Eigenvalues and eigenstates of the $s\ell_q(2)$-invariant Universal $R$-operator defined for cyclic representations at roots of unity.

D.R. Karakhanyan

Yerevan Physics Institute, Armenia

(Alikhanian Brs. Str. 2, Yerevan 375036, Armenia)

E-mail: karakhan@lx2.yerphi.am

Abstract

The $s\ell_q(2)$ representations are realized in the space of polynomials for general and exceptional values of deformation parameter $q$ and on finite set of theta-functions for cyclic representation corresponding to $q^N = \pm 1$, which are a natural extension of the polynomials. The complete set of eigenstates of the Universal R-matrix are constructed and corresponding eigenvalues are calculated.
1 Introduction

The representation of the symmetry groups in the functional space seems to be interesting in the applications of theory of integrable systems for construction of the Universal R-matrix [3], which intertwins two arbitrary representations, because it provides a unique approach to the representations regardless to their dimensions [12].

The representations of $q$-deformed (quantum) groups [2], [7] for positive real values of deformation parameter are, in fact, the same as for non-deformed case. When $q$ takes arbitrary complex values $q$-deformed universal enveloping algebra becomes complex with non-unitary representations, which are less interesting from the physical point of view. However, the realization of the representation in the space of functions does not differ from the case of positive real values of $q$.

The only exception is the case of complex roots of unity, which has been considered at the early stage of studying of quantum groups [1]. The new type of representations (cyclic) appears in this case. The unitary representations in this case exists and appear in physical applications [11]. The complete set of irreducible representations of $sl(2)$ was presented in [10].
The eigenvalues and eigenvectors of Universal R-matrix were calculated using functional representations for Heisenberg magnet invariant with respect to $s\ell(2)$, $s\ell(2|1)$ and $s\ell_q(2)$ symmetry groups. The Universal R-matrix in mentioned cases was realized also as an integral operator [12].

2 The case of isotropic (XXX) Heisenberg model

It is reasonable to start the description of the proposed method with discussion of more simple case of the isotropic, i.e. $s\ell(2)$-invariant, Heisenberg spin chain. The integrability of that model is based on the fundamental Yang-Baxter relation (YBE):

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v),$$

(2.1)

here operators $R_{ij}$ act on tensor product of two representations of $s\ell(2)$ algebra: $V_i \otimes V_j$. In the simplest case, when all of spaces $V_i$ correspond to the fundamental (two-dimensional) representation, the solution to the Yang-Baxter equation is given by the $R$-matrix of minimal possible dimension $4 \times 4$:

$$R(u) = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix},$$

(2.2)

where non-zero matrix elements are:

$$a = u + \eta, \quad b = u, \quad c = \eta.$$

Here $u$ is a spectral parameter of the theory and $\eta$ is some model parameter, which can be set equal to unity. The next in complexity solution is the Lax operator, which corresponds to the case when YBE (2.1) is defined on an arbitrary and two fundamental spaces.

$$R_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)R(u-v).$$

(2.3)

Originally the Lax operator appeared in Kortevg-de Vries equation [8] and some other famous problems [9]. In present context it has form:

$$L(u) = \begin{pmatrix} u + \eta S & \eta S^- \\ \eta S^+ & u - \eta S \end{pmatrix} = u + \eta \sigma a^a,$$

(2.4)
where $\sigma_a, a = 1, 2, 3$ are Pauli matrices. When spin of the operators $S$, entering into the $L$-operator, is equal to $1/2$, i.e. third space also is two-dimensional, they can be identified with Pauli matrices: $S^a = \frac{1}{2}\sigma^a$ and that definition coincides with (2.2). The relation (2.4) takes place due to the commutation relations of the $\mathfrak{sl}(2)$ algebra for operators $S^a$.

\[ [S, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = 2S. \quad (2.5) \]

An arbitrary representation of this algebra can be realized in the space of polynomials of one real or complex variable $x$. The general representation of $\mathfrak{sl}(2)$ is specified by one parameter $\ell$, which is the spin of the representation. The generators (2.5) can be realized as differential operators:

\[ S^- = \partial, \quad S = x\partial - \ell, \quad S^+ = 2\ell x - x^2\partial. \quad (2.6) \]

In this realization the lowest weight vector always exists. It is annihilated by the lowering generator $S^-$ and is obviously given by a constant function. The representation space is given then by linear combinations of monomials:

\[ \{x^n\}. \quad (2.7) \]

This monomials appear after repeating action of rising generator $S^+$ on the lowest weight vector. If parameter $\ell$ is specified by half integer number: $2\ell = N$, this procedure is ended on $N$-th step as it can be seen from definition (2.6). In this case the representation possesses also the highest weight vector and is finite-dimensional $d = N+1$. Otherwise the process of creating of new vectors is not finished and for general complex values of $\ell$ the representation $V_\ell$ is infinite-dimensional. The tensor product of two such representations can be built in similar way in the space of functions of two variables. The $\mathfrak{sl}(2)$ generators are given then by the sum of corresponding generators:

\[ S^- = \partial_1 + \partial_2, \quad S = x_1\partial_1 + x_2\partial_2 - \ell_1 - \ell_2, \quad (2.8) \]

\[ S^+ = 2\ell_1x_1 + 2\ell_2x_2 - x^2\partial_1 - x^2\partial_2. \]

The lowest weight vector $\varphi_0(x_1, x_2)$ is defined again as a solution to the equation:

\[ S^-\varphi_0(x_1, x_2) = 0, \]

and is given by the monomials of translational invariant difference: $(x_1 - x_2)^n$. The whole space of tensor product $V_{\ell_1} \otimes V_{\ell_2}$ is reduced to the space of
polynomials of variable $x_1$ with degree not more than $2\ell_1$ and of variable $x_2$
with the degree no more than $2\ell_2$ if both parameters $2\ell_i$ are given by positive
integer numbers. Otherwise it is infinite-dimensional and given by the linear
space with basis:

$$\varphi_n^m(x_1, x_2) = (S^+)^m \varphi_n(x_1, x_2), \quad \varphi_n(x_1, x_2) = (x_1 - x_2)^n. \quad (2.9)$$

The next in complexity solution is related to the case when one of vect or
spaces in (2.3) corresponds to the fundamental representation of $s\ell(2)$ while
two others are arbitrary ones. Corresponding Yang-Baxter equation can be
used as a defining relation to determine the universal $R$-operator. Let third
space to be fundamental. Lax operators are $2 \times 2$ matrices with respect to
the third space and are differential operators with respect to the first and the
second spaces, while the Universal $R$-operator is inert (scalar) with respect
to the third space and is complicated pseudo-differential operator acting on
the second and third spaces. So YBE (2.3) has to be understood as a matrix
relation $2 \times 2$ both sides of which act on some function, belonging to tensor
product $V_1 \otimes V_2$.

The authors of [5] used this approach to determine a recurrent relation on
the matrix elements of the universal $R$-matrix in finite-dimensional case. To
carry out the same procedure in this case we shall use a manifest form of
$L$-operator and shall separate dependence on $u + v$ and $u - v$. Then YBE
takes the form:

$$R_{12}(u) S^a = S^a R_{12}(u),$$
$$R_{12}(u) K(u) = \bar{K}(u) R_{12}(u),$$

where we denoted

$$K(u) = \begin{pmatrix} \frac{u}{2}(S_2 - S_1) + S_1 S_2 + S_1^- S_2^+ & \frac{u}{2}(S_2^- - S_1^-) + S_1 S_2^- - S_1^- S_2^- \\ \frac{u}{2}(S_2^+ - S_1^+) - S_1 S_2^+ + S_1^+ S_2 & \frac{u}{2}(S_1 - S_2) + S_1 S_2 + S_1^+ S_2^- \end{pmatrix},$$

and

$$\bar{K}(u) = \begin{pmatrix} \frac{u}{2}(S_2 - S_1) + S_1 S_2 + S_1^- S_2^- & \frac{u}{2}(S_2^- - S_1^-) + S_1 S_2 - S_1^- S_2^- \\ \frac{u}{2}(S_2^+ - S_1^+) + S_1 S_2^+ - S_1^+ S_2 & \frac{u}{2}(S_1 - S_2) + S_1 S_2 + S_1^+ S_2^+ \end{pmatrix}.$$
defined on all vectors of basis \((S^+)^m \varphi_n(x_1, x_2), (m, n = 0, 1, 2, 3...). However due to commutativity of \(R\)-operator with \(S^+\) one can restrict oneself only to consideration of the lowest weight vectors \(\varphi_n(x_1, x_2)\), because on vectors created from the lowest weight ones by \(S^+\) YBE will take place automatically. It follows from the symmetry relations that \(R\)-operator commutes with Casimir operator:

\[ C = (S_1^a + S_2^a)^2, \]

i.e. these operators have common set of eigenstates. In other words, the lowest weight vectors and vectors created from them by the action of rising generator \((S^+)^m \varphi_n(x_1, x_2)\) are eigenstates of the \(R\)-operator with eigenvalues which independent on \(m\). Then, the matrices \(K(u)\) and \(\bar{K}(u)\) transform covariantly with respect to the algebra \(s\ell(2)\):

\[ [\sigma^a - S^a, K(u)] = 0, \quad [\sigma^a + S^a, \bar{K}(u)] = 0. \]

