ON THE ATTRACTOR ZERO OF A SEQUENCE OF POLYNOMIALS

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ABSTRACT. The main purpose of this paper is to study the roots of a family of polynomials that arise from a linear recurrences associated to Pascal’s triangle and their zero attractor, using an analytical methods based on conformal mappings.

1. INTRODUCTION

Fibonacci numbers can be recovered as the sum of the main rays of Pascal’s triangle, that is, each element of Fibonacci sequence \((F_n)_n\) is the sum of binomial coefficients \(\binom{n-k}{k}: F_{n+1} = \sum_k \binom{n-k}{k}\). In this context, one way to extend the work of Goh et al (cf. [9]) is to generalize to the linear recurrence sequence \((T_n)_n\), associated to different directions of the rays in Pascal’s triangle, defined for \(n, p, q, r \in \mathbb{Z}\) with \(n \geq 0, r \geq 1, 0 \leq p \leq r - 1\) and \(q + r > 0\), by:

\[
T_{n+1}^{(r,q,p)} = \sum_{k=0}^{\lfloor (n-p)/(q+r) \rfloor} T_{r,q,p}^{(n,k)} = \sum_{k \geq 0} \binom{n-qk}{p+rk} x^{n-p-(q+r)k} y^{p+rk},
\]

with the convention \(T_0 = 0\).

(Notice that because a sum over empty set is zero)

\((T_0 = 0)T_1 = \cdots = T_p = 0\),

and

\[
T_j = \binom{j-1}{p} x^{j-p-1} y^p, \quad p + 1 \leq j \leq r + q + p - 1,
\]

which is studied by Belbachir et al (cf. [4])

\[
\begin{array}{cccccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1
\end{array}
\]

Theorem 1. The sequence defined in (1.2) satisfy the linear recurrence relation:
\[
\sum_{k=0}^{r} (-x)^k \binom{r}{k} T_{n-k} = y^r T_{n-r-q}
\]

and its generating function (cf. [5]) is given by:

\[
G(t) = \sum_{n \geq 0} T_{n+1}^{(r,q,p)} t^n = \frac{y^p t^{r+1} (1 - xt)^r - y^q t^{r+q}}{1 - xt} - y^r T_{n-r} - q
\]

When \( q \leq 0 \), the sequence \((T_n)_n\) is of order \( r \) for any \( q (-r < q \leq 0) \), and the coefficient \( y^r \) of \( T_{n-r-q} \) is subtracted from one of coefficients of the terms \( T_{n-1}, \ldots, T_{n-r} \), such that \(-r < q \leq 0\), we get

\[
T_n - x \binom{r}{1} T_{n-1} + \ldots + \left( (-x)^{r+q} \binom{r}{r+q} - y^r \right) T_{n-r-q} + \ldots + (-x)^r \binom{r}{r} T_{n-r} = 0
\]

Thus, the coefficient of this terms changes status. This is what we call the Morgan-Voyce phenomenon.

Then the sequence is defined as:

\[
T_n = 2xT_{n-1} - x^2 T_{n-2} + y^2 T_{n-3},
\]

We focus our work in a particular case, first \( r = 2, q = 1, p = 1 \)

In this case we have: \( T_j = (j-1) x^j - p^{-1} y^p \), \( 2 \leq j \leq 3 \)

The table below gives us the 10 terms defined in (1.4)

| \( n \) | \( T_0 \) | \( T_1 \) | \( T_2 \) | \( T_3 \) | \( T_4 \) | \( T_5 \) | \( T_6 \) | \( T_7 \) | \( T_8 \) | \( T_9 \) | \( T_{10} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|
| \( 0 \) | 0 | 0 | \( y \) | \( 2xy \) | \( 3x^2 y \) | \( y (4x^3 + y^2) \) | \( xy (5x^3 + 4y^2) \) | \( 2x^2 y (3x^3 + 5y^2) \) | \( y (7x^6 + 20x^3 y^3 + y^4) \) | \( xy (8x^6 + 35x^3 y^2 + 6y^4) \) | \( x^2 y (9x^6 + 56x^3 y^2 + 21y^4) \) |

