Non-adiabatic emission of ultrastrongly-coupled oscillators: signatures of the $A^2$ term

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We investigate the emission of a polaritonic system, where the coupling between a large number of two-level emitters and a single-mode cavity field is non-adiabatically switched on. Counter-rotating terms as well as the so-called $A^2$ term are included in the light-matter interaction, where $A$ is the vector potential. We find that the Thomas-Reiche-Kuhn sum rule enforces qualitative constraints on the quantum statistics of the resulting vacuum radiation, which consists of two spectrally resolved output modes. For ideal two-level emitters the populations of the two modes are always equal. This result cannot be recovered if $A^2$ is neglected, or even if it is included perturbatively via renormalization of the cavity frequency. We then extend our study to imperfect two-level emitters, featuring residual couplings to higher levels, and find that a naive application of the two-level approximation alters these predictions incorrectly. We discuss how a refined two-level approximation may be obtained by rescaling the $A^2$ term.

PACS numbers: 42.50.-p, 42.50.Pq

Introduction — Quantum technologies exploit intense interactions between field and matter degrees of freedom [1], and it is a typical experimental goal in this context to maximize the coupling between the two. Traditional cavity QED setups have been extremely successful in this regard, yet they result in coupling frequencies that are only a tiny fraction of that of the system components [2]. Experimental advances, for example in semiconductor microcavities and circuit QED, have now pushed the strength of field-matter interactions into the ultrastrong-coupling regime (USC) [3–7]. This regime is characterized by interactions where the coupling frequency $\lambda$ is a non-negligible fraction of the bare frequency of the matter degree of freedom, say $\omega_b$. The theoretical description of the USC goes beyond the rotating wave approximation (RWA), demanding the inclusion in the Hamiltonian of terms that do not conserve the excitation numbers of the individual components — the so-called ‘counter-rotating terms’ (CR) [8–10]. This regime has been studied extensively due to the lure of exotic phenomena such as the existence of virtual excitations in the ground state [3], dynamical Casimir effects [11], quantum phase transitions [12, 13], and counter-intuitive radiation statistics [14, 15] among others.

In this regime, however, the CR terms may not be the only extra terms that should be taken into account. Another important ingredient is the diamagnetic — or $A^2$ — term, which is proportional to the squared vector potential and ensures gauge invariance in the non-relativistic minimal coupling Hamiltonian [10]. The effects associated with this term, and the related Thomas-Reiche-Kuhn (TRK) sum rules, are of crucial importance in the research on superradiant phase transitions, and are still under active investigation and debate [12, 15, 27]. A further point deserving attention is that the two-level approximation for the description of quantum emitters may fail in the USC [28]. Finally, even the multi-mode nature of the cavity field is known to play a role in the “deep strong coupling” regime $\lambda \gtrsim \omega_b$ [29].

In most of the above examples the physics beyond the CR terms, in particular $A^2$, becomes crucial as the strength of light-matter interactions is pushed towards the extreme regime $\lambda \sim \omega_b$. In contrast, to the best of our knowledge, a clearcut qualitative signature of the $A^2$ term has not been identified in the currently experimentally relevant regime $\lambda/\omega_b \lesssim 0.2$: the necessity of such a term is typically assessed by fitting the experimental data [10]. The present work aims at giving a contribution towards filling this void. We study the evolution of a polaritonic system, initially in the bare modes vacuum, following a non-adiabatic ‘switch-on’ of the coupling between a large number of two-level emitters and a single-mode cavity field. While this process is known to yield quantum vacuum radiation [11], we show that a closer look at the quantum statistics of the latter reveals striking signatures of the $A^2$ term, a possibility that has gone unnoticed so far. Specifically, as the system relaxes to the ground state of the coupled Hamiltonian, photons are radiated into two spectrally distinct output modes. We find that $A^2$, in combination with the TRK rules, affects qualitatively the statistics of these modes, regardless of how small $\lambda/\omega_b$ is. Refining our model beyond the two-level approximation to include the possibility of transitions to far detuned levels of the emitters, we then show how suitable parameter regimes can be identified in which an imperfect two-level system can be approximated by an ideal one, upon appropriately rescaling $A^2$. More generally, the examples we studied suggest that a consistent description of physics beyond the RWA should display a balance between the weight of the $A^2$ term and the number of dipole transitions that are included when modelling the quantum emitters.

