Abstract—In this work, we study and analyze a class of Duhem hysteresis operators that can exhibit butterfly loops. We study first the consistency property of such operator, which corresponds to the existence of an attractive periodic solution when the operator is subject to a periodic input signal. Subsequently, we study the two defining functions of the Duhem operator such that the corresponding periodic solutions can admit a butterfly input–output phase plot. We present a number of examples where the Duhem butterfly hysteretic operators are constructed using two zero-level set curves that satisfy some mild conditions.

Index Terms—Hysteresis, magnetic hysteresis, nonlinear control systems.

I. INTRODUCTION

HYSTERESIS is a natural phenomenon that was originally investigated in the study of magnetic field and magnetic flux density in ferromagnetic materials [1]. In the following centuries, the hysteresis phenomena are well-documented and studied in numerous systems originating from various disciplines, from biology [2], [3], physics [4], astronomy [5], economics [6] to experimental psychology [7]. The hysteresis is typically characterized by the presence of memory in its (dynamic) behavior and has attracted the attention of scientists for its intrinsic complexity. The multitude of domains, where hysteresis can be found, has led most of the works in the literature to describe it using phenomenological models, which are rather independent of the specific process underlying it. In this regard, the Duhem model [8] is one of the well-known generic models of hysteresis. Its mathematical formulation encompasses many of other phenomenological models, for instance, the Dahl model, the Bouc–Wen model, and the Maxwell-slip model [9]. Another large class of popular models is the Preisach models [9], [10], which will not be considered in this article.

The Duhem model has been extensively studied in the literature and several mathematical properties have been established. Roughly speaking, the Duhem model maps input signals to output signals via switched nonlinear differential equations, where the switch signal depends on the sign of the derivative of its input signal. Mathematical properties of the resulting Duhem operator (with time-independent vector fields) have been presented in the literature that include the existence and uniqueness of the solutions, as well as, monotonicity, semigroup, and rate-independent properties. A thorough exposition of these properties and other fundamental mathematical properties can be found in [8], [11], and [12]. Control systems properties, where the Duhem operator is feedback interconnected with other nonlinear systems, have been studied in the literature. For instance, the study of dissipativity in a class of Duhem operators is presented in [13], [14], and [15] where the associated storage functions and supply rate functions depend on the specific hysteresis loops obtained from the Duhem models. In recent decades, attention has also been given to the convergent systems property [16] or consistency and strong-consistency [12], [17] of the Duhem model, where the output converges to a periodic signal when a periodic input signal is given. Such property in the literature of hysteresis is known as the accommodation property, as presented, for instance, in [18], which investigates the hysteresis modeling in ferromagnetic material. In this case, the phase plot of input and output signals will show loops that converge to a limit cycle around the so-called anhysteresis curve.

This convergent systems property has been shown for the semilinear Duhem model [19] and for the Babuška’s model [20], which is a class of the Duhem model where each vector field in the integro-differential equations can be expressed as the multiplication of two single variable functions. Moreover, Naser and Ifkhouane [12] presented the analysis of the convergence properties of the LuGre friction model, which is based on the introduced concept of strong-consistency of the hysteresis map. Loosely speaking, strong-consistency refers to the property of the hysteresis map to approach a time-periodic when the input is periodic. Although the generality of this formulation is suitable for the study of rate-dependent hysteresis operators, it is beyond the scope of this work.

In this article, we extend the aforementioned results to a class of Duhem models that can exhibit asymmetric butterfly loops.
Here, the butterfly loops refer to presence of closed orbits with two or multiple loops in the input–output phase plot. While the standard hysteresis operators produce either counterclockwise or clockwise loops, the butterfly ones comprise of both clockwise and counterclockwise loops. The presence of butterfly loops has been shown and observed in practice for decades, e.g., in piezoeactuator systems [21] and in magnetostrictive materials [22]. The first simple mathematical modeling, analysis, and identification of hysteresis with butterfly loops is presented in [23], where a convex function is added to the output of standard hysteresis operator in order to enforce two inflection points to the standard loop and thereby creating butterfly loops. A general modeling and analysis of a butterfly hysteresis operator based on the Preisach model is presented in [24] and [25]. This class of Preisach operator was first reported in [26], where it is used to describe the shape memory property of a newly purposely-designed piezoelectric materials. This framework has been used in the development of a deformable mirror with high-density programmable actuators [27], [28], [29]. An microelectromechanical systems (MEMS) mirror with programmable tilt whose actuators are modeled by Preisach hysteresis operators have also been studied in [30]. With the aim of providing an alternative modeling and analysis framework to describe the butterfly hysteresis loops of this piezoelectric material as well as other classes of actuators based on smart material or memory material, in this work, we study conditions that enable the Duhem model to exhibit multiple loops, including the butterfly loop. We note that the presence of multiloops hysteresis behavior in advanced materials has recently been shown in [31]. To the best of authors’ knowledge, the modeling and analysis of a Duhem model that can exhibit multiple loops is still largely open.

As our first main contribution in the extension of previous results to the butterfly hysteresis operator using the Duhem model, we investigate the applicability of Babuška’s conditions used in [20] as sufficient conditions for guaranteeing the convergence of the input–output phase plot to a closed orbit when the input signal is simple periodic1 in Section III. These conditions correspond to the monotonicity of the vector fields in the Duhem model when the input argument is fixed. Using only these Babuška’s conditions, we can relax the strong sign-definite assumption on these vector fields that are typically assumed in the literature. Furthermore, we show that if we have strict monotonicity conditions then the closed orbit is unique for any initial value of the output. We have applied these conditions to the Bouc–Wen model and validated that in fact they are equivalent to some other known conditions in [32] that guarantee the bounded-input, bounded-output (BIBO) stability of the model. In Section IV, we present our second main contribution, where we study a class of Duhem models whose vector fields are sign-indefinite but satisfy the aforementioned Babuška’s conditions. Under some additional mild assumptions on the vector fields, we show that the input–output phase plot of this Duhem model converges to a closed orbit with two or more loops, for example, it can exhibit a butterfly loop. We remark that the conditions presented in this section focus on the phenomenological description of the butterfly hysteresis loops and on finding vector functions $f_1$ and $f_2$ that have specific physical interpretation. Such characterization of Duhem models that can produce butterfly hysteresis loops is, to the best authors’ knowledge, still an open problem. At the end of Section IV, we provide illustrative examples of this class of Duhem models using simple vector functions composed of polynomial.

## II. Preliminaries

**Notation:** We denote by $C(U, Y), AC(U, Y), C_{pw}(U, Y)$, the spaces of continuous, absolute continuous, and piece-wise continuous functions $f : U \to Y$, respectively. We denote $\mathbb{R}_+ := [0, \infty)$. Moreover, we call a function $f : U \to Y$ monotonically increasing (resp. decreasing) if $u_1 < u_2$ implies $f(u_1) \leq f(u_2)$ (resp. $f(u_1) \geq f(u_2)$), and strictly monotonically increasing (resp. decreasing) if the inequality is strict.

We define the next two auxiliary operators, which are used throughout this work. The right-shift operator $\mathcal{S}_\tau : AC(\mathbb{R}_+, \mathbb{R}) \to AC(\mathbb{R}_+, \mathbb{R})$ parameterized by $\tau \in \mathbb{R}$ is defined by

$$[\mathcal{S}_\tau(v)](t) := v(t + \tau).$$

(1)

The right-continuation operator $\mathcal{R}_\tau : AC(\mathbb{R}_+, \mathbb{R}) \to AC(\mathbb{R}_+, \mathbb{R})$ parameterized by $\tau \in \mathbb{R}_+$ is defined by

$$[\mathcal{R}_\tau(v)](t) := \begin{cases} v(t) & \text{if } t \in [0, \tau] \\ v(\tau) & \text{if } t \in (\tau, \infty). \end{cases}$$

(2)

The Duhem hysteresis operator is a mapping $\Phi : AC(\mathbb{R}_+, \mathbb{R}) \times \mathbb{R} \to AC(\mathbb{R}_+, \mathbb{R})$ such that $y = \Phi(u, y_0)$ satisfies

$$\dot{y}(t) = \begin{cases} f_1(u(t), y(t)) \dot{u}, & \text{if } \dot{u}(t) \geq 0 \\ f_2(u(t), y(t)) \dot{u}, & \text{if } \dot{u}(t) < 0 \end{cases}$$

$$y(0) = y_0$$

(3)

at almost every $t \geq 0$ and with $f_1, f_2 \in C_0^0(\mathbb{R}^2, \mathbb{R})$. Given an arbitrary input $u \in AC(\mathbb{R}_+, \mathbb{R})$ and initial condition $y_0 \in \mathbb{R}$, the existence and uniqueness of $y \in AC(\mathbb{R}_+, \mathbb{R})$ satisfying (3) at almost every $t \in [0, T]$ with $T > 0$ is studied in [9] and [11] and guaranteed when $f_1$ and $f_2$ satisfy

$$(f_1(v, \gamma_1) - f_1(v, \gamma_2)) (\gamma_1 - \gamma_2) \leq \lambda_1(u)(\gamma_1 - \gamma_2)^2$$

(4)

$$(f_2(v, \gamma_1) - f_2(v, \gamma_2)) (\gamma_1 - \gamma_2) \geq -\lambda_2(u)(\gamma_1 - \gamma_2)^2$$

(5)

for every $v, \gamma_1, \gamma_2 \in \mathbb{R}$ and some for non-negative functions $\lambda_1, \lambda_2 \in C(\mathbb{R}, \mathbb{R}_+)$.

