Abstract

The negative dimensional integration method (NDIM) is a technique where several difficulties concerning loop integration can be overcome. From usual covariant gauges to complicated Coulomb gauge integrals, and even the trickiest light-cone integrals one can apply the methodology of NDIM. In this work we show how to construct a general formula — we mean arbitrary exponents of propagators, off-shell external momenta and distinct massive propagators — for one-loop scalar integrals, for covariant gauges, and apply it to one through six-point loop integrals. We present detailed analysis of pentagon and hexagon scalar integrals for massive/massless internal particles being external momenta on or off mass shell.
I. INTRODUCTION

Important mathematical methods have been required to evaluation of the complex Feynman integrals in the calculations of scattering amplitudes in QED and QCD, in the radiative corrections, study of Green function behavior, renormalization group and others problems in quantum field theory. The integration using Mellin-Barnes representation [1, 2, 3], the Gegenbauer polynomial technique [4], integration by parts [5], negative dimensional integration (NDIM) [6, 7], string inspired methods [8], differential equation approach [9] and several others [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21], are some of the technique that have been currently used.

In the present article we make use of the NDIM [6, 7]. Its implementation transfer the complexities of the performing $D$-dimensional Feynman integrals to resolution of a system of linear algebraic equations that we call system of constraint equations. Therefore, when we choose one solution to those constraint equations we are obtaining one specific solution to Feynman integral in question that is defined in a specific kinematic region. The others solutions to system constraint equations represent the possibilities of analytic continuation of such solution. In other words, the NDIM solve the Feynman integral as well as obtain solving the constraint system equation, all the analytic continuation possibilities to the solution found. We note that after arrive the final formula to Feynman integral is necessary yet carry out other but straightforward analytic continuation, now to negative value of the powers of the propagators in the scalar integral.

We implement the NDIM in this paper to obtain the one-loop general massive $n$-point function. As already mentioned above we obtain, by solving the system constraints equation, all the other solutions analytically continued and show that the number of this possibilities is a function of number of external lines $n$. We present then applications to cases of 1 to 6-point functions, considering only scalar integrals since any tensorial integral can be reduced to scalar integral according to [22]. The hypergeometric series we choose to present in this work are such that can be used in the dimensional or analytic regularization schemes [23, 24, 25], always preserving gauge symmetry, a fact well-known from quantum field theory. Similar results for N-point scalar integrals were obtained in [1, 2] using Mellin-Barnes approach, however Davydychev quote only the hypergeometric series he call "symmetric". Here we write such results and several others, an interesting feature of NDIM, providing a very large
number of hypergeometric series that represent the original Feynman integral.

This work is organized as follows. In the early section we present a detailed approach to the implementation of the NDIM to one-loop n-point function. In the subsequent sections we have applied the results from section one, the general formula, starting from one-point integrals, two, three, four (box), pentagon and finally hexagon scalar integrals. These results are exact, i.e., no approximations was made. The solutions obtained in this sections are given in terms of the hypergeometric functions (see appendix) and compared with known ones, when they are available, in the literature.

II. ONE-LOOP N-POINT FUNCTION: GENERAL FORMULA

In this section we present the calculations to evaluate the one-loop scalar integrals with the NDIM. Consider a one-loop Feynman diagram with \( n, n = 1, 2, ..., \) internal momenta \( l_0, l_0 - l_1, l_0 - l_2, ..., l_0 - l_{n-1} \) and masses \( m_0, m_1, m_2, ..., m_{n-1} \), where \( l_1, l_2, ..., l_{n-1} \) are given in terms of a linear combination of the external momenta. Its scalar integral associated is

\[
\int \frac{d^Dl_0}{[l_0^2 - m_0^2]^\alpha_0[(l_0 - l_1)^2 - m_1^2]^\alpha_1[...[(l_0 - l_{n-1})^2 - m_{n-1}^2]^\alpha_{n-1}]. \tag{1}
\]

Consider now the gaussian integral

\[
I = \int d^Dl_0 \exp\{-\alpha_0[l_0^2 - m_0^2] - \sum_{i=1}^{n-1} \alpha_i[(l_0 - l_i)^2 - m_i^2]\}. \tag{2}
\]

where \( \alpha_0, \alpha_i \) are positive parameters. Then, the exponential function above can be expanded and we get

\[
I = \sum_{\alpha_0, ..., \alpha_{n-1}=0}^{\infty} (-1)^{\sum_{i=0}^{n-1} \alpha_i} \frac{\alpha_0^{\alpha_0} \alpha_1^{\alpha_1} ... \alpha_{n-1}^{\alpha_{n-1}}}{\alpha_0! \alpha_1! ... \alpha_{n-1}!} \times \int d^Dl_0[l_0^2 - m_0^2]^\alpha_0[(l_0 - l_1)^2 - m_1^2]^{\alpha_1}...[(l_0 - l_{n-1})^2 - m_{n-1}^2]^{\alpha_{n-1}}. \tag{3}
\]

Using the definition

\[
\alpha = \alpha_0 + \sum_{i=1}^{n-1} \alpha_i, \tag{4}
\]

we can rewrite the \( I \) integral of form

\[
I = \int d^Dl_0 \exp\{-\alpha[l_0^2 - \sum_{i=1}^{n-1} \alpha_i l_0 \cdot l_i + \sum_{i=1}^{n-1} \alpha_i l_i^2 + \sum_{i=0}^{n-1} \alpha_i m_i^2]\} + \frac{(\sum_{i=1}^{n-1} \alpha_i l_i)^2}{\alpha} - \sum_{i=1}^{n-1} \alpha_i l_i^2 + \sum_{i=0}^{n-1} \alpha_i m_i^2}. \tag{5}
\]
After evaluate the integral we have

\[
I = \left(\frac{\pi}{\alpha}\right)^{D/2} \exp\left\{ \frac{\left(\sum_{i=1}^{n-1} \alpha_i l_i\right)^2}{\alpha} - \sum_{i=1}^{n-1} \alpha_i l_i^2 + \sum_{i=0}^{n-1} \alpha_i m_i^2 \right\}.
\]  

(6)

Yet can us rewritten this expression of form

\[
I = \left(\frac{\pi}{\alpha}\right)^{D/2} \exp\left\{ \sum_{i>j=1}^{n-1} \alpha_i \alpha_j l_i \cdot l_j - \alpha_0 \sum_{i=1}^{n-1} \alpha_i l_i^2 - \sum_{i=1}^{n-1} \alpha_i \alpha_j l_i^2 \right\}
+ \sum_{i=0}^{n-1} \alpha_i m_i^2 \right\}.
\]  

(7)

Also can be show that

\[
\sum_{i,j=1}^{n-1} \alpha_i \alpha_j l_i \cdot l_j = 2 \sum_{i>j=1}^{n-1} \alpha_i \alpha_j l_i \cdot l_j + \sum_{i=1}^{n-1} \alpha_i^2 l_i^2,
\]  

(8)

\[
\sum_{i,j=1}^{n-1} \alpha_i \alpha_j l_i^2 = \sum_{i>j=1}^{n-1} \alpha_i \alpha_j (l_i^2 + l_j^2) + \sum_{i=1}^{n-1} \alpha_i^2 l_i^2.
\]  

(9)

Performing the substitution of this results in (33), we get

\[
I = \left(\frac{\pi}{\alpha}\right)^{D/2} \exp\left\{ -\sum_{i>j=1}^{n-1} \alpha_i \alpha_j l_{ij}^2 - \alpha_0 \sum_{i=1}^{n-1} \alpha_i l_i^2 \right\}
+ \sum_{i=0}^{n-1} \alpha_i m_i^2 \right\}
\]  

(10)

where \(l_{ij} = l_i - l_j\). From exponential argument above we have \(w = \frac{n^2-3n+2}{2}\) terms with different coefficients \(\frac{\alpha_i \alpha_j}{\alpha}\), namely

\[
\alpha_1 \alpha_2, \alpha_1 \alpha_3, \alpha_1 \alpha_4, \ldots \alpha_1 \alpha_{n-1},
\]

\[
\alpha_2 \alpha_3, \alpha_2 \alpha_4, \ldots \alpha_2 \alpha_{n-1},
\]

\[
\alpha_3 \alpha_4, \ldots \alpha_3 \alpha_{n-1},
\]

\[
\ldots \alpha_{n-2} \alpha_{n-1}.
\]

Also, there are \(n - 1\) terms with different coefficients \(\alpha_0 \alpha_i\) and \(n\) terms to coefficients \(\alpha_i\). This result in \(w + 2n - 1 = \frac{n^2+n}{2}\) terms with different coefficients. After expansion of the exponential above we can write

\[
I = \left(\frac{\pi}{\alpha}\right)^{D/2} \sum_{j_1,\ldots,j_{w+2n-1}}^{\infty} \frac{1}{\alpha_{j_1+j_2+\ldots+j_w+n-1}}
\]

\[
\times \alpha_0^{j_1+j_2+\ldots+j_n-1} \alpha_1^{j_1+j_n+\ldots+j_{2n-2}}
\]
\begin{align*}
&\times \alpha_2^{j_2+j_n+j_{2n-2}+\ldots+j_{3n-4}} \ldots \alpha_{n-1}^{j_{2n-2}+j_{3n-4}+j_{4n-7}+\ldots+j_w}
&\times \frac{(-l_2^2)^{j_1}}{j_1!} \frac{(-l_2^2)^{j_2}}{j_2!} \ldots \frac{(-l_2^2)^{j_{n-1}}}{j_{n-1}!}
&\times \frac{(-l_2^2)^{j_n}}{j_n!} \frac{(-l_2^2)^{j_{n+1}}}{j_{n+1}!} \ldots \frac{(-l_2^2)^{j_w+n-1}}{j_{w+n-1}!}
&\times \frac{(m_0^2)^{j_w+n}}{j_w+n!} \frac{(m_2^2)^{j_w+n+1}}{j_{w+n+1}!} \ldots \frac{(m_2^2)^{j_w+2n-1}}{j_{w+2n-1}!}.
\end{align*}

(11)

If we take the multinomial expansion in the exponents of the \( \alpha \), given by (6), we get

\[ \frac{1}{\alpha^{D/2+j_1+j_2+\ldots+j_{w+n-1}}} = \frac{1}{[\alpha_0 + \sum_{i=1}^{n-1} \alpha_i]^{D/2+j_1+j_2+\ldots+j_{w+n-1}}} = \sum_{j_w+2n, \ldots, j_w+3n-1} \Gamma(1-D/2-j_1-j_2-\ldots-j_{w+n-1}) \times \frac{\alpha_0^{j_w+2n}}{j_w+2n!} \frac{\alpha_1^{j_w+n+1}}{j_{w+n+1}!} \ldots \frac{\alpha_{n-1}^{j_w+3n-1}}{j_{w+3n-1}!} .
\]

(12)

with the constraint

\[ D/2 = -j_1 - j_2 - \ldots - j_{w+n-1} - j_{w+2n} - j_{w+2n+1} - \ldots - j_{w+3n-1} . \]

(13)

Performing the substitution of this expression in (11) and compare the exponents of the parameters \( \alpha_i \) in (3), we have follow constraint equations

\[ a_0 = j_1 + j_2 + \ldots + j_{n-1} + j_{w+n} + j_{w+2n}, \]

(14)

\[ a_1 = j_1 + j_n + \ldots + j_{2n-2} + j_{w+n+1} + j_{w+2n+1}, \]

(15)

\[ a_2 = j_2 + j_n + j_{2n-2} + \ldots + j_{3n-6} + j_{w+n+2} + j_{w+2n+2}, \]

(16)

\[ a_3 = j_3 + j_{n+1} + j_{2n-2} + j_{3n-5} + \ldots + j_{4n-10} + j_{w+n+3} + j_{w+2n+3} \]

(17)

\[ \ldots \]

\[ a_{n-1} = j_{n-1} + j_{2n-2} + j_{3n-6} + j_{4n-10} + \ldots + j_{w+n-2} + j_{w+2n} + j_{w+3n-1} \]

(18)

\[ D/2 = -j_1 - j_2 - \ldots - j_{w+n-1} - j_{w+2n} - j_{w+2n+1} - \ldots - j_{w+3n-1} . \]

(19)

Using the results we have \( S = w + 3n - 1 = \frac{n^2 + 3n}{2} \) sums with \( n + 1 \) constraint equations and \( C_{S,n+1} \) different forms to evaluate. But, if there are \( F_i \) internal lines or \( F_e \) external lines
associated to no massive fields, the number of the sums reduce to \( S - F_i - F_e \). Then, the number of the different forms to perform the \( S - F_i - F_e \) sums is

\[
C_{S-F_i-F_e,n+1} = \frac{(n^2+3n - F_i - F_e)!}{(n+1)!(\frac{n^2+n-2}{2} - F_i - F_e)!}. \tag{20}
\]

Finally we obtain the final expression to integral given in (3), that is

\[
J(n) = J(n)(D, a_0, ..., a_{n-1}, l_1, l_2, ..., l_{n-1}, m_0, m_1, ...m_{n-1})
\]

\[
= \int d^D l_0 l_0^2 - m_0^2 \alpha_0 ((l_0 - l_1)^2 - m_1^2) ... ((l_0 - l_{n-1})^2 - m_{n-1}^2)^{a_{n-1}}
\]

\[
= \pi^{D/2} (-1)^{\sum_{i=0}^{n-1} a_i} \Gamma(1 + a_0) \Gamma(1 + a_1) ... \Gamma(1 + a_{n-1})
\]

\[
\times \sum_{j_1,...,j_S} \Gamma(1 - D/2 - j_1 - j_2 - ... - j_{w+n-1})
\]

\[
\times \frac{(-l_{12}^2)^{j_1} (-l_{23}^2)^{j_2} ... (-l_{n-1,n-2}^2)^{j_{n-1}}}{j_1! j_2! ... j_{n-1}!}
\]

\[
\times \frac{(-l_{12}^2)^{j_n} (-l_{23}^2)^{j_{n+1}} ... (-l_{n-1,n-2}^2)^{j_{w+n-1}}}{j_n! j_{n+1}! ... j_{w+n-1}!}
\]

\[
\times \frac{(m_0^2)^{j_{w+n}} (m_1^2)^{j_{w+n+1}} ... (m_{n-1})^{j_{w+2n-1}}}{j_{w+n}! j_{w+n+1}! ... j_{w+2n-1}!}.
\] \tag{21}

This expression only represent the one-loop n-point function after the analytic continuation in the parameters \( a_0, a_1, ..., a_{n-1} \) to negative value.

