An estimate for the Morse index of a Stokes wave

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Abstract

Stokes waves are steady periodic water waves on the free surface of an infinitely deep irrotational two dimensional flow under gravity without surface tension. They can be described in terms of solutions of the Euler-Lagrange equation of a certain functional. This allows one to define the Morse index of a Stokes wave. It is well known that if the Morse indices of the elements of a set of non-singular Stokes waves are bounded, then none of them is close to a singular one. The paper presents a quantitative variant of this result.

1 Introduction

A Stokes wave is a steady periodic wave, propagating under gravity with constant speed on the surface of an infinitely deep irrotational two dimensional flow. Its free surface is determined by the Laplace equation, kinematic and periodic boundary conditions and by a dynamic boundary condition given by the requirement that pressure in the flow at the surface should be constant (Bernoulli’s theorem). A mathematical model for Stokes waves can be described as follows.

Let \( \Omega \subset \mathbb{C} \) denote the domain below a curve \( \mathcal{S} \) in the \( (X,Y) \)-plane, where

\[
\mathcal{S} := \{ (x(s),y(s)) : s \in \mathbb{R} \},
\]

\( (x,y) \) is injective and absolutely continuous,

\( x'(s)^2 + y'(s)^2 > 0 \) for almost all \( s \),

\( s \mapsto (x(s) - s, y(s)) \) is \( 2\pi \)-periodic.

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The Stokes waves are solutions of the following free boundary problem: find $S$ for which there exists $\psi \in C(\Omega) \cap C^2(\Omega)$ such that

\begin{align*}
\Delta \psi &= 0 \text{ in } \Omega, \\
\psi &\text{ is } 2\pi \text{-periodic in } X, \\
\nabla \psi(X, Y) &\to (0, 1) \text{ as } Y \to -\infty \text{ uniformly in } X, \\
\psi &= 0 \text{ on } S, \\
|\nabla \psi(X, Y)|^2 &= 1 - 2\mu Y \text{ almost everywhere on } S.
\end{align*}

Here $\mu^{-1/2} > 0$ is the Froude number, a dimensionless combination of speed, wavelength and gravitational acceleration. If $(S, \psi)$ is a solution of (1)-(5) such that $1 - 2\mu Y > 0$ everywhere on $S$, then $S$ and $\psi$ are real analytic ([42], see also [37]). We call such solutions regular or non-singular. The famous Stokes wave of extreme form which has a stagnation point and a corner containing an angle of $120^\circ$ at its crest is a singular solution, and $1 - 2\mu Y = 0$ at the crest. Non-singular solutions of (1)-(5) are in one-to-one correspondence with the critical points $v \in W^{1,2}_{2\pi}$ of the functional

$$
J(v) = J_\mu(v) := \int_{-\pi}^{\pi} \left( v(t) Cu(t) - \mu v^2(t)(1 + Cu(t)) \right) dt, \quad v \in W^{1,2}_{2\pi}
$$

(see [8, 37, 44]), where $Cu$ denotes the periodic Hilbert transform of a $2\pi$-periodic function $u : \mathbb{R} \to \mathbb{R}$:

$$
Cu(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(s) \cot \frac{t - s}{2} \, ds
$$

(see Section 2 for the notation of function spaces and for more information on $C$). This allows one to define the Morse index of a non-singular Stokes wave. Let $v$ be a critical point of $J$. Consider the quadratic form of the second Fréchet derivative $J''(v)$ (the Hessian):

$$
Q_v[u] := 2 \int_{-\pi}^{\pi} \left( (1 - 2\mu v(t))u(t)Cu'(t) - \mu(1 + Cu(t))u^2(t) \right) dt.
$$

The Morse index $\mathcal{M}(v)$ of $v$ and of the corresponding Stokes wave is the number $N_-(Q_v)$ which is defined as follows.

Let $\mathcal{H}$ be a Hilbert space and let $\mathbf{q}$ be a Hermitian form with a domain $\text{Dom}(\mathbf{q}) \subseteq \mathcal{H}$. Set

$$
N_-(\mathbf{q}) := \sup \{ \dim \mathcal{L} \mid \mathbf{q}[u] < 0, \forall u \in \mathcal{L} \setminus \{0\} \},
$$

where $\mathcal{L}$ denotes a linear subspace of $\text{Dom}(\mathbf{q})$. If $\mathbf{q}$ is the quadratic form of a self-adjoint operator $A$ with no essential spectrum in $(-\infty, 0)$, then
$N_-(q)$ is the number of negative eigenvalues of $A$ repeated according to their multiplicity (see, e.g., [1] S1.3 or [6] Theorem 10.2.3).

Every critical point $v \in W_{2\pi}^{1,3}$ of (6) is a real analytic function corresponding to a nonsingular Stokes wave $(S, \psi)$ and

$$\min_{t \in \mathbb{R}} (1 - 2\mu v(t)) = \min_{(X,Y) \in S} (1 - 2\mu Y) > 0$$

(see [8, 37]). Let

$$\nu(v) := \max_{t \in \mathbb{R}} \frac{2\mu}{1 - 2\mu v(t)} = \frac{2\mu}{\min_{t \in \mathbb{R}} (1 - 2\mu v(t))},$$

$$\nu_0(v) := \frac{1}{\min_{t \in \mathbb{R}} (1 - 2\mu v(t))} = \frac{\nu(v)}{2\mu}.$$  \hspace{1cm} (10)

Let $v_k \in W_{2\pi}^{1,3}$ be a critical point of $J_{\mu_k}$, $k \in \mathbb{N}$ (see (6)). It is well known that if $\nu(v_k) \to \infty$, then $M(v_k) \to \infty$ as $k \to \infty$ (see [9, 31, 37]). In other words, if the Morse indices of the elements of a set of non-singular Stokes waves are bounded, then none of them is close to a singular one. The following quantitative version of this statement is the main result of the paper.

**Theorem 1.1.** There exist constants $M_1, M_2 > 0$ such that

$$M_1 \ln^{1/3}(1 + \nu(v)) \leq M(v) \leq 1 + M_2 \nu(v) \ln(2 + \nu_0(v))$$  \hspace{1cm} (11)

holds for every critical point $v \in W_{2\pi}^{1,3}$ of (6) and every $\mu > 0$.

