A local gauge invariant infrared regularization of the Yang-Mills theory.

A.A.Slavnov
Steklov Mathematical Institute
and Moscow State University
Moscow
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Abstract
A local gauge invariant infrared regularization for the Yang-Mills theory is constructed on the basis of a higher derivative formulation of the model.

1 Introduction.
In spite of great achievements in the study of nonabelian gauge theories the problem of construction of a gauge invariant infrared regularization remained unsolved. Of course in the Yang-Mills theory a scattering matrix connecting the free asymptotic states, which include massless particles, does not exist. However some gauge invariant infrared regularization allowing to make sense of formal manipulations with it certainly would be welcome. Moreover in nondiagonal gauges even the Green functions are plagued with infrared divergencies which have to be regularized in a gauge invariant way.

In this paper I propose an infrared regularization for the Yang-Mills theory, which may be described by a local gauge invariant Lagrangian. This Lagrangian contains higher derivatives and hence the regularized theory includes nonpositive norm states, however in the limit when the regularization is removed the nonphysical states decouple.

In the next section a path integral formulation of such a regularization will be presented. The third section deals with the field theoretical realization of this construction.

2 A path integral regularization.
To illustrate the main idea I present a heuristic derivation of the regularized action for the $SU(2)$ gauge theory. Generalization to other groups is straightforward.
The following formal equality obviously holds:

\[ \int \exp \left\{ i \int \left[ L_{YM} + m^2 \varphi^* \varphi \right] dx \right\} d\mu = \int \exp \left\{ i \int \left[ L_{YM} + m^{-2} (D^2 \varphi')^* (D^2 \varphi') - d^* D^2 b - b^* D^2 d \right] dx \right\} d\mu' \]

(1)

Here \( L_{YM} \) is the usual Yang-Mills Lagrangian

\[ L_{YM} = -\frac{1}{4} F^j_{\mu\nu} F^j_{\mu\nu} \]

(2)

and the complex scalar fields \( \varphi \) form the \( SU(2) \) doublet, which may be conveniently parameterised as follows:

\[ \varphi_1 = \frac{iB^1 + B^2}{\sqrt{2}}; \quad \varphi_2 = \frac{B^0 + iB^3}{\sqrt{2}} \]

(3)

Anticommuting complex scalar fields \( b, d \) form the similar doublets. The measure \( d\mu \) includes differentials of all fields as well as gauge fixing factors. The measure \( d\mu' \) differs by the presence of the differentials of the fields \( b, b^*, d, d^* \). The operator \( D^2 \) denotes the sum \( \sum_{\mu} D_\mu D_\mu \), where \( D_\mu \) is the covariant derivative

\[ D_\mu \varphi = (\partial_\mu + \frac{ig\tau^j}{2} A^j_\mu) \varphi \]

(4)

The boundary conditions for the Yang-Mills field \( A_\mu \) are the standard ones. The fields \( \varphi \) are fastly decreasing, and the fields \( \varphi', b, d \) satisfy the Feynman boundary conditions. The integral over \( \varphi \) obviously produces a trivial constant, so that l.h.s. of the eq. (1) is just the path integral for the Yang-Mills theory. Performing explicitly the integration over \( \varphi', b, d \) in the r.h.s. of the eq. (1), we get the same result.

The equation (1) is formal, as neither l.h.s. nor r.h.s. exist because of infrared divergencies. We define the infrared regularized theory in the following way. Let us add to the action in the r.h.s. the gauge invariant term

\[ \int \left\{ \alpha(D_\mu \varphi')^* (D_\mu \varphi') - \alpha m^2 (d^* b + b^* d) \right\} dx \]

(5)

The integral in the r.h.s. of eq. (1) is still infrared divergent. However if we make the shift

\[ \varphi' \to \varphi' + \hat{a}, \quad \hat{a}_1 = 0, \quad \hat{a}_2 = a, \]

(6)

the regularized action acquires a form

\[ A_R = \int \left\{ -\frac{1}{4} F^j_{\mu\nu} F^j_{\mu\nu} - m^{-2} (D^2 \varphi')^* (D^2 \varphi') - (D_\mu b)^* D_\mu b - (D_\mu b)^* D_\mu d \right. \]

\[ - \frac{a^2 g^2}{4m^2} (\partial_\mu A_\mu)^2 + \frac{ag}{\sqrt{2}m^2} \partial^j B^j \partial_\mu A^j_\mu + \]

\[ + \alpha(D_\mu \varphi)^* (D_\mu \varphi) + \frac{ag a^2}{4} A^2_\mu - \frac{ag}{2\sqrt{2}} B^j \partial_\mu A^j_\mu - \alpha m^2 (d^* b + b^* d) + \ldots \]

(7)
Here ... denote the interaction terms which arise due to shift \( (6) \).