### III. ONE-POINT FUNCTION

One-point functions at one-loop level are the simplest Feynman loop integrals and we start with them for completeness. The integral associated to one-loop one-point function, case \( n = 1 \), given by

\[
J^{(1)}(D, \alpha_1, m) = \int d^D l_0 (l_0^2 - m^2)^{\alpha_1}, \tag{22}
\]

can be evaluated by method above, obtain, after analytic continuation to \( i < 0 \), the known result

\[
J^{(1)}(D, \alpha_1, m) = \pi^{D/2} (-\alpha_1 - D/2)(-m^2)^{\alpha_1 + D/2} \tag{23}
\]

that is to according to [26], when \( \alpha_1 = -1 \), and the Pochhammer symbol is defined as,

\[
(a|b) = (a)_b = \frac{\Gamma(a+b)}{\Gamma(a)}, \tag{24}
\]
we will turn to the left form when we deal with pentagon and hexagon integrals, because the number of sum indices will be large (more than 10 sometimes) and then the first notation becomes better to read.

IV. TWO-POINT FUNCTION

Two-point integrals are needed in order to study radiative corrections such as self-energy and vacuum polarization. These integrals raises no difficulties and their results are well-known of quantum field theory courses. The integral associated to one-loop two-point function, case $n = 2$ given by

$$J^{(2)}(D, \alpha_1, \alpha_2, l_1, m_0, m_1) = \int d^D l_0 (l_0^2 - m_0^2)^{\alpha_1} [(l_0 - l_1)^2 - m_1^2]^{\alpha_2},$$  \hspace{1cm} (25)

can be evaluated by method above, one obtains ten different solutions analytically continued that can be calculated by general expression

$$J^{(2)} = J^{(2)}(D, \alpha_1, \alpha_2, l_1, m_0, m_1)$$

$$= \pi^{D/2}(-1)^{\alpha_1+\alpha_2}\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)$$

$$\times \sum_{j_1,...,j_5=0}^{\infty} \frac{\Gamma(1-D/2-j_1)}{(1)_{j_4}(1)_{j_5}} \frac{(-l_1^2)(m_0^2)(m_1^2)}{(1)_{j_1}(1)_{j_2}(1)_{j_3}},$$  \hspace{1cm} (26)

using the constraints equations

$$D/2 = -j_1 - j_4 - j_5,$$  \hspace{1cm} (27)

$$\alpha_1 = j_1 + j_2 + j_3,$$  \hspace{1cm} (28)

$$\alpha_2 = j_1 + j_3 + j_5.$$  \hspace{1cm} (29)

The two point function will be obtained after the analytic continuation of the each solution to $\alpha_1, \alpha_2 < 0$.

We choose one convenient solution given by (consider $\sigma_2 = \alpha_1 + \alpha_2 + D/2$)

$$J^{(2)} = J^{(2)}(D, \alpha_1, \alpha_2, l_1, m_0, m_1)$$

$$= \pi^{D/2}(-m_1^2)^{\alpha_2} \Gamma\{-\alpha_2\} \frac{(-\alpha_2)_{-\alpha_1-D/2}(D/2)_{\alpha_1}}{(1)_{\sigma_2}(1)_{1-\alpha_1-D/2}(1)_{1-\alpha_2-D/2}}$$

$$\times F_4[-\sigma_2, -\alpha_1; D/2, 1 - \alpha_1 - D/2; m_0^2/m_1^2],$$
\[ J^{(2)} = J^{(2)}(D, \alpha_1, \alpha_2, l_1, 0, m) \]
\[ = \Delta - \pi^2 \left[ \log(-m^2) + \log(1 - \frac{l_1^2}{m^2}) + \frac{m^2}{l_1^2} \log(1 - \frac{l_1^2}{m^2}) - 1 \right]. \]
\[ (32) \]

where \( \Delta = \pi^2 \left( \frac{1}{\epsilon} + \gamma - \log \pi \right) \). This result is to according to [26]. Other case of interest we have when \( m_0 = m_1 = m \). This case is associated with the diagram of the vacuum polarization. The expression (30) is then given by

\[ J^{(2)} = J^{(2)}(D, \alpha_1, \alpha_2, l_1, m, m) \]
\[ = \pi^{D/2}(-m^2)^{\sigma_2}(-\alpha_1 - \alpha_2)^{-D/2} \times 3F_2 \left[ \begin{array}{c} -\sigma_2, -\alpha_1 - \alpha_2 \\ -\frac{l_1^2}{m^2} \end{array} \left| \frac{1}{4m^2} \right. \right], \]
\[ (33) \]

where \( 3F_2 \) is a hypergeometric function. The above solutions as well as its analytic continuation is to according to [1] and [28].
V. THREE-POINT FUNCTION

We turn to scalar three-point functions, increasing a little bit the difficulties, we one put the external momenta off-shell and consider three propagators with distinct masses. The one-loop three-point function is given by

\[
J^{(3)} = J^{(3)}(D, \alpha_1, \alpha_2, \alpha_3, l_1, l_2, m_0, m_1, m_2)
= \int d^D l_0 [l_0^2 - m_0^2]^{\alpha_1} [(l_0 - l_1)^2 - m_1^2]^{\alpha_2} [(l_0 - l_2)^2 - m_2^2]^{\alpha_3},
\]  

(34)

performing the method presented in Section 1, we obtain the sixty nine non-trivial independent solutions, all analytically continued. We select the convenient solution (consider \( \sigma_3 = \alpha_1 + \alpha_2 + \alpha_3 + D/2 \))

\[
J^{(3)} = J^{(3)}(D, \alpha_1, \alpha_2, \alpha_3, l_1, l_2, m_0, m_1, m_2)
= \pi^{D/2}(-m_2^2)^{\sigma_3}(D/2)^{\alpha_1 + \alpha_2 - \sigma_2} \times \Psi_3 \left[ -\sigma_3, -\alpha_1, -\alpha_2, \frac{l_1^2}{m_2^2}, \frac{l_2^2}{m_2^2}, \frac{l_{12}^2}{m_2^2}, \frac{m_0^2}{m_2^2}, \frac{m_1^2}{m_2^2} \right],
\]

(35)

where \( \Psi_3 \) is a hypergeometric function, that can be given in terms of the generalized Lauricella functions [27], (see appendix A). With this solution we can obtain some cases of physical interest. The case \( m_0 \neq 0 \) and \( m_1 = m_2 = m \), that is applied for example in Higgs \( \to \gamma \gamma \) decay, is given by

\[
J^{(3)} = J^{(3)}(D, \alpha_1, \alpha_2, \alpha_3, l_1, l_2, m_0, m)
= \pi^{D/2}(-m_2^2)^{\sigma_3} \frac{(D/2)^{\alpha_1}}{(-\sigma_3)_{\alpha_1 + D/2}}
\times R_{11} \left[ -\sigma_3, -\alpha_1, -\alpha_3, -\alpha_2, \frac{l_1^2}{m_2^2}, \frac{m_2^2}{m_2^2}, -\frac{l_{12}^2}{m_2^2}, \frac{l_2^2}{m_2^2} \right],
\]

(36)

where \( R_{11} \) is hypergeometric-type function given in the Appendix A. If we make \( m_0 = 0 \) and \( m_1 \neq m_2 \neq 0 \), in (35), we have the integral that used also in the H decay. The expression to this solution is

\[
J^{(3)} = J^{(3)}(D, \alpha_1, \alpha_2, \alpha_3, l_1, l_2, m_1, m_2)
= \pi^{D/2}(-m_2^2)^{\sigma_3} (D/2)^{\alpha_1 + \alpha_2 - \sigma_2}
\times R_6 \left[ -\sigma_3, -\alpha_2, -\alpha_1, \frac{l_1^2}{m_2^2}, \frac{m_2^2}{m_2^2}, -\frac{l_{12}^2}{m_2^2}, \frac{l_2^2}{m_2^2} \right],
\]

(37)
where also \( R_6 \) is a hypergeometric function, that can be given by generalized Lauricella functions, expressed in the Appendix A. The others cases to one-loop three-point function, namely: 1) \( m_0 = 0 \) and \( m_1 = m_2 \); 2) \( m_0 = m_1 = m_2 \neq 0 \); 3) \( m_0 = m \) and \( m_1 = m_2 = m \); 4) \( m_0 = m \) and \( m_1 = m_2 = 0 \); 5) \( m_0 = m_1 = m_2 = 0 \), are studied in [1, 29].

VI. FOUR-POINT FUNCTION

Usually four-point integrals are the most complicated in quantum field theory courses. They represent the scattering (2 \( \rightarrow \) 2) and for this reason are very important in phenomenology [30]. In a previous work [7] two of us studied such integrals — the ones that contribute to photon-photon scattering in QED — and did show several hypergeometric series representing it. Two of them were calculated before, using Mellin-Barnes approach, by Davydychev [31], the functions given by Appel’s \( F_3 \) and \( \Sigma F_2 \), the first one a single hypergeometric function and the second a sum of four ones.

The integral associated to one-loop four-point function, case \( n = 4 \), is given

\[
J^{(4)} = J^{(4)}(D, \alpha_1, \alpha_2, \alpha_3, \alpha_4, l_1, l_2, l_3, m_0, m_1, m_2, m_3)
\]

\[
= \int d^D l_0 \left[ l_0^2 - m_0^2 \right]^{\alpha_1} \left[ (l_0 - l_1)^2 - m_1^2 \right]^{\alpha_2} \left[ (l_0 - l_2)^2 - m_2^2 \right]^{\alpha_3} \left[ (l_0 - l_3)^2 - m_3^2 \right]^{\alpha_4},
\]

(38)

and is far more general than that we considered in [7] and Duplančić and Nižić presented in [30]. This integral can be evaluated by method described in the Section 1 and we obtain one thousand and twelve non trivial solution all analytically continued, as can be read from table-1. We choose again only the convenient solution, it is written as (consider \( \sigma_4 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + D/2 \))

\[
J^{(4)} = J^{(4)}(D, \alpha_1, \alpha_2, \alpha_3, \alpha_4, l_1, l_2, l_3, m_0, m_1, m_2, m_3)
\]

\[
= \pi^{D/2} (-\alpha_4)_{-\sigma_4 + \alpha_4} (D/2)^{\sigma_4 - \alpha_4} (D/2)^{-\alpha_4} \Phi
\]

\[
\times \Phi \left[ -\sigma_4, -\alpha_1, -\alpha_2, -\alpha_3, -l_2^2; \frac{m_0^2}{m_3^2}; \frac{l_2^2}{m_3^2}; \frac{-l_2^2 - l_3^2}{m_3^2}; \frac{l_3^2}{m_3^2}; \frac{l_2^2 - l_3^2}{m_3^2}; \frac{m_0^2}{m_3^2}; \frac{m_1^2}{m_3^2}; \frac{m_2^2}{m_3^2} \right]
\]

(39)

where \( \Phi \) is a hypergeometric function (see Appendix A), that can be written in terms of generalized Lauricella functions. We observe that above hypergeometric series does not allow
one to take limit of vanishing $m_3$ since this it is not defined on that kinematic region. In the above solution is possible take out the limits: $m_0 = m_1 = m_2 = 0$; $m_0 = m_1 = 0$ and $m_2 \neq 0$; $m_0 = 0$ and $m_1 = m_2$ or $m_1 \neq m_2 \neq 0$; $m_0 = m_1 = m_2 = m_3 = m$; and on-shell cases, see also \[7, 31\]. To the special case where $m_0 = m_1 = m_2 = m_3 = 0$ we obtain the solution

\[
J^{(4)} = J^{(4)}(D, \alpha_1, \alpha_2, \alpha_3, \alpha_4, l_1, l_2, l_3) \]

\[
= \pi^{D/2} (p_3^2)^{\sigma_4 (-\alpha_1 - \sigma_4 (-\alpha_2 - \sigma_4))} ((\sigma_4)^{D/2})
\]

\[
\times \Psi_3 \begin{bmatrix}
\sigma_4, -\alpha_2, -\alpha_3 \\
1 - \sigma_4 + \alpha_1, 1 - \sigma_4 + \alpha_4
\end{bmatrix}
\begin{bmatrix}
l_1^2, l_2^2, l_3^2, l_1^2, l_2^2, l_3^2
\end{bmatrix}.
\]

\[
(40)
\]

VII. GENERATING FUNCTIONAL FOR PENTAGON INTEGRAL

Pentagon integrals were studied by Melrose\[32]\ and recently by Bern\[33]\ and Weinzierl and Kosower\[34]\, and several authors \[35\], in order to study scattering process where 2 particles go to 3 particles.

