Special classes of solutions for linear string baryon configuration

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Abstract

For the linear string baryon model with three material points (three quarks) joined sequentially by the relativistic strings, the class of motions admitting linearizable boundary conditions is investigated. These motions may be represented as the Fourier series with eigenfunctions of some boundary-value problem. The two types of rotational motions are found among the mentioned class of solutions.

Introduction

Four various string models of baryon were suggested by X. Artru [1]. They differ from each other by geometric character of junction of three massive points (quarks) by relativistic strings. Four variants are possible: a) the “three-string” model or Y configuration with three strings from three quarks joined in the fourth massless point [2, 3]; b) the “triangle” model or Δ-configuration with pairwise connection of three quarks by three relativistic strings [4, 5]; c) the quark-diquark model $q^3q$ [6] (from the point of view of classical dynamics it coincides with the meson model of relativistic string with massive ends [7, 8]); d) the linear configuration $q^3$ with quarks connected in series [9, 10] (see Fig. 1).

Figure 1: Linear string baryon configuration $q^3$.

In the present work the latter model is considered. It was not studied quantitatively before Ref. [9] where we solved the initial-boundary value problem for classical motion of this configuration and investigated the stability problem for the rotational motion of this system. It was shown that this motion (a flat uniform rotation of the rectilinear string with the middle quark at rest at a center of rotation [6, 10]) is unstable. This instability results in the complicated motion of the middle material point (quark) with quasi-periodical varying of

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the distance between the nearest two quarks [9]. But the system $q$-$q$-$q$ is not transformed in quark-diquark $(q$-$qq)$ one, as was supposed formerly in Ref. [6].

In this paper for the system $q$-$q$-$q$ we study the class of motions admitting linearizable boundary conditions. The similar motions for the string with massive ends were first considered in Refs. [11, 12] and they were exhaustively classified in Ref. [13].

For the $q$-$q$-$q$ string baryon configuration after the review of its classical dynamics in Sect. 1 we consider in Sects. 2 the classes of motions with linearizable boundary conditions.

1. Classical dynamics of the linear string baryon configuration

Let’s consider an open relativistic string with the tension $\gamma$ carrying three pointlike masses $m_1, m_2, m_3$ (the masses $m_1$ and $m_3$ are placed at the ends of the string). The action for this system is

$$S[X^\mu] = -\int_{\tau_1}^{\tau_2} d\tau \left\{ \gamma \int_{\sigma_1(\tau)}^{\sigma_3(\tau)} \left[ \dot{X}(\tau) \cdot \dot{X}(\tau) - \dot{X}(\tau)^2 \right] \frac{d\sigma}{\gamma(\tau)} + \sum_{i=1}^{3} m_i \sqrt{X^\mu_i(\tau)^2} \right\}. \quad (1)$$

Here $X^\mu(\tau, \sigma)$ are coordinates of a point of the string in $D$-dimensional Minkowski space $R^{1,D-1}$ with signature $(+, -, -, \ldots)$, the speed of light $c = 1$, $(\tau, \sigma) \in \Omega = \Omega_1 \cup \Omega_2$ (Fig. 2), $(a, b) = a^b b_\mu$ — (pseudo)scalar product, $\dot{X}^\mu = \partial_\tau X^\mu$, $X'^\mu = \partial_\sigma X^\mu$, $X'^\mu(\tau) = \frac{dX^\mu(\tau)}{d\tau}(\tau, \sigma(\tau))$; $\sigma_i(\tau)$ ($i = 1, 2, 3$) — inner coordinates of world lines of pointlike masses (quarks).

![Figure 2: Domain of integration in Eq. (1).](image)

The equations of motion of the string and the boundary conditions are derived from the action [1] [2]. They have the simplest form if with the help of nondegenerate reparametrization $\tau = \tau(\tilde{\tau}, \tilde{\sigma})$, $\sigma = \sigma(\tilde{\tau}, \tilde{\sigma})$ the induced metric on the world surface of the string is made continuous and conformally-flat [4], i.e., satisfies the orthonormality conditions.

