On Fuzzy soft Regularly nowhere dense sets

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Abstract. In this paper, several characterizations of fuzzy soft regularly nowhere dense sets, fuzzy soft regularly dense, fuzzy soft regularly residual, several examples are given to illustrate the concepts introduced in this paper.

Keywords: Fuzzy soft regularly dense; fuzzy soft regularly nowhere dense set; fuzzy soft regularly first category; fuzzy soft regularly residual.

Introduction
Modeling uncertain data has been an interesting subject in engineering, eco-nomics, environmental and social sciences. Crisp set theory helped a little for formal modeling, reasoning and computing uncertain data. However, there are many com-plex problems in various branches of science that involve data which is not always crisp. We can not successfully use classical methods because of various types of uncertainties present in these problems. In 1999 Molodtsov[8] introduced the Soft set theory (SST) as a new mathematical tool to deal with uncertain data which is free from such difficulties. This theory has proved to be useful in many different fields such as: decision making, data analysis forecasting and simulation. Later on Maji et al. [7] introduced several operations of soft sets. Pei and Miao [10], M. I. Ali et al. [1] introduced and studied several soft set operations. The concept of FSS is introduced and studied [3-6, 9] a more generalized concept, which is a combination of fuzzy set and fuzzy soft set and studied its properties. The aim of this paper is to introduce the concepts of fuzzy soft regularly nowhere dense sets.

2. Preliminaries
2.1[3].Definition
The fuzzy soft set $F_{\phi}$$\in$$FS(U, E)$ is said to be null fuzzy soft set and it is denoted by $\phi$, if for all $e\in E$, $F(e)$ is the null fuzzy soft set $\bar{0}$ of $U$, where $\bar{0}(x) = 0$ for all $x\in U$.

2.2 [3].Definition
Let $F_{E}$$\in$ FS $(U, E)$ and $F_{E}(e)$ = $\bar{1}$ for all $e\in E$, where $\bar{1}(x) = 1$ for all $x\in U$. Then $F_{E}$ is called absolute fuzzy soft set. It is denoted by $\overline{E}$.

2.3[3].Definition
A fuzzy soft set $F_{A}$ is said to be a fuzzy soft nowhere dense set if there exists no non–zero fuzzy soft open set if there exists no non–zero fuzzy soft open set $G_{B}$ in $(U, E, \psi)$ such that $GB$$<$$cl_{fs}(F_{A})$. that is, $int_{fs}cl_{fs}(F_{A})=0$.
2.5[11]. Definition

A FSS \( F_A \) is a FSTS \((U, E, \psi)\) is called a fuzzy soft dense if there exists no fuzzy soft closed set \( G_B \) in \((U, E, \psi)\), such that \( F_A \leq G_B < 1 \), that is \( \text{cl}^s(F_A) = 1 \).

2.6[11]. Definition

Let \((U, E, \psi)\) be a fuzzy soft topology. A Fuzzy soft set \( F_A \) in \((U, E, \psi)\) is called fuzzy soft first category. If \( V_{F} \subseteq (F_A) \) where \( V_{F} \subseteq (F_A) \), where \( F_A \)'s are fuzzy soft nowhere dense sets in \((U, E, \psi)\). Any other fuzzy soft set in \((U, E, \psi)\) is said to be of fuzzy soft second category.

2.7[12]. Definition

Let \( \psi \) be the collection of fuzzy soft sets over \( U \). Then \( \psi \) is called a fuzzy soft topology on \( U \) if \( \psi \) satisfies the following axioms:

- \( \phi, \overline{E} \) belong to \( \psi \).
- The union of any number of fuzzy soft sets in \( \psi \) belongs to \( \psi \).
- The intersection of any two fuzzy soft sets \( \psi \) belongs to \( \psi \).

The triplet \((U, E, \psi)\) is called a fuzzy soft topological space over \( U \). The members of \( \psi \) are called fuzzy soft open sets in \( U \) and complements of them are called fuzzy soft closed sets in \( U \).

2.8[12]. Definition

The union of all fuzzy soft open subsets of \( F_A \) over \((U, E)\) is called the interior of \( F_A \) and is denoted by \( \text{int}^s(F_A) \).

2.9[12]. Proposition

\[ \text{int}^s(F_A \Delta G_B) = \text{int}^s(F_A) \Delta \text{int}^s(G_B) \].

2.10. [12]. Definition

Let \( F_A \in \text{FS}(U, E) \) be a fuzzy soft set. Then the intersection of all closed sets, each containing \( F_A \), is called the closure of \( F_A \) and is denoted by \( \text{cl}^s(F_A) \).

