DYNAMICS UNDER GEOMETRIC DISSIPATION

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Abstract

We give sufficient conditions for asymptotic stabilization of equilibrium points and periodic orbits of a dynamical system when we add a geometric dissipation of gradient type. We also describe the domain of attraction in the case of asymptotic stability.

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1 Introduction

We consider the dynamical system

\[ \dot{x} = X(x), \tag{1.1} \]

where \( X \in \mathcal{X}(M) \) with \((M, g)\) a smooth finite dimensional Riemannian manifold. We will denote by \( x_{un}(\cdot, x_0) \) the solution of (1.1) with the initial condition \( x_0 \). Suppose that we have \( F_1, \ldots, F_k, G \in C^\infty(M) \) conserved quantities for dynamics (1.1). In [3] have been constructed a perturbation vector field that conserves \( F_1, \ldots, F_k \) and dissipates \( G \) after a prescribed rule given by \( h \in C^\infty(M) \). If we take \( h(x) = \det \Sigma_{(F_1, \ldots, F_k, G)}(x) \), then the perturbation is given by the standard control vector field

\[ \nu_0 = \sum_{i=1}^{k} (-1)^{i+k+1} \det \Sigma_{(F_1, \ldots, F_k)}(F_i, \ldots, F_k, G) \nabla F_i + \det \Sigma_{(F_1, \ldots, F_k)} \nabla G. \tag{1.2} \]

For \( f_1, \ldots, f_r, g_1, \ldots, g_s : M \to \mathbb{R} \) smooth functions on the manifold \((M, g)\) and \( \langle \cdot, \cdot \rangle \) the scalar product induced by Riemannian metric \( g \), we use the notation

\[ \Sigma_{(g_1, \ldots, g_s)}^{(f_1, \ldots, f_r)}(x) = \begin{pmatrix} \langle \nabla g_1, \nabla f_1 \rangle & \ldots & \langle \nabla g_1, \nabla f_r \rangle \\ \vdots & \ddots & \vdots \\ \langle \nabla g_s, \nabla f_1 \rangle & \ldots & \langle \nabla g_s, \nabla f_r \rangle \end{pmatrix}. \tag{1.3} \]

In [3] has been given three different formulations for the standard control vector field \( \nu_0 \):

(i) the covariant formulation, \( \nu_0 = (-1)^{n+1} \frac{\partial}{\partial g} (dF_1 \wedge \cdots \wedge dF_k \wedge dG) \);

(ii) the contravariant formulation, \( \nu_0 = i_{\mathbf{T}} \mathbf{d}g \), where \( \mathbf{T} : \Omega^1(M) \times \Omega^1(M) \to \mathbb{R} \) is the symmetric contravariant 2-tensor given by

\[ \mathbf{T} := \sum_{i,j=1}^{k} (-1)^{i+j+1} \det \Sigma_{(F_1, \ldots, F_k)}^{(F_i, \ldots, F_k)}(F_i, \ldots, F_k) \nabla F_i \otimes \nabla F_j + \det \Sigma_{(F_1, \ldots, F_k)}^{(F_i, \ldots, F_k)} g^{-1}; \tag{1.4} \]

(iii) the formulation with orthogonal projection, \( \nu_0(x) = \det \Sigma_{(F_1, \ldots, F_k)}^{(F_1, \ldots, F_k)}(x) P_{T_x L_c}(\nabla G(x)) \), where \( L_c \) is the regular leaf of \( \mathcal{F} = (F_1, \ldots, F_k) : M \to \mathbb{R}^k \) which contains \( x \) and \( P_{T_x L_c} : T_x M \to T_x M \) is the orthogonal projection.
The aim of this paper is to study the dynamics of the \textbf{geometrically dissipated system}

\[
\dot{x} = X(x) - v_0(x).
\]  

(1.5)

We will denote by \(x_p(t, x_0)\) the solution of (1.5) with the initial condition \(x_0\). By construction of the standard control vector field \(v_0\) we have that the function \(G\) decreases along the solutions of the geometrically dissipated system (1.5), i.e.

\[
dG{dt}(x_p(t, x)) = - \det \Sigma^{\{F_1, \ldots, F_k, G\}}(x_p(t, x)) \leq 0. \tag{1.6}
\]

The standard control vector field can be formally written as

\[
v_0(x) = \det \begin{pmatrix}
< \nabla F_1(x), \nabla F_1(x) > & \ldots & < \nabla F_k(x), \nabla F_1(x) > & < \nabla G(x), \nabla F_1(x) > \\
< \nabla F_1(x), \nabla F_k(x) > & \ldots & < \nabla F_k(x), \nabla F_k(x) > & < \nabla G(x), \nabla F_k(x) > \\
\nabla F_1(x) & \ldots & \nabla F_k(x) & \nabla G(x)
\end{pmatrix}
\]

and consequently, we have the following result.

