A SPECTRAL SEQUENCE FOR KHOVANOV HOMOLOGY WITH AN APPLICATION TO \((3, q)\)-Torus Links

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ABSTRACT. A spectral sequence converging to Khovanov homology is constructed which is applied to calculate the rational Khovanov homology of \((3, q)\)-torus links.

1. INTRODUCTION

There is a lack of theoretical computational tools for Khovanov homology when compared to, say, the homology of spaces where one has a range of long exact sequences and spectral sequences at hand. There is one long exact sequence used in Khovanov homology, the skein exact sequence, implicit in [2] and explicit in [5] which is formed as follows. A given crossing of an link diagram \(D\) can be resolved in two ways: to the 0-smoothing giving a new diagram \(D'\) and to the 1-smoothing giving a new diagram \(D''\). There is then a short exact sequence of Khovanov complexes

\[ 0 \to C(D'') \to C(D) \to C(D') \to 0 \]

which gives rise to a long exact sequence in homology. One can repeatedly apply this long exact sequence, but it requires careful book keeping. This is essentially what leads to the spectral sequence defined in this paper.

We start with a collection of \(m\) crossings of a diagram \(D\). For \(1 \leq k \leq m\) let \(D_{(k)}\) be the diagram obtained from \(D\) by resolving the crossings \(1, \ldots, k\) to 1-smoothings and let \(\tilde{D}_{(k)}\) be the diagram obtained from \(D\) by resolving the crossings \(1, \ldots, k - 1\) to 1-smoothings and crossing \(k\) to a 0-smoothing. The idea is that the diagrams \(\tilde{D}_{(k)}\) and \(D_{(m)}\) might be simpler to handle than the diagram \(D\). By defining appropriate constants \(a_k, b_k, \tilde{a}_k, \tilde{b}_k, A_k, \) and \(B_k\) we arrive at the following result where \(j\) is fixed.

**Proposition 2.2** There is a spectral sequence \((E_{r,s}^{s,t}, d_r: E_{r,s}^{s,t} \to E_{r,s+r,t-r+1}^{s,t})\) converging to \(KH^{s,j}(D)\) with \(E_1\)-page given by

\[
E_1^{s,t} = \begin{cases} 
KH^{s+t+A_s+s+1,j+B_s+b_{s+1}}(\tilde{D}_{(s+1)}) & s = 0, \ldots, m - 1 \\
KH^{m+t+A_m,j+B_m}(D_{(m)}) & s = m \\
0 & s < 0 \text{ or } s > m
\end{cases}
\]

As an application of this spectral sequence we compute the rational Khovanov homology of \((3, q)\)-torus links. It is easy to guess what the result is, based on available computer calculations, but by combining the above spectral sequence with another spectral sequence (Lee’s spectral sequence) we prove the result for all \(q\).
2. The spectral sequence

Let $R$ be a commutative ring with unit and let $D$ be an oriented link diagram with $n$ crossings. As is now familiar one can construct the Khovanov complex by placing the $2^n$ smoothings of $D$ on the vertices of the cube $\{0, 1\}^n$. To each smoothing $\alpha$ one then assigns the $R$-module $V_\alpha = V^{\otimes k_\alpha} \{r_\alpha\}$ where $k_\alpha$ is the number of circles in the smoothing, $r_\alpha$ is the number of 1’s in $\alpha$ and shifts are defined by $(W\{i\})^m = W^{m−i}$. The module $V$ is the graded, rank two, free $R$-module with generators 1 and $x$ in degree 1 and $−1$ respectively. Now let us consider a collection of $2^n$ smoothings of $D$ and in fact the Khovanov homology of $D$ uses a Frobenius algebra structure on $V$.

Bi-graded complexes may be shifted in each of the degrees and for a bi-graded module $W^{**}$ we define

$$(W^{*,*}[l]\{m\})^{i,j} = W^{i−l,j−m}.$$ 

Suppose $D$ has $n^+$ positive crossings and $n^−$ negative crossings, then the normalised Khovanov complex $C^{*,*}(D)$ is defined by

$$C^{i,j}(D) = (C^{*,*}(D)[−n^−]\{n^+ − 2n^−\})^{i,j}$$

and the Khovanov homology of $D$ is defined as the homology of this complex.

