The Hochschild-Serre property for some $p$-adic analytic group actions

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Abstract

Let $H \subseteq G$ be an inclusion of $p$-adic Lie groups. When $H$ is normal or even subnormal in $G$, the Hochschild-Serre spectral sequence implies that any continuous $G$-module whose $H$-cohomology vanishes in all degrees also has vanishing $G$-cohomology. With an eye towards applications in $p$-adic Hodge theory, we extend this to some cases where $H$ is not subnormal, assuming that the $G$-action is analytic in the sense of Lazard.

1 Introduction

Let $H \subseteq G$ be an inclusion of groups and let $M$ be a $G$-module. If $H$ is normal, then the Hochschild-Serre spectral sequence \cite{5} has the form

$$E_2^{p,q} = H^p(G/H, H^q(H, M)) \implies H^{p+q}(G, M). \quad (1.0.1)$$

(This is sometimes also called the Lyndon spectral sequence in recognition of a similar prior result \cite{9} which did not explicitly exhibit the spectral sequence.) If $H$ is not normal, one can still ask to what extent the $G$-cohomology of $M$ is determined by the $H$-cohomology. In particular, one can ask whether for any morphism $M \to N$ of $G$-modules such that $H^i(H, M) \to H^i(H, N)$ is an isomorphism for all $i \geq 0$, $H^i(G, M) \to H^i(G, N)$ is also an isomorphism for all $i \geq 0$; in this case, we say that the inclusion $H \subseteq G$ of groups has the HS (Hochschild-Serre) property. Thanks to (1.0.1), the HS property holds when $H$ is subnormal in $G$, i.e., there exists a finite sequence $H = H_0 \subset H_1 \subset \cdots \subset H_m = G$ in which each inclusion $H_i \subset H_{i+1}$ is normal. On the other hand, it is not difficult to produce examples of inclusions of finite groups for which the HS property fails; see for instance Example \ref{2.7}.

One can also exhibit an analogue of the Hochschild-Serre spectral sequence for normal inclusions of topological groups, which again implies the HS property for subnormal inclusions; see \cite{4}. The main result of this paper (Theorem \ref{4.1}) is a restricted analogue of the HS property for certain non-subnormal inclusions of $p$-adic Lie groups, which applies only to the category of topological modules which are of characteristic $p$ and analytic in the sense of
Lazard [8]. It is crucial that the cohomology groups of such modules can be computed using either continuous or analytic cochains; this makes it possible to quantify the statement that an analytic group action of a \( p \)-adic Lie group is “approximately abelian.”

We illustrate this theorem with some examples which arise from \( p \)-adic Hodge theory. To be precise, these examples come from upcoming joint work with Liu [7] on generalizations of the Cherbonnier-Colmez theorem on descent of \((\varphi, \Gamma)\)-modules [3], in the style of our new approach to the original theorem of Cherbonnier and Colmez [6].

# 2 The HS property for discrete groups

For context, we begin with some remarks on the HS property for discrete groups.

**Definition 2.1.** For \( G \) a group and \( M \) a \( G \)-module, we say that \( M \) has *totally trivial* \( G \)-cohomology if \( H^i(G, M) = 0 \) for all \( i \geq 0 \). Note that for given \( G, H, \mathcal{C} \), the HS property can be formulated as the statement that any \( M \in \mathcal{C} \) with totally trivial \( H \)-cohomology also has totally trivial \( G \)-cohomology.

**Remark 2.2.** If \( H \subset G \) is a proper inclusion of groups and \( M \) is a \( G \)-module with totally trivial \( G \)-cohomology, \( M \) need not have totally trivial \( H \)-cohomology.

**Proposition 2.3.** Let \( G \) be a finite \( p \)-group and let \( M \) be a \( G \)-module. The following conditions are equivalent.

(a) The \( G \)-module \( M \) has totally trivial \( G \)-cohomology.

(b) The group \( M \) is uniquely \( p \)-divisible (i.e., is a module over \( \mathbb{Z}[p^{-1}] \)) and \( H^0(G, M) = 0 \).

