Entanglement renormalization for chiral topological phases

Zhi Li and Roger S. K. Mong

Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, PA, 15260, United States

We considered the question of applying the multiscale entanglement renormalization ansatz (MERA) to describe chiral topological phases. We defined a functional for each layer in the MERA, and rigorously proved its monotonicity with respect to adjacent layers, and the existence of a lower bound for non-trivial states. Using this theorem, we show the number of orbitals per cell (which roughly corresponds to the bond dimension) should grow with the height. Conversely, if we restrict the bond dimensions to be constant, then there is an upper bound of the height. Specifically, we established a No-Go theorem stating that we won’t approach a renormalization fixed point in this case.

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I. INTRODUCTION

Renormalization group (RG) is one of the most important concepts in condensed matter physics for studying long-distance behaviors and topological features. In real space, RG procedures by blocking several sites into one effective site, accompanied by a block-decimation—a reduction in the local degrees of freedom per site such that it does not increase exponentially with renormalization steps.

Entanglement renormalization [1] provides a concrete realization of a real space RG for quantum states. Crucial to entanglement renormalization is the application of dis-entanglers before each coarse-graining step, removing the short-ranged entanglement which then allows the local Hilbert space to decrease. Entanglement renormalization has been employed in many systems, to critical phenomena [2–5], topological ordered phases [6, 7], and quantum fields [8]. Applied to a typical (non-critical) state, this RG procedure yields a fixed-point wavefunction—a state with zero effective correlation length. These zero-correlation length states have the property that any connected correlation function is exactly zero beyond some finite distance. These fixed-point wavefunctions are often the “model wavefunctions” for the corresponding topological phase [9–11].

The multiscale entanglement renormalization ansatz (MERA) [12] is a tensor network description of the entanglement RG procedure. By keeping track of the distanglers and decimations at each RG step, a MERA network can be reversed to recover the original quantum state from a coarse-grained one. In other words, a MERA—considered as a quantum circuit—can be used to recover the short-distance physics from the long-distance physics.

In this paper, we investigate the possibility to use a 2D MERA to describe chiral topological states. We show that there are no IR fixed-points to chiral Chern insulators on the 2D lattice; any Chern insulator on a lattice with local dimension $D$ must admit a finite correlation length $\xi$, and we argue that there must be a tradeoff between $D$ and $\xi$.

Specifically, we consider a fermionic Gaussian MERA along with an IR wavefunction for a Chern insulator. We define a functional $B$ for each layer in the MERA which captures its correlation length, and rigorously proved that it obeys monotonicity with respect to adjacent layers. In addition, we prove the existence of a lower bound for $B$ when the Chern number is nonzero, and that such bound is a decreasing function of the bond dimension.

Our results can be interpreted as follows: Consider a wavefunction for the ground state of a Chern insulator $\psi_0$, undergoing a series of entanglement renormalization steps to generate coarse-grained wavefunctions $\psi_1, \psi_2, \ldots$. Naturally, as in any RG procedure, we expect the correlation length $\xi(\psi_n)$ to decrease exponentially with the number of renormalization steps $n$. Our results imply that we need more orbitals per cell as we continue the renormalization process, or (2) the Chern number must change for some large $n$. The former case imply that the bond dimension of a MERA must grow with the number of layers, while the latter scenario imply that RG procedure has failed to capture the topological properties of the state.

This article is organized as follows. In Sec. II, we give a short review of MERA and define the notation used in the paper. In Sec. III, we state the main theorem of this article and discuss its physical implications. Then in Sec. IV (and App. A for details), we prove the theorem. Finally, in Sec. V, we give some discussions and outlooks.

II. ENTANGLEMENT RENORMALIZATION AND MERA

In this section we briefly describe the entanglement renormalization and multi-scale entanglement renormalization ansatz (MERA). While our ultimate goal will use a 2D MERA network, here we review 1D MERA for simplicity.

We view entanglement renormalization as a process which takes a short-distance (UV) description of an object to a long-distance (IR) description. For this work, we want to restrict to entanglement renormalization processes that are reversible, in the sense that the UV limit can be recovered from the IR. A MERA is a prescri-
tion, specifically a tensor network, which allows the UV physics (e.g., correlation functions) of the wavefunction to be reconstructed from the IR physics (i.e., symmetry breaking, topological order).

In the ordinary real space renormalization, we simply group several sites into one effective site, resulting in a tree tensor network (TTN). Taking Fig. 1, we get a binary TTN if we ignore the blue rectangular blocks. Here, the coarse-graining process is represented by the green triangles, called isometries, denoted by \( W \). Each layer represents a physical degree of freedom (i.e., a spin on a lattice site). The layers (labelled by \( n \), counted from below, as shown in the picture) represents intermediate steps of the RG process. Regarded as a quantum circuit, each green triangles enlarges the Hilbert space, and describes an isometric embedding from layer \( n+1 \) to layer \( n \).

