THE COMPUTATION OF ZEROS OF AHLFORS MAP FOR DOUBLY CONNECTED REGIONS

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Abstract. The relation between the Ahlfors map and Szegö kernel $S(z,a)$ is classical. The Szegö kernel is a solution of a Fredholm integral equation of the second kind with the Kerzman-Stein kernel. The exact zeros of the Ahlfors map are unknown except for the annulus region. This paper presents a numerical method for computing the zeros of the Ahlfors map of any bounded doubly connected region. The method depends on the values of $S(z(t),a)$, $S'(z(t),a)$ and $\theta'(t)$ where $\theta(t)$ is the boundary correspondence function of Ahlfors map. A formula is derived for computing $S'(z(t),a)$. An integral equation is constructed for solving $\theta'(t)$. The numerical examples presented here prove the effectiveness of the proposed method.

1. Introduction

The conformal mapping from a multiply connected region onto the unit disk is known as the Ahlfors map. If the region is simply connected then the Ahlfors map reduces to the Riemann map. Many of the geometrical features of a Riemann mapping function are shared with Ahlfors map. The Riemann mapping function can be regarded as a solution of the following extremal problem: For a simply connected region $\Omega$ and canonical region $D$ in the complex plane $\mathbb{C}$ and fixed $a$ in $\Omega$, construct an extremal analytic map $F : \Omega \to D$ with $F'(a) > 0$.

The Riemann map is the solution of this problem. It is unique conformal, one-to-one and onto map with $F(a) = 0$.

For a multiply connected region $\Omega$ of connectivity $n > 1$, the answer to the same extremal problem above becomes the Ahlfors map. It is unique analytic map $f : \Omega \to D$ that is onto, $f'(a) > 0$ and $f(a) = 0$. However it has $2n - 2$ branch points in the interior and is no longer one-to-one there. In fact it maps $\Omega$ onto $D$ in an $n$-to-one fashion, and maps each boundary curve one-to-one onto the unit circle (see [5] and [17]). Therefore the Ahlfors map can be regarded as the Riemann mapping function in the multiply connected region.

Conformal mapping of multiply connected regions can be computed efficiently using the integral equation method. The integral equation method has been used by many authors to compute the one-to-one conformal mapping from multiply connected regions onto some standard canonical regions [4,6,8,11,13,15,19,23,27].

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Some integral equations for Ahlfors map have been given in [1,7,14,16,17]. In [3], Kerzman and Stein have derived a uniquely solvable boundary integral equation for computing the Szegö kernel of a bounded region and this method has been generalized in [1] to compute Ahlfors map of bounded multiply connected regions without relying on the zeros of Ahlfors map. In [7,14] the integral equations for Ahlfors map of doubly connected regions requires knowledge of zeros of Ahlfors map, which are unknown in general.

In this paper, we extend the approach of Sangawi [19,22–24] to construct an integral equation for solving $\theta'(t)$ where $\theta(t)$ is the boundary correspondence function of Ahlfors map of multiply connected region onto a unit disk.

The plan of this paper is as follows: Section 2 presents some auxiliary materials. We shall derive in Section 3 a boundary integral equation satisfied by $\theta'(t)$, where $\theta(t)$ is the boundary correspondence function of Ahlfors map of bounded multiply connected regions onto a disk. From the computed values of $\theta'(t)$, we then determine the Ahlfors map. In Section 4 we present a formula for computing the derivative of Szegö kernel which we used for finding the derivative of Ahlfors map giving another way of computing $\theta'(t)$ analytically. In Section 5 we present a method for computing the second zero of the Ahlfors map for doubly connected regions. In Section 6 we present some examples to illustrate our boundary integral equation method for finding the zeros of Ahlfors map for general doubly connected region. The numerical examples are first restricted to annulus region for which the exact Ahlfors map is known, then verified on general doubly connected region and obtained accurate results. Finally, Section 7 presents a short conclusion.

2. Auxiliary Materials

Let $\Omega$ be a bounded multiply connected region of connectivity $M + 1$. The boundary $\Gamma$ consists of $M + 1$ smooth Jordan curves $\Gamma_0, \Gamma_1, \ldots, \Gamma_M$ such that $\Gamma_1, \ldots, \Gamma_M$ lie in the interior of $\Gamma_0$, where the outer curve $\Gamma_0$ has counterclockwise orientation and inner curves $\Gamma_1, \ldots, \Gamma_M$ have clockwise orientation. The positive direction of the contour $\Gamma = \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_M$ is usually that for which $\Omega$ is on the left as one traces the boundary as shown in Figure 1.

