Lax monads, equipments and generalized multicategory theory

Dimitri Chikladze

Abstract

Generalized multicategories, also called $T$-monoids, are well known class of mathematical structures, which include diverse set of examples. In this paper we construct a generalization of the adjunction between strict monoidal categories and multicategories, where the latter are replaced by $T$-monoids. To do this we introduce lax monads in a 3-category, and establish their relationship with equipments, which are bicategory like structures appropriate for the generalized multicategory theory.

1 Introduction

The idea of generalized multicategories is well known in category theory. It allows abstract expression of several mathematical structures, including ordinary and enriched multicategories, topological spaces and different notions of space with a metric structure. One starts with a category $X$ and a bicategory $A$ of arrows of which one may think of as relations between objects of $X$, and a monad like structure $T$. One uses Kleisli construction which produces a “bicategory like” structure $A_T$. A generalized multicategory, also called a $T$-monoid, is defined as a monad in $A_T$. There is also a certain interaction between $X$ and $A_T$ which allows to consider maps between generalized multicategories. Indeed, it is more appropriate to say that the Kleisli construction produces a “data” $(X, A_T)$ from the data $((X, A), T)$, and the category of $T$-monoids is defined as the category of monads in $(X, A_T)$ in a certain sense. In the case leading to the ordinary multicategories $(X, A) = (\text{Set, Span})$, and $T$ is a free-monoid monad $(T, m, e)$ on Set together with its extension to Span. An ordinary multicategory $(x, a)$ has a set of object $x$, and a set of multimorphisms $a$, which is a span $x \to T x$.

In ordinary and enriched multicategory theory of fundamental importance is an adjunction which exists between multicategories and monoidal categories (see [4]). Let us write a strict monoidal category as $((x, b), h)$, where $x$ is its set of objects, $b : x \to x$ is the span with $b$ the set of morphisms, and $h : T x \to x$ is a set map which gives the monoidal product on objects. The functor from the multicategories to the monoidal categories on objects is given by:

$$(x, a) \mapsto ((x, m_x T a), m_x),$$
Its right adjoint is given by:

\(( (x, b), h ) \mapsto (x, a(h^r)) \) 

where \( h^r : X \rightarrow Tx \) is a right adjoint of \( h \) in \( \text{Span} \). An abstract analogue of a strict monoidal category is a \( T \)-algebra. The motivation of this paper is to construct an adjunction between \( T \)-monoids and \( T \)-algebras in a way which emphasizes the monad (or monoid) nature of \( T \)-monoids.

When one considers generalized multicategories from the abstract point of view, first one has to specify what is the data \((X, A)\) one works with. It should be general enough to allow a definition of Kleisli construction. An obvious choice is to work with pseudofunctors \( X \rightarrow A \) from a category \( X \) to a bicategory \( A \). But in general, \( A_T \) is not a bicategory, and there is no appropriate pseudofunctor from \( X \) to it. The only work in the literature where an abstract generalized multicategory theory is developed should be [3]. There \((X, A)\) are certain kind of double categories.

The data \((X, A)\) which we will work with in this paper will be called an equipment. It consists of a category \( X \), and for any pair of objects \( x, y \) of it, a category \( A(x, y) \), objects and morphisms of which can be composed over objects of \( X \). This composition is lax associative and lax unitive. We formalize our equipments as lax monads in a tricategory of 2-sided indexed categories. Lax functors between equipments and lax transformation between these, are defined as certain kind of lax maps between lax monads and lax transformations between theses.

Various concepts and constructions of multicategory theory can be expressed in terms of lax monads in a tricategory. The Kleisli equipment \((X, A_T)\) is defined as a composite monad of the lax monad \( A \) with the monad \( T \). A monad in an equipment \((X, A)\) can be defined as a lax functor between equipments:

\[(I, I^*) \rightarrow (X, A)\]

Where \((I, I^*)\) is a terminal equipment. Hence a \( T \)-monoid is a lax functor:

\[(I, I^*) \rightarrow (X, A_T)\]

\( T \)-algebras in general are not monads anywhere. However they can be expressed as morphisms:

\[((I, I^*), 1_{(I, I^*)}) \rightarrow ((X, A), T)\]

in the 2-category \( \text{Mnd}(\mathcal{E}) \) of monads (in the sense of [5]) in the 2-category of equipments \( \mathcal{E} \). As we have written above the adjunction between multicategories and monoidal categories depends on the fact that every set map \( h \) has a right adjoint \( h^r \) in \( \text{Span} \). For an abstract equipment \((X, A)\), this fact corresponds to the existence of restrictions, which can be expressed by existence of right adjoints of certain 2-morphism in the tricategory of 2-sided indexed categories. The adjunction between \( T \)-monoids and \( T \)-algebras is obtained from an adjunction constructed within the framework of lax monads, by a procedure involving manipulation with adjoint 2-cells in a 3-category.

The organization of the paper is simple. In the first few sections we consider lax monads in a 3-category and constructions on them. These are then applied to equipments and multicategory theory in the rest of the paper. It should be admitted that the writing is condensed, with details hidden, and many computations omitted.
2 Lax monads in a 3-category

Let $\mathcal{B}$ be a 3-category. For the horizontal composition in $\mathcal{B}$ we will use the dot symbol. The vertical composition of 2-cells will be represented by juxtaposition. A 2-cell of $\mathcal{B}$ of the form $T.A \to S.B$ we will sometimes call a square, envisioning $A$ and $B$ as its horizontal edges and $T$ and $S$ as its vertical edges. Such squares can be pasted horizontally and vertically. Horizontal pastings of squares will be written using the dot symbol, and the vertical pastings of squares will be written using juxtaposition.

