New Representations for the Curvature Tensor of a Surface with Application to Theories of Elastic Shells

Nathaniel N. Goldberg, Oliver M. O’Reilly

1 Department of Mechanical Engineering, University of California, Berkeley, CA 94720-1740, USA

Abstract

Consider two points $P$ and $Q$ on a surface. Modulo rotations about the normal vector to the surface at $P$ and the normal vector to the surface at $Q$, a rotation can be defined that maps the unit normal vector to the surface at $Q$ to the corresponding unit normal vector at $P$. With the help of Weingarten’s formulae, new representations are established for the components of the curvature tensor of a surface and the associated mean and Gaussian curvatures in terms of components of a pair of vectors associated with the rotation. The formulae are shown to be helpful in demonstrating how different strain measures for Kirchhoff-Love shell theory are equivalent.

Keywords Theory of shells · Finite rotations · Weingarten’s formulae

Mathematics Subject Classification 53A01 · 73L05

1 Introduction

Given a surface $S$, Weingarten’s well-known formulae express the evolution of a unit normal vector $\mathbf{n}$ to a two-dimensional surface in terms of the coefficients $b_{\alpha\beta}$ of second-fundamental form of the surface and the covariant basis vectors $\mathbf{a}_\gamma$ (cf. Fig. 1). In this paper, we use his formulae to establish new representations for the components of the curvature tensor $\mathbf{b} = b_{\alpha\beta} \mathbf{a}_\alpha \otimes \mathbf{a}_\beta = b^\beta_\alpha \mathbf{a}_\beta \otimes \mathbf{a}_\alpha$, the mean curvature $H$, and the Gaussian curvature $K$:

$$b_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{n}) \cdot \mathbf{\omega}_\beta, \quad b^\beta_\alpha = (\mathbf{a}^\alpha \times \mathbf{n}) \cdot \mathbf{\omega}_\beta,$$

$$H = \frac{1}{2\sqrt{a}} (\mathbf{\omega}_2 \cdot \mathbf{a}_1 - \mathbf{\omega}_1 \cdot \mathbf{a}_2), \quad K = \frac{1}{\sqrt{a}} (\mathbf{\omega}_1 \times \mathbf{\omega}_2) \cdot \mathbf{n},$$

(1.1)

where the angular rate vectors $\mathbf{\omega}_\beta$ are used to compute the partial derivatives of $\mathbf{n}$:

$$\mathbf{n}_{,\beta} = \mathbf{\omega}_\beta \times \mathbf{n}.$$  

(1.2)
As an illustrative example, the case of a spherical surface is considered. The representations (1.1) are then shown to be helpful in relating two different formulations for some of the strain measures used in Kirchhoff-Love shell theory. We also note that the discussion in this paper complements alternative representations for $b_{\alpha\beta}$, $H$, and $K$, including those using Cartan’s moving frames that can be found in texts on differential geometry (see, e.g., [1, 5, 12]).

2 Background

Consider a two-dimensional surface $S$ that is embedded in three-dimensional Euclidean space $\mathbb{E}^3$. The surface is parameterized using a curvilinear coordinate system $\{\theta_1, \theta_2\}$ such that every point $P$ on the surface can be uniquely identified by a position vector (relative to a fixed origin):

$$r = r(\theta^1, \theta^2).$$  \hfill (2.1)

We now define a covariant basis $\{a_1, a_2\}$ for the tangent space to a point on $S$:

$$a_\beta = r_\beta = \frac{\partial r}{\partial \theta^\beta} , \quad (\beta = 1, 2).$$  \hfill (2.2)

Here, and in the sequel, lower-case Greek letters range from 1 to 2, the summation convention on repeated indices is employed, and the comma denotes partial derivative. A unit normal $n$ is assumed to be defined at every point $P$ on the surface where $(a_1 \times a_2) \cdot n > 0$.

