CONSISTENCY TESTS OF $\rho^0(770) - f_0(980)$ MIXING IN $\pi^- p \to \pi^- \pi^+ n$

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Abstract
Analytical solutions of the $S$- and $P$-wave subsystem in $\pi^- p \to \pi^- \pi^+ n$ and $\pi^+ n \to \pi^+ \pi^- p$ measured on polarized targets at CERN reveal evidence for $\rho^0(770) - f_0(980)$ spin mixing. We study the response of these analytical solutions to the presence of small $D$ wave amplitudes with helicity $\lambda \leq 1$ (Response analysis A) and $\lambda \leq 2$ (Response analysis B) which contaminate the input data. In both Response analyses the $\rho^0(770) - f_0(980)$ spin mixing effect is clearly consistent with the presence of the $D$-wave amplitudes provided they are not too large below 750 MeV. This result at low momentum transfer $t$ is in agreement with the evidence for $\rho^0(770) - f_0(980)$ mixing in the presence of $D$-wave amplitudes from the amplitude analysis of the CERN data on $\pi^- p \to \pi^- \pi^+ n$ at high $t$. We also show that the $\rho^0(770) - f_0(980)$ mixing is consistent with isospin relations for the $S$-wave intensities measured in $\pi^- p \to \pi^+ n$, $\pi^- p \to \pi^0 \pi^0 n$, and $\pi^+ p \to \pi^+ \pi^+ n$ processes. These results strengthen the experimental evidence for the $\rho^0(770) - f_0(980)$ spin mixing found in the analytical solutions and are in agreement with recent theoretical expectations. We present a survey of moduli of the $S$-wave amplitudes and $S$-wave intensities from all amplitude analyses of the five measurements of $\pi^- p \to \pi^- \pi^+ n$ and $\pi^+ n \to \pi^+ \pi^- p$ on polarized targets. All analyses are in a remarkable agreement that shows a clear evidence for a resonant structure at $\rho^0(770)$ mass in the $S$-wave moduli and intensities in a broad confirmation of the $\rho^0(770) - f_0(980)$ spin mixing.

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I. INTRODUCTION.

The evidence for a rho-like state in the $S$-wave amplitudes in $\pi^- p \rightarrow \pi^- \pi^+ n$ dates back to 1960's [1,5] and was confirmed later in CERN measurements on polarized targets in $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c [7–12, 15–17] and in $\pi^- n \rightarrow \pi^- \pi^+ p$ at 5.98 and 11.85 [15,17]. Additional evidence came from the ITEP data on $\pi^- p \rightarrow \pi^- \pi^+ n$ on polarized target at 1.78 GeV/c [18]. These findings were controversial because the measurements of $\pi^- p \rightarrow \pi^0 \pi^0 n$ at CERN in 1972 [19] and at BNL in 2001 [20] found no evidence for the rho-like meson in the $S$-wave amplitudes. Using three different methods we show in a recent work [21] that the rho-like resonance in the $S$-wave transversity amplitudes arises entirely from the contribution of the $\rho^0(770)$ resonance. In addition, there is a dip at the $f_0(980)$ mass in the $P$-wave amplitude $|L_d|^2$. These results present evidence for a $\rho^0(770) – f_0(980)$ mixing in $\pi^- p \rightarrow \pi^- \pi^+ n$. Since there is no $P$-wave in $\pi^- p \rightarrow \pi^0 \pi^0 n$ this explains why there is no rho-like resonance observed in this process. The theoretical interpretation of the evidence for $\rho^0(770) – f_0(980)$ spin mixing is developed in Ref. [22,23].

In this work we strengthen the evidence for the $\rho^0(770) – f_0(980)$ mixing in two ways. First, we show that the observed mixing is not generated by the admixture of small $D$-wave amplitudes in the input data. The spin mixing effect is consistent with the presence of the $D$-wave amplitudes at low as well as at high momentum transfers. Second, we show that the $\rho^0(770) – f_0(980)$ mixing is consistent with the isospin relations between the observed amplitudes in $\pi^- p \rightarrow \pi^- \pi^+ n$ and $\pi^- p \rightarrow \pi^0 \pi^0 n$.

The paper is organized as follows. In Section II, we define the observables $t^L_M, p^L_M, r^L_M, q^L_M$ measured in $\pi^- p \rightarrow \pi^- \pi^+ n$ on polarized target when the polarization of the recoil nucleon is not observed. In Section III, we define new observables with definite nucleon transversity $\tau$ called $a_{k,\tau}, k = 1, 1.5$ in terms of $t^L_M, p^L_M$ and express them in terms of $S_\tau, P_\tau$ and $D_\tau$ partial wave transversity amplitudes. In Section IV, we discuss the solvability of this system of equations for the $S$, $P$ and $D$ wave amplitudes and focus on the response of the analytical solutions of the $S$- and $P$-wave subsystem to the presence of $D$-wave amplitudes. We also present evidence for $\rho^0(770) – f_0(980)$ mixing from an amplitude analysis at high $t$ which includes $D$-wave amplitudes. In Section V, we derive isospin relations between $S$-wave intensities in $\pi^- p \rightarrow \pi^- \pi^+ n, \pi^- p \rightarrow \pi^0 \pi^0 n$ and $\pi^+ p \rightarrow \pi^+ \pi^+ n$ processes and present experimental evidence for the consistency of the $\rho^0(770) – f_0(980)$ mixing with these relations.

In Section VI, we survey the results for the moduli of the $S$-wave transversity amplitudes and for $S$-wave intensities from all amplitude analyses of five measurements of $\pi N \rightarrow \pi^- \pi^+ N$ on polarized targets and compare the latest results for relative phases. All analyses are in a remarkable agreement that shows a clear evidence for a resonant structure at $\rho^0(770)$ mass in the $S$-wave moduli and intensities in a broad confirmation of $\rho^0(770) – f_0(980)$ spin mixing. The paper closes with a summary in Section VII.

II. THE OBSERVABLES IN $\pi^- p \rightarrow \pi^- \pi^+ n$ ON POLARIZED TARGET.

Consider the pion production process $\pi^- p \rightarrow \pi^- \pi^+ n$ with four-momenta $p_a + p_b = p_1 + p_2 + p_d$. The invariant mass of the dipion system is $m^2 = (p_1 + p_2)^2$. The angular distribution of the produced dipion system is described by the direction of $\pi^-$ in the two-pion center-of-mass system and its solid angle $\Omega = \theta,\phi$. When the polarization of the recoil nucleon is not measured the angular intensity takes the form [21,24,25]

$$I(\Omega, \psi) = I_U(\Omega) + P_T \cos \psi I_C(\Omega) + P_T \sin \psi I_S(\Omega) + P_L I_L(\Omega)$$  \hspace{1cm} (2.1)
Here $\vec{P} = (P_x, P_y, P_z) = (P_T \sin \psi, P_T \cos \psi, P_L)$ is the target polarization vector where $P_T$ and $P_L$ are transverse and longitudinal polarization components perpendicular and parallel to the $z$-axis, respectively. The angle $\psi$ is the angle between $\vec{P}_T$ and the $y$-axis. In the laboratory system of the reaction the $+z$ axis has the direction opposite to the incident pion beam. The $+y$ axis is perpendicular to the scattering plane and has the direction of $\vec{p}_a \times \vec{p}_c$ where $p_c = p_1 + p_2$.