A (normal) lax monad $(X, A)$ in $\mathcal{B}$ consists of an object $X$, a 1-cell $A : X \to X$, for every $n > 0$, a 2-cell $\Pi_n : A^n \to A$, with $\Pi_1$ an identity, and for every partition $m = n_1 + \cdots + n_j$, a 3-cell

$$\xi_{n_1,\ldots,n_j} : \Pi_m \to \Pi_j(\Pi_{n_1},\ldots,\Pi_{n_j}) : A^{n_1}\cdots A^{n_j} \to A,$$

(1)
called associativity comparison maps, satisfying the coherence condition:

$$\Pi_l \downarrow \downarrow \Pi_k((\Pi_j(\Pi_{n_1},\cdots,\Pi_{n_{11}})),\cdots,(\Pi_{j_k}(\Pi_{n_{k_1}},\cdots,\Pi_{n_{k_{j_k}}}))$$

$$\Pi_h(\Pi_{n_{11}},\cdots,\Pi_{n_{k_{j_k}}}) \xrightarrow{\kappa_l} \Pi_k(\Pi_{m_1},\cdots,\Pi_{m_j})$$

$$\Pi_k(\Pi_{j_1},\cdots,\Pi_{j_k})(\Pi_{n_{11}},\cdots,\Pi_{n_{k_{j_k}}})$$

(2)

for $l = m_1 + \cdots + m_k$, $m_i = n_{i_1} + \cdots + n_{i_{j_i}}$, $h = j_1 + \cdots + j_{k}$, as well as $\xi_{1,1,\ldots,1}$ and $\xi_n$ required to be identities. A lax monad is a pseudomonad or a strong monad when the associativity comparison maps are isomorphisms. It is a strict monad when these are equalities.

A lax map of lax monads $(F, \Phi) : (X, A) \to (Y, B)$ consists of a 1-cell $F : X \to Y$, a square $\Phi : B.F \to F.A$, and for every $n \geq 0$, a 3-cell

$$\kappa_n : (F.\Pi_n)\Phi^n \to \Phi(\Pi_n.F),$$

(3)
called lax comparison maps, which for every $m = n_1 + \cdots + n_i$ satisfy the axiom:
(\(\kappa_m\)) \(\Phi(\Pi_{m}, F)\)

(\(\kappa_{n_1 \ldots n_j}\)) \(\Phi((\Pi_j(\Pi_{n_1} \cdots \Pi_{n_j})).F)\)

(\(\kappa_n\)) \(\Phi(\Pi_{n}.F)\)

(\(\kappa_1\)) \(\kappa_{n_1} \cdots \kappa_{n_j}\)

(\(\kappa_{n_1} \cdots \kappa_{n_j}\)) \(\Phi(\Pi_{n}.F)\)

(\(\kappa_{n_1} \cdots \kappa_{n_j}\)) \(\Phi(\Pi_{n}.F)\)

\(\kappa_1\) is required to be an identity. Note that, if \((X, B)\) is a strong monad, then \(\kappa_n\)-s are determined only by \(\kappa_0\) and \(\kappa_2\). A colax map between lax monads is defined similarly except that its lax comparison maps \(\kappa_n\)-s have the opposite direction and satisfy the axiom obtained from (4) by reversing the arrows involving \(\kappa_n\)-s.

A lax transformation of lax maps of lax monads \(\Omega : (F, \Phi) \rightarrow (G, \Psi) : (X, A) \rightarrow (Y, B)\) consists of a 2-cell \(\Omega : F \rightarrow G\), and a 3-cell

\[\alpha : (\Omega. A)\Phi \rightarrow \Psi(B. \Omega)\]

satisfying the axiom:

\[(\Omega. A)(\Omega. A)\Phi \rightarrow (\Omega. A)\Phi(\Pi_{n}. F) \rightarrow \alpha(\Pi_{n}. F) \rightarrow (G. \Psi)(\Pi_{n}. F)\]

\[(G. \Psi)(\Pi_{n}. F) \rightarrow (G. \Psi)(\Pi_{n}. F) \rightarrow \alpha(\Pi_{n}. \Phi) \rightarrow (G. \Psi)(\Pi_{n}. F)\]

A colax transformation of lax maps is defined in the same way, except that the 3-cell \(\alpha\) takes the opposite direction, and it is required to satisfy an axiom obtained from (5) by reversing the arrows involving \(\alpha\)-s. By leaving the direction of \(\alpha\) unchanged, but reversing the directions of \(\kappa_n\)-s we obtain a definition of lax transformation of colax maps. Changing both directions of \(\alpha\) and \(\kappa_n\)-s we get a definition of colax transformation of colax maps.

A modification between lax transformations \(\lambda : \Omega \rightarrow \Sigma : (F, \Phi) \rightarrow (G, \Psi) : (X, A) \rightarrow (Y, B)\) is a 3-cell \(\lambda : \Omega \rightarrow \Sigma\) satisfying the equation

\[(\lambda. \Phi)\Phi \rightarrow \Psi(B. \Sigma)\]

Modifications between all other types of (co)lax transformations are defined similarly.
There are 3-categories:

\[ \text{lMnd}^{1/l}(\mathcal{B}), \quad \text{lMnd}^{1/\text{col}}(\mathcal{B}), \quad \text{lMnd}^{\text{col}/l}(\mathcal{B}), \quad \text{lMnd}^{\text{col}/\text{col}}(\mathcal{B}). \]

All of these have as their objects the lax monads in \( \mathcal{B} \). Morphism of the first two are lax maps. Morphism of the last two are colax maps. 2-morphisms for the first and the third are lax transformations. 2-morphisms of the second and the fourth are colax transformations. 3-cells for all of them are modifications. Compositions are defined straightforwardly.

