Post-Newtonian expansion of gravitational waves from a particle in circular orbits
around a rotating black hole: Up to $O(v^8)$ beyond the quadrupole formula

Hideyuki Tagoshi
California Institute of Technology, Theoretical Astrophysics, Pasadena, CA 91125,
and National Astronomical Observatory, Mitaka, Tokyo 181, Japan

Masaru Shibata, Takahiro Tanaka, Misao Sasaki
Department of Earth and Space Science, Osaka University, Toyonaka, Osaka 560, Japan
(draft January 1996)

Abstract
Extending a method developed by Sasaki in the Schwarzschild case and by Shibata, Sasaki, Tagoshi, and Tanaka in the Kerr case, we calculate the post-Newtonian expansion of the gravitational wave luminosities from a test particle in circular orbit around a rotating black hole up to $O(v^8)$ beyond the quadrupole formula. The orbit of a test particle is restricted on the equatorial plane. We find that spin dependent terms appear in each post-Newtonian order, and that at $O(v^6)$ they have a significant effect on the orbital phase evolution of coalescing compact binaries. By comparing the post-Newtonian formula of the luminosity with numerical results we find that, for $30 M < \sim r < \sim 100M$, the spin dependent terms at $O(v^6)$ and $O(v^7)$ improve the accuracy of the post-Newtonian formula significantly, but those at $O(v^8)$ do not improve.

PACS numbers: 04.25.Nx, 04.30.-w, 04.30.Db, 04.70.-s

I. INTRODUCTION

Among the possible sources of gravitational waves, coalescing compact binaries are considered to be the most promising candidates for detection by near-future, ground based laser interferometric detectors such as LIGO [1], VIRGO [2], GEO600, TAMA and AIGO. There are two reasons for this: first, we can expect sufficiently large amplitude of gravitational waves from these systems. Second, the estimated event rate, for neutron star binaries, is several/yr within 200Mpc [3]. Furthermore, the observations of coalescing compact binaries are potentially important because they bring us new physical and astronomical information. They can be used to test general relativity [4], to measure cosmological parameters [5] and neutron star radii. It may even be possible to obtain information about the equation of state of neutron stars [6]. If a neutron star or a small black hole spirals into a massive black hole with mass $< 300 M_\odot$, the inspiral wave form will be detected by the above detectors. Such wave forms carry detailed information about the spacetime geometry around the black hole, and therefore may be used to test the black hole no hair theorem [7].

When a gravitational wave signal is detected, matched filtering will be used to extract the binary’s parameters (i.e. masses, spins, etc.) [8]. In this method, the parameters are determined by cross-correlating the noisy signal from the detectors with theoretical templates. If the signal and the templates lose phase with each other by one cycle over $\sim 10^3 - 10^4$ cycles as the waves sweep through the LIGO/VIRGO band, their cross correlation will be significantly reduced. This means that we need to construct theoretical templates which are accurate to better than one cycle during entire sweep through the LIGO/VIRGO band [1]. If we have accurate templates, we can, in principle, determine the mass of the systems within 1% error [8]. Thus, much effort has been expended to construct accurate theoretical templates [8].

The standard method to calculate inspiraling wave forms from coalescing binaries is the post-Newtonian expansion of the Einstein equations, in which the orbital velocity $v$ of the binaries are assumed to be small compared to the speed of light. Since, for coalescing binaries, the orbital velocity is not so small when the frequency of gravitational waves is in LIGO/VIRGO band, it is necessary to carry the post-Newtonian expansion up to extremely high order in $v$. A post-Newtonian wave generation formalism which can handle the high order calculation has been developed by Blanchet, Damour and Iyer [10,11]. Based on this formalism, calculations have been carried out up to post $5/2$-Newtonian order, or $O(v^5)$ beyond the leading order quadrupole formula [12-20]. An another formalism is also developed up to $O(v^4)$ by Will and Wiseman [10,21] which is based on Epstein-Wagoner’s formalism [21,22].

Although the post-Newtonian calculation technique will be developed and applied to the higher order calculation, it will become more difficult and complicated. Thus, it would be very helpful if we could have another reliable method
to calculate the higher order post-Newtonian corrections. Recently the post-Newtonian expansion based on black hole perturbation formalism is developed. In this analysis, one considers gravitational waves from a particle of mass $\mu$ orbiting a black hole of mass $M$ when $\mu \ll M$. Although this method is restricted to the case when $\mu \ll M$, one can calculate very high order post-Newtonian corrections to gravitational waves using a relatively simple algorithm in contrast with the standard post-Newtonian analysis. This direction of research was first done analytically by Poisson [24] who worked to $O(v^3)$ and numerically by Cutler et.al. [25] to $O(v^5)$. Subsequently, a highly accurate numerical calculation was carried out by Tagoshi and Nakamura [26] to $O(v^8)$ in which they found the appearance of $\log v$ terms in the energy flux at $O(v^6)$ and at $O(v^8)$. They also clarified that the accuracy of the energy flux to at least $O(v^6)$ is needed to construct template wave forms for coalescing binaries. Tagoshi and Sasaki [27], using the formulation built up by Sasaki [27], performed analytic calculations which confirmed the result of Tagoshi and Nakamura. These calculations were extended to a rotating black hole case by Shibata, Sasaki, Tagoshi and Tanaka (SSTT) [28] to $O(v^5)$. They calculated gravitational waves from a particle in circular orbit with small inclination from the equatorial plane to see the effect of spin at high post-Newtonian orders. They found that the effect of spin on the orbital phase is important at $O(v^5)$ order when one of the stars is a rapidly rotating neutron star with its pulse period less than 2 ms or a rapidly rotating black hole with $q = J_{BH}/M^2 \geq 0.2$. This analysis was extended to the case of slightly eccentric orbits by Tagoshi [27]. The absorption of gravitational waves into the black hole horizon, appearing at $O(v^8)$, was also calculated by Poisson and Sasaki in the case when a test particle is in a circular orbit around a Schwarzschild black hole [29].

In this paper, we extend these analyses in the rotating black hole case to $O(v^8)$ order. Once again, the calculation is based on the formalism developed by Sasaki [27] to treat a Schwarzschild black hole. Based on the post-Newtonian expansion of the luminosity in the test particle limit when the central body is a Schwarzschild black hole (23,24), Cutler and Flanagan [31] estimated that we will have to calculate post-Newtonian expansion of gravitational wave luminosity at least up to $O(v^8)$ in order to obtain the theoretical templates which cause less systematic errors than statistical errors for the LIGO detector. Further, in a previous paper [28], we suggested that the effect of spin at $O(v^6)$ to the orbital phase of coalescing binaries wouldn’t be negligible if spin of the black hole was large (i.e. $|q| \sim 1$). Also, the perturbation study can provide accurate templates for binaries with $M \gg \mu$(for example, binaries of 100$M_\odot$ black hole-1.4$M_\odot$ neutron star). Since LIGO and VIRGO will be able to detect gravitational wave signals from binaries with masses less than $\sim 300M_\odot$, it is important to construct templates for such binaries. The frequency of gravitational waves from such a massive binary, however, comes into the frequency band for LIGO and VIRGO at $r/M \sim 16(100M_\odot/M)^{2/3}$, i.e., highly relativistic region. We do not know whether the convergence property of the post-Newtonian approximation is good or not in such a highly relativistic motion. Hence, it is an urgent problem to clarify at what point the convergence property of the post-Newtonian expansion is good. For these purposes, we study the effect of spin beyond $O(v^6)$ order in this paper.

The paper is organized as follows. In section 2, we present the basic formalism to perform the post-Newtonian expansion in our perturbative approach. First we perform the post-Newtonian expansion of the Teukolsky radial function using the Sasaki-Nakamura equation. We also show the post-Newtonian expansion of the angular equation, which is given in Appendix F. In section 3, we first describe the post-Newtonian expansion of the source terms. We consider circular orbits in the equatorial plane around a Kerr black hole. Then the gravitational wave luminosities to $O(v^8)$ beyond the quadrupole formula are derived. In section 4, we compare post-Newtonian formulas with numerical data which gives the exact value of gravitational wave luminosity and investigate the convergence property of the post-Newtonian expansion. Section 5 is devoted to summary and discussion.

Throughout this paper we use the units of $c=G=1$.

II. GENERAL FORMULATION

A. The Teukolsky equation

We consider the case when a test particle of mass $\mu$ travels in a circular orbit around a Kerr black hole of mass $M \gg \mu$. We follow the notation used by SSTT [28], but for definiteness, we recapitulate necessary formulas and definitions.

To calculate gravitational radiation from a particle orbiting a Kerr black hole, we start with the Teukolsky equation [24,33]. We focus on the radiation going out to infinity described by the fourth Newman-Penrose quantity, $\psi_4$ [34], which may be expressed as

$$\psi_4 = (r - i a \cos \theta)^{-4} \int d\omega e^{-i\omega t} \sum_{\ell, m} \frac{c^{\ell m \varphi}}{\sqrt{2\pi}} (-2S^{\ell m}_{\ell m}(\theta)) R_{\ell m \omega}(r),$$

(2.1)
where $-2S_{lm}^{\omega}$ is the spheroidal harmonic function of spin weight $s = -2$, which is normalized as
\[ \int_0^\pi |-2S_{lm}^{\omega}|^2 \sin \theta d\theta = 1. \] (2.2)

The radial function $R_{\ell m \omega}(r)$ obeys the Teukolsky equation with spin weight $s = -2$,
\[ \Delta^2 \frac{d}{dr} \left( \frac{1}{\Delta} \frac{dR_{\ell m \omega}}{dr} \right) - V(r) R_{\ell m \omega} = T_{\ell m \omega}(r), \] (2.3)
where $T_{\ell m \omega}(r)$ is the source term whose explicit form will be shown later, and $\Delta = r^2 - 2Mr + a^2$. The potential $V(r)$ is given by
\[ V(r) = -\frac{K^2 + 4i(r - M)K}{\Delta} + 8i\omega r + \lambda, \] (2.4)
where $K = (r^2 + a^2)\omega - ma$ and $\lambda$ is the eigenvalue of $-2S_{lm}^{\omega}$.

The solution of the Teukolsky equation at infinity ($r \to \infty$) is expressed as
\[ R_{\ell m \omega}(r) \to \frac{r^3 e^{i\omega r^*}}{2\omega B_{\ell m \omega}^{\text{in}}} \int_{r_+}^\infty dr' \frac{T_{\ell m \omega}(r') B_{\ell m \omega}^{\text{in}}(r')}{\Delta^2(r')} \equiv \tilde{Z}_{\ell m \omega} r^3 e^{i\omega r^*}, \] (2.5)
where $r_+ = M + \sqrt{M^2 - a^2}$ denotes the radius of the event horizon and $B_{\ell m \omega}^{\text{in}}$ is the homogeneous solution which satisfies the ingoing-wave boundary condition at horizon,
\[ R_{\ell m \omega}^{\text{in}} \to \begin{cases} D_{\ell m \omega} \Delta^2 e^{-ikr^*} & \text{for } r^* \to -\infty, \\ r^3 B_{\ell m \omega}^{\text{out}} e^{i\omega r^*} + r^{-1} B_{\ell m \omega}^{\text{in}} e^{-i\omega r^*} & \text{for } r^* \to +\infty, \end{cases} \] (2.6)
where $k = \omega - ma/2Mr_+$ and $r^*$ is the tortoise coordinate defined by
\[ \frac{dr^*}{dr} = \frac{r^2 + a^2}{\Delta}. \] (2.7)

For definiteness, we fix the integration constant such that $r^*$ is given explicitly by
\[ r^* = \int \frac{dr^*}{dr} dr = r + \frac{2Mr_+}{r_+ - r_-} \ln \frac{r - r_+}{2M} - \frac{2Mr_-}{r_+ - r_-} \ln \frac{r - r_-}{2M}, \] (2.8)
where $r_\pm = M \pm \sqrt{M^2 - a^2}$.

**B. Post-Newtonian expansion of the homogeneous solution**

In the previous papers [27, 28], the post-Newtonian expansion of the homogeneous solution was performed to $O(\epsilon^2)$ in the Schwarzschild case and $O(\epsilon)$ in the Kerr case, where $\epsilon \equiv 2M\omega$. In this section, we extend those methods, performing the expansion of homogeneous solutions up to $O(\epsilon^3)$.

In order to calculate gravitational waves emitted to infinity from a particle in a circular orbit, we need to know the explicit form of the source term $T_{\ell m \omega}(r)$, which has support only at $r = r_0$ where $r_0$ is orbital radius in the Boyer-Lindquist coordinate, the ingoing-wave Teukolsky function $B_{\ell m \omega}^{\text{in}}(r)$ at $r = r_0$, and its incident amplitude $B_{\ell m \omega}^{\text{in}}$ at infinity. We consider the expansion of these quantities in terms of a small parameter, $v^2 \equiv M/r$. In addition, we need to expand those quantity in terms of $\epsilon \equiv 2M\omega$ since $\omega \sim O(\Omega)$ where $\Omega$ is the orbital angular velocity of the particle and $M\omega \sim O(v^3)$. In the case of a Kerr black hole, other combination of parameters $a\omega$ appears in the Teukolsky equation. We define $q \equiv a/M$ and we have $a\omega = q\epsilon/2 \sim O(\epsilon^3)$.

