact, the Lefschetz theorem is applicable even for subspaces $P$ of smaller dimension). We conclude that $Y_{k-l} \not\subset |R_{k-l-1}|$ and construct an irreducible variety $Y_{k-l+1}$ satisfying the inequality

$$\frac{\text{mult}_o}{\text{deg}} Y_{k-l+1} > \frac{2^{k+1}}{d}.$$ 

Then we argue exactly as in the non-singular case, producing a sequence of irreducible subvarieties $Y_{k-l+2}, \ldots, Y_N$ whose last term satisfies the bound

$$\frac{\text{mult}_o}{\text{deg}} Y_N > \gamma_l = \frac{4}{3} \beta(l)^{-1},$$
where
\[ \beta(\ell) = \prod_{i=N_l+1}^{M-\ell} \beta_{i,i} \]  
(7)
(recall that \( N_l = M - [2 \log k] + \ell \) when \( \ell \leq [2 \log k] \) and \( N_l = M - \ell \) otherwise). The product (7) contains fewer terms than (6) and it is easy to see that
\[ \beta_{i,M-\ell-j} = \beta_{M-j} \quad \text{for} \quad j = 0, 1, \ldots, M - \ell - N_l - 1. \]
Hence \( \beta(\ell) < \beta \) when \( \ell \geq 1 \) and, therefore, \( \gamma_l > \gamma > 1 \). This gives us the desired contradiction. The multi-quadratic case for \( 1 \leq \ell \leq k - 2 \) is excluded.

We finally assume that \( \ell \in \{k - 1, k\} \). Then the subvariety \( Y_2 \) (an irreducible component of the self-intersection \( Z \) with the maximal value of the ratio \( \text{mult}_o / \deg \)) satisfies the inequality
\[ \frac{\text{mult}_o}{\deg} Y_2 > \frac{2^{k+1}}{d} \]
by Proposition 1.4. In this case we omit the part of our argument dealing with the tangent divisors and proceed straight to the second part, repeating verbatim the argument in the case \( \ell \leq k - 2 \).

The multi-quadratic case is excluded.

This completes the proof of Theorem 0.3.

§ 2. Irreducible factorial complete intersections

In this section we prove Theorem 0.2. In § 2.1 we explain the strategy of our proof and consider the case of a hypersurface. Then we begin the inductive part of the proof (§ 2.2) and first study the easier question as to whether the complete intersections are irreducible and reduced. Finally, in § 2.3 we complete the proof by considering complete intersections with correct multi-quadratic singularities.

2.1. Complete intersections with correct multi-quadratic singularities.

Let
\[ \mathcal{P}^{\geq j} = \prod_{i=j}^{k} \mathcal{P}_{d_i,M+k+1} \]
be the space of truncated tuples \((f_j, \ldots, f_k)\), and let \( \mathcal{P}^{\geq j}_{mq} \) be the set of tuples such that
\[ V(f_j, \ldots, f_k) = \{f_j = \cdots = f_k = 0\} \subset \mathbb{P} \]
is an irreducible reduced complete intersection of codimension \( k - j + 1 \) with at most correct multi-quadratic singularities in the sense of Definition 0.1 with \( k \) replaced by \( k - j + 1 \). Note that \( \mathcal{P}^{\geq 1} = \mathcal{P}(d) \) and \( \mathcal{P}^{\geq 1}_{mq} = \mathcal{P}_{mq}(d) \). We will prove Theorem 0.2 by downward induction on \( j = k, k - 1, \ldots, 1 \) in the following form:
\[ \text{codim}((\mathcal{P}^{\geq j} \setminus \mathcal{P}^{\geq j}_{mq}) \subset \mathcal{P}^{\geq j}) \geq \frac{(M - 4k + 1)(M - 4k + 2)}{2} - (k - 1). \]  
(8)
The base of the induction is the case of a hypersurface $V(f_k) \subset \mathbb{P}$ of degree $d_k$. It is easy to calculate that the closed subset of reducible or non-reduced polynomials of degree $d_k$ has codimension
\[
\left( \frac{M + k + d_k - 1}{d_k} \right) - (M + k + 1)
\]
in $\mathcal{P}_{d_k,M+k+1}$ (this corresponds to the case when $f_k$ has a linear factor) and the closed subset of polynomials $f_k$ such that the hypersurface $V(f_k)$ has at least one singular point which is not a quadratic singularity of rank at least 7 has codimension
\[
\frac{(M + k - 6)(M + k - 5)}{2} + 1
\]
in $\mathcal{P}_{d_k,M+k+1}$; see [8] for a similar detailed calculation in the case of rank at least 5. Hence the inequality (8) holds when $k = 1$.

We now proceed to describe the inductive argument.

2.2. The induction step: irreducibility. We assume that (8) has already been proved for $j+1$. Given a fixed tuple $(f_{j+1}, \ldots, f_k) \in \mathcal{P}_{\geq j+1}^{\mathbb{N}}$, our task is to estimate the codimension of the set of polynomials $f_j \in \mathcal{P}_{d_j,M+k+1}$ such that $V(f_j, \ldots, f_k)$ does not satisfy the required condition, that is, $(f_j, f_{j+1}, \ldots, f_k) \notin \mathcal{P}_{\geq j}^{\mathbb{N}}$.