Let \((-)^{\text{op}}\) stand for the operation on 3-categories which inverts 1-morphisms, but leaves 2- and 3-cells unchanged. Define:

\[ \text{lMnd}^{\text{op}/l}(\mathcal{B}) = (\text{lMnd}^{1/l}(\mathcal{B}^{\text{op}}))^{\text{op}}. \]

\( \text{lMnd}^{\text{op}/l}(\mathcal{B}) \) is a 3-category whose objects again are the lax monads. We define a lax opmap (relative to \( \mathcal{B} \)) between lax monads to be a morphism of this 3-category. Essentially, a lax opmap \((F, \Phi) : (X, A) \rightarrow (Y, B)\) consists of a 1-cell \( F : X \rightarrow Y \), a square \( \Phi : F.A \rightarrow B.F \) and 3-cells

\[ \kappa_n : (\Pi_n.F)^{\Phi^n}_n \rightarrow \Phi(F.\Pi_n), \quad (7) \]

for every \( n \geq 0 \), satisfying the axiom obtained from (1) by flipping sides of \( F \)-s and \( \Pi \)-s. 2-morphisms of \( \text{lMnd}^{\text{op}/l}(\mathcal{B}) \) by definition are lax transformations of lax opmaps. In the same way, we define 3-categories

\[ \text{lMnd}^{\text{op}/\text{col}}(\mathcal{B}), \quad \text{lMnd}^{\text{col}/\text{op}}(\mathcal{B}), \quad \text{lMnd}^{\text{col}/\text{col}}(\mathcal{B}), \]

as well as a notion of colax opmap, and various notions of (co)lax transformations between (co)lax monad opmaps. 3-cells of all these 3-categories will be called again modifications.

### 3 Distributive laws of lax monads

By a distributive law of lax monads \( ((X, A), T) \) we will mean a lax monad \( ((X, A), (T, \Theta)) \) in \( \text{lMnd}^{1/l}(\mathcal{B}) \) or equivalently a lax monad \( ((X, T), (A, \Theta)) \) in \( \text{lMnd}^{\text{op}/l}(\mathcal{B}) \). This amounts to two lax monads \( (X, A) \) and \( (X, T) \) together with a 2-cell \( \Theta : A.T \rightarrow T.A \) and 3-cells

\[ \kappa_n : (T.\Pi_n)^{\Theta^n}_n \rightarrow \Theta(\Pi_n.T), \quad (8) \]

\[ \alpha_n : (\Pi_n.A)^{(\Theta^n)}_n \rightarrow \Theta(A.\Pi_n) \quad (9) \]

satisfying a set of axioms.

Given a distributive law as above define a lax monad \( \text{Cmp}((X, A), T) = (X, T.A) \), with its \( n \)-ary multiplication \( (T.A)^n \rightarrow T.A \) defined as the composite:

\[ (T.A)^n \xrightarrow{T.\Theta^{(n-1)}_n.A} T^2(A.T)^{(n-1)}_n.A^2 \xrightarrow{T.2.\Theta^{(n-1)}_n.A^2} \cdots \xrightarrow{T^n.A^n} \Pi_n.\Pi_n \rightarrow T.A, \]

5
and its associativity comparison maps built using $\kappa_n$-s, $\alpha_n$-s and the associativity comparison maps of $(X, A)$ and $(X, T)$ in the straightforward way. This construction extends to a 3-functor:

$$\text{Cmp} : \text{lMnd}^{l/}(\text{lMnd}^{l/}(B)) \to \text{lMnd}^{l/}(B),$$

by setting

$$\text{Cmp}((F, \Phi, \Psi)) = (F, \Psi.\Phi)$$

$$\text{Cmp}(\Omega) = \Omega,$$

with lax comparison maps for $(F, \Psi.\Phi)$ and the lax transformation structure for $\Omega$ defined in the straightforward way.

4 **Cmp** construction

Now suppose that every 2-cell $\Phi$ of $B$ has a right adjoint $\Phi^r$ in the hom 2-category in which it resides.

Let $(F, \Phi) : (X, A) \to (Y, B)$ be a colax opmap of monads. Then $(F, \Phi^r)$ becomes a lax map of monads $(X, A) \to (Y, B)$, with the lax comparison maps for it

$$(F.\Pi_n)(\Phi^r).n \to \Phi^r(\Pi_n.F),$$

defined from the lax comparison maps of $\Phi$

$$\Phi(F.\Pi_n) \to (\Pi_n.F)\Phi.n$$

by transposing the adjoint 2-cells. Given a colax transformation of colax opmaps $\Omega : (F, \Phi) \to (G, \Psi) : (X, A) \to (Y, B)$ we get a lax transformation $\Omega : (F, \Phi^r) \to (G, \Psi^r) : (X, A) \to (Y, B)$ with the lax structure

$$(\Omega.A)\Phi^r \to \Psi^r(B.\Omega)$$

obtained from the colax structure of $\Omega$

$$\Psi(\Omega.A) \to (B.\Omega)\Phi$$

by transposing the adjoint 2-cells. These constructions define a pseudo 3-functor:

$$\text{Tr} : \text{lMnd}^{\text{colop}/\text{col}}(B) \to \text{lMnd}^{l/}(B).$$

Define a pseudo 3-functor

$$\text{Cmp}^* : \text{lMnd}^{\text{colop}/\text{col}}(\text{lMnd}^{l/}(B)) \xrightarrow{\text{Cmp}} \text{lMnd}^{l/}.$$
On objects $\text{Cmp}^*$ agrees with $\text{Cmp}$. On morphisms and 2-morphisms

$$\text{Cmp}^*((F, \Phi), \Psi) = (F, \Psi^r \Phi)$$

$$\text{Cmp}^*(\Omega) = \Omega.$$

5 The adjunction proposition

Further we will restrict ourselves to those distributive laws $((X, A), T)$ which are strict monads in $\text{Mnd}^{l/l}(B)$. This implies that, the lax monad $(X, T)$ has equalities as its associativity comparison 3-cells, while its multiplications $\Pi_n$ are determined only by $\Pi_0$ and $\Pi_2$. Furthermore, all the 3-cells $\alpha_n$ are determined by $\alpha_0$ and $\alpha_2$.