$$\dot{X}^2 + X'^2 = 0, \quad (\dot{X}, X') = 0. \quad (2)$$

Under conditions (2) the equations of motion become linear

$$\ddot{X}^\mu - X'^\mu = 0 \quad (3)$$

and the boundary conditions take the simplest form

$$m_i \frac{d}{d\tau} U_i^{\mu}(\tau) + \epsilon_i \gamma [X'^{\mu} + \sigma'_i(\tau) \dot{X}^\mu] \bigg|_{\sigma = \sigma_i(\tau)} = 0, \quad i = 1, 3, \quad (4)$$

$$m_2 \frac{d}{d\tau} U_2^{\mu}(\tau) - \gamma [X'^{\mu} + \sigma'_2(\tau) \dot{X}^\mu] \bigg|_{\sigma = \sigma_2(\tau)} + \gamma (X'^{\mu} + \sigma'_2(\tau) \dot{X}^\mu) \bigg|_{\sigma = \sigma_2(\tau)} = 0. \quad (5)$$
Here $\epsilon_1 = -1$, $\epsilon_3 = 1$ and

$$ U_i^\mu(\tau) = \frac{X_i^\mu(\tau)}{\sqrt{X_i^2(\tau)}} = \frac{\dot{X}^\mu + \sigma_i'(\tau)X_i^\mu}{\sqrt{\dot{X}^2 - (1 - \sigma_i'^2)\sigma_i(\tau)}}, \quad i = 1, 2, 3 $$

are the unit $R^{1,D-1}$-velocity vector of $i$-th quark.

Derivatives of $X^\mu(\tau, \sigma)$ can have discontinuities on the line $\sigma = \sigma_2(\tau)$. However, the function $X^\mu(\tau, \sigma)$ and the tangential derivatives $\frac{d}{d\tau}X^\mu(\tau, \sigma_2(\tau))$ are continuous. Therefore the jumps of the functions $\dot{X}^\mu$ and $X''^\mu$ are related by the condition

$$ [\dot{X}^\mu] + \sigma_2'(\tau)[X''^\mu] = 0. $$

Using this relation the boundary condition (3) may be rewritten in the form

$$ m_2 \frac{d}{d\tau} U_2^\mu(\tau) + \gamma(1 - \sigma_2'^2)[X^\mu(\tau, \sigma_2 - 0) - X^\mu(\tau, \sigma_2 + 0)] = 0. \quad (6) $$

1. Motions with linearizable boundary conditions

The present work is focused on world surfaces supposing the parametrization satisfying the conditions (2) and also the following conditions:

$$ \sqrt{\dot{X}^2}_{\sigma = \sigma_i} = C_i = \text{const}, \quad \sigma_i(\tau) = \sigma_i = \text{const}, \quad i = 1, 2, 3. \quad (7) $$

To fulfill these conditions the parameter $\tau$ for the quark trajectories is to be proportional to the natural parameter $s = \int(\dot{X}^2)^{1/2}d\tau$ (the proper time). Let’s assume, without loss of generality, that $\sigma_1 = 0$ and $\sigma_3 = \pi$ \[1]. For the class of surfaces under consideration, constraint (7) leads to linearization of boundary conditions (3), (4):

$$ (\dot{X}^\mu - Q_1X^\mu)_{\sigma = 0} = 0, \quad (\dot{X}^\mu + Q_3X^\mu)_{\sigma = \pi} = 0. \quad (8) $$

$$ \dot{X}^\mu(\tau, \sigma_2) + Q_2[X''^\mu(\tau, \sigma_2 - 0) - X''^\mu(\tau, \sigma_2 + 0)] = 0. \quad (9) $$

Here the notation

$$ Q_i = \gamma C_i/m_i, \quad i = 1, 2, 3; \quad Q_i > 0 $$

is used. The solutions to Eq.(3) with boundary conditions (8), (9) are constructed by the method of separation of variables (the Fourier method), i.e., the solution is sought in the form of a linear combination of the expressions $X^\mu(\tau, \sigma) = \alpha^\mu u(\sigma) T(\tau)$, where $\alpha^\mu$ is an arbitrary constant vector. The substitution of this expression into Eq.(3) results in the equations

$$ T''(\tau) + \omega^2 T = 0, \quad u''(\sigma) + \omega^2 u = 0. \quad (10) $$

We write the nontrivial solution $u(\sigma)$ of Eq.(10) for $\omega \neq 0$ in the form

$$ u(\sigma) = \begin{cases} 
A_1 \cos \omega \sigma + B_1 \sin \omega \sigma, & \sigma \in [\sigma_1, \sigma_2], \\
A_2 \cos \omega \sigma + B_2 \sin \omega \sigma, & \sigma \in [\sigma_2, \sigma_3]. 
\end{cases} \quad (11) $$

From the physical point of view the function $u(\sigma)$ is continuous in the segment $[\sigma_1, \sigma_3]$ but the $u'(\sigma)$ may have discontinuities for $\sigma = \sigma_2$. The function $\alpha^\mu u(\sigma)e^{i\omega \tau}$ is a solution of Eq.(3). Boundary conditions (3), (4) impose constraints on the $A_1, B_1, A_2, B_2$. 


