$N = 1$ Supergravity BPS Domain Walls on Kähler-Ricci Soliton

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ABSTRACT

This paper provides a study of some aspects of flat and curved BPS domain walls together with their Lorentz invariant vacua of four dimensional chiral $N = 1$ supergravity. The scalar manifold can be viewed as a one-parameter family of Kähler manifolds generated by a Kähler-Ricci flow equation. Consequently, a vacuum manifold characterized by $(m, \lambda)$ where $m$ and $\lambda$ are the dimension and the index of the manifold, respectively, does deform with respect to the flow parameter related to the geometric soliton. Moreover, one has to carry out the renormalization group analysis to verify the existence of such a vacuum manifold in the ultraviolet or infrared regions. At the end, we discuss a simple model with linear superpotential on $U(n)$ symmetric Kähler-Ricci orbifolds.

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1 Introduction

A Kähler-Ricci flow equation has gained both mathematical and physical interests due to several developments. In the mathematical context, particularly in compact Kähler manifolds with the first Chern class $c_1 = 0$ or $c_1 < 0$, this equation could give a new proof of the well-known Calabi conjecture [1]. Moreover, it has been shown the existence of solutions of the Kähler-Ricci equation with $U(n)$ symmetry on $\mathbb{C}^n$ [2, 3] and line bundles over $\mathbb{C}P^{n-1}$ [4].

In the physical context, for example, four dimensional chiral $N = 1$ supersymmetries on Kähler-Ricci solitons generating a one-parameter family of Kähler manifolds together with their domain wall solutions have been studied by two of the authors in serial papers [5, 6]. Those works provide preliminary studies of the Anti de Sitter/Conformal Field Theory (AdS/CFT) correspondence (for a review, see for example [7]) in four dimensions and its evolution on special cases of Kähler-Ricci solitons, namely Kähler-Einstein manifolds, $U(n)$ symmetric cone-dominated Kähler-Ricci solitons on line bundles over $\mathbb{C}P^{n-1}$ [4], and two dimensional Kähler-Ricci solitons.

The aim of this paper is to study flat and curved BPS domain walls of four dimensional $N = 1$ supergravity coupled to chiral multiplets whose scalar manifold is considered to be a one-parameter family of Kähler geometries generated by the Kähler-Ricci flow equation. Then, several aspects of their Lorentz invariant vacua will also be discussed. Our study here is in the context of Morse(-Bott) theory and the renormalization group (RG) flow analysis that generalizes the previous works for flat domain walls on Kähler-Einstein manifolds and the $U(n)$ symmetric cone-dominated Kähler-Ricci solitons [5], and also, curved (AdS sliced) domain walls on two dimensional Kähler-Ricci solitons [6]. At the end, we apply the study to a case of $U(n)$ symmetric Kähler-Ricci solitons admitting an orbifold-type singularity at the origin (called $U(n)$ symmetric Kähler-Ricci orbifolds) [4].

We organize this paper as follows. Section 2 is devoted to review general descriptions of Kähler-Ricci soliton and the construction of an example, namely $U(n)$ symmetric Kähler-Ricci orbifolds. Then, some possible cases of the soliton near the orbifold point (the origin) which can be viewed as a one-parameter family of Kähler manifolds are summarized in Theorem 1 and Lemma 2. In Section 3 we provide a review of $N = 1$ supergravity coupled to chiral multiplets where the scalar manifold is considered to be a one-parameter family of Kähler manifolds generated by a Kähler-Ricci soliton. In Section 4 flat domain walls are reviewed and then, we study some aspects of their vacuum structure on general Kähler-Ricci soliton related to AdS/CFT correspondence. Some results are written down in Theorem 3 and Theorem 4. Then, we extend the previous case to curved domain walls in Section 5. In Section 6 we discuss properties of vacua of $N = 1$ theory on $U(n)$ symmetric Kähler-Ricci orbifolds near the origin. Finally, we put our conclusions in Section 7.

2 Kähler-Ricci Flow

The structure of this section is as follows. Firstly, we review some general aspects of the the Kähler-Ricci flow. Secondly, the construction of Kähler-Ricci solitons on complex line bundles over $\mathbb{C}P^{n-1}$, $n \geq 2$, with orbifold-type singularities called Kähler-Ricci orbifolds will also be reviewed [4]. Then, we particularly discuss some possible cases where the soliton near the orbifold point can be viewed as a one-parameter family of Kähler manifolds.
2.1 General Picture

Let us first consider a general picture of the Kähler-Ricci soliton. A complex Kähler manifold \((M, g(\tau))\) satisfying
\[
\frac{\partial g_{i\bar{j}}(z, \bar{z}; \tau)}{\partial \tau} = -2R_{i\bar{j}}(z, \bar{z}; \tau), \quad 0 \leq \tau < T,
\]
(2.1)
is called Kähler-Ricci soliton where \((z, \bar{z}) \in M\) and the metric
\[
g(\tau) \equiv g_{i\bar{j}}(z, \bar{z}; \tau) \, dz^i d\bar{z}^\bar{j}.
\]
(2.2)
In particular, the metric \(g(\tau)\) can be written as
\[
g(\tau) = \sigma(\tau) \psi_\tau^*(g(0)), \quad 0 \leq \tau < T,
\]
(2.3)
with \(\sigma(\tau) \equiv (1 - 2\Lambda \tau)\) and the diffeomorphism map \(\psi_\tau\). The initial metric at \(\tau = 0\), namely \(g(0)\), fulfills the following relation
\[
-2R_{i\bar{j}}(0) = \nabla_i Y_{\bar{j}}(0) + \bar{\nabla}_{\bar{j}} Y_i(0) - 2\Lambda g_{i\bar{j}}(0),
\]
(2.4)
for \(\Lambda \in \mathbb{R}\) and some holomorphic vector fields
\[
Y(0) = Y^i(z,0) \partial_i + Y^\bar{i}(\bar{z},0) \bar{\partial}_i,
\]
(2.5)
on \(M\) where \(i, j = 1, \ldots, \dim\mathbb{C}(M)\). Moreover, using the vector field \(Y(0)\), we can define a \(\tau\)-dependent vector field \(X(\tau)\)
\[
X(\tau) = \frac{1}{\sigma(\tau)} Y(0),
\]
(2.6)
generating a family of diffeomorphisms \(\psi_\tau\). In addition, the vector field \(X(\tau)\) satisfies the following equation
\[
\frac{\partial \hat{z}^i}{\partial \tau} = X^i(\hat{z}, \tau),
\]
(2.7)
where
\[
\hat{z} \equiv \psi_\tau(z).
\]
(2.8)

2.2 An Example: \(U(n)\) Symmetric Kähler-Ricci Orbifold

Before constructing Kähler-Ricci orbifolds, we firstly discuss the construction of a Kähler metric on \(\mathbb{C}^n \setminus \{0\}\). Our starting point is to define the initial Kähler potential at \(\tau = 0\), namely
\[
K(z, \bar{z}, 0) = \phi(u),
\]
(2.9)

