High-Dynamic-Range Mode-Decomposition for Interferometric Gravitational Wave Detectors and Associated Alignment Considerations

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Accurate readout of low-power optical higher-order spatial modes is of increasing importance to the precision metrology community. Such devices may prevent mode mismatches from degrading quantum and thermal noise mitigation strategies. Direct mode analysis sensors are a promising technology for real time monitoring of arbitrary higher-order modes. We demonstrate this technology with photo-diode readout to mitigate the typically low dynamic range of CCDs. We experimentally demonstrate the effect of a relative misalignment between the phase-plate and the photo-diode on the readout of higher-order modes, then derive this response. We then compute the residual effects of the finite photo-diode aperture on the dynamic range and discuss implementation considerations for future sensors.

I. INTRODUCTION

Two fundamentally limiting noise sources in ground based interferometric gravitational wave (GW) detectors and optical clocks are thermal noise [1, 2] and Quantum (Projection) Noise [3–5]. Advanced GW detectors, operating at high power, have implemented a squeezed vacuum as a quantum noise reduction technique [6–8]. There are proposals to use a spatial Higher-Order Mode (HOM) as the carrier beam to mitigate thermal noise [9, 10].

Squeezed vacuum is very sensitive to optical loss, thus requiring careful sensing and control of the 6 mode matching parameters between the optical resonators. Furthermore, the high power used can lead to Parametric Instabilities [11]. Lastly, if a HOM is used as a carrier, mismatches cause extensive scattering into other modes, which places tight requirements on mirror astigmatism [12]; mitigation strategies include in-situ actuation [13].

GW detectors use interference between reflected first order modes and RF sidebands for minimization of resonator translation and tilt mismatches [14] which is well developed [15, 16] and references therein. Direct detection of waist position and size mismatch is less well developed, but of increasing importance [17]. Such methods include: Bulls Eye photo detectors [18], Mode Converters [19], Hartmann Sensors [20] and the clipped photo-diode array discussed in [21] could be modified to be a direct mismatch sensor. Sensors beyond second order include scanning, lock in and Spatial Light Modulator (SLM) based phase cameras [22–24], as well as optical cavities [25–27].

In contrast, direct mode analysis sensors (MODANs) extract the phase and amplitude for each of higher order mode [28] breaking degeneracy between modes of the same order. Furthermore, MODANs do not need a reference beam and can be used with an SLM to allow in-situ adjustment of the basis and mode of interest [29]. The resulting sensor output can be readily and intuitively compared against models, offering substantial insight into the structure of the beam and easing mode matching.

Previous proposals [28–30] use a CCD as a light-sensor, allowing easily calibration of relative alignment between the phase-plate and light-sensor, but limiting the dynamic range of MODAN. CCD blooming and streaking from light scattered by the phase-plate limits the exposure time and dark noise is typically high.

This paper demonstrates MODAN with commercial low noise, high dynamic range, high bandwidth photo-diodes and 1064 nm wavelength light. A pinhole is used as a spatial filter to extract the signal from the scattered light. We then explore the alignment of the pinhole-photodiode light-sensor assembly relative to the phase-plate and derive an expression for the output of a mispositioned light-sensor. Lastly, we compute the finite photo-diode aperture effects, estimate the resulting bandwidth and discuss design considerations.

This work demonstrates the feasibility of high dynamic range mode analysis, an enabling technology for Quantum and Thermal Noise reduction strategies. It can easily be extended to multi-branch MODANs. Furthermore, we note that our methodology is similar to mode division multiplexing with Multi-Mode Fibers [31], which is of increasing interest for increasing communications bandwidth [32].

