Entanglement Test of Third-Order Coupling in Gravitating Quantum Systems

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We describe a method for precise study of gravitational interaction between two nearby quantum masses. Since the displacements are much smaller than initial separation between the two masses, the displacement-to-separation ratio is a natural parameter in which the gravitational potential can be expanded. We show that entanglement in such experiments is sensitive to the initial relative momentum only when the system evolves into non-gaussian states, i.e., the potential is expanded at least to the third-order (cubic) term. A closed-form expression for the amount of entanglement is established, which shows a linear dependence on the initial relative momentum.

Due to the weakness of gravitational coupling all quantum experiments up to date in which gravity plays a role utilised the field of the Earth, see Refs. [1–6] for milestone examples. Since this field undergoes practically undetectable back-action from quantum particles, it effectively admits a classical description, either in terms of a fixed background Newtonian field [3–5] or as a fixed background spacetime [1, 2, 6]. This argument strongly motivates theoretical and experimental research towards a demonstration of gravitation between two quantum particles, as this is one of the simplest scenarios where quantum properties of gravity could be observed. Along this line, a number of recent proposals studied the possibility of the generation of quantum entanglement between two massive objects [7–14]. Our aim here is to extend these methods and provide a simple precision test of gravitational coupling.

The experiment we have in mind could be realised within the field of optomechanics [15], which already succeeded in cooling individual massive particles near their motional ground state [16–18], and in entangling cantilevers to light and between themselves [19–21]. In such a setup the centers of mass between two microscopic particles are separated much more than their displacements. For example, two Osmium spheres (the densest natural material) each of mass 100 µg (radius 0.1 mm) with an initial inter-surface distance of 0.1 mm move by less than a nanometer within 1 second of evolution [11], vividly illustrating the weakness of gravity. Therefore, given that the initial relative momentum is small, a natural parameter in which the interaction potential can be expanded is the ratio of displacements to the initial separation between the centers of two spheres. Since the situation under consideration is a non-relativistic one, the relevant interaction is characterised by the quantum Newtonian potential. The goal of precision test is to identify phenomena that can only occur if the potential is expanded to a particular order, thus witnessing relevance of at least this order in the experiment.

From this perspective, the gravitational entanglement proposals, in addition to providing clues about the quantum nature of gravity, also supply tests of the form of gravitational interaction. For example, in the Bose-Marletto-Vedral setup the masses are initially prepared in an extended spatial superposition, and in this case already linear term in the expansion leads to gravitational entanglement [7, 8]. However, when the masses are initially well confined, the linear term acts completely locally, and at least the second-order term is required for entanglement gain [11]. Here we show a method that witnesses the third-order term and has an advantage of a simple modification of the entanglement scheme with confined particles. Hence, an experiment designed to probe gravitational entanglement can also be used to effortlessly witness even weaker gravitational coupling.

Our basic idea is to initially push the particles towards (or in principle against) each other. We demonstrate theoretically that quantum entanglement generated by gravitational potential truncated at the second order is insensitive to the initial momentum of the masses. This is shown explicitly by analytically solving the time evolution of the corresponding covariance matrix. One ex-

![FIG. 1. Setup under consideration. Two identical spheres of mass \( m \) are released from the ground state of identical harmonic traps with an equal and opposite momentum \( p_0 \) along the line joining their centers. The centers are initially separated by a distance \( L \), and displacements from them are denoted by \( x_A \) and \( x_B \). After time \( t \) entanglement is estimated with the help of the probing lasers.](attachment:entanglement-test-diagram.png)
pects that masses pushed towards each other gather more entanglement than in the absence of initial momentum due to an ever-increasing interaction strength. This is indeed the case but provided the potential contains at least the third-order term, i.e., when the system evolves into a non-gaussian state. Closed-form expression for entanglement is established, which agrees with numerical simulations showing a linear dependence on the initial relative momentum and the key role played by the force gradient across the reduced mass wave packet. On the technical side, the methods introduced can be applied to any central force, and they show remarkable robustness, e.g., even the impact of the fourth-order term on non-gaussianity quantifier can be captured despite an astonishingly weak gravitational interaction.