\textbf{Lemma 1.1.} For \(x \in M\) the following are equivalent:

(i) \(v_0(x) = 0\);

(ii) \(\nabla F_1(x), \ldots, \nabla F_k(x), \nabla G(x)\) are linear dependent;

(iii) \(\det \Sigma^{\{F_1, \ldots, F_k, G\}}(x) = 0\).

\textbf{Proof.} For the implication (ii) \(\Rightarrow\) (i) we have that in the formal determinant that defines \(v_0\) one of the column is a linear combination of the remaining columns. The implication (i) \(\Rightarrow\) (ii) is obvious from the definition of \(v_0\). The equivalence between (ii) and (iii) is a well known result in linear algebra. \(\square\)

\section{Equilibrium points}

In this section we study the equilibrium points for the geometrically dissipated system (1.5).

\textbf{Proposition 2.1.} We have \(X(x) - v_0(x) = 0\) if and only if \(X(x) = 0\) and \(v_0(x) = 0\).

\textbf{Proof.} The implication "\(\Leftarrow\)" is trivial.

For the other implication, from (1.6) we have that \(dG{dt}(x_p(t, x)) = - \det \Sigma^{\{F_1, \ldots, F_k, G\}}(x_p(t, x))\). If \(x\) is an equilibrium point of the vector field \(X - v_0\) then \(\det \Sigma^{\{F_1, \ldots, F_k, G\}}(x) = 0\) and consequently, \(v_0(x) = 0\) and also \(X(x) = 0\). \(\square\)

We denote by \(E_{un}\) and \(E_p\) the sets of equilibrium points for the unperturbed system (1.4), respectively the geometrically dissipated system (1.5). A relevant set for the perturbed dynamics is given by

\[
\text{Inv} := \{ x \in M \mid \det \Sigma^{\{F_1, \ldots, F_k, G\}}(x) = 0 \} = \{ x \in M \mid v_0(x) = 0 \}. \tag{2.1}
\]

Using Lemma 1.1 and Proposition 2.1 we obtain the following characterization of equilibrium points for the perturbed system.

\textbf{Theorem 2.2.} The set of equilibria for the geometrically dissipated system is characterized by the equality

\[
E_p = E_{un} \cap \text{Inv}.
\]
By perturbing the initial dynamics (1.1) with the standard control vector field $v_0$, some of unperturbed equilibrium points will not remain equilibrium points for the geometrically dissipated system. We loose exactly that equilibrium points $x_e$ for which the vectors $\nabla F_1(x_e), \ldots, \nabla F_k(x_e)$ and $\nabla G(x_e)$ are linear independent. The set $\text{Inv}$ is invariant under the unperturbed dynamics (1.1) (see Corollary 2.4 in [3]). Next we will prove that the set $\text{Inv}$ is also an invariant set for the geometrically dissipated dynamics (1.5). Moreover, a solution of the unperturbed system which start from $\text{Inv}$ is also a solution for the geometrically dissipated system.

**Theorem 2.3.** We have the following properties:

(i) For an initial condition $x \in \text{Inv}$ we have $x_{un}(t, x) = x_p(t, x)$.

(ii) The set $\text{Inv}$ is invariant under the geometrically dissipated system (1.5).

(iii) If $x \notin \text{Inv}$, then $\frac{dG}{dt}(x_p(t, x)) < 0$ for all $t$.

**Proof.** (i) Let $x \in \text{Inv}$ and $x_{un}(t, x)$ be the solution of (1.1) starting from initial condition $x$. From the invariance of $\text{Inv}$ under the unperturbed dynamics, we obtain that $x_{un}(t, x) \in \text{Inv}$ for all $t$. By Lemma 1.1, we have that $v_0(x_{un}(t, x)) = 0$ for all $t$. This shows that $x_{un}(t, x)$ is also a solution of the geometrically dissipated system (1.5).

