Differential Passivity based Dynamic Controllers

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Abstract—In this paper, we develop two passivity based control methods by using variances of passivity techniques; they are applicable for a class of systems for which the standard passivity based controllers may be difficult to design. As a preliminary step, we establish the connections among four relevant passivity concepts, namely differential, incremental, Krasovskii’s and shifted passivity properties as follows: differential passivity $\Rightarrow$ incremental passivity $\Rightarrow$ shifted passivity, and differential passivity $\Rightarrow$ Krasovskii’s passivity. Then, based on our observations, we provide two novel dynamic controllers based on Krasovskii’s and shifted passivity properties.

I. INTRODUCTION

Passivity as a tool enables us to develop various types of passivity based control (PBC) techniques, and moreover as a property, it helps us to understand these techniques in the standard engineering parlance. Lyapunov analysis is always discussed with respect to an equilibrium or an operating point. However, notions like incremental stability and contraction analysis [2], [3] study the convergence between the pair of trajectories. These differences have resulted in diverse stability definitions, which further resulted in disparate passivity definitions such as incremental passivity and differential passivity. There are several papers that describes these relatively new passivity concepts [4]–[6]. Apart from the elegance of analysis, it is not well understood how differential passivity can be used either as a tool or as a property although there is a few differential passivity based control techniques [7]–[10]. This is significantly different from the successive development of passivity analysis or relevant control techniques.

If the system has an operating point, incremental passivity results into the so-called shifted passivity at the operating point. Shifted passivity can be interpreted as a generalization of standard passivity for a system whose operating point is not necessarily the origin. Thus by removing the assumption that the operating point is the origin, shifted passivity is replacing standard passivity. This has been applied to various situations, see, e.g. [11]–[13]. However, for differential passivity, there has been no relevant passivity concept at an operating point until the preliminary conference version [1] of this paper. As for the shifted passivity, this missing passivity concept can also be useful for analysis and controller design.

In this paper, we establish a new passivity concept, which we call Krasovskii’s passivity. Then, we marshal aforementioned relevant four passivity concepts: differential passivity, Krasovskii’s passivity, incremental passivity, and shifted passivity. Especially, we show that differential passivity with respect to a constant metric implies the other three passivity properties. Furthermore, we provide novel dynamic control techniques based on Krasovskii’s passivity. Finally, our results are illustrated by the stabilization problem of a DC-Zeta converter. It is worth mentioning that for this converter, a passivity based controller has not been designed in the literature.

In the preliminary conference version [1], we have proposed Krasovskii’s passivity, provided sufficient conditions for port-Hamiltonian and gradient systems to be Krasovskii’s passivity, and gave a brief introduction of Krasovskii’s passivity based control techniques. However, the relation between the four passivity concepts has not been well investigated. Moreover, in this paper, we present a more general Krasovskii’s passivity based dynamic controller, and newly provide a shifted passivity based dynamic controller.

The remainder of this paper is organized as follows. In Section II, we define Krasovskii’s passivity and establish the connection among differential passivity, Krasovskii’s passivity, incremental passivity, and shifted passivity. In Section III, we design two novel passivity based dynamic controllers based on Krasovskii’s and shifted passivity properties. In Section IV, our dynamic controllers are illustrated by the stabilization problem of a DC-Zeta converter. Finally, Section V concludes this paper.

Notation: The set of real numbers and non-negative real numbers are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively. For a vector $x \in \mathbb{R}^n$ and a symmetric and positive semidefinite matrix $M \in \mathbb{R}^{n \times n}$, define $\|x\|_M := (x^T M x)^{1/2}$. If $M$ is identity, this is nothing but the Euclidean norm and is simply denoted by $\|x\|$. For symmetric matrices $P, Q \in \mathbb{R}^{n \times n}$, $P \leq Q$ implies that $Q - P$ is positive semidefinite.

II. ANALYSIS OF PASSIVITY PROPERTIES

A. Preliminaries and Motivating Examples

Consider the following input-affine nonlinear system:

$$\dot{x} = f(x,u) := g_0(x) + \sum_{i=1}^m g_i(x)u_i,$$  

(1)

where $x : \mathbb{R} \to \mathbb{R}^n$ and $u = [u_1, \ldots, u_m]^T : \mathbb{R} \to \mathbb{R}^m$ denote the state and input, respectively. Functions $g_i: \mathbb{R}^n \to \mathbb{R}^n$, $i = 0, 1, \ldots, m$ are of class $C^1$, and define $g := [g_1, \ldots, g_m]$ by using the latter $m$ vector valued functions. Denote $\psi(t, x_0, u)$ by the solution to the system (1) at time $t$ starting from initial condition $x(0) = x_0 \in \mathbb{R}^n$ with the control input $u$.

