MULTIVARIATE GENERATING FUNCTIONS BUILT OF
CHEBYSHEV POLYNOMIALS AND SOME OF ITS
APPLICATIONS AND GENERALIZATIONS.

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Abstract. We sum multivariate generating functions composed of products of
Chebyshev polynomials of the first and the second kind. That is, we find closed
forms of expressions of the type
\[ \sum_{j \geq 0} \rho^j \prod_{m=1}^{k} T_{j + t_m}(x_m) \prod_{m=k+1}^{n+k} U_{j + t_m}(x_m), \]
for different integers \( t_m, m = 1, \ldots, n+k \). We also find a Kibble-Slepian formula
of \( n \) variables with Hermite polynomials replaced by Chebyshev polynomials
of the first or the second kind. In all the considered cases, the obtained closed
forms are rational functions with positive denominators. We show how to
apply the obtained results to integrate some rational functions or sum some
related series of Chebyshev polynomials. We hope that the obtained formulae
will be useful in the so-called free probability. We expect also that the obtained
results should inspire further research and generalizations. In particular, that,
following methods presented in this paper, one would be able to obtain similar
formulae for the so-called \( q \)-Hermite polynomials. Since the Chebyshev poly-
nomials of the second kind considered here are the \( q \)-Hermite polynomials for
\( q = 0 \). We have applied these methods in the one- and two-dimensional cases
and were able to obtain nontrivial identities concerning \( q \)-Hermite polynomi-
als.

1. Introduction

In this work we obtain closed forms of the following expressions:

Case I. The multivariate generating functions:

\[ \chi_{k,n}^{(t_1,\ldots,t_{k+n})}(x_1,\ldots,x_{n+k}|\rho) = \sum_{j \geq 0} \rho^j \prod_{m=1}^{k} T_{j + t_m}(x_m) \prod_{m=k+1}^{n+k} U_{j + t_m}(x_m), \]

where \(|t_{m}|, k, n \in \{0, 1, \ldots\}, k + n \geq 1, |\rho| < 1, |x_m| \leq 1\) and \( T_j, U_j \) denote \( j \)-th
Chebyshev polynomials respectively of the first and second kind.
Case II. The so-called Kibble–Slepian formula for Chebyshev polynomials i.e. closed forms of the expressions:

\[ f_T(x|K_n) = \sum_{S} \prod_{1 \leq i < j \leq n} (\rho_{ij})^{s_{ij}} \prod_{m=1}^{n} T_{\sigma_m}(x_m), \]  
\[ f_U(x|K_n) = \sum_{S} \prod_{1 \leq i < j \leq n} (\rho_{ij})^{s_{ij}} \prod_{m=1}^{n} U_{\sigma_m}(x_m), \]

where \( x = (x_1, ..., x_n) \). \( K_n \) denotes the symmetric, non-singular, \( n \times n \) matrix with ones on its diagonal and with \( \rho_{ij} \) as its non-diagonal \( ij \)-th entry. \( \sum_{S} \) denotes summation over all \( n(n-1)/2 \) non-diagonal entries of a symmetric \( n \times n \) matrix \( S_n \) with zeros on the main diagonal and entries \( s_{ij} \) being nonnegative integers, while \( \sigma_m \) is the sum of the entries \( s_{ij} \) along the \( m \)-th row of the matrix \( S_n \).

We will show that in the case I. all functions \( \chi_{k,n} \) are rational with common denominator \( w_{n+k}(x_1, ..., x_{n+k}|\rho) \) which is a symmetric polynomial in \( x_1, ..., x_{n+k} \) of degree \( 2n+k-1 \) as well as in \( \rho \) of degree \( 2n+k \) defined recursively by (3.2).

In case II. both functions \( f_T(x|K_n) \) and \( f_U(x|K_n) \) are rational with the same denominator

\[ V_n(x|K_n) = \prod_{j=1}^{n-1} \prod_{k=j+1}^{n} w_2(x_k, x_j|\rho_{kj}), \]

where \( w_2 \) is defined by (2.14), below.

The fact that these functions are rational, is not very surprising, given the fact that Chebyshev polynomials could be expressed by the trigonometric functions and the fact that by the Euler formulae the series (1.1), (1.2) and (1.3) are sums of some geometric series. However, to get the exact forms of the denominators and especially the numerators, is nontrivial.

Both statements will be proved in the sequel. The first one in the Section 3 and the second in the Section 4.

Chebyshev polynomials of the second kind (that are orthogonal with respect to the semicircle distribution) have played a similar role in the rapidly recently developing "free probability", as the Hermite polynomials (that are orthogonal with respect to the normal distribution) play in classical probability. This is so because the central role in the free probability is played by the semicircle distribution, while in the classical one the central role is played by the normal distribution. Hence the results presented below are of significance for the free probability theory.

The possible other applications of the results of the paper can, for example, help in the following:

(1) To simplify calculations of some of the multiple integrals of the form

\[ \int \ldots \int_{k-fold} v_m(x_1, ..., x_n|p) \prod_{j=1}^{k}(1 - x_j^2)^{m_j/2} dx_1 \ldots dx_k, \]

where \( v_m \) denotes some polynomial in variables \( x_1, ..., x_n \) and numbers \( m_j \in \{-1, 1\} \), \( p \) denotes a set of parameters. Thus, this set might be different in cases I. or II. \( \Omega_n \) is equal to \( w_n \) in the case I. (see iterative formula (3.2)) or \( V_n \) in the case II (see formula (1.4)). This is based on the observation that the closed forms in Case I and Case II are the rational functions with
the denominators of the form $\Omega_n$ while the numerators are, depending on the case and on numbers $t_m$, $m = 1, \ldots, n$, polynomials of degree at most $\sum_{m=1}^{n}(t_m + 1)$. For example, for $n = 2$ see Proposition \ref{prop}. Hence, one could imagine expanding $\frac{v_m(x_1, \ldots, x_n)}{\Omega_m(x_1, \ldots, x_n)}$ into the linear combinations of the series of the forms (1.1), (1.2) or (1.3) depending on the cases considered Case I or Case II. Now notice that having an absolute uniform convergence of the appropriate series ($|\rho|, |\rho_j| < 1$ and $|T_i(x)|, |U_i(x)| \leq i, |x| \leq 1, i \geq 0$) one can perform integrations of each summand separately, which is very easy.

Below we present a few examples illustrating this idea. In the first three of these examples we will use the fact that following Proposition \ref{prop} iii), the numerators of the functions $\chi_{0,0}^{(0)}(x, y, \rho)$ and $\chi_{0,2}^{(0)}(x, y, \rho)$ are equal respectively

$$1 - \rho^2 \text{ and } (4x^2 - 4xy - 1 + \rho^2).$$

Thus for $|x|, |y| \leq 1$ and $|\rho| < 1$ we get

$$\int_{-1}^{1} \frac{2(1 - \rho^2)\sqrt{1 - y^2}dy}{\pi((1 - \rho^2)^2 - 4xy(1 + \rho^2) + 4\rho^2(x^2 + y^2))} = 1,$$

$$\int_{-1}^{1} \frac{2(4x^2 - 4xy - 1 + \rho^2)\sqrt{1 - y^2}dy}{\pi((1 - \rho^2)^2 - 4xy(1 + \rho^2) + 4\rho^2(x^2 + y^2))} = 4x^2 - 1,$$

since $U_2(x) = 4x^2 - 1$. In the next example we use the (2.4) to sum

$$\sum_{j \geq 0} \rho^{2j}U_{2j}(x) = \chi_{0,2}^{(0)}(x, 0, i\rho) = \frac{1 + \rho^2}{(1 + \rho^2)^2 - 4\rho^2x^2}$$

and then (2.8) and the form of $\chi_{0,2}^{(0)}(x, y, \rho)$ to get the following result :

$$\int_{-1}^{1} \frac{(4x^2 - 4xy - 1 + \rho^2)dy}{\pi\sqrt{1 - y^2((1 - \rho^2)^2 - 4xy(1 + \rho^2) + 4\rho^2(x^2 + y^2))}} = \frac{4x^2 - 1 - \rho^2}{(1 + \rho^2)^2 - 4x^2\rho^2}.$$
Theorem 1 applied to the well-known cases and leading to the non-obvious identities like the ones shown by (5.14), (5.12) and (5.13) is presented in Subsection 5.1.

To get these identities we used formulae given in (2.4), (2.5), (2.6) as well as

(1.13) \[
\sum_{j \geq 0} t^j T_{2j+1}(x)T_{2j+1}(y) = \frac{(1-t)xy(1+6t + t^2 - 4t(x^2 + y^2))}{(1-t)^4 + 8t(1-t)^2(x^2 + y^2) - 16t(1+t^2)x^2y^2 + 16t^2(x^4 + y^4)}.
\]

To illustrate this idea we have the following examples:

(1.12) \[
\sum_{j=0}^{\infty} (j+1)\rho^j U_j(x)U_j(y) = \frac{(1+\rho^2)(1-\rho^2)^2 - 4\rho^2(1+\rho^2)(x^2 + y^2) + 16\rho^2 xy}{((1-\rho^2)^2 - 4xy\rho(1+\rho^2) + 4\rho^2(x^2 + y^2))^2},
\]

\[\text{we have here } w_3(x,y,z) \text{ is given by (3.3).}\]

(2) To derive several expansions of the type (1.3) and (1.2) for the special choices of the parameters \(x_j\). To illustrate this idea we have the following examples:

(1.11) \[
\frac{1}{\pi} \int_{-1}^{1} \frac{(1+\rho^2)^3 + 16\rho^3xyz - 4\rho^2(1+\rho^2)(x^2 + y^2 + z^2)}{\sqrt{1 - z^2w_3(x,y,z)\rho}} \, dz
= \frac{(1-\rho^2)^3 + 4\rho^2(1-\rho^2)(x^2 + y^2)}{(1-\rho^2)^4 + 16\rho^4(x^4 + y^4) + 8\rho^2(1-\rho^2)^2(x^2 + y^2) - 16\rho^2(1+\rho^4)x^2y^2},
\]

There is one more reason for which the results are important. Namely, the Chebyshev polynomials of the second kind are, as stated above, identical with the so-called \(q\)-Hermite polynomials for \(q = 0\). Thus the results of the paper can be an inspiration to obtain similar results for the \(q\)-Hermite polynomials. All these ideas are explained and made more precise in the sequence of observations, remarks, hypothesis and conjectures presented in Section 5.

An interesting, nontrivial example of an application of the method presented in Theorem 1 applied to the well-known cases and leading to the non-obvious identities like the ones shown by (5.14), (5.12) and (5.13) is presented in Subsection 5.1.