Now let us consider a collection of $m$ crossings of the diagram $D$ and number these $1,\ldots, m$. For $k = 1,\ldots, m$ let $D(\{k\})$ be the diagram obtained from $D$ by resolving the crossings $1,\ldots, k$ to 1-smoothings and let $\tilde{D}(\{k\})$ be the diagram obtained from $D$ by resolving the crossings $1,\ldots, k−1$ to 1-smoothings and crossing $k$ to a 0-smoothing. We also define $D(\{0\})$ and $\tilde{D}(\{0\})$ to be the original diagram $D$.

There is a decomposition of modules

$$C^{i,j}(D(\{k−1\})) = C^{i,j}(\tilde{D}(\{k\})) \oplus C^{i−1,j−1}(D(\{k\}))$$

and in fact $C^{*,−1,*−1}(D(\{k\}))$ is a sub-complex of $C^{*,*}(D(\{k−1\}))$. Thus, there is a short exact sequence

$$(1) \quad 0 \longrightarrow C^{*,*}(D(\{k\}))[1]\{1\} \longrightarrow C^{*,*}(D(\{k−1\})) \longrightarrow C^{*,*}(\tilde{D}(\{k\})) \longrightarrow 0.$$

This is just the usual short exact sequence giving the long exact sequence mentioned in the introduction for the diagram $D(\{k\})$ resolving the $k$’th crossing in our set of $m$ crossings.

We now discuss orientations for the diagrams $D(\{k\})$ and $\tilde{D}(\{k\})$. Suppose that we already have an orientation for $D(\{k−1\})$. If the $k$’th crossing is positive then $\tilde{D}(\{k\})$ inherits an orientation because for positive crossings the 0-smoothing is the oriented resolution. There is no orientation of $D(\{k\})$ consistent with the orientation of $D(\{k−1\})$ so choose any orientation for $D(\{k\})$. Similarly if the $k$’th crossing is negative then $D(\{k\})$ inherits an orientation and we choose any orientation for $\tilde{D}(\{k\})$. The diagram $D = D(\{0\}) = \tilde{D}(\{0\})$ comes with an orientation so the process above has somewhere to start.
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Now for \(k = 0, \ldots, m\) define
\[
\begin{align*}
    n_k^+ &= \text{number of positive crossings in } D_k \\
    n_k^- &= \text{number of negative crossings in } D_k \\
    \bar{n}_k^+ &= \text{number of positive crossings in } \tilde{D}_k \\
    \bar{n}_k^- &= \text{number of negative crossings in } \tilde{D}_k
\end{align*}
\]

We define additional constants associated to \(D_k\) and \(\tilde{D}_k\) as follows.

If the \(k\)'th crossing is positive then set
\[
a_k = n_{k-1}^- - n_k^- - 1 \quad \text{and} \quad \bar{a}_k = 0.
\]

If the \(k\)'th crossing is negative then set
\[
a_k = 0 \quad \text{and} \quad \bar{a}_k = n_{k-1}^- - \bar{n}_k^-.
\]

For convenience we also define (for positive and negative crossings)
\[
    b_k = 3a_k + 1 \quad \text{and} \quad \bar{b}_k = 3\bar{a}_k - 1.
\]

These constants help us to write down the short exact sequence\(^1\) in terms of normalised Khovanov homology.

**Proposition 2.1.** For each \(k = 1, \ldots, m\) there is a short exact sequence of complexes
\[
0 \longrightarrow C^{*,*}(D_k)[-a_k][-b_k] \longrightarrow C^{*,*}(D_{k-1}) \longrightarrow C^{*,*}(\tilde{D}_k)[-\bar{a}_k][-\bar{b}_k] \longrightarrow 0
\]

**Proof.** We shift the entire sequence\(^1\) by \([-n_k^-]\{n_k^+ - 2n_k^-\}\}. One can then readily verify (treating positive and negative crossings separately) that
\[
\begin{align*}
-n_{k-1}^- + 1 &= -n_k^- - a_k, \\
-n_{k-1}^- &= -n_k^- - \bar{a}_k, \\
n_{k-1}^+ - 2n_k^- + 1 &= n_k^+ - 2n_k^- - b_k, \\
n_{k-1}^+ - 2n_k^- &= \bar{n}_k^+ - 2\bar{n}_k^- - \bar{b}_k.
\end{align*}
\]

We now define
\[
A_k = \sum_{i=1}^k a_i, \quad \text{and} \quad B_k = \sum_{i=1}^k b_i = 3A_k + k.
\]

and set \(A_0 = B_0 = 0\).

