Anisotropic CMB distortions from non-Gaussian isocurvature perturbations

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Received December 25, 2014
Revised February 5, 2015
Accepted February 12, 2015
Published March 9, 2015

Abstract. We calculate the CMB $\mu$-distortion, $\langle \mu \rangle$, and the angular power spectrum of its cross-correlation with the temperature anisotropy, $\langle \mu T \rangle$, in the presence of the non-Gaussian neutrino isocurvature density (NID) mode. While the pure Gaussian NID perturbations give merely subdominant contributions to $\langle \mu \rangle$ and do not create $\langle \mu T \rangle$, we show large $\langle \mu T \rangle$ can be realized in case where, especially, the NID perturbations $S(\mathbf{x})$ are proportional to the square of a Gaussian field $g(\mathbf{x})$, i.e. $S(\mathbf{x}) \propto g^2(\mathbf{x})$. Such Gaussian-squared perturbations contribute to not only the power spectrum, but also the bispectrum of CMB anisotropies. The constraints from the power spectrum is given by $P_{SS}(k_0) \sim P^2_g(k_0) \lesssim 10^{-10}$ at $k_0 = 0.05 \text{Mpc}^{-1}$. We also forecast constraints from the CMB temperature and E-mode polarisation bispectra, and show that $P_g(k_0) \lesssim 10^{-5}$ would be allowed from the Planck data. We find that $\langle \mu \rangle$ and $|l(l+1)C^\mu T_l|$, can respectively be as large as $10^{-9}$ and $10^{-14}$ with uncorrelated scale-invariant NID perturbations for $P_g(k_0) = 10^{-5}$. When the spectrum of the Gaussian field is blue-tilted (with spectral index $n_g \simeq 1.5$), $\langle \mu T \rangle$ can be enhanced by an order of magnitude.

Keywords: CMBR theory, non-gaussianity

ArXiv ePrint: 1412.4517
1 Introduction

Inflationary scenario is a successful prescription to solve the initial condition problems of the hot Big Bang universe, and to determine the concrete theoretical model is one of the most important themes in the current observational cosmology [1–3]. Most of the models predict the generation of nearly scale-invariant and Gaussian primordial fluctuations, which can successfully explain a variety of cosmological observations including anisotropies in the Cosmic Microwave Background (CMB). The next step to sort out theoretical models furthermore would be to investigate deviations from Gaussian statistics, that is, to find non-Gaussianity [4]. The latest space mission Planck has revealed that the non-linearity parameter $f_{\text{NL}}$ should be at most $\mathcal{O}(1)$, which has ruled out a number of inflationary models [5]. However, we should note that the primary CMB anisotropies are visible only up to the multipole $l \sim 3000$ since they lack a power due to Silk damping. As observations are reaching this resolution, information of primordial perturbations available from the primary CMB anisotropies will saturate sooner or later.

CMB spectrum distortions have been recently considered to be alternative probes of the inflationary universe [6–9]. The distortions can be created from small-scale density perturbations, without introducing new physics, and hence use of CMB distortions in conjunction with the CMB anisotropies on large scales may complementarily allow us to test the nature of primordial perturbations over a wide range of scales. The distortions are basically classified into two types, $y$- and $\mu$-distortions, depending on whether the system is in kinetic equilibrium or not [12, 13]. Although kinetic equilibrium of the photon system is established in the early universe, once the Compton scattering becomes inefficient around redshift $z \sim 10^5$, deviations from the kinetic equilibrium can no longer vanish since the Thomson scattering never transfers the photon energy through non-relativistic electrons and the Compton scattering can no longer equilibrate the system. Then the deviation from the kinetic equilibrium is parameterized by the Compton $y$-parameter and hence called $y$-distortion. Another one is $\mu$-distortion, which corresponds to the chemical potential in the Bose-Einstein distribution function of the photon system. Such a deviation from the Planck distribution is a consequence

1 Some people discuss the intermediate type distortions [10, 11].
of realization of kinetic equilibrium under number conserving process such as the Compton scattering, so we can investigate the Compton-dominant era around $z \sim 10^6$ [10, 11, 14–19]. Here we would focus only on this type of distortion. Typical magnitude of the $\mu$-distortion which originates from the nearly scale-invariant primordial adiabatic curvature perturbations is $10^{-8}$ [16, 18, 20–22] and the contribution from primordial tensor perturbations would be subdominant [23, 24]. Constraints on these distortions are given by COBE FIRAS as $\mu < 9 \times 10^{-5}$ and $y < 1.5 \times 10^{-5}$ (95% C.L.) [25–27], and future space missions such as PIXIE [28] and PRISM [29] have potential to improve the constraints up to $10^{-8}$ or $10^{-9}$. Therefore, such distortions can be useful tools to study primordial fluctuations.

Recently, $\mu T$ cross-correlation is also proposed as a probe of primordial non-Gaussianities down to small scales [30–32]. Roughly speaking, $\mu$ is proportional to the square of dimensionless temperature perturbations and then, the cross-correlation originates from the primordial three-point function. For the local-type non-Gaussianity, the constraints on the non-linear parameter $f_{NL}^{loc}$ by PIXIE’s sensitivity are estimated as $f_{NL}^{loc} \lesssim 10^3$ in ref. [30]. In this paper we calculate the $\mu T$ cross-correlation in the presence of not only adiabatic but also isocurvature modes. The isocurvature modes do not necessarily follow the Gaussian distribution unlike the adiabatic modes and then they can cause large $\mu T$ signals. Accordingly, we calculate contribution from Gaussian-squared isocurvature perturbations to the total cross-correlation as an example of the non-Gaussian case. Here we focus on the neutrino isocurvature density (NID) mode, which is not suppressed on small scales compared to adiabatic perturbations, in contrast to the matter isocurvature perturbations [6, 8]. We organize this paper as follows. In section 2, we present the formalism of the power spectrum and bispectrum of the non-Gaussian isocurvature perturbations, with particular focus on the Gaussian-squared ones. Sections 3 and 4 show the calculations for $\mu$ and $\mu T$ cross-correlations, respectively. We summarize the discussions and conclude in the final section. In appendix A, we comment on the constraints from CMB angular bispectrum.

2 Gaussian-squared type isocurvature perturbations

Models of generating the NID mode have been proposed in the literature [37–39]. So far, most of the models can be categorised into two types. In one type, there assumed to be a large lepton asymmetry ($n_L/n_\gamma = O(0.01) \gg n_B/n_\gamma = O(10^{-9})$) in the universe. If fields sourcing the lepton asymmetry spatially fluctuate differently from inflaton (or in general fields reheating the universe), isocurvature perturbations inevitably arise between neutrino and photon. In the other type, there is assumed to exist dark radiation in the universe other than neutrino. In the context of structure formation, there is no distinction between dark radiation and neutrino, and they in effect consist a single fluid of neutrino species. Therefore, the NID mode is sourced if dark radiation is produced from fields which have isocurvature perturbations.

The NID mode can be non-Gaussian when so are source fields themselves. In addition, even when the source fields are Gaussian, the so-called local-type non-Gaussianity in the NID...
mode can be induced in the similar fashion as in e.g. the curvaton and the modulated reheating models. In ref. [40], several concrete models generating the local-type non-Gaussian NID mode are discussed. A model of the type with dark radiation can be realised by generalising the curvaton model. When a curvaton field which creates non-Gaussian curvature perturbations decays into dark radiation with some branching ratio, non-Gaussian NID perturbations should also be created. On the other hand, a model of the type with large lepton asymmetry can be realised in the Affleck-Dine baryogenesis [41] with Q-ball formation [42]. In this case, the non-linear dependence of the amount of the lepton asymmetry on the initial value of the Affleck-Dine field leads to non-Gaussianity in the NID mode. Generation of non-Gaussian NID perturbations in the modulated reheating scenario can also be realized [39]. In this paper, we in particular focus on non-Gaussian isocurvature perturbations which are proportional to the square of Gaussian field (see (2.1)–(2.2)). Such Gaussian-squared perturbations can be realized in the ungaussiton model [43–45].