**Proof.** For \( i > 0 \), \( H^i(G, M) \) is a torsion group killed by the order of \( G \) [10, §2.4, Proposition 9]; hence (b) implies (a). Conversely, the \( p \)-torsion subgroup \( M[p] \) of \( M \) has the property that \( H^0(G, M[p]) = M[p] \) injects into \( H^0(G, M) \). Consequently, if \( M \) has totally trivial \( G \)-cohomology, then on one hand multiplication by \( p \) is injective on \( M \); on the other hand, the same is then true for \( pM \) (which is isomorphic to \( M \) as a \( G \)-module) and \( M/pM \) (by the long exact sequence in cohomology), but the latter forces \( M/pM = 0 \). Hence (a) implies (b). 

**Remark 2.4.** Proposition 2.3 implies the HS property for inclusions of finite \( p \)-groups, although this is already clear because such inclusions are always subnormal. An immediate corollary is that if \( H \) is a subgroup of a normal subgroup \( P \) of \( G \) which is a finite \( p \)-group, then \( H \subseteq G \) has the HS property.

**Example 2.5** (Serre). For \( G \) a semisimple algebraic group over \( \mathbb{F}_q \) and \( P \) a \( p \)-Sylow subgroup, the Steinberg representation of \( G \) restricts to a free \( \mathbb{F}_q[P] \)-module and thus has totally trivial \( G \)-cohomology.

Here are some examples to show that the HS property does not always hold. We start with a minimal example.
Example 2.6 (Naumann). Put $G = S_3$, let $H$ be the subgroup generated by a transposition, and take $M = \mathbb{F}_3$ with the action of $G$ being given by the sign character. It is apparent that $M$ has vanishing $H$-cohomology. On the other hand, the groups $H^i(A_3, M)$ are all $\mathbb{F}_3$-vector spaces and are hence $H$-acyclic, so (1.0.1) yields $H^1(S_3, M) = H^1(A_3, M) = \mathbb{F}_3$. Explicitly, a nonzero class is represented by the crossed homomorphism taking one element of order 3 to $+1$ and the other to $-1$, mapping the other elements to 0.

A similar example exists in any odd characteristic $p$ using the dihedral group of order $2p$.

Example 2.7 (Serre). Let $M'$ be a 5-dimensional vector space over $\mathbb{F}_2$ equipped with a nondegenerate quadratic form $q$. The associated bilinear form $b$ has rank 4; let $K$ be its kernel and put $M = M'/K$. The action of $G = SO(M', q) (\cong S_6)$ preserves $K$ and the induced action on $M$ defines an isomorphism $SO(M', q) \cong Sp(M, b) \cong Sp_4(\mathbb{F}_2)$. The exact sequence

$$0 \to K \to M' \to M \to 0$$

of $G$-modules does not split, so $H^1(G, M)$ is nonzero.

Now split $M$ as a direct sum $M_1 \oplus M_2$ of nonisotropic subspaces and put $H_i = SL(M_i)$ and $H = H_1 \times H_2 (\cong S_3 \times S_3)$. As in Example 2.5, $M_1$ has no nonzero $H_1$-invariants and restricts to a free module over $\mathbb{F}_2[P_1]$ for $P_1$ a 2-Sylow subgroup of $H_1$; it follows that $M_1$ has totally trivial $H_1$-cohomology, hence also totally trivial $H$-cohomology by (1.0.1). Similarly, $M_2$ has totally trivial $H$-cohomology, as then does $M$. We conclude that the inclusion $H \subseteq G$ does not have the HS property.

## 3 Analytic group actions

We now introduce the class of group actions to which our main result applies. The basic setup is taken from the work of Lazard [8].

**Hypothesis 3.1.** Throughout [3] let $\Gamma$ be a profinite $p$-analytic group in the sense of [8 III.3.2.2]. For example, we may take $\Gamma$ to be a compact $p$-adic Lie group.

**Definition 3.2.** For $M$ a $\Gamma$-module, let $C^*(\Gamma, M)$ be the complex of inhomogeneous cochains, so that $C^n(\Gamma, M) = \text{Map}(\Gamma^n, M)$ and for $h \in C^n(\Gamma, M)$ and $\gamma_0, \ldots, \gamma_n \in \Gamma$,

$$(dh)(\gamma_0, \ldots, \gamma_n) = \gamma_0 h(\gamma_1, \ldots, \gamma_n) + \sum_{i=1}^n (-1)^i h(\gamma_0, \ldots, \gamma_{i-2}, \gamma_{i-1} \gamma_i, \gamma_{i+1}, \ldots, \gamma_n) + (-1)^{n+1} h(\gamma_0, \ldots, \gamma_{n-1}).$$