FIG. 1: Optical Convolution System. The light is incident on a phase-plate resulting in the field just after the phase-plate being, \(U(x, y) = U_{0}(x, y)T_{0}(x, y)\). The light propagates a distance of \(2f\) to a sensor, with a lens of focal length \(f\) placed half way between the sensor and the phase-plate.
II. MODE ANALYZERS

The methodology of mode decomposition is discussed extensively in [33]. In summary, the device consists of an optical convolution processor preceded by a phase-plate as shown in Fig. 1. For given scalar input field \( U_{in}(x, y, z_0) \), phase-plate transmission function, \( T(x, y) \), and lens focal length, \( f \), the field at the light-sensor is,

\[
U(x, y, z_0 + 2f) \approx \frac{\exp(i(2k_f + \frac{\pi}{2}))}{f \lambda} \int \int d\xi d\eta \ U_{in}(\xi, \eta, z_0) e^{i(\xi x + \eta y)},
\]

as determined by repeated application of the Rayleigh Som-merfeld equation and all parameters defined as per [34].

We employ the modal model by setting,

\[
T(\xi, \eta) = b_{n,m}u_{n,m}(\xi, \eta)
\]

\[
U_{in}(\xi, \eta, z_0) = \sum_{n', m'} a_{n', m'}^{*} u_{n', m'}(x, y, z_0) e^{i(\omega t + k_{z_0})}.
\]

Further, we assume that the mode basis functions, \( u \), form a complete, orthonormal basis set and recognize the inner product; and neglect common phase factors, then the on-axis field at the sensor is,

\[
U(0, 0, z_0 + 2f) \approx \frac{a_{n,m}b_{n,m} e^{i\omega t}}{f \lambda},
\]

where \( a \) is the amplitude of the mode and \( b \) is the grating efficiency. During detection the intermodal phase information is typically lost, but, by designing the phase pattern to overlap two fields, \( T^{\text{sum}} = u_{n,m_0} + u_{n,m_1} \) and \( T^{\text{sin}} = u_{n,m_0} + i u_{n,m_1} \) the inter-modal phases can be recovered.

III. EXPERIMENTAL DESIGN

The experimental layout is shown in Fig. 2. A laser source excites the eigenmodes of a resonator producing high-purity zeroth and first order Hermite Gauss (HG) modes [35]. A triangular resonator, based on a Pre-Mode-Cleaner design [36, 37], was used due to its natural HG basis, good mode separation and \( \pi \) radians Gouy phase difference between HG01 and HG10 modes. The light is then incident on a steering mirror before being split between a witness Quadrant Photodiode (QPD) and a MODAN. The beam radius at the SLM was \( w_{\text{SLM}} = 1.2 \) mm.

The pure first order modes were used to initially verify the mode analyzers operation. For small excitations of HG10 relevant to GW detectors, we misaligned a spatially fundamental beam to the phase-plate, since it can be described as an aligned beam with a small excitation of first order modes [14]. This misalignment could either be added in software with the beam centered on the SLM (e.g. Fig. 5), or the steering mirror could add a misalignment which was witnessed with the QPD (e.g. Fig. 8).

A blazed grating was added to the phase-plate and programmed onto a liquid crystal SLM (HOLOEYE PLUTO-2-NIR-015). This grating separated light which interacted with the phase-plate from specular reflections.

HG phase only plates were designed with transmission function,

\[
T_{n,m}^{\text{PO}}(x, y) = \exp\left[i \mod \left\{ \arg(u_{n,m}(x, y, z)) + 2\pi \frac{x \cos(\phi_s) + y \sin(\phi_s)}{\Lambda_s}, 2\pi \right\} \right],
\]

where: \( \phi_s \) is the grating angle; \( \Lambda_s \) is the grating period;
FIG. 4: Phase-plates with various software offsets. Upper plates are $T_{P0}^{PS}$ and lower plates are $T_{Q0}^{PS}$. The grating period has been increased from 80 $\mu$m (10 pixels) which was used in the experiment, to 1536 $\mu$m and the number of pixels decreased by a factor 10 in both directions, to provide a legible figure.