We shall use the parametrization of the angular components $I_U, I_C, I_S, I_L$ due to Lutz and Rybicki [10, 11, 24, 25]

$$I_U(\Omega) = \sum_{L,M} t_{LM}^I \Re Y_{LM}^I(\Omega)$$ (2.2)
$$I_C(\Omega) = \sum_{L,M} p_{LM}^I \Re Y_{LM}^I(\Omega)$$
$$I_S(\Omega) = \sum_{L,M} r_{LM}^I \Im Y_{LM}^I(\Omega)$$
$$I_L(\Omega) = \sum_{L,M} q_{LM}^I \Im Y_{LM}^I(\Omega)$$

The parametrization (2.2) assumes $P$-parity conservation. In terms of density matrix elements the parameters $t, p, r, q$ read [24, 25]

$$t_{LM}^I = \sum J \sum_{J', \lambda'} K_{jJ'\lambda'\lambda}^L \Re (R^0_{0j})_{J'\lambda'\lambda}^{J\lambda}$$ (2.3)
$$p_{LM}^I = \sum J \sum_{J', \lambda'} K_{jJ'\lambda'\lambda}^L \Re (R^0_{0y})_{J'\lambda'\lambda}^{J\lambda}$$
$$r_{LM}^I = \sum J \sum_{J', \lambda'} K_{jJ'\lambda'\lambda}^L \Im (R^0_{0x})_{J'\lambda'\lambda}^{J\lambda}$$
$$q_{LM}^I = \sum J \sum_{J', \lambda'} K_{jJ'\lambda'\lambda}^L \Im (R^0_{0z})_{J'\lambda'\lambda}^{J\lambda}$$

where

$$K_{jJ'\lambda'\lambda}^L = (-1)^{\lambda'} \sqrt{\frac{(2J + 1)(2J' + 1)}{4\pi(2L + 1)}} < J J' 00 | L 0 > < J J' M + \lambda' - \lambda | L M >$$ (2.4)

General expressions for the full set of density matrix elements $(R_{k\lambda_\tau}^{J_{\lambda_\tau}})$ for $j = u, y, x, z$ including recoil nucleon polarization $j = 1, 2, 3$ in terms of the unnatural and natural exchange transversity amplitudes $U_{\lambda_\tau}^J$ and $N_{\lambda_\tau}^J$ are given in Ref. [22, 24]. Here $\tau = +\frac{1}{2}, -\frac{1}{2} = up(u), down(d)$ is the target nucleon transversity. It is these amplitudes which are most suitable for the amplitude analysis of measurements on both polarized and unpolarized targets.

III. THE S-, P- AND D-WAVE SUBSYSTEM.

The $S$-, $P$- and $D$-wave subsystem is described by parameters $t, p, r, q$ for $L \leq 4$ and $M \leq 4$. The CERN measurements on transversely polarized target did not measure the parameters $q_{LM}^I$. Expressions for $t, p, r$ in terms of the transversity amplitudes for $L \leq 4$ and $M \leq 2$ corresponding to $J \leq 2$ and $\lambda \leq 1$ were given by Lutz and Rybicki in Ref. [24]. Expressions for $t, p, r$ for $L \leq 4$ and $M \leq 4$ corresponding to $J \leq 2$ and $\lambda \leq 2$ were given by Sakrejda in Ref. [25].
In this work we focus on the parameters $t_M^{L}$ and $p_M^{L}$. These parameters organize themselves into two groups: $t_M^{L} + p_M^{L}$ are expressed in terms of bilinear terms $Re(A_u B_u^{*})$ with transversity $u$, while $t_M^{L} - p_M^{L}$ are expressed in terms of bilinear terms $Re(A_d B_d^{*})$ with transversity $d$. We define the following convenient set of observables $a_{i,\tau}, i = 1, 15$

$$a_{1,\tau} = \sqrt{\pi}(t_0^0 + p_0^0), \quad a_{2,\tau} = \sqrt{\pi}(t_0^0 + p_0^0)\sqrt{5} \quad (3.1)$$

$$a_{3,\tau} = \sqrt{\pi}(t_0^2 + p_0^2)(-\sqrt{\frac{5}{6}}), \quad a_{4,\tau} = \sqrt{\pi}(t_0^1 + p_0^1)\frac{1}{2} \quad (3.2)$$

$$a_{5,\tau} = \sqrt{\pi}(t_0^2 + p_0^2)(\frac{1}{2}\sqrt{\frac{5}{6}}), \quad a_{6,\tau} = \sqrt{\pi}(t_1^1 + p_1^1)(\frac{1}{2}\sqrt{\frac{1}{2}})$$

$$a_{7,\tau} = \sqrt{\pi}(t_0^3 + p_0^3)(\frac{1}{6}\sqrt{\frac{35}{3}}), \quad a_{8,\tau} = \sqrt{\pi}(t_1^3 + p_1^3)(\frac{1}{8}\sqrt{\frac{35}{3}})$$

$$a_{9,\tau} = \sqrt{\pi}(t_0^3 + p_0^3)(\frac{1}{2}\sqrt{\frac{7}{6}}), \quad a_{10,\tau} = \sqrt{\pi}(t_0^4 + p_0^4)\frac{7}{2} \quad (3.3)$$

$$a_{11,\tau} = \sqrt{\pi}(t_1^4 + p_1^4)(\frac{7}{4}\sqrt{\frac{1}{35}}), \quad a_{12,\tau} = \sqrt{\pi}(t_2^4 + p_2^4)(\frac{7}{2}\sqrt{\frac{1}{10}})$$

$$a_{13,\tau} = \sqrt{\pi}(t_3^5 + p_3^5)(\frac{7}{3}), \quad a_{14,\tau} = \sqrt{\pi}(t_3^5 + p_3^5)(\frac{7}{5})$$

$$a_{15,\tau} = \sqrt{\pi}(t_4^5 + p_4^5)(\sqrt{\frac{14}{5}})$$

In (3.1)-(3.3) $\tau = u$ for the $+$ sign and $\tau = d$ for the $-$ sign. Next we express the observables $a_{i,\tau}$ in terms of $S$, $P$- and $D$-wave amplitudes defined as follows:

$$\begin{align*}
U_0^{0,\tau} & = S_{\tau} \\
U_1^{0,\tau} & = L_{\tau} \\
U_1^{1,\tau} & = U_{\tau} \\
U_2^{0,\tau} & = D_0^{u,\tau} \\
U_1^{2,\tau} & = D_0^{u,\tau} \\
U_2^{2,\tau} & = D_2^{u,\tau} \\
N_1^{1,\tau} & = N_{1,\tau} \\
N_2^{1,\tau} & = N_{2,\tau} \\
N_2^{2,\tau} & = D_2^{2,\tau} \\
& \quad (3.4)
\end{align*}$$

