On separability of the unbounded norm topology

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Abstract
In this paper we continue the investigation of topological properties of the unbounded norm (un-)topology in normed lattices. We characterize separability and second countability of the un-topology in terms of properties of the underlying normed lattice. We apply our results to prove that an order continuous Banach function space $X$ over a semi-finite measure space is separable if and only if it has a $\sigma$-finite carrier and is separable with respect to the topology of local convergence in measure. We also address the question when a normed lattice is a normal space with respect to the un-topology.

Keywords Normed lattice · Banach function spaces · Un-topology · Separability · Semi-finite measures

Mathematics Subject Classification Primary 46B42; Secondary 46A40 · 46E30

1 Introduction

Recently, the study of unbounded convergences in vector and normed lattices attracted many researchers in the area. Unbounded order convergence was already introduced by Nakano [27] and studied by DeMarr [8] and Kaplan [19], but its systematic study was initiated by Gao, Troitsky and Xanthos (see [12–14]). While the unbounded norm (un-)convergence and topology were formally introduced in [7], the study of topo-
logical properties of the un-topology began in [17]. In [16], the authors extended
the un-topology beyond normed lattices and studied its properties. Generalizations
of the un-topology to locally solid vector lattices and other instances of unbounded
convergences were studied in e.g. [5, 18, 30, 31, 34]. In this paper we continue the
investigation of topological properties of the un-topology. In particular, we are inter-
ested in separability, second countability, and normality.

The paper is structured as follows. In Sect. 2 we introduce the notation and basic
notions needed throughout the text. In Sect. 3 we consider the question when a given
normed lattice \( X \) equipped with its unbounded norm topology \( \tau_{\text{un}} \) is a separable or
even a second countable space. We first prove that \( X \) contains a countable quasi-
interior set \( \{ u_n : n \in \mathbb{N} \} \) whenever \( (X, \tau_{\text{un}}) \) is separable. This result enables us to
prove (see Theorem 3.2) that separability and second countability of \( (X, \tau_{\text{un}}) \) are both
equivalent to separability of the normed lattice \( X \) itself. This result is applied to prove
that separability of the unbounded norm topology passes up and down between \( X \) and
its Banach space completion \( \hat{X} \). In Sect. 4 we go one step further. Instead of considering
separability of \( X \) or equivalently separability of the principal ideals \( I_{u_n} \) where the \( u_n \)
are as above, we rather consider separability of the sets \( C_{u_n} \) of components of the
vectors \( u_n \) equipped with the metric topology inherited from \( X \). For a normed lattice
with the projection property we prove in Theorem 4.2 that it is separable if and only
if the set of components \( C_{u_n} \) is separable for each \( n \in \mathbb{N} \). We also provide an example
that this statement does not hold in general for normed lattices without the principal
projection property.

In Sect. 5 we consider separability of order continuous Banach function spaces over
semi-finite measure spaces. In Theorem 5.9 we prove that such a Banach function space
is separable if and only if it has a \( \sigma \)-finite carrier and it is separable with respect to the
topology of local convergence in measure.

In Sect. 6 we discuss the separation axioms for a normed lattice equipped with
the unbounded norm topology. Since \( (X, \tau_{\text{un}}) \) satisfies the Hausdorff axiom, by the
general theory of topological vector spaces the space \( (X, \tau_{\text{un}}) \) is completely regular.
We prove that \( (X, \tau_{\text{un}}) \) is a paracompact and hence a normal space whenever \( X \) is an
atomic KB-space.

2 Preliminaries

Throughout the paper we will assume that all vector lattices are Archimedean. Let \( X \)
be a vector lattice. The set of all positive elements is denoted by \( X^+ \). For a subset \( A \)
we denote by \( I_A \) and \( B_A \) the ideal and the band in \( X \) generated by \( A \), respectively. In
the case when the set \( A = \{ u \} \) is a singleton set we write \( I_A = I_u \) and \( B_A = B_u \). Ideals and
bands generated by a single element are called principal. For a given set \( A \subseteq X \) we
denote the set \( \{ x \in X : |x| \wedge |a| = 0 \text{ for all } a \in A \} \) by \( A^d \) and we call it the disjoint
complement of \( A \) in \( X \). It turns out that \( A^d \) is always a band in \( X \). If a band \( B \) satisfies
\( X = B \oplus B^d \), then \( B \) is called a projection band. To a projection band \( B \) we associate
the band projection \( P_B \) which is the identity mapping on \( B \) and the zero mapping on
\( B^d \). It is well known that band projections always commute. If every principal band in
\( X \) is a projection band, then \( X \) is said to have the principal projection property. A vector
lattice $X$ is said to be Dedekind complete or order complete if non-empty bounded from above subsets of $X$ have suprema. If non-empty countable bounded from above subsets of $X$ have suprema, then $X$ is said to be $\sigma$-Dedekind complete or $\sigma$-order complete. A vector lattice is said to satisfy the countable sup property whenever every non-empty subset possessing a supremum contains an at most countable subset possessing the same supremum. The countable sup property is equivalent to the following fact: for each net $(x_\alpha)$ in $X$ that satisfies $0 \leq x_\alpha \uparrow x$ there is an increasing sequence of indices $(\alpha_n)_{n \in \mathbb{N}}$ such that $0 \leq x_{\alpha_n} \uparrow x$ (this follows from [24, Theorem 23.2(iii)]). An ideal $I$ of $X$ is said to be order dense in $X$ if for each positive non-zero vector $x \in X$ there exists a positive vector $y \in I$ such that $0 < y \leq x$. A positive non-zero vector $a$ in $X$ is said to be an atom whenever the principal ideal $I_a$ is one-dimensional (see [24, Theorem 26.4]). Since $X$ is Archimedean, the principal ideal $I_a$ is a projection band in $X$. A vector lattice $X$ is called atomic if the linear span of the set of all atoms, which is an ideal in $X$ by [33, Theorem 7.6], is order dense. It easily follows from the definition that a vector lattice is atomic if and only if for each positive non-zero vector $x \in X$ there exists an atom $a \in X$ such that $0 < a \leq x$.

A lattice norm $\| \cdot \|$ on a vector lattice is a norm which satisfies $\|x\| \leq \|y\|$ for all elements $x$ and $y \in X$ with $0 \leq |x| \leq |y|$. A vector lattice equipped with a lattice norm is said to be a normed lattice. A Banach lattice is a normed lattice which is also a Banach space. The Banach space completion of a normed lattice $X$, which is denoted by $\hat{X}$, is always a Banach lattice. A normed lattice is order continuous whenever $x_\alpha \downarrow 0$ implies $x_\alpha \rightarrow 0$ in norm.

A set $Q$ of positive vectors of a normed lattice $X$ is said to be a quasi-interior set if the ideal generated by $Q$ in $X$ is dense in $X$. A positive vector $u \in X$ is said to be a quasi-interior point whenever the set $\{u\}$ is a quasi-interior set of $X$. A positive vector $u \in X$ is called a weak unit if $|x| \wedge u = 0$ implies $x = 0$. Every separable Banach lattice has a quasi-interior point. Since dense ideals are order dense, every quasi-interior point is a weak unit. Although the converse statement does not hold in general, in order continuous normed lattices these two notions coincide. The remaining unexplained facts about vector and normed lattices can be found in [1, 3, 24, 26, 29, 33].

