Algebra of Constraints and Solutions of Quantum Gravity

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Abstract

We construct the regularized Wheeler–De Witt operator demanding that the algebra of constraints of quantum gravity is anomaly free. We find that for only a small subset of all wavefunctions being integrals of scalar densities this condition can be satisfied. It turns out that the resulting operator is much simpler than the one used in [6] to find exact solutions of Wheeler–De Witt equation. We proceed to finding exact solutions of quantum gravity and we discuss their interpretation making use of the quantum potential approach to quantum theory.

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1 Introduction

One of the most outstanding problems of modern theoretical physics is the construction of quantum theory of gravity [1]. Indeed, it have been claimed many times that various unsolved problems like the cosmological constant problem, the problem of origin of the universe, the problem of black holes radiation will find their ultimate solution once this theory is finally constructed and properly understood. Some [2], claim that the theory of quantum gravity will also shed some light on the fundamental problems of quantum mechanics and even on the origin of mind. These all prospects are very exciting indeed, however the sad fact remain that the shapes of the future theory are still very obscured.

Nowadays there are two major ways of approaching the problem of quantum gravity. The first one is associated with the broad term “superstrings”. In this approach the starting point is a two-dimensional quantum field theory which yields quantum gravity as one of resulting low-energy effective theories. It is clear that in superstrings, as in other, less developed approaches in whose gravity appears as an effective theory, it does not make sense to try to “quantise” classical gravity.

In the canonical approach one does something opposite: the idea is to pick up some structures which appear already at the classical level and then promote them to define the quantum theory. In both the standard canonical approach in metric representation, which we will follow here, and in the approach based on loop variables [3], these fundamental structures are constraints of the classical canonical formalism reflecting the symmetries of the theory and their algebra. There are good reasons for such an approach. The equivalence principle is the main physical principle behind the classical theory of gravity; this principle leads to the general coordinate invariance and selects the Einstein–Hilbert action as the simplest possible one.

Another building block of the quantum theory is the quantisation procedure. Here one encounters the problem as to if a generalization of the standard Dirac procedure of quantisation of gauge theories is necessary. This would be the case if one shows that the standard approach is not capable of producing any interesting results. It is not excluded that this may be eventually the result of possible failure of investigations using standard techniques, however, in our opinion, at the moment there is no reason to modify the basic principles of quantum theory.
Our starting point consists therefore of

(i) The classical constraints of Einstein’s gravity: the diffeomorphism constraint generating diffeomorphism of the spatial three-surface “of constant time”

\[ D_a = \nabla_b \pi^{ab} \]  

and the Hamiltonian constraint generating “pushes in time direction”:

\[ \mathcal{H} = \kappa^2 G_{abcd} \pi^{ab} \pi^{cd} - \frac{1}{\kappa^2} \sqrt{h} (R + 2\Lambda) \]  

In the formulas above \( \pi^{ab} \) are momenta associated with the three-metric \( h_{ab} \),

\[ G_{abcd} = \frac{1}{2\sqrt{h}} (h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}) \]

is the Wheeler–De Witt metric, \( R \) is the three-dimensional curvature scalar, \( \kappa \) is the gravitational constant, and \( \Lambda \) the cosmological constant. The constraints satisfy the following algebra

\[ [\mathcal{D}, \mathcal{D}] \sim \mathcal{D}, \]  

\[ [\mathcal{D}, \mathcal{H}] \sim \mathcal{H}, \]  

\[ [\mathcal{H}, \mathcal{H}] \sim \mathcal{D}. \]  

(ii) The rules of quantisation given by the metric representation of the canonical commutational relations

\[ [\pi^{ab}(x), h_{cd}(y)] = -i \delta^{(a}_{c} \delta^{b)_{d}} \delta(x, y), \]

\[ \pi^{ab}(x) = -i \frac{\delta}{\delta h_{ab}(x)}. \]

Sadly, in the canonical approach, the points (i) and (ii) above encompass the whole of the input in our disposal in construction of the quantum theory. In particular, we do not know what is the correct physical inner product, and thus we do not know if the relevant operators are hermitean or not. Besides, we do not even know if we should demand these operators to be hermitean: the Hamiltonian annihilates the physical states (the famous time problem
and thus unitary evolution does not play the privileged role anymore. It follows that we cannot distinguish “relevant” wave functions by demanding that they are normalizable, as in the case of quantum mechanics, in fact, since the probabilistic interpretation of the “wavefunction of the universe” is doubtful, it is not clear if the norm of this wavefunction is to be 1.