\(^2\)We also recommend [8, 9, 10] for a review of the Ricci flow equation.
\(^3\)Holomorphicity of \(Y(0)\) is coming from the fact that we impose the condition \(\psi_\tau^*(J) = J\) on \(M\) where \(J\) is the complex structure on \(M\).
where
\[ u \equiv 2 \ln(\delta_{i\bar{j}} z^i \bar{z}^j) = 2 \ln|z|^2 , \] (2.10)
on \mathbb{C}^n \setminus \{0\}. Thus, the ansatz (2.9) maintains a \( U(n) \) symmetry. For a shake of simplicity we take the vector field \( Y^i(0) \) to be holomorphic and linear
\[ Y^i(0) = \mu z^i , \] (2.11)
where \( \mu \in \mathbb{R} \). Then, the Kähler potential (2.9) implies that the metric is given by
\[ g(0) = g_{i\bar{j}}(0) dz^i d\bar{z}^j = \left[ 2 e^{-u/2} \phi_u \delta_{i\bar{j}} + 4 e^{-u} (\phi_{uu} - \frac{1}{2} \phi_u) \bar{z}^i z^j \right] dz^i d\bar{z}^j , \] (2.12)
together with its inverse
\[ g^{-1}(0) = g^{i\bar{j}}(0) \bar{\partial}_j \partial_i = \frac{e^{u/2}}{2 \phi_u} \left[ \delta^{i\bar{j}} - e^{-u/2} \frac{\phi_{uu} - \frac{1}{2} \phi_u}{\phi_u} \bar{z}^i z^j \right] \bar{\partial}_j \partial_i , \] (2.13)
where the positivity of the metric (2.12) implies
\[ \phi_u \equiv \frac{d\phi}{du} > 0 , \quad \phi_{uu} \equiv \frac{d^2 \phi}{du^2} > 0 . \] (2.14)
Inserting (2.11) and (2.12) into (2.4), and defining \( \Phi \equiv \phi_u \) and \( \Phi_u \equiv F(\Phi) \), we then obtain
\[ \frac{dF}{d\Phi} + \left( n - 1 - 4\mu \right) F - \left( n - 2\Lambda \Phi \right) = \frac{A_0}{\Phi} e^{(1-n)u/2} , \] (2.15)
with \( A_0 \) is an arbitrary constant. Taking simply \( A_0 = 0 \), the solution of (2.15) has the form
\[ \Phi_u = F(\Phi) = A_1 e^{4\mu \Phi} \Phi^{n-1} + \frac{\Lambda}{2\mu} \Phi + \frac{2(\Lambda - \mu)}{(4\mu)^n} \sum_{j=0}^{n-1} \frac{n!}{j!} (4\mu)^j \Phi^{1+n-j} , \] (2.16)
where \( A_1 \) is also an arbitrary constant. Note that for \( \mu = 0 \) we obtain the Kähler-Einstein geometry as the solution of (2.15).

Recalling (2.6) and (2.11) we find that the vector field \( X^i(\tau) \) has the form
\[ X^i(\tau) = \frac{\mu}{\sigma(\tau)} z^i , \] (2.17)
which generates the diffeomorphisms
\[ \hat{z} \equiv \psi_\tau(z) = \sigma(\tau)^{-\mu/2} z . \] (2.18)
Thus, the complete \( \tau \)-dependent Kähler-Ricci soliton on \( \mathbb{C}^n \setminus \{0\} \) is given by
\[ g(z, \hat{z}; \tau) = \sigma(\tau)^{1-\mu/\Lambda} g_{i\bar{j}} (\sigma(\tau)^{-\mu/2} z) \, dz^i d\hat{z}^j . \] (2.19)
Now, we turn to construct the gradient Kähler-Ricci soliton on line bundles over \( \mathbb{C} P^{n-1} \) with \( n \geq 2 \). Let us first define the metric (2.12) on \( (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_\ell \), where \( \mathbb{Z}_\ell \) acting on \( \mathbb{C}^n \setminus \{0\} \) by \( z \mapsto e^{2\pi i/\ell} z \) with \( \ell \) is a positive integer and \( \ell \neq 0 \). A positive line bundle over \( \mathbb{C} P^{n-1} \), denoting by \( L^\ell \), can then be constructed by gluing \( \mathbb{C} P^{n-1} \) into \( (\mathbb{C}^n \setminus \{0\})/\mathbb{Z}_\ell \) at infinity. Note that for \( \ell = 1 \), \( L \) is called hyperplane bundle which is dual to the tautological
line bundle $L^{-1}$. Moreover, we replace the above coordinates $z$ by new coordinates of the form

$$\xi^i \equiv (z^i)\ell, \quad (2.20)$$

parameterizing $(\mathbb{C}^n\setminus\{0\})/\mathbb{Z}_\ell$. This follows that around infinity, the initial metric (2.12) can be changed into the form

$$g(0) = \Phi g_{FS} + \Phi_v dw d\bar{w}, \quad (2.21)$$

where $g_{FS}$ is the standard Fubini-Study metric of $\mathbb{C}P^{n-1}$

$$g_{FS} = \left(\frac{\delta_{\bar{a}b}}{1 + \zeta^c\bar{\zeta}^c} - \frac{\zeta^b\bar{\zeta}^a}{(1 + \zeta^c\bar{\zeta}^c)^2}\right) d\zeta^a d\bar{\zeta}^b, \quad (2.22)$$

with $\zeta^a \equiv \xi^a / \xi^n$, $\zeta^n = 1$ and $a, b, c = 1, \ldots, n - 1$. Here, $w \equiv w(\xi, \bar{\xi})$ is a nonholomorphic coordinate and $v \equiv 2 \ln|\xi|^2$. So, in this case we have

$$\lim_{v \to +\infty} \Phi(v) = a > 0, \quad F(a) = 0, \quad \frac{dF}{d\Phi}(a) < 0, \quad (2.23)$$

with

$$a = \frac{1}{4\Lambda} (n + \ell). \quad (2.24)$$

The condition $F(a) = 0$ implies

$$A_1(\Lambda, a; \mu) = -e^{4\mu a} \left[ \frac{2}{4\mu} \frac{\Lambda}{a^n} + 2n!(\Lambda - \mu) \sum_{j=0}^{n-1} \frac{(4\mu)^j}{j!} a^j \right]. \quad (2.25)$$

For the case at hand we only have $\Lambda > 0$ and $\ell > 0$ which describe a shrinking Kähler-Ricci soliton.

