WEAKLY MULTIPLICATIVE ARITHMETIC FUNCTIONS AND
THE NORMAL GROWTH OF GROUPS

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Abstract. We show that an arithmetic function which satisfies some weak
multiplicativity properties and in addition has a non-decreasing or log-uniformly
continuous normal order is close to a function of the form \( n \mapsto n^c \). As an ap-
plication we show that a finitely generated, residually finite, infinite group,
whose normal growth has a non-decreasing or a log-uniformly continuous nor-
mal order is isomorphic to \((\mathbb{Z}, +)\).

1. Introduction and results

A function \( f : \mathbb{N} \to \mathbb{R} \) is called multiplicative, if for all coprime positive integers\( n, m \) we have \( f(nm) = f(n)f(m) \). P. Erdős [2] showed that a non-decreasing
multiplicative function \( f \) is of the form \( f(n) = n^c \) for some \( c \geq 0 \). Birch [1] showed
that the same conclusion holds, if we assume that \( f \) has a non-decreasing normal
order (see Definition 2). Following these results there has been a lot of activity
dealing with similar statements for other regularity properties of multiplicative
functions; however, the question whether “multiplicative” can be replaced by a
weaker statement has received much less attention. In [6] it was shown that a
function \( f \) is of the form \( f(n) = n^c \) for some \( c \), provided that \( f \) has the following
property: \( f \) is monotonic, non-vanishing, and for all \( n \in \mathbb{N} \) and all \( \epsilon > 0 \) there is
some \( x_0 > 0 \) such that for all \( x > x_0 \) the interval \([x, (1 + \epsilon)x]\) contains some \( m \)
with \( f(nm) = f(n)f(m) \). This statement was motivated by the fact that, if \( G \) is
a finitely generated group and if \( f(n) \) denotes the number of normal subgroups of
index \( n \) in \( G \), then \( f \) satisfies some weak multiplicativity properties. In this note
we will deal in a similar way with functions having a smooth normal order.

Definition 1. A function \( f : \mathbb{N} \to [0, \infty) \) is weakly super-multiplicative, if for all
\( n \in \mathbb{N} \) and all \( \epsilon > 0 \) there exists some \( x_0 > 0 \) and some \( \delta > 0 \) such that for all
\( x > x_0 \) we have
\[
\#\{m \in [x, (1 + \epsilon)x] : f(nm) \geq (1 - \epsilon)f(n)f(m)\} \geq \delta x.
\]

Note that being weakly super-multiplicative is a very weak property. Clearly
multiplicative functions are weakly super-multiplicative. A more striking example is
the fact that if the values of \( f(n) \) are chosen as the values of independent identically
distributed random variables with values in \([0, 1]\), then \( f \) is almost surely weakly
super-multiplicative. To see this note that, as \( f(m) \leq 1 \) for all \( m \), we have for every
fixed \( n \) that
\[
\{m : f(nm) \geq f(n)f(m)\} \subseteq \{m : f(nm) \geq f(n)\}.
\]
Our claim now follows from the fact that for each \( m \) the event \( f(nm) \geq f(n) \) has
positive probability.
Definition 2. (1) A function \( f : \mathbb{N} \to [0, \infty) \) has normal order \( g \), if for all \( \epsilon > 0 \) the set \( \{ n : |f(n) - g(n)| \geq \epsilon g(n) \} \) has upper density 0.

(2) A function \( g : (0, \infty) \to (0, \infty) \) is log-uniformly continuous, if for every \( \epsilon > 0 \) there exists some \( \delta > 0 \) such that for all \( x, y > 0 \) with \( \frac{x}{y} - 1 < \delta \) we have \( \frac{g(x)}{g(y)} - 1 < \epsilon \).

(3) The essential limit \( \lim \text{ess} a_n \) of a sequence \( (a_n) \) exists and is equal to \( a \), if for all \( \epsilon > 0 \) the set \( \{ n : |a_n - a| > \epsilon \} \) has density 0. We say the essential limit is \( \infty \), if for all \( M \in \mathbb{R} \) the set \( \{ n : a_n < M \} \) has density 0.

