The number of solutions of an equation related to a product of multilinear polynomials

T. Narayansamy, D.-J. Mercier, J.-P. Cherdieu

Laboratoire A.O.C. de l’Université des Antilles et de la Guyane, 97159 Pointe-à-Pitre cedex, France.

March 1st, 2006

Abstract

We look at the number of solutions of an equation of the form $f_1 f_2 \ldots f_k = a$ in a finite field, where each $f_i$ is a multilinear polynomial. We use two methods to construct a solution of this problem for the cases $a = 0$, $a \neq 0$, and we generally get a semi-explicit formula. We show that this formula can generate a more efficient algorithm than the traditional algorithm which consists to make a systematic computation. We also give explicit formulas in some special cases, and an application of our main result to the search of the weight hierarchy of the multilinear code with separated variables.

Keywords

Linear systems, multilinear polynomials, finite fields, separated variables, exponential sums.

MSC Codes

Primary classification: 11T06; Secondary classification: 11T23, 11T71.

I INTRODUCTION

Let $\mathbb{F}_q$ be a finite field with $q$ elements, $(m, n, k) \in (\mathbb{N} - \{0\})^3$ with $m \leq n$, $\{J_1, J_2, \ldots, J_m\}$ a partition of $\{1, \ldots, n\}$, $(f, a) \in \mathbb{F}_q[X_1, \ldots, X_n] \times \mathbb{F}_q$, $A$ an $k \times m$ matrix with entries in $\mathbb{F}_q$. We assume that $f$ satisfies the following:

\begin{align}
(1.1) \quad f &= f_1 f_2 \ldots f_k, \\
\text{where:} \quad f_i(X_1, \ldots, X_n) &= \sum_{j=1}^{m} a_{ij} \prod_{\tau \in J_j} X_\tau, \quad A = (a_{ij})_{1 \leq i \leq k, 1 \leq j \leq m}.
\end{align}

The polynomials $f_i$ are called multilinear polynomials, ([3] p. 32). In this present paper, the problem of determining the number of solutions in $\mathbb{F}_q^n$ of the following equation is considered:

\begin{align}
(1.3) \quad f(X_1, \ldots, X_n) &= a.
\end{align}
The solution of this problem is known only for special cases of (1.1) ([1], [2], [3]). We determine for the general case a semi-explicit formula for the solution of this problem. We obtain an explicit formula in special cases. When \( a \neq 0 \), the considered method is based on a generalization of the following formula:

\[
N(h(X)g(X) = a) = \sum_{u \in \mathbb{F}_q^*} \left( N(h(X) = u \& g(X) = \frac{a}{u}) \right),
\]

where \( N(h(X) = u \& g(X) = a/u) \) denotes the number of common solutions of the equations \( h(X) = u \) and \( g(X) = a/u \).

When \( a = 0 \), the considered method is based on the so-called "inclusion - exclusion principle". Note that if \( m \geq 2 \) and if \( a_{ij} \neq 0 \) for at least two entries \( j \), the polynomials of the form (1.2) are strictly contained in the class of irreducible polynomials. Then, it could be interesting in a future study to determine a sufficient condition to have the form (1.1). When this form exists, there exist algorithms to find these irreducible factors [4]. In our study, we suppose that the associated irreducible factors to \( f \) are known and our aim is to determine an expression for the number of solutions of the equation (1.3).

The organisation of this paper is the following: In section II, we continue the presentation of the considered notations, we recall basic results concerning linear systems and exponential sums. The "inclusion - exclusion principle" is also recalled. We compute formulas for the numbers of solutions of special systems of equations in Section III. These formulas will be useful in the treatment of the considered problem. The special case \( k \in \{1, 2\} \) is considered in this Section.

The main result for the general case is Proposition IV.1. In section V, we apply the obtained result to two "numerical" examples such that \( k \in \{2, 3\} \), \( a_{ij} \in \{0, 1\} \), \( m \in \{3, 4\} \), \( n \in \{7, 8\} \). We show from these examples that the obtained formula can generate a clearly more efficient algorithm than the "traditional" one which consists to make a systematic computation of all the solutions of this equation.

