Cyclic homology and quantum orbits

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Abstract

A natural isomorphism between the homological invariants of the cyclic objects that correspond to the homogeneous quotient coalgebra-Galois extensions, and those of the cyclic objects associated to module coalgebras and SAYD modules is obtained as a homological counterpart of the Takeuchi-Galois correspondence between the left coideal subalgebras and the quotient right module coalgebras. A Pontryagin type self-duality of this correspondence is revealed, which is combined with the cyclic duality of Connes to obtain dual results about the invariant cyclic homology with SAYD coefficients of the algebras of invariants in homogeneous quotient coalgebra-Galois extensions.

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∗The paper was partially supported by the NCN grant 2011/01/B/ST1/06474
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1 Introduction

It is well known due to Takeuchi [28] (vastly extended by van Oystaeyen-Zhang [29] and Schauenburg [27]) that given a Hopf algebra $\mathcal{H}$ there is a Galois 1-1 correspondence between its left coideal subalgebras, for which $\mathcal{H}$ is a faithfully flat algebra extension, and its quotient right module coalgebras, for which $\mathcal{H}$ is a faithfully coflat coalgebra coextension.

This can be viewed as a relation between some extensions of comodule algebras and some coextensions of module coalgebras. Both related subjects have their specific homological invariants (Hochschild, cyclic, periodic cyclic and negative cyclic homology and homology with coefficients) computed from appropriate cyclic objects.

On one hand, for algebra extensions, it is a relative cyclic object introduced by Kadison [21] and for comodule algebras it is a cyclic object with stable anti-Yetter-Drinfeld (SAYD) coefficients introduced by Hajac-Khalkhali-Rangipour-Sommerhäuser [19] [18], and independently by Jara-Ştefan [20] (in a cyclic dual version).

On the other hand, for module coalgebras, it is a cyclic object with SAYD coefficients which is cyclic dual to the cocyclic object also introduced in [18], and for coalgebra extensions it is the cyclic dual of the Pontryagin dual analogue of the Kadison’s cyclic object.

Therefore it is very natural to ask whether these different types of cyclic objects are related in the context of the aforementioned Takeuchi-Galois correspondence.

This question goes far beyond Galois theory (herein the theory of so called homogeneous quotient coalgebra-Galois extensions [9]) and reaches topology. Adapted to the case of smooth functions on compact Lie groups and their homogeneous spaces, relative periodic cyclic homology computes the vector bundle of de Rham cohomology of the stabilizer over the homogeneous space. This bundle is equipped with the Gauss-Manin connection (see [15] for noncommutative fibrations with commutative base) determining a local system of coefficients whose cohomology appears in the second page of the Leray spectral sequence interpolating between cohomology of the compact Lie group and cohomology of the homogeneous space. Then the above Takeuchi-Galois 1-1 correspondence boils down to that one between stabilizers and orbits, and the algebra extension describes the orbital map.

On the other hand, the cyclic object with SAYD coefficients of a module coalgebra is cyclic dual to the cocyclic object generalizing the one used by Connes and Moscovici in the proof of a generalized transversal local index theorem for foliated spaces, where the index computation relies on a symmetry governed by a Hopf algebra [12] [18].

The aim of the present paper is to prove in the aforementioned context of Takeuchi-Galois correspondence that, after choosing appropriate SAYD coefficients, the corresponding two types of homological invariants become isomorphic in a natural way.

It is worth noticing that even in the case of such a basic homogeneous $\mathcal{H}$-Galois extension as $k \to \mathcal{H}$ computing the left hand side, being then Hochschild homology of...
the Hopf algebra itself, is nontrivial and of fundamental interest \[14, 16, 17, 5, 10, 2\]. Using our result we give an independent proof of the classical computation of Feng-Tsygan \[14\], reproved by another argument by Bichon \[2\]. Therefore our natural isomorphism can be viewed as a generalization of their results to the case of arbitrary homogeneous coalgebra-Galois extensions.

The Podleś quantum deformation of the Hopf fibration by circles of a 3-sphere over a 2-sphere, carrying at the same time two Pontryagin dual structures, provides an example of a homogeneous coalgebra-Galois extension of algebras, and a co-homogeneous algebra-Galois coextension of coalgebras \[7\], illustrating the Pontryagin duality principle discussed here.

2 Preliminaries

In this section we recall the material we will use in the sequel. In the first subsection we recall coalgebra-Galois extensions. In the second subsection we recall the Pontryagin like duality between discrete and linearly compact vector spaces and after recalling the Takeuchi-Galois correspondence between left comodule subalgebras and right module quotient coalgebras of a Hopf algebra we show Pontryagin self-duality of this correspondence. In the third subsection we recall the relative cyclic homology of algebra extensions. Finally, the Hopf-cyclic homology of coalgebras with coefficients is recalled in the fourth subsection.

2.1 Coalgebra-Galois extensions

In this subsection we recall the definition and basic properties of coalgebra-Galois extensions from \[9\].

Let \( C \) be a coalgebra, and \( A \) an algebra. Let \( A \) also be a right \( C \)-comodule via \( \rho : A \to A \otimes C \), which we denote by \( \rho(a) = a_{(0)} \otimes a_{(1)} \). Then the coaction invariants of \( A \) is defined to be the algebra of endomorphisms of \( A \) regarded as a right \( A \)-module right \( C \)-comodule

\[
A^{coC} := \text{End}_A^C(A) = \{ b \in A | \rho(ba) = b\rho(a) \}. \tag{2.1}
\]

It follows, from the second equality, that \( A^{coC} \) is a subalgebra of \( A \).

An algebra extension \( B \to A \) is called a \( C \)-extension if \( B = A^{coC} \). Finally, a \( C \)-extension \( B \to A \) is said to be Galois if the left \( A \)-linear right \( C \)-colinear map

\[
\text{can} : A \otimes_B A \to A \otimes C, \quad a \otimes_B a' \mapsto aa'_{(0)} \otimes a'_{(1)} \tag{2.2}
\]

is bijective. The map \(2.2\) is called the canonical map of the extension.

An interesting, due to the geometric examples it covers, class of coalgebra-Galois extensions is the quotient coalgebra-Galois extensions. In this setting, one lets \( \mathcal{H} \) to
be a Hopf algebra, $\mathcal{I} \subseteq \mathcal{H}$ a coideal right ideal, and $\mathcal{A}$ a right $\mathcal{H}$-comodule algebra via $\nabla : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{H}$. Then the composition

$$\rho : \mathcal{A} \xrightarrow{\nabla} \mathcal{A} \otimes \mathcal{H} \xrightarrow{\Delta} \mathcal{A} \otimes \mathcal{H} / \mathcal{I}$$

(2.3)

determines a right $\mathcal{H} / \mathcal{I}$-coaction on $\mathcal{A}$. Finally, a Galois $\mathcal{H} / \mathcal{I}$-extension $\mathcal{A}^{\text{co} \mathcal{H} / \mathcal{I}} \rightarrow \mathcal{A}$ is called a quotient coalgebra-Galois extension. An important property of quotient coalgebra-Galois extensions shared with Hopf-Galois extensions is that the algebra of invariants of the coaction can be expressed diagrammatically in vector spaces as

$$\mathcal{A}^{\text{co} \mathcal{H} / \mathcal{I}} = Eq(\mathcal{A} \Rightarrow \mathcal{A} \otimes \mathcal{H} / \mathcal{I})$$

(2.4)

where one arrow in the parallel pair in the equalizer is $\rho$ defined as in (2.3), and another one is the composition

$$\mathcal{A} \xrightarrow{\cong} \mathcal{A} \otimes k \xrightarrow{\Delta \otimes \eta} \mathcal{A} \otimes \mathcal{H} \xrightarrow{\Delta \otimes \pi} \mathcal{A} \otimes \mathcal{H} / \mathcal{I}.$$  

(2.5)

The quantum instanton bundle introduced in [4] is a quotient coalgebra-Galois extension, [3], and it uses the full generality of the quotient coalgebra-Galois extensions: $\mathcal{A} \neq \mathcal{H}$ and $\mathcal{I} \neq 0$.