In fact, we prove in Section 5 a more general result which holds for Bernoulli free-boundary problems. Plotnikov’s transformation ([31], see also [9]) allows one to pass from (6) to a simpler quadratic form

$$q_V[u] := \int_{-\pi}^{\pi} ((Cu'(t))u(t) - V(t)u^2(t)) dt, \quad u \in H_{2\pi}^{1/2,2} \cap C_{2\pi},$$  \hspace{1cm} (12)

where the potential $V \geq 0$, $V \in L_{2\pi}^1$ is determined by $v$. It is convenient for us to extend the domain of $q_V$ from $W_{2\pi}^{1,2}$ to $H_{2\pi}^{1/2,2} \cap C_{2\pi}$. An easy approximation argument shows that this does not affect the Morse index.

In order to state the estimate for $N_-(q_V)$, we need some notation from the theory of Orlicz spaces (see [19, 33]). Let $(\Omega, \Sigma, \mu)$ be a measure space, let $\Phi$ and $\Psi$ be mutually complementary $N$-functions, and let $L_\Phi(\Omega)$, $L_\Psi(\Omega)$ be the corresponding Orlicz spaces. (These spaces are denoted by $L_{\Phi}(\Omega)$, $L_{\Psi}(\Omega)$ in [19], where $\Omega$ is assumed to be a closed bounded subset of $\mathbb{R}^d$ equipped
with the standard Lebesgue measure.) We will use the following norms on \( L^\Psi(\Omega) \)
\[
\|f\|_\Psi = \|f\|_{\Psi,\Omega} = \sup \left\{ \left| \int_\Omega fg \, d\mu \right| : \int_\Omega \Phi(g) \, d\mu \leq 1 \right\}.
\]
(13) and
\[
\|f\|_{(\Psi)} = \|f\|_{(\Psi,\Omega)} = \inf \left\{ \kappa > 0 : \int_\Omega \Psi\left(\frac{f}{\kappa}\right) \, d\mu \leq 1 \right\}.
\]
(14)
These two norms are equivalent
\[
\|f\|_{(\Psi)} \leq \|f\|_\Psi \leq 2\|f\|_{(\Psi)} , \quad \forall f \in L^\Psi(\Omega).
\]
(15)
We will need the following pair of mutually complementary \(N\)-functions
\[
A(s) = e^{[s]} - 1 - |s|, \quad B(s) = (1 + |s|) \ln(1 + |s|) - |s|, \quad s \in \mathbb{R}.
\]
(16)

**Theorem 1.2.** There exist constants \( C_1, C_2 > 0 \) such that
\[
C_1\|V\|_{L_{2\pi}^\frac{1}{2}} \leq N_{(q_V)} \leq C_2\|V\|_{B_{[-\pi,\pi]}} + 1, \quad \forall V \in L_{2\pi}^1, \quad V \geq 0.
\]
(17)
The above theorem answers the question posed by the author at the LMS Durham Symposium “Operator Theory and Spectral Analysis” in 2005: is there a function \( h : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( h(\tau) \to +\infty \) as \( \tau \to +\infty \) and
\[
N_{(q_V)} \geq h\left(\|V\|_{L_{2\pi}^1}\right), \quad \forall V \in C_{2\pi}^\infty, \quad V \geq 0?
\]
Although he offered a “decent bottle of wine” for the first correct solution, no one came up with an answer. After repeating the question and the offer in several later talks, the author decided in 2007 to rise the stakes and to upgrade a “decent bottle of wine” to a bottle of Chateau Latour which is arguably the best wine in the world. Although this much more attractive offer was repeated in several talks since then and also in [35], the bottle of Chateau Latour remained unclaimed. It turns out that the latter was just pure luck as the lower estimate in (17) can be derived from [14] (see also [15]). The proof of the upper estimate in (17) is more difficult and relies upon methods developed in [39] (see Section 3 below).

## 2 Notation and auxiliary results

The subscript \( 2\pi \) is used throughout the paper to denote spaces of \( 2\pi \)-periodic real valued functions: \( C_{2\pi} \) is the Banach space of real valued continuous \( 2\pi \)-periodic functions; \( C_{2\pi}^\infty \) is the subspace of \( C_{2\pi} \) consisting of infinitely smooth functions; \( L_{2\pi}^p, \quad p \geq 1 \) is the Banach space of real valued
locally $p^{th}$-power summable $2\pi$–periodic functions; $W^{1,p}_{2\pi}$ is the Banach space of absolutely continuous, $2\pi$–periodic functions $u$ with weak first derivatives $u' \in L^p_{2\pi}$; $H^{1/2,2}_{2\pi} = W^{1/2,2}_{2\pi}$ is the subspace of $L^2_{2\pi}$ consisting of functions $u$ such that

$$
\|u\|_{W^{1,2}}^2 := \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \left( \frac{u(t) - u(s)}{\sin \frac{1}{2}(t - s)} \right)^2 dt \, ds + \|u\|_{L^2_{2\pi}}^2 < +\infty.
$$

The following is an equivalent norm on $H^{1/2,2}_{2\pi}$:

$$
\|u\|_{H^{1/2,2}} := \left( \sum_{n \in \mathbb{Z}} (1 + |n|) |\hat{u}(n)|^2 \right)^{1/2},
$$

where $\hat{u}(n)$ are the Fourier coefficients of $u$ (see, e.g., [13]). The periodic Hilbert transform (7) is bounded in $L^p_{2\pi}$ if $1 < p < \infty$ (M. Riesz theorem), and

$$
\widehat{Cu}(0) = 0, \quad \widehat{Cu}(n) = -i \text{sign}(n) \hat{u}(n), \quad n \in \mathbb{Z} \setminus \{0\}, \quad u \in L^p_{2\pi},
$$

(18)

$$
C^2 u = -u + \hat{u}(0).
$$

(19)

It follows from (18) that

$$
\widehat{C u'}(n) = |n| \hat{u}(n), \quad n \in \mathbb{Z}, \quad u \in W^{1,p}_{2\pi},
$$

(20)

and hence

$$
\left( \int_{-\pi}^{\pi} \left( (Cu'(t))u(t) + u^2(t) \right) dt \right)^{1/2}
$$

is an equivalent norm on $H^{1/2,2}_{2\pi}$.