2.11. [12]. Remarks

- For any fuzzy soft set \( F_A \) in a fuzzy soft topological space \((U, E, \psi)\), it is easy to see that \( \text{cl}^s(F_A) = \text{int}^s(F_A) \). (\( \text{int}^s(F_A) = \text{cl}^s(F_A) \).)
- For any fuzzy soft subset \( F_A \) of a fuzzy soft topological space \((U, E, \psi)\), the definition of fuzzy soft subspace topology \( \text{FS}(F_A) \) on \( F_A \) by \( K_{B} \subseteq \text{FS}(F_A) \) if \( K_{D} = F_A \Delta G_B \) for some \( G_B \in \psi \).
- For any fuzzy soft set \( H_E \) in \( F_A \) fuzzy soft subspace of a fuzzy soft topological space, we denote to the interior and closure of \( H_E \) in \( F_A \) by \( \text{int}^s(F_A(H_E)) \) and \( \text{cl}^s(F_A(H_E)) \), respectively.

2.12[2]. Definition

Let \((X, T, E)\) be a fuzzy soft topological space and \( f_A \in \text{FSOS}(X) \).

- Arbitrary fuzzy soft union of fuzzy semi open soft set is fuzzy semi open soft.
- Arbitrary fuzzy soft intersection of fuzzy semi closed soft set is fuzzy semi closed soft.

2.14[2]. Theorem

Let \((X, T, E)\) be a fuzzy soft topological space and \( f_A \in \text{FSOS}(X) \).

Then:

- \( f_A \in \text{FSOS}(X) \) if and only if \( \text{Fcl}(f_A) = \text{Fcl}(\text{Fin}(f_A)) \).
- If \( g_B \in T \), then \( g_B \subseteq \text{Fcl}(f_A) \cup \text{Fcl}(g_B) \).

2.15[2]. Theorem

Let \((X, T, E)\) be a fuzzy soft topological space and \( f_A \in \text{FSOS}(X) \).

- \( f_A \in \text{FSOS}(X) \) if and only if there exists \( g_B \in T \) such that \( g_B \subseteq \text{Fcl}(g_B) \).
- If \( f_A \in \text{FSOS}(X) \) and \( f_A \subseteq \text{Fcl}(f_A) \), then \( h_D \in \text{FSOS}(X) \).

2.16[2]. Definition

Let \((X, T, E)\) be a fuzzy soft topological space, \( f_A \in \text{FSOS}(X) \), and \( f_A \in \text{FSOS}(X) \).

- \( f_A \) is called fuzzy semi interior soft point of \( f_A \) if \( \exists g_B \in \text{FSOS}(X) \) such that \( f_A \subseteq g_B \).
- The set of all fuzzy semi interior soft points of \( f_A \) is called the fuzzy semi soft interior of \( f_A \) and is denoted by \( \text{FSint}(f_A) \) consequently, \( \text{FSint}(f_A) = \left\{ g_B : g_B \subseteq f_A : g_B \in \text{FSOS}(X) \right\} \).
Let \((X, T, E)\) be fuzzy soft topological space and \(f_A, g_B \in \text{FSS}(X)\). Then the following properties are satisfied for the fuzzy semi interior operator, denoted by \(\text{FSint}\).

- \(\text{FSint}(\overline{E}) = \overline{E}\) and \(\text{FSint}(\overline{E}) = \overline{E}\).
- \(\text{FSint}(f_A) = \text{FSint}(f_A)\).

### 3. Fuzzy soft regularly nowhere dense sets

#### 3.1. Definition

A fuzzy soft nowhere dense \(F_A\) is called a fuzzy soft regularly nowhere dense set in the FSTS \((U, E, \psi)\), if \(F_A \subseteq \text{cl}(\text{int}(G_B)) \leq \text{cl}(H_C)\) and \(G_B \leq 1\) in \(H_C\) where \(G_B, H_C \in \psi\).

#### 3.1. Example

Let \(U = \{a, b, c\}\). The fuzzy soft sets \(F_A, G_B, H_C\) defined on \(U\) are as follows:

- \(F_A: U \rightarrow \{0, 1\}\) is defined as \(F_A(a) = 0.6, F_A(b) = 0.4, F_A(c) = 0.5\)
- \(G_B: U \rightarrow \{0, 1\}\) is defined as \(G_B(a) = 0.4, G_B(b) = 0.5, G_B(c) = 0.6\)
- \(H_C: U \rightarrow \{0, 1\}\) is defined as \(H_C(a) = 0.6, H_C(b) = 0.6, H_C(c) = 0.6\)

Clearly, \(\psi = \{0, F_A, G_B, H_C, F_A \lor G_B\} \) is a FST on \(U\).