Given a normed lattice $X$, a positive number $\varepsilon > 0$ and a positive vector $x \in X^+$ we define the set

$$U_{x,\varepsilon} = \{y : \|y\land x\| < \varepsilon\}.$$

Then the collection of all sets of this form is a base of neighborhoods of zero for a locally solid Hausdorff linear topology on $X$. This topology is called the unbounded norm topology (un-topology for short) on $X$ induced by the norm, and it is denoted by $\tau_{un}$. It is easy to see that un-topology is weaker than the norm topology and that they coincide on order bounded sets. A net $(x_\alpha)$ in $X$ is said to un-converge to a vector $x \in X$ whenever for each $y \in X^+$ we have $|x_\alpha - x| \land y \rightarrow 0$ in norm. It should be clear that a net $(x_\alpha)$ un-converges to $x$ if and only if it converges to $x$ with respect to $\tau_{un}$. Taylor proved that in the case of an order continuous Banach lattice the corresponding un-topology is the so-called Hausdorff uo-Lebesgue topology and that it coincides with the unbounded absolute weak topology (see [31, Example 5.6]).
The notion of the un-topology was extended in [16] beyond normed lattices as follows. Suppose that \( X \) is a normed lattice which is contained as an order ideal in a vector lattice \( Y \). Given a positive \( \varepsilon > 0 \) and a positive vector \( x \in X \) we define
\[
U_{x,\varepsilon} = \{ y \in Y : \| y \wedge x \| < \varepsilon \}.
\]

It was observed in [16] that the collection of all sets of this form is a base of neighbourhoods of zero for a locally solid linear topology on \( Y \), which is called the un-topology on \( Y \) induced by \( X \). It follows immediately from the definition that the relative topology on \( X \) agrees with the original un-topology. By [16, Proposition 1.4], the induced topology is Hausdorff if and only if \( X \) is order dense in \( Y \).

3 Separability of the un-topology for general normed lattices

In this section we consider the question under which conditions a normed lattice equipped with the unbounded norm topology \( \tau_{\text{un}} \) is a separable or even a second countable topological space. Although in general the unbounded norm topology behaves differently than the norm topology, the answer is surprisingly simple. In fact, we prove in Theorem 3.2 that separability and second countability of \( (X, \tau_{\text{un}}) \) are both equivalent to separability of \( X \). We start with the following proposition which is an improvement of [1, Exercise 4.2.3]. It will be used to prove Theorem 3.2.

Proposition 3.1 Let \( X \) be a normed lattice such that \( (X, \tau_{\text{un}}) \) is separable.

(1) Then \( X \) has a countable quasi-interior set.
(2) If \( X \) is a Banach lattice, then \( X \) has a quasi-interior point.

Proof

(1) Let \( \{x_n : n \in \mathbb{N}\} \) be a countable \( \tau_{\text{un}} \)-dense subset of \( X \). Since the inequality
\[
0 \leq |x^+ - y^+| \wedge |z| \leq |x - y| \wedge |z|
\]
holds for all \( x, y, z \in X \), the set \( \{x_n^+ : n \in \mathbb{N}\} \) is \( \tau_{\text{un}} \)-dense in \( X^+ \). Let \( F \) be the set of all finite subsets of \( \mathbb{N} \). For \( F \in F \) we define \( x_F = \sum_{n \in F} x_n^+ \) and we observe that the family \( \mathcal{D} := \{x_F : F \in F\} \) is countable and upward directed. The set \( \mathcal{G} \) of all finite sums of elements from \( \mathcal{D} \) is countable, additive, upward directed and it obviously contains \( \mathcal{D} \). Since \( \mathcal{G} \) is upward directed, it can be considered as an increasing net.

Since \( \mathcal{G} \) is additive, in order to prove that \( \mathcal{G} \) is a quasi-interior set, by [9, Proposition 2.2] it is enough to show that for every positive vector \( x \) the net \( (x - x \wedge u)_{u \in \mathcal{G}} \) converges to zero in norm. Pick a positive vector \( x \in X^+ \), and let \( \varepsilon > 0 \) be given. Choose a set \( F_1 \in F \). Then \( x + U_{|x - x_{F_1}|,\varepsilon} \) is a \( \tau_{\text{un}} \)-neighborhood of \( x \), and since \( \mathcal{D} \) is \( \tau_{\text{un}} \)-dense in \( X^+ \), there exists \( F_2 \in F \) such that \( x_{F_2} \in x + U_{|x - x_{F_1}|,\varepsilon} \), from which it follows that
\[
\| x - x_{F_2} \| \wedge |x - x_{F_1}| \| < \varepsilon.
\]

Then \( x_{F_1} + x_{F_2} \in \mathcal{G} \), and for any \( u \in \mathcal{G} \) such that \( u \geq x_{F_1} + x_{F_2} \) we have that \( x - u \leq x - x_{F_1} \) and \( x - u \leq x - x_{F_2} \), from which it follows that \( (x - u)^+ \leq (x - x_{F_1})^+ \).
and $(x - u)^+ \leq (x - x_{F_2})^+$. Since each vector $a \in X$ satisfies $0 \leq a^+ \leq |a|$, we conclude that

$$
\| (x - u)^+ \| = \| (x - u)^+ \lor (x - u)^+ \| \leq \| (x - x_{F_1})^+ \lor (x - x_{F_2})^+ \| \\
\leq \| |x - x_{F_1}| \lor |x - x_{F_2}| \| < \varepsilon.
$$

From this it follows that the net $((x - u)^+)_{u \in G}$ converges to zero in norm. Due to the equality $(x - u)^+ = x - x \lor u$, we conclude that for each $x \in X^+$ the net $(x \lor u)_{u \in G}$ converges to $x$ in norm.

(2) By (1), there exists a countable quasi-interior set $D = \{u_n : n \in \mathbb{N}\}$ in $X$. Since the series of positive vectors

$$
\sum_{n=1}^{\infty} \frac{u_n}{2^n (\|u_n\| + 1)}
$$

converges absolutely, it converges to some positive vector $u \in X$ in norm. Since for each $n \in \mathbb{N}$ we have $0 \leq u_n \leq 2^n (\|u_n\| + 1)u$, the principal ideal $I_u$ contains the quasi-interior set $D$, from which it follows that $I_u$ is dense in $X$. $\square$

The following theorem which follows from Proposition 3.1 is the main result of this section.

**Theorem 3.2** The following assertions about a normed lattice $X$ are equivalent.

1. $X$ is second countable.
2. $X$ is separable.
3. $(X, \tau_{un})$ is separable.
4. $(X, \tau_{un})$ is second countable.

**Proof** The equivalence (1)$\iff$(2) holds for general normed spaces.

(2)$\implies$(3) If $X$ is separable, then $X$ is separable with respect to any topology weaker than the norm topology. In particular, $(X, \tau_{un})$ is separable.

(3)$\implies$(4) Suppose that $(X, \tau_{un})$ is separable. By Proposition 3.1 we conclude that $X$ has a countable quasi-interior set, so that $(X, \tau_{un})$ is metrizable by [18, Theorem 4.3]. Since $(X, \tau_{un})$ is metrizable and separable, it is second countable.

(4)$\implies$(2) Since $(X, \tau_{un})$ is second countable, it is separable, so that by Proposition 3.1 we conclude that $X$ has a countable quasi-interior set $G$. By enumerating the vectors in $G$ as $\{u_1, u_2, \ldots\}$ and after replacing the $n$-th vector by the sum $u_1 + \cdots + u_n$, we may assume without loss of generality that $u_1 \leq u_2 \leq \ldots$

Let us consider the order ideal $I_G$ generated by $G$. Since $G$ is a quasi-interior set, the ideal $I_G$ is dense in $X$. We claim that the un-topology on $I_G$ equals the relative topology induced by the un-topology from $X$. To see this, note first that the equality

$$
U_{x,\varepsilon}^{I_G} := \{y \in I_G : \|y\| \lor x \| < \varepsilon\} = U_{x,\varepsilon} \cap I_G
$$

for all positive $x \in I_G$ yields that every basis neighborhood $U_{x,\varepsilon}^{I_G}$ of zero for the un-topology on $I_G$ is open in the relative topology. To prove the converse, pick any $x \in X^+$ and $\varepsilon > 0$. Then $U_{x,\varepsilon} \cap I_G$ is a basis open neighborhood for zero in the relative topology on $I_G$ induced by the un-
topology of $X$. Since $I_G$ is dense in $X$, there exists a positive vector $y \in I_G$ such that $\|x - y\| < \frac{\varepsilon}{2}$. Pick any $z \in U_{\frac{\varepsilon}{2}}^{I_G}$. From the inequality

$$\|z\wedge x\| \leq \|z\wedge x - y\| + \|z\wedge y\| < \varepsilon$$

we conclude that $U_{\frac{\varepsilon}{2}}^{I_G} \subseteq U_{x,\varepsilon} \cap I_G$ which proves the claim.

