RIGIDITY OF MANIFOLDS ADMITTING STABLE SOLUTIONS OF AN ELLIPTIC PROBLEM

M. BATISTA AND J. I. SANTOS

Abstract. In this paper, we study geometric rigidity of Riemannian manifolds admitting stable solutions of certain elliptic problems (stability in a variational sense), that is, under suitable hypotheses, we are able to characterize the Riemannian manifold which admits a stable solution. Furthermore, under the non-negativity of the weighted Ricci curvature, we deduce several data about the stable solution and a splitting result for the manifold.

1. Introduction

The study of metric measure spaces has flourished in last few years, and a much better understanding of their geometric structure has evolved. We emphasize that the study of metric measure spaces and its generalized curvatures go back to Lichnerowicz [7, 8] and more recently by Bakry and Émery [1], in the setting of diffusion process, and it has been an active subject in recent years. For an overview, see for instance [2, 6, 11]. As the approach allows us to explore the setting of metric measure spaces and this theme is very attractive, see for instance the following list [9, 10, 13, 14, 15] of interesting articles about this subject, we are led to study stable solutions of a class of elliptic problems on metric measure spaces.

Throughout the paper, we use some notions which we introduce at this moment. We recall that a metric measure space is a Riemannian manifold $(\Sigma^n, g)$ endowed with a real-valued smooth function $f : \Sigma \to \mathbb{R}$ which is used as density in the following way: $d\text{Vol}_f = e^{-f}d\text{Vol}$, where Vol is the Riemannian measure of $\Sigma$.

Associated to this structure we have a second order differential operator defined by

$$\Delta_f u = e^f \text{div}(e^{-f} \nabla u),$$

acting on space of smooth functions. This operator is known in the literature as Drift Laplacian. Following [1], the natural generalization of Ricci curvature is defined by

$$\text{Ric}_f = \text{Ric} + \text{Hess} f,$$

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which is known as Bakry-Émery Ricci tensor or simply by weighted Ricci curvature. Given $g \in C^\infty(\mathbb{R})$, we consider the closed Dirichlet problem

$$\Delta f u + g(u) = 0.$$  

A solution of this problem is a critical point of an energy functional, which we denote by $E_f$, see the details in section 2. We say that a solution $u$ of (1.1) is stable if the second variation of $E_f$ at $u$ is non-negative on $H^1_c(\Sigma)$. Lastly, we say that a metric measure space is $f$-parabolic if there exists no nonnegative non-constant function which is $f$-superharmonic. Now, we are able to introduce our results.

The first one is read as follows:

**Theorem 1.1.** Let $\Sigma^n$ be a complete and non-compact metric measure space without boundary. Assume that $u$ is a smooth non-constant stable solution of (1.1) and the weighted Ricci curvature satisfies

$$\text{Ric}_f(\nabla u, \nabla u) \geq -\frac{H^2}{n-1} |\nabla u|^2,$$

on the regular part of each level set $\Sigma_t = \{u = t\}$ of $u$ and $H_t$ is the mean curvature of $\Sigma_t$.

If either

(i) $\Sigma$ is $f$-parabolic and $\nabla u \in L^\infty(M)$,

or

(ii) the function $|\nabla u|$ satisfies

$$\int_{B_R} |\nabla u|^2 d\text{Vol}_f = o(R^2 \log R) \quad \text{as } R \to +\infty,$$

then, $u$ has no critical points and $\Sigma = \mathbb{R} \times N$ is furnished with a warped product metric

$$ds^2 = dt^2 + \exp \left(-2 \int_0^t \lambda(s) ds \right) g_N,$$  

for a smooth function $\lambda$ which depends on $\nabla u$.

We would like to point out that as far as we know, the result above is new even in the Riemannian case and so we improve [3, Theorem 1].

An interesting conclusion happens whether we assume that the space is non-negatively curved. The result is the following:

**Theorem 1.2.** Let $\Sigma^n$ be a complete and non-compact metric measure space without boundary and the weighted Ricci curvature is nonnegative. Assume that $u$ is a smooth non-constant stable solution of (1.1).

If either

(i) $\Sigma$ is $f$-parabolic and $\nabla u \in L^\infty(M)$,

or

$$\int_{B_R} |\nabla u|^2 d\text{Vol}_f = o(R^2 \log R) \quad \text{as } R \to +\infty,$$  

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$$ds^2 = dt^2 + \exp \left(-2 \int_0^t \lambda(s) ds \right) g_N,$$  

for a smooth function $\lambda$ which depends on $\nabla u$. 

