On the problem of classifying solvable Lie algebras having small codimensional derived algebras

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ABSTRACT
In this paper, we study the classification of finite-dimensional real solvable Lie algebras whose derived algebras are of codimension 1 or 2. We present an effective method to classify \((n+1)\)-dimensional real solvable Lie algebras having 1-codimensional derived algebras provided that a full classification of \(n\)-dimensional nilpotent Lie algebras is given. In addition, the problem of classifying \((n+2)\)-dimensional real solvable Lie algebras having 2-codimensional derived algebras is proved to be wild. In this case, we classify a subclass of the considered Lie algebras which are extended from their derived algebras by a pair of derivations containing at least one inner derivation.

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1. Introduction

About 1972, Donovan and Freislich [8, 9] introduced the notion of wildness. Namely, a classification problem is called to be wild if it contains the problem of classifying pairs of matrices up to simultaneous similarity. According to Belitskii and Sergeichuk [4, Section 1], wild problems are hopeless in a certain sense. Several classification problems were pointed out to be wild (see [1–4, 12, 30] and references therein). Unfortunately, the problem of classifying solvable Lie algebras is wild since it contains the problem of classifying nilpotent Lie algebras which is wild (see [1, Theorem 4]). As a consequence, the problem of classifying solvable Lie algebras is very difficult.

From historical point of view, real nilpotent Lie algebras of dimensions 5 and 6 were classified by Dixmier [6] and Morozov [19], respectively. With respect to solvable Lie algebras, Mubarakzyanov [20–22] and Turkowski [34] classified solvable Lie algebras up to dimension 6 over the real field. Beyond dimension 6, there are partial results such as Turkowski [35], Gong [13], Parry [26], Hindeleh and Thompson [15]. Furthermore, classification problems for solvable Lie algebras over arbitrary fields have also been studied. Solvable Lie algebras of dimension up to 4 over perfect fields have been classified by Patera and Zassenhaus [27]. Besides, de Graaf [7] proposed the Gröbner basis technique to classify solvable Lie algebras in low dimensions over fields of any characteristic. Further partial results in low dimensions can be found in [5, 33] and references therein. However, the classification of solvable Lie algebras is far from being completed.
It is well-known that one of the most popular ways to classify solvable Lie algebras is to classify solvable extensions of nilradicals, that is, we start with a nilpotent Lie algebra and then classify solvable Lie algebras which admit it as their nilradicals. This method was initialized in 1963 by a series of articles of Mubarakzyanov [20, 21] when he classified solvable Lie algebras of dimensions 4 and 5 over a field of characteristic zero. By using the same method, the results for the case of dimension 6 were also achieved by Mubarakzyanov [22] and Turkowski [34]. Furthermore, the results of Shabanskaya and Thompson [31, 32] as well as Ndogmo and Winternitz [23, 24] show that this method is also effective for classifying arbitrarily finite-dimensional solvable Lie algebras.

In this paper, we present another approach to classify solvable Lie algebras which are solvable extensions of derived algebras. The problem of classifying solvable Lie algebras with a given derived algebra has been extensively studied recently. Real solvable Lie algebras with 1-dimensional derived algebras are completely classified by Schöbel [28]. Partial results on the classification of solvable Lie algebras with 2-dimensional-derived algebras was obtained in [10, 28]. Afterwards, based on the well-known formula of the maximal dimension of commutative subalgebras contained in the matrix Lie algebra which established by Schur [29] and Jacobson [17], a full classification for real solvable Lie algebras with 2-dimensional derived algebras was achieved in [18].

To the best of our knowledge, the problem of classifying real solvable Lie algebras with the derived algebras of dimension different from 1 and 2 still remains open. This paper aims to study the problem of classifying solvable Lie algebras with high dimensional derived algebras. For convenience, we denote by Lie(n + 1, n) (resp., Lie(n + 2, n)) the class of all (n + 1)-dimensional (resp., (n + 2))-dimensional real solvable Lie algebras with n-dimensional derived algebras. This paper contains three main results which are as follows.

First of all, for a given n-dimensional real nilpotent Lie algebra K, each Lie algebra in Lie(n + 1, n) admitting K as its derived algebra is an extension of K by a derivation of K. We point out that the derivation of this extension must be an outer derivation. However, a Lie algebra extended from K by an outer derivation of K is not necessary in Lie(n + 1, n). We give necessary and sufficient conditions for the derivation so that the extension is in Lie(n + 1, n) (see Proposition 3.1). Furthermore, we prove that:

**Theorem 1.** For an arbitrary n-dimensional nilpotent Lie algebra K, the problem of classifying all Lie algebras in Lie(n + 1, n) with derived algebra isomorphic to K is equivalent to the problem of classifying outer derivations in the first cohomology space $H^1(K, K)$ satisfying condition 3 in Proposition 3.1, up to proportional similarity by the automorphism group of K.

Similarly, for a given n-dimensional real nilpotent Lie algebra H, each Lie algebra in Lie(n + 2, n) admitting H as its derived algebra is an extension of H by a pair of derivations. However, not every extension of H by a pair of derivations is in Lie(n + 2, n). We give necessary and sufficient conditions for the pair of derivations so that the extension is always in Lie(n + 2, n) (see Proposition 4.1). Based on the conditions, we prove that:

**Theorem 2.** The problem of classifying Lie(n + 2, n) is wild.

The wildness of the problem of classifying Lie(n + 2, n) motivates us to consider special cases (see Belitskii et al. [2, Section 3]). Based on the proof of Theorem 2, we consider the subclass $\text{Lie}_{ad}(n + 2, n) \subseteq \text{Lie}(n + 2, n)$ containing the Lie algebras satisfying the condition: “the pair of derivations of the extension contains at least one inner derivation”. For this subclass, we have:

**Theorem 3.** For an arbitrary n-dimensional nilpotent Lie algebra H, the problem of classifying all Lie algebras in $\text{Lie}_{ad}(n + 2, n)$ with derived algebra isomorphic to H is equivalent to the problem of classifying outer derivations in the first cohomology space $H^1(\mathbb{R} \oplus H, \mathbb{R} \oplus H)$ satisfying condition 3 in Proposition 4.1, up to proportional similarity by the automorphism group of $\mathbb{R} \oplus H$. 

We emphasize that the problem of classifying \( \text{Lie}(n+1,n) \) and \( \text{Lie}_{ad}(n+2,n) \) is essentially wild since it involves the problem of classifying \( n \)-dimensional real nilpotent Lie algebras which is wild as mentioned above. However, Theorems 1 and 3 can be applied to classify \( \text{Lie}(n+1,n) \) and \( \text{Lie}_{ad}(n+2,n) \), respectively, whenever a classification of \( n \)-dimensional real nilpotent Lie algebras is given. We illustrate these applications in Algorithms 1 and 2 as well as Examples 3.5 and 5.5.

The paper is structured as follows. In Section 2, we recall necessary definitions and facts from Lie algebras. Afterwards, we prove Theorem 1 and illustrate its applications in Section 3. Section 4 is devoted to proving Theorem 2. Finally, in Section 5, the proof of Theorem 3 and its applications will be expressed.

### 2. Preliminaries

Throughout this paper, the base field is always the field \( \mathbb{R} \) of real numbers and \( n \geq 2 \) is an integer. We also use the following notations:

- \( \text{span}\{x_1,\ldots,x_n\} \) is the vector space with basis \( \{x_1,\ldots,x_n\} \).
- For two vector subspaces \( U \) and \( V \), we denote their sum and direct sum by \( U + V \) and \( U \perp V \), respectively.
- For a Lie algebra \( L \), the notations \( L^1 := [L,L] \) and \( L^2 := [L^1,L^1] \) are respectively the first and second derived ideals in its derived series (note that \( L^1 \) is also called the derived algebra of \( L \)). Besides, \( \mathcal{Z}(L) \), \( \text{Der}(L) \), \( \text{ad}(L) \) and \( \text{Aut}(L) \) indicate the center, the set of derivations, inner derivations and automorphisms of \( L \), respectively. It is also well-known that the quotient space \( \text{Der}(L)/\text{ad}(L) \) can be identified with \( H^1(L,L) \) which is the first cohomology space of \( L \) with coefficients in \( L \) (see [14, Proposition 2.2]).
- \( a_x := \text{ad}_x|_{L^1} \) is the restriction of the adjoint operator \( \text{ad}_x \) on \( L^1 \).
- \( \text{Mat}_n(\mathbb{R}) \) is the set of \( n \)-square matrices with real entries and \( \text{GL}_n(\mathbb{R}) \) denotes the group of all invertible matrices in \( \text{Mat}_n(\mathbb{R}) \).
- \( H \oplus K \) is the direct sum of two Lie algebras \( H \) and \( K \), whereas \( \mathbb{R} \oplus dK \) is the semi-direct sum of Lie algebras by \( d \in \text{Der}(K) \), i.e., \( [z,x] = d(x) \) for all \( x \in K \).
- \( b_3 = \text{span}\{x_1,x_2,x_3 \mid [x_2,x_3] = x_1\} \) is the 3-dimensional Heisenberg Lie algebra.
- \( E_{ij} \) is the square matrix whose only non-zero entry is 1 in row \( i \) and column \( j \).

**Definition 2.1.**

- Two matrices \( A,B \in \text{Mat}_n(\mathbb{R}) \) are **proportionally similar**, denoted by \( A \sim_{\rho} B \), if there exist \( c \neq 0 \) and \( C \in \text{GL}_n(\mathbb{R}) \) such that \( cA = C^{-1}BC \).
- Let \( K \) be a Lie algebra. Two derivations \( d_1,d_2 \in \text{Der}(K) \) are **proportionally similar by the automorphism group of \( K \)**, denoted by \( d_1 \sim^K_{\rho} d_2 \), if there exist \( \alpha \neq 0 \) and \( \sigma \in \text{Aut}(K) \) such that \( \alpha d_1 = \sigma^{-1}d_2\sigma \). Two classes \( \bar{a}, \bar{b} \in H^1(K,K) \) are **proportionally similar by the automorphism group of \( K \)**, denoted by \( \bar{a} \sim^K_{\rho} \bar{b} \), if \( d_1 \sim^K_{\rho} d_2 \) for some \( d_1 \in \bar{a} \) and \( d_2 \in \bar{b} \).

