A State-Dependent Updating Period For Certified Real-Time Model Predictive Control

Mazen Alamir

Abstract—In this paper, a state-dependent control updating period framework is proposed that leads to real-time implementable Model Predictive Control with certified practical stability results and constraints satisfaction. The scheme is illustrated and validated using new certification bound that is derived in the case where the Fast Gradient iteration is used through a penalty method to solve generally constrained convex optimization problems. Both the certification bound computation and its use in the state-dependent updating period framework are illustrated in the particular case of linear MPC. An illustrative example involving a chain of four integrators is used to show the explicit computation of the state-dependent control updating scheme.

Index Terms—MPC, Certification, Real-time, Stability.

I. INTRODUCTION

Modern control paradigms such as Model Predictive Control [10], Moving-Horizon Observers [1] or adaptive identification of varying models [16] to cite but few issues involve the real-time, on-line solution of constrained optimization problems. In such applications, the output of the optimizer (namely the sub-optimal solution of the optimization problem) is fed to some neighboring modules in order to achieve some engineering tasks. The quality of the global task may strongly depend on the quality of the sub-optimal solution and the frequency with which it can be updated by the optimizer and since this solution has to be delivered in finite and probably short time, it is important to be able to precisely link the quality of the suboptimal solution to the available computation time for a predefined embedded computation power. When the latter is not yet defined, such insight enables to choose the appropriate computational power given the required quality of the sub-optimal solution.

The last few years witnessed an increasing interest in the certification issue [13], [9], [5]. These almost simultaneous works proposed certification bounds for fast gradient-based iterations (regardless of their inherent computational cost) which accounts for the total number of elementary operations until convergence which is generically erroneous and that in realistic situations, the overall convergence analysis and trade-off handling. This feature if absent from recent works on the certification issue such as [14] where the number of iteration is induced from the required precision on the solution and the corresponding number of iteration is derived as a consequence. This argumentation suggests that provided that one uses sufficiently high number of iterations, convergence of the real-time MPC will be guaranteed. This paper shows that this argument is generically erroneous and that in realistic situations, the appropriate updating period shows lower and upper bounds.

Regard the other alternatives, active set iterations [8], while computationally efficient and while showing a provably finite number of iterations to converge (for QP problems), seem to resist to the derivation of convergence rates which makes impossible the computation of certification bounds. As for interior point methods [7], [15], [4], certification bounds exist [11] but seem to be systematically over pessimistic [14]. Nevertheless, for many problems, it might still be more appropriate to use these efficient although uncertified or pessimistically certified algorithms rather than to use a slow certified iterations. The right choice is problem-dependent.

The first part of this paper belongs to the family of works that address the derivation of certification bounds for fast gradient-based iterations in the presence of general constraints. This is motivated by the nice properties mentioned above, namely the reduced complexity of the single associated iteration that enables the use of extremely short updating period. As it has been recently shown [2], [3], [6], this last property may compensate the drawback of potentially higher number of iterations when compared to some alternative methods, especially in uncertain context (which includes perfectly known systems under unpredictable set-point dynamics). In such situations, as underlined by [13], it is important to distinguish between the concepts of analytical complexity which involves only the number of iterations (regardless of their inherent computational cost) and the arithmetical complexity which accounts for the total number of elementary operations until convergence which is obviously the appropriate indicator in real-time context and this is precisely why fast gradient is an interesting option.

The second part of the paper proposes a general framework to explicitly account for the arithmetical complexity by including the computation time for a single iteration in the overall convergence analysis and trade-off handling. This feature if absent from recent works on the certification issue such as [14] where the number of iteration is induced from the required precision on the solution and the corresponding number of iteration is derived as a consequence. This argumentation suggests that provided that one uses sufficiently high number of iterations, convergence of the real-time MPC will be guaranteed. This paper shows that this argument is generically erroneous and that in realistic situations, the appropriate updating period shows lower and upper bounds.
consider the following optimization problem in the decision variable $p$:

$$ \min_{p \in \mathbb{R}^{n+p}} f_0(p) \quad | c_i(p) \leq 0 \quad \forall i \in I_h \cup I_s := \{1, \ldots, n_c\} \quad (1) $$

where $I_s$ and $I_h$ are the disjoint subsets of $\{1, \ldots, n_c\}$ that define a partition of the set of constraints into soft and hard constraints respectively. $f_0(\cdot)$ is the cost to be minimized while $c_i : \mathbb{R}^{n+p} \rightarrow \mathbb{R}$ defines the $i$-th inequality constraint. Note that saturation constraints on $p$ are supposed to be included in the set of inequality constraints. It is assumed that $f_0$ and $c_i$ are differentiable for all $i$.

The algorithm proposed in this paper invokes the following penalty induced augmented cost:

$$ f(p) := f_0(p) + \rho \times \psi(p) \quad (2) $$

where $\rho$ is called the penalty parameter while $\psi : \mathbb{R}^{n+p} \rightarrow \mathbb{R}_+$ is the constraints induced cost given by:

$$ \psi(p) := \sum_{i \in I_s} [\max\{0, c_i(p)\}]^2 + \sum_{i \in I_h} [\max\{0, c_i(p) + \varepsilon\}]^2 $$

For a given pair $\varepsilon := (\varepsilon_0, \varepsilon_\psi)$ of strictly positive reals, a candidate value $p$ is called an $\varepsilon$-suboptimal solution of (1) if the following two conditions hold:

$$ |f_0(p) - f^{\text{opt}}| \leq \varepsilon_0 \quad \text{and} \quad \psi(p) \leq \varepsilon_\psi^2 \quad (3) $$

where $f^{\text{opt}}$ denotes the optimal value of (1).

The relevance of the second constraint in (3) lies in the fact that when satisfied, this constraint implies that all the hard constraints are rigorously satisfied while the maximum violation of any soft constraint is lower than $\varepsilon_\psi$.

The first aim of the present paper is to derive the necessary relations that enable for a given precision $\varepsilon$ to choose the appropriate penalty coefficient $\rho$ and the stopping condition for the fast gradient iteration to be used in the unconstrained minimization of the cost function $f$ defined by (2). Moreover, the bound on the minimum number of iterations that guarantees an $\varepsilon$-suboptimal solution to the original problem is derived. This is done in sections III and IV.

The second aim is to show that this certification result (or any similar one for possibly another algorithm) can then be used to design a real-time constrained MPC implementation in which a state-dependent control updating period is used to yield certified convergence properties. This is done in section V.

The results are proved in a rather general convex settings and for both goals, the expressions enabling the parameters involved in the statements of the results to be computed are explicitly given in the specific case of QP problems and linear MPC design.

### III. Assumptions and Preliminary Results

#### A. Definitions and Notation

In what follows, $f'(p), f'_0(p)$ and $\psi'(p)$ denote the gradients of the functions w.r.t the decision variable $p$. The euclidian norm of $f'(p)$ is denoted by $g(p) = \|f'(p)\|$. For a scalar continuously differentiable function $\ell$ defined on $\mathbb{R}^n$, the notation $\ell \in S^1_1$ states that $\ell$ is a $\mu$-strongly convex function, namely for all $(p_1, p_2)$:

$$ \ell(p_2) \geq \ell(p_1) + \langle \ell'(p_1), p_2 - p_1 \rangle + \frac{\mu}{2} \|p_2 - p_1\|^2 \quad (4) $$

where $\mu$ is called the convexity parameter of $\ell$ [13]. Similarly, the notation $\ell \in F^1_\mu$ indicates that the continuously differentiable function $\ell$ satisfies for all $(p_1, p_2)$:

$$ \ell(p_2) \leq \ell(p_1) + \langle \ell'(p_1), p_2 - p_1 \rangle + \frac{L}{2} \|p_2 - p_1\|^2 \quad (5) $$

When $\ell$ satisfies both (4)-(5), the notation $\ell \in S^1_{\mu, L}$ is used. The set $\mathcal{C}$ denotes the set of singular points of $f'(\cdot)$, namely...
the set of \( p \) such that \( q(p) = 0 \). Given a subset \( A \subset \mathbb{R}^{n_x} \), the notation \( d(p, A) \) refers to the distance from \( p \) to \( A \), namely \( d(p, A) := \min_{z \in A} \| z - p \| \). The short notation \( d(p) := d(p, C) \) is used for the specific set \( C \). The set \( A_{\psi=0} \) is the set of \( p \) such that \( \psi(p) = 0 \). Given a bounded subset \( P \), \( \delta_p \) denote the radius of \( P \) namely \( \delta_p := \sup_{x_1, x_2 \in P} \| x_1 - x_2 \| \). For a compact set \( X \), the notation \( \varrho(X) \) denotes the maximum norm of elements in \( X \), namely \( \varrho(X) := \sup_{x \in X} \| x \| \).

**B. Working Assumptions**

**Assumption 3.1:** The cost function value \( f_0(p) \) is nonnegative for all \( p \).

This assumption can be made satisfied by adding sufficiently high positive constant. It is quite common in MPC context where the cost function refers quite often to the integral of the tracking error that is added to some positive terminal term.

**Assumption 3.2:** There are two reals \( L_0 \geq 0 \) and \( L_\psi \geq 0 \) such that \( f_0 \in F_{L_0}^1 \) and \( \psi \in F_{L_\psi}^1 \).

**Assumption 3.3:** There is \( \mu_0 > 0 \) such that \( f_0 \in S_{\mu_0}^1 \). Moreover, \( \Psi \) is convex.

Note that this assumption implies that \( f \in S_{\mu_0}^1 \) and that there is a unique critical point for \( f \) which is denoted hereafter by \( p^* \in C \). Therefore, according to the definition of \( d(p) \), one has \( d(p) := \| p - p^* \| \).