(ii) the function $|\nabla u|$ satisfies
\begin{equation}
\int_{B_R} |\nabla u|^2 d\Vol_f = o(R^2 \log R) \quad \text{as } R \to +\infty,
\end{equation}
then, $u$ has no critical point and $\Sigma = \mathbb{R} \times N$ is furnished with the product metric
\[ ds^2 = dt^2 + g_N, \]
and $N$ is $f$-parabolic complete totally geodesic hypersurface and its weighted Ricci curvature is nonnegative. Moreover, $u$ depends only on $t$, has no critical points, $\langle \nabla f, \nabla u \rangle = k|\nabla u|$ for a constant $k$ and writing $u = y(t)$, $u$ satisfies
\[ -y'' + ky' = g(y). \]
Furthermore, if (ii) is met,
\[ \Vol_f(B_R^N) = o(R^2 \log R) \quad \text{as } R \to +\infty, \]
and
\[ \int_{-R}^R |y'(t)|^2 dt = o \left( \frac{R^2 \log R}{\Vol_f(B_R^N)} \right) \quad \text{as } R \to +\infty. \]

We would like to highlight that the Allen-Cahn equation fit at our study, to see this choose $f = 0$ and $g(t) = (1-t^2)t$. This equation is very interesting and well-understand on closed manifolds. Here, under suitable assumptions, our results provide some information about it on complete non-compact manifolds without boundary.

**Outline of the paper:** In section 2 we briefly survey some concepts and equivalences that we use in the paper. In section 3 we prove some technical lemmata that will be the heart of the proof of the results. In sections 4 and 5 we give the proof of them.

### 2. Background

Throughout the paper $\Sigma$ will denote a connect metric measure space of dimension $n \geq 2$ without boundary. We briefly fix some notation. Having fixed an origin $p_0$, we set $r(x) = \text{dist}(x,p_0)$, and we write $B_R$ for geodesic ball centered at $p_0$. If we need to emphasize the set under consideration, we will add a superscript symbol, so that, for instance, we will also write $\text{Ric}^f_\Sigma$ and $B_R^\Sigma$.

The Riemannian $n$-dimensional volume will be indicated with $\Vol$, and the measure with density by $d\Vol_f = e^{-f}d\Vol$. Furthermore, in a similar way, we write $\mathcal{H}^{n-1}$ for the induced $(n-1)$-dimensional Hausdorff measure and $d\mathcal{H}^{n-1}_f = e^{-f}d\mathcal{H}^{n-1}$ for the weighted associated measure. Lastly, we
use the symbol \{\Omega_j\} \uparrow \Sigma for indicate a family \{\Omega_j\}_{j \in \mathbb{N}} of relativity compact, open sets with smooth boundary and satisfying

$$ \Omega_j \subseteq \Omega_{j+1} \subseteq \Sigma, \quad \Sigma = \bigcup_{j=0}^{+\infty} \Omega_j, $$

where \( A \subseteq B \) means \( \overline{A} \subseteq B \). Such a family will be called an exhaustion of \( \Sigma \). Hereafter, we consider 

$$ g \in C^\infty(\mathbb{R}), $$

and a solution \( u \) on \( \Sigma \) of

$$ \Delta_f u + g(u) = 0 \quad \text{in} \ \Sigma. $$

We recall that \( u \) is characterized, on each open subset \( U \subseteq \Sigma \), as a critical point of the energy functional acting on functions in the Sobolev space which has compact support on \( \Sigma \), that is, \( E_f : H^1_c(\Sigma) \rightarrow \mathbb{R} \) given by

$$ E_f(u) = \frac{1}{2} \int_{\Sigma} |\nabla u|^2 d\text{Vol}_f - \int_{\Sigma} G(u) d\text{Vol}_f, \quad \text{where} \ G(t) = \int_0^t g(s) ds, $$

with respect to compactly variation in \( U \). Indeed, after a straightforward computation and using the integration by parts we get:

$$ E'_f(u)h = -\int_{\Sigma} (\Delta_f u + g(u))h d\text{Vol}_f, $$

for all \( h \in C^\infty_c(U) \). In a similar way, we have that

$$ E''_f(u)(h,k) = -\int_{\Sigma} (\Delta_f h + g'(u)h)k d\text{Vol}_f, $$

for \( h,k \in C^\infty_c(U) \). So, the stability operator associated to \( E_f \) at \( u \) is given by

$$ J_f h = -\Delta_f h - g'(u)h, \quad h \in C^\infty_c(\Sigma). $$

**Definition 1.** A function \( u \) that solves (1.1) is said to be a stable solution if the stability operator at \( u \) is nonnegative on \( C^\infty_c(\Sigma) \), that is, if

$$ (2.1) \quad \int_\Sigma g'(u)h^2 d\text{Vol}_f \leq \int_\Sigma |\nabla h|^2 d\text{Vol}_f, \quad \text{for all} \ h \in C^\infty_c(\Sigma). $$

By density, we can replace \( C^\infty_c(\Sigma) \) in (2.1) by \( \text{Lip}_c(\Sigma) \). By a simple adaptation of [4, Theorem 1], the stability of \( u \) turns out to be equivalent to the existence of a positive \( w \in C^\infty(\Sigma) \) solving \( \Delta_f w + g'(u)w = 0 \) in \( \Sigma \).