![Figure 1](image.png)

**Figure 1.** A bounded multiply connected region $\Omega$ of connectivity $M + 1$.

The curves $\Gamma_j$ are parameterized by $2\pi$-periodic triple continuously differentiable complex-valued functions $z_j(t)$ with non-vanishing first derivatives

$$z'_j(t) = dz_j(t)/dt \neq 0, \quad t \in J_j = [0, 2\pi], \quad j = 0, 1, \ldots, M.$$
The total parameter domain $J$ is defined as the disjoint union of $M + 1$ intervals $J_0, \ldots, J_M$. The notation

$$z(t) = z_j(t), \quad t \in J_j, \quad j = 0, 1, \ldots, M.$$ 

must be interpreted as follows [14]: For a given $\hat{t} \in [0, 2\pi]$, to evaluate the value of $z(t)$ at $\hat{t}$, we should know in advance the interval $J_j$ to which $\hat{t}$ belongs, i.e., we should know the boundary $\Gamma_j$ contains $z(\hat{t})$, then we compute $z(\hat{t}) = z_j(\hat{t})$.

The generalized Neumann kernel formed with a complex continuously differentiable $2\pi$-periodic function $A(t)$ for all $t \in J$, is defined by [25,26]

$$N(t, s) = \frac{1}{\pi} \text{Im} \left( \frac{z'(s) z'(t)}{A(s) z(s) - A(t) z(t)} \right).$$

The classical Neumann kernel is the generalized Neumann kernel formed with $A(t) = 1$, i.e.

$$N(t, s) = \frac{1}{\pi} \text{Im} \left( \frac{z'(s)}{z(s) - z(t)} \right).$$

The kernel is continuous which takes on the diagonal the values

$$N(t, t) = \frac{1}{\pi} \left( \frac{1}{2} \text{Im} \frac{z''(t)}{z'(t)} - \text{Im} \frac{A'(t)}{A(t)} \right).$$

The adjoint of the function $A$ is given by

$$\hat{A} = \frac{z'(t)}{A(t)}.$$

The generalized Neumann kernel $\tilde{N}(s, t)$ formed with $\hat{A}$ is given by

$$\tilde{N}(s, t) = \frac{1}{\pi} \text{Im} \left( \frac{\hat{A}(s) z'(t)}{\hat{A}(t) z(t) - z(s)} \right),$$

which implies

$$\tilde{N}(s, t) = -N^*(s, t),$$

where $N^*(s, t) = N(t, s)$ is the adjoint kernel for the generalized Neumann kernel $N(s, t)$ (see [25]).

In the remaining part of this paper, we shall only be dealing with $A(t) = 1$, which the classical Neumann kernel. The adjoint kernel $N^*(s, t)$ of the classical Neumann kernel is given by

$$N^*(t, s) = N(s, t) = \frac{1}{\pi} \text{Im} \left( \frac{z'(t)}{z(t) - z(s)} \right).$$

It is known that $\lambda = 1$ is an eigenvalue of the kernel $N$ with multiplicity 1 and $\lambda = -1$ is an eigenvalue of the kernel $N$ with multiplicity $M$ [26]. The eigenfunctions of $N$ corresponding to the eigenvalue $\lambda = -1$ are $\{\chi^1, \chi^2, \ldots, \chi^M\}$, where

$$\chi^{[j]}(\xi) = \begin{cases} 1, & \xi \in \Gamma_j, \\ 0, & \text{otherwise}, \end{cases} \quad j = 1, 2, \ldots, M.$$

Let $H^*$ be the space of all real Hölder continuous $2\pi$-periodic functions $\omega(t)$ of the parameter $t$ on $J_j$ for $j = 0, 1, \ldots, M$, i.e.

$$\omega(t) = \omega_k(t), \quad t \in J_j, \quad k = 0, 1, \ldots, M, \quad j = 0, 1, \ldots, M.$$
Define also the integral operator $J$ by

$$J^\nu = \int_J \frac{1}{2\pi} \sum_{j=1}^M \chi^{|j|}(s)\chi^{|\nu|}(t)\nu(s)ds.$$ 

We also define the Fredholm integral operator $N^*$ by

$$N^*\psi(t) = \int_J N^*(t,s)\psi(s)ds, \quad t \in J.$$ 

A complex-valued function $P(z)$ is said to satisfy the interior relationship if $P(z)$ is analytic in $\Omega$ and satisfies the non-homogeneous boundary relationship

$$P(z) = \frac{b(z)T(z)}{G(z)}P(z) + H(z), \quad z \in \Gamma,$$

where $G(z)$ is analytic in $\Omega$ and Hölder continuous on $\Gamma$, $G(z) \neq 0$ on $\Gamma$, and $T(z) = z'(t)/|z'(t)|$ denotes the unit tangent in the direction of increasing parameters at the point $z = z(t) \in \Gamma$. The boundary relationship (2.1) also has the following equivalent form:

$$G(z) = \frac{b(z)T(z)}{|P(z)|^2} + \frac{G(z)H(z)}{P(z)}, \quad z \in \Gamma.$$

The following theorem gives an integral equation for an analytic function satisfying the non-homogeneous boundary relationship (2.1) or (2.2).