This relation means the linear dependence of corresponding equations. In order to determine eigenvalues of the \(R\)-operator it is enough to solve the simplest equation corresponding to the right upper corner of matrices \(K\) on lowest weight vectors. One has:

\[ R(u)K(u)(x_1 - x_2)^N = (\ell_1 + \ell_2 + 1 - N - u)R(u)(x_1 - x_2)^{N-1} = \bar{K}R(u)(x_1 - x_2)^N, \]

and as vectors \((x_1 - x_2)^N\) are eigenstates of the \(R\)-operator one can deduce from here the following recurrent relation on eigenvalues of the \(R\)-operator:

\[ R_N(u) = -R_{N-1}(u) \frac{\ell_1 + \ell_2 + 1 - N - u}{\ell_1 + \ell_2 + 1 - N + u}, \quad (2.10) \]

\[ R_N(u) = (-1)^N R_0(u) \prod_{n=1}^{N} \frac{\ell_1 + \ell_2 + 1 - n - u}{\ell_1 + \ell_2 + 1 - n + u}. \]

Using this recurrent relation one can restore explicit form of the universal \(R\)-matrix for given value of parameters \(\ell_1\) and \(\ell_2\). Some simple examples are given in Appendix.

3 \(s\ell_q(2)\) algebra.

Arguments presented in previous section work almost without any changes in other, more complicated cases. The case of usual so called \(q\)-deformation,
which corresponds to violation of three-dimensional rotational symmetry of XXX Heisenberg model to cylindrical symmetry of XXZ model is considered below. The case of Heisenberg chain invariant with respect to the superalgebra \( s\ell(2|1) \) was considered in \([12]\). The work devoted to consideration of the model deformed by the dimensionful parameter, which is more interesting from the physical point of view is in progress \([13]\).

Mentioned symmetry violation in XXZ model described by introduction a new parameter \( \Delta = \cos \lambda \) into Hamiltonian. This new model can be considered as deformation of isotropic model and namely in that context the notion of quantum group was appeared \([4]\). However there exists another point of view \([6]\), according to which general integrable two-dimensional quantum field theory at classic level is described by Yang-Baxter equation corresponding to rational dependence of the \( R \)-matrix on spectral parameter (analog of XXX chain), while upon quantization the quantum fluctuations leads to the appearance of quantum anomalies, describing violation, more correctly deformation of some classic symmetries of the initial theory. It is equivalent to deformation of the representation space structure, i.e. eigenvalues and eigenstates of the physical quantities such as anomalous dimensions.

So in this way, \( q \)- or quantum deformation of \( s\ell(2) \) has two aspects. The first one consists of deformation of algebra itself:

\[
[S, S^\pm] = \pm S^\pm, \quad [S^+, S^-] = \frac{q^{2S} - q^{-2S}}{q - q^{-1}}.
\] (3.11)

In the space of polynomials \( s\ell_q(2) \) generators can be realized through differential operators as follows:

\[
S^- = \frac{1}{x} \frac{q^\vartheta - q^{-\vartheta}}{q - q^{-1}}, \quad S = x\vartheta - \ell, \quad S^+ = x\frac{q^{2\vartheta - \vartheta} - q^{\vartheta - 2\ell}}{q - q^{-1}}.
\] (3.12)

These operators more correctly should be called finite-difference rather than differential taking into account that on test function \( f(x) \) these act according to formulae:

\[
q^{\alpha S} f(x) = q^{-\alpha \ell} f(q^\alpha x),
\]

\[
S^- f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})},
\]

\[
S^+ f(x) = x \frac{q^{2\ell} f(q^{-1}x) - q^{-2\ell} f(qx)}{q - q^{-1}}.
\]
One can pass from multiplicative action of these operators to the additive one using simple change of variables $x = q^t$:

$$q^S = q^{-\ell}e^\partial_t \quad q^S f(t) \equiv q^{-\ell}f(t + 1)$$

$$S^- = q^{-\ell}e^\partial_t - e^{-\partial_t}/q - q^{-1}, \quad S^- f(t) \equiv q^{-\ell}f(t + 1) - f(t - 1)/q - q^{-1} \tag{3.13}$$

$$S^+ = q^{2\ell}e^{-\partial_t} - q^{-2\ell}e^\partial_t/q - q^{-1}, \quad S^+ f(t) \equiv q^{2\ell}f(t - 1) - q^{-2\ell}f(t + 1)/q - q^{-1}.$$ 

In this form the origin of name finite-difference operator becomes absolutely clear.

As for the case of non-deformed symmetry the representation of the algebra is realized in the space of polynomials. This argument was crucial for the choice (3.12) of spin generators. Indeed, the operator of finite-difference derivative $S^-$ annihilates a constant function, which can be chosen as the lowest weight vector. Then acting repeatedly by rising generator one creates higher powers of variable $x$ and that process is ended only if parameter $2\ell$ is given by positive integer. In this case representation possesses also highest weight vector $x^{2\ell}$ and is finite-dimensional. Otherwise it is infinite-dimensional.

Algebra (3.11) possesses the central element, Casimir operator. (In this section we consider the case of general complex values of deformation parameter $q$ different from the roots of unity.)

$$\mathbf{C} = S^+ S^- + [S]_q[S - 1]_q = S^- S^+ + [S]_q[S + 1]_q f(x). \tag{3.14}$$

here notation $[N]_q$ is used for $q$-deformed numbers: $[N]_q = \frac{q^N - q^{-N}}{q - q^{-1}}$. It is easy to see that in above mentioned realization of spin operators (3.12) the Casimir is proportional to unity:

$$\mathbf{C} f(x) = [\ell]_q[\ell + 1]_q.$$

So, for the general values of deformation parameter $q$ the finite-dimensional representations are given by the same spaces $\mathbf{C}^{2\ell+1}$ as in non-deformed case. The case of arbitrary complex $\ell$ corresponds to the infinite-dimensional space of polynomials. The case $|q| = 1$ requires more careful analysis, in particular for $q^N = \pm 1$, the new type representations appear. Those representations called cyclic ones and have no classic (non-deformed) analog. In this section
we restrict ourselves to case $|q| \neq 0$. Such kind of deformation reduces complex plane to a strip. It becomes obvious by considering spin operators in terms of variable $t$ (3.13).
In order to construct the representation with lowest (highest) weight vector it is necessary to solve the following equation:
\[
S^- \varphi_0 = 0, \quad (S^+ \varphi_0 = 0),
\]
for the lowest (highest) vector. One has:
\[
S^- \varphi_0(t) = \frac{q^{-t}}{q - q^{-1}} [\varphi_0(t + 1) - \varphi_0(t - 1)] = 0, \quad \varphi_0(t + 1) = \varphi_0(t - 1).
\]
In other words the lowest weight vector is given by functions of the form:
\[
\varphi_0(t) = \sum_k a_k e^{i\pi k t},
\]
with arbitrary complex coefficients $a_k$. Then the lowest weight vector creates the whole representation space:
\[
\{ \varphi_n(t) \} = \{(S^+)^n \varphi_0(t) \}, \quad (3.15)
\]
\[
\varphi_n(t) = (S^+)^n \varphi_0(t) = [2\ell]_q [2\ell - 1]_q \ldots [2\ell - n + 1]_q \sum_k (-1)^n k a_k e^{(i\pi k + n \log q) t}.
\]
The lowest weight vectors $\varphi^{(k)}_0(t) \equiv a_k e^{i\pi k t}$ appears to be eigenstates of operator of third projection of spin:
\[
q S^- \varphi^{(0)}_0(t) = (-1)^k q^{-\ell} \varphi^{(k)}_0(t).
\]
These create the equivalent representations: $\{ \varphi^{(k)}_0(t) \} = \{ q^{nt} e^{i\pi k t} \}$. However remaining in the frame of the strip $\text{arg } q \in [0, 2\pi]$ one can ignore that difference and consider the representation realized in terms of variable $x$.
Another aspect of transition from $s\ell(2)$ to $s\ell_q(2)$ is related to deformation of tensor product. The simple rule of spin addition $S^a = S^a_1 + S^a_2$ is not working, because the sum of generators does not satisfy to the algebra (3.11). The simple scaling arguments lead to the following modification:
\[
S = S_1 + S_2,
\]
(this is because the symmetry with respect to rotations around the z-axis preserves)

\[ S^+ = q^{aS_1} S_2^+ + q^{bS_2} S_1^+ , \quad S^- = q^{cS_1} S_2^- + q^{dS_2} S_1^- . \]

