PRINCIPLES AND APPLICATIONS OF DIFFERENCE IMAGING

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The principles of difference imaging outlined and the technique of Alard and Lupton (1997) is generalised to generate the best possible difference images to within the limits of measurement error. It is shown how for a large database of images we can approach diffraction-limited spatial resolution to within measurement error of a combined image. This is achieved through an iterative procedure of difference imaging and deconvolution. The design of the ideal imager to approach this limit is discussed.
1. INTRODUCTION

This work reviews a number of techniques for image processing and suggests how they can be put together to maximise spatial resolution in generalised imaging systems.

The field of microlensing has made tremendous progress since Paczynski’s 1986 paper on the subject. This has largely driven new techniques to improve the detection efficiency of microlensing events (e.g., Tomaney & Crotts 1996, based on the earlier work of Ciardullo, Tamblyn & Phillips 1990 and Phillips & Davis 1995) which have general application to detecting variability of any kind in an image. Recently Alard & Lupton (1997) outlined an elegant algorithm for difference imaging. Here we develop these ideas further.

2. PRINCIPLES OF DIFFERENCE IMAGING

We review the general principles of difference imaging and mention some specifics in the context of microlensing.

1. There is no object in the sky that is not blended to some finite extent.

2. However, if this object varies and we perform a difference image analysis it can be isolated as being one or some combination of the following things:

   (a) a single star that has intrinsically varied or has been microlensed,

   (b) a proper motion of a star on the sky. This might be seen as a bipolar residual Point Spread Function (PSF) that has spatially separate negative and positive components in the Difference Image (DI) (this effect depends on the flux ratio of the star in the pre-subtracted image and reference image and is largest when the ratio is unity),

   (c) an apparent astrometric shift due to microlensing redistributing the star’s light on the sky as the lens transverses the line of sight,
(d) a superposition of more than one star that is photometrically and/or astrometrically varying (because of any of the above or simply the intrinsic variability of the star(s) within the detected PSF on the DI),

(e) a purely atmospheric effect which shifts the star’s light in ways that depend on its spectrum (i.e. “colour”) and the observation taken at a given airmass and parallactic angle (see Filippenko 1982). (The atmospheric effect, however, can be addressed to some extent for every pixel in the DI analysis since we can know each pixel’s colour, airmass, and parallactic angle and can correct for the first order effect of the location of the object centroid in the bandpass observed in which depends on the spectrum as well as the second order effect associated with the PSF shape dependence on the object’s colour, airmass and parallactic angle. However, the accuracy is limited by how much of the pixel’s light actually comes from one star and the problems of finite bandpasses where multi-band photometry indicates two stars which actually have different spectra are measured to have the same apparent colour.)

(f) a purely local phenomenon: examples would be detector defects (especially including non-linearity in detector response), satellite tracks and many other artificial phenomena.

Note that a residual that is significant on a DI can be distinguished from a star by its consistency or otherwise by the PSF appropriate for the DI. This can be used as a discriminator between cosmic rays and CCD defects in a formal manner, except a star that has varied during the exposure will have a PSF that is inconsistent with the mean PSF for all non-varying stars in the frame (an extreme example is a cosmic ray). However, this discrimination is only true if the star varies very slowly during the duration of the exposure, since the atmospheric conditions and instrument focus are constantly changing during the exposure.
To the extent that many of the effects in (a) to (e) are small, a DI largely removes blending with other stars so the photometric effect of apparent colour shifting in the microlensing case (e.g., Kamionkowski 1995) and the associated apparent astrometric centroid shift is removed.

2. Note that for the two extremes of variability: periodic and single exclusions (such as microlensing) we can get centroids that are good to the accuracy of centroid measurement on a single frame, $dx, dy$, divided by $\sqrt{N_{\text{frames}}}$ in the first case and $\sqrt{N_{\text{peak-frames}}}$ in the second (assuming $dx, dy$ are the same for every measurement) allowing for accurate centroiding depending on how large $N$ is in each case.

