The quantum configuration space of loop quantum cosmology

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Abstract

This paper gives an account of several aspects of the space known as the Bohr compactification of the line, featuring as the quantum configuration space in loop quantum cosmology, as well as of the corresponding configuration space realization of the so-called polymer representation. Analogies with loop quantum gravity are explored, providing an introduction to (part of) the mathematical structure of loop quantum gravity, in a technically simpler context.

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1. Introduction

Loop quantum cosmology (LQC) is a canonical approach to the quantization of symmetry reduced, typically homogeneous, gravitational models. LQC originated from the works of Bojowald (see [Bo] and references therein) was further developed by Ashtekar, Bojowald and Lewandowski [ABL], and more recently by Ashtekar, Pawlowski and Singh [APS]. (See also [BCK] for new developments.)

LQC is inspired by loop quantum gravity1 (LQG), importing (whenever possible) ideas, methods and structures used in LQG to the context of quantum cosmology. In particular, in the quantization of the gravitational sector of homogeneous models, LQC uses a representation of the (standard finite dimensional) Weyl algebra which fails to produce a continuous representation of the configuration part of the Heisenberg–Weyl group2. The LQC quantization is therefore non-equivalent to the usual Schrödinger quantization.

1 For reviews of loop quantum gravity, see [AL5, T1, T2].
2 Representations of Weyl relations violating continuity conditions are sometimes called non-regular in the mathematical-physics literature. Discussions and applications of such representations, both for infinite- and finite-dimensional systems, can be found, e.g., in [AMS, CMS].
The representation of the Weyl relations used in LQC—called in this context the polymer representation—is typically presented in its momentum space version, with momentum operators such as the triads acting diagonally, and connection-related configuration variables acting as translation operators.

On the other hand, it is known that a configuration space realization of the polymer representation exists. However, even though only a finite number of degrees of freedom are involved, the polymer representation requires an extension of the classical configuration space, in order that configuration operators can be realized as functions or multiplication operators. Restricting attention to isotropic models, so that the classical configuration space is the real line \( \mathbb{R} \), the role of such a ‘quantum configuration space’ is in this case played by the so-called Bohr compactification of the line.

The Bohr compactification of the line, which we will denote by \( \mathbb{R}_\circ \), is a well known space within the mathematics community (see, e.g., [R]), with several equivalent characterizations. Not surprisingly, some of these characterizations are analogous to those of the space of generalized connections \( \mathcal{A} \), featuring as the (kinematical) quantum configuration space in LQG\(^3\).

The present paper provides a unified account of several aspects of the quantum configuration space \( \mathbb{R}_\circ \), together with an explicit construction of the configuration space version of the polymer representation\(^4\). Moreover, we explore the analogies with the space \( \mathcal{A} \), aiming at an introduction to (part of) the mathematical structure of LQG, in a technically simpler context.

Necessarily, the paper revisits constructions and arguments previously put forward in the context of LQG. On the other hand, we will obviate detailed technical proofs, putting the emphasis on displaying typical structures instead. We will focus exclusively on kinematical aspects and corresponding structures, leaving aside the Hamiltonian constraint operator and the final picture provided by it.

The paper is organized as follows. Section 2 reviews the basics of the polymer representation used in LQC (see [ABL, AFW, ALS, Ve2, CVZ] for more details and discussion). A review of the emergence of the Bohr compactification and its natural structures is also given. Section 3.1 describes \( \mathbb{R}_\circ \) as a space of (homo)morphisms. Up to some point, this corresponds to a similar characterization of \( \mathcal{A} \). However, \( \mathbb{R}_\circ \) is a group, so it is equipped with more structure, and in particular with the Haar measure. The configuration space realization of the polymer representation is then presented in section 3.2. In section 4, we discuss the projective characterization of \( \mathbb{R}_\circ \), directly related to an important structure in LQG. In section 5, the space \( \mathbb{R}_\circ \) is seen as the spectrum of a classical configuration algebra, corresponding to the original construction of \( \mathcal{A} \). Finally, one appendix is included, where the usual Schrödinger representation is presented in this context, by means of a different, non-equivalent measure on \( \mathbb{R}_\circ \).

2. Polymer representation

2.1. Momentum space formulation

Let us consider the phase space \( T^*\mathbb{R} \equiv \mathbb{R}^2 \) of a particle in the line coordinatized by a configuration variable \( x \) and a canonical momentum variable \( p \). The so-called polymer representation of the associated Weyl relations is defined as follows, using a momentum space

\(^3\) For reviews giving a detailed account of the space \( \mathcal{A} \), see, e.g., [T1, Ve1].

\(^4\) Descriptions of the configuration version of the polymer representation more detailed than previous ones in the literature appeared recently [A, CVZ].
formulation. Let $\mathcal{H}_P$ be the non-separable Hilbert space spanned by mutually orthogonal vectors $\langle p \rangle$, $p \in \mathbb{R}$, with $\langle p' | p \rangle = \delta_{p',p}$, where $\delta_{p,p}$ is the Kronecker delta. A general element of $\mathcal{H}_P$ is of the form
\[ \sum_{p \in \mathbb{R}} \psi(p) \langle p \rangle \quad \text{with} \quad \sum_{p \in \mathbb{R}} |\psi(p)|^2 < \infty. \] 
Thus, the polymer Hilbert space $\mathcal{H}_P$ can also be described as the space of complex functions on $\mathbb{R}$ that are square integrable with respect to the discrete measure. Necessarily, any wavefunction $\psi(p)$ can be non-zero only on a countable subset of $\mathbb{R}$.

The momentum operator $\hat{p}$ is defined in this representation by
\[ \hat{p} \langle p \rangle = p \langle p \rangle \quad \text{or} \quad \hat{p} \psi(p) = p \psi(p). \]
Since the discrete measure is translation invariant, there are also well defined unitary operators $U(k)$ implementing translations in momentum space
\[ U(k) \langle p \rangle = \langle p + k \rangle \quad \text{or} \quad U(k) \psi(p) = \psi(p + k), \quad k \in \mathbb{R}. \]

The exponentiation of the operator $\hat{p}$ together with the operators $U(k)$ provides a representation of the Weyl relations, which is moreover irreducible. However, the representation of $\mathbb{R}$ given by $k \mapsto U(k)$ is clearly not continuous. In fact, for arbitrarily small $k$, a vector $\langle p \rangle$ is mapped by $U(k)$ to an orthogonal one $\langle p + k \rangle$. Thus, the generator which would correspond to the configuration operator $\hat{x}$ is not defined.