For $\mathcal{I} = 0$, the quotient coalgebra-Galois extensions recover the Hopf-Galois extensions, and in case of $\mathcal{A} = \mathcal{H}$ they are called homogeneous coalgebra-Galois extensions. In particular, viewing $\mathcal{H}$ as a right $\mathcal{H}$-comodule algebra via its comultiplication, one obtains the homogeneous $\mathcal{H} / \mathcal{H}$-Galois extensions

$$\rho : \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{\Delta \otimes \pi} \mathcal{H} \otimes \mathcal{H} / \mathcal{I}, \quad h \mapsto h_{(1)} \otimes h_{(2)}. \quad (2.6)$$

In the context of faithfully flat homogeneous extensions the diagrammatical definition of invariants of the coaction [2-4] plays a role in the context of Pontryagin self-duality of the Takeuchi-Galois correspondence considered in the next subsection.

Such Galois extensions correspond to the quantum homogeneous spaces considering $\mathcal{H} = \mathcal{O}(G)$ as the algebra of functions on a quantum group $G = \text{Spec}(\mathcal{H})$, and $\mathcal{B} = \mathcal{O}(X)$ as the algebra of functions on a quantum space $X = \text{Spec}(\mathcal{B})$. Then the $G$-action on $X$ is encoded by the $\mathcal{H}$-coaction, [13 Sect. 1]. There is a celebrated example of this construction due to Podleś [6, 25] which is a quantum spherical fibration $SU_q(2) \rightarrow S_{q,\mu,\nu}^2$, see [7], a quantum deformation of the classical Hopf fibration of a 3-sphere over a 2-sphere into circles.

### 2.2 Formal Pontryagin duality and Galois-Takeuchi correspondence

In this subsection we will first summarize the basic facts on the dualization functor on vector spaces from [1]. We will then recall a one-to-one correspondence between the coideal subalgebras and quotient coalgebras, known as Galois-Takeuchi correspondence [28].
From the point of view of linear topological vector spaces with continuous linear mappings as morphisms, the dualization functor defines an equivalence between the opposite category of discrete vector spaces and the category of linearly compact vector spaces [1, Prop. 24.8]. Moreover, it transforms naturally the algebraic tensor product of discrete vector spaces into a completed one of linearly compact spaces [1, Cor. 24.25]. In other words, dualization is a strong monoidal functor. Therefore one can regard on equal footing all structures defined by diagrams in vector spaces together with their dual counterparts obtained by reversing all arrows in all necessary diagrams. This regards linear subspaces and quotient spaces, (co)algebras and Hopf algebras, as well as their bi(co)modules or one sided (co)modules, one-sided and two-sided (co)ideals, and their (co)tensor products. Hence, having a diagrammatical proof of a theorem in a symmetric monoidal category of linear (topological) spaces, one has automatically a dual theorem after an appropriate dualization of the structures. It is well known that the notion of Hopf algebra is self-dual. In particular, it transforms the Hopf group-algebra of a discrete group into a linearly compact topological Hopf algebra of functions on that group, customary regarded as a group algebra of a dual compact quantum group. Therefore we will call this duality Pontryagin to distinguish it from cyclic duality which will appear later on. We will show that the notion of SAYD module is Pontryagin self-dual up to interchanging left and right. According to [7, 8] the notion of coalgebra-Galois extension of algebras is Pontryagin dual to the notion of algebra-Galois coextension of coalgebras. Both dualities play a role in the present paper.

For every Hopf algebra, with the multiplication $\mu : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$, the unit $\eta : k \to \mathcal{H}$, the comultiplication $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$, the counit $\varepsilon : \mathcal{H} \to k$, and the antipode $S : \mathcal{H} \to \mathcal{H}$, Takeuchi introduces [28] the one-to-one (Galois) correspondence

$$\{ \mathcal{B} \subseteq \mathcal{H} \mid \mathcal{B} \text{ is a left coideal subalgebra, } \mathcal{H} \text{ is faithfully flat over } \mathcal{B} \}$$

$$\uparrow \downarrow$$

$$\{ \mathcal{H}/\mathcal{I} \mid \mathcal{I} \text{ is a coideal right ideal, } \mathcal{H} \text{ is faithfully coflat over } \mathcal{H}/\mathcal{I} \},$$

where $\mathcal{B} \mapsto \mathcal{H}/\mathcal{B}^+\mathcal{H}$ and $\mathcal{H}/\mathcal{I} \mapsto \mathcal{H}^{\text{co}\mathcal{H}/\mathcal{I}}$. Let us denote by $\mathcal{A}$ the underlying left $\mathcal{H}$-comodule algebra and by $\mathcal{D}$ the underlying right $\mathcal{H}$-module coalgebra of the Hopf algebra $\mathcal{H}$. Then the above correspondence can be written categorically as an equivalence between the category of left $\mathcal{H}$-comodule flat extensions $i : \mathcal{B} \to \mathcal{A}$ and the category of right $\mathcal{H}$-module coflat coextensions $\pi : \mathcal{D} \to \mathcal{C}$, which is given as

$$\mathcal{B} \mapsto \text{Coeq}(\mathcal{H} \rightrightarrows \mathcal{B} \otimes \mathcal{H}), \quad \mathcal{C} \mapsto \text{Eq}(\mathcal{H} \rightrightarrows \mathcal{H} \otimes \mathcal{C}), \quad (2.7)$$

where the parallel pair of left arrows in the coequalizer and the parallel pair of right arrows in the equalizer read as composites

$$\mathcal{H} \leftarrow \mathcal{H} \otimes \mathcal{H} \xrightarrow{\otimes \mathcal{H}} \mathcal{B} \otimes \mathcal{H}, \quad \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{\mathcal{H} \otimes \eta} \mathcal{H} \otimes \mathcal{C},$$

$$\mathcal{H} \xrightarrow{\varepsilon} k \otimes \mathcal{H} \xrightarrow{\varepsilon \otimes \mathcal{H}} \mathcal{B} \otimes \mathcal{H}, \quad \mathcal{H} \xrightarrow{\varepsilon} \mathcal{H} \otimes k \xrightarrow{\mathcal{H} \otimes \eta} \mathcal{H} \otimes \mathcal{C},$$

respectively. It is hence evident that this correspondence is Pontryagin self-dual up to an interchange of left and right in dual structures.
2.3 Relative cyclic homology of algebra extensions

In this subsection we recall the homological preliminaries that will be needed in the sequel. More precisely, we shall first include a very brief summary of the homological bar complex and its relative version, as well as their relations with Hochschild homology \[22, 24\]. We will then provide with a short discussion on the relative cyclic homology of algebras from \[20, 26\].

Let \( A \) be an associative algebra, \( X \) a right \( A \)-module, and \( Y \) a left \( A \)-module. Then,

\[
CB_s(X, A, Y) := \bigoplus_{n \geq 0} CB_n(X, A, Y), \quad CB_n(X, A, Y) := X \otimes A \otimes^n Y \tag{2.8}
\]

together with the differential \( d : CB_n(X, A, Y) \rightarrow CB_{n-1}(X, A, Y) \) given by

\[
d(x \otimes a_1 \otimes \ldots \otimes a_n \otimes y) = x \cdot a_1 \otimes a_2 \otimes \ldots \otimes a_n \otimes y + \sum_{j=1}^{n-1} (-1)^j x \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes y + (-1)^n x \otimes a_1 \otimes \ldots \otimes a_{n-1} \otimes a_n \cdot y, \tag{2.9}
\]

for \( n \geq 1 \), is called the two sided bar complex. If \( i : B \rightarrow A \) is a morphism of associative unital algebras, then \( A \) becomes a \( B \)-bimodule via the map \( i : B \rightarrow A \), and we can consider the relative bar complex

\[
CB_s(X, A | B, Y) := \bigoplus_{n \geq 0} CB_n(X, A | B, Y), \quad CB_n(X, A | B, Y) := X \otimes_B A \otimes^n_B Y \tag{2.10}
\]

with the differential \( (2.9) \).