Next, let us consider a condition for adding a point to $(\mathbb{C}^n\setminus\{0\})/\mathbb{Z}_\ell$ at $z = 0$. We begin by mentioning a similar condition as (2.23), namely

$$\lim_{v \to -\infty} \Phi(v) = 0, \quad F(0) = 0, \quad \frac{dF}{d\Phi}(0) > 0. \quad (2.26)$$

From $F(0) = 0$, it follows

$$A_1(\Lambda, 0; \mu) = -2n!(\Lambda - \mu) / (4\mu)^{n+1}. \quad (2.27)$$

After some computations, we then obtain

$$\Phi(v) = |\xi|^2 B(|\xi|^2), \quad (2.28)$$

where $B(0) > 0$. It is easy to see that around $z = 0$, the initial metric (2.12) simplifies to

$$g(0) = 2 \left[ B(|\xi|^2) \delta_{ij} + \bar{B}(|\xi|^2) \bar{\xi}^i \xi^j \right] d\xi^i d\bar{\xi}^j. \quad (2.29)$$

The complete $\tau$-dependent soliton has the form

$$g(\tau) = 2 \sigma(\tau)^{-1-\mu/\Lambda} \left[ B(\sigma(\tau)^{-\mu/\Lambda} |\xi|^2) \delta_{ij} + \sigma(\tau)^{-\mu/\Lambda} \bar{B}(\sigma(\tau)^{-\mu/\Lambda} |\xi|^2) \bar{\xi}^i \xi^j \right] d\xi^i d\bar{\xi}^j. \quad (2.30)$$
Some comments are as follows. Firstly, taking (2.25) and (2.27) satisfy simultaneously, we find that for $\Lambda \geq 1$ and small $\mu > 0$, one positive root $\mu$ indeed exists in $0 < \mu < \Lambda$. Secondly, the condition (2.26) means $L^\ell \cup \{0\} \simeq \mathbb{C}P^n/\mathbb{Z}_\ell$. For $\ell = 1$, we have $\mu = 0$ which means that the soliton is the Kähler-Einstein geometry $\mathbb{C}P^n$. In other words, we smoothly add a point at $z = 0$ and the initial metric (2.29) becomes flat. In the other case, namely the $\ell \geq 2$ case, the metric (2.29) vanishes at $z = 0$ showing that this is an orbifold singularity at the origin. Finally, the flow (2.30) also diverges at $\tau = 1/2\Lambda$.

For the rest of the paper we will consider the case where the Kähler-Ricci soliton can be viewed as one-parameter family of Kähler manifolds. In order to obtain a consistent picture we take the functions $B(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2)$ and $\dot{B}(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2)$ to be real valued for $\tau \geq 0$ and $\tau \neq 1/2\Lambda$. For the case at hand, namely the soliton (2.30), the first possibility can be written down in the following statements.

**Theorem 1** Let us consider the Kähler-Ricci soliton around the origin (2.30) where $p/q \equiv 1 - \mu/\Lambda$ is taken to be a rational number with $0 < p < q$. Assuming that there exist such $p$ and $q$ for $\Lambda \geq 1$ and $0 < \mu < \Lambda$. Then, the possible cases are as follows.

1. If both $p$ and $q$ are odd integers, one has then

$$\begin{align*}
\sigma(\tau)^{p/q} > 0 & \quad \text{for } \tau < 1/2\Lambda, \\
\sigma(\tau)^{p/q} < 0 & \quad \text{for } \tau > 1/2\Lambda.
\end{align*}$$

The functions $B(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2)$ and $\dot{B}(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2)$ are positive definite for $\tau \geq 0$ and $\tau \neq 1/2\Lambda$.

2. If $p$ and $q$ are even and odd integers respectively, then $\sigma(\tau)^{p/q} > 0$ for $\tau \geq 0$ and $\tau \neq 1/2\Lambda$. But, the sign of both $B(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2)$ and $\dot{B}(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2)$ might be altered after hitting the singularity at $\tau = 1/2\Lambda$.

3. If $p$ and $q$ are odd and even respectively, then $\sigma(\tau)^{p/q}$ becomes imaginary for $\tau > 1/2\Lambda$. In other words, no Kähler manifold exists for $\tau > 1/2\Lambda$.

The second possibility is then

**Lemma 2** If $1 - \mu/\Lambda$ is an irrational number, then it does not exist any Kähler manifold for $\tau > 1/2\Lambda$ because $\sigma(\tau)^{1-\mu/\Lambda}$ turns into a complex number for $\tau > 1/2\Lambda$.

We put the proof of Theorem 1 and Lemma 2 in Appendix A. In addition, since $\xi \approx 0$ we could have

$$\begin{align*}
B(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2) & \approx B(0), \\
\dot{B}(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2) & \approx \dot{B}(0),
\end{align*}$$

for some finite $\tau$ that far away from $\tau = 1/2\Lambda$. This is the simplest case of Theorem 1 and Lemma 2 where the functions $B$ and $\dot{B}$ are positive definite. Some of these results will be applied to study a vacuum structure of the $N = 1$ theory in Section 6.
3 $N=1$ Chiral Supergravity on Kähler-Ricci Soliton

This section is devoted to review properties of four dimensional $N=1$ chiral supergravity on a one-parameter family of Kähler manifolds generated by a Kähler-Ricci soliton [5]. Here, we only consider some aspects which are useful for our analysis for the rest of the paper. Some excellent references for a review of $N=1$ supergravity in four dimensions can be found, for example in [11, 12].

The $N=1$ theory is simply a gravitational multiplet coupled with $n$ chiral multiplets. The gravitational multiplet has a vierbein $e^a_\mu$ and a vector spinor $\psi_\mu$ where $a = 0, \ldots, 3$ and $\nu = 0, \ldots, 3$ are the flat and the curved indices, respectively. The chiral multiplet consists of a complex scalar $z$ and a spin-$\frac{1}{2}$ fermion $\chi$.

The construction of the local $N=1$ theory on a Kähler-Ricci is as follows. First, we consider the Lagrangian in [13] as the initial Lagrangian at $\tau = 0$. Then, by replacing all couplings that depend on the geometric quantities such as the metric $g_{ij}(0)$ by the soliton $g^{ij}(\tau)$, the bosonic parts of the Lagrangian has the form

$$\mathcal{L}^{N=1} = -\frac{M_P^2}{2} R + g_{ij}(z, \bar{z}; \tau) \partial_\nu z^i \partial^\nu \bar{z}^j - V(z, \bar{z}; \tau),$$

where $M_P$ is the Planck mass. The quantity $R$ is the Ricci scalar of the four dimensional spacetime; the scalar fields $(z, \bar{z})$ span a Hodge-Kähler manifold endowed with metric $g_{ij}(z, \bar{z}; \tau) \equiv \partial_i \partial_j K(z, \bar{z}; \tau)$ satisfying (2.1); and $K(z, \bar{z}; \tau)$ is a real function, called the Kähler potential. Then, the $N=1$ scalar potential $V(z, \bar{z}; \tau)$ has the form

$$V(z, \bar{z}; \tau) = e^{K(\tau)/M_P^2} \left( g^{ij}(\tau) \nabla_i W \nabla_j \bar{W} - \frac{3}{M_P^2} W \bar{W} \right),$$

where $W$ is a holomorphic superpotential and $\nabla_i W \equiv \partial_i W + (K_i(\tau)/M_P^2) W$. Furthermore, the Lagrangian (3.1) admits a symmetry realized in realized in the following supersymmetry transformations up to three-fermion terms