FIG. 5: Camera images for several phase-plate offsets. $O_x$ is the phase-plate offset with respect to the SLM. The central spot is the first diffraction order, with the specular and the second diffraction orders either side.

$$u_{n,m}(x,y,z) \equiv u_n(x,z)u_m(y,z); \text{ and,}$$

$$u_n(x,z_0) = \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \sqrt{\frac{\exp(i(2n+1)\Psi(z_0))}{2^nn!w_0}} \times H_n\left(\frac{\sqrt{2}x}{w_0}\right) \exp\left(-\frac{x^2 + y^2}{w_0^2}\right),$$

is the spatial mode distribution function at the waist. All other parameters are defined as per [34]. This plate was compared in simulation to a phase-plate produced with phase and effective amplitude encoding [38]. The transmission function was,

$$T_{n,m}^{PA}(x,y) = \exp\left[\mathcal{M}(x,y) \mod \left[F(x,y) + \frac{2\pi(x \cos(\phi_x) + y \sin(\phi_y))}{\Lambda_x}, 2\pi\right]\right],$$

(7)

where,

$$\mathcal{M} = 1 + \frac{\text{arcsinc} |u_{n,m}(x,y,z_0)|}{\pi}$$

(8)

$$\mathcal{F} = \arg(u_{n,m}(x,y,z_0)) - \pi \mathcal{M}.$$  (9)

Aside from an overall reduction in grating efficiency when using $T^{PA}$, the features in our results obtained by FFT simulation [39] and experimentally were very similar (e.g. Fig 8).

IV. EFFECT OF A MIS-POSITIONED LIGHT-SENSOR

The mode analyzer is a three component device, requiring careful relative alignment of each of these components for optimal performance. In this section, we generate a controlled amount HG10 mode by scanning the phase-plate position on the SLM and study the response of the MODAN at several light-sensor positions. Light-sensor refers to the pinhole and photo-diode assembly.

We define the possible beam and plate misalignments: $O_{x,y}$, $d$, $\sigma_{x,y}^{SLM}$ as per Fig. 3. We define, $\sigma_{x,y}^{OPD}$, to be the difference between the center of the SLM and the center of the QPD and $S_z$ to be the light-sensor misalignment.

The first order, HG, phase-only plates, shown in Fig. 4, do not depend on the beam parameter, and the HG01 and HG10 modes are orthogonal. Thus, by working with these plates and modes we mitigate the effect of vertical alignment and beam radius mismatches, allowing a controlled study of the effect of horizontal light-sensor mis-positioning on HG10 readout.

For a first order phase only grating, $T_{P0}^{PS}$, and misaligned TEM00 input beam, when $d > w_{SLM}$, little light interacts with the phase discontinuity, so the MODAN acts like a simple blazed grating, as shown in Fig. 5 for $O_x = 1600 \mu$m. When the phase discontinuity is brought nearer the center of the beam, the device works as a mode analyzer and thus the intensity is,

$$I \propto |U(0,0,z_0 + 2f)|^2 \propto |a_{1,0}|^2 \propto d^2,$$

(10)

which is symmetric in $d$. Fig. 6 shows a scan of $O_x$ with a HG10 plate and $O_y$ with a HG01 plate for several light-sensor positions and constant $\sigma_{x,y}^{SLM}, \sigma_{x,y}^{OPD}$. When $S_z = 80$ and $90 \mu$m the response of the fraction of light detected at the photo-diode is nearly symmetric about the minima. These minima are the lowest of all the measurements measuring $\sim 19$ ppm at $O_x \sim$ $-145 \mu$m and $29$ ppm at $O_x \sim 37 \mu$m respectively.

When the light-sensor is moved away from this position the response of the MODAN becomes asymmetric, thus the MODAN incorrectly determines the power in the HG10 mode.
The light-sensor y position was optimized by eliminating the asymmetry in the response prior to collection of the data shown and both peaks are approximately the same height throughout the measurement. All the HG01 minima are \(< 25\ \text{ppm}\) and occur when \(O_y = (710 \pm 15)\ \mu\text{m}\), illustrating the orthogonality of the analysis. The maximal HG01 signal occurs when the light-sensor position is \((80 < S_x < 90)\ \mu\text{m}\) and decreases as the light-sensor moves away from this range.

