Nonclassical correlations between photon number and quadrature components of the light field

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Abstract

Finite resolution quantum nondemolition (QND) measurements allow a determination of light field properties while preserving some of the original quantum coherence of the input state. It is thus possible to measure correlations between the photon number and a quadrature component of the same light field mode. Nonclassical features emerge as photon number quantization is resolved. In particular, a strong anti-correlation of quantization and coherence is observable in QND measurements of photon number, and a correlation between measurement induced quantum jumps and quadrature component measurement results is obtained in QND measurements of quadrature fluctuations in the photon vacuum. Such nonclassical correlations represent fundamental quantum properties of the light field and may provide new insights into the nature of quantization itself.

Keywords: Nonclassical correlations, Quantum nondemolition measurements

1 Photon number and quadrature components

When Max Planck introduced the concept of quantization one hundred years ago [1], he was painfully aware that this theory did not fit in with Maxwell’s highly successful theory of electromagnetic radiation. Specifically, the assumption that the energy of the continuous light field can only have a discrete set of values appears to be in contradiction with the necessary continuity of interference phenomena. In the more complete formalism of quantum mechanical operators and states, this strange relationship of a discrete light field intensity given by a photon number \( \hat{n} \) and the continuous quadrature components \( \hat{x} \) and \( \hat{y} \) is expressed by the operator equation

\[
\hat{n} + \frac{1}{2} = \hat{x}^2 + \hat{y}^2.
\]

(1)

If \( \hat{n} \), \( \hat{x} \), and \( \hat{y} \) were given by real numbers, continuous shifts in \( \hat{x} \) and \( \hat{y} \) such as the ones caused by interference with another field mode should cause continuous changes in \( \hat{n} \). However, \( \hat{n} \), \( \hat{x} \), and \( \hat{y} \) are operators with non orthogonal eigenstates. Precise knowledge of the eigenvalue of \( \hat{n} \) therefore restricts the possible knowledge about \( \hat{x} \) and \( \hat{y} \). This uncertainty between the photon number \( \hat{n} \) and the quadrature components \( \hat{x} \) and \( \hat{y} \) is a necessary requirement for the discreteness of the eigenvalues of the photon number \( \hat{n} \).

However, uncertainty cannot explain the quantization of photon number. Even arbitrarily small changes in \( \hat{x} \) and \( \hat{y} \) cause either no change in photon number, or seemingly discontinuous “quantum jumps” changing the photon number by at least one full photon. The randomness of these quantum jumps may seem to defy further analysis. Yet, equation (1) indicates some correlation between the photon number \( \hat{n} \) and the quadrature components \( \hat{x} \) and \( \hat{y} \). In order to investigate this correlation, it is useful to consider quantum nondemolition (QND) measurements with a finite measurement resolution [2, 3, 4, 5, 6, 7, 8, 9]. Since the measured light field is not absorbed in a QND measurement, further measurements performed on a different property of the same field are possible [10, 11]. In this manner, photon number measurements may be combined with quadrature component measurements. Even though the measurement interaction introduces some noise into the field, the finite measurement resolution permits a limitation of noise to the minimum required by the uncertainty relations. The measurement result of the QND measurement can then be correlated with the outcome of the final measurement performed on the same light field mode [2, 3]. Since the noise introduced in the QND measurement should not depend on the measurement result, this correlation re-
veals fundamental quantum mechanical properties of
the original light field state.

2 General properties of QND measurements

Optical quantum nondemolition measurements probe
the light field by nonlinear interactions between a me-
ter field $M$ and the signal field $S$. Usually, the meter
field is initially in a coherent state and the quadrature
component $\hat{x}_M$ of the meter field serves as pointer vari-
able. The interaction between the QND variable $\hat{A}_S$ of
the system and the meter components $\hat{x}_M$ and $\hat{y}_M$ is
then given by

$$
\begin{align*}
\hat{x}_M^{\text{out}} &= \hat{x}_M^{\text{in}} + \frac{\hat{A}_S^{\text{in}}}{2\delta A} \\
\hat{y}_M^{\text{out}} &= \hat{y}_M^{\text{in}} \\
\hat{A}_S^{\text{in}} &= \hat{A}_S^{\text{in}}.
\end{align*}
$$

(2)

