THE SO(3)-INSTANTON MODULI SPACE AND TENSOR PRODUCTS
OF ADHM DATA

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ABSTRACT. Let $\mathcal{M}_n^K$ be the moduli space of framed $K$-instantons with instanton number $n$ when $K$ is a compact simple Lie group of classical type. Due to Donaldson’s theorem [3], its scheme structure is given by the regular locus of a GIT quotient of $\mu^{-1}(0)$ where $\mu$ is the moment map on the associated symplectic vector space of ADHM data.

A main theorem of this paper asserts that $\mu$ is flat for $K = SO(3, \mathbb{R})$ and any $n \geq 0$. Hence we complete the interpretation of the K-theoretic Nekrasov partition function for the classical groups [5] in term of Hilbert series of the instanton moduli spaces together with the author’s previous results [1][2].

We also write ADHM data for the second symmetric and exterior products of the associated vector bundle of an instanton. This gives an explicit quiver-theoretic description of the isomorphism $\mathcal{M}_n^K \cong \mathcal{M}_n^{K'}$ for all the pairs $K, K'$ with isomorphic Lie algebras.

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4.4. Construction of the tensor, symmetric and exterior product morphisms in terms of ADHM data

Appendix A. Character of $\mathbb{C}[\rho^{-1}(0)]^{\text{Sp}(1)}$

A.1. Torus character of $\mathbb{C}[\rho^{-1}(0)]^{\text{Sp}(1)}$

A.2. Proof of Corollary A.1

References

1. Introduction

1.1. Main result. Let $K$ be a compact classical group. The K-theoretic Nekrasov partition function is defined by Nekrasov and Shadchin as the formal sum of the equivariant integrations of K-theory classes on the ADHM quiver representation space associated to $K$-instantons over the four-sphere [5]. By our previous results [1][2], it is the generating function of the Hilbert series of the coordinate rings of the framed $K$-instanton moduli spaces $M^K_n$ over all instanton numbers $n$ when $K = \text{USp}(N/2)$, $N \in 2\mathbb{Z}_{\geq 0}$ (the real symplectic group) or $K = \text{SO}(N,\mathbb{R})$, $N \geq 5$ (the real special orthogonal group). If $K = \text{SO}(4,\mathbb{R})$, it is not the generating function of Hilbert series for $M^K_n$, but that of the symplectic quotient of the associated ADHM space given below. In fact this symplectic quotient becomes the Uhlenbeck partial compactification (Uhlenbeck space, for short) for $K = \text{USp}(N/2)$, $N \in 2\mathbb{Z}_{\geq 0}$ or $K = \text{SO}(N,\mathbb{R})$, $N \geq 5$. In this paper we give a similar geometric interpretation for the remaining case $K = \text{SO}(3,\mathbb{R})$.

Let us state the main theorem for $\text{SO}(3,\mathbb{R})$. We use the following ADHM description of the $\text{SO}(3,\mathbb{R})$-instantons with instanton number $n \geq 0$ due to Donaldson [3]. Let $k := 4n$. Let $W := \mathbb{C}^3$, $V := \mathbb{C}^k$ with the standard orthogonal and symplectic forms respectively. First we consider the space of ordinary ADHM quiver representations $M = \text{End}(V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$. The orthogonal and symplectic forms induce the right adjoint maps $\text{End}(V) \rightarrow \text{End}(V)$, $B \mapsto B^*$ and $\text{Hom}(W, V) \rightarrow \text{Hom}(V, W)$, $i \mapsto i^*$. The $*$-invariant subspace in $\text{End}(V)$ is denoted by $p(V)$. We define a subspace of $M$ as

$$N := N_{V,W} := \{(B_1, B_2, i, j) \in M| B_1, B_2 \in p(V), j = i^*\}.$$ 

Let $\mu$ be the moment map on $N$ with respect to the natural Hamiltonian $\text{Sp}(V)$-action. Now the $\text{SO}(3,\mathbb{R})$-instantons with instanton number $n$ are bijectively corresponds to the stable-costable ADHM quiver representations in $\mu^{-1}(0)$ via the monad construction. The above pair $(N, \mu)$ is also defined even if $k = \dim V \in 2\mathbb{Z}$. We call the ADHM data in $\mu^{-1}(0)$ as $\text{SO}(3)$-data. This construction works for $\text{SO}(N,\mathbb{R})$-instantons (resp. $\text{USp}(N/2)$-instantons) by replacing $W = \mathbb{C}^3$ to $\mathbb{C}^N$ (resp. further by exchanging the orthogonal/symplectic structures).

A scheme structure of $M^K_n$ is given by $\mu^{-1}(0)\text{reg}/\text{Sp}(V)$ where ‘reg’ denotes the stable-costable locus. It is the smooth locus of the singular symplectic quotient $\mu^{-1}(0)/\text{Sp}(V)$. Since the K-theoretic partition function is the $\text{Sp}(V)$-invariant part of the equivariant push-forward of the Koszul complex class on $N$, it encodes only coherent sheaves over $N/\text{Sp}(V)$, but not over the genuine instanton space $M^K_n$. Therefore the K-theoretic partition function coincides with the Hilbert series of $\mu^{-1}(0)/\text{Sp}(V)$ if the Koszul complex
becomes the free resolution of the structure sheaf $\mathcal{O}_{\mu^{-1}(0)}$ as an $\mathcal{O}_N$-module. This amounts to $\mu^{-1}(0)$ is a complete intersection. See the details on the definition of K-theoretic partition function in [1, §1]. The following is the first main theorem on the scheme structures of $\mu^{-1}(0)$ and $\mu^{-1}(0)//Sp(V)$.

**Theorem 1.1.** Let $\mu$ be the moment map for SO(3)-data with $k = \dim V \in 2\mathbb{Z}_{\geq 1}$. Then we have the followings:

1. $\mu^{-1}(0)$ is a non-reduced complete intersection of dimension $\frac{k^2 + 3k}{2}$ with the equi-dimensional $\left\lfloor \frac{k}{2} \right\rfloor + 1$ irreducible components.
2. The GIT quotient $\mu^{-1}(0)//Sp(V)$ is also a non-reduced of dimension $k$ with the equi-dimensional $\left\lfloor \frac{k}{2} \right\rfloor + 1$ irreducible components. Each irreducible component is birational to the product of symmetric products $S_{\left\lfloor \frac{k}{2} \right\rfloor} \mathbb{A}^4 \times S_{\left\lfloor \frac{k}{2} \right\rfloor + 1} (\mathbb{A}^2 \times \mathbb{F})$ where $k'$ runs over the nonnegative integers in the set $k - 4\mathbb{Z}_{\geq 0}$.

Here $\lfloor r \rfloor$ denotes the largest integer in $\mathbb{R}_{\leq r}$, where $r \in \mathbb{R}$ and $\mathbb{F}$ denotes the fat point $\text{Spec} \mathbb{C}[x, y, z]/(x, y, z)^2$.

The second main theorem is the quiver description of the scheme-theoretic isomorphisms $\mathcal{M}_n^K \cong \mathcal{M}_n^{K'}$ in terms of ADHM data (Theorem 4.13). Here the isomorphisms mean the obvious identifications of instanton spaces for the pairs $(K, K')$ with the same Lie algebras. There are precisely five pairs $(K, K')$ among the classical groups

- $(\text{USp}(1), \text{SU}(2), (\text{SU}(2), \text{SO}(3, \mathbb{R})), (\text{USp}(1) \times \text{USp}(1), \text{SO}(4, \mathbb{R})), (\text{USp}(2), \text{SO}(5, \mathbb{R})), (\text{SU}(4), \text{SO}(6, \mathbb{R})))$.

For each pair $(K, K')$, the ADHM data of instantons are different. This implies that the $K$-instanton space has two different scheme structures. Nevertheless the two different scheme-scheme structures are isomorphic as follows: The natural isomorphism of Lie algebras $\text{Lie}(K) \cong \text{Lie}(K')$ defines an isomorphism between the spaces of associated vector bundles, hence induces a scheme-theoretic isomorphism $\mathcal{M}_n^K \cong \mathcal{M}_n^{K'}$ as Gieseker spaces. The Gieseker space is scheme-theoretically isomorphic to the space of ADHM data via the monad construction.

### 1.2 Flatness of moment map

In this section we explain a new idea for flatness of the moment map $\mu$ for SO(3) compared to the previous cases SO(N), $N \geq 4$ in [2]. Note that $\mathcal{N}$ is a $G$-Hamiltonian vector space of the form $T^*X \oplus Y$ where $G = \text{Sp}(V)$, $X = p(V)$, $Y = \text{Hom}(W, V)$. For $x \in X$, we denote the induced moment map for $(Y, G^x)$ by $\mu_x$, where $G^x$ denotes the stabilizer subgroup of $x$. A basic principle is that $\mu$ is flat if so is $\mu_x$ for all $x \in X$. This was used for SO(N), $N \geq 4$ ([2, Theorem 1.1]). A significant difference for SO(3) is that $\mu_x$ is not flat. In fact the principle can be sharpened by a dimension formula involving $\dim \mu_x^{-1}(0)$ (Lemma 2.3). Thus flatness amounts to a fibre dimension estimate of $\mu_x$.

In the case SO(3), $Y$ is also of the form $T^*X' \oplus Y'$ where $X' = Y' = V$. Therefore a similar sharpened dimension formula also works. Hence the flatness problem is reduced to a fibre dimension estimate of the moment map on $V$. The last moment map, say $\mu_{(x, x')}$, is defined for the stabilizer group of a pair $(x, x') \in p(V) \times V$. For instance if $x$ is a generic nilpotent endomorphism, the Lie algebra $g^x := \text{Lie}(G^x)$ is a truncated current algebra.
we prove Theorem 1.1 and then for the second symmetric and exterior products of the ADHM data for tensor products of vector bundles. First we do for the case 

$$\mathfrak{g}^{(x,x')} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{C}[z]/(z^{k/2}).$$

Hence $\mu_{(x,x')}(0) = (1,0) \otimes \mathbb{C}[z]/(z^{k/2}) + \mathbb{C}^2 \otimes z^{[k/4]+1}\mathbb{C}[z]/(z^{k/2}).$ In the study of truncated current algebras in §3, we give a systematic way to identify the zero fibre of $\mu_{(x,x')}$. The approach to flatness using current algebras was introduced in [2] in order to evaluate the fibre dimension of $\mu_x$. Since flatness in the cases $N \geq 5$ comes from $N = 4$, it suffices to consider $\mu_x$ defined over the cotangent space $T^*V^\oplus 2$. In contrast the study of $\mu_{(x,x')}$ is a new aspect in this paper.

1.3. Tensor products of ADHM data. In [2], the isomorphism $\mathcal{M}_n^K \cong \mathcal{M}_n^{K'}$ in terms of ADHM data was constructed for $(K, K') = (\text{USp}(1) \times \text{USp}(1), \text{SO}(4, \mathbb{R}))$ by giving the ADHM data of tensor products of vector bundles. In this section we explain a new idea in the construction of the isomorphism for the other pairs $(K, K')$.

First the case $(\text{USp}(1), \text{SU}(2))$ is immediate from the inclusion $\mathcal{N}_{V,W} \subset \mathcal{M}_{V,W}$ where $W = \mathbb{C}^2$. Recall that for $(\text{USp}(1) \times \text{USp}(1), \text{SO}(4, \mathbb{R}))$, the isomorphism $\mathcal{M}_n^K \cong \mathcal{M}_n^{K'}$ comes as a first step. And then the ADHM data of $\Lambda^2 F$ are quiver subrepresentations of the ADHM datum of the self-tensor product $F^\otimes 2$. The explicit form of the ADHM datum of $F^\otimes 2$ is given in (4.3) after substitution $t = 0$ when $\dim V = 1$. The explicit form for $\dim V \geq 2$ is deduced from the factorization property (Proposition 4.6). The explicit forms of the ADHM data of $S^2 F, \Lambda^2 F$ are given in Theorem 4.13.

1.4. Contents. This paper is organized as follows. In §2 we recollect the moment map $\mu$ defining the SO(3)-data and deduce an estimate of fibre dimension of $\mu$.

In §3 we prove Theorem 1.1. We need to study vector representation of the current algebra $\mathfrak{sl}_2[z]$ in §§3.1–3.2 mentioned in §1.2. With this preliminary, Theorem 1.1 comes from combination of the above study of vector representation and the fibre dimension estimate in §2.

In §4 we express the ADHM data for tensor products of vector bundles. First we do for the self-tensor product in §4.2 and then for the second symmetric and exterior products in §4.3. In §4.4 the isomorphism $\mathcal{M}_n^K \cong \mathcal{M}_n^{K'}$ is written in terms of these ADHM data.Acknowledgement. The author would like to express the deepest gratitude to Professor Hiraku Nakajima for leading to the instanton spaces and sharing his ideas.
2. Moment maps for SO-data and the flatness

We derive moment maps defining SO-data from the usual moment maps defining ADHM data in §2.1 and then modify further them in different forms in §2.2. As a result the moment maps are always canonically defined on symplectic spaces of the form $T^*X \times Y$. In the last subsection §2.3, we obtain a dimension estimate for the zero fibre of the moment maps. This gives flatness criteria of the moment maps.