We know that the sequence defined by (1.3) is \( r \)-periodic

Then by using periodicity we have:

\[
T_i = y h (x^3, y^2) \text{ if } i \equiv -1 \mod 3
\]
\[
T_i = x y h (x^3, y^2) \text{ if } i \equiv 0 \mod 3
\]
\[
T_i = x^2 y h (x^3, y^2) \text{ if } i \equiv 1 \mod 3
\]

Claim 1. The set of roots satisfy

\[
y^2 = -x^3
\]
The corresponding generating function is:

\[
G(t) : = \sum_{n \geq 0} T_{n+1}^{(2,1,1)} t^n = \frac{yt^2}{(1-xt)^2 - y^2t^3}
\]

Then integral representation formula give us:

**Lemma 1.** The polynomial sequence \( T_n(x, y) \) given by (1.4) has the integral representation:

For all \( x, y \) nonzero real parameters there exist a non negative real number \( r_{x,y} > 0 \) such that:

\[
(1.5) \quad T_n(x, y) = \frac{1}{2\pi i} \int_{|t|=r_{x,y}} \frac{y}{(1-xt)^2 - y^2t^3} \frac{dt}{t^{n-1}}
\]

**Proof.** Since \( \frac{y}{(1-xt)^2 - y^2t^3} \to y \) if \( t \to 0 \) then we can find \( r_{x,y} > 0 \) such that:

\[
|1 - x t^2| > 0.9 \quad \text{for} \quad |t| = r_{x,y}
\]

Once the integral is well-defined, denoted the integral by \( \tilde{T}_n(x, y) \).

\[
\tilde{T}_n(x, y) = \frac{1}{2\pi i} \int_{|t|=r_{x,y}} \frac{y}{(1-xt)^2 - y^2t^3} \frac{dt}{t^{n-1}}
\]

We can directly verify that \( \tilde{T}_n(x, y) \) satisfies \( (1.1) \) for \( n \geq 1 \). Next, since the Taylor expansion of \( \frac{y}{(1-xt)^2 - y^2t^3} = y + 2yt + O(t^2) \), by residue theorem we have \( \tilde{T}_0(x, y) = 0, \tilde{T}_1(x, y) = 0 \) and \( \tilde{T}_2(x, y) = y \). Hence the initial conditions in \( (1.1) \) are satisfied. Thus the integral representation \( \tilde{T}_n(x, y) \) is a solution to a recursion. Since the solution to the recursion is unique, hence \( \tilde{T}_n(x, y) = T_n(x, y) \). This completes the proof of the lemma. \( \square \)

By a change of variable \( y = x\sqrt{x} \) in equation \( (1.5) \) where \( x > 0 \) we get:

\[
(1.1) \quad T_n(x, y) = \frac{1}{2\pi i} \int_{|t|=r_{x,y}} \frac{y}{(1-xt)^2 - y^2t^3} \frac{dt}{t^{n-1}} = \frac{1}{2\pi i} \int_{|t|=r_x} \frac{1}{(1-xt)^2 - x^3t^3} \frac{dt}{t^{n-1}}
\]

On replacing \( t \) by \( \frac{t}{x} \) in the equation \( (1.1) \) we obtain:

\[
T_n(x, y) = \frac{-x^{n-\frac{1}{2}}}{2\pi i} \int_{|t|=r_x} \frac{1}{t^3 - (1-t)^2} \frac{dt}{t^{n-1}}
\]

Where

\[
(1.7) \quad P(t) = t^3 - (1-t)^2
\]

We put

\[
(1.8) \quad \tau_n(x) = \frac{-1}{2\pi i} \int_{|t|=r_x} \frac{1}{P(t)} \frac{dt}{t^{n-1}}
\]
\( \tau_n(x) = x^{(\frac{1}{2} - n)} T_n(x, y) \)

**Lemma 2.**
1. The polynomial \( P(t) \) defined in (1.7) does not have zeros of order 3.
2. \( P(t) \) has zeros of order 2 if and only if \( t_1 = \frac{1 - i\sqrt{5}}{3} \quad t_2 = \frac{1 + i\sqrt{5}}{3} \).