Let us consider two fields $R_G$ and $g$ both of which obey Gaussian statistics. In the above model, the curvature $R$ and the residual isocurvature perturbation $S$ can be written as

$$R(x) = R_G(x) + \gamma_1 (g^2(x) - \langle g^2 \rangle),$$  \hspace{1cm} (2.1)$$
$$S(x) = \gamma_2 (g^2(x) - \langle g^2 \rangle),$$  \hspace{1cm} (2.2)$$

where $\gamma_1$ and $\gamma_2$ are the model-dependent constant parameters. In this paper, we adopt the convention in ref. [46], where $S$ is defined to be the density contrast of neutrino in the synchronous gauge of CDM, with curvature perturbations being set to vanish. The two-point correlation and the cross-correlation functions are

$$\langle R(x)R(0) \rangle = \langle R_G(x)R_G(0) \rangle + 2\gamma_1^2 \langle g(x)g(0) \rangle^2,$$  \hspace{1cm} (2.3)$$
$$\langle R(x)S(0) \rangle = 2\gamma_1 \gamma_2 \langle g(x)g(0) \rangle^2,$$  \hspace{1cm} (2.4)$$
$$\langle S(x)S(0) \rangle = 2\gamma_2^2 \langle g(x)g(0) \rangle^2,$$  \hspace{1cm} (2.5)$$

and their Fourier transformations have the following form:

$$P_{SS}(k) = 2\gamma_2^2 \int \frac{d^3 k_1}{(2\pi)^3} P_g(k_1) P_g(|k - k_1|),$$  \hspace{1cm} (2.6)$$
$$P_{RR}(k) = P_G(k) + \frac{\gamma_1^2}{\gamma_2} P_{SS}(k),$$  \hspace{1cm} (2.7)$$
$$P_{RS}(k) = \frac{\gamma_1}{\gamma_2} P_{SS}(k),$$  \hspace{1cm} (2.8)$$

where $P_G$ and $P_g$ are the power spectra of the above two Gaussian fields,

$$P_G(k) = \int \frac{d^3 k}{(2\pi)^3} e^{-ik \cdot x} \langle R_G(x)R_G(0) \rangle,$$  \hspace{1cm} (2.9)$$
$$P_g(k) = \int \frac{d^3 k}{(2\pi)^3} e^{-ik \cdot x} \langle g(x)g(0) \rangle.$$  \hspace{1cm} (2.10)$$

While we can also consider weakly non-Gaussian (i.e. local-type) NID perturbations, difference in results (i.e. $\mu$ and $\mu T$ cross-correlation) are rather trivial due to the similarity in the transfer functions. Indeed, as we will discuss at the ends of sections 3.1 and 4, both $\mu$-distortion and $\mu T$ cross-correlation change only by a constant multiplicative factor from the adiabatic case [30].
Dimensionless power spectra are also defined as usual, i.e. \( P(k) := \frac{k^3}{2\pi^2} P(k) \), for each perturbation. Here, we assume that the power spectrum of the adiabatic curvature perturbations is dominated by the part of a Gaussian fluctuation \( R_G \), and we parameterize the dimensionless power spectrum of \( R_G \) as

\[
P_G(k) = A_G \left( \frac{k}{k_0} \right)^{n_s-1},
\]

where we take constant parameters \( A_G \) and \( n_s \) to be \( A_G = 2.196 \times 10^{-9} \) and \( n_s = 0.96 \) [5], with \( k_0 = 0.05 \text{ Mpc}^{-1} \) being the pivot scale. The fraction of the isocurvature perturbations is given by

\[
\beta_{\text{iso}} := \frac{P_{SS}(k_0)}{P_{RR}(k_0) + P_{SS}(k_0)}.
\]

In the case with the NID mode, its contribution is constrained by Planck as \( \beta_{\text{NID}} < 0.27 \) or \( P_{SS}/P_{RR} < 0.37 \) [5].

By using the Feynman parameters, (2.6) can be explicitly integrated and \( P_{SS}(k) \) can be recast into

\[
P_{SS}(k) = \gamma^2_2 P_g^2(k) \frac{2^{1-n_g} \pi \Gamma \left( \frac{5}{2} - n_g \right) \Gamma \left( \frac{n_s-1}{2} \right)}{\Gamma^2 \left( 2 - \frac{n_s}{2} \right) \Gamma \left( \frac{n_s}{2} \right)}, \quad \text{for } 1 < n_g < 2.5,
\]

where we have assumed that \( P_g \) is a power of wavenumber with a spectral index \( n_g \), i.e. \( P_g(k) = P_g(k_0) (k/k_0)^{n_g-1} \). We can see \( P_{SS}(k) \propto P_g^2(k) \), so that \( P_{SS}(k) \) can be parameterized as

\[
P_{SS}(k) = P_{SS}(k_0) \left( \frac{k}{k_0} \right)^{2n_g-2}.
\]

This suggests that the blue-tilted original Gaussian field induces bluer isocurvature perturbations. \( P_{SS}(k_0) \) in the above expression apparently has the IR logarithmic divergence at \( n_g = 1 \), therefore, let us introduce an IR cut-off which is motivated by the horizon size \( L = 14 \text{ Gpc} \) to obtain a physically reasonable power spectrum. For an almost flat spectrum, most contributions are from the two IR regions, and they are equivalent by the translation and variable transformation (see figure 1). Then we can combine these regions into one and obtain the following form from (2.6):

\[
P_{SS}(k_0) \simeq 4 \gamma^2_2 P_g^2(k_0) \int_{1/(k_0L)}^{k_{\text{max}}/k_0} dt \frac{t^{n_g-1}}{t^{n_g-1}}
\]

\[
= \frac{4}{n_g - 1} \gamma^2_2 P_g^2(k_0) \left[ \left( \frac{k}{k_0} \right)^{n_g-1} \right]_{1/(k_0L)}^{k_{\text{max}}},
\]

where \( k_{\text{max}} \) is the upper limit of the IR region. Expanded by \( n_g - 1 \), \( P_{SS} \) can be expressed around the flat spectrum as

\[
P_{SS}(k_0) \sim 4 \gamma^2_2 P_g^2(k_0) \log(k_{\text{max}}L) \left[ 1 + \frac{n_g - 1}{2} \log \left( \frac{k_{\text{max}}}{k_0L} \right) + \cdots \right].
\]
Figure 1. The schematic diagram of IR regions for the convolutional integration of $\mathcal{P}_{SS}(k_0)$. There are two IR singularities at $k_1 = 0$ and $k_0$, and the IR regions around them, $\Omega_1$ and $\Omega_2$, are equivalent as they can be interchanged by variable transformation. It can be seen the maximum radii of them, $k_{\text{max}}$, are about $k_0/2$. Hereafter we take $k_{\text{max}} = k_0$ for simplicity.

Figure 2. $g^2$ isocurvature power spectrum at pivot scale $k_0 = 0.05\text{Mpc}^{-1}$ in units of $\mathcal{P}_g^2(k_0)$ with $\gamma_2 = 1$. In the latter sections, we will assume $n_g \lesssim 1.5$ in which the IR formulae can be used safely.