Let $\text{Mnd}(\text{Mnd}^{l/l}(B))$ denote the 2-category whose object are strict monads in $\text{Mnd}^{l/l}(B)$, the morphisms are strict monad maps, and the 2-morphisms are the strict transformations between these. Let $\text{Mnd}^{op}(\text{Mnd}^{l/l}(B))$ denote the 2-category whose object are strict monads in $\text{Mnd}^{l/l}(B)$, the morphisms are strict monad opmaps, and the 2-morphisms are strict transformations between monad opmaps. Since $\text{Mnd}(\text{Mnd}^{l/l}(B))$ is a sub 2-category of $\text{Mnd}^{l/l}(\text{Mnd}^{l/l}(B))$, and $\text{Mnd}^{op}(\text{Mnd}^{l/l}(B))$ is a sub 2-category of $\text{Mnd}^{colop/col}(\text{Mnd}^{l/l}(B))$ we have pseudo 3-functors:

$$\text{Cmp} : \text{Mnd}(\text{Mnd}^{l/l}(B)) \to \text{Mnd}^{l/l}(B)$$

$$\text{Cmp}^* : \text{Mnd}^{op}(\text{Mnd}^{l/l}(B)) \to \text{Mnd}^{l/l}(B).$$

Let

$$\text{Inc} : \text{Mnd}^{l/l}(B) \to \text{Mnd}^{op}(\text{Mnd}^{l/l}(B))$$

be the canonical inclusion 2-functor which sends a lax monad $(X, A)$ to $((X, A), 1_X) = ((X, A), (1_X, 1_A))$.

For an object $((X, A), T)$ of $\text{Mnd}^{op}(\text{Mnd}^{l/l}(B))$ and an object $(Y, B)$ of $\text{Mnd}^{l/l}(B)$, define a 2-functor:

$$R : \text{Mnd}^{op}(\text{Mnd}^{l/l}(B))(((X, A), T), \text{Inc}(Y, B)) \to \text{Mnd}^{l/l}(B)(\text{Cmp}^*((X, A), T), (Y, B)).$$

by

$$\varepsilon_{(Y,B)}(\text{Cmp}^*(-)),$$

where $\varepsilon$ stands for the equality $\text{Cmp}^*(\text{Inc}(Y, B)) = (Y, B)$.

Suppose that the distributive law $((X, A), T)$ is such that the 3-cell $\alpha_2$ is invertible. The squares $\Pi_2 : T.T \to T.1_X$ and $\Theta : A.T \to T.A$ together with the 3-cell

$$(\alpha_2)^{-1} : \Theta(A.\Pi_2) \to (\Pi_2.A)(T.\Theta)(\Theta.T)$$

determine a morphism in $\text{Mnd}(\text{Mnd}^{l/l}(B))$:

$$((T, \Theta), \Pi_2) : ((X, A), 1_X) \to ((X, A), T).$$
Applying the Cmp to this we get a morphism in $\text{Mnd}^{1/1}(\mathcal{B})$:

$$(T, \Pi_2 \Phi) : (X, A) \to (X, T.A) = \text{Cmp}(X, A).$$

Further, the squares $\Pi_2 : T.T \to 1_X.T$ and $\Pi_2.\Theta : (T.A).T \to T.(T.A)$ together with the 3-cell

$$(\Pi_2.A)(T.\Theta)(\Pi_2.A.T)(T.\Theta.T) = (\Pi_2.A)(T.\Pi_2.A)(T.\Theta.T)(T.\Theta.T)\xrightarrow{\Pi_2.A(T.A_2)} (\Pi_2.A)\Theta(A.\Pi_2)$$

determine a morphism in $\text{Mnd}^{\text{op}}(\text{Mnd}^{1/1}(\mathcal{B}))$:

$$(T, \Pi_2.\Theta), \Pi_2 : ((X, A), T) \to ((X, T.A), 1_X) = \text{Inc}(\text{Cmp}((X, A), T)).$$

We denote this morphism by $\mathbf{n}_{(X, A, T)}$. Define a functor

$$L : \text{Mnd}^{1/1}(\mathcal{B})(\text{Cmp}^*(X, A), (Y, B)) \to \text{Mnd}^{\text{op}}(\text{Mnd}^{1/1}(\mathcal{B}))(\text{Mnd}^{1/1}(\mathcal{B}))/\text{Inc}(Y, B))$$

by

$$\text{Inc}(-)\mathbf{n}_{(X, A, T)}.$$

Before stating the adjunction proposition let us add the following data to the family $\mathbf{n}$. For a morphism $((F, \Phi), \Psi) : ((X, A), T) \to ((Y, B), S)$ of $\text{Mnd}^{\text{op}}(\text{Mnd}^{1/1})$ define a 2-morphism

\[
\begin{array}{ccc}
((X, A), T) & \xrightarrow{((F, \Phi), \Psi)} & (Y, B), S) \\
\text{Inc}(\text{Cmp}^*((X, A), T)) & \xrightarrow{\mathbf{n}_{(F, \Phi), \Psi}} & \text{Inc}(\text{Cmp}^*((Y, B), S))
\end{array}
\]

in $\text{Mnd}^{\text{op}}(\text{Mnd}^{1/1}(\mathcal{B}))$ to be given by the 2-morphism of $\text{Mnd}^{1/1}(\mathcal{B})$

$$(S, \Pi_2.\Theta)(F, \Phi) \to (F, \Psi^r.\Phi)(T, \Pi_2.\Theta) : (X, A) \to (Y, S.B)$$

which consists of a 2-cell $\Psi : S.F \to F.T$, and a 3-cell

$$((\Psi.A)(F.\Pi_2.A)(F.T.\Theta)(\Psi^r.A.T)(S.\Phi.T)) \to ((\Psi(F.\Pi_2)(\Psi^r.T)).A)(S.\Phi.\Theta)(S.T)$$

