Weakly versus highly nonlinear dynamics in 1D systems

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Abstract. – We analyze the morphological transition of a one-dimensional system described by a scalar field, where a flat state loses its stability. This scalar field may for example account for the position of a crystal growth front, an order parameter, or a concentration profile. We show that two types of dynamics occur around the transition: weakly nonlinear dynamics, or highly nonlinear dynamics. The conditions under which highly nonlinear evolution equations appear are determined, and their generic form is derived. Finally, examples are discussed.

In the study of pattern formation, weakly nonlinear equations play a central role. By construction, these equations catch the main effects of nonlinearities via a limited number of nonlinear terms added to a linear equation. This approach has been used for a wide variety of physical systems such as crystal growth [1], reaction-diffusion systems [2], flame fronts [3], or phase separation [4]. Weakly nonlinear equations can be derived from a multi-scale analysis when separation of scales is possible. This is for example the case in the vicinity of an instability threshold, where the system is weakly unstable, or in the analysis of amplitude and phase dynamics of modulated structures [5]. These equations are also obtained from renormalization techniques [6].

Some analysis and attempt of classification of generic nonlinear equations based on symmetry or geometry have already been reported in the literature [5,7,8]. The most systematic approach up to now was that of Ref. [8], where nonlinear equations result from the expansion in Cartesian coordinates of dynamics expressed in intrinsic coordinates. We here present a more general approach based on a multi-scale analysis. For the sake of simplicity, we assume that dynamics is local and that an instability appears at long wavelength at the instability threshold. From the assumption that the stabilizing or nonlinear terms do not scale with the small parameter of the expansion $\epsilon$, we find that the Benney, and the sand ripple [9] equations are expected in systems with translational invariance, and that the convective Ginzburg-Landau equation is expected in absence of translational invariance.

Furthermore, our approach determines the range of validity of weakly nonlinear expansions even when $\epsilon$ is present in the stabilizing or nonlinear terms. As a central result, we show that the weakly nonlinear approach breaks down for a large class of front dynamics. The main
property of these systems is to be in the vicinity of a variational steady-state (which may be thermodynamic equilibrium). We obtain highly non-linear evolution equations in two cases: (1) Conserved dynamics with translational invariance. We provide explicit illustrations of this case in the context of Molecular Beam Epitaxy. (2) Dynamics without translational invariance, but with a translationally invariant stabilization process (such as surface tension). This situation for example applies to phase separation [4], or to amplitude equations for modulated patterns [5].

Unexpectedly, highly non-linear equations exhibit the “central” symmetry in case (1), and the left-right symmetry in case (2), which are not present in the original physical problem. Furthermore, a Lyapunov functional may be found in case (2), although we are in a fully non-equilibrium situation.

Let us first consider the case where the front obeys translational invariance. This means that $\partial_t h$ does not change when the whole front is translated via $h \rightarrow h + h_0$, where $h_0$ is a constant. Then, dynamics does not depend on the position of the front $h(x,t)$, but only on its derivatives with respect to time $t$ and space $x$. For a linear analysis, we consider a small perturbation corresponding to a unique Fourier mode $h(x,t) = h_{\omega k} \exp(\omega t + ikx)$. Inserting this relation in any front dynamics model then provides $\omega$ as a specific function of $k$. Assuming that dynamics is local, $\Re[\omega]$ and $\Im[\omega]$ respectively only involve even and odd powers of $k$. A long wavelength (i.e. small $k$) expansion then leads to:

$$
\Re[\omega] = L_2 k^2 + L_4 k^4 + o(k^6); \quad \Im[\omega] = L_3 k^3 + o(k^5)
$$

From translational invariance, the mode $k = 0$ is marginally stable, and thus, there is no constant term in $\Re[\omega]$. Moreover, we have performed a Galilean transform $x \rightarrow x + L_1 t$ in order to eliminate the linear term $L_1 k$ in $\Im[\omega]$. As $k \rightarrow 0$, one has $L_2 k^2 \gg L_4 k^4$. Therefore, the criterion for an instability to occur (i.e. $\Re[\omega] > 0$) at long wavelength is simply $L_2 > 0$. Since we restrict the analysis to instabilities occurring at long wavelength at the threshold, one should generically require $L_4 < 0$.