First we perform the expansion of the spherical harmonics $-2S_{lm}^{\omega}$ and their eigenvalues $\lambda$ in terms of $a\omega$. Since $a\omega = O(v^3)$, we have to calculate $-2S_{lm}^{\omega}$ and $\lambda$ up to $O((a\omega)^2)$. The eigenvalue $\lambda$ has already been evaluated up to $O((a\omega)^2)$ in a previous paper [28]. We calculate the expansion of $-2S_{lm}^{\omega}$ at $O((a\omega)^2)$ in the Appendix F. As a result, the spheroidal harmonics $-2S_{lm}^{\omega}$ are given by
\[-2S_{\ell m}^{2 \omega} = -2P_{\ell m} + a \omega S_{\ell m}^{(1)} + (a \omega)^2 S_{\ell m}^{(2)} + O((a \omega)^3), \tag{2.9}\]

where \(-2P_{\ell m}\) are the spherical harmonics of spin weight \(s = -2\) and

\[S_{\ell m}^{(1)} = \sum_{\ell'} c_{\ell m}^{\ell'} - 2P_{\ell m}. \tag{2.10}\]

Here \(c_{\ell m}^{\ell'}\) are non-zero only for \(\ell' = \ell \pm 1\), explicitly

\[c_{\ell m}^{\ell + 1} = \frac{2}{(\ell + 1)^2} \left[ \frac{(\ell + 3)(\ell - 1)(\ell + m + 1)(\ell - m + 1)}{(2\ell + 1)(2\ell + 3)} \right]^{1/2}, \]
\[c_{\ell m}^{\ell - 1} = \frac{2}{\ell^2} \left[ \frac{\ell + 2)(\ell - 2)(\ell + m)(\ell - m)}{(2\ell + 1)(2\ell - 1)} \right]^{1/2}. \]

\(S_{\ell m}^{(2)}\) is given by

\[S_{\ell m}^{(2)} = \sum_{\ell'} d_{\ell m}^{\ell'} - 2P_{\ell m}, \tag{2.11}\]

where the non-zero components of \(d_{\ell m}^{\ell'}\) are given by

\[d_{\ell m}^{\ell} = -\frac{1}{2} \left( (c_{\ell m}^{\ell+1})^2 + (c_{\ell m}^{\ell-1})^2 \right), \tag{2.12}\]

for any \(\ell\), by

\[d_{\ell m}^{\ell+1} = \frac{m}{324\sqrt{7}}(3 - m)^{1/2}(3 + m)^{1/2}, \]
\[d_{\ell m}^{\ell+2} = \frac{11}{1764\sqrt{3}}(3 - m)^{1/2}(3 + m)^{1/2}(4 - m)^{1/2}(4 + m)^{1/2}, \]

for \(\ell = 2\), and by

\[d_{\ell m}^{\ell+1} = \frac{m}{120\sqrt{21}}(4 - m)^{1/2}(4 + m)^{1/2}, \]
\[d_{\ell m}^{\ell+2} = \frac{1}{180\sqrt{11}}(4 - m)^{1/2}(4 + m)^{1/2}(5 - m)^{1/2}(5 + m)^{1/2}, \]
\[d_{\ell m}^{\ell-1} = -\frac{m}{324\sqrt{7}}(3 - m)^{1/2}(3 + m)^{1/2}, \]

for \(\ell = 3\). We don’t need \(S_{\ell m}^{(2)}\) for \(\ell = 4\) in this paper.

The eigenvalue \(\lambda\) is given by

\[\lambda = \lambda_0 + a \omega \lambda_1 + a^2 \omega^2 \lambda_2 + O((a \omega)^3) \tag{2.13}\]

where \(\lambda_0 = (\ell - 1)(\ell + 2)\), \(\lambda_1 = -2m(\ell^2 + \ell + 4)/(\ell^2 + \ell)\) and

\[\lambda_2 = -2(\ell + 1)(c_{\ell m}^{\ell+1})^2 + 2d_{\ell m}^{\ell+2} + \frac{2}{3} \frac{(\ell + 4)(\ell - 3)(\ell^2 + \ell - 3m^2)}{\ell(\ell + 1)(2\ell + 3)(2\ell - 1)}. \tag{2.14}\]

Next we calculate the homogeneous solution \(R_{\ell m}^{n \omega}\). Here we only consider the case when \(\omega > 0\). We must treat the case \(\omega \leq 0\) separately. The Teukolsky equation is transformed into the Sasaki-Nakamura equation \(36\), which is given by

\[\left[ \frac{d^2}{dr^2} - F(r) \frac{d}{dr} - U(r) \right] X_{\ell m \omega} = 0. \tag{2.15}\]

The explicit forms of \(F(r)\) and \(U(r)\) are given in the Appendix A. The relation between \(R_{\ell m \omega}\) and \(X_{\ell m \omega}\) is
\[ R_{\ell m \omega} = \frac{1}{\eta} \left\{ \left( \alpha + \frac{\beta_r}{\Delta} \right) \chi_{\ell m \omega} - \frac{\beta}{\Delta} \chi_{\ell m \omega, r} \right\}, \tag{2.16} \]

where \( \chi_{\ell m \omega} = X_{\ell m \omega} \Delta / (r^2 + a^2)^{1/2} \), and the functions \( \alpha, \beta \) and \( \eta \) are shown in Appendix A. Conversely, we can express \( X_{\ell m \omega} \) in terms of \( R_{\ell m \omega} \) as

\[ X_{\ell m \omega} = (r^2 + a^2)^{1/2} r J_+ \left[ \frac{1}{r^2} R_{\ell m \omega} \right], \tag{2.17} \]

where \( J_+ = (d/dr) - i(K/\Delta) \). Then the asymptotic behavior of the ingoing-wave solution \( X_{\ell m \omega}^{\text{in}} \) which corresponds to Eq.(2.6) is

\[ X_{\ell m \omega}^{\text{in}} \rightarrow \begin{cases} A_{\ell m \omega}^{\text{out}} e^{i \omega r^*} + A_{\ell m \omega}^{\text{in}} e^{-i \omega r^*} & \text{for } r^* \to \infty, \\ C_{\ell m \omega} e^{-ik r^*} & \text{for } r^* \to -\infty. \end{cases} \tag{2.18} \]

The coefficient \( A_{\ell m \omega}^{\text{in}}, A_{\ell m \omega}^{\text{out}} \) and \( C_{\ell m \omega} \) are respectively related to \( B_{\ell m \omega}^{\text{in}}, B_{\ell m \omega}^{\text{out}} \) and \( D_{\ell m \omega} \), defined in Eq.(2.6), by

\[ B_{\ell m \omega}^{\text{in}} = -\frac{1}{4 \omega^2} A_{\ell m \omega}^{\text{in}}, \tag{2.19} \]
\[ B_{\ell m \omega}^{\text{out}} = -\frac{4 \omega^2}{c_0} A_{\ell m \omega}^{\text{out}}, \]
\[ D_{\ell m \omega} = \frac{1}{d_{\ell m \omega}} C_{\ell m \omega}, \]

where \( c_0 \) is given in Eq.(A.3) of Appendix A and

\[ d_{\ell m \omega} = \sqrt{2Mr}(8 - 4iM \omega - 16M^2 \omega^2) r_+^2 + (12 \omega m - 16M + 16amM + 24iM^2 \omega) r_+ - 4a^2 m^2 - 12iMm + 8M^2]. \]

Now we introduce the variable \( z = \omega r \) and

\[ z^* = z + \epsilon \left[ \frac{z_+}{z_+ - z_-} \ln(z - z_+) - \frac{z_-}{z_+ - z_-} \ln(z - z_-) \right] = \omega r^* + \epsilon \ln \epsilon, \tag{2.20} \]

where \( z_\pm = \omega r_\pm \). To solve \( X_{\ell m \omega}^{\text{in}} \) by expanding it in terms of \( \epsilon \), we set

\[ X_{\ell m \omega}^{\text{in}} = \sqrt{z^2 + a^2 \omega^2} \xi_{\ell m}(z) \exp(-i \phi(z)), \tag{2.21} \]

where

\[ \phi(z) = \int dr \left( \frac{K}{\Delta} - \omega \right) = z^* - z - \frac{\epsilon}{2} mq \frac{1}{z_+ - z_-} \ln \frac{z - z_+}{z - z_-}, \tag{2.22} \]

which generalizes the phase function \( \omega (r^* - r) \) of the Schwarzschild case. This prescription makes it easy to implement the ingoing-wave boundary condition on \( X_{\ell m \omega}^{\text{in}} \).

Inserting Eq.(2.21) into Eq.(2.15) and expanding it in powers of \( \epsilon = 2M \omega \), we obtain

\[ L^{(0)}[\xi_{\ell m}] = \epsilon L^{(1)}[\xi_{\ell m}] + \epsilon Q^{(1)}[\xi_{\ell m}] + \epsilon^2 Q^{(2)}[\xi_{\ell m}] + \epsilon^3 Q^{(3)}[\xi_{\ell m}] + \epsilon^4 Q^{(4)}[\xi_{\ell m}] + O(\epsilon^5), \tag{2.23} \]

where \( L^{(0)}, L^{(1)}, Q^{(1)} \) and \( Q^{(2)} \) are differential operators given by
we obtain from Eq.(2.23) the iterative equations, 

\[ L^{(0)} = \frac{d^2}{dz^2} + \frac{2}{z} \frac{d}{dz} + \left( 1 - \frac{\ell(\ell + 1)}{z^2} \right), \]  

\[ L^{(1)} = \frac{1}{z} \frac{d^2}{dz^2} + \left( \frac{1}{z^2} + \frac{2i}{z} \right) \frac{d}{dz} - \left( \frac{4}{z^3} - \frac{i}{z^2} + \frac{1}{z} \right), \]  

\[ Q^{(1)} = \frac{iq\lambda_1}{2z^2} \frac{d}{dz} - \frac{4imq}{l(l+1)z^3}, \]  

and \( Q^{(2)}, Q^{(3)} \) and \( Q^{(4)} \) are given in Appendix C. Note that the real part of \( Q^{(1)} \) vanishes when we insert the expression for \( \lambda_1 \). There are \( \lambda_3 \) or \( \lambda_4 \) in the formulas for \( Q^{(3)} \) and \( Q^{(4)} \). However, it is straightforward to show that both \( \lambda_3 \) and \( \lambda_4 \) do not influence the results in this paper.

By expanding \( \xi \) in terms of \( \epsilon \) as

\[ \xi = \sum_{n=0}^{\infty} \epsilon^n \xi^{(n)}(z), \]  

we obtain from Eq.(2.23) the iterative equations,

\[ L^{(0)}[\xi^{(0)}] = 0, \]  

\[ L^{(0)}[\xi^{(1)}] = L^{(1)}[\xi^{(0)}] + Q^{(1)}[\xi^{(0)}] \equiv W^{(1)} \]  

\[ L^{(0)}[\xi^{(2)}] = L^{(1)}[\xi^{(1)}] + Q^{(1)}[\xi^{(1)}] + Q^{(2)}[\xi^{(0)}] \equiv W^{(2)} \]  

\[ L^{(0)}[\xi^{(3)}] = L^{(1)}[\xi^{(2)}] + Q^{(1)}[\xi^{(2)}] + Q^{(2)}[\xi^{(1)}] + Q^{(3)}[\xi^{(0)}] \equiv W^{(3)} \]  

\[ L^{(0)}[\xi^{(4)}] = L^{(1)}[\xi^{(3)}] + Q^{(1)}[\xi^{(3)}] + Q^{(2)}[\xi^{(2)}] + Q^{(3)}[\xi^{(1)}] + Q^{(4)}[\xi^{(0)}] \equiv W^{(4)} \]  

The general solution to Eq.(2.28) is immediately obtained as

\[ \xi^{(0)} = \alpha^{(0)} j_{\ell} + \beta^{(0)} n_{\ell}, \]  

where \( j_{\ell} \) and \( n_{\ell} \) are the usual spherical Bessel functions. As we discuss later, the boundary condition for \( n \leq 2 \) is that \( \xi^{(n)} \) is regular at \( z = 0 \). Hence \( \beta^{(0)} = 0 \) and we set \( \alpha^{(0)} = 1 \) for convenience.

To calculate \( \xi^{(n)} \) for \( n \geq 1 \), we rewrite Eqs.(2.28) - (2.32) in the indefinite integral form by using the spherical Bessel functions as

\[ \xi^{(n)} = n_{\ell} \int dz z^2 j_{\ell} W^{(n)}_{\ell m} - j_{\ell} \int dz z^2 n_{\ell} W^{(n)}_{\ell m} \quad (n = 1, 2). \]  

The calculation for \( n = 1 \) was done in a previous paper 28 and we have

\[ \xi^{(1)} = \alpha^{(1)} j_{\ell} + \frac{(\ell - 1)(\ell + 3)}{2(\ell + 1)(2\ell + 1)} j_{\ell+1} - \frac{\ell^2 - 4}{2(\ell + 1)^2} j_{\ell-1} + \frac{z^2(n_{\ell} j_0 - j_{\ell} n_0)}{2} \sum_{k=1}^{\ell-1} \left( \frac{1}{k} + \frac{1}{k + 1} \right) z^2 (n_{\ell} j_k - j_{\ell} n_k) j_k + n_{\ell} (\text{Ci}(2z) - \gamma \ln 2z) - j_{\ell} \text{Si}(2z) + i j_{\ell} \ln z + \frac{imq}{2} \left( \frac{\ell^2 + 4}{\ell^2 (2\ell + 1)} \right) j_{\ell-1} + \frac{imq}{2} \left( \frac{(\ell + 1)^2 + 4}{(\ell + 1)^2 (2\ell + 1)} \right) j_{\ell+1}, \]  

where \( \text{Ci}(x) = -\int_0^\infty dt \cos t/t \) and \( \text{Si}(x) = \int_0^x dt \sin t/t \) are cosine and sine integral functions, \( \gamma \) is the Euler constant, and \( \alpha^{(1)} \) is an integration constant which represents the arbitrariness of the normalization of \( X^{in}_{\ell m \omega} \). We set \( \alpha^{(1)} = 0 \) for simplicity.

