On the infimum convolution inequalities with improved constants

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Abstract

The goal of the article is to improve constants in the infimum convolution inequalities (IC for short) which were introduced by R. Latała and J.O. Wojtaszczyk. We show that the exponential distribution satisfies IC with constant 2 but not with constant 1, which implies that linear functions are not extremal in Maurey’s property (τ). Using transport of measure we use this result to better constants in the IC inequalities for product symmetric log-concave measures as well as in the Talagrand’s two level concentration inequality for the exponential distribution.

Keywords infimum convolution inequalities, property (τ), concentration of measure, log-concave measures

1 Introduction

In the seminal paper [8] B. Maurey introduced the property (τ) for a probability measure µ with a cost function W (see Definition [1]) and established its connections with the concentration of measure phenomenon (see Proposition [2]). Later in [5] R. Latała and J.O. Wojtaszczyk showed that if a pair (µ, W) satisfies property (τ), where µ is a symmetric probability measure and W is a convex cost function then W ≤ Λ∗ µ, where Λ∗ µ is the Cramer transform of µ. This observation led to the definition of the so called infimum convolution inequality, IC for short. Namely a measure µ satisfies IC(β) if the pair (µ, Λ∗ µ(·/β)) satisfies property (τ).

Latała and Wojtaszczyk proved that the symmetric exponential distribution dν = 1/2 e−|x| dx satisfies IC(9) and used that result to prove that any symmetric product log-concave fully supported probability measure satisfies IC(48). Moreover using the connection of IC with the concentration of measure phenomenon the authors proved the two level concentration inequality for product exponential distribution νn with constants C1 = 18, C2 = 6√2, obtained previously with rather large constants by Talagrand in [9].

The goal of this paper is to improve constants in the inequalities obtained in [5]. We show that any Gaussian measure satisfies IC(1) (Theorem 1), while one-sided and symmetric exponential distributions satisfy IC(2) (Theorem 2) but not IC(1) (Theorem 3). The latter result comes as a surprise as it shows that linear functions are not extremal in the property (τ) for the exponential measure. Next we prove that any symmetric product log-concave fully supported probability distribution satisfies IC(9.61929...) (Theorem 4). Finally we obtain Talagrand’s two level concentration inequality with constants C1 = 4, C2 = 8 (Theorem 5).

1.1 Notation

In the whole paper µ denotes a probability measure on the Euclidean space Rn with scalar product (x, y) = Σi=1n xi yi. We assume that all functions that are considered are Lebesgue measurable. Moreover we use the following notation

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2 Preliminaries

2.1 Infimum convolution and property (\(\tau\))

**Definition 1** (Infimum convolution operator \(\square\)). For functions \(f, g: \mathbb{R}^n \to (-\infty; \infty]\) the infimum convolution of \(f\) and \(g\) is
\[
(f \square g)(x) = \inf\{f(x-y) + g(y) : y \in \mathbb{R}^n\}. \tag{1}
\]

In the next Proposition we collect the properties of the infimum convolution operator.

**Proposition 1.** For functions \(f, g, h: \mathbb{R}^n \to (-\infty; \infty]\) one has
1. \(f \square g = g \square f\) (commutativity).
2. \(f \square e = f\), where \(e(x) = \begin{cases} 0 & x = 0 \\ \infty & x \neq 0 \end{cases}\) (existence of neutral element).
3. \((f \square g) \square h = f \square (g \square h)\) (associativity).
4. \(f \square 0 = \inf f\).
5. \(f \square g + \inf h \leq f \square (g + h) \leq f \square g + \sup h\).
6. If \(f_n \Rightarrow f\) then \(g \square f_n \Rightarrow g \square f\) where \(\Rightarrow\) denotes uniform convergence.
7. If \(f\) is convex then \((f \square f)(x) = 2f(x/2)\).
8. If \(g\) is convex and \(f(x) = g(2x)/2\) then \(f \square f = g\).

The next definition was introduced by B. Maurey in [3].

**Definition 2** (Property \((\tau)\)). An ordered pair \((\mu, W)\), where \(\mu\) is a probability measure on \(\mathbb{R}^n\) and \(W: \mathbb{R}^n \to [0; \infty]\) is a cost function satisfies property \((\tau)\) if for every bounded function \(f\)
\[
\int_{\mathbb{R}^n} e^{W \square f} \, d\mu \int_{\mathbb{R}^n} e^{-f} \, d\mu \leq 1. \tag{2}
\]
The motivation for the Definition 2 comes from the following Proposition from [5] which connects property (τ) with the concentration of measure phenomenon (see [6]).

**Proposition 2.** Assume that $(\mu, W)$ satisfies property $(\tau)$ then for every Borel set $A$

1. $\forall_{t>0} \mu(A + B_W(t)) \geq \frac{e^t\mu(A)}{\|\mu\|_{TV}} \geq 1 - \mu(A)^{-1}e^{-t}.$
2. If $\mu(A) = \nu(-\infty; x]$, then $\forall_{t>0} \mu(A + B_W(2t)) \geq \nu(-\infty; x + t].$

First three parts of the next Proposition are from [8], fourth part is a straightforward consequence of the second part, while the fifth part is a generalization of the result from [3].