We now perform a multi-scale analysis in the vicinity of the instability threshold based on an expansion with the small parameter $\epsilon \sim L_2$ (1)\(^1\). From Eq. (1), the unstable modes are those for which $0 \leq k \leq k_c = (-L_2/L_4)^{1/2} \sim \epsilon^{1/2}$, and the most unstable mode is $k_m = k_c/2^{1/2}$. Therefore, the unstable modes have $k \sim \epsilon^{1/2}$ and $\Re[\omega] \sim \epsilon^2$, so that the relevant spatio-temporal scales are $x \sim k^{-1} \sim \epsilon^{-1/2}$, and $t \sim \Re[\omega]^{-1} \sim \epsilon^{-2}$ (2). Using Eq. (1) for the linear part, the general form of a weakly nonlinear equation is defined as:

$$
\partial_t h = -L_2 \partial_{xx} h - L_3 \partial_{xxxx} h + L_4 \partial_{xxxxx} h + \epsilon^\gamma [\partial_x]^n [\partial_t]^l |h|^m
$$

In the last term, the brackets mean that we account at the same time for all terms containing $n$ spatial and $l$ temporal derivatives, and $m$ times $h$ (where $m > 1$), with arbitrary numerical prefactors. Translational invariance imposes that $n + l \geq m$. Moreover, we do not know the value of $\gamma$ a priori, which is a consequence of the specific properties of the system.

We first analyse the self consistency of Eq. (2), using power counting arguments. Indeed, nonlinear terms and linear terms relevant to stability should be of the same order in $\epsilon$. Stating that $h \sim \epsilon^\alpha$, we then find:

$$
\alpha = [2 - \gamma - n/2 - 2l]/(m - 1).
$$

\(^1\) We could assume that $L_4 \sim \epsilon^\delta$, as long as the instability occurs at long wavelength (i.e. $\delta > 1$). Defining $t' = \epsilon^{t_1} t$, $\gamma = [\gamma + \delta l(l-1)]/(1 - \delta l)$, and $\delta' = \epsilon^{\delta l}$, we then have again an equation of the form Eq. (3).

\(^2\) Although we do not present it here for the sake of clarity, we should introduce a propagative time-scale $\sim \epsilon^{-3/2} L_4^{-1}$, related to the dispersive term. This would not change the results.
The weakly nonlinear equation (2) is the result of an expansion of \( \partial_t h \) which is an unknown function of the derivatives of \( h \) in the limit \( \epsilon \to 0 \). In order to perform this expansion, we need all the derivatives of \( h \) to be smaller than one. Since \( h \sim \epsilon^\alpha \) and \( x \sim \epsilon^{-1/2} \), we need \( \partial_t^{\alpha_1} \partial_x^{\alpha_2} h \sim \epsilon^{2\alpha_1 + \alpha_2/2 + \alpha} \ll 1 \). A sufficient condition is that the largest derivative \( \partial_x h \) is smaller than one. This reads: \( \partial_x h \sim \epsilon^{\alpha + 1/2} \ll 1 \). We shall thus require that \( \alpha > -1/2 \). This condition allows the usual gradient expansion. Combining the different constraints, we find the condition:

\[
2\gamma + 3(l - 1) < m - n - l \leq 0,
\]

which determines all possible terms which may intervene in a weakly nonlinear equation.

We have now determined the conditions (4) under which a weakly nonlinear expansion is well defined. We shall now deal with the more subtle question of the self-consistency of the dynamics resulting from a weakly nonlinear equation. The central question is the interplay between dominant and subdominant terms. We define the rescaled variables \( \tilde{h} = \epsilon^{-\alpha} h \), \( \tilde{x} = \epsilon^{1/2} x \), and \( \tilde{t} = \epsilon^2 t \). Let us then consider two nonlinear terms \( N_1 \) and \( N_2 \) (with their rescaled forms \( \tilde{N}_1 \) and \( \tilde{N}_2 \)) and the related values of \( \alpha \) from Eq.(3) such that \( \alpha_1 > \alpha_2 \) (3).

Assuming that \( \alpha = \alpha_1 \), one finds that

\[
N_2/N_1 \sim \epsilon^{(\alpha_2 - \alpha_1)}(\tilde{N}_2/\tilde{N}_1)
\]