The IR singular points which are originally at $k_1 = 0$, $k_0$ are separated by the distance of $k_0$. Therefore $k_{\text{max}}$ should be about $k_0$. For simplicity, we take $k_{\text{max}} = k_0$ hereafter. Figure 2 shows the comparison between the exact formula and the one with an IR cut-off. We can see that they are in good agreement up to $n_g \simeq 1.7$.

Three-point correlation functions are defined in the same manner and bispectra can be obtained by the Fourier transformation. For instance, the bispectrum of the isocurvature perturbations is written as follows:

$$B_{SSS}(k_1, k_2, k_3) = \int d^3x \int d^3y e^{-i k_1 \cdot x} e^{-i k_2 \cdot y} \langle S(x) S(y) S(0) \rangle = 8 \gamma_2^3 \int \frac{d^3q}{(2\pi)^3} P_g(q) P_g(|q - k_1|) P_g(|q - k_1 - k_2|). \quad (2.17)$$
The IR regions of the integration for the bispectrum in the squeezed limit \((k_1 \ll k_2, k_3)\). Since \(P_g(k_1) \gg P_g(k_2), P_g(k_3)\) in this case, the contributions from \(\Omega_1\) and \(\Omega_3\), which are approximately proportional to \(P_g(k_1)P_g(k_2)\) and \(P_g(k_1)P_g(k_3)\), are much larger than that from \(\Omega_2\), which are approximately proportional to \(P_g(k_2)P_g(k_3)\). Therefore it is enough to consider the expansions of power spectrum only in \(\Omega_1\) and \(\Omega_3\). At this time, the upper limit of these IR regions is around \(k_1/2\).

From now on, we take \(k_{\text{max}} = k_1\).

We note that the other bispectra can be related to \(B_{SSS}\) as

\[
B_{RRR}(k_1, k_2, k_3) = \left(\frac{\gamma_1}{\gamma_2}\right)^3 B_{SSS}(k_1, k_2, k_3),
\]

\[
B_{RRS}(k_1, k_2, k_3) = \left(\frac{\gamma_1}{\gamma_2}\right)^2 B_{SSS}(k_1, k_2, k_3),
\]

\[
B_{RSS}(k_1, k_2, k_3) = \frac{\gamma_1}{\gamma_2} B_{SSS}(k_1, k_2, k_3).
\]

\(B_{SSS}\) also has IR singularities, and here we treat them in the same way as the power spectrum. Let us assume a squeezed configuration, which we will consider in the latter sections. Let the wavenumbers satisfy \(k_1 \ll k_2 \simeq k_3\). In this limit, the contributions of two IR regions which include \(k_1\) are dominant (see figure 3). Therefore (2.17) can be rewritten as

\[
B_{SSS}(k_1, k_2, k_3) \simeq \frac{8}{n_g - 1} \gamma_2^3 P_g(k_0) \left[\left(\frac{k}{k_0}\right)^{n_g-1}\right]^{k_{\text{max}}} \times \left[P_g(k_1)P_g(k_2) + P_g(k_1)P_g(k_3)\right], \quad \text{for } k_1 \ll k_2, k_3,
\]

where \(k_{\text{max}} \lesssim k_1\) and we will use \(k_{\text{max}} = k_1\) hereafter. Here we also introduce IR cut-off to be the current horizon size \(\sim L\).

3 CMB \(\mu\)-distortion

3.1 Homogeneous distortions

We usually assume that the photon system is locally kinetic equilibrium in the early universe. In other words, the photon fluid generally has a local Bose-Einstein distribution function, defined as

\[
f(x, \omega) = \frac{1}{e^{\omega \beta_{\text{BE}(x)}} + \mu(x) - 1},
\]
where $\omega$ is the frequency. Note that the temperature parameter of the Bose-Einstein distribution function $T_{\text{BE}}$ is different from that of the Planck distribution function $T_{\text{Pl}}$. Assuming that the deviations from the Planck distribution function are induced by the mixing of different local blackbodies, $T_{\text{BE}} - T_{\text{Pl}}$ and $\mu$ are the second-order quantities of the first-order dimensionless temperature perturbation which is defined as [33]

$$\Theta(x) = \frac{T_{\mu}(x) - \langle T_{\mu}\rangle}{\langle T_{\mu}\rangle}.$$  \hfill (3.2)

Then, using the conservation laws of both average energy and number, we can derive the evolution equation for the averaged $\mu$-distortion as [8, 18, 23]

$$\frac{d}{d\eta} \langle \mu \rangle = \frac{\langle \mu \rangle}{t_\mu} - 1.4 \times 4 \left( \Theta \frac{d\Theta}{d\eta} \right) + O(\Theta^2),$$  \hfill (3.3)

where $\eta$ is the conformal time and the first term in right hand side is added by hand to take into account the effect of the process which do not conserve the number of photon such as the double Compton effect or electron-positron pair annihilation. $t_\mu$ is the time-scale of decreasing of the chemical potential by the above processes, which become inefficient at $z \lesssim 2 \times 10^6$ [34, 35]. $d\Theta/d\eta$ is immediately calculated by the linear Boltzmann equations.

Here let us follow the notation in Ma and Bertschinger [36]. In Fourier space, the brightness functions for intensity and linear polarization are given by

$$F_\gamma = \frac{\int q^2dq f^{(0)}(q)\Psi}{\int q^2dq f^{(0)}(q)};$$  \hfill (3.4)

$$G_\gamma = \frac{\int q^2dq f^{(0)}(q)\Psi_P}{\int q^2dq f^{(0)}(q)},$$  \hfill (3.5)

where $q$ is comoving momentum and $f^{(0)}$ is the background Planck distribution function. $\Psi$ and $\Psi_P$ are fractional perturbations in $\rho_{11} + \rho_{22}$ and $\rho_{11} - \rho_{22}$ of the photon density matrix, respectively. We expand these quantities with respect to multipoles as $F_\gamma = \sum_l (\gamma_l \Psi^{(0)} + \gamma_{2l} \Psi^{(2)} + \gamma_{4l} \Psi^{(4)}).$ To obtain the solutions order by order. To the linear order, $F_\gamma = 4\Theta$ is always satisfied. Then we can derive the following formula for the $\mu$-distortion [18]:

$$\langle \mu \rangle = 1.4 \cdot \frac{1}{4} \int_0^{\eta_f} d\eta' JDC(\eta') \int d(ln k) \left[ P_G(k) + \left( f + \frac{\gamma_1}{\gamma_2} \right)^2 P_{SS}(k) \right]$$

$$\times n_e\sigma T a \left[ \frac{3}{4} (F_{\gamma 1} - F_{b1})^2 - \frac{F_{\gamma 2}^2}{2} (-9F_{\gamma 2}^2 + G_{\gamma 2} + G_{\gamma 0}) + \sum_{l \geq 3} (2l + 1) F_{\gamma l} F_{\gamma l} \right],$$  \hfill (3.6)

where $J_{DC}$ is a window function induced by the double Compton scattering and $\eta_f$ is the end time of $\mu$ era. $F_{b1}$ is the velocity perturbation of baryons. Here, $f := -R_\nu/(4R_\nu)$ with $R_\nu = \rho_\nu/ (\rho_\nu + \rho_\gamma)$ and $R_\gamma = 1 - R_\nu$ is the ratio of the NID mode to the adiabatic mode [46]. We replace $F_{\gamma l} \rightarrow F_{\gamma 1} - F_{b1}$ to manifest gauge invariance of the first term, which can be omitted in the radiation dominated period. Note that Legendre coefficients and $f$ only depend on the magnitude of $k$. In the tight-coupling regime, we can approximately solve the Boltzmann equations for the photon sector. In the case with the adiabatic condition, the solution is given by [47]