Let \( A \) be an algebra, and \( M \) an \( A \)-bimodule. Then the Hochschild homology of \( A \) with coefficients in \( M \), denoted by \( H_s(A, M) \), is defined to be the homology of the complex

\[
C(A, M) = \bigoplus_{n \geq 0} C_n(A, M), \quad C_n(A, M) := M \otimes A \otimes^n \tag{2.11}
\]

with the differential

\[
b(m \otimes a_1 \otimes \ldots \otimes a_n) = m \cdot a_1 \otimes a_2 \otimes \ldots \otimes a_n + \sum_{k=1}^{n-1} (-1)^k m \otimes a_1 \otimes \ldots \otimes a_k a_{k+1} \otimes \ldots \otimes a_n + (-1)^{n+1} a_n \cdot m \otimes a_1 \otimes \ldots \otimes a_{n-1}. \tag{2.12}
\]

We will denote by \( HH_s(A) \) the Hochschild homology of the algebra \( A \), with coefficients in itself. If \( A \) is flat as a left \( B \)-module and as a right \( B \)-module, then we call such an extension \( B \rightarrow A \) flat, and the Hochschild homology of the extension can be computed from the bar complex \( (2.8) \). Indeed, \( A^e := A \otimes A^{\text{op}} \) being the enveloping algebra of \( A \),

\[
H_n(A, M) = \text{Tor}_n^{A^e}(M, A) = H_n(CB_s(M, A^e, A), d).
\]
We proceed with a quick detour on the relative cyclic homology of algebra extensions. To this end we shall first need to recall the notion of cyclic tensor product from [26]. The cyclic tensor product of $B$-bimodules $\mathcal{M}_1, \ldots, \mathcal{M}_n$ is defined by

$$\mathcal{M}_1 \circ_B \cdots \circ_B \mathcal{M}_n := (\mathcal{M}_1 \otimes_B \cdots \otimes_B \mathcal{M}_n) \otimes_B B.$$  \hfill (2.13)

As it is remarked in [20], for any $B$-bimodule $\mathcal{M}$ one has $\mathcal{M} \otimes_B B \cong [\mathcal{M}]_B$, where $[\mathcal{M}]_B = \mathcal{M}/[\mathcal{M}, B]$ and $[\mathcal{M}, B]$ is the subspace of $\mathcal{M}$ generated by all commutators $[m, b] := m \cdot b - b \cdot m$.

The relative cyclic homology of an algebra extension $B \to A$ is computed by the cyclic object

$$C_n(A \mid B) := A \hat{\otimes}_B n + 1,$$  \hfill (2.14)

equipped, for all $n \geq 0$, with the morphisms

$$d_i : C_n(A \mid B) \longrightarrow C_{n-1}(A \mid B), \quad 0 \leq j \leq n$$

$$d_i(a^0 \hat{\otimes}_B a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^n) =$$

$$\begin{cases} 
  a^0 a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^n & i = 0 \\
  a^0 \hat{\otimes}_B a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^{j+1} \hat{\otimes}_B \cdots \hat{\otimes}_B a^n & 1 \leq i \leq n - 1 \\
  a^n a^0 \hat{\otimes}_B a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^{n-1} & i = n,
\end{cases}$$

$$s_j : C_n(A \mid B) \longrightarrow C_{n-1}(A \mid B), \quad 0 \leq j \leq n - 1$$

$$s_j(a^0 \hat{\otimes}_B \cdots \hat{\otimes}_B a^n) = a^0 \hat{\otimes}_B a^1 \hat{\otimes}_B \cdots \hat{\otimes}_B a^{j+1} \hat{\otimes}_B a^j \hat{\otimes}_B \cdots \hat{\otimes}_B a^n$$

and

$$t : C_n(A \mid B) \longrightarrow C_n(A \mid B),$$

$$t(a^0 \hat{\otimes}_B \cdots \hat{\otimes}_B a^n) = a^n \hat{\otimes}_B a^0 \hat{\otimes}_B \cdots \hat{\otimes}_B a^{n-1}.$$  \hfill (2.17)

Hochschild homology of the complex (2.14) is called the Hochschild homology of the extension, and is denoted by $HH_s(A \mid B)$. Similarly, cyclic (resp. periodic cyclic, negative cyclic) homology of the cyclic object (2.14) is called the relative cyclic (resp. periodic cyclic, negative cyclic) homology of the extension $B \to A$, and it is denoted by $HC_s(A \mid B)$ (resp. $HP_s(A \mid B)$, $HN_s(A \mid B)$).

### 2.4 Hopf-cyclic homology of $\mathcal{H}$-module coalgebras

In this subsection we recall the relative Hopf-cyclic homology with coefficients, for coalgebras, using the cyclic duality principle [11 23].

Let us first recall the definition of a left-right stable anti-Yetter-Drinfeld module (SAYD module) over a Hopf algebra $\mathcal{H}$ from [19]. Let a linear space $\mathcal{M}$ be a left $\mathcal{H}$-module by $\lambda : \mathcal{H} \otimes \mathcal{M} \longrightarrow \mathcal{M}$ given by $\lambda(h \otimes m) = h \cdot m$, and a right $\mathcal{H}$-comodule via $\rho : \mathcal{M} \longrightarrow \mathcal{M} \otimes \mathcal{H}$ given by $\rho(m) = m_{(0)} \otimes m_{(1)}$. Then $\mathcal{M}$ is called a left-right
anti-Yetter-Drinfeld module (AYD module) over \( \mathcal{H} \) if the \( \mathcal{H} \)-action and \( \mathcal{H} \)-coaction are compatible as

\[
\rho(h \cdot m) = h_{(2)} \cdot m_{(0)} \otimes h_{(3)} m_{(1)} S(h_{(1)}). \tag{2.18}
\]

Similarly, \( \mathcal{M} \) is called stable if

\[
m_{(1)} \cdot m_{(0)} = m. \tag{2.19}
\]

On the other hand, a right-left SAYD module is defined by

\[
(m \cdot h)_{(-1)} \otimes (m \cdot h)_{(0)} = S(h_{(3)}) \cdot m_{(-1)} \otimes h_{(1)} m_{(0)} h_{(2)}, \tag{2.20}
\]

\[
m_{(0)} \cdot m_{(-1)} = m. \tag{2.21}
\]

Note that these conditions are Pontryagin dual to each other. Indeed, written as commutativity of diagrams in the category \( \mathcal{V}ect \) of vector spaces (\( \sigma \) being the transposition of the corresponding tensorands) the left-right SAYD module compatibility reads

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{M} \otimes \mathcal{H} & \xrightarrow{\Delta^3 \otimes \rho} & \mathcal{H} \otimes \mathcal{M} \otimes \mathcal{H} \otimes \mathcal{H} \\
\mathcal{H} \otimes \mathcal{M} & \xrightarrow{\lambda} & \mathcal{M} & \xrightarrow{\rho} & \mathcal{M} \otimes \mathcal{H},
\end{array}
\tag{2.22}
\]

where the upper horizontal arrow admits two decompositions

\[
\begin{array}{ccc}
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{M} \otimes \mathcal{H} & \xrightarrow{\sigma_{H,H} \otimes H \otimes \sigma_{H,M} \otimes S} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{M} \otimes \mathcal{H} \\
\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{M} \otimes \mathcal{H} & \xrightarrow{S \otimes H \otimes \sigma_{H,M} \otimes \mathcal{H}} & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{M} \otimes \mathcal{H},
\end{array}
\tag{2.23}
\]

and

\[
\begin{array}{ccc}
\mathcal{M} \otimes \mathcal{H} & \xrightarrow{\sigma_{M,H}} & \mathcal{H} \otimes \mathcal{M} \\
\mathcal{M} & \xrightarrow{\rho} & \mathcal{M} & \xrightarrow{\lambda} & \mathcal{M},
\end{array}
\tag{2.24}
\]

Now we see that reversing the arrows and interchanging the pairs \( \Delta \) and \( \mu \), \( \lambda \) and \( \rho \), and finally left and right, we obtain the right-left SAYD module compatibility.