3. If we compare two bandpasses at wavelengths $\lambda_1$ and $\lambda_2$ then, barring finite source effects and limb darkening (Alcock et al. 1997), the following must be true for microlensing:

$$\frac{F(\lambda_1, t) - F_0(\lambda_1, t_{\text{baseline}})}{F(\lambda_2, t) - F_0(\lambda_2, t_{\text{baseline}})} = \text{constant},$$

(1.1)

to within the limits of measurement, in contrast to the blended case which has an extra term (the zero-point) to be solve for when plotting $F(\lambda_1, t)$ against $F(\lambda_2, t)$.

4. We can speak only in terms of the one parameter we can measure in the DI Light Curve (DILC), the FWHM of the curve, $t_{\text{fwhm}}$, not the Einstein crossing time, $t_E$.

5. We can use 1-3 to attempt to solve for the amplitude, $A$, of the event with the best available high-resolution image.

6. We can use the DILC together with the DI Astrometric Curve (DIAC) to attempt to break the degeneracies involved and solve for the projected Einstein radius, $\theta_E$.

7. We still have blending when we go to the “source” in the baseline image to solve for $A$. It helps if we can get an high-resolution image as the star is being lensed to solve for $A$. 
We need to measure the Luminosity Function (LF) to determine an amplitude detection efficiency for a given surface brightness, $S$, unless the LF is a monotonic power law (Gould 1996). The LF can be measured from fits to the skewness of the intensity value histogram of the surface brightness fluctuations (Tomaney and Crotts 1996). However, for those light curves in which we have large $S/N (> 100)$ the timescale/amplitude degeneracy can be broken (Gould 1996).

3. DIFFERENCE IMAGING

Consider a Gedanken experiment involving an image, $r$, taken at a location above the atmosphere with a detector $D$ which is to be compared to a similar image, $i$, taken with the same detector at the same time where the centroids of the sources imaged in both cases are spatially coincident on the two detectors. There must be a convolution kernel, $k$, describing the mapping of photons on one image to the other (largely describing the smearing effects of the atmosphere and instrumental focus on the photon paths). This convolution kernel is spatially dependent in the image plane due to atmospheric effects and aberrations of the imager. Thus,

$$i = r \otimes k,$$  \hspace{1cm} (3.1)

If the Fourier Transform (FT) of $r$, $i$ and $k$ are $R$, $I$ and $K$ respectively, the Convolution Theorem states,

$$I = KR,$$  \hspace{1cm} (3.2)

Thus, if $K^{-1}$ is the inverse of the matrix, $K$,
\[ K^{-1}I = K^{-1}KR = R, \]  
(3.3)

and,

\[ r = \text{FT}(K^{-1}I), \]  
(3.4)

Generally,

\[ i(x') = [k(x) \otimes r(x)]s(x) + b(x), \]  
(3.5)

where \( x \) are the reference image coordinates and \( x' \) are the image coordinates. The transformation,

\[ \text{dx} = x - x'; \text{dy} = y - y', \]  
(3.6)

represents the geometric registration of the image \( r \) to \( i \). This transform generally is of low order for most imaging telescopes. Note that for a detector, \( D \), that images simultaneously in two bandpasses every pixel can be defined with a colour (bandpass flux ratio after subtraction of the sky), then to first order (blending issues aside) this transformation may also contain for every pixel the correction for airmass (and parallactic angle, since airmass has a dual degeneracy in this term) to align pixels taken at different airmasses to that above the atmosphere.

The remaining terms of equation (3.5) above are \( s(x) \), the photometric (scaling) normalisation and \( b(x) \) the sky difference (zero-point correction) and the convolution kernel, \( k(x) \) to bring the reference image to the same Point Spread Function (PSF) as the
ground-based image, \( i \). All these terms are functions of \( x \), but note that like the PSF for a given location in \( x \) the convolution kernel is infinite in spatial extent.