(we recall that \( m_2 > 1 \). Depending on the dynamics in presence of \( N_1 \) in Eq.(2), three cases may be observed: (i) The solution \( \tilde{h}(\tilde{x}, \tilde{t}) \) (or its derivatives for a system with translational invariance) is bounded. Then \( \tilde{N}_1 \) are bounded. It follows that \( N_2 \ll N_1 \). (ii) \( \tilde{h}(\tilde{x}, \tilde{t}) \) scales with time, i.e. the typical scales of the patterns evolve as \( \tilde{h} \sim \tilde{t}^{\kappa_2} \) and \( \tilde{x} \sim \tilde{t}^{\kappa_1} \). Then \( \tilde{N}_1 \sim \tilde{t}^{\kappa_1} \) and \( N_2/N_1 \sim \epsilon^{(\alpha_2 - \alpha_1)}(\kappa_2 - \kappa_1) \). Therefore, if \( \kappa_1 \geq \kappa_2 \), we have \( N_1 \gg N_2 \) at all times. If \( \kappa_1 < \kappa_2 \), then \( N_2 \sim N_1 \) after a time \( t_c \sim t_L \epsilon^{-(\alpha_2 - \alpha_1)} \), where \( t_L \) is the time for appearance of the instability from the linear analysis. As \( \epsilon \to 0 \), one finds that \( t_c \gg t_L \). (iii) \( \tilde{h}(\tilde{x}, \tilde{t}) \) exhibits an exponential –or even faster– increase of the amplitude (including the case of singularities in finite time). Then \( N_2 \) may become of the same order as \( N_1 \) after a time \( t_c \sim t_L \) (with possible logarithmic corrections).

From an inspection of the three cases mentioned above, we see that the only self-consistent choice in order to describe the dynamics at timescales that are much longer than \( t_c \) is to find, if it exists, the nonlinearity with the biggest value of \( \alpha \). The dynamics may then be: (i) bounded, and the time \( t_c \) under which the weakly nonlinear equation is valid is infinite; (ii) power-law in time. The time \( t_c \) is then either infinite, or increasing as \( \epsilon \) to some negative power when \( \epsilon \to 0 \). But in the case (iii), where the growth of the amplitude is exponential or faster, the weakly nonlinear equation does not provide a satisfactory description of the dynamics in the nonlinear regime.

We conclude that a nonlinear analysis in the regime where \( \epsilon \) is small requires an expansion of the form (2), but also relies on the knowledge of the dynamics of (2) with the dominant nonlinearities. There is to our knowledge no general analytical tool which could systematically determine whether the nonlinear dynamics belongs to one of these 3 classes of dynamics. Therefore, a numerical solution is in general needed.

Let us now consider some precise examples. In general, the dominant contribution depends on the precise values of \( \alpha \), which themselves depend on \( \gamma \). Therefore, the specific physical ingredients of the system will determine the most relevant terms. Nevertheless, we shall first consider the simplified case where \( \gamma = 0 \). Relations (4) then readily show that \( l = 0 \).
From Eq. 4, we are now able list all possible nonlinearities in a weakly nonlinear expansion: 

\[ (\partial_x h)^m; \partial_x (\partial_x h)^m; \partial_{xxx}(\partial_x h)^m; (\partial_x h)^{m-2}(\partial_{xxx} h)^2, \] 

where \( m > 1 \). Using this result we find that the expected equation is the Benney equation:

\[ \partial_t h = -\partial_{xxx} h + \beta \partial_{xxx} h - \partial_{xxxx} h + (\partial_x h)^2 \]  

(6)

because it leads to the largest possible value of \( \alpha \), which is \( \alpha = 1 \), and because the solution of Eq. 3 leads to a saturation of the amplitude. The variables \( x, t, h \) have been normalized in Eq. 3 so that only one constant remains. Eq. 3 is non-variational and exhibits order or chaos when the parameter \( \beta \) is larger or smaller than one respectively. In the chaotic limit \( \beta = 0 \), it is called the Kuramoto-Sivashinsky equation. The Benney and Kuramoto-Sivashinsky equations have been derived from multi-scale analysis in many physical situations, such as flame fronts [3], crystal step meandering [1] and bunching [10], or ion sputtering [11].

In Molecular Beam Epitaxy (MBE), atoms land on the surface but can usually not go back to the atmosphere. One then has from mass conservation:

\[ \partial_t h = F - \partial_x j \]  

(7)

where \( F \) is the incoming flux, and \( j \) is a mass flux along the surface. The constant term is eliminated by the transformation \( \partial_t h \rightarrow h + Ft \). Many other systems obey a conservation law, such as wave dynamics in thin liquid layers [12], or sand ripple formation [9].