$$F_{\gamma 1} \sim -\frac{4}{\sqrt{3}} \sin(k r_s) \exp \left( -\frac{k^2}{k_B^2} \right).$$  \hfill (3.7)
where \( r_s \) is the sound horizon and \( k_D \) is the Silk damping scale. In the tight-coupling regime we can write \( G_{\gamma 0} + G_{\gamma 2} = 3F_{\gamma 2}/2 \), then (3.6) can be approximated as

\[
\langle \mu \rangle = -2.8 \int_0^{\eta_f} d\eta D_C(\eta) \int d(\ln k) \left[ P_G(k) + \left( f + \frac{\gamma_1}{\gamma_2} \right)^2 P_{SS}(k) \right] \partial_\eta \exp \left( -\frac{2k^2}{k_D^2} \right)
\]

\[
\sim -2.8 \int d(\ln k) \left[ P_G(k) + \left( f + \frac{\gamma_1}{\gamma_2} \right)^2 P_{SS}(k) \right] \left[ \exp \left( -\frac{2k^2}{k_D^2} \right) \right]^f,
\]

where we have used the relation \( F_{\gamma 2} = 8kF_{\gamma 2}/(15\pi) \) and replaced \( \sin^2(kr_s) \) with \( 1/2 \), and in addition we have adopted the relation \( \partial_\eta k_{D}^2 = -8/(45\pi) \) in the limit of full radiation domination. Let us divide \( \langle \mu \rangle \) into parts originating from \( R_G \) and \( g^2 \), that is, \( \langle \mu \rangle = \langle \mu \rangle_G + \langle \mu \rangle_{g^2} \). Then (3.8) yields

\[
\langle \mu \rangle_G = 2.8P_G(k_0) \log \left( \frac{k_{Di}}{k_{Df}} \right) \left[ 1 + \frac{n_s - 1}{2} \left( \log \left( \frac{k_{Di}k_{Df}}{2k_0^2} \right) - C \right) + \cdots \right]
\]

\[
= 3.36 \times 10^{-8}[1 + 8.91(n_s - 1) + \cdots],
\]

(3.9)

where \( C = 0.577 \ldots \) is the Euler-Mascheroni constant and \( \log(k_{Di}/k_{Df}) \simeq 5.477 \). If we take into account terms of \( O(n_s - 1) \) with \( n_s = 0.96, \langle \mu \rangle \) becomes \( 2.2 \times 10^{-8} \), which is consistent with the values derived in the previous works [16, 20–22]. The second-order correction is smaller than 10%. On the other hand, the contribution from \( g^2 \) can be calculated as

\[
\langle \mu \rangle_{g^2} = 1.4P_{SS}(k_0) \left( \frac{R_\nu}{4R_\gamma} + \frac{\gamma_1}{\gamma_2} \right)^2 \left[ \left( \frac{k_D}{\sqrt{2k_0}} \right)^{2n_g-2} \right] f \Gamma(n_g - 1)
\]

\[
= \frac{5.6\gamma_2^2P_g^2(k_0)}{n_g - 1} \left( \frac{R_\nu}{4R_\gamma} + \frac{\gamma_1}{\gamma_2} \right)^2 \left[ \left( \frac{k}{k_0} \right)^{n_g-1} \right] \left[ \left( \frac{k_D}{\sqrt{2k_0}} \right)^{2n_g-2} \right] f \Gamma(n_g - 1),
\]

(3.10)

where we have used (2.16). In particular, for the uncorrelated case with \( \gamma_1 = 0 \) and \( \gamma_2 = 1 \), expanding around the flat case, we obtain

\[
\langle \mu \rangle_{g^2}^{uncor} \simeq 12.1P_g^2(k_0)[1 + 13.9(n_g - 1) + \cdots],
\]

(3.11)

where we have assumed \( k_{\max} = k_0 \). Figure 4 shows \( \langle \mu \rangle_{g^2}^{uncor} \) as a function of \( n_g \). Current constraints on the NID isocurvature mode roughly give \( P_{SS}(k_0) \sim P_g^2(k_0) \lesssim 10^{-10} \) (we obtain similar constraints from the CMB angular bispectrum and see appendix A for details).\(^4\) For example, assuming \( P_g(k_0) \sim 10^{-5} \), it can be shown from (3.11) that \( \langle \mu \rangle_{g^2}^{uncor} \sim 10^{-9} \) for the case of flat spectrum, i.e. \( n_g \sim 1 \). As shown in figure 4, if we consider a blue-tilted power spectrum of \( g \), we can realize a large enhancement of \( \langle \mu \rangle \). From this figure, by employing the COBE FIRAS constraint, that is, \( \mu < 9 \times 10^{-5} \), we find an upper limit on \( n_g \) as \( n_g \lesssim 1.5 \).

\(^4\)To be more precise, the top-right panel of figure 22 in [5] gives the current constraint on amplitudes of the blue-tilted power-law isocurvature power spectrum at two different scales, \( k = 0.002 \) Mpc\(^{-1} \) and 0.1 Mpc\(^{-1} \). Although the constraint is in the case of generally correlated isocurvature perturbations, the bounds would be almost the same or less stringent in the case of uncorrelated ones, where the bound should be least stringent. Given \( \gamma_2P_g(k_0) \) and \( n_g \), we can compute the amplitudes of the isocurvature power spectrum at these two scales using (2.15). Thus we can translate the constraint into that on \( \gamma_2P_g(k_0) \) and \( n_g \). Indeed, we found that \( \gamma_2P_g(k_0) \) should be less than \( 0.5 \times 10^{-5} \) and \( 0.9 \times 10^{-5} \) at 2\( \sigma \) level for \( n_g = 1 \) and \( n_g = 1.5 \), respectively. In conjunction with forecasted constraints from bispectrum in appendix A, current observations would marginally allow \( \gamma_2P_g(k_0) = 10^{-6} \) for \( 1 \lesssim n_g \lesssim 1.5 \).
Incidentally, the $\mu$-distortion originating from the nearly Gaussian uncorrelated NID mode is also calculated straightforwardly and the difference from the case of the adiabatic perturbation arises only from that of the transfer functions of them. Specifically, the $\mu$-distortion from the isocurvature mode becomes $O(0.01)$ times smaller than that from the standard adiabatic one (see [8] for more details).

