Evolution of Magnetic Fields in Intra Cluster Media

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ABSTRACT

Intra Cluster Media (ICMs) located at galaxy clusters is in the state of hot, tenuous, magnetized, and highly ionized X-ray emitting plasmas. This overall collisionless, viscous, and conductive magnetohydrodynamic (MHD) turbulence in ICM is simulated using hyper magnetic diffusivity with weak background magnetic field. The result shows that fluctuating random plasma motion amplifies the magnetic field, which cascades toward the diffusivity scale passing through the viscous scale. The kinetic eddies in the subviscous scale are driven and constrained by the magnetic tension which is eventually balanced with the highly damping effect of the kinetic eddies. Simulation results show the saturated kinetic energy spectrum is $\sim k^{-3}$, deeper than that of the incompressible or compressible fluid. To explain this unusual field profile we set up two simultaneous differential equations for the kinetic and magnetic energy spectrum using an Eddy Damped Quasi Normal Markovianized (EDQNM) approximation. The analytic solution shows that the magnetic energy in addition to the viscous damping effect constrains the plasma motion leading to the power spectra: kinetic energy spectrum $E^k_v \sim k^{-3}$ and corresponding magnetic energy spectrum $E^k_M \sim k^{-1/2}$. Also the comparison of simulation results with different resolutions implies the role of small scale magnetic energy in the nonlocal energy transfer from kinetic to magnetic eddy.

Subject headings: galaxies: clusters: intracluster medium, magnetic fields
1. Introduction

ICM located at the center of galaxy cluster is composed of fully ionized hot plasmas ($T \sim 10^8$K). The gas includes most of the cluster baryons (> 85%) and heavy elements, but overall density ‘$n$’ is very low ($n < 10^3$ cm$^{-3}$). As a result ICM has very small diffusivity ‘$\eta$’ ($\sim T^{-3/2}$) while viscosity ‘$\nu$’ ($\sim 1/n$) is very large (Schekochihin et al. 2005). Dynamo effect from the turbulent flow motions in ICM exceeds the dissipation of such high viscous plasmas causing the growth of seed magnetic fields, which react back the plasma motion through magnetic tension ($\mathbf{B} \cdot \nabla \mathbf{B}$). Besides, the amplified magnetic field also constrains some intrinsic properties in ICM. For example, magnetic field makes the pressure anisotropic leading to the anisotropic thermal conductivity, and viscosity in consequence. Non-magnetized thermal Spitzer conductivity ‘$\kappa_{SP}$’ is defined like (Narayan & Medvedev 2001):

$$\kappa_{SP} \sim \frac{\lambda_e^2}{t_{col}} \sim 4 \times 10^{32} T_1^{5/2} n^{-1} \text{cm}^2 \text{s}^{-1},$$

where $T_1 = kT/10$ Kev, $\lambda_e$ is the mean free path of an electron and $t_{col}$ is the time between coulomb collisions. If magnetic field is injected, $\kappa_\perp$, conductivity perpendicular to the field is reduced to $\sim (\rho_e/\lambda_e)^2 \kappa_{SP} (\ll \kappa_{SP})$ so that the conductivity becomes anisotropic with one third of the thermal Spitzer conductivity: $\kappa = \kappa_\perp + \kappa_\parallel \to \kappa_\parallel$. This anisotropic thermal conductivity brings about temperature distribution that depends on the direction and strength of magnetic field. In addition as the conservation of magnetic moment $\mu_B = u_\perp^2/B$ and kinetic energy ($\sim u_\parallel^2 + u_\perp^2$) implies, magnetic field makes pressure tensor $P_{ij} = \sum u_i u_j$ anisotropic along the field. The ratio of pressure anisotropy ‘$\triangle$’ is implicitly related to the viscosity and stability of ICM (Schekochihin et al. 2010):

$$\triangle \equiv \frac{p_\perp - p_\parallel}{p} \sim \frac{1}{\nu_{ii}} \frac{1}{B} \frac{dB}{dt} \in \left[ -\frac{2}{\beta}, \frac{1}{\beta} \right],$$

($$\beta = 8\pi p/B^2$$).

Here, kinematic viscosity $\nu \sim u_{th}^2/\nu_{ii}$; ion-ion collision frequency $\nu_{ii} = 4\pi n e^4 \ln \Lambda m_i^{-1/2} T^{3/2}$. If $\triangle$ is smaller than $-2/\beta$ or larger than $1/\beta$, firehose or mirror instability occurs in the range between ion Larmor radius ($\rho_i \sim 10^4 - 10^6$ km) and the mean free path ($\lambda_{mfp} \sim 10^{15}$ km) (Schekochihin et al. 2008). Since ICM has high ‘$\beta$’, stability range is very narrow, i.e., practically unstable. Instability is thought to redistribute the plasma motions faster than coulomb collision does, but the role of instability is not fully understood yet (Jones 2008).

We have pointed out the relation between magnetic field and anisotropic properties in ICM. However even in case of a unit magnetic prandtl number ($Pr_M = \nu/\eta = 1$), where magnetic fields are not so strongly frozen, the large scales driven by a random isotropic force have a tendency of remaining isotropic if the guide or background magnetic field ‘$\mathbf{b}_{ext}$’ is not too
strong (Cho et al. 2009). With $\eta \to 0$ the magnetic fields are more strongly frozen into the plasma, and the anisotropy in small scale cannot decisively affect the energy spectrum because of the small eddy turnover time. So, as long as the random isotropic force continuously drives the system, especially large scales, the fluid theoretical model assuming the isotropy is valid except the case of microscale instability or very strong $b_{ext}$. We can focus on the properties of isotropic MHD turbulence dynamo in ICM.

