Freeze-in Dark Matter and $\Delta N_{\text{eff}}$ via Light Dirac Neutrino Portal

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Abstract

We propose a scenario where dark matter (DM) and dark radiation ($\Delta N_{\text{eff}}$) can be generated non-thermally due to the presence of a light Dirac neutrino portal between the standard model (SM) and dark sector particles. The SM is minimally extended by three right handed neutrinos ($\nu_R$), a Dirac fermion DM candidate ($\psi$) and a complex scalar ($\phi$), transforming non-trivially under an unbroken $\mathbb{Z}_4$ symmetry while being singlets under the SM gauge group. While DM and $\nu_R$ couplings are considered to be tiny in order to be in the non-thermal or freeze-in regime, $\phi$ can be produced either thermally or non-thermally depending upon the strength of its Higgs portal coupling. We consider both these possibilities and find out the resulting DM abundance and $\Delta N_{\text{eff}}$ via freeze-in mechanism to constrain the model parameters in the light of Planck 2018 data. We find that the scenario where $\phi$ remains out of equilibrium throughout or after a certain epoch allows more parameter space consistent with DM phenomenology and $\Delta N_{\text{eff}}$ in view of Planck 2018 data. The next generation experiments like CMB-S4, SPT-3G etc. will have the required sensitivities to probe a major portion of the entire model parameter space, offering a promising way of probing such non-thermal DM scenario which typical direct detection experiments are not much sensitive to.

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I. INTRODUCTION

As suggested by irrefutable evidences from astrophysics and cosmology based experiments gathered over several decades, we live in a universe whose matter content is dominated by a non-baryonic, non-luminous form of matter, known as dark matter (DM) [1, 2]. While it is approximately five times more dominant than ordinary baryonic matter, its total contribution to present universe’s energy density is around 26%. Present abundance of DM is often quoted in terms of density parameter $\Omega_{\text{DM}}$ and reduced Hubble parameter $h = \text{Hubble Parameter}/(100 \text{ km s}^{-1}\text{Mpc}^{-1})$ as $\Omega_{\text{DM}}h^2 = 0.120 \pm 0.001$ at 68% CL. In spite of so many observational evidences, the particle nature of DM is not yet known. However, it is known for sure that none of the standard model (SM) particles can satisfy the criteria for being a particle DM candidate, leading to several beyond standard model (BSM) proposals in the literature. Among these proposals, the weakly interacting massive particle (WIMP) paradigm is one of the most well studied one. In WIMP paradigm, a particle DM candidate having mass and interaction strength (with SM particles) typically around the electroweak ballpark can give rise to the observed DM abundance after thermal freeze-out, a remarkable coincidence often referred to as the WIMP Miracle [3]. The same interactions responsible for thermal freeze-out of WIMP can also lead to its promising direct detection prospects like observable DM-nucleon scattering. However, the direct detection experiments have not seen any such scattering yet leading to tighter bounds on DM-nucleon couplings. Similar null results have also been reported at indirect detection as well as collider experiments. A recent review on the status of WIMP type DM models can be found in [4]. The null results in WIMP detection have also motivated the particle physics community to look for other viable alternatives like freeze-in or feebly interacting massive particle (FIMP) dark matter [5–17] where DM, due to its feeble interactions with SM bath, never enters equilibrium in the early universe. A recent review of such models can be found in [18]. While FIMP offers a viable alternative to WIMP, such models are often difficult to probe due to tiny DM interactions except some special cases [19, 20].

In this work, we propose a possible way of probing a FIMP DM scenario via measuring additional relativistic degrees of freedom or dark radiation at cosmic microwave background (CMB) experiments. We consider a FIMP model where the same interactions responsible for freeze-in production of DM also produces additional relativistic species or dark radia-
tion. Existing data from CMB experiments like Planck constraints such additional light species by putting limits on the effective degrees of freedom for neutrinos during the era of recombination ($z \sim 1100$) as \[2\]
\[
N_{\text{eff}} = 2.99^{+0.34}_{-0.33}
\] at 2$\sigma$ or 95% CL including baryon acoustic oscillation (BAO) data. At 1$\sigma$ CL it becomes more stringent to $N_{\text{eff}} = 2.99 \pm 0.17$. Similar bound also exists from big bang nucleosynthesis (BBN) $2.3 < N_{\text{eff}} < 3.4$ at 95% CL \[21\]. All these bounds are consistent with SM predictions $N_{\text{eff}}^{\text{SM}} = 3.045$ \[22–24\]. Future CMB experiment CMB Stage IV (CMB-S4) is expected reach a much better sensitivity of $\Delta N_{\text{eff}} = N_{\text{eff}} - N_{\text{eff}}^{\text{SM}} = 0.06$ \[25\], taking it closer to the SM prediction.

We also connect it to neutrino physics by considering light neutrinos to be of Dirac nature such that right chiral part of Dirac neutrinos can act like dark radiation. After getting produced from the SM bath, they can contribute to the radiation energy density of the universe. For some recent studies on light Dirac neutrinos and enhanced $\Delta N_{\text{eff}}$ in different contexts, please see \[26–37\]. Keeping this in mind, we consider a minimal framework where the SM is extended by three right handed neutrinos, one singlet fermion DM candidate and one additional singlet scalar to facilitate the coupling of DM with right handed neutrinos. Additional discrete symmetry $Z_4$ is imposed in order to forbid unwanted couplings while keeping DM stable. The same model was studied in our earlier work \[36\] where DM as well as $\Delta N_{\text{eff}}$ were generated thermally. In the present work, we consider the alternative possibility of freeze-in production with contrasting results and constraints on the model parameters.

This paper is organised as follows. In section II we discuss our basic setup including the model description and relevant Boltzmann equations required to compute the abundance of DM as well as $\Delta N_{\text{eff}}$. We discuss the details of our numerical results in section III and finally conclude in section IV.

II. THE BASIC SETUP

There have been several BSM proposals to realise light Dirac neutrinos, see \[12, 16, 28, 38–64\], for example, and references therein. In order to keep our framework minimal, we consider
only three types of BSM particles sufficient to highlight the interesting phenomenology. They are namely, right handed neutrinos $\nu_R$, fermion singlet DM $\psi$ and a complex scalar singlet $\phi$ transforming non-trivially under an unbroken discrete $\mathbb{Z}_4$ symmetry. The right handed neutrinos couple to left handed lepton doublets via SM Higgs with fine-tuned Dirac Yukawa couplings to generate sub-eV Dirac neutrino masses. All SM leptons as well as $\nu_R$ have $\mathbb{Z}_4$ charge $i$ which keep the Majorana mass terms away. The $\mathbb{Z}_4$ charges of $\psi, \phi$ are chosen to be $-1, i$ respectively which ensures DM has only one tree level coupling of the form $y_\phi \overline{\psi} \nu_R \phi$.

On the other hand, $\nu_R, \phi$ can have other couplings as well. For example $\nu_R$ couples to SM lepton doublet $\ell$ and Higgs $H$ as $y_H \overline{\ell} H \nu_R$. On the other hand, the scalar singlet $\phi$ can have quartic interactions with the SM Higgs as $\lambda_H \phi (H^\dagger H)(\phi \dagger \phi)$. Thus, the Lagrangian involving the newly introduced fermions can be written as

$$L_{\text{fermion}} = i \overline{\nu}_R \gamma^\mu \partial_\mu \nu_R + i \overline{\psi} \gamma^\mu \partial_\mu \psi - m_\psi \overline{\psi} \psi - \left( y_H \overline{\ell} H \nu_R + y_\phi \overline{\psi} \nu_R \phi + \text{h.c.} \right).$$

Similarly, the scalar Lagrangian of the model is

$$L_{\text{scalar}} = (D_{H\mu} H)^\dagger (D^\mu_H H) + (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \lambda_H (H^\dagger H)^2 + \frac{\lambda_H}{2} (\phi \dagger \phi) + \lambda_\phi (\phi \dagger \phi)^2 + \lambda_H \phi (H^\dagger H)(\phi \dagger \phi) + \lambda_\phi \phi^4 + (\phi \dagger \phi)^4,$$

where, the covariant derivative for $H$ is defined as

$$D_{H\mu} H = \left( \partial_\mu + ig^a \sigma_a W^a_\mu + ig'^a \frac{1}{2} B_\mu \right) H.$$
$y_\phi, \lambda_{H_\phi}$ play crucial roles along with the masses of $\phi, \psi$ denoted by $m_\phi, m_\psi$ respectively. Therefore, the relevant free parameters of this model are the following couplings and the masses,

$$m_\phi, m_\psi, y_\phi, \lambda_{H_\phi}.$$  \hspace{1cm} (7)

Since both DM and $\nu_R$ will be dominantly produced from $\phi$, it is important to track the evolution of $\phi$ in the early universe. Depending upon coupling of $\phi$ with SM Higgs and its mass $m_\phi$, production of DM, $\nu_R$ can occur while $\phi$ is either in equilibrium or out of equilibrium. In order to discuss the our results in details, we consider three different scenarios and write the corresponding Boltzmann equations as follows. For the detailed derivations of the Boltzmann equations for each of these scenarios, please refer to appendix A.

A. Case I: $\phi$ in equilibrium

In this case, $\phi$ remains in equilibrium with the SM bath during DM and $\nu_R$ production from $\phi$ decay. Thus $\phi$ abundance can be considered to be its equilibrium abundance throughout while for the other two species $\psi, \nu_R$, the relevant Boltzmann equations, in terms of comoving number densities, are given by

$$\frac{dY_\psi}{dx} = \frac{\beta}{xH} \frac{K_1(x)}{K_2(x)} Y_{\phi}^{eq},$$  \hspace{1cm} (8)

$$\frac{dY_{\nu_R}}{dx} = \frac{\beta}{H s^{1/3} x} \langle E \Gamma \rangle Y_{\phi}^{eq},$$  \hspace{1cm} (9)

where $x = m_\phi/T$ and

$$\beta = \left[1 + \frac{T d g_s/d T}{3 g_s}\right],$$  \hspace{1cm} (10)

$$\langle E \Gamma \rangle = g_\phi g_{\nu_R} \frac{\left| \mathcal{M}_{{\phi} \rightarrow {\nu_R} \psi} \right|^2 (m_\phi^2 - m_\psi^2)^2}{32 \pi m_\phi^2}.$$  \hspace{1cm} (11)

Here $H$ is the Hubble parameter in radiation dominated universe and $K_i$ is modified Bessel function of i-th order. While $Y_\psi = n_\psi/s$ as usual for massive particles, the comoving density of $\nu_R$ which remain relativistic, is defined in terms of its energy density as $Y_{\nu_R} = \rho_{\nu_R}/s^{4/3}$. 
B. Case II: freeze-out of $\phi$

For certain choices of model parameters, one can have a scenario where $\phi$ gets thermally produced first followed by its freeze-out and only after that dominant production of DM and $\nu_R$ take place from decay of $\phi$. Since $\phi$ can no longer be taken to be in equilibrium throughout, we need to track its evolution using the corresponding Boltzmann equation. The system of Boltzmann equations in this case is given by

$$\frac{dY_\phi}{dx} = \frac{\beta s}{\mathcal{H} x} \left( -\langle \sigma v \rangle \left( (Y_\phi)^2 - (Y_\phi^{eq})^2 \right) - \frac{\Gamma_\phi}{s} \frac{K_1(m_\phi/T)}{K_2(m_\phi/T)} Y_\phi \right), \quad (12)$$

$$\frac{dY_\psi}{dx} = \frac{\beta}{x \mathcal{H}} \Gamma_\phi \frac{K_1(x)}{K_2(x)} Y_\phi, \quad (13)$$

$$\frac{dY_{\nu R}}{dx} = \frac{\beta}{\mathcal{H} s^{1/3} x} \langle ET \rangle Y_\phi. \quad (14)$$

Here $\langle \sigma v \rangle$ is the thermally averaged annihilation cross-section [65] of $\phi$ into SM particles via Higgs portal interactions while the definition of other parameters remain same as in Case I discussed earlier.