In general, one need more and more “local degrees of freedom” (i.e., the local Hilbert space grows with each iteration) to compensate the coarse-graining due to the entanglement structure, see Ref. 13 for an argument using the entanglement entropy. The way to fix this problem is to apply “disentanglers” between coarse-graining steps to reduce the cross-site entanglement. They are simply some unitary transformations among adjacent sites, denoted by \( U \), represented by the blue rectangular blocks in Fig. 1. The resulting tensor network is called the multiscale entanglement renormalization ansatz (MERA).

To maintain translational invariance, we will assume the disentanglers and isometries within one layer are the same (but the may differ from layer to layer). Then the states in all layers are translational invariant if the state in at least one layer is translational invariant (with different periods in general). Formally speaking, a (translational invariant) MERA with \( L \) layers are specified by the following data

- isometries \( W_n \),
- disentanglers \( U_n \),
- bond dimensions \( D_n \),
- top level wavefunction \( \psi_L \).

Note that we only label those layers below disentanglers, as shown in Fig. 1. The bond dimension \( D \) referred to here is the non-interacting one, which is equal to the number of orbitals in the site. The conventional (interacting) bond dimension for a tensor network is the dimension of the local Hilbert space, which is equal to \( 2^D \) (if the physical degree freedom in a site is a qubit).

The generalization in higher dimension is evident [14, 15]. Note that even in 1D, we can have different type of MERA: for example, we may construct a ternary MERA where each isometry has three legs [16]. In 2D or more, the choices of isometries and disentanglers are more diverse.

III. MAIN THEOREM: STATEMENT AND IMPLICATIONS

We would like to see what will happen if we want to apply MERA to describe chiral states (for example, a Chern insulator with Chern number \( c \neq 0 \)).

Consider a Chern insulator living on a 2D lattice. In the following, we will call the minimal geometrical translational invariant unit as a site. The sites must form a \( \mathbb{Z}^2 \) lattice. There may be additional degrees of freedom per site (such as sublattice structure, orbitals, spin), which we collectively refer to as orbitals. The total number of orbitals per site is what we call the bond dimension \( D \), so there is a vector of annihilation operators for each site \( x: \phi_x = (\phi_{x1}, \ldots, \phi_{xD}) \).

In this article, we only consider translational invariant states. We will call the minimal translational invariant unit for a state as a cell, denoted by \( C \). In general, a cell may contain multiple sites.

$$\text{orbitals} \subseteq \text{sites} \subseteq C. \quad (1)$$

As usual, one define the correlation matrix \( P_{x,y} = \langle \phi_x \phi_y \rangle \) for each layer, where \( x \) and \( y \) label sites. For a non-interacting fermionic system, the \( P \) matrix is a projector onto filled bands, and encodes all the information of the state, including its topological property.

We define a functional \( B \) for each layer as:

$$B = \frac{1}{|C|} \sum_{y \in C} \sum_{x \in \mathbb{Z}^2} a_{x-y} ||P_{x,y}||^2. \quad (2)$$

Here \( |C| \) is the size of the unit cell (the number of sites in \( C \)), \( \{ a_x \} \) are nonnegative constants to be specified below, \( ||\cdot|| \) is the Hilbert-Schmidt norm. For gapped states, \( P_{x,y} \) decays exponentially (or faster) with respect to \( |x-y| \), so we demand \( a_x \) to be asymptotic polynomial to guarantee the convergence. The factor \( 1/|C| \) makes \( B \) independent of the choice of the unit cell. It is appropriate to think of \( B \) as a proxy for the correlation length.
**Theorem** For each number \( s > 2 \), there exist a constant \( A > 1 \) and a function \( a_x \) and such that \( a_x \rightarrow |x|^s \) as \( x \rightarrow \infty \) and that the functional \( B \) satisfies the following properties:

1. (monotonicity) \( \forall l \), we have \( B^{(n)} \geq AB^{(n+1)} \). Here, \( B^{(n)} \) represents the value of \( B \) for \( n+1 \) layer.

2. (lower bound) If Chern number \( c \neq 0 \), then \( B \) has a strictly positive lower bound \( \epsilon \). The bound \( \epsilon \) will depends on the Chern number \( c \) and the number of orbitals per cell \( N = D|C| \). Note that although \( B \) does not dependent on how we identify the unit cell, \( N \) does. The strongest lower bound is given by the minimal unit cell, as we will see.

The choice of \( a_x \) is as follows: we pick a finite region \( F_{s,A} \subset \mathbb{Z}^2 \), which includes the origin, then define

\[
a_x = \begin{cases} 0, & x \in F_{s,A} \\ |x|^s, & x \notin F_{s,A} . \end{cases}
\]

Before proving the theorem, we discuss its interpretations and implications.