The substitution of $X^\mu(\tau, \sigma) = \alpha^\mu u(\sigma) e^{i\omega \tau}$ into conditions (8), (14) results to the system of equations with respect to the $A_1, B_1, A_2, B_2$.

$$\begin{align*}
\omega A_1 + Q_1 B_1 &= 0, \\
(\omega C + Q_3 S) A_2 + (\omega S - Q_3 C) B_2 &= 0, \\
(Q_3 + \omega C_2 S_2 Q_2) A_1 + \omega S_2^2 B_1 - Q_2 A_2 &= 0, \\
\omega C_2^2 A_1 - (Q_2 - \omega C_2 S_2) B_1 + Q_2 B_2 &= 0.
\end{align*}$$  
(12)

Here the notations are used: $C = \cos \pi \omega$, $S = \sin \pi \omega$, $C_2 = \cos \omega \sigma_2$, $S_2 = \sin \omega \sigma_2$, $C_s = \cos \omega(\pi - \sigma)$, $S_s = \sin \omega(\pi - \sigma)$.

System (12) has nontrivial solution only if the following condition is executed

$$\omega^3 S_2 S_s - \omega^2 (Q_1 C_2 S_s + Q_2 S + Q_3 S_2 C_s) + \omega [Q_2 (Q_1 + Q_3) C + Q_1 Q_3 C_2 C_s] + Q_1 Q_2 Q_3 S = 0.$$  
(13)

This condition is equation with respect to the $\omega$.

Transcendental Eq.(13) has the countable set of real roots and $\lim_{n \to \infty} \omega_n = +\infty$ if $\omega = \omega_n$ are numerated in the order of increasing.

For every value of $\omega = \omega_n$ and an arbitrary value of $A_1$ we find $B_1, A_2, B_2$ from the system (12). In this case the function $\alpha^\mu u(\sigma) e^{i\omega_n \tau}$ is the solution of Eq.(13) and it also satisfies boundary conditions (8), (14).

These facts let us search the solution of the problem (14), (8), (14) in the form of a series

$$X^\mu(\tau, \sigma) = x^\mu_0 + p^\mu_0 \tau + \sum_{n \neq 0} \alpha^\mu_n u_n(\sigma) e^{i\omega_n \tau}.$$  
(14)

where $\alpha^\mu_n u_{-n}(\sigma) = \overline{\alpha^\mu_n u_n(\sigma)}$, if $X^\mu$ is real.

Now we show that world surfaces of the form (14) with the single frequency $\omega \neq 0$ is possible

$$X^\mu(\tau, \sigma) = \alpha^\mu_0 + p^\mu_0 \tau + B_1 u_n(\sigma) [\alpha^\mu_n \cos \omega \tau + \beta^\mu_n \sin \omega \tau].$$  
(15)

Let’s require fulfillment of the condition: $u(\sigma) \in C^1(0, \pi)$. Taking into account (11) that it condition is satisfied only if $A_1 = A_2 = A$, $B_1 = B_2 = B$, we rewrite the solution (11) in the form

$$u(\sigma) = A \cos \omega \sigma + B \sin \omega \sigma, \quad \sigma \in [\sigma_1, \sigma_3].$$

The last two equations of system (12) will be equivalent to one equation

$$A \cos \omega \sigma_2 + B \sin \omega \sigma_2 = 0.$$  
(16)

The first two equations of system (12) may be represented as follows

$$\begin{align*}
\omega A + Q_1 B &= 0, \\
(\omega C + Q_3 S) A + (\omega S - Q_3 C) B &= 0.
\end{align*}$$  
(17)

System (17) has nontrivial solutions only if $\omega$ is a root of the equation (13)

$$(Q_1 + Q_3) \omega \cos \pi \omega = (\omega^2 - Q_1 Q_3) \sin \pi \omega.$$  
(18)
Transcendental equation (18) has the countable set of real roots \( \omega = \omega_n, n \in \mathbb{Z} \). If \( \omega = \omega_n \) the solutions \( u_n(\sigma) \) take the form

\[
u_n(\sigma) = A\left( \cos \omega_n \sigma - \frac{\omega_n}{Q_1} \sin \omega_n \sigma \right), \quad \sigma \in [0, \pi]. \tag{19}\]