C. Case III: non-thermal $\phi$

Finally, we consider the remaining possibility where $\phi$ can be out-of-equilibrium throughout due to tiny couplings with the SM Higgs. Thus, the initial abundance of $\phi$ remains negligible, like FIMP DM, and then it starts to populate the universe due to decay or annihilation of SM bath particles. Since $\phi$ has only Higgs portal couplings, the relevant production mechanism is from Higgs decay or Higgs annihilation depending upon $m_\phi$. The distribution function for $\phi$ can be calculated by solving the following equation

$$\frac{\partial f_\phi}{\partial t} - \mathcal{H} p_1 \frac{\partial f_\phi}{\partial p_1} = C^{h \rightarrow \phi \phi^\dagger} + C^{hh \rightarrow \phi \phi^\dagger} + C^{\phi \rightarrow \nu_R \psi}, \quad (15)$$

the details of the collision terms on the RHS are given in Appendix A. Once the distribution function $f_\phi$ is evaluated, it can be used to find the evolution of DM and $\nu_R$ densities by solving the following Boltzmann equations

$$\frac{dY_\psi}{dr} = \frac{g_\phi \beta}{r \mathcal{H} s} \frac{\Gamma_\phi m_\phi}{2 \pi^2} \int \frac{(A m_{r \nu})^3 \xi^2 f_\phi(\xi, r)}{\sqrt{(\xi A m_{r \nu})^2 + m_\phi^2}} d\xi, \quad (16)$$

$$\frac{dY_{\nu R}}{dr} = \frac{g_\phi \beta}{r \mathcal{H} s^{1/3}} \langle ET \rangle \frac{1}{2 \pi^2} \int_0^\infty \left( A m_{r \nu}^0 \right)^3 \xi^2 f_\phi(\xi, r) d\xi, \quad (16)$$
where \( r = \frac{m_0}{T} \) with \( m_0 \) being an arbitrary mass scale and details of \( A, \xi \) are given in Appendix A.

### III. NUMERICAL RESULTS

In this section, we discuss our numerical results for all the three cases mentioned above. After solving the Boltzmann equations for comoving densities of dark sector species, we can find the observable quantities like DM abundance \( \Omega_{DM} h^2 \) and \( \Delta N_{\text{eff}} \) by following the procedure shown in Appendix B. Since the region of validity for these three cases crucially depends upon the parameters involving complex scalar singlet \( \phi \), we first show the parameter space in terms of its mass and Higgs portal couplings in left panel of Fig. 1 indicating the region excluded by the constraints from the large hadron collider (LHC) on invisible decay of the SM Higgs boson into a pair of \( \phi \). The ATLAS and the CMS collaboration have put the limit on invisible Higgs branching ratio as \( \text{BR}_{h \rightarrow \text{inv}} < 14.6\% \) [66] and \( \text{BR}_{h \rightarrow \text{inv}} < 18\% \) [67] respectively, of which we use the stronger ATLAS bound in the left panel of Fig. 1.

In the right panel of Fig. 1, we show the interaction rate of \( \phi \) (\( \Gamma \)) in comparison to the Hubble expansion rate for three benchmark values of \( m_\phi, \lambda_{H\phi} \) to indicate typical Higgs portal couplings required to consider thermal production of \( \phi \) in the early universe. Clearly, for Higgs portal coupling \( \lambda_{H\phi} \leq 10^{-8} \) validates the non-thermal nature of \( \phi \) as we consider while discussing details of case III. In the following, we will choose the benchmark points as well as the scan range while keeping Fig. 1 in mind.

In addition to bounds on \( \Omega_{DM} h^2, \Delta N_{\text{eff}} \) and \( (m_\phi, \lambda_{H\phi}) \) plane mentioned above, we also note the model independent bounds on DM mass. If DM is very light, it can remain relativistic for a long time after being produced from \( \phi \) decay resulting in large free-streaming length. While hot dark matter is ruled out, a warm dark matter (WDM) component is still allowed provided certain bounds are satisfied. Depending upon the details of production mechanism, warm dark matter mass below a few keV is ruled out as shown in several works incorporating different observations [68–70]. Coincidentally, similar lower bound exists on fermion DM mass from galactic phase space arguments [71, 72]. While these lower bounds can vary slightly depending upon the production scenario and observational constraint imposed, we consider a lower bound of \( \mathcal{O}(1) \) keV in our analysis. We also consider a conservative upper bound on \( \phi \) lifetime such that its decay is complete before the BBN epoch \( T_{\text{BBN}} \sim \mathcal{O}(10) \)
MeV. This ensures the production of dark matter as well as dark radiation before the onset of BBN epoch.

\[ \text{Br}_{h \rightarrow \text{inv}} > 14.5\% \]

\[ m_\phi = 50 \text{ GeV}, \lambda_{H\phi} = 10^{-8} \]

\[ m_\phi = 1 \text{ TeV}, \lambda_{H\phi} = 10^{-4} \]

\[ m_\phi = 1 \text{ GeV}, \lambda_{H\phi} = 10^{-3} \]

\[ m_\psi \]

\[ y_\phi \]

\[ \Omega_{\text{DM}} h^2 \]

\[ T \lesssim m_\phi \]

\[ \text{FIG. 1: Left panel: LHC constraint in } m_\phi - \lambda_{H\phi} \text{ plane showing the region excluded by upper limit on invisible decay width of the SM Higgs. Right panel: Interaction rates of } \phi \text{ in comparison to the Hubble expansion rate for benchmark choices of } m_\phi - \lambda_{H\phi} \text{ used in our analysis.} \]

**A. Case I**

In this case, \( \phi \) remains in equilibrium while DM and \( \nu_R \) production takes place. This is the simplest scenario where we need to solve only two coupled Boltzmann equations for \( \psi, \nu_R \) while using equilibrium abundance for \( \phi \) throughout. Fig. 2 shows the evolution of dark sector particles as functions of temperature for two different choices of benchmark parameters. The magenta, blue and green lines correspond to the comoving number densities of \( \phi \) (in equilibrium), \( \psi \), and \( \nu_R \) respectively. The parameters \( m_\psi, y_\phi \) and \( m_\psi \) are taken in such a way that 2\( \sigma \) Planck 2018 bound on \( \Delta N_{\text{eff}} \) and \( \Omega_{\text{DM}} h^2 \) are satisfied. While \( \phi \) abundance follows the equilibrium abundance as shown by the magenta line, DM and \( \nu_R \) freeze in from decay of \( \phi \) and gets saturated after \( \phi \) abundance gets Boltzmann suppressed for \( T \lesssim m_\phi \).

Now, let us discuss the difference due to two different choices of \( y_\phi \) as shown in the left panel plot of Fig. 2. As expected in freeze-in production [5], a decrease in mother particle’s
FIG. 2: Evolution of dark sector particles ($\phi, \psi, \nu_R$) in case I considering $\phi$ to be in equilibrium throughout. The left and right panel plots show the change in evolution for two different choices of $y_\phi, m_\phi$ respectively. Chosen benchmark points including DM mass $m_\psi$ keep DM abundance as well as $\Delta N_{\text{eff}}$ within Planck limits.

Coupling reduces the number densities of daughter particles. As seen by comparing the solid and dashed lines in Fig. 2 (left panel), a decrease in $y_\phi$ leads to decrease $Y_\psi$ and $Y_{\nu_R}$. In order to keep DM density within Planck limits, lower value of $y_\phi$ which further lowers $Y_\psi$ requires a higher value of DM mass $m_\psi$. For $m_\phi \gg m_\psi$, $m_\psi$ will have negligible effect on $Y_\psi$ and $Y_{\nu_R}$. Thus, DM abundance will be linearly proportional to $\psi$ mass. For the choice of $y_\phi$ which almost saturates Planck $2\sigma$ bound on $\Delta N_{\text{eff}}$, we also have a large production of $Y_\psi$ requiring very light DM mass below a keV to keep $\Omega_{\text{DM}} h^2$ within Planck limits. If we lower $y_\phi$, to bring $\Delta N_{\text{eff}}$ close to sensitivity of future experiment like CMB-S4, we can get slightly heavier DM mass but still ruled out due to WDM bounds mentioned before. If we want to bring DM mass in the allowed regime $\geq O(1)$ keV, its coupling with the mother particle needs to be brought further down. However, lowering $y_\phi$ will also lower $\Delta N_{\text{eff}}$, keeping it below future experimental sensitivities. While we keep $m_\phi$ fixed at 0.5 GeV in Fig. 2, the general conclusions reached here for case I do not change significantly for a different choice of $m_\phi$. This is clearly seen from the right panel plot of Fig. 2 which compares the change in evolution for two different choices of $m_\phi$. As expected, a heavier $m_\phi$ leads to Boltzmann suppression in equilibrium abundance of $\phi$ at much earlier epoch, leading to production of $\nu_R$ as well as $\psi$ at earlier epochs compared to the case for lighter $\phi$. The decrease in yield for heavier $\phi$ can be understood by solving the Boltzmann equation in Eq. (8) analytically.
which gives the final yield as $Y_\psi(x \to \infty) \propto \Gamma_\phi/m_\phi^2$ \cite{5}. Since $\Gamma_\phi \propto m_\phi$ for $m_\psi \ll m_\phi$, heavier $\phi$ leads to smaller yield $Y_\psi$. Similarly, one can explain the decrease in $\nu_R$ yield too for heavier $\phi$. Thus, for $\phi$ in equilibrium, it is not possible to find a suitable set of $m_\phi, y_\phi, m_\psi$, such that $\Delta N_{\text{eff}}$ remains within experimental reach while DM abundance agreeing to Planck limits. This is not surprising, as the equilibrium condition on $\phi$ leaves us with a fewer number of free parameters with limited scope of choosing our parameters to get $\Omega_{\text{DM}}h^2$ and $\Delta N_{\text{eff}}$ in the region of our interest. Therefore, although case I remains consistent with current experimental limits, it is of least interest as it does not offer a future CMB probe of our model.