**Proposition 3.** Let $\mu, \mu_1, \mu_2$ be measures on $\mathbb{R}^n, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}$ and let $T: \mathbb{R}^n \to \mathbb{R}^k$. Assume that pairs $(\mu, W), (\mu_1, W_1), (\mu_2, W_2)$ satisfy property $(\tau)$. Then

1. Pair $(\mu_1 \otimes \mu_2, W)$ satisfies property $(\tau)$, where $W(x_1, x_2) = W_1(x_1) + W_2(x_2)$.
2. If $V: \mathbb{R}^k \to [0; \infty]$ satisfies for all $x, y \in \mathbb{R}^n$ the condition
   
   $$V(T(x) - T(y)) \leq W(x - y),$$

   then pair $(T\#\mu, V)$ satisfies property $(\tau)$.
3. If $n_1 = n_2$, then pair $(\mu_1 * \mu_2, W_1 \square W_2)$ satisfies property $(\tau)$.
4. If $L: \mathbb{R}^n \to \mathbb{R}^k$ is an affine map $(L(x) = Ax + b$, where $A \in \mathbb{R}^{n \times k}$ and $b \in \mathbb{R}^k$) such that
   
   $$V(Ax) \leq W(x),$$

   then pair $(L\#\mu, V)$ satisfies property $(\tau)$.
5. If $W(x) = \hat{W}(|x|)$ where $\hat{W}: [0; \infty) \to [0; \infty]$ is nondecreasing, then pair $(T_{\#\mu}, (\hat{W} \circ \omega_T)(|\cdot|))$ satisfies property $(\tau)$, where for $h \geq 0$

   $$\omega_T(h) = \inf\{|x - y| : |T(x) - T(y)| \geq h\}, \quad \inf\emptyset = \infty.$$  

### 2.2 Transforms

In this section we recall definitions and basic properties of Laplace, Legendre and Cramer transforms. These operators are used in convex analysis (see [7]) and in the theory of large deviations (see [2]).

**Definition 3** (Laplace transform). Laplace transform of a measure $\mu$ on $\mathbb{R}^n$ is

$$M_\mu(x) = \int_{\mathbb{R}^n} e^{<x,y>} d\mu(y).$$

**Proposition 4.** For probability distributions $\mu, \mu_1, \mu_2$ on $\mathbb{R}^n, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}$ and $x \in \mathbb{R}^n, x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$ one has

1. $M_{\mu_1 \otimes \mu_2}(x_1, x_2) = M_{\mu_1}(x_1)M_{\mu_2}(x_2).$
2. If $T(x) = Ax + b$, where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$ then for $y \in \mathbb{R}^m$ there is $M_{T\#\mu}(y) = e^{<y,b>} M_{\mu}(Ay)$.
3. If $n_1 = n_2 = n$, then $M_{\mu_1 * \mu_2}(x) = M_{\mu_1}(x)M_{\mu_2}(x).$
Definition 4 (Legendre transform). Legendre transform of a function $f: \mathbb{R}^n \to (-\infty; \infty]$ is
\[
f^*(x) = \sup_y \{ \langle x, y \rangle - f(y) \}.
\]

Proposition 5. For arbitrary $f, g: \mathbb{R}^n \to (-\infty; \infty]$

1. $f^*$ is convex.
2. $f^{**} \leq f$.
3. If $f$ is convex and lower semicontinuous then $f^{**} = f$.
4. If $f \leq g$, then $f^* \geq g^*$.
5. If $C$ is a real number then $(Cf)^*(x) = Cf^*(x/C)$ and $(f(\cdot/C))^*(x) = f^*(Cx)$.
6. If $f, g$ are convex then $(f / \square g)^* = f^* + g^*$.

Definition 5 (Cramer transform). Cramer transform of a measure $\mu$ is $\Lambda_\mu^*$, where $\Lambda_\mu = \ln M_\mu$.

Proposition 6. For probability distributions $\mu, \mu_1, \mu_2$ on $\mathbb{R}^n, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}$ and $x \in \mathbb{R}^n, x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{n_2}$ one has

1. $\Lambda_{\mu_1 \otimes \mu_2}(x_1, x_2) = \Lambda_{\mu_1}(x_1) + \Lambda_{\mu_2}(x_2)$.
2. $\Lambda_{\mu_1 \otimes \mu_2}^*(x_1, x_2) = \Lambda_{\mu_1}^*(x_1) + \Lambda_{\mu_2}^*(x_2)$
3. If $T(x) = Ax + b$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, then $\Lambda_{T_\# \mu}(y) = \langle y, b \rangle + \Lambda_\mu(A^t y)$ for $y \in \mathbb{R}^m$.
4. If $n_1 = n_2 = n$, then $\Lambda_{\mu_1 \ast \mu_2}(x) = \Lambda_{\mu_1}(x) + \Lambda_{\mu_2}(x)$.
5. $\Lambda_\mu$ is convex.
6. $\Lambda_\mu^*$ is convex and nonnegative.
7. $\Lambda_\mu^*(0) = 0$.
8. If $\mu$ is a symmetric probability measure, then $\Lambda_\mu^*$ is even and $\Lambda_\mu^*(x) = 2\Lambda_\mu^*(x/2)$.