We can define a contravariant basis $\{a^1, a^2\}$ where $a_\alpha \cdot a^\beta = \delta_\alpha^\beta$ and $n \cdot a^\beta = 0$. The Kronecker delta $\delta_\alpha^\beta = 0$ if $\beta \neq \alpha$ and $= 1$ if $\alpha = \beta$. Solving $a_\alpha \cdot a^\beta = \delta_\alpha^\beta$ for $a^\beta$:

$$\sqrt{a} a^1 = a_2 \times n, \quad \sqrt{a} a^2 = -a_1 \times n.$$  \hfill (2.3)

where $\sqrt{a} = \|a_1 \times a_2\| = (a_1 \times a_2) \cdot n$. We note that

$$a_\beta = a_{\beta\alpha} a^\alpha, \quad a^\beta = a^{\beta\alpha} a_\alpha,$$  \hfill (2.4)

where

$$a_{\alpha\beta} = a_\alpha \cdot a_\beta, \quad a^{\alpha\beta} = a^\alpha \cdot a^\beta.$$  \hfill (2.5)

Here, $a_{\alpha\beta}$ are the coefficients of the first fundamental form of $S$. 

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3 A Rotation of the Unit Normal Vector and a Pair of Axial Vectors

We consider a point $Q$ on the surface and denote the normal to the surface at this point by $n_0$. As the normal vector is a unit vector, we can define a rotation that transforms $n_0$ to the normal $n$ at any point on the surface. The resulting rotation tensor, which we denote by $Q$, is a function of the coordinates $\theta^\beta$:

$$n = n(\theta^1, \theta^2) = Q(\theta^1, \theta^2)n_0. \quad (3.1)$$

As discussed in the sequel, the tensor $Q$ is not uniquely defined. We also note that various representations, including Euler angles, axis-angle, Euler parameters, unit quaternions, and Cayley parameters for $Q$ can be employed but this specification is not needed for present purposes.

Differentiating the identity $n = Qn_0$ with respect to $\theta^\beta$, using the facts that $Q^T Q = I$ and $n_0$ is constant, we conclude that

$$n_\beta = Q_{\beta \gamma} Q^\gamma_n. \quad (3.2)$$

However, as $Q_{\beta \gamma} Q^\gamma$ is skew-symmetric, we can define an axial vector $\omega_\beta$ such that

$$\left(Q_{\beta \gamma} Q^\gamma\right) a = \omega_\beta \times a \quad (3.3)$$

for any vector $a$. The vector $\omega_\beta$ has the representations

$$\omega_\beta = \omega_\beta^1 a_1 + \omega_\beta^2 a_2 + \omega_\beta^n n = \omega_{\beta 1} a^1 + \omega_{\beta 2} a^2 + \omega_{\beta n} n. \quad (3.4)$$

As the vectors $\omega_\beta$ associated with $Q$ are identical to those associated with $QQ_1$ where $Q_1$ is an arbitrary constant rotation tensor, the choice of $Q$, as anticipated, does not effect the forthcoming results.

The prescription (3.1) for $Q$ does not define a unique rotation tensor. Indeed, if $Q$ satisfies (3.1) then so does

$$R = L(\nu, n)QL(\nu_0, n_0), \quad (3.5)$$

where the tensor $L(\theta, p)$ represents a rotation through an angle $\theta$ about the unit vector $p$: \(^1\)

$$Rn_0 = L(\nu, n)QL(\nu_0, n_0)n_0 = L(\nu, n)Qn_0 = L(\nu, n)n = n. \quad (3.6)$$

The angle of rotation $\nu_0$ is a constant. The axial vectors associated with the rotation $R$ can be computed using relative angular velocity vectors [3]:

$$\left(\frac{\partial R}{\partial \theta^\alpha} R^\gamma\right) a = (\omega_\alpha + \nu_\alpha n) \times a. \quad (3.7)$$

\(^1\)Representations for the rotation $L$ using a wide range of parameterizations can be found in [8]. A discussion of how to compute $\omega_1$ and $\omega_2$ using relative angular velocity vectors [3] is presented in [8].
Thus, the non-uniqueness of $Q$ implies that the $n$ component of $\omega_\alpha$ are not uniquely prescribed. Examining (1.1), we find that the components of $b$ and $K$ and $H$ are independent of $\omega_\alpha \cdot n$. In conclusion, the curvature tensor $b$ is insensitive to the fact that $Q$ is defined modulo an arbitrary rotation about $n$.

### 4 Formulae for the Curvature Tensor, the Mean Curvature, and the Gaussian Curvature

The curvature tensor $b$ has the representations (see, e.g., [2, 9]):

$$b = b_\beta^\alpha a_\alpha \otimes a^\beta = b_{\alpha\beta} a^\alpha \otimes a_\beta,$$

where

$$b_{\alpha\beta} = a_{\alpha,\beta} \cdot n. \quad b_\alpha^\gamma = a^\gamma a_{\alpha\beta}.$$  

(4.1)

and $\otimes$ is the tensor product: $(a \otimes b) c = a (b \cdot c)$ for any vectors $a$, $b$, and $c$.