Next we consider \( \xi^{(2)} \). From Eqs.(2.34) and (2.33), and by using formulas in the paper 27, we obtain \( \xi^{(2)} \) as

\[ \xi^{(2)} = f^{(2)}(\epsilon) + g^{(2)}(\epsilon) + k^{(2)}_{\ell m}(q) + \alpha^{(2)} n_{\ell} + \beta^{(2)} n_{\ell}, \]  

6
where \( f_\ell^{(2)} \) and \( g_\ell^{(2)} \) are the real and imaginary part of \( \xi_\ell^{(2)} \) in the Schwarzschild case respectively, \( k_{\ell m}^{(2)}(q) \) exists only in Kerr case and \( \alpha_{\ell m}^{(2)} \) and \( \beta_{\ell m}^{(2)} \) are arbitrary constants. The explicit forms of \( f_\ell^{(2)} \) and \( g_\ell^{(2)} \) are given in a previous paper \cite{26}. The term \( k_{\ell m}^{(2)}(q) \) is given for \( \ell = 2 \) by

\[
k_{2m}^{(2)}(q) = \frac{191 i}{180} m q j_0 - \frac{m^2 q^2 j_0}{30} - \frac{m q j_1}{10} - \frac{68 i}{63} m q j_2 - \frac{q^2 q_j}{392} + \frac{73 m^2 q^2 j_2}{1764} - \frac{7 m q j_3}{180} - \frac{i}{72} q^2 j_3 + \frac{i}{324} m^2 q^2 j_3 + \frac{11 i}{420} m q j_4 - \frac{q^2 q_{j4}}{392} - \frac{71 m^2 q^2 j_4}{8820} + \frac{13 i}{6} m q n_1 + \frac{5}{90} (m q j_1 - \frac{13 m q j_3}{90}) \ln z + \left( -\frac{i}{5} m q j_1 - \frac{13 i}{90} m q j_3 \right) S(z) + \left( \frac{i}{5} m q n_1 + \frac{13 i}{90} m q n_3 \right) C(z),
\]

(2.37)

and for \( \ell = 3 \)

\[
k_{3m}^{(2)}(q) = \frac{3527 i}{840} m q j_1 - \frac{2 m^2 q^2 j_1}{315} - \frac{m q j_2}{36} - \frac{5 i}{504} q^2 j_2 + \frac{5 i}{2268} m^2 q^2 j_2 - \frac{379 i}{360} m q j_3 - \frac{q^2 q_{j3}}{30} + \frac{7 m^2 q^2 j_3}{720} + \frac{3 m q j_4}{160} - \frac{i}{140} q^2 j_4 + \frac{i}{1120} m^2 q^2 j_4 + \frac{97 i}{5040} m q j_5 - \frac{q^2 q_{j5}}{360} - \frac{17 m^2 q^2 j_5}{5040} - \frac{103 i}{48} m q n_0 + \frac{25 i}{8} m q n_2 - \frac{13 m q j_2}{126} + \frac{5 m q j_4}{56} \ln z + \left( -\frac{13 i}{126} m q j_2 - \frac{5 i}{56} m q j_4 \right) S(z) + \left( \frac{13 i}{126} m q n_2 + \frac{5 i}{56} m q n_4 \right) C(z),
\]

(2.38)

where \( C(z) = \text{Ci}2z - \gamma - \ln 2z \) and \( S(z) = \text{Si}2z \). Note that to obtain above two formulas, we have added terms proportional to \( j_r \) to simplify the formulas of \( A_{\ell m}^{in} \) below. As noted previously, the source term \( T_{\ell m}^{in} \) has support only at \( r = r_0 \) and \( \omega r_0 = O(v) \). Hence we only need \( X_{\ell m}^{in} \) at \( z = O(\nu^2) \) to evaluate the source integral, apart from the value of the incident amplitude \( A_{\ell m}^{in} \). Hence the post-Newtonian expansion of \( X_{\ell m}^{in} \) corresponds to the expansion not only in terms of \( \epsilon = O(\nu^3) \), but also \( z \) by assuming \( \epsilon \ll z \ll 1 \). In order to evaluate the gravitational wave luminosity to \( O(\nu^8) \) beyond the leading order, we must calculate the series expansion of \( \xi_{\ell m}^{(n)} \) in powers of \( z \) for \( n = 0 \) to \( \ell = 6 \), for \( n = 1 \) to \( \ell = 5 \), for \( n = 2 \) to \( \ell = 4 \), for \( n = 3 \) to \( \ell = 3 \) and for \( n = 4 \) to \( \ell = 2 \) (See Appendix C of SSTT).

When we evaluate \( A_{\ell m}^{in} \), we examine the asymptotic behavior of \( \xi_{\ell m}^{(n)} \) at infinity. Since the accuracy of \( A_{\ell m}^{in} \) we need is \( O(\epsilon^2) \), we don’t have to calculate \( \xi_{\ell m}^{(3)} \) and \( \xi_{\ell m}^{(4)} \) in closed analytic form. We need only the series expansion formulas for \( \xi_{\ell m}^{(3)} \) and \( \xi_{\ell m}^{(4)} \) around \( z = 0 \), which is easily obtained by \text{Eq. (2.33)}. Inserting \( \xi_{\ell m}^{(n)} \) into \text{Eq. (2.21)} and expanding it by \( z \) and \( \epsilon \) assuming \( \epsilon \ll z \ll 1 \), we obtain

\[
\xi_{2m}^{(3)} = \frac{-q^2}{30 z} - \frac{i}{30} m q^3 + \frac{i}{30} m^2 q^2 + \frac{7 m q}{180} - \frac{i}{60} m^2 q^2 + \frac{m^3 q^3}{30} - \frac{m^3 q^3}{90} - \frac{m q \ln z}{30} - \frac{i}{30} m^2 q^2 \ln z + z \left( \frac{319}{6300} + \frac{100637}{441000} m q - \frac{q^2}{180} + \frac{17 m^2 q^2}{1134} + \frac{83 i}{5880} m q^3 \right) - \frac{61 i}{13230} m^3 q^3 + \frac{\ln z}{15} - \frac{106 i}{1575} m q \ln z - \frac{i}{30} m q \ln z^2) + O(z^2) + \alpha_{2m}^{(3)} j_2 + \beta_{2m}^{(3)} n_2,
\]

(2.39)

\[
\xi_{2m}^{(4)} = \frac{q^4}{80 z^2} + O(z^{-1}) + \alpha_{2m}^{(4)} j_2 + \beta_{2m}^{(4)} n_2,
\]

(2.40)

\[
\xi_{3m}^{(3)} = \frac{-i}{1260} m q + \frac{m^2 q^2}{1890} - \frac{i}{1260} m q^3 - \frac{i}{3780} m^3 q^3 + O(z)
\]
The boundary condition of \( \xi^{(n)}_{\ell m} \) that correctly represent the boundary condition of \( X_{\ell m}^{in} \) (Eq.(2.18)) is that \( ze^{(n)}_{\ell m} \) must be no more singular than \( z^{(\ell+1-n)} \) at \( z \to 0 \). Since we need \( \xi^{(n)}_{\ell m} \) only up to \( n = 4 \), we set \( \beta^{(n)}_{\ell m} = 0 \) for all of \( \ell \) and \( n \) in this paper. As for \( \alpha^{(n)}_{\ell m} \), they still remain arbitrary and we set \( \alpha^{(n)}_{\ell m} = 0 \) for all of \( \ell \), \( m \) and \( n = 3, 4 \).

Inserting \( \xi^{(n)} \) into Eq.(2.22) and expanding it in terms of \( \epsilon = 2M\omega \), we obtain \( X_{\ell m}^{in} \), which are shown in Appendix C. Using the transformation of Eq.(2.16), we obtain \( R_{\ell m}^{in} \), which are also shown in Appendix D.

Next, we consider \( A_{\ell m}^{in} \) at \( O(\epsilon^2) \). Using the relation \( j_{\ell+1} \sim -j_{\ell-1} \sim (-1)^{\ell+1}n_{2\ell-1} \) at \( z \to \infty \), etc., we obtain the asymptotic behavior of \( \xi^{(1)}_{\ell m} \) and \( \xi^{(2)}_{\ell m} \) at \( z \sim \infty \) as

\[
\begin{align*}
\xi^{(1)}_{\ell m} &\sim p^{(1)}_{\ell m}j_{\ell} + (q^{(1)}_{\ell m} - \ln z)n_{\ell} + i\omega j_{\ell}\ln z, \\
\xi^{(2)}_{\ell m} &\sim \left(p^{(2)}_{\ell m} + q^{(2)}_{\ell m}\ln z - (\ln z)^2\right)j_{\ell} + (q^{(2)}_{\ell m} - p^{(1)}_{\ell m}\ln z)n_{\ell} + i\omega p^{(1)}_{\ell m}\ln z + i(q^{(1)}_{\ell m} - \ln z)n_{\ell}\ln z
\end{align*}
\]

(2.43)

where

\[
\begin{align*}
p^{(1)}_{\ell m} &= -\frac{\pi}{2}, \\
p^{(2)}_{\ell m} &= \frac{457}{210} - \frac{\gamma^2}{2} + \frac{5\pi^2}{24} - \frac{i}{18} \ln 2 + \frac{52}{21} \ln 2 - \frac{\ln(2)^2}{2}, \\
q^{(1)}_{\ell m} &= \frac{457}{210} - \frac{\gamma^2}{2} + \frac{5\pi^2}{24} + \frac{i}{36} \ln 2 - \frac{q^2}{72} + \frac{i}{2} q^2 + \frac{17 i}{324} m^2 q^2 + \frac{\pi}{2} \ln 2, \\
p^{(2)}_{\ell m} &= \frac{52}{21} + \frac{\gamma^2}{2} - \frac{5\pi^2}{24} - \frac{i}{12} m \ln 2 - \frac{210}{72} q^2 + \frac{17 i}{360} m^2 q^2 + \frac{\pi}{2} \ln 2, \\
q^{(2)}_{\ell m} &= \frac{1440}{21} + \frac{\gamma^2}{2} + \frac{67 m q}{1440} + \frac{i}{144} m \ln 2 + \frac{i}{360} q^2 + \frac{17 i}{12960} m^2 q^2 + \frac{\pi}{2} \ln 2.
\end{align*}
\]

(2.47, 2.48, 2.49, 2.50)

Then noting that \( \exp(-i\phi) \sim \exp(-i(z^* - z)) \) at \( z \to \infty \), the asymptotic form of \( X_{\ell m}^{in} \) is expressed as

\[
X_{\ell m}^{in} = \sqrt{z + a^2} \omega z^2 \exp(-i\phi) \left\{ f^{(1)}_{\ell m} + \epsilon f^{(2)}_{\ell m} + \cdots \right\}
\]

(2.51)

where \( h^{(1)}_{\ell} \) and \( h^{(2)}_{\ell} \) are the spherical Hankel functions of the first and second kinds, respectively, which are given by

\[
h^{(1)}_{\ell} = j_{\ell} + in_{\ell} \to (-1)^{\ell + 1} e^{i\pi/2}, \quad h^{(2)}_{\ell} = j_{\ell} - in_{\ell} \to (-1)^{\ell + 1} e^{-i\pi/2}.
\]

(2.52)

From these equations, noting \( \omega^* = z^* - \epsilon \ln e \), we obtain

\[
A_{\ell m}^{in} = \frac{1}{2} e^{i\epsilon \ln e} \left(1 + \epsilon f^{(1)}_{\ell m} + \epsilon^2 f^{(2)}_{\ell m} + \cdots \right).
\]

(2.53)

The corresponding incident amplitude \( B_{\ell m}^{in} \) for the Teukolsky function are obtained from Eq.(2.19).
III. GRAVITATIONAL WAVE LUMINOSITY TO $O(V^8)$

A. The geodesic equations

In this section, we solve the geodesic equation for circular motion in the equatorial plane. The geodesic equations in the Kerr geometry are given by

\[
\begin{align*}
\Sigma \frac{d\theta}{dr} &= \pm \left[ C - \cos^2 \theta \left\{ a^2 (1 - E^2) + \frac{l_z^2}{\sin^2 \theta} \right\} \right]^{1/2} \equiv \Theta(\theta), \\
\Sigma \frac{d\varphi}{dr} &= - \left( aE - \frac{l_z}{\sin^2 \theta} \right) + \frac{a}{\Delta} \left( E (r^2 + a^2) - al_z \right) \equiv \Phi, \\
\Sigma \frac{dt}{dr} &= - \left( aE - \frac{l_z}{\sin^2 \theta} \right) a \sin^2 \theta + \frac{r^2 + a^2}{\Delta} \left( E (r^2 + a^2) - al_z \right) \equiv T, \\
\Sigma \frac{dr}{dt} &= \pm \sqrt{\Delta},
\end{align*}
\]

(3.1)

where $E$, $l_z$ and $C$ are the energy, the $z$-component of the angular momentum and the Carter constant of a test particle, respectively. $\Sigma = r^2 + a^2 \cos^2 \theta$ and

\[
R = [E (r^2 + a^2) - al_z]^2 - \Delta [(Ea - l_z)^2 + r^2 + C].
\]

(3.2)

Since we consider a motion of a particle in the equatorial plane $\theta = \pi/2$, we can set $C = 0$. We define the orbital radius as $r = r_0$. Then $E$ and $l_z$ are determined by $R(r_0) = 0$ and $\partial R/\partial r |_{r=r_0} = 0$ as

\[
E = \frac{1 - 2v^2 + qv^3}{(1 - 3v^2 + 2q^2v^3)^{1/2}}, \\
l_z = \frac{r_0 v (1 - 2q^3 + q^2 v^4)}{(1 - 3v^2 + 2q^3)^{1/2}},
\]

where $v = (M/r_0)^{1/2}$. After these preparations, we can easily obtain $\varphi(t)$ as

\[
\begin{align*}
\varphi(t) &= \Omega t, \\
\Omega &= \frac{M^{1/2}}{r_0^{3/2}} \left[ 1 - qv^3 + q^2 v^6 + O(v^9) \right].
\end{align*}
\]

(3.3)

B. Integration of the source term

Using results of the previous section, we can now derive the source term of the Teukolsky equation and integrate it to give the amplitude of the Teukolsky function at infinity.

