Poisson summation formula and noncommutative tori

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Abstract

An analog of the Poisson summation formula for two-dimensional noncommutative tori is established and applied to the Gauss circle problem.

Key words and phrases: complex and noncommutative tori

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A. The Poisson summation formula is an elementary and fundamental fact of harmonic analysis and representation theory. The simplest case of such a formula says that for each function $f \in C_0^\infty(\mathbb{R})$ it holds

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n),$$

(1)

where $C_0^\infty(\mathbb{R})$ is the set of smooth functions on $\mathbb{R}$ with compact support and $\hat{f}(\nu) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i \nu x} dx$ is the Fourier transform of function $f$. An analog of formula (1) for the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$ can be written as

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(m, n) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \hat{f}(m, n),$$

(2)

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where function \( f \in C_0^\infty(\mathbb{R}^2) \) and \( \hat{f} \) is the Fourier transform of \( f \). In what follows we restrict to the radially symmetric functions on \( \mathbb{R}^2 \), i.e. the functions \( f(u,v) = f(r) \), where \( r = u^2 + v^2 \geq 0 \). Let \( \tau \) be a complex number such that \( \Im(\tau) > 0 \); for the radially symmetric function \( f(r) \) on the lattice \( L = \mathbb{Z} + \mathbb{Z}\tau \) formula (2) becomes

\[
\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(m^2 + 2mn \Re(\tau) + n^2|\tau|^2) = \sum_{r=0}^{\infty} \mu(r)\hat{f}(r),
\]

where \( \mu(r) = |\{(m,n) \in \mathbb{Z}^2 : m^2 + n^2 = r\}| \) and \( \hat{f} \) the Fourier transform of \( f \) given by the formula

\[
\hat{f}(r) = \frac{\pi}{\Im(\tau)} \int_0^{\infty} f(s)J_0 \left[ 2\pi\sqrt{s} \sqrt{\frac{m^2|\tau|^2 - 2mn \Re(\tau) + n^2}{\Im^2(\tau)}} \right] ds,
\]

with \( J_0(z) = \frac{1}{\pi} \int_0^\pi \cos(z\cos\alpha)d\alpha \) being the Bessel function.

**B.** Let \( 0 < \theta < 1 \) be an irrational number. A noncommutative torus is the universal \( C^* \)-algebra generated by the unitaries \( u \) and \( v \) satisfying the commutation relation \( vu = e^{2\pi i\theta}uv \); we shall denote such an algebra by \( \mathcal{A}_\theta \). The \( \mathcal{A}_\theta \) and \( \mathcal{A}_{\theta'} \) are said to be stably isomorphic (Morita equivalent) whenever \( \mathcal{A}_\theta \otimes \mathcal{K} \cong \mathcal{A}_{\theta'} \otimes \mathcal{K} \), where \( \mathcal{K} \) is the \( C^* \)-algebra of compact operators; such an isomorphism occurs if and only if \( \theta' = (a\theta + b)/(c\theta + d) \), where \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \). The algebraic \( K \)-theory of \( \mathcal{A}_\theta \) is Bott periodic with \( K_0(\mathcal{A}_\theta) \cong \mathbb{Z} + \theta\mathbb{Z} \) and \( K_1(\mathcal{A}_\theta) \cong \mathbb{Z}^2 \). The abelian group \( \mathbb{Z} + \theta\mathbb{Z} \subset \mathbb{R} \) is called a pseudo-lattice and will be denoted by \( \Lambda \). The \( \mathcal{A}_\theta \) has a canonical realization as algebra of bounded linear operators on \( L_2(S^1) \), where \( S^1 \) is the unit circle. To get such a realization denote by \( S \) a rotation of \( S^1 \) through the angle \( 2\pi\theta \) and by \( M_f \) the pointwise multiplication by a continuous function on \( S^1 \); then \( \mathcal{A}_\theta \) is generated by the operators \( S \) and \( M_f \) for all \( f \in C(S^1) \).

