Quantum physics with a hidden variable

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Abstract

Every quantum physical system can be considered the "shadow" of a special kind of classical system.

The system proposed here is classical mainly because each observable function has a well precise value on each state of the system: an hypothetical observer able to prepare the system exactly in an assigned state and able to build a measuring apparatus perfectly corresponding to a required observable gets always the same real value.

The same system considered instead by an unexpert observer, affected by the ignorance of a hidden variable, is described by a statistical theory giving exactly and without exception the states, the observables, the dynamics and the probabilities prescribed for the usual quantum system.

1. INTRODUCTION

In the following will be presented a system essentially classical: its states are points of a manifold, its observables are functions on the manifold, its dynamics are generated by functions and its differentiable dynamics come from differentiable "hamiltonian" vector fields.

This system, on the other hand, is not perfectly classical because its observable functions are, in general, far from being continuous or differentiable and do not make a vector space; however when you consider their "mean value functions" (that is integrated with respect to the hidden variable) you get the "expected value functions" of the usual quantum theory and these, when defined everywhere, are differentiable functions making a vector space.

The system differs from a classical one also because you need two functions to define a general dynamic: one is the hamiltonian "expected value function" and the other is a differentiable function measuring an increment in the speed of changement of the "hidden variable": when these two functions are both smooth the dynamic comes from a smooth vector field.

The system to be examined will be introduced following a list of remarks:

A. The states of the system live inside an infinite dimensional real Hilbert space $\mathcal{H}$ and the state space to consider is the infinite dimensional spheric hypersurface $\mathbb{S} = \mathbb{S}(\sqrt{2})$ of radius $\sqrt{2}$ with its geometry.

This radius is chosen in such a way to have:

$$\hbar = 1$$
in the Schrödinger equation (choosing a different radius \( r \) would imply a Schrödinger equation with the constant \( \frac{1}{2}r^2 \) instead of 1 in the place of \( \hbar \)).

**B.** For every state \( \varphi \) in \( \mathbb{S} \) there is family of states ”equivalent” to \( \varphi \) and differing by \( \varphi \) only by a ”phase”. More precisely there is an action of the group \( \mathbb{S}^1 \) on \( \mathbb{S} \) making the vector \( \varphi \) to travel along the \( \mathbb{S}^1 \)-orbits of states \([\varphi] = \{\rho \varphi\} \) (or \( \{e^{i\theta} \cdot \varphi\} \)).

This action and in particular the operator \( J = \rho_{\pi/2} \) allows to consider, if we need it, the vector space \( \mathcal{H} \) as a complex vector space.

These \( \mathbb{S}^1 \)-orbits have a twofold structure: ”from the outside” they are riemannian circles embedded in \( \mathbb{S} \), ”from the inside” they are only probability spaces, that is sets \([\varphi]\) furnished with a \( \sigma \)-algebra of measurable subsets and a measure function \( \mu_{[\varphi]} \) taking value 1 on the whole set \([\varphi]\).

The states \( \rho \varphi \) in the present theory will play the role of the ”hidden states” behind the ”apparent state” \([\varphi]\).

**C.** On the system can be carried out some propositions, that is observable functions taking only the values 0 or 1; a proposition is obviously characterized by the set \( L \) where the function takes value 1.

We will consider as propositions for our system the subsets \( L \) of \( \mathbb{S} \) satisfying the following conditions:

- The measure \( \mu_{[\varphi]}(L \cap [\varphi]) \) varies differentiably with the vector \( \varphi \) (or the class \([\varphi]\))
- Given two orthogonal vectors \( \varphi \) and \( \psi \) of \( \mathbb{S} \) (not in the same \( \mathbb{S}^1 \)-orbit) consider the vectors:
  \[ \varphi(t) = \cos t \cdot \varphi + \sin t \cdot \psi \]
  (the superpositions of \( \varphi \) and \( \psi \) in \( \mathbb{S} \)) we require that \( \mu_{[\varphi(t)]}(L \cap [\varphi(t)]) \), as a function of the variable \( t \), must take the form:
  \[ \mu_{[\varphi(t)]}(L \cap [\varphi(t)]) = a \cdot \cos^2 t + b \cdot \sin t \cos t + c \cdot \sin^2 t \]
- If \( \mu_{[\varphi(t)]}(L \cap [\varphi(t)]) \) is not always 0 or always 1, given \( \varphi \) in \( \mathbb{S} \), for some orthogonal vector \( \psi \) of \( \mathbb{S} \) (not in \([\varphi]\)) the function \( \mu_{[\varphi(t)]}(L \cap [\varphi(t)]) \) takes all the values in the interval \([0, 1]\).

All these subsets \( L \) make a family \( \mathcal{L} \) of subsets in \( \mathbb{S} \) (called the logic of the system).

The family \( \mathcal{L} \) is closed by complementation but is not a boolean algebra (or a logic as in \([V]\)), however it contains infinite boolean algebras corresponding to the boolean algebras of commuting projectors in \( \mathcal{H} \).

**D.** A function \( f : \mathbb{S} \to \mathbb{R} \) will be called an observable function if it allows, for every borel subset \( B \) of \( \mathbb{R} \), to check whether or not the function \( f \), on a given vector \( \varphi \), takes its value in \( B \); that is if \( f^{-1}(B) \) of \( \mathbb{S} \) is a proposition in \( \mathcal{L} \) for every borel subset \( B \).

All these functions make a family \( \mathcal{O} \).

Each observable function has a well precise value on each state of the system independently from the corresponding ”measuring process”\; an hypothetical expert observer able to prepare the system exactly in the state \( \varphi \) and able to build a
measuring apparatus perfectly corresponding to the observable function \( f \) would get always the real value \( f(\varphi) \).

The family \( \mathcal{O} \) is not an algebra or a vector space, however it contains infinite commutative functions algebras corresponding to the commutative algebras of self-adjoint operators.

To an observable function \( f \) it’s associated its "mean value" function:

\[
\langle f \rangle(\varphi) = \int_{[\varphi]} f(\psi) \cdot d\mu_{[\varphi]}
\]

In general \( \langle f \rangle \) is not defined on all \( S \), however when this happens the function \( \langle f \rangle \) is smooth (a smooth kaehlerian) function. All these functions (called kaehlerian) make a space \( K(S) \) definable through the geometry of the space \( S \).

E. What is a symmetry for the system \( S \)? The bijective maps \( \Phi : S \to S \) respecting the scalar products, the sovrappositions of orthogonal vectors and commuting with the actions \( \rho_\theta \) are good candidates: it is not difficult to verify that these maps are exactly the unitary maps of \( \mathcal{H} \) making the group \( Unit(\mathcal{H}) \).

But maybe is more suitable here to consider as symmetries the diffeomorphisms \( \Phi : S \to S \) that are characterized by the properties to bring \( S^1 \)-orbits in \( S^1 \)-orbits, couples of orthogonal \( S^1 \)-orbits in couples of orthogonal \( S^1 \)-orbits and to commute with the actions \( \rho_\theta \) (plus a technical condition): in this way a much wider group \( Aut(S) \) of symmetries is obtained.

With respect to the group \( Aut(S) \) the dynamics of the system are the one-parameter (continuous) groups \( \Phi : \mathbb{R} \to Aut(S) \). It can be proved that with a natural topology on \( Aut(S) \) all the one-parameter continuous groups in \( Aut(S) \) are given by two functions: a kaehlerian function \( l \) giving origin to a one-parameter unitary group that "moves the \( S^1 \)-orbits" and a differentiable function \( h \) on \( S \) (constant on the \( S^1 \)-orbits), incrementing the speed "inside the \( S^1 \)-orbits".

F. The system \( S \) is essentially a classical system to the eyes of an hypothetical observer that we will call the precise observer.

How appears the same system to an imprecise observer not having under control the phase? Let’s suppose that this observer is not able, for practical limits or intrinsic reasons, to produce the state \( \varphi \) rather than its rotated state \( \rho_\theta \varphi \).

When he tries to prepare the system in the state \( \varphi \) he can be precise enough to prepare a state inside the \( S^1 \)-orbit \( [\varphi] = \{\rho_\theta \varphi : 0 \leq \theta < 2\pi\} \) but he does not know which state in the \( S^1 \)-orbit is the outcome; he cannot avoid to the state produced to be completely random in its \( S^1 \)-orbit.

When he tries to measure the observable \( f \) on the state \( \varphi \) he can get anyone of the values \( \{f(\psi) ; \psi \in [\varphi]\} \). After a large number of trials he gets his outcomes distributed on the real line and, in the end, what he gets are only the numbers \( \pi(\varphi, f, B) \) expressing the probabilities that the outcome falls in a general borel subset \( B \) (varying in the family \( \mathcal{B}(\mathbb{R}) \) of all borelian subsets of \( \mathbb{R} \)).

This probability, from the other side, measures the frequency for \( \varphi \), moving randomly in \( [\varphi] \), to fall in the set \( f^{-1}(B) \cap [\varphi] \). Therefore:

\[
\pi(\varphi, f, B) = \mu_{[\varphi]}(f^{-1}(B) \cap [\varphi])
\]
The imprecise observer is compelled to consider the measuring process intrinsically statistic; the space $S$ keeps for him only the meaning of the set of all possible preparations of the system and $O$ keeps only the meaning of the set of all realizable measuring apparatuses. All his experimental knowledge reduces to a map:

$$\pi : S \times O \times B(\mathbb{R}) \rightarrow [0, 1]$$

But now why he should consider different two preparations $\varphi_1$ and $\varphi_2$ if:

$$\pi(\varphi_1, f, B) = \pi(\varphi_2, f, B)$$

for every apparatus $f$ and every borelian subset $B$? Dually why he should consider different two apparatuses $f_1$ and $f_2$ if:

$$\pi(\varphi, f_1, B) = \pi(\varphi, f_2, B)$$

for every preparation $\varphi$ and every borelian subset $B$?

His imprecision generates an equivalence relation $R_S$ among states in $S$ and an equivalence relation $R_O$ among observables in $O$: for the imprecise observer the "states" he can define through his experiments are the equivalence classes of $R_S$ in $S$ and his "state space" is the quotient space $\hat{S} = S/R_S$, analogously his "observable space" is $\hat{O} = O/R_O$. Over these objects he has a well defined probability map:

$$\hat{\pi} : \hat{S} \times \hat{O} \times B(\mathbb{R}) \rightarrow [0, 1]$$

Analogously he will consider his symmetries on $\hat{S}$ and not on $S$: since the "hidden" symmetries respect the equivalence relation $R_S$ the symmetry group $\text{Aut}(S)$ acts naturally on $\hat{S}$ and so if we introduce the subgroup $\text{Aut}_I(S)$ of $\text{Aut}(S)$ made by all the symmetries acting identically on $\hat{S}$ the quotient group $\widehat{\text{Aut}}(S) = \text{Aut}(S)/\text{Aut}_I(S)$ acts effectively on $\hat{S}$ and it can be proved that gives the right group of "apparent" symmetries for the imprecise observer.

Moreover every continuous dynamic in $\text{Aut}(S)$ induces a dynamic in $\hat{S}$, made of transformations in $\widehat{\text{Aut}}(S)$, continuous for the induced topology of $\widehat{\text{Aut}}(S)$ and conversely it is possible to prove that every continuous dynamic in $\text{Aut}(S)$ comes from a continuous dynamic in $\text{Aut}(S)$.

Briefly starting with the ingredients of "classical" system $S$:

$$(S, L, \mu, \mathcal{O}, \text{Aut}(S))$$

in consideration by the precise observer, the ignorance of the phase induces the imprecise observer to consider instead the statistical system:

$$(\hat{S}, \hat{O}, \hat{\pi}, \widehat{\text{Aut}}(S)).$$

of states, observables, probabilities and symmetries.

What it is proved in this paper, in short, it is exactly that this last system is (it is isomorphic to) the usual quantum system.

Precisely the state space $\hat{S}$ is the complex projective space of $\mathcal{H}$, the observable space $\hat{O}$ is naturally isomorphic to the set of all self-adjoint operators of $\mathcal{H}$ and the probability map $\hat{\pi}$ becomes the probability map $\tilde{\pi}(\varphi, A, B) = \langle E_A^B \rangle_{\varphi}$ (where $E_A^B$ is the projector associated to the borel subset $B$ in the projector valued measure defined by the self-adjoint operator $A$) of the canonical quantum theory; note
also that every observable function takes its essential values in the set of the true outcomes of the corresponding quantum observable (the spectrum of its associated self-adjoint operator).

Moreover $\text{Aut}(\mathbb{S})$ becomes the symmetry group of $\mathbb{P}_\mathbb{C}(\mathcal{H})$, that is the group of unitary transformations of $\mathcal{H}$ modulo the multiples of the identity.

This statement gives a rigorous content to the assertion that a quantum physical system can be considered the "shadow" of a classical system with a "hidden variable".

Some of the characteristic features of the present hidden variable theory are to be declared:

1. **This theory is non local**: there is no room for (non-banal) properties with the special independence required in the proof of the Bell inequalities and wishful to represent apparatuses acting independently in two spatially separated regions of the spacetime.

2. **This theory is contextual**: behind a quantum proposition there are in $\mathbb{S}$ infinite "hidden" classical propositions, therefore the truth value 0 or 1 of the classical proposition on a "hidden" classical state depends not only on the "hidden variable" of the state but also on the "experimental context" defined by the particular classical proposition in consideration. The same holds for the observables.

3. **This theory is "relativistically invariant"**: it works equally well either if you have assigned for the quantum system a unitary representation of the Galilei group or a unitary representation of the Poincaré group; it works also in the general relativistic case as long as you have a quantum theory via a Hilbert space.

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2. **THE SYSTEM $\mathbb{S}$, ITS STATES AND OBSERVABLES**

In the following $(\mathcal{H}, \langle ., . \rangle)$ will denote an infinite dimensional real Hilbert space furnished with an effective action $\rho : S^1 \to \text{Isom}(\mathcal{H}, \langle ., . \rangle)$ of the group $S^1$ on $\mathcal{H}$ via isometries.

For the action $\rho$ we suppose valid the property:

$$\rho \theta \varphi = \cos \theta \cdot \varphi + \sin \theta \cdot \rho_{\pi/2} \varphi$$

where we denote, for simplicity, an element of $S^1$ as a real number $\theta$ (implicitly modulo $2\pi$) and we write $\rho \theta \varphi$ instead of $\rho(e^{i\theta})(\varphi)$. 

The special isometry $ρ_{π/2}$ will be denoted by $J : H → H$; obviously we get $J^2 = −i12$ and $⟨φ, Jφ⟩ = 0$ for every $φ$ in $H$. Through $J$ the space $H$ becomes a complex Hilbert space with $ρφ = e^{iθ}φ$; we can also consider on it the sesquilinear form defined by:

$$⟨⟨φ, ψ⟩⟩ = ⟨φ, ψ⟩ + i · ⟨Jφ, ψ⟩$$

However in general it appears preferable to consider $H$ a real Hilbert space with the action $ρ$ instead of a complex Hilbert space.

Obviously two vectors $φ, ψ$ in $H$ are orthogonal with respect to the sesquilinear form $⟨⟨·, ·⟩⟩$ if and only if $ψ$ is orthogonal to $φ$ and $Jφ$ with respect to the real scalar product $⟨·, ·⟩$.

Afterwards we will refer always to to the orthogonalily with respect to the real scalar product $⟨·, ·⟩$, unless dealing with complex objects like complex projectors or complex linear subspaces.

**Definition 1.** Our space of states will be the (hypersurface) sphere of radius $\sqrt{2}$:

$$S = S(\sqrt{2}) = \{ φ ∈ H : ∥φ∥ = \sqrt{2} \}$$

The space $S$ is an infinite dimensional riemannian manifold modelled on a Hilbert space; its radius $\sqrt{2}$ is chosen in such a way to have:

$$h = 1$$

in the Schroedinger equation; choosing a different radius $r$ would imply a Schroedinger equation with the constant $1/r^2$ in the place of the Plank constant, for example the more elegant choice $r = 1$ would imply the uncommon choice for the Plank constant $h = 1/2$.

The space $H$ contains spherical hypersurfaces where, formally, the Plank constant $h$ takes all possible positive values. In this perspective on $H$ is defined a non-negative "quadratic" function $\tilde{ρ}(·) : H → \mathbb{R}$ inducing a norm $∥φ∥ = \sqrt{2\tilde{ρ}(φ)}$ and a real scalar product $⟨φ, ψ⟩ = \tilde{ρ}(φ + ψ) − \tilde{ρ}(φ) − \tilde{ρ}(ψ)$.

The real tangent space $T_φS = (φ) \perp$ contains the vector $Jφ$ and can be splitted in a "vertical part" $Ver_φ = R · Jφ$ and in a "horizontal part" $Hor_φ = \{ φ, Jφ \} \perp$. The map $J$ sends $Hor_φ$ in itself, we will denote this restriction by $J_φ$. Analogously a map: $ρ_φ : Hor_φ → Hor_φ$ can be defined as $ρ_φ(X) = \cos θ · X + \sin θ · J_φX = e^{iθ} · X$.

The group $S^1$ acts on $S$ describing the $S^1$-orbits $[φ] = \{ ρθφ : θ ∈ \mathbb{R} \}$; on each $S^1$-orbit $[φ]$ there is only one measure $μ_φ$ on the natural borelian subsets having total measure equal to 1 and making measure preserving the natural correspondence $θ → ρθφ$ between $S^1$, with the normalized Haar measure, and $[φ]$ . Note that $T_φ[φ] = Ver_φ = R · Jφ$.

Given two vectors $φ$ and $ψ$ in the same $S^1$-orbit we will denote by $ψ/φ$ the unique complex number $u ∈ S^1$ such that $ψ = ρ(u)(φ)$.

In each $S^1$-orbit is well defined a metric $d_φ : [φ] × [φ] → [0, π]$ given by: $d_φ(φ, ψ) = |Arg(ψ/φ)|$ (where $Arg : S^1 → [−π, π]$ verifies $Arg(e^{iθ}) = θ$ for every $θ$ in $[−π, π]$). We will refer to this metric as the phase distance in $[φ]$. 
Definition 2. A subset \( B \) of \( S \) will be called a **pseudo-borel subset** if every intersection \( B \cap [\varphi] \) is Borel in \( [\varphi] \); a pseudo-borel subset of \( S \) will be called a **pseudo-borel null subset** if every intersection \( B \cap [\varphi] \) is Borel null in \( [\varphi] \).

Definition 3. Two pseudo-borel subsets \( B \) and \( C \) of \( S \) will be called **equivalent up to a pseudo-borel null subset** (or **null equivalent**) if their symmetric difference is a pseudo-borel null subset; moreover \( B \) and \( C \) will be called **equivalent in measure** if

\[
\mu_{[\varphi]}(B \cap [\varphi]) = \mu_{[\varphi]}(C \cap [\varphi])
\]

for every \( S^1 \)-orbit \( [\varphi] \).

Given two orthogonal vectors \( \varphi \) and \( \psi \) in \( S \) the map: \( \gamma_{\varphi\psi} : \mathbb{R} \to S \) defined by:

\[
\gamma_{\varphi\psi}(t) = \cos t \cdot \varphi + \sin t \cdot \psi
\]

parametrizes a **maximal circle** (a geodesics curve) in \( S \).

Definition 4. A pseudo-borel subset \( L \) of \( S \) will be called a **proposition** of \( S \) if:

- the function \( \varphi \mapsto \mu_{[\varphi]}(L \cap [\varphi]) \) is differentiable between \( S \) and \( [0, 1] \)
- given two orthogonal vectors \( \varphi \) and \( \psi \) in \( S \) the function \( \mu_{[\gamma_{\varphi\psi}(t)]}(L \cap [\varphi_{\psi}(t)]) \) in the variable \( t \) is a function of the form

\[
a \cdot \cos^2 t + b \cdot \sin t \cos t + c \cdot \sin^2 t
\]

- unless \( L \) is equivalent in measure to \( \emptyset \) or to \( S \), for every \( \varphi \) in \( S \) there is a \( \psi \) in \( S \) orthogonal to \( \varphi \) such that the function \( t \mapsto \mu_{[\gamma_{\varphi\psi}(t)]}(L \cap [\varphi_{\psi}(t)]) \) takes all the values of \( [0, 1] \).