The operators $U(k)$ can nevertheless be seen as giving a quantization of the classical configuration functions $e^{ikx}$. Reality conditions are satisfied, since $U^*(k) = U(-k)$. Thus, the polymer representation provides a quantization, in the usual Dirac sense, of the Poisson algebra of phase-space functions made of finite linear combinations of the functions $p$ and $e^{ikx}$, $k \in \mathbb{R}$. In particular, the configuration part of the Poisson algebra is the linear space of continuous and bounded complex functions in $\mathbb{R}$ of the form
\[ \mathbb{R} \ni x \mapsto f(x) = \sum_j c_j e^{ik_j x}, \] 
where the sums are finite, $k_j$ are arbitrary real numbers and $c_j$ are complex coefficients. The set of functions (4) clearly separates points in $\mathbb{R}$, i.e. given $x, x' \in \mathbb{R}, x \neq x'$, one can find a function $f$ such that $f(x) \neq f(x')$. In fact, two functions are sufficient to separate points, e.g., $e^{ik_1 x}$ and $e^{ik_2 x}$, with $k_1/k_2$ being an irrational number.

In this representation, the spectrum of the momentum operator $\hat{p}$ is the real line, but each point of $\mathbb{R}$ belongs to the discrete spectrum, rather than to the continuous spectrum, since well defined eigenvectors $\langle p \rangle$ exist. In this sense, the ‘quantum momentum space’ in the polymer representation can be seen as $\mathbb{R}$ equipped with the discrete topology.

Note that by switching the roles of configuration and momentum one obtains a non-equivalent representation, where the Bohr compactification corresponds to momentum space. With the variable $x$ corresponding to connections and $p$ to triads, the polymer representation used in homogeneous and isotropic LQC [ABL] is the one described above.

Further regarding the correspondence of the present case to LQG, the functions $x \mapsto e^{ikx}$ are playing the role of holonomies, indexed by real numbers $k$, which then correspond to edges.

2.2. From momentum space to configuration space: an overview

Before we start displaying the structure of the Bohr compactification of the line, let us first review its emergence from the perspective of $C^*$-algebras, which is the approach followed in loop quantization. A brief overview of the remaining part of the paper is also given.
The configuration version of the polymer representation is, of course, a unitarily equivalent representation such that the configuration variables (4) are realized as multiplication operators. Note first that the linear space of variables (4) is actually a commutative $^\ast$-algebra with identity, with respect to multiplication of functions and complex conjugation. This algebra, let us call it $C$, is obviously represented by bounded operators in $H_P$. Moreover, the representation is cyclic, i.e. there is a vector, e.g. $|0\rangle \in H_P$, such that the action of (the representation of) $C$ on $|0\rangle$ produces a dense set.

One can now take advantage of the general theory of commutative $C^\ast$-algebras. In a generic situation, e.g. when looking for representations, one would need to complete our $^\ast$-algebra into a $C^\ast$-algebra, by means of an appropriate norm. In the present case, we already have the (equivalence class of the) representation, so the obvious thing to do is to consider the operator norm in the polymer representation, and the corresponding $C^\ast$-algebra of bounded operators generated by the unitaries $U(k)$. Not surprisingly, this $C^\ast$-algebra coincides (up to isomorphism) with the Bohr algebra of almost periodic functions, i.e. the algebra of (bounded and continuous) functions in $\mathbb{R}$ obtained by completing $C$ with respect to the supremum norm. Let us denote this $C^\ast$-algebra by $\overline{C}$, corresponding to the holonomy algebra in LQG.

General results now tell us that any commutative $C^\ast$-algebra (with identity) $A$ can be seen as the algebra $C(\Delta)$ of all continuous functions on a compact space $\Delta$, called the spectrum of the algebra. The isomorphism that maps elements of $A$ to functions in the spectrum is called the Gelfand transform. When, as in the present case, the algebra $A$ is an algebra of functions in a space $S$, and the algebra separates points, the Gelfand transform also gives an injective map $S \to \Delta$, whose image is dense. The Bohr compactification can be seen as a particular case of this Gelfand compactification. In the present case, the spectrum of the algebra $\overline{C}$ is the space $\mathbb{R}$, the Bohr compactification of the line, which then contains $\mathbb{R}$ as a dense subset.

Finally, any cyclic representation of the algebra of continuous functions $C(X)$ on a compact space $X$ can be naturally realized in a Hilbert space $L^2(X, \mu)$ of square integrable functions, for some normalized measure $\mu$ on $X$.

Putting it all together, it is guaranteed to exist a measure $\mu_0$ on $\mathbb{R}$ such that the polymer representation of $\overline{C}$ is unitarily equivalent to the representation in $L^2(\mathbb{R}, \mu_0)$, with elements of $\overline{C}$ acting as multiplication operators.

Note that the knowledge of the spectrum of the configuration algebra alone does not tell us much about the actual 'quantum configuration space' for a given representation. This is determined by the support of the corresponding measure. In the case of the polymer representation the extension from the classical space $\mathbb{R}$ to $\mathbb{R}$ is in fact essential. This situation is typical of quantum field theories, including LQG. On the other hand, one can check that the measure in $\mathbb{R}$ corresponding to the Schrödinger representation is supported on the classical configuration space $\mathbb{R}$.

As in LQG, a projective description of $\mathbb{R}$ can be given. This is related to the inductive structure of the algebra $C(4)$. This structure is provided by the family of the finitely generated $^\ast$-subalgebras, whose union is the whole algebra $C$. In a natural way, the spectrum of (the completion of) each of these algebras is a torus, its dimension being equal to the number of generators. As we consider larger and larger subalgebras, one can see the spectrum $\mathbb{R}$ as the limit of a family of tori, of growing dimensions. (Note, however, that projective–inductive techniques do not play in LQC the same prominent role as in LQG.)