In section VI we show that the research of the number of zeros of products \( f = f_1f_2...f_k \) of the form (1.1) is important when we compute the weight hierarchy of a multilinear code. To conclude in Section VII, we give three examples of problems that can be investigated from the present study.

II NOTATIONS, BASIC RESULTS

Recall that \( m \leq n \). We assume:

\[
(2.1) \quad k \leq m,
\]

\[
(2.2) \quad A \text{ is an } k \times m \text{ matrix of rank of } k.
\]

Let \( 1 \leq i_1 < i_2 < ... < i_l \leq k, \ (b_1, ..., b_l) \in \mathbb{F}_q^l \).

The system:

\[
\begin{align*}
\begin{cases}
  f_{i_1}(X_1, ..., X_n) = b_1, \\
  f_{i_2}(X_1, ..., X_n) = b_2, \\
  \quad \vdots \\
  f_{i_l}(X_1, ..., X_n) = b_l,
\end{cases}
\end{align*}
\]

is equivalent to:
We will use the following notations:

\[ A_{[i_1,i_2,...,i_l]} \left( \prod_{\tau \in J_1} X_\tau, \prod_{\tau \in J_2} X_\tau, ..., \prod_{\tau \in J_m} X_\tau \right)^T = (b_1, ..., b_l)^T , \]

where

\[ A_{[i_1,i_2,...,i_l]} = (a_{i,t})_{1 \leq i \leq l, 1 \leq j \leq m} . \]

We can write:

\[ (2.4a) \quad B_{[i_1,i_2,...,i_l]} \left( \prod_{\tau \in J_{\nu_1}} X_\tau, \prod_{\tau \in J_{\nu_2}} X_\tau, ..., \prod_{\tau \in J_{\nu_l}} X_\tau \right)^T = (b_1, ..., b_l)^T - \]

\[ C_{[i_1,i_2,...,i_l]} \left( \prod_{\tau \in J_{\nu_1}} X_\tau, \prod_{\tau \in J_{\nu_2}} X_\tau, ..., \prod_{\tau \in J_{\nu_{m-1}}} X_\tau \right)^T , \]

where

\[ (2.4b) \quad B_{[i_1,i_2,...,i_l]} = (a_{i,t})_{j \in \{\nu_1,...,\nu_l\} \subset \{1,...,m\}} \]

is an invertible \( l \times l \) submatrix of \( A_{[i_1,i_2,...,i_l]} \),

\[ (2.4c) \quad C_{[i_1,i_2,...,i_l]} = (a_{i,t})_{j \in \{1,...,m\} - \{\nu_1,...,\nu_l\}} \]

is an \( l \times (m - l) \) submatrix of \( A \),

\[ (2.4d) \quad \{\nu_1',...,\nu_{m-l}'\} = \{1,...,m\} - \{\nu_1,...,\nu_l\} . \]

We will use the following notations:

\[ (2.5a) \quad \left\{ \begin{array}{l}
B^{-1}_{[i_1,i_2,...,i_l]} = (a'_{i,t})_{j \in \{\nu_1,...,\nu_l\} \subset \{1,...,m\}} , \\
B^{-1}_{[i_1,i_2,...,i_l]} C_{[i_1,i_2,...,i_l]} = (\sigma'_{i,t})_{1 \leq i \leq l} , 1 \leq j \leq m-l , \\
I_{\nu'} = J_{\nu_1'} \cup J_{\nu_2'} \cup \ldots \cup J_{\nu_{m-l}'} .
\end{array} \right. \]

We denote by

\[ (2.5b) \quad N (f_1, ..., f_i, b_1, ..., b_l) , \]

the number of solutions in \( \mathbb{F}_q^n \) of the system of equations (2.3). The number of solutions in \( \mathbb{F}_q^n \) of the equation (1.3) is denoted by

\[ (2.5c) \quad N (f, a) . \]