2.3 Infimum convolution inequality - IC

The next Proposition which was proved in [5] gives an upper bound for any convex cost function from the Definition 2.

Proposition 7. If a pair $(\mu, W)$ satisfies property $(\tau)$ and $W$ is convex, then $W \leq \Lambda_\mu^*$.

It motivates the following definition

Definition 6 (Infimum convolution inequality - IC). A probability measure $\mu$ on $\mathbb{R}^n$ satisfies the infimum convolution inequality with constant $\beta > 0$ if the pair $(\mu, \Lambda_\mu^*(\cdot/\beta))$ satisfies property $(\tau)$.

In the next Proposition we collect properties of the infimum convolution inequalities

Proposition 8. For any probability measures $\mu, \mu_1, \mu_2$ on $\mathbb{R}^n, \mathbb{R}^{n_1}, \mathbb{R}^{n_2}$ satisfying IC$(\beta), IC(\beta_1), IC(\beta_2)$ there holds
1. If $L$ is an affine map, then $L\#\mu$ satisfies $\text{IC}(\beta)$.
2. The product $\mu_1 \otimes \mu_2$ satisfies $\text{IC}(\beta_1 \lor \beta_2)$.
3. If $n_1 = n_2$, then convolution $\mu_1 \ast \mu_2$ satisfies $\text{IC}(\beta_1 \lor \beta_2)$.
4. Symmetrization $\overline{\mu}$ satisfies $\text{IC}(\beta)$.

Proof.

1. Denote $L(x) = Ax + b$. Thanks to part 4 of Proposition 3 it is enough to check that

$$\Lambda_{L\#\mu}^*\left(\frac{Ax}{\beta}\right) \leq \Lambda_{\overline{\mu}}^*\left(\frac{x}{\beta}\right).$$

We will show that

$$L\#\mu = A\#\overline{\mu}.$$  

Indeed if $X_1, X_2$ are independent random variables with distribution $\mu$ then $A(X_1 - X_2)$ has distribution $A\#\overline{\mu}$ while $L(X_1) - L(X_2)$ has distribution $L\#\mu$ and the desired equality follows from $A(X_1 - X_2) = L(X_1) - L(X_2)$.

$$\Lambda_{L\#\mu}^*\left(\frac{Ax}{\beta}\right) = \Lambda_{\overline{\mu}}^*\left(\frac{x}{\beta}\right).$$

2. Since for $i = 1, 2$ pair $(\mu_i, \Lambda_{\overline{\mu}}^*\left(\cdot/\beta_i\right))$ has $(\tau)$ property, so using Proposition 3 the pair $(\mu_1 \otimes \mu_2, W)$ has property $(\tau)$, where

$$W(x_1, x_2) = \Lambda_{\overline{\mu}}^*\left(\frac{x_1}{\beta_1}\right) + \Lambda_{\overline{\mu}}^*\left(\frac{x_2}{\beta_2}\right).$$

The claim follows from

$$\Lambda_{\overline{\mu}_1 \otimes \overline{\mu}_2}^*\left(\frac{x_1, x_2}{\beta_1 \lor \beta_2}\right) = \Lambda_{\overline{\mu}_1}^*\left(\frac{x_1}{\beta_1}\right) + \Lambda_{\overline{\mu}_2}^*\left(\frac{x_2}{\beta_2}\right) = W(x_1, x_2).$$

3. $\mu_1 \ast \mu_2 = T_{\#}(\mu_1 \otimes \mu_2)$ for $T(x_1, x_2) = x_1 + x_2$ so it is enough to use Part 1 and Part 2.

4. $\overline{\mu} = S_{\#}(\mu \otimes \mu)$, for $S(x_1, x_2) = x_1 - x_2$.

\[\square\]

3 Results

3.1 IC for Gaussian and exponential distributions

We start by analizing the Gaussian distributions.
Theorem 1. Every Gaussian distribution on $\mathbb{R}^n$ satisfies IC(1).

Proof. Since any Gaussian distribution is an affine transport of $\gamma^n$, thus using Proposition it suffices to prove that $\gamma$ satisfies IC(1). Standard calculations show that $\Lambda^*_\gamma(x) = x^2/2$. Since $\gamma$ is a symmetric distribution thus $\Lambda^*_\gamma(x) = 2\Lambda^*_\gamma(x/2) = x^2/4$. The claim follows from the fact that for $G(x) = x^2/4$ the pair $(\gamma, G)$ satisfies property $(\tau)$ which was shown in [8].

Next we turn our attention to exponential distributions.

Theorem 2.

1. One sided exponential distribution $\nu_+$ satisfies IC(2).
2. Symmetric exponential distribution $\nu$ satisfies IC(2).