### 3.2 Inhomogeneous distortions

In the case with inhomogeneous distortions, (3.3) does not work since the equation is a result of conservation laws in the homogeneous and isotropic background. In other words, the quantities in the equation are globally averaged. Therefore, generally we should consider local conservation laws of energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$ and number flux $\nabla_\mu N^\mu = 0$ to compute inhomogeneous distortions [48]. Nevertheless approximately we can just remove $\langle \cdots \rangle$ of (3.3) since the thermodynamic variables are locally averaged quantities. Therefore we again use (3.8) with the slight change

$$
\int d(\ln k) P_{XY}(k) \rightarrow \int \frac{d^3 k_1}{(2\pi)^3} \frac{d^3 k_1'}{(2\pi)^3} e^{i(k+k') \cdot x} \mathcal{X}(k) \mathcal{Y}(k') 5 P_2(\hat{k} \cdot \hat{n}) P_2(\hat{k}' \cdot \hat{n}) \text{, with } \mathcal{X}, \mathcal{Y} = \mathcal{R}, \mathcal{S}, \quad (3.12)
$$

where $\hat{k}$ and $\hat{k}'$ are unit vectors of $k$ and $k'$, respectively. $\hat{n}$ is a tangential unit vector of the line-of-sight of the photons. Then we obtain the $\mu$ distortion in Fourier space as follows:

$$
\mu(\eta_f, k) = 16\alpha \int \frac{d^3 k_1}{(2\pi)^3} \left[ R_{\mathcal{G}k_1} R_{\mathcal{G}k_2} + \left( \frac{R_\nu}{4R_\gamma} \right)^2 S_{k_1} S_{k_2} \right] \\
\times 5 P_2(\hat{k}_1 \cdot \hat{n}) P_2(\hat{k}_2 \cdot \hat{n}) \langle \sin k_1 r_s \sin k_2 r_s \rangle_p \left[ \exp \left( -\frac{k_1^2 + k_2^2}{k_D^2} \right) \right]_f, \quad (3.13)
$$

where $\alpha = -1.4 \times 1/4$, $k_2 = k - k_1$ and $\langle \cdots \rangle_p$ is a periodic average.
4 \(\mu T\) angular cross-correlation

The harmonic coefficients of observed anisotropies in \(\Theta\) and \(\mu\) can be written as

\[
a_{T,lm} = \int d\mathbf{n} Y^*_l(m)(\eta_0, \mathbf{x}, \mathbf{n}),
\]

\[
a_{\mu,lm} = \int d\mathbf{n} Y^*_l(m)(\eta_0, \mathbf{x}, \mathbf{n}),
\]

where 0 is the conformal time at present. Without loss of generality, we can take \(\mathbf{x} = 0\). Then the angular power spectrum of the \(\mu T\) cross-correlation function is defined as usual,

\[
C^\mu_T = \frac{1}{2l+1} \sum_m \langle a^*_{\mu,lm} a_{T,lm} \rangle.
\]

In Fourier space, the temperature perturbations are

\[
\Theta(\eta, \mathbf{k}, \mathbf{n}) = \Theta^R(\eta; k, \lambda) R_{\mathbf{k}} + \Theta^S(\eta; k, \lambda) S_{\mathbf{k}},
\]

where \(\lambda := \cos(\mathbf{k} \cdot \mathbf{n})/k\), and the \(\mu\)-distortion is written as

\[
\mu(\eta, \mathbf{k}, \mathbf{n}) = \Delta_\mu(\eta; \eta_f, k, \lambda) \mu(\eta_f, \mathbf{k}),
\]

where \(\mu(\eta_f, \mathbf{k})\) is given by (3.13) and the transfer function, \(\Delta_\mu(\eta; \eta_f, k, \lambda)\), is given in ref. [48]. Substituting (4.4) and (3.13) into (4.1) and (4.2), we obtain

\[
a_{T,lm} = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} Y^*_l(m)(\hat{k}) \left[ \Theta^R_l(\eta_0, k) R_{\mathbf{k}} + \left( \Theta^S_l(\eta_0, k) + \frac{\gamma_1}{\gamma_2} \Theta^R_l(\eta_0, k) \right) S_{\mathbf{k}} \right],
\]

\[
a_{\mu,lm} = 4\pi(-i)^l \cdot 16\alpha \int \frac{d^3k}{(2\pi)^3} \frac{d^3k_1}{(2\pi)^3} Y^*_l(m)(\hat{k}) \Delta_\mu(\eta_0, k) \sin k_1 r_s \sin k_2 r_s \frac{5}{p} P_2(\hat{k}_1 \cdot \mathbf{n}) P_2(\hat{k}_2 \cdot \mathbf{n})
\]

\[
\times \left[ \exp \left( -\frac{k_1^2 + k_2^2}{k_D^2} \right) \right]^j \left[ R_{\mathbf{k}_1} R_{\mathbf{k}_2} + \left( \frac{R_{\mathbf{k}_1}}{4R_{\mathbf{r}}} + \frac{\gamma_1}{\gamma_2} \right)^2 S_{\mathbf{k}_1} S_{\mathbf{k}_2} \right],
\]

where \(\Theta^X_l(X = R, S)\) and \(\Delta_\mu\) are respectively the Legendre coefficients of the transfer functions for the temperature perturbations and the \(\mu\)-distortion. On large angular scales, where the Sachs-Wolfe effect is dominant, \(\Theta_l\) is approximately given by

\[
\Theta_l(\eta_0, k) \sim [\Theta_0(\eta_*) + \psi(\eta_*))] j_l(k(\eta_0 - \eta_*)),
\]

where \(\eta_*\) is the conformal time at recombination and \(\psi\) is the gravitational potential in the conformal Newtonian gauge. \(\Theta_0(\eta_*) + \psi(\eta_*)\) depends on the initial conditions. Using the numerical code CLASS [49], \(\Theta_0(\eta_*) + \psi(\eta_*) \sim -0.24\) for the adiabatic perturbation and \(\Theta_0(\eta_*) + \psi(\eta_*) \sim -0.175\) for the neutrino isocurvature density mode, respectively. \(\Delta_\mu\) is also given by the line-of-sight integral method as shown in ref. [48]. Using the above equations, \(\mu T\) angular power spectrum is obtained as

\[
|C^\mu_T| = \left( \frac{4\pi}{2l+1} \right)^2 \frac{1}{2l+1} \left( 0.175 + 0.24 \frac{\gamma_1}{\gamma_2} \right) \left( \frac{R_{\mathbf{r}}}{R_{\mathbf{r}}} + \frac{4\gamma_1}{\gamma_2} \right)^2 \int \frac{d^3k d^3k_1 d^3k_2}{(2\pi)^6} \sum_{m=-l}^{l} Y^*_m(\hat{k}) Y_m(\frac{k_1 + k_2}{|k_1 + k_2|})
\]

\[
\times 5 P_2(\hat{k}_1 \cdot \mathbf{n}) P_2(\hat{k}_2 \cdot \mathbf{n}) (S_{\mathbf{k}_1} S_{\mathbf{k}_2}) j_l(k(\eta_0)) j_l(|k_1 + k_2|\eta_0) \sin k_1 r_s \sin k_2 r_s \exp \left( -\frac{k_1^2 + k_2^2}{k_D^2} \right)^j,
\]

(4.9)
where we have used $\eta_0 \gg \eta_*$. The three-point function $\langle SSS \rangle$ is nonzero since $S$ is non-Gaussian. From the reality condition, we obtain

$$
\langle S_k S^*_k, S_{k_2} \rangle = \langle S_k S_{-k_1}, S_{-k_2} \rangle = (2\pi)^3 \delta^{(3)}(k - k_1 - k_2) B_{SSS}(k, k_1, k_2).
$$

(4.10)

Note that the periodic average only picks up the configurations of an isosceles triangle, i.e. $k_1 = k_2$, then, we can also replace it by $1/2$. Let us consider a transformation $k_\pm = k_1 \pm k_2$. For low-$l$, the integrations of $k$ and $k_\pm$ are negligible except around CMB scales $k = k_+ \sim \eta_0^{-1}$, because of the behaviour of the spherical Bessel functions. On the other hand, due to the exponential suppression factor the contribution around $k^2_\pm + k^2_\pm \sim k^2_\pm \gg k^2_\pm$ is dominant. Therefore, we obtain a hierarchical relation $k_1 \sim k_2 \sim k_-/2 \gg k_+$. From the above results, noting that the Jacobian of coordinate transformation to $k_\pm$ is 1/8 and focusing on the squeezed configuration here, we obtain

$$
B_{SSS}(k_+, k_-/2, k_-/2)
\sim \frac{2\pi^2}{k_+^3} \left( \frac{k_-/2}{k_0} \right)^{n_g-1} \frac{16 \mathcal{P}^3_g(k_0)}{n_g - 1} \left[ \left( \frac{k}{k_0} \right)^{n_g-1} \right]^{k_{\text{max}}} L^{-1},
$$