**C.** Consider the upper half-plane \( \mathbb{H} = \{ x + iy \in \mathbb{C} \mid y > 0 \} \). It is known that each \( \tau \in \mathbb{H} \) defines a lattice \( L = \mathbb{Z} + \tau\mathbb{Z} \) and hence the complex torus \( \mathbb{C}/L \); two such tori are isomorphic whenever \( \tau' = (a\tau + b)/(c\tau + d) \), where \( a, b, c, d \in \mathbb{Z} \) and \( ad - bc = 1 \). The departure point of present study is a functor \( F \) between the isomorphism classes of complex tori and stable isomorphism classes of noncommutative tori; the details of this construction are in [3]. To give an idea let \( \phi \) be a closed form on torus \( T^2 \); the trajectories \( \phi = 0 \) define a
measured foliation on $T^2$. The measured foliations are known to parametrize the Teichmüller space $\mathbb{H}$ of $T^2$; such a claim is the Hubbard-Masur Theorem which we take for granted here. Our map $F: \mathbb{H} \to \partial \mathbb{H}$ is defined by the formula $\tau \mapsto \theta = \int_{\gamma_2} \phi / \int_{\gamma_1} \phi$, where $\gamma_1$ and $\gamma_2$ are generators of the first homology of the torus. The following is true: (i) $\mathbb{H} = \partial \mathbb{H} \times (0, \infty)$ is a trivial fiber bundle whose projection map coincides with $F$; (ii) $F$ is a functor, which sends isomorphic complex tori to the stably isomorphic noncommutative tori.

We shall refer to $F$ as the Teichmüller functor. Clearly $F$ takes every lattice $L = \mathbb{Z} + \tau \mathbb{Z}$ to a (scaled) pseudo-lattice $\Lambda = \mu (\mathbb{Z} + \theta \mathbb{Z})$, where $\mu \in (0, \infty)$.

D. Let $f \in C_0^\infty(\mathbb{R})$ be a function on a lattice $L$ and $\Lambda = F(L)$; consider the Poisson summation $\sum f(L) := \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(m^2 + 2mn \Re(\tau) + n^2 |\tau|^2)$ on $L$. Our objective is to express $\sum f(L)$ in terms of the pseudo-lattice $\Lambda$. Let $\text{Trace}$ be the canonical trace on the $\mathcal{C}^*$-algebra $\mathcal{A}_\theta \otimes \mathcal{K}$; our main result is as follows.

**Theorem 1** For every (radially symmetric) function $f(u^2 + v^2) \in C_0^\infty(\mathbb{R})$ there exists a unique self-adjoint operator $\hat{f} \in \mathcal{A}_\theta \otimes \mathcal{K}$ such that

$$\sum f(L) = \text{Trace} (\hat{f}). \quad (5)$$

**Proof of theorem 7.** We shall split the proof in a series of lemmas.

**Lemma 1** For every function $f \in C_0^\infty(\mathbb{R})$ there exists an element $b_0 \in \mathcal{A}_\theta$ such that $\text{Trace} (b_0) = f(\theta)$.

**Proof.** Let us recall the Rieffel construction of projection $p \in \mathcal{A}_\theta$ such that $\text{Trace} (p) = \theta$ as given in [4], p.418-419. In particular, it was proved that the required projection has the form

$$p = M h_1 S^{-1} + M_{\mathcal{R}} + M h_2 S \quad (6)$$

for certain periodic functions $h_1, h_2, \mathcal{R} \in C^\infty(\mathbb{R})$ and it holds $\int_0^1 \mathcal{R}(t) dt = \theta$; we shall refer to $\mathcal{R}$ as the Rieffel function. Denote by $\rho(x) = \int_0^x \mathcal{R}(t) dt$ the antiderivative of the Rieffel function; it is a monotone growing function on $\mathbb{R}$. We shall introduce a function $\varphi(x) \in C^\infty(\mathbb{R})$ defined by the equations:

$$f(\varphi(x)) = \rho(x), \quad f(\varphi(0)) = 0, \quad \varphi(1) = \theta, \quad \varphi(\tau) = \theta, \quad (7)$$

where $f \in C_0^\infty(\mathbb{R})$; the reader can verify that $\varphi(x)$ is a correctly defined function. Indeed, $\varphi(x)$ is uniquely defined by the functions $f$ and $\rho$; since
\[ f(\varphi(0)) = \rho(0) = \int_0^1 R(t)dt = 0, \] one concludes that the first condition in (7) is satisfied. To prove the second condition notice that \[ f(\varphi(1)) = \rho(1) = \int_0^1 R(t)dt = \theta; \] therefore the second condition is equivalent to the condition \[ f(\theta) = \theta. \] If the latter is not already the case, one can consider a smooth approximation, \( \tilde{f}(x) \), of the function

\[ g(x) = \begin{cases} f(x), & \text{if } x \neq \theta \\ \theta, & \text{if } x = \theta. \end{cases} \]  (8)