$$\delta \psi_{1\nu} = M_P \left( D_\nu \epsilon_1 + \frac{i}{2} e^{K(\tau)/2M_P^2} W \gamma_\nu \epsilon_1 + \frac{i}{2M_P} Q_\nu(\tau) \epsilon_1 \right),$$

$$\delta \chi^i = i \partial_\nu z^i \gamma^\nu \epsilon_1 + N^i(\tau) \epsilon_1,$$

$$\delta e^a_\nu = -\frac{i}{M_P} (\bar{\psi}_{1\nu} \gamma^a \epsilon_1 + \bar{\psi}_{1\nu} \gamma^a \epsilon_1),$$

$$\delta z^i = \chi^i \epsilon_1,$$

where $N^i(\tau) \equiv e^{K(\tau)/2M_P^2} g^{ij}(\tau) \nabla_j \bar{W}$, $g^{ij}(\tau)$ is the inverse of $g_{ij}(\tau)$, and the $U(1)$ connection $Q_\nu(\tau) \equiv -(K_i(\tau) \partial_\nu z^i - K_i(\tau) \partial_\nu \bar{z}^i)$. In addition, we have also introduced $\epsilon_1 \equiv \epsilon_1(x, \tau)$.

4 Flat BPS Domain Walls on Kähler-Ricci Soliton

The organization of this section is as follows. First, we provide a short review of flat BPS domain walls of the four dimensional $N=1$ supergravity coupled to arbitrary chiral multiplets whose the scalar fields span a one-parameter family of Kähler manifolds generated by the Kähler-Ricci flow (2.1). These have originally discussed in [5] 4. Here, we
use similar convention as in [5, 16, 17]. Second, we describe general aspects of supersymmetric vacua of the $N = 1$ theory on general Kähler-Ricci soliton in the context of Morse theory. Finally, some aspects of the vacua on the Kähler-Ricci orbifold are discussed. We particularly consider those aspects near the origin (which is the orbifold point), since around $\xi \to +\infty$ the soliton becomes the Kähler-Einstein geometry $\mathbb{C}P^{n-1}$ which has been studied in [5].

4.1 BPS Equations for Flat Domain Walls

The flat background domain walls can be viewed as

$$ds^2 = a^2(u, \tau) \eta_{\mu \lambda} dx^\mu dx^\lambda - du^2,$$  

(4.1)

where $\mu, \lambda = 0, 1, 2$, $\eta_{\mu \lambda}$ is the metric on the three dimensional Minkowskian spacetime $\mathbb{R}^{1,2}$, and then, the components of the corresponding Ricci tensor of the metric (4.1) are given by

$$R_{\mu \nu} = \left[ \left( \frac{a'}{a} \right)' + 3 \left( \frac{a'}{a} \right)^2 \right] a^2 \eta_{\mu \nu},$$

$$R_{33} = -3 \left[ \left( \frac{a'}{a} \right)' + \left( \frac{a'}{a} \right)^2 \right],$$  

(4.2)

and the Ricci scalar has the form

$$R = 6 \left[ \left( \frac{a'}{a} \right)' + 2 \left( \frac{a'}{a} \right)^2 \right],$$

(4.3)

where $a' \equiv \partial a/\partial u$. Here, $a(u, \tau)$ is the warped factor taken to be $\tau$ dependent.

In order to derive a set of equations that preserves partially supersymmetry on the walls, we firstly have to consider the supersymmetry transformation (3.3) on the background (4.1). Setting $\psi_\mu = \chi^i = 0$, $z = z(u, \tau)$, and correspondingly, solving the equations $\delta \psi_\mu = 0$, $\delta \chi^i = 0$, it then results in [5]

$$\frac{a'}{a} = \pm W(\tau),$$

$$z^{ii} = \mp 2 g^{ij}(\tau) \partial_j W(\tau),$$

$$\bar{z}^{\bar{i}i} = \pm 2 g^{\bar{i}\bar{j}}(\tau) \partial_j W(\tau),$$  

(4.4)

where $W(\tau) \equiv e^{K(\tau)}|W(z)|$, $K(\tau) \equiv K(z, \bar{z}; \tau)$ is the Kähler potential, and $W(z)$ is a holomorphic superpotential. In this case, the warped factor $a$ is monotonically decreasing related to the c-function in the holographic correspondence [18]. The second and the third equations in (4.4) are the BPS equations for flat domain walls describing supersymmetric gradient flows. Another relevant supersymmetric flows for our analysis is the renormalization group flow given by the beta functions

$$\beta^i \equiv a \frac{\partial z^i}{\partial a} = -2 g^{ij}(\tau) \partial_j W/W,$$

$$\bar{\beta}^{\bar{i}} \equiv a \frac{\partial \bar{z}^{\bar{i}}}{\partial a} = -2 g^{\bar{i}\bar{j}}(\tau) \partial_j W/W,$$  

(4.5)
describing the behavior of the couplings \((z^i, \bar{z}^\bar{i})\) with respect to the energy scale \(a\) in the context of the conformal field theory (CFT) on the boundary \(\mathbb{R}^{1,2}\) \cite{16, 18, 19}.

Furthermore, the scalar potential (3.2) can be cast into the form

\[
V(z, \bar{z}; \tau) = 4g^{ij}(\tau) \partial_i W \partial_j \bar{W} - \frac{3}{M_p^2} \mathcal{W}^2, \tag{4.6}
\]

and then, its first derivative with respect to \((z, \bar{z})\) is

\[
\partial_i V = 4g^{jk} \nabla_i \partial_j W \partial_k \bar{W} + 4g^{jk} \partial_i W \partial_j \bar{W} \partial_k \bar{W} - \frac{6}{M_p^2} \mathcal{W} \partial_i \bar{W}, \tag{4.7}
\]

where \(\nabla_i \partial_j W = \partial_i \partial_j W - \Gamma^k_{ij} \partial_k W\). All above quantities are useful to study properties of supersymmetric Lorentz invariant vacua in the next subsection \(^5\).

### 4.2 General Picture of Supersymmetric Vacua

Let us first discuss general properties of vacua of the \(N = 1\) theory. A point \(p_0 \equiv (z_0, \bar{z}_0)\) is a vacuum if

\[
\partial_i V(p_0) = \partial_i \bar{V}(p_0) = 0. \tag{4.8}
\]

Supersymmetry further demands that \(p_0\) defines a critical point of the real function \(W(\tau)\), namely

\[
\partial_i \bar{W}(p_0) = \partial_i W(p_0) = 0, \tag{4.9}
\]

which can be regarded as a fixed point of the BPS equations in (4.4). At \(p_0\), the scalar potential (4.6) becomes

\[
V(p_0; \tau) = -\frac{3}{M_p^2} \mathcal{W}^2(p_0; \tau) \equiv -\frac{3}{M_p^2} \mathcal{W}_0^2, \tag{4.10}
\]

which shows that the spacetime is AdS with negative cosmological constant (or \(W_0 \neq 0\)) and the warped factor

\[
a(u, \tau) = a_0(\tau) e^{\pm \mathcal{W}_0 u}. \tag{4.11}
\]