Since $(X, \tau_{un})$ is second countable, the un-topology on $I_G$ is second countable as well. This yields that for each $n \in \mathbb{N}$ the interval $[-nu_n, nu_n]$ is second countable with respect to the relative un-topology induced from $I_G$. Since on order intervals the un-topology agrees with the norm topology, each order interval $[-nu_n, nu_n]$ is separable. If $\mathcal{F}_n$ is a countable dense subset of $[-nu_n, nu_n]$, then the set $\mathcal{F} := \bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a countable dense subset of $I_G$, so that the norm density of $I_G$ in $X$ yields the norm density of $\mathcal{F}$ in $X$. This proves that $X$ is separable.

Although in metric spaces separability is equivalent to second countability, this is not the case even for locally convex Hausdorff spaces. For example, the space $\mathbb{R}^\mathbb{R}$ equipped with the product topology is separable by [28, Theorem 1], yet the space is not second countable by [32, 16.2 Theorem c])

The following result tells that second countability and separability of the unbounded norm topology pass up and down between a normed lattice and its norm completion.

**Corollary 3.3** For a normed lattice $X$ the following assertions are equivalent.

1. $(\hat{X}, \tau_{un})$ is second countable.
2. $(X, \tau_{un})$ is second countable.
3. $(\hat{X}, \tau_{un})$ is separable.
4. $(X, \tau_{un})$ is separable.

**Proof** The equivalences (1)$\iff$(3) and (2)$\iff$(4) follow directly from Theorem 3.2.

(1)$\implies$(2) By Theorem 3.2, the Banach lattice $\hat{X}$ is separable, so that $X$ is also separable. Another application of Theorem 3.2 yields that $(X, \tau_{un})$ is second countable. The converse implication can be proved similarly.

The following corollary follows from Theorem 3.2 and from the fact that $X$ is separable whenever its norm dual $X^*$ is.

**Corollary 3.4** Let $X$ be a normed lattice. If $(X^*, \tau_{un})$ is separable, then $(X, \tau_{un})$ is separable.

We conclude this section with a characterization of separability of atomic normed lattices with order continuous norm.

**Corollary 3.5** For an atomic normed lattice $X$ with order continuous norm the following statements are equivalent.

1. $X$ is separable.
2. $(X, \tau_{un})$ is second countable.
3. $(X, \tau_{un})$ is separable.
(4) \((X, \tau_{\text{un}})\) is first countable.

(5) \((X, \tau_{\text{un}})\) is metrizable.

(6) \(X\) has a countable quasi-interior set.

(7) Every disjoint set in \(X\) is at most countable.

(8) Every disjoint set of atoms in \(X\) is at most countable.

**Proof** The equivalences (1)\(\iff\)(2)\(\iff\)(3) follow from Theorem 3.2 and (5)\(\iff\)(6) follows from [18, Theorem 4.3]. Since \(\tau_{\text{un}}\) is a linear Hausdorff topology on \(X\), (4)\(\iff\)(5) follows from [20, p. 49]. Furthermore, while Proposition 3.1 yields the implication (3)\(\implies\)(6), the implication (7)\(\implies\)(8) is obvious.

(6)\(\implies\)(7) Suppose that \(\mathcal{D}\) is a countable quasi-interior set in \(X\) and pick a disjoint family \(S\) of positive vectors in \(X\). Then for each \(u \in \mathcal{D}\) the family \(\{u \wedge x : x \in S\}\) is an order bounded disjoint family in \(X\). For all \(u \in \mathcal{D}\) and \(m, n \in \mathbb{N}\) we consider the set

\[ S_{u,m} := \left\{ x \in S : \|u \wedge x\| \geq \frac{1}{m} \right\}. \]

Fix \(u \in \mathcal{D}\) and suppose that \(S_{u,m_0}\) is infinite for some \(m_0 \in \mathbb{N}\). Take any sequence \((y_k)_{k \in \mathbb{N}}\) in \(\{u \wedge x : x \in S_{u,m_0}\}\) of distinct vectors. Since \(X\) is order continuous, by [23, Theorem 64.3] the norm completion \(\hat{X}\) is also order continuous, so that the disjoint sequence \((y_k)_{k \in \mathbb{N}}\) converges to zero in norm by [3, Theorem 4.14]. This contradicts the fact that \(\|y_k\| \geq \frac{1}{m_0}\) for each \(k \in \mathbb{N}\). Therefore, for all \(u \in \mathcal{D}\) and \(m, n \in \mathbb{N}\) the set \(S_{u,m}\) is finite.

Now consider the set \(S_0 := \bigcup_{u \in \mathcal{D}} \bigcup_{m=1}^{\infty} S_{u,m}\) which is clearly at most countable. If \(x \in S_0\), then \(x \wedge u = 0\) for each \(u \in \mathcal{D}\). Since \(\mathcal{D}\) is a quasi-interior set, we conclude that \(x = 0\), from which it follows that \(S\) is at most countable.

(8)\(\implies\)(1) Since finite-dimensional normed spaces are separable, we may assume that \(X\) is infinite-dimensional. Pick a maximal set \(A\) of atoms in \(X\). Since \(X\) is infinite-dimensional, we may enumerate the elements of \(A\) as \((e_n)_{n \in \mathbb{N}}\). For each \(n \in \mathbb{N}\) consider the linear span \(\mathcal{J}_n\) of the set \(\{e_1, \ldots, e_n\}\). Since \(\mathcal{J}_n\) is spanned by finitely many atoms, \(\mathcal{J}_n\) is a projection band in \(X\). Since each \(\mathcal{J}_n\) is finite-dimensional, the union \(F := \bigcup_{n=1}^{\infty} \mathcal{J}_n\) is a separable subspace of \(X\). Since \(X\) is atomic, \(F\) is order dense in \(X\), and since the norm on \(X\) is order continuous, \(F\) is dense in \(X\). This proves separability of \(X\).

\(\blacksquare\)

**Remark 3.6** Some implications in Corollary 3.5 do not hold if we relax certain assumptions.

(1) The implications (6)\(\implies\)(7) and (8)\(\implies\)(1) in Corollary 3.5 do not necessarily hold if we replace order continuity of the norm with Dedekind completeness. To see this, for a given index set \(I\) consider the Dedekind complete Banach lattice \(\ell^\infty(I)\) of all bounded sequences over the index set \(I\). Clearly, \(\ell^\infty(I)\) contains a strong unit.

If \(I\) is uncountable, then \(\ell^\infty(I)\) contains uncountably many pairwise disjoint atoms. On the other hand, in \(\ell^\infty := \ell^\infty(\mathbb{N})\) every disjoint set of atoms is at most countable, yet \(\ell^\infty\) is not separable.