We first consider the issues of irreducibility and reduceness. By the inductive assumption and the Grothendieck theorem [4], $V(f_{j+1}, \ldots, f_k)$ is a factorial complete intersection and we have an isomorphism
\[
\text{Cl} V(f_{j+1}, \ldots, f_k) \cong \text{Pic} V(f_{j+1}, \ldots, f_k) \cong \mathbb{Z} H,
\]
where $H$ is the class of a hyperplane section. Moreover, the restriction map
\[
r_a: H^0(\mathbb{P}, \mathcal{O}_\mathbb{P}(a)) \rightarrow H^0(V_{j+1}, \mathcal{O}_{V_{j+1}}(a))
\]
is surjective for every $a \in \mathbb{Z}_+$ (here we write $V_{j+1}$ instead of $V(f_{j+1}, \ldots, f_k)$ to simplify the notation). When $a < d_{j+1}$, this map is also injective, and when $a = d_{j+1}$ we have
\[
\dim \ker r_a = \# \{ i \in \{ j+1, \ldots, k \} \mid d_i = d_{j+1} \}.
\]
Easy calculations now show that the set of polynomials $f_j \in \mathcal{P}_{d_j,M+k+1}$ such that $V(f_j, f_{j+1}, \ldots, f_k)$ is either reducible or non-reduced, is of codimension at least
\[
\left( \frac{M + k + d_j - 1}{d_j} \right) - (M + k + 1) - (k - j)
\]
(again, this corresponds to the case when the divisor $\{ f_j \mid V_{j+1} = 0 \}$ has a hyperplane section of $V_{j+1}$ as a component). This estimate is higher (and, in fact, much higher) than we need, so we may assume that $V(f_j, f_{j+1}, \ldots, f_k)$ is irreducible and reduced.

Finally, we need to consider the condition for the singularities of the complete intersection $V(f_j, f_{j+1}, \ldots, f_k)$ to be multi-quadratic. To avoid cumbersome formulae, we will consider only the final case $j = 1$ when the weakest estimate occurs. The same argument works for larger values of $j$ with an appropriate adjustment of indices and dimensions.
2.3. Multi-quadratic singularities. We fix a point \(o \in \mathbb{P}\) and consider a tuple \((f_1, \ldots, f_k) \in \mathcal{P}^{\geq 1}\) with \(o \in V = V(f_1, \ldots, f_k)\). Let \((z_1, \ldots, z_{M+k})\) be a system of affine coordinates on an affine chart \(\mathbb{C}^{M+k} \subset \mathbb{P}\) with origin at \(o\). We write the corresponding dehomogenized polynomials (denoted by the same symbols) in the form

\[
\begin{align*}
    f_1 &= q_{1,1} + q_{1,2} + \cdots + q_{1,d_1}, \\
    \vdots \\
    f_k &= q_{k,1} + q_{k,2} + \cdots + q_{k,d_k},
\end{align*}
\]

where \(q_{i,j}\) is a homogeneous polynomial in \(z_*\) of degree \(j\). Assume that \(\dim \langle q_{1,1}, \ldots, q_{k,1} \rangle = k - l\), where \(l \geq 0\). Let \(I \subset \{1, \ldots, k\}\) be a subset of cardinality \(|I| = k - l\) such that the linear forms \(\{q_{i,1} \mid i \in I\}\) are linearly independent. We define a subspace \(\Pi \subset \mathbb{C}^{M+k}\) by putting

\[
\Pi = \{q_{i,1} = 0 \mid i \in I\} \cong \mathbb{C}^{M+l}.
\]

By assumption, for every \(j \in J = \{1, \ldots, k\} \setminus I\) there are (uniquely determined) constants \(\beta_{j,i}, i \in I\), such that

\[
q_{j,1} = \sum_{i \in I} \beta_{j,i} q_{i,1}.
\]

For every \(j \in J\) we put

\[
q_{j,2}^* = \left. \left( q_{j,2} - \sum_{i \in I} \beta_{j,i} q_{i,2} \right) \right|_{\Pi}.
\]

In the following assertion, the condition for \(o\) to be a correct multi-quadratic singularity is restated in terms of properties of the quadratic forms \(q_{j,2}^*\) defined above.

**Proposition 2.1.** Assume that for a general subspace \(\Theta \subset \mathbb{P}(\Pi)\) of dimension

\[
b = \max\{k + l + 1, 4l + 2\}
\]

the set of quadratic equations

\[
\{q_{j,2}^*|_{\Theta} = 0 \mid j \in J\}
\]

determines a non-singular complete intersection of type \(2^l\). Then \(o \in V\) is a correct multi-quadratic singularity of type \(2^l\).

**Proof.** It is easy to see that the germ \(o \in V\) is analytically equivalent to the closed set in \(\Pi\) defined by \(l\) equations

\[
0 = q_{j,2}^* + \cdots, \quad j \in J,
\]

where the dots stand for higher order terms. The rest is obvious. \(\square\)
Remark 2.1. In the notation of Definition 0.1, the exceptional divisor $Q_P$ is precisely the complete intersection of $l$ quadrics $\{q_{j,2}^* | \Theta = 0\}$, $j \in J$, in the $b$-dimensional space $\Theta$. Proposition 2.1 gives a sufficient condition for $o$ to be a correct multi-quadratic singularity. We now use this criterion to estimate the codimension of the set of tuples violating the conditions of Definition 0.1 at a given point $o \in V$.

**Definition 2.1.** We say that an $l$-tuple of quadratic forms $(q_{j,2}^* | j \in J)$ is correct if its zero set in $\mathbb{P}(\Pi)$ is an irreducible reduced complete intersection $Q_\Pi$ satisfying the inequality

$$\text{codim}(\text{Sing} \ Q_\Pi \subset Q_\Pi) \geq b.$$ 

**Corollary 2.1.** Assume that $(q_{j,2}^* | j \in J)$ is a correct $l$-tuple. Then $o \in V$ is a correct multi-quadratic singularity of type $2^l$.