This definition takes into account the behaviour of quantum probabilities for a quantum property; infact if the property is represented by the (complex) projector \( E \) you get:

- the map \( \varphi \mapsto \langle \gamma_{\varphi} \rangle = \frac{1}{2} \langle \varphi, E \varphi \rangle \) is differentiable in \( \varphi \)
- if you take two orthogonal elements \( \varphi \) and \( \psi \) in \( S \) and consider the states parametrized by the path \( \gamma(t) = \cos t \cdot \varphi + \sin t \cdot \psi \) (the superposition states of \( \varphi \) and \( \psi \) in \( S \)) then the function of \( t \) given by \( \langle E \rangle_{\gamma(t)} \) is:

\[
\frac{1}{2} \langle \gamma(t), E(\gamma(t)) \rangle = \cos^2 t \cdot \langle \gamma_{\varphi} \rangle + \sin t \cos t \cdot \langle \varphi, E\psi \rangle + \sin^2 t \cdot \langle E \rangle_{\psi}
\]

- when \( \varphi \) is in \( \text{Im} E \) and \( \psi \) is in \( \ker E \) then the function above becomes equal to \( \cos^2 t \) and takes all the values of \( [0, 1] \). If \( E \) is not 0 or I given \( \varphi \) you can always find an orthogonal \( \psi \) in such a way that \( \gamma_{\varphi\psi}(t) \) meets \( \text{Im} E \) in \( \sigma \) and \( \ker E \) in \( \tau \), then \( \gamma_{\varphi\psi} \) as \( \gamma_{\sigma\tau} \) takes all the values of \( [0, 1] \).

Definition 5. The set \( \mathcal{L} \) of all the propositions of \( S \) will be called the **logic** of \( S \).
The functions of the form $a \cdot \cos^2 t + b \cdot \sin t \cdot \cos t + c \cdot \sin^2 t$ make a three dimensional vector space (with base $\{\cos^2 t, \sin t \cdot \cos t, \sin^2 t\}$ or $\{1, \sin t \cdot \cos t, \sin^2 t\}$). With respect to this last base a function $h(t)$ of this space can be written as:

$$h(t) = h(0) + \dot{h}(0) \cdot \sin t \cdot \cos t + \frac{1}{2} \ddot{h}(0) \cdot \sin^2 t$$

A function in this space takes all the values of the interval $[0, 1]$ if and only if can be expressed as $\cos^2(t + \beta)$.

**Remark 1.**
- The empty set $\emptyset$ and all the sphere $\mathbb{S}$ are propositions
- If $L$ is a proposition its complement $\mathbb{S} \setminus L$ is also a proposition
- Every pseudo-borel subset of $\mathbb{S}$ equivalent in measure to a proposition is a proposition
- Every pseudo-borel subset of $\mathbb{S}$ null equivalent to a proposition is a proposition
- If $L$ is a proposition and $U : \mathbb{S} \to \mathbb{S}$ is a unitary map then $U(L)$ is also a proposition.
- In general the intersection or the union of two propositions *is not* a proposition.
- Later on it will be proved that if $L$ and $M$ are propositions with $L \subset M$ then also $M \setminus L$ is a proposition.

**Definition 6.** A function $f : \mathbb{S} \to \mathbb{R}$ will be called a *pseudo-borelian* function on $\mathbb{S}$ if for every borel subset $B$ in $\mathbb{R}$ the inverse image $f^{-1}(B)$ is a pseudo-borel subset of $\mathbb{S}$. Two pseudo-borelian functions on $\mathbb{S}$ will be called *null equivalent* if they differ only on a null pseudo-borel subset of $\mathbb{S}$.

**Definition 7.** A function $f : \mathbb{S} \to \mathbb{R}$ will be called an *observable* on $\mathbb{S}$ if for every borel subset $B$ in $\mathbb{R}$ its inverse image $f^{-1}(B)$ is a proposition of $\mathbb{S}$.

**Notation 1.** The symbol $\mathcal{O}$ will denote the set of all the observable functions on the space $\mathbb{S}$.

**Remark 2.**
- The characteristic function $\chi_A$ of a pseudo-borel subset $A$ in $\mathbb{S}$ is an observable if and only if $A$ is a proposition
- Every constant function on $\mathbb{S}$ is an observable
- If $f : \mathbb{S} \to \mathbb{R}$ is an observable on $\mathbb{S}$ and $b : \mathbb{R} \to \mathbb{R}$ is a borel function, the function $b \circ f : \mathbb{S} \to \mathbb{R}$ is also an observable
- If $f : \mathbb{S} \to \mathbb{R}$ is an observable on $\mathbb{S}$ then $|f|$, $f^+$ and $f^-$ are observable functions
- If $f : \mathbb{S} \to \mathbb{R}$ is a never zero observable then the function $1/f$ is an observable (in fact $1/f = b \circ f$ where $b : \mathbb{R} \to \mathbb{R}$ is the function defined by $b(0) = 1$ and $b(x) = 1/x$ for $x \neq 0$)
- If $f : \mathbb{S} \to \mathbb{R}$ is an observable and $g : \mathbb{S} \to \mathbb{R}$ is a (pseudo-borelian) function null equivalent to $f$ then $g$ is also an observable
• if \( f : S \to \mathbb{R} \) is an observable and \( k \) is a constant the functions \( k \cdot f \) and 
\( k + f \) are observables.
• if \( f : S \to \mathbb{R} \) is an observable and \( U : S \to S \) is a unitary map then \( f \circ U^{-1} \)
is an observable.
• in general the sum or the product of two observable functions is not an 
observable function.

**Theorem 1.** For every observable function \( f \) there exists a unique open subset \( \Omega_f \)
of \( \mathbb{R} \) such that \( f^{-1}(\Omega_f) \) is pseudo-borel null and maximal among the open subsets 
of \( \mathbb{R} \) with this property.

**Proof.** The family of all the open subsets \( U \) of \( \mathbb{R} \) such that \( f^{-1}(U) \) is pseudo-borel null subset of \( S \) is not empty and contains a maximum element (the union of all its elements since it can be expressed as a countable union).

**Definition 8.** Let \( f \) be an observable function the **essential image** of \( f \) is the 
(non empty) closed subset \( \text{Im}_e(f) = \mathbb{R} \setminus \Omega_f \) of \( \mathbb{R} \).

A real value \( y \) is not in \( \text{Im}_e(f) \) if and only if there is a neighborhood \( [y-\varepsilon, y+\varepsilon] \)
of \( y \) such that \( f^{-1}([y-\varepsilon, y+\varepsilon]) \) is a pseudo-borel null subset of \( S \).
\( \text{Im}_e(f) \subseteq f(S) \); if \( y_0 \) is an isolated value of \( \text{Im}_e(f) \) then \( y_0 \) is a value in \( f(S) \).
Two null equivalent observable functions \( f, g \) have the same essential image.
It is always possible to modify an observable function \( f \) on a pseudo-borel null subset in such a way to have \( f^{-1}(\Omega_f) = \emptyset \) and then \( \text{Im}_e(f) = f(S) \).
When \( \text{Im}_e(f) \) is a discrete subset of \( \mathbb{R} \) and \( f^{-1}(\Omega_f) = \emptyset \) then \( \text{Im}_e(f) = f(S) \).

**Notation 2.** Let \( f : S \to \mathbb{R} \) be an observable, the family:
\[ \mathcal{A}_f = \{ f^{-1}(B); B \text{ is a borel subset of } \mathbb{R} \} \]
is a \( \sigma \)-algebra of subsets of \( S \) contained in \( \mathcal{L} \). Let's denote by \( \mathcal{A}_f \) the bigger \( \sigma \)-algebra in \( \mathcal{L} \):
\[ \mathcal{A}_f = \{ A : A \text{ is null equivalent to a } f^{-1}(B) \text{ where } B \text{ is a borel subset of } \mathbb{R} \} \]

**Notation 3.** Given two propositions \( L \) and \( M \) in \( \mathcal{L} \) the boolean algebra of subsets 
of \( S \) generated by \( L \) and \( M \) will be denoted by \( \mathcal{B}_{L,M} \):
\[ \mathcal{B}_{L,M} = \{ \emptyset, L, M, L \cap M, L \cup M, C(L), C(M), L \cap C(M), C(L \cap M), \}
\[ \{ C(L \cup M), L \cup C(M), C(L \cup M), C(L \cap C(M), S \} \}

**Definition 9.** Two propositions \( L \) and \( M \) in \( \mathcal{L} \) will be called **compatible** if their 
boolean algebra of subsets \( \mathcal{B}_{L,M} \) is contained in \( \mathcal{L} \).

**Theorem 2.** Given two propositions \( L \) and \( M \) there is an observable function \( f \)
such that \( \mathcal{A}_f \) contains \( L \) and \( M \) if and only if the propositions are compatible.
Proof. \( (\Rightarrow) \) Obvious. 

\( (\Leftarrow) \) The sets 
\( X_1 = L \cap \mathbb{C}M, \ X_2 = \mathbb{C}L \cap M, \ X_3 = L \cap M, \ X_4 = \mathbb{C}L \cap \mathbb{C}M \) 
are pairwise disjoint (possibly empty) in \( L \) and every element of \( B_{L,M} \) is obtainable as a union of them.

The function \( f : S \to \mathbb{R} \) defined by \( f(\varphi) = n \) if \( \varphi \in X_n \) is well defined, is an observable with \( L = f^{-1}\{1,3\} \) and \( M = f^{-1}\{2,3\} \). \( \square \)

Given two propositions \( L \) and \( M \) we will prove later that are compatible if and only if \( L \cap M \) is contained in \( L \), therefore when \( L \cap M \) is not contained in \( L \) the characteristic functions \( \chi_L, \chi_M \) are observable functions but the functions \( \chi_L + \chi_M, \chi_L \cdot \chi_M \) are not. Anyway (exercise) it is already possible to prove now that if \( B_{L,M} \) is not contained in \( L \) then there exist two propositions \( P \) and \( Q \) in \( B_{L,M} \cap L \) with \( P \cap Q \not\in L \); the corresponding characteristic functions then \( \chi_P, \chi_Q \) are observable functions but \( \chi_P + \chi_Q, \chi_P \cdot \chi_Q \) are not.

Inside \( O \) there are some natural algebras of functions:

**Notation 4.** Let \( \mathcal{A} \) be a \( \sigma \)-algebra of subsets of \( S \) contained in \( L \), the symbol:
\[
\mathcal{O}_\mathcal{A} = \{ f \in \mathcal{O} : f^{-1}(B) \in \mathcal{A} \text{ for every borel subset } B \text{ of } \mathbb{R} \}
\]

will denote the space of all the observables performable in the context of the "classical logic" assigned by the family \( \mathcal{A} \).

**Theorem 3.** Let \( \mathcal{A} \) be a \( \sigma \)-algebra of subsets of \( S \) contained in \( L \), the space \( \mathcal{O}_\mathcal{A} \) is an algebra (over \( \mathbb{R} \)) of observable functions of \( S \) containing the constants. Moreover \( \mathcal{O}_\mathcal{A} \) is closed with respect to the left composition with the borel functions of \( \mathbb{R} \) and if \( f : S \to \mathbb{R} \) is a never zero observable in \( \mathcal{O}_\mathcal{A} \) the function \( 1/f \) is also in \( \mathcal{O}_\mathcal{A} \).

Proof. Taken two functions \( f, g \) in \( \mathcal{O}_\mathcal{A} \) let’s consider the map \( (f,g) : S \to \mathbb{R}^2 \) since \( \mathcal{A} \) is closed by intersections the inverse image with respect to \( (f,g) \) of every open rectangle of \( \mathbb{R}^2 \) is in \( \mathcal{A} \) and then also the inverse image with respect to \( (f,g) \) of every borel subset of \( \mathbb{R}^2 \) is in \( \mathcal{A} \).

Denoted by \( S : \mathbb{R}^2 \to \mathbb{R} \) and \( P : \mathbb{R}^2 \to \mathbb{R} \) the two borel functions given by the operations, respectively, of addition and multiplication on the real numbers we can deduce that the two functions: \( f + g = S \circ (f,g) \) and \( f \cdot g = P \circ (f,g) \) are in \( \mathcal{O}_\mathcal{A} \). \( \square \)

If \( \mathcal{A} \) is a \( \sigma \)-algebra of subsets of \( S \) contained in \( L \) closed with respect to the passage to a null equivalent element in \( L \) then \( \mathcal{O}_\mathcal{A} \) is closed with respect to the passage to a null equivalent function.

**Definition 10.** A map \( \nu : S \to S \) will be called a measure equivalence if:

- is bijective
- sends every \( S^1 \)-orbit \( [\varphi] \) in itself
- every map \( \nu|_{[\varphi]} : [\varphi] \to [\varphi] \) is a borel equivalence preserving the measure \( (\nu|_{[\varphi]}^* \mu_{[\varphi]} = \mu_{[\varphi]}). \)
Theorem 4. Two propositions $L$ and $M$ are equivalent in measure if and only if there exists a measure equivalence $\nu$ of $S$ such that $\nu(L)$ and $M$ are null equivalent.

Proof. ($\Longleftrightarrow$) Obvious.

($\Rightarrow$) Since the sets $L \cap [\varphi]$ and $M \cap [\varphi]$ have the same measure in $[\varphi]$ there exists a borel equivalence $\nu|_{[\varphi]} : [\varphi] \to [\varphi]$ preserving the measure such that $\nu|_{[\varphi]}(L \cap [\varphi])$ and $M \cap [\varphi]$ are equivalent up to a borel null subset (it is a consequence, for example, of Thm. 9 p. 327 in [R]). The family $\{\nu|_{[\varphi]}\}$ makes a a measure equivalence $\nu$ of $S$.

3. OBSERVABLES AND KAHELIAN FUNCTIONS

Definition 11. A function $l : S \to \mathbb{R}$ is called smooth kaehlerian on $S$ if:

- is smooth on $S$
- $l \circ \rho_\theta = l$ for every $\theta$ in $\mathbb{R}$
- for every couple of orthogonal vectors $\varphi$ and $\psi$ in $S$, $l(\cos t \cdot \varphi + \sin t \cdot \psi)$ is a function of the form $a \cdot \cos^2 t + b \cdot \sin t \cos t + c \cdot \sin^2 t$ in the variable $t$.

The smooth kaehlerian functions on $S$ are defined in such a way to be exactly the liftings to $S$ of the (smooth) kaehlerian functions defined on $\mathbb{P}_C(\mathcal{H})$ in [CMP], [G] and [CGM].

Notation 5. Let's denote by $KS(S)$ the vector space of all smooth kaehlerian functions on the space $S$.

For every smooth function $l : S \to \mathbb{R}$ such that $l \circ \rho_\theta = l$ for every $\theta$ in $\mathbb{R}$ the vectors $\text{Grad}_\varphi^\theta l$ and $J_\varphi \text{Grad}_\varphi^\theta l$ are horizontal.

For every proposition $L$ the function $\varphi \mapsto \mu_{[\varphi]}(L \cap [\varphi])$ is smooth kaehlerian.

The function $\langle A \rangle : S(\sqrt{2}) \to \mathbb{R}$ defined by $\langle A \rangle(\varphi) = \frac{1}{2} \langle \varphi, A\varphi \rangle$, for a bounded self-adjoint complex operator $A$ on $\mathcal{H}$, is a smooth kaehlerian function such that for every $X$ in $T_\varphi S$ it holds: $X_\varphi(A) = \langle A\varphi, X \rangle$.

We have $\text{Grad}_\varphi^\theta(\langle A \rangle) = \text{pr}_{\varphi^\perp} (A\varphi)$ because: $\langle \text{Grad}_\varphi^\theta(\langle A \rangle), X \rangle = \langle A\varphi, X \rangle = \langle \text{pr}_{T_\varphi S} (A\varphi), X \rangle$ for every $X$ in $T_\varphi S$, then $\text{Grad}_\varphi^\theta(\langle A \rangle) = A\varphi - \langle A \rangle \varphi \cdot \varphi$.

Definition 12. A function $l : W \cap S \to \mathbb{R}$, where $W$ is a complex dense linear subspace of $\mathcal{H}$, is called kaehlerian if:

- for every $\varphi$ in $W \cap S$ the map $d_\varphi l : W \cap T_\varphi S \to \mathbb{R}$ given by $d_\varphi l(X) = X_\varphi l$ is well defined, linear and continuous.
• for every complex closed linear subspace \( F \) of \( W \) the restriction \( l|_F \) is a smooth kaehlerian function on \( F \cap S \).
• the couple \((l, W)\) is maximal with respect to the two properties given above.

Notation 6. Let’s denote by \( K(S) \) the set of all kaehlerian functions on \( S \).

A smooth function \( l : S \to \mathbb{R} \) such that for some couple of orthogonal vectors \( \phi \) and \( \psi \) in \( S \) the function \( l((\cos t \cdot \phi + \sin t \cdot \psi)) \) has the form \( a \cdot \cos^2 t + b \cdot \sin t \cdot \cos t + c \cdot \sin^2 t \) can be expressed as:

\[
l(\cos t, \phi + \sin t, \psi) = l(\phi) + d_\phi l(\psi) \cdot \sin t \cdot \cos t + \frac{1}{2} H^l_\psi(\phi, \psi) \cdot \sin^2 t
\]

Where \( H^l \) is the hessian of the function \( l \) (it is enough to remember that \( d_\phi l(\psi) = (l \circ \gamma_\psi)'(0) \) and \( H^l_\psi(\phi, \psi) = (l \circ \gamma_\psi)''(0) \)

\[\text{since } \gamma_\psi \text{ is a geodesic curve).}\]

It is not difficult to prove that two smooth functions \( l_1, l_2 : S \to \mathbb{R} \) such that for every couple of orthogonal vectors \( \phi \) and \( \psi \) in \( S \) the functions \( l_1(\cos t \cdot \phi + \sin t \cdot \psi) \), \( l_2(\cos t \cdot \phi + \sin t \cdot \psi) \) have the form \( a \cdot \cos^2 t + b \cdot \sin t \cdot \cos t + c \cdot \sin^2 t \) are equal if (and only if) it is possible to find a vector \( \phi_0 \) in \( S \) where:

\[
l_1(\phi_0) = l_2(\phi_0), \quad d_\phi l_1 = d_\phi l_2, \quad H^l_\phi(\phi_0) = H^l_\phi(\phi_0)
\]

Theorem 5. Let \( l : S \to \mathbb{R} \) be a function, \( l \) is a smooth kaehlerian function if and only if there exists one (and only one) bounded self-adjoint complex linear operator \( A : H \to H \) such that:

\[
l(\phi) = \langle A \phi, \phi \rangle = \frac{1}{2} \langle \phi, A \phi \rangle
\]

for every \( \phi \) in \( S \).

Proof. \((\Leftarrow \Rightarrow)\) Obvious.

\((\Rightarrow\Leftarrow)\) (The proof mimics the proof given in [G]). Let’s fix a vector \( \phi_0 \) in \( S \).

Since the map \( d_{\phi_0} : T_{\phi_0}S \to \mathbb{R} \) is a continuous linear map there exists a vector \( Z (= \text{Grad}_{\phi_0} l) \) in \( T_{\phi_0}S \) such that \( d_{\phi_0} l(X) = \langle Z, X \rangle \) for every vector \( X \) in \( T_{\phi_0}S \);

moreover the Hessian \( H^l_{\phi_0} : T_{\phi_0}S \times T_{\phi_0}S \to \mathbb{R} \) is a continuous bilinear symmetric map and therefore there exists a bounded self-adjoint map \( B : T_{\phi_0}S \to T_{\phi_0}S \) such that \( H^l_{\phi_0}(X, Y) = \langle X, B(Y) \rangle \) for every \( X, Y \) in \( T_{\phi_0}S \).

There exists a unique continuous real linear map \( A : H \to H \) with the assigned value \( A(\phi_0) = l(\phi_0) \cdot \phi_0 + Z \) and such that: \( A(X) = \frac{1}{2} \langle Z, X \rangle \cdot \phi_0 + l(\phi_0) \cdot X + B(X) \) for every \( X \) in \( T_{\phi_0}S \).

It is not difficult to prove that \( A \) is symmetric.

The function \( \langle A \rangle : S \to \mathbb{R} \) defined by \( \langle A \rangle(\phi) = \frac{1}{2} \langle \phi, A \phi \rangle \) is smooth and for every couple of orthogonal vectors \( \phi, \psi \) in \( S \) the function \( \langle A \rangle(\gamma_\psi(t)) \) has the form \( a \cdot \cos^2 t + b \cdot \sin t \cdot \cos t + c \cdot \sin^2 t \). With some more calculation it is possible to verify that:

\[
\langle A \rangle(\phi_0) = l(\phi_0), \quad d_{\phi_0} \langle A \rangle = d_{\phi_0} l, \quad H^l_{\phi_0}(A) = H^l_{\phi_0}
\]

therefore \( \langle A \rangle = l \) and in particular \( \langle A \rangle \circ J = \langle A \rangle \).