Let the external legs of the pentagon be $(p_1, p_2 - p_1, p_3 - p_2, p_4 - p_3, p_4)$, see figure, and let us consider firstly the on-shell case. The generating functional for the scalar negative-dimensional integrals is written as a special case of the general functional of section 1,

\[
G_{ON}^5 = \int d^Dq \exp \left[ -\alpha q^2 - \beta (q + p_1)^2 - \gamma (q + p_2)^2 - \theta (q + p_3)^2 - \omega (q + p_4)^2 \right],
\]

then completing the square one can easily integrate and simplify several terms. Eventually one get,

\[
G_{ON}^5 = \left( \frac{\pi}{\lambda} \right)^{D/2} \exp \left[ -\frac{1}{\lambda} \left( \alpha \gamma p_2^2 + \alpha \theta p_3^2 + \beta \theta s_{13} + \beta \phi s_{14} + \gamma \phi s_{24} \right) \right],
\]

\[
(42)
\]

where we define $\lambda = \alpha + \beta + \gamma + \theta + \omega$, and three of the Mandelstam variables for the on-shell pentagon,

\[
s_{13} = (p_1 - p_3)^2, \quad s_{14} = (p_1 - p_4)^2, \quad s_{24} = (p_2 - p_4)^2.
\]

\[
(43)
\]

Following the usual procedure applied in NDIM we count the total number of solutions we will have to deal with. Exponential gives us five sums (Taylor expansion for each argument), $\lambda$ other five (multinomial expansion), and the equations are the number of propagators plus...
Then, there are \( C_{10,6} = 210 \) possible ways to solve the \( 6 \times 6 \) system. The ones that have null determinant do not have solution, they are 85, the remaining 125 after properly solved will provide us hypergeometric series representing the original Feynman integral.

On the other hand, projecting out powers of the exponential in (41),

\[
G_{5}^{ON} = \sum_{\alpha_1, \ldots, \alpha_5=0}^{\infty} \frac{(-1)^{\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5} \alpha_1! \alpha_2! \alpha_3! \alpha_4! \alpha_5!}{\alpha_1! \alpha_2! \alpha_3! \alpha_4! \alpha_5!} J_{5}^{ON} (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5), \quad (44)
\]

where

\[
J_{5}^{ON} (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \int d^D q \left( q^2 \right)^{\alpha_1} (q + p_1)^{2\alpha_2} (q + p_2)^{2\alpha_3} (q + p_3)^{2\alpha_4} (q + p_4)^{2\alpha_5}, \quad (45)
\]

is the negative-dimensional integral.

We will present two kinds of hypergeometric functions since several others can be written immediately if one observe the symmetries of the diagram (or alternatively the symmetries of the generating functional),

\[
\{ \alpha_3 \leftrightarrow \alpha_4, p_1 \leftrightarrow p_4 - p_3, p_2 - p_1 \leftrightarrow p_1 \}, \{ \alpha_5 \leftrightarrow \alpha_4, p_1 \leftrightarrow p_3 - p_2, p_4 \leftrightarrow p_2 - p_1 \},
\]

\[
\{ \alpha_1 \leftrightarrow \alpha_5, p_1 \leftrightarrow p_3 - p_2, p_2 - p_1 \leftrightarrow p_4 - p_3 \}, \quad (46)
\]

and so forth, there are several other symmetries of the integral (42). When we have a hypergeometric series one can obtain some others merely interchanging exponents of propagators and external momenta, where three such sets are given in the above list.

A. Hypergeometric functions for massless on-shell pentagon

Let us define \( \sigma_5 = \alpha_{12345} + D/2 \) and the shorthand notation we will use hereafter in order to have a more compact notation,

\[
\hat{j}_{ab} = \hat{j}_{a} + \hat{j}_{b}
\]

and so on. Observe that we use this compact notation only for sum index. Mandelstam variables are given by (12).

The first hypergeometric function representing the Feynman loop integral is given

\[
J_{5}^{ON} (\{\alpha_i\}) = \Gamma_5^{(1)} S_5^{(1)} \left[ \begin{array}{cccc}
-\alpha_1, -\alpha_2, -\alpha_5, -\sigma_5 - D/2 & p_2^2 & 0 & 0 \\
1 + \alpha_3 - \sigma_5 & s_{24} & s_{14} & s_{13} \\
1 + \alpha_5 - \sigma_5 & -s_{24} & -s_{24} & -s_{24}
\end{array} \right], \quad (47)
\]
the possible poles are contained in the factor
\[
\Gamma_5^{(1)} = \pi^{D/2}(-\alpha_3|\sigma_5)(-\alpha_5|\sigma_5)(\sigma_5 + D/2 - 2\sigma_5 - D/2)s_{24}^{\sigma_5},
\] (48)
where the Pochhammer symbol is given by eq. (24). In the appendix we define all hypergeometric functions we use in this paper.

The above hypergeometric function is the "symmetric" solution calculated by Davydychev [1], using Mellin-Barnes integral representation approach.

However, in the negative-dimensional approach we obtain, in general, several hypergeometric functions, which represent it and are related by analytic continuation (direct or indirect). We write down other hypergeometric function for the massless scalar pentagon,
\[
\mathcal{J}_5^{ON}(\{\alpha_i\}) = \Gamma_5^{(2)} \mathcal{T}_4^{(5)} \left[ -\alpha_1, -\alpha_3, -\alpha_4, D/2 + \alpha_{123} \right] \left[ \begin{array}{c} \frac{p_2^2}{s_{21}}, \frac{p_3^2}{s_{13}}, -\frac{s_{14}}{s_{24}}, \frac{s_{14}}{s_{24}} \end{array} \right],
\] (49)
where
\[
\Gamma_5^{(2)} = \pi^{D/2}(-\alpha_2|\sigma_5 - k)(-\alpha_5|\sigma_5 - m)(\sigma_5 + D/2 - 2\sigma_5 - D/2 + \alpha_{34}) \frac{s_{14}^{\sigma_5 - \alpha_34}}{s_{13}^{\alpha_34} s_{24}^{\alpha_3}},
\] (50)
the same set of permutations of exponents of propagators and external momenta can be applied to the above result, in order to generate several others.

B. Hypergeometric functions for massive on-shell pentagon

We now turn to the case where the five propagators have masses, and let them to be distinct. We will present a hypergeometric function that allows us to consider interesting particular cases: 2, 3, 4 and 5 equal masses.

When we deal with massive propagators the system we must solve is greater than the related to the massless diagram. Let the masses be \(m_1^2\) attached to the propagators labelled as \(\alpha_1\), \(m_2^2\) attached to the one labelled as \(\alpha_2\) and so on.

The generating functional is simply the massless one (12) times the exponential containing the masses,
\[
G_{5M}^{ON} = G_5^{ON} \exp \left( \alpha m_1^2 + \beta m_2^2 + \gamma m_3^2 + \theta m_4^2 + \omega m_5^2 \right),
\] (51)
the total number of systems we must solve now is given by the combinatorics \(C_{15,6} = 5005\). Among them we pick the most convenient one, i.e., the hypergeometric series that will allow us to study several important special cases. We choose to present the result
where

\[ z_1 = \frac{p_2^2}{m_5^2}, \quad z_2 = \frac{p_3^2}{m_5^2}, \quad z_3 = -\frac{s_{14}}{m_5^2}, \quad z_4 = \frac{s_{13}}{m_5^2}, \quad z_5 = -\frac{s_{24}}{m_5^2}, \quad z_6 = \frac{m_1^2}{m_5^2}, \quad z_7 = \frac{m_2^2}{m_5^2}, \quad z_8 = \frac{m_3^2}{m_5^2}, \quad z_9 = \frac{m_4^2}{m_5^2}, \]

which is valid for five different masses, such that \( m_5 \neq 0 \) and we define

\[ \Gamma_5^{(\text{mass})}(\alpha_5; m_5) = \pi^D/2 (-\alpha_5 | \sigma_5)(D/2) - \sigma_5 - D/2 (m_5^2)^{\sigma_5}, \]

observe that the above 9-fold hypergeometric series allows us to study the cases where any of the masses \( m_1, m_2, m_3, m_4 \) vanish, or any of the cases where they are equal to \( m_5 \) (which in this kinematical region can not vanish).

We proceed to special cases now. First, let \( m_1 = m_5 \). Then, the hypergeometric series in the \( l_1 \) index can be rewritten as \( _2F_1(\cdot|\cdot|1) \). Using Gauss’ summation formula \[27\],

\[ _2F_1(a, b, c|1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \]

we can exactly sum up the series. Rewriting the Pochhammer symbols involving \( l_1 \) as,

\[
\sum_{l_1=0}^{\infty} \frac{(-\alpha_1 j_{12} + l_1)(-\sigma_5 j_{12345} + l_{1234})}{l_1!(1 + \alpha_5 - \sigma_5 j_{124} + l_{1234})} = \frac{(-\alpha_1 j_{12})(-\sigma_5 j_{12345} + l_{1234})}{(1 + \alpha_5 - \sigma_5 j_{124} + l_{1234})} \\
\times \sum_{l_1=0}^{\infty} \frac{(-\sigma_5 + j_{12345} + l_{234}|l_1)(-\alpha_1 + j_{12}|l_1)}{l_1!(1 + \alpha_5 - \sigma_5 + j_{124} + l_{234}|l_1)},
\]

where we used the property of Pochhammer symbols,

\[ (a|b + c) = (a|b)(a + b|c), \]

thus the series in \( l_1 \) is recast as a \( _2F_1 \) of unity argument, and then can be summed,

\[
\sum_{l_1=0}^{\infty} \left[ \frac{(-\alpha_1 j_{12})(-\sigma_5 j_{12345} + l_{1234})(A_1 - j_{1235})}{(A_2 - j_{35})(A_3 j_{4} + l_{234})} \right] \frac{\Gamma(1 + \alpha_5 - \sigma_5)\Gamma(A_1)}{\Gamma(A_2)\Gamma(A_3)},
\]

where

\[ A_1 = 1 + \alpha_{15}, \quad A_2 = 1 + \alpha_5, \quad A_3 = 1 + \alpha_{15} - \sigma_5, \]
the same procedure of summing up series of hypergeometric type was applied in [30].

Substituting the above result in (52) one obtain the result for the pentagon integral having four different masses is given by,

\[ J_5^{ON}(m_i^2; 4) = \Gamma_5^{(mass)}(\alpha_{15}; m_5)S_8^{(5)} \left[ \begin{array}{c} -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, -\alpha_5, -\sigma_5 \\ D/2, -\alpha_{15}, 1 + \alpha_{15} - \sigma_5 \end{array} \right] \left( z_1; \ldots; z_8 \right), \quad (60) \]

where

\[ z_1 = -\frac{p_2^2}{m_5^2}, \quad z_2 = -\frac{p_3^2}{m_5^2}, \quad z_3 = -\frac{s_{14}}{m_5^2}, \quad z_4 = \frac{s_{13}}{m_5^2}, \quad z_5 = -\frac{s_{24}}{m_5^2}, \quad z_6 = \frac{m_3^2}{m_5^2}, \quad z_7 = \frac{m_4^2}{m_5^2}, \quad z_8 = \frac{m_2^2}{m_5^2}. \]  \( (61) \]

where the argument "4" is to remember one the number of different masses and the factor involving gamma functions, given by [32], is greatly simplified, and we perform the analytic continuation in the final step of the calculation (sum the series and then analytically continue the Pochhammer symbols).