Model — The system under examination consists of a confined photonic mode of bare frequency $\omega_a$ and an annihilation operator $\hat{a}$ (cavity mode for brevity), and a collection of $n$ two-level emitters with ground state $|g\rangle$.
and excited state $|e\rangle$, separated by a frequency $\omega_b$, and identically coupled to the cavity mode via electric dipole. The detailed microscopic model is discussed in [30]. For convenience, we introduce the collective operator $b = n^{-1/2} \sum_{k=1}^{n} |g\rangle \langle e|_k$ (matter mode for brevity), and consider the Holstein-Primakoff (HP) regime $[\hat{b}^\dagger \hat{b}] \ll n$, where $[\hat{b}, \hat{b}^\dagger] \simeq 1$. The cavity mode is weakly coupled to a continuum of external modes $\tilde{\alpha}_\omega$ where it can radiate. To simplify the discussion we shall neglect losses in the matter system, and assume that the modes $\tilde{\alpha}_\omega$ are fully accessible for measurement. Deviations from this idealized scenario will be discussed in a future work.

While we are adopting a terminology typical of polaritonic systems, our treatment may be applied to other scenarios. The Hamiltonian is $H = H_{\text{sys}} + H_{\text{ext}} + H_I$, where $(h = 1)$

$$H_{\text{sys}} = \omega_0 \hat{a}^\dagger \hat{a} + \omega_b \hat{b}^\dagger \hat{b} + \lambda (\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger) + D(\hat{a} + \hat{a}^\dagger)^2 \quad (1)$$

is the system Hamiltonian, while $H_{\text{ext}} = \int d\omega \omega \hat{a}^\dagger \hat{a}_\omega \hat{\omega}$ and $H_I = \int d\omega J(\omega) (\hat{a} + \hat{a}^\dagger)(\hat{a}_\omega + \hat{a}^\dagger_\omega)$ model the continuum of external modes and their coupling to the system, through a smooth coupling profile $J(\omega)$. The term proportional to $D$ is due to the $A^2$ term in the minimal coupling Hamiltonian. For a collection of ideal two-level systems, the TRK sum rule for the ground state imposes $D = \lambda^2/\omega_b$ [30, 32]. Nevertheless, we shall leave $D$ implicit for later convenience. When the coupling $\lambda$ is a significant fraction of the bare frequencies $\omega_a, \omega_b$, the open dynamics of the system is better described in terms of the eigenmodes diagonalizing $H_{\text{sys}}$. We shall refer to these as the upper (U) and lower (L) polariton, and they are expressed as

$$\hat{p}_k = w_k \hat{a} + x_k \hat{b} + y_k \hat{a}^\dagger + z_k \hat{b}^\dagger, \quad (2)$$

where $k \in \{L, U\}$. The Hopfield coefficients, $(w_k, x_k, y_k, z_k)$, and the polariton frequencies, $\omega_k$, are determined by inserting Eq. (2) into $[\hat{p}_k, H_{\text{sys}}] = \omega_k \hat{p}_k$ and solving the resulting eigenvalue problem [3, 33]. These coefficients are normalized by imposing the bosonic commutation relations $[\hat{p}_k, \hat{p}^\dagger_{k'}] = \delta_{kk'}$. By recasting $H$ in terms of the polariton operators, and assuming that a RWA can be performed in $H_I$, we obtain

$$H \simeq \sum_{k=L,U} \omega_k \hat{p}_k^\dagger \hat{p}_k + \int d\omega \omega \hat{a}^\dagger_\omega \hat{a}_\omega + \int d\omega \sum_{k=L,U} J_k(\omega) (\hat{a}^\dagger_k \hat{p}_k + \hat{a}_k \hat{p}^\dagger_k). \quad (3)$$

where $J_k(\omega) = J(\omega)(x_k^2 + z_k^2)$. Note that the RWA must be performed only after the change into the polaritonic basis [34, 35], since it is the operators $p_L, p_U$ that oscillate harmonically in the interaction picture.

