An introduction to Wishart matrix moments

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Abstract

This article provides a comprehensive, rigorous, and self-contained introduction to the analysis of Wishart matrix moments. This article may act as an introduction to some aspects of random matrix theory, or as a self-contained exposition of Wishart matrix moments.

Random matrix theory plays a central role in nuclear and statistical physics, computational mathematics and engineering sciences, including data assimilation, signal processing, combinatorial optimization, compressed sensing, econometrics and mathematical finance, among numerous others. The mathematical foundations of the theory of random matrices lies at the intersection of combinatorics, non-commutative algebra, geometry, multivariate functional and spectral analysis, and of course statistics and probability theory. As a result, most of the classical topics in random matrix theory are technical, and mathematically difficult to penetrate for non-experts and regular users and practitioners.

The technical aim of this article is to review and extend some important results in random matrix theory in the specific context of real random Wishart matrices. This special class of Gaussian-type sample covariance matrix plays an important role in multivariate analysis and in statistical theory. We derive non-asymptotic formulae for the full matrix moments of real valued Wishart random matrices. As a corollary, we derive and extend a number of spectral and trace-type results for the case of non-isotropic Wishart random matrices. We also derive the full matrix moment analogues of some classic spectral and trace-type moment results. For example, we derive semi-circle and Marchenko-Pastur-type laws in the non-isotropic and full matrix cases. Laplace matrix transforms and matrix moment estimates are also studied, along with new spectral and trace concentration-type inequalities.

Keywords: Wishart matrices, sample covariance matrices, non-crossing partitions, Gaussian orthogonal ensemble, semi-circle law, Marchenko-Pastur law, random matrix moments, exponential concentration inequalities.

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1 Introduction

Let $X$ be a centred Gaussian random column vector with covariance matrix $P$ on $\mathbb{R}^r$, for some dimension parameter $r \geq 1$. The rescaled sample covariance matrix associated with $(N+1)$ independent copies $X_i$ of $X$ is given by the random matrix

$$P_N = \frac{1}{N} \sum_{1 \leq i \leq N+1} (X_i - m^N) (X_i - m^N)' \quad \text{with} \quad m^N := \frac{1}{N+1} \sum_{1 \leq i \leq N+1} X_i$$

The random matrix $P_N$ has a Wishart distribution $\mathcal{W}_r (N, N^{-1}P)$ with $N$ degrees of freedom and covariance matrix $N^{-1}P$. In addition, we have the equivalent formulations

$$P_N \overset{law}{=} N^{-1} \sum_{1 \leq i \leq N} X_i = X X' \quad \text{with the} \quad (r \times N)\text{-random matrix} \quad X = \frac{1}{\sqrt{N}} [X_1, \ldots, X_N]$$

In the above display, $X_i$ stand for $N$ independent copies of the single rank random matrix $X = XX'$. The superscript $(,)'$ denotes the transposition operation.

Sample covariance matrices, and more particularly Wishart random matrices, play a crucial role in statistics, physics, finance and engineering sciences. Their interpretation depends on
the application model motivating their study. For example, in the context of multiple-input
multiple-output systems, the matrix $X$ may represent the channel gain matrix $[59, 76]$. In data
assimilation problems and filtering theory, sample covariance matrices arise as the control gain
in ensemble (Kalman-type) filters; see e.g. $[16, 77]$ and the references therein. In finance, sample
covariance matrices arise in risk management and asset allocation; in this context, they represent
the correlated fluctuations of assets $[46, 9, 20]$. Numerous other applications and motivators
exist; see e.g. $[74, 75, 19, 23, 47, 69]$. Typical questions arising in practice revolve around the
calculation of the spectrum distribution, and the corresponding eigenvector distribution of these
random matrices.

The analysis of Wishart matrices started in 1928 with the pioneering work of J. Wishart $[77]$. Since
this, the theory of random matrices has been a fruitful contact point between statistics,
pure and applied probability, combinatorics, non-commutative algebra, as well as differential
geometry and functional analysis.

The joint distribution of the eigenvalues of real valued Wishart matrices is only known for
full rank and isotropic models; that is when the sample size is greater than the dimension, and
the covariance matrix $P \propto I$ is proportional to the identity matrix $I$; see for instance $[26, 50]$. In
this situation, the matrix of random eigenvectors is uniformly distributed on the manifold of
unitary matrices equipped with the Haar measure. In this context, marginal distributions for
these uncorrelated models can also be computed in a tractable form only for the smallest and the
largest eigenvalues. Sophisticated integral formulae for the marginal distribution of intermediate
eigenvalues are provided by Zanella-Chiani-Win $[81]$. Upper bounds on the marginal distribution
of the ordered eigenvalues are given in $[52]$. The cumulative distribution of the largest eigenvalue of real
Wishart random matrices can be expressed explicitly in terms of the hypergeometric function of a matrix argument. These functionals can also be described in terms of zonal polynomials. The smallest and largest
eigenvalue distributions can also be expressed in terms of Tricomi functions $[19]$. For a detailed
discussion on these objects we refer the reader to the book of Muirhead $[50]$. As shown in $[36]$, these hypergeometric functions depends on alternating series involving zonal polynomials which converge very slowly even in low dimensions. Some explicit calculations for $r = 1, 2, 3$ can be found in $[68]$. Non-necessarily isotropic Wishart models can be considered if we restrict our attention to
linear transforms and other trace-type mathematical objects. We refer to the articles of Letac
and his co-authors $[22, 67, 38, 70]$ and the tutorial $[39]$. See also $[27, 40]$ for a description of the
joint distribution of traces of Wishart matrices.

To bypass the complexity of finding computable and tractable closed form solutions, one
natural and common way to obtain useful information is to derive limit distributions when the
dimension tends to $\infty$. In this context, one can analyze the convergence of the histogram of the
eigenvalues when the dimension tends to $\infty$. This approach is central in random matrix theory.
We refer the reader to the pioneering article by E. Wigner $[75]$ published in 1955, the lectures
notes of A. Guionnet $[23]$, the research monographs by M.L. Mehta $[47]$ and T. Tao $[69]$, and the
references therein. This commonly used limiting theory has some drawbacks. Firstly, as the name
suggests, these limiting techniques cannot capture nor control the non-asymptotic fluctuations
arising in practical problems. Moreover, the limiting techniques developed in the literature
often yield information only on the limiting behaviour of trace or spectral-type properties of
random matrix powers. In addition, in the context of Wishart matrices, these limiting spectral-
type techniques only apply to asymptotically isotropic-type models. To be more precise, the
convergence analysis relies on strong hypotheses on the bias and the variance of the random
matrix entries which are satisfied only for Wishart matrices with a covariance matrix close to
the identity (up to some ad-hoc scaling factor). When $P \neq I$, the distribution of the eigenvalues
and the corresponding eigenvectors is much more involved. The distribution of the sample
eigenvalues depends on sophisticated Harish-Chandra integrals $[24]$. 

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Importantly, all the spectral and trace-type approaches discussed above (whether in the limit or not) give no information on the random matrix moments themselves, but rather on their eigenvalues, etc. Conversely, in many practical situations, such as in data assimilation theory and signal processing (e.g. ensemble Kalman filter theory [1]), we are typically interested in the direct analysis of full matrix moments of interacting-type (non-independent) covariance matrices. This article concerns a step toward this latter direction. Specifically, we derive formulae for the full matrix moments of real valued Wishart random matrices. As a corollary, we derive and extend a number of spectral and trace-type results for the case of non-isotropic Wishart random matrices. Laplace matrix transforms and matrix moment estimates are also studied, along with new spectral and trace concentration-type inequalities.

The rest of the article is organized as follows: Section [1.1] is concerned with the description of real random Wishart matrices and their fluctuation analysis. We also review some central result in random matrix theory, such as the semi-circle law and the Marchenko-Pastur law for isotropic Wishart matrices.

Section [1.2] provides a brief description of the main results of this article. We provide a closed form Taylor-type formula to compute the matrix moments of $P_N$, and its fluctuations defined in (1.1), w.r.t. the precision parameter $1/N$. We also present the matrix version of the semi-circle law and the Marchenko-Pastur law for non-necessarily isotropic Wishart matrices. Non-asymptotic matrix moments and exponential Laplace transforms are also provided. The last part of the section is concerned with exponential concentration inequalities for operator norms of fluctuation matrices and the eigenvalues of sample covariance matrices.

The rest of the article is concerned with the precise statement and proof of the theorems in Section [1.2]. Some auxiliary outcomes and discussion surrounding these results are also given.

Section [2] reviews some useful mathematical background on Laplace and exponential inequalities, matrix norms, spectral analysis, tensor products, Fréchet derivatives, and fluctuation-type results. This section also contains a brief review of non-crossing partitions, Catalan, Narayana, and Riordan numbers, Bell polynomials, and Murasaki and circular-type representations of non-crossing partitions. The last part of this section discusses Fréchet differentiable functionals on matrix spaces and Taylor-type approximations.

Section [3] concerns closed form polynomial formulae for computing the matrix moments of $P_N$, and its fluctuations in (1.1), in terms of the precision parameter $1/N$ and some partition-type matrix moments. The isotropic semi-circle law (1.7) and the Marchenko-Pastur law (1.9) are simple consequences of these matrix moments expansions. New matrix versions of the semi-circle and the Marchenko-Pastur law for non-necessarily isotropic Wishart matrices are discussed in Section [3.2] and in Section [3.3].

Section [4] is concerned with some matrix moment estimates and Laplace matrix transforms of the fluctuation matrix. Spectral and trace-type concentration-type inequalities are discussed in Section [5]. Section [6] is dedicated to the proof of the main theorems. An Appendix is also given containing the proof of a number of technical results required throughout the article.

1.1 Description of the models
We recall the the multivariate Central Limit theorem
\[
P_N = P + \frac{1}{\sqrt{N}} \mathcal{H}_N \quad \text{with} \quad \mathcal{H}_N \overset{\text{law}}{=} \frac{1}{\sqrt{N}} \sum_{1 \leq i \leq N} (X_i - P) \longrightarrow_{N \to \infty} \mathcal{H} \tag{1.1}
\]
where $\mathcal{H}$ is a symmetric $(r \times r)$-matrix with centered Gaussian entries equipped with a symmetric Kronecker covariance structure
\[
(\mathcal{H} \otimes \mathcal{H})^\sharp = 2 (P \otimes P) = \mathbb{E} \left[ (\mathcal{H}_N \otimes \mathcal{H}_N)^\sharp \right] \tag{1.2}
\]
where $(A \otimes B)^\sharp$ and $(A \otimes B)$ are the entry-wise and the symmetric tensor product of matrices $A$ and $B$. These products are defined at the beginning of Section [2.3].
A detailed discussion on the fluctuation result (1.1) can be found [32]; see also [10] for non-necessarily Gaussian variables. The fluctuation result (1.1) can also be deduced from the Laplace matrix transform estimates stated in theorem 1.5 and corollary 4.5.

Combining a perturbation analysis with the continuous mapping theorem, the central limit result (1.1) can be used to analyze the fluctuation of smooth matrix functional of the sample covariance matrix. Roughly speaking, given some smooth Fréchet differentiable mapping \( \mathcal{Y} : \mathcal{S}_r \mapsto \mathcal{B} \) from symmetric matrices \( \mathcal{S}_r \) to some Banach space \( \mathcal{B} \), we have the Taylor expansion

\[
\mathcal{H}_N^X := \sqrt{N} [\mathcal{Y}(P_N) - \mathcal{Y}(P)] = \nabla \mathcal{Y}(P) \cdot \mathcal{H}_N + \frac{1}{2 \sqrt{N}} \nabla^2 \mathcal{Y}(P) \cdot (\mathcal{H}_N, \mathcal{H}_N) + \ldots
\]

Using the unbiasedness properties of the sample covariance matrix, the second order term gives the bias of the estimate \( \mathcal{Y}(P_N) \); that is we have that

\[
\mathbb{E} [\mathcal{H}_N^X] = \frac{1}{2 \sqrt{N}} \mathbb{E} [\nabla^2 \mathcal{Y}(P) \cdot (\mathcal{H}, \mathcal{H})] + O \left( \frac{1}{N} \right)
\]

\[\iff\]

\[
\mathbb{E} [\mathcal{Y}(P_N)] = \mathcal{Y}(P) + \frac{1}{2N} \mathbb{E} [\nabla^2 \mathcal{Y}(P) \cdot (\mathcal{H}, \mathcal{H})] + O \left( \frac{1}{N^{3/2}} \right)
\]

For a more precise statement and several illustrations we refer the reader to Section 2.6, theorem 2.2. For instance, for power functions \( \mathcal{Y}_n(Q) := Q^n \), for any \( 1 \leq m \leq n \) we have

\[
\mathbb{E} [\nabla^m \mathcal{Y}_n(P) \cdot \mathcal{H}_N^\otimes m] = m! \sum_{0 \leq i_1 < \ldots < i_m \leq n} \mathbb{E} \left[ \prod_{1 \leq k \leq m} P^{i_k - i_{k-1} - 1} \mathcal{H}_N \right] = (n)_m \mathbb{E}(\mathcal{H}_N^m) \iff P = I
\]

with the Pochhammer symbol \((n)_m := n!/(n-m)!\), and the convention \((i_0,i_{m+1}) = (0,n)\). In this context, the \(m\)-moments of the fluctuation matrices \( \mathcal{H}_N \) represents the mean-error of order \( m \). This property also holds for rational powers. For instance, we have the non-asymptotic estimate on the Frobenius norm,

\[
\left\| \mathbb{E} \left( \sqrt{P_N^r} \right) - \sqrt{P} + \frac{1}{2N} \sqrt{P} \left[ \frac{1}{4} I + \int_0^\infty t \text{Tr} \left( P e^{-t\sqrt{P}} \right) e^{-t\sqrt{P}} dt \right] \right\|_F \leq \frac{r}{4} \frac{1}{N^{1/2}} \lambda_{\min}(P)^{-5/2} \left[ \text{Tr}(P^2) + \text{Tr}(P)^2 \right]
\]

The proof of this assertion is provided in Section 2.6.

In summary, to analyze these approximations at any order it is therefore necessary to be able to compute the \(m\)-moments of the fluctuation matrices \( \mathcal{H}_N \).

Gaussian approximation techniques also require one to estimate the fluctuations of the moment \( \mathbb{E}(\mathcal{H}_N^m) \) around the ones \( \mathbb{E}(\mathcal{H}_N^m) \) of the limiting Gaussian matrix \( \mathcal{H} \), with respect to the sample size parameter. Last but not least, it is essential to estimate the behavior of these mathematical objects when the dimension parameter tends to \( \infty \).

All of these questions are central in random matrix theory. The limiting random matrix model \( \mathcal{H} \) discussed above is closely related to Gaussian orthogonal ensembles arising in random matrix theory. To be more precise, we check that

\[
\mathcal{H} \overset{\text{law}}{=} P^{1/2} \left( \frac{\mathcal{W} + \mathcal{W}'}{\sqrt{2}} \right) P^{1/2}
\]

where \( \mathcal{W} = (\mathcal{W}_{i,j})_{1 \leq i,j \leq r} \) is a matrix of independent, centred Gaussian elements of unit variance.
When $P = I$, the random matrix $\mathcal{H}$ introduced in (1.5) resumes to a Gaussian orthogonal ensemble. In this situation, we have

$$
 r^{-2} \mathbb{E} \left( \text{Tr} \left( \mathcal{H}^2 \right) \right) = 1 + r^{-1} \quad \text{and} \quad r^{-3} \mathbb{E} \left( \text{Tr} \left( \mathcal{H}^4 \right) \right) = 2 + 5 r^{-2} + 5 r^{-1} \quad (1.6)
$$

The trace of the higher moments $\mathbb{E}(\mathcal{H}^n)$ can be estimated using the below semi-circle law (1.7) in large dimensions. This celebrated limiting result is proved using the convergence of moment property

$$
\forall n \geq 1, \quad r^{-1} \mathbb{E} \left( \text{Tr} \left( \left[ \frac{\mathcal{H}}{r} \right]^n \right) \right) = \frac{1}{2\pi} \int_{-2}^{2} x^n \sqrt{4 - x^2} \, dx + O(1/r) \quad (1.7)
$$

with the Catalan numbers

$$
C_n := \frac{1}{n+1} \binom{2n}{n} \quad (1.8)
$$

A proof of the above assertion via Wick’s theorem, including detailed reference pointers is given in [23, Section 1.4], see also [48, Chapter 1]. When $P = I$ and $N = r/\rho$ for some parameter $\rho > 0$, another important result in random matrix theory is the Marchenko-Pastur law

$$
\lim_{r \to \infty} r^{-1} \mathbb{E} \left[ \text{Tr} \left( P^n \right) \right] = \frac{\rho^m}{m+1} \binom{n}{m} \binom{n-1}{m} \quad (1.9)
$$

$$
= \int_{a_- (\rho)}^{a_+ (\rho)} x^n \left( 1 - \frac{1}{\rho} \right) \delta_0 (dx) + \frac{1}{2\pi \rho x} \sqrt{\rho (\rho + x) \left( \rho - x \right)} \left( \rho - x \right) dx
$$

with the parameters

$$
a_- (\rho) := \left( 1 - \sqrt{\rho} \right)^2 \quad \text{and} \quad a_+ (\rho) := \left( 1 + \sqrt{\rho} \right)^2
$$

The proof of the above integral formula can be found in [78, lemma 5.2], see also [21, 51] and the pioneering article by Vladimir Marchenko and Leonid Pastur [45]. A new proof of this result follows from the matrix version of the Marchenko-Pastur law given in corollary 3.3 of this article.

When $P \neq I$, formula (1.5) can be combined with Isserlis’ theorem [24] (or Wick’s theorem [2]) to compute the matrix moments of the random matrix $\mathcal{H}$. For instance, we have $\mathbb{E}(\mathcal{H}^{2n+1}) = 0$, for any $n \geq 0$. After some lengthy combinatorial computations we also find the matrix polynomials

$$
\mathbb{E}(\mathcal{H}^2) = P^2 + \text{Tr}(P) \, P
$$

$$
\mathbb{E}(\mathcal{H}^4) = 5 P^4 + 3 \text{Tr}(P) \, P^3 + \left[ \text{Tr}(P^2) + \text{Tr}(P)^2 \right] \, P^2 + \left[ \text{Tr}(P^3) + \text{Tr}(P) \, \text{Tr}(P^2) \right] \, P
$$

Although these matrix moments are given by a closed form formula, its complex combinatorial structure cannot be used by simple calculations. For example, the calculation of $\mathbb{E}(\mathcal{H}^{2n})$ requires the matrix moments associated with $2^{-n} (2n)!/n!$ partitions over $\{1, \ldots, 2n\}$ with $n$-blocks. For $n = 4$, more than one hundred moments need to be computed. The computational complexity to numerically compute the central moments of the multivariate normal distribution is discussed in [53]; see also [2, p. 49], [30, proposition 1], [50, p. 46], and the matrix derivative formula in [71].

The above formulae also show that we cannot expect to have a semi-circle-type law as in (1.6) for any covariance matrix. Different types of behaviour can be expected depending on the behavior of the eigenvalues of $P$ w.r.t. the dimension parameter $r$. For instance, if the largest eigenvalue is $\lambda_1 (P) = r$, we have $1 \leq r^{-1} \text{Tr}(P) \leq 2$ but

$$
\mathbb{E}(\text{Tr} \left( \mathcal{H}^4 \right)) \geq 5 \, r^4 \implies r^{-1} \mathbb{E} \left( \text{Tr} \left( \frac{\mathcal{H}}{r} \right)^4 \right) \xrightarrow{r \to \infty} \infty
$$
1.2 Statement of some main results

One of the main objectives of the article is to analyze the properties of real Wishart matrix moments. Let \( \mathcal{P}_n \) be the set of all partitions \( \pi \) of \([n] := \{1, \ldots, n\}\), \( \mathcal{P}_{n,m} \subset \mathcal{P}_n \) be the subset of all partitions with \( m \) blocks \( \pi_1 \leq \ldots \leq \pi_m \) ordered w.r.t. their smallest element. Let \( \mathcal{Q}_n \subset \mathcal{P}_n \) and \( \mathcal{Q}_{n,m} \subset \mathcal{P}_{n,m} \) be the subset of partitions without the singleton. Also let \( \alpha^π := \sum_{1 \leq i \leq m} i \pi_i \).

The \( \pi \)-matrix moments \( M_{π}^{|Q|}(P) \) and \( M_{π}^{\circ |Q|}(P) \) associated with some collection of \((r \times r)\) matrices \((Q_i)_{i \geq 1}\) are defined by

\[
M_{π}^{|Q|}(P) := \mathbb{E} \left( [X - P]^Q_π \right) \quad \text{and} \quad M_{π}^{\circ |Q|}(P) := \mathbb{E} \left( X^Q_π \right) \quad (1.10)
\]

with the random matrices

\[
X^Q_π := \prod_{1 \leq i \leq n} \left[ Q_i X_{\alpha^*(i)} \right] \quad \text{and} \quad [X - P]^Q_π := \prod_{1 \leq i \leq n} \left[ Q_i (X_{\alpha^*(i)} - P) \right]
\]

We also consider the matrix moments

\[
M_{n,m}^{|Q|}(P) := \sum_{\pi \in \mathcal{Q}_{n,m}} M_{\pi}^{|Q|}(P) \quad \text{and} \quad M_{n,m}^{\circ |Q|}(P) := \sum_{\pi \in \mathcal{Q}_{n,m}} M_{\pi}^{\circ |Q|}(P)
\]

Our first main result provides polynomial formulae w.r.t. the precision parameter \( 1/N \).

**Theorem 1.1.** For any collection of matrices \( Q_n \), and any \( 2N \geq n \geq 1 \) we have the polynomial formulae

\[
\mathbb{E} \left[ (Q_1 \mathcal{H}_N) \ldots (Q_n \mathcal{H}_N) \right] = \sum_{1 \leq m \leq \lfloor n/2 \rfloor} \frac{1}{N^{n/2-m}} \partial_{n,m}^{|Q|}(P) \quad (1.11)
\]

with \( \partial_{n,m}^{|Q|}(P) := \sum_{m \leq \ell \leq \lfloor n/2 \rfloor} s(l,m) M_{n,l}^{|Q|}(P) \)

as well as

\[
\mathbb{E} \left[ (Q_1 P_N) \ldots (Q_n P_N) \right] = \sum_{1 \leq m \leq n} \frac{1}{N^{n-m}} \partial_{n,m}^{\circ |Q|}(P) \quad (1.12)
\]

with \( \partial_{n,m}^{\circ |Q|}(P) = \sum_{m \leq \ell \leq n} s(l,m) M_{n,l}^{\circ |Q|}(P) \)

In the above displayed formulae, \( s(l,m) \) are the Stirling numbers of the first kind.

For the detailed discussion of these matrix moments, including several corollaries and examples, we refer to Section 3.1; see for example theorem 3.1 when \( Q_i = I \).

To simplify notation, for homogeneous models \( Q_n = I \) we suppress the index \( (.)^{[l]} \), and write

\[
(\partial_{n,m}(P), \partial_{n,m}^{\circ}(P), M_{n,l}(P), M_{n,l}^{\circ}(P)) \quad \text{instead of} \quad (\partial_{n,m}^{[l]}(P), \partial_{n,m}^{\circ,[l]}(P), M_{n,l}^{[l]}(P), M_{n,l}^{\circ,[l]}(P))
\]

The polynomial formula \((1.11)\) differs from the invariant moments which can be derived using the algorithm presented in [37]. In the latter, the authors provide an elegant spectral technique to interpret these moments in terms of spherical polynomials and matrix-eigenfunctions of Wishart integral operators; see [37] e.g. proposition 4.3. A drawback of this spectral method is that it requires one to diagonalize and invert complex combinatorial matrices. It is difficult to use this technique to derive estimates w.r.t. the sample size parameter.

Besides the fact that the matrix moments \( M_{n,l}^{[l]}(P) \) can be computed using Isserlis’ theorem, to be the best of our knowledge no explicit and closed form polynomial formulae in terms of \( P \) are known. In further development of this article, we provide estimates of the fluctuation...
matrix moments w.r.t. the sample size in terms of the dominating term of the sum (1.11). A brief description of these estimates are provided in theorem 1.3 below.

To move one step further in our discussion we assume that $Q_i = I$ and $n = 2m$. In this situation the single dominating term in (1.11) is given by the central matrix moments

$$
\theta_{2m,m}(P) = M_{2m,m}(P) \implies \mathbb{E}\left[\mathcal{H}_{N}^{2m}\right] = M_{2m,m}(P) + O\left(\frac{1}{N}\right) I
$$

(1.13)

We say a partition crosses whenever we can find $i < j < k < l$ with $i, k$ in a block and $j, l$ in the other block. Let $\mathcal{N}_n \subset \mathcal{P}_n$ and $\mathcal{N}_{n,m} \subset \mathcal{P}_{n,m}$ be the subsets of non-crossing partitions. Let $\Sigma_n(P)$ be the matrix polynomial given by

$$
\Sigma_n(P) := \text{Tr}(P) \sum_{\pi \in \mathcal{N}_n} \left[ \prod_{i \geq 0} \text{Tr}\left(P^{1+i}r_i(\pi)\right) \right] \frac{P_{|\pi|}}{\text{Tr}(P^{1+|\pi|})}
$$

(1.14)

In the above display, $r_0(\pi) := n - \sum_{i \geq 1} r_i(\pi)$ where $r_i(\pi)$ is the number of blocks of size $i \geq 1$ in the partition $\pi$. Also let $\Sigma_{n,m}^o(P)$ be the matrix polynomial given by

$$
\Sigma_{n,m}^o(P) := \sum_{\pi \in \mathcal{N}_{n,m}} \left[ \prod_{i \geq 1} \text{Tr}(P^i r_i(\pi)) \right] \frac{P^i(\pi)}{\text{Tr}(P^i(\pi))}
$$

(1.15)

In the above display, $\iota(\pi)$ stands for the number of blocks visible from above in the Murasaki diagram associated with $\pi$; see Section 2.3 for examples.

The partition $\Xi(\pi) \in \mathcal{N}_{n+1-m}$ is defined in terms of a circular representation of $\pi$. Firstly, we subdivide the $n$ arcs of $\pi \in \mathcal{N}_{n,m}$ by a new series of $n$ nodes placed clockwise. Then $\Xi(\pi)$ is the coarsest non-crossing partition of these nodes whose chords don’t cross the ones of $\pi$. For a detailed description of the mapping $\Xi$, and examples, we refer Section 2.3, see e.g. (2.10).

The next theorem relates the matrix moments $(\Sigma_n(P), \Sigma_{n,m}^o(P))$ with the matrix moments $(M_{n,m}(P), M_{n,m}^o(P))$ and the ones of the Gaussian matrix $\mathcal{H}$.

**Theorem 1.2.** The central matrix moments $M_{2n,n}(P)$ coincide with the ones of the limiting Gaussian matrix. In addition, we have the matrix moment estimates

$$
M_{2n,n}(P) = \mathbb{E}\left[\mathcal{H}_{2n}^{2n}\right] = \Sigma_n(P) + O\left(r^{n-1}\right) I \quad \text{and} \quad M_{n,m}^o(P) = \Sigma_{n,m}^o(P) + O\left(r^{n-m-1}\right) I
$$

for any $n \geq m \geq 1$, as soon as

$$
\forall n \geq 1, \quad r^{-1} \text{Tr}(P^n) = \tau_n(P) + O\left(r^{-1}\right)
$$

For a more detailed discussion on these matrix moment relations we refer to Section 3.2 and Section 3.3 see e.g. theorem 3.3, theorem 3.5 and theorem 3.7.

Theorem 1.2 together with (1.13) yields the estimates

$$
\begin{align*}
&\quad r^{-(n+1)} \mathbb{E}\left[\mathcal{H}_{2n}^{2n}\right] = \Sigma_n(P) + O\left(r^{-1}\right) I_r \\
&\mathbb{E}\left[\mathcal{H}_{2n}^{2n}\right] = \mathbb{E}\left[\mathcal{H}_{2n}^{2n}\right] + O\left(N^{-1}\right) I \\
&\quad r^{-(n+m+1)} M_{n,m}^o(P) = \Sigma_{n,m}^o(P) + O\left(r^{-1}\right) I_r \quad \text{with} \quad I_r := r^{-1} I
\end{align*}
$$

with the matrix polynomials $(\Sigma_n(P), \Sigma_{n,m}^o(P))$ defined by $(\Sigma_n(P), \Sigma_{n,m}^o(P))$ with the trace operator replaced by the normalized traces $\text{Tr}(Q) := r^{-1} \text{Tr}(Q)$. A more refined estimate between $\mathbb{E}\left[\mathcal{H}_{N}^{2m}\right]$ and $\mathbb{E}\left[\mathcal{H}_{2n}^{2n}\right]$ can be found in theorem 1.5.

The first assertion in theorem 1.2 provides a semi-circle-type asymptotic theorem when the dimension parameter tends to $\infty$. 