One sees that the shift \( (6) \) generates the mass term for the vector field, the term \( (\partial_\mu A_\mu)^2 \), the mixing of \( B^j \) with \( \partial_\mu A^i_\mu \) and additional interaction terms. For simplicity in the following we choose \( g^2 a^2 = 2m^2 \). Then the mass of the Yang-Mills field is \( \sqrt{\alpha m} \).

The theory described by the action \( (7) \) is free of infrared singularities. At the same time the action is local and invariant with respect to the gauge transformations

\[
A^j_\mu \to A^j_\mu - g\varepsilon^{jik} A^i_\mu \eta^k + \partial_\mu \eta^j \\
B^0 \to B^0 + g(B^j \eta^j) \\
B^j \to B^j - m \eta^j - \frac{g}{2} \varepsilon^{jik} B^i \eta^k - \frac{g}{2} B^0 \eta^j
\]

(8)

This invariance allows to use in the corresponding path integral any admissible gauge condition. Particularly convenient is the Lorentz gauge \( \partial_\mu A_\mu = 0 \). In this gauge the mixing between \( A^j_\mu \) and \( B^j \) is absent and renormalizability is manifest.

One has to understand that the transformation \( (6) \) is not a simple change of variables. It changes the boundary conditions in the path integral. Rather it is a definition of the infrared regularized Yang-Mills theory. More precisely

\[
\int \exp\{i \int L_{YM} dx \} d\mu_{\text{reg}} = \int \exp\{iA_R \} d\mu'
\]

(9)

The equation \( (9) \) gives a definition of the infrared regularized scattering matrix for the Yang-Mills theory as a path integral of the exponent of a local gauge invariant action. It also allows to give a sensible definition of the correlation functions as in the regularised theory one can perform the Wick rotation in all Feynman integrals making the transition \( \alpha \to 0 \) legitimate.

In the next section we shall show that this path integral regularization admits an elegant field theoretical realization, similar to the BRST quantization of gauge invariant models.

3 Canonical quantization and unitarity of regularized theory.

It was shown in our papers \( [1], [2] \) that a change of variables in a path integral which introduces higher derivatives may be interpreted as a transition to a field theory model including unphysical ghost fields. This theory possesses a (super)symmetry which leads via Noether theorem to existence of a conserved nilpotent charge \( Q \). Existence of such a charge allows to separate the physical states by imposing the condition

\[
Q|\psi >_{\text{phys}} = 0
\]

(10)

These states have nonnegative norms and the scattering matrix is unitary in the subspace \( (10) \).

Below we shall show that a similar construction may be done in the present model. A peculiar feature of our model is related to the fact that contrary to
the cases considered before the conserved charge $Q$ is not nilpotent. Nilpotency is recovered only in the limit $\alpha \to 0$, and this limit, when it exists, determines the Yang-Mills theory. The limit $\alpha \to 0$ for the on-shell $S$-matrix does not exist due to infrared divergencies, but the formal expression for the matrix elements in the limit when the regularization is removed coincides with the $S$-matrix elements of original Yang-Mills theory.

Our starting point is the regularized action

$$A_R = \int \left\{ -\frac{1}{4} F_{\mu\nu}^j F_{\mu\nu}^j - m^{-2} (D^2 (\varphi + \hat{\alpha}))^* D^2 (\varphi + \hat{\alpha}) + (D_\mu d)^* D_\mu b \\
+ (D_\mu b)^* D_\mu d + \alpha [(D_\mu (\varphi + \hat{\alpha}))^* D_\mu (\varphi + \hat{\alpha}) - m^2 (d^* b + b^* d)] \right\} dx$$

This action is invariant with respect to the gauge transformations (8) and the supersymmetry transformations

$$\delta \varphi = \varepsilon b \\
\delta d = m^{-2} D^2 (\varphi + \hat{\alpha}) \varepsilon$$

where $\varepsilon$ is an anti-Hermitean parameter anticommuting with $b, d$. In terms of the components this transformation looks as follows:

$$\delta \varphi = \varepsilon b \\
\delta d_1 = [m^{-2} (D^2 \varphi)_1 + \frac{1}{\sqrt{2m}} (i \partial_\mu A^1_\mu + \partial_\mu A^2_\mu)] \varepsilon \\
\delta d_2 = [m^{-2} (D^2 \varphi)_2 - \frac{i}{\sqrt{2m}} (\partial_\mu A^3_\mu - g^2 mA^2_\mu)] \varepsilon$$

Note that these transformations are not nilpotent: $\delta^2 d \neq 0$. The nilpotency is restored only in the limit $\alpha = 0$.