The energy momentum tensor of a test particle of mass $\mu$ is given by

\[
T^{\mu\nu} = \frac{\mu}{\Sigma \sin \theta \omega dt/d\tau} \frac{dz^\mu}{dz^\varphi} \delta(r - r_0) \delta(\theta - \pi/2) \delta(\varphi - \varphi(t)).
\]

(3.4)

The source term of the Teukolsky equation is given by

\[
T_{\ell m \omega} = 4 \int d\Omega dt \rho^{-5} \rho^{-1} (B_2' + B_2^{*}) e^{-im\varphi + i\omega t} - \frac{2S_{\ell m \omega}}{\sqrt{2\pi}},
\]

(3.5)

where
Substituting Eq. (3.6) into Eq. (3.5) and integrating by parts, we obtain

\[ B_2' = -\frac{1}{2} \rho^2 \rho L_{-1}[\rho^{-4} L_0(\rho^{-2} \bar{\rho}^{-1} T_{nn})] \\
- \frac{1}{2\sqrt{2}} \rho^2 \bar{\rho} \Delta^2 L_{-1}[\rho^{-4} \bar{\rho}^2 J_+(\rho^{-2} \bar{\rho}^{-2} \Delta^{-1} T_{\bar{m}\bar{m}})], \]

\[ B_2'' = -\frac{1}{4} \rho^2 \rho \Delta^2 J_+[\rho^{-4} J_+(\rho^{-2} \bar{\rho} T_{\bar{m}\bar{m}})] \\
- \frac{1}{2\sqrt{2}} \rho^2 \bar{\rho} \Delta^2 J_+[\rho^{-4} \bar{\rho}^2 \Delta^{-1} L_{-1}(\rho^{-2} \bar{\rho}^{-2} T_{\bar{m}\bar{m}})], \]  

(3.6)

with

\[ \rho = (r - ia \cos \theta)^{-1}, \]

\[ L_s = \partial_\theta + \frac{m}{\sin \theta} - a \omega \sin \theta + s \cot \theta, \]

\[ J_+ = \partial_r + iK/\Delta, \]  

(3.7)

and \( \bar{\rho} \) denotes the complex conjugate of \( \rho \).

In the present case, the tetrad components of the energy momentum tensor, \( T_{nn}, T_{\bar{m}\bar{m}} \) and \( T_{\bar{m}\bar{m}} \), take the form,

\[ T_{nn} = \frac{C_{nn}}{\sin \theta} \delta(r - r_0) \delta(\theta - \pi/2) \delta(\varphi - \varphi(t)), \]

\[ T_{\bar{m}\bar{m}} = \frac{C_{\bar{m}\bar{m}}}{\sin \theta} \delta(r - r_0) \delta(\theta - \pi/2) \delta(\varphi - \varphi(t)), \]

\[ T_{\bar{m}\bar{m}} = \frac{C_{\bar{m}\bar{m}}}{\sin \theta} \delta(r - r_0) \delta(\theta - \pi/2) \delta(\varphi - \varphi(t)), \]  

(3.8)

where

\[ C_{nn} = \frac{\mu}{4\Sigma^4 t} \left[ E(r^2 + a^2) - al_z \right]^2, \]

\[ C_{\bar{m}\bar{m}} = \frac{\mu \rho}{2\sqrt{2} \Sigma^2 t} \left[ E(r^2 + a^2) - al_z \right] \left[ i \sin \theta \left( aE - \frac{l_z}{\sin^2 \theta} \right) \right], \]

\[ C_{\bar{m}\bar{m}} = \frac{\mu \rho^2}{2\Sigma t} \left[ i \sin \theta \left( aE - \frac{l_z}{\sin^2 \theta} \right) \right]^2, \]  

(3.9)

and \( \dot{t} = dt/dr \).

Substituting Eq. (3.6) into Eq. (3.3) and integrating by parts, we obtain

\[ T_{tmw} = \frac{4}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt \int d\theta e^{i\omega t - im\varphi(t)} \]

\[ \times \left[ \frac{1}{2} L_s^0 \left( \rho^{-4} L_s^0 \right) C_{nn} \rho^{-2} \bar{\rho}^{-1} \delta(r - r_0) \delta(\theta - \pi/2) \right] \\
+ \frac{\Delta^2 \rho^2}{\sqrt{2} \rho} \left( L_s^0 S + ia(\bar{\rho} - \rho) \sin \theta S \right) J_+ \left\{ C_{\bar{m}\bar{m}} \rho^{-2} \bar{\rho}^{-2} \Delta^{-1} \delta(r - r_0) \delta(\theta - \pi/2) \right\} \\
+ \frac{1}{2\sqrt{2}} L_s^0 \left( \rho^3 S \rho^{-4} \right) C_{\bar{m}\bar{m}} \Delta \rho^{-2} \bar{\rho}^{-2} \delta(r - r_0) \delta(\theta - \pi/2) \\
- \frac{1}{4} \rho^3 \Delta^2 S J_+ \left\{ \rho^{-4} J_+ \left( \bar{\rho} \rho^{-2} C_{\bar{m}\bar{m}} \delta(r - r_0) \delta(\theta - \pi/2) \right) \right\}. \]  

(3.10)

where

\[ L_s^0 = \partial_\theta - \frac{m}{\sin \theta} + a \omega \sin \theta + s \cot \theta, \]

(3.11)
and $S$ denotes $-2S_{\ell m}^{\omega}(\theta)$ for simplicity.

We further rewrite Eq. (3.10) as

$$T_{\ell m \omega} = \int_{-\infty}^{\infty} dt e^{i\omega t} \Delta^2 [(A_{n n 0} + A_{\bar{m} \bar{m} 0}) \delta(r - r_0)$$

$$+ \{(A_{\bar{m} n 1} + A_{\bar{m} \bar{m} 1}) \delta(r - r_0)\}_{r} + \{(A_{\bar{m} \bar{m} 2} \delta(r - r_0)\}_{rr}]_{\theta = \pi/2},$$

(3.12)

where $A_{n n 0}$ etc. are given in Appendix B. Inserting Eq. (3.12) into Eq. (2.5), we obtain $\tilde{T}_{\ell m \omega}$ as

$$\tilde{T}_{\ell m \omega} = \frac{2\pi \delta(\omega - m\Omega)}{2i\omega D_{\ell m \omega}^m} \left[ R_{\ell m \omega}^m (A_{n n 0} + A_{\bar{m} \bar{m} 0})ight.$$

$$- \frac{dR_{\ell m \omega}^m}{dt} \{A_{\bar{m} n 1} + A_{\bar{m} \bar{m} 1}\} + \frac{d^2 R_{\ell m \omega}^m}{dt^2} A_{\bar{m} \bar{m} 2} \left.]_{r = r_0, \theta = \pi/2},ight.$$

$$\equiv \delta(\omega - m\Omega) Z_{\ell m \omega}.\quad (3.13)$$

Using characters of $-2S_{\ell m}^{\omega}(\theta)$ at $\theta = \pi/2$, it is straightforward to show that $\tilde{T}_{\ell, -m, -\omega} = (-1)^{\ell} \tilde{T}_{\ell, m, \omega}$ where $\tilde{T}_{\ell, m, \omega}$ is the complex conjugate of $T_{\ell, m, \omega}$. Since the homogeneous Teukolsky equation is invariant under the complex conjugate followed by $m \to -m$ and $\omega \to -\omega$, we have $\tilde{T}_{\ell, -m, -\omega} = (-1)^{\ell} \tilde{T}_{\ell, m, \omega}$.

C. Results

In this section, we calculate the gravitational wave luminosity up to $O(v^8)$ beyond the quadrupole formula. From Eq. (2.1), $\psi_4$ at $r \to \infty$ takes a form,

$$\psi_4 = \frac{1}{r} \sum_{\ell = 2}^{6} \sum_{m = -\ell}^{\ell} Z_{\ell m \omega_0} \frac{-2S_{\ell m}^{\omega_0}}{2\pi} e^{i\omega_0 (r - t) + im\varphi},$$

(3.14)

where $\omega_0 = m\Omega$. At infinity, $\psi_4$ is related to the two independent modes of gravitational waves $h_+$ and $h_\times$ as

$$\psi_4 = \frac{1}{2} (\tilde{h}_+ - i\tilde{h}_\times).$$

(3.15)

From Eqs. (3.13), (3.14) and (3.15), the gravitational wave luminosity is given by

$$\left\langle \frac{dE}{dt} \right\rangle = \sum_{\ell, m} \frac{|Z_{\ell m \omega_0}|^2}{4\pi \omega_0^2} \equiv \sum_{\ell, m} \left( \frac{dE}{dt} \right)_{\ell m}.$$

(3.16)

In order to express the post-Newtonian corrections to the luminosity, we define $\eta_{\ell m}$ as

$$\left( \frac{dE}{dt} \right)_{\ell m} = \frac{1}{2} \left( \frac{dE}{dt} \right)_N \eta_{\ell m},$$

(3.17)

where $(dE/dt)_N$ is the Newtonian quadrupole luminosity:

$$\left( \frac{dE}{dt} \right)_N = \frac{32\mu^2 M^3}{5v_0^5} = \frac{32}{5} \left( \frac{\mu}{M} \right)^2 v^{10}.$$

We only show $\eta_{\ell m}$ for $m > 0$ mode since $\eta_{\ell, m} = \eta_{\ell, -m}$.

$$\eta_{2, 2} = 1 - \frac{107}{21} v^2 + (4\pi - 6q) v^3 + \left( \frac{4784}{1323} + 2q^2 \right) v^4$$

$$+ \left( \frac{-428\pi}{21} + \frac{4216q}{189} \right) v^5 + \left( \frac{99210071}{1091475} - \frac{1712\gamma}{105} + \frac{16\pi^2}{3} \right) v^6.$$
\(-28 \pi q + \frac{8830 q^2}{567} - \frac{3424 \ln 2}{105} - \frac{1712 \ln v}{105}\) \(v^6\)
\(+ \left( \frac{19136 \pi}{1323} + \frac{163928 q}{11907} + 8 \pi q^2 - 12 q^3 \right) v^7\)
\(+ \left( -\frac{27956920577}{81265275} + 183184 \gamma - \frac{1712 \pi^2}{63} + \frac{20716 \pi q}{189} \right) \left( \frac{9261}{456028 q^2 + q^4 + \frac{366368 \ln 2}{2205} + \frac{183184 \ln v}{2205} \right) v^8,\)
\(\eta_{2,1} = \frac{v^2}{36} - \frac{q v^3}{12} + \left( -\frac{17}{504} + \frac{q^2}{16} \right) v^4\)
\(+ \left( \frac{\pi}{18} - \frac{793 q}{9072} \right) v^5 + \left( -\frac{2215}{254016} \pi q \frac{959 q^2}{6} + \frac{859 q^2}{1512} \right) v^6\)
\(+ \left( -\frac{17 \pi}{252} + \frac{11861 q}{190512} + \frac{\pi q^2}{8} - \frac{7 q^3}{12} \right) v^7\)
\(+ \left( -\frac{17 \pi}{252} + \frac{11861 q}{190512} + \frac{\pi q^2}{8} - \frac{7 q^3}{12} \right) \left( \frac{1045 \pi q}{4536} \right) v^8.\)

Putting together the above results, we obtain \((dE/dt)_{\ell} = \sum_m (dE/dt)_{\ell m}\) for \(\ell = 2\) as
\[
(dE/dt)_2 = (dE/dt)_N \left\{ 1 - \frac{1277 v^2}{252} + \left( 4 \pi - \frac{73 q}{12} \right) v^3 + \left( \frac{37915}{10584} + \frac{33 q^2}{16} \right) v^4 \right. \\
+ \left( -\frac{2561 \pi}{126} + \frac{201575 q}{9072} \right) v^5 + \left( \frac{2116278473}{23284800} - \frac{1712 \gamma}{105} + \frac{16 \pi^2}{3} \right) v^6 \\
- \left( \frac{169 \pi q}{6} + \frac{73217 q^2}{4536} - \frac{3424 \ln 2}{105} - \frac{1712 \ln v}{105} \right) v^6 \bigg\}.
\]