The paper is organized as follows. In the next section we present some elementary observations, we recall the basic properties of Chebyshev polynomials as well as we prove some important auxiliary results. The main results of the paper are presented in the two successive Sections 3 and 4 presenting respectively closed forms of the one-parameter multivariate generating functions and the closed form of the analogue of Kibble–Slepian formula. The next Section 5 presents generalization, observations, conjectures and examples. Finally the last Section 6 contains longer proofs.
2. Auxiliary results and elementary observations

Let us recall (following [2]), the definitions of the Chebyshev polynomials:

\[ U_n(\cos(\alpha)) = \sin((n+1)\alpha)/\sin(\alpha) \text{ and } T_n(\cos(\alpha)) = \cos(n\alpha) \]

and the orthogonality relations they satisfy:

\[
\int_{-1}^{1} T_i(x)T_j(x) \frac{1}{\pi \sqrt{1-x^2}} dx = \left\{ \begin{array}{ll}
0 & \text{if } i \neq j \\
1/2 & \text{if } i = j \neq 0 \\
1 & \text{if } i = j = 0
\end{array} \right.
\]

\[
\int_{-1}^{1} U_i(x)U_j(x) \frac{2}{\pi} \sqrt{1-x^2} dx = \left\{ \begin{array}{ll}
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{array} \right.
\]

We have also some simple properties of Chebyshev polynomials that we will be useful in obtaining examples (1.6-1.11) and (1.12,1.13):

\[ T_j(0) = U_j(0) = \left\{ \begin{array}{ll}
0 & \text{if } j \text{ is odd} \\
(-1)^{j/2} & \text{if } j \text{ is even}
\end{array} \right.
\]

\[ T_1(1) = 1, T_j(-1) = (-1)^{j-2}[j/2], \]

\[ U_j(\pm 1) = \pm (j+1), \]

for \( j \geq 0, \)

\[
\int_{-1}^{1} T_j(x) \frac{2\sqrt{1-x^2}}{\pi} dx = \left\{ \begin{array}{ll}
1 & \text{if } j = 0 \\
-1/2 & \text{if } j = 2 \\
0 & \text{if } j \notin \{0,2\}
\end{array} \right.
\]

and

\[
\int_{-1}^{1} U_j(x) \frac{1}{\pi \sqrt{1-x^2}} dx = \left\{ \begin{array}{ll}
0 & \text{if } j \text{ is odd} \\
1 & \text{if } j \text{ is even}
\end{array} \right.
\]

In the sequel, if all integer parameters \( t_1, \ldots, t_{n+k} \) will be equal to zero, then they will be dropped from function \( \chi \). Notice also that the functions \( \chi \) are known for \( n = 1 \) and \( n = 2 \) and \( t_1 = 0, t_2 = 0 \). By (1.11) we have:

\[
\chi_{0,1}(x|\rho) = \frac{1}{w_1(x|\rho)}; \chi_{1,0}(x|\rho) = \frac{1-\rho x}{w_1(x|\rho)},
\]

\[
\chi_{0,2}(x,y|\rho) = \sum_{n \geq 0} \rho^n U_n(x)U_n(y) = \frac{1-\rho^2}{w_2(x,y|\rho)},
\]

\[
\chi_{2,0}(x,y|\rho) = \sum_{n \geq 0} \rho^n T_n(x)T_n(y) = \frac{1-\rho^2 + 2\rho^2(x^2+y^2) - (\rho^2+3)\rho xy}{w_2(x,y|\rho)},
\]

\[
\chi_{1,1}(x,y|\rho) = \sum_{n \geq 0} \rho^n U_n(x)T_n(y) = \frac{1-\rho^2 - 2\rho xy + 2\rho^2y^2}{w_2(x,y|\rho)},
\]

where:

\[
w_1(x|\rho) = 1 - 2\rho x + \rho^2;
\]

\[
w_2(x,y|\rho) = (1-\rho^2)^2 - 4xy\rho(1+\rho^2) + 4\rho^2(x^2+y^2).
\]

Notice also that both \( \chi_{2,0} \) and \( \chi_{0,2} \) are positive on \([-1,1] \times [-1,1]\). The formulae in (2.9) are well known within e.g. theory of Poisson kernel. The formula in (2.10)
it is famous Poisson-Mehler formula for \( q \)-Hermite polynomials where we set \( q = 0 \). Both can be found in [2]. The second formula in (2.11) and in (2.12) have been recently obtained in [8].

To calculate the functions \( \chi_{k,n}^{(t_1, \ldots, t_{k+n})} \) we need the following auxiliary results. They are very simple, based on the elementary properties of the trigonometric functions. We present them for the sake of the completeness of the paper. We have:

**Proposition 1.**

(2.15) \( w_1(\cos(\alpha + \beta)|\rho)w_1(\cos(\alpha - \beta)|\rho) = w_2(\cos(\alpha), \cos(\beta)|\rho) \).

**Proof.** We have

\[
(1 - 2\rho \cos(\alpha + \beta) + \rho^2)((1 - 2\rho \cos(\alpha - \beta) + \rho^2) = (1 + \rho^2)^2 - 2\rho(1 + \rho^2)(\cos(\alpha + \beta) + \cos(\alpha - \beta)) + 4\rho^2 \cos(\alpha + \beta) \cos(\alpha - \beta).
\]

Now recall that \( \cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos(\alpha) \cos(\beta) \) and \( \cos(\alpha + \beta) \cos(\alpha - \beta) = \cos^2 \alpha + \cos^2 \beta - 1. \) \( \square \)

**Proposition 2.**

(2.16) \[
\prod_{j=1}^{k} \cos(\alpha_j) = \frac{1}{2^n} \sum_{i_1 \in \{-1,1\}} \cdots \sum_{i_k \in \{-1,1\}} \cos\left(\sum_{l=1}^{k} i_l \alpha_l\right),
\]

(2.17) \[
\prod_{j=1}^{n} \sin(\alpha_j) \prod_{j=n+1}^{n+k} \cos(\alpha_j) = \begin{cases} 
(-1)^{(n+1)/2} \frac{1}{2^{n+1}} \sum_{i_1 \in \{-1,1\}} \cdots \sum_{i_{n+k} \in \{-1,1\}} \sin\left(\sum_{l=1}^{n+k} i_l \alpha_l\right) & \text{if } n \text{ is odd} \\
(-1)^{n/2} \frac{1}{2^n} \sum_{i_1 \in \{-1,1\}} \cdots \sum_{i_{n+k} \in \{-1,1\}} \sin\left(\sum_{l=1}^{n+k} i_l \alpha_l\right) & \text{if } n \text{ is even}
\end{cases}.
\]

**Proof.** See section 6 \( \square \)

**Lemma 1.** Let us take \( n \in \mathbb{N}, |\rho_i| < 1, \alpha_i \in \mathbb{R}, i \in S_n = \{1, \ldots, n\}. \) Let \( M_{i,n} \) denote a subset of the set \( S_n \) containing \( i \) elements. Let us denote by \( \sum_{M_{i,n} \subseteq S_n} \) summation over all \( M_{i,n} \) contained in \( S_n \). We have:

(2.18) \[
\sum_{k_i \geq 0} \cdots \sum_{k_n \geq 0} \left( \prod_{i=1}^{n} \rho_i^{k_i} \right) \cos(\beta + \sum_{i=1}^{n} k_i \alpha_i) = \frac{\sum_{j=0}^{n}(-1)^j \sum_{M_{j,n} \subseteq S_n} \left( \prod_{k \in M_{j,n}} \rho_k \right) \cos(\beta - \sum_{k \in M_{j,n}} \alpha_k)}{\prod_{j=1}^{n} (1 + \rho_i^2 - 2\rho_i \cos(\alpha_i))},
\]

(2.19) \[
\sum_{k_i \geq 0} \cdots \sum_{k_n \geq 0} \left( \prod_{i=1}^{n} \rho_i^{k_i} \right) \sin(\beta + \sum_{i=1}^{n} k_i \alpha_i) = \frac{\sum_{j=0}^{n}(-1)^j \sum_{M_{j,n} \subseteq S_n} \left( \prod_{k \in M_{j,n}} \rho_k \right) \sin(\beta - \sum_{k \in M_{j,n}} \alpha_k)}{\prod_{j=1}^{n} (1 + \rho_i^2 - 2\rho_i \cos(\alpha_i))}.
\]

**Proof.** See section 6 \( \square \)

We will also need the following almost trivial special cases of formulae (2.18) and (2.19). We will formulate them as corollary.
Corollary 2. For all $|\rho| < 1$ we have

\begin{align}
(2.20) & \quad \sum_{n \geq 0} \rho^n \sin(n\alpha + \beta) = (\sin(\beta) - \rho \sin(\beta - \alpha))/(1 - 2\rho \cos(\alpha) + \rho^2), \\
(2.21) & \quad \sum_{n \geq 0} \rho^n \cos(n\alpha + \beta) = (\cos(\beta) - \rho \cos(\beta - \alpha))/(1 - 2\rho \cos(\alpha) + \rho^2).
\end{align}

Proof. Set $n = 1$ and $\alpha = \alpha_1$ \[\text{[2.10]}\] and \[\text{[2.18]}\].

3. One parameter sums. Multivariate generating functions of Chebyshev polynomials

The theorem below is obtained by very elementary methods. Given the definition of the function $l_{k,n}^{(t_1,\ldots,t_{n+k})}(x_1,\ldots,x_{n+k}|\rho)$ presented by \[\text{[1.1]}\] it is obvious that it must be in the form of a rational function. Even many properties of the denominator of these functions can be more or less deduced from the definition. However the exact forms of the numerators of these functions are not trivial. For the sake of completeness of the paper, we present all these trivial and nontrivial observations in one theorem.

Theorem 1. For all integers $n, k \geq 0$, $|x_s| < 1, t_s \in \mathbb{Z}$, $s = 1, \ldots, n + k$, we have:

\begin{equation}
\chi_{k,n}^{(t_1,\ldots,t_{n+k})}(x_1,\ldots,x_{n+k}|\rho) = \frac{l_{k,n}^{(t_1,\ldots,t_{n+k})}(x_1,\ldots,x_{n+k}|\rho)}{w_{n+k}(x_1,\ldots,x_{n+k}|\rho)},
\end{equation}

where $w_m(x_1,\ldots,x_m|q)$ is a symmetric polynomial of degree $2m - 1$ in $x_1,\ldots,x_m$ and of degree $2m$ in $\rho$ defined by the following recurrence :

\begin{equation}
w_m+1(x_1,\ldots,x_{m-1},\cos(\alpha), \cos(\beta)|\rho) = w_m(x_1,\ldots,x_{m-1},\cos(\alpha + \beta)|\rho)w_m(x_1,\ldots,x_{m-1},\cos(\alpha - \beta)|\rho),
\end{equation}

$n \geq 1$, with $w_1(x|q)$ given by \[\text{[2.13]}\], $l_{k,n}^{(t_1,\ldots,t_{n+k})}(x_1,\ldots,x_{n+k}|\rho)$ is another polynomial given by the relationship:

\begin{equation}
l_{k,n}^{(t_1,\ldots,t_{n+k})}(x_1,\ldots,x_{n+k}|\rho) = \sum_{j=0}^{2^{n+k}-1} \rho^j \sum_{m=0}^j \frac{1}{m!} \left. \frac{d^m}{d\rho^m} w_{k+n}(x_1,\ldots,x_{k+n}|\rho) \right|_{\rho=0} \\
\times \prod_{s=1}^{k} T_{(j-m)+t_s}(x_s) \prod_{s=1+k}^{n+k} U_{(j-m)+t_s}(x_s).
\end{equation}

Proof. See section \[\text{[6]}\].