**Experimental setup.** Consider the setup schematically represented in Fig. 1, where we also introduce our notation. The initial wave function is assumed to describe two independent masses, each in a natural Gaussian state with position spread \( \sigma \): \( \Psi(x_A, x_B, t = 0) = \psi_A(x_A)\psi_B(x_B) \), where

\[
\psi_A(x_A) = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} \exp \left( -\frac{x_A^2}{4\sigma^2} + \frac{p_0 x_A}{\hbar} \right),
\]

\[
\psi_B(x_B) = \left( \frac{1}{2\pi\sigma^2} \right)^{1/4} \exp \left( -\frac{x_B^2}{4\sigma^2} - \frac{p_0 x_B}{\hbar} \right).
\]

Note that without loss of generality we choose the momenta to be opposite and equal. The gravitational Hamiltonian is given by

\[
\hat{H} = \frac{\hat{p}_A^2}{2m} + \frac{\hat{p}_B^2}{2m} - \frac{Gm^2}{L + (x_B - x_A)}.
\]

Since this is a two-body problem, it is well known that the center-of-mass (COM) motion separates from the relative motion. Accordingly, we introduce the usual change of variables from the LAB frame to the COM frame: \( R = (x_A + x_B)/2 \) and \( r = x_B - x_A \), where \( R \) and \( r \) denote the respective displacements of the COM and the reduced mass from their initial average positions. As a result, the initial wave function separates as \( \Psi(x_A, x_B, t = 0) = \phi(R, t = 0)\psi(r, t = 0) \), where

\[
\phi(R, t = 0) = \left( \frac{1}{\pi\sigma^2} \right)^{1/4} \exp \left( -\frac{R^2}{2\sigma^2} \right),
\]

\[
\psi(r, t = 0) = \left( \frac{1}{4\pi\sigma^2} \right)^{1/4} \exp \left( -\frac{r^2}{8\sigma^2} - \frac{\hbar r}{2m} \right).
\]

The wave functions \( \phi \) and \( \psi \) describe the motion of the COM and the reduced mass, respectively. Compared to the original wave packets, the COM wave packet admits a smaller width of \( \sigma/\sqrt{2} \), and the reduced mass wave packet admits a larger width of \( \sigma\sqrt{2} \). The corresponding relations are illustrated in Fig. 2. In this frame the Hamiltonian decouples as

\[
\hat{H} = \hat{H}_R + \hat{H}_r = \left( \frac{\hat{p}_r^2}{2m} \right) + \left( \frac{\hat{p}_r^2}{m} - \frac{Gm^2}{L + r} \right),
\]

where \( \hat{p} = -i\hbar \partial/\partial R \) and \( \hat{p} = -i\hbar \partial/\partial r \) are the momentum operators for the COM and the reduced mass, respectively. Therefore, the two-body wave function retains its product form at all times: \( \Psi(x_A, x_B, t) = \phi(R, t)\psi(r, t) \). Furthermore, the COM wave packet evolves like a free particle, i.e., its gaussianity is preserved [22, 23], and the first two statistical moments characterize the quantum state fully. For completeness, they are given in the Appendix.