(4.11)

with use of (2.21). Finally we get the following form

$$
|C^\mu_T| \sim 0.583 \left( 1 + 1.4 \frac{\gamma_1}{\tilde{\gamma}_2} \right) \left( 1 + 5.8 \frac{\gamma_1}{\tilde{\gamma}_2} \right)^2 \frac{\gamma_2^3 \mathcal{P}^3_g(k_0)}{n_g - 1} \left[ \left( \frac{k}{k_0} \right)^{n_g-1} \right]^{k_{\text{max}}} L^{-1}
\times \left[ \left( \frac{\sqrt{2} k_D}{k_0^2 L} \right)^{n_g-1} \right]^f_i \frac{\Gamma(f + \frac{n_g}{2} - 1)}{\Gamma(f + \frac{3}{2} - \frac{n_g}{2})} \Gamma(3 - n_g) \Gamma \left( \frac{n_g-1}{2} \right).
$$

(4.12)

where we have used $P_2(\hat{k}_- \cdot \nu) = P_2(\hat{k}_- \cdot \nu)$ in the integration with respect to $\hat{k}_-$ and $\eta_0 = L$. For the flat and uncorrelated spectrum with $\mathcal{P}_g \sim 10^{-5}$, which would be marginally allowed by CMB bispectrum from Planck (See appendix A for the Planck forecast), and taking $k_{\text{max}} = k_0$, we obtain $|l(l + 1)C^\mu_T| \sim 10^{-14}$. This level of signal corresponds to $f^l_{\text{NL}} \sim 100$ in the case of the local-type non-Gaussianity in adiabatic perturbations [30], which is 10 times smaller than the expected sensitivity of PIXIE and comparable to that of PRISM. For example, when $k_{\text{max}} = k_0$ and $\gamma_1 = 0$, at $l = 10$ we find

$$
\frac{10 \cdot 11 \times |C^\mu_T|_{10}}{\mathcal{P}_g^3(k_0)} \sim 3.4 \left[ 1. + 1.75514(n_g - 1) + \cdots \right].
$$

(4.13)

Figure 5 shows the $n_g$-dependence of $l(l + 1)C^\mu_T \mathcal{P}_g^{-3}(k_0)$ with $l = 10$. In the case with $n_g \sim 1.5$, the signal is almost 10 times bigger than that in the flat case, and can be also detected by PIXIE.

For the sake of completeness, let us consider the case with the weakly non-Gaussian NID mode, $S = S^\nu_4 + \frac{3}{5} f_{\text{NL}}^{\text{loc}, \nu}(S^2_0 + \langle S^2_0 \rangle)$. Local-type bispectrum of the uncorrelated isocurvature perturbation is parameterized as

$$
B_S(k_+, k_-/2, k_-/2) = -\frac{6}{5} f_{\text{NL}}^{\text{loc}, \nu} \left[ P_S(k_+) P_S(k_-/2) + 2 \text{perm.} \right] - \frac{6}{5} f_{\text{NL}}^{\text{loc}, \nu} P_S(k_+) P_S(k_-/2).
$$

(4.14)
Figure 5. $n_g$ dependence of $l(l+1)C_{l}^{TT}P_{g}^{-3}(k_0)$ with $l = 10$.

As discussed in (3.10) and (4.9), the transfer functions of the adiabatic and NID modes differ only by a multiplicative constant factor. Thus by taking into account this difference in the results of ref. [30], we obtain

$$l(l+1)C_{l}^{TT} \lesssim 8.04 \times 10^{-19} f_{NL}^{loc,\nu} \left[ 1 + \left( 5.37 + \frac{1}{l} + \frac{1}{l+1} + 2\psi(0)(l) \right) \frac{n_s - 1}{2} + \cdots \right], \quad (4.15)$$

where we have assumed $\beta_{NID} = 0.27$ and $\psi(0)(x) = d \log \Gamma(x)/dx$ is a poly-gamma function. When $f_{NL} = f_{NL}^{loc,\nu}$, the $\mu T$ cross-correlation from the non-Gaussian neutrino isocurvature perturbations should be $10^2$ times smaller than that from the adiabatic ones calculated in ref. [30]. However as is shown in appendix A, the expected observational constraints on $f_{NL}^{loc,\nu}$ would be $10^4$ at around $2\sigma$ level when $\beta_{NID} \simeq P_{SS}/P_{RR} = 10^{-1}$. When we take $f_{NL}^{loc,\nu}$ to be $10^4$, the cross-correlation from the non-Gaussian neutrino isocurvature perturbations can be as large as that from local-type adiabatic ones with $f_{NL} = 100$. The size of signal here is the same as in the Gaussian-squared case, which we have presented before. This is by no means surprising since both the amplitudes of $\mu T$ cross-correlation and the CMB bispectrum are determined by the primordial bispectrum, whose spectral shape can be approximated with the local-type one both in the cases of weakly non-Gaussian and Gaussian-squared isocurvature perturbations.

5 Conclusions

In this paper, we have calculated the mean $\mu$ and the cross-correlation of its anisotropy with primary CMB temperature one in the presence of non-Gaussian neutrino isocurvature perturbations. In particular, we have focused on the Gaussian-squared perturbations, and explicitly shown that the primordial bispectrum of Gaussian-squared type perturbations can be approximated by the local-type one. We have found that when the power spectrum of the isocurvature perturbations is nearly scale-invariant, the mean $\mu$ and the $\mu T$ cross-correlation can be respectively as large as $10^{-9}$ and $10^{-14}$, with the present constraints from CMB power spectrum and bispectrum on NID mode being satisfied. In particular,
µT cross-correlation from NID perturbations is potentially observed by the PRISM surveys, which is contrastive to the cases of adiabatic local-type ones, which requires $f_{NL}$ an order of magnitude larger than the upper bound from current CMB bispectrum measurements. If the power spectrum of isocurvature perturbations is allowed to be blue-tilted, the µT cross-correlation can be enhanced by an order of magnitude and expected to be observed by PIXIE.

Acknowledgments

We would like to thank Masahide Yamaguchi for the helpful discussions and comments. This work was supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. Y.T. is supported by an Advanced Leading Graduate Course for Photon Science grant. T.S. is supported by the Academy of Finland grant 1263714. S.Y. acknowledges the support by Grant-in-Aid for JSPS Fellows No. 242775.

A Constraints from CMB angular bispectrum

In this appendix, we summarize the Fisher matrix analysis of CMB bispectrum and forecast for the Planck constraints on non-Gaussian isocurvature perturbations, following refs. [40, 50, 51].