In the proof of Theorem 2 we will use the following

Lemma 1. If a function $W \geq 0$ satisfies the following two conditions

1. $2|W'| \leq 1$,
2. $e^{W(1 - 4(W')^2)} \geq 1$,

then the pair $(\nu_+, W)$ satisfies property $(\tau)$.

Proof. In [8] it was shown that the pair $(\nu, U)$ satisfies property $(\tau)$, where

$$U(x) = \begin{cases} \frac{1}{36}x^2, & |x| \leq 4 \\ \frac{4}{3}(|x| - 2), & |x| > 4 \end{cases}.$$ 

From that proof it follows that conditions given in the Lemma are sufficient for the pair $(\nu_+, W)$ to have the property $(\tau)$.

Proof of Theorem 2. To prove the first part we need to show that $W(x) = \Lambda^*_\nu_+(x/2) = \Lambda^*_\nu(x/2)$ satisfies conditions given in the Lemma[1]. Denote

$$\phi(x) = W(2x) = \Lambda^*_\nu(x).$$

We calculate

$$\phi(x) = \sqrt{x^2 + 1} - 1 - \ln \left( \frac{\sqrt{x^2 + 1} + 1}{2} \right), \quad \phi'(x) = \frac{x}{\sqrt{x^2 + 1} + 1},$$

from which $2|W'(x)| = |\phi'(x/2)| \leq 1$.

The second condition of Lemma[1] follows from the following estimation:

$$e^{W(2x)}(1 - 4(W'(2x))^2) = e^{\phi(x)}(1 - (\phi'(x))^2) = e^{\sqrt{x^2 + 1} - 1} \frac{2}{\sqrt{x^2 + 1} + 1} \left( 1 - \left( \frac{x}{\sqrt{x^2 + 1} + 1} \right)^2 \right)$$

$$= e^{\sqrt{x^2 + 1} - 1} \frac{4}{(\sqrt{x^2 + 1} + 1)^2} = \left( \frac{e^y}{y + 1} \right)^2 \geq 1,$$

where $y = \sqrt{x^2 + 1} - 1$.

The second part of the Theorem 2 is a consequence of $\nu = \nu_+$ and the fourth part of Proposition[8].

Remark 1. It can be shown that $\Lambda^*_\nu(x/2) > U(x)$ and thus Theorem 2 improves the result from [8].

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The next Theorem gives a negative answer to the hypothesis that for the exponential distribution linear functions are extremal in the property \( \tau \).

**Theorem 3.**

1. \( \nu_+ \) does not satisfy IC(1).
2. \( \nu \) does not satisfy IC(1).

To prove Theorem 3 we will use the following

**Lemma 2.** If \( \mu \) satisfies IC(1), then

\[
\int e^{2\Lambda^*_\mu(x/2)} d\mu(x) \int e^{-\Lambda^*_\mu(x)} d\mu(x) \leq 1.
\]

**Proof.** We substitute \( f = W = \Lambda^*_\mu \) in the Definition 2 and use the identity \( f \square W = W \square W = 2W(\cdot/2) \), which is a consequence of convexity of \( W \) and Part 7 of Proposition 1).

**Proof of Theorem 3.** Using Lemma 2 it is enough to show that

\[
\int e^{2\Lambda^*_\nu(x/2)} d\nu_+(x) \int e^{-\Lambda^*_\nu(x)} d\nu_+(x) > 1,
\]

and

\[
\int e^{2\Lambda^*_\nu(x/2)} d\nu(x) \int e^{-\Lambda^*_\nu(x)} d\nu(x) > 1.
\]

Denote

\[
f(x) = \Lambda^*_\nu(x) = \sqrt{x^2 + 1} - 1 - \ln \left( \frac{\sqrt{x^2 + 1} + 1}{2} \right).
\]

Then

\[
\Lambda^*_\nu_+(x) = \Lambda^*_\nu(x) = f(x), \quad \Lambda^*_\nu(x) = 2\Lambda^*_\nu(x/2) = 2f(x/2),
\]

which after change of variables is equivalent to

\[
2 \int_0^\infty e^{2(f(y)-y)} dy \int_0^\infty e^{-(f(y)+y)} dy > 1,
\]

\[
8 \int_0^\infty e^{4(f(y)-y)} dy \int_0^\infty e^{-(2f(y)+y)} dy > 1.
\]

The above inequalities were verified using numerical integration in Mathematica software. We include the computations

\[
f[y_] := \text{Sqrt}[1 + y^2] - 1 - \text{Log}[(\text{Sqrt}[1 + y^2] + 1)/2]
\]

\[
I1 = \text{NIntegrate}[\text{Exp}[2(f[y] - y)], \{y, 0, \text{Infinity}\}]
\]

\[
0.822119
\]

\[
I2 = \text{NIntegrate}[\text{Exp}[-(f[y] + y)], \{y, 0, \text{Infinity}\}]
\]

\[
0.787272
\]

\[2 \times I1 \times I2\]
3.2 IC for log-concave distributions

The next Theorem deals with the behaviour of IC under transport of measure by certain special class of maps.