The conventions in the entire part of this paper are as follows. We are working over $\mathbb{C}$. Schemes are of finite type and vector spaces are finite dimensional unless stated otherwise. Dimension of a scheme of finite type is defined to be the maximum of dimensions of irreducible components.

2.1. Moment maps for SO-data. In this subsection we define moment maps for SO-data. Some facts will be left as exercises if they come from direct computation, but the details can be found in [2, §2].

Let $(X, \omega)$ be a symplectic manifold with a Hamiltonian $G$-action. We denote by $\mu_X : X \to \mathfrak{g}^\vee$ a moment map. If $X$ is a linear $G$-representation and $\mu_X(0) = 0$, $\mu_X$ is given uniquely as $\mu_X(x) = \frac{1}{2} \omega(x, x)$.

Let $V, W$ be vector spaces. The usual ADHM quiver representations form a symplectic vector space

$$M_{V,W} = \text{End}(V)^{\oplus 2} \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W).$$

Here the symplectic structures are natural one on the cotangent space $M_{V,W} = T^*(\text{End}(V) \oplus \text{Hom}(W, V))$, where $\text{End}(V), \text{Hom}(W, V)$ are identified with their duals via trace respectively. The moment map with respect to the natural $\text{GL}(V)$-action is given as

$$\mu_{M_{V,W}}(B_1, B_2, i, j) = [B_1, B_2] + ij.$$

Here we identified the target space $\mathfrak{gl}(V)^\vee$ of $\mu_{M_{V,W}}$ with $\mathfrak{gl}(V)$ via trace pairing.

We further assume that there are symplectic and orthogonal forms $(,)_V, (,)_W$ on $V, W$ respectively. For $i \in \text{Hom}(W, V)$ we denote the right adjoint by $i^* \in \text{Hom}(V, W)$ (i.e. $(i(w), v)_V = (w, i^*(v))_W$ for $v \in V, w \in W$). Via $i \mapsto (i, i^*)$, $\text{Hom}(W, V)$ is a symplectic subspace of $T^*\text{Hom}(W, V) = \text{Hom}(W, V) \oplus \text{Hom}(V, W)$ because we can check directly $\langle (i, i^*), (-j^*, j) \rangle_{T^*\text{Hom}(W, V)} = 2\text{tr}(ij)$ where $j \in \text{Hom}(V, W)$. This subspace is invariant under the natural $\text{Sp}(V)$-action since $g^* = g^{-1}$.

For $B \in \text{End}(V)$, $B^*$ denotes the right adjoint. Let

$$p(V) := \{B \in \text{End}(V) | B = B^* \}$$

(the space of symmetric forms). This is also invariant under the natural $\text{Sp}(V)$-action. The trace pairing gives an identification $p(V) = \mathfrak{p}(V)^\vee$. Note that the Lie algebra $\mathfrak{sp}(V)$ of $\text{Sp}(V)$ is the subspace $\{B \in \text{End}(V) | B = -B^* \}$ (the space of antisymmetric forms).

As a result the vector space

$$N_{V,W} = p(V)^{\oplus 2} \oplus \text{Hom}(W, V)$$

is a $\text{Sp}(V)$-invariant symplectic subspace of $M_{V,W}$. The moment map $\mu_{N_{V,W}}$ with respect to $\text{Sp}(V)$ is given as $\mu_{N_{V,W}}(B_1, B_2, i) = [B_1, B_2] + ii^*$. Note that this is the composite of the restriction of $\mu_{M_{V,W}}$ and the projection $\mathfrak{gl}(V)^\vee \to \mathfrak{sp}(V)^\vee$. 

2.2. Another form of moment maps. We modify \( \mu_{N,W} \) in another forms. The reason for the modifications is that we need to understand geometry of the zero-fibre of \( \mu_{N,W} \). Recall that \( \mu_{N,W} \) is the sum of two moment maps \( \mu_{T^*p(V)} \) and \( \mu_{\text{Hom}(W,V)} \) and thus that the zero-fibre is the fibre product of \( T^*p(V) \) and \( \text{Hom}(W,V) \) over \( \text{sp}(V)^\vee \) via \( -\mu_{T^*p(V)} \) and \( \mu_{\text{Hom}(W,V)} \). Now the standard base change argument can apply once we know the fibres of one of the moment maps explicitly (cf. [2]). We will modify the latter moment map. The computational details in this subsection can be found in [2, §2.4].

The first modification is as follows. Let us fix any orthogonal basis of \( W \) and then identify \( \text{Hom}(W,V) = V^\oplus N \). The moment map \( \mu_V \) with respect to \( \text{Sp}(V) \) is given by \( \mu_V(v) = \frac{1}{2}(v,v) \). And the moment map \( \mu_{V^\oplus N} \) with respect to \( \text{Sp}(V) \) is the \( N \)-sum of \( \mu_V \).

The second modification is as follows. For simplicity we assume \( N = \dim W = 3 \). Let \( W = W_1 \oplus W_2 \) be an orthogonal decomposition such that \( \dim W_1 = 2, \dim W_2 = 1 \). Let \( \{e_1, e_2\} \) be a basis of \( W_1 \) such that \( (e_1,c_1)_W = (e_2,c_2)_W = 0 \). We take the right adjoints of linear maps in \( \text{Hom}(W_1,V) \) as

\[
*: \text{Hom}(W_1,V) \cong T^*\text{Hom}(C e_1,V) = \text{Hom}(C e_1,V) \oplus \text{Hom}(C e_2,V) \to \text{Hom}(V,C e_1) \oplus \text{Hom}(V,C e_2),
\]

\[i = (i_1, i_2) \mapsto (i_2^*, i_1^*)\]

This induces an isomorphism

\[\text{Hom}(W_1,V) \cong T^*\text{Hom}(C e_1,V) = \text{Hom}(C e_1,V) \oplus \text{Hom}(V,C e_1), \quad (i_1, i_2) \mapsto (i_1^*, i_2^*)\]

which pull backs the natural symplectic form of the cotangent space \( T^*\text{Hom}(C e_1,V) \) to the half of the original one of \( \text{Hom}(W_1,V) \). As a result we identify

\[\mu_{\text{Hom}(W,V)} = 2\mu_{T^*V} + \mu_V.\]

Remark 2.1. (1) Recall the modification of \( \mu_{\text{Hom}(W,V)} \) in the case \( N = \dim W \geq 4 \) in [2]. If \( N = \dim W = 4 \), we identified \( \mu_{\text{Hom}(W,V)} = 2\mu_{T^*V^\oplus 2} \) using a decomposition of \( W \) into two complementary maximal isotropic subspaces. For \( N \geq 5 \) we used an orthogonal decomposition \( W = W_1 \oplus W_2 \) where \( \dim W_1 = 4 \) and then the base change argument.

(2) In the case of Sp-data, \( V, W \) are orthogonal and symplectic respectively. By [6], \( \mu_{T^*p(V)} \) is flat and thus we used the base change argument.

2.3. Flatness criteria. The moment maps for SO-data or for \( \text{Hom}(W,V) \) are defined on symplectic spaces of the form \( T^*X \times Y \). In this section we give criteria for the moment maps defined on those spaces.

Let \( X \) be a \( G \)-representation and \( Y \) be a symplectic smooth scheme with a Hamiltonian \( G \)-action. Let \( \mu : T^*X \times Y \to g^\vee \) be the moment map. It is the sum of the two moment maps on \( T^*X \) and \( Y \) (denoted by \( \mu_{T^*X}, \mu_Y \) respectively). For any \( x \in X \) we denote by \( \mu_x : Y \to (g^x)^\vee \) the moment map on \( Y \) with respect to \( g^x = \text{Lie}(G^x) \). It is given by the composite of the moment map on \( Y \) with the natural projection \( g^\vee \to (g^x)^\vee \).

Lemma 2.2. We fix any \( x \in X \).

(1) The restriction of \( \mu_{T^*X} \) to \( \{x\} \times X^\vee \) has the image \((g^x)^\perp \) in \( g^\vee \). In particular its kernel has dimension \( \dim X - \dim g_x \).

(2) \( (\mu_Y)^{-1}(g^x)^\perp = \mu_{T^*X}^{-1}(0) \).

Proof. (1) is immediate from the fact that the dual of the restriction coincides with the \( g \)-action map \( g \to X, \xi \mapsto \xi.x \).
Corollary 2.4.  \( \mu_{x} \) is flat if \( \dim X_{(s)} - s + \dim \mu_{x}^{-1}(0) \leq \dim X + \dim Y - \dim G \) for any \( x \in X_{(s)} \) and \( s \geq 0 \).

In particular \( \mu \) is flat if so is \( \mu_{x} \) for any \( x \in X \).

Proof. We need to show \( \dim \mu_{x}^{-1}(0) \leq 2 \dim X + \dim Y - \dim G \). This inequality follows from the above lemma and the obvious inequality

\[
\dim X_{[G']} - \dim G.x_{0} + \dim X + \dim \mu_{x_{0}}^{-1}(0) \leq \dim X_{(s)} - s + \dim X + \max_{x \in X_{(s)}} \dim \mu_{x}^{-1}(0)
\]

for any \( x_{0} \in X_{[G']} \) and any \([G'] \in \mathcal{C}(G)\) such that \( X_{[G']} \subset X_{(s)} \).

Let us prove the second statement. By flatness of \( \mu_{x} \) we have \( \dim \mu_{x}^{-1}(0) = \dim Y - \dim g^{x} \). Then the statement follows from plugging this into the inequality of the first criterion.

Definition 2.5. For a \( G \)-variety \( X \) we define modality by

\[
\text{mod}(G : X) := \max_{s \geq 0}(\dim X_{(s)} - s).
\]

Remark 2.6. The second criterion of the above corollary was used for the flatness of \( \mu \) for \( \text{SO}(N) \)-data, \( N \geq 4 \). In the case we set \( X = p(V) \) and \( Y = \text{Hom}(W, V) \). Then \( \mu_{x} : Y \to (g^{x})^{\vee} \) is flat for any generic nilpotent endomorphism \( x \in p(V) \) ([2, Corollary 3.12]). However this does not hold for \( \text{SO}(3) \)-data (see [1, Remark 5.2]).
Let us recall the above flatness of \( \mu_x \) when \( N \in 2\mathbb{Z}_{>2} \). We write \( \text{Hom}(W, V) = T^*\text{Hom}(\mathbb{C}^r, V) \) where \( \mathbb{C}^r \) is a maximal isotropic subspace of \( \mathbb{W} \). So the flatness amounts to the inequality \( \text{mod}(\mathfrak{sp}(V)^s : \text{Hom}(\mathbb{C}^r, V)) \leq \dim \text{Hom}(\mathbb{C}^r, V) - \dim \mathfrak{sp}(V)^s = (2r - 3)k \). This was shown in [2, Theorem 3.6].

3. Flatness of the moment map for SO(3)-data: the proof of Theorem 1.1

In this section we prove Theorem 1.1. The description on \( \mu^{-1}(0)/\text{Sp}(V) \) in this theorem follows from geometry of \( \mu^{-1}(0) \).

**Theorem 3.1.** (1) \( \mu \) is flat.

(2) \( \mu^{-1}(0) \) has the \( \lfloor \frac{k}{2} \rfloor + 1 \) irreducible components. Moreover each irreducible component has an étale dense locally closed subset \( \mu^{-1}(0) \times_{\text{Sp}(\mathbb{C})_0} (\Delta S^n \mathbb{C})_0 \) where \( n \) is a partition of \( k \) consisting of only 4 or 2.

(3) If \( k \in 4\mathbb{Z}_{>0} \), the regular locus \( \mu^{-1}(0)^{\text{reg}} \) is Zariski dense open in only one irreducible component. Otherwise \( \mu^{-1}(0)^{\text{reg}} = \emptyset \).

The above notation are defined as follows: \( S^k \mathbb{C} \) denotes the \( k \)th symmetric product of \( \mathbb{C} \). \( \Delta S^k \mathbb{C} \) denotes the diagonal (i.e. the set of unordered \( k \)-tuples with one-point support). Let \( \eta = (\eta_1, \eta_2, ..., \eta_e) \) be a partition of \( k \), i.e. a decreasing sequence of nonnegative integers with sum \( k \). \( S^n \mathbb{C} \) denotes the product \( \prod_{n=1}^e S^{n_1} \mathbb{C} \). \( \Delta S^n \mathbb{C} \) denotes \( \prod_{n=1}^e \Delta S^{n_1} \mathbb{C} \). \( (S^n \mathbb{C})_0 \) denotes the set of \( (z_1, z_2, ..., z_e) \) such that the supports of \( z_n \) are mutually disjoint. Let \( (\Delta S^n \mathbb{C})_0 := (\Delta S^n \mathbb{C}) \cap (S^n \mathbb{C})_0 \).