Let \( \Omega \) be an open set on \( \Sigma \) and \( K \) be a compact set in \( \Omega \). We call the pair \( (K,\Omega) \) of a \( f \)-capacitor and define the \( f \)-capacity \( \text{cap}_f(K,\Omega) \) by

$$ \text{cap}_f(K,\Omega) = \inf_{\phi \in \mathcal{L}(K,\Omega)} \int_{\Omega} |\nabla \phi|^2 d\text{Vol}_f, $$

where \( \mathcal{L}(K,\Omega) \) is the set of Lipschitz functions \( \phi \) on \( \Sigma \) with compact support in \( \overline{\Omega} \) such that \( 0 \leq \phi \leq 1 \) and \( \phi|_K = 1 \).
For an open precompact set $K \subset \Omega$, we define its $f$-capacity by
\[
\text{cap}_f(K, \Omega) := \text{cap}_f(K, \Omega).
\]
In case that $\Omega$ coincide with $\Sigma$, we write $\text{cap}_f(K)$ for $\text{cap}_f(K, \Omega)$. It is obvious from the definition that the set $L^p(K, \Omega)$ increases on expansion of $\Omega$ (and on shrinking of $K$). Therefore, the capacity $\text{cap}_f(K, \Omega)$ decreases on expanding of $\Omega$ (and on shrinking of $K$). In particular, we can prove that, for any exhaustion sequence $\{\mathcal{E}_k\}$
\[
\text{cap}_f(K) := \lim_{k \to \infty} \text{cap}_f(K, \mathcal{E}_k).
\]
Moreover, the limit is independent of the exhaustion.

In the next definition we provide an analytical concept that is related with the notion of capacity. It shall be clear at next proposition.

**Definition 2.** A metric measure space is $f$-parabolic if there exists no non-constant nonnegative $f$-superharmonic function $u$, that is, if $\Delta_f u \leq 0$ and $u \geq 0$, then $u$ is constant.

Hence, we have the following characterization of $f$-parabolicity. For a proof see for instance [6].

**Proposition 2.1.** Let $\Sigma$ be a complete metric measure space. Then, the following are equivalent:

(i) $\Sigma$ is $f$-parabolic.

(ii) $\text{cap}_f(K) = 0$ for some (then any) compact set $K \subset \Sigma$.

The following criterion of $f$-parabolicity is well known, for more details see for instance [5, Proposition 3.4] or [6, Theorem 11.14].

**Proposition 2.2.** Let $p_o$ be a fixed point in a metric measure space $\Sigma$ and let
\[
L(r) = \int_{B(p_o, r)} d\text{Vol}_f \quad \text{and} \quad V(r) = \int_{B(p_o, r)} d\text{Vol}_f.
\]
If
\[
\int_1^\infty \frac{dr}{L(r)} = +\infty \quad \text{or} \quad \int_1^\infty \frac{rdr}{V(r)} = +\infty,
\]
then $\Sigma$ is $f$-parabolic.

3. **Key Lemmata**

The first lemma is a simplified Picone type identity for metric measure spaces.

**Lemma 1.** Let $\Sigma$ be a complete manifold without boundary and let $u \in C^3(\Sigma)$ be a solution of $\Delta_f u + g(u) = 0$ on $\Sigma$. Let $w \in C^2(\Sigma)$ be a supersolution of $\Delta_f w + g'(u)w \leq 0$ such that $w > 0$ on $\Sigma$. Then the following inequality holds true for every $h \in \text{Lip}_c(\Sigma)$,
\[
\int_\Sigma w^2 \left| \nabla \left( \frac{h}{w} \right) \right|^2 d\text{Vol}_f \leq \int_\Sigma \left| \nabla h \right|^2 d\text{Vol}_f - \int_\Sigma g'(u)h^2 d\text{Vol}_f.
\]
The inequality is indeed an equality if \( w \) solves \( \Delta_f w + g'(u)w = 0 \) on \( \Sigma \).

**Proof.** Multiplying \( \Delta_f w + g'(u)w \) by the test function \( \frac{h^2}{w} \), integrating and using integration by parts we deduce

\[
- \int_{\Sigma} (\Delta_f w + g'(u)w) \frac{h^2}{w} \, dVol_f = \int_{\Sigma} \langle \nabla \left( \frac{h^2}{w} \right), \nabla w \rangle \, dVol_f - \int_{\Sigma} g'(u)h^2 \, dVol_f.
\]