For \(\ell = 3\), we obtain
\[
\eta_{3,3} = \frac{1215 v^2}{896} - \frac{1215 v^4}{112} + \left( \frac{3645 \pi}{448} - \frac{1215 q}{112} \right) v^5 \\
+ \left( \frac{243729}{9856} + \frac{3645 q^2}{896} \right) v^6 + \left( \frac{-3645 \pi}{56} + \frac{131949 q}{1792} \right) v^7 \\
+ \left( \frac{25037019729}{125565440} + \frac{47385 \pi}{1568} + \frac{3645 \pi^2}{224} - \frac{32805 \pi q}{448} \right) v^8,\)
\(\eta_{3,2} = \frac{5 v^4}{63} - \frac{4 q v^5}{189} + \left( \frac{-193}{567} + \frac{80 q^2}{567} \right) v^6 \\
+ \left( \frac{20 \pi}{63} + \frac{352 q}{1701} \right) v^7 + \left( \frac{86111}{280665} - \frac{160 \pi q}{189} + \frac{40 q^2}{27} \right) v^8,\)
\(\eta_{3,1} = \frac{v^2}{8064} - \frac{v^4}{1512} + \left( \frac{\pi}{4032} - \frac{17 q}{9072} \right) v^5.\)
\[ + \left( \frac{437}{266112} + \frac{17 q^2}{24192} \right) v^6 + \left( -\frac{\pi}{756} + \frac{3601 q}{435456} \right) v^7 \]
\[ + \left( \frac{1137077}{50854003200} - \frac{13 \gamma}{42336} + \frac{\pi^2}{6048} + \frac{145 \pi q}{36288} + \frac{41183 q^2}{3483648} \right) v^8 - \frac{13 \ln 2}{42336} - \frac{13 \ln v}{42336} \] \quad (3.23)

Then we obtain
\[
\left( \frac{dE}{dt} \right)_3 = \left( \frac{dE}{dt} \right)_N \left\{ \frac{1367 v^2}{1008} - \frac{32567 v^4}{3024} + \left( \frac{16403 \pi}{2016} - \frac{896 q}{81} \right) v^5 \right. \\
+ \frac{152122}{6237} + \frac{341 q^2}{81} \left) v^6 \right. \\
+ \left. \left( \frac{5712521850527}{28605376800} - \frac{79963 \gamma}{2646} + \frac{6151 \pi^2}{378} + \frac{192005 \pi q}{2592} \right) v^7 \right. \\
+ \left. \frac{11168371 q^2}{435456} - \frac{79963 \ln 2}{2646} + \frac{47385 \ln 3}{1568} - \frac{79963 \ln v}{2646} \right) v^8 \} . \quad (3.24)

For \( \ell = 4 \), we have
\[
\eta_{4,4} = \frac{1280 v^4}{567} - \frac{151808 v^6}{6237} + \left( \frac{10240 \pi}{567} - \frac{12800 q}{567} \right) v^7 \\
+ \left( \frac{560069632}{6243237} + \frac{5120 q^2}{567} \right) v^8 , \quad (3.25)
\]
\[
\eta_{4,3} = \frac{729 v^6}{4480} - \frac{729 q v^7}{1792} + \left( \frac{28431}{24640} + \frac{3645 q^2}{14336} \right) v^8 , \quad (3.26)
\]
\[
\eta_{4,2} = \frac{5 v^4}{3969} - \frac{437 v^6}{43659} + \left( \frac{20 \pi}{3969} - \frac{80 q}{3969} \right) v^7 \\
+ \left( \frac{7199152}{218513295} + \frac{200 q^2}{27783} \right) v^8 , \quad (3.27)
\]
\[
\eta_{4,1} = \frac{v^6}{282240} - \frac{q v^7}{112896} + \left( \frac{101}{4656960} + \frac{5 q^2}{903168} \right) v^8 . \quad (3.28)
\]

Then we obtain
\[
\left( \frac{dE}{dt} \right)_4 = \left( \frac{dE}{dt} \right)_N \left\{ \frac{8965 v^4}{3996} - \frac{84479081 v^6}{3492720} + \left( \frac{23900 \pi}{1323} - \frac{59621 q}{2592} \right) v^7 \\
+ \left( \frac{51619996697}{582702120} + \frac{66084895 q^2}{7112448} \right) v^8 \right. \} . \quad (3.29)
\]

For \( \ell = 5 \) we have
\[
\eta_{5,5} = \frac{9765625 v^6}{2433024} - \frac{2568359375 v^8}{47443968} , \quad (3.30)
\]
\[
\eta_{5,4} = \frac{4096 v^8}{13365} , \quad (3.31)
\]
\[
\eta_{5,3} = \frac{2187 v^6}{450560} - \frac{150003 v^8}{2928640} , \quad (3.32)
\]
\[
\eta_{5,2} = \frac{4 v^8}{40095} , \quad (3.33)
\]
\[
\eta_{5,1} = \frac{2851 v^6}{127733760} - \frac{179 v^8}{2490808320} . \quad (3.34)
\]

Then we have
\[
\left( \frac{dE}{dt} \right)_5 = \left( \frac{dE}{dt} \right)_N \left\{ \frac{1002569 v^6}{249480} - \frac{3145396841 v^8}{58378320} \right\}. \tag{3.35}
\]

For \( \ell = 6 \) we have
\[
\eta_{6,6} = \frac{26244 v^8}{3575}, \tag{3.36}
\]
\[
\eta_{6,4} = \frac{131072 v^8}{9555975}, \tag{3.37}
\]
\[
\eta_{6,2} = \frac{4 v^8}{573585}, \tag{3.38}
\]
and \( \eta_{6,5}, \eta_{6,3}, \eta_{6,1} \) become \( O(v^9) \). Then we have
\[
\left( \frac{dE}{dt} \right)_6 = \left( \frac{dE}{dt} \right)_N \frac{210843872 v^8}{28667925}. \tag{3.39}
\]

Finally, gathering all the above results, the total luminosity up to \( O(v^8) \) is expressed as
\[
\left\langle \frac{dE}{dt} \right\rangle = \left. \left( \frac{dE}{dt} \right)_N \right\{ 1 - \frac{1247 v^2}{336} + \left( 4 \pi - \frac{73 q}{12} \right) v^3 + \left( -\frac{44711}{9072} + \frac{33 q^2}{16} \right) v^4 \\
+ \frac{16 \pi^2}{3} - \frac{169 \pi q}{6} + \frac{3419 q^2}{16} - \frac{3424 \ln 2}{105} - \frac{1712 \ln v}{105} \right\} v^6 \\
+ \left( -\frac{16285 \pi}{504} + \frac{83819 q}{1296} + \frac{65 \pi q^2}{8} - \frac{151 q^3}{12} \right) v^7 \\
+ \left( -\frac{323105549467}{3178375200} + \frac{232597 \gamma}{4410} - \frac{1369 \pi^2}{126} + \frac{3389 \pi q}{96} - \frac{124091 q^2}{9072} \right) v^8 \right\}. \tag{3.40}
\]

In Appendix G, we present formulas for \( \eta_{\ell,m} \) and \( dE/dt \) in terms of \( v' \equiv (M\Omega)^{1/3} \) for the sake of convenience to calculate the phase function for a inspiraling wave form \[8\].

Setting \( q = 0 \), above reproduces the previous results \[25\] \[26\] in a Schwarzsc hild case. Up to \( O(v^5) \), the results agree with those obtained by SSTT \[28\] in the case when the test particle moves a circular orbit in the equatorial plane. For \( \ell = 5 \) and 6, there are no contributions due to the black hole spin and the results are identical to the Schwarzschild case.

In Eq.(3.40), the numerical value of terms at order \( O(v^6) \) is given by \( v^6 (115.7 - 88.48 q + 20.35 q^2 - 16.30 \ln v) \). We find that the spin dependent terms are not so small compared to the other two terms if \( |q| \) is of order unity. Thus, we see that spin dependent terms at \( O(v^6) \) will give a significant effect to template wave forms of coalescing binaries when spin of a black hole is large.

Finally we note that the angular momentum flux can be easily calculated from
\[
\left\langle \frac{dJ}{dt} \right\rangle = \frac{1}{\Omega} \left\langle \frac{dE}{dt} \right\rangle. \tag{3.41}
\]

**IV. COMPARISON WITH NUMERICAL RESULTS**

As discussed in section I, it is important to investigate the detailed convergence property of the post-Newtonian approximation. Therefore we compare the formula for \( dE/dt \), derived above, with numerical results and investigate the accuracy of the post-Newtonian expansion of \( dE/dt \).

In this section, we consider the total mass of the binary systems including black holes \( \sim 2 - 300 M_\odot \) because gravitational waves from such binaries can be detected by LIGO and VIRGO. In particular, we pay attention to the
accuracy of post-Newtonian formula for $dE/dt$ when $r \leq 100M$ (or $v \geq 0.1$), because gravitational waves from these binary systems will be detected when the orbital separation becomes less than $r \simeq 100M$. Here, we ignore the effect of absorption of gravitational waves by the black hole. We will briefly discuss its effect in the next section.

A numerical study of $dE/dt$ from a particle in a circular orbit in the equatorial plane around a Kerr black hole has been performed by Shibata [39]. Since nothing was assumed about the velocity of a test particle, those results are correct relativistically in the limit $\mu \ll M$. In that work, $dE/dt$ was calculated with accuracy $\lesssim 10^{-4}$. However we found that this accuracy is not sufficient to compare it with the post-Newtonian formula for $dE/dt$ including terms up to $O(v^8)$. Thus, in this paper, we calculate $dE/dt$ again requiring the accuracy to be $\sim 10^{-5}$. In the numerical calculations, we have taken into account the contribution from the $\ell = 2$ through $\ell = 6$ modes in $dE/dt$ which is consistent with the post-Newtonian formula.

In figs.1(a–e), we show the error in the post-Newtonian formulas as a function of the Boyer-Lindquist coordinate radius when $q = -0.9$, -0.5, 0, 0.5 and 0.9. In these figures, we show the error for $6 \leq r/M \leq 100$. Since the radius of the inner stable circular orbit for $q = 0.9$ is $r_{rso} \simeq 2.32M$ and a stable circular orbit is possible for $r > r_{rso}$, we also show the errors in the case when $q = 0.9$ for $2.5 \leq r/M \leq 12$ in fig.2. The error in the post-Newtonian formula is defined as

$$\text{Error} = \left| 1 - \frac{(dE/dt)_{\text{PN}}}{(dE/dt)_{\text{NR}}} \right|,$$

(4.1)

where $(dE/dt)_{\text{PN}}$ and $(dE/dt)_{\text{NR}}$ denote the post-Newtonian formula and the numerical results respectively. As for $(dE/dt)_{\text{PN}}$, we have used 2-PN, 2.5-PN, 3-PN, 3.5-PN and 4-PN formulas. Here, we define $n$-PN formula as the expression for $dE/dt$ which includes post-Newtonian terms up to $O(v^{2n})$ beyond the quadrupole formula. In each figure, open square, filled triangle, open triangle, filled circle, and open circle denote the error of 2-PN, 2.5-PN, 3-PN, 3.5-PN and 4-PN formulas, respectively. We note that in fig.2, the errors in the 2.5-, 3- and 4-PN formulas become greater than unity for very small radius, because in such a region, $dE/dt$ for those PN formulas becomes negative.

From these figures, we find the following.

(1) If we use the 2-PN or 2.5-PN formula, the error is always greater than $10^{-4}$ when $r \lesssim 100M$ irrespective of $q$. If we use the 3-PN formula, however, the error decreases significantly, and it becomes less than $10^{-4}$ for $r > 60M$, and less than $10^{-3}$ for $r > 30M$ irrespective of $q$.

(2) If we adopt the 3.5-PN formula, the accuracy becomes better than that of 3-PN formula. The error is always less than $10^{-4}$ when $r > 30M$ and less than $10^{-5}$ when $r > 60M$. This feature does not depend on $q$. However, if we use the 4-PN formula, the accuracy is not improved compared with the 3.5-PN formula. In particular, this tendency is remarkable for smaller radius.

(3) The accuracy of the 3.5-PN or 4-PN formula is not always better than that of the lower-PN one inside $r_c$, where $r_c \lesssim 5M$ for $q = 0.5$ and $0.9$, $r_c \sim 10M$ for $q = 0$ and $-0.5$, and $r_c \sim 15M$ for $q = -0.9$. Thus, the convergence of the post-Newtonian expansion seems rather poor around $r_c$.

Using the above results, we investigate the accuracy of the post-Newtonian formulas as templates for various binary systems. As explained in section 1, to investigate the accuracy of the post-Newtonian formulas as templates, it is useful to check if they can predict the number of cycles of the gravitational waves, $N$, with accuracy less than 1. compact binary systems, the cycles are mainly accumulated around $\sim 10\text{Hz}$ which is the lowest frequency region in the LIGO band, and $N$ is approximately given by

$$N \sim 1.9 \times 10^3 \left( \frac{10M_\odot}{M} \right)^{5/3} \left( \frac{M}{4\mu} \right),$$

(4.2)

where $M$ and $\mu$ are the total mass and reduced mass, respectively. This means that the template must have an accuracy less than

$$\sim 5 \times 10^{-4} \left( \frac{M}{10M_\odot} \right)^{5/3} \left( \frac{4\mu}{M} \right),$$

(4.3)

when the frequency of gravitational wave becomes 10Hz.

First we consider equal mass binary systems, that is $M = 4\mu$. At 10Hz, the orbital separation of a binary of total mass $M$ is approximately given by $r/M \simeq 347(M_\odot/M)^{2/3}$. We find that the 2-PN and 2.5-PN formulas are insufficient if $M \gtrsim 5M_\odot$, and the 3-PN formula is needed. The 3-PN formula seems adequate irrespective of $q$.