B. Case II

We now discuss the results for the intermediate scenario where $\phi$ gets produced thermally followed by its freeze-out. This requires solving the Boltzmann equation for $\phi$ as well together with the ones for $\psi, \nu_R$. Therefore, in addition to $m_\phi, y_\phi, m_\psi$, the Higgs portal coupling $\lambda_{H\phi}$ can play crucial role in deciding DM abundance as well as $\Delta N_{\text{eff}}$. We show the evolution of dark sector particles for case II in Fig. 3. The top left, top right and bottom panels in this figure show the comparisons for two different choices of $y_\phi, \lambda_{H\phi}, m_\phi$ respectively. Similar to case I, the magenta, blue and green lines correspond to the comoving number densities of $\phi$ (in equilibrium), $\psi$, and $\nu_R$ respectively. The red line corresponds to the actual comoving number density of $\phi$ which undergoes thermal freeze-out at an intermediate epoch followed by complete decay at later epochs. In all these plots, one can clearly see the production of $\psi, \nu_R$ to be taking place during equilibrium as well as frozen out phases of $\phi$ separated by a kink in between, as seen from the blue and green lines. The Higgs portal coupling of $\phi$ is chosen in such a way that the freeze-out abundance of $\phi$ is non-negligible in order to play substantial role in $\psi, \nu_R$ production. This is clearly visible from the plots shown in Fig. 3, where the production of $\psi, \nu_R$ from frozen out $\phi$ appear to be significant. Another significant improvement from case I is that mass of DM can satisfy the lower limits discussed earlier even when $\Delta N_{\text{eff}}$ saturates Planck upper bound, keeping the scenario verifiable at CMB experiments.

In the top left panel plot of Fig. 3, we show the evolution for two different values of $y_\phi$ while keeping other parameters fixed. Since $y_\phi$ dictates the decay width of $\phi$, a lower value
of $\phi$ delays the decay of frozen out $\phi$. Change in $y_\phi$, however, keeps DM density same as the number of $\phi$ gets transferred to number of $\psi$, both of which behave as non-relativistic particles. On the other hand, a lower value of $\phi$ or delayed production of $\nu_R$ from frozen out $\phi$ increases the comoving density of $\nu_R$ which behaves as radiation with comoving density defined as $Y_{\nu_R} = \frac{\rho_{\nu_R}}{s^{4/3}}$. This can be understood if we solve the coupled Boltzmann equations given in Eqs. (12), (13), (14) analytically after the freeze-out of $\phi$. Once $\phi$ freezes out with a freeze-out abundance $Y_{\phi}^{fo}$, it decreases due to decay as $Y_{\phi} = Y_{\phi}^{fo} e^{-r(x^2-x_F^2)/2}$ where $x_F = m_\phi/T_F$ is a constant depending upon freeze-out temperature $T_F$ and $r$ is a constant proportional to decay width $\Gamma_\phi \propto y_\phi^2$. Thus, although the production rate of $Y_{\nu_R}$ is directly proportional to $y_\phi^2$ as inferred from Eq. (14), the exponential suppression in $\phi$ abundance after its freeze-out leads to a decrease in $\nu_R$ production from frozen out $\phi$ for larger values of $y_\phi$. On the other hand, if we solve the Boltzmann equation for DM namely Eq. (13) analytically after $\phi$ freezes out, we find that $Y_{\phi}^{fo}$ just gets converted into $Y_\psi$ as evident from Fig. 3.

In top right panel plot of Fig. 3, we show the evolution for two different choices of Higgs portal coupling $\lambda_{H\phi}$. As expected from freeze-out mechanism of WIMP type particles, a larger value of $\lambda_{H\phi}$ leads to smaller freeze-out abundance of $\phi$ and hence smaller yield of $\psi, \nu_R$ at later epochs. On the other hand, for larger benchmark value of $\lambda_{H\phi}$ resulting in smaller yield of $Y_\psi$, we choose a heavier DM mass in order to keep $\Omega_{DM}h^2$ within Planck bounds. Finally, in the bottom panel plot of Fig. 3, we show the evolution of dark sector particles for two different choices of $\phi$ mass. Due to change in Boltzmann suppression, the equilibrium evolution also changes for these two values. Since annihilation cross section decreases with increase in mass, we see larger freeze-out abundance for heavier $\phi$. Naturally, a larger freeze-out abundance for heavier $\phi$ leads to enhancement in comoving densities of DM and $\nu_R$ as well. The benchmark values of $m_\phi, m_\psi$ are chosen in such a way that DM abundance $\Omega_{DM}h^2$ remains within Planck limit while heavier (lighter) benchmark of $m_\phi$ keep $\Delta N_{\text{eff}}$ close to Planck upper bound (CMB-S4 sensitivity). It should also be noted that increasing $\phi$ mass also increases its decay width (for $m_\psi \ll m_\phi$) and hence we notice a delay in production of $\psi, \nu_R$ for lighter $\phi$ mass. Although we noticed enhanced $Y_{\nu_R}$ from such delayed production in top left panel plot of Fig. 3, in bottom panel plot of the same figure, this effect remains sub-dominant. The expected increase in $Y_{\nu_R}$ for lighter $m_\phi$ due to delayed production remains subdominant compared to decrease in in $Y_{\nu_R}$ for lighter $m_\phi$ due
to reduced freeze-out abundance of the latter. Therefore, we only notice an overall increase in $Y_{\nu R}$ for heavier $\phi$ having larger freeze-out abundance. In each of these plots shown in Fig. 3, the two benchmark parameter values (that is, $y_\phi$ in top left, $\lambda_{H\phi}$ in top right, $m_\phi$ in bottom) are chosen in such a way that one of them leads to $\Delta N_{\text{eff}}$ close to Planck 2$\sigma$ upper limit while the other pushes it close to CMB-S4 sensitivity limit.

After highlighting the interesting features of case II with benchmark choices of key parameters, we perform a numerical scan over the parameter space. The relevant parameters
are varied in the following range:

\[ 200 \text{ GeV} \leq m_\phi \leq 2000 \text{ GeV}, \]
\[ 10^{-5} \leq \lambda_{H\phi} \leq 10^{-3.5}, \]
\[ 1 \text{ keV} \leq m_\psi \leq 10 \text{ MeV}. \]

The value of \( y_\phi \) is kept constant and remains fixed at \( 10^{-10} \), which also ensures that the decay of \( \phi \) occurs before the BBN epoch. The resulting parameter space is shown in \( \Delta N_{\text{eff}} \) vs \( m_\phi \) plane in Fig. 4. The colour bar in left and right panel plots show the variation in \( \lambda_{H\phi} \) and \( m_\psi \) respectively. While all the points satisfy the Planck bound on DM relic abundance, the corresponding upper bound on \( \Delta N_{\text{eff}} \) is shown by magenta shaded region. The future sensitivity of CMB-S4 experiment is shown as grey shaded region. From the left panel of Fig. 4, we can clearly see that for decrease in \( \lambda_{H\phi} \), while keeping \( m_\phi \) constant, \( \Delta N_{\text{eff}} \) decreases. This is expected as a smaller value of Higgs portal coupling \( \lambda_{H\phi} \) leads to a larger freeze-out abundance of \( \phi \) followed by enhanced production of \( \nu_R \) from \( \phi \) decay. Since the same decay also produces DM, we need to choose lower values of DM masses in order to keep its relic abundance within Planck limits. This can be noticed from the right panel plot of Fig. 4 where the points with large \( \Delta N_{\text{eff}} \) correspond to smaller DM masses. Additionally, for fixed \( \lambda_{H\phi} \), if we increase \( m_\phi \), the corresponding \( \Delta N_{\text{eff}} \) increases. Once again, this is due to larger freeze-out abundance of \( \phi \) for heavier masses, as noticed while discussing the evolution plots in Fig. 3. Accordingly, for heavier \( m_\phi \) with fixed \( \lambda_{H\phi} \), we need to choose lighter DM masses in order to keep its relic abundance within observed limits, as seen from the right panel plot of Fig. 4. Thus, FIMP type DM candidate in our setup with masses all the way upto a few tens of keV can already get disfavoured by Planck 2018 limit (2\( \sigma \)) on \( \Delta N_{\text{eff}} \). Future CMB experiments like CMB-S4 will be able to improve it at least by an order of magnitude, having the potential to probe DM mass all the way upto a few hundred keV.

C. Case III

In this subsection, we discuss the results for the last subclass of scenarios mentioned earlier where the mother particle \( \phi \) never enters equilibrium due to feeble Higgs portal coupling. In order to simplify the analysis, we consider \( \phi \) production to be taking place dominantly from the SM Higgs, either via decay or via annihilation. For \( m_\phi < m_h/2 \), the decay process
FIG. 4: Parameter space plot for case II obtained from numerical scans, shown in terms of $\Delta N_{\text{eff}}$ vs $m_\phi$ while $\lambda H \phi$ (left panel) and $m_\psi$ (right panel) are shown in colour code. The other relevant parameter $y_\phi$ is kept fixed at $10^{-10}$. The magenta and grey shaded regions indicate the current and future bound on $\Delta N_{\text{eff}}$ from Planck 2018 ($2\sigma$) and CMB-S4 respectively.

$(h \to \phi\phi)$ dominates while in the other limit only annihilation $(hh \to \phi\phi)$ can contribute to $\phi$ production. To show the roles of decay and annihilation separately, we discuss these two limits separately.