Note that the measurement resolution $\delta A$ is a function
of the initial meter fluctuation of $\delta x_M = 1/2$ and the
coupling strength provided by the nonlinearity. Signal
variables $\hat{B}_S$ which do not commute with the QND
variable $\hat{A}_S$ are subject to random changes induced by
the quantum fluctuations of $\hat{y}_M$ according to

$$
\hat{B}_S^{\text{out}} = \hat{U}^{-1}\hat{B}_S^{\text{in}}\hat{U}
$$

(3)

with

$$
\hat{U} = \exp\left(-i\frac{\hat{A}_S^{\text{in}}\hat{y}_M^{\text{in}}}{\delta A}\right).
$$

After the interaction, the meter field and the signal
field are entangled. A measurement of $\hat{A}_S$ can still
be avoided by reading out the noise variable $\hat{y}_M$. The
change of the system is then found to be a unitary trans-
nformation and no information about system prop-
erties is obtained. If the meter variable $\hat{x}_M$ is read
out, however, information about the system variable
$\hat{A}_S$ is obtained while the uncertainty of the noise term
$\hat{y}_M$ causes an uncontrollable change in all other system
properties $\hat{B}_S$.

By identifying the measurement readout of $\hat{x}_M$ di-
rectly with the most likely value $A_m$ of $\hat{A}_S$, the mea-
surement can be represented by a generalized measure-
ment operator $\hat{P}_{\delta A}(A_m)$ given by

$$
\hat{P}_{\delta A}(A_m) = (2\pi\delta A^2)^{-1/4} \exp\left(-\frac{(\hat{A}_S - A_m)^2}{4\delta A^2}\right).
$$

(4)

Note that the measurement values $A_m$ are continuous
even if $\hat{A}_S$ has only discrete eigenvalues. In this sense,
the generalized measurement operator $\hat{P}_{\delta A}(A_m)$ over-
comes the limitation to eigenvalues inherent in the con-
tventional measurement postulate \[13\]. The whole effect
of a measurement of $A_m$ with a resolution $\delta A$ can now
be described by the operator $\hat{P}_{\delta A}(A_m)$. For an initial
state $|\psi_S^{\text{in}}\rangle$ of the signal field, the probability dis-
bution $P(A_m)$ over measurement results $A_m$ and the
state $|\psi_S(A_m)\rangle$ conditioned by a measurement result
of $A_m$ are given by

$$
P(A_m) = \langle\psi_S^{\text{in}}|\hat{P}_{\delta A}^2(A_m)|\psi_S^{\text{in}}\rangle
$$

$$
|\psi_S(A_m)\rangle = \frac{1}{\sqrt{P(A_m)}}\hat{P}_{\delta A}(A_m)|\psi_S^{\text{in}}\rangle.
$$

(5)

It is then possible to derive correlations between the
measurement result $A_m$ and further measurements by
referring to the statistical properties of the conditioned
output state $|\psi_S(A_m)\rangle$.

3 Anti-correlation of quantiza-
tion and coherence

QND measurements of photon number have been real-
ized experimentally using fiber optics \[3, 4\]. In these
setups, a third order nonlinearity shifts the phase of
the coherent meter field by an amount proportional to
the intensity fluctuations of the signal field, while the intensity fluc-
tuations of the coherent meter field $|\psi_B\rangle$ in a coherent
state $|\psi_a\rangle$ is given by

$$
\delta \phi = \frac{1}{2\delta n}.
$$

(6)

The expectation value of the signal field amplitude $\hat{a} =
\hat{x} + i\hat{y}$ is consequently reduced to

$$
\langle\hat{a}\rangle^\text{out} = \exp\left(-\frac{1}{8\delta n^2}\right)\langle\hat{a}\rangle^\text{in},
$$

(7)

as illustrated in figure \[1\].