Assume there is a MERA (finite-layer or infinite-layer) generating a given chiral state. From monotonicity, \( B^{(n)} \leq A^{-1}B^{(n-1)} \leq \cdots \leq A^{-n}B^{(0)} \) for all \( n \). The Chern number, denoted by \( c \), must be the same for each layer [17]. So we have

\[
A^{-n}B^{(0)} \geq \epsilon(c, N_n).
\]

\( \epsilon \) is a decreasing function of \( N \) (because by definition it’s a lower bound and we can embed a small cell into a larger one by adding empty bands). Thus, the above inequality gives us a lower bound of \( N_n \):

\[
N_n \geq \epsilon^{-1}(c, A^{-n}B^{(0)}),
\]

where \( \epsilon^{-1}(c, \cdot) \) is the inverse function of \( \epsilon(c, N) \) with respect to the second argument; this lower bound is an increasing function of \( n \).

Physically, it means that for a given chiral state \( \psi_0 \) (with \( c \neq 0 \) at the bottom, there will be a lower bound of orbitals per cell \( N \) for each layer, and the bound will increase with the MERA’s depth. Note that this statement is only about the lower bound of \( N_n \); for a specific MERA, the actual number \( N_n \) in each layer does not necessarily increase with the layer index. Equivalently, given a chiral state \( \psi_0 \), if we want to use an \((L+1)\)-layer MERA to generate it, we need in general more orbitals per cell on the top layer compared with an \( L \)-layer MERA. In particular, if we want the bond dimension to be asymptotically constant, Eq. (4) gives us an upper bound of the depth \( L \). So we have a No-Go theorem: no infinite-layer MERA with asymptotically constant bond dimension could represent a gapped translational invariant chiral state.

On the other hand, let’s fix the bond dimension on the top layer, then Eq. (4) (valid for \( \forall n \), in particular, for \( n = L \)) leads to the following conclusion: for MERAs with a given bond dimension \( D \) such that the unit cell is a single top layer site, the value of \( B^{(0)} \) for the UV layer will diverge with the number of layers \( L \) and hence the wavefunction \( \psi_0 \) must also have diverging correlation length. In the case of infinite-layer MERA with asymptotically constant bond dimension, the same logic show that not only is it impossible to represent a chiral state (the above No-Go theorem), no such infinite-layer MERA can even provide a good approximation in the sense of \( B \). Note that it might be possible to approximate a chiral PEPS and the exact state is the “tail behavior” (power vs exponential), which is hard to distinguish by a naive norm, but can be distinguished by \( B \).

**IV. SKETCH OF THE PROOF**

Now we sketch the proof of this theorem. The details will be given in the Appendix A.

The proof of monotonicity is straightforward. To keep the basic idea as clear as possible, we will use words like “exists a constant” and “when |x| is large enough”, the technical details can be found in the appendix. We use the standard 2D MERA for example, the general case is similar.

Since the second-quantization operator \( \phi^{(n)}_x \) in the \( n \)th layer is linearly related to those in the \((n+1)\)th layer by \( W \) and \( U \), we can represent the correlation matrix \( P^{(n+1)} \) using \( P^{(n)} \). The tensors in a MERA are local: each block (\( W \) or \( U \)) has at most 4 legs in each side, so it’s easy to show \( x \) in the \( n \)th layer only talks to \( 2x + i \) where the norm of each components of \( i \) is 0 or 1 or 2, so that \( P^{(n+1)}_{x,y} \) is only related to \( P^{(n)}_{2x+i,2y+j} \) where \( i \) and \( j \) are valued in a finite set. Plugging the linear relation between \( P^{(n+1)} \) and \( P^{(n)} \) into Eq. (2), one can obtain an inequality with the following form:

\[
B^{(n+1)} \leq C_1 \sum a_{x-y} ||P^{(n)}_{2x+i,2y+j}||^2.
\]

for some constant \( C_1 \).

To prove \( B^{(n)} > AB^{(n+1)} \) for some \( A \), we only need \( a_x < C_2a_{(2x+i)-i-(2y+j)} \), so the right hand side of Eq. (6) goes to \( B^{(n)} \). This is obvious from Eq. (3) provided that \( s \) is large enough.

In order to prove the existence of the lower bound (Thm. 2), we proceed in two steps. First, we will prove that \( B \neq 0 \) as long as the state is chiral (\( c \neq 0 \)) no matter how we choose \( a_x \). Recall our definition of \( B \) and \( a_x \) in Eq.(2,3), what we need is: for any finite region \( F \), \( P_{x,y} \) cannot simultaneously vanish for all \( x - y \notin F \). This statement has essentially been proved in Ref. 20 in
different language. In Appendix A.2, we will give a different and more compact proof. Physically, this means although the correlation is short-ranged in the sense that it decays exponentially, it cannot be strictly local (the correlation length cannot be 0).

While $B(P)$ must be positive, the infimum (best lower bound) might be 0. We use a continuity argument to rule out this possibility: suppose not, there will be a sequence of maps such that $B(P) \to 0$. A limit $\bar{P}$ (to be specified in the appendix) of these sequence will satisfy $B(\bar{P}) = 0$ (for a slightly larger $F$) hence the Chern number $c(\bar{P}) = 0$ according to the first step. However the Chern number, as an integer, should not change when taking the limit, which provides a contradiction.

