GOODNESS-OF-FIT ANALYSIS OF RADIAL VELOCITY SURVEYS

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ABSTRACT

Using eigenmode expansion of the Mark III and SFI surveys of cosmological radial velocities, a goodness-of-fit analysis is applied on a mode-by-mode basis. This differential analysis complements the Bayesian maximum likelihood analysis that finds the most probable model given the data. Analyzing the surveys with their corresponding most likely models from the CMB-like family of models, as well as with the currently popular ΛCDM model, reveals a systematic inconsistency of the data with these “best” models. There is a systematic trend of the cumulative χ² to increase with the mode number (where the modes are sorted by decreasing order of the eigenvalues). This corresponds to a decrease of the χ² with the variance associated with a mode and hence with its effective scale. It follows that the differential analysis finds that on small (large) scales the global analysis of all the modes “puts” less (more) power than actually required by the data. This observed trend might indicate one of the following: (1) the theoretical model (i.e., power spectrum) or the error model (or both) have an excess of power on large scales, (2) velocity bias, or (3) the velocity data suffers from systematic errors that have not yet been corrected.

Subject headings: cosmology: observations — cosmology: theory — galaxies: distances and redshifts — large-scale structure of universe — methods: statistical

1. INTRODUCTION

Surveys of radial velocities of galaxies have played a major role in the study of the large-scale structure. The analysis of such surveys has been conducted in two main directions: the mapping of the local cosmography and the estimation of the cosmological parameters (see Dekel 1994 for a review). The Bayesian framework provides one with very elegant and powerful tools for conducting both the mapping and parameter estimation, where the recovery of the large-scale structure is done by means of the Wiener filter and the parameters are estimated by maximum likelihood analysis (Zaroubi et al. 1995). In the case in which the deviations from a homogeneous and isotropic universe constitute a Gaussian random field, the Wiener filter and the maximum likelihood analysis are the optimal tools for performing such an analysis (Zaroubi et al. 1995). Indeed, the MARK III catalog of radial velocities (Willick et al. 1995, 1996, 1997) has been recently analyzed by Wiener filtering (Zaroubi, Hoffman, & Dekel 1999) and by maximum likelihood (Zaroubi et al. 1997). The SFI survey of da Costa et al. (1996) has been studied by maximum likelihood analysis by Freudling et al. (1999) and by Wiener filtering (Y. Hoffman & S. Zaroubi, unpublished). Both surveys seem to yield similar results.

In the Bayesian maximum likelihood analysis, one calculates the posterior probability of a model being correct given the data (Zaroubi et al. 1995; Vogeley & Szalay 1996). Thus, the model that maximizes the likelihood function, over a given parameter (or model) space, is the most likely model in that space. The maximum likelihood analysis cannot guarantee, however, that the most probable model is indeed consistent with the data. It provides only a relative measure for models to be correct. It is common to adopt an independent measure for the goodness of fit, which is often given by the requirement that the reduced χ² is close to unity. Often, when the most likely model (given the data) also passes the goodness-of-fit test, one assumes that the “correct” model has been nailed down.

Here, the χ² test is expanded and a much more critical test is suggested and then applied to the Mark III and SFI surveys. For SFI we adopt the catalog version used in Freudling et al. (1997), and for Mark III we use the catalog version described in Willick et al. (1997). Both catalogs have been Malmquist bias corrected. The reader is referred to the above-mentioned papers for a detailed description of the data sets.

The χ² “goodness of fit” is based on the assumption that all the random variables that affect the observables are normally distributed. In the cosmological context, this applies to both the underlying dynamical (e.g., density and velocity) field and the statistical errors. Thus, for a survey containing \( N \) data observables (e.g., radial velocities), the \( χ² \) of the system of \( N \) degrees of freedom (dof) is calculated given the model that maximizes the likelihood function. The goodness of fit is measured by how close the \( χ²/dof \) is to unity. This provides a global measure for the consistency of the data with the model, since it includes all the observables. A situation might occur of some “conspiracy” in which different parts of the data deviate from the predictions of the model, but when combined together they “conspire” to yield a reasonable \( χ² \). A much stronger test on the model is to decompose the data into statistical independent eigenmodes and observe the \( χ² \) behavior of the independent modes. Eigenmode analysis, also known as principal component analysis (PCA) and the Karhunen-Loeve transform, is not a new tool in the field. It has been applied to studies of redshift surveys (Vogeley & Szalay 1996), the cosmic microwave background (Bunn 1997; Bond 1995), and more recently to radial velocities surveys (Hoffman 2000). The latter study is extended here to perform the goodness-of-fit test on a mode-by-mode basis. The basic formalism is presented in § 2, and its application to the Mark III and SFI surveys is given in § 3. Our results are discussed and the conclusions are summarized in § 4.