Note that some authors include the monotonicity of \( g \) in the definition of a normal order, however, we do not do so here. With these notations we state the following.

Theorem 1. Let \( f \) be a weakly super-multiplicative function, which has a strictly positive normal order \( g \), where \( g \) is either non-decreasing or log-uniformly continuous. Then
\[
\sup \frac{\log f(n)}{\log n} = \lim \text{ess} \frac{\log f(n)}{\log n}.
\]

In particular \( f(n) \) either tends super-polynomially to \( \infty \), or it approaches \( n^c \) for some constant \( c \) from below. Note that a more precise statement is impossible, since for any function \( \delta(n) \) which decreases monotonically to 0, the function \( f(n) = n^{1-\delta(n)} \) is both strictly increasing and super-multiplicative, i.e. we have \( f(nm) \geq f(n)f(m) \) for all \( n, m \). This example shows that even if in Theorem 1 we replace “non-decreasing normal order” by “strictly increasing”, and “weakly super-multiplicative” by “super-multiplicative”, the convergence to the limit can still be arbitrarily slow.

As a first application we recover a strengthening of Birch’s result.

Corollary 1. Let \( f : \mathbb{N} \to (0, \infty) \) be a function such that both \( f \) and \( f^{-1} \) are weakly super-multiplicative. If \( f \) has a normal order that is monotonic or log-uniformly continuous, then there is some \( c \) such that \( f(n) = n^c \) holds for all \( n \).

As a second application we prove the following.

Corollary 2. Let \( G \) be a finitely generated residually finite group, and let \( f(n) \) be the number of normal subgroups of \( G \) of index \( n \). If \( f \) has a strictly positive normal order that is monotonic or log-uniformly continuous, then \( G \cong (\mathbb{Z}, +) \).

This result shows that the normal subgroup growth behaves completely different from subgroup growth. For the latter monotonicity has been established in a variety of cases, see e.g. [3], [4].

2. Proof of the Theorem

For the proof we first deduce a growth condition for \( g \), given in equation (4) below. The deduction of this condition depends on whether \( g \) is supposed to be non-decreasing or log-uniformly continuous. From that point onwards the proof of the two cases runs completely parallel.

A growth condition for monotonic \( g \). Let \( n \) be an integer and \( \epsilon > 0 \) a real number. Let \( x_0 > 0 \) and \( \delta > 0 \) be real numbers such that for \( x > x_0 \) we have \( f(nm) \geq (1 - \epsilon)f(n)f(m) \) holds for \( \geq \delta x \) integers \( m \in [x, (1 + \epsilon)x] \). Let \( x_1 > 0 \)...
be a real number such that for $x > x_1$ we have that $|f(t) - g(t)| < \epsilon g(n)$ holds for all integers $t \in [x, (1 + \epsilon)x]$ with at most $\frac{\epsilon}{3\delta} x$ exceptions. We conclude that for $x > \max(x_0, x_1)$ the interval $[x, (1 + \epsilon)x]$ contains at least $(1 - \frac{\delta}{3\epsilon}) x \geq \frac{2\delta}{3} x$ integers $m$ with

$$f(nm) \geq (1 - \epsilon)f(n)f(m) \geq (1 - 2\epsilon)f(n)g(m) \geq (1 - 2\epsilon)f(n)g(x),$$

where in the last step we used the monotonicity of $g$. In the interval $[nx, n(1 + \epsilon)x]$ there are at most $\frac{\delta}{3\epsilon} \cdot (nx) = \frac{\delta}{3} x$ integers $q$ with $|f(q) - g(q)| > \epsilon g(q)$, thus, for at least $\frac{2\delta}{3} x$ integers $m \in [x, (1 + \epsilon)x]$ we have

$$g(n(1 + \epsilon)x) \geq g(nm) \geq (1 - \epsilon)f(n)g(x) \geq (1 - 3\epsilon)f(n)g(x)$$

