Self-dual perturbiner in Yang-Mills theory

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Abstract

The perturbiner approach to the multi-gluonic amplitudes in Yang-Mills theory is reviewed.
1 Definition and motivation

Perturbiner is a solution of field equations which can be defined in any field theory [1]. Definition. (I use for a moment the scalar field theory) Consider linear part of the equations (that is, with no interactions), and take its solution of the type of $\phi^{(1)} = \sum_j^n E_j$, where $E_j = a_j e^{ik_j x}$, $k$’s are on-shell momenta, $k^2 = m^2$ and $a_j$ is assumed to be nilpotent, $a_j^2 = 0$. The perturbiner is a (complex) solution of the nonlinear field equations which is: i) polynomial in $E_j$ with constant coefficients and ii) whose first order in $E_j$ part is precisely $\phi^{(1)}$.

Generalization for higher spins is obvious: one should put a polarization factor in front of $E_j$ in $\phi^{(1)}$.

Motivation. The perturbiner so defined is a generating function for the tree form-factors. The set of the plane waves entering the perturbiner is essentially the set of asymptotic states in the amplitudes which the perturbiner is the generating function for. The nilpotency condition $a_j^2 = 0$ means that one considers only amplitudes with no multiple particles in the same state.

Notice: i) due to this definition one actually works with a finite-dimensional space of polynomials in $N$ nilpotent variables instead of an infinite-dimensional function space; ii) this definition is different from the one traditionally considered in the stationary phase approach to the $S$-matrix [2].

In general case, one cannot proceed in a different from the usual perturbation theory way. However in some theories and/or for some sets of the asymptotic states included one can use other powerful methods. In the Yang-Mills (YM) theory one can consider only the same helicity states which leads to considering only the self-duality (SD) equations instead of the full YM ones. This type of SD solutions has been discussed in refs. [4], [5], [6]. In ref. [4] and independently in ref. [5], the tree like-helicity amplitudes were related to solutions of the SD equations. In [4] it was basically shown that the SD equations reproduce the recursion relations for the tree form-factors (also called one-gluonic currents) obtained originally in ref. [9] from the Feynman diagrams; the corresponding solution of SD equations was obtained in terms of the solution of ref. [9] of the recursion relations for the “currents”. In ref. [4] an example of SD perturbiner was obtained in the SU(2) case by a ’tHooft anzatz upon further restriction on the asymptotic states included. The consideration of ref. [4] is based on solving recursion relations analogous to refs. [4]. In [11] the YM SD perturbiner was constructed by the twistor methods [7] which also allowed us to obtain perturbiner with one opposite-helicity gluon and thus to obtain a generating function for the so-called maximally helicity violating Parke-Taylor amplitudes [8], [9]. In [10] we also constructed the SD perturbiner in the background of an arbitrary instanton solution.

Briefly, the twistor approach to the SD equations goes as follows. One introduces an auxiliary twistor variable $p^\alpha, \alpha = 1,2$, which can be viewed on as a pair of complex numbers, and form objects $A_\dot{\alpha} = p^\alpha A_{\dot{\alpha} \alpha}, \bar{\partial}_\dot{\alpha} = p^\alpha \frac{\partial}{\partial x^\alpha}, \bar{\nabla}_\dot{\alpha} = \bar{\partial}_\dot{\alpha} + A_\dot{\alpha}$. In terms of $\bar{\nabla}_\dot{\alpha}$, the SD equations turn to a zero-curvature condition,
\[ \overline{\nabla}_\alpha, \overline{\nabla}_\beta \] = 0, at any \( p^\alpha, \alpha = 1, 2 \), which can be solved for \( A_\dot{\alpha} = p^\alpha A_{\alpha \dot{\alpha}} \) as

\[ A_\dot{\alpha} = g^{-1} \overline{\partial}_\alpha g \] (1)

where \( g \) is a function of \( x^{\alpha \dot{\alpha}} \) and \( p^\alpha \) with values in the complexification of the gauge group. \( g \) must depend on \( p^\alpha \) in such a way that the resulting \( A_\dot{\alpha} \) is a linear function of \( p^\alpha \), \( A_\dot{\alpha} = p^\alpha A_{\alpha \dot{\alpha}} \). Actually, \( g \) is sought for as a homogeneous of degree zero rational function of \( p^\alpha \). Such function necessary has singularities in the \( p^\alpha \)-space and it is subject to condition of regularity of \( A_\dot{\alpha} \). Then by construction, \( A_\dot{\alpha} \) is a homogeneous of degree one regular rational function of two complex variables \( p^\alpha, \alpha = 1, 2 \). As such, it is necessary just linear in \( p^\alpha \).