The action (11) is invariant both with respect to the gauge transformations and the transformations (13). The supersymmetry transformations do not change the fields $A_\mu$, so it is convenient to choose for quantization a manifestly supersymmetric and renormalizable gauge $\partial_\mu A_\mu = 0$.

In this gauge the Lagrangian may be written in terms of the components $B^a, B^0$, and the similar components for the fields $b, d$

$$b_1 = \frac{ib^1 + b^2}{\sqrt{2}}; \quad b_2 = \frac{b^0 + ib^3}{\sqrt{2}} \\
d_1 = \frac{d^1 - id^2}{\sqrt{2}}; \quad d_2 = \frac{-id^0 + d^3}{\sqrt{2}}$$

as follows

$$L_R = -\frac{m^{-2}}{2} \partial^2 B^\rho \partial^2 B^\rho - i \partial_\mu b^\rho \partial_\mu d^\rho - \frac{1}{4} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2 \\
+ \alpha [\partial_\mu B^\rho \partial_\mu B^\rho - im^2 b^\rho d^\rho] + \ldots$$

where $\rho = 0, 1, 2, 3$ and $\ldots$ denote the interaction terms.
The quantization of the Yang-Mills fields $A_\mu^a$ is performed in a standard way and requires the introduction of the Faddeev-Popov ghosts $\bar{c}, c$. The scalar fields $b, d$ also make no problems.

The fields $B^\rho$ are described by the higher derivative Lagrangian and for their quantization we shall use Ostrogradsky canonical formalism ([3], [1], [2]). As we are working in the framework of perturbation theory it is sufficient to consider the quantization of the free theory.

In the Ostrogradsky formalism the system is described by the canonical coordinates

$$X_1^\rho = B^\rho; \quad X_2^\rho = \dot{B}^\rho$$

and conjugate momenta

$$P_1^\rho = \frac{\delta L}{\delta \dot{B}^\rho} - \partial_0 (\frac{\delta L}{\delta B^\rho}) = \alpha \partial_0 B^\rho + m^{-2} \partial_0 \partial^2 B^\rho$$

$$P_2^\rho = \frac{\delta L}{\delta B^\rho} = -m^{-2} \partial^2 B^\rho$$

The Hamiltonian for the $B^\rho$ fields, being written in terms of Fourier components is given by the expression

$$H = P_1^\rho X_2^\rho + P_2^\rho \dot{X}_2^\rho - L =$$

$$P_1^\rho X_2^\rho - \frac{m^2}{2} (P_2^\rho)^2 - k^2 P_2^\rho X_1^\rho - \frac{\alpha}{2} (X_2^\rho)^2 + \frac{\alpha k^2}{2} (X_1^\rho)^2 + \ldots$$

Introducing the creation and annihilation operators one can write the free hamiltonian in the form

$$H_0 = \int [\omega_1(k) q_1^{\rho+}(k) q_1^{\rho-}(k) - \omega_2(k) q_2^{\rho+}(k) q_2^{\rho-}(k)] dk$$

where

$$q_1^{\rho\pm} = \pm i \omega_1 \alpha X_1^\rho \mp \omega_1 P_2^\rho \sqrt{2 \alpha \omega_1}; \quad q_2^{\rho\pm} = -\alpha X_2^\rho \mp \omega_2 P_2^\rho \sqrt{2 \alpha \omega_2}$$

In these equations $\omega_1 = \sqrt{k^2}$; $\omega_2 = \sqrt{k^2 + \alpha m^2}$, and the operators $q_1^{\rho\pm}$ satisfy the commutation relations

$$[q_1^{\rho-}(k), q_1^{\rho+}(k')] = \delta^{\rho\sigma} \delta(k - k') \quad [q_2^{\rho-}(k), q_2^{\rho+}(k')] = -\delta^{\rho\sigma} \delta(k - k')$$

One sees that the operators $q_2^{\rho+}$ create negative norm states.