Here $\chi : \Psi(F.\Pi_2)(\Psi^r.T) \to (\Pi_2.F)(S.\Phi)$ is a 3-cell obtained from the equality $\Psi(F.\Pi_2) = (\Pi_2.F)(S.\Phi)(\Psi.T)$ by transposing adjoints. While, $\phi : (\Psi.A)(F.\Theta)(\Phi.T) \to (F.\Theta)(\Phi.T)(B.\Psi)$ is the part of the structure of $((F, \Phi), \Psi)$. 
Proposition 1. The functor $L$ is a left adjoint to the functor $R$.

Proof. The counit of the adjunction is defined by the family $n_{((F,\Phi),\Psi)}$. To construct the unit, define a 2-morphism in $\text{Mnd}^{1/1}(\mathcal{B})$

$$ (1_X, 1_{T.A}) \Rightarrow R(n_{(X,A,T)}) = (T, \Pi_2^r, \Pi_2.\Theta) $$

(10)

to consist of a 2-cell $\Pi_0 : 1_X \to T$ and a 3-cell:

$$ \Pi_0.T.A \xrightarrow{\zeta.A} (\Pi_2^r, (\Pi_2.A)(\Pi_0.T.A)) \xrightarrow{(\Pi_2.A)(\Pi_2.A)(T.\alpha_0)} (\Pi_2^r, (\Pi_2.A)(T.\Theta)(T.A.\Pi_0)). $$

Here the 3-cell $\zeta : \Pi_0.T \Rightarrow \Pi_2^r, \Pi_2(T.\Pi_0)$ is obtained from the equality $\Pi_2^r(\Pi_0.T) = \Pi_2^r(T.\Pi_0)$ by transposing adjoint 2-cells. The unit of the adjunction is defined from this morphism. The triangle identities are then verified straightforwardly.

In fact what we have here is that the 2-functor $\text{Cmp}^*$ is a colax 2-adjoint to the canonical inclusion $\text{Inc}$. A colax adjunction between 2-categories has colax natural transformations for its unit and counit, while the triangle identities are replaced by appropriately directed 2-cells. In our situation the counit $(\text{Cmp}^*)(\text{Inc}) \Rightarrow 1$ is an equality. The unit $1 \Rightarrow (\text{Inc})(\text{Cmp}^*)$ is given by the family $n$. One of the triangle 2-cells is an equality, and the other one is given by (10).

6 Equipments

Let $X$ and $Y$ be categories. By a two-sided indexed category we will mean a pseudofunctor $A : X^{\text{op}} \times Y \to \text{Cat}$ and write it as $A : X \to Y$. Pseudonatural transformations between such pseudofunctors will be called indexed functors, and modifications between these will be call indexed natural transformations. Two sided indexed categories are essentially the same as double fibrations internal to $\text{Cat}$ in the sense of [5].

Let $\mathcal{M}$ denote the tricategory whose objects are categories, and for any pair of categories $X$ and $Y$ the homcategory $\mathcal{M}(X, Y)$ is the 2-category whose objects are the two-sided indexed categories $X \to Y$, the morphisms are indexed functors and the 2-morphisms are indexed natural transformations. The horizontal composition is defined using pseudo coends in $\text{Cat}$. Alternatively, one can pass to double fibrations internal to $\text{Cat}$, and then the horizontal composition is defined as the operation constructed in [5]. The identities are given by the two-sided indexed categories $X(\cdot, \cdot)$, which will be further denoted by $X^*$. Of course $\mathcal{M}$ is not a strict 3-category. This means that we can not directly apply constructions of the previous sections. However, all those constructions can be mimicked, and that is what we rely on further in the paper.
By an equipment $(X, A)$ we mean a lax monad in $B$. So, an equipment has a category $X$ and a two-sided indexed category $A : X^{op} \times X \to \text{Cat}$. Objects of $X$ will be called objects of the equipment. For a pair of objects $x, y$, the objects of $A(x, y)$ will be called morphisms of the equipment, and will be written as $a : x \to y$. The maps of $A(x, y)$ will be called 2-cells of the equipment and will be written as $a : x \to y$. The equipment also has functors:

$$X(z, x) \times X(y, w) \times A(x, y) \to A(z, y)$$

We view these as left-right actions, and for their value at $(f, g, a)$ we write $gaf$.

For each $n \geq 2$, the monad multiplication $\Pi_n : A^n \to A$ is determined by functors

$$A(x_1, x_2) \times A(x_2, x_3) \times \cdots \times A(x_n, x_{n+1}) \to A(x_1, x_{n+1}).$$

We call these $n$-ary compositions of the equipment, and for their value at $(a_1, a_2, \ldots, a_n)$ we write $a_1a_2\cdots a_n$. $\Pi_0 : X^* \to A$ is determined by functors

$$X(x, x) \to A(x, x).$$

These give objects $u_x$ in each $A(x, x)$, which we call units of the equipment. The associativity comparison maps $\Pi$ amount to coherent 2-cells

$$(a_{11} \cdots a_{jn}) \to (a_{11} \cdots a_{1n_1}) \cdots (a_{j1} \cdots a_{jn_j}).$$

Note that here strings of zero lengths can also appear, with nullary composites being the units $u_x$.

There is a locally faithful pseudofunctor:

$$\text{Cat}^{op} \to \mathcal{M}$$

which is identity on objects, and takes the functor $F : X \to Y$ to $F^* = Y(-, F-)$.