An important property of the Duhem operator $\Phi$ as defined in (3) is that it is rate-independent. In other words, for every $\phi \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that $\phi(0) = 0$, increasing and radially unbounded (i.e., $\phi(t) \to \infty$ as $t \to \infty$) we have

$$[\Phi(u \circ \phi, y_0)](t) = [\Phi(u, y_0) \circ \phi](t).$$

Moreover, following the work of [11], we consider hysteresis operator that satisfies the semigroup property, which means that if $y = \Phi(u, y_0)$ then

$$\mathcal{S}_\tau(\Phi(u, y_0)) = \Phi(\mathcal{S}_\tau(u), \mathcal{S}_\tau(y)).$$

---

1 A periodic signal is called simple if it admits only one maximum and one minimum within its periodic interval.
Throughout this work, we assume that the implicit function \( v \mapsto \{ \gamma \in \mathbb{R} \mid f_1(v, \gamma) - f_2(v, \gamma) = 0 \} \) admits an explicit solution
\[
\gamma = \alpha(v)
\]
with \( \alpha \in C^0(\mathbb{R}, \mathbb{R}) \), which we call the anhysteresis function and the corresponding curve (generated by \( \alpha \)) given by
\[
A = \{(v, \gamma) \in \mathbb{R}^2 \mid \gamma = \alpha(v)\}
\]
is called the anhysteresis curve. By definition, the curve \( A \) divides the input–output plane into two regions where \( f_1(v, \gamma_1) - f_2(v, \gamma_1) \geq 0 \) whenever \( \gamma_1 \geq \gamma = \alpha(v) \), and \( f_1(v, \gamma_1) - f_2(v, \gamma_1) \leq 0 \) whenever \( \gamma_1 \leq \gamma = \alpha(v) \).

### III. Duhem Operator Accommodation Property

As briefly described in Section I, the accommodation property of the Duhem operator \( \Phi \) refers to the property where the input–output phase plot always converges to a periodic closed orbit when the input signal is periodic [18]. In this section, we formally study this property and prove that when the input is periodic with a single maximum and a single minimum in its periodic interval, the input–output phase plot approaches a unique periodic closed-loop which revolves in a neighborhood of the anhysteresis curve \( A \). We begin studying the input–output phase plot produced by the application of monotonic inputs and then we extend our analysis to periodic inputs. For simplicity of notation, in what follows we use \( Y_u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R} \), which we define by
\[
Y_u(t, y_0) := [\Phi(u, y_0)](t)
\]
to refer to the output of the Duhem operator \( \Phi \) parameterized by the input signal \( u \) and with the time \( t \) and initial condition \( y_0 \) as independent variables.

#### A. Duhem Operator With Monotonic Inputs

Let \( u_+ \in AC(\mathbb{R}^+, \mathbb{R}) \) be an input which is monotonically increasing in \([0, \infty)\) and consider a subinterval \([0, \tau_1]\) with \( \tau_1 > 0 \) such that \( u(0) = u_{\min} < u_{\max} = u(\tau_1) \). Since the Duhem operator \( \Phi \) defined with (3) is rate-independent as shown in [8], [11], and [19], for every \( t \in [0, \tau_1] \) we have that
\[
Y_{u_+}(t, y_0) - y_0 = \int_0^t f_1(u_+(t), Y_{u_+}(t, y_0)) \, d\tau = \int_{u_{\min}}^{u_+(t)} f_1(v, Y_{u_+}(v, y_0)) \, dv = Y_{u_+}(u_+(t), y_0) - Y_{u_+}(y_{\min}, y_0) \tag{8}
\]
where
\[
Y_{u_+} : [u_{\min}, u_{\max}] \times \mathbb{R} \to \mathbb{R}
\]
is the parameterization of the corresponding solution \( Y_{u_+}(t, y_0) \) with the instantaneous value of the input \( u_+ \) and the initial condition \( y_0 \) as independent variables (i.e., \( Y_{u_+}(u_+(t), y_0) = Y_{u_+}(t, y_0) \) for every \( t \in [0, \tau_1] \)).

Analogously, let \( u_- \in AC(\mathbb{R}^+, \mathbb{R}) \) be an input which is monotonically decreasing in \([0, \infty)\) and consider a subinterval \([0, \tau_2]\) with \( \tau_2 > 0 \) such that \( u(0) = u_{\max} > u_{\min} = u(\tau_2) \). By the rate-independent property of the Duhem operator, we have that, for every \( t \in [0, \tau_2] \),
\[
Y_{u_-}(t, y_0) - y_0 = \int_0^t f_2(u_-(\tau), Y_{u_-}(\tau, y_0)) \, d\tau = \int_{u_{\max}}^{u_-(t)} f_2(v, Y_{u_-}(v, y_0)) \, dv = Y_{u_-}(u_-(t), y_0) - Y_{u_-}(u_{\max}, y_0) \tag{9}
\]
where in this case
\[
Y_{u_-} :! [u_{\min}, u_{\max}] \times \mathbb{R} \to \mathbb{R}
\]
is the parameterization of the corresponding solution \( Y_{u_-}(t, y_0) \) with the instantaneous value of the input \( u_- \) and the initial condition \( y_0 \) as independent variables (i.e., \( Y_{u_-}(u_-(t), y_0) = Y_{u_-}(t, y_0) \) for every \( t \in [0, \tau_2] \)).

In what follows, we present a series of auxiliary lemmas necessary to prove the accommodation property. First, using the parameterizations \( Y_{u_+} \) and \( Y_{u_-} \) of the output, we state formally in the first lemma that two input–output phase plots obtained with the same monotonic input but from different initial conditions never cross each other.

**Lemma 3.1:** The next statements are true.

a) If two initial conditions satisfy \( \gamma_a \leq \gamma_b \), then we have
\[
Y_{u_+}(v, \gamma_a) \leq Y_{u_+}(v, \gamma_b)
\]
and
\[
Y_{u_-}(v, \gamma_a) \leq Y_{u_-}(v, \gamma_b)
\]
for every \( v \in [u_{\min}, u_{\max}] \).

b) If we have that
\[
Y_{u_+}(v, \gamma_a) < Y_{u_+}(v, \gamma_b)
\]
(resp. \( Y_{u_-}(v, \gamma_a) < Y_{u_-}(v, \gamma_b) \))
then
\[
Y_{u_+}(v, \gamma_a) < Y_{u_+}(v, \gamma_b)
\]
(resp. \( Y_{u_-}(v, \gamma_a) < Y_{u_-}(v, \gamma_b) \))
for every \( v \in [u_{\min}, u_{\max}] \).

**Proof:** In what follows, we prove both statements for \( Y_{u_+} \) by contradiction. The proof for \( Y_{u_-} \) can be obtained in a similar way by vis-a-vis arguments and it is omitted.

**Part a)** Let \( \gamma_a \leq \gamma_b \) and assume that \( Y_{u_+}(v, \gamma_a) > Y_{u_+}(v, \gamma_b) \) for some \( v \in [u_{\min}, u_{\max}] \).

Let \( \tau_c \in [0, \tau_1] \) be a time instance such that \( Y_{u_+}(v, \gamma_a) \geq Y_{u_+}(\tau_c, \gamma_a) = Y_{u_+}(v, \gamma_a) \) for \( v \in [u_{\min}, u_{\max}] \).

By continuity of \( Y_{u_+} \), there exists \( v_c \in [u_{\min}, v_c] \) such that \( Y_{u_+}(v_c, \gamma_a) = Y_{u_+}(v_c, \gamma_a) \) [see Fig. 1(a)]. Let \( \tau_c \in [0, \tau_c] \) be a time instance such that \( \gamma_x = Y_{u_+}(\tau_x, \gamma_a) = Y_{u_+}(v_c, \gamma_a) \).

We can create a right-shifted input \( u_+ = S_{\tau_x}(u_+) \) and note that by the semigroup property of the Duhem operator we must
have that
\[ Y_{u_+}(t + \tau_x, \gamma_a) = Y_{u_+}(t, \gamma_x) = Y_{u_+}(t + \tau_x, \gamma_b) \]
for every \( t \in [0, \tau_1 - \tau_x] \), which implies a contradiction to the uniqueness of solution \( Y_{u_+} \) since
\[ Y_{u_+}(\tau_x, \gamma_a) > Y_{u_+}(\tau_x, \gamma_b). \]
Therefore, \( y_{u_+}(v, \gamma_a) \leq y_{u_+}(v, \gamma_b) \) for every \( v \in [v_{\min}, v_{\max}] \).

**Part b** Let \( y_{u_+}(v_{\max}, \gamma_a) < y_{u_+}(v_{\max}, \gamma_b) \) and assume that \( \gamma_x = y_{u_+}(v_x, \gamma_a) = y_{u_+}(v_x, \gamma_b) \) for some \( v_x \in [v_{\min}, v_{\max}] \) [see Fig. 1(b)]. Letting \( \tau_x \in [0, \tau_1) \) be a time instance such that \( \gamma_x = Y_{u_+}(\tau_x, \gamma_a) = Y_{u_+}(\tau_x, \gamma_b) \) and creating right-shifted input \( u_+ = S_{\tau_x}(u_+) \), we can obtain the same contradiction as in **Part a**. Therefore, \( y_{u_+}(v, \gamma_a) < y_{u_+}(v, \gamma_b) \) for every \( v \in [v_{\min}, v_{\max}] \).  