Then written \( A = B + C \) with \( B = \frac{1}{2}(A - JA) \) and \( C = \frac{1}{2}(A + JA) \) it holds \( BJ = JB, CJ = -JC, \langle B \rangle \circ J = \langle B \rangle \) and \( \langle C \rangle \circ J = -\langle C \rangle \). Then \( \langle A \rangle \circ J = \langle A \rangle \).
implies \((C) = 0\), \(C = 0\) and \(AJ = JA\), that is \(A\) is a (bounded) self-adjoint complex operator.

If another complex linear self-adjoint operator \(B\) verifies \(<B>\varphi = <A>\varphi\) on \(\mathcal{S}\) then \(<X,AX> = <X,BX>\) for every \(X \neq 0\) and then \(A = B\).

**Theorem 6.** Let \(W\) be a complex linear dense subspace of \(\mathcal{H}\) and \(l : W \cap \mathcal{S} \to \mathbb{R}\) a function, the couple \((W,l)\) defines a kaehlerian function if and only if there exists one (and only one) self-adjoint operator \(A : W \to \mathcal{H}\) such that:

\[
\mathcal{D}(A) = W \quad \text{and} \quad l(\varphi) = <A>\varphi = \frac{1}{2} <\varphi,A\varphi>
\]

for every \(\varphi\) in \(W \cap \mathcal{S}\).

**Proof.** \((\Rightarrow)\) Taken \(\varphi\) in \(W \cap \mathcal{S}\) the map \(\lambda_{\varphi} : W \to \mathbb{R}\) defined by:

\[
\lambda_{\varphi}(X) = \left[ X - \frac{1}{2} <X,\varphi> \cdot \varphi \right] (l) + <X,\varphi> \cdot l(\varphi)
\]

is linear and continuous, therefore admits a continuous linear extension \(\tilde{\lambda}_{\varphi} : \mathcal{H} \to \mathbb{R}\) and is possible to find a unique element \(A_0(\varphi)\) in \(\mathcal{H}\) such that: \(<X,A_0(\varphi)> = \lambda_{\varphi}(X)\) for every \(X\) in \(\mathcal{H}\).

The map so obtained \(A_0 : W \cap \mathcal{S} \to \mathcal{H}\) can be extended to a map \(\hat{A}_0 : W \to \mathcal{H}\) with the position \(\hat{A}_0(\varphi) = \frac{\|\varphi\|_2}{\sqrt{2}} A_0(\frac{\varphi}{\|\varphi\|_2})\) for every \(\varphi \neq 0\) and \(\hat{A}_0(0) = 0\).

Fixed a closed complex linear subspace \(F\) of \(W\) we know, by the previous theorem, that there exists a (complex) self-adjoint linear operator \(A_F : F \to F\) such that

\[
l(\varphi) = \frac{1}{2} <\varphi,A_F(\varphi)>
\]

for every \(\varphi\) in \(F \cap \mathcal{S}\).

Then because \(X = \left[X - \frac{1}{2} <X,\varphi> \cdot \varphi \right] + \frac{1}{2} <X,\varphi> \cdot \varphi\) for \(\varphi\) in \(F \cap \mathcal{S}\) and \(X\) in \(F\), remembering that \(Y_{\varphi}l = <Y_{\varphi},A_F(\varphi)>\), for every \(\varphi\) in \(F \cap \mathcal{S}\) and \(Y\) in \(F \cap (\varphi)^{\perp}\), we can prove that \(<X,A_F(\varphi)> = <X,\hat{A}_0(\varphi)>\). That is \(A_F(\varphi) = pr_F(\hat{A}_0(\varphi))\). Hence:

\[
pr_F(\hat{A}_0(\varphi)) = A_F(\varphi) \quad \text{for every} \quad F \subset W \quad \text{and every} \quad \varphi \in F.
\]

Using this property systematically and the fact that \(pr_F(Z) = 0\) for every \(F \subset W\) implies \(Z = 0\) it is possible to prove that \(\hat{A}_0\) is a complex, linear and hermitian operator defined on a dense linear subspace of \(\mathcal{H}\). Its closure \(A = \overline{A_0}\) is the desired self-adjoint operator since on every \(\varphi\) in \(W\) taken a closed suspace \(F \subset W\) with \(\varphi\) in \(F\) we have

\[
\frac{1}{2} <\varphi,\overline{A_0}(\varphi)> = \frac{1}{2} <\varphi,pr_FA(\varphi)> = \frac{1}{2} <\varphi,A_F(\varphi)> = l(\varphi).
\]

Therefore \((\mathcal{D}(A) \cap \mathcal{S},<A>)\) extends \((W \cap \mathcal{S},l)\) but for the maximality it must hold \(\mathcal{D}(A) \cap \mathcal{S} = W \cap \mathcal{S}\) and then \(\mathcal{D}(A) = W\).

As in the proof of the previous theorem it is possible to prove the unicity on \(W\) of such operator.

\((\Leftarrow)\) For every self-adjoint complex operator \(A : W \to \mathcal{H}\) the first two properties in the definition of a (non smooth) kaehlerian function are easily verified for the function \(<A> : \mathcal{D}(A) \cap \mathcal{S}(\sqrt{2}) \to \mathbb{R}\) defined by \(<A>(\varphi) = \frac{1}{2} <\varphi,A\varphi>\). About the maximality: if the couple \((W,l)\) extends \((\mathcal{D}(A),<A>)\) reasoning as above it is possible
to find a self-adjoint operator $A'$ with $(\mathcal{D}(A'), \langle A' \rangle)$ extending $(W, l)$; but $A$ and $A'$
are self-adjoint therefore $\mathcal{D}(A) = \mathcal{D}(A')$ and $(W, l) = (\mathcal{D}(A), \langle A \rangle)$.

\textbf{Notation 7.} The map $A \mapsto \langle A \rangle$ between self-adjoint operators and kaehlerian functions on $\mathcal{S}$ is bijective; the restricted map between bounded self-adjoint operators and smooth kaehlerian functions is a linear isomorphism. Denoted by $SA(\mathcal{H})$ the set of all (bounded or unbounded) self-adjoint operators of $\mathcal{H}$ and by $SAB(\mathcal{H})$ its subset (a vector space) of all bounded operators, let’s denote by $\alpha : K(\mathcal{S}) \rightarrow SA(\mathcal{H})$ the bijective map defined by the previous theorem ($\alpha(l)$ is the self-adjoint operator associated to the kaehlerian function $l$). Its inverse is the map $\langle \cdot \rangle : SA(\mathcal{H}) \rightarrow K(\mathcal{S})$. This map induces an isomorphism $\alpha| : KS(\mathcal{S}) \rightarrow SAB(\mathcal{H})$.

Since we have proved that for every bounded self-adjoint operator $A$ we have: $\text{Grad}_\varphi \langle A \rangle = A\varphi - \langle A \rangle \varphi \cdot \varphi$ for every smooth kaehlerian function $l$ it holds:

$$\alpha(l)(\varphi) = \text{Grad}_\varphi l + l(\varphi) \cdot \varphi$$

\textbf{Theorem 7.} For every proposition $L$ of $\mathcal{S}$ there exists one and only one (complex) orthogonal projector $E$ of $\mathcal{H}$ such that for every $\varphi$ in $\mathcal{S}$ it holds:

$$\mu_{[\varphi]}(L \cap [\varphi]) = \langle E \rangle$$

Conversely every orthogonal projector $E$ of $\mathcal{H}$ comes in this way from a proposition $L$ and two propositions give origin to the same projector if and only if they are measure equivalent.

\textbf{Proof.} When $L$ is null equivalent to $\emptyset$ or to $\mathcal{S}$ the thesis follows immediately taking, respectively, $E = 0$ and $E = I$. We will then suppose $L$ not null equivalent to $\emptyset$ or $\mathcal{S}$.

Since the function $\varphi \mapsto \mu_{[\varphi]}(L \cap [\varphi])$ is smooth kaehlerian there exists a bounded self-adjoint operator $E$ such that $\mu_{[\varphi]}(L \cap [\varphi]) = \langle E \rangle$; we have only to prove that $E$ verifies $E^2 = E$.

We have already proved in a remark above that for every $\varphi$ in $\mathcal{S}$ we have:

$$\text{Grad}_\varphi \langle E \rangle = E\varphi - \langle E \rangle \varphi.$$ Therefore $\|\text{Grad}_\varphi \langle E \rangle\|^2 = 2 \cdot (\langle E \rangle \varphi - \langle E \rangle^2 \varphi)$ and if we prove that $\|\text{Grad}_\varphi \langle E \rangle\|^2 = 2 \cdot (\langle E \rangle \varphi - \langle E \rangle^2 \varphi)$ for every $\varphi$ in $\mathcal{S}$ we get $E^2 = E$.

Now $\|\text{Grad}_\varphi \langle E \rangle\|^2 =
= \max_{\|\tau\| = 1} \left\|\langle \text{Grad}_\varphi \langle E \rangle, \tau \rangle \right\| = \max_{\psi \perp \varphi, \|\psi\| = \sqrt{2}} \left[\left\|\langle \text{Grad}_\varphi \langle E \rangle, \frac{1}{\sqrt{2}} \psi \rangle \right\|\right] =
= \frac{1}{\sqrt{2}} \max_{\psi \perp \varphi, \|\psi\| = \sqrt{2}} \left[\left|\langle E \rangle \circ \gamma_{\varphi \psi} \right|(0)\right|.$

Written $\langle E \rangle \circ \gamma_{\varphi \psi}(t) = \frac{1}{2} [M(\psi) + m(\psi)] + (M(\psi) - m(\psi)) \cdot \cos(2t + \alpha(\psi))]$ where

$$0 \leq m(\psi) = \min(\langle E \rangle \circ \gamma_{\varphi \psi}) \leq \max(\langle E \rangle \circ \gamma_{\varphi \psi}) = M(\psi) \leq 1$$

It is not difficult to calculate:

$$\left|\langle E \rangle \circ \gamma_{\varphi \psi} \right|(0)^2 = (M(\psi) - m(\psi))^2 \cdot \sin^2(\alpha(\psi))$$

and

$$4(\langle E \rangle \varphi - \langle E \rangle^2 \varphi) \geq (M(\psi) - m(\psi))^2 \cdot \sin^2(\alpha(\psi)).$$
By the definition of proposition we know there exists a vector \( \psi_0 \) in \( S \) orthogonal to \( \varphi \) where \( m(\psi_0) = 0 \) and \( M(\psi_0) = 1 \); we have: \( \langle E \rangle_\varphi = \frac{1}{2}[1 + \cos(\alpha(\psi_0))] \) and \( |(\langle E \rangle_\varphi \varphi(\psi_0))'(0)|^2 = \sin^2(\alpha(\psi_0)) = 4(\langle E \rangle_\varphi \varphi(\psi_0) - \langle E \rangle_\varphi)^2 \) with respect to this vector.

Therefore \( \| \text{Grad}_{\varphi} \langle E \rangle \|^2 = 2(\langle E \rangle_\varphi - \langle E \rangle_{\varphi}^2) \) and we have proved that \( E \) is a projector. Since \( L \) prescribes \( \langle E \rangle \) on \( S \) the projector \( E \) is the only one.

Conversely let \( E \) be a projector on a complex closed linear subspace \( F \) of \( H \), if \( E = 0 \) or \( E = I \) the thesis follows immediately. We will then suppose \( E \neq 0 \) and not \( I \).

For every \( S^1 \)-orbit \([\varphi]\) choose a borelian subset \( L_{[\varphi]} \) such that \( \mu_{[\varphi]}(L_{[\varphi]}) = \langle E \rangle_\varphi \).
The set \( L = \bigcup_{[\varphi]} L_{[\varphi]} \) is pseudo-borelian in \( S \) and the map \( \varphi \mapsto \mu_{[\varphi]}(L \cap [\varphi]) \) is smooth as the map \( \varphi \mapsto \langle E \rangle_\varphi \).

For a vector \( \psi \) in \( S \) orthogonal to \( \varphi \) the map \( t \mapsto \gamma_{[\varphi]}(L \cap [\varphi]) \) has the form \( a \cos^2 t + b \sin t \cos t + c \sin^2 t \).

Moreover if \( \varphi \) is in \( F \) we can take \( \psi \) in \( F^\perp \) with \( \gamma_{[\varphi]}(L \cap [\varphi]) = \cos^2 t \) and analogously if \( \varphi \) is in \( F^\perp \).

When \( \varphi \) is not in \( F \) or in \( F^\perp \) we can find a vector \( \alpha \) in \( F \cap S \) and a vector \( \beta \) in \( F^\perp \cap S \) in such a way that \( \varphi = \cos t_0 \alpha + \sin t_0 \beta \). If we consider the vector \( \psi = -\sin t_0 \alpha + \cos t_0 \beta \) we get \( \gamma_{[\varphi]}(L \cap [\varphi]) = \cos^2(t + \theta) \).

Two propositions \( L \) and \( L' \) with \( \mu_{[\varphi]}(L \cap [\varphi]) = \langle E \rangle_\varphi = \mu_{[\varphi]}(L' \cap [\varphi]) \) are obviously measure equivalent.

To give an explicit proposition \( L \) associated to a projector \( E \) we can proceed as follows: let’s fix a map \( \sigma : \mathbb{F}_C(H) \rightarrow S \) such that \( \sigma([\varphi]) \in [\varphi] \) for every \( S^1 \)-orbit \([\varphi]\), the pseudo-borelian subset \( L^\varphi \) is \( \sigma([\varphi]) \in S \) is a proposition associated to \( E \). All the others propositions associated to \( E \) are the \( \nu(L^\varphi) \) (where \( \nu \) is a measure equivalence of \( S \)) and their null equivalent. Therefore in this theory it is possible to claim that “behind” a quantum state \([\varphi]\) there are infinite “hidden” states: the Hilbert vectors \( e^{i\theta} \sigma([\varphi]) \) (where \( e^{i\theta} \) varies in the group \( S^1 \)) and “behind” a quantum proposition \( E \) there are infinite “hidden” classical propositions: essentially the propositions \( \nu(L^\varphi) \) (where \( \nu \) varies in the group of all measure equivalences of \( S \)). The truth value 0 or 1 of one of these hidden classical proposition on the hidden classical state is \( \chi_{[\nu(L^\varphi)]}(e^{i\theta} \sigma([\varphi])) \) and depends not only on the “hidden variable” \( e^{i\theta} \) but also on the “experimental context” defined by the measure equivalence \( \nu \).

The suggestion that to each quantum measurement there could correspond a collection of several deterministic “hidden measurements” has appeared in several moments in the history of the hidden variable theories. For a direction of development of this idea cfr. [A] and [CM].

**Notation 8.** Denoted by \( PR(H) \) the set of all projector operators of \( H \) the previous theorem claims there is a surjective map \( \varepsilon : L \rightarrow PR(H) \) associating to each proposition \( L \) a projector \( \varepsilon(L) \) such that \( \langle \varepsilon(L) \rangle_\varphi = \mu_{[\varphi]}(L \cap [\varphi]) \) for every \( \varphi \) in \( S \).

A pseudo-borel subset \( A \) of \( S \) such that \( \mu_{[\varphi]}(A \cap [\varphi]) = \langle E \rangle_\varphi \) (for every \( \varphi \) in \( S \)) for a projector \( E \) is necessarily a proposition (is measure equivalent to a proposition).
Theorem 8. If \( L \) and \( M \) are propositions with \( L \subset M \) then also \( M \setminus L \) is a proposition.

Proof. \( L \subset M \) implies \( \varepsilon(L) \leq \varepsilon(M) \) therefore \( \varepsilon(L) \) and \( \varepsilon(M) \) commute and the difference \( \varepsilon(M) - \varepsilon(L) \) is also a projector. Therefore there exists a proposition \( N \) such that \( \varepsilon(N) = \varepsilon(M) - \varepsilon(L) \) and the pseudo-borel subset \( M \setminus L \), because is measure equivalent to \( N \), is a proposition. \( \square \)

Theorem 9. Given two propositions \( L \) and \( M \) if (and only if) \( L \cap M \) is contained in \( L \) then all the family \( B_{L,M} \) is contained in \( L \) (and \( M \) are compatible).

Proof. We already know that \( \emptyset, S, L, M, L \cap M, CL, CM \) and \( CL \cup CM \) are in \( L \).
Moreover \( L \cap M \subset L \) implies \( L \setminus L \cap M = L \cap CM \) and \( CL \cup CM \) are in \( L \). \( L \cap M \subset M \) implies \( CL \cap M = CL \) and \( L \cup CM \) are in \( L \). \( CL \cap M \subset CL \) implies \( CL \cap CM \) and \( L \cup M \) are in \( L \). For a proposition \( L \) we have \( \varepsilon(L) = 0 \) if and only if \( L \) is a pseudo-borel null subset.

The following properties are easily proved: \( \varepsilon(\emptyset) = 0 \), \( \varepsilon(S) = I \), \( \varepsilon(CL) = I - \varepsilon(L) \), \( \varepsilon(L) = \varepsilon(\nu L) \) for every measure equivalence \( \nu \) on \( S \). \( \varepsilon(L) = \varepsilon(L') \) if \( L \) and \( L' \) are null equivalent, \( L \subset M \) implies \( \varepsilon(L) \leq \varepsilon(M) \) and \( \varepsilon(M \setminus L) = \varepsilon(M) - \varepsilon(L) \). \( L \cap M = \emptyset \) implies \( \varepsilon(L) \perp \varepsilon(M) \).

Theorem 10. For every proposition \( L \) and every unitary transformation \( U \) it holds:
\[
\varepsilon(U(L)) = U \circ \varepsilon(L) \circ U^{-1}
\]

Proof. \( \langle U \circ \varepsilon(L) \circ U^{-1} \rangle_\varphi = \langle \varepsilon(L) \rangle_{U^{-1},\varphi} = \mu_{[U^{-1},\varphi]}(L \cap [U^{-1},\varphi]) = \mu_{[\varphi]}(U(L) \cap [\varphi]) \)
therefore \( \varepsilon(U(L)) = U \circ \varepsilon(L) \circ U^{-1} \). \( \square \)

Theorem 11. For every observable function \( f \) on \( S \) there exists one and only one self-adjoint operator \( T \) on \( H \) such that for every state \( \varphi \) in \( S \) and every borel subset \( B \) of \( R \) it holds:
\[
\mu_{[\varphi]}(f^{-1}(B) \cap [\varphi]) = \langle E_B^T \rangle_\varphi
\]

Conversely every self-adjoint operator \( T \) on \( H \) comes, in this way, from an observable function.

Proof. For every \( s \) in \( R \) let \( L_s = f^{-1}((-\infty, s]) \) and \( E_s = \varepsilon(L_s) \). It is not difficult to prove that the family \( \{E_s\}_{s \in R} \) is a spectral family of \( H \), therefore there is a self-adjoint complex operator \( T \) such that \( E_{(-\infty, s]}^T = E_s \) for every \( s \) in \( R \). That is \( \mu_{[\varphi]}(f^{-1}((-\infty, s]) \cap [\varphi]) = \langle E_{(-\infty, s]}^T \rangle_\varphi \) for every \( \varphi \) in \( S \) and every \( s \) in \( R \). Then, for the usual properties of borelian subsets of \( R \), it is possible to prove that the analog properties hold for an interval \( [r, s] \) and a general borel subset \( B \) of \( R \):
\[
\mu_{[\varphi]}(f^{-1}([r, s]) \cap [\varphi]) = \langle E_{[r,s]}^T \rangle_\varphi, \quad \mu_{[\varphi]}(f^{-1}(B) \cap [\varphi]) = \langle E_B^T \rangle_\varphi
\]

If \( T' \) is another operator with the same property, from \( \langle E_B^T \rangle_\varphi = \langle E_{B'}^T \rangle_\varphi \) for every \( \varphi \) and \( B \) it follows first \( E_B^T = E_{B'}^T \) for every \( B \) and then \( T = T' \).
Conversely let’s suppose a self-adjoint operator $T$ is given.

Let’s fix a map $\sigma : \mathbb{P}_C(\mathcal{H}) \to \mathbb{S}$ such that $[\sigma \varphi ] \in \{ \varphi \}$ for every $\mathbb{S}$-orbit $[\varphi ]$, it is possible to define a bijective map $\delta_{[\varphi ]} : [\varphi ] \to \mathbb{S}$ with $\delta_{[\varphi ]}(\psi) = \psi/\sigma [\varphi ]$; such a map is a borel isomorphism preserving the measure $(\delta_{[\varphi ]})_* \mu_{[\varphi ]} = \mu_{[\sigma \varphi ]})$.

