Photon-ALP oscillations inducing modification on $\gamma$-ray polarization

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Axion-like particles (ALPs) are very light, neutral, spin zero bosons predicted by many theories which try to extend and complete the standard model of elementary particles. ALPs interact primarily with two photons and can generate photon-ALP oscillations in the presence of an external magnetic field. They are attracting increasing interest since photon-ALP oscillations produce deep consequences in astrophysics particularly in the very-high-energy (VHE) band, where they increase the transparency of the Universe to VHE photons by partially preventing absorption caused by the extragalactic background light. Furthermore, ALPs explain why photons coming from flat spectrum radio quasars (a particular class of active galactic nuclei, AGN) have been observed for energies above 20 GeV – which represents a first hint for the existence of an ALP. In addition, ALPs solve an anomalous redshift dependence of blazar (an AGN class) spectra – which represents a second hint for the existence of an ALP. In this paper, we study another effect of the photon-ALP interaction: the change of the polarization state of photons. In particular, we study the propagation of the photon-ALP beam starting where photons are produced – we consider photons generated in a galaxy cluster or in the jet of a blazar – crossing several magnetized media (blazar jet, host galaxy, galaxy cluster, extragalactic space, Milky Way) up to their arrival at the Earth, where photons can be detected. In the presence of photon-ALP interaction, we analyze the photon survival probability $P_{\gamma\rightarrow\gamma}$ and the corresponding photon degree of linear polarization $\Pi_L$ for observed energies in the range $(1-10^{15})$ eV dividing it into three energy bands: (i) X-ray band ($10^{-3}-10^3$ keV), (ii) high-energy (HE) band ($10^{-1}-10^5$ MeV), (iii) VHE band ($10^{-2}-10^3$ TeV). We observe that photons, which are expected as unpolarized in the absence of ALPs, are made partially polarized by photon-ALP interaction, which generally modifies the initial photon degree of linear polarization $\Pi_{L,0}$ in a sizable and measurable way. Our findings about the X-ray and HE bands can be tested by current and planned observatories like IXPE, Polstar, COSI, e-ASTROGAM and AMEGO. A possible detection of a departure of the photon polarization from the standard expectations would represent an additional hint for the existence of an ALP. We also discover a peculiar feature in the VHE band, where photons at energies above $\sim 1$ TeV are fully polarized because of photon-ALP interaction. A possible detection of this feature would represent a proof for the existence of an ALP, but, unfortunately, current technologies do not allow yet to detect photon polarization up to so high energies.

I. INTRODUCTION

The understanding of the forces governing our Universe and of the nature of the constituent particles is at least incomplete: much evidence has been established for the existence of dark matter and dark energy as dominant elements of our Universe. Very promising candidates for dark matter are axion-like particles (ALPs), for a review see e.g. [5, 6], which are hypothetical very light particles predicted among many theories by the superstring theory [7–14]. ALPs are a generalization of the axion, the pseudo-Goldstone boson associated with the global Peccei-Quinn symmetry $U(1)_{PQ}$ which was proposed as a natural solution to the strong CP problem (see e.g. [15–18]).

ALPs differ from axions in two aspects: (i) while the mass and the coupling constant to photons are related quantities for the axion, the ALP mass $m_a$ and two-photon coupling constant $g_{\alpha\gamma\gamma}$ are unrelated parameters, (ii) while the axion necessarily couples to fermions and to gluons in order for the Peccei-Quinn mechanism to work, ALPs interact primarily with two photons – other interactions are subdominant and can be safely discarded. Thus, photons traveling in a magnetized medium mix with ALPs through the coupling $g_{\alpha\gamma\gamma}$ and the magnetic field is necessary in order to compensate for the spin mismatch between photons and ALPs – producing two different effects on the photon propagation: (i) photon-ALP oscillations [19, 20] similar to the oscillations of different flavor massive neutrinos, (ii) the change of the polarization state of photons [20, 21]. Therefore, ALPs have a huge impact in astrophysics and especially in the very-high-energy (VHE) band: whenever the medium crossed by photons is filled by intense magnetic fields and/or the path inside a magnetized medium is long (for an incomplete review, see [22]), photon propagation gets modified producing a long list of effects.

Photon-ALP conversion can occur inside different magnetic fields: in that of the jet of an active galactic nucleus (AGN), where a sizable amount of ALPs can be produced [23] explaining why flat spectrum radio quasars (FSRQs, a particular class of AGN) have been observed for energies above 20 GeV [24] – which repre-
observed photon polarization. We calculate the photon-ALP beam analyzing the ALP-induced modification on e-ASTROGAM [48, 49] and AMEGO [50]. In addition, a second hint for ALP existence comes from other astrophysical sources have been studied e.g. bursts have been analyzed in [38], and photon-ALP con-
fusions on possible ALP indirect detection. In fact, the polarization of the detected photons differs with respect to conventional physics expectations, this fact may represent a hint for new physics in the form of ALPs. Consequences of photon-ALP interaction on the polarization of photons produced by gamma-ray bursts have been analyzed in [35], and photon-ALP conversion effects on the polarization of photons originated from other astrophysical sources have been studied e.g. in [39–41]. In addition, new attention on this topics has been recently paid because of some existing or proposed experiments that measure the polarization of cosmic photons in the X-ray band like IXPE [45] and Polstar [46] and in the high-energy (HE) band such as COSI [47], e-ASTROGAM [48] [49] and AMEGO [50].

In this paper, we study the propagation of the photon-ALP beam analyzing the ALP-induced modification on observed photon polarization. We calculate the photon survival probability \( P_{\gamma \rightarrow \gamma} \) and the photon degree of linear polarization \( \Pi_\gamma \) of the photon-ALP beam while crossing different magnetized environments using state-of-the-art knowledge (see following Sections). In particular, we con-
ter the case where photons are generated and oscillate into ALPs inside the magnetic field of the jet when a blazar is present and the alternative case of photon production in the central region of a galaxy cluster. We then study the propagation of the photon-ALP beam inside the Kolmogorov-type turbulent galaxy cluster magnetic field. We consider both the cases of cool-core (CC) and non-cool-core (nCC) galaxy clusters. Furthermore, for the propagation inside the extragalactic space two possibilities are taken into account: low extragalactic magnetic field strength, \( B_{\text{ext}} < 10^{-15} \) G and the higher value \( B_{\text{ext}} = 1 \) nG. While in the former case photon-ALP conversion is negligible, it is efficient in the latter. At the end, we add photon-ALP interaction inside the magnetic field of the Milky Way. Then, we analyze the photon degree of linear polarization in order to investigate possible features indicating hints for the ALP existence. We study the behavior of the photon survival probability and of the photon degree of linear polarization of the photon-ALP beam in the energy range \((1 - 10^{15})\) eV dividing it into three bands: (i) X-ray band \((10^{-3} \text{ keV} - 10^{2}\text{ keV})\), (ii) HE band \((10^{-1} \text{ MeV} - 10^{4}\text{ MeV})\), (iii) VHE band \((10^{-2}\text{ TeV} - 10^{3}\text{ TeV})\). While our findings can be tested by current and planned observatories in the X-ray and HE band, our results about the VHE band are nowadays only theoretical since present technologies are currently unable to detect photon polarization up to so high energies.

The paper is organized as follows. In Sect. II we briefly introduce ALPs and the photon-ALP system, while in Sect. III we deal with polarization and recall some results linking conversion/survival probability and particle polarization. Then, in Sect. IV we discuss the photon-ALP beam propagation crossing different magnetized media, in Sect. V we present our results in the three considered energy ranges, while in Sect. VI we draw our conclusions.

II. AXION-LIKE PARTICLES

ALPs are spin-zero, neutral, and extremely light pseudo-scalar bosons interacting primarily with photons (interactions with fermions are subdominant and therefore safely negligible) through the Lagrangian:

\[
\mathcal{L}_{\text{ALP}} = \frac{1}{2} \partial^\mu a \partial_\mu a - \frac{1}{2} m_a^2 a^2 - \frac{1}{4} g_{a\gamma\gamma} F_{\mu\nu} \tilde{F}^{\mu\nu} a
\]

\[
= \frac{1}{2} \partial^\mu a \partial_\mu a - \frac{1}{2} m_a^2 a^2 + g_{a\gamma\gamma} \mathbf{E} \cdot \mathbf{B} a ,
\]

where \( a \) represents the ALP field, \( F_{\mu\nu} \) is the electromagnetic tensor whose electric and magnetic components are \( \mathbf{E} \) and \( \mathbf{B} \), respectively and \( \tilde{F}^{\mu\nu} \) is the \( F_{\mu\nu} \) dual. Concerning the photon-ALP coupling \( g_{a\gamma\gamma} \) and the ALP mass \( m_a \) many bounds exist in the literature such as those derived in [25] [51–59]. The firmest one reads \( g_{a\gamma\gamma} < 0.66 \times 10^{-10}\text{ GeV}^{-1} \) for \( m_a < 0.02\text{ eV} \) at the 2\( \sigma \) level from no detection of ALPs from the Sun [51].

In the presence of a strong external magnetic field, we must also consider the photon one-loop vacuum polarization effects accounted by the Heisenberg-Euler-Weisskopf (HEW) effective Lagrangian which reads

\[
\mathcal{L}_{\text{HEW}} = \frac{2 \alpha^2}{45 m_e^2} \left[ (\mathbf{E}^2 - \mathbf{B}^2)^2 + 7 (\mathbf{E} \cdot \mathbf{B})^2 \right] ,
\]

where \( \alpha \) is the fine-structure constant and \( m_e \) is the electron mass [60,62].

We study a photon-ALP beam of energy \( E \) propagating in the \( y \)-direction and crossing a magnetized medium...
whose external magnetic field entering Eq. [1] is denoted by $\mathbf{B}$, while $\mathbf{E}$ pertains to a propagating photon. Since the mass matrix of the $\gamma - a$ system is off-diagonal, the propagation eigenstates differ from the interaction eigenstates, producing $\gamma \leftrightarrow a$ oscillations in a similar way as oscillations of different flavor massive neutrinos with the only difference that in the case of the photon-ALP system an external $B$ field is necessary in order to compensate for the spin mismatch between photons and ALPs. From the form of the photon-ALP coupling in Eq. [1], we infer that $a$ couples only with the component $B_T$ of $\mathbf{B}$ which is transverse to the photon momentum $k$ (see also [28]). The photon-ALP beam propagation equation following from $\mathcal{L}_{\text{ALP}}$ of Eq. [1] reads

$$\left( i \frac{d}{dy} + E + \mathcal{M}(E, y) \right) \psi(y) = 0 \,,$$

with

$$\psi(y) = \left( \begin{array}{c} A_x(y) \\ A_z(y) \\ a(y) \end{array} \right) \,,$$

where $\mathcal{M}(E, y)$ represents the photon-ALP mixing matrix, while $A_x(y)$ and $A_z(y)$ are the two photon linear polarization amplitudes along the $x$ and $z$ axis, respectively and $a(y)$ denotes the ALP amplitude. In Eq. [3] we have employed the short-wavelength approximation [29], which stands since we are working in the regime $E \gg m_a$ (as it will be clear in the following because of the chosen parameters). As a consequence, the photon-ALP beam propagation equation becomes a Schrödinger-like equation with the coordinate $y$ along the beam in place of the time $t$: thus, the relativistic beam can formally be treated as a three-level nonrelativistic quantum system.