In this paper, our objective is to design controllers based on variants of passivity concepts. For standard passivity, there are plenty of rich results useful for analysis and controller design. However, for some classes of systems, the standard passivity
concepts cannot be established easily as demonstrated in [1].
Our approach to deal with such systems is to investigate
different passivity properties. These passivity properties are
developed by appropriately applying the following dissipativity
concept.

Definition 1: ([14]–[16]) The system (1) is said to be dissi-
pative with respect to a supply rate \( w : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) if
there exists a class \( \mathcal{C}^1 \) storage function \( S : \mathbb{R}^n \to \mathbb{R}_+ \) such
that \( S(x^*) = 0 \) at some \( x^* \in \mathbb{R}^n \) and

\[
\frac{\partial S(x)}{\partial x} f(x, u) \leq w(x, u)
\]

(2)

for all \((x, u) \in \mathbb{R}^n \times \mathbb{R}^m\).

In the above definition, if there is a function \( h : \mathbb{R}^n \to \mathbb{R}^m \)
such that \( w(x, u) = h^\top(x)u \), then the system (1) is passive
with respect to the input \( u \) and the output \( y = h(x) \) in the
standard sense [15], [16].

B. Differential and Incremental Passivity Properties

First, we provide the definition of differential passivity and
its necessary and sufficient condition given by [5]. Differential
passivity is introduced by using the so-called variational
system associated with the nonlinear system (1):

\[
\dot{x} := \frac{d(\delta x)}{dt} = F(x, u)\delta x + \sum_{i=1}^{m} g_i(x)\delta u_i,
\]

(3)

\[
F(x, u) := \left( \frac{\partial g_0(x)}{\partial x} + \sum_{i=1}^{m} \frac{\partial g_i(x)}{\partial x} u_i \right),
\]

where \( \delta x : \mathbb{R} \to \mathbb{R}^n \) and \( \delta u = [\delta u_1, \ldots, \delta u_m]^\top : \mathbb{R} \to \mathbb{R}^m \)
denote the state and input of the variational system, respecti-
vely. Hereafter, we call the system (1) together with (3) the
prolonged system of (1).

Definition 2 (Differential passivity [3]): Let \( h_D : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \). Then the nonlinear system (1) is said to be differentially passive if its prolonged system is dissipative
with respect to the supply rate \( \delta u^\top h_D(x, \delta x) \) with a storage
function in the form \( S_D(x, \delta x) \).

As a specific case of [5] Proposition 4.1 with a constant
metric, we have the following necessary and sufficient condi-
tion for differential passivity.

Proposition 3: Let \( M \in \mathbb{R}^{n \times n} \) be a symmetric and positive
semidefinite matrix. A system (1) is differentially passive with
respect to supply rate \( \|M\|\delta x \)

\[
w_D(\delta x, \delta u) := \delta u^\top g^\top(x)M\delta x
\]

(4)

with storage function

\[
S_D(x, \delta x) := \frac{1}{2} \delta x^\top M \delta x
\]

(5)

if and only if

\[
M_{g_0}(x) := M \frac{\partial g_0(x)}{\partial x} + \frac{\partial^\top g_0(x)}{\partial x} M \leq 0,
\]

(6)

\[
M_{g_i}(x) := M \frac{\partial g_i(x)}{\partial x} + \frac{\partial^\top g_i(x)}{\partial x} M = 0, \quad \forall i = 1, \ldots, m,
\]

(7)

for all \( x \in \mathbb{R}^n \).

In contraction analysis, it is clarified that differential proper-
ties have strong connections with the corresponding incremen-
tal properties such as stability [6]. Motivated by these analysis,
we also consider incremental passivity, which is defined by
using a pair \( ((x^1, u^1), (x^2, u^2)) \) of the states and inputs of the
system (1) as follows.

Definition 4 (Extended Incremental Passivity): Let \( h_I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \).
Then, the system (1) is said to be extended incremen-
tally passive if it is dissipative with respect to the supply-rate \( (u_1 - u_2)^\top h_I(x_1, x_2) \) with a storage function in the form \( S_I(x_1, x_2) \).

This incremental passivity is an extension of the concept
introduced by [4], [13] as \( h_I \) is not necessarily an incremental
function \( h(x_1) - h(x_2) \) of some function \( h : \mathbb{R}^n \to \mathbb{R}^m \).
The generalization is done to establish a connection between
differential and incremental passivity properties, which is
crucial for developing our passivity based controller design.

In general, differential passivity does not imply incremen-
tal passivity. However, if one considers a constant metric, we have
the following implication.