In the recent paper [6] a class of exact solutions of the Wheeler–De Witt equation was found. In that paper we used the heat kernel to regularize the hamiltonian operator and inserted the particular ordering. The question arises what is the level of arbitrariness in this construction. In other words, could we construct other (possibly simpler) regularized hamiltonian operators and what would be their properties? This question is the subject of the present paper.

It is clear from the discussion above that the only principle, we can base our construction on is the principle that the algebra of constraints is to be anomaly–free, that is, the corresponding algebra of commutators of quantum constraints is weakly identical with the classical one. This means that the structure of the Poisson bracket algebra (3–5) is to be preserved, in the sense which will be explained below, on the quantum level. The following section is devoted to the analysis of this problem. In section 3 we investigate solutions of the resulting equations, and in section 4 we seek interpretation of the wavefunctions making use of the quantum potential approach to quantum mechanics. In the final section we draw our conclusions and describe the open problems.

2 The commutator algebra and construction of regularized operators

As we explained in Introduction, our starting point in construction of the quantum hamiltonian operator (the Wheeler–De Witt operator) is the algebra (3–5) and we demand that the same algebra holds on the quantum level. At this point we encounter immediately the problem, well known from the investigations of anomalies in quantum field theories, that the sole algebra of regularized operators is meaningless unless the space of states on which
these operators act is defined \textit{a priori}. This follows from the fact that the transition from regularised to renormalised action of an operator depends crucially on what particular state this operator acts (see below.) We will chose our starting space of states to be the space of integrals over compact three-space $M$ of scalar densities like $V = \int_M \sqrt{h}$, $R = \int_M \sqrt{h}R$, etc.;

$$\Psi = \Psi(V, R, \ldots).$$ (6)

We choose the following representation of the diffeomorphism constraint

$$D_a(x) = -i\nabla^x_b \frac{\delta}{\delta h_{ab}(x)},$$

where we employed the notation $\nabla^x_b$ meaning that the covariant derivative acts at the point $x$. Then we see that diffeomorphism constraint annihilates all the states and the commutator relation (3) is identically satisfied. Moreover we see that the relation (4) reduces to the formal relation

$$D(\mathcal{H}\Psi) \sim \mathcal{H}\Psi.$$ (7)

Now we must turn to the heart of the problem, the construction of the Wheeler–De Witt operator. It is well known that second functional derivative acting at the same point on a local functional produces divergent result. We deal with this problem by making the point split in the kinetic term, to wit

$$G_{abcd}(x)\pi^{ab}(x)\pi^{cd}(x) \Rightarrow \int dx' K_{abcd}(x, x'; t) \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} ,$$

where $K_{abcd}(x, x'; t)$ satisfies

$$\lim_{t \to 0^+} K_{abcd}(x, x'; t) = \delta(x, x').$$

By virtue of the correspondence principle, we take

$$K_{abcd}(x, x'; t) = G_{abcd}(x') \Delta(x, x'; t) (1 + K(x, t)),$$ (8)

where

$$\Delta(x, x'; t) = \exp\left(\frac{-1}{4t} h_{ab}(x-x')^a(x-x')^b\right)$$

$4\pi t^{3/2}$

\footnote{Here our approach differs from the one used in, for example [3], where the authors choose to analyze the algebra of quantum constraints without defining the space of states.}
and $K(x,t)$ is analytic in $t$.