It is easy to see that for the function \( \tilde{f} \in C_0^\infty(\mathbb{R}) \) the second condition of (7) is now satisfied. The Rieffel function takes the form \( R(x) = \rho'(x) = f'(\varphi(x))\varphi'(x) \) using the chain rule. One can now evaluate the integral

\[ \int_0^1 R(x)dx = \int_0^1 f'(\varphi(x))\varphi'(x)dx = \int_0^1 f'(\varphi(x))d\varphi(x) = f(\varphi(1)) - f(\varphi(0)) = f(\theta), \]  (9)

where the last line follows from conditions (7). Consider an element \( b_0 \in A_\theta \) given by the formula

\[ b_0 = M_{h_1}S - 1 + M_{f'(\varphi(x))\varphi'(x)}S. \]  (10)

Notice that \( b_0 = p \) is the Rieffel projection if and only if \( f = Id \). It follows from the construction that \( \text{Trace} \ (b_0) = \int_0^1 f'(\varphi(x))\varphi'(x)dx = f(\theta) \). Lemma 1 is proved. □

**Lemma 2** For every function \( f \in C_0^\infty(\mathbb{R}) \) and every \( \beta \in \mathbb{Z} + \theta\mathbb{Z} \cap [0, 1] \) there exists an element \( b \in A_\theta \) such that \( \text{Trace} \ (b) = f(\beta) \).

**Proof.** Suppose that \( m \in \mathbb{Z} \) is a non-zero integer and let us define a function \( \{m\theta\} := m\theta - [m\theta] \), where \([m\theta]\) is the integer part of the irrational number \( m\theta \). Following Rieffel [4] p.409, we construct a subalgebra \( A_{\{m\theta\}} \subseteq A_\theta \); it is known that the trace functions of \( A_\theta \) and \( A_{\{m\theta\}} \) coincide ibid. Lemma 1 says that there exists an element \( b_0 \in A_{\{m\theta\}} \) such that

\[ \text{Trace} \ (b_0) = f(\{m\theta\}). \]  (11)

Since \( A_{\{m\theta\}} \subseteq A_\theta \) is an inclusion of the \( C^* \)-algebras, we conclude that \( b_0 \in A_\theta \). Notice that points \( \beta \in \mathbb{Z} + \theta\mathbb{Z} \cap [0, 1] \) are in one-to-one correspondence with the real numbers \( \{m\theta\} \) as \( m \) runs the set \( \mathbb{Z} - \{0\} \). Therefore, one can set \( b = b_0 \) in each case and lemma 2 follows. □
Lemma 3  For every function \( f \in C_0^\infty(\mathbb{R}) \) and every \( \beta \in \mathbb{Z} + \theta \mathbb{Z} \) there exists an element \( b \in A_\theta \otimes K \) such that \( \text{Trace} (b) = f(\beta) \).

Proof. (i) First let us assume that \( \beta = n\theta \) for some integer \( n \in \mathbb{Z} - \{0\} \). One can introduce a function \( g(x) = \frac{1}{n} f(nx) \); by lemma 1 there exists an element \( a \in A_\theta \) such that \( \text{Trace} (a) = g(\theta) \). Let us consider an element \( b \in A_\theta \otimes M_n \) of the form

\[
b = \begin{pmatrix}
a \\
\vdots \\
a
\end{pmatrix}.
\]

Note that the trace function on the \( C^* \)-algebra \( A_\theta \) can be canonically extended to such on the \( C^* \)-algebra \( A_\theta \otimes M_n \); therefore, one arrives at the following formulas

\[
\text{Trace} (b) = n \text{Trace} (a) = ng(\theta) = n \frac{1}{n} f(n\theta) = f(n\theta).
\]

One concludes in this case that for each \( n \in \mathbb{Z} - \{0\} \) there exists an element \( b \in A_\theta \otimes K \) such that \( \text{Trace} (b) = f(\beta) \).