**Proof.**
1. The derivative polynomial \( P'(t) \) has no zeros of order 2, consequently \( P(t) \) has not zeros of order 3.
2. \( P'(t) \) has two complex zeros of order 1: \( t_1 = \frac{1 - i\sqrt{5}}{3} \), \( t_2 = \frac{1 + i\sqrt{5}}{3} \).

When \( t = t_1 \) the zeros of \( P(t) \) are: \( \frac{1}{3} - i\frac{\sqrt{5}}{3}, \frac{1}{3} + i\frac{\sqrt{5}}{3}, -\frac{1}{2} - i\frac{\sqrt{5}}{5} \).

Let \( t_1, t_2, t_3 \) be the zeros of \( P(t) \) arranged via their magnitudes:

\[
|t_1| \leq |t_2| \leq |t_3|
\]

After developing the partial fraction decomposition for \( \frac{1}{P(t)} \) we obtain:

\[
\frac{1}{P(t)} = \frac{1}{P'(t_1)} \frac{1}{t - t_1} + \frac{1}{P'(t_2)} \frac{1}{t - t_2} + \frac{1}{P'(t_3)} \frac{1}{t - t_3}
\]

By using equation (1.8) the integration term by term and by using the residue theorem we get:

\[
\tau_n(x) = \left[ \frac{t_1^{1-n}}{P'(t_1)} + \frac{t_2^{1-n}}{P'(t_2)} + \frac{t_3^{1-n}}{P'(t_3)} \right]
\]

**Remark 1.** To study the zeros of \( \tau_n(x) \) it suffices to study the zeros of \( P(t) \).

2. **Conformal mappings**

In this section we describe how the zeros of \( P(t) \) can be obtained by a sequence of conformal mappings.

If we set \( t = z + \frac{1}{3} \) then \( P(t) = 0 \) return to its canonical form:

\[
z^3 + \frac{5}{3} z - \frac{11}{27} = 0
\]

When \( z = \lambda q \) where \( \lambda = \frac{i\sqrt{5}}{3} \).

Then we have \( q^3 - 3q + \beta = 0 \) with \( \beta = \frac{-11i\sqrt{5}}{25} \).
Next if
\[ q = p + \frac{1}{p} \]
Then the equation in \( p \) is:
\[ p^6 + \beta p^3 + 1 = 0 \]

Again if
\[ p^3 = s \]
then we get:
\[ s^2 + \beta s + 1 = 0 \]

which implies
\[ \beta = -\left( s + \frac{1}{s} \right) \]

Then it is clear how to obtain the zeros of \( P(t) \) by going through a sequence of conformal mappings starting from the \( \beta \)-plane and subsequently ending in the \( t \)-plane .

**Definition 1.** The map : \( J(\zeta) = \zeta + \frac{1}{\zeta} \) is called Joukowski map
This map is conformal in the regions \( |\zeta| < 1 \) and \( |\zeta| > 1 \) (cf. [12]).

The \( \beta \)-plane is mapped into the exterior region to the unit circle in the \( s \)-plane under \( J^{-1}(-\beta) \). The region is mapped into the exterior region to the unit circle in the \( p \)-plane under \( p = s^\frac{1}{3} \).

**Remark 2.** The map \( p = s^\frac{1}{3} \) is multiple-valued.

Then we follow \( q = J(p) \), \( z = \frac{i\sqrt{3}}{2} q \) and \( t = z + \frac{1}{3} \) to recover the zeros of \( P(t) \) in the \( p \)-plane and finally \( t \) to \( \frac{1}{t} \) in the \( x \)-plane.

We can expressed this situation symbolically as:
\[
\beta \rightarrow s \rightarrow p \downarrow t \leftarrow z \leftarrow q
\]

The Joukowski map : \( J(\zeta) = \zeta + \frac{1}{\zeta} \) is conformal on \( |\zeta| < 1 \) and \( |\zeta| > 1 \). We focus the behaviour of \( J(\zeta) \) on \( |\zeta| > 1 \).