For the purposes of our analysis we shall split the observables $a_{i,\tau}$ into three parts

$$a_{i,\tau} = c_{i,\tau} + d_{i,\tau} + e_{i,\tau} \quad (3.5)$$

where $c_{i,\tau}$ involve only $S$- and $P$-wave amplitudes, $d_{i,\tau}$ involve terms with $D$-wave amplitudes with only helicity $\lambda \leq 1$, and $e_{i,\tau}$ involve terms with $D$-wave amplitudes with $\lambda = 2$ (rank 2 amplitudes). The expressions for the $D$-wave terms $d_{i,\tau}$ and $e_{i,\tau}$ in terms of the transversity amplitudes are given in the Table I. The expressions for $c_{i,\tau}$ read as follows

$$\begin{align*}
c_{1,\tau} & = |S_{\tau}|^2 + |L_{\tau}|^2 + |U_{\tau}|^2 + |N_{\tau}|^2 \\
c_{2,\tau} & = 2|L_{\tau}|^2 - |U_{\tau}|^2 - |N_{\tau}|^2 \\
c_{3,\tau} & = |N_{\tau}|^2 - |U_{\tau}|^2 \\
c_{4,\tau} & = |L_{\tau}|S_{\tau}| \cos \Phi(L_{\tau}S_{\tau}^{*}) \\
c_{5,\tau} & = |L_{\tau}|U_{\tau}| \cos \Phi(L_{\tau}U_{\tau}^{*}) \\
c_{6,\tau} & = |U_{\tau}|S_{\tau}| \cos \Phi(U_{\tau}S_{\tau}^{*}) \\
& \quad (3.6)
\end{align*}$$

where the cosines of relative phases

$$\cos \Phi(A_{\tau}B_{\tau}^{*}) = \cos(\Phi(A_{\tau}) - \Phi(B_{\tau})) \quad (3.7)$$

All $c_{i,\tau} = 0$ for $i = 7, 15$. 
TABLE I: D-wave contributions $d_{i_1, \tau}$ and $e_{i_1, \tau}$ to the observables $a_{i_1, \tau}$ corresponding to D-wave transversity amplitudes with helicities $\lambda \leq 1$ and $\lambda \leq 2$, respectively. The transversity index $\tau$ is omitted for the sake of brevity and the bilinear terms $AB^* \equiv Re(AB^*)$. Table from Ref. [24, 25].

| $a_{i_1, \tau}$ | $d_{i_1, \tau}$ | $e_{i_1, \tau}$ |
|-----------------|-----------------|-----------------|
| a1 | $|D^0|^2 + |D^U|^2 + |D^N|^2$ | $|D^{2U}|^2 + |D^{2N}|^2$ |
| a2 | $2\sqrt{3}D^0S^* + \frac{2}{3}(2|D^0|^2 + |D^U|^2 + |D^N|^2)$ | $-\frac{2}{3}(|D^{2U}|^2 + |D^{2N}|^2)$ |
| a3 | $\frac{2}{3}(D^0|^2 + |D^U|^2)$ | $-2\sqrt{\frac{2}{3}}SD^{2U} + 20\sqrt{\frac{1}{3}}D^0D^{2U}$ |
| a4 | $\sqrt{\frac{2}{3}}D^0L^* + \sqrt{\frac{2}{3}}(D^U U^* + D^N N^*)$ | 0 |
| a5 | $\sqrt{\frac{2}{3}}D^U S^* + \frac{2}{3}\sqrt{\frac{2}{3}}D^U D^0s$ | $\frac{2}{3}(D^U D^{2U} + D^N D^{2N})$ |
| a6 | $\sqrt{\frac{2}{3}}D^0 L^* - \sqrt{\frac{2}{3}}D^0 U^*$ | $\sqrt{\frac{2}{3}}(UD^{2U} + ND^{2N})$ |
| a7 | $D^0L^* - \sqrt{\frac{1}{3}}(D^U U^* + D^N N^*)$ | 0 |
| a8 | $D^U L^* + \sqrt{\frac{2}{3}}D^0 U^*$ | $-\frac{1}{3}(UD^{2U} + ND^{2N})$ |
| a9 | $D^U U^* - D^N N^*$ | $LD^{2U}$ |
| a10 | $3|D^0|^2 - 2(|D^U|^2 + |D^N|^2)$ | $\frac{1}{2}(D^U|^2 + |D^{2N}|^2)$ |
| a11 | $D^U D^0s$ | $-\frac{1}{2}\sqrt{\frac{1}{3}}(UD^U D^{2U} + D^N D^{2N})$ |
| a12 | $|D^U|^2 - |D^N|^2$ | $\sqrt{3}D^0D^{2U}$ |
| a13 | 0 | $UD^{2U} - ND^{2N}$ |
| a14 | 0 | $D^U D^{2U} - D^N D^{2N}$ |
| a15 | 0 | $|D^U|^2 - |D^N|^2$ |

IV. CONSISTENCY OF $\rho^0(770) - f_0(980)$ MIXING WITH THE D-WAVE AMPLITUDES.

A. Assessment of the data and solvability

The full S-, P- and D-wave system $a_{i_1, \tau}, i = 1, 15$ with $d_{i_1, \tau} \neq 0$ and $e_{i_1, \tau} \neq 0$ is not analytically solvable even when additional information on $r_M^x$ and $q_M^x$ is added. To obtain the moduli and relative phases of the amplitudes $\chi^2$ fits to the data are needed bin by bin [11]. Assuming $D^{2U} = D^{2N} = 0$ all $e_{i_1, \tau} = 0$. The truncated S-, P- and D-wave system $a_{i_1, \tau}, i = 1, 12$ with $d_{i_1, \tau} \neq 0$ is in general not analytically solvable. However assuming phase coherence of unnatural exchange amplitudes

$$\Phi(U_\tau) - \Phi(L_\tau) = \pi$$

$$\Phi(D^{2U}_\tau) - \Phi(D^{2U}_\tau) = \pi$$

the simplified system is analytically solvable [10, 24]. Finally, assuming all $d_{i_1, \tau} = 0, e_{i_1, \tau} = 0$ the S- and P-wave subsystem is analytically solvable [10, 21, 24] for the moduli and relative phases of the S- and P-wave transversity amplitudes.

The evidence for $\rho^0(770) - f_0(980)$ mixing comes from the amplitude analyses of CERN data on polarized target at low $|t|$ below 1080 MeV assuming S- and P-wave dominance and $c_{i_1, \tau} = a_{i_1, \tau}$ [21]. Ideally we would like to separate the S- and P-wave amplitudes from the D-wave amplitudes to ensure that the $\rho^0(770) - f_0(980)$ mixing is not generated by the assumption $c_{i_1, \tau} = a_{i_1, \tau}$. To assess the CERN data on $\pi^- p \rightarrow \pi^- \pi^+ n$ from the point of view of evidence for $\rho^0(770) - f_0(980)$ mixing the Table II summarizes the measurements below 1080 MeV.

While the measurement 1 on unpolarized target presents evidence for small D-wave amplitudes with helicity $\lambda \leq 1$, the measurements 2 on polarized target are not sensitive to such amplitudes below 960 MeV. Measurements 1 and 2 at low $|t|$ below 960 MeV thus cannot be separately used to
separate $S$- and $P$-wave amplitudes from the $D$-wave amplitudes. However the two measurements could be combined for observables with $L \leq 4, M \leq 2$ if we set $p_M^L = r_M^L = 0$ for $L = 3, 4$ and $M \leq 2$ in measurement 2, and if we assume the phase coherence (4.1) to solve analytically the approximate $S$-, $P$- and $D$-wave system.