Let us assume that our detector is a photon counting device (such as a CCD detector) which detects photons in spatially discrete bins (henceforth referred to as pixels). For simplicity we choose an array of such pixels of \( N \times N \) in size. We can now follow the technique of Alard and Lupton (1997) and use equation (3.5) to express the intensity of every pixel in image \( i \) as a function of the intensity of every pixel another ground-based image, \( i' \). (Note that this approach is working directly on the raw data of both images respectively.) If we use \( N_{dx}, N_{dy}, N_s, N_b, \) and \( N_k \) coefficients to solve equation (3.5), then the number of elements which can be solved for in the \( k \) matrix is given by,

\[
\frac{N^2}{2N_{dx}N_{dy}N_sN_bN_k^2},
\]

In the experience of this author for a 2K by 2K CCD image in a number of wide-field telescopes all of the terms in the denominator are quite small compared with the extent of the kernel which can be solved for up to a maximum of 1K by 1K if all of these terms are unity. Given the complexity of a typical PSF in an imaging system (due to diffraction annuli from the optical components as well as diffraction spikes from “spiders” holding the secondary mirror in place in a typical Cassegrain design telescope) it would appear sensible to make few assumptions about the actual shape of the PSF (and consequently the matching kernel, \( k \)). However, we can probably converge on a solution for \( k \) more quickly by making reasonable assumptions about the actual shape of the PSF and the kernel by expanding its polynomial expression with a basis function that best describes the gross properties of the PSF; in an astronomical image this is typically a Gaussian (Alard and Lupton 1997).
3.1. Algorithm for the Practical Estimation of $k$

We can estimate $k$ to the extent that we are limited by the following four critical aspects of measurement,

1) The truncation effect of $N^2$ pixels on a detector which effectively constrains our ability to measure the infinite extent of $k$ to $(N/2)^2$ pixels in size.

2) The photon and detector noise (typically the read-out noise in a CCD, but often will include the systematic errors in the calibration of the data).

3) The discrete sampling of the PSF by the detector elements.

4) A source which is spatially or photometrically variable during the exposure.

Let us consider differencing our best spatial resolution image, $i_r$, against all other $(n - 1)$ images that we have in a time-series. The steps for doing this are,

(i) Solve for $k_n$ for image $n$ (performed only on the raw data of both $i_r$ and $i_n$).

(ii) Calculate the transformation for $i_r$ to $i_n$ using equation (3.5).

(iii) Generate a Difference Image (DI) by subtracting the transformed $i_r$ and $i_n$ (note that at this point we have introduced a resampling noise penalty due to the necessity of interpolating that data).

(iv) To the extent to which the residuals in the DI are statistically significant those measurements are no longer considered, reducing the number of equations to be solved for from $N^2$ to $N'$. This statistical significance test must be made relative to one of the two images, since the DI contains structure due to the noise of sources above sky, even in the ideal case of perfect subtraction.

(v) The total number of elements which $k$ now can be solved for is given by equation
(3.7) with $N^2$ replaced by $N'$. Thus $k$ is recalculated with these measurements.

(vi) The $N^2 - N'$ bad pixels are interpolated over in both the raw images $i_r$ and $i_n$ at the geometric location $x$ in the former and $x'$ in the latter image using the $N'$ good pixels.

(vii) Steps (ii) to (v) are repeated using the “cleaned” images from step (vi) until all remaining measurements are statistically insignificant (step iv). Thus we iterate no further beyond step (v) and have solved for $k$ to within the four limitations above.

4. IMAGE DECONVOLUTION

We now consider how to improve the spatial resolution of individual images and then a combined image.

4.1. Algorithm for deconvolving images to the spatial resolution of the $i_r$ image

(i) Take the FT of the “cleaned” $i_n$ raw image to yield $I_n$.

(ii) Take the FT of the kernel, $k_n$, determined for $i_n$ to yield $K_n$.

(iii) Invert $K_n$ to $K_n^{-1}$.

(iv) Derive an image $(i_r)_n$ which is the FT of $(K_n^{-1}I_n)$, i.e., the equivalent measurement of the $n$’th image at the spatial resolution of the $i_r$ image.

To test whether $K$ has been solved for to within the limits of measurement error the significance of the residuals in the DI of $i_r$ minus $(i_r)_n$ can be tested. However, to do this properly we must geometrically and photometrically transform the image simultaneously (using equation 3.5), thereby introducing some systematic noise. Nevertheless these effects
can be made very small.