The free Hamiltonian for the supersymmetry ghosts $b^\rho, d^\rho$ may be obtained in a standard way

$$H_0' = i \int [\omega_2 (d^{\rho+}(k) b^{\rho-}(k) - b^{\rho+}(k) d^{\rho-}(k))] dk$$

where the creation and annihilation operators are given by the equations

$$d^{\rho\pm} = \frac{d^\rho \omega_2 \pm p_d^\rho}{\sqrt{2 \omega_2}}; \quad b^{\rho\pm} = \frac{b^\rho \omega_2 \pm p_b^\rho}{\sqrt{2 \omega_2}}$$

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They satisfy the anticommutation relations

\[ [b^\rho(k), d^{\sigma+}(k')]_+ = -i\delta^{\rho\sigma}\delta(k-k'); \quad [d^{\rho-}(k), b^{\sigma+}(k')]_+ = i\delta^{\rho\sigma}\delta(k-k') \quad (24) \]

The space of states includes many unphysical excitations, like the supersymmetry ghost states, states corresponding to the fields \( B^\rho \), unphysical components of \( A^\mu \) and Faddeev-Popov ghosts. The real physical states including only transversal components of the Yang-Mills field may be separated by imposing on the asymptotic states the conditions

\[ Q_0 |\psi >_{phys} = 0 \]
\[ Q_0^{BRST} |\psi >_{phys} = 0 \quad (26) \]

and taking the limit \( \alpha \to 0 \). Here \( Q_0 \) is the free charge associated with the supersymmetry transformations (13), and \( Q_0^{BRST} \) is the free BRST charge.

The invariance of the action \( (11) \) with respect to the supersymmetry transformations (13) generates via Noether theorem the conserved current, whose asymptotic form is

\[ J_\mu = m^{-2}(\partial_\mu B^\rho b^\rho - B^\rho \partial_\mu b^\rho) \quad (27) \]

The corresponding conserved charge may be written as follows

\[ Q_0 = \frac{1}{\sqrt{2\omega_2}} \int \left\{ b^{\rho+}\left( P_1^\rho + i\omega_2 P_2^\rho - \alpha X_2^2 \right) + (P_1^\rho - i\omega_2 P_2^\rho - \alpha X_2^2)b^{\rho-} \right\} dk \]

\[ \sim \text{const} \int \left\{ \frac{b^{\rho+}(k)q_1^{\rho-}(k) + q_2^{\rho-}(k)}{2} + \frac{q_1^{\rho+}(k) + q_2^{\rho+}(k)}{2}b^{\rho-}(k) \right\} dk + O(\alpha) \quad (28) \]

One sees that although for a finite \( \alpha \) the charge \( Q_0 \) is not nilpotent, in the limit \( \alpha \to 0 \) the nilpotency is recovered as the operators \( b^{\rho+}, b^{\rho-} \) and \( q_1^{\rho\pm} = q_1^{\rho\pm} + q_2^{\rho\pm} \) are mutually (anti)commuting.

Any vector annihilated by \( Q_0 \) may be presented in the form

\[ |\varphi >= |\varphi >_A + Q_0 |\chi > + O(\alpha) \quad (29) \]

Here \( |\varphi >_A \) is a vector which does not include the excitations, corresponding to the ghost fields \( q_{1,2}^\rho \) and \( b^\rho, c^\rho \). This vector depends only on the Yang-Mills field excitations and the Faddeev-Popov ghosts. Imposing on it the condition (26), which is compatible with the condition (25), we conclude that the vectors \( |\psi >_{phys} \) have a form

\[ |\psi >_{phys} = |\psi >_{tr} + |N > + O(\alpha) \quad (30) \]

where \( |\psi >_{tr} \) depends only on transversal polarizations of the Yang-Mills field, and \( |N > \) is a zero norm vector. Hence in the limit \( \alpha \to 0 \) we recover the usual Yang-Mills theory. It completes the proof.

### 4 Discussion.

In the present paper we proposed a local gauge invariant infrared regularization of the Yang-Mills theory. Our construction is based on the mechanism different from
the mechanism commonly used in the process of regularization. Usually one intro-
duces in a regularized theory some unphysical exitations which disappear from the
spectrum when the regularization is removed. In our scheme unphysical exitations
do not disappear when the regularization is removed, but decouple completely from
the physical exitations, which is sufficient for a physical interpretation of the theory.

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References

[1] A.A.Slavnov, Phys.Lett.B, 258 (1991), 391.
[2] A.A.Slavnov, Phys.Lett.B, 620 (2005), 97.
[3] A.A.Slavnov, Nucl.Phys.B, 31 (1971), 301.