Complex Berry curvature pair and quantum Hall admittance in non-Hermitian systems

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Abstract

We propose complex Berry curvatures associated with the non-Hermitian Hamiltonian and its Hermitian adjoint and use these to reveal new physics in non-Hermitian systems. We give the complex Berry curvature and Berry phase for the two-dimensional non-Hermitian Dirac model. The imaginary part of the complex Berry phase induces susceptance so that the quantum Hall conductance is generalized to admittance for non-Hermitian systems. This implies that the non-Hermiticity of physical systems can induce intrinsic capacitive or inductive properties, depending on the non-Hermitian parameters. We analyze the complex energy band structures of the two-dimensional non-Hermitian Dirac model, determine the point and line gaps, and identify the conditions for their closure. We find that closure is associated with the exceptional degeneracy of the energy bands in the parameter space, which, in turn, is associated with topological phase transitions. In the continuum limit, we obtain the complex Berry phase in the parameter space.

1. Introduction

Geometric and topological descriptions of quantum systems provide powerful mathematical tools to reveal new quantum phases in condensed matter physics [1–7]. The quantum Hall effect and quantum spin Hall effects can be understood in terms of the topological Chern classes, \( \mathbb{Z} \) and \( \mathbb{Z}_2 \) [7–10]. These discoveries have inspired many attempts to study topological phases [9, 10], which can be classified in terms of ten topological classes, based on the symmetry group and homological invariance of the system [9]. The design of robust topological states is of vital importance for various applications in quantum materials and quantum technology [11–14].

Recently, non-Hermitian systems exhibiting unconventional characteristics have attracted growing attention, with a view to potential applications [13–18]. Many efforts have been devoted to exploring topological states in non-Hermitian systems [19–23]. Non-Hermiticity appears in non-equilibrium open systems with dissipative phenomena, such as energy gain or loss, or the non-conservation of probability associated with electron currents [24]. The non-Hermitian matrix representation of quantum mechanics involves unconventional characteristics not, present in the canonical theory, which is based on Hermitian operators, including the existence of complex energy spectra and their nonorthogonal eigenvectors. In the current literature, the problems addressed include: (1) How to construct a non-Hermitian generalization of the canonical (Hermitian) quantum formalism by introducing the inner product, pseudo-Hermiticity, and non-unitary similarity transformations, in order to reformulate the mathematical framework of the theory in a self-consistent way [25–28]. (2) How to generalize the classification of topological phases in Hermitian systems, which is based on Altland-Zirnbauer (AZ) symmetry, to non-Hermitian systems [22, 23].

Recently, a complete topological classification of non-Hermitian systems was proposed which generalizes the ten-fold topological classification of Hermitian systems to a 38-fold classification scheme. This includes
generalized AZ symmetry, sublattice symmetry, and pseudo-Hermiticity, which together cover all the internal symmetries of non-Hermitian systems [22, 23]. The topological classes are characterized by the winding number, vorticity, Berry curvature, and Chern number of their quantum states [20–23, 29]. The vorticity defined in the complex-energy plane has been shown to be equivalent to the winding number for the classification of the topological invariants [20, 23]. In particular, it was found that the complex energy band gap for non-Hermitian systems could form a point or line gap that preserves its topological invariants under unitary or Hermitian-flattening transformations [23]. The energy band gap closes to form exceptional points and lines, which act as reference points (or lines) associate with topological phases [22, 23]. The non-Hermiticity deforms the Bloch-wave behavior, yielding a skin effect in lattice models [30–33]. The biorthogonal polarization was introduced to modify the conventional bulk-boundary correspondence and to show the zero modes [34–36], the topological edge states, and finite-size effects in the non-Hermitian Su-Schrieffer-Heeger (SSH) model [21, 37–40]. In particular, Chen and Zhai studied the Hall conductance of a non-Hermitian Chern insulator and found some deviations from the quantized Chern number of canonical quantum mechanics, due to non-Hermitian effects [41].

In general, the non-Hermiticity of a system implies that the energy bands are complex. The band gaps exhibit exceptional degeneracy that can induce rich new phenomena beyond that obtained in conventional Hermitian systems. It is natural to ask: What role does the Hermitian adjoint of the Hamiltonian play in such systems? What new physical observables and phenomena arise from non-Hermiticity?

In this paper, we determine the complex Berry curvature and Berry phase for the two-dimensional (2D) non-Hermitian Dirac model. In section 2 we redefine the complex Berry curvature of the non-Hermitian Hamiltonian and its Hermitian adjoint to reveal new features of the associated quantum states. We then generalize the quantum Hall conductance to quantum Hall admittance, which is given by the complex Berry phase for non-Hermitian systems, in section 3. The presence of quantum Hall admittance implies that the non-Hermiticity of the system induces an intrinsic quantum Hall suscetptance, generating new capacitive and inductive properties that depend on the non-Hermitian parameters. In section 4 we analyze the energy band structure and determine the point- and line-gap exceptional degeneracies, together with their associated Dirac cones and Weyl nodes, which imply the existence of a topological phase. In the continuum limit we obtain the complex Berry phase of the model in the parameter space. Our conclusions are summarized in section 5. For the reader’s convenience, we present the full derivation of the complex Berry curvature and Berry phase for 2D non-Hermitian systems in the appendix.