Corollary 2. Theorem\[\text{[7]}\] provides for free the following important set of identities involving Chebyshev polynomial of the first and the second kind. Namely we have:

\begin{equation}
\forall n, k \geq 0 : n + k \geq 1, \forall t_1, \ldots, t_{n+k} \geq 0, \forall j \geq 2^{n+k}, \forall(x_1,\ldots,x_{k+n}) \in (-1,1)^{n+k}
\end{equation}

\begin{equation}
\sum_{m=0}^j \frac{1}{m!} \left. \frac{d^m}{d\rho^m} w_{k+n}(x_1,\ldots,x_{k+n}|\rho) \right|_{\rho=0} \times \prod_{s=1}^{k} T_{(j-m)+t_s}(x_s) \prod_{s=1+k}^{n+k} U_{(j-m)+t_s}(x_s) = 0.
\end{equation}

In particular we have for $n + k = 1$ :

$$U_k(x) - 2xU_{k+1}(x) + U_{k+2}(x) = 0,$$
which is nothing else as the well-known three-term recurrence satisfied by the Chebyshev polynomials. However for say \( k = 0 \) and \( n = 2 \) we get for all \( s, m \geq 0 \)

\[
-4xyU_s(y)U_m(x) + 2(2x^2 + 2y^2 - 1)U_{s+1}(y)U_{m+1}(x) \\
-4xyU_{s+2}(y)U_{m+3}(x) + U_{s+3}(y)U_{m+3}(x) = 0,
\]

which is, to my knowledge, unknown.

**Proof.** Since \( l_{k,n}^{(t_1,...,t_{n+k})}(x_1,...,x_{n+k}|\rho) \) is a polynomial of degree \( 2^{k+n} - 1 \) in \( \rho \) all its derivatives with respect to \( \rho \) of higher than \( 2^{k+n} - 1 \) should be equal to zero. \( \square \)

**Corollary 3.** For \( n \geq 1 \), after swapping \( x_1 \) and \( x_n \), taking \( \beta = 0 \), \( \cos(\alpha) = x_2 \) we get:

\[
w_n(1,...x_{n-1},x_n|\rho) = (w_{n-1}(x_2,...x_n|\rho))^2.
\]

In particular

\[
w_3(x_1,\cos(\alpha_2),\cos(\alpha_3)|\rho) = w_2(x_1,\cos(\alpha_3 + \alpha_2)|\rho)w_2(x_1,\cos(\alpha_3 - \alpha_2)|\rho),
\]

which, after replacing \( \cos(\alpha_2) \) by \( x_2 \) and \( \cos(\alpha_3) \) by \( x_3 \) and with the help of Mathematica, yields:

\[
(3.6) \quad w_3(x_1,x_2,x_3|\rho) = 16\rho^4(x_1^4 + x_2^4 + x_3^4) - 8\rho^2(1 + \rho^2)^2(x_1^2 + x_2^2 + x_3^2) \\
+ 16\rho^2(1 + \rho^4)(x_1^2x_2^2 + x_1^2x_3^2 + x_2^2x_3^2) + 64\rho^4x_1^2x_2^2x_3^2 - 32\rho^3(1 + \rho^2)x_1x_2x_3(\rho^2 + x_1^2 + x_2^2 + x_3^2) \\
- 8\rho(1 + \rho^2)(1 + \rho^4 - 6\rho^2)x_1x_2x_3 + (1 + \rho^2)^4.
\]

**Remark 1.** Notice that from Theorem 7 we deduce that for all integers \( t_1,...,t_{k+n} \) the ratio

\[
\frac{\chi_{k,n}^{(t_1,...,t_{k+n})}(x_1,...,x_{n+k}|\rho)}{\chi_{k,n}^{(0,...,0)}(x_1,...,x_{n+k}|\rho)}
\]

is a rational function of arguments \( x_1,...,x_{n+k},\rho \).

Such observation for was first made by Carlitz for \( k+n = 2, \) nonnegative integers \( t_1 \) and \( t_2 \) concerning the so-called Rogers–Szegö polynomials and two variables \( x_1 \) and \( x_2 \) in [1] (formula 1.4). Later it was generalized by Szabowski in [4] for the so-called \( q \)-Hermite polynomials, also for the two variables. Now, it turns out that for \( q = 0 \) the \( q \)-Hermite polynomials are equal to Chebyshev polynomials of the second kind, hence one can state that so far the above-mentioned observation was known for \( k = 0 \) and \( n = 2 \). Hence we deal with far-reaching generalization both in the number of variables as well as for the Chebyshev polynomials of the first kind.

**Corollary 4.** For \( |x_i| \leq 1 \) and \( |\rho| < 1 \), \( n \geq 1 \):

\[
\int_{-1}^{1}... \int_{-1}^{1} \left( \prod_{s=1}^{n} \frac{1}{\pi \sqrt{1-x_s^2}} \right) \chi_{n,0}(x_1,...,x_n|\rho)dx_1...dx_j = \prod_{s=j+1}^{n} \frac{1}{\pi \sqrt{1-x_s^2}},
\]

for \( j = 1,...,n. \)
Proof. For the first assertion recall that based on Theorem we have

\[ \chi_{n,0}(\cos(\alpha_1), \ldots, \cos(\alpha_n)|\rho) = \sum_{k \geq 0} \rho^k \prod_{j=1}^{n} T_k(\cos(\alpha_j)) = \]

\[ \frac{1}{2^n} \sum_{i_i \in \{-1,1\}} \cdots \sum_{i_n \in \{-1,1\}} \frac{(1 - \rho \cos(\sum_{k=1}^{n} i_k \alpha_k))}{(1 - 2\rho \cos(\sum_{k=1}^{n} i_k \alpha_k) + \rho^2)}, \]

which is nonnegative for all \( \alpha_i \in \mathbb{R}, \ i = 1, \ldots, n \) and \(|\rho| < 1\).

The remaining part follows directly the definition of \( \chi_{n,0} \) and the properties of polynomials \( T_i \).

Let us now finish the case \( n = 2 \). That is let us calculate \( \chi_{2,0}^{n,m}(x, y|\rho), \chi_{1,1}^{n,m}(x, y|\rho) \). The case of \( \chi_{0,2}^{n,m}(x, y|\rho) \) has been solved in e.g. [5] (Lemma 3, with \( q = 0 \)).

Proposition 3. i)

\[ \chi_{1,0}^{m,0}(x|\rho) = \sum_{i=0}^{\infty} \rho^i T_{i+m}(x) = \frac{T_m(x) - \rho T_{m-1}(x)}{w_1(x|\rho)}, \]

\[ \chi_{0,1}^{0,m}(x|\rho) = \sum_{i=0}^{\infty} \rho^i U_{i+m}(x) = \frac{U_m(x) - \rho U_{m-1}(x)}{w_1(x|\rho)}, \]

ii)

\[ \chi_{2,0}^{n,m}(x, y|\rho) = \sum_{k \geq 0} \rho^k T_{k+n}(x)T_{k+m}(y) = \]

\[ (T_n(x)T_m(y)(w_2(x, y|\rho) - \rho^4) \]

\[ + \rho T_{n+1}(x)T_{m+1}(y)(1 - 2\rho^2 + 4\rho^2(x^2 + y^2) - 4\rho xy) \]

\[ + \rho^2 T_{n+2}(x)T_{m+2}(y)(1 - 4\rho xy) + \rho^3 T_{n+3}(x)T_{m+3}(y))/w_2(x, y|\rho), \]

iii)

\[ \chi_{0,2}^{n,m}(x, y|\rho) = \sum_{j \geq 0} \rho^j U_{j+n}(x)U_{j+m}(y) = \]

\[ (U_n(x)U_m(y)(w_2(x, y|\rho) - \rho^4) \]

\[ + \rho U_{n+1}(x)U_{m+1}(y)(1 - 2\rho^2 + 4\rho^2(x^2 + y^2) - 4\rho xy) \]

\[ + \rho^2 U_{n+2}(x)U_{m+2}(y)(1 - 4\rho xy) + \rho^3 U_{n+3}(x)U_{m+3}(y))/w_2(x, y|\rho) \]

iv)

\[ \chi_{1,1}^{n,m}(x, y|\rho) = \sum_{j \geq 0} \rho^j U_{m+j}(x)T_{n+j}(y) = \]

\[ (T_n(y)U_m(x)(w_2(x, y|\rho) - \rho^4) \]

\[ + \rho T_{n+1}(y)U_{m+1}(x)(1 - 2\rho^2 + 4\rho^2(x^2 + y^2) - 4\rho xy) \]

\[ + \rho^2 T_{n+2}(y)U_{m+2}(x)(1 - 4\rho xy) + \rho^3 T_{n+3}(y)U_{m+3}(x))/w_2(x, y|\rho). \]
**Proof.** We apply a formula (3.3). For i) we take \( n = 1 \) and notice that values of derivatives of \( w_1 \) with respect to \( \rho \) at \( \rho = 0 \) are \( 1, -2x, 2 \).

To get ii) we notice that subsequent derivatives of \( w_2 \) with respect to \( \rho \) at \( \rho = 0 \) are \( 1, -4xy, 8z^2 + 8y^2 - 4, -24xy \). Having this and applying directly (3.3) we get certain defined formula expanded in powers of \( \rho \). Now it takes Mathematica to get this form.

iii) and iv) We argue similarly getting expansions in powers of \( \rho \). Then using Mathematica we try to get more friendly form. \( \square \)

As a corollary we get formulae presented in (2.10) and (2.11) when setting \( n = m = 0 \) and remembering that \( T_{-1}(x) = T_1(x), U_{-1}(x) = -U_{-2}(x) \), for \( i = 0, 1, 2 \).

**Corollary 5.** \( \forall x, y, z \in [-1, 1], |\rho| < 1 \):

i) \[
\chi_{3,0}(x, y, z|\rho) = \sum_{i \geq 0} \rho^i T_i(x) U_i(y) T_i(z) = ((1 + \rho^2)^3 + 8\rho^4 (x^4 + y^4 + z^4) + 32\rho^4 x^2 y^2 z^2 - 2 (\rho^2 + 3) \rho^2 (x^2 + y^2 + z^2) + 4 (\rho^2 + 3) \rho^2 (x^2 y^2 + x^2 z^2 + y^2 z^2) - 4 (3\rho^2 + 5) \rho^3 x y z (x^2 + y^2 + z^2) - (\rho^4 - 15\rho^4 - 25\rho^2 + 7) \rho^4 x y z)/w_3(x, y, z|\rho),
\]

ii) \[
\chi_{0,3}(x, y, z|\rho) = \sum_{i \geq 0} \rho^i U_i(x) U_i(y) U_i(z) = ((1 + \rho^2)^3 + 16\rho^3 x y z - 4\rho^2 (1 + \rho^2)(x^2 + y^2 + z^2))/w_3(x, y, z|\rho),
\]

iii) \[
\chi_{1,2}(x, y, z|\rho) = \sum_{i \geq 0} \rho^i T_i(x) U_i(y) U_i(z) = ((\rho^2 + 1)^3 + 8\rho^4 x^4 - 16\rho^3 x^3 y z - 2 (\rho^2 + 1) (\rho^2 + 3) \rho^2 x^2 + 8\rho^2 x^2 (y^2 + z^2) - 4\rho (5 - (\rho^2 + 2)) x y z - 4 (\rho^2 + 1) \rho^2 (y^2 + z^2))/w_3(x, y, z|\rho),
\]

iv) \[
\chi_{2,1}(x, y, z|\rho) = \sum_{i \geq 0} \rho^i T_i(x) T_i(y) U_i(z) = ((\rho^2 + 1)^3 + 8\rho^4 (x^4 + y^4) - 2 (\rho^2 + 1) (\rho^2 + 3) \rho^2 (x^2 + y^2) + 4 (\rho^4 + 3) \rho^2 x^2 y^2 + 16\rho^4 x^2 y^2 z^2 + 8\rho^2 z^2 (x^2 + y^2) - 8 (\rho^2 + 2) \rho^3 x y z (x^2 + y^2) - 8\rho^3 x y z^3 - 2 (-5\rho^4 - 10\rho^2 + 3) \rho^3 x y z - 4 (\rho^2 + 1) \rho^2 z^2)/w_3(x, y, z|\rho),
\]

where \( w_3(x, y, z|\rho) \) is given by (3.6).