Above 960 MeV the measurements 3 on polarized target at low $|t|$ are sensitive to the $D$-wave amplitudes with $\lambda \leq 1$ and the amplitude analyses [9, 10, 31] separate $S$-, $P$- and $D$-wave amplitudes for $m \geq 980$ MeV. No measurement at low $|t|$ has detected the $\lambda = 2$ $D$-wave amplitudes. However the measurement 4 on polarized target shows that these amplitudes are present at high $|t|$ even below 960 MeV. In this case the $S$- and $P$-wave amplitudes can be separated and present evidence for $\rho^0(770) - f_0(980)$ mixing [11] (Section IV.C).

**B. Response analysis at low $t$**

The $S$- and $P$-wave subsystem (3.6) is formally solvable for the moduli and cosines of relative phases in terms of the unknown observables $c_{i,\tau}, i = 1, 6$. With

$$c_{i,\tau} = a_{i,\tau} - b_{i,\tau} = a_{i,\tau} - d_{i,\tau} - e_{i,\tau} \quad (4.2)$$

and omitting the index $\tau$ for the sake of brevity, the solution reads

$$|S|^2 = c_1 + c_2 - 3|L|^2 = a_1 + a_2 - 3|L|^2 - b_1 - b_2 \quad (4.3)$$

$$|U|^2 = |L|^2 - \frac{1}{2}(c_2 + c_3) = |L|^2 - \frac{1}{2}(a_2 + a_3) - \frac{1}{2}(b_2 + b_3)$$

$$|N|^2 = |L|^2 - \frac{1}{2}(c_2 - c_3) = |L|^2 - \frac{1}{2}(a_2 - a_3) - \frac{1}{2}(b_2 - b_3)$$

$$\cos \Phi(LS^*) = \frac{c_4}{|L||S|} = \frac{a_4 - b_4}{|L||S|} \quad (4.4)$$

$$\cos \Phi(LU^*) = \frac{c_5}{|L||U|} = \frac{a_5 - b_5}{|L||U|}$$

$$\cos \Phi(US^*) = \frac{c_6}{|U||S|} = \frac{a_6 - b_6}{|U||S|}$$

The phase condition $\Phi(LS^*) + \Phi(LU^*) + \Phi(US^*)$ implies a cosine condition

$$\cos \Phi(LS^*)^2 + \cos \Phi(LU^*)^2 + \cos \Phi(US^*)^2 - 2 \cos \Phi(LS^*) \cos \Phi(LU^*) \cos \Phi(US^*) = 1 \quad (4.5)$$

Substituting from (4.4) and (4.3) we obtain a cubic equation for $|L|^2$ [21, 24].

Assuming $c_{i,\tau} = a_{i,\tau}$ we neglect the unknown $D$-wave contributions $b_{i,\tau}$ but the system is analytically solvable with two physical solutions for the amplitudes with both transversities. Instead of solving the $S$-, $P$- and $D$-wave subsystem our strategy is to examine the response of the solutions of the equations (4.3),(4.4) and (4.5) to assumed $D$-wave contributions $b_{i,\tau}$. We want to

| Exp. | Observables | $m[\text{MeV}]$ | $|t|[(\text{GeV/c})^2]$ | $L, M$ | $J = 2, \lambda = 1$ | $J = 2, \lambda = 2$ | Ref. |
|------|-------------|----------------|-----------------|--------|----------------|----------------|------|
| 1    | $t_M^L$     | 300 – 1080     | 0.00 < $|t|$ < 0.15 | $L \leq 6, M \leq 2$ | yes | no | [20] |
| 2    | $t_M^L, p_M^L, r_M^L$ | 580 – 960 | 0.005 < $|t|$ < 0.20 | $L \leq 2, M \leq 2$ | no | no | [9, 10] |
| 3    | $t_M^L, p_M^L, r_M^L$ | 960 – 1080 | 0.005 < $|t|$ < 0.20 | $L \leq 4, M \leq 2$ | yes | no | [9, 10] |
| 4    | $t_M^L, p_M^L, r_M^L$ | 580 – 1080 | 0.20 < $|t|$ < 1.0 | $L \leq 4, M \leq 4$ | yes | yes | [11] |

 TABLE II: The CERN measurements of $\pi^- p \rightarrow \pi^\pm \pi^\mp n$ on unpolarized and polarized target at 17.2 GeV/c below dipion mass 1080 MeV.
find out if the solutions with $b_{i,\tau} = 0, i = 1, 6$ change so much for small $D$-wave amplitudes that the $\rho^0(770) - f_0(980)$ mixing "disappears". If this "disappearance" does not happen, then we can be confident that the observed $\rho^0(770) - f_0(980)$ mixing is a genuine effect not generated by our assumption $e_{i,\tau} = a_{i,\tau}$.

We perform two types of response analysis. In Response Analysis A we assume $D_{\tau}^{SU} = D_{\tau}^{SN} = 0$ so that $b_{i,\tau} = d_{i,\tau}$. In Response Analysis B these amplitudes are no longer vanishing. In both cases the response analysis is feasible only when the system of equations (4.3),(4.4) and (4.5) remains analytically solvable. This solvability requires that the phases of $D$-wave amplitudes decouple from the $S$- and $P$-wave amplitudes such that

$$Re(D_{\tau}^0 S_{\tau}^*) = 0 \quad (4.6)$$

$$d_{4,\tau} = d_{5,\tau} = d_{6,\tau} = 0$$

$$e_{3,\tau} = e_{5,\tau} = e_{6,\tau} = 0$$

In both cases the critical amplitude to watch is the Solution 2 for the $S$-wave amplitude $|S_d|^2$.

1. Response analysis A

In this analysis we assume

$$b_{1,\tau} = |D_0^0|^2 + |D_\tau^U|^2 + |D_\tau^N|^2 \quad (4.7)$$

$$b_{2,\tau} = \frac{5}{7} (2|D_\tau^0|^2 + |D_\tau^U|^2 + |D_\tau^N|^2)$$

$$b_{3,\tau} = \frac{5}{7} (|D_\tau^N|^2 - |D_\tau^U|^2)$$

For dipion masses $m > 980$ MeV we know the $D$-wave intensities $I(A) = |A_u|^2 + |A_d|^2$, $A = D^0, D^U, D^N$ from the amplitude analysis of the CERN measurement 3 [10]. We linearly extrapolate these intensities from their values $I_2(A)$ at $m_2 = 990$ MeV to value $I_1(A) = TI_2(A)$ at $m_1 = 590$ MeV where the fraction $T$ defines the slope parameter. The extrapolated intensities at mass $m$ are

$$I(A, m) = TI_2(A) + \frac{(1 - T)I_2(A)}{m_2 - m_1}(m - m_1) \quad (4.8)$$

Below 980 MeV there is a fairly constant ratio of the moduli $|A_u|^2 : |A_d|^2 \approx 1 : 3$ for all $S$- and $P$-wave amplitudes. Using this ratio we reconstruct the moduli of the $D$-wave amplitudes from the intensities

$$|A_u(m)|^2 = 0.25I(A, m)F \quad (4.9)$$

$$|A_d(m)|^2 = 0.75I(A, m)F$$

where the factor $F$ accounts for the sudden decrease of the $D$-wave moments with $L = 3, 4$ below the $K\bar{K}$ threshold. We assume $F = 0.5$. We vary the slope parameter $T$ in the range from 0.05 to 1.00 to estimate the $D$-wave amplitudes below 980 MeV. Above 980 MeV we used the amplitudes (4.9) calculated from the measured intensities of the analysis [10].