Denoting by $\mathcal{U}$ the transfer matrix of the photon-ALP beam propagation equation, which is the solution of Eq. [3] with initial condition $\mathcal{U}(E; y_0, y_0) = 1$, a generic wave function possesses solution

$$\psi(y) = \mathcal{U}(E; y, y_0)\psi(y_0) \,,$$

with $y_0$ accounting for the initial position of the beam. In the case of a non-polarized beam we have to use the density matrix $\rho(y)$ satisfying the Von Neumann-like equation linked to Eq. [4], which reads

$$i \frac{d\rho(y)}{dy} = \rho(y) \mathcal{M}^\dagger(E, y) - \mathcal{M}(E, y) \rho(y) \,,$$

whose solutions can be represented in terms of $\mathcal{U}(E; y, y_0)$ as

$$\rho(y) = \mathcal{U}(E; y, y_0) \rho_0 \mathcal{U}^\dagger(E; y, y_0) \,.$$

Hence, the probability describing a beam in the initial state $\rho_0$ at position $y_0$ and in the final state $\rho$ at position $y$ reads

$$P_{\rho_0 \rightarrow \rho}(E, y) = \text{Tr} \left[ \rho \mathcal{U}(E; y, y_0) \rho_0 \mathcal{U}^\dagger(E; y, y_0) \right] \,,$$

with $\text{Tr} \rho_0 = \text{Tr} \rho = 1$ [28].

By defining $\phi$ the angle that $B_T$ forms with the $z$ axis, the mixing matrix $\mathcal{M}$ in Eq. [3] can be written as

$$\mathcal{M}(E, y) = \left( \begin{array}{ccc} \Delta_{xx}(E, y) & \Delta_{xz}(E, y) & \Delta_{a\gamma}(y) \sin \phi \\ \Delta_{zx}(E, y) & \Delta_{zz}(E, y) & \Delta_{a\gamma}(y) \cos \phi \\ \Delta_{a\gamma}(y) \sin \phi & \Delta_{a\gamma}(y) \cos \phi & \Delta_{aa}(E) \end{array} \right) \,,$$

with

$$\Delta_{xx}(E, y) \equiv \Delta_{zz}(E, y) \equiv \frac{\Delta_{a\gamma}(y)}{2} \sin^2 \phi + \Delta_{\parallel}(E, y) \sin^2 \phi \,,$$

$$\Delta_{zz}(E, y) \equiv \frac{\Delta_{a\gamma}(y)}{2} \cos^2 \phi + \Delta_{\parallel}(E, y) \cos^2 \phi \,,$$

$$\Delta_{a\gamma}(y) = \frac{1}{2} g_{a\gamma\gamma} B_T(y) \,,$$

$$\Delta_{aa}(E) = \frac{m_a^2}{2E} \,,$$

and

$$\Delta_{\parallel}(E, y) = \frac{i}{2 \lambda_\gamma E(y, y)} - \frac{\omega_{\text{pl}}^2(y)}{2E}$$

$$+ \frac{2a_0}{45\pi} \left( \frac{B_T(y)}{B_{\text{cr}}} \right)^2 E + \rho_{\text{CMB}} E \,,$$

$$\Delta_{\perp}(E, y) = \frac{i}{2 \lambda_\gamma E(y, y)} - \frac{\omega_{\text{pl}}^2(y)}{2E}$$

$$+ \frac{7a_0}{90\pi} \left( \frac{B_T(y)}{B_{\text{cr}}} \right)^2 E + \rho_{\text{CMB}} E \,,$$

where $B_{\text{cr}} \approx 4.41 \times 10^{13}$ G is the critical magnetic field and $\rho_{\text{CMB}} = 0.522 \times 10^{-42}$ G Eq. [13] accounts for the photon-ALP interaction, while Eq. [14] describes the ALP mass effect. The first term in Eqs. [15] and [16] accounts for absorption (e.g. due to the EBL) and $\lambda_\gamma$ is the $\gamma \gamma \rightarrow e^+e^-$ mean free path [63]. The second term of Eqs. [15] and [16] $\omega_{\text{pl}}$ is the plasma frequency, which is related to the electron number density $n_e$ by $\omega_{\text{pl}}^2 = (4\pi n_e/e^2)^{1/2}$. The third term in Eqs. [15] and [16] accounts for the photon one-loop vacuum polarization coming from $\mathcal{L}_{\text{HEW}}$ of Eq. [2] and which produces polarization variation and birifrangence on the beam, while the fourth term represents the contribution from photon dispersion on the cosmic microwave background (CMB) [67] which produces sizable effects inside the extragalactic space [29].

In order to understand the different regimes defined by the relative importance of the $\Delta$ terms in Eq. [9] that the
photon-ALP system can experience, we consider: (i) the case of fully polarized photons, (ii) no photon absorption i.e. $\lambda_\gamma \to \infty$, (iii) a homogeneous and constant $B$ field, so that $B(y) \equiv B, \forall y$ having thus the freedom to choose the $z$ axis along the direction of $B_T$ – this fact translates to set $\phi = 0$ in Eq. (9). With these assumptions the $\gamma \to a$ conversion probability reads

$$P_{\gamma \to a}(E, y) = \left(\frac{g a_{\gamma\gamma} B_T l_{\text{osc}}(E)}{2\pi}\right)^2 \sin^2\left(\frac{\pi y}{l_{\text{osc}}(E)}\right),$$

where

$$l_{\text{osc}}(E) \equiv \frac{2\pi}{\left[\left(\Delta_{zz}(E) - \Delta_{aa}(E)\right)^2 + 4 \Delta_a^2\right]^{1/2}}$$

is the photon-ALP beam oscillation length. It is useful to define the low-energy threshold $E_L$ and the high-energy threshold $E_H$ as

$$E_L \equiv \frac{|m_a^2 - \omega_p^2|}{2g a_{\gamma\gamma} B_T},$$

and

$$E_H \equiv g a_{\gamma\gamma} B_T \left[\frac{7a_{\gamma\gamma}}{90\pi} \left(\frac{B_T}{B_c}\right)^2 + \rho_{\text{CMB}}\right]^{-1},$$

respectively. For $E_L \lesssim E \lesssim E_H$ the strong-mixing regime takes place and the plasma contribution, the ALP mass term, the QED one-loop effect and the photon dispersion on the CMB are negligible. In such a situation the $P_{\gamma \to a}$ is maximal, energy independent and reads

$$P_{\gamma \to a}(y) = \sin^2\left(\frac{g a_{\gamma\gamma} B_T}{2} y\right).$$

For $E \lesssim E_L$ the plasma contribution and/or the ALP mass term dominate and the same is true for $E \gtrsim E_H$ concerning the QED one-loop effect and/or the photon dispersion on the CMB: in both the cases we are in the weak-mixing regime and $P_{\gamma \to a}$ becomes energy dependent and progressively vanishes.

Everything we have discussed above in the case of no absorption and homogeneous and constant $B$ field can be translated in the general case: however, the analytic expressions of the equations would be unacceptably cumbersome and would shed no light on what is going on, so that we have decided to report the considered simplified case. In the following Sections the complete spatial-dependent expressions are considered.

### III. POLARIZATION EFFECTS

Whenever the polarization of the photon-ALP beam is not measurable – as in the VHE band – or the beam is expected to be unpolarized, the generalized polarization density matrix $\rho$ must be used: the matrix $\rho$ associated to the photon-ALP beam can be written as

$$\rho(y) = \begin{pmatrix} A_x(y) \\ A_y(y) \\ a(y) \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & |a(y)|^2 \end{pmatrix},$$

which allows to treat unpolarized, partially-polarized and totally polarized beams (pure states), at once. Pure photon states in the $x$ and $z$ direction read

$$\rho_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho_z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

the ALP state can be expressed by

$$\rho_a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix},$$

while unpolarized photons are described by

$$\rho_{\text{unpol}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$}

Partially polarized photons are associated to a polarization density matrix that possesses an intermediate functional expression between Eqs. (23) and Eq. (25).

We can express the $2 \times 2$ photon polarization density matrix – which is the 1-2 submatrix of the density matrix for the photon-ALP system of Eq. (22) – in terms of the Stokes parameters as [65]

$$\rho_{\gamma} = \frac{1}{2} \begin{pmatrix} I + Q & U + iV \\ U - iV & I - Q \end{pmatrix},$$

while the definition of the photon degree of linear polarization $\Pi_L$ reads [66]

$$\Pi_L \equiv \frac{(Q^2 + U^2)^{1/2}}{I},$$

which in terms of the photon polarization density matrix elements $\rho_{ij}$ with $i, j = 1, 2$ can be expressed as

$$\Pi_L = \frac{[(\rho_{11} - \rho_{22})^2 + (\rho_{12} + \rho_{21})^2]^{1/2}}{\rho_{11} + \rho_{22}}.$$
In any isolated system consisting of photons interacting with ALPs only, where photons are not absorbed and with initial condition of only photons with initial degree of linear polarization $\Pi_{L,0}$, the conversion probability satisfies the inequality $P_{\gamma \rightarrow a} \leq (1 + \Pi_{L,0})/2$, while $P_{\gamma \rightarrow \gamma} \geq (1 - \Pi_{L,0})/2$.

In the previous conditions but in the case of initially unpolarized photons ($\Pi_{L,0} = 0$), we observe $P_{\gamma \rightarrow a} \leq 1/2$ and $P_{\gamma \rightarrow \gamma} \geq 1/2$.

(iii) In the conditions of item (i) $\Pi_{L,0}$ represents the measure of the overlap between the values assumed by $P_{\gamma \rightarrow a}$ and $P_{\gamma \rightarrow \gamma}$.