**Theorem 2.1.** [19] If the function $P(z)$ satisfies the interior relationship with (2.1) or (2.2), then

$$T(z)P(z) + \int_\Gamma K(z,w)T(w)P(w)dw|$$

$$+ b(z) \left[ \sum_{\alpha_j, \text{inside}} \text{Res}_{w=\alpha_j} \frac{P(w)}{(w-z)G(w)} \right] = -L_R^-(z),$$

where

$$K(z,w) = \frac{1}{2\pi i} \left[ \frac{T(z)}{z-w} - \frac{b(z)T(w)}{b(w)(z-w)} \right],$$

and

$$L_R^-(z) = \frac{-1}{2} \frac{H(z)}{T(z)} + \text{PV} \frac{1}{2\pi i} \int_\Gamma \frac{b(z)H(w)}{b(w)(w-z)T(w)}dw.$$

The symbol “$-$” in the superscript denotes the complex conjugate and the sum is over all those zeros that lie inside $\Omega$. If $G$ has no zeros in $\Omega$, then the term containing the residue will not appear.
3. Integral equation method for computing $\theta'$

Let $f(z)$ be the Ahlfors function which maps $\Omega$ conformally onto a unit disc. The mapping function $f$ is determined up to a factor of modulus 1. The function $f$ could be made unique by prescribing that

$$f(a_j) = 0, \quad f'(a_0) > 0, \quad j = 0, 1, 2, \ldots, M.$$ 

where $a_j \in \Omega, j = 0, 1, \ldots, M$ are the zeros of the Ahlfors map. The boundary values of $f$ can be represented as

$$f(z_j(t)) = e^{i\theta_j(t)}, \quad \Gamma_j : z = z_j(t), \quad 0 \leq t \leq \beta_j,$$

where $\theta_j(t), j = 0, 1, \ldots, M$ are the boundary correspondence functions of $\Gamma_j$. Also we have

$$f'(z_j(t)) \frac{z'(t)}{f(z(t))} = i\theta'(t), \quad (3.1)$$

The unit tangent to $\Gamma$ at $z_j(t)$ is denoted by

$$T(z_j(t)) = \frac{z'(t)}{|z'(t)|}.$$ 

Thus it can be shown that

$$f(z_j(t)) = \frac{1}{i} T(z_j(t)) \frac{\theta_j'(t)}{|\theta_j'(t)|} \frac{f'(z_j(t))}{|f'(z_j(t))|}, \quad z_j \in \Gamma_j. \quad (3.2)$$

By the angle preserving property of conformal map, the image of $\Gamma_0$ remains in counter-clockwise orientation so $\theta_0'(t) > 0$ while the images of inner boundaries $\Gamma_j$ in clockwise orientation so $\theta_j'(t) < 0$, for $j = 1, \ldots, M$. Thus

$$\text{sign}(\theta_j'(t)) = \begin{cases} +1, & j = 0, \\ -1, & j = 1, \ldots, M. \end{cases}$$

The boundary relationship $(3.3)$ can be written briefly as

$$f(z) = \text{sign}(\theta'(t)) \frac{1}{i} T(z) \frac{f'(z)}{|f'(z)|}, \quad z \in \Gamma. \quad (3.4)$$

Since the Ahlfors map can be written as $f(z) = \prod_{j=0}^{M} (z - a_j) \hat{g}(z)$, where $\hat{g}(z)$ is analytic in $\Omega$ and $\hat{g}(z) \neq 0$ in $\Omega$, we have

$$\frac{f'(z)}{f(z)} = \frac{\hat{g}'(z)}{\hat{g}(z)} + \sum_{j=0}^{M} \frac{1}{z - a_j}, \quad z \in \Gamma, \quad (3.5)$$

so that

$$D(z) = \frac{f'(z)}{f(z)} - \sum_{j=0}^{M} \frac{1}{z - a_j} = \frac{\hat{g}'(z)}{\hat{g}(z)}. \quad (3.6)$$

is analytic in $\Omega$. Squaring both sides of $(3.4)$, gives

$$f(z)^2 = -T(z)^2 \frac{f'(z)^2}{|f'(z)|^2}, \quad (3.7)$$

which implies

$$\frac{f'(z)}{f(z)} = -T(z)^2 \left( \frac{f'(z)}{f(z)} \right). \quad (3.8)$$
From (3.6), we have

\[(3.9) \quad f'(z) f(z) = D(z) + \sum_{j=0}^{M} \frac{1}{z - a_j}.\]