(ii) It is an immediate consequence of (i).

(iii) From (1.11) we have that $\frac{dG}{dt}(x_p(t, x)) = -\det \Sigma_{(F_1, \ldots, F_k, G)}(x_p(t, x)) \leq 0$. If $x \notin \text{Inv}$, then from (ii) we have that $x_p(t, x) \notin \text{Inv}$, for all $t$. By the definition of $\text{Inv}$, we obtain the strict inequality.

An immediate consequence of Theorem 2.3 (i) and (iii) is the following result.

**Corollary 2.4.** If the geometrically dissipated system (1.5) has a periodic orbit or a homoclinic orbit or a heteroclinic cycle, they are contained in $\text{Inv}$ and they are also a periodic orbit, respectively homoclinic orbit or heteroclinic cycle for the unperturbed system (1.1).

For the invariant set $\text{Inv}$ we have a further decomposition

$$\text{Inv} = G_* \cup Y,$$

where $G_* := \{x \in M \mid \nabla G(x) = 0\}$ and $Y$ is the complementary set of $G_*$ in $\text{Inv}$. The subsets $G_*$ and $Y$ are also invariant subsets for the perturbed dynamics. Indeed, $G_*$ is invariant under the unperturbed dynamic (see [3], [8]) and if $x \in G_*$ is an initial condition then $x_p(t, x) = x_{un}(t, x) \in G_*$ for all $t \in \mathbb{R}$, which implies that $G_*$ is invariant under the perturbed dynamics. Consequently, $Y$ is also an invariant set for the perturbed dynamics. By construction we have

$$Y = \{x \in M \mid \nabla G(x) \neq 0 \text{ and } \nabla F_1(x), \ldots, \nabla F_k(x), \nabla G(x) \text{ are linear dependent}\}.$$

### 3 Perturbed dynamics on the regular leaves

In this section we will study the geometrically dissipated dynamics (1.5) restricted to a regular leaf $L_c := F^{-1}(c)$ generated by a regular value of the function $F := (F_1, \ldots, F_k) : M \to \mathbb{R}^k$. Every regular leaf $L_c$ is invariant under perturbed dynamics (1.5) as both vector fields $X$ and $v_0$ are tangent vector fields to the leaves.

Next we will give a characterization of the invariant set $\text{Inv}_c \cap L_c$ for the perturbed dynamics restricted to the regular leaf $L_c$. It has been proved in [8], Theorem 4.5, that $v_0|_{L_c} = \nabla x_c G|_{L_c}$, where $x_c = \frac{1}{\det \Sigma_{(F_1, \ldots, F_k)}(x_c)} x_c^* g$ is a conformal metric with the induced metric $x_c^* g$ on $L_c$. Consequently, we have

$$\text{Inv}_c \cap L_c = \{x \in L_c \mid \nabla x_c G|_{L_c}(x) = 0\}.$$
The derivative of the function $G_{\mid L_c}$ along the solution of the geometrically dissipated dynamics \([1.5]\) restricted to the regular leaf $L_c$ is given by

$$
\dot{G}_{\mid L_c}(x) = L(X_{\mid L_c} - v_0_{\mid L_c})G_{\mid L_c}(x) = \tau_c(x)(X_{\mid L_c}(x) - v_0_{\mid L_c}(x), \nabla_{\tau_c}G_{\mid L_c}(x))
$$

$$
= \tau_c(x)(X_{\mid L_c}(x), \nabla_{\tau_c}G_{\mid L_c}(x)) - \tau_c(x)(v_0_{\mid L_c}(x), \nabla_{\tau_c}G_{\mid L_c}(x))
$$

$$
= L X_{\mid L_c} G_{\mid L_c}(x) - \tau_c(x)(\nabla_{\tau_c}G_{\mid L_c}(x), \nabla_{\tau_c}G_{\mid L_c}(x))
$$

$$
= -||\nabla_{\tau_c}G_{\mid L_c}(x)||^2_{\tau_c},
$$

for any $x \in L_c$. In the above computations we have used the fact that the function $G$ is a conserved quantity for the vector field $X$ which implies $L X_{\mid L_c} G_{\mid L_c} = 0$. Consequently, we obtain the set equality

$$
\text{Inv} \cap L_c = \{ x \in L_c \mid \dot{G}_{\mid L_c}(x) = 0 \}. \tag{3.1}
$$