The Hessian matrix of the scalar potential (4.6) evaluated at \(p_0\) is given by

\[
\begin{align*}
\partial_i \partial_j V(p_0; \tau) &= -\frac{1}{M_p^2} \mathcal{M}_{ij}(p_0; \tau) \bar{L}(p_0; \tau), \\
\bar{\partial}_i \bar{\partial}_j V(p_0; \tau) &= -\frac{1}{M_p^2} \bar{\mathcal{M}}_{ij}(p_0; \tau) L(p_0; \tau), \\
\partial_i \bar{\partial}_j V(p_0; \tau) &= \frac{2}{M_p^4} \mathcal{G}_{ij}(p_0; \tau) \mathcal{W}_0^2,
\end{align*}
\]

\(^5\)For the rest of paper we mention Lorentz invariant vacuum (vacua) as vacuum (vacua) or ground state.
where the quantities

\[ L(p_0; \tau) = e^{K(p_0; \tau)/2M_P^2}W(z_0), \]
\[ M_{ij}(p_0; \tau) = e^{K(p_0; \tau)/2M_P^2}\left( \partial_i \partial_j W(z_0) + \frac{1}{M_P^2}K_{ij}(p_0; \tau)W(z_0) + \frac{1}{M_P^2}K_j(p_0; \tau)\partial_i W(z_0) \right), \]

(4.13)

with their complex conjugate are related to the couplings in two-fermions terms in the

\[ N = 1 \]

Lagrangian providing the masses of the gravitino field and the masses of the spin-half fermions, respectively. Moreover, using real coordinates defined as \[ z^i \equiv x^i + ix^{i+n} \] one can then show that the Hessian matrix (4.12) in this case is indeed a Hermitian matrix, and therefore, it only admits real eigenvalues.

The eigenvalues of the Hessian matrix (4.12) determine the stability of domain walls in the context of dynamical system. For negative eigenvalues we have unstable walls since the gradient flow provided by the last two equations in (4.4) is unstable along this direction, whereas stable walls are for positive eigenvalues. In addition, the number of negative eigenvalues of (4.12) is called the Morse index of a vacuum.

Then, the vacuum \( p_0 \) could also related to the CFT on \( \mathbb{R}^{1,2} \) which can be checked by using the first order expansion of the beta function (4.5) at \( p_0 \), namely

\[ U \equiv - \left( \begin{array}{cc} \partial_j \beta^i & \partial_j \bar{\beta}^i \\ \bar{\partial}_j \beta^i & \bar{\partial}_j \bar{\beta}^i \end{array} \right) (p_0; \tau), \]

(4.14)

where

\[ \partial_j \beta^i(p_0; \tau) = -2g^{ik}(p_0; \tau)\frac{\partial_j \partial_k W_0}{W_0} = -\frac{1}{M_P^2}\delta_j^i, \]
\[ \bar{\partial}_j \bar{\beta}^i(p_0; \tau) = -2g^{ik}(p_0; \tau)\frac{\bar{\partial}_j \bar{\partial}_k W_0}{W_0}, \]

(4.15)

together with their complex conjugate with \( \partial_j \partial_k W_0 \equiv \partial_j \bar{\partial}_k W(p_0; \tau) \). It is easy then to see that the matrix (4.14) has real eigenvalues since it is also a Hermitian matrix. Thus, we have a consistent theory.

In the ultraviolet (UV) region we have the energy scale \( a \to +\infty \) and the matrix (4.14) must have at least a positive eigenvalue because the RG flow is flowing away from the region in this direction. On the other side, namely in the infrared (IR) region, the energy scale \( a \to 0 \) and the RG flow approaches the vacuum in the direction of negative eigenvalue of (4.14).

Here, we consider a general case, but the results are similar as in the Kähler-Einstein case [5]. First of all, we consider a case where only nondegenerate vacuum exists as follows.

**Theorem 3** Let the scalar potential \( V(\tau) \) be a Morse function, i.e. no degenerate vacuum exists. Defining

\[ V_{ij}(p_0; \tau) \equiv -\frac{\varepsilon(\sigma)}{M_P^2}M_{ij}(p_0; \tau)\bar{L}(p_0; \tau), \]
\[ V_{ij}(p_0; \tau) \equiv |g^{kl}(p_0; \tau)|M_{ik}(p_0; \tau)\bar{M}_{lj}(p_0; \tau) - \frac{2}{M_P^2}|g_{ij}(p_0; \tau)|W_0^2, \]

(4.16)
where
\[ \varepsilon(\sigma) = \begin{cases} 
1 & \text{if } 0 \leq \tau < 1/2\Lambda, \\
-1 & \text{if } \tau > 1/2\Lambda,
\end{cases} \]
with \( \Lambda > 0 \) for \( \tau \geq 0 \) and taking the inequalities
\[
\begin{align*}
\text{Re}(V_{ij}(p_0; \tau)) &> 0, & \text{Im}(V_{ij}(p_0; \tau)) &> 0, \\
\text{Re}(\bar{V}_{ij}(p_0; \tau)) &> 0, & \text{Im}(\bar{V}_{ij}(p_0; \tau)) &> 0,
\end{align*}
\] for \( \tau \geq 0 \) and \( \tau \neq 1/2\Lambda \), then there exists a parity transformation of the Hessian matrix (4.12) caused by the Kähler-Ricci soliton (2.19). In other words, if \( p_0 \) is a vacuum of the index \( \lambda \) in \( \tau < 1/2\Lambda \), then it becomes \( \hat{p}_0 \) of the index \( 2n - \lambda \) in \( \tau > 1/2\Lambda \).

The proof of the above theorem has similar way with the case of Kähler-Einstein geometries discussed in [5]. We have assumed that the Kähler-Ricci soliton is well defined for \( \tau \geq 0 \) unless at \( \tau = 1/2\Lambda \). Note that since \( \mathcal{V}(\tau) \) is Morse function, then no index modification of the index \( \lambda \) in \( \tau < 1/2\Lambda \) and of the index \( 2n - \lambda \) in \( \tau > 1/2\Lambda \). If the conditions (4.18) do not hold, then there is no parity transformation and the index remains unchanged.

To make the above statements clearer, we now take a look at a case where all spin-1/2 fermions are massless at the ground states. This means
\[ M_{ij}(p_0; \tau) = 0 \Rightarrow \partial_i \partial_j \mathcal{W}_0 = 0. \] Then, the Hessian matrix (4.12) simplifies to
\[
\partial_i \bar{\partial}_j V(p_0; \tau) = -\frac{2}{M_P^2} g_{ij}(p_0; \tau) \mathcal{W}_0^2,
\] and the matrix (4.14) becomes
\[
U \equiv \frac{1}{M_P^2} \begin{pmatrix}
\delta^i_j & 0 \\
0 & \delta^i_j
\end{pmatrix},
\] showing that these vacua exist only in UV region. If \( g_{ij}(p_0; \tau) \) is positive definite for finite \( \tau \), then the walls are unstable in this case. The other case, namely for negative definite \( g_{ij}(p_0; \tau) \) we have only stable walls. If \( g_{ij}(p_0; \tau) \) vanishes, then we have singularity of the theory.

