ON A MIXED ARITHMETIC-GEOMETRIC MEAN INEQUALITY

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ABSTRACT. We extend a result of Holland on a mixed arithmetic-geometric mean inequality.

1. INTRODUCTION

Let $M_{n,r}(q,x)$ be the generalized weighted power means: $M_{n,r}(q,x) = (\sum_{i=1}^{n} q_i x_i^r)^{\frac{1}{r}}$, where $q = (q_1, q_2, \cdots, q_n)$, $x = (x_1, x_2, \cdots, x_n)$, $q_i > 0, 1 \leq i \leq n$ with $\sum_{i=1}^{n} q_i = 1$. Here $M_{n,0}(q,x)$ denotes the limit of $M_{n,r}(q,x)$ as $r \to 0^+$. Unless specified, we always assume $x_i > 0, 1 \leq i \leq n$. When there is no risk of confusion, we shall write $M_{n,r}$ for $M_{n,r}(q,x)$ and we also denote $A_n, G_n$ for the arithmetic mean $M_{n,1}$, geometric mean $M_{n,0}$, respectively.

For fixed $x = (x_1, \cdots, x_n), w = (w_1, \cdots, w_n)$ with $w_1 > 0, w_i \geq 0$, we define $x_i = (x_1, \cdots, x_i)$, $w_i = (w_1, \cdots, w_i), W_i = \sum_{j=1}^{i} w_j, M_{i,r} = M_{i,r}(w_i/W_i, x_i), M_{i,r} = (M_{1,r}, \cdots, M_{i,r})$. The following result on mixed mean inequalities is due to Nanjundiah [5] (see also [1]):

**Theorem 1.1.** Let $r > s$ and $n \geq 2$. If for $2 \leq k \leq n - 1$, $W_n w_k - W_k w_n > 0$. Then

$$M_{n,s}(M_{n,r}) \geq M_{n,r}(M_{n,s}),$$

with equality holding if and only if $x_1 = \cdots = x_n$.

It is easy to see that the case $r = 1, s = 0$ of Theorem 1.1 follows from the following Popoviciu-type inequalities established in [4] (see also [1, Theorem 9]):

**Theorem 1.2.** Let $n \geq 2$. If for $2 \leq k \leq n - 1$, $W_n w_k - W_k w_n > 0$, then

$$W_n^{-1} (\ln M_{n-1,0}(M_{n-1,1}) - \ln M_{n-1,1}(M_{n-1,0})) \leq W_n (\ln M_{n,0}(M_{n,1}) - \ln M_{n,1}(M_{n,0}))$$

with equality holding if and only if $x_n = M_{n-1,0} = M_{n-1,1}(M_{n-1,0})$.

In [6], the following Rado-type inequalities were established:

**Theorem 1.3.** Let $s < 1$ and $n \geq 2$. If for $2 \leq k \leq n - 1$, $W_n w_k - W_k w_n > 0$, then

$$W_n^{-1} (M_{n-1,s}(M_{n-1,1}) - M_{n-1,1}(M_{n-1,s})) \leq W_n (M_{n,s}(M_{n,1}) - M_{n,1}(M_{n,s}))$$

with equality holding if and only if $x_1 = \cdots = x_n$ and the above inequality reverses when $s > 1$.

The above theorem is readily seen to imply Theorem 1.1. In [3], Holland further improved the condition in Theorem 1.3 for the case $s = 0$ by proving the following:

**Theorem 1.4.** Let $n \geq 2$. If $W_n^2 \geq w_n \sum_{i=1}^{n-2} W_i$ with the empty sum being 0, then

$$(1.1) \quad W_n^{-1} (M_{n-1,0}(M_{n-1,1}) - M_{n-1,1}(M_{n-1,0})) \leq W_n (M_{n,0}(M_{n,1}) - M_{n,1}(M_{n,0}))$$

with equality holding if and only if $x_1 = \cdots = x_n$.