Let us now recall the Hopf-cyclic cohomology of a Hopf-module coalgebra with coefficients. Let \( \mathcal{H} \) be a Hopf algebra, \( \mathcal{C} \) a right \( \mathcal{H} \)-module coalgebra, \( \text{i.e.,} \), a right \( \mathcal{H} \)-module such that

\[
\Delta(c \cdot h) = c_{(1)} \cdot h_{(1)} \otimes c_{(2)} \cdot h_{(2)}, \quad \varepsilon(c \cdot h) = \varepsilon(c) \varepsilon(h), \quad \forall c \in \mathcal{C}, \forall h \in \mathcal{H}, \tag{2.25}
\]

and \( \mathcal{M} \) a left-right SAYD module over \( \mathcal{H} \). Then the collection of vector spaces

\[
C^n(\mathcal{C}, \mathcal{M})_\mathcal{H} := \mathcal{C} \otimes \mathcal{H} \otimes \mathcal{M} \tag{2.26}
\]
for all \( n \geq 0 \), is a cocyclic module with the operators
\[
\delta_i : C^{n-1}(\mathcal{C}, \mathcal{M})_{\mathcal{H}} \to C^n(\mathcal{C}, \mathcal{M})_{\mathcal{H}}, \quad 0 \leq i \leq n
\]
\[
\delta_i(c^0 \otimes \ldots \otimes c^{n-1} \otimes_{\mathcal{H}} m) = c^0 \otimes \ldots \otimes c^{i(1)} \otimes \delta_i(c^{i(2)} \otimes \ldots \otimes c^{n-1} \otimes_{\mathcal{H}} m), \quad 0 \leq i \leq n - 1
\]
\[
\delta_n(c^0 \otimes \ldots \otimes c^{n-1} \otimes_{\mathcal{H}} m) = c^0 \otimes \ldots \otimes c^{n-1} \otimes c^{0(1)} \cdot S^{-1}(m_{(\langle}) \otimes_{\mathcal{H}} m_{(\rangle)},
\]
\[
(2.27)
\]
\[
\sigma_j : C^{n+1}(\mathcal{C}, \mathcal{M})_{\mathcal{H}} \to C^n(\mathcal{C}, \mathcal{M})_{\mathcal{H}}, \quad 0 \leq i \leq n
\]
\[
\sigma_j(c^0 \otimes \ldots \otimes c^{n+1} \otimes_{\mathcal{H}} m) = c^0 \otimes \ldots \otimes \varepsilon(c^{i+1}) \otimes \ldots \otimes c^{n+1} \otimes_{\mathcal{H}} m,
\]
\[
(2.28)
\]
and
\[
\tau : C^n(\mathcal{C}, \mathcal{M})_{\mathcal{H}} \to C^n(\mathcal{C}, \mathcal{M})_{\mathcal{H}}
\]
\[
\tau(c^0 \otimes \ldots \otimes c^n \otimes_{\mathcal{H}} m) = c^1 \otimes \ldots \otimes \varepsilon \otimes \ldots \otimes c^{n+1} \otimes_{\mathcal{H}} m_{(\langle}) \otimes_{\mathcal{H}} m_{(\rangle)},
\]
\[
(2.29)
\]
Cyclic cohomology of this cocyclic module is called the Hopf-cyclic cohomology of the \( \mathcal{H} \)-module coalgebra \( \mathcal{C} \) with coefficients in the left-right SAYD module \( \mathcal{M} \) over \( \mathcal{H} \), and is denoted by \( HC^*(\mathcal{C}, \mathcal{M})_{\mathcal{H}} \).

Applying the cyclic duality procedure \cite{11, 23} we obtain on \((2.26)\) the cyclic module structure given by the faces
\[
d_i : C_{n+1}(\mathcal{C}, \mathcal{M})_{\mathcal{H}} \to C_n(\mathcal{C}, \mathcal{M})_{\mathcal{H}}, \quad 0 \leq i \leq n + 1
\]
\[
d_i(c^0 \otimes \ldots \otimes c^{n+1} \otimes_{\mathcal{H}} m) = c^0 \otimes \ldots \otimes \varepsilon(c^i) \otimes \ldots \otimes c^{n+1} \otimes_{\mathcal{H}} m,
\]
\[
(2.30)
\]
degeneracies
\[
s_i : C_{n-1}(\mathcal{C}, \mathcal{M})_{\mathcal{H}} \to C_n(\mathcal{C}, \mathcal{M})_{\mathcal{H}}, \quad 0 \leq i \leq n - 1
\]
\[
s_i(c^0 \otimes \ldots \otimes c^{n-1} \otimes_{\mathcal{H}} m) = c^0 \otimes \ldots \otimes c^{i(1)} \otimes c^{i(2)} \otimes \ldots \otimes c^{n-1} \otimes_{\mathcal{H}} m,
\]
\[
(2.31)
\]
and the cyclic operator
\[
t : C_n(\mathcal{C}, \mathcal{M})_{\mathcal{H}} \to C_n(\mathcal{C}, \mathcal{M})_{\mathcal{H}}
\]
\[
t(c^0 \otimes \ldots \otimes c^n \otimes_{\mathcal{H}} m) = c^n \cdot m_{(\langle}) \otimes \ldots \otimes c^{n-1} \otimes_{\mathcal{H}} m_{(\rangle)}
\]
\[
(2.32)
\]
Similarly, the cyclic homology of this cyclic module is called the cyclic homology of the \( \mathcal{H} \)-module coalgebra \( \mathcal{C} \), with coefficients in the left-right SAYD module \( \mathcal{M} \) over \( \mathcal{H} \), and is denoted by \( HC_* (\mathcal{C}, \mathcal{M})_{\mathcal{H}} \).

3 Relative cyclic homology as Hopf-cyclic homology with coefficients

In this section we achieve our main result identifying the cyclic homology of a homogeneous \( \mathcal{C} \)-Galois extension \( \mathcal{B} \to \mathcal{H} \) with the Hopf-cyclic homology, with coefficients, of the right \( \mathcal{H} \)-module coalgebra \( \mathcal{C} \). Then, by the Pontryagin duality, we automatically have the identification of the cyclic homology of a \( \mathcal{B} \)-Galois coextension \( \mathcal{H} \to \mathcal{C} \) with the cyclic homology of the Hopf-cyclic homology, with (the Pontryagin dual) coefficients, of the \( \mathcal{H} \)-comodule algebra \( \mathcal{B} \). We shall conclude the section developing spectral sequences to shed further light on the relative homology groups.
3.1 Isomorphisms of cyclic objects

In this subsection we will construct an explicit isomorphism from the relative homology complex of a homogeneous \( C := H/I \)-Galois extension \( B := H^\text{coH}/I \rightarrow H \) to the Hopf-cyclic homology complex of the \( H \)-module coalgebra \( C = H/I \). Thus, in view of the Pontryagin duality sketched in Section 2.2, we will automatically have an isomorphism between the relative homology complex of a \( B \)-Galois coextension \( D := H \rightarrow C := H/B^+H \) and the Hopf-cyclic homology complex of the \( H \)-comodule algebra \( B \).

Let \( H \) be a Hopf algebra and \( I \subseteq H \) a coideal right ideal of \( H \). Then \( H/I \) becomes a coaugmented quotient coalgebra in a canonical way,

\[
\Delta(h) := \overline{h}_{(1)} \otimes \overline{h}_{(2)}, \quad \varepsilon(h) := \varepsilon(h), \quad \forall h \in H,
\]

where \( \overline{h} := h + I \), and the canonical coaugmentation of \( H/I \) is given by the group-like \( \overline{1} \).