The blazing was in the x plane, the motion of the blazing over the SLM causes a small periodic shifts in the optimal light-sensor position which is not present in the HG01 scan. Additionally the data shown was filtered with a low pass filter to reduce noise cause by the refresh of the SLM and motion of the blazing.

V. LIGHT-SENSOR POSITION ERROR SIGNALS

Given that a mispositioned light-sensor can cause systematic errors in the modal readout, it is important to develop error signals to control this degree of freedom.

The mode basis is set entirely by parameters on the phase-plate, therefore, the light-sensor must be aligned with respect to this. In the first demonstration of direct mode analysis, four adjustment branches were produced \([28]\). These adjustment branches contained the unperturbed beam and provided a coordinate reference system on the CCD. The single branch analogue of this is to place the light-sensor at the position of maximal intensity for a mode matched \((n = n', m = m')\) input beam and plate.

In the case of a HG00 input beam and plate, the resulting power at the light-sensor is a Gaussian, \(I(x) \propto \exp((-x^2))\), since \(\frac{dI}{dx}|_{x=0} = 0\), small levels of light-sensor mis-positioning are difficult to detect and directional information is missing.

In contrast, in Fig. 6 when the peak corresponding to a \(O_x > 0\) is larger than the peak corresponding \(O_x < 0\), the light-sensor x position needs to be increased. Thus, by continuously scanning \(O_x\) and adjusting the light-sensor position to balance the response of the MODAN, the light-sensor can be aligned with respect to the beam and phase-plate.

To analytically confirm this effect, consider Eq. 1, use the transmission function for a phase and amplitude encoded HG10 plate, assume the incoming beam contains only horizontal misalignment modes, exploit the separability of the HG modes and assume the light-sensor is vertically aligned, then
We then assume the wavefront curvature at the SLM is 0, \( \frac{\lambda}{w_{SLM}} = 0 \), and determine that the beam radius at the phase-plate is,

\[
\frac{A f}{R_{WSLM}} = \frac{2 f}{k_{WSLM}}.
\]

By assuming the beam has a waist at the phase-plate, including the Gouy phase in the complex mode amplitudes and recognizing the \( w_{2f} \) terms, we find that,

\[
U(x, 0, z_0 + 2f) \approx \frac{b_1 e^{i(2k f z_0)}}{f A} \exp \left( \frac{i (2k f + \frac{\pi}{2}) - \frac{x^2}{2w_{2f}^2}}{f} \right) \left( - a_0^H \frac{\sqrt{2} x}{2w_{2f}} + a_1^H \left( 1 - \frac{x^2}{w_{2f}^2} \right) \right).
\]

We then compute the intensity as \( I = U U^* \) and find that,

\[
I(x, 0, z_0 + 2f) = \frac{|b|^2}{f^2 A^2} e^{-x^2/w_{2f}^2} \left[ |a_0^H|^2 \left( \frac{\sqrt{2} x}{2w_{2f}} \right)^2 + |a_1^H|^2 \left( 1 - \frac{x^2}{w_{2f}^2} \right)^2 \right. \\
\left. + 2 |a_1^H| |a_0^H| \left( 1 - \frac{x^2}{w_{2f}^2} \right) \frac{\sqrt{2} x}{2w_{2f}} \sin \left( \arg \left( a_0^H \right) - \arg \left( a_1^H \right) \right) \right].
\]

The field at the light-sensor is,

\[
U(x, 0, z_0 + 2f) \approx \frac{b_1 e^{i(2k f z_0)}}{f A} \int_{-\infty}^{\infty} \left( a_0^H u_0(\xi, z) + a_1^H u_1(\xi, z) \right) \left( u_1^*(\xi, z) \right) \exp \left( \frac{-i k x \xi}{f} \right) d\xi.
\]

where \( b_{am} = b_a^H b_m^V \) and similar for \( a_{mn} \). We now construct the relevant ABCD matrix to describe the system as,

\[
M_{2f} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ \frac{f}{2} & \frac{f}{2} & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & f \\ f & 0 \end{bmatrix}.
\]