**Proposition 9.** Assume that \( T : \mathbb{R} \to \mathbb{R} \) satisfies the following conditions:

1. is odd and nondecreasing,
2. is concave on \([0; \infty)\),
3. there exists finite \( T'(0) \),
4. \( \int x^2 d\# T\# \nu = 1 \).

If \( c \geq T'(0) \) then measure \( T\# \nu \) satisfies \( \text{IC}(2c\delta) \), where \( \delta > 0 \) and \( \Lambda^*(\delta) = \ln 2 + 1/c \).

To prove Proposition 9 we will use four Lemmas. Lemmas 4 and 5 were proved in [5].

**Lemma 3.** Under the assumptions of Proposition 9 one has \( \omega_T(2T(x)) = 2x \) for \( x \geq 0 \).

**Proof.** We will show first that

\[ |T(x) - T(y)| \leq 2T\left(\frac{|x - y|}{2}\right). \]

Without loss of generality we may assume that \( x \geq y \).

If \( x \geq y \geq 0 \) then

\[ 2T\left(\frac{|x - y|}{2}\right) = 2T\left(\frac{x - y}{2}\right) \geq 2\left(\frac{T(x) x - y}{x^2}\right) = \frac{T(x) x}{x} (x - y) \geq T(x) - T(y) = |T(x) - T(y)|. \]

The first inequality follows from the fact that the graph of the concT lies above the line passing through points \((0, T(0))\), \((x, T(x))\). The second inequality is a consequence of the that the gradient of the line passing through points \((0, T(0)), (x, T(x))\) is larger than the gradient of the line passing through points \((y, T(y)), (x, T(x))\).

If \( 0 \geq y \geq x \), then \( -y \geq -x \geq 0 \), so using the previous case one gets

\[ |T(x) - T(y)| = |T(-y) - T(-x)| \leq 2T\left(\frac{|-y + x|}{2}\right) = 2T\left(\frac{|x - y|}{2}\right). \]

If \( x \geq 0 \geq y \), then

\[ |T(x) - T(y)| = 2\left(\frac{1}{2}T(x) + \frac{1}{2}T(-y)\right) \leq 2T\left(\frac{x - y}{2}\right) = 2T\left(\frac{|x - y|}{2}\right). \]
To finish the proof let us observe that
\[
\omega_T(2T(x)) = \inf\{|x' - y'| : |T(x') - T(y')| \geq 2T(x)\} \geq \inf\{|x' - y'| : 2T\left(\frac{|x' - y'|}{2}\right) \geq 2T(x)\}
\]
\[
= \inf\{|x' - y'| : |x' - y'| \geq 2x\} \geq 2x,
\]
and the equality holds for \(x' = -y' = x\). \(\square\)

**Lemma 4.** If \(\mu\) is a symmetric, probability measure on \(\mathbb{R}\) such that \(\int x^2 d\mu(x) = 1\) then for \(0 \leq x \leq 1\) the following holds
\[
\Lambda^*_\mu(x) \leq (\ln(\cosh))^*(x) = \frac{1}{2}((1 + x) \ln(1 + x)) + (1 - x) \ln(1 - x)].
\]

**Lemma 5.** If \(\mu\) is a symmetric, probability measure on \(\mathbb{R}\) then \(\Lambda^*_\mu(x) \leq -\ln(\mu[x; \infty))\).

**Lemma 6.** Cramer transform of the symmetric exponential distribution \(\nu\) satisfies for \(0 \leq x \leq 1\)
\[
\Lambda^*_\nu(\theta x) \geq (\ln(\cosh))^*(x),
\]
where \(\theta > 0\) is such that \(\Lambda^*_\nu(\theta) = (\ln(\cosh))^*(1) = \ln 2\).

**Proof.** For \(0 \leq x \leq 1\) define
\[
H(x) = \Lambda^*_\nu(\theta x) - (\ln(\cosh))^*(x)
\]
\[
= \sqrt{1 + \theta^2 x^2} - 1 - \ln \left(\frac{\sqrt{1 + \theta^2 x^2} + 1}{2}\right) - \frac{1}{2}((1 + x) \ln(1 + x) + (1 - x) \ln(1 - x)]).
\]

From standard calculations we get
\[
H'(x) = \frac{\theta^2 x}{\sqrt{1 + \theta^2 x^2}} - \frac{1}{2} \ln \left(\frac{1 + x}{1 - x}\right), \quad H''(x) = \frac{\theta^2}{\sqrt{1 + \theta^2 x^2}} + \frac{1 + \theta^2 x^2}{2} - \frac{1}{1 - x^2}.
\]
Since \(\theta > \sqrt{2}\) (because \(\ln 2 = \Lambda^*_\nu(\theta) > \Lambda^*_\nu(\sqrt{2})\)) we check that \(H''(x) > 0\) for \(x \in [0; x_0)\) and \(H''(x) < 0\) for \(x \in (x_0, 1]\), where \(x_0 = \sqrt{\frac{10\theta^2}{3} - \frac{\sqrt{8\theta^2 + 9}}{8\theta^2}}\). Since \(H(0) = H'(0) = 0\) we conclude that \(H\) is increasing on \([0; x_0]\), in particular \(H \geq 0\) on \([0; x_0]\). Inequality \(H \geq 0\) on \([x_0; 1]\) follows from \(H(x_0) \geq 0\), \(H(1) = 0\) and concavity of \(H\) on \([x_0; 1]\). \(\square\)