On the other hand, the situation is slightly different in the case when a neutron star of mass $\sim 1.4M_\odot$ spirals into a larger black hole. In such a case, the number of the cycles of the gravitational waves is large compared with the equal mass case when the total mass is the same. Thus, it seems that we need at least the 3.5-PN formula for binaries
of mass greater than $\sim 30M_\odot$ to obtain the required accuracy. Also, for binaries of mass greater than $\sim 70M_\odot$, we need higher post-Newtonian corrections beyond 4-PN order.

Binary systems of total mass greater than $\sim 100M_\odot$ can be detected when $r$ is smaller than $\sim 15M$. However, as mentioned in (3) above, the convergence property of the post-Newtonian expansion becomes bad for small orbital separations. In particular, for $q \sim -1$, the accuracy of the post-Newtonian expansion seems bad at $r \sim 15M$. Thus, it may not be appropriate to use the post-Newtonian approximation for binaries of total mass $\sim 100M_\odot$ with large mass ratio $\mu \ll M$. A more detailed investigations of the convergence of the post-Newtonian expansion will require the calculation to be carried beyond 4-PN order.

V. SUMMARY AND DISCUSSION

In this paper, we have performed a post-Newtonian expansion of gravitational waves from a particle in a circular orbit around a Kerr black hole. The orbit lies in the equatorial plane and the calculations are accurate to $O(v^8)$ beyond the quadrupole level. We have performed the post-Newtonian expansion of the Sasaki-Nakamura equation and obtained the Green function of the radial Teukolsky equation up to $O(\ell^2)$ using methods developed previously. Then we obtained all the necessary radial functions to the required accuracy. We have also calculated the spin weighted spheroidal harmonics up to $O((a\omega)^2)$. The outgoing wave amplitude of the Teukolsky function and the gravitational wave luminosities were derived up to $O(v^8)$ beyond the quadrupole formula.

It is worth noting that in the formula for $\eta_{22}$ in Appendix G, there are terms such as $(-8/3)q^2 v^5$, $2q^2 v^4$, $(-8/3)q^3 v^7$ and $q^4 v^8$. In a previous paper [28], we pointed out that the terms $2q^2 v^4$ can be explained in terms of the quadrupole formula as the contribution of the quadrupole moment of the Kerr black hole to the orbit of the test particle. A similar explanation is possible for $(-8/3)q^3 v^7$ and $q^4 v^8$. We can derive those terms by using the quadrupole formula $\frac{dE}{dt} = 32\sqrt{\mu} \tilde{r}^2 \tilde{M}_\odot\Omega$, where $\tilde{r}$ is the orbital radius of a test particle in de Donder coordinates. If multipole moments of the black hole exist, the orbital radius is changed due to the influence of those multipole moments (or if we fix the orbital radius, $\Omega$ is changed due to the multipole moments of black hole). We can calculate the leading order effect of the multipole moments to the orbital radius by using multipole expansion of the Kerr metric (Eq.(10.6) of Ref. [44]). In this way, we find that the dominant effect of the multipole moments of a Kerr black hole to $\frac{dE}{dt}$ can be expressed as

$$\frac{dE}{dt} = \frac{32}{5} \left( \frac{\mu}{M} \right)^2 v^{10} \left\{ 1 - \frac{8}{3} S_1 \ell v^3 - 2M_2 v^4 + 4S_3 v^7 + \left( -\frac{3}{2} M_2^2 + \frac{5}{2} M_4 \right) \ell v^8 \right\},$$

where $M_\ell$ and $S_\ell$ are mass and current multipole moments of a Kerr black hole given by $M_\ell + i S_\ell = M(ia)^\ell$. Now we can interpret the term $-12q^2 v^7$ as the effect of the current octopole moment of a black hole and the term $q^4 v^8$ as the effect of both the mass quadrupole moment and $\ell = 4$ mass multipole moment of a black hole.

As for $\ell = 2$ and $m = 1$ mode, there are terms $-q^5 v^7/18$, $q^3 v^4/6$, $-q v^2 1/24$ and $q^4 v^8/16$. The terms $-q^5 v^7/18$ and $q^3 v^4/6$ can be explained as the correction to the radiative current quadrupole moment $[12]$ $[12]$. We expect that the terms $-q^5 v^7/18$ and $q^4 v^8/16$ can also be derived simply in a similar way.

In section 4, by comparing post-Newtonian formulas for $\frac{dE}{dt}$ with numerical data, we indicated that the convergence of the post-Newtonian expansion seems bad when orbital radii of binaries become less than $\sim 15M$. This suggests that the post-Newtonian expansion may not be appropriate to construct theoretical templates for large mass ratio binaries where the total mass is greater than $\sim 100M_\odot$ because gravitational waves from such binaries enter the LIGO/VIRGO frequency band when $r \lesssim 15M$. Nevertheless, the higher order post-Newtonian terms gradually improve the accuracy of the templates. Hence, it is very natural to ask whether the post-Newtonian expansion is always appropriate or not, and if appropriate, up to what order we need the post-Newtonian terms to construct accurate templates. Fortunately, it is possible to obtain the formulas for $\frac{dE}{dt}$ which include post-Newtonian order terms beyond $O(v^8)$ by extending techniques developed in this paper. Extension of the present work up to the higher post-Newtonian order, beyond $O(v^8)$, is very important and that is our future work.

The analysis, in this paper, has been restricted to the case when a test particle moves in a circular orbit on the equatorial plane. However, as shown in a previous paper [28], inclination of the orbital plane from the equatorial plane will significantly affect the orbital phase evolution. Hence, the present work should be considered as a first step toward the complete calculation of the energy and angular momentum luminosities including the orbital inclination.

Finally, we comment on the effect of absorption of gravitational waves by the black hole event horizon which should be taken into account when we consider the orbital evolution of black hole binaries. According to Gal’tsov [12], the lowest order contribution of the gravitational wave absorption to $dE/dt$ is given by

$$\frac{dE}{dt} = \left( \frac{dE}{dt} \right)_N \frac{v^5}{2} \left\{ v^3 \left( 1 + \sqrt{1 - q^2} - \frac{q}{2} \right) (1 + 3q^2) \right\}.$$
Thus, the effect of absorption appears from $O(v^5)$ if $q \neq 0$. Although the coefficient is small compared with that of $dE/dt$ for the outgoing wave even in the case $|q| \sim 1$, we need the expression for $dE/dt$ due to the black hole absorption to obtain an accurate template up to $O(v^8)$. Therefore, to obtain the higher order post-Newtonian corrections to the black hole absorption is a problem for the future.

Acknowledgments

H.T. thanks F. Ryan for useful comments and Kip Thorne for continuous encouragement. We thank P.R. Brady for careful reading of the manuscript. H.T. was supported by Research Fellowship of the Japan Society for the Promotion of Science for Young Scientists and by NSF Grant No. AST-9417371 and NASA Grand No. NAGW-4268. This work was also supported in part by the Japanese Grant-in-Aid for Scientific Research of the Ministry of Education, Science and Culture, No. 07740355 and No. 04234104.

APPENDIX A: THE FORMULAE OF $F$ AND $U$

In this Appendix we show the potential functions $F$ and $U$ of the SN equation (2.15). Details of the derivation are given in Ref. [36].

The function $F(r)$ is given by

$$F(r) = \frac{\eta_r \Delta}{\eta r^2 + a^2},$$

(A1)

where

$$\eta = c_0 + c_1/r + c_2/r^2 + c_3/r^3 + c_4/r^4,$$

(A2)

with

$$c_0 = -12i\omega M + \lambda(\lambda + 2) - 12a\omega(a\omega - m),$$
$$c_1 = 8i[a[3a\omega - \lambda(a\omega - m)]],$$
$$c_2 = -24iaM(a\omega - m) + 12a^2[1 - 2(a\omega - m)^2],$$
$$c_3 = 24ia^3(a\omega - m) - 24Ma^2,$$
$$c_4 = 12a^4.$$  

(A3)

The function $U(r)$ is given by

$$U(r) = \frac{\Delta U_1}{(r^2 + a^2)^2} + G^2 + \frac{\Delta G_r}{r^2 + a^2} - FG,$$

(A4)

where

$$G = -\frac{2(r - M)}{r^2 + a^2} + \frac{r\Delta}{(r^2 + a^2)^2},$$
$$U_1 = V + \frac{\Delta^2}{\beta} \left[ \left( \frac{2\alpha + \beta_r}{\Delta} \right)_r - \frac{\eta_r}{\eta} \left( \alpha + \beta_r \right) \right],$$
$$\alpha = -i\frac{K\beta}{\Delta^2} + 3iK_r + \lambda + \frac{6\Delta}{r^2},$$
$$\beta = 2\Delta \left( -iK + r - M - \frac{2\Delta}{r} \right).$$

(A5)
APPENDIX B: FUNCTIONS IN THE SOURCE TERM

In this Appendix, we show the $A$'s in Eq. (13).

\[
A_{n\ell m} = \frac{-2}{\sqrt{2\pi\Delta^2}} C_{n\ell m} \mu^{-2} \rho^{-1} L_1^{\ell} \left( \rho^{-4} L_2^{\ell} (\rho^3 S) \right),
\]

\[
A_{m\ell n} = \frac{2}{\sqrt{\pi\Delta}} C_{m\ell n} \rho^{-3} \left[ (L_2^\ell S) \left( \frac{i K}{\Delta} + \rho + \bar{\rho} \right) \right. \\
\left. - a \sin \theta S \frac{K}{\Delta} (\bar{\rho} - \rho) \right],
\]

\[
A_{\bar{m}\bar{m} n} = -\frac{1}{\sqrt{2\pi}} \rho^{-3} \bar{\rho} C_{\bar{m}\bar{m} n} S \left[ -i \left( \frac{K}{\Delta} \right) \right. \\
\left. - \frac{K^2}{\Delta^2} + 2i \rho \frac{K}{\Delta} \right],
\]

\[
A_{\bar{m} \bar{m} 1} = \frac{2}{\sqrt{\pi\Delta}} \rho^{-3} C_{\bar{m} \bar{m} 1} \left[ L_2^\ell S + ia \sin \theta (\bar{\rho} - \rho) S \right],
\]

\[
A_{\bar{m} \bar{m} 1} = -\frac{2}{\sqrt{2\pi}} \rho^{-3} \bar{\rho} C_{\bar{m} \bar{m} 1} \left( \frac{K}{\Delta} + \rho \right),
\]

\[
A_{\bar{m} \bar{m} 2} = \frac{1}{\sqrt{2\pi}} \rho^{-3} \bar{\rho} C_{\bar{m} \bar{m} 2} S,
\]

where $S$ denotes $-2S_{\ell m}^{q\omega}$.

APPENDIX C: $Q^{(2)}, Q^{(3)}, Q^{(4)}$

\[
Q^{(2)} = \left[ \left(-28 i m q - \frac{32 i m q}{\ell} + 8 i \ell m q + 4 i \ell^2 m q - 13 q^2 - \frac{6 q^2}{\ell} - 12 \ell q^2 - \ell^2 q^2 + 6 \ell^3 q^2 + 2 \ell^4 q^2 + 8 m^2 q^2 + \frac{32 m^2 q^2}{\ell^2} + \frac{8 m^2 q^2}{\ell} \right) \frac{1}{z^4} + \right.
\]

\[
\left( 16 m q + \frac{24 m q}{\ell} + \frac{20 m q}{\ell^2} - 8 \ell m q - 4 \ell^2 m q - 14 i q^2 - \frac{16 i q^2}{\ell} + 4 i \ell q^2 + 2 i \ell^2 q^2 + 2 i \ell \right. \\
\left. \frac{3 \ell \lambda_2 q^2}{4} + \frac{\ell^2 \lambda_2 q^2}{4} + \frac{3 \ell^3 \lambda_2 q^2}{4} + \frac{\ell^4 \lambda_2 q^2}{4} - 2 \lambda_1 m q^2 + \frac{4 \lambda_1 m q^2}{4} - \frac{2 \lambda_1 m q^2}{\ell} - \frac{24 m^2 q^2}{\ell^2} \right) \frac{1}{z^3} + \\
\left[ \left( 24 i m q - 17 q^2 + \frac{10 q^2}{\ell} - 13 \ell q^2 - 9 \ell^2 q^2 - 3 \ell^3 q^2 - 4 \ell^4 q^2 + \frac{\ell^4 \lambda_2 q^2}{4} - 2 \lambda_1 m q^2 + \frac{4 \lambda_1 m q^2}{4} - \frac{2 \lambda_1 m q^2}{\ell} - \frac{24 m^2 q^2}{\ell^2} \right) \frac{1}{z^2} \right]
\]

\[
\left. \frac{1}{\ell^4 (\ell^2 + \ell - 2)} \left( -24 i \ell \lambda_0 m q - 4 i \lambda_2 m q + 4 i \lambda_0^3 m q - 12 \lambda_0 q^2 - 6 \lambda_0^2 q^2 + 24 \lambda_0 m^2 q^2 - 4 \lambda_0^2 m^2 q^2 \right) \frac{1}{z^3} + \frac{24 \lambda_0 m q - 12 i \lambda_0 q^2 - 2 i \lambda_0^2 q^2 + 2 i \lambda_0^3 q^2 + 2 i \lambda_0^2 \lambda_1 m q^2 + 24 i \lambda_0 m q^2}{z^3} \right]
\]

\[
Q^{(3)}_{\ell = 2} = \left[ \left( \frac{1}{2} m q + \frac{5 q^2}{8} - \frac{5 m^2 q^2}{9} + \frac{11 i m q^3}{24} - \frac{11 i m^3 q^3}{54} \right) \frac{1}{z^4} + \right.
\]

\[
\left. \left( -\frac{(m q)}{24} + \frac{5 i}{48} q^2 - \frac{65 i}{216} m^2 q^2 - \frac{m q^3}{2} + \frac{16 m^3 q^3}{81} \right) \frac{1}{z^3} \right]
\]