A lax functor between equipments $(F, \Phi) : (Y, B) \to (X, A)$ is defined to be a lax map of monads of the form $(F^*, \Phi) : (X, A) \to (Y, B)$ (note the reversal of the direction). So, $F$ is a functor $Y \to X$, and $\Phi$ is an indexed functor $F^*B \to A.F^*$. The latter amounts to giving functors

$$B(x, y) \to A(F(x), F(y)).$$

For the value of these at $a$ we write $F(a)$. The lax comparison maps give the 2-cells:

$$\kappa_{a_1, \ldots, a_n} : F(a_1) \cdots F(a_n) \to F(a_1 \cdots a_n).$$

compatible with the compositions. Note that if the equipment $(Y, B)$ is a strong monad in $\mathcal{M}$ then these 2-cells are determined by those with $n = 2$ or $n = 0$.

A lax transformation between lax functors $t : (F, \Phi) \to (G, \Psi) : (Y, B) \to (X, A)$ is defined to be a lax transformation of lax monad maps $(F^*, \Phi) \to (G^*, \Psi)$ of the form $t^*$. So, $t$ itself is a natural transformation $F \to G$. The lax transformation structure on $t^*$ amounts to 2-cells

$$t_y F(a) \to G(b)t_x$$
compatible with the compositions and the $\kappa$-s.

There is a 2-category $\mathcal{E}$ of equipments, lax functors and their lax transformations. It is a sub 2-category of $(\text{LMnd}^{1/1}(B))^{\text{op}}$.

Now consider a monad $T$ on an equipment $(X, A)$ in $\mathcal{E}$ (we can speak only of strict monads since $\mathcal{E}$ is only a 2-category). This means that we have a monad $(T, e, m)$ on the category $X$, we have functors

$$T : A(x, y) \to A(T(x), T(y)),$$

which come with the comparison 2-cells

$$\kappa_{a_1, \ldots, a_n} : T(a_1) \cdots T(a_n) \to T(a_1 \cdots a_n),$$

and we have 2-cells

$$\alpha_0 : e_x a \to T(a)e_x,$$

$$\alpha_2 : m_x T^2(a) \to T(a)m_x,$$

the data satisfying a set of axioms.

Define a Kleisli equipment $(X, A_T)$ of the monad $T$ by

$$(X, A_T) = \text{Cmp}((X, A), T^*).$$

We identify it as follows. For objects $x$ and $y$

$$A_T(x, y) = A(x, Ty).$$

the left and right actions on $a : x \to Ty$ by morphisms of $X$ are given by formulas:

$$T(f)a,$$

$$ag.$$

The units and the compositions are given by

$$e_x,$$

$$m_{x_1} T^{(n-1)}(a_1) \cdots T(a_{n-1})a_n.$$  

**Examples.** Any pseudofunctor $(-) : X \to A$ from a category $X$ to a bicategory $A$ defines an equipment: $A$ defines a 2-sided indexed category by

$$X^{\text{op}} \times X \xrightarrow{(-)^{\text{op}} \times (-)} A^{\text{op}} \times A \xrightarrow{\text{Hom}} \text{Cat}.$$  

The bicategory composition and identities of $A$ give the lax monad structure on it. Indeed $(X, A)$ is a strong monad in $\mathcal{M}$. Our examples of interest belong to this situation:
Let $C$ be a category with finite limits. There are two canonical pseudofunctors $C \to \text{Span}(C)$ and $C \to \text{Span}(C)^{\text{op}}$, correspondingly we have equipments $(C, \text{Span}(C))$ and $(C, \text{Span}(C)^{\text{op}})$. Any monad $T$ on the category $C$ can be extended to a monad on these equipments.

Let $V$ be a closed monoidal category. There are two canonical pseudofunctors $\text{Set} \to \text{Mat}(V)$, and $\text{Set} \to \text{Mat}(V)^{\text{op}}$. So, there are equipments $(\text{Set}, \text{Mat}(V))$ and $(\text{Set}, \text{Mat}(V)^{\text{op}})$. A monad on either of these equipments with insignificant differences is a set monad $T$ together with a lax extension to $\text{Mat}(V)$ of $[2]$.

The equipment $(\text{Set}, \text{Span})$ is a special case of both of these situations.

Set theoretic issues deserve a comment here. We defined an equipment as a pseudofunctor $X^{\text{op}} \times X \to \text{Cat}$. Here $X$ is a category internal to some universe $\mathcal{U}$, and $\text{Cat}$ is a category of all categories internal to $\mathcal{U}$. To work with examples such as $(C, \text{Span}(C))$ and $(\text{Set}, \text{Mat}(V)^{\text{op}})$ we needs to chose a universe $\mathcal{U}$ which contains $C$ and $\text{Set}$ respectively.

7 $T$-monoids and $T$-algebras

Let $I$ denote the terminal category. There is a terminal equipment $(I, I^*)$. Note that $(I, I^*)$ is a strong monad in $\mathcal{M}$.

Define the category of monoids in an equipment $(X, A)$ by

$$\text{Mon}(X, A) = \mathcal{E}((I, I^*), (X, A)).$$

Objects of this category are called monoids in $(X, A)$. Its morphisms are called maps of monoids. A monoid amounts to an object $x$ of $X$, an arrow $a : x \to x$ and 2-cells $aa \to a$ and $1_x \to a$ satisfying three axioms. A map between monoids $(x, a) \to (y, b)$ amounts to an arrow $f : x \to y$ and a 2-cell $fa \to bf$ satisfying two axioms.

