Ward identities for extended objects

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(Dec. 28, 2012)

Abstract

Ward identities for extended objects are discussed. In the limit of dc transport it is rigorously proved that charge-density and spin-density fluctuations do not couple to electromagnetic field.

1 Introduction

In most cases of condensed matter physics Ward identities have been discussed for single-particle vertices [1, 2]. Recently I have reported the discussions of Ward identities for particle-particle or particle-hole pairs [3, 4, 5].

In this note I will summarize the essential point for extended pairs. Following description is based on refs. [3, 4, 5].

2 Ward identity for electric current vertex

The central quantity in the discussions of Ward identity for electric current vertex is the equal-time commutation relation between the electric charge

\[ j^e_{\vec{k}} = e \sum_{\vec{p}} \left( a^\dagger_{\vec{p} - \vec{k}} a_{\vec{p}} + b^\dagger_{\vec{p} - \vec{k}} b_{\vec{p}} \right), \]

and the object which comes in or go out of the vertex.

If the object is an electron, then the equal-time commutation relation is estimated as

\[ [j^e_{\vec{k}}, a^\dagger_{\vec{p}}] = ea^\dagger_{\vec{p} - \vec{k}}. \]

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where \( a^\dagger_p \) is the creation operator of an electron. If the object is a local
Cooper pair,
\[
[j^e_{\vec{k}}, P^\dagger_q] = 2eP^\dagger_{q-\vec{k}},
\]  
(3)
where \( P^\dagger_q \) is the creation operator of a pair. In the right-hand side of these
equations the charge carried by the object appears; \( e \) in the case of the
electron and \( 2e \) in the case of the Cooper pair.

Next we consider the case of extended Cooper pairs. An extended Cooper
pair is represented as
\[
\Psi(\vec{R}) = \int d\vec{r} \chi_1(\vec{r}) \psi_\downarrow(\vec{r}_1) \psi_\uparrow(\vec{r}_2),
\]  
(4)
where
\[
\chi_1(\vec{r}) = \sum_{\vec{p}} e^{i\vec{p} \cdot \vec{r}} \chi_1(\vec{p}), \quad \psi_\downarrow(\vec{r}_1) = \sum_{\vec{p}_1} e^{i\vec{p}_1 \cdot \vec{r}_1} b_{\vec{p}_1}, \quad \psi_\uparrow(\vec{r}_2) = \sum_{\vec{p}_2} e^{i\vec{p}_2 \cdot \vec{r}_2} a_{\vec{p}_2}.
\]  
(5)
Here \( \vec{R} \) is the center-of-mass coordinate of the pair and \( \vec{r} \) is the relative
coordinate. Namely, \( \vec{r}_1 = \vec{R} + \vec{r}/2 \) and \( \vec{r}_2 = \vec{R} - \vec{r}/2 \). Performing the integral
in terms of the relative coordinate \( \vec{r} \) we obtain
\[
\Psi(\vec{R}) = \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{R}} P_{\vec{q}},
\]  
(6)
with
\[
P_{\vec{q}} = \sum_{\vec{p}} \chi_1(\vec{p}) b_{\frac{\vec{q}}{2} + \frac{\vec{p}}{2} + \vec{p}} a_{\frac{\vec{q}}{2} + \frac{\vec{p}}{2} - \vec{p}}, \quad P^\dagger_{\vec{q}} = \sum_{\vec{p}} \chi_1(\vec{p}) a^\dagger_{\frac{\vec{q}}{2} + \frac{\vec{p}}{2} - \vec{p}} b^\dagger_{\frac{\vec{q}}{2} + \frac{\vec{p}}{2} + \vec{p}}.
\]  
(7)