Let $\Psi$ be an $N$-function ([19, 33]). Then

$$
\int_{\Omega} \Psi \left( \frac{f}{\kappa_0} \right) d\mu \leq C_0, \quad C_0 \geq 1 \implies \|f\|_{(\Psi)} \leq C_0 \kappa_0.
$$

(21)

Indeed, since $\Psi$ is even, convex and increasing on $[0, +\infty)$, and $\Psi(0) = 0$, we get for any $\kappa \geq C_0 \kappa_0$,

$$
\int_{\Omega} \Psi \left( \frac{f}{\kappa} \right) d\mu \leq \int_{\Omega} \Psi \left( \frac{f}{C_0 \kappa_0} \right) d\mu \leq \frac{1}{C_0} \int_{\Omega} \Psi \left( \frac{f}{\kappa_0} \right) d\mu \leq 1.
$$

We will use the following standard notation

$$
a_+ := \max\{0, a\}, \quad a \in \mathbb{R}.
$$
Lemma 2.1. $\frac{1}{2} s \ln s \leq B(s) \leq s + 2s \ln s, \ \forall s \geq 0.$

Proof. Integrating the inequality

$$1 + \ln s = \ln(es) < \ln(1 + 3s) \leq 2 \ln(1 + s), \quad s \geq 1,$$

one gets $s \ln s \leq 2B(s)$.

If $s \geq 1$, then

$$\mathcal{B}(s) = (1 + s) \ln(1 + s) - s \leq 2s \ln(2s) - s = (2 \ln 2 - 1)s + 2s \ln s < s + 2s \ln s.$$ 

If $s \in [0, 1)$, then integrating the inequality $\ln(1 + s) \leq s$ one gets

$$(1 + s) \ln(1 + s) - s \leq \frac{1}{2} s^2 \leq \frac{1}{2} s \leq s.$$ 

Suppose $g \in L^\infty_{2\pi}$ and let $\kappa_0 := \|g\|_{L^1_{2\pi}}$. Then it follows from Lemma 2.1 and (22) that

$$\int_{-\pi}^{\pi} \mathcal{B} \left( \frac{g(t)}{\kappa_0} \right) dt \leq \frac{2 \|g\|_{L^1_{2\pi}}}{\kappa_0} \ln \left( \frac{2 \|g\|_{L^\infty_{2\pi}}}{\kappa_0} \right) < 2 \left( 1 + \ln \left( 1 + 2\pi \|g\|_{L^\infty_{2\pi}} \right) \right) < 4 \ln \left( 2 + \frac{2\pi \|g\|_{L^\infty_{2\pi}}}{\|g\|_{L^1_{2\pi}}} \right).$$

Hence

$$\|g\|_{\mathcal{B}_{[-\pi, \pi]}} \leq 2\|g\|_{(\mathcal{B}_{[-\pi, \pi]})} \leq 8\|g\|_{L^1_{2\pi}} \ln \left( 2 + \frac{2\pi \|g\|_{L^\infty_{2\pi}}}{\|g\|_{L^1_{2\pi}}} \right)$$

(see (15) and (21)).

According to Zygmund’s theorem ([46, Ch. VII, Theorem (2.8)]), there exist constants $A, B > 0$ such that

$$\|Cf\|_{L^1_{2\pi}} \leq A \int_{-\pi}^{\pi} |f(t)| \ln_+ |f(t)| dt + B$$

(24)

for any $f$ with $f \ln_+ |f| \in L^1_{2\pi}$. In fact, a necessary and sufficient condition that there exists $B > 0$ for which (24) holds is that $A > 2/\pi$ (see [30]). Taking $\kappa = \|f\|_{(\mathcal{B}_{[-\pi, \pi]})}$ in the inequality

$$\left\| C \left( \frac{f}{\kappa} \right) \right\|_{L^1_{2\pi}} \leq 2A \int_{-\pi}^{\pi} \mathcal{B} \left( \frac{f(t)}{\kappa} \right) dt + B$$

(25)
(see (24) and Lemma 2.1) one arrives at
\[ \|Cf\|_{L^2_{2\pi}} \leq (2A + B)\|f\|_{L^2_{[-\pi,\pi]}} \leq (2A + B)\|f\|_{L^2_{[-\pi,\pi]}}. \] (25)

Sharp inequalities of this type involving other equivalent norms on \( L_B \) (see [2]) can be found in [3, 28].

If \( f \geq 0 \), then
\[ \int_{-\pi}^{\pi} (1 + f(t)) \ln(1 + f(t)) \, dt \leq \pi \|Cf\|_{L^2_{2\pi}} + 2\pi \left(1 + \frac{1}{2\pi}\|f\|_{L^2_{2\pi}}\right) \ln \left(1 + \frac{1}{2\pi}\|f\|_{L^2_{2\pi}}\right) \]
(see [46] Ch. VII, (2.25)). Let
\[ \kappa_1 := \max\{\|Cf\|_{L^1_{2\pi}}, \|f\|_{L^1_{2\pi}}\}. \]

Applying the above inequality to \( f/\kappa_1 \) one gets
\[ \int_{-\pi}^{\pi} B \left(\frac{f(t)}{\kappa_1}\right) \, dt \leq \frac{\pi}{2} + (2\pi + 1) \ln \left(1 + \frac{1}{2\pi}\right) =: A_0. \]

Hence
\[ \|f\|_{L^1_{[-\pi,\pi]}} \leq 2 \|f\|_{L^1_{[-\pi,\pi]}} \leq 2A_0 \max\{\|Cf\|_{L^1_{2\pi}}, \|f\|_{L^1_{2\pi}}\}, \quad f \geq 0 \] (26)
(see [15] and [21]).

3 A Solomyak type estimate for two dimensional Schrödinger operators with singular potentials

A mapping \( F \) from a metric space \( (X_1, d_1) \) into a metric space \( (X_2, d_2) \) is called bi-Lipschitz if there exists a constant \( M > 0 \) such that
\[ d_1(x, y)/M \leq d_2(F(x), F(y)) \leq Md_1(x, y), \quad \forall x, y \in X_1. \]

We say that a curve \( \ell \) in \( \mathbb{C} \) is a Lipschitz arc if it is a bi-Lipschitz image of \([0, 1]\). It is clear that a Lipschitz arc is non-self-intersecting and rectifiable. Using the arc length parametrisation, one can easily show that a simple rectifiable curve \( \ell \) is a Lipschitz arc if and only if it is a chord-arc curve, i.e. if there exists a constant \( K \geq 1 \) such that the length of the subarc of \( \ell \) joining any
two points is bounded by $K$ times the distance between them. When dealing
with function spaces on $\ell$, we will always assume that $\ell$ is equipped with the
arc length measure.

Let $\ell_j, j = 1, \ldots, J$ be Lipschitz arcs. Any two arcs $\ell_j, \ell_l$ may intersect, have
common subarcs or even coincide. The most important case for us is when $J = 2$ and $\ell_1$ and $\ell_2$ are two halves of the unit circle $T$. The main result of
this section is a Cwikel-Lieb-Rozenblum type estimate for a two dimens ional
Schrödinger operator with a potential supported by $\Sigma := \bigcup_{j=1}^J \ell_j$.