(ii) Let us assume now that \( \beta = \theta + m \) for some integer \( m \in \mathbb{Z} \). Again one can introduce a function \( g(x) = f(x + m) - m \); by lemma 1 there exists an element \( a \in A_\theta \) such that \( \text{Trace} (a) = g(\theta) \). One can consider an element \( b \in A_\theta \otimes M_m \) of the form \( b = a + I_m \), where \( I_m \) is the unit matrix of dimension \( m \); then one gets

\[
\text{Trace} (b) = \text{Trace} (a) + m = g(\theta) + m = f(\theta + m) = m + m = f(\theta + m).
\]

Therefore in this case for each \( m \in \mathbb{Z} \) there exists an element \( b \in A_\theta \otimes M_m \) such that \( \text{Trace} (b) = f(\beta) \). Since every point of the pseudo-lattice \( \mathbb{Z} + \theta \mathbb{Z} \) can be written as \( m + n\theta \) the conclusion of lemma 3 follows from items (i) and (ii). \( \square \)

Lemma 4  For every function \( f \in C_0^\infty(\mathbb{R}) \) and every \( \beta \in \mu(\mathbb{Z} + \theta \mathbb{Z}) \) there exists an element \( b \in A_\theta \otimes K \) such that \( \text{Trace} (b) = f(\beta) \).

Proof. Let us consider a function \( g(x) = f(\mu x) \); it is obvious that for every real number \( \mu \in (0, \infty) \) the function \( g(x) \in C_0^\infty(\mathbb{R}) \). Repeating the argument of lemmas 1 and 3 one arrives at the conclusion of lemma 4. \( \square \)
Lemma 5 If \( F(z + \tau z) = \mu(z + \theta z) \) then for every function \( f \in C_0^\infty(\mathbb{R}) \) there exists an element \( \hat{f} \in A_\theta \otimes K \) such that

\[
\text{Trace} (\hat{f}) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(m^2 + 2mn \Re(\tau) + n^2|\tau|^2).
\]  

(15)

Proof. Notice that function \( f \in C_0^\infty(\mathbb{R}) \) maps the lattice \( L \cong \mathbb{Z}^2 \) into the real line \( \mathbb{R} \); in view of the bijective correspondence \( F \) between the points of lattice \( L \) and such of the pseudo-lattice \( \Lambda \cong \mathbb{Z}^2 \) one can extend the map \( f \) to the points of \( \Lambda \) by the formula \( f(\beta) := f(F^{-1}(\beta)) \), where \( \beta \in \Lambda \). Therefore applying the Poisson summation to the lattice \( L \) one gets the following equality

\[
\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(L) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(\Lambda).
\]  

(16)

Recall that by lemma 4 for each point \( \beta \in \Lambda = \mu(z + \theta z) \) there exists an element \( b \in A_\theta \otimes K \) such that \( \text{Trace} (b) = f(\beta) \). We shall introduce an element \( \hat{f} \) of the \( C^* \)-algebra \( A_\theta \otimes K \) as the formal sum of elements \( b_j \) as index \( j \) runs all points of the pseudo-lattice \( \Lambda \); in other words, \( \hat{f} := \sum_{\beta \in \Lambda} b_j \). Notice that \( \text{Trace} \) is an additive function on the \( C^* \)-algebra \( A_\theta \otimes K \); therefore one concludes that

\[
\text{Trace} (\hat{f}) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} f(\Lambda).
\]  

(17)

The conclusion of lemma 5 follows from equations (16) and (17). □

It remains to show that \( \hat{f} \in A_\theta \otimes K \) is a self-adjoint operator. Indeed notice that the operator \( b_0 \) given by formula (10) is self-adjoint since the involved functions \( f, h_1, h_2 \) and \( \varphi \) are real-valued functions. Thus \( \hat{f} \) is also a self-adjoint operator being a sum of the self-adjoint operators. Notice that in the class of all self-adjoint operators of algebra \( A_\theta \otimes K \) the operator \( \hat{f} \) is uniquely defined by the function \( f \in C_0^\infty(\mathbb{R}) \). Indeed, function \( \varphi(x) \) is a unique solution of the functional equation (7); therefore no other self-adjoint operator \( b_0 \in A_\theta \otimes K \) satisfies conditions of lemma 4. This argument finishes the proof of theorem 1. □