For $M$ a topological $\Gamma$-module, let $C^{\text{cont}}_*(\Gamma, M)$ be the subcomplex of $C^*(\Gamma, M)$ consisting of continuous cochains, so that $C^n_{\text{cont}}(\Gamma, M) = \text{Cont}(\Gamma^n, M)$. Let $H^{\text{cont}}_*(\Gamma, M)$ be the cohomology groups of $C^{\text{cont}}_*(\Gamma, M)$, topologized as subquotients for the compact-open topology; for a more intrinsic interpretation of these groups, see [4 Proposition 9.4].
For normal subgroups of $\Gamma$, we again have a Hochschild-Serre spectral sequence.

**Lemma 3.3.** For any closed normal subgroup $\Gamma'$ of $\Gamma$ and any topological $\Gamma$-module $M$, there is a spectral sequence

$$E_2^{p,q} = H^p_{\text{cont}}(\Gamma/\Gamma', H^q_{\text{cont}}(\Gamma', M)) \implies H^{p+q}_{\text{cont}}(\Gamma, M).$$

For our purposes, convergence of the spectral sequence may be interpreted at the level of bare abelian groups, but it also makes sense at the level of topological groups: starting from $E_2$, each stage of the spectral sequence induces a subquotient topology on the subsequent stage, and $H^{p+q}_{\text{cont}}(\Gamma, M)$ admits a filtration by subgroups (not guaranteed to be closed) whose subquotients are homeomorphic to the corresponding terms of $E_\infty$.

**Proof.** Since $\Gamma$ and $\Gamma'$ are profinite, the surjection of topological spaces $\Gamma \to \Gamma/\Gamma'$ admits a continuous section. Consequently, the explicit construction of the spectral sequence for finite groups given in [5, §2] carries over without change. For further discussion, see [8, §V.3.2]. □

**Definition 3.4.** Let $A$ be the completion of the group ring $\mathbb{Z}_p[\Gamma]$ with respect to the $p$-augmentation ideal $\ker(\mathbb{Z}_p[\Gamma] \to \mathbb{F}_p)$. Put $I = \ker(A \to \mathbb{F}_p)$; we view $A$ as a filtered ring using the $I$-adic filtration. We also define the associated valuation: for $x \in A$, let $w(A; x)$ be the supremum of those nonnegative integers $i$ for which $x \in I^i$.

**Definition 3.5.** An analytic $\Gamma$-module is a left $A$-module $M$ complete with respect to a valuation $w(M; \bullet)$ for which there exist $a > 0, c \in \mathbb{R}$ such that

$$w(M; xy) \geq aw(A; x) + w(M; y) + c \quad (x \in A, y \in M).$$

Equivalently, there exist an open subgroup $\Gamma_0$ of $\Gamma$ and a constant $a > 0$ such that

$$w(M; (\gamma - 1)y) \geq w(M; y) + a \quad (\gamma \in \Gamma_0, y \in M).$$

**Example 3.6.** Let $M$ be a torsion-free $\mathbb{Z}_p$-module of finite rank on which $\Gamma$ acts continuously. Then $M$ is an analytic $A$-module for the valuation defined by any basis; see [8, Proposition V.2.3.6.1].

**Definition 3.7.** Let $M$ be a continuous $\Gamma$-module. A cochain $\Gamma^i \to M$ is analytic if for every homeomorphism between an open subspace $U$ of $\Gamma^i$ and an open subspace $V$ of $\mathbb{Z}_p^n$ for some nonnegative integer $n$, the induced function $V \to M$ is locally analytic (i.e., locally represented by a convergent power series expansion). Let $C^i_{\text{an}}(\Gamma, M) \subseteq C^i_{\text{cont}}(\Gamma, M)$ be the space of analytic cochains.

Suppose now that $M$ is an analytic $\Gamma$-module. Then by the proof of [8, Proposition V.2.3.6.3], $C^i_{\text{an}}(\Gamma, M)$ is a subcomplex of $C^i_{\text{cont}}(\Gamma, M)$; we thus obtain analytic cohomology groups $H^i_{\text{an}}(\Gamma, M)$ and natural homomorphisms $H^i_{\text{an}}(\Gamma, M) \to H^i_{\text{cont}}(\Gamma, M)$.