Let $V_j \in L^B(\ell_j), j = 1, \ldots, J$ (see (16)), $V_j \geq 0,$ and

\[ E_V[w] := \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx - \sum_{j=1}^J \int_{\ell_j} V_j(x)w^2(x)ds(x), \]  

(27)

\[ \text{Dom (} E_V) = \bigoplus_{j=1}^J (L^2(\mathbb{R}^2, \ell_j)) \cap C (\mathbb{R}^2). \]

**Theorem 3.1.** There exists a constant $C(\Sigma) > 0$ such that

\[ N_-(E_V) \leq C(\Sigma) \sum_{j=1}^J \|V_j\|_{L^B,\ell_j} + 1, \quad \forall V_j \in L^B(\ell_j), \; V_j \geq 0. \] (28)

The proof of the theorem is an adaptation of the argument in [39] to the case
of singular potentials. In particular, we use the following equivalent n orm on
$L^2(\Omega)$ with $\mu(\Omega) < \infty$ which was introduced in [39]:

\[ \|f\|_{L^2(\Omega)}^{(av)} = \|f\|_{L^2(\Omega, \mu)}^{(av)} = \sup \left\{ \left( \int_\Omega f g d\mu \right) : \int_\Omega \Phi(g) d\mu \leq \mu(\Omega) \right\}. \]

Let $Q := (0,1)^2$ and $I := (0,1)$. For any subinterval $I \subseteq \mathbb{I}$, we denote the
square $I \times (0,|I|) \subseteq Q$ by $S(I)$. We will also use the following notation:

\[ w_S := \frac{1}{|S|} \int_S w(x) dx, \]

where $S \subseteq \mathbb{R}^2$ is a set of a finite positive two dimensional Lebesgue measure $|S|$.

**Lemma 3.2.** (Cf. [39, Lemma 2]) There exists $C_3 > 0$ such that for any
$I \subseteq \mathbb{I}$, any $w \in W^1_2(S(I)) \cap C \left( S(I) \right)$ with $w_{S(I)} = 0$ and any $V \in L^B(I), \; V \geq 0$ the following inequality holds:

\[ \int_I V(t)w^2(t,0) dt \leq C_3\|V\|_{L^B,\ell_j}^{(av)} \int_{S(I)} |\nabla w(x)|^2 dx. \] (29)
Proof. Let us start with the case $I = \mathbb{I}$. There exists $C_4 > 0$ such that
\[ \|w^2(\cdot, 0)\|_{A, I} \leq C_4 \|w\|_{W_{2}^1(Q)}^2, \quad \forall w \in W_{2}^1(Q) \cap C(Q) \]
(see (16)). This can be proved by applying the trace theorem (see, e.g., [1, Theorems 4.32 and 7.53]) and then using the Yudovich–Pohozhaev–Trudinger embedding theorem for $H^{1/2,2}$ (see [16], [29] and [25, Lemma 1.2.4], [1, 8.25]) or, in one go, by applying a trace inequality of the Yudovich–Pohozhaev–Trudinger type (see [24, Corollary 11.8/2]; a sharp result can be found in [10]).

Next, we use the Poincaré inequality (see, e.g., [24, 1.11.1]): there exists $C_5 > 0$ such that
\[ \int_Q (w(x) - w_Q)^2 \, dx = \inf_{a \in \mathbb{R}} \int_Q (w(x) - a)^2 \, dx \leq C_5 \int_Q |\nabla w(x)|^2 \, dx, \quad \forall w \in W_2^1(Q). \]
Hence
\[ \|w^2(\cdot, 0)\|_{A, I} \leq C_3 \int_Q |\nabla w(x)|^2 \, dx, \quad \forall w \in W_{2}^1(Q) \cap C(Q), \quad w_Q = 0, \]
where $C_3 = C_4(C_5 + 1)$. Now, the Hölder inequality (see [19, Theorem 9.3]) implies
\[ \int_{\mathbb{I}} V(t) w^2(t, 0) \, dt \leq \|V\|_{B, I} \|w^2(\cdot, 0)\|_{A, I} \leq C_3 \|V\|_{B, I} \int_Q |\nabla w(x)|^2 \, dx, \]
i.e. (29) holds for $I = \mathbb{I}$. For an arbitrary subinterval $I \subseteq \mathbb{I}$, (29) is proved by applying a simple affine transformation and arguing as in the proof of Lemma 2 in [39].

Lemma 3.3. (Cf. [39, Theorem 1]) For any $V \in L_B(\mathbb{I})$, $V \geq 0$ and any $n \in \mathbb{N}$ there exists a finite cover of $\mathbb{I}$ by intervals $I_k$, $k = 1, \ldots, n_0$ such that $n_0 \leq n$ and
\[ \int_{\mathbb{I}} V(t) w^2(t, 0) \, dt \leq 4C_3 n^{-1} \|V\|_{B, I} \int_Q |\nabla w(x)|^2 \, dx \]
for all $w \in W_2^1(Q) \cap C(Q)$ with $w_{S(I_k)} = 0$, $k = 1, \ldots, n_0$. (The constant $C_3$ here is the same as in Lemma 3.2.)
Proof. Arguing as in the proof of Theorem 1 in [39] (see also the proof of Lemma 7.6 in [36]) and taking into account that the best constant in the Besicovitch covering lemma equals 2 in the one dimensional case (see, e.g., [17, Theorem 7] or [41, Ch. 3, Section 6, Problem 2]), one can prove the existence of a finite cover of $I$ by intervals $I_k$, $k = 1, \ldots, n_0$ such that $n_0 \leq n$ and $\|V\|_{B, I_k} = 2n^{-1}\|V\|_{B, I}$. Then Lemma 3.2 implies the following for all $w \in W^1_2(Q) \cap C(Q)$ with $w_{S(I_k)} = 0$, $k = 1, \ldots, n_0$

$$
\int_{I} V(t) w^2(t, 0) dt \leq \sum_{k=1}^{n_0} \int_{I_k} V(t) w^2(t, 0) dt \\
\leq \sum_{k=1}^{n_0} C_3 \|V\|_{B, I_k} \int_{S(I_k)} |\nabla w(x)|^2 dx \\
= 2C_3 n^{-1} \|V\|_{B, I} \sum_{k=1}^{n_0} \int_{S(I_k)} |\nabla w(x)|^2 dx \\
\leq 4C_3 n^{-1} \|V\|_{B, I} \int_Q |\nabla w(x)|^2 dx.
$$

In the last inequality, we use the fact that the intervals $I_k$, $k = 1, \ldots, n_0$ can be divided into two groups in such a way that any two intervals from the same group are disjoint.