E. In conclusion, we shall consider an application of theorem 1. Let \( N \geq 1 \) be an integer and denote by

\[
\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}) \mid a, d \equiv 1 \mod N, \quad c \equiv 0 \mod N \right\}
\]  

(18)
a finite index subgroup of the matrix group $GL(2, \mathbb{Z})$. For an open disk $D = \{ z \in \mathbb{C} | |z - 1| < \frac{3}{2} \}$ let $L(D)$ be the Hilbert space of functions holomorphic in $D$ and continuous on $\overline{D}$. For a complex parameter $s \in \mathbb{C}$ and a matrix $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(N)$ consider an operator $\mathcal{L}_\alpha(s)$ on the space $L(D)$ given by the formula

$$\mathcal{L}_\alpha(s)f(z) := \frac{1}{(cz + d)^2s}f(\frac{az + b}{cz + d}).$$

(19)

Following Mayer [2] (when $N = 1$) and Manin-Marcolli [1] (when $N \geq 1$) one can define a matrix $\beta \in \Gamma_1(N)$ such that

$$\beta := \sum_{i=1}^{\infty} \begin{pmatrix} 0 & 1 \\ 1 & d_i \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} 0 & 1 \\ 1 & d_i \end{pmatrix} \in \Gamma_1(N).$$

(20)

Consider an operator $\mathcal{L}_N(s) := \mathcal{L}_\beta(s)$ which we shall call a Mayer-Ruelle operator; such an operator has many remarkable properties. For instance, $\det(1 - \mathcal{L}_N^2(s)) = Z_{\Gamma_1(N)}(s)$, where $Z_{\Gamma_1(N)}(s)$ is the Selberg zeta function of the group $\Gamma_1(N)$, while $\mathcal{L}_1(1)$ has an eigenvalue $\lambda_1 = 1$ such that the corresponding eigenfunction $f(z) = \frac{1}{z}$ coincides with the density of the Gauss-Kuzmin distribution of integers in the regular continued fraction [1]. A relation of $\mathcal{L}_1(s)$ with the Riemann Hypothesis is recorded [2], p.59.

Let $L_{CM} = \mathbb{Z} + \tau_D \mathbb{Z}$ be a lattice with complex multiplication by the ring of integers in the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$; denote by $H(x) \in C^\infty_0(\mathbb{R})$ a smooth approximation of the step function on $\mathbb{R}$. In view of theorem [1] there exists a self-adjoint operator $\hat{H}_D \in \mathcal{A}_\theta \otimes \mathcal{K}$ such that

$$\text{Trace} \left( \hat{H}_D \right) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} H(m^2 + 2mn \Re(\tau_D) + n^2|\tau_D|^2).$$

(21)

Notice that the double sum at the RHS of formula (21) is the counting function $N(r)$ for the number of integer points of the lattice $L_{CM}$ inside an ellipse

$$\frac{1 + \Re^2(\tau_D)}{3^2(\tau_D)}u^2 + 2\frac{\Re(\tau_D)}{3(\tau_D)}uv + v^2 = r, \quad r > 0.$$  

(22)

When $D = 1$ the evaluation of function $N(r)$ is known as the Gauss circle problem. Finally, for each $s \in \mathbb{C}$ we shall denote by $\hat{H}_D(s) \in \mathcal{A}_\theta \otimes \mathcal{K}$ such that $\text{Trace} \left( \hat{H}_D(s) \right) = \sum_{i=1}^{\infty} \lambda_i s^i$, where $\{\lambda_i\}_{i=1}^{\infty}$ are real numbers representing spectrum of the operator $\hat{H}_D$. The following conjecture links $\hat{H}_D(s)$ with the Mayer-Ruelle operator.

**Conjecture 1** $e^{-\hat{H}_D(s)} \equiv \mathcal{L}_D(s)$ whenever $\Re(s) > \frac{1}{2}$.  

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