Next we must resolve the ordering ambiguity in the operator $H$. To this end we add the new term $L_{ab}(x)\frac{\delta}{\delta h_{ab}(x)}$, where $L_{ab}$ is a tensor to be derived along with $K(x,t)$. Thus the final form of the Wheeler–De Witt operator is

$$
H(x) = \kappa^2 \int dx' \; K_{abcd}(x, x'; t) \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} + L_{ab}(x) \frac{\delta}{\delta h_{ab}(x)} + \frac{1}{\kappa^2} \sqrt{\hbar(R + 2\Lambda)}.
$$

To set the stage, we still need to define the action of operators on states. To this end we must discuss the issue of regularization and renormalization. The operator (9) acting on a state (defined as an integral of a scalar density) produces, in general, terms with arbitrary (positive and negative) powers of $t$. This provides the regularized version of the operator since all the terms are finite, and singularities of the form $\delta(0)$ are traded for terms which are singular for $t \to 0$. Observe that, as noted above, the singular part of the action of the operator on a state depends on this state. To renormalize, we follow the procedure proposed by Mansfield [7] which result in the following: the terms with positive powers of $t$ are dropped, and the singular terms of the form $t^{-k/2}$ are replaced by the renormalization coefficients $\rho^k$. This procedure provides us with the finite action of the Wheeler–De Witt operator on any state.

Now we can turn to the interpretation of equation (7). We understand it in the following way. An operator acts on a state and after renormalization gives another state depending on renormalization constants. On this resulting state the second operator acts. Thus the formal relation (7) is defined to mean

$$
D(H\Psi)_{\text{ren}} \sim (H\Psi)_{\text{ren}},
$$

and, similarly, for the hamiltonian–hamiltonian commutator

$$
(H[N](H[M]\Psi)_{\text{ren}} - (H[M](H[N]\Psi))_{\text{ren}} = 0
$$

since $\Psi$ is diffeomorphism invariant. In the formula above we used the smeared form of the Wheeler–De Witt operator

$$
H[M] = \int dx \; M(x)H(x).
$$

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In the paper [8] we took $\tilde{K}_{abcd}(x, x') = G_{abcd}(x)\tilde{K}(x, x')$, where $\tilde{K}$ is a heat kernel, and $L_{ab}$ was taken to be the functional derivative of $\tilde{K}_{abcd}$ with respect to $h_{cd}$. 

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Let us turn back to equation (10). Since the action of diffeomorphism is standard, it suffices to check that $(\mathcal{H}\Psi)_{\text{ren}}$ is a scalar density. But this is clearly the case: the first functional derivative acting on a state produces a tensor density $T^{ab}(x')$. After acting by the second derivative and contracting indices, we obtain the terms of the form

$$T_0(x')\delta(x', x) + T_1(x')\circ \nabla x'\circ \nabla x'\delta(x', x) + T_2(x')\circ \nabla x'\circ \nabla x'\circ \nabla x'\circ \nabla x'\delta(x', x) + \ldots$$

where $\circ$ denotes various indices contractions, and $T_n$ are tensor densities. These terms are multiplied by $\Delta(x, x'; t)$ and integrated over $x'$. Now we integrate by parts which results in replacing covariant derivatives acting on $K$ with appropriate powers of $t^{-1}$ multiplied by some coefficients. After renormalization we obtain a scalar density as required. The action of the $L$ term clearly gives the same result. Thus

For the states being integrals of scalar densities there is no anomaly in the diffeomorphism — hamiltonian commutator

This result is quite important because the anomaly in the string theory appears in the diffeomorphism — hamiltonian commutator.

Now we turn to the most complicated problem, the hamiltonian — hamiltonian commutator (11). Our goal will be to find the maximal space of states together with conditions defining coefficients $K$ and $L_{ab}$ of the Wheeler–DeWitt operator. Our approach is based on the following

**Claim.** If $(\mathcal{H}\Psi)_{\text{ren}}$ contains terms which contain four or more derivatives of the metric like $R^2$, $R_{ab}R^{ab}$ etc., then (11) cannot be satisfied.

We have checked this claim for terms proportional to square of three-curvature; it is clear from this computation that the claim holds for higher order terms as well unless there are some miraculous cancellations. We leave it as an open problem to check if the claim above is generally valid.

Let us start with the simplest state $\Psi = 1$. Then the action of the first smeared operator gives

$$(\mathcal{H}[M]\Psi)_{\text{ren}} = \int dx \sqrt{h(x)}M(x)(R(x) + 2\Lambda).$$
After rather tedious computation one finds in the commutator the term proportional to $N \nabla^a M - M \nabla^a N$ which must vanish, to wit

$$
\rho^{(1)} \frac{3}{2} \nabla_a K - \frac{1}{\kappa^2} \left( \nabla_a L + \nabla_b L^b \right) = 0,
$$

(12)

where $L = h_{ab} L^{ab}$.