Proof of Theorem 3.1. Let $\varphi_j : [0, 1] \to \mathbb{R}^2$ be a bi-Lipschitz mapping such that $\varphi_j([0, 1]) = \ell_j$, $j = 1, \ldots, J$. The mapping $\varphi_j$ can be extended to a bi-Lipschitz homeomorphism $\varphi_j : \mathbb{R}^2 \to \mathbb{R}^2$ ([45], see also [12, 18, 23] and [32, Theorem 7.10]). Using Lemma 3.3 one can easily prove that for any $n_j \in \mathbb{N}$ there exist intervals $I_{j,k}$, $k = 1, \ldots, n_{j,0}$ such that $n_{j,0} \leq n_j$ and

$$
\int_{I_{\ell_j}} V_j(x) w^2(x) ds(x) \leq C(\ell_j) n_{j}^{-1} \|V_j\|_{B, \ell_j} \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx
$$

for all $w \in W^1_2(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ with

$$
\int_{\varphi_j(S(I_{j,k}))} w(x) \left|J_{\varphi_j^{-1}}(x)\right| dx = 0, \quad k = 1, \ldots, n_{j,0},
$$

where $J_{\varphi_j^{-1}}$ is the Jacobian determinant of $\varphi_j^{-1}$, and $C(\ell_j)$ is independent of $V_j$ and $n_j$.

Let $n_j = [C(\ell_j) J\|V_j\|_{B, \ell_j}] + 1$, $j = 1, \ldots, J$. Take any linear subspace $\mathcal{L} \subset W^1_2(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ such that

$$
\dim \mathcal{L} > \sum_{j=1}^{J} n_j = J + \sum_{j=1}^{J} [C(\ell_j) J\|V_j\|_{B, \ell_j}].
$$
Since \( n_{j,0} \leq n_j \), there exists \( w \in \mathcal{L} \setminus \{0\} \) which satisfies (31) for all \( j = 1, \ldots, J \). Then

\[
\mathcal{E}_V[w] = \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx - \sum_{j=1}^J \int_{\ell_j} V_j(x) w^2(x) ds(x)
\]

\[
\geq \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx - \sum_{j=1}^J \frac{C(\ell_j)}{J} \|V_j\|_{B,\ell_j} \frac{1}{\|V_j\|_{B,\ell_j}} + 1 \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx
\]

\[
\geq \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx - \sum_{j=1}^J \frac{1}{J} \int_{\mathbb{R}^2} |\nabla w(x)|^2 dx = 0.
\]

Hence

\[
N_-(\mathcal{E}_V) \leq J + \sum_{j=1}^J \left[ C(\ell_j) \|V_j\|_{B,\ell_j} \right] \leq J \left( \sum_{j=1}^J C(\ell_j) \|V_j\|_{B,\ell_j} + 1 \right). \tag{32}
\]

Take \( R > 0 \) such that

\[
\Sigma \subset B_R := \{ x \in \mathbb{R}^2 : |x| < R \}. \tag{33}
\]

Using the Yudovich–Pohožaev–Trudinger embedding theorem and the Hölder inequality for Orlicz spaces as in the proof of Lemma 3.2 one can prove the existence of a constant \( C_0(\Sigma) > 0 \) such that

\[
\sum_{j=1}^J \int_{\ell_j} V_j(x) w^2(x) ds(x) \leq C_0(\Sigma) \sum_{j=1}^J \|V_j\|_{B,\ell_j} \|w\|^2_{W^1_2(B_R)},
\]

\[
\forall w \in W^1_2(\mathbb{R}^2) \cap C(\mathbb{R}^2).
\]

Next, we use the Poincaré inequality as in the proof of Lemma 3.2 there exists a constant \( C_R > 0 \) such that

\[
\|w\|^2_{W^1_2(B_R)} \leq C_R \int_{B_R} |\nabla w(x)|^2 dx, \quad w \in W^1_2(B_R), \ w_{B_R} = 0.
\]

If \( \sum_{j=1}^J \|V_j\|_{B,\ell_j} \leq 1/(C_0(\Sigma)C_R) \), then

\[
\mathcal{E}_V[w] \geq 0
\]

for all \( w \in W^1_2(\mathbb{R}^2) \cap C(\mathbb{R}^2) \) with \( w_{B_R} = 0 \), and hence \( N_-(\mathcal{E}_V) \leq 1 \). Combining this with (32) one gets the existence of a constant \( C(\Sigma) > 0 \) for which (28) holds.
Remark 3.4. Let $R > 0$ satisfy (33). Then it follows from the above proof that (28) holds, with a constant depending on $R$, for the form
\[
\mathcal{E}_{V,R}[w] := \int_{B_R} |\nabla w(x)|^2 dx - \sum_{j=1}^J \int_{\ell_j} V_j(x) w^2(x) ds(x),
\]
\[
\text{Dom} (\mathcal{E}_{V,R}) = W^2_2(B_R) \cap C (\overline{B_R}).
\]

Remark 3.5. It might be interesting to give an explicit estimate of the constant $C(\Sigma)$ in (28) in terms of geometric properties of $\ell_j$, $j = 1, \ldots, J$ (see [11, 22] for some related results in the case $V = \text{const}$).

Remark 3.6. Using conformal mappings like $z \mapsto \frac{1}{z - z_0}$ one can extend Theorem 3.1 to some unbounded Lipschitz curves (see [36] where this trick has been applied in the case $V \in L_B(\mathbb{R}^2)$). One can also combine Theorem 3.1 with the known results on Schrödinger operators with locally integrable on $\mathbb{R}^2$ potentials (see [5, 13, 20, 26, 27, 36, 39, 40] and the references therein). It would be interesting to extend these results to potentials of the form $\varrho \mu$, where $\mu$ is a Radon measure and $\varrho$ is a suitable function (see [4]).