In what follows, the notation \( p_u \) and \( p_a \) refer to two vectors such that:

\[
p_u := \min_{p \in \mathbb{R}^p} f_0(p) : \quad \psi(p_u) \leq 0 \tag{6}
\]

namely, \( p_u \) is the unconstrained minimum of \( f_0 \) while \( p_a \) is any admissible point. Having \( p_a \), the following definition can be stated since \( f_0 \) is supposed to be continuously differentiable and because \( f_0 \in S_{\mu_0}^1 \) [Assumption 3.3].

**Definition 3.1:** Define \( D_0 \) by:

\[
D_0 := \sup_{f_0(p) \leq f_0(p_u)} \| f_0'(p) \| \geq 0 \tag{7}
\]

**Remark 3.1:** In fact, the knowledge of the admissible point \( p_u \) is only required to compute \( D_0 \), therefore, if an upper bound of \( D_0 \) can be found, the knowledge of \( p_u \) is not mandatory. This is clearly shown in section C in the specific case of QP problems [see inequality (23)]. This is crucial since in the MPC context the constraints are state dependent and it may become cumbersome to compute \( p_u \) for each current state.

The next assumption concerns the behavior of the penalty map outside the admissible set.

**Assumption 3.4:** There is \( \beta > 0 \) such that the following inequality:

\[
\psi(p) \geq \beta \times \left[ d(p, A_{\psi=0}) \right]^2 \tag{8}
\]

holds for all \( p \).

The expressions of the parameters \( L_0, L_\psi, \mu_0, D_0 \) and \( \beta \) in the specific case of quadratic cost \( f_0 \) and \( \psi \) in \( p \) constraints are expressed in section C.

**C. Preliminary results**

In this section some preliminary results are stated. For better readability, all the proofs are given in the appendix. The first result gives a property of the gradient of \( f_0 \) at the stationary point \( p^* \):

**Lemma 3.1:** The following inequality holds

\[
\| f_0'(p^*) \| \leq D_0 \tag{9}
\]

**Proof.** See Appendix A.

The following result characterizes the behavior of the penalty term \( \psi \) in terms of the penalty coefficient \( \rho \):

**Lemma 3.2:** If \( \rho > L_0/\beta \) then the following inequality:

\[
\psi(p) \leq \frac{L_\psi}{2} \left[ d(p) + \frac{\kappa_0}{\sqrt{\rho}} \right]^2 \quad \text{where} \quad \kappa_0 := \frac{2L_0}{\beta} \sqrt{\frac{2}{\mu_0} \psi(p_u)} \tag{10}
\]

holds for all \( p \). In particular, for \( p^* \) one has:

\[
\psi(p^*) \leq \frac{L_\psi \kappa_0^2}{2 \rho} \tag{11}
\]

**Proof.** See Appendix B.

Note that Lemma 3.2 quantifies how increasing \( \rho \) leads to a smaller constraint violation depending on the amount of violation \( \psi(p_u) \) at the unconstrained minimum \( p_u \) of \( f_0 \).

The following corollary gives a bound on the difference in the cost \( f_0 \) evaluated at the unconstrained optimum \( p^* \) of \( f = f_0 + \rho \psi \) and the true optimal cost as a function of constraint violation:

**Lemma 3.3:** Let \( p^{opt} \) be the optimal solution of the original problem (1), \( p^* \) the unconstrained minimum of \( f \). The following inequality holds:

\[
|f_0(p^{opt}) - f_0(p^*)| \leq D_0 \left[ \frac{\psi(p^*)}{\beta} \right]^\frac{1}{2} + \frac{L_0}{2} \left[ \frac{\psi(p^*)}{\beta} \right] \tag{12}
\]

**Proof.** See Appendix C.

Using Lemma 3.3 one can prove the following result:

**Corollary 1:** If the following inequality holds:

\[
\left[ \frac{\psi(p^*)}{\beta} \right]^\frac{1}{2} \leq Z_1(\epsilon) := \frac{D_0}{L_0} \left[ \left( 1 + \frac{2L_0}{D_0} \right)^\frac{1}{2} - 1 \right] \tag{13}
\]

then the stationary solution \( p^* \) satisfies:

\[
|f_0(p^{opt}) - f_0(p^*)| \leq \epsilon \tag{14}
\]
such an upper bound in terms of the value of the function $\psi(p^*)/\beta$. Writing that this polynomial is equal to $\epsilon$ and solving for it gives the result. □

Note however that $p^*$ is never reached exactly. Instead, the fast gradient iteration will be used to reach an iterate $p$ that is close to $p^*$. Now since the available certification bounds on the fast gradient iterations concern the guaranteed value of $|f(p^*) - f(p)|$ while the $\bar{\epsilon}$-suboptimality is defined in terms of the original cost $f_0$, the following lemma gives a link between these two indicators:

**Lemma 3.4:** The following implication holds for all $\epsilon$:

$$
\left\{ |f(p) - f(p^*)| \leq \epsilon \right\} \rightarrow 
\left\{ |f_0(p) - f_0(p^*)| \leq D_0 \frac{2\epsilon}{\mu_0} + \frac{L_0}{2} \frac{\epsilon}{\mu_0} \right\}
$$

(15)

**Proof.** See Appendix D.

Here again, Lemma 3.4 gives the condition on the precision $\epsilon_1$ required on $f$ in order to induce a precision $\epsilon_2$ on $f_0$, namely:

**Corollary 2:** If $p$ is such that $|f(p) - f(p^*)| \leq \epsilon_1$ with

$$
\left[ \frac{2\epsilon_1}{\mu_0} \right]^\frac{1}{2} \leq Z_1(\epsilon_2)
$$

(16)

where $Z_1$ is the function defined by 13 then, one has $|f_0(p) - f_0(p^*)| \leq \epsilon_2$.

**Proof.** Use the same arguments as before since the r.h.s of 15 involves the same polynomial as in 12.

The certification bound of the fast gradient needs an upper bound on the distance between the initial guess $p$ and the minimizer of $f$, namely $p^*$. The following lemma gives such an upper bound in terms of the value of the function $f$ at the initial guess $p$:

**Lemma 3.5:** The following inequality is satisfied for all $p$:

$$
|p^* - p| \leq \left[ \frac{2f(p)}{\mu_0} \right]^\frac{1}{2} := r(p)
$$

(17)

**Proof.** This is a direct consequence of the inclusion $f \in S_{p_0}^\dagger$ and the fact that $f_0$ (and hence $f$) is positive.

**IV. THE ALGORITHM**

**A. Recalls on the Fast Gradient iteration**

The fast gradient algorithm proposed in 13 is commonly used to perform unconstrained minimization of a function $f \in S_{\mu,L}$. It is briefly recalled through Algorithm 1 for which the following convergence result holds

**Algorithm 1** $[p_N, q_N, \alpha_N] = F^{(N)}(p_0, q_0, \alpha_0)$

1: for $i = 1 : N$ do
2: \quad $p_{i+1} \leftarrow q_i - f'(q_i)/L$
3: \quad Compute $\alpha_{i+1} \in (0,1)$ solution of $\alpha_i^2 = (1 - \alpha_i)(1 - \alpha_{i+1})/(\alpha_i^2 + \alpha_{i+1})$
4: \quad $q_{i+1} \leftarrow p_i + \beta_i(p_{i+1} - p_i)$
5: end for

**Proposition 4.1:** (13), page 80 The successive iterates of Algorithm 1 starting from the initial guess $p_0, \alpha_0 = \sqrt{\mu_0/L}$ and $q_0 = p_0$ satisfy the following inequality:

$$
L + \frac{\mu_0}{2} \times \min \left\{ (1 - \gamma)^2, \frac{1}{(1 + i\eta)^2} \right\} \times \|p_0 - p^*\|^2
$$

(18)

where $c := \sqrt{\mu_0/L}$ and where $p^*$ stands for the unconstrained minimin of $f$.

The following is a direct consequence of Proposition 4.1.

**Corollary 3:** If the initial guess satisfies $\|p_0 - p^*\| \leq \delta$ then for any $\epsilon > 0$, the integer:

$$
N(c, \gamma) := \max \left\{ 0, \min \left( \frac{\log(\gamma)}{\log(1 - c)}, \frac{1}{c} \sqrt{\frac{\gamma - 1}{\gamma}} \right) \right\}
$$

(19)

where $\gamma := 2\epsilon/(L + \mu_0)^2$ ; $c = \sqrt{\mu_0/L}$ is an upper bound of the number of iterations $N$ needed by Algorithm 1 to deliver a sub-optimal solution $p_N$ satisfying $|f(p_N) - f(p^*)| \leq \epsilon$.

**Proof.** Inject $\|p_0 - p^*\| \leq \epsilon$ in 18 and impose that the r.h.s is $\leq \epsilon$.

Now using the bound on $\|p_0 - p^*\| \leq r(p_0)$ given by 17, the following result follows:

**Corollary 4:** Given any initial value $p_0$, let $\gamma_0 := \epsilon \mu_0/[(L + \mu_0)f(p_0)]$ then $N(c, \gamma_0)$ is an upper bound of the number of iterations $N$ needed by Algorithm 1 to deliver a sub-optimal solution $p_N$ satisfying $|f(p_N) - f(p^*)| \leq \epsilon$.