Once again, we shall first assume that \( \gamma = 0 \). From Eqs. 4 and 11, the only nonlinear terms which are allowed are: \( \partial_x (\partial_x h)^m; \partial_{xxx}(\partial_x h)^m \). Following the same line as above, we find that the first dominant nonlinear term is \( \partial_x (\partial_x h)^2 \). But this term leads to an increase of the amplitude which is faster than power-law [9]. This nonlinearity can be absent if it is proportional to \( \epsilon^2 \), with \( \gamma > 1/2 \), but also if some symmetry (such as \( h \rightarrow -h \), or \( x \rightarrow -x \)) is imposed. We then find:

\[ \partial_t h = \partial_x \left[ -\partial_x h + \beta \partial_{xxx} h - \partial_{xxxx} h + c_3 (\partial_x h)^3 + c_2 \partial_x (\partial_x h)^2 \right] \]  

(8)

which corresponds to \( \alpha = 0 \). In Eq. 5, \( t, x, \) and \( h \) have been normalized, and \( c_1 \) are constants. This equation was first obtained by Csaok et al [9]. When \( \beta = 0 \) and \( c_2 = 0 \) (e.g. when dynamics is variational, or when it exhibits the \( (h, x) \rightarrow (-h, -x) \) symmetry), and when \( c_3 < 0 \), one recovers the Cahn-Hilliard [4] equation with a double well potential, which is known to lead to logarithmic coarsening [13]. In this case, the amplitude \( \partial_x h \) remains finite at all times. Therefore, the expansion is self consistent, and higher order terms are negligible. As shown in Ref. [9], this behavior seems to persist when \( \beta \) and \( c_2 \) are both non-zero. But when \( c_3 = 0 \) and \( c_2 = 0 \), power law coarsening is found, with the wavelength \( \sim t^{1/2} \) and the amplitude \( \sim t^{3/2} \) [14]. When \( c_3 > 0 \), the dynamics leads to a local blow up of the slope \( \partial_x h \). Eq. 5 describes a wide variety of systems, such as phase separation [4], sand ripple formation [9], and step bunching [14].

Up to this point, we have found that weakly nonlinear expansions are generically obtained from multi-scale analysis. This result was based on the assumption that \( \gamma = 0 \). Nevertheless, when \( \gamma \neq 0 \), the set of nonlinear terms which must be kept can change. For example, in Ref. [14], the step bunching instability on a vicinal surface is studied under growth and mobile atom migration. In this case, \( \epsilon \) is proportional to the growth rate for some given value of the migration rate. The terms \( \partial_x (\partial_x h)^2 \) and \( \partial_x (\partial_x h)^3 \) are absent in Eq. 5, because they are multiplied by \( \epsilon \gamma \) with \( \gamma = 1 \). A more drastic consequence of a non-vanishing \( \gamma \) is the possible break-down of the weakly nonlinear expansion. In the following, we indeed show
that conserved dynamics, in the vicinity of thermodynamic equilibrium, forbids any weakly nonlinear expansion.

Let us first consider the conserved model: \( \partial_t h = \partial_x [M \partial_x (\delta \mathcal{F}/\delta h)] \), where \( \delta \) denotes the functional derivative, and \( \mathcal{F} = \int dx \phi \). \( M > 0 \) and \( \phi \) are function of the spatial derivatives of \( h \). \( \mathcal{F} \), which plays the role of an energy, and is a Lyapunov functional (i.e. \( \partial_t \mathcal{F} < 0 \)). Assuming that \( \mathcal{F} \) is minimum for a straight front (\( \partial_x h = 0 \)), any initial perturbation would decay. This model may for example account for relaxation towards thermodynamic equilibrium. When it is driven by a small non-equilibrium force \( f > 0 \) (not breaking mass conservation), the system will respond by an additional flux \( J \). To leading order in \( f \), the new dynamics reads:

\[
\partial_t h = \partial_x \left[ M \partial_x \frac{\delta \mathcal{F}}{\delta h} + f J \right] + o(f^2),
\]

where \( J \) is a function of the spatial and temporal derivatives of \( h \). Linearizing Eq. (9), we find:

\[
\partial_t h = f J_1 \partial_{xx} h + f J_2 \partial_{xxx} h - (M_0 - f J_3) \partial_{xxxx} h + ...
\]