To treat the considered problem, we use a method based on the following relations:
\[ N(f, a) = \sum_{(a_1, \ldots, a_k) \in \mathbb{F}_q^k \atop a_1 \ldots a_k = a} N(f_1, \ldots, f_k, a_1, \ldots, a_k), \]

when \( a \neq 0 \),

\[ N(f, 0) = N(f_1, 0) + N(f_2, 0) + \ldots + N(f_k, 0) - \sum_{1 \leq i_1 < i_2 \leq k} N(f_{i_1}, f_{i_2}, 0, 0) + \ldots + (-1)^{j+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_j \leq k} N(f_{i_1}, f_{i_2}, \ldots, f_{i_j}, 0, 0, \ldots, 0) + \ldots + (-1)^{k+1} N(f_1, f_2, \ldots, f_k, 0, 0, \ldots, 0). \]

The considered method in [2] to solve this problem in the particular case \( k = 1, a = 0 \), is based on exponential sums. We shall use a similar method in Section III to solve the more general case \( k = 1 \).

For all \( u \in \mathbb{F}_q \), the following function is an additive character on \( \mathbb{F}_q \)

\[ \Psi_u(v) = \exp \left( \frac{2i\pi}{p} \text{Tr}_{\mathbb{F}_q}\left( uv \right) \right), \]

where \( \text{Tr}_{\mathbb{F}_q}(uv) \) is the absolute trace of \( uv \).

We will use the following basic results:

\[ \sum_{u \in \mathbb{F}_q} \Psi_u(v) = \begin{cases} 0 & \text{if } v \neq 0, \\ q & \text{if } v = 0. \end{cases} \]

\[ \sum_{v \in \mathbb{F}_q} \Psi_u(v) = \begin{cases} 0 & \text{if } u \neq 0, \\ q & \text{if } u = 0. \end{cases} \]

We also need the following result given in [2]:

**PROPOSITION II-1**

Let \((\alpha, u, d) \in (\mathbb{F}_q - \{0\})^2 \times (\mathbb{N} - \{0\})\), then:

\[ \sum_{(X_1, X_2, \ldots, X_d) \in \mathbb{F}_q^d} \Psi_u \left( \alpha \prod_{i \in \{1, \ldots, d\}} X_i \right) = q^d - q(q-1)^{d-1}. \]

We conclude this present Section by recalling the wellknown map [1]:

\[ \kappa : \mathbb{F}_q \to \{-1, q - 1\}, \]

\[ \kappa(X) = \begin{cases} -1 & \text{if } X \neq 0, \\ q - 1 & \text{if } X = 0. \end{cases} \]

**III NUMBER OF SOLUTIONS OF SPECIAL SYSTEMS**

In the following Proposition, we determine the solution of the problem for the special case \( k = 1 \).

**PROPOSITION III-1**

Assume that \( k = 1 \) in (1.1), then:
(3.1) \[ N(f,a) = q^{n-1} + \kappa(a) q^{n-1} \left[ \prod_{\{l/a_{1l} \neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_l|-1} \right) \right], \]

(Recall that \( N(f,a) \) is the number of solutions of (1.3)).

**Proof**

First, (2.8) and (2.9) imply that:

\[
N(f,a) = \frac{1}{q} \left( \sum_{(X_1,X_2,...,X_n) \in \mathbb{F}_q^n} \sum_{u \in \mathbb{F}_q} \Psi_u \left( f(X_1, X_2, ..., X_n) - a \right) \right),
\]

thus:

\[
N(f,a) = \frac{1}{q} \left( \sum_{u \in \mathbb{F}_q} \left( \sum_{(X_1,X_2,...,X_n) \in \mathbb{F}_q^n} \Psi_u \left( f(X_1, X_2, ..., X_n) \right) \right) \right)
= \frac{1}{q} \left( \sum_{u \in \mathbb{F}_q} \left( \sum_{(X_1,X_2,...,X_n) \in \mathbb{F}_q^n} \Psi_u \left( \sum_{j=1}^m a_{1j} \prod_{\tau \in J_j} X_\tau \right) \right) \right)
= \frac{1}{q} \left( \sum_{u \in \mathbb{F}_q} \left[ \Psi_u(-a) \prod_{\{k/a_{1k} \neq 0\}} \sum_{(X_1,X_2,...,X_{|J_k|}) \in \mathbb{F}_q^{|J_k|}} \Psi_u(a_{1k} \prod_{\tau \in J_k} X_\tau) \right] \right).
\]