Since

\[
\langle \nabla \left( \frac{h^2}{w} \right), \nabla w \rangle = 2\frac{h}{w} \langle \nabla h, \nabla w \rangle - \frac{h^2}{w^2} |\nabla w|^2,
\]

using the identity

\[
w^2 \left| \nabla \left( \frac{h}{w} \right) \right|^2 = |\nabla h|^2 + \frac{h^2}{w^2} |\nabla w|^2 - 2\frac{h}{w} \langle \nabla w, \nabla h \rangle,
\]

we infer that

\[
\langle \nabla \left( \frac{h^2}{w} \right), \nabla w \rangle = |\nabla h|^2 - w^2 \left| \nabla \left( \frac{h}{w} \right) \right|^2.
\]

Inserting the latter equality into the integral equation at this lemma and using our hypothesis we conclude the desired result. \( \square \)

**Lemma 2.** Under the assumptions of the former Lemma, for every \( h \in C^\infty(\Sigma) \) the following integral inequality holds true:

\[
\int_{\Sigma} \left[ |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) \right] h^2 \, dVol_f - \int_{\Sigma} h^2 |\nabla |\nabla u||^2 \, dVol_f \leq \int_{\Sigma} |\nabla h|^2 |\nabla u|^2 \, dVol_f - \int_{\Sigma} w^2 \left| \nabla \left( \frac{h|\nabla u|}{w} \right) \right|^2 \, dVol_f.
\]

The inequality is indeed an equality if \( \Delta_f w + g'(u)w = 0 \) on \( \Sigma \).

**Proof.** Recall the Böchner formula

\[
\frac{1}{2} \Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla u, \nabla (\Delta_f u) \rangle + Ric_f(\nabla u, \nabla u),
\]

for all \( u \in C^3(\Sigma) \). Since \( u \) solves \( -\Delta_f u = g(u) \), we get

\[
\frac{1}{2} \Delta_f |\nabla u|^2 = |\nabla^2 u|^2 - g'(u)|\nabla u|^2 + Ric_f(\nabla u, \nabla u).
\]

Integrating the latter equality against the test function \( h^2 \) and setting

\[
I = \int_{\Sigma} \left( |\nabla^2 u|^2 + Ric_f(\nabla u, \nabla u) \right) h^2 \, dVol_f,
\]
we deduce that
\[
I = \int_{\Sigma} g'(u)|\nabla u|^2 h \text{dVol}_f + \frac{1}{2} \int_{\Sigma} h^2 \Delta_f |\nabla u|^2 \text{dVol}_f
= \int_{\Sigma} g'(u)|\nabla u|^2 h \text{dVol}_f - \frac{1}{2} \int_{\Sigma} \langle \nabla h^2, \nabla |\nabla u|^2 \rangle \text{dVol}_f
= \int_{\Sigma} g'(u)|\nabla u|^2 h \text{dVol}_f - \int_{\Sigma} h \langle \nabla h, \nabla |\nabla u|^2 \rangle \text{dVol}_f.
\]

Now, we plugging the test function \( h|\nabla u| \) into \( (3.1) \) in Lemma 1 to obtain:
\[
0 \leq \int_{\Sigma} |\nabla (h|\nabla u|)|^2 \text{dVol}_f - \int_{\Sigma} g'(u) h^2 |\nabla u|^2 \text{dVol}_f - \int_{\Sigma} w^2 \left| \nabla \left( \frac{h|\nabla u|}{w} \right) \right|^2 \text{dVol}_f
= \int_{\Sigma} |\nabla h|^2 |\nabla u|^2 \text{dVol}_f + \int_{\Sigma} h^2 |\nabla |\nabla u||^2 \text{dVol}_f + 2 \int_{\Sigma} h |\nabla u| \langle \nabla h, \nabla |\nabla u| \rangle \text{dVol}_f
- \int_{\Sigma} g'(u)h^2 |\nabla u|^2 \text{dVol}_f - \int_{\Sigma} w^2 \left| \nabla \left( \frac{h|\nabla u|}{w} \right) \right|^2 \text{dVol}_f.
\]

Recalling that \( |\nabla |\nabla u||^2 = 2 |\nabla u| |\nabla |\nabla u| \) weakly on \( \Sigma \) and putting the integral \( I \) into the former inequality we conclude \((3.2)\) as desired.

For the next result, we fix the following notation. Denote the level set of \( u \) by \( \Sigma_t = \{ u = t \} \) and by \( A_t \) its second fundamental form. Assuming that \( p \) is a regular point of \( u \), that is, \( \nabla u(p) \neq 0 \) we are able to prove:

**Lemma 3** (Kato’s inequality for functions). Under the above notation holds:
\[
|\nabla^2 u|^2 - |\nabla |\nabla u||^2 = |\nabla u|^2 |A_t|^2 + |\nabla T|\nabla u|^2
\]
at \( p \in \Sigma_t \), where \( \nabla T \) is the tangential gradient on the level set \( \Sigma_t \).