Given a monad $T$ on an equipment $(X, A)$, a $T$-monoid in $(X, A)$ is a monoid in $(X, A_T)$. The category of $T$-monoids is:

$$T\text{-Mon}(X, A) = \mathcal{E}((I, I^*), (X, A_T)).$$

A $T$-monoid amounts to an object $x$ of $X$, an arrow $a : x \to T(a)$ and 2-cells $m_x T(a) a \to a$ and $e_x \to a$ satisfying three axioms. A map of $T$-monoids amounts to an arrow $f : x \to y$ and a 2-cell $T(f)a \to bf$ satisfying two axioms.

The monad $T$ induces a monad on the category $\text{Mon}(X, A)$. A $T$-algebra is a strict algebra for this monad. Essentially, a $T$-algebra is a strict monad map $((I, I^*), 1_I) \to ((X, A), T)$. The category of $T$-algebras is:

$$T\text{-Alg}(X, A) = \text{Mnd}(\mathcal{E})((I, I^*), 1_I), ((X, A), T)).$$

A more concrete description is as follows. The induced monad on $\text{Mon}(X, A)$ takes a monoid $(x, a)$ into a monoid $(T(x), T(a))$. A $T$-algebra amounts to monoid $(x, a)$, a morphism $h : T(x) \to x$ in $X$ and a 2-cell $\sigma : hT(a) \to ah$ satisfying few axioms. Particularly, $(x, h)$ becomes an algebra for the monad $T$ on the category $X$, and $(h, \sigma)$ becomes a monoid map $(T(x), T(a)) \to (x, a)$. 

12
Examples.

A $T$-monoid in $(C,\operatorname{Span}(C))$ is a $T$-multicategory of common categorical knowledge.

A $T$ monoid in $(\text{Set},\operatorname{Mat}(V)^{op})$ is essentially a $(T,V)$-category of $[2]$.

$T$ monoids in $(\text{Set},\operatorname{Span})$, where $T$ is the free-monoid monad are ordinary multicategories. $T$-algebras in this case are monoidal categories.

8 Restrictions

We say that a two-sided indexed category $A : X \rightarrow Y$ has restriction if for any natural transformation $t : H \rightarrow K : Z \rightarrow X$, $t^*A$ has a right adjoint $(t^*A)^r$ in $A(Z,Y)$.

Suppose that $A,B : X \rightarrow Y$ have restrictions. An indexed functor $\Phi : A \rightarrow B : X \rightarrow Y$ will be said to preserve restrictions, if for any natural transformation $f : H \rightarrow K : Z \rightarrow Y$ the canonical arrow
\[(H^*\Phi)(t^*A)^r \rightarrow (t^*B)^r(K^*\Phi)\]

obtained from the interchange equality $(t_*B)(H_*\Phi) = (K_*\Phi)(t_*A)$ by transposing the adjoints is invertible.

We will say that the equipment $(X,A)$ has restrictions if $A$ has restrictions and the composition maps $\Pi_n$ for $n > 2$ preserve them. A lax functor of equipments $(F,\Phi)$ will be said to preserve restriction if $\Phi$ preserves restrictions.

An arrow $f : x \rightarrow y$ of $X$ can be seen as a transformation between functors of the form $I \rightarrow X$. If an equipment $(X,A)$ has restriction then $(f^*A)^r$ give functors

\[A(z,y) \rightarrow A(z,x)\]

For the value of these on $a$ we write $f^*a$. If a lax functor of equipments $(F,\Phi)$ preserves equipments, then we have $F(f)^rF(a) = F(f^*a)$.

An equipment $(X,A)$ which arises from a pseudofunctor $X \rightarrow A$ from a category $X$ into a bicategory $A$ has restrictions if and only if the image of any map $f$ in $X$ has a right adjoint in $A$.

9 The adjunction between $T$-monoids and $T$-algebras

We want to extend the Kleisli construction to the morphisms and 2-cells of $\text{Mnd}(\mathfrak{E})$. As noted $\mathfrak{E}$ is a sub 2-category of $(\text{Mnd}^{lh}(B)^{op})$. Then, it can be easily seen that $\text{Mnd}(\mathfrak{E})$ is a sub 2-category of $(\text{Mnd}^{op}(\text{Mnd}^{lh}(B)))^{op}$. Our $B$ does not have right adjoints hence we can not directly apply $\text{Cmp}^*$ defined in Section 4. For the solution we resort to a heuristic argument, which however may be made precise using some ad hoc reasoning.

Assume that $((X,A),T)$ is a monad in $\mathfrak{E}$, such that $(X,A)$ has restrictions and $(T,\Theta)$ preserves them.

For a morphism $((F,\Phi),h) : ((Y,B),S) \rightarrow ((X,A),T)$ in $\text{Mnd}(\mathfrak{E})$ define

\[\text{Cmp}^*((F,\Phi),h) = ((F,h^*\Phi)\]
where \( h^r \Phi \) is given by:

\[
S^* B F^* \xrightarrow{S^* \Phi} S^* F^* A \xrightarrow{(h^r A)^r} F^* T^* A.
\]

For a 2-morphism \( t : ((F, \Phi), h) \rightarrow ((F, \Phi'), h) \) of \( \text{Mnd}(\mathcal{E}) \) define

\[\text{Cmp}^*(t) = t.\]

Indeed \( \text{Cmp}^*((F, \Phi), h) \) is a lax equipment functor, and \( \text{Cmp}^*(t) \) is a lax transformation of lax functors. The lax comparision maps for the first and the lax transformation structure for the second are defined in much the same way as in the abstract situation of \( \text{Cmp}^* \). More precisely by this is meant the following. One could “unpackage” the \( \text{Cmp}^* \) constructions of Section 4 on morphisms and 2-cells of \( \text{Mnd}^{\text{op}}(\text{Mnd}^{1/}) \), writing out all the details. Then, one can imitate these procedure for the current situation, with the only difference being that in the more abstract case \( h^r \Phi \) is an actual pasting of squares in \( B \), while here it is defined by the formula above. The fact that \( \Pi_n \) and \( \Theta \) preserve restriction guarantee that we can carry out the same computations in both cases. We rely on the same argument further in the paper too, thus avoiding direct verifications which are long and tedious calculations.