Theorem 5: Let \( M \in \mathbb{R}^{n \times n} \) be a symmetric and positive
semidefinite matrix. If a system (1) is differentially passive
with respect to supply rate in (4) with storage function in (5),
then it is extended incrementally passive for

\[
h_I(x_1, x_2) := \int_0^1 g^\top(\gamma(s))M(x_1 - x_2)ds,
\]

(8)

where \( \gamma(s) = x_2 + s(x_1 - x_2) \) for \( s \in [0, 1] \) and \( x_1, x_2 \in \mathbb{R}^n \).

Proof: It suffices to show that

\[
S_I(x_1, x_2) = \frac{1}{2} \|x_1 - x_2\|_M
\]

(9)

is a storage function for incremental passivity, i.e. we need to show that

\[
(x_1 - x_2)^\top M \left( g_0(x_1) - g_0(x_2) + \sum_{i=1}^{m} (g_i(x_1)u_{1,i} - g_i(x_2)u_{2,i}) \right)
\]

\[
\leq (u_1 - u_2)^\top h_I(x_1, x_2)
\]

(10)

holds for all \( (x_1, u_1), (x_2, u_2) \in \mathbb{R}^n \times \mathbb{R}^m \).

We use Proposition 3. By using the straight line \( \gamma(s) \)
and (6), compute

\[
(x_2 - x_1)^\top M(\gamma(s)) = (x_2 - x_1)^\top M \int_0^1 \frac{dg_0(\gamma(s))}{ds} ds
\]

\[
= (x_2 - x_1)^\top M \int_0^1 \frac{dg_0(\gamma(s))}{ds} \frac{d\gamma(s)}{ds} ds
\]

\[
= (x_2 - x_1)^\top \int_0^1 M_{g_0}(\gamma(s))ds(x_2 - x_1) \leq 0
\]

(11)

for all \( x_1, x_2 \in \mathbb{R}^n \).

For any two points \( u_1, u_2 \in \mathbb{R}^m \), consider the straight line
parameterized by \( s, \mu(s) = u_2 + s(u_1 - u_2) \). By using two
straight lines $\gamma(s)$ and $\mu(s)$, the product rule of the derivative, and (7), compute

$$
\sum_{i=1}^{m} (x_1 - x_2)^\top M (g_i(x_1) u_{1,i} - g_i(x_2) u_{2,i}) \\
= \sum_{i=1}^{m} (x_1 - x_2)^\top M \int_0^1 \frac{d(g_i(\gamma(s))\mu_i(s))}{ds} ds \\
= \sum_{i=1}^{m} (x_1 - x_2)^\top M \int_0^1 g_i(\gamma(s))(x_1 - x_2)\mu_i(s) ds \\
+ \sum_{i=1}^{m} (x_1 - x_2)^\top M \int_0^1 g_i(\gamma(s))(u_{1,i} - u_{2,i}) ds \\
= \sum_{i=1}^{m} (x_1 - x_2)^\top M \int_0^1 g_i(\gamma(s))(u_{1,i} - u_{2,i}) ds. \\
$$

(12)

Therefore, (10) for $h_I$ in (8) follows from (11) and (12).

Suppose that each $g_i^\top(x) M dx$, $i = 1, \ldots, m$ is an exact differential one-form, i.e., there exists a function $h_i : \mathbb{R}^n \to \mathbb{R}$ such that

$$
g_i^\top(x) M = \frac{\partial h_i(x)}{\partial x}. \\
$$

(13)

Then $h_I$ in (9) becomes $h_I(x_1, x_2) = [h_1(x_1) - h_2(x_1), \ldots, (h_m(x_1) - h_m(x_2))]^\top$, and our incremental passivity matches the incremental passivity in literature (1).

Remark 6: In fact, $g_i^\top(x) M$ satisfying both (7) and (13) can be shown to be a constant. To see this, consider the derivatives of both sides of (13) with respect to $x$, which yields

$$
\frac{\partial^2 h_I(x)}{\partial x^2} = \frac{\partial g_i(x)}{\partial x} M. \\
$$

(14)

Since $\partial^2 h_I(x)/\partial x^2$ is symmetric, it follows from (7) that $\partial^2 h_I(x)/\partial x^2 = 0$, and consequently $g_i^\top(x) M$ is constant. Moreover, if $M$ is positive definite, $g_i(x)$ is constant. Indeed, let constant $g_i^\top(x) M$ denote by $b_i^\top$. Then, $g_i^\top = b_i^\top M^{-1}$.

In [3], we consider the straight line as a path connecting $x_1$ and $x_2$. One can however use an arbitrary class $C^1$ path, then the integral depends on the considered path. However, as well known [17] if $g_i^\top(x) M dx$ is exact, the path integral does not depend on the choice of a path.