We conclude that for all $n$, all $\epsilon > 0$ and all $x > x_0(n, \epsilon)$ we have

$$g(n(1 + \epsilon)x) \geq (1 - 3\epsilon)f(n)g(x). \quad (1)$$

**A growth condition for** log-uniformly continuous $g$. Let $n$ be an integer, $\epsilon > 0$ be a real number, and let $0 < \gamma \leq \epsilon$ be a real number such that $\left| \frac{\epsilon}{y} - 1 \right| < \gamma$ implies $\left| \frac{g(x)}{g(y)} - 1 \right| < \epsilon$. Let $x_0 > 0$ and $\delta > 0$ be a real numbers such that for $x > x_0$ we have that $f(nm) \geq (1 - \epsilon)f(n)f(m)$ holds for $n \leq \delta x$ integers $m \in [x, (1 + \epsilon)x]$. As in the case $g$ non-decreasing we conclude that for $x$ sufficiently large we deduce

$$g(nm) \geq (1 - \epsilon)f(nm) \geq (1 - \epsilon)^2 f(n)f(m) \geq (1 - \epsilon)^3 f(n)g(m)$$

for at least $\frac{2\delta}{3} x$ integers $m \in [x, (1 + \epsilon)x]$. Using the fact that $g$ is log-uniformly continuous and our definition of $\gamma$ we have for $m$ in this range the estimates

$$\left| \frac{g(nm)}{g((1 + \gamma)nx)} - 1 \right| \leq \epsilon \quad \text{and} \quad \left| \frac{g(m)}{g(x)} - 1 \right| < \epsilon,$$

thus

$$g(n(1 + \gamma)x) \geq \frac{1}{1 + \epsilon} g(nm) \geq \frac{(1 - \epsilon)^3 f(n)g(m)}{1 + \epsilon} \geq \frac{(1 - \epsilon)^4}{1 + \epsilon} f(n)g(x) \geq (1 - 5\epsilon)f(n)g(x). \quad (2)$$

**Conclusion of the theorem.** Comparing (1) and (2) we find in either case that for every $n$ and every $\epsilon > 0$ there exists some $\gamma$ in the range $0 < \gamma \leq \epsilon$ and some $x_0 = x_0(n, \epsilon)$ such that for $x > x_0$ we have

$$g(n(1 + \gamma)x) \geq (1 - 5\epsilon)f(n)g(x). \quad (3)$$

Iterating (2) we obtain for $x > x_0(n, \epsilon)$ and an integer $k \geq 1$ the bound

$$g(n^k(1 + \gamma)^k x) \geq (1 - 5\epsilon)^k f(n)g(x).$$

Put $\mu = \inf \{g(t) : 1 \leq t \leq n(1 + \gamma)\}$. If $g$ is non-decreasing, then $mu = g(1)$. If $g$ is log-uniformly continuous, than in particular $g$ is continuous, thus $g$ attains its minimum in this interval. Since $g$ is strictly positive, in both cases we obtain $\mu > 0$. Then we get for $y \in [n^k(1 + \gamma)^k, n^{k+1}(1 + \gamma)^{k+1}]$ the estimate

$$g(y) \geq (1 - 5\epsilon)^k f(n)g(x),$$

thus

$$\liminf_{y \to \infty} \frac{\log g(y)}{\log y} \geq \liminf_{k \to \infty} \frac{\log ((1 - 5\epsilon)^k f(n)^k m)}{\log (n^{k+1}(1 + \gamma)^{k+1})} = \frac{\log ((1 - 5\epsilon)f(n))}{\log (n(1 + \gamma))}.$$
As $\epsilon \to 0$, and $n$ ranges over all integers, we obtain $\lim \inf \frac{\log g(y)}{\log y} \geq \sup \frac{\log f(n)}{\log n}$. By the definition of a normal order we have

$$
\limsup_{y \to \infty} \frac{\log g(y)}{\log y} \leq \sup \frac{\log f(n)}{\log n} \leq \liminf_{y \to \infty} \frac{\log g(y)}{\log y},
$$

thus $\lim \frac{\log g(y)}{\log y}$ exists and equals $\sup \frac{\log f(n)}{\log n}$. Again from the definition of the normal order we see that we can replace $\lim \frac{\log g(y)}{\log y}$ by $\lim \inf \frac{\log f(n)}{\log n}$, and the theorem follows.