Turning to the states depending of $V = \int_M d^3 x \sqrt{h}$, we find, taking the Claim above into account that $K$ and $L_{ab}$ can only contain terms at most linear in Ricci tensor. Given that, there is no anomaly. Thus we take

$$
L_{ab} = \frac{1}{\kappa} \alpha h_{ab} + \kappa (\beta h_{ab} R + \gamma R_{ab})
$$

where in the first term we included the gravitational constant for dimensional reasons.

All states depending on integrals of scalars constructed from powers of curvature tensor will necessarily lead to terms excluded by virtue of the Claim. This means that not all states of this form will lead to the anomaly-free algebra: the wavefunction will have to satisfy equations guaranteeing that such terms are absent. These equations, for the case of states depending on $\mathcal{R} = \int_M d^3 x \sqrt{h} \mathcal{R}$ will further restrict the form of the regularized Wheeler–De Witt operator.

Now we turn to the wavefunction $\Psi = \Psi(\mathcal{R})$. As we argued above, in the action of the Wheeler–De Witt operator on this state all terms with four derivatives must vanish. These terms are

$$
\kappa^2 \frac{\partial^2 \Psi}{\partial \mathcal{R}^2} \left( R_{ab} R^{ab} - \frac{3}{8} R^2 \right) + \kappa \frac{\partial \Psi}{\partial \mathcal{R}} \left( -\gamma R_{ab} R^{ab} + \frac{1}{2} (\gamma + \beta) R^2 \right) = 0,
$$

(13)

from which we obtain conditions on the coefficients

$$
\beta = -\frac{1}{4} \gamma,
$$

(14)

and from (13) we find that $\Psi(\mathcal{R})$ must be of the form

$$
\Psi(\mathcal{R}) = \exp \left( \frac{\gamma}{\kappa} \mathcal{R} \right).
$$

Thus $L_{ab} = \frac{\kappa}{\alpha} h_{ab} + \kappa \gamma (R_{ab} - \frac{1}{3} h_{ab} R)$. But then it follows from (12) that $K(x, t)$ must be constant. It is possible, in principle, to construct $K$ from
global integrals like $V$ and/or $R$, for example $K = t \frac{\mathcal{R}}{\gamma}$, however we will not pursue this (interesting) possibility here.

Thus the final form of the regularised Wheeler–De Witt operator is

$$ H(x) = \kappa^2 \int dx' G_{abcd}(x') \Delta(x, x'; t) \frac{\delta}{\delta h_{ab}(x)} \frac{\delta}{\delta h_{cd}(x')} + $$

$$ + \left( \frac{1}{\kappa} \alpha h_{ab} + \kappa \gamma \left( -\frac{1}{4} h_{ab} R + R_{ab} \right) \right)(x) \frac{\delta}{\delta h_{ab}(x)} + \frac{1}{2} \kappa^2 \sqrt{h}(R + 2\Lambda). \quad (15) $$

The formula (15) completes our construction of the Wheeler–De Witt operator. As compared to the choice made in the paper \cite{6}, where we used the heat kernel and $L_{ab}$ was its functional derivative, here we gained much more freedom in the form of two independent constants. In particular, we can make the constants $\alpha$ and $\gamma$ complex. This fact is very important in view of the quantum potential interpretation of our results (see Section 4), where it turns out that only complex wavefunctions lead to time-evolving universes.

3 Solutions

From the previous section we know that the most general form of the Wheeler–De Witt operator satisfying our criteria is given by equation (15), with coefficients $\alpha$ and $\gamma$ being still not fixed. Now, employing this operator, we will try to find a class of solutions of the Wheeler–De Witt equation. It should be stressed at this point that we regard the existence of a maximal possible space of solutions as an ultimate condition fixing the operator completely. Thus our goal is twofold: to find solutions and to fix the operator as to allow for the maximal possible number of them.