4 Proof of Theorem 1.2

Let
\[
\mathcal{E}_0(u,v) := \int_{-\pi}^{\pi} (\mathcal{C}u'(t))v(t) dt, \quad \mathcal{E}_0[u] := \mathcal{E}_0(u,u).
\]

Proof of the lower estimate in (17). Let $\mathbb{D}$ denote the unit disk. For any $w \in W^2_2(\mathbb{D}) \cap C (\overline{\mathbb{D}})$, define $w^*(t) := w(e^{it})$, $t \in \mathbb{R}$. According to the trace theorem (see, e.g., [21 Ch. 1, Theorem 8.3] or [11 Theorem 7.53]), there exists $c_1 > 0$ such that
\[
\mathcal{E}_0[w^*] \leq c_1 \mathcal{E}[w], \quad \forall w \in W^2_2(\mathbb{D}) \cap C (\overline{\mathbb{D}}),
\]
where
\[
\mathcal{E}[w] := \|w\|^2_{W^2_2(\mathbb{D})}, \quad \text{Dom} (\mathcal{E}) = W^1_2(\mathbb{D}) \cap C (\overline{\mathbb{D}}).
\]

Let $X$ be the closed unit disk. Define a Radon measure $\sigma$ on $X$ by
\[
\sigma(E) := \int_{(F \cap \mathbb{T})^*} V(t) dt,
\]
where
\[
F^* := \{ t \in (-\pi, \pi) : e^{it} \in F \}, \quad F \subseteq \mathbb{T}.
\]
The corresponding Hermitian form is
\[
\sigma(w, h) := \int_X wh \, d\sigma = \int_{-\pi}^{\pi} V(t)w^*(t)h^*(t) \, dt.
\]

It follows from [14, Theorem 4.1] that there is a universal constant \(c > 0\) for which
\[
N_-(\mathcal{E} - \frac{1}{c_1} \sigma) \geq \left[ \frac{c}{c_1} \sigma(X) \right] = \left[ \frac{c}{c_1} \|V\|_{L^1_{2\pi}} \right],
\]
where \([\cdot]\) denotes the integer part.

Let \(L \subset \text{Dom } (\mathcal{E})\) be a linear subspace such that
\[
c_1 \mathcal{E}(w) - \sigma[w] < 0, \quad \forall w \in L \setminus \{0\}.
\]

Then the linear mapping \(w \mapsto w^*\) is one-to-one on \(L\). Otherwise there would exist \(w_0 \in L \setminus \{0\}\) such that \(w_0^* = 0\) and
\[
c_1 \mathcal{E}(w_0) - \sigma[w_0] = c_1 \|w_0\|_{W^2_{2\pi}}^2 - \int_{-\pi}^{\pi} V(t) (w_0^*(t))^2 dt = c_1 \|w_0\|_{W^2_{2\pi}}^2 > 0,
\]
which is a contradiction. Now, it follows from (35) that
\[
N_-(q_V) \geq N_-(c_1 \mathcal{E} - \sigma) = N_-(\mathcal{E} - \frac{1}{c_1} \sigma) \geq \left[ \frac{c}{c_1} \|V\|_{L^1_{2\pi}} \right].
\]

Note that \(N_-(q_V) \geq 1\) if \(V \neq 0\). Indeed,
\[
q_V[1] = \mathcal{E}_0[1] - \int_{-\pi}^{\pi} V(t) dt = -\|V\|_{L^1_{2\pi}} < 0.
\]

Hence
\[
N_-(q_V) \geq \frac{c}{2c_1} \|V\|_{L^1_{2\pi}}
\]
(cf. the proof of Theorem 4.17 in [14]). \(\square\)

**Remark 4.1.** (i) The reason for applying Theorem 4.1 of [14] to the form \(\mathcal{E}\) rather than directly to \(\mathcal{E}_0\) in the above proof is that the former is local while the latter is not.

(ii) One can take \(X = \mathbb{R}^2\) instead of \(X = \overline{D}\) in the above proof. The former is more convenient in the proof of the upper estimate for \(N_-(q_V)\) below.

**Proof of the upper estimate in (17).** Since \(\left( \mathcal{E}_0[u] + \|u\|_{L^2_{2\pi}}^2 \right)^{1/2}\) is an equivalent norm on \(H_{2\pi}^{1/2}\) (see [20], [33]), it follows from the trace theorem (see,
e.g., [21, Ch. 1, Theorems 8.1 and 8.3] or [1, Theorems 4.28 and 7.53]) that there exists a linear operator $R : H^{1/2,2}_{2\pi} \cap C_{2\pi} \rightarrow W^1_2(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ and a constant $c_2 > 0$ which satisfy

$$\|Ru\|_{W^1_2(\mathbb{R}^2)}^2 \leq c_2 \left( \mathcal{E}_0[u] + \|u\|_{L^2_{2\pi}}^2 \right), \quad \forall u \in H^{1/2,2}_{2\pi} \cap C_{2\pi},$$

and $(Ru)(e^{it}) = u(t)$, $t \in \mathbb{R}$.

Consider the form

$$\mathcal{E}_{1,V}[w] := \int_{\mathbb{R}^2} |\nabla w(x)|^2 \, dx - c_2 \int_{-\pi}^{\pi} (V(t) + 1)w^2(e^{it}) \, dt,$$

$$\text{Dom}(\mathcal{E}_{1,V}) = W^1_2(\mathbb{R}^2) \cap C(\mathbb{R}^2).$$

Let $\mathcal{L}_0$ be a linear subspace of $H^{1/2,2}_{2\pi} \cap C_{2\pi}$ such that $\dim \mathcal{L}_0 = N - (q_{V})$ and $q_{V}[w] < 0$, $\forall u \in \mathcal{L}_0 \setminus \{0\}$ (see [32], [12]), and let $\mathcal{L} = R\mathcal{L}_0$. Then

$$\mathcal{E}_{1,V}[w] < 0, \quad \forall w \in \mathcal{L} \setminus \{0\}.$$ 

Indeed,

$$\mathcal{E}_{1,V}[w] := \int_{\mathbb{R}^2} |\nabla w(x)|^2 \, dx - c_2 \int_{-\pi}^{\pi} (V(t) + 1)w^2(e^{it}) \, dt$$

$$\leq c_2 \left( \mathcal{E}_0[w^*] + \|w^*\|_{L^2_{2\pi}}^2 \right) - c_2 \int_{-\pi}^{\pi} (V(t) + 1)w^2(e^{it}) \, dt$$

$$= c_2 q_{V}[w^*] < 0, \quad \forall w \in \mathcal{L} \setminus \{0\},$$

where $w^*(t) := w(e^{it})$, $t \in \mathbb{R}$ as in the proof of the lower estimate in (17).

Since $\dim \mathcal{L} = \dim \mathcal{L}_0 = N - (q_{V})$, Theorem 3.1 implies the existence of a constant $C > 0$ such that

$$N - (q_{V}) \leq N - (\mathcal{E}_{1,V}) \leq C\|c_2(V + 1)\|_{B([-\pi,\pi])} + 1$$

$$\leq Cc_2\|V\|_{B([-\pi,\pi])} + Cc_2\|1\|_{B([-\pi,\pi])} + 1. \quad (36)$$

(According to [19] (9.11), $\|1\|_{B([-\pi,\pi])} = 2\pi A^{-1}(\frac{1}{2\pi})$.)