Again on the algebraic side, a natural characterization of $\mathbb{R}$ is related to the properties of the basic elements of $C$, namely the exponential functions. These are labelled by real numbers $k$, and map sums into products. Thus, exponentials define (homo)morphisms from $\mathbb{R}$ to the unit circle in $\mathbb{C}$, which are in particular continuous. It turns out that $\mathbb{R}$ coincides with the set of all, not necessarily continuous, homomorphic maps from $\mathbb{R}$ to the unit circle.
This characterization of \( \mathbb{R} \) is well known in harmonic analysis, where it is seen as the dual group of the discrete group \( \mathbb{R} \) (see, e.g., [R]). The relation with the \( \mathcal{C}^{*} \)-algebra approach is that \( \mathcal{C} \) is essentially generated by this discrete group. Thus, in particular, \( \mathbb{R} \) is a commutative compact group, and comes equipped with the Haar measure, which is precisely the measure \( \mu_{0} \) corresponding to the polymer representation. From this perspective, the unitary transformation from \( \mathcal{H}_{P} \) to \( L^{2}(\mathbb{R}, \mu_{0}) \) is a Fourier transformation, and it is actually quite easy to perform.

Although this group structure is not present in the space of generalized connections \( \mathcal{A} \) in LQG\(^{5} \), its characterization as a set of (homo)morphisms is, in fact, one of the three known characterizations. So, this description also has an analogue in LQG, and we actually start from it, as its early introduction facilitates further discussion.

Note also that introducing the \( \mu_{0} \) measure by projective–inductive techniques (corresponding to the original introduction of the Ashtekar–Lewandowski measure in LQG), although possible, is not natural here, and we therefore start from the given Haar measure in \( \mathbb{R} \). On the other hand, we do explicitly confirm that the Gelfand spectrum of \( \mathcal{C} \) coincides with the dual of the discrete group \( \mathbb{R} \). In particular concerning finitely generated subalgebras (which is the essence of the argument), this discussion provides an easy-to-follow example of the Gelfand compactification, in finite dimensions.

3. Configuration space realization

The space \( \mathbb{R} \) is introduced as a set of (homo)morphisms, corresponding to a similar characterization of \( \mathcal{A} \) (see [AL1, AL2, AL3, Ba] for the origins and [Ve4, Ve1] for reformulation and review). The role of the group of hoops (or the groupoid of paths) is here played by the discrete group \( \mathbb{R} \). The group \( SU(2) \) is replaced by \( T \), the unit circle in the complex plane \( \mathbb{C} \).

As mentioned, the present characterization of \( \mathbb{R} \) is precisely that of a dual group. Although we do not insist on this perspective, the group structure of \( \mathbb{R} \) is introduced from the start. Furthermore, basic results in harmonic analysis will necessarily emerge, in particular in section 3.2, where the configuration version of the polymer representation is presented. In the following, we will refer to homomorphisms simply as morphisms.

3.1. Quantum configuration space as a compact group

Let us consider the real line \( \mathbb{R} \) equipped with the commutative group structure given by addition of real numbers. The Bohr compactification \( \mathbb{R} \) can be described as the set \( \text{Hom}[\mathbb{R}, T] \) of all, not necessarily continuous, group morphisms from the group \( \mathbb{R} \) to the multiplicative group \( T \) of unitaries in \( \mathbb{C} \). Since we are going to introduce the topology of \( \mathbb{R} \) immediately after, we identify it with \( \text{Hom}[\mathbb{R}, T] \) right from the start

\[ \mathbb{R} \equiv \text{Hom}[\mathbb{R}, T]. \quad (5) \]

The generic element of \( \mathbb{R} \) will be denoted by \( \bar{x} \). So, every \( \bar{x} \in \mathbb{R} \) is a map, \( \bar{x} : \mathbb{R} \to T \) such that

\[ \bar{x}(0) = 1 \quad \text{and} \quad \bar{x}(k_{1} + k_{2}) = \bar{x}(k_{1})\bar{x}(k_{2}), \quad \forall k_{1}, k_{2} \in \mathbb{R}. \quad (6) \]

Since \( T \) is a commutative group, it is clear that

\[ \bar{x} \bar{x}'(k) := \bar{x}(k)\bar{x}'(k) \quad (7) \]

\(^{5} \) It is present in the case of generalized \( U(1) \) connections [AL1, M, Ve3, AL4], which is then much closer to the present case.
defines a group structure on $\mathbb{R}$, which is moreover commutative. The group $\mathbb{R}$ is a subgroup of the group of all (not necessarily morphisms) maps from $\mathbb{R}$ to $T$. Since this group of maps can be identified with the product group $\times_{k\in\mathbb{R}} T$, and $T$ is compact, it carries the Tychonoff product topology, with respect to which $\times_{k\in\mathbb{R}} T$ becomes a compact (Hausdorff) group. This structure descends to the subgroup $\mathbb{R}$, making it a topological group. Moreover, from the fact that $\mathbb{R}$ contains only morphisms, one can see that it is a closed subset of $\times_{k\in\mathbb{R}} T$, and it is therefore compact.

Thus, $\mathbb{R}$ is a commutative compact group, with respect to the group operation (7) and the Tychonoff topology. A more explicit description of the topology is the following. For each $k \in \mathbb{R}$, let us consider the function $F_k : \mathbb{R} \rightarrow T$ defined by

$$F_k(\bar{x}) = \bar{x}(k).$$

Functions (8) are continuous and, in fact, the Tychonoff topology in $\mathbb{R}$ is precisely the weakest topology such that all functions (8), $\forall k \in \mathbb{R}$, are continuous.

As in the case of connections in LQG, there is a natural map from the classical configuration space to the quantum configuration space, let us call it $\Theta_1$, defined as follows (we will confirm in section 5 that $\Theta_1$ is indeed the Gelfand–Bohr compactification):

$$\Theta : \mathbb{R} \rightarrow \mathbb{R}, \quad y \mapsto \bar{x}_y, \quad \bar{x}_y(k) := e^{i y k} \quad \forall k \in \mathbb{R}. \quad (9)$$

Since this map is injective, the set $\mathbb{R}$ can be seen as an extension of $\mathbb{R}$. The classical configuration space then appears as a subspace, of continuous morphisms, of the space of all morphisms $\mathbb{R}$. In addition, the image $\Theta(\mathbb{R})$ is a dense subset. It is worth mentioning that the injection $\Theta$ is continuous, but is not a homeomorphism into its image. The topology induced on $\mathbb{R}$ as a subset of $\mathbb{R}$, which can be seen as the weakest topology such that all functions $x \mapsto e^{ikx}, k \in \mathbb{R}$ are continuous, is weaker than the usual topology.

