LIFESPAN OF SOLUTIONS TO A DAMPED PLATE EQUATION WITH LOGARITHMIC NONLINEARITY

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Abstract. This paper is devoted to the lifespan of solutions to a damped plate equation with logarithmic nonlinearity

\[ u_{tt} + \Delta^2 u - \Delta u - \Delta u_t + u_t = |u|^{p-2} u \ln |u|. \]

Finite time blow-up criteria for solutions at both lower and high initial energy levels are established and an upper bound for the blow-up time is given for each case. Moreover, by constructing a new auxiliary functional and making full use of the strong damping term, a lower bound for the blow-up time is also derived.

1. Introduction. In this paper, we are concerned with the following initial boundary value problem for a damped plate equation with logarithmic nonlinearity

\[
\begin{align*}
&u_{tt} + \Delta^2 u - \Delta u - \Delta u_t + u_t = |u|^{p-2} u \ln |u|, \quad (x, t) \in \Omega \times (0, T), \\
u(x, t) &= \Delta u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n (n \geq 1) \) is a bounded domain with smooth boundary \( \partial \Omega \), \( T \in (0, +\infty) \) is the maximal existence time of the solution \( u(x, t) \), and the exponent \( p \) satisfies

\[ 2 < p < 2^*_s, \]

where \( 2^*_s = +\infty \) if \( n \leq 4 \) and \( 2^*_s = \frac{2n}{n - 4} \) if \( n \geq 5 \).

Problems like (1) have their roots in many branches of physics such as nuclear physics, optics and geophysics. They may also be used to describe some phenomena of granular materials such as the longitudinal motion of an elastic-plastic bar. Interested reader may refer to [1, 2, 3, 4, 10] for more background of problems like (1). It is well known that the damping terms (both strong \( \Delta u_t \) and weak \( u_t \)) prevent solutions from blowing up while the nonlinear terms force solutions to blow up. So it is of great interest to investigate how one dominates the other, and much effort

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The first eigenvalue of the operator in a bounded domain of $\mathbb{R}^d$ damping to (2) for initial data at different energy levels. As for the damped fourth-order [23, 24], they obtained the existence of global and finite time blow-up solutions condition. By using the potential well method first proposed by Sattinger et al. [9] investigated the following damped wave equation 

$$u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t = f(u)$$

in a bounded domain of $\mathbb{R}^n$ with $\omega > 0$. Under certain conditions on the initial data and on the nonlinearity $f$, they proved the existence of global weak solutions and global strong solutions by using the classical potential well method.

When the nonlinearity $f(u)$ grows super-linearly with respect to $u$ as $u$ tends to infinity, the solutions to (3) may blow up in finite time. In 2018, Wu [25] considered the following initial boundary value problem 

$$u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t = f(u)$$

where $\omega > 0$, $\alpha(t) : [0, \infty) \to [0, \infty)$ is a nonincreasing bounded differentiable function and $p$ satisfies the so-called subcritical condition, i.e.,

$$p \in (2, \infty) \quad if \quad n \leq 4; \quad p \in \left(2, \frac{2n - 4}{n - 4}\right) \quad if \quad n \geq 5.$$ 

After showing that the unstable set is invariant under the flow of (4), he proved a blow-up result for problem (4) with initial energy smaller than the depth of the potential well, by applying concavity argument. Moreover, a lower bound for the blow-up time is derived. Later, problem (4) was reconsidered by Guo et al. [10] and the results of [25] were extended in two aspects. The first is that they obtained a blow-up result for high initial energy, and the second is that lower bound for the blow-up time is also derived for some supercritical $p$, with the help of inverse Hölder’s inequality and interpolation inequality.

On the other hand, evolution equations with logarithmic nonlinearity have also attracted more and more attention in recent years, due to their wide applications to quantum field theory and other applied sciences. Among the huge amount of interesting literature, we only refer the interested reader to [5, 6, 7, 8, 11, 12, 14, 15, 16, 19, 22], where qualitative properties of solutions to hyperbolic or parabolic equations with logarithmic nonlinearities were studied. In particular, Di et al. [8] considered the following initial boundary value problem for a semilinear wave equation with strong damping and logarithmic nonlinearity

$$u_{tt} - \Delta u - \Delta u_t = |u|^{p-2} u \ln |u|, \quad (x, t) \in \Omega \times (0, T),$$

$$u(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T),$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,$$

when $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded domain with smooth boundary $\partial \Omega$, $2 < p < +\infty$ if $n = 1, 2$ and $2 < p < \frac{2n}{n-2}$ if $n \geq 3$. The existence of global or finite time blow-up
solutions to problem (5) with initial energy less than or equal to the depth of the potential well was investigated by using the potential well method. Moreover, the decay rate of the energy functional was obtained for global solutions and upper and lower bounds for the blow-up time were also derived for blow-up solutions. However, the case that the initial energy is larger than the depth of the potential well was not considered in [8], and we do not know whether or not problem (5) admits finite time blow-up solutions for this case. In addition, the lower bound for the blow-up time was obtained when \( p \) is subcritical, i.e., \( p < \frac{2n-2}{n-2} \). When \( p \in \left( \frac{2n-2}{n-2}, \frac{2n}{n-2} \right) \) for \( n \geq 3 \), whether a lower bound for the blow-up time can be obtained is still open. Moreover, as far as we know, there is no blow-up result for plate equation with both damping terms and logarithmic nonlinearity like problem (1).

Motivated mainly by [8, 10, 25], we will consider problem (1) and investigate how the damping terms and logarithmic nonlinearity determine the blow-up conditions and blow-up time of the solutions. More precisely, we shall present some sufficient conditions for the solutions to problem (1) to blow up in finite time with both lower and high initial energy and derive an upper bound for the blow-up time for each case. Moreover, we also estimate a lower bound for the blow-up time, which, thanks to the strong damping term, also includes some supercritical case.