C. Krasovskii’s and Shifted Passivity Properties

Next, we define Krasovskii’s and shifted passivity properties by assuming that the following set representing the steady-state solution of (1) is not empty.

$$
\mathcal{E} := \{(x^*, u^*) \in \mathbb{R}^n \times \mathbb{R}^m : f(x^*, u^*) = 0\}. \\
$$

(15)

The main motivation of introducing these two passivity concepts is developing passivity-based control techniques.

When the considered metric is constant in contraction (differential) analysis, the so-called differential Lyapunov function is related with the Krasovskii’s method [16]. This connection can be extended to passivity properties. Motivated by the construction of a Lyapunov function by Krasovskii’s method, we newly introduce Krasovskii’s Passivity by using the so-called extended system [18]:

$$
\begin{align*}
\dot{x} &= f(x,u), \\
\dot{u} &= u_d,
\end{align*}
$$

(16)

where $u_d : \mathbb{R} \to \mathbb{R}^m$. Krasovskii passivity is defined as the standard passivity for the mapping from $u_d$ (instead of $u$) to some function of $x$.

Definition 7 (Krasovskii’s passivity): Suppose that $\mathcal{E}$ is not empty. Let $h_K : \mathbb{R}^n \to \mathbb{R}^m$. Then the nonlinear system (16) is said to be Krasovskii passive if the extended system (16) is dissipative with respect to the supply rate $u_d^\top h_K(x)$ with a storage function having a specific structure $S_K(x, u) = (1/2)\|f(x, u)\|_Q^2(u_{x,u})$, where $Q : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^{n \times n}$ is symmetric and positive semidefinite for each $(x, u)$.

Note that $\|f(x^*, u^*)\|_Q(x^*, u^*) = 0$ for $(x^*, u^*) \in \mathcal{E}$, and thus $S_K$ satisfies the property of the storage function. The name “Krasovskii” comes from the structure of the storage function. It then follows easily that differential passivity implies Krasovskii’s passivity, i.e.,

Proposition 8: Suppose that $\mathcal{E}$ is not empty. Let $M \in \mathbb{R}^{n \times n}$ be a symmetric and positive semidefinite matrix. If a system (1) is differentially passive with respect to supply rate in (4) with storage function in (5), then it is Krasovskii passive.

Proof: By taking the Lie derivative of the storage function (1/2)$\|f(x, u)\|_M^2$ along the vector field of (16), one obtains the statement of this proposition from (6) and (7).

One notices that $h_K$ in the above proposition can also be written as $h_K = g^\top(x) M \dot{x}$. This has a similar structure of $h_D(x, \delta x) = g^\top(x) M \delta x$ as differential passivity. The reason is that the dynamics of $\dot{x}$ are

$$
\frac{d\dot{x}}{dt} = F(x,u) \dot{x} + \sum_{i=1}^{m} g_i(x) u_{d,i},
$$

(17)

which is very similar to the variational system (3). This interpretation is helpful for our controller design. The main difference between differential and Krasovskii’s passivity properties is that from $\dot{x} = f(x, u)$, two variables $x$ and $\dot{x}$ are dependent in contrast to $x$ and $\delta x$. Therefore, we have the implication only for one direction.

As shown, Krasovskii passivity has a strong connection with differential passivity. As a counterpart, we have a similar relation between incremental and shifted passivity properties. For incremental passivity, we consider a pair $((x_1, u_1), (x_2, u_2))$ of the states and inputs. By fixing $(x_2, u_2)$ on $(x^*, u^*)$, we define the shifted passivity as follows.

Definition 9 (Shifted Passivity): Suppose that $\mathcal{E}$ is not empty. Let $h_S : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$. Then, the system (1) is said to be shifted passive if the system is dissipative with respect to supply-rate $(u - u^*)^\top h_S(x, x^*)$ for any $(x^*, u^*) \in \mathcal{E}$.

From their definitions, it follows that incremental passivity implies shifted passivity, which we formally state as follows.

Proposition 10: Suppose that $\mathcal{E}$ is not empty. Let $M \in \mathbb{R}^{n \times n}$ be a symmetric and positive semidefinite matrix. If a system (1) is incrementally passive with respect to supply...
rate \((u_1 - u_2)^\top h_I(x_1, x_2)\), then it is shifted passive with respect to supply rate \((u - u^*)^\top h_I(x, x^*)\).

Again our shifted passivity is an extension of \([11], [12]\) as \(h_S\) is not necessarily to be an incremental function.

III. PASSIVITY BASED CONTROLLER DESIGNS

For passive systems, it is known that one can design a feedback control law to shape the closed-loop storage function such that it takes minimum at the desired operating point \([19]\).