**Theorem 3.8** (Lazard). If $M$ is an analytic $\Gamma$-module, then the inclusion $C^i_{\text{an}}(\Gamma, M) \to C^i_{\text{cont}}(\Gamma, M)$ is a quasi-isomorphism. That is, the continuous cohomology of $M$ can be computed using analytic cochains.
Proof. In the context of Example 3.6, this is the statement of [8, Théorème V.2.3.10]. However, the proof of this statement only uses the stronger hypothesis in the proof of [8, Proposition V.2.3.6.1], which we have built into the definition of an analytic Γ-module. The remainder of the proof of [8, Théorème V.2.3.10] thus carries over unchanged.

Remark 3.9. In considering Theorem 3.8, it may help to consider the first the case of 1-cocycles: every 1-cocycle is cohomologous to a crossed homomorphism, which is analytic because of how it is determined by its action on topological generators.

4 The HS property for some analytic group actions

We now establish our main result, which gives an analogue of the HS property for certain analytic group actions.

Theorem 4.1. Let Γ be a profinite p-analytic group. Let H be a pro-p procyclic subgroup of Γ (i.e., it is isomorphic to $\mathbb{Z}_p$). Let M be a analytic Γ-module which is a Banach space over some nonarchimedean field of characteristic $p$ with a nontrivial absolute value. (It is not necessary to require Γ to act on this field.) If $H^i_{\text{cont}}(H, M) = 0$ for all $i \geq 0$, then $H^i_{\text{cont}}(\Gamma, M) = 0$ for all $i \geq 0$.

Proof. Let $\eta$ be a topological generator of H. The vanishing of $H^0_{\text{cont}}(H, M)$ and $H^1_{\text{cont}}(H, M)$ means that $\eta - 1$ is a bijection on M; by the Banach open mapping theorem [2, §I.3.3, Théorème 1], $\eta - 1$ admits a bounded inverse. Since M is of characteristic $p$, for each nonnegative integer n the actions of $\eta^{p^n} - 1$ and $(\eta - 1)^{p^n}$ coincide; hence $\eta^{p^n} - 1$ also has a bounded inverse.

We next make some reductions. Recall that M has been assumed to be an analytic Γ-module. We may thus choose a pro-$p$-subgroup $\Gamma_0$ of $\Gamma$ on which the logarithm map defines a bijection with $\mathbb{Z}_p^h$ for some $h$, such that for some $c_0 \in (0, 1)$ we have

$$|(\gamma - 1)y| \leq c_0 |y| \quad (\gamma \in \Gamma_0, y \in M).$$

By the previous paragraph, we may also assume $\eta \in \Gamma_0$. Using Lemma 3.3 we may also assume $\Gamma = \Gamma_0$. By Theorem 3.8 to check that $H^i_{\text{cont}}(\Gamma, M) = 0$ it suffices to check that $H^i_{\text{an}}(\Gamma, M) = 0$.

Let $\Gamma_n$ be the subgroup of $\Gamma_0$ which is the image of $p^n\mathbb{Z}_p^h$ under the exponential map. For $c_0$ as above, we have

$$|(\gamma - 1)y| \leq c_0^{p^n} |y| \quad (n \geq 0, \gamma \in \Gamma_n, y \in M).$$

(4.1.1)

For $c \in (0, c_0]$, we say that a cochain $f : \Gamma^n \to M$ is $c$-analytic if there exists $d > 0$ such that

$$|f(\gamma_1, \ldots, \gamma_n) - f(\gamma_1\eta_1, \ldots, \gamma_n\eta_n)| \leq d^{p^{i_1} + \cdots + i_n} \quad (\gamma_1, \ldots, \gamma_n \in \Gamma; i_1, \ldots, i_n \geq 0; \eta_j \in \Gamma_i).$$

(4.1.2)

Using the fact that M is of characteristic p, one may check that any analytic cochain in the sense of Lazard is $c$-analytic for some $c > 0$. This means that $C^n_{\text{an}}(\Gamma, M)$ can be written as

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the union of the subspaces $C^m_{\text{an},c}(\Gamma, M)$ of $c$-analytic cochains over all $c \in (0, c_0]$. Moreover, using (4.1.1) we see that $C^m_{\text{an},c}(\Gamma, M)$ is a subcomplex of $C^m_{\text{an}}(\Gamma, M)$, so to prove the theorem it suffices to check the acyclicity of each $C^m_{\text{an},c}(\Gamma, M)$.