**Proof of Proposition 7** Using Part 2 of Theorem 2 the pair \((\nu, \Lambda^*_\nu(\theta/2))\) has property (\(\tau\)). Hence using Part 5 of Proposition \(\square\) the pair \((T_{\mu\nu}, \Lambda^*_\nu(\omega_T(|\cdot|)/2))\) has property (\(\tau\)). To finish the proof it is enough to show that for \(\beta = 2c\) there is
\[
\Lambda^*_{T_{\mu\nu}}(\frac{y}{\beta}) \leq \Lambda^*_\nu(\frac{\omega_T(|y|)}{2}).
\]
Since functions which are present in the above inequality are even we can assume without loss of generality that \(y \geq 0\). Due to the symmetry of measures \(\nu\) and \(T_{\mu\nu}\) the inequality is equivalent to
\[
\Lambda^*_{T_{\mu\nu}}(\frac{y}{2\beta}) \leq \Lambda^*_\nu(\frac{\omega_T(y)}{4}).
\]
Let us observe that if \(y \notin 2T(\mathbb{R}) = \{2T(x) : x \in \mathbb{R}\}\), then \(\{(x', y') : |T(x') - T(y')| \geq y\} = \emptyset\) hence \(\omega_T(y) = \infty\) and the inequality is true. If \(y \in 2T(\mathbb{R})\) then \(y = 2T(x)\), so due to Lemma \(\square\) it is enough to show that for \(x \geq 0\) the following inequality holds
\[
\Lambda^*_{T_{\mu\nu}}(\frac{T(x)}{\beta}) \leq \Lambda^*_\nu(\frac{\omega_T(2T(x))}{4}) = \Lambda^*_\nu(\frac{x}{2}).
\]
We consider two cases

**Case 1.** \(0 \leq \frac{cx}{\beta} \leq 1\)

Using concavity of \(T\) on \([0; \infty)\) and \(T(0) = 0\) we have

\[
\frac{T(x)}{\beta} = \frac{1}{\beta}T(x) + \left(1 - \frac{1}{\beta}\right)T(0) \leq T\left(\frac{x}{\beta}\right) \leq c \frac{x}{\beta},
\]

thus using Lemma 4 and Lemma 6 we get

\[
\Lambda^*_\nu\left(\frac{T(x)}{\beta}\right) \leq \Lambda^*_\nu\left(\frac{cx}{\beta}\right) \leq (\ln \cosh)^*\left(\frac{cx}{\beta}\right) \leq \Lambda^*_\nu\left(\frac{x}{2}\right),
\]

since

\[
\frac{\theta c}{\beta} = \frac{\theta}{2\delta} \leq \frac{1}{2}.
\]

**Case 2.** \(\frac{cx}{\beta} \geq 1\)

From the fact that \(\Lambda^*_\nu\) is nondecreasing on \([0; \infty)\) (because it is even and convex) and Lemma 5 we have

\[
\Lambda^*_\nu\left(\frac{T(x)}{\beta}\right) \leq h^*_\nu\left(T\left(\frac{x}{\beta}\right)\right) = -\ln \left(\nu\left[\frac{x}{\beta}; \infty\right)\right) = \frac{x}{\beta} + \ln 2.
\]

To finish the proof it suffices to show that for \(x \geq \frac{\beta}{c}\)

\[
\frac{x}{\beta} + \ln 2 \leq \Lambda^*_\nu\left(\frac{x}{2}\right). \tag{3}
\]

Denote \(a(x) = \frac{x}{\beta} + \ln 2\) and \(b(x) = \Lambda^*_\nu\left(\frac{x}{2}\right)\). Then (3) is a consequence of the fact that \(a\) is an affine function, \(b\) is convex and increasing and

\[
a(0) = \ln 2 > 0 = b(0), \quad a(\beta/c) = 1/c + \ln 2 = \Lambda^*_\nu(\delta) = b(\beta/c).
\]

**Definition 7** (Logarithmically concave measure). We call a measure \(\mu\) on \(\mathbb{R}^n\) logarithmically concave (log-concave) if for any nonempty compact sets \(A, B\) and \(t \in [0; 1]\),

\[
\mu(tA + (1 - t)B) \geq \mu(A)^t \mu(B)^{1-t}.
\]

The following Proposition (see [II]) gives a full characterisation of log-concave measures with a fully dimensional support.

**Proposition 10.** A measure \(\mu\) on \(\mathbb{R}^n\) with fully dimensional support (i.e. there does not exist a proper affine subspace containing the support of the measure) is log-concave if and only if it is absolutely continuous with respect to the Lebesgue measure and has a log-concave density \((g_\mu(x) = e^{-W(x)}, \text{ where } W : \mathbb{R}^n \to (-\infty; \infty]}\) is convex).