**Remark 3.1.** The set $\{ x \in L_c \mid \dot{G}_{\mid L_c}(x) = 0 \}$ is the key set that appears in LaSalle Invariance Principle. The largest invariant set contained in $\{ x \in L_c \mid \dot{G}_{\mid L_c}(x) = 0 \}$ is the set that contains informations about asymptotic behaviour of certain solutions. For our case the the largest invariant set contained in $\{ x \in L_c \mid \dot{G}_{\mid L_c}(x) = 0 \}$ is the set itself as being equal with $\text{Inv} \cap L_c$.

In what follows we study the asymptotic behaviour of the solutions for the geometrically dissipated system \([1.5]\) restricted to a regular leaf $L_c$. We suppose that any solutions of the geometrically dissipated system \([1.5]\) are defined on $\mathbb{R}$. The $\omega$-limit set of $x_0$ is

$$
\omega(x_0) := \{ z \in L_c \mid \exists \tau_1, \tau_2, ... \to \infty \quad s.t. \quad x_p(t_k, x_0) \to z \quad as \quad k \to \infty \}
$$

The $\omega$-limit sets have the following properties that we will use later. For more details, see \([10]\).

(i) If $x_p(t, y) = z$ for some $t \in \mathbb{R}$, then $\omega(y) = \omega(z)$.

(ii) $\omega(x_0)$ is a closed subset and both positively and negatively invariant (contains complete orbits).

We have the following LaSalle type result.

**Theorem 3.1.** (Invariance Principle for the geometrically dissipated system)

Let $x_0$ be an arbitrary point in $L_c$, then the following holds:

(i) If $a, b \in \omega(x_0)$ then $G(a) = G(b) \leq G(x_0)$. Equality holds if and only if $x_0 \in \text{Inv} \cap L_c$.

(ii) We have the following set inclusion $\omega(x_0) \subset \text{Inv} \cap L_c$.

(iii) If $\{ x_p(t, x_0) \mid t \geq 0 \}$ is bounded then $\omega(x_0)$ is compact and nonempty and moreover

$$
\lim_{t \to \infty} d_{\tau_c}(x_p(t, x_0), \text{Inv} \cap L_c) = 0,
$$

where $d_{\tau_c}$ is the distance function on $L_c$ induced by the Riemannian metric $\tau_c$.

**Proof.** For (i), let $a, b \in \omega(x_0)$. There exists two sequences $(t_n)_{n \in \mathbb{N}}$ and $(s_n)_{n \in \mathbb{N}}$ such that $t_n < s_n < t_{n+1}$ with $t_n \to \infty$ and $x_p(t_n, x_0) \to a$, $x_p(s_n, x_0) \to b$. Consequently, as $G$ is a decreasing function along the solution $x_p(\cdot, x_0)$ we have the inequality $G(x_p(t_{n+1}, x_0)) \leq G(x_p(s_n, x_0)) \leq G(x_p(t_n, x_0))$. Taking the limit we obtain $G(a) = G(b) \leq G(x_0)$. If $x_0 \in \text{Inv}$ then $x_p(t, x_0) \in \text{Inv}$ and $G(x_p(t, x_0)) = G(x_0)$ for all $t \in \mathbb{R}$. Reciprocally, if $G(a) = G(b) = G(x_0)$ then using Theorem 2.3 \((iii)\) we obtain the enounced result.

(ii) Let $a \in \omega(x_0)$, then $x_p(t, a) \in \omega(x_0)$ for all $t$. From (i) we have that $G(x_p(t, a)) = G(a)$ for all $t$ and consequently, $\frac{dG}{dt}(x_p(t, a)) = 0$. We obtain that $x_p(t, a) \in \text{Inv}$ for all $t$ and in particular $a \in \text{Inv}$.