However, for general differentially passive systems, the control design methodologies for set-point regulation has not been well explored yet. The bottleneck is that if one simply applies techniques of the set-point regulation, then the controller is not designed for the original system but for the variational system. The main idea to address this problem is to use the fact that \(\dot{x}\) is a specific solution to the variational system when \(\delta u = u_d\) as shown in \([17]\). That is, we use the extended system \([16]\) for differential, more precisely Krasovskii’s passivity based controller design. As a counterpart, we also provide a shifted passivity based controller.

For differentially passive systems, we provide the following stabilizing controller. This controller is obtained by using the relation between differential and Krasovskii’s passivity properties in Proposition \([8]\).

**Theorem 11:** Suppose that \(\mathcal{E}\) is not empty, and the system \([16]\) satisfies \([6]\) and \([7]\) for some symmetric and positive definite matrix \(M \in \mathbb{R}^{n \times n}\). Then, consider the system \([16]\) with the following dynamic controller:

\[
K_1\dot{u}_d = \nu_1 - K_2u_d - K_3(u - u^*) - g^\top(x)Mf(x, u),
\]

where \(\nu_1 : \mathbb{R} \to \mathbb{R}^m\), and symmetric and positive definite matrix \(K_1 \in \mathbb{R}^{m \times m}\) and positive semidefinite matrices \(K_2, K_3 \in \mathbb{R}^{m \times m}\) are free tuning parameters. Then, the following two statements hold:

(a) The closed-loop system consisting of \([16]\) and \([18]\) is dissipative with respect to the supply rate \(\nu_1^\top \dot{u}_d\).

(b) Let \(\nu_1 = 0\), and \((x^*, u^*) \in \mathcal{E}\) be an isolated equilibrium of the system \([16]\). If \(K_3\) is positive definite, then there exists an open subset \(\bar{D} \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m\) containing \((x^*, u^*)\) in its interior such that any solution to the closed-loop system starting from \(D\) converges to the largest invariant set contained in

\[
\{(x, u, u_d) \in D : \|f(x, u)\|_{M_{\nu_0}} = 0, \|u_d\|_{K_2} = 0\}.
\]

**Proof:** Consider the following storage function:

\[
S(x, u) := \frac{1}{2}\|f(x, u)\|_{M_{\nu_0}}^2 + \frac{1}{2}\|u_d\|_{K_1}^2 + \frac{1}{2}\|u - u^*\|_{K_3}^2
\]

By using \([7]\), \([16]\) and \([18]\), compute the Lie derivative of \(S\) along the vector field of the closed-loop system, simply denoted by \(dS/dt\) as follows,

\[
\frac{dS}{dt} = (f^\top(x, u)Mg(x) + \dot{u}_d^\top K_1 + (u - u^*)^\top K_3)v + \|f(x, u)\|_{M_{\nu_0}}^2 - \|u_d\|_{K_2}^2 + \nu_1^\top \dot{u}_d.
\]

From \([6]\), (a) holds.

Next, since \((x^*, u^*) \in \mathcal{E}\) is isolated, there exists a bounded open subset \(D \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m\) containing \((x^*, u^*)\) in its interior such that \(S(x, u)\) is positive definite on \(D\). Then, by substituting \([18]\) and \(\nu_1 = 0\) into the above, (b) follows from LaSalle’s invariance principle.

We can provide an interpretation of the proposed controller as follows. Suppose that each \(g_i^\top(x)Mdx, i = 1, \ldots, m\) is exact, i.e., \([13]\) holds, and let \(y_{K} = h(x) - h(x^*)\). Then the controller is a linear system. Moreover, if \(\nu_1 = 0\), then in the frequency domain, the controller \([19]\) can be described as

\[
U(s) = -(K_1s^2 + K_2s + K_3)^{-1}sY_{K}(s),
\]

where \(U(s)\) is the Laplace transformation of \(u - u^*\). Our controller is an extension of this type of strictly proper controllers to nonlinear controllers, i.e., when \(g_i^\top(x)Mdx\) is not necessarily integrable.

If \(K_1 = 0\), then the above controller can be viewed as an approximate derivative feedback controller. Although \(K_1\) is supposed to be positive definite in Theorem \([11]\) we have another result when \(K_1 = 0\) as its corollary, which is a generalization of differential passivity based controller design for boost converters in DC microgrids \([8], [9]\) to general nonlinear systems. Since the proof is similar as Theorem \([11]\) we omit it.

**Corollary 12:** Supposing that \(\mathcal{E}\) is not empty, and the system \([16]\) satisfies \([6]\) and \([7]\) for some symmetric and positive definite matrix \(M \in \mathbb{R}^{n \times n}\). Then, consider the system \([16]\) with the following dynamic controller:

\[
K_2\dot{u}_d = \nu_1 - K_3(u - u^*) - g^\top(x)Mf(x, u),
\]

where \(\nu_1 : \mathbb{R} \to \mathbb{R}^m\), and symmetric and positive definite matrix \(K_2 \in \mathbb{R}^{m \times m}\) and positive semidefinite matrices \(K_3 \in \mathbb{R}^{m \times m}\) are free tuning parameters. Then, the following two statements hold:

(a) The closed-loop system consisting of \([16]\) and \([22]\) is dissipative with respect to the supply rate \(\nu_1^\top \dot{u}_d\).