It is our goal in this paper to extend the above result of Holland by considering the validity of inequality (1.1) for the case $W_n^2 < w_n \sum_{i=1}^{n-2} W_i$. Note that this only happens when $n \geq 3$. In the next section, we apply the approach in [2] to prove the following

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Theorem 1.5. Let \( n \geq 3 \). Inequality (1.1) holds when the following conditions are satisfied:

\[
0 < \frac{w_n \sum_{i=1}^{n-2} W_i}{W_n^2} - 1 \leq \frac{w_1}{w_n}, \quad \frac{W_n-1}{W_n} \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{W_i} \right)^{w_i w_n} \leq 1,
\]

\[
\left( \frac{W_n-1 w_n}{W_n w_1} \left( \sum_{i=1}^{n-2} \frac{W_i w_n}{W_n^2} - 1 \right) + \frac{w_n}{W_n} \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{w_{i+1}} \right)^{w_{i+1} w_n} \right) \leq 1.
\]

In the above theorem, we do not give the condition for the equality in (1.1) to hold. In Section 3 we show that there do exist sequences \( \{w_i\}_{i=1}^{n} \) that satisfy the conditions of Theorem 1.5.

2. Proof of Theorem 1.5

We may assume that \( x_i > 0, w_i > 0, 1 \leq i \leq n \) and the case \( x_i = 0 \) or \( w_i = 0 \) for some \( i \) follows by continuity. We recast (1.1) as

\[
G_n(A_n) - \frac{W_n-1}{W_n} G_{n-1}(A_{n-1}) - \frac{w_n}{W_n} G_n \geq 0.
\]

Note that

\[
G_n(A_n) = \left( G_{n-1}(A_{n-1}) \right)^{W_n-1/W_n} A_n^{w_n/W_n}, \quad G_{n-1}(A_{n-1}) = A_n^{\prod_{i=1}^{n-1} \left( \frac{A_i}{A_{i+1}} \right)^{W_i/W_n}}.
\]

Dividing \( G_n(A_n) \) on both sides of (2.1) and using (2.2), we can recast (2.1) as:

\[
\frac{W_n-1}{W_n} \prod_{i=1}^{n-1} \left( \frac{A_i}{A_{i+1}} \right)^{W_i w_n/(W_n-1 W_n)} + \frac{w_n}{W_n} \prod_{i=1}^{n} \left( \frac{x_i}{A_i} \right)^{w_i/W_n} \leq 1.
\]

We express \( x_i = (W_i A_i - W_{i-1} A_{i-1})/w_i, 1 \leq i \leq n \) with \( W_0 = A_0 = 0 \) to recast (2.3) as

\[
\frac{W_n-1}{W_n} \prod_{i=1}^{n-1} \left( \frac{A_i}{A_{i+1}} \right)^{w_i w_n/(W_n-1 W_n)} + \frac{w_n}{W_n} \prod_{i=1}^{n} \left( \frac{W_i A_i - W_{i-1} A_{i-1}}{w_i A_i} \right)^{w_i/W_n} \leq 1.
\]

We set \( y_i = A_i/A_{i+1}, 1 \leq i \leq 2 \) to further recast the above inequality as

\[
\frac{W_n-1}{W_n} \prod_{i=1}^{n-1} y_i^{w_i w_n/(W_n-1 W_n)} + \frac{w_n}{W_n} \prod_{i=1}^{n} \left( \frac{W_i + 1}{w_{i+1} y_i} - \frac{W_i}{w_i y_i} \right)^{w_{i+1}/W_n} \leq 1.
\]

We now regard the right-hand side expression above as a function of \( y_{n-1} \) only and define

\[
f(y_{n-1}) = \frac{W_n-1}{W_n} c \cdot y_{n-1}^{w_n/W_n} + \frac{w_n}{W_n} c' \cdot \left( \frac{W_n}{w_n} - \frac{W_n-1}{w_n} y_{n-1} \right)^{w_n/W_n},
\]

where

\[
c = \prod_{i=1}^{n-2} y_i^{w_i w_n/(W_n-1 W_n)}, \quad c' = \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{w_{i+1}} - \frac{W_{i}}{w_{i+1}} \right)^{y_{i+1}/W_n}.
\]

On setting \( f'(y_{n-1}) = 0 \), we find that

\[
y_{n-1} = \left( \frac{W_n-1}{W_n} + \frac{w_n}{W_n} \left( \frac{c'}{c} \right)^{w_n/W_n} \right)^{-1}.
\]

It is easy to see that \( f(y_{n-1}) \) is maximized at the above value with its maximal value being

\[
\left( \frac{W_n-1}{W_n} c^{W_n/W_n} + \frac{w_n}{W_n} \left( \frac{c'}{c} \right)^{W_n/W_n} \right)^{W_n-1/W_n}.
\]
Thus, in order for inequality (2.4) to hold, it suffices to have
\[
\frac{W_{n-1}}{W_n} e^{W_n/W_{n-1}} + \frac{w_n}{W_n} (c')^{W_n/W_{n-1}} \leq 1.
\]