The organization of this paper is as follows. In Section 2, as preliminaries, some notations, definitions and lemmas that will be used in the sequel are introduced. Finite time blow-up of solutions and upper bound for the blow-up time with lower and high initial energy will be considered in Section 3 and Section 4, respectively. In Section 5 we derive a lower bound for the blow-up time. In Section 6, we point out some possible generalizations of our main results and propose some open problems related to problem (1).

2. Preliminaries. In this section, we introduce some notations and lemmas which will be used in the sequel. In what follows, we denote by \( \| \cdot \|_r \) the \( L^r(\Omega) \)-norm \( (1 \leq r \leq \infty) \), by \( (\cdot, \cdot) \) the \( L^2(\Omega) \)-inner product and by \( \lambda_1 > 0 \) the first eigenvalue of \(-\Delta\) in \( \Omega \) under homogeneous Dirichlet boundary condition. We use \( \langle \cdot, \cdot \rangle \) to denote the inner product in \( H^1(\Omega) \), i.e.,

\[
\langle u, v \rangle = \int_\Omega (uv + \nabla u \cdot \nabla v) dx, \quad u, v \in H^1(\Omega).
\]

The norm induced by the inner product is denoted by \( \| \cdot \|_2 \), i.e.,

\[
\|u\| = \sqrt{\|u\|_2^2 + \|\nabla u\|_2^2}, \quad u \in H^1(\Omega).
\]

Set

\[
H = \{ u \in H^2(\Omega) \cap H^1_0(\Omega) : u = \Delta u = 0 \text{ on } \partial \Omega \},
\]

and equip it with the norm

\[
\|u\|_H = \sqrt{\|\Delta u\|_2^2 + \|\nabla u\|_2^2}.
\]

For any \( u \in H \), it follows from the definition of \( \lambda_1 \) that

\[
\lambda_1 \|u\|_2^2 \leq \|\nabla u\|_2^2.
\]
Furthermore, by using integration by parts, Cauchy inequality and (6), we have, for any \( \varepsilon > 0 \),
\[
\int_{\Omega} |\nabla u|^2 \, dx = \int_{\Omega} \frac{\partial u}{\partial \nu} \, d\sigma - \int_{\Omega} u \Delta u \, dx = -\int_{\Omega} u \Delta u \, dx \\
\leq \frac{\varepsilon}{2} \int_{\Omega} |u|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} |\Delta u|^2 \, dx \leq \frac{\varepsilon}{2\lambda_1} \int_{\Omega} |\nabla u|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} |\Delta u|^2 \, dx.
\]
Taking \( \varepsilon = \lambda_1 \), we obtain
\[
\lambda_1 \| \nabla u \|_2^2 \leq \| \Delta u \|_2^2. \tag{7}
\]
Combining (6) with (7) one sees, for any \( u \in H \), that
\[
\lambda_1 \| u \|_H^2 \leq \| u \|_2^2 \quad \text{and} \quad (\lambda_1 + \lambda_2^2) \| u \|_2^2 \leq \| u \|_H^2. \tag{8}
\]
For any \( u \in H \), define
\[
J(u) = \frac{1}{2} \| u \|_H^2 - \frac{1}{p} \int_{\Omega} |u|^p \ln |u| \, dx + \frac{1}{p^2} \| u \|_p^p, \tag{9}
\]
\[
I(u) = \| u \|_H^2 - \int_{\Omega} |u|^p \ln |u| \, dx, \tag{10}
\]
\[
\mathcal{N} = \{ u \in H \setminus \{ 0 \} : I(u) = 0 \}, \tag{11}
\]
\[
d = \inf_{u \in H \setminus \{ 0 \}} \sup_{\lambda > 0} J(\lambda u) = \inf_{u \in \mathcal{N}} J(u), \tag{12}
\]
where \( \mathcal{N} \) is called the Nehari manifold and \( d \) is the depth of the potential well (also called mountain pass level). In what follows, we shall show that \( \mathcal{N} \) is non-empty and \( d \) is positive.

The following lemma gives some properties of the so-called fibering map \( J(\lambda u) \). Since the proof is more or less standard (see [8] for example), we omit it here.

**Lemma 2.1.** Let \( p \) satisfy (A). Then for any \( u \in H \setminus \{ 0 \} \), we have
(i) \( \lim_{\lambda \to 0^+} J(\lambda u) = 0 \), \( \lim_{\lambda \to +\infty} J(\lambda u) = -\infty \).
(ii) there exists a unique \( \lambda^* = \lambda^*(u) > 0 \) such that \( \frac{d}{d\lambda} J(\lambda u) \big|_{\lambda = \lambda^*} = 0 \). \( J(\lambda u) \) is increasing on \( 0 < \lambda < \lambda^* \), decreasing on \( \lambda^* < \lambda < +\infty \) and takes its maximum at \( \lambda = \lambda^* \).
(iii) \( I(\lambda u) > 0 \) on \( 0 < \lambda < \lambda^* \), \( I(\lambda u) < 0 \) on \( \lambda^* < \lambda < +\infty \) and \( I(\lambda^* u) = 0 \).

Let \( \sigma \) be any positive number such that \( p + \sigma < 2_\ast \). Then it is well known that the embedding from \( H \) to \( L^{p+\sigma}(\Omega) \) is compact and there is a positive constant \( B_\sigma \) such that
\[
\| u \|_{p+\sigma} \leq B_\sigma \| u \|_H, \quad \forall \ u \in H. \tag{13}
\]

**Lemma 2.2.** There is a positive constant \( C_\ast \) such that \( \| u \|_H \geq C_\ast \) for any \( u \in \mathcal{N} \).