18
\[ Q^{(3)}_{\ell=3} = \left[ \left( \frac{i}{10} - \frac{7}{80} \right) + \frac{23}{1200} \cdot \frac{m^2 q^2}{m^3 q^3} - \frac{7}{72} \cdot \frac{5}{8} \right] d \]
\[
\frac{101 q^4}{288} - \frac{3991 m^2 q^4}{3024} + \frac{15853 m^4 q^4}{40824} \left( \frac{1}{z^4} \right) \\
+ \left( \frac{-i}{32} q^2 + \frac{53 i}{432} m^2 q^4 - \frac{2 m q^3}{27} + \frac{431 m^2 q^3}{3888} + \frac{349 i}{3024} q^4 + \right. \\
\frac{i}{72} \lambda_3 m q^4 - \frac{7235 i}{13608} m^2 q^4 + \frac{2549 i}{15309} m^4 q^4 \left( \frac{1}{z^4} \right) \\
+ \left. \frac{-i}{96} m q + \frac{q^2}{192} - \frac{11 m^2 q^2}{288} + \frac{5 i}{108} m^3 q^3 - \frac{i}{3888} m^3 q^3 - \frac{7 q^4}{432} + \frac{\lambda_4 q^4}{16} \right) \\
\frac{\lambda_3 m q^4}{72} + \frac{1349 m^2 q^4}{13608} - \frac{509 m^4 q^4}{15309} \left( \frac{1}{z^4} \right) .
\]

**APPENDIX D: \(X_{\ell m\omega}^{in}\)**

(a) \(\ell = 2\)

\[
X_{2m\omega}^{in} = \frac{z^3}{15} - \frac{z^5}{210} + \frac{z^7}{7560} - \frac{z^9}{498960} + \frac{z^{11}}{51891840} \\
+ \epsilon \left( \frac{i}{30} m q z^2 - \frac{13 z^4}{630} - \frac{11 i}{3780} m q z^4 + \frac{z^6}{810} + \frac{13 i}{136080} m q z^6 \right. \\
- \frac{53 z^8}{1782000} - \frac{i}{598752} m q z^8 \right) \\
+ \epsilon^2 \left( \frac{i}{60} m q + \frac{q^2}{120} - \frac{m^2 q^2}{120} \right) z - \frac{m q z^2}{30} \\
+ z^3 \left( \frac{26743}{110250} - \frac{433 i}{22680} m q - \frac{3 q^2}{3920} + \frac{79 m^2 q^2}{105840} - \frac{107 \ln z}{3150} \right) \\
+ \frac{\left( m q \right)}{270} - \frac{i}{7560} q^2 + \frac{i}{34020} m^2 q^3 \right) z^4 \\
+ z^5 \left( \frac{-140953}{9261000} + \frac{17 i}{12960} m q + \frac{11 q^2}{423360} - \frac{19 m^2 q^2}{762048} + \frac{107 \ln z}{44100} \right) \\
+ \epsilon^3 \left( \frac{-i}{90} m q - \frac{q^2}{30} + \frac{m^2 q^2}{40} - \frac{19 i}{720} m q^3 + \frac{7 i}{720} m^3 q^3 \right. \\
+ \frac{\left( -\frac{m q}{36} - \frac{i}{120} q^2 + \frac{m q^3}{36} - \frac{m^3 q^3}{90} \right)}{z} \\
+ z^2 \left( \frac{319}{6300} - \frac{2074 i}{18375} m q - \frac{41 q^2}{5040} + \frac{m^2 q^2}{648} + \frac{2887 i}{211680} m^3 q^3 \right. \\
- \frac{1153 i}{211680} m^3 q^3 - \frac{107 i}{6300} m q \ln z \right) \\
+ \epsilon^4 \left( \frac{-i}{120} m q + \frac{17 m^2 q^2}{1440} + \frac{11 i}{480} m q^3 - \frac{i}{480} m^3 q^3 + \frac{23 q^4}{1920} \right. \\
- \frac{11 m^2 q^4}{576} + \frac{17 m^4 q^4}{5760} \right) \frac{1}{z} .
\]

(b) \(\ell = 3\)

\[
X_{3m\omega}^{in} = \frac{z^4}{105} - \frac{z^6}{1890} + \frac{z^8}{83160} - \frac{z^{10}}{6486480} \\
+ \epsilon \left( \frac{-z^3}{126} + \frac{2 i}{945} m q z^3 - \frac{z^5}{630} - \frac{i}{7560} m q z^5 + \frac{221 z^7}{2494800} + \frac{i}{299376} m q z^7 \right) .
\]

20
\[ + e^2 \left( \frac{-i}{630} m q + \frac{q^2}{840} + \frac{m^2 q^2}{7560} \right) z^2 + \left( -\frac{m q}{540} - \frac{i}{1512} q^2 + \frac{i}{6804} m^2 q^2 \right) z^3 \]
\[ + e^4 \left( \frac{76369}{1852200} - \frac{299 i}{453600} m q - \frac{q^2}{10800} - \frac{m^2 q^2}{75600} - \frac{13 \ln z}{4410} \right) \]
\[ + e^3 \left( \frac{-i}{2520} m q - \frac{q^2}{1008} + \frac{m^2 q^2}{2160} - \frac{i}{7560} m q^3 + \frac{i}{7560} m^3 q^3 \right) z \].

(c) \( \ell = 4 \)

\[ X_{4m\omega}^\text{in} = \frac{z^5}{945} - \frac{z^7}{20790} + \frac{z^9}{1081080} + \epsilon \left( \frac{-z^5}{630} + \frac{i}{7560} m q z^4 - \frac{z^6}{9900} - \frac{i}{154000} m q z^6 \right) \]
\[ + e^2 \left( \frac{z^3}{1764} - \frac{i}{5040} m q z^3 + \frac{q^2 z^3}{4410} + \frac{m^2 q^2 z^3}{52920} \right). \]

(d) \( \ell = 5 \)

\[ X_{5m\omega}^\text{in} = \frac{z^6}{10395} - \frac{z^8}{270270} + \epsilon \left( -\frac{z^5}{4950} + \frac{2 i}{259875} m q z^5 \right). \]

(d) \( \ell = 6 \)

\[ X_{6m\omega}^\text{in} = \frac{z^7}{135135}. \]

**APPENDIX E:** \( R_{\ell m\omega}^\text{in} \)

(a) \( \ell = 2 \)

\[ \omega R_{2m\omega}^\text{in} = \frac{z^4}{30} + \frac{i}{45} z^5 - \frac{11 z^6}{1260} - \frac{i}{420} z^7 + \frac{23 z^8}{45360} + \frac{i}{11340} z^9 \]
\[ - \frac{13 z^{10}}{997920} - \frac{i}{5987752} z^{11} + \frac{59 z^{12}}{311351040} \]
\[ + \epsilon \left( -\frac{z^3}{15} - \frac{i}{60} m q z^3 - \frac{i}{60} z^4 + \frac{m q z^3}{45} - \frac{41 z^5}{3780} + \frac{277 i}{22680} m q z^5 \right) \]
\[ - \frac{31 i}{3780} z^6 - \frac{7 m q z^6}{1620} - \frac{17 z^7}{54432} + \frac{61 i}{54432} m q z^7 \]
\[ + \frac{41 i}{54432} z^8 + \frac{47 m q z^8}{204120} - \frac{1579 z^9}{17962560} + \frac{703 i}{17962560} m q z^9 \right). \]
\[ + e^2 \left( \frac{z^2}{30} + \frac{i}{40} m q z^2 + \frac{q^2 z^2}{60} - \frac{m^2 q^2 z^2}{240} - \frac{i}{60} z^3 - \frac{m q z^3}{30} + \frac{i}{90} q^2 z^3 \right) \]
\[ - \frac{i}{120} m^2 q^2 z^3 + \frac{7937 z^4}{55125} + \frac{53 i}{9072} m q z^4 - \frac{101 q^2 z^4}{35280} + \frac{4213 m^2 q^2 z^4}{635040} \]
\[ + \frac{4673 i}{55125} z^5 - \frac{13 m q z^5}{2835} + \frac{5 i}{63504} q^2 z^5 + \frac{3503 i}{1143072} m^2 q^2 z^5 - \frac{1665983 z^6}{55566000} \]
\[ - \frac{1777 i}{544320} m q z^6 - \frac{q^2 z^6}{5040} - \frac{643 m^2 q^2 z^6}{653184} - \frac{107 z^7}{16656000} + \ln z \]
\[ - \frac{107 i}{9450} z^7 \ln z + \frac{1177 z^7 \ln z}{264600} \right) \]
\[ + e^3 \left( -\frac{i}{180} m q - \frac{q^2}{240} + \frac{i}{144} m q^3 + \frac{i}{1440} m^3 q^3 \right) z \]
\begin{align*}
&+ \left( \frac{i}{120} + \frac{2mq}{135} - \frac{i}{360}q^2 + \frac{19i}{1440}m^2q^2 + \frac{11mq^3}{1080} - \frac{m^3q^3}{540} \right) z^2 \\
&+ z^3 \left( -\frac{10933}{49000} - \frac{578569i}{793800}mq - \frac{677q^2}{52920} - \frac{529m^2q^2}{63504} \\
&+ \frac{317i}{63504}m^3q^3 - \frac{167i}{84672}m^3q^3 + \frac{107lnz}{3150} + \frac{107i}{12600}mqlnz \right) \\
&+ \epsilon^4 \left( -\frac{i}{720}mq + \frac{m^2q^2}{2880} + \frac{i}{288}m^3q + \frac{q^4}{480} - \frac{m^2q^4}{720} + \frac{m^4q^4}{11520} \right).
\end{align*}

(b) $\ell = 3$

$$\omega R_{3m\omega}^{\text{lin}} = \frac{z^5}{630} + \frac{i}{1260} \frac{z^6}{3780} - \frac{i}{16200} z^8 + \frac{29z^9}{2494800} + \frac{i}{554400}z^{10} - \frac{i}{194594400}z^{11}$$

$$+ \epsilon \left( -\frac{z^4}{252} - \frac{i}{1890}mqz^4 - \frac{i}{756}z^5 + \frac{11mqz^5}{22680} + \frac{19i}{90720}mqz^6 \\
- \frac{i}{9450}q^7 - \frac{m^2q^7}{16200} + \frac{647q^8}{14968800} - \frac{247i}{17962560}mqz^8 \right)$$

$$+ \epsilon^2 \left( \frac{z^3}{315} + \frac{i}{945}mqz^3 + \frac{q^2z^3}{1260} - \frac{m^2q^2z^3}{15120} + \frac{i}{2520}z^4 - \frac{17mqz^4}{15120} \\
+ \frac{i}{2160}q^3z^4 - \frac{31i}{272160}m^2q^2z^4 + \frac{81409z^5}{11113200} \\
- \frac{313i}{907200}mqz^5 - \frac{41q^2z^5}{226800} + \frac{617m^2q^2z^5}{8164800} - \frac{13z^5lnz}{26460} \right)$$

$$+ \epsilon^3 \left( -\frac{z^2}{1260} - \frac{i}{1680}mqz^2 - \frac{q^2z^2}{840} + \frac{m^2q^2z^2}{10080} - \frac{i}{5040}mq^3z^2 \right).$$

(c) $\ell = 4$

$$\omega R_{4m\omega}^{\text{lin}} = \frac{z^6}{11340} + \frac{i}{28350} \frac{z^7}{1247400} - \frac{i}{467775}z^9 + \frac{71z^{10}}{194594400}$$

$$+ \epsilon \left( -\frac{z^5}{3780} - \frac{i}{45360}mqz^5 - \frac{11i}{136080}z^6 + \frac{mqz^6}{64800} \\
+ \frac{131z^7}{18711000} + \frac{697i}{124740000}mqz^7 \right)$$

$$+ \epsilon^2 \left( \frac{z^4}{3528} + \frac{i}{18144}mqz^4 + \frac{q^2z^4}{21168} - \frac{m^2q^2z^4}{635040} \right).$$

(d) $\ell = 5$

$$\omega R_{5m\omega}^{\text{lin}} = \frac{z^7}{207900} + \frac{i}{623700}z^8 - \frac{z^9}{2316600}$$

$$+ \epsilon \left( -\frac{z^6}{59400} - \frac{i}{1039500}mqz^6 \right).$$

(e) $\ell = 6$

$$\omega R_{6m\omega}^{\text{lin}} = \frac{z^8}{4054050}. $$
APPENDIX F: SPHEROIDAL HARMONICS

In this appendix, we describe the expansion of the spheroidal harmonics $-2S_{\ell m}^{0\omega}$ at order $O((a\omega)^2)$. The spheroidal harmonics of spin weight $s = -2$ obey the equation,

$$\left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - a^2 \omega^2 \sin^2 \theta - \frac{(m - 2\cos \theta)^2}{\sin^2 \theta} \right.$$
$$\left. + 4a\omega \cos \theta - 2 + 2ma\omega + \lambda \right] \mathcal{L}_0 S_{\ell m}^{0\omega} = 0. \quad (F1)$$