We will consider only the states of the form $\Psi = \Psi(\mathcal{V}, \mathcal{R})$. Since we have already taken care of the terms proportional to squares of Ricci tensor in (14), we have two equations for multipliers of $\sqrt{h}R(x)$ and $\sqrt{h}$. They read,

$$ - \frac{1}{4} \kappa^2 \frac{\partial^2 \Psi}{\partial \mathcal{V} \partial \mathcal{R}} + \frac{7}{8} \kappa^2 \rho^{(3)} \frac{\partial \Psi}{\partial \mathcal{R}} + \frac{1}{8} \kappa \gamma \frac{\partial \Psi}{\partial \mathcal{V}} + \frac{1}{2} \kappa \alpha \frac{\partial \Psi}{\partial \mathcal{R}} + \frac{1}{\kappa^2} \Psi = 0, \quad (16) $$

for $\sqrt{h}R$: 
and for $\sqrt{h}$:

$$-\frac{3}{8} \kappa^2 \frac{\partial^2 \Psi}{\partial V^2} - \frac{21}{8} \kappa^2 \rho^{(3)} \frac{\partial \Psi}{\partial V} - \frac{3}{4} \kappa^2 \rho^{(5)} \frac{\partial \Psi}{\partial R} + \frac{3}{2\kappa} \alpha \frac{\partial \Psi}{\partial V} + \frac{2}{\kappa^2} \Lambda \Psi = 0. \quad (17)$$

Now we must consider two cases:

1. $\Psi$ does not depend on $R$. Then from equations above we find

$$\frac{1}{8} \kappa \gamma \frac{\partial \Psi}{\partial V} + \frac{1}{\kappa^2} \Psi = 0, \quad (18)$$

$$-\frac{3}{8} \kappa^2 \frac{\partial^2 \Psi}{\partial V^2} - \left(\frac{21}{8} \kappa^2 \rho^{(3)} - \frac{3}{2\kappa} \alpha\right) \frac{\partial \Psi}{\partial V} + \frac{2}{\kappa^2} \Lambda \Psi = 0. \quad (19)$$

To make equations (18) and (19) consistent with each other, the coefficients must satisfy the relation

$$\frac{2\Lambda}{\kappa^2} \gamma^2 + \left(\frac{21}{8} \kappa^2 \rho^{(3)} - \frac{3}{2\kappa} \alpha\right) \gamma - 24 \frac{1}{\kappa^4} = 0 \quad (20)$$

and the solution is

$$\Psi(V) = \exp\left(-\frac{8}{\kappa^3} V\right). \quad (21)$$

2. In the case when $\Psi$ depends on both $V$ and $R$, we must take into account the fact that $\Psi = \tilde{\Psi}(V) \exp\left(\frac{2}{\kappa} R\right)$. Then we find the following equations

$$-\frac{1}{8} \kappa \gamma \frac{\partial \tilde{\Psi}}{\partial V} + \left(\frac{7}{8} \kappa \gamma \rho^{(3)} + \frac{1}{2\kappa^2} \alpha \gamma + \frac{1}{\kappa^2}\right) \tilde{\Psi} = 0, \quad (22)$$

$$-\frac{3}{8} \kappa^2 \frac{\partial^2 \tilde{\Psi}}{\partial V^2} - \left(\frac{21}{8} \kappa^2 \rho^{(3)} - \frac{3}{2\kappa} \alpha\right) \frac{\partial \tilde{\Psi}}{\partial V} - \left(\frac{3}{4} \rho^{(5)} \frac{\gamma}{\kappa} - \frac{2}{\kappa^2} \Lambda\right) \tilde{\Psi} = 0. \quad (23)$$

From equation (22) we find the solution for $\tilde{\Psi}$; thus

$$\Psi(V, R) = \exp\left\{-\frac{7}{\kappa} \rho^{(3)} + \frac{4\alpha}{\kappa^3} + \frac{8}{\kappa^3 \gamma}\right\} \exp\left\{\frac{2}{\kappa} R\right\}. \quad (24)$$

Substituting (24) into (23) we find another condition on the coefficients $\gamma$ and $\alpha$. Together with (20) it forms a system of equations which turns into a sixth order algebraic equation for $\gamma$. Each of the solutions of the latter defines unambiguously the operator and the set of its solutions.
4 Quantum potential interpretation

In the previous section we found a class of solutions of the Wheeler - De Witt equation. However the physical interpretation of these "wavefunctions of the universe" is quite obscured.