**B. Duhem Operator With a Periodic Input**

We can now analyze the behavior of the Duhem operator \( \Phi \) when the applied input signal is periodic. For this, let \( u_p \in AC(\mathbb{R}_+, \mathbb{R}) \) be a periodic input with period \( T > 0 \) and with one minimum and one maximum \( v_{\min} < v_{\max} \) in its periodic interval. Without loss of generality, we assume that \( u_p(0) = v_{\min} \) and \( u_p(t_1) = v_{\max} \) for some \( t_1 \in (0, T) \). In other words, \( 0 < t_1 < T \) is a monotonic partition of \([0, T]\). We can split \( u_p \) into its two monotonic intervals using the right-shift and right-continuation operators (1) and (2), which are formalized using two functions \( u_{p+}, u_{p-} \in AC(\mathbb{R}_+, \mathbb{R}) \) given by
\[
\begin{align*}
  u_{p+} &= \mathcal{R}_{t_1}(u_p) \\
  u_{p-} &= S_{t_1}(\mathcal{R}_T(u_p))
\end{align*}
\]
whose corresponding outputs when applied to the Duhem operator are given by \( Y_{u_{p+}}(t, \gamma) \) and \( Y_{u_{p-}}(t, \zeta) \) for some initial conditions \( \gamma, \zeta \in \mathbb{R} \).

Following the same argumentation as before to obtain (8) and (9), we can parameterize \( Y_{u_{p+}}(t, \gamma) \) and \( Y_{u_{p-}}(t, \zeta) \) by two mappings
\[
\begin{align*}
  y_{u_{p+}} : [v_{\min}, v_{\max}] \times \mathbb{R} &\rightarrow \mathbb{R} \\
  y_{u_{p-}} : [v_{\min}, v_{\max}] \times \mathbb{R} &\rightarrow \mathbb{R}
\end{align*}
\]
respectively, where the instantaneous values of the inputs \( u_{p+} \) and \( u_{p-} \), and initial conditions \( \gamma \) and \( \zeta \) are the independent variables.

For arbitrary \( \gamma \in \mathbb{R} \), let us define two sequences \( (\zeta_n)_{n \in \mathbb{N}_0} \) and \( (\gamma_n)_{n \in \mathbb{N}_0} \) recursively by
\[
\begin{align*}
  \zeta_n &= y_{u_{p+}}(v_{\max}, \gamma_n) \\
  \gamma_{n+1} &= y_{u_{p-}}(v_{\min}, \zeta_n).
\end{align*}
\]
Note then that making \( \gamma_0 = y_0 \), the output \( Y_{u_+}(t, y_0) \) can be constructed recursively by
\[
Y_{u_+}(t, y_0) = \begin{cases} 
  y_{u_{p+}}(u_+(t), \gamma_n), & \text{if } n \leq t < t_1 + nT \\
  y_{u_{p-}}(u_-(t), \gamma_n), & \text{if } t_1 + nT \leq t < (n + 1)T
\end{cases}
\]
where \( n \in \mathbb{N}_0 \) and \( t_1 > 0 \) is such that \( 0 < t_1 < T \) is a monotonic partition of \([0, T]\). Therefore, we study the convergence of the solution \( Y_{u_+} \) to a periodic solution using these sequences.

The following three lemmas present properties of the sequences \( (\zeta_n)_{n \in \mathbb{N}_0} \) and \( (\gamma_n)_{n \in \mathbb{N}_0} \) generated by the recursive composition of the function \( y_{u_{p+}} \) and \( y_{u_{p-}} \), that will be used in the proof of the main result of this section.

**Lemma 3.2**: Let \( \gamma_0 \in \mathbb{R} \). The sequences \( (\zeta_n)_{n \in \mathbb{N}_0} \) and \( (\gamma_n)_{n \in \mathbb{N}_0} \) generated by (10) and (11) are monotonic in the same direction (i.e., both increasing or both decreasing).

**Proof**: By induction, let \( \gamma_i \geq \gamma_{i+1} \) and note that by Lemma 3.1, we have
\[
\begin{align*}
  \zeta_i &= y_{u_{p+}}(v_{\max}, \gamma_i) \geq y_{u_{p+}}(v_{\max}, \gamma_{i+1}) = \zeta_{i+1} \\
  \gamma_i &= y_{u_{p-}}(v_{\min}, \zeta_i) \geq y_{u_{p-}}(v_{\min}, \zeta_{i+1}) = \gamma_{i+1}
\end{align*}
\]
which proves that both sequences are increasing. Reversing all previous inequalities proves that both sequences are decreasing.

**Lemma 3.3**: Let \( \gamma_0 \in \mathbb{R} \) and consider the sequences \( (\zeta_n)_{n \in \mathbb{N}_0} \) and \( (\gamma_n)_{n \in \mathbb{N}_0} \) generated by (10) and (11). Then, the next statements are true.

a) If \( \gamma_i = \gamma_{i+1} \) for some \( i \in \mathbb{N}_0 \) then for every \( k \geq i \) we have
\[
\zeta_k = \zeta_{k+1} \quad \text{and} \quad \gamma_k = \gamma_{k+2}.
\]

b) If \( \zeta_i = \zeta_{i+1} \) for some \( j \in \mathbb{N}_0 \) then for every \( k \geq j \) we have
\[
\gamma_{k+1} = \gamma_{k+2} \quad \text{and} \quad \zeta_{k+1} = \zeta_{k+2}.
\]

**Proof**: Let \( \gamma_i = \gamma_{i+1} \) and note that the uniqueness of solution \( Y_{u_{p+}} \) implies \( y_{u_{p+}}(v, \gamma_i) = y_{u_{p+}}(v, \gamma_{i+1}) \) for every \( v \in [v_{\min}, v_{\max}] \), and consequently \( \zeta_i = y_{u_{p+}}(v_{\max}, \gamma_i) = y_{u_{p+}}(v_{\max}, \gamma_{i+1}) = \zeta_{i+1} \). Therefore, we have that
\[
\gamma_i = \gamma_{i+1} \quad \Rightarrow \quad \zeta_i = \zeta_{i+1}.
\]
Similarly, when \( \zeta_j = \zeta_{j+1} \), the uniqueness of solution \( Y_{u_{p-}} \) implies \( y_{u_{p-}}(v, \zeta_j) = y_{u_{p-}}(v, \zeta_{j+1}) \) for every \( v \in [v_{\min}, v_{\max}] \), and consequently \( \gamma_{j+1} = y_{u_{p-}}(v_{\min}, \zeta_j) = y_{u_{p-}}(v_{\min}, \zeta_{j+1}) = \gamma_{j+2} \). Thus, we have that
\[
\zeta_j = \zeta_{j+1} \quad \Rightarrow \quad \gamma_{j+1} = \gamma_{j+2}.
\]
It follows that combining both implications proves both statements.

**Lemma 3.4**: Let \( \gamma_0 \in \mathbb{R} \) and consider the sequences \( (\zeta_n)_{n \in \mathbb{N}_0} \) and \( (\gamma_n)_{n \in \mathbb{N}_0} \) generated by (10) and (11). The sequence \( (\zeta_n)_{n \in \mathbb{N}_0} \) is unbounded if and only if \( (\gamma_n)_{n \in \mathbb{N}_0} \) is unbounded. Moreover, when they are unbounded, they are strictly monotonic.
in the same direction (i.e., both strictly increasing or both strictly decreasing).

**Proof:** To prove the if part, let \((\gamma_n)_{n \in \mathbb{N}_0}\) be unbounded and note that by Lemma 3.2 we have that both sequences \((\gamma_n)_{n \in \mathbb{N}_0}\) and \((\zeta_n)_{n \in \mathbb{N}_0}\) are monotonic in the same direction. Moreover, by Lemma 3.3, assuming that \(\gamma_i = \gamma_{i+1}\) or \(\zeta_j = \zeta_{j+1}\) for some \(i, j \in \mathbb{N}_0\) implies that \((\gamma_n)_{n \in \mathbb{N}_0}\) is not unbounded, which is a contradiction. Therefore, \((\zeta_n)_{n \in \mathbb{N}_0}\) is also unbounded and both are strictly monotonic.

To prove the only if part, let \((\zeta_n)_{n \in \mathbb{N}_0}\) be unbounded and note that by Lemma 3.2, we have that both sequences \((\gamma_n)_{n \in \mathbb{N}_0}\) and \((\zeta_n)_{n \in \mathbb{N}_0}\) are monotonic in the same direction. Moreover, by Lemma 3.3, assuming that \(\gamma_i = \gamma_{i+1}\) or \(\zeta_j = \zeta_{j+1}\) for some \(i, j \in \mathbb{N}_0\) implies that \((\zeta_n)_{n \in \mathbb{N}_0}\) is not unbounded, which is a contradiction. Therefore, \((\gamma_n)_{n \in \mathbb{N}_0}\) is also unbounded and both are strictly monotonic.

In the next two propositions, we introduce the main results of this section where sufficient conditions are presented such that the sequences \((\gamma_n)_{n \in \mathbb{N}_0}\) and \((\zeta_n)_{n \in \mathbb{N}_0}\) generated by (10) and (11) are convergent. We remark that if the sequences \((\gamma_n)_{n \in \mathbb{N}_0}\) and \((\zeta_n)_{n \in \mathbb{N}_0}\) are convergent to some pair \(\gamma \in \mathbb{R}\) and \(\zeta \in \mathbb{R}\), respectively, then due to the continuity and uniqueness of solutions of the Duhem operator we must have that

\[
y_{u^+}(v, \gamma_n) = \zeta_n \quad \text{and} \quad y_{u^-}(v, \gamma_n) = \gamma_n
\]

and consequently both parameterized solutions \(y_{u^+}(v, \gamma_n)\) and \(y_{u^-}(v, \gamma_n)\) form a periodic closed orbit in the phase plot. With the first proposition we present a pair of inequalities that ensure the convergence to some periodic orbit. These inequalities have been previously presented in [20] and are used together with other sets of conditions to prove the convergence of the output to a periodic function for a specific version of the Duhem model known as Babuška’s model. We show that only these two conditions are sufficient to ensure the convergence of the output to a periodic function in the scalar rate-independent Duhem model. Subsequently, with the second proposition we show that the strict versions of the inequalities ensure the uniqueness of the pair \(\gamma \in \mathbb{R}\) and \(\zeta \in \mathbb{R}\) and consequently the uniqueness of the closed periodic orbit.