**Quantum statistics of emission** — To describe non-adiabatic emission, it is useful to determine the statistics of the external field modes, as a function of the initial system conditions. To this end we express the initial system operators, in the Heisenberg picture, as

$$\hat{p}_k(0) = \sum_n v_{nk}(t) \hat{p}_k(t) + \int d\omega \phi_k(\omega, t) \tilde{\alpha}_\omega(t), \quad (4)$$

where the time-independence of the left-hand side implies the differential equations $\dot{v}_{nk}(t) = i \omega v_{nk} \delta_{kk'} + i \int d\omega J_k(\omega) \phi_k(\omega, t) + \int d\omega \phi_k(\omega, t) \delta_{kk'}$. For a given form of $J(\omega)$, $v$ and $\phi$ could be calculated in principle, e.g. numerically, by Fano-like techniques or Laplace transforms [35, 38]. Such details, however, are largely unimportant for our purposes. In standard scenarios, Hamiltonian [34] will induce a dissipative dynamics of the polaritonic system, such that one has $v \to 0$ for sufficiently long times, and hence the full quantum statistics of the initial system state will be retrieved in specific combinations of the external field modes. These can be formally expressed as

$$\hat{f}_k \equiv \lim_{t \to \infty} \int d\omega \phi_k(\omega, t) \tilde{\alpha}_\omega(t) = \hat{p}_k(0). \quad (5)$$

We note that the main message expressed by Eq. (5) does not depend on the details of the interaction between cavity and external fields, but each asymptotic amplitude $\phi_k(\omega, t) \equiv \lim_{t \to \infty} \phi_k(\omega, t) e^{-i\omega t}$ does, and needs to be evaluated on a case-by-case basis. Typically, the timescales of emission are long as compared to $\omega^{-1}$, so that $|\phi_k(\omega)|^2$ is sharply peaked around the corresponding polaritonic frequency $\omega_k$, and $\hat{f}_U$ and $\hat{f}_L$ are spectrally well resolved. As an example, in Fig.1 we plot $|\phi_{U,L}(\omega)|^2$ for the simplest case of a frequency-independent coupling to the continuum. The results are however not qualitatively changed by considering more realistic expressions for $J(\omega)$.

**Non-adiabatic emission** — The impact of $A^2$ on the emission properties of our polaritonic system can be appreciated with a simple and yet interesting example, in
which the coupling \( \lambda \) is non-adiabatically “switched-on” from an initially negligible value. Such a modulation has been experimentally demonstrated, for example, in USC intersubband-cavity systems [40–42]. The system is thus unable to respond to the perturbation and remains in its initial state, which we assume to be the vacuum state \(|0\rangle\) of the bare modes \((\hat{a}|0\rangle = \hat{b}|0\rangle = 0)\). Since \( \hat{p}_k|0\rangle \neq 0 \), for \( t > 0 \) the ultrastrongly coupled system will start dissipating towards the vacuum of the polaritonic modes and in order to do so it must radiate into the continuum. According to Eq. (4), the quantum properties of such radiation are fully captured by the output modes \( \hat{f}_U, \hat{f}_L \).

To begin with, we turn our attention to the mean populations \( n_k \equiv \langle \hat{f}_k^\dagger \hat{f}_k \rangle = |y_k|^2 + |z_k|^2 \). A nonzero value of these quantities is perhaps the simplest signature of the influence of CR terms on the dynamics. Fig. 2 illustrates the behaviour of \( n_k \) as a function of the coupling \( \lambda \), the bare frequency difference \( \omega_b - \omega_a \) and, most importantly, the parameter \( D \). When the correct value for a two-level system, \( D = \lambda^2/\omega_b \), is taken, we observe that the excitations are distributed equally between \( \hat{f}_U \) and \( \hat{f}_L \). Setting instead \( D = 0 \), which corresponds to neglecting \( A^2 \) but not the CR terms, results in a higher population being predicted for the lower frequency mode \( \hat{f}_L \).

We note that, as far as the population of the two modes is not negligible (which would correspond to the RWA), this discrepancy holds in all the explored range of the remaining parameters \( \lambda, \omega_b, \omega_a \). This, in particular, rules out the explanation of this effect via a simple renormalization of the cavity frequency, corresponding to treating \( A^2 \) perturbatively and neglecting the terms proportional to \( \hat{a}^2 \) and \( (\hat{a})^2 \) in Eq. (1).