From now on we fix \(j\). We define a filtration on \(C^{*,j}(D)\) by
\[
F^kC^{*,j}(D) = C^{*,j}(D_k)[-A_k][-B_k] \quad k = 0, \ldots, m.
\]

It follows immediately from Proposition 2.1 that \(F^kC^{*,j}(D) \subset F^{k-1}C^{*,j}(D)\) and for \(k > m\) we set \(F^kC^{*,j}(D) = 0\). It is clear that the filtration is bounded and so there is an associated spectral sequence.
Proposition 2.2. There is a spectral sequence \( (E^s,t_r, d_r; E^s,t_r \to E^{s+r,t-r+1}) \) converging to \( KH^{*,j}(D; R) \) with \( E_1 \)-page given by

\[
E_{s,t}^1 = \begin{cases} 
KH^{s+t+A_s+a_{s+1},j+B_s+b_{s+1}}(\tilde{D}_{(s+1)}); R & s = 0, \ldots, m - 1 \\
KH^{m+t+A_m,j+B_m}(D_{(m)}); R & s = m \\
0 & s < 0 \text{ or } s > m
\end{cases}
\]

Proof. By using the filtration above there is a spectral sequence with

\[
E_{s,t}^0 = \frac{F^s C^{s+t,j}(D)}{F^{s+1} C^{s+t,j}(D)}.
\]

By applying Proposition 2.1 there is a short exact sequence for \( 0 \leq s < m \) as follows.

\[
0 \longrightarrow F^{s+1} C^{s+t,j}(D) \longrightarrow F^s C^{s+t,j}(D) \longrightarrow C^{s+t+A_s+a_{s+1},j+B_s+b_{s+1}}(\tilde{D}_{(s+1)}) \longrightarrow 0
\]

Since this is a short exact sequence of complexes the differential \( d_0 \), which is induced by the differential on \( F^s C^{s+t,j}(D) \), can be identified with the differential on the right hand side, that is, in the complex \( C^{*,*}(\tilde{D}_{(s+1)}) \). In particular the homology of \( E_{0}^{*,*} \) is given by the homology (in suitable gradings) of \( C^{*,*}(\tilde{D}_{(s+1)}) \), namely the Khovanov homology of \( \tilde{D}_{(s+1)} \).

When \( s = m \) we have \( E_{m,t}^0 = E^m C^{m+t,j}(D) = C^{m+t+A_m,j+B_m}(D_{(m)}) \) and so \( E_1 \) is again as claimed.

The differential \( d_1 \) on the \( E_1 \)-page can be understood as follows. There is a decomposition (of modules)

\[
C^{*,*}(D_{(s)}) = C^{*,a_{s+1},*,b_{s+1}}(\tilde{D}_{(s+1)}) \\
\oplus C^{*,a_{s+1},a_{s+2},*,b_{s+1},b_{s+2}}(\tilde{D}_{(s+2)}) \\
\oplus C^{*,a_{s+1},a_{s+2},a_{s+3},*,b_{s+1},b_{s+2},b_{s+3}}(D_{(s+2)})
\]

and with respect to this the differential on \( C^{*,*}(D_{(s)}) \) can be written as a matrix

\[
\begin{pmatrix}
\delta_{s+1} & 0 & 0 \\
\delta & \tilde{\delta}_{s+2} & 0 \\
\delta' & \delta'' & \delta_{s+2}
\end{pmatrix}
\]

The differential on the \( E_1 \)-page of the spectral sequence is the map

\[
\delta: C^{*,a_{s+1},*,b_{s+1}}(\tilde{D}_{(s+1)}) \to C^{*,a_{s+1},a_{s+2},*,b_{s+1},b_{s+2}}(\tilde{D}_{(s+2)})
\]
in the above matrix.

Note that if \( m = 1 \) then the \( E_1 \)-page is concentrated in columns \( s = 0 \) and \( s = 1 \) and so collapses at the \( E_2 \)-page for dimensional reasons. The differential on the \( E_1 \)-page is precisely the boundary map in the usual long exact sequence. Indeed one can always assemble such a situation into a long exact sequence.