The fuzzy soft nowhere dense sets in \((U, E, \psi)\) are \{1- \(G_B, 1\)- \(H_C, 1\)- \(F_A \lor G_B\)\}, but the fuzzy soft regularly nowhere dense sets in \((U, E, \psi)\) are \{1- \(H_C, 1\)- \(F_A \lor G_B\)\}. The Fuzzy Soft nowhere dense set in \((U, E, \psi)\) are \(1- \(G_B \leq \text{cl}(H_C)\) \land \text{cl}(G_B)\). Therefore the fuzzy soft regularly nowhere dense sets in \((U, E, \psi)\) are \(1- \(H_C, 1\)- \(F_A \lor G_B\)\).

#### 3.1. Proposition

If \(F_A \subseteq \text{cl}(\text{int}(G_B)) \land \text{cl}(H_C)\) and \(F_A \leq 1\)-\(H_C\) where \(G_B, H_C \in \psi\), for a fuzzy soft closed set with \(\text{int}^{fs}(F_A) = 0\) in a FSTS \((U, E, \psi)\), then \(F_A\) is a fuzzy soft regular nowhere dense set in \((U, E, \psi)\).

**Proof:**

Suppose that \(F_A \subseteq \text{cl}(\text{int}(G_B)) \land \text{cl}(H_C)\) and \(F_A \leq 1\)-\(H_C\) where \(F_A, H_C \in T\). Since \(F_A\) is a fuzzy soft closed set with \(\text{int}^{fs}(F_A) = 0\) in \((U, E, \psi)\), then \(\text{int}^{fs}[\text{cl}^{fs}(F_A)] = \text{int}^{fs}(F_A) = 0\). Therefore, \(F_A\) is a fuzzy soft regular nowhere dense set in \((U, E, \psi)\).

#### 3.2. Proposition

If \(F_A \subseteq \text{cl}(\text{int}(G_B)) \land \text{cl}(H_C)\) and \(G_B \leq 1\)-\(H_C\) where \(G_B, H_C \in \psi\), for a Fuzzy Soft Closed Set with \(\text{int}^{fs}(F_A) = 0\) in a FSTS \((U, E, \psi)\), then \(\text{cl}^{fs}(F_A)\) is a fuzzy soft regularly nowhere dense set in \((U, E, \psi)\).

**Proof:**

By proposition 3.1, \(F_A\) is a fuzzy soft regular nowhere dense set in \((U, E, \psi)\). Since \(F_A\) is a FSCS (short form: fuzzy soft closed set) with \(\text{int}^{fs}(F_A) = 0\) in \((U, E, \psi)\), then \(\text{cl}^{fs}[\text{int}^{fs}(F_A)] = \text{cl}^{fs}(F_A) = \text{Fsc}(F_A) = 0\) (since \(\text{cl}^{fs}(F_A) = F_A\)). Thus \(\text{cl}^{fs}(F_A)\) is a Fuzzy soft regular nowhere dense set in \((U, E, \psi)\).

#### 3.3. Proposition

If \(J_A \geq \text{int}^{fs}(G_B) \lor \text{int}^{fs}(H_C)\) and \(G_B \geq 1\)-\(H_C\) where \(G_B, H_C \in \psi\), for a fuzzy soft open set with \(\text{cl}^{fs}(J_A) = 1\) in a FSTS \((U, E, \psi)\), then \(J_A\) is a fuzzy soft dense set in \((U, E, \psi)\).

**Proof:**

Let us assume that \(J_A \geq \text{int}^{fs}(G_B) \lor \text{int}^{fs}(H_C)\) and \(G_B \geq 1\)-\(H_C\) where \(G_B, H_C \in \psi\), since \(J_A\) is a fuzzy soft open set with \(\text{cl}^{fs}(J_A) = 1\) in \((U, E, \psi)\), then \(\text{cl}^{fs}[\text{int}^{fs}(F_A)] = \text{cl}^{fs}(F_A) = 1\) (since \(\text{cl}^{fs}(J_A) = J_A\)). Thus \(J_A\) is a fuzzy soft dense set in \((U, E, \psi)\).