**Remark 2.2.** We would like to explain two points here.

- If \( A \) and \( B \) are two \( n \)-square matrices, \( A \sim_{\rho} B \) means that the Jordan canonical forms (hereafter, JCFs) of \( cA \) and \( B \) coincide for some non-zero constant \( c \). Therefore, the quotient set \( \text{Mat}_n(\mathbb{R})/\sim_{\rho} \) can be determined by using JCFs of square matrices.
- Assume that \( \dim K = n \) and \( d_1,d_2 \in \text{Der}(K) \). If \( K \) is abelian then \( \text{Aut}(K) \equiv \text{GL}_n(\mathbb{R}) \) and thus \( d_1 \sim_{\rho} d_2 \) and \( d_1 \sim^K_{\rho} d_2 \) are the same. In case \( K \) is non-abelian, we have that \( d_1 \sim^K_{\rho} d_2 \) implies \( d_1 \sim_{\rho} d_2 \) but the converse, in general, is not true.
Definition 2.3 ([12, Introduction]). A classification problem is called wild if it contains the problem of classifying pairs of $n \times n$ matrices up to similarity transformations $(M, N) \rightarrow S^{-1}(M, N)S := (S^{-1}MS, S^{-1}NS)$ with nonsingular $S$.

It is noted that this concept was introduced by Donovan and Freislich [8, 9]. Moreover, wild problems are considered as hopeless since they contain the problem of classifying an arbitrary system of linear mappings, that is, representations of an arbitrary quiver [12].

Definition 2.4 ([12, Introduction]). Let $A, B, A', B' \in \text{Mat}_n(F)$. Two matrix pairs $(A, B)$ and $(A', B')$ are called weakly similar if there exist $S \in \text{GL}_n(F)$ and $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{GL}_2(F)$ such that $(A', B') = S^{-1}(\alpha A + \beta B, \gamma A + \delta B)S$.

Remark 2.5. In [12, Theorem 1], the authors proved that the problem of classifying pairs of commuting matrices up to weak similarity is wild.

Next, we recall some well-known facts and present some preliminary results before entering to the main results. First of all, let us state two lemmas of de Graaf [7].

Proposition 2.6 ([7, Lemma 2.1]). Let $L$ be a real solvable Lie algebra. Then there is a subalgebra $K \subset L$ of codimension 1 and a derivation $d$ of $K$ such that $L = \mathbb{R}x \oplus dK$. Moreover, if $L$ is not abelian, then $d$ and $K$ can be chosen such that $d$ is an outer derivation of $K$.

Proposition 2.7 ([7, Lemma 2.2]). Let $K$ be a solvable Lie algebra and $d_1, d_2$ derivations of $K$. Set $L_i = \mathbb{R}x \oplus d_i K$ for $i = 1, 2$. Suppose that there exists $\sigma \in \text{Aut}(K)$ such that $\sigma d_1 \sigma^{-1} = \lambda d_2$ for some scalar $\lambda \neq 0$. Then $L_1$ and $L_2$ are isomorphic.

We also note that the result in Proposition 2.7 is just a sufficient condition, not a necessary one.

Proposition 2.8. Let $K$ be a real solvable Lie algebra and $d_1, d_2 \in \text{Der}(K)$. Set $L_i := \mathbb{R}x \oplus d_i K$ for $i = 1, 2$. If $d_1 = d_2 + \text{ad}_u$ for some $u \in K$, then $L_1$ and $L_2$ are isomorphic.

Proof. We define a map $\tilde{\sigma} : \mathbb{R}x \oplus d_1 K \rightarrow \mathbb{R}x \oplus d_2 K$ such that:

\[
\begin{align*}
\tilde{\sigma}(y) &= y, \quad y \in K, \\
\tilde{\sigma}(x) &= x + u.
\end{align*}
\]

It is obvious that $\tilde{\sigma}$ is a linear isomorphism. We show that it also preserves Lie brackets. In fact, for arbitrary $y \in K$, we have

\[
\tilde{\sigma}([x, y]) = [\tilde{\sigma}(x), \tilde{\sigma}(y)]
\]

\[
\iff [x, y] = [\tilde{\sigma}(x), \tilde{\sigma}(y)]
\]

\[
\iff d_1(y) = [x + u, y]
\]

\[
\iff d_1(y) = (d_2 + \text{ad}_u)(y).
\]

The last equation is obvious which completes the proof. \hfill \square

By combining Propositions 2.7 and 2.8, we have the following corollary.

Corollary 2.9. Let $K$ be a real solvable Lie algebra and $d_1, d_2 \in \text{Der}(K)$. Set $L_i := \mathbb{R}x \oplus d_i K$ for $i = 1, 2$. If there is $\sigma \in \text{Aut}(K)$ such that $\sigma d_1 \sigma^{-1} = \lambda d_2 + \text{ad}_u$ for some $\lambda \in \mathbb{R}\setminus\{0\}$ and $u \in K$, then $L_1$ and $L_2$ are isomorphic.
Corollary 2.10. Let $K$ be a real solvable Lie algebra and $d \in \text{Der}(K)$. Set $L := \mathbb{R}x \oplus dK$. If $d \in \text{ad}(K)$ then $L$ is decomposable. In addition, we can decompose $L$ into the form $L = \mathbb{R}x' \oplus K$ for some $x' \in L\backslash K$.

Proof. If $d = \text{ad}_u$ for some $u \in K$ then we change $x' = x - u$ to get $L = \mathbb{R}x' + K$. Besides, we have $[x', z] = [x - u, z] = d(z) - \text{ad}_u(z) = 0$ for all $z \in K$. Therefore, $L = \mathbb{R}x' \oplus K$ with $x' \in L\backslash K$. □

3. The problem of classifying $\text{Lie}(n + 1, n)$

We will prove Theorem 1 in this section. Recall that $\text{Lie}(n + 1, n)$ is the class of all $(n + 1)$-dimensional real solvable Lie algebras having $n$-dimensional derived algebras.

3.1. Description of $\text{Lie}(n + 1, n)$

Since the derived algebra of a solvable Lie algebra is nilpotent (see [16, Chapter II, Section 7, Corollary 1]), we can classify $\text{Lie}(n + 1, n)$ by applying the following steps:

Step 1. Start with an $n$-dimensional real nilpotent Lie algebra $K$;
Step 2. Extend $K$ to all $(n + 1)$-dimensional real solvable Lie algebras $L$ which admits $K$ as derived algebra, i.e., $L^1 = K$. Afterward, classify such Lie algebras;
Step 3. Repeat two steps above for all possibilities of $K$.

First of all, we fix an arbitrary $n$-dimensional real nilpotent Lie algebra $K$. Assume that $L \in \text{Lie}(n + 1, n)$ with $L^1 = K$. Without loss of generality, we can choose a basis $\{x_1, ..., x_n, y\}$ of $L$ in which $L^1 = K = \text{span}\{x_1, ..., x_n\}$. By Proposition 2.6, there exists $d \in \text{Der}(K)$ satisfying

$$L = \mathbb{R}y \oplus dK.$$  

It also notes that $d(u) = [y, u]$ for $u \in K$, i.e., $d = a_y$.

In general, a Lie algebra of the form $L = \mathbb{R}y \oplus dK$ for some $d \in \text{Der}(K)$ does not necessarily belong to $\text{Lie}(n + 1, n)$. Therefore, we first point out a necessary and sufficient condition of $d$ such that $L$ belongs to $\text{Lie}(n + 1, n)$.

Proposition 3.1. Let $K$ be an $n$-dimensional real nilpotent Lie algebra, and $L = \mathbb{R}y \oplus dK$ for $d \in \text{Der}(K)$ as above. By renumbering, if necessary, we can assume that $K^1 = \text{span}\{x_1, ..., x_m\}$ for $0 \leq m < n$. Then the following conditions are equivalent:

1. $L \in \text{Lie}(n + 1, n)$;
2. $L^1 = K$;
3. $\text{rank}(d_{ij})_{i > m} = n - m$, where $(d_{ij}) \in \text{Mat}_n(\mathbb{R})$ is the matrix of $d$ with respect to the basis $\{x_1, ..., x_n\}$.

Proof. It is obvious that

$$L^1 = \text{span}\{y, u \mid u \in K\} + \text{span}\{v, u \mid v, u \in K\} = d(K) + K^1 \subset K.$$  

Therefore, $L \in \text{Lie}(n + 1, n)$ if and only if

$$L^1 = K$$

$$\iff d(K) + K^1 = K$$

$$\iff d(K) + K^1 = \text{span}\{x_{m+1}, ..., x_n\} + K^1$$

$$\iff \text{span}\{x_{m+1}, ..., x_n\} \subset d(K) + K^1 \quad \text{(since span}\{x_1, ..., x_m\} = K^1)$$

$$\iff \text{rank}(d_{ij})_{i > m} = n - m.$$  

This completes the proof. □
**Proposition 3.2.** Let $K$ be an $n$-dimensional real nilpotent Lie algebra. Then we have the following assertions:

1. All Lie algebras in $\text{Lie}(n+1, n)$ are indecomposable.
2. No $L \in \text{Lie}(n+1, n)$ is an extension of $K$ by an inner derivation $d$ of $K$. In other words, if $d \in \text{ad}(K)$ then $L = \mathbb{R}y \oplus dK \not\subset \text{Lie}(n+1, n)$.