The results for the critical Solution 2 of $|S_d|^2$ are shown in Figures 1 and 2 for $T = 0.05, 0.10, 0.15, 0.20$ and $T = 0.30, 0.50, 0.75, 1.00$, respectively. We find that for $T < 0.70$ the $\rho^0(770)$ structure survives and is largely insensitive to the presence of the $D$-wave amplitudes for $T \lesssim 0.30$. However, below 749 MeV the analysis is compatible only with small $D$-wave amplitudes for $T \lesssim 0.30$, which is an expected result from the data of measurements 1.
2. Response analysis B

The measurements 3 on polarized target \( |t| > 0 \) indicate that \( t_M^L = p_M^L = 0 \) for \( L = 3, 4 \) and \( M = 3, 4 \). From the Table I, we see that \( t_4^L = p_4^L = 0 \) implies \( |D_\tau^{2U}|^2 = |D_\tau^{2N}|^2 \). Then

\[
\begin{align*}
    b_{1,\tau} &= |D_\tau^0|^2 + |D_\tau^U|^2 + |D_\tau^N|^2 + 2|D_\tau^U|^2 \\
    b_{2,\tau} &= \frac{5}{I}(2|D_\tau^0|^2 + |D_\tau^U|^2 + |D_\tau^N|^2 - 4|D_\tau^U|^2) \\
    b_{3,\tau} &= \frac{5}{I}(|D_\tau^N|^2 - |D_\tau^U|^2)
\end{align*}
\]

We assume \( |D_\tau^{2U}|^2 \) is a fraction of \( |D_\tau^U|^2 \)

\[
|D_\tau^{2U}|^2 = T_2|D_\tau^U|^2
\]

We examine the sensitivity of the solutions on \( |D_\tau^{2U}|^2 \) for small values of \( T \). The results for the Solution 2 of \( |S_d|^2 \) are shown in Figures 3 and 4 for \( T = 0.05 \) and \( T = 0.15 \), respectively, for a broad range of \( T_2 \). Again, the \( \rho^0(770) \) structure survives and is largely insensitive to \( |D_\tau^{2U}|^2 \). Above 1000 MeV the solutions require \( |D_\tau^{2U}|^2 = |D_\tau^{2N}|^2 \approx 0 \) in excellent agreement with the absence of these amplitudes in the amplitude analyses \( 9, 10 \) of the measurements 3 up to dipion mass 1780 MeV.

C. Amplitude analysis at high \( t \)

Measurements 4 of \( t_M^L, p_M^L, r_M^L \) with \( L \leq 4, M \leq 4 \) at high momentum transfers \( 0.20 \leq |t| \leq 1.00 \) (GeV/c)^2 and dipion mass \( 580 \leq m \leq 1500 \) MeV revealed the presence of all \( D \)-wave amplitudes even below 960 MeV. Amplitude analysis \( 11 \) used \( \chi^2 \) fits of the data to determine the amplitudes. Two solutions were found for the \( S \)- and \( P \)-wave amplitudes in the \( \rho^0(770) \) mass region. The solutions for the \( D \)-wave amplitudes are unique for all dipion masses.

Figure 5 shows the corresponding two solutions for the \( S \)-wave intensity \( I(S) \) and \( D \)-wave intensities \( I(D_0) \) and \( I(D^{2U}) \) below 1100 MeV. The other \( D \)-wave intensities are similar to \( I(D_0) \). The Solution 2 of \( I(S) \) suggests the presence of \( \rho^0(770) \) in the \( S \)-wave indicating \( \rho^0(770) - f_0(980) \) mixing at high \( |t| \) even in the presence of a large \( D \)-wave amplitude \( D^{2U} \). Unfortunately, some details of the \( \rho^0(770) - f_0(980) \) mixing may be lost in both solutions for \( I(S) \) since this analysis is done in a single broad bin \( 0.20 \leq |t| \leq 1.00 \) (GeV/c)^2 and with 40 MeV mass bins.

V. CONSISTENCY OF \( \rho^0(770) - f_0(980) \) MIXING WITH ISOSPIN RELATIONS BETWEEN S-WAVE AMPLITUDES IN \( \pi^- p \rightarrow \pi^- \pi^+ n \) AND \( \pi^- p \rightarrow \pi^0 \pi^0 n \).

A. Isospin relations between \( S \)-wave intensities in \( \pi^- p \rightarrow \pi^- \pi^+ n \) and \( \pi^+ \pi^- n \) production

If \( \rho^0(770) - f_0(980) \) mixing is to be a genuine physical effect than it must be compatible with other \( \pi N \rightarrow \pi \pi N \) processes whose amplitudes are related by isospin. Such isospin relations are the same for the \( S \)-matrix amplitudes and for the measured dephased amplitudes. An experimental test of these relations for \( S \)-wave amplitudes provides an independent evidence for or against the existence of the \( \rho^0(770) - f_0(980) \) mixing in \( \pi^- p \rightarrow \pi^- \pi^+ n \).

Generalized Bose-Einstein statistics is an extension of Bose-Einstein statistics to all particles belonging to an isospin multiplet of isospin \( I_1 \) which are regarded as \( 2I_1 + 1 \) charge states of the same particle. By incorporating the isospin quantum numbers in the state vector the symmetrization properties are extended to the interchange of particles belonging to the same multiplet. The
FIG. 1: Response of S-wave amplitude $|S_d|^2$ Solution 2 to $|D_0|^2, |D_U|^2, |D_N|^2$ at low $T$.

FIG. 2: Response of S-wave amplitude $|S_d|^2$ Solution 2 to $|D_0|^2, |D_U|^2, |D_N|^2$ at high $T$. 
FIG. 3: Response of $S$-wave amplitude $|S_d|^2$ Solution 2 to $|D_2^{2U}|^2, |D_2^{2N}|^2$ at $T = 0.05$.

FIG. 4: Response of $S$-wave amplitude $|S_d|^2$ Solution 2 to $|D_2^{2U}|^2, |D_2^{2N}|^2$ at $T = 0.15$. 
Generalized Bose-Einstein statistics requires that for two-pion state $J + I = \text{even}$ where $J$ is the dipion spin and $I$ is the total spin [27].

Consider the two-pion state $|\pi^- \pi^+\rangle$. It can be written in the form

$$|\pi^- \pi^+\rangle = \frac{1}{\sqrt{2}}|S\rangle + \frac{1}{\sqrt{2}}|A\rangle \quad \text{(5.1)}$$

where $|S\rangle$ and $|A\rangle$ are symmetric and antisymmetric $\pi^- \pi^+$ isospin states

$$|S\rangle = \frac{1}{\sqrt{2}}(|\pi^- > |\pi^+ > + |\pi^+ > |\pi^- >) = \frac{1}{\sqrt{3}}(\sqrt{2}|0,0> + |2,0>) \quad \text{(5.2)}$$

$$|A\rangle = \frac{1}{\sqrt{2}}(|\pi^- > |\pi^+ > - |\pi^+ > |\pi^- >) = |1,0>$$

where we used the convention $|\pi^+ >= |1,1>$. In $\pi^- \pi^+$ channel the $J = \text{even}$ and $J = \text{odd}$ transversity amplitudes thus involve isospin states $\frac{1}{\sqrt{2}}|S\rangle$ and $\frac{1}{\sqrt{2}}|A\rangle$, respectively. The two-pion states $|\pi^0 \pi^0\rangle$ and $|\pi^+ \pi^+\rangle$ correspond to even $I$

$$|\pi^0 \pi^0\rangle = \frac{1}{\sqrt{3}}(|0,0> - \sqrt{2}|2,0>) \quad \text{(5.3)}$$

$$|\pi^+ \pi^+\rangle = |2,2>$$

which implies only $J = \text{even}$ dipion states are allowed.