The morphisms \( S^n \mathbb{C} \rightarrow S^k \mathbb{C} \) and \( \mu^{-1}(0) \rightarrow S^k \mathbb{C} \) are given respectively by the sum map \( (z_1, z_2, ..., z_e) \mapsto z_1 + z_2 + \cdots + z_e \) and \( (B_1, B_2, i) \mapsto E(B_1) \) where \( E(B_1) \) denotes the unordered \( k \)-tuple of eigenvalues.

This section is organized as follows: In §3.1 we study the vector representation \( \mathbb{C}^2 \otimes \mathbb{C}[z] \) of the current algebra \( \mathfrak{sl}_2[z] \). This is preliminary step to estimate \( \dim \mu^{-1}(0) \) in §3.2. In §3.3 we prove Theorem 3.1. The flatness of \( \mu \) will be proven by the dimension estimate of \( \mu^{-1}(0) \). Combining the flatness with factorization property, we deduce the description of irreducible components. In §3.4 we prove Theorem 1.1 using Theorem 3.1. In §3.5 we compare \( \mu^{-1}(0)/\text{Sp}(V) \) and the Uhlenbeck space via their natural stratifications. One can cross-check the description of irreducible components in Theorem 1.1 at set-theoretic level from the stratification.

3.1. Vector representation of the current algebra \( \mathfrak{sl}_2[z] \). As a preliminary step for the flatness of \( \mu \) when \( \dim W = 3 \), we need to study vector representation of the current algebra \( \mathfrak{sl}_2[z] \). This appeared in [2, §3.4] for the case \( \dim W = 4 \). But there is an essential difference in the case \( \dim W = 3 \): the moment map \( \mu_x : \text{Hom}(W, V) \rightarrow (\mathfrak{g}^*)^\vee \) is no more flat (Remark 2.6). After the preliminary we will perform a new dimension estimate of the fibres \( \mu_x^{-1}(0) \) in §3.2.

Let \( T := \mathbb{C}^2 = \mathbb{C}\langle e_1, e_2 \rangle \) with the standard symplectic structure. We denote some tensor products as \( \mathbb{C} \)-vector spaces by

\[
T[z] := T \otimes \mathbb{C}[z], \mathfrak{sl}_2[z] := \mathfrak{sl}(T) \otimes \mathbb{C}[z], \mathfrak{gl}_2[z] := \mathfrak{gl}(T) \otimes \mathbb{C}[z].
\]
Thus their elements have polynomial expression \( x = x_0 + x_1 z + \cdots + x_d z^d \) where \( x_i \in T \) or \( \mathfrak{sl}_2 \) and \( \mathfrak{g}_1 \). Both \( T[z] \), \( \mathfrak{g}_1 \) are left \( \mathfrak{g}_2 \)-modules and thus left \( \mathbb{C}[z] \)- and \( \mathfrak{sl}_2 \)-modules. We call \( T[z] \) vector representation of the current algebras \( \mathfrak{sl}_2 \), \( \mathfrak{g}_1 \).

We define
\[
V_d := T[z]/z^{d+1}T[z], \quad \mathfrak{g}_d := \mathfrak{sl}_2[z]/z^{d+1}\mathfrak{sl}_2[z]
\]
where \( d \geq 0 \). We can also write them as \( V_d = T \otimes \mathbb{C}[z]/(z^{d+1}) \) and \( \mathfrak{g}_d = \mathfrak{sl}_2 \otimes \mathbb{C}[z]/(z^{d+1}) \) where \( \mathbb{C}[z]/(z^{d+1}) \) is the truncated polynomial algebra. Note that any \( \mathfrak{sl}_2 \)-orbit in \( V_d \) coincides with the \( \mathfrak{g}_d \)-orbit since \( z^{d+1} \) annihilates any element.

Let \( (\ ,\ )_{T[z]} \) be the \( T[z] \)-linear extension of the symplectic form \( (\ ,\ )_T \). The induced symplectic structure on \( V_d \) is defined by
\[
(f, g)_{V_d} := \text{Res}_{z=0} (f, g)_{T[z]}/z^{d+1}.
\]
We identify \( V = V_d \) where \( k = \dim V = 2(d + 1) \).

**Lemma 3.2.** ([2, Lemma 3.13]) If \( B \) is a generic nilpotent element of \( \mathfrak{p}(V) \). Then the Lie algebra \( \text{Lie}(\text{Sp}(V)^B) \) equals \( \mathfrak{g}_d \).

Let \( x = x_0 + x_1 z + \cdots + x_d z^d \in V_d \) or \( \mathfrak{g}_d \). We define minimal degree
\[
\text{min.deg } x := \min \{ n \geq 0 | x_n \neq 0 \}.
\]
In particular if \( x = 0 \), \( \text{min.deg } x = d + 1 \).

**Proposition 3.3.** Let \( x \in V_d \) and \( n := \text{min.deg } x \).

1. There exists \( \xi \in \mathfrak{g}_d^x \) with \( \text{min.deg } \xi = 0 \).

2. For any such \( \xi \), \( \mathfrak{g}_d^\xi \) is given as
\[
\mathfrak{g}_d^\xi = \mathbb{C}[z] \xi + z^{d+1-n} \mathfrak{g}_d
\]
\[
= \mathbb{C} \xi \oplus \mathbb{C} \xi z \oplus \cdots \oplus \mathbb{C} \xi z^{d-n} \oplus \mathfrak{sl}_2 z^{d+1-n} \oplus \mathfrak{sl}_2 z^{d+2-n} \oplus \cdots \oplus \mathfrak{sl}_2 z^d.
\]

3. For \( s = 2(d + 1 - n) \) \( \mathfrak{g}_d^x \cap (V_d)_s = \{ x' \in V_d | \text{min.deg } x' = n \} = \{ T \setminus 0 \} z^n \oplus T z^{n+1} \oplus T z^{n+2} \oplus \cdots \oplus T z^d \).

For the other values of \( s \), \( (V_d)_s = \emptyset \).

In particular \( \text{mod}(\mathfrak{g}_d : V_d) = 0 \) and it is attained by all the nonempty \( (V_d)_s \).

**Proof.** (1) We will find \( \xi = \xi_0 + \xi_1 z + \cdots + \xi_d z^d \in \mathfrak{g}_d^x \) with \( \xi_0 \neq 0 \). First we set \( \xi_0 \in \mathfrak{sl}_2 \) with \( \mathfrak{sl}_2^{z^n} = \mathbb{C} \xi_0 \). We will find \( \xi_1, \xi_2, \ldots, \xi_d \in \mathfrak{sl}_2 \) inductively. Assume the induction hypothesis up to \( m \), i.e.
\[
\left( \sum_{i=0}^{m} \xi_i z^i \right) x = 0 \mod z^{m+n+1} = 0
\]
as far as \( m + n \leq d \). The initial step \( m = 0 \) is already done as \( \xi_0 x = 0 = \xi_0 x_n z^n = 0 \mod z^{n+1} \). We need to find a solution \( \xi_{m+1} \) of the equation
\[
\xi_{m+1} x_n + \xi_m x_{n+1} + \cdots + \xi_0 x_{n+m+1} = 0.
\]
This is always solvable because \( x_n \neq 0 \) so that it can be transformed to any vector of \( T \) by the \( \mathfrak{sl}_2 \)-action. If \( m + n > d \), we set \( \xi_m \) to be any element of \( \mathfrak{sl}_2 \).
(2) It is clear that for all \( m \geq d + 1 - n \), \( \text{Coef}_{x=0}(g_d^x) = sl_2 z^m \) because \( sl_2 z^m \) itself annihilates \( x \). So it suffices to prove that if \( \xi' \in g_d^x \), we have \( \xi' \in \mathbb{C}[z] \xi \mod z^{d+1-n} \). Let \( \xi' \in g_d^x \) with \( n' := \min \deg \xi' \). If \( n' \geq d - n + 1 \), there is nothing to show. So we suppose \( n' \leq d - n \). Since \( \xi'' \) annihilates \( x_n \), there exists \( c \in \mathbb{C}^* \) such that \( \xi'' = c\xi_0 \). Here we used \( sl_2^\omega = \mathbb{C} \xi_0 \). Now \( \xi' - c\xi'' \) has minimal degree \( > n' \) and annihilates \( x \). By the induction on minimal degree \( n' \), it is contained in \( \mathbb{C}[z] \xi \mod z^{d+1-n} \). Therefore so is \( \xi' \).

(3) By (2), we have

\[
(V_d)_s = \{x' \in V_d|\min \deg x' = n \}, \quad s = \dim g_d - \dim g_d^x = 2(d + 1 - n).
\]

Note also that \( \dim(V_d)_s = s \) unless \( (V_d)_s = \emptyset \).

**Remark 3.4.** (1) If \( n = \min \deg x = 0 \) in the proposition, we obtain

\[
g_d^x = \mathbb{C}[z] \xi
\]

(a principally generated \( \mathbb{C}[z] \)-module). Even for \( x \in \text{Hom}(\mathbb{C}^r, V_d) \), \( r \geq 2 \), as far as rank\( x_0 = 1 \), \( g_d^x \) is also a principally generated \( \mathbb{C}[z] \)-module ([2, Lemma 3.9]).

(2) The modality 0 in (3) of the proposition can be proven slightly differently: \( (V_d)_s \) is always one \( G_d \)-orbit where \( G_d \) denotes the adjoint group of \( g_d \). This is immediate from the corollary in the below.

**Corollary 3.5.** For the adjoint group \( G_d = \exp(g_d) \), we have

\[
G_d.e_1 z^n = \{x \in V_d|\min \deg x = n \} \quad (= (V_d)_s)
\]

where \( n = 0, 1, ..., d + 1 \) (and \( s = 2(d + 1 - n) \)).

**Proof.** The inclusion \( \subset \) is obvious. To check the opposite we observe that the RHS and any \( G_d \)-orbit in it have the same dimension \( 2(d + 1 - n) \) by Proposition 3.3 (2). Since the RHS is an irreducible Zariski locally closed set, it is one orbit. \( \square \)

### 3.2. Dimension estimate of \( \mu^{-1}(0) \) for the moment map \( \mu: \text{Hom}(\mathbb{C}^3, V_d) \to g_d^\vee \).

By identification \( \text{Hom}(\mathbb{C}^3, V_d) = V_d^\otimes 3 \) by the standard orthogonal basis of \( \mathbb{C}^3 \), \( \mu \) is the sum of the three identical moment maps defined on the symplectic vector space \( (V_d, ( , )_{V_d}) \) with respect to \( g_d \). The main theorem in this subsection is the following:

**Theorem 3.6.** For the moment map \( \mu: \text{Hom}(\mathbb{C}^3, V_d) \to g_d^\vee \) with respect to the \( g_d \)-action, we have \( \dim \mu^{-1}(0) = 4d - 2\lfloor \frac{d}{2} \rfloor + 3 \).

In particular if \( d = 0, 1, \) \( \dim \mu^{-1}(0) = 2 \dim V_d - 1 \). For \( d \geq 2 \), \( \dim \mu^{-1}(0) \leq 2 \dim V_d - 3 \).

The proof will appear at the end of this subsection. We will use Lemma 2.3 in the proof, so we need to rewrite \( \mu \) as the moment map defined on a space of the form \( T^*X \times Y \). This was done in §2.2 where \( X = Y = V_d \).

To use Lemma 2.3 we need dimension estimate of \( \mu_x^{-1}(0) \) for each \( x \in V_d \), where \( \mu_x: V_d \to (g_d^x)^\vee \) is the moment map with respect to \( g_d^x \). By Corollary 3.5 we may set \( x = e_1 z^n \). Let \( E, F, H \) be the standard \( sl_2 \)-triple:

\[
E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]
Proposition 3.7. Let \( x = e_1 z^n, 0 \leq n \leq d + 1 \). Then the reduced scheme
\[
\left( \mu_x^{-1}(0) \right)_{\text{red}} = z^{1\over 2}\left[z^{d+1} + C[z]e_1 + z^{d+1}V_d \right].
\]

In particular if \( d = 0, 1 \) and \( n = 0 \), \( \mu_x^{-1}(0) \) has codimension 1 in \( V_d \). Otherwise it has codimension \( \geq 2 \).