Next, we consider a generalized case where the theory possibly admits degenerate vacua. Or in other words, the scalar potential (4.6) is a Morse-Bott function. The construction can be structured as follows. Let \( S \) be a \( m \) dimensional be a vacuum submanifold of a Kähler geometry \( M \). Then at any \( p_0 \in S \) we can split the tangent space \( T_{p_0}M \) as
\[ T_{p_0}M = T_{p_0}S \oplus N_{p_0}S, \] where \( T_{p_0}S \) is the tangent space of \( S \) and \( N_{p_0}S \) is the normal space \( S \). Moreover, the Hessian matrix (4.12) is non-degenerate in the normal direction to \( S \). So we have a rich and complicated structure of vacua. In the following we list some possibilities.
Theorem 4 Let $S$ be an $m$-dimensional submanifold of $M$ with index $\lambda$ in $0 \leq \tau < \tau_0$ and $\tau_0 < 1/2\Lambda$. Then we have the following cases.

1. $S$ deforms to an $m_1$-dimensional vacuum submanifold $S_1 \subseteq M$ of the index $\lambda_1$ in $\tau_0 \leq \tau < 1/2\Lambda$. If $m_1 \neq m$, then the index $\lambda_1 \in \{0, \ldots, 2n - m_1\}$. On the other hand, if $m_1 = m$, then $\lambda_1 \neq \lambda$ and $\lambda_1 \in \{0, \ldots, 2n - m_1\}$.

2. If the inequalities (4.18) hold, then there exists a parity pair of $S$, namely an $m$-dimensional submanifold $\hat{S} \subseteq M$ of the index $2n - \lambda$ in $\tau > 1/2\Lambda$.

3. If the inequalities (4.18) do not hold, then $S$ deforms to an $n_1$-dimensional vacuum submanifold $\hat{S}_1 \subseteq M$ of the index $\lambda_2$ in $\tau > 1/2\Lambda$. If $n_1 \neq m$, then we have $\lambda_2 \in \{0, \ldots, 2n - n_1\}$. However, if $n_1 = m$, then $\lambda_2 \neq 2n - \lambda$ and $\lambda_2 \in \{0, \ldots, 2n - n_1\}$.

Furthermore, $\hat{S}$ is the parity pair of $S$, and the others, namely $S_1$ and $\hat{S}_1$, are not the parity pair. All $S$, $S_1$, $\hat{S}$, and $\hat{S}_1$ may exist in the UV or IR regions.

We leave the proof of Theorem 4 in appendix B.

We close this section by pointing out flat Minkowskian vacua. In this case we have the condition

$$\partial_i W(z_0) = W(z_0) = 0,$$

which implies that the matrix (4.14) diverges. Therefore, these vacua do not correspond to the AdS/CFT correspondence. This case is excluded in the paper.

5 Generalization to Curved BPS Domain Walls

In this section we generalize the previous case to the curved (AdS sliced) domain walls. The structure of this section is as follows. Firstly, we discuss shortly some aspects of the curved domain walls on arbitrary dimensional Kähler-Ricci soliton. These have been studied for two dimensional case in [6]. Secondly, general properties of vacua will be discussed.

5.1 BPS Equations For Curved Domain Walls

Similar as in the flat case, we consider the curved domain walls by taking the ansatz metric of the four dimensional spacetime as

$$ds^2 = a^2(u, \tau) g_{\lambda\nu} dx^\lambda dx^\nu - du^2,$$

where $\lambda, \nu = 0, 1, 2$, $a(u, \tau)$ is again the warped factor, and $g_{\lambda\nu}$ is the metric on the three dimensional AdS spacetime. Therefore, the corresponding components of the Ricci tensor of the metric (5.1) are given by

$$R_{\lambda\nu} = \left[ \left( \frac{a'}{a} \right)' + 3 \left( \frac{a'}{a} \right)^2 - \frac{\Lambda_3}{a^2} \right] a^2 g_{\lambda\nu},$$

$$R_{33} = -3 \left[ \left( \frac{a'}{a} \right)' + \left( \frac{a'}{a} \right)^2 \right],$$

(5.2)
and the Ricci scalar has the form
\[ R = 6 \left( \left( \frac{a'}{a} \right)' + 2 \left( \frac{a'}{a} \right)^2 \right) - \frac{3\Lambda_3}{a^2}, \] (5.3)
where \( \Lambda_3 \) is the negative three dimensional cosmological constant.

A set of equations that describes curved domain walls with residual supersymmetry, can be derived by writing the supersymmetry transformation (3.3) on the background (5.1). Using similar way as in the flat case, it then results in [6]
\[
\begin{align*}
\frac{a'}{a} &= \pm \left| e^{K(\tau)/2M_P^2} W(z) - \ell/a \right|, \\
z^{i'} &= \mp 2e^{i\theta(\tau)} g^{ij}(\tau) \bar{\partial}_j W(\tau), \\
\bar{z}^{i'} &= \mp 2e^{-i\theta(\tau)} g^{ij}(\tau) \partial_j W(\tau),
\end{align*}
\] (5.4)
where the phase function \( \theta(z, \bar{z}; u, \tau) \) has been introduced with
\[
\begin{align*}
e^{i\theta(\tau)} &= \frac{\left( 1 - \ell e^{-K(\tau)/2M_P^2} (aW)^{-1} \right)}{\left( 1 - \ell e^{-K(\tau)/2M_P^2} (aW)^{-1} \right)},
\end{align*}
\] (5.5)
Note that at \( \theta = 0 \) the flat domain wall case is regained, which corresponds to \( \ell = 0 \). The second and the third equations in (5.4) are called the BPS equations for curved domain walls. Again, the renormalization group (RG) flow is given by the beta functions
\[
\begin{align*}
\beta_z^i(\tau) &\equiv a \frac{\partial z^i}{\partial a} = -\frac{2e^{i\theta(\tau)}}{e^{K(\tau)/2M_P^2} W(z) - \ell/a} g^{ij}(\tau) \bar{\partial}_j W(\tau), \\
\bar{\beta}_{\bar{z}}^i(\tau) &\equiv a \frac{\partial \bar{z}^i}{\partial a} = -\frac{2e^{-i\theta(\tau)}}{e^{K(\tau)/2M_P^2} W(z) - \ell/a} g^{ij}(\tau) \partial_j W(\tau),
\end{align*}
\] (5.6)
which give a description of a conformal field theory (CFT) on the three dimensional AdS spacetime. Therefore, the scalars \( (z^i, \bar{z}^i) \) and the warped factor \( a \) can be viewed as coupling constants and an energy scale, respectively [16, 18, 19].

5.2 Generalized Vacuum Structure

As mentioned in the preceding section, vacua of the \( N = 1 \) theory are in general defined by (4.8). Then, in order to maintain supersymmetry one has to add the condition (4.9) which can be viewed as the critical point of the BPS equations in (5.4). The scalar potential (4.6) evaluated at a vacuum \( p_0 \) has the form (4.10), however it is not the cosmological constant of the four dimensional spacetime.