**Proof.** First, it follows from Lemma 2.1 (iii) that \( \mathcal{N} \) is non-empty. For any \( u \in \mathcal{N} \), using (13) and the basic inequality \( \ln s \leq \frac{1}{e\sigma} s^\sigma \) for \( s \geq 1 \) and \( \sigma > 0 \), we have
\[
\| u \|_H^2 = \int_{\Omega} |u|^p \ln |u| \, dx = \int_{\Omega_1} |u|^p \ln |u| \, dx + \int_{\Omega_2} |u|^p \ln |u| \, dx \\
\leq \int_{\Omega_2} |u|^p \ln |u| \, dx \leq \frac{1}{e\sigma} \int_{\Omega_2} |u|^{p+\sigma} \, dx \\
\leq \frac{1}{e\sigma} \| u \|_{p+\sigma} \leq \frac{B_{p+\sigma}^\sigma}{e\sigma} \| u \|_H^{p+\sigma}, \tag{14}
\]
where
there is a subsequence of without loss of generality, we may assume that \( v \). Hence, we obtain
\[
\int |v|^p \ln |v_0| dx = \lim_{k \to \infty} \int |v_k|^p \ln |v_k| dx, \tag{17}
\]
\[
\int |v_0|^p dx = \lim_{k \to \infty} \int |v_k|^p dx. \tag{18}
\]
Moreover, by the weak lower semicontinuity of \( \| \cdot \|_H \), we have
\[
\|v_0\|_H \leq \liminf_{k \to \infty} \|v_k\|_H. \tag{19}
\]
Therefore, it follows from (17)-(19) that
\[
J(v_0) \leq \liminf_{k \to \infty} J(v_k) = d, \tag{20}
\]
and
\[
I(v_0) \leq \liminf_{k \to \infty} I(v_k) = 0. \tag{21}
\]
It remains to show that \( v_0 \neq 0 \) and \( I(v_0) = 0 \) to complete the proof. By (17) and Lemma 2.2 we know
\[
\int |v_0|^p \ln |v_0| dx \leq \lim_{k \to \infty} \int |v_k|^p \ln |v_k| dx
\]
\[
= \lim_{k \to \infty} \|v_k\|_H^2 \geq C_*^2,
\]
which implies that \( v_0 \neq 0 \). By Lemma 2.1 (iii) we know that there exists a unique \( \lambda^* > 0 \) such that \( I(\lambda^* v_0) = 0 \), \( I(\lambda v_0) > 0 \) when \( \lambda \in (0, \lambda^*) \) and \( I(\lambda v_0) < 0 \) when
\( \lambda > \lambda^* \). Therefore, if \( I(v_0) < 0 \), then we must have \( \lambda^* \in (0,1) \). By the definition of \( d \), we see
\[
d \leq J(\lambda^* v_0) = \frac{(p-2)\lambda^{*2}}{2p} \|v_0\|_H^2 + \frac{\lambda^{*p}}{p^2} \|v_0\|_p^p \\
= \lambda^{*2} \left[ \frac{p-2}{2p} \|v_0\|_H^2 + \frac{\lambda^{p-2}}{p^2} \|v_0\|_p^p \right] \\
< \lambda^{*2} \liminf_{k \to \infty} \left[ \frac{p-2}{2p} \|v_k\|_H^2 + \frac{1}{p^2} \|v_k\|_p^p \right] \\
\leq \lambda^{*2} \liminf_{k \to \infty} J(v_k) = \lambda^{*2} d,
\]
a contradiction. Therefore, \( I(v_0) = 0 \) and \( v_0 \neq 0 \), which means that \( v_0 \in \mathcal{N} \). Recalling (20) and the definition of \( d \) again one sees that \( J(v_0) = d \). The proof is complete.

In this paper, we consider weak solutions to problem (1). For completeness, we state, without proof, the local existence theorem which can be established by slightly modifying the argument in [21]. Sometimes \( u(x,t) \) will be simply written as \( u(t) \) if no confusion arises.

**Theorem 2.4.** (\([10, 25]\)) Let \( u_0 \in H \) and \( u_1 \in L^2(\Omega) \). Then the problem (1) admits a unique weak solution \( u \in L^\infty(0,T;H), u_t \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)) \), for \( T > 0 \) suitably small. Moreover, the energy functional satisfies
\[
E'(t) = -\|u_t\|^2 \leq 0,
\]
where
\[
E(t) = E(u(t)) = \frac{1}{2} \|u_t\|_2^2 + J(u(t)).
\]

At the end of this section, we present the well-known concavity lemma which will play essential role in proving the blow-up result.

**Lemma 2.5.** (See [13, 17]) Suppose that a positive, twice-differentiable function \( \psi(t) \) satisfies the inequality
\[
\psi''(t)\psi(t) - (1 + \theta)(\psi'(t))^2 \geq 0, \quad t > 0,
\]
where \( \theta > 0 \). If \( \psi(0) > 0, \psi'(0) > 0 \), then there exists a time \( t_\ast \leq t^* = \frac{\psi(0)}{\theta \psi'(0)} \) such that
\[
\lim_{t \to t_\ast} \psi(t) = +\infty.
\]

3. **Blow-up for lower initial energy.** In this section, we will investigate the blow-up phenomena of solutions to problem (1) with lower initial energy. We first show that the unstable set \( \mathcal{U} \) is invariant under the flow of problem (1), where
\[
\mathcal{U} = \{ u \in H : I(u) < 0, J(u) < d \},
\]
and \( d \) is the depth of the potential well defined in (12).

**Lemma 3.1.** Let \( u_0 \in \mathcal{U} \) and \( u_1 \in L^2(\Omega) \) such that \( E(0) < 0 \). Then \( u(t) \in \mathcal{U} \) for all \( t \in [0,T) \) and
\[
\frac{p-2}{2p} \|u(t)\|_H^2 + \frac{1}{p^2} \|u(t)\|_p^p > d, \quad \forall \ t \in [0,T).
\]
Theorem 3.2. Problem (1) with lower initial energy.