Since our subsequent arguments (up to the end of this section) use only the quadratic forms $q_{i,2}$, we may assume without loss of generality that

$$J = \{1, \ldots, l\}$$

and $I = \{l + 1, \ldots, k\}$. Fixing the forms $q_{i,2}$ for $i \in I$, we work with $l$-tuples

$$(q_{j,2}^* | j = 1, \ldots, l) \in \mathcal{P}_{2,M+l}^{\times l}.$$ 

Theorem 0.2 is an obvious corollary of the following proposition.

**Proposition 2.2.** The codimension of the closed set $X \subset \mathcal{P}_{2,M+l}^{\times l}$ of incorrect $l$-tuples is not smaller than

$$\frac{(M + 3 - b)(M + 4 - b)}{2} - (l - 1).$$ 

We recall that $b = \max\{k + l + 1, 4l + 2\}$.

**Proof.** Elementary computations show that the codimension of the closed subset $X_\ast \subset \mathcal{P}_{2,M+l}^{\times l}$ of linearly dependent $l$-tuples is greater than (9), so we may assume that the forms $q_{j,2}^*$, $j = 1, \ldots, l$, are linearly independent. We denote their common zero set by $Q_\Pi$ and write $\text{Sing} \ Q_\Pi$ for the closed set of points $p \in Q_\Pi$ such that the linear terms of the dehomogenized polynomials $q_{j,2}^*$ with respect to any system of affine coordinates with origin at $p$ are linearly independent. (We argue in this way to avoid discussing the irreducibility and reduceness of the zero scheme of the forms $q_{j,2}^*$, $j = 1, \ldots, l$, at this stage of the proof.) For every

$$\lambda = (\lambda_1 : \cdots : \lambda_l) \in \mathbb{P}^{l-1}$$

let

$$W(\lambda) = \{\lambda_1 q_{1,2}^* + \cdots + \lambda_l q_{l,2}^* = 0\} \subset \mathbb{P}^{M+l-1}$$

be the corresponding quadric in the linear system generated by $(q_{j,2}^*)$. We will use the following simple observation, which was used in [7] for $k = 2$.

**Lemma 2.1.** For every point $p \in \text{Sing} Q_\Pi$ there is a point $\lambda \in \mathbb{P}^{l-1}$ such that $p \in \text{Sing} W(\lambda)$.
Proof. Obvious computations. □

Corollary 2.2. The following inclusion holds:

$$\text{Sing } Q_\Pi \subset \bigcup_{\lambda \in \mathbb{P}^{l-1}} \text{Sing } W(\lambda).$$

Let $\mathcal{R}_{\leq a} \subset \mathcal{P}_{2,M+l}$ be the closed subset of quadratic forms of rank at most $a$. It is well known that

$$\text{codim}(\mathcal{R}_{\leq a} \subset \mathcal{P}_{2,M+l}) = \frac{(M + l + 1 - a)(M + l + 2 - a)}{2}.$$

For every $e = 1, \ldots, l$ we consider the closed subset $\mathcal{X}_{e,a} \subset \mathcal{P}_{2,M+l}^e$ consisting of $e$-tuples $(g_1, \ldots, g_e)$ whose linear span $\langle g_1, \ldots, g_e \rangle$ has a positive-dimensional intersection with $\mathcal{R}_{\leq a}$.

Lemma 2.2. The following estimate holds:

$$\text{codim}(\mathcal{X}_{e,a} \subset \mathcal{P}_{2,M+l}^e) \geq \text{codim}(\mathcal{R}_{\leq a} \subset \mathcal{P}_{2,M+l}) - (e - 1).$$

Proof. Consider the natural projections of $\mathcal{P}_{2,M+l}^e = \mathcal{P}_{2,M+l}^{(e-1)} \times \mathcal{P}_{2,M+l}$ onto the last factor and onto the direct product $\mathcal{P}_{2,M+l}^{(e-1)}$ of the first $e - 1$ factors.

For every tuple

$$(g_1, \ldots, g_e) \in \mathcal{P}_{2,M+l}^e$$

such that

$$(g_1, \ldots, g_{e-1}) \notin \mathcal{X}_{e-1,a},$$

the condition $(g_1, \ldots, g_e) \in \mathcal{X}_{e,a}$ implies that the quadratic form $q_e$ belongs to the cone with base $\mathcal{R}_{\leq a}$ and vertex space $\langle g_1, \ldots, g_{e-1} \rangle$. The dimension of this cone does not exceed $\dim \mathcal{R}_{\leq a} + (e - 1)$. Arguing by increasing induction on $e = 1, \ldots, l$, we complete the proof. □

We can now complete the proof of Proposition 2.2. Consider an $l$-tuple $(q_{j,2}^* | j = 1, \ldots, l)$ such that

$$\text{codim}(\text{Sing } Q_\Pi \subset Q_\Pi) \leq b - 1$$

or, equivalently,

$$\dim \text{Sing } Q_\Pi \geq M + l - b.$$

By Corollary 2.2 we conclude that

$$\max_{\lambda \in \mathbb{P}^{l-1}} \{\dim \text{Sing } W(\lambda)\} \geq M + 1 - b,$$

which in its turn implies that

$$(g_1, \ldots, g_e) \notin \mathcal{X}_{l,a}$$

for $a = l + b - 2$. Lemma 2.2 now shows that in the proof of Proposition 2.2 we can consider only $l$-tuples satisfying the inequality