(iv) In the conditions of item (ii) $\Pi_{L,0} = 0$ establishes that $P_{\gamma \rightarrow a}$ and $P_{\gamma \rightarrow \gamma}$ possess the common value of 1/2, at most.

IV. PHOTON-ALP BEAM PROPAGATION

In this Section we analyze the photon-ALP beam propagation in all the media considered in this paper: the blazar jet, the host galaxy, the galaxy cluster, the extragalactic space and the Milky Way. Hereafter, we cursorily recall the main properties and consequences of photon-ALP propagation in such media letting the details to the specific papers cited below, which are dedicated to that particular subject. About the photon-ALP interaction we take $g_{a\gamma \gamma} = 0.5 \times 10^{-11}$ GeV$^{-1}$ and two cases concerning the ALP mass: (i) $m_a \lesssim 10^{-14}$ eV, (ii) $m_a = 10^{-10}$ eV, for definiteness (more about this, later). These choices allow us to stay within the current firmest bound (see Sect. V for more details). In any case, we want to stress that all existing bounds about $g_{a\gamma \gamma}$ and $m_a$ have to be viewed as indications at most and even a choice of such parameters beyond these limits is perfectly allowed.

A. Active galactic nuclei

Active galactic nuclei (AGN) are basically extragalactic supermassive black holes (SMBHs) accreting matter from neighborhood and in which two collimated relativistic jets develop in opposite directions. When one of the jets occasionally points towards us, AGN are called blazars. Blazars are divided into two groups: flat spectrum radio quasars (FSRQs) and BL Lac objects (BL Lacs). While FSRQs are more powerful and characterized by strong optical emission lines and by the presence of high absorption zones for VHE energy photons (broad line region, torus; see e.g. [65]), BL Lacs are less powerful and possess neither sizable emission lines nor the above-mentioned absorption regions. BL Lacs have a harder spectrum reaching observed energies up to $\sim 20$ TeV for close sources (see e.g. Markarian 501 [69]). We study BL Lacs in this paper.

By closely following the results obtained in [23], we study here the propagation of the photon-ALP beam inside the magnetic field $B^\text{jet}$ of the jet, whose axis is supposed to coincide with the direction $y$. We start from the photon emission region placed at a distance of about $y_{\text{em}} = (10^{16} - 10^{17})$ cm from the central SMBH – for definiteness we take $y_{\text{em}} = 3 \times 10^{16}$ cm – up to the distance where the jet ends at about 1 kpc, entering the host galaxy. Concerning $B^\text{jet}$ what is relevant is its toroidal part which is transverse to the jet axis [71, 72]. Its profile reads

$$B^\text{jet}(y) = B_0^\text{jet} \left( \frac{y_{\text{em}}}{y} \right),$$

where $B_0^\text{jet}$ is the jet magnetic field strength at the photon emission position $y_{\text{em}}$. Because of the conical shape of the jet, the electron number density $n_e^\text{jet}$ profile is expected to be represented by

$$n_e^\text{jet}(y) = n_{e,0}^\text{jet} \left( \frac{y_{\text{em}}}{y} \right)^2,$$

where $n_{e,0}^\text{jet}$ is the jet electron number density at the photon emission position $y_{\text{em}}$. Synchrotron Self Compton (SSC) diagnostics applied to blazar spectra can give information about realistic values for $B_0^\text{jet}$ and $n_{e,0}^\text{jet}$ [73]. For definiteness, we take the average values $B_0^\text{jet} = 0.5$ G and $n_{e,0}^\text{jet} = 5 \times 10^4$ cm$^{-3}$.

Once all the above quantities are fixed, the whole propagation process of the photon-ALP beam within the jet can be evaluated and we can calculate its transfer matrix $U_{\text{jet}}$ (for more details see [23]).

By defining $\gamma$ as the Lorentz factor, since we calculate the photon-ALP beam propagation crossing the jet in its comoving frame, we must apply the transformation $E \rightarrow \gamma E$ to the beam in order to translate it to the fixed frames of the following regions. We take $\gamma = 15$.

B. Host galaxy

BL Lacs are normally located in elliptical galaxies, where the magnetic field $B_{\text{host}}$ is believed to be of turbulent nature. A domain-like model is commonly used to describe the $B_{\text{host}}$ behavior, while its average strength and coherence length are $B_{\text{host}} \simeq 5 \mu$G and $L_{\text{dom}} \simeq 150$ pc, respectively [74].

Since the $\gamma \rightarrow a$ oscillation length is much larger than $L_{\text{dom}}$, photon-ALP conversion turns out to be totally inefficient in this region, so that the effect of the host galaxy on the whole photon-ALP beam propagation process can be safely neglected, as shown in [24]. Yet, we carefully calculate the transfer matrix in the host galaxy $U_{\text{host}}$. 
C. Galaxy cluster

Faraday rotation measurements and synchrotron radio emissions establish the existence of $\mathcal{O}(1-10)\,\mu G$ magnetic fields $B^{\text{clu}}$ inside galaxy clusters [73, 76]. While old models have described $B^{\text{clu}}$ with a domain-like structure, it is nowadays established that $B^{\text{clu}}$ is turbulent. In particular, $B^{\text{clu}}$ is of isotropic gaussian turbulent nature and possesses a Kolmogorov-type turbulence power spectrum $M(k) \propto k^q$ with $k$ the wave number in the interval $[k_l, k_H]$ and index $q = -11/3$ [77]. For definiteness, we take $k_l = 0.2\,\text{kpc}^{-1}$ and $k_H = 3\,\text{kpc}^{-1}$. The behavior of $B^{\text{clu}}$ and of the cluster electron number density $n_e^{\text{clu}}$ with respect to the radial distance reads [77–79]

$$
B^{\text{clu}}(y) = B \left( B_0^{\text{clu}}, k, q, y \right) \left( \frac{n_e^{\text{clu}}(y)}{n_e,0} \right)^{\eta_{\text{clu}}},
$$

and

$$
n_e^{\text{clu}}(y) = n_e^{\text{clu},0} \left( 1 + \frac{y^2}{r_{\text{core}}} \right)^{-\frac{2}{3} \beta_{\text{clu}}},
$$

respectively, where $B$ represents the spectral function accounting for the Kolmogorov-type turbulence of the cluster magnetic field (see e.g. [80] for more details), $B_0^{\text{clu}}$ and $n_e^{\text{clu},0}$ are the central cluster magnetic field strength and the central electron number density, respectively, while $\eta_{\text{clu}}$ and $\beta_{\text{clu}}$ are two parameters of the cluster and $r_{\text{core}}$ is the cluster core radius. In the following, we employ average values for the above cluster parameters, by considering $B_0^{\text{clu}} = 15\,\mu G$, $\eta_{\text{clu}} = 0.75$ and the typical values $\beta_{\text{clu}} = 2/3$ and $r_{\text{core}} = 100\,\text{kpc}$ [77] [79] [81].

The choice of the value of $n_e^{\text{clu},0}$ is more involved. Two main categories of galaxy clusters exist: cool-core (CC) and non-cool-core (nCC) galaxy clusters (see also note [82]). While CC galaxy clusters usually host an AGN, the SMBH in the center of nCC galaxy clusters is generally not active. Some studies propose an interplay between active/quiescent SMBHs and CC/nCC galaxy clusters suggesting that the two systems are linked and with the one influencing the evolution of the other [83]. Although CC and nCC galaxy clusters differ in many aspects (see e.g. [84]), what is important for our studies is their central electron number density $n_e^{\text{clu},0}$. We consider $n_e^{\text{clu},0} = 5 \times 10^{-2}\,\text{cm}^{-3}$ for CC galaxy clusters and $n_e^{\text{clu},0} = 0.5 \times 10^{-2}\,\text{cm}^{-3}$ for nCC ones, which represent the average values for the two classes [85].

As an example, we plot the component along the $x$-axis of the galaxy cluster turbulent magnetic field $B^{\text{clu}}$ with respect to the cluster radial distance $y$ in Fig. 1 with the above-reported choice of the cluster parameters.

Then, by propagating the photon-ALP beam in the cluster starting from the central region up to its external border – we take 1 Mpc – we obtain the transfer matrix $\mathcal{U}_{\text{clu}}$ of the photon-ALP system inside the cluster.

D. Extragalactic space

The extragalactic magnetic field $B_{\text{ext}}$ affects the photon-ALP beam propagation in an amount which depends on the strength and morphology of $B_{\text{ext}}$. However, our knowledge of $B_{\text{ext}}$ is nowadays very poor: $B_{\text{ext}}$ is restricted by current limits to the range $10^{-7}\,\text{nG} \leq B_{\text{ext}} \leq 1.7\,\text{nG}$ on the scale of $\mathcal{O}(1)\,\text{Mpc}$ [84, 85]. Although several models for $B_{\text{ext}}$ exist in the literature [87, 88], $B_{\text{ext}}$ is believed to possess a domain-like structure: $B_{\text{ext}}$ keeps a constant strength in each domain and the same direction over an entire domain of size $L_{\text{dom}}$, which is equal to the magnetic field coherence length, but it randomly and discontinuously varies its direction crossing from one domain to the following one [87, 88]. By turbulence amplified outflows from primeval galaxies [89, 91] predict values for the extragalactic magnetic field in the upper range of the existing limits: $B_{\text{ext}} = \mathcal{O}(1)\,\text{nG}$ for a coherence length equal to the size of the magnetic domains $L_{\text{dom}} = \mathcal{O}(1)\,\text{Mpc}$. Especially in the VHE range where the $\gamma \leftrightarrow$ a oscillation length $l_{\text{osc}}$ can become smaller than $L_{\text{dom}}$ – in this case we can have $E \gtrsim E_H$ because of the photon dispersion on the CMB [64] (see Eq. (20) in Sect. II) – the simple discontinuous domain-like model for $B_{\text{ext}}$ produces unphysical results about the photon-ALP propagation since the system becomes sensitive to the $B_{\text{ext}}$ substructure. This is the reason why an improved physically consistent continuous domain-like model has been developed in [95], where $B_{\text{ext}}$ maintains the same strength in all domains and its orientation is constant in the central part of the domain but continuously and smoothly changes direction passing – still randomly – from a domain to the following one. This procedure preserves the domain-like structure of $B_{\text{ext}}$ correcting the unphysical
behavior at the domain edge crossing and still permits an analytical even if cumbersome solution of Eq. (3) [95].