\section*{V. DISCUSSION}

While our results are phrased in terms of a MERA tensor network, the statements we make are applicable to entanglement renormalization as a whole. Particularly, entanglement renormalization fails for a Chern insulator on a lattice, provided one demand the RG procedure is reversible.

Our proof of Thm. 2 is not constructive; it doesn’t provide an explicit expression for the lower bound. However, one can give a very rough estimation of $\epsilon(c,N)$ by the following argument.

Consider the case where $|C| = 1$, $D = N$ (the general case will be similar). Let us group $i^2$ lattices into an effective cell, so that $N \to i^2 N$, where $N$ is the linear size of each matrix $P(k)$. One gets a new series $P_x = (P_{i+k-i})$, where $i, j$ are labels in the new cell (now with linear size $l$). For a $x$ at the boundary of the region $F$, due to the fast decay of $P_x$, we can apply the saddle point method to estimate $||P_x||^2$:

$$||P_x||^2 = \sum_{i,j} ||P_{i+k-i-j}||^2 \sim e^{-\alpha |F|l}. \quad (8)$$

Here, the first $\sim$ is because only the largest element (when $i, j$ are at some corners of the new cell) in the summation contributes, the second $\sim$ assumes $P_x$ indeed decays exponentially with $\alpha$ as the decay rate. $|F|$ is the radius of $F$. Also by the saddle point approximation, one has:

$$B(P') \sim \sum_{\mathbf{x} \in \partial(F)} ||\mathbf{x}||^s ||P_x||^2 \sim ||\partial F|| |F|^s e^{-\alpha |F|l}. \quad (9)$$

Here, $|\partial F|$ is the perimeter of the boundary. Eq. (9) is valid when the linear size of $P$ is $l^2 N$, so in general for $P$ of linear size $N$, we have $B(P) \sim e^{-\beta \sqrt{N}}$, where $\beta$ is another constant. In particular, $\epsilon(c,N) \lesssim e^{-\beta \sqrt{N}}$.

This is just the crudest estimation. One could obtain a better estimation given a faster (than exponential) decayed $P_x$ for some $P(k)$. From another point of view, this argument gives a refinement of Proposition 1 in Sec. A.2: not only $P_x$ cannot simultaneously vanish for large $\mathbf{x}$, but it cannot decay faster than a bound settled by $\epsilon(c,N)$. We don’t know what $\epsilon(c,N)$ is exactly. If we assume $\epsilon(c,N) \sim e^{-\beta \sqrt{N}}$, then Eq. (5) tells us $N_n \gtrsim n^2$ (ignore all the coefficients). Note again that this is not a proved bound of $N_n$: if $\epsilon$ decays faster, $N_n$ grows more slowly.

Part of our conclusions can be understood from another way. It was shown in Ref. 21 that a MERA with bounded bond dimension $D_M$ (they use $\chi_M$) can be mapped into a PEPS with bounded bond dimension $D_P$ which is a polynomial of $D_M$ and independent of the system size and the number of layers (they call this property efficiency). One can generalize their proof to the case of infinite-size MERA, and hence obtain an infinite-size PEPS with bounded bond dimension and no “input” (since it comes from an infinite layer which has no “input” on the top). However, according to Ref. 19 and 20, PEPS (with no input) cannot generate exact ground states of gapped, local parent Hamiltonians. So we conclude that no infinite-layer MERA with bounded bond dimension could represent a gapped chiral state. Compared to the above argument, the treatment in our article emphasizes on the renormalization point of view where MERA originates from.

At last, we mention some possible generalizations. In this article, we focused on the 2D non-interacting translational invariant chiral states. The generalization to higher dimensions is straightforward: in $d$ dimension, we need $s > d$ to guarantee both the monotonicity and the convergence in the proof of the theorem.

One possible generalization is the case without translational invariance. Here, the state is also determined by the correlation matrix $P$, but one cannot use Fourier transformation due to the lack of translational invariance. Instead, one should, for example, proceed in the spirit of Ref. 22, 23 to define the Chern number. The first part of our theorem is still valid with almost no changes in the proof. It is plausible that similar construction as our functional $B(P)$ also has a nonzero lower bound and one can proceed similarly to show the obstruction provided by the topology.

The generalization to the interacting case is certainly worth exploring. We conjecture that the same result holds in the presence of any chiral anomaly. In particular the U(1) chiral anomaly (e.g. in the case of the U(1) boson SPT phase [24], which manifest itself in the form of a quantized Hall conductance), would prevent a lattice fixed-point IR state to be constructed. In addition, the gravitational chiral anomaly, which arises from a non-zero chiral central charge, may also provide such obstruction.

\section*{VI. ACKNOWLEDGMENT}

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Appendix A: Proof of the theorem

1. Determination of $F$ and the monotonicity of $F$

In order to prove the monotonicity of $F$, we compare $P_{x,y}^{(n+1)}$ with $P_{x,y}^{(n)}$. To simplify our notation, we denote the $(n+1)^{th}$ layer to be $R$, $n^{th}$ layer to be $S$, the layer between them (below the isometry, above the disentangler) to be $T$. See Fig. 2 for illustration.