Note that, since we know in advance that $\mathbb{R}$ is the spectrum of the $C^*$-algebra $\mathcal{C}$, it is guaranteed that any almost periodic function in $\mathbb{R}$ can be extended from the dense set $\Theta(\mathbb{R})$ to a continuous function in $\mathbb{R}$ (this is the Gelfand transform in this case). In particular, the functions $F_k$ correspond to the exponential functions $\mathbb{R} \ni x \mapsto e^{ikx}$. However, especially upon introduction of the measure defining the representation, this correspondence is to be seen in the sense of operators, i.e. it is when considered as a multiplication operator that $F_k$ corresponds to the classical function $e^{ikx}$, giving in fact its quantization.

### 3.2. Polymer representation in configuration space

Being a compact group, $\mathbb{R}$ is equipped with a normalized invariant (under the group operation) measure, namely the Haar measure, which we denote by $\mu_0$. The Haar measure gives precisely the configuration space realization of the polymer representation, as we now show.

Let us consider the Hilbert space of square integrable functions $L^2(\mathbb{R}, \mu_0)$, which we will denote by $H_0$. Since the functions $F_k$ (8) are continuous, they are in particular integrable. As expected, they form a complete orthonormal set in $H_0$. One can easily see from the invariance of the measure that the set is orthonormal. In fact, for every fixed $\bar{x}' \in \mathbb{R}$ we have

$$\int_{\mathbb{R}} F_k(\bar{x}') d\mu_0(\bar{x}) = \int_{\mathbb{R}} F_k(\bar{x}) d\mu_0(\bar{x}),$$

leading to

$$(1 - \bar{x}'(k)) \int_{\mathbb{R}} F_k d\mu_0 = 0. \quad (11)$$

6 All spaces considered are Hausdorff spaces, so we will refrain from mentioning it.
Since this is true \( \forall x' \in \mathbb{R} \), we conclude that
\[
\int_{\mathbb{R}} F_k \, d\mu_0 = \delta_{k0}, \quad k \in \mathbb{R},
\] (12)
where \( \delta_{k0} \) is the Kronecker delta. From \( F_0^* = F_{-k} \) and \( F_k F_{-k} = F_{k+k} \) it follows that \([F_k, k \in \mathbb{R}]\) is an orthonormal set. It is also straightforward to confirm that this set is complete, since the space of finite linear combinations of functions \( F_k \) is a \( * \)-subalgebra of the algebra of all continuous functions \( C(\mathbb{R}) \), contains the identity function and separates points in \( \mathbb{R} \). The Stone–Weierstrass theorem then ensures that this linear space is dense in \( C(\mathbb{R}) \), with respect to the supremum norm. (Alternatively, this linear space corresponds to \( C \), which is dense in \( \mathcal{C} \) by construction.) Standard arguments show that it is necessarily dense with respect to the \( L^2 \)-norm\(^7\).

Thus, the Hilbert space \( \mathcal{H}_0 \) is isomorphic to the polymer space \( \mathcal{H}_0 \), the unitary transformation \( T : \mathcal{H}_0 \to \mathcal{H}_0 \) being given by the map between bases
\[
T : |p\rangle \mapsto F_p, \quad \forall p \in \mathbb{R}.
\] (13)

The Hilbert space \( \mathcal{H}_0 \) provides a representation of the \( C^* \)-algebra \( C(\mathbb{R}) \) which is faithful, since the invariant Haar measure is faithful (i.e. every non-empty open set has non-zero measure). Denoting in particular the representative of \( F_k \in C(\mathbb{R}) \) by \( \Pi(k) \) we therefore have
\[
\Pi(k) \psi(x) = F_k(x) \psi(x), \quad \psi(x) \in \mathcal{H}_0.
\] (14)

In addition, since the Haar measure is invariant, it is in particular invariant under the action of the classical subgroup \( \Theta(\mathbb{R}) \). This action thus naturally gives rise to a one-parameter unitary group \( V(y) \)
\[
V(y) \psi(x) = \psi(x, xy), \quad y \in \mathbb{R}, \quad \psi(x) \in \mathcal{H}_0.
\] (15)

One can easily check that the unitary transformation \( T \) maps the operators \( \mathcal{U}(k) \) (3) to \( \Pi(k) \) and \( e^{iyP} \) (2) to \( V(y) \). The quantization of the momentum variable can be defined, e.g., on the dense subspace of finite linear combinations of functions \( F_k \), by
\[
\Pi(p) F_k = k F_k.
\] (16)

The \( \mathcal{H}_0 \) representation defined by (14) and (15) is then a unitarily equivalent version of the polymer representation.

It is interesting to see how irreducibility is achieved in the configuration representation, despite the fact that we have only one momentum operator. This is possible due to the denseness of the orbits of the action of the classical group \( \Theta(\mathbb{R}) \). Thus, there is no non-trivial function in \( \mathbb{R} \) that remains invariant under this action, and therefore no configuration operator commutes with \( \Pi(p) \).

Finally, note that finite linear combinations of the form
\[
\sum c_k F_k(x) = \sum c_k x(k)
\]
are called cylindrical functions. A more general definition will be given in the following section.

4. Projective aspects

The projective characterization of \( \mathbb{R} \) is now presented, corresponding to a similar and very important structure in LQG [AL1, AL2, AL3, Ba, MM, ALMMT]. As mentioned

\(^7\) For compact \( X \), \( C(X) \) is \( L^2 \)-dense for any regular Borel measure. On the other hand, uniform convergence implies \( L^2 \)-convergence.
(see, moreover, section 5), this structure is directly related to the family of finitely generated \(\ast\)-subalgebras of the configuration \(\ast\)-algebra \(C(4)\). To finitely generated subalgebras correspond finitely generated subgroups of \(R\), which then appear naturally, playing the role of finitely generated subgroupoids in LQG. These subgroups, or alternatively sets of real numbers playing the role of graphs, are used as a set of labels for both projective and inductive structures.