It is worth commenting that the essential ingredient for the construction of the spectral sequence is the cube construction of link homology, not the particular variant of link homology we choose to consider. Thus for example one may set up similar spectral sequences in Khovanov-Rozansky homology.
3. The rational Khovanov homology of \((3, q)\)-torus links.

In this section we work over \(\mathbb{Q}\) and write \(KH^{*,*}(D)\) for \(KH^{*,*}(D; \mathbb{Q})\). Our interest is with the torus links \(T(3, q)\) which we take to have negative crossings. We consider the diagram for \(T(3, q)\) obtained as the closure of a three stranded braid as shown in Figure 1. When \(q\) is a multiple of 3 then \(T(3, q)\) is a three component link, otherwise \(T(3, q)\) is a knot.

![Figure 1.](image)

**Theorem 3.1.** Let \(N\) be an integer, \(N \geq 1\).

(i) The rational Khovanov homology of the \((3, 3N)\)-torus link is given in Figure 2.

![Figure 2.](image)
(ii) The rational Khovanov homology of the $(3, 3N + 1)$-torus knot is given in Figure 3.

(iii) The rational Khovanov homology of the $(3, 3N - 1)$-torus knot is given in Figure 4.
This has the following corollary.

**Corollary 3.2.** The rational Khovanov homology of the torus links $T(3, 3N)$, $T(3, 3N + 1)$ and $T(3, 3N + 2)$ occupy exactly $N + 2$ diagonals.

Before proving the theorem we need to make some recollections about another spectral sequence defined by Lee [3] (see also [4]). Recall that Lee theory is a variant of rational Khovanov homology obtained from the same underlying vector spaces but using a different differential (based on a different Frobenius algebra). Lee theory is a singly graded theory and we denote it by $\text{Lee}^*(L)$. We summarise the results we need about Lee theory in the following proposition.

**Proposition 3.3.** Let $L$ be an oriented link with $k$ components $L_1, L_2, \ldots, L_k$.

(i) $\dim(\text{Lee}^*(L)) = 2^k$.

(ii) For every orientation $\theta$ of $L$ there is a generator of homology in degree

$$2 \times \sum_{l \in E, m \in \overline{E}} \text{lk}(L_l, L_m)$$

where $E \subset \{1, 2, \ldots, k\}$ indexes the set of components of $L$ whose original orientation needs to be reversed to get the orientation $\theta$ and $\overline{E} = \{1, \ldots, k\} \setminus E$. The linking numbers $\text{lk}(L_l, L_m)$ are the linking number (for the original orientation) between component $L_l$ and $L_m$.

(iii) There is a spectral sequence converging to $\text{Lee}^*(L)$ with $E_1^{s,t} = KH^{s+4i}H^t(L)$.

If we index the $E_1$-page of the spectral sequence by the usual indexing of the Khovanov homology (rather than of the spectral sequence) we note that the differential is of bi-degree $(1, 4)$. Indexing the $E_i$-page similarly, the differential has bi-degree $(1, 4i)$. We note that for a knot Lee theory has two generators in degree zero. For the $(3, 3N)$-torus link (a three component link) Lee theory has two generators in degree zero and six generators in degree $-4N$.

**Proof. (of Theorem 3.1)** The proof consists of three claims:

Claim 1: if the result is true for $T(3, 3N - 1)$ then the result is true for $T(3, 3N)$.

Claim 2: if the result is true for $T(3, 3N)$ then the result is true for $T(3, 3N + 1)$.

Claim 3: if the result is true for $T(3, 3N + 1)$ then the result is true for $T(3, 3(N + 1) - 1)$.

Each claim is proved by the same technique, namely we use the $E_1$-page of the spectral sequence defined in Section 2 to produce some generators and also to produce some additional possible generators. It is not sure that the possible generators are in fact generators because there may be higher differentials in the spectral sequence killing them. We then use Lee’s spectral sequence to determine whether or not these possible generators are killed or not. By playing off one spectral sequence against another in this way, we do not actually have to explicitly compute any differentials in either spectral sequence.