where \( J_1 = \partial_{t_0} h J, J_2 = \partial_{t_1} h J, J_3 = \partial_{t_2} h J, \) and \( M_0 = M \partial_{t_0} h \partial_{t_0} x \phi \) in the steady-state configuration. From the analogy between Eqs. (10) and (1), we conclude that an instability occurs if \( J_1 < 0 \), and that \( \epsilon \sim f \). From the last term of Eq. (10) one then finds that \( L_4 \) does not scale with \( \epsilon \). We now look for possible weakly nonlinear terms. For the first term in the brackets of Eq. (9), \( l = 0, \gamma = 0, \) and \( n \geq m + 3 \), which is in contradiction with (4). If a term comes from the non-equilibrium contribution \( J \), one has \( \gamma = 1 \), and \( n \geq m + 1 \), which is again in contradiction with (4). Finally higher order terms have \( \gamma \geq 2 \), which is also in contradiction with (4). Therefore, no nonlinear term satisfies (4), and the small gradient constraint (\( \partial_x h \ll 1 \)) has to be waived. Thus, we choose \( \alpha = -1/2 \), and since \( \partial_x h \sim 1 \), the full nonlinear dependence of \( M, \delta \mathcal{F}/\delta h, \) and \( J \) on \( \partial_x h \) must be kept, while terms such as \( \partial_{xx} h \) or \( \partial_x h \) are negligible. Therefore, \( J \to A, M \to B, \) and \( \delta \mathcal{F}/\delta h \to \partial_x C, \) where \( A, B \) and \( C \) are functions of \( \partial_x h \) only. To leading order in \( \epsilon \), Eq. (10) then takes the highly nonlinear form: (4)

\[
\partial_t h = \partial_x \left[ B \partial_x x^2 C + \epsilon A \right]
\]

Unexpectedly, Eq. (11) is invariant under the “central” symmetry \( (h, x) \to (-h, -x) \), although the starting point Eq. (9) is not. We shall notice that a formal expansion of \( A, B, \) and \( C \) with respect to \( \partial_x h \) in Eq. (11) leads to an equation of the form (2) with an infinite number of nonlinear terms, which all have the same value of \( \alpha \). Therefore, highly nonlinear equations can be considered as a special case of (2), and the classification of the dynamics into the 3 classes, bounded, power-law, and exponential, is still valid.

An equation of the form (11) was first derived from a multi-scale analysis of crystal step meandering during MBE [15]. Although Eq. (11) is not variational (except in some special cases, e.g. when \( B \partial_x h = A \) or \( C' = B \)), the simple structure of its steady-states allows one to analyze in details its coarsening dynamics [19] as found in the case of step meandering from several recent works [15–17]. Eq. (11) was also introduced as a model for mound formation during MBE [18]. Our analysis now provides a frame to understand its origin.

We now turn to the dynamics of a front without translational invariance. This situation occurs when a front, such as the free surface of an thin adsorbate, is subject to an external field which is a function of \( h \) (due for example to the substrate). It may also account for unstable concentration profiles in reaction diffusion systems [2], or for phase separating systems [4].

\[ (*') \text{A more general form for conserved dynamics when } \alpha = -1/2 \text{ is } \partial_t h = \partial_x [B_1 \delta_{xx} m + B_2 (\partial_x m)^2 + B_3 \partial_x m + \epsilon B m], \text{ where } B_i \text{ are function of } m = \partial_x h \text{ such that } B_i \to \text{ constant when } m \to 0. \]
Since we use a similar approach to that presented above, we will be more concise here. We shall assume that there exist a flat steady state $h = 0$. At long wavelength,

$$\Re [\omega] \approx L_0 + L_2 k^2.$$  \hspace{1cm} (12)

The instability now occurs if $L_0 > 0$, and we expect $L_2 < 0$. We therefore choose $\epsilon \sim L_0$ and the relevant spatio-temporal scales are: $x \sim \epsilon^{-1/2}$ and $t \sim \epsilon^{-1}$. As in the previous case, the term $L_1 k$, which appears in $\Im [\omega]$, can be cancelled with the help of a Galilean transform. The contribution $\Im [\omega] \approx L_3 k^3 \sim \epsilon^{3/2}$ is negligible when $L_3$ does not scale with a negative power of $\epsilon$. The general form of a weakly nonlinear equation is now:

$$\partial_t h = L_0 h - L_2 \partial_x^2 h + \epsilon^2 [\partial_x^n] [\partial_t^l] [h]^m$$ \hspace{1cm} (13)

with $m > 1$. The nonlinear term will be of the same order as the linear terms if: $\alpha = (1 - n/2 - \gamma - l)/(m - 1)$. The condition for having weakly nonlinear dynamics is $h \ll 1$, which implies that $\alpha > 0$. This leads to some restriction, namely:

$$\gamma < 1 - n/2 - l.$$ \hspace{1cm} (14)