Let \( \zeta = r \exp i\theta \)

then
\[
J(\zeta) = r \exp (i\theta) + r^{-1} \exp (-i\theta) = (r + r^{-1}) \cos \theta + i (r - r^{-1}) \sin \theta
\]

We set: \( \left\{ \begin{array}{l}
u = (r + r^{-1}) \cos \theta \\v = (r - r^{-1}) \sin \theta \end{array} \right. \)

We obtain:
\[
\frac{u^2}{(r + r^{-1})^2} + \frac{v^2}{(r - r^{-1})^2} = 1
\]

this show that \( J(\zeta) \) maps the circles \( r = \text{constants} \) onto the ellipses of semi axes \( r + r^{-1} \) and \( |r - r^{-1}| \) and they have common foci \( \pm 2 \).

In a similar way, \( J(\zeta) \) maps the rays \( \theta = \text{constants} \) onto hyperbolas with the same foci \( \pm 2 \).
2.1. **Zero analysis.** Recall that $t_1, t_2, t_3$ are the zeros of $P(t) = 0$ such that they satisfied (1.8).

In this subsection we study the set $A$ defined as:

$$ (2.2) \quad A = \{ |t_1(x)| = |t_2(x)| \} $$

2.1.1. **Structure in the $p$–plane.** To depict the set $A$ in the $x$–plane, it is better to depict the set of points in the $p$–plane that leads to $|t_1(x)| = |t_2(x)|$ under (2.1).

Let $p = r \exp (i \theta), \ r \geq 1$ be a point in the $p$–plane.

The image of $p$ in the $t$–plane is determined as follows:

$$ q = J(p) = r \exp (i \theta) + r^{-1} \exp (-i \theta) = (r + r^{-1}) \cos \theta + i \left( r - r^{-1} \right) \sin \theta $$

$$ z = \lambda q = \frac{i \sqrt{5}}{3} (r + r^{-1}) \cos \theta + \frac{\sqrt{5}}{3} (-r + r^{-1}) \sin \theta $$

and

$$ t = z + \frac{1}{3} = \frac{i \sqrt{5}}{3} (r + r^{-1}) \cos \theta + \frac{\sqrt{5}}{3} (-r + r^{-1}) \sin \theta + \frac{1}{3} $$

Then

$$ |t|^2 = \left( \frac{\sqrt{5}}{3} (-r + r^{-1}) \sin \theta + \frac{1}{3} \right)^2 + \left( \frac{\sqrt{5}}{3} (r + r^{-1}) \cos \theta \right)^2 $$

$$ = \frac{5}{9} (r^2 + r^{-2}) + \frac{10}{9} \cos 2 \theta + \frac{1}{9} + \frac{2 \sqrt{5}}{9} (r^{-1} - r) \sin \theta $$

Now let, $r \exp (i \theta_0), r \exp (i (\theta_0 + \frac{2 \pi}{3}))$ and $r \exp (i (\theta_0 + \frac{4 \pi}{3}))$ be the images of a point $x$ in the $P$–plane.

For example we may assume that: $r \exp (i \theta_0)$ leads to $t_1$ in the $t$–plane and $r \exp (i (\theta_0 + \frac{4 \pi}{3}))$ leads to $t_2$.

Thus $|t_1|^2 = |t_2|^2$ implies

$$ \frac{5}{9} (r^2 + r^{-2}) + \frac{10}{9} \cos 2 \theta_0 + \frac{1}{9} + \frac{2 \sqrt{5}}{9} (r^{-1} - r) \sin \theta_0 = \frac{5}{9} (r^2 + r^{-2}) + $$

$$ + \frac{10}{9} \cos 2 \left( \theta_0 + \frac{4 \pi}{3} \right) + $$

$$ + \frac{1}{9} + \frac{2 \sqrt{5}}{9} (r^{-1} - r) \sin \left( \theta_0 + \frac{4 \pi}{3} \right) $$

Then we get after simplification:
\[
\frac{10}{9} \left[ \cos 2\theta_0 - \cos 2 \left( \theta_0 + \frac{4\pi}{3} \right) \right] = \frac{2\sqrt{5}}{9} (r^{-1} - r) \left[ -\sin \theta_0 + \sin \left( \theta_0 + \frac{4\pi}{3} \right) \right]
\]

By using the trigonometric identity
\[
\cos 2a - \cos 2b = -2 \sin a \sin b - 2 \sin a \sin b
\]