**Proof.** Again we apply formula (3.3). Besides we take \( n = 3, k = 0 \) for i), \( n = 0, k = 3 \) for ii), \( n = 1, k = 2 \) for iii) and \( n = 2, k = 1 \) for iv). Now we have to remember that successive derivatives of \( w_3 \) with respect to \( \rho \) taken at \( \rho = 0 \) are respectively \( 1, -8xyz, 8(1 - (x^2 + y^2 + z^2) + 4(x^2 y^2 + x^2 z^2 + y^2 z^2)), 48xyz(5 - 4(x^2 + y^2 + z^2)), 48(3 - 8(x^2 + y^2 + z^2) + 8(x^2 y^2 + y^2 + z^2) + 32x^2 y^2 z^2), 960xyz(5 - 4(x^2 + y^2 + z^2)), 2880(1 - (x^2 + y^2 + z^2) + 4(x^2 y^2 + x^2 z^2 + y^2 z^2)), -40320xyz \). Then we get certain formulae by applying directly formula (3.3). The expression are long and not very legible. We applied Mathematica to get forms presented in i), ii) iii) and iv). \( \square \)
4. Kibble–Slepian formula and related sums for Chebyshev polynomials

Let \( f_n(x_1, \ldots, x_n|K_n) \) denote the density of the normal distribution with zero expectations and non-singular covariance matrix \( K_n \) such that \( \text{var}(X_i) = 1 \) for \( i = 1, \ldots, n \), i.e. having 1’s on the diagonal. Let \( \rho_{ij} \) denote the \( ij \)-th entry of matrix \( K_n \). Consequently, the one-dimensional marginals \( f_1 \) are given by:

\[
f_1(x) = \exp(-x^2/2)/\sqrt{2\pi}.
\]

Let us also denote by \( S_n \) a symmetric \( n \times n \) matrix with zeros on the diagonal and nonnegative integers as off-diagonal entries. Let us denote the \( ij \)-th entry of the matrix \( S_n \) by \( s_{ij} \). Recall that Kibble in the 40s and Slepian in the 70s presented the following formula:

\[
(4.1) \quad \frac{f_n(x_1, \ldots, x_n|K_n)}{\prod_{m=1}^n f_1(x_m)} = \sum_S \left( \prod_{1 \leq i < j \leq n} (\rho_{ij})^{s_{ij}} \right) \frac{\prod_{m=1}^n H_{\sigma_m}(x_m))}{s_{ij}!},
\]

where \( H_i(x) \) denotes \( i \)-th (so called probabilistic) Hermite polynomial i.e. forming the orthonormal base of the space of functions square integrable with respect to the weight \( f_1(x) \), \( \sigma_m = \sum_{j=1}^{m-1} s_{jm} + \sum_{j=1+m}^n s_{mj} \), \( \sum_S \) denotes, as before, summation over all \( n(n-1)/2 \) non-diagonal entries of the matrix \( S_n \). To see more details on Kibble–Slepian formula see e.g. recent paper by Ismail [3]. A partially successful attempt was made by Szablowski in [6] where for \( n = 3 \) the author replaced polynomials \( H_n \) by the so called \( q \)-Hermite polynomials \( H_n(x|q) \) and \( s_{ij}! \) substituted by \( [s_{ij}]_q! \) where \( [n]_q = (1 - q^n)/(1 - q) \) for \( |q| < 1 \), \( [n]_q! = \prod_{i=1}^n [i]_q ! \) with \( [0]_q! = 1 \). Taking into account that \( H_n(0) = U_n(x/2) \) and \( [0]_q! = 1 \) we see that (4.1) has been generalized and summed already for other polynomials. The intention of summing in [6] was to find a generalization of the normal distribution that has compact support. The attempt was partially successful since also one has obtained a relatively closed form for the sum, however the obtained sum was not positive for the suitable values of parameters \( \rho_{ij} \) and all values of parameters \( |q| < 1 \).

In the present paper, we are going to present closed form of the sum (4.1) where polynomials \( H_n \) are replaced by Chebyshev polynomials of both the first and second kind and \( s_{ij}! \) are replaced by 1. This last replacement is justified by the fact that \( [s_{ij}]_q! = 1 \) if \( q = 0 \). For more details, see publications on the so-called \( q \)-series and also brief introduction at the beginning of the Section 5 below.

In other words, we are going to find closed forms for the sums (1.2) and (1.3), where \( x \) and \( K_n \), used below, mean, as before, \( x = (x_1, \ldots, x_n) \) while \( K_n \) denotes symmetric \( n \times n \) matrix with ones on its diagonal and \( \rho_{ij} \) as its \( ij \)-th entry. We will assume that all \( \rho \)'s are from the segment \((-1, 1)\) and additionally that matrix \( K_n \) is positive definite.

We have the following result:

**Theorem 2.** Let us denote \( K_n = \{(i, j) : 1 \leq i < j \leq n\}, \beta_{n,m} = \beta_{n,m}(i_n, i_m) = i_n \alpha_n + i_m \alpha_m \). For \( S \subseteq K_n \) let \( p_S = \prod_{(n,m) \in S} \rho_{nm}, b_S = \sum_{(n,m) \in S} \beta_{n,m}, B_{i_1, \ldots, i_n} = B(i_1, \ldots, i_n) = \sum_{j=1}^n i_j \alpha_j \).
We have i)

\[ f_T(\cos(\alpha_1), ..., \cos(\alpha_n)|K_n) = \frac{1}{2^n} \sum_{i_j \in \{-1, 1\}} \cdots \sum_{i_n \in \{-1, 1\}} \frac{\sum_{k=0}^{n} (-1)^k \sum_{S_k \subseteq K_n} \rho_{S_k} \cos(b_{S_k})}{\prod_{j=1}^{n} \prod_{m=j+1}^{n} (1 - 2\rho_{jm} \cos(\beta_{j,m}(i_j,i_m)) + \rho_{jm}^2)}, \]

ii) If \( n \) is even then

\[ f_U(\cos(\alpha_1), ..., \cos(\alpha_n)|K_n) = \frac{(-1)^{n/2}}{2^n} \prod_{j=1}^{n} \sin(\alpha_j) \sum_{i_1 \in \{-1, 1\}} \cdots \sum_{i_n \in \{-1, 1\}} (-1)^{\sum_{i=1}^{n} (i_{i+1})/2} \frac{\sum_{k=0}^{n} (-1)^k \sum_{S_k \subseteq K_n} \rho_{S_k} \cos(B_1, ..., n - b_{S_k})}{\prod_{j=1}^{n} \prod_{m=j+1}^{n} (1 - 2\rho_{jm} \cos(\beta_{j,m}(i_j,i_m)) + \rho_{jm}^2)} \]

while if \( n \) is odd then

\[ f_U(\cos(\alpha_1), ..., \cos(\alpha_n)|K_n) = \frac{(-1)^{n/2}}{2^n} \prod_{j=1}^{n} \sin(\alpha_j) \sum_{i_1 \in \{-1, 1\}} \cdots \sum_{i_n \in \{-1, 1\}} (-1)^{\sum_{i=1}^{n} (i_{i+1})/2} \frac{\sum_{k=0}^{n-1} (-1)^k \sum_{S_k \subseteq K_n} \rho_{S_k} \sin(B_1, ..., n - b_{S_k})}{\prod_{j=1}^{n} \prod_{m=j+1}^{n} (1 - 2\rho_{jm} \cos(\beta_{j,m}(i_j,i_m)) + \rho_{jm}^2)} \]

where \( S_k \) denotes any subset of \( K_n \) that contains \( k \) elements and \( \sum'_{S_k \subseteq K_n} \) means summation over all \( S_k \).

**Proof.** Let us consider (1.2) first. Keeping in mind assertions of Proposition 2 we see that \( f_T(\cos(\alpha_1), ..., \cos(\alpha_n)|K_n) \) is the sum of \( 2^n \) summands depending on different arrangement of values of variables \( i_k \in \{-1, 1\}, k = 1, ..., n \). Each summand is equal to cosine taken at \( \sum_{j=1}^{n} i_j s_j \alpha_j \). Recalling the definition of numbers \( s_j \) we see that in such sum \( s_{m,j}, 1 \leq m < j \leq n \) appears twice, once as \( s_{m,j} \alpha_m i_m \) and secondly as \( s_{m,j} \alpha_j i_j \). Or in other words, we have \( \sum_{j=1}^{n} i_j s_j \alpha_j = \sum_{m=1}^{n-1} \sum_{j=m+1}^{n} s_{m,j} (\alpha_m i_m + \alpha_j i_j) \). Having this in mind, we can now apply summation formula (2.18) with \( \beta = 0 \) and have summed each cosine with a particular system of values of the set \( \{i_j : j = 1, ..., n\} \). Now it remains to sum over, all such systems of values.

As far as other assertions are concerned, we use the definition of Chebyshev polynomials of the second kind, formulae presented in Proposition 2. We have in this case \( \sum_{j=1}^{n} i_j (s_j + 1) \alpha_j = \sum_{j=1}^{n} i_j \alpha_j + \sum_{m=1}^{n-1} \sum_{j=m+1}^{n} s_{m,j} (\alpha_m i_m + \alpha_j i_j) \). As the result we deal with signed sum of either sines or cosines depending on the fact if \( n(n-1)/2 \) (the number of different \( s_{m,j}, 1 \leq m < j \leq n \)) is odd or even. Now again we refer to either (2.19) or (2.18) depending on the parity of \( n(n-1)/2 \) this time with \( \beta = \sum_{j=1}^{n} i_j \alpha_j \). \( \square \)