(2) The implication (7)\(\implies\)(1) does not hold for order continuous Banach lattices without atoms. To see this, consider the circle \(\mathbb{S}^1\) equipped with the Borel \(\sigma\)-algebra and the arc-length measure. Furthermore, consider an uncountable product of circles equipped with the product \(\sigma\)-algebra and the product probability measure \(\mu\). Since the
constant one function is a quasi-interior point in $L^2(\mu)$, similar arguments as in the proof of implication (6)$\Rightarrow$(7) in Corollary 3.5 show that every pairwise disjoint set in $L^2(\mu)$ is at most countable. Since the coordinate functions are pairwise orthogonal, the Hilbert space $L^2(\mu)$ is not separable.

4 Components of positive vectors and separability of the un-topology

In 3 we considered separability of the un-topology in comparison with general topological properties of the norm topology and the un-topology. In this section we study separability of the un-topology of normed lattices with the principal projection property in terms of separability of “special” metric spaces which can be considered as their building blocks. These metric spaces are the spaces of components of positive vectors equipped with the metric induced by the norm of the underlying normed lattice. The motivation comes from Measure Theory, which will be now explained.

Given a finite measure space $(\Omega, \mathcal{F}, \mu)$ one can define the map $d_\mu : \mathcal{F} \times \mathcal{F} \to [0, \mu(\Omega)]$ as

$$d_\mu(A, B) := \|\chi_{A \triangle B}\|_1 = \mu(A \setminus B) + \mu(B \setminus A).$$

It is easy to see that $d_\mu$ is a semi-metric on $\mathcal{F}$ called the Fréchet-Nikodym semi-metric (see [4]). On $\mathcal{F}$ we introduce the equivalence relation $\sim$ by

$$A \sim B \iff d_\mu(A, B) = 0.$$

Then the semi-metric $d_\mu$ induces the well-defined metric $d$ on the metric Boolean algebra $\mathcal{F}/\sim$ defined as

$$d([A], [B]) := d_\mu(A, B).$$

It should be clear that the semi-metric space $(\mathcal{F}, d_\mu)$ is separable if and only if the metric space $(\mathcal{F}/\sim, d)$ is separable. It should be noted that separability of the semi-metric space $(\mathcal{F}, d_\mu)$ is sometimes in the literature referred to as separability of $\mu$.

The following proposition which immediately follows from Theorem 46.4 and Theorem 46.5 proved by Luxemburg in [22] presents an important connection between separability of the norm topology (or equivalently, separability of the un-topology) and the local structure of normed lattices.

**Proposition 4.1** Suppose $(\Omega, \mathcal{F}, \mu)$ is a finite measure space. Then $L^1(\mu)$ is separable if and only if the semi-metric space $(\mathcal{F}, d_\mu)$ is separable.

Since each $L^1$-function is a limit of a sequence of step functions, the key role in the proof of Proposition 4.1 is actually played by characteristic functions in $L^1(\mu)$ or equivalently by their representing sets in $\mathcal{F}$. This concept of a characteristic function can be fruitfully generalized to the realm of vector lattices. Given a positive vector $u$ of a vector lattice $X$, a positive vector $x$ is a component of $u$ if $0 \leq x \leq u$ and $x \perp (u - x)$. Since for each measurable set $A \subseteq X$ we have $\chi_A + \chi_{\Omega \setminus A} = 1$ and
On separability of the unbounded norm topology

\( \chi_A \wedge \chi_{\Omega \setminus A} = 0 \), the notion of a component of a positive vector naturally extends the notion of a characteristic function to vector lattices. The set of all components of a given vector \( u \) is denoted by \( C_u \). It should be clear that we always have \( 0, u \in C_u \). The set \( C_u \) inherits the metric \( \rho \) induced by the norm of \( X \). Hence, \( (C_u, \rho) \) is a metric space. A component \( x \in C_u \) is called non-trivial if \( x \) is neither 0 nor \( u \). A linear combination of components of \( u \) is called a \( u \)-step function. It is easy to see that we can always take pairwise disjoint components when working with a particular \( u \)-step function. For more information about components and step functions we refer the reader to [3, p. 40 and p. 86], respectively.

The following result is clearly a generalization of Proposition 4.1. It will be used in Theorem 5.6 in order to obtain a characterization of separability of order continuous Banach function spaces over semi-finite measure spaces.

**Theorem 4.2** For a normed lattice \( X \) with the principal projection property the following statements are equivalent.

1. \( X \) is separable.
2. \( (X, \tau_{un}) \) is separable.
3. \( X \) contains a countable quasi-interior set and for each quasi-interior set \( Q \) and each vector \( u \in Q \) the metric space \( (C_u, \rho) \) is separable.
3' \( X \) contains a countable quasi-interior set such that for each vector \( u \in Q \) the metric space \( (C_u, \rho) \) is separable.

Moreover, if \( X \) is a Banach lattice, then (1)-(3') are equivalent to each of the following statements.

4. \( X \) contains a quasi-interior point and for each quasi-interior point \( u \) the metric space \( (C_u, \rho) \) is separable.
4' \( X \) contains a quasi-interior point \( u \) such that the metric space \( (C_u, \rho) \) is separable.

**Proof** While the equivalence (1)\( \Leftrightarrow \) (2) follows from Theorem 3.2, the implications (3)\( \Rightarrow \) (3') and (4)\( \Rightarrow \) (4') are obvious.

(2)\( \Rightarrow \) (3) Suppose that \( X \) is \( \tau_{un} \)-separable. By Proposition 3.1, \( X \) contains a countable quasi-interior set. Let \( Q \) be any quasi-interior set and pick any positive vector \( u \) from \( Q \). Since \( X \) is separable by Theorem 3.2, the metric space \( (C_u, \rho) \) is separable as well.

(3')\( \Rightarrow \) (1) We first prove that for a positive vector \( u \in X^+ \) the principal ideal \( I_u \) is separable whenever the metric space of components \( C_u \) is separable. Suppose that \( C_u \) is separable, and let \( D \) be an at most countable dense subset of \( C_u \). Let \( \mathcal{G} \) be the set of all linear combinations of vectors from \( D \) with rational coefficients. We claim that \( \mathcal{G} \) is dense in \( I_u \). To this end, pick any positive vector \( x \in I_u \) and \( \varepsilon > 0 \). By Freudenthal’s spectral theorem (see e.g. [33, Theorem 33.2]) there exists an increasing sequence \( (s_n)_{n \in \mathbb{N}} \) of \( u \)-step functions such that \( 0 \leq s_n \uparrow x \) and that \( s_n \to u \)-uniformly. In particular, \( s_n \to x \) in norm, so that the positive part of the linear span of \( C_u \) is dense in \( I_u^+ \). Due to the equality \( x = x^+ - x^- \) we immediately obtain that the linear span of \( C_u \) is dense in \( I_u \). Since \( C_u \) is separable, by a standard approximation argument one can conclude that \( I_u \) is separable as well.

After this preparation, we prove (1). Let us enumerate the vectors from \( \mathcal{G} \) as \( \{u_1, u_2, \ldots \} \). For each \( n \in \mathbb{N} \) we denote by \( w_n \) the vector \( u_1 + \cdots + u_n \). From the
identity $I_{w_n} = I_{u_1} + \cdots + I_{u_n}$, which follows from [33, Exercise 7.7], and the fact that the ideal $I_{u_k}$ is separable for each $k \in \mathbb{N}$, we conclude that the ideal $I_{w_n}$ is separable as well. From this we immediately derive that the increasing union $I_G = \bigcup_{n=1}^{\infty} I_{w_n}$ is separable in $X$. Since $I_G$ is dense in $X$, we obtain separability of $X$.

Suppose now that $X$ is also norm complete. For the moreover statement it is enough to see that (1) implies (4). This follows immediately from the fact that every separable Banach lattice has a quasi-interior point and from the implication (1)$\Rightarrow$(3).  □

The following example shows that Theorem 4.2 does not hold if one does not assume that the underlying normed lattice has the principal projection property.