3. Proof of the Corollaries

To prove Corollary 1 note that the conclusion of Theorem 1 can be reformulated as stating that either $\lim \inf \frac{\log f(n)}{\log n} = \infty$, or there exists a constant $c$ and a non-negative function $\omega$, tending to 0, such that $f(n) = n^c - \omega(n)$ holds for all $n$, and $f(n) = n^c - \omega(n)$ holds for almost all $n$. Hence, if $f$ and $f^{-1}$ are both weakly super-multiplicative, and $f$ has a strictly positive normal order which is either non-decreasing or log-uniformly continuous, then there exist two constants $c_1, c_2$, and two non-negative functions $\omega_1, \omega_2$, tending to 0, such that $n^{c_1} \leq f(n) \leq n^{c_2}$ holds true for all $n$, and $n^{c_1 + \omega_1(n)} = f(n) = n^{c_2 - \omega_2(n)}$ holds for almost all $n$. But then $c_1 + \omega_1(n) = c_2 - \omega_2(n)$, since $\omega_i \to 0$, we deduce $c_1 = c_2$ and $\omega_1(n) = \omega_2(n) = 0$. This in turn is equivalent to the statement that $f(n) = n^c$ for all $n$.

To prove Corollary 2 we first recall some properties of the number of normal subgroups of a finitely generated group.

**Proposition 1.** Let $G$ be an $r$-generated group, $f(n)$ be the number of normal subgroups of index $n$.

- (1) If $(n, m) = 1$, then $f(nm) \geq f(n)f(m)$.
- (2) For all $\epsilon > 0$ we have that for almost all $n$ the inequality $f(n) \leq n^{r-1+\epsilon}$ holds.
- (3) If $n$ is an integer, $p$ a prime number, $(n, p(p-1)) = 1$, and $n$ has no non-trivial divisor $d \equiv 1 \pmod{p}$, then $f(np) = f(n)$.

**Proof.** The first statement follows from the fact that if $N, M$ are normal subgroups of $G$ of coprime index $m$ and $n$, then $M \cap N$ is a normal subgroup of index $mn$. Moreover, the map $(M, N) \mapsto M \cap N$ is injective, since in this case $G/(M \cap N) \cong (G/N) \times (G/M)$. The second statement is [5, Theorem 2 (i)].

For the third statement let $H$ be a group of order $np$, where $n$ and $p$ satisfy the conditions of the proposition. By Sylow’s theorem $H$ has a normal $p$ Sylow subgroup $P$, which is cyclic of order $p$. Hence, $h \in H$ acts on $P$ by conjugation. The order of $h$ divides $n$, and is therefore coprime to $|\text{Aut}(C_p)| = p - 1$, thus $h$ acts trivially on $P$. We conclude that $P$ is central in $H$. Since $(n, p) = 1$, Zassenhaus’ theorem implies that $P$ has a complement, and since $P$ is central, this complement is normal. We conclude that every group of order $np$ is the direct product of a group of order $n$ and a group of order $p$. This implies that in $G$ every normal subgroup of index $np$ is the intersection of a normal subgroup of index $n$ with a normal subgroup of index $p$, thus the map $(M, N) \mapsto M \cap N$ used to prove the first statement is actually a bijection, thus $f(np) = f(n)f(p)$.

For an integer $n$, denote by $P^+(n)$ the largest prime divisor of $n$. Then we have the following.
Proposition 2. The set of integers $n$ such that $P^+(n) > \sqrt{n}$ and $(P^+(n) - 1, n) = 1$, has natural density $(\log 2) \prod_p \left(1 - \frac{1}{p(p-1)}\right)$.