**Proposition 3.5:** If the functions \(f_1\) and \(f_2\) in (3) satisfy

\[
(f_1(v, \gamma) - f_1(v, \gamma_2)) (\gamma - \gamma_2) \leq 0 \quad \text{(12)}
\]

\[
(f_2(v, \gamma) - f_2(v, \gamma_2)) (\gamma - \gamma_2) \geq 0 \quad \text{(13)}
\]

for every \(\gamma \neq \gamma_2\) and \(v \in \mathbb{R}\), then for every \(\gamma_0 \in \mathbb{R}\) the sequences \((\gamma_n)_{n \in \mathbb{N}_0}\) and \((\gamma_n)_{n \in \mathbb{N}_0}\) generated by (10) and (11) are convergent.

Prior to proving the proposition above, we need the following technical lemma.

**Lemma 3.6:** Suppose that (12) and (13) hold with divergent sequences \((\gamma_n)_{n \in \mathbb{N}_0}\) and \((\gamma_n)_{n \in \mathbb{N}_0}\) generated by (10) and (11), both of which are unbounded and strictly monotonic in the same direction according to Lemma 3.4. Then, for fixed \(v \in [v_{\min}, v_{\max}]\), the sequences

\[
y_{u^+}(v, \gamma_n) \quad \text{and} \quad y_{u^-}(v, \zeta_n)
\]

are also unbounded and strictly monotonic in the same direction as the sequences \((\gamma_n)_{n \in \mathbb{N}_0}\) and \((\zeta_n)_{n \in \mathbb{N}_0}\).

**of Lemma 3.6:** From (8), we can obtain

\[
y_{u^+}(v, \gamma) - y_{u^-}(v, \gamma_{i-1}) = \int_{v_{\min}}^{v_{\max}} \left\{ f_1(v, y_{u^+}(v, \gamma)) - f_1(v, y_{u^-}(v, \gamma_{i-1})) \right\} d\nu + \gamma_i - \gamma_{i-1}.
\]

Moreover, we have that

\[
\xi_i = \int_{v_{\min}}^{v_{\max}} f_1(v, y_{u^+}(v, \gamma)) d\nu + \gamma_i.
\]

Therefore, we can solve the previous expression for \(\gamma_i\) and \(\gamma_{i-1}\) and replace both into (14) to obtain

\[
y_{u^+}(v, \gamma) - y_{u^-}(v, \gamma_{i-1}) = -\int_{v}^{v_{\max}} \left\{ f_1(v, y_{u^+}(v, \gamma)) - f_1(v, y_{u^-}(v, \gamma_{i-1})) \right\} d\nu + \xi_i - \xi_{i-1}.
\]

and by (12), the integrand in the previous expression is negative or zero when \(\gamma_i > \gamma_{i-1}\) and positive or zero when \(\gamma_i < \gamma_{i-1}\). Consequently, for every \(v \in [v_{\min}, v_{\max}]\) we have

\[
y_{u^+}(v, \gamma) \geq \xi_i - \xi_{i-1} + y_{u^-}(v, \gamma_{i-1}), \text{ when } \gamma_i > \gamma_{i-1}
\]

\[
y_{u^+}(v, \gamma) \leq \xi_i - \xi_{i-1} + y_{u^-}(v, \gamma_{i-1}), \text{ when } \gamma_i < \gamma_{i-1}
\]

and fixing the terms \(\gamma_{i-1}\) and \(\xi_{i-1}\), then it follows that \(y_{u^+}(v, \gamma)\) is unbounded in the same direction than \(\xi_i\).

In a similar way, from (9), we can obtain

\[
y_{u^-}(v, \gamma) - y_{u^-}(v, \zeta_{i-1}) = -\int_{v}^{v_{\max}} \left\{ f_2(v, y_{u^-}(v, \gamma)) - f_2(v, y_{u^-}(v, \zeta_{i-1})) \right\} d\nu + \xi_i - \xi_{i-1}.
\]

Again, note that we have

\[
\gamma_i = -\int_{v_{\min}}^{v_{\max}} f_2(v, y_{u^-}(v, \zeta_{i-1})) d\nu + \xi_i - \xi_{i-1}.
\]

Therefore, we can solve the previous expression for \(\xi_{i-1}\) and \(\xi_i\) and replace both into (13) to obtain

\[
y_{u^-}(v, \gamma) \geq \xi_{i-1} - \xi_i + y_{u^-}(v, \zeta_{i-1}), \text{ when } \xi_i > \xi_{i-1}
\]

and by (13), the integrand in the previous expression is positive or zero when \(\xi_i > \xi_{i-1}\) and negative or zero when \(\xi_i < \xi_{i-1}\). Consequently, for every \(v \in [v_{\min}, v_{\max}]\) we have

\[
y_{u^-}(v, \zeta) \geq \gamma_{i-1} + y_{u^-}(v, \zeta_{i-1}), \text{ when } \gamma_{i-1} > \zeta_{i-1}
\]
\( y_{u_{p,-}}(v, \zeta) \leq \gamma_i+1 - \gamma_i + y_{u_{p,-}}(v, \zeta_{i-1}) \) when \( \zeta_i < \zeta_{i-1} \) and fixing the terms \( \gamma_i \) and \( \zeta_{i-1} \), then it follows that \( y_{u_{p,-}}(v, \zeta) \) is unbounded in the same direction than \( \gamma_i+1 \).

**Proof of Proposition 3.5**

It follows from (8) and (9) that the difference between two consecutive elements in \( (\gamma_n)_{n \in \mathbb{N}_0} \) is given by

\[
\gamma_i+1 - \gamma_i = \int_{v_{\min}}^{v_{\max}} \left\{ f_1(v, y_{u_{p,\bar{p}}} (v, \gamma_i)) - f_2(v, y_{u_{p,-}} (v, \zeta)) \right\} \, dv.
\]

Moreover, since by the definition of anhysteresis function \( \alpha \), we have \( f_1(v, \alpha(v)) = f_2(v, \alpha(v)) \) for every \( v \in [v_{\min}, v_{\max}] \), then we can add and subtract these terms inside the integral and obtain

\[
\gamma_i+1 - \gamma_i = \int_{v_{\min}}^{v_{\max}} \left\{ f_1(v, y_{u_{p,\bar{p}}} (v, \gamma_i)) - f_1(v, \alpha(v)) \right\} \, dv
- \int_{v_{\min}}^{v_{\max}} \left\{ f_2(v, y_{u_{p,-}} (v, \zeta)) - f_2(v, \alpha(v)) \right\} \, dv.
\]

(16)

We prove the proposition by contradiction. Assume that any of the sequences \( (\zeta_n)_{n \in \mathbb{N}_0} \) or \( (\gamma_n)_{n \in \mathbb{N}_0} \) is not convergent. Thus, by Lemmas 3.2–3.4 both are unbounded and strictly monotonic in the same direction.

On the one hand, if both are strictly increasing, then by Lemmas 3.1 and 3.6, we can find two pairs \( \gamma_i > \gamma_{i+1} \) and \( \zeta_i < \zeta_{i+1} \) such that both \( y_{u_{p,\bar{p}}} (v, \gamma_i) \) and \( y_{u_{p,-}} (v, \zeta_i) \) lie completely below the anhysteresis curve \( A \) [see Fig. 2(b)] and we have

\[
y_{u_{p,\bar{p}}} (v, \gamma_i) < \alpha(v) \quad \text{and} \quad y_{u_{p,-}} (v, \zeta_i) < \alpha(v)
\]

for every \( v \in [v_{\min}, v_{\max}] \) and some \( i \in \mathbb{N}_0 \). Similar as before, from (12) and (13), we have that

\[
f_1(v, y_{u_{p,\bar{p}}} (v, \gamma_i)) - f_1(v, \alpha(v)) \leq 0
- \int_{v_{\min}}^{v_{\max}} \left\{ f_2(v, y_{u_{p,-}} (v, \zeta)) - f_2(v, \alpha(v)) \right\} \, dv
\]

(17)

for every \( v \in [v_{\min}, v_{\max}] \). Consequently, the right term of (17) is positive or zero, which is a contradiction since by assumption the sequence is strictly decreasing and \( \gamma_i+1 - \gamma_i < 0 \) for every \( i \in \mathbb{N}_0 \).

Therefore, both sequences are bounded, and since by Lemma 3.2 they are monotonic, then they are convergent.

**Proposition 3.7:** If the functions \( f_1 \) and \( f_2 \) in (3) satisfy the strict version of inequalities (12) and (13) given by

\[
(f_1(v, y_{u_{p,\bar{p}}} (v, \gamma_i)) - f_1(v, \alpha(v))) < 0
(f_2(v, y_{u_{p,-}} (v, \zeta)) - f_2(v, \alpha(v))) \leq 0
\]

(18)

(19)

for every \( v \in [v_{\min}, v_{\max}] \). Consequently, the right term of (17) is positive (resp. negative) when \( \gamma_i < \gamma_{i+1} \) and \( \zeta_i > \zeta_{i+1} \), respectively. Then, by (18) and (19) and Lemma 3.1, the right term of the last expression is positive (resp. negative) when \( \gamma_i < \gamma_{i+1} \) and \( \zeta_i > \zeta_{i+1} \), respectively. The conditions are presented in the proposition as follows.