In this case the warped factor at \( p_0 \) is given by
\[
a(u, \tau) = \frac{l}{W_0^2} \pm \left( \frac{l^2}{W_0^2} - \frac{l^2}{W_0^1} \right)^{1/2} \left[ A_0 e^{\pm W_0 u} - A_0^{-1} e^{\mp W_0 u} \right],
\] (5.7)
where \( l \equiv \ell e^{K(p_0; \tau)/2M_P^2} \Re W(z_0) \) and \( A_0 \neq 0 \). Since \( a \) is real, then \( W_0 > |l|/\ell \). Moreover, we have \( (a'/a)' \neq 0 \) near \( p_0 \). Inserting (5.7) into (5.2), one can then show that the
spacetime in general is non-Einstein whose components of the Ricci tensor are given by

\[
R_{\lambda\nu} = \left( \pm k' + 3k^2 + 2\ell^2 e^{\mp 2f k du} \right) e^{\pm 2f k du} g_{\lambda\nu},
\]

\[
R_{33} = -3 \left( \pm k' + k^2 \right),
\]

(5.8)

where

\[
k \equiv \left| e^{K(p_0;\tau)/2M_P^2} W(z_0) - \frac{\ell}{a} \right|.
\]

(5.9)

Note that if we take \( \ell \to 0 \) and \( A_0 \to \pm \infty \), then we obtain the flat wall case [5, 17]. For \( k = 0 \), one has

\[
\text{Im} W(z_0) = 0,
\]

(5.10)

and the spacetime becomes simply AdS_3 × R which corresponds to the singularity of the beta function (5.6). So, this vacuum is not related to the CFT on AdS_3.

Since the ground states are defined by (4.8), the next step would be to analyze the Hessian matrix of the scalar potential given by (4.12). These have been described by Theorem 3 for only nondegenerate cases and Theorem 4 for general cases. Thus, this level is the common step.

Next, similar as in the flat case, we also need to look at the RG flows described by the first order expansion of the beta function (5.6) at \( p_0 \), namely

\[
\mathcal{U}_c \equiv - \begin{pmatrix}
\partial_j \beta^i_c & \partial_j \bar{\beta}^i_c \\
\bar{\partial}_j \beta^i_c & \bar{\partial}_j \bar{\beta}^i_c
\end{pmatrix}(p_0; \tau),
\]

(5.11)

where

\[
\partial_j \beta^i_c(p_0; \tau) = -\frac{e^{i\theta_0(\tau)}}{k M_P^2} W_0 \delta^i_j,
\]

\[
\bar{\partial}_j \beta^i_c(p_0; \tau) = -\frac{2e^{i\theta_0(\tau)}}{k} g^{ik}(p_0; \tau) \bar{\partial}_j \bar{\partial}_k W_0,
\]

(5.12)

together with their complex conjugate with \( \theta_0(\tau) \equiv \theta(p_0; \tau) \). In general, the matrix (5.11) is not Hermitian and thus, has complex eigenvalues. In order to have a consistent theory, one has to diagonalize (5.11) and then, in this basis imposes Hermiticity condition on it. This will results a consistency condition for RG flows.

In two dimensional cases, it is much easier to achieve the consistency condition in which the eigenvalues of (5.11) are real. As studied in [6], such condition does exist, namely

\[
|\text{Tr} \mathcal{U}_c| \geq 2 |\text{Det} \mathcal{U}_c|.
\]

(5.13)

6 Vacuum Structure Near Orbifold Point

Before turning to the main discussion of this section, let us first mention two quantities which are useful for our analysis around the orbifold point, namely the origin. These are
the $U(1)$ connection and the Kähler potential related to the metric (2.30) of the theory given by

\[
Q(\xi, \bar{\xi}; \tau) = -\frac{2i}{M_P^2} \sigma(\tau)^{-\mu/\Lambda} B (\sigma(\tau)^{-\mu/\Lambda}|\xi|^2) \left[ \bar{\xi}^i d\xi^i - \xi^i d\bar{\xi}^i \right],
\]

\[
K(\xi, \bar{\xi}; \tau) = \sigma(\tau)^{-\mu/\Lambda} \int e^{v/2} B (\sigma(\tau)^{-\mu/\Lambda} e^{v/2}) dv + c,
\]

(6.1)

respectively where $v \equiv 2 \ln|\xi|^2$ and $c$ is a real constant. Here, the order $1 - \mu/\Lambda = p/q$ satisfies the conditions given in Theorem 1.

Firstly, we consider the case where the scalar potential (4.6) is Morse function. Let $p_0 \equiv (\xi_0, \bar{\xi}_0)$ be a ground state and exists around the origin. We assume that $p_0 \neq 0$ and it is not an orbifold point. Then, for the cases 1 and 2 of Theorem 1, it might be possible to have a situation described in Theorem 3 if the condition (4.18) is fulfilled. In the massless case, the case 1 can be easily seen, but it might be not possible for the case 2 if the sign of both $B (\sigma(\tau)^{-\mu/\Lambda}|\xi|^2)$ and $\dot{B} (\sigma(\tau)^{-\mu/\Lambda}|\xi|^2)$ could not be altered after $\tau = 1/2\Lambda$. For the case 3 of Theorem 1 and Lemma 2, the condition (4.18) does not exist since there is no Kähler geometry in $\tau > 1/2\Lambda$.

Secondly, if the scalar potential (4.6) is Morse-Bott function, then we might have the situations stated in Theorem 4 for the cases 1 and 2 of Theorem 1. However, only the situation 1 in Theorem 4 are possible for the case 3 of Theorem 1 and Lemma 2 because no Kähler geometry exists after the singularity at $\tau = 1/2\Lambda$.

In the following we simply take a simple case near $\tau \to +\infty$. So, it is valid only for the cases 1 and 2 of Theorem 1. Since $\xi \approx 0$, then the quantities in (6.1) are simplified to

\[
Q(\xi, \bar{\xi}; \tau) \approx -\frac{2i}{M_P^2} \sigma(\tau)^{p/q} B (0) \left[ \bar{\xi}^i d\xi^i - \xi^i d\bar{\xi}^i \right],
\]

\[
K(\xi, \bar{\xi}; \tau) \approx 2\sigma(\tau)^{p/q} B (0) |\xi|^2 + c.
\]

(6.2)

Moreover, for a shake of simplicity the form of the superpotential $W(\xi)$ is taken to be linear, namely

\[
W(\xi) = a_0 + a_i \xi^i,
\]

(6.3)

with $a_0, a_i \in \mathbb{R}$. For $a_i = 0$ and $a_0 \neq 0$, we find that the solution of (4.9) is the origin which is the orbifold point. This further implies that all spin-$\frac{1}{2}$ fermions are massless, but it is an ill defined $N = 1$ theory. On the other case, for finite $a_i > 0, a_0 > 0$, and $a_0 \gg a_i$, the equation (4.9) gives

\[
\xi_0^i(\tau) \approx -\frac{1}{2} M_P^2 \sigma(\tau)^{-p/q} \frac{a_i}{a_0},
\]

(6.4)

which can then be shown that all spin-$\frac{1}{2}$ fermions are massive. In this case we may have a well defined theory.