**Proof.** In this proof we will omit the sub-index \( t \). Fix a local orthonormal frame \( \{ e_i \} \) on \( \Sigma \), and let \( \nu = \nabla u/|\nabla u| \) be the normal vector. For every vector field \( X \in \mathfrak{X}(\Sigma) \),
\[
\nabla^2 u(\nu, X) = \frac{1}{|\nabla u|^2} \nabla^2 u(\nabla u, X) = \frac{1}{2|\nabla u|^2} \langle \nabla |\nabla u|^2, X \rangle = \langle \nabla |\nabla u||, X \rangle.
\]

Moreover, for a level set
\[
A = -\frac{\nabla^2 u|_{T\Sigma \times T\Sigma}}{|\nabla u|}.
\]

Therefore
\[
|\nabla^2 u|^2 = \sum_{i,j}(\nabla^2 u(e_i, e_j))^2 + 2 \sum_j(\nabla^2 u(\nu, e_j))^2 + (\nabla^2 u(\nu, \nu))^2
= |\nabla u|^2 |A|^2 + 2 \sum_j(\nabla |\nabla u|, e_j)^2 + (\nabla |\nabla u|, \nu)^2
= |\nabla u|^2 |A|^2 + |\nabla T|\nabla u|^2 + |\nabla |\nabla u||^2,
\]
and so we conclude the Lemma.

\( \Box \)
Lemma 4. For all $h \in C^\infty(\Sigma)$ holds:
\[
\int_\Sigma \left( |\nabla^2 u|^2 - |\nabla |\nabla u||^2 + \text{Ric}_f(\nabla u, \nabla u) \right) h^2 \text{dVol}_f \leq 2 \int_\Sigma |\nabla h|^2 |\nabla u|^2 \text{dVol}_f.
\]
The equality holds if and only if $|\nabla u| = cw$ for some real constant $c$.

Proof. Using the formula (3.2) in Lemma 2, we have for all test function $h$:
\[
\int_\Sigma \left( |\nabla^2 u|^2 - |\nabla |\nabla u||^2 + \text{Ric}_f(\nabla u, \nabla u) \right) h^2 \text{dVol}_f + \int_\Sigma (1 - \delta) |\nabla h|^2 |\nabla u|^2 \text{dVol}_f \leq \frac{1}{\delta} \int_\Sigma |\nabla h|^2 |\nabla u|^2 \text{dVol}_f.
\]
Using the weighted Cauchy inequality for vectors we get
\[
|X + Y|^2 \geq |X|^2 + |Y|^2 - 2|X||Y| \geq (1 - \delta)|X|^2 + (1 - \delta^{-1})|Y|^2,
\]
for each $\delta > 0$, and so the last term of the right hand side can be arranged as
\[
w^2 \left| \nabla \left( \frac{h|\nabla u|}{w} \right) \right|^2 \geq (1 - \delta^{-1})|\nabla u|^2 |\nabla h|^2 + (1 - \delta)h^2 w^2 \left| \nabla \left( \frac{|\nabla u|}{w} \right) \right|^2.
\]
Plugging this into (3.3) yields
\[
\int_\Sigma \left( |\nabla^2 u|^2 - |\nabla |\nabla u||^2 + \text{Ric}_f(\nabla u, \nabla u) \right) h^2 \text{dVol}_f + \int_\Sigma (1 - \delta) h^2 w^2 \left| \nabla \left( \frac{|\nabla u|}{w} \right) \right|^2 \text{dVol}_f \leq \frac{1}{\delta} \int_\Sigma |\nabla h|^2 |\nabla u|^2 \text{dVol}_f.
\]
Choosing $\delta = \frac{1}{2}$ we conclude the desired result. \(\square\)

4. Proof of Theorem 1.1

Proof of Theorem 1.1. We claim that, for a suitable family $\{h_j\}_{j \in \mathbb{N}}$, it holds
\[
\{h_j\} \text{ is monotone increasing to } 1, \quad \lim_{j \to +\infty} \int_\Sigma |\nabla h_j|^2 |\nabla u|^2 \text{dVol}_f = 0.
\]
Choose $h_j$ as follows, according to the case.

In the first case, \(i\), fix $\Omega \subset \Sigma$ with smooth boundary and let $\{\Omega_j\} \uparrow \Sigma$ be a smooth exhaustion with $\Omega \subset \Omega_1$. Choose $h_j \in \text{Lip}_c(M)$ to be identity 1 on $\Omega$, 0 on $M \setminus \Omega_j$ and the $f$-harmonic capacitor on $\Omega_j \setminus \Omega$, that is, the solution of
\[
\begin{align*}
\Delta_f h_j &= 0 & \text{on } \Omega_j \setminus \Omega \\
h_j &= 1 & \text{on } \partial \Omega, \\
h_j &= 0 & \text{on } \partial \Omega_j,
\end{align*}
\]
which exists by variational arguments. By Maximum principle and since $\Sigma$ is $f$-parabolic, $\{h_j\}$ is monotonically increasing and pointwise convergent to 1, and furthermore

$$\int_{\Omega_j} |\nabla \phi_j|^2 |\nabla u|^2 d\text{Vol}_f \leq |\nabla u|_{L^2}^2 \text{cap}_f(\Omega, \Omega_j) \to |\nabla u|_{L^2}^2 \text{cap}_f(\Omega) = 0,$$

where we have used the Proposition 2.1. This conclude (4.1) under the hypothesis (i).