Proof. We partition the set $\mathcal{A}$ of all integers $n \leq x$ with $P^+(n) > \sqrt{n}$ and $(P^+(n) - 1, n) = 1$ into three subsets, depending on the size of $P^+(n)$. Put
\[
\mathcal{A}_1 = \{ n \in \mathcal{A} : P^+(n) > \sqrt{x} \},
\mathcal{A}_2 = \{ n \in \mathcal{A} : \frac{\sqrt{x}}{\log x} < P^+(n) \leq \sqrt{x} \},
\mathcal{A}_3 = \{ n \in \mathcal{A} : P^+(n) \leq \frac{\sqrt{x}}{\log x} \}.
\]

As usual $\mathcal{A}_2$ and $\mathcal{A}_3$ are negligible, we therefore begin with estimating $|\mathcal{A}_1|$. Fix a parameter $y$, and let $Q$ be the product of all prime numbers $\leq y$. Let $d$ be a divisor of $Q$. The Siegel-Walfisz-theorem implies that for $A$ fixed and $d < \log A x$ we have
\[
\sum_{p \leq x} \frac{1}{p} = \frac{1}{\varphi(d)} \log \log x + C_d + O\left(\frac{1}{\log x}\right).
\]

Therefore the number of integers $n \leq x$ such that the largest prime divisor $p$ of $n$ is larger than $\sqrt{x}$, and $d | (n, p-1)$ equals
\[
\sum_{p \leq x} \#\{ n \leq \frac{x}{p} : d | n \} = \sum_{p \leq \sqrt{x}} \left( \frac{x}{dp} + O(1) \right)
\]
\[
= \frac{x}{d} \sum_{p \leq \sqrt{x}} \frac{1}{p} + O\left(\frac{x}{\log x}\right) = \frac{x}{d \varphi(d)} \log 2 + O\left(\frac{x}{\log x}\right).
\]

Since the product of all primes below $\log x$ is $(\log x)^{1+o(1)}$, this implies that for $y \leq \log \log x$ the number of integers $n \leq x$ such that $P^+(n) > \sqrt{x}$ and $(n, P^+(n) - 1, Q) = 1$ is
\[
\sum_{d | Q} \mu(d) \frac{x}{d \varphi(d)} \log 2 + O\left(\frac{x}{\log x}\right) = x (\log 2) \prod_{p \leq y} \left(1 - \frac{1}{p(p-1)}\right) + O\left(\frac{\tau(Q)x}{\log x}\right)
\]
\[
= x (\log 2) \prod_{p \leq y} \left(1 - \frac{1}{p(p-1)}\right) + O\left(\frac{2^y x}{\log x}\right).
\]

For modulus $d > \log^4 x$ the prime number theorem for arithmetic progression might not hold anymore, we therefore switch to the Brun-Titchmarsh inequality in the form $\pi(x, q, a) \leq \frac{2x}{\varphi(q) \log(x/q)}$, which holds for all choices of $x$ and $q$. If $q \leq \sqrt{x}$, we
obtain by partial summation
\[
\#\{n \leq x : P^+(n) > \sqrt{x}, q|(P^+(n) - 1, n)\} = \sum_{\sqrt{x} \leq p \leq x} \left\lfloor \frac{x}{pq} \right\rfloor
\]
\[
\leq \frac{\pi(x, q, 1)}{xq} + \sum_{\sqrt{x} \leq t \leq x} \frac{\pi(t, q, 1) - \pi(\sqrt{x}, q, 1)}{qt(t-1)} \leq \frac{2x \log \sqrt{x}}{q(q-1) \log(\sqrt{x}/q)} \ll \frac{x}{q^2}.
\]
For larger values of \(q\) we omit the condition that \(p\) be prime, and obtain similarly
\[
\#\{n \leq x : P^+(n) > \sqrt{x}, q|(P^+(n) - 1, n)\} = \sum_{\sqrt{x} \leq p \leq x} \left\lfloor \frac{x}{q^\nu} \right\rfloor
\]
\[
\leq \frac{x}{xq} + \sum_{\sqrt{x} \leq t \leq x} \frac{t - \sqrt{x}}{qt(t-1)} \leq \frac{2x \log x}{q(q-1)} \ll \frac{x \log x}{q^2}.
\]
Merging these ranges we find that the number of integers \(n \leq x\) such that \((P^+(n) - 1, n) = 1\) and \(P^+(n) > \sqrt{x}\) equals
\[
x(\log 2) \prod_{p \leq y} \left(1 - \frac{1}{p(p-1)}\right) + O\left(\frac{2^y x}{\log x}\right) + O\left(\sum_{y \leq q \leq \sqrt{x}} \frac{x}{q^2}\right) + O\left(\sum_{\sqrt{x} \leq q \leq \sqrt{y}} \frac{x \log x}{q^2}\right)
\]
\[
= x(\log 2) \prod_{p \leq y} \left(1 - \frac{1}{p(p-1)}\right) + O\left(\frac{2^y x}{\log x}\right) + O\left(\frac{\nu}{y}\right)
\]
For \(y \geq 3\) we have
\[
1 > \prod_{p > y} \left(1 - \frac{1}{p(p-1)}\right) \geq \exp\left(-\sum_{p > y} \frac{2}{{p^2}}\right) \geq \exp(-\frac{2}{y}) \geq 1 - \frac{2}{y},
\]
thus we can extend the product over all primes without enlarging the error term. Taking \(y = \log \log x\) we obtain
\[
|A_1| = x(\log 2) \prod_p \left(1 - \frac{1}{p(p-1)}\right) + O\left(\frac{x}{\log \log x}\right).
\]
Next we give upper bounds for \(|A_2|\) and \(|A_3|\). We have
\[
|A_2| \leq \sum_{\frac{x}{\log x} \leq p \leq \sqrt{x}} \left\lfloor \frac{x}{p} \right\rfloor \sim x \left(\log \log \sqrt{x} - \log \log \frac{\sqrt{x}}{\log x}\right) \sim \frac{2x \log \log x}{\log x},
\]
Finally if \(n \in A_3\), then \(\sqrt{n} \leq P^+(n) \leq \frac{\sqrt{x}}{\log x}\), thus \(n \leq \frac{x}{\log^2 x}\), and therefore \(|A_3| \leq \frac{x}{\log^2 x}\).