Proof. First, it follows from (9), (23) and (22) that
\[ J(u(t)) \leq E(t) \leq E(0) < d, \quad \forall \ t \in [0, T). \]

Therefore, in order to prove \( u(t) \in U \) for all \( t \in [0, T) \), it suffices to show that \( I(u(t)) < 0 \) for all \( t \in [0, T) \). Assume by contradiction that there exists a \( t_1 \in (0, T) \) such that \( u(t_1) \in N \). Then by the variational definition of \( d \), we obtain
\[ d \leq J(u(t_1)) \leq E(t_1) \leq E(0) < d, \]
a contradiction.

For any \( t \in [0, T) \), since \( I(u(t)) < 0 \), it follows from Lemma 2.1 (iii) that there exists a \( \lambda(t) \in (0, 1) \) such that \( I(\lambda(t)u(t)) = 0 \), i.e., \( \lambda(t)u(t) \in N \). By the definition of \( d \) and (15), we have
\[
\frac{p-2}{2p} \|u(t)\|^2_H + \frac{1}{p^2} \|u(t)\|^p_p \geq \frac{(p-2)}{2p} \lambda^2(t) \|u(t)\|^2_H + \frac{\lambda^p(t)}{p^2} \|u(t)\|^p_p = J(\lambda^2(t)u(t)) \geq d.
\]

The proof is complete. \( \square \)

With the preliminaries given above, we can show the first blow-up results for problem (1) with lower initial energy.

Theorem 3.2. Let \( p \) satisfy (A), \( u_0 \in U \) and \( u_1 \in L^2(\Omega) \) such that \( E(0) < d \). Then the solution \( u(x, t) \) to problem (1) blows up at a finite time \( T \) in the sense that
\[
\lim_{t \to T^-} \left( \|u(t)\|_H^2 + \int_0^t \|u(s)\|_p^p \right) = \infty. \tag{27}
\]

Moreover, the blow-up time \( T \) can be estimated from above as follows
\[
T \leq \frac{4 \left( \left( a^2 + p \right)^2 + 2 \right) \|u_0\|_H^2 \|u_1\|_H^2}{(p-2)^2 b}, \tag{28}
\]
where \( a, b \) are constants that will be fixed in the proof.

Proof. Assume by contradiction that the solution \( u \) exists globally. As was done in [25], fix \( T^* > 0 \) and define the functional
\[
G(t) = \|u(t)\|_H^2 + \int_0^t \|u(s)\|_H^2 + (T^* - t) \|u_0\|^2 + b(t + \tau)^2, \quad t \in [0, T^*], \tag{29}
\]
where \( T^* \), \( b \) and \( \tau \) are positive constants to be fixed later. Taking derivative we have
\[
G'(t) = 2 \langle u, u_t \rangle + 2 \int_0^t \langle u, u_s \rangle ds + 2b(t + \tau). \tag{30}
\]
Taking derivative again and using (22), (23) and Lemma 3.1, we obtain
\[
G''(t) = 2 \langle u_t, u_t \rangle + 2 \langle u, u_{tt} \rangle + 2 \langle u, u_t \rangle + 2b
\]
\[
= (p + 2) \|u_t\|^2_H + (p - 2) \|u\|_H^2 - 2pE(t) + 2b + \frac{2}{p} \|u\|_p^p
\]
\[
\geq (p + 2) \|u_t\|^2_H + 2pd - 2pE(t) + 2b
\]
\[
= (p + 2) \|u_t\|^2_H + 2p(d - E(0)) + 2p \int_0^t \|u_s(s)\|_H^2 ds + 2b. \tag{31}
\]
Choosing $b = 2(d - E(0)) > 0$ and noticing $p > 2$ we get
\[ G''(t) \geq (p + 2)[\|u_t\|_2^2 + \int_0^t \|u_s(s)\|^2 ds + b] > 0. \tag{32} \]

In order to apply Lemma 2.5 to $G(t)$, it remains to show that
\[ G(t)G''(t) - \frac{p + 2}{4}(G'(t))^2 \geq 0, \quad t \in [0, T^*]. \]

For this, set
\[ A(t) = \left( \int_0^t \|u(s)\|^2 ds \right)^{1/2}, \quad B(t) = \left( \int_0^t \|u(s(s))\|^2 ds \right)^{1/2}. \]

By Cauchy-Schwarz inequality and Hölder’s inequality, we have
\[ \|u\|_2 \leq \|u(t)\|_2 \|u_t(t)\|_2, \quad \left| \int_0^t \langle u, u_s \rangle ds \right| \leq \int_0^t \|u(s)\| \cdot \|u_s(s)\| ds \leq A(t)B(t). \tag{34} \]

Therefore, by combining (29), (30) with (32)-(34) we obtain, for any $t \in [0, T^*]$, that
\[
\frac{1}{4} (G'(t))^2 \leq \|u\|^2_2 + A^2(t)B^2(t) + b^2(t + \tau)^2 + 2A(t)B(t)\|u\|_2 \|u_t\|_2 \\
+ 2A(t)B(t)b(t + \tau) + 2b(t + \tau)\|u_t\|^2_2 \\
\leq \|u\|^2_2 + A^2(t)B^2(t) + b^2(t + \tau)^2 + B^2(t)\|u\|^2_2 + A^2(t)\|u_t\|^2_2 \\
+ bA^2(t) + bB^2(t)(t + \tau)^2 + b\|u\|^2_2 + b(t + \tau)^2\|u_t\|^2_2 \\
= \left[\|u(t)\|^2_2 + A^2(t) + b(t + \tau)^2\right] \cdot \left[\|u_t\|^2_2 + B^2(t) + b\right] \\
\leq \frac{1}{p + 2} G(t)G''(t),
\]

that is
\[ G(t)G''(t) - \frac{p + 2}{4}(G'(t))^2 \geq 0, \quad t \in [0, T^*]. \]