Also the map $\rho : \mathbb{S} \to [0,1]$ defined by $\rho(u) = \frac{1}{2\pi} (\pi + Arg(u))$ is a borel isomorphism preserving the measure $(\rho_* \mu_{[\mathbb{S}]}) = \lambda_{[0,1]}$.

The cumulative function $F_{[\varphi ]} : \mathbb{R} \to [0,1]$ defined by $F_{[\varphi ]}(s) = \langle E_{T}^{T}(\cdot) \rangle \varphi$ is a monotone non-decreasing function with the property:

$$(F_{[\varphi ]})_* \lambda(B) = \nu_{F_{[\varphi ]}}(B)$$

(cfr. [K-S] thm. 4 p. 94) (the symbol $\nu_{F}$ denotes the Borel measure associated to the monotone function $F$). The function $\widetilde{F_{[\varphi ]}}$ can be extended monotonically to a function (denoted in the same way) $\widetilde{F_{[\varphi ]}} : [0,1] \to \mathbb{R} \cup \{ +\infty \}$ with the position $\widetilde{F_{[\varphi ]}}(1) = +\infty$ and has the same property provided that we decide that $\nu_{\widetilde{F_{[\varphi ]}}}(\{ +\infty \}) = 0$.

The function $f : \mathbb{S} \to \mathbb{R} \cup \{ +\infty \}$ defined on each $[\varphi ]$ by $f_{[\varphi ]} = \widetilde{F_{[\varphi ]}} \circ \rho \circ \delta_{[\varphi ]}$ is a pseudo-borelian function on $\mathbb{S}$ with the property:

$$(f_{[\varphi ]})_* \mu_{[\varphi ]} = \nu_{F_{[\varphi ]}}$$

that is: $\mu_{[\varphi ]}(f^{-1}(r,s) \cap [\varphi ]) = \widetilde{F_{[\varphi ]}}(s) - \widetilde{F_{[\varphi ]}}(r) = \langle E_{T}^{T}(r) \rangle \varphi$. This implies as usual: $\mu_{[\varphi ]}(f^{-1}(B) \cap [\varphi ]) = \langle E_{B}^{T} \rangle \varphi$ for every borel subset $B$ of $\mathbb{R}$.

Since $f^{-1}(\{ +\infty \}) = \{ -\sigma [\varphi ]; [\varphi ] \in \mathbb{P}_C(\mathcal{H}) \}$ is a pseudo-borel null subset of $\mathbb{S}$ we can assign to $f$ on $f^{-1}(\{ +\infty \})$ finite values in such a way to obtain a pseudo-borel function $f : \mathbb{S} \to \mathbb{R}$ keeping the property:

$\mu_{[\varphi ]}(f^{-1}(B) \cap [\varphi ]) = \langle E_{B}^{T} \rangle \varphi$ for every borel subset $B$ of $\mathbb{R}$.

As observed in a remark following the previous theorem this implies that each $f^{-1}(B)$ is a proposition, that is $f$ is an observable function.

Remembering the definition of a quasi-inverse function (cfr. [K-S] thm. 4 p. 94) we can give the following explicit expression (out of the null pseudo-borel subset $-\sigma (\mathbb{P}_C(\mathcal{H}))$ of the function $f$ associated to the operator $T$ (with the help of $\sigma$) in the preceding proof:

$$f(\varphi) = \min \left\{ s \in \mathbb{R}; \langle E_{T}^{T}(\cdot) \rangle \varphi \geq \frac{1}{2\pi} (\pi + Arg(\varphi/\sigma [\varphi ])) \right\}$$

We will denote this function by $f_{T}^{\sigma}$.

Notation 9. Let’s denote by $\tau : \mathcal{O} \to SA(\mathcal{H})$ the (surjective) map defined by the previous theorems.

In the proof of the previous theorem we have proved in particular that for an observable function $f$ it holds: $f_{[\varphi ]} \mu_{[\varphi ]} = \nu_{f_{[\varphi ]}}$. 
Note that: \( E^{\tau(f)}_B = \varepsilon(f^{-1}(B)) \) for every borel subset \( B \) of \( \mathbb{R} \) and moreover \( \varepsilon(L) = \tau(x_L) \).

For every projector \( E \) we have that \( L^E_\sigma \) is null equivalent to \( (f^E_\sigma)^{-1}(\{1\}) \) (with the notations chosen above).

Moreover: \( \tau(f) = 0 \) if only if \( f \) is (essentially) zero and \( \tau(k) = k \cdot I \) for every constant function \( k \) on \( \mathbb{S} \).

**Theorem 12.** For every observable function \( f \) and every unitary transformation \( U \) it holds:
\[
\tau(f \circ U^{-1}) = U \circ \tau(f) \circ U^{-1}
\]

**Proof.** \( E^U_{(-\infty,s]} = U \circ E^{\tau(f)}_{(-\infty,s]} \circ U^{-1} = U \circ \varepsilon(f^{-1}((-\infty,s])) \circ U^{-1} = \varepsilon((f \circ U^{-1})^{-1}((-\infty,s])) = E^{\tau(f \circ U^{-1})}_{(-\infty,s]} \) for every \( s \) in \( \mathbb{R} \).

\[\square\]

**Theorem 13.** If \( f \) is an observable function and \( \nu \) is a measure equivalence then \( \tau(f \circ \nu) = \tau(f) \)

**Proof.** Infact \( \mu_\varphi((f \circ \nu)^{-1}(B) \cap [\varphi]) = \mu_\varphi(f^{-1}B \cap [\varphi]) \) for every \( \varphi \) and every \( B \).

\[\square\]

Given a self-adjoint operator \( T \) there are infinite observable functions \( f \) such that \( \tau(f) = T \), infact for every measure equivalence \( \nu \) we have \( \tau(f \circ \nu) = T \). These however are not the only ones since it is enough to take \( \nu \) in such a way to have \( \nu \cdot \mu_\varphi = \mu_\varphi \) for every \( \mathbb{S}^1 \)-orbit \( [\varphi] \) : it is not necessary for \( \nu \) to be a borel isomorphism for every \( \mathbb{S}^1 \)-orbit.

Therefore "behind" a quantum observable \( T \) there are infinite "hidden" classical observable functions \( f \); the choice of a particular function depends on the "experimental context" choosen and gives the value \( f(\varphi) \) of the hidden classical observable on the hidden classical state \( \varphi \).

**Theorem 14.** Given two compatible propositions \( L \) and \( M \) the corresponding projectors \( \varepsilon(L) \) and \( \varepsilon(M) \) commute.

**Proof.** Infact there is an observable function \( f \) and two borel subsets \( B,C \) of \( \mathbb{R} \) such that \( L = f^{-1}(B) \) and \( M = f^{-1}(C) \), therefore \( \varepsilon(L) = E^{\tau(f)}_B \) and \( \varepsilon(M) = E^{\tau(f)}_C \) are in the same spectral measure.

\[\square\]

In other words given two non commuting projectors \( E \) and \( F \) it is not possible to find two propositions \( L \) and \( M \) with \( \varepsilon(L) = E \) and \( \varepsilon(M) = F \) and moreover with \( L \cap M \) in \( \mathcal{L} \); this means that in this theory it is never possible to check if a state \( \varphi \) in \( \mathcal{S} \) verifies, in the same time, two non compatible properties.

If \( \varepsilon(L) \) and \( \varepsilon(M) \) don't commute the "precise observer" can build an apparatus checking the property \( L \) and separately an apparatus checking the property \( M \) but he will never be able to build an apparatus checking the classical property \( L \ AND \ M \) and satisfying the properties stated for a proposition!
The decision to add to \( \mathcal{L} \) all the intersections \( L \cap M \) or, better, to consider all the boolean algebra (\( \sigma \)-algebra) generated by \( \mathcal{L} \) is equivalent to consider possible some behaviour for the probabilities not predicted by the usual Quantum Mechanics.

Assigned two projectors \( E \) and \( F \) we know that it is possible to find two propositions \( L \) and \( M \) with \( E = \varepsilon(L) \), \( F = \varepsilon(M) \); it is possible to find \( L \) and \( M \) and a pseudo-borelian subset \( N \) in \( \mathcal{S} \) such that for every \( \varphi \) in \( \mathcal{S} \) it holds:

\[
\mu_{\{\varphi\}}(N \cap [\varphi]) = \mu_{\{\varphi\}}(L \cap [\varphi]) \cdot \mu_{\{\varphi\}}(M \cap [\varphi])
\]

The answer is yes, it is not difficult and you can also take \( N = L \cap M \); let's fix a map \( \sigma : \mathbb{F}_C(\mathcal{H}) \to \mathcal{S} \) such that \( \sigma([\varphi]) \in [\varphi] \) for every \( S^1 \)-orbit \([\varphi] \), the two pseudo-borelian subsets

\[
L = \bigcup [\varphi] e^{-i2(E)_{\varphi} + i2(F)_{\varphi}} (F)_{\varphi} = \bigcup [\varphi] e^{-i2(E)_{\varphi} + i2(F)_{\varphi}} (F)_{\varphi}
\]

are propositions with \( \varepsilon(L) = E \), \( \varepsilon(M) = F \) and \( \mu_{\{\varphi\}}(L \cap M \cap [\varphi]) = \langle E \rangle_\varphi \cdot \langle F \rangle_\varphi \).

The pseudo-borelian subset \( L \cap M \) is not, in general, a proposition because if you take two orthogonal elements \( \varphi \) and \( \psi \) in \( \mathcal{S} \) (not in the same \( S^1 \)-orbit) and consider the states parametrized by the path \( \gamma(t) = t \cdot \varphi + \sin t \cdot \psi \) (the superposition states of \( \varphi \) and \( \psi \) in \( \mathcal{S} \)) then the function: \( \mu_{\gamma(t)}(L \cap M \cap [\gamma(t)]) = \langle E \rangle_{\gamma(t)} \cdot \langle F \rangle_{\gamma(t)} \) is the product of two functions of the form \( a \cos^2 t + b \sin t \cdot \cos t + c \sin^2 t \) and is not of the same form (unless one of the factor is constant).

If we stay in \( \mathcal{L} \) in particular there is not hope to use the informations coming from two apparatuses corresponding to \( L \) and \( M \) "acting in two spatially separable regions of the spacetime" to decide whether or not a state \( \varphi \) is in \( L \cap M \)!

This is connected with the absence of meaningful propositions furnished with the following special independence (the one considered by Bell to prove his inequalities):

**Definition 13.** Two propositions \( L \) and \( M \) will be called **totally independent** if there exists a proposition \( N \) such that for every \( \varphi \) in \( \mathcal{S} \) it holds:

\[
\mu_{\{\varphi\}}(N \cap [\varphi]) = \mu_{\{\varphi\}}(L \cap [\varphi]) \cdot \mu_{\{\varphi\}}(M \cap [\varphi])
\]

Two propositions \( L \) and \( M \) will be called **banally independent** if one of them is null equivalent to \( \emptyset \) or to \( \mathcal{S} \).

**Theorem 15.** Two propositions (in \( \mathcal{L} \)) are totally independent if and only if are banally independent.

**Proof.** (\( \Longleftarrow \)) Obvious.

(\( \Longrightarrow \)) Let \( E = \varepsilon(L) \), \( F = \varepsilon(M) \) and \( G = \varepsilon(N) \); we have \( \langle G \rangle_\varphi = \langle E \rangle_\varphi \cdot \langle F \rangle_\varphi \) for every non zero vector in \( \mathcal{H} \) by hypothesis.

This implies \( \ker G = \ker E \cup \ker F \) but this is possible only for \( \ker E \subset \ker F \) or \( \ker F \subset \ker E \).

In the first case we have: \( G = F \) and \( \langle F \rangle_\varphi \cdot (1 - \langle E \rangle_\varphi) = 0 \), therefore or \( F = 0 \) or \( E \neq 0 \) and \( E = I \).

In the other case analogously or \( E = 0 \) or \( F = I \).  

\( \square \)
Theorem 16. For every observable function $f$ we have: $\text{Im}_c(f) = \text{spec}(\tau(f))$.

Proof. A value $y$ is not in the spectrum of $\tau(f)$ if and only if there a neighborhood $|y - \sigma, y + \sigma|$ of $y$ such that $E^\tau(f)|_{[y - \sigma, y + \sigma]} = 0$ (cfr. [W] thm. 7.22 pag. 200), therefore if and only if $\varepsilon(f^{-1}([y - \sigma, y + \sigma])) = 0$ or, equivalently, when $f^{-1}([y - \sigma, y + \sigma])$ is a pseudo-borel null subset of $\mathbb{S}$; but this is precisely the condition for the value $y$ not to lie in $\text{Im}_c(f)$. \hfill \Box

The observable functions are rarely continuous, for example when $\text{spec}(\tau(f))$ is not an interval of $\mathbb{R}$ the observable function $f : \mathbb{S} \to \mathbb{R}$ is not continuous.

Theorem 17. Fixed an $\mathbb{S}^1$-orbit $[\varphi_0]$, for every borel function $b : [\varphi_0] \to \mathbb{R}$ there is an observable function $f$ such that $f|[\varphi_0] = b$.

Proof. Let’s suppose first $\mathcal{H} = L^2_{\mathbb{C}}(\mathbb{R}, \lambda)$ and $\varphi_0$ a positive real continuous function in $\mathbb{S}$.

As in the proof of a previous theorem let’s fix a map $\sigma : \mathbb{P}_{\mathbb{C}}(\mathcal{H}) \to \mathbb{S}$ such that $\sigma([\varphi]) \in [\varphi]$ for every $\mathbb{S}^1$-orbit $[\varphi]$; associated to the self-adjoint position operator $Q$ there is the cumulative function $F_{[\varphi_0]} : \mathbb{R} \to [0, 1]$ defined by the equality: $F_{[\varphi_0]}(s) = \langle E_{(-\infty, s]}^Q \rangle_{\varphi_0} = \frac{1}{2} f^*_{\sigma_{-\infty}} \varphi_0(r)^2 \cdot d\lambda(r)$. This function is a homeomorphism between $\mathbb{R}$ and $[0, 1]$, therefore its quasi-inverse is equal to its inverse and is again a homeomorphism and is a homeomorphism too the restricted function $f^0 : [\varphi_0] \setminus \{\varphi_0\} \to \mathbb{R}$. Let $f_1$ be an observable function on $\mathbb{S}$ extending $f^0_{\sigma}$ to the pseudo-borel null subset $-\sigma \mathbb{P}_{\mathbb{C}}(\mathcal{H})$.

The real function $c = b \circ (f^0_{\sigma})^{-1}$ is a borel function and then the composed function $f_2 = c \circ f_1$ is an observable function with $f_2|[\varphi_0] = b$ out of $-\sigma [\varphi_0]$, then the observable function $f$ defined as $f_2$ out of $-\sigma \mathbb{P}_{\mathbb{C}}(\mathcal{H})$ and defined as $b(-\sigma [\varphi_0])$ on $-\sigma \mathbb{P}_{\mathbb{C}}(\mathcal{H})$ is the required function.

For any other vector $\varphi_1$ in $\mathbb{S}$ taken a unitary map $U$ such that $U(\varphi_0) = \varphi_1$ the function $b \circ f \circ U^{-1}$ will have the desired property with respect to $\varphi_1$.

The proof above extends easily to any separable Hilbert space (with infinite dimension).

For a general Hilbert space (with infinite dimension) it is possible to proceed in an analogous way on a single addend remembering that any such space is isomorphic to a direct Hilbert sum of a family $\{L^2_{\mathbb{C}}(\mathbb{R}, \lambda)\}_\beta$ of $L^2$-spaces. \hfill \Box

Remark 3. In particular taken $\varphi_0$ in $\mathbb{S}$ there exists an observable function coincident on $[\varphi_0]$ with the function: $d(\varphi_0, \varphi) = |\text{Arg}(\varphi/\varphi_0)|$ (the "phase distance from the state $\varphi_0$").
Definition 14. Let \( f \) be an observable function on \( S \), let’s denote by:
\[
\mathcal{D}(\langle f \rangle) = \{ \varphi \in S; f|_{[\varphi]} \in L^2([\varphi], \mu_{[\varphi]}) \}
\]
the domain of the mean value function of \( f \):
\[
\langle f \rangle : \mathcal{D}(\langle f \rangle) \to \mathbb{R}
\]
defined by \( \langle f \rangle(\varphi) = \int_{[\varphi]} f|_{[\varphi]} \cdot d\mu_{[\varphi]} \).

Note that since \( \mu_{[\varphi]}([\varphi]) \) is finite it holds: \( L^2([\varphi], \mu_{[\varphi]}) \subset L^1([\varphi], \mu_{[\varphi]}) \), then \( \langle f \rangle \) is well defined with: \( \langle f \rangle(\varphi) = \int_{[\varphi]} f|_{[\varphi]} \cdot d\mu_{[\varphi]} = e^{i\theta} \int_0^{2\pi} f(e^{i\theta} \cdot \varphi) \cdot d\lambda(\theta) \).

Let’s remember that we have: \( D(T) = \{ \varphi \in \mathcal{H}; \int_{\mathbb{R}} t^2 \cdot d\nu_{F^T_\varphi} < +\infty \} \) for every self-adjoint operator \( T \) (cfr. [W] where \( F^T_\varphi : \mathbb{R} \to \mathbb{R} \) is the function defined by \( F^T_\varphi(r) = \int_{E^T_{-\infty, r]} \varphi \).

Theorem 18. For every observable function \( f \) we have:
- \( \mathcal{D}(\langle f \rangle) = \mathcal{D}(\tau(f)) \cap S \)
- \( \langle f \rangle = \langle \tau(f) \rangle \quad \mathcal{D}(\tau(f)) \cap S \)

Proof. \( \int_{\mathbb{R}} t^2 \cdot d\nu_{F^T_\varphi(t)} = \int_{\mathbb{R}} t^2 \cdot d(f|_{[\varphi]} \cdot \mu_{[\varphi]}) = \int_{[\varphi]} (f|_{[\varphi]} \cdot \mu_{[\varphi]}) \) (cfr. [K-S] pag. 93) therefore the integrals are finite together.

For the equality in Thm. 7.14 (e) of [W] we have for \( \varphi \) in \( \mathcal{D}(\tau(f)) \cap S \) the equalities: \( \langle \tau(f) \rangle \varphi = \int_{\mathbb{R}} t \cdot d\nu_{F^T_\varphi(t)} = \int_{\mathbb{R}} t \cdot d(f|_{[\varphi]} \cdot \mu_{[\varphi]}) = \int_{[\varphi]} f|_{[\varphi]} \cdot \mu_{[\varphi]} = \langle f \rangle(\varphi) \).

Definition 15. An observable function \( f : S \to \mathbb{R} \) is essentially bounded if there is a real number \( M \) such that \( f^{-1}((-\infty, -M \cup M, +\infty)) \) is a pseudo-borel null subset.

An observable function \( f \) is essentially bounded if and only if its essential image is a bounded subset of \( \mathbb{R} \).

Notation 10. Let’s denote by \( \text{OB} \) the set of all essentially bounded observable functions on \( S \).

Let \( f \) be an observable function, a real value \( y \) is not in \( \text{Im}_e(f) \) if and only if there exists an essentially bounded observable \( g \) such that \( g \cdot (f - y) \) is null equivalent to \( 1 \), in other words if and only if the function \( \frac{1}{f - y} \) is defined almost everywhere (exercise).

Theorem 19. For an observable function \( f \) are equivalent:
1. the function \( f \) is essentially bounded
2. the operator \( \tau(f) \) is a bounded operator
3. \( \mathcal{D}(\langle f \rangle) = S \)
Proof. \(2 \iff 3\) For the Hellinger-Toeplitz thm. \(\tau(f)\) is a bounded operator if and only if \(\mathcal{D}(\tau(f)) = \mathcal{H};\) since \(\mathcal{D}(\langle f \rangle) = \mathcal{D}(\tau(f)) \cap \mathbb{S}\) this is equivalent to \(\mathcal{D}(\langle f \rangle) = \mathbb{S}\).

\(1 \implies 3\) If the function \(f\) is essentially bounded then each function \(f|_{\phi}\) is in \(L^2([\phi], \mu_{[\phi]})\). This is equivalent to \(\mathcal{D}(\langle f \rangle) = \mathbb{S}\).

\(2 \implies 1\) If \(\tau(f)\) is a bounded operator then its spectrum is bounded, therefore so is the essential image of \(f\). \(\square\)

The map \(\tau : \mathcal{O} \to SA(\mathcal{H})\) sends \(\mathcal{O}\) in \(SAB(\mathcal{H})\).

**Theorem 20.** The mean value function \(\langle f \rangle\) of an observable function \(f\) is a kaehlerian function, if the function \(f\) is essentially bounded then the function \(\langle f \rangle\) is smooth kaehlerian.