The same algebraic manipulations can be made to obtain the cases where the pentagon has three different masses \( m_5, m_4, m_3 \),

\[ J_5^{ON}(m_i^2; 3) = \Gamma_5^{(mass)}(\alpha_{125}; m_5)S_7^{(5)} \left[ \begin{array}{c} -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, -\alpha_5, -\sigma_5 \\ D/2, -\alpha_{125}, 1 + \alpha_{125} - \sigma_5 \end{array} \right] \left( z_1; \ldots; z_7 \right), \quad (62) \]

where the variables are,

\[ z_1 = -\frac{p_2^2}{m_5^2}, \quad z_2 = -\frac{p_3^2}{m_5^2}, \quad z_3 = -\frac{s_{14}}{m_5^2}, \quad z_4 = -\frac{s_{13}}{m_5^2}, \quad z_5 = -\frac{s_{24}}{m_5^2}, \quad z_6 = \frac{m_3^2}{m_5^2}, \quad z_7 = \frac{m_4^2}{m_5^2}. \]  \( (63) \]

When only two masses are different we can sum the \( l_3 \) index,

\[ J_5^{ON}(m_i^2; 2) = \Gamma_5^{(mass)}(\alpha_{1235}; m_5)S_6^{(5)} \left[ \begin{array}{c} -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, -\alpha_5, -\sigma_5 \\ D/2, -\alpha_{1235}, 1 + \alpha_{1235} - \sigma_5 \end{array} \right] \left( z_1; \ldots; z_6 \right), \quad (64) \]

where its six variables are,

\[ z_1 = \frac{p_2^2}{m_5^2}, \quad z_2 = -\frac{p_3^2}{m_5^2}, \quad z_3 = \frac{s_{14}}{m_5^2}, \quad z_4 = \frac{s_{13}}{m_5^2}, \quad z_5 = \frac{s_{24}}{m_5^2}, \quad z_6 = \frac{m_3^2}{m_5^2}. \]  \( (65) \]

finally, the very special case where all five masses are equal,

\[ J_5^{ON}(m_i^2; 1) = \pi^{D/2}(m_5^2)^{\sigma_5}(-\sigma_5 + D/2) - D/2 \times S_5^{(5)} \left[ \begin{array}{c} -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, -\alpha_5, -\sigma_5 \\ -\alpha_{12345} \end{array} \right] \left( z_1; \ldots; z_5 \right), \quad (66) \]
where,
\[ z_1 = -\frac{p_2^2}{m_5^2}, \quad z_2 = \frac{p_3^2}{m_5^2}, \quad z_3 = -\frac{s_{14}}{m_5^2}, \quad z_4 = \frac{s_{13}}{m_5^2}, \quad z_5 = -\frac{s_{24}}{m_5^2}, \]
observe that two Pochhammer symbols cancelled. The above result is a generalization of our previous study of box integrals pertaining to photon-photon scattering.[7]

C. Hypergeometric functions for massless off-shell pentagon

Massless pentagon integrals were necessary to Bern, Dixon and Kosower[33] in order to study \( 2 \rightarrow 3 \) scattering, such as \( e^+e^- \rightarrow 3 \) jets. In that work two of the external momenta were considered to be massive and the remaining ones massless.

Here we will present the whole five possibilities: from one to five massive external particles. In our approach the difficulty to carry out the integrals with different masses or equal ones is the same. So, we prefer the most general case, five distinct masses.

Hereafter we consider massive external particles,
\[ p_1^2 = M^2, \quad (p_1 - p_2)^2 = M_{12}^2, \quad (p_2 - p_3)^2 = M_{23}^2, \quad (p_3 - p_4)^2 = M_{34}^2, \quad p_4^2 = M_4^2. \]
(68)

1. One external leg off-shell

We present two four-fold hypergeometric series representing the scalar pentagon integral with one massive external particle.

\[ \mathcal{J}_5^{(1)}(\{\alpha_i\}) = \Gamma_5(\alpha_1, \alpha_2; M^2)S_5^{(5-1off)} \left[ -\alpha_3, -\alpha_4, -\alpha_5, -\sigma_5 \left| 1 + \alpha_1 - \sigma_5, 1 + \alpha_2 - \sigma_5 \right. \right] z_1; \ldots; z_5, \]
(69)
where its five variables are,
\[ z_1 = \frac{p_2^2}{M^2}, \quad z_2 = \frac{p_3^2}{M^2}, \quad z_3 = \frac{s_{14}}{M^2}, \quad z_4 = \frac{s_{13}}{M^2}, \quad z_5 = -\frac{s_{24}}{M^2}, \]
(70)
where the superscript (1) means one leg off-shell and
\[ \Gamma_5(\alpha_1, \alpha_2; M^2) = \pi^{D/2}(-\alpha_1|\sigma_5)(-\alpha_2|\sigma_5)(\sigma_5 + D/2) - 2\sigma_5 - D/2)(M^2)^{\sigma_5}, \]
(71)
observe that the above hypergeometric series does not allow one to take the limit of vanishing \( M \) since this it is not defined on that point.

16
However, there are others multiple series that allow us to take such limit. One example
is,

\[
\mathcal{J}_5^{(1)}(\{\alpha_i\}) = \Gamma_5(\alpha_2, \alpha_4; s_{13}) \mathcal{J}_5^{(6-off)} \left[ \begin{array}{c}
-\alpha_1, -\alpha_3, -\alpha_5, -\sigma_5 \\
1 + \alpha_2 - \sigma_5, 1 + \alpha_4 - \sigma_5
\end{array} \right]_{z_1; \ldots; z_6},
\]

define,

\[
z_1 = -\frac{p_2^2}{s_{13}}, \quad z_2 = \frac{p_3^2}{s_{13}}, \quad z_3 = \frac{s_{14}}{s_{13}}, \quad z_4 = -\frac{s_{24}}{s_{13}}, \quad z_5 = \frac{M^2}{s_{13}}, \quad (72)
\]
in the limit of \(M = 0\) the series on \(j_6\) index reduces to its first term, unity, so we are left
with a four-fold hypergeometric series.

2. Two external legs off-shell

Next we put another external leg off mass shell, namely, one consider \((p_1 - p_2)^2 = M_{12}^2\).
The first hypergeometric series we collect from the total 924 ones (see table II) , is

\[
\mathcal{J}_5^{(2)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \Gamma_5(\alpha_1, \alpha_2; M^2) \mathcal{J}_6^{(5-2-off)} \left[ \begin{array}{c}
-\alpha_3, -\alpha_4, -\alpha_5, -\sigma_5 \\
1 + \alpha_1 - \sigma_5, 1 + \alpha_2 - \sigma_5
\end{array} \right]_{z_1; \ldots; z_6} \]
its six variables are given by,

\[
z_1 = \frac{p_2^2}{M^2}, \quad z_2 = \frac{p_3^2}{M^2}, \quad z_3 = \frac{s_{13}}{M^2}, \quad z_4 = -\frac{s_{14}}{M^2}, \quad z_5 = \frac{s_{24}}{M^2}, \quad z_6 = \frac{M_{12}^2}{M^2}, \quad (74)
\]
where the superscript \((2)\) means two legs off-shell. Observe again that the above hypergeometric series
does not allow one to take the limit of vanishing \(M\) but admits the limit of \(M_{12} = 0\) to be taken.

We can consider also the case where the two masses are equal. Then the hypergeometric
function in the \(j_7\) index reduces to a gaussian one, namely \(_2F_1\), that can be exactly summed
when its argument is unity.

Another 6-fold hypergeometric series

\[
\mathcal{J}_5^{(2)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \Gamma_5^\prime(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \mathcal{J}_6^{(5-2-off)} \left[ \begin{array}{c}
\alpha_2 - \sigma_5, -\alpha_2, -\alpha_4, -\alpha_5, D/2 + \alpha_4 \alpha_5 \\
1 + \alpha_3 - \sigma_5
\end{array} \right]_{z_1; \ldots; z_6} \]

\[
\times \Gamma_5^{(6-off)} \left[ \begin{array}{c}
\alpha_2 - \sigma_5, -\alpha_2, -\alpha_4, -\alpha_5, D/2 + \alpha_4 \alpha_5 \\
1 + \alpha_3 - \sigma_5
\end{array} \right]_{z_1; \ldots; z_6} \]

\[
(76)
\]

17
where
\[ z_1 = \frac{p_3}{p_2}, \quad z_2 = \frac{s_{13}}{M_{12}^2}, \quad z_3 = \frac{s_{14}}{M_{12}^2}, \quad z_4 = -\frac{s_{24}}{p_2}, \quad z_5 = \frac{M^2}{M_{12}^2}, \quad z_6 = -\frac{p_2}{M_{12}^2}, \quad (77) \]
where the factor is defined by,
\[ \Gamma'_5(\alpha_1, \alpha_2, \alpha_3; M_{12}^2/p_2^2) = (-\alpha_1|\sigma_5 - \alpha_2)(-\alpha_3|\sigma_5)(\sigma + D/2)|-2\sigma_5 - D/2 + \alpha_2) \left( \frac{M_{12}^2}{p_2^2} \right)^{\alpha_2} (p_2^2)^{\sigma_5}, \quad (78) \]
taking \( M_{12} = M \) we can sum up, using Gauss' summation formula\[^{27}\], the \( j_6 \) series and get,
\[ \mathcal{J}_5^{(2)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \Gamma_M \Gamma'_5(\alpha_1, \alpha_2, \alpha_3; M^2/p_2^2) \times \mathcal{U}_5^{(5-2off)}(\alpha_2 - \sigma_5 - \alpha_2, -\alpha_4, -\alpha_5, D/2 + \alpha_{345}) \left| \begin{array}{c} z_1; \ldots; z_5 \end{array} \right. \quad (79) \]
where
\[ z_1 = \frac{p_3}{p_2^2}, \quad z_2 = \frac{s_{13}}{M^2}, \quad z_3 = \frac{s_{14}}{M^2}, \quad z_4 = -\frac{s_{24}}{p_2}, \quad z_5 = -\frac{p_2}{M^2}, \quad (80) \]
and
\[ \Gamma_M = \frac{1}{(-\alpha_3 + \sigma_5 - \alpha_2)(\sigma_5 + \alpha_{45} + D/2 - \alpha_2)}. \quad (81) \]

3. Three external legs off-shell

Continue to study the off-shell pentagon, now where three external particles are massive, the third one being \((p_2 - p_3)^2 = M_{23}^2\), and one such hypergeometric series representation,
\[ \mathcal{J}_5^{(3)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \Gamma_5(\alpha_3, \alpha_4; M^2) \times \mathcal{U}_5^{(5-3off)}(\alpha_1, \alpha_2, -\alpha_5, -\sigma_5) \left| \begin{array}{c} z_1; \ldots; z_7 \end{array} \right. \], \quad (82) \]
where
\[ z_1 = \frac{p_3^2}{M_{23}^2}, \quad z_2 = \frac{p_3^2}{M_{23}^2}, \quad z_3 = \frac{s_{13}}{M_{23}^2}, \quad z_4 = -\frac{s_{14}}{M_{23}^2}, \quad z_5 = \frac{s_{24}}{M_{23}^2}, \quad z_6 = -\frac{M^2}{M_{23}^2}, \quad z_7 = \frac{M_{12}^2}{M_{23}^2}, \quad (83) \]
that also admits one to take two masses to vanish. Observe that \( M_{23} \) can never be zero.
4. Four external legs off-shell

The last step before we put all five legs off mass shell is to make \((p_3 - p_4)^2 = M_{34}^2\). Then we present two sample series: the first is suitable for equal masses limit but non-vanishing,

\[
\mathcal{J}_5^{(4)}(\{\alpha_i\}) = \Gamma_5(\alpha_1, \alpha_2; M^2) S_8^{(5-4\text{off})} \left[ \begin{array}{c} -\alpha_3, -\alpha_4, -\alpha_5, -\sigma_5 \\ 1 + \alpha_1 - \sigma_5, 1 - \alpha_2 - \sigma_5 \end{array} \right] z_1; \ldots; z_8, \tag{84}
\]

where its variables are,

\[
\begin{align*}
z_1 &= \frac{p_2^2}{M^2}, \\
z_2 &= \frac{p_3^2}{M^2}, \\
z_3 &= \frac{s_{13}}{M^2}, \\
z_4 &= \frac{s_{14}}{M^2}, \\
z_5 &= -\frac{s_{24}}{M^2}, \\
z_6 &= \frac{M_{12}^2}{M^2}, \\
z_7 &= -\frac{M_{23}^2}{M^2}, \\
z_8 &= -\frac{M_{34}^2}{M^2},
\end{align*}
\]

and the second one suitable for vanishing masses limit,

\[
\mathcal{J}_5^{(4)}(\{\alpha_i\}) = \Gamma_5(\alpha_3, \alpha_5; s_{24}) T_8^{(5-4\text{off})} \left[ \begin{array}{c} -\alpha_1, -\alpha_2, -\alpha_4, -\sigma_5 \\ 1 + \alpha_3 - \sigma_5, 1 - \alpha_5 - \sigma_5 \end{array} \right] z_1; \ldots; z_8, \tag{86}
\]

being

\[
\begin{align*}
z_1 &= \frac{p_2^2}{s_{24}}, \\
z_2 &= -\frac{p_3^2}{s_{24}}, \\
z_3 &= -\frac{s_{13}}{s_{24}}, \\
z_4 &= \frac{s_{14}}{s_{24}}, \\
z_5 &= -\frac{M^2}{s_{24}}, \\
z_6 &= \frac{M_{12}^2}{s_{24}}, \\
z_7 &= \frac{M_{23}^2}{s_{24}}, \\
z_8 &= \frac{M_{34}^2}{s_{24}},
\end{align*}
\]

or when the masses are such that \(s_{24} >> M_j^2\).