The next Theorem was proved in [5]. Our proof improves the constant significantly.

**Theorem 4.** Every symmetric, product, log-concave probability measure on \(\mathbb{R}^n\) with fully dimensional support satisfies \(IC(C)\) with a universal constant \(C = 2\sqrt{3}\delta \approx 9.61929\ldots\), where \(\delta > 0\) is such that \(\Lambda^*_\nu(\delta) = \ln 2 + 1/\sqrt{3}\).
To prove Theorem 4 we will use the following Proposition which is a modification of the result obtained by Hensley (see [4]).

**Proposition 11.** If $g: \mathbb{R} \to [0; \infty)$ is even, nonincreasing on $[0; \infty)$ and satisfies

1. $\int g(x) dx = 1$,
2. $\int x^2 g(x) dx = 1$,

then $g(0) \geq \frac{1}{2\sqrt{3}}$.

**Proof.** For $c > 0$ we denote by $A(c)$ the set of functions $g: \mathbb{R} \to [0; \infty)$ such that

1. $g$ is even, nonincreasing on $[0; \infty)$;
2. $g(0) = c$;
3. $\int g(x) dx = 1$.

We will find $m(c) = \inf \{ \int x^2 g(x) dx : g \in A(c) \}$. Denote $u(x) = c \int_{[-1/2c; 1/(2c)]} W(x)$. Observe that $u \in A(c)$. Moreover for any function $g \in A(c)$, using integration by parts we obtain

$$
\int x^2 g(x) dx = 2 \int_0^\infty x^2 g(x) dx = 2 \int_0^\infty x^2 \left( \int_x^\infty -g(s) ds \right) dx = 2 \int_0^\infty (x^2)' \left( \int_x^\infty g(s) ds \right) dx
$$

$$
= 4 \int_0^\infty x \left( \frac{1}{2} - \int_x^\infty g(s) ds \right) dx \geq 4 \int_0^\infty x \left( \frac{1}{2} - \int_0^x u(s) ds \right) dx = \int x^2 u(x) dx = \frac{1}{12c^2},
$$

hence $m(c) = \frac{1}{12c^2}$. Assume now that $g$ satisfies the assumptions of the Proposition. Then $g \in A(g(0))$ and

$$
1 = \int x^2 g(x) dx \geq m(g(0)) = \frac{1}{12g(0)^2},
$$

which finishes the proof. \hfill \Box

**Proof of Theorem 4.** Using Proposition 8 one can assume that $\mu$ is one dimensional and isotropic (i.e. $\int x^2 d\mu = 1$). Using Proposition 10 the density of $\mu$ is $g_{\mu}(x) = e^{-W(x)}$, for certain even, convex function $W$. Let $T: \mathbb{R} \to \mathbb{R}$ be the increasing rearrangement transporting $\nu$ to $\mu$ i.e. $T = F_{\mu}^{-1} \circ F_\nu$, where $F_\nu$ and $F_{\mu}$ are cumulative distribution functions. Then $T$ is nondecreasing, odd and concave on $[0; \infty)$ and $T'(0) = 1/(2g_{\mu}(0)) \leq \sqrt{3}$, where the last inequality follows from Proposition 11. Thus $T$ fulfills assumptions of Theorem 4 with constant $c = \sqrt{3}$ which finishes the proof. \hfill \Box

### 3.3 Talagrand’s two level concentration inequality for exponential distribution

The next theorem with rather large constants goes back to Talagrand (see [4]). The same result with better constants ($C_1 = 18, C_2 = 6\sqrt{2}$) was obtained in [5]. The proof that we present improves them even further.

**Theorem 5.** There exist constants $C_1, C_2$ such that for every $n \geq 1$ and Borel set $A \subset \mathbb{R}^n$,

$$
\nu^n(A) = \nu(-\infty; x] \implies \forall t \geq 0 \quad \nu^n(A + C_1 t B_1^n + C_2 \sqrt{t} B_2^n) \geq \nu(-\infty; x + t],
$$

moreover one can put $C_1 = 4, C_2 = 8$.

In the proof of Theorem 5 we will use two lemmas
Lemma 7. Assume that $W: \mathbb{R} \to [0; \infty]$ satisfies for certain constants $a, C_1, C_2 > 0$

$$\forall t > 0 \quad B_W(at) \subset C_1 t B_1^1 + C_2 \sqrt{t} B_2^1,$$

then for every $n \geq 1$ one has

$$\forall t > 0 \quad B_{W_n}(at) \subset C_1 t B_1^n + C_2 \sqrt{t} B_2^n,$$

where $W_n(x) = \sum_{i=1}^n W(x_i)$.