Let \( B \rightarrow H \) be a homogeneous \( H/I \)-Galois extension given by the canonical right \( H/I \)-coaction

\[
\xymatrix{ H \ar[r]^\Delta & H \otimes H \ar[r] & H \otimes (H/I), \quad h \mapsto h_{(1)} \otimes \overline{h}_{(2)}}
\]

on \( H \). In view of the definition \( B := H^\text{coH}/I \), which reads as \( b_{(1)} \otimes \overline{b}_{(2)} = b \otimes \overline{1} \), applying the counit to the left tensorands we deduce \( \overline{b} = \varepsilon(b) \overline{1} \). The latter is written as \( \overline{1}(b - \varepsilon(b)1) = 0 \), and proves for \( B^+ := B \cap \ker(\varepsilon) \) a containment \( B^+H \subseteq I \) of right ideals. By the left coideal property of \( B \subseteq H \) (see for instance [3]), i.e.,

\[
\Delta(B) \subseteq H \otimes B,
\]

one has

\[
\Delta((b - \varepsilon(b)1)h) = b_{(1)}h_{(1)} \otimes (b_{(2)} - \varepsilon(b_{(2)})h_{(1)}) + (b - \varepsilon(b)1)h_{(1)} \otimes h_{(2)} \in B \otimes H + H \otimes B.
\]

On the other hand, it is clear that \( \varepsilon(B^+H) = 0 \), hence \( B^+H \) is a coideal in \( H \).

Now,

\[
\begin{align*}
S(((b - \varepsilon(b)1)h)_{(1)} \otimes_B ((b - \varepsilon(b)1)h)_{(2)}) &= S((bh)_{(1)} \otimes_B (bh)_{(2)} - \varepsilon(b)S(h_{(1)}) \otimes_B h_{(2)}) \\
&= S(b_{(1)}h_{(1)} \otimes_B b_{(2)}h_{(2)} - \varepsilon(b)S(h_{(1)}) \otimes_B h_{(2)}) \\
&= S(h_{(1)})S(b_{(1)} \otimes_B b_{(2)}h_{(2)} - \varepsilon(b)S(h_{(1)}) \otimes_B h_{(2)}) \\
&= S(h_{(1)})S(b_{(1)} \otimes_B b_{(2)} - \varepsilon(b)S(h_{(1)}) \otimes_B 1)h_{(2)} \\
&= S(h_{(1)})S(b_{(1)})h_{(2)} \otimes_B 1 - \varepsilon(b)S(h_{(1)}) \otimes_B 1)h_{(2)} = 0.
\end{align*}
\]

Therefore if \( I = B^+H \) the map

\[
H \otimes C \longrightarrow H \otimes_B H, \quad h \otimes g \mapsto hS(g_{(1)}) \otimes_B g_{(2)},
\]

is a coaugmented coaction for which the canonical counit holds, i.e.,

\[
\varepsilon(hS(g_{(1)}) \otimes_B g_{(2)}) = \varepsilon(hg_{(1)}) \otimes_B \varepsilon(g_{(2)}).
\]
is well defined and provides the inverse to the canonical map

$$\text{can} : \mathcal{H} \otimes B \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{C}, \quad h \otimes_B g \mapsto hg_{(1)} \otimes g_{(2)}$$  \hspace{1cm} (3.6)

hence the extension \( \mathcal{B} \to \mathcal{H} \) is a \( \mathcal{H}/\mathcal{I} \)-Galois extension. The canonical map and its inverse can be inductively extended to \( \mathcal{H} \)-bimodule isomorphisms

$$\text{can}_n : \mathcal{M} \otimes_B \mathcal{H}^{\otimes B^n} \longrightarrow \mathcal{M} \otimes \mathcal{C}^{\otimes n},$$

$$\text{can}_n(m \otimes_B h^1 \otimes_B \ldots \otimes_B h^n) = mh^1(1) \otimes_B h^1_{(1)} \otimes_B \ldots \otimes_B h^n(1) \otimes_B h^n_{(1)} \otimes_B h^n_{(n+1)}, \quad \text{and}$$

$$\text{can}_n^{-1} : \mathcal{M} \otimes \mathcal{C}^{\otimes n} \longrightarrow \mathcal{M} \otimes_B \mathcal{H}^{\otimes B^n},$$

$$\text{can}_n^{-1}(m \otimes_g g^1 \otimes_g \ldots \otimes_g g^n) = mS(g^1(1)) \otimes_B g^1(2)S(g^2(1)) \otimes_B \ldots \otimes_B g^n(1) \otimes_B g^n(2),$$

respectively, for any \( \mathcal{H} \)-bimodule \( \mathcal{M} \).

Next, recall for the translation map

$$\tau : \mathcal{H}/\mathcal{I} \longrightarrow \mathcal{H} \otimes_B \mathcal{H}$$

$$\tau_k \mapsto S(h_{(1)}) \otimes_B h_{(2)}$$  \hspace{1cm} (3.9)

that \( \tau(\mathcal{H}/\mathcal{I}) \subset (\mathcal{H} \otimes_B \mathcal{H})^B \subset \mathcal{H} \otimes_B \mathcal{H} \), that is,

$$bS(h_{(1)}) \otimes_B h_{(2)} = S(h_{(1)}) \otimes_B h_{(2)} b$$  \hspace{1cm} (3.10)

for any \( h \in \mathcal{H} \). The latter implies also that under the isomorphism

$$\mathcal{H} \otimes_B \mathcal{H} \cong \mathcal{H} \otimes_B \mathcal{H}, \quad [h \otimes h'] \mapsto h' \otimes_B h$$  \hspace{1cm} (3.11)

we have

$$[h_{(2)} \otimes BS(h_{(1)}) - bh_{(2)} \otimes S(h_{(1)})]_B \mapsto bS(h_{(1)}) \otimes_B h_{(2)} - S(h_{(1)}) \otimes_B h_{(2)} b = 0, \quad (3.12)$$

and hence

$$[h_{(2)} b \otimes S(h_{(1)})]_B = [h_{(2)} \otimes BS(h_{(1)})]_B,$$  \hspace{1cm} (3.13)

making the left \( \mathcal{H} \)-action

$$h \triangleright [h']_B := [h_{(2)} h' S(h_{(1)})]_B$$  \hspace{1cm} (3.14)

on \( \mathcal{H} \otimes_B \mathcal{H} \) well-defined. We then have

$$[h_{(2)} \triangleright [h' h_{(1)}]_B = [h_{(2)}(2) h' h_{(1)} S(h_{(2)}(1))]_B = [h_{(3)} h' h_{(1)} S(h_{(2)})]_B$$

$$= [h_{(2)} h' \varepsilon(h_{(1)})]_B = [hh']_B.$$  \hspace{1cm} (3.15)