We conclude that \(|A| \sim |A_1| \sim x(\log 2) \prod_p \left(1 - \frac{1}{p(p-1)}\right)|\), and our claim follows.

\[\Box\]

To prove Corollary 2 note first that Proposition 1 (1) implies that we can apply Theorem 1. From Proposition 1 (2) we find that a normal order of \(f\) grows at most polynomially, and conclude that there exists a constant \(c\) and a non-negative function \(\omega(n)\), tending to 0, such that \(f(n) = n^c - \omega(n)\) for almost all \(n\).
If $n$ is an integer, $p$ the largest prime divisor of $n$, and $p > \sqrt{n}$, then $n/p$ has no divisor $d \neq 1$ that satisfies $d \equiv 1 \pmod{p}$. If in addition $(n,p-1) = 1$, then Proposition[1](3) implies $f(n) = f(n/p)f(p)$. Proposition[2] shows that for a positive proportion of all integers $n$ we have $f(n) = f(n/P^+(n))f(P^+(n))$. Neglecting a set of integers $n$ of density 0 we may assume that $f(n) = n^{c-\omega(n)}$, and $f(n/p) = (n/p)^{r-\omega(n/p)}$. We obtain $f(p) = p^{r+o(1)}$ for infinitely many prime numbers $p$. On the other hand we know that every normal subgroup of prime index in $G$ contains the commutator of $G$, thus the number of normal subgroups of index $p$ in $G$ equals the number of subgroups of index $p$ in $G/G'$, where $G'$ is the commutator subgroup fo $G$. Being a finitely generated abelian group, this quotient is isomorphic to $A \oplus \mathbb{Z}^r$, where $A$ is some finite abelian group. Hence, for all but finitely many $p$ we have $f(p) = \frac{p^r-1}{p-1} = p^{r-1+o(1)}$. Comparing these two bounds we conclude that $c = r - 1$. Hence, $\frac{p^r-1}{p-1} \leq f(p) \leq p^r-1$, which is only possible if $r = 1$ and $A$ is trivial. We conclude that $f(n) \leq 1$ and $G/G' \cong \mathbb{Z}$. In particular, all normal subgroups of finite index contain $G'$. Since $G$ is residually finite, we conclude $G' = 1$, and finally obtain $G \cong \mathbb{Z}$.

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