It turns out that there exists a nice device which makes it possible to interpret a wavefunction in terms of modified particle or field dynamics. This approach is an extension of works of David Bohm on interpretation of quantum mechanics [8] and was presented in [9] (see also [10].) It should be stressed at this point that we use here the quantum potential approach solely as a technical device to picture the wavefunction and we do not attempt to discuss the issue of interpretation of quantum theory.

As compared with the work [9] here we have to do with one important modification resulting from the presence of the $L$ term in our Wheeler - De Witt operator. Therefore we repeat first the most important steps, referring the reader to the original paper [9] for more details.

Assume that the wavefunction of the universe is of the form

$$\Psi = e^\Gamma e^{i\Sigma}.$$  \hfill (25)

The idea is to substitute this wavefunction to the Wheeler - De Witt equation and consider only the real part of the resulting equation. We obtain

\[-\kappa^2 G_{abcd}(x) \frac{\delta \Sigma}{\delta h_{ab}(x)} \frac{\delta \Sigma}{\delta h_{cd}(x)} + \frac{1}{\kappa} \sqrt{\hbar(x)(R(x) + 2\Lambda)} + \Re(L)_{ab}(x) \frac{\delta \Gamma}{\delta h_{ab}(x)} - \Im(L)_{ab}(x) \frac{\delta \Sigma}{\delta h_{ab}(x)} + e^{-\Gamma} \kappa^2 \left( \frac{\delta^2 e^\Gamma}{\delta h^2} \right)_{\text{ren}}(x) = 0, \hfill (26)\]

where $\Re(L)_{ab}$ and $\Im(L)_{ab}$ denote the real and imaginary part of $L_{ab}$, respectively. In the last term we used the abbreviated notation to indicate that the action of the second functional derivative is renormalised. Now one identifies momenta with the (functional) gradient of $\Sigma$, to wit

$$p^{ab}(x) = \frac{\delta \Sigma}{\delta h_{ab}(x)}. \hfill (27)$$

Then the first two terms in (26) are identical with the hamiltonian constraint of classical general relativity. The remaining terms are understood as quantum corrections (if we reintroduced $\hbar$ all these terms would become multiplied...
by $h^2$.) The wave function is subject to the second set of equations, namely the ones enforcing the three-dimensional diffeomorphism invariance. These equations read (for imaginary part)

$$\nabla^a \frac{\delta \Sigma}{\delta h_{ab}(x)} = \nabla^a p_{ab} = 0$$

Thus our theory is defined by two equations (26) with functional derivatives of $\Sigma$ replaced by $p_{ab}$ as in (27), and (28). Now we can follow without any alternations the derivation of Gerlach [11] to obtain the full set of ten equations governing the quantum gravity theory in quantum potential approach

$$0 = H^a = \nabla_a p_{ab},$$

$$0 = H_\perp = -\kappa^2 G_{abcd}(x)p^{ab}p^{cd} + \frac{1}{\kappa^2} \sqrt{h(x)}(R(x) + 2\Lambda)$$

$$+ \Re(L)_{ab}(x) \frac{\delta \Gamma}{\delta h_{ab}(x)} - \Im(L)_{ab}(x)p^{ab} + \kappa^2 e^{-\Gamma} \left( \frac{\delta^2 e^\Gamma}{\delta h^2} \right)_{\text{ren}}(x),$$

$$\dot{h}_{ab}(x, t) = \{h_{ab}(x, t), H[N, \vec{N}]\},$$

$$\dot{p}^{ab}(x, t) = \{p^{ab}(x, t), H[N, \vec{N}]\}.$$  

In equations above, $\{\star, \star\}$ is the usual Poisson bracket, and

$$H[N, \vec{N}] = \int d^3 x (N(x)H^a(x) + N^a(x)H_a(x))$$

is the total Hamiltonian (which is a combination of constraints). It might seem puzzling at the first sight why to a single wavefunction there corresponds a set of equations with, clearly, many solutions. The resolution of this problem is that the wavefunction, as a rule, is sensitive only to some aspects of the configuration. For example, the wavefunction (21) depends on $V$ only, and thus any configuration with given volume leads to the same numerical value of it. The above dynamical equations provide us with much more detailed information concerning the dynamics of the system than the wavefunction alone.