2. Complex Berry curvature and Berry phase in the Non-Hermitian two-band model

Let us consider a non-Hermitian two-band model with translation invariance. The effective Hamiltonian is described by the 2D non-Hermitian Dirac model in momentum space [20],

$$\mathcal{H}(k) = \mathbf{h}(k) \cdot \sigma,$$

where $\mathbf{h}(k)$ is a complex function and $k$ is the generalized crystal momentum in the complex domain. $\mathbf{h}(k)$ may be regarded as a generalized Zeeman-like magnetic field and $\sigma$ is the spin 1/2 Pauli operator. Thus, the Hamiltonian is not Hermitian, i.e., $\mathcal{H}^\dagger(k) = \mathbf{h}^\ast(k) \cdot \sigma = \mathcal{H}(k)$, where $^\ast$ is the complex conjugated operator. The eigen equations of the Hamiltonian and its Hermitian adjoint are

$$\mathcal{H}\lvert \psi^R \rangle = E^R \lvert \psi^R \rangle,$$

$$\mathcal{H}^\dagger\lvert \phi^L \rangle = E^L \lvert \phi^L \rangle,$$

where the eigen vectors of $\phi^L$ and $\psi^R$ form a biorthonormal basis, \{ $\phi^L$, $\psi^R$ \}. The completeness relations for $\mathcal{H}$ and $\mathcal{H}^\dagger$ can be written as

$$I = P_+ + P_- \equiv \lvert \phi^+ \rangle \langle \phi^+ \rvert + \lvert \psi^+ \rangle \langle \psi^+ \rvert,$$

$$I = P_+^\dagger + P_-^\dagger \equiv \lvert \phi^- \rangle \langle \phi^- \rvert + \lvert \psi^- \rangle \langle \psi^- \rvert,$$

where $P_+$ and $P_-^\dagger$ are projection operators for $\mathcal{H}$ and $\mathcal{H}^\dagger$, respectively.

The spectral representations of $\mathcal{H}$ and $\mathcal{H}^\dagger$ are

$$\mathcal{H} = E^R \lvert \psi^R \rangle \langle \phi^+ \rvert + E^- \lvert \psi^- \rangle \langle \phi^- \rvert,$$

$$\mathcal{H}^\dagger = E^+ \lvert \phi^+ \rangle \langle \psi^R \rvert + E^- \lvert \phi^- \rangle \langle \psi^- \rvert.$$

Using the spectral representation of the Hamiltonian in (6) and (7), the projection operators can then be rewritten as

\[ \]
In order to explore the new physics generated by the complex Berry curvature and complex Berry phase, let us consider electric and magnetic fields applied to a condensed matter system. The electric field is perpendicular to the magnetic field. Using the fluctuation-dissipative theorem with the current-current correlation function, the current density of a filled magnetic band is given by [43]

\[ J = \frac{e}{4\pi} \int_{BZ} \mathbf{h} \cdot \left( \frac{\partial \mathbf{h}}{\partial \mathbf{k}_x} \times \frac{\partial \mathbf{h}}{\partial \mathbf{k}_y} \right) \, dk_x \, dk_y \]
where $\Omega_\kappa$ is the Berry curvature. For a 2D Hall system with a constant electric field along the $x$ direction, $E_x$ the Hall current density is defined as $J_x = \sigma_H E_x$, where the Hall conductance can be expressed as $\sigma_H = \frac{e^2}{h} \gamma_\kappa$ and $\gamma_\kappa$ is the Berry phase. Note that, for non-Hermitian systems, $\Omega^2$ is complex. Thus, the quantum Hall conductance for Hermitian systems is generalized to the quantum Hall admittance,

$$Y = \sigma_H + iB,$$

where $\sigma_H = \frac{e^2}{h} \gamma_\kappa$ is the quantum Hall conductance and $\gamma_\kappa$ is the real part of the Berry phase in (20), which corresponds to the Hermitian Hamiltonian. $B = \frac{e^2}{h} \gamma_\kappa$ is the quantum Hall susceptance and $\gamma_\kappa$ is the imaginary components of the Berry phase in (21), which comes from the non-Hermitian part of the Hamiltonian. A positive susceptance $B > 0$ implies that the system has capacitive properties and a negative susceptance $B < 0$ implies inductive properties. Thus, because of the energy bands of the non-Hermitian system are complex, the quantum Hall conductance is generalized to quantum Hall admittance.