These dephasing characteristics have been discussed pre-
viously and correspond well with the measurement
dynamics observed experimentally \[10, 11\]. However,
the averaged results hide a peculiar correlation be-
tween the dephasing statistics and the measurement
results which can be obtained by applying the general-
fied measurement operator \[12\]. If the initial signal
field is in a coherent state $|\alpha\rangle$ given by

$$
|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle,
$$

(8)

the probability distribution over measurement results
$n_m$ is given by

$$
P(n_m) = \langle\alpha|\hat{P}_{\delta A}^2(n_m)|\alpha\rangle
$$
and the expectation value of the coherent amplitude \( \hat{a} \) after the measurement is given by

\[
\langle \hat{a} \rangle_f(n_m) = \frac{\langle \alpha | \hat{P}_\alpha(n_m) \hat{a} \hat{P}_\alpha(n_m) | \alpha \rangle}{\langle \alpha | \hat{P}_\alpha^2(n_m) | \alpha \rangle} = \alpha \exp \left( -\frac{1}{8\delta n^2} \right) \sum_n \frac{|\alpha|^{2n}}{n!} \exp \left( -\frac{(n - n_m)^2}{2\delta n^2} \right)
\]

(10)

The results for \( \alpha = 3 \) and a resolution of \( \delta n = 0.3 \) are shown in figure 2. After the measurement, the expectation values of the coherent amplitude \( \hat{a} \) are maximal if the measurement result \( n_m \) was a half integer value and minimal if it was an integer value. Therefore, the accidental measurement of a properly quantized integer photon number causes additional phase noise, while the accidental observation of a half-integer photon number preserves the original phase coherence of the field.

In order to quantify this property, it is useful to define the quantization \( Q \) of the measurement result \( n_m \) as

\[
Q(n_m) = \cos (2\pi n_m).
\]

(11)

It is then possible to determine the correlation \( C(Q; \langle \hat{a} \rangle_f) \) between quantization \( Q \) and coherence \( \langle \hat{a} \rangle_f \) by

\[
C(Q; \langle \hat{a} \rangle_f) = \frac{Q \langle \hat{a} \rangle_f - \overline{Q \langle \hat{a} \rangle_f}}{\delta_n \langle \hat{a} \rangle_f} \exp \left( -\frac{1}{8\delta n^2} \right) \alpha,
\]

(12)

where the over-lined quantities are averages over all possible measurement values \( n_m \). The resulting correlation is always negative, since the coherence is maximal at half-integer photon numbers which have a quantization \( Q \) of minus one. Figure 2 shows the dependence of this anti-correlation between quantization and coherence as a function of measurement resolution. At resolutions \( \delta n \gg 0.3 \), there is no correlation because quantization is not resolved. At resolutions \( \delta n \ll 0.3 \), there is no correlation because the phase is completely randomized and the average coherence \( \langle \hat{a} \rangle_f \) is reduced to zero. In the intermediate regime, however, quantization and coherence are clearly anti-correlated properties of the light field.

While the correlation \( C(Q; \langle \hat{a} \rangle_f) \) between quantization and coherence is definitely an observable property of the light field, an experimental verification in the op-
tical region is difficult because of the relative weakness of the available nonlinearities. In the QND measurements using fiber optics, the resolutions achieved are still far below the quantum limit (e.g. \( \delta n \approx 10^{-4} \) in [4]). Alternatively, it is possible to investigate nonclassical correlations by first realizing a QND measurement of coherent field properties, followed by a precise photon number measurement.

4 Correlation of field fluctuations and quantum jumps

QND measurements of quadrature components have been realized using the phase sensitive nonlinear interaction in optical parametric amplifiers (OPAs) [7, 8, 9]. By exploiting the interference properties of two phase sensitive amplification steps, it is possible to shift one quadrature component of the meter field by an amount proportional to the signal field component \( \hat{x} \). Since the uncertainty relation between the quadrature components is given by

\[
\delta x \delta y \geq \frac{1}{4}
\]  

(13)

this measurement increases the noise in the \( \hat{y} \) component as illustrated in figure 4. While there appears to be no correlation between the measurement result \( x_m \) and the change in \( \hat{y} \), it is possible to establish a connection between the measurement result \( x_m \) and changes in the photon number \( n \). The natural input state for this investigation is the vacuum field \( |0\rangle \) with its well defined photon number of zero. The total probability distribution \( P(x_m) \) over measurement results \( x_m \) is then given by

\[
P(x_m) = |\langle 0 | \hat{P}_x(x_m) | 0 \rangle|^2
\]

(14)