Substituting (3.9) into (3.8)

\[(3.10) \quad D(z) + \sum_{j=0}^{M} \frac{1}{z - a_j} = -\overline{T(z)}^2 D(z) - \sum_{j=0}^{M} \frac{T(z)^2}{z - a_j}\]

or

\[(3.11) \quad D(z) = -\overline{T(z)}^2 D(z) - \sum_{j=0}^{M} \frac{T(z)^2}{z - a_j} - \sum_{j=0}^{M} \frac{1}{z - a_j}.\]

Comparing (3.11) with (2.1), we get

\[(3.12) \quad P(z) = D(z), b(z) = -\overline{T(z)}, G(z) = 1, H(z) = -\sum_{j=0}^{M} \frac{T(z)^2}{z - a_j} - \sum_{j=0}^{M} \frac{1}{z - a_j}.\]

Substituting these assignments into Theorem 2.1, we obtain

\[(3.13) \quad T(z)D(z) + \int_{\Gamma} K(z, w) T(w) D(w) |dw| + 0 = -\overline{L_R}(z),\]

where

\[K(z, w) = \frac{1}{2\pi i} \left[ \frac{T(z)}{z - w} - \frac{\overline{T(z)}}{\overline{z} - \overline{w}} \right] = \frac{1}{\pi} \text{Im} \left( \frac{T(z)}{z - w} \right) = N(z, w)\]

and

\[L_R(z) = -\frac{1}{2T(z)} \left[ \sum_{j=0}^{M} \frac{T(z)^2}{z - a_j} - \sum_{j=0}^{M} \frac{1}{z - a_j} \right]\]

\[+ \text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{T(z)}{(w - z)T(w)^2} \left[ \sum_{j=0}^{M} \frac{T(w)^2}{w - a_j} - \sum_{j=0}^{M} \frac{1}{w - a_j} \right].\]

Using the fact that \[20\]

\[\text{PV} \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{(w - z)(w - a_j)} dw = -\frac{1}{2(z - a_j)} ,\]

dw = T(w)|dw|, and after some simplifications, (3.13) becomes

\[(3.14) \quad T(z) \frac{f'(z)}{f(z)} + \int_{\Gamma} N(z, w) \frac{f'(w)}{f(w)} T(w) |dw| = 2i \text{Im} \left[ \sum_{j=0}^{M} \frac{T(z)}{z(t) - a_j} \right].\]
where

\[(3.15)\quad N(z, w) = \begin{cases} \frac{1}{2\pi i} \left[ T(z) - T(w) \right], & z \neq w \in \Gamma, \\ \frac{1}{2\pi |z'(t)|} \text{Im} \left( \frac{z''(t)}{z'(t)} \right), & z = w \in \Gamma, \end{cases}\]

with

\[(3.16)\quad \frac{1}{2\pi i} \int_{\Gamma_j} \frac{f'(w)T(w)}{f(w)} |dw| = 1, \quad j = 1, 2, \ldots, M.\]

In integral equation \((3.14)\), letting \(z = z(t), \ w = z(s)\) and multiplying both sides by \(|z'(t)|\), gives

\[(3.17)\quad f'(z(t)) f(z(t)) z'(t) + \int_{\Gamma} N(z(t), z(s)) f'(z(s)) z'(s)ds = 2i \text{Im} \left( \sum_{j=0}^{M} \frac{z'(t)}{z(t) - a_j} \right).\]

Using \((3.2)\), the above equation becomes

\[(3.18)\quad \theta'(t) + \int_{\Gamma} N(t, s) \theta'(s)ds = 2i \text{Im} \left( \sum_{j=0}^{M} \frac{z'(t)}{z(t) - a_j} \right).\]

Since \(N^*(s, t) = N(t, s)\), the integral equation \((3.18)\) in the operator form is

\[(3.19)\quad (I + N^*)\theta' = \phi, \quad j = 0, 1, \ldots, M,\]

where

\[(3.20)\quad \phi = 2i \text{Im} \left( \sum_{j=0}^{M} \frac{z'(t)}{z(t) - a_j} \right).\]