(iii) The first part is a classical result, see \([2]\), \([10]\). We have $\omega(x_0) \subset \text{Inv} \cap L_c$ and consequently, $d_{\tau_c}(x_p(t, x_0), \text{Inv} \cap L_c) \leq d_{\tau_c}(x_p(t, x_0), \omega(x_0))$. But $\lim_{t \to \infty} d_{\tau_c}(x_p(t, x_0), \omega(x_0)) = 0$. 

\[\square\]
Our next purpose is to study the change of stability for equilibrium points of the unperturbed dynamics restricted to a regular leaf \( L_c \) when we add the geometric dissipation of gradient type \( -\nu_0 \). Because the added dissipation \( -\nu_0 \) is of gradient type when restricted to a regular leaf \( L_c \), it is to be expected that the stability of an equilibrium point for the perturbed system to be dictated by the nature of the equilibrium point as a critical point for the function \( G_{|L_c} \).

**Theorem 3.2.** Let \( x_e \in L_c \) be a locally strict minimum for \( G_{|L_c} \). Then the following holds:

(i) \( x_e \) is an asymptotically stable equilibrium for the geometrically dissipated system \([15]\) restricted to \( L_c \).

(ii) There exists \( k > G(x_e) \) such that \( cc_{x_e} G_{|L_c}^{-1}([G(x_e), k]) \cap Inv = \{ x_e \} \). (The set \( cc_{x_e} G_{|L_c}^{-1}([G(x_e), k]) \) is the connected component of \( G_{|L_c}^{-1}([G(x_e), k]) \) that contains the point \( x_e \).)

(iii) If \( G_{|L_c} : L_c \to \mathbb{R} \) is a proper function, then for any \( k > G(x_e) \) for which \( cc_{x_e} G_{|L_c}^{-1}([G(x_e), k]) \cap Inv = \{ x_e \} \) the set \( cc_{x_e} G_{|L_c}^{-1}([G(x_e), k]) \) is included in the domain of attraction of the asymptotically stable equilibrium point \( x_e \).

**Proof.** (i) We prove that if \( x_e \) is a locally strict minimum for \( G_{|L_c} \), then \( x_e \) is isolated in \( Inv \cap L_c \). Indeed, we have \( cc_{x_e} \{ x \in L_c \mid \nabla \tau_{G_{|L_c}}(x) = 0 \} \subset cc_{x_e} (Inv \cap L_c) \). Using Sard Theorem, see \([11]\), Lemma 10, we have the inclusion \( cc_{x_e} \{ x \in L_c \mid \nabla \tau_{G_{|L_c}}(x) = 0 \} \subset G_{|L_c}^{-1}([G(x_e), k]) \). But \( x_e \) is a locally strict minimum for \( G_{|L_c} \) and consequently, it is isolated in \( G_{|L_c}^{-1}(G(x_e)) \) which also implies that it is isolated in \( Inv \cap L_c \).

From the invariance of \( Inv \cap L_c \) for the perturbed dynamics \([15]\) restricted to \( L_c \) and the fact that \( x_e \) is isolated in \( Inv \cap L_c \), we obtain that \( x_e \) is an equilibrium point. From locally strict minimality of \( x_e \) and the fact that \( x_e \) is isolated in \( Inv \cap L_c \), there exists a small neighborhood \( U \) in \( L_c \) such that \( G_{|L_c}(x) < G(x_e) \) for any \( x \in U \setminus \{ x_e \} \). By Lyapunov theorem we obtain that \( x_e \) is an asymptotically stable equilibrium for the perturbed dynamics \([15]\) restricted to \( L_c \).

(ii) We have shown that \( x_e \) is isolated in the set \( Inv \cap L_c \) and consequently, there exists a closed ball \( B(x_e, r) \) with \( G_{|L_c}(x) > G_{|L_c}(x_e), \forall x \in \overline{B(x_e, r)} \cap L_c \) and \( B(x_e, r) \cap Inv \cap L_c = \{ x_e \} \). There exists \( k_0 = \min_{S(x_e, r) \cap L_c} G_{|L_c}(x) \geq G_{|L_c}(x_e) \), where \( S(x_e, r) \) is the sphere with the radius \( r \) and centered at \( x_e \). If \( k \in (G(x_e), k_0) \) then \( cc_{x_e} G_{|L_c}^{-1}([G(x_e), k]) \subset \overline{B(x_e, r)} \cap L_c \). We will prove this by contradiction, indeed suppose there exists \( y \in cc_{x_e} G_{|L_c}^{-1}([G(x_e), k]) \) and \( y \notin \overline{B(x_e, r)} \cap L_c \). There exists a continuous arc \( a_{y, x_e} \) included in \( cc_{x_e} G_{|L_c}^{-1}([G(x_e), k]) \) connecting \( y \) and \( x_e \). Therefore, there exists \( z \in a_{y, x_e} \cap S(x_e, r) \cap L_c \). Consequently, \( k_0 > k \geq G(z) \) which is a contradiction with the fact that \( k_0 \) is the minimum value of \( G_{|L_c} \) for points in the sphere \( S(x_e, r) \cap L_c \).