Take
\[ \tau = \max \left\{ 0, \frac{2\|u_0\|^2 - (p - 2)(u_0, u_1)}{(p - 2)b} \right\}, \tag{35} \]

then
\[ G(0) = \|u_0\|^2 + T^*\|u_0\|^2 + b\tau^2 > 0, \]
\[ G'(0) = 2(u_0, u_1) + 2b\tau > 0, \]

and
\[ \frac{4G(0)}{(p - 2)G'(0)} = \frac{2\|u_0\|^2 + T^*\|u_0\|^2 + b\tau^2}{(p - 2)(u_0, u_1) + b\tau} \leq T^*, \tag{36} \]

for suitably large $T^*$. According to Lemma 2.5, there exists a $T_0 > 0$ satisfying
\[ T_0 \leq \frac{4G(0)}{(p - 2)G'(0)} \tag{37} \]

such that
\[ G(t) \to \infty \text{ as } t \to T_0^- \]

This contradicts with the assumption that $G(t)$ is well defined on the closed $[0, T^*]$ for any $T^* > 0$. 
To derive an upper bound for the blow-up time, we proceed as follows: Let \( T \) be the maximal existence time of \( u(x, t) \) (which is finite by the above argument) and let \( G(t) \) be given in (29), with the exception that \( T^* \) is replaced by \( T \) and \( t \in [0, T] \), where \( T \in (0, T) \). Similarly to the foregoing arguments, one can show that

\[
T \leq \frac{2[||u_0||^2 + T^*||u_0||^2 + b\tau^2]}{(p-2)(||u_0|| + b\tau)},
\]

where we require that \( \tau \), which is independent of \( T \), still satisfies (35). It then follows from the arbitrariness of \( T < T^* \) that

\[
T \leq \frac{2[||u_0||^2 + T^*||u_0||^2 + b\tau^2]}{(p-2)(||u_0|| + b\tau)},
\]

which guarantees

\[
T \leq T(\tau) := \frac{2(||u_0||^2 + b\tau^2)}{(p-2)(||u_0|| + b\tau)} - 2||u_0||^2.
\]

Set \( a = 2||u_0||^2 - (p-2)(||u_0|| + b\tau) \) and \( \tau_0 = \frac{[a^2 + (p-2)^2b||u_0||^2 + a]}{(p-2)b} \). Then it is an easy matter to verify that \( \tau_0 \) satisfies (35), \( T(\tau) \) attains its minimum at \( \tau_0 \) and

\[
T(\tau_0) = \frac{4\left[a^2 + (p-2)^2b||u_0||^2 + a\right]}{(p-2)^2b}.
\]

Therefore,

\[
T \leq 4\left[a^2 + (p-2)^2b||u_0||^2 + a\right] / (p-2)^2b.
\]

The proof is complete.

**Remark 1.** By (23) and (15) and recalling that \( p > 2 \) one sees \( E(0) < 0 \) implies \( I(u_0) < 0 \). Therefore, Theorem 3.2 implies that the solution \( u(x, t) \) to problem (1) blows up in finite time for negative initial energy.

4. **Blow-up for high initial energy.** In this section we shall build a blow-up criterion for problem (1) at high initial energy level. Some ideas used in this section are borrowed from [10] and [18]. As a preliminary, we first establish a lemma that will play a fundamental role.

**Lemma 4.1.** Let \( p \) satisfy (A). Assume that \( u_0 \in H \) and \( u_1 \in L^2(\Omega) \) such that

\[
0 < E(0) < \frac{C_0}{p} (u_0, u_1).
\]

Then the solution \( u(x, t) \) to problem (1) satisfies

\[
(u, u_t) - \frac{p}{C_0} E(t) \geq \left[(u_0, u_1) - \frac{p}{C_0} E(0)\right] e^{C_0 t}, \quad t \in [0, T).
\]

Here

\[
C_0 = \min \left\{ p + 2, p(p-2)\lambda_1, \frac{(p-2)(\lambda_1 + \lambda_2^2)}{2}\right\} > 0.
\]
Proof. Set \( F(t) = (u, u_t) \). By direct calculations and recalling (23) we have
\[
F'(t) = \frac{p}{2} \|u_t\|^2_H + \frac{p - 2}{2} \|u\|^2_H - \frac{C_0}{4p} \|u\|^2 - \frac{p}{C_0} \|u_t\|^2 - pE(t).
\]

By using Cauchy inequality, we can estimate the third and fourth terms in the last inequality as follows
\[
\left| \int \nabla u \cdot \nabla u_t \, dx \right| \leq \frac{C_0}{4p} \|\nabla u\|^2 + \frac{p}{C_0} \|\nabla u_t\|^2, \tag{44}
\]
\[
\frac{1}{2} \|u_t\|^2 + \frac{p - 2}{2} \|u\|^2_R \leq \frac{C_0}{4p} \|u\|^2 - \frac{p}{C_0} \|u_t\|^2 - pE(t). \tag{45}
\]

Substituting (44) and (45) into (43) we arrive at
\[
F'(t) \geq \frac{p}{2} \|u_t\|^2 + \frac{p - 2}{2} \|u\|^2_R - \frac{C_0}{4p} \|u\|^2 - \frac{p}{C_0} \|u_t\|^2 - pE(t). \tag{46}
\]

Set \( H(t) = F(t) - \frac{p}{C_0} E(t) \). Then in view of (8), (22), (42) and (46) we obtain
\[
H'(t) = F'(t) - \frac{p}{C_0} E'(t) = F'(t) + \frac{p}{C_0} \|u_t\|^2
\]
\[
\geq \frac{p}{2} \|u_t\|^2 + \frac{p - 2}{2} \|u\|^2_R - \frac{C_0}{4p} \|u\|^2 - pE(t)
\]
\[
\geq \frac{p}{2} \|u_t\|^2 + \left[ \frac{(p - 2)\lambda_1}{2} - \frac{p(2 - 2)\lambda_1}{4p} \right] \|u\|^2 - pE(t)
\]
\[
= \frac{p}{2} \|u_t\|^2 + \left[ \frac{(p - 2)\lambda_1}{2} - \frac{p(2 - 2)\lambda_1}{4p} \right] \|u\|^2 - pE(t)
\]
\[
\geq \frac{p}{2} \|u_t\|^2 + \left[ \frac{(p - 2)\lambda_1 + \lambda_1^2}{4} \right] \|u\|^2 - pE(t)
\]
\[
\geq C_0 \left[ \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u\|^2 - \frac{p}{C_0} E(t) \right] \geq C_0 H(t). \tag{47}
\]

Since \( H(0) > 0 \) by (40), (41) follows after an application of Gronwall’s inequality to \( H(t) \). The proof is complete. \qed

With Lemma 4.1 at hand, we are now in the position to prove high initial energy blow-up and estimate an upper bound for the blow-up time for problem (1).