4.1. Quantum configuration space as a projective limit

For arbitrary \(n \in \mathbb{N}\), a finite set of real numbers \(\gamma = \{k_1, \ldots, k_n\}\) will be said to be independent if \(k_1, \ldots, k_n\) are algebraically independent with respect to the additive group operation in \(R\), i.e. if the condition

\[
\sum_{i=1}^{n} m_i k_i = 0, \quad m_i \in \mathbb{Z}
\]

implies \(m_i = 0, \forall i\). The set of all such independent sets \(\gamma\) will be denoted by \(\Gamma_1\). A partial order relation making \(\Gamma_1\) a directed set can be introduced as follows. Let \(G_\gamma\) denote the subgroup of \(R\) freely generated by the set \(\gamma = \{k_1, \ldots, k_n\}\),

\[
G_\gamma := \left\{ \sum_{i=1}^{n} m_i k_i, m_i \in \mathbb{Z} \right\}.
\]

A set \(\gamma'\) is said to be greater than \(\gamma\), and we write \(\gamma' \geq \gamma\), if \(G_\gamma\) is a subgroup of \(G_{\gamma'}\). It is clear that given \(\gamma\) and \(\gamma'\) one can always find \(\gamma''\) such that \(\gamma'' \geq \gamma\) and \(\gamma'' \geq \gamma'\), and so \(\Gamma_1\) becomes a directed set. Also \(\Gamma_1\) has no maximal element.

We are now in a position to describe the projective structure of \(\bar{R}\). For each \(\gamma \in \Gamma_1\), let us consider the space (again a group) \(R_\gamma\) of all morphisms from \(G_\gamma\) to \(T\),

\[
R_\gamma := \text{Hom}[G_\gamma, T].
\]

For any pair \(\gamma, \gamma'\) such that \(\gamma' \geq \gamma\) there are surjective projections

\[
p_{\gamma'\gamma} : R_{\gamma'} \to R_\gamma
\]

defined by restrictions, i.e. an element \(\bar{x}_{\gamma'} \in R_{\gamma'}\) is mapped to its restriction to \(G_\gamma\). Projections (20) clearly satisfy the consistency conditions

\[
p_{\gamma''\gamma} p_{\gamma'\gamma'} = p_{\gamma''\gamma'}, \quad \forall \gamma'' \geq \gamma' \geq \gamma.
\]

Such a family of spaces and projections, labelled by a directed set, is called a projective family.

Since each \(G_\gamma\) is freely generated by an independent set \(\gamma = \{k_1, \ldots, k_n\}\), each \(R_\gamma\) is essentially a space of the form \(T^n\), where \(n\) is the cardinality of the corresponding set \(\gamma\). In fact, every element \(\bar{x}_\gamma\) of \(R_\gamma\) is uniquely determined by the images of the generators, and one gets a bijection between \(R_\gamma\) and \(T^n\), given by \(\bar{x}_\gamma \mapsto (\bar{x}_\gamma(k_1), \ldots, \bar{x}_\gamma(k_n))\). From now on we identify each \(R_\gamma\) with the corresponding space \(T^n\) with the standard topology, or with the \(n\)-torus \(\mathbb{R}/(2\pi \mathbb{Z})^n\). (We will not distinguish between the two, i.e. the correspondence \(\mathbb{R}/(2\pi \mathbb{Z}) \ni x \leftrightarrow e^{i x} \in \mathbb{C}\) is freely used.) Each space \(R_\gamma\) is then a compact space. It can be seen that projections (20) are continuous, and so the family \(\{R_\gamma\}_{\gamma \in \Gamma_1}\) forms a compact projective family.

There is a well-defined notion of projective limit of a family of spaces, which in the case of a compact family is again a compact space. The projective limit of the family \(\{R_\gamma\}_{\gamma \in \Gamma_1}\) is the

\[\text{Directed sets can be used as labelling sets for families of objects, generalizing discrete sequences and allowing consistent definitions of limits.}\]
subset of the Cartesian product $\times_{\gamma \in \Gamma} \mathbb{R}_\gamma$ of those elements $(\tilde{x}_\gamma)_{\gamma \in \Gamma}$ that satisfy the consistency conditions

$$p_{\gamma'\gamma} \tilde{x}_\gamma = \tilde{x}_{\gamma'}, \quad \forall \gamma' \geq \gamma.$$  \hspace{1cm} (22)

Not surprisingly, there is a bijection between the projective limit of the family $\{\mathbb{R}_\gamma\}_{\gamma \in \Gamma}$ and the set $\bar{\mathbb{R}} = \text{Hom}[\mathbb{R}, T]$, given by

$$\bar{\mathbb{R}} \ni \bar{x} \mapsto (\bar{x}_\gamma)_{\gamma \in \Gamma},$$  \hspace{1cm} (23)

where $\bar{x}_\gamma$ denotes the restriction of $\bar{x}$ to the subgroup $G_\gamma$. Moreover, the natural topology on the projective limit, namely the weakest topology such that all projections

$$p_\gamma : \bar{\mathbb{R}} \to \mathbb{R}_\gamma, \quad \bar{x} \mapsto \bar{x}_\gamma,$$  \hspace{1cm} (24)

are continuous, coincides with the Tychonoff topology introduced in section 3.

Thus, the compact space $\bar{\mathbb{R}}$ is naturally homeomorphic to the projective limit of the family $\{\mathbb{R}_\gamma\}_{\gamma \in \Gamma}$. Since each space $\mathbb{R}_\gamma$ is in turn homeomorphic to a torus, one can see the space $\bar{\mathbb{R}}$ as a limit of a family of tori of growing dimensions.

Although the projective description of the quantum configuration space is analogous to the one encountered in LQG (with tori $T^n$ corresponding to manifolds $SU(2)^n$), there is a difference worth mentioning, related to the fact that we now have a one-dimensional classical configuration space $\mathbb{R}$, while most of the spaces $\mathbb{R}_\gamma$ are multi-dimensional. Thus, the projection $p_\gamma \Theta(R)$ to a generic $\mathbb{R}_\gamma$ (e.g. with $\gamma$ containing at least two elements) of the image $\Theta(\mathbb{R}) \subseteq \bar{\mathbb{R}}$ does not cover $\mathbb{R}_\gamma$. The image of $\mathbb{R}$ on $\mathbb{R}_\gamma$ is nevertheless dense. In fact, writing $\gamma = \{k_1, \ldots, k_n\}$ and using the identification of $\mathbb{R}_\gamma$ with the $T^n$ torus $\mathbb{R}^n/(2\pi \mathbb{Z})^n$, one can see that $\mathbb{R}$ is mapped to $T^n$ as follows:

$$\mathbb{R} \ni x \mapsto (k_1 x, \ldots, k_n x) \in T^n.$$

(25)

We therefore obtain an injective map $p_\gamma \Theta : \mathbb{R} \to T^n \cong \mathbb{R}_\gamma$ with dense image, since the frequencies $[k_1, \ldots, k_n]$ are algebraically independent.