Every kaehlerian function is the mean value of an observable function and every smooth kaehlerian function is the mean value of a bounded observable function.

Proof. Since \(\langle f \rangle = \langle \tau(f) \rangle\) the function \(\langle f \rangle\) is kaehlerian; when \(f\) is essentially bounded the function \(\langle f \rangle\) is smooth kaehlerian as \(\langle \tau(f) \rangle\).

If \(l\) is a kaehlerian function there exists a self-adjoint operator \(T\) such that \(l = \langle T \rangle\), therefore taken an observable function \(f\) such that \(\tau(f) = T\) we have \(\langle f \rangle = \langle T \rangle = l\). When \(l\) is smooth kaehlerian \(T\) is bounded and \(f\) can be taken essentially bounded. \(\square\)

The map \(\langle \cdot \rangle : \mathcal{O} \to K(\mathcal{H})\) sends \(\mathcal{O}\) in \(K\mathcal{S}(\mathcal{H})\).

\[\alpha(\langle f \rangle) = \tau(f).\]

**Theorem 21.** Let \(f\) be an observable function for every borelian function \(b : \mathbb{R} \to \mathbb{R}\) it holds:

\[b(\tau(f)) = \tau(b \circ f)\]

Proof. The operator \(\tau(f)\) has spectral measure: \(\{E^\tau(f)_B\}_{B \subseteq B(\mathbb{R})} = \{\varepsilon(f^{-1}(B))\}_B\)

and \(\tau(b \circ f)\) has spectral measure given by \(\{\varepsilon((b \circ f)^{-1}(B))\}_B = \{\varepsilon(f^{-1}(b^{-1}(B)))\}_B\)

but this is exactly the spectral measure of \(b(\tau(f))\) (cfr. [W] prop. pag. 196), therefore \(b(\tau(f)) = \tau(b \circ f)\). \(\square\)

We have: \(\chi_B \circ \tau(f) = \varepsilon(f^{-1}B)\) and \(\langle b(\tau(f)) \rangle = \langle b \circ f \rangle\).
4. Algebras of Propositions and Observables

Theorem 22. Let $\mathcal{A}$ be a boolean algebra of subsets of $\mathcal{S}$ contained in $\mathcal{L}$ and let $L$, $M$ be two elements of $\mathcal{A}$:

- $\varepsilon(L \cap M) = \varepsilon(L) \cap \varepsilon(M) = \varepsilon(L) \cap \varepsilon(M)$ and $\varepsilon(L \cup M) = \varepsilon(L) \cup \varepsilon(M) = \varepsilon(L) + \varepsilon(M) - \varepsilon(L) \cdot \varepsilon(M)$
- $\varepsilon(\mathcal{A})$ is a boolean algebra of commuting projectors of $\mathcal{H}$ and $\varepsilon : \mathcal{A} \to \varepsilon(\mathcal{A})$ is a boolean algebra morphism
- $\varepsilon(L) = \varepsilon(M)$ if and only if $L$ and $M$ are null equivalent

Proof. Since $L$ and $M$ are compatible we know that there exists an observable function $f$ and two borel subsets $B$ and $C$ of $\mathbb{R}$ such that $L = f^{-1}(B)$ and $M = f^{-1}(C)$; in particular this proves that $\varepsilon(L)$ and $\varepsilon(M)$ commute, then:

- $\varepsilon(L \cap M) = E_B^{(f)} \cdot E_C^{(f)} = E_B^{(f)} \cdot E_C^{(f)} = \varepsilon(L) \varepsilon(M) = \varepsilon(L) \cap \varepsilon(M)$ and also $\varepsilon(L \cup M) = E_B^{(f)} + \varepsilon(L) \varepsilon(M) = \varepsilon(L) \cup \varepsilon(M)$
- we already know that $\varepsilon(\mathbb{C}L) = I - \varepsilon(L)$
- if $\varepsilon(L) = \varepsilon(M)$ then $\varepsilon(L \Delta M) = \varepsilon(L \cup M) - \varepsilon(L \cap M) = \varepsilon(L) - \varepsilon(L) = 0$, therefore $L \Delta M$ is a pseudo-borel null subset.

\[ \Box \]

Theorem 23. Let $\mathcal{B}$ be a boolean algebra of (commuting) projectors in a separable Hilbert space $\mathcal{H}$, there exists a boolean algebra $\overline{\mathcal{B}}$ of subsets of $\mathcal{S}$ in $\mathcal{L}$ such that:

- $\varepsilon(\overline{\mathcal{B}}) = \mathcal{B}$
- $\varepsilon : \overline{\mathcal{B}} \to \mathcal{B}$ is a morphism of boolean algebras.

Proof. Let $\mathcal{B} = \{E_i\}_{i \in I}$, since the $E_i$ commute pairwise there exist a self-adjoint operator $T$ and a family of borel functions $\{b_i\}_{i \in I}$ such that $E_i = b_i \circ T$ for every $i \in I$ (cfr. [V] Thm. 3.9 p. 56).

Taken an observable function $f$ such that $\tau(f) = T$ let’s consider the boolean $\sigma$-algebra $\mathcal{A}_f$ of subsets of $\mathcal{S}$ in $\mathcal{L}$, the functions $b_i \circ f$ are all in $\mathcal{O}_{\mathcal{A}_f}$ and the propositions $L_i = f^{-1}(b_i^{-1}(1))$ are all in $\mathcal{A}_f$. Obviously $\tau(b_i \circ f) = E_i$, moreover $\tau((b_i \circ f)^2 - b_i \circ f) = 0$ therefore there exists a null pseudo-borel subset $\mathcal{N}_i$ of $\mathcal{S}$ such that $(b_i \circ f)^2 - b_i \circ f = 0$ on of $\mathcal{S}\setminus\mathcal{N}_i$. That is $b_i \circ f$ is null equivalent to $\chi_{L_i}$, this implies $E_i = \tau(b_i \circ f) = \tau(\chi_{L_i}) = \varepsilon(L_i)$ for every $i \in I$ and therefore $\varepsilon(\mathcal{A}_f) \supset \mathcal{B}$.

It is enough now to choose $\overline{\mathcal{B}} = (\varepsilon|_{\mathcal{A}_f})^{-1}(\mathcal{B})$. \[ \Box \]

Remark 4. A boolean algebra of pairwise commuting projectors can always be realized in $\mathcal{L}$ through a boolean algebra of compatible propositions. This realization is not unique.

Corollary 1. When $\mathcal{H}$ is a (separable) Hilbert space two projectors $E$ and $F$ commute if and only if there exist two compatible propositions $L$ and $M$ such that $\varepsilon(L) = E$ and $\varepsilon(M) = F$. 


Theorem 24. Let $\mathcal{H}$ be a (separable) Hilbert space, two propositions $L$ and $M$ have commuting associated projectors $\varepsilon(L)$ and $\varepsilon(M)$ if and only if there exists a measure equivalence $\nu$ such that $L$ and $\nu(M)$ are compatible.

Proof. If $L$ and $\nu(M)$ are compatible then $\varepsilon(L)$ and $\varepsilon(\nu M) = \varepsilon(M)$ commute.

Conversely if $\varepsilon(L)$ and $\varepsilon(M)$ commute there are two propositions $L'$ and $M'$ with $L' \cap M' \in \mathcal{L}$ and $\varepsilon(L') = \varepsilon(L)$, $\varepsilon(M') = \varepsilon(M)$. Therefore there exist two measure equivalences $\rho$ and $\sigma$ such that $L'$ is null equivalent to $\rho(L)$ and $M'$ is null equivalent to $\sigma(M)$, then $\rho L \cap \sigma M$ and $L \cap \rho^{-1} \sigma M$ are in $\mathcal{L}$. \hfill $\square$

Theorem 25. Let $\mathcal{A}$ be a $\sigma$-algebra of subsets in $\mathcal{S}$ contained in $\mathcal{L}$ and $\{L_n\}_{n \geq 1} \subset \mathcal{A}$, there exists an observable $f$ in $\mathcal{O}_\mathcal{A}$ and a sequence $\{B_n\}_{n \geq 1} \subset \mathcal{B}(\mathbb{R})$ such that $f^{-1}(B_n) = L_n$ for every $n \geq 1$.

Proof. Let $\chi : \mathcal{S} \to \{0, 1\}^{\mathbb{N}^+}$ be the map defined by $\chi(\varphi) = (\chi_{L_n}(\varphi))_{n \geq 1}$, taken a borel equivalence $\beta : \{0, 1\}^{\mathbb{N}^+} \to \mathbb{R}$ between $\{0, 1\}^{\mathbb{N}^+}$ and a borel subset of $\mathbb{R}$ it is not difficult to prove that the function $f = \beta \circ \chi : \mathcal{S} \to \mathbb{R}$ is an observable in $\mathcal{O}_\mathcal{A}$.

Taken a borel subset $B_n$ of $\mathbb{R}$ such that $\beta^{-1}(B_n) = \prod_{k \neq n} \{0, 1\} \times \{1\}_n$ it is possible to check that $f^{-1}(B_n) = L_n$. \hfill $\square$

Corollary 2. Let $\mathcal{A}$ be a $\sigma$-algebra of subsets in $\mathcal{S}$ contained in $\mathcal{L}$ and $\{L_n\}_{n \geq 1} \subset \mathcal{A}$, then $\varepsilon(\bigcup_{n \geq 1} L_n) = \bigvee_{n \geq 1} \varepsilon(L_n)$.

Proof. $\varepsilon(\bigcup_{n \geq 1} L_n) = \varepsilon(\bigcap_{n \geq 1} B_n)) = E^{\sigma(f)}_{\bigcup_{n \geq 1} B_n} = \bigvee_n E^{\sigma(f)}_{B_n} = \bigvee_n \varepsilon(L_n)$. \hfill $\square$

Theorem 26.

Theorem 27. When $\mathcal{H}$ is a separable Hilbert space assigned a sequence $\mathcal{E} = \{E_n\}_{n \in \mathbb{N}}$ of pairwise orthogonal (complex) projectors with $\sum_{n \in \mathbb{N}} E_n = I$ it is possible to find a partition $\{L_n\}_{n \in \mathbb{N}}$ of $\mathcal{S}$ by propositions with $\varepsilon(L_n) = E_n$ for every $n \in \mathbb{N}$.

Proof. Proceeding as in the proof of a previous theorem we find an observable $f$, a sequence of borel functions $\{b_n\}_{n \in \mathbb{N}}$ and propositions $L'_n = f^{-1}(b_n^{-1}(1))$ such that $\varepsilon(L'_n) = E_n$ for every $n \in \mathbb{N}$.

Since the $E_n$ are pairwise orthogonal the propositions have a pairwise pseudo-borel null intersection $N_{nm} = L'_n \cap L'_m$ (whenever $n \neq m$); since $\sum_{n \in \mathbb{N}} E_n = I$ the union $\bigcup_{n \in \mathbb{N}} L'_n$ has a pseudo-borel null complement $N_0$ in $\mathcal{S}$.

The set $N = N_0 \cup \bigcup_{n \neq m} N_{nm}$ is a pseudo-borel null subset, taken $L_1 = L'_1 \cup N$ and $L_n = L'_n \setminus N$ for every $n \geq 2$, we have $L_n \cap L_m = \emptyset$ (whenever $n \neq m$) and $\bigcup_{n \in \mathbb{N}} L_n = \mathcal{S}$. \hfill $\square$

Remark 5. The theorem above defines a map $\delta : \mathcal{E} \to \mathcal{L}$ (defined by $\delta(E_n) = L_n$) such that $(\varepsilon \circ \delta)(E) = E$ for every $E$ in $\mathcal{E}$ and transforming the sequence $\{E_n\}_{n \in \mathbb{N}}$ of pairwise orthogonal (complex) projectors with $\sum_{n \in \mathbb{N}} E_n = I$ in a partition $\{\delta(E_n)\}_{n \in \mathbb{N}}$ of $\mathcal{S}$. This construction can not be generalized too much:
Theorem 28. Let $\mathcal{H}$ be a separable Hilbert space (of complex dimension at least three), $\mathcal{E}$ a family of (complex) projectors and $\delta : \mathcal{E} \to \mathcal{L}$ a map such that:

1. $(\varepsilon \circ \delta)(E) = E$ for every $E$ in $\mathcal{E}$
2. if $E$ and $F$ are orthogonal in $\mathcal{E}$ then $\delta(E)$ and $\delta(F)$ are disjoint
3. $\delta\left(\sum_{n \in \mathbb{N}} E_n\right) = \bigcup_n \delta(E_n)$ if the $E_n$ are pairwise orthogonal in $\mathcal{E}$
4. $\delta(1) = \mathbb{S}$

then $\mathcal{E}$ cannot contain the family of all projectors on (complex) lines.

Proof. Let’s suppose the thesis is false; fixed a vector $\varphi_0$ in $\mathbb{S}(\sqrt{2})$ let’s consider the function $G : \mathbb{S}(1) \to \{0, 1\}$ defined by $G(\varphi) = \chi_{\delta(p_{\mathcal{C}u_n})}(\varphi_0)$. For every orthonormal basis $\{u_n\}_{n \geq 1}$ in $\mathcal{H}$ the family $\{\delta(p_{\mathcal{C}u_n})\}_{n}$ is a partition of $\mathbb{S}(\sqrt{2})$, therefore the vector $\varphi_0$ belongs to one and only to one of the sets $\delta(p_{\mathcal{C}u_n})$. This implies that $\sum_n G(u_n) = 1$ for every orthonormal basis $\{u_n\}_{n \geq 1}$, that is the function $G$ is a Gleason frame function (of weight 1) (cfr. [Gl]).

Since we are in a separable Hilbert space of dimension at least three there must exist a bounded self-adjoint operator $T$ such that $G(\varphi) = \langle T, \varphi \rangle$ for every $\varphi$ in $\mathbb{S}(1)$ (cfr. Thm. 3.5 of [Gl]), but this implies that the function $T$ is constantly 0 or constantly 1 bringing in both cases to an absurd since the vector $\varphi_0$ must belong to some of the sets $\delta(p_{\mathcal{C}u_n})$ but cannot belong to all of them. \qed

Remark 6. The previous theorem is strictly connected with the necessity of considering the observables in their context (cfr. Ghirardi in [B] 4.6.5, [G-D] and [KS]).

Theorem 29. Let $\mathcal{A}$ be a (non empty) $\sigma$-algebra of subsets in $\mathbb{S}$ contained in $\mathcal{L}$ and $\{f_n\}_{n \geq 1} \subset \mathcal{O}_\mathcal{A}$, there exists an observable $f$ in $\mathcal{O}_\mathcal{A}$ such that $\mathcal{A}_{f_n} \subset \mathcal{A}_f$ for every $n \geq 1$.

Proof. For the countable family $\{f_n^{-1}(r) \mid n \geq 1 \text{ and } r \text{ in } \mathbb{Q}\}$ it is possible, for a previous theorem, to find an observable $f$ in $\mathcal{O}_\mathcal{A}$ and a countable family $\{B_{n,r} \mid n \geq 1 \text{ and } r \text{ in } \mathbb{Q}\}$ in $\mathcal{B}(\mathbb{R})$ such that $f^{-1}(B_{n,r}) = f_n^{-1}(r)$ for every $n$, $r$. Therefore $f_n^{-1}(B) \in \mathcal{A}_f$ for every $n \geq 1$ and every borel subset $B$ of $\mathbb{R}$. \qed

Theorem 30. Let $f$, $g$ be two observable functions, if $\mathcal{A}_g \subset \mathcal{A}_f$ then there exists a borelian function $b : \mathbb{R} \to \mathbb{R}$ such that $g = b \circ f$.

Proof. Let $\mathbb{Q} = \{r_n\}_{n \geq 1}$ the set of rational numbers in a sequence, since $\mathcal{A}_g \subset \mathcal{A}_f$ it is possible to find a borel subset $B_1$ in $\mathbb{R}$ such that $f^{-1}(B_1) = g^{-1}(-\infty, r_1)$; then, taken $B$ such that $f^{-1}(B) = g^{-1}(-\infty, r_2)$ it is possible to "correct" it by defining $B_2 = B \cap B_1$ if $r_2 < r_1$ and $B_2 = B \cup B_1$ if instead $r_1 < r_2$.

In both cases we get $f^{-1}(B_2) = g^{-1}(-\infty, r_2)$ and, after the "correction", we have moreover that $B_1$ and $B_2$ are ordered as $r_1$, $r_2$.

Let $\{r_1, r_2\} = \{r_k, r_{k_2}\}$ with $r_{k_1} < r_{k_2}$, taken $B$ with $f^{-1}(B) = g^{-1}(-\infty, r_3)$ it is possible to "correct" it by defining $B_3 = B \cap B_k$ if $r_3 < r_{k_1}$, $B_3 = B_{k_1}$ if instead $r_{k_1} < r_{k_2}$ and $B_3 = B \cup B_{k_2}$ if instead $r_{k_1} < r_{k_2} < r_3$.

Again we get $f^{-1}(B_3) = g^{-1}(-\infty, r_3)$ and $B_1, B_2, B_3$ are ordered as $r_1, r_2, r_3$. \qed
Proceeding in this way it is possible to find a sequence \( \{B_n\}_{n \geq 1} \subset \mathcal{B}(\mathbb{R}) \) such that \( f^{-1}(B_n) = g^{-1}(-\infty, r_n) \) or every \( n \geq 1 \) and \( B_n \subset B_{n+1} \) whenever \( r_n < r_{n+1} \).

It is not difficult to check that \( f(S) \subset X = \bigcup_n B_n \setminus \bigcap_n B_n \).

The function \( b : \mathbb{R} \to \mathbb{R} \) defined by \( b(x) = \inf \{ r_n : x \in B_n \} \) when \( x \in X \) and \( 0 \) elsewhere is well defined and it is a borel function since it holds the equality: 
\[
b^{-1}(-\infty, s) \cap X = (\bigcup_{r_n < s} B_n) \cap X \text{ for every } s \in \mathbb{R}.
\]

In the end it is possible to check that \( f^{-1}(b^{-1}(-\infty, s)) = g^{-1}(-\infty, s) \) for every \( s \) in \( \mathbb{R} \); this proves that \( g = b \circ f \).

\[\square\]

**Corollary 3.** Let \( \mathcal{A} \) be a (non empty) \( \sigma \)-algebra of subsets in \( \mathbb{S} \) contained in \( \mathcal{L} \) and \( \{f_n\}_{n \geq 1} \subset \mathcal{O}_A \), there exists an observable \( f \) in \( \mathcal{O}_A \) and a sequence \( \{b_n\}_{n \geq 1} \) of borelian functions such that \( f_n = b_n \circ f \) for every \( n \geq 1 \).

**Proof.** It follows from the last two theorems. \[\square\]

**Theorem 31.** Let \( \mathcal{A} \) be a (non empty) \( \sigma \)-algebra of subsets in \( \mathbb{S} \) contained in \( \mathcal{L} \)

- \( \tau(\mathcal{O}_A) \) is a commutative algebra of operators
- the map \( \tau : \mathcal{O}_A \to SA(\mathcal{H}) \) is an algebra homomorphism
- \( \ker(\tau) \) is the set of observable functions that are zero out of a null pseudo-borel subset of \( \mathcal{A} \).

**Proof.** Taken two functions \( f, g \) in \( \mathcal{O}_A \) the two self-adjoint operators \( \tau(f) \) and \( \tau(g) \) commute since all the projectors \( E_B^{\tau(f)} \) and \( E_C^{\tau(g)} \) in their spectral measures commute; infact: since \( f^{-1}B \) and \( g^{-1}C \) are in \( \mathcal{A} \) the projectors \( E_B^{\tau(f)} = \varepsilon(f^{-1}B) \) and \( E_C^{\tau(g)} = \varepsilon(g^{-1}C) \) commute.

The \( \sigma \)-algebras \( \mathcal{A}_f \) and \( \mathcal{A}_g \) are in \( \mathcal{A} \) therefore there exists a function \( h \) in \( \mathcal{O}_A \) and two borel functions \( b, c : \mathbb{R} \to \mathbb{R} \) such that \( f = b \circ h \) and \( g = c \circ h \). Then we have: \( \tau(f + g) = \tau((b + c) \circ h) = (b + c)(\tau(h)) = \tau(f) + \tau(g) \) and analogously \( \tau(f \cdot g) = \tau(f) \cdot \tau(g) \).