![Diagram](image)

FIG. 2. The illustration of our notation in 1D case. (a) We denote those three relevant layers to be $R$, $S$, $T$. (b) The definition of $s(x) + i$. (c) The definition of $x' + i$ and $t(x') + k$. In this case, $x, y$ are scalars valued in $Z$, $s(x) = 2x$, $i, j, k, l$ are scalars valued in $\{0, 1\}$.

For the isometry $W = (W_i)$ (arranged in one column) between $R$ and $S$, we have:

$$\phi^R_x = \phi^S_{s(x) + i}W_i,$$

so $P^R_{x,y} = \langle \phi^R_{x}, \phi^S_{y} \rangle = W^\dagger_i P^S_{s(x) + i, s(y) + j}W_j = W^\dagger_i P^S_{x,y}W_j$. Here $s(x)$ is a representative point under the isometry starting with $x$ (for the standard binary MERA in 2D, we can just choose $s(x) = 2x$). $W$ is an isometry in the sense that $W^\dagger W = 1_R$ (so $WW^\dagger$ is a projection in $S$). $P^S_{x,y}$ is the matrix with elements $P^S_{s(x) + i, s(y) + j}$.

By the definition of the Hilbert-Schmidt norm, we have:

$$||P^R_{x,y}||^2 = ||W^\dagger P^S_{x,y}W||^2 = tr(P^S_{x,y}WW^\dagger P^S_{x,y}W^\dagger)$$

$$\leq ||P^S_{x,y}W||^2 \cdot ||W^\dagger P^S_{x,y}|| \leq ||P^S||^2$$

$$= \sum_{ij} ||P^S_{s(x) + i, s(y) + j}||^2.$$  \hfill (A1)

Here, we’re using the Cauchy inequality for Hilbert-Schmidt norm and the fact that $WW^\dagger$ is a projection (hence $||WW^\dagger P|| \leq ||P||$).

In practice, in a MERA we have

$$\phi^S = (\phi^R, \ast)\tilde{W}^\dagger = \phi^R\tilde{W}^\dagger,$$

where $\tilde{W}$ is a unitary augmentation of $W$, where the "*" denotes some local (no correlation with $\phi^R$ and other stars) degrees of freedom, so $\tilde{W}^S x, y \tilde{W}^\dagger = P^R_{x,y}$. Hence Eq.((A1)) is actually an equality when $x \neq y$. However, we don’t need this result.

For the disentangler $U$ between $S$ and $T$, we have, similarly:

$$\sum_{ij} ||P^S_{x', y' + i, y' + j}||^2 = \sum_{kl} ||P^T_{t(x') + k, t(y') + l}||^2$$  \hfill (A2)

where $t(x)$ is a representative point under the disentangler that contains $x$. Here, the summation of $i,j$ and $k,j$ is over the disentangler containing $x,y$. See Fig. 2(c).

Now we can compare $B^R$ and $B^T$ as follows:

$$B^R = \frac{1}{|C_{n+1}|} \sum_{y \in C_{n+1}} \sum_{x \in Z^2} a_{x-y}||P^R_{x,y}||^2$$

$$\leq \frac{1}{|C_{n+1}|} \sum_{y \in C_{n+1}} \sum_{x \in Z^2} a_{x-y} ||P^S_{s(x) + i, s(y) + j}||^2$$

$$= \frac{w}{|C_n|} \sum_{y' \in C_n} \sum_{x' \in Z^2} a_{x-y} ||P^S_{x', y'}||^2$$  \hfill (A3)

Here $x', y'$ denote bonds in the $S$ layer, $w$ is the number of components under the isometry $W$ (for standard binary MERA in 2D, $w = 2^2 = 4$), so that $|C_n| = w|C_{n+1}|$. Note that we have the freedom to enlarge the unit cell so we can assume this relation without loss of generality. We can also assume that the cell $C_n$ exactly contains several disentanglers.

Denote $x, x', x''$ to be general bonds in layer $R, S, T$ respectively. Denote $x'$ to be the disentangler that $x$ belongs to. We would like to have the following property for each pair of disentanglers $(x', y')$:

$$a_{x-y} \leq \frac{1}{M} a_{x''-y''} \text{ for } \forall x, x''((y, y'')) \text{ connected to } x'(y').$$ \hfill (A4)

($x$ is connected to $x'$ means some $s(x) + i$ belongs to $x'$; $x''$ is connected to $x'$ means $x''$ is a leg of $x'$). If so, from Eq.(A2) we know:

$$\sum_{ij} a_{x-y} ||P^S_{x', y'}||^2 \leq \frac{1}{M} \sum_{k,l} a_{x''-y''} ||P^T_{x'', y''}||^2$$  \hfill (A5)

so we can continue as follows:

$$B^R \leq \frac{w}{M|C_n|} \sum_{x', y'} \sum_{k,l} a_{x''-y''} ||P^T_{x'', y''}||^2$$

$$= \frac{w}{M|C_n|} \sum_{y' \in C_n} \sum_{x' \in Z^2} a_{x-y} ||P^T_{x', y'}||^2$$  \hfill (A6)

$$= \frac{w}{M} B^T.$$
As long as
\[ A \overset{\text{def}}{=} \frac{M}{w} > 1, \] (A7)
we will get the desired inequality.