Note that the structure of the above series is similar but not equal. In the first one, only one index \(j_9\) appears five times (in other words, the series in \(j_9\) can be rewritten as a \(3F_2\) function) the other eight indices appear only three times. The second 9-fold hypergeometric series has the indices \(j_3\) and \(j_6\) appearing five times, what turns it to be more complicated than the former. They (equations (84) and (86)) are very similar but are not the same 9-fold series.

5. Five external legs off-shell

Now we turn to the general case where all five legs are off-shell. Despite there are a great number of hypergeometric 9-fold series, see table I, we present here just two of them,

\[
\mathcal{J}_5^{(5)}(\{\alpha_i\}) = \Gamma_5(\alpha_1, \alpha_5; M_4^2) S_9^{(5-5\text{off})} \left[ \begin{array}{c} -\alpha_2, -\alpha_3, -\alpha_4, -\sigma_5 \\ 1 + \alpha_1 - \sigma_5, 1 + \alpha_5 - \sigma_5 \end{array} \right] z_1; \ldots; z_9, \tag{88}
\]
FIG. 1: Scalar massless pentagon. Labels \((i, j, k, l, m)\) represent exponents of propagators and the arrows show momentum flow. In the case where the external legs are off-shell we define \(p_1^2 = M^2\), \(s_{12} = M_{12}^2\), \(s_{23} = M_{23}^2\), \(s_{34} = M_{34}^2\), and \(p_4^2 = M_4^2\).

its nine variables are given as follows,

\[
\begin{align*}
& z_1 = \frac{p_2^2}{M_4^2}, \quad z_2 = \frac{p_3^2}{M_4^2}, \quad z_3 = -\frac{s_{13}}{M_4^2}, \quad z_4 = \frac{s_{14}}{M_4^2}, \quad z_5 = \frac{M_2^2}{M_4^2}, \quad z_6 = -\frac{M_{12}^2}{M_4^2}, \\
& z_7 = -\frac{M_{23}^2}{M_4^2}, \quad z_8 = \frac{M_{34}^2}{M_4^2}, \quad z_9 = \frac{M_{34}^2}{M_4^2}.
\end{align*}
\]  

(89)

other hypergeometric 9-fold series that have two masses in the denominator

\[
\mathcal{J}_5^{(5)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = \Gamma_5'(\alpha_4, \alpha_1, \alpha_5; M_{4/34}^2) \\
\times \mathcal{T}_9^{(5\text{off})} \left[ -\alpha_1, -\alpha_2, -\alpha_3, \alpha_1 - \sigma_5 \right| \\
1 + \alpha_1 - \sigma_5, 1 - \alpha_5 + \sigma_5 \bigg| z_1; \ldots; z_9 \right],
\]

(90)

where we define,

\[
\begin{align*}
& z_1 = \frac{p_2^2}{M_4^2}, \quad z_2 = -\frac{p_3^2}{M_4^2}, \quad z_3 = \frac{s_{13}}{M_{34}^2}, \quad z_4 = \frac{s_{14}}{M_{34}^2}, \quad z_5 = \frac{s_{24}}{M_{34}^2}, \quad z_6 = \frac{M_2^2}{M_4^2}, \\
& z_7 = -\frac{M_{12}^2}{M_{34}^2}, \quad z_8 = \frac{M_{23}^2}{M_{34}^2}, \quad z_9 = \frac{M_{24}^2}{M_4^2}.
\end{align*}
\]  

(91)

The last we present is a hypergeometric series that allows one to take the limit of vanishing masses (any of them),

\[
\mathcal{J}_5^{(5)}(\{\alpha_i\}) = \Gamma_5'(\alpha_5, \alpha_1, \alpha_3; p_3^2/s_{24})
\]

20
\[ \times U_9^{(5-off)} \left[ \begin{array}{c} \alpha_1 - \sigma_5, -\alpha_1, -\alpha_2, \alpha_4 \\ 1 + \alpha_3 - \sigma_5, 1 - \alpha_{15} + \sigma_5 \end{array} \right] \{z_1; \ldots; z_9\} \], \quad (92) \]

where
\[ z_1 = \frac{p_3^2}{p_2^2}, \quad z_2 = -\frac{s_{13}}{s_{24}}, \quad z_3 = \frac{s_{14}}{s_{24}}, \quad z_4 = \frac{M^2}{p_2^2}, \quad z_5 = \frac{M_{12}^2}{s_{24}}, \quad z_6 = \frac{M_{23}^2}{s_{24}}, \quad z_7 = \frac{M_{34}^2}{s_{24}}, \quad z_8 = -\frac{s_{24}^2}{p_2^2}, \quad z_9 = -\frac{s_{24}^2}{p_2^2}. \] \quad (93)

In the negative-dimensional approach we can select the kinematical region one would like to study and then work with hypergeometric series defined on that region. All hypergeometric series presented in this paper are exact, there are no approximations. They converge very fast as we have verified in [37], a precision of 20 digits can be achieved very quickly.

VIII. HEXAGON FEYNMAN LOOP INTEGRAL

Recently Bern, Dixon and Kosower [38] studied amplitudes of process involving six particles, namely, \( e^+e^- \rightarrow 4 \) partons. Also, QCD corrections to \( e^+e^- \rightarrow 4 \) jets were calculated by Weinzierl and Kosower [34]; Binoth, Guillet, Heinrich and Schubert [39] show how to reduce the hexagon diagrams in order to study Yukawa model at one-loop level. Their works motivates us to perform such integrals in general cases, i.e., arbitrary exponents of propagators, massive external legs and propagators.

The general formula we calculated in section I gives us the generating functional,
\[ G_{Hex} = \left( \frac{\pi}{\zeta} \right)^{D/2} \exp \left[ -\frac{1}{\zeta} \left( \alpha \gamma p_2^2 + \alpha \theta p_2^2 + \alpha \phi p_1^2 + \beta \theta s_{13} + \beta \phi s_{14} + \beta \omega s_{51} + \gamma \phi s_{24} + \gamma \omega s_{25} + \theta \omega s_{35} \right) \right], \] \quad (94)

define \( \sigma_6 = \alpha_{123456} + D/2 \) and \( \zeta = \lambda + \phi \). Total number of systems \( C_{15,7} = 6435 \), such that, 2790 of them have solutions, the remaining 3645 do not have interest at all.

The first series we write is of a kind Davydychev [1, 2] called ”symmetric”,
\[ \mathcal{J}_6^{(0)}(\{\alpha_i\}) = \Gamma_6(\alpha_4, \alpha_6; s_{35}) S_8^{(6-on)} \left[ \begin{array}{c} -\alpha_1, -\alpha_2, -\alpha_3, \alpha_5, -\sigma_6 \\ 1 + \alpha_4 - \sigma_6, 1 + \alpha_6 - \sigma_6 \end{array} \right] \{z_1; \ldots; z_8\}, \] \quad (95)
FIG. 2: Scalar massless hexagon. Labels \((\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)\) represent exponents of propagators and the arrows show momentum flow. In the case where the external legs are off-shell we define

\[ p_1^2 = M^2, \quad s_{12} = M_{12}^2, \quad s_{23} = M_{23}^2, \quad s_{34} = M_{34}^2, \quad s_{45} = M_{45}^2, \quad p_5^2 = M_5^2. \]

where the function has eight variables,

\[
\begin{align*}
    z_1 &= -\frac{p_2^2}{s_{35}}, \quad z_2 = \frac{s_{35}}{s_{35}}, \quad z_3 = -\frac{p_4^2}{s_{35}}, \quad z_4 = \frac{s_{13}}{s_{35}}, \quad z_5 = -\frac{s_{14}}{s_{35}}, \quad z_6 = \frac{s_{51}}{s_{35}}, \quad z_7 = -\frac{s_{24}}{s_{35}}, \\
    z_8 &= \frac{s_{35}}{s_{35}},
\end{align*}
\]

(96)

the superscript \((0)\) means that no external leg is off mass shell. In the appendix we define all hypergeometric functions we use in this paper in terms of generalized Lauricella functions.

The second one pertain to another kind, another kinematical region,

\[
\mathcal{J}_6^{(0)}(\{\alpha_i\}) = \Gamma_6(\alpha_6, \alpha_5, \alpha_3; s_{24}/s_{25})T_8^{(6-on)} \left[ \frac{\alpha_5 - \sigma_6 - \alpha_1 - \alpha_2 - \alpha_4 - \alpha_5}{1 + \alpha_3 - \sigma_6, 1 + \alpha_56 - \sigma_6} \right] \{z_1; \ldots; z_8\},
\]

(97)

define,

\[
\begin{align*}
    z_1 &= \frac{p_2^2}{s_{25}}, \quad z_2 = -\frac{p_3^2}{s_{25}}, \quad z_3 = \frac{s_{13}}{s_{25}}, \quad z_4 = -\frac{s_{14}}{s_{25}}, \quad z_5 = \frac{s_{51}}{s_{25}}, \quad z_6 = \frac{s_{24}}{s_{25}}, \quad z_7 = \frac{s_{35}}{s_{25}}, \\
    z_8 &= \frac{s_{25}}{s_{24}},
\end{align*}
\]

(98)

where in both hypergeometric series the factor \(\Gamma_6\) means that in equation (71) we change \(\sigma_5\) by \(\sigma_6\).
A. One massive external leg

Considering one of the external legs of the hexagon to be off mass shell means that one must to deal with almost two times more systems of algebraic equations, see table (III).

However, some hypergeometric series follow some pattern and we think it can be generalized for scalar \( N \)-point functions.

The first one we write does not allow one to take the mass to vanish,

\[
\mathcal{J}_6^{(1)}(\{\alpha_i\}) = \Gamma_6(\alpha_1, \alpha_2; M_1^2) \mathcal{S}_9^{(6-\text{off})} \begin{bmatrix} -\alpha_3, -\alpha_4, -\alpha_5, \alpha_6, -\sigma_6 & 1 + \alpha_1 - \sigma_6, 1 + \alpha_2 - \sigma_6 \end{bmatrix} \begin{bmatrix} z_1; \ldots; z_9 \end{bmatrix},
\]

where

\[
z_1 = \frac{p_2^2}{M_1^2}, \quad z_2 = \frac{p_3^2}{M_1^2}, \quad z_3 = \frac{p_4^2}{M_1^2}, \quad z_4 = \frac{s_{13}}{M_1^2}, \quad z_5 = \frac{s_{14}}{M_1^2}, \quad z_6 = \frac{s_{51}}{M_1^2}, \quad z_7 = -\frac{s_{24}}{M_1^2}, \quad z_8 = -\frac{s_{25}}{M_1^2}, \quad z_9 = -\frac{s_{35}}{M_1^2},
\]

the superscript (1) means that one external is massive.

Another one, similar to the second we show for the on-shell case,

\[
\mathcal{J}_6^{(1)}(\{\alpha_i\}) = \Gamma'_6(\alpha_1, \alpha_6, \tilde{j}; s_{51}/M_1^2) \mathcal{T}_9^{(6-\text{off})} \begin{bmatrix} -\alpha_3, -\alpha_4, -\alpha_5, \alpha_6, \alpha_6 - \sigma_6 & 1 + \alpha_2 - \sigma_6, 1 + \alpha_6 - \sigma_6 \end{bmatrix} \begin{bmatrix} z_1; \ldots; z_9 \end{bmatrix},
\]

where it has as the previous one nine variables,

\[
z_1 = \frac{p_2^2}{M_1^2}, \quad z_2 = \frac{p_3^2}{M_1^2}, \quad z_3 = \frac{p_4^2}{M_1^2}, \quad z_4 = \frac{s_{13}}{M_1^2}, \quad z_5 = \frac{s_{14}}{M_1^2}, \quad z_6 = \frac{s_{24}}{M_1^2}, \quad z_7 = \frac{s_{25}}{s_{51}}, \quad z_8 = \frac{s_{35}}{s_{51}}, \quad z_9 = \frac{M_1^2}{s_{51}},
\]

where \( \Gamma'_6(.) = \Gamma'_5(.) \) substituting \( \sigma_5 \) by \( \sigma_6 \).