In general, the energy bands of non-Hermitian systems are complex. This indicates the existence of non-equilibrium processes, such as energy and probability exchange between system and environment. These dissipative effects lead to an intrinsic susceptance. Interestingly, the non-Hermitian parameters can tune the susceptance to be positive or negative, which corresponds to the electric capacitive or inductive properties.

4. The non-Hermitian Dirac model

4.1. Energy band structure

As a typical example, we consider the two-dimensional (2D) non-Hermitian Dirac model, in which the Zeeman-like magnetic field $h(k) = k_x + i k_y$ is also regarded as a generalized complex crystal momentum in the first Brillouin zone (BZ), where $k = (k_x, k_y, m)$ and $\kappa = (\kappa_x, \kappa_y, \delta)$. The $\kappa$ breaks the Hermiticity and time-reversal symmetry of the Hamiltonian. The energy bands are obtained as [20]

$$E_{\pm} = \pm \sqrt{k^2 - \kappa^2 + 2ik \cdot \kappa},$$

where $k = \sqrt{k_x^2 + k_y^2 + m^2}$ and $\kappa = \sqrt{\kappa_x^2 + \kappa_y^2 + \delta^2}$. In general, the roots $E_{\pm}$ can be rewritten as the sum of the real and imaginary parts, $E_{\pm} = \pm \frac{1}{2} \sqrt{E'' + \mu E'}$, where $\mu = +1$ for $k \cdot \kappa > 0$ and $\mu = -1$ for $k \cdot \kappa < 0$, respectively, yielding

$$E' = \sqrt{k^2 - \kappa^2 + \sqrt{(k^2 - \kappa^2)^2 + 4(k \cdot \kappa)^2}}$$

$$E'' = \sqrt{-(k^2 - \kappa^2) + \sqrt{(k^2 - \kappa^2)^2 + 4(k \cdot \kappa)^2}}.$$  

In the polar representation, the the energy bands can be expressed as $E_{\pm} = \pm E^\frac{1}{4} e^{i\phi}$, where

$$E = [(k^2 - \kappa^2)^2 + 4(k \cdot \kappa)^2]^{1/4}$$

$$\phi = \arctan \left( \frac{\sqrt{1 + \eta^2}}{\sqrt{1 + \eta^2}} \right),$$

and $\eta = \frac{2k \cdot \kappa}{4(k \cdot \kappa)^2}$. The angle $\phi$ describes the phase of the energy bands. The energy bands of non-Hermitian systems are complex and the energy band structures play an important role in the quantum phase of these systems.

In general, the energy gap protects the quantum phase as the parameters vary and the energy bands deform, which can give rise to associated topological phases. The complex energy band structure leads to two kinds of energy gaps associated with the preservation of symmetry [23]. The first is called the point gap and is defined by a reference point in the complex energy plane. This obstructs the deformation of the complex energy bands and prohibits them from continuously crossing the reference point [23]. The point gap closes as the the parameters are varied which is associated with a phase transition between the trivial and non-trivial topological phases.

The second is called the line gap and is defined by a reference line in the complex energy plane. This obstructs deformation of the the complex energy bands and prevents them from continuously crossing the reference line [23]. More precisely, a point gap occurs when $\text{det} H(k) = 0$ in all BZ, so that all eigenenergies are nonzero, and the line gap occurs when $\text{det} H(k) = 0$ in all BZ and either $E^+_{\pm} = 0$ or $E^-_{\pm} = 0$ [23].

When the point gap occurs its energy bands can be continuously deformed by a unitary flattening transformation which keeps the gap structure invariant. For the line gap, the energy bands can be continuously deformed by either Hermitian or anti-Hermitian flattening transformations which also preserve the gap.
structure. These invariants imply the existence of a topological phase, by the unitary and Hermitian flattening theorems [23]. Supposing that the BZ is within the range $-1 \leq (k_x, k_y) \leq 1$ (here we ignore the factor of $\pi$ for convenience), the condition $\det H(k) \neq 0$ implies $k \neq \kappa$ or $k \cdot \kappa \neq 0$, so that a point or line gap occurs.

We now investigate the energy band structures of a few typical cases. Figure 1 shows two cases, (a) and (b), in which both real and imaginary gaps occur. In (c) there is a real band gap, but the imaginary band gap closes and inverts (has a $\pi$ phase flip), which corresponds to $k \cdot \kappa > 0$ and $k \cdot \kappa < 0$. The structure of the energy band gap is preserved under unitary or Hermitian flattening transformations and is associated with the topological phase [23].