Proof. Note first that \( \mu_x^{-1}(0) = \{ v \in V_d | (v, g_d^x v) = 0 \} \) set-theoretically. Let \( n = 0 \) first. By Proposition 3.3 (2), \( g_d^x = \mathbb{C}[z]E \) and thus
\[
\mu_x^{-1}(0) = \{ v \in V_d | (v, E.v)_{V_d} = (v, zE.v)_{V_d} = \cdots = (v, z^d E.v)_{V_d} = 0 \}
\]
\[
\{ v \in V_d | (v, E.v)_{T[z]} = 0 \mod z^{d+1} \}.
\]

Using \( v = v_0 + v_1 z + \cdots v_d z^d \), the constraint \( (v, E.v)_{T[z]} = 0 \mod z^{d+1} \) amounts to that for each \( 0 \leq l \leq d \) the following is zero:
\[
\text{Coeff}_z(v, E.v)_{T[z]} = (v_0, E v_0) + (v_1, E v_{l-1}) + \cdots + (v_l, E v_0)_T
\]
\[
\text{Coeff}_z(v, E.v)_{T[z]} = \begin{cases} (v_{l/2}, E v_{l/2})_T & l: \text{even} \\ 2(v_{(l+1)/2}, E v_{(l+1)/2})_T & l: \text{odd} \end{cases}
\]

We claim the solution space for \( v \) of this system of quadratic equations is \( \mathbb{C}[z]e_1 + z^{1\over 2}\left[z^{d+1} + T[z] \right] \mod z^{d+1} \). We use the induction on \( d \). When \( d = 0 \), by putting \( v_0 = ae_1 + be_2 \) the equation \( (v_0, E v_0)_T = 0 \) gives \( b^2 = 0 \). So \( v_0 \in C e_1 \). Let \( d \geq 1 \). By the induction on \( d \) the truncated quadratic equation system
\[
\text{Coeff}_z(v, E.v)_{T[z]} = 0 \mod z^d, \quad 0 \leq l \leq d - 1
\]
has the solution space \( \mathbb{C}[z]e_1 + z^{1\over 2}\left[z^{d+1} + T[z] \right] \mod z^d \). The original quadratic equation system has one more equation \( \text{Coeff}_z(v, E.v)_{T[z]} = 0 \) than (3.2). Looking at (3.1), when \( d \) is odd there is no new equation because \( v_m \in C e_1 \) for \( m \leq \frac{d-1}{2} \) and \( E v_m = 0 \). Only when \( d \) is even, the equation \( (v_{(d+1)/2}, E v_{(d+1)/2})_T = 0 \) becomes new. This gives \( v_{(d+1)/2} \in C e_1 \). The solution space is now given as in the claim. This completes the proof of the proposition when \( n = 0 \).

Secondly we assume \( n \geq 1 \). Then there is the additional defining equation of \( \mu_x^{-1}(0) \)
\[
(v, z^{n'} g_d^x v)_{T[z]} = 0 \mod z^{d+1}, \quad n' := d + 1 - n.
\]
This amounts to that (3.1) vanishes for each \( l \) with \( E \) replaced by \( z^{n'} E, z^{n'} F \) and \( z^{n'} H \). By a similar argument as above, (3.3) imposes the constraints \( v_l = 0 \) for each \( l \geq 0 \) satisfying \( 2l + n' \leq d \) (i.e. \( l \leq \left\lfloor \frac{d-1}{2} \right\rfloor \)). This finishes the proof of the proposition.

\[\square\]

Remark 3.8. From the computation \( b^2 = 0 \) where \( v_0 = ae_1 + be_2 \) in the above proof, we see that \( \mu_x^{-1}(0) \) is always a non-reduced scheme.

Now we can prove Theorem 3.6.

Proof of Theorem 3.6. By Lemma 2.3 we need to compute \( \dim X_{[G']} - \dim G.x \) \( x \in X_{[G']} \) and \( \dim \mu_x^{-1}(0) \) where \( [G'] \in C(G) \) and \( x \in X_{[G']} \). By Proposition 3.3 (3), \( \dim X_{[G']} - \dim G.x = 0 \) for any \( [G'] \in C(G) \) and \( x \in X_{[G']} \). By Proposition 3.7, \( \dim \mu_x^{-1}(0) \) attains the maximum \( 2(d - \lfloor \frac{d-1}{2} \rfloor) + 1 \) precisely when \( \min. \deg x = 0 \). Therefore by the dimension formula in Lemma 2.3 we obtain \( \dim \mu^{-1}(0) = 4d - 2\lfloor \frac{d}{2} \rfloor + 3 \). \[\square\]
3.3. **Proof of Theorem 3.1.** We get back to the setting of Theorem 3.1: \( \mu: T^*X \times Y \to \text{sp}(V)^\vee \) is the moment map with respect to \( \text{Sp}(V) \) where \( X = \mathfrak{p}(V), \ Y = \text{Hom}(W, V) \) and \( W = \mathbb{C}^3 \) (the orthogonal vector space).

First we prove \( \mu \) is flat. We need factorization property. To emphasize \( k = \dim V \) we use the notation \( \mu_{(k)} \) instead of \( \mu \).

**Lemma 3.9.** [2, Lemma 2.7 (2)] Let \( \eta = (\eta_1, \eta_2, \ldots, \eta_e) \) be a partition of \( k \). Then there is a surjective smooth morphism

\[
\sigma: \text{Sp}(V) \times \left( \mu_{(\eta_1)}^{-1}(0) \times \mu_{(\eta_2)}^{-1}(0) \times \cdots \times \mu_{(\eta_e)}^{-1}(0) \right) \times_{S^e \mathbb{C}} (S^e \mathbb{C})_0 \to \mu_{(k)}^{-1}(0) \times_{S^k \mathbb{C}} (S^k \mathbb{C})_0
\]

with the (scheme-theoretic) fibres

\[
\sigma^{-1}(\sigma(g, x)) = \{(gh^{-1}, h. x) | h \in \text{Sp}(\eta_1/2) \times \text{Sp}(\eta_2/2) \times \cdots \times \text{Sp}(\eta_e/2)\}.
\]

Moreover \( \sigma \) satisfies \( \sigma(g, x) = g.\sigma(e, x) \) where \( e \) denotes the identity of \( \text{Sp}(V) \) and the \( \text{Sp}(V) \)-action is trivial on the factor \( S^e \mathbb{C} \).

This lemma holds similarly for the ordinary ADHM data by replacing \( \text{Sp}(V) \) into \( \text{GL}(V) \). But in our case, only when \( \eta_1, \eta_2, \ldots, \eta_e \) are even, \( \sigma \) has the nonempty domain and target spaces.

If \( e \geq 2 \) and \( \eta_e \neq 0 \), this factorization property gives the induction hypothesis on \( k = \dim V \) assuring the étale open subset associated to \( \eta \), \( \mu_{(k)}^{-1}(0) \times_{S^k \mathbb{C}} (S^k \mathbb{C})_0 \) has the expected dimension \( \dim \mathfrak{p}(V) + 2 \dim V \). Thus to prove flatness of \( \mu_{(k)} \) we need only the dimension estimate when \( e = 1 \), i.e.,

\[
\dim \mu_{(k)}^{-1}(0) \cap (S \times X \times Y) \leq \dim \mathfrak{p}(V) + 2 \dim V,
\]

where

\[
S := \{ B \in \mathfrak{p}(V) | B \text{ has only one eigenvalue} \}.
\]

By the dimension formula (2.1) this amounts to

\[
\dim S_{[G']} - \dim \text{Sp}(V).x + \dim \mu_{x}^{-1}(0) \leq 2 \dim V
\]

where \([G'] \in \mathcal{C}(S)\) and \( x \in S_{[G']}\).

Let us estimate the LHS of (3.4). Let \( x \in S \). Thus \( x \) has only one eigenvalue, say \( a \). It is well-known that there is a \( x \)-stable decomposition into symplectic subspaces \( V = V_1 \oplus V_2 \oplus \cdots \oplus V_{l_0} \) such that the nilpotent endomorphism \( x - a \) on \( V_l \) corresponds to the partition \( \left( \frac{\dim V_1}{2}, \frac{\dim V_1}{2} \right) \) for each \( l \) ([2, Corollary C.3]). Note that the conjugacy class of stabilizers \( G' = \text{Sp}(V)^x \) is uniquely given by the unordered \( l_0 \)-tuple of such partitions. This implies \( \text{Sp}(V) \)-orbits in \( S_{[G']} \) are in one-to-one correspondence with the eigenvalues \( \mathbb{C} \), which means

\[
\dim S_{[G']} - \dim \text{Sp}(V).x = 1
\]

for any \( x \in S_{[G']} \).

On the other hand \( \dim \mu_{x}^{-1}(0) \) is estimated as follows. Let \( x_l := x|_{V_l} \in \mathfrak{p}(V_l) \). Since the product \( \prod_{l=1}^{l_0} \text{Sp}(V_l)^{x_l} \) is a subgroup of \( G' \), \( \mu_{x}^{-1}(0) \) is contained in the product \( \prod_{l=1}^{l_0} \mu_{x_l}^{-1}(0) \).
where $\mu_{x_j} : \text{Hom}(W, V_i) \to \text{(sp}(V_i)^{x_i})^\vee$ is the moment map with respect to $\text{Sp}(V_i)^{x_i}$. By Theorem 3.6, we obtain

\[
(3.6) \quad \dim \mu_{x}^{-1}(0) \leq \sum_{l=1}^{l_0} \dim \mu_{x_i}^{-1}(0) \leq 2 \dim V - l_0.
\]

By adding (3.5) and (3.6), we obtain an estimate of the LHS of (3.4): $\text{LHS} \leq 2 \dim V - l_0 + 1$. So (3.4) is checked. The proof of Theorem 3.1 (1) is done.

We prove Theorem 3.1 (2). We need to check

\[
\dim \mu^{-1}(0) \times_{S^4} (\Delta S^0 \mathbb{C})_0 = \dim \mu^{-1}(0) \text{ if } \eta_1 \leq 4,
\]
\[
\dim \mu^{-1}(0) \times_{S^4} (\Delta S^0 \mathbb{C})_0 < \dim \mu^{-1}(0) \text{ if } \eta_1 \geq 6.
\]

As in the proof of (1) we use the dimension formula (2.1). By the second statement of Theorem 3.6, (3.6) is further modified to

\[
(3.7) \quad \dim \mu_{x}^{-1}(0) \leq 2 \dim V - \# \{1 \leq l \leq l_0 | \dim V_i = 2 \text{ or } 4\} - 3\# \{1 \leq l \leq l_0 | \dim V_i > 4\}.
\]

Therefore we have

\[
\dim \mu_{x}^{-1}(0) = 2 \dim V - 1 \text{ if } l_0 = 1 \text{ and } \dim V \leq 4,
\]
\[
\dim \mu_{x}^{-1}(0) \leq 2 \dim V - 2 \text{ otherwise}.
\]

By (2.1) combined with the factorization property (Lemma 3.9), the étale open subset $\mu^{-1}(0) \times_{S^4} (S^0 \mathbb{C})_0$ has dimension equal to $\dim \mu^{-1}(0)$ (resp. strictly less than $\dim \mu^{-1}(0)$) if $\eta_1 \leq 4$ (resp. otherwise).

It remains to show $\mu_{(4)}^{-1}(0)$ has two irreducible étale locally closed subsets $\mu_{(4)}^{-1}(0) \times_{S^4} (S^{(2,2)} \mathbb{C})_0$ and $\mu_{(4)}^{-1}(0) \times_{S^4} \Delta S^{(4)} \mathbb{C}$. This is equivalent to the following:

**Lemma 3.10.** ([1, Corollary 8.9]) *There are precisely two irreducible components of $\mu_{(4)}^{-1}(0)$. They are set-theoretically the $(\mathbb{A}^2 \times \text{SL}_2) \times \text{Sp}(2) \times \text{SO}(3)$-orbit closures of some elements $(B_1^{(i)}, B_2^{(i)}, i^{(i)}), (B_1^{(II)}, B_2^{(II)}, i^{(II)})$ such that $B_1^{(i)}$ has only one eigenvalue while $B_1^{(II)}$ has two distinct eigenvalues.*

Here $\mathbb{A}^2 \times \text{SL}_2$ acts on $\mathfrak{p}(V)^{\oplus 2}$ by

\[
\begin{pmatrix} (a_1, a_2), (a_1, a_2) \end{pmatrix} \cdot (B_1, B_2) = (a_1, a_2) + (aB_1 + bB_2, cB_1 + dB_2).
\]

The explicit forms of $(B_1^{(i)}, B_2^{(i)}, i^{(i)}), (B_1^{(II)}, B_2^{(II)}, i^{(II)})$ are given in [1, Lemma 8.8 and pp.298–299]. This completes the proof of (2).