1. $m_\phi < m_h/2$

In this case, $\phi$ freezes in from Higgs decay and then decays into $\psi$ and $\nu_R$. Similar to earlier cases, we first show the evolution of dark sector particles for suitable choices of model parameters such that both DM abundance as well as $\Delta N_{\text{eff}}$ remain within Planck $2\sigma$ limits. The corresponding evolution plots are shown in Fig. 5. We maintain similar colour codes as before namely, magenta, red, blue, green to show the evolution of comoving number densities of $\phi$ (equilibrium), $\phi$ (actual), DM $\psi$, $\nu_R$ respectively. In sharp contrast to case I, II discussed earlier, here we see that the initial abundance of $\phi$ remains negligible and then it slowly freezes in from decay of SM Higgs. In the top left panel of Fig. 5, we show the differences in these evolution for two different choices of $y_\phi$. As usual, a smaller value of $\phi$ delays the decay of $\phi$. While final DM density remains same for both the values of $y_\phi$, the
smaller value of $y_\phi$ leads to enhancement in $\nu_R$ density. Similar observation was noted in case II as well. In the top right panel of Fig. 5, we show the variation due to two different choices of Higgs portal coupling $\lambda_{H\phi}$. In sharp contrast to case II, here we get smaller abundance of $\phi$ for smaller value of $\lambda_{H\phi}$ which also highlights the generic difference between freeze-in and freeze-out production mechanisms [5]. Consequently, smaller $\lambda_{H\phi}$ leads to smaller yields in $\psi, \nu_R$ as clearly seen from the same plot in top right panel. Finally, in the bottom panel plot of Fig. 5, we show the variation due to two different choices of $m_\phi$. We see a marginal decrease in freeze-in abundance of $\phi$ for larger $m_\phi$ due to the fact that as $m_\phi$ approaches $m_h/2$, the corresponding partial decay width $\Gamma_{h \rightarrow \phi \phi^\dagger}$ decreases suppressing the production of $\phi$ slightly. On the other hand, a larger $m_\phi$ corresponds to larger decay width of $\phi$ in the limit $m_\psi \ll m_\phi$ leading to depletion in $\phi$ abundance earlier. The increase in $\phi$ decay width for larger $m_\phi$ also results in increased initial production of $\psi, \nu_R$. While final DM abundance decreases slightly for larger $m_\phi$ due to smaller freeze-in abundance of heavier $\phi$, the abundance of $\nu_R$ gets slightly enhanced for larger $m_\phi$ due to larger decay width. Thus, there exists a competition between two effects: (i) decrease in $\nu_R$ production due to decrease in freeze-in production of $\phi$ for larger $m_\phi$ and (ii) increase in $\nu_R$ production due to increase in $\phi$ decay width for larger $m_\phi$ and the final results will be decided by the dominance of either of these, to be discussed below. In all the plots shown in Fig. 5, we notice an intermediate plateau region for $\phi$ abundance. This arises when the freeze-in production rate of $\phi$ from Higgs decay and decay rate of $\phi$ into $\psi, \nu_R$ remain comparable.

We then perform a numerical scan to show the parameter space assuming $\phi$ to be out-of-equilibrium throughout which freezes in only from the SM Higgs decay. In the scan, we vary the relevant parameters in the following range:

$$5 \text{ GeV} \leq m_\phi \leq 60 \text{ GeV},$$
$$10^{-9} \leq \lambda_{H\phi} \leq 10^{-8},$$
$$1 \text{ keV} \leq m_\psi \leq 1 \text{ MeV}.$$ 

Here also $y_\phi$ is kept fixed at $10^{-10}$. The resulting parameter space is shown in $\Delta N_{\text{eff}}$ vs $m_\phi$ plane in Fig. 6 with the colour bars in left and right panel plots showing the variation in $\lambda_{H\phi}$ and $m_\psi$ respectively. Similar to case II, here also the scattered points satisfy the Planck bound on DM relic abundance while the corresponding upper bound (future sensitivity) on $\Delta N_{\text{eff}}$ is shown by magenta (grey) shaded region. With an increase in $\lambda_{H\phi}$ while keeping
FIG. 5: Evolution of dark sector particles ($\phi, \psi, \nu_R$) in case III considering $\phi$ to freeze in from Higgs decay and then decaying into ($\psi, \nu_R$). Top left, top right and bottom panel plots show the change in evolution for two different choices of $y_\phi$, $\lambda_{H\phi}$, $m_\phi$ respectively. Chosen benchmark points including DM mass $m_\psi$ keep DM abundance as well as $\Delta N_{\text{eff}}$ within Planck limits.

$m_\phi$ fixed, we get enhancement in $\Delta N_{\text{eff}}$ as seen from the left panel plot of Fig. 6, in sharp contrast with the corresponding results in case II. As discussed above, this trend is expected as increase in $\lambda_{H\phi}$ leads to increased freeze-in production of $\phi$. Since DM number density also increases from the same $\phi$ decay, we need to choose lighter DM masses for larger $\lambda_{H\phi}$ in order keep $\Omega_{\text{DM}} h^2$ within observed limits, as seen from the right panel plot of Fig. 6. On the other hand, if $\phi$ mass increases for fixed $\lambda_{H\phi}$, we first see an increase in $\Delta N_{\text{eff}}$ followed by decrease for $m_\phi$ closer to $m_h/2$. The initial rise in $\Delta N_{\text{eff}}$ can be explained by noting the increase in $\phi$ decay width for larger $m_\phi$. However, if we continue to increase $m_\phi$, taking it closer to $m_h/2$, the partial decay width of the SM Higgs $\Gamma_{h\to\phi\phi^*}$ decreases leading to suppression in freeze-in abundance of $\phi$. Consequently, this leads to decrease in $\nu_R, \psi$
FIG. 6: Parameter space plot for case III (considering $\phi$ to freeze in from Higgs decay) obtained from numerical scans, shown in terms of $\Delta N_{\text{eff}}$ vs $m_\phi$ while $\lambda_{H\phi}$ (left panel) and $m_\psi$ (right panel) are shown in colour code. The other relevant parameter $y_\phi$ is kept fixed at $10^{-10}$. The magenta and grey shaded regions indicate the current and future bound on $\Delta N_{\text{eff}}$ from Planck 2018 ($2\sigma$) and CMB-S4 respectively.

densities. Correct DM abundance can be obtained by choosing heavier DM masses in the high $m_\phi$ regime, as seen from the right panel plot of Fig. 6. Similar to case II discussed before, here also we can probe FIMP DM masses upto a few tens of keV from CMB-S4 observations in future.

2. $m_\phi > m_h/2$

We now briefly discuss the essential features of the non-thermal $\phi$ scenario where its freeze-in production is dominated by annihilations only and decay is forbidden kinematically due to $m_\phi > m_h/2$. The evolution of dark sector particles in this case are shown in Fig. 7. Once again, the choice of benchmark parameters is made in such a way that the final DM abundance and $\Delta N_{\text{eff}}$ remain within Planck 2018 limits. In top left panel of Fig. 7, we show the variation in evolution for two different choices of $y_\phi$. As expected, this only alters the decay width of $\phi$ and hence the production of $\nu_R, \psi$. While final DM density remains same for both the choices, late production of $\nu_R$ due to smaller $y_\phi$ leads to an enhancement in $Y_{\nu_R}$,
an observation which was also made in other scenarios discussed above. In top right panel of Fig. 7, we show the difference in evolution due to variation in Higgs portal coupling $\lambda_{H\phi}$. Naturally, a smaller $\lambda_{H\phi}$ results in smaller freeze-in abundance of $\phi$ from annihilation and hence smaller yields in $\nu_R, \psi$. Variation due to change in $m_\phi$ is shown in the bottom panel plot of Fig. 7. We do not see much difference between the two values except for the fact that a larger $m_\phi$ increase $\phi$ decay width leading to early depletion. Since the overall features in this case remains similar to the earlier case where $\phi$ is produced from decay only, we expect the parameter space to remain similar. Therefore, we do not perform any numerical scan in this case.

\[ m_\phi = 100 \text{ GeV} \]
\[ m_\psi = 72.5 \text{ keV} \]
\[ \lambda_{H\phi} = 3.5 \times 10^{-8} \]

FIG. 7: Evolution of dark sector particles ($\phi, \psi, \nu_R$) in case III considering $\phi$ to freeze in from Higgs annihilations and then decaying into ($\psi, \nu_R$). Top left, top right and bottom panel plots show the change in evolution for two different choices of $y_\phi, \lambda_{H\phi}, m_\phi$ respectively. Chosen benchmark points including DM mass $m_\psi$ keep DM abundance as well as $\Delta N_{\text{eff}}$ within Planck limits.
IV. CONCLUSION

We have studied a minimal scenario where the origin of neutrino mass and dark matter remain connected with interesting observational prospects at CMB experiments. Assuming light neutrinos to be of Dirac nature necessitates the inclusion of right handed neutrinos $\nu_R$ which can also act like a portal to dark sector comprising of a fermion singlet DM and a scalar singlet $\phi$. While the scalar singlet can be directly coupled to the SM bath via Higgs portal coupling, fermion singlet DM can couple only to $\nu_R$ via $\phi$. We study in details, the freeze-in production of $\psi$ and $\nu_R$ from $\phi$ decay, by considering three different possibilities with (i) $\phi$ in equilibrium, (ii) $\phi$ undergoing thermal freeze-out and (iii) $\phi$ getting produced via freeze-in. Since $\nu_R$ couples to SM leptons very feebly due to the requirement of generating sub-eV scale Dirac neutrino mass, the corresponding freeze-in production of $\nu_R$ directly from the SM bath remains suppressed. Since the same coupling with $\phi$ leads to freeze-in production of both DM and $\nu_R$ with the latter remaining relativistic throughout, we show the possibility of correlating DM parameter space with effective relativistic degrees of freedom $\Delta N_{\text{eff}}$. We find that the scenario with $\phi$ in equilibrium throughout leads to tiny enhancement in $\Delta N_{\text{eff}}$ out of reach from future experiments. However, for the other two scenarios, due to one additional free parameter in the form of Higgs portal coupling $\lambda_{H\phi}$ at play, we can have correct DM phenomenology while getting a sizeable enhancement in $\Delta N_{\text{eff}}$ at the same time. While freeze-in or FIMP type DM have limited observational scopes, we show that DM mass upto a few hundred keV can be probed via future CMB measurements by experiments like CMB-S4 as well as other planned experiments like SPT-3G [73], Simons Observatory [74]. Additionally, depending upon the choice of parameters, existing bounds from the Planck experiment can also rule out DM mass upto a few tens of keV. Since the scalar singlet can be light in these scenarios opening up the possibility of SM Higgs decaying invisibly into a pair of $\phi$, future LHC measurements will be able to constrain the Higgs portal coupling further from measurements of Higgs invisible decay rates. In addition to these specific signatures of our model keeping it very predictive, one can also pursue such neutrino portal dark matter scenarios from the point of view of easing cosmological tensions between early and late universe cosmological observations [75]. There have been a few works already in this direction [76, 77] which we plan to explore in future works.
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Appendix A: Derivation of Boltzmann equations

The Boltzmann equation in differential form can be written as

\[
\frac{\partial f}{\partial t} - \mathcal{H} p \frac{\partial f}{\partial p} = C[f]
\]

(A1)

where \(\mathcal{H}\) is the Hubble expansion rate and \(C[f]\) is the collision term for a species with distribution function \(f\). In this section, we discuss the derivation of the Boltzmann equations for the relevant species \((\phi, \psi, \nu_R)\) in all the cases considered in this work.