On the other hand $b_{ext}$ is not indispensable for the growth of magnetic fields. The turbulence by cosmological shocks can amplify the weak seed field of any origin (Ryu et al. 2008). They also pointed out that subsonic turbulence could develop with a very weak seed magnetic field (Ryu et al. 2012). Moreover it was shown that turbulence could amplify a localized seed magnetic field in (Cho & Yoo 2012). Going one step further, Cho (Cho 2014) investigated the origin of seed magnetic fields and insisted that the origin of the seed field should be more like astrophysical.

Simulation results show that as the excited magnetic fields (energy) are instilled into the damped kinetic eddies, the viscous damping scale $\sim 1/l_\nu$ is extended toward resistivity scale $\sim 1/l_\eta$ ($l_\eta \ll l_\nu$). Then, these two coupled velocity and magnetic field ($E_M \gg E_V$) generate specific power spectra which are different from the typical spectrum of Kolmogorov’s incompressible fluid or Burger’s compressible fluid. So sort of stochastic method is introduced to derive two simultaneous differential equations for the kinetic and magnetic energy. We show how the coupled momentum and magnetic induction equation explain the simulation results.

In chapter 2, we briefly introduce simulation tool and analytic method used in this paper. Simulational and analytic results are introduced in chapter 3. And in the final chapter we discuss about the results, their physical meanings, and future topics. Detailed analytic calculations are discussed in the appendix.

2. Numerical and analytic method

We have solved the incompressible MHD equations using a pseudo-spectral code with a periodic box of size $(2\pi)^3$:

$$\frac{\partial \mathbf{v}}{\partial t} = -\mathbf{v} \cdot \nabla \mathbf{v} - \nabla P + (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{v} + \mathbf{f},$$  \hspace{1cm} (3)

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \mathbf{b}_{ext},$$  \hspace{1cm} (4)
where ‘\(f\)’ is a random mechanical force driving a system at \(k \sim 2 - 3\) in fourier space, and ‘\(b_{\text{ext}}\)’ is a weak background (guide) magnetic field (\(b_{\text{ext}} = 0.0001\)) covering the whole magnetic eddy scales. Here, ‘\(B\)’, magnetic field divided by ‘\((4\pi\rho)^{1/2}\)’, has the unit of Alfvén velocity, and ‘\(v\)’ is ‘rms velocity’, and ‘\(t\)’ has the unit of large scale eddy turnover time ‘\(L/v\)’. For example, if ‘\(L\)’ of a cluster is \(\sim 400\) kpc and ‘\(v\)’ is \(\sim 400\) km/s, then ‘\(L/v\)’ is \(\sim 10^9\) year. The time ‘\(t\)’ has the unit of ‘\(10^9\)’ year. The system realizes the state of ICM of the high viscous plasma state with ideally frozen magnetic fields. We have tested the simulation with various resolutions: 256\(^3\), 512\(^3\), 1024\(^3\), and 1536\(^3\). Comparing the results of different resolutions will clearly show the influence of magnetic energy on the small scale dynamo and power spectrum.

To understand the evolution of field profile and the power spectrum, we use an Eddy Damped Quasi Normal Markovianization method EDQNM, (Kraichnan & Nagaranjan 1967; McComb & Park 2013) and dimensional approach in a limited way. We simplified the resultant, simultaneous differential equations from the quasi normal approximation to find a solution.

### 3. Results

#### 3.1. Simulation results

Fig.1(a) shows evolving kinetic energy \(E_V(t)\) and magnetic energy \(E_M(t)\) of resolution 1024\(^3\). When forcing begins, \(E_V(t)\) grows and gets saturated first. At \(t \sim 18\) \(E_M(t)\) starts growing with \(E_V(t)\) losing its energy. While \(E_M(t)\) is not large enough yet \(t < 18\), energy transfer from \(E_V(t)\) to \(E_M(t)\) is ignobably small. Most of the energy stays near the injection scale of kinetic eddies balanced with the energy damping by viscosity, local advective energy transfer, and tiny leakage of \(E_V(t)\) to magnetic eddies. However, as \(E_M(t)\) approaches a critical level \(t \sim 18\), the nonlocal energy transfer from kinetic to magnetic eddies gets accelerated. As ‘\(B \cdot \nabla v\)’ in magnetic induction equation indicates, both \(E_V(t)\) and \(E_M(t)\) are necessary for the nonlocal energy transport. The growing \(E_M(t)\) does not only boost the nonlocal energy transfer, but also accompanies a backward energy transfer from magnetic to kinetic eddy through magnetic tension \((B \cdot \nabla B)\). This magnetic back reaction does not look so conspicuous, but the tension effect plays an important role in subviscous scale on behalf of the quickly diminishing effect of external forcing. For the local and nonlocal energy transfer in magnetic eddies, the geometrical property of ‘\(v\)’ and ‘\(B\)’ is also an important factor. Only magnetic field parallel to the gradient of velocity field \((B \cdot \nabla V \sim B d/dr|V|)\) contributes to the energy transfer, whereas the perpendicular components do not. In other words, magnetic field should be perpendicular to the equipotential line of velocity field, which causes
stretch, twist, fold, and merge of magnetic field. For \( \sim 5 < t < \sim 18 \) \( E_V(t) \) is already highly saturated, but \( E_M(t) \) still stays in the weak stage. This resting stage before the onset of magnetic energy implies some processes occur in the meantime. By the time \( t \sim 40 \), \( E_V(t) \) and \( E_M(t) \) get saturated with the ratio of \( E_M(t)/E_V(t) \sim 2' \).