Let us start by generalizing the form of CMB anisotropies in (4.1) into

$$a_{lm}^P = 4\pi(-i)^l \int \frac{d^3k}{(2\pi)^3} \sum_X g_{l}^{XP}(k) Y_{lm}^X(\hat{k}) Y_{lm}^X(\hat{k}),$$

(A.1)

where $\hat{k}$ is the unit vector of $k$, and $X$ and $P$ respectively represent types of initial perturbations and CMB anisotropies, i.e. $X$ is either the adiabatic ($R$) or neutrino isocurvature ($S$) mode and $P$ is either the temperature (T) or E-mode polarization (E) anisotropy. $g_{l}^{XP}$ is the Legendre coefficient of the transfer function of $P$ from $X$ and numerically evaluated using the CAMB code [53]. The CMB bispectrum in harmonic space is given as

$$B_{l_1m_1l_2m_2l_3m_3}^{P_1P_2P_3} \equiv \langle a_{l_1m_1}^{P_1} a_{l_2m_2}^{P_2} a_{l_3m_3}^{P_3} \rangle,$$

(A.2)

Given a primordial bispectrum

$$\langle \mathcal{X}_1^X(k_1) \mathcal{X}_2^X(k_2) \mathcal{X}_3^X(k_3) \rangle = B_{l_1}^{X_1} X_2 X_3(k_1, k_2, k_3)(2\pi)^3 \delta^3(k_1 + k_2 + k_3),$$

(A.3)

eq. (A.2) can be rewritten as

$$B_{l_1m_1l_2m_2l_3m_3}^{P_1P_2P_3} = \sum_{X_1X_2X_3} \prod_{i=1}^3 \left[ 4\pi (-i)^l_i \int \frac{d^3k_i}{(2\pi)^3} g_i^{XP}(k_i) Y_{lm_i}^X(\hat{k}_i) \right]$$

$$\times B_{l_1}^{X_1} X_2 X_3(k_1, k_2, k_3)(2\pi)^3 \delta^3(k_1 + k_2 + k_3).$$

(A.4)

By Fourier transforming the delta function and using a formula for the partial wave decomposition

$$e^{ikr} = \sum_{lm} 4\pi i^l j_l(kr) Y_{lm}(\hat{k}) Y_{lm}^*(\hat{r}),$$

(A.5)

We defer it for future work to derive constraints on non-Gaussian neutrino isocurvature perturbations from the actual Planck data here. Constraints from the WMAP data can be found in ref. [52].
eq. (A.4) can be further recast into

\[ B_{l_1m_1l_2m_2l_3m_3}^{P_1P_2P_3} = G_{l_1l_2l_3}^{m_1m_2m_3} b_{l_1m_1l_2m_2l_3m_3}^{P_1P_2P_3}, \]  

(A.6)

where

\[ b_{l_1m_1l_2m_2l_3m_3}^{P_1P_2P_3} = \sum_{X_1X_2X_3} \int r^2 dr \prod_{i=1}^{3} \left[ \frac{2}{\pi} \int k_i^2 dk_i g_{l_i}^{P_i}(k_i) j_i(k_i r) \right] B^{X_1X_2X_3}(k_1, k_2, k_3) \]  

(A.7)

is the reduced bispectrum and

\[ G_{l_1l_2l_3}^{m_1m_2m_3} = \int d\hat{r} \prod_{i=1}^{3} Y_{l_i m_i}^{*}(\hat{r}) \]  

(A.8)

is the Gaunt integral, which can be represented in terms of the Wigner-3j symbol as

\[ G_{l_1l_2l_3}^{m_1m_2m_3} = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}. \]  

(A.9)

Now let us consider the non-Gaussian perturbations given in (2.1)–(2.2). As is shown in (2.21), the bispectrum of these Gaussian-squared type perturbations can be approximated in the form of the local-type bispectrum,

\[ B^{X_1 X_2 X_3}(k_1, k_2, k_3) \approx 8 A \gamma_{X_1} \gamma_{X_2} \gamma_{X_3} P_g(k_0) \left[ P_g(k_1) P_g(k_2) + (2 \text{ cyclic perms}) \right], \]  

(A.10)

where the factor \( \gamma_X \) should be \( \gamma_1 \) for \( X = R \) and \( \gamma_2 \) for \( X = S \). Here, the factor \( A \approx \left[(k/k_0)^{n_g-1}\right]_{L_1^{-1}}^{L_2^{-1}}/(n_g-1) \) depends on wave numbers weakly, and the scale dependence of \( A \) can be safely neglected as long as CMB anisotropies (i.e. T and E) are concerned. In the following analysis, we simply replace \( A \) with unity. Then the reduced bispectrum in (A.6) can be rewritten as

\[ b_{l_1l_2l_3}^{X_1P_1X_2P_2X_3P_3} = \sum_{X_1X_2X_3} 8 \gamma_{X_1} \gamma_{X_2} \gamma_{X_3} P_g(k_0) [b_{l_1l_2l_3}^{X_1P_1X_2P_2X_3P_3} + (2 \text{ cyclic perms})]. \]  

(A.11)

Here, \( b_{l_1l_2l_3}^{X_1P_1X_2P_2X_3P_3} \) is given as

\[ b_{l_1l_2l_3}^{X_1P_1X_2P_2X_3P_3} \equiv \int r^2 dr \alpha_{l_1}^{X_1P_1}(r) \beta_{l_2}^{X_2P_2}(r) \beta_{l_3}^{X_3P_3}(r), \]  

(A.12)

with \( \alpha_{l}^{XP}(r) \) and \( \beta_{l}^{XP}(r) \) being defined as

\[ \alpha_{l}^{XP}(r) \equiv \frac{2}{\pi} \int k^2 dk g_{l}^{XP}(k) j_l(k r), \]  

(A.13)

\[ \beta_{l}^{XP}(r) \equiv \frac{2}{\pi} \int k^2 dk P_g(k) g_{l}^{XP}(k) j_l(k r). \]  

(A.14)

For later convenience, we define the following set of non-linearity parameters

\[ f_{NL}^{(1)} = 8 \gamma_1^3 P_g(k_0), \]  

(A.15)

\[ f_{NL}^{(2)} = 8 \gamma_1^2 \gamma_2 P_g(k_0), \]  

(A.16)

\[ f_{NL}^{(3)} = 8 \gamma_1 \gamma_2^2 P_g(k_0), \]  

(A.17)

\[ f_{NL}^{(4)} = 8 \gamma_2^3 P_g(k_0). \]  

(A.18)
Then the reduced bispectrum can be given as

\[ b^{P_iP_jP_k}_{l_1l_2l_3} = \sum_{j=1}^{4} f^{(j)}_{NL} b^{P_iP_jP_k}_{l_1l_2l_3}, \]  

(A.19)

where \( \{b^{P_iP_jP_k}_{l_1l_2l_3}\} \) are template bispectra for the nonlinearity parameters \( f^{(j)}_{NL} \), which are defined as

\[
\begin{align*}
  b^{P_iP_jP_k}_{l_1l_2l_3}(1) &= b^{R_iR_jR_k}_{l_1l_2l_3} + (2 \text{ cyclic perms}), \\
  b^{P_iP_jP_k}_{l_1l_2l_3}(2) &= [b^{SP_iR_jR_k}_{l_1l_2l_3} + b^{R_iSP_jR_k}_{l_1l_2l_3} + b^{R_iR_jSP_k}_{l_1l_2l_3}] + (2 \text{ cyclic perms}), \\
  b^{P_iP_jP_k}_{l_1l_2l_3}(3) &= [b^{R_iSP_jR_k}_{l_1l_2l_3} + b^{SP_iR_jR_k}_{l_1l_2l_3} + b^{SP_iSP_jR_k}_{l_1l_2l_3}] + (2 \text{ cyclic perms}), \\
  b^{P_iP_jP_k}_{l_1l_2l_3}(4) &= b^{SP_iSP_jR_k}_{l_1l_2l_3} + (2 \text{ cyclic perms}). 
\end{align*}
\]

(A.20)

(A.21)

(A.22)

(A.23)