**Proposition 3.8:** If the functions \( f_1 \) and \( f_2 \) in the Duhem model (3) satisfy the reversed inequalities to (18) and (19), which
are given by

\[
(f_1(v, \gamma_1) - f_1(v, \gamma_2)) (\gamma_1 - \gamma_2) > 0 \tag{21}
\]

\[
(f_2(v, \gamma_1) - f_2(v, \gamma_2)) (\gamma_1 - \gamma_2) < 0 \tag{22}
\]

for every \( \gamma_1 \neq \gamma_2 \) and \( v \in \mathbb{R} \), then for every \( \gamma_0 \in \mathbb{R} \) such that

\[
\int_{v_{\min}}^{v_{\max}} \left\{ f_1 (\nu, Y_{u^+}(\nu, \gamma_0)) - f_2 (\nu, Y_{u^-}(\nu, \gamma_0)) \right\} d\nu 
eq 0 \tag{23}
\]

with \( \zeta_0 = Y_{u^+}(v_{\max}, \gamma_0) \), the sequences \( (\zeta_n)_{n \in \mathbb{N}_0} \) and \( (\gamma_n)_{n \in \mathbb{N}_0} \) generated by (10) and (11) are divergent.

**Proof:** Note that when (23) does not holds, then from (16) we have \( \gamma_1 - \gamma_0 = 0 \) and by Lemma 3.3 all the terms of both sequences are equal.

Therefore, let (23) hold. We prove the case when both sequences are increasing and unbounded. Assume that \( \gamma_i > \gamma_{i-1} \). Replacing \( v = v_{\max} \) into (14), we can obtain

\[
\zeta_i = \zeta_{i-1} - \int_{v_{\min}}^{v_{\max}} \left\{ f_1 (\nu, Y_{u^+}(\nu, \gamma_i)) - f_1 (\nu, Y_{u^+}(\nu, \gamma_{i-1})) \right\} d\nu + \gamma_i - \gamma_{i-1}
\]

where the integral is positive following (21) and Lemma 3.1. Therefore, we obtain \( \zeta_i - \zeta_{i-1} > \gamma_i - \gamma_{i-1} > 0 \). In a similar form, replacing \( v = v_{\min} \) into (15), we have

\[
\gamma_{i+1} - \gamma_i = - \int_{v_{\min}}^{v_{\max}} \left\{ f_2 (\nu, Y_{u^-}(\nu, \zeta_i)) - f_2 (\nu, Y_{u^-}(\nu, \zeta_{i-1})) \right\} d\nu + \zeta_i - \zeta_{i-1},
\]

where the integral is negative following from (22) and Lemma 3.1. Thus, we obtain \( \gamma_{i+1} - \gamma_i > \zeta_i - \zeta_{i-1} \). Combining both inequalities, we obtain \( \gamma_{i+1} - \gamma_i > \gamma_i - \gamma_{i-1} > 0 \), and consequently the sequence \( (\gamma_n)_{n \in \mathbb{N}_0} \) is increasing and unbounded. Moreover, by Lemma 3.4 \( (\zeta_n)_{n \in \mathbb{N}_0} \) is also increasing and unbounded.

Similar argumentation holds for the case when both sequences are decreasing and unbounded by reversing all inequalities above.

**C. Case Example: The Bouc–Wen Model**

We use now the propositions presented in this section to study a particular case of the Bouc–Wen hysteresis model [32], [33], [34]. The Bouc–Wen model is commonly used to describe relations between displacement and restoring force as input and output in piezoactuated mechanical systems and it is defined by

\[
y(t) = \alpha \ddot{u}(t) - \beta |y(t)|^{n-1} \dddot{u}(t) - \zeta y(t) |y(t)|^{n-1} |\dot{u}(t)|
\]

where \( \alpha, \beta, \zeta \in \mathbb{R} \) are model parameters. The equation above can be also written as a Duhem model of the form (3) whose vector field functions \( f_1 \) and \( f_2 \) are defined by

\[
f_1(v, \gamma) := \alpha - \beta |\gamma|^n - \zeta \gamma |\gamma|^{n-1}
\]

\[
f_2(v, \gamma) := \alpha - \beta |\gamma|^n + \zeta \gamma |\gamma|^{n-1}.
\]

Using these \( f_1 \) and \( f_2 \) into (12) and (13) of Proposition 3.5, we have

\[
\begin{align*}
[(\beta + \zeta \text{sign} (\gamma_1)) |\gamma_1|^n - (\beta + \zeta \text{sign} (\gamma_2)) |\gamma_2|^n] (\gamma_1 - \gamma_2) & \geq 0 \\
[(\beta - \zeta \text{sign} (\gamma_1)) |\gamma_1|^n - (\beta - \zeta \text{sign} (\gamma_2)) |\gamma_2|^n] (\gamma_1 - \gamma_2) & \leq 0.
\end{align*}
\]

Assuming without loss of generality that \( \gamma_1 > \gamma_2 \), we obtain

\[
\begin{align*}
[(\beta + \zeta \text{sign} (\gamma_1)) |\gamma_1|^n - (\beta + \zeta \text{sign} (\gamma_2)) |\gamma_2|^n] & \geq 0 \tag{24} \\
[(\beta - \zeta \text{sign} (\gamma_1)) |\gamma_1|^n - (\beta - \zeta \text{sign} (\gamma_2)) |\gamma_2|^n] & \leq 0. \tag{25}
\end{align*}
\]

Note now that when \( \gamma_1 > \gamma_2 \geq 0 \) or \( \gamma_1 > \gamma_2 \), we can reduce (24) and (25) to

\[
\begin{align*}
(\beta + \zeta) (|\gamma_1|^n - |\gamma_2|^n) & \geq 0 \\
(\beta - \zeta) (|\gamma_1|^n - |\gamma_2|^n) & \leq 0
\end{align*}
\]

respectively, which are trivially satisfied when

\[
\begin{align*}
\beta + \zeta & \geq 0 \tag{26} \\
\beta - \zeta & \leq 0. \tag{27}
\end{align*}
\]

Moreover, when \( \gamma_1 > 0 > \gamma_2 \), we have

\[
\begin{align*}
\beta (|\gamma_1|^n - |\gamma_2|^n) + \zeta (|\gamma_1|^n + |\gamma_2|^n) & \geq 0 \\
\beta (|\gamma_1|^n - |\gamma_2|^n) - \zeta (|\gamma_1|^n + |\gamma_2|^n) & \leq 0
\end{align*}
\]

which are also satisfied for (26) and (27).

Therefore, the sequences defined by (10) and (11) are convergent for every initial value \( \gamma_0 \in \mathbb{R} \), or equivalently, the input–output phase plot of the Bouc–Wen model will converge to a periodic orbit from every initial point when conditions in (26) and (27) are satisfied. In fact, it can be checked that these conditions are equivalent to the ones presented in [32, Table 1] corresponding to BIBO stable Bouc–Wen models of class I, III, and V. As an illustrative example, the input–output phase plot of a Bouc–Wen model whose parameters satisfy the convergence conditions in (26) and (27) with \( \alpha = 1, \beta = 1, \) and \( \zeta = 2 \) is shown in Fig. 3.

Conversely, based on Proposition 3.8, when we have the reversed inequalities

\[
\beta + \zeta < 0 \tag{28}
\]
\[ f(29) \]
\[ \gamma \text{ and } \{0 \implies B_2 \} \]
\[ \beta - \zeta > 0. \]  

then the sequences defined by (10) and (11) will diverge which means that the input–output phase plot Bouc–Wen model will not exhibit a hysteresis loop. An example of this case is illustrated in Fig. 4 with the divergent input–output phase plot of a Bouc–Wen model whose parameters are \( \alpha = 0.1, \beta = 0.1, \) and \( \zeta = -0.2. \)

### IV. DUHEM BUTTERFLY MODEL

In this section, we introduce a special class of Duhem operators, which we call the Duhem butterfly operators. This operator is characterized by its capability in producing complex periodic hysteresis loops with self-intersections. In this class of operators both functions \( f_1 \) and \( f_2 \) in (3) can assume positive and negative values as long as they satisfy the conditions (18) and (19), respectively, to guarantee the existence of a unique periodic solution.

We assume now that the implicit functions \( v \mapsto \{ \gamma \mid f_1(v, \gamma) = 0 \} \) and \( v \mapsto \{ \gamma \mid f_2(v, \gamma) = 0 \} \) admit explicit solutions

\[
\gamma = c_1(v) \quad \text{and} \quad \gamma = c_2(v)
\]

respectively, with \( c_1, c_2 \in AC(\mathbb{R}, \mathbb{R}) \) such that \( f_1(v, c_1(v)) = 0 \) and \( f_2(v, c_2(v)) = 0 \) for every \( v \in \mathbb{R}. \) In other words, the curves described by \( c_1 \) and \( c_2 \) are the zero level set of the functions \( f_1 \) and \( f_2, \) respectively. Note that by conditions (18) and (19), each of the curves \( c_1 \) and \( c_2 \) split the input–output plane \( u - y \) into two regions such that

\[
\begin{align*}
\text{if } & f_1(v, \gamma) < 0 \text{ whenever } \gamma > c_1(v) \\
\text{if } & f_1(v, \gamma) > 0 \text{ whenever } \gamma < c_1(v) \\
\text{if } & f_2(v, \gamma) > 0 \text{ whenever } \gamma > c_2(v) \text{ and} \\
\text{if } & f_2(v, \gamma) < 0 \text{ whenever } \gamma < c_2(v).
\end{align*}
\]

In the following, we will prove that when the functions \( f_1 \) and \( f_2, \) and the zero-level set functions \( c_1 \) and \( c_2 \) satisfy some mild assumptions, there is a periodic hysteresis loop with a self-intersection which gives the existence of a butterfly hysteresis loop. Prior to this, we need to introduce the following notations.