The state \( \psi \) evolves in the gravitational potential, which we now expand in the powers of \( r/L \):

\[
\hat{H}_r \approx \frac{\hat{p}_r^2}{m} - \frac{1}{4} m \omega^2 \sum_{n=0}^{N} \frac{(-1)^n}{L^{n-2}} r^n,
\]

where \( N \) is the order of approximation, and we defined \( \omega^2 = 4Gm/L^3 \) for later convenience. The gaussianity is preserved if this series is truncated at the second order, i.e., \( N \leq 2 \). In such cases, we have derived exact analytical expressions for the covariance matrix by solving the related Ehrenfest’s equations and present them in the Appendix. With the inclusion of higher-order terms in the potential, i.e., \( N > 2 \), the corresponding Ehrenfest’s equations cannot be solved analytically due to the emergence of an infinite set of coupled differential equations involving ever-increasing statistical moments. We therefore resort to numerical methods and calculate the time evolution of \( \psi \) by implementing Cayley’s form of evolution operator [24]. In order to deal with weak gravitational interaction we improve on the accuracy of Cayley’s method by utilizing the five-point stencil to approximate the second derivative in the Hamiltonian. This way the wave function is discretised onto a pentadiagonal Crank-Nicolson scheme, which is solved by the LU factorisation techniques. The code is publicly available at GitHub [25].

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**FIG. 2.** From LAB frame to COM frame. The gaussianity of the initial state is preserved as well as the product form. The widths, however, are different in different frames as marked.
with the corresponding documentation in Ref. [26] where we demonstrate our superior accuracy as compared to the standard tridiagonal solutions. We also implemented a dynamic grid allocation as described in Ref. [27] to avoid any reflections from numerical boundaries.

**Results.** The methodology just described returns analytical form of the quantum state (covariance matrix) at time $t$ for potentials truncated at $N = 2$, and a numerical form for all $N$. By implementing the inverse coordinate change we compute entanglement between the masses. In particular, we use logarithmic negativity and entropy of entanglement as entanglement quantifiers. While logarithmic negativity is known to be a faithful entanglement quantifier for gaussian states [28–30] we will also discuss non-gaussian pure states and hence the inclusion of the entropy of entanglement. We first give the results for $N = 2$ with emphasis on the independence of initial relative momentum and its origin. Then we move to $N = 3$ and demonstrate that entanglement does depend on the initial momentum, and in the relevant regime it is linear in the relative momentum. We also analyse an indicator of non-gaussianity (skewness) and demonstrate the precision of our methods by calculating the marginal impacts of the fourth-order term in the potential expansion.

**Quadratic interactions:** Consider the gravitational potential truncated at the second order. We obtained exact analytical solutions for the independent elements of the covariance matrix and present them explicitly in the Appendix, where we also recall how to obtain the logarithmic negativity and subsystem entropy from these quantities. The solutions simplify if they are written in terms of already introduced $\omega$ and in terms of $\omega_0 = h/2ma^2$, the frequency of harmonic trap for which the initial state is the ground state. The logarithmic negativity for $p_0 = 0$, in the regime $\omega \ll \omega_0$ and $\omega t \ll 1$, was already approximated in Ref. [11] to:

$$E_N(\sigma) \approx -\log_2 \sqrt{1 + 2\Omega^6 t^6 - 2\Omega^3 t^3 \sqrt{1 + \Omega^6 t^6}},$$

where $\Omega^3 = \omega_0^2/6$. We verified that this formula indeed matches our solution, and emphasize that the solutions calculated in this work are exact as long as the quadratic truncation is valid, and hence they can be used to quantify entanglement outside of the constraints that led to Eq. (8). An example is given below.

Perhaps the most striking feature of the covariance matrix is its insensitivity to the initial momentum $p_0$. Accordingly, all quantities derived from the covariance matrix, say entanglement, are independent of the initial momentum. Evidently, in this approximation, the two masses moving towards each other or away from each other accumulate the same amount of entanglement as when they started from rest. This is confirmed by the simulations presented in Fig. 3. Not only there is no initial momentum dependence in the dynamics of logarithmic negativity and entropy of entanglement, they also perfectly overlap with analytical results, showing that our methods are reliable and consistent. We note that Eq. (8) is not applicable to the configuration considered in Fig. 3 because $\omega_0 \approx 25\omega$.