We get:
\[
\frac{2\sqrt{5}}{9} (r^{-1} - r) \left[ 2 \sin \frac{2\pi}{3} \cos \left( \theta_0 + \frac{2\pi}{3} \right) \right] = \frac{-20}{9} \left[ 2 \sin \frac{2\pi}{3} \cos \left( \theta_0 + \frac{2\pi}{3} \right) \right] \times \left[ 2 \cos \frac{2\pi}{3} \sin \left( \theta_0 + \frac{2\pi}{3} \right) \right]
\]

Two cases to discuss, the first one is:

If \( \cos \left( \theta_0 + \frac{2\pi}{3} \right) = 0 \) this implies that \( \theta_0 = \frac{-\pi}{6} \) or \( \theta_0 = \frac{-7\pi}{6} \), then any \( r \geq 1 \) satisfies (2.1)

If \( \cos \left( \theta_0 + \frac{2\pi}{3} \right) \neq 0 \) then we get after cancelling the common factor \( \cos \left( \theta_0 + \frac{2\pi}{3} \right) \) leads to
\[
\frac{\sqrt{5}}{9} (r^{-1} - r) = \frac{10}{9} \sin \left( \theta_0 + \frac{2\pi}{3} \right)
\]

Which is equivalent to the rectangular equation:
\[
\left( x + \frac{\sqrt{15}}{2} \right)^2 + \left( y - \frac{\sqrt{5}}{2} \right)^2 = 6
\]

we study this result as a lemma

**Lemma 3.** The condition for \( |t_1| = |t_2| \) in the \( p \)-plane where \( re^{i\theta_0} = p_1 \) corresponds to \( t_1 \) and \( re^{i\left( \theta_0 + \frac{2\pi}{3} \right)} = p_2 \) corresponds to \( t_2 \) is:

1. \( p_1 = re^{-i\frac{\pi}{3}} \) or \( re^{-i\frac{11\pi}{6}} \), \( r \geq 1 \) or
2. \( p_1 \) lies in the circular arc of the circle
\[
\left( x + \frac{\sqrt{15}}{2} \right)^2 + \left( y - \frac{\sqrt{5}}{2} \right)^2 = 6
\]

in the region \( r \geq 1 \).

We need to impose \( |t_1| \leq |t_3| \) to make sure that \( |t_1| \leq |t_2| \leq |t_3| \)

Recall that \( t_3 \) corresponds to \( re^{i\left( \theta_0 + \frac{2\pi}{3} \right)} \) in the \( p \)-plane

Since \( |t_1| < |t_3| \) this implies \( |t_1|^2 < |t_3|^2 \) then we have:
\[
\frac{2\sqrt{5}}{9} (r^{-1} - r) \left[ 2 \sin \left( -\frac{\pi}{3} \right) \cos \left( \theta_0 + \frac{\pi}{3} \right) \right] > \frac{-20}{9} \left[ 2 \sin \left( -\frac{\pi}{3} \right) \cos \left( \theta_0 + \frac{\pi}{3} \right) \right] \times \left[ 2 \sin \left( \theta_0 + \frac{\pi}{3} \right) \cos \left( -\frac{\pi}{3} \right) \right]
\]

To solve this inequality two cases to discuss:
1- If \( \cos \left( \theta_0 + \frac{\pi}{3} \right) > 0 \) which is equivalent to

\[
\frac{-5\pi}{6} < \theta_0 < \frac{\pi}{6}
\]

Then after cancelling \( \cos \left( \theta_0 + \frac{\pi}{3} \right) \) we get:

\[
\frac{2\sqrt{5}}{9} \left( r^{-1} - r \right) < -\frac{20}{9} \sin \left( \theta_0 + \frac{\pi}{3} \right)
\]

or in rectangular form

\[
\left( x + \frac{\sqrt{15}}{2} \right)^2 + \left( y - \frac{\sqrt{5}}{2} \right)^2 > 6
\]

The corresponding region in this case follows after combining (2.4) and (2.5). Namely

\[
\left( x + \frac{\sqrt{15}}{2} \right)^2 + \left( y - \frac{\sqrt{5}}{2} \right)^2 > 6, \quad r \geq 1, \quad \frac{-5\pi}{6} < \theta_0 < \frac{\pi}{6}
\]