Let \( u_+, u_- \in AC(\mathbb{R}^+, \mathbb{R}) \) be inputs which are monotonically increasing and decreasing, respectively, and radially unbounded, i.e., \( u_+(t) \to \infty \) as \( t \to \infty, \) respectively.

Similarly to (8) and (9), we define the solutions of the Duhem model (3) parameterized by the instantaneous value of the inputs \( u_+ \) and \( u_- \) by \( \gamma_{u_+} \) and \( \gamma_{u_-} \), respectively. The next lemma shows, under mild assumptions on the functions \( c_1 \) and \( c_2, \) the positive invariance of the region below the curves \( c_1 \) and \( c_2 \) with respect to the solutions of \( \gamma_{u_+} \) and \( \gamma_{u_-} \), respectively.

**Lemma 4.1:** If \( 0 \leq \frac{dc_1(v)}{du} \leq L_1 \) for all \( v \geq u_+(0) \) then for all \( \gamma_0 \leq c_1(u^+(0)) \), \( \gamma_{u_+}(v, \gamma_0) \leq c_1(v) \) for all \( v \geq u_+(0) \).

Analogously, if \( -L_2 \leq \frac{dc_2(v)}{du} \leq 0 \) for all \( v \leq u_-(0) \) then for all \( \gamma_0 \leq c_2(u^-(0)) \), \( \gamma_{u_-}(v, \gamma_0) \leq c_2(v) \) for all \( v \leq u_-(0) \).

**Proof:** We prove now the first claim of the lemma. Let us define the domain under the curve \( c_1 \) as follows:

\[
\mathcal{E}_{c_1^+} := \{ (v, \gamma) \in \mathbb{R}^2 \mid \gamma \leq c_1(v) \}.
\]

It can be checked that \( \mathcal{E}_{c_1^+} \) is positively invariant with respect to the solutions of Duhem model (3) with monotonically increasing input \( u_+ \) and with initial conditions in \( \mathcal{E}_{c_1^+}. \) Indeed, for every point \( x \in \mathcal{E}_{c_1^+} \) we can construct the tangent cone to this set as defined in [35, Def. 3.1], which is given by

\[
\mathcal{T}_{\gamma_{u_+}}(x) = \left\{ z \in \mathbb{R}^2 : \liminf_{h \to 0} \frac{\text{dist}(x + h z, \mathcal{E}_{c_1^+})}{h} = 0 \right\}
\]

where we take \( \text{dist}(\cdot) \) to be the Euclidian distance from \( x \) to the closest point \( y \in \mathcal{E}_{c_1^+}. \) Let \( \nu_+(v_0) \in \mathbb{R}^2 \) be the tangent vector to the solution \( \gamma_{u_+} \) which is given by

\[
\nu_+(v_0) = \left( 1, \frac{dy}{dv} \Big|_{v=v_0} \gamma_{u_+}(v, c_1(v_0)) \right).
\]

Now, we show that \( \nu_+(v_0) \in \mathcal{T}_{\gamma_{u_+}}(x_1) \) with \( x_1 = (v_0, c_1(v_0)) \in \mathcal{E}_{c_1^+} \) for every \( v_0 \in \mathbb{R} \) so that the solutions of \( \gamma_{u_+} \) do not escape \( \mathcal{E}_{c_1^+} \) on the boundary [see Fig. 5(a)]. In other words, we show that the tangent vector to the solution \( \gamma_{u_+} \) belongs to the tangent cone of the set \( \mathcal{E}_{c_1^+} \) at every point of the boundary. For this let us consider a point \( w = (v_0 + h, c_1(v_0)) \) and note that since \( c_1(v) \) is monotonically increasing we have

\[ f_1(v, \gamma) < 0 \]
\[ c_1(v) \]
\[ c_2(v) \]
\[ f_2(v, \gamma) > 0 \]

\[ (a) \]
\[ (b) \]
\( w \in \mathcal{C}_1^+ \) for every \( h > 0 \). Then, we can check
\[
\lim_{h \to 0^+} \left\| x_1 + h \nu_\pm(v_0) - w \right\| = \lim_{h \to 0^+} \frac{h|f_1(v_0, c_1(v_0)) - |f_1(v_0, c_1(v_0))| = 0
\]
which proves that \( \nu_\pm(v_0) \in \mathcal{F}_{\mathcal{C}_1}(x_1) \). Consequently, following from the Nagumo’s theorem [35, Th. 3.1] the set \( \mathcal{C}_1^+ \) is positively invariant and \( y_{u_+}(v, \gamma_0) \leq c_1(v) \) for every \( v \geq v_0 \).

For proving the second claim of the lemma, we consider the set
\[
\mathcal{C}_2^- := \left\{ (v, \gamma) \in \mathbb{R}^2 \mid \gamma \leq c_2(v) \right\}
\]
which consists of all the points below the curve parameterized by \( \gamma = c_2(v) \). We let \( \nu_-(v_0) \in \mathbb{R}^2 \) be the tangent vector to the solution \( y_{u_-} \) given by
\[
\nu_-(v_0) = \left(-1, \frac{d}{dv} \bigg|_{v=v_0} y_{u_-}(v, c_2(v_0)) \right) = (-1, f_2(v_0, c_2(v_0))).
\]
In this case, we show that the tangent vector \( \nu_-(v_0) \) to the solution \( y_{u_-} \) belongs to the tangent cone of the set \( \mathcal{C}_2^- \) at every point of the boundary [see Fig. 5(b)]. We consider in this case a point \( w = (v_0 - h, c_2(v_0)) \) and note that since \( c_2(v) \) is monotonically decreasing we have \( w \in \mathcal{C}_2^- \) for every \( h > 0 \). Then, we can check analogously that
\[
\lim_{h \to 0^+} \left\| (x_2 + h \nu_-(v_0)) - w \right\| = \lim_{h \to 0^+} \frac{h|f_2(v_0, c_2(v_0)) - |f_2(v_0, c_2(v_0))| = 0
\]
which proves that \( \nu_-(v_0) \in \mathcal{F}_{\mathcal{C}_2}(x_2) \) and following again from the Nagumo’s theorem the set \( \mathcal{C}_2^- \) is positively invariant and \( y_{u_-}(v, \gamma_0) \leq c_2(v) \) for every \( v \leq v_0 \).

We remark that Lemma 4.1 proves invariance of the solutions only for the case when the slopes of the level set functions \( c_1 \) and \( c_2 \) in (30) are positive and negative, respectively. Nevertheless, it is also possible to prove invariance for the opposite case corresponding to the level set functions \( c_1 \) and \( c_2 \) having negative and positive slopes, respectively. In this opposite case, the invariant set for \( y_{u_+} \) and \( y_{u_-} \) correspond to the closure of the complement of \( \mathcal{C}_1^+ \) in (31) and \( \mathcal{C}_2^- \) in (32), respectively. In the following lemma, we prove that under mild assumptions regarding the monotonicity in the first argument of the functions \( f_1 \) and \( f_2 \), the extended solutions \( y_{u_+} \) and \( y_{u_-} \) in the reverse direction (when the input signal \( u_+ \) and \( u_- \) as defined before Lemma 4.1 are extended from \( \mathbb{R}_+ \) to the whole real \( \mathbb{R} \)) intersect with the zero level set curve \( c_2 \) and \( c_1 \), respectively.

**Lemma 4.2:** Assume that the hypotheses in Lemma 4.1 hold. Suppose that \( f_1 \) satisfy
\[
(f_1(v_1, \gamma) - f_1(v_2, \gamma)) (v_1 - v_2) < 0
\]
for every \( v_1, v_2, \gamma \in \mathbb{R} \) and let \( v_0, \gamma_0 \in \mathbb{R} \) be such that \( \gamma_0 = c_1(v_0) < c_2(v_0) \). Then, there exists \( v_b \) such that \( y_{u_+}(v_b, \gamma_0) = c_2(v_b) \).

**Proof:** Let us first prove the existence of a point \( v_0 \) such that \( y_{u_+}(v, \gamma_0) \) for every \( v \geq v_0 \) and using the Nagumo’s theorem the set \( \mathcal{C}_1^+ \) is positively invariant and \( y_{u_+}(v, \gamma_0) \leq c_1(v) \) for every \( v \geq v_0 \).

Analogously, suppose that \( f_2 \) satisfy
\[
(f_2(v_1, \gamma) - f_2(v_2, \gamma)) (v_1 - v_2) > 0
\]
for every \( v_1, v_2, \gamma \in \mathbb{R} \) and let \( v_0, \gamma_0 \in \mathbb{R} \) be such that \( \gamma_0 = c_1(v_0) > c_2(v_0) \). Then, there exists \( v_b > v_0 \) such that \( y_{u_+}(v_b, \gamma_0) = c_1(v_b) \).