Fig.1(b) shows the saturated kinetic energy spectrum \( E_V(k) \) and magnetic energy spectrum \( E_M(k) \) in fourier space. This plot shows not only typical energy spectrum but also peculiar properties of high \( Pr_M \) small scale dynamo. \( E_V(k) \) follows the power law of \( k^{-3} \) covering a long subviscous scale range while \( E_M(k) \) smoothly and continuously changes. The scaling factor of \( E_V(k) \) is quite different from Kolmogorov's power law ' \( k^{-5/3} \) ' (Kolmogorov 1941) for the incompressible fluid system or Burger's power law ' \( k^{-2} \) ' (Lesieur 1987) for the extremely compressible fluid system. Also the results are different from those of previous MHD turbulence simulations with high \( Pr_M \): \( E_V(k) \sim k^{-4}, E_M(k) \sim k^3 - k^{-1} \) (Cho et al. 2003; Lazarian et al. 2004; Schekochihin et al. 2004). In (Cho et al. 2003; Lazarian et al. 2004), a strong background magnetic field ' \( b_{ext}' \) = 1 was used. And balance relation and simulation data were referred to find out the scaling factors of energy spectra. (Schekochihin et al. 2004) used a Kazantsev theory (Kazantsev 1968) and simulation but with low resolution. The range of \( k^{-3} \) is in fact an extended viscous damping scale by the magnetic tension. This inert viscous scale range does not transfer \( E_V(k) \) to magnetic eddies. But nontrivial \( E_V(k) \) is necessary for the local advective magnetic energy transfer through \( -v \cdot \nabla B \) in this extended subviscous scale. We will come back to this problem soon.

Fig.2(a) shows the evolving \( E_V(k) \) and \( E_M(k) \) in the early time regime \( (t = 0.0007 - 6.3, \) from bottom to top). Also Fig.2(b) includes the de facto same plots for the later time regime \( (t = 16.1 - 29.4) \). When random mechanical energy starts driving the system \( (k = 2.5) \), \( E_V \) (black dashed-star line, \( t = 0.0007 \)) grows first and then followed by \( E_M \) (red dashed-star line). The magnitudes of \( E_V \) and \( E_M \) in small scale get reversed soon, but the reverse of the two energy spectra in large scale, more exactly, near the injection scale does not occur. At \( t = 0.0007 \) \( E_V(k) \) grows first and then cascades toward the range ' \( k \sim 10' \) where magnetic eddies are not excited (' \( E_M(k) \sim 0) \). As ' \(-\nabla P' \) and ' \( v \cdot \nabla v' \) distributing energy in kinetic eddies show, \( E_M(k) \) is not necessarily required for the energy transfer in kinetic eddies. In contrast magnetic eddies cannot receive or transfer energy without the help of \( E_V(k) \). Nonetheless, \( E_M(k) \) in small scale grows much faster due to the negligible resistivity and short eddy turnover time once magnetic eddies see nontrivial \( E_V \) of similar scales. Small scale \( E_M(k) \) begins to surpass \( E_V(k) \) at \( t \sim 1.8 \), and gets past it soon. As \( E_M(k) \) grows, kinetic eddies in \( k <\sim 16 \) lose energy while smaller scale kinetic eddies \( (k >\sim 16) \) receive energy through magnetic tension \( B \cdot \nabla B \) (Fig.2(b), 1(a)). With growing \( E_M(k) \), its
peak migrates forward and backward, and finally settles down. We will see that the unusual spectrum $k^{-3}$ in the inert subviscous scale is not only due to the viscosity and but also due to the suppression effect of $E_M$.

Fig. 3(a) shows the influence of resolution; more exactly, the influence of small scale magnetic energy on the evolution of $E_M(k)$ and $E_V(k)$. The lowest resolution is $256^3$ $(k_{max} = 128$, thinnest line) and the highest one is $1536^3$ $(k_{max} = 768$, thickest line). As the plot shows, at $t \sim 18$ $E_M(k)$ of the largest resolution arises first in line with the drop of $E_V(k)$. In contrast $E_M(k)$ with the smallest resolution shows the most lagging evolution. The split of $E_M(k)$ begins earlier $(t \sim 3 - 4$, Fig. 3(b)), but the difference is not recognizable until $t \sim 18$. A close look of $E_V(k)$ in Fig. 2(b) and 1(a) for $0.18 < t < 1.8$ indicates that kinetic eddies in the inert subviscous scale gains energy from magnetic eddies, but large scale kinetic eddies at $k = 1$ and $k = 3$ lose energy. The field behaviors in these plots are closely related to those of Fig. 1(a). Fig. 4(b) supplementally shows $E_M/E_K$ also depends on the resolutions. The ratio of $1536^3$ which has the largest amount of $E_M$ in small scale grows and reaches the saturation fastest, which is consistent with the previous results.

Fig. 5(a), 5(b) show the compensated $k^3E_V$ and $k^{1/2}E_M$. The scaling factor of $E_V(k)$ is clearly $k^{-3}$ in the subviscous scale. But since the scaling factor of $E_M(k)$ does not clearly appear, it is ambiguous to pinpoint an exact inertial range scaling factor. Instead, we choose a scaling factor $'k^{-1/2}'$, an approximate median value of the already known results. Analytic results derives quite exact scaling factor of $E_V(k)$ with that $E_M(k)$ spectrum. Nonetheless we do not insist $'k^{-1/2}'$ be the scaling factor of inertial range $E_M(k)$. Rather it is sort of a representative $E_M$ corresponding to $E_V$ in the inert subviscous scale. Additionally a reference line $'k^{-1.0}'$ for the scaling invariant factor (Ruzmaikin et al. 1982, Kleeorin et al. 1996) is drawn together for reference. But we do not discuss this concept from renormalization group theory at this time.