2- If: \( \cos \left( \theta_0 + \frac{2\pi}{3} \right) < 0 \) this is equivalent to:

\[
\frac{\pi}{6} < \theta_0 < \frac{7\pi}{6}
\]

And the inequality to solve is:

\[
\frac{2\sqrt{5}}{9} \left( r^{-1} - r \right) > -\frac{10}{9} \sin \left( \theta_0 + \frac{\pi}{3} \right)
\]

In a similar way we get the region for the case 2:

\[
\left( x + \frac{\sqrt{15}}{2} \right)^2 + \left( y - \frac{\sqrt{5}}{2} \right)^2 < 6, \quad r \geq 1, \quad \frac{\pi}{6} < \theta_0 < \frac{7\pi}{6}
\]

Let region \( \Gamma_1 \) correspond to (2.6) and region \( \Gamma_2 \) correspond to (2.8).

Then:

\[
\Gamma_1 \cup \Gamma_2 = \{ \text{points in } p-\text{plane that corresponds to } |t_1| < |t_3| \}
\]

Now, imposing the condition stated in lemma 4, we get the point set \( L_1 \) of the point \( p_1 \) that corresponds to root \( t_1 \).

Explicitly, \( L_1 \) is the point set defined as:

\[
L_1 = \{ \Gamma_1 \cup \Gamma_2 \} \cap \left\{ \left\{ \text{ray} : \theta = \frac{-\pi}{6} \right\} \cup \left\{ \left( x + \frac{\sqrt{15}}{2} \right)^2 + \left( y - \frac{\sqrt{5}}{2} \right)^2 = 6 \right\} \right\}
\]

Similarly, we defined \( L_2 \) as the rotation of \( L_1 \) through an angle of \( \frac{4\pi}{3} \) and \( L_3 \) through \( \frac{2\pi}{3} \).

It is clear that \( L_2 \) consists of points in the \( p-\text{plane} \) that corresponds to \( t_2 \) and \( L_3 \) corresponds to \( t_3 \). Thus we have proved the following:

**Theorem 2.** The condition for the points in the \( p-\text{plane} \) for which \( |t_1| = |t_2| < |t_3| \) is \( p_i \in L_i \), for \( i = 1, 2, 3 \).
The pull back action

**Theorem 3.** The curve represented by set \( A \) in the \( x \)-plane is the curve which is obtained by pulling the symmetric curve in the \( p \)-plane through the conformal mappings back to the \( x \)-plane.

Let \( Z(\tau_n) \) denoted the zero attractor of \( \tau_n(x) \)

To see the zero attractor of \( T_n(x,y) \), by using \((1.9)\) we have \( \tau_n(x) = x^{(\frac{1}{3} - \frac{1}{n})}T_n(x,y) \)

Assume \( x_0 \in Z(\tau_n) \) again by equation \((1.9)\) \( \tau_n(x_0) = x_0^{\frac{2}{3} - n}T_n(x_0,y_0) \) where \( y_0 = x_0^{\frac{2}{3}} \). Since \( \tau_n(x_0) = 0 \), we get \( T_n(x_0,y_0) = 0 \). Hence \( x_0^{\frac{2}{3} - n} \in Z(T_n) \).

3. Zero attractor

The concept of attractors was introduced in 1965 by Auslander et al. (2).

According to Milnor (11) the attractors have played an increasingly important role in thinking about dynamical systems.

**Definition 2.** Let \( \{q_n(x)\}_{n \geq 0} \) be a sequence of polynomials, where the degree of \( q_n(x) \) increases to infinity as \( n \to \infty \).

A set \( A \) in the \( x \)-plane is called the asymptotic zero attractor of zeros of \( \{q_n(x)\}_{n \geq 0} \) if the following two conditions holds:

1. Let \( A_\varepsilon = \cup_{x \in A} B(x, \varepsilon) \) where \( B(x, \varepsilon) \) is the open disc centred at \( x \) with radius \( \varepsilon \), \( A_\varepsilon \) is just a neighborhood of \( A \), \( \exists n_0(\varepsilon) \) such that \( \forall n \geq n_0 \) all the zeros of \( q_n(x) \) are in \( A_\varepsilon \).

2. For all \( x \in A, \forall \varepsilon > 0, \exists n_1(x, \varepsilon) \) and there exist a zero \( r \) of \( \{q_n(x)\}_{n \geq 0} \) such that \( r \in B(x, \varepsilon) \).

The condition for \( x \) values for which \( |t_1| = |t_2| \) is a curve. Let \( L \) be this curve. Let \( L_\varepsilon \) be the \( \varepsilon \)-neighborhood of \( L \) in the \( x \)-plane.

In this section we will give a justification that, for all large \( n \), all the zeros of \( \tau_n(x) \) are contained in \( L_\varepsilon \).

According to \((1.11)\), the asymptotics of \( \tau_n(x) \) depends on the magnitudes of the zeros of \( P(t) = 0 \).

Let \( B = \{x \in x - \text{plane} : |t_1(x)| < |t_2(x)|\} \)

Obviously, \( B \) is an open region in the \( x \)-plane.

Then we have

**Lemma 4.** There exists a non negative real number \( \rho \) such that for all large \( n \), the zeros of \( \tau_n(x) \) are contained in the disc \( D_\rho = \{x : |x| \leq \rho\} \)

**Proof.** The point infinity point in the extended \( x \)-plane is mapped to 0 under the mapping \( \frac{1}{z} \)

By the sequence of mappings defined in \((2.1)\) 0 is mapped to \( \beta = \frac{-1 + i\sqrt{3}}{2} \) corresponds to \( t_1 = 0, t_2 = \frac{-1 + i\sqrt{3}}{3} \) and \( t_3 = \frac{-1 + i\sqrt{3}}{3} \)

The choice of \( t_2 \) and \( t_3 \) is arbitrary since they have the same magnitude.

In this situation we have:

\[
|t_1| = 0 \\
|t_2| = \frac{2}{3} \\
|t_3| = \frac{2}{3}
\]
and

\[ |P'(t_1)| = 2 \]
\[ |P'(t_2)| = \frac{14 + 4\sqrt{5}}{3} \]
\[ |P'(t_3)| = \frac{14 - 4\sqrt{5}}{3} \]

Hence \( x = \infty \in B \).

Since \( B \) is open region, there exists a non negative real number \( \rho > 0 \), \( \{x : |x| \geq \rho\} \subseteq B \) such that for all \( x \in \{x : |x| \geq \rho\} \), we have:

\[ P'(t_1) = 2 \]
\[ 1 \leq |P'(t_2)| \]

and

\[ 1 \leq |P'(t_3)| \]

Now from \((1.11)\) we get

\[ \tau_n(x) = \frac{t_2^{t_{n-1}}}{P'(t_2)} \left[ 1 + \frac{P'(t_2)}{P'(t_1)} \left( \frac{t_1}{t_2} \right)^{n-1} - \frac{P'(t_2)}{P'(t_3)} \left( \frac{t_2}{t_3} \right)^{n-1} \right] \]

This gives the estimate

\[ |\tau_n(x)| = \left| \frac{t_2^{t_{n-1}}}{P'(t_2)} \right| \left[ 1 + \frac{P'(t_2)}{P'(t_1)} \left( \frac{t_1}{t_2} \right)^{n-1} + \frac{P'(t_2)}{P'(t_3)} \left( \frac{t_2}{t_3} \right)^{n-1} \right] \]

\[ = \left( \frac{3}{2} \right)^{1-n} \left[ 1 - \left( \frac{2}{3} \right)^{n-1} \right] \]

\[ = \left( \frac{3}{2} \right)^{1-n} \left[ 1 - \left( \frac{2}{3} \right)^{n-1} \right] \]

\[ \geq \frac{1}{2} \frac{3^n}{2} > 0 \]

for all large \( n \). This completes the proof of the lemma.

We know that the zeros of \( \tau_n(x) \) are contained in the disk \( \{x : |x| \leq \rho\} \), then the next lemma allow us to know where the zeros are going.

**Lemma 5.** Let \( K \) be a compact subset of \( B \). Then \( \tau_n(x)t_2^{n-1}P'(t_2) \rightarrow 1 \) uniformly for all \( x \in K \) as \( n \rightarrow \infty \).

**Proof.** By lemma 3, if \( x \in B \), then \( P(t) \) does not have any repeated zeros so that \( P'(t_i) \neq 0 \) for \( i = 1, 2, 3 \), by assumption \( K \) is compact, there must exist a \( \lambda > 0 \) and a number \( M \) such that \( M \geq |P'(t_i)| \geq \lambda > 0 \) uniformly for \( x \in K \) and \( i = 1, 2, 3 \).

Again from \((1.11)\) we have
\[ \tau_n(x) t_2^{n-1} P'(t_2) = 1 + O \left( \frac{t_2}{t_3} \right) \]

The big \( O \) terms approach zero uniformly because \( K \) is a compact set. Hence the result follows. \( \square \)

**Corollary 1.** The region \( B \) contains no points of \( Z(\tau_n) \).

**Proof.** By lemma 5, \( t_2^{n-1} P'(t_2) \) is never zero for \( x \in K \), \( \tau_n(x) \) and \( \tau_n(x) t_2^{n-1} P'(t_2) \) have the same zero set. Let \( x_0 \) be an arbitrary point in \( Z(\tau_n) \). Then \( x_0 \) is an accumulation point of zeros of \( \tau_n(x) \). Suppose \( x_0 \in B \), since \( B \) is open there exists a neighborhood \( N(x_0) \) of \( x_0 \) such that \( \overline{N(x_0)} \subseteq B \). With respect to this neighborhood, there exists an infinite sequence of integers \( n_j, j \geq 1 \) and a zero \( r_{n_j} \) of \( \tau_n(x) \) such that \( r_{n_j} \in N(x_0) \) for all \( j \geq 1 \). So for all sufficiently large \( j \) we have \( \tau_{n_j}(r_{n_j}) = 0 \), but this violates the asymptotic estimate in (3.1) with \( K \) chosen as \( \overline{N(x_0)} \), which is a contradiction, hence \( x_0 \notin B \). Hence the proof of corollary is completed. \( \square \)

**Corollary 2.** For all large \( n \), all zeros of \( \tau_n(x) \) are contained in \( L_\varepsilon \), that is \( Z(\tau_n) \subseteq L_\varepsilon \).

**References**

[1] Ait-Amrane L, Belbachir H, Betina K.: Periods of Morgan-Voyce and elliptic curves Math. Slovaca 66 (2016), No. 6, 1267-1284.

[2] Auslander, J., Bhatia, N.P., Seibert, P.: Attractors in dynamical systems. Bol. Soc. Mat. Mex. 9, 55-66 (1964).

[3] Belbachir, H. Bencherif, F: Linear recurrent sequences and powers of a square matrix. Electronic journal of combinatorial number theory, Integers 6 (2006), A12, 17 pp.

[4] Belbachir, H. Komatsu, T. Szalay, L.: Characterization of linear recurrences associated to rays in Pascal’s triangle. In: Diophantine analysis and related fields 2010. AIP Conf. Proc., 1264, Amer. Inst. Phys., Melville, NY, 2010, pp. 90–99.

[5] Belbachir, H. Komatsu, T. Szalay, L.: Linear recurrences associated to rays in Pascal’s triangle and combinatorial identities Math. Slovaca 64 (2014), 287–300.

[6] Boyer, R., Goh, W.: On the zero attractor of the Euler polynomials. Adv. Appl. Math. 38, 97–132 (2007)

[7] Boyer, R., Goh, W.: On the zero attractor of the partition polynomials. http://arxiv.org/abs/0809.1266

[8] He, M.X., Ricci, P.E., Simon, D.: Numerical results on the zeros of generalized Fibonacci polynomials. calcolo 34, 25-40 (1997)

[9] Goh, W. He, M X. Ricci P E.: On the universal zero attractor of the Trinacci-related polynomials. Calcolo 46, 95–129 (2009)

[10] Koshy, T.: Fibonacci and Lucas Numbers with applications (Wiley, New York, 2001)

[11] Milnor, J.: On the Concept of Attractor. Commun. Math. Phys. 99,177-195 (1985)

[12] Nehari, Z.: Conformal mappings (McGraw-Hill, New York, 1952)

[13] Stanley, R.: http://www-math.mit.edu/~rstan/zeros

[14] Vince, A.: Period of a linear recurrence, Acta Arith. 39 (1981), 303–311.

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