In the second case, (ii), we apply a logarithmic cut-off argument. For fixed $R > 0$, choose the following radial function $h(x) = h_R(r(x))$:

$$h_R(r) = \begin{cases} 
1 & \text{if } r \leq \sqrt{R}, \\
2 - 2 \frac{\log r}{\log R} & \text{if } r \in [\sqrt{R}, R], \\
0 & \text{if } r \geq R.
\end{cases}$$

Note that

$$|\nabla h(x)|^2 = \frac{4}{r(x)^2 \log^2 R} \chi_{B_R \setminus B_r}(x),$$

where $\chi_A$ is the characteristic function of a subset $A \subseteq M$. Choose $R$ in such a way that $\log R/2$ is an integer. Then

$$\int_M |\nabla h|^2 |\nabla u|^2 d\text{Vol}_f = \int_{B_R \setminus B_r} |\nabla h|^2 |\nabla u|^2 d\text{Vol}_f$$

$$= \frac{4}{\log^2 R} \sum_{k=\log R/2}^{\log R-1} \int_{B_{e^{2k+1}} \setminus B_{e^{2k}}} \frac{|\nabla u|^2}{r(x)^2} d\text{Vol}_f$$

$$\leq \frac{4}{\log^2 R} \sum_{k=\log R/2}^{R} \frac{1}{e^{2k}} \int_{B_{e^{2k+1}}} |\nabla u|^2 d\text{Vol}_f.$$

By assumption

$$\int_{B_{e^{2k+1}}} |\nabla u|^2 d\text{Vol}_f \leq (k + 1)e^{2(k+1)} \gamma(k),$$

for some $\gamma(k)$ satisfying $\gamma(k) \to 0$ as $k \to +\infty$. Without loss of generality, we can assume $\gamma(k)$ to be decreasing as function of $k$. Hence,
\[
\frac{4}{\log^2 R} \sum_{k=\log R/2}^{\log R} \frac{1}{e^{2k}} \int_{B_{e^{k+1}}} |\nabla u|^2 d\text{Vol}_f \leq \frac{8}{\log^2 R} \sum_{k=\log R/2}^{\log R} e^{2(k+1)} (k + 1) \gamma(k) \\
\leq \frac{8e^2}{\log^2 R} \gamma(\log R/2) \sum_{k=0}^{\log R} (k + 1) \\
\leq \frac{C}{\log^2 R} \gamma(\log R/2) \log^2 R \\
= C \gamma(\log R/2),
\]

for some constant \( C > 0 \). Combining (4.2) and (4.3) and letting \( R \to +\infty \) we deduce (4.1). Therefore, in both cases, from Lemma 6 we infer

\[
\int_{\Sigma} (|\nabla^2 u|^2 - |\nabla|\nabla u|^2 + Ric_f(\nabla u, \nabla u)) d\text{Vol}_f \leq 0.
\]

Now we are able to apply the Co-area formula and Lemma 3 and so we obtain

\[
\int_{\Sigma} \int_{\Sigma_t} \left( |A_t|^2 - \frac{H_t^2}{n-1} \right) |\nabla u| d\text{H}^{n-1}_f dt \leq \int_{\Sigma} (|\nabla^2 u|^2 - |\nabla|\nabla u|^2 + Ric_f(\nabla u, \nabla u)) d\text{Vol}_f \leq 0.
\]

Since for all operator \( T \) acting on a vector space of dimension \( n \) we have that \( |T|^2 \geq \frac{\text{trace}(T)^2}{n-1} \) and the equality holds if and only if \( T \) is multiple of the identity, we obtain:

\[
(4.4) \quad |\nabla| = cw, \quad \text{for some } c > 0, \quad |\nabla^2 u|^2 = |\nabla|\nabla u|^2, \quad \Sigma_t \text{ is totally umbilical.}
\]

Consider \( \Phi \) the flow associated of \( \nu = \nabla u/|\nabla u| \), which is well-defined on \( \mathbb{R} \times \Sigma \) because \( \Sigma \) is complete and \( |\nu| = 1 \). By (4.4) and Lemma 3 \( |\nabla u| \) is constant on each connected component of a level set \( \Sigma_t \). Therefore, in an adapted orthonormal frame \( \{e_j, \epsilon_n = \nu\} \) for the level set \( \Sigma_t \), we have that

\[
(4.5) \quad \begin{cases} 
|A_t|^2 = \frac{H_t^2}{n-1} \text{ implies } \nabla^2 u(e_i, e_j) = \lambda \delta_{ij}, \\
0 = \langle \nabla|\nabla u|, e_j \rangle = \nabla^2 u(\nu, e_j).
\end{cases}
\]