The equal time commutation relation is calculated as
\[
[j^e_{\vec{k}}, P^\dagger_q] = e \sum_{\vec{p}} \sum_{\vec{p}'} \chi_1(\vec{p}) \left( a^\dagger_{\frac{\vec{p}'}{2} - \vec{k}} \left( a_{\frac{\vec{p}}{2} + \vec{p}} a^\dagger_{\frac{\vec{p}}{2} + \vec{p}} + a^\dagger_{\frac{\vec{p}}{2} + \vec{p}} a_{\frac{\vec{p}}{2} + \vec{p}} \right) b^\dagger_{\frac{\vec{q}}{2} - \vec{p}} \right.
\]
\[
- \left. b^\dagger_{\frac{\vec{q}}{2} + \vec{p}} \left( b_{\frac{\vec{p}}{2} + \vec{p}} b^\dagger_{\frac{\vec{q}}{2} + \vec{p}} + b^\dagger_{\frac{\vec{q}}{2} + \vec{p}} b_{\frac{\vec{p}}{2} + \vec{p}} \right) a_{\frac{\vec{q}}{2} + \vec{p}} \right)
\]
\[
e e \sum_{\vec{p}} \chi_1(\vec{p}) \left( a^\dagger_{\frac{\vec{p}}{2} + \vec{p} - \vec{k}} b^\dagger_{\frac{\vec{q}}{2} - \vec{p}} + a^\dagger_{\frac{\vec{p}}{2} + \vec{p}} b^\dagger_{\frac{\vec{q}}{2} - \vec{p} - \vec{k}} \right)
\]
\[
e e \sum_{\vec{p}} \chi_1(\vec{p}) \left( a^\dagger_{\frac{\vec{q}}{2} + \vec{p} - \vec{k}} b^\dagger_{\frac{\vec{q}}{2} + \vec{p} - \vec{k}} + a^\dagger_{\frac{\vec{q}}{2} + \vec{p} - \vec{k}} b^\dagger_{\frac{\vec{q}}{2} + \vec{p} - \vec{k}} \right).
\]  
(8)
By shifting the variable of the summation we obtain
\[
[j^e_{\vec{k}}, P^\dagger_q] \to 2e \sum_{\vec{p}} \chi_1(\vec{p}) a^\dagger_{\frac{\vec{q}}{2} + \vec{p}} b^\dagger_{\frac{\vec{q}}{2} - \vec{p}} = 2eP^\dagger_{\frac{\vec{q}}{2} - \vec{k}},
\]  
(9)
\footnote{Eq. (18) in ref. \cite{4} should be replaced by (6) in this note.}
in the limit of vanishing external momentum $\vec{k} \to 0$, where the shift in the argument of the form factor $\chi_l(\vec{p}^\prime \pm \frac{\vec{k}}{2})$ is negligible. Thus even in the case of extended Cooper pair the commutation relation in the limit of vanishing external momentum picks up the integrated charge of the object so that we obtain the same Ward identity as in the case of local pair.

Next we consider the case of extended particle-hole pairs. An extended particle-hole pair is represented as

$$A^\dagger(\vec{R}) = \int d\vec{r} \chi(\vec{r}) \psi^\dagger_\uparrow(\vec{r}_1) \psi_\downarrow(\vec{r}_2),$$

(10)

where

$$\chi(\vec{r}) = \sum_{\vec{p}} e^{i\vec{p} \cdot \vec{r}} \chi(\vec{p}), \quad \psi^\dagger_\uparrow(\vec{r}_1) = \sum_{\vec{p}_1} e^{-i\vec{p}_1 \cdot \vec{r}_1} a^\dagger_{\vec{p}_1}, \quad \psi_\downarrow(\vec{r}_2) = \sum_{\vec{p}_2} e^{i\vec{p}_2 \cdot \vec{r}_2} b_{\vec{p}_2}. \quad \text{(11)}$$

Performing the integral in terms of the relative coordinate $\vec{r}$ we obtain

$$A^\dagger(\vec{R}) = \sum_{\vec{q}} e^{i\vec{q} \cdot \vec{R}} A^\dagger_{\vec{q}},$$