B. Six massive external legs

Instead of presenting the particular cases – as we have done for the pentagon – where two, three, four and five external are massive, we jump toward the most general case, namely, six massive external legs,

\[
\mathcal{J}_6^{(6)}(\{\alpha_i\}) = \Gamma_6(\alpha_1, \alpha_6; M_5^2) \mathcal{S}_{14}^{(6-\text{off})} \begin{bmatrix} -\alpha_2, -\alpha_3, -\alpha_4, -\alpha_5, -\sigma_6 & 1 + \alpha_1 - \sigma_6, 1 + \alpha_6 - \sigma_6 \end{bmatrix} \begin{bmatrix} z_1; \ldots; z_{14} \end{bmatrix},
\]

23
where

\[
\begin{align*}
  z_1 &= \frac{p_2^2}{M_5^2}, \quad z_2 = \frac{p_3^2}{M_6^2}, \quad z_3 = \frac{p_4^2}{M_5^2}, \quad z_4 = -\frac{s_{13}}{M_5^2}, \quad z_5 = -\frac{s_{14}}{M_5^2}, \quad z_6 = \frac{s_{51}}{M_5^2}, \quad z_7 = -\frac{s_{24}}{M_5^2}, \\
  z_8 &= \frac{s_{25}}{M_5^2}, \quad z_9 = \frac{s_{35}}{M_6^2}, \quad z_{10} = -\frac{M_1^2}{M_5^2}, \quad z_{11} = -\frac{M_2^2}{M_5^2}, \quad z_{12} = -\frac{M_3^2}{M_5^2}, \quad z_{13} = \frac{M_3^2}{M_5^2}, \quad z_{14} = \frac{M_4^2}{M_5^2}, \quad (104)
\end{align*}
\]

where \(\Sigma_j = j_{123456789}\) and \(\Sigma_n = n_{12345}\). From the above result we can infer several special cases, namely, on-shell external legs, equal masses external legs and so on. To work out these particular cases one can proceed on the same way we did in the previous sections and subsections.

### C. On-shell hexagon with 6 massive propagators

In this subsection, the last one, we present for completeness a result for the hexagon scalar integral, where all its external legs are massless and on-shell and its propagators have distinct masses, \(M_1, \ldots, M_6\).

We pick a sample hypergeometric series, that allows us to obtain several particular cases (equal masses, or some of them null). The one that is not contained in this kinematical region is the case that has \(M_6 = 0\).

Let us call such 14-fold series \(J_{6m}\)

\[
J_{6m}(\{\alpha_i\}) = \Gamma_6^{(mass)}(\alpha_6; M_6) S_{14}^{(6-mass)} \left[ -\alpha_1, -\alpha_2, -\alpha_3, -\alpha_4, -\alpha_5, -\sigma_6 \middle| 1 + \alpha_6 - \sigma_6, D/2 \right]_{z_1; \ldots; z_{14}}, \quad (105)
\]

where

\[
\begin{align*}
  z_1 &= -\frac{p_2^2}{M_5^2}, \quad z_2 = -\frac{p_3^2}{M_6^2}, \quad z_3 = -\frac{p_4^2}{M_5^2}, \quad z_4 = -\frac{s_{13}}{M_5^2}, \quad z_5 = -\frac{s_{14}}{M_5^2}, \quad z_6 = \frac{s_{51}}{M_5^2}, \quad z_7 = -\frac{s_{24}}{M_5^2}, \\
  z_8 &= \frac{s_{25}}{M_5^2}, \quad z_9 = \frac{s_{35}}{M_6^2}, \quad z_{10} = \frac{M_1^2}{M_5^2}, \quad z_{11} = \frac{M_2^2}{M_5^2}, \quad z_{12} = \frac{M_3^2}{M_5^2}, \quad z_{13} = \frac{M_3^2}{M_5^2}, \quad z_{14} = \frac{M_4^2}{M_5^2}, \quad (106)
\end{align*}
\]

where \(\Gamma_6^{(mass)}\) is given by equation (104) changing \(\sigma_5\) by \(\sigma_6\). So we finish our study on six-point functions. Special cases where two or more masses are equal can be obtained summing up the series, as we did in previous sections.
IX. CONCLUSIONS

Higher one-loop n-point integrals are becoming more important to scattering process\[33, 39, 40\] in the standard model. We presented in this work a general formula for such integrals, and did show how to apply it to several cases of interest, from one through six-point integrals. Being its external legs on or off mass shell, its propagators massless or massive, and their exponents arbitrary. Working with different masses or equal ones raises the same difficulties in the NDIM approach, as well as exponents of propagators, taking the most general case does not imply in additional technical difficulties. We choose sample hypergeometric solutions, the ones in which one can take interesting particular cases. Of course, it is not possible to present all of them (see tables I, II and III). The same procedure can be followed in order to calculate even higher \((N \geq 7)\) scalar integrals.

Acknowledgments

ATS and AGMS gratefully acknowledge CNPq and ESS wishes to thank CAPES for financial support.

APPENDIX A: SOME HYPERGEOMETRIC FUNCTION USED

We present here some of hypergeometric or hypergeometric-type functions used in this paper all in terms of the generalized Lauricella functions $[11]$, that is expressed by

\[
F^{A:B_1,\ldots:B_N}_{C:D_1,\ldots:D_N} \left[ \begin{array}{c} [a : \alpha^{(1)}, \ldots, \alpha^{(N)}] ; [b^{(1)} : \beta^{(1)}] ; \ldots ; [b^{(N)} : \beta^{(N)}] \\ [c : \gamma^{(1)}, \ldots, \gamma^{(N)}] ; [d^{(1)} : \delta^{(1)}] ; \ldots ; [d^{(N)} : \delta^{(N)}] \end{array} \right] \\
\frac{z_1^{j_1} \cdots z_N^{j_N}}{j_1! \cdots j_N!}
\]

\[
= \sum_{j_1,\ldots,j_N=0}^{\infty} \prod_{i=1}^{A} (a_i)_{\alpha_i^{(1)}_{j_1} + \ldots + \alpha_i^{(N)}_{j_N}} \prod_{i=1}^{B_1} (b_i^{(1)})_{\beta_i^{(1)}_{j_1} + \ldots + \beta_i^{(N)}_{j_N}} \prod_{i=1}^{B_2} (b_i^{(N)})_{\beta_i^{(N)}_{j_1} + \ldots + \beta_i^{(N)}_{j_N}} \prod_{i=1}^{C} (c_i)_{\gamma_i^{(1)}_{j_1} + \ldots + \gamma_i^{(N)}_{j_N}} \prod_{i=1}^{D_1} (d_i^{(1)})_{\delta_i^{(1)}_{j_1} + \ldots + \delta_i^{(N)}_{j_N}} \prod_{i=1}^{D_2} (d_i^{(N)})_{\delta_i^{(N)}_{j_1} + \ldots + \delta_i^{(N)}_{j_N}} \times \frac{z_1^{j_1} \cdots z_N^{j_N}}{j_1! \cdots j_N!} \tag{A1}
\]

where

\[
[a : \alpha^{(1)}, \ldots, \alpha^{(N)}] = (a_1 : \alpha_1^{(1)}, \ldots, \alpha_1^{(N)}), \ldots, (a_A : \alpha_A^{(1)}, \ldots, \alpha_A^{(N)}),
\]

\[
[b^{(1)} : \beta^{(k)}] = (b^{(k)}_1 : \beta_1^{(k)}), \ldots, (b^{(k)}_{B(k)} : \beta_{B(k)}^{(k)}),
\]

\[
[c : \gamma^{(1)}, \ldots, \gamma^{(N)}] = (c_1 : \gamma_1^{(1)}, \ldots, \gamma_1^{(N)}), \ldots, (c_C : \gamma_C^{(1)}, \ldots, \gamma_C^{(N)}),
\]

\[
[d^{(1)} : \delta^{(k)}] = (d^{(k)}_1 : \delta_1^{(k)}), \ldots, (d^{(k)}_{D(k)} : \delta_{D(k)}^{(k)}),
\]

\[
[d^{(N)} : \delta^{(N)}] = (d^{(N)}_1 : \delta_1^{(N)}), \ldots, (d^{(N)}_{D(N)} : \delta_{D(N)}^{(N)}).
\]
with $k = 1, ..., N$. The parameters $\alpha, \beta, \gamma, \delta$ can be non-negative integers in hypergeometric functions and too negative integers in hypergeometric-type functions. From (A1) we can extract all the functions used in this paper, that is,

\[
R_6 = R_6 \begin{bmatrix} x_1, x_2, x_3 \\ x_4, x_5 \end{bmatrix} \mid z_1; z_2; z_3; z_4
\]

\[
= F_{1:0;0:0;0}^{4:0:0:0:0} \begin{bmatrix} (x_1 : 1, 1, 1, 1), (x_2 : 1, 1, 1, 0), (x_3 : 1, 0, 0, 0) \\ (x_4 : 1, 0, 1, 1), (x_5 : 1, 1, 0, 0) \end{bmatrix} \mid z_1; z_2; z_3; z_4 \] (A2)

\[
R_{11} = R_{11} \begin{bmatrix} x_1, x_2, x_3, x_4 \\ x_5, x_6, x_7 \end{bmatrix} \mid z_1; \ldots; z_4
\]

\[
= F_{3:0}^{4:0} \begin{bmatrix} (x_1 : 1), (x_2 : 1, 1, 0, 1), (x_3 : 1, 0, 1, 1), (x_4 : 1, 0, 1, 0) \\ (x_5 : 1, 0, 1, 1), (x_6 : 1, 0, 2, 1), (x_7 : 0, 1, -1, 0) \end{bmatrix} \mid z_1; \ldots; z_4 \] (A3)

where $F_{3:0}^{4:0} = F_{3:0;0:0;0:0}^{4:0:0:0:0}$ and $(x_1 : 1) = (x_1 : 1, 1, 1, 1),

\Psi_3 = \Psi_3 \begin{bmatrix} x_1, x_2, x_3 \\ x_4, x_5 \end{bmatrix} \mid z_1; \ldots; z_5
\]

\[
= F_{2:0}^{3:0} \begin{bmatrix} (x_1 : 1, 1, 1, 1, 1), (x_2 : 1, 1, 0, 1, 0), (x_3 : 1, 0, 1, 0, 1) \\ (x_4 : 1, 0, 0, 1, 1), (x_5 : 1, 1, 1, 0, 0) \end{bmatrix} \mid z_1; \ldots; z_5 \] (A4)

where $F_{2:0}^{3:0} = F_{2:0;0:0;0:0}^{3:0:0:0:0}$,

\[
\Phi = \Phi \begin{bmatrix} x_1, x_2, x_3, x_4 \\ x_5, x_6 \end{bmatrix} \mid z_1; \ldots; z_9
\]

\[
= F_{2:0}^{4:0} \begin{bmatrix} (x_1 : a_1), (x_2 : a_2), (x_3 : a_3)(x_4 : a_4) \\ (x_5 : a_5), (x_6 : a_6) \end{bmatrix} \mid z_1; \ldots; z_9 \] (A5)

where $F_{2:0;0:0;0}^{4:0:0:0;0;0:0:0:0} = F_{2:0;0:0;0;0:0:0:0:0}^{4:0:0:0;0:0;0:0:0:0}$ and $(x_1 : a_1) = (x_1 : 1, 1, 1, 1, 1, 1, 1, 1, 1), (x_2 : a_2) = (x_2 : 1, 1, 0, 1, 0, 0, 1, 0, 0), (x_3 : a - 3) = (x_3 : 1, 0, 0, 1, 1, 0, 0, 1, 0), (x_4 : a_4) = \ldots$
\( (x_4 : 0, 1, 0, 1, 0, 0, 1), (x_5 : a_5) = (x_5 : 1, 1, 1, 1, 1, 0, 0, 0) \) and \( (x_6 : a_6) = (x_6 : 1, 1, 0, 1, 0, 0, 0, 1) \).

For the pentagon integrals we define the hypergeometric series, \( S_{A}^{(5)} \), where "5" and \( A \) represent pentagon and the number of variables the series has, respectively.