Bare mode resonance — For the special case \( \omega_a = \omega_b = \omega_0 \), a thorough analytical understanding of the above finding is possible. In addition, we fully characterize the output state for this case as a product of two single-mode squeezed vacuum states. To do this, we explicitly diagonalize \( H_{\text{sys}} \) in two simple steps. First, we consider the ‘number-conserving’ transformation \( \hat{a} = \cos \theta \hat{r}_U - \sin \theta \hat{r}_L, \quad \hat{b} = \cos \theta \hat{r}_L + \sin \theta \hat{r}_U \) where \( \hat{r}_U, \hat{r}_L \) are independent bosonic modes that annihilate the same vacuum state as \( \hat{a}, \hat{b} \). With the choice \( \theta = \frac{1}{2} \tan^{-1}(\lambda/D) \), we can recast \( H_{\text{sys}} = \sum_k \omega_k \hat{r}_k^\dagger \hat{r}_k + \frac{\lambda}{2}(\hat{r}_k + \hat{r}_k^\dagger)^2 \), where \( \eta_k = D + \sqrt{\lambda^2 + D^2}, \eta_L = D - \sqrt{\lambda^2 + D^2} \). Note that \( \eta_k > 0 \), while \( \eta_L < 0 \). The \( \hat{r}_U, \hat{r}_L \) modes do not interact, and are intimately related to the polaritonic modes \( \hat{p}_U, \hat{p}_L \) respectively. To complete the diagonalization we consider the squeezing transformation

\[
\hat{r}_k = \cosh \xi_k \hat{p}_k - \sinh \xi_k \hat{p}_k^\dagger, \quad k = U, L, \quad (6)
\]

which, choosing \( \xi_k = \frac{1}{4} \log \left( 1 + \frac{2\eta_k}{\omega_0} \right) \), reduces \( H_{\text{sys}} \) to the diagonal form employed in Eq. (3), with the polaritonic frequencies given by \( \omega_k = \sqrt{\omega_0(\omega_0 + 2\eta_k)} \). We finally note that \( \omega_U > \omega_L \) and \( \xi_U > 0 \), while \( \xi_L < 0 \), signifying that \( \hat{p}_U \) and \( \hat{p}_L \) are obtained by squeezing the modes \( \hat{r}_U \) and \( \hat{r}_L \) in orthogonal quadratures. Eq. (5) illustrates that the vacuum state of \( \hat{a}, \hat{b} \) (hence of \( \hat{r}_U, \hat{r}_L \)) corresponds to a product of single-mode squeezed states for \( \hat{p}_U, \hat{p}_L \), and hence for the output modes \( \hat{f}_U, \hat{f}_L \) — see Fig. 2. Noting that the degree of squeezing and population of each mode is a monotone function of \( |\xi_k| \), it is useful to calculate

\[
|\xi_U| - |\xi_L| = \frac{1}{4} \log \left[ 1 + \frac{4}{\omega_0} \left( D - \frac{\lambda^2}{\omega_0} \right) \right], \quad (7)
\]

which proves in a transparent manner that the two populations are equal when \( D \) assumes the TRK value for a two-level system. This simple and yet striking signature of \( A^2 \) on the system emission, and more generally on the relationship between bare modes and polaritonic operators, constitutes the main result of our work.

Imperfect two-level emitters — As anticipated, a further commodity that may need to be given up beyond the RWA is the two-level approximation [28]. We thus find it useful to assess how the results of the previous section are modified by the presence of unwanted transitions to higher levels. To investigate this issue in our context we enrich the structure of each emitter by considering, in addition to the levels \(|g\rangle,|e\rangle\), the existence

\[
\begin{align*}
|\xi_U| - |\xi_L| &= \frac{1}{4} \log \left[ 1 + \frac{4}{\omega_0} \left( D - \frac{\lambda^2}{\omega_0} \right) \right],
\end{align*}
\]
of higher excited states $|e'_j\rangle$. Each transition $|g\rangle \leftrightarrow |e'_j\rangle$ has a frequency $\omega_j > \omega_0$ and is dipole coupled to the cavity field. Again, we consider collective operators $\hat{e}_j = n^{-1/2} \sum_{k=1}^{n} |g\rangle_k \langle e'_k|$, which together with $\hat{b}$ describe a set of mutually independent annihilation operators when $\langle \hat{b} \hat{b} \rangle + \sum_j \langle \hat{e}_j \hat{e}_j \rangle \ll n$, a regime that also allows us to neglect transitions between excited states \cite{30}. Thus, the system Hamiltonian in Eq. (1) is modified as

$$H'_{\text{sys}} = H_{\text{sys}} + \sum_j \left[ \omega_j \hat{c}^\dagger_j \hat{c}_j + \lambda_j (\hat{c}_j + \hat{c}^\dagger_j) (\hat{a} + \hat{a}^\dagger) \right], \quad (8)$$