$$\text{codim}(\text{Sing } Q_\Pi \subset Q_\Pi) \geq b.$$

(10)
The rest is very easy. If $Q_\Pi$ is an irreducible reduced complete intersection, then (10) guarantees that the tuple of quadratic forms under consideration is correct. Moreover, if for some $e \geq 1$ the system of quadratic equations

$$q_{1,2}^* = \cdots = q_{e,2}^* = 0$$

determines an irreducible reduced complete intersection of $e$ quadrics, then this variety is factorial by (10). Arguing as in §2.2, we can now estimate the codimension of the set of tuples whose zero set is not an irreducible reduced complete intersection. It is easy to check that this codimension is equal to

$$\frac{(M + l - 1)(M + l - 2)}{2} - e. \tag*{\square}$$

This also completes the proof of Theorem 0.2 since the minimum value of the estimate obtained in Proposition 2.2 occurs when $l = k$.

§ 3. Regular complete intersections

In this section we prove Theorem 0.4. In § 3.1 we give estimates for the codimension of the set of non-regular tuples of polynomials using the projection method. This reduces the proof of Theorem 0.4 to establishing a purely analytical fact: estimating the minimum of an integral sequence of certain binomial coefficients depending on several integer parameters. The required computations are by no means trivial. We perform them in several steps. In § 3.2 we simplify the problem by making several reductions. In § 3.3, the expression to be minimized is approximated with high accuracy by a smooth function with the help of Stirling’s classical formula, and then we study this function using the standard tools of calculus. In §§ 3.4 and 3.5 we complete the proof by establishing the required estimates.

3.1. The projection method. We use the notation in § 0.3. Since an elementary dimension count relates the codimension of the set of globally non-regular tuples $f$ (which is what Theorem 0.4 estimates) to the codimension of the set of tuples $f$ that are non-regular at a fixed point $o \in V(f)$ (see Theorem 3.1 and the subsequent comments), we concentrate on the local problem: fix a point $o \in \mathbb{P}$ and a system of affine coordinates $z_1, \ldots, z_{M+k}$ with origin at $o$ and consider (non-homogeneous) tuples $f$ such that $o \in V(f)$.

Next, we fix an $l \in \{0,1,\ldots,k\}$ and assume that the rank of the set of linear forms $q_{i,1}, i = 1,\ldots,k$, is equal to $k - l$, so that exactly the first $k - l$ polynomials in the sequence (3) are linear forms. We fix them also. Then the vector subspace

$$\Pi = \{h_1 = \cdots = h_{k-l} = 0\} \cong \mathbb{C}^{M+l}$$

of the space $\mathbb{C}^{M+k}$ is also fixed. Recall the notation

$$N_l = M - \max\{[2 \log k],l\}$$

introduced in § 1.2. Put

$$g_i = h_{k-l+i}|_{\Pi},$$
This is a sequence of $N_l$ homogeneous polynomials of non-decreasing degrees $m_i = \deg g_i$ on the projective space $\mathbb{P}(\Pi) \cong \mathbb{P}^{M+l-1}$. We define the space of such sequences:

$$G(d, l) = \prod_{i=1}^{N_l} \mathcal{P}_{m_i, M+l}.$$  

The point $o \in V$ is clearly regular (as a multi-quadratic point of type $2^l$ in the sense of Definition 0.3) if and only if the sequence $g_1, \ldots, g_{N_l}$ is regular, that is, the closed algebraic set

$$\{g_1 = \cdots = g_{N_l} = 0\} \subset \mathbb{P}(\Pi)$$

has codimension $N_l$. Let $\mathcal{Y} = \mathcal{Y}(d, l) \subset G(d, l)$ be the closed set of non-regular tuples.

**Theorem 3.1.** Assume that $M \geq 8k \log k$ and $k \geq 20$. Then

$$\text{codim}(\mathcal{Y} \subset G(d, l)) \geq \frac{(M-5k)(M-6k)}{2} + M + k.$$  

Taking into account that the point $o$ varies in the $(M+k)$-dimensional projective space $\mathbb{P}$ and the original tuple $f$ satisfies the conditions $f_1(o) = \cdots = f_k(o) = 0$ and $\dim \langle g_i, 1 \mid 1 \leq i \leq k \rangle = k - l$, we obtain by an elementary dimension count that Theorem 0.4 follows immediately from Theorem 3.1.

The rest of this section is a proof of Theorem 3.1. Our main tool is the projection method developed in [9]. It was explained and applied to solving similar problems in [1], Ch. 3, and many other papers, for example, [7], [8]. The idea is to represent

$$\mathcal{Y} = \prod_{e=1}^{N_l} \mathcal{Y}_e$$

as a disjoint union of constructive subsets $\mathcal{Y}_e$ consisting of tuples $(g_1, \ldots, g_{N_l})$ such that the closed set

$$\{g_1 = \cdots = g_{e-1} = 0\} \subset \mathbb{P}(\Pi)$$

is of codimension $e - 1$, but $g_e$ vanishes on some irreducible components of this set (when $e = 1$, this simply means that the quadratic form $g_1$ is identically zero). The projection method estimates the codimension of $\mathcal{Y}_e$ in $G(d, l)$ as follows:

$$\text{codim}(\mathcal{Y}_e \subset G(d, l)) \geq \gamma(e, d, l) = h^0(\mathbb{P}^{M+l-e}, \mathcal{O}_{\mathbb{P}^{M+l-e}}(m_e)) = \binom{M + l - e + m_e}{M + l - e},$$