**Proof of Claim 1** We will calculate the Khovanov homology of the link $T(3, 3N)$ under the assumption that the Khovanov homology of $T(3, 3N - 1)$ is as given in the statement of the theorem. Consider the set of crossings consisting of the two top crossings in the braid diagram (so $m = 2$). We have diagrams as presented in Figure 5.
Note that $D_{(2)} = T(3, 3N - 1)$ and it is easy to see that $\tilde{D}_{(1)} \sim U \sqcup U$ and $\tilde{D}_{(2)} \sim U$, where $U$ is the unknot. Using the orientations shown in Figure 5 one computes

$$\tilde{n}_1^+ = 4n - 1, \quad \tilde{n}_1^- = 2N, \quad \tilde{a}_1 = 4N, \quad \tilde{b}_1 = 12N - 1$$

and

$$\tilde{n}_2^+ = 4n - 1, \quad \tilde{n}_2^- = 2N - 1, \quad \tilde{a}_2 = 4N, \quad \tilde{b}_2 = 12N - 1.$$ 

From Proposition 2.2 we have

$$E_{0,t}^{0,t} = KH^{t+4N,j+12N-1}(U \sqcup U),$$

$$E_{1,t}^{1,t} = KH^{t+4N+1,j+12N}(U),$$

$$E_{2,t}^{2,t} = KH^{t+2,j+2}(T(3, 3N - 1)).$$

When $s = 0$ we see that $E_{0,t}^{0,t} = 0$ unless $t = -4N$ and $j = -12N - 1$, $j = -12N + 1$ or $j = -12N + 3$. Similarly, $E_{1,t}^{1,t} = 0$ unless $t = -4N - 1$ and $j = -12N - 1$ or $j = -12N + 1$ and $E_{2,t}^{2,t}$ is zero unless $-12N + 1 \leq j \leq -6N + 3$.

For $j > -12N + 3$ the $E_1$-term of the spectral sequence is concentrated in the column $s = 2$ and hence collapses for dimensional reasons. Thus,

$$KH^{i,j}(T(3, 3N)) \cong E_{2,i-2}^{2,i-2} = KH^{i+2,j+2}(T(3, 3N - 1)).$$

We need to consider the three cases $j = -12N - 1$, $j = -12N + 1$ and $j = -12N + 3$. The $E_1$ pages are give in Figure 6.

For $j = -12N - 1$ and $j = -12N + 3$ there are no differentials for dimensional reasons thus the spectral sequence collapses at $E_1$. For $j = -12N + 1$ there is a possible $d_1$ and a possible $d_2$ (but not both) as shown in Figure 6. Thus for $T(3, 3N)$ we have the situation presented in Figure 7 where possible generators are circled.

The three possible generators in bi-degree $(-4N, -12N + 1)$ must all indeed be generators because we require at least six generators in homological degree $-4N$. This is because Lee theory in this degree has six generators and due to Lee’s spectral sequence these must show up in Khovanov homology.
The possible generator in bi-degree \((-4N + 1, -12N + 1)\) is also a generator. If we look at the \(E_1\) page for \(j = -12N + 1\) then since the three generators on the line \(s + t = -4N\) survive until \(E_\infty\) (as shown in the previous paragraph) then there is nothing to kill the remaining generator. (Alternatively, the generator in bi-degree \((-4N + 2, -12N + 5)\) must be killed in Lee’s spectral sequence and the only possible way this can happen is for the possible generator in bi-degree \((-4N + 1, -12N + 1)\) to be present. To see this, recall that indexed this way the differential \(d_i\) in Lee’s spectral sequence has bi-degree \((1, 4i)\).)

Thus we end up computing \(KH^{*-s}(T(3, 3N))\) as presented in the theorem.

**Proof of Claim 2** Consider the link \(T(3, 3N + 1)\) and as above take the set of crossings to be the two top crossings in the braid diagram. We have diagrams as presented in Figure 8.

Note that \(D_{(2)} = T(3, 3N)\) and it is easy to show \(\tilde{D}(1) \sim U\) and \(\tilde{D}_{(2)} \sim U \sqcup U\). Using the orientations shown in Figure 8 one computes

\[
\begin{align*}
\bar{n}_1^+ &= 4N, & \bar{n}_1^- &= 2N + 1, & \bar{a}_1 &= 4N + 1, & \bar{b}_1 &= 12N + 2 \\
\bar{n}_2^+ &= 4N, & \bar{n}_2^- &= 2N, & \bar{a}_2 &= 4N + 1, & \bar{b}_2 &= 12N + 2.
\end{align*}
\]