Now we apply this formalism to the case of the wavefunction depending only on $V$, (21) Then $\Gamma = -\frac{8}{\kappa \gamma \gamma} \Re(\gamma) V$. Let us inspect equation (30). The
term
\[ \left( \frac{\delta^2 e^{-\Gamma}}{\delta h^2} \right)_{\text{ren}} (x) \]

provides us only with modification of the cosmological constant. The term
\[ \mathcal{R}(L)_{ab}(x) \frac{\delta \Gamma}{\delta h_{ab}(x)} \]

modifies both the cosmological constant and the coefficient of the term \( \sqrt{h} R \). Taken together these modification can be written as
\[ \frac{1}{\kappa^2} \sqrt{h}(R + 2\Lambda) \Rightarrow \frac{1}{\tilde{\kappa}^2} \sqrt{h}(R + 2\tilde{\Lambda}), \]

where \( \tilde{\kappa} \) and \( \tilde{\Lambda} \) are modified gravitational and cosmological constants, respectively. Our final, modified, hamiltonian constraint reads therefore
\begin{equation}
\mathcal{H}_{\perp}(x) = -\kappa^2 G_{abcd}(x)p^{ab}p^{cd} + \frac{1}{\tilde{\kappa}^2} \sqrt{h}(x)(R(x) + 2\tilde{\Lambda})
- \frac{\mathcal{C}(\alpha)}{\kappa} h_{ab}p^{ab} - \kappa \mathcal{C}(\gamma) \left( R_{ab}p^{ab} - \frac{1}{4} h_{ab}p^{ab} R \right). \tag{34} \end{equation}

Now it is clear that the modified hamiltonian above is not a first class constraint. This follows from the presence of the terms linear in \( p \). Thus the effective theory has less symmetries than the classical general relativity. It follows then that the parameter \( N \) in the definition of total hamiltonian is not free anymore, rather it should be fixed by the requirement that the hamiltonian is time independent, to wit,
\[ \{ \mathcal{H}_{\perp}(x), \mathcal{H}[N, \vec{N}] \} = 0 \text{ (weakly).} \tag{35} \]

It should also be noted that even if the terms linear in \( p \) were not present (if the Poisson bracket constraint algebra would close), we still would not be able to recover the standard four dimensional action \( \int d^4x \sqrt{g}(R + 2\Lambda) \) from the hamiltonian action
\[ \int d^3x dt \left( p^{ab} \dot{h}_{ab} - \mathcal{H}[N, \vec{N}] \right). \]

The reason is that the coefficients \( \kappa \) and \( \tilde{\kappa} \) are not identical and therefore the three curvature and external curvature components of the four dimensional curvature scalar would be multiplied by different coefficients.
It should be stressed that the situation described for the case of this particular solution is quite generic. The conclusion, we draw from these computation is that the four dimensional general coordinate invariance seems to be broken by quantum corrections (besides, this provides the ultimate solution of the celebrated problem of time.) This symmetry is restored when quantum effects are negligible (because the ordering problems leading to introducing the $L$-term disappears in the semiclassical limit.)

5 Conclusions

The main problem, we address in this paper was to find a set of conditions which would make it possible to construct the regularized Wheeler–De Witt operator and a class of states for which the algebra of quantum constraints is anomaly free. It turned out that this space of states is quite modest but we were able to find some physical states (solutions of quantum gravity.) We then tried to find an interpretation of one of the solutions employing the method of quantum potential. We found that the resulting modified 3+1 theory does not possess the symmetry of time translation anymore. It is hard to say at this moment if this is just an artefact of employing the canonical quantisation method, where, by construction of the formalism, the time translation symmetry is very vulnerable from the very beginning or if it signifies some real physical effect at the Planck scale. It may also happen to be an artefact of the quantum potential interpretation of the wavefunction. These questions should be certainly further investigated.

Another direction of research is to include the coupling of gravity to matter fields like the scalar field or the supergravity. We are now in position to construct relevant regularised operators in both cases, but it turns out that to find solutions in these cases is (surprisingly?) hard.

These open problems are subject of intensive investigations and we hope to be able to present the results soon.

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