An essential moment is that in the case of perturbiner \( g^{ptb} \) can only be a polynomial in the variables \( E_j \). First order in \( E_j \) term in \( g^{ptb} \) is fixed by the plane wave solution of the free equation (that is by the set of asymptotic states included), while the demand of regularity of \( A_\dot{\alpha} \) fixes \( g^{ptb} \) up to the gauge freedom.

## 2 The plane wave solution of the free equation

A solution of the free (i.e. linearized) SD equation consisting of \( N \) plane waves looks as follows

\[ A_{\alpha \dot{\alpha}}^{(1)N} = \sum_j^{N} \epsilon_{\alpha \dot{\alpha}}^{+j} \hat{E}_j \] (2)

where the sum runs over gluons, \( N \) is the number of gluons, \( \epsilon_{\alpha \dot{\alpha}}^{+j} \) is a four-vector defining a polarization of the \( j \)-th gluon, \( \hat{E}_j = t_j E_j = t_j a_j e^{ik_j x} \), \( t_j \) is a matrix defining color orientation of the \( j \)-th gluon. \( k_j^{\alpha \dot{\alpha}} \), as a light-like four-vector, decomposes into a product of two spinors \( k_j^{\alpha \dot{\alpha}} = \lambda_\dot{\alpha}^j \lambda_\alpha^j \]. The polarization \( \epsilon_{\alpha \dot{\alpha}}^{+j} \), as a consequence of the linearized SD equations, also decomposes into a product of spinors, such that the dotted spinor is the same as in the decomposition of momentum \( k \), \( \epsilon_{\alpha \dot{\alpha}}^{+j} = \lambda_\alpha^j \lambda_\dot{\alpha}^j \) where normalization factor is defined with use of a convolution \( (\omega^j, q^j) = \epsilon^{\alpha \dot{\alpha}} \omega_\alpha^j q_\dot{\alpha}^j = \epsilon^{\alpha \dot{\alpha}} \omega_\alpha^j q_\dot{\alpha}^j \). Indexes are raised and lowered with the \( \varepsilon \)-tensors. The free anti-SD equation would give rise to a polarization \( \epsilon_{\alpha \dot{\alpha}}^+ = \frac{\omega_\alpha^j q_\dot{\alpha}^j}{\omega^j q^j} \). The auxiliary spinors \( q_\alpha \) and \( \bar{q}_{\dot{\alpha}} \) form together a four-vector \( q_{\alpha \dot{\alpha}} = q_\alpha \bar{q}_{\dot{\alpha}} \) usually called a reference momentum.

The normalization was chosen so that \( \epsilon^+ \cdot \epsilon^- = \epsilon^{\alpha \beta} \epsilon_{\dot{\alpha} \dot{\beta}} \epsilon^{+\alpha} \epsilon^{-\dot{\alpha}} = -1 \)

## 3 The solution for \( g^{ptb} \) and \( A_{\dot{\alpha}}^{ptb} \)

First order in \( E \) terms in \( g^{ptb} \) are easily found from equation (1), first order

\(^1\)the reality of the four momentum in Minkowski space assumes that \( \lambda_\alpha = \bar{\lambda}_\alpha \)
version of which reads $A^{\text{ptb}(1)}_{\alpha} = \tilde{\partial}_\alpha g^{\text{ptb}(1)}$ and hence, with use of (2),

$$g^{\text{ptb}(1)} = 1 + \sum_j \frac{(p, q_j)}{(p, \tilde{\alpha}_j)} \frac{\hat{E}_j}{(\tilde{\alpha}_j, q_j)}$$  \hspace{1cm} (3)

Thus $g^{\text{ptb}(1)}$ has simple poles on the auxiliary space at the points $p_\alpha = \tilde{\alpha}_\alpha$. Actually, one can see that the condition of regularity of $A^{\text{ptb}}_\alpha$ dictates that the full $g^{\text{ptb}}$ has only simple poles at the same points as $g^{\text{ptb}(1)}$. Moreover, it also fixes residues of $g^{\text{ptb}}$ to all orders in $\mathcal{E}$ in terms of the residues of $g^{\text{ptb}(1)}$ Eq. (3).