By Theorem 12 in \([26]\), \(\lambda = -1\) is an eigenvalue of \(N^*\) with multiplicity \(M\), therefore the integral equation \((3.19)\) is not solvable. To overcome this problem, we note from \((3.16)\) and \((3.2)\) that

\[(3.21)\quad \mathbf{J} \theta' = \psi,\]

where

\[(3.22)\quad \psi = (0, 1, 1, \ldots, 1).\]

By adding \((3.19)\) and \((3.21)\), we get

\[(3.23)\quad (I + N^* + \mathbf{J})\theta' = \phi + \psi\]

which implies

\[(3.24)\quad \phi = (I + N^* + \mathbf{J})\theta' - \psi.\]

In the next section, we derive another formula for computing \(\theta'\) from which the second zero of the Ahlfors map for doubly connected region can be calculated using \((3.24)\).
4. An Alternative Formula for Computing $\theta'$

The Ahlfors map is related to the Szegö kernel $S(z, a_0)$ and the Garabedian kernel $L(z, a_0)$ by [1]

\[ f(z) = \frac{S(z, a_0)}{L(z, a_0)}, \quad z \in \Omega \cup \Gamma. \]

The Szegö kernel $S(z, a_0)$ and Garabedian kernel $L(z, a_0)$ are related on $\Gamma$ as

\[ L(z, a_0) = -iT(z)S(z, a_0), \quad z \in \Gamma, \]

so that (4.1) becomes

\[ f(z) = \frac{1}{i} \frac{S(z, a_0)T(z)}{S(z, a_0)}, \quad z \in \Gamma, \]
or

\[ f(z(t)) = \frac{1}{i} \frac{S(z(t), a_0)T(z(t))}{S(z(t), a_0)}, \quad z(t) \in \Gamma. \]

From Bell [1], we have the integral equation for the Szegö kernel

\[ S(z, a_0) + \int_{\Gamma} A(z, w)S(w, a_0)|dw| = g(z), \quad z \in \Gamma, \]

where

\[ A(z, w) = \begin{cases} \frac{1}{2\pi i} \left[ \frac{T(w)}{z-w} - \frac{T(z)}{z-w} \right], & \text{if } z \neq w \in \Gamma \\ 0, & \text{if } z = w, \end{cases} \]

and

\[ g(z(t)) = \frac{1}{2\pi i} \frac{T(z(t))}{z(t) - a_0}. \]

With $z = z(t)$ and $w = z(s)$, (4.4) becomes

\[ S(z(t), a_0) + \int_{\Gamma} A(z(t), z(s))S(z(s), a_0)|z'(s)|ds = g(z(t)). \]

Differentiate both sides with respect to $t$, we get

\[ \frac{d}{dt} S(z(t), a_0) + \frac{d}{dt} \int_{\Gamma} A(z(t), z(s))S(z(s), a_0)|z'(s)|ds = \frac{d}{dt} g(z(t)), \]

which is equivalent to

\[ S'(z(t), a_0)z'(t) = g'(z(t))z'(t) - \int_{\Gamma} \left[ \frac{d}{dt} A(z(t), z(s))z'(s) \right] S(z(s), a_0)|z'(s)|ds. \]

Differentiate (4.9) with respect to $t$, gives

\[ g'(z(t))z'(t) = -\frac{1}{2\pi i} \left[ \frac{T'(z(t))z'(t)}{z(t) - a_0} - \frac{z'(t)}{z(t) - a_0} \frac{T(z(t))}{(z(t) - a_0)^2} \right]. \]

Since

\[ T(z(t)) = \frac{z'(t)}{|z'(t)|}, \]
we get

\[ T'(z(t))z'(t) = \frac{z''(t)}{2|z'(t)|} - \frac{z'^2(t)z''(t)}{2|z'(t)|^3}. \tag{4.8} \]

Since

\[ A(z(t), z(s)) = \frac{1}{2\pi i} \left[ \frac{T(z(s))}{z(t) - z(s)} \right. \]

\[ \left. \frac{T'(z(t))z'(t)}{z(t) - z(s)} \right], \quad \text{for } t \neq s, \]

differentiating with respect to \( t \), we get

\[ \frac{d}{dt}A(z(t), z(s)) = \frac{1}{2\pi i} \left[ \frac{-z'(t)T(z(s))}{(z(t) - z(s))^2} \right. \]

\[ \left. \frac{-T'(z(t))z'(t)}{(z(t) - z(s))^2} \right] \frac{z'(s)}{z(t) - z(s)} \]

\[ - \frac{2}{z'(t)} \left. \frac{z'(s)}{z(t) - z(s)} \right]. \tag{4.9} \]