We prove Theorem 3.1 (3). By [1, Lemma 8.10 (2)], $(B_1^{(II)}, B_2^{(II)}, i^{(II)})$ is a regular element. By [2, Lemma 2.5], any tuple of regular elements maps to a regular element in $\mu^{-1}(0)$ via $\sigma$ in Lemma 3.9. Therefore if $k \in 4\mathbb{Z}_{>0}$, the product of $\mu_{(4)}^{-1}(0)^{\text{reg}}$ maps via $\sigma$ into a Zariski dense open subset of an irreducible component of $\mu^{-1}(0)^{\text{reg}}$. To complete the proof of (3) we observe that any element of $\mu_{(2)}^{-1}(0)^{\text{reg}}$ has nontrivial stabilizer in $\text{Sp}(1)$. For, if $(B_1, B_2, i) \in \mu_{(2)}^{-1}(0)$, both $B_1, B_2$ are scalars and $i$ is of rank 1 (see [1, Corrigendum and addendum: Remark 2.4]). Hence for any partition $\eta$ with $\eta_e = 2$ and...
$x \in \mu_{(0)_2}^{-1}(0) \times \mu_{(0)_2}^{-1}(0) \times \cdots \times \mu_{(0)_2}^{-1}(0)$, $\sigma(e, x)$ has nontrivial stabilizer in $\text{Sp}(V)$ by the second statement of Lemma 3.9. Hence it cannot be a regular element. This finishes the proof of Theorem 3.1 (3).

3.4. Proof of Theorem 1.1. The first item (1) of the theorem, except the non-reducedness is immediate from the flatness of $\mu$ as $\mu$ has the equi-dimensional fibres. The non-reducedness comes from the binational description of the irreducible components in (2).

We prove (2). Note first that the sum map $(\Delta_{\eta} \text{Sp}(2)) / \text{Sp}(2)$ onto the image where $\eta = (4^{n_1}, 2^{n_2})$, $4n_1 + 2n_2 = k$. By Theorem 3.1 (2), each irreducible component of $\mu^{-1}(0)$ contains the $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$-quotient of $\mu^{-1}(0) \times S_k \mathbb{C} (\Delta S^n \mathbb{C})_0$ as a Zariski dense locally closed subset.

To describe the above Zariski dense locally closed subset, we consider first $\mu^{-1}(0) \times S_k \mathbb{C} (\Delta S^n \mathbb{C})_0$. By the factorization property (Lemma 3.9), it is isomorphic to an affine open subset of the free GIT quotient

$$\left(\text{Sp}(V) \times \mu_{(4)}^{-1}(0)^{n_1} \times \mu_{(2)}^{-1}(0)^{n_2}\right) / \text{Sp}(2)^{n_1} \times \text{Sp}(1)^{n_2}.$$  

Recall the quotient is defined with respect to the $\text{Sp}(2)^{n_1} \times \text{Sp}(1)^{n_2}$-action $h.(g, x) = (gh^{-1}, h.x)$. Since the actions of $\text{Sp}(V)$ and $\text{Sp}(2)^{n_1} \times \text{Sp}(1)^{n_2}$ on $\text{Sp}(V) \times \mu_{(4)}^{-1}(0)^{n_1} \times \mu_{(2)}^{-1}(0)^{n_2}$ commute, the GIT $\text{Sp}(V)$-quotient of the above affine scheme (3.8) is written as

$$\left(\mu_{(4)}^{-1}(0) / \text{Sp}(2)\right)^{n_1} \times \left(\mu_{(2)}^{-1}(0) / \text{Sp}(1)\right)^{n_2}.$$  

We recollect from [1] the scheme structures of the factors $\mu_{(4)}^{-1}(0) / \text{Sp}(2)$ and $\mu_{(2)}^{-1}(0) / \text{Sp}(1)$. Let $\rho : \text{Hom}(W, V) \to \text{sp}(V)$, $i \mapsto i^*$ where dim $V = 2$. Let $\mathcal{N}$ be the minimal nilpotent $SO(3)$-orbit in $\mathfrak{o}(3)$ and $\overline{\mathcal{N}}$ be its Zariski closure.

**Lemma 3.11.** There are (natural) isomorphisms

$$\mu_{(2)}^{-1}(0) / \text{Sp}(1) \cong A^2 \times \rho^{-1}(0) / \text{Sp}(1) \cong A^2 \times F;$$

$$\mathcal{M}_{1}^{SO(3, \mathbb{R})} \cong A^2 \times \mathcal{N} \cong A^2 \times (A^2 \setminus 0).$$

**Proof.** The first isomorphism is given in [1, Corrigendum and addendum: §3.1]. The second one will be proven in Corollary A.1. The third one is given in [1, Lemma 8.13]. The last one comes from $\overline{\mathcal{N}} \cong \text{Spec } \mathbb{C}[x^2, xy, y^2]$ ([1, Lemma 8.11]).

As a result the affine GIT $\text{Sp}(V)$-quotient $\mu^{-1}(0) \times S_k \mathbb{C} (\Delta S^n \mathbb{C})_0 / \text{Sp}(V)$ is birational to $(A^4)^{n_1} \times (A^2 \times F)^{n_2}$. Now the $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$-quotient $S^{n_1} A^4 \times S^{n_2} (A^2 \times F)$ is birational to the irreducible component of $\mu^{-1}(0) / \text{Sp}(V)$ indexed by $\eta = (4^{n_1}, 2^{n_2})$. This completes the proof of Theorem 1.1.

3.5. Uhlenbeck space. Let us consider the Uhlenbeck space of $\mathcal{M}_{k/4}^{SO(3, \mathbb{R})}$ in the case $k \in 4 \mathbb{Z}$. It has the stratification

$$\bigcup_{0 \leq k' \leq k, k' \in 4 \mathbb{Z}} \mu_{(k')}^{-1}(0)^{\text{reg}} / \text{Sp}(V_{k'}) \times S_{k-k'} \mathbb{A}^2.$$
On the other hand by identification of closed $\text{Sp}(V)$-orbits we have stratification (cf. [1, Theorem 2.6 (1)]):

\[ \mu^{-1}(0)/G = \bigsqcup_{0 \leq k' \leq k, k' \in 4\mathbb{Z}} \mu_{(k')}^{-1}(0)^{\text{reg}}/\text{Sp}(V_{k'}) \times S_{k-k'/2}^{4} A^{2}. \]  

Thus $\mu^{-1}(0)/G$ is different from the Uhlenbeck space unlike the case $\text{SO}(N), N \geq 5$.

From (3.10) one can check that each stratum indexed by $k'$ is birational to the product of symmetric products $S_{k}^{4} A^{4} \times S_{k-k'/2}^{4} A^{2}$. This reconfirms Theorem 1.1 (2) when we consider only the reduced scheme structures. The case when $k \in 2\mathbb{Z} \setminus 4\mathbb{Z}$ is similar.

4. Tensor products of ADHM data

The following is the complete list of pairs of simple compact classical groups having the isomorphic Lie algebras:

\[(K, K') = (\text{USp}(1), \text{SU}(2)), (\text{SU}(2), \text{SO}(3, \mathbb{R})), (\text{USp}(2), \text{SO}(5, \mathbb{R})), (\text{SU}(4), \text{SO}(6, \mathbb{R})).\]

Therefore there are isomorphisms $M_{n}^{K} \cong M_{n}^{K'}$ mapping the associated vector bundles

\[ F \mapsto F, \text{ad} F, (\Lambda^{2} F)_{0}, \Lambda^{2} F \]

respectively. Here $\text{ad} F$ is the trace-free part of $\text{End}(F)$, and $(\Lambda^{2} F)_{0}$ is the kernel of the natural symplectic form $\Lambda^{2} F \rightarrow O$. For a rank 4 vector bundle $F$ with $O \cong \Lambda^{4} F$, the natural orthogonal structure on $\Lambda^{2} F$ is given by wedge product.

In this section we interpret the above isomorphisms as the morphisms between the spaces of ADHM data (Theorem 4.13). Except the obvious case $(\text{USp}(1), \text{SU}(2))$ the isomorphisms involve self-tensor products of vector bundles $F \otimes^{2}$. The ADHM datum of tensor product $F \otimes F'$ in terms of ADHM data of $F, F'$ is called tensor product of ADHM data. But we hope the readers do not confuse this with the usual tensor product as representations of the quiver algebra with respect to the diagonal action.

In §4.1 we recollect the construction of tensor product in [2, §2.8]. In §4.2 we construct the ADHM data of the self-tensor product. In §4.3, using the self-tensor product ADHM data, we construct the symmetric product and exterior product of ADHM data. In §4.4 we construct the above claimed morphisms in terms of ADHM data.

4.1. Recollection on tensor product of ADHM data. First we recollect tensor product of ADHM data in [2, §2.8]. This was considered for the semisimple classical group pair $(\text{USp}(1) \times \text{USp}(1), \text{SO}(4, \mathbb{R}))$.

The prototype construction is for the ordinary ADHM data. We are given two pairs of vector spaces $(V, W)$ and $(V', W')$ as the representation spaces of the ADHM quiver algebra. Let

\[ \widetilde{V} := V \otimes W' \oplus W \otimes V', \quad \widetilde{W} := W \otimes W'. \]
We define a rational map
\[(4.1)\]
\[T: M_{V,W} \times M_{V',W'} \rightarrow M_{V',W'},\]
\[(x, x') = ((B_1, B_2, i, j), (B'_1, B'_2, i', j')) \mapsto \]
\[(\begin{pmatrix} B_1 \otimes \text{Id}_{W'} & 0 \\ 0 & \text{Id}_W \otimes B'_1 \end{pmatrix}, \begin{pmatrix} B_2 \otimes \text{Id}_{W'} & \tilde{B}^{(1,2)} \\ \text{Id}_W \otimes B'_2 & \text{Id}_{W'} \otimes i' \end{pmatrix}, \begin{pmatrix} i \otimes \text{Id}_{W'} \\ j \otimes \text{Id}_{W'} \end{pmatrix})\]
where \(\tilde{B}^{(1,2)}\) and \(\tilde{B}^{(2,1)}\) satisfy the ADHM equations
\[(4.2)\]
\[(B_1 \otimes \text{Id}_{W'})\tilde{B}^{(1,2)} - (\text{Id}_W \otimes B'_1)\tilde{B}^{(1,2)} + i \otimes j' = 0,\]
\[(\text{Id}_W \otimes B'_1)\tilde{B}^{(2,1)} - (B_1 \otimes \text{Id}_{W'})\tilde{B}^{(2,1)} + j \otimes i' = 0.\]
Looking at the first equation, \(\tilde{B}^{(1,2)}\) is uniquely defined if and only if \(B_1, B'_1\) do not have a common eigenvalue. Its matrix elements are rational functions in the matrix elements of \(B_1, B'_1, i, j'\). Similarly the matrix elements of \(\tilde{B}^{(2,1)}\) are rational functions in the ones of \(B_1, B'_1, i', j\) defined over the locus where \(B_1, B'_1\) do not have a common eigenvalue.

**Proposition 4.1.** ([2, Theorem 2.15]) The rational map \(T\) induces the tensor product map \(M_{\text{SU}(N)}^n \times M_{\text{SU}(N')}^{n'} \rightarrow M_{\text{SU}(N+N')}^{n+n'}\).

We extend \(T\) to the locus where \(i \otimes j' = j \otimes i' = 0\) by assigning \(\tilde{B}^{(1,2)} = \tilde{B}^{(2,1)} = 0\). In particular if one of \(V, V', W, W'\) is zero, \(T(x, x')\) is defined. Note that the extended map is not a morphism, but only a set-theoretic map.

We define the dual of \(x\) by
\[x^\vee := (B_1^\vee, B_2^\vee, -j^\vee, i^\vee) \in M_{V^\vee,W^\vee}.
\]
Let \(F\) be the associated monad to \(x\) (= a complex of vector bundles on \(\mathbb{P}^2\) via the monad construction [4, Chap. 2]). Let \(\mathcal{D}^b(\mathbb{P}^2)\) be the bounded derived category of coherent sheaves on \(\mathbb{P}^2\). It is well-known that the derived dual \(F^\vee\) in \(\mathcal{D}^b(\mathbb{P}^2)\) coincides with the monad associated to \(x^\vee\).

The following two lemmas come from direct calculation of the duals and tensor products:

**Lemma 4.2.** \(x\) is stable (resp. costable) if and only if \(x^\vee\) is costable (resp. stable).

**Lemma 4.3.** If \(B_1, B'_1\) have no common eigenvalue,
\[T(x^\vee, x'^\vee) = T(x, x')^\vee.
\]

**Corollary 4.4.** For framed vector bundles \(F, F'\), the ADHM datum of \((F \otimes F')^\vee\) is \(T(x^\vee, x'^\vee)\).

**Proof.** The ADHM datum of \((F \otimes F')^\vee\) is \(T(x, x')^\vee = T(x^\vee, x'^\vee)\) by Lemma 4.3. \(\square\)

We end up with a lemma, which will be not used in the sequel.