If \( \tau(f) = 0 \) then \( f^{-1}(\mathbb{R} \setminus \{0\}) \) is a pseudo-borel null subset in \( \mathcal{A} \).

\[\tau(1/f) = \tau(f)^{-1} \text{ for an never zero observable } f.\]

**Theorem 32.** Let \( \mathcal{H} \) be a separable Hilbert space and let \( \mathcal{R} \) be a commutative algebra in the family of self-adjoint operators of \( \mathcal{H} \). There exists in \( \mathcal{O} \) a commutative algebra \( \mathcal{R} \) of observables such that:

- \( \tau(\mathcal{R}) = \mathcal{R} \)
- \( \tau : \mathcal{R} \to \mathcal{R} \) is an algebra homomorphism
- \( \ker(\tau) = \{ f \in \mathcal{R} : f \text{ is null equivalent to } 0 \} \)

**Proof.** Let \( \mathcal{R} = \{T_i\}_{i \in I} \), since the operators commute there exist a self-adjoint operator \( T \) and a family of borel functions \( \{b_i\}_{i \in I} \) such that \( T_i = b_i \circ T \) for every \( i \in I \) (cfr. [V] Thm. 3.9 p. 56).

Taken an observable function \( f \) such that \( \tau(f) = T \) let’s consider the \( \sigma \)-algebra \( \mathcal{A} = \mathcal{A}_f \), the functions \( b_i \circ f \) are all in \( \mathcal{O}_A \). Obviously \( \tau(b_i \circ f) = T_i \) therefore \( \tau(\mathcal{O}_A) \supset \mathcal{R} \). The proof is concluded taking \( \mathcal{R} = (\tau|_{\mathcal{O}_A})^{-1}(\mathcal{R}). \)

\[\square\]
The algebra homomorphism $\tau|: \tilde{\mathcal{R}} \to \mathcal{R}$ is essentially injective since measurable functions are usually identified when differ only on a null subset. Therefore the theorem above asserts that you can always realize a commutative operators algebra through a commutative algebra of observable functions.

This realization is not unique and the choice of the algebra $\tilde{\mathcal{R}}$ is a way to declare the "context" of your observables.

Sometime, however, you can ask the algebra homomorphism $\tau|$ to be properly injective:

**Theorem 33.** Let $T$ be a self-adjoint operator and let $\mathcal{B}$ be an algebra of borelian functions $b$ on $\mathbb{R}$ with the following property:

$$b^{-1}(0) \supset \text{spec}(T) \implies b = 0$$

It is possible to find a (non empty) $\sigma$-algebra $\mathcal{A}$ of subsets in $\mathbb{S}$ and an injective algebra homomorphism:

$$\omega : \mathcal{B} \circ T = \{b \circ T; \ b \in \mathcal{B}\} \to \mathcal{O}_\mathcal{A}$$

such that $\tau(\omega(b \circ T)) = b \circ T$ for every $b$ in $\mathcal{B}$.

**Proof.** Let $f$ in $\mathcal{O}$ such that $\tau(f) = T$, the family $\mathcal{B} \circ f = \{b \circ f; \ b \in \mathcal{B}\}$ is an algebra of observable functions contained in $\mathcal{O}_\mathcal{A}$ where $\mathcal{A} = \mathcal{A}_f$ and $\tau| : \mathcal{B} \circ f \to \mathcal{B} \circ T$ is a surjective algebra homomorphism.

If $\tau(b \circ f) = 0$ then $f^{-1}(\mathcal{B}^{-1}0)$ is pseudo-borel null and also $f^{-1}(\mathcal{B}^{-1}0)$ is pseudo-borel null, therefore $b^{-1}0 \supset \text{Im}(f) = \text{spec}(T)$ and by hypothesis $b \circ f = 0$. That is $\tau|$ is an algebra isomorphism, its inverse is the desired injective homomorphism $\omega$. \[\square\]

**Example 1.**

1. If $\text{spec}(T) = \mathbb{R}$ the theorem hypothesis is verified for any algebra $\mathcal{R}$ of continuous functions on $\mathbb{R}$.
2. If $\text{spec}(T)$ has a non-empty interior part the theorem hypothesis is verified for the algebra $\mathcal{R}$ of analytic functions on $\mathbb{R}$.
3. If $\text{spec}(T)$ is an infinite subset the theorem hypothesis is verified for the algebra $\mathcal{R}$ of polynomial functions.

### 5. Uncertainty relations

This section adapts to the space $\mathbb{S}$, with some modifications, several results contained primarily in [CMP], [G], [CGM] and is inserted with the main goal to show that the uncertainty relations follow by themself (essentially because the dispersion is given by the norm of a suitable vector).

**Definition 16.** We will call **pre-symplectic form** on $\mathbb{S}$ the smooth 2-form $\omega$ defined on each tangent space by $\omega_\varphi : T\varphi \mathbb{S} \times T\varphi \mathbb{S} \to \mathbb{R}$ given by: $\omega_\varphi(X,Y) = \langle JX,Y \rangle$. 

The form $\omega$ is bilinear, antisymmetric and closed but is degenerate on the 1-dimensional subspace of vertical vectors.

**Definition 17.** Taken two smooth functions $h, l$ on $\mathbb{S}$ we can define the following two smooth functions $h \circ l, \{h, l\} : \mathbb{S} \to \mathbb{R}$ on $\mathbb{S}$ by the following expressions:

\[
(h \circ l)(\varphi) = \frac{1}{2} \langle \text{Grad}_h \varphi, \text{Grad}_l \varphi \rangle + h(\varphi) \cdot l(\varphi)
\]

\[
\{h, l\}(\varphi) = \omega(\text{Grad}_h \varphi, \text{Grad}_l \varphi)
\]

Occasionally we will write $l^n$ instead of $l \circ l \circ ... \circ l$.

**Theorem 34.** For every couple of smooth kaehlerian functions $h, l$ on $\mathbb{S}$ it holds:

1. $h \circ l$ and $\{h, l\}$ are smooth kaehlerian functions
2. $a(h \circ l) = \frac{1}{2} [\alpha(h) \cdot \alpha(l) + \alpha(l) \cdot \alpha(h)]$
3. $a(\{h, l\}) = -i [\alpha(h) \cdot \alpha(l) - \alpha(l) \cdot \alpha(h)]$

**Proof.** 2. Written $A = \alpha(h)$, $B = \alpha(l)$, $A = \langle A \rangle_\varphi$ and $b = \langle B \rangle_\varphi$, it is possible to prove, with some calculations, that: $\langle \frac{1}{2} [AB + BA] \rangle_\varphi = ab + \frac{1}{2} \langle (A - aI) \varphi, (B - bI) \varphi \rangle$.

3. In an analogous way it is possible to prove that:

\[
\langle -i [AB - BA] \rangle_\varphi = \text{Im} \langle (A - aI) \varphi, (B - bI) \varphi \rangle.
\]

1. follows from 2. and 3. \qed

Obviously $\mathcal{KS(H)}$ with the operations $(\cdot) \circ (\cdot)$ and $(\cdot, \cdot)$ becomes a Jordan-Lie algebra (cfr. [E]).

Each states $\varphi$ in $\mathbb{S}$ is trivially dispersion free for the "precise observer", using the logic $\mathcal{L}$, since his evaluation map $\hat{\varphi} : \mathcal{L} \to [0, 1]$, defined by $\hat{\varphi}(L) = \chi_L(\varphi)$, takes only the values: 0, 1 (the dispersion is defined by: $\Delta_\varphi(L) = \sqrt{\hat{\varphi}(L) - \hat{\varphi}(L)^2}$ cfr. [J] ch. 6.3).

The situation is different for the "imprecise observer"; his "evaluation map" is $\hat{\varphi} : \mathcal{L} \to [0, 1]$ defined by $\hat{\varphi}(L) = \langle \chi_L \rangle_\varphi$ (therefore he gets a dispersion: $\delta_\varphi(L) = \sqrt{\hat{\varphi}(L) - \hat{\varphi}(L)^2}$ $\sqrt{\langle \varepsilon(L) \rangle_\varphi - \langle \varepsilon(L) \rangle_\varphi^2} = \sqrt{\langle [\varepsilon(L) - \langle \varepsilon(L) \rangle_\varphi, I]^2 \rangle_\varphi}$ generally positive.

**Definition 18.** Let $f$ be an essentially bounded observable function and let $\varphi$ be a state in $\mathbb{S}$, the dispersion of $f$ in $\varphi$ is given by the following expression:

\[
\delta_\varphi(f) = \sqrt{\langle [f - \langle f \rangle_\varphi]^2 \rangle_\varphi}
\]

Let $l$ be a smooth function on $\mathbb{S}$ and let $\varphi$ be a state in $\mathbb{S}$, the dispersion of $l$ in $\varphi$ is given by the following expression:

\[
\delta_\varphi(l) = \sqrt{\langle [l - l(\varphi)]^2 \rangle_\varphi}
\]
Let $T$ be a bounded self-adjoint operator and let $\varphi$ be a state in $\mathcal{S}$, the dispersion of $T$ in $\varphi$ is given by the following expression:

$$\delta_\varphi(T) = \sqrt{\left\langle \left[ T - \langle T \rangle_\varphi \right] \cdot I \right\rangle_\varphi}$$

**Theorem 35.** For every essentially bounded observable function $f$ it holds:

$$\delta_\varphi(f) = \delta_\varphi(\langle f \rangle) = \delta_\varphi(\alpha(\langle f \rangle))$$

**Proof.** It is enough to use the definitions. □

**Theorem 36.** For every essentially bounded observable function $f$ it holds:

$$\delta_\varphi(f) = \frac{1}{\sqrt{2}} \cdot \| \text{Grad}_\varphi (f) \|$$

**Proof.** Written $A = \alpha(\langle f \rangle)$ and $a = \langle A \rangle_\varphi$, it is not difficult to prove that: $(\delta_\varphi(f))^2 = (\delta_\varphi(A))^2 = \frac{1}{2} \langle (A - aI)\varphi, (A - aI)\varphi \rangle$. □

**Theorem 37.** (Heisenberg uncertainty relation) For every couple of smooth kaehlerian functions $h, l$ on $\mathcal{S}$ it holds:

$$\delta_\varphi(h) \cdot \delta_\varphi(l) \geq \sqrt{\left| (h \circ l)(\varphi) - h(\varphi) \cdot l(\varphi) \right|^2 + \frac{1}{4} \left\{ \langle h, l \rangle(\varphi) \right\}^2} \geq \frac{1}{2} \left\{ h, l \right\}(\varphi)$$

**Proof.** By the Cauchy-Schwartz inequality (applied to the sesquilinear scalar product $\langle \cdot, \cdot \rangle$) we have:

$$(\delta_\varphi(h) \cdot \delta_\varphi(l))^2 = \frac{1}{4} \| \text{Grad}_\varphi h \|^2 \cdot \| \text{Grad}_\varphi l \|^2 \geq \frac{1}{4} |\langle \text{Grad}_\varphi h, \text{Grad}_\varphi l \rangle|^2$$

and the proof is concluded observing that:

$$(h \circ l)(\varphi) - h(\varphi) \cdot l(\varphi) = \frac{1}{2} \text{Re} \langle \langle \text{Grad}_\varphi h, \text{Grad}_\varphi l \rangle \rangle$$

$$\left\{ h, l \right\}(\varphi) = \text{Im} \langle \langle \text{Grad}_\varphi h, \text{Grad}_\varphi l \rangle \rangle$$

□
6. SYMMETRIES AND DYNAMICS

Definition 19. A diffeomorphism \( \nu : S \to S \) will be called an \textbf{internal equivalence} if:

- it sends every \( S^1 \)-orbit in itself
- \( \nu \circ \rho_\theta = \rho_\theta \circ \nu \) for every \( \rho_\theta \)

Remark 7. Every internal equivalence is a measure equivalence.

Notation 11. We will denote by \( \text{Aut}_I(S) = \{ \nu \;|\; \nu \text{ is an internal equivalence} \} \) the set of all internal equivalences, it is a group of transformations of \( S \) containing \( S^1 \cdot I \).

Theorem 38. Let \( \nu : S \to S \) be a diffeomorphism, the following properties are equivalent:

1. \( \nu \) is an internal equivalence
2. there exists a differentiable map \( \varsigma : S \to S^1 \) constant on the \( S^1 \)-orbits such that \( \nu(\varphi) = \varsigma(\varphi) \cdot \varphi \)
3. there exists a differentiable function \( h : S \to \mathbb{R} \) constant on the \( S^1 \)-orbits such that \( \nu(\varphi) = \exp(-ih(\varphi)) \cdot \varphi \)

Proof. 3)\(\Rightarrow\)2) and 2)\(\Rightarrow\)1) are obvious.
1)\(\Rightarrow\)3) Since \( \varphi \) and \( \nu(\varphi) \) are in the same \( S^1 \)-orbit there exists \( \varsigma(\varphi) \) in \( S^1 \) such that \( \nu(\varphi) = \varsigma(\varphi) \cdot \varphi \). The map \( \varsigma : S \to S^1 \) so defined verifies \( \varsigma(u \cdot \varphi) = \varsigma(\varphi) \) and is differentiable; therefore the map \( \hat{\varsigma} : \mathbb{P}_C(H) \to S^1 \) defined by \( \hat{\varsigma}([\varphi]) = \varsigma(\varphi) \) is well defined and differentiable and admits a continuous lifting \( \hat{h} : \mathbb{P}_C(H) \to \mathbb{R} \) such that \( \hat{\varsigma}([\varphi]) = \exp(-i\hat{h}([\varphi])) \) for every \([\varphi]\) in \( \mathbb{P}_C(H) \); the function \( \hat{h} \) is moreover differentiable since the map \( t \mapsto \exp(it) \) is a local diffeomorphism.

The function \( h = \hat{h} \circ \pi \), where \( \pi : S \to \mathbb{P}_C(H) \) is the natural map, is the function required. \(\square\)

Definition 20. A \textbf{Hilbert automorphism} of \( S \) is a diffeomorphism \( U : S \to S \) with the following properties:

- \( U \circ \rho_\theta = \rho_\theta \circ U \) (\( U \) respects the action of \( S^1 \))
- \( U \) sends orthogonal vectors in orthogonal vectors (\( U \) respects the orthogonality)
- \( U(\cos t \cdot \varphi + \sin t \cdot \psi) = \cos t \cdot U(\varphi) + \sin t \cdot U(\psi) \) for every couple \( \varphi, \psi \) of orthogonal vectors in \( S \) (\( U \) respects the sovrapposizioni).

Remark 8. A Hilbert automorphism sends \( S^1 \)-orbits in \( S^1 \)-orbits and couples of orthogonal \( S^1 \)-orbits in couples of orthogonal \( S^1 \)-orbits.

Theorem 39. A diffeomorphism \( U : S \to S \) is a Hilbert automorphism if and only if there exists a unitary trasformation \( U : \mathcal{H} \to \mathcal{H} \) such that \( U = U|_S \).
there exists a continuous (and then differentiable) function $h$

Let $\varsigma$

Theorem 5.1) there exists a unitary or antiunitary transformation is well defined, bijective and preserves the antipodality relation. Therefore (cfr. $[U]$)

$c$ of measure equivalences of $S$

The set $\Theta$ of transformations of $S$

$\Theta$ Aut$(S)$ = $\{\Phi : \Phi$ is a semi-automorphism of $S\}$ is a group of transformations of $S$. Every Hilbert automorphism and every internal equivalence is a semi-automorphism.

There exists a unique group homomorphism $\sigma : \Theta$ Aut$(S)$ $\rightarrow$ $\{1, -1\}$ such that $\sigma(\Phi) = 1$ if and only if $\Phi$ is expressible as $\Phi = \Psi^2 \circ \Gamma^2$ with $\Psi$ and $\Gamma$ in $\Theta$ Aut$(S)$.

Proof. Let $\Phi : S$ $\rightarrow$ $S$ be a differentiable map, its horizontal differential is the linear map $\Phi^\text{Hor}_\varphi : \text{Hor}_\varphi$ $\rightarrow$ $\text{Hor}_{\Phi(\varphi)}$ defined by: $\Phi^\text{Hor}_\varphi(X) = \Phi^\text{Hor}(\Phi(\Phi^\text{Hor}(X)))$.

Note that if $\Phi^\text{Hor}_\varphi$ is injective then $\Phi^\text{Hor}_\varphi$ is not zero and if $h : S$ $\rightarrow$ $R$ is a differentiable function then $(e^{-i\theta} \cdot \Phi)^\text{Hor}_\varphi = e^{-i\theta(\Phi(\varphi))} \cdot \Phi^\text{Hor}_\varphi$.

If $\Phi$ is a semi-automorphism the map $\Phi^\text{Hor} : \text{PC}(\mathcal{H})$ $\rightarrow$ $\text{PC}(\mathcal{H})$ given by $\Phi^\text{Hor}([\varphi] = [\Phi(\varphi)]$ is well defined, bijective and preserves the antipodality relation. Therefore (cfr. $[U]$ Thm. 5.1) there exists a unitary or antiunitary transformation $U : \mathcal{H}$ $\rightarrow$ $\mathcal{H}$ such that $\hat{U} = \Phi$; that is there is a map $\varsigma : S$ $\rightarrow$ $S^1$ verifying the equality: $\Phi(\varphi) = \varsigma(\varphi) \cdot U \varphi$.

The map $\varsigma : S$ $\rightarrow$ $S^1$ is necessarily differentiable and, since $S$ is simply connected, there exists a continuous (and then differentiable) function $h : S$ $\rightarrow$ $R$, constant on the $S^1$-orbits when $U$ is unitary, such that $\Phi = e^{-ih} \cdot U$.

It’s easy to check that a unitary or antiunitary transformation $\Phi : S$ $\rightarrow$ $S$ verifies the equality:

$$\Phi^\text{Hor}_\varphi \circ J_\varphi = \sigma(\Phi) \cdot J_{\Phi(\varphi)} \circ \Phi^\text{Hor}_\varphi$$

Proof. ($\Rightarrow$) is obvious.

($\Leftarrow$) The map $\hat{U} : \text{PC}(\mathcal{H})$ $\rightarrow$ $\text{PC}(\mathcal{H})$ defined by $\hat{U}[\varphi] = [U \varphi]$ is a well defined bijective map respecting the antipodality, therefore (cfr. $[U]$ Thm. 5.1) there exists a unitary or antiunitary (linear) map $U : \mathcal{H}$ $\rightarrow$ $\mathcal{H}$ and a map $\varsigma : S$ $\rightarrow$ $S^1$ such that $U \varphi = \varsigma(\varphi) \cdot U(\varphi)$. The map $\varsigma$ is differentiable with $\varsigma(\varphi \cdot \varphi) = \varsigma(\varphi)$ when $U$ is unitary and $\varsigma(\varphi \cdot \varphi) = u^2 \cdot \varsigma(\varphi)$ when $U$ is antiunitary.

Since $U$ respects the sovrappositions for every couple $\varphi$, $\psi$ of orthogonal vectors and every $(c,s) = (\cos \theta, \sin \theta)$ we have:

$$c\varsigma(c\varphi + s\psi)U \varphi + sc(c\varphi + s\psi)U \psi = U(c\varphi + s\psi) = c\varsigma(\varphi)U \varphi + sc(\varphi)U \psi$$

therefore: $c \cdot [\varsigma(\varphi) - \varsigma(c\varphi + s\psi)] = 0$ and $s \cdot [\varsigma(\psi) - \varsigma(c\varphi + s\psi)] = 0$ that is

$$\varsigma(\varphi) = \varsigma(c\varphi + s\psi) = \varsigma(\psi)$$

for every couple $\varphi$, $\psi$ of orthogonal vectors and every $(c,s) = (\cos \theta, \sin \theta)$.

This proves that $\varsigma$ is constant and $\varsigma(\varphi \cdot \varphi)$ cannot be equal to $u^2 \cdot \varsigma(\varphi)$ for a general $u$; therefore $U$ is unitary. □
with \( \sigma(\Phi) = 1 \) if \( \Phi \) is unitary or with \( \sigma(\Phi) = -1 \) if \( \Phi \) is antiunitary.

For a general semi-automorphism \( \Phi = e^{-ih} \cdot U \) if we take the sign \( \sigma(\Phi) = \sigma(U) \) the equality above is still verified (the sign \( \sigma(\Phi) \) is well defined since \( \Phi^* \) cannot be zero).

Using the equality it’s easy to check that \( \sigma \) is a group homomorphism and therefore \( \sigma(\Psi^2 \circ \Gamma^2) = 1 \) for every \( \Psi \) and \( \Gamma \) in \( \Theta \text{Aut}(S) \).