### 3.2. Analytic methods & results

#### 3.2.1. Eddy Damped Quasi Normal Markovianization

We start from the momentum (Eq. 3) and magnetic induction equation (Eq. 4). Taking divergence of the momentum equation we can replace pressure by convection and magnetic
tension (Leslie and Leith 1975, Yoshizawa 2011)\footnote{\(v(k, t)\) and \(B(k, t)\) depend on time ‘t’ and wavenumber ‘k’, but ‘t’ will be omitted for simplicity.}:

\[
\frac{\partial v_i(k, t)}{\partial t} = \sum_{p+r=k} M_{iqm}(k)[v_q(p, t)v_m(r, t) - B_q(p, t)B_m(r, t)] + \nu \nabla^2 v_i(k, t),
\]

(5)

\[
\frac{\partial B_i(k, t)}{\partial t} = \sum_{p+r=k} M_{iqm}^B(k)v_q(p, t)B_m(r, t),
\]

(6)

with the definition of algebraic multipliers

\[
M_{iqm}(k) = -\frac{i}{2}(k_m\delta_{iq} + k_q\delta_{im} - \frac{2k_i k_q k_m}{k^2}),
\]

\[
M_{iqm}^B(k) = i(k_m\delta_{iq} - k_q\delta_{im}).
\]

(7)

Then we can get the evolving second order correlation equations of \(\langle v_i(k)v_i(-k)\rangle\) and \(\langle B_i(k)B_i(-k)\rangle\):

\[
\left(\frac{\partial}{\partial t} + 2\nu k^2\right)\langle v_i(k)v_i(-k)\rangle = \sum_{p+r=k} M_{iqm}(k)\left[\langle v_q(p)v_m(r)v_i(-k)\rangle - \langle v_q(-p)v_m(-r)v_i(k)\rangle\right.

\left.\langle B_q(p)B_m(r)v_i(-k)\rangle - \langle B_q(-p)B_m(-r)v_i(+k)\rangle\right].
\]

(8)

\[
\frac{\partial}{\partial t}\langle B_i(k)B_i(-k)\rangle = \sum_{p+r=k} M_{iqm}^B(k)\left[\langle v_q(p)B_m(r)B_i(-k)\rangle - \langle v_q(-p)B_m(-r)B_i(+k)\rangle\right].
\]

(9)

The third order correlation, ‘\(A1, A2, ..., B2\)’, is called a transport function. These triple correlations play a role of transfer and dissipation of energy to decide the field profiles of the system. If the field is helical, ‘\(\alpha\)’ coefficient (\(\sim \langle j \cdot b \rangle - \langle v \cdot \omega \rangle\)) for the inverse cascade of magnetic energy can be derived. Also other terms for the forward cascade of energies are derived (Krause & Rädler 1980; Moffatt 1978; Park & Blackman 2012a,b; Pouquet et al. 1976). However the helical field is not assumed in our system, which means ‘\(\alpha\)’ coefficient or the inverse cascade of magnetic energy is excluded. We use a quasi normalization approximation, sort of an iterative method, to find the transport function (appendix).
The formal representations of $E_V(k)$ and $E_M(k)$ are as follows:

\[
\frac{\partial E_V(k)}{\partial t} = +\frac{1}{2} \int dp \, dr \, \Theta_{kpr}^{\nu\nu}(t) \frac{k^3}{p \, r} (1 - 2y^2z^2 - xyz) E_V(p) E_V(r) \\
- \int dp \, dr \, \Theta_{kpr}^{\nu\nu}(t) \frac{p^2}{r} (xy + z^3) E_V(r) E_V(k) - 2\nu k^2 E_V(k) \\
+ \frac{1}{2} \int dp \, dr \, \Theta_{kpr}^{\eta\eta}(t) \frac{k^3}{p \, r} (1 - 2y^2z^2 - xyz) E_V(p) E_V(r) \\
+ \int dp \, dr \, \Theta_{kpr}^{\eta\eta}(t) \frac{p^2}{r} (y^2z - z) E_M(r) E_V(k),
\]

(10)

and

\[
\frac{\partial E_M(k)}{\partial t} = -\int dp \, dr \, \Theta_{kpr}^{\nu\nu}(t) \frac{p^2}{r} z(1 - x^2) E_M(r) E_M(k) \\
- \int dp \, dr \, \Theta_{kpr}^{\nu\nu}(t) \frac{r^2}{p} (y + xz) E_V(p) E_M(k) \\
+ \int dp \, dr \, \Theta_{kpr}^{\eta\eta}(t) \frac{k^3}{p \, r} (1 + xyz) E_M(p) E_M(r).
\]

(11)

The integral variables ‘p’, ‘r’, i.e., wavenumbers, are constrained by the relation of $p + r = k$. ‘x’, ‘y’, and ‘z’ are cosines of the angles formed by three vectors ‘k’, ‘p’, and ‘r’. (Fig.6(a)). Algebraically ‘k’, ‘p’, and ‘r’ should satisfy a condition like (Leslie and Leith 1975):

\[|k - r| \leq p \leq k + r.\]  

(12)

To derive analytically solvable equations from Eq.(10), (11), we need to simplify these two equations considering the interaction between ‘k’ and its close wave number. So, we take account of only two cases: large $p$ ($k \sim p \gg r$) and large $r$ ($k \sim r \gg p$). In principle ‘$k/2 \sim p \sim r$’ should also be included. But since the interaction between the close wave vectors, local energy transfer, is dominant in the nonhelical small scale dynamo, the latter case can be ignored.