We expand $-2S_{\ell m}^{0\omega}$ and $\lambda$ as

$$-2S_{\ell m}^{0\omega} = -2P_{\ell m} + a\omega S_{\ell m}^{(1)} + (a\omega)^2 S_{\ell m}^{(2)} + O((a\omega)^3),$$
$$\lambda = \lambda_0(\ell) + a\omega \lambda_1(\ell) + a^2 \omega^2 \lambda_2(\ell) + O((a\omega)^3), \quad (F2)$$

where $-2P_{\ell m}$ are the spherical harmonics of spin weight $s = -2$ and $\lambda_n$ are given in section 2.2. Here we explicitly represent the $\ell$-dependence of $\lambda_n$ for later convenience. We set the normalization of $-2P_{\ell m}$ as

$$\int_0^\pi |-2P_{\ell m}|^2 \sin \theta d\theta = 1. \quad (F3)$$

Inserting Eq. (F2) into Eq. (F1) and collecting the terms of order $(a\omega)^2$, we obtain

$$\mathcal{L}_0 S_{\ell m}^{(2)} + \lambda_0(\ell)S_{\ell m}^{(2)} = -(4\cos \theta + 2m + \lambda_1(\ell))S_{\ell m}^{(1)} - (\lambda_2(\ell) - \sin^2 \theta) \mathcal{L}_0 S_{\ell m}^{0\omega} - 2P_{\ell m}, \quad (F4)$$

where $\mathcal{L}_0$ is the operator for the spin-weighted spherical harmonics,

$$\mathcal{L}_0[-2P_{\ell m}] = \left[ \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - \frac{(m - 2\cos \theta)^2}{\sin^2 \theta} - 2 \right] \mathcal{L}_0 S_{\ell m}^{0\omega} = -\lambda_0[-2P_{\ell m}], \quad (F5)$$

$$\mathcal{L}_0 S_{\ell m}^{(1)} = \sum_{\ell'} c_{\ell m}^{\ell'} -2P_{\ell' m},$$
$$\mathcal{L}_0 S_{\ell m}^{(2)} = \sum_{\ell'} d_{\ell m}^{\ell'} -2P_{\ell' m}, \quad (F6)$$

By setting

$$S_{\ell m}^{(1)} = \sum_{\ell'} c_{\ell m}^{\ell'} \mathcal{L}_0 S_{\ell' m},$$
$$S_{\ell m}^{(2)} = \sum_{\ell'} d_{\ell m}^{\ell'} \mathcal{L}_0 S_{\ell' m}, \quad (F7)$$

we insert it into Eq. (F3), multiply it by $-2P_{\ell m}$ and integrate it over $\theta$. Then we have

$$d_{\ell m}^{\ell'} = \frac{1}{\lambda_0(\ell) - \lambda_0(\ell')} \left[ -(2m + \lambda_1(\ell)) \left( c_{\ell m}^{\ell+1} \delta_{\ell', \ell+1} + c_{\ell m}^{\ell-1} \delta_{\ell', \ell-1} \right) - \delta_{\ell', \ell} \lambda_2(\ell) \right.$$
$$\left. -4c_{\ell m}^{\ell+1} \int d(\cos \theta) (-2)P_{\ell' m-2}P_{\ell+1 m} \cos \theta - 4c_{\ell m}^{\ell-1} \int d(\cos \theta) (-2)P_{\ell' m-2}P_{\ell-1 m} \cos \theta \right.$$
$$\left. + \int d(\cos \theta) (-2)P_{\ell' m-2}P_{\ell m} \sin^2 \theta \right]. \quad (F8)$$

The integrals in this equation are given by\[37,38\]

$$\int d(\cos \theta) (-2)P_{\ell' m-2}P_{\ell m} \cos \theta = \sqrt{\frac{2\ell + 1}{2\ell' + 1}} \delta_{\ell', \ell} \sin \theta < \ell, 1, m, 0|\ell', m > < \ell, 1, 2, 0|\ell', 2 >,$$
$$\int d(\cos \theta) (-2)P_{\ell' m-2}P_{\ell m} \sin^2 \theta = \frac{2}{3} \delta_{\ell', \ell} \left[ \sin \theta < \ell, 2, m, 0|\ell', m > < \ell, 2, 2, 0|\ell', 2 >, \right.$$
where \(<j_1, j_2, m_1, m_2|J, M >\) is a Clebsch-Gordan coefficient. Then, for \(\ell = 2\) and 3, we obtain \(d_{\ell m}' (\ell' \neq \ell)\) which are given in section II.B. As for \(d_{\ell m}\), we consider the normalization of \(\pm_2 P_{\ell m}\) (Eq. (2.2)). Inserting Eq. (F2) into Eq. (2.2), and using the orthogonality of \(\pm_2 P_{\ell m}\), we obtain

\[
1 = \int_0^\pi d\theta \sin \theta |\pm_2 S_{\ell m}|^2
= \int_0^\pi d\theta \sin \theta \left\{ (\pm_2 P_{\ell m})^2 + 2a\omega \sum_{\ell'} c_{\ell m}^{\ell'\ell} c_{\ell m}^{\ell'\ell} P_{\ell'\ell m} - 2P_{\ell m}\right\}
+ 2(a\omega)^2 \sum_{\ell''} d_{\ell m}^{\ell''\ell m} P_{\ell''m} + O ((a\omega)^3)
= 1 + (a\omega)^2 \sum_{\ell'} \left( c_{\ell m}^{\ell'\ell} \right)^2 + 2(a\omega)^2 d_{\ell m}^2 + O((a\omega)^3).
\]

Then we have

\[
d_{\ell m} = -\frac{1}{2} \left\{ \left( c_{\ell m}^{\ell + 1} \right)^2 + \left( c_{\ell m}^{\ell - 1} \right)^2 \right\}.
\]

(F9)

**APPENDIX G: THE EXPRESSION OF THE LUMINOSITY BY MEANS OF THE ORBITAL ANGULAR FREQUENCY**

For the sake of convenience to calculate the orbital phase error, we describe the formula of gravitational wave luminosity by means of \(v' \equiv (M\Omega)^{1/3}\). In this appendix, we define \(\eta_{\ell, m}\) as

\[
\left( \frac{dE}{dt} \right)_{\ell m} \equiv \frac{16}{5} \left( \frac{\mu}{M} \right)^2 v^{10} \eta_{\ell, m}.
\]

(G1)

\[
\eta_{2, 2} = 1 - \frac{107}{21} v'^2 + \frac{4\pi}{3} \left( 8q \right) v'^3 + \frac{4784}{1323} + 2q^2 v'^4
+ \frac{52q}{27} v'^5
\]

\[
+ \frac{99210071}{1091475} \left( -\frac{1712\gamma}{105} - \frac{16\pi^2}{3} - \frac{32\pi q}{3} \right) v'^6

+ \frac{3424 \ln 2}{105} \left( -\frac{1712 \ln v'}{105} \right) v'^6

+ \frac{19136 \pi}{1323} + \frac{634856q}{11907} + 8\pi q^2 - \frac{8q^3}{3} \right) v'^7
\]

\[
+ \frac{105922q^2}{9261} + q^4 + \frac{366368\ln 2}{2205} + \frac{183184 \ln v'}{2205} \right) v'^8.
\]

(G2)

\[
\eta_{2, 1} = \frac{v'^2}{36} - \frac{q}{12} v'^3 + \left( -\frac{17}{504} + \frac{q^2}{16} \right) v'^4
\]

\[
+ \frac{\pi}{18} + \frac{215q}{9072} v'^5
\]

\[
- \frac{2215}{254016} \left( -\frac{\pi q}{6} + \frac{313q^2}{1512} \right) v'^6
\]

\[
- \frac{17\pi}{252} + \frac{18127q}{190512} + \frac{\pi q^2}{8} - \frac{7q^3}{24} \right) v'^7.
\]
\[
\eta_{3.3} = \frac{1215 \nu^2}{896} - \frac{1215 \nu^4}{112} + \left( \frac{3645 \nu}{448} - \frac{1215 \nu q}{224} \right) \nu^5
\]
\[
+ \left( \frac{243729}{9856} + \frac{3645 q^2}{896} \right) \nu^6 + \left( \frac{-3645 \nu + 41229 q}{56} + \frac{1215 \nu q}{1792} \right) \nu^7
\]
\[
+ \left( \frac{25037019729}{125565440} - \frac{47385 \gamma}{1568} + \frac{3645 \nu^2}{224} - \frac{3645 \nu q}{112} \right) \nu^8.
\]

\[
\eta_{3.2} = \frac{5 \nu^4}{63} - \frac{40 q \nu^5}{189} + \left( \frac{193 \nu^2}{567} + \frac{80 q^2}{567} \right) \nu^6
\]
\[
+ \left( \frac{20 \nu}{63} + \frac{98 q}{1701} \right) \nu^7 + \left( \frac{86111}{280665} - \frac{160 \nu q}{189} + \frac{80 q^2}{189} \right) \nu^8.
\]

\[
\eta_{3.1} = \frac{5 \nu^4}{8064} - \frac{\nu^6}{1512} + \left( \frac{\pi}{4032} - \frac{25 q}{18144} \right) \nu^5
\]
\[
+ \left( \frac{437}{266112} + \frac{17 q^2}{24192} \right) \nu^6 + \left( \frac{-\pi}{756} + \frac{2257 q}{435456} \right) \nu^7
\]
\[
+ \left( -\frac{1137077}{50854003200} - \frac{13 \gamma}{42336} + \frac{\pi^2}{6048} \right)
\]
\[\quad\quad - \frac{25 \nu q}{9072} + \frac{12863 \nu^2}{3483648} + \frac{13 \ln 2}{42336} - \frac{13 \ln \nu'}{42336} \right) \nu^8.
\]

\[
\eta_{4.4} = \frac{1280 \nu^4}{567} - \frac{151808 \nu^6}{6237} + \left( \frac{10240 \nu}{567} - \frac{20480 q}{1701} \right) \nu^7
\]
\[
+ \left( \frac{560069632}{6243237} + \frac{5120 q^2}{567} \right) \nu^8.
\]

\[
\eta_{4.3} = \frac{729 \nu^6}{4480} - \frac{729 \nu \nu^7}{1792} + \left( -\frac{28431}{24640} + \frac{3645 q^2}{14336} \right) \nu^8.
\]

\[
\eta_{4.2} = \frac{5 \nu^4}{3969} - \frac{437 \nu^6}{34369} + \left( \frac{20 \nu}{3969} - \frac{170 q}{11907} \right) \nu^7
\]
\[
+ \left( \frac{7199152}{218513295} + \frac{200 q^2}{27783} \right) \nu^8.
\]

\[
\eta_{4.1} = \frac{282240}{282240} - \frac{q \nu^7}{112896} + \left( -\frac{101}{4656960} + \frac{5 q^2}{903168} \right) \nu^8.
\]

\[
\eta_{5.5} = \frac{9765625 \nu^6}{2433924} - \frac{2568359375 \nu^8}{47443968}.
\]

\[
\eta_{5.4} = \frac{4096 \nu^8}{15365}.
\]

\[
\eta_{5.3} = \frac{2187 \nu^6}{450560} - \frac{150903 \nu^8}{2928640}.
\]

\[
\eta_{5.2} = \frac{4 \nu^8}{40095}.
\]

\[
\eta_{5.1} = \frac{\nu^8}{127733760} - \frac{179 \nu^8}{2490808320}.
\]

\[
\eta_{6.6} = \frac{26244 \nu^8}{3575}.
\]
\[ \eta_{6.4} = \frac{131072}{9555975} v^8. \]  
(G17)

\[ \eta_{6.2} = \frac{4}{5733585} v^8. \]  
(G18)

In total,

\[
\langle \frac{dE}{dt} \rangle = \frac{32}{5} \left( \frac{\mu}{M} \right) v^{10} \left( 1 - \frac{1247}{336} v^2 + \left( \frac{4 \pi - 11}{4} \right) v^3 \right) \\
+ \left( \frac{44711}{9072} + \frac{33 q^2}{16} \right) v^4 \right) \\
\quad + \left( \frac{6643739519}{69854400} - \frac{1712 \gamma}{105} + \frac{16 \pi^2}{3} - \frac{65 \pi q}{6} + \frac{661 q^2}{504} \right) v^5 \\
- \left( \frac{1712 \ln v'}{105} \right) v^6 \right) \\
\quad + \left( \frac{17 q^4}{16} + \frac{39931 \ln 2}{294} - \frac{47385 \ln 3}{1568} + \frac{232597 \ln v'}{4410} \right) v^8. \]  
(G19)
Figure Captions

Figs.1(a-e) : Error of the post-Newtonian formulas as a function of the Boyer-Lindquist coordinate radius \( r \) for \( 6 \leq r/M \leq 100 \) in the case \( q = -0.9, -0.5, 0, 0.5 \) and 0.9. In each figure, open square, filled triangle, open triangle, filled circle, and open circle denote the error of 2-PN, 2.5-PN, 3-PN, 3.5-PN and 4-PN formulas, respectively.

Fig.2 : Error of the post-Newtonian formula of \( q = 0.9 \) for \( 2.5 \leq r/M \leq 12 \). Open square, filled triangle, open triangle, filled circle, and open circle denote the error of 2-PN, 2.5-PN, 3-PN, 3.5-PN and 4-PN formulas, respectively.
Error versus $r/M$ for $q=0.9$. Different symbols represent different orders of post-Newtonian (PN) approximation: 
- Open circles: 4PN
- Filled circles: 3.5PN
- Open triangles: 3PN
- Filled triangles: 2.5PN
- Filled squares: 2PN

The error decreases with increasing $r/M$, indicating improving accuracy as the PN order increases.