**B. The Proposed Algorithm**

The proposed algorithm involves the quantities defined by 20-21 that depend on:

- the problem’s intrinsic properties $(\mu_0, L_0, L_\psi, \beta, D_0)$
- the unconstrained solution-dependent parameter $\kappa_0$ [see 10]
- the desired precision pair $\bar{\epsilon} := (\epsilon_0, \epsilon_\psi)$

$$
\rho_1 := \frac{2L_\psi \kappa_0^2}{\epsilon_\psi}
\rho_2 := \frac{L_0 \kappa_0^2}{2 \beta Z_1(\epsilon_0/2)}
\rho_3 := L_0/\beta
$$

(20)

$$
\eta_1 := \frac{\mu_0 \epsilon_0^2}{2 Z_1(\epsilon_0/2)}
\eta_2 := \frac{\mu_0 \epsilon_\psi}{4 L_\psi}
$$

(21)
These quantities are used in Algorithm 2 below:

**Algorithm 2** \( \hat{p}^* = A(p_0, \varepsilon := (\varepsilon_0, \varepsilon_{\psi})) \)

1: \( \alpha_0 = (\mu_0/L)^{\frac{1}{2}} \), \( q_0 := p_0 \)
2: \( \rho = \max\{\rho_1, \rho_2, \rho_3\} \)
3: \( \eta = \min\{\eta_1, \eta_2\} \)
4: \( c = \sqrt{\rho_0/L} \)
5: \( \gamma_0 = \eta \mu_0 / (|L + \mu_0| f_0(p_0)) \)
6: \( N_{\text{max}} = \bar{N}(c, \gamma_0) \)
7: \( g_{\text{min}} = \mu_0 \sqrt{2\eta L} \)
8: \( \text{again} = \text{true} \)
9: while (again) do
10: \( [p_{i+1}, q_{i+1}, \alpha_{i+1}] = FG(1)(p_i, q_i, \alpha_i) \)
11: if \( [(i \geq N_{\text{max}}) \text{ or} \ (g(p_i) \leq g_{\text{min}}) ] \) then
12: \( \text{again} = \text{false} \)
13: else
14: \( i = i + 1 \)
15: end if
16: end while
17: \( \hat{p}^* = p_i \)

The following result gives a certification bound on the number of iterations needed by Algorithm 2 to achieve an \( \varepsilon \)-suboptimal solution of the original problem.

**Proposition 4.2:** Let be given a precision pair \( \varepsilon := (\varepsilon_0, \varepsilon_{\psi}) \), an initial guess \( p_0 \). Let \( \gamma_0 := \eta \mu_0 / (|L + \mu_0| f_0(p_0)) \) where \( \eta := \min\{\eta_1, \eta_2\} \) with the \( \eta \)s given by (21). The algorithm in which \( \rho = \max\{\rho_1, \rho_2, \rho_3\} \) is used with the \( \rho \)s defined by (20) involves at most \( N(c, \gamma_0) \) unconstrained gradient elementary iterations before it delivers an estimate \( \hat{p}^* \) that is an \( \varepsilon \)-suboptimal solution of the original constrained optimization problem (1).

PROOF. See Appendix [E]

In the remainder of the paper, the maximum number of iterations that guarantee the precision as expressed in Proposition 4.2 is denoted by:

\[
N(p_0, \varepsilon_0, \varepsilon_{\psi}) := \bar{N}(c, \gamma_0)
\]

as the arguments of \( N \) completely determine \( c \) and \( \gamma_0 \).

**C. Case of Quadratic Programming (QP) problems**

Here, the expressions of \( L_0, L_{\psi}, \mu_0, D_0 \) and \( \beta \) are given in the specific case of QP problems where the cost function and the constraints take the form:

\[
f_0(p) = \frac{1}{2} p^T H p + F^T p + s_0 \quad ; \quad c_i(p) = A_i p - B_i
\]

In this case, Assuming that \( s_0 \) is such that assumption 3.1 holds, it is straightforward that Assumptions 3.2 and 3.3 holds with \( L_0 = \lambda_{\text{max}}(H), L_{\psi} = \sigma_{\text{max}}(A) \) and \( \mu_0 = \lambda_{\text{min}}(H) \). Moreover, one has \( p_u := -H^{-1} F \). Now according to remark 3.1, \( p_u \) is not needed provided that an upper bound for \( D_0 \) can be derived. This is the aim of the following proposition:

**Proposition 4.3:** Provided that the set of inequalities \( A p \leq B \) implies the condition \( p \in \mathbb{P} \), the following inequality holds:

\[
D_0 \leq \left[ \lambda_{\text{max}}(H) \right] \hat{p} + \| F \| \tag{23}
\]

where

\[
\hat{p} := \frac{\| F \| + \sqrt{\| F \|^2 + 2\lambda_{\text{min}}(H) \| f \|}}{\lambda_{\text{min}}(H)} \tag{24}
\]

in which

\[
\| f \| := \frac{1}{2} \lambda_{\text{max}}(H) (\varphi(\mathbb{P}))^2 + \| F \| \cdot \varphi(\mathbb{P}) \tag{25}
\]

PROOF. See Appendix [E]

Assumption 3.4 is satisfied with \( \beta := \sigma_{\text{min}}(A) \) which is the lowest non zero singular value of the constraints matrix \( A \). The coefficient \( \kappa_0 \) involved in lemma 3.2 and the expressions (20) and (21) used to compute \( \rho \) and \( \eta \) is obtained using the values of \( L_0, \beta, \mu_0 \) and \( p_u \) mentioned above.

**Numerical Experiments**

In order to check the validity of the certification bound \( N(p_0, \varepsilon_0, \varepsilon_{\psi}) \), 500 random QP problems have been generated with \( n = 10 \) decision variables and \( n_c = 20 \) constraints. More precisely, \( H := CC^T + \sigma I \) is used where \( C \in \mathbb{R}^{n \times 1} \) and \( \sigma \in [10^{-3}, 1] \). \( F \) and \( s_0 \) has been computed so that the cost is \( \| p - p_u \|_H^2 + 1 \) where \( p_u \) is randomly generated. The constraints matrices \( A \in \mathbb{R}^{n \times n_c} \) and \( B \in \mathbb{R}^{n_c} \) has been randomly generated so that a feasible solution exists. The precision \( \varepsilon_0 = 10^{-2} \) has been used while \( \varepsilon_0 \) has been systematically taken equal to 1% of the true optimal cost that is obtained by QUADPROG-MATLAB solver. The initial guess is systematically taken equal to 0 as one might use in cold start MPC context.

The results are shown in Figure 1 where the histogram over the 500 trials of the ratio between the effectively needed number of iterations \( N \) and the maximal computed certification bound \( N_{\text{max}} \) (step 6 of Algorithm 2) is plotted. The results suggest that for this class of QP problems, the bounds is not that conservative and that since some scenarios lead to a ratio between 0.5 and 0.6, as far as certification is needed, it cannot be strongly reduced.

**V. APPLICATION TO REAL-TIME MPC**

In this section, it is assumed that a certification bound \( N(p_0, \varepsilon_0, \varepsilon_{\psi}) \) is given for some algorithm. Based on such a bound, a real-time MPC implementation framework is proposed using a state-dependent control updating period leading to provable practical convergence. It is therefore important to underline that the results of this section does not necessarily relate to the use of the fast-gradient algorithm as they can apply to any algorithm for which a certification can be associated that depends on the initial guess \( p_0 \) and some required precision pair \( (\varepsilon_0, \varepsilon_{\psi}) \) in the sense of (3)
A. Definition, notation and working assumptions

In this section, a set of assumptions are stated. Not all of them are used in all the subsequent results. That is why in the statement of each result, the assumptions that are needed are explicitly mentioned.

In MPC framework, the controller disposes of a model of the form
\[ \dot{x} = F(x, u) \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^m \] (26)
where the following assumption is used regarding the definition of the vector \( x \):

**Assumption 5.1:** The state vector \( x \) involved in (26) gathers the physical state of the system together with the current set-point and current estimation of the disturbance. The model also incorporates the assumption on the future behavior of these exogenous variables.

We consider that the future control profiles are parametrized through a finite dimensional vector \( p \) of degrees of freedom such that at each instant \( t \), the future profile depends on \( p(t) \) according to:
\[ u(t + s) := \mathcal{U}(s, p(t)) \quad s \in [0, T] \] (27)
where \( \mathcal{U} \) is some predefined map and \( T \) is the prediction horizon.

Since the MPC has to be computed based on the prediction of the future state (in the sense of Assumption 5.1), the following assumption is needed to characterize the state prediction error:

**Assumption 5.2:** For each compact set \( \mathbb{C} \) to which belongs the pair \((p(t), x(t))\), the prediction \( \hat{x}(t + \tau) \) of the future state starting from \( x(t) \) and under the control profile \( \mathcal{U}(\cdot, p(t)) \) can be affected by an error satisfying
\[ \| \hat{x}(t + \tau) - x(t + \tau) \| \leq E_0^2 + E_C^1 \times \tau \] (28)
Note that \( E_C^1 \) in (28) accommodates for unpredictable set-point changes while \( E_0^2 \) accommodates for the presence of disturbances that affects the input of some integrator in the system or for the presence of unpredictable time-varying set-point.

The cost function is defined at instant \( t \) based on the knowledge of the state \( x(t) \) (including the current value of the set point and the disturbance estimation and prediction). This leads to a constrained optimization problem of the form (1) in which both \( f_0 \) and \( c_i \) are dependent on the current value \( v(t) \) of the state, namely:
\[ f_0(p, x(t)) \ ; \ \psi(p, x(t)) \]
Consequently, the call of Algorithm 2 as well as the bound (22) on the number of iterations must now incorporate the state \( x(t) \) as an argument, namely:
\[ \bar{p}^* = A(p_0, \varepsilon_0, \varepsilon_r, x) \ ; \ N(p_0, \varepsilon_0, \varepsilon_r, x) \] (29)
In order to use the results of the preceding section, one needs to assume that for all \( x \), there are positive reals \( L_0(x) \), \( L_\psi(x) \) and \( \beta(x) \) and a strictly positive \( \mu_0(x) > 0 \) that play the roles of \( L_0 \), \( L_\psi \), \( \beta \) and \( \mu_0 \) as defined in the preceding section.

Now if for some reasons, one knows that the pair \((p_0, x)\) involved in (29) belongs to some compact set \( \mathbb{C} := \mathbb{P} \times \mathbb{X} \), then one can obtain a certification bound that depends only on the precision parameters \( \varepsilon_0 \) and \( \varepsilon_\psi \) as defined in the preceding section.