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2. EIGENMODE ANALYSIS OF RADIAL VELOCITIES

Consider a database of radial velocities \( \{u_i\}_{i=1, \ldots, N} \), where

\[
u_i = \mathbf{v}(\mathbf{r}_i) \cdot \hat{\mathbf{r}}_i + \epsilon_i,
\]

\( \mathbf{v} \) is the three-dimensional velocity, \( \mathbf{r}_i \) is the position of the \( i \)th data point, and \( \epsilon_i \) is the statistical error associated with the \( i \)th radial velocity. The assumptions made here are of a cosmological model which well describes the data, that systematic errors have been properly dealt with, and that the statistical errors are well understood. The data autocovariance matrix is then written as

\[
R_{gg} \equiv \langle u_i u_j \rangle = \mathbf{r}_i \cdot \langle \mathbf{v}(\mathbf{r}_i) \mathbf{v}(\mathbf{r}_j) \rangle \hat{r}_i + \sigma_i^2.
\]

(Here angle brackets denote an ensemble average.) The last term is the error covariance matrix. The velocity covariance tensor that enters this equation was derived by Górski (1988; see also Zaroubi et al. 1999), and it depends on the power spectrum and cosmological parameters.

The eigenmodes of the data covariance matrix provide a natural representation of the data:

\[
R n^{(i)} = \lambda_i n^{(i)}. \]

The set of \( N \) eigenmodes \( \{n^{(i)}\} \) constitutes an orthonormal basis, and the eigenvalues \( \lambda_i \) are arranged in decreasing order (in absolute values). A new representation of the data is given by

\[
\tilde{a}_i = n^{(i)} u_i. \tag{4}
\]

This provides a statistical orthogonal representation, namely

\[
\langle \tilde{a}_i \tilde{a}_j \rangle = \delta_{ij}. \tag{5}
\]

The normalized transformed variables are defined by

\[
a_i = \frac{\tilde{a}_i}{\sqrt{\lambda_i}}. \tag{6}
\]

Equation (5) is written now as

\[
\langle a_i a_j \rangle = \delta_{ij}. \tag{7}
\]

Note that since the modes are statistically independent one can measure the \( \chi^2 \) of a given mode, \( \chi_i^2 = \tilde{a}_i^2 \), and the cumulative reduced \( \chi^2 \) is given by

\[
\chi_{\hat{u}}^2 = \frac{1}{M} \sum_{i=1}^M a_i^2. \tag{8}
\]

For normally distributed errors and a Gaussian random velocity field, the \( a_i \) are normally distributed with zero mean and a variance of unity.

In addition, the probability of finding such \( \chi_{\hat{u}}^2 \) is calculated as well. The probability is defined by

\[
P_{\chi_{\hat{u}}^2}(M) = \begin{cases} P_{\chi^2}(MX_{\hat{u}}^2, M) & \text{for } P_{\chi^2}(MX_{\hat{u}}^2, M) < 0.5, \\ 1 - P_{\chi^2}(MX_{\hat{u}}^2, M) & \text{otherwise,} \end{cases} \tag{9}
\]
where $P(x; M)$ is the probability that a random variable drawn from a $\chi^2$ distribution with $M$ degrees of freedom is less than a given value $x$.

3. DIFFERENTIAL $\chi^2$ ANALYSIS

Here the goodness of fit of the Mark III and SFI surveys is studied. The models studied here are the maximum likelihood solutions for these surveys, which are slightly different from one another. The most likely model given Mark III is a tilted-CDM of $Q_0 = 0.4$, $h = 0.6$, and $n = 10$ where $Q_0$ is the cosmological density parameter, $h$ is Hubble’s constant in units of $100$ km s$^{-1}$ Mpc$^{-1}$, and $n$ is the power spectrum index (Zaroubi et al. 1997). The most likely model given SFI is an open CDM of $Q_0 = 0.79$, $h = 0.6$, and $n = 0.92$ (Friedling et al. 1999). For both cases, the maximum likelihood best model has a total $\chi^2$ very close to unity. Thus, from the point of view of the integral $\chi^2$, the maximum likelihood solutions seem to be very consistent with the data. This is extended to perform a differential $\chi^2$ analysis, namely to study the $\chi^2$ behavior across the modes spectrum.