**Corollary 6.** Both functions \( f_T(x|K_n) \) and \( f_U(x|K_n) \) are rational functions of all its arguments. Moreover, they have the same denominators given by the following formula:

\[ V_n(x|K_n) = \prod_{j=1}^{n-1} \prod_{k=j+1}^{n} w_2(x_j, x_k, \rho_{ij}), \]

where \( w_2 \) is given by the formula (2.14).
Proof. First of all, notice that following formulae given in Theorem 2 the functions $f_T(x|K_n)$ and $f_U(x|K_n)$ are rational functions of $x_1 = \cos(\alpha_1), \ldots, x_n = \cos(\alpha_n)$. Moreover, it is easy to notice that all formulae have the same denominators. To find these denominators notice that the factors in each denominator referring to $(i_j, i_m)$ and $(-i_j, i_m)$ are the same since cosine is an even function and that cosines appear solely in denominators. Further, we can group factors $(1 - 2\rho_{jm} \cos(\beta_{j,m}(i_j, i_m)))$ and $(1 - 2\rho_{jm} \cos(\beta_{j,m}(i_j, -i_m)))$ with $(1 - 2\rho_{jm} \cos(\beta_{j,m}(i_j, i_m))) + \rho_{jm}^2$ and apply (2.15)

$$(1 - 2\rho_{jm} \cos(\beta_{j,m}(i_j, i_m))) + \rho_{jm}^2(1 - 2\rho_{jm} \cos(\beta_{j,m}(i_j, -i_m))) + \rho_{jm}^2 = w_2(\cos(\alpha_j), \cos(\alpha_m)|\rho_{jm}).$$

since $\beta_{n,m}(i_n, i_m) = i_n\alpha_n + i_m\alpha_m$. □

Corollary 7. Let us denote $\beta_{k,j} = i_k\alpha_k + i_j\alpha_j$, $k = 1, 2$, $j = 2, 3$, $k < j$, $p = \rho_{12}\rho_{13}\rho_{23}$. Then $B_{1,2,3} = \sum_{j=1}^{3} i_j a_j$,

$$c(i_1, i_2, i_3, \alpha_1, \alpha_2, \alpha_3, \rho_{12}, \rho_{13}, \rho_{23}) = (1 - \sum_{1 \leq k < j \leq 3} \rho_{k,j} \cos(\beta_{k,j}) + \rho_{k,j}^{-1} \cos(2B_{1,2,3} - \beta_{k,j}) - p \cos(2B_{1,2,3}))/ \prod_{1 \leq k < j \leq 3} (1 - \rho_{k,j} \cos(\beta_{k,j}) + \rho_{k,j}^2),$$

$$s(i_1, i_2, i_3, \alpha_1, \alpha_2, \alpha_3, \rho_{12}, \rho_{13}, \rho_{23}) = (\sin(B_{1,2,3})(1 + p)$$

$$- (\rho_{12}\sin(i_3\alpha_3) + \rho_{13}\sin(i_2\alpha_2) + \rho_{23}\sin(i_1\alpha_2))$$

$$- (\rho_{12}\rho_{13}\sin(i_1\alpha_3) + \rho_{12}\rho_{23}\sin(i_2\alpha_1) + \rho_{13}\rho_{23}\sin(i_3\alpha_3))$$

$$\prod_{1 \leq k < j \leq 3} (1 - \rho_{k,j} \cos(\beta_{k,j}) + \rho_{k,j}^2).$$

Then:

i) $f_T(\cos(\alpha_1), \cos(\alpha_2), \cos(\alpha_3), \rho_{12}, \rho_{13}, \rho_{23}) = \frac{1}{8} \sum_{i_2 \in \{-1,1\}} \sum_{i_3 \in \{-1,1\}} c(1, i_2, i_3, \alpha_1, \alpha_2, \alpha_3, \rho_{12}, \rho_{13}, \rho_{23})$,

ii) $f_U(\cos(\alpha_1), \cos(\alpha_2), \cos(\alpha_3), \rho_{12}, \rho_{13}, \rho_{23}) = \frac{1}{8} \sum_{i_1 \in \{-1,1\}} \sum_{i_2 \in \{-1,1\}} \sum_{i_3 \in \{-1,1\}} s(i_1, i_2, i_3, \alpha_1, \alpha_2, \alpha_3, \rho_{12}, \rho_{13}, \rho_{23})$ $\prod_{1 \leq k < j \leq 3} (1 - \rho_{k,j} \cos(\beta_{k,j}) + \rho_{k,j}^2)$

in case of $\rho_{kj} = 0$ is understood as the limit when $\rho_{kj} \to 0$.

iii) $f_T(x, y, z, \rho_{12}, \rho_{13}, \rho_{23}) = (4\rho_{12}\rho_{13}(\rho_{23} - \rho_{12}\rho_{13})(1 - \rho_{23}^2)x^2 + 4\rho_{12}\rho_{23}(\rho_{13} - \rho_{12}\rho_{23})(1 - \rho_{13}^2)y^2 + 4\rho_{13}\rho_{23}(\rho_{12} - \rho_{13}\rho_{23})(1 - \rho_{13}^2)z^2 - 4(\rho_{12} - \rho_{12}\rho_{13})(\rho_{23} - \rho_{13}\rho_{23})1(1 + \rho_{12}\rho_{13}\rho_{23})(\rho_{12} - \rho_{13}\rho_{23})x^2 - 4(\rho_{12}\rho_{13}\rho_{23})(\rho_{13} - \rho_{12}\rho_{23})z^2 - 4(\rho_{12} - \rho_{13}\rho_{23})(\rho_{13} - \rho_{12}\rho_{23})y^2 - 4(\rho_{12}\rho_{13}\rho_{23})(\rho_{13} - \rho_{23})(\rho_{23} - \rho_{13})(\rho_{12} - \rho_{13})(\rho_{12} - \rho_{13}\rho_{23}(\rho_{13} - \rho_{12})(\rho_{12} - \rho_{13}\rho_{23}))$ $/(w_2(x, y, \rho_{12})w_2(x, z, \rho_{13})w_2(y, z, \rho_{23}))$

Proof. First of all, notice that $\sum_{k=1}^{2} \sum_{j=k+1}^{3} \beta_{k,j} = 2B_{1,2,3}$ hence in particular $B_{1,2,3} - \sum_{k=1}^{2} \sum_{j=k+1}^{3} \beta_{k,j} = -B_{1,2,3}$. Then the formula i) is clear based on (2.18) with $\beta = B_{1,2,3}$. To get ii) notice that $B_{1,2,3} - \beta_{12} = i_3\alpha_3$ and $B_{1,2,3} - \beta_{12} - \beta_{13} = -i_1\alpha_1$, similarly for the other pairs (1, 3) and (2, 3). Recall also that $B_{1,2,3} - \sum_{k=1}^{2} \sum_{j=k+1}^{3} \beta_{k,j} = -B_{1,2,3}$. Now based on (2.19) ii) is also clear.

iii) was obtained with the help of Mathematica. □
Remark 2. With the help of Mathematica one can show, for example, that the numerator of $f_T(x,y,z|K_3)$ is a polynomial of degree 6 and it consists of 265 monomials. Numerical simulation suggest that it is a nonnegative on $(-1,1)^3$. Unfortunately $f_U(x,y,z|K_3)$ is not nonnegative there since we have for example

$$f_U(-.9, -.95, .94, [\begin{array}{ccc} 0 & .6 & .8 \\ .6 & 0 & .9 \\ .8 & .9 & 0 \end{array}]) = -0.0912121.$$ Besides notice that it happens in the case when matrix $[\begin{array}{ccc} 1 & .6 & .8 \\ .6 & 1 & .9 \\ .8 & .9 & 1 \end{array}]$ is positive definite. This observation is in accordance with the general negative result presented in [8] Theorem 1. Recall that [8] concerns something like generalization of $f_U$ to all parameters $q \in (-1,1)$ taking into account that $q$-Hermite polynomials $H_n(x|q)$ can be identified for $q = 0$ with polynomials $U_n(x/2)$. The example presented in [8] concerns the case (adopted to $q = 0$) when say $\rho_{12} = 0$. Hence we see that there are many sets of 6 tuples $x,y,z,\rho_{12},\rho_{13},\rho_{23}$ leading to negative values of $f_U$.

5. Remarks on generalization

In this section, we are going firstly to present $q-$generalization of the Chebyshev of the first kind and secondly present some remarks and observations that might help to obtain formulæ similar to the ones presented in Theorem 1 with Chebyshev polynomials replaced by the so-called $q-$Hermite $\{h_n\}$ and related polynomials. $q$ is here a certain real (in general) number such that $|q| < 1$. Since in the previous chapters we considered, so to say, the case $q = 0$ we will assume in this chapter that $q \neq 0$.

To proceed further we need to recall certain notions used in $q-$series theory: $[0]_q = 0; [n]_q = 1 + q + \ldots + q^{n-1}, [n]_q != \prod_{j=1}^{n} [j]_q$, with $[0]_q ! = 1$,

$$\binom{n}{k}_q = \begin{cases} [n]_q! / [n-k]_q! \cdot [k]_q! & , 0 \leq k \leq n \\ 0 & , otherwise \end{cases}.$$ $inom{n}{k}$ will denote ordinary, well known binomial coefficient.

It is useful to use the so-called $q-$Pochhammer symbol for $n \geq 1$

$$(a|q)_n = \prod_{j=0}^{n-1} (1 - aq^j), (a_1, a_2, \ldots, a_k|q)_n = \prod_{j=1}^{k} (a_j|q)_n .$$

with $(a|q)_0 = 1$. Note that $n$ can be equal to $\infty$, then the $q-$Pochhammer symbol is well defined provided $|q| < 1$.

Often $(a|q)_n$, as well as, $(a_1, a_2, \ldots, a_k|q)_n$ will be abbreviated to $(a)_n$ and $(a_1, a_2, \ldots, a_k)_n$, if it will not cause misunderstanding.

It is easy to notice that $(q)_n = (1 - q)^n [n]_q !$ and that

$$\binom{n}{k}_q = \begin{cases} \frac{(a)}{(q)_{n-k}(q)_k} & , n \geq k \geq 0 \\ 0 & , otherwise \end{cases}.$$ 

The above mentioned formula is just an example where direct setting $q = 1$ is senseless however, the passage to the limit $q \rightarrow 1^-$ makes sense.
Notice that in particular \([n]_1 = n\), \([n]_1! = n!\), \([n]_1 = (\frac{n}{1})\), \((a)_1 = 1 - a\), \((a;1)_n = (1 - a)^n\) and \([n]_0 = \begin{cases} 1 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases}\), \([n]_0! = 1\), \([n]_1 = 1\), \((a;0)_n = \begin{cases} 1 & \text{if } n = 0 \\ 1 - a & \text{if } n \geq 1 \end{cases}\).

\[i\] will denote, as before, the imaginary unit, unless otherwise clearly stated. In the sequel we will need also the so-called \(q\)-Hermite polynomials. There exists a very large literature on the properties as well as applications of these polynomials. Let us recall only that the three-term recurrence satisfied by these polynomials is the following

\[ h_{n+1}(x|q) = 2xh_n(x|q) - (1 - q^n)h_{n-1}(x|q), \]

with \(h_{-1}(x|q) = 0\), \(h_0(x|q) = 1\). It is well known that the density, which makes these polynomials orthogonal is the following

\[ f_n(x|q) = \frac{2(q)_\infty \sqrt{1 - x^2}}{\pi} \prod_{k=1}^{\infty} l(x|q^k), \]

where \(l(x|a) = (1 + a)^2 - 4x^2a\). Moreover, generating functions of these polynomials, are equal to:

\[ \sum_{j=0}^{\infty} \frac{t^j}{(q)_j} h_j(x|q) = \frac{1}{\prod_{k=0}^{\infty} v(x|tq^k)}, \]

where \(v(x|a) = 1 - 2ax + a^2\).

**Remark 3.** For the sake of completeness of the paper, let us recall that \(h_n(x|0) = U_n(x)\), for \(n \geq -1\).

5.1. **Conjectures, remarks and interesting identities.** Theorem \[\text{1}\] suggests the new method of summing characteristic functions. One can formulate it in the following way.