With the assumption of $E_V(k) \sim k^\nu$ and $E_M(k) \sim k^m$, we simplify the first term in Eq.(11) like below:

(i) Large $p$ (small $r$, i.e., $k \sim p \gg r$, $\eta \sim 0$)

Eddy damping function $\Theta_{kpr}^{\nu\nu}(k,t)$ is approximately

\[
\Theta_{kpr}^{\nu\nu}(k,t) = \left. \frac{1 - e^{-[\nu p^2 + \eta(k^2 + r^2) + \mu kpr]t}}{\nu p^2 + \eta (k^2 + r^2) + \mu kpr} \right|_{\eta = 0} \sim \frac{1}{\nu p^2}; \quad (t \to \infty).
\]

(13)
Also as Fig. 6(a), 6(b) show, we can use the relations 
\[ x \sim y \sim 0, \ z \sim 1, \text{ and } r = k - p. \]
Then,
\[ -\int dp \frac{1}{\nu p^2} (1 - x^2) (k - p)^m E_M(k) \sim -\frac{1}{\nu} k^m E_M(k). \] (14)

(ii) Small \( p \) (large \( r \), i.e., \( k \sim r \gg p, \ x \sim z \sim 0, \ y \sim 1 \))
In this case only the triad relaxation time \( \mu_{kpr} \) is left. We assume that it is independent of time, so we can write
\[ \Theta_{kpr}^{\eta\eta}(k, t) \sim \frac{1}{\nu p^2 + \mu_{kpr}} \sim \frac{1}{\mu_{kpr}}. \] (15)

Then,
\[ \sim -\int dp \frac{1}{\mu_{kpr}} (k^2 r^{m-1} - 2k r^m + r^{m+1}) x(1 - x^2) E_M(k) \sim 0. \] (16)

The results, Eq.(14), (16) represent the first term in ‘\( D \)’ in Eq.(18). The other terms can be found in a similar way.

Then, the coupled equations of \( E_V(k) \) and \( E_M(k) \) are
\[
\frac{\partial E_V(k)}{\partial t} = - \underbrace{\left( a_1 k^n + 2\nu k^2 \right)}_A E_V(k) + \underbrace{\left( b_1 k^m - b_2 k^v \right)}_B E_M(k), 
\] (17)
\[
\frac{\partial E_M(k)}{\partial t} = \underbrace{(c_1 k^m + c_2 k^{m+2})}_C E_V(k) - \underbrace{(d_1 k^m + d_2 k^{v+2})}_D E_M(k). 
\] (18)

The coefficients ‘\( a_i \)', ‘\( b_i \)', ‘\( c_i \)', and ‘\( d_i \)' are independent of ‘\( k \)', and assumed to be independent of time for simplicity. The matrix form of these simultaneous differential equations is simply
\[
\begin{bmatrix}
E'_V(k) \\
E'_M(k)
\end{bmatrix} = \begin{bmatrix}
-A & B \\
C & -D
\end{bmatrix} \begin{bmatrix}
E_V(k) \\
E_M(k)
\end{bmatrix}.
\] (19)

Eq.(19) can be solved diagonalizing the matrix ‘\( \mathcal{M} \)’. For this, the bases \( E_V(k) \) and \( E_M(k) \) need to be transformed using a transition matrix ‘\( \mathcal{P} \)’ which is composed of eigenvectors of ‘\( \mathcal{M} \)’:
\[
\begin{bmatrix}
E_V(k) \\
E_M(k)
\end{bmatrix} = \mathcal{P} \begin{bmatrix}
V(k) \\
M(k)
\end{bmatrix}, \quad \mathcal{P} = \begin{bmatrix}
B & B \\
A + \lambda_1 & A + \lambda_2
\end{bmatrix}.
\] (20)
Then,

\[
\begin{bmatrix}
V'(k) \\
M'(k)
\end{bmatrix} = P^{-1}MP \begin{bmatrix}
V(k) \\
M(k)
\end{bmatrix} = \begin{bmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{bmatrix} \begin{bmatrix}
V(k) \\
M(k)
\end{bmatrix}
\]

\[\Rightarrow \begin{bmatrix}
V(k) \\
M(k)
\end{bmatrix} = \begin{bmatrix}
V_0(k)e^{\lambda_1 t} \\
M_0(k)e^{\lambda_2 t}
\end{bmatrix}. \tag{21}\]

\(\lambda_1\) and \(\lambda_2\) are eigenvalues:

\[\lambda_1 = -\frac{(A + D) + \sqrt{(A + D)^2 - 4AD + 4BC}}{2}, \tag{22}\]

\[\lambda_2 = -\frac{(A + D) - \sqrt{(A + D)^2 - 4AD + 4BC}}{2}. \tag{23}\]

We choose a leading term in each elements with the consideration of their coefficients \(k \sim p\) or \(k \sim r\). Then the coefficients are like

\[A \sim 2\nu k^2, \quad B \sim b_1 k^{m}, \quad C \sim c_2 k^{m+2}, \quad D \sim d_2 k^{v+2}. \tag{24}\]

Here\(^2\), since \(AD - BC \sim k^{v+4} - k^{2m+2} \sim 0\) as \(k \to \infty\), the eigenvalues converge to \(\lambda_1 \sim 0, \quad \lambda_2 \sim -2\nu k^2\) as ‘\(k\)’ increases.