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Corollary 1.3. Under the assumptions of theorem 1.2 we have the extended semi-circle law

\[ r^{-1} \mathbb{E} \left( \text{Tr} \left( \frac{\mathcal{M}}{r} \right)^{2n} \right) = \sigma_n(P) + O(r^{-1}) \]

with

\[ \sigma_n(P) := 2 \sum_{\mu} \left( \mu_1^{\mu_2} \cdots \mu_n^{\mu} \right) \tau_{\mu}(P) \quad \text{and} \quad \tau_{\mu}(P) := \prod_{1 \leq i \leq n} \tau_i(P)^{\mu_i} \quad (1.16) \]

In the above display, the summation is taken over all collection of non-negative indices \( \mu = (\mu_1, \ldots, \mu_n) \) such that

\[ \sum_{1 \leq i \leq n} \mu_i = n + 1 \quad \text{and} \quad \sum_{1 \leq i \leq n} i \mu_i = 2n \quad (1.17) \]

For instance, we have

\[
\begin{align*}
\sigma_1(P) &= \tau_1(P)^2 \\
\sigma_2(P) &= 2 \tau_1(P)^2 \tau_2(P) \\
\sigma_3(P) &= 2 \tau_1(P)^3 \tau_3(P) + 3 \tau_1(P)^2 \tau_2(P)^2 \\
\sigma_4(P) &= 2 \tau_1(P)^4 \tau_4(P) + 8 \tau_1(P)^3 \tau_2(P) \tau_3(P) + 4 \tau_1(P)^2 \tau_2(P)^3 \\
\sigma_5(P) &= 2 \tau_1(P)^5 \tau_5(P) + 10 \tau_1(P)^4 \tau_2(P) \tau_4(P) + 5 \tau_1(P)^3 \tau_3(P)^2 \\
&\quad + 20 \tau_1(P)^3 \tau_2(P)^2 \tau_3(P) + 5 \tau_1(P)^2 \tau_2(P)^4
\end{align*}
\]

Observe that \( \sigma_n(\alpha I) = C_n \alpha^{2n} \), for any \( \alpha \geq 0 \). These formulae can be checked combining (3.17) and (3.13) with corollary 3.6. To the best of our knowledge the extended and matrix version of the semi-circle law stated in the above theorem have not been considered in the literature. See also Section 3.2.

Also recall that Carleman’s condition

\[ \sum_{n \geq 1} \sigma_n(P)^{-1/(2n)} = \infty \]

ensures the existence and uniqueness of a random variable with null odd moments and the \((2n)\)-moments \( \sigma_n(P) \) defined in corollary 1.3 (cf. [1,11] and p. 296 in [60]). For instance, when \( P = I \) we have

\[ \sigma_n(I) = C_n \sim \frac{2^{2n}}{n^{3/2} \sqrt{n}} \leq 2^{2n} \quad \implies \quad \sigma_n(I)^{-1/(2n)} \geq \frac{1}{2} \quad \implies \quad \sum_{n \geq 1} \sigma_n(I)^{-1/(2n)} = \infty \]

In this case, the random variable with null odd moments and the \((2n)\)-moments \( \sigma_n(I) \) is given by the semi-circle law (1.1).

Another direct consequence of the above theorem is the Marchenko-Pastur law for non-isotropic Wishart matrices due to Y.Q. Yin [79,80], see also [13] and [62].

Corollary 1.4 ([79, 80]). Consider a collection \( P : r \mapsto P(r) \) of possibly random matrices satisfying the condition stated in (3.14). Let \( N = r/\rho \) be a scaling of the sample size in terms of the dimension associated with some parameter \( \rho > 0 \). For any \( n \geq 1 \) we have the Kreweras-type formula

\[ \lim_{r \to \infty} r^{-1} \text{Tr} \left( \mathbb{E} \left[ P_N^r \right] \right) = \sum_{1 \leq m \leq n} \rho^{n-m} \sum_{\mu^{-}[n] : m + |\mu| = n + 1} K(\mu) \tau_{\mu}(P) \]

with the trace parameters \( \tau_{\mu}(P) \) and the Kreweras numbers \( K(\mu) \) defined in (1.16) and (2.12).
When $P = I$ the above limit result resumes to
$$
\lim_{r \to \infty} r^{-1} \mathbb{E} \left[ \text{Tr} \left( P_N^n \right) \right] = \sum_{1 \leq m \leq n} \rho^{n-m} N_{n,m}
$$
In this situation, we also have the centered version limiting result
$$
\lim_{r \to \infty} r^{-1} \mathbb{E} \left( \text{Tr} \left( (P_N - I)^n \right) \right) = \sum_{1 \leq m \leq \lfloor n/2 \rfloor} \rho^{n-m} R_{n,m}
$$
In the above display, $N_{n,m}$ and $R_{n,m}$ stand for the Narayana and the Riordan numbers defined in (2.12) and (2.14). The matrix version of these isotropic results can be found in corollary 3.3 with the trace-type Marchenko-Pastur law (1.9) a simple corollary. See Section 3.3 for a new matrix version of a non-isotropic Marchenko-Pastur law.

Our third main result concerns moment estimates. We let $\| \cdot \|_F$ and $\| \cdot \|_{op}$ the operator norm and the Frobenius norm. In this notation, we have the following theorem.

**Theorem 1.5.** For some sufficiently small time horizon and for any sufficiently large sample size and any $n \geq 1$ we have the estimates

$$
N \| \mathbb{E} \left[ \mathcal{H}_N^2 \right] - \mathbb{E} \left[ \mathcal{H}_N^{2n} \right] \|_F \leq c_1 \sqrt{N} \left( 2n \right) \text{Tr}(P)^{2n} \\
\sqrt{N} \| \mathbb{E} \left( \exp (t \mathcal{H}_N) \right) - \mathbb{E} \left( \exp (t \mathcal{H}) \right) \|_F \leq c_2 \left( t \text{Tr}(P) \right)^3 \\
\mathbb{E} \| \mathcal{H}_N \|_{op} \wedge \mathbb{E} \| \mathcal{H}_N \|_{op} \leq c_3 \sqrt{r} \lambda_1(P)
$$

for some finite universal constants $c_1, c_2, c_3 < \infty$ whose values don’t depend on the dimension parameter, nor on the parameter $n$.

A more precise statement with a more detailed description of the constants is provided in Section 4 and Section 5; see for instance theorem 4.1, theorem 4.4, corollary 4.5, and theorem 5.4. The operator norm estimate stated in the above theorem extends the norm estimate for isotropic random vectors presented in [43] in the context of Gaussian random matrices. These norm-type bounds are based on non-commutative versions of Khintchine-type inequalities for Rademacher series presented in [43, 44]. More sophisticated approaches based on Burkholder/Rosenthal martingale-type inequalities are also developed in [28, 29]. Nevertheless these inequalities cannot be used to estimate random operator norms and the constants are often not explicit.

The last part of the article is concerned with non-asymptotic exponential concentration inequalities for traces and the operator norm of the fluctuation matrix. In this context, our main results can be stated as follows.

**Theorem 1.6.** For any symmetric matrix $A$, any $\delta \geq 0$ and any sufficiently large sample size the probability of the following events

$$
\left| \text{Tr} \left( A \mathcal{H}_N \right) \right| \leq c_1 \sqrt{(\delta + 1) \left[ \text{Tr}(AP)^2 + \| AP \|_F^2 \right]} \\
\| \mathcal{H}_N \|_{op} \leq c_2 \lambda_1(P) \sqrt{\delta + r} \\
\sup_{1 \leq k \leq r} | \lambda_k(P_N) - \lambda_k(P) | \leq c_2 \lambda_1(P) \sqrt{\delta + r}/N \text{ are greater than } 1 - e^{-\delta}
$$

are greater than $1 - e^{-\delta}$

where $c_1, c_2$ are for some universal constants.

For a precise statement of this result and a detailed description of the constants $c_1, c_2$ we refer to Section 6; see in particular Section 5.2 and theorem 5.2, theorem 5.4, and corollary 5.5.

The proof of the trace-type concentration inequality is based on sub-Gaussian Laplace estimates of well known Wishart trace-type Laplace transforms [55, 56]. See Section 5.2 for a description of these sub-Gaussian estimates; e.g. see (5.7) and the first assertion in theorem 5.2. The operator norm concentration inequality comes from the variational formulation

$$
\| Q \|_{op} := \sup_{x,y \in \mathbb{B}} \langle Qx, y \rangle = \sup_{A \in \mathcal{A}} \text{Tr}(QA)
$$

(1.18)
where \( B \) is the unit ball in \( \mathbb{R}^r \) equipped with the Euclidean distance and \( A \) is the set of matrices
\[
A := \{ A = (xy' + y'x) / 2 : x, y \in B \} \tag{1.19}
\]
The last spectral concentration estimate is a direct consequence of Weyl’s inequality (2.3).

When end this section with some comparisons of the above concentration inequalities with existing results in random matrix theory. When \( P = I \) the joint density of the random eigenvalues of \( P_N \) is explicitly known; see e.g. [3]. Elegant Sanov-type large deviation principles for the spectral empirical measures have been developed by G. Ben Arous and A. Guionnet [4].

The literature also consists of non-asymptotic concentration inequalities for sums of independent random matrices. We refer to the seminal book of J. Tropp [72] for the state of the art on these topics.

Other types of models have been considered in the literature leading to different results. For example, Gaussian concentration inequalities have been derived for Rademacher and Gaussian series associated with deterministic self-adjoint matrices; see e.g. theorem 2.1 in [72]. Matrix Hoeffding, Bernstein and Azuma-type inequalities have been derived for almost surely bounded random matrices; see theorem 2.8 and theorem 8.1 in [72]. The concentration results developed in the present article provide more refined estimates, but they are restricted to random Wishart matrix models.

Lastly, we emphasize that the Laplace transform-type techniques developed in the present article differ from the ones based on Lieb’s inequality (4.13). The latter are often used to control the largest eigenvalue of a random matrix using trace-type estimates; see proposition 4.4 and Section 4.5 in [72].

2 Some preliminary results

2.1 Moments, Laplace transforms and exponential inequalities

We recall that for any non-negative random variable \( Z \) such that
\[
\mathbb{E}(Z^n)^{1/n} \leq z \quad n \quad \text{(with } z = \sqrt{\frac{\pi}{2}}y) \iff \mathbb{E}(Z^n) \leq y^n n! \quad \text{(with } y = e^z)\]

for some parameters \( y, z \neq 0 \) and any \( n \geq 1 \), the probability of the event
\[
\frac{Z}{z} \leq \frac{e^2}{\sqrt{2}} \left[ \frac{1}{2} + \left( \delta + \sqrt{\delta} \right) \right] \quad \text{is greater than } 1 - e^{-\delta}, \text{ for any } \delta \geq 0 \tag{2.1}
\]
To check this claim, notice that for any \( n \geq 1 \) we have
\[
\mathbb{E}((Z/z)^n) \leq n^n \leq \frac{e}{\sqrt{2}} \left( \frac{e}{2} \right)^{\frac{n}{2}} \mathbb{E}(V^{2n})
\]
for some Gaussian and centered random variable \( V \) with unit variance. We check this claim using Stirling approximation
\[
\mathbb{E}(V^{2n}) = 2^{-n} \frac{(2n)!}{n!} \geq e^{-1} 2^{-n} \frac{\sqrt{4\pi n} (2n)^{2n} e^{-2n}}{\sqrt{2\pi n} n^n e^{-n}} = \sqrt{2} e^{-1} \left( \frac{2}{e} \right)^n n^n
\]
The estimate (2.1) is now a direct consequence of [15, Proposition 11.6.6].

Tail probability estimates can be used to obtain moments and Laplace transform estimate. The transfer of exponential tail estimates to these statistical models relies on the integral formulae
\[
\mathbb{E}(Z^n) = n \int_0^\infty z^{n-1} \mathbb{P}(Z \geq z) \, dz \quad \text{and} \quad \mathbb{E}(\exp[tZ]) = 1 + t \int_0^\infty \exp[tz] \mathbb{P}(Z \geq z) \, dz
\]
which are valid for any \( n \geq 1 \) and \( t \in \mathbb{R} \). Laplace transform estimates can also be used to estimate moments using the formula

\[
\mathbb{E}(Z^n) \leq \left( \frac{n}{e} \right)^n \mathbb{E}(\exp[tZ])
\]

which are valid for any \( n \geq 1 \) and \( t \geq 0 \). To check this claim we use the decomposition

\[
\mathbb{E}(Z^n) = \mathbb{E}(Z^n e^{-tZ} e^{tZ}) \leq \sup_{z \geq 0} \left[ z^n e^{-tz} \right] \mathbb{E}(\exp[tZ])
\]

= \exp \left[ -\inf_{z \geq 0} (tz - n \log z) \right] \mathbb{E}(\exp[tZ])

and we note that the infimum is attained at \( z = n/t \).

We end this section with some basic Cramér-Chernov tools to derive quantitative concentration inequalities. We associate with any non-negative convex function \( L \) defined on some domain \( \text{Dom}(L) \subset \mathbb{R}_+ \), \( L(0) = 0 \), the Legendre-Fenchel transform \( L^*(\lambda) := \sup_{t \in \text{Dom}(L)} (\lambda t - L(t)) \) defined for any \( \lambda \geq 0 \). Note that \( L^* \) is a convex increasing function with \( L^*(0) = 0 \) and its inverse \( (L^*)^{-1} \) is a concave increasing function. We let \( L_Z(t) := \log \mathbb{E}(\exp(tZ)) \) be the log-Laplace transform of a random variable \( Z \) defined on some domain \( \text{Dom}(L_Z) \subset \mathbb{R}_+ \). Using the Cramér-Chernov-Chebychev inequality, for any \( \lambda \geq 0 \) and any \( \delta \geq 0 \) we find that

\[
\log \mathbb{P}(Z \geq \lambda) \leq -L_Z^*(\lambda) \iff \mathbb{P}(Z \geq (L_Z^*)^{-1}(\delta)) \leq e^{-\delta}
\]

### 2.2 Matrix norms and spectral analysis

Firstly, we note some general matrix notation. We let \( \mathcal{M}_r \) be the space of \((r \times r)\)-matrices with real entries. We denote by \( S_r \subset \mathcal{M}_r \) the subspace of symmetric \((r \times r)\)-matrices, \( S^0_r \subset S_r \) the subset of semi-definite matrices, and \( S^+_r \subset S^0_r \) the subset of positive semi-definite matrices. We let \( \lambda_i(A) \in \mathbb{C} \), with \( 1 \leq i \leq r \), denote the eigenvalues of \( A \in \mathcal{M}_r \). When \( A \in S_r \) we assume that \( \lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_r(A) \). For any collection \( A = (A_r)_{r \geq 1} \) of \((r \times r)\)-matrices we set

\[
\lambda^*(A_r) = \max_{1 \leq i \leq r} |\lambda_i(A_r)| \quad \text{and} \quad \lambda_*(A) = \sup_{r \geq 1} \lambda^*(A_r) \in [0, \infty]
\]

We have already defined \( \|\cdot\|_{op} \) and \( \|\cdot\|_F \) as the operator norm and the Frobenius norm. The \( n \)-Schatten norm of a matrix \( Q \in S_r \), with \( n \geq 1 \), is defined by

\[
\|Q\|_n := \left[ \sum_{1 \leq k \leq r} |\lambda_k(Q)|^n \right]^{1/n} \Rightarrow \|Q\|_{2n}^2 = \text{Tr}(Q^{2n}) = \|Q\|_F^2
\]

The 2-Schatten norm resumes to the Frobenius norm.

For any \( P, Q \in S^0_r \) and any \( m, n \geq 0 \) we have the estimate

\[
0 \leq \text{Tr}((PQ)^{n+m}) \leq \text{Tr}((PQ)^n) \text{Tr}((PQ)^m)
\]

We check this claim using the fact that

\[
0 \leq \text{Tr}((Q^{1/2}PQ^{1/2})^{n+m}) = \text{Tr}((PQ)^{n+m}) \leq \text{Tr}((Q^{1/2}PQ^{1/2})^m) \text{Tr}((Q^{1/2}PQ^{1/2})^n) = \text{Tr}((PQ)^m) \text{Tr}((PQ)^n)
\]

For any \((r \times r)\)-symmetric matrices \( P \) and \( H \) we have the \( n \)-Weilandt-Hoffman inequality

\[
\sum_{1 \leq k \leq r} |\lambda_k(P + H) - \lambda_k(P)|^n \leq \|H\|_n^n
\]
The case \( n = 2 \) is sometimes called Mirsky’s inequality. In the above display, \( \lambda_k(Q) \) stands for the eigenvalues of a given symmetric matrix \( Q \in S_r \) ranked in decreasing order. We also have Weyl’s inequality

\[
\sup_{1 \leq k \leq r} |\lambda_k(P + H) - \lambda_k(P)| \leq \|H\|_{op}
\]

(2.4)

For a further discussion on the perturbation theory of eigenvalues we refer to [73] and [49].

### 2.3 Tensor products and Fréchet derivatives

The tensor products \((A \otimes B)\) and \((A \bar{\otimes} B)\) and the entry-wise tensor product \((A \otimes B)^\sharp\) of matrices \(A\) and \(B\) with appropriate dimensions are defined by the formulae

\[
(A \otimes B)_{(i,j),(k,l)} = (A \bar{\otimes} B)_{(i,j),(k,l)} = (A \otimes B)^\sharp_{(i,k),(j,l)} = A_{i,k} B_{j,l}
\]

(2.5)

We also consider the symmetric tensor product

\[
4(A \otimes B) = (A \otimes B) + (B \otimes A) + (A \bar{\otimes} B) + (B \bar{\otimes} A)
\]

(2.6)

Let \( B_1, B_2 \) be a couple of Banach spaces equipped with some norm \( \|\cdot\|_{B_i} \), with \( i = 1, 2 \); also let \( \mathcal{O}_1 \subset B_1 \) be an open subset of \( B_1 \). We let \( \mathcal{L}(B_1, B_2) \) be the set of bounded linear functional from \( B_1 \) into \( B_2 \).

We recall that a mapping \( \Upsilon : B_1 \rightarrow B_2 \) is Fréchet differentiable at some \( A \in \mathcal{O}_1 \) if there exists a continuous linear functional \( \nabla \Upsilon(A) \in \mathcal{L}(B_1, B_2) \) such that

\[
\lim_{\|H\|_{B_1} \rightarrow 0} \frac{\|H\|_{B_1}^{-1} \|\Upsilon(A + H) - \Upsilon(A) - \nabla \Upsilon(A) \cdot H\|_{B_2}}{\|H\|_{B_1}} = 0
\]

(2.7)

The mapping is said to be twice Fréchet differentiable at \( A \in \mathcal{O}_1 \) when \( \Upsilon \) and the mapping

\[
\nabla \Upsilon : A \in \mathcal{O}_1 \mapsto \nabla \Upsilon(A) \in \mathcal{L}(B_1, B_2)
\]

is also Fréchet differentiable, and so on. Given some Fréchet differentiable mapping \( \Upsilon \) at the third order, at some \( A \in \mathcal{O}_1 \) for any \( H \in B_1 \) we have

\[
\Upsilon(A + H) = \Upsilon(A) + \nabla \Upsilon(A) \cdot H + \frac{1}{2} \nabla^2 \Upsilon(A) \cdot (H, H) + \nabla^3 \Upsilon[A, H]
\]

with the remainder functional in the Taylor expansion given

\[
\nabla^3 \Upsilon[A, H] := \frac{1}{3!} \int_0^1 (1 - \epsilon)^2 \nabla^3 \Upsilon(A + \epsilon H) \cdot (H, H, H) \, d\epsilon
\]

We also consider the multi-linear operator norm

\[
\|\nabla^n \Upsilon(P)\| = \sup_{\|H\|_{B_1} = 1} \|\nabla^n \Upsilon(P) \cdot H^\otimes n\|_{B_2}
\]

We says \( \Upsilon \) satisfies the third-order polynomial growth condition when we have

\[
\forall n = 1, 2 \quad \|\nabla^n \Upsilon(P)\| \leq \kappa_n(A) \quad \text{and} \quad \|\nabla^3 \Upsilon[A, H]\| \leq \kappa_3(A) \quad (1 + \|H\|_{B_1}^{\alpha}) \quad \|H\|_{B_1}^{\frac{3}{2}}\]

(2.8)

for some parameter \( \alpha > 0 \) and some constants \( \kappa_n(A) \) whose values only depends on \( A \). For instance, the principal square root functional

\[
\psi : Q \in S_r^+ \mapsto \psi(Q) = Q^{1/2} \in S_r^+
\]
is Fréchet differentiable at any order and satisfies the third-order polynomial growth condition stated above. More precisely, as shown in [11] the first and second order derivative given for any \((A, H) \in (S_r^+ \times S_r)\) s.t. \(A + \epsilon H \in S_r^0\) for any \(\epsilon \in [0, 1]\) by
\[
\nabla \psi(A) \cdot H = \int_0^\infty e^{-t\psi(A)} H e^{-t\psi(A)} \, dt
\]
\[
2^{-1} \nabla^2 \psi(A) \cdot (H, H) := -\nabla \psi(A) \cdot (\nabla \psi(A) \cdot H)^2
\]  
(2.9)

In addition, we have the second order approximation
\[
\|\psi(A + H) - \psi(A) - \nabla \psi(A) \cdot H - 2^{-1} \nabla^2 \psi(A) \cdot (H, H)\| \leq \frac{c^2}{4}\lambda_{\min}(A)^{-5/2} \|H\|^3
\]  
(2.10)

where \(c = \sqrt{r}\) for the Frobenius norm \(\|\cdot\| = \|\cdot\|_F\), and \(c = 1\) for the \(L_2\)-norm \(\|\cdot\| = \|\cdot\|_2\). The first and second derivatives can be estimated by the operator norm inequalities:
\[
\|\nabla \psi(A)\| \leq 2^{-1} \lambda_{\min}(A)^{-1/2} \quad \text{and} \quad \|\nabla^2 \psi(A)\| \leq \frac{K}{4} \lambda_{\min}(A)^{-3/2}
\]

Thus we can identify \(\nabla \Upsilon(Q)\) with the tensor
\[
\nabla \Upsilon(Q)_{(i,j),(k,l)} := (\nabla \Upsilon(Q) \cdot e_{i,j})_{k,l} = e_k \cdot (\nabla \Upsilon(Q) \cdot e_{i,j}) \cdot e_l
\]

In this notation, we have
\[
H = \sum_{i,j} H_{i,j} e_{i,j} \implies (\nabla \Upsilon(Q) \cdot H)_{(k,l)} = \sum_{i,j} H_{i,j} \nabla \Upsilon(Q)_{(i,j),(k,l)} = (H \nabla \Upsilon(Q))_{(k,l)}
\]

In the same vein, we can identify \(\nabla^2 \Upsilon(Q)\) with the tensor
\[
\nabla^2 \Upsilon(Q)_{((i,j),(k,l)),(m,n)} := (\nabla^2 \Upsilon(Q) \cdot (e_{i,j}, e_{k,l}))_{m,n}
\]
\[
\implies (\nabla^2 \Upsilon(Q) \cdot (H_1, H_2)) = (H_1 \otimes H_2)^\Upsilon \nabla^2 \Upsilon(Q)
\]  
(2.11)

### 2.4 Partitions and Bell polynomials

Let \(P_{n,m}(\mu) \subset P_{n,m}\) be the subset of partitions of type \(\mu \vdash n\), with \(|\mu| = \mu_1 + \ldots + \mu_n = m\) blocks. The cardinality of these set partitions are given by the Bell numbers \(B(n)\) and the Stirling numbers \(S(n, m)\) of the second kind; that is, we have that
\[
B(n) := |P_n| = \sum_{1 \leq m \leq n} S(n, m) \quad \text{with} \quad S(n, m) := |P_{n,m}|
\]

Let \(N_{n,m} \subset P_{n,m}\) the subset of non-crossing partitions, and \(N_{n,m}(\mu) \subset P_{n,m}(\mu)\) the subset of non-crossing partitions of type \(\mu\). The number of non-crossing partitions of \(n\) are computed using the Narayana and the Kreweras numbers
\[
N_{n,m} := |N_{n,m}| = \frac{1}{n} \binom{n}{m-1} \binom{m}{n} = \frac{1}{m} \binom{m-1}{m-1} \binom{n}{m-1} = |N_{n+1,m-1}|
\]
\[
K(\mu) := |N_{n,m}(\mu)| = \frac{1}{(n+1-m)!} \binom{n}{\mu_1 \mu_2 \ldots \mu_n}
\]  
(2.12)

for any \(\mu \vdash n\), with \(|\mu| = m \leq n\). A proof of the above formula can be found in [31], e.g. corollary 9.12, see also [11, 11], the pioneering article by G. Kreweras [35] and the survey of R. Simion on non-crossing partitions [63].
Let $Q^+_{n,m} \subset Q_{n,m}$ be the subset of non-crossing partitions, and $Q^-_{n,m} \subset Q_{n,m}$ the subset of crossing partitions without singleton. Notice that $Q^+_{n,m} = 0$ as soon as $m > \lfloor n/2 \rfloor$. In addition we have

$$\pi \in Q_{n,m} \quad \text{and} \quad m > \lfloor n/2 \rfloor \implies \exists i \in [m] : |\pi_i| = 1 \quad (2.13)$$

The number of non-crossing partitions of $[n]$ without singletons is given by the Riordan numbers

$$R_n = \sum_{1 \leq m \leq \lfloor n/2 \rfloor} R_{n,m} \quad \text{with} \quad R_{n,m} = |Q^+_{n,m}| = \frac{1}{n+1} \binom{n+1}{m} \binom{n-1-m}{m-1} \quad (2.14)$$

A proof of the above assertion can be found in [33].

For a centered Gaussian random variable $V$ with unit variance we have the estimates

$$R_{2n,n} = N_{2n,n}(0, n, 0, \ldots, 0) = C_n \leq |Q_{2n,n}| = 2^{-n} (2n)_n = \mathbb{E}(V^{2n}) = v_{2n}$$

with the Catalan numbers $C_n$ defined in [1,3].

We also recall that Stirling numbers of the first kind are denoted by $s(n, m)$ and the Pochhammer symbol is defined by $(n)_m := n!/(n-m)!$. We also denote by $r_i(\pi)$ the number of blocks of size $i \geq 1$ in the partition $\pi$.

For a detailed discussion, and the combinatorial interpretation of non-crossing partitions, Catalan, Riordan and Narayana numbers, we refer to the series of articles [33, 35, 53, 61, 65, 66, 67, 78]. For instance, Narayana numbers $N_{n,m}$ count the number of expressions with $n$ pairs of matched parenthesis and $m$ different nestings. These numbers also count the number of ordered trees with $n$ edges and $m$ leaves; as well as the number of Dicks paths of length $n$ and $m$ distinct picks. Recall that Dicks paths are staircase walks with $2n$ steps in the plane starting at the origin $(0, 0)$ and ending in $(n, n)$, and staying above the diagonal (they may touch the diagonal). The Riordan number $R_n$ coincides with the number of plane trees with $n$ edges, in which no vertex has outdegree one. The $n$-th Riordan number also counts Motzkin paths of length $n$ with no horizontal steps of null height. Recall that Motzkin paths are lattice paths in the plane starting from the origin up to $(n, 0)$, using up steps $(1, 1)$, down steps $(1, -1)$, and horizontal steps $(1, 0)$, and never going below the $x$-axis. Identifying $k \in [n]$ with $e^{-2ik\pi/n}$, then non-crossing partitions can also be seen as compact subsets of the unit disk [34].

We also consider the complete Bell polynomials on a commutative algebra $A$ given by $B_0 = I$ the unit neutral state, and for any $n \geq 1$ and any $x = (x_1, \ldots, x_n) \in A^n$ by the formula

$$B_n(x) = \sum_{1 \leq k \leq n} B_{k,n}(x) = n! \sum_{1 \leq i \leq n} \prod_{1 \leq i \leq n} \left[ \frac{1}{i!} \left( \frac{x_i}{i!} \right)^{l_i} \right] \quad (2.15)$$

with the partial Bell polynomials

$$B_{k,n}(x) = \frac{n!}{k!} \sum_{m_1 + \ldots + m_k = n, m_i \geq 1} \frac{x_{m_1} \ldots x_{m_k}}{m_1! \ldots m_k!} \quad (2.16)$$

We recall that

$$\exp \left[ \sum_{n \geq 1} \frac{t^n}{n!} x_n \right] = \sum_{n \geq 0} \frac{t^n}{n!} y_n \quad \text{with} \quad y_n = B_n(x_1, \ldots, x_n)$$

When $y_0 = I$ we also have the inverse relations

$$x_n = \sum_{1 \leq k \leq n} (-1)^{k-1} (k-1)! B_{n,k}(y_1, \ldots, y_n) \quad (2.17)$$

We have the following technical lemma whose proof is given in the Appendix.
Lemma 2.1. Assume that $1 \leq k < p$ we have $x_k = 0$ and $\forall n \geq p$ we have $|x_n| \leq n! \rho \beta^n$ for some $p \geq 1$ and non-negative parameters $\rho, \beta \geq 0$. Then, for any $1 \leq q < p$ we have

$$|B_{pn}(x) - \frac{(np)!}{n!} \left( \frac{x_p}{p!} \right)^n| \leq \frac{(np)!}{2} \left( 2\beta \right)^{pn} \sum_{1 \leq k < n} \frac{1}{k!} \left( \frac{\rho}{2p-1} \right)^k$$

$$|B_{pn+q}(x)| \leq \frac{(np+q)!}{2} \left( 2\beta \right)^{pn+q} \sum_{1 \leq k < n} \frac{1}{k!} \left( \frac{\rho}{2p-1} \right)^k$$

In addition, when $x_n \geq 0$ for any $n \geq 1$ we have $B_n(x) \geq 0$ and $B_{pn}(x) \geq \frac{(np)!}{n!} \left( \frac{x_p}{p!} \right)^n$.