The function \( u_n(\sigma) \) has \( n \) roots on the interval \((0, \pi)\). Let’s require that the value \( \sigma_2 \) coincides with any of these roots:

\[
\cos \omega_n \sigma_2 - \frac{\omega_n}{Q_1} \sin \omega_n \sigma_2 = 0. \tag{20}\]

Equality (20) for function (19) results in the fulfillment of condition (16). Therefore, function (19) is the solution of system (16), (17). Here \( \omega_n \) is the root of Eq. (18) and \( \sigma_2 \) is the root of Eq. (20). It means that function (15) satisfies to boundary conditions (8)-(9).

The substitution shows that function (15) with \( u_n(\sigma) \) in form (19) is the solution of Eq. (3) and satisfies to conditions (2), (7) if \( p_0^2 = (AB_n)^2 \omega_n^2 (1 + \omega_n^2/Q_1^2) \) and the vectors \( p_n^\mu, e_n^\mu \) and \( g_n^\mu \) are pairwise orthogonal and \( e_n^2 = g_n^2 = -1 \).

The solution (15), (19) describes the uniform rotation of the \( n \) times folded rectilinear string in the plain with the vectors \( e_n^\mu, g_n^\mu \). The massive endpoints \( m_1, m_3 \) move along the circles and the middle point is at the rotational center, i.e. at the zero point \( \sigma = \sigma_2 \) of the function \( u_n(\sigma) \), satisfying Eq. (21). The world surface of the solution (15), (19) is helicoid.

The uniform rotations of the folded rectilinear open (massless) string were founded by Y. Nambu [14], and for the string with massive ends they were classified in Ref. [13].

Note that in the particular case \( \sigma_2 = \pi/2 \) the equation (13) after division by \( S_2 = S_* = \sin^2(\pi \omega/2) \) may be solved with respect to \( \cot(\pi \omega/2) \):

\[
\cot \frac{\pi \omega}{2} = \frac{(Q_1 + 2Q_2 + Q_3) \omega^2 - 2Q_1Q_2Q_3 \pm \sqrt{D(\omega)}}{2Q_2^2 \omega},
\]

where \( D(\omega) = [(Q_1 - Q_3)^2 + 4Q_2^2] \omega^4 + 4Q_2^2(Q_1^2 + Q_3^2) \omega^2 + 4Q_1^2Q_2^2Q_3^2 \).

And in the symmetrical case \( m_1 = m_3, Q_1 = Q_3, \sigma_2 = \pi/2 \) these two equations (or Eq. (13)) take the form:

\[
\cot \frac{\pi \omega}{2} = \frac{\omega}{Q_1}, \quad \cot \frac{\pi \omega}{2} = \frac{\omega^2 - 2Q_1Q_3}{(Q_1 + 2Q_2) \omega}. \tag{21}\]

The roots of the first Eq. (21) correspond to the solutions (13), (19) with the middle quark at the rotational center. But the roots of the second Eq. (21) result in the rotations of twice folded string with the quark \( m_2 \) at one end (at the bend point) and two other massive points at the other end.

**Conclusion**

In the present work the we obtain the class of solutions for the linear string baryon model \( q-q-q \), generalizing the simple rotational motion, that may be presented as (15), (19) with \( n = 1 \). In Ref. [4] the stability of this solution was investigated numerically and was shown that it is unstable in Lyapunov’s sense — any small asymmetric perturbation grows and results in complicated motion with quasi-periodical varying of the distance between the nearest two quarks. However the minimal value of the mentioned distance \( \Delta R \) does not equal zero, in
other words, the system $q$-$q$-$q$ is not transformed in quark-diquark ($q$-$qq$) one, as was supposed
in Ref. [6].

The class of motions (13), (19) obtained in this work has the same nature with the men-
tioned simple motion. So they are to be unstable too, but the rotations corresponding the roots
of the second Eq. (21) are candidates to be unique stable solutions for this string configuration.

This analysis is important for choosing the most adequate string baryon model and for
the progress in quantization of nonlinear string models carrying masses on the basis of their
quasirotational states [15].

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