1. Case I: \(\phi\) in equilibrium

a. For \(\psi\) abundance:

For the process: \(\phi(K) \rightarrow \psi(P_1) + \nu_R(P_2)\)

Integrating both sides of Eq. (A1) over the three momentum \(p_1\) of species \(\psi\), we get

\[
\int g_\psi \frac{d^3 p_1}{(2\pi)^3} \left[ \frac{\partial f_\psi}{\partial t} - \mathcal{H} p_1 \frac{\partial f_\psi}{\partial p_1} \right] = \int g_\psi \frac{d^3 p_1}{(2\pi)^3} C[f_\psi].
\]

(A2)

Using the definition of \(n_\psi\) and integration by parts method for the term proportional to \(\mathcal{H}\), the LHS of Eq. (A2) becomes

\[
\frac{dn_\psi}{dt} + 3\mathcal{H} n_\psi,
\]

(A3)

where,

\[
n_\psi = \int g_\psi \frac{d^3 p_1}{(2\pi)^3} f_\psi,
\]

(A4)
with \( g_\psi \) being the internal degree of freedom of \( \psi \). The RHS of Eq. (A2) is

\[
\int g_\psi \frac{d^3p_1}{(2\pi)^3} C[f_\psi] = \int g_\psi \frac{d^3p_1}{(2\pi)^3} 2E_1 \int g_\nu R \frac{d^3p_2}{(2\pi)^3} 2E_2 g_\phi \frac{d^3k}{(2\pi)^3} 2E_k \times (2\pi)^4 \delta^4(K - P_1 - P_2)|M|_{\phi \to \nu R \psi}^2 f_\phi^{eq}. \tag{A5}
\]

Now using the definition of decay width of \( \phi \) in the rest frame of \( \phi \) i.e.

\[
\Gamma_\phi = \frac{1}{2m_\phi} \int g_\psi g_\nu R \frac{d^3p_1}{(2\pi)^3} 2E_1 \int g_\nu R \frac{d^3p_2}{(2\pi)^3} 2E_2 (2\pi)^4 \delta^4(K - P_1 - P_2)|M|_{\phi \to \nu R \psi}^2 \Gamma_\phi f_\phi^{eq}. \tag{A6}
\]

we get

\[
\text{RHS} = g_\phi \int \frac{d^3k}{(2\pi)^3} 2m_\phi \Gamma_\phi f_\phi^{eq}. \tag{A7}
\]

Here, the decay width \( \Gamma_\phi \) is given by

\[
\Gamma_\phi = \frac{g_\phi g_\nu R}{16\pi m_\phi} |M|^2_{\phi \to \nu R \psi} \left( 1 - \frac{m_\phi^2}{m_\psi^2} \right)
\]

and

\[
|M|^2_{\phi \to \nu R \psi} = \frac{1}{g_\phi g_\nu R g_\psi} g_\psi^2 (m_\phi^2 - m_\psi^2). \tag{A8}
\]

Using \( f_\phi^{eq} = e^{-E_\phi/T} \), the Maxwell-Boltzmann distribution, we get,

\[
\text{RHS} = g_\phi \Gamma_\phi \int \frac{d^3k}{(2\pi)^3} 2m_\phi e^{-E_\phi/T} = g_\phi \Gamma_\phi \frac{T}{2\pi^2} m_\phi^2 K_1(m_\phi/T). \tag{A9}
\]

Putting \( n_\phi^{eq} = \frac{g_\phi}{2\pi^2} m_\phi^2 TK_2(m_\phi/T) \), the RHS becomes

\[
\text{RHS} = \Gamma_\phi \frac{K_1(m_\phi/T)}{K_2(m_\phi/T)} n_\phi^{eq}. \tag{A10}
\]

Finally, after equating LHS and RHS of Eq. (A2), the Boltzmann equation for \( n_\psi \) becomes

\[
\frac{dn_\psi}{dt} + 3\mathcal{H}n_\psi = \Gamma_\phi \frac{K_1(m_\phi/T)}{K_2(m_\phi/T)} n_\phi^{eq}. \tag{A11}
\]

Now, instead of \( n_\psi \), we can write the equation in terms of a new variable \( Y_\psi = n_\psi/s \), known as comoving number density. Using the fact that \( sa^3 = \text{constant} \) with \( s, a \) being the entropy density, cosmic scale factor of the FLRW metric respectively, the LHS of Eq. (A11) becomes

\[
s \frac{dY_\psi}{dt} = \frac{dn_\psi}{dt} + 3\mathcal{H}n_\psi
\]

\[
\implies \frac{dY_\psi}{dT} = -\frac{1}{3\mathcal{H}s} \left[ \frac{3}{T} + \frac{dg_s/dT}{g_s} \right] \left( \frac{dn_\psi}{dt} + 3\mathcal{H}n_\psi \right)
\]

\[
= -\frac{1}{3\mathcal{H}} \left[ \frac{3}{T} + \frac{dg_s/dT}{g_s} \right] g_\psi g_\nu R g_\phi \Gamma_\phi \frac{K_1(m_\phi/T)}{K_2(m_\phi/T)} Y_\phi^{eq}
\]

\[
= -\frac{1}{\mathcal{H}T} \left[ 1 + \frac{Tdg_s/dT}{3g_s} \right] \Gamma_\phi \frac{K_1(m_\phi/T)}{K_2(m_\phi/T)} Y_\phi^{eq}. \tag{A12}
\]
Now defining $x = m_\phi/T$, we can write the above equation in terms of dimensionless variables $x$

$$\frac{dY_\phi}{dx} = \frac{\beta}{x\mathcal{H}} \frac{K_1(x)}{K_2(x)} Y_{eq}^\phi,$$

where,

$$\beta = \left[ 1 + \frac{T d g_s}{dT} \right].$$

b. For $\nu_R$ energy density:

Let us start with the differential Boltzmann equation for $\nu_R$

$$\frac{\partial f_{\nu_R}}{\partial t} - \mathcal{H} p_2 \frac{\partial f_{\nu_R}}{\partial p_2} = C[f_{\nu_R}].$$

(A15)

Integrating both sides with $\int g_{\nu_R} E_2 \frac{d^3p_2}{(2\pi)^3}$, we get

$$\int g_{\nu_R} E_2 \frac{d^3p_2}{(2\pi)^3} \left( \frac{\partial f_{\nu_R}}{\partial t} - \mathcal{H} p_2 \frac{\partial f_{\nu_R}}{\partial p_2} \right) = \int g_{\nu_R} E_2 \frac{d^3p_2}{(2\pi)^3} C[f_{\nu_R}].$$

(A16)

The LHS, after simplification becomes -

$$\int g_{\nu_R} E_2 \frac{d^3p_2}{(2\pi)^3} \left( \frac{\partial f_{\nu_R}}{\partial t} - \mathcal{H} p_2 \frac{\partial f_{\nu_R}}{\partial p_2} \right) = \frac{d\rho_{\nu_R}}{dt} + 4\mathcal{H}\rho_{\nu_R},$$

(A17)

where,

$$\rho_{\nu_R} = \int g_{\nu_R} \frac{d^3p_2}{(2\pi)^3} E_2 f_{\nu_R}. $$

(A18)

Expanding the collision term, the RHS becomes

$$\int g_{\nu_R} E_2 \frac{d^3p_2}{(2\pi)^3} C[f_{\nu_R}] = g_{\nu_R} \int \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} \int g_\psi \frac{d^3p_1}{(2\pi)^3} E_1 g_\phi \frac{d^3k}{(2\pi)^3} E_k$$

$$\times E_2 (2\pi)^4 \delta^4 (K - P_1 - P_2) |\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi} f_\phi^eq. $$

(A19)

Let us do the following integral first.

$$I = \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} E_2 (2\pi)^4 \delta^4 (K - P_1 - P_2) |\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi}$$

$$= \frac{1}{4(2\pi)^2} \int \frac{d^3p_1}{E_1} d^3p_2 \delta^4 (K - P_1 - P_2) |\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi}. $$

(A20)

We first do the integration over $p_2$ using the Dirac delta function,

$$I = \frac{1}{4(2\pi)^2} \int \frac{d^3p_1}{E_1} \delta (E_k - E_1 - E_{k-1}) |\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi}$$

$$= \frac{2\pi}{4(2\pi)^2} \int \frac{p_2^2 d p_1 d (\cos \theta)}{E_1} \delta (f(\theta)) |\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi}. $$

(A21)
Here, $\theta$ is the angle between $\vec{k}$ and $\vec{p}_1$ and $f(\theta) = E_k - E_1 - E_{k-1}$ with $E_{k-1} = \sqrt{(\vec{k} - \vec{p}_1)^2 + m^2}$. Now to find the root of $f(\theta)$, we set -

$$f(\theta) = 0$$

$$\implies E_k - E_1 - E_{k-1} = 0$$

$$\implies \cos \theta = \frac{2E_kE_1 - (m^2 + m^2 - m^2)}{2|\vec{k}||\vec{p}_1|} \equiv \cos \theta_0. \quad (A22)$$

Also,

$$\left. \frac{df}{d\cos \theta} \right|_{\cos \theta = \cos \theta_0} = \frac{|\vec{k}||\vec{p}_1|}{E_k - E_1}. \quad (A23)$$

Thus, the integral $I$ reduces to

$$I = \frac{1}{4(2\pi)} \int \frac{p_1^2 dp_1}{E_1} \int d(\cos \theta) \frac{\delta(\cos \theta - \cos \theta_0)}{E_1} |\mathcal{M}|^2_{\phi \rightarrow \varphi \nu \psi}$$

$$= \frac{|\mathcal{M}|^2_{\phi \rightarrow \varphi \nu \psi}}{8\pi} \int \frac{p_1^2 dp_1}{E_1} \frac{E_k - E_1}{|\vec{k}||\vec{p}_1|}$$

$$= \frac{|\mathcal{M}|^2_{\phi \rightarrow \varphi \nu \psi}}{8\pi|\vec{k}|} \int_{E_1^{\min}}^{E_1^{\max}} dE_1 (E_k - E_1). \quad (A24)$$

In the above, $|\mathcal{M}|'$ implies $|\mathcal{M}|$ at $\theta = \theta_0$. The limits of the integration will come from the condition

$$-1 \leq \cos \theta_0 \leq 1. \quad (A25)$$

Working through it, we get

$$E_1^{\min} = \frac{E_k(m^2 + m^2 - m^2)}{2m^2} - \sqrt{E_k^2(m^2 + m^2 - m^2)^2 - m^2(\Lambda + 4E_k^2 m^2)} \equiv g_1(E_k)$$

$$E_1^{\max} = \frac{E_k(m^2 + m^2 - m^2)}{2m^2} + \sqrt{E_k^2(m^2 + m^2 - m^2)^2 - m^2(\Lambda + 4E_k^2 m^2)} \equiv g_2(E_k), \quad (A26)$$

where,

$$\Lambda = (m^2 + m^2 - m^2)^2 - 4m^2m^2. \quad (A27)$$
Hence, $I$ becomes

$$I = \frac{g_2(E_k) - g_1(E_k)}{8\pi|\vec{k}|} |\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi} \left( E_k - \frac{g_2(E_k) + g_1(E_k)}{2} \right)$$

$$= \sqrt{\frac{E_k^2(m_\phi^2 + m_\psi^2 - m_\nu^2)^2 - m_\phi^2(\Lambda + 4E_k^2m_\psi^2)}{8\pi|\vec{k}|m_\phi^2}} |\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi} \left( E_k - \frac{E_k(m_\phi^2 + m_\psi^2 - m_\nu^2)}{2m_\phi^2} \right)$$

$$= |\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi} \left( \frac{E_k^2(m_\phi^2 + m_\psi^2 - m_\nu^2)^2 - m_\phi^2(\Lambda + 4E_k^2m_\psi^2)}{8\pi|\vec{k}|m_\phi^2} \right) E_k \left( \frac{m_\phi^2 + m_\psi^2 + m_\nu^2}{2m_\phi^2} \right). \quad (A28)$$