### 4.2. Cylindrical functions and measure theoretical aspects

We will now consider the pull-back of projections (24). For any given $\gamma$ and any given continuous function $f \in C(\mathbb{R}_\gamma)$, the pull-back $p_\gamma^* f(\bar{x}) = f(p_\gamma \bar{x})$ gives a function in $\bar{\mathbb{R}}$. Since the projections are continuous and surjective, the function $p_\gamma^* f$ is again continuous and the map $p_\gamma^* : C(\mathbb{R}_\gamma) \to C(\bar{\mathbb{R}})$ is injective. Thus, functions on any given $\mathbb{R}_\gamma$ or corresponding tori, are faithfully mapped to functions in $\bar{\mathbb{R}}$. These so-called cylindrical functions are therefore functions in $\bar{\mathbb{R}}$ that are essentially living on a torus. Moreover, there is a ‘sufficient’ supply of these functions, in the sense that the set of all such functions, for all $\gamma \in \Gamma$, is dense in $C(\bar{\mathbb{R}})$, and therefore $L^2$-dense. Of course, one cannot stop at any fixed $\gamma$, but note that finite linear combinations of cylindrical functions corresponding to different $\gamma$’s, say $\gamma_1, \ldots, \gamma_n$, can again be seen as functions on a higher dimensional torus, since there is $\gamma'$ such that $\gamma' \geq \gamma_1, \ldots, \gamma' \geq \gamma_n$.

Furthermore, the pull-back extends from a map between continuous functions to a map between $L^2$-spaces. To see this and for subsequent discussion, let us introduce a simple notion from measure theory.

Since $\bar{\mathbb{R}}$ projects to every space $\mathbb{R}_\gamma$, the Haar measure $\mu_0$ defines a measure $\mu_{0\gamma}$ on each $\mathbb{R}_\gamma$. Explicitly, each measure $\mu_{0\gamma}$ is defined by the push-forward of $\mu_0$, with respect to the projection $p_\gamma : \bar{\mathbb{R}} \to \mathbb{R}_\gamma$, i.e.

$$\mu_{0\gamma}(B) = \mu_0(p_\gamma^{-1} B) \quad \forall \text{ measurable set } B \subseteq \mathbb{R}_\gamma.$$  \hspace{1cm} (26)


where $p^{-1}_γ B$ denotes the inverse image of the set $B$. An equivalent definition of $μ_{0γ}$ can be given in terms of integration of continuous functions, namely $μ_{0γ}$ is defined by
\[\int_{Rγ} f \, dμ_{0γ} = \int_{R} p^{-1}_γ f \, dμ_0, \quad ∀ f ∈ C(Rγ). \quad (27)\]

As expected, measures $μ_{0γ}$ coincide with Haar measures on the groups $Rγ$, or the usual (Haar–Lebesgue) measures on corresponding tori\(^9\). (This can be easily checked, e.g., from (27).)

One can then form the Hilbert spaces $L^2(Rγ, μ_{0γ})$, which are essentially the usual Hilbert spaces of square integrable functions on tori. It is easily seen (again from (27)) that, for each $γ$, the pull-back defines a unitary transformation between $L^2(Rγ, μ_{0γ})$ and the separable Hilbert subspace $H_{0γ} := p^*_γ L^2(Rγ, μ_{0γ})$ of $H_0$.

In particular, one can consider the restriction of the polymer representation to a given $H_{0γ}$, with $γ = \{k_1, \ldots, k_n\}$. Of course, operators $Π(k)$ (14) with $k$ outside the group $G_γ$ no longer act on $H_{0γ}$, but those who do, i.e. such that $k = \sum m_j k_j$, are seen to correspond to the usual quantization of the functions exp $(i \sum m_j x_j)$ on the torus $T^n$. On the other hand, the momentum operator $Π(p)$ (16) is defined on every $H_{0γ}$, and corresponds to derivation along the direction of the torus defined by $p_γ Θ(R)$ as above.

Finally, let us see that the image $Θ(R)$ of the classical configuration space is a zero Haar measure set in $R$. This is particularly easy to see in the present case, since it already happens for generic $Rγ$. Consider then a set $γ = \{k_1, k_2\}$ with two elements (or more), so that the space $Rγ$ can be seen as $T^2$ (or $T^n, n ≥ 2$.) Since the number of times that the line $R$ is wrapped around the torus is clearly countable, the measure of the set $p_γ Θ(R)$ is zero. (The situation is similar to that of the set of lines of constant rational $y$ in the square $(x, y) ∈ [0, 1] × [0, 1]$.) It follows immediately from (26) that $Θ(R)$ is a zero Haar measure set in $R$.

As a consequence, the restriction of a generic element of $H_0 = L^2(R, μ_0)$ to $Θ(R)$ is not defined. In this sense, the completion of the linear space $C(4)$ with respect to the inner product $⟨e^{ikx}, e^{ilx}⟩ = δ_{kl}$ leads to a space of functions in $R$, not on $R\(^10\). Thus, the extension of the configuration space from $R$ to $R$ is essential in the polymer representation.

5. $C^*$-algebra aspects

We will now see explicitly that $R$ is in fact the spectrum of the $C^*$-algebra of almost periodic functions in $R$ [R]. This characterization of the quantum configuration space corresponds to the original introduction of the space of generalized connections $A$ as the spectrum of the holonomy algebra [Al, AL1].