**Example 4.3** Let $X$ be the Banach lattice $C(K)$ where $K$ is a product of uncountably many copies of the interval $[0, 1]$. Since $K$ is connected, we have $C_{\perp} = \{0, 1\}$, so that $C_{\perp}$ is separable. On the other hand, $K$ is not metrizable (see [32, 22.3 Theorem]) hence $C(K)$ is not separable (see [2, Theorem 4.1.3]).

### 5 Applications to Banach function spaces

In this section we characterize separable Banach function spaces over semi-finite measure spaces in terms of measure-theoretical properties of the underlying measure spaces. Given an arbitrary measure space $(\Omega, \mathcal{F}, \mu)$, a function space $X$ is an (order) ideal in $L^0(\mu)$. A function space $X$ is a Banach function space if it is equipped with a complete lattice norm. Luxemburg proved in [22, Theorem 46.2] that for a separable Banach function space $X$ over a $\sigma$-finite measure $\mu$ we have $X = X^a$. This means by definition that for every element $x \in X$ the element $[x]$ is order continuous, and so $X$ is order continuous as a Banach lattice. For more details see [26, Proposition 2.4.10]. When $\mu$ is not $\sigma$-finite, the vector lattice $L^0(\mu)$ might not be order complete. However, it is always $\sigma$-order complete. Hence, by [3, Corollary 4.52] (see also [21] and [25]) every separable Banach function space is always order continuous.

**Lemma 5.1** Every separable Banach function space in $L^0(\mu)$ is order continuous.

Let $X$ be an order dense order continuous Banach function space in $L^0(\mu)$ where $\mu$ is finite. Separability of $X$ is equivalent to separability of $X$ equipped with the un-topology. An application of [16, Theorem 5.2] yields that $X$ is separable if and only if $X$ is separable with respect to the topology $\tau_\mu$ of convergence in measure. In fact, separability of $(X, \tau_{un})$, or equivalently, separability of $X$, depends only on $L^0(\mu)$.

**Lemma 5.2** Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and let $X$ be an order dense order continuous Banach function space in $L^0(\mu)$. Then $X$ is separable if and only if $L^0(\mu)$ equipped with the topology of convergence in measure is separable.

**Proof** Suppose first that $L^0(\mu)$ is separable with respect to the topology of convergence in measure. Since $\mu$ is finite and $X$ is order continuous, by [16, Theorem 5.2] the topology of convergence in measure is the un-topology on $L^0(\mu)$ induced by $X$.

By [1, Corollary 5.22] there exists a weak unit $g$ in $X$. Due to the order density of $X$ in $L^0(\mu)$ we have that $\varphi$ is also a weak order unit in $L^0(\mu)$. Since the norm of $X$
is order continuous, \( \varphi \) is a quasi-interior point in \( X \). Hence, by [16, Theorem 3.3] we conclude that the topology of convergence in measure on \( L^0(\mu) \) is metrizable.

Since the un-topology on \( X \) is just the restriction of the un-topology induced by \( X \) on \( L^0(\mu) \), the un-topology on \( X \) is precisely the topology of convergence in measure restricted to \( X \). Since \((X, \tau_\mu|X)\) is separable as a subspace of a separable metric space, Theorem 3.2 shows that \( X \) is separable.

To prove the converse statement, assume that \( X \) is separable. By Theorem 3.2 it follows that \( X \) is \( \tau_{\text{un}} \)-separable. To finish the proof it suffices to prove that \( X \) is \( \tau_\mu \)-dense in \( L^0(\mu) \). Pick any non-negative function \( f \in L^0(\mu) \). Since \( X \) is order dense in \( L^0(\mu) \) and \( L^0(\mu) \) has the countable sup property (see [26, Lemma 2.6.1]), there is an increasing sequence \( (f_n)_{n \in \mathbb{N}} \) in \( X \) such that \( f_n \uparrow f \) in \( L^0(\mu) \). This yields that \( f_n \to f \) \( \mu \)-almost everywhere, and since the measure \( \mu \) is finite, by a consequence of the well-known Egorov theorem we conclude that \( f_n \to f \) in measure. \( \square \)

**Proposition 5.3** Let \((\Omega, \mathcal{F}, \mu)\) be a finite measure space. For an order continuous order dense Banach function space \( X \) in \( L^0(\mu) \) the following assertions are equivalent.

1. \( X \) is separable.
2. \( X \) is separable with respect to the topology of convergence in measure.
3. The semi-metric space \((\mathcal{F}, d_\mu)\) is separable.
4. The metric space \((C_f, \rho)\) is separable for each almost everywhere strictly positive function \( f \in X \).

If \( \mathcal{F} \) is countably generated, then all statements (1)-(4) hold.

**Proof** While the equivalence between (1) and (2) immediately follows from Theorem 4.2, the equivalence between (1) and (4) follows from Theorem 4.2 together with the fact that the classes of quasi-interior points and almost everywhere strictly positive functions coincide in the case of order continuous Banach function spaces.

(3) \iff (1) If \((\mathcal{F}, d_\mu)\) is separable, then \((C_1, \rho)\) is separable in \( L^1(\mu) \) by Proposition 4.1. By using the already proven implication (4) \( \Rightarrow \) (1) in the case of the Banach lattice \( L^1(\mu) \) we conclude that \( L^1(\mu) \) is separable. Therefore, by an application of Lemma 5.2 we conclude that \( X \) is separable. The proof of the converse statement can be proved similarly and is therefore omitted.

Although the proof that the additional assertion implies (3) is well known and is given as an exercise in [4, Exercise 1.12.102], we will provide its sketch for the sake of completeness. Suppose that an at most countable set \( \mathcal{F}_0 \) generates \( \mathcal{F} \). Let \( \mathcal{F}_1 \) be the family of all sets which are obtained after finitely many steps of effecting Boolean operations on the sets of \( \mathcal{F}_0 \). Since \( \mathcal{F}_0 \) is at most countable, \( \mathcal{F}_1 \) is at most countable as well. Let \( \mathcal{F}_2 \) be the closure of \( \mathcal{F}_1 \) in \( \mathcal{F} \) with respect to the semi-metric \( d_\mu \). The sets from \( \mathcal{F}_2 \) are precisely the sets which are approximable by sets from \( \mathcal{F}_1 \). Since the measure \( \mu \) is finite, the complement of an approximable set is again an approximable set. Also, countable unions of approximable sets are also approximable. Hence, \( \mathcal{F}_2 \) is a \( \sigma \)-subalgebra in \( \mathcal{F} \). Since \( \mathcal{F}_2 \) contains \( \mathcal{F}_0 \), we conclude \( \mathcal{F}_2 = \mathcal{F} \), and so \( \mathcal{F} \) is separable as it is the closure of the at most countable set \( \mathcal{F}_1 \). \( \square \)

In Theorem 5.6 we will extend Proposition 5.3 to order continuous Banach function spaces over semi-finite measures. First we need a general result about semi-finite measures.
Lemma 5.4 Let $(\Omega, \mathcal{F}, \mu)$ be a semi-finite measure space which is not $\sigma$-finite. Then there exists an uncountable disjoint family of measurable sets in $\Omega$ of finite positive measure.

Proof Let $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be the family of all families of measurable pairwise disjoint subsets of $\Omega$ with finite positive measure. Since $\mu$ is semi-finite, the family $\mathcal{A}$ is non-empty. The family $\mathcal{A}$ ordered by set inclusion is a partially ordered set. If $\mathcal{C}$ is a chain in $\mathcal{A}$, then it is obvious that the union of $\mathcal{C}$ is again in $\mathcal{A}$. Hence, by Zorn’s lemma there exists a maximal element $\mathcal{M}$ in $\mathcal{A}$.