**Lemma 4.5.** Suppose \(B_1, B'_1\) have no common eigenvalue. If \(x, x'\) are stable (resp. costable), so is \(T(x, x')\).
Proof. It is enough to prove only the statement for costability thanks to the above two lemmas. We use the notation $T(x, x') = (\tilde{B}_1, \tilde{B}_2, \tilde{i}, \tilde{j})$. Let $K$ be a $\tilde{B}_1, \tilde{B}_2$-invariant subspace in $\text{Ker} \tilde{j}$. By $\tilde{B}_1$-invariance we have decomposition

$$K = (K \cap (V_1 \otimes W_2)) \oplus (K \cap (W_1 \otimes V_2)).$$

For each generalized eigenspace of $K$ with respect to $\tilde{B}_1$ is a subspace of either $V_1 \otimes W_2$ or $W_1 \otimes V_2$ due to the assumption on the eigenvalues.

Now $K \cap (V_1 \otimes W_2)$ is a $B_1 \otimes \text{Id}_{W_2}, B_2 \otimes \text{Id}_{W_2}$-invariant subspace of $\text{Ker}(j \otimes \text{Id}_{W_2})$. We claim that $K \cap (V_1 \otimes W_2) = 0$. Otherwise there is a nonzero $v \in K \cap (V_1 \otimes W_2)$. Since $\text{Ker}(j \otimes \text{Id}_{W_2}) = \text{Ker}(j) \otimes W_2$, we can write $v = \sum l v_{1}^{(l)} \otimes w_{2}^{(l)}$ where $v_{1}^{(l)} \in \text{Ker}(j) \setminus 0$ for each $l$ and $w_{2}^{(l)} \in W_2$ are linearly independent. We fix any $l_0$ among the indices $l$. Since $x$ is costable there exists a 2-variable polynomial $f$ such that $f(B_1, B_2)v_{1}^{(l_0)}$ is not contained in $\text{Ker}(j)$. Thus $f(\tilde{B}_1, \tilde{B}_2)v$ has a summand $(f(B_1, B_2)v_{1}^{(l_0)}) \otimes w_{2}^{(l_0)}$ which does not lie in $\text{Ker}(j \otimes \text{Id}_{W_2})$. This is contradiction which proves the claim.

Similarly we have $K \cap (W_1 \otimes V_2) = 0$. Therefore $K = 0$, which shows $T(x, x')$ is costable. $\square$

4.2. Self-tensor product of ADHM data. Now we deduce the self-tensor product of ADHM data. Thus we need to consider the case when $x = x'$ and the first factor $B_1$ has only multiplicity 1 eigenvalues. The idea to define $T(x, x) \in M_{\tilde{V}, \tilde{W}}$ associated to the self-tensor product framed vector bundle $F^\otimes 2$ is as follows: First we consider the ADHM datum $x_t := (B_1 + t, B_2, i, j)$ where the first factor is translated by $t \in \mathbb{C}$. It is clear that $x_t$ is regular and $T(x_t, x)$ is defined for $t \neq 0$, but not defined for $t = 0$. Let $F_t$ denote the associated framed vector bundle to $x_t$ so that $T(x_t, x)$ corresponds to $F_t \otimes F (t \neq 0)$ by Proposition 4.1. Next we find a family $\Phi(t)$ in $\text{GL}(\tilde{V})$ parametrized by $\mathbb{C}^*$ such that there exists the limit regular ADHM datum $\lim_{t \to 0} \Phi(t), T(x_t, x) \in M_{\tilde{V}, \tilde{W}}$. Our main result Proposition 4.6 asserts that this limit corresponds to $F^\otimes 2$ as framed vector bundles.

We assume $V = \mathbb{C}$ first for simplicity. Thus we have $\tilde{V} = W^\otimes 2$ and $\tilde{W} = W^\otimes 2$. Let $b_1, b_2$ denote the scalars $B_1, B_2$ respectively. Let $f := j(1) \in W$ and $e \in W$ be a nonzero vector with $i(e) = 1$. Note that $i$ is nonzero by stability and that $e, f$ are linearly independent since $i(f) = ij(1) = 0$. We decompose $W = \mathbb{C}<e, f> \oplus W_0$ where $W_0 \subset \text{Ker}(i)$. This gives identification

$$W^\otimes 2 = \mathbb{C}<e \otimes e, f \otimes f, e \otimes f, f \otimes e> \oplus (e \otimes W_0) \oplus (W_0 \otimes e) \oplus (f \otimes W_0) \oplus (W_0 \otimes f) \oplus W_0^\otimes 2.$$

We define a $\mathbb{C}^*$-action $\Phi(t)$ by giving the weight subspaces as

$$(\tilde{V})_{wt-1} = \mathbb{C}<e, -e>, \hspace{1em} (\tilde{V})_{wt_1} = \mathbb{C}<f, -f>, \hspace{1em} (\tilde{V})_{wt_0} = \mathbb{C}<(e, e), (f, f)> \oplus W_0^\otimes 2.$$

We write $T(x_t, x) = (\tilde{B}_1(t), \tilde{B}_2(t), \tilde{i}, \tilde{j})$. Note that

$$\tilde{B}_1(t) = \begin{pmatrix} b_1 + t & 0 \\ 0 & b_1 \end{pmatrix}, \hspace{1em} \tilde{B}_2(t) = \begin{pmatrix} b_2 & t^{-1}i \otimes j \\ -t^{-1}j \otimes i & b_2 \end{pmatrix}.$$
where all the block matrices are endomorphisms in $\text{End}(W)$. A direct calculation shows

$$
\Phi(t).\widetilde{B}_1 - b_1 : \begin{cases}
(e, -e) \mapsto \frac{t}{2}(e, -e) + \frac{t^2}{2}(e, e), & (f, -f) \mapsto \frac{t}{2}(f, -f) + \frac{1}{2}(f, f), \\
(e, e) \mapsto \frac{t}{2}(e, -e) + \frac{1}{2}(e, e), & (f, f) \mapsto \frac{t^2}{2}(f, -f) + \frac{1}{2}(f, f),
\end{cases}
$$

$$
\Phi(t).\widetilde{B}_2 - b_2 : \begin{cases}
(e, -e) \mapsto -(f, f), & (f, -f) \mapsto 0, \\
(e, e) \mapsto (f, -f), & (f, f) \mapsto 0,
\end{cases}
$$

\begin{equation}
(4.3)
\end{equation}

$$
\Phi(t)\widetilde{i} : e \otimes e \mapsto (e, e), \quad f \otimes f \mapsto 0,
$$

$$
\widetilde{j}\Phi(t)^{-1} : (e, -e) \mapsto t(f \otimes e - e \otimes f), \quad (f, -f) \mapsto 0,
$$

$$
(e, e) \mapsto f \otimes e + e \otimes f, \quad (f, f) \mapsto 2f \otimes f.
$$

The images of the other basis elements do not change by the $\Phi(t)$-action. In particular $\Phi(t).\widetilde{B}_2|_{W_0} = 0$. As a result $\lim_{t \to 0} \Phi(t).T(x, x)$ exists. We denote the limit ADHM datum by $((\widetilde{B}_1)_0, (\widetilde{B}_2)_0, \tilde{i}_0, \widetilde{j}_0)$.

In the general case $k = \dim V \geq 1$, we identify $V = \mathbb{C}^k$ using the eigenspace decomposition, hence $\widetilde{V} = (W^\oplus 2)^{\oplus k}$. Now we define a $\mathbb{C}^*$-action $\Phi(t)$ on $\widetilde{V}$ as the diagonal $\mathbb{C}^*$-action on each summand $W^\oplus 2$ of $\widetilde{V}$ as before. More precisely, let $p_i : V \to \mathbb{C}$, $q_i : \mathbb{C} \to V$ be the $i$th projection and the inclusion as the $i$th summand respectively. By the factorization property, $x$ is the $\sigma$-image of the $k$-tuple $(B_{1i}^1, B_{1i}^2, i_1, j_1)$ where $(B_{1i}^1, B_{1i}^2, i_1, j_1) := (p_i B_{1i}^1 q_i, p_i B_{1i}^2 q_i, p_i i_1, j_1)$ an ADHM datum in $M_{\mathbb{C}, W}$. Instead of $e, f$, we use $e_1, e_2, ..., e_k$ and $f_1, f_2, ..., f_k$ in $W$ using $(B_{1i}^1, B_{1i}^2, i_1, j_1)$. Instead of $W_0$, we use $W_0^l \subset \text{Ker}(i_l) \subset W$ complementary to $\mathbb{C}\langle e_l, f_l \rangle$ for each $l$. Then $\Phi(t)$ is set to be the $\mathbb{C}^*$-action on $\widetilde{V}$ with the weight spaces

$$
(\widetilde{V})_{\text{wt} - 1} = \mathbb{C}\langle (e_1, -e_1), (e_2, -e_2), ..., (e_k, -e_k) \rangle,
$$

$$
(\widetilde{V})_{\text{wt} 1} = \mathbb{C}\langle (f_1, -f_1), (f_2, -f_2), ..., (f_k, -f_k) \rangle,
$$

$$
(\widetilde{V})_{\text{wt} 0} = \bigoplus_{l=1}^k \mathbb{C}\langle (e_l, f_l), (f_l, f_l) \rangle \oplus (W_0^l)^{\oplus 2}
$$

Due to the above calculation (4.3) applied to each summand $W^\oplus 2$ of $\widetilde{V}$, we also obtain the well-defined $t \to 0$ limit ADHM datum $T(x, x) := ((\widetilde{B}_1)_0, (\widetilde{B}_2)_0, \tilde{i}_0, \widetilde{j}_0)$.

**Proposition 4.6.** The ADHM datum of the framed vector bundle $F^\oplus 2$ is the above $((\widetilde{B}_1)_0, (\widetilde{B}_2)_0, \tilde{i}_0, \widetilde{j}_0)$.

**Proof.** By (4.3) there is no nonzero $(\widetilde{B}_1)_0, (\widetilde{B}_2)_0$-invariant subspace in

$$
\text{Ker}(\widetilde{j}_0) = \bigoplus_{l=1}^k \mathbb{C}\langle (e_l, -e_l), (f_l, -f_l) \rangle.
$$

Thus $((\widetilde{B}_1)_0, (\widetilde{B}_2)_0, \tilde{i}_0, \widetilde{j}_0)$ is costable.

By (4.3) any $(\widetilde{B}_1)_0, (\widetilde{B}_2)_0$-invariant subspace containing

$$
\text{Im}(\tilde{i}_0) = \bigoplus_{l=1}^k \mathbb{C}\langle (e_l, e_l), (f_l, f_l) \rangle \oplus (W_0^l)^{\oplus 2}
$$

...
should be $\tilde{V} = (W^\otimes 2)^{\oplus k}$. Thus $((\tilde{B}_1)_0, (\tilde{B}_2)_0, \tilde{i}_0, \tilde{j}_0)$ is stable. Hence it is regular and corresponds to a framed vector bundle. Since this framed bundle is isomorphic to a limit of $F_l \otimes F$, it is isomorphic to $F \otimes F$ as framed bundles. \hfill $\square$

In general an ADHM datum in $M_{V,W}$ of a given framed vector bundle is unique up to $GL(V)$-action. And the $C^*$-action $\Phi(t)$ is absorbed in $GL(\tilde{V})$-action.

We recall that if $F$ has a symplectic structure, $V, W$ are orthogonal and symplectic vector spaces respectively and $B_1 = B_1^1, B_2 = B_2^2, j = i^*$. Thus $\tilde{V}, \tilde{W}$ are naturally symplectic and orthogonal vector spaces. We identify $V = C^k$ the standard orthogonal vector space using the eigenspace decomposition. Note that $(e_i, f_l)_W = 1$ since $j = i^*$. We set $W'_0 := C(e_l, f_l)^\perp$ the orthogonal complement in $W$. Then the above $C^*$-action $\Phi(t)$ preserves the symplectic form of $\tilde{V}$. Thus the limit ADHM datum $((\tilde{B}_1)_0, (\tilde{B}_2)_0, \tilde{i}_0, \tilde{j}_0)$ is contained in $N_{\tilde{V}, \tilde{W}}$. Hence Proposition 4.6 can be adapted to the symplectic version:

Proposition 4.7. Suppose further $F$ is a symplectic bundle. The ADHM datum of the framed orthogonal bundle $F^\otimes 2$ is the above $((\tilde{B}_1)_0, (\tilde{B}_2)_0, \tilde{i}_0, \tilde{j}_0)$.

4.3. Symmetric product and exterior product of ADHM data. We find the ADHM datum of the second symmetric product $S^2F$ and the second exterior product $\Lambda^2F$ of a framed vector bundle $F$. We also use the assumption that the first factor $B_1$ of the ADHM datum $x$ corresponding to $F$ has only multiplicity 1 eigenvalues.