To study the robustness of this probe, it is first applied to a linear mock catalog of Mark III, constructed from an unconstrained realization of the velocity field. This field is sampled at the location of the Mark III data points, to which normally distributed errors are added according to Mark III’s error covariance matrix. The cumulative $\chi^2$ of such a catalog should oscillate around unity, given that the model used to generate the catalog is known. Indeed, this has been confirmed by an analysis of a few linear mock catalogs of Mark III. The probabilities of obtaining such a $\chi^2$ distribution lies comfortably within the 90% confidence level. The nontrivial result of this test is that the very poor sampling of the long-wavelength Fourier waves, i.e., cosmic variance, does not affect the goodness-of-fit test.

The differential $\chi^2$ and its associated probability of the Mark III and SFI surveys are presented in Figure 1, each case analyzed in its maximum likelihood solution. A clear trend is noticed: namely, over almost the entire mode spectrum the cumulative $\chi^2$ increases monotonically. When all modes are included, the total $\chi^2$/dof is indeed close to 1, but if we had to take half the modes, starting from the top or the bottom, a very different $\chi^2$ would have been obtained.

The differential $\chi^2$ analysis is repeated for the currently popular model of $\Lambda$CDM ($Q_0 = 0.4$, $h = 0.6$, and $n = 1$; Fig. 2). Indeed, the same trend is found in this case as well, but the total $\chi^2$ converges to a value outside the 90% confidence level.

The conclusion that follows is that for both data sets (Mark III and SFI) and for a variety of theoretical models, the differential $\chi^2$ increases monotonically with the mode number (with the exception of the first 10 modes of the Mark III). The theoretical expectation is that if indeed the data is consistent with the assumed model, then $\chi^2_M$ will fluctuate around unity. The probability of observing such a trend given a model is very small across most of the mode number range.

4. DISCUSSION

What have we learned from the differential $\chi^2$ analysis? It has been found that even the most probable CDM-like model, the one that maximizes the likelihood function given the data, is not fully consistent with the data. The cumulative $\chi^2$ has been calculated both downward and upward (namely starting from the modes with the largest and smallest eigenmode, respectively). Over more than 90% of the modes, the cumulative $\chi^2$ lies well outside the 90% confidence level, indicating a very small probability of measuring such data given the assumed model. Over most of the mode number range, $\chi^2_M$ increases monotonically. It is this behavior of the $\chi^2$ that indicates a systematic inconsistency of the model with the data. The assumed model actually contains two ingredients, the theoretical power spectrum and the error model. However, the present
analysis cannot indicate which one is to be “blamed” for the systematic trend. It should be noted here that apart from the first few (10–20) modes, there is a clear correlation of the eigenvalues with its weighted mean distance (of data points of the given mode). Namely, the variance associated with a mode (i.e., its eigenvalue) increases with its mean distance (I. Zehavi 1999, private communication; L. Silverman et al. 2000, in preparation). It follows that the $\chi^2$ trend seen here is closely correlated with the distance and that the data “asks” for less power on large scales than the model (power spectrum and noise) provides. A detailed study of the power spectrum and error model possible modifications is to be given elsewhere (Silverman et al. 2000, in preparation). (Note that these first 10–20 modes are the ones dominated by the underlying velocity field and not the noise [Hoffman 2000].)

The cosmological implications of the present findings are that either the error and/or the theoretical model need to be modified. The theoretical model assumed in the analysis of large-scale radial velocity surveys is that the velocities are drawn from a Gaussian random field defined by a given power spectrum. The present study might indicate the inconsistency of the power spectrum with the data. A less likely possibility is that it indicates a departure from the Gaussian statistics. Alternatively, the present work might indicate a systematic error that has not been accounted for that causes this trend, e.g., spatial error correlations. Still another possibility is that of an indication for a velocity bias.

The conclusions reached here should not be taken as a contradiction of the results of Zaroubi et al. (1997) and Fruedling et al. (1999), but rather as extending and complementing them. The Bayesian maximum likelihood analysis can be performed only within the assumed parameter/model space. The differential $\chi^2$ allows one to go beyond this and analyze the nature of the agreement, or the lack of it, between a given model and the data on a mode-by-mode basis.

The PCA transforms the data to a statistically independent representation and enables the study of the compatibility of the data with the model on a mode-by-mode basis. This differential analysis is in contrast to the more traditional approach in which a data set is analyzed as a whole. The differential $\chi^2$ analysis should be performed together with the Bayesian maximum likelihood analysis and complement it. This should be useful in fields in which the maximum likelihood analysis is the basic tool of analysis such as the mapping of the CMB angular fluctuations and the study of redshift surveys as well as all radial velocity surveys. The present analysis can prove to be very useful and powerful in those fields in which systematic errors play crucial roles, such as redshift and radial velocity surveys.

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