(b) Let \(\nu_1 = 0\), and \((x^*, u^*) \in \mathcal{E}\) be an isolated equilibrium of the system \([16]\). If \(K_3\) is positive definite, then there exists an open subset \(\bar{D} \subset \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m\) containing \((x^*, u^*)\) in its interior such that any solution to the closed-loop system starting from \(D\) converges to the largest invariant set contained in

\[
\{(x, u, u) \in \bar{D} : \|f(x, u)\|_{M_{\nu_0}} = 0, \|K_3(u - u^*) + g^\top(x)Mf(x, u)\|_{K_2} = 0\}.
\]

Above, we have provided controllers based on the newly introduced Krasovskii’s passivity. One notices that it is not always easy to compute the maximal invariant sets \([19]\) and \([23]\). However, for some systems as in the examples presented in Section IV the invariant set contains only the desired equilibrium point.

Next, we provide a different controller based on shifted passivity. As shown below, for this controller, analysis of the maximal invariant set is easier.

**Theorem 13:** Suppose that \(\mathcal{E}\) is not empty. Also, suppose that there exist symmetric and positive semidefinite matrices
\[ M, P \in \mathbb{R}^{n \times n} \text{ and function } h_I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \text{ such that } h_I(x^*, x^*) = 0 \text{ and} \]
\[ (x - x^*)^T M f(x, u) \leq -\|x - x^*\|^2_P + (u - u^*)^T h_I(x, x^*). \]  
(24)

Then, consider the system \((1)\) with the following PI type feedback controller:
\[
\begin{align*}
\{ u & = u^* - K_4 h_I(x, x^*) + K_5 v, \\
\dot{v} & = \nu_2 - K_6 (u - u^*) - h_I(x, x^*),
\end{align*}
\]
(25)
where \(\nu_2 : \mathbb{R} \to \mathbb{R}^m\), and symmetric and positive semidefinite matrices \(k_3, k_4, k_5 \in \mathbb{R}^{m \times m}\) are free tuning parameters. Then, the following two statements hold:
(a) The closed-loop system consisting of \((1)\) and \((25)\) is dissipative with respect to the supply rate \(\nu_2 K_5 v\);
(b) Let \(\nu_2 = 0 \text{ and } K_5 \neq 0\). If \(M\) and the following symmetric matrix \(K \in \mathbb{R}^{2m \times 2m}\),
\[
K := \begin{bmatrix} K_4 & K_4 K_6 / 2 \\ K_6 K_4 / 2 & K_6 \end{bmatrix}
\]
(26)
is positive semidefinite, then any solution to the closed-loop system converges to the largest invariant set contained in
\[
\{ (x, u) \in \mathbb{R}^n \times \mathbb{R}^m : ||x - x^*||^2_P + ||h_I(x, x^*) - u - u^*||_K^2 = 0 \}. \]
(27)

Proof: Consider the following storage function:
\[
S(x, u) := \frac{1}{2} \|x - x^*\|^2_M + \frac{1}{2} \|v\|^2_{K_{v}}.
\]
By using \((24)\) and \((25)\), compute the Lie derivative of \(S\) along the vector field of the closed-loop system, simply denoted by \(dS/dt\) as follows,
\[
\frac{dS}{dt} \leq -\|x - x^*\|^2_P + (u - u^*)^T h_I(x, x^*)
\]
\[
+ v^T K_5 (\nu_2 - K_6 (u - u^*) - h_I(x, x^*))
\]
\[
= -\|x - x^*\|^2_P - ||h_I(x, x^*)||_{K_4}^2 - ||u - u^*||_{K_6}^2
\]
\[
+ h_I^T (x, x^*) K_4 K_6 (u - u^*) + \nu_2 v^T K_5 v.
\]
If \(K\) in \((26)\) is positive semidefinite, then
\[
\frac{dS}{dt} \leq -\|x - x^*\|^2_P - ||h_I(x, x^*) - u - u^*||_K^2 + \nu_2 v^T K_5 v.
\]
Thus, (a) holds. Also (b) follows from LaSalle’s invariance principle. In a similar manner as (b), one can also confirm (c).