From now on, fix $c \in (0, c_0]$. We define a norm on $C^m_{\text{an},c}(\Gamma, M)$ assigning to each cochain $f$ the minimum $d \geq 0$ for which (4.1.2) holds; note that $C^m_{\text{an},c}(\Gamma, M)$ is complete with respect to this norm. For $m \geq 0$, we define a chain homotopy $h_m$ on $C^m_{\text{an},c}(\Gamma, M)$ by the following formula: for $f_n \in C^m_{\text{an},c}(\Gamma, M)$,

$$h_m(f_n)(\gamma_1, \ldots, \gamma_{n-1}) = (\eta^{p^m} - 1)^{-1} \sum_{i=1}^{n} (-1)^{i-1} f_n(\gamma_1, \ldots, \gamma_{i-1}, \eta^{p^m}, \gamma_i, \ldots, \gamma_{n-1}).$$

We then compute that

$$(d \circ h_m + h_m \circ d - 1)(f_n)(\gamma_1, \ldots, \gamma_n)$$

$$= (\gamma_1(\eta^{p^m} - 1)^{-1} - (\eta^{p^m} - 1)^{-1}\gamma_1) \sum_{i=1}^{n} (-1)^{i-1} f_n(\gamma_2, \ldots, \gamma_i, \eta^{p^m}, \gamma_{i+1}, \ldots, \gamma_n)$$

$$- \sum_{i=1}^{n} (\eta^{p^m} - 1)^{-1}(f_n(\gamma_1, \ldots, \gamma_{i-1}, \eta^{p^m} \gamma_i, \gamma_{i+1}, \ldots, \gamma_n) - f_n(\gamma_1, \ldots, \gamma_{i-1}, \gamma_i \eta^{p^m}, \gamma_{i+1}, \ldots, \gamma_n)).$$

To bound the right side of this equality, write

$$\gamma(\eta^{p^m} - 1)^{-1} - (\eta^{p^m} - 1)^{-1}\gamma = (\eta^{p^m} - 1)^{-1}(\eta^{p^m} \gamma)(1 - \gamma^{-1} \eta^{p^m} \gamma \eta^{p^m})(\eta^{p^m} - 1)^{-1}.$$

Then note that if $\gamma_i \in \Gamma_j$, then $\eta^{p^m} \gamma_i$ and $\gamma_i \eta^{p^m}$ differ by an element of $\Gamma_{m+j+1}$. Finally, let $t > 0$ be the operator norm of the inverse of $\eta - 1$ on $M$; then $\eta^{p^m} - 1$ has an inverse of operator norm at most $tp^m$. Fix $\epsilon \in (0, 1)$; for $m$ sufficiently large, we have

$$\max \{t^{2p^m} \epsilon^{2m}, t^{p^m} \epsilon^{p^m+1} \} < 1 - \epsilon.$$

For such $m$, the map $d \circ h_m + h_m \circ d - 1$ acts on $C^m_{\text{an},c}(\Gamma, M)$ with operator norm at most $1 - \epsilon$; consequently, there is an invertible map on $C^m_{\text{an},c}(\Gamma, M)$ which is homotopic to zero. This proves the claim.

Note that Example 2.6 and Example 2.7 show that Theorem 4.1 cannot remain true if we drop the condition that $H$ be pro-$p$. However, it does not resolve the following question.

**Question 4.2.** Does Theorem 4.1 remain true if we drop the condition that $H$ be pro-cyclic? This does not follow from Theorem 4.1 because the hypothesis of the theorem is not preserved upon replacing $H$ with a subgroup (Remark 2.2).

5 **Examples from $p$-adic Hodge theory**

We conclude with some examples of Theorem 4.1 which are germane to $p$-adic Hodge theory.
Definition 5.1. For any ring $R$ of characteristic $p$, let $\varphi : R \to R$ denote the $p$-power Frobenius endomorphism.

Remark 5.2. We will frequently use the “Leibniz rule” for group actions, in the form of the identity

$$\gamma - 1)(\overline{x}y) = (\gamma - 1)(\overline{x})\overline{y} + \gamma(\overline{x})(\gamma - 1)(\overline{y}).$$  \hspace{1cm} (5.2.1)

For instance, this holds if $\gamma$ acts on a ring containing $\overline{x}$ and $\overline{y}$, or if it acts compatibly on a ring containing $\overline{x}$ and a module containing $\overline{y}$ (or vice versa).