We point out that, by first equality in (4.4) \( u \) has no critical points. Now on, we will verify the property claimed in the Theorem.
Let $\gamma$ be any integral curve of $\nu$, we will prove that is a geodesic. Indeed, let $X \in \mathfrak{X}(\Sigma)$ be a vector field, we have that

$$\langle \nabla_{\gamma'}^\gamma, X \rangle = \frac{1}{|\nabla u|^2} \langle \nabla \nabla u \nabla u, X \rangle - \frac{1}{|\nabla u|^3} \langle \nabla |\nabla u| \nabla u, X \rangle$$

$$= \frac{1}{|\nabla u|^2} \text{Hess } u(\nabla u, X) - \frac{1}{|\nabla u|^3} \langle \nabla |\nabla u|, \nabla u \rangle \langle \nabla u, X \rangle$$

$$= \frac{1}{|\nabla u|^2} \text{Hess } u(\nu, X) - \frac{1}{|\nabla u|^3} \text{Hess } u(\nu, \nu) \langle \nu, X \rangle = 0,$$

where we have used (4.5). So, $\nabla_{\gamma'}^\gamma = 0$ and $\gamma$ is a geodesic as claimed.

Following the arguments in the proof of [12, Theorem 9.3], step-by-step we will prove the topological splitting. Since $|\nabla u|$ is constant on level sets of $u$, $|\nabla u| = \beta(u)$ for some function $\beta$. Evaluating $u$ along curves $\Phi_t(x)$, since $u \circ \Phi_t$ is a local bijection, we deduce that $\beta$ is continuous.

**Claim 4.1.** $\Phi_t$ moves level sets of $u$ to level sets of $u$.

Indeed, integrating $\frac{d}{dt}(u \circ \Phi_t) = |\nabla u| \circ \Phi_t = \beta(u \circ \Phi_t)$ we get

$$t = \int_{u(x)}^{u(\Phi_t(x))} \frac{d\xi}{\beta(\xi)},$$

thus $u(\Phi_t(x))$ is independent of $x$ varying in a level set. As $\beta(\xi) > 0$, this also show that flow lines starting from a level set of $u$ do not touch the same level set and we conclude the claim. □

Let $N$ be a connected component of a level set $\Sigma_t$ of $u$.

**Claim 4.2.** $\Phi|_{\mathbb{R} \times N}$ is surjective.

Indeed, since the flow of $\nu$ is through geodesics, for each $x \in N$, $\Phi_t$ coincides with the normal exponential map $\exp_{\nu}(t\nu(x))$. Moreover, since $N$ is closed in $\Sigma$ and $\Sigma$ is complete, the normal exponential map is surjective because each geodesic from $x \in \Sigma$ to $N$ minimizing distance $\text{dist}(x, N)$ is perpendicular to $N$ (by variational arguments). □

**Claim 4.3.** $\Phi|_{\mathbb{R} \times N}$ is injective.

Suppose that $\Phi(t_1, x_1) = \Phi(t_2, x_2)$. Then, since $\Phi$ moves level sets to level sets, necessarily $t_1 = t_2 = t$. If by contradiction $x_1 \neq x_2$, two distinct flow lines of $\Phi_t$ would intersect at the point $\Phi_t(x_1) = \Phi_t(x_2)$, contradicting the fact that $\Phi_t$ is a diffeomorphism on $\Sigma$ for every $t$, as desired. □

From the former arguments, we conclude that $\Phi : \mathbb{R} \times N \to \Sigma$ is a diffeomorphism. In particular, each level set $\Phi_t(N)$ is connected. This proves the topological part of the splitting.
To conclude the proof, we shall verify that $\Phi$ is isometry whether we consider the product space, $\mathbb{R} \times N$, endowed with a warped metric with warping function depends of $\nabla u$. Indeed, we consider the Lie derivative of the metric in the direction of $\nu$:

$$(\mathcal{L}_\nu g_\Sigma)(X, Y) = \langle \nabla_X \nu, Y \rangle + \langle X, \nabla_Y \nu \rangle$$

$$= \frac{2}{|\nabla u|} \nabla^2 u(X, Y) + X \left( \frac{1}{|\nabla u|} \right) \langle \nabla u, Y \rangle + Y \left( \frac{1}{|\nabla u|} \right) \langle \nabla u, X \rangle.$$

So, using that $|\nabla u|$ is constant on $N$ and the properties of $\nabla^2 u$, we obtain that

$$(\mathcal{L}_\nu g_\Sigma)(X, Y) = \frac{2}{|\nabla u|} \nabla^2 u(X, Y) = \left\{ \begin{array}{ll} -2\lambda g_\Sigma(X, Y), & \text{if } X, Y \in T\Sigma_t, \\ 0, & \text{if } X \text{ or } Y \text{ are not in } T\Sigma_t. \end{array} \right.$$