So the question is to choose \( \{ a_x \} \) such that Eq. (A4,A7) hold. This is always possible. For example, for the standard MERA, \( s(x) = 2x, \ w = 4 \). One can easily see that the distance between \( x'' \) and \( 2x \) is bounded by \( 2\sqrt{2} \), so that the distance between \( (x'' - y'') \) and \( 2(x - y) \) is bounded by \( 4\sqrt{2} \). We can demand \( a_x = |x|^s \) asymptotically so that
\[ \lim_{|x - y| \to \infty} \frac{a_x - y^s}{a_x - y} = 2^s. \] (A8)
We demand \( s > 2 \), hence for all \( M \) such that \( w = 4 < M < 2^s \), Eq. (A4) hold when \( |x - y| \) is large (depends on \( M \)). Problems may happen when \( |x - y| \) is small, but we can simply demand \( a_x = 0 \) for \( x \) in some finite region \( F \) to fix the problem.

The region \( F \) will depend on \( s \) and \( A \). Indeed, since
\[ (x'' - y'') \geq 2|x - y| - 4\sqrt{2}, \] Eq. (A4) holds when
\[ |x - y| \geq \frac{4\sqrt{2}}{2 - (4A)^s}. \] So we can simply demand \( F \) contains the disc \( D(0, \frac{4\sqrt{2}}{2 - (4A)^s}) \).

2. Existence of the lower bound

Firstly, we recall how the Chern number is defined from the projection matrix \( (P_x,y) \).

We start with a special case: when we have a translational invariant Hamiltonian:
\[ H = \sum_{x,y} \phi_x^T H_{x,y} \phi_y^T = \int \frac{d^2 k}{(2\pi)^2} \phi_k^T H(k) \phi_k^T, \]
and the state is the ground state. Here the integral is taken over the Brillouin zone \( T^2 \), each \( H(k) \) is a \( D \times D \) Hermitian matrix with \( p \) positive eigenvalues (corresponds to the \( p \) empty bands). It’s straightforward to show that the ground state correlation function is closely related to \( H(k) \):
\[ P_{kk'} = \langle \phi_k^4 \phi_{k'} \rangle = (2\pi)^2 \delta(k - k') P_{+}(H(k)). \]
Here, \( P_{+}(H) = \bigoplus_{\lambda > 0} P_{\lambda} \), where \( P_{\lambda} \) is the projection matrix on the eigenspace of \( H \) with eigenvalue \( \lambda \). The assignment for \( k \) to the corresponding empty subspace given by \( H(k) \) (thus by the ground state correlation \( P \)) determines a map from the Brillouin zone \( T^2 \) to the Grassmanian \( \text{Gr}(D, q) \), which is classified by the Chern number \( c \).

In the general case where the (minimal) unit cell \( C \) may not equal to the unit site and the Hamiltonian is not given in advance, one can proceed exactly as above. Regard the matrix \( (P_x,y) \) as a block matrix \( (P_{x\lambda,y\mu}) \) (here \( \lambda, \mu \) are labels in a unit cell, \( \lambda, \mu \in \{1, 2, \cdots, |C|\} \), which only depends on \( x - y \) and \( \lambda, \mu \). Then taking the Fourier transform with respect to \( x - y \), one obtains projection matrices \( P(k) = (P_{\lambda \mu}(k)) \) and hence a map from \( T^2 \) to \( \text{Gr}(N, q) \) where \( N = D|C| \). Here \( \text{Gr}(N, q) \) is naturally identified as the space of \( N \times N \) projection matrices with \( q \) positive eigenvalues. In other words, \( P(k) : T^2 \to \text{Gr}(N, q) \subseteq \mathbb{C}^{N^2} \). The Chern number \( c \) of the state is just the Chern number of this map.

Go back to our theorem. As a first step, we prove the following weaker statement which says \( B(P) \neq 0 \) for chiral state.

**Proposition 1.** If \( P_{x,y} = 0 \) for all \( x, y \) such that \( |x - y| \) is large enough, then \( c(P) = 0 \).

**Proof:** To proceed, we write the matrix-valued map \( P(k) : T^2 \to \text{Gr}(N, q) \subseteq \mathbb{C}^{N^2} \) in components \( p_{ij}(k) \) (\( i, j \in \{1, 2, \cdots, N^2\} \)). The condition of this proposition is all the matrix elements \( p_{ij}(k) \) have only finite many Fourier modes.