\(E_V(k)\) and \(E_M(k)\) are expressed like

\[
\begin{bmatrix}
E_V(k) \\
E_M(k)
\end{bmatrix} = \begin{bmatrix}
b_1 k^m & b_1 k^m \\
2\nu k^2 & 0
\end{bmatrix} \begin{bmatrix}
V_0(k)e^{-\nu t} \\
M_0(k)e^{-2\nu k^2 t}
\end{bmatrix}. \tag{25}\]

If ‘\(V_0\)’ and ‘\(M_0\)’ are replaced by \(E_{V0}(k) \sim k^v\) and \(E_{M0}(k) \sim k^m\), we find the saturated solutions:

\[E_M(k) = 2\nu k^2 E_0(k) \sim 2\nu k^2 \frac{E_{M0}(k)}{2\nu k^2} \sim k^m, \tag{26}\]

\[E_V(k) = b_1 k^m \frac{E_{M0}(k)}{2\nu k^2} + b_1 k^m \left[ -\frac{E_{M0}(k)}{2\nu k^2} + \frac{E_{V0}(k)}{b_1 k^m} \right] e^{-2\nu k^2 t} \sim k^{2m-2}. \tag{27}\]

For complete energy spectra, ‘\(m\)’ is required. But it is difficult to pinpoint a representative magnetic power spectrum because \(E_M(k)\) is a continuously changing curve. (Schekochihin et al. 2004) found ‘\(m = 0\)’, peak of \(E_M(k)\), but we do not think a peak can be a representative power spectrum that drops continuously with the wavenumber ‘\(k\)’. Moreover EDQNM approximation in this paper is for the range of ‘\(k_{peak} \ll k \ll k_{max}\)’. On the other hand,

\(^2\)The dimensional analysis of Eq.(17) implies \(2m \sim v + 2\) when the system is saturated.
(Lazarian et al. 2004) found ‘$m = -1$’ using a simulation done with a strong background magnetic field and balance relation $\mathbf{B} \cdot \nabla \mathbf{B} \sim \nu \nabla^2 \mathbf{v}$. So we infer the index ‘$m = -1/2$’ for the magnetic scaling factor in a system under the influence of a weak background magnetic field. Then, from Eq. (27) we get $E_V(k) \sim k^{-3}$, which matches the simulation results well. Also this makes ‘$AD - BC \sim 0$’, i.e., ‘$\lambda_1 \sim 0$’ so that the energy spectra become independent of time when they are saturated. If ‘$m = -1$’ is chosen, ‘$E_V(k) \sim k^{-4}$’ and $E_M(k) \sim k^{-1}$, coincident with the results of (Cho et al. 2003, Lazarian et al. 2004). A simple relation $E_M^2/E_V = k^2$ can be derived in high $Pr_M$.

4. Discussion

The analytic and simulation job in this paper are to realize the high $Pr_M$ plasma state in ICM. Magnetic field affects the plasma motion through Lorentz force $q \mathbf{v} \times \mathbf{B}$ causing the rotation of ionized particles around magnetic field with frequency $\Omega = qB/m$. The essential reason of anisotropic viscosity, electric and thermal conductivity is the cross product in Lorentz force. As a result, temperature gradient, viscous damping, magnetic diffusion depend on the nontrivial magnetic field in ICM. Moreover, the tendency of anisotropy is in inverse proportion to the eddy scale. Nonetheless we assume an isotropic system, which is reasonable for the system driven by a random isotropic force in large scale.

Simulation results tell us some important features. The viscous scale $k_\nu$ is extended to the much smaller diffusivity scale $k_\eta$. This extension of $k_\nu$, or nontrivial $E_V$ in the subviscous scale is necessary for the growth of $E_M$. Large enough $E_M$ in small scale makes kinetic eddies near the injection scale ($k \sim 2 - 3$) transfer energy to the magnetic eddies. In contrast, kinetic eddies in the inert subviscous scale receive energy from the magnetic eddies. The saturated energy spectra are $E_V \sim k^{-3}$ and smoothly changing $E_M$ in consequence of the interaction between the kinetic and magnetic energy. Analytic analysis using a quasi normal approximation shows the viscosity $\sim \nu k^2$ coupled with $E_M^2$ induces the unusual spectra (Eq. 26, 27).

As mentioned, the features of small scale cannot affect the large scale in general. However, the possibility still exists, especially when there is an instability in microscale due to the anisotropic pressure. Making clear the influences of anisotropic pressure, viscosity, or conductivity on the system is essential to understand ICM further. But, we will leave this topic for the future research.
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A. Appendix

For ‘A1’ in Eq.(8), we differentiate this triple correlation term over time to use Eq.(5), (6). Then, we have

$$\frac{\partial}{\partial t} + \nu (k^2 + p^2 + r^2) \langle v_q(p)v_m(r)v_i(-k) \rangle_{A1} =$$

$$\langle \left[ \frac{\partial}{\partial t} + \nu k^2 \right] v_i(-k) v_q(p) v_m(r) \rangle + \langle v_i(-k) \left[ \frac{\partial}{\partial t} + \nu p^2 \right] v_q(p) v_m(r) \rangle$$

$$+ \langle v_i(-k) v_q(p) \left[ \frac{\partial}{\partial t} + \nu r^2 \right] v_m(r) \rangle = \langle vvvv \rangle + \langle vvBB \rangle \ldots$$  \hspace{1cm} (A1)

If we see the first term, for example,

$$\langle \left[ \frac{\partial}{\partial t} + \nu k^2 \right] v_i(-k) v_j(p) v_m(r) \rangle \delta_{p+r,k} = \sum_{j,l} \left[ M_{ins}(-k) \langle v_n(j) v_s(l) v_q(p) v_m(r) \rangle \right]$$

$$- M_{ins}(-k) \langle B_n(j) B_s(l) v_q(p) v_m(r) \rangle \delta_{j+l,-k}. \hspace{1cm} (A2)$$

the differentiation generates the fourth order correlation. Another differentiation just induces the fifth order correlation. So we need an assumption to close this equation. It is known that statistically turbulent quantities follow a normal distribution. And the fourth-order term $\langle uuuu \rangle$ can be decomposed into the combination of second-order correlation terms like below: quasi-normal approximation \cite{Proudman et al. 1954; Tatsumi 1957}:

$$\langle u_1 u_2 u_3 u_4 \rangle = \langle u_1 u_2 \rangle \langle u_3 u_4 \rangle + \langle u_1 u_3 \rangle \langle u_2 u_4 \rangle + \langle u_1 u_4 \rangle \langle u_2 u_3 \rangle. \hspace{1cm} (A3)$$