2.5 Some combinatorial constructions

Given some partitions $\pi \in \mathcal{P}_{n,m}$ and $\pi' \in \mathcal{P}_{n',m'}$ we let $\pi \oplus \pi' \in \mathcal{P}_{n+n',m+m'}$ the partition of $[n+n']$ with $m+m'$ blocks given by

$$\forall i \in [m+m'] \quad (\pi \oplus \pi')_i = \begin{cases} \pi_i & \text{if } i \in [n] \\ n+\pi'_i & \text{if } i \in n+[n'] \end{cases}$$

We also consider the closure operator

$$\text{cl} : \pi \in \mathcal{N}_{n,m} \mapsto \text{cl} (\pi) \in \mathcal{N}_{n+1,m}$$

defined by the blocks

$$\text{cl}(\pi)_{i+1} = \begin{cases} 1+\pi_i & \text{if } n \notin \pi_i \text{ and } 1 \leq i \leq m \\ \{1\} \cup (1+\pi_j) & \text{if } n \in \pi_j \text{ and } i = 0 \end{cases}$$

For example, the partition $\pi = (\pi_1, \pi_2, \pi_3, \pi_4) \in \mathcal{N}_{7,4}$ with blocks,

$$\pi_1 := \{1, 4\} \leq \pi_2 := \{2, 3\} \leq \pi_3 := \{5\} \leq \pi_4 := \{6, 7\}$$

is defined by the Murasaki diagram presented in Figure 1. We recall that $\iota(\pi)$ is the number of blocks visible from above in the Murasaki diagram associated with $\pi$

![Figure 1: Murasaki diagram representation of $\pi$](image)

Observe that the partition $\pi$ in (2.18) has $\iota(\pi) = 3$ blocks visible from above. Its closure

$$7 \in \pi_4 \implies \text{cl}(\pi) = (\{1, 7, 8\}, \{2, 5\}, \{3, 4\}, \{6\}) \in \mathcal{N}_{8,4}$$

is defined by the Murasaki diagram presented in Figure 2

![Figure 2: Murasaki diagram representation of $\pi$ and $\text{cl}(\pi)$](image)

Writing clockwise the nodes the partition $\pi$ defined in (2.18) also has the circle representation given in Figure 3.
Given a circular representation of a partition \( \pi \in \mathcal{N}_{n,m} \), we subdivide each of the \( n \) arcs by a new blue node. The blue node which subdivides the arc \((1,2)\) is labelled with 1. The remaining subdivision of nodes \(2, \ldots, n\) are placed clockwise. We also consider the mapping

\[
\Xi : \pi \in \mathcal{N}_{n,m} \mapsto \Xi(\pi) \in \mathcal{N}_{n,n+1-m}
\]

(2.19)
defined by taking the coarsest non-crossing partition \( \Xi(\pi) \) of the blue nodes whose chords don’t cross the chord representation of \( \pi \). For instance the blue nodes and the circular representations of \( \Xi(\pi) \) associated with the partition \( \pi \) defined in Figure 3 are given in Figure 4. Up to a change of numbering \( \Xi \) coincides with the order-reversing involution introduced in [65].

![Figure 3: Circular representation of \( \pi \)](image)

![Figure 4: Circular representations of \( \pi \) and \( \Xi(\pi) \)](image)

Observe that for any partition \( \pi \) of \([n]\) the block of \( \Xi(\pi) \) containing the node with label \( n \) is a block of size \( \iota(\pi) \). In other words, the size of the block of \( \Xi(\pi) \) containing the node with label \( n \) characterizes the number of visible blocks of \( \pi \) from above.

Next we analyze the difference between \( \Xi(\pi \oplus \pi') \) and \( \Xi(\pi) \oplus \Xi(\pi') \). We start with a simple example. Consider the partitions with Murasaki diagrams in Figure 5 and Figure 6.

![Figure 5: Murasaki diagram of \( \pi \in \mathcal{N}_{12,6} \), with \( \iota(\pi) = 3 \)](image)

![Figure 6: Murasaki diagram of \( \pi' \in \mathcal{N}_{11,6} \), with \( \iota(\pi') = 3 \)](image)

The Murasaki diagram of \( \pi \oplus \pi' \in \mathcal{N}_{23,12} \) is given as in Figure 7.
The Murasaki diagram of $\Xi(\pi) \in \mathcal{N}_{12,6}$ and $\Xi(\pi') \in \mathcal{N}_{11,6}$ are defined in Figure 8 and Figure 9. Observe that the size of the blue block of $\Xi(\pi)$ containing the node “12” and the one of $\Xi(\pi')$ containing the node “11” coincides with $\iota(\pi)$ and respectively $\iota(\pi')$.

In terms of Murasaki diagrams, $\Xi(\pi \oplus \pi')$ is deduced from $\Xi(\pi) \oplus \Xi(\pi')$ by merging the blue blocks of sizes $\iota(\pi)$ and $\iota(\pi')$. This shows that $\Xi(\pi \oplus \pi')$ and $\Xi(\pi) \oplus \Xi(\pi')$ have the same combinatorial structure, but $\Xi(\pi \oplus \pi')$ loses a block of size $\iota(\pi)$ and a block of size $\iota(\pi')$, and gains a new block of size $\iota(\pi) + \iota(\pi')$ (so that the total number of blocks is kept unchanged). See Figure 11 (and compare with Figure 7 and Figure 10).

Non-crossing partitions can be generated sequentially using the formula

$$\forall 0 \leq m \leq n \quad \mathcal{N}_{n+1,m+1} = [\mathcal{N}_{n,m} \oplus \text{cl}(\mathcal{N}_{0,0})] \cup \bigcup [\mathcal{N}_{n_1,m_1} \oplus \text{cl}(\mathcal{N}_{n_2,m_2})] \quad (2.20)$$

In the above display, the second set-union is taken over all non-negative parameters $0 \leq m_1 \leq n_1$ and $0 \leq m_2 \leq n_2$ such that $n_1 + n_2 = n$ and $m_1 + m_2 = m + 1$ (2.21).

We also use the conventions $\mathcal{N}_{0,0} = \{\pi_0\}$, where $\pi_0 = \{0\}$ stands for the null partition on $[0]$, as well as $\{0\} \oplus \pi = \pi$. In addition for any $n \geq 1$ we set $\mathcal{N}_{n,0} = \emptyset$ so that $\emptyset \oplus \pi = \emptyset$. 

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For instance, the sets of non-crossing partitions with one or two elements are given by the formulae

\[ N_{1,1} = N_{0,0} \oplus cl(N_{0,0}) = cl(N_{0,0}) = \{\{1\}\}, \]

\[ N_{2,1} = N_{0,0} \oplus cl(N_{1,1}) = cl(N_{1,1}) = \{\{1, 2\}\}, \]

\[ N_{2,2} = N_{1,1} \oplus cl(N_{0,0}) = N_{1,1} \oplus N_{1,1} = \{\{1, 2\}\} \]

In the same vein, the sets of non-crossing partitions on \([3]\) are given by the decompositions

\[ N_{3,1} = N_{0,0} \oplus cl(N_{2,1}) = cl(N_{2,1}) = \{\{1, 2, 3\}\}, \]

\[ N_{3,2} = [N_{2,1} \oplus cl(N_{0,0})] \cup [N_{1,1} \oplus cl(N_{1,1})] \cup [N_{0,0} \oplus cl(N_{2,2})] \]

\[ = \begin{array}{ccc}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{array} \]

\[ N_{3,3} = N_{2,2} \oplus cl(N_{0,0}) = \begin{array}{ccc}
2 & 3 \\
1 & 2 \\
1 & 2 \\
\end{array} \]

We also have the decompositions

\[ N_{4,1} = N_{0,0} \oplus cl(N_{3,1}) \]

\[ N_{4,2} = [N_{3,1} \oplus cl(N_{0,0})] \cup [N_{2,1} \oplus cl(N_{1,1})] \cup [N_{1,1} \oplus cl(N_{2,1})] \cup [N_{0,0} \oplus cl(N_{3,2})] \]

\[ N_{4,3} = [N_{3,2} \oplus cl(N_{0,0})] \cup [N_{2,2} \oplus cl(N_{1,1})] \cup [N_{1,1} \oplus cl(N_{2,2})] \cup [N_{0,0} \oplus cl(N_{3,3})] \]

\[ N_{4,4} = N_{3,3} \oplus cl(N_{0,0}) \]

as well as the partitioning of non-crossing partitions on \([3]\) given by

\[ N_{5,1} = N_{0,0} \oplus cl(N_{4,1}) \]

\[ N_{5,2} = [N_{4,1} \oplus cl(N_{0,0})] \cup [N_{3,1} \oplus cl(N_{1,1})] \cup [N_{2,1} \oplus cl(N_{2,1})] \]

\[ \quad \cup [N_{1,1} \oplus cl(N_{3,1})] \cup [N_{0,0} \oplus cl(N_{4,2})] \]

\[ N_{5,3} = [N_{2,2} \oplus cl(N_{2,1})] \cup [N_{1,1} \oplus cl(N_{2,2})] \cup [N_{1,1} \oplus cl(N_{3,2})] \cup [N_{0,0} \oplus cl(N_{4,3})] \]

\[ N_{5,4} = [N_{3,3} \oplus cl(N_{1,1})] \cup [N_{2,2} \oplus cl(N_{2,2})] \cup [N_{1,1} \oplus cl(N_{3,3})] \cup [N_{0,0} \oplus cl(N_{4,4})] \]

\[ N_{5,5} = N_{4,4} \oplus cl(N_{0,0}) \]

### 2.6 Fluctuation and bias approximations

Let \( \mathcal{B} \) be some Banach space equipped with some norm \( \| \cdot \|_{\mathcal{B}} \).

**Theorem 2.2.** For any Fréchet differentiable mapping \( \Upsilon : S_r \to \mathcal{B} \) at the third order, and satisfying the third-order polynomial growth condition (2.8), we have

\[ \sqrt{N} \left[ \Upsilon(P_N) - \Upsilon(P) \right] \overset{\mathcal{D}}{\to}_{N \to \infty} \nabla \Upsilon(P) \cdot \mathcal{H} \]

In addition, we have the bias estimate

\[ \| B (\Upsilon(P_N)) - \Upsilon(P) - N^{-1} (P \otimes P) \nabla^2 \Upsilon(P)\|_{\mathcal{B}} \leq c \kappa_3(P) N^{-3/2} \]

with the tensor composition (2.11), and the parameter \( \kappa_3(A) \) introduced in (2.3).
Proof. The first assertion is a direct consequence of the continuous mapping theorem. In terms of the tensor composition \((2.11)\), we also have the bias estimate

\[
N^{3/2} \left\| \mathbb{E} (\Upsilon(P_N)) - \Upsilon(P) - N^{-1} (P \otimes P) \nabla^2 \Upsilon(P) \right\|_B \leq c_1 \kappa_3(A) \left( 1 + N^{-\alpha} \mathbb{E} \left[ \|\mathcal{H}_N\|_{F_0}^{1/p} \right] \right) \mathbb{E} \left[ \|\mathcal{H}_N\|_{F_0}^{3 \cdot p^{1/q}} \right] \leq c_2 \kappa_3(A)
\]

This ends the proof of the theorem. \(\square\)

For instance, when \((B, \|\cdot\|_B) = (\mathcal{S}_r^+, \|\cdot\|_F)\) and \(\Upsilon = \psi\) is the square-root functional, we find that

\[
\sqrt{N} \left[ \sqrt{P_N} - \sqrt{P} \right] \xrightarrow{N \to \infty} \int_0^\infty e^{-t\psi(P)} \mathcal{H} e^{-t\psi(P)} \, dt
\]

and the negative bias estimate

\[
N^{3/2} \left\| \mathbb{E} \left( \sqrt{P_N} - \sqrt{P} \right) + N^{-1} \nabla \psi(P) \cdot \mathbb{E} \left[ (\nabla \psi(P) \cdot (X - P))^2 \right] \right\|_F \leq \frac{r}{4} \lambda_{\min}(P)^{-5/2} \mathbb{E} \left( \|\mathcal{H}_N\|^3_{F_0} \right) \leq \frac{r}{4} \lambda_{\min}(P)^{-5/2} \left( 1 + \mathbb{E} \left( \text{Tr}(\mathcal{H}_N^2) \right) \right)
\]

We conclude that

\[
\left\| \mathbb{E} \left( \sqrt{P_N} - \sqrt{P} \right) + \frac{1}{N} \nabla \psi(P) \cdot \mathbb{E} \left[ (\nabla \psi(P) \cdot (X - P))^2 \right] \right\|_F \leq \frac{r}{4} \frac{1}{N \sqrt{N}} \lambda_{\min}(P)^{-5/2} \left[ \text{Tr}(P^2) + \text{Tr}(P)^2 \right]
\]

We check \((1.4)\) using the matrix moment formula \((6.6)\). Choosing

\[(B, \|\cdot\|_B) = (\mathbb{R}, |\cdot|) \quad \text{and} \quad \Upsilon(A) = \text{Tr}(AB) \Rightarrow \nabla \Upsilon(A) \cdot H = \text{Tr}(HB)\]

for some given \((r \times r)\)-matrix \(B\), we find that

\[
\sqrt{N} \left[ \text{Tr}(P_NB) - \text{Tr}(PB) \right] \xrightarrow{\text{law}} \frac{1}{N} \sum_{1 \leq i \leq N} \text{Tr}([X_i - P]B) = \frac{1}{N} \sum_{1 \leq i \leq N} [X_i'BX_i - \mathbb{E} \left[ \text{Tr}(X_i'BX) \right]] \xrightarrow{N \to \infty} \text{Tr}(HB) \sim \mathcal{N} \left( 0, 2 \text{Tr}((PB)^2) \right)
\]

3 Matrix moment polynomial formulae

3.1 Fluctuation and partition-type matrix moments

Note the collection of matrix moments \(M_{n,m}(P)\) and \(M_{n,m}^\pm(P)\) defined for any \(1 \leq m \leq n\) by

\[
M_{n,m}(P) := M_{n,m}^n(P) + M_{n,m}^+(P) \quad \text{with} \quad M_{n,m}^+(P) := \sum_{\pi \in \mathcal{Q}_{n,m}^+} M_\pi(P)
\]

Notice that \(M_{n,m}(P)\) is symmetric but \(M_{\pi}(P)\) is not necessary symmetric. We also consider the “un-centered” matrix moments \(M_{n,m}(P)\), \(M_{n,m}^\pm(P)\) defined as \(M_{n,m}(P)\) and \(M_{n,m}^\pm(P)\) by replacing \(M_\pi(P)\) by the \(\pi\)-matrix “un-centered” moment \(M_\pi^0(P)\) defined in \((1.10)\).
For any $1 \leq m \leq \lfloor n/2 \rfloor$ we also have the rather crude estimates
\[
\|M_{2n,n}(P)\|_F \leq v_{2n} \left[ \text{Tr}(P^2) + \text{Tr}(P)^2 \right]^n \quad \text{and} \quad \|M_{n,m}(P)\|_F \leq \frac{n!}{m!} 2^{3n-(2m+1)} \text{Tr}(P)^n
\] (3.2)

A derivation of this estimate is technical, thus it is housed in the Appendix.

When $P = I$, integrating sequentially adjacent non-crossing blocks we find that
\[
M_{2n,n}^+(I) := C_n (1 + r)^n I \implies \tag{3.3}
\]
This formula can also be proved using the Catalan-type recursive formulae (3.10) provided in theorem [3.3]. For any $\pi \in \mathcal{N}_{n,m}(\mu)$ we also have
\[
M_\pi^0(I) = \prod_{1 \leq k \leq n} E(\chi^k)^{\mu_k}
= r^{n-m} \prod_{1 \leq k \leq n} \left[ \prod_{0 \leq i < k} \left( 1 + \frac{2i}{r} \right) \right]^{\mu_k} I = r^{n-m} (1 + O(1/r)) I
\] (3.4)

The proof of the above assertion follows standard calculations for isotropic matrix moments. For instance we can use the $\chi$-square decomposition (6.30) discussed in the Appendix.

For crossing partitions we also have
\[
\pi \in \mathcal{P}_{n,m} - \mathcal{N}_{n,m} \implies M_\pi^0(I) = c_\pi r^{n-m} (1 + O(1/r)) r^{-1} I
\] (3.5)

for some finite constant $c_\pi$ whose values doesn’t depend on $r$. This implies that
\[
r^{-(n-m+1)} \sum_{\pi \in \mathcal{N}_{n,m}(\mu)} \text{Tr}(M_\pi^0(I)) = N_{n,m}(\mu) + O(1/r) \tag{3.6}
\]
\[
\implies r^{-(n-m+1)} \sum_{\pi \in \mathcal{N}_{n,m}} \text{Tr}(M_\pi^0(I)) = N_{n,m} + O(1/r) = r^{-(n-m+1)} \text{Tr}(M_{n,m}^0(I))
\]

The proof of (3.3) is provided in the Appendix.

In the same vein, for any $\pi \in \mathcal{N}_{n,m}(\mu)$ we have
\[
M_\pi(I) = 1_{\mu_1=0} \prod_{2 \leq k \leq n} \left[ \left( \frac{1}{r} B_k \left( r - 1, r 2^{k-1} (2-1)! \ldots, r 2^{k-1} (k-1)! \right) + (-1)^k \left( 1 - \frac{1}{r} \right) \right) \right]^{\mu_k} I
\] (3.7)

with the complete Bell polynomials (cf. (2.13)). In addition, for crossing partitions we have
\[
\pi \in \mathcal{Q}_{n,m}^- \implies M_\pi(I) = c_\pi r^{n-m} (1 + O(1/r)) r^{-1} I
\] (3.8)

for some finite constant $c_\pi$ whose values doesn’t depend on $r$. The proof of (3.8) is provided in the Appendix. This implies that
\[
r^{-(n-m+1)} \sum_{\pi \in \mathcal{N}_{n,m}(\mu)} \text{Tr}(M_\pi(I)) = 1_{\mu_1=0} N_{n,m}(\mu) + O(1/r) \tag{3.9}
\]
\[
\implies r^{-(n-m+1)} \sum_{\pi \in \mathcal{Q}^+_{n,m}} \text{Tr}(M_\pi(I)) = R_{n,m} + O(1/r) = r^{-(n-m+1)} \text{Tr}(M_{n,m}(I))
\]

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with the Narayana and the Riordan numbers introduced in (2.12) and (2.14). The proof of (3.7) is provided in the Appendix.

The computation of the matrix moments of the fluctuation matrices \( \mathcal{H}_N \) requires some more sophisticated combinatorial techniques.

Next theorem provides a centered matrix moment interpretation of \( \mathbb{E}(\mathcal{H}^{2n}) \), a polynomial decomposition of centered matrix moments.

**Theorem 3.1.** For any \( n \geq 1 \) and \( N \geq 1 \) we have matrix polynomial formulae

\[
\mathbb{E} \left( (P_N - P)^n \right) = \frac{1}{N^n} \sum_{1 \leq m \leq [n/2]} (N)_m \ M_{n,m}(P) = \sum_{1 \leq m \leq [n/2]} \frac{1}{N^{n-m}} \ \partial_{n,m}(P)
\]

with the differential-type matrix moments \( \partial_{n,m}(P) \) defined in (1.11) and (1.13). In addition, for any \( 1 \leq m \leq [n/2] \) we also have the inversion formula

\[
M_{n,m}(P) = \sum_{1 \leq k \leq m} (-1)^{m-k} \ \frac{k^{n/2}}{k!(m-k)!} \ \mathbb{E}(H^n_k) \quad \text{and} \quad M_{2n,n}(P) = \mathbb{E}(\mathcal{H}^{2n}) \quad (3.10)
\]

The proof of the above theorem is provided in Section 6.1. Combining (1.3) with the above theorem we compute the covariance matrix moments.

**Corollary 3.2.** For any \( 2N \geq n \geq 1 \) we have

\[
\mathbb{E} \left( P_N^n \right) = \frac{1}{N^n} \sum_{1 \leq m \leq n} (N)_m \ M^o_{n,m}(P) = \sum_{1 \leq m \leq n} \frac{1}{N^{n-m}} \ \partial^o_{n,m}(P)
\]

with the differential-type matrix moments given by

\[
\partial^o_{n,m}(P) := \sum_{1 \leq k \leq (n-m) \wedge m} (n)_{k+n-m} \ \partial_{k+n-m,k}(P) = \sum_{m \leq k \leq n} s(k,m) \ M^o_{n,k}(P)
\]

for any \( 1 \leq m \leq n \) with the convention \( \partial^o_{n,n}(P) = P^n \) when \( m = n \).

The proof of the corollary is provided in the Appendix.

**Corollary 3.3.** Let \( P = I \) and let \( N = r/\rho \) be a scaling of the sample size in terms of the dimension associated with some parameter \( \rho > 0 \).

\[
\mathbb{E} \left( P_N^n \right) = \sum_{1 \leq m \leq n} \rho^{n-m} \ N_{n,m} \ (1 + O(1/r)) \ I
\]

\[
\mathbb{E} \left( (P_N - I)^n \right) = \sum_{1 \leq m \leq [n/2]} \rho^{n-m} \ R_{n,m} \ (1 + O(1/r)) \ I
\]

The first assertion in corollary 3.3 is a direct consequence of (3.4) and (3.5). The second assertion is a consequence of (3.7) and (3.8). Using (2.12) and taking the trace in the first estimate we obtain the Marchenko-Pastur law (1.9).

The matrix polynomial formulae (1.11) and corollary 3.2 can be extended without further work to matrix moments associated with some collection of matrices \( Q_n \) and given in (1.11).

When \( N = 1 \) and \( P = I \) the matrix moments

\[
\mathbb{E}(H^n_1) = \mathbb{E}((X - I)^n) = M_{n,1}(I)
\]

resume to the formula

\[
\mathbb{E} \left( (X - I)^n \right)
\]

\[
= \left[ \frac{1}{r} \ B_n \left( r - 1, \ r \ 2^{2-1} \ (2 - 1)!, \ldots, \ r \ 2^{n-1} \ (n-1)! \right) \right] \ I \underset{r \to \infty}{\longrightarrow} 1_{n \geq 1}
\]

(3.11)
where $B_n$ stands for the complete Bell polynomials. For more general covariance matrices we have the matrix polynomial formula

$$0 \leq \mathbb{E}((X - P)^n) = \sum_{1 \leq k \leq n} \rho_{k,n}(P) P^k \leq \mathbb{E}(X^n) - P^n \quad (3.12)$$

for some non-negative parameters $\rho_{k,n}(P)$ which can be computed explicitly in terms of the traces of the covariance matrix powers. For instance, we find that

$$\begin{align*}
\mathbb{E}((X - P)^2) &= P^2 + P \text{Tr}(P) \\
\mathbb{E}((X - P)^3) &= 4 P^3 + 2 P^2 \text{Tr}(P) + P \left[ \text{Tr}(P)^2 + \text{Tr}(P^2) \right] \\
\mathbb{E}((X - P)^4) &= 25 P^4 + 15 \text{Tr}(P) P^3 + 2 \left[ 2 \text{Tr}(P)^2 + 3 \text{Tr}(P^2) \right] P^2 \\
&+ \left[ \text{Tr}(P)^3 + 4 \text{Tr}(P) \text{Tr}(P^2) + 5 \text{Tr}(P^3) \right] P
\end{align*}$$

The matrix moments of the rank-one matrices $X$ can also be computed by the formula

$$\begin{align*}
\frac{2^{-(n-1)}}{(n-1)!} \mathbb{E}(X^n) &= \sum_{0 \leq k < n} \frac{2^{-k}}{k!} B_k(\text{Tr}(P), 2^{2-1}(2-1)! \text{Tr}(P^2), \ldots, 2^{k-1}(k-1)! \text{Tr}(P^k)) \ P^{n-k}
\end{align*}$$

The proof of (3.12) and the above formulae, including a more refined analysis of these rank-one matrix moments can be found in Section 2 in [6].

The first matrix moments of the fluctuation matrix $\mathcal{H}_N$ are given by the formulae

$$\begin{align*}
\mathbb{E}(\mathcal{H}_N^2) &= \mathbb{E}(\mathcal{H}^2) = \mathbb{E}((X - P)^2) \\
\mathbb{E}(\mathcal{H}_N^3) &= N^{-1/2} \mathbb{E}((X - P)^3) \quad \text{and} \quad \mathbb{E}(\mathcal{H}_N^4) = N^{-1} \mathbb{E}((X - P)^4) + (1 - N^{-1}) \mathbb{E}(\mathcal{H}^4)
\end{align*}$$

This yields the fourth polynomial matrix moment formula

$$\begin{align*}
N \left[ \mathbb{E}(\mathcal{H}_N^4) - \mathbb{E}(\mathcal{H}^4) \right] &= 20 P^4 + 12 \text{Tr}(P) P^3 + \left[ 3 \text{Tr}(P)^2 + 5 \text{Tr}(P^2) \right] P^2 \\
&+ \left[ \text{Tr}(P)^3 + 3 \text{Tr}(P) \text{Tr}(P^2) + 4 \text{Tr}(P^3) \right] P \quad (3.13)
\end{align*}$$

The explicit description of all matrix moments $M_{n,m}(P)$ in terms of traces $\text{Tr}(P^k)$ and matrix powers $P^l$ is not known.

### 3.2 A semi-circle law for non-isotropic Wishart matrices

The semi-circle law (1.7) combined with (3.3) and (3.10) imply that

$$P = I \quad \implies \quad \lim_{r \to \infty} \mathbb{E} \left( \text{Tr} \left( \left[ M_{2n,n}(I) - M_{2n,n}^+(I) \sqrt{r} \right] \right) \right) = 0$$

This shows that the dominating terms in the circle law fluctuation are encapsulated in the matrix moments $M_{2n,n}^+(I)$.

Our second main result is a recursive formula for computing the centered moments associated with non-crossing partitions for a general covariance matrix. To describe with some precision this result we need to introduce some notation. For any $m \geq 0$ and any $v = (v_1, \ldots, v_m)$ with $v_i \geq 0$ we set

$$\text{Tr}_v(P) := \prod_{1 \leq i \leq m} \text{Tr} \left( P^{v_i} \right) \quad \text{with the convention} \quad \text{Tr}_0(P) = 1$$
We let $\mathcal{T}_{p,q}$, $p \geq q$, be the vector space of matrix polynomials in the variable $P$ with maximal degree $p$ and at most $q$ trace-product; that is the vector space spanned by the monomials

$$\text{Tr}_v(P)^{P^l} \quad \text{with} \quad \sum_{1 \leq i \leq k} v_i \leq q \quad \sum_{1 \leq i \leq k} iv_i + l = p \quad \text{and} \quad k \geq 0 \quad 1 \leq l \leq p$$

We also let $\mathcal{T}_{p,q}^+ \subset \mathcal{T}_{p,q}$ be the subset defined as above and considering exactly $q$ trace product terms. For instance we have

$$E(H^2) = E(H_N^2) \in \mathcal{T}_{2,1} \quad \text{and} \quad E(H_4) \in \mathcal{T}_{4,2} \quad \text{but} \quad E(H_N^3) \in \mathcal{T}_{4,3}$$

We also have the decomposition

$$E(H^d) - \left[ \text{Tr}(P)^2 P^2 + \text{Tr}(P) \text{Tr}(P^2) P \right] = 5 P^4 + 3 \text{Tr}(P) P^3 + \text{Tr}(P^2) P^2 + \text{Tr}(P^3) P \in \mathcal{T}_{4,1}$$

Consider a collection $P : r \mapsto P(r)$ of possibly random matrices satisfying the condition

$$\forall n \in \mathbb{N} , \quad r^{-1} \tau_n(P(r)) := r^{-1} \text{Tr}(P(r)^n) \xrightarrow[r \to \infty]{} \tau_n(P) \quad (3.14)$$

In this situation we have

$$r^{-3} E\left(\text{Tr}(H^4)\right) \xrightarrow[r \to \infty]{} \sigma_2(P) := C_2 \tau_3(P)^2 \tau_2(P)$$

Our next objective is to extend this result to any trace moment. The next theorem provides an explicit description of the dominating terms in the matrix moments $M_{2n,n}$.