**Proof.**

1. Let $L \in \text{Lie}(n+1, n)$. If $L$ is decomposable then $L = L_1 \oplus L_2$ which implies $L^1 = L^1_1 \oplus L^1_2$ and $\dim L^1 = \dim L^1_1 + \dim L^1_2$. Since $L_1$ and $L_2$ are solvable, we have

$$\begin{cases} \dim L^1_1 \leq \dim L_1 - 1 \\ \dim L^1_2 \leq \dim L_2 - 1 \end{cases}$$

and thus $\dim L^1 \leq \dim L_1 + \dim L_2 - 2 = \dim L - 2$. This contradicts the assumption $L \in \text{Lie}(n+1, n)$. Therefore, $L$ is indecomposable.

2. A direct consequence of Corollary 2.10 and part 1 above.

The proof of Proposition 3.2 is complete. \(\square\)

### 3.2. Proof of Theorem 1

As mentioned in Section 3.1, the classification of $\text{Lie}(n+1, n)$ consists of three steps initializing by an $n$-dimensional real nilpotent Lie algebra $K$. Results in this subsection perform Step 2, that is, to extend $K$ to all $L \in \text{Lie}(n+1, n)$ with $L^1 = K$ and then classify such Lie algebras.

According to Proposition 3.1, the problem of classifying $L \in \text{Lie}(n+1, n)$ with $L^1 = K$ is reduced to find out the conditions of $d_1$ and $d_2$ satisfying Proposition 3.1 such that two Lie algebras $\mathbb{R}y \oplus d_1 K$ and $\mathbb{R}y \oplus d_2 K$ are isomorphic. Namely, we have the following result.

**Proposition 3.3.** Let $L_1 = \mathbb{R}y \oplus d_1 K$ and $L_2 = \mathbb{R}y \oplus d_2 K$ be extensions of $K$ by outer derivations $d_1$ and $d_2$ which satisfy Proposition 3.1, respectively. Then $L_1$ and $L_2$ are isomorphic if and only if there exist $x \neq 0$ and $\sigma \in \text{Aut}(K)$ such that $\sigma d_1 \sigma^{-1} = xd_2 + ad_u$ for some $u \in K$.

**Proof.** ($\Rightarrow$) It follows directly from Corollary 2.9.

($\Leftarrow$) Assume that $L_1$ and $L_2$ are isomorphic by $\tilde{\sigma} : \mathbb{R}y \oplus d_1 K \to \mathbb{R}y \oplus d_2 K$. By Proposition 3.1, $L^1_1 = L^1_2 = K$. Thus $\sigma := \tilde{\sigma}|_K : K \to K$ is also an isomorphism. Set $\sigma(y) = xy + u$ with $x \neq 0$ and $u \in K$. Then

$$\tilde{\sigma}([y, x]) = [\tilde{\sigma}(y), \tilde{\sigma}(x)], \quad \forall x \in K$$

$$\iff \sigma([y, x]) = [xy + u, \sigma(x)], \quad \forall x \in K$$

$$\iff \sigma d_1(x) = (xd_2 + ad_u)\sigma(x), \quad \forall x \in K$$

$$\iff \sigma d_1 \sigma^{-1} = xd_2 + ad_u.$$ 

The proof of Proposition 3.3 is complete. \(\square\)

**Proof of Theorem 1.** As we have seen, for a given $n$-dimensional real nilpotent Lie algebra $K$, all $L \in \text{Lie}(n+1, n)$ with $L^1 = K$ are of the forms $\mathbb{R}y \oplus dK$ in which $d \in \text{Der}(K) \setminus \text{ad}(K)$ satisfies condition 3 in Proposition 3.1. Now, let us consider two extensions

$$L_i := \mathbb{R}y \oplus d_i K; \quad i = 1, 2,$$

where $d_1, d_2 \in \text{Der}(K) \setminus \text{ad}(K)$ satisfying condition 3 in Proposition 3.1. By Proposition 3.3, $L_1 \cong L_2$ if and only if there exist $x \neq 0$ and $\sigma \in \text{Aut}(K)$ such that $\sigma d_1 \sigma^{-1} = xd_2 + ad_u$ for some $u \in K.$
Equivalently, we have that

\[ \sigma d_1 \sigma^{-1} = \alpha (d_2 + \text{ad}_{\alpha}) \, . \]

Since \( d_1 \in \tilde{d}_1 \) and \( d_2 + \text{ad}_{\alpha} \in \tilde{d}_2 \), where \( \tilde{d}_1, \tilde{d}_2 \in H^1(K, K) \) are the \( \text{ad}(K) \)-modulo classes of \( d_1 \) and \( d_2 \), the above equation means that \( \tilde{d}_1 \sim^K_p \tilde{d}_2 \). Therefore, the problem of classifying all \( L \in \text{Lie}(n + 1, n) \) with \( L^1 = K \) is equivalent to the problem of classifying all equivalence classes \( \tilde{d} \in H^1(K, K) \) of outer derivations \( \tilde{d} \) satisfying condition 3 in Proposition 3.1 up to proportional similarity by the automorphism group of \( K \).

For \( L = \mathbb{R}y \oplus dK \) with \( d \in \text{Der}(K) \backslash \text{ad}(K) \), we set

\[ H_1^s(K, K) := \{ \tilde{d} \in H^1(K, K) \mid \text{d satisfies condition 3 in Proposition 3.1} \} \, . \]

Then Proposition 3.3 implies that classifying \( L \in \text{Lie}(n + 1, n) \) is equivalent to determining \( H^1_s(K, K) / \sim^K_p \). If \( K \) is abelian then \( H^1_s(K, K) \equiv \text{GL}_n(\mathbb{R}) \) and \( \sim^K_p \equiv \sim^p \). Hence, \( H^1_s(K, K) / \sim^K_p = \text{GL}_n(\mathbb{R}) / \sim^p \). Due to Remark 2.2, we have that:

**Corollary 3.4.** The problem of classifying \( L \in \text{Lie}(n + 1, n) \) with \( L^1 \equiv \mathbb{R}^n \) is equivalent to the problem of classifying \( \text{GL}_n(\mathbb{R}) \) up to proportional similarity which can be done easily by using JCFs of invertible \( n \times n \) square matrices.

If \( K \) is non-abelian, then determining \( H^1_s(K, K) / \sim^K_p \) is much harder. Note that each class of \( H^1_s(K, K) / \sim^K_p \) in this case is constituted by some classes of \( H^1_s(K, K) / \sim^K_p \). To determine \( H^1_s(K, K) / \sim^K_p \), we first use JCFs to determine completely \( H^1_s(K, K) / \sim^K_p \). Afterwards, by using a computer algebra tool proposed in [25], we can find out a complete list of representatives for equivalence classes of \( H^1_s(K, K) / \sim^K_p \) (see Example 3.5 for an illustration).

For convenience, we summarize the procedure of classifying \( \text{Lie}(n + 1, n) \) as in Algorithm 1 below.

**Algorithm 1.** Classification of \( \text{Lie}(n + 1, n) \)

- **Input:** A list \( \mathcal{N} \) of non-isomorphic \( n \)-dimensional nilpotent Lie algebras
- **Output:** A list \( \mathcal{S} \) of non-isomorphic \( (n + 1) \)-dimensional solvable Lie algebras whose derived algebras belong to \( \mathcal{N} \).

1. \( \mathcal{S} := \{ \} \);
2. **while** \( K \in \mathcal{N} \)**
3. \( L_d := \mathbb{R}_n \oplus dK \) with \( d \in \text{Der}(K) \backslash \text{ad}(K) \);
4. \( H_1^s(K, K) := \{ \tilde{d} \in H^1(K, K) \mid \text{d satisfies condition 3 in Proposition 3.1} \} \);
5. \( \mathcal{D} := H^1_s(K, K) / \sim^K_p \);
6. **for** \( \tilde{d} \in \mathcal{D} \)**
7. **Isomorphism verification for** \( L_d \) **by basis transformations of** \( K \) **which leave** \( K \) **fixed**;
8. \( \mathcal{I} := \{ \text{Isomorphism classes of} \ L_d \} \);
9. \( \mathcal{S} := \mathcal{S} \cup \mathcal{I} \);
10. **Return** \( \mathcal{S} \).

### 3.3. Demonstrations

In this subsection, we give an example in case \( n = 3 \) to demonstrate how Theorem 1 can be applied in the computation of \( \text{Lie}(n + 1, n) \).
Example 3.5. Let $K = h_3$. We will classify all $L \in \text{Lie}(4,3)$ which admit $K$ as derived algebras. Note that all such $L$ are always indecomposable by Proposition 3.2.