### 7 Conclusions

First of all, we particularly have studied the $U(n)$ Kähler-Ricci soliton that admits an orbifold-type singularity at the origin. This soliton may be considered as a one-parameter family of Kähler geometries. As mentioned in Theorem 1, near the orbifold point (or the origin) if there exists a rational number $p/q = 1 - \mu/\Lambda$ with $0 < p < q$, then there
are three possible situations. In the situations 1 and 2, we possibly have a family of Kähler geometries even after the singularity at $\tau = 1/2\Lambda$. For the situation 3, the Kähler geometry exists only in $\tau < 1/2\Lambda$. Such situation also occurs in Lemma 2.

Next, some aspects of the $N = 1$ supergravity domain walls on Kähler-Ricci soliton have been discussed. Firstly, we considered flat domain walls together with their vacuum structure. For the case at hand, it is natural that both the Hessian matrix (4.12) and the matrix (4.14) describing RG flows are Hermitian matrices and therefore, they have real eigenvalues. Thus, we have a consistent theory for the flat case. In addition, a ground state is characterized by the pair $(m, \lambda)$ where $m$ and $\lambda$ are the dimension and the Morse index of a ground state, respectively. As stated in Theorems 3 and 4, the pair $(m, \lambda)$ may in general be deformed with respect to the flow parameter of the Kähler-Ricci soliton. The results here is the same as in the previous results for Kähler-Einstein geometry [5].

Secondly, curved domain walls with their vacuum structure have also been discussed. In this case, the pair $(m, \lambda)$ also characterizes a vacuum and possibly has a deformation caused by the Kähler-Ricci soliton. However, the matrix (5.11) describing RG flows has in general complex eigenvalues. Then, we have to impose Hermiticity condition in the basis where the matrix (5.11) has the diagonal form.

Finally, we have considered some aspects of vacuum structure on the $U(n)$ Kähler-Ricci orbifold near the origin. The cases 1 and 2 of Theorem 1 have a possibility of having the situations described in Theorems 3 and 4. But in the case 3 of Theorem 1 and Lemma 2, the possible case is only the case 1 of Theorem 4. At the end, we simply considered a simple case when the flow parameter $\tau \rightarrow +\infty$ and the superpotential $W(\xi) = a_0 + a_i \xi^i$.

We find that the vacuum structure related to $a_0 \neq 0$ and $a_i = 0$ is an ill defined massless $N = 1$ theory at the origin (which is the orbifold point). On the other side, for finite and positive definite $a_0$, $a_i$ and $a_0 \gg a_i$ we may have a well defined theory which is valid only for the cases 1 and 2 of Theorem 1.

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A Proof of Theorem 1 and Lemma 2

Let us first look at the functions $B(|\xi|^2)$ and $\dot{B}(|\xi|^2)$ defined in (2.29). Since $\xi \approx 0$ those functions can be expanded as

\[
B(|\xi|^2) = B(0) + |\xi|^2 \dot{B}(0) + O(\xi, \bar{\xi}) , \\
\dot{B}(|\xi|^2) = \dot{B}(0) + |\xi|^2 \ddot{B}(0) + \ddot{O}(\xi, \bar{\xi}) .
\]  

(A.1)

Positivity of the metric (2.29) implies $\dot{B}(|\xi|^2) > B(|\xi|^2)$. Employing the diffeomorphism (2.18) we obtain

\[
B(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2) = B(0) + \sigma(\tau)^{-\mu/\Lambda}|\xi|^2 \dot{B}(0) + O(\xi, \bar{\xi}; \tau) , \\
\dot{B}(\sigma(\tau)^{-\mu/\Lambda}|\xi|^2) = \dot{B}(0) + \sigma(\tau)^{-\mu/\Lambda}|\xi|^2 \ddot{B}(0) + \ddot{O}(\xi, \bar{\xi}; \tau) .
\]  

(A.2)

Thus, we have a consistent theory if the above functions are real valued. Assuming that there exists a rational number $p/q = 1 - \mu/\Lambda$ and $0 < p < q$. For the case 3 there are no
Kähler manifolds or $\tau > 1/2\Lambda$ since the functions (A.2) become complex valued. In the case 1 the functions (A.2) are always positive definite for $\tau \geq 0$ and $\tau \neq 1/2\Lambda$ because the order $\mu/\Lambda$ has the form $2n/2m + 1$ for some positive integer $m, n$. In the case 2 for $\tau \to +\infty$ the functions (A.2) are positive definite since the leading terms are $B(0)$ and $\dot{B}(0)$. However, around $\tau = 1/2\Lambda$ the second and the higher order terms would be dominant and the sign of the functions (A.2) could be changed after $\tau = 1/2\Lambda$.

Using the same statement as the case 3 of Theorem 1, we have thus proved Lemma 2.

B Proof of Theorem 4

Our starting point is to consider the Hessian matrix of the scalar potential $H_V$ whose components are given in (4.12). In particular, $H_V$ is a $2n \times 2n$ matrix in real coordinates such that $z^i \equiv x^i + iz^{n+i}$. First of all, we want to mention that the proof of the case 2 is similar like in Theorem 3 which has been proved for Kähler-Einstein geometries. Generalization to any Kähler-Ricci soliton is straightforward. So, the rest step is only to prove that the cases 1 and 3 are possible. As we will see, the proof of the case 1 has a similar logical way as the case 3. Therefore, we just have to prove the case 1.

Suppose at $\tau = \tau_0 < 1/2\Lambda$, we have an $m$-dimensional submanifold $S$ of the index $\lambda = 2n - m$. This corresponds to the existence of $m$ numbers zero eigenvalues and of $2n - m$ numbers negative eigenvalues of $H_V$. Let us denote each eigenvalue by $\alpha_i$, and in this case there are some $\alpha_i < 0$, $i = 1, \ldots, 2n - m$.

i. Then, at $\tau = \tau_1$ with $\tau_0 < \tau_1 < 1/2\Lambda$ one of the eigenvalues vanishes, say $\alpha_{2n-m} = 0$. Thus, $S$ deforms to another $m+1$-dimensional submanifold $S_1$ of the index $\lambda_1 = 2n - m - 1$. This proves $m_1 \neq m$ and $\lambda_1 \neq \lambda$.

ii. Then, at $\tau = \tau_2$ with $\tau_1 < \tau_2 < 1/2\Lambda$ one of the eigenvalues becomes positive, say $\alpha_{2n-m} > 0$. So, $S$ deforms to another $m$-dimensional submanifold $S_1$ of the index $\lambda_1 = 2n - m - 1$. This proves $m_1 = m$ and $\lambda_1 \neq \lambda$.

iii. Comparing i. and ii., we have proved $m_1 \neq m$ and $\lambda_1 = \lambda$.

The case 3 can be straightforwardly proved using the above steps.

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