(iii) We prove that the set \( D_{x_e} := cc_{x_e} G_{|L_c}^{-1}([k, G(x_e)]) \) is an invariant set for the perturbed dynamics \([15]\) under the hypothesis of (iii).

We will proceed by contradiction, we suppose that \( D_{x_e} \) is not invariant under \([15]\). There exists \( x_0 \in D_{x_e} \setminus \{ x_e \} \) and \( t^* > 0 \) such that \( G_{|L_c}(x_p(t^*, x_0)) = G(x_e) \) and this is a consequence of the fact that \( G_{|L_c} \) is an increasing function along the solutions of \([15]\). As \( x_e \) is a critical point for \([15]\) we have that \( x_p(t^*, x_0) \neq x_e \). We have the following partition

\[
D_{x_e} \cap G_{|L_c}^{-1}(G(x_e)) = \{ x_e \} \cup Y,
\]

where \( x_e \notin Y \), and \( x_p(t^*, x_0) \in Y \) and \( Y \) is a compact set in the relative topology of \( D_{x_e} \). The compactness of \( Y \) is a consequence of the the compactness of \( D_{x_e} \cap G_{|L_c}^{-1}(G(x_e)) \) and the fact that \( x_e \) is isolated in \( G_{|L_c}^{-1}(G(x_e)) \) as being a locally strict maximum.

Because \( D_{x_e} \) has \( T_3 \) separability property there exists two open neighborhoods \( V_{x_e} \) and \( V_Y \) (in the relative topology of \( D_{x_e} \) of \( x_e \) and respectively \( Y \) such that \( V_{x_e} \cap Y = \emptyset \).

Let \( S := D_{x_e} \setminus (V_{x_e} \cup Y) \). The set \( S \) is a closed set in the compact set \( D_{x_e} \), and consequently it is compact and by construction separates \( x_e \) and \( x_p(t^*, x_0) \in Y \). By the Mountain Pass Theorem (see
there exists a point $x^* \in D_{x_e}$ which is a local maximum or a mountain pass point for $G_{|D_{x_e}}$ with $G(x^*) < G(x_e)$. According to Lemma 1.1 (see Annexe), we have that $x^*$ is a local maximum or a mountain pass point for $G_{|L_c}$ restricted to the set $cc_{x_e}G_{|L_c}^{-1}(\{(k- \varepsilon, G(x_e))\})$, where $\varepsilon > 0$ is small. Because $G(x^*) < G(x_e)$ we have that $x^*$ is a local maximum or a mountain pass point for $G_{|L_c}$ restricted to the open set $cc_{x_e}G_{|L_c}^{-1}(\{(k- \varepsilon, G(x_e))\})$. Consequently, $x^*$ is a critical point for $G_{|L_c}$ which implies that $x^* \in Inv$. We have obtained a contradiction which shows that $D_{x_e}$ is a compact invariant set for the perturbed dynamics (1.5).

Let $x_0 \in D_{x_e}$ be arbitrary. Because $D_{x_e}$ is invariant and compact we obtain that $\omega(x_0) \neq \emptyset$ and $\omega(x_0) \in D_{x_e}$. But also $\omega(x_0) \in Inv \cap L_c$ by Theorem 3.1 (ii). By hypothesis $\omega(x_0) = \{x_e\}$ and as $x_e$ is also asymptotically stable we obtain $\lim_{t \to \infty} x_p(t, x_0) = x_e$.