First of all, we set

$$L = L_d := \mathbb{R}x_4 \oplus dK, \quad d \in \text{Der}(K) \setminus \text{ad}(K).$$

Since $d \in \text{Der}(K)$, we have $d([x_i, x_j]) = [d(x_i), x_j] + [x_i, d(x_j)]$ for $1 \leq i < j \leq 3$. By computing these three equalities, the matrix of $d$ with respect to the basis $\{x_1, x_2, x_3\}$ is of the following form

$$d = \begin{bmatrix}
  a + b & f & g \\
  0 & a & c \\
  0 & e & b
\end{bmatrix}; \quad a, b, c, e, f, g \in \mathbb{R}.$$

Besides, $\text{ad}(K) = \text{span}\{\text{ad}_{x_2} = E_{13}, \text{ad}_{x_3} = -E_{12}\}$ and $d \in \text{Der}(K) \setminus \text{ad}(K)$ imply that $a^2 + b^2 + c^2 + e^2 \neq 0$. Hence,

$$H^1(K, K) = \text{span}\left\{ d = \begin{bmatrix} a + b & 0 & 0 \\
  0 & a & c \\
  0 & e & b \end{bmatrix} \middle| a^2 + b^2 + c^2 + e^2 \neq 0 \right\}.$$

For $K = h_3$, we have $K^1 = \text{span}\{x_1\}$. Thus, condition 3 of Proposition 3.1 becomes $\det \begin{bmatrix} a & c \\
  e & b \end{bmatrix} \neq 0$ which implies that

$$H^1_+(K, K) = \text{span}\left\{ d = \begin{bmatrix} a + b & 0 & 0 \\
  0 & a & c \\
  0 & e & b \end{bmatrix} \middle| \det \begin{bmatrix} a & c \\
  e & b \end{bmatrix} \neq 0 \right\}.$$

Now, we determine $H^1_+(K, K)/\sim_p$. It is easy to see that the possible JCFs of nonsingular matrices $\begin{bmatrix} a & c \\
  e & b \end{bmatrix}$ are as follows (we omit zeros):

$$\begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} (\lambda_1 \lambda_2 \neq 0) \sim_p \begin{bmatrix} 1 & 0 \\
  \lambda & 1 \end{bmatrix} (\lambda = \frac{\lambda_2}{\lambda_1} \neq 0),$$

$$\begin{bmatrix} \lambda & 1 \\
  \lambda & \lambda \end{bmatrix} (\lambda \neq 0) \sim_p \begin{bmatrix} 1 & 0 \\
  1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} \lambda & 1 \\
  -1 & \lambda \end{bmatrix} (\lambda \in \mathbb{R}).$$

These JCFs lead to three classes in $H^1_+(K, K)/\sim_p$ as follows:

$$d_1 := \text{diag}(1 + \lambda, 1, \lambda)(\lambda \neq 0), \quad d_2 := \begin{bmatrix} 1 & 1 \\
  1 & 1 \end{bmatrix}, \quad d_3 := \begin{bmatrix} 2\lambda & \lambda & 1 \\
  -1 & -1 & \lambda \end{bmatrix} (\lambda \in \mathbb{R}).$$

This means that $H^1_+(K, K)/\sim_p = \{d_1, d_2, d_3\}$.

Next, we determine $H^1_+(K, K)/\sim_p$. According to [11, Section 4], all automorphisms $\sigma$ of $K$ are of the following form:

$$\sigma = \begin{bmatrix} xy - zt & p & q \\
  0 & x & z \\
  0 & t & y \end{bmatrix}; \quad xy - zt \neq 0.$$
Let $\bar{d} \in H^1_s(K, K)$. We claim that $\bar{d}$ is always proportionally similar by the automorphism group of $K$ to $d_1$, $d_2$, or $d_3$. In other words, for a given $\bar{d} \in H^1_s(K, K)$, there always exists an index $i \in \{1, 2, 3\}$ such that the following system

$$\sigma^{-1}\bar{d}\sigma = x d_i$$

admits real root $x \neq 0$ and $\sigma \in \text{Aut}(K)$. In fact, set $\Delta := (a - b)^2 + 4ce$. Then, by using a computer algebra tool in [25], we have that

$$L \sim_p \begin{cases} 
  d_1, & \Delta > 0, \\
  d_2, & \Delta = 0, \\
  d_3, & \Delta < 0.
\end{cases}$$

Therefore, we also have $H^1_s(K, K)/\sim_p = \{d_1, d_2, d_3\}$. Hence, there exist three families of Lie algebras in $\text{Lie}(4, 3)$ with derived algebra $K = b_3$, namely, we have

$$L^1 = \mathbb{R}x_4 \oplus d_1 K \; (\lambda \neq 0), \quad L_2 = \mathbb{R}x_4 \oplus d_2 K, \quad L^3 = \mathbb{R}x_4 \oplus d_3 K.$$ 

Finally, by using [25] we can check that $L^1 \cong \tilde{L}^1_1$ and $L^3 \cong \tilde{L}^{-\lambda}_3$ by isomorphisms

$$\begin{bmatrix}
-1 & 0 & 1 \\
1 & 0 & 0 \\
\lambda & & 
\end{bmatrix}, \quad \text{diag}(-1, -1, 1, -1),$$

respectively. Therefore, the parameters $\lambda$ of $L^1$ and $L^3$ can be reduced respectively to $0 < |\lambda| \leq 1$ and $\lambda \geq 0$. We also note that these Lie algebras coincide with those of Mubarakzyanov [20, §5], namely, $L^1 \cong g_{4,8}$, $L_2 \cong g_{4,7}$ and $L^3 \cong g_{4,9}$.

**4. The problem of classifying Lie($n + 2, n$)**

The goal of this section is to prove Theorem 2. Before giving the proof of Theorem 2, we first explore some properties of $\text{Lie}(n + 2, n)$.

**4.1. Description of Lie($n + 2, n$)**

Recall that $\text{Lie}(n + 2, n)$ is the class of all $(n + 2)$-dimensional real solvable Lie algebras having $n$-dimensional derived algebras. To classify $\text{Lie}(n + 2, n)$, we proceed in a similar way as described in Subsection 3.1:

**Step 1.** Start with an $n$-dimensional real nilpotent Lie algebra $H$;

**Step 2.** Extend $H$ to all $(n + 2)$-dimensional real solvable Lie algebras $L$ which admits $H$ as derived algebra, i.e., $L^1 = H$. Afterward, classify such Lie algebras;

**Step 3.** Repeat two steps above for all possibilities of $H$.

First of all, we fix an arbitrary $n$-dimensional real nilpotent Lie algebra $H$. Let $L \in \text{Lie}(n + 2, n)$ with $L^1 = H$. Without loss of generality, we can assume

$$L = \text{span}\{x_1, \ldots, x_n, y, z\}, \quad L^1 = H = \text{span}\{x_1, \ldots, x_n\}.$$

By Proposition 2.6, $L$ can be represented in the following form:

$$L = \mathbb{R}z \oplus dK = \mathbb{R}z \oplus d(\mathbb{R}y \oplus dH), \quad d \in \text{Der}(K), d' \in \text{Der}(H).$$
It is not true that all Lie algebras of the above forms belong to \(\text{Lie}(n+2, n)\). We give a necessary and sufficient condition of the pair \((d, d')\) for which \(L \in \text{Lie}(n+2, n)\).

**Proposition 4.1.** Let \(H\) be an \(n\)-dimensional real nilpotent Lie algebra, and \(L = \mathbb{R}z \oplus_d K = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus_{d'} H)\) with \(d \in \text{Der}(K)\), \(d(K) \subset H\) and \(d' \in \text{Der}(H)\) as above. By renumbering, if necessary, we can assume that \(H^1 = \text{span}\{x_1, \ldots, x_m\}\) for some \(0 \leq m < n\). Then, the following conditions are equivalent:

1. \(L \in \text{Lie}(n+2, n)\);
2. \(L^1 = H\);
3. \(H/H^1 = \text{Im} \, \tilde{d} + \text{Im} \, \tilde{d}'\), where \(\tilde{d} : K/H^1 \to K/H^1\) and \(\tilde{d}' : H/H^1 \to H/H^1\) are homomorphisms induced from \(d\) and \(d'\) on the quotient Lie algebras \(K/H^1\) and \(H/H^1\), respectively.

**Proof.** First of all, we note that:

\[
L^1 = \text{span}\{[z, u] | u \in K\} + \text{span}\{[y, v] | v \in H\} + \text{span}\{[u', v'] | u', v' \in H\} = d(K) + d'(H) + H^1 \subset \text{Lie}\ n \quad \text{because } d(K) \subset H.
\]

Therefore, \(L \in \text{Lie}(n+2, n)\) if and only if

\[
L^1 = H \iff d(K) + d'(H) + H^1 = H \\
\iff d(K) + d'(H) + H^1 = \text{span}\{x_{m+1}, \ldots, x_n\} + H^1 \\
\iff \text{span}\{x_{m+1}, \ldots, x_n\} \subset d(K) + d'(H) + H^1 \quad \text{(since span}\{x_1, \ldots, x_m\} = H^1)\]

By the Leibniz formula, the derived algebra \(H^1\) is invariant with respect to \(d\) and \(d'\), i.e., \(d(H^1), d'(H^1) \subset H^1\). Therefore, \(\tilde{d} : K/H^1 \to K/H^1\) and \(\tilde{d}' : H/H^1 \to H/H^1\) are well-defined, and the last equation above means that \(H/H^1 = \text{Im} \, \tilde{d} + \text{Im} \, \tilde{d}'\).

Thus, each Lie algebra \(L = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus_{d'} H) \in \text{Lie}(n+2, n)\) with \(L^1 = H\) can be seen as an extension of \(H\) by a pair of derivations \((d, d')\) satisfying condition 3 in **Proposition 4.1**. Furthermore, an immediate consequence of **Proposition 4.1** and **Corollary 2.10** is as follows.

**Corollary 4.2.** If \(L = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus_{d'} H)\) with \(d \in \text{ad}(\mathbb{R}y \oplus_{d'} H)\) and \(d' \in \text{ad}(H)\) then \(L \notin \text{Lie}(n+2, n)\). In other words, all Lie algebras extended from \(H\) by a pair of inner derivations cannot belong to \(\text{Lie}(n+2, n)\).

**Proposition 4.3.** An \(L \in \text{Lie}(n+2, n)\) is decomposable if and only if there exist \(L_1 \in \text{Lie}(m_1 + 1, m_1)\) and \(L_2 \in \text{Lie}(m_2 + 1, m_2)\) such that \(L = L_1 \oplus L_2\) for some \(m_1, m_2 \geq 0\) and \(m_1 + m_2 = n\).

**Proof.** Assume that \(L\) is decomposable, i.e., \(L = L_1 \oplus L_2\) for certain two real solvable Lie algebras \(L_1\) and \(L_2\). In particular, \(\dim L_1 + \dim L_2 = \dim L = n + 2\). For convenience, we set

\[
\dim L_1 = m_1 + 1, \quad \dim L_2 = m_2 + 1,
\]

with \(m_1, m_2 \geq 0\) and \(m_1 + m_2 = n\). Since \(L_1\) and \(L_2\) are solvable, we have

\[
\begin{cases}
\dim L_1^1 \leq \dim L_1 - 1 = m_1, \\
\dim L_2^1 \leq \dim L_2 - 1 = m_2.
\end{cases}
\]

Besides, \(L^1_1 \oplus L^1_2\) implies

\[
n = \dim L^1 = \dim L^1_1 + \dim L^1_2 \leq m_1 + m_2 = n.
\]

Equality holds if \(\dim L^1_1 = m_1\) and \(\dim L^1_2 = m_2\) which lead to
Let \( L_1 \in \text{Lie}(m_1 + 1, m_1), \ L_2 \in \text{Lie}(m_2 + 1, m_2) \).