Using the Yudovich–Pohožaev–Trudinger embedding theorem for $H^{1/2,2}$ and the Hölder inequality for Orlicz spaces as in the proofs of Lemma 3.2 and Theorem 3.1, one can prove the existence of a constant $c_3 > 0$ such that

$$\int_{-\pi}^{\pi} V(t)u^2(t) \, dt \leq c_3 \|V\|_{B([-\pi,\pi])} \left( \mathcal{E}_0[u] + \|u\|_{L^2_{2\pi}}^2 \right), \quad \forall u \in H^{1/2,2}_{2\pi} \cap C_{2\pi}.$$
If $\|V\|_B \leq 1/(2c_3)$, then

$$q_V[u] \geq E_2[u] := \frac{1}{2} \left( E_0[u] - \|u\|_{L^2}^2 \right), \quad \forall u \in H^{1/2,2}_{2\pi} \cap C_{2\pi}$$

and

$$N_-(q_V) \leq N_-(E_2) = 1,$$

where the last equality holds because the spectrum of $u \mapsto Cu'$ consists of a simple eigenvalue 0 and of double eigenvalues $m \in \mathbb{N}$ (see (20)). Combining this estimate with (36) one gets the existence of a constant $C_2 > 0$ for which the upper estimate in (17) holds.

Sketch of an alternative proof of the upper estimate in (17). Let $P u$ be the harmonic in $\mathbb{D}$ function such that $(P u)(e^{it}) = u(t), t \in \mathbb{R}$, i.e. let $P u$ be the Poisson integral of $u$. If $u \in C^\infty_{2\pi}$, then the Cauchy–Riemann equations allow one to express the normal derivative of $P u$ at $z = e^{it}$ in terms of the tangential derivative of the harmonic conjugate of $P u$. Applying Green’s identity one gets

$$\int_{\mathbb{D}} |\nabla P u(x)|^2 dx = \mathcal{E}_0[u],$$

which can be extended to all $u \in H^{1/2,2}_{2\pi}$ by continuity.

Let us extend $P u$ to $\mathbb{R}^2 = \mathbb{C}$ by

$$P u(z) := P u \left( \frac{1}{z} \right), \quad z \in \mathbb{C}, \quad |z| > 1.$$ 

An easy calculation shows that

$$\int_{\mathbb{R}^2 \setminus \mathbb{D}} |\nabla P u(x)|^2 dx = \int_{\mathbb{D}} |\nabla P u(x)|^2 dx.$$

Since

$$\lim_{\mathbb{R}^2 \cap \mathbb{D} \ni y \to x} P u(y) = \lim_{\mathbb{D} \ni y \to x} P u(y), \quad |x| = 1,$$

the extended function $P u$ belongs to $W^1_2(B_2) \cap C(\mathbb{R}^2)$ (see (33)) and

$$\int_{B_2} |\nabla P u(x)|^2 dx \leq \int_{\mathbb{R}^2} |\nabla P u(x)|^2 dx = 2E_0[u].$$

Let

$$\mathcal{E}_{3, V}[w] := \int_{B_2} |\nabla w(x)|^2 dx - 2 \int_{-\pi}^{\pi} V(t) w^2(e^{it}) dt, \quad \text{Dom} (\mathcal{E}_V) = W^1_2(B_2) \cap C(B_2).$$
Using Remark 3.4 and arguing as above, one gets

\[ N_- (q_V) \leq N_- (E_{3,V}) \leq 2C \| V \|_{\mathcal{B}([-\pi, \pi])} + 1, \]

and there is no need for an additional treatment of the case of a small \( \| V \|_{\mathcal{B}([-\pi, \pi])} \) like in the final part of the above proof.

**Remark 4.2.** Repeating the argument from Example 2.7 in [13] one can show that no estimate of the type

\[ N_- (q_V) \leq \text{const} + \int_{-\pi}^{\pi} V(t) W(t) \, dt \]

can hold, provided the weight function \( W \) is bounded in a neighborhood of at least one point.

**Remark 4.3.** If \( V \in L_{\mathcal{B}}([-\pi, \pi]), V \geq 0 \), then

\[ \lim_{\alpha \to +\infty} \frac{N_- (q_{\alpha V})}{\alpha} = \frac{1}{\pi} \int_{-\pi}^{\pi} V(t) \, dt. \] (37)

Indeed, suppose first that \( V \in C^\infty_{2\pi}, V > 0 \) and consider the form

\[ \mathcal{E}_{\alpha,V}[u] := \mathcal{E}_0 [V^{-1/2}u] - \alpha \| u \|_{L^2_{2\pi}}^2, \quad u \in H^{1/2,2}_{2\pi} \cap C_{2\pi}. \]

Since the operator of multiplication by \( V^{-1/2} \) is invertible on \( H^{1/2,2}_{2\pi} \cap C_{2\pi} \), we get

\[ N_- (q_{\alpha V}) = N_- (\mathcal{E}_{\alpha,V}). \]

The right-hand side equals the value at \( \alpha \) of the eigenvalue counting function of the operator \( u \mapsto V^{-1/2} \mathcal{C} (V^{-1/2}u)' \), which can be viewed as a first-order pseudodifferential operator on the unit circle. Formula (37) follows from the well known results on spectral asymptotics for elliptic pseudodifferential operators on compact manifolds (see, e.g., [38, §15]; sharp asymptotic results for operators on the circle can be found in [34]). One can extend (37) to arbitrary \( V \in L_{\mathcal{B}}([-\pi, \pi]), V \geq 0 \) by a standard perturbation-theoretic argument (see, e.g., the proof of Theorem 2.2 in [40]).