Finally, the RHS becomes -

$$\text{RHS} = \frac{g_\phi g_\psi g_{\nu_R}}{32\pi^3} \frac{|\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi} (m_\phi^2 - m_\psi^2 + m_\nu^2)}{2m_\phi^4}$$

$$\times \int_{m_\phi}^\infty E_k f_{\phi}^{eq} \sqrt{E_k^2(m_\phi^2 + m_\psi^2 - m_\nu^2)^2 - m_\phi^2(\Lambda + 4E_k^2m_\psi^2)} dE_k$$

$$= \frac{g_\phi g_\psi g_{\nu_R}}{32\pi^3} \frac{|\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi} (m_\phi^2 - m_\psi^2 + m_\nu^2)^2}{2m_\phi^4} \int_{m_\phi}^\infty E_k f_{\phi}^{eq} \sqrt{E_k^2 - m_\phi^2} dE_k \quad (\therefore m_\nu \simeq 0) \quad (A29)$$

$$= \frac{g_\phi g_\psi g_{\nu_R}}{32\pi^3} \frac{|\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi} (m_\phi^2 - m_\psi^2 + m_\nu^2)^2}{2m_\phi^4} \int_{m_\phi}^\infty E_k e^{-E_k/T} \sqrt{E_k^2 - m_\phi^2} dE_k$$

$$= \frac{g_\phi g_\psi g_{\nu_R}}{32\pi^3} \frac{|\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi} (m_\phi^2 - m_\psi^2 + m_\nu^2)^2}{2m_\phi^4} m_\phi^2 TK_2(m_\phi/T)$$

$$= \langle ET \rangle n_{\phi}^{\text{eq}}, \quad (A30)$$

where

$$\langle ET \rangle = g_\phi g_\psi g_{\nu_R} \frac{|\mathcal{M}|^2_{\phi \rightarrow \bar{\nu}_R \psi} (m_\phi^2 - m_\psi^2)^2}{32\pi} \frac{m_\phi^4}{m_\psi^4}. \quad (A31)$$

So, the final form of the evolution equation of $\rho_{\nu_R}$ is

$$\frac{d\rho_{\nu_R}}{dt} + 4\mathcal{H}\rho_{\nu_R} = \langle ET \rangle n_{\phi}^{\text{eq}}$$

$$\Rightarrow \frac{dY_{\nu_R}}{dT} = -\frac{\beta}{\mathcal{H}T s^{4/3}} \langle ET \rangle n_{\phi}^{\text{eq}} \quad (\text{where } Y_{\nu_R} = \frac{\rho_{\nu_R}}{s^{4/3}}). \quad (A32)$$

In terms of $x = m_\phi/T$, the above equation becomes

$$\frac{dY_{\nu_R}}{dx} = \frac{\beta}{\mathcal{H} s^{1/3} x} \langle ET \rangle Y_{\phi}^{\text{eq}}. \quad (A33)$$

2. Case II

In this case, $\phi$ is not in equilibrium always. It is produced in equilibrium and at some epoch it goes out of equilibrium due to thermal freeze-out.
a. For $\psi$ abundance:

The procedure to obtain the Boltzmann equation for $\psi$ in this case is same as the above case from Eq. (A2) to Eq. (A7) except that $f^{eq}_\phi$ is now replaced by $f_\phi$. Thus, the Boltzmann equation for $\psi$ is

$$\frac{dn_\psi}{dt} + 3Hn_\psi = g_\phi \int \frac{d^3k}{(2\pi)^3} \frac{2m_\phi}{2E_k} \Gamma_\phi f_\phi .$$

(A34)

Since $\phi$ was in equilibrium earlier and goes out of equilibrium after freeze-out, we can write the general form of the Maxwell-Boltzmann distribution function for $\phi$ with a chemical potential that is nonzero only after the freeze-out of $\phi$ i.e. $f_\phi = e^{\mu/T} e^{-E_k/T}$. The chemical potential $\mu$ is defined as $\mu = T \ln \left( \frac{n_\phi(T)}{n^{eq}_\phi(T)} \right)$. Substituting $f_\phi$ in Eq. (A34), the Boltzmann equation becomes

$$\frac{dn_\psi}{dt} + 3Hn_\psi = g_\phi e^{\mu/T} \int \frac{d^3k}{(2\pi)^3} \frac{2m_\phi}{2E_k} \Gamma_\phi e^{-E_k/T} .$$

(A35)

The RHS of the equation is same as Eq. (A9) in case I except for the $e^{\mu/T}$ factor. Hence, following the same procedure as Eq. (A9) to Eq. (A11) and replacing $\mu$ by number density, we get

$$\frac{dn_\psi}{dt} + 3Hn_\psi = e^{\mu/T} \Gamma_\phi \frac{K_1(m_\phi/T)}{K_2(m_\phi/T)} n^{eq}_\phi ,$$

$$= \Gamma_\phi \frac{K_1(m_\phi/T)}{K_2(m_\phi/T)} n_\phi .$$

(A36)

We can write the above equation in terms of $Y_\psi = n_\psi/s$, $Y_\phi = n_\phi/s$ and $x = m_\phi/T$. In terms of these dimensionless quantities the above equation takes the following form

$$\frac{dY_\psi}{dx} = \frac{\beta}{xH} \Gamma_\phi \frac{K_1(x)}{K_2(x)} Y_\phi .$$

(A37)

b. For $\nu_R$ energy density:

To find the energy density of $\nu_R$ in this case, we will follow the same procedure as in the previous case, the only difference will be that now $f^{eq}_\phi$ will be replaced by $f_\phi = e^{\mu/T} e^{E_k/T}$. Hence starting from Eq. (A29), the R.H.S. of the Boltzmann equation for $\rho_{\nu_R}$ can be written
as
\[
\frac{d\rho_{\nu R}}{dt} + 4H \rho_{\nu R} = g_{\phi} g_{\psi} g_{\nu_R} \frac{|M|_{\phi \rightarrow \nu_R \psi}^2 (m_{\phi}^2 - m_{\psi}^2)}{2m_{\phi}} \int_{m_{\phi}}^{\infty} E_k f_\phi \sqrt{E_k^2 - m_{\phi}^2} dE_k,
\]
\[
= g_{\phi} g_{\psi} g_{\nu_R} \frac{|M|_{\phi \rightarrow \nu_R \psi}^2 (m_{\phi}^2 - m_{\psi}^2)}{2m_{\phi}} \int_{m_{\phi}}^{\infty} E_k e^{\mu/T} T^{E_k/T} \sqrt{E_k^2 - m_{\phi}^2} dE_k,
\]
\[
= g_{\phi} g_{\psi} g_{\nu_R} \frac{|M|_{\phi \rightarrow \nu_R \psi}^2 (m_{\phi}^2 - m_{\psi}^2)}{2m_{\phi}} e^{\mu/T} m_{\phi}^2 T K_2(m_{\phi}/T),
\]
\[
= \langle E \Gamma \rangle e^{\mu/T} n_{eq}^\phi,
\]
\[
\implies \frac{d\rho_{\nu R}}{dt} + 4H \rho_{\nu R} = \langle E \Gamma \rangle n_{\phi}^\phi. \tag{A38}
\]

In terms of $T$ and $x = m_{\phi}/T$, the above equation becomes
\[
\frac{dY_{\nu R}}{dT} = - \frac{\beta}{\mathcal{H} T^{4/3}} \langle E \Gamma \rangle Y_{\phi},
\]
\[
\frac{dY_{\nu R}}{dx} = \frac{\beta}{\mathcal{H} T^{4/3}} \langle E \Gamma \rangle Y_{\phi}. \tag{A39}
\]

c. For comoving number density of non-thermal $\phi$:

The calculation of the number density of $\phi$ will involve two processes: $X(K_1) + \bar{X}(K_2) \rightarrow \phi(K_1) + \phi^\dagger(K_2)$ and $\phi(K_1) \rightarrow \psi(P_1) + \nu_R(P_2)$. Hence, the differential form of the Boltzmann equation is ($X$ is any SM particle) -
\[
\frac{\partial f_\phi}{\partial t} - \mathcal{H} k_1 \frac{\partial f_\phi}{\partial k_1} = C^{X \bar{X} \rightarrow \phi \phi^\dagger} [f_{\phi}] - C^{\phi \rightarrow \psi \nu_R} [f_{\phi}]
\]
\[
\int g_\phi \frac{d^3 k_1}{(2\pi)^3} \left( \frac{\partial f_\phi}{\partial t} - \mathcal{H} k_1 \frac{\partial f_\phi}{\partial k_1} \right) = \int g_\phi \frac{d^3 k_1}{(2\pi)^3} \left( C^{X \bar{X} \rightarrow \phi \phi^\dagger} [f_{\phi}] - C^{\phi \rightarrow \psi \nu_R} [f_{\phi}] \right). \tag{A40}
\]