The converse is straightforward. The proof is complete. \( \Box \)

Thanks to Proposition 4.3, we only need to pay attention to indecomposable Lie algebras. For completeness, the next result presents detailed images of decomposable Lie algebras in \( \text{Lie}(n + 2, n) \).

**Proposition 4.4.** Let \( H \) be an \( n \)-dimensional real nilpotent Lie algebra and \( L \in \text{Lie}(n + 2, n) \) with \( L^1 = H \). Then \( L \) is decomposable if and only if there exists a basis \( \{x_1, \ldots, x_n, y, z\} \) of \( L \) such that \( L = \mathbb{R}z \oplus d(\mathbb{R}y \oplus dH) \), \( [z, y] = 0 \) and the pair \( (d, d') \) satisfy the following condition: there exist Lie algebras \( H_1, H_2 \subseteq H \) such that

\[
\begin{cases}
  d(H_1) \subset H_1, & d(H_2) = 0, \\
  d'(H_1) = 0, & d'(H_2) \subset H_2,
\end{cases}
\]

and \( H_1 \cap H_2 = H \).

In particular, if \( H \) is indecomposable then \( L \) is decomposable if and only if \( L \cong \mathbb{R} \oplus \bar{L} \) with \( \bar{L} \in \text{Lie}(n + 1, n) \) and \( \bar{L}^1 = L^1 = H \).

**Proof.** (\( \Leftarrow \)) It is obvious. More precisely, we have \( L = (\mathbb{R}z \oplus dH_1) \oplus (\mathbb{R}y \oplus dH_2) \).

(\( \Rightarrow \)) By Proposition 4.3, there exist \( L_1 \in \text{Lie}(m_1 + 1, m_1) \) and \( L_2 \in \text{Lie}(m_2 + 1, m_2) \) such that

\[
L = L_1 \oplus L_2, \quad m_1, m_2 \geq 0, m_1 + m_2 = n.
\]

Then \( H = L_1^1 \oplus L_2^1 := H_1 \oplus H_2 \). Assume that

\[
H_1 = \text{span}\{x_1, \ldots, x_{m_1}\}, \quad H_2 = \text{span}\{x_{m_1+1}, \ldots, x_n\}.
\]

We can always find completions \( \{x_1, \ldots, x_{m_1}\} \) of \( \{z\} \) and \( \{x_{m_1+1}, \ldots, x_n\} \) of \( \{y\} \) to get bases of \( L_1 \) and \( L_2 \), respectively. By this way, we have \( [z, y] = 0 \), \( H = \text{span}\{x_1, \ldots, x_n\} \) and \( L = \mathbb{R}z \oplus d(\mathbb{R}y \oplus dH) \) in which

\[
\begin{cases}
  [z, H_1] = d(H_1) \subset H_1, & [z, H_2] = d(H_2) = 0, \\
  [y, H_1] = d'(H_1) = 0, & [y, H_2] = d'(H_2) \subset H_2.
\end{cases}
\]

The proof of Proposition 4.4 is complete. \( \Box \)

### 4.2. Proof of Theorem 2

We will show that the problem of classifying \( \text{Lie}(n + 2, n) \) contains a wild problem. In fact, let us consider the following class:

\[
\text{Lie}_c(n + 2, n) := \{ L \in \text{Lie}(n + 2, n) \mid L^1 = \mathbb{R}^n \} \subset \text{Lie}(n + 2, n).
\]

We will prove that the problem of classifying \( \text{Lie}_c(n + 2, n) \) is wild.

Let \( L = \text{span}\{x_1, \ldots, x_n, y, z\} \in \text{Lie}_c(n + 2, n) \) such that

\[
L^1 = \text{span}\{x_1, \ldots, x_n\} = \mathbb{R}^n.
\]

By Proposition 2.6, we represent \( L \) in the following form:

\[
L = \mathbb{R}z \oplus dK = \mathbb{R}z \oplus d(\mathbb{R}y \oplus d\mathbb{R}^n),
\]

where \( d \in \text{Der}(K) \) and \( d' = \text{ad}_y|_{\mathbb{R}^n} = a_y \in \text{Der}(\mathbb{R}^n) \) satisfy conditions 3 in Proposition 4.1. For simplicity, we assume additionally two conditions as follows:

- \( [z, y] = 0 \) which allows us to identify

\[
d = \text{ad}_z|_{\mathbb{R}^n} \oplus 0 = a_z \oplus 0 \equiv a_z,
\]

and consider the pair \( (d, d') \) as derivations of \( \mathbb{R}^n \).
• $d$ and $d'$ are outer derivations of $\mathbb{R}^n$. Moreover, they must be non-proportional, in order to guarantee that $L$ is indecomposable.

Even if we treat this simpler case, the result is as follows.

**Proposition 4.5.** Let $L_i = \mathbb{R}z \oplus \mathbb{R}(\mathbb{R}y \oplus \mathbb{R}^n)$ for $i = 1, 2$, be two Lie algebras in $\text{Lie}_e(n + 2, n)$ which satisfy all of the above conditions. Then $L_1 \cong L_2$ if and only if there exists $\sigma \in \text{Aut}(\mathbb{R}^n)$ and 
\[
\begin{bmatrix}
\alpha & \beta \\
\gamma & \delta
\end{bmatrix} \in \text{GL}_2(\mathbb{R}) \text{ such that }
\begin{align*}
\sigma d_1 \sigma^{-1} &= \gamma d_2 + \alpha d'_2 \\
\sigma d'_1 \sigma^{-1} &= \delta d_2 + \beta d'_2
\end{align*}
\] (4.1)

**Proof.** ($\Rightarrow$) If $\tilde{\sigma} : L_1 \to L_2$ is an isomorphism, then so is $\sigma := \tilde{\sigma}|_{\mathbb{R}^n}$. Setting
\[
\begin{align*}
\tilde{\sigma}(z) &= \gamma z + \alpha y + u, \quad u \in \mathbb{R}^n, \\
\tilde{\sigma}(y) &= \delta z + \beta y + v, \quad v \in \mathbb{R}^n.
\end{align*}
\]
Since $d_i$ and $d'_i$ ($i = 1, 2$) are non-proportional, $\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \neq 0$, i.e., $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \text{GL}_2(\mathbb{R})$. Now, for arbitrary $x \in \mathbb{R}^n$, we have:
\[
\tilde{\sigma}([z, x]) = [\tilde{\sigma}(z), \tilde{\sigma}(x)] \\
\Leftrightarrow \sigma([z, x]) = [\gamma z + \alpha y + u, \sigma(x)] \\
\Leftrightarrow \sigma([z, x]) = [\gamma z + \alpha y, \sigma(x)] \\
\Leftrightarrow \sigma d_1(x) = (\gamma d_2 + \alpha d'_2)\sigma(x).
\]
Thus, $\sigma d_1 \sigma^{-1} = \gamma d_2 + \alpha d'_2$. Similarly, replacing $z$ by $y$ we get $\sigma d'_1 \sigma^{-1} = \delta d_2 + \beta d'_2$.

($\Leftarrow$) Assume that there exists $\sigma \in \text{Aut}(\mathbb{R}^n)$ and $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \text{GL}_2(\mathbb{R})$ satisfying (4.1). We define $\tilde{\sigma} : L_1 \to L_2$ as follows:
\[
\begin{align*}
\tilde{\sigma}(x) &= \sigma(x), \quad x \in \mathbb{R}^n, \\
\tilde{\sigma}(z) &= \gamma z + \alpha y, \\
\tilde{\sigma}(y) &= \delta z + \beta y.
\end{align*}
\]
Since $\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \neq 0$, $\tilde{\sigma}$ is a linear isomorphism. Moreover, it also preserves Lie brackets. In fact, the equation $\sigma d_1 \sigma^{-1} = \gamma d_2 + \alpha d'_2$ (resp., $\sigma d'_1 \sigma^{-1} = \delta d_2 + \beta d'_2$) implies $\tilde{\sigma}([z, x]) = [\tilde{\sigma}(z), \tilde{\sigma}(x)]$ (resp., $\tilde{\sigma}([y, x]) = [\tilde{\sigma}(y), \tilde{\sigma}(x)]$) for every $x \in \mathbb{R}^n$. Besides, $\tilde{\sigma}([z, y]) = [\tilde{\sigma}(z), \tilde{\sigma}(y)]$ is equivalent to $[z, y]\det \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = 0$ which is obviously true because of the assumption $[z, y] = 0$. Thus $\tilde{\sigma}$ is an isomorphism and $L_1 \cong L_2$.

The proof of Proposition 4.5 is complete. $\square$

**Proof of Theorem 2.** According to Definition 2.4, two pairs $(d_1, d'_1)$ and $(d_2, d'_2)$ satisfying (4.1) in Proposition 4.5 are weakly similar. By Remark 2.5, the problem of classifying pairs of matrices up to weak similarity, even if the pairs of commuting matrices, is wild. That means the problem of classifying $\text{Lie}_e(n + 2, n)$ is wild. So is the problem of classifying $\text{Lie}(n + 2, n)$. $\square$

**Remark 4.6.** The wildness of the problem of classifying $\text{Lie}(n + 2, n)$ is slightly different from that of $\text{Lie}(n + 1, n)$ because it is wild not only in Step 2 but also in Step 3 in Subsection 4.1.
5. A Special case of Lie\((n + 2, n)\)

This section is devoted to considering a special case of Lie\((n + 2, n)\).