Then compound generators satisfy to commutation relations (3.11) upon conditions \( d = -a, \) \( c = -b \) and \( a - b = \pm 2. \) In this way two different co-product can be introduced (remaining parameter \( a \) can be absorbed by redefinition of operators):

\[ S^+ = S_1^+ q^S_2 + q^{-S_1} S_2^+ , \]

\[ \Delta : \quad S = S_1 + S_2 , \quad (3.16) \]

\[ S^- = S_1^- q^S_2 + q^{-S_1} S_2^- , \]

and

\[ \bar{S}^+ = S_1^+ q^{-S_2} + q^{S_1} S_2^+ , \]

\[ \bar{\Delta} : \quad \bar{S} = S_1 + S_2 , \quad (3.17) \]

\[ \bar{S}^- = S_1^- q^{-S_2} + q^{S_1} S_2^- . \]

Note that introduced co-products do not preserve symmetry with respect to exchange \( q \leftrightarrow q^{-1} \). It means that building the tensor product using of them one will deal with two equivalent but different realizations of the same space \( V \) (\( \bar{V} \)), corresponding to \( \Delta \) (\( \bar{\Delta} \)). The spaces \( V \) and \( \bar{V} \) built from linear combinations of vectors: \( (S^+)^m \varphi_N(x_1, x_2|q) \equiv \varphi_N^m(x_1, x_2|q) \) and \( (\bar{S}^+)^m \varphi_N(x_1, x_2|q) \equiv \bar{\varphi}_N^m(x_1, x_2|q) \) correspondingly (so called Drinfel'd quantum double) [7]. Here we introduce the lowest weight vectors

\[ \varphi_N(x_1, x_2|q) = \prod_{n=1}^{N} (q^{\ell_1+1-n} x_1 - x_2 q^n q^{-\ell_2} ) , \quad (3.18) \]

which correspond to solutions to the equations:

\[ S^- \varphi_N(x_1, x_2|q) = 0 , \]

and similarly for over barred quantities: \( \bar{\varphi}_N(x_1, x_2|q) = \varphi_N(x_1, x_2|q^{-1}) \). The vectors \( \varphi_N^m(x_1, x_2|q) \) (\( \varphi_N^m(x_1, x_2|q) \)), form basis in the space \( V \) (\( \bar{V} \)) and are eigenstates of Casimir operators

\[ C^{(2)} = q^{S_2-S_1+1} S_1^+ S_2^- + q^{S_2-S_1-1} S_1^- S_2^+ -(q-q^{-1})^{-2} ((q+q^{-1}) (1 + q^{2S_2-2S_1}) + \]

\[ 10 \]
\[ q^{2S_2}[\ell_1]_q[\ell_1 + 1]_q + q^{-2S_1}[\ell_2]_q[\ell_2 + 1]_q \]

and

\[
\bar{C}^{(2)} = (q^{S_1-S_2-1}S_1^+S_2^- + q^{S_1-S_2+1}S_1^-S_2^+) - (q-q^{-1})^{-2} \left( (q + q^{-1}) \left( 1 + q^{2S_1-2S_2} \right) \right) + q^{-2S_2}[\ell_1]_q[\ell_1 + 1]_q + q^{2S_1}[\ell_2]_q[\ell_2 + 1]_q,
\]
correspondingly

\[
C(S^+)^m \varphi_N(x_1, x_2|q) = [N - \ell_1 - \ell_2]_q[N - \ell_1 - \ell_2 - 1]_q(S^+)^m \varphi_N(x_1, x_2|q).
\]
The same is true for over barred quantities. Note that like in non-deformed case the eigenstates are degenerate over index \( m \).

## 4 Yang-Baxter relation

The solution to the YB relation for XXZ spin \( \frac{1}{2} \) chain has the same form as for isotropic chain (2.2) but with trigonometric dependence on spectral parameter \( u \):

\[
a = q^{u+1} - q^{-u-1}, \quad b = q^u - q^{-u}, \quad c = q - q^{-1},
\]

. The deformation parameter \( q = e^{i\lambda} \) enters here through anisotropy parameter of XXZ chain: (cos \( \lambda = \Delta \)). We use notations of article [14]. The fundamental representation is again given by the Pauli matrices as for \( 2 \times 2 \) matrices the relations (3.11) coincide with non-deformed ones. The Lax operator of XXZ model is given by the expression

\[
L_{\ell, a}(u) = \begin{pmatrix}
q^{u+S} - q^{-u-S} & (q - q^{-1})S^- \\
(q - q^{-1})S^+ & q^{u-S} - q^{-u+S}
\end{pmatrix}. \tag{4.19}
\]

The Yang-Baxter relation for two fundamental and one arbitrary representation take place due to commutation relations (3.11) for spin operators. When \( q \to 1 \) these relations go to the non-deformed (2.5).

In order to determine the universal \( R \)-operator we consider YBE for spins \( \ell_1, \ell_2 \) and \( \frac{1}{2} \):

\[
R_{\ell_1, \ell_2}(u-v) \begin{pmatrix}
q^{u+S_1} - q^{-u-S_1} & (q - q^{-1})S_1^- \\
(q - q^{-1})S_1^+ & q^{u-S_1} - q^{-u+S_1}
\end{pmatrix} \begin{pmatrix}
q^{v+S_2} - q^{-v-S_2} & (q - q^{-1})S_2^- \\
(q - q^{-1})S_2^+ & q^{v-S_2} - q^{-v+S_2}
\end{pmatrix} =
\]

\[
\begin{pmatrix}
q^{u+S_1} - q^{-u-S_1} & (q - q^{-1})S_1^- \\
(q - q^{-1})S_1^+ & q^{u-S_1} - q^{-u+S_1}
\end{pmatrix} \begin{pmatrix}
q^{v+S_2} - q^{-v-S_2} & (q - q^{-1})S_2^- \\
(q - q^{-1})S_2^+ & q^{v-S_2} - q^{-v+S_2}
\end{pmatrix} \begin{pmatrix}
q^{u+S_1} - q^{-u-S_1} & (q - q^{-1})S_1^- \\
(q - q^{-1})S_1^+ & q^{u-S_1} - q^{-u+S_1}
\end{pmatrix} \begin{pmatrix}
q^{v+S_2} - q^{-v-S_2} & (q - q^{-1})S_2^- \\
(q - q^{-1})S_2^+ & q^{v-S_2} - q^{-v+S_2}
\end{pmatrix}^{-1} \tag{4.20}
\]
\begin{align*}
(q^{v_{i}^j} S^1_j - q^{u_{i}^j} S^1_j)(q^{u_{i}^j} - q^{v_{i}^j}) S^1_j \quad (q^{u_{i}^j} - q^{v_{i}^j}) S^1_j (q^{u_{i}^j} - q^{v_{i}^j}) S^1_j (q^{u_{i}^j} - q^{v_{i}^j}) S^1_j \quad (q^{u_{i}^j} - q^{v_{i}^j}) S^1_j R_{\ell_1, \ell_2}(u-v).
\end{align*}

As mentioned above, the main intertwining property of the $R$-operator consists in that it intertwines two different, corresponding to co-products $\Delta$ and $\bar{\Delta}$ realizations of the spaces of representations $V$ and $\bar{V}$:

\begin{equation}
R(u)\Delta = \bar{\Delta} R(u).
\end{equation}

Separating in YBE dependence on $u-v$ and $u+v$ one obtains that it equivalent to the following set of equations:

\begin{equation}
[R_{\ell_1, \ell_2}(u), q^{S^1_i S^2_j}] = 0,
\end{equation}

\begin{align*}
R(u) \left( q^{v_{i}^j} S^1_i S^2_j + q^{u_{i}^j} S^1_i S^2_j \right) &= \left( q^{u_{i}^j} S^1_i S^2_j + q^{v_{i}^j} S^1_i S^2_j \right) R(u), \\
R(u) \left( q^{d_{i}^j - S^1_i S^2_j} + q^{d_{i}^j + S^1_i S^2_j} \right) &= \left( q^{d_{i}^j + S^1_i S^2_j} + q^{d_{i}^j - S^1_i S^2_j} \right) R(u), \\
R(u) \left( q^{u_{i}^j S^1_i S^2_j} - q^{u_{i}^j S^1_i S^2_j} \right) &= \left( q^{u_{i}^j S^1_i S^2_j} + q^{u_{i}^j S^1_i S^2_j} \right) R(u), \\
R(u) \left( q^{v_{i}^j S^1_i S^2_j} + q^{v_{i}^j S^1_i S^2_j} \right) &= \left( q^{v_{i}^j S^1_i S^2_j} + q^{v_{i}^j S^1_i S^2_j} \right) R(u).
\end{align*}

One deduces from this set that it compatible with general unitarity property of $R$-operator:

\begin{equation}
R^{-1}(u) = R(-u).
\end{equation}

Indeed, multiplying (4.25) by $R^{-1}(u)$ from the right and from the left one sees that operator $R^{-1}(u)$ satisfies to the relation (4.23) with exchange $u \leftrightarrow -u$, i.e. $R^{-1}(u)$ and $R(-u)$ satisfy to the same equations (the same is true for eqs. (4.26) and (4.24), (4.27) and (4.28)).