After, from Proposition II-1 we get:

\[
N(f_1,a) = q^{n-1} + q \sum_{\{k/a_{1k} \neq 0\}} |J_k| \left( \sum_{u \in \mathbb{F}_q^{|J_k|}} \left[ \Psi_u(-a) \prod_{\{k/a_{1k} \neq 0\}} \left( q^{|J_k|} - q(q-1)^{|J_k|-1} \right) \right] \right)
= q^{n-1} + q q^{n-1} \left[ \prod_{\{k/a_{1k} \neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_k|-1} \right) \right] \left[ \sum_{u \in \mathbb{F}_q} \Psi_u(-a) \right].
\]

From (2.8) and (2.11) we deduce \( \sum_{u \in \mathbb{F}_q} \Psi_u(-a) = \kappa(a) \), thus

\[
N(f,a) = q^{n-1} + \kappa(a) q^{n-1} \prod_{\{l/a_{1l} \neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_l|-1} \right),
\]

which is the desired result.\( \blacksquare \)

The following Corollary which is a consequence of Proposition III-1, will be useful for instance in the proof of Proposition III-3.

**COROLLARY III-2**

Let \((w,\alpha,d) \in \mathbb{F}_q \times (\mathbb{F}_q - \{0\}) \times (\mathbb{N} - \{0\})\), then the number \( N(\alpha \prod_{\tau=1}^d X_\tau, w) \) of solutions in \( \mathbb{F}_q^d \) of the equation:

\[
\alpha \prod_{\tau=1}^d X_\tau = w
\]

is
\[
N \left( \alpha \prod_{\tau=1}^{d} X_{\tau}, w \right) = q^{d-1} + \kappa \left( \hat{w} \right) q^{d-1} \left( 1 - \left( \frac{q-1}{q} \right)^{d-1} \right).
\]

**PROPOSITION III-3**

Let \( 1 \leq i_1 < i_2 < \ldots < i_l \leq k \), \((b_1, \ldots, b_l) \in \mathbb{F}_q^l \), then

\[
N \left( f_{i_1}, f_{i_2}, \ldots, f_{i_l}, b_1, b_2, \ldots, b_l \right) =
\]

\[
\sum_{\left( X_{\tau} \right)_{i \in I_{\nu}}} \prod_{t=1}^{l} \left( q \left| J_{\nu} \right| - 1 + \kappa \left( \sum_{j=1}^{l} a_{i_t,j} b_j - \sum_{j=1}^{m-l} \sigma_{i_t,j} \prod_{\tau \in J_{\nu}^t} X_{\tau} \right) q \left| J_{\nu} \right| - 1 \left( 1 - \left( \frac{q-1}{q} \right) \left| J_{\nu} \right|^{-1} \right) \right),
\]

\[
\left( I_{\nu}, J_{\nu}, a_{i_t,j}, \sigma_{i_t,j} \right) \text{ are defined in (2.4b), (2.4c), (2.4d), (2.5a)}.\]

**Proof**

First recall that the matrix \( A \) is of rank \( k \), then the existence of the \( l \times l \) invertible matrix \( B_{i_1,i_2,\ldots,i_l} \) introduced in (2.4a), (2.4b) is justified. Consequently, the equality (2.4) is equivalent to:

\[
\left( \prod_{\tau \in J_{\nu_1}} X_{\tau}, \prod_{\tau \in J_{\nu_2}} X_{\tau}, \ldots, \prod_{\tau \in J_{\nu_{m-l}}} X_{\tau} \right)^T = B_{[i_1,i_2,\ldots,i_l]}^{-1} (b_1, \ldots, b_l)^T -
\]

\[
B_{[i_1,i_2,\ldots,i_l]}^{-1} C_{[i_1,i_2,\ldots,i_l]} \left( \prod_{\tau \in J_{\nu_1}} X_{\tau}, \prod_{\tau \in J_{\nu_2}} X_{\tau}, \ldots, \prod_{\tau \in J_{\nu_{m-l}}} X_{\tau} \right)^T,
\]

with \( \{\nu_1, \ldots, \nu_l\} \cup \{\nu_1', \ldots, \nu_{m-l}'\} = \{1, \ldots, m\} \), (see (2.4d)).