We now consider $S$-wave transversity amplitudes $S_{\tau}(\pm)$, $S_{\tau}(00)$ and $S_{\tau}(++)$ in three processes \( \pi^- p \to \pi^- \pi^+ n \), \( \pi^- p \to \pi^0 \pi^0 n \) and \( \pi^+ p \to \pi^+ \pi^+ n \). Using (5.2) and (5.3) we can express these amplitudes in terms of amplitudes $S_{I I_3}^{I_3}$ with definite total isospin $I$ and $I_3$. 

FIG. 5: $S$-wave and $D$-wave intensities from amplitude analysis at high $t$ [11].
\[ S_\tau(-+) = -\frac{1}{\sqrt{3}} \{ S_{\tau}^{00} + \frac{1}{2} \sqrt{2} S_{\tau}^{20} \} \]  
(5.4a)

\[ S_\tau(00) = -\frac{1}{\sqrt{3}} \{ S_{\tau}^{00} - \sqrt{2} S_{\tau}^{20} \} \]  
(5.4b)

\[ S_\tau(++) = S_{\tau}^{22} \]  
(5.4c)

Assuming the invariance of the amplitudes \( S_\tau(c) \), \( c = (-+), (00), (++) \) under the rotations in isospin space, the isospin amplitudes \( S_I^{3I_3} \) then do not depend on the component \( I_3 \) and we have

\[ S_{\tau}^{20} = S_{\tau}^{22} = S_\tau(++) \]  
(5.5)

From (5.4a) and (5.4b) we then get the isospin relation between the \( S \)-wave amplitudes

\[ S_\tau(00) = S_\tau(-+) + \sqrt{\frac{3}{2}} S_\tau(++) \]  
(5.6)

It is useful to write the following combinations of this equation

\[ \sqrt{\frac{3}{2}} S_\tau(++) = S_\tau(00) - S_\tau(-+) \]  
(5.7a)

\[ S_\tau(-+) = S_\tau(00) - \sqrt{\frac{3}{2}} S_\tau(++) \]  
(5.7b)

\[ S_\tau(00) = S_\tau(-+) + \sqrt{\frac{3}{2}} S_\tau(++) \]  
(5.7c)

and calculate the \( S \)-wave intensities

\[ I_S(c) = |S_u(c)|^2 + |S_d(c)|^2 \]  
(5.8)

for \( c = (++), (-+), (00) \) using expressions on r.h.s. of (5.7). With

\[ I_S(2) = \frac{3}{2} I_S(++) \]  
(5.9)

we then obtain

\[ I_S(-+) + I_S(00) - I_S(2) = +2 \sum_\tau \text{Re}[S_\tau(-+) S_\tau^*(00)] \]  
(5.10a)

\[ I_S(-+) - I_S(00) - I_S(2) = -2 \sqrt{\frac{3}{2}} \sum_\tau \text{Re}[S_\tau(00) S_\tau^*(++)] \]  
(5.10b)

\[ I_S(-+) - I_S(00) + I_S(2) = -2 \sqrt{\frac{3}{2}} \sum_\tau \text{Re}[S_\tau(-+) S_\tau^*(++)] \]  
(5.10c)

The interference terms on the r.h.s. of (5.10) are scalar products \( <A_S(c)|A_S(c')> \) of four-vectors

\[ |A_S(c)> = (\text{Re} S_u(c), \text{Im} S_u(c), \text{Re} S_d(c), \text{Im} S_d(c)) \]  
(5.11)

in a Euclidian 4-dimensional space with the norm \( <A_S(c)|A_S(c)> = I_S(c) \) and scalar product

\[ <A_S(c)|A_S(c')> = \sqrt{I_S(c)} \sqrt{I_S(c')} \cos \Omega_{cc'}(S) \]  
(5.12)
FIG. 6: Comparison of intensity $I_S(-+)$ from Ref. [21] with the intensity $I_S(00)$ from Ref. [20].

FIG. 7: Intensities $I_S(2)$ and $I_{D^0}(2)$ from CERN data on $\pi^+\pi^+$ at 12.5 GeV/c [30] scaled to 17.2 GeV/c.
where $\Omega_{cc'}(S)$ is an angle between the vectors $A_S(c)$ and $A_S(c')$. The relations (5.10) then read

\begin{align}
I_S(-+) + I_S(00) - I_S(2) &= +2\sqrt{I_S(-+)\sqrt{I_S(00)}} \cos \Omega_1(S) \\
I_S(-+) - I_S(00) - I_S(2) &= -2\sqrt{I_S(00)\sqrt{I_S(2)}} \cos \Omega_2(S) \\
I_S(-+) - I_S(00) + I_S(2) &= -2\sqrt{I_S(-+)\sqrt{I_S(2)}} \cos \Omega_3(S)
\end{align}
(5.13a, b, c)

The equations (5.13) represent three linearly independent constraints on the measured spectra $I_S(-+), I_S(00)$ and $I_S(++ ) = \frac{2}{3}I_S(2)$ imposed by the requirement that the cosines have physical values. While the cosines are linearly independent, they satisfy a non-linear constraint

$$\cos^2 \Omega_1(S) + \cos^2 \Omega_2(S) + \cos^2 \Omega_3(S) - 2 \cos \Omega_1(S) \cos \Omega_2(S) \cos \Omega_3(S) = 1$$
(5.14)

The constraint (5.14) implies that for physical values of the cosines the phases satisfy a phase condition

$$\Omega_1(S) + \Omega_2(S) + \Omega_3(S) = 0$$
(5.15)

B. Data used in the test of isospin relations for the $S$-waves

To test the constraints (5.13) we need data on the intensities $I_S(c)$ from $\pi^-\pi^+, \pi^0\pi^0$ and $\pi^+\pi^-$ production in the dipion mass interval $580 \leq m \leq 1080$ MeV where we observe $\rho^0(770) - f_0(980)$ mixing. The available data allow to perform the tests at small momentum transfers $0.005 < |t| < 0.20$ (GeV/c)$^2$.

For the $\pi^-\pi^+$ channel we used our high resolution analysis using Monte Carlo method presented in Ref. [21]. It produced two solutions for the moduli $|S_u(i)|^2, i = 1, 2$ and $|S_d(j)|^2, j = 1, 2$. Two solutions for the $S$-wave intensity $I_S(-+)$ were used in this mass range corresponding to combinations (1,1) and (2,2) of solutions for the moduli

$$I_S(-+) \text{ Solution (1,1)} = |S_u(1)|^2 + |S_d(1)|^2$$
$$I_S(-+) \text{ Solution (2,2)} = |S_u(2)|^2 + |S_d(2)|^2$$
(5.16)

The results for $I_S(-+)$ from our analysis are shown in Figure 6. The unit for $d^2\sigma/dt dm$ in Ref. [26] used in our analysis can be converted to $\mu b/20 \text{ MeV}$ using a conversion factor $0.109 \mu b/20 \text{ MeV} = 1000 \text{ events}/20 \text{ MeV}$.