When the band gaps close as the parameters vary, the topological phases reduce to the trivial phases. Thus, the gap closure in the BZ corresponds to the existence of exceptional degenerate states. The critical points or curves of the topological phase transition then satisfy two equations,

$$k_x^2 + k_y^2 - \kappa_x^2 - \kappa_y^2 = \delta^2 - m^2$$
(29)

$$k_x\kappa_x + k_y\kappa_y = -m\delta.$$  
(30)

When the parameters satisfy equations (29) and (30) both energy band gaps, $E^r_k$ and $E^i_k$, close and the topological phases disappear [23]. We tune the parameters and show a few typical cases in figure 2. In figures 2(a) and (b), we see that the real energy band generates a Dirac cone and that the imaginary bands invert along the line $k_y = k_\Sigma$ in the BZ, for $\delta = 0$. Conversely, in figure 2 (c) and (d), the real band gap closes and inverts along the line $k_y = k_{\Sigma}$ for $m = 0$. The imaginary band touches near to the $\Sigma$ points but opens near the $\Gamma$ point in the BZ, as expected from equation (29). These gap closures corresponding to exceptional degenerate states play a reference-state role for the point or line gap to associate with the topological phase [23]. Consequently, the necessary condition for the existence of the band gap is $m\delta \neq 0$.

We plot a few particular cases in figure 3 to illustrate the simultaneous solutions of the equation (29), where the energy band gaps close:

(a) When $\kappa_x = 0$, $k_x = -m\delta/\kappa_x$, the gap closes at $(k_x, k_y) = (-m\delta/\kappa_x, 0)$, which corresponds to two lines $(k_x = -m\delta/\kappa_x > 0 \text{ or } < 0)$ parallel to the $k_x$ axis.

(b) When $\kappa_x = \kappa_y$, the gap closes at $(k_x, k_y) = (0, \pm m\delta/\kappa)$ and $(k_x, k_y) = (\pm m\delta/\kappa_x, 0)$. These two parallel lines are at an angle of $3\pi/4$, relative to the $k_x$ axis.
When \( m = 0 \) or \( \delta = 0 \), the real band gap closes at \( (k_x, k_y) = \frac{\sqrt{2}}{2} (\kappa^2 - m^2, \kappa^2 - m^2) \).

Interestingly, both the real and imaginary energy bands could change by a \( \pi \) phase (band inversion) at the line \( \mathbf{k} \cdot \mathbf{\kappa} = 0 \). This is the critical line that demarcates the transition between energy-gain and energy-loss and which acts as a reference line associated with the topological phase of the non-Hermitian system.

Figure 2. Online color: The complex energy band in the BZ. In (a), the real part of the band exhibits a Dirac cone. In (b) the imaginary parts of the band inverts. In (c) the real bands invert along the line \( k_y = k_x \), corresponding to the angle \( \frac{\pi}{4} \) in BZ. This is a Weyl node corresponding to \( m = 0 \). In (d) the imaginary part of the band closes at the \( \Sigma \) point in the BZ.

Figure 3. Online color: The typical cases of energy gap closure in the BZ. The cross points of the circle and lines are the critical points that enable the energy gap to close when the parameters satisfy equation (29).
4.2. Complex Berry connection, Berry curvature, and Berry phase

The simultaneous eigen equations of the 2D non-Hermitian Dirac model are written as

$$\mathcal{H}|\psi^{(R)}_g\rangle = E^g_\pm |\psi^{(R)}_g\rangle, \quad \mathcal{H}|\phi^{(L)}_g\rangle = E^g_\pm |\phi^{(L)}_g\rangle,$$

where the normalized eigenstates are given by

$$|\psi^{(R)}_\pm\rangle = \frac{1}{2E^g_\pm (E^g_\pm + h_\pm)} \left( \begin{array}{c} h_x - i h_y \\ E^g_\pm - h_\pm \end{array} \right),$$

$$|\phi^{(L)}_\pm\rangle = \frac{1}{2E^g_\pm (E^g_\pm + h_\pm)} \left( \begin{array}{c} h_x^* - i h_y^* \\ E^g_\pm - h_\pm^* \end{array} \right).$$

These form a complete biorthonormal basis, \{\psi^{(R)}_g, \phi^{(L)}_g\}, where \langle \phi^{(L)}_\alpha | \psi^{(R)}_\beta \rangle = \delta_{\alpha\beta} and \alpha, \beta = \pm .

The complex Berry connection is defined as

$$A_{\pm} = \langle \phi^{(L)}_\pm | \nabla_k \mathcal{H} |\psi^{(R)}_\pm\rangle \cdot dk$$

giving

$$A_{\pm} = -h_d dk_x + \frac{h_d dk_y}{2E^g_\pm (E^g_\pm + h_\pm)}.$$

Note that, since \(\hbar \cdot \left( \frac{\partial}{\partial k_x} \times \frac{\partial}{\partial k_y} \right) \frac{h_d}{E^g_\pm}(E^g_\pm + h_\pm)\), we obtain the complex Berry curvatures for the occupied band as

$$\Omega^x = \frac{h_x}{2E^g_\pm}, \quad \Omega^y = \frac{h_y}{2E^g_\pm},$$

where \(E = \sqrt{k^2 + \kappa^2 + 2ik \cdot \kappa}\). After some algebra, the complex Berry curvatures can be simplified to