where $m_e = \deg g_e$; see, for example, [1], Ch. 3. Therefore, to prove Theorem 3.1, we need to show that the numbers $\gamma(e, d, l)$ for $e = 1, \ldots, N_l$ are not smaller than the right-hand side of the inequality in Theorem 3.1. This is what we are going to do. The task is by no means trivial. We first do some preparatory work for simplifying the inequalities to be proved and decreasing the number of integer parameters on which the numbers $\gamma(e, d, l)$ depend.
3.2. Reductions. If the original tuple $f$ of defining polynomials consists of $k_2$ quadrics, $k_3$ cubics, ..., $k_m$ polynomials of degree $m = d_k \geq 8 \log k$, then

$$k_2 + k_3 + \cdots + k_m = k$$

and

$$2k_2 + 3k_3 + \cdots + mk_m = |d| = d_1 + \cdots + d_k.$$ 

It is easy to see that

$$m_e = \deg g_e = \min \left\{ j \mid \sum_{\alpha=2}^j \left( \sum_{\beta=\alpha}^m k_\beta \right) \geq e \right\}.$$ 

This explicit representation gives us the first reduction.

**Proposition 3.1.** The following estimate holds:

$$\gamma(e, d^*, l) \geq \gamma(e, d^*, l),$$

where the $k$-tuple $d^* = (d^*_1, \ldots, d^*_k)$ is defined by the equalities

$$d^*_1 = \cdots = d^*_r = a + 1, \quad d^*_{r+1} = \cdots = d^*_k = a + 2$$

and $M = ka + (k-r)$, where $0 \leq r \leq k - 1$.

**Proof.** Explicitly, the proposition asserts that

$$\left( \frac{M + l - e + m_e}{M + l - e} \right) \geq \left( \frac{M + l - e + m_e^*}{M + l - e} \right),$$

where $m_e^*$ is calculated for the tuple $d^*$. It is easy to see that $m_e \geq m_e^*$. $\square$

The second reduction further simplifies the situation and enables us to consider only the case when all the degrees $d_i$ are equal.

**Proposition 3.2.** For the tuple $d^+ = (d^+_1, \ldots, d^+_k)$ such that $d^+_1 = \cdots = d^+_k$ with $M^+ + k = |d^+|$ and $M^+ \geq 8k \log k - k$, we have

$$\gamma(e, d^+, l) \geq \frac{(M^+ - 4k)(M^+ - 5k)}{2} + M^+ + 2k$$

for all $e = 1, \ldots, N^+_t = M^+ - \max\{[2 \log k], l\}$.

We claim that Theorem 3.1 follows from Propositions 3.1 and 3.2. Indeed, by Proposition 3.1 it suffices to prove the inequality

$$\gamma(e, d^*, l) \geq \frac{(M - 5k)(M - 6k)}{2} + M + k$$

for $e = 1, \ldots, N_t$. Consider the tuple $d^+$ with

$$d^+_1 = \cdots = d^+_k = a + 1,$$
where \( a \) is the constant defined in Proposition 3.1. Put \( M^+ = ka \). Clearly, 
\[
\gamma(e, d^*, l) \geq \gamma(e, d^+, l)
\]
for \( e = 1, \ldots, N_l^+ \) since \( M \geq M^+ \). If \( N_l > N_l^+ \), then
for \( i = 0, \ldots, N_l - N_l^+ - 1 \) we have a similar estimate ‘from the other end’:
\[
\gamma(N_l - i, d^*, l) \geq \gamma(N_l^+ - i, d^+, l)
\]
(note that \( N_l - N_l^+ = M - M^+ \leq k \)). Therefore,
\[
\gamma(d^*, l) = \min_{1 \leq e \leq N_l} \{ \gamma(e, d^*, l) \} \geq \gamma(d^+, l) = \min_{1 \leq e \leq N_l^+} \{ \gamma(e, d^+, l) \}.
\]
Applying Proposition 3.2 and taking into account that \( M^+ > M - k \), we obtain
Theorem 3.1.

The third reduction enables us to remove the integer parameter \( l \in \{0, 1, \ldots, k\} \).
To simplify the notation, we write \( d_i \) instead of \( d^+_i \), thus assuming that \( d_1 = \cdots = d_k = a + 1 \), so that \( M = ka \). We use the notation \( \gamma(d, l) \) for the minimum of the numbers \( \gamma(e, d, l) \), \( e = 1, \ldots, N_l \), introduced above.

**Proposition 3.3.** We have
\[
\gamma(d, l) \geq \gamma(d, 0)
\]
for all \( l = 0, 1, \ldots, k \).

**Proof.** Since \( N_0 > N_l \) for \( l \geq 1 \), it suffices to compare the integers \( \gamma(e, d, l) \) and \( \gamma(e, d, 0) \) for the same values of \( e = 1, \ldots, N_l \). These numbers are
\[
\left( \frac{M + l - e + m_e}{M + l - e} \right) \quad \text{and} \quad \left( \frac{M - e + m_e}{M - e} \right),
\]
whence the assertion becomes obvious. \( \square \)

**Remark 3.1.** We could start with the third reduction by showing that the minimum of the integers \( \gamma(e, d, l) \) is attained at \( l = 0 \) (this corresponds to regular non-singular points of \( V \)) and then proving that the worst estimates correspond to the case (11).