We consider mutually complementary subspaces $S^2W$, $\Lambda^2W$ in $W^\otimes 2$ spanned by the vectors $w \otimes w' + w' \otimes w$, $w \otimes w' - w' \otimes w$ respectively. Let

$$V_S := \sum h((\tilde{B}_1)_0, (\tilde{B}_2)_0)\tilde{i}_0(S^2W), \quad V_E := \sum h((\tilde{B}_1)_0, (\tilde{B}_2)_0)\tilde{i}_0(\Lambda^2W)$$

where $h$ runs over the 2-variable polynomials. We denote the involutive and anti-involutive subspaces in $W^\otimes 2$ by

$$\Delta^\pm W := \{(w, \pm w) \in W^\otimes 2\}.$$

We define similar subspaces in $W^\otimes 2$ using the notation $\Delta^\pm$.

Lemma 4.8. $V_S, V_E$ are identified as

$$V_S = \bigoplus_{l=1}^k C(e_l, f_l)^{\otimes 2} \oplus \Delta^+ W^l_0, \quad V_E = \bigoplus_{l=1}^k \Delta^- W^l_0.$$

Proof. We check the first identification. For each $l$ there is decomposition

$$S^2W = C(e_l \otimes e_l, f_l \otimes f_l, e_l \otimes f_l + f_l \otimes e_l) \oplus \{e_l \otimes w + w \otimes e_l | w \in W^l_0\}$$

$$\oplus \{f_l \otimes w + w \otimes f_l | w \in W^l_0\} \oplus S^2W^l_0$$

We denote by $(S^2W)_m$, $m = 1, 2, 3, 4$ the above summands in order. Let $p_l: \tilde{V} = (W^\otimes 2)^{\oplus k} \to W^\otimes 2$ be the $l$th projection.

Via $p_l$, the sum $\sum h((\tilde{B}_1)_0, (\tilde{B}_2)_0)\tilde{i}_0((S^2W)_1)$ projects onto $C(e_l \otimes e_l, f_l \otimes f_l, e_l \otimes f_l, f_l \otimes e_l)$. The sum for $(S^2W)_2$ projects onto $\Delta^+ W^l_0$ since it coincides with the set of $p_l(e_l \otimes$
Remark 4.9. \( V_S = c_2(S^2F) = (\text{rank} F + 2)c_2(F) \) and \( V_E = c_2(\Lambda^2F) = (\text{rank} F - 2)c_2(F) \)

Theorem 4.10. The restrictions of \((\tilde{\mathcal{B}}_1)_0, (\tilde{\mathcal{B}}_2)_0, \tilde{\mathcal{I}}_0, \tilde{\mathcal{J}}_0)\) to \((V_S, S^2W), (V_E, \Lambda^2W)\) are the ADHM data of \(S^2F, \Lambda^2F\) respectively.

Proof. We first need to check the restrictions are well-defined, i.e., \( \tilde{\mathcal{J}}_0(V_S) \subset S^2W, \tilde{\mathcal{J}}_0(V_E) \subset \Lambda^2W \). By (4.3), for any \( l \) we have \( \tilde{j}_0(\mathbb{C}(e_l \otimes f_i))^{\oplus 2} = \mathbb{C}(f_i \otimes e_l + e_l \otimes f_i, f_i \otimes f_i) \). We also have
\[
\tilde{j}_0(\Lambda^+ W_0^l) = \tilde{j}(\Lambda^+ W_0^l) = \{ f_i \otimes w - w \otimes f_i | w \in W_0 \}.
\]
Thus \( \tilde{j}_0(V_S) \subset S^2W \) by Lemma 4.8. Similarly
\[
\tilde{j}_0(V_E) \subset \Lambda^2W.
\]

Now \((\tilde{\mathcal{B}}_1)_0, (\tilde{\mathcal{B}}_2)_0, \tilde{\mathcal{I}}_0, \tilde{\mathcal{J}}_0)\) is the direct sum of the two restrictions to \((V_S, S^2W), (V_E, \Lambda^2W)\) as quiver representations. Since \((\tilde{\mathcal{B}}_1)_0, (\tilde{\mathcal{B}}_2)_0, \tilde{\mathcal{I}}_0, \tilde{\mathcal{J}}_0)\) is a regular element (Proposition 4.6), these restrictions are also regular. Let \( F_S, F_E \) be the corresponding framed vector bundles. We have \( F^{\oplus 2} = F_S \oplus F_E \). On the other hand there is an inclusion \( F_S \subset S^2F \) as framed sheaves, because the frames of \( F_S, S^2F \) coincide and \( F_S \) is the minimal framed subsheaf in \( F^{\oplus 2} \) with such a frame due to Lemma 4.8. Similarly \( F_E \subset \Lambda^2F \). By the decomposition \( F^{\oplus 2} = S^2F \oplus \Lambda^2F \) these inclusions are nothing but \( F_S = S^2F, F_E = \Lambda^2F \). \( \square \)

4.4. Construction of the tensor, symmetric and exterior product morphisms in terms of ADHM data. We construct the morphisms \( F \mapsto \text{ad}F, (\Lambda^2F)_0, \Lambda^2F \) in terms of ADHM data \( x = (B_1, B_2, i, j) \) of \( F \) introduced in the beginning of this section. To be precise we will give the explicit forms of the morphisms only over the locus
\[
\mu^{-1}(0)_{\text{reg}} := \{ x \in \mu^{-1}(0)_{\text{reg}} | B_1 \text{ has only the multiplicity 1 eigenvalues} \}.
\]
These morphisms extend to the morphisms between the framed vector bundles due to the factorization property since the quasi-affine GIT quotient of \( \mu^{-1}(0)_{\text{reg}} \) is Zariski dense open in \( \mathcal{M}_n^K \).
We denote the moment map on \(M_{\tilde{\mathcal{V}}, \tilde{W}}\) or \(N_{\tilde{\mathcal{V}}, \tilde{W}}\) by \(\tilde{\mu}\).

**Proposition 4.11.** (1) The morphism \(\mathcal{M}_{n}^{\SU(N)} \to \mathcal{M}_{2nN}^{\SU(N^2)}, F \mapsto F \otimes^2 \) is induced by the morphism \(\mu^{-1}(0)^0 \to \tilde{\mu}^{-1}(0)^0 / \GL(\tilde{V}), x \mapsto [T(x, x)]\).

(2) The morphism \(\mathcal{M}_{n}^{\USp(N/2)} \to \mathcal{M}_{2nN}^{\SO(N^2, \mathbb{R})}\), \(F \mapsto F \otimes^2 \) is induced by the morphism \(\mu^{-1}(0)^0 \to \tilde{\mu}^{-1}(0)^0 / \Sp(\tilde{V}), x \mapsto [T(x, x)]\).

**Proof.** It suffices to observe that \(\Phi(f)\) is absorbed in \(\GL(\tilde{V})\) or \(\Sp(\tilde{V})\) by construction (Proposition 4.7). \(\square\)

We construct first the morphisms \(F \mapsto S^2 F, \Lambda^2 F\) in terms of ADHM data. We also use the same notation \(\tilde{\mu}\) for the moment map on \(M_{V_S, S^2W}^\circ\) or \(M_{V_E, \Lambda^2W}^\circ\). We fix a triple \((x, V_S, V_E)\) where \(x = (B_1, B_2, i_1, j_1) \in \mu^{-1}(0)^0 \subset M_{V,W}\) and \(V_S, V_E\) are subspaces of \(\tilde{V}\) constructed from \(x\) and choices of \(e_i, f_j\) as before. For each triple \((x, V_S, V_E)\) we choose any \(\phi \in \GL(\tilde{V})\) with \(V_S = \phi(V_S), V_E = \phi(V_E)\). The restrictions of \(\phi, T(x, x)\) to \((V_S, S^2W)\) and \((V_E, \Lambda^2W)\) do not depend on the choice of \(\phi\). This induces morphisms between instanton spaces:

**Proposition 4.12.** (1) The symmetric product morphism \(\mathcal{M}_{n}^{\SU(N)} \to \mathcal{M}_{n'}^{\SU(N')}, F \mapsto S^2 F\) is induced by the morphism \(\mu^{-1}(0)^0 \to \tilde{\mu}^{-1}(0)^0 / \GL(V_S^\circ), x \mapsto [\phi.T(x, x)]_{(V_S^\circ, S^2W)}\) where \(N' = \frac{N(N+1)}{2}\) and \(n' = n(N + 2)\).

(2) The exterior product morphism \(\mathcal{M}_{n}^{\SU(N)} \to \mathcal{M}_{n'}^{\SU(N')}, F \mapsto \Lambda^2 F\) is induced by the morphism \(\mu^{-1}(0)^0 \to \tilde{\mu}^{-1}(0)^0 / \GL(V_E^\circ), x \mapsto [\phi.T(x, x)]_{(V_E^\circ, \Lambda^2W)}\) where \(N' = \frac{N(N-1)}{2}\) and \(n' = n(N - 2)\).

We get back to the construction of morphisms \(F \mapsto \ad F, (\Lambda^2 F)_{0}, \Lambda^2 F\) in terms of ADHM data.

**Theorem 4.13.** (1) The isomorphism \(\mathcal{M}_{n}^{\SU(2)} \cong \mathcal{M}_{n}^{\SO(3, \mathbb{R})}\), \(F \mapsto \ad F\) is given by the symmetric product morphism in Proposition 4.12 (1).

(2) The isomorphism \(\mathcal{M}_{n}^{\USp(2)} \cong \mathcal{M}_{n}^{\SO(5, \mathbb{R})}\) is given by \(x \mapsto \phi.T(x, x)|_{(V_E^\circ, \Ker(\omega))}\) where \(\omega: \Lambda^2 W \to \mathbb{C}\) is the symplectic form on \(W\).

(3) The isomorphism \(\mathcal{M}_{n}^{\SU(4)} \cong \mathcal{M}_{n}^{\SO(6, \mathbb{R})}\) is given by \(x \mapsto \phi.T(x, x)|_{(V_E^\circ, \Lambda^2W)}\).

In the statements (2) and (3) we need to explain additional structures on \(V_E^\circ, \Lambda^2 W, \phi, \) etc. In the item (2), \(V, W\) are orthogonal and symplectic vector spaces respectively. Recall that \(\tilde{V}, \tilde{W}\) have the induced symplectic and orthogonal structures respectively. Recall also that for a given \(x \in \mu^{-1}(0)^0\), further choice of \(e_i\) defines \(V_S, V_E\).

**Lemma 4.14.** \(V_S, V_E\) are symplectic subspaces of \(\tilde{V}\).

**Proof.** We check \(V_E\) is a symplectic subspace of \(\tilde{V}\). We identify \(V = \mathbb{C}^k\) the standard orthogonal vector space and \(\tilde{V} = (W \otimes \mathbb{C}^2)^{\oplus k}\). For each \(l, W_0^l\) is a symplectic subspace in \(W \otimes \mathbb{C}^2\) by the definition. Thus so is \(V_E = \bigoplus_l \Lambda^{-} W_0^l\) in \(\tilde{V}\).
One can check similarly that $V_S$ is a symplectic subspace. \hfill \Box

The orthogonal structure on $\widetilde{W} = W^{\otimes 2}$ is given by the induced isomorphism $W^{\otimes 2} \cong (W^\vee)^{\otimes 2} \cong (W^{\otimes 2})^\vee$. The subspace $\Lambda^2 W$ is given by the 4-form $\Lambda^2 \omega$. $\text{Ker}(\omega)$ is an orthogonal subspace of $\Lambda^4 W$. In (2) we need the restriction $T(x, x)$ to $(V_E, \text{Ker}(\omega))$ is well-defined. This amounts to $\tilde{j}_0(V_E) \subset \text{Ker}(\omega)_x$, which follows immediately from (4.4). We set $\phi$ to be any symplectic isomorphism of $\widetilde{V}$ satisfying $V_S^x = \phi(V_S), V_E^x = \phi(V_E)$.

In the case (3) we give any orthogonal structure on $V$ and then identify $V = C^k$ the standard orthogonal vector space. For each ADHM datum $x \in \mu^{-1}(0)_0$ we use the notation $f_1^x, f_2^x, ..., f_k^x \in W$ for the $j$-images of the standard basis elements of $V$ to emphasize $x$. We choose a generic symplectic structure $\omega^x: \Lambda^2 W \to C$ such that there exist $e_1^x, e_2^x, ..., e_k^x \in W$ satisfying $\omega^x(e_i^x, f_i^x) = 1$. The existence of such $\omega^x$ is left as an exercise. Note that $\omega^x(e_i^x, f_i^x) = 1$ is no more automatic because $j$ is not necessarily $i^*$ with respect to $\omega^x$. We further impose a condition on $\omega^x$: the 4-forms $\Lambda^2 e_i^x$ do not depend on $x$. This condition is satisfied simply by scalar multiplications of $\omega^x$ and $e_i^x$ since $\Lambda^4 W$ is 1-dimensional. Thus $\Lambda^2 W$ attains the orthogonal structure independent of $x$. It is now easy to check that $T(x, x)$ restricts to an element of $N_{V_E, \Lambda^2 W}$. We choose $\phi \in \text{Sp}(\tilde{V})$ with $V_S^x = \phi(V_S), V_E^x = \phi(V_E)$ as before. Hence we have the morphism in the statement $\mu^{-1}(0)_0 \to N_{V_E^x, \Lambda^2 W}/\text{Sp}(\tilde{V})$.