Suppose, that we can guess, that the form of certain multivariate characteristic function, say for example

\[ \chi_n^{(l_1, \ldots, l_n)}(x_1, \ldots, x_n|\rho, q) = \sum_{j \geq 0} \frac{\rho^j}{(q)_j} \prod_{k=1}^{n} h_{j+l_k}(x_k|q), \]

where numbers \(l_1, \ldots, l_n\) are integer and \(|\rho|, |q| < 1\), is of the form of the ratio of two functions. Moreover, suppose that we can guess the form of the denominator \(W_n(x_1, \ldots, x_n|\rho, q)\) of this ratio. Then the numerator can be obtained by the formula similar to \(\text{(5.2)}\) i.e. by:

\[ \sum_{j=0}^{\infty} \rho^j \sum_{k=0}^{j} \frac{1}{k! \, d\rho^k} W_n(x_1, \ldots, x_{n+m}|\rho, q) \bigg|_{\rho = 0} \frac{1}{(q)_{j-k}} \prod_{s=1}^{n} h_{j-k+l_s}(x_s|q). \]

**Remark 4.** There are classes of characteristic functions that have common denominators like for example bivariate ones described in \[\text{1}\], Proposition 7 (iv) or, more generally, bivariate functions of the form similar to \(\text{(5.2)}\) that were considered by Carlitz in \[\text{1}\]. The point is that all these functions are at most bivariate. There are no results concerning more variables. Thus we have the following conjecture.
Conjecture 1. Functions $\chi_n^{(l_1,\ldots,l_n)}(x_1,\ldots,x_n|\rho,q)$ for all $n,m,l_1,\ldots,l_n$ are the ratios of some functions with the common denominators of the form

$$W_n(x_1,\ldots,x_n|\rho,q) = \prod_{i=0}^{\infty} w_n(x_1,\ldots,x_n|\rho q^i),$$

where functions $w_n(x_1,\ldots,x_n|\rho)$ are given by the iterative relationship [5.2].

5.1.1. One-dimensional case. Now we will present a one-dimensional example, in order to show that even in this simplest case we obtain interesting identities. In this example, we will, so to say, derive once more formula (5.1). First of all, notice that $$(1 - ae^{i\varphi})(1 - ae^{-i\varphi}) = 1 + a^2 - 2ax d f_n(x|a)$$ where $x = \cos \varphi$. Moreover, we have:

$$W_1(x|\rho,q) = \prod_{j=0}^{\infty} v(x|\rho q^j) = (\rho e^{i\varphi})_\infty (\rho e^{-i\varphi})_\infty.$$

Let us denote indirectly function $d_n(x|q)$ by the relationship:

$$\frac{d^n}{(q)_n} W_1(x|\rho,q) \bigg|_{\rho=0}.$$ Notice that $d_n(x|q)$ are coefficients of the expansion of $W_1(x|\rho,q)$ in the following series

$$W_1(x|\rho,q) = \sum_{n\geq 0} \frac{\rho^n}{(q)_n} d_n(x|q).$$

For the sake of symmetry let us also denote by $f_n(x|q)$ coefficients of the expansion $1/W_1(x|\rho,q)$ in the following series

$$1/W_1(x|\rho,q) = \sum_{n\geq 0} \frac{\rho^n}{(q)_n} f_n(x|q).$$

Remark 5. Let us recall polynomials $\{b_n\}$ defined in [10] and later analyzed in [9] (2.43). These polynomials satisfy the following three term recurrence:

$$b_{n+1}(x|q) = -2q^n x b_n(x|q) + q^{n-1}(1 - q^n)b_{n-1}(x|q),$$

with $b_1(x|q) = 0$, $b_1(x|q) = 1$. Moreover, as it follows from [5] (3.18) after some trivial transformation polynomials $\{b_n\}$ satisfy the following identity:

$$\sum_{j=1}^{n} \binom{n}{j} q^{n-j} b_{n-j}(x|q) h_{j+k}(x|q) = \begin{cases} 0 & \text{if } k < n \\ (-1)^n q^{\binom{n}{2}} \frac{(q)_{n-k}}{(q)_{k-n}} h_{k-n}(x|q) & \text{if } k \geq n \end{cases}.$$  

Recall also that the two families of polynomials $\{h_n\}$ and $\{b_n\}$ are related to one another by

$$b_n(x|q) = (-1)^n q^{\binom{n}{2}} h_n(x|q^{-1}),$$

for $q \neq 0$ and for $q = 0$ we have $b_{-1}(x|0) = b_n(x|0) = 0$ for $n \geq 3$, $b_1(x|q) = -2x$, $b_2(x|0) = 1$.

In the sequel when considering the case $q = 0$ we will understand as the limit with $q \to 0$ in the function in question.

One can notice that, we have

$$\frac{n!}{(q)_n} f_n(x|q) = \frac{d^n}{d\rho^n} W_1^{-1}(x|\rho,q) \bigg|_{\rho=0}.$$  

We have the following lemma.
Lemma 2. For \(|x| \leq 1, |q| < 1\), we have

\begin{align}
5.5 & \\
5.6 & \\
\end{align}

To get \((5.6)\) we recall \((5.1)\). The separate proof is needed for the case \(q = 0\). Then \(W_1(x, \rho, 0) = v(x|\rho) = 1 - 2x\rho + \rho^2\) which confronted with our definition of polynomials \(b_n\) for \(q = 0\) shows that the \((5.5)\) is true for this case also. \(\square\)

Now we see that following, adapted to the present situation, formula \((5.5)\) we have, for \(|q|, |\rho| < 1\) and \(|x| \leq 1\).

\[
\chi_1^{(1)}(x|\rho, q) = \sum_{j=0}^{\infty} \frac{\rho^j}{(q)_j} h_{j+t}(x|q) = \frac{1}{W_1(x|\rho, q)} \times \frac{1}{W_1(x|\rho, q)} \sum_{j=0}^{\infty} \frac{\rho^j}{(q)_j} \sum_{m=0}^{j} \frac{(j-m)!}{(j-m)! (q)_j (q)_m} b_{j-m}(x|q) h_{m+t}(x|q)
\]

In particular, for \(t = 0\), we get once more formula \((5.1)\). This can be regarded as yet another prove of this formula since we started from \((5.3)\).

5.1.2. Two-dimensional case. Again, as before, let us denote

\[
\frac{n!}{(q)_n} d_n^{(2)}(x, y|q) = \left. \frac{d^n}{d\rho^n} W_2(x, y|\rho, q) \right|_{\rho=0}, \quad \frac{n!}{(q)_n} f_n^{(2)}(x, y|q) = \left. \frac{d^n}{d\rho^n} W_2^{-1}(x, y|\rho, q) \right|_{\rho=0},
\]

where \(W_2(x, y|\rho, q) = \sum_{j=0}^{\infty} w_2(x, y|\rho q^j)\), with \(w_2(x, y|a)\) defined by \((2.14)\).

Lemma 3. For \(\theta, \varphi \in [0, 2\pi), |q| < 1\), we have

\begin{align}
5.7 & \\
5.8 & \\
\end{align}

Proof. First of all, notice that \(w_2(\cos \theta, \cos \varphi|\rho)\) can be decomposed as

\[
w_2(\cos \theta, \cos \varphi|\rho) = w_1(\cos (\theta + \varphi)|\rho) w_1(\cos (\theta - \varphi)|\rho)
\]
hence, taking into account Leibniz rule, we get:

\[ d_n^{(2)}(x, y|q) = \left( \frac{q}{n!} \right) \frac{d^n}{d\rho^n} (W_1(\cos(\theta + \varphi)|\rho, q)W_1(\cos(\theta - \varphi)|\rho, q) \bigg|_{\rho=0} \]

\[ = \left( \frac{q}{n!} \right) \sum_{m=0}^{n} \left( \frac{n}{m} \right) \frac{d^m}{d\rho^m} (W_1(\cos(\theta + \varphi)|\rho, q) \bigg|_{\rho=0} \frac{d^{n-m}}{d\rho^{n-m}} (W_1(\cos(\theta - \varphi)|\rho, q) \bigg|_{\rho=0} \]

\[ = \left( \frac{q}{n!} \right) \sum_{m=0}^{n} \left( \frac{n}{m} \right) m! b_m(\cos(\theta + \varphi)|q) (n-m)! b_{n-m}(\cos(\theta - \varphi)|q). \]

To get (5.8), we argue in a similar way using Lemma 2 on the way. \qed

**Theorem 3.** We have for \(|x|, |y|, |q| \in \mathbb{R}\) and all \(n \geq 0\):

\[ f_n^{(2)}(x, y|q) = \sum_{j=0}^{\lfloor n/2 \rfloor} \left( \frac{n}{j} \right) \frac{1}{(q)_j} h_{n-2j}(x|q) h_{n-2j}(y|q). \]

**Proof.** Is shifted to Section 6. \qed

**Remark 6.** Notice that, in accordance with our agreement that the case \(q = 0\) will be understood as the limit when \(q \to 0\), we have \(d_0^{(2)}(x, y|0) = 1\), \(d_1^{(2)}(x, y|0) = -4xy\), \(d_2^{(2)}(x, y|0) = 4(x^2 + y^2) - 2\), \(d_3^{(2)}(x, y|0) = -4xy\), \(d_4^{(2)}(x, y|0) = 1\), \(d_n^{(2)}(x, y|0) = 0\) for all \(n \geq 4\).

As a corollary we get the following interesting nontrivial identity involving polynomials \(\{b_n\}\) and \(\{h_n\}\).

**Corollary 8.** For all complex \(x, y, q, k \geq 0\) and \(t, s \in \mathbb{N} \cup \{0\}\), we get

\[ \sum_{m=0}^{k} \left[ \frac{k}{m} \right] d_m^{(2)}(x, y|q) h_{k-m+t}(x|q) h_{k-m+s}(y|q) = P_{t,s}^{(k)}(x, y|q) \]

where \(P_{t,s}^{(k)}(x, y|q)\) is a polynomial of order \(t + s\) in \(x\) and \(y\).

**Proof.** Knowing that

\[ \sum_{j=0}^{\infty} \frac{\rho^j}{(q)_j} h_{j+t}(x|q) h_{j+s}(y|q) = \frac{(\rho^2)^\infty V_{t,s}(x, y|\rho, q)}{W_2(x, y|\rho, q)}, \]

for \(t, s \in \mathbb{N} \cup \{0\}\), which is a modification of the formula given in assertion i) of Lemma 3 in [5], where \(V_{t,s}(x, y|\rho, q)\) denotes certain polynomial of the degree \(t + s\)
in \(x\) and \(y\), our expansion of \(W_2(x, y|\rho, q)\) and then applying Cauchy multiplication of series get the identity
\[
\sum_{j=0}^{\infty} \frac{\rho^j}{(q)_j} \sum_{m=0}^{j} \frac{[j]_m}{m!} d_m^{(2)}(x, y|q) h_{j-m+t}(x|q) h_{j-m+s}(y|q) = V_{t,s}(x, y|\rho, q)(\rho^2)_\infty, \tag{5.14}
\]
true for all \(|x|, |y| \leq 1\), \(|\rho|, |q| < 1\). Now knowing the form of the polynomial \(V_{t,s}\) given either in [5], [7] or [9], we deduce that the expansion of the polynomial \(V_{t,s}\) in the power series of \(\rho\) is of a form of the sum of infinite power series only in \(\rho\) times polynomials of \(x\) and \(y\) of order at most \(t + s\). Hence it is of the form of the power series in \(\rho\) with coefficients being polynomials in \(x\) and \(y\) of order at most \(t + s\). Since the linear combination of polynomials of order \(t + s\) is a polynomial of order \(t + s\). A similar argument can be applied to the product \(V_{t,s}(x, y|\rho, q)(\rho^2)_\infty\).

Now comparing the coefficients of the powers of \(\rho\) on the two sides of (5.14), one proves the first part of the statement.