We notice that the condition $x_e \in L_c$ being a locally strict minimum for $G_{|L_c}$ is equivalent with the following two conditions: $x_e \in L_c$ is a local minimum for $G_{|L_c}$ and $x_e$ is isolated in $Inv \cap L_c$. We can summarize as follows:

1. Suppose $x_e$ is stable for the unperturbed dynamics (1.1) restricted to the leaf $L_c$.

   1.1) If $x_e$ is strict local minimum for $G_{|L_c}$ (which implies that it is isolated in $Inv \cap L_c$), then $x_e$ is an asymptotically stable equilibrium for the geometrically dissipated system (1.5) restricted to the leaf $L_c$.

   1.2) If $x_e$ is not a strict local minimum for $G_{|L_c}$ but it is still an isolated point in the set $Inv \cap L_c$, then $x_e$ is an unstable equilibrium for the geometrically dissipated system (1.5) restricted to the leaf $L_c$.

2. Suppose $x_e$ is unstable for the unperturbed dynamics (1.1) restricted to the leaf $L_c$, then $x_e$ remains an unstable equilibrium for the geometrically dissipated system (1.5) restricted to the leaf $L_c$.

Also, if $x_e \in L_c$ is a locally strict extremum for $G_{|L_c}$ we obtain that $x_e$ is a stable equilibrium point for the unperturbed dynamics (1.1). This is a consequence of the algebraic method for stability, see [5], [6], and [7], i.e. $x_e$ is an isolated solution of the algebraic system

$$F_1(x) = F_1(x_e), \ldots, F_k(x) = F_k(x_e), G(x) = G(x_e).$$

The passage from asymptotic stability of equilibrium points of the geometrically dissipated system (1.5) to the asymptotic stability of periodic orbits for the geometrically dissipated system (1.5) is allowed by Theorem 3.1. More precisely, for periodic orbits we have the following stability result.

**Theorem 3.3.** Let $x_p(\mathbb{R}, x_0)$ be a periodic orbit for the geometrically dissipated system (1.5). Assume that $x_p(\mathbb{R}, x_0) = cc_{x_0}(Inv \cap L_c)$ and all $y \in x_p(\mathbb{R}, x_0)$ are local minima for $G_{|L_c}$. Then the following holds:

(i) The periodic orbit $x_p(\mathbb{R}, x_0)$ is asymptotically stable for the geometrically dissipated system (1.5) restricted to $L_c$. (Asymptotic stability is understood in the sense of asymptotic stability of an invariant set of a dynamical system).

(ii) There exists $k > G(x_0)$ such that $cc_{x_0}G_{|L_c}^{-1}([G(x_0), k]) \cap Inv = x_p(\mathbb{R}, x_0)$.

(iii) If $G_{|L_c} : L_c \to \mathbb{R}$ is a proper function, then for any $k > G(x_0)$ for which $cc_{x_0}G_{|L_c}^{-1}([G(x_0), k]) \cap Inv = x_p(\mathbb{R}, x_0)$ the set $cc_{x_0}G_{|L_c}^{-1}([G(x_0), k])$ is included in the domain of attraction of the asymptotically stable periodic orbit $x_p(\mathbb{R}, x_0)$. 6
4 Annexe

The following results are taken from book [9].

**Definition 4.1.** Let $X$ be a topological space and $f : X \to \mathbb{R}$ be a continuous function. A point $x \in X$ is called a mountain pass point (in the sense of Katriel) if for every neighborhood $N$ of $x$, the set

$$N \cap \{ y \in X \mid f(y) > f(x) \}$$

is disconnected.

**Lemma 4.1.** Let $f : \tilde{X} \to \mathbb{R}$ be a continuous function and let $X \subseteq \tilde{X}$ be a subset with the property that

$$f(y) \leq \inf_{z \in X} f(z), \ \forall y \in \tilde{X} \setminus X.$$

If $x \in X$ is a mountain pass point for $f|_X$ then $x$ is a mountain pass for $f : \tilde{X} \to \mathbb{R}$.

**Proof.** Let $x$ be a mountain pass point for $f|_X$. Let $\tilde{N}$ be an arbitrary neighborhood of $x$ in $\tilde{X}$. By definition of induced topology we have that $N := \tilde{N} \cap X$ is a neighborhood of $x$ in $X$. Using the hypothesis we obtain the set equality

$$N \cap \{ y \in X \mid f(y) > f(x) \} = \tilde{N} \cap \{ y \in \tilde{X} \mid f(y) > f(x) \}.$$

This shows that $x$ is also a mountain pass point for $f : \tilde{X} \to \mathbb{R}$.

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