**Theorem 3.4.** For any $n \geq 1$, the matrix polynomial $M_{2n,n}^+(P) \in \mathcal{T}_{2n,n}$ are given by the Catalan-type recursive formulæ

$$M_{2n,n}^+(P) = \sum_{k+l=n-1} \left[ M_{2k,k}^+(P)P + \text{Tr}(M_{2k,k}^+(P)P) I \right] M_{2l,l}^+(P)P \quad \text{with} \quad M_{0,0}^+(P) = I \quad (3.15)$$

We also have the decompositions

$$M_{2n,n}^-(P) \in \mathcal{T}_{2n,n-2} \quad \text{and} \quad M_{2n,n}^+(P) \in \Sigma_n(P) + \mathcal{T}_{2n,n-1}$$

**with the matrix polynomial** $\Sigma_n(P) \in \mathcal{T}_{2n,n}^+$** defined by the Catalan-type recursive formulæ

$$\Sigma_{n+1}(P) = \sum_{k+l=n} \text{Tr}(\Sigma_k(P) \Sigma_l(P)) P \quad \text{with} \quad \Sigma_0(P) = I \quad (3.16)$$

The proof of this theorem is provided in Section 6.2. The above theorem clearly yields the convergence result

$$\lim_{r \to \infty} r^{-1} \text{E}\left(\text{Tr}\left(\left[ \frac{H_{M_{2n,n}^+}}{r^2} \right]^{2n}\right)\right) = \lim_{r \to \infty} r^{-(n+1)} \text{E}\left(\text{Tr}(\Sigma_n(P))\right)$$

with the matrix polynomials $\Sigma_n(P)$ which can be computed sequentially using formulæ (3.16). For instance, we have

$$\Sigma_1(P) = \text{Tr}(P) P$$
$$\Sigma_2(P) = \text{Tr}(P)^2 P^2 + \text{Tr}(P^2) \text{Tr}(P) P$$
$$\Sigma_3(P) = \text{Tr}(P)^3 P^3 + 2 \text{Tr}(P^2) \text{Tr}(P)^2 P^2 + \left[ \text{Tr}(P^2)^2 \text{Tr}(P) + \text{Tr}(P^3) \text{Tr}(P) P \right] P$$
$$\Sigma_4(P) = \text{Tr}(P)^4 P^4 + 3 \text{Tr}(P^2) \text{Tr}(P)^3 P^3 + \left[ 2 \text{Tr}(P^3) \text{Tr}(P)^2 P + \text{Tr}(P^2)^2 \text{Tr}(P)^2 P^2 \right. \right.$$
$$+ \left. \left[ \text{Tr}(P^4) \text{Tr}(P)^3 + 3 \text{Tr}(P^3) \text{Tr}(P)^2 P + \text{Tr}(P^2)^3 \text{Tr}(P) \right] P$$

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and
\[ \Sigma_5(P) = \text{Tr}(P)^5 P^5 + 4 \text{Tr}(P^2) \text{Tr}(P)^4 P^4 + [6 \text{Tr}(P)^3 \text{Tr}(P^2)^2 + 3 \text{Tr}(P^3) \text{Tr}(P)^2] P^3 \\
+ [2 \text{Tr}(P^4) \text{Tr}(P)^4 + 8 \text{Tr}(P^3) \text{Tr}(P^2) \text{Tr}(P)^3 + 4 \text{Tr}(P^2)^3 \text{Tr}(P)^2] P^2 \\
+ [\text{Tr}(P^5) \text{Tr}(P)^4 + 4 \text{Tr}(P^4) \text{Tr}(P^2) \text{Tr}(P)^3 + 2 \text{Tr}(P^3)^2 \text{Tr}(P)^3 \\
+ 6 \text{Tr}(P^3) \text{Tr}(P^2)^2 \text{Tr}(P)^2 + \text{Tr}(P^2)^4 \text{Tr}(P)] P \]
(3.17)

Next theorem provides a closed form multinomial formula to compute the matrix polynomial functions \( \Sigma_n(P) \).

**Theorem 3.5.** The matrix polynomial \( \Sigma_n(P) \in \mathcal{T}_{2n,n}^+ \) defined in (3.16) are given by the formula (1.17). In addition, we have the multinomial-trace-type formula
\[ \text{Tr} \left( \Sigma_n(P) \right) = 2 \sum_{\mu} \left( \prod_{1 \leq i \leq n} \mu_i \right) \text{Tr} \mu(P) \quad \text{with} \quad \text{Tr} \mu(P) := \prod_{1 \leq i \leq n} \text{Tr} \left( P^i \right)^{\mu_i} \]
(3.18)

In the above display, the summation is taken over all \( \mu \) satisfying (1.17).

The proof of this theorem is given in Section 6.3.

**Corollary 3.6.** Let \( P : \tau \mapsto P(\tau) \) possibly random matrices satisfying condition (3.14). For any \( n \geq 1 \), the functional \( \sigma_n(P) \) defined by (1.16) in corollary 1.3, obeys,
\[ \sigma_n(P) = \lim_{\tau \to \infty} \text{Tr} \left( \Sigma_n(P) \right) \]

### 3.3 A Marchenko-Pastur law for non-isotropic Wishart matrices

The aim of this section is to provide a matrix version of the Marchenko-Pastur law for non-isotropic Wishart matrices. For any non-crossing partition \( \pi \in \mathcal{N}_{n,m} \) we have the matrix moment formulae
\[ X_{\pi} \overset{\text{law}}{=} Q_1 X Q_2 \cdots X Q_{|\pi|} X \implies X_{\pi^{(\pi)}} \overset{\text{law}}{=} X Q_1 X Q_2 \cdots X Q_{|\pi|} X \implies M_{\pi^{(\pi)}} \overset{\text{law}}{=} E \left( X Q_1 X Q_2 \cdots X Q_{|\pi|} X \right) = E \left( \prod_{1 \leq i \leq |\pi|} \langle X, Q_i X \rangle \right) \]
for some (non-necessarily symmetric) random matrices \( (Q_i)_{1 \leq i \leq |\pi|} \) independent of \( X \). On the other hand, for any symmetric matrices \( (S_i)_{1 \leq i \leq k} \) we also have the reduction formula
\[ E \left( X S_1 X S_2 \cdots X S_k X \right) = \text{Tr} \left( E \left[ X S_1 X S_2 \cdots X S_k X \right] \right) P + \sum_{1 \leq i \leq k} E \left( X S_1 \cdots X S_{i-1} X S_{i+1} \cdots X S_k X S_i \right) P \]
A proof of this decomposition can be found in [6]. This implies that
\[ M_{\pi^{(\pi)}} \overset{\text{law}}{=} \text{Tr} \left( M^{(\pi)} \right) P + \sum_{1 \leq i \leq |\pi|} M_{\varphi_i(\pi)} \overset{\text{law}}{=} \text{Tr} \left( M^{\varphi_i(\pi)} P \right) \]
for some non-crossing partitions \( \varphi_i(\pi) \in \mathcal{N}_{n,m} \). Using (2.20) and arguing as in the proof of theorem (3.3) we have
\[ M_{n,m} \overset{\text{law}}{=} \Sigma_{n,m} \overset{\text{law}}{=} O(r^{n-m}) \]
with the matrices \( \Sigma_{n,m} \) defined sequentially by the formula
\[ \Sigma_{n+1,m+1} = \Sigma_{n,m} P + \sum_{n_1,n_2,m_1,m_2} \Sigma_{n_1,m_1} \text{Tr} \left( \Sigma_{n_2,m_2} \right) P \]
(3.19)
The summation in the above display is taken over all non-negative parameters \( n_1, n_2, m_1, m_2 \) satisfying (2.21).
Theorem 3.7. The matrices $\Sigma_{n,m}^\alpha(P)$ defined in (3.19) are given by formula (1.13).

The proof of the above theorem is provided in Section 6.4.

3.4 Some combinatorial formulae

The theorems stated above can be used to derive several new combinatorial formulae. We end this section around this theme. Firstly using (3.18) we have

$$C_n = \sum_{1 \leq m \leq n} N_{n,m} = \sum_{1 \leq m \leq n} \prod_{\nu + n = \mu - n} K(\nu) = 2 \sum_{\mu} \left( \frac{n}{\mu_1 \mu_2 \ldots \mu_n} \right)$$

with the last sum running over the same set of indices as in (3.18).

We fix the covariance matrix $P$ and we set $u(i) = \text{Tr}(P^{1+i})$ for any $i \geq 0$. In this notation, by (1.14) for any $n \geq 0$ we have

$$\Sigma_n(P) = \sum_{0 \leq m \leq n} \alpha_n(m) P^m \text{ with } \alpha_n(m) := \frac{u(0)}{u(m)} \sum_{\pi \in \mathcal{N}_n : |\pi_1| = m} \prod_{i \geq 0} u(i)^r(\pi)$$

(3.20)

Observe that $\alpha_n(0) = 1_{n=0}$. In addition, using (3.16) we check that

$$\alpha_{n+1}(1) = \sum_{1 \leq m \leq n} \alpha_n(m) u(m)$$

and the matrix formula

$$\begin{pmatrix}
\alpha_{n+1}(2) \\
\alpha_{n+1}(3) \\
\alpha_{n+1}(4) \\
\vdots \\
\alpha_{n+1}(n+1)
\end{pmatrix} = \begin{pmatrix}
\alpha_n(1) & \alpha_{n-1}(1) & \ldots & \alpha_3(1) & \alpha_2(1) & \alpha_1(1) \\
\alpha_2(1) & \alpha_{n-1}(2) & \ldots & \alpha_3(2) & \alpha_2(2) & 0 \\
\alpha_3(1) & \alpha_{n-1}(3) & \ldots & \alpha_3(3) & 0 & \vdots \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\
\alpha_n(n) & 0 & \ldots & 0 & 0 & \alpha_1(1)
\end{pmatrix}$$

This provides a rather simple way to compute the coefficients $\alpha_n(m)$ in terms of $\alpha_n(1)$; that is we have that

$$\forall 1 \leq m \leq n \quad \alpha_n(m) = \sum_{k_1 + \ldots + k_m = n, k_i \geq 1} \prod_{1 \leq i \leq m} \alpha_k(1)$$

For $m = 1$ the above formula doesn’t give any information. To compute the evolution of the terms $\alpha_n(1)$ we use the “highly nonlinear” Bell-type induction

$$\alpha_{n+1}(1) = \sum_{1 \leq m \leq n} u(m) \sum_{k_1 + \ldots + k_m = n, k_i \geq 1} \prod_{1 \leq i \leq m} \alpha_k(1) = \frac{m!}{n!} \frac{2n}{n-m} B_{n,m}(1! \alpha_1(1), 2! \alpha_2(1), \ldots, n! \alpha_n(1))$$

with the partial Bell polynomials defined in (2.10).

By construction, the solution of these equations is given by (3.20), for any given sequence $u = (u(i))_{i \geq 0}$. For instance when $u(i) = 1$ we obtain the formulæ

$$C_n = \sum_{1 \leq m \leq n} C_{n,m}$$

with the Catalan triangle $C_{n,m}$ defined for any $1 \leq m \leq n$ by

$$C_{n,m} := \sum_{m+l_1+\ldots+l_m=n} \prod_{1 \leq i \leq m} C_{l_i} = \frac{m!}{n!} \frac{2n}{n-m}$$

In the above display, the summation run over non-negative indices $l_i \geq 0$. The Catalan triangle has been introduced by Shapiro in [58], see also [12] for a review of some combinatorial properties.
4 Laplace matrix transforms

4.1 Some matrix moment estimates

From the practical viewpoint, the combinatorial complexity of the matrix moments requires to find some useful computable estimates. We set

\[ \epsilon_n(N) := \frac{1}{N^{n/2-[n/2]}} \sum_{1 \leq m \leq [n/2] \land N} \left( \frac{4}{N} \right)^{[n/2]-m} \frac{[n/2]!}{m!} \]  

(4.1)

(4.2)

and

\[ v_n := \frac{1}{2^{[n/2] [n/2]!}} \]

In this notation, we have the following estimates.

**Theorem 4.1.** For any \( n \geq 1 \) and any \( N \geq 1 \) we have the estimates

\[ \| E(\mathcal{H}_N^n) \|_F \leq \frac{\nu_n}{2^{1+[n/2]-3n}} \text{Tr}(P)^n \epsilon_n(N) \]

(4.2)

In addition, for any \( N \geq n \) we have

\[ \text{Tr} \left( E \left[ \mathcal{H}_N^{2n} \right] \right) \leq \text{Tr} \left( E \left[ \mathcal{H}_N^{2n} \right] \right) + \sqrt{r} 2^{12n-1} v_{2n} \text{Tr}(P)^{2n} [\epsilon_{2n}(N) - 1] \]

(4.3)

and

\[ N \| E \left[ \mathcal{H}_N^{2n} \right] - E \left[ \mathcal{H}_N^{2n} \right] \|_F \leq (n-1)^2 v_{2n} E \left[ \| \mathcal{H} \|_F^2 \right]^n + 2^{6n-1} v_{2n} \text{Tr}(P)^{2n} \epsilon_{2n}(N) \]

(4.4)

The proof of this theorem is provided in Section 6.5.

4.2 Single rank matrices

Recalling that \( X \) has a Wishart distribution with a single degree of freedom and covariance matrix \( P \) we check the integral matrix formula

\[ E(\exp [tX]) = I + \int_0^t \exp \left[ \frac{1}{2} \sum_{n \geq 1} \frac{(2s)^n}{n} \text{Tr}(P^n) \right] [I - 2sP]^{-1} P \, ds \]

(4.5)

\[ = \left( 1 + \frac{1}{r (1-2t)^{r/2}} \right) I \quad \iff \quad P = I \]

for any \( t \in \mathbb{R} \) s.t. \( I - 2tP \) is invertible. The proof of (4.5) is housed in the Appendix.

Our next objective is to compute the log-Laplace matrix transforms

\[ L_P(t) := \log E_P(t) \quad \text{with} \quad E_Q(t) := E(\exp [t(X - Q)]) \]

We consider the positive mappings

\[ \varpi_1(Q) := Q \left[ \frac{1}{5} I + \frac{1}{3} Q + Q^2 \right] \geq \varpi(Q) := Q \left[ \frac{1}{5} I + \frac{5}{24} Q + Q^2 \right] \]

and the convex function \( L : t \in [0,1] \mapsto \mathbb{R}_+ \) defined by

\[ L(t) := \frac{t^2}{1-t} \quad \Rightarrow \quad u \in \mathbb{R}_+ \mapsto L^*(u) := \sup_{t \in [0,1]} (ut - L(t)) = (\sqrt{u+1} - 1)^2 \]

\[ \Rightarrow \quad u \in \mathbb{R}_+ \mapsto (L^*)^{-1}(u) = u + 2\sqrt{u} \]
Theorem 4.2. For any $0 \leq t < 2\mathrm{Tr}(P)$ we have

$$I \leq E_0(t) \exp \left[ -tP \right] \leq \exp \left[ L\left(2t\mathrm{Tr}(P)\right) \circ_1 \left(P/\Tr(P)\right)\right] \tag{4.7}$$

In addition we have the centered Laplace estimates

$$I \leq E_0(t) \leq \exp \left[ L\left(2t\mathrm{Tr}(P)\right) \circ (P/\Tr(P))\right] \tag{4.8}$$

The proof of this theorem is provided in Section 6.6. Recalling that $0 < A \leq B \implies \log(A) \leq \log(B)$, and $A = \log \exp(A)$ for any symmetric matrix $A$ we prove the following corollary.

Corollary 4.3. For any $0 \leq t < 2\mathrm{Tr}(P)$ we have the Laplace matrix inequalities

$$0 \leq L_P(t) - tP \leq L\left(2t\mathrm{Tr}(P)\right) \circ_1 \left(P/\Tr(P)\right) \tag{4.9}$$
$$0 \leq L_P(t) \leq L\left(2t\mathrm{Tr}(P)\right) \circ (P/\Tr(P)) \tag{4.10}$$

4.3 Laplace matrix estimates

The objective of this section is to estimate the Laplace matrix transforms defined by

$$\mathcal{E}_N(t) := \mathbb{E}\left(\exp\left(t\mathcal{H}_N\right)\right) \quad \text{and} \quad \mathcal{E}(t) := \mathbb{E}\left(\exp\left(t\mathcal{H}\right)\right)$$

The existence of $\mathcal{E}(t)$ is ensured by the estimate

$$\log \|\mathcal{E}(t)\|_F \leq \frac{t^2}{2} \left[\Tr(P^2) + \Tr(P)^2\right]$$

which is valid for any $t \in \mathbb{R}$. This assertion is a consequence of the technical lemma 6.3.

Theorem 4.4. For any $0 \leq 4\sqrt{2}t\Tr(P) < 1$ and $N \geq 8$ we have the estimates

$$\|\mathbb{E}\left[\cosh(t\mathcal{H}_N)\right] - \mathcal{E}(t)\|_F \leq \left(8t\right)^4 \left[\Tr(P^2) + \Tr(P)^2\right]^2 / N\tag{4.11}$$
$$\|\mathbb{E}\left[\sinh(t\mathcal{H}_N)\right]\|_F \leq 4 \left(t\Tr(P)^3\right) / \sqrt{N}$$

The proof of this theorem is provided in Section 6.7. Recalling that

$$\mathbb{E}\left[\exp(t\mathcal{H}_N)\right] = \mathbb{E}\left[\cosh(t\mathcal{H}_N)\right] + \mathbb{E}\left[\sinh(t\mathcal{H}_N)\right]$$

the estimates (4.11) readily yields the following corollary.

Corollary 4.5. For any $0 \leq 4\sqrt{7}t\Tr(P) < 1$ and $N \geq 8$ we have

$$\|\mathcal{E}_N(t) - \mathcal{E}(t)\|_F \leq \frac{4^6}{N} t^4 \left[\Tr(P^2) + \Tr(P)^2\right]^2 + \frac{4}{\sqrt{N}} \left(t\Tr(P)^3\right) \tag{4.12}$$

The Laplace matrix estimate (4.12) differs from trace-type estimates which can be derived using Lieb’s inequality

$$\mathbb{E}\left(\Tr\left(\exp\left[A + B\right] \mid A\right)\right) \leq \Tr\left(\exp\left[A + \log \mathbb{E}\left(e^B \mid A\right)\right]\right) \tag{4.13}$$

which is valid for any possibly random symmetric matrices $A, B$. Applying sequentially this estimate and using the spectral mapping theorem we obtain the following spectral-sub-additivity estimate.

Lemma 4.6 (Tropp [72]). For any sequence of independent $(r \times r)$ symmetric matrices $A_i$ indexed by some finite set $i \in I$ we have

$$\Tr\left[\mathbb{E}\left(\exp \sum_{i \in I} A_i\right)\right] \leq \Tr\left(\exp \sum_{i \in I} \log \mathbb{E}\left(\exp(A_i)\right)\right) \tag{4.14}$$
Observe that
\[ \varpi(Q) \leq \varpi_+(Q) I \quad \text{with} \quad \varpi_+(Q) := \lambda_1(\varpi(Q)) \]

Applying this lemma to
\[ A_i = \frac{t}{\sqrt{N}} (X_i - P) \implies \sum_{1 \leq i \leq N} A_i = t H_N \]

\[ \implies \text{Tr} (\mathcal{E}_N(t)) \leq \text{Tr} \left( \exp \left( N L_p(t/\sqrt{N}) \right) \right) \]
\[ \leq r \exp \left( NL \left[ 2 t \text{Tr}(P)/\sqrt{N} \right] \varpi_+ [P/\text{Tr}(P)] \right) \quad \iff \quad 4.10 \]

These trace estimates are often used to derive exponential concentration estimates. For instance, using conventional large deviation techniques we check that for any \( N \geq 1 \) and \( \delta > 0 \) the probability of the event

\[ \lambda_1(H^N) \leq 2 \text{Tr}(P) \left[ \frac{\delta + \log(r)}{\sqrt{N}} + 2 \sqrt{\left( \delta + \log(r) \right) \varpi_+ \left( \frac{P}{\text{Tr}(P)} \right)} \right] \quad (4.15) \]

is greater than \( 1 - e^{-\delta} \). For the convenience of the reader a proof of this assertion is housed in the Appendix.

5 Concentration and trace-type estimates

5.1 A fluctuation theorem

For any symmetric matrix \( A \) and for any \( t \in \mathbb{R} \) we have

\[ \log \mathbb{E} (\exp [t \text{Tr}(A H)]) = t^2 \text{Tr}((AP)^2) \quad (5.1) \]

In other words, \( \text{Tr}(A H) \) is a centered Gaussian random variable with variance \( 2 \text{Tr}((AP)^2) \). We also have the \( \chi^2 \)-square Laplace formula

\[ \log \mathbb{E} (\exp (t\|H\|_F^2)) = \frac{1}{4} \sum_{n \geq 1} \frac{(4t)^n}{n} \left[ \text{Tr}(P^n)^2 + \text{Tr}(P^{2n}) \right] \quad (5.2) \]

In addition, there exists some \( t > 0 \) and some finite constant \( c < \infty \) such that

\[ \sup_{N \geq 1} \mathbb{E} [\exp (t \|H_N\|_F)] < \infty \iff \forall n \geq 1 \sup_{N \geq 1} \mathbb{E} [\|H_N\|_F^n] \leq c^n n! \quad (5.3) \]

The first two assertions comes from the fact that \( H \) is a matrix with Gaussian entries.

The proofs of (5.3) and the Laplace formulae (5.1) and the Hardy’s exponential property (5.2) are provided in the Appendix.

The Laplace estimate (5.1) readily implies that any \( \delta \geq 0 \) the probability of the event

\[ \text{Tr}(A H) \leq 2 \sqrt{\delta \text{Tr}((AP)^2)} \] is greater than \( 1 - e^{-\delta} \). (5.4)

The proof of this assertion is based on conventional Cramér-Chernov arguments. A brief summary of large deviation techniques is provided in Section 2.1.

Recalling that \( P_N \) has a Wishart distribution with \( N \) degree of freedom and covariance matrix \( N^{-1} P \), for any \( 2\|t\|_F \leq \sqrt{N} \) we have

\[ \log \mathbb{E} (\exp (t \text{Tr}(A H_N))) = t^2 \text{Tr}((AP)^2) + t^2 \sum_{n \geq 1} \frac{t^n}{n + 2} \frac{2^{n+1}}{N^{n/2}} \text{Tr}((AP)^{2+n}) \quad (5.5) \]
Explicit formulae for $\text{Tr}((AP)^n)$ are rarely available. For rank one matrices $A = (xy' + yx')/2$ we have the formula

$$2^{n+1} \text{Tr}[(AP)^{n+2}] = \sum_{0 \leq l \leq [(n+1)/2]} \binom{n+1}{2l} \langle x, Py \rangle^{n+2-2l} \langle (x, Px) \langle y, Py \rangle \rangle^l \tag{5.6}$$

$$+ \sum_{1 \leq l \leq [n/2]-1} \binom{n+1}{2l-1} \langle x, Py \rangle^{n+2-2l} \langle (x, Px) \langle y, Py \rangle \rangle^l$$

$$\leq 2^{n+2} \lambda_1(P)^n$$

as soon as $x, y \in \mathbb{B}$, where $\mathbb{B}$ is the unit ball in $\mathbb{R}^r$. A proof of (5.6) is provided in the Appendix. In this situation, we obtain the Laplace estimate

$$\mathbb{E} \left( \exp \left( t \text{Tr} (AH_N) - t^2 \text{Tr}((AP)^2) \right) \right) \leq \exp \left( -(2t\lambda_1(P))^2 \log \left( 1 - \frac{1}{N^{1/2}} (2t\lambda_1(P)) \right) \right)$$

Whenever $x = u_i$ and $y = u_j$ are two orthonormal eigenvectors associated with some different eigenvalues $\lambda_i(P)$ and $\lambda_j(P)$ with $i \neq j$ we check that

$$2^{n-1} \text{Tr}[(AP)^n] = (\lambda_i(P)\lambda_j(P))^{n/2}$$

**Theorem 5.1.** For any function $N = N(r)$ s.t. $rN(r) \to_{r \to \infty} \infty$ and any symmetric matrix $A$ we have

$$\sup_{r \geq 1} \lambda_1(A) < \infty \quad \text{and} \quad \sigma_A^2 : = r^{-1} \text{Tr}((AP)^2) \to_{r \to \infty} \sigma_A^2$$

$$\implies \frac{1}{\sqrt{2r}} \text{Tr} (AH_{N(r)}) \to_{r \to \infty} \mathcal{N}(0, \sigma_A^2(P))$$

Assume that $0 \leq r^{-1} |\text{Tr}((AP)^n)| \leq \alpha_A(P) \beta_A(P)^n$, for some parameters $\alpha_A(P) \geq 1$ and $\beta_A(P) \geq 0$ whose values doesn’t depend on $n$. In this situation, we have the estimates

$$\frac{1}{(2n)!} \mathbb{E} \left( \left( \frac{1}{\sqrt{2r}} \text{Tr} (AH_N) \right)^{2n} \right) - \mathbb{E} \left( \left( \frac{1}{\sqrt{2r}} \text{Tr} (AH) \right)^{2n} \right) \leq \frac{4(e - 1)}{rN} \alpha_A(P)^{n-1} \beta_A(P)^{2n} \left( 1 + \frac{8}{rN} \right)^{n-1}$$

as well as

$$\frac{1}{(2n + 1)!} \mathbb{E} \left( \left( \frac{1}{\sqrt{2r}} \text{Tr} (AH_N) \right)^{2n+1} \right) \leq \frac{(e - 1)\sqrt{2}}{\sqrt{rN}} \alpha_A(P)^{n} \beta_A(P)^{2n+1} \left( 1 + \frac{8}{rN} \right)^n$$

In addition if $A \geq 0$ then the terms in absolute value $|.|$ in the above display are positive.

Under the assumptions of the above theorem we have

$$\frac{1}{\sqrt{2r}} \text{Tr} (AH_N) \to_{N \to \infty} \frac{1}{\sqrt{2r}} \text{Tr} (AH) \to_{r \to \infty} \mathcal{N}(0, \sigma_A^2(P))$$

The proof of theorem 5.1 is provided in Section 6.8.

### 5.2 Spectral vs Trace-type concentration inequalities

For any symmetric matrix $A$ we have the sub-Gaussian Laplace estimate

$$4|t| ||AP||_F \leq \sqrt{N} \implies \log \mathbb{E} \left( \exp (t \text{Tr} (AH_N))) \right) \leq t^2 \left[ \text{Tr}((AP)^2) + 2 ||AP||_F^2 \right]$$

$$4|t| \text{Tr}(AP) < \sqrt{N}, \ A \geq 0 \implies \log \mathbb{E} \left( \exp (t \text{Tr} (AH_N)) \right) \leq 3t^2 \text{Tr}((AP)^2)$$

The proof of these estimates are essentially based on the Laplace formula (5.5). The details of the proof including more refined logarithmic-type estimates are provided in the Appendix.
Theorem 5.2. For any symmetric matrix $A$ and any $0 \leq 8\delta \leq N$ the probability of the event
\[
|\text{Tr}(A\mathcal{H}_N)| \leq 2\sqrt{(\delta + 1) - \text{Tr}((AP)^2) + 2\|AP\|^2_2} \quad \text{is greater than} \quad 1 - e^{-\delta}
\] (5.8)

In addition, if $A \geq 0$ then for $0 \leq 8\delta \leq N$ the probability of the event
\[
\text{Tr}(A\mathcal{H}_N) \leq 2\sqrt{3} \sqrt{\delta \text{Tr}((AP)^2)} \quad \text{is greater than} \quad 1 - e^{-\delta}
\]

Furthermore for any $0 \leq 4\delta \leq N$ $\text{Tr}((AP)^2)/\text{Tr}(AP)^2$ the probability of the event
\[
\text{Tr}(A\mathcal{H}_N) \geq -2\sqrt{3} \sqrt{\delta \text{Tr}((AP)^2)} \quad \text{is greater than} \quad 1 - e^{-\delta}
\]

The proof of the above theorem is provided in Section 6.9. We also have the immediate corollary.

Corollary 5.3. For any $0 \leq 4\delta \leq N$ $[1/2 \land \text{Tr}((AP)^2)/\text{Tr}(AP)^2]$ and any matrix $A \geq 0$ the probability of the event
\[
|\text{Tr}(AP^N) - \text{Tr}(AP)| \leq 2\sqrt{\frac{\delta + 1}{N} - 2\text{Tr}(AP)^2 + \text{Tr}((AP)^2)} \quad \text{is greater than} \quad 1 - e^{-\delta}
\]

We end this section with some operator norm estimates which can be deduced almost directly combining the trace concentration inequalities (5.8) with the variational formulation (1.18) of the operator norm.

Theorem 5.4. For any $N \geq 8r \log (31)$ we have
\[
\mathbb{E} [\|H\|_{op}] \land \mathbb{E} [\|\mathcal{H}_N\|_{op}] \leq 6 \sqrt{r} \lambda_1(P) \implies \mathbb{E} [\lambda_1(P_N)/\lambda_1(P)] \leq 1 + 6 \sqrt{r/N}
\]

For any $N$ and $\delta$ such that $N/8 \geq \delta + 7r$, the probability of the event
\[
\|\mathcal{H}_N\|_{op} \leq 5\sqrt{3} \sqrt{\delta + 7r} \lambda_1(P) \quad \text{is greater than} \quad 1 - e^{-\delta}
\] (5.9)

Corollary 5.5. For any $N$ and $\delta$ such that $N/8 \geq \delta + 7r$, the probability of the event
\[
\sup_{1 \leq k \leq r} |\lambda_k(P_N) - \lambda_k(P)| \leq 5\sqrt{3} \sqrt{\frac{\delta + 7r}{N}} \lambda_1(P) \quad \text{is greater than} \quad 1 - e^{-\delta}
\]

The proof of theorem 5.4 and corollary 5.5 is given in Section 6.10.