where $\lambda_j$ are the coupling strengths of the newly introduced transitions, and for simplicity we shall keep the assumption $\omega_j = \omega_0 \equiv \omega_0$. Note that the TRK sum rule for the ground state now imposes a larger weight for the coefficient of $A^2$, that is $D = \lambda^2/\omega_0 + \sum_j \lambda_j^2/\omega_j$ \cite{30, 32}. The construction of the normal modes of $H'_{\text{sys}}$, and the determination of the output fields statistics following a non-adiabatic introduction of the coupling, follow the same lines as before. We shall again focus on the statistics of the output operators $\hat{f}_L, \hat{f}_U$, being in this case defined as the two output modes with the lowest ($L$), and second-lowest ($U$) carrier frequency, respectively.

We are interested here in regimes where $\lambda_j \ll \omega_j$, such that transitions to the higher levels are suppressed and each emitter can be expected to approximate a two-level system. Thus it may be tempting at this point to simply neglect the modes $\hat{c}_j$ and recover the Hamiltonian in Eq. (1), valid for two level emitters, with the only difference that $D > \lambda^2/\omega_0$, since the $|g\rangle \leftrightarrow |e\rangle$ transition does not saturate the sum rule anymore. This, for example, would predict a slightly higher population and squeezing in the output mode $\hat{f}_U$ according to Eq. (7). In Fig. 3 we compare this approach with the full calculation employing Hamiltonian (5), for the simplest case in which only one extra mode $\hat{c}_1$ is considered. We find qualitative discrepancies whenever the difference of the populations is non-negligible: we can indeed observe that the full Hamiltonian can result in $n_L > n_U$. Such a naive two-level approximation, therefore, cannot explain this behaviour, and even worse, its comparison with the observed populations may lead to the erroneous conclusion that the TRK sum rule has been violated, as Hamiltonian (1) requires $D < \lambda^2/\omega_0$ to achieve a higher population in the lower frequency mode [see Eq. (7)]. Still, in the regime $D \gtrsim \lambda^2/\omega_0$, one may be content to neglect the fine details associated to the modes $\hat{c}_j$, and seek a refined two-level approximation. This is indeed obtained by adiabatically eliminating the modes $\hat{c}_j$ from Hamiltonian (5) \cite{43, 44}. At lowest order, one has

$$H'_{\text{sys}} \approx H_{\text{sys}} - \sum_j \frac{\lambda_j^2}{\omega_j} (\hat{a} + \hat{a}^\dagger)^2, \quad (9)$$

where the contribution of higher levels is effectively removed from the $A^2$ term, such that $D \rightarrow D'_{\text{off}} = \lambda^2/\omega_0$, the TRK value for an ideal two-level system. This exactly corresponds to recovering our idealized two-level description with good approximation — see Fig. 3. A similar line of reasoning can be adopted if one is interested in including a small number of extra modes $\hat{c}_j$ in the model, in which case the sum in Eq. (9) will extend only to those modes that are neglected. This indicates that, in a consistent description of two- or multi-level emitters beyond the RWA, one should remove from the $A^2$ term the contributions of any neglected transitions.

**Outlook** — We remark that our results are based on a number of assumptions. One of these is the single-mode treatment of the field. This would ideally require a cavity whose mirrors are highly reflective only nearby its fundamental frequency. It should also hold at low enough light-matter coupling in a standard cavity, e.g. if the emitters are placed at a node of the next allowed field mode. Still, multimode effects may be unavoidable as $\lambda$ is increased to higher and higher values \cite{29}. Furthermore, we have neglected losses in the matter mode as well as in the output light, resulting in particularly straightforward relationships between intra- and extracavity observables. In a realistic system, featuring losses, more elaborate strategies must be adopted to access the intra-cavity statistics. We believe it worthwhile to refine our model in future studies to address these limitations.

**Conclusions** — We have identified a qualitative signature of $A^2$ in the quantum vacuum emission of a very general polaritonic model, valid at arbitrarily low coupling and hence potentially of interest to current experiments. In doing so we have highlighted a companionship between the CR terms, $A^2$, and the robustness of the two-level approximation, points that can be of be of general interest in future studies of the USC regime.