**Relevance of force gradient:** We now move to explanations of the observed results. Mathematically, it is clear that a non-zero force gradient across the size of the wave packet is a necessary condition for entanglement. Without it the potential is effectively truncated at $N = 1$, and the total Hamiltonian is the sum of local terms. Physically, entanglement is caused by correlations in complementary variables, here between positions and momenta. Due to the force gradient the parts of the wave packets that are closer are gravitationally attracted more than the parts which are further apart. Hence a moment later different momentum is developed across different positions within the wave packets, leading to quantum entanglement.

Furthermore, assuming that the force gradient is the main contributor to entanglement gain explains the independence of initial momentum. Since the potential is truncated at $N = 2$, the force gradient is constant in space. It is therefore irrelevant if in the meantime the particle has moved to a different location, and accordingly the initial momentum does not play a role in entanglement dynamics. Quantitatively, the force gradient is $F_2'' = -V''(\hat{r}) = ma^2/2$, and therefore it fully describes entanglement in Eq. (8) since now $\Omega^3 = \omega_0^2/6 \equiv (\omega_0^2/3m)F_2''$. In the next section we provide further evidence for a pivotal role of the gradient in entanglement dynamics also in the case of higher-order interactions.

**Higher-order interactions:** Let us continue with the working hypothesis that the force gradient is dominant in entanglement dynamics. For the cubic potential, $N = 3$, the gradient is given by $F_3''(\hat{r}) = (1 - 3\hat{r}/L)ma^2/2$ and

![FIG. 3. Accumulation of entanglement with gravitational potential truncated at the quadratic term ($N = 2$). The configuration consists of identical Osmium spheres with $m = 0.25$ pg, $L = 2.5R$, and $\sigma = 2.5$ nm. $p_0$ is the initial momentum in keV/c. Analytical results are calculated from the closed form of covariance matrix in the Appendix. $E_N(\sigma)$ denotes the logarithmic negativity of covariance matrix, and $S(\rho_A)$ is the entanglement entropy.](image)
importantly it admits a position dependence. Accordingly, entanglement should be sensitive to the initial momentum as this time the gradients are different at different locations. This is indeed observed in Fig. 4a for gravitational potential truncated at the cubic term. Furthermore, when the two masses are moving towards each other, $p_0 > 0$ and $\langle \dot{r} \rangle < 0$, and consequently, the gradient increases, matching the growing entanglement. Conversely, when the masses are moving away, $p_0 < 0$ and $\langle \dot{r} \rangle > 0$, the force gradient decreases, matching the slower entanglement gain. The quantitative statements can again be achieved.

Fig. 4b shows experimentally friendly plots of the ratio of entanglement accumulated within time $t$ with non-zero initial momentum to entanglement gained from rest. The numerically calculated linear dependence (solid lines) can be explained with closed expressions (dotted lines) that we now explain. The force gradients for the quadratic and the cubic interactions are related by the following conversion factor: $F'_N(\dot{r}) = (1 - 3\dot{r}/L)F'_N$. The average factor therefore reads

$$1 - \frac{3}{L} \langle \dot{r} \rangle = 1 + \frac{6p_0}{mL}t \equiv 1 + \epsilon_3(t). \tag{9}$$

Fig. 4a shows that for vanishing initial momentum, $p_0 = 0$, the entanglement obtained with cubic and quadratic potentials is practically the same.\textsuperscript{2} We therefore extrapolate that entanglement for non-zero initial momentum (supposedly caused by $F'_3$) is related to entanglement from rest by a simple function of the conversion factor. The plots of Fig. 4b are fitted with

$$S(\rho_A) = \left[1 + \epsilon_3(t)\right] S(\rho_A, p_0 = 0), \tag{10}$$

$$E_N(\sigma) = \left[1 + \epsilon_3(t)/2\right] E_N(\sigma, p_0 = 0). \tag{11}$$