For \( s \) sufficiently close to \( t \), we have

\[ z(s) = z(t) + z'(t)(s - t) + \frac{1}{2} z''(t)(s - t)^2 + \frac{1}{6} z'''(t)(s - t)^3 + O((s - t)^4) \]

which implies

\[ \frac{z'(t)}{z(s) - z(t)} = \frac{1}{s - t} \left[ 1 + \frac{1}{2} \frac{z''(t)}{z'(t)}(s - t) + \frac{1}{6} \frac{z'''(t)}{z'(t)}(s - t)^2 + O((s - t)^4) \right]^{-1}. \tag{4.10} \]

Using the fact that

\[ \frac{1}{1 + \varrho} = 1 - \varrho + \varrho^2 + O(\varrho^3), \]

for \( \varrho \) close to zero, \( \tag{4.10} \) becomes

\[ \frac{z'(t)}{z(s) - z(t)} = \frac{1}{s - t} - \frac{1}{2} \frac{z''(t)}{z'(t)}(s - t) + \frac{1}{6} \frac{z'''(t)}{z'(t)}(s - t)^2 + O((s - t)^2). \tag{4.12} \]

We observe that

\[ \frac{z'(s)}{z'(t)} = 1 + \frac{z''(t)}{z'(t)}(s - t) + \frac{1}{2} \frac{z'''(t)}{z'(t)}(s - t)^2 + O((s - t)^3). \tag{4.13} \]

Next observe that

\[ \left| \frac{z'(s)}{z'(t)} \right|^2 = 1 + 2 \text{Re} \left( \frac{z''(t)}{z'(t)} \right) (s - t) \]

\[ + \left( \text{Re} \left( \frac{z'''(t)}{z'(t)} \right) + \left| \frac{z''(t)}{z'(t)} \right|^2 \right) (s - t)^2 + O((s - t)^3). \tag{4.14} \]

Using the fact that

\[ \sqrt{1 + \tilde{x}} = 1 + \frac{1}{2} \tilde{x} - \frac{1}{8} \tilde{x}^2 + \ldots \]

\[ \sqrt{1 + \tilde{x}} = 1 + \frac{1}{2} \tilde{x} - \frac{1}{8} \tilde{x}^2 + \ldots \]
for small \( \hat{\delta} \), (4.14) becomes
\[
\left| \frac{z'(s)}{z'(t)} \right| = 1 + \text{Re} \left( \frac{z''(t)}{z'(t)} \right) (s-t) + \frac{1}{2} \left( \text{Re} \left( \frac{z''(t)}{z'(t)} \right) + \left| \frac{z'(t)}{z'(t)} \right| \right)^2 (s-t)^2 + O((s-t)^3).
\]
Substituting (4.16), (4.13), (4.12) into (4.9) and then taking the limit \( s \to t \) to both sides, we obtain
\[
\frac{d}{dt} A(z(t), z(s)) = \begin{cases} 
1 \left[ \frac{-z'(t) T(z(s))}{(z(t) - z(s))^2} - \frac{T'(z(t)) z'(t)}{(z(t) - z(s))^2} \right] 
& t \neq s \in \Gamma, \\
\frac{1}{12\pi |z'(t)|} \text{Im} \left( \frac{z''(t)}{z'(t)} \right) 
& t = s \in \Gamma.
\end{cases}
\]
Using the values from (4.4) and (4.7), we can find the derivative of Szegö kernel from (4.6). By defining the following terms,
\[
f_p = f'(z(t))z'(t), \quad S_p = S'(z(t), a_0)z'(t),
\]
(4.17)
\[
T_p = T'(z(t))z'(t), \quad T_z = T(z(t)).
\]
and differentiating both sides of (4.3) with respect to \( t \), the result is
\[
f_p = \frac{1}{i} \left[ \frac{S_p T_z + S_z T_p}{S_z} - \frac{S_p (S_z T_z)}{(S_z)^2} \right].
\]
By using the values from (4.2) and (4.18) in (3.2), we get
\[
\theta'(t) = 2\text{Im} \left( \frac{S_p}{S_z} \right) + \text{Im} \left( \frac{z''(t)}{z(t)} \right)
\]
which is the second formula for computing \( \theta' \).