**FIG. 3:** (a) Ratio ($n_U/n_L$) of the mean populations of $\hat{f}_U$ and $\hat{f}_L$ in the case of (b) three-level emitters, where the third level is weakly coupled to the light mode ($\lambda_1/\omega_1 < 1$). $n_U/n_L$ is plotted as a function of the third level coupling frequency normalized to the third level transition frequency $\lambda_1/\omega_1$. The blue solid line represents the model in which the full three-level system is accounted for. The red dashed line represents a two-level approximation which naively incorporates the third level in the $A^2$ term. We also include the model for a perfect two-level system as a qualitative comparison (black dotted line). We see that the deviation from the ideal two-level result, as $\lambda_1$ is increased, is qualitatively incorrect in the naive treatment, as well as being overestimated in magnitude. Parameters: $\lambda/\omega_0 = 0.1$ and $\omega_1/\omega_0 = 2.5$. 

In Fig. 3 we compare this approach with the full calculation employing Hamiltonian (5), for the simplest case in which only one extra mode $\hat{c}_1$ is considered. We find qualitative discrepancies whenever the difference of the populations is non-negligible: we can indeed observe that the full Hamiltonian can result in $n_L > n_U$. Such a naive two-level approximation, therefore, cannot explain this behaviour, and even worse, its comparison with the observed populations may lead to the erroneous conclusion that the TRK sum rule has been violated, as Hamiltonian (1) requires $D < \lambda^2/\omega_0$ to achieve a higher population in the lower frequency mode [see Eq. (7)]. Still, in the regime $D \gtrsim \lambda^2/\omega_0$, one may be content to neglect the fine details associated to the modes $\hat{c}_j$, and seek a refined two-level approximation. This is indeed obtained by adiabatically eliminating the modes $\hat{c}_j$ from Hamiltonian (5) \cite{43, 44}. At lowest order, one has

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Acknowledgements — This work was supported by the Leverhulme Trust and the Qatar National Research Fund (Grant NPRP 4 - 426 554 - 1 - 084). We thank S. Barnett, J. Iles-Smith, M.-J. Hwang, S. De Liberato and P. L. Knight for fruitful discussions.

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SUPPLEMENTAL MATERIAL

Remarks on notation

In the main text, we found convenient to use a non-uniform notation in which the symbols $|g\rangle, |e\rangle$ denoted the ground and first excited state of a single emitter, assumed to take central role in the dynamics of interest, while the remaining levels were indicated as $|e_j\rangle$. The corresponding bosonic modes were also grouped as $\tilde{b}$, associated with the $|g\rangle \leftrightarrow |e\rangle$ transition, and $\tilde{c}_j$ associated with $|g\rangle \leftrightarrow |e_j\rangle$. For the purposes of this appendix, however, it shall be more convenient to adopt a uniform notation where we label all levels as $|\varepsilon_l\rangle$ through a single discrete index $l \geq 0$. It is assumed that the corresponding energies $\varepsilon_l$ are non-decreasing in $l$, with $l = 0$ labelling the ground state. Similarly, the collective operator associated with $|\varepsilon_0\rangle \leftrightarrow |\varepsilon_l\rangle$ shall be indicated as $\tilde{b}_l$. This has the advantage of rendering the derivations to follow more compact. It is straightforward to recover the notation of the main text by associating $|g\rangle \equiv |\varepsilon_0\rangle$, $|e\rangle \equiv |\varepsilon_1\rangle$ and $\tilde{b} \equiv \tilde{b}_1$.

Appendix A: Model derivation for a single emitter

We assume that each emitter can be microscopically described as a collection of non-relativistic particles of mass $m_j$ and charge $q_j$, subject to a potential $\tilde{V}$ that includes trapping forces as well as inter-particle interactions. The effective linear size of the emitter is assumed to be much smaller than the wavelength of light under consideration. The corresponding minimal coupling Hamiltonian reads

$$H_{\text{mic}} = \sum_j \frac{(\hat{p}_j - q_j \hat{A})^2}{2m_j} + \hat{V}(\hat{x}_1, \hat{x}_2, \ldots),$$  \hspace{1cm} (A1)

where $\hat{p}_j$ is the momentum of the $j$-th particle, $\hat{x}_j$ being its position, and $\hat{A}$ is the vector potential operator whose spatial dependence across the emitter is neglected, that is, we adopt the dipole approximation. Under this assumption the components of $\hat{A}$ commute with all particle operators, hence we can expand the Hamiltonian as