In equation (17), the result was written in terms of,

\[
S_{5}^{(5)} = S_{6}^{(5)} \left[ \begin{array}{c|c c c c c c}
    x_1, x_2, x_3, x_4 & z_1; & \ldots; & z_4 \\
    x_5, x_6 & \end{array} \right]
\]

\[
= F_{2;0}^{4;0} \left[ \begin{array}{c|c c c c c c}
    (x_1 : 1, 1, 0, 0), (x_2 : 0, 0, 1, 1), (x_3 : 0, 1, 0, 1), (x_4 : 1) & z_1; & \ldots; & z_4 \\
    (x_5 : 0, 1, 1, 1), (x_6 : 1, 1, 0, 1) & \end{array} \right],
\]

(A6)

Equation (49) was written as a series defined by,

\[
T_{4}^{(5)} = T_{5}^{(5)} \left[ \begin{array}{c|c c c c c c}
    x_1, x_2, x_3, x_4 & z_1; & \ldots; & z_4 \\
    x_5, x_6 & \end{array} \right]
\]

\[
= F_{2;0}^{4;0} \left[ \begin{array}{c|c c c c c c}
    (x_1 : 1, 1, 0, 0), (x_2 : 1, 0, 1, 0), (x_3 : 0, 1, 0, 1), (x_4 : -1, 0, 0, 1) & z_1; & \ldots; & z_4 \\
    (x_5 : 0, 0, 1, 1), (x_6 : 0, 1, -1, 0) & \end{array} \right],
\]

(A7)

Five variables series was obtained as a result for the massive pentagon in the special case where all five masses are equal,

\[
S_{5}^{(5)} = S_{5}^{(5)} \left[ \begin{array}{c|c c c c c c}
    x_1, x_2, x_3, x_4, x_5, x_6 & z_1; & \ldots; & z_5 \\
    x_7 & \end{array} \right]
\]

\[
= F_{1;0}^{6;0} \left[ \begin{array}{c|c c c c c c}
    (x_1 : 11000), (x_2 : 00110), (x_3 : 10001), (x_4 : 01010), (x_5 : 00101), (x_6 : 1) & z_1; & \ldots; & z_5 \\
    (x_7 : 2) & \end{array} \right],
\]

(A8)

where we did not write the commas, i.e., \( (x_4 : 01010) = (x_4 : 0, 1, 0, 1, 0) \) and so forth.

Six variables is one of the representation for the pentagon with two distinct masses,

\[
S_{6}^{(5)} = S_{6}^{(5)} \left[ \begin{array}{c|c c c c c c}
    x_1, x_2, x_3, x_4, x_5, x_6 & z_1; & \ldots; & z_6 \\
    x_7, x_8, x_9 & \end{array} \right]
\]

\[
= F_{3;0}^{6;0} \left[ \begin{array}{c|c c c c c c}
    (x_1 : 110000), (x_2 : 001100), (x_3 : 100010), (x_4 : 010101), (x_5 : 001010), (x_6 : 1) & z_1; & \ldots; & z_6 \\
    (x_7 : 111110), (x_8 : 212120), (x_9 : -1, 0, -1, 0, -1, 1) & \end{array} \right],
\]

(A9)
The reader can observe that the following result and previous two follow a pattern, seven variables, for three different masses,

$$S_7^{(5)} = S_7^{(5)} \begin{bmatrix} x_1, x_2, x_3, x_4, x_5, x_6 & \vdots & z_1, \ldots, z_7 \\ x_7, x_8, x_9 & \end{bmatrix}$$

$$= F_{3,0}^{6,0} \begin{bmatrix} (x_1 : 11000000), (x_2 : 00110000), (x_3 : 10001110), (x_4 : 01010010), (x_5 : 00101000), (x_6 : 1) & \vdots & z_1, \ldots, z_7 \\ (x_7 : 11100000), (x_8 : 11111100), (x_9 : 00, -1, 0011) & \end{bmatrix},$$

(A10)

The eight-fold series representing the case where the pentagon has four different masses,

$$S_8^{(5)} = S_8^{(5)} \begin{bmatrix} (a_1), (a_2), (a_3), (a_4), (a_5), (a_6) & \vdots & z_1, \ldots, z_8 \\ (a_7), (a_8), (a_9) & \end{bmatrix}$$

$$= F_{3,0}^{6,0} \begin{bmatrix} (a_1), (a_2), (a_3), (a_4), (a_5), (a_6) & \vdots & z_1, \ldots, z_8 \\ (a_7), (a_8), (a_9) & \end{bmatrix},$$

(A11)

where, $a_1 = (x_1 : 11000000), a_2 = (x_2 : 00110000), a_3 = (x_3 : 10001010), a_4 = (x_4 : 01010001), a_5 = (x_5 : 00101000), a_6 = (x_6 : 1).

Finally the nine-fold series, representing the on-shell pentagon with five distinct masses,

$$S_9^{(5)} = S_9^{(5)} \begin{bmatrix} (b_1), (b_2), (b_3), (b_4), (b_5) & \vdots & z_1, \ldots, z_9 \\ (x_7), (x_8) & \end{bmatrix}$$

$$= F_{2,0}^{5,0} \begin{bmatrix} (b_1), (b_2), (b_3), (b_4), (b_5) & \vdots & z_1, \ldots, z_9 \\ (x_7 : 111110000), (x_8 : 110101111) & \end{bmatrix},$$

(A12)

where, $b_1 = (x_1 : 11000000), b_2 = (x_2 : 00110000), b_3 = (x_3 : 10001010), b_4 = (x_4 : 01010001), b_5 = (x_5 : 1).

The off-shell pentagon starts with a five variables Lauricella function, in the case where one leg is off mass shell,

$$S_5^{(5-\text{off})} = S_5^{(5-\text{off})} \begin{bmatrix} x_1, x_2, x_3, x_4 & \vdots & z_1, \ldots, z_5 \\ x_5, x_6 & \end{bmatrix}$$

$$= F_{2,0}^{4,0} \begin{bmatrix} (x_1 : 10001), (x_2 : 01100), (x_3 : 00011), (x_4 : 1) & \vdots & z_1, \ldots, z_5 \\ (x_5 : 01110), (x_6 : 1001) & \end{bmatrix},$$

(A13)
the following function represents eq. (72),
\[
T^{(5-1\text{off})}_5 = T^{(5-1\text{off})}_5 \begin{bmatrix} x_1, x_2, x_3, x_4, x_5, x_6 \end{bmatrix}
= \begin{bmatrix} F_{4:0}^{4:0} \end{bmatrix}
\begin{bmatrix} (x_1 : 11001), (x_2 : 10010), (x_3 : 00110), (x_4 : 1) \end{bmatrix},
\]
(A14)

we presented two results when two external legs are off-shell, the first one was given in terms of,
\[
S^{(5-2\text{off})}_6 = S^{(5-2\text{off})}_6 \begin{bmatrix} x_1, x_2, x_3, x_4, x_5, x_6 \end{bmatrix}
= \begin{bmatrix} F_{2:0}^{4:0} \end{bmatrix}
\begin{bmatrix} (x_1 : 100011), (x_2 : 011000), (x_3 : 000110), (x_4 : 1) \end{bmatrix},
\]
(A15)

and the second one,
\[
T^{(5-1\text{off})}_6 = T^{(5-1\text{off})}_6 \begin{bmatrix} a_1, a_2, a_3, a_4, a_5 \end{bmatrix}
= \begin{bmatrix} F_{1:0}^{5:0} \end{bmatrix}
\begin{bmatrix} (a_1) \end{bmatrix},
\]
(A16)

where \((a_1) = (x_1 : 10010, -1), \ (a_2) = (x_2 : 010011), \ (a_3) = (x_3 : 110000), \ (a_4) = (x_4 : 001100), \ (a_5) = (x_5 : 000, -1, 11), \ (a_6) = (x_6 : 111010).\)

The special case of \(M = M_{12}\) gives rise to,
\[
U^{(5-2\text{off})}_5 = U^{(5-2\text{off})}_5 \begin{bmatrix} x_1, x_2, x_3, x_4, x_5, x_6, x_7 \end{bmatrix}
= \begin{bmatrix} F_{2:0}^{5:0} \end{bmatrix}
\begin{bmatrix} (x_1 : 1001, -1), (x_2 : 011101), (x_3 : 11000), (x_4 : 001100), (x_5 : -1, 1) \end{bmatrix},
\]
(A17)

Three external legs off shell was written as a function,
\[
S^{(5-3\text{off})}_7 = S^{(5-3\text{off})}_7 \begin{bmatrix} x_1, x_2, x_3, x_4, x_5, x_6 \end{bmatrix}
\]
\[
= F^{4:0}_{2:0} \left[ \begin{array}{c}
(x_1 : 1100010), (x_2 : 0011011), (x_3 : 00011000), (x_4 : 1) \\
(z_1; \ldots; z_7)
\end{array} \right].
\]

(A18)

Four external legs, the first result presented,

\[
S^{(5-4\text{off})}_8 = S^{(5-4\text{off})}_8 \left[ \begin{array}{c}
a_1, a_2, a_3, a_4 \\
x_5, x_6
\end{array} \right] z_1; \ldots; z_8
\]

\[
= F^{4:0}_{2:0} \left[ \begin{array}{c}
(a_1), (a_2), (a_3), (a_4) \\
(x_5 : 00111111), (x_6 : 10010111)
\end{array} \right],
\]

(A19)

where \((a_1) = (x_1 : 10001110)\), \((a_2) = (x_2 : 01100011)\), \((a_3) = (x_3 : 00011001)\), \((a_4) = (x_4 : 1)\), and the second is written as the generalized Lauricella function,

\[
T^{(5-4\text{off})}_8 = T^{(5-4\text{off})}_8 \left[ \begin{array}{c}
a_1, a_2, a_3, a_4 \\
x_5, x_6
\end{array} \right] z_1; \ldots; z_8
\]

\[
= F^{4:0}_{2:0} \left[ \begin{array}{c}
(a_1), (a_2), (a_3), (a_4) \\
(x_5 : 00111001), (x_6 : 11001011)
\end{array} \right],
\]

(A20)

where \((a_1) = (x_1 : 11001000)\), \((a_2) = (x_2 : 01111100)\), \((a_3) = (x_3 : 01100011)\), \((a_4) = (x_4 : 1)\).

Finally, we define the hypergeometric functions for the five external legs off-shell pentagon,

\[
S^{(5-5\text{off})}_9 = S^{(5-5\text{off})}_9 \left[ \begin{array}{c}
a_1, a_2, a_3, a_4 \\
x_5, x_6
\end{array} \right] z_1; \ldots; z_9
\]

\[
= F^{4:0}_{2:0} \left[ \begin{array}{c}
(a_1), (a_2), (a_3), (a_4) \\
(x_5 : 00111011), (x_6 : 11100110)
\end{array} \right],
\]

(A21)

where \((a_1) = (x_1 : 001101100)\), \((a_2) = (x_2 : 100010110)\), \((a_3) = (x_3 : 01100011)\), \((a_4) = (x_4 : 1)\), the second one,
\[ F_{2,0}^{4,0} \begin{bmatrix} (a_1), (a_2), (a_3), (a_4) \\ (x_5 : 0, -1, 011010, -1), (x_6 : 111001110) \end{bmatrix} z_1; \ldots; z_9, \tag{A22} \]

where \((a_1) = (x_1 : 110001001), \ (a_2) = (x_2 : 001101100), \ (a_3) = (x_3 : 100010110), \ (a_4) = (x_4 : 00111011, -1), \) and the third,

\[ \mathcal{U}_9^{(5-5off)} = \mathcal{U}_9^{(5-5off)} \begin{bmatrix} a_1, a_2, a_3, a_4, a_5 \\ x_6, x_7 \end{bmatrix} z_1; \ldots; z_9 \\
= F_{2,0}^{5,0} \begin{bmatrix} (a_1), (a_2), (a_3), (a_4) \\ (x_6 : 111100110), (x_7 : 0100110, -1, -1) \end{bmatrix} z_1; \ldots; z_9, \tag{A23} \]

where \((a_1) = (x_1 : 01101110, -1), \ (a_2) = (x_2 : 100100111), \ (a_3) = (x_3 : 011110000), \ (a_4) = (x_4 : 110001100)\)

The hexagon integrals are also defined in terms of generalized Lauricella functions, the first result we presented for the on-shell case is given as,

\[ \mathcal{S}_8^{(6-on)} = \mathcal{S}_8^{(6-on)} \begin{bmatrix} a_1, a_2, a_3, a_4, a_5 \\ x_6, x_7 \end{bmatrix} z_1; \ldots; z_8 \\
= F_{2,0}^{5,0} \begin{bmatrix} (a_1), (a_2), (a_3), (a_4) \\ (x_6 : 101011111), (x_7 : 11111010) \end{bmatrix} z_1; \ldots; z_8, \tag{A24} \]

where \((a_1) = (x_1 : 111000000), \ (a_2) = (x_2 : 000111100), \ (a_3) = (x_3 : 100000111), \ (a_4) = (x_4 : 001010100), (a_5) = (x_5 : 1). \) The second one,

\[ \mathcal{T}_8^{(6-on)} = \mathcal{T}_8^{(6-on)} \begin{bmatrix} a_1, a_2, a_3, a_4, a_5 \\ x_6, x_7 \end{bmatrix} z_1; \ldots; z_8 \\
= F_{2,0}^{5,0} \begin{bmatrix} (a_1), (a_2), (a_3), (a_4), (a_5) \\ (x_6 : 01111110), (x_7 : 1101000, -1) \end{bmatrix} z_1; \ldots; z_8, \tag{A25} \]

where \((a_1) = (x_1 : 11010111, -1), \ (a_2) = (x_2 : 111000000), \ (a_3) = (x_3 : 000111000), \ (a_4) = (x_4 : 010100100), (a_5) = (x_5 : 00101001). \)
When one of the external particles is massive, or an external leg is off mass shell, the integral is written in terms of 9-fold Lauricella function,

\[ S_{9}^{(6-1\text{off})} = S_{9}^{(6-1\text{off})} \left[ \begin{array}{c|c|c} a_1, a_2, a_3, a_4, a_5 \\ x_6, x_7 \end{array} \right] z_1; \ldots; z_9 \]

\[ = F_{5:0}^{2:0} \left[ \begin{array}{c} (a_1), (a_2), (a_3), (a_4), (a_5) \\ x_6 : 00011111, x_7 : 111000111 \end{array} \right] z_1; \ldots; z_9, \]