Note that the factor of $1/2$ next to $\epsilon_3$ in the logarithmic negativity is causing a departure from the exact conversion factor between the force gradients. These formulae are in remarkable agreement with the computational results in the regime of positive initial momentum (masses moving towards each other, the regime of experimental interest) and also work quite well for negative initial momenta. This again affirms that the force gradient is the main driver of gravitational entanglement. Furthermore, these closed forms can now be used in a plethora of configurations to estimate the amplification of entanglement for a nonzero initial momentum given entanglement from rest. The higher-order terms can also be readily incorporated with an appropriate conversion factor between the force gradients: $F'_N(\dot{r}) = \sum_{n=2}^{N} (-1)^n n(n-1)(\dot{r}/L)^{n-2}F'_2/2, \forall N > 2$.

**Discussion.** The results presented so far could also be seen as a simple momentum-based witness of non-gaussianity in a quantum state. Indeed the cubic term is responsible for evolution into non-gaussian states, and we now describe in more detail the non-gaussian characteristics of states induced by the gravitational interaction.

One of the necessary constraints of gaussian dynamics is the vanishing skewness. Fig. 5 presents the skewness $\bar{\mu}_3$ in the evolution of the reduced mass wave function $\psi$. While $\bar{\mu}_3$ vanishes for $N = 2$, as it should, it rises steeply for $N = 3$. The physical reason is clear from Fig. 2 describing the change of variables between LAB and COM frames. The left end of wave function $\psi$ is attracted towards the center of mass more than the right end. Over

\textsuperscript{1} With a Newtonian approximation $\langle \dot{r} \rangle \approx r_{\downarrow} = -2p_0 t/m$.

\textsuperscript{2} Even in extreme cases when the wave packet expands quickly to larger displacements, the difference is marginal.
time this makes the probability density function negatively skewed, which is what is indicated by $\tilde{\mu}_3 < 0$. Fig. 5 also demonstrates the precision of our numerical methods which capture very small contributions of the fourth-order term to the skewness.

Finally, we would like to address the question whether a simpler method for detecting the third-order coupling exists than based on measurements of entanglement. Indeed, note that solely the momentum signal could be used for such purposes. One verifies that by truncating the potential in Eq. (7) at $N = 2$ that the relative momentum satisfies the following condition $\langle \tilde{p} \rangle / \langle p \rangle = \omega^2$, i.e., it is time independent. Any time dependence of this ratio reveals third-order coupling. In cases where the center of mass is stationary instead of the relative momentum the local momentum of any particle could be used.

Conclusions. We have shown that experiments aimed at observation of gravitational entanglement can also be used as precision tests of gravitational coupling. In particular, entanglement dependence on the initial relative momentum of interacting particles indicates third-order coupling. Furthermore, the amount of entanglement accumulated in a fixed time interval grows linearly with the initial relative momentum, when the particles are pushed towards each other. We presented a closed expression for the amount of entanglement as a function of initial relative momentum based on derived exact covariance matrix for gaussian dynamics extended to the third-order coupling. The methods introduced are also applicable to higher-order couplings and work for arbitrary central interactions.

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APPENDIX

The Ehrenfest’s dynamics. The Ehrenfest’s theorem relates the time derivative of the expectation value of an operator $\hat{a}$ to the expectation value of its commutator with Hamiltonian [31]:

$$\frac{d}{dt} \langle \hat{a} \rangle = \frac{1}{i\hbar} \left[ \langle \hat{a}, \hat{H} \rangle \right] + \langle \frac{\partial \hat{a}}{\partial t} \rangle. \quad (12)$$

Evolution of the COM: The COM Hamiltonian is $\hat{H}_R = \hat{P}^2/4m$, and the exact solutions of the coupled Ehrenfest’s equations imply

$$\Delta R^2 = \frac{1}{2} \sigma^2 (1 + \omega_0 t^2), \quad (13)$$

$$\Delta P^2 = \frac{\hbar^2}{2m} \sigma^2, \quad (14)$$

$$\text{Cov}(\hat{R}, \hat{P}) = \frac{1}{2} \hbar \omega_0 t, \quad (15)$$

where $\Delta$ and $\text{Cov}$ denote the variance and the covariance, respectively. Alternatively, one arrives at the exact same results by utilising the functional form of time-dependent wave function [32].