$$\Omega_x^x = \frac{mE^3 - \delta E^{13} - 3mE^2E^1 + 3\delta E^3E^2}{E^3},$$

$$\Omega_y^y = \frac{mE^3 + \delta E^{13} - 3mE^2E^1 - 3\delta E^3E^2}{E^3}.$$

It can be seen that both the real and imaginary Berry curvatures acquire a singularity at the exceptional point \(E = 0\). We now investigate the behavior of the complex Berry curvature at the exceptional points. When the parameters satisfy the circle equation (29), \(k_x^2 + k_y^2 = \kappa^2 - m^2\), the complex Berry curvature is reduced to

$$\Omega_x^x = \Omega_y^y = -\frac{m + \delta}{4(k \cdot \kappa)^{3/2}}.$$\hspace{1cm}(38)

Consequently, when \(m = \delta\), the Berry curvature vanishes, \(\Omega_x^x = \Omega_y^y = 0\). However, when \(m = \delta\) and \(k \cdot \kappa = 0\), both real and imaginary parts of the complex Berry curvature are divergent. The exceptional points of the solution of equation (29) represent singularities in the Berry curvature, which indicates the emergence of the topological phase.

When \(k \cdot \kappa = 0\), or, equivalently, \(\phi = 0\), the complex Berry curvature is reduced to

$$\Omega^x = \frac{m + i\delta}{2(k^2 - \kappa^2)^{3/2}}.$$

Similarly, at the exceptional points, i.e., the cross points between the circle and lines in (29), both the real and imaginary parts of the complex Berry curvature are divergent. When \(\kappa = \delta = 0\), the complex Berry curvature is real, \(\Omega^x = \frac{m}{2m}\), which is consistent with the previous results obtained in [6, 10]. The exceptional degeneracy plays the role of a defect in the energy band, forming a topological phase labeled by winding number, so that the topological phase depends on the non-Hermitian parameters.

The energy band structures near the exceptional points play an important role. The energy of Bloch electrons is no longer conserved for non-Hermitian systems, which yields some interesting physical properties.

To explore the low-energy topological properties of the model, we study the Berry phase of the model in the continuum limit of the long-wavelength regime, i.e., we extend the integral region to infinity. Thus, we obtain the Berry phase as

$$\gamma_\omega = -\frac{i}{2\pi} \frac{m + i\delta}{(m + i\delta)^2} \ln \left( \begin{array}{c} \kappa_x + m + i\delta \\ \kappa_x - m - i\delta \end{array} \right)$$

$$+ \ln \left( \begin{array}{c} -\kappa_y - (m + i\delta)(\kappa_x - (m + i\delta)) + i\kappa_y \sqrt{\kappa_x^2 - \kappa_y^2 + (m + i\delta)^2} \\ -\kappa_y + (m + i\delta)(\kappa_x + (m + i\delta)) + i\kappa_y \sqrt{\kappa_x^2 - \kappa_y^2 + (m + i\delta)^2} \end{array} \right),$$

where the factor on the left-hand side of the logarithm turns out to be \(\pm 1\), depending on the values of \(m\) and \(\delta\). In general, the Berry phase is complex. We now analyze some special cases.
The Berry phase becomes complex and depends on the non-Hermitian parameters. The quantum Hall conductance depends on the real part of the Berry phase, whereas the quantum Hall susceptance depends on the imaginary part. The positive imaginary part of the Berry phase implies that the system exhibits the electric conductance depends on the real part of the Berry phase, whereas the quantum Hall susceptance depends on the imaginary part.

When $\kappa_x = 0$ or $\kappa_y = 0$, the Berry phase becomes

$$
\gamma^- = \frac{m + i\delta}{2\sqrt{(m + i\delta)^2}} = \begin{cases} 
\frac{1}{2} \frac{m}{|m|}, & \text{for } \delta = 0 \\
\frac{1}{2} \frac{\delta}{|\delta|}, & \text{for } m = 0.
\end{cases}
$$

(41)

The Berry phase reduces to $\gamma^- = \pm \frac{1}{2}$ once $\kappa_x = 0$ or $\kappa_y = 0$, which is consistent with the previous results obtained in [6, 10, 44]. Interestingly, for either $\kappa_x = 0$ or $\kappa_y = 0$, the Berry phase is either positive or negative one-half, depending on the signs of $m$ and $\delta$, whereas it is independent of the values of $m$ and $\delta$. This is a topological property of quantum Hall effect, even though some non-Hermitian effects also exist.

When $m = 0$, the Berry phase can be expressed as

$$
\gamma^- = \frac{1}{\pi |\delta|} \left[ \arctan \left( \frac{\delta}{\kappa_x} \right) + \arctan \left( \frac{\delta \kappa_x}{\kappa_y^2 + \delta^2 + \kappa_x \sqrt{\kappa_y^2 + \kappa_x^2 + \delta^2}} \right) \right].
$$

(42)

The Berry phase becomes real and depends on the non-Hermitian parameters. When $\kappa_x \rightarrow \pm 0$ the Berry phase shows a step with $\pm \frac{1}{2}$, which is associated with a topological phase transition. When the real energy gap closes ($m = 0$) the quantum Hall effect depends on the non-Hermitian parameters.