We also recall that \( \mathcal{H} \otimes_B \mathcal{H} \), equipped with the (left) \( \mathcal{H} \)-action (3.14) and the (right) \( \mathcal{H} \)-coaction given by the comultiplication, is a left-right SAYD module over \( \mathcal{H} \), see [20] Ex. 4.3. All that is used in the following lemma.
Lemma 3.1. Let $\mathcal{B} \to \mathcal{H}$ be a homogeneous $\mathcal{H}/\mathcal{I}$-Galois extension. Then for any $n \geq 0$ we have the isomorphism of vector spaces $[\mathcal{H}^{\otimes B} n+1]_B \cong (\mathcal{H}/\mathcal{I})^{\otimes n+1} \otimes \mathcal{B} [\mathcal{H}]_B$ implemented by

$$\varphi_n : [\mathcal{H}^{\otimes B} n+1]_B \xrightarrow{\cong} (\mathcal{H}/\mathcal{I})^{\otimes n+1} \otimes \mathcal{B} [\mathcal{H}]_B$$

$$\varphi_n (\left[ b^0 \otimes_B \cdots \otimes_B b^n \right]_B) = \left( h^1 (2) \cdots h^n (2) \otimes \cdots \otimes h^{n-1} (n) h^n (n) \otimes h^{n+1} (n+1) \otimes I \right) \otimes \mathcal{B} [h^0 h^1 (1) \cdots h^n (1)]_B$$

with the inverse

$$\psi_n : (\mathcal{H}/\mathcal{I})^{\otimes n+1} \otimes \mathcal{B} [\mathcal{H}]_B \xrightarrow{\cong} [\mathcal{H}^{\otimes B} n+1]_B$$

$$\psi_n \left( g^0 \otimes \cdots \otimes g^n \right) \otimes \mathcal{B} [h]_B) = [ g^0 (2) h S (g^0 (1)) \otimes_B g^0 (2) g^1 (1) \otimes_B \cdots \otimes_B g^n (2) S (g^n (1))]_B.$$  

Proof. First of all, we have to prove that the maps are well defined. Let us begin with (3.16). We observe that

$$\varphi_n (\left[ bh^0 \otimes_B \cdots \otimes_B h^n \right]_B) = \left( h^1 (2) \cdots h^n (2) \otimes \cdots \otimes h^{n-1} (n) h^n (n) \otimes h^{n+1} (n+1) \otimes I \right) \otimes \mathcal{B} [bh^0 h^1 (1) \cdots h^n (1)]_B = \left( h^1 (2) \cdots h^n (2) \otimes \cdots \otimes h^{n-1} (n) h^n (n) \otimes h^{n+1} (n+1) \otimes I \right) \otimes \mathcal{B} [h^0 h^1 (1) \cdots h^n (1) h^1 (b_1)]_B = \left( h^1 (2) \cdots h^n (2) \otimes \cdots \otimes h^{n-1} (n) h^n (n) \otimes h^{n+1} (n+1) \otimes I \right) \mathcal{B} [h^0 h^1 (1) \cdots h^n (1) b_1]_B = \left( h^1 (2) \cdots h^n (2) \otimes \cdots \otimes h^{n-1} (n) h^n (n) \otimes h^{n+1} (n+1) \otimes I \right) \mathcal{B} [h^0 h^1 (1) \cdots h^n (1) b_1]_B = \varphi_n (\left[ h^0 \otimes_B \cdots \otimes_B h^n \right]_B).$$  

(3.18)
As for (3.17), on the other hand, we have

\[
\psi_n \left( (g^0 \otimes \cdots \otimes g^n) \otimes \mathcal{H} \left[ bh \right]_B \right) = \\
[g^n \otimes h S(g^0(1)) \otimes B g^0(2) S(g^1(1)) \otimes B \cdots \otimes B g^n(1) S(g^n(1)) \otimes B g^{n-2}(2) S(g^{n-1}(1)) \otimes B g^n(1) S(g^n(1)) \otimes B]_B = \\
[g^n \otimes h S(g^0(1)) \otimes B g^0(2) S(g^1(1)) \otimes B \cdots \otimes B g^n(1) S(g^n(1)) \otimes B g^{n-2}(2) b S(g^{n-1}(1)) \otimes B g^n(1) S(g^n(1)) \otimes B]_B = \\
\vdots \\
[g^n \otimes h S(g^0(1)) \otimes B g^0(2) S(g^1(1)) \otimes B \cdots \otimes B g^n(1) S(g^n(1)) \otimes B g^{n-2}(2) b S(g^{n-1}(1)) \otimes B g^n(1) S(g^n(1)) \otimes B]_B = \\
\psi_n \left( (g^0 \otimes \cdots \otimes g^n) \otimes \mathcal{H} \left[ bh \right]_B \right).
\]

(3.19)

Next we observe for any \( p \in \mathcal{H} \) that

\[
\psi_n \left( (g^0 \otimes \cdots \otimes g^n) \cdot p \otimes \mathcal{H} \left[ h \right]_B \right) = \\
\psi_n \left( (g^0 p(1) \otimes \cdots \otimes g^n p(n+1)) \otimes \mathcal{H} \left[ h \right]_B \right) = \\
[g^n \otimes h S(g^0(1) p(1)) \otimes B g^0(2) S(g^1(1) p(2)) \otimes B \cdots \otimes B g^n(1) S(g^n(1) p(n+1)) \otimes B]_B = \\
[g^n \otimes h S(g^0(1) p(1)) S(g^0(1) p(2)) \otimes B g^0(1) S(g^1(1) p(1)) \otimes B \cdots \otimes B g^n(1) S(g^n(1)) \otimes B]_B = \\
\psi_n \left( (g^0 \otimes \cdots \otimes g^n) \otimes \mathcal{H} \left[ p \triangleright h \right]_B \right).
\]

(3.20)

Accordingly, \( \psi_n \) is well-defined for any \( n \geq 0 \).

Finally, we prove that \( \varphi_n \) and \( \psi_n \) are inverses to each other for any \( n \geq 0 \). On one hand we have,

\[
(\psi_n \circ \varphi_n) \left[ (h^0 \otimes B \cdots \otimes B h^n) \right]_B = \\
\psi_n \left( \left( \left( h^1(2) \cdots h^n(2) \otimes \cdots \otimes h^{n-1}(n) h^{n}(n) \otimes \bar{h}^{n(n+1)} \otimes \bar{1} \right) \otimes \mathcal{H} \left[ h^0 h^1(1) \cdots h^n(1) \right]_B \right) = \\
[1(2) h^0 h^1(1) \cdots h^n(1) S \left( (h^1(2) \cdots h^n(2))(1) \otimes B (h^1(2) \cdots h^n(2))(2) S \left( (h^2(3) \cdots h^n(3))(1) \otimes B \cdots \otimes B (h^{n-1}(n) h^n(n)) \otimes B h^n(n+1)(1) S(1(1)) \right) \right)_B = \\
\psi_n \left( \left( \left( h^1(2) \cdots h^n(2) \otimes \cdots \otimes h^{n-1}(n) h^n(n) \otimes \bar{h}^{n(n+1)} \otimes \bar{1} \right) \otimes \mathcal{H} \left[ h^0 h^1(1) \cdots h^n(1) \right]_B \right) = \\
[h^0 h^1(1) \cdots h^n(1) S \left( (h^1(2) \cdots h^n(2))(1) \otimes B (h^1(2) \cdots h^n(2))(2) S \left( (h^2(3) \cdots h^n(3))(1) \otimes B \cdots \otimes B (h^{n-1}(n) h^n(n)) \otimes B h^n(n+1)(2) \right) \right)_B = \\
[h^0 h^1(1) \cdots h^n(1) S \left( (h^1(2) \cdots h^n(2))(1) \otimes B (h^1(2) \cdots h^n(2))(2) S \left( (h^2(3) \cdots h^n(3))(1) \otimes B \cdots \otimes B h^n(n+1)(1) \right) \otimes \mathcal{H} h^n(n+1) \right)_B = \\
[h^0 \otimes B \cdots \otimes B h^n]_B.
\]

(3.21)
while on the other hand
\[
(\varphi_n \circ \psi_n) \left( (\overline{g}^0 \otimes \cdots \otimes \overline{g}^n) \otimes_H [h]_B \right) = \\
\varphi_n \left( [g_n^1] h S(g_n^1) \otimes_B g_0^0 S(g_0^1) \otimes_B \cdots \otimes_B g_{n-1}^1 S(g_{n-1}^1) \right) \otimes_H [g_n^1] h S(g_n^1) \otimes_H [h]_B = \\
\left( \overline{g}^0 S(g_0^1) \otimes_B \cdots \otimes_B g_{n-1}^1 S(g_{n-1}^1) \otimes_B \overline{1} \right) \otimes_H [g_n^1] h S(g_n^1) \otimes_H [h]_B = \\
\left( \overline{g}^0 \otimes_B \cdots \otimes_B \overline{g}^{n-1} \right) \cdot \overline{S(g_n^1) \otimes 1} \otimes_H [g_n^1] h S(g_n^1) \otimes_H [h]_B = \\
\left( \overline{g}^0 \otimes_B \cdots \otimes_B \overline{g}^{n-1} \right) \otimes_H [h]_B.
\]

\[\square\]

We are now ready to state our main result.