The last (fourth) reduction makes the computations more compact. We recall that all the degrees \( d_i \) are now equal to \( a + 1 \). Define an integer-valued function \( \beta: \{2, \ldots, a\} \to \mathbb{Z}_+ \) by the formula
\[
\beta(t) = \left( \frac{k(a - t + 1) + t}{t} \right) = \left( \frac{kb(t) + t}{t} \right),
\]
where \( b(t) = a - t + 1 \). We also put
\[
\alpha = \alpha(M, k) = \left( \frac{a + 1 + [2 \log k]}{a + 1} \right).
\]

**Proposition 3.4.** We have
\[
\gamma(d, 0) \geq \min \left\{ \min_{t \in \{2, \ldots, a\}} \{ \beta(t) \}, \alpha \right\}.
\]
Proof. This follows immediately since for a special tuple $d$ of equal degrees one has

$$m_{ki+1} = m_{ki+2} = \cdots = m_{ki+k} = i + 2$$

for $i = 0, \ldots, a - 1$. □

Thus Theorem 3.1 is a corollary of the following two facts, where we assume that $M > 8k \log k - k$ and $k > 20$.

**Proposition 3.5.** The minimum of the function $\beta(t)$ on the set $\{2, 3, \ldots, a\}$ is attained at $t = 2$.

**Proposition 3.6.** We have

$$\alpha(M, k) \geq A(M, k) = \frac{(M - 4k)(M - 5k)}{2} + (M + 2k).$$

**Remark 3.2.** The proof of Proposition 3.5 requires only that $k > 10$. It is Proposition 3.6 that requires $k > 20$; see Remark 3.3 for more details.

The rest of this section is a proof of the last two propositions. It requires some (quite non-trivial) analytic arguments.

### 3.3. Stirling’s formula

The strategy of the proof of Proposition 3.5 is as follows. Using Stirling’s formula, we construct a smooth function $\varepsilon: \mathbb{R}_+ \to \mathbb{R}$ such that $\varepsilon(t) \leq \beta(t)$ for $t = 2, \ldots, a$ and $\varepsilon$ approximates $\beta$ with high accuracy. Then we show that the minimum of the function $\varepsilon(t)$ on the interval $[2, a]$ occurs at an endpoint (either $t = 2$, or $t = a$). Using this, we deduce Proposition 3.5.

Recall that by Stirling’s formula one has

$$n! = \sqrt{2\pi n} n^n \exp(-n) \exp\left(\frac{\theta_n}{12n}\right)$$

for some $\theta_n$ between 0 and 1. The integer parameter $e$ labelling the polynomials $g_e$ will never be used again in this paper, so we use the symbol $e$ for the number $\exp(1)$.

We put

$$\varepsilon(t) = \frac{\sqrt{2\pi}}{e^2} (kb(t) + t)^{kb(t)+t+1/2} (kb(t))^{-(kb(t)+1/2)} t^{-(t+1/2)}.$$  

By Stirling’s formula, $\beta(t) \geq \varepsilon(t)$.

**Lemma 3.1.** The smooth function $\varepsilon(t)$ for $k \geq 3$ has only one critical point on $[2, a]$. It is a maximum point. Hence the minimum of this function is attained at an endpoint.

**Proof.** This is shown by demonstrating that

(a) the function $\log \varepsilon(t)$ is strictly convex for $2 \leq t \leq (M + k)/(2k)$;

(b) it is strictly decreasing for $(M + 1)/(k + 1) \leq t \leq a = M/k$;

(c) the second derivative of $\log \varepsilon(t)$ is strictly negative for $(M + k)/(2k) \leq t \leq (M + 1)/(k + 1)$ (this is where the maximum lies).
The first derivative \(d \log \varepsilon(t)/dt\) is equal to
\[
\frac{t^2 - kb(t)^2}{2b(t)t(kb(t) + t)} - k \log \left(1 + \frac{t}{kb(t)}\right) + \log \left(1 + \frac{kb(t)}{t}\right).
\] (12)

The second derivative \(d^2 \log \varepsilon(t)/dt^2\) is given by the formula
\[
\frac{1}{b(t)t} + \frac{(t^2 - kb(t)^2)^2}{2b(t)^2t^2(kb(t) + t)^2} + \frac{(k-1)(t^2 - kb(t)^2)}{b(t)t(kb(t) + t)^2} - \frac{k(t+b(t))^2}{tb(t)(kb(t) + t)}.
\] (13)

The derivatives are written in this way for convenience in using the inequality
\[
\left|\frac{t^2 - kb(t)^2}{2b(t)t(kb(t) + t)}\right| \leq \frac{1}{2b(t)}.
\] (14)

We now consider the domains (a)–(c) separately.

(a) Assume that \(2 \leq t \leq (M+k)/(2k)\). Note that \(b(t) \geq 2\) on this interval.

Hence,
\[
\left|\frac{t^2 - kb(t)^2}{2b(t)t(kb(t) + t)}\right| \leq \frac{1}{4}.
\]

The last term in (12) can be estimated as
\[
\log \left(1 + \frac{kb(t)}{t}\right) \geq \log(1 + k) \geq \log 4 > 3
\]
since we have \(t \leq b(t)\) on the interval \([2, (M + k)/(2k)]\). For the second term in (12) we have
\[
-k \log \left(1 + \frac{t}{kb(t)}\right) \geq -\frac{t}{b(t)} \geq -1.
\]

Combining these estimates, we obtain an inequality
\[
\frac{d}{dt} \log \varepsilon(t) \bigg|_{2 \leq t \leq (M+k)/(2k)} \geq -\frac{5}{4} + 3 > 0,
\]
whence \(\varepsilon(t)\) is indeed increasing on the interval under consideration.