**Theorem 4.2.** Let all the assumptions in Lemma 4.1 hold. Then the solution \( u(x, t) \) to problem (1) blows up at some finite time \( T \) in the sense of (27). Moreover, if
\[
E(0) < \frac{C_0}{p} \|u_0\|^2, \tag{48}
\]
then
\[
T \leq \frac{2(\|u_0\|^2 + \beta T^2)}{(p - 2)(\|u_0, u_1\| + \beta T) - 2\|u_0\|^2}. \tag{49}
\]
Here $C_0$ is the positive constant given in (42),
\[
\beta = 2 \left[ \frac{C_0}{p} \|u_0\|_2^2 - E(0) \right] > 0,
\]
(50)
to is suitably large such that
\[
(p - 2) \left[ (u_0, u_1) + \beta t_0 \right] > 2 \|u_0\|^2.
\]
(51)

**Proof.** We divide the proof into two steps.

**Step I: Finite time blow-up.** Suppose by contradiction that (27) will not happen for any finite $T$. Then $\|u(\cdot, t)\|_2$ is well-defined for all $t \geq 0$. Without loss of generality, we may assume that $E(t) \geq 0$ for all $t \geq 0$. Otherwise by Remark 1 we know that $u(x, t)$ blows up in finite time.

On one hand, it follows from Lemma 4.1 that
\[
\frac{d}{dt} \|u(t)\|_2^2 = 2(u, u_t) \geq 2 H(0)e^{C_0t} + \frac{2p}{C_0} E(t) \geq 2 H(0)e^{C_0t}.
\]
(52)
Integration of (52) over $[0, T]$ yields
\[
\|u(t)\|_2^2 = \|u_0\|_2^2 + 2 \int_0^t \int_\Omega uu_t \, dx \, d\tau \geq \|u_0\|^2_2 + 2 \int_0^t H(0)e^{C_0\tau} \, d\tau
\]
\[
= \|u_0\|_2^2 + \frac{2H(0)}{C_0}(e^{C_0t} - 1).
\]
(53)
On the other hand, by virtue of Minkowski inequality, Hölder inequality, (22), the definition of $\lambda_1$ and the fact $E(t) \geq 0$ one gets
\[
\|u(t)\|_2 \leq \|u_0\|_2 + \|u(t) - u_0\|_2 = \|u_0\|_2 + \| \int_0^t u_\tau \, d\tau \|_2
\]
\[
\leq \|u_0\|_2 + \int_0^t \|u_\tau\|_2 \, d\tau \leq \|u_0\|_2 + \frac{1}{\sqrt{1 + \lambda_1}} \int_0^t \|u_\tau\| \, d\tau
\]
\[
\leq \|u_0\|_2 + \sqrt{\frac{1}{1 + \lambda_1}} \left( \int_0^t \|u_\tau\|^2 \, d\tau \right)^{1/2} = \|u_0\|_2 + \sqrt{\frac{1}{1 + \lambda_1}} (E(0) - E(t))^{1/2}
\]
\[
\leq \|u_0\|_2 + \sqrt{\frac{E(0)}{1 + \lambda_1}} t^{1/2},
\]
which contradicts (53) when $t$ is sufficiently large. Therefore, $u(x, t)$ blows up in finite time.

**Step II: Upper bound for the blow-up time.** From now on, we assume that $T > 0$ is the blow-up time of $u(x, t)$, which is finite by Step I. According to Lemma 4.1 and the assumption that $E(t) \geq 0$, we see that
\[
\frac{d}{dt} \|u(t)\|_2^2 = 2(u, u_t) \geq 2 H(0)e^{C_0t} + \frac{2p}{C_0} E(t) > 0, \quad t \in [0, T),
\]
which implies $\|u(t)\|_2^2$ is increasing with respect to $t$. To estimate $T$ from above, as was done in the proof of Theorem 3.2, we define
\[
K(t) = \|u(t)\|_2^2 + \int_0^t \|u(s)\|^2 ds + (T - t) \|u_0\|^2 + \beta(t + t_0)^2, \quad t \in [0, T),
\]
(54)
where $\beta$ and $t_0$ are given in (50) and (51), respectively. By using (23), (22), (8), (42), (50) and recalling the monotonicity of $\|u(t)\|_{L^2}^2$, we obtain
\[
K''(t) = 2\|u_t\|_{L^2}^2 + 2\langle u, u_{tt} \rangle + 2\langle u, u_t \rangle + 2\beta \\
\geq (p + 2)\|u_t\|_{L^2}^2 + (p - 2)\|u_t\|_{H^1}^2 - 2pE(t) + 2\beta + \frac{2}{p}\|u_t\|_{p^*}^p \\
\geq (p + 2)\|u_t\|_{L^2}^2 + (p - 2)\|u_t\|_{H^1}^2 - 2pE(t) + 2\beta \\
\geq (p + 2)\|u_t\|_{L^2}^2 + (p - 2)(\lambda_1 + \lambda_2^2)\|u\|_{L^2}^2 - 2pE(0) + 2p\int_0^t \|u_s(s)\|_{L^2}^2 ds + 2\beta \\
\geq (p + 2)\|u_t\|_{L^2}^2 + 2C_0\|u\|_{L^2}^2 - 2pE(0) + 2p\int_0^t \|u_s(s)\|_{L^2}^2 ds + 2\beta \\
\geq (p + 2)\|u_t\|_{L^2}^2 + 2p\int_0^t \|u_s(s)\|_{L^2}^2 ds + (p + 2)\beta \\
\geq (p + 2)\left[\|u_t\|_{L^2}^2 + \int_0^t \|u_s(s)\|_{L^2}^2 ds + \beta \right].
\]
(55)