Suppose that $\mathcal{M}$ is at most countable. Since $E := \bigcup_{F \in \mathcal{M}} F$ is a measurable subset of $\Omega$ and since $\mu$ is not $\sigma$-finite, the set $\Omega \setminus E$ has positive measure. Due to the semi-finiteness of $\mu$ there exists a measurable set $F \subseteq \Omega \setminus E$ with finite positive measure. This is in contradiction with the maximality of $\mathcal{M}$. \qed

For a function space $X$ over a measure space $(\Omega, \mathcal{F}, \mu)$ and a measurable set $A \in \mathcal{F}$ we define the set $X_A = \{ \lambda f : f \in X \}$. Since $X$ is an ideal in $L^0(\mu)$, $X_A$ is an ideal in $X$, so that $X_A$ is a function space in $L^0(\mu)$. If $X$ is a Banach function space, then $X_A$ is a Banach function space as well. Note that $X_A$ is order continuous when $X$ is, and $X_A$ is order dense in $L^0(\mu|A)$ when $X$ is order dense in $L^0(\mu)$. By $\mathcal{F}_A$ we denote the relative $\sigma$-algebra on $A$, i.e., the $\sigma$-algebra of all measurable sets contained in $A$. Whenever $\mu(A) < \infty$, the restriction $d_{\mu|A}$ of $d_\mu$ on $\mathcal{F}_A$ induces a semi-metric.

If $X$ is an order continuous order dense Banach function space over a $\sigma$-finite measure space, then [16, Theorem 5.2] yields that $\tau_{\text{un}}$ agrees with the topology $\tau_\mu$ of local convergence in measure. An examination of its proof actually reveals that one can replace $\sigma$-finite measure spaces with semi-finite ones. Recall that a net $(f_\lambda)$ converges to $f$ in $L^0(\mu)$ with respect to the topology of local convergence in measure if $f_\lambda X_A \to f X_A$ in measure for each subset $A \in \mathcal{F}$ with finite measure.

Lemma 5.5 Let $(\Omega, \mathcal{F}, \mu)$ be a semi-finite measure space and let $X$ be an order dense order continuous Banach function space in $L^0(\mu)$. Then a net $(f_\alpha)$ in $X$ $\tau_{\text{un}}$-converges to zero if and only if it converges to zero with respect to the topology of local convergence in measure.

Proof For the proof of the forward implication, pick a set $A$ of finite measure. Then $f X_A \in X_A$, and since $f_\alpha X_A$ $\tau_{\text{un}}$-converges to zero in $X$, it also $\tau_{\text{un}}$-converges to zero in $X_A$. Since $\mu(A) < \infty$, [16, Theorem 5.2] yields that $(f_\alpha X_A)$ converges to zero in measure on $A$. This proves the forward implication.

For the proof of the backward implication, assume that $(f_\alpha)$ converges to zero with respect to the topology of local convergence in measure. As in the proof of [16, Theorem 5.2] let $Z$ be the set of all functions in $X$ which vanish outside of a set of finite measure. We claim that $Z$ is an order dense ideal in $X$. The fact that $Z$ is an ideal in $X$ is clear. To prove that $Z$ is order dense in $X$, pick a non-zero function $f \in X^+$. Then there exists $n \in \mathbb{N}$ such that the set $A_n := \{ x \in \Omega : f(x) \geq \frac{1}{n} \}$ has a positive measure. Since $\mu$ is semi-finite, there exists a measurable set $A \subseteq A_n$ of finite positive measure. Hence, $\chi_A \leq \chi_{A_n} \leq nf$ and since $X$ is an ideal in $L^0(\mu)$, we have that $\chi_A \in X$. This yields that $\chi_A \in Z$ and so $Z$ is order dense in $X$.

Now we claim that $|f_\alpha| \wedge g \to 0$ in norm for each $g \in Z^+$. Since $g$ can be approximated by an increasing sequence of step functions and since $X$ is order continuous, it
suffices to consider the case when $g$ itself is a step function. Furthermore, an application of the triangle inequality for the norm implies that we need to consider only the case when $g = \chi_A$ for some set $A$ of positive finite measure. Since $|f_\alpha| \wedge \chi_A \to 0$ in measure on $A$, by [16, Theorem 5.2] we have that $|f_\alpha| \wedge \chi_A = (|f_\alpha| \wedge \chi_A) \wedge \chi_A \to 0$ in norm in $X_A$ and so in $X$. This proves that $|f_\alpha| \wedge g \to 0$ in norm for each $g \in Z^+$.

To conclude the proof, due to the standard approximation argument we need to show that $Z$ is dense in $X$. Since $X$ is order continuous, it suffices to prove that $Z$ is order dense in $X$. Pick a positive vector $f \in X^+$ and find a step function $s \in X^+$ such that $0 < s \leq f$. Then there exists a measurable set $A$ and a scalar $\alpha > 0$ such that $0 < \alpha \chi_A \leq s \leq f$. Since $\mu$ is semi-finite, there is a subset $A'$ of $A$ of finite positive measure, from where it follows $0 < \alpha \chi_{A'} \leq f$. To conclude the proof, note that we have $\chi_{A'} \in Z$.

It should be noted that Lemma 5.5 follows from the combination of [6, Proposition 4.10] and [6, Theorem 6.3] which is proved more generally.

The following two theorems are the main results of this section. They provide a characterization of separability of order dense order continuous Banach function spaces.

**Theorem 5.6** Let $(\Omega, \mathcal{F}, \mu)$ be a semi-finite measure space and let $X$ be an order dense order continuous Banach function space in $L^0(\mu)$. The following statements are equivalent.

1. $X$ is separable.
2. $X$ is separable with respect to the topology of local convergence in measure.
3. The measure $\mu$ is $\sigma$-finite and the semi-metric space $(\mathcal{F}_A, d_{\mu|A})$ is separable for each set $A$ of finite measure.
4. The measure $\mu$ is $\sigma$-finite and the Banach function space $X_A$ is separable for each set $A$ of finite measure.
5. $X$ admits almost everywhere strictly positive functions and the metric space $(C_f, \rho)$ is separable for any strictly positive function $f \in X$.

**Proof** While the equivalence (1)$\iff$(2) follows from Theorem 3.2 and Lemma 5.5, the equivalence (3)$\iff$(4) follows from Proposition 5.3.

(1)$\implies$(4) Assume that $\mu$ is not $\sigma$-finite. By Lemma 5.4 there exists an uncountable family $M$ of pairwise disjoint sets of finite positive measure. Since $X$ is order dense in $L^0(\mu)$, for each $E \in M$ there exists a function $g_E \in X$ such that $0 < g_E \leq \chi_E$. It is clear that functions $g_E$ and $g_E'$ are disjoint whenever $E \cap E' = \emptyset$. For each $n \in \mathbb{N}$ we introduce the family of functions in $X$ as

$$\mathcal{F}_n := \{g_E : E \in M \text{ and } \|g_E\|_X \geq \frac{1}{n}\}.$$ 

Since $M$ is uncountable, there exists $m \in \mathbb{N}$ such that $\mathcal{F}_m$ is uncountable. For each $g_E \in \mathcal{F}_m$ consider the open ball $U_E$ with center in $g_E$ and radius $\frac{1}{2m}$. We claim that for different sets $E$ and $E'$ the balls $U_E$ and $U_{E'}$ are disjoint. Indeed, if $f$ is in their intersection, then $\|g_E - g_E'\| \leq \|g_E - f\| + \|f - g_E'\| < \frac{1}{m}$. However, this is in contradiction with

$$\|g_E - g_E'\| = \|g_E - g_{E'}\| = \|g_E + g_{E'}\| \geq \|g_E\| \geq \frac{1}{m}.$$
Since $X$ admits an uncountable family of pairwise disjoint open sets, it cannot be separable.