Conversely if \( \sigma(\Phi) = \sigma(e^{-ih} \cdot U) = 1 \) then \( U \) is a unitary transformation expressible as \( U = e^{-iA} \) for a self-adjoint endomorphism \( A \) on \( \mathcal{H} \), therefore \( \Phi = \Psi^2 \circ \Gamma^2 \) with \( \Psi(\varphi) = e^{-\frac{i}{2}A} \varphi \) and \( \Gamma(\varphi) = e^{-\frac{i}{2}h(\varphi)} \cdot \varphi \).

The unicity of the homomorphism \( \sigma \) follows from the characterization of the elements with \( \sigma(\Phi) = 1 \).

\( \square \)

**Remark 9.** If \( \Phi, \Psi \) are in \( \Theta \text{Aut}(S) \) with \( \Psi = e^{-ih} \cdot \Phi \) then \( \sigma(\Psi) = \sigma(\Phi) \).

**Definition 22.** A semi-automorphism \( \Phi \) of \( S \) will be called an automorphism of \( S \) if \( \sigma(\Phi) = 1 \).

**Remark 10.** The set \( \text{Aut}(S) = \{ \Phi; \Phi \) is an automorphism of \( S \} \) is a normal subgroup of \( \Theta \text{Aut}(S) \) containing \( \text{Unit}(\mathcal{H}) \) and \( \text{Aut}_I(S) \). For every \( \Psi \) in \( \Theta \text{Aut}(S) \) the element \( \Psi^2 \) is in \( \text{Aut}(S) \); if \( \Phi : \mathbb{R} \rightarrow \Theta \text{Aut}(S) \) is a one-parameter group then every \( \Phi_t \) is in \( \text{Aut}(S) \), infact \( \Phi_t = (\Phi_{t/2})^2 \) for every \( t \) in \( \mathbb{R} \).

**Theorem 41.** A diffeomorphism \( \Phi : S \rightarrow S \) is an automorphism of \( S \) if and only if it can be expressed as the composition \( \Phi = U \circ \nu \) of a Hilbert automorphism \( U \) and an internal equivalence \( \nu \).

**Proof.** From the proof of the preceding theorem we know that, when \( \sigma(\Phi) = 1 \), then \( \Phi = e^{-ih} \cdot U \) with \( U \) unitary and \( h \) constant on the \( S^1 \)-orbits. Therefore taken \( \nu : S \rightarrow S \) defined by: \( \nu(\varphi) = e^{-ih(\varphi)} \cdot \varphi \), the map \( \nu \) is an internal equivalence and \( \Phi = U \circ \nu \). Conversely every composition \( \Phi = U \circ \nu \) of a Hilbert automorphism \( U \) and of an internal equivalence \( \nu \) is a semi-automorphism that can be written as \( \Phi = e^{-ih} \cdot U \). Therefore \( \sigma(\Phi) = \sigma(U) = 1 \).

\( \square \)

**Corollary 4.** A diffeomorphism \( \Phi : S \rightarrow S \) is an automorphism of \( S \) if and only if there exists a self-adjoint operator \( A \) of \( \mathcal{H} \) (or, equivalently, a kaehlerian function \( l \) with \( \alpha(l) = A \)) and a differentiable function \( h : S \rightarrow \mathbb{R} \) constant on the \( S^1 \)-orbits such that:

\[
\Phi(\varphi) = e^{-ih(\varphi)} \cdot e^{-iA} \varphi = e^{-ih(\varphi)} \cdot e^{-i\alpha(l)} \varphi
\]

**Proof.** It follows from the definitions.

\( \square \)

**Remark 11.**
1. \( \text{Aut}(S) \) is the smallest group of diffeomorphism of \( S \) containing \( \text{Unit}(\mathcal{H}) \) and \( \text{Aut}_I(S) \)
2. \( \text{Aut}_I(S) \) is a normal subgroup of \( \text{Aut}(S) \)
• if \( \Phi = U \circ \nu \) and \( \Psi = V \circ \varpi \) with \( U \) and \( V \) unitary and \( \nu \) and \( \varpi \) internal equivalence then \( \Phi \circ \Psi = U \circ V \circ \rho \) for a suitable internal equivalence \( \rho 
\)
• every automorphism is a measure equivalence
• an automorphism \( \Phi \) sends every proposition \( L \) in a proposition \( \Phi(L) \)
• if \( \Phi \) is an automorphism then for every observable \( f \) the function \( f \circ \Phi^{-1} \)
  is also an observable
• an automorphism respects the phase distance of the \( S^1 \)-orbits

**Remark 12.** On \( \text{Aut}(S) \) we will consider the topology induced by \( S^0 \) (the topology of "pointwise convergence").

**Definition 23.** A dynami in \( S \) is a continuous 1-parameter group of automorphisms of \( S \).

**Theorem 42.** A differentiable map \( \Phi : \mathbb{R} \times S \rightarrow S \) is a dynamic in \( S \) if and only if there exists a self-adjoint operator \( A \) on \( H \) and a differentiable function \( h : S \rightarrow \mathbb{R} \)
constant on the \( S^1 \)-orbits such that:

\[
\Phi_t(\varphi) = e^{-i \int_0^t h(e^{-itA} \varphi) \cdot dr} \cdot e^{-itA} \varphi
\]

**Proof.** (\( \Leftarrow \)) Obvious.

(\( \Rightarrow \)) The family \( \{ \Phi_t \}_{t \in \mathbb{R}} \) is a differentiable 1-parameter group of symmetries of \( PC(\mathcal{H}) \) therefore (cfr. [Ba]) there exists a self-adjoint operator \( A \) on \( H \) such that \( \Phi_t = U^t \) for every \( t \) (where \( U^t = e^{-itA} \)).

Therefore there exists a map \( \varsigma : \mathbb{R} \times S \rightarrow S^1 \) such that \( \Phi_t(\varphi) = \varsigma(t, \varphi) \cdot U^t \varphi \) for every \( \varphi \) in \( S \) and every \( t \) in \( \mathbb{R} \); the map \( \varsigma \) is constant on the \( S^1 \)-orbits and is necessarily differentiable.

We can find a continuous lifting \( \tilde{\varsigma} : \mathbb{R} \times S \rightarrow \mathbb{R} \) with respect to the space \( S \) and the covering map \( \varepsilon : \mathbb{R} \rightarrow S^1 \) (given by \( \varepsilon(r) = e^{-ir} \)) such that \( \tilde{\varsigma}(0, \varphi) = 0 \) for every \( \varphi \); since \( \varepsilon \) is a local diffeomorphism the lifting \( \tilde{\varsigma} \) is differentiable.

Using \( \Phi_{t+s} = \Phi_t \circ \Phi_s \) we have:

\[
\varsigma(t+s, \varphi) = \varsigma(t, U^s \varphi) \cdot \varsigma(s, \varphi) = \varsigma(s, U^t \varphi) \cdot \varsigma(t, \varphi)
\]

therefore:

\[
\tilde{\varsigma}(t+s, \varphi) = \tilde{\varsigma}(t, U^s \varphi) + \tilde{\varsigma}(s, \varphi) + 2\pi \cdot k(t, s, \varphi)
\]

where \( k(t, s, \varphi) \) is an integer. Since \( k(\cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \times S \rightarrow \mathbb{Z} \) is continuous, the function \( k \) must be constant and then equal to 0.

In the same way it is possible to prove that \( \tilde{\varsigma} \) is constant on the \( S^1 \)-orbits.

Therefore for the differentiable function \( \eta(\cdot, \cdot) : \mathbb{R} \times S \rightarrow \mathbb{R} \) defined by \( \eta(t, \varphi) = \frac{1}{2\pi} \frac{\partial}{\partial t}(t, \varphi) \) we get:

\[
\int_{t+s}^{t+s} \eta(r, \varphi) \cdot dr = \tilde{\varsigma}(t, U^s \varphi) \quad \text{and} \quad \eta(s, \varphi) = \eta(0, U^s \varphi);
\]

then the function \( h(\cdot) : S \rightarrow \mathbb{R} \) defined by \( h(\varphi) = \eta(0, \varphi) \) is differentiable, constant on the \( S^1 \)-orbits and:

\[
\varsigma(t, \varphi) = e^{-i \int_0^t h(U^r \varphi) \cdot dr}
\]

\( \square \)
Notation 14. Let \( l \) be a kaehlerian function on \( \mathbb{S} \) and let \( h \) be a differentiable function constant on the \( \mathbb{S}^1 \)-orbits we will denote by \( \Phi_{l,h} \); the smooth flow (the Hamiltonian flow defined by \( l \) and \( h \)) given by:

\[
\Phi_{l,h}(t) = e^{-i \int_0^t h(e^{-i \alpha(l)} \varphi) dt} \cdot e^{-i \alpha(l)} \varphi
\]

When \( l \) is a smooth kaehlerian function we will denote by \( X_{l,h} \) the smooth vector field (the Hamiltonian field defined by \( l \) and \( h \)) given by:

\[
X_{l,h} |_{\psi} = -J \varphi \text{Grad}_l \varphi - [l(\varphi) + h(\varphi)] \cdot J \varphi = -i \cdot \alpha(\varphi) - i \cdot h(\varphi) \cdot \varphi
\]

Remark 13. It’s easy to check that \((l_1, h_1)\) and \((l_2, h_2)\) define the same hamiltonian field if and only if there exists a real constant \( c \) such that:

\[
l_2 = l_1 + c \quad \text{and} \quad h_2 = h_1 - c.
\]

A little bit more complicated is to prove \( \Phi_{l_1,h_1} \) and \( \Phi_{l_2,h_2} \) define the same hamiltonian flow under the same condition.

Theorem 43. Let \( l \) be a smooth kaehlerian function on \( \mathbb{S} \) and let \( h \) be a differentiable function constant on the \( \mathbb{S}^1 \)-orbits, the field \( X_{l,h} \) is a complete field defining the Hamiltonian flow \( \Phi_{l,h} \).

Proof. In the Hilbert space \( \mathcal{H} \), written \( \Phi_t = \Phi_{l,h,t} \), we have for every vector \( \varphi \) in \( \mathbb{S} : \Phi_{l,h} |_{\psi} = \Phi_{l,h,t} \) = \( -i \cdot \alpha(l) \cdot \varphi - i \cdot h(\varphi) \cdot \varphi \). Therefore each curve \( t \mapsto \Phi_{l,h,t} \) is an integral curve for the field \( X_{l,h} \). \( \square \)

Remark 14. Let \( l \) be a smooth kaehlerian function on \( \mathbb{S} \) with \( \alpha(l) = \lambda \) and let \( h \) be a differentiable function constant on the \( \mathbb{S}^1 \)-orbits, the evolution \( \{ \psi_t \} \) of a state \( \psi_0 \) follows the (non generally linear) differential equation:

\[
\dot{\psi} = -i \lambda \psi - i h(\psi) \cdot \psi = -J \varphi \text{Grad}_l \varphi - [l(\psi) + h(\psi)] \cdot J \varphi
\]

7. The imprecise observer

In this section new structures are defined using the heuristic hypothesis that the system considered previously is described now by an observer intrinsically unable to distinguish between states in the same \( \mathbb{S}^1 \)-orbit (states differing only by "the phase").

To better understand the situation let’s describe in more detail what could be a precise observer.

Imagine an observer with a very efficient and completely automatized laboratory: all the precise observer has to do is to give to his main computer a program prescribing what is to be done and to press the ENTER key!

In particular, given a physical system, for each state \( \varphi \) of the system the precise observer has a program \( \text{SP}(\varphi) \) describing to the computer how to prepare the system exactly in the state \( \varphi \). Among these programs there are some act to prepare the state \( \rho_\theta \varphi \) from the state \( \varphi \) or to prepare a specific assigned state in the \( \mathbb{S}^1 \)-orbit \( [\varphi] = \{ \rho_\theta \varphi : 0 \leq \theta < 2\pi \} \).
Moreover for each observable \( f \) measurable on the system the precise observer has a program \( \text{OP}(f) \) describing to the computer how to prepare or activate the corresponding measuring apparatus.

Given a state \( \varphi \) and an observable \( f \) the precise observer has the great satisfaction to check that every time the measuring procedure \( \text{OP}(f) \) is executed on the system prepared by \( \text{SP}(\varphi) \) the measure displayed is always the same real number \( f(\varphi) \): for the precise observer the measuring process is completely deterministic.

Among his observables there are some phase distance observables allowing the precise observer to distinguish between states in the same \( S^1 \)-orbit.

The precise observer, in particular, concludes that, almost all the times, the phase is decisive in the measuring process: he makes the experience that a small change in the phase may change completely the measurement outcome.

Let’s consider now another observer studying the same physical system but with a poorer ability (let’s call him the **imprecise observer**). The imprecise observer has all the programs of the precise observer but the procedures of his laboratory and his computer are under a curse: they can never reach the precision necessary to distinguish between two different states in the same \( S^1 \)-orbit.

When the imprecise observer runs the procedure \( \text{SP}(\varphi) \) his laboratory can be precise enough to prepare a state in the \( S^1 \)-orbit \( [\varphi] = \{ \rho \varphi : 0 \leq \theta < 2\pi \} \) but he does not know which state in the \( S^1 \)-orbit is the outcome; the imprecise observer cannot avoid the state produced to be completely random in its \( S^1 \)-orbit.

When the imprecise observer runs the measure procedure \( \text{OP}(f) \) after \( \text{SP}(\varphi) \) he can get anyone of the values \( \{ f(\psi) ; \psi \in [\varphi] \} \). After a large number of trials the imprecise observer gets his outcomes distributed on the real line and, in the end, all he gets is representable by the numbers \( \pi(\varphi, f, B) = \mu([\varphi] \cap f^{-1}(B)) \) expressing the probability that the outcome falls in a general borel subset \( B \) of \( \mathbb{R} \).

The precise observer could let us know what’s wrong with the imprecise observer; he could say that all the science and technology of the imprecise observer ignore how to deal with the phases: his computer simply skips over the program’s lines prescribing some action able to define the phase of a state of the system.

In the imprecise observer’s laboratory the phase of the system simply comes from the past evolution of the system, therefore when a phase is involved in a measure (and this happens almost all the times) the imprecise observer gets the consequent result but without any control: from his viewpoint the results come by chance.

The precise observer could tell us what happens, for example, in the idealized experiment of a lamp producing isolated photons in a state assigned up to the phase and directed toward a half-silvered mirror reflecting exactly half of them:

"I am able to control the phase of the photon’s wave function and I checked that the passage of the photon through the mirror is strictly deterministic depending on this photon’s phase.

The imprecise observer instead is completely unaware of all these phases coming and going and in any case is not able to measure or to use them; therefore his photons are produced with a phase decided randomly or, better, by the history of lamp+photon system and when the photon interacts with the mirror it is this phase (its evolution) that decides what happens next: the photon passes or does not pass the mirror".
If the imprecise observer is left unaware of his ignorance then he comes to the drastic decision to consider his measuring process intrinsically statistic.

This implies a series of consequences. The space \( S \leftrightarrow \{ SP(\varphi) \} \) keeps for him only the meaning of set of all theoretically possible preparations of the system and \( O \leftrightarrow \{ OP(f) \} \) keeps only the meaning of set of all theoretically possible measuring apparatuses. All his experimental knowledge reduces to a probability map:

\[ \pi : S \times O \times B(\mathbb{R}) \to [0, 1] \]

remembering the probability \( \pi(\varphi, f, B) \) that the outcome of the measuring apparatus associated to \( OP(f) \) on the system prepared following \( SP(\varphi) \) falls in the borel set \( B \).

But now why he should consider different two preparations \( \varphi_1 \) and \( \varphi_2 \) if:

\[ \pi(\varphi_1, f, B) = \pi(\varphi_2, f, B) \]

for every apparatus \( f \) and every borelian subset \( B \)? And, dually, why he should consider different two apparatuses \( f_1 \) and \( f_2 \) if:

\[ \pi(\varphi, f_1, B) = \pi(\varphi, f_2, B) \]

for every preparation \( \varphi \) and every borelian subset \( B \)?

The ignorance of the phase generates therefore for the imprecise observer an equivalence relation \( R_S \) among the states in \( S \) and an equivalence relation \( R_O \) among the observables in \( O \): for the imprecise observer the real "states" he can distinguish through his experiments are the equivalence classes of \( R_S \) in \( S \) and his "state space" is the quotient space \( \hat{S} = S/R_S \). Analogously his "observable space" is \( \hat{O} = O/R_O \).

Over these objects is still well defined the probability map:

\[ \hat{\pi} : \hat{S} \times \hat{O} \times B(\mathbb{R}) \to [0, 1] \]

given by: \( \hat{\pi}([\varphi], [f], B) = \pi(\varphi, f, B) \) and now distinguishes the "states" through the "observables" and conversely.

In the following we will show that the physical system experimented by the imprecise observer with its states, observables, probabilities, symmetries and dynamics is naturally isomorphic to the usual quantum system and this gives a rational basis to our claim that the imprecise observer, because of his ignorance of the phases, experiments the physical system \( S \) as a quantum system.

Notation 15. Since the probability \( \pi(\varphi, f, B) = \mu_{[\varphi]}(f^{-1}(B) \cap [\varphi]) = \langle E_B^{\tau(f)} \rangle_{\varphi} \)
depends on the class \([\varphi]\) and not from its representative \( \varphi \) we have:

\[ \varphi R_S(\rho \varphi) \]

for every \( \theta \) in \( \mathbb{R} \)

therefore the map denoted by \( \chi : \mathbb{P}_C(H) \to \hat{S} \) and defined by \( \chi([\varphi]) = [\varphi]_{R_S} \) is well defined. If \( f_1 R_O f_2 \) then

\[ \langle E_B^{\tau(f_1)} \rangle_{\varphi} = \langle E_B^{\tau(f_2)} \rangle_{\varphi} \]

for every \( \varphi \) in \( S \) and every \( B \) in \( B(\mathbb{R}) \) then \( \tau(f_1) = \tau(f_2) \) and therefore the map denoted by \( \eta : \hat{O} \to SA(H) \) and defined by \( \eta([f]_{R_O}) = \tau(f) \) is well defined.

Theorem 44. The map \( \chi : \mathbb{P}_C(H) \to \hat{S} \) is bijective.
Proof. The map is clearly surjective; if \( \varphi, \psi \) are in \( S \) with \( \langle E^T_B \rangle_\varphi = \langle E^T_B \rangle_\psi \) for every self-adjoint operator \( T \) and every borel subset \( B \) of \( \mathbb{R} \) then \( \varphi \) is an element of a complex linear subspace \( F \) of \( \mathcal{H} \) if and only if \( \psi \) is in \( F \). Therefore \( \psi = e^{i\theta} \cdot \varphi \). □

**Remark 15.** In short the equivalence classes of \( \mathcal{R}_S \) are the \( S^1 \)-orbits. If we consider on \( \widehat{S} \) the quotient topology induced by \( S \) (this is topology that the precise observer assigns to \( \widehat{S} \)) the map \( \chi \) is a homeomorphism.

The map \( \beta : S \to \mathbb{P}_{\mathbb{C}}(\mathcal{H}) \) defined by \( \beta(\varphi) = [\varphi] \) is a submersion where each linear map \( \beta_{\varphi} : \text{Hor}_\varphi \to T_{[\varphi]}\mathbb{P}_{\mathbb{C}}(\mathcal{H}) \) is a topological isomorphism. It is not difficult to check that if we consider on each \( T_{[\varphi]}\mathbb{P}_{\mathbb{C}}(\mathcal{H}) \) the scalar product moved from \( \text{Hor}_\varphi \) by \( \beta_{\varphi} \), we get on \( \mathbb{P}_{\mathbb{C}}(\mathcal{H}) \) a well defined metric tensor coincident with the Kähler-Fubini-Study metric tensor \( g_\nu \) with \( \nu = 1 \) considered in [CMP].

Each tangent space \( T_{[\varphi]}\mathbb{P}_{\mathbb{C}}(\mathcal{H}) \) inherits from \( \text{Hor}_\varphi \) an isometric endomorphism \( J_{[\varphi]} : T_{[\varphi]}\mathbb{P}_{\mathbb{C}}(\mathcal{H}) \to T_{[\varphi]}\mathbb{P}_{\mathbb{C}}(\mathcal{H}) \) given by \( \beta_{\varphi} \circ J_{\varphi} \circ (\beta_{\varphi})^{-1} \) (not depending on the representative \( \varphi \) chosen) defining on \( \mathbb{P}_{\mathbb{C}}(\mathcal{H}) \) an (integrable) almost-complex structure.