6 Proof of the main results

6.1 Proof of theorem 3.1

When $m > n/2$ at least one block of $\pi \in \mathcal{P}_{n,m}$ is of unit size and the matrix moment $M_\pi(P) = 0$ resume to the null matrix. This implies that
\[
\forall \{n/2\} < m \leq n \quad M_{n,m}(P) = 0
\] (6.1)

Let $\langle p, N \rangle$ be the set of all the $\{N\}_p := N!/(N - p)!$ one to one mappings $\beta$ from $\{p\} := \{1, \ldots, p\}$ into $[N]$. In this notation we have the decomposition
\[
[N]^{[n]} = \bigcup_{1 \leq p \leq (n\land N)} \bigcup_{\pi : \pi_i = p} \{\alpha_\beta^n : \beta \in \langle p, N \rangle\} \quad \text{with} \quad \alpha_\beta^n = \sum_{1 \leq i \leq |\pi|} \beta(i) 1_{\pi_i}
\] (6.2)

where $\pi_1, \ldots, \pi_{|\pi|}$ stand for the $|\pi|$ blocks of a partition of $[n]$ ordered w.r.t. their smallest element. When $\beta(i) = i$ we write $\alpha^\pi$. A detailed proof of this decomposition can be found in
Section 8.6 in [14]. For a further discussion on partitioning and counting onto mappings we also refer to [18]. This decomposition implies that

$$N^{n/2} \mathbb{E}(H^N_k) = \sum_{1 \leq m \leq \lfloor n/2 \rfloor} (N)_m M_{n,m}(P)$$

$$= \sum_{1 \leq l \leq \lfloor n/2 \rfloor} N^l \sum_{l \leq m \leq \lfloor n/2 \rfloor} s(m,l) M_{n,m}(P)$$

(6.3)

The second polynomial formulae in (6.3) comes from the formula

$$(N)_m = \sum_{1 \leq l \leq m} s(m,l) N^l$$

This ends the proof of the first assertion.

Using (6.3), for any given $n \geq 1$ we also have

$$\forall 1 \leq k \leq n \quad \mathbb{E}
\left[
\left(\sqrt{k} H_k\right)^{2n}\right]
= f_n(k) = \sum_{1 \leq l \leq k} \binom{k}{l} g_n(l) \text{ with } g_n(l) := l! M_{2n,l}(P)$$

Using the binomial inversion formula we find that

$$\forall 1 \leq k \leq n \quad g_n(k) = \sum_{1 \leq l \leq k} (-1)^{k+l-1} \binom{k}{l} f_n(l)$$

Thus, for any $1 \leq m \leq \lfloor n/2 \rfloor$ we also have the matrix moment inversion formula

$$M_{n,m}(P) = \sum_{1 \leq k \leq m} (-1)^{m-k} \frac{1}{k!(m-k)!} \mathbb{E} \left(\sum_{1 \leq i \leq k} (X_i - P)^m \right)$$

This ends the proof of the theorem.

6.2 Proof of theorem 3.4

For $n = 2$ we have $C_2 = 2$ non-crossing partitions and a single crossing partition

$$N_{2n,n}(0,0,...,0) = \{\pi^1, \pi^2\} \quad \text{and} \quad \mathcal{P}_{2n,n}(0,0,...,0) - N_{2n,n}(0,0,...,0) = \{\pi^3\}$$

given by

$$\pi^1 = \{1,2\} \quad \pi^2 = \{1,4\} \quad \pi^3 = \{1,3\} \quad \{2,4\} \quad \text{(6.4)}$$

Non-crossing pair partitions can be interpreted in various ways. Firstly, we can interpret non-crossing pair partitions in terms of connections of the ordered set of integers $[2n]$ lying on a horizontal line in the plane with $n$ nonintersecting arcs/lines. In the above example we have the Murasaki diagram

$$\pi^1 \quad \pi^2$$

A given crossing pair partition can be interpreted as balanced parentheses on the ordered set $[2n]$ (with $n$ left and $n$ right). When $n = 2$ we have

$$\pi^1 = \{1,2\} \quad \{3,4\} = () () \quad \text{and} \quad \pi^2 = \{1,4\} \quad \{2,3\} = (())$$
They can also be seen as the number of ways of grouping $n + 1$ letters with parenthesis

$$\pi^1 = \{1, 2\} \{3, 4\} = (()) = ((a_1 \cdot a_2) \cdot a_3) \quad \text{and} \quad \pi^2 = \{1, 4\} \{2, 3\} = (()) = (a_1 \cdot (a_2 \cdot a_3))$$

(6.5)

We obtain the balanced parenthesis by removing the letters and the left parenthesis “(“ and then replacing each “)” by a left parenthesis “(“. This construction provides a simple way to count the number of non-crossing partitions using Catalan recursion

$$C_2 = C_0 \ C_1 + C_1 \ C_0$$

For $n = 3$ we have 5 non-crossing pair partitions of 6 elements into pairs. The first $C_2 \ C_0 = 2$ partitions

$$\begin{align*}
\pi_1 &= \{1, 2\} \{3, 4\} \{5, 6\} = (()) = (((a_1 \cdot a_2) \cdot a_3) \cdot a_4) \\
\pi_2 &= \{1, 3\} \{2, 4\} \{5, 6\} = (()()) = ((a_1 \cdot (a_2 \cdot a_3)) \cdot a_4)
\end{align*}$$

Then $C_1 \ C_1 = 1$ partition given by

$$\pi_3 = \{1, 2\} \{2, 6\} \{3, 4\} = ()(()()) = ((a_1 \cdot a_2) \cdot (a_3 \cdot a_4))$$

Finally, the last $C_0 \ C_2 = 2$ partitions

$$\begin{align*}
\pi_4 &= \{1, 6\} \{2, 3\} \{4, 5\} = (()()) = (a_1 \cdot (((a_2 \cdot a_3) \cdot a_4)) \\
\pi_5 &= \{1, 6\} \{2, 5\} \{3, 4\} = (((()))) = (a_1 \cdot (a_2 \cdot (a_3 \cdot a_4)))
\end{align*}$$

The matrix moments $M_\pi(P)$ associated with pair partition can be sequentially computed using the positive maps defined for any $Q \in S_r$ by the matrix linear functionals

$$\begin{align*}
\Omega(Q) &:= \mathbb{E}((X - P)\text{Tr}[Q(X - P)]) = 2PQP \\
\Gamma(Q) &:= \mathbb{E}((X - P)Q(X - P)) = \frac{1}{2} \Omega(Q) + \bar{\Gamma}(Q) \quad \text{with} \quad \bar{\Gamma}(Q) = \text{Tr}(PQ) \ P
\end{align*}$$

Observe that

$$P = I \implies \Omega(Q) = 2Q \quad \bar{\Gamma}(Q) = \text{Tr}(Q) \ P = (1 + r) \ I \quad (6.7)$$

The matrix moments $M_\pi(P)$ associated with non-crossing pair partition are expressed in terms of powers and compositions of the mapping $\Gamma$. For instance, for $n = 2$ the matrix moments associated with the partitions (6.5) are given by adding a $\Gamma$ in front of each left balanced parenthesis and replacing the inside by the identity; that is we have

$$\begin{align*}
M_{\pi^1}(P) &= \mathbb{E}((X - P)^2) = \Gamma(I) = \Gamma(I)^2 = P^4 + 2\text{Tr}(P) \ P^3 + \text{Tr}(P)^2 \ P^2 \\
M_{\pi^2}(P) &= \mathbb{E}((X - P)\Gamma(I)(X - P)) \\
&= \Gamma(\Gamma(I)) = \Gamma^2(I) - \text{Tr}(P) \ P^3 + \text{Tr}(P)^2 \ [\text{Tr}(P^3) + \text{Tr}(P) \text{Tr}(P^2)] \ P
\end{align*}$$

This shows that

$$\begin{align*}
M_{2,1}(P) &= M_{0,0}(P) \Gamma(M_{0,0}^+(P)) = \Gamma(I) \quad \text{with} \quad M_{0,0}^+(P) = I \\
&= P^2 + \text{Tr}(P) \ P \\
M_{4,2}(P) &= M_{0,0}(P) \Gamma(M_{2,1}^+(P)) + M_{2,1}^+(P) \Gamma(M_{0,0}^+(P)) = \Gamma(I) + \Gamma(I) \Gamma(I) \\
&= 2P^4 + 3\text{Tr}(P) \ P^3 + \text{Tr}(P)^2 \ P^2 + [\text{Tr}(P^3) + \text{Tr}(P^2) \text{Tr}(P)] \ P
\end{align*}$$

In the same vein, for $n = 3$ we have the sum of the $C_3 = 5$ terms associated with the balanced parenthesis

$$(()()), (())(), ()(()), ()(()), ()()()$$
These balanced strings give the value of the mapping \((\alpha^\pi(1), \ldots, \alpha^\pi(6))\) for the non-crossing partitions of \([6]\) into 3 blocks:

\[
(1, 2, 3, 3, 2, 1) , (1, 2, 2, 3, 3, 1) , (1, 1, 2, 3, 3, 2) , (1, 2, 2, 1, 3, 3) , (1, 1, 2, 2, 3, 3)
\]

This implies that

\[
M_{6,3}^+(P) = \Gamma (\Gamma^2(I) + \Gamma(I)^2) + \Gamma(I) \Gamma^2(I) + \left[\Gamma^2(I) + \Gamma(I)^2\right] \Gamma(I)
\]

\[
= M_{0,0}^+(P) \Gamma(M_{6,3}^+(P)) + M_{2,1}^+(P) \Gamma(M_{4,2}^+(P)) + M_{1,2}^+(P) \Gamma(M_{2,1}^+(P)) + M_{6,3}^+(P) \Gamma(M_{0,0}^+(P))
\]

\[
= 5 P^6 + 10 \text{Tr}(P) P^5 + 6 \text{Tr}(P)^2 P^4 + [3 \text{Tr}(P^3) + 3 \text{Tr}(P^2) \text{Tr}(P) + \text{Tr}(P)^3] P^3
\]

\[
+ 2 \left[\text{Tr}(P^2) \text{Tr}(P^2) \text{Tr}(P)^2\right] P^2
\]

\[
+ P \left[2 \text{Tr}(P^5) + 3 \text{Tr}(P^4) \text{Tr}(P) + \text{Tr}(P^3) \text{Tr}(P)^2 + \text{Tr}(P^3) \text{Tr}(P)^2 + \text{Tr}(P^2) \text{Tr}(P)\right]
\]

For \(n = 4\) we have the sum of the \(C_4 = 14\) terms. The first \(C_0 C_3 = 5\) terms

\[
((())()) , (((()))()) , ((())()) , ((())()) , ((())())
\]

The \(C_1 C_2 = 2\) and the \(C_2 C_1 = 2\) terms

\[
((()())() , (())()()) \quad \text{and} \quad ((())())() , (())(()())
\]

and finally the \(C_3 C_0 = 5\) terms

\[
((())()) , ((())()) , ((())()) , ((())()) , ((())())
\]

This yields the decomposition

\[
M_{8,4}^+(P)
\]

\[
= M_{0,0}^+(P) \Gamma(M_{6,3}^+(P)) + M_{2,1}^+(P) \Gamma(M_{4,2}^+(P)) + M_{1,2}^+(P) \Gamma(M_{2,1}^+(P)) + M_{6,3}^+(P) \Gamma(M_{0,0}^+(P))
\]

\[
= \Gamma^2 (\Gamma^2(I) + \Gamma(I)^2) + \Gamma(I) \Gamma^2(I) + \left[\Gamma^2(I) + \Gamma(I)^2\right] \Gamma(I)
\]

\[
+ \Gamma(I) \Gamma (\Gamma^2(I) + \Gamma(I)^2) + (\Gamma^2(I) + \Gamma(I)^2) \Gamma(I)
\]

More generally, we have the Catalan-type matrix polynomial recursion

\[
M_{2n,n}^+(P) := \sum_{k+l=n-1} M_{2k,k}^+(P) \Gamma(M_{2l,l}^+(P)) \quad (6.8)
\]

The matrix polynomial \(M_{2n,n}^+(P)\) is a linear combination of monomials of the form

\[
\left[ \prod_{1 \leq i \leq m} \text{Tr}(P^i)^{v_i} \right] P^w \quad \text{with} \quad \begin{cases} 1 \leq w \leq 2n & 0 \leq m \leq n \\ \sum_{1 \leq i \leq m} v_i \leq n & (\implies m \leq n) \\ \sum_{1 \leq i \leq m} i v_i + w = 2n \end{cases}
\]

In addition, the coefficients don’t depend on the matrix \(P\). We check this property by induction w.r.t. the parameter \(n\) using the recursion (6.8). This ends the proof of (3.15).
We let $M^{[Q_0,\ldots,Q_{2n}]}_{2n,n,+}(P)$ the matrix moments associated with a collection of matrices $Q_i$ defined by

$$M^{[Q_0,\ldots,Q_{2n}]}_{2n,n,+}(P) := \sum_{\pi \in \mathbb{Q}^+_{2n,n}} \mathbb{E} \left( Q_0(X_{\alpha^1(1)} - P)Q_1(X_{\alpha^2(2)} - P)Q_2 \ldots (X_{\alpha^m(2n)} - P)Q_{2n} \right)$$

(6.9)

For homogeneous sequences $Q_i = Q$ we write $M^{[Q]}_{2n,n,+}(P)$ instead of $M^{[Q,\ldots,Q]}_{2n,n,+}(P)$.

Extending the arguments given above to the matrix polynomials (6.9), we obtain the following technical lemma.

**Lemma 6.1.** We have the functional recursion

$$M^{[Q_0,\ldots,Q_{2n}]}_{2n,n,+}(P) = \sum_{k+i=n-1} M^{[Q_0,\ldots,Q_{2k}]}_{2k,k,+}(P) \Gamma \left[ M^{[Q_{2k+1},\ldots,Q_{2n-1}]}_{2n-1,1,+}(P) \right] Q_{2n}$$

(6.10)

with the initial condition $M^{Q,+}_{0,0}(P) = Q$.

For instance we have

$$M^{[Q_0,Q_1,Q_2]}_{2,1,+}(P) = Q_0 \Gamma(Q_1)Q_2$$

$$M^{[Q_0,\ldots,Q_{2n}]}_{2n,n,+}(P) = Q_0 \Gamma[Q_1 \Gamma(Q_2)Q_3] Q_4 + Q_0 \Gamma(Q_1)Q_2 \Gamma(Q_3)Q_4$$

$$M^{[Q_0,\ldots,Q_{2n}]}_{6,3,+}(P) = Q_0 \Gamma(Q_1)Q_2 \Gamma(Q_3)Q_4 \Gamma(Q_5)Q_6 + Q_0 \Gamma(Q_1)Q_2 \Gamma(Q_3)Q_4 \Gamma(Q_5)Q_6$$

$$+ Q_0 \Gamma(Q_1)Q_2 \Gamma(Q_3)Q_4 \Gamma(Q_5)Q_6 + Q_0 \Gamma(Q_1)Q_2 \Gamma(Q_3)Q_4 \Gamma(Q_5)Q_6$$

For any $0 \leq m \leq n$ we let $\Xi_{n,m}$ be the set of partition $\tau = (\tau_1,\ldots,\tau_{m+1})$ of $[2n-1]$ be into $(m+1)$ increasing sequences

$$\tau_j = (i_1^j,\ldots,i_k^j) \quad \text{with} \quad 0 < i_1^j < \ldots < i_k^j < 2n$$

We also set

$$(PQ)_{\tau_j} = (PQ_{i_1^j})(PQ_{i_2^j})\ldots(PQ_{i_k^j})$$

In this notation, using the induction (6.10) and recalling that $\Gamma$ only increases the number of traces by 1, we check that for any $n \geq 1$ the matrix polynomial $M^{[Q_0,\ldots,Q_{2n}]}_{2n,n,+}(P)$ is a linear combination of monomials of the form

$$\left[ \prod_{1 \leq i \leq m} \text{Tr}((PQ)_{\tau_i}) \right] Q_0(PQ)_{\tau_{m+1}}Q_{2n} \quad \text{with} \quad 0 \leq m \leq n \quad \text{and} \quad \tau \in \Xi_{n,m}$$

(6.11)

Arguing as above, we check that the matrix polynomial $M^{[P]}_{2n,n,+}(P)$ is a linear combination of monomials of the form

$$\text{Tr}_v(P)P^w \quad \text{with} \quad \begin{cases} 3 \leq w \leq 4n+1 \\ 0 \leq m \leq n \\ \sum_{1 \leq i \leq m} v_i \leq n \\ \sum_{1 \leq i \leq m} iv_i + w = 4n+1 \end{cases}$$
Rewritten in terms of the matrix functional $\Gamma$ the Catalan-type recursive formulae (5.10) resumes to the formula

$$\Sigma_n(P) := \sum_{k+l=n-1} \Sigma_k(P) \Gamma(\Sigma_l(P)) \quad \text{with} \quad \Sigma_0(P) = I$$

A simple induction w.r.t. the parameter $n$ shows that the matrix polynomial $\Sigma_n(P)$ is a linear combination of monomial for $n$ trace terms given by of the form

$$\text{Tr}_v(P) P^w \quad \text{with} \quad \begin{cases} 1 \leq w \leq n & 0 \leq m \leq n & \sum_{1 \leq i \leq m} v_i = n \\ \sum_{1 \leq i \leq m} \ i v_i + w = 2n \end{cases}$$

This assertion is also a direct consequence of the formula (6.9) stated in theorem 5.5. Let $\tilde{N}_{2n,n}(P) := M_{2n,n}^+(P) - \Sigma_{2n,n}(P) \implies \tilde{M}_{0,0}(P) = 0$

Combining the properties discussed above with the easily checked induction

$$\tilde{M}_{2n,n}(P) = \sum_{k+l=n-1} \tilde{M}_{2k,k}(P) \Gamma(\tilde{M}_{2l,l}(P)) + \frac{1}{2} \sum_{k+l=n-1} M_{2k,k}(P) \Omega(M_{2l,l}(P))$$

we check that that the matrix polynomial $\tilde{M}_{2n,n}(P)$ is a linear combination of monomial for $n$ trace terms given by

$$\text{Tr}_v(P) P^w \quad \text{with} \quad \begin{cases} 1 \leq w \leq 2n & 0 \leq m \leq n & \sum_{1 \leq i \leq m} v_i < n \\ \sum_{1 \leq i \leq m} \ i v_i + w = 2n \end{cases}$$

The matrix moments $M_x(P)$ associated with crossing pair partitions require more sophisticated calculations. Nevertheless they can be computed in terms of powers and compositions of the mappings $(\Gamma, \Omega)$ using sequentially conditioning formulae. For instance, the $\pi_3$-matrix moment associated with the partition $\pi_3$ discussed in (6.4) is given by

$$M_{\pi_3}(P) = \mathbb{E}((\mathbb{I}_1 - P)Q_1(\mathbb{I}_2 - P)Q_2(\mathbb{I}_1 - P)Q_3(\mathbb{I}_2 - P))$$

$$= \mathbb{E}((\mathbb{I}_1 - P)Q_1(\Gamma_1(\mathbb{I}_2 - P)Q_2(\mathbb{I}_1 - P)Q_3(\mathbb{I}_2 - P) \mathbb{I}_1))$$

$$= \mathbb{E}((\mathbb{I}_1 - P)Q_1(\Gamma_2(\mathbb{I}_2 - P)Q_3))$$

$$= \mathbb{E}((\mathbb{I}_1 - P)Q_1PQ_2(\mathbb{I}_1 - P)) Q_3P + \mathbb{E}(\text{Tr}((\mathbb{I}_1 - P)(Q_3PQ_2)) (\mathbb{I}_1 - P)) Q_1P$$

This yields the formula

$$\mathbb{E}(Q_0(\mathbb{I}_1 - P)Q_4(\mathbb{I}_2 - P)Q_2(\mathbb{I}_1 - P)Q_3(\mathbb{I}_2 - P)Q_4)$$

$$= \mathbb{E}[Q_0PQ_1PQ_2PQ_3PQ_4] + 2 \mathbb{E}[Q_0PQ_3PQ_2PQ_1PQ_4] + \mathbb{E}[\text{Tr}(Q_1PQ_2P)Q_0PQ_3PQ_4]$$

(6.14)
for any random matrices $Q_i$, with $0 \leq i \leq 4$, independent of $X_1$ and $X_2$. Any $\pi$-matrix moment $M_\pi(P)$ associated with a crossing partition $\pi \in \mathcal{Q}_{2n,m} - \mathcal{Q}^+_{2n,m}$ has the form (6.14) with some random matrix $Q_0$ associated with a non-crossing partition $\pi^{[m]} \in \mathcal{Q}^+_{2m,m}$ for some $0 \leq m \leq n-2$. When $m = 0$ we use the convention $Q_0 = I$.

Observe that

$$E [Q_0 P Q_1 P Q_2 P Q_3 P Q_4] \quad \text{and} \quad E [Q_0 P Q_3 P Q_2 P Q_1 P Q_4]$$

can be interpreted as a matrix moment

$$M_\pi^{[R_0 R_1 R_2 ((n-m)-2)]}(P) = E \left( R_0 (X^*_{\alpha^*(1)} - P) R_1 (X^*_{\alpha^*(2)} - P) R_2 \cdots (X^*_{\alpha^*(2((n-m)-2))} - P) R_{2((n-m)-2)} \right)$$

associated with some deterministic matrices $R_i \in \{ P^k : 0 \leq k \leq 4 \}$ and some partition $\pi \in \mathcal{Q}_{2((n-m)-2),((n-m)-2)}$. By (6.11) when $\pi \in \mathcal{Q}^+_{2((n-m)-2),((n-m)-2)}$, the matrix moment $M_\pi^{[R_0 R_1 R_2 ((n-m)-2)]}(P)$ is a linear combination of monomials of the form

$$\text{Tr}_v(P) \ P^w \quad \text{with} \quad \begin{cases} 1 \leq w \leq 2(n-m) & 0 \leq m \leq (n-m) - 2 \\
\sum_{1 \leq i \leq m} v_i \leq (n-m) - 2 \\
\sum_{1 \leq i \leq m} iv_i + w = 2(n-m) \end{cases}$$

Assume that $(Q_1, Q_2)$ and $(Q_3, Q_4)$ are independent. In this case we have

$$E [\text{Tr}(Q_1 P Q_2 P) \ Q_0 P Q_3 P Q_4] = \text{Tr}(E [Q_1 P Q_2 P]) \ E [Q_0 P Q_3 P Q_4]$$

Arguing as above

$$E [Q_1 P Q_2 P] \quad \text{and} \quad E [Q_0 P Q_3 P Q_4]$$

can be can be interpreted as the matrix moments $M_{\alpha}^{[S_0 \ldots S_k]}(P)$ and $M_{\beta}^{[T_0 \ldots T_l]}(P)$ associated with a collection of deterministic partitions $S_i \in \{ P^k : 0 \leq k \leq 2 \}$ and $T_j \in \{ P^k : 0 \leq k \leq 2 \}$ and some partitions $\alpha \in \mathcal{Q}_{2k,k}$ and $\beta \in \mathcal{Q}_{2l,l}$ with $k + l = (n-m) - 2$. By (6.11) when $\alpha \in \mathcal{Q}^+_{2k,k}$, the matrix moment $M_{\alpha}^{[S_0 \ldots S_k]}(P)$ is a linear combination of monomials of the form

$$\text{Tr}_v(P) \ P^w \quad \text{with} \quad \begin{cases} 1 \leq w \leq 2(k+1) & 0 \leq m \leq k \\
\sum_{1 \leq i \leq m} v_i \leq k \\
\sum_{1 \leq i \leq m} iv_i + w \leq 2(k+1) \end{cases}$$

In addition, when $\beta \in \mathcal{Q}^+_{2k,k}$, the matrix moment product

$$M_{\alpha}^{[S_0 \ldots S_k]}(P) \ M_{\beta}^{[T_0 \ldots T_l]}(P)$$

is a linear combination of monomials of the form

$$\text{Tr}_v(P) \ P^w \quad \text{with} \quad \begin{cases} 1 \leq w \leq 2(n-m) & 0 \leq m \leq k \\
\sum_{1 \leq i \leq m} v_i \leq n-m-2 \\
\sum_{1 \leq i \leq m} iv_i + w = 2(n-m) \end{cases}$$

Whenever $(Q_1, Q_2)$ and $(Q_3, Q_4)$ are not independent we have

$$\text{Tr}(Q_1 P Q_2 P) \ Q_0 P Q_3 P Q_4 \overset{\text{law}}{=} \text{Tr}(\overline{Q}_1 (X - P) \overline{Q}_2) \ Q_0 \overline{Q}_3 (X - P) \overline{Q}_4$$

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The quantity $2$ associated with some partitions $\pi$ (crossing partition for some $\alpha, \beta$) and/or non-crossing between them. For instance for an even number $n$ can be expressed as a sum of matrix moments products $\alpha \in Q_{2(n-m)-3}, (n-m)-3$, the matrix moment $M_\alpha(\alpha, \beta)(P)$ is a linear combination of monomials of the form

$$\text{Tr}_v(P) \, P^w \quad \text{with} \quad \begin{cases} 1 \leq w \leq 2((n-m)-1) & 0 \leq m \leq (n-m)-3 \\ \sum_{1 \leq i \leq m} v_i \leq (n-m)-3 \\ \sum_{1 \leq i \leq m} iv_i + w = 2((n-m)-1) \end{cases}$$

In summary, we have proved that the $\pi$-matrix moment $M_\pi(P)$ associated with a given crossing partition $\pi \in Q_{2n,n}$ can be expressed as a sum of matrix moments products

$$M_\beta^{[R_0, \ldots, R_{2k}]}(P) \, M_\beta^{[S_0, \ldots, S_{2l}]}(P)$$

associated with some partitions $(\alpha, \beta) \in (Q_{2k, k} \times Q_{2l, l})$ with $0 \leq k+l \leq n-2$, and some matrices

$$R_i, S_i \in \{P^k \mid 0 \leq k \leq 4\}$$

whose values depend on $\pi$. Iterating this procedure whenever $\alpha$ and/or $\beta$ are crossing, we can write $M_\pi(P)$ as a sum of matrix moments products

$$M_{\gamma_1}^{[T_0^{(1)}, \ldots, T_{2k_1}^{(1)}]}(P) \, \ldots \, M_{\gamma_q}^{[T_0^{(m)}, \ldots, T_{2k_q}^{(m)}]}(P)$$

for some $2 \leq q \leq n-2$ and some non-crossing partitions $\gamma_i \in Q_{2k_i, k_i}$ with $0 \leq k_1 + \ldots + k_q \leq n-2$. In addition, $M_{2k_i, k_i}^{[T_0^{(1)}, \ldots, T_{2k_i}^{(1)}]}(P)$ is a linear combination of monomials of the form

$$\text{Tr}_v(P) \, P^w \quad \text{with} \quad \begin{cases} 1 \leq w \leq l_i & 0 \leq m \leq k_i \\ \sum_{1 \leq j \leq m} v_j \leq k_i \\ \sum_{1 \leq j \leq m} jv_j + w \leq l_i \end{cases}$$

for some $l_i \geq 2k_i$. We conclude that $M_{2n,n}(P)$ is a linear combination of monomials of the form

$$\text{Tr}_v(P) \, P^w \quad \text{with} \quad \begin{cases} 1 \leq w \leq 2n & 0 \leq m \leq n-2 \\ \sum_{1 \leq j \leq m} v_j \leq n-2 \\ \sum_{1 \leq j \leq m} jv_j + w \leq 2n \end{cases}$$

The quantity $2n$ corresponds to maximal power of $P$ we can obtain using crossing partitions. It arises when we consider partitions $\pi$ with the maximal number of pairwise crossing pairs, non-crossing between them. For instance for an even number $n$ of pairs, pairwise crossing and non-overlapping we have

$$M_\pi(P) = (3 \, P^4 + \text{Tr}(P^2) \, P^2)^{n/2}$$

The proof of theorem 3.4 is now complete. \qed
6.3 Proof of theorem 3.5

We check that (1.14) satisfy (3.16). Observe that

\[
\Sigma_n(P) = \text{Tr}(P) \sum_{\pi \in \mathcal{N}_n} \prod_{i \geq 0} \text{Tr} \left( P^{i+1} \right)^{r_i(\pi)} \] \Rightarrow \text{Tr} \left( \Sigma_n(P) \right) = \text{Tr}(P) \sum_{\pi \in \mathcal{N}_n} \prod_{i \geq 0} \text{Tr} \left( P^{i+1} \right)^{r_i(\pi)} \frac{P^{\pi_0} \pi + 1}{\text{Tr}(P^{\pi_0} + 1)}
\]

Thus, for any \( k + l = n \), we have

\[
\text{Tr} \left[ \Sigma_k(P)P \right] \cdot \text{Tr} \left[ \Sigma_l(P) \right] = \text{Tr}(P)^2 \sum_{(\pi, \tau) \in (\mathcal{N}_k \times \mathcal{N}_l)} \prod_{i \geq 0} \text{Tr} \left( P^{i+1} \right)^{r_i(\pi) + r_i(\tau)} \frac{P^{\pi_0 + 1}}{\text{Tr}(P^{\pi_0 + 1})}
\]

We associate with a pair of partitions \((\pi, \tau) \in (\mathcal{N}_k \times \mathcal{N}_l)\) a partition \(\tau \in \mathcal{N}_{n+1}\) with blocks \(\tau_1 := \pi_1 \cup \{n + 1\} \leq \tau_i := \pi_i \leq \tau_{i+1} := (k + 1) + u : u \in \tau_j\)
for any \(2 \leq i \leq |\pi|\) and \(1 \leq j \leq |\tau|\). Observe that

\[
r_{\tau_{i+1}}(\tau) = r_{\tau_{i+1}}(\pi) + r_{\tau_{i+1}}(\tau) + 1 \quad r_{\tau_{i+1}}(\tau) = r_{\tau_{i+1}}(\pi) + r_{\tau_{i+1}}(\tau) - 1
\]

\[
r_i(\tau) = r_i(\pi) + r_{\tau_{i+1}}(i) \quad \forall i \geq 1 \quad \text{and} \quad i \notin \{|\pi|, |\pi| + 1\}
\]

\[
\Rightarrow \sum_{i \geq 1} r_i(\tau) = \sum_{i \geq 1} [r_i(\pi) + r_i(\tau)]
\]

On the other hand, we have

\[
\text{Tr} \left( P^{n+1} - \sum_{i \geq 1} r_i(\pi) + r_i(\tau) \right) \left[ \prod_{i \geq 1} \text{Tr} \left( P^{i+1} \right)^{r_i(\pi) + r_i(\tau)} \right] \frac{P^{\pi_0 + 1}}{\text{Tr}(P^{\pi_0 + 1})}
\]

\[
= \text{Tr}(P)^{r_0(\tau)} \left[ \prod_{i \geq 1, i \notin \{|\pi|, |\pi| + 1\}} \text{Tr} \left( P^{i+1} \right)^{r_i(\tau)} \right]
\]

\[
\times \text{Tr} \left( P^{\tau_1} \right)^{r_{\tau_1 - 1}(\tau) + 1} \text{Tr} \left( P^{\tau_1} \right)^{r_{\tau_1 - 1}(\tau) - 1} \frac{P^{\tau_1}}{\text{Tr}(P^{\tau_1})}
\]

This implies that

\[
\sum_{k+l=n} \text{Tr} \left[ \Sigma_k(P)P \right] \cdot \text{Tr} \left[ \Sigma_l(P) \right] = \text{Tr}(P) \sum_{\tau \in \mathcal{N}_{n+1}} \text{Tr}(P)^{r_0(\tau)} \left[ \prod_{i \geq 1} \text{Tr} \left( P^{i+1} \right)^{r_i(\tau)} \right] \frac{P^{\tau_1}}{\text{Tr}(P^{\tau_1} + 1)} = \Sigma_{n+1}(P)
\]

This ends the proof of the matrix polynomial formula (1.14).