**Proposition 5.1:** Let a compact set \( \mathbb{C} := \mathbb{P} \times \mathbb{X} \) be given, the bound \( N_\mathbb{C}(\varepsilon_0, \varepsilon_\psi) \) defined by (30) can be computed by the following steps:
1) Compute \( \psi_{\text{max}} \) according to:
\[ \psi_{\text{max}} := \max_{x \in \mathbb{X}} \left\{ \psi(p_u, x) \mid f_0^*(p_u, x) = 0 \right\} \] (31)
2) Compute \( L_0 \), \( L_\psi \) as the maximum of \( L_0(x) \) and \( L_\psi(x) \) over \( x \in \mathbb{X} \)
3) Compute \( \beta \) and \( \mu_0 \) as the minimums of \( \beta(x) \) and \( \mu_0(x) \) over \( x \in \mathbb{X} \)
4) Compute \( \kappa_0^{\text{max}} := \frac{2L_0}{\beta} \sqrt{2\psi_{\text{max}}/\mu_0} \)
5) Compute \( \rho_{\text{max}} \) using (20) in which \( \kappa_0^{\text{max}} \) replaces \( \kappa_0 \)
6) Compute \( \eta_{\text{min}} := \min\{\eta_1, \eta_2\} \) where the \( \eta_i \) are computed by (21) in which \( \rho_{\text{max}} \) replaces \( \rho \)
7) Compute \( f_0^{\text{max}} := \max_{(p,x) \in \mathbb{C}} f_0(p, x) \)
8) Compute \( \gamma_0^{\text{min}} := \eta_{\text{min}} \mu_0/(\|L_{\text{max}}(\rho_{\text{max}}) + \mu_0\|f_0^{\text{max}}) \)
9) Compute \( c_{\text{min}} := \psi_0/L_{\text{max}}(\rho_{\text{max}}) \)

Finally compute the desired quantity:
\[ N_{\mathbb{C}}(\varepsilon_0, \varepsilon_\psi) := \tilde{N}(c_{\text{min}}, \gamma_0^{\text{min}}) \] (32)
where \( \tilde{N} \) is defined by (19).

**Proof:** Straightforward as the computation systematically takes the worst case towards the increase of \( N \). □
Regarding the formulation of the MPC, the following make the following inequality satisfied:

\[
\{ f_0(p, x) \leq \phi \} \Rightarrow \{(p, x) \in \mathbb{C}_\phi \}
\]  
(33)

Regarding the dependence of \( f_0 \) and \( \psi \) on \( x \), the following assumption is considered:

**Assumption 5.4:** For any compact set \( \mathbb{C} \), there are positive real \( K_0^p, K_x^p > 0 \) such that:

\[
\| f_0(p, x_1) - f_0(p, x_2) \| \leq K_0^p \cdot \| x_1 - x_2 \|  
\]  
(34)

\[
\| \psi(p, x_1) - \psi(p, x_2) \| \leq K_x^p \cdot \| x_1 - x_2 \|  
\]  
(35)

for all \((p, x_1), (p, x_2) \in \mathbb{C}\).

A typical formulation of \( f_0(p, x_0) \) in MPC is given by:

\[
f_0(p, x_0) := \Omega(\bar{x}(T, p, x_0)) + \int_0^T \ell(s, p, x_0, p, s)ds  
\]

\[= \Omega(\bar{x}(T, p, x_0)) + \int_0^T \ell(s, p, x_0)ds  
\]  
(36)

where \( \bar{x}(s, p, x_0) \) is the predicted state value at instant \( s \) starting from \( x_0 \) at instant 0.

Regarding the formulation of the MPC, the following (commonly satisfied) assumption is needed in the sequel:

**Assumption 5.5:** The MPC formulation is based on a cost function of the form \( \ell \) with the necessary constraints that make the following inequality satisfied:

\[
f_0(p_{opt}(t + \tau), \bar{x}(t + \tau)) - f_0(p_{opt}(t), x(t)) \leq -\Delta(\tau, x(t)) := -\int_0^T \ell(s, p_{opt}(t), x(t))ds  
\]  
(37)

where \( p_{opt}(t) \) is the optimal solution of the problem defined for the state \( x(t) \) while \( p_{opt}(t + \tau) \) is the optimal solution of the problem defined by the predicted future state \( \bar{x}(t + \tau) \) starting from \( x(t) \) under the optimal control \( u(\cdot, p_{opt}(t)) \) that is applied on the interval \([t, t + \tau]\).

Note that \( p_{opt}(t) \) does not appear as an argument of \( \Delta \) since \( p_{opt}(t) \) is assumed to be uniquely determined by \( x(t) \).

**Remark 5.1:** Note that the inequality (37) is satisfied only for the ideal predicted future state \( \bar{x}(t + \tau) \) since otherwise the bad knowledge of uncertainties and/or the set-point changes may invalidate the inequality if the true value \( x(t + \tau) \) of the state is used.

**Remark 5.2:** Note that inequality (37) is commonly satisfied in the standard provably stable MPC formulations. Moreover, the r.h.s \( \Delta(\tau, x(t)) \) is generally exhibited through the corresponding stability proof (see [10]).

In section [V.C] Explicit computation of all the quantities involved in Proposition 5.1 is given for the specific case of state-dependent QP optimization problems that arise in the linear MPC context.

It is also assumed that the cost function \( f_0 \) is proper in both \( p \) and \( x \) in the following sense:

**Assumption 5.3:** For any positive real \( \phi > 0 \), there is a compact set \( \mathbb{C}_\phi \) such that the following implication holds:

\[
\{ f_0(p, x) \leq \phi \} \Rightarrow \{(p, x) \in \mathbb{C}_\phi \}
\]  
(33)

Regarding the penalty function \( \ell \), the following assumption is used:

**Assumption 5.6:** [Figure 2] For any compact set \( \mathbb{C} \), there is a positive real \( D_C > 0 \) and a positive function \( q(\cdot) \) such that:

\[
\bar{\ell}(s, p, x) \geq \max \{0, q(x) - D_C s\}
\]  
(38)

Note that condition (38) simply states that with bounded control, there is a limitation on the speed with which the state can be steered to the desired region. With this respect, \( q(x) \) is simply a state dependent term in \( \ell \) that expresses how far does \( x \) lie from the desired region. This notation enables many situations to be handled as \( x \) includes set-point definition and therefore, measures of the difference between the physical state of the system and their desired value can take the simple form expressed by \( q(x) \).

Finally, the following assumption is used to characterize the available computational facility:

**Assumption 5.7:** The system is controlled with a computational facility that performs a single elementary iteration of the fast gradient (step 9 of Algorithm 2) in \( \tau_c \) time units.

Note that if another certified algorithm than the fast gradient is used, \( \tau_c \) used hereafter denotes the time necessary to perform a single iteration of that specific algorithm.

**B. Certified MPC by state-dependent updating period**

Assume that a scheme is based on the iterative on-line definition of a sequence of updating instants and a sequence of precision parameters denoted by:

\[
t_{k+1} = t_k + \tau_k : \{ \tau_k \}^{\infty}_{k=0}
\]  
(39)

which are linked through the definition of the updating periods \( \tau_k \) according to:

\[
\tau_k := \tau_c \times N_C(\varepsilon_0^{(k+1)}, \varepsilon^{(k+1)})
\]  
(40)

where \( \mathbb{C} \) is some compact subset of \( \mathbb{R}^{n_u} \times \mathbb{R}^n \) and \( \tau_c \) is the computation time needed for a single fast gradient iteration (see Assumption 5.7).
More precisely, given the current state \( x(t_k) \) and a control \( u(\cdot, \hat{p}^*(t_k)) \) that is applied during the sampling period \([t_k, t_{k+1}]\). Algorithm 2 is used to compute the control parameter \( \hat{p}^*(t_{k+1}) \) (that is to be applied during the next sampling period) with the hot start \( [\hat{p}^*(t_k)]^{+\tau_k} \) and the precision parameters \( (\varepsilon_0^{(k+1)}, \varepsilon_{\psi}^{(k+1)}) \). Note that by the very definition (40) of \( \tau_k \), the value of the control parameter \( \hat{p}^*(t_{k+1}) \) is obtained by Algorithm 2 (before \( t_{k+1} \) necessarily meets the precision requirements, namely:

\[
\begin{align*}
& f_0(\hat{p}^*(t_{k+1}), \dot{x}(t_{k+1})) - f_0(p_0^{\text{opt}}(t_{k+1}), \dot{x}(t_{k+1})) \leq \varepsilon_0^{(k+1)} \\
& c_i(\hat{p}^*(t_{k+1}), \dot{x}(t_{k+1})) \leq 0, & i \in I_h \\
& c_i(\hat{p}^*(t_{k+1}), \dot{x}(t_{k+1})) \leq \varepsilon_{\psi}^{(k+1)}, & i \in I_s
\end{align*}
\]

Using the first inequality, one can prove the following result:

**Lemma 5.1:** If the following conditions hold

1) \( \tau_k \) is defined by (40) for some compact set \( C := \mathbb{P} \times \mathbb{X} \)
2) For all \( k \), \( [\hat{p}^*(t_k)]^{+\tau_k} \in \mathbb{P} \)
3) For all \( k \), \( x(t_k) \in \mathbb{X} \)
4) Assumptions 5.2, 5.4 and 5.5 are satisfied

then the following inequality holds for all \( k \):

\[
f_0(\hat{p}^*(t_{k+1}), x(t_{k+1})) - f_0(\hat{p}^*(t_k), x(t_k)) \leq \varepsilon_0^{(k)} + K^0(E_0^{\text{c}} + E_1^{\text{c}}\tau_k) + \varepsilon_0^{(k+1)} - \Delta(\tau_k, x(t_k)) \quad (42)
\]

**PROOF.** See Appendix G.

Note that the term \( f_0(\hat{p}^*(t_k), x(t_k)) \) represents the value of the cost function at the effectively visited pairs \( (\hat{p}(t_k), x(t_k)) \). Therefore, the difference expressed in the l.h.s of (42) is relevant for the stability assessment of the resulted truncated MPC implementation. On the other hand, using the definition (40) of \( \tau_k \), the r.h.s of (42) can be viewed as a function of the precision parameter \((\varepsilon_0^{(k+1)}, \varepsilon_{\psi}^{(k+1)})\). The stability issue is therefore dependent on the possibility to define these precision parameters in such a way that the r.h.s of (42) is negative. This is the aim of the following development.