The LHS is
\[
\int g_\phi \frac{d^3 k_1}{(2\pi)^3} \left( \frac{\partial f_\phi}{\partial t} - \mathcal{H} k_1 \frac{\partial f_\phi}{\partial k_1} \right) = \frac{dn_{\phi}}{dt} + 3\mathcal{H} n_{\phi}. \tag{A41}
\]
The first term of RHS is
\[ \int \frac{d^3k_1}{(2\pi)^3} C^{X \phi \rightarrow \phi \phi} [f_\phi], \]
\[ = \int \frac{g_\phi}{(2\pi)^3} \frac{d^3k_1}{2E_{k_1}} \int g_\phi \frac{d^3k_1'}{(2\pi)^3} \frac{1}{2E_{k_1'}} \int g_\phi \frac{d^3k_2'}{(2\pi)^3} \frac{1}{2E_{k_2'}} \int g_\phi \frac{d^3k_2}{(2\pi)^3} \frac{1}{2E_{k_2}} (2\pi)^4 \delta^4(K_1' + K_2' - K_1 - K_2) \]
\[ \times |M|^2_{X \phi \rightarrow \phi \phi} (f_\phi f_{k_1} f_{k_2} - f_{k_1} f_{k_2}) \]
\[ = (n^q)^2 \langle \sigma v \rangle \left( \left( \frac{n_X}{n^q} \right)^2 - \left( \frac{n_\phi}{n^q} \right)^2 \right), \quad \therefore f_i = e^{\mu_i/T} e^{-E_i/T} = \frac{n_i}{n^q}, \] (A42)

where,
\[ \langle \sigma v \rangle = \frac{1}{(n^q)^2} \int g_\phi \frac{d^3k_1}{(2\pi)^3} \frac{1}{2E_{k_1}} \int g_\phi \frac{d^3k_1'}{(2\pi)^3} \frac{1}{2E_{k_1'}} \int g_\phi \frac{d^3k_2'}{(2\pi)^3} \frac{1}{2E_{k_2'}} \int g_\phi \frac{d^3k_2}{(2\pi)^3} \frac{1}{2E_{k_2}} (2\pi)^4 \delta^4(K_1' + K_2' - K_1 - K_2) |M|^2_{X \phi \rightarrow \phi \phi} e^{-(E_{k_1} + E_{k_2})/T}, \]
\[ = \frac{1}{(n^q)^2} \int g_\phi \frac{d^3k_1}{(2\pi)^3} \int g_\phi \frac{d^3k_2}{(2\pi)^3} \frac{1}{2E_{k_1}} \int g_\phi \frac{d^3k_1'}{(2\pi)^3} \frac{1}{2E_{k_1'}} \int g_\phi \frac{d^3k_2'}{(2\pi)^3} \frac{1}{2E_{k_2'}} (2\pi)^4 \delta^4(K_1' + K_2' - K_1 - K_2) |M|^2_{X \phi \rightarrow \phi \phi} e^{-(E_{k_1} + E_{k_2})/T}, \]
\[ = g_\phi^2 \left( \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} \langle \sigma v \rangle_{\phi \rightarrow X \phi \phi} e^{-(E_{k_1} + E_{k_2})/T} \right), \]
\[ = \frac{1}{8m_\phi^2 T K_2^2 (m_\phi/T)} \int_{4m_\phi^2}^\infty (s - 4m_\phi^2) \sqrt{s} K_1(\sqrt{s}/T) ds. \] (A43)

We have obtained the last expression following the prescription given in [65]. Now the second term in the RHS is
\[ \int g_\phi \frac{d^3k_1}{(2\pi)^3} C^{\phi \rightarrow \nu_R \nu_R} [f_\phi] = \int g_\phi \frac{d^3k_1}{(2\pi)^3} \frac{1}{2E_{k_1}} \int g_\phi \frac{d^3p_1}{(2\pi)^3} \frac{1}{2E_1} \int g_\phi \frac{d^3p_2}{(2\pi)^3} \frac{1}{2E_2} (2\pi)^4 \delta^4(K_1 - P_1 - P_2)f_\phi. \] (A44)

Here due to non-thermal nature of $\psi$ and $\nu_R$, we have omitted the back reaction term which otherwise will be there in Eq. (A44) and is proportional to $f_\phi f_{\nu_R}$. This is the same decay process that we have worked through the section A2a when $\phi$ is not in equilibrium. Therefore, from Eq. (A36) we obtain
\[ \int g_\phi \frac{d^3k_1}{(2\pi)^3} C^{\phi \rightarrow \nu_R \nu_R} [f_\phi] = \Gamma_\phi \frac{K_1(m_\phi/T)}{K_2(m_\phi/T)} n_\phi. \] (A45)
Finally, the full equation for the evolution of \( n_\phi \) is
\[
\frac{dn_\phi}{dt} + 3 H n_\phi = -\langle \sigma v \rangle ((n_\phi)^2 - (n_\phi^{eq})^2) - \frac{\Gamma_1 K_1(m_\phi/T)}{K_2(m_\phi/T)} n_\phi. \tag{A46}
\]

In terms of comoving number density \( Y_\phi \),
\[
\frac{dY_\phi}{dT} = -\frac{\beta_s}{H T} \left( -\langle \sigma v \rangle ((Y_\phi)^2 - (Y_\phi^{eq})^2) - \frac{\Gamma_1 K_1(m_\phi/T)}{s K_2(m_\phi/T)} Y_\phi \right),
\]
\[
\Longrightarrow \frac{dY_\phi}{dx} = \frac{\beta_s}{H x} \left( -\langle \sigma v \rangle ((Y_\phi)^2 - (Y_\phi^{eq})^2) - \frac{\Gamma_1 K_1(m_\phi/T)}{s K_2(m_\phi/T)} Y_\phi \right). \tag{A47}
\]

3. Case III

a. Distribution function of \( \phi \):

The Case-III, where \( \phi \) never attains thermal equilibrium with the SM bath, has the same forms of Boltzmann equations for \( n_\phi \) and \( \rho_{\nu_R} \) as those are in Case-I except here we need to replace the thermal distribution function of \( \phi \) by the non-thermal distribution function. The differential form of the Boltzmann equation to find the distribution function of \( \phi \), \( f_\phi \) is given by \([78, 79]\)
\[
\frac{\partial f_\phi}{\partial t} - H p_1 \frac{\partial f_\phi}{\partial p_1} = C^{h\rightarrow\phi\phi^\dagger} + C^{hh\rightarrow\phi\phi^\dagger} + C^{\phi\rightarrow\nu_R\psi}. \tag{A48}
\]

Here \( C^{h\rightarrow\phi\phi^\dagger} \) is the collision term for production of \( \phi\phi^\dagger \) pair from the decay of the SM Higgs boson \( h(K) \rightarrow \phi(P_1) + \phi^\dagger(P_2) \). The expression of \( C^{h\rightarrow\phi\phi^\dagger} \) is given by
\[
C^{h\rightarrow\phi\phi^\dagger} = \frac{1}{2 E_{p_1}} \int \frac{d^3 p_2}{2 E_{p_2}(2\pi)^3} \frac{d^3 k}{2 E_{k}(2\pi)^3} (2\pi)^4 \delta^4(K - P_1 - P_2)
\times |M|_{h\rightarrow\phi\phi^\dagger}^2 \left( f_{\phi}(k) - f_{\phi}(p_1) f_{\phi^\dagger}(p_2) \right),
\]
\[
= \frac{1}{2 E_{p_1}(2\pi)^3} \int \frac{d^3 p_2}{4 E_{p_2} E_{p_1 + p_2}} \delta(E_{p_1+p_2} - E_{p_1} - E_{p_2})
\times |M|_{h\rightarrow\phi\phi^\dagger}^2 \left( f_{\phi}(k) - f_{\phi}(p_1) f_{\phi^\dagger}(p_2) \right). \tag{A49}
\]

Now we can write \( d^3 p_2 = p_2^2 dp_2 d(\cos \theta) d\phi \), where \( \theta \) is the angle between \( \vec{p}_1 \) and \( \vec{p}_2 \). Therefore, the Dirac delta function \( \delta(E_{p_1+p_2} - E_{p_1} - E_{p_2}) \) actually fixes the angle \( \theta \). So, from the condition \( E_{p_1+p_2} = E_{p_1} + E_{p_2} \), we will get
\[
\cos \theta = \frac{2 m_\phi^2 - m_h^2 + 2 E_{p_1} E_{p_2}}{2 p_1 p_2} \equiv \cos \theta_0. \tag{A50}
\]
Therefore,
\[
C^{h \rightarrow \phi^+} = \frac{1}{2E_{p_1}(2\pi)^2} \int \frac{p_2 dp_2 (2\pi)}{4E_{p_2}} \int_{-1}^{1} \frac{d(cos \theta)\delta(cos \theta - \cos \theta_0)}{E_{p_1+p_2}} \left| \frac{df}{d \cos \theta} \right|_{\theta=\theta_0}^{\theta=\theta_0} \times |M|_{h \rightarrow \phi^+}^2 \left( f_{h}(E_{p_1+p_2}) - f_{\phi}(p_1)f_{\phi^+}(p_2) \right),
\]
where \( f(cos \theta) = E_{p_1+p_2} - E_{p_1} - E_{p_2} \) with \( E_{p_1+p_2} = \sqrt{p_1^2 + p_2^2 + m_h^2} \) and
\[
\frac{df}{d \cos \theta} \Big|_{\theta=\theta_0} = \frac{p_1 p_2}{E_{p_1} + E_{p_2}},
\]
\[
E_{p_1+p_2} \Big|_{\theta=\theta_0} = E_{p_1} + E_{p_2}.
\]
After some simplification, the collision term takes the following form
\[
C^{h \rightarrow \phi^+} = \frac{1}{16\pi E_{p_1} p_1} \int_{p_{2}^{\min}}^{p_{2}^{\max}} \frac{p_2 dp_2}{E_{p_2}} |M|_{h \rightarrow \phi^+}^2 \left( f_{\phi}^e(E_{p_1})f_{\phi^+}^e(E_{p_2}) - f_{\phi}(p_1)f_{\phi^+}(p_2) \right).
\]
The limits of the integration is obtained from the condition \(-1 \leq \cos \theta_0 \leq 1\). This condition translates to
\[
p_{2}^{\min} = \frac{p_1(m_h^2 - 2m_{\phi}^2) - m_h \sqrt{(m_h^2 - 4m_{\phi}^2)(p_1^2 + m_{\phi}^2)}}{2m_{\phi}^2},
\]
\[
p_{2}^{\max} = \frac{p_1(m_h^2 - 2m_{\phi}^2) + m_h \sqrt{(m_h^2 - 4m_{\phi}^2)(p_1^2 + m_{\phi}^2)}}{2m_{\phi}^2}.
\]
Here we have neglected the inverse decay term in Eq. (A54) as it is substantially smaller compared to the decay term as long as \( \phi \) is non-thermal. Therefore, the collision term \( C^{h \rightarrow \phi^+} \) becomes
\[
C^{h \rightarrow \phi^+} = \frac{1}{16\pi E_{p_1} p_1} \int_{p_{2}^{\min}}^{p_{2}^{\max}} \frac{p_2 dp_2}{E_{p_2}} |M|_{h \rightarrow \phi^+}^2 e^{-E_{p_1}/T} e^{-E_{p_2}/T},
\]
\[
= \frac{|M|_{h \rightarrow \phi^+}^2 T e^{-E_{p_1}/T} e^{-E_{p_2}/T}}{16\pi E_{p_1} p_1} \left( e^{-E_{p_1} \min/T} - e^{-E_{p_2} \max/T} \right),
\]
and \( E_{p_2}^{\max(\min)} = \sqrt{(p_{2}^{\max(\min)})^2 + m_{\phi}^2} \).