Using \( \text{Cmp}^* \) we define a 2-functor:

\[
R : \text{Mnd}(\mathcal{E})(((Y, B), 1_Y), ((X, A), T)) \rightarrow \mathcal{E}((Y, B, (X, A_T))
\]
as \( R \) was defined in Section 5. Taking \( (Y, B) = (I, I) \) we obtain a functor

\[
R : T\text{-Alg}(X, A) \rightarrow T\text{-Mon}(X, A).
\]

To have a more concrete characterization, \( R \) takes a \( T \)-algebra \( ((x, b), h) \) to a \( T \)-monoid \( (x, h^r b) \).

In the other direction we have a functor

\[
L : \mathcal{E}((Y, B), (X, A_T)) \rightarrow \text{Mnd}(\mathcal{E})((Y, B), 1_Y), ((X, A), T))
\]
defined in the same way as \( L \) was defined in Section 5. For this we need to assume invertibility of \( \alpha_2 \). Taking \( (Y, B) = (I, I) \) we obtain a functor:

\[
L : T\text{-Mon}(X, A) \rightarrow T\text{-Alg}(X, A).
\]

This takes a \( T \)-monoid \( (x, a) \) to the \( T \) algebra \( ((Tx, m_x T(a)), m_x) \).

**Theorem 1.** Given a monad \( T \) on the equipment with restrictions \( (X, A) \), such that \( \alpha_2 \) is invertible, the functor \( L \) is a left adjoint to the functor \( R \).

For more detailed descriptions of the functors \( R \) and \( L \) see [1], where the case \( (\text{Set}, (\text{Mat}(V))^{\text{op}}) \) is considered.
10 Adding 2-cells

Let us briefly return to the abstract context of the first chapters. Suppose that $\Omega$ and $\Delta$ are lax natural transformations between lax maps of monads $(F, \Phi) \rightarrow (G, \Psi)$ : $(Y, B) \rightarrow (X, A)$. Suppose that $(Y, B)$ is a strong monad. A generalized modification $\Omega \rightarrow \Delta$ is a 3-cell

$$\lambda : (\Delta.A)\Phi \rightarrow \Psi(B.\Theta)$$

satisfying the axiom:

$$((G.\Pi_2)(\Delta.B^2)(\Phi.B)(A.\Phi)) \rightarrow ((G.\Pi_2)(\Psi.B)(A.\Omega.B)(A.\Phi))$$

$$((G.\Pi_2)(\alpha.B)(A.\Phi)) \rightarrow ((G.\Pi_2)(\Psi.B)(A.\alpha))$$

$$((G.\Pi_2)(\Psi.B)(A.\Delta.B)(A.\Phi)) \rightarrow ((G.\Pi_2)(\Psi.B)(A.\Psi)(\Omega.A^2))$$

Relying on the strongness of $(Y, B)$, a generalized modification $\lambda$ is completely determined by the 3-cell:

$$\lambda(\Pi_0.F) : (\Delta.A)\Phi(\Pi_0.F) \rightarrow \Psi(B.\Delta)(\Pi_0.F).$$

Composition of generalized modifications over morphisms of $\text{Mnd}^{l/l}(B)$ (i.e. composition of the third dimensional arrows over one dimensional arrows) are defined in a straightforward way. Composition of generalized modification over 2-morphisms is defined using the alternative characterization in terms of 3-cells $\lambda(\Pi_0.F)$. Also, one defines whiskering form the left of generalized modifications by morphisms and 2-morphisms of $\text{Mnd}^{l/l}(B)$. For every strong monad $(Y, B)$

$$\text{Mnd}^{l/l}((X, A), (Y, B))$$

becomes a 2-category. Whiskering from the left by morphisms of $\text{Mnd}^{l/l}(B)$ become functors on such categories. Whiskering from the left by 2-morphisms of $\text{Mnd}^{l/l}(B)$ become natural transformations. Moreover, the functors $L$ and $R$ of Section 5 become 2-functors, while the adjunction between them becomes a 2-adjunction.

Now take $B = M$. Define a modification between lax transformations of lax functors of equipments as a generalized modification. Such a modification $t \rightarrow s$ amount to 2-cells

$$t_xG(f) \rightarrow F(f)s_y$$

for any morohism $f : x \rightarrow y$ of the equipment. Or, using the alternative characterization, to 2-cells

$$t_x \rightarrow F(1_x)s_y.$$
**Theorem 2.** Given a monad $T$ on the equipment with restrictions $(X, A)$, such that $\alpha_2$ is invertible, the functor $L$ is a left 2-adjoint to the functor $R$.

Indeed the monad induced on $T$-$\text{Mon}$ by this adjunction is lax idempotent monad, also called a KZ monad.

**References**

[1] D. Chikladze, M. Clementino, D. Hofmann, Representable $(T, V)$-categories, in preparation.

[2] M. M. Clementino, W. Tholen, Metric, Topology and Multicategory - A Common Approach, Journal of Pure and Applied Algebra 179 (2003) 13–47.

[3] G. Cruttwell, M. Shulman, A unified framework for generalized multicategories. Theory and Applications of Categories 24 (2010), No. 21, 580–655.

[4] C. Hermida, Representable multicategories, Advances in Mathematics, 151 (1999), 164–225.

[5] R. Street, Fibrations in bicategories, Cahiers de topologie et geometrie differentielle 21 (1980) 111–160.

[6] R. Street, Formal theory of monads, Journal of Pure Applied Algebra 2 (1972) 149–168.