**Proof:** Let us first prove the existence of a point \( v_0 \) where the curve \( y_{u_+}(v, \gamma_0) \) intersects with \( c_2 \) at \( v_0 \). By extending \( u_+ \) from \( \mathbb{R}_+ \) to \( \mathbb{R} \) while still satisfying the monotonicity and radial unbounded assumption of \( u_+ \) (e.g., \( \lim_{t \to -\infty} u_+(t) = -\infty \) and \( \lim_{t \to +\infty} u_+(t) = \infty \)), the solution \( y_{u_+}(v, \gamma_0) \) can be extended in the negative direction (i.e., \( u < v < v_0 \)) and the equation
\[
y_{u_+}(v, \gamma_0) = \int_{v_0}^v f_1(v', y_{u_+}(v', \gamma_0)) dv + \gamma_0 = -\int_v^{v_0} f_1(v', y_{u_+}(v', \gamma_0)) dv + \gamma_0
\]
is still valid [see Fig. 6(a)]. Moreover, since \( f_1(v, \gamma) < 0 \) whenever \( \gamma > c_1(v) \) we have that
\[
y_{u_+}(v, \gamma_0) = \int_{v_0}^v f_1(v', y_{u_+}(v', \gamma_0)) dv + \gamma_0
\]
for every \( v < v_0 \), which means that the extension of the solution \( y_{u_+}(v, \gamma_0) \) in the negative direction remains above the curve parameterized by \( \gamma = c_1(v) \). By the assumption (33) and using the bound \( L_2 \) of \( \frac{d c_2(v)}{dv} \) as in the hypotheses of Lemma 4.1, we have that there exists \( v_{L_2} \leq v_0 \) such that for every \( v < v_{L_2} \) we have
\[
f_1(v, y_{u_+}(v, \gamma_0)) < f_1(v_{L_2}, y_{u_+}(v_{L_2}, \gamma_0)) = -L_2.
\]
Since we have that
\[
c_2(v) = \int_{v_0}^v \frac{d c_2(v)}{dv} dv + c_2(v_0) \leq L_2(v_0 - v) + c_2(v_0)
\]
the solution \( y_{u_+}(v, \gamma_0) \) and the curve parameterized by \( \gamma = c_2(v) \) intersect each other at some \( v_b < v_{L_2} \). Indeed, this can be
observed from the fact that
\[
\begin{align*}
\gamma_{u+}(v, \gamma) - c_2(v) = \gamma_a - c_2(v_a)
\end{align*}
\]
\[
+ \int_{v_a}^{\nu} \left\{ f_1(v, \gamma_{u+}(v, \gamma)) - \frac{dc_2}{dv} \right\} \, dv
\]
\[
= \int_{v_a}^{\nu_1} \left\{ f_1(v, \gamma_{u+}(v, \gamma)) - \frac{dc_2}{dv} \right\} \, dv
\]
\[
+ \int_{\nu_1}^{\nu} \left\{ f_1(v, \gamma_{u+}(v, \gamma)) - \frac{dc_2}{dv} \right\} \, dv
\]
where the last term grows radially unbounded for \( v > \nu_1 \).

We can prove analogously the second claim of the lemma as illustrated in Fig. 6(b). Similar as before, the solution \( \gamma_{u+}(v, \gamma) \) can be extended in the positive direction (i.e., \( v > v_a \)) when \( u_- \) is extended from \( \mathbb{R}_+ \) to \( \mathbb{R} \) satisfying the monotonicity and radial unbounded assumption of \( u_- \). In this case
\[
\gamma_{u-}(v, \gamma) = \int_{v}^{v_a} f_2(v, \gamma_{u-}(v, \gamma)) \, dv + \gamma_a
\]
and since \( f_2(v, \gamma) > 0 \) whenever \( \gamma > c_2(v) \), we have that
\[
\gamma_{u-}(v, \gamma) = \int_{v}^{v_a} f_2(v, \gamma_{u-}(v, \gamma)) \, dv + \gamma_a
\]
for every \( v > v_a \), which means that the extension of the solution \( \gamma_{u-}(v, \gamma) \) in the positive direction remains above the curve parameterized by \( \gamma = c_2(v) \). In this case, using (34) and the bound \( L_1 \) of \( \frac{dc_2}{dv} \), we have that there exists \( v_{L1} \geq v_a \) such that for every \( v > v_{L1} \) we have
\[
f_2(v, \gamma_{u-}(v, \gamma)) > f_2(v_{L1}, \gamma_{u-}(v_{L1}, \gamma)) = L_1.
\]
Since we have that
\[
c_1(v) = \int_{v}^{v_a} \frac{dc_1}{dv} \, dv + c_1(v_a) \leq L_1(v - v_a) + c_1(v_a)
\]
the solution \( \gamma_{u-}(v, \gamma) \) and the curve parameterized by \( \gamma = c_1(v) \) intersect each other at some \( v_b > v_{L1} \). It follows from the fact that
\[
\gamma_{u-}(v, \gamma) - c_1(v) = \gamma_a - c_1(v_a)
\]
\[
+ \int_{v_a}^{\nu} \left\{ f_2(v, \gamma_{u-}(v, \gamma)) - \frac{dc_1}{dv} \right\} \, dv
\]
\[
= \int_{v_a}^{\nu_1} \left\{ f_2(v, \gamma_{u-}(v, \gamma)) - \frac{dc_1}{dv} \right\} \, dv
\]
\[
+ \int_{\nu_1}^{\nu} \left\{ f_2(v, \gamma_{u-}(v, \gamma)) - \frac{dc_1}{dv} \right\} \, dv
\]
where the last term grows radially unbounded for \( v > \nu_1 \).

As with Lemma 4.1, we also remark that Lemma 4.2 proves that the extension of the solutions in the negative direction of their corresponding input intersect with the zero level set functions \( c_1 \) and \( c_2 \) only for the case when their slopes are positive and negative, respectively. However, vis-a-vis arguments can prove the opposite case when the extended solutions in the negative direction of their corresponding input intersect with the level set functions \( c_1 \) and \( c_2 \) have negative and positive slopes, respectively.

In the following proposition, we present the main result of this section, where we prove constructively the existence of outputs \( \gamma_{u+} \) and \( \gamma_{u-} \) with intersections.

**Proposition 4.3:** Assume that the hypotheses in Lemma 4.2 are satisfied (which include those in Lemmas 4.1). Let \( v_f \in \mathbb{R} \) be such that \( c_1(v_f) = c_2(v_f) \). Then, for every \( v_{a+} < v_f \) there exist
\[
u_{min} < v_{a+} < v_x < v_{a-} < v_{max}
\]

such that
\[
\gamma_{u+}(\nu_{min}, c_1(v_{a+})) = \gamma_{u-}(\nu_{min}, c_2(v_{a+}))
\]
\[
\gamma_{u+}(v_x, c_1(v_{a+})) = \gamma_{u-}(v_x, c_2(v_{a+}))
\]
\[
\gamma_{u+}(v_{max}, c_1(v_{a+})) = \gamma_{u-}(v_{max}, c_2(v_{a+})).
\]

In other words, the solutions \( \gamma_{u+}(\cdot, c_1(v_{a+})) \) and \( \gamma_{u-}(\cdot, c_2(v_{a+})) \), which intersect \( c_1 \) and \( c_2 \) at \( v_{a+} \) and \( v_{a-} \), respectively, intersect also each other at \( v_x \), \( \nu_{min} \) and \( v_{max} \).

**Proof:** For a better understanding of the constructive proof of this proposition, we refer the reader to Fig. 7.

Consider the solution \( \gamma_{u+}(v, c_1(v_{a+})) \). By Lemma 4.2 there exists \( v_{b+} < v_{a+} \) where this solution intersects the curve \( c_2 \), i.e.,
\[
\gamma_{u+}(v_{b+}, c_1(v_{a+})) = c_2(v_{b+}).
\]

Additionally, by Lemma 4.1, we have that the solution \( \gamma_{u+} \) remains below the curve \( c_1 \) for every \( v > v_{a+} \) but always increasing as \( v \) increases since \( f_2(v, \gamma) > 0 \) when \( \gamma < c_1(v) \). Therefore, since \( \frac{dc_2}{dv} \leq 0 \), then the solution \( \gamma_{u+} \) must also intersect the curve \( c_2 \) at some \( v_{c+} > v_{a+} \), i.e.,
\[
\gamma_{u+}(v_{c+}, c_1(v_{a+})) = c_2(v_{c+}).
\]

Let us define now \( v_{a+} = v_{c+} + \varepsilon \) with \( \varepsilon > 0 \) being arbitrarily small and consider the solution \( \gamma_{u-}(v, c_2(v_{a-})) \). As in the previous case, by Lemma 4.2 there exists \( v_{b-} > v_{a-} \) where this
solution intersects the curve $c_1$, i.e.,
\[ y_{u_+}(v_{b_-}, c_2(v_{b_-})) = c_1(v_{b_-}). \]

We can also note that by Lemma 4.1 the solution $y_{u_-}$ remains below the curve $c_2$ but always increasing as $v$ decreases given that $f_2(v, \gamma) > 0$ for every $v < v_{a_-}$. Consequently, since $\frac{dv}{dt} \geq 0$, then the solution $y_{u_-}$ must also intersect the curve $c_1$ at some $v_{c_-} < v_{a_-}$, i.e.,
\[ y_{u_+}(v_{c_-}, c_2(v_{c_-})) = c_1(v_{c_-}). \]

If the value $v_{c_-}$ satisfies $v_{c_-} > v_{a_-}$ it is clear that the solution $y_{u_-}$ intersects with $y_{u_+}$ at some $v_x$ such that $v_{a_-} < v_x < v_{a_+}$. In the opposite case that $v_{c_-} < v_{a_-}$ and the solution $y_{u_+}$ does not intersect with $y_{u_+}$ at some $v$ such that $v_{a_-} < v < v_{a_+}$, then we can decrease arbitrarily $\varepsilon$ as long as it is positive and since $v_{a_+} = v_{c_-} + \varepsilon$, then there must exists $v_{a_-} < v_x < v_{a_-}$ such that
\[ y_{u_+}(v_x, c_1(v_{a_-})) = y_{u_-}(v_x, c_2(v_{a_-})). \]

Note now that since $y_{u_-}$ intersects with $c_1$ at $v_{b_-}$, and $y_{u_-}$ always increases but remains below $c_1$ as $v$ increases, then there must exists $v_{a_-} < v_{\text{max}} < v_{a_-}$ such that
\[ y_{u_+}(v_{\text{max}}, c_1(v_{a_-})) = y_{u_-}(v_{\text{max}}, c_2(v_{a_-})). \]

Finally, by converse arguments, since $y_{u_-}$ intersects with $c_2$ at $v_{b_+}$, and $y_{u_+}$ always increases but remains below $c_2$ as $v$ decreases, then there must exists $v_{b_+} < v_{\text{min}} < v_{a_-}$ such that
\[ y_{u_+}(v_{\text{min}}, c_1(v_{a_-})) = y_{u_-}(v_{\text{min}}, c_2(v_{a_-})). \]

It should be immediately noted from Proposition 3.7 on the accommodation property and from Proposition 4.3 on the existence of an invariant butterfly loop that applying a simple periodic input $v_p \in AC(\mathbb{R}_+, \mathbb{R})$ with only one maximum and one minimum in its periodic interval whose values are $v_{\text{min}}$ and $v_{\text{max}}$, then the input-output phase plot will converge to the butterfly hysteresis loop for every initial value of the output $\gamma_0 \in \mathbb{R}$.