Now let us move on to the Fisher matrix analysis. According to refs. [54, 55], given template bispectra \( b^{P_iP_jP_k}_{l_1l_2l_3} \), the Fisher matrix for the nonlinearity parameters \( f^{(j)}_{NL} \) can be approximately given as

\[
F_{jj'} = f_{\text{sky}} \sum_{l_1 \leq l_2 \leq l_3} \frac{(2l_1 + 1)(2l_2 + 1)(2l_3 + 1)}{4\pi} \binom{l_1 \; l_2 \; l_3}{0 \; 0 \; 0}^2 
\times \sum_{P_1P_2P_3P_1'P_2'P_3'} b^{P_iP_jP_k}_{l_1l_2l_3} \left[ \text{Cov}_{l_1l_2l_3}^{-1} \right]_{P_1P_2P_3|P_1'P_2'P_3'} f^{(j)}_{NL} f^{(j')}_{NL},
\]

(A.24)

(A.25)

where \( f_{\text{sky}} \) is the fraction of the sky covered by observations and \( \left[ \text{Cov}_{l_1l_2l_3}^{-1} \right]_{P_1P_2P_3|P_1'P_2'P_3'} \) is the inverse of covariance matrix. The covariance matrix \( \left[ \text{Cov}_{l_1l_2l_3} \right]_{P_1P_2P_3|P_1'P_2'P_3'} \) should be given in the limit of weak non-Gaussianity as

\[
\left[ \text{Cov}_{l_1l_2l_3} \right]_{P_1P_2P_3|P_1'P_2'P_3'} = \Delta_{l_1l_2l_3} C_{l_1}^{P_1} C_{l_2}^{P_2} C_{l_3}^{P_3'},
\]

(A.26)

where \( C^{PP'}_l = C^{PP'}_l + N^{PP'}_l \) is the sum of signal (\( C^{PP'}_l \)) and noise (\( N^{PP'}_l \)) power spectra, and \( \Delta_{l_1l_2l_3} \) takes values 6, 2 and 1 for the cases that all \( l's \) are the same, only two of them are the same and otherwise, respectively. For the noise power spectrum, we adopt the Knox’s formula [56],

\[
N^{PP'}_l = \delta_{PP'} \theta_{\text{FWHM}}^2 \sigma^2_P \exp \left[ l(l+1) \frac{\theta_{\text{FWHM}}^2}{8 \ln 2} \right],
\]

(A.27)

where \( \theta_{\text{FWHM}} \) is the full width at half maximum of the Gaussian beam, and \( \sigma_P \) is the root mean square of the instrumental noise per pixel. For cases of multi-frequency observations, \( N^{PP'}_l \) is given via quadrature sum over the frequency channels. In Table 1, we summarize the survey parameters for Planck that we adopt in what follows. In addition, we set the sky coverage \( f_{\text{sky}} \) to 0.7.

To translate the forecasted constraints on \( f^{(j)}_{NL} \) from the Fisher matrix into those on \( \gamma_1, \gamma_2 \) and \( P_g(k_0) \), we define an effective \( \Delta \chi^2 \) as follows

\[
\Delta \chi^2 \equiv \sum_{jj'} f^{(j)}_{NL} F_{jj'} f^{(j')}_{NL}.
\]

(A.28)
| bands [GHz] | $\theta_{\text{FWHM}}$ [arcmin] | $\sigma_T$ [$\mu$K] | $\sigma_P$ [$\mu$K] |
|------------|-------------------------------|----------------|----------------|
| 30         | 33.0                          | 2.0            | 2.8            |
| 44         | 24.0                          | 2.7            | 3.9            |
| 70         | 14.0                          | 4.7            | 6.7            |
| 100        | 10.0                          | 2.5            | 4.0            |
| 143        | 7.1                           | 2.2            | 4.2            |
| 217        | 5.0                           | 4.8            | 9.8            |
| 353        | 5.0                           | 14.7           | 29.8           |

Table 1. Survey parameters adopted in our analysis for Planck. We here assume 1-year duration of observation.

The expected allowed regions in the $\gamma_1$ and $\gamma_2$ plane are shown in figure 6. Note that $P_g(k_0)$ is degenerate with $\gamma_1$ and $\gamma_2$, and we can constrain only the combinations $\gamma_1 P_g(k_0)$ and $\gamma_2 P_g(k_0)$. As can be read from the figure, when $\gamma_1 = 0$ and $\gamma_2 = 1$, $P_g(k_0) = 10^{-5}$ would be marginally allowed at 2 $\sigma$ level by Planck for $1.5 \lesssim n_g \lesssim 2$.

In addition, we can also compute the expected constraint on $f_{\text{NL}}^{\text{loc,\nu}}$ from Planck by performing the Fisher matrix analysis in the similar manner. Given the primordial bispectrum of (4.14), the template bispectrum $\tilde{b}_{l_1 l_2 l_3}^{P_1 P_2 P_3,\nu}$ of the non-linearity parameter $f_{\text{NL}}^{\text{loc,\nu}}$ should be given as

$$
\tilde{b}_{l_1 l_2 l_3}^{P_1 P_2 P_3,\nu} = -\frac{6}{5} \int r^2 dr \tilde{a}_{l_1}^{P_1}(r) \tilde{a}_{l_2}^{P_2}(r) \tilde{a}_{l_3}^{P_3}(r),
$$

(A.29)

Since the bispectrum is the sum of cubic products of $\gamma_1$ and $\gamma_2$ as can be seen in eq. (A.10), its derivatives with respect to $\gamma_1$ and $\gamma_2$ and hence the Fisher matrix for $\gamma_1$ and $\gamma_2$ vanish when the fiducial model is assumed to be Gaussian, i.e. $\gamma_1 = \gamma_2 = 0$. To obtain constraints on these parameters, here we adopt the effective $\Delta \chi^2$ based on the least-squared method [57]. By deeming $\Delta \chi^2$ as a two-parameter function of $\gamma_1 P_g(k_0)$ and $\gamma_1 P_g(k_0)$, we plot contours at $\Delta \chi^2 = 2.3$ and 6.18 in figure 6 as the constraints at 1 and 2 $\sigma$ levels. Because $\Delta \chi^2$ is not a quadratic form of $\gamma_1$ and $\gamma_2$ but a sextic one, the contours do not form ellipses.
with \( \tilde{\alpha}_i^P(r) \) and \( \tilde{\beta}_i^P(r) \) being defined as

\[
\tilde{\alpha}_i^P(r) \equiv \alpha_i^{SP}(r), \\
\tilde{\beta}_i^P(r) \equiv \frac{2}{\pi} \int k^2 dk P_S(k) y_i^{SP}(k) y_i(k r).
\]

(A.30) \hspace{1cm} (A.31)

The reduced bispectrum is given by \( b_{i_1 i_2 i_3}^{P, P, \nu} = \tilde{f}^{loc, \nu}_{i_1 i_2 i_3} \tilde{g}^{SP}_{i_1 i_2 i_3} \). When \( P_S \) is scale-invariant, using the Planck survey parameters given in table 1, we find the 1\( \sigma \) error on \( \tilde{f}^{loc, \nu}_{i_1 i_2 i_3} \tilde{g}^{2}_{i_1 i_2 i_3} \) is expected to be 41. Thus, if \( \beta_{iso} \) is assumed to be \( 10^{-1} \), \( \tilde{f}^{loc, \nu}_{i_1 i_2 i_3} \tilde{g}^{2}_{i_1 i_2 i_3} \) can be as large as \( 10^4 \) at around 2\( \sigma \) level.

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