Once again, we assume that $\gamma = 0$. Then $l = 0$, and the only possible nonlinear terms are: $h^m; \partial_x(h^m)$. The generic first contribution is $h^2$. It leads to the Fisher-Kolmogorov equation, which has been extensively used for its traveling wave solutions [20]. But here, we must start from a front with zero average height, in which case $h$ locally diverges in finite time. As before, the presence of a prefactor $\epsilon^\gamma$, with $\gamma > 1/2$, or the existence of symmetries such as $h \rightarrow -h$ or $(h,x) \rightarrow (-h,-x)$ may forbid this term. One then finds the convective Ginzburg-Landau equation

$$\partial_t h = h + \partial_x^2 h + \sigma h^3 + \mu h \partial_x h$$ \hspace{1cm} (15)

where $x$, $h$, and $t$ are normalized. When $\sigma > 0$, $h$ again locally diverges in finite time. We therefore focus on the case where $\sigma < 0$. Although the Time Dependent Ginzburg-Landau equation ($\mu = 0$) is well known for phase separating systems [4] and as a generic amplitude equation [5], we are not aware of previous work incorporating the convective term proportional to $\mu$. This term breaks the variational character of the dynamics and the $x \rightarrow -x$ symmetry. Since there exist two asymmetric kink solutions for any value of $\mu$, we do not expect qualitative changes in the asymptotic dynamics of Eq. (15) when $\mu$ varies. Hence, we conjecture that the known result of logarithmic coarsening of Eq. (15) for $\mu = 0$ [13] extends to arbitrary $\mu$. At short times, when $\mu$ is large enough, Eq. (15) shares similarities with Burgers’ equation, and its solution exhibits shocks.

Let us now consider dynamics having a stable steady state $h = 0$, and a Lyapunov functional $F = \int d\phi$. The simplest dynamics which has this property (usually referred to as model A [4]), is: $\partial_t h = -\Gamma(\delta F/\delta h)$, where $\Gamma$ is a function of $h$ and its spatial derivatives. In presence of a small destabilizing non-equilibrium force $f$, we have:

$$\partial_t h = -\Gamma \frac{\delta F}{\delta h} + f K + o(f^2),$$ \hspace{1cm} (16)

where $K$ depends on $h$ and its spatial and temporal derivatives. If $F$ is translationally invariant, then $\phi$ is a function of $\partial_x h$ and its spatial derivatives only. Such a situation may be found in phase separation [4], where a gradient energy $\sim \int d\phi(\partial_x h)^2$ is used, or in the non-conserved dynamics of a thin film stabilized by surface tension. Linearizing Eq. (16), we then find that
$\epsilon \sim f$. Once again, an inspection of Eq. (16) proves that the dynamics is highly nonlinear (i.e. $\alpha = 0$). To leading order in $\epsilon$, the evolution equation reads:
\begin{equation}
\partial_t h = P \partial_{xx} h + \epsilon Q ,
\end{equation}
where $P$ and $Q$ are functions of $h$ only. We shall notice here two unexpected and strongly restrictive properties of this equation: (i) it has the $x \rightarrow -x$ symmetry, although the dynamics of the original problem does not necessarily have it. (ii) Although the front is not at equilibrium, Eq. (17) exhibits a Lyapunov functional
\begin{equation}
U = \int dx [(\partial_x h)^2/2 + \epsilon R],
\end{equation}
where $R$ is a function of $h$ defined by the relation $R' = Q/P$. Indeed one can write $\partial_t h = -P \partial U / \partial h$, and $\partial_t U \leq 0$. Since Eq. (17) is variational, one can use the concepts of dynamical scaling developed for the study of phase transitions [4] to study the coarsening. One should once again rely on the structure of the steady-states in order to analyze the dynamics [19].

To conclude, we have presented two scenarios for the destabilization of a 1D system: weakly nonlinear, and highly nonlinear dynamics. During weakly nonlinear dynamics, the evolution of the front morphology is described by an nonlinear expansion at small amplitudes. The amplitude then tends to zero as one gets closer to the threshold. During highly nonlinear dynamics, nonlinearities come into play only when the amplitude becomes finite. A small amplitude expansion is then not justified anymore. We have shown that the form of the evolution equation can still be found in this case.

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(\footnote{A more general form when $\alpha = 0$ is $\partial_t h = P_0 \partial_{xx} h + P_1 (\partial_x h)^2 + \epsilon P_0 h$, where $P_i$ are functions of $h$ with $P_i \rightarrow$ constant when $h \rightarrow 0$. As opposed to Eq. (17), this equation does not have a Lyapunov functional.})