When $\delta = 0$, the Berry phase can be obtained as

$$
\gamma^- = \begin{cases} 
\frac{i}{2} \frac{m}{|m|} \ln \left( \frac{\kappa_x + m}{\kappa_x - m} \right) + \ln \left( \frac{\kappa_x^2 + m(\kappa_x - m) + \kappa_y \sqrt{\kappa_x^2 + \kappa_y^2}}{\kappa_x^2 - m(\kappa_x - m) + \kappa_y \sqrt{\kappa_x^2 + \kappa_y^2}} \right), & \text{for } \kappa_x^2 + \kappa_y^2 \geq m^2 \\
\frac{i}{2} \frac{m}{|m|} \ln \left( \frac{\kappa_x + m}{\kappa_x - m} \right) + \ln \left( \frac{\left|m^2(m^2 - \kappa_x^2 - \kappa_y^2) + 4m^2\kappa_y^2\kappa_x^2(m^2 - \kappa_x^2 - \kappa_y^2)\right|}{\kappa_x^2 - m(\kappa_x + m)\kappa_y^2 + \kappa_x\kappa_y \sqrt{m^2(m^2 - \kappa_x^2 - \kappa_y^2)}} \right) & \text{for } \kappa_x^2 + \kappa_y^2 < m^2.
\end{cases}
$$

(43)

The Berry phase becomes complex and depends on the non-Hermitian parameters. The quantum Hall conductance depends on the real part of the Berry phase, whereas the quantum Hall susceptance depends on the imaginary part. The positive imaginary part of the Berry phase implies that the system exhibits the electric capacity and the negative imaginary part indicates electric induction.

In general, the quantum Hall conductance depends on the Berry phase in Hermitian systems. For non-Hermitian systems, the Berry phase becomes complex, so that the quantum Hall conductance is extended to the complex domain, yielding admittance (23).

Figure 4 shows the complex Berry phase in the parameter space based on (40), in which we set $\kappa_x = \kappa_y$ and $m = \delta$ for convenience, as some typical cases. The real part of the Berry phase is positive (red part) and negative (blue part), for positive (negative) values of $\kappa_x(\kappa_y)$, whereas, conversely, the imaginary components of the complex Berry phase are negative (positive) for positive (negative) values of $\kappa_x(\kappa_y)$. These results tell us the
relationship between the non-Hermitian parameters and quantum Hall admittance, in which the positive and negative imaginary Berry phases correspond to the electric capacitive and inductive properties, respectively. This implies that the electric properties of non-Hermitian systems vary between capacitative and inductive regimes, according to the values of the non-Hermitian parameters. This gives us an important clue about how to develop potential applications in nanoelectronics.

It should be remarked that even though the bulk states characterized by the complex Berry phase cannot capture directly the edge modes due to the non-Hermiticity of systems breaking the bulk-boundary correspondence for Hermitian systems, the bulk-boundary correspondence for non-Hermitian systems can be recovered by a similarity transformation \[30, 35\]. In other words, the bulk states characterized by the complex Berry phase can still trace the edge modes for non-Hermitian systems.

5. Conclusions and outlooks

In summary, the geometric and topological properties of non-Hermitian systems exhibit novel quantum phenomena \[20, 22, 23\]. The exceptional degeneracy of the energy bands indicates rich new physics which is of potential relevance for both fundamental concepts and applications \[11–14\]. In particular, the quantum states of non-Hermitian systems can be classified into different topological classes based on symmetry and topology \[20, 22, 23\]. A natural question is, therefore, what new physics underlies the existence of the topological phase in non-Hermitian systems?

We have proposed complex Berry curvatures for the non-Hermitian Hamiltonian and its Hermitian adjoint and used these to reveal new phenomena in non-Hermitian systems. We have given the complex Berry curvature and Berry phase for the 2D non-Hermitian Dirac model and found that the imaginary part of the Berry phase can induce quantum Hall susceptance. This demonstrates that the quantum Hall conductance is generalized to quantum Hall admittance. The quantum Hall conductance was found to deviate from a quantized value in a specific non-Hermitian Dirac Model \[41\]. Our results further extend the fractional quantized Hall conductance to the complex domain, namely to quantum Hall admittance. When the non-Hermitian parameter vanishes the quantum Hall admittance reduces to the standard quantum Hall conductance, as required, which is consistent with previous results \[6, 10, 44\].