Let \(H\) be an \(n\)-dimensional real nilpotent Lie algebra. By Subsection 4.1, all \(L \in \text{Lie}_{ad}(n + 2, n)\) with \(L^1 = H\) are of the following form

\[
L = \mathbb{R}z \oplus_d K = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus_d H),
\]

where \(d \in \text{Der}(K)\) with \(d(K) \subset H\) and \(d' \in \text{Der}(H)\) satisfy condition 3 in Proposition 4.1. As we have seen in Section 4.2, if the pair \((d, d')\) consists of outer derivations then the classification problem is wild. Therefore, it is natural to consider a subclass \(\text{Lie}_{ad}(n + 2, n)\) which consists of all Lie algebras of the forms:

\[
L = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus_d H),
\]

where \(H\) is an arbitrary \(n\)-dimensional real nilpotent Lie algebra, and the following conditions hold:

- \((d, d')\) satisfies condition 3 in Proposition 4.1, or equivalently, \(L \in \text{Lie}(n + 2, n)\);
- \((d, d')\) contains at least one inner derivation.

5.1. Proof of Theorem 3

The proof of Theorem 3 will take place through certain steps in which we need some auxiliary results as follows.

**Proposition 5.1.** For any indecomposable Lie algebra \(L \in \text{Lie}_{ad}(n + 2, n)\) with \(L^1 = H\), there exist \(y, z \in L \setminus H\) and \(d \in \text{Der}(K) \setminus \text{ad}(K)\) such that \(L = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus H)\) where \(K := \mathbb{R}y \oplus H\).

**Proof.** Since \(L \in \text{Lie}_{ad}(n + 2, n)\) with \(L^1 = H\), there exist \(y', z \in L \setminus H\) as well as \(d' \in \text{Der}(H)\) and \(d \in \text{Der}(\mathbb{R}y' \oplus_d H)\) such that

\[
L = \mathbb{R}z \oplus_d (\mathbb{R}y' \oplus_d H).
\]

Set \(K := \mathbb{R}y' \oplus_d H\). First of all, we note that

- By Corollary 4.2, \(d\) and \(d'\) cannot be inner derivations simultaneously;
- If \(d \in \text{ad}(K)\) then \(L\) is decomposable by Corollary 2.10 which conflicts with the indecomposability of \(L\).

Therefore, the pair \((d, d')\) contains one and only one inner derivation which is exactly \(d'\), i.e., \(d' \in \text{ad}(H)\). Taking Corollary 2.10 into account, we have

\[
K = \mathbb{R}y' \oplus_d H = \mathbb{R}y \oplus H, \quad \text{for some } y \in K \setminus H.
\]

Thus \(L = \mathbb{R}z \oplus_d K = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus H)\). The proof is complete.

According to Proposition 5.1, in order to classify \(\text{Lie}_{ad}(n + 2, n)\), we need to point out conditions of \(d_1\) and \(d_2\) such that two Lie algebras

\[
L_i = \mathbb{R}z \oplus_d K = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus H), \quad i = 1, 2,
\]

determined by \(d_1, d_2 \in \text{Der}(K) \setminus \text{ad}(K)\) are isomorphic. To that end, we next explore some additional properties of \(d\).

**Proposition 5.2.** If \(L = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus H) \in \text{Lie}_{ad}(n + 2, n)\) then \(d|_H = a_z\) is an outer derivation of \(H\).
Proof. Assume that \( d|_H = a_z = \text{ad}_z \) with \( u \in H \). By changing \( z' = z - u \) we have \([z', x] = 0\) for all \( x \in H \). Therefore, without loss of generality, we can assume that \([z, H] = 0\). In this way, all Lie brackets of \( L \) are determined by the original ones of \( H \) and \([z, y] = d(y) \in H\). Taking \( L^1 = H \) into account, this means that

\[
H = H^1 + \text{span}\{[z, y]\}.
\]

(5.1)

Since \( \dim H = n \) and \( \dim H^1 < n \), it implies \( \dim H^1 = n - 1 \) and \([z, y] \notin H^1\). Therefore, \( H \in \text{Lie}(n, n-1) \) and \([z, y] \in H \setminus H^1\).

On the other hand, it follows from the Jacobi identity that

\[
[[z, y], x] = [[x, y], z] + [[z, x], y] = 0,
\]

i.e., \([z, y] \in \mathcal{Z}(H)\), and (5.1) turns into

\[
H = H^1 \oplus \text{span}\{[z, y]\}
\]

which contradicts to \( H \in \text{Lie}(n, n-1) \) and Proposition 3.2. So \( d|_H = a_z \) cannot be an inner derivation of \( H \). The proof is complete. \( \Box \)

Proposition 5.3. Let \( L = \mathbb{R}z \oplus d(\mathbb{R}y \oplus H) \in \text{Lie}_{ad}(n + 2, n) \). Then the following assertions are equivalent:

1. \( L \) is decomposable;
2. \( L \cong \mathbb{R} \oplus L, \) where \( L \in \text{Lie}(n+1, n) \) and \( \bar{L}^1 = L^1 = H; \)
3. \([z, y] \in d(\mathcal{Z}(H))\).

In particular, if \( H = \mathbb{R}^n \) then \( L \) is decomposable if and only if \( d|_{\mathbb{R}^n} = a_z \) is nonsingular.

Proof. We will prove that \( 1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1 \).

(1 \Rightarrow 2) Assume that \( L = L_1 \oplus L_2 \). Then \( H = H_1 \oplus H_2 := L_1^1 \oplus L_2^1 \), where \( L_1 \in \text{Lie}(m_1 + 1, m_1) \), \( L_2 \in \text{Lie}(m_2 + 1, m_2) \) and \( m_1 + m_2 = n \). We assert that one of two summands \( m_1 \) or \( m_2 \) must be zero. In fact, assume that \( m_1 \geq 1 \) and \( m_2 \geq 1 \). Then, without loss of generality, we can assume that

\[
H_1 = \text{span}\{x_1, \ldots, x_{m_1}\}, \quad H_2 = \text{span}\{x_{m_1+1}, \ldots, x_n\}.
\]

We can always supplement two elements \( z', y' \in L \setminus H \) such that

\[
L = (\mathbb{R}z' \oplus_H H_1) \oplus (\mathbb{R}y' \oplus_H H_2).
\]

Since \( L_1 \in \text{Lie}(m_1 + 1, m_1) \), there exists \( x_{i_0} \in H_1 \) such that \([z', x_{i_0}] \notin H_1^1\) (similarly, we can also choose \( x_{i_0} \in H_2 \) such that \([y', x_{i_0}] \notin H_2^1\)). Put

\[
\begin{cases}
  z' = a_1z + b_1y + u_1, & u_1 \in H, \\
  y' = a_2z + b_2y + u_2, & u_2 \in H.
\end{cases}
\]

Since \( L_1 \in \text{Lie}(m_1 + 1, m_1) \) and \( L_2 \in \text{Lie}(m_2 + 1, m_2) \), we have \( a_1a_2 \neq 0 \). Then

\[
[z, x_{i_0}] = \frac{1}{a_1} \left( [z', x_{i_0}] - [u_1, x_{i_0}] \right) \notin H_1^1,
\]

which implies \([y', x_{i_0}] = a_2[z, x_{i_0}] + [u_2, x_{i_0}] \neq 0\). However, it conflicts with the decomposable form of \( L \). This contradiction shows that \( m_1 = 0 \) or \( m_2 = 0 \), i.e., \( L \cong \mathbb{R} \oplus L \). It is obvious that \( \bar{L} \in \text{Lie}(n+1, n) \) and \( \bar{L}^1 = L^1 = H \).

(2 \Rightarrow 3) Assume that \( L = \mathbb{R}z' \oplus \bar{L} := \mathbb{R}z' \oplus (\mathbb{R}y' \oplus_H H) \) with \( \bar{L} \in \text{Lie}(n+1, n) \) and \( \bar{L}^1 = L^1 = H \). Put
\[
\begin{align*}
&\left\{ \begin{array}{l}
z' = x_0 z + \beta_1 y + v_1, \quad v_1 \in H, \\
y' = x_2 z + \beta_2 y + v_2, \quad v_2 \in H.
\end{array} \right.
\end{align*}
\]

We have \( z_2 \neq 0 \) since on the contrary, it conflicts with \( L \in \text{Lie}(n + 1, n) \). By similar arguments as above, there exists \( x_{i_0} \in H \) such that \([y', x_{i_0}] \notin H^1\). Hence, \([z, x_{i_0}] = \frac{1}{x_2} ([y', x_{i_0}] - [v_2, x_{i_0}]) \notin H^1\). Then

\[
0 = [z', x_{i_0}] = x_1 [z, x_{i_0}] + [v_1, x_{i_0}] \iff \begin{cases} 
\alpha_1 = 0 \\
[v_1, x_{i_0}] = 0.
\end{cases}
\]

This implies \([v_1, x] = [z', x] = 0\) for all \( x \in H \), i.e., \( v_1 \in Z(H) \). Now, \( \beta_1 x_2 \neq 0 \) since on the contrary, \( z' \notin L \). Therefore, we have

\[
0 = [z', y'] = [\beta_1 y + v_1, x_2 z + \beta_2 y + v_2] = -\beta_1 x_2 [z, y] - x_2 [z, v_1]
\]

which leads to \([z, y] = -\frac{1}{\beta_1} [z, v_1] = -\frac{1}{\beta_1} d(v_1) \in d(Z(H))\).

(3 \Rightarrow 1) If \([z, y] = x \in d(Z(H))\) then there exists \( x' \in Z(H) \) such that \( d(x') = [z, x'] = x \). By changing \( y' = y - x' \) we have \([z, y'] = 0\) and \([y', u] = 0\) for all \( u \in H \). This means that \( L = R y' \oplus \text{span}\{x_1, \ldots, x_n, z\} := R y' \oplus L \), i.e., \( L \) is decomposable.