Now we come to the proof of the Kreweras trace-type formula (3.18). For any \(\tau \in \mathcal{N}_{n+1}\) we set

\[
k_{\tau_1 - 1}(\tau) := r_{\tau_1 - 1}(\tau) + 1 \quad k_1(\tau) = 1 + r_0(\tau)
\]

\[
k_{\tau_1 + 1}(\tau) := r_{\tau_1}(\tau) - 1 \quad \text{and} \quad k_{i+1}(\tau) := r_i(\tau) \quad \forall i \geq 0 : i \notin \{0, |\tau_1| - 1, |\tau_1|\}
\]
Observe that
\[
\sum_{i \geq 1} k_i(\tau) = 1 + r_0(\tau) + \sum_{i \geq 1} r_i(\tau) = 1 + n
\]
\[
\sum_{i \geq 1} i k_i(\tau) = 1 + r_0(\tau) + \sum_{j \geq 1, j \notin \{|\tau_1|-1, |\tau_1|\}} (j + 1) r_j(\tau) + |\tau_1| (|\tau_1|-1) + (|\tau_1| + 1) (|\tau_1| - 1)
\]
\[
= r_0(\tau) + \sum_{j \geq 1} r_j(\tau) + \sum_{j \geq 1} j r_j(\tau) = 2n
\]

Taking the trace in the above matrix polynomial formula we find that
\[
\text{Tr}(\Sigma_n(P)) = \sum_{\tau \in \mathcal{N}_n} \text{Tr}(P) k_1(\tau) \left[ \prod_{i \geq 1, i \notin \{|\tau_1|-1, |\tau_1|\}} \text{Tr}(P^{1+i})^{k_{i+1}(\tau)} \right] 
\times \text{Tr}\left(P^{1+|\tau_1|}\right)^{r_{|\tau_1|}(\tau)-1} \text{Tr}\left(P^{|\tau_1|}\right)^{r_{|\tau_1|-1}(\tau)+1} = \sum_{\tau \in \mathcal{N}_n} \left[ \prod_{i \geq 1} \text{Tr}(P^i)^{k_i(\tau)} \right]
\]

This ends the proof of \(3.18\). The proof of the theorem is now completed. \(\square\)

### 6.4 Proof of theorem 3.7

Observe that \(\text{cl}(\mathcal{N}_{n,m})\) coincides with the subset of partitions \(\pi \in \mathcal{N}_{n+1,m}\) s.t. \(\iota(\pi) = 1\). Also observe that for any given \(\pi \in \mathcal{N}_{n,m}\) the partition \(\Xi(\text{cl}(\pi))\) has the same combinatorial structure as \(\Xi(\pi)\) but it contains one more singleton than \(\Xi(\pi)\). This yields the formula

\[
\sum_{\pi \in \mathcal{N}_{n,m}} \left[ \prod_{i \geq 1} \text{Tr}(P^i)^{r_i(\Xi(\pi))} \right] P
\]

\[
= \sum_{\pi \in \mathcal{N}_{n,m}} \left[ \prod_{i \geq 1} \text{Tr}(P^i)^{r_i(\Xi(\text{cl}(\pi)))} \right] \frac{P^{i(\text{cl}(\pi))}}{\text{Tr}(P^{i(\text{cl}(\pi))})} P
\]

from which we prove that

\[
\sum_{\pi_1 \in \mathcal{N}_{n_1,m_1}} \sum_{\pi_2 \in \mathcal{N}_{n_2,m_2}} \left[ \prod_{i \geq 1} \text{Tr}(P^i)^{r_i(\Xi(\pi_1)) + r_i(\Xi(\pi_2))} \right] \frac{P^{i(\pi_1)}}{\text{Tr}(P^{i(\pi_1)})} P
\]

\[
= \sum_{\pi_1 \in \mathcal{N}_{n_1,m_1}} \sum_{\pi_2 \in \mathcal{N}_{n_2,m_2}} \left[ \prod_{i \geq 1} \text{Tr}(P^i)^{r_i(\Xi(\pi_1) \oplus \Xi(\text{cl}(\pi_2)))} \right] \frac{P^{i(\pi_1) + (\text{cl}(\pi_2))}}{\text{Tr}(P^{i(\pi_1)}) \text{Tr}(P^{i(\pi_2)})} P
\]

Following the arguments and construction in Section 2.5 we have
\[
\forall i \notin \{\iota(\pi_1), \iota(\text{cl}(\pi_2))\} \quad r_i(\Xi(\pi_1) \oplus \Xi(\text{cl}(\pi_2))) = r_i(\Xi(\pi_1 + \text{cl}(\pi_2)))
\]
as well as
\[
\forall i \in \{\iota(\pi_1), \iota(\text{cl}(\pi_2))\} \quad r_i(\Xi(\pi_1) + \Xi(\text{cl}(\pi_2))) - 1 = r_i(\Xi(\pi_1 + \text{cl}(\pi_2)))
\]

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and
\[ r_{i(\pi_1) + i(\text{cl}(\pi_2))}(\Xi(\pi_1 \oplus \text{cl}(\pi_2))) = r_{i(\pi_1) + i(\text{cl}(\pi_2))}(\Xi(\pi_1) \oplus \Xi(\pi_2)) + 1 \]

This implies that
\[ \sum_{\pi_1 \in \mathbb{N}_{n_1}, m_1} \sum_{\pi_2 \in \mathbb{N}_{n_2}, m_2} \left[ \prod_{i \geq 1} \text{Tr}(P^i r_i(\Xi(\pi_1)) + r_i(\Xi(\pi_2))) \right] \frac{P^i(\pi_1)}{\text{Tr}(P^i(\pi_1))} P = \sum_{\pi_1 \in \mathbb{N}_{n_1}, m_1} \sum_{\pi_2 \in \mathbb{N}_{n_2}, m_2} \left[ \prod_{i \geq 1} \text{Tr}(P^i r_i(\Xi(\pi_1) \oplus \text{cl}(\pi_2))) \right] \frac{P^i(\pi_1 \oplus \text{cl}(\pi_2))}{\text{Tr}(P^i(\pi_1 \oplus \text{cl}(\pi_2)))} P \]

Using (2.20), this shows that (1.15) satisfies (3.19).

6.5 Proof of theorem 4.1

Combining (6.3) with the estimates (3.2) we check that
\[ \| E(H^n_{N}) \|_F \leq n! (8 \text{Tr}(P))^n N^{-n/2} \sum_{1 \leq m \leq (n/2) \land N} \frac{(N)_m}{m!} 2^{-(2m+1)} \]

This ends the proof of (4.2). Also observe that for any \( n \leq N \) we have
\[ E[H^N_{2n}] = \frac{(N)_n}{N^n} M_{2n,n}(P) + \sum_{1 \leq m < n} \frac{(N)_m}{N^n} M_{2n,m}(P) \]  
(6.15)

We check (4.3) using the estimates (3.2). We also have
\[ 1 - \frac{(n-1)^2}{N} \leq \frac{(N)_n}{N^n} \leq 1 \]

Using the estimates (3.2) we readily check (4.4). The proof is now completed.

6.6 Proof of theorem 4.2

Using the fact that
\[ (I - 2sP)^{-1} P = \partial_s \log (I - 2sP)^{-1/2} = -\frac{1}{2} \partial_s \log (I - 2sP) \]

formula (4.5) can alternatively be rewritten as
\[ E(\exp[t X]) = I + \int_0^t \exp \left[ \text{Tr} \left( \log (I - 2sP)^{-1/2} \right) \right] \partial_s \log (I - 2sP)^{-1/2} ds \]
\[ \leq I - \frac{1}{2} \log (I - 2tP) \exp \left[ -\frac{1}{2} \text{Tr} (\log (I - 2tP)) \right] \]

The proof of theorem 4.2 is based on the following technical lemma.

Lemma 6.2. For any \( 0 \leq t \leq 1/(2 \text{Tr}(P)) \) we have
\[ E \left( \exp \left[ \frac{t X}{2 \text{Tr}(P)} \right] \right) \leq I + \frac{t}{2} \frac{P}{\text{Tr}(P)} + \frac{t^2}{4} \left[ \left( \frac{P}{\text{Tr}(P)} \right)^2 + \frac{1}{2} \frac{P}{\text{Tr}(P)} \right] + t L(t) \approx_0 \left( \frac{P}{\text{Tr}(P)} \right) \]  
(6.16)
By (6.27) and (6.28) this yields the estimate

$$w_0(Q) := Q \left[ \frac{1}{6\sqrt{2}} I + \frac{1}{12} Q + Q^2 \right]$$

In addition, we have

$$\mathbb{E} \left( \exp \left[ \frac{t(X - P)}{2 \text{Tr}(P)} \right] \right) \leq I + \frac{t^2}{8} \left[ \left( \frac{P}{\text{Tr}(P)} \right)^2 + \frac{P}{\text{Tr}(P)} \right] + tL(t) w_0 \left( \frac{P}{\text{Tr}(P)} \right) \quad (6.17)$$

Proof. Using (6.26) we check that

$$\Rightarrow \forall t \geq 0 \quad \exp [tP] \leq \mathbb{E} (\exp [tX]) \leq I + Pt + \frac{t^2}{2} \left[ 2P^2 + P \text{Tr}(P) \right] + \sum_{n \geq 3} \frac{t^n}{n!} \mathbb{E}(X^n)$$

as well as

$$\mathbb{E} (\exp [t(X - P)]) \leq I + \frac{t^2}{2} \left[ P^2 + P \text{Tr}(P) \right] + \sum_{n \geq 3} \frac{t^n}{n!} \mathbb{E}(X^n)$$

On the other hand, using (6.26) we also check that

$$\sum_{n \geq 3} \frac{t^n}{n!} \mathbb{E}(X^n)$$

$$\leq \sum_{n \geq 3} \frac{(2t)^n}{2n} \left[ \frac{(2(n-1))!}{2^{(n-1)}(n-1)!} \text{Tr}(P)^{n-1} P + \frac{(2(n-2))!}{2^{(n-2)}(n-2)!^2} \text{Tr}(P)^{n-2} P^2 \right]$$

$$+ (2tP)^3 \sum_{n \geq 0} \frac{(2t)^n}{2(n+3)} \sum_{0 \leq k \leq n} \frac{(2k)!}{2^{2k}k!^2} \text{Tr}(P)^k P^{n-k}$$

By (6.27) and (6.28) this yields the estimate

$$\sum_{n \geq 3} \frac{t^n}{n!} \mathbb{E}(X^n) \leq \frac{(2t)^3}{2} \sum_{n \geq 0} \frac{(2t \text{Tr}(P))^n}{n + 3} \frac{1}{\sqrt{3n+4}} P \text{Tr}(P)^2$$

$$+ \frac{(2t)^3}{2} \sum_{n \geq 0} \frac{(2t \text{Tr}(P))^n}{n + 3} \frac{1}{\sqrt{3n+4}} P^2 \text{Tr}(P)$$

$$+ (2tP)^3 \sum_{n \geq 0} \frac{(2t \text{Tr}(P))^n}{2(n+3)} \frac{2n + 1}{\sqrt{3n+1}} \quad (6.18)$$

This implies that

$$\sum_{n \geq 3} \frac{t^n}{n!} \mathbb{E}(X^n) \leq \frac{(2t)^2}{2} \sum_{n \geq 1} \frac{(2t \text{Tr}(P))^n}{n + 2} \frac{1}{\sqrt{3n+4}} P \text{Tr}(P)$$

$$+ \frac{(2t)^2}{2} \sum_{n \geq 1} \frac{(2t \text{Tr}(P))^n}{n + 2} \frac{1}{\sqrt{3n+1}} P^2 + \frac{(2tP)^3}{1 - 2t \text{Tr}(P)}$$

$$\leq - (2t \text{Tr}(P))^2 \left( \frac{1}{4 \text{Tr}(P)} + \frac{1}{2} \left( \frac{P}{\text{Tr}(P)} \right)^2 \right) \log (1 - 2t \text{Tr}(P)) + \frac{(2tP)^3}{1 - 2t \text{Tr}(P)}$$

$$\leq \frac{(2t \text{Tr}(P))^3}{1 - 2t \text{Tr}(P)} \left[ \frac{1}{4 \text{Tr}(P)} + \frac{1}{2} \left( \frac{P}{\text{Tr}(P)} \right)^2 + \left( \frac{P}{\text{Tr}(P)} \right)^3 \right]$$
In the last assertion we have used the estimate \(- \log(1 - x) \leq x/(1 - x)\). This estimate can be slightly improved. Indeed, using (6.18) we check that
\[
\sum_{n \geq 3} \frac{t^n}{n!} \mathbb{E}(X^n) \leq \frac{(2t \text{Tr}(P))^3}{1 - 2t \text{Tr}(P)} \left[ \frac{1}{6\sqrt{7}} \frac{P}{\text{Tr}(P)} + \frac{1}{12} \left( \frac{P}{\text{Tr}(P)} \right)^2 + \left( \frac{P}{\text{Tr}(P)} \right)^3 \right]
\]
This ends the proof of the lemma follows elementary manipulations, thus it is skipped.

We now return to prove theorem 4.2. The estimate (6.16) can be rewritten as follows
\[
\mathbb{E} \left( \exp \left[ \frac{t}{2} \text{Tr}(P) \right] \right) \leq I + \frac{t}{2} \frac{P}{\text{Tr}(P)} + t^2 \left[ \frac{1}{4} \left( \frac{P}{\text{Tr}(P)} \right)^2 + \frac{1}{8} \frac{P}{\text{Tr}(P)} \right]
\]
\[
+ \left( \frac{1}{1 - t} - 1 \right) t^2 \left[ \frac{1}{6\sqrt{7}} \frac{P}{\text{Tr}(P)} + \frac{1}{12} \left( \frac{P}{\text{Tr}(P)} \right)^2 + \left( \frac{P}{\text{Tr}(P)} \right)^3 \right]
\]
\[
\leq I + \frac{t}{2} \frac{P}{\text{Tr}(P)} + \frac{t^2}{1 - t} \left[ \frac{1}{4} \left( \frac{P}{\text{Tr}(P)} \right)^2 + \frac{1}{8} \frac{P}{\text{Tr}(P)} \right]
\]
\[
+ \frac{t^2}{1 - t} \left[ \frac{1}{6\sqrt{7}} \frac{P}{\text{Tr}(P)} + \frac{1}{12} \left( \frac{P}{\text{Tr}(P)} \right)^2 + \left( \frac{P}{\text{Tr}(P)} \right)^3 \right]
\]
\[
\leq I + \frac{t}{2} \frac{P}{\text{Tr}(P)} + \frac{t^2}{1 - t} \left[ \frac{1}{3} \left( \frac{P}{\text{Tr}(P)} \right)^2 + \frac{1}{5} \frac{P}{\text{Tr}(P)} + \left( \frac{P}{\text{Tr}(P)} \right)^3 \right]
\]
This yields
\[
\mathbb{E} \left( \exp \left[ \frac{t}{2} \text{Tr}(P) \right] \right) \leq I + \frac{t}{2} \frac{P}{\text{Tr}(P)} + \frac{t^2}{1 - t} \omega_1 \left( \frac{P}{\text{Tr}(P)} \right) \quad (6.19)
\]
The exponential matrix estimate (6.17) is now a consequence of (3.12) and the estimate (6.19).
In the same vein, we have
\[
\mathbb{E} \left( \exp \left[ t (X - I) \right] \right) \leq I + \frac{(2t \text{Tr}(P))^2}{1 - 2t \text{Tr}(P)} \left[ \frac{1}{5} \frac{P}{\text{Tr}(P)} + \frac{5}{24} \left( \frac{P}{\text{Tr}(P)} \right)^2 + \left( \frac{P}{\text{Tr}(P)} \right)^3 \right]
\]
This ends the proof of (4.8). The proof of the theorem is completed.

6.7 Proof of theorem 4.4

The proof of theorem 4.4 is partly based on the following technical lemma of its own interest.

**Lemma 6.3.** For any \( t \in \mathbb{R} \) we have
\[
\log \|\mathcal{E}(t)\|_P \leq \frac{t^2}{2} \mathbb{E}(\|\mathcal{H}\|_P^2) \quad (6.20)
\]
In addition, for any \( m \geq 1 \) we have the estimate
\[
\left\| E(t) - \sum_{n \leq m} \frac{t^{2n}}{(2n)!} E(\mathcal{H}^{2n}) \right\|_F 
\leq \exp \left[ \frac{t^2}{2} \mathbb{E}(\|\mathcal{H}\|^2_F) \right] \frac{1}{\sqrt{2\pi (m + 1)}} \left( \frac{ct^2}{2} \mathbb{E}(\|\mathcal{H}\|^2_F) \right)^{m+1}
\tag{6.21}
\]

**Proof.** By (3.2) we have
\[
\frac{t^{2n}}{(2n)!} \left\| E(\mathcal{H}^{2n}) \right\|_F \leq \frac{t^{2n}}{(2n)!} \nu_{2n} \mathbb{E} \left[ \| \mathcal{H}^2 \|_F^n \right] = \frac{1}{n!} \left( \frac{t^2}{2} \mathbb{E} \left[ \| \mathcal{H}^2 \|_F \right] \right)^n
\]
This ends the proof of (6.20). On the other hand, for any \( x \geq 0 \) we have
\[
\sum_{n \geq m} \frac{x^n}{n!} \leq e^x \frac{x^m}{m!} \left( \leq e^x \frac{1}{\sqrt{2\pi m}} \left( \frac{ex}{m} \right)^m \right)
\]
This implies that
\[
\sum_{n > m} \frac{t^{2n}}{(2n)!} \left\| E(\mathcal{H}^{2n}) \right\|_F \leq \exp \left[ \frac{t^2}{2} \mathbb{E}(\|\mathcal{H}\|^2_F) \right] \frac{1}{\sqrt{2\pi (m + 1)}} \left( \frac{ct^2}{2m} \mathbb{E}(\|\mathcal{H}\|^2_F) \right)^{m+1}
\]
This ends the proof of (6.21). The proof of the lemma is completed.

We have
\[
\mathbb{E} \left[ \cosh \left( t \mathcal{H}_N \right) \right] - \mathbb{E} \left( \exp \left( t \mathcal{H}_n \right) \right) = \sum_{2 \leq n \leq N} \frac{t^{2n}}{(2n)!} \left[ \mathbb{E}(\mathcal{H}_N^{2n}) - \mathbb{E}(\mathcal{H}_n^{2n}) \right] + \sum_{n > N} \frac{t^{2n}}{(2n)!} \left( \mathbb{E}(\mathcal{H}_N^{2n}) - \mathbb{E}(\mathcal{H}_n^{2n}) \right)
\]
On the other hand, for any \( n \leq N \) we have
\[
\epsilon_{2n}(N) = \sum_{0 \leq m < n} \frac{4^m}{N^m} \frac{n!}{(n - m)!}
\]
Using (4.3) and (4.4) we obtain the estimate
\[
N \left\| \mathbb{E} \left[ \cosh \left( t \mathcal{H}_N \right) \right] - \mathbb{E} \left( \exp \left( t \mathcal{H}_n \right) \right) \right\|_F 
\leq \left( \frac{t^2}{2} \mathbb{E}(\|\mathcal{H}\|^2_F) \right)^2 \sum_{2 \leq n \leq N} \frac{1}{(n - 2)!} \frac{(n - 1)^2}{n(n - 1)} \left( \frac{t^2}{2} \mathbb{E}(\|\mathcal{H}\|^2_F) \right)^{n-2}
\]
\[
+ \frac{1}{2} \sum_{2 \leq n \leq N} (4\sqrt{2} t \text{Tr}(P))^{2n} \sum_{0 \leq m < n} \frac{(4/N)^m}{(n - m)!} + N \left\| \sum_{n > N} \frac{t^{2n}}{(2n)!} \mathbb{E}(\mathcal{H}^{2n}) \right\|_F 
\]
\[
+ \frac{1}{N} \left( 2t \text{Tr}(P) \right)^{2N} \left( \sqrt{2} t \text{Tr}(P) \right)^{2N} \frac{1}{1 - (2\sqrt{2} t \text{Tr}(P))^{2N}} \sum_{0 \leq m < N} \left( \frac{4}{N} \right)^m \frac{1}{(N - m)!}
\]
as soon as \( \sqrt{N} > 2 \sqrt{2} t \text{Tr}(P) \). We use the decomposition
\[
\sum_{2 \leq n \leq N} a_n \sum_{0 \leq m < n} b_{n,m} = \sum_{2 \leq k \leq N} a_k b_{k,0} + \sum_{1 \leq l < N} \sum_{k=l+1}^N a_k b_{k,l}
\]
which is valid for any collection of numbers \( a_n \) and \( b_{n,m} \), to check that

\[
\sum_{2 \leq n \leq N} (4\sqrt{2} t \text{Tr}(P))^{2n} \sum_{0 \leq m < n} \frac{(4/N)^m}{(n-m)!}
= \sum_{2 \leq k \leq N} \frac{(4\sqrt{2} t \text{Tr}(P))^{2k}}{k!} + \sum_{1 \leq l < N} \left( \frac{1}{N} (8\sqrt{2} t \text{Tr}(P))^2 \right) \sum_{k=1}^{N-l} \frac{(4\sqrt{2} t \text{Tr}(P))^{2k}}{k!} \frac{1}{k!}
\leq \sum_{2 \leq k \leq N} \frac{(4\sqrt{2} t \text{Tr}(P))^{2k}}{k!} + \frac{4(e-1)}{N} \frac{(4\sqrt{2} t \text{Tr}(P))^4}{1 - \frac{1}{N} (4\sqrt{2} t \text{Tr}(P))^2}
\]
as soon as \( 4\sqrt{2} t \text{Tr}(P) < 1 \). In addition, when \( N \geq 8 \) we have

\[
\leq \left[ \frac{e}{2} + \frac{8(e-1)}{N} \right] (4\sqrt{2} t \text{Tr}(P))^4
\]

Choosing \( (4\sqrt{2} t \text{Tr}(P) \leq 4\sqrt{2} t \mathbb{E} [\|H\|_F^2]^{1/2} < 1 \) and \( N \geq 8 \) we check that

\[
N \| \mathbb{E} [\cosh (t \|H\|_N)] - \mathbb{E} (\exp (t \|H\|_N)) \|_F
\leq \left( \frac{t^2}{2} \mathbb{E} [\|H\|_F^2] \right)^2 \exp \left( \frac{t^2}{2} \mathbb{E} [\|H\|_F^2] \right) + \left[ \frac{e}{4} + \frac{4(e-1)}{N} \right] (4\sqrt{2} t \text{Tr}(P))^4
+ N \| \sum_{n>N} \frac{t^{2n}}{(2n)!} \mathbb{E} [\|H\|_F^{2n}] \|_F + \frac{(e-1)}{4^2 N} \left( 4\sqrt{2} t \text{Tr}(P) \right)^4 \left( \sqrt{2} t \text{Tr}(P) \right)^{2(N-1)}
\leq \left[ \frac{e}{4} + \frac{5(e-1)}{N} \right] (4\sqrt{2} t \text{Tr}(P))^4
+ \left( \frac{t^2}{2} \mathbb{E} [\|H\|_F^2] \right)^2 \exp \left( \frac{t^2}{2} \mathbb{E} [\|H\|_F^2] \right) \left[ 1 + \frac{1}{N^{3/2}} \frac{e^2}{\sqrt{2\pi}} \left( \frac{et^2}{2(N+1)} \mathbb{E} [\|H\|_F^2] \right)^{N-1} \right]
\leq t^4 \mathbb{E} [\|H\|_F^2] \left( 4^4 e + \frac{e}{4} + \frac{5(e-1)}{N} \right) 4^5 + \frac{1}{N^{3/2}} \frac{e^3}{4\sqrt{2\pi}}
\]

We also have

\[
\mathbb{E} [\sinh (t \|H\|_N)] = \sum_{n \geq 1} \frac{t^{2n+1}}{(2n+1)!} \mathbb{E} [\|H\|_N^{2n+1}]\]

Using \( [4.2] \) we find that

\[
\| \mathbb{E} [\sinh (t \|H\|_N)] \|_F \leq \frac{t}{2} \text{Tr}(P) \sum_{n \geq 1} \frac{(2t^2 \text{Tr}(P))^n}{N^{1/2}} \sum_{1 \leq m \leq n \land N} \left( \frac{4}{N} \right)^{n-m} \frac{1}{m!}
\leq t^3 (e-1) \frac{\text{Tr}(P)^3}{N^{1/2}} \frac{1}{1 - 2t^2 \text{Tr}(P)^2} \leq 2t^3 (e-1) \frac{\text{Tr}(P)^3}{N^{1/2}}
\]
as soon as \( N \geq 4 \) and \( 4t^2\text{Tr}(P)^2 \leq 1 \). This ends the proof of (4.11). This ends the proof of the theorem.