Our analysis shows that non-Hermiticity can induce intrinsic electric capacitance or inductance, which is associated with positive or negative values of the electric susceptance, respectively. The latter depends on the values of the non-Hermitian parameters. Interestingly, the intrinsic quantum Hall susceptance induced by the non-Hermitian effects is beyond the common electric susceptance realized by device design (e.g. in capacitors or inductors). This hints at the existence of new fundamental physics in non-Hermitian quantum mechanics, which is yet to be explored, and is potentially useful for applications.

The quantum transport problem in non-Hermitian systems has been studied as a nonequilibrium stochastic dynamics \[24, 45\]. Non-equilibrium stochastic currents in non-Hermitian systems exhibit nontrivial topological phases and their topological phase transitions can be classified by braid groups, together with the winding number \[24, 45\]. The quantized and fractionally-quantized currents in stochastic systems depend on the vector potential \[24\], which may be connected to the quantum Hall admittance we have revealed in the non-Hermitian Dirac model. The quantum transport in non-Hermitian systems manifests interesting geometric and topological properties, which promises potential applications.

As a typical example, we have analyzed the complex energy band structures of the 2D non-Hermitian Dirac model and determined the energy band gaps together with their corresponding Dirac cones and Weyl nodes. The exceptional degeneracy of the energy bands in the BZ was investigated, and found to be associated with the topological phase.

These results demonstrate the predictions of the point and line gaps associated with the topological phase, based on the unitary flattening and Hermitian flattening theorems for point- and line-gaps in non-Hermitian systems \[23\]. In the continuum limit, we determined the complex Berry phase in the parameter space. The quantized and non-analytic behaviors of the Berry phase in the parameter space are associated with topological phases and quantum phase transitions. These findings reveal novel phenomena in non-Hermitian systems and inspire further exploration of their application in designing quantum devices.

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Appendix

A.1. Derivation of Berry curvature of non-Hermitian systems

Let us consider a generic non-Hermitian quantum system. Its Hamiltonian $\mathcal{H} \equiv \mathcal{H}$ acts on an N-dimensional separable Hilbert space. The corresponding Schrödinger equations are:

$$i\hbar \frac{\partial}{\partial t}|\Psi\rangle = \mathcal{H}|\Psi\rangle$$  \hspace{1cm} (44)

$$i\hbar \frac{\partial}{\partial t}|\Phi\rangle = \mathcal{H}'|\Phi\rangle.$$  \hspace{1cm} (45)

For the 2D non-Hermitian Dirac model in the crystal momentum space [20], we have

$$\mathcal{H}(k) = \hbar (k) \cdot \sigma,$$  \hspace{1cm} (46)

where is the generalized Zeeman magnetic field and $\sigma$ is the spin-1/2 Pauli operator. In the adiabatic approximation, the simultaneous eigen equations are written as

$$\mathcal{H}|\psi_{\pm}^R\rangle = E_{\pm}|\psi_{\pm}^R\rangle, \quad \mathcal{H}'|\phi_{\pm}^L\rangle = E_{\pm}^L|\phi_{\pm}^L\rangle.$$  \hspace{1cm} (47)

The sequences of eigenvectors $\{\psi_{\pm}^R, \phi_{\pm}^L\}$ form a complete biorthonormal basis, $\langle \phi_{\alpha}^L|\psi_{\beta}^R\rangle = \delta_{\alpha\beta}$, where $\alpha, \beta = \pm$. The completeness relation for $\mathcal{H}$ is expressed as

$$I = P_+ + P_-,$$  \hspace{1cm} (48)

where $P_{\pm} = |\psi_{\pm}^R\rangle \langle \phi_{\pm}^L|$ are the projection operators. The wave function can be expanded as

$$|\Psi\rangle = c_+|\psi_{+}^R\rangle + c_-|\psi_{-}^R\rangle.$$  \hspace{1cm} (49)

Substituting the wave function to the Schrödinger equation (44) and taking the inner product by acting with the bra $\langle \phi_{\pm}\rangle$, we have

$$\frac{d}{dt} (c_+^* c_-) = \begin{pmatrix} iA_{++} + \frac{1}{i\hbar}H_{++} & iA_{+-} + \frac{1}{i\hbar}H_{+-} \\ iA_{-+} + \frac{1}{i\hbar}H_{-+} & iA_{--} + \frac{1}{i\hbar}H_{--} \end{pmatrix} (c_+ c_-)^T.$$  \hspace{1cm} (50)

where

$$A_{\pm\pm} = i \begin{pmatrix} \phi_{\pm}^L \frac{d}{dt} \psi_{\pm}^R \end{pmatrix}, \quad H_{\pm\pm} = \langle \phi_{\pm}^L|\mathcal{H}|\psi_{\pm}^R\rangle.$$  \hspace{1cm} (51)