$$H_{\text{mic}} = H_{\text{mic}}^0 + H_{\text{int}} + H_{\text{diam}},$$  \hspace{1cm} (A2)

$$H_{\text{mic}}^0 = \sum_j \frac{\hat{p}_j^2}{2m_j} + \hat{V},$$  \hspace{1cm} (A3)

$$H_{\text{int}} = -\sum_j \frac{q_j \hat{p}_j \cdot \hat{A}}{m_j},$$  \hspace{1cm} (A4)

$$H_{\text{diam}} = \sum_j \frac{q_j^2}{2m_j} \hat{A}^2.$$  \hspace{1cm} (A5)

To connect such microscopic description to the effective models employed in the main text we assume that, within the energy scale of interest, $H_{\text{mic}}^0$ gives rise to a discrete level structure:

$$H_{\text{mic}}^0 = \sum_l \varepsilon_l |\varepsilon_l\rangle \langle \varepsilon_l|,$$  \hspace{1cm} (A6)

where we have introduced the dipole operator $\hat{d} = \sum_j q_j \hat{x}_j$. Assuming that transitions between excited states can be neglected in our problem, and applying appropriate phase rotations to the matrix elements for convenience, we can rewrite

$$H_{\text{int}} \simeq \sum_l \varepsilon_l \left( |\varepsilon_l\rangle \langle \varepsilon_l| \hat{A} |\varepsilon_0\rangle \langle \varepsilon_0| + |\varepsilon_0\rangle \langle \varepsilon_l| \hat{A}^\dagger |\varepsilon_l\rangle \langle \varepsilon_l| \right),$$  \hspace{1cm} (A7)

where the ground state energy $\varepsilon_0$ has been set to zero without loss of generality, so that $\varepsilon_l$ now coincides with the frequency of the $|\varepsilon_0\rangle \leftrightarrow |\varepsilon_l\rangle$ transition. We now consider a single-mode quantized field, whose vector potential at the emitter location can be written $\hat{A} = A_0 (\hat{a} + \hat{a}^\dagger)$, $A_0$ being a constant vector. We introduce the abbreviations $g_l \equiv \varepsilon_l |\varepsilon_l\rangle \langle \varepsilon_l|, \alpha \equiv \sum_j \frac{q_j^2}{2m_j} A_0^2$. Hence we can rewrite Eq. (A1) as

$$H_{\text{mic}} \simeq \sum_l \left[ \varepsilon_l |\varepsilon_l\rangle \langle \varepsilon_l| + g_l (|\varepsilon_l\rangle \langle \varepsilon_l| + |\varepsilon_0\rangle \langle \varepsilon_l|) (\hat{a} + \hat{a}^\dagger) \right] + \alpha (\hat{a} + \hat{a}^\dagger)^2,$$  \hspace{1cm} (A9)
The TRK rule for the ground state (see Ref. 32 of the main text) can now be used to derive a strict relationship between the couplings \( g_l \), the transition frequencies \( \epsilon_l \), and the coefficient \( \alpha \), characterizing the strength of the diamagnetic term. One has

\[
\sum_l \epsilon_l \left| \langle \epsilon_l | \hat{d} \cdot A_0 | \epsilon_0 \rangle \right|^2 = \sum_j \frac{g_j^2}{2m_j} A_0^2 \tag{A10}
\]

which is easily seen to correspond to the equality

\[
\sum_j \frac{g_j^2}{\epsilon_l} = \alpha. \tag{A11}
\]

**Appendix B: Bosonic approximation for many emitters**

Having established the key parameters of single emitters that emerge from the microscopic model, we can move on to the description of a large number \( n \gg 1 \) of these that interact with the same field. Following the same steps as before, the corresponding microscopic Hamiltonian can be written as

\[
H_{\text{mic}}^{(n)} \approx \sum_{l,k} \left[ \epsilon_l |\epsilon_l \rangle \langle \epsilon_l |_{k} + g_l (|\epsilon_l \rangle \langle \epsilon_0 |_{k} + |\epsilon_0 \rangle \langle \epsilon_l |_{k})(\hat{a} + \hat{a}^\dagger) \right] + D(\hat{a} + \hat{a}^\dagger)^2, \tag{B1}
\]