Moreover, being physically observable quantity $R$-matrix has to be single-valued function of $q$, i.e. has to be periodic function of $\log q$. In other words $R$-matrix has to be inert to the choice of a strip $\arg q$ on complex plane.
It follows from the YBE (4.22-4.28) that algebra $s\ell_q(2)$ is realized by the spin operators which are in addition to definitions (3.16) and (3.17) twisted by spectral parameter:

\[
S_u^- = q^{\frac{u}{2}+S_1} S_{1}^- + q^{-\frac{u}{2}-S_1} S_{2}^-,
S_u^+ = q^{-\frac{u}{2}+S_1} S_{1}^+ + q^{\frac{u}{2}-S_1} S_{2}^+,
\]

(4.29)

\[
S_{\bar{u}}^- = q^{-\frac{u}{2}-S_1} S_{1}^- + q^{\frac{u}{2}+S_1} S_{2}^-,
S_{\bar{u}}^+ = q^{\frac{u}{2}-S_1} S_{1}^+ + q^{-\frac{u}{2}+S_1} S_{2}^+,
\]

(4.30)

This extension is compatible with $s\ell_q(2)$-invariance of co-products. The Yang-Baxter equations in terms of these operators take more compact form:

\[
\begin{align*}
[R(u), S] &= 0, \\
R(u)S_u^- &= S_{-u} R(u), \\
R(u)S_u^+ &= S_{+u} R(u), \\
R(u)S_{\bar{u}}^- &= \tilde{S}_{-u} R(u), \\
R(u)S_{\bar{u}}^+ &= \tilde{S}_{+u} R(u).
\end{align*}
\]

It allows to express the main intertwining property of $R$-operator as follows:

\[
R(u)\Delta_u = \tilde{\Delta}_{-u} R(u)
\]

(4.31)

which implies

\[
C_{-u} R(u) = R(u) \tilde{C}_u, \quad \tilde{C}_{-u} R(u) = R(u) C_u.
\]

These relations express $s\ell_q(2)$-invariance of $R$-operator, the eigenproblem for which can be formulated as follows:

\[
\begin{align*}
R(u)\varphi_N^m(x_1, x_2 | q, u) &= R_N \varphi_N^m(x_1, x_2 | q, -u), \\
R(u)\varphi_N^m(x_1, x_2 | q, u) &= \tilde{R}_N \varphi_N^m(x_1, x_2 | q, -u),
\end{align*}
\]

(4.32)
where vectors $\varphi$ are the eigenfunctions of Casimir operators constructed from spin operators (4.29) and (4.30). In other words, operator $R(u)$ acts on Drinfel’d quantum double $V \oplus \overline{V}$ in skew-symmetric way:

$$R(u) \left( \frac{\varphi^m_N}{\varphi^m_N} \right) = \left( \begin{array}{cc} 0 & \tilde{R}_N \\ R_N & 0 \end{array} \right) \left( \frac{\varphi^m_N}{\varphi^m_N} \right).$$

If one arranges equations:

$$Rq^{\pm(S_1+S_2)} = q^{\pm(S_1+S_2)}R, \quad (4.33)$$
$$S^-_u R = R\overline{S}^-, \quad (4.34)$$
$$S^+_u R = R\overline{S}^+, \quad (4.35)$$

to be the symmetry relations of the $R$-operator and denoting remaining equations as follows:

$$RK^- = K^- R, \quad (4.36)$$
$$RK^+ = K^+ R, \quad (4.37)$$
$$RK^{+-} - K^{+-} = (q - q^{-1})^{-1} (q^{-S_1-S_2} S^+_1 S^+_2 - q^{-S_1+S_2} S^+_1 S^+_2), \quad (4.38)$$

one obtains:

$$[\overline{S}^-, K^-] = 0 = [S^-, \overline{K}^-], \quad [\overline{S}^+, K^+] = 0 = [S^+, \overline{K}^+],$$
$$[\overline{S}^-, K^+] = (q - q^{-1})^{-1} (q^{-S_1-S_2} S^+_1 S^+_2 - q^{-S_1+S_2} S^+_1 S^+_2),$$
$$[S^-, \overline{K}^+] = (q - q^{-1})^{-1} (-q^{-S_1-S_2} \overline{K}^{+-} + q^{S_1+S_2} \overline{K}^{+-}),$$
$$[\overline{S}^+, K^-] = (q - q^{-1})^{-1} (-q^{-S_1-S_2} K^{+-} + q^{S_1+S_2} K^{+-}),$$

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\[ [S^-_u, \tilde{K}^+] = (q - q^{-1})^{-1} \left( q^{-S_1} - S_2 \tilde{K}^{++} - q^{S_1} + S_2 \tilde{K}^{+-} \right) . \]

So like in non-deformed case quantities \( K \) (\( \tilde{K} \)'s) are linearly dependent and one can restrict oneself to consideration of symmetry relations (4.33-4.35) and the simplest (4.36) equation among remaining ones.

Let us turn now to the "eigenproblem" of the \( R \)-operator in sense of definition (4.32) or that is the same, to the problem of finding eigenstates of Casimir operators \( C_u \) and \( \tilde{C}_u \). It follows from the relation (4.31) that mentioned eigenvectors have the form:

\[ \varphi^m_N(x_1, x_2|q, u) = (S^+_u)^m \varphi_N(x_1, x_2|q, u), \quad (4.40) \]

and

\[ \tilde{\varphi}^m_N(x_1, x_2|q, u) = \varphi^m_N(x_1, x_2|q^{-1}, u), \]

where the lowest weight vectors \( \varphi_N(x_1, x_2|q, u) \) are defined as solutions to the equation

\[ S^-_u \varphi(x_1, x_2) = 0, \]

in form of polynomials of degree not more than \( N \) over variables \( x_1 \) and \( x_2 \):

\[ \varphi_N(x_1, x_2|q, u) = \prod_{n=1}^N (q^{\ell_1 + 1 - n - \frac{n}{2}} x_1 - q^{\frac{n}{2} + n - 1 - \ell_2}), \quad (4.41) \]

\[ \tilde{\varphi}_N(x_1, x_2|q, u) = \varphi_N(x_1, x_2|q^{-1}, u) = \prod_{n=1}^N (q^{-\ell_1 + 1 + \frac{n}{2}} x_1 - q^{-\frac{n}{2} + n + 1 + \ell_2}). \]

It is easy to see that functions \( \varphi(x_1, x_2|q, u) \) differ from \( \varphi_N(x_1, x_2|q, u) \) by formal exchange \( (x_1, \ell_1) \leftrightarrow (x_2, \ell_2) \). It can be shown also

\[ S^-_u \left( \varphi_N(x_1, x_2|q, u) \right) = \left( \varphi_N(x_1, x_2|q, u) \right) = \begin{pmatrix} 0 \\ \tilde{\varphi}_{N-1}(x_1, x_2|q, u) \end{pmatrix}, \]

\[ S^-_u \left( \varphi_N(x_1, x_2|q, u) \right) = \left( \begin{pmatrix} \ell_1 + \ell_2 + 1 - N - u \end{pmatrix} \right) \]

\[ \begin{pmatrix} (q - q^{-1})[u]_q [N]_q q^{\ell_1 - \ell_2} \varphi_{N-1}(q^{-1} x_1, q x_2, u) \\ (q - q^{-1})\ell_2 [N]_q q^{\ell_1 - \ell_2} \varphi_{N-1}(x_1, x_2|q, u) \end{pmatrix}. \]