Since a high $B_{\text{ext}}$ strength scenario is favored but not certain, we consider two cases in this paper: (i) $B_{\text{ext}} = 1 \text{nG}$ with the length of $L_{\text{dom}}$, randomly varying according to a power-law distribution function $\propto (L_{\text{dom}}^{-1})^{1.2}$ in the range $(0.2-10)\text{Mpc}$ and with $(L_{\text{dom}}) = 2 \text{Mpc}$ – which is consistent with present bounds [95]; (ii) $B_{\text{ext}} < 10^{-15} \text{G}$. In the former case photon-ALP conversion is efficient and produces sizable effects on the photon-ALP beam propagation reducing the VHE photon absorption caused by interaction with the EBL photons [29]; we consider the EBL model of Franceschini and Rodighiero [30]. Instead, the photon-ALP interaction is totally negligible in the latter case, so that propagation in the extragalactic space is dominated by EBL absorption, when present. In both the previous cases we can calculate the transfer matrix of the photon-ALP beam in the extragalactic space $U_{\text{ext}}$ by following the above-discussed strategy and developed in [29] [95].

E. Milky Way

The knowledge of the Milky Way magnetic field $B_{\text{MW}}$ has greatly improved in the last years: it is well known that it possesses a strength of the order $O(1) \mu\text{G}$ and presents both a turbulent and a regular component. The regular part of $B_{\text{MW}}$ produces the dominant effects on the photon-ALP beam propagation, while the contribution of the turbulent part can often be discarded since the coherence length of the turbulent field is much smaller than the $\gamma \leftrightarrow a$ oscillation length. Nevertheless, accurate maps concerning the profile of $B_{\text{MW}}$ and its behavior with respect to the observational direction and distance nowadays exist in the literature [96, 97]. For this reason in this paper we calculate the photon-ALP beam propagation inside the Milky Way by closely following the strategy developed in [31] by using the model of Jansson and Farrar [96, 97], which takes into account a disk and a halo component, both parallel to the Galactic plane, and poloidal ‘X-shaped’ component at the galactic center. In addition, newer data about polarized synchrotron and different models of the cosmic ray and thermal electron distribution are described in the newer version [98]. We improve the description of the turbulent component of $B_{\text{MW}}$ by using the model developed in [100].

We have tested that our results are qualitatively unchanged by using the model of Pshirkov et al. [98] but we have preferred the model of Jansson and Farrar [96, 97] for our calculation since the one of Pshirkov et al. [98] does not determine the Galactic halo component of $B_{\text{MW}}$ with accuracy. The electron number density inside the Milky Way disk is $n_{e,0}^{\text{MW}} \approx 1.1 \times 10^{-2} \text{cm}^{-3}$, as inferred from the model developed in [101], which we employ in this paper.

By using the strategy developed in [31] and the model [96, 97, 100] concerning $B_{\text{MW}}$, we calculate the transfer matrix $U_{\text{MW}}$ of the photon-ALP system inside the Milky Way for a specific direction. In order to be conservative, we consider our source as placed in the direction of the Galactic pole, where $B_{\text{MW}}$ is smaller and photon-ALP conversion is less efficient than in other directions.

F. Overall photon-ALP beam propagation

By knowing all transfer matrices in each region, we can calculate the total transfer matrix $U$ of the photon-ALP system both in the case where photons are generated in the central region of a galaxy cluster with $U$ reading

$$U = U_{\text{MW}} U_{\text{ext}} U_{\text{clu}},$$

(33)

and the alternative case of photons produced in the jet of a blazar, in which case $U$ is

$$U = U_{\text{MW}} U_{\text{ext}} U_{\text{clu}} U_{\text{host}} U_{\text{jet}}.$$  

(34)

As discussed in Sect. II, the whole photon survival probability can be expressed as

$$P_{\gamma \rightarrow \gamma} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{l=0}^{\infty} \text{Tr} [\rho_{0} U \rho_{l} U^{l}] ,$$

(35)

where $\rho_{0}$ and $\rho_{l}$ read from Eqs. [23] while $\rho_{l}$ is the beam initial polarization density matrix which can be in general unpolarized, partially polarized or fully polarized (see also Sects. II and III for definition and Sect. V for the chosen values). Instead, the whole final photon degree of linear polarization $P_{l}$ is expressed by Eq. (28) where $\rho_{ij}$ are the elements of the final photon density matrix $\rho$ that reads from Eq. (7) by considering its 1-2 submatrix of $2 \times 2$ dimension.

V. PHOTON SURVIVAL PROBABILITY AND PHOTON POLARIZATION

In this Section we analyze the final photon survival probability $P_{\gamma \rightarrow \gamma}$ and the corresponding photon degree of linear polarization $P_{l}$ resulting from the propagation of the photon-ALP beam crossing the several different magnetized media discussed in Sect. IV (blazar jet, host galaxy, galaxy cluster, extragalactic space and Milky Way). In particular, we consider two different scenarios: (i) photons are produced in the central region of a nCC galaxy cluster ($n_{e,0}^{\text{clu}} = 0.5 \times 10^{-2} \text{cm}^{-3}$), (ii) photons are generated at the blazar jet base and will propagate in a CC galaxy cluster ($n_{e,0}^{\text{clu}} = 5 \times 10^{-2} \text{cm}^{-3}$). In both cases we contemplate two possibilities: (i) high value of the extragalactic magnetic field with $B_{\text{ext}} = 1 \text{nG}$ and consequently efficient photon-ALP conversion in the extragalactic space, (ii) $B_{\text{ext}} < 10^{-15} \text{G}$ resulting in a negligible photon-ALP interaction so that photons are actually subjected to EBL absorption only.
As benchmark values concerning the photon-ALP system we take: $g_{\gamma\gamma} = 0.5 \times 10^{-11} \text{GeV}^{-1}$ and the two values for the ALP mass: (i) $m_a \lesssim 10^{-14} \text{eV}$, (ii) $m_a = 10^{-10} \text{eV}$. In the former case ($m_a \lesssim 10^{-14} \text{eV}$), we can see from Eqs. (14) and (16) that the ALP mass term is smaller than the plasma term inside the AGN jet, in the cluster and in the Milky Way for the chosen parameters of the model for all the considered energy range. Instead, in the latter case ($m_a = 10^{-10} \text{eV}$) the mass term dominates over the plasma term in each region apart from the central zone of the blazar jet.

We calculate $P_{\gamma \rightarrow \gamma}$ and $\Pi_L$ along with its corresponding probability density function $f_{\Pi}$ for photons with observed energies $E_0$ in the range $1 \text{eV} < E_0 < 10^{15} \text{eV}$ with $E_0 = E/(1 + z)$ and $z$ being the redshift where photons are produced. In the following, we divide the above energy range into three intervals: (i) X-ray band ($10^{-3}\text{keV} - 10^2 \text{keV}$), (ii) HE band ($10^{-1}\text{MeV} - 10^4 \text{MeV}$), (iii) VHE band ($10^{-2}\text{TeV} - 10^4 \text{TeV}$).

### A. X-ray band

In the energy range ($10^{-3} - 10^2$) keV, if $m_a \lesssim 10^{-14} \text{eV}$, the photon-ALP beam propagates in the weak mixing regime and goes close to the strong mixing regime only in the upper part of the band, as Eq. (19) shows. If $m_a = 10^{-10} \text{eV}$, the ALP mass term effect is very strong, so that photon-ALP conversion is very inefficient to an extent that $P_{\gamma \rightarrow \gamma} \rightarrow 0$. As a result, ALP induced effects on the photon survival probability and on the photon final degree of linear polarization are negligible for $m_a = 10^{-10} \text{eV}$. This is the reason why we consider only the case $m_a \lesssim 10^{-14} \text{eV}$ in this Subsection.

We show our results for the case of photons produced in the galaxy cluster in Figs. 2 and 3 while the results concerning the alternate scenario of photons generated inside the jet of a blazar are reported in Figs. 4 and 5.

We start from the case of photon production in the galaxy cluster central zone. Photons are generated in the cluster via thermal Bremsstrahlung [102, 103]: we take $n_{e,0}^{\text{clus}} = 0.5 \times 10^{-2} \text{cm}^{-3}$ corresponding to a nCC galaxy cluster (see also Sect. IV.C) and we assume photons as initially unpolarized, i.e. with initial degree of linear polarization $\Pi_{L,0} = 0$ [69]. In Fig. 2 we show $P_{\gamma \rightarrow \gamma}$ and the corresponding $\Pi_L$ in the case where the extragalactic magnetic field strength is very small with $B_{\text{extr}} < 10^{-15} \text{G}$ corresponding to an inefficient photon-ALP conversion inside the extragalactic space and in the case $B_{\text{extr}} = 1 \text{nG}$ with sizable photon-ALP conversion. In the energy band considered here ($10^{-3}\text{keV} - 10^2 \text{keV}$), the Universe is transparent to the photon propagation with a very good accuracy, so that in the case $B_{\text{extr}} < 10^{-15} \text{G}$ the transfer matrix of the photon-ALP system in the extragalactic space reduces to $U_{\text{extr}} = \text{diag}(\exp(i\phi_x), \exp(i\phi_y), \exp(i\phi_z))$, where $\phi_x$ and $\phi_x$ are two phases, as there is no substantial mixing between photons and ALPs. As a result, the photon-ALP system is statistically insensible to the source redshift.

Instead, for $B_{\text{extr}} = 1 \text{nG}$ the transfer matrix $U_{\text{extr}}$ remains unitary because there is still no photon absorption but $U_{\text{extr}}$ is no more a diagonal matrix for the increased photon-ALP conversion efficiency: now, the photon-ALP system becomes sensible to the source distance. This is the reason why we consider the two redshifts $z = 0.03$ and $z = 0.4$ when $B_{\text{extr}} = 1 \text{nG}$.

As a general finding of Fig. 2 we observe that the weak mixing regime extends for more than three energy decades ($10^{-2}\text{keV} - 10\text{keV}$): this fact reflects the big variation of the cluster magnetic field strength $B_{\text{clus}}$ and electron number density $n_{e,0}^{\text{clus}}$ expressed by Eq. (31) and (32), respectively, starting from their values in the cluster core up to its border. As a result, the low-energy threshold of the photon-ALP system $E_L$ of Eq. (19) fails to be a single reference energy – as would instead happen in the case of constant magnetic fields and electron number densities – and becomes an interval of energies. We can observe from the first row of Fig. 2 that $P_{\gamma \rightarrow \gamma}$ never decreases down 0.5 as assured by item (ii) of Sect. III (see 67 for more details). In addition, from Fig. 2 we infer that for $E_0 \gtrsim 10^{-2} \text{keV}$ the photon-ALP interaction is efficient (see the first row where $P_{\gamma \rightarrow \gamma}$ is plotted) and produces sizable effects on the final $\Pi_L$ as the second row shows. In addition, it seems that the case of $B_{\text{extr}} < 10^{-15} \text{G}$, of $B_{\text{extr}} = 1 \text{nG}$ with $z = 0.03$ and of $B_{\text{extr}} = 1 \text{nG}$ with $z = 0.4$ are qualitatively similar.