Remark 14: If the system \((1)\) satisfies \((6)\) and \((7)\) for some symmetric and positive semidefinite matrix \(M \in \mathbb{R}^{n \times n}\), then \((24)\) holds for \(h_I\) in \((8)\) and \(P = 0\). Moreover, if there exists a symmetric and positive semidefinite \(P \in \mathbb{R}^{n \times n}\) such that \(M_{g_{0}}(x) \leq P\) for all \(x \in \mathbb{R}^n\), then \((24)\) holds for such a \(P\).

Let \(y_1 = h_I(x, x^*)\). Recall that if each \(g_i^T(x) M dx, i = 1, \ldots, m\) is exact, i.e., \((13)\) holds, then \(y_1 = h_I(x, x^*) = h(x) - h(x^*) = y_K\). If \(\nu_1 = 0\), then in the frequency domain, the new controller \((25)\) can be described as
\[
U(s) = -(s I_m + K_5 K_6)^{-1} (K_4 s + K_5) Y_I(s),
\]
where \(U(s)\) is the Laplace transformation of \(u - u^*\). This controller has a different structure from \((21)\). If \(K_4 = 0\), one has a structure of the low pass filter. If \(K_5 = 0\), one has a standard passivity based controller. If \(K_6 = 0\), one has a PI feedback controller, which is an extension of one presented in \((11)\) as we do not require \(h_I(x, x^*)\) as an incremental function \(h(x) - h(x^*)\). It is worth mentioning in \((25)\) that \(K_4\) and \(K_6\) can also be chosen as functions of \(x\) and \((x, u)\), respectively.

For the passivity based controller, asymptotic stability of an equilibrium point is guaranteed under the detectability assumption \((10)\), see e.g. the Krassovski-Barbashin’s theorem. We have similar conclusions. Suppose that for the system \((1)\),
\[
\begin{cases}
\begin{aligned}
u(0) = 0, & \Rightarrow x(\cdot) = x^*,
\end{aligned}
\end{cases}
\]
(29)
which is nothing but the detectability property. If \(K\) in \((26)\) is positive definite, the largest invariant set contained in \((27)\) is \((x, w^*)\). Also, the largest invariant set contained in \((28)\) is \(x^*\). Therefore, global asymptotic stability of the equilibrium is guaranteed for the closed-loop system under the detectability assumption \((29)\).

IV. EXAMPLE

In this example, we consider the average model of a DC-Zeta converter. It has the capability of both buck and boost converters, i.e., it can amplify and reduce the supply voltage while maintaining the polarity. The schematic of Zeta converter is given in Fig. 1. As shown, it contain four storage elements, namely two inductors \(L_1, L_2\) and two capacitors \(C_1, C_2\), an ideal switching element \(u\) and an ideal diode. Further, \(V_s\) and \(G\) denote the constant supply voltage and the load, respectively. The objective of the converter is to maintain a desired voltage across the load \(G\). After some changes of state and time variables, one obtains the following normalized model for the converter; for more details about changes of variables, see \([20]\) Chapter 2.8.
\[
\begin{bmatrix}
1 + x_2 \\
-x_2 \\
x_1 \\
-x_4 \\
-x_3 \\
1 \\
1 \\
1 \\
1 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6 \\
x_7 \\
x_8 \\
x_9 \\
x_{10}
\end{bmatrix}
\begin{bmatrix}
-x_2 \\
x_1 \\
-x_4 \\
-x_3 \\
1 \\
1 \\
1 \\
1 \\
0 \\
\end{bmatrix}
\begin{bmatrix}
u_1 \\
u_2 \\
u_3 \\
u_4 \\
u_5 \\
u_6 \\
u_7 \\
u_8 \\
u_9 \\
u_{10}
\end{bmatrix}
\]
(30)
where $\alpha_1, \alpha_2$ and $\alpha_3$ are positive constants depending on the system parameters. It is worth pointing out that a (standard) passivity based controller has not been provided for this class of systems because it is difficult (or maybe impossible) to find a storage function for passivity. However, we demonstrate that our proposed two passivity based control techniques are useful for stabilizing controller design.

For this system, the set $\mathcal{E}$ in (15) is obtained as

$$
\mathcal{E} = \left\{ (x^*, u^*) \in \mathbb{R}^4 \times \mathbb{R} : x^* = \left( \frac{(v^*)^2}{\alpha_3}, v^*, \frac{v^*}{\alpha_3}, v^* \right), \right. \\
\left. u^* = \frac{v^*}{v^* + 1}, \forall v^* \in \mathbb{R}_+ \right\}. 
$$

(31)

One notices that if $v^*$ is fixed, then $\mathcal{E}$ has a unique element.