6.8 Proof of theorem 5.1

Observe that

\[
\mathbb{E} \left( \exp \left( \frac{t}{\sqrt{2r}} \text{Tr}(A\mathcal{H}_N) \right) \right)
= \exp \left[ \frac{t^2}{2} \text{Tr}((AP)^2) + \left( \frac{2}{rN} \right)^{1/2} \sum_{n \geq 3} \frac{t^n}{n} \left( \frac{2}{rN} \right)^{(n-3)/2} \text{Tr}((AP)^n) \right]
\]

Also recall that \( \text{Tr}[(AP)^{2n}] \geq 0 \) and \( \text{Tr}[(AP)^{2n+1}] \in \mathbb{R} \) for any \( n \geq 0 \). We also have the estimates

\[
\begin{align*}
\text{Tr}((AP)^n) &\leq r^{-1}|\lambda^*(AP)|^n + r^{-1} \sum_{2 \leq i \leq r-1} \left[ \frac{\lambda_i(AP)}{\lambda^*(AP)} \right]^n \leq r^{-1} |\lambda^*(AP)|^n + 1
\end{align*}
\]

We conclude that

\[
\sup_{r \geq 1} \lambda^*(AP) < \infty \implies \sum_{n \geq 3} \frac{t^n}{n} \sup_{r \geq 1} \text{Tr}((AP)^n) | < \infty
\]

The end of the proof of the first assertion of the theorem is now completed. Using (2.2), for any \( n \geq 2 \) we check that

\[
\left| \frac{1}{n} \left( \frac{2}{rN} \right)^{(n-2)/2} \text{Tr}((AP)^n) \right| \leq \frac{1}{2} \left( \frac{2}{rN} \right)^{(n-2)/2} \alpha_A(P) \beta_A(P)^n = \rho \beta^n
\]

with

\[
\rho = \frac{rN}{4} \alpha_A(P) \quad \text{and} \quad \beta = \sqrt{\frac{2}{rN}} \beta_A(P)
\]

The end of the proof is a direct consequence of lemma 2.1. For instance we check that

\[
\begin{align*}
\left| \mathbb{E} \left( \left( \frac{1}{\sqrt{2r}} \text{Tr}(A\mathcal{H}_N) \right)^{2n} \right) - \frac{(2n)!}{2^n n!} \left[ \text{Tr}((AP)^2) \right]^n \right|
\leq \frac{(2n)!}{2} \beta_A(P)^{2n} \left( \frac{8}{rN} \right)^n \sum_{1 \leq k < n} \frac{1}{k!} \left( \frac{rN}{8} \right)^k \alpha_A(P)^k
\leq \frac{(2n)!}{2} \beta_A(P)^{2n} \alpha_A(P)^{n-1} \left( \frac{8}{rN} \right)^n \sum_{1 \leq k < n} \frac{1}{k!} \left( \frac{rN}{8} \right)^k
\leq \frac{(e-1)}{2} \frac{1}{2} (2n)! \alpha_A(P)^{n-1} \beta_A(P)^{2n} \left( \frac{8}{rN} \right)^n \left( 1 + \frac{rN}{8} \right)^{n-1}
\end{align*}
\]

This end the proof of (3.7).
6.9 Proof of theorem 5.2

Using the sub-Gaussian estimate (5.7) we check that

$$\mathbb{P}(\pm \text{Tr}(A^H N) \geq u) \leq \exp \left( -\frac{1}{4} \frac{u^2}{\text{Tr}((A^2) + 2\|AP\|_F^2)} \right)$$

for any

$$u \leq \sqrt{N} \frac{2\|AP\|_F^2 + \text{Tr}((A^2) + 2\|AP\|_F^2)}{2\|AP\|_F^2}$$

We end the proof of the first assertion by choosing

$$\delta = \frac{1}{4} \frac{u^2}{\text{Tr}((A^2) + 2\|AP\|_F^2)} \leq \frac{N}{8} \left( 1 + \frac{\text{Tr}((A^2) + 2\|AP\|_F^2)}{2\|AP\|_F^2} \right) \iff \delta \leq \frac{N}{8}$$

By lemma 6.6 we have

$$2t\text{Tr}(AP) < \sqrt{N} \implies \log \mathbb{E} (\exp (-t \text{ Tr} (A^H N))) \leq \frac{t^2}{2} \left( 2\text{Tr}((A^2) + 2\|AP\|_F^2) \right)$$

Thus, for any \( u \geq 0 \) and any \( t \geq 0 \) s.t. \( 2t\text{Tr}(AP) < \sqrt{N} \) we have

$$\mathbb{P} (-\text{Tr}(A^H N) \geq u) \leq \mathbb{E} [\exp (-ut - t \text{Tr}(A^H N))] \leq \exp (-ut + t^2 \text{Tr}((A^2) + 2\|AP\|_F^2))$$

Choosing \( N \) and \( t \) such that

$$t = \frac{u}{2\text{Tr}((A^2) + 2\|AP\|_F^2)} \leq \frac{\sqrt{N}}{2\text{Tr}(AP)} \iff \sqrt{N} \geq u \frac{\text{Tr}(AP)}{\text{Tr}((A^2) + 2\|AP\|_F^2)}$$

we find that

$$\mathbb{P} (\text{Tr}(A^H N) \leq -u) \leq \exp \left( -\frac{1}{4} \frac{u^2}{\text{Tr}((A^2) + 2\|AP\|_F^2)} \right)$$

This implies that

$$0 \leq \delta \leq \frac{N}{4} \frac{\text{Tr}((A^2) + 2\|AP\|_F^2)}{\text{Tr}(AP)} \implies \mathbb{P} \left( \text{Tr}(A^H N) \leq -2 \sqrt{\delta \text{Tr}((A^2) + 2\|AP\|_F^2)} \right) \leq \exp (-\delta)$$

In the same vein, using the sub-Gaussian Laplace estimate (5.7) we have

$$\mathbb{P} (\text{Tr}(A^H N) \geq u) \leq \mathbb{E} [\exp (-ut + t \text{Tr}(A^H N))] \leq \exp \left( -ut + \frac{t^2}{2} \left[ 6\text{Tr}((A^2) + 2\|AP\|_F^2) \right] \right)$$

for any \( t < \frac{\sqrt{N}}{4\text{Tr}(AP)} \). Choosing \( N \) and \( t \) such that

$$t = \frac{u}{6\text{Tr}((A^2) + 2\|AP\|_F^2)} \leq \frac{N^{1/2}}{4\text{Tr}(AP)} \iff \sqrt{N} \geq \frac{2}{3} u \frac{\text{Tr}(AP)}{\text{Tr}((A^2) + 2\|AP\|_F^2)}$$

we conclude that

$$\mathbb{P} \left( \text{Tr}(A^H N) \geq u \right) \leq \exp \left( -\frac{u^2}{12\text{Tr}((A^2) + 2\|AP\|_F^2)} \right)$$

Replacing \( u \) by

$$\delta = \frac{u^2}{12\text{Tr}((A^2) + 2\|AP\|_F^2)} \leq \frac{3}{4} u \frac{\text{Tr}((A^2) + 2\|AP\|_F^2)}{\text{Tr}(AP)}$$

we prove that

$$\mathbb{P} \left( \text{Tr}(A^H N) \geq 2 \sqrt{3\delta \text{Tr}((A^2) + 2\|AP\|_F^2}) \right) \leq \exp (-\delta)$$

This ends the proof of the theorem. \( \square \)
6.10 Proof of theorem 5.4

We need the following lemma.

**Lemma 6.4.** For any collection of symmetric matrices \((A_i)_{i \in I}\) indexed by some finite set \(I\) we have

\[
4 \log |I| \leq N \implies \mathbb{E} \left( \max_{i \in I} \text{Tr}(A_i \mathcal{H}_N) \right) \leq 2 \sqrt{\log |I|} \max_{i \in I} \sqrt{\left[ \text{Tr}((A_i P)^2) + 2 \|A_i P\|_F^2 \right]} \quad (6.22)
\]

In addition, we have

\[
\mathbb{E} \left( \max_{i \in I} \text{Tr}(A_i \mathcal{H}) \right) \leq 2 \sqrt{\log |I|} \max_{i \in I} \sqrt{\left[ \text{Tr}((A_i P)^2) \right]}
\]

**Proof.** Applying Jensen’s inequality to the logarithm we check that

\[
0 < t \leq \frac{\sqrt{N}}{4 \max_{i \in I} \|A_i P\|_F} \implies \mathbb{E} \left( \max_{i \in I} \text{Tr}(A_i \mathcal{H}_N) \right) \leq \frac{1}{t} \log \sum_{i \in I} \mathbb{E} (\exp (t \text{Tr}(A_i \mathcal{H}_N)))
\]

In this situation, using (6.22) we have the estimate

\[
\mathbb{E} \left( \max_{i \in I} \text{Tr}(A_i \mathcal{H}_N) \right) \leq \frac{1}{t} \log \sum_{i \in I} \exp \left( t^2 \left[ \text{Tr}((A_i P)^2) + 2 \|A_i P\|_F^2 \right] \right)
\]

\[
\leq \frac{\log |I|}{t} + t \max_{i \in I} \left[ \text{Tr}((A_i P)^2) + 2 \|A_i P\|_F^2 \right]
\]

as soon as \(0 \leq t \leq \frac{\sqrt{N}}{4 \max_{i \in I} \|A_i P\|_F}\). We prove (6.22) by choosing

\[
t = \sqrt{\frac{\log |I|}{\max_{i \in I} \left[ \text{Tr}((A_i P)^2) + 2 \|A_i P\|_F^2 \right]}} \leq \frac{\sqrt{N}}{4 \max_{i \in I} \|A_i P\|_F} \quad \iff \quad 4 \log |I| \leq N
\]

This ends the proof of (6.22). Using the Laplace trace estimate (5.1) the last assertion follows the same line of arguments, thus it is skipped. This ends the proof of the lemma. □

We now move to prove theorem 5.4. We recall that,

\[
\|\mathcal{H}_N\|_{op} = \sup_{x,y \in \mathbb{B}} \langle \mathcal{H}_N x, y \rangle = \sup_{A \in \mathcal{A}} \text{Tr}(A \mathcal{H}_N)
\]

with the sets \((\mathcal{A}, \mathbb{B})\) introduced in (1.1). Observe that for any \(x, y \in \mathbb{B}\)

\[
\text{Tr}((xy' P)^2) + 2 \|xy' P\|_F^2 = \langle x, Py \rangle^2 + 2 \langle x, y \rangle \langle x, P^2 y \rangle \leq 3 \lambda_1(P)^2
\]

(6.23)

\[
\Rightarrow \max_{A \in \mathcal{A}} \sqrt{\left[ \text{Tr}((AP)^2) + 2 \|AP\|_F^2 \right]} \leq \sqrt{3} \lambda_1(P)
\]

We recall that \(\mathbb{B}\) can be covered by an \(\epsilon\)-net \(\mathbb{B} \subset \mathbb{B}\) of cardinality \(|\mathbb{B}| \leq (1 + 2/\epsilon)^r\), in the sense that for any \(x \in \mathbb{B}\) there exists some \(x_\epsilon \in \mathbb{B}_\epsilon\) such that \(\|x - x_\epsilon\| \leq \epsilon\). Using the decomposition

\[
xy' - x_\epsilon y' = (x - x_\epsilon)(y - y_\epsilon)' + x_\epsilon(y - y_\epsilon)' + (x - x_\epsilon)y_\epsilon'
\]

we check that \(\mathcal{A}\) can be covered by an \(\epsilon\)-net \(\mathcal{A}_\epsilon \subset \mathcal{A}\) w.r.t. the Frobenius norm of cardinality \(|\mathcal{A}_\epsilon| \leq (1 + 6/\epsilon)^{2r}\). In addition, we have

\[
|\text{Tr}(\mathcal{H}_N(xy' - x_\epsilon y_\epsilon'))| \leq \frac{2}{3} \epsilon \sup_{A \in \mathcal{A}} \text{Tr}(A \mathcal{H}_N)
\]

(6.24)

\[
\Rightarrow \left( 1 - \frac{2}{3} \epsilon \right) \sup_{A \in \mathcal{A}} \text{Tr}(A \mathcal{H}_N) \leq \sup_{A \in \mathcal{A}_\epsilon} \text{Tr}(A \mathcal{H}_N)
\]
Taking the expectation we conclude that
\[ \mathbb{E}[\|H_N\|_{op}] \leq 2\sqrt{3} \left( 1 - \frac{2}{3} \epsilon \right)^{-1} \sqrt{\log (1 + 6/\epsilon)} \sqrt{r} \lambda_1(P) \]
as soon as \( N \geq 8r \log (1 + 6/\epsilon) \). Choosing \( \epsilon = 2/10 \) we find that
\[ N \geq 8r \log (31) \implies \mathbb{E}[\|H_N\|_{op}] \leq 6 \sqrt{r} \lambda_1(P) \implies \mathbb{E}[\lambda_1(P_N)/\lambda_1(P)] \leq 1 + 6 \sqrt{r/N} \]
This ends the proof of the first assertion.

Now we come to the proof of (5.9). Combining (5.8) with the estimate (6.23) we check that
for any \( A \in \mathcal{A} \) the probability of the event
\[ \text{Tr}(AH^N) \geq 2 \sqrt{3} \delta \lambda_1(P) \]is less than \( e^{-\delta} \) as soon as \( 0 \leq 8\delta \leq N \). Using the estimate
\[ \mathbb{P}\left( \max_{i \in I} \text{Tr}(A_iH_N) \geq 2 \sqrt{3} \delta \lambda_1(P) \right) \leq \sum_{i \in I} \mathbb{P}\left( \text{Tr}(A_iH_N) \geq 2 \sqrt{3} \delta \lambda_1(P) \right) \leq (1 + 6/\epsilon)^{2r} e^{-\delta} \]
which is valid for any \( \epsilon \)-net \( \mathcal{A}_\epsilon = \{ A_i : i \in I \} \subset \mathcal{A} \), we find that the probability of the event
\[ \sup_{A \in \mathcal{A}_\epsilon} \text{Tr}(AH^N) \leq 2 \sqrt{3} \delta \lambda_1(P) \]is greater than \( 1 - (1 + 6/\epsilon)^{2r} e^{-\delta} \). Using (6.24) we also check that the probability of the event
\[ \|H_N\|_{op} \leq 2\sqrt{3} \left( 1 - \frac{2}{3} \epsilon \right)^{-1} \sqrt{\delta + 2r \log (1 + 6/\epsilon)} \lambda_1(P) \]is greater than \( 1 - e^{-\delta} \) for any \( (N, \delta, \epsilon) \) s.t. \( N/8 \geq \delta + 2r \log (1 + 6/\epsilon) \). We end the proof of (5.9) by choosing \( \epsilon = 2/10 \).

We come to the proof of corollary 5.5. Following the proof of proposition 5.4 and using (5.4), we check that the probability of the event
\[ \|H\|_{op} \leq \lambda_1(P) \left( 1 - \frac{2}{3} \epsilon \right)^{-1} 2\sqrt{\delta + 2r \log (1 + 6/\epsilon)} \lambda_1(P) \]is greater than \( 1 - e^{-\delta} \) for any \( \delta > 0 \) and \( \epsilon \in ]0, 1[ \). Choosing \( \epsilon = 2/10 \) we conclude that for any \( \delta \geq 0 \) the probability of the event
\[ \|H\|_{op} \leq \frac{5}{2} \lambda_1(P) \sqrt{\delta + 7r} \]is greater than \( 1 - e^{-\delta} \) (6.25)

By Weyl’s inequality (2.4) we have
\[ \sup_{1 \leq k \leq r} |\lambda_k(P_N) - \lambda_k(P)| \leq \|P_N - P\|_{op} = \frac{1}{\sqrt{N}} \|H_N\|_{op} \]
This completes the proof of the corollary.

**Appendix**

This Appendix contains the proofs of a number of results and formulae in the order in which the appear in the main text. In particular, the proof of lemma 2.1, formula (3.2), formula (3.7), formulae (3.5) and (3.8), corollary 3.2, formula (4.15), formulae (5.1), (5.2), and (5.3), formula (5.6), and formula (5.7) are given here in order.
Proof of lemma [2.1]

For any $n \geq 0$ and $0 \leq q < p$ we have

$$p - 1 \text{ terms}$$

$$B_{pn + q}(\underbrace{0, \ldots, 0}_{n}, x_p, x_{p+1}, \ldots, x_{pn + q})$$

$$= \sum_{1 \leq k \leq n} \frac{(pn + q)!}{k!} \sum_{m_1 + \ldots + m_k = p(n-k) + q} \prod_{1 \leq i \leq k} \frac{x_{p+m_i}}{(p+m_i)!}$$

We check this claim applying the summation formula

$$\sum_{k \geq 1} \sum_{n \geq k} a_n b_{n,k} = \sum_{1 \leq n} \sum_{0 \leq q < p} a_{np+q} \sum_{1 \leq k} b_{np+q,k}$$

to the equation

$$\exp \left[ \sum_{n \geq p} \frac{t^n}{n!} x_n \right]$$

$$= 1 + \sum_{k \geq 1} \sum_{n \geq kp} \frac{t^n}{n!} \sum_{m_1 + \ldots + m_k = p(n-k) + q} \prod_{1 \leq i \leq k} \frac{x_{p+m_i}}{(p+m_i)!}$$

$$= 1 + \sum_{1 \leq n} \sum_{0 \leq q < p} \frac{t^{np+q}}{(np+q)!} \sum_{1 \leq k \leq n} \frac{(np+q)!}{k!} \sum_{m_1 + \ldots + m_k = p(n-k) + q} \prod_{1 \leq i \leq k} \frac{x_{p+m_i}}{(p+m_i)!}$$

Also observe that

$$\exp \left[ \sum_{n \geq p} \frac{t^n}{n!} x_n \right]$$

$$= 1 + \sum_{1 \leq n} \frac{t^{np}}{(np)!} \left[ \frac{(np)!}{n!} \left( \frac{x_p}{p!} \right)^n + \sum_{1 \leq k < n} \frac{(np)!}{k!} \sum_{m_1 + \ldots + m_k = p(n-k)} \prod_{1 \leq i \leq k} \frac{x_{p+m_i}}{(p+m_i)!} \right]$$

$$+ \sum_{1 \leq n} \sum_{1 \leq q < p} \frac{t^{np+q}}{(np+q)!} \sum_{1 \leq k \leq n} \frac{(np+q)!}{k!} \sum_{m_1 + \ldots + m_k = p(n-k) + q} \prod_{1 \leq i \leq k} \frac{x_{p+m_i}}{(p+m_i)!}$$

Under our assumption, we have

$$\sum_{1 \leq k < n} \frac{(np)!}{k!} \sum_{m_1 + \ldots + m_k = p(n-k)} \prod_{1 \leq i \leq k} \frac{x_{p+m_i}}{(p+m_i)!}$$

$$\leq \beta^{pn} \sum_{1 \leq k < n} \frac{(np)!}{k!} \rho^k \sum_{m_1 + \ldots + m_k = p(n-k)} 1$$

$$\leq \beta^{pn} \sum_{1 \leq k < n} \frac{(np)!}{k!} \rho^k 2^{p(n-k)+k-1} = \frac{1}{2} (2\beta)^{pn} \sum_{1 \leq k < n} \frac{(np)!}{k!} \left( \frac{\rho}{2^{p-1}} \right)^k$$

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In the same vein, we have
\[
\sum_{1 \leq k \leq n} \frac{(np + q)!}{k!} \sum_{m_1 + \ldots + m_k = p(n-k) + q, m_i \geq 0} \prod_{1 \leq i \leq k} \frac{x^{p + m_i}}{(p + m_i)!}
\]
\[
\leq \beta^{pn + q} \sum_{1 \leq k \leq n} \frac{(np + q)!}{k!} \rho^k \sum_{m_1 + \ldots + m_k = p(n-k) + q, m_i \geq 0} 1
\]
\[
\leq \frac{1}{2} (2\beta)^{pn + q} \sum_{1 \leq k \leq n} \frac{(np + q)!}{k!} \left(\frac{\rho}{2^{p-1}}\right)^k
\]
This completes the proof of this lemma.

\[\square\]

**Proof of (3.2)**

The proof is partly based on the estimate
\[
\frac{2^{-n}}{n!} \mathbb{E}(X^n) \leq \frac{1}{2n} \sum_{0 \leq k < n} \frac{2^{-k}}{k!} \nu_{2k} \operatorname{Tr}(P)^k P^{n-k}
\]  
(6.26)

For \( n = 1, 2 \) the inequality (6.26) resumes to an equality. The proof of the above formulae can be found in [5]. We also have the identity
\[
\sum_{0 \leq k \leq n} \frac{(2k)!}{2^{2k}k!^2} = \frac{(2n + 1)!}{2^{2n}n!^2} \leq \frac{2n + 1}{\sqrt{3n + 1}}
\]  
(6.27)

The proof of the first assertion in the above display can be found in [51] (cf. formula (1.38) in Section 1.2). The r.h.s. estimate comes from the central binomial estimates
\[
\frac{1}{\sqrt{4n}} \leq \frac{(2n)!}{2^{2n}n!^2} \leq \frac{1}{\sqrt{3n + 1}}
\]  
(6.28)

The proof of the above estimate can be found in [31]. Using (6.26) we check the trace estimates
\[
\operatorname{Tr}[\mathbb{E}(X^n)] = \mathbb{E}(\|X\|^{2n}) \leq \frac{(2n)!}{2^{2n}n!^2} \operatorname{Tr}(P)^n \leq \frac{2^n}{\sqrt{3n + 1}} n! \operatorname{Tr}(P)^n \leq 2^{n-1} n! \operatorname{Tr}(P)^n
\]

On the other hand, we have the rather crude estimates
\[
\|\mathbb{E}[M_{2n,n}(P)]\| \leq \frac{(2n)!}{2^{n}n!} \mathbb{E}(\|X - P\|^2)^n
\]  
(6.29)

for any compatible matrix norm. For instance, for the Frobenius norm we find that
\[
\|M_{2n,n}(P)\|_F \leq \frac{(2n)!}{2^{n}n!} \left[\operatorname{Tr}(\mathbb{E}(\mathcal{H}^2))\right]^n = \frac{(2n)!}{2^{n}n!} \left[\operatorname{Tr}(P^2) + \operatorname{Tr}(P)^2\right]^n
\]

Using (6.26) and (6.28) we have the rather crude estimate
\[
\mathbb{E}[\|X - P\|^2_F] \leq 2^{n-1} \left[\mathbb{E}[\|X\|^{2n}] + \|P\|^2_F\right] = 2^{n-1} \left[2^{n-1} n! + 1\right] \operatorname{Tr}(P)^n \leq \frac{n!}{2} (4\operatorname{Tr}(P))^n
\]

Therefore, for any partition \( \pi \) of \([n]\) with \( m \) blocks \( \pi_i \) of size \( |\pi_i| = n_i \geq 2 \) we have
\[
\|M_{\pi}(P)\|_F \leq \prod_{1 \leq i \leq m} \mathbb{E}[\|X - P\|^2_F] \leq 2^{-m} (4\operatorname{Tr}(P))^n \prod_{1 \leq i \leq m} n_i!
\]
This implies that
\[
\|M_{n,m}(P)\|_F \leq \frac{2^{-m}}{m!} \sum_{n_1 + \ldots + n_m = n} \frac{n!}{n_1! \ldots n_m!} \prod_{1 \leq i \leq m} \mathbb{E} [\|X - P\|_{F}^{n_i}]
\]
\[
\leq \frac{n!}{m!} 2^{-m} (4 \text{Tr}(P))^n \left( \frac{n - 2m + m - 1}{n - 2m} \right) \leq \frac{n!}{m!} (8 \text{Tr}(P))^n 2^{-2m+1}
\]

The last assertion comes from the fact that the number of solutions to the equation
\[
j_1 + j_2 + \ldots + j_p = q \quad \text{with} \quad j_i \geq 0 \quad \text{is given by} \quad \binom{p+q-1}{q}
\]
and \(\binom{n}{p} \leq 2^n\) for any \(0 \leq p \leq n\). This ends the proof of (3.2). \(\square\)

**Proof of (3.7)**

Recall that
\[
P = I \implies X \overset{\text{law}}{=} \chi_r \text{ Unif} \implies X - I = (\chi_r - 1) \text{ Unif} + (\text{Unif} - I)
\]
(6.30)

where \(\text{Unif} := \overline{X} \overline{X}'\) stands for the random projection matrix associated with an uniform random vector \(\overline{X} := X/\|X\|\) on the unit sphere, and \(\chi_r = \|X\|^2\) is an independent \(\chi^2\)-random variable with \(r\) degrees of freedom.

We use the binomial formula
\[
\mathbb{E} [(X - I)^n] = (-1)^n \left( 1 - \frac{1}{r} \right) I + \mathbb{E} \left[ \left( \|X\|^2 - 1 \right)^n \right] \frac{1}{r} I \quad \text{and} \quad \mathbb{E} \left[ \|X\|^{2k} \right] = \prod_{0 \leq l < k} (r + 2l)
\]

The above formula can be checked using (6.30). Notice that \(\text{Unif} = \text{Unif} + (\text{Unif} - I) \implies \text{Unif}(\text{Unif} - I)^n = 0 = \text{Unif}(\text{Unif} - I)^n\) for any \(n \geq 1\). This yields
\[
\mathbb{E} [(X - I)^n] = r^{n-1} \left[ (-1)^n \left( \frac{1}{r} \right)^{n-1} + \sum_{1 \leq l \leq n} \left( \begin{array}{c} n \\ l \end{array} \right) \left( -\frac{1}{r} \right)^{n-l} \prod_{0 \leq j < l} \left( 1 + \frac{2j}{r} \right) \right] I
\]

Integrating sequentially the non-crossing indices, for any partition of \([n]\) of type \(\mu \vdash n\) we find that
\[
M_\mu(I)
\]
\[
= 1_{\mu_1 = 0} \prod_{2 \leq k \leq n} \mathbb{E} \left( (X - I)^k \right)^{\mu_k}
\]
\[
= r^{n-m} 1_{\mu_1 = 0} \prod_{2 \leq k \leq n} \left[ (-1)^n \left( \frac{1}{r} \right)^{k-1} + \sum_{1 \leq l \leq k} \left( \begin{array}{c} k \\ l \end{array} \right) \left( -\frac{1}{r} \right)^{k-l} \prod_{0 \leq j < l} \left( 1 + \frac{2j}{r} \right) \right]^{\mu_k} I
\]

Also observe that for any \(2t < 1\) we have
\[
\log \mathbb{E} \left( e^{t\|X\|^2 - 1} \right) = (r - 1) t + \frac{r}{2} \sum_{n \geq 2} \frac{(2t)^n}{n}
\]
\[
\implies \mathbb{E} \left( \left( \|X\|^2 - 1 \right)^n \right) = B_n (r - 1, r 2^{-1} (2 - 1)!, \ldots, r 2^{n-1} (n - 1)!) \geq 1
\]

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In this situation, the polynomial formula (3.12) resumes to
\[
\mathbb{E}[(X - I)^n] = \left[ \frac{1}{r} \right] B_n \left( r - 1, r 2^{2-1} (2 - 1)!, \ldots, r 2^{n-1} (n - 1)! \right) + (-1)^n \left( 1 - \frac{1}{r} \right) I
\]
A more thorough discussion on more general rank-one matrix moments can be found in [6]. Finally observe that
\[
r^{-\langle n - 1 \rangle} \mathbb{E}[(X - I)^n] = (-1)^n \left( \frac{1}{r} \right)^{n - 1} + \prod_{0 \leq j < n} \left( 1 + \frac{2j}{r} \right) + \sum_{1 \leq l < n} \left( \frac{n}{l} \right) \left( \frac{1}{r} \right)^{n - l} \prod_{0 \leq j < l} \left( 1 + \frac{2j}{r} \right) I
\]
\[
\longrightarrow_{r \to \infty} (1_{n > 1} + O(1/r)) I
\]
The end of the proof of (3.9) is now easily completed. This ends the proof of (3.7). □

**Proof of** (3.5) **and** (3.8)

For any \( n \geq 1 \) and any collection of symmetric matrices \( Q_1, \ldots, Q_n \) we have the reduction formula
\[
\mathbb{E}((XQ_2)(XQ_3) \ldots (XQ_n)X) = \mathbb{E}((XQ_1) \text{Tr} [(XQ_2)(XQ_3) \ldots (XQ_n)]) \quad \text{with} \quad Q_1 = I
\]
\[
= \sum_{1 \leq m \leq n} \sum_{\tau \in \mathcal{P}_{n,m}} 2^{n-m} \prod_{1 \leq k \leq n} (k - 1)! r_k(\tau) \prod_{2 \leq i \leq m} \text{Tr}(Q_{\tau_i}) \left( Q_{\tau_i} \right)_{\text{sym}} \quad \text{(6.31)}
\]
In the above display, \( \tau_1 \leq \ldots \leq \tau_m \) stands for the \( m \) ordered blocks of the partition \( \tau (\Rightarrow \tau_1 \ni 1) \), \( r_k(\tau) \) stands for the number of blocks of size \( k \), and
\[
\tau_i = \{ j_1^i, \ldots, j_{|\tau_i|}^i \} \quad \text{with} \quad j_1^i \leq \ldots \leq j_{|\tau_i|}^i \quad \Rightarrow \quad Q_{\tau_i} = Q_{j_1^i} Q_{j_2^i} \cdots Q_{j_{|\tau_i|}^i}
\]
The proof of the above formulae can be found in [6]. We further assume that \( \pi \in \mathcal{P}_{n,m} - \mathcal{N}_{n,m} \) is an \( X \)-connected crossing partition of \( [n] \) in the sense that \( X_{\pi} \) cannot be written in terms of the product of two independent random matrices. In this situation, we have
\[
X_{\pi} = [(X_1 Q_2)(X_1 Q_3) \ldots (X_1 Q_{m_1})X_1] Q_{m_1 + 1} \quad \text{with} \quad m_1 = |\pi_1|
\]
and some random matrices \( Q_j \) independent of \( X_1 \) and such that
\[
h(\pi) := |\{ 2 \leq i \leq m_1 : Q_i = I \}| \leq m_1 - 2
\]
For any single rank matrix \( X = XX^t \), for some column vector \( X \in \mathbb{R}^r \), and for any collection of matrices \( (R_i)_{1 \leq i \leq p} \) and \( (S_i)_{0 \leq i \leq q} \) we have
\[
\text{Tr} [(XR_1) \ldots (XR_p)] \quad \text{S}_0 \quad [(XS_1) \ldots (XS_p)] = \quad \text{S}_0 \quad [(XR_1) \ldots (XR_p)] \quad [(XS_1) \ldots (XS_p)]
\]
Combining the above decomposition with the reduction formula (6.31) we check that
\[
\mathbb{E}(X_{\pi}) = \sum_{l=0}^{h(\pi)} \sum_{k \in J_l} r^l \alpha_{k,l}(\pi) \mathbb{E}(X_{\varphi_{k,l}(\pi)})
\]
for any \( X \)-connected crossing partition \( \pi \) of \( [n] \) with \( b \) blocks.
In the above display, \( \alpha_{k,l}(\pi) \geq 0 \) stands for some non-negative parameters whose values don’t depend on \( r \), \( \varphi_{k,l}(\pi) \) a collection of partitions on \([n - m_1] \) with \((b - 1)\) blocks, and \( \lambda \) is a set with cardinality less than \( 2B(m_1) \), with the Bell number \( B(m_1) := \sum_{1 \leq k \leq m_1} S(m_1, k) \).