However, this is only superficially true. In Fig. 2 we plot a single realization of the photon-ALP propagation process, which depends on the particular realization of $B_{\text{clus}}$ and $B_{\text{extr}}$ (the variation of $B_{\text{host}}$ and $B_{\text{MW}}$ is subdominant). Since the exact orientation of these fields is unknown and only their statistical properties are, the photon-ALP beam propagation becomes a stochastic process. Nevertheless, we want to stress that what we actually observe is a single realization of the propagation process. This is the reason why we calculate several realizations of the photon-ALP propagation in order to infer its statistical properties. Thus, in Fig. 3 we plot the probability density function $f_{\Pi}$ for the final $\Pi_L$ associated to the different realizations for the two benchmark energies $E_0 = 1 \text{keV}$ and $E_0 = 10 \text{keV}$. As a general result, we observe from Fig. 3 that the photon-ALP interaction produces a variation of the initial $\Pi_{L,0} = 0$ in all the cases and the final value $\Pi_L = 0$ is never the most probable one. The effect of a high $B_{\text{extr}} = 1 \text{nG}$ is to broaden $f_{\Pi}$ and to translate the expectation for the final $\Pi_L$ to larger values. This fact is more evident for $z = 0.4$ with respect to $z = 0.03$ since photons oscillate into ALPs longer in the former case.

In the case of photons generated inside the magnetic field of the jet of a blazar, we take $n_{e,0}^{\text{clus}} = 5 \times 10^{-2} \text{cm}^{-3}$ corresponding to a CC galaxy cluster (see also Sect. IV.C). Photons emitted at the blazar jet base in the energy range considered here ($10^{-3}\text{keV} - 10^2 \text{keV}$) are produced via synchrotron emission with a resulting initial polarization. Nevertheless, photons are not fully polar-
Figure 2: Photon survival probability $P_{\gamma \rightarrow \gamma}$ (upper panels) and corresponding final photon degree of linear polarization $\Pi_L$ (lower panels) in the energy range $(10^{-3} - 10^{2})$ keV from the cluster, where photons are produced, up to us by taking $g_{a\gamma\gamma} = 0.5 \times 10^{-11}$ GeV$^{-1}$, $m_a \lesssim 10^{-14}$ eV and $n_{e,0}^{\text{clu}} = 0.5 \times 10^{-2}$ cm$^{-3}$. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. In the first column an extragalactic magnetic field $B_{\text{ext}} < 10^{-15}$ G is assumed. In the second column we take $B_{\text{ext}} = 1$ nG and a redshift $z = 0.03$. In the third column we consider $B_{\text{ext}} = 1$ nG and $z = 0.4$.

Figure 3: Probability density function $f_{\Pi}$ arising from the plotted histogram for the final photon degree of linear polarization $\Pi_L$ at 1 keV (upper panels) and 10 keV (lower panels) by considering the system described in Fig. 2. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. In the first column an extragalactic magnetic field $B_{\text{ext}} < 10^{-15}$ G is assumed. In the second column we take $B_{\text{ext}} = 1$ nG and a redshift $z = 0.03$. In the third column we consider $B_{\text{ext}} = 1$ nG and $z = 0.4$. 
Figure 4: Same as Fig. 2 but with also photon-ALP conversion within the blazar jet, where photons are produced. Thus, we accordingly take $n_{\text{clu},0} = 5 \times 10^{-2} \text{ cm}^{-3}$. The initial photon degree of linear polarization is $\Pi_{L,0} = 0.3$.

Figure 5: Same as Fig. 3 but with also photon-ALP conversion within the blazar jet by considering the system described in Fig. 4. The initial photon degree of linear polarization is $\Pi_{L,0} = 0.3$. 

The initial photon degree of linear polarization is $\Pi_{L,0} = 0.3$. 

ized and a realistic degree of linear polarization for such photons is expected to be \( \Pi_{\gamma,0} = 0.2 - 0.4 \) as discussed e.g. in [103]. Thus, we assume photons as initially partially polarized with initial degree of linear polarization \( \Pi_{\gamma,0} = 0.3 \). In Fig. 4 we show \( P_{\gamma \rightarrow \gamma} \) and the corresponding \( \Pi_L \) in the case \( B_{\text{ext}} < 10^{-15} \) G and when \( B_{\text{ext}} = 1 \) nG with the source placed at redshifts \( z = 0.03 \) and \( z = 0.4 \), in a similar way as we have done for the case of photon production inside the cluster. What we have previously discussed about \( U_{\text{ext}} \) is still valid in this case so that we do not consider a redshift dependence in the case \( B_{\text{ext}} < 10^{-15} \) G.

From Fig. 4 we observe that the photon-ALP beam propagates in the weak mixing regime in the interval \((10^{-2} - 10) \) keV for the same reason discussed in the case of photon production inside the cluster. Moreover, we note an additional energy dependence also in the \((10 - 100) \) keV decade caused by the behavior of the blazar jet magnetic field \( B_{\text{ext}} \) and of the electron number density \( n_e^{\text{ext}} \) of Eq. (20) and (30), respectively: confirmation of this fact comes from Eq. (19) about the value of \( E_L \) in the jet. From the first row of Fig. 4 we observe that \( P_{\gamma \rightarrow \gamma} \) never decreases down 0.35 as assured by item (i) of Sect. III (see [67] for more details). Furthermore, for \( E_0 = 10^{-2} \) keV the first row of Fig. 4 – where \( P_{\gamma \rightarrow \gamma} \) is plotted – shows that the photon-ALP interaction is efficient and produces sizable effects on the final \( \Pi_L \), modifying the initial \( \Pi_{L,0} = 0.3 \) (see the second row of Fig. 4). At a first sight, the cases of \( B_{\text{ext}} < 10^{-15} \) G, of \( B_{\text{ext}} = 1 \) nG with \( z = 0.03 \) and of \( B_{\text{ext}} = 1 \) nG with \( z = 0.4 \) look qualitatively similar.

What happens in the present situation is totally analogous to the case of photon production inside the cluster: thus, we calculate several realizations of the total stochastic photon-ALP propagation process from the blazar jet base up to the Earth and we report our results about its statistical properties in Fig. 5, where we plot \( f_{\Pi} \) associated to the different realizations for the two benchmark energies \( E_0 = 1 \) keV and \( E_0 = 10 \) keV. Fig. 5 shows that the photon-ALP interaction produces a broadening of the initial \( \Pi_{L,0} = 0.3 \) in all the cases but the final value \( \Pi_L = 0.3 \) still remains the most probable result. For \( E_0 = 1 \) keV the broadening effect on \( f_{\Pi} \) increases by passing from the case \( B_{\text{ext}} < 10^{-15} \) G to that \( B_{\text{ext}} = 1 \) nG and \( z = 0.03 \) and it is even more evident for \( B_{\text{ext}} = 1 \) nG and \( z = 0.4 \), while for \( E_0 = 10 \) keV this trend is less visible.

Concerning the real detectability of the above described features, we must be aware that polarization measurements are more difficult with respect to the flux ones, so that a lower energy resolution is likely: we empirically consider an energy resolution worse by a factor 4 – 5 for polarization observations with respect to flux surveys. Hence, by considering the energy resolution of current flux-measuring X-ray observatories, we expect that 15 – 20 energy bins per decade can be resolved by polarimeters in the X-ray band [104]. Therefore, we expect observatories like IXPE [45] and Polstar [46] to possess enough energy resolution to be able to detect ALP effects on photon polarization and its features especially for \( E_0 \gtrsim (1 - 5) \) keV.

### B. High-energy band

In the energy range \((10^{-1} - 10^{4}) \) MeV, the \( \gamma \gamma \) absorption of HE photons is totally negligible as in the X-ray band, so that what we have stated above about \( U_{\text{ext}} \) still holds true: thus, we have \( U_{\text{ext}} = \text{diag}[\text{exp}(i\phi_\gamma), \text{exp}(i\phi_\gamma), \text{exp}(i\phi_\gamma)] \) in the case \( B_{\text{ext}} < 10^{-15} \) G, while \( U_{\text{ext}} \) is still unitary but \( U_{\text{ext}} \) is no more a diagonal matrix in the case \( B_{\text{ext}} = 1 \) nG for the efficiency of photon-ALP conversion in the extragalactic space. This is the reason why only in the latter situation we consider two possibilities by placing the source (in both the cases of photon emission either in the cluster or inside the blazar jet) at redshifts \( z = 0.03 \) and \( z = 0.4 \). In both the cases of photons produced either inside the cluster or in the blazar jet we assume them as initially unpolarized, i.e. with initial degree of linear polarization \( \Pi_{\gamma,0} = 0.0 \). Several emission mechanisms are believed to contribute to photon production inside the cluster such as inverse Compton scattering and neutral pion decay produced in several ways (see e.g. [106][109]). Concerning the case of emission at the blazar jet base, photons are likely produced via an inverse Compton process, where lower energy photons are boosted to energies in the HE and VHE bands [110][113]. Photons produced by such processes are expected to be unpolarized [114].