For high resolutions (\( \delta x \to 0 \)), this Gaussian distribution reproduces the quantum noise level of \( \langle \hat{x}^2 \rangle = 1/4 \). At low resolutions (\( \delta x > 1/2 \)), the measurement uncertainty dominates. The measurement induced changes in the quantum state of the signal field are described by \( \hat{P}_x(x_m) \). At low resolution, this change is small and the vacuum component is still dominant in the output field. However, a photon counting measurement in the signal output analyzes this slight change in terms of quantum jumps from zero to one photon. The joint probability \( P_1(x_m) \) of measuring \( x_m \) and observing a photon in the output signal is given by

\[
P_1(x_m) = |\langle 1 | \hat{P}_x(x_m) | 0 \rangle|^2
\]

(15)

For low resolutions, this symmetric, double peaked probability distribution has its maxima near \( x_m = \pm \sqrt{2} \delta x \). A comparison of \( P_1(x_m) \) and \( P(x_m) \) at a resolution of \( \delta x = 1 \) is shown in figure 5. Note that the total probability shown in figure 5 is reduced by 1/16, since the total probability of observing a photon in the output at \( \delta x = 1 \) is equal to 1/16. The double peaked distribution has its peaks at the flanks of the total distribution, indicating that quantum jump events
are always associated with high field fluctuations. The photon number after the measurement is therefore correlated with the measurement result. This correlation can be written as

\[ C(x_m^2; \langle \hat{n} \rangle_f) = \int (0 | \hat{P}_m(x_m) \hat{n} \hat{P}_m(x_m) | 0) x_m^2 \, dx_m. \] (16)

The integral over the measurement results \( x_m \) and the expectation values of \( \hat{n} \) after the measurement can be solved by making use of the operator properties given by equation (4). It can then be written as an operator correlation which does not depend on the measurement resolution,

\[ C(x_m^2; \langle \hat{n} \rangle_f) = \frac{1}{4} \langle \hat{x}^2 \hat{n} + 2 \hat{x} \hat{n} \hat{x} + \hat{n} \hat{x}^2 \rangle - \langle \hat{x}^2 \rangle \langle \hat{n} \rangle = \frac{1}{8}. \] (17)

The characteristics of the quantum jump statistics illustrated by figure 5 are therefore directly related to fundamental properties of the operator formalism. In particular, equation (17) shows that an operator correlation between photon number and field fluctuations is possible even if the field is in a photon number eigenstate. This correlation is a direct consequence of the non-commutativity of operators, since the sandwiching of the photon number operator \( \hat{n} \) between the field operators \( \hat{x} \) makes the eigenvalue of \( n = 0 \) in the photon number state irrelevant. Indeed, the photon number of the vacuum is only zero with respect to actual photon number measurements. It cannot be considered a measurement independent physical property of the system. Much of the confusion surrounding the interpretation of quantum mechanics arises from an erroneous identification of eigenvalues with such measurement independent “elements of reality”.

5 Implications for the interpretation of quantization

Both the anti-correlation of quantization and coherence and the correlation of field fluctuations and quantum jumps indicate that the discreteness of photon number is not an intrinsic property of the light field itself but a property of the specific measurement interaction. If the photon number is not resolved, it should not be considered an integer number. Eigenvalues of operator variables do not represent the “real” physical values of that property. The EPR paradox [15] and Bell’s inequalities [16] clearly illustrate the fallacy of attempting an identification of eigenvalues with “elements of reality”. In particular, there is every reason to reject the assumption that “If, without in any way disturbing a system, we can predict with certainty the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.” [15]. If the predicted experiment is never performed, it is pointless to demand an “element of reality” for something that might have been. The anti-correlation of quantization and coherence shows that half integer photon numbers are a relevant part of the correlated photon-field statistics. The quantum jump correlation shows that the photon number of the vacuum is effectively nonzero when field measurements are performed first. The nature of the quantum mechanical formalism itself thus demands a dependence of reality on the actual measurement situation. Planck’s problem of reconciling the discreteness of photon number with the continuity of interference in the light field can only be overcome by admitting the context dependence of quantum mechanical reality expressed by the operator formalism. An operator does not represent a numerical value. Rather, it represents a potential interaction with its environment. Instead of abstractly analyzing states and eigenvalues, it is therefore necessary to explore quantum mechanical properties from the perspective of a realistic measurement context.

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