Since the only negative term in the r.h.s of (42) is \(-\Delta(\tau_k, x(t_k))\), we need a lower bound on \( \Delta(\tau_k, x(t_k)) \). The following straightforward lemma gives such a lower bound:

**Lemma 5.2:** If the following conditions hold:

1) \((\hat{p}^*(t_k), x(t_k)) \in C\)
2) Assumption 5.6 is satisfied

then a computable lower bound of the quantity \( \Delta(\tau, x(t_k)) \) can be obtained by:

\[
\Delta(\tau, x(t_k)) \geq \Gamma_C(\tau, q(x(t_k))) \quad (43)
\]

where \( \Gamma_C(\tau, q) \) is given by (see Figure 3):

\[
\Gamma_C(\tau, q) := \begin{cases} 
q^2 - \frac{1}{2} D_C \tau^2 & \text{if } \tau \leq q/D_C \\
\frac{q^2}{2D_C} & \text{otherwise}
\end{cases} \quad (44)
\]

**PROOF.** See Appendix H.

Using the definition (40) of \( \tau_k \) and the r.h.s of (43) in (42) the following computable function can be defined:

\[
R_{\tau_k}(\varepsilon_0, \varepsilon_{\psi}, \bar{q}) := K^0(E_0^{\text{c}} + \tau_k E_1^{\text{c}} N(\varepsilon_0, \varepsilon_{\psi})) + \varepsilon_0 - \Gamma_C(\tau_k, \bar{N}(\varepsilon_0, \varepsilon_{\psi}, \bar{q})) \quad (45)
\]

so that the following corollary of Lemma 5.1 can be stated:

**Corollary 5:** If the following conditions hold

1) The requirements of Lemma 5.1 are satisfied
2) Assumption 5.6 holds
3) \( q(x(t_k)) \geq \bar{q} \)

then the following inequality holds:

\[
f_0(\hat{p}^*(t_{k+1}), x(t_{k+1})) - f_0(\hat{p}^*(t_k), x(t_k)) \leq \varepsilon_0^{(k)} + R_{\tau_k}(\varepsilon_0^{(k+1)}, \varepsilon_{\psi}^{(k+1)}, \bar{q}) \quad (46)
\]

where \( R_{\tau_k}(\cdot) \) is defined by (45).

Figure 4 presents a typical situation showing that for a given past achieved precision \( \varepsilon_0^{(k)} \), a given computational power
leading to the computation time $\tau_c$ and a given precision $\varepsilon_\psi$ on the soft constraints satisfaction, either there is no $\varepsilon_0^{(k+1)}$ making the r.h.s of equation (45) involved in corollary 5 negative or there is an interval of successful values of $\varepsilon_0^{(k+1)}$ which does not contain 0 and which depends on the current value of $q(x(t_k)) = \bar{q}$.

Note that corollary 5 involves quantities that depend on some compact set to which belong all the pair $(\hat{p}^*(t_k))^{+\tau_c}, \hat{x}(t_{k+1})$. Using assumption 5.3 it is possible to prove that such compact set is linked to a set of initial conditions for which a certified convergence result can be derived for the resulting real-time MPC. This is stated in the following proposition which is the main contribution of the paper:

**Proposition 5.2:** Consider a positive real $\phi_0 > 0$ and the corresponding compact subset $C_{\phi_0} \subset \mathbb{R}^{np} \times \mathbb{R}^n$ defined according to assumption 5.3. Let be given a precision $\varepsilon_\psi > 0$ on the soft constraints satisfaction.

If the following conditions hold with $C = C_{\phi_0}$:

1) Assumptions 5.2, 5.4, 5.5 and 5.6 are satisfied
2) $\exists \bar{q}_{\min} > 0$ and $\gamma_c > 0$ such that the inequality:

$$R_{\tau_c}(\varepsilon_0, \varepsilon_\psi, \bar{q}) \leq \left[ \frac{\gamma_c \bar{q}_{\min}^2}{3D_{C_{\phi_0}}} \right]$$  \hspace{1cm} (47)

admits a solution $\varepsilon_0^{sol}(\bar{q}) \in [0, \gamma_c \bar{q}_{\min}^2/(2D_{C_{\phi_0}})]$ for all $\bar{q} \geq \bar{q}_{\min}$

then the truncated MPC design based on the adaptive sampling period defined by:

$$\tau_k := \tau_c \times \hat{N}(\varepsilon_0^{sol}(q(x(t_k))), \varepsilon_\psi) \hspace{1cm} (48)$$

steers the system to the set:

$$X_{\min} := \{ x \in \mathbb{R}^n \mid q(x) \leq \bar{q}_{\min} \} \hspace{1cm} (49)$$

provided that the initial condition satisfies:

$$f_0(\hat{p}^*(t_0), x(t_0)) \leq \phi_0 \hspace{1cm} (50)$$

Moreover, if the hard constraints depend only on $p$, then along the closed-loop trajectory, one has:

$$\max_{i \in I_k, k \geq 0} [c_i(\hat{p}^*(t_k), x(t_k))] \leq 0$$

$$\max_{i \in I_k, k \geq 0} [c_i(\hat{p}^*(t_k), x(t_k))] \leq \varepsilon_\psi + R_0^{\psi} \cdot (E_{C_{\phi_0}}^0 + E_{C_{\phi_0}}^1 \tau_k) \hspace{1cm} (51)$$

**Proof.** See Appendix 1.

**C. Case of linear MPC**

Linear MPC formulation applies to system of the form:

$$\dot{z} = A_0 z + B_0 u \hspace{1cm} (52)$$

in order to stabilize the physical state $z$ around some desired value $z_d$. We assume for the sake of simplicity that $z_d$ is a steady state for (52) that corresponds to the steady control $u_d = 0$. Using the extended system with the extended state $x = (z^T, z_d^T)$ and the extended dynamic built up using (52) with $\dot{z}_d = 0$, one obtains the controlled system model given by:

$$\dot{x} = A_s x + B_s u \hspace{1cm} (53)$$

where $x$ is an extended state containing the set-point and disturbance model state and where the cost function (56) is given by:

$$\tilde{\ell}(s, p, x) := \frac{1}{2} [q(\hat{x}(s, p, x)) + ||U(s, p)||^2_R] \hspace{1cm} (54)$$

where $q(x)$ is given by:

$$q(x) = ||z - z_d||^2_Q := ||C x||^2_Q \hspace{1cm} (55)$$

The control parametrization map $U(\cdot, p)$ used in (56) gives the control profile over the prediction horizon as a function of the finite dimensional parameter vector $p$.

This formulation leads to state-dependent QP where the cost function and the constraints are given by:

$$f_0(p, x) = \frac{1}{2} p^T H p + (F_1 x)^T p + x^T S x \hspace{1cm} (56)$$

$$A_p \leq B^{(0)} + B^{(1)} x \hspace{1cm} (57)$$

It results that the definition of $L_0$, $L_\psi$ and $\mu_0$ remains unchanged since these parameters depends only on the state independent quantities $H$ and $A$.

It is also assumed that the formulation involves appropriate final constraints such that (57) of Assumption 5.3 holds with $\Delta(\tau, x)$ satisfying:

$$\Delta(\tau, x) \geq \int_0^{\tau} q(\hat{x}(s, p^{opt}, x))ds \hspace{1cm} (58)$$

This can be obtained through appropriate final equality constraints that can be explicitly embedded in the control parametrization map $U(\cdot, p)$ or through softened final inequality constraints as suggested in [10].

Given a set of interest $X$, the upper bound on $D_0$ defined by (23)-(25) can be used provided that $||F||$ is replaced by:

$$\sup_{x \in X} ||F_1 x|| \leq ||F_1|| \times \varrho(X) \hspace{1cm} (59)$$

The computation of $\psi^{max}$ invoked in (31) of proposition 5.1 is obtained according to:

$$\psi^{max} := \max_{x \in X} \left[ \sum_{i=1}^{n_x} \max\{0, M_i x - L_i\} \right]^2 \hspace{1cm} (60)$$

where

$$M_i := - \left[ A_i H^{-1} F_1 + B_i^{(1)} \right]$$

$$L_i := B_i^{(0)}$$

and $A_i$ and $B_i^{(j)}$ denote the $i$-th line of $A$ and $B^{(j)}$ respectively. Note that the optimization problems (60) can be computed once for all using available NLP solvers for a
Once $\psi^{\text{max}}$ is computed, the resulting $\kappa^{\text{max}}_0$ involved in Proposition 5.1 (item (4)) can be computed and used in the computation of $\rho^{\text{max}}$. Finally, the parameter $f_0^{\text{max}}$ involved in Proposition 5.1 is computed according to:

$$f_0^{\text{max}} := \max_{(p,x) \in \mathbb{P} \times \mathbb{X}} \begin{bmatrix} p \\ x \end{bmatrix}^T \begin{pmatrix} H & F_1 \\ F_1^T & S \end{pmatrix} \begin{bmatrix} p \\ x \end{bmatrix} := W$$

which admits the upper bound:

$$f_0^{\text{max}} \leq [\lambda^{\text{max}}(W)] \times \max_{x \in \mathbb{P} \times \mathbb{X}} \|z\|^2 = [\lambda^{\text{max}}(W)] \times \varrho(\mathbb{P} \times \mathbb{X})$$

It remains to give explicit computation of $D_\mathcal{C}$ in (38) of Assumption 5.6. This is given by the following proposition:

**Proposition 5.3:** If the constraints $p \in \mathbb{P}$ implies that $\mathcal{U}(s,p) \in \mathcal{U}$ for some compact set $\mathcal{U}$, then the following expression of $\mathcal{D}_\mathcal{C}$ meets the requirement of Assumption 5.6:

$$D_\mathcal{C} = \lambda^{\text{max}}(Q) \times \varrho(\mathbb{X}) \times \|\langle A_s \varrho(\mathbb{X}) \rangle + \|B_s\| \varrho(\mathbb{U})\|$$

**Proof.** Compute the derivative of $\|\bar{C}x(s,p,x)\|_Q^2$ which takes the values $\varrho(x)$ at $s = 0$ and derive a lower bound on the speed with which this term may converge to 0 given the compact set to which belongs the arguments $x$ and $p$. □

The next result concerns the explicit derivation of the compact set $\mathcal{C}_\phi$ given an initial cost function level $\phi$ as described in Assumption 5.3. This is the aim of the following result:

**Proposition 5.4:** If the possible set points $z_d$ belong to a compact set $\mathcal{Z}_d$, then given $\phi$, the compact set $\mathcal{C}_\phi$ involved in (33) of Assumption 5.3 is given by $\mathcal{C}_\phi := \mathbb{P} \times \mathbb{X}$ where:

$$\mathbb{P} := \left\{ p \ s.t \ |p| \leq \frac{\varrho/\lambda^{\text{min}}(W_0)}{2} \right\}$$

$$\mathbb{X} := \left\{ x \ s.t \ |x| \leq \varrho(\mathcal{Z}_d) + \left|\frac{\varrho/\lambda^{\text{min}}(W_0)}{2}\right| \right\}$$

where $W_0$ is the matrix given by:

$$W_0 = \begin{pmatrix} H & F_{11} \\ F_{11}^T & S_{11} \end{pmatrix}$$

where $F_{11} \in \mathbb{R}^{n_z \times n_z}$ and $S_{11} \in \mathbb{R}^{n_z \times n_z}$ are the corresponding sub-matrices of $F_1$ and $S$ involved in (56) respectively.

**Proof.** See Appendix II.

Once the compact set $\mathcal{C}_\phi$ is computed for a given initial value $\phi$ of the cost function, the constants $K^0_\mathcal{C}$ and $K^1_\mathcal{C}$ involved in (34) and (35) of Assumption 5.4 can be explicitly computed using the following proposition:

**Proposition 5.5:** For the cost function $f_0$ defined by (56) and the constraints defined by (57), given a compact set $\mathcal{C} := \mathbb{P} \times \mathbb{X}$, the constants $K^0_{\mathcal{C}}$ and $K^1_{\mathcal{C}}$ involved in (34) and (35) of Assumption 5.4 can be explicitly computed using the following proposition:

$$K^0_{\mathcal{C}} := \|F_1^T\| \times \varrho(\mathbb{P}) + 2\lambda^{\text{max}}(S) \times \varrho(\mathbb{X})$$

$$K^1_{\mathcal{C}} := 2n_c[\psi^{\text{max}}] \times \|(B_1)^T\|$$

where $\psi^{\text{max}}$ is computed by (60).

**Proof.** See Appendix II.

The only remaining parameters are $E^0_\mathcal{C}$ and $E^1_\mathcal{C}$ involved in (28) of Assumption 5.2 and which describe the prediction error on the extended state $x$ as a function of $\tau$. Note that if the model is perfectly known, the only prediction error comes from the fact that the future evolution of the set-point $z_d$ is unknown. Two cases can be distinguished:

- If the set point if filtered, then

$$E^0_{\mathcal{C}} = 0 \text{ and } E^1_{\mathcal{C}} = \max_t(\|\dot{z}_d(t)\|)$$

- Otherwise

$$E^0_{\mathcal{C}} = \varrho(\mathcal{Z}_d) \text{ and } E^1_{\mathcal{C}} = \max_t(\|\dot{z}_d(t)\|)$$

In case other sources of prediction errors prevail, then an additional positive term $e_1$ has to be added so that $E^1_{\mathcal{C}} = \max_t(\|\dot{z}_d(t)\|) + e_1$ is used.

1) Illustrative example: MPC control of a chain of integrators: Let us consider MPC control of a chain of $n$ integrators given by:

$$\dot{z}_i = z_{i+1} \quad \text{for } i = 1, \ldots, n - 1$$

$$\dot{z}_n = u \quad \text{under } |u| \leq \bar{u} = 10$$

in which the objective is to track a reference trajectory on $z_1$ under the state constraints:

$$\begin{pmatrix} -2 \\ -1 \end{pmatrix} \leq \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} \leq \begin{pmatrix} +2 \\ +1 \end{pmatrix}$$

using the formulation of section VC. This is obviously a very important sub-class of systems that is heavily used in Mechatronics.

We consider a parametrization of the form:

$$\mathcal{U}(s,p_u(t)) = [\Phi_u(s)]p_u(t) : p_u \in \mathbb{R}^m$$

in which a final constraint on the state is imposed:

$$\|z(T) - Z_d\| = Cx(T) = 0$$

By doing this, the stability of the ideal perfect scheme is guaranteed with Assumption 5.3 satisfied. The final constraint satisfaction can be imposed through the reduced parametrization:

$$p_u = Kp + Mx_0$$

where the matrices $K$ and $M$ depend on the function basis $\phi_u$ involved in (72) and the prediction horizon $T$ such that taking $p = 0$ always leads to $p_u$ that satisfies the final constraint. This means that because of the saturation constraint (70), the final
constraint \( \mathbf{73} \) can be feasible through \( \mathbf{44} \) only for initial state \( x_0 \) such that
\[
\| M x_0 \| \leq \bar{u}
\] (75)
leading to the bound \( \| x_0 \| \leq \bar{u}/\| M \| \) (\( \approx 11.3 \) in the case \( n = 4 \)). By doing this, the number of decision variables is given by \( n_p = m - n \). In the following results, the weighting matrices \( Q = I_n \) and \( R = 0.001 \) are systematically used in \( \mathbf{55} \) and \( \mathbf{56} \). A prediction horizon \( T = 10 \) is used in the sequel while \( m = 10 \) dimensional parametrization variable is used in control parametrization \( \mathbf{72} \). This leads to a number of free decision variables of dimension \( n_p = 6 \).

The computation time for a single iteration \( \tau_c = 0.1 \mu s \) is used (time needed for a matrix-vector multiplication in Step 10 of Algorithm 2). It is supposed that the reference set-point defined by \( q \geq \bar{q}_{\text{min}} \) is satisfied for \( \tau_c \) given by \( \mathbf{47} \).

Note that in order to check the existence of \( q_{\text{min}} \) satisfying the condition \( \mathbf{47} \) of Proposition \( \mathbf{52} \), one can check the existence of solution \( \varepsilon_0^* \) to the inequality
\[
R_{\tau_c}(\varepsilon_0, \varepsilon, \bar{q}_{\text{min}}) \leq -\left[ \frac{\varepsilon q_{\text{min}}^2}{3D_{\varepsilon_0}} \right]
\] (76)
since if \( \mathbf{76} \) is satisfied for \( \bar{q}_{\text{min}} \) in the l.h.s, it will be satisfied for any \( \bar{q} \geq \bar{q}_{\text{min}} \) used in the l.h.s while \( \bar{q}_{\text{min}} \) is used in the r.h.s. An additional condition invoked in Proposition \( \mathbf{52} \) states that this solution \( \varepsilon_0^* \) must be such that:
\[
\varepsilon_0^* \leq \left[ \frac{2c q_{\text{min}}^2}{D_{\varepsilon_0}} \right]
\] (77)

Note also that the parameter \( \phi_0 \) invoked in \( \mathbf{76} \) defines an upper bound on the possible initial value of the cost \( f_0(p, x) \). Therefore, in the case of hot starts, the size of \( \phi_0 \) can define the quality of the hot start. Otherwise, one can take an upper bound pessimistic value \( \phi_0 \) by starting from \( p = 0 \) and taking the upper value of \( f_0(0, x) \) over the set of admissible initial state defined by \( \mathbf{75} \), namely:
\[
\phi_0 \leq \lambda_{\text{max}}(S) \left[ \frac{\bar{u}}{\| M \|} \right]
\] (78)

Based on the knowledge of \( \phi_0 \) given by \( \mathbf{78} \), the condition \( \mathbf{76} \) can be checked for different candidate values of \( \bar{q}_{\text{min}} \).

Figures \( \mathbf{5} \) and \( \mathbf{6} \) shows the results for the cases \( n = 4 \) and \( n = 2 \) respectively. More precisely, Figure \( \mathbf{5} \) shows that for the quadruple integrator system under an unknown future behavior of the set-point characterized by \( E^1_c = 0.05 \), the certification conditions \( \mathbf{76} \) and \( \mathbf{77} \) are satisfied with \( \bar{q}_{\text{min}} = 0.36 \) and \( \gamma_c = 0.2 \).

Figure \( \mathbf{6} \) shows that the certification is possible for the double integrator system with the unknown behavior of the set-point defined by \( E^2_c = 0.3 \) provided that \( \bar{q}_{\text{min}} = 0.18 \) and

![Checking conditions \( \mathbf{76} \) and \( \mathbf{77} \)](image)

\( \bar{K}^0_cE^1_c\tau_c\bar{N}(\varepsilon_0) \)

\( \bar{N} = 0 \)

\( \Gamma(\tau_c\bar{N}(\varepsilon_0), \bar{q}_{\text{min}}) \)

l.h.s of \( \mathbf{76} \)

r.h.s of \( \mathbf{76} \)

r.h.s of \( \mathbf{77} \)

\( K \)

\( E \)

\( C \)

\( q \)

\( \varepsilon \)

\( \bar{u} \)

\( M \)

\( D \)

\( x \)

\( f_0 \)

\( \lambda \)

\( \phi_0 \)

\( \bar{q}_{\text{min}} \)

\( \gamma_c \)

\( \varepsilon_0 \)

\( \varepsilon_0^{\text{sol}}(x) \)

\( (1 - \lambda)\varepsilon_0(q(x)) + \lambda \varepsilon_0(q(x)) \) (79)

\( \lambda \in [0.5, 0.9] \)

\( \bar{N} \)

\( q_{\text{min}} \)

VI. CONCLUSION

In this paper a certification bound on the convergence of the fast gradient algorithm when applied to solve convex
Fig. 6. Checking of the certification feasibility for the chain of
integrators.