For the $\pi^0\pi^0$ channel we used the BNL data at 18.3 GeV/c [20]. The BNL data were converted from native BNL units ”intensity/40 MeV” into our units ”1000 events/20 MeV” using a conversion factor $F = 0.6700 \times 10^{-4}$. We obtained this factor by comparing the $f_2(1270)$ peak value in their Figure 5F given in units ”intensity/40 MeV” with the value of corresponding 4 bins at $f_2(1270)$ peak in their Figure 4a given in units ”events/10 MeV”. The data in two bins $0.01 < |t| < 0.10$ (GeV/c)$^2$ and $0.10 < |t| < 0.20$ (GeV/c)$^2$ were combined by addition to a single bin $0.01 < |t| < 0.20$ (GeV/c)$^2$ corresponding to the CERN measurements. The data were then interpolated to 20 MeV bins and scaled to 17.2 GeV/c using phase and flux factor $K(s, m^2)$ given by [17]

$$K(s, m^2) = \frac{G(s, m^2)}{\text{Flux}(s)}$$
(5.17)

$$G(s, m^2) = \frac{1}{(4\pi)^3} \frac{q(m^2)}{\sqrt{|s - (M + \mu)^2| |s - (M - \mu)^2|}}$$

$$\text{Flux}(s) = 4M \rho_{lab}$$
FIG. 8: Test of isospin relations with Solution (1,1) for $S$-wave intensity $I_S(-\pm)$ from analysis [21].

FIG. 9: Test of isospin relations with Solution (2,2) for $S$-wave intensity $I_S(\pm)$ from analysis [21].
ANALYSES ON POLARIZED TARGETS.

VI. SURVEY OF EVIDENCE FOR $\rho^0(770) - f_0(980)$ SPIN MIXING FROM AMPLITUDE ANALYSES ON POLARIZED TARGETS.

There are five measurements of production $\pi N \to \pi^- \pi^+ N$ on polarized target at four energies: (1) CERN measurement of $\pi^- p \to \pi^- \pi^+ n$ at 17.2 GeV/c at $0.005 < |t| < 0.20$ (GeV/c)$^2$ [8][10].
FIG. 10: $S$-wave moduli $|S_r|^2$ in $\pi^-p \to \pi^-\pi^+n$ at 17.2 GeV/c at low $t$. Data from Ref. [8].

FIG. 11: $S$-wave moduli $|S_r|^2$ in $\pi^-p \to \pi^-\pi^+n$ at 17.2 GeV/c at low $t$. Data from Ref. [12].
FIG. 12: Moduli of $S$-wave amplitudes $|S_r|^2$ and $|S_0|^2$ from Analyses I and II (with line) [21].

FIG. 13: $S$-wave moduli $|S_r|^2$ in $\pi^+ n \rightarrow \pi^+ \pi^- n$ at 11.85 GeV/c at high $t$. Data from Ref [13].
FIG. 14: Intensities $I(S)$ and $I(L)$ in $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c at high $t$. Data from Ref [11].

FIG. 15: Intensities $I(S)$ and $I(L)$ in $\pi^- p \rightarrow \pi^- \pi^+ n$ at 1.78 GeV/c at low $t$. Data from Ref [18].
FIG. 16: Relative phases $\Phi(L_\tau) - \Phi(S_\tau)$ in $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c at low $t$ from Ref. [12].

FIG. 17: Phases $\Phi_{L_\tau} - \Phi_{S_\tau}$ and $\Phi_{L_\tau}^0 - \Phi_{S_\tau}^0$ from Analyses I and II (with line) at 17.2 GeV/c at low $t$ [21].
FIG. 18: Intensities $I(S)$ in $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c at low $t$ from analysis [12] (top) and Analysis I [21] (bottom).

(2) CERN measurement of $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c at $0.20 < |t| < 1.0$ (GeV/c)$^2$ [11].

(3) ITEP measurement of $\pi^- p \rightarrow \pi^- \pi^+ n$ at 1.78 GeV/c at $0.0 < |t| < 0.20$ (GeV/c)$^2$ [18].

(4) CERN-Saclay measurement of $\pi^+ n \rightarrow \pi^+ \pi^- p$ at 5.98 GeV/c at $0.20 < |t| < 0.40$ (GeV/c)$^2$ [13].

(5) CERN-Saclay measurement of $\pi^+ n \rightarrow \pi^+ \pi^- p$ at 11.85 GeV/c at $0.20 < |t| < 0.40$ (GeV/c)$^2$ [13].

There are several amplitude analyses of the CERN measurement at low $t$ [8–10, 16, 17] with the most recent in Ref. [12, 21, 31]. Amplitude analyses of the CERN measurement at high $t$ and of the ITEP measurement are presented in Ref. [11] and Ref. [18], respectively. Amplitude analyses of the CERN-Saclay (CS) data are presented in Ref. [14, 16, 17] and in this work (Figure 17). The analyses used different methods to determine the amplitudes and their errors: $\chi^2$ fits [8–11, 31], Monte Carlo analytical solutions [16, 17, 21] and analytical solutions with error propagation [14, 18]. All amplitude analyses are mutually consistent in providing a clear evidence for $\rho^0(770) - f_0(980)$ mixing in the $S$-wave amplitudes.

Figures 10-12 show the $S$-wave moduli from three analyses [8, 12, 21] of CERN measurements at low $t$. Authors of Ref. [8] present normalized moduli $|S_\tau|^2$. Figure 10 presents the corresponding unnormalized moduli $|S_\tau|^2 = |\tilde{S}_\tau|^2 d^2\sigma/dtdm$. Authors of Ref. [8, 12] present $S$-wave intensity $I(S) = |S_u|^2 + |S_d|^2$ and the ratio of the moduli $R = |S_u|/|S_d|$.

From this data it is a simple matter to reconstruct the moduli $|S_\tau|^2$. The moduli from the earlier analysis [8] in 40 MeV bins from 600-1520 MeV are presented in Figure 5 of Ref. [17]. These results agree with our analysis [17] and are not reproduced here. The moduli from the later analysis [12] in 20 MeV bins are shown in Figure 11 below 1080 MeV. Figure 12 presents our latest results for the moduli from analyses on polarized target (Analysis I = $|S_\tau|^2$) and unpolarized target (Analysis II = $|S_\tau^0|^2$) using the same input data as the analysis in the Figure 11. Our Monte Carlo Analysis I and the $\chi^2$ fit analysis in Figure 11.
are very similar. Figures 10-12 from all three analyses show a clear presence of a $\rho^0(770)$ structure in both Solutions for the amplitude $|S_d|^2$. A comparison with Figure 1 of the new analysis [32] using spin mixing mechanism [23] shows a suppression of the structure near 930 MeV seen in the Solution 2 of $|S_d|^2$ in Figures 11 and 12. Spin mixing mechanism excludes the Solution 1 for $|S_t|^2$.

Figure 13 shows our results for $S$-wave moduli from our Monte Carlo amplitude analysis of the CERN-Saclay data on $\pi^+n \to \pi^-\pi^+p$ at 11.85 GeV/c. At these larger $t$ the Solution 1 is small for both transversity amplitudes while the Solution 2 shows a clear $\rho^0(770)$ structure in both $|S_u|^2$ and $|S_d|^2$. The results at 5.98 GeV/c are very similar.