Now knowing that \(V_{0,0} = 1\), expanding \((\rho^2)_\infty\) in a standard way and finally comparing coefficients by equal powers of \(\rho\) we arrive to (5.12).

\[\square\]

6. Proofs

Proof of Proposition 2. We will be using well known formulae for the product of sines and cosines. The proof is by induction. For \(n = 1\) and \(k = 1\) we have in case of (2.16) and \(k = 0\) \(\cos(\alpha) = \frac{1}{2}(\cos(\alpha) + \cos(-\alpha))\) while in case of (2.17) we get
\[
\sin(\alpha_1) \cos(\alpha_2) = -\frac{1}{4}(\sin(-\alpha_1 - \alpha_2) + \sin(-\alpha_1 + \alpha_2) - \sin(\alpha_1 - \alpha_2) - \sin(\alpha_1 + \alpha_2))
= \frac{1}{2}(\sin(\alpha_1 + \alpha_2) + \sin(\alpha_1 - \alpha_2)).
\]

Hence, let us assume that they are true for \(n = m\).

In the case of the first one, we have
\[
\prod_{j=1}^{m+1} \cos(\xi_j) = \cos(\xi_{m+1}) \prod_{j=1}^{m} \cos(\xi_j) = \\
\frac{1}{2^m} \sum_{i_1 \in \{-1, 1\}} \cdots \sum_{i_m \in \{-1, 1\}} \cos(\sum_{k=1}^{m} i_k \xi_k) \cos(\xi_{m+1})
= \frac{1}{2^{m+1}} \times \sum_{i_1 \in \{-1, 1\}} \cdots \sum_{i_m \in \{-1, 1\}} \left(\cos(\sum_{k=1}^{m} i_k \xi_k + \xi_{m+1}) + \cos(\sum_{k=1}^{m} i_k \xi_k - \xi_{m+1})\right).
\]

Along the way we used the fact that \(\cos(\alpha) \cos(\beta) = (\cos(\alpha - \beta) + \cos(\alpha + \beta))/2\). Let us also observe that the product \(\prod_{j=1}^{m} \cos(\xi_j)\) is a sum of cosines of a certain linear combination of arguments \(\xi_j, j = 1, \ldots, m\) multiplied by \(2^{m-1}\).
In the case of the second one we first consider the case of \( k = 0 \). Assuming that \( m \) is even we get:

\[
\prod_{j=1}^{m+1} \sin(\xi_j) = \sin(\xi_{m+1}) \prod_{j=1}^{m} \sin(\xi_j) = (-1)^{m/2} \frac{1}{2^m} \times \\
\sum_{i_1 \in \{-1, 1\}} \cdots \sum_{i_m \in \{-1, 1\}} (-1)^\sum_{k=1}^{m+1} (i_k + 1)/2 \cos(\sum_{k=1}^{m} i_k \xi_k) \sin(\xi_{m+1})
\]

\[
= (-1)^{m/2} \frac{1}{2^m+1} \sum_{i_{m+1} \in \{-1\}} \cdots \sum_{i_1 \in \{-1, 1\}} (-1)^\sum_{k=1}^{m+1} (i_k + 1)/2 \sin(\sum_{k=1}^{m} i_k \xi_k)
\]

\[
= (-1)^{m/2} \frac{1}{2^m+1} \sum_{i_{m+1} \in \{-1\}} \cdots \sum_{i_1 \in \{-1, 1\}} (-1)^\sum_{k=1}^{m+1} (i_k + 1)/2 \sin(\sum_{k=1}^{m} i_k \xi_k).
\]

We used the fact that \( \sin(\alpha) \cos(\beta) = (\sin(\alpha - \beta) + \sin(\alpha + \beta))/2 \). The case of \( m \) odd is treated in the similar way.

Now to consider general case we expand both products of sines and cosines.

**Proof of Lemma** \[2.18\] Using the Euler’s identity \( \cos(\theta) = (e^{i\theta} + e^{-i\theta})/2 \) we get

\[
\cos(\beta + \sum_{j=1}^{n} k_j \alpha_j) = \exp(i\beta + \sum_{j=1}^{n} ik_j \alpha_j)/2 + \exp(-i\beta - \sum_{j=1}^{n} ik_j \alpha_j)/2.
\]

So

\[
\sum_{k_1 \geq 0} \cdots \sum_{k_n \geq 0} (\prod_{j=1}^{n} \rho_j^{k_j}) \exp(i\beta + \sum_{j=1}^{n} ik_j \alpha_j)/2 = \frac{1}{2} \exp(i\beta) \prod_{j=1}^{n} \frac{1}{1 - \rho_j \exp(i\alpha_j)}.
\]

Similarly:

\[
\sum_{k_1 \geq 0} \cdots \sum_{k_n \geq 0} (\prod_{j=1}^{n} \rho_j^{k_j}) \exp(-i\beta - \sum_{j=1}^{n} ik_j \alpha_j)/2 = \frac{1}{2} \exp(-i\beta) \prod_{j=1}^{n} \frac{1}{1 - \rho_j \exp(-i\alpha_j)}.
\]

Thus

\[
\sum_{k_1 \geq 0} \cdots \sum_{k_n \geq 0} (\prod_{j=1}^{n} \rho_j^{k_j}) \cos(\beta + \sum_{j=1}^{n} ik_j \alpha_j) = \frac{\exp(i\beta) \prod_{j=1}^{n} (1 - \rho_j \exp(-i\alpha_j)) + \exp(-i\beta) \prod_{j=1}^{n} (1 - \rho_j \exp(i\alpha_j))}{2 \prod_{j=1}^{n} (1 + \rho_j^2 - 2\rho_j \cos(\alpha_j))}.
\]
Now, notice that we convert products Chebyshev polynomials to the products of $\sin(\theta)$.

Proof of Theorem 1. So

$$\exp(-i\beta) \prod_{j=1}^{n} (1 - \rho_j \exp(i\alpha_j)) =$$

$$\sum_{j=1}^{n} (-1)^j \sum_{M_{j,n} \subseteq S_n} \prod_{k \in M_{j,n}} \rho_k \exp(-i\beta + i \sum_{k \in M_{j,n}} \alpha_k).$$

To verify (2.19), we use the fact that $\sin(\theta) = (e^{i\theta} - e^{-i\theta})/2$ getting

$$\sin(\beta + \sum_{j=1}^{n} k_j \alpha_j) = \exp(i\beta + \sum_{j=1}^{n} i k_j \alpha_j)/2i - \exp(-i\beta - \sum_{j=1}^{n} i k_j \alpha_j)/2i.$$

So we have:

$$\sum_{k_1 \geq 0} \ldots \sum_{k_n \geq 0} (\prod_{j=1}^{n} \rho_j^{k_j}) \exp(i\beta + i \sum_{j=1}^{n} k_j \alpha_j)/2i = \exp(i\beta) \prod_{j=1}^{n} \frac{1}{2i} \frac{1}{1 - \rho_j \exp(i\alpha_j)}.$$

Similarly we get

$$\sum_{k_1 \geq 0} \ldots \sum_{k_n \geq 0} (\prod_{j=1}^{n} \rho_j^{k_j}) \exp(-i\beta - i \sum_{j=1}^{n} k_j \alpha_j)/2i = \exp(-i\beta) \prod_{j=1}^{n} \frac{1}{2i} \frac{1}{1 - \rho_j \exp(-i\alpha_j)}.$$

So

$$\sum_{k_1 \geq 0} \ldots \sum_{k_n \geq 0} (\prod_{j=1}^{n} \rho_j^{k_j}) \sin(\beta + \sum_{j=1}^{n} k_j \alpha_j) =$$

$$\frac{1}{2i} \exp(i\beta) \prod_{j=1}^{n} (1 - \rho_j \exp(-i\alpha_j)) - \exp(-i\beta) \prod_{j=1}^{n} (1 - \rho_j \exp(i\alpha_j)).$$

\[ \square \]

Proof of Theorem 1. The proof is based on the following observation. First one is that we convert products Chebyshev polynomials to the products of $\sin(j\alpha_s + (t_s + 1)\alpha_s)$ and $\cos(j\alpha_s + t_s\alpha_s)$ according to (2.11). Secondly we change these products to sums of either cosines if $n$ is even or zero or sines if $n$ is odd according to the assertion of the Proposition 2. The arguments of these sines and cosines are the linear combinations of the arguments of sines and cosines that were participating in the products. The coefficients of these linear combinations are $j \geq 0$ and $i_m \in \{-1, 1\}, \quad m = 1, \ldots, n + k$. Thus we can sum first with respect to $j$ and apply Corollary 1. There the rôle of $\alpha$ plays now $\sum_{s=1}^{k+n} i_s \alpha_s$ for chosen combination of $i'$s while the rôle of $\beta$ similar combination $\sum_{s=1}^{n} i_s (t_s + 1)\alpha_s + \sum_{s=n+1}^{n+k} i_s t_s \alpha_s$. The point is that the sum of such sines or cosines with respect to $j$, is a ratio of two trigonometric expressions. Moreover all these the expressions in the denominators depend only on $\sum_{s=1}^{k+n} i_s \alpha_s$, i.e. do not depend on indices $t_s$ (note that denominators of sums in Corollary 1 do not depend on $\beta$). For $\alpha_s \in \mathbb{R}$, $t_s \in \mathbb{Z}$, $s = 1, \ldots, n + k$, $|\rho| < 1$ we have, depending on the parity of $n$, the following equations.
If $n$ is odd then,

$$
\sum_{j \geq 0} \rho^j \prod_{s=1}^{n} U_{j+s}(\cos(\alpha_s)) \prod_{s=n+1}^{n+k} T_{j+s}(\cos(\alpha_s)) =
$$

$$
\frac{(-1)^{(n+1)/2}}{2^{n+k}} \prod_{l=1}^{n} \sin(\alpha_l) \sum_{i_1 \in \{-1,1\}} \ldots \sum_{i_{n+k} \in \{-1,1\}} (-1)^{\sum_{i=1}^{n+k}(i_k+1)/2} \times
$$

$$
\frac{(\sin(\sum_{s=1}^{n} i_s(t_s + 1)\alpha_s + \sum_{s=n+1}^{n+k} i_s(t_s\alpha_s) - \rho \sin(\sum_{s=1}^{n} i_s t_s\alpha_s + \sum_{s=n+1}^{n+k} i_s(t_s - 1)\alpha_s))}{(1 - 2\rho \cos(\sum_{s=1}^{n+k} i_s\alpha_s) + \rho^2)}
$$

while, when $n$ is even or zero, we get:

$$
\sum_{j \geq 0} \rho^j \prod_{s=1}^{n} U_{j+s}(\cos(\alpha_s)) \prod_{s=n+1}^{n+k} T_{j+s}(\cos(\alpha_s)) =
$$

$$
\frac{(-1)^{n/2}}{2^{n+k}} \prod_{l=1}^{n} \sin(\alpha_l) \sum_{i_1 \in \{-1,1\}} \ldots \sum_{i_{n+k} \in \{-1,1\}} (-1)^{\sum_{i=1}^{n+k}(i_k+1)/2} \times
$$

$$
\frac{\cos(\sum_{s=1}^{n} i_s(t_s + 1)\alpha_s + \sum_{s=n+1}^{n+k} i_s(t_s\alpha_s) - \rho \cos(\sum_{s=1}^{n} i_s t_s\alpha_s + \sum_{s=n+1}^{n+k} i_s(t_s - 1)\alpha_s))}{(1 - 2\rho \cos(\sum_{s=1}^{n+k} i_s\alpha_s) + \rho^2)}
$$

To justify it, we use \[2.1\] first, then based on Proposition \[2\] we convert products to sums of sines or cosines (if $n$ is odd sines if $n$ is even cosines) that are of the following arguments:

$$
\sum_{s=1}^{n} l_s((j + 1)\alpha_s + t_s\alpha_s) + \sum_{s=n+1}^{n+k} l_s(j\alpha_s + t_s\alpha_s)
$$

$$
= j \sum_{s=1}^{n+k} l_s\alpha_s + \sum_{s=1}^{n} l_s(t_s + 1)\alpha_s + \sum_{s=n+1}^{n+k} l_s t_s\alpha_s.
$$

Then, we change the order of summation and we sum over $j$ first. We identify $\alpha$ with $\sum_{s=1}^{n+k} l_s\alpha_s$ and $\beta$ with $\sum_{s=1}^{n} l_s(t_s + 1)\alpha_s + \sum_{s=n+1}^{n+k} l_s t_s\alpha_s$ and apply formulae \[2.20\] or \[2.21\] depending on on the case of parity of $n$.