Explicitly, the above inequality is
\[
g(y_1, y_2, \ldots, y_{n-2}) := \frac{W_{n-1}}{W_n} \prod_{i=1}^{n-2} y_i^{W_i/W_{n-1}} + \frac{w_n}{W_n} \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_i} \right)^{w_{i+1}/W_{n-1}} \leq 1.
\]

Let \((a_1, a_2, \ldots, a_{n-2}) \in [0, W_2/W_1] \times [0, W_3/W_2] \times \cdots \times [0, W_{n-1}/W_{n-2}]\) be the point in which the absolute maximum of \(g\) is reached. If one of the \(a_i\) equals 0 or \(W_{i+1}/W_i\), then it is easy to see that we have
\[
(2.5)\quad g(a_1, a_2, \ldots, a_{n-2}) \leq \max \left( \frac{W_{n-1}}{W_n} \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{W_i} \right)^{w_{i+1}/W_{n-1}}, \frac{w_n}{W_n} \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{w_{i+1}} \right)^{w_{i+1}/W_{n-1}} \right).
\]

If the point \((a_1, a_2, \ldots, a_{n-2})\) is an interior point, then we have
\[
\nabla g(a_1, a_2, \ldots, a_{n-2}) = 0.
\]

It follows that for every \(1 \leq i \leq n - 2\), we have
\[
(2.6)\quad \frac{\prod_{i=1}^{n-2} a_i^{W_i w_i/W_{n-1}}}{\prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{w_{i+1}} - \frac{W_i}{w_i} \right)^{w_{i+1}/W_{n-1}}} = \frac{a_i}{W_{i+1}/W_i - 1} := \frac{1}{d},
\]
where \(d > 0\) is a constant (depending on the \(w_i\)). In terms of \(d\), we have
\[
a_i = \frac{W_{i+1}}{d w_{i+1} + W_i}.
\]

We use this to recast the first equation in (2.6) as
\[
\prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{d w_{i+1} + W_i} \right)^{W_i w_i/W_{n-1}} = \frac{1}{d} \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{w_{i+1}} + \frac{d w_{i+1}}{d w_{i+1} + W_i} \right)^{w_{i+1}/W_{n-1}}.
\]

We recast the above equality as
\[
\ln \left( \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{W_{n-1}} \right)^{W_i w_i/W_{n-1} - w_{i+1}/W_{n-1}} \right) = \sum_{i=1}^{n-2} \left( \frac{W_i w_n}{W_{n-1}^2} - \frac{w_{i+1}}{W_{n-1}} \right) \ln (d w_{i+1} + W_i) - \frac{w_1}{W_{n-1}} \ln d := h(d).
\]

Note that the above equality holds when \(d = 1\) and we have
\[
h'(d) = \sum_{i=1}^{n-2} \left( \frac{W_i w_n}{W_{n-1}^2} - \frac{w_{i+1}}{W_{n-1}} \right) \frac{1}{d + W_i/w_{i+1}} - \frac{w_1}{W_{n-1}} \frac{1}{d}.
\]

As
\[
\frac{W_i w_n}{W_{n-1}^2} - \frac{w_{i+1}}{W_{n-1}} \geq 0 \iff \frac{W_i}{w_{i+1}} \geq \frac{W_{n-1}}{W_n},
\]
it follows that
\[
h'(d) \leq \sum_{i=1}^{n-2} \left( \frac{W_i w_n}{W_{n-1}} - \frac{w_{i+1}}{w_{n-1}} \right) \frac{1}{d + W_{n-1}/w_n} - \frac{w_1}{W_{n-1} d}
\]
\[
\leq \frac{\left( \sum_{i=1}^{n-2} \frac{W_i w_n}{W_{n-1}} - 1 \right) d - \frac{w_n}{w_{n-1}}}{d(d + W_{n-1}/w_n)}.
\]

Therefore, when
\[
\sum_{i=1}^{n-2} \frac{W_i w_n}{W_{n-1}} - 1 \leq 0,
\]
the function \( h(d) \) is a decreasing function of \( d \) so that \( d = 1 \) is the only value that satisfies (2.6) and we have \( a_i = 1 \) correspondingly with \( g(1, 1, \ldots, 1) = 1 \) and this allows us to recover Theorem 1.4 by combining the observation that the right-hand side expression of (2.4) is an increasing function of \( w_n \) for fixed \( w_i, 1 \leq i \leq n - 1 \) with the discussion in the next section.