We further assume that for any \( \pi \in \mathcal{P}(m, m) - \mathcal{N}(m, m) \) we have

\[
\mathbb{E}(X_\pi) = c_{\pi} m^{-1} (1 + O(1/r)) r^{-1} I
\]

for any \( 1 \leq m < n \) and some finite constant \( c_{\pi} \) whose values don’t depend on \( r \). This result is clearly met for \( n = 5 \). In this case \( \mathcal{P}(4, 2) - \mathcal{N}(4, 2) \) resumes to the partition with \( b = 2 \) blocks

\[
\pi_1 = \{1, 3\} \leq \pi_2 = \{2, 4\} \implies \mathbb{E}(X_1X_2X_1X_2) = 3 (r + 2) I
\]

Under the above induction hypothesis and applying (3.4) when \( \varphi_{k,l}(\pi) \in \mathcal{N}(n, m) \) we have

\[
\mathbb{E}(X_\pi) = c_{\pi} m^{-1} r^{m_1 - 2} r^{(n - m_1) - (b - 1)} (1 + O(1/r)) I = c_{\pi} m^{-1} r^{n - b} (1 + O(1/r)) r^{-1} I
\]

When \( \pi \) is crossing but non-necessarily \( \mathcal{X} \)-connected we have

\[
\mathbb{E}(X_\pi) = \mathbb{E}(C_1) \ldots \mathbb{E}(C_k)
\]

with some \( \mathcal{X} \)-connected components \( C_i \), with \( 1 \leq i \leq k \). Each of them, say \( C_i \), is associated with a partition of \([n_i]\) with \( b_i \) blocks for some parameters \( n = n_1 + \ldots + n_k \) and \( b = b_1 + \ldots + b_k \). Since \( \pi \) is crossing at least of of the components, say \( C_j \) is crossing. In this case we have

\[
\mathbb{E}(X_\pi) = c_{\pi} m^{n - b_1} r^{n_k - b_k} (1 + O(1/r)) r^{-1} I = c_{\pi} m^{-b} (1 + O(1/r)) r^{-1} I
\]

This ends the proof of (3.5).

The proof of (3.8) relies on the decomposition

\[
(X_{\alpha(1)} - I) \ldots (X_{\alpha(n)} - I) = \sum_{\epsilon=(\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n} (-1)^{n-S_{\epsilon}} X_{\pi}^{\epsilon} \quad \text{with} \quad X_{\pi}^{\epsilon} := X_{\alpha(1)}^{\epsilon_1} \ldots X_{\alpha(n)}^{\epsilon_n}
\]

Using (2.13) we check that \( \pi \in \mathcal{Q}_{n,b} \) we have

\[
\pi \in \mathcal{Q}_{n,b} \quad \text{and} \quad b > \lfloor n/2 \rfloor \implies \mathbb{E}(M_\pi(P)) = 0
\]

We further assume that \( \pi \in \mathcal{Q}_{n,b} \) is a crossing partition with a number of blocks \( b \leq \lfloor n/2 \rfloor \). For any \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0, 1\}^n \) with \( \sum_{1 \leq i \leq n} \epsilon_i = m \) we have

\[
X_{\pi} \overset{\text{law}}{=} X_{\varphi(\pi)}
\]

for some partition \( \varphi(\pi) \) of \([m] \) with a number of blocks

\[
b - \lfloor (n - m)/2 \rfloor \leq b_m := |\varphi(\pi)| \leq b \wedge m
\]

For any \( m > n - 2b \) the partition \( \varphi_{(\pi)} \) has at least \( b_m \geq b - \lfloor (n - m)/2 \rfloor \geq 1 \) blocks. In this situation, since \( \varphi(\pi) \) may be a crossing partition by (3.4) and (3.5) we have

\[
\mathbb{E}(X_{\varphi_{(\pi)}}) = r^{m - b_m} (1 + O(1/r)) I \leq r^{m + \lfloor (n - m)/2 \rfloor - b} (1 + O(1/r)) I \leq r^{n - b} (1 + O(1/r)) r^{-1} I
\]

In the last assertion we have used the fact that \( m \leq n \) and \( |k/2| \leq k - 1 \), which is valid for any \( k \geq 1 \). In the reverse angle, when \( n \leq m - 2b \) the partition \( \varphi(\pi) \) may have a single block. In this situation, arguing as above we have

\[
\mathbb{E}(X_{\varphi_{(\pi)}}) = r^{n - 1} (1 + O(1/r)) I \leq r^{n - 2b - 1} (1 + O(1/r)) I \leq r^{n - b} (1 + O(1/r)) r^{-1} I
\]

This ends the proof of (3.8).
Proof of corollary 3.2

Combining (1.3) with theorem 3.1, for any $2N \geq n \geq 1$ we have

$$
\mathbb{E} [ P_N^n ] = P^n + \sum_{1 \leq k \leq n} \sum_{1 \leq l \leq \lfloor k/2 \rfloor} \frac{1}{N^{k-l}} (n)_k \partial_{k,l}(P)
$$

$$
= P^n + \sum_{1 \leq l \leq \lfloor n/2 \rfloor} \sum_{2l \leq k \leq n} \frac{1}{N^{k-l}} (n)_k \partial_{k,l}(P)
$$

The last assertion comes from the iterative series formula

$$
\sum_{1 \leq k \leq n} \sum_{1 \leq l \leq \lfloor k/2 \rfloor} a_{k,l} = \sum_{1 \leq l \leq \lfloor n/2 \rfloor} \sum_{2l \leq k \leq n} a_{k,l} \quad \text{applied to} \quad a_{k,l} := \frac{1}{N^{k-l}} (n)_k \partial_{k,l}(P)
$$

A proof of the above formula can be found in [54, 55]. Assume that $n = 2m$. In this case, we have

$$
\sum_{1 \leq l \leq \lfloor n/2 \rfloor} \sum_{2l \leq k \leq n} a_{k,l} = \sum_{1 \leq l \leq m} \sum_{2l \leq k \leq n} a_{l+(k-l),l}
$$

$$
= \sum_{1 \leq l \leq m} \sum_{k=l}^{2m-l} b_{k,l} \quad \text{with} \quad b_{k,l} := a_{l+k,l} = \frac{1}{N^{k-l}} (n)_{l+k} \partial_{l+k,l}(P)
$$

Observe that

$$
\sum_{1 \leq l \leq m} \sum_{k=l}^{2m-l} b_{k,l} = \sum_{1 \leq p \leq m} \sum_{1 \leq k \leq p} b_{p,k} + \sum_{1 \leq p < n} \sum_{1 \leq k \leq m-p} b_{m+p,k}
$$

$$
= \sum_{1 \leq p \leq m} \frac{1}{N^p} \sum_{1 \leq k \leq p} (n)_{p+k} \partial_{p+k,k}(P)
$$

$$
+ \sum_{1 \leq p < m} \frac{1}{N^{m+p}} \sum_{1 \leq k \leq m-p} (n)_{k+m+p} \partial_{m+p+k,k}(P)
$$

This yields the formula

$$
\sum_{1 \leq l \leq m} \sum_{k=l}^{2m-l} b_{k,l} = \sum_{1 \leq p \leq m} \frac{1}{N^p} \sum_{p < k \leq 2p} (n)_k \partial_{k,k-p}(P)
$$

$$
+ \sum_{m < p \leq 2m} \frac{1}{N^p} \sum_{p < k \leq 2m} (n)_k \partial_{k,k-p}(P)
$$

$$
= \sum_{1 \leq p < n} \frac{1}{N^p} \sum_{p < k \leq (2p) \wedge n} (n)_k \partial_{k,k-p}(P)
$$

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For odd parameters \( n = (2m + 1) \) we have

\[
\sum_{1 \leq k \leq 2m + 1} \sum_{1 \leq l \leq \lfloor k/2 \rfloor} a_{k,l} \\
= \sum_{1 \leq l \leq m} a_{2m+1,l} + \sum_{1 \leq l \leq m} \sum_{2l \leq k \leq 2m} a_{k,l} \\
= \sum_{1 \leq p \leq m} \frac{1}{N^p} \sum_{p < k \leq 2p} (n)_k \partial_{k,k-p}(P) \\
+ \frac{1}{N-1} \delta_{n,1}(P) + \sum_{m < p \leq 2m} \frac{1}{N^p} \left[ \partial_{n,n-p}(P) + \sum_{p < k \leq 2m} (n)_k \partial_{k,k-p}(P) \right] \\
= \sum_{1 \leq p \leq m} \frac{1}{N^p} \sum_{p < k \leq 2p} (n)_k \partial_{k,k-p}(P) \\
+ \frac{1}{N-1} \delta_{n,1}(P) + \sum_{m < p \leq 2m} \frac{1}{N^p} \sum_{p < k \leq n} (n)_k \partial_{k,k-p}(P)
\]

This yields the formula

\[
\sum_{1 \leq k \leq n} \sum_{1 \leq l \leq \lfloor k/2 \rfloor} a_{k,l} = \sum_{1 \leq p \leq m} \frac{1}{N^p} \sum_{p < k \leq (2p) \wedge n} (n)_k \partial_{k,k-p}(P) \\
+ \sum_{m < p \leq n} \frac{1}{N^p} \sum_{p < k \leq (2p) \wedge n} (n)_k \partial_{k,k-p}(P)
\]

This ends the proof of the first assertion. The last differential-type matrix moment formula comes from the fact that

\[
E(P^m_N) = \frac{1}{N^n} \sum_{1 \leq m \leq n} (N)_m M_{m,m}^o(P) = \frac{1}{N^n} \sum_{1 \leq m \leq n} s(m,l) N^l M_{m,m}^o(P) \\
= \sum_{1 \leq l \leq n-1} \frac{1}{N^{n-l}} \sum_{l \leq m \leq n} s(m,l) M_{m,m}^o(P) = \sum_{0 \leq k < n} \frac{1}{N^n} \sum_{n-k \leq m \leq n} s(m,n-k) M_{m,m}^o(P)
\]

This ends the proof of the corollary.

**Proof of (4.5)**

Observe that for any \( n \geq 1 \) and \( m \geq 0 \) we have

\[
\mathbb{X}^{n+m} = \|X\|^{2m} \mathbb{X}^n = (\text{Tr} \mathbb{X})^m \mathbb{X}^n \implies \mathbb{X}^n \exp[t \mathbb{X}] = \mathbb{X}^n \exp[t \text{Tr} \mathbb{X}]
\]

In addition, we have the exponential formulae

\[
\mathbb{X}^m = \|X\|^{2(n-1)} \mathbb{X} = \frac{\mathbb{X}}{\text{Tr}(\mathbb{X})} \|X\|^{2n} \implies \exp[t \mathbb{X}] = I + \frac{\mathbb{X}}{\text{Tr}(\mathbb{X})} \exp[t \text{Tr} \mathbb{X}] \quad (6.32)
\]

Assume that \( P = I \). In this case the projection matrix \( \mathbb{X}/\text{Tr}(\mathbb{X}) \) and the \( \chi^2 \)-square variable \( \text{Tr}(\mathbb{X}) \) are independent. This yields for any \( t < 1/2 \) the formula

\[
\mathbb{E}(\exp[t \mathbb{X}]) = I + \mathbb{E}(\mathbb{X}/\text{Tr}(\mathbb{X})) \mathbb{E}(\exp[t \text{Tr} \mathbb{X}]) = \left( 1 + \frac{1}{r (1-2t)^{r/2}} \right) I
\]
For any \( n \geq 1 \) and any parameter \( t \) s.t. \( I - 2tP \) is invertible we have the change of variable formula 
\[
\mathbb{E}(X^n \exp[tX]) = \mathbb{E}(X^n \exp[t\operatorname{Tr}[X]]) = \mathbb{E}(\exp[t\operatorname{Tr}[X]]) \mathbb{E}[Y_n]
\]
(6.33)
where \( Y_n \) stands for Wishart distribution with a single degree of freedom and covariance matrix \( P_t := (I - 2tP)^{-1}P \). We check (6.33) recalling that \( X \) has a Wishart distribution with a single degree of freedom and covariance matrix \( P \).

Using formulae (6.26), the matrix moments (6.33) can be computed explicitly in terms of the covariance matrix \( P \). For instance, we have
\[
\partial_t \mathbb{E}(\exp[tX]) = \mathbb{E}(\exp(t\operatorname{Tr}[X])) (I - 2tP)^{-1}P
\]
(4.36)

\[\square\]

**Proof of (4.15)**

Applying (4.14) we prove that
\[
\mathbb{E} \left( \operatorname{Tr} \left[ \exp \left( \frac{t}{2} \frac{\mathcal{H}_N}{\operatorname{Tr}(P)} \right) \right] \right) \leq r \exp \left( NL \left( \frac{t}{\sqrt{N}} \right) \right) \mathbb{E} \left( \left[ \left( \frac{P}{\operatorname{Tr}(P)} \right)^{1/2} \right] \right)
\]

Combining the above estimate with the Laplace transform method for random matrices presented in proposition 3.1 in [72] we check that
\[
\mathbb{P} \left( \frac{1}{2} \frac{\lambda_1(\mathcal{H}_N)}{\operatorname{Tr}(P)} \geq v \right) \leq \inf_{t \in [0,1]} \left\{ e^{-vt} \mathbb{E} \left( \operatorname{Tr} \left[ \exp \left( \frac{t}{2} \frac{\mathcal{H}_N}{\operatorname{Tr}(P)} \right) \right] \right) \right\}
\]
\[
\leq r \exp \left[ -N v \mathbb{E} \left( \left( \frac{P}{\operatorname{Tr}(P)} \right)^{1/2} \right) \mathbb{E} \left( \left( \frac{P}{\operatorname{Tr}(P)} \right)^{1/2} \right) \right]
\]

We end the proof of (4.15) using (4.6). This ends the proof of the estimate (4.15).

Combining (4.15) with the almost sure estimate
\[
\lambda_1(P_N) \leq \lambda_1(P) + \frac{1}{\sqrt{N}} \lambda_1(\mathcal{H}_N)
\]
we also check that the probability of the event
\[
\lambda_1(P_N) \leq \lambda_1(P) + 2 \operatorname{Tr}(P) \left[ \frac{\delta + \log(r)}{N} + 2 \sqrt{\frac{\delta + \log(r)}{N}} \mathbb{E} \left( \left( \frac{P}{\operatorname{Tr}(P)} \right)^{1/2} \right) \right]
\]
(6.34)
is greater than \( 1 - e^{-\delta} \).

\[\square\]

**Proof of (5.1), (5.2), and (5.3)**

We use the fact that
\[
\operatorname{Tr}([A\mathcal{H}]) = \sum_{u \in [r] \times [r]} A(u) \mathcal{H}(u)
\]
where \( \mathcal{H}(u) \) are Gaussian random variables with covariance matrix
\[
C(u, v) = \mathbb{E}(\mathcal{H}(u)\mathcal{H}(v)) = 2(P \otimes P)_{u,v}
\]
\[\implies \log \mathbb{E} \left( \exp \left( t \operatorname{Tr}[A\mathcal{H}] \right) \right) = \frac{t^2}{2} \sum_{u,v \in [r] \times [r]} C(u,v) A(u) A(v)
\]

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This ends the proof of (5.1). □

To check (5.2) we observe that

\[ \|\mathcal{H}\|_{F}^{2} = \sum_{u \in \{r\} \times \{r\}} \mathcal{H}(u)^{2} \implies \log \mathbb{E} \left( \exp \left( t \|\mathcal{H}\|_{F}^{2} \right) \right) = \frac{1}{2} \sum_{n \geq 1} \frac{(2t)^{n}}{n} \text{Tr}(C^{n}) \]

This assertion comes from the fact that \( \sum_{u \in \{r\} \times \{r\}} \mathcal{H}(u)^{2} \) is a \( \chi^{2} \)-square random variable associated with a centered Gaussian \( r^{2} \)-dimensional vector with covariance matrix \( C \). On the other hand, using (1.2) we have

\[ C = 2 \left( P \otimes P \right) \quad \text{and} \quad (P \otimes P)^{n} = (P^{n} \otimes P^{n}) \]

\[ \implies \text{Tr}(C^{n}) = 2^{n} \text{Tr}(P^{n} \otimes P^{n}) = 2^{n-1} \left[ \text{Tr}(P^{n})^{2} + \text{Tr}(P^{2n}) \right] \]

This ends the proof of (5.2). □

Also observe that

\[ \mathbb{E} \left( \|\mathcal{H}\|_{F}^{2n} \right) = B_{n} \left( \text{Tr}(P)^{2} + \text{Tr}(P^{2}), 4^{2-1} (2 - 1)! \left[ \text{Tr}(P^{2})^{2} + \text{Tr}(P^{4}) \right], \ldots, 4^{n-1} (n - 1)! \left[ \text{Tr}(P^{n})^{2} + \text{Tr}(P^{2n}) \right] \right) \]

with the complete Bell polynomials \( B_{n} \) defined in (4.15).

Now we come to the proof of (5.3). Let \( e_{k} \) be the \( r \)-column vectors with null entries but the \( k \)-th unit one. Choosing \( A = e_{k,l} \) in (5.3), we find that

\[ \text{Tr} \left( A\mathcal{H}_{N} \right) = \mathcal{H}_{N}(k,l) \]

\[ \text{Tr}(AP)^{2} = \frac{P(k,l)^{2} + P(k,l)P(l,l)}{2} = (P \otimes P)_{(k,l),(k,l)} = \mathbb{E}(\mathcal{H}(k,l)^{2})/2 \]

and

\[ \|AP\|_{F}^{2} = \sigma_{k,l}^{2} := \frac{P^{2}(k,k) + P^{2}(l,l)}{4} + 2^{-1}P^{2}(k,k) \sum_{k=l} \]

This implies that

\[ \mathbb{E} \left( \cosh \left( t \mathcal{H}(k,l) - \frac{t^{2}}{2} \mathbb{E}(\mathcal{H}(k,l)^{2}) \right) \right) \leq \exp \left[ -2t^{2} \sigma_{k,l}^{2} \log \left( 1 - \frac{2|t|}{N^{1/2}} \sigma_{k,l} \right) \right] \]

On the other hand, for any \( 0 < t \leq (2\sigma_{k,l})^{-1} \) we have

\[ \mathbb{E} \left( \exp \left( t \|\mathcal{H}(k,l)\| \right) \right) \leq 2 \mathbb{E} \left( \cosh \left( t \mathcal{H}(k,l) \right) \right) \]

Using Hölder’s inequality we find the rather crude estimate

\[ \|\mathcal{H}_{N}\|_{F} \leq \sum_{1 \leq k,l \leq r} |\mathcal{H}_{N}(k,l)| \]

\[ \implies \mathbb{E} \left( \exp \left[ t \|\mathcal{H}_{N}\|_{F} - \frac{(rt)^{2}}{2} \mathbb{E}(\|\mathcal{H}\|_{F}^{2}) \right] \right) \leq 2 \exp \left[ -2(rt)^{2} \left( r\sigma \right)^{2} \log \left( 1 - \frac{2r^{2}t}{N^{1/2}} \sigma \right) \right] \]

as soon as \( 0 < t \leq (2r\sigma)^{-1} \) with \( \sigma := \text{max}_{k,l} \sigma_{k,l} \). We conclude that

\[ \exists t > 0 \quad \text{ s.t. } \sup_{N \geq 1} \mathbb{E} \left[ \exp \left( t \|\mathcal{H}_{N}\|_{F} \right) \right] < \infty \iff \exists c < \infty \quad \text{ s.t. } \sup_{N \geq 1} \mathbb{E} \left[ \|\mathcal{H}_{N}\|_{F}^{p} \right] \leq c^{p} n! \]

for some constant \( c \) whose value depends on \( P \). The proof of the equivalence between the growth of the moments and the Hardy’s exponential condition can be found in [42] [61] (see for instance Section 5, theorem 3 in [61]). This ends the proof (5.3). □
Proof of (5.6)

The proof of (5.6) is a direct consequence of the following lemma.

**Lemma 6.5.** For any $n \geq 1$, $A = (x'y + y'x)/2$ with $x, y \in \mathbb{R}^r$ we have

$$4 \, (AP)^n A = a_n(x, y) \left( x'y + y'x \right) + b_n(x, y) \left[ \langle x, Px \rangle \, yy' + \langle y, Py \rangle \, xx' \right]$$

with the symmetric functions

$$a_n(x, y) = 2^{-(n-1)} \sum_{0 \leq l \leq \lfloor n/2 \rfloor} \binom{n}{2l} \left( \langle x, Py \rangle^{n-2l} \langle \langle x, Px \rangle \langle y, Py \rangle \rangle^l \right)$$

$$b_n(x, y) := 2^{-(n-1)} \sum_{0 \leq l \leq \lfloor (n-1)/2 \rfloor} \binom{n}{2l+1} \left( \langle x, Py \rangle^{n-1-2l} \langle \langle x, Px \rangle \langle y, Py \rangle \rangle^l \right)$$

**Proof.** For any $n \geq 1$, $A = (x'y + y'x)/2$ with $x, y \in \mathbb{R}^r$ we have

$$(AP)^n = p_n \, AP + q_n \, BP$$

with $B = \langle x, Px \rangle \, yy' + \langle y, Py \rangle \, xx'$

and some polynomial functions $p_n, q_n$ of maximal degree $(n - 1)$ in the couple of variables $u := \langle x, Py \rangle$ and $v := \langle x, Px \rangle \langle y, Py \rangle$. By Cauchy Schwartz inequality we have $u^2 \leq v$.

We check this assertion by induction w.r.t. the parameter $n \geq 1$. For $n = 1$ the result is clear with $p_n = 1$ and $q_n = 0$. Assume the assertion has been checked up to rank $(n - 1)$. In this case we have

$$(AP)^n = p_{n-1} \, (AP)^2 + q_{n-1} \, BPAP$$

Observe that

$$4 \, (AP)^2 = (x'y + y'x)P(x'y + y'x)P = 2u \, AP + BP$$

and

$$2 \, BPAP = (\langle x, Px \rangle \, yy'P + \langle y, Py \rangle \, xx'P) \left( x'yP + y'xP \right)$$

$$= \langle x, Px \rangle \langle x, Py \rangle \, yy'P + \langle x, Px \rangle \langle y, Py \rangle \, (yx' + y'x)P + \langle y, Py \rangle \langle x, Py \rangle \, xx'P$$

$$= u \, BP + 2v \, AP$$

This implies that

$$(AP)^n = p_{n-1} \left( \frac{u}{2} \, AP + \frac{BP}{4} \right) + q_{n-1} \left[ \frac{u}{2} \, BP + v \, AP \right]$$

from which we find the recursion

$$\begin{bmatrix} p_n \\ q_n \end{bmatrix} = \begin{bmatrix} u/2 & v \\ 1/4 & u/2 \end{bmatrix} \begin{bmatrix} p_{n-1} \\ q_{n-1} \end{bmatrix}$$

The eigenvalues of the above matrix are given by

$$\alpha_- = \frac{1}{2} [u - \sqrt{v}] \leq 0 \leq \alpha_+ = \frac{1}{2} [u + \sqrt{v}]$$

The eigenvectors $U_-$ and $U_+$ are given by

$$U_- = \begin{bmatrix} 2 \sqrt{\sigma} \\ -1 \end{bmatrix} \text{ and } U_+ = \begin{bmatrix} 2 \sqrt{\sigma} \\ 1 \end{bmatrix}$$
This implies that
\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} p_1 \\ q_1 \end{bmatrix} = a_- U_- + a_+ U_+ = \frac{1}{4\sqrt{v}} (U_- + U_+)
\]
\[\implies \begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = a_- \alpha_- U_- + a_+ \alpha_+ U_+ = \frac{1}{4\sqrt{v}} \begin{bmatrix} 2\sqrt{v}(\alpha_-^n + \alpha_+^n) \\ \alpha_-^n - \alpha_+^n \end{bmatrix} \]

On the other hand we have
\[
2^n (\alpha_+^n + \alpha_-^n) = \sum_{0 \leq k \leq n} \binom{n}{k} u^{n-k} (1 + (-1)^k) v^{k/2} = 2 \sum_{0 \leq l \leq \lfloor n/2 \rfloor} \binom{n}{2l} u^{n-2l} v^l
\]

In the same vein, we have
\[
2^n (\alpha_+^n - \alpha_-^n) = \sum_{0 \leq k \leq n} \binom{n}{k} u^{n-k} (1 - (-1)^k) v^{k/2} = 2 \sqrt{v} \sum_{0 \leq l \leq \lfloor (n-1)/2 \rfloor} \binom{n}{2l+1} u^{n-1-2l} v^l
\]

This implies that
\[p_{n+1} = \frac{1}{2^n} \sum_{0 \leq l \leq \lfloor n/2 \rfloor} \binom{n}{2l} u^{n-2l} v^l\]
and
\[q_{n+1} = \frac{1}{2^{n+1}} \sum_{0 \leq l \leq \lfloor (n-1)/2 \rfloor} \binom{n}{2l+1} u^{n-1-2l} v^l\]

This ends the proof of the lemma. \(\square\)

We end the proof of the spectral estimate stated in (5.6) using the formula
\[
\sum_{0 \leq l \leq \lfloor n/2 \rfloor} \binom{n}{2l} = 2^{n-1} = \sum_{0 \leq l \leq \lfloor (n-1)/2 \rfloor} \binom{n}{2l+1}
\]
which can be found in e.g. [54, 55]. This ends the proof of (5.6). \(\square\)

Proof of (5.7)

The proof of the sub-Gaussian estimates (5.7) is based on the following technical lemma.

Lemma 6.6. For any symmetric matrix \(A\) and any \(2|t|\|AP\|_F \leq \sqrt{N}\) we have
\[
\log \mathbb{E} (\exp (t \text{ Tr}(A\mathcal{H}_N))) \\
\leq t^2 \text{ Tr}((AP)^2) - 2t^2 \|AP\|_F^2 \log \left(1 - 2|t| N^{-1/2} \|AP\|_F \right)
\]

In addition, for any \(A \geq 0\) and any \(2t\text{Tr}(AP) < \sqrt{N}\) we have
\[
\log \mathbb{E} (\exp (t \text{ Tr}(A\mathcal{H}_N)))) \leq t^2 \text{ Tr}((AP)^2) \left[1 - 2 \log \left(1 - \frac{2t}{N^{1/2}} \text{ Tr}(AP) \right) \right]
\]
and
\[
\log \mathbb{E} (\exp (-t \text{ Tr}(A\mathcal{H}_N))) \\
\leq t^2 \text{ Tr}((AP)^2) + \frac{t}{2} \sqrt{N} \text{Tr}(AP) \left(1 - \frac{2t}{N} \text{Tr}(AP) \right) \log \left[1 - \frac{1}{\sqrt{N}} \left(2t\lambda_r \left(A^{1/2} PA^{1/2}\right)\right)^2 \right]
\]
Proof. Using (5.5) we find the estimate

$$
\mathbb{E} \left( \exp \left( t \text{Tr} (A H_N) - t^2 \text{Tr}((AP)^2)) \right) \right) \leq \exp \left[ 2t^2 \|AP\|_F^2 \sum_{n \geq 1} \frac{1}{n} \left( \frac{2|t|}{N^{1/2}} \|AP\|_F \right)^n \right]
$$

$$
= \exp \left[ -2t^2 \|AP\|_F^2 \log \left( 1 - \frac{2|t|}{N^{1/2}} \|AP\|_F \right) \right]
$$

Whenever $A \geq 0$ using (2.2) we have

$$
1 \leq \mathbb{E} \left( \exp \left( t \text{Tr} (A H_N) - t^2 \text{Tr}((AP)^2)) \right) \right) \leq \exp \left[ \sum_{n \geq 3} \frac{t^n}{n} \frac{2^{n-1}}{N^{n/2-1}} \text{Tr}(AP)^n \right]
$$

$$
\leq \exp \left[ -2t^2 \text{Tr}((AP)^2) \log \left( 1 - \frac{2t}{N^{1/2}} \text{Tr}(AP) \right) \right]
$$

as soon as $2t \text{Tr}(AP) < \sqrt{N}$.

On the other hand, we have the decomposition

$$
\sum_{m \geq 1} \frac{(-t)^{m+2}}{m+2} \frac{1}{N^{m/2}} \frac{1}{2^m} \text{Tr}((AP)^m+2)
$$

$$
= \frac{2t}{\sqrt{N}} \sum_{m \geq 1} \frac{t^{2m+1}}{2m+2} \frac{1}{N^{m-1/2}} 2^m \text{Tr}((AP)^{2m+2})
$$

$$
- \sum_{m \geq 1} \frac{t^{2m+1}}{2m+1} \frac{1}{N^{m-1/2}} 2^m \text{Tr}((AP)^{2m+1})
$$

$$
\leq - \left( 1 - \frac{2t}{\sqrt{N}} \text{Tr}(AP) \right) \sum_{m \geq 1} \frac{t^{2m+1}}{2m+1} \frac{1}{N^{m-1/2}} 2^{2m} \text{Tr}((AP)^{2m+1})
$$

This yields the estimate

$$
\mathbb{E} \left( \exp \left( -t \text{Tr} (A H_N) - t^2 \text{Tr}((AP)^2)) \right) \right)
$$

$$
\leq \exp \left[ -\frac{1}{\sqrt{N}} \left( 1 - \frac{2t}{\sqrt{N}} \text{Tr}(AP) \right) \sum_{m \geq 1} \frac{t^{2m+1}}{2m+1} \frac{1}{N^{m-1/2}} 2^{2m} \text{Tr}((AP)^{2m+1}) \right]
$$

$$
\leq \exp \left[ -\frac{t}{2} \sqrt{N} \text{Tr}(AP) \left( 1 - \frac{2t}{\sqrt{N}} \text{Tr}(AP) \right) \sum_{m \geq 1} \frac{t^{2m+1}}{m} \frac{\lambda_m (A^{1/2}PA^{1/2})^{2m}}{N^m} \right]
$$

as soon as $2t \text{Tr}(AP) < \sqrt{N}$. This ends the proof of the lemma. \hfill \Box

Combining lemma 6.6 with the estimate $- \log (1 - x) \leq t \frac{2x}{1 - x} \leq 2x \leq 1$, which is valid for any $0 \leq x \leq 1/2$, we obtain the sub-Gaussian property (5.7). \hfill \Box

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