We start by considering the case of an ALP with mass \( m_a \lesssim 10^{-14} \) eV. In such a situation, the calculation of \( E_L \) from Eq. (19) and of \( E_H \) from Eq. (20) with the parameters and corresponding profiles considered in Sect. IV concerning the magnetic field and the electron number density in the various crossed regions leads to the conclusion that the photon-ALP beam propagates in the strong mixing regime in almost the entire energy range \((10^{-1} - 10^{4}) \) MeV. Since the photon-ALP system is in the strong mixing regime for \( m_a \lesssim 10^{-14} \) eV, both \( P_{\gamma \rightarrow \gamma} \) and \( \Pi_L \) are energy independent in the HE band, so that only \( f_{\Pi} \), which we plotted in Fig. 6, is really informative and gives us the statistical properties of the several realizations of the propagation process. The strong mixing regime assures that the behavior of \( f_{\Pi} \) is the same for all energies in the range \((10^{-1} - 10^{4}) \) MeV. From Fig. 6 we observe a general trend that is common to both the cases of photon production inside the cluster (first row, where we take \( n_e^{\text{cl}} = 0.5 \times 10^{-2} \) cm\(^{-3} \) corresponding to a nCC galaxy cluster – see also Sect. IV.C) and in the blazar jet (second row, where we assume \( n_e^{\text{cl}} = 5 \times 10^{-2} \) cm\(^{-3} \) corresponding to a CC galaxy cluster – see also Sect. IV.C) and to the various choices of \( B_{\text{ext}} \) and redshifts. In particular, since the system is in the strong mixing regime, the conversion probability is maximal and this fact produces a sizable modification of \( \Pi_L \) with respect to the initial \( \Pi_{L,0} = 0.0 \). For all the cases
The magnetic field \( B \) we take is assumed, while in the second row the photon-ALP beam propagates also inside the blazar jet, where photons are emitted, and we report in Figs. 9 and 10 results for the case of photons produced in the cluster are as we can infer by the calculation of \( E \). In the first column an extragalactic magnetic field \( B_{\text{ext}} < 10^{-15} \text{ G} \) is assumed. In the second column we take \( B_{\text{ext}} = 1 \text{ nG} \) and a redshift \( z = 0.4 \). In the third column we consider \( B_{\text{ext}} = 1 \text{ nG} \) and \( z = 0.03 \). In the first row photons are produced in the galaxy cluster and \( n_{e}^{\text{clu}} = 0.5 \times 10^{-2} \text{ cm}^{-3} \), which corresponds to a nCC galaxy cluster (see also Sect. IV.C). In Fig. [6] we report \( P_{\gamma \rightarrow a \gamma} \) and the corresponding \( \Pi_L \) for the different choices of \( B_{\text{ext}} \) and redshifts. From Fig. [6] we observe that the weak mixing regime extends for the entire energy range analyzed here \((10^{-1} \text{ MeV} - 10^{4} \text{ MeV})\). The very large extent of the energy range, where the weak mixing takes place, is due to the high variety of the properties of the media crossed by the photon-ALP beam (cluster, extragalactic space, Milky Way): as a result, \( E_L \) greatly varies in the different zones. We note from the first row of Fig. [7] that \( P_{\gamma \rightarrow a \gamma} \) never decreases down 0.5 as stated by item (ii) of Sect. III (see [67] for more details). The second row of Fig. [7] shows that the final \( \Pi_L \) turns out to be greatly modified with respect to \( \Pi_{L,0} \) in almost all the energy band by the photon-ALP interaction. As \( P_{\gamma \rightarrow a \gamma} \) confirms, we note a strong energy dependence of \( \Pi_L \): the case of \( B_{\text{ext}} < 10^{-15} \text{ G} \) is qualitatively similar to the ones of \( B_{\text{ext}} = 1 \text{ nG} \) with \( z = 0.03 \) and of \( B_{\text{ext}} = 1 \text{ nG} \) with \( z = 0.4 \) for \( E \lesssim 100 \text{ MeV} \), while they differ for higher energies, where photon-ALP interaction in the extragalactic space produces a sizable effect for \( B_{\text{ext}} = 1 \text{ nG} \).

Figure 6: Probability density function \( f_{\Pi_L} \) arising from the plotted histogram for the final photon degree of linear polarization \( \Pi_L \) for photons in the energy range \((10^{-1} - 10^{4}) \text{ MeV}\) after propagation from the emission zone up to us by taking \( g_{a \gamma} = 0.5 \times 10^{-11} \text{ GeV}^{-1} \) and \( m_a \lesssim 10^{-14} \text{ eV} \). In the first row photons are produced in the galaxy cluster and \( n_{e}^{\text{clu}} = 0.5 \times 10^{-2} \text{ cm}^{-3} \) is assumed, while in the second row the photon-ALP beam propagates in the weak mixing regime in all the energy band considered here \((10^{-1} \text{ MeV} - 10^{4} \text{ MeV})\), as we can infer by the calculation of \( E_L \) of Eq. [19]. Our results for the case of photons produced in the cluster are shown in Figs. [7] and [8] while we report in Figs. [9] and [10] our findings concerning the alternate scenario of photons produced inside the jet of a blazar.

We now move to the case of an ALP with mass \( m_a = 10^{-10} \text{ eV} \). In the present situation, the photon-ALP beam propagates in the weak mixing regime in all the energy band considered here \((10^{-1} \text{ MeV} - 10^{4} \text{ MeV})\), as we can infer by the calculation of \( E_L \) of Eq. [19].
Figure 7: Photon survival probability $P_{\gamma \rightarrow \gamma}$ (upper panels) and corresponding final photon degree of linear polarization $\Pi_L$ (lower panels) in the energy range $(10^{-1} - 10^4)$ MeV from the cluster, where photons are produced, up to us by taking $g_{\nu e} = 0.5 \times 10^{-11}$ GeV$^{-1}$, $m_\nu = 10^{-10}$ eV and $n_{\text{H}} = 0.5 \times 10^{-2}$ cm$^{-3}$. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. In the first column an extragalactic magnetic field $B_{\text{ext}} < 10^{-15}$ G is assumed. In the second column we take $B_{\text{ext}} = 1$ nG and a redshift $z = 0.03$. In the third column we consider $B_{\text{ext}} = 1$ nG and $z = 0.4$.

Figure 8: Probability density function $f_{\Pi}$ arising from the plotted histogram for the final photon degree of linear polarization $\Pi_L$ at 10 MeV (upper panels) and 100 MeV (lower panels) by considering the system described in Fig. 7. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. In the first column an extragalactic magnetic field $B_{\text{ext}} < 10^{-15}$ G is assumed. In the second column we take $B_{\text{ext}} = 1$ nG and a redshift $z = 0.03$. In the third column we consider $B_{\text{ext}} = 1$ nG and $z = 0.4$. 
Figure 9: Same as Fig. 7 but with also photon-ALP conversion within the blazar jet, where photons are produced. Thus, we accordingly take $n_{e,0}^{\text{clu}} = 5 \times 10^{-2} \text{ cm}^{-3}$. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$.

Figure 10: Same as Fig. 8 but with also photon-ALP conversion within the blazar jet by considering the system described in Fig. 9. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. 

\[
\text{Figure 9: Same as Fig. 7 but with also photon-ALP conversion within the blazar jet, where photons are produced. Thus, we accordingly take } n_{e,0}^{\text{clu}} = 5 \times 10^{-2} \text{ cm}^{-3}. \text{ The initial photon degree of linear polarization is } \Pi_{L,0} = 0. 
\]

\[
\text{Figure 10: Same as Fig. 8 but with also photon-ALP conversion within the blazar jet by considering the system described in Fig. 9. The initial photon degree of linear polarization is } \Pi_{L,0} = 0. 
\]
Because of the stochastic nature of the photon-ALP propagation process (see Sect. V.A) and in order to understand the impact of photon-ALP interaction on $\Pi_L$, we plot the probability density function $f_{\Pi_L}$ for the final $\Pi_L$ associated to different realizations for the two benchmark energies $E_0 = 10\, \text{MeV}$ and $E_0 = 100\, \text{MeV}$ in Fig. 8 (we recall that $P_{\gamma\rightarrow\gamma}$ and $\Pi_L$ in Fig. 7 are associated to one realization of the propagation process). From Fig. 8 we infer that the photon-ALP interaction produces a sizable variation of the initial $\Pi_{L,0} = 0$ in all the cases. In addition, the most probable value of $\Pi_L$ is never $\Pi_L = 0$ for $E_0 = 10\, \text{MeV}$, while it remains the most probable one for $E_0 = 100\, \text{MeV}$, when photon-ALP interaction in the extragalactic space is not efficient ($B_{\text{ext}} < 10^{-15} \, \text{G}$ and $B_{\text{ext}} = 1 \, \text{nG}$ with $z = 0.03$).

We now move to the case of photons emitted at the blazar jet base, and we accordingly take $n_{e,0}^{\text{ch}} = 5 \times 10^{-2} \, \text{cm}^{-3}$ which corresponds to a CC galaxy cluster (see also Sect. IV.C). Our results about $P_{\gamma\rightarrow\gamma}$ and the corresponding $\Pi_L$ are reported in Fig. 9 for the different choices of $B_{\text{ext}}$ and redshifts. What we have stated about the extent of the weak mixing regime in the case of photon production inside the cluster still holds true in the present case and similar conclusions about $P_{\gamma\rightarrow\gamma}$ and $\Pi_L$ can be achieved: $\Pi_L$ is greatly modified with respect to the initial value $\Pi_{L,0}$ (see above for more details). We just have to add that photon-ALP interaction inside the magnetic field of the jet modifies in a sizable way the behavior of $P_{\gamma\rightarrow\gamma}$ (first row of Fig. 9) and the corresponding $\Pi_L$ (second row of Fig. 9) for energies smaller than $\sim 0.5\, \text{MeV}$ with respect to the corresponding cases of photon production in the cluster. The reason for this modification is that photon-ALP conversion is efficient inside the blazar jet also for $E_0 \lesssim 0.5\, \text{MeV}$, but the same is not true inside the galaxy cluster. As the first row of Fig. 9 shows, we can check that $P_{\gamma\rightarrow\gamma}$ never decreases down 0.5 as assured by item (ii) of Sect. III (see [67] for more details).

In Fig. 10 we report the probability density function $f_{\Pi_L}$ for the final $\Pi_L$ associated to several realizations of the propagation process (see also Sect. V.A for discussion about the stochastic behavior of the system) for the two benchmark energies $E_0 = 10\, \text{MeV}$ (upper panels) and $E_0 = 100\, \text{MeV}$ (lower panels). The behavior of $f_{\Pi_L}$ is almost independent on the value of $B_{\text{ext}}$ and of the redshift apart from a small broadening increase of $f_{\Pi_L}$ as the redshift grows. For $E_0 = 10\, \text{MeV}$ (upper panels of Fig. 10) $\Pi_L = 0$ is never the most probable value, while for $E_0 = 100\, \text{MeV}$ (lower panels of Fig. 10) the most probable value of $\Pi_L$ turns out to be $\Pi_L \gtrsim 0.6$. From our findings about such strong polarization features, we can conclude that in the HE band, the case of photon production inside the blazar jet represents a better opportunity with respect to photon production inside the galaxy cluster, in order to search for ALP-induced effects on $\Pi_L$.

What we have discussed in the X-ray energy band extends in the HE range about the possibility to detect the polarization features described in this subsection. Thus, by considering the energy resolution of planned flux-measuring observatories in the HE range [48,49], we expect a polarimeter energy resolution of $8 - 10$ energy bins per decade. Therefore, observatories like COSI [77], e-ASTROGAM [48,49] and AMEGO [50] are expected to be able to detect the ALP-induced modifications to photon polarization.