Suppose now
\[
\sum_{i=1}^{n-2} \frac{W_i w_n}{W_{n-1}} - 1 > 0,
\]
then the function \( h(d) \) is a decreasing function of \( d \) for
\[
d \leq \frac{\frac{w_n}{w_{n-1}}}{\sum_{i=1}^{n-2} \frac{W_i w_n}{W_{n-1}}} := d_0.
\]
If follows that if \( d_0 \geq 1 \), then \( d = 1 \) is the only value \( \leq d_0 \) that satisfies (2.6) and we have \( a_i = 1 \) correspondingly with \( g(1, 1, \ldots, 1) = 1 \). We further note that for any \( d \geq d_0 \) satisfying (2.6), the value of \( g \) at the corresponding \( a_i \) satisfies
\[
g(a_1, a_2, \ldots, a_{n-2}) = \left( \frac{W_{n-1}}{dW_n} + \frac{w_n}{W_n} \right) \prod_{i=1}^{n-2} \left( \frac{W_i + 1}{w_i + 1} \cdot \frac{d w_i + 1}{d w_i + W_i} \right)^{w_i + 1/W_{n-1}}
\]
\[
\leq \left( \frac{W_{n-1}}{dW_n} + \frac{w_n}{W_n} \right) \prod_{i=1}^{n-2} \left( \frac{W_i + 1}{w_i + 1} \right)^{w_i + 1/W_{n-1}}
\]
\[
\leq \left( \frac{W_{n-1}}{d_0W_n} + \frac{w_n}{W_n} \right) \prod_{i=1}^{n-2} \left( \frac{W_i + 1}{w_i + 1} \right)^{w_i + 1/W_{n-1}}.
\]
Combining this with (2.5), we see that inequality (2.4) holds when the conditions in (1.2) are satisfied and this completes the proof of Theorem 1.5.

3. A Further Discussion

We show in this section that there does exist sequences \( \{w_i\}_{i=1}^{n} \) satisfying the conditions of Theorem 1.5. To see this, we note that the left-hand side expression of (2.7) vanishes when
\[
w_n = \frac{W_{n-1}^2}{\sum_{i=1}^{n-2} W_i}.
\]
It follows by continuity that such sequences \( \{w_i\}_{i=1}^{n} \) satisfying the conditions of Theorem 1.5 exist as long as the positive sequence \( \{w_i\}_{i=1}^{n} \) with \( w_i, 1 \leq i \leq n - 1 \) being arbitrary and \( w_n \) defined
by (3.1) satisfies the last two inequalities of (1.2) with strict inequalities there. It is readily checked that these inequalities become

\[
(\sum_{i=1}^{n-2} W_i) \sum_{i=1}^{n-2} W_i W_{n-1} < \prod_{i=1}^{n-2} W_i^{w_i},
\]

(3.2)

\[
W_{n-1} \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{w_{i+1}} \right)^{w_{i+1}/W_{n-1}} < 1.
\]

(3.3)

Now, it is easy to see that inequality (3.2) holds when \( n = 3 \). It follows that it holds for all \( n \geq 3 \) by induction as long as we have

\[
(\sum_{i=1}^{n-2} W_i) \sum_{i=1}^{n-2} W_i W_{n-1} \geq \prod_{i=1}^{n-2} W_i^{w_i} W_{n-1}.
\]

(3.4)

The right-hand side expression above when regarded as a function of \( W_n \) only is maximized at

\[
W_n = \frac{W_{n-1} \sum_{i=1}^{n-1} W_i}{\sum_{i=1}^{n-1} W_i}.
\]

As the inequality in (3.4) becomes an equality with this value of \( W_n \), we see that inequality (3.2) does hold for all \( n \geq 3 \).

Note that it follows from the arithmetic-geometric mean inequality that

\[
\frac{W_{n-1}}{\sum_{i=1}^{n-1} W_i} \prod_{i=1}^{n-2} \left( \frac{W_{i+1}}{w_{i+1}} \right)^{w_{i+1}/W_{n-1}} \leq \frac{W_{n-1}}{\sum_{i=1}^{n-1} W_i} \left( \sum_{i=0}^{n-2} \frac{W_{i+1}}{w_{i+1}} \cdot \frac{w_{i+1}}{W_{n-1}} \right) = 1.
\]

As one checks easily that the above inequality is strict in our case, we see that inequality (3.3) also holds. We therefore conclude the existence of sequences \( \{w_i\}_{i=1}^{n} \) satisfying the conditions of Theorem 1.5.

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