Another relations appearing in eigenproblem (4.32) can be obtained from this one by exchange \( q \leftrightarrow q^{-1} \) or \( u \leftrightarrow -u \). Taking into account that

\[ R(u) \begin{pmatrix} \varphi_N(x_1, x_2|q, u) \\ \tilde{\varphi}_N(x_1, x_2|q, u) \end{pmatrix} = \begin{pmatrix} 0 & \tilde{R}_N \\ R_N & 0 \end{pmatrix} \begin{pmatrix} \varphi_N(x_1, x_2|q, u) \\ \tilde{\varphi}_N(x_1, x_2|q, u) \end{pmatrix}, \quad (4.42) \]
and acting by the relation (4.36) to $\varphi_N(x_1, x_2|q, u)$ one obtains zero in both sides, while acting to $\bar{\varphi}_N(x_1, x_2|q, u)$ one obtains desirable recurrent relation on eigenvalues $R_N$'s:

$$R_N = -R_{N-1} \frac{[\ell_1 + \ell_2 + 1 - N - u]_q}{[\ell_1 + \ell_2 + 1 - N + u]_q}$$  \hspace{1cm} (4.43)$$

multiplied on $\varphi_{N-1}(x_1, x_2|q, -u)$. Then acting by the relation (4.34) to $\bar{\varphi}_N(x_1, x_2|q, u)$ one obtains identity $0 = 0$ and acting to $\varphi_N(x_1, x_2|q, u)$ one obtains another relation:

$$\tilde{R}_N = -\tilde{R}_{N-1} \frac{[\ell_1 + \ell_2 + 1 - N - u]_q}{[\ell_1 + \ell_2 + 1 - N + u]_q},$$

multiplied on $\bar{\varphi}_{N-1}(x_1, x_2|q, -u)$. Notice that due to the relation $\varphi_0(x_1, x_2|q, u) = \bar{\varphi}_0(x_1, x_2|q, u) = 1$ the initial conditions of both recurrent relations are the same too $R_0 = \tilde{R}_0$ so both quantities coincide for all values $n$

$$R_N = \tilde{R}_N = (-1)^N R_0 \prod_{n=1}^{N} \frac{[\ell_1 + \ell_2 + 1 - n - u]_q}{[\ell_1 + \ell_2 + 1 - n + u]_q}.$$  

That means that eigenvalues of universal $R$-operator are inert with respect to exchange $q \rightarrow q^{-1}$. Notice also that (4.43) differs from corresponding non-deformed expression consist of the replacement of usual number $s$ to quantum ones.

### 5 Universal $R$-operator for the case of exceptional values of $q$.

In this section we consider the case of exceptional values of deformation parameter, corresponding to complex roots of unity of degree $N$. We set for definiteness $q = \exp(2\pi i/N)$ where $N$ is odd number.

In this case the center of algebra $s\ell_q(2)$ is extended and three more Casimir operator appear: $q^NS$ and $(S^\pm)^N$. Indeed the relation (3.11) in this case gives:

$$(q^S)^N S^\pm = q^{\pm N} S^\pm q^{NS} = S^\pm (q^S)^N,$$

$$q^S (S^-)^N = q^N (S^-)^N q^S = (S^-)^N q^S,$$
\[ [S^+, (S^-)^N] = \sum_{n=0}^{N-1} (S^-)^{N-n-1} [S^+, S^-] (S^-)^n = \]
\[
\frac{1}{q - q^{-1}} \sum_{n=0}^{N-1} (S^-)^{N-n-1} \left( q^{-2n} (S^-)^n q^{2S} - q^{2n} (S^-)^n q^{-2S} \right) = \]
\[
\frac{(S^-)^{N-1}}{q - q^{-1}} \left( q^{2S} \sum_{n=0}^{N-1} q^{-2n} - q^{-2S} \sum_{n=0}^{N-1} q^{2n} \right) = 0. \]

Here we took into account that for mentioned values of \( q \) the both sums turn to be zero due to the formula of geometric progression: \( \sum_{n=0}^{N-1} q^{2n} = \frac{q^{2N-1}}{q^2-1} = 0 \). Similarly one has:

\[ [(S^+)^N, S^-] = 0, \quad (S^+)^N q^S = q^S (S^+)^N. \]

That means that for mentioned operators the relations:

\[ (S^+)^N = \alpha^\pm I, \quad q^{NS} = q^{N\gamma} I, \]

must take place on all vectors of representation. So the functions which form the representation specified by parameters \( \ell, \gamma \) and \( \alpha^\pm \) are determined as the solutions to the set of equations:

\[ (S^\pm)^N f(t) = \alpha^\pm f(t), \]
\[ q^{NS} f(t) = q^{N\gamma} f(t). \]

So for exceptional values of deformation parameter the center of \( sl_q(2) \) algebra is extended, i.e. the representations are specified by more number of parameters related to the eigenvalues of new Casimir operators, which are constrained by one algebraic condition. It was shown in famous work of Kac and De Concini [17] that in general case the representations of the \( sl_q(2) \) algebra with deformation parameter \( q \) is given by complex root of unity are specified by three parameters. Indeed definition (3.12) is not most general one. Commutation relations (3.11) can be realized by operators:

\[ S^- = q^{-\frac{1}{2}} x \frac{q^{\alpha - \beta} - q^{\beta - \alpha}}{q - q^{-1}}, \quad q^S = q^{-\frac{1}{2}(\alpha + \beta)} q^x, \quad S^+ = x q^\frac{1}{2} q^{\alpha - \beta} - q^{\beta - \alpha} q^{-1}. \]

However in the case of general values of \( q \) using of this expression is meaningless because algebra possesses only one free parameter \( 2\ell = \alpha + \beta \) related
to the unique Casimir operator. It follows from the definition (5.46) that Casimir operator \( q^{NS} \) acts trivially in functional space:

\[
q^{NS} f(x) = q^{-\frac{\alpha+\beta}{2}} f(q^N x) = q^{-\frac{\alpha+\beta}{2}} f(x),
\]

and is multiple to unity operator i.e. does not provide any restriction on the form of functions which form representation.

The condition \( \alpha^+(\alpha^-) = 0 \) means that operator \( S^+(S^-) \) is nilpotent, i.e. there exists vector, which annihilated by that operator and representation possesses highest (lowest) weight vector. Vise versa, if one supposes that highest (lowest) weight vector does not exist then it would contradict to nilpotency of operator \( S^+(S^-) \). So the representation corresponding to the case \( \alpha^+(\alpha^-) = 0 \) can be obtained by setting parameter \( q \) equal to the root of unity in already considered case of representations with highest (lowest) weight vector for general values of \( q \). So below we suppose that both parameters \( \alpha^\pm \) differ from zero. Consider first the equation for operator \((S^-)^N\) (equation for \((S^+)^N\) can be considered similarly). Commuting multipliers \( q^{-t} = x^{-1} \) to the left one obtains:

\[
(S^-)^N f(x) = q^{-Nt} \prod_{n=0}^{N-1} [x \partial + n]q f(x) = \alpha^- f(x).
\]

Consider the function:

\[
\Phi_N(\alpha) \equiv \prod_{n=0}^{N-1} [\alpha + n]q.
\]

(5.47)

From the simple identity

\[
[\alpha + kN]q = [\alpha]q, \quad k \in \mathbb{Z}
\]

one can establish periodicity property:

\[
\Phi_N(\alpha + 1) = \Phi_N(\alpha).
\]

It follows from this property that in expansion of the product (5.47) (for \( N \) odd) to the sum only terms multiple to \( q^{-N\alpha} \) can survive

\[
\Phi_N(\alpha) = (q - q^{-1})^{-N}(q^{\alpha N} - q^{-\alpha N}).
\]

(5.48)
Then equations \((S^\pm)^N f(t) = \alpha^\pm f(t)\) take the following simple form:

\[
(S^+)^N f(t) = (q - q^{-1})^{-N} q^{Nt}(q^\alpha f(t - N) - q^{-\alpha N} f(t + N)),
\]

and

\[
(S^-)^N f(t) = (q - q^{-1})^{-N} q^{-Nt}(q^{-N\beta} f(t + N) - q^{N\beta} f(t - N)).
\]

The shift \(t \rightarrow t \pm N\) in terms of variable \(x\) means multiplication of the argument by \(q^\pm N = 1\), i.e. identity transformation. So we come to the following restriction for functions realizing cyclic representation of the \(s\ell_q(2)\) algebra in form of finite-difference equations:

\[
x^\pm N f(x) = const f(x).
\] (5.49)

It is well known [18] that theta-functions possesses such kind quasi-periodicity property. Theta-functions play the role of building blocks for functions defining on Riemann surfaces, which is very similar to the role of monomials \(x^n\) for building functions on complex plane. However in theory of Riemann surfaces dependence of theta-functions on the second argument \(\tau\) has crucial significance, in this context the only meaning of second argument is that it provides convergence of corresponding series.