It follows that:

\[
(3.2) \quad \prod_{\tau \in J_{\nu_t}} X_{\tau} = \sum_{j=1}^{l} a_{i_t,j} b_j - \sum_{j=1}^{m-l} \sigma_{i_t,j} \prod_{\tau \in J_{\nu_j}} X_{\tau}, \quad t = 1, \ldots, l,
\]

applying Corollary III-2 to (3.2), we get:

\[
N \left( f_{i_1}, f_{i_2}, \ldots, f_{i_l}, b_1, b_2, \ldots, b_l \right) =
\]

\[
\sum_{\left( X_{\tau} \right)_{i \in I_{\nu}}} \prod_{t=1}^{l} \left( q \left| J_{\nu_t} \right| - 1 + \kappa \left( \sum_{j=1}^{l} a_{i_t,j} b_j - \sum_{j=1}^{m-l} \sigma_{i_t,j} \prod_{\tau \in J_{\nu_j}} X_{\tau} \right) q \left| J_{\nu_t} \right| - 1 \left( 1 - \left( \frac{q-1}{q} \right) \left| J_{\nu_t} \right|^{-1} \right) \right),
\]

which is the announced result.\[\square\]

Now in the following Proposition, we give a semi-explicit formula for the number of solutions of (1.3) for the case \( k = 2 \).

**PROPOSITION III-4**

Assume that \( k = 2 \) in (1.1). With the notations (2.5a), (2.5b) and (2.5c), we have:
\[ N(f, 0) = 2q^{n-1} + (q - 1)q^{n-1} \left( \prod_{\{l/a_{1i}\neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_l| - 1} \right) + \prod_{\{l/a_{2i}\neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_l| - 1} \right) \right) \]

\[ - \sum_{(X_i)_{i \in I', \sigma \in \mathbb{F}_q^*}} \sum_{l/a \in I'} \prod_{t=t_1}^{2} q^{|J_{l,t}| - 1} \left( 1 + \kappa \left( a_{11}u + a_{21}a - \sum_{j=1}^{m-2} \sigma_{12,j} \prod_{\tau \in J_{l,t}} X_{\tau} \right) q^{|J_{l,t}| - 1} \right), \]

\[ N(f, a) = \sum_{u \in \mathbb{F}_q^*} \sum_{(X_i)_{i \in I', \sigma \in \mathbb{F}_q^*}} \prod_{t=t_1}^{2} q^{|J_{l,t}| - 1} \left( 1 + \kappa \left( a_{11}u + a_{21}a - \sum_{j=1}^{m-2} \sigma_{12,j} \prod_{\tau \in J_{l,t}} X_{\tau} \right) q^{|J_{l,t}| - 1} \right), \]

**Proof**

First consider the case \( a = 0 \). Then, from (2.6b) we have:

\[ N(f, 0) = N(f_1, 0) + N(f_2, 0) - N(f_1, f_2, 0, 0). \]

Using (3.1) we get:

\[ (3.3) \quad N(f_1, 0) + N(f_2, 0) = \]

\[ q^{n-1} + \kappa(0)q^{n-1} \left[ \prod_{\{l/a_{1i}\neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_l| - 1} \right) \right] + q^{n-1} + \kappa(0)q^{n-1} \left[ \prod_{\{l/a_{2i}\neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_l| - 1} \right) \right] \]

\[ = 2q^{n-1} + (q - 1)q^{n-1} \left[ \prod_{\{l/a_{1i}\neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_l| - 1} \right) \right] \]

\[ + \prod_{\{l/a_