Let us consider again the configuration $*$-algebra $C$ of functions given by finite sums of the form
\[^9\] The Ashtekar–Lewandowski measure in LQG was actually introduced the other way around, using a (suitably compatible) family of Haar measures on finite-dimensional spaces of a projective family to define a measure on the projective limit $A$. This is the general procedure to introduce measures in projective limit spaces with no further structure. In amenable cases, including linear field theories and LQG, expressions like (26), seen as the definition of the measure on the projective limit, indeed lead to proper $σ$-additive measures. In the case of compact families, the formalism of $C^*$-algebras representation theory, corresponding to (27), is also available. (See [AL1, AL2, MM] for measure theoretical aspects in LQG.)

\[^10\] In other words, the non-continuous function $R \ni k \mapsto δ_{k0}$ is not the Fourier transform of a measure in $R$, but in $R$. On the momentum side, the discrete measure is defined on $R$, but it is not $σ$-finite, allowing it to produce a unitary implementation of translations and be non-equivalent to the Lebesgue measure.
\[ \mathbb{R} \ni x \mapsto f(x) = \sum_j c_j e^{ik_j x}, \] 

(28)

and its \( C^\ast \)-completion \( \bar{\mathcal{C}} \), with respect to the supremum norm, \( \| f(x) \| = \sup_{x \in \mathbb{R}} |f(x)| \).

The spectrum \( \Delta(\bar{\mathcal{C}}) \) of the algebra \( \bar{\mathcal{C}} \) is the set of all non-zero multiplicative linear functionals on \( \bar{\mathcal{C}} \), i.e. non-zero linear functionals \( \varphi : \bar{\mathcal{C}} \to \mathbb{C} \) such that

\[ \varphi(fg) = \varphi(f)\varphi(g), \quad \forall f, g \in \bar{\mathcal{C}}. \] 

(29)

(Such functionals are necessarily continuous.) One can easily check that \( \varphi(e^{ikx}) \) takes values in \( T \), \( \forall \varphi \in \Delta(\bar{\mathcal{C}}) \), \( \forall k \), and it follows that the assignment \( \mathbb{R} \ni k \mapsto \varphi(e^{ikx}) \) determines a group morphism, i.e. an element of \( \mathbb{R} \). There is thus a map, injective, from \( \Delta(\bar{\mathcal{C}}) \) to \( \mathbb{R} \). Conversely, any element \( \bar{x} \in \bar{\mathbb{R}} \) defines a non-zero multiplicative linear functional on \( \bar{\mathcal{C}} \), by \( e^{ikx} \mapsto \bar{x}(k) \).

To establish a bijection between \( \Delta(\bar{\mathcal{C}}) \) and \( \bar{\mathbb{R}} \) it remains to prove that these functionals can be extended to functionals on \( \bar{\mathcal{C}} \), for which it is sufficient to show that they are continuous on \( \bar{\mathcal{C}} \).

This is another point where one finds a difference to a corresponding argument in LQG, again due to the above-mentioned fact that \( p_\gamma/\Theta^1(\mathbb{R}) \) does not cover \( \mathbb{R} \gamma \), for generic \( \gamma \). However, the denseness of \( p_\gamma/\Theta^1(\mathbb{R}) \) leads to the same result, as we now confirm\(^{11}\).

We begin by displaying the inductive structure of the algebra \( \mathcal{C} \). For each \( \gamma \in \Gamma \), \( \gamma = \{k_1, \ldots, k_n\} \), let \( \mathcal{C}_\gamma \subset \mathcal{C} \) denote the \( * \)-subalgebra generated by the set of functions \( \{e^{ik_1 x}, \ldots, e^{ik_n x}\} \), whose elements are finite sums of the form

\[ f(x) = \sum_k c_k e^{ikx} \quad \text{with} \quad k \in G_\gamma. \] 

(30)

It is clear that \( \mathcal{C}_\gamma \) is a subalgebra of \( \mathcal{C}_{\gamma'} \), whenever \( \gamma' \supseteq \gamma \). Moreover, any element of \( \mathcal{C} \) belongs to some subalgebra \( \mathcal{C}_\gamma \). In fact, since the number of different frequencies \( k_j \) in \( (28) \) is finite \( \forall f \in \mathcal{C} \), they all belong to some group \( G_\gamma \). Thus, the algebra \( \mathcal{C} \) is the union of all the subalgebras \( \mathcal{C}_\gamma \).

Let then \( f(x) = \sum_j c_j e^{ik_j x} \) be an arbitrary element of \( \mathcal{C} \), and \( \gamma = \{\lambda_1, \ldots, \lambda_n\} \) an independent set such that \( k_j \in G_\gamma \forall j \), i.e.

\[ k_j = \sum_{l=1}^n m^j_l \lambda_l, \quad \forall j, \] 

(31)

for some integers \( m^j_l \). We thus have

\[ \| f(x) \| = \sup_{x \in \mathbb{R}} \left| \sum_j c_j \prod_{l=1}^n (e^{i\lambda_l x})^{m^j_l} \right|. \] 

(32)

For each value of \( x \), \( (e^{i\lambda_1 x}, \ldots, e^{i\lambda_n x}) \) is an element of \( T^n \cong \mathbb{R}_\gamma \), and the image of \( T^n \) by this map is a dense set. Since functions in \( T^n \) of the form

\[ T^n \ni (x_1, \ldots, x_n) \mapsto \sum_j c_j \prod_{l=1}^n (e^{i\lambda_l x})^{m^j_l} \] 

(33)

(with finite sums) are continuous, one can see that the supremum over \( \mathbb{R} \) in \( (32) \) can be replaced by the supremum over \( T^n \). Thus

\[ \| f(x) \| = \sup_{(x_1, \ldots, x_n) \in T^n} \left| \sum_j c_j \prod_{l=1}^n (e^{i\lambda_l x})^{m^j_l} \right|. \] 

(34)

\(^{11}\) Similar arguments nevertheless appear in appendix A of [AL1], where \( U(1) \) holonomies associated with piecewise \( C^1 \) (rather than piecewise analytic) loops are considered.
It is now easy to see that linear functionals on $\mathcal{C}$ defined by elements of $\overline{\mathbb{R}}$ are continuous. Let then $\varphi_{\bar{x}}$ denote the functional defined by $\varphi_{\bar{x}}(e^{ikx}) = \bar{x}(k)$, $\bar{x} \in \overline{\mathbb{R}}$. We obtain
\[ \varphi_{\bar{x}}(f(x)) = \sum_j c_j \bar{x}(k_j) = \sum_j c_j \prod_{l=1}^n \bar{x}(\lambda_l)^{m_l}, \tag{35} \]
which is simply the evaluation of the function appearing in (34), at the point $(\bar{x}(\lambda_1), \ldots, \bar{x}(\lambda_n)) \in T^n$. Thus, $|\varphi_{\bar{x}}(f(x))| \leq \|f(x)\|$, $\forall f(x) \in \mathcal{C}$, proving that the functionals $\varphi_{\bar{x}}$ are continuous.