#### 3.4. Proposition

If \(J_A\) and \(K_A\) are fuzzy soft regularly nowhere dense set in \((U, E, \psi)\), then \((J_A \land K_A)\) is a fuzzy soft regularly nowhere dense set in \((U, E, \psi)\).

**Proof:**

Let \(J_A\) and \(K_A\) are fuzzy soft closed sets with \(\text{int}^{fs}(J_A) = 0\) and \(\text{int}^{fs}(K_A) = 0\), then \(\text{int}^{fs}[\text{cl}^{fs}(J_A)] = \text{int}^{fs}(J_A) = 0\), and \(\text{int}^{fs}[\text{cl}^{fs}(K_A)] = \text{int}^{fs}(K_A) = 0\) [since \(\text{cl}^{fs}(J_A) = J_A\); \(\text{cl}^{fs}(K_A) = K_A\)]. Now \(J_A \land K_A\)
\[ \leq \text{int}^i \left[ \mathcal{C}^i(J_A) \right] \land \mathcal{C}^i(K_A) \leq \text{int}^i \left[ \mathcal{C}^i(J_A) \setminus \text{int}^i \left[ \mathcal{C}^i(K_A) \right] \right] = 0 \land 0 = 0. \] Thus \( J_A \land K_A \) is a fuzzy soft nowhere dense set in \((U,E,\psi)\), which implies that \( J_A \land K_A \) is a fuzzy soft regularly nowhere dense set in \((U,E,\psi)\).

### 3.5. Definition

Let \((U,E,\psi)\) be a FSTS. A fuzzy soft set \( F_A \) in \((U,E,\psi)\) is called a fuzzy soft regularly first category set, if \( F_A = V_{i=1}^{\infty} (F_A) \), where \((F_A)_i\)'s are FSRnwds in \((U,E,\psi)\). Any other FSS in \((U,E,\psi)\) is said to be of [short form fuzzy soft regularly FSR|FSR] second category.

### 3.5. Proposition

If \( F_A \) is a FSRFCS in the FSTS \((U,E,\psi)\), then \( 1 - F_A = \lambda_{i=1}^{\infty} (G_B) \), where \( \text{cl}^i \left[ \text{int}^i (G_B) \right] = 1 \) in \((U,E,\psi)\).

**Proof:**

Let \( F_A \) be a FSRFCS in \((U,E,\psi)\), then \( F_A = V_{i=1}^{\infty} (G_B) \), where \((V_{i=1}^{\infty} (F_A))_i\)'s are FSRnwds set in \((U,E,\psi)\), since \((F_A)_i\)'s are fuzzy soft regularly nowhere dense sets in \((U,E,\psi)\), then \( 1 - (F_A)_i \)'s are fuzzy soft open sets with \( \text{cl}^i(G_B) = 1 \) in \((U,E,\psi)\). Let \( 1 - (F_A)_i = (G_B)_i \), and hence \( 1 - F_A = 1 - V_{i=1}^{\infty} (F_A) = \lambda_{i=1}^{\infty} (1 - F_A) = \lambda_{i=1}^{\infty} G_B \), where \( \text{cl}^i [\lambda_{i=1}^{\infty} (G_B)] = \text{cl}^i [\text{int}^i (G_B)] = \text{cl}^i (G_B) = 1 \).

### 3.6. Definition

Let \( F_A \) be a FSRFCS in a FSTS \((U,E,\psi)\), then \( 1 - F_A \) is called a fuzzy soft residual set in \((U,E,\psi)\).

### 3.6. Proposition

If \( F_A \) is a FSRFCS in the FSTS \((U,E,\psi)\), then \( 1 - F_A \) is a FSR residual set in \((U,E,\psi)\).

**Proof:**

Let \( F_A \) be a FSRFCS in \((U,E,\psi)\), then \( F_A = V_{i=1}^{\infty} (F_A) \), where \((F_A)_i\)'s are FSR nowhere dense set in \((U,E,\psi)\). By proposition 3.5, \( 1 - F_A \) is a fuzzy soft regularly residual set in \((U,E,\psi)\).

### 3.7. Proposition

If \( F_A \) is a fuzzy soft residual set in \((U,E,\psi)\), then \( F_A \) is a FSR residual set in \((U,E,\psi)\).