It is reasonable to start the construction of representation after little step back. \(s\ell_q(2)\) algebra is closely related to the Weyl algebra which is formed by two generators \(X\) and \(Z\):

\[
ZX = qXZ,
\] (5.50)

their arbitrary powers and unity. In the space of functions generators (5.50) can be realized in many ways. We mention two of them: using of our finite-difference operators the generators of Weyl algebra can be realized in most simple way as follows:

\[
X = x, \quad Z = q^{x\partial}.
\] (5.51)

Another realization in which we are interested in related to the theory of theta-functions. Recall [18], that theta-function is the analytic, which determined by its Fourier series:

\[
\theta(t, \tau) = \sum_{n=-\infty}^{\infty} \exp i\pi(n^2 \tau + 2nt), \quad Im \tau > 0.
\] (5.52)
Theta-functions with characteristics are introduced using quasi-translation operators $S_a$ and $T_b$:

\[ S_b f(t) = f(t + b), \quad T_a f(t) = e^{i\pi(a^2\tau + 2at)} f(t + a\tau), \quad (5.53) \]

which form following algebra:

\[ S_a S_b = S_{a+b}, \quad T_a T_b = T_{a+b}, \quad S_b T_a = e^{2\pi iab} T_a S_b. \]

Then theta-functions with characteristics $a, b$ is defined as result of action of these operators on usual theta-function:

\[ \theta_{a,b}(t, \tau) \equiv S_b T_a \theta(t, \tau) = \sum_{n=-\infty}^{+\infty} e^{i\pi[(n+a)^2\tau + 2(n+a)(t+b)]}. \quad (5.54) \]

Easy to see that quasi-translations operators (5.53) form the Weyl algebra if one sets $q = e^{2\pi iab}$. Hence theta-functions with characteristics form the space of representation of the Weyl algebra. In the case when parameter $q$ is equal to the root of unity, the generators of Weyl algebra become nilpotent while algebra itself becomes cyclic. In matrix form in this case generator $Z$ is given by diagonal square matrix with powers $q^k$ $k = 0, 1, 2, \ldots N - 1$ while $X$ is square matrix with unities over diagonal and in lower left corner. The representation space is given by the set of $N$ theta-functions with characteristics $b = 0$ and $a_k = \frac{k}{N}$. Indeed setting:

\[ Z = S_{\frac{1}{N}}, \quad X = T_{\frac{1}{N}} \quad (5.55) \]

one can check that these operators obey to the Weyl algebra (5.50) with $q = e^{2\pi i/N}$ and act on mentioned set of theta-functions:

\[ \theta_k(t, \tau) \equiv (T_{\frac{1}{N}})^k \theta(t, \tau) = \sum_{n=-\infty}^{+\infty} e^{i\pi[(n+\frac{k}{N})^2\tau + 2(n+\frac{k}{N})(t+b)]}, \quad (5.56) \]

in following way:

\[ Z\theta_k(z, \tau) = e^{\frac{2\pi i}{N}} \theta_k(t, \tau) = q\theta_k(t, \tau), \quad X\theta_k(z, \tau) = \theta_{k+1}(t, \tau). \]

Choosing basis of representation as follows:

\[ \theta_k(x|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2} x^{Nn+k}, \quad (5.57) \]
one can establish relations:
\[ \theta_k(q^\pm x|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2} x^{Nn+k} q^{+}(Nn+k) = q^{\pm k} \theta_k, \]
\[ x^\pm \theta_k(x|\tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2} x^{Nn+k\pm 1} = \theta_{k\pm 1}(x|\tau). \]

It follows now that spin operators act on basis as:
\[ S^- \theta_k = q^{-\frac{1}{2}} [k - \beta] q^k \theta_{k-1}, \]
\[ q^S \theta_k + q^{-\frac{1}{2}} [\alpha + \beta] \theta_k \]
\[ S^+ \theta_k = q^{\frac{1}{2}} [\alpha - k] q^k \theta_{k+1}. \] (5.58)

It can be checked that basis theta-functions still unchanged upon multiplication by \( x^\pm N \). This is because being multiplied to power function theta-function divides its power index by module \( N \).

6 Tensor product of cyclic representations.

In order to determine an action of the \( R \)-operator it is necessary to realize tensor product of two representations on which Casimir operators corresponding to \( N \)'s power of operators (4.29), (4.30) would multiple of unit operator. The tensor products of finite-dimensional irreducible representations of \( s\ell_q(2) \) at roots of unity were studied and decomposed into direct sum of irreducible representations by D.Arnaudon [19]. However in present context we are interested in a little bit different approach. Let us define basis elements in this space as follows:
\[ \theta_{k_1,k_2} = \sum_{n_i=-\infty}^{\infty} e^{i\pi n_i \tau_{ij} n_j} x_1^{Nn_1+k_1} x_2^{Nn_2+k_2}, \] (6.59)

here symmetric matrix \( \tau_{ij} \) is defined such that its imaginary part sets a negative defined quadratic form. The generators \( Z_i \) and \( X_i \) do not lead out from this set of functions:
\[ Z_i \theta_{k_1,k_2} = q^{k_i} \theta_{k_1,k_2}, \quad X_1 \theta_{k_1,k_2} = \theta_{k_1+1,k_2}, \quad X_2 \theta_{k_1,k_2} = \theta_{k_1,k_2+1}. \]
In fact, as it was shown by V. Bazhanov and Yu. Stroganov [20], the form of L-operator (4.19) is not the most general one when deformation parameter is given by the root of unity $q = e^{2\pi i/N}$, where $N$ is primary number. However their extension, which depends on six free parameters was mainly intended to describe the Chiral Potts Model in the approach of Yang-Baxter equation do not leads to any significant changes in present context. The statement of Bazhanov and Stroganov consists of following: the Lax operator (4.19) with finite-difference operators subjected to Bogolyubov transformation for positive- and negative-frequency operators shifting additive variable $t$ in positive and negative direction with arbitrary coefficients also obeys to Yang-Baxter equation with standard $r$-matrix and $q$ given by a root of unity. So using of this more complicated Lax operator leads only to redefinition of eigenvalues of Casimir operators.

It is easy to see that our basis elements are eigenstates of new Casimirs. Indeed, one has

\[
S^\pm_{k_1,k_2} = q^{\frac{1}{2} (\lambda_1 - \alpha_2 - \beta_2) + k_2}[k_1 - \beta_1]_q \theta_{k_1 - 1,k_2} + q^{\frac{1}{2} (\alpha_1 + \beta_1 - \alpha_2 - \lambda_2 - k_1)}[k_2 - \beta_2]_q \theta_{k_1,k_2 - 1},
\]

Then for their $N$’s powers one obtains using arguments similar to those above:

\[
(S_u^-)^N = q^{N(\frac{u}{2} + S_2)}(S_1^-)^N + q^{-N(\frac{u}{2} + S_1)}(S_2^-)^N,
\]

\[
(S_u^+)^N = q^{N(-\frac{u}{2} + S_2)}(S_1^+)^N + q^{-N(\frac{u}{2} - S_1)}(S_2^+)^N,
\]

from which desirable statement is easily deduced:

\[
(S_u^-)^N \theta_{k_1,k_2} = (q^{\frac{N}{2}(u - \alpha_2 - \beta_2 - \lambda_2)}(q^{-N\beta_1} - q^{N\beta_1}) + q^{\frac{N}{2}(-u + \alpha_1 + \beta_1 - \lambda_2)}(q^{-N\beta_2} - q^{N\beta_2})) \theta_{k_1,k_2},
\]

and in similar manner for operators $S_u^+$ and $\bar{S}_u^\pm$. So the set of $N^2$ functions (6.59) realize the space of tensor product of two representations. We are interested in construction of the eigenvalues of $R$-operator. Again, like as for general values of $q$ a crucial role play Yang-Baxter relations, which express commutativity of the $R$-operator with operators (4.29) and (4.30). Let us look for eigenstates of the $R$-operator as a linear combinations of the basis vectors (6.59)

\[
\varphi_m \equiv \sum_{k=0}^{N-1} a_k \theta_{m-k,k} \quad m = 0, 1, 2 \ldots N - 1.
\]
The operator $S_u^-$ acts on these states as follows:

$$S_u^- \varphi_m = \sum_{k=0}^{n-1} a_k \left( q^{k+\frac{1}{2}(u-\alpha_2-\beta_2-\lambda_1)}[m-k-\beta_1]q^{\theta_{m-1-k,k}} + q^{k-m+\frac{1}{2}(\alpha_1+\beta_1-\lambda_2-u)}[k-\beta_2]q^{\theta_{m-k,k-1}} \right).$$

One can deduce that choosing:

$$a_{k+1} = a_k q^{-2+\frac{1}{2}(\beta_1+\beta_2-\alpha_1+\lambda_2-\lambda_1)}, \quad a_k = a_0 q^{k-2+\frac{1}{2}(\beta_1+\beta_2-\alpha_1+\lambda_2-\lambda_1)}, \quad (6.61)$$

one will obtain:

$$S_u^- \varphi_m(\alpha_i, \beta_i, \lambda_i|x_i) = q^{-1+\frac{1}{2}(u-\lambda_1+\beta_2-\alpha_2)}[m+1-\beta_1-\beta_2]q^{\varphi_{m-1}(\alpha_i, \beta_i, \lambda_i|x_i)}. \quad (6.62)$$