Weingarten’s formulae relate the evolution of $n$ to the components of $b$:

$$n_{,\beta} = -b_{\alpha\beta} a^\alpha.$$  

(4.2)

The standard derivation of this formula starts by differentiating the identities $n \cdot n = 1$ and $n \cdot a_\beta = 0$. For our purposes, it is fruitful to consider the components of $n_{,\beta}$:

$$n_{,\beta} \cdot a_\alpha = \left(\omega_\beta \times n\right) \cdot a_\alpha = \omega_\beta \cdot \left(n \times a_\alpha\right) = \epsilon_\alpha\gamma \omega_\beta \cdot a_\gamma,$$

(4.3)

where $\epsilon_{11} = \epsilon_{22} = 0$ and $\epsilon_{12} = -\epsilon_{21} = \sqrt{a}$.

We can use (4.3) along with (4.4) to show that

$$b_{\alpha\beta} = -n_{,\beta} \cdot a_\alpha = -\left(\omega_\beta \times n\right) \cdot a_\alpha = -\omega_\beta \cdot \left(n \times a_\alpha\right).$$

(4.4)

Thus, we find the sought-after representations (1.1)\textsubscript{1,2} for the components $b_{\alpha\beta}$ and $b_\alpha^\gamma$:

$$b_{\alpha\beta} = \left(a_\alpha \times n\right) \cdot \omega_\beta, \quad b_\alpha^\gamma = \left(a^\gamma \times n\right) \cdot \omega_\beta.$$  

(4.5)

The identities $b_\alpha^\gamma = a^\gamma b_{\beta\delta}$ and $a^\alpha = a^\alpha a_\beta$ were used to establish the second representation. The identity (4.6)\textsubscript{1} along with the symmetry $b_{12} = b_{21}$ implies that

$$\omega_1 \cdot a^1 + \omega_2 \cdot a^2 = 0.$$  

(4.6)

Thus, the components of $\omega_1$ and $\omega_2$ are not all independent.

The representation (1.1)\textsubscript{3} for $H$ can be obtained by first substituting for $b_\alpha^\gamma$ in the definition of $H$ (cf. [9]) and then appealing to (2.3) and (4.6):
We observe that the representation for $K$ provides an elegant representation for $KdA$ where $dA$ is the element of area on $S$:

$$KdA = (\omega_1 \times \omega_2) \cdot n d\theta^1 d\theta^2.$$  \hfill (4.10)

The integral of $KdA$ is central to the Gauss-Bonnet theorem.

## 5 Application to a Sphere

As an example, we consider a sphere of radius $R$ and use a set of polar angles $\theta^1 = \varphi$ and $\theta^2 = \vartheta$ to parameterize the sphere (cf. Fig. 2). The representation for the position vector $\mathbf{r}$ of any point on the sphere is

$$\mathbf{r} = R \mathbf{e}_R,$$  \hfill (5.1)

where

$$\mathbf{e}_r = \cos (\vartheta) \mathbf{E}_1 + \sin (\vartheta) \mathbf{E}_2, \quad \mathbf{e}_R = \sin (\varphi) \mathbf{e}_r + \cos (\varphi) \mathbf{E}_3.$$  \hfill (5.2)
Differentiating \( \mathbf{r} \), the covariant basis vectors can be computed along with their contravariant counterparts:

\[
\mathbf{a}_1 = R \mathbf{e}_\varphi, \quad \mathbf{a}_2 = R \sin(\varphi) \mathbf{e}_\vartheta, \quad \mathbf{a}^1 = \frac{1}{R} \mathbf{e}_\varphi, \quad \mathbf{a}^2 = \frac{1}{R \sin(\varphi)} \mathbf{e}_\vartheta, \tag{5.3}
\]

where

\[
\mathbf{e}_\varphi = \cos(\varphi) \mathbf{e}_r - \sin(\varphi) \mathbf{E}_3, \quad \mathbf{e}_\vartheta = \cos(\vartheta) \mathbf{E}_2 - \sin(\vartheta) \mathbf{E}_1. \tag{5.4}
\]

Clearly, \( \mathbf{n} = \mathbf{e}_R \) and \( \sqrt{\mathbf{a}} = R^2 \sin(\varphi) \).