A bijection between the spectrum $\Delta(\bar{\mathcal{C}})$ and $\overline{\mathbb{R}}$ is therefore established. Moreover, $\Delta(\bar{\mathcal{C}})$ and $\overline{\mathbb{R}}$ are homeomorphic. In fact, the topology on the spectrum $\Delta(\bar{\mathcal{C}})$, namely the Gelfand topology defined as the weakest one such that all maps $\Delta(\bar{\mathcal{C}}) \to \mathbb{C}$, $\varphi \mapsto \varphi(f)$, $f \in \overline{\mathbb{C}}$, are continuous, is again seen to correspond to the Tychonoff topology in $\overline{\mathbb{R}}$ introduced in section 3. It is also clear that the Gelfand compactification $\mathbb{R} \to \Delta(\bar{\mathcal{C}})$, given by $y \mapsto \varphi_y$ such that $\varphi_y(e^{ikx}) = e^{iky}$, corresponds to the map $\Theta(9)$.

Finally, let us make the relation between inductive structures and projective structures more explicit.

Considering the completions $\bar{\mathcal{C}}_\gamma$ of the algebras $\mathcal{C}_\gamma$, it is again true that $\bar{\mathcal{C}}_\gamma \subset \bar{\mathcal{C}}_{\gamma'}$ for $\gamma' \geq \gamma$, but the union $\bigcup_{\gamma} \bar{\mathcal{C}}_\gamma$ is not a complete space. The family $\{\bar{\mathcal{C}}_\gamma\}_{\gamma \in \Gamma}$, with natural inclusions, is a particular case of an inductive family of $C^*$-algebras. The completion of the union, in this case $\bar{\mathcal{C}}$, is called in this context the inductive limit $C^*$-algebra. Taking now into account that the set of all functions of the form (33) is dense in $C(T^n)$, the above arguments immediately lead to the conclusion that each algebra $\bar{\mathcal{C}}_\gamma$ is isomorphic to the algebra $C(\bar{\mathbb{R}}_{\gamma})$. In other words, $\bar{\mathbb{R}}_{\gamma}$ is the spectrum of $\bar{\mathcal{C}}_\gamma$. As we take the limit over the set $\Gamma$, we find that $\overline{\mathbb{R}}$ is the spectrum of $\bar{\mathcal{C}}$.\footnote{There is a perfect duality between compact projective families and inductive families of $C^*$-algebras with identity, in the sense that there is a one-to-one (Gelfand) correspondence and the compact projective limit is the spectrum of the corresponding inductive limit $C^*$-algebra (see [AL2]).}

Summarizing, from the point of view of $C^*$-algebras, the configuration $C^*$-algebra $\bar{\mathcal{C}}$ and the algebra $C(\overline{\mathbb{R}})$ of continuous functions in $\overline{\mathbb{R}}$ are one and the same. As in the finite-dimensional cases seen above, this can be understood from the denseness of $\Theta(\mathbb{R})$, allowing for the restriction of continuous functions to be an isomorphism.

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Appendix. The Schrödinger representation

We will now see how the usual Schrödinger representation is obtained in the present context. It is given, of course, by a different measure in $\overline{\mathbb{R}}$. Though technically much simpler, the introduction of this measure corresponds to the construction of the so-called $r$-Fock measures in the loop quantization of $U(1)$ connections [Va, Ve3, AL4] (see also [ALS]).

Let us start by considering the Gaussian measure $d\nu_G = e^{-x^2/2} \frac{dx}{\sqrt{2\pi}}$ in $\mathbb{R}$, and the corresponding space of square integrable functions $L^2(\mathbb{R}, \nu_G)$. The Hilbert space $L^2(\mathbb{R}, \nu_G)$
The quantum configuration space of loop quantum cosmology carries (a representation unitarily equivalent to) the usual Schrödinger representation of the Weyl relations.

We will now use the natural dense injection \( \Theta : \mathbb{R} \to \bar{\mathbb{R}} \) in order to push forward the Gaussian measure, thus obtaining a measure \( \mu_G \) in \( \bar{\mathbb{R}} \). The measure \( \mu_G \) is defined by

\[
\mu_G(B) = \nu_G(\Theta^{-1} B) \quad \forall \text{ measurable set } B \subset \bar{\mathbb{R}}.
\]

The corresponding Hilbert space \( L^2(\bar{\mathbb{R}}, \mu_G) \) is then isomorphic to \( L^2(\mathbb{R}, \nu_G) \). To confirm it, let us see that the mapping between dense subspaces defined by

\[
L^2(\mathbb{R}, \nu_G) \ni e^{ikx} \mapsto F_k(\bar{x}), \quad k \in \mathbb{R},
\]

preserves the inner product, thus defining a unitary transformation. With \( \Theta^* \) denoting pull-back, we obtain

\[
\int_{\mathbb{R}} F_k^* F_{k'} d\mu_G = \int_{\mathbb{R}} F^*_{|k'-k|} d\mu_G = \int_{\mathbb{R}} \Theta^* F^*_{|k'-k|} d\nu_G,
\]

where the second equality follows from definition (A.1) of \( \mu_G \). Recalling definitions (8) and (9), it is clear that \( \Theta^* F_k \) is the exponential function \( e^{ikx} \), and therefore

\[
\int_{\mathbb{R}} F_k^* F_k d\mu_G = \int_{\bar{\mathbb{R}}} e^{i(k'-k)x} d\nu_G.
\]

The measure \( \mu_G \) provides a representation of the configuration algebra which is obviously different from the polymer representation. The functions \( x \mapsto e^{ikx} \) are again quantized as the multiplication operators \( F_k \), as in (14), with the crucial difference that the measure \( \mu_G \) is now supported on the image \( \Theta(\mathbb{R}) \). Thus, essentially, only points \( x \in \mathbb{R} \) of the classical configuration space contribute to the measure, the functions \( F_k \) reassert the form \( e^{ikx} \), and continuity with respect to \( k \) is restored. It is clear that the measure \( \mu_G \) gives a representation of the Weyl relations which is equivalent to the Schrödinger one, the unitary map being transformation (A.2). The Haar measure \( \mu_0 \) and the measure \( \mu_G \) are mutually singular, since \( \Theta(\mathbb{R}) \) is a set of zero Haar measure.

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