The condition (76) is satisfied by an interval of values of \( \varepsilon_0 \) including values
satisfying (77). Successful values: \( q_{\text{min}} = 0.18, \gamma_c = 0.2, E_1^{\varepsilon_0} = 0.3 \).

Fig. 7. Quadruple integrator: Evolution of the bounding values \( \varepsilon_0(q(x)) \) and
\( \bar{\varepsilon}_0(q(x)) \) and a possible state dependent precision \( \varepsilon_0^{\text{opt}}(x) \) defined by (79).

optimization problems with general inequality constraints with
a prescribed level of sub-optimality is first given. The resulting
bound is then used to derive a real-time implementation of MPC with state-dependent updating period leading to certified
convergence of the resulting closed-loop to a neighborhood
of the desired set-point. The proposed results clearly showed
that the time needed to perform the elementary iteration is
a key parameter in the resulting MPC implementation.

To this respect, the proposed results can be used to afford
limited computational power or to compute, for a given control
problem and a given specification in terms of optimality and
constraints fulfillment, the admissible computation power that
need to be assigned.

APPENDIX

A. Proof of Lemma 3.7

This comes from the fact that \( p^* \) is the unconstrained optimum of \( f \) which means that:

\[ f_0(p^*) + \rho \psi(p^*) \leq f_0(p_u) + \psi(p_u) = f_0(p_u) \]  \hspace{1cm} (80)

and since \( \psi(p^*) \geq 0 \), the last inequality gives \( f_0(p^*) \leq f_0(p_u) \). The inequality to be proved is therefore a simple
consequence of the definition 3.1 of \( D_0 \).

B. Proof of Lemma 3.8

Let us denote by \( p_\psi \) the closest element of \( A_{\psi=0} \) to \( p^* \).
The triangular inequality implies:

\[ \| p - p_\psi \| \leq \| p - p^* \| + \| p^* - p_\psi \| \leq (d(p) + \| p^* - p_\psi \| \]  \hspace{1cm} (81)

and because \( \psi \in \mathcal{F}_{L_\psi} \):

\[ \psi(p) \leq \psi(p_\psi) + \langle \psi'(p_\psi), p - p_\psi \rangle + \frac{L_\psi}{2} \| p - p_\psi \|^2 \]  \hspace{1cm} (82)

but since \( p_\psi \in A_{\psi=0} \), one has that \( \psi(p_\psi) = 0 \) and \( \psi'(p_\psi) = 0 \)
[because of the particular structure of the penalty], therefore
(82) becomes (because of (81)):

\[ \psi(p) \leq \frac{L_\psi}{2} \| p - p_\psi \|^2 \leq \frac{L_\psi}{2} [d(p) + \| p^* - p_\psi \| \]  \hspace{1cm} (83)

It remains to prove that the term \( \| p^* - p_\psi \| \) can be bounded so that the inequality (10) holds. Note that since \( p^* \) minimizes
\( f \), one has:

\[ f_0(p_\psi) + \rho \psi(p_\psi) = f_0(p^*) + \rho \psi(p^*) \]

and since \( \psi(p_\psi) = 0 \) the last inequality leads to:

\[ \psi(p^*) \leq \frac{1}{\rho} \left[ f_0(p_\psi) - f_0(p^*) \right] \]  \hspace{1cm} (84)

\[ \leq \frac{1}{\rho} \left[ \| p^* - p_\psi \| \cdot \| p^* - p_\psi \| + \frac{L_0}{2} \| p^* - p_\psi \| \right] (85)

Now let \( p_u \) be the unconstrained minimizer of \( f_0 \), namely
\( f_0(p_u) = 0 \). Note that \( p_u \) is uniquely defined since \( \mu_0 > 0 \) by assumption. Now by definition of \( p^* \) and \( p_u \), one has:

\[ f_0(p^*) + \rho \psi(p^*) \leq f_0(p_u) + \rho \psi(p_u) \]  \hspace{1cm} (86)
on the other hand,
\[
    f_0(p^*) \geq f_0(p_a) + \frac{\mu_0}{2} p^* - p_a \quad (87)
\]
By combining (86)-(87), it comes that:
\[
    \|p^* - p_a\| \leq \sqrt{\frac{2p_0}{\mu_0}} p(p_a)
\]
This with the Lyapitsch induced inequality gives:
\[
    \|f_0'(p^*) - 0\| \leq L_0 \|p^* - p_a\| \leq L_0 \sqrt{\frac{2p_0}{\mu_0}} p(p_a) =: \kappa_0 \sqrt{\beta}
\]
where \(\kappa_0 := L_0 \sqrt{2p_0}/\mu_0\). This last inequality together with (85) implies:
\[
    \psi(p^*) \leq \frac{1}{\rho} \left[ \kappa_0 \sqrt{\beta} \|p^* - p_\psi\| + \frac{L_0}{2} \|p^* - p_\psi\|^2 \right] \quad (88)
\]
Now using (8) in which \(d(p^*, \mathcal{A}_{\psi=0}) = \|p^* - p_\psi\|\) gives:
\[
    \beta \|p^* - p_\psi\|^2 \leq \frac{1}{\rho} \left[ \kappa_0 \sqrt{\beta} \|p^* - p_\psi\| + \frac{L_0}{2} \|p^* - p_\psi\|^2 \right]
\]
and after straightforward manipulations, it comes that:
\[
    \left[ \beta - \frac{L_0}{2\rho} \right] \|p^* - p_\psi\| \leq \frac{\kappa_0}{\sqrt{\beta}} \quad (89)
\]
Now assuming that \(\rho \geq L_0/\beta\), one obtains:
\[
    \|p^* - p_\psi\| \leq \frac{2\kappa_0}{\beta \sqrt{\rho}} = \frac{2L_0 \sqrt{2p_0(p_a)/\mu_0}}{\beta \sqrt{\rho}}
\]
which together with (83) clearly ends the proof since the inequality (11) is a direct consequence of the fact that \(d(p^*) = 0\) by definition.

\[\square\]

C. Proof of Lemma 3.3

Since \(p_\psi\) is admissible and \(p^{opt}\) is the optimal solution of the constrained problem, one necessarily has:
\[
    f_0(p^{opt}) \leq f_0(p_\psi) \quad (90)
\]
Moreover, since \(f_0 \in \mathcal{X}_{L_0}\), the following inequality holds:
\[
    f_0(p_\psi) \leq f_0(p^*) + \|f_0'(p^*)\| \cdot \|p^* - p_\psi\| + \frac{L_0}{2} \|p^* - p_\psi\|^2
\]
and since \(f_0'(p^*)\) \leq D_0 (Lemma 3.1):
\[
    f_0(p_\psi) \leq f_0(p^*) + D_0 \|p^* - p_\psi\| + \frac{L_0}{2} \|p^* - p_\psi\|^2 \quad (91)
\]
which together with (90) and (8) of Assumption 3.4 gives:
\[
    |f_0(p^{opt}) - f_0(p^*)| \leq D_0 \left[ \frac{\psi(p^*)}{\beta} \right]^\frac{1}{2} + \frac{L_0}{2} \left[ \frac{\psi(p^*)}{\beta} \right] \quad (92)
\]
This obviously ends the proof.

\[\square\]

D. Proof of Lemma 3.4

Assume that for some \(p\) the following inequality hold:
\[
    |f(p) - f(p^*)| \leq \varepsilon \quad (93)
\]
this means that \((f \in \mathcal{S}_{\varepsilon})\):\[
    \|p - p^*\| \leq \left[ \frac{2\varepsilon}{\mu_0} \right]^\frac{1}{2} \quad (94)
\]
on the other hand:
\[
    |f_0(p) - f_0(p^*)| \leq D_0 \|p - p^*\| + \frac{L_0}{2} \|p - p^*\|^2 \quad (95)
\]
this together with (94) gives the result. \[\square\]

E. Proof of Proposition 4.2

PROOF. We shall first prove that when the algorithm stops, one has:
\[
    |f(\hat{p}^*) - f(p^*)| \leq \eta \quad (96)
\]
then we prove that when (96) holds then \(\hat{p}^*\) is an \(\varepsilon\)-suboptimal solution of the original problem. To prove (96), we shall distinguish two situations depending on the exit condition of step 10. Indeed, either \(g(\hat{p}_1) \leq g_{\min} := \mu_0 \sqrt{2\eta}/L\) in which case (96) is satisfied since \(f \in \mathcal{S}_{\varepsilon}\). Or the algorithm stops after \(N(\varepsilon, \gamma_0)\) iterations where \(\gamma_0 := \eta \mu_0/([L + \mu_0] f(\hat{p}_0))\) which implies (96) by virtue of Corollary 4.

We shall now prove that when (96) holds, one necessarily has:
\[
    |f_0(\hat{p}^*) - f_0(p^{opt})| \leq \varepsilon_0 \quad ; \quad \psi(\hat{p}^*) \leq \varepsilon_0^2 \quad (97)
\]
Proof of \(\psi(\hat{p}^*) \leq \varepsilon_0^2\).