Now, we will briefly discuss the derivation of the collision term \( C^{hh \rightarrow \phi^+} \) for the production of \( \phi^+ \) pair due the scattering of the Higgs boson \( h(K_1) + h(K_2) \rightarrow \phi(P_1) + \phi^+(P_2) \).
\[
C^{hh \rightarrow \phi^+} = \frac{1}{2E_{p_1}} \int \frac{d^3k_1}{2E_{k_1}(2\pi)^3} \frac{d^3k_2}{2E_{k_2}(2\pi)^3} \frac{d^3p_2}{2E_{p_2}(2\pi)^3} (2\pi)^4 \delta^4(K_1 + K_2 - P_1 - P_2) |M|_{h \rightarrow \phi^+}^2 \left( f_{h}(k_1) f_{h}(k_2) - f_{\phi}(p_1)f_{\phi^+}(p_2) \right),
\]
\[
= \frac{1}{2E_{p_1}} \int \frac{d^3p_2}{2E_{p_2}(2\pi)^3} \left[ \int \frac{d^3k_1}{2E_{k_1}(2\pi)^3} \frac{d^3k_2}{2E_{k_2}(2\pi)^3} (2\pi)^4 \delta^4(K_1 + K_2 - P_1 - P_2) \right] |M|_{h \rightarrow \phi^+}^2 \left( f_{h}(k_1) f_{h}(k_2) - f_{\phi}(p_1)f_{\phi^+}(p_2) \right).
\]
The term inside the square bracket is Lorentz invariant, and we can do that integration easily in the centre of momentum frame. Here, for the calculational simplification, we assume that the matrix amplitude square $|\mathcal{M}|_{hh \to \phi \phi}^2$ depends only on the Mandelstam variable $s$ which is true for $s$-channel scatterings and contact interactions. For a general matrix amplitude square depending on all three Mandelstam variables one can use the prescription given in [80].

$$I = \int \frac{d^3k_1}{2E_{k_1}(2\pi)^3} \frac{d^3k_2}{2E_{k_2}(2\pi)^3}(2\pi)^4\delta^4(K_1 + K_2 - P_1 - P_2).$$ \hspace{1cm} (A58)

This will give -

$$I = \frac{1}{8\pi} \sqrt{1 - \frac{4m_h^2}{s}},$$ \hspace{1cm} (A59)

Now, since $I$ is a Lorentz invariant quantity, we can use this result in any inertial frame of reference with proper definition of $s$. In any arbitrary reference frame, the Mandelstam variable $s(p_1, p_2, \cos \alpha) = (P_1 + P_2)^2 = 2m_{\phi}^2 + 2E_{p_1}E_{p_2} - 2|\vec{p}_1||\vec{p}_2|\cos \alpha$, $\alpha$ is the angle between $\vec{p}_1$ and $\vec{p}_2$ which is $\pi$ in the centre of momentum frame. Hence, the collision term in an arbitrary inertial frame of reference is given by

$$C_{hh \to \phi \phi}^1 = \frac{1}{16\pi E_{p_1}} \int \frac{d^3p_2}{2E_{p_2}(2\pi)^3} \sqrt{1 - \frac{4m_h^2}{s(p_1, p_2, \cos \alpha)}} \times |\mathcal{M}|_{hh \to \phi \phi}^2(s) \left(f_h(k_1)f_h(k_2) - f_{\phi}(p_1)f_{\phi}(p_2)\right),$$

$$= \frac{2\pi}{16\pi E_{p_1}2(2\pi)^3} \int \frac{p_2^2 dp_2 d(\cos \alpha)}{E_{p_2}} \sqrt{1 - \frac{4m_h^2}{s(p_1, p_2, \cos \alpha)}} \times |\mathcal{M}|_{hh \to \phi \phi}^2(s)f_h(k_1)f_h(k_2),$$ \hspace{1cm} (A60)

where, in the last step we have neglected the back scattering term. Now using the Maxwell-Boltzmann distribution function for the SM Higgs boson and $f_{h}(k_1)f_{h}(k_2) = e^{-(E_{k_1} + E_{k_2})/T} = e^{-(E_{p_1} + E_{p_2})/T}$, we obtain

$$C_{hh \to \phi \phi}^1 = \frac{e^{-E_{p_1}/T}}{16E_{p_1}(2\pi)^3} \int_0^\infty \frac{p_2^2 dp_2}{\sqrt{p_2^2 + m_{\phi}^2}} e^{-E_{p_2}/T}$$

$$\times \int \cos \alpha_{max} \cos \alpha d(\cos \alpha) \sqrt{1 - \frac{4m_h^2}{s(p_1, p_2, \cos \alpha)}} |\mathcal{M}|_{hh \to \phi \phi}^2(s).$$ \hspace{1cm} (A61)
The limit on $\cos \alpha$ will come from the condition that $\sqrt{1 - \frac{4m_h^2}{s(p_1,p_2,\cos \alpha)}}$ is real. This is possible only when $s \geq 4m_h^2$ and therefore

$$\cos \alpha \leq \frac{2m^2 - 4m_h^2 + 2Ep_1Ep_2}{2|p_1||p_2|} \equiv \cos \alpha_0.$$  \hfill (A62)

Thus the upper limit of the integration is

$$\cos \alpha_{\text{max}} = \text{Min} \left[ \text{Max} \left[ \cos \alpha_0, -1 \right], 1 \right].$$  \hfill (A63)

And, lastly, the collision term $C^\phi \to \bar{\nu}_R \psi$ is for the decay of $\phi$ into $\bar{\nu}_R$ and $\psi$ ($\phi(P_1) \to \bar{\nu}_R(q) + \psi(q')$) and it has the following expression \cite{79}

$$C^\phi \to \bar{\nu}_R \psi = -f_\phi \frac{m_\phi}{p_1^2 + m_\phi^2} \Gamma_{\phi \to \bar{\nu}_R \psi}. \hfill (A64)$$

The LHS of Eq. (A48), can be greatly simplified in we transform the variables from $p_1$ and $T$ to new variables $r = m_0/T$ and $\xi = \left( \frac{g_s(T_0)}{g_s(T)} \right)^{1/3} \frac{p_1}{T}$ where $m_0$ is any arbitrary mass scale. In terms of the two new variables, the LHS of Eq. (A48) depends only on $r$ \cite{78, 79}

$$\frac{\partial f_\phi}{\partial t} - \mathcal{H} \frac{\partial f_\phi}{\partial p_1} = r\mathcal{H} \left( 1 + \frac{Tg_s'(T)}{3g_s(T)} \right)^{-1} \frac{\partial f_\phi}{\partial r}. \hfill (A65)$$

Therefore, the full Boltzmann equation for $f_\phi$ is

$$\frac{\partial f_\phi(\xi, r)}{\partial r} = \left( 1 - \frac{r}{3g_s(r)} \frac{dg_s(r)}{dr} \right) \frac{1}{r\mathcal{H}} \left( C^{h \to \phi \phi^\dagger}(\xi, r) + C^{h h \to \phi \phi^\dagger}(\xi, r) + C^\phi \to \bar{\nu}_R \psi(\xi, r) \right). \hfill (A66)$$

Now, the number density of $\phi$ can be written as

$$n_\phi(r) = \frac{g_\phi}{2\pi^2} A(r)^3 \left( \frac{m_0}{r} \right)^3 \int d\xi \xi^2 f_\phi(\xi, r), \hfill (A67)$$

where

$$A(r) = \left( \frac{g_s(m_0/r)}{g_s(m_0/T_0)} \right)^{1/3}. \hfill (A68)$$

After solving the Eq. (A66) for the non-thermal distribution function $f_\phi(\xi, r)$, we can now calculate comoving number density of $\psi$ and $Y_{\nu_R}$ using the following Boltzmann equations

$$\frac{dY_\psi}{dr} = \frac{g_\phi \beta}{r\mathcal{H}s} \frac{\Gamma_{\phi \psi} m_\phi}{2\pi^2} \int_0^\infty \left( A_{\phi r} \frac{m_0}{r} \right)^3 \xi^2 f_\phi(\xi, r) d\xi,$$

$$\frac{dY_{\nu_R}}{dr} = \frac{g_\phi \beta}{r\mathcal{H}s^{4/3} \langle E \rangle} \frac{1}{2\pi^2} \int_0^\infty \left( A_{\phi r} \frac{m_0}{r} \right)^3 \xi^2 f_\phi(\xi, r) d\xi. \hfill (A69)$$
Appendix B: Equations for $\Omega_{\text{DM}} h^2$ and $\Delta N_{\text{eff}}$

The effective number of relativistic degrees of freedom $N_{\text{eff}}$ can be defined as

$$N_{\text{eff}} = \frac{8}{7} \left(\frac{11}{4}\right)^{4/3} \left(\frac{\rho_{\text{rad}} - \rho_\gamma}{\rho_\gamma}\right)$$  \hspace{1cm} (B1)$$

where $\rho_{\text{rad}}, \rho_\gamma$ denote total radiation and photon densities respectively. The change in $N_{\text{eff}}$ is defined as $\Delta N_{\text{eff}} = N_{\text{eff}} - N_{\text{SM}}^{\text{eff}}$. While the expected value in the SM is close to 3 due to three left handed neutrinos, in our scenario this can increase due to the presence of three right handed neutrinos $\nu_R$ which are relativistic. Thus, taking $\rho_{\nu_R}$ to be part of $\rho_{\text{rad}}$, we can write $\Delta N_{\text{eff}}$ as

$$\Delta N_{\text{eff}} = 2 \times 3 \left(\frac{\rho_{\nu_R}}{\rho_{\nu_L}}\right)_{\text{CMB}} = 2 \times 3 \left(\frac{\rho_{\nu_R}}{\rho_{\nu_L}}\right)_{10 \text{MeV}} \left(\therefore \rho_{\nu_L} \propto \frac{1}{a^4}; \rho_{\nu_R} \propto \frac{1}{a^4}\right)$$

$$= 2 \times 3 \left(\frac{s^{4/3}Y_{\nu_R}}{\rho_{\nu_L}}\right)_{10 \text{MeV}},$$ \hspace{1cm} (B2)$$

where in the second step, we equate the ratio $\rho_{\nu_R}/\rho_{\nu_L}$ at the scale of recombination or CMB to that of BBN $\sim O(10)$ MeV. This is possible as we ensure the production of $\nu_R$ is complete before the BBN epoch.

Similarly, final DM abundance $\Omega_{\text{DM}} h^2$ can be written in terms of corresponding comoving number density as

$$\Omega_{\text{DM}} h^2 = 2 \times \frac{\rho_0^\psi}{\rho_c^0} h^2 = 2 \times \frac{m_\psi s^0 Y_\psi^0}{\rho_c^0} h^2 = 2 \times \frac{m_\psi s^0 (Y_\psi)_{10}}{\rho_c^0} h^2.$$ \hspace{1cm} (B3)$$

Since we have taken $g_\phi = 1$ throughout (the value of $g_\psi$ and $g_{\nu_R}$ are taken as 2), this implies that we are considering either the equations for $\phi$ or $\phi^\dagger$. Hence, $Y_\psi$ and $Y_{\nu_R}$ are only for either particles or anti-particles. So, in the expressions for $\Delta N_{\text{eff}}$ and $\Omega_{\text{DM}} h^2$ above, we have included a factor of 2 to incorporate both particles and antiparticles. Also a factor of 3 is included in $\Delta N_{\text{eff}}$ for three flavours of $\nu_R$.

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