(b) Assume that \((M + 1)/(k+1) \leq t \leq M/k\). Here \(t \geq kb(t) \geq k\). First of all, we have
\[
\left|\frac{t^2 - kb(t)^2}{2b(t)t(kb(t) + t)}\right| \leq \frac{1}{2}.
\]

The other two terms in (12) satisfy the estimates
\[
-k \log \left(1 + \frac{t}{kb(t)}\right) \leq -k \log 2 \quad \text{and} \quad \log \left(1 + \frac{kb(t)}{t}\right) \leq \log 2.
\]

Combining these inequalities, we see that
\[
\frac{d}{dt} \log \varepsilon(t) \leq \frac{1}{2} - (k - 1) \log 2 < 0
\]
for \(t \in [(M + 1)/(k + 1), M/k]\), as required.
Let $G$ be the expression in brackets in the numerator is not larger than $-2a^2-a$. Hence the whole numerator is not larger than

$$t^2 - kb(t)(2a^2 + a) \leq kb(t)(t - 2a^2 - a) < 0.$$ 

Thus $d^2 \log \varepsilon(t)/dt^2 < 0$ for $t \in [(M + k)/(2k), (M + 1)/(k + 1)]$. □

3.4. Proof of Proposition 3.5. In view of the inequality $\varepsilon(t) \leq \beta(t)$ and Lemma 3.1, Proposition 3.5 follows from the two lemmas stated below.

Lemma 3.2. We have $\beta(2) \leq \varepsilon(3)$.

Lemma 3.3. We have $\beta(2) \leq \varepsilon(a)$.

Proof of Lemma 3.2. To use $\beta(3)$ instead of $\varepsilon(3)$, we need to estimate the error in Stirling’s approximation. The number $\beta(3)$ is a polynomial in $M$ and $k$, and this makes the task easier. By Stirling’s formula we have

$$1.126 \cdot \varepsilon(3) \leq \beta(3) \leq 1.132 \cdot \varepsilon(3).$$

The lemma will be proved if we can show that $1.14 \cdot \beta(2) < \beta(3)$, and this is equivalent to the inequality $G_1(M, k) = 6(\beta(3) - 1.14 \cdot \beta(2)) > 0$. Here $G_1(M, k)$ is given explicitly by the expression

$$M^3 + M^2(2.58 - 6k) + M(12k^2 - 17.16k + 0.74) - 8k^3 + 20.58k^2 - 11.74k - 0.84.$$ 

It is easy to check that $G_1(8k \log k, k)$ and the partial derivative $\partial G_1(M, k)/\partial M$ are positive for $k \geq 20$ and $M \geq 8k \log k$. □

Proof of Lemma 3.3. The assertion of the lemma is equivalent to the inequality

$$G_2(M, k) = \log \varepsilon(a) - \log \beta(2) \geq 0.$$ 

A direct calculation shows that $G_2(160 \log 20 - 20, 20) > 0$. Put

$$G_3(t) = \frac{d}{dt} G_2(8t \log t - t, t).$$

Lemma 3.4. $G_3(t) > 0$ for $t \geq 20$. 

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(c) Assume that $(M + k)/(2k) \leq t \leq (M + 1)/(k + 1)$. One has $b(t) \leq t \leq kb(t)$ on this interval. We claim that the second derivative (13) is negative. Indeed, using (14), we obtain that on the interval considered, $d^2 \log \varepsilon(t)/dt^2$ is at most

$$\frac{1}{b(t) t} + \frac{1}{2b(t)^2} + \frac{(k - 1)}{b(t)(kb(t) + t)} - \frac{k(t + b(t))^2}{tb(t)(kb(t) + t)} = \frac{t^2 + kb(t)(-2t^2 - 4b(t)t + 3t - 2b(t)^2 + 2b(t))}{2tb(t)^2(kb(t) + t)}.$$

Taking into account that $t \leq a$, we see from elementary computations that the expression in brackets in the numerator is not larger than $-2a^2 - a$. Hence the whole numerator is not larger than

$$t^2 - kb(t)(2a^2 + a) \leq kb(t)(t - 2a^2 - a) < 0.$$ 

Thus $d^2 \log \varepsilon(t)/dt^2 < 0$ for $t \in [(M + k)/(2k), (M + 1)/(k + 1)]$. □
Proof. We write the function $G_3$ explicitly:

$$G_3(t) = \log \left(1 + \frac{8 \log t - 1}{t}\right) + \frac{8}{t} \log \left(1 + \frac{t}{8 \log t - 1}\right) - \frac{1}{2t} + H_1(t) + H_2(t),$$

where

$$H_1(t) = \left(\frac{8}{t} + 1\right) \frac{8 \log t + t - 0.5}{8 \log t + t - 1} - \frac{8 \log t - 0.5}{t},$$

$$H_2(t) = -(8 \log t + 6) \left(\frac{1}{8t \log t - 2t + 2} + \frac{1}{8t \log t - 2t + 1}\right).$$

Using the power series expansion of $\log(1 + x)$, we obtain that

$$G_3(t) > -\frac{1}{2t} + \frac{8 \log t - 1}{t} - \frac{(8 \log t - 1)^2}{2t^2} + \frac{8}{t} \log \left(1 + \frac{t}{8 \log t - 1}\right) + H_1(t) + H_2(t).$$

When $t \geq 20$ we have $H_1(t) \geq 0$ and $H_2(t) \geq -4/t$ (this can be checked directly). Hence we get

$$G_3(t) > \frac{16 \log t - 11}{2t} - \frac{(8 \log t - 1)^2}{2t^2} + \frac{8}{t} \log \left(1 + \frac{t}{8 \log t - 1}\right).$$

The right-hand side of this inequality is greater than

$$\frac{1}{2t^2}(16t \log t - 11t - 64(\log t)^2 + \log t - 1).$$

This expression is positive when $t \geq 20$. □

We conclude that $G_2(8t \log t - t, t) > 0$ when $t \geq 20$. The assertion of Lemma 3.3 will be proved if we can establish that $G_2(M, k)$ is an increasing function of $M$ when $k \geq 20$ and $M \geq 8k \log k - k$. Put

$$G_4(s, t) = \frac{\partial}{\partial s} G_2(s, t).$$

Lemma 3.5. $G_4(s, t) > 0$ when $t \geq 20$ and $s \geq 8t \log t - t$.