First, we illustrate Krasovskii’s passivity based controller (18) in Theorem 11. One can confirm that the Zeta converter (30) satisfies the conditions (6) and (7) for stabilizing controller design. Illustrated techniques based on Krasovskii’s and shifted passivity properties for the dynamic stabilizing controller design. Illustrated techniques are useful when traditional methods are hard to use. Moreover, throughout the note we conduct our analysis and establish new techniques using a constant $M$. As a future direction, we plan to explore these results with a state-dependent $M(x)$.

V. CONCLUSION

In this paper, we introduce the concept of Krasovskii’s passivity. Then, we show that differential passivity with respect to a constant metric implies Krasovskii’s, incremental, and shifted passivity properties. Finally, we propose new PBC techniques based on Krasovskii’s and shifted passivity properties for the dynamic stabilizing controller design. Illustrated techniques are useful when traditional methods are hard to use. Moreover, throughout the note we conduct our analysis and establish new techniques using a constant $M$. As a future direction, we plan to explore these results with a state-dependent $M(x)$.

REFERENCES

[1] K. C. Kosaraju, Y. Kawano, and J. M. A. Scherpen, “Krasovskii’s passivity,” 2019, (submitted).
[2] D. Angeli, “A Lyapunov approach to incremental stability properties,” IEEE Transactions on Automatic Control, vol. 47, no. 3, pp. 410–421, 2002.
[3] W. Lohmiller and J. J. E. Slotine, “On contraction analysis for non-linear systems,” Automatica, vol. 34, no. 6, pp. 683–696, 1998.
[4] A. Pavlov and L. Marconi, “Incremental passivity and output regulation,” Systems & Control Letters, vol. 57, no. 5, pp. 400 – 409, 2008.
[5] A. J. van der Schaft, “On differential passivity,” vol. 47, no. 3, pp. 410–421, 2002.
[6] F. Forni and R. Sepulchre, “A differential Lyapunov framework for contraction analysis,” IEEE Transactions on Automatic Control, vol. 59, no. 3, pp. 614–628, 2014.
[7] K. C. Kosaraju, V. Chinde, R. Pasumarthy, A. Kelkar, and N. M. Singh, “Differential passivity like properties for a class of nonlinear systems,” in Proc. 2018 Annual American Control Conference. IEEE, 2018, pp. 3621–3625.

[8] K. C. Kosaraju, M. Cucuzzella, J. M. A. Scherpen, and R. Pasumarthy, “Differentiation and Passivity for Control of Brayton-Moser Systems,” ArXiv e-prints, 2018.

[9] M. Cucuzzella, R. Lazzari, Y. Kawano, K. C. Kosaraju, and J. M. A. Scherpen, “Voltage control of boost converters in DC microgrids with ZIP loads,” ArXiv e-prints, 2019.

[10] R. Reyes-Báez, A. Donaire, A. J. van der Schaft, B. Jayawardhana, and T. Perez, “Tracking control of marine craft in the port-Hamiltonian framework: A virtual differential passivity approach,” arXiv preprint arXiv:1803.07938, 2018.

[11] B. Jayawardhana, R. Ortega, E. García-Canseco, and F. Castaños, “Passivity of nonlinear incremental systems: Application to PI stabilization of nonlinear RLC circuits,” Systems & Control Letters, vol. 56, no. 9, pp. 618 – 622, 2007.

[12] N. Monshizadeh, P. Monshizadeh, R. Ortega, and A. J. van der Schaft, “Conditions on shifted passivity of port-Hamiltonian systems,” Systems & Control Letters, vol. 123, pp. 55 – 61, 2019.

[13] J. W. Simpson-Porco, “A Hill-Moylan lemma for equilibrium-independent dissipativity,” in 2018 Annual American Control Conference, 2018, pp. 6043–6048.

[14] J. C. Willems, “Dissipative dynamical systems part ii: Linear systems with quadratic supply rates,” Archive for rational mechanics and analysis, vol. 45, no. 5, pp. 352–393, 1972.

[15] A. J. van der Schaft, L2-Gain and Passivity Techniques in Nonlinear Control. Springer, 2000, vol. 2.

[16] H. K. Khalil, Nonlinear Systems. New Jersey: Prentice-Hall, 1996.

[17] D. Bao, S. S. Chern, and Z. Shen, An Introduction to Riemann-Finsler Geometry. New York: Springer-Verlag, 2012.

[18] A. J. van der Schaft, “Observability and controllability for smooth nonlinear systems,” SIAM Journal on Control and Optimization, vol. 20, no. 3, pp. 338–354, 1982.

[19] R. Ortega, A. J. van der Schaft, I. Mareels, and B. Maschke, “Putting energy back in control,” IEEE Control Systems Magazine, vol. 21, no. 2, pp. 18–33, Apr. 2001.

[20] H. J. Sira-Ramirez and R. Silva-Ortigoza, Control Design Techniques in Power Electronics Devices. Springer Science & Business Media, 2006.