C. Very-high-energy band

First of all, we want to stress that, while for the X-ray and the HE bands our findings can be tested by current and planned observatories [48,50], our results about the photon polarization in the VHE range are nowadays purely theoretical. However, as it will be clear below, some important features about photon polarization linked to the photon-ALP interaction arise. Such features can be used to detect ALPs and/or constrain ALP parameters in case new techniques will hopefully be available in the future to measure photon polarization even in the VHE band.

The calculation of $E_H$ from Eq. (20) is made complicated by the fact that in the energy range ($10^{-2} - 10^3$) TeV VHE photons are absorbed because of their interaction with the EBL photons producing an $e^+e^-$ pair through the process $\gamma\gamma \rightarrow e^+e^-$. This process limits the $\gamma$-ray horizon more and more as the observed VHE photon energy $E_0$ grows [115]. Nevertheless, by inspection of $P_{\gamma\rightarrow\gamma}$ in the following figures we infer that the photon-ALP system is never in the strong mixing regime in almost all the energy band considered here and we can observe that both $P_{\gamma\rightarrow\gamma}$ and the corresponding $\Pi_L$ become energy dependent. In addition, because of the $\gamma\gamma$ absorption $U_{\text{ext}}$ is no more unitary and in the case of both $B_{\text{ext}} < 10^{-15} \, \text{G}$ and $B_{\text{ext}} = 1 \, \text{nG}$ the photon-ALP system is sensible to the distance traveled in the extragalactic space, so that in both the situations we consider the two redshifts $z = 0.03$ and $z = 0.4$. In particular, our results for the case of photons produced in the cluster are plotted in Figs. 11 and 12 for $B_{\text{ext}} < 10^{-15} \, \text{G}$, and in Figs. 13 and 14 for $B_{\text{ext}} = 1 \, \text{nG}$. Similarly, we plot our findings for the case of photons generated in the blazar jet in Figs. 15 and 16 for $B_{\text{ext}} < 10^{-15} \, \text{G}$, and in Figs. 17 and 18 for $B_{\text{ext}} = 1 \, \text{nG}$. As for the X-ray and HE energy band, when photons are emitted in the cluster, we consider $n_{e,0}^{\text{ch}} = 5 \times 10^{-2} \, \text{cm}^{-3}$, which corresponds to a nCC galaxy cluster, while if photons are produced in the blazar jet, we correspondingly take a CC galaxy cluster with $n_{e,0}^{\text{ch}} = 5 \times 10^{-2} \, \text{cm}^{-3}$ (see also Sect. IV.C). Throughout this section we consider an ALP with mass $m_a = 10^{-10} \, \text{eV}$. The case $m_a \lesssim 10^{-14} \, \text{eV}$ slightly differs from the previous one for $E_0 \lesssim 50 \, \text{GeV}$ only, where the system is in the strong mixing regime and $P_{\gamma\rightarrow\gamma}$ and $\Pi_L$ are energy independent. Therefore, what we have found in the HE band in the case of $m_a \lesssim 10^{-14} \, \text{eV}$ perfectly extends here in the VHE range for $E_0 \lesssim 50 \, \text{GeV}$ and
Figure 11: Photon survival probability $P_{\gamma\rightarrow\gamma}$ (upper panels) and corresponding final photon degree of linear polarization $\Pi_L$ (lower panels) in the energy range $(10^{-2} - 10^{3})$ TeV from the cluster, where photons are produced, up to us by taking $g_{\nu\gamma\gamma} = 0.5 \times 10^{-11}$ GeV$^{-1}$, $m_a = 10^{-10}$ eV and $n_{\text{clu},0} = 0.5 \times 10^{-2}$ cm$^{-3}$. An extragalactic magnetic field $B_{\text{ext}} < 10^{-15}$ G is assumed. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. In the first column we take a redshift $z = 0.03$, while in the second column we consider $z = 0.4$.

Figure 12: Probability density function $f_{\Pi_L}$ arising from the plotted histogram for the final photon degree of linear polarization $\Pi_L$ at different energies (see subfigures) by considering the system described in Fig. 11. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. In the first column we take a redshift $z = 0.03$, while in the second column we consider $z = 0.4$. 
Figure 13: Same as Fig. 11 but by considering an extragalactic magnetic field $B_{\text{ext}} = 1 \, \text{nG}$. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$.

Figure 14: Same as Fig. 12 but by considering the system described in Fig. 13. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. 
Figure 15: Same as Fig. 11 but with also photon-ALP conversion within the blazar jet, where photons are produced. Thus, we accordingly take $n_{\text{clu},0} = 5 \times 10^{-2} \text{ cm}^{-3}$. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$.

Figure 16: Same as Fig. 12 but with also photon-ALP conversion within the blazar jet by considering the system described in Fig. 15. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. 
Figure 17: Same as Fig. 11 but with also photon-ALP conversion within the blazar jet, where photons are produced. Thus, we accordingly take $n_{e,0} = 5 \times 10^{-2}$ cm$^{-3}$. Here we consider $B_{\text{ext}} = 1$ nG. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$.

Figure 18: Same as Fig. 12 but with also photon-ALP conversion within the blazar jet by considering the system described in Fig. 17. The initial photon degree of linear polarization is $\Pi_{L,0} = 0$. 

\( m_a \lesssim 10^{-14} \text{eV} \). Hereafter, we concentrate on the case \( m_a = 10^{-10} \text{eV} \).

What we have discussed for the HE band about the initial photon degree of linear polarization \( \Pi_{L,0} \) still holds true for the VHE range, so that we assume photons as initially unpolarized with initial degree of linear polarization \( \Pi_{L,0} = 0 \) in both the cases of photons produced either inside the cluster or in the blazar jet.

In all figures concerning \( P_{\gamma \rightarrow \gamma} \) we observe that \( P_{\gamma \rightarrow \gamma} \) starts to decrease in a sizable way because of the EBL \( \gamma \gamma \) absorption for \( E_0 \gtrsim 3 \text{ TeV} \) at \( z = 0.03 \) and for \( E_0 \gtrsim 200 \text{ GeV} \) at \( z = 0.4 \). Correspondingly, we observe an increase of \( \Pi_L \) up to the limit value \( \Pi_L = 1 \) – with photons totally polarized – for \( E_0 \gtrsim 30 \text{ TeV} \) in the case \( z = 0.03 \) and for \( E_0 \gtrsim 5 \text{ TeV} \) in the case \( z = 0.4 \): this fact takes place where the associated \( P_{\gamma \rightarrow \gamma} \lesssim 10^{-2} \). The reason for this behavior concerning \( \Pi_L \) lies in the growing \( \gamma \gamma \) absorption due to the EBL as \( E_0 \) increases. For energies where absorption is not dramatic – \( E_0 \lesssim 3 \text{ TeV} \) at \( z = 0.03 \) and \( E_0 \lesssim 200 \text{ GeV} \) at \( z = 0.4 \) – what happens in the VHE range is totally similar to the X-ray and HE band. In fact, we observe that, when absorption is not too high, \( \Pi_L \) moderately increases above the initial value \( \Pi_{L,0} = 0 \) showing an energy-dependent behavior, since the photon-ALP system is in the weak mixing regime. In the case of high absorption, instead, what takes place can be visualized as follows. When photons are produced (either in the cluster or in the blazar jet), they partially convert into ALPs while crossing the magnetized media close to the source (blazar jet magnetic field \( B^\text{ext} \) and/or galaxy cluster magnetic field \( B^\text{clu} \)), so that before the photon-ALP beam propagates inside the extragalactic space is made of both photons and ALPs. Conversion inside the extragalactic space may take place \( (B^\text{ext} = 1 \text{nG}) \) or not \( (B^\text{ext} < 10^{-15} \text{G}) \), but it is in any case not fully efficient because the system lies in the weak mixing regime. Therefore, while photons are almost totally absorbed, a sizable amount of ALPs survives up to the Milky Way, where they can reconvert back to photons inside the magnetic field of the Milky Way \( B^\text{MW} \). Since what is efficient for photon-ALP conversion inside the Milky Way is the coherent part of \( B^\text{MW} \), photons reconverted back from ALPs inside the Milky Way are fully polarized. This behavior is valid at all energies, in the X-ray, HE and VHE band, but in case of no/low absorption (X-ray and HE band) it is hidden by the presence of the photons that oscillate into ALPs in other regions outside the Milky Way. Instead, when absorption is very high – i.e. in the VHE band – almost all photons apart from those reconverted back from ALPs in the Milky Way are absorbed in the extragalactic space because of their interaction with the EBL. This is the reason why \( \Pi_L \) grows towards the limit value \( \Pi_L = 1 \) as the photon energy grows, in the same energy range where \( \gamma \gamma \) absorption due to the EBL grows as well.

We can observe that Figs. 15 and 17 showing \( P_{\gamma \rightarrow \gamma} \) and \( \Pi_L \) for photon production in the cluster for the cases of \( B^\text{ext} < 10^{-15} \text{G} \) and \( B^\text{ext} = 1 \text{nG} \), respectively and Figs. 13 and 14 showing \( P_{\gamma \rightarrow \gamma} \) and \( \Pi_L \) for photon production in the blazar jet for the cases of \( B^\text{ext} < 10^{-15} \text{G} \) and \( B^\text{ext} = 1 \text{nG} \), respectively are all qualitatively similar and described by the behavior discussed above. We note that, when \( B^\text{ext} = 1 \text{nG} \), the photon-ALP conversion in the extragalactic space produces more oscillations in \( P_{\gamma \rightarrow \gamma} \) and \( \Pi_L \) with respect to the energy if compared to the case \( B^\text{ext} < 10^{-15} \text{G} \).