**Proof:**

Let \( F_A \) be a fuzzy soft residual set in \((U,E,\psi)\), then \( F_A = \lambda_{i=1}^{\infty} (F_A) \) where \( \text{cl}^i [\text{int}^i (F_A)] = 1 \), since \( F_A \) is a fuzzy soft open set in \((U,E,\psi)\), then \( \text{int}^i (F_A) = F_A \). Now \( \text{cl}^i [\text{int}^i (F_A)] = \text{cl}^i [\lambda_{i=1}^{\infty} \text{int}^i (F_A)] = \text{cl}^i [\text{int}^i (F_A)] = \text{cl}^i (F_A) = 1 \). Thus \( F_A \) is a FSR residual set in \((U,E,\psi)\).

### 3.8. Proposition

If \( F_A \) is a fuzzy soft semi closed set with \( \text{int}^i (F_A) \neq 0 \) in a FSTS \((U,E,\psi)\), then \( \text{int}^i [\text{cl}^i (F_A)] \neq 0 \).

**Proof:**

Let \( F_A \) be a fuzzy soft semi – closed set with \( \text{int}^i (F_A) \neq 0 \) in \((U,E,\psi)\), then \( \text{int}^i [\text{cl}^i (F_A)] \leq \text{int}^i (F_A) \). implies that \( \text{int}^i [\text{cl}^i (F_A)] \neq 0 \).

### 3.9. Proposition

If \( F_A \) is a fuzzy soft semi – closed set with \( \text{int}^i (F_A) \neq 0 \) in a FSTS \((U,E,\psi)\), then \( F_A \) is not a fuzzy soft nowhere dense set in \((U,E,\psi)\).

**Proof:**

Let \( F_A \) be a fuzzy soft semi- closed set with \( \text{int}^i (F_A) \neq 0 \) in \((U,E,\psi)\). By proposition 3.8, \( \text{int}^i [\text{cl}^i (F_A)] \neq 0 \), implies that \( F_A \) is not a fuzzy soft nowhere dense set in \((U,E,\psi)\).

### 3.10. Proposition

If \( F_A \) is a fuzzy soft closed set with \( \text{int}^i (F_A) \neq 0 \) in a FSTS \((U,E,\psi)\), then \( \text{int}^i (F_A) \neq 0 \) in a FSTS \((U,E,\psi)\), then \( \text{int}^i [\text{cl}^i (F_A)] \neq 0 \).

**Proof:**

Let \( F_A \) be a fuzzy soft closed set with \( \text{int}^i (F_A) \neq 0 \) in \((U,E,\psi)\). Since \( \text{cl}^i (F_A) = F_A \), then \( \text{int}^i [\text{cl}^i (F_A)] \leq \text{int}^i (F_A) \).
Proof:
Let \( F_A = \bigvee_{i=1}^{\infty} (F_{A_i}) \), where \( (F_{A_i})'s \) are fuzzy soft closed set with \( \text{int}^b(F_{A_i}) \neq 0 \). Since \( F_A \) is a fuzzy soft closed set in \( (U,E,\psi) \), then by proposition 3.11, \( F_A \) is not a fuzzy soft nowhere dense set in \( (U,E,\psi) \).

3.13. Definition
Let \( (U,E,\psi) \) be a FSTS. A FSS \( F_A \) in \( (U,E,\psi) \) is called a FSR residual set, if \( F_A = \bigcup_{i=1}^{\infty} (F_{A_i}) \), where \( (F_{A_i})'s \) are such that \( \text{cl}^b[\text{int}^b(F_{A_i})] = 1 \) in \( (U,E,\psi) \).

3.14. Definition
A fuzzy soft dense \( G_B \) is called a FSR dense set in the FSTS \( (U,E,\psi) \), if \( G_B \geq \text{int}(H_C) \lor \text{int}(J_D) \) and \( H_C \geq 1 - K_E \) where \( H_C, J_D \in \psi \).

3.15. Proposition
If \( F_A \) is a fuzzy soft closed set with \( \text{int}^b(F_A) = 0 \) in a FSTS \( (U,E,\psi) \), then \( F_A \) is a FSR nowhere dense set in \( (U,E,\psi) \).

Proof:
Let \( F_A \) is a fuzzy soft closed set with \( \text{int}^b(F_A) \neq 0 \) in \( (U,E,\psi) \). Since \( F_A \) is a fuzzy soft closed in \( (U,E,\psi) \), then \( \text{cl}^b(F_A) = F_A \) implies that \( \text{int}^b[\text{cl}^b(F_A)] = \text{int}^b(F_A) = 0 \). Thus \( F_A \) is a fuzzy soft nowhere dense set and hence \( F_A \) is a fuzzy soft residual nowhere dense set in \( (U,E,\psi) \).

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