Proof. Let $n \geq 1$, $t > 0$ and $x \in B_{W_n}(at)$. Observe that

$$x_i \in B_W \left( \frac{W(x_i)}{a} \right) \subset C_1 \frac{W(x_i)}{a} B_1^1 + C_2 \sqrt{\frac{W(x_i)}{a}} B_2^1.$$

Thus $x_i = y_i + z_i$, where $|y_i| \leq C_1 \frac{W(x_i)}{a}$ and $|z_i| \leq C_2 \sqrt{\frac{W(x_i)}{a}}$

Moreover

$$|y| = \sum_{i=1}^n |y_i| \leq \sum_{i=1}^n C_1 \frac{W(x_i)}{a} = C_1 \frac{W_n(x)}{a} \leq C_1 \frac{at}{a} = C_1 t,$$

so $y \in C_1 t B_1^n$ and

$$|z|^2 = \sum_{i=1}^n z_i^2 \leq \sum_{i=1}^n C_2^2 \frac{W(x_i)}{a} \leq C_2^2 \frac{W_n(x)}{a} \leq C_2 \sqrt{\frac{at}{a}} = C_2 \sqrt{t},$$

so $z \in C_2 \sqrt{t} B_2^n$, hence $x = y + z \in C_1 t B_1^n + C_2 \sqrt{t} B_2^n$.

\[\square\]

Lemma 8. The Cramer transform of a symmetric exponential distribution satisfies

$$\Lambda^*_a(x) \geq \left( \sqrt{1 + |x|} - 1 \right)^2.$$

Proof. Denote $H(x) = \Lambda^*_a(x) - \left( \sqrt{1 + |x|} - 1 \right)^2$. We need to show that $H \geq 0$. Since $H$ is even we can assume that $x \geq 0$. Standard computation gives

$$H'(x) = \frac{x}{1 + \sqrt{x^2 + 1}^2} - 1 + \frac{1}{\sqrt{1 + x}}.$$

We will show that $H' \geq 0$, from which using $H(0) = 0$ the claim follows.

Case 1. ($0 < x \leq 1$)

We compute

$$H'(x) = \frac{x}{1 + \sqrt{1 + x^2}^2} - 1 + \frac{1}{\sqrt{1 + x}} = \frac{x}{1 + \sqrt{1 + x}^2} - 1 + \frac{1}{\sqrt{1 + x}} = \frac{x(\sqrt{1 + x} - 1)}{(1 + \sqrt{1 + x})\sqrt{1 + x}} \geq 0.$$

Case 2. ($x > 1$)

Using $\sqrt{1 + x^2} \leq \sqrt{2} - 1 + x$, we obtain

$$H'(x) = \frac{x}{1 + \sqrt{1 + x^2}} - 1 + \frac{1}{\sqrt{1 + x}} \geq \frac{x}{x + \sqrt{2}} - 1 + \frac{1}{\sqrt{1 + x}} = \frac{x + \sqrt{2} - \sqrt{2}\sqrt{x + 1}}{1 + x(x + \sqrt{2})}$$

$$= \frac{(\sqrt{x + 1} - \frac{\sqrt{2}}{2})^2 + \sqrt{2} - \frac{3}{2}}{\sqrt{1 + x(x + \sqrt{2})}} \geq 0,$$

since

$$\left( \sqrt{x + 1} - \frac{\sqrt{2}}{2} \right)^2 + \sqrt{2} - \frac{3}{2} \geq \left( \sqrt{1 + 1} - \frac{\sqrt{2}}{2} \right)^2 + \sqrt{2} - \frac{3}{2} = \sqrt{2} - 1 > 0.$$

\[\square\]
Proof of Theorem 4 Let $n \geq 1$ and $A \subset \mathbb{R}^n$ be such that $\nu^n(A) = \nu(-\infty; x]$. Using Proposition 2 and Theorem 2 we obtain that

$$\forall t > 0 \quad \nu^n(A + B_{W_n}(2t)) \geq \nu(-\infty; x + t],$$

where

$$W_n(x) = \sum_{i=1}^{n} W(x_i), \quad W(x) = \Lambda^*_n(x/2) = 2\Lambda^*_n(x/4).$$

To finish the proof it suffices to show that

$$\forall t > 0 \quad B_{W_n}(2t) \subset 4t B_1^n + 8\sqrt{t} B_2^n,$$

which by the Lemma (3) reduces to

$$\forall t > 0 \quad B_{W}(2t) \subset 4t B_1^1 + 8\sqrt{t} B_2^1.$$

The last condition is equivalent to

$$\forall t > 0 \forall x \quad W(x) \leq 2t \implies \exists y \quad |x - y| \leq 4t, y \leq 8\sqrt{t},$$

which follows from

$$\forall x \exists y \quad |x - y| \leq 2W(x), y^2 \leq 32W(x).$$

To finish the proof it suffices to show that

$$\forall x \exists y \quad \frac{1}{2} W(4x) = \Lambda^*_n(x) \geq \max \left\{ \left| x - \frac{y}{4} \right|, \left( \frac{y}{8} \right)^2 \right\}.$$ 

Since for $y(x) = 8\text{sgn}(x)(\sqrt{|x|} + 1 - 1)$ we get

$$\left| x - \frac{y(x)}{4} \right| = \left( \frac{y(x)}{8} \right)^2 = \left( \sqrt{1 + |x|} - 1 \right)^2,$$

thus the last inequality follows from Lemma (8). \qed

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