Consider the pullback bundle \( P^*(\tau) \), where \( \tau \) is the tautological bundle over \( \text{Gr}(N, q) \). We want to show the existence of \( q \) everywhere-linear-independent global sections of \( P^*(\tau) \), hence the bundle \( P^*(\tau) \) is trivial, hence the map \( P \) is trivial. Denote \( x = e^{ik_1}, y = e^{ik_2} \), where \( k = (k_1, k_2) \). Since \( p_{ij} \) has finite Fourier modes, we know \( p_{ij} \in \mathbb{C}^{N \times N} \subseteq \mathbb{C}[x, x^{-1}, y, y^{-1}] \). From now on, we extend \( T^2 \) to the complex tori \( (\mathbb{C}^*)^2 \). We still have \( P^2 = P \) since it can be reduced to a polynomial identity and is valid on the real torus. Moreover, the rank of \( P \) on the entire \( (\mathbb{C}^*)^2 \) is always \( q \). Indeed, \( P^2 = P \) means the extended map is into the space of projection matrices, which is \( \bigcup_{r=1}^{N} \text{Gr}(N, r) \). Each \( \text{Gr}(N, r) \) is closed, thus disconnected to each other. However, the extended map \( P \) is continuous, so the image is actually contained in \( \text{Gr}(N, q) \).

Denote \( S = \{ u \in \mathbb{R}^N | (P - 1)u = 0 \} \),
\[
(A9)
\]
which is the \( R \)-module of global Laurent sections (each component is a Laurent polynomial of \( x, y \)). We need to study the structure of this module. First of all, we study its local property:

**Lemma.** For \( \forall(x_0, y_0) \in (\mathbb{C}^*)^2 \), there exists \( u_1, \cdots, u_q \in S \) such that \( u_1(x, y), \cdots, u_q(x, y) \) are linear independent in a neighborhood of \( (x_0, y_0) \).

**Proof of Lemma:** For a fixed \( (x_0, y_0) \), take a similarity transformation so that \( P(x_0, y_0) = \text{diag}(1, \cdots, 1, 0, \cdots, 0) \). Equivalently, choose bases of the 1.0 eigenspaces of \( P(x_0, y_0) \) respectively and combine them into a basis of \( \mathbb{C}^N \). Under this basis, we write \( P(x, y) - 1 \) as a block matrix:

\[
P(x, y) - 1 = \frac{1}{x^s y^r} \begin{bmatrix} A(x, y) & B(x, y) \\ C(x, y) & D(x, y) \end{bmatrix},
\]
where \( A, B, C, D \) are matrices with polynomial elements. By continuity, we have \( \det D \neq 0 \) in a neighborhood of
It’s easy to see $P - 1$ is a projection matrix with rank $N - q$, hence the $q$ linear-independent vectors in $\ker(P - 1)$ is given by the columns in the following matrix:

$$(P - 1) \begin{bmatrix} (\det D)I_q \\ -(\det D)D^{-1}C \end{bmatrix} = 0.$$ 

So we’ve proved the lemma.

The above lemma and its proof tell us $S$ is a locally free module. Indeed, $(\mathbb{C}^*)^2$ is an affine variety with coordinate ring $R$, so every maximal ideal $\mathfrak{m}$ of $R$ corresponds to a point in $(\mathbb{C}^*)^2$. For $\forall \mathfrak{m} \in \text{MaxSpec}(R)$, consider the point corresponds to it and denote $\det D$ in the above proof to be $f_{\mathfrak{m}}$. Denote $v_1, \ldots, v_q$ to be the image of $u_1, \ldots, u_q$ in $S_{\mathfrak{m}}$. Then for $\forall v \in S_{\mathfrak{m}}$, it’s obvious that $v$ can be uniquely written as a linear combination of $v_i$ with coefficients in $R_{\mathfrak{m}}$. So $S_{\mathfrak{m}}$ is a free $R_{\mathfrak{m}}$ module with rank $q$.

Back to the original question. Since $S \subseteq R^N$ and $R$ (as a quotient of a polynomial ring) is a Noetherian ring, $S$ is a Noether $R$-module. Thus $S$ is a projective module[25]. According to a generalization of the Quillen-Suslin theorem[26] on the Laurent polynomial ring [27], $S$ must be a free module. Fix a basis of $S$, then each element of the basis must be an everywhere-nonzero section (otherwise the lemma breaks down at those points). So we have found the desired set of global sections. □

Now we can use a continuity argument to prove that as long as $c \neq 0$, $B(P)$ will have a lower bound $\epsilon$ (which may depend on $c$ and $N$). The idea is: if not, there will be a sequence of maps such that $B(P) \to 0$. The limit $\bar{P}$ of these $P$ will satisfy $B(\bar{P}) = 0$. However the Chern number should not change and thus non-zero, which contradicts Proposition 1.

**Proposition 2.** Fix exponent $s > 2$, region $F$, Chern number $c \neq 0$, number of orbitals per site $D$ and number of sites per cell $|C|$, then $\exists \epsilon > 0$ such that $B(P) > \epsilon$ for $\forall P \in C^o(T^2, Gr(N, q))$ (Gr means real analytic) such that $c(P) = c \neq 0$.