Evolution of the reduced mass: The reduced mass Hamiltonian can be represented by a binomial series, see Eq. (7). For $N = 2$ we solve the coupled Ehrenfest’s equations and derive:

$$\Delta r^2 = 2\sigma^2 \left( \cosh^2(\omega t) + \frac{\omega_0^2}{\omega^2} \sinh^2(\omega t) \right),$$

$$\Delta p^2 = \frac{\hbar^2}{8\sigma^2} \left( \cosh^2(\omega t) + \frac{\omega_0^2}{\omega^2} \sinh^2(\omega t) \right),$$

$$\text{Cov}(\hat{r}, \hat{p}) = \frac{\hbar}{4} \left( \frac{\omega_0}{\omega} + \frac{\omega}{\omega_0} \right) \sinh(2\omega t).$$

Quantification of entanglement. We have employed the formalism based on the covariance matrix to quantify entanglement gain via logarithmic negativity, and additionally used density matrix to compute the entropy of entanglement.

Negativity of covariance matrix: The covariance matrix formalism is based on the first two statistical moments of a quantum state. The covariance matrix for a bipartite system $AB$ is defined by [28–30]:

$$\sigma_{ij} = \frac{1}{2} \left( \langle \hat{u}_i \hat{u}_j \rangle + \langle \hat{u}_j \hat{u}_i \rangle \right), \quad (16)$$

where $\hat{u} = (\hat{x}_A, \hat{p}_A, \hat{x}_B, \hat{p}_B)$. In the block form

$$\sigma = \begin{pmatrix} \alpha & \gamma^T \\ \gamma & \beta \end{pmatrix}, \quad (17)$$

where $\alpha (\beta)$ contains the local mode correlation for $A (B)$, and $\gamma$ describes the intermodal correlation. In our setting the local modes are identical, i.e., $\alpha = \beta$, and a coordinate change to the COM frame implies

$$\sigma_{00}(\sigma_{03}) = \Delta R^2 + (-) \frac{1}{4} \Delta \nu^2, \quad (18)$$

$$\sigma_{11}(\sigma_{33}) = \frac{1}{4} \Delta P^2 + (-) \Delta \nu^2, \quad (19)$$

$$\sigma_{01}(\sigma_{03}) = \frac{1}{2} \text{Cov} (\hat{R}, \hat{P}) + (-) \frac{1}{2} \text{Cov} (\hat{r}, \hat{p}). \quad (20)$$

Given the symmetry of the problem we have $\sigma_{22} = \sigma_{00}$, $\sigma_{33} = \sigma_{11}$, $\sigma_{23} = \sigma_{01}$, and $\sigma_{21} = \sigma_{03}$. The negativity of partially transposed density matrix is a necessary and sufficient condition for entanglement in two-mode gaussian states [33]. As a result of the partial transposition, the covariance matrix is transformed to $\tilde{\sigma}$, which differs from $\sigma$ by a sign-flip of $\text{Det} \gamma$ [30]. The symplectic eigenvalues of $\tilde{\sigma}$ are given by $2\tilde{\nu}^2 = \tilde{\Sigma}(\sigma) \mp \sqrt{\Sigma^2(\sigma) - 4 \text{Det}(\sigma)}$, where $\Sigma(\sigma) = \text{Det} \alpha + \text{Det} \beta - 2 \text{Det} \gamma$ [28, 29]. Entanglement is quantified by the $\nu$ via logarithmic negativity:

$$E_N(\sigma) = \max \left[ 0, -\log_2 \left( \frac{2\tilde{\nu}}{\hbar} \right) \right]. \quad (21)$$