## 5 Proof of Theorem 1.1

We prove a generalisation of Theorem 1.1 to Bernoulli free-boundary problems ([37]) which are obtained from (1)-(5) by substituting the boundary condition (5) with the following one:

\[ |\nabla \psi (X,Y)|^2 = \lambda (Y) \text{ almost everywhere on } \mathcal{S}, \] (38)
where \( \lambda : \mathbb{R} \to \mathbb{R} \) is a given continuous function. We assume that \( \lambda \) is non-constant and real analytic on the open set of full measure where it is non-zero, and that the following holds with some \( \varrho > 0 \):

\[
\text{if } \lambda(y_0) = 0, \text{ then } |\lambda(y)| \leq \text{const} |y - y_0|^{\varrho}, \quad \forall y \in \mathbb{R}, \quad (39)
\]

\[
\ln |\lambda| \text{ is concave, and } \lambda' \leq 0 \text{ where } \lambda \neq 0. \quad (40)
\]

If \((S, \psi)\) is a solution of \((1)-(4), (38)\), then, clearly, \(\lambda \geq 0\) almost everywhere on \(S\). If \(\lambda > 0\) everywhere on \(S\), then \(S\) and \(\psi\) are real analytic \((37)\), and the solution is called regular or non-singular. Let

\[
p(\varrho) := \frac{\varrho + 2}{\varrho}.
\]

Non-singular solutions of \((1)-(4), (38)\) are in one-to-one correspondence with the critical points \(v \in W^{1,p(\varrho)}_{2\pi}\) of the functional

\[
\mathcal{J}(v) = \int_{-\pi}^{\pi} \left( \Lambda(v(t))(1 + \mathcal{C}v'(t)) - v(t) \right) dt, \quad v \in W^{1,2}_{2\pi},
\]

where \(\Lambda\) is a primitive of \(\lambda\) (see \([37]\)).

Let \(v\) be a critical point of \(\mathcal{J}\). The Morse index \(M(v)\) of \(v\) and of the corresponding solution to \((1)-(4), (38)\) is the number \(N_-(Q_v)\), where \(Q_v\) is the quadratic form of the second Fréchet derivative \(\mathcal{J}''(v)\) (the Hessian):

\[
Q_v[u] := \int_{-\pi}^{\pi} \left( 2\lambda(v(t))u(t)\mathcal{C}u'(t) + \lambda'(v(t))(1 + \mathcal{C}v'(t))u^2(t) \right) dt.
\]

Every critical point \(v \in W^{1,p(\varrho)}_{2\pi}\) of \((11)\) is a real analytic function and

\[
\min_{t \in \mathbb{R}} \lambda(v(t)) > 0
\]

(see \([37]\)). Let

\[
\nu(v) := \max_{t \in \mathbb{R}} \frac{|\lambda'(v(t))|}{\lambda(v(t))}, \quad \nu_0(v) := \max_{t \in \mathbb{R}} \frac{1}{\lambda(v(t))} = \frac{1}{\min_{t \in \mathbb{R}} \lambda(v(t))}.
\]

Suppose there exist constants \(m_1, m_2 > 0\) such that

\[
\frac{m_1}{\lambda(y)^{1/\varrho}} \leq \frac{|\lambda'(y)|}{\lambda(y)} \leq \frac{m_2}{\lambda(y)^{1/\varrho}} \quad \text{for all } y \in \mathbb{R} \text{ with } \lambda(y) \neq 0. \quad (44)
\]

It is easy to see that \((44)\) implies \((39)\).

In the case of Stokes waves, \(\lambda(y) = 1 - 2\mu y\) and \((10), (14)\) hold with \(\varrho = 1\) and \(m_1 = m_2 = 2\mu\) (note also that \(p(1) = 3\)). Hence, Theorem \((11)\) is a special case of the following result.
Theorem 5.1. Suppose (10) and (14) hold. Then there exist constants $M_1, M_2 > 0$ which depend only on $\varrho$ and are such that

$$M_1 \frac{m_1}{m_2} \ln \frac{v}{\varrho^2} (1 + \nu(v)) \leq \mathcal{M}(v) \leq 1 + M_2 \nu(v) \ln (2 + \nu_0(v))$$

(45)

holds for every critical point $v \in W_{2\pi}^{1,p(\varrho)}$ of (11).

Using Plotnikov’s transformation, one can show that $M(v) = N_-(q_v)$ (see (12)), where

$$2V(t) = \mathcal{C} \left( \frac{\lambda'(v(t))}{\lambda(v(t))} v'(t) \right) - \frac{\lambda'(v(t))}{\lambda(v(t))} (1 + C v'(t)) > 0, \quad t \in \mathbb{R}$$

(46)

(see [37]). Hence, Theorems 5.1 and 1.1 follow from Theorem 1.2 if one has suitable estimates for $\|V\|_{L^1_{2\pi}}$ and $\|V\|_{B_{[-\pi,\pi]}}$. These are provided by the following two lemmas.

Lemma 5.2. There exists a constant $M_3 > 0$ which depends only on $\varrho$ and is such that

$$\|V\|_{L^1_{2\pi}} \geq M_3 \frac{m_1}{m_2} \ln \frac{v}{\varrho^2} (1 + \nu(v)).$$

Proof. This is an easy corollary of Lemma 4.25 in [37].

Lemma 5.3. There exists a constant $M_4 > 0$ which depends only on $\varrho$ and is such that

$$\|V\|_{B_{[-\pi,\pi]} \leq M_4 \nu(v) \ln (2 + \nu_0(v)).$$

Proof. Since $v \in W_{2\pi}^{1,p(\varrho)}$ is a critical point of (11), $1 + C v' > 0$ and $\|1 + C v'\|_{L^\infty_{2\pi}} \leq \nu_0(v)$ (see [37] Theorems 2.4, 3.3 and 3.5). Then it follows from (46), (18) and (43) that

$$\|V\|_{L^1_{2\pi}} = \int_{-\pi}^{\pi} V(t) \, dt = \int_{-\pi}^{\pi} \frac{\lambda'(v(t))}{2\lambda(v(t))} (1 + C v'(t)) \, dt \leq \pi \nu(v)$$

(see [37] Lemma 4.24), and (23) implies

$$\|1 + C v'\|_{B_{[-\pi,\pi]}} \leq 8 \ln \left( 2 + \frac{2\pi \sqrt{\nu_0(v)}}{\|1 + C v'\|_{L^1_{2\pi}}} \right) \int_{-\pi}^{\pi} (1 + C v'(t)) \, dt$$

$$= 16\pi \ln \left( 2 + \sqrt{\nu_0(v)} \right).$$

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Hence

\[ \|CV\|_{L^2_{\pi}} \leq \frac{1}{2} \left\| \frac{\lambda'(v)}{\lambda(v)} v' \right\|_{L^2_{\pi}} + \frac{1}{2} \left\| C \left( \frac{\lambda'(v)}{\lambda(v)} (1 + Cv') \right) \right\|_{L^2_{\pi}} \]

\[ \leq \frac{\nu(v)}{2} \|v'\|_{L^2_{\pi}} + \text{const} \nu(v) \|1 + Cv'\|_{B_{\pi,\pi}} \]

\[ \leq \text{const} \nu(v) \|1 + Cv'\|_{B_{\pi,\pi}} \leq \text{const} \nu(v) \ln(2 + \nu_0(v)) \]

with constants independent of \( v \) and \( \lambda \) (see (19) and (25)). It is now left to apply (26).

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