So if these second order correlation terms are replaced by energy spectrum expressions as follows:

$$4\pi k^2 \langle v_i(k) v_q(k') \rangle = P_{iq}(k) E_V(k) \delta_{k+k',0}, \hspace{1cm} (A4)$$

$$4\pi k^2 \langle B_i(k) B_q(k') \rangle = P_{iq}(k) E_M(k) \delta_{k+k',0},$$

$$4\pi k^2 \langle B_i(k) v_q(k') \rangle = P_{iq}(k) H_{BV}(k) \delta_{k+k',0},$$

$$(P_{iq} = \delta_{iq} - \frac{k_i k_q}{k^2})$$
Eq. (A1) can be rewritten like
\[
\left[ \frac{\partial}{\partial t} + \nu (k^2 + p^2 + r^2) \right] \langle v_q(p)v_m(r)v_i(-k) \rangle_{A1} =
2M_{ins}(-k) \sum_{p,r} (4\pi p^2)^{-1}(4\pi r^2)^{-1} P_{mq}(p) P_{ms}(r) \left[ E_V(p) E_V(r) - H_{BV}(p) H_{BV}(r) \right] +
2M_{qns}(p) \sum_{p,r} (4\pi k^2)^{-1}(4\pi r^2)^{-1} P_{in}(k) P_{ns}(r) \left[ E_V(k) E_V(r) - H_{BV}(k) H_{BV}(r) \right] +
2M_{mnq}(r) \sum_{p,r} (4\pi k^2)^{-1}(4\pi r^2)^{-1} P_{in}(p) P_{js}(p) \left[ E_V(k) E_V(p) - H_{BV}(k) H_{BV}(p) \right]
= L_{iqm}^{vv1}(k, p, r; t) .
\] (A5)

However, when the fourth-order correlation is decomposed into the combinations of second order terms, the summation of decomposed ones, i.e., right hand side of Eq. (A3), can be larger than its actual value. This can cause a negative energy spectrum, which cannot be allowed (Ogura 1963). So (Orszag 1970) introduced an eddy damping coefficient \(\mu_{kpr}\) of which dimension is \(\sim 1/t\). Its more detailed expression (Pouquet et al. 1976) can be contrived, but the dimension \(\sim 1/t\) does not change. We assume it to be sort of a reciprocal of time constant for simplicity in this paper. Then, with a simple integration we can find the third correlation term:
\[
\langle v_q(p)v_m(r)v_i(-k) \rangle_{A1} = \int_{t}^{t} e^{-\nu (k^2 + p^2 + r^2) + \mu_{kpr}}(t-\tau) L_{iqm}^{vv1}(k, p, r; \tau) d\tau .
\] (A6)

We can also calculate the representation of ‘A3’ in the same way:
\[
\langle B_q(p)B_m(r)v_i(-k) \rangle_{A3} = \int_{t}^{t} e^{-\nu (k^2 + \mu_{kpr})} E_V(p) E_V(r) L_{iqm}^{vv1}(k, p, r; \tau) d\tau .
\] (A7)

Since ‘A2’ and ‘A4’ are ‘-A1’ and ‘-A3’ respectively, Eq.(5) are
\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) \langle v_i(k)v_i(-k) \rangle = \sum_{p+r=k} 2M_{iqm}(k) \left[ \int_{t}^{t} e^{-\nu (k^2 + p^2 + r^2) + \mu_{kpr}}(t-\tau) L_{iqm}^{vv1}(k, p, r; \tau) d\tau 
- \int_{t}^{t} e^{-\nu (k^2 + \mu_{kpr})} E_V(p) E_V(r) L_{iqm}^{vv1}(k, p, r; \tau) d\tau \right].
\] (A8)

If \(L_{iqm}^{vv1}(k, p, r; \tau)\) or \(L_{iqm}^{vv3}(k, p, r; \tau)\) is larger than \((\nu k^2 + \mu_{kpr})^{-1}\) or \((\nu k^2 + \mu_{kpr})^{-1}\), this equation can be markovianized. Thus, with the definition of a triad relaxation time \(\Theta(t)\)
(Frisch et al. 1975) we get

\[
\left( \frac{\partial}{\partial t} + 2\nu k^2 \right) E_V(k) = \sum_{p+r=k} 4\pi k^2 M_{iqm}(k) \left[ \frac{1 - e^{-\nu(k^2+p^2+r^2)+\mu_{kpr} t}}{\nu(k^2 + p^2 + r^2) + \mu_{kpr}} \right] L_{iqm}^{vv1}(k, p, r; t) \\
- \left( \frac{1 - e^{-\nu k^2 + \mu_{kpr} t}}{\nu k^2 + \mu_{kpr}} \right) L_{iqm}^{vv3}(k, p, r; t) \\
eq \sum_{p+r=k} 4\pi k^2 M_{iqm}(k) \left[ \Theta_{kpr}^{vvv}(t) L_{iqm}^{vv1}(k, p, r; t) - \Theta_{kpr}^{vnu}(t) L_{iqm}^{vv3}(k, p, r; t) \right].
\]

(A9)

Using a trigonometric relation: \( k \cdot p = kpz, \ k \cdot r = kry, \ p \cdot r = -prx \) (Fig.6(a)) and \( dp \, dr = \frac{2\pi pr}{k} \, dp \, dr \), we can simplify this expression:

\[
\frac{1}{2} \int dp \, dr \, \Theta_{kpr}^{vvv}(t) \frac{k^3}{pr} (1 - 2y^2z^2 - xyz)[E_V(p)E_V(r) - H_{BV}(p)H_{BV}(r)].
\]

(A10)

Eq.(10), (11) can be derived using a similar way.