5. Computing the Second Zero of the Ahlfors Map for Doubly Connected Region

In particular, if \( \Omega \) is a doubly connected region, i.e. \( M = 1 \), then (3.20) becomes
\[
\phi = 2\text{Im} \left[ \frac{z'(t)}{z(t) - a_0} + \frac{z'(t)}{z(t) - a_1} \right].
\]
As \( \phi \) is known from (3.21), and the zero \( a_0 \in \Omega \) can be freely prescribed, the only unknown in (5.1) is the second zero \( a_1 \) of Ahlfors map. The above equation can be written as
\[
\text{Im} \left[ \frac{z'(t)}{z(t) - a_1} \right] = k_1(t).
\]
where
\[
k_1(t) = \frac{1}{2} \left[ \phi - 2\text{Im} \left( \frac{z'(t)}{z(t) - a_0} \right) \right].
\]
Now we suppose that
\[
z(t) = x(t) + iy(t),
\]
\[
a_1 = \alpha + i\beta.
\]
Then (5.2) becomes

\[ y'(t)(x(t) - \alpha) + x'(t)(\beta - y(t)) = k_1(t) \]

After some algebraic manipulations, we obtain

\[ k_2(t)\alpha + k_3(t)\beta + k_1(t)(\alpha^2 + \beta^2) = -k_4(t), \]

where

\[ k_2(t) = y'(t) - 2k_1(t)x(t), \]

\[ k_3(t) = -2k_1(t)y(t) - x'(t), \]

\[ k_4(t) = k_1(t)x^2 + k_1(t)y^2 + x'(t)y(t) - y'(t)x(t). \]

Equation (5.5) can be viewed as containing three unknowns namely \( \alpha, \beta \) and \( \alpha^2 + \beta^2 \). To determine them, we need three equations. For this we can choose any three values of parameter \( t \) in the given interval. At these three random values of \( t \), we can find the values of the coefficients \( k_1(t), k_2(t), k_3(t) \) and \( k_4(t) \). By solving the obtained system of three equations with three unknowns, we can find the value of \( \alpha \) and \( \beta \), which are the real and imaginary parts of the second zero of Ahlfors map for doubly connected region.

### 6. Numerical Examples

For solving the integral equation (4.4) numerically, the reliable procedure is by using the Nyström method with the trapezoidal rule with \( n \) equidistant nodes in each interval \( J_j, j = 1, \ldots, M \). The trapezoidal rule is the most accurate method for integrating periodic functions numerically \([2, pp.134-142]\). By solving the integral equation (4.4) for \( S(z(t), a_0) \) gives \( S_p = S'(z(t), a_0)z'(t) \) from (4.6) and \( \theta'(t) \) from (4.19). By (3.24) we get the value of \( \phi \) and then from (5.1) we get the value of \( a_1 = \alpha + i\beta \). We then apply (4.3) and the Cauchy integral formula to compute \( f(\alpha + i\beta) \). For evaluating the Cauchy integral formula \( f(z) = (1/(2\pi i)) \int_{\Gamma} f(w)/(w-z)dw \) numerically, we use the equivalent form

\[ f(z) = \frac{\int_{\Gamma} f(w)dw}{\int_{\Gamma} \frac{1}{w-z}dw}, \quad z \in \Omega, \]

which also works very well for \( z \in \Omega \) near the boundary \( \Gamma \). When the trapezoidal rule is applied to the integrals in (6.1), the term in the denominator compensates the error in the numerator (see [22]). All the computations were done using MATLAB 7.12.0.635(R2011a).

**Example 6.1.** Consider an annulus region bounded by the two circles

\[ \Gamma_0: \{z(t) = e^{it}\}, \]

\[ \Gamma_1: \{z(t) = re^{-it}\}, \quad t: 0 \leq t \leq 2\pi, 0 < r < 1. \]

In [17], Tegtmeyer and Thomas computed the Ahlfors map using Szegő and Garabedian kernels for the annulus region, where the authors have used series representations of both Szegő kernel and Garabedian kernel. With these representations,
they found the two zeros $a_0$ and $a_1 = \frac{-r}{a_0}$ for Ahlfors map, where $r$ is the radius of the inner circle. They have also considered the symmetry case where the zeros are $a_0 = \sqrt{r}$ and $a_1 = -\sqrt{r}$. This example has also been considered in [14] where Ahlfors map was computed using a boundary integral equation related to a Riemann-Hilbert problem. Here we shall use these values of zeros of Ahlfors map for comparison with our numerical zeros with proposed method in the annulus $r < |z| < 1$. In this example we have chosen $r = 0.1$ and the first zero $a_0 = 0.6$. See Table 1 for numerical comparison of the computed second zeros $a_{1n}$ from (5.5) and the exact second zeros of the Ahlfors map, $a_1 = -r/a_0$.