By the \(\mu_0\)-strong convexity of \(f\), equation (96) implies that
\[
    \|\hat{p}^* - p^*\| \leq \sqrt{2\eta/\mu_0}\varepsilon.
\]
Injecting this in (10) gives:
\[
    \psi(\hat{p}^*) \leq \frac{L_0}{2} \left[ \sqrt{\frac{2\eta}{\mu_0}} + \frac{\kappa_0}{\sqrt{\rho}} \right]^2 \quad (98)
\]
So in order to prove that \(\psi(\hat{p}^*) \leq \varepsilon_0^2\), it is sufficient to prove the following two inequalities:
\[
    \sqrt{\frac{2\eta}{\mu_0}} \leq \frac{\varepsilon}{2\varepsilon_0^2} \quad \text{and} \quad \frac{\kappa_0}{\sqrt{\rho}} \leq \frac{\varepsilon}{2\varepsilon_0^2} \sqrt{\frac{2}{L_0}} \quad (99)
\]
But the first inequality is satisfied because \(\eta \leq \eta_2\) while the second is satisfied because \(\rho \geq \rho_1\).

Proof of \(|f(\hat{p}^*) - f_0(p^{opt})| \leq \varepsilon_0\).

Using the triangular inequality:
\[
    |f_0(\hat{p}^*) - f_0(p^{opt})| \leq |f_0(\hat{p}^*) - f_0(p^*)| + |f_0(p^*) - f_0(p^{opt})| \quad (100)
\]
and using (15) and (12) the last inequality gives:
\[
    |f_0(\hat{p}^*) - f_0(p^{opt})| \leq D_0 \left[ \frac{2\eta}{\mu_0} \right]^\frac{1}{2} + \frac{L_0}{2} \left[ \frac{2\eta}{\mu_0} \right] + D_0 \left[ \frac{\psi(p^*)}{\beta} \right]^\frac{1}{2} + \frac{L_0}{2} \left[ \frac{\psi(p^*)}{\beta} \right] \quad (101)
\]
The first inequality is satisfied since $\eta \leq \eta_1$ while the second is satisfied if:

$$\left[ \frac{1}{\beta} \right] \leq Z_1 \left( \frac{\varepsilon_0}{2} \right) \tag{100}$$

But thanks to (11) [satisfied since $\rho \geq \rho_3$] this can be proved if the following inequality holds:

$$\frac{L_\beta \rho_0^2}{2 \beta \rho} \leq Z_2 \left( \frac{\varepsilon_0}{2} \right) \tag{101}$$

which is satisfied because $\rho \geq \rho_2$. \hfill $\square$

\section*{F. Proof of Proposition 4.3}

Recall that in the specific case of QP problem, the definition of $D_0$ becomes

$$D_0 := \sup_{f_0(p) \leq f_0(p_a)} \| H p + F \|$$

But we have by assumption $\|p_a\| \leq p_{max}$, which enables to write:

$$f_0(p_a) \leq \frac{1}{2} \lambda_{max}(H) [\hat{p}(P)]^2 + \| F \| \cdot \hat{p}(P) + \phi_0 =: f$$

and since $f_0(p) \geq \frac{1}{2} \lambda_{min}(H) \| p \|^2 - \| F \| \| p \| + \phi_0$, the last inequality implies:

$$\| p \| \leq \frac{\| F \| + \sqrt{\| F \|^2 + 2 \lambda_{min}(H) [f - \phi_0]}}{\lambda_{min}(H)} =: \hat{p}$$

which obviously gives the result. \hfill $\square$

\section*{G. Proof of Lemma 5.1}

Using Assumptions 5.4 and 5.2, it comes that:

$$f_0(\hat{p}^*(t_{k+1}), x(t_{k+1})) \leq f_0(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})) +$$

$$+ K_{0} \times \left[ E_{c}'' + E_{k}^{1} \times \tau_{k} \right] \tag{102}$$

Now by definition of $\tau_{k}$, the solution $\hat{p}^*(t_{k+1})$ satisfies

$$f_0(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})) \leq f_0(p_{opt}^*(t_{k+1}), \hat{x}(t_{k+1})) + \varepsilon_{0}^{(k+1)}$$

which together with Assumption 5.5 gives:

$$f_0(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})) \leq$$

$$\Delta(\tau_{k}, x(t_{k})) - \Delta(\tau_{k}, x(t_{k}))$$

$$\leq$$

$$f_0(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})) + \varepsilon_{0}^{(k+1)} + \varepsilon_{0}^{(k+1)} - \Delta(\tau_{k}, x(t_{k})) \tag{103}$$

Using the last inequality in (102) gives the result. \hfill $\square$

\section*{H. Proof of Lemma 5.2}

By definition of (37) of $\Delta$ and using (38) of Assumption 5.6, it comes that:

$$\Delta(\tau, x) \geq \int_{0}^{\tau} \max\{0, q(x) - D_{C}s)\} ds$$

$$= \int_{0}^{\min\{\tau, q(x)/D_{C}\}} (q(x) - D_{C}s) ds$$

$$= \left[ q(x)\tau - \frac{1}{2} D_{C}\tau^{2} \right]_{0}^{\min\{\tau, q(x)/D_{C}\}} \tag{104}$$

which can be expressed using $\Gamma_{C}(\tau, q)$ given by (44). \hfill $\square$

\section*{I. Proof of Proposition 5.2}

The first inequality in (50), together with Assumption 5.3, imply that Corollary 5 applies with $k = 0$, $C = C_{\phi_0}$ and $\bar{q} := q(x(t_k))$, therefore one has:

$$f_0(\hat{p}^*(t_{1}), x(t_{1})) - f_0(\hat{p}^*(t_{0}), x(t_{0}))$$

$$\varepsilon_{0}^{(0)} + R_{\tau}(\varepsilon_{0}^{(1)}, \varepsilon_{0}, q(x(t_{0}))) \tag{105}$$

and since $\varepsilon_{0}^{(1)} = \varepsilon_{0}^{(0)}(q(x(t_{0})))$, if $q(x(t_{0})) > \bar{q}_{min}$ the inequality (47) gives:

$$R_{\tau}(\varepsilon_{0}^{(1)}, \varepsilon_{0}, q(x(t_{0}))) \leq -\frac{\gamma \varepsilon_{0}^{2}}{3D_{C_{\phi_0}}} \tag{106}$$

This together with (104) implies that $f_0(\hat{p}^*(t_{1}), x(t_{1}))$ decreases meaning that the new pair is still in $C_{\phi_0}$ and since $\varepsilon_{0}^{(1)}$ satisfies by assumption the second inequality in (50), the argumentation can be repeated to derive the properties of the next pair $(\hat{p}^*(t_{2}), x(t_{2}))$ meaning that the following inequality:

$$f_0(\hat{p}^*(t_{k+1}), x(t_{k+1})) - f_0(\hat{p}^*(t_{k}), x(t_{k})) \leq -\frac{\gamma \varepsilon_{0}^{2}}{6D_{C_{\phi_0}}} \tag{107}$$

is satisfied as far as $q(x(t_{k}))$ remains greater than $\bar{q}_{min}$. This clearly implies that $x(t_{k})$ converges to the limit set $\bar{x}_{min}$ defined (49).

regarding the constraints, note that the hard constraints are necessarily satisfied since they depend only on $p$ by assumption and that $\hat{p}^*(t_{k+1})$ satisfies by construction the hard constraints while allowing only for a violation of the soft constraints by an amount which is lower than $\varepsilon_{\phi}$, therefore, one has:

$$\varepsilon_{i}(\hat{p}^*(t_{k+1}), \hat{x}(t_{k+1})) \leq \varepsilon_{\phi} \quad \forall i \in I_s \tag{107}$$

which obviously gives (51) by Assumptions 5.2 and 5.4. \hfill $\square$
J. Proof of Proposition \[5.4\]

Given \( z_d \), one rewrite the cost function using the change of variable \( y = z - z_d \) enables to write the cost function \[56\] in the form

\[
 f_0(p, y) = \frac{1}{2} \left( y^T \begin{bmatrix} H & F_{11} \\ F_{11}^T & S_{11} \end{bmatrix} \right) \begin{bmatrix} F_{11}^T \\ S_{11} \end{bmatrix} \begin{bmatrix} y \end{bmatrix} \]

which means that if \( f_0(p, x) \leq \phi \) then the following inequalities hold:

\[
 \| p \| \leq \frac{\phi}{\lambda_{\text{min}}(W_0)} \quad ; \quad \| y \| \leq \frac{\phi}{\lambda_{\text{min}}(W_0)} \]

The first inequality obviously gives \[62\] while the second leads to:

\[
 \| z \| \leq \| z_d \| + \frac{\phi}{\lambda_{\text{min}}(W_0)} \]

which gives \[65\]. \[\square\]

K. Proof of Proposition \[5.5\]

In order to prove that \[65\] satisfies \[34\], we use the definition of \( f_0 \) to write:

\[
 \| f_0(p, x_1) - f_0(p, x_2) \| = \| (F_1(x_1 - x_2))^T p + \| x_1 \|^2 - \| x_2 \|^2 \| \\
 \leq \| (F_1^T p)^T (x_1 - x_2) \| + 2\lambda_{\text{max}}(S) \times \phi(X) \times \| x_1 - x_2 \| \\
 \leq \| F_1^T \| \times \phi(P) + 2\lambda_{\text{max}}(S) \times \phi(X) \times \| x_1 - x_2 \|
\]

which proves \[65\].

It remains to prove that \( K_{\epsilon}^1 \) defined by \[66\] satisfies \[35\] we first note that:

\[
 \psi(x) := \sum_{i=1}^{n_c} \| r_i(p, x) \|^2
\]

with \( r_i(p, x) = \max \{ 0, A_i p - B_i^0 - B_i^1 x \} \). Therefore:

\[
 \| \frac{\partial \psi}{\partial x} \| \leq 2 \sum_{i=1}^{n_c} \| r_i(p, x) \| \times \| B_i^{(1)} \|
\]

and using the inequalities expressing the equivalence of the \( L_2 \) and \( L_1 \) norms, the last inequality gives:

\[
 \| \frac{\partial \psi}{\partial x} \| \leq 2n_c \sum_{i=1}^{n_c} \| r_i(p, x) \|^2 \times \| (B^{(1)})^T \| \\
 \leq 2n_c \psi_{\text{max}} \times \| (B^{(1)})^T \|
\]

which obviously gives \[66\]. \[\square\]

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