It is easy to check, that operator $S_u^+$ shifts the same combination to up by unity:

$$S_u^+ \varphi_m(\alpha_i, \beta_i, \lambda_i|x_i) = q^{-1-\frac{1}{2}(u-\lambda_1+\beta_2-\alpha_2)}[\alpha_1+\alpha_2+1-m]q^{-1+\frac{1}{2}(u+\lambda_2+\beta_1-\alpha_1)}\varphi_{m+1}(\alpha_i, \beta_i, \lambda_i|x_i). \quad (6.63)$$

The similar formulae take place also for operators $\bar{S}_u^\pm$:

$$\bar{S}_u^- \bar{\varphi}_m(\alpha_i, \beta_i, \lambda_i|x_i, u) = q^{-1+\frac{1}{2}(u+\lambda_1+\beta_2-\alpha_2)}[m+1-\beta_1-\beta_2]q^{-1+\frac{1}{2}(u+\lambda_2+\beta_1-\alpha_1)}\bar{\varphi}_{m-1}(\alpha_i, \beta_i, \lambda_i|x_i), \quad (6.64)$$

$$\bar{S}_u^+ \bar{\varphi}_m(\alpha_i, \beta_i, \lambda_i|x_i, u) = q^{\frac{1}{2}(u+\lambda_1+\beta_2-\alpha_2)-1}[\alpha_1-\alpha_2+1-m]q^{\bar{\varphi}_{m+1}(\alpha_i, \beta_i, \lambda_i|x_i, u)},$$

where

$$\bar{\varphi}_m(\alpha_i, \beta_i, \lambda_i|x_i, u) \equiv \sum_{k=0}^{N-1} q^{k[2-u+\frac{1}{2}(\alpha_1+\alpha_2-\beta_1-\beta_2+\lambda_2-\lambda_1)]}\theta_{m-k,k}. \quad (6.65)$$

So these relations are compatible each to other and to Yang-Baxter equations:

$$R_m(u) = q^{2-u+\alpha_2-\beta_2-\lambda_1}R_{m-1} = q^{m(2-u+\alpha_2-\beta_2-\lambda_1)}R_0(u), \quad (6.66)$$

if one sets

$$R(u)\varphi_m(\alpha_i, \beta_i, \lambda_i|x_i, u) = R_m(u)\bar{\varphi}_m(\alpha_i, \beta_i, \lambda_i|x_i, u),$$

and

$$R(u)\bar{\varphi}_m(\alpha_i, \beta_i, \lambda_i|x_i, u) = R_m(u)\varphi_m(\alpha_i, \beta_i, \lambda_i|x_i, u).$$

In fact the eigenvalues of the $R$-operator can be turn to unity by redefinition of eigenstates $\varphi_m$ (by absorption of phase multipliers).
7 Acknowledgements

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8 Conclusion

In this way realizing the generators of the $sl_q(2)$ algebra generators in form of differential, more precisely finite-difference operators (3.12) or (3.13) it is appears to be possible to construct the functional representation (??), which allows to deal with representations of any dimension and to construct tensor product of such representations and define the universal $R$-operator on this product. Also it is possible to find eigenstates of the $R$-operator in that space and corresponding eigenvalues for general and for exceptional values of deformation parameter. Presented method works both for finite-dimensional both for infinite-dimensional representations. In particular it is applicable to the cyclic representations at roots of unity, which have no classic analogs.

9 Appendix

In this appendix we show how obtained formulae can be used to construct $R$-matrix for given representations. For general values of deformation parameter the representations are given by the same spaces $\mathbb{C}^{2\ell+1}$, as in non-deformed case. So one has for $\ell = \frac{1}{2}$:

$$S^+ = x \frac{q^{1-x^2} - q^{-x^2} - 1}{q - q^{-1}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \frac{1}{x^2} \frac{q^{x^2} - q^{-x^2} - 1}{q - q^{-1}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$S = x \partial - \frac{1}{2} = 1/2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad q^{aS} = \begin{pmatrix} q^{1/2} & 0 \\ 0 & q^{-1/2} \end{pmatrix}$$

normalized vectors are:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1, \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = x.$$
The tensor product of such representations is built to be:

\[
1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad x_1 x_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
\]

the spin operators are:

\[
S^+_u = S^+_1 \otimes q^{-\frac{u}{2}} + S^+_2 \otimes \frac{u}{2} - S^+_1, \quad S^-_u = \left( \begin{array}{cccc}
0 & q^{-\frac{u}{2}} & q^{\frac{u}{2}} & 0 \\
0 & 0 & 0 & q^{-\frac{u+1}{2}} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array} \right),
\]

Casimirs are calculated to be:

\[
C_u = \left( \begin{array}{cccc}
q + q^{-1} & 0 & 0 & 0 \\
0 & q^{-1} & q^{-u} & 0 \\
0 & q^u & q & 0 \\
0 & 0 & 0 & q + q^{-1}
\end{array} \right), \quad \bar{C}_u = \left( \begin{array}{cccc}
q + q^{-1} & 0 & 0 & 0 \\
0 & q & q^u & 0 \\
0 & q^{-u} & q^{-1} & 0 \\
0 & 0 & 0 & q + q^{-1}
\end{array} \right).
\]

and have following eigenvalues:

\[
\varphi_0 = \varphi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \varphi_1(u) = \begin{pmatrix} 0 \\ q^{\frac{1-u}{2}} - q^{-\frac{u+1}{2}} \\ 0 \end{pmatrix}, \quad \varphi_2^1(u) = \begin{pmatrix} 0 \\ q^{-\frac{u+1}{2}} \\ 0 \end{pmatrix}, \quad \varphi_2^2(u) = (q+q^{-1}) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\]

Over barred quantities are obtained by replacement \( q \rightarrow q^{-1} \) and by simultaneous change of sign of argument \( \varphi(-u) \). Looking at the form of eigenvectors it is easy to guess that \( R(u) \)-matrix has block-diagonal form:

\[
R(u) = \begin{pmatrix}
g & 0 & 0 & 0 \\
0 & a & b & 0 \\
0 & c & d & 0 \\
0 & 0 & 0 & h
\end{pmatrix}.
\]
The recurrence relation gives:

\[ R_1 = R_0 \frac{[u-1]_q}{[u+1]_q} \]

then relations \( R(u)\varphi_1(u) = R_1\varphi_1(-u) \) and \( R(u)\varphi_1(u) = \bar{R}_1\bar{\varphi}_1(-u) \) lead to the following conditions:

\[
(q^u - q^{1-u})b = (q - q^{-1})R_1 = (q^{u-1} - q^{1-u})c,
\]

\[
(q^{u-1} - q^{1-u})a = (q^u - q^{-u})R_1 = (q^{u-1} - q^{1-u})d,
\]

while \( R(u)\varphi_0 = R_0\varphi_0 \) gives \( g = h = R_0 \). In this way:

\[
R(u) = \frac{R_1}{q^{u-1} - q^{1-u}} \begin{pmatrix}
q^{u+1} - q^{u-1} & 0 & 0 & 0 \\
0 & q^u - q^{-u} & q - q^{-1} & 0 \\
0 & q - q^{-1} & q^u - q^{-u} & 0 \\
0 & 0 & 0 & q^{u+1} - q^{u-1}
\end{pmatrix},
\]

in accordance with the initial relation. This \( R \)-matrix could be obtained in other way too:

\[
R(u) = [u+1]_q \left( \varphi_0(-u) \times \varphi_0^T(u) + \frac{1}{q + q^{-1}} S_+ \varphi_0(-u) \times \varphi_0^T(u) S_-' \right) - \frac{[u-1]_q}{q + q^{-1}} \bar{\varphi}_1(-u) \times \bar{\varphi}_1^T.
\]

here we omitted the factor: \( \frac{R_0}{[u+1]_q} \). Consider now the case \( \ell_1 = \frac{1}{2}, \ell_2 = 1 \). Spin operators take form:

\[
S_2 = x_2 \partial_2 - 1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & -1
\end{pmatrix}, \quad q^a S = \begin{pmatrix}
q^a & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & q^{-a}
\end{pmatrix},
\]

\[
S^+ = x_2 \frac{q^{2-x_2 \partial_2} - q^{-x_2 \partial_2-2}}{q - q^{-1}} = \sqrt{q + q^{-1}} \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix},
\]

\[
S^- = \frac{1}{x_2} \frac{q^{x_2 \partial_2} - q^{-x_2 \partial_2}}{q - q^{-1}} = \sqrt{q + q^{-1}} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0
\end{pmatrix}.
\]
These act on tensor product according to formulae:

\[ S^+ = q^{-\frac{\lambda}{2}+S_2^+} + q^{\frac{\lambda}{2}-S_1^+} = \]

\[
\begin{pmatrix}
0 & q^{\frac{\lambda}{2}\sqrt{1+q^{-2}}} & 0 & q^{1-\frac{\lambda}{2}} \\
0 & 0 & q^{\frac{\lambda}{2}\sqrt{1+q^{-2}}} & q^{-\frac{\lambda}{2}} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[ S^- = q^{\frac{\lambda}{2}+S_1^+} + q^{-\frac{\lambda}{2}-S_2^-} = \]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
q^{1+\frac{\lambda}{2}} & q^{\frac{\lambda}{2}\sqrt{1+q^{-2}}} & 0 & 0 \\
q^{\frac{\lambda}{2}} & q^{-\frac{\lambda}{2}\sqrt{1+q^{-2}}} & 0 & 0
\end{pmatrix}.
\]