**Proof:** If not, there exist a sequence of maps $P^i(k)$ with the same Chern number $c^i = c \neq 0$ such that $B(P^i) \to 0$. We Fourier expand each $P^i(k)$:

$$P^i_{\lambda, \mu}(k) = \sum_{x \in F} P^i_{x, \lambda, \mu}e^{ik \cdot x} + \sum_{x \notin F} P^i_{x, \lambda, \mu}e^{ik \cdot x}. \quad (A11)$$

Here $\bar{F}$ a set of cells such that $x \notin \cup \bar{F}$ implies $x - y \notin F$ for $\forall y \in C_0$ ($C_0$ is the unit cell containing 0) (see Fig. 3). Obviously $F \subseteq \cup \bar{F}$. $\bar{x}$ is the cell containing $x$. One can understand $e^{ik \cdot x}$ (and $|x|^s$ in the following) as $e^{ik \cdot \bar{x}}$ (and $|\bar{x}|^s$) where $x \in \bar{x}$.

Since $P^i_{\lambda, \mu}(k)$ is uniformly bounded (the Grassmanian is compact), $P^i_{x, \lambda, \mu}$ are bounded, thus there is a converging subsequence of $P^i_{0, \lambda=0, \mu=0}$. We pick up this subsequence and do the same thing for each point in $\cup \bar{F}$. As a result, we can assume without loss of generality that $P^i_{x, \lambda, \mu}$ converge for $\forall \bar{x} \in \bar{F}$ and $\forall \lambda, \mu$. Denote the limit as $\tilde{P}_{x, \lambda, \mu}$. Define $\tilde{P}(k)$ as:

$$\tilde{P}_{\lambda, \mu}(k) = \sum_{x \in F} \tilde{P}_{x, \lambda, \mu}e^{ik \cdot x}. \quad (A12)$$

We claim $P^i = \tilde{P}$ as maps into $\mathbb{C}^{N^2}$ (uniformly converge, in other words, converge in the $L^\infty$ norm). Indeed, denote $G' = P^i - \tilde{P}$, using the Cauchy inequality, we have:

$$||G'(k)||^2 = \sum_{x \in \bar{F}} |G'_x|^2 \lesssim \sum_{x \in \bar{F}} |G'_x||G'_x||G'_x|,$$

$$\lesssim \sum_{x \in \bar{F}} \begin{bmatrix} 1 + \sum_{x \notin \bar{F}} \frac{1}{|x|^s} \end{bmatrix} \sum_{x \in \bar{F}} |G'_x|^2 + \sum_{x \notin \bar{F}} |G'_x|^2. \quad (A13)$$

The first factor converges when $s > 2$. The second factor converges to 0 since $\bar{F}$ is finite, $G'_x = P_x - \tilde{P}_x \to 0$ for $\forall x \in \bar{F}$ by construction, and

$$\sum_{x \notin \bar{F}} |G'_x|^2 = \sum_{x \notin \bar{F}} |G'_x|^2 = \sum_{y \in C, x \notin \bar{F}} |G'_{x,y}|^2 \lesssim \sum_{y \in C, x \notin \bar{F}} |x - y|^s |G'_{x,y}|^2 \leq |C|B(P^i) \to 0.$$

FIG. 3. The definition of $\bar{F}$. In this figure, the each block is a unit cell, the green one is $C_0$. $F$ is bounded by the black circle and $\cup \bar{F}$ is bounded by the red lines. Roughly speaking, $\cup \bar{F}$ is the extension of $F$ by two cells.
It’s easy to show $\tilde{P}(k) \in \text{Gr}(N, q)$ for $\forall k$, so $\text{Im}(\tilde{P}) \subseteq \text{Gr}(N, q)$ and we can define its Chern number $\tilde{c}$. According to Proposition 1, $\tilde{c} = 0$.

On the other hand, we claim that $\tilde{c} = \lim c^i = c$. Indeed, since $P^i \Rightarrow \tilde{P}$, there $\exists I$ such that $|P^i(k) - \tilde{P}(k)| < \delta$ when $i > I$ ($\delta$ is chosen small enough as follows). Since $\text{Gr}(N, q)$ is compact, there exists a small $r$ such that if we pick up the disk with radius $r$ in the normal subspace of each point $p \in \text{Gr}(N, q)$, the resulting bundle is isomorphic to $\text{Gr}(N, q) \times D(r)$. Also due to the compactness, we can choose $\delta$ small enough so that whenever $p, q \in \text{Gr}(N, q)$ and $|p - q| < \delta$, each point in the segment between $p, q$ belongs to a unique normal disk. Now we can construct a homotopy $\Psi(t, k)$ between $P^i$ and $\tilde{P}$ as: $\Psi(t, k) = $ the center of the disk that $(1 - t)P^i(k) + t\tilde{P}(k)$ belongs to. Thus, $\tilde{c} = c^i$ when $i > I$, which means $\tilde{c} = c \neq 0$.

This contradiction shows the existence of the lower bound. □