Finally, if \( H = \mathbb{R}^n \) then \( L \) is decomposable if and only if \([z, y] \in d(Z(\mathbb{R}^n)) = d(\mathbb{R}^n)\). By Proposition 4.1, we have

\[
L \in \text{Lie}_{\text{ad}}(n + 2, n) \iff d(R y \oplus \mathbb{R}^n) = \mathbb{R}^n
\]

\[
\iff \text{span}\{[z, y]\} + d(\mathbb{R}^n) = \mathbb{R}^n
\]

\[
\iff d(\mathbb{R}^n) = \mathbb{R}^n \quad \text{(since} \quad [z, y] \in d(\mathbb{R}^n)\text{)}
\]

\[
\iff d|_{\mathbb{R}^n} = a_z \text{ is nonsingular.}
\]

The proof of Proposition 5.3 is complete. \( \square \)

**Proposition 5.4.** Let

\[
L_i = R z \oplus_{d_i} K = R z \oplus_{d_i} (R y \oplus H) \in \text{Lie}_{\text{ad}}(n + 2, n), \quad i = 1, 2.
\]

Then \( L_1 \cong L_2 \) if and only if \( \tilde{a}_1 \cong_{\rho} \tilde{a}_2 \), where \( \tilde{a}_i \in H^1(K, K) \) are the equivalence classes of \( a_i \).

**Proof.** (\( \Leftarrow \)) A direct application of Corollary 2.9.

(\( \Rightarrow \)) Let \( \tilde{a} : R z \oplus_{d_i} K \to R z \oplus_{d_i} K \) be an isomorphism. We claim that \( K \) is \( \tilde{a} \)-invariant, i.e., \( \tilde{a}(K) = K \). Indeed, set \( \tilde{a}(y) = \alpha z + \beta y + u \) for \( u \in H \). Then, for an arbitrary \( x \in H \), we have

\[
\tilde{a}(z') = \alpha z + \beta y + u,
\]

\[
\iff 0 = [\alpha z + \beta y + u, x] = \alpha [z, x] + [u, x]
\]

\[
\iff \alpha a_z(x) = -\text{ad}_u(x)
\]

\[
\iff \alpha = 0 \quad \text{(since} \quad a_z \text{ is an outer derivation of} \quad H \text{ by Proposition 5.2).}
\]

Since \( K \) is \( \tilde{a} \)-invariant, \( \sigma := \tilde{a}|_{K} \) is well-defined, and hence, an isomorphism. Set \( \tilde{a}(z) = z' z + v \) where \( z' \neq 0 \) and \( v \in K \). Then, for every \( x \in K \), we have

\[
\tilde{a}(z, x) = [\tilde{a}(z), \tilde{a}(x)]
\]

\[
\iff \sigma([z, x]) = [z' z + v, \sigma(x)]
\]

\[
\iff \sigma d_1(x) = (z' d_2 + \text{ad}_v)\sigma(x)
\]

\[
\iff \sigma d_1 \sigma^{-1} = z' d_2 + \text{ad}_v
\]

\[
\iff \sigma d_1 \sigma^{-1} = z' (d_2 + \text{ad}_v).
\]
Since \( d_1 \in \bar{d}_1 \) and \( d_2 + \text{ad}_{\bar{x}_2} \in \bar{d}_2 \), where \( \bar{d}_1, \bar{d}_2 \in H^1(K, K) \), the last equation above means that \( \bar{d}_1 \overset{\sim}{\sim}_p \bar{d}_2 \) in \( H^1(K, K) \).

The proof of Proposition 5.4 is complete. \( \square \)

**Proof of Theorem 3.** As a direct consequence of Propositions 5.1, 5.2 and 5.4, the problem of classifying \( \text{Lie}_{\text{ad}}(n + 2, n) \) is equivalent to the problem of classifying equivalence classes in \( H^1(K, K) \) of outer derivations of \( K \) satisfying conditions 3 in Proposition 4.1 up to proportional similarity by the automorphism group of \( K \), where \( H \) is an arbitrary \( n \)-dimensional real nilpotent Lie algebra and \( K = \mathbb{R} \oplus H \).

For \( L = \mathbb{R}z \oplus dK = \mathbb{R}z \oplus_d (\mathbb{R}y \oplus H) \) with \( d \in \text{Der}(K) \setminus \text{ad}(K) \), we set
\[
H^1_s(K, K) := \left\{ \bar{d} \in H^1(K, K) \mid d \text{ satisfies condition 3 in Proposition 4.1} \right\}.
\]

Then Proposition 5.4 implies that classifying \( L \in \text{Lie}_{\text{ad}}(n + 2, n) \) is equivalent to determining \( H^1_s(K, K) / \overset{\sim}{\sim}_p \) which is essentially similar to that of Algorithm 1. Therefore, we can summarize the procedure of classifying \( \text{Lie}_{\text{ad}}(n + 2, n) \) as in Algorithm 2.

**Algorithm 2.** Classification of \( \text{Lie}_{\text{ad}}(n + 2, n) \)

**Input:** A list \( \mathcal{N} \) of non-isomorphic \( n \)-dimensional nilpotent Lie algebras \( H 

**Output:** A list \( \mathcal{S} \) of non-isomorphic \((n + 2)\)-dimensional indecomposable solvable Lie algebras \( \mathbb{R} \oplus_d (\mathbb{R} \oplus H) \) with derived algebras \( H 

1. \( \mathcal{S} := \{ \} \);
2. while \( H \in \mathcal{N} \) do
3. \( K := \mathbb{R}x_{n+1} \oplus H \);
4. \( L_d := \mathbb{R}x_{n+2} \oplus_d K \) with \( d \in \text{Der}(K) \setminus \text{ad}(K) \);
5. \( H^1_s(K, K) := \left\{ \bar{d} \in H^1(K, K) \mid d \text{ satisfies condition 3 in Proposition 4.1} \right\} \);
6. \( D := H^1_s(K, K) / \overset{\sim}{\sim}_p \);
7. for \( \bar{d} \in D \) do
8. Isomorphism verification for \( L_d \) by basis transformations of \( K \) which leave \( H \) fixed;
9. \( \mathcal{I} := \{ \text{isomorphism classes of } L_d \} \);
10. \( \mathcal{S} := \mathcal{S} \cup \mathcal{I} \);
11. Return \( \mathcal{S} \).

**5.2. Demonstrations**

In the following, we give an example in case \( n = 3 \) to demonstrate how Theorem 3 can be applied in the computation of \( \text{Lie}_{\text{ad}}(n + 2, n) \).

**Example 5.5.** Take \( H = \mathfrak{h}_3 \). We will classify all \( L \in \text{Lie}_{\text{ad}}(5, 3) \) which admits \( H \) as derived algebras. There are two cases as follows.

1. By Proposition 5.3, if \( L \) is decomposable then \( L \cong \mathbb{R} \oplus \bar{L} \), where \( \bar{L} \in \text{Lie}(4, 3) \) with \( \bar{L}^1 = \mathfrak{h}_3 \). According to Example 3.5, we have \( \bar{L} \in \{ \bar{L}^1, L_2, L_3^2 \} \).

2. Assume that \( L \) is indecomposable. In this case, we set
\[
L = L_d := \mathbb{R}x_5 \oplus_d K = \mathbb{R}x_5 \oplus_d (\mathbb{R}x_4 \oplus H),
\]
where \( d \in \text{Der}(K) \setminus \text{ad}(K) \) with \( d(K) \subset H \) and \( \bar{d} : K/H^1 \to K/H^1 \) induced from \( d \) are of the following forms:
First of all, we have
\[ H_1(K, K) = \text{span} \left\{ \tilde{d} = \begin{bmatrix} a + b & 0 & 0 & h \\ 0 & a & c & 0 \\ 0 & e & b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} / C_2 \right\}, \]
\[ \tilde{d} = \begin{bmatrix} a & c & 0 \\ e & b & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

Next, condition (3) of Proposition 4.1 becomes \( \det \begin{bmatrix} a & c \\ e & b \end{bmatrix} \neq 0 \). Furthermore, \( d(Z(H)) = \text{span}\{(a + b)x_1\} \) since \( Z(H) = \text{span}\{x_1\} \). By Proposition 5.3, \( L_d \) is indecomposable if and only if
\[ [x_5, x_4] = hx_1 \notin \text{span}\{(a + b)x_1\} \iff h \neq 0 = a + b. \]

Therefore, we have
\[ H_1^*(K, K) = \text{span} \left\{ \tilde{d} = \begin{bmatrix} 0 & 0 & 0 & h \\ 0 & a & c & 0 \\ 0 & e & -a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} / C_2 \right\}, \]
\[ \det \begin{bmatrix} a & c \\ e & -a \end{bmatrix} \neq 0 \text{ and } h \neq 0. \]

The possible JCFs of nonsingular matrix \( \begin{bmatrix} a & c \\ e & -a \end{bmatrix} \) are
\[ \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} (\lambda \neq 0) \sim_p \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \]
which lead to two classes in \( H_1^*(K, K) / \sim_p \) as follows:
\[ d_4 := \begin{bmatrix} 0 & h \\ 1 & 0 \\ -1 & 0 \end{bmatrix} \text{ and } d_5 := \begin{bmatrix} 0 & 1 \\ 0 & h \\ -1 & 0 \end{bmatrix}. \]

This means that \( H_1^*(K, K) / \sim_p = \{d_4, d_5\} \). Now, we determine \( H_1^*(K, K) / \tilde{\sim}_p \). According to [11, Theorem 3.4], all automorphisms \( \sigma \) of \( K = \mathbb{R}x_4 \oplus H \) are of the following form:
\[ \sigma = \begin{bmatrix} xy - zt & p & q & r \\ 0 & x & z & 0 \\ 0 & t & y & 0 \\ 0 & u & v & w \end{bmatrix} ; \ (xy - zt)w \neq 0. \]

Let \( \tilde{d} \in H_1^*(K, K) \). Set \( \Delta := a^2 + ce \neq 0 \). In a similar way as in Example 3.5, i.e., the same way in [25], we have that
\[ \tilde{d} \tilde{\sim}_p \begin{cases} d_4, & \Delta > 0, \\ d_5, & \Delta < 0. \end{cases} \]

Thus, we also have \( H_1^*(K, K) / \tilde{\sim}_p = \{d_4, d_5\} \) and there are two families of indecomposable Lie algebras in \( \text{Lie}_{ad}(5, 3) \) derived algebra \( H = h_3 \) as follows:
\[ L_h^4 = \mathbb{R}x_5 \oplus d_4(\mathbb{R}x_4 \oplus H), \quad L_h^5 = \mathbb{R}x_5 \oplus d_5(\mathbb{R}x_4 \oplus H), \quad (h \neq 0). \]
Finally, we also use [25] to have that \( L^h_4 \cong L^1_4 \) and \( L^h_5 \cong L^1_5 \) by isomorphisms \( \text{diag}(h,1,h,1,1) \) and \( \text{diag}(h^2,h,h,h,1) \), respectively. This allows us to reduce parameters \( h \) in both two families above to 1, i.e., we only have two indecomposable Lie algebras, namely, \( L_4 := L^1_4 \) and \( L_5 = L^1_5 \).