Thus we have

\[
\begin{align*}
E_{1,0}^t &= KH^{t+4N+1,j+12N+2}(U) \\
E_{1,1}^t &= KH^{t+4N+2,j+12N+3}(U \sqcup U) \\
E_{2,1}^t &= KH^{t+2,j+2}(T(3, 3N))
\end{align*}
\]

For \(s = 0\) we must have \(j\) in the range \(-12N - 3 \leq j \leq -12N - 1\), for \(s = 1\) in the range \(-12N - 5 \leq j \leq -12N - 1\) and for \(s = 2\) in the range \(-12N - 3 \leq j \leq -6N + 1\).
For $j > -12N - 1$, as in the previous case, we instantly see that the result is as claimed. For the three remaining $j$-values we have $E_1$-pages as given in Figure 9.

For $j = -12N - 5$ there are no differentials for dimensional reasons, but for $j = -12N - 3$ and $j = -12N - 1$ there are possible differentials. The situation is presented in Figure 10, where, as above, possible generators are circled.
Consider the two possible generators in bi-degree \((-4N, -12N - 3)\). There cannot be generators in this bi-degree since they would appear in the \(E_\infty\)-page of Lee’s spectral sequence. However, \(T(3, 3N + 1)\) is a knot so the \(E_\infty\)-page has only two generators and these lie on the line \(s + t = 0\).

Now look at the \(E_1\)-page for \(j = -12N - 3\). We have just argued that the two generators on the line \(s + t = -4N\) must be killed. There are two possible ways this might happen, but either way one is left with one generator on the line \(s + t = -4N - 1\) and this must survive to \(E_\infty\).

A similar argument holds for the two possible generators in bi-degree \((-4N - 1, -12N - 1)\) and one is left with one generator in bi-degree \((-4N, -12N - 1)\).

**Proof of Claim 3** This is very similar to the previous arguments so we present this case only briefly. We follow the same orientation convention as above for the diagrams. We have

\[
E_{0,t}^1 = KH_{t+4N+3,j+12N+8}(U), \\
E_{1,t}^1 = KH_{t+4N+3,j+12N+6}(U), \\
E_{2,t}^2 = KH_{t+2,j+2}(T(3, 3N + 1)).
\]

For \(j > -12N - 5\) there is nothing to do and for the remaining \(j\)-values of interest we have \(E_1\)-pages as given in Figure 11 leading to the generators and possible generators presented in Figure 12.

\[
\begin{array}{ccc}
0 & 1 & 2 \\
\hline
-4N - 2 & & \\
-4N - 3 & 1 & \\
\end{array}
\quad
\begin{array}{ccc}
0 & 1 & 2 \\
\hline
-4N - 2 & & \\
-4N - 3 & 1 & 1 & 1 \\
\end{array}
\quad
\begin{array}{ccc}
0 & 1 & 2 \\
\hline
-4N - 2 & & \\
-4N - 3 & 1 & \\
\end{array}
\]

\(j = -12N - 9\) \quad \(j = -12N - 7\) \quad \(j = -12N - 5\)

**Figure 11.**

\[
\begin{array}{ccc}
0 & 1 & 2 \\
\hline
-4N - 3 & & \\
1 & 1 & 1 \\
4 & 1 & 1 \\
1 & 1 & 1 \\
-12N - 9 & 1 & \\
\end{array}
\]

**Figure 12.**

As above it is easy to see that the possible generator in bi-degree \((-4N - 3, -12N - 7)\) cannot survive to \(E_\infty\) in Lee’s spectral sequence so must be killed. There is only one
possibility which leaves one generator in homological degree $-4N - 1$. When $j = -12N - 5$ the two generators both survive because they are needed in Lee’s spectral sequence to kill the generators in bi-degree $(-4N - 3, -12N - 9)$ and $(-4N, -12N - 1)$.

Finally we note that the inductive process of the above three claims has a beginning because the Khovanov homology of $T(3, 2)$ is easily calculated (even by hand), and $T(3, 3)$, $T(3, 4)$ and $T(3, 5)$ can also be computed (by computer or using the spectral sequence - the computations are similar, though not identical, to those above). These cases are seen to have the required form.

The Khovanov homology of positive crossing $(3, q)$-torus links can be computed from the above by recalling that the rational Khovanov homology of the mirror image $L^!$ of a link $L$ can be computed as $KH^{i,j}(L^!) = KH^{-i,-j}(L)$.

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