| Table 1. Maximum error norm $||a_{1n} - a_1||$ with $r = 0.1$ for Example 6.1 |
|----------------------------------------------------------|
| n | $||a_{1n} - a_1||$ |
|---|------------------|
| 64 | 2.66(-15) |

| Symmetric case ($a_0 = \sqrt{r}, a_1 = -\sqrt{r}$) |
|--------------------------------------------------|
| n | $||a_{1n} - a_1||$ |
|---|------------------|
| 64 | 5.56(-14) |

**Example 6.2.** Consider a doubly connected region $\Omega$ bounded by the two non-concentric circles

$$\Gamma_0: \{ z(t) = 2e^{it} \},$$
$$\Gamma_1: \{ z(t) = c + re^{-it} \}, \quad t: 0 \leq t \leq 2\pi,$$

with $c = 0.2 + 0.6i$ and radius $r = 0.3$. The test region is shown in Figure 2. Given a first zero $a_0$ of the Ahlfors map, the exact second zero $a_1$ is unknown for this region. We compute $a_{1n}$ from (5.5) which is the approximate value of $a_1$. The accuracy is measured by computing $f(a_{1n})$ from (6.1). The theoretical value of $f(a_1)$ is zero. See Table 2 for the numerical values of $a_{1n}$ and $f(a_{1n})$ involving two choices of $a_0$.

![Figure 2. The region Ω for Example 6.2](image-url)
Table 2. Numerical values of $a_{1n}$ and $f(a_{1n})$ for Example 6.2

| $n$  | $a_{1n}$          | $f(a_{1n})$ |
|------|-------------------|-------------|
| 16   | 0.7135 + 1.0357i  | 2.2(−03)    |
| 32   | 0.7125 + 1.0342i  | 1.12(−06)   |
| 64   | 0.7125 + 1.0342i  | 2.9(−13)    |
| 128  | 0.7125 + 1.0342i  | 4.8(−15)    |

Example 6.3. Consider a doubly connected region $\Omega$ bounded by $\Gamma_0$ and $\Gamma_1$

$$\Gamma_0: \{z(t) = -0.1 - 0.4i + (6 + 0.8 \cos(18t))e^{it}\},$$
$$\Gamma_1: \{z(t) = \xi + (1.2 + 0.4 \cos(4t))e^{(-it)}\}, \quad t: 0 \leq t \leq 2\pi,$$

The test regions are shown in Figure 3 for two different values of $\xi$. Given a first zero $a_0$ of the Ahlfors map, the exact second zero $a_1$ is also unknown for this region. We compute $a_{1n}$ from (5.5), which is the approximate value of $a_1$. The accuracy is measured by computing $f(a_{1n})$ from (6.1). The theoretical value of $f(a_1)$ is zero. See Table 3 for the numerical values of $a_{1n}$ and $f(a_{1n})$.

![Figure 3](image_url)
Table 3. Numerical values of $a_{1n}$ and $f(a_{1n})$ for Example 6.3

| $\xi$                     | $n$ | $a_{1n}$        | $f(a_{1n})$ |
|---------------------------|-----|----------------|-------------|
| $0.6452 - 0.8655i$ & $a_0 = -1.3088 + 1.8012i$ | 128 | $2.0567 - 2.4889i$ | $3.0(-02)$  |
|                           | 256 | $2.0625 - 2.5444i$ | $5.8(-05)$  |
|                           | 512 | $2.0624 - 2.5445i$ | $8.9(-08)$  |
|                           | 1024| $2.0624 - 2.5445i$ | $9.8(-12)$  |
| $-2.4516 + 2.3626i$ & $a_0 = 2.2673 - 2.0351i$ | 256 | $-3.0647 + 3.2599i$ | $6.9(-04)$  |
|                           | 512 | $-3.0648 + 3.2600i$ | $1.8(-07)$  |
|                           | 1024| $-3.0648 + 3.2600i$ | $5.6(-12)$  |

7. Conclusion

In this paper, we have constructed a numerical method for finding second zero of the Ahlfors map of doubly connected regions. We derived two formulas for the derivative of the boundary correspondence function $\theta(t)$ of the Ahlfors map. These formulas were then used to find the second zero of the Ahlfors map for any smooth doubly connected regions. Analytical method for computing the exact zeros of Ahlfors map for annulus region is presented in [16] and [17] but the problem of finding zeros for arbitrary doubly connected regions is the first time presented in this paper. The numerical examples presented have illustrated that our method involving boundary integral equation has high accuracy.

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