**Theorem 3.2.** Let \( \mathcal{H} \to C \) be a right module quotient coalgebra of a Hopf algebra \( \mathcal{H}, B := H_{\mathcal{H}} C \to \mathcal{H} =: A \) the corresponding \( C \)-Galois extension, and \( M = [H]_B \) be the left-right SAYD module as above. Then

\[\varphi_n : C_\mathcal{H}(A \mid B) \longrightarrow C_\mathcal{H}(C, M), \quad \text{(3.23)}\]

defined by \( \text{(3.10)} \), is an isomorphism of cyclic modules.

**Proof.** Let \( C = H/I \) where \( I \subseteq H \) is a coideal right ideal and let us first go through the face operators. For \( i = 0, \)

\[\varphi_{n-1} d_0([h^0 \otimes_B \cdots \otimes_B h^n]_B) = \varphi_{n-1}([h^0 h^1 \otimes_B \cdots \otimes_B h^n]_B) = \\
\left( h^1_{(2)} \cdots h^n_{(2)} \otimes_B \cdots \otimes_B h^n_{(n-1)} \otimes_B h^n_n \otimes_B \overline{1} \right) \otimes_H [h^0 h^1 h^2 \cdots h^n]_B = \\
d_0 \varphi_n([h^0 \otimes_B \cdots \otimes_B h^n]_B). \quad \text{(3.24)}\]

For \( 1 \leq i \leq n - 1 \), we observe that

\[\varphi_{n-1} d_i([h^0 \otimes_B \cdots \otimes_B h^n]_B) = \varphi_{n-1}([h^0 \otimes_B \cdots \otimes_B h^i h^{i+1} \otimes_B \cdots \otimes_B h^n]_B) = \\
\left( h^1_{(2)} \cdots h^i h^{i+1} \cdots h^n_{(i+1)} \otimes_B \cdots \otimes_B h^n_{(i+1)} \otimes_B h^{i+2} \cdots h^n_{(i+2)} \otimes_B \cdots \right. \\
\left. \cdots \otimes h^n_{(n-1)} h^n_n \otimes_B \overline{1} \right) \otimes_H [h^0 h^1 \cdots h^i h^{i+1} \cdots h^n]_B = \\
\left( h^1_{(2)} \cdots h^i_{(2)} \otimes_B \cdots \otimes_B h^i_{(i+1)} h^{i+1}_{(i+1)} \cdots h^n_{(i+1)} \otimes_B \cdots \otimes_B h^{i+2}_{(i+2)} \cdots h^n_{(n+2)} \otimes_B \cdots \right. \\
\left. \cdots \otimes h^n_{(n-1)} h^n_n \otimes_B \overline{1} \right) \otimes_H [h^0 h^1 \cdots h^n]_B = \\
d_i \varphi_n([h^0 \otimes_B \cdots \otimes_B h^n]_B). \quad \text{(3.25)}\]
Finally for the last face operator we have

\[
\varphi_{n-1} d_n ([h^0 \otimes_B \cdots \otimes_B h^n]) = \varphi_{n-1} ([h^n h^0 \otimes_B \cdots \otimes_B h^{n-1}]) = \\
\left( h_1^{(2)} \cdots h_{n-1}^{(2)} \otimes \cdots \right) \\
\hspace{1cm} \cdots \otimes h_{n-2}^{(n-1)} h_{n-1}^{(n-1)} \otimes h_{n-1}^{(n)} \otimes \mathcal{I} \right) \otimes_{\mathcal{H}} \left[ h^n h^1 \cdots h^{n-1} \right] = \\
\left( h_1^{(2)} \cdots h_{n-1}^{(2)} \otimes \cdots \right) \\
\hspace{1cm} \cdots \otimes h_{n-2}^{(n-1)} h_{n-1}^{(n-1)} \otimes h_{n-1}^{(n)} \otimes \mathcal{I} \right) \otimes_{\mathcal{H}} \left[ h^n h^1 \cdots h^{n-1} \right] = \\
\left( h_1^{(2)} \cdots h_{n-1}^{(2)} \otimes \cdots \right) \\
\hspace{1cm} \cdots \otimes h_{n-2}^{(n-1)} h_{n-1}^{(n-1)} \otimes h_{n-1}^{(n)} \otimes \mathcal{I} \right) \otimes_{\mathcal{H}} \left[ h^n h^1 \cdots h^{n-1} \right] = \\
d_n \varphi_n ([h^0 \otimes_B \cdots \otimes_B h^n]).
\tag{3.26}
\]

We next investigate the interaction between the degeneracy operators. To this end, for \(0 \leq j \leq n-1\) we observe

\[
\varphi_{n+1} s_j ([h^0 \otimes_B \cdots \otimes_B h^n]) = \\
\varphi_{n+1} ([h^0 \otimes_B \cdots \otimes_B h^j \otimes_B h^{j+1} \otimes_B \cdots \otimes_B h^n]) = \\
\left( h_1^{(2)} \cdots h_{n-1}^{(2)} \otimes \cdots \right) \\
\hspace{1cm} \cdots \otimes h^{(j+1)} h_{j+1}^{(j+2)} \cdots h^{(j+2)} h_{j+2}^{(j+3)} \cdots h^{(j+3)} \otimes \cdots \\
\hspace{1cm} \cdots \otimes h_{n-1}^{(n-1)} h_{n-1}^{(n-2)} \otimes h_{n-1}^{(n-2)} \otimes \mathcal{I} \right) \otimes_{\mathcal{H}} \left[ h^0 h^1 \cdots h^n \right] = \\
\left( h_1^{(2)} \cdots h_{n-1}^{(2)} \otimes \cdots \right) \\
\hspace{1cm} \cdots \otimes h^{(j+1)} h_{j+1}^{(j+2)} \cdots h^{(j+2)} h_{j+2}^{(j+3)} \cdots \right) \otimes_{\mathcal{H}} \left[ h^0 h^1 \cdots h^n \right] = \\
s_j \varphi_n ([h^0 \otimes_B \cdots \otimes_B h^n]).
\tag{3.27}
\]
Let us finally check the cyclic operators. To this end we have

\[
\varphi_n t_n( [h_0 \otimes_B \cdots \otimes_B h^n ]_{B}) = \varphi_{n+1}( [h_{n+1} \otimes_B \cdots \otimes_B h_{n+1} ]_{B}) = \biggl( h_{0}^{(2)} \cdots h_{n+2}^{(n)} \otimes_{H} \cdots \otimes_{H} h_{n+1}^{(n+1)} \otimes_{H} h_{0}^{(1)} \cdots h_{n+1}^{(n+1)} \biggr)_{B} =
\]

\[
\biggl( h_{0}^{(2)} h_{1}^{(2)} \cdots h_{n+1}^{(2)} \otimes_{H} \cdots \otimes_{H} h_{0}^{(1)} h_{1}^{(1)} \cdots h_{n+1}^{(1)} \biggr)_{B} =
\]

\[
t_n \varphi_n( [h_0 \otimes_B \cdots \otimes_B h^n ]_{B}).
\]

(3.28)

We will conclude this subsection with a Pontryagin dual of this result.