If, however, $X$ and $Y$ are normal (w.l.o.g. take $X = Y = \nabla u$), we have

$$(\mathcal{L}_\nu g_\Sigma)(X, Y) = \frac{2}{|\nabla u|} \nabla^2 u(\nabla u, \nabla u) + 2\nabla u \left( \frac{1}{|\nabla u|} \right) |\nabla u|^2$$

$$= \frac{2}{|\nabla u|} \nabla^2 u(\nabla u, \nabla u) - 2\nabla u(|\nabla u|)$$

$$= \frac{2}{|\nabla u|} \nabla u(|\nabla u|^2) - 2\langle \nabla |\nabla u|, \nabla u \rangle = 0.$$

Summarizing, we get

$$(\mathcal{L}_\nu g_\Sigma)(X, Y) = -2\lambda(\langle X, Y \rangle - (\nu \otimes \nu)(X, Y)),$$

and $\lambda = -\frac{1}{n-1} \text{div}_\Sigma (\nabla u/|\nabla u|)$.

Thus, it is a standard computation to verify that $\Phi$ is isometry from $\mathbb{R} \times N$ endowed with the warped metric $ds^2 = dt^2 + e^{-2\int_0^t \lambda(s) ds} g_N$ to $\Sigma$.

\[ \Box \]

5. Proof of Theorem 1.2

**Proof of Theorem 1.2** Under this assumption, the level sets of the function $u$ are totally geodesic and so $\lambda$ obtained in former result vanishes. Thus we conclude that $\Sigma$ splits as a Riemannian product, as desired. In particular, the weighted Ricci curvature of $N$ is nonnegative.

Lastly, we shall verify the properties of the function $u$. Let $\gamma$ be any integral curve of $\nu$. Then

$$\frac{d}{dt}(u \circ \gamma) = \langle \nabla u, \nu \rangle = |\nabla u| \circ \gamma > 0,$$

since $|\nabla u| > 0$. As $\Sigma$ splits isometrically in the direction of $\nabla u$, we get that $\text{Ric}(\nu, \nu) = 0$ and this imply that $\text{Hess} f(\nu, \nu) = 0$. Consequently $\langle \nabla f, \nu \rangle$ is a constant $k$ (depending of $f$) in the splitting direction.
By the other hand,
\[-g(u \circ \gamma) = \Delta_f u(\gamma) = \text{Hess } u(\nu, \nu)(\gamma) - \langle \nabla f, \nabla u \rangle(\gamma)
\]
\[= \langle \nabla \nabla u, \nu \rangle(\gamma) - \langle \nabla f, \nu \rangle \nabla u(\gamma)
\]
\[= \frac{d}{dt}(|\nabla u| \circ \gamma) - k|\nabla u| \circ (\gamma)
\]
\[= \frac{d^2}{dt^2}(u \circ \gamma) - k \frac{d}{dt}(u \circ \gamma),
\]
and thus \(y = u \circ \gamma\) solves the ODE \(-y'' + ky' = g(y)\) with \(y' > 0\).

We next address the \(f\)-parabolicity. Under assumption (i), \(\Sigma\) is \(f\)-parabolic and so \(N\) is necessarily \(f\)-parabolic too. We are going to deduce the same under assumption (ii). Note that the chain of inequalities
\[
\left( \int_{-R}^{R} |y'(t)|^2 dt \right) \text{Vol}_f(B_R^N) \leq \int_{[-R,R] \times B_R^N} |y'(t)|^2 dt \text{ dVol}_f^N
\]
\[\leq \int_{B_{R^2}} |\nabla u|^2 \text{ dVol}_f = o(R^2 \log R)
\]
gives immediately the desired result, since \(|y'| > 0\) everywhere. Thus, since \(\text{Vol}_f(B_R^N) = o(R^2 \log R)\), we know that there is a constant \(A\) such that \(\text{Vol}_f(B_R^N) \leq AR^2 \log R\), that is,
\[
\frac{R}{\text{Vol}_f(B_R^N)} \geq \frac{1}{AR \log R},
\]
and so
\[
\lim_{t \to \infty} \int_{1}^{t} \frac{R \text{ dR}}{\text{Vol}_f(B_R^N)} \geq A^{-1} \lim_{t \to \infty} \int_{1}^{t} \frac{\text{ dR}}{R \log R} = A^{-1} \lim_{t \to \infty} \log(\log t) = \infty,
\]
and thus, by Proposition 2.2, \(N\) is \(f\)-parabolic.

\[\square\]

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IM, Universidade Federal de Alagoas, Maceió, AL, CEP 57072-970, Brazil
E-mail address: mhbs@mat.ufal.br

Instituto Federal de Alagoas, Campus Piranhas, Av. Sergipe, Xingó, Piranhas, AL, 57460-000, Brazil.
E-mail address: jissivan@gmail.com