**A. Example of a Duhem Butterfly Operator**

As an illustrative example, we introduce now a Duhem butterfly operator, which we build constructively by: i) defining arbitrary curves $c_1(v, \gamma)$ and $c_2(v, \gamma)$ satisfying conditions of Lemma 4.1; and ii) selecting the functions $f_1$ and $f_2$ such that these curves correspond to the zero level set (i.e., $f_1(v, c_1(v)) = f_2(v, c_2(v)) = 0$) and satisfy the hypotheses in Lemma 4.2 and Proposition 4.3. In general, any functions $f_1$ and $f_2$ satisfying hypotheses in Lemmas 4.2, 4.1 and Proposition 4.3, which can be constructed using particular kernel functions or identified using existing models in literature, will produce Duhem butterfly operators.

Let us first define the curves $c_1$ and $c_2$ by
\[ c_1(v, \gamma) := a_1 + a_2 v + a_3 v^3 \quad (35) \]
\[ c_2(v, \gamma) := -b_1 - b_2 v - b_3 v^3. \quad (36) \]

In order to assign these curves as the zero level sets we can define $f_1$ and $f_2$ as the signed vertical distance between the curve $c_1(v, \gamma)$ and the point $(v, \gamma)$, and respectively, between $c_1(v, \gamma)$ and the point $(v, \gamma)$. Here, we need to take care that the convergence conditions (18) and (19) are satisfied. Accordingly, we can define $f_1$ and $f_2$ by

\[ f_1(v, \gamma) := c_1(v) - \gamma \]
\[ = (a_1 + a_2 v + a_3 v^3 - \gamma) \quad (37) \]
\[ f_2(v, \gamma) := -(c_2(v) - \gamma) \]
\[ := (b_1 + b_2 v + b_3 v^3 + \gamma). \quad (38) \]

Substituting the functions defined above into (18) and (19) we obtain that
\[ -(\gamma_1 - \gamma_2)^2 < 0 \quad \text{and} \quad (\gamma_1 - \gamma_2)^2 > 0 \]
which are trivially satisfied.

In Fig. 8(a), we present the simulation results of a Duhem butterfly operator (3) defined with (37) and (38) when a periodic input, whose maximum and minimum are $u_{\text{max}} = 5$ and $u_{\text{min}} = -5$, is applied. As remarked before, our main results in Lemma 4.1, Lemma 4.2, and Proposition 4.3 hold also for the case when the signs are reversed. Correspondingly, in this section, we present an example of a Duhem operator that satisfies all the opposite conditions to Lemmas 4.1–4.2 and to Proposition
4.3. We modify slightly the previous example in Section IV-A
by defining $f_1$ and $f_2$ as follows:

$$f_1(v, \gamma) := (-c_1(v) - \gamma)$$

$$= (-a_1 - a_2 v - a_3 v^3 - \gamma)$$

$$f_2(v, \gamma) := -(-c_2(v) - \gamma)$$

$$= (-b_1 - b_2 v - b_3 v^3 + \gamma).$$

(39)

By vis-à-vis arguments to the ones of Proposition 4.3, a Duhem operator with the above $f_1$ and $f_2$ can also produce a hysteresis loops with self-intersections. This will result in the reversion of the loop orientation. Fig. 8(b) shows a simulation result of a Duhem butterfly operator (3) defined by (39) and (40).

B. Modeling of Doped Lead Zirconate Tinate (PZT)
Material by Duhem Butterfly Operator

In this section, we present an example of a Duhem butterfly operator fitted to experimental data of the sample of piezoelectric material whose relation between electric-field and strain exhibited an asymmetric butterfly hysteresis loop. The sample was made out of doped PZT, which has been developed for the design of hysteretic deformable mirror [28]. The strain measurements were taken by laser interferometer. The applied input signals were triangular periodic signals of 1400 V of amplitude with frequency of 1 Hz. For numerical fitting purpose, we use the same kernel vector functions as the ones in Example IV-A using the signed distance between the point $(v, \gamma)$ to the two curves $c_1(v, \gamma)$ and $c_2(v, \gamma)$, as follows:

$$c_1(v, \gamma) := \sum_{n=0}^{5} a_n v^n; \quad c_2(v, \gamma) := -\sum_{n=0}^{5} b_n v^n.$$ 

Therefore, the functions $f_1$ and $f_2$ are given by

$$f_1(v, \gamma) := -\gamma + \sum_{n=0}^{5} a_n v^n, \quad f_2(v, \gamma) := \gamma + \sum_{n=0}^{5} b_n v^n.$$ 

(41)

The identified numerical parameters were $a_0 = -5.1171, a_1 = -1.1072, a_2 = 0.0608, a_3 = 0.0321, a_4 = 0.0009, a_5 = -0.0001, b_0 = 0.2223, b_1 = 1.6194, b_2 = -0.0806, b_3 = -0.0068, b_4 = 0.0016$, and $b_5 = -0.0001$. These fitted parameters were obtained by minimizing the least square error given by the difference between the model output and experimental data. Fig. 9 shows the input–output phase plot with the experimental data, the simulation results, the level curves $c_1(v, \gamma)$ and $c_2(v, \gamma)$, which are constructed based on $f_1$ and $f_2$, respectively, and the corresponding anhysteress curve.

C. Counter-Example of Duhem Operator With Multiloop Behavior

In this final section, we present an example of a Duhem operator that can exhibit multiloop hysteresis behavior. This is analogous behavior presented in [29] when the Preisach operator weighting function is allowed to have more than two regions of positive and negative values over its domain. In the case of the Duhem operator, its vector functions $f_1$ and $f_2$ are no longer restricted to the hypotheses in Lemmas 4.1–4.2 and Proposition 4.3 while they still satisfy the hypotheses in Proposition 3.7.

When the Duhem operator in this example is subjected to a periodic input signal, the input–output phase plot converges to a periodic orbit as expected and additionally the orbit can exhibit multiloop hysteresis behavior. For constructing this example, we define the zero level set curves $c_1$ and $c_2$ by

$$c_1(v, \gamma) := 10 \sin \left( 6\pi v + \frac{\pi}{8} \right)$$

(42)

$$c_2(v, \gamma) := -8 \sin \left( 6\pi v - \frac{\pi}{8} \right)$$

(43)

and as presented in Section IV-A, the functions $f_1$ and $f_2$ are defined as the signed vertical distance between these curves (i.e., $c_1(v, \gamma)$ and $c_2(v, \gamma)$) and the point $(v, \gamma)$, respectively. Explicitly, they are given by

$$f_1(v, \gamma) := (c_1(v) - \gamma)$$

$$= 10 \sin \left( 6\pi v + \frac{\pi}{8} \right) - \gamma$$

(44)

$$f_2(v, \gamma) := -(c_2(v) - \gamma)$$

FIG. 9. Duhem model whose gradient functions $f_1$ and $f_2$ are given by (41) with the parameters fitted to experimental data obtained from a sample of piezoelectric material exhibiting asymmetric butterfly hysteresis behavior. The initial point $(u(0), y(0))$ is marked by a circle.

FIG. 10. Multiloop hysteresis loop obtained from a model Duhem model whose gradient functions $f_1$ and $f_2$ are given by (37) and (38), respectively, when a periodic input whose minimum and maximum are $v_{\min} = -12$ and $v_{\max} = 12$. The initial point $(u(0), y(0)) = (v_{\min}, y_0)$ is marked by a circle.
The simulation results of such Duhem operator (3) with \( f_1 \) as in (44) and \( f_2 \) as in (45) are shown in Fig. 10 where multiloop hysteresis behavior is exhibited.

V. Conclusion

In this article, we have studied and presented sufficient conditions for a class of Duhem hysteresis operators that admit butterfly loops. First, we studied general conditions on the functions \( f_1 \) and \( f_2 \) so that the Duhem operator has the accommodation property. In particular, we do not impose positive definiteness or an explicit form on these functions. Based on the sufficient conditions for the accommodation property, we presented sufficient conditions on \( f_1 \) and \( f_2 \) such that the corresponding Duhem hysteresis operator is capable of exhibiting butterfly hysteresis loops. Numerical simulations show also the possibility of having multiloop behavior when these conditions are not satisfied. The work presented in this article can be the basis for the development of systems identification methods to model butterfly or multiloop hysteresis phenomena in many electro-mechanical applications based on the use of integro-differential Duhem models.

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