Proof. We write the function $G_4$ explicitly:

$$G_4(s, t) = \frac{1}{t} \log \left(1 + \frac{t^2}{s}\right) - \frac{t^2}{2s(t^2 + s)} - \frac{2s^3 - 2t}{s^2 + (3 - 2t)s + t^2 - 3t + 2}.$$

First consider the case when $s \leq t^2$. Then we have

$$G_4\big|_{s \leq t^2} \geq \frac{1}{t} \log 2 - \frac{t^2}{2s(t^2 + s)} - \frac{2s + 3 - 2t}{s^2 + (3 - 2t)s + t^2 - 3t + 2}.$$

It is easy to see that the minimum of the right-hand side is attained at $s = 8t \log t - t$, that is, at the smallest possible value of $s$. Hence, when $s \leq t^2$, the function $G_4(s, t)$ is bounded below by the expression

$$\frac{1}{t} \log 2 - \frac{1}{(16 \log t - 2)(t + 8 \log t - 1)} - \frac{16t \log t + 3 - 4t}{t^2(8 \log t - 1)^2 + (3 - 2t)(8 \log t - t) + t^2 - 3t + 2},$$

which is positive when $t \geq 20$. 
We now consider the domain \( s \geq t^2 \). Here we get
\[
G_4(s, t) = \frac{t}{s} - \frac{t^3}{2s^2} - \frac{t^2}{2s(t^2 + s)} - \frac{2s + 3 - 2t}{s^2 + (3 - 2t)s + t^2 - 3t + 2}.
\]
A direct verification shows that the right-hand side is positive when \( t > 20 \). This completes the proof of Lemmas 3.5, 3.3 and Proposition 3.5. \( \square \)

3.5. Proof of Proposition 3.6. This proof is obtained in the same way as the proof of Proposition 3.5 and we only point out the main steps in the computation, leaving the details to the reader. To prove the inequality \( \alpha(M, k) \geq A(M, k) \), we use Stirling’s approximation for \( \alpha(M, k) \). Namely, we introduce a function \( G_5(s, t, r) \) of three real variables by the formula
\[
G_5(s, t, r) = \left( \frac{s}{t} + r + \frac{3}{2} \right) \log \left( \frac{s}{t} + r + 1 \right) - \left( r + \frac{1}{2} \right) \log r
- \left( \frac{s}{t} + \frac{3}{2} \right) \log \left( \frac{s}{t} + 1 \right) + \log \frac{\sqrt{2\pi}}{e^2} - \log A(s, t).
\]
By Stirling’s formula, Proposition 3.6 follows from the inequality
\[
G_5(M, k, [2 \log k]) \geq 0.
\]
It is easy to see that
\[
G_5(M, k, [2 \log k]) \geq G_5(M, k, 2 \log k - 1).
\]
Therefore we put \( G_6(s, t) = G_5(s, t, 2 \log t - 1) \) and prove that
\[
G_6(s, t) \geq 0
\]
for \( s \geq 8 t \log t - t, t \geq 20 \). First of all, explicit computations show that
\[
G_6(8 t \log t - t, t) \geq 0
\]
for \( t \geq 20 \). We put
\[
G_7(s, t) = \frac{\partial}{\partial s} G_6(s, t).
\]
It remains to prove that \( G_7(s, t) \geq 0 \) for \( s \geq 8 t \log t - t \) and \( t \geq 20 \). The function \( G_7(s, t) \) is given explicitly by the expression
\[
\frac{1}{t} \log \left( 1 + \frac{2 \log t - 1}{s/t + 1} \right) - \frac{2 \log t - 1}{2t(s/t + 1)(s/t + 2 \log t)} - \frac{2s - 9t + 2}{s^2 - 9ts + 2s + 20t^2 + 4t}.
\]
The inequality \( G_7(s, t) \geq 0 \) is now obtained by tedious but straightforward computations using the estimate
\[
\frac{1}{t} \log \left( 1 + \frac{2 \log t - 1}{s/t + 1} \right) > \frac{2 \log t - 1}{t(s/t + 1)} - \frac{(2 \log t - 1)^2}{2t(s/t + 1)^2}.
\]
We leave the details to the reader. This completes the proof of Proposition 3.6 and Theorem 3.1. \( \square \)
Remark 3.3. (i) It is clear from the computations in this section and in the proof of Lemma 1.3 that much weaker lower bounds for \( k \) would suffice in certain parts of our argument. For example, Lemma 3.2 requires only that \( k \geq 5 \), and Lemmas 3.3 and 3.4 hold for \( k \geq 10 \). The proof of the inequality \( G_7(s, t) \geq 0 \) at the end of § 3.5 requires only that \( t \geq 10 \). However, the inequality 

\[
G_6(8t \log t - t, t) \geq 0
\]

requires that \( t \geq 20 \). To perform the whole argument, we have to select the strongest restriction.

(ii) Another paper [23] on birational superrigidity of non-singular Fano complete intersections of index 1 was put on the archive after our paper was finalized. We add it to the list of references.

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