(12)

with

$$A^\dagger_{\vec{q}} = \sum_{\vec{p}} \chi(\vec{p}) a^\dagger_{\vec{p} - \frac{\vec{q}}{2}} b_{\vec{p} + \frac{\vec{q}}{2}}. \quad \text{(13)}$$

The equal time commutation relation is calculated as

$$[j_{\vec{k}}^e, A^\dagger_{\vec{q}}] = e \sum_{\vec{p}^\prime} \sum_{\vec{p}} \chi(\vec{p}) \left( a^\dagger_{\vec{p}^\prime - \vec{k}} \left( a_{\vec{p}^\prime - \frac{\vec{q}}{2}} + a_{\vec{p}^\prime - \frac{\vec{q}}{2}}^\dagger \right) b_{\vec{p} + \frac{\vec{q}}{2}} \right)$$

$$- a^\dagger_{\vec{p}^\prime - \frac{\vec{q}}{2}} \left( b^\dagger_{\vec{p}^\prime - \vec{k}} + b^\dagger_{\vec{p}^\prime + \frac{\vec{q}}{2}} \right) b_{\vec{p} + \frac{\vec{q}}{2}}$$

$$= e \sum_{\vec{p}} \chi(\vec{p}) \left( a^\dagger_{\vec{p} - \frac{\vec{q}}{2} - \vec{k}} b_{\vec{p} + \frac{\vec{q}}{2}} - a^\dagger_{\vec{p} - \frac{\vec{q}}{2}} b_{\vec{p} + \frac{\vec{q}}{2} + \vec{k}} \right)$$

$$= e \sum_{\vec{p}} \chi(\vec{p}) \left( a^\dagger_{(\vec{p} - \frac{\vec{q}}{2}) - \vec{k}} b_{\vec{p} + \frac{\vec{q}}{2}} - a^\dagger_{\vec{p} - \frac{\vec{q}}{2}} b_{\vec{p} + \frac{\vec{q}}{2} + \vec{k}} \right). \quad \text{(14)}$$

By the same procedure as in the case of the Cooper pair we obtain

$$[j_{\vec{k}}^e, A^\dagger_{\vec{q}}] \to e \sum_{\vec{p}^\prime} \chi(\vec{p}^\prime) \left( a^\dagger_{\vec{p}^\prime - \frac{\vec{q}}{2} + \vec{k}} b_{\vec{p}^\prime + \frac{\vec{q}}{2}} - a^\dagger_{\vec{p}^\prime - \frac{\vec{q}}{2}} b_{\vec{p}^\prime + \frac{\vec{q}}{2} + \vec{k}} \right) = 0. \quad \text{(15)}$$

Since the integrated charge of the particle-hole pair vanishes, the commutation relation vanishes. The current vertex for dc conductivity is obtained from the Ward identity in the limit of vanishing external momentum $k \to 0$. Thus the Ward identity derived from this commutation relation tells us that particle-hole pairs have no contribution to dc conductivity. Such a conclusion is natural, since particle-hole pairs are charge-neutral and carry no charge.
3 Ward identity for heat current vertex

The central quantity in the discussions of Ward identity for heat current vertex in the limit of vanishing external momentum \( k \to 0 \) is the equal-time commutation relation between the Hamiltonian and the object which comes in or go out of the vertex. As the discussion in the previous section, we obtain the same Ward identity as in the case of local pair in this limit.

4 Conclusion

In the discussion of the Ward identity an equal-time commutation relation plays the central role.

In the case of the electric current vertex it picks up the integrated charge of the object in the limit of vanishing external momentum. In this limit the wavelength of the electromagnetic field exceeds the size of the object so that the object can be treated as a point with its integrated charge in the discussion of the electromagnetic response. Thus it is concluded that charge-neutral pairs do not couple to electromagnetic field. Namely, charge- and spin-density fluctuations do not carry charge. On the other hand, Cooper pairs carrying charge \( 2e \) couple to electromagnetic field.

In the case of the heat current vertex it picks up the energy of the object.
References

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