To prove the remaining claim of (4), pick a set $A$ of finite measure in $\Omega$ and note that separability of the normed space $X$ passes down to the Banach function space $X_A$.

(4)$\Rightarrow$(1) Since $\mu$ is $\sigma$-finite, there exists an increasing sequence $(A_n)_{n \in \mathbb{N}}$ of sets of finite positive measure such that $\bigcup_{n=1}^{\infty} A_n = \Omega$. Since each space $X_{A_n}$ is separable by assumption, a simple topological argument yields separability of $Y := \bigcup_{n=1}^{\infty} X_{A_n}$.

We claim that $Y$ is dense in $X$. To this end, it suffices to prove that $Y^+$ is dense in $X^+$. Pick a function $f \in X^+$. Then the sequence $(f \chi_{A_n})_{n \in \mathbb{N}}$ is increasing pointwise to $f$, so that $f_n \to f$ in order. Since $X$ is order continuous, we conclude that $f_n \to f$ in norm which proves the claim. Separability of $X$ follows now from separability of $Y$ and its density in $X$.

(1)$\iff$(5) If $X$ is separable, then (1)$\Rightarrow$(3) implies that the measure $\mu$ is $\sigma$-finite, so that by [1, Corollary 5.22] the function space $X$ contains a function $f$ which is strictly positive almost everywhere on $\Omega$. Separability is inherited by $(C_f, \rho)$.

On the other hand, if $X$ admits a function $f$ which is strictly positive almost everywhere on $\Omega$, then $f$ is a quasi-interior point in $X$. To finish the proof we apply Theorem 4.2. $\square$

The equivalent statements (1) and (3) from Theorem 5.6 yield the following dichotomous result about separability of order continuous order dense Banach function spaces.

**Corollary 5.7** Let $(\Omega, \mathcal{F}, \mu)$ be a semi-finite measure space. Then either all order continuous order dense Banach function spaces are separable or none is.

Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space and let $X$ be an ideal in $L^0(\mu)$. If there exists a smallest measurable subset $C_X \subseteq \Omega$ with respect to $\mu$-almost everywhere set inclusion such that every $f \in X$ vanishes $\mu$-almost everywhere on $\Omega \setminus C_X$, then this set is called the carrier (or the support) of $X$. If $\mu$ is $\sigma$-finite, then for each ideal $X$ in $L^0(\mu)$ the carrier $C_X$ exists by [1, Theorem 1.92]. For a general measure, the carrier of an ideal may not exist. Nevertheless, every separable Banach function space has the carrier as the following lemma shows.

**Lemma 5.8** Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space. Then for every separable Banach function space $X$ in $L^0(\mu)$ its carrier $C_X$ exists.

**Proof** Since $X$ is separable, $X$ contains a weak unit $\varphi$. Let

$$A := \{ x \in \Omega : \varphi(x) > 0 \}.$$

Since for each $g \geq 0$ in $X$ we have $g \wedge n\varphi \uparrow g$ for almost every $x \in X$, the function $g$ is zero almost everywhere on $X \setminus A$ and so $g \in X_A$. This proves that $X = X_A$. Since $\varphi$ is strictly positive on $A$, the set $A$ is the carrier of $X$. $\square$

The existence of the carrier of a separable Banach function space $X$ allows us to characterize separable Banach function spaces which are not necessarily order dense.
Theorem 5.9 For an order continuous Banach function space over a semi-finite measure the following assertions are equivalent.

(1) $X$ is separable.
(2) $X$ is separable with respect to the topology of local convergence in measure and the carrier $C_X$ is $\sigma$-finite.

Proof (1) $\Rightarrow$ (2) Since $X$ is separable, by Lemma 5.8 and its proof the carrier $C_X$ of $X$ exists, and it is equal to the set $A = \{ x \in \Omega : \varphi(x) > 0 \}$ for any weak unit $\varphi$ in $X$. We claim that $X$ is order dense in $L^0(\mu|_A)$. To see this, pick any function $0 < f \in L^0(\mu|_A)$. Since $f$ is non-zero, there exists $n \in \mathbb{N}$ such that the set $B := \{ x \in A : f(x) \geq \frac{1}{n} \}$ has a positive measure. Then $\chi_B \leq nf$, and since $\varphi \wedge \chi_B \neq 0$, from $0 \leq \frac{1}{n}(\varphi \wedge \chi_B) \leq f$ it follows that $X_A$ is order dense in $L^0(\mu|_A)$. Since $X = X_A$ is separable and $\mu|_A$ is semi-finite, by Theorem 5.6 we conclude that the measure $\mu|_A$ is $\sigma$-finite and that $X = X_A$ is separable with respect to the topology of local convergence in measure on $A$. We conclude the proof of (2) by observing that each function in $X$ is zero almost everywhere on $\Omega \setminus A$.

(2) $\Rightarrow$ (1) The same argument as in the proof of the previous implication shows that $X = X_A$ is order dense in $L^0(\mu|_A)$, Hence, by Theorem 5.6 we conclude that $X = X_A$ is separable.

For a Banach function space $X$ over a $\sigma$-finite measure space $(\Omega, \mathcal{F}, \mu)$ Luxemburg proved in [22, Theorem 46.4] that $(\mathcal{F}, d_\mu)$ is separable whenever $X$ is separable. Moreover, he also proved in [22, Theorem 46.5] that $X^\sigma$ is separable whenever $(\mathcal{F}, d_\mu)$ is separable. In particular, if $X = X^\sigma$, that is, if $X$ is order continuous, then $X$ is separable if and only if the semi-metric space $(\mathcal{F}, d_\mu)$ is separable. Therefore, Theorem 5.6 and Theorem 5.9 are extensions of aforementioned Luxemburg’s results to Banach function spaces over semi-finite measure spaces.

The one-dimensional Banach function space $L^\infty(\mu)$ over a singleton set with infinite measure shows that neither Theorem 5.6 nor Theorem 5.9 need hold when the underlying measure space is not semi-finite.

The equivalence between (1) and (3) in Theorem 5.6 directly shows that for a semi-finite measure $\mu$ the space $L^p(\mu)$ is separable if and only if the $L^q(\mu)$ is separable for $1 \leq p, q < \infty$. While this result in the case of finite measures can be found in [4, Exercise 4.7.63], the general case follows from [11, Theorem 366B] and [11, Exercise 366X(c)]. Since the solutions are not provided, we include our own proof for the general case as an application of the developed theory.

Corollary 5.10 For every measure space $(\Omega, \mathcal{F}, \mu)$ and $1 \leq p, q < \infty$ the Banach lattice $L^p(\mu)$ is separable if and only if $L^q(\mu)$ is separable.

Proof Suppose that $L^p(\mu)$ is separable. Then $L^p(\mu)$ contains a weak unit $\varphi$. As before, let us denote the set $\{ x \in \Omega : \varphi(x) > 0 \}$ by $A$. It is easy to see that $A$ is $\sigma$-finite. Since $\varphi$ is a weak unit in $L^p(\mu)$, we have $L^p(\mu) = L^p(\mu|_A)$, and since the function $\varphi^\frac{p}{q}$ is a weak unit in $L^q(\mu)$ we also have $L^q(\mu) = L^q(\mu|_A)$. The result follows from Corollary 5.7 for the measurable space $(A, \mathcal{F}|_A, \mu|_A)$.