Now let us analyze polynomial $w_n$. Notice that denominator in both \[6.1\] and \[6.2\] is of the form

$$
\prod_{i_1 \in \{-1,1\}} \ldots \prod_{i_{n+k} \in \{-1,1\}} (1 - 2\rho \cos(\sum_{s=1}^{n+k} i_s\alpha_s) + \rho^2).
$$

To get \[6.3\] we will argue by induction. Let us replace $n+k$ by $m$ to avoid confusion.

To start with $m = 1$ for $m = 2$ we recall \[2.15\]. Hence \[6.3\] is true for $m = 1, 2$. 
Let us assume that the formula is true for \( m = k + 1 \). Hence, taking \( \alpha = \alpha_{k+1} \) and \( \beta = \sum_{s=1}^{k} i_s \alpha_s \) and noting that \( i_k^2 = 1 \) we get:

\[
w_{k+1}(\cos(\alpha_1), ..., \cos(\alpha_{k+1})|\rho) = \prod_{i_s \in \{-1, 1\}} \prod_{i_k \in \{-1, 1\}} ((1 - 2\rho \cos(\sum_{s=1}^{k-1} i_s \alpha_s + i_k (\alpha_k - i_k \alpha_{k+1})) + \rho^2)
\]

\[\times (1 - 2\rho \cos(\sum_{s=1}^{k-1} i_s \alpha_s + i_k (\alpha_k + i_k \alpha_{k+1})) + \rho^2))
\]

\[= w_{k}(\cos(\alpha_1), ..., \cos(\alpha_k + \alpha_{k+1})|\rho)w_{k}(\cos(\alpha_1), ..., \cos(\alpha_k - \alpha_{k+1})|\rho).
\]

by induction assumption. Now it is elementary to see that polynomials \( w_n \) satisfy relationship \( (3.2) \). Similarly, the remarks concerning degree of symmetry and the degree of polynomials \( w_n \) follow directly \( (6.3) \).

Now, let us multiply both sides of \( (6.1) \) and \( (6.2) \) by \( w_{n+k}(x_1, ..., x_{n+k}|\rho) \). We see that this product is equal to the right hand sides of these equalities with an obvious replacement \( \cos(\alpha_s) \rightarrow x_s, s = 1, ..., n + k \). Inspecting \( (6.1) \) and \( (6.2) \), we notice that these right hand sides are polynomials of degree \( 2(2^{n+k-1} - 1) + 1 = 2^{n+k} - 1 \) in \( \rho \). Thus, these polynomials can be regained by using well known formula:

\[
p_n(x) = \sum_{i=0}^{n} x^n a_n = \sum_{j=0}^{n} \frac{x^j}{j!} \frac{d^j}{dx^j} p_n(x) \bigg|_{x=0}.
\]

This leads directly to the differentiation of the products of \( w_{n+k}(x_1, ..., x_{n+k}|\rho) \) and right hand side of \( (1.1) \). Now we apply the Leibniz formula:

\[
\frac{d^n}{dx^n} [f(x)g(x)] \bigg|_{x=0} = \sum_{j=0}^{n} \binom{n}{j} \frac{d^j}{dx^j} f(x) \bigg|_{x=0} \frac{d^{n-j}}{dx^{n-j}} g(x) \bigg|_{x=0}.
\]

and notice that

\[
\frac{d^k}{d\rho^k} \sum_{j=0}^{n} \rho^j \prod_{s=1}^{n} T_{j+s}(x_s) \prod_{s=1+n}^{n+k} U_{j+s}(x_s) \bigg|_{\rho=0} = k! \prod_{s=1}^{n} T_{k+s}(x_s) \prod_{s=1+n}^{n+k} U_{k+s}(x_s).
\]

Having this we get directly \( (3.1) \).

**Proof of the Theorem** \( \Box \) The proof consists of several steps. First, we prove that for all \( \theta, \varphi \in \mathbb{R} \) we have

\[
(6.4) \quad \sum_{m=0}^{n} \binom{n}{m} h_m(\cos(\theta + \varphi)|q)h_{n-m}(\cos(\theta - \varphi)|q) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(q)_n}{(q)_j (q)_{n-2j}} h_{n-2j}(\cos \theta|q)h_{n-2j}(\cos \varphi|q).
\]

This formula follows, firstly from the fact that we have

\[
\frac{d^n}{d\rho^n} W_1^{-1}(x|\rho, q) \bigg|_{\rho=0} = \frac{n!}{(q)_n} h_n(x|q),
\]
which follows directly from (5.1). Secondly, arguing in the similar way as in the proof of Lemma 3 we deduce that
\[
\frac{d^n}{d\rho^n} W_1^{-1}(\cos(\theta + \varphi)|\rho, q) W_1^{-1}(\cos(\theta - \varphi)|\rho, q) \bigg|_{\rho=0} = \sum_{n=0}^{\infty} \frac{n!}{(q)_n} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q h_m(\cos(\theta + \varphi)|q) h_{n-m}(\cos(\theta - \varphi)|q).
\]

Thirdly, we notice that
\[
\frac{1}{W_1(\cos(\theta + \varphi)|\rho, q) W_1(\cos(\theta - \varphi)|\rho, q)} = \frac{1}{W_2(\cos(\theta), \cos(\varphi)|\rho, q)},
\]

which follows directly from (5.9).

Now, let us calculate
\[
\sum_{n \geq 0} \frac{\rho^n}{(q)_n} \sum_{j \geq 0} \frac{(q)_n}{(q)_j (q)_{n-2j}} h_{n-2j}(\cos(\theta)|q) h_{n-2j}(\cos(\varphi)|q).
\]

After changing the order of summation, we get
\[
\sum_{j \geq 0} \rho^{2j} \sum_{n \geq 2j} \frac{(q)_{n-2j}}{(q)_n} h_{n-2j}(\cos(\theta)|q) h_{n-2j}(\cos(\varphi)|q) = \frac{1}{(\rho^2)_\infty} \frac{(\rho^2)_\infty}{W_2(\cos(\theta), \cos(\varphi)|\rho, q)},
\]

by the binomial and Poisson-Mehler summation theorems. Thus we have proved (6.4) as well as (5.11) at least for \(|q| < 1\). The formula can be easily extended to all values of \(q \neq 1\) since both sides are polynomials in \(q\). Similarly, we can extend it to all values of \(x\) and \(y\) by substitution \(\cos(\theta)\) by \(x\) and \(\cos(\varphi)\) by \(y\). Now, having proven (6.4) we recall the definition of polynomials \(b_n(x|q)\) given in Lemma 2 above. Recall also that
\[
\left(\frac{1}{q} \frac{1}{q}\right)_n = (-1)^n q^{-\binom{n+1}{2}}_n(q)_n,
\]

and consequently that we have:
\[
\left[ \begin{array}{c} n \\ j \end{array} \right]_{1/q} = \left[ \begin{array}{c} n \\ j \end{array} \right] q^{-j(n-j)}.
\]

Hence, for the left hand side of (6.4), we have after changing \(q\) to \(1/q\)
\[
\sum_{n=0}^{\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_{1/q} h_m(\cos(\theta + \varphi)|\frac{1}{q}) h_{n-m}(\cos(\theta - \varphi)|\frac{1}{q}) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-m(n-m)} (-1)^m q^{-\binom{n}{2}}
\]
\[
\times b_m(\cos(\theta + \varphi)|q) (-1)^{n-m} q^{-\binom{n-m}{2}} b_{n-m}(\cos(\theta - \varphi)|q) = (-1)^n q^{-\binom{n}{2}} \sum_{m=0}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix}_q b_m(\cos(\theta + \varphi)|q) b_{n-m}(\cos(\theta - \varphi)|q).
\]
Now let us consider the right hand side of (6.4) and change $q$ by $1/q$. We have

$$\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (-1)^j q^{-\frac{n+1}{2}} b_{n-2j}(x|q^{-1}) b_{n-2j}(y|q^{-1})$$

$$= \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} (-1)^j q^{-\frac{n+1}{2}} b_{n-2j}(x|q^{-1}) b_{n-2j}(y|q^{-1}).$$

We deduce that (6.10) is true since we have $\binom{n}{2} + n = \binom{n+1}{2}$. □

**References**

[1] Carlitz, L. Generating functions for certain $q$-orthogonal polynomials. *Collect. Math.* 23 (1972), 91–104. MR0316773 (47 #5321)

[2] Mason, J. C.; Handscomb, D. C. *Chebyshev polynomials*. Chapman & Hall/CRC, Boca Raton, FL, 2003. xiv+341 pp. ISBN: 0-8493-0355-9 MR1937591

[3] Ismail, Mourad E. H.; Zhang, Ruiming. Kibble-Slepian formula and generating functions for 2D polynomials. *Adv. in Appl. Math.* 80 (2016), 70–92.

[4] Szabowski, Paweł J. Multidimensional $q$-normal and related distributions—Markov case. *Electron. J. Probab.* 15 (2010), no. 40, 1296–1318. MR2678392

[5] Szabowski, Paweł J. On the structure and probabilistic interpretation of Askey-Wilson densities and polynomials with complex parameters. *J. Funct. Anal.* 261 (2011), no. 3, 635–659. MR2799574, [http://arxiv.org/abs/1011.1541](http://arxiv.org/abs/1011.1541)

[6] Szabowski, Paweł J. Towards a $q$-analogue of the Kibble-Slepian formula in 3 dimensions. *J. Funct. Anal.* 262 (2012), no. 1, 210–233. MR2852260 (2012i:42029), [http://arxiv.org/abs/1011.4929](http://arxiv.org/abs/1011.4929)

[7] Szabowski, Paweł J. Around Poisson-Mehler summation formula. *Hacet. J. Math. Stat.* 45 (2016), no. 6, 1729–1742. MR3699734, [http://arxiv.org/abs/1108.3024](http://arxiv.org/abs/1108.3024)

[8] Szabowski, Paweł J. On probabilistic aspects of Chebyshev polynomials, Statistics and Probability Letters 145 (2019) 205–215, [http://arxiv.org/abs/1508.04450](http://arxiv.org/abs/1508.04450)

[9] Szabowski, Paweł J. On the $q$–Hermite polynomials and their relationship with some other families of orthogonal polynomials, *Dem. Math.* 66(2013) no.4, 679-708, [http://arxiv.org/abs/1101.2873](http://arxiv.org/abs/1101.2873)

[10] Bryc, Włodzimierz; Matysiak, Wojciech; Szabowski, Paweł J. Probabilistic aspects of Al-Salam-Chihara polynomials, *Proc. Amer. Math. Soc.* 133 (2005), no. 4, 1127–1134 (electronic). MR2117214 (2005m:33033)

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