Figure 14 shows the $S$-wave intensity from the amplitude analysis of the CERN data at high $t$ [11]. Again at these larger $t$ the Solution 1 is small while the Solution 2 suggests a pronounced $\rho^0(770)$ structure. Figure 15 shows the $S$-wave intensity from the analysis of the ITEP data [18] at low $t$. Here both Solutions indicate evidence for $\rho^0(770)$ structure but this structure is clearly prominent in the Solution 2. Note that in these two analyses $I(S) \approx I(L)$ in the Solutions 2 at $\rho^0(770)$ mass.

Figures 16 and 17 compare the relative phases $\Phi(L_{\tau}S_{\tau}^*) = \Phi(L_{\tau}) - \Phi(S_{\tau})$ from the $\chi^2$ fit analysis [12] and Monte Carlo analysis [21], respectively. Figure 17 includes results on polarized target (Analysis I = $\Phi_L - \Phi_S$) and unpolarized target (Analysis II = $\Phi_L^0 - \Phi_S^0$). The $\chi^2$ fits suggest a zero structure of these relative phases near 700-800 MeV which allows for the change of sign of $\Phi(L_{\tau}) - \Phi(S_{\tau})$ but does not require it. Such structure is in tension with the Monte Carlo Analyses I and II which allow for no change of sign of the phases. In fact, a detailed histogram analysis in steps of 1° showed that below 5° all degree bins for the Solution 2 of the phases $\Phi(L_uS_u^*)$ and $\Phi(L_dS_d^*)$ are empty for all mass bins except for just several events in a few mass bins below 800 MeV.
in both Analyses I and II. The results of the \( \chi^2 \) fits are in tension also with our new analysis \cite{32} using the spin mixing mechanism \cite{23} which shows even clearer non-zero structure of \( \Phi(L_\tau S_\tau^*) \) at all dipion masses.

The early analyses of the CERN data at low \( t \) \cite{9,10} were later updated in a new analysis by the Cracow group \cite{12}. It is instructive to compare the two solutions "up" and "down" for the \( S \)-wave intensity in the Cracow analysis with the corresponding two solutions \((1,1)\) and \((2,2)\) from the Monte Carlo analysis \cite{21}. The results are shown in Figure 18. A comparison shows that the Solution "up" is nearly identical to the Solution \((1,1)\), and that the Solution "down" is nearly identical to the Solution \((2,2)\).

The ITEP analysis is particularly interesting in that it shows a clear rho peak in both Solutions for the \( S \)-wave intensity \( I_S \) providing a support for the 1960's low energy analyses \cite{1,2}. A review of these analyses is given in \cite{6}. In analyses of the CERN data on polarized target at 17.2 GeV/c shown in Figure 18 the rho peak is somewhat obscured in the Solution "down"/\((2,2)\). Approximate methods used in 1970's to determine \( I_S \) from unpolarized CERN-Munich data appear to have missed the rho peak in this solution. The two solutions of the exact analysis of the same unpolarized target data both show the rho peak and are nearly identical to the two solutions for \( I_S(1,1) \) and \( I_S(2,2) \) from the analysis on polarized target (Figure 4 of Ref. \cite{21}).

While the Solutions "up"/\((1,1)\) show a pronounced \( \rho^0(770) \) structure this structure is less pronounced in the Solutions "down"/\((2,2)\) due to a broad structure near 930 MeV. This structure is absent in Solution 2 for the amplitude \( |S_d|^2 \) in our new analysis \cite{32} using spin mixing mechanism in which the observed amplitudes are a certain mixture of \( S \)-matrix amplitudes with different spins \cite{22}. Spin mixing mechanism allows to extract information on the \( D \)-wave amplitudes and the amplitude analysis can be performed without and with the determination of the \( D \)-waves. Results for the Solution \((2,2)\) of the \( S \)-wave and \( P \)-wave intensities \( I(S) \) and \( I(L) \) without and with \( D \)-wave determination are presented in Figure 19. The evidence for \( \rho^0(770) \) in the Solution \( I_S(2,2) \) becomes clear. The sudden drop of \( I(S) \) and the sudden rise of \( I(L) \) at and above \( f_0(980) \) mass is due to a strong mixing at these masses of \( S \)-wave and \( P \)-wave \( S \)-matrix amplitudes. Spin mixing mechanism excludes the Solution \((1,1)\) of \( I(S) \) and \( I(L) \).

VII. CONCLUSIONS.

We have studied the response of the analytical solutions of the \( S \)- and \( P \)-wave subsystem in \( \pi^- p \to \pi^- \pi^+ n \) on polarized target to the presence of \( D \)-wave amplitudes. We have found that the \( \rho^0(770) \) \(- \) \( f_0(980) \) mixing is consistent with the presence of \( D \)-wave amplitudes with helicities \( \lambda \leq 1 \) (Response analysis A) as well as with helicities \( \lambda \leq 2 \) (Response analysis B) provided the \( D \)-wave amplitudes are not too large below 750 MeV, as expected from the CERN measurements on unpolarized and polarized targets. Above 750 MeV the spin mixing effect is consistent with larger \( D \)-waves amplitudes. This result is in agreement with the amplitude analysis of \( S \), \( P \) and \( D \) wave subsystem at high momentum transfer \( t \) below 960 MeV. The observed \( \rho^0(770) \) \(- \) \( f_0(980) \) mixing is thus a real effect not generated by the small \( D \)-wave contamination of the input data. We have also shown that the spin mixing \( S \)-wave amplitudes in \( \pi^- p \to \pi^- \pi^+ n \) are consistent with isospin relations between the \( S \)-wave amplitudes in \( \pi^- \pi^+, \pi^0 \pi^0 \) and \( \pi^+ \pi^- \) channels and thus with the data on \( \pi^- p \to \pi^0 \pi^0 n \) and \( \pi^+ p \to \pi^+ \pi^+ n \).

Finally we have presented a complete survey of evidence for \( \rho^0(770) \) \(- \) \( f_0(980) \) mixing. We conclude that all amplitude analyses of all five measurements on polarized targets show a clear evidence for \( \rho^0(770) \) \(- \) \( f_0(980) \) mixing in the \( S \)-wave moduli and intensities. Apart from the tension in the zero structure of relative phases \( \Phi_L - \Phi_S \) near 700-800 MeV the relative phases have similar magnitudes. The tension in the zero structure of \( \Phi_L - \Phi_S \) may reflect the fact that while in our
analysis the cosine conditions are imposed on solutions for the amplitudes for every Monte Carlo sampling of the data error volume they are not imposed on the amplitudes by the $\chi^2$ minimization program. This difference appears significant only near 700-800 MeV.

The consistency of $\rho^0(770) - f_0(980)$ mixing with the presence of $D$-wave amplitudes, with the amplitude analysis of $\pi^- p \rightarrow \pi^0 \pi^0 n$ BNL data and the mutual consistency of all amplitude analyses on polarized target significantly strengthen the experimental evidence for $\rho^0(770) - f_0(980)$ spin mixing $[21]$. This evidence is further supported by our new amplitude analysis $[32]$ using spin mixing mechanism developed in Ref. $[22, 23]$. The origin of $\rho^0(770) - f_0(980)$ spin mixing is in a new non-standard interaction of particle scattering processes with a quantum environment $[22, 23]$. In Ref. $[32]$ we propose to identify this environment with dark matter.

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