REFERENCES

Cho, J., Lazarian, A., & Vishniac, E., 2003, ApJ, 595, 812

Cho, J., Vishniac, E., Beresnyak, A., Lazarian, A., & Ryu, D., 2009, ApJ, 693, 1449

Cho, J., & Yoo, H., 2014, ApJ, 780, 99

Cho, J., 2014, 797, 133

Frisch, U., Pouquet, A., Leorat, J. & Mazure, A., 1975, J. Fluid Mech., 68, 769

Jones, T. W., 2008, Astronomical Society of the Pacific Conference Series, 386, 398

Kazantsev, A., P., 1968, JETP, 26, 1031

Kleeorin, N., Mond, M., & Rogachevskii, I., 1996, A&A, 307, 293

Kolmogorov, A., 1941, Akademiia Nauk SSSR Doklady, 30, 301

Kraichnan R. H. & Nagarajan, S., 1967, Physics of Fluids, 10, 859
Krause, F. & Rädler, K. H., 1980, Mean-field magnetohydrodynamics and dynamo theory
Lazarian, A., Vishniac, E. T., & Cho, J., ApJ, 2004, 603, 180
Lesieur, M., 1987, Turbulence in fluids: Stochastic and numerical modeling
Leslie, D. C. & Leith, C. E., 1975, Physics Today, 28, 59
McComb, W. D., 1990, The physics of fluid turbulence
Moffatt, H. K., 1978, Magnetic Field Generation in Electrically Conducting Fluids
Narayan, R., & Medvedev, M. V., 2001, ApJ, 562, L129
Ogura, Y., 1963, J. Fluid Mech., 16, 33
Orszag, S. A., 1970, J. Fluid Mech., 41, 363
Park, K., 2013, MNRAS, 434, 2020
Park, K., & Blackman, E. G., 2012a, MNRAS, 419, 913
Park, K., & Blackman, E. G., 2012b, MNRAS, 423, 2120
Pouquet, A., Frisch, U., & Leorat, J., 1976, J. Fluid Mech., 77, 321
Proudman, I., & Reid, W. H., 1954, Royal Society of London Philosophical Transactions Series A, 247, 163
Ruzmaikin, A. A., & Shukurov, A. M., 1982, Astrophysics and Space Science, 82, 397
Ryu, D., Kang, H., Cho, J., & Das, S., 2008, SCIENCE, 320, 909
Ryu, D., Porter, D. H., Cho, J., & Jones, T., W., 2012, AAS...21933803R
Schekochihin, A. A., Cowley, S. C., Taylor, S. F., Maron, J. L., & McWilliams, J. C., 2004, ApJ, 612, 276
Schekochihin, A. A., Cowley, S. C., Kulsrud, R. M., Hammett, G. W., & Sharma, P., 2005, ApJ, 629, 139
Schekochihin, A. A., Cowley, S. C., Kulsrud, R. M., Hammett, G. W., & Sharma, P., 2005, ApJ, 629, 139
Schekochihin, A. A., Cowley, S. C., Kulsrud, R. M., Rosin, M. S., & Heinemann, T., 2008, Physical Review Letters, 100(8), 081301
Schekochihin, A. A., Cowley, S. C., Rincon, F., & Rosin, M. S., 2010, MNRAS, 405, 291

Tatsumi, T., 1957, Royal Society of London Proceedings Series A, 239, 16

Yoshizawa, A., 2011, Hydrodynamic and Magnetohydrodynamic Turbulent Flows: Modelling and Statistical Theory (Fluid Mechanics and Its Applications)
Fig. 1.— (a) Time evolution of $E_V(t)$ and $E_M(t)$. Resolution $1024^3$. (b) Averaged energy spectra of $E_V(k)$ and $E_M(k)$ ($56 < t < 70$). Resolution $1024^3$.

Fig. 2.— (a) $E_V(k)$ and $E_M(k)$ in the early time regime ($t \leq 6.3$) (b) $E_V(k)$ and $E_M(k)$ ($16.1 \leq t \leq 29.4$) As $E_M$ grows, large scale kinetic energy is transferred to magnetic eddies, but kinetic eddies in subviscous scale receive energy from the magnetic eddies.
Fig. 3.— (a) Time evolution of $E_V(t)$ and $E_M(t)$ with different resolutions (b) $E_V(t)$ and $E_M(t)$ in the early time regime.

Fig. 4.— (a) $E_V(t)$ and $E_M(t)$ at ‘$t \sim 18.5$’. This is the onset position of $E_M(t)$ with resolution $1536^3$. (b) The evolving ratio of the magnetic energy to the kinetic energy.
Fig. 5.— (a) Compensated energy spectrum \(k^3 E_V(k)\). The flat reference line \(k^0\) actually indicates \(k^{-3}\) (b) \(k^{0.5} E_M(k)\). The peak means \(k^{-1/2}\) in \(E_M(k)\), and another line \(k^{-0.5}\) is for the scale invariant line \(k^{-1}\).

Fig. 6.— (a) The summation of three wave numbers should satisfy the condition \(p + r + k = 0\) (b) Upper plot is for the case of large \(p\) (small \(r\)): \(k \sim p \gg r, x \sim y, z \sim 1\), lower plot is for large \(r\) (small \(p\)): \(k \sim r \gg p, x \sim z, y \sim 1\).