where \( D = n \alpha \) and the dummy index \( k \) labels the individual emitters, which we have assumed identically coupled to the field \( \hat{a} \) for simplicity. However, the treatment can be easily adapted when this constraint is lifted and the parameters \( g_l, \alpha \) become \( k \)-dependent. We can exploit the symmetries of Eq. (B1) to define the collective operators

\[
\hat{b}_l \equiv \frac{1}{\sqrt{n}} \sum_k |\epsilon_0 \rangle \langle \epsilon_l |_{k}, \tag{B2}
\]

allowing us to rewrite

\[
H_{\text{mic}}^{(n)} = \sum_l \left[ \epsilon_l \hat{b}_l \hat{b}_l^\dagger + \lambda_l (\hat{b}_l \hat{b}_l^\dagger + \hat{b}_l^\dagger \hat{b}_l) (\hat{a} + \hat{a}^\dagger) \right] + D(\hat{a} + \hat{a}^\dagger)^2, \tag{B3}
\]

where \( \lambda_l = \sqrt{n} g_l \), and we have neglected a large portion of the Hilbert space corresponding to the so-called dark modes, collective excitations of the \( n \) emitters that do not couple to the field and are hence irrelevant for our purposes. Comparing the definitions of \( \lambda_l \) and \( D \) with Eq. (A11), we find that the TRK sum rule translates in this case to

\[
\sum_l \frac{\lambda_l^2}{\epsilon_l} = D \tag{B4}
\]

Finally, we now show that the operators \( \hat{b}_l \) are approximately bosonic in the limit of diluted excitations. Their exact commutator reads

\[
[\hat{b}_l, \hat{b}_l^\dagger] = \frac{1}{n} \sum_k \left( |\epsilon_0 \rangle \langle \epsilon_0 |_{k} \delta_{l,l'} - |\epsilon_l \rangle \langle \epsilon_l |_{k} \right). \tag{B5}
\]

The approximate bosonic behaviour of the above commutator is then obtained if the dynamics of interest involves states \( |\Psi \rangle \) such that most emitters are found in their ground state \( |\epsilon_0 \rangle \). Indeed, if one assumes that the average number of ground state emitters in the state \( |\Psi \rangle \) is at least \( n(1-\epsilon) \), where \( \epsilon \ll 1 \), it follows trivially that

\[
\langle \Psi | [\hat{b}_l, \hat{b}_l^\dagger] | \Psi \rangle = \delta_{l,l'} + O(\epsilon). \tag{B6}
\]

In practice, the condition of diluted excitations can be verified a posteriori, by tracking the average value \( \langle \sum_l \hat{b}_l \hat{b}_l^\dagger \rangle \) during the dynamics and verifying that it is well below \( n \) at all times. This is precisely the regime considered in the main text.

**Appendix C: Neglecting excited-state transitions**

In this final section we argue that the regime of diluted excitations also justifies the neglect of the terms of the form \( |\epsilon_l \rangle \langle \epsilon_l^\prime |_{k} \) with both \( l, l' \neq 0 \) in Eq. (A7). These will appear, in the full Hamiltonian for \( n \) emitters, as collective operators of the form

\[
\hat{n}_{l,l'} = \frac{1}{\sqrt{n}} \sum_k \epsilon_l |\epsilon_l \rangle \langle \epsilon_l^\prime |_{k}. \tag{C1}
\]

For the lower excited states, which we can expect to provide the most significant contribution to the dynamics of interest, these operators will be multiplied by coefficients with weights comparable to the \( \lambda_l \)'s. Since the operators \( \hat{n}_{l,l'} \) cannot change the number of emitters in the ground state, we can estimate their effect by comparing the matrix elements with those of the \( \hat{b}_l \)'s in the low-excitation sector of the Hilbert space. Hence, taking any two states \( |\Psi \rangle, |\Phi \rangle \) with an average number of ground state emitters \( \geq n(1-\epsilon) \), we can perform easy calculations to provide the rough estimates

\[
|\langle \Phi | \hat{n}_{l,l'} | \Psi \rangle| \propto \epsilon \quad l, l' \neq 0, \tag{C2}
\]

\[
|\langle \Phi | \hat{b}_l | \Psi \rangle| \propto \sqrt{\epsilon} \quad l \neq 0, \tag{C3}
\]

thus illustrating that the probability amplitudes for transitions between excited states scale less favourably with \( \epsilon \), and can be suppressed for sufficiently diluted excitations (\( \epsilon \ll 1 \)).