Entropy of entanglement: For a pure bipartite system described by the density matrix $\rho_{AB}$, the entanglement entropy is defined as the von Neumann entropy for any one of the subsystems, e.g., $S(\rho_A) = -\text{Tr}[\rho_A \log_2(\rho_A)]$, where $\rho_A = \text{Tr}_B(\rho_{AB})$ is the reduced density matrix for subsystem $A$. In order to calculate $S(\rho_A)$ we start with the the two-body wave function:

$$\Psi(x_A, x_B, t) = \phi \left( \frac{x_A + x_B}{2}, t \right) \psi(x_B - x_A, t), \quad (22)$$

where $\phi$ is derived analytically [32], and $\psi$ is calculated numerically [25, 26]. We thereafter perform a singular value decomposition [34, 35]: $\Psi(x_A, x_B) = \sum_i \sqrt{\lambda_i} \ u_i(x_A) \ v_i(x_B)$, where $u_i(x_A)$ and $v_i(x_B)$ are orthonormal states in subsystems $A$ and $B$, respectively,
and $\lambda_i$ are the Schmidt coefficients. A numerical implementation uses Google TensorNetwork [36–38]. With this decomposition the entanglement entropy reduces to

$$S(\rho_A) = -\sum_i \lambda_i \log_2(\lambda_i).$$

In case of evolution into Gaussian states, $S(\rho_A)$ is calculable directly from the covariance matrix [39].

$$S(\rho_A) = \int \left( \frac{\sqrt{\text{Det} \alpha}}{\hbar} \right),$$

where $f(x) = (x + \frac{1}{2}) \log_2(x + \frac{1}{2}) - (x - \frac{1}{2}) \log_2(x - \frac{1}{2})$.

The case of a quadratic interaction: In the special case where the gravitational interaction is truncated up to the second order in Eq. (7), we derive the covariance matrix in a closed form:

$$\sigma_{00} = \frac{1}{2} \sigma^2 \left[ 2 + \omega_0^2 t^2 + \left( 1 + \frac{\omega_0^2}{\omega^2} \right) \sinh^2(\omega t) \right],$$

$$\sigma_{02} = \frac{1}{2} \sigma^2 \left[ \omega_0^2 t^2 - \left( 1 + \frac{\omega_0^2}{\omega^2} \right) \sinh^2(\omega t) \right],$$

$$\sigma_{11} = \frac{\hbar^2}{8} 2 + \left( 1 + \frac{\omega^2}{\omega_0^2} \right) \sinh^2(\omega t),$$

$$\sigma_{13} = -\frac{\hbar^2}{8} \left( 1 + \frac{\omega^2}{\omega_0^2} \right) \sinh^2(\omega t),$$

$$\sigma_{01} = \frac{\hbar}{8} 2 \omega_0 t + \left( \frac{\omega_0}{\omega} + \frac{\omega}{\omega_0} \right) \sinh(2\omega t).$$

### Galilean relativity and a moving COM

We made a change of reference frames so as to dissect the bipartite evolution into two independent single-particle dynamics. The first one is the free evolution of the COM, and the second one is the evolution of reduced mass in gravitational potential. Under the assumption that two spheres are imparted with equal and opposite momentum, the COM is stationary on average. While this simplifies our theoretical framework substantially, such a configuration may be difficult to achieve in an actual experiment. It is much easier to push one of the masses while the other one is kept at rest. In such a case the COM moves rectilinearly with a constant velocity.

The Galilean principle of relativity demands that the laws of physics must be invariant in all inertial frames of reference. Consequently, the centered moments of the moving COM evolve in the same way as for the stationary one [32]. This is readily cross-checked as we get the same correlation and variances after incorporating a non-zero momentum in the initial conditions for solving COM Ehrenfest’s equations. In conclusion, a uniformly moving COM has no role in generating quantum (or, for that matter, classical) correlations. Only the relative momentum matters, and as long as it remains the same, the individual momenta can be tweaked as per convenience.