By applying the same argument as that in the proof of the first part of Theorem 3.2, we obtain
\[
K(t)K''(t) - \frac{p + 2}{4}(K'(t))^2 \geq 0, \quad t \in [0, T).
\]
(56)

Besides, $K(0) = \|u_0\|_{L^2}^2 + T\|u_0\|^2 + \beta t_0^2 > 0$ and $K'(0) = 2\langle u_0, u_1 \rangle + 2\beta t_0 > 0$ by (51). Applying Lemma 2.5 to $K(t)$ yields
\[
T \leq \frac{4K(0)}{(p - 2)K'(0)} = \frac{2\langle \|u_0\|_{L^2}^2 + T\|u_0\|^2 + \beta t_0^2 \rangle}{(p - 2)(\langle u_0, u_1 \rangle + \beta t_0)}. \]
Since
\[
\frac{2\|u_0\|_{L^2}^2}{(p - 2)(\langle u_0, u_1 \rangle + \beta t_0)} < 1 \quad \text{by (51)},
\]
we further obtain
\[
T \leq \frac{2\langle \|u_0\|_{L^2}^2 + \beta t_0^2 \rangle}{(p - 2)(\langle u_0, u_1 \rangle + \beta t_0) - 2\|u_0\|^2}.
\]

The proof of Theorem 4.2 is complete.

**Remark 2.** As was done in deriving (28), one can also minimize the right-hand side term of (49) for $t_0$ satisfying (51) to obtain a more accurate upper bound for $T$. Interested reader may check it.

**Remark 3.** Theorem 4.2 implies that problem (1) may admit finite time blow-up solutions at arbitrarily high initial energy level. Indeed, by applying similar argument to that of the proof of Theorem 3.13 in [9] or Proposition 3.2 in [12], one sees that for any $M > 0$, especially for any $M > d$, there exist initial data $u_0$ and $u_1$ satisfying $E(0) > M$ as well as all the conditions in Theorem 4.2. Therefore, the blow-up result in Theorem 4.2 is usually referred to as high initial energy blow-up or supercritical initial energy blow-up.
5. **Lower bound for the blow-up time.** Since the lower bound for the blow-up time provides a safe time interval for the system under consideration, it is more important in practice to estimate $T$ from below. In this section, our aim is to determine a lower bound for the blow-up time of problem (1) by constructing a new auxiliary functional. Throughout this section we shall use $C, C_1, C_2, \cdots$, to denote generic positive constants which may depend on $\Omega, p, n$, but are independent of the solution $u(x, t)$.

**Theorem 5.1.** Assume that $p$ satisfies \( \frac{2n}{n+2} < \frac{2n(p-1)}{n+2} < 2^*, \) i.e.,

\[
p \in (2, \infty) \quad \text{if} \quad n \leq 4; \quad p \in \left(2, \frac{2n-2}{n-4}\right) \quad \text{if} \quad n \geq 5.
\]

Let $u(x, t)$ be a weak solution to problem (1) that blows up at $T$ in the sense of (27). Then

\[
T \geq \int_{N(0)}^\infty \frac{ds}{C_4 + C_5 s^{p-1+\mu}},
\]

where $N(0) = \|u_1\|_2^2 + \|u_0\|_H^2$ and $\mu > 0$ is suitably small.

**Proof.** For simplicity, we only prove this theorem for $n \geq 3$. The case for $n = 1, 2$ is similar (and simpler, since $H$ can be embedded into $L^q(\Omega)$ for any $q \in (1, \infty)$ and the positive constant $\mu$ in (58) can be chosen arbitrarily). We aim to determine a time interval $(0, T_0)$ on which the quantity $\|u(t)\|_H^2$ is bounded. Clearly $T_0$ is a lower bound for $T$ since both $\|u(t)\|_2^2$ and $\|u(t)\|_H^2$ can be bounded by $\|u(t)\|_H^2$.

Define

\[
N(t) = \|u_t(t)\|_2^2 + \|u(t)\|_H^2, \quad t \in [0, T_0).
\]

Then

\[
\lim_{t \to T_0^-} N(t) = +\infty.
\]

Differentiating (59) and making use of Green’s second identity, we obtain

\[
N'(t) = 2[(u_t, u_{tt}) + (\Delta u, \Delta u_t) + (\nabla u, \nabla u_t)]
= 2(u_t, u_{tt} + \Delta^2 u - \Delta u)
= 2(u_t, \Delta u_t - u_t + |u|^{p-2} u - 2\int_{\Omega} u_t |u|^{p-2} u \ln |u| dx).
= -2\|u_t\|^2 + 2\int_{\Omega} u_t |u|^{p-2} u \ln |u| dx.
\]

Set $\Omega_1 = \Omega_1(t) = \{x \in \Omega : |u(x, t)| < 1\}$ and $\Omega_2 = \Omega_2(t) = \{x \in \Omega : |u(x, t)| \geq 1\}$. Since $p$ satisfies (57), we can choose $\mu > 0$ suitably small such that \( \frac{2n(p-1+\mu)}{n+2} < 2^* \), which implies that $H$ can be embedded into $L^{\frac{2n(p-1+\mu)}{n+2}}(\Omega)$ continuously. We use $B_\mu$ to denote the embedding constant from $H$ to $L^{\frac{2n(p-1+\mu)}{n+2}}(\Omega)$, i.e.,

\[
\|v\|_{L^{2n(p-1+\mu)/(n+2)}} \leq B_\mu \|v\|_H, \quad \forall \ v \in H.
\]