Proposition 5.3. Let $F$ be a complete discretely valued field of characteristic $p$. Let $\Gamma$ be an affinoid algebra over $F$. Let $M$ be a finitely generated $R$-module. Let $\Gamma$ be a profinite $p$-analytic group acting compatibly on $F, R, M$, and suppose that there is an open subgroup of $\Gamma$ fixing the residue field of $F$. Then $M$ is an analytic $\Gamma$-module.

Proof. Let $\sigma_F$ be the valuation subring of $F$. Let $\overline{\pi}$ be a uniformizer of $\sigma_F$. By hypothesis, there exists an open subgroup $\Gamma_0$ on $\Gamma$ fixing $\sigma_F/(\overline{\pi})$. Then for any $\gamma \in \Gamma_0$ and any positive integer $n$, $\gamma^p$ fixes $\sigma_F/(\overline{\pi}^{n+1})$, so $F$ itself is an analytic $\Gamma$-module.

By definition, $R$ is a quotient of the Tate algebra $F\{T_1, \ldots, T_n\}$ for some nonnegative integer $n$. Equip $R$ with the quotient norm for some such presentation. Let $r_i \in R$ be the image of $T_i$. Since the action of $\Gamma$ on $R$ is continuous, for any $c > 0$ there exists an open subgroup $\Gamma_0$ of $\Gamma$ such that

$$| (\gamma - 1)(f) | \leq \frac{c}{2} |f|, \quad | (\gamma - 1)(r_i) | \leq \frac{c}{2}$$

for all $\gamma \in \Gamma_0$, $i \in \{1, \ldots, n\}$, $f \in F$. Then for any $x \in R$, we can lift it to some $y = \sum_{i_1, \ldots, i_n=0}^{\infty} y_{i_1, \ldots, i_n} T_1^{i_1} \cdots T_n^{i_n} \in F\{T_1, \ldots, T_n\}$ with $|y| \leq 2|x|$, and then observe that

$$| (\gamma - 1)(x) | \leq \max\left\{ |(\gamma - 1)(y_{i_1, \ldots, i_n} T_1^{i_1} \cdots T_n^{i_n}) | : i_1, \ldots, i_n \geq 0 \right\}$$

$$\leq \frac{c}{2} \max\left\{ |y_{i_1, \ldots, i_n} | : i_1, \ldots, i_n \geq 0 \right\} \quad \text{(by (5.2.1))}$$

$$= \frac{c}{2} |y| \leq c |x|.$$  

It follows that the action of $\Gamma$ on $R$ is analytic.

Since $R$ is noetherian, $M$ may be viewed as a finite Banach module over $R$ by [1, Proposition 3.7.3/3, Proposition 6.1.1/3]. By choosing topological generators for $M$ as an $R$-module, we may repeat the argument of the previous paragraph to deduce that the action of $\Gamma$ on $M$ is analytic. \hfill \Box

Example 5.4. The action of $\Gamma = \mathbb{Z}_p^\times$ on $F = \mathbb{F}_p((\overline{\pi}))$ via the substitution $\pi \mapsto (1 + \pi)^{\gamma} - 1$ is analytic. By contrast, the induced action on the completion of the perfect closure of $F$ is continuous but not analytic.

Now take $R = F$ and $M = \varphi^{-1}(R)/R$. By Proposition 5.3, the action of $\Gamma$ on $M$ is analytic.
Put $\gamma = 1 + p^2 \in \Gamma$; this element generates the pro-$p$ procyclic subgroup $H = 1 + p^2 \mathbb{Z}_p$ of $\Gamma$. As an $H$-module, $M$ splits as a direct sum $\bigoplus_{j=1}^{d-1}(1 + \pi)^{j/p}F$. Choose $j \in \{1, \ldots, p-1\}$ and put $\overline{y} = (1 + \pi)^{j/p}$. We have
\begin{equation}
(\gamma - 1)(\pi) = (\gamma - 1)(1 + \pi) = ((1 + \pi)^p - 1)(1 + \pi).
\end{equation}
Thus on one hand,
\begin{equation}
|\gamma - 1)(\pi)| \leq p^2 |\pi| (\pi \in F);
\end{equation}
on the other hand,
\begin{equation}
|\gamma - 1)(\pi)| = |\pi|^p |\overline{y}|
\end{equation}
and by (5.2.1), we see that for all $\pi \in \overline{y}F$ we have
\begin{equation}
|\gamma - 1)(\pi)| = |\pi|^p |\pi|.
\end{equation}
In particular, $\gamma - 1$ is bijective on $\overline{y}F$ for each $j$, so $H^i_{\text{cont}}(H, M) = 0$ for all $i \geq 0$. In this example, $H$ is normal in $\Gamma$, so we may invoke Lemma 3.3 to deduce that $H^i_{\text{cont}}(\Gamma, M) = 0$ for all $i \geq 0$. This calculation plays an essential role in the proof of the Cherbonnier-Colmez theorem described in [7].