In order to infer the statistical properties of the photon-ALP system, we analyze the probability density function \( f_{\Pi} \) of \( \Pi_L \) associated to several realizations of the photon-ALP propagation process. Thus, for different energies, we plot in Figs. 12 and 14 \( f_{\Pi} \) for photon production in the cluster in the cases of \( B^\text{ext} < 10^{-15} \text{G} \) and \( B^\text{ext} = 1 \text{nG} \), respectively, and we report in Figs. 10 and 15 \( f_{\Pi} \) for photon generation in the blazar jet in the cases of \( B^\text{ext} < 10^{-15} \text{G} \) and \( B^\text{ext} = 1 \text{nG} \), respectively. In particular, in all the above-mentioned figures we consider \( E_0 = 500 \text{ GeV} \) and \( E_0 = 30 \text{ TeV} \) when \( z = 0.03 \) and \( E_0 = 100 \text{ GeV} \) and \( E_0 = 2 \text{ TeV} \) when \( z = 0.4 \). We take lower energies when the redshift grows since EBL \( \gamma \gamma \) absorption increases with the enhancement of both energy and distance, so that the behavior of \( f_{\Pi} \) at the two redshifts becomes comparable for the energies considered. Correspondingly, in all the figures about \( f_{\Pi} \) we have low absorption in both the situations of \( E_0 = 500 \text{ GeV} \) with \( z = 0.03 \) and \( E_0 = 100 \text{ GeV} \) with \( z = 0.4 \): in the present situation photon-ALP conversion broadens the final \( \Pi_L \) making \( \Pi_L > 0 \) the most probable value in all the figures apart from the case of photon production in the cluster and \( B^\text{ext} < 10^{-15} \text{G} \). Instead, in both the situations of \( E_0 = 30 \text{ TeV} \) with \( z = 0.03 \) and \( E_0 = 2 \text{ TeV} \) with \( z = 0.4 \) EBL absorption is very high, so that the greatest part of detectable photons are those reconverted back from ALPs inside the Milky Way, as discussed above. In fact, in the present case the most probable value for \( \Pi_L \) becomes \( \Pi_L \gtrsim 0.8 \). Obviously, by increasing \( E_0 \) also \( \Pi_L \) grows up to its limit value \( \Pi_L = 1 \). In the presence of low absorption the most probable value for the final \( \Pi_L \) is higher in the case of photon generation inside the blazar jet than in the case of photon production in the cluster. As already observed in the X-ray and HE band, the effect of \( B^\text{ext} = 1 \text{nG} \) is to broaden the value of \( \Pi_L \).

VI. CONCLUSIONS

In this paper, we have studied the propagation of the photon-ALP beam up to the Earth when photons are produced in the central region of a nCC galaxy cluster \( (n^\text{clu} = 0.5 \times 10^{-2} \text{ cm}^{-3}) \) and when they are generated in the jet of a blazar by accordingly considering a hosting CC galaxy cluster \( (n^\text{clu} = 5 \times 10^{-2} \text{ cm}^{-3}) \). We have analyzed all the magnetized media crossed by the beam: the blazar jet, the host galaxy, the galaxy cluster, the extragalactic space and the Milky Way. We have considered the case of both efficient \( (B^\text{ext} = 1 \text{nG}) \) and negligible \( (B^\text{ext} < 10^{-15} \text{G}) \) photon-ALP conversion in the
extragalactic space. In the presence of photon-ALP interaction, we have then calculated the photon survival probability \( P_{\gamma\gamma} \) and the corresponding photon degree of linear polarization \( \Pi_L \) by taking physically consistent values for the parameters concerning both the crossed media (magnetic field, electron number density and their profiles) and the photon-ALP system with \( g_{a\gamma\gamma} = 0.5 \times 10^{-11} \, \text{GeV}^{-1} \) and two cases concerning the ALP mass: (i) \( m_a \lesssim 10^{-14} \, \text{eV} \), (ii) \( m_a = 10^{-10} \, \text{eV} \). We have considered three energy ranges: (i) X-ray band \((10^{-5} \text{keV} - 10^2 \text{keV})\), (ii) HE band \((10^{-1} \text{MeV} - 10^4 \text{MeV})\), (iii) VHE band \((10^{-2} \text{TeV} - 10^3 \text{TeV})\). While our results about the first two energy ranges can be tested by current and planned observatories \([45, 50]\), our findings in the VHE band are currently of theoretical nature. We have checked that our results about \( P_{\gamma\gamma} \) and \( \Pi_L \) satisfy the theorems linking conversion/survival probability and photon polarization, which have been enunciated and demonstrated in \([67]\). Our results can be summarized as follows.

(i) In the X-ray band, we take an initial photon degree of linear polarization \( \Pi_{L,0} = 0 \) for the case of photon production in the cluster and \( \Pi_{L,0} = 0.3 \) for the case of photon generation in the blazar jet, as explained in Sect. V.A. If \( m_a = 10^{-10} \, \text{eV} \), the photon-ALP conversion is very inefficient so that \( P_{\gamma\gamma} \rightarrow 0 \) and ALP induced effects on the photon final polarization are negligible. If \( m_a \lesssim 10^{-14} \, \text{eV} \) the photon-ALP beam propagates in the weak mixing regime for almost all the energy interval: \( P_{\gamma\gamma} \) and the corresponding \( \Pi_L \) show oscillations with respect to the observed energy \( E_0 \). The probability density function \( f_{\Pi} \) of \( \Pi_L \) associated to several realizations of the photon-ALP propagation process shows that in all considered cases \( \Pi_L \) is modified, broadened and its most probable expectation translates to a higher value with respect to the initial \( \Pi_{L,0} \).

(ii) In the HE band, we consider \( \Pi_{L,0} = 0 \) for both the cases of photon production in the cluster and in the blazar jet, as explained in Sect. V.B. If \( m_a \lesssim 10^{-14} \, \text{eV} \) in this energy interval the photon-ALP beam propagates in the strong mixing regime, so that \( P_{\gamma\gamma} \) and \( \Pi_L \) are energy independent. In all considered cases, \( f_{\Pi} \) shows a modification and broadening of the values assumed by \( \Pi_L \) with respect to the initial \( \Pi_{L,0} \). The most probable expectation for the final \( \Pi_L \) is \( \Pi_L \gtrsim 0.8 \) but with a wide broadening. This fact can be understood because of the efficiency of the photon-ALP conversion that occurs in the strong mixing regime. Instead, if \( m_a = 10^{-10} \, \text{eV} \) the photon-ALP system lies in the weak mixing regime with a resulting oscillatory behavior with respect to the observed energy \( E_0 \) of both \( P_{\gamma\gamma} \) and \( \Pi_L \). The probability density function \( f_{\Pi} \) of \( \Pi_L \) shows that, in the case \( m_a = 10^{-10} \, \text{eV} \), \( \Pi_L \) is less modified.

(iii) In the VHE band, we take \( \Pi_{L,0} = 0 \) for both the cases of photon production in the cluster and in the blazar jet, as discussed in Sect. V.C. For almost all the considered energy interval, the photon-ALP beam propagates in the weak mixing regime for both the cases \( m_a \lesssim 10^{-14} \, \text{eV} \) and \( m_a = 10^{-10} \, \text{eV} \), so that \( P_{\gamma\gamma} \) and \( \Pi_L \) show oscillations with respect to \( E_0 \). In addition, \( f_{\Pi} \) still shows a modification and broadening of the values assumed by \( \Pi_L \) with respect to the initial \( \Pi_{L,0} \) but with a difference with respect to the previous energy intervals. In the VHE band \( \gamma\gamma \) absorption caused by the EBL decreases the amount of photons which can be detected at the Earth. Therefore, we find a peculiar feature: when absorption is very high, all photons are absorbed in the extragalactic space, so that only photons reconverted back from ALPs in the Milky Way can be detected. In this case, the corresponding \( \Pi_L \) increases to very high values up to the limit \( \Pi_L = 1 \) with almost no broadening, as shown by \( f_{\Pi} \). Thus, a detection of fully polarized photons would represent a proof for the existence of ALPs with the properties discussed in this paper. However, the possibility of such a detection is nowadays only a hope for the future, since current techniques to measure photon polarization reach a few tens of GeV at most \([110]\).

We want to stress that we have assumed physically consistent parameters about the media crossed by the photon-ALP beam. By considering different values and profiles concerning the magnetic field and the electron number densities (e.g. in the galaxy cluster), all our findings still hold true but with a translation to lower/higher energies of the weak mixing regime.

As discussed above, observatories in the X-ray and HE bands \([45, 50]\) are expected to possess a sufficient energy resolution to be able to detect the photon polarization features induced by ALPs and analyzed in this paper (see also \([117]\)). Still, we plan to study the actual detectability in a further publication.

When photon polarization accurate data will be available, their analysis will be crucial, in order to understand their physical origin and to distinguish among several possibilities. In particular, since photons originated in the central region of a galaxy cluster are expected to be unpolarized both in the X-ray and HE bands, a detection of \( \Pi_L > 0 \) would represent a hint for new physics. Although Lorentz invariance violation (LIV) induces a variation to the final \( \Pi_L \), LIV has the tendency of reducing \( \Pi_L \) \([118]\), so that a detection of \( \Pi_L > 0 \) for photons produced in the central zone of a galaxy cluster would invariably represent a hint for the existence of an ALP. In the case of photon generation in the blazar jet, the situation is more involved: for photons in the HE range everything we have just stated above still holds true since \( \Pi_{L,0} = 0 \). Instead, in the X-ray band since \( \Pi_{L,0} = 0.2 - 0.4 \), a final \( \Pi_L \lesssim 0.1 - 0.2 \) would represent a hint for LIV, while a detected \( \Pi_L \gtrsim 0.4 - 0.5 \) would imply an indication for the existence of an ALP.

Finally, ALPs with the properties considered in this paper can be observed by the new genera-
tion of gamma-ray observatories such as CTA [119], HAWC [120], GAMMA 400 [121], LHAASO [122], TAIGA-HiSCORE [123] and HERD [124]. Moreover, these ALPs can be directly detected by laboratory experiments like the upgrade of ALPS II at DESY [125], the planned IAXO [126, 127] and STAX [128], and with other techniques developed by Avignone and collaborators [129, 131]. In addition, if ALPs are the greatest constituents of the dark matter, they can also be detected by the planned ABRACADABRA experiment [132].

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Other processes discussed in [133] are totally irrelevant.

Concerning photon-ALP system parameters ($m_a$, $g_{a\gamma\gamma}$), although the only firm bound is that derived by CAST [51], there are indications that the model ($m_a = 10^{-10} \text{eV}$, $g_{a\gamma\gamma} = 0.5 \times 10^{-11}$ GeV$^{-1}$) is preferable with respect to that ($m_a \lesssim 10^{-14} \text{eV}$, $g_{a\gamma\gamma} = 0.5 \times 10^{-11}$ GeV$^{-1}$). Yet, the model ($m_a \lesssim 10^{-14} \text{eV}$, $g_{a\gamma\gamma} = 0.5 \times 10^{-11}$ GeV$^{-1}$) cannot be excluded.

Other models with a bigger number of parameters with respect to those entering Eq. (32) are employed to describe $n_{k,cl}$ especially concerning CC clusters (see e.g. [79, 81]). However, we have checked that the final results about $P_{\gamma \gamma}$ and $I_{\gamma \gamma}$ in the presence of photon-ALP interaction are not substantially affected by the utilized model. Thus, we always employ Eq. (32) to describe $n_{k,cl}$ in order to reduce the number of the system parameters.

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