Using Hölder’s inequality, Cauchy inequality, (62) and the basic inequalities $|s^{p-2} \ln s| \leq (e(p-1))^{-1}$ for $0 < s < 1$ and $\ln s \leq \frac{1}{e\mu} s^\mu$ for $s \geq 1$, we can estimate the second
term on the right-hand side of (61) as follows
\[
\int_{\Omega} u_t |u|^{p-2} u \ln |u| \, dx = \int_{\Omega_1} u_t |u|^{p-2} u \ln |u| \, dx + \int_{\Omega_2} u_t |u|^{p-2} u \ln |u| \, dx
\leq \left( \int_{\Omega_1} |u_t|^{\frac{2\alpha}{n-2}} \, dx \right)^{\frac{n-2}{2\alpha}} \left( \int_{\Omega_1} |u|^{p-2} u \ln |u| \, dx \right)^{\frac{2\alpha}{n-2}}
+ \left( \int_{\Omega_2} |u_t|^{\frac{2\alpha}{n-2}} \, dx \right)^{\frac{n-2}{2\alpha}} \left( \int_{\Omega_2} |u|^{p-2} u \ln |u| \, dx \right)^{\frac{2\alpha}{n-2}}
\leq \|u_t\|^{\frac{2\alpha}{n-2}} \left[ (e(p-1))^{-1} |\Omega_1|^{\frac{n+2}{2n}} + (e\mu)^{-1} \left( \int_{\Omega} |u|^{\frac{2(n-1+\mu)}{n+2}} \, dx \right)^{\frac{n+2}{2n}} \right]
\leq C\|u_t\| \left[ (e(p-1))^{-1} |\Omega_1|^{\frac{n+2}{2n}} + (e\mu)^{-1} B^{p-1+\mu}_\mu \|u\|^{p-1+\mu}_H \right]
\leq \varepsilon \|u_t\|^2 + C(\varepsilon) \left[ C_1 + C_2 \|u\|_H^{2(p-1+\mu)} \right]
\leq \varepsilon \|u_t\|^2 + C(\varepsilon) \left[ C_1 + C_3 N^{p-1+\mu}(t) \right].
\] (63)

Therefore, it follows by taking \( \varepsilon \leq 1 \) and substituting (63) into (61) that
\[ N'(t) \leq C_4 + C_5 N^{p-1+\mu}(t). \] (64)

Integrating (64) over \([0, t]\), we have
\[ \int_0^t \frac{N'(\tau)}{C_4 + C_5 N^{p-1+\mu}(\tau)} \, d\tau \leq t. \] (65)

Letting \( t \to T_0^- \) and recalling (60), we obtain
\[ \int_{N(0)}^{\infty} \frac{ds}{C_4 + C_5 s^{p-1+\mu}} \leq T_0 \leq T. \] (66)

Recalling that \( p - 1 + \mu > 1 \), the left-hand side term in (66) is finite. The proof is complete. \( \Box \)

**Remark 4.** By making full use of the damping term, we obtain the lower bound for the blow-up time not only for subcritical exponent \( p \), but also for some supercritical ones. We point out that this observation can also be applied to problem (5) considered in [8].

6. **Possible generalizations and some open problems.** In this section we present some possible generalizations of our main results and propose some related open problems.

- **Possible generalizations**
  
The main results in this paper (Theorems 3.2, 4.2 and 5.1) can be generalized to the following problem
\[
\begin{cases}
  u_{tt} + \Delta^2 u - \Delta u - \omega \Delta u_t + \alpha(t) u_t = |u|^{p-2} u \ln |u|, & (x, t) \in \Omega \times (0, T), \\
  u(x, t) = \Delta u(x, t) = 0, & (x, t) \in \partial \Omega \times (0, T), \\
  u(x, 0) = u_0(x), & x \in \Omega,
\end{cases}
\] (67)
where \( \omega > 0, \alpha(t) : [0, \infty) \to [0, \infty) \) is a nonincreasing bounded differentiable function and \( p \) satisfies (A). Indeed, to show that Theorems 3.2 and 4.2 still hold for
problem (67), one only need to replace the functionals $G(t)$ and $K(t)$, respectively, by

$$
G_1(t) = \int_0^t \int_\Omega \left( \alpha(s)u^2(s) + \omega|\nabla u(s)|^2 \right) dx ds + \int_0^t \int_\Omega (s-t)\alpha_s(s)u^2(s) dx ds \\
+ (T^* - t) \int_\Omega \left( \alpha(0)u_0^2 + \omega|\nabla u_0|^2 \right) dx + b(t+\tau)^2 + \|u(t)\|_2^2
$$

and

$$
K_1(t) = \int_0^t \int_\Omega \left( \alpha(s)u^2(s) + \omega|\nabla u(s)|^2 \right) dx ds + \int_0^t \int_\Omega (s-t)\alpha_s(s)u^2(s) dx ds \\
+ (T - t) \int_\Omega \left( \alpha(0)u_0^2 + \omega|\nabla u_0|^2 \right) dx + \beta(t + t_0)^2 + \|u(t)\|_2^2,
$$

and then shows that the concavity inequality (24) holds for $G_1(t)$ and $K_1(t)$, under the corresponding conditions. The lower bound for the blow-up time (Theorem 5.1) can also be obtained for problem (67) by using the same auxiliary functional $N(t)$ as defined in (59).

- Some open problems
  1. Do Theorems 3.2 and 4.2 extend to the case of nonlinear weak damping such as $|u_t|^{m-2}u_t$ with $m > 2$ in place of $u_t$?
  2. Do these results extend to such equations with nonlinear strong damping such as $-\Delta_m u_t$ (the $m$-Laplace operator)?

Since the linearity of the damping terms plays an important role in performing the reduction to an ordinary differential inequality in time (which is necessary in applying the concavity argument), the proof would become too involved for nonlinear damping terms.

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