This example generalizes as follows.

**Example 5.5.** Put $F = \mathbb{F}_p((\pi))$ and $R = F\{\overline{t}_1, \ldots, \overline{t}_d\}$ for some $d \geq 0$. The ring $R$ admits a continuous action of $\Gamma = \mathbb{Z}_p^\times \triangleright \mathbb{Z}_p^d$ in which $\gamma \in \mathbb{Z}_p^\times$ acts as in Example 5.4 fixing $\mathbb{Z}_p^d$, while for $j = 1, \ldots, d$ an element $\gamma_j$ in the $j$-th copy of $\mathbb{Z}_p^\times$ sends $\overline{t}_j$ to $(1 + \pi)^{j/p}\overline{t}_j$ and fixes $\pi$ and $\overline{t}_k$ for $k \neq j$. Put $M = \overline{\pi}^{-1}(R)/R$. By Proposition 5.3, the actions of $\Gamma$ on $F, R, M$ are analytic.

Put $\Gamma_0 = (1 + p^2 \mathbb{Z}_p) \triangleright p\mathbb{Z}_p^d$. We then have a decomposition
\begin{equation}
M \cong \bigoplus (1 + \pi)^{e_0/p}\overline{t}_1^{e_1/p} \cdots \overline{t}_d^{e_d/p} R
\end{equation}
of $R$-modules and $\Gamma_0$-modules, in which $(e_0, \ldots, e_d)$ runs over $\{0, \ldots, p-1\}^{d+1} \setminus \{(0, \ldots, 0)\}$.

Choose a tuple $(e_0, \ldots, e_d) \neq (0, \ldots, 0)$ and put $\overline{y} = (1 + \pi)^{e_0/p}\overline{t}_1^{e_1/p} \cdots \overline{t}_d^{e_d/p}$. Suppose first that $e_j \neq 0$ for some $j > 0$. Let $\gamma$ be the canonical generator of the $j$-th copy of $p\mathbb{Z}_p^d$. Then
\begin{equation}
|\gamma - 1)(\pi)| = |\pi|^p |\overline{y}|
\end{equation}
on the other hand,
\begin{equation}
|\gamma - 1)(\pi)| \leq p^2 |\pi| (\pi \in R),
\end{equation}
so using (5.2.1) again we see that $\gamma - 1$ acts invertibly on $\overline{y}R$. By Lemma 3.3 we have $H^i_{\text{cont}}(\Gamma_0, \overline{y}R) = 0$ for all $i \geq 0$.

Suppose next that $e_0 \neq 0$ but $e_1 = \cdots = e_d = 0$. Put $\gamma = 1 + p^2 \in \mathbb{Z}_p^\times$. As in Example 5.4 we see that $\gamma - 1$ acts invertibly on $\overline{y}R$. Since $\mathbb{Z}_p^\times$ is not normal in $\Gamma$, we must now apply Theorem 4.1 instead of Lemma 3.3 to deduce that $H^i_{\text{cont}}(\Gamma_0, \overline{y}R) = 0$ for all $i \geq 0$.

Putting everything together, we deduce that $H^i_{\text{cont}}(\Gamma_0, M) = 0$ for all $i \geq 0$. By Lemma 3.3 once more, we see that $H^i_{\text{cont}}(\Gamma, M) = 0$ for all $i \geq 0$. This calculation plays an essential role in a generalization of the Cherbonnier-Colmez theorem described in [7].
Remark 5.6. Another class of examples to be considered in [7], based on Lubin-Tate towers, yields cases in which $\Gamma = \text{GL}_d(\mathbb{Z}_p)$ and the vanishing of cohomology can again be checked using Theorem 4.1.

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