The known singularities of $g^{\text{ptb}}$ fix it up to an independent of the auxiliary variables $p^{\alpha}, \alpha = 1, 2$ matrix, i.e., up to a gauge freedom. The problem of reconstructing $g^{\text{ptb}}$ from its singularities essentially simplifies if one considers color ordered highest degree monomials in $g^{\text{ptb}}$, for which one obtains ([10])

$$g^{\text{ptb}}_{N(N, \ldots, 1)} = \frac{(p, q_N)(\tilde{\alpha}_N, q_N^{-1}) \ldots (\tilde{\alpha}_2, q_1)}{(p, \tilde{\alpha}_N)(\tilde{\alpha}_N, \tilde{\alpha}_N^{-1}) \ldots (\tilde{\alpha}_2, \tilde{\alpha}_1)} \hat{E}_N \ldots \hat{E}_1$$  \hspace{1cm} (4)

This is, essentially, a solution of the problem. Substituting $g^{\text{ptb}}$ (4) into equation (1) determines the perturbiner $A^{\text{ptb}}_{\alpha}$ (see ref.[10]).

4 The Parke-Taylor amplitudes

The SD perturbiner can be used as a base point for a perturbation procedure of adding one-by-one gluons of the opposite helicity, or other particles, say, fermions, interacting with gluons.. The explicit expression for $g^{\text{ptb}}$ (4) is very useful in this procedure. The SD perturbiner itself describes the tree form-factors - objects including an arbitrary number of on-shell SD gluons and one arbitrary off-shell gluon. To obtain the Parke-Taylor amplitudes, those with two gluons of the opposite helicity, one essentially needs to construct the perturbiner including one on-shell gluon of the opposite helicity, that is, to solve the linearized YM equation in the background of SD perturbiner. Details of this solution can be found in ref.[10]. The resulting generating function for the Parke-Taylor amplitudes reads

$$M(k'', k', \{a_j\}) = -i(\tilde{\alpha}''(\tilde{\alpha}')^2 \int d^4x \tr \hat{E}'(g^{\text{ptb}}_{\{p=\tilde{\alpha}', q=\tilde{\alpha}''\}})^{-1} \hat{E}'(g^{\text{ptb}}_{\{p=\tilde{\alpha}', q=\tilde{\alpha}''\}})$$  \hspace{1cm} (5)

Considering cyclic ordered terms in this expression with $g^{\text{ptb}}$ (4) one easily reproduces the Parke-Taylor maximally helicity violating amplitudes [8], [9].

5 The SD perturbiner in a topologically non-trivial sector

The concept of perturbiner can be generalized to a topologically nontrivial sector (see [10]). In the latter case it provides a framework for the instanton
mediated multi-particle amplitudes. All what we need to know about the instanton, $A^\text{inst}_\alpha$, that it can be represented in the twistor-spirit form $A^\text{inst}_\alpha = g^{-1}_\text{inst} \bar{\partial}_\alpha g^\text{inst}_\alpha$ is assumed to be a rational function of the auxiliary variables $p^\alpha$, such that $A^\text{inst}_\alpha$ is a linear homogeneous function of $p^\alpha$. Then the SD topologically nontrivial perturbiner $A^\text{iptb}_\alpha$ is represented in the form $A^\text{iptb}_\alpha = (g^\text{iptb})^{-1} \bar{\partial}_\alpha g^\text{iptb}$ and the corresponding $g^\text{iptb}$ is found to be

$$g^\text{iptb}(\hat{\mathcal{E}}_1, \hat{\mathcal{E}}_2, \ldots) = g^\text{inst} g^\text{iptb}(\hat{\mathcal{E}}_g, \hat{\mathcal{E}}_g, \ldots)$$

where $\hat{\mathcal{E}}^j_g$ stand for twisted harmonics $\hat{\mathcal{E}}^j_g = (g^\text{inst}|_{(p=\alpha^j)})^{-1}\hat{\mathcal{E}}^j|_{(p=\alpha^j)}$ and $g^\text{iptb}$ as in Eq. (4). This is the sought for SD perturbiner in an arbitrary instanton background.

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