**Theorem 45.** The map \( \eta : \hat{\mathcal{O}} \to SA(\mathcal{H}) \) is bijective.

**Proof.** Every self-adjoint operator comes from an observable function therefore the map \( \eta \) is surjective. Let \( [f_1]_{\mathcal{R}_\omega}, [f_2]_{\mathcal{R}_\omega} \) in \( \hat{\mathcal{O}} \) with \( \tau(f_1) = \tau(f_2) \), then we have the following equalities: \( \pi(\varphi, f_1, B) = \langle E^T_B \rangle_\varphi = \langle E^T_B \rangle_\varphi = \pi(\varphi, f_2, B) \) for every \( \varphi \) and every borel subset \( B \) of \( \mathbb{R} \); that is \( [f_1]_{\mathcal{R}_\omega} = [f_2]_{\mathcal{R}_\omega} \). □

If we identify \( \mathbb{P}_{\mathbb{C}}(\mathcal{H}) \) with \( \widehat{S} \) (through \( \chi \)) and \( \hat{\mathcal{O}} \) with \( SA(\mathcal{H}) \) (through \( \eta \)) the probability map \( \hat{\pi} : \widehat{S} \times \mathcal{O} \times B(\mathbb{R}) \to [0,1] \) becomes the usual quantum probability map \( p : \mathbb{P}_{\mathbb{C}}(\mathcal{H}) \times SA(\mathcal{H}) \times B(\mathbb{R}) \to [0,1] \) given by \( p([\varphi], A, B) = \langle E^A_B \rangle_\varphi \), infact:

\[
\hat{\pi}(\chi([\varphi], [f], B) = \mu([\varphi])([\varphi] \cap f^{-1}(B)) = \langle E^T_B \rangle_\varphi = p([\varphi], \eta([f]), B)
\]

The couple \((\chi^{-1}, \eta)\) is an isomorphism between \( (\mathbb{S}, \hat{\mathcal{O}}, \hat{\pi}) \) and \( (\mathbb{P}_{\mathbb{C}}(\mathcal{H}), SA(\mathcal{H}), p) \).

For the precise observer the symmetries of the physical system are the automorphisms of \( S \), that is the elements of the group \( \text{Aut}(S) \). Let’s make clear what is a symmetry for the imprecise observer:

**Definition 24.** A **semi-symmetry** for \((\mathbb{S}, \hat{\mathcal{O}}, \hat{\pi})\) is a couple \((\Lambda, \Omega)\) of a bijective map \( \Lambda : \mathbb{S} \to \mathbb{S} \) and a bijective map \( \Omega : \hat{\mathcal{O}} \to \hat{\mathcal{O}} \) such that:

\[
\hat{\pi}(\Lambda([\varphi], \Omega([f]), B) = \hat{\pi}([\varphi], [f], B)
\]

for every \([\varphi]\) in \( \mathbb{S} \), every \([f]\) in \( \hat{\mathcal{O}} \) and every borel subset \( B \) of \( \mathbb{R} \).

**Notation 16.** We will denote by \( \Theta \text{Sym}(\hat{\mathbb{S}}, \hat{\mathcal{O}}, \hat{\pi}) \) the set of all the semi-symmetries of \((\mathbb{S}, \hat{\mathcal{O}}, \hat{\pi})\); it is a group with respect to composition of couples of bijective maps.
Theorem 46. For every semi-symmetry \((\Lambda, \Omega)\) there exists a unitary or antiunitary transformation \(U : \mathbb{S} \rightarrow \mathbb{S}\) such that:

- \(\Lambda [\varphi] = \hat{U} [\varphi] = [U \varphi]\) for every \([\varphi]\) in \(\tilde{\mathbb{S}}\)
- \(\Omega[f] = \hat{U}^* [f] = [f \circ U^{-1}]\) for every \([f]\) in \(\hat{\mathbb{O}}\)

Proof. Since \((\tilde{\mathbb{S}}, \hat{\mathbb{O}}, \hat{\pi})\) is isomorphic to \((\mathbb{P}_C(\mathcal{H}), SA(\mathcal{H}), p)\) we can suppose to be in this case. Note that if \(E\) is a projector in \(\mathcal{H}\) the self-adjoint operator \(\Omega(E)\) is also a projector because \(E_{\mathbb{P}_C\{0,1\}} = 0\).

Taken two antipodal classes \([\varphi]\) and \([\psi]\) in \(\mathbb{P}_C(\mathcal{H})\) it is possible to find a projector \(E\) such that \(\langle E_{\{1\}}^E \rangle_{[\varphi]} = 0\) and \(\langle E_{\{1\}}^E \rangle_{[\psi]} = 1\), therefore \(\langle E_{\{1\}}^{\Omega(E)} \rangle_{\Lambda[\varphi]} = 0\) and \(\langle E_{\{1\}}^{\Omega(E)} \rangle_{\Lambda[\psi]} = 1\) and then the classes \(\Lambda [\varphi]\) and \(\Lambda [\psi]\) are antipodal too; that is \(\Lambda\) is a bijective transformation of \(\mathbb{P}_C(\mathcal{H})\) preserving the antipodality.

This implies that (cfr. [U] Thm. 5.1) there exists a unitary or antiunitary map \(U : \mathcal{H} \rightarrow \mathcal{H}\) such that \(\Lambda = \hat{U}\), that is \(\Lambda [\varphi] = [U \varphi]\) for every \([\varphi]\) in \(\mathbb{P}_C(\mathcal{H})\).

It holds: \(\langle E_{\{1\}}^{B \circ \sigma \circ U^{-1}} \rangle_{[U \varphi]} = \langle E_{\{1\}}^B \rangle_{[\varphi]} = \langle E_{\{1\}}^{\Omega(A)} \rangle_{\Lambda[\varphi]} = \langle E_{\{1\}}^B \rangle_{[U \varphi]}\) for every \(\varphi\) and every borel subset \(B\) of \(\mathbb{R}\); then \(\Omega(A) = U \circ A \circ U^{-1}\) for every self-adjoint operator \(A\). Remembering that \(\tau(f \circ U^{-1}) = U \circ \tau(f) \circ U^{-1}\) we get the thesis for \((\tilde{\mathbb{S}}, \hat{\mathbb{O}}, \hat{\pi})\). 

\(\square\)

Remark 16. If \((\chi^{-1}, \eta)\) and \((\chi_1^{-1}, \eta_1)\) are two isomorphism between \((\tilde{\mathbb{S}}, \hat{\mathbb{O}}, \hat{\pi})\) and \((\mathbb{P}_C(\mathcal{H}), SA(\mathcal{H}), p)\) then the couple \((\chi_1^{-1} \circ \chi^{-1} \circ \eta^{-1})\) is an automorphism of \((\mathbb{P}_C(\mathcal{H}), SA(\mathcal{H}), p)\) and therefore, reasoning as in the proof of the previous theorem, comes from a unitary or antiunitary map \(U : \mathcal{H} \rightarrow \mathcal{H}\). Hence on \(\tilde{\mathbb{S}}\) there is a unique natural topological and differentiable structure; in particular this means that the imprecise observer is able to reconstruct the topology of \(\tilde{\mathbb{S}}\) assigned by the precise observer.

Theorem 47. There exists a unique homomorphism \(\sigma : \Theta Sym(\tilde{\mathbb{S}}, \hat{\mathbb{O}}) \rightarrow \{1, -1\}\) such that \(\sigma(\Lambda, \Omega) = 1\) if and only if there is another element \((\Lambda_1, \Omega_1)\) such that \((\Lambda_1, \Omega_1)^2 = (\Lambda, \Omega)\).

Proof. It is clear that the unicity follows from the characterization of the elements where \(\sigma\) takes the value +1. Since \((\tilde{\mathbb{S}}, \hat{\mathbb{O}}, \hat{\pi})\) is isomorphic to \((\mathbb{P}_C(\mathcal{H}), SA(\mathcal{H}), p)\) we can suppose to be in this case. As in the proof of a previous theorem we can prove that for each \(\Lambda = \hat{U}\) where \(U\) is unitary or antiunitary, there is a well defined sign \(\sigma(\Lambda) = \sigma(U) = +1\) for \(U\) unitary and \(\sigma(\Lambda) = \sigma(U) = -1\) for \(U\) antiunitary such that in \(\tilde{\mathbb{S}}\) it holds the property:

\[ U_{\varphi}^{\text{Hor}} \circ J_{\varphi} = \sigma(U) \cdot J_{U \varphi} \circ U_{\varphi}^{\text{Hor}} \]

and therefore in \(\mathbb{P}_C(\mathcal{H})\) it holds the property:

\[ \Lambda_{\Lambda[\varphi]} \circ J_{\Lambda[\varphi]} = \sigma(\Lambda) \cdot J_{\Lambda[\varphi]} \circ \Lambda \]

It’s easy to prove then that the map \(\sigma : \Theta Sym(\mathbb{P}_C(\mathcal{H}), SA(\mathcal{H})) \rightarrow \{1, -1\}\) defined by \(\sigma(\Lambda, \Omega) = \sigma(\Lambda)\) is a group homomorphism.

If \((\Lambda, \Omega) = (\Lambda_1, \Omega_1)^2\) for an element \((\Lambda_1, \Omega_1)\) then \(\sigma(\Lambda, \Omega) = [\sigma(\Lambda_1, \Omega_1)]^2 = 1\).
Conversely if \( \sigma(\Lambda, \Omega) = \sigma(\Lambda) = 1 \) then \( \Lambda = \widehat{U} \) where \( U = e^{-iA} \) for a self-adjoint transformation \( A \) and \( \Omega(T) = \widehat{U}[T] = U \circ T \circ U^{-1} \); therefore taken \( \Lambda_1 = e^{-iA} \) and \( \Omega_1(T) = e^{-iA} \circ T \circ e^{iA} \) we have an element in \( \Theta \text{Sym}(\mathbb{P}_C(\mathcal{H}), \mathbb{SA}(\mathcal{H})) \) with \((\Lambda_1, \Omega_1)^2 = (\Lambda, \Omega)\). \(\square\)

**Definition 25.** A semi-symmetry \((\Lambda, \Omega)\) will be called a symmetry of \((\widehat{S}, \widehat{O}, \widehat{\pi})\) if \(\sigma(\Lambda, \Omega) = 1\).

**Notation 17.** We will denote by \(\text{Sym}(\widehat{S}, \widehat{O})\) the set of all symmetries of \((\widehat{S}, \widehat{O}, \widehat{\pi})\); \(\text{Sym}(\widehat{S}, \widehat{O})\) is a normal subgroup of \(\Theta \text{Sym}(\widehat{S}, \widehat{O})\).

**Notation 18.** Every automorphism \(\Phi\) of \(S\) brings equivalence classes of \(\mathcal{R}_S\) (the \(S^1\)-orbits) in equivalence classes, therefore defines an induced transformation \(\widehat{\Phi}\) of \(\widehat{S}\) by \(\widehat{\Phi}[\varphi] = [\Phi(\varphi)]\). Since an automorphism \(\Phi\) of \(S\) is a measure equivalence the map \(\Phi^* : \mathcal{O} \to \mathcal{O}\), given by \(\Phi^*(f) = f \circ \Phi^{-1}\), brings equivalence classes of \(\mathcal{R}_O\) in equivalence classes of \(\mathcal{R}_O\) and therefore is well defined an induced transformation \(\widehat{\Phi}^*\) of \(\widehat{\mathcal{O}}\) by \(\widehat{\Phi}^*[f] = [f \circ \Phi^{-1}]\).

**Remark 17.** Note that \(\widehat{\pi}(\widehat{\Phi}[\varphi], \widehat{\Phi}^*[f], B) = \widehat{\pi}([\varphi], [f], B)\) and then \((\widehat{\Phi}, \widehat{\Phi}^*)\) is a semi-symmetry for \((\widehat{S}, \widehat{O}, \widehat{\pi})\) and using the definitions it’s easy to check that \(\sigma(\widehat{\Phi}, \widehat{\Phi}^*) = 1\). We express this fact by saying that ”a symmetry for the precise observer appears as a symmetry to the imprecise observer”.

**Remark 18.** An automorphism \(\Phi\) of \(\text{Aut}(S)\) induces on \(\widehat{S}\) the identity map if and only if is in \(\text{Aut}_1(S)\), then the actions \(\widehat{\Phi}\) on \(\widehat{S}\) belong to the transformation group \(\text{Aut}(\widehat{S}) = \text{Aut}(S)/\text{Aut}_1(S)\) (acting effectively on \(\widehat{S}\)).

Remember that in quantum mechanics a symmetry of \(\mathbb{P}_C(\mathcal{H})\) can be described as a diffeomorphism \(\Lambda : \mathbb{P}_C(\mathcal{H}) \to \mathbb{P}_C(\mathcal{H})\) induced by unitary transformations of \(\mathcal{H}\); therefore the symmetry group of \(\mathbb{P}_C(\mathcal{H})\) is:

\[
\text{PROJ}(\mathbb{P}_C(\mathcal{H})) = \{\Lambda; \ \Lambda\text{ is a symmetry of }\mathbb{P}_C(\mathcal{H})\}
\]

(with the topology of pointwise convergence) and is naturally isomorphic to the group \(\text{Unit}(\mathcal{H})/\langle S^1, I\rangle\).

The following theorem proves that the symmetries induced on \(\widehat{S}\) by the automorphisms of \(S\) are precisely the natural symmetries for the imprecise observer and through the identification \(\chi : \mathbb{P}_C(\mathcal{H}) \to \widehat{S}\) the symmetries for the imprecise observer become the symmetries of \(\mathbb{P}_C(\mathcal{H})\):

**Theorem 48.** (1) The map \(\Sigma : \text{Aut}(S) \to \text{Sym}(\widehat{S}, \widehat{O})\) defined by the expression: \(\Sigma([\Phi]) = (\widehat{\Phi}, \widehat{\Phi}^*)\) is a group isomorphism.
(2) The map \( \Pi : \text{Sym}(\hat{S}, \hat{O}) \to \text{PROJ}(\mathbb{P}_C(\mathcal{H})) \) defined by: \( \Pi(\Lambda, \Omega) = \chi^{-1} \circ \Lambda \circ \chi \) is a group isomorphism.

**Proof.** 1. If \( \Sigma([\Phi]) = e_{\text{Sym}(\hat{S}, \hat{O})} \) then \( \Phi \) sends every \( S^1 \)-orbit in itself and therefore \( [\Phi] = e_{\hat{\text{Aut}}(\hat{S})} \). Moreover \( \Sigma \) is surjective since every symmetry in \( \text{Sym}(\hat{S}, \hat{O}) \) comes from a unitary transformation.

2. Since \( (\hat{S}, \hat{O}, \hat{\pi}) \) is isomorphic to \( (\mathbb{P}_C(\mathcal{H}), \text{SA}(\mathcal{H}), p) \) we can suppose to be in this case with \( \Pi(\Lambda, \Omega) = \Lambda \). Since every element in \( \text{PROJ}(\mathbb{P}_C(\mathcal{H})) \) comes from a unitary transformation the map \( \Pi \) is surjective. If \( \Pi(\Lambda, \Omega) = \Lambda = id_{\mathbb{P}_C(\mathcal{H})} \) then for every self-adjoint operator \( A \) we have \( \langle E_B^{\Omega(A)} \rangle_{\phi} = \langle E_B^A \rangle_{\phi} \) for every \( \phi \) in \( S \) and every borel set \( B \); therefore \( \Omega(A) = A \) and \( (\Lambda, \Omega) = (id, id) \). \( \square \)

**Remark 19.** The isomorphisms \( \Sigma \) and \( \Pi \) respect also the actions of the "symmetries" on the "observables".

**Remark 20.** We will consider on \( \hat{\text{Aut}}(\hat{S}) \) the topology induced by \( \hat{S} \) (the topology of "pointwise convergence") and on \( \text{Sym}(\hat{S}, \hat{O}) \) the topology that makes the isomorphism \( \Sigma \) a homeomorphism. From now on we will simply identify \( \text{Sym}(\hat{S}, \hat{O}) \) with \( \hat{\text{Aut}}(\hat{S}) \).

**Definition 26.** A **dynamic** in \((\hat{S}, \hat{O})\) is a continuous 1-parameter group in \( \hat{\text{Aut}}(\hat{S}) \) (in \( \text{Sym}(\hat{S}, \hat{O}) \)).

It is easy to check that a one-parameter continuous group \( \Phi : \mathbb{R} \to \hat{\text{Aut}}(\hat{S}) \) induces a one-parameter continuous group: \( \hat{\Phi} : \mathbb{R} \to \hat{\text{Aut}}(\hat{S}) \), therefore every dynamic in the physical system \( S \) induces an "apparent" dynamic on \( \hat{S} \) seen by the imprecise observer. Two dynamics \( \Phi \) and \( \Psi \) on \( S \) give origin to the same dynamic on \( \hat{S} \) for the imprecise observer (\( \hat{\Phi} = \hat{\Psi} \)) if and only if there is a continuous map \( \nu : \mathbb{R} \to \hat{\text{Aut}}_1(\hat{S}) \) such that \( \Psi_t = \Phi_t \circ \nu_t \) for every \( t \in \mathbb{R} \).

In this way on the set \( \{ \Phi : \mathbb{R} \to \hat{\text{Aut}}(\hat{S}) \} \) of all dynamics on \( \hat{S} \) is defined an equivalence relation \( \mathcal{R}_{\text{dyn}} \).

Let’s remember that all the continuous dynamics on \( \mathbb{P}_C(\mathcal{H}) \) are expressible as: \( t \to e^{-itH} \) where \( H \) is a self-adjoint operator on \( \mathcal{H} \) and that two operators \( H_1 \) and \( H_2 \) define the same dynamic in \( \mathbb{P}_C(\mathcal{H}) \) if and only if \( H_2 - H_1 \) is in \( \mathbb{R} \cdot I \). Therefore the set of all the dynamics of \( \mathbb{P}_C(\mathcal{H}) \) can be represented by the set: \( \text{SA}(\mathcal{H})/\mathbb{R} \cdot I \).

The following theorem proves that the apparent dynamics induced on \( \hat{S} \) by the dynamics of \( S \) are precisely the natural dynamics for the imprecise observer and moreover that, through the identification \( \chi : \mathbb{P}_C(\mathcal{H}) \to \hat{S} \), the dynamics for the imprecise observer become the dynamics of \( \mathbb{P}_C(\mathcal{H}) \):

**Theorem 49.** (1) The map \( \Delta : \text{SA}(\mathcal{H})/\mathbb{R} \cdot I \to \{ \hat{\Phi} : \hat{\Phi} \text{ is a dyn. on } \hat{S} \} \) defined by \( \Delta[H] = (e^{-itH})_{t \in \mathbb{R}} \) is bijective.
(2) The map: \( \delta : \{ \Phi : \Phi. \text{ is a dyn. on } S \}/\mathcal{R}_{\text{Dyn}} \rightarrow \{ \hat{\Phi} : \hat{\Phi}. \text{ is a dyn. on } \hat{S} \} \)
defined by \( \delta [\Phi] = (\hat{\Phi}) \) is well defined and bijective.

Proof. 1. If \( \Theta : \mathbb{R} \rightarrow \hat{\text{Aut}}(\hat{S}) \) is a one-parameter continuous group then for every \( t \) there is an automorphism \( \Phi_t \) such that \( \Theta_t = [\Phi_t]_{\hat{\text{Aut}}(\hat{S})} \). This implies that for each \( \varphi \) in \( S \) the map: \( t \rightarrow [\Phi_t(\varphi)]_{\hat{S}} \) is continuous and then is continuous the map:
\( t \rightarrow \hat{\Phi}_t [\varphi]_{\mathcal{P}(\mathcal{H})} = [\Phi_t(\varphi)]_{\mathcal{P}(\mathcal{H})} \). Therefore the map \( \hat{\Phi} : \mathbb{R} \rightarrow \text{PROJ}(\mathcal{P}(\mathcal{H})) \) is a one-parameter continuous group and by a theorem (cfr. [Ba]) there is a self-adjoint operator \( H \) such that \( \hat{\Phi}_t [\varphi]_{\mathcal{P}(\mathcal{H})} = [e^{-itH}\varphi]_{\mathcal{P}(\mathcal{H})} \) for every \( [\varphi]_{\mathcal{P}(\mathcal{H})} \). This proves the surjectivity of \( \Delta \).

2. The map \( \delta \) is obviously injective; if \( \Theta : \mathbb{R} \rightarrow \hat{\text{Aut}}(\hat{S}) \) is a one-parameter continuous group we already know that exists a self-adjoint operator \( H \) such that \( \Theta_t = e^{-itH} \) for every \( t \) in \( \mathbb{R} \), therefore \( t \mapsto e^{-itH} \) is a one-parameter continuous group in \( \text{Aut}(S) \) whose class is sent by \( \delta \) in \( \Theta \).

\[ \square \]

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