The rotation tensor \( \mathbf{Q} \) can be defined as the product of two rotations: one about the \( \mathbf{E}_3 \) axis through an angle \( \vartheta \) followed by a rotation about \( \mathbf{e}_\vartheta \) through an angle \( \varphi \):

\[
\mathbf{Q} = \mathbf{L}(\varphi, \mathbf{e}_\vartheta) \mathbf{L}(\vartheta, \mathbf{E}_3). \tag{5.5}
\]

Thus,

\[
\mathbf{\omega}_1 = \mathbf{e}_\vartheta, \quad \mathbf{\omega}_2 = \mathbf{E}_3, \quad \mathbf{\omega}_1 \times \mathbf{\omega}_2 = \mathbf{e}_r. \tag{5.6}
\]

Noting that \( \mathbf{e}_\vartheta \times \mathbf{e}_R = \mathbf{e}_\varphi \), we can now compute the components of \( \mathbf{b} \) using either (1.1)\(_1\) or (1.1)\(_2\):\(^2\)

\[
\mathbf{b} = -\frac{1}{R} \left( \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \mathbf{e}_\vartheta \otimes \mathbf{e}_\vartheta \right). \tag{5.7}
\]

The curvatures \( H \) and \( K \) can be computed using (1.1)\(_3,4\):

\[
H = \frac{1}{2 \sqrt{\mathbf{a}}} (\mathbf{\omega}_2 \cdot \mathbf{a}_1 - \mathbf{\omega}_1 \cdot \mathbf{a}_2) = -\frac{1}{R}, \quad K = \frac{1}{\sqrt{\mathbf{a}}} (\mathbf{\omega}_1 \times \mathbf{\omega}_2) \cdot \mathbf{n} = \frac{1}{R^2}, \tag{5.8}
\]

as anticipated.

### 6 Comments on Applications to Kirchhoff-Love Shell Theory

The representation (1.1)\(_1\) is of particular use when demonstrating the equivalence of formulations of Kirchhoff-Love shell theory. The strain energy density function for this shell theory has the representations (cf., e.g., [7, 11]):

\[
W = \hat{W}(a_{\alpha\beta}, b_{\gamma\delta}, \theta^1, \theta^2) \tag{6.1}
\]

and (cf., e.g., [4, 6])

\[
W = \tilde{W}(a_{\alpha\beta}, \mathbf{\omega}_\gamma, \theta^1, \theta^2). \tag{6.2}
\]

\(^2\)That is, \( b_{11} = -R, b_1^1 = -\frac{1}{R}, b_2^1 = b_1^2 = b_{12} = b_{21} = 0, b_{22} = -R \sin^2(\varphi) \), and \( b_2^2 = -\frac{1}{R} \).
The formula (1.1) can be used to establish the equivalence of $\tilde{W}$ and $\hat{W}$:

$$
\begin{align*}
    b_{11} &= -\omega_1 \cdot a^2, \\
    b_{22} &= \omega_2 \cdot a^1, \\
    b_{12} &= b_{21} = \omega_1 \cdot a^1 = -\omega_2 \cdot a^2.
\end{align*}
$$

It should also be evident from (6.3) that the strain energy function $\tilde{W}$ should be independent of the components $\omega_\beta \cdot n$. This restriction is in complete agreement with our earlier remarks that $Q$ is defined modulo a rotation about $n$.

Although the tensor $Q$ plays a central role in some formulations of Kirchhoff-Love shell theory (see [6]), we have not found a previous discussion of the non-uniqueness of $Q$. In some shell theories (cf. [4, 10]) where a rotation tensor $\Psi$ is associated with each point of the material surface, a pair of right-handed orthonormal triads (or adapted frames) are chosen: one comprised of $n_0$ and two unit tangent vectors $s_{01}$ and $s_{02}$ in the reference configuration and the other comprised of $n$ and two unit tangent vectors $s_1$ and $s_2$ in the present configuration. The rotation tensor $\Psi$, where

$$
\Psi = s_1 \otimes s_{01} + s_2 \otimes s_{02} + n \otimes n_0,
$$

is uniquely defined in this case. The fact that there are infinitely many choices of the triads $\{n_0, s_{01}, s_{02}\}$ and $\{n, s_1, s_2\}$ provides another explanation for the non-uniqueness of $Q$.\(^3\)

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Declarations

Declaration of Competing Interests

The authors declare no conflict of interest.

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\(^3\)We are indebted to an anonymous reviewer for pointing out this alternative explanation.
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