So far the results in this section provided a connection between separability of a given Banach function space and separability of the underlying measure space. In the
following proposition we establish a suitable topological version for \( L^p \)-spaces where \( 1 \leq p < \infty \). Before we formulate and prove it, we recall some standard terminology from Measure Theory. Given a topological space \( \Omega \), by \( B_\Omega \) we denote the Borel \( \sigma \)-algebra which is generated by all open sets from \( \Omega \). Any measure defined on \( B_\Omega \) is called a Borel measure. Furthermore, a Borel measure \( \mu \) is outer regular if for each Borel set \( E \) in \( \Omega \) we have

\[
\mu(E) = \inf\{\mu(U) : U \supseteq E, U \text{ is open}\}.
\]

**Proposition 5.11** Let \( \Omega \) be a second countable topological space. If \((\Omega, B_\Omega, \mu)\) is a measure space with an outer regular Borel measure \( \mu \), then \( L^p(\mu) \) is separable for each \( 1 \leq p < \infty \).

**Proof** By Corollary 5.10 it is enough to prove that \( L^1(\mu) \) is separable. If there are no sets of positive finite measure in \( \Omega \), then \( L^p(\mu) = \{0\} \). Otherwise, let \( \mathcal{W} \) be an at most countable basis for \( \Omega \) and let \( \mathcal{W}' \) be the family of all finite unions of elements of \( \mathcal{W} \) of finite measure. We will show that the non-empty countable set

\[
S = \left\{ \sum_{W \in K} q_w \chi_W : K \subset \mathcal{W}' \text{ is a finite set and } q_w \in \mathbb{Q} \right\}
\]

is dense in \( L^1(\mu) \). Let \( f \in L^1(\mu) \) and \( \varepsilon > 0 \). Then there exists a step function \( \sum_{k=1}^n q_k \chi_{A_k} \in L^1(\mu) \) with pairwise disjoint sets \( A_1, \ldots, A_n \) such that \( \|f - \sum_{k=1}^n q_k \chi_{A_k}\|_1 < \frac{\varepsilon}{2} \) where \( q_k \in \mathbb{Q}\setminus\{0\} \). Since \( \mu \) is outer regular and \( \mu(A_k) < \infty \) there exists an open set \( V_k \supseteq A_k \) such that \( \mu(V_k \setminus A_k) < \frac{\varepsilon}{2^{k+1} \max(1, |q_k|)} \). Since \( V_k \) is an at most countable union of elements from \( \mathcal{W} \), there exists \( W_k \in \mathcal{W}' \), such that \( W_k \subseteq V_k \) and \( \mu(V_k \setminus W_k) < \frac{\varepsilon}{2^{k+1} \max(1, |q_k|)} \). Then

\[
\|\chi_{A_k} - \chi_{W_k}\|_1 = \mu(A_k \setminus W_k) + \mu(W_k \setminus A_k) \leq \mu(V_k \setminus W_k) + \mu(V_k \setminus A_k) < \frac{\varepsilon}{2^{k+1} \max(1, |q_k|)}.
\]

and therefore

\[
\left\| f - \sum_{k=1}^n q_k \chi_{W_k} \right\|_1 \leq \left\| f - \sum_{k=1}^n q_k \chi_{A_k} \right\|_1 + \left\| \sum_{k=1}^n q_k \chi_{A_k} - \sum_{k=1}^n q_k \chi_{W_k} \right\|_1 < \varepsilon.
\]

\[ \square \]

The following example shows that there exist a topological space that is not second countable, yet the corresponding \( L^1 \)-space is separable.

**Example 5.12** Consider the Sorgenfrey line \( \mathbb{R}_S \), i.e., the real line \( \mathbb{R} \) equipped with the topology defined by the basis \( \{(a, b) : a, b \in \mathbb{R}\} \). It is well known that \( \mathbb{R}_S \) is separable but not second countable. The corresponding Borel \( \sigma \)-algebra \( B_S \) is the standard Borel \( \sigma \)-algebra on \( \mathbb{R} \), and hence, \( L^1(\mathbb{R}_S, B_S, m) \) is separable.
In comparison with the Sorgenfrey line one can also find a non-separable topological space \( \Omega \) for which the function space \( L^1(\mu) \) is still separable.

**Example 5.13** Let \( \tau_E \) be the Euclidean topology on \( \mathbb{R} \) and let 

\[
\mathcal{B} = \{U \setminus C : U \in \tau_E \text{ and } C \text{ is at most countable}\}
\]

be a basis of a topology \( \tau \) on \( \mathbb{R} \). Let \( \mathcal{B}_\tau \) and \( \mathcal{B}_E \) be the Borel \( \sigma \)-algebras generated by \( \tau \) and \( \tau_E \), respectively. Because \( \tau \) is stronger than \( \tau_E \), we have \( \mathcal{B}_E \subseteq \mathcal{B}_\tau \). On the other hand, every set \( U \setminus C \) is in \( \mathcal{B}_E \), so that \( \mathcal{B} \subseteq \mathcal{B}_E \), and therefore, \( \mathcal{B}_\tau \subseteq \mathcal{B}_E \). As in the previous example we have that \( L^1(\mathbb{R}, \mathcal{B}_\tau, m) \) is separable. However, the space \( (\mathbb{R}, \tau) \) is not separable, since for every at most countable set \( D \) we have \( D \cap (\mathbb{R} \setminus D) = \emptyset \) and \( \mathbb{R} \setminus D \in \tau \).

### 6 Some remarks on the normality of the unbounded topology

We conclude this paper with this short section where we consider the question under which conditions a given Banach lattice \( X \) equipped with its unbounded norm topology is a normal topological space. Since metrizable spaces are normal, \( (X, \tau_{un}) \) is certainly normal in this case. By [18, Theorem 4.3], if \( X \) is a normed lattice, then \( \tau_{un} \) is metrizable if and only if \( X \) contains a countable quasi-interior set. In particular, if \( X \) is a Banach lattice with a quasi-interior point, then \( (X, \tau_{un}) \) is metrizable.

The following theorem whose proof is a combination of results from general topology, topological vector spaces and the unbounded norm topology provides a particular case when \( (X, \tau_{un}) \) is a paracompact space. In order to do that we first recall the following notions. A Banach lattice \( X \) is a **KB-space** whenever every increasing norm bounded sequence in \( X^+ \) converges in norm. A Hausdorff topological space is **paracompact** whenever every open cover of the space has a locally finite refinement.

**Theorem 6.1** If \( X \) is an atomic KB-space, then \( (X, \tau_{un}) \) is a paracompact topological space.

**Proof** Since \( X \) is an atomic KB-space, the closed unit ball \( B_X \) is un-compact by [17, Theorem 7.5], so that \( (X, \tau_{un}) \) is a \( \sigma \)-compact topological space. By [7], the unbounded norm topology on \( X \) is a linear Hausdorff topology, so that \( (X, \tau_{un}) \) is regular (see e.g. [29, 1.3 Corollary] or [15, Theorem 21.5]). Since \( (X, \tau_{un}) \) is \( \sigma \)-compact, it is clearly a Lindelöf space, so that paracompactness of the unbounded norm topology finally follows from Morita’s theorem [10, Theorem VIII.6.5].

Since paracompact Hausdorff spaces are normal by [10, Theorem VIII.2.2], by Theorem 6.1 every atomic KB-space equipped with the un-topology is normal.

**Corollary 6.2** If \( X \) is an atomic KB-space, then \( (X, \tau_{un}) \) is a normal topological space.

The proof of Theorem 6.1 heavily depends on Morita’s theorem from general topology. It would be of interest to find a functional analytical proof. We conclude the paper with the following open questions.
Problem 6.3 Under which conditions on a Banach lattice $X$ is the topological space $(X, \tau_{un})$ normal?

Problem 6.4 Under which conditions on a Banach lattice $X$ is the topological space $(X, \tau_{un})$ paracompact?

At the time of writing, the answers appear to be unknown.

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