In the adiabatic approximation, the off-diagonal elements tend to zero, $\langle \phi_{\pm}^L|\mathcal{H}|\psi_{\pm}^R\rangle \rightarrow 0$ and $\langle \phi_{\pm}^L|\mathcal{H}'|\psi_{\pm}^R\rangle \rightarrow 0$. The adiabatic evolution of states as time varies in the range $0 < t \leq \tau$ can be mapped to the boundary of the BZ, $-1 \leq (k_x, k_y) \leq 1$. The wave function of the Bloch electron states can be expressed as

$$|\psi_{\pm}^R(\tau)\rangle = c_{\pm}(0)e^{i\gamma_{\pm}^R}e^{i\Omega_{\pm}^R}|\psi_{\pm}^R(0)\rangle,$$  \hspace{1cm} (52)

where $\gamma_{\pm}^R$ is the complex Berry phase and $\gamma_{\pm}^D$ is the complex dynamical phase,

$$\gamma_{\pm}^R = \oint_{\partial \Omega_{\pm}} A_{\pm}, \quad \text{and} \quad \gamma_{\pm}^D = \int_{0}^{\tau} E_{\pm} \frac{d\tau}{\tau},$$  \hspace{1cm} (53)

Here, $A_{\pm} = i \langle \phi_{\pm}^L|d|\psi_{\pm}^R\rangle$ is the generalized Berry connection for the non-Hermitian system. The imaginary parts of $\gamma_{\pm}^R$ and $\gamma_{\pm}^D$ describe the phenomena of energy dissipation and the non-conservation of probability. The complex Berry curvature is defined as $\Omega_{\pm} = dA_{\pm}$, yielding

$$\Omega_{\pm} = i (d\phi_{\pm}^L \wedge |d\psi_{\pm}^R\rangle) = \Omega_{\pm}^x dk_x \wedge dk_y,$$  \hspace{1cm} (54)

where

$$\Omega_{\pm}^x = i [d\phi_{\pm}^L \wedge |d\psi_{\pm}^R\rangle] = \Omega_{\pm}^x dk_x \wedge dk_y,$$  \hspace{1cm} (55)

Note that $\partial_{\pm} = (\partial_{k_x}, \partial_{k_y})$ is the derivative operator, so that

$$[\partial_{k_x} \phi_{\pm}^L, \partial_{k_y} \phi_{\pm}^L] = [\partial_{k_x} \psi_{\pm}^R \phi_{\pm}^L, \partial_{k_y} \phi_{\pm}^L \psi_{\pm}^R] = [\partial_{k_y} \phi_{\pm}^L, \partial_{k_x} \psi_{\pm}^R \phi_{\pm}^L] + [\partial_{k_y} \psi_{\pm}^R \phi_{\pm}^L, \partial_{k_x} \phi_{\pm}^L]$$.  \hspace{1cm} (56)

and $\partial_{k_x} \langle \phi_{\pm}^L|\psi_{\pm}^R\rangle = 0$, where $\alpha = x, y$. The trace is defined as $\text{Tr}(\cdot) = \sum_{\pm} \langle \phi_{\pm}^L|\psi_{\pm}^R\rangle$ for the biorthonormal basis. We then have
Thus, the Berry curvature can be rewritten as\cite{46}
\[
\Omega^ \pm_\mathbf{k} = i \text{Tr}(P_\mathbf{k} [\partial_\mathbf{k} P_\mathbf{k}, \partial_\mathbf{k} \psi^\pm_\mathbf{k}]).
\] (58)
In the spectral representation, the Hamiltonian is expressed as
\[
\mathcal{H} = E_+ |\psi^R_+\rangle \langle \phi^L_+ | + E_- |\psi^R_-\rangle \langle \phi^L_- |.
\] (59)
Using the completeness relation \cite{48} and the spectral representation of the Hamiltonian \cite{59}, the projection operators can be rewritten as
\[
P_\pm = \frac{1}{2} (I \pm \hbar \cdot \sigma).
\] (60)
where $\hbar = \frac{h}{\hbar}$ is the dimensionless crystal momentum. Note that $\partial_\mathbf{k} P_\mathbf{k} = \pm \frac{1}{2} \hbar \hbar \cdot \sigma$. Using the mathematical identity $(\mathbf{a} \cdot \sigma)(\mathbf{b} \cdot \sigma) = \mathbf{a} \cdot \mathbf{b} + i \sigma \cdot (\mathbf{a} \times \mathbf{b})$ we obtain the Berry curvature,
\[
\Omega^ \pm_\mathbf{k} = \hbar \cdot \left( \frac{\partial \hbar}{\partial k_x} \times \frac{\partial \hbar}{\partial k_y} \right)
\] (61)
and the Berry phase for the non-Hermitian Dirac model is
\[
\gamma = \frac{1}{4\pi} \int d^2 k \hbar \cdot \left( \frac{\partial \hbar}{\partial k_x} \times \frac{\partial \hbar}{\partial k_y} \right) dk_x dk_y.
\] (62)
The Berry curvature and Berry phase for the Hermitian adjoint of the Hamiltonian can be obtained in a similar way.

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