4.2. Algorithm for deconvolving images to approach the diffraction limited spatial resolution of the detector

Now imagine we have a library of many images taken under many observing different conditions, but we have deconvolved (or convolved) all the images to a common spatial resolution using the steps outlined above. We wish to use these to try to achieve diffraction limited spatial resolution. One of the biggest hurdles to overcome is that many images are heavily undersampled compared with the diffraction limit. Here we suggest a simple method to overcome this effect.

The angular size, $\theta$, of the minimum of flux between the centre of a PSF and the first diffraction order for a mirror or lens is given by the expression,

$$\theta = B \frac{\lambda}{d}, \quad (3.8)$$

where $\lambda$ is the wavelength of the photons being measured and $d$ is the physical size of detector. For a circular mirror or lens the constant $B$ has a value of roughly 1.22. Within this ring we cannot distinguish between two point sources (represented by delta functions) and an extended source.

Unfortunately detectors do not sample $\theta$ with infinitely discrete detecting elements. We can, however, extrapolate our images to arbitrarily high resolution and use the information contained in $n$ deconvolved $(i_r)_n$ images taken at $n$ $\mathbf{x'}$ discrete locations to compensate for this effect. A high signal-to-noise $(S/N)$ combined image, $R$, can be made by selecting those subpixels that correspond to locations in $\mathbf{x}$ that suffer a minimal resampling noise penalty in the transformation from $\mathbf{x'}$ to $\mathbf{x}$ (e.g., Mukai 1990) (in addition to corresponding
to original image $i$ pixels that have passed a DI statistical insignificance test). An optimal choice must be made between the degree of subpixel sampling and the attainment of the best S/N for each subpixel in the final combined image in $x$ coordinates for the $n$ images considered.

By averaging together our measurements at the $x$ pixel or subpixel location of the $(i_r)_n$ image (or extrapolated image) and appropriately weighting the measurements with the inverse variance of the noise associated with each pixel, we can derive an $R$ of optimum S/N which can suppress some systematic effects associated with the detector such as the variation of the quantum efficiency across an individual detector pixel as well as the resampling noise and other sources of systematic noise associated with each subpixel for the $(i_r)_n$ images comprising $R$.

The intrinsic spatially varying PSF ($x$) needs to be determined for an $R$ image. In astronomical images this can be done with standard algorithms such as DAOPHOT or DoPHOT which are described in detail in Stetson 1987 and Schechter, Mateo & Saha 1993 respectively. Rather than detail the algorithms here we point out that after sources are detected on the image and an estimate for the PSF is made at location $x$ the sources must be subtracted from the image to within the measurement errors in order for the best possible estimate for the PSF at ($x$) to be made. Again this is a similar iterative procedure involving detection, PSF estimation and source subtraction until all sources are removed to within the limits of measurement. Once we have made an estimate of PSF ($x$) we may proceed with deconvolution of the image $R$ at that $x$ location to allow us to approach a diffraction-limited image to within the limits of our measurement error (e.g., Lucy 1992).
5. SUMMARY

Given the above arguments, the ideal imaging system comprises simply a re-imaging optic which brings to focus (although not necessarily a perfect focus) a beam of photons onto a detector. In the case where the re-imaging optic is a mirror (thereby suffering from minimal achromatic aberration) this detector is best removed from the incoming photon path by tilting the mirror to bring the image to a focus off-axis. This removes the strongly complicating effects on the final PSF and matching convolution kernel of diffraction due to other optical elements, but does introduce strong spatially dependent aberrations in the PSF. Image distortion due to this tilt can be reduced with a compensating tilt of the detector in the the focal plane of the imager. Diffraction-limited images can then be obtained to within the limits of measurement error by an iterative process of image differencing and deconvolution for a large database of images. It should be made clear that in both a difference image and a deconvolved image we have not in any way improved the signal-to-noise of a detection (and may have paid a penalty due to “imperfect” image processing) so we are still motivated to increase our signal-to-noise ratio of our final combined and deconvolved image by ensuring the tightest possible focus of a point source on the detector in an original image.

It is interesting to note that this re-imaging optic may comprise a gravitating body, which acts as an achromatic (but spherically aberrated) lens.

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