The morphisms in Theorem 4.13 are now well-defined. We are ready to prove the theorem.

\textbf{Proof of Theorem 4.13.} (1) Since $SU(2) = USp(1)$, there is an isomorphism $F \cong F^\vee$ corresponding to the symplectic structure of $F$. Thus we have the isomorphism $\text{End}(F) \cong F^{\otimes 2}$ and this restricts to $\text{ad}F \cong S^2 F$. Hence the map $F \mapsto \text{ad}F$ is given by Proposition 4.12 (1).

(2), (3) The proof of the item (3) is a first half of that of (2), so we prove only (2). By similar arguments in the proofs of Proposition 4.11 (2) and Proposition 4.12 (2), we obtain a morphism $\mu^{-1}(0)_{0^{\text{reg}}} \to \mu^{-1}(0)_{0^{\text{reg}}}/\text{Sp}(V_E)$ which induces $M_{n}^{\text{USp}(2)} \to M_{2n}^{\text{SO}(6, \mathbb{R})}$, $F \mapsto \Lambda^2 F$. The rest follows from the fact that $(\Lambda^2 F)_{0} = \text{Ker}(\Lambda^2 F \to \mathcal{O})$ has the ADHM datum $T(x, x)|_{\tilde{V}_E, \text{Ker}(\omega)}$. To check this fact, we notice that the morphism between the ADHM data induced by $V_E \oplus \Lambda^2 W \overset{0^\oplus\mathbb{C}}{\to} 0 \oplus \mathbb{C}$ becomes the morphism $\Lambda^2 F \to \mathcal{O}$. \hfill \Box

\textbf{APPENDIX A. Character of $\mathbb{C}[\rho^{-1}(0)]^{\text{Sp}(1)}$.}

In this section we compute the Hilbert series of $\rho^{-1}(0)/\text{Sp}(1)$ where $\rho: \text{Hom}(\mathbb{C}^3, \mathbb{C}^2) \to \mathfrak{sp}(1), i \mapsto i i^*$ and $\mathbb{C}^3, \mathbb{C}^2$ are the standard orthogonal and symplectic vector spaces respectively.

Since $\mathfrak{o}(3)$ is spanned by the weight $-1, 0, 1$ vectors with respect to a maximal torus $\text{SO}(3)$, we denote by $x, y, z$ the dual basis elements. Thus we identify $\mathbb{C}[\mathfrak{o}(3)] = \mathbb{C}[x, y, z]$. By the first fundamental theorem of invariant theory $\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)/\text{Sp}(1)$ is $\text{SO}(3)$-equivariantly isomorphic to $\mathfrak{o}(3)$. Hence $\mathbb{C}[\rho^{-1}(0)/\text{Sp}(1)]$ is $\text{SO}(3)$-equivariantly isomorphic to a quotient algebra of $\mathbb{C}[\mathfrak{o}(3)]$. By the above computation of Hilbert series we will deduce the following:
Corollary A.1. There is an $\SO(3)$-equivariant isomorphism

$$\mathbb{C}[\rho^{-1}(0)]^{\Spin(1)} \cong \mathbb{C}[x, y, z]/(x, y, z)^2.$$ 

The proof will appear in §A.2 after identifying the $\SO(3)$-representation $\mathbb{C}[\rho^{-1}(0)]^{\Spin(1)}$ as a torus character.

A.1. Torus character of $\mathbb{C}[\rho^{-1}(0)]^{\Spin(1)}$. For an algebraic group $G$ we denote by $R(G)$ the ring of isomorphism classes of finite dimensional $G$-representations. For a given $G$-representation $V$, its $G$-character is denoted by $\chi_V : G \to \mathbb{C}, g \mapsto \text{tr}(g : V \to V)$. Thus we have character map $\chi_t : R(G) \to \mathbb{C}[G]$. If $G$ is a reductive group, let $T$ be a maximal torus. The composite of the character map $\chi_t : R(G) \to \mathbb{C}[G]$ and the restriction $\mathbb{C}[G] \to \mathbb{C}[T]$ is known to be injective. And its image is the Weyl group-invariant ring $\mathbb{C}[T]^W$. We denote this composite by the same symbol $\chi$ and call torus character.

We want to add some infinitely dimensional representations to $R(G)$, e.g. coordinate ring of a scheme with nonzero dimension. We narrowly focus on the example $\mathbb{C}[\rho^{-1}(0)]$. We set $G = \Spin(1) \times \SO(3)$ from now on. Let $T = T^\Spin(1) \times T^\SO(3)$ where $T^\Spin(1), T^\SO(3)$ are maximal tori of $\Spin(1), \SO(3)$ respectively. Regarding $\mathbb{C}^3, \mathbb{C}^2$ as the vector representations of $\SO(3), \Spin(1)$ respectively and then defining a $C^*$-action with only weight 1, $\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)$ is a $G \times C^*$-representation. So is its coordinate ring $\mathbb{C}[\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)] = \text{Sym}(\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)^\vee)$ the total symmetric product. The torus character $\chi_{\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)} = (1 + t + t^{-1})(z + z^{-1})q$. Here $t, z, q$ are torus characters corresponding to the 1-dimensional representations of $T^\SO(3), T^\Spin(1), C^*$ with weight 1 after identifying $T^\SO(3) \cong C^*, T^\Spin(1) \cong C^*$. Note that $\mathbb{C}[T] = \mathbb{Z}[t^{\pm 1}, z^{\pm 1}, q^{\pm 1}]$.

Each $T$-weight space of $\text{Sym}(\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)^\vee)$ is finite dimensional. Hence $\chi_{\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)}$ is an element in both completed rings

$$\hat{R}(T \times T') := R(T \times T')[[q^{-1}]], \quad \hat{R}(G \times T') := R(G \times T')[[q^{-1}]].$$

Here and hereafter we identified $z, t, q$ with their corresponding representations. In $\hat{R}(T \times T')$ we have

$$\chi_{\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)} = \frac{1}{P(zq^{-1})P(z^{-1}q^{-1})},$$

where $P(x) := (1 - x)(1 - tx)(1 - t^{-1}x)$ and the RHS is understood as the formal series expansion in $q^{-1}$.

We regard the adjoint $\Spin(1)$-representation $\mathfrak{sp}(1)$ as a $G \times T'$-representation with the trivial $\SO(3)$-action and the weight 2 $T'$-action. Thus $\rho$ becomes $G \times T'$-equivariant. By finite dimensionality of $T'$-weight spaces, $\chi_{\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)}$ is an element in both $\hat{R}(T \times T'), \hat{R}(G \times T')$. The pull-backs of $\mathfrak{sp}(1)^\vee$ via $\rho$ generates the defining ideal of $\rho^{-1}(0)$. These can be seen as sections of the trivial vector bundle $V := \mathfrak{sp}(1)^\vee \times \text{Hom}(\mathbb{C}^3, \mathbb{C}^2)$ over $\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)$. Since $\rho^{-1}(0)$ is a complete intersection ([1, Theorem 4.1 (2)]), the Koszul complex $\Lambda^*V$ of $\mathcal{O}_{\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)}$-modules is equal to $\mathcal{O}_{\rho^{-1}(0)}$ as classes of the $G \times T'$-equivariant Grothendieck
group \( R^{G \times T'}(\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)) \). Hence we have

\[
X_{\mathbb{C}[^\rho^{-1}(0)]} = \sum_{l} (-1)^l X_{\mathbb{C}[^\rho^{-1}(0)]^l} = \sum_{l} (-1)^l X_{\mathbb{C}[^\rho^{-1}(0)]^l} \cdot X_{\mathbb{C}[\text{Hom}(\mathbb{C}^3, \mathbb{C}^2)]}
\]

\[
= \frac{(1 - q^{-2}) (1 - z^2 q^{-2}) (1 - z^{-2} q^{-2})}{P(z q^{-1}) P(z^{-1} q^{-1})}
\]

in \( \tilde{R}(T \times T') \) where \( H^0(\mathcal{A}' \mathcal{V}) \) denotes the space of sections of \( \mathcal{A}' \mathcal{V} \) over \( \text{Hom}(\mathbb{C}^3, \mathbb{C}^2) \).

The character of the invariant subspace \( \mathbb{C}[^\rho^{-1}(0)]^{\text{Sp}(1)} \) becomes an element of a completed ring \( \tilde{R}(T_{\text{SO}(3)} \times T') = R(T_{\text{SO}(3)} \times T')[[q^{-1}]] \). It is computed by Weyl’s integration formula as

\[
X_{\mathbb{C}[^\rho^{-1}(0)]^{\text{Sp}(1)}} = \frac{1}{2} \oint_{|z| = 1} \frac{dz}{2\pi \sqrt{-1} z} \cdot N(1) \cdot \frac{(1 - q^{-2}) N(q^{-2})}{P(z q^{-1}) P(z^{-1} q^{-1})}
\]

(the counter-clockwise integral) where \( N(x) := (1 - x z^2)(1 - x z^{-2}) \) the Jacobian of the finite map \( T_{\text{Sp}(1)} \times \text{Sp}(1)/T \rightarrow \text{Sp}(1), (t, gT) \mapsto g^{-1} t g \). We notice that the integrand formal series in \( q^{-1} \) converges in the range \( |q| \gg 1 \). Thus it is also regarded as a rational function in \( |q| \gg 1 \). The denominator of this rational function has the zeros \( z = 0, \frac{1}{q^2}, \frac{1}{q} \) inside \( |z| = 1 \). By direct residue computation we get

(A.1) \[
X_{\mathbb{C}[^\rho^{-1}(0)]^{\text{Sp}(1)}} = 1 + t q^{-2} + q^{-2} + t^{-1} q^{-2}
\]

in \( \tilde{R}(T_{\text{SO}(3)} \times T') \). Since this character is a finite sum, it is also defined in \( R(T_{\text{SO}(3)} \times T') \).

Remark A.2. By the second fundamental theorem of invariant theory, the reduced scheme \( \mathbb{C}[^\rho^{-1}(0)]^{\text{Sp}(1)} \) coincides with the zero in \( \mathfrak{a}(3) \). But in the above we checked \( \rho^{-1}(0)/\text{Sp}(1) \) is not reduced.

A.2. Proof of Corollary A.1. We give the weight 2 \( T' \)-action on \( \mathfrak{a}(3) \). Then we have

\[
X_{\mathbb{C}[\mathfrak{a}(3)]} = \frac{1}{(1 - t q^{-2})(1 - q^{-2})(1 - t^{-1} q^{-2})} \text{ in } \tilde{R}(T_{\text{SO}(3)} \times T')
\]

In the statement of Corollary A.1, the variables \( x, y, z \) can be set to be the bases of the 1-dimensional subrepresentations \( t q^{-2}, q^{-2}, t^{-1} q^{-2} \) respectively.

By (A.1), \( \mathbb{C}[\mathfrak{a}^{-1}(0)]^{\text{Sp}(1)} \) coincides the subspace of \( \mathbb{C}[\mathfrak{a}(3)] \) spanned by \( 1, x, y, z \). On the other hand by the first fundamental theorem of invariant theory, \( \text{Hom}(\mathbb{C}^2, \mathbb{C}^3)/\text{Sp}(1) \) is an \( \text{SO}(3) \times T' \)-invariant closed subscheme of \( \mathfrak{a}(3) \). As a quotient ring of \( \mathbb{C}[\mathfrak{a}(3)] \), \( \mathbb{C}[\text{Hom}(\mathbb{C}^2, \mathbb{C}^3)]^{\text{Sp}(1)} \) should be \( \mathbb{C}[\mathfrak{a}(3)]/(x, y, z)^2 \). This proves the corollary.

References

[1] J. Choy, Moduli spaces of framed symplectic and orthogonal bundles on \( \mathbb{P}^2 \) and the K-theoretic Nekrasov partition functions, J. Geometry Phys. 106 (2016), 284–304, Corrigendum and addendum, Submitted.

[2] ———, Geometry of Uhlenbeck partial compactification of orthogonal instanton spaces and the K-theoretic Nekrasov partition functions, arXiv:1606.00707.

[3] S.K. Donaldson, Instantons and geometric invariant theory, Comm. Math. Phys. 93 (1984), no. 4, 453–460.

[4] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, Univ. Lect. Ser. 18, AMS, 1999.

[5] N. Nekrasov and S. Shadchin, ABCD of instantons, Comm. Math. Phys. 252 (2004), no. 1-3, 359–391; arXiv:math/0404225v2.
[6] D.I. Panyushev, *The Jacobian modules of a representation of a Lie algebra and geometry of commuting varieties*, Compositio Math. 94 (1994), no. 2, 181–199.

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