**Theorem 3.3.** Let \( \mathcal{B} \hookrightarrow \mathcal{H} \) be a left comodule subalgebra in a Hopf algebra \( \mathcal{H} \), \( \mathcal{D} = \mathcal{H} \twoheadrightarrow \mathcal{H} / \mathcal{B} \) be the corresponding \( \mathcal{B} \)-Galois coextension, and \( \mathcal{M} = (\mathcal{H})^C \) be the right-left SAYD module with the right action given by the multiplication of \( \mathcal{H} \), and the left coadjoint coaction. Then there exist an isomorphism

\[
\pi_n : C_n(\mathcal{D} \mid \mathcal{C}) \longrightarrow C_n(\mathcal{B}, \mathcal{M})^H
\]

of cyclic modules.

**Proof.** The proof of Theorem 3.2, using only the structural maps and relations equivalent to the commutativity of appropriate diagrams, is in fact diagrammatical. Hence, in view of the Pontryagin duality, interchanging the left and right, reversing the arrows and applying the cyclic duality we obtain a diagrammatical proof of the claim.

\( \square \)

### 3.2 Spectral sequences

We note that the left hand side of Theorem 3.2, resp. Theorem 3.3, compute the homology of the extension \( \mathcal{A} \) relative to \( \mathcal{B} \), resp. coextension \( \mathcal{D} \) corelative to \( \mathcal{C} \), while the right hand side computes the cyclic dual homology of the Pontryagin dual objects. In order to be able to investigate the latter homologies, in this subsection we shall develop computational tools.

We will focus on the Hochschild homology groups of the relative homology of the extension.

**Theorem 3.4.** Let \( \mathcal{B} \hookrightarrow \mathcal{H} \) be a homogeneous \( \mathcal{C} = \mathcal{H} / \mathcal{I} \)-Galois extension, and \( \mathcal{M} = [\mathcal{H}]_B \). Then there exists a spectral sequence (constructed in the proof) such that

\[
HH_\ast(\mathcal{A} \mid \mathcal{B}) = E_{2,0}^2, \quad E_{\ast, \ast}^2 \Longrightarrow \text{Tor}_\ast^H(k, \mathcal{M}).
\]

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where Tor is computed for the left $\mathcal{H}$-module structure on $\mathcal{M}$ coming from its SAYD module structure. In particular, we have a five-term exact sequence

$$\text{Tor}^2(k, \mathcal{M}) \to HH_2(\mathcal{A} \mid \mathcal{B}) \to H_0(\text{Tor}^1(C^{\otimes \bullet+1}, \mathcal{M})) \to \text{Tor}^1(k, \mathcal{M}) \to HH_1(\mathcal{A} \mid \mathcal{B}) \to 0.$$  

Proof. Let us consider the cyclic dual $C_{\bullet}$ to the standard cocyclic object of the coalgebra $C$, consisting of the tensor powers of $C$, $C_p = C^{p+1}$, with the boundary map $\partial: C_p \to C_{p-1}$

$$\partial(c^0 \otimes \cdots \otimes c^p) = \sum_{i=0}^{p} (-1)^i c^0 \otimes \cdots \otimes \varepsilon(c^i) c^{i+1} \otimes \cdots \otimes c^p$$

where the cyclic order of length $p + 1$ is assumed. Note that $\partial$ is a morphism of right $\mathcal{H}$-modules. The operator $h: C_p \to C_{p+1}$

$$h(c^0 \otimes \cdots \otimes c^p) := 1 \otimes c^0 \otimes \cdots \otimes c^p$$

is a homotopy contracting this complex to $k$ concentrated at zero degree, see for instance [23]. Let $M_{\bullet}$ be a flat resolution of the left $\mathcal{H}$-module $\mathcal{M}$, let us consider the total complex $C_{\bullet} \otimes_H M_{\bullet}$. We have two spectral sequences abutting to the total homology. The first page of the first one reads

$$E^{1}_{p,q} = H_q(C_{p} \otimes_H M_{\bullet}) = \text{Tor}^q(C_{p}, \mathcal{M})$$

hence its second page is computed as

$$E^{2}_{p,q} = H_p(\text{Tor}^q(C_{\bullet}, \mathcal{M})).$$

By Theorem 3.2 we are interested in

$$HH_n(\mathcal{A} \mid \mathcal{B}) = E^{2}_{n,0}. \quad (3.30)$$

Now, let us compute the first page of the second spectral sequence, the transposed analogue of the first one, abutting to the total homology, which by flatness of the resolution $M_{\bullet}$ and the acyclicity of $C_{\bullet}$ can be rewritten as

$$^\top E^{1}_{p,q} = H_p(C_{\bullet} \otimes_H M_q) = H_p(C_{\bullet}) \otimes_H M_q = \begin{cases} \text{Tor}^H(k, \mathcal{M}) & \text{for } p = 0, \\
0 & \text{for } p > 0 \end{cases}$$

As a result, degenerating at the second page

$$^\top E^{2}_{p,q} = \begin{cases} \text{Tor}^H(k, \mathcal{M}) & \text{for } p = 0, \\
0 & \text{for } p > 0 \end{cases}$$

this spectral sequence yields the total cohomology

$$H_n(C_{\bullet} \otimes_H M_{\bullet}) = \text{Tor}^H_n(k, \mathcal{M}). \quad (3.31)$$

Finally, we use (3.30), (3.31) and the canonical homological five-term exact sequence

$$H_2 \to E^2_{2,0} \xrightarrow{d} E^2_{0,1} \to H_1 \to E^2_{1,0} \to 0$$

to finish the proof. The second arrow is the boundary map of the second page of the first spectral sequence, the next one is induced by the augmentation $C_{\bullet} \to k$. \(\square\)
Now we show that the above theorem generalizes the classical result of Feng-Tsygan [14] (reproved by another argument by Bichon [2]), from the case of the homogeneous $\mathcal{H}$-Galois extension $k \to \mathcal{H}$ to arbitrary homogeneous quotient coalgebra-Galois extensions $\mathcal{B} \to \mathcal{H}$. The following can be regarded as a third independent proof of this classical result.

**Corollary 3.5.** We have

$$HH_\bullet(\mathcal{H}) = \text{Tor}_\bullet^k(k, \mathcal{H}),$$

(3.32)

where on the right hand side the left $\mathcal{H}$-module structure on $\mathcal{H}$ comes from its canonical SAYD module structure.

**Proof.** Note that for $\mathcal{C} = \mathcal{H}$, or equivalently $\mathcal{B} = k$, the SAYD module of coefficients $\mathcal{M}$ coincides with $\mathcal{H}$ with its canonical SAYD structure. Moreover, since for every right $\mathcal{H}$-module $\mathcal{N}$ the invertible linear map

$$\mathcal{N} \otimes \mathcal{H} \to \mathcal{N} \otimes \mathcal{H}, \quad n \otimes h \mapsto nS(h(1)) \otimes h(2)$$

makes the diagonal right $\mathcal{H}$-module $\mathcal{N} \otimes \mathcal{H}$ free, hence flat, then the diagonal right $\mathcal{H}$-module $\mathcal{C}_\bullet = \mathcal{H}^{\otimes \bullet+1}$ is flat. This implies that

$$E^2_{p,q} = H_p(\text{Tor}_q^\mathcal{H}(\mathcal{C}_\bullet, \mathcal{M})) = \begin{cases} H_p(\mathcal{C}_\bullet \otimes_{\mathcal{H}} \mathcal{M}) & \text{for } q = 0, \\ 0 & \text{for } q > 0 \end{cases}$$

(3.33)

hence the first spectral sequence degenerates at the second page as well, and therefore we obtain

$$HH_n(\mathcal{H}) = HH_n(\mathcal{H} \mid k) \xrightarrow{3.30} E^2_{n,0} = H_n(\mathcal{C}_\bullet \otimes_{\mathcal{H}} \mathcal{M}_\bullet) \xrightarrow{5.31} \text{Tor}_n^\mathcal{H}(k, \mathcal{H}).$$

(3.34)

**Acknowledgements:** S. Sütlü would like to thank his former PhD advisor B. Rangipour for drawing his attention to the homology of the coalgebra-Galois extensions.

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