To sum up, there are five families of Lie algebras in \( \text{Lie}_{ad}(5,3) \) with derived algebra \( H = h_3 \), namely, we have

\[
\begin{align*}
\mathbb{R}x_5 \oplus L^1_1, & \quad \mathbb{R}x_5 \oplus L_2, & \quad \mathbb{R}x_5 \oplus L^2_2, & \quad (\text{decomposable}) \\
\mathbb{R}x_5 \oplus d(H), & \quad \mathbb{R}x_5 \oplus d(H), & \quad (\text{indecomposable}).
\end{align*}
\]

These Lie algebras, one again, coincide with those of Mubarakzyanov [21, §10], namely, \( L_4 \cong g_{5,20} \) and \( L_5 \cong g_{6,26} \).

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**References**

[1] Belitskii, G. R., Dmytryshyn, A. R., Lipyanski, R., Sergeichuk, V. V., Tsurkov, A. (2009). Problems of classifying associative or Lie algebras over a field of characteristic not two and finite metabelian groups are wild. *Electron. J. Linear Algebra*. 18:516–529.

[2] Belitskii, G. R., Lipyanski, R., Sergeichuk, V. V. (2005). Problems of classifying associative or Lie algebras and triples of symmetric or skew-symmetric matrices are wild. *Linear Algebra Appl.* 407:249–262. DOI: 10.1016/j.laa.2005.05.007.

[3] Bondarenko, V. M., Petravchuk, A. P. (2019). Wildness of the problem of classifying nilpotent Lie algebras of vector fields in four variables. *Linear Algebra Appl.* 568:165–172. DOI: 10.1016/j.laa.2018.07.031.

[4] Belitskii, G. R., Sergeichuk, V. V. (2003). Complexity of matrix problems. *Linear Algebra Appl.* 361:203–222. DOI: 10.1016/S0024-3795(02)00391-9.

[5] Cicalo, S., de Graaf, W. A., Schneider, C. (2012). Six-dimensional nilpotent Lie algebras. *Linear Algebra Appl.* 436(1):163–189. DOI: 10.1016/j.laa.2011.06.037.

[6] Dixmier, J. (1958). Sur les représentations unitaires des groupes de Lie nilpotents III. *Can. J. Math*. 10: 321–348. DOI: 10.4153/CJM-1958-033-5.

[7] de Graaf, W. A. (2005). Classification of solvable Lie algebras. *Experiment. Math.* 14(1):15–25. DOI: 10.1080/10586458.2005.10128911.

[8] Donovan, P., Freislich, M. R. (1972). Some evidence for an extension of the Brauer–Thrall conjecture. *Sonderforschungsbereich Theor. Math.* 26. DOI: 10.1016/j.laa.2011.06.037.

[9] Donovan, P., Freislich, M. R. (1973). *The Representation Theory of Finite Groups and Associated Algebras*. Carleton Mathematical Lecture Notes, No. 5. Ottawa, ON: Carleton University.

[10] Eberlein, P. (2003). The moduli space of 2-step nilpotent Lie algebras of type \((p, q)\). In: Bland, J., Kim, K-T., Krantz, S. G., eds. *Explorations in Complex and Riemannian Geometry: A Volume Dedicated to Robert E. Greene*. Contemporary Mathematics, Vol. 332. Providence, RI: American Mathematical Society, pp. 37–72.

[11] Fisher, D. J., Gray, R. J., Hydon, P. E. (2013). Automorphisms of real Lie algebras of dimension five or less. *J. Phys. A: Math. Theor.* 46(22):225204–2218pp. DOI: 10.1088/1751-8113/46/22/225204.

[12] Futorny, V., Klymchuk, T., Petravchuk, A. V., Sergeichuk, V. V. (2018). Wildness of the problems of classifying two-dimensional spaces of commuting linear operators and certain Lie algebras. *Linear Algebra Appl.* 536:201–209. DOI: 10.1016/j.laa.2017.09.019.

[13] Gong, M.-P. (1998). Classification of nilpotent Lie algebras of dimension 7 (Over algebraically closed fields and \( \mathbb{R} \)). Waterloo, Ontario, CA: University of Waterloo. http://hdl.handle.net/10012/1148.

[14] Hilton, P. J., Stammbach, U. (1997). *A Course in Homological Algebra*. New York: Springer-Verlag.

[15] Hindeleh, F., Thompson, G. (2008). Seven dimensional Lie algebras with a four-dimensional nilradical. *Algebras Groups Geom.* 25:243–265.

[16] Jacobson, N. (1962). *Lie Algebras*. New York: Dover.

[17] Jacobson, N. (1944). Schur’s Theorem on commutative matrices. *Bull. Amer. Math. Soc.* 50(6):431–436. DOI: 10.1090/S0002-9904-1944-08169-X.
[18] Le, A. V., Nguyen, A. T., Nguyen, T. C. T., Nguyen, T. M. T., Vo, N. T. (2020). Applying matrix theory to classify real solvable Lie algebras having 2-dimensional derived ideals. *Linear Algebra Appl.* 588:282–303. DOI: 10.1016/j.laa.2019.11.031.

[19] Morozov, V. V. (1958). Classification of nilpotent Lie algebras of dimension 6. *Izv. Vyssh. Uchebn. Zaved. Mat.* 4:161–171.

[20] Mubarakzyanov, G. M. (1963). On solvable Lie algebras. *Izv. Vyssh. Uchebn. Zaved. Mat.* 1:114–123.

[21] Mubarakzyanov, G. M. (1963). Classification of real structures of Lie algebras of fifth order. *Izv. Vyssh. Uchebn. Zaved. Mat.* 3:99–106.

[22] Mubarakzyanov, G. M. (1963). Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element. *Izv. Vyssh. Uchebn. Zaved. Mat.* 4:104–116.

[23] Ndogmo, J. C. (1994). Sur les fonctions invariantes sous laction coadjointe dune algèbre de Lie résoluble avec nilradical abélien [PhD dissertation]. Montréal, Québec, CA: Université de Montréal. https://umontreal.on.worldcat.org/v2/oclc/53598383.

[24] Ndogmo, J. C., Winternitz, P. (1994). Solvable Lie algebras with Abelian nilradicals. *J. Phys. A: Math. Gen.* 27(2):405–423. DOI: 10.1088/0305-4470/27/2/024.

[25] Nguyen, A. T., Le, A. V., Vo, N. T. (2021). Testing isomorphism of complex and real Lie algebras. 14pp. https://arxiv.org/abs/2102.10770arXiv:2102.10770.

[26] Parry, A. R. (2007). A classification of real indecomposable solvable Lie algebras of small dimension with codimension one nilradicals [Master thesis]. Logan, UT: Utah State University. https://digitalcommons.usu.edu/etd/7145.

[27] Patra, J., Zassenhaus, H. (1990). Solvable Lie algebras of dimension \( \leq 4 \) over perfect fields. *Linear Algebra Appl.* 142:1–17.

[28] Schöbel, C. (1993). A classification of real finite-dimensional Lie algebras with a low-dimensional derived algebra. *Rep. Math. Phys.* 33(1-2):175–186. DOI: 10.1016/0034-4877(93)90053-H.

[29] Schur, J. (1905). Zur Theorie vertauschbaren Matrizen. *J. Reine Angew. Math.* 1905(130):66–76. DOI: 10.1515/crll.1905.130.66.

[30] Sergeichuk, V. V. (2000). Canonical matrices for linear matrix problems. *Linear Algebra Appl.* 317(1–3):53–102. DOI: 10.1016/S0024-3795(00)00150-6.

[31] Shabanskaia, A., Winternitz, G. (2013). Solvable extensions of a special class of nilpotent Lie algebras. *Arch. Math. (Brno).* 49(3):141–159. DOI: 10.5817/AM2013-3-141.

[32] Shabanskaia, A. (2016). Solvable indecomposable extensions of two nilpotent Lie algebras. *Commun. Algebra.* 44(8):3626–3667. DOI: 10.1080/00927987.2014.925117.

[33] Snobl, L., Winternitz, P. (2014). *Classification and Identification of Lie Algebras*, Vol. 33 of CRM Monograph Series. Providence, RI: American Mathematical Society.

[34] Turkowski, P. (1990). Solvable Lie algebras of dimension six. *J. Math. Phys.* 31(6):1344–1350. DOI: 10.1063/1.528721.

[35] Turkowski, P. (1992). Structure of real Lie algebras. *Linear Algebra Appl.* 171:197–212. DOI: 10.1016/0024-3795(92)90259-D.