The lowest weight vectors are determined to be:

\[ \varphi_0 = \bar{\varphi}_0 = \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}, \quad \varphi_1(u) = \begin{pmatrix}
0 \\
0 \\
\frac{q^{\frac{\lambda}{2}}}{\sqrt{q+q^{-1}}} \\
\frac{q^{\frac{\lambda}{2}}}{\sqrt{q+q^{-1}}}
\end{pmatrix}.\]

Comparing with the non-deformed case one can see that matrix structure of the representation does not changed and all difference consist of the replacement of ordinary numbers to the quantum ones:

\[ R^q_{\frac{\lambda}{2},1} = \begin{pmatrix}
g & 0 & 0 & 0 \\
0 & a & 0 & b \\
0 & 0 & d & 0 & b & 0 \\
0 & b & 0 & d' & 0 & 0 \\
b & 0 & a' & 0 \\
0 & 0 & 0 & g
\end{pmatrix}.\]
The recurrence relation for present case takes form:

\[
R_1 = \frac{[u - \frac{3}{2}]q}{[u + \frac{3}{2}]q} R_0,
\]

Taking into account that \( \phi_0 \) and \( \phi_3^0 \) are constant vectors, i.e. are independent on \( u \) and inert with respect to the replacement \( q \) to \( q^{-1} \) one immediately obtains:

\[ g = R_0. \]

Acting by the \( R \)-matrix on the lowest weight vectors one obtains the following set of equations:

\[
d - \frac{cq^u}{\sqrt{1 + q^2}} = q^{-1} R_1, \\
b - \frac{aq^u}{\sqrt{1 + q^2}} = - \frac{q^u R_1}{\sqrt{1 + q^2}},
\]

\[
d - \frac{cq^{1-u}}{\sqrt{1 + q^{-2}}} = q R_1, \\
b - \frac{aq^{1-u}}{\sqrt{1 + q^{-2}}} = - \frac{q^{-u} R_1}{\sqrt{1 + q^2}},
\]

which leads to

\[ c = \frac{\sqrt{q + q^{-1} R_1}}{[u - \frac{3}{2}]q} = c', \quad d = \sqrt{q + q^{-1} R_1} \left[ \frac{[u - 1]q}{[u - \frac{3}{2}]q} \right], \quad a = \sqrt{q + q^{-1} R_1} \left[ \frac{[u + 1]q}{[u - \frac{3}{2}]q} \right]. \]

The remaining relations give:

\[
R^q_{\ell_1 \ell_2}(u) = \frac{R_1}{[u - \frac{3}{2}]q} \left( \begin{array}{cccc}
[u + \frac{3}{2}]q & [u + \frac{1}{2}]q & 0 & 0 \\
0 & \sqrt{q + q^{-1}} & [u - \frac{1}{2}]q & 0 \\
0 & 0 & \sqrt{q + q^{-1}} & [u + \frac{1}{2}]q \\
0 & 0 & 0 & [u + \frac{3}{2}]q
\end{array} \right).
\]

which is also coincides with non-deformed result under replacement of ordinary numbers to the quantum ones.

Let us consider then more consistent case \( \ell_1 = \ell_2 = 1 \). The vectors \( \phi_N^q(u) \)
are given by the following matrix expressions:

\[
\begin{align*}
\varphi_0(u) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \varphi_0^4(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \varphi_0^1(u) = \begin{pmatrix} 0 & 0 \\ 0 & q^{-\frac{u}{2} - 1} \end{pmatrix}, \quad \varphi_0^3(u) = \begin{pmatrix} 0 & q^{\frac{u}{2} + 1} \\ q^{-\frac{u}{2} - 1} & 0 \end{pmatrix}, \\
\varphi_1(u) &= \begin{pmatrix} 0 & q^{1-\frac{u}{2}} \\ 0 & 0 \\ q^{1-\frac{u}{2}} & 0 \\ -q^{\frac{u}{2} - 1} & 0 \end{pmatrix}, \quad \varphi_1^2(u) = \begin{pmatrix} q^{1-\frac{u}{2}} & 0 \\ 0 & 0 \\ 0 & -q^{\frac{u}{2} - 1} \\ 0 & 0 \end{pmatrix}, \quad \varphi_1^1(u) = \begin{pmatrix} 0 & q^{-u} \\ 0 & 0 \\ q^{-u} & 0 \\ 0 & 0 \end{pmatrix}, \\
\varphi_2(u) &= \begin{pmatrix} 0 & q^{1-u} \\ q^{1-u} & 0 \\ 0 & -1 \\ q^{u-1} & 0 \end{pmatrix}, \quad \varphi_2^2(u) = \begin{pmatrix} 0 & q^{-2-u} \\ q^{-2-u} & 0 \\ 0 & q + q^{-1} \\ 0 & q^{u+2} \end{pmatrix}.
\end{align*}
\]

The \(R\)-matrix again has the same block form as in the non-deformed case as it can be seen from the form of its eigenvectors:

\[
R_1(u) = [u-2]_q[u+1]_q \frac{R_0(u)}{[u+1]_q[u+2]_q}, \quad R_2(u) = [u-2]_q[u-1]_q \frac{R_0(u)}{[u+1]_q[u+2]_q}.
\]

From the condition \(R(u)\varphi_0(u) = R_0(u)\varphi_0(-u)\) one obtains:

\[
g = R_0(u).
\]
The same is true for $\varphi_0$. The next eigenvalue leads for $\varphi_1(u)$ leads to the set of equations:

$$q^{1-\frac{q}{2}} e - q^{\frac{q}{2}-1} f = q^{1-\frac{q}{2}} R_1(u), \quad q^{1-\frac{q}{2}} f - q^{\frac{q}{2}-1} e = -q^{1+\frac{q}{2}} R_1(u),$$

and the same for $\varphi_2(u)$. One has:

$$e = [u]_q[u + 1]_q \frac{R_0(u)}{[u + 1]_q[u + 2]_q}, \quad f = [2]_q[u + 1]_q \frac{R_0(u)}{[u + 1]_q[u + 2]_q}.$$

The equations for $\varphi_0^1(u)$ and $\varphi_0^3(u)$ lead to the same values for $e$ and $f$. While equations for $\varphi_2(u)$, $\varphi_1^1(u)$ and $\varphi_0^2(u)$ result to:

$$bq^{1-u} - d + cq^{u-1} = q^{-u-1} R_2(u),$$
$$dq^{1-u} - a + dq^{u-1} = -R_2(u),$$
$$cq^{1-u} - d + bq^{u-1} = q^{u+1} R_2(u),$$
$$bq^{-u} + d(q - q^{-1}) - cq^u = q^{-u} R_1(u),$$
$$dq^{-u} + a(q - q^{-1}) - dq^u = -(q - q^{-1}) R_1(u),$$
$$cq^{-u} + d(q - q^{-1}) - bq^u = -q^{u-1} R_1(u),$$
$$bq^{-u-2} + d(q + q^{-1}) + cq^u = q^{2-u} R_0(u),$$
$$dq^{-u-2} + a(q + q^{-1}) + dq^u = (q + q^{-1}) R_0(u),$$
$$cq^{-u-2} + d(q + q^{-1}) + bq^u = q^{u-2} R_0(u).$$

This set is consistent and has following solution:

$$d = [u]_q[2]_q \frac{R_0(u)}{[u + 1]_q[u + 2]_q}, \quad c = [2]_q \frac{R_0(u)}{[u + 1]_q[u + 2]_q},$$
$$b = [u]_q[u - 1]_q \frac{R_0(u)}{[u + 1]_q[u + 2]_q}, \quad a = ([u]_q[u + 1] + [2]_q) \frac{R_0(u)}{[u + 1]_q[u + 2]_q}.$$
\[
\begin{pmatrix}
[u + 1][u + 2] & [u][u + 1] & [2][u + 1] & [u][u - 1] & [2][u] & [2]
\end{pmatrix}
\begin{pmatrix}
[2][u + 1] & [u][u + 1] & [2][u] & [2][u + 1] & [u][u - 1] & [2][u + 1]
\end{pmatrix}
\begin{pmatrix}
[2][u] & [u][u + 1] + [2][u] & [2][u + 1] & [u][u - 1] & [2][u + 1]
\end{pmatrix}
\begin{pmatrix}
[2] & [2][u] & [2][u + 2] & [u + 1][u + 2]
\end{pmatrix}
\].

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