We can then compute the relevant mode amplitudes for an offset, \( d \), between the phase-plate and beam \([40]\),

\[
d_0^H = \langle u_0(\xi - O_x) | u_0(\xi - \sigma_{xSLM}) \rangle = \exp \left( \frac{-d^2}{2 w_{SLM}^2} \right)
\]

\[
d_1^H = \langle u_1(\xi - O_x) | u_1(\xi - \sigma_{xSLM}) \rangle = - \frac{d}{w_{SLM}} \exp \left( \frac{-d^2}{2 w_{SLM}^2} \right),
\]

with the inter-modal phase depending on the distance from the waist. Substituting this into Eq. 14 yields the anticipated response of the system to a plate misalignment scan at several light-sensor positions, plotted in Fig 7. As expected, when the light-sensor is centered, the ideal response peaks when the first order mode power is maximum, \( d = w_{SLM} \). Furthermore, when the plate-beam misalignment becomes very large, \( d \rightarrow \pm \infty \), or the first order mode amplitude is very small \( d \rightarrow 0 \) the response goes to zero. When the light-sensor becomes mis-centered, the cross talk and interference described above lead to an offset and asymmetry in the response.

We can then fit data to Eq. 14 to determine the light-sensor offset, \( S_x \), during operation. We misaligned the light-sensor position, centered the phase-plate on the SLM \( (O_x = O_y = 0) \) and added a small translational misalignment using a steering mirror. The light was then split between the mode analyzer and a witness QPD as shown in Fig. 2.

The response of the MODAN is then plotted against the beam misalignment measured with the QPD in Fig. 8. A Levenberg-Marquardt least squares regression [41, 42] is used to extract the results shown in Table I.

![FIG. 7: Ideal response of alignment phase-plate to a relative misalignment between the beam and the plate, for several light-sensor positions. This is computed using Eq. 14, with \( a_0^H, a_1^H \) from 15, 16 and inter-modal phase difference \( \phi_0 - \phi_1 = \frac{\pi}{4} \).](image-url)

### TABLE I: Positioning offsets determined from fit.

| Light-Sensor Mis-position, \( S_x \) [w_{SLM}] | \( T_{10}^{PD} \) | \( T_{01}^{PD} \) |
|-------------------------------------------------|----------------|----------------|
| \( 0.539 \pm 0.007 \) | \( 0.595 \pm 0.003 \) |
| Inter-modal Phase, \( \phi_0 - \phi_1 \) [deg] | \( 11 \pm 1 \) | \( 3.8 \pm 0.4 \) |
| QPD Offset, \( \sigma_{xQPD} \) [w_{SLM}] | \( -0.027 \pm 0.015 \) | \( 0.019 \pm 0.008 \) |

As we would expect, the sensitivity to misalignments is normalized by the waist size at the light-sensor, and this gives us an important insight when choosing a focal length for low noise mode analyzers.

We then note some interesting effects: \( a_0^H \) couples into the signal, and there is a reduction in \( a_1^H \), which are both proportional to the square of the waist normalized light-sensor mis-position. There is also a global reduction in total intensity which is exponentially sensitive to waist normalized light-sensor mis-position. Lastly and most importantly, there is interference between the zeroth and first order modes, which is proportional to the sine of the inter-modal phase difference; due to the factor \( i \) acquired by the \( u_0 \) beam in Eq. 13. This interference shifts the apparent minima by a small amount proportional to the light-sensor mis-position and causes the asymmetry which we observe in Fig. 6.
and HG00 incoming beam and \( T_{10}^{PA} \) phase-plate is,

\[
U(x, y, z_0 + 2f) \approx \frac{b_0 e^{i(2k_f + \gamma)}}{f\lambda} \int_{-\infty}^{\infty} a_0^T u_0(x, y, z_0) (-i k\eta) d\eta,
\]

\[
\int_{-\infty}^{\infty} (a_0^H u_0(x, z) + a_1^H u_1(x, z)) \left( u_1^*(x, z) \right) \exp \left( \frac{-i k x \xi}{f} \right) d\xi.
\]