(A26)

where \((a_1) = (x_1 : 100000110), \) \((a_2) = (x_2 : 010100001), \) \((a_3) = (x_3 : 001010100), \) \((a_4) = (x_4 : 000001011), \) \((a_5) = (x_5 : 1), \) the next we presented is also for the case of one massive external particle,

\[ T_{9}^{(6-1\text{off})} = T_{9}^{(6-1\text{off})} \left[ \begin{array}{c|c|c} a_1, a_2, a_3, a_4, a_5 \\ x_6, x_7 \end{array} \right] z_1; \ldots; z_9 \]

\[ = F_{5:0}^{2:0} \left[ \begin{array}{c} (a_1), (a_2), (a_3), (a_4), (a_5) \\ x_6 : 111001110, x_7 : 00111100 -1 \end{array} \right] z_1; \ldots; z_9, \]

(A27)

where \((a_1) = (x_1 : 100001100), \) \((a_2) = (x_2 : 010100010), \) \((a_3) = (x_3 : 001011000), \) \((a_4) = (x_4 : 000000111), \) \((a_5) = (x_5 : 11111100, -1). \)

The most complicated integral, the hexagon with all external legs massive was given as a 14-fold series,

\[ S_{14}^{(6-6\text{off})} = S_{14}^{(6-6\text{off})} \left[ \begin{array}{c|c|c} a_1, a_2, a_3, a_4, a_5, a_6 \\ x_6, x_7 \end{array} \right] z_1; \ldots; z_{14} \]

\[ = F_{5:0}^{2:0} \left[ \begin{array}{c} (a_1), (a_2), (a_3), (a_4), (a_5), (a_6) \\ x_6 : 00011111101111, x_7 : 11111010011110 \end{array} \right] z_1; \ldots; z_{14}, \]

(A28)

where \((a_1) = (x_1 : 00011111000), \) \((a_2) = (x_2 : 1000011001100), \) \((a_3) = (x_3 : 0101000100110), \) \((a_4) = (x_4 : 0010101000011), \) \((a_5) = (x_5 : 1). \)

Finally, the last generalized Lauricella function, for on-shell hexagon with six different massive propagators,

\[ S_{14}^{(6-6\text{mass})} = S_{14}^{(6-6\text{mass})} \left[ \begin{array}{c|c|c} a_1, a_2, a_3, a_4, a_5, a_6 \\ x_6, x_7 \end{array} \right] z_1; \ldots; z_{14} \]
\[ F_{2;0} \left( \begin{array}{c}
(a_1), (a_2), (a_3), (a_4), (a_5), (a_6) \\
(x_7 : 11111111001111), (x_8 : 11111111100000)
\end{array} \right) \]
\[ z_1; \ldots; z_{14} \]  
(A29)

where 
\( a_1 = (x_1 : 11100000010000) \), 
\( a_2 = (x_2 : 00011100001000) \), 
\( a_3 = (x_3 : 10000011000100) \), 
\( a_4 = (x_4 : 01010000100010) \), 
\( a_5 = (x_5 : 00101010000001) \), 
\( a_6 = (x_6 : 1) \).

[1] A.I.Davydychev, J.Math.Phys. 32, 1052 (1991).
[2] A.I.Davydychev, J.Math.Phys. 33, 358 (1992).
[3] É. É. Boos and A. I. Davydychev, Theor. Math. Phys. 89 (1991) 1052.
[4] A.E.Terrano, Phys.Lett.B 93, 424 (1980).
[5] F.V.Tkachov, Phys.Lett.B 100, 65 (1981). K.G.Chetyrkin, F.V.Tkachov, Nucl.Phys.B 192, 159 (1981).
[6] I. G. Halliday, R. M. Ricotta, Phys.Lett.B 193, 241 (1987). G.V.Dunne, I. G. Halliday, Phys.Lett.B 193, 247 (1987).
[7] A.T.Suzuki, A.G.M.Schmidt, J. Phys. A31, 8023 (1998).
[8] M.G.Schmidt, C.Schubert, Phys. Rev. D 53, 2150 (1996).
[9] T. Gehrmann, E. Remiddi, Nucl. Phys. B 601 (2001) 287; Nucl. Phys. Proc. Suppl. 89 (2000) 251. *Ibid*, preprint [hep-ph/0207020].
[10] V. A. Smirnov, Nucl.Phys.B 566, 469 (2000); Phys.Lett.B 460, 397 (1999).
[11] S. Laporta, E. Remiddi, Phys. Lett. B 356 (1995) 390; Phys. Lett. B 379 (1996) 283. V. W. Hughes, T. Kinoshita, Rev. Mod. Phys. 71 (1999) S133. S Laporta, E. Remiddi, Acta Phys. Pol. B 28 (1997) 959.
[12] C. Anastasiou, E. W. N. Glover, C. Oleari, Nucl. Phys. B 575 (2000) 416. Erratum-ibid B 585 (2000) 763. L.W.Garland, T.Gehrmann, E.W.N.Glover, A.Koukoutsakis, E.Remiddi, hep-ph/0206067 .
[13] Z. Bern, L. Dixon, D. A. Kosower, JHEP 1 (2000) 27.
[14] K. Chetyrkin, M. Misiak, M. Münz, Nucl. Phys. B 518 (1998) 473.
[15] E. W. N. Glover, J. B. Tausk, J. J. van der Bij, Phys. Lett. B 516 (2001) 33.
[16] J. B. Tausk, Phys. Lett. B 469 (1999) 225.
[17] J. Fleischer, V. A. Smirnov, A. Frink, J. Körner, D. Kreimer, K. Schilcher, J. B. Tausk, Eur. Phys. J. C2 (1998) 747.
[18] M. C. Bergére, C de Calan and A. P. C. Malbouisson, Commun. Math. Phys. 62 (1978) 137.
[19] L.G. Cabral-Rosetti, M.A. Sanchis-Lozano, hep-ph/0206081.
[20] A. Bashir, R. Delbourgo, M. L. Roberts, J. Math. Phys. 42 (2001) 5553. A. Bashir, A. Kızılersü, M. R. Pennington, Phys. Rev. D62 (2000) 085002. A. Bashir, A. Raya, Phys. Rev. D64 (2001) 105001.
[21] Z. Bern, preprint [gr-qc/0206071].
[22] G. Passarino and M. Veltman, Nucl. Phys. B 160 (1979) 151. O. V. Tarasov, Nucl. Phys. Proc. Suppl. 89 (2000) 237.
[23] C. G. Bollini and J. J. Giambiagi, Nuovo Cimento B 12 (1972) 20.
[24] G. Leibbrandt, Rev. Mod. Phys. 47 (1975) 849.
[25] H-C. Lee, M. S. Milgram, J. Math. Phys. 26 (1985) 1793. ibid, J. Comput. Phys. 59 (1985) 331; Phys. Lett. B133 (1983) 320.
[26] G. ’t Hooft and M. Veltman, Nucl. Phys. B 153 (1979) 365.
[27] Y. L. Luke, The Special Functions and their Approximations, Vol. I, (Academic Press, 1969). L. J. Slater, Generalized Hypergeometric Functions, (Cambridge Univ. Press, 1966). P. Appel, J. Kampé de Feriet, Fonctions Hypergémétriques et Hypersphériques. Polynômes d’Hermite, Gauthier-Villars, Paris (1926). A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions (McGraw-Hill, 1953).
[28] I. G. Halliday, R. M. Ricotta and A. T. Suzuki, Proceedings of the XVII Brazilian Meeting of Particles and Fields, Caxambu, 1996.
[29] A. T. Suzuki, E. S. Santos, A. G. M. Schmidt, hep-th/0205158.
[30] G. Duplančić, B. Nižić preprint [hep-ph/0201306]. To be published in EPJC.
[31] A. I. Davydiev, International Seminar on Quarks, Zvenigorod, 260 (1992). hep-ph/9307323.
[32] D. B. Melrose, Nuovo Cim. 40 (1965) 181.
[33] Z. Bern, L. Dixon, D. A. Kosower, Nucl. Phys. B 412, 751 (1994). Z. Bern, L. Dixon, C. Schmidt, preprint [hep-ph/0206194].
[34] S. Weinzierl, J. Phys. G26 (2000) 654. S. Weinzierl, D. A. Kosower, Phys. Rev. D60 (1999) 054028.
[35] Y. Yasui, preprint [hep-ph/0203163]; Phys. Rev. D61 (2000) 094502. V. Del Duca, F. Maltoni,
Z.Trocsanyi, JHEP 0205:005,2002. L.W. Garland, T. Gehrmann, E.W.N. Glover, A. Koukoutsakis, E. Remiddi, Nucl.Phys.B627 (2002) 107. Z.Nagy, Z.Trocsanyi, Phys.Rev.Lett.87 (2001) 082001. M. Cacciari, V. Del Duca, S. Frixione, Z. Trocsanyi, JHEP 0102:029,2001.

[36] A.T.Suzuki, A.G.M.Schmidt, JHEP 09 (1997) 002. ibid, Eur.Phys.J.C5 (1998) 175.

[37] A.T.Suzuki, E.S.Santos, A.G.M.Schmidt, in preparation.

[38] Z.Bern, L.Dixon, D.A.Kosower, Nucl.Phys.B513 (1998) 3.

[39] T. Binoth, J.P. Guillet, G. Heinrich, C. Schubert, Nucl.Phys.B615 (2001) 385. G. Heinrich, T. Binoth, Nucl.Phys.Proc.Suppl.89 (2000) 246. T. Binoth, J.P. Guillet, G. Heinrich, Nucl.Phys.B572 (2000) 361. See also A. Ferroglia, G. Passarino, M. Passera, S. Uccirati, hep-ph/0209219.

[40] Z.Bern, A. de Freitas, L.Dixon, JHEP 09, 037 (2001).

[41] A. A. Inayat-Hussain, J. Phys. A 20 (1987) 4109.

[42] Z.Bern, L.Dixon, D.A.Kosower, Phys.Lett.B302 (1993) 299.
### TABLE I: Off-shell Scalar Box Integral: number of systems, solutions and kind of series

| Off-shell Box | Total | Solutions | No solution | Hypergeom. series |
|---------------|-------|-----------|-------------|-------------------|
| Massless      | 252   | 162       | 90          | 5-fold            |
| 1 mass        | 462   | 267       | 195         | 6-fold            |
| 2 masses      | 792   | 426       | 366         | 7-fold            |
| 3 masses      | 1287  | 663       | 624         | 8-fold            |
| 4 masses      | 2002  | 1012      | 990         | 9-fold            |

### TABLE II: Pentagon Integral: number of systems, solutions and kind of series

| Pentagon      | Total | Solutions | No solution | Hypergeom. series |
|---------------|-------|-----------|-------------|-------------------|
| On-shell      | 210   | 125       | 85          | 4-fold            |
| 1 off-shell   | 462   | 247       | 215         | 5-fold            |
| 2 off-shell   | 924   | 474       | 450         | 6-fold            |
| 3 off-shell   | 1716  | 855       | 861         | 7-fold            |
| 4 off-shell   | 3003  | 1518      | 1485        | 8-fold            |
| 5 off-shell   | 5005  | 2530      | 2475        | 9-fold            |

### TABLE III: Hexagon Integral: number of systems, solutions and kind of series

| Hexagon       | Total  | Solutions | No solution | Hypergeom. series |
|---------------|--------|-----------|-------------|-------------------|
| On-shell      | 6435   | 2790      | 3645        | 8-fold            |
| 1 off-shell   | 11440  | 4736      | 6704        | 9-fold            |
| 2 off-shell   | 19448  | 7155      | 12293       | 10-fold           |
| 3 off-shell   | 31824  | 12408     | 19416       | 11-fold           |
| 4 off-shell   | 50388  | 19484     | 30904       | 12-fold           |
| 5 off-shell   | 77520  | 30410     | 47110       | 13-fold           |
| 6 off-shell   | 116280 | 45615     | 70665       | 14-fold           |