Solving, simplifying, substituting to cylindrical coordinates and integrating between 0 \( \leq \theta \leq 2\pi \) and 0 \( \leq r \leq r_0 \), yields,

\[
P_T(r_0) = \left| \frac{a_0^T}{a_1^T} \right|^2 \left( \frac{\pi w_2^2}{2} \right) \left( 1 - \frac{r_0^2}{w_2^2} \right) \left( 1 + \frac{r_0^2}{w_2^2} \right) \left( 1 - \frac{3 r_0^2}{2w_2^2} \right) + 3 \left( 1 - \frac{r_0^2}{w_2^2} \right).
\]

We note that the interference terms in Eq. 14 integrate away for a centered, finite size aperture, leaving terms that are either proportional to \( a_0^T \) or \( a_1^T \). Defining the crosstalk, \( P_0 \), to be the sum of all terms proportional to \( a_0^T \) and the signal, \( P_1 \), to be the sum of all terms proportional to \( a_1^T \).

Fig. 9 shows Eq. 14 plotted for some reasonable experimental parameters. The lightest line has all the power in the fundamental mode and the darkest line has all the power in the HG10 mode. When the pinhole aperture is much smaller than the beam-size at the light-sensor, \( r_0 << w_2f \), the cross talk is very low \( P_0/P_1 << 1 \), but at the cost of reduced power. As \( r_0 \) increases the fraction of cross coupling rapidly increases.

When \( r_0 = w_2f \), with 50:50 power split between the \( a_{10}^T \) and \( a_{11}^T \), 23.6% of the light at the light-sensor is from crosstalk.

If we then apply energy conservation by setting, \( (a_{0}^T)^2 = 1 - (a_{1}^T)^2 \), then solve \( 0 = P_0 - P_1 \), for \( (a_{0}^T)^2 \), we obtain the expression for the HG10 fraction with equal signal and crosstalk contributions,

\[
(a_{1}^T)^2 = \frac{4w_2^2 \left( r_0^2 - w_2^2 e^{2\gamma} + w_2^2 \right)}{3r_0^4 + 2r_0^2 w_2^2 - 10w_2^4 e^{2\gamma} + 10w_2^4}.
\]

When evaluated for our experimental parameters \( r_0 = 5 \mu m, w_2f = 54 \mu m \), then \( (a_{1}^T)^2 \approx 0.002 \). The maximum signal occurs when all of the light is in the HG10 modes, \( a_{10} = 1 \), thus the dynamic range is 500. This can be increased further by increasing \( w_2f \), and decreasing \( r_0 \).

VI. FINITE APERTURE EFFECTS

At any point other than, \( x = 0 \), \( a_0 \) couples into the signal. Thus the finite size of the pixel in the CCD, or photo-diode aperture, will experience this coupling, reducing the dynamic range. We compute this effect for a centered light-sensor of radius \( r_0 \). The field at the light-sensor for a vertically aligned

and HG00 incoming beam and \( T_{10}^{PA} \) phase-plate is,

\[
U(x, y, z_0 + 2f) \approx \frac{b_0 e^{i(2k_f + \gamma)}}{f\lambda} \int_{-\infty}^{\infty} a_0^T u_0(x, y, z_0) (-i k\eta) d\eta,
\]

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\int_{-\infty}^{\infty} (a_0^H u_0(x, z) + a_1^H u_1(x, z)) \left( u_1^*(x, z) \right) \exp \left( \frac{-i k x \xi}{f} \right) d\xi.
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\[
(a_{1}^T)^2 = \frac{4w_2^2 \left( r_0^2 - w_2^2 e^{2\gamma} + w_2^2 \right)}{3r_0^4 + 2r_0^2 w_2^2 - 10w_2^4 e^{2\gamma} + 10w_2^4}.
\]

When evaluated for our experimental parameters \( r_0 = 5 \mu m, w_2f = 54 \mu m \), then \( (a_{1}^T)^2 \approx 0.002 \). The maximum signal occurs when all of the light is in the HG10 modes, \( a_{10} = 1 \), thus the dynamic range is 500. This can be increased further by increasing \( w_2f \), and decreasing \( r_0 \).

VII. CONSIDERATIONS FOR HIGHER ORDER MODANS

In this paper, we demonstrate the use of a pinhole and photo-diode as a light-sensor for high dynamic range mode
Optical Power at PD

\[ P_T (r_a, a_{10}) \ [W] \]

Power in HG10 Mode, \( a_{10}^2 \)

- 0.0W
- 0.3W
- 0.6W
- 0.9W
- 0.1W
- 0.4W
- 0.7W
- 1.0W
- 0.2W
- 0.5W
- 0.8W

FIG. 9: The upper plot shows the total optical power on the light-sensor as a function of aperture radius, for 1W total power and different amounts of HG10 power. The lower plot shows the fraction of this light which is crosstalk from the HG00 mode. The parameters used were: \( \lambda = 1064 \text{ nm}, f = 0.2 \text{ m}, b_{10} = w_{\text{SLM}} = 1.2 \text{ mm}, a_{10}^2 = 1 - a_{10}^2 \). \( w_{2f} \) is given by Eq. 12.

analysis. Our analysis is restricted to first order modes due to existence of good witness sensors and ability to generate controlled small amounts of HG10, however, the methods described may be used generally for higher order sensors. Specifically, in the case of an SLM based MODAN monitoring arbitrary higher order modes, the light-sensor should be positioned using the phase-plates and methods shown, before collecting data on other modes.

We studied the horizontal and vertical position of the light-sensor with respect to the phase-plate, however, mode analysis requires that the longitudinal position is also tuned. The longitudinal position of the light-sensor was not tuned in this work, which introduced an additional gowy phase. If the Rayleigh range is suitably large at the light-sensor, then profiling the beam may suffice. If not, then a similar approach to the one presented, scanning the beam parameter used during the phase-plate generation and the longitudinal position of the light-sensor, may be required.

The dynamic range of the MODAN is limited by the ratio of the aperture radius and beam radius at the light-sensor. Stock pinholes exist down to 1 µm and beam radius can be increased by increasing the focal length of the lens or decreasing the beam radius at the SLM. Furthermore, a photo-diode with low dark noise and offset is strongly recommended for future work. Additionally, sufficient light must be collected to mitigate shot noise.

Commercial photo-diodes exist with very broad bandwidths, however, SLMs generate noise at their display refresh rates which is typically 60Hz. For a GW detector implementation, this noise can be trivially filtered because mode mismatches and parametric instability growth typically occurs at thermal timescales and parametric instabilities oscillate at kHz timescales.

VIII. CONCLUSIONS

MODAN is a promising technology for high-dynamic-range spatial-mode analysis in GW detectors. In a single branch MODAN, it is possible to increase the dynamic range by using photo-diode readout instead of a camera.

A relative misalignment between the photo-diode and phase-plate causes a reduced dynamic range and introduces systematic errors. This can be characterized and eliminated by scanning the first-order Hermite Gauss mode content as shown in V. With a suitable SLM, this scan may be done in software allowing easy calibration of the device as frequently as desired, before exploring another mode of interest.

The finite aperture of the photo-diode causes an optical offset to the measurement. Equation 18 can be used to determine the optical offset and additional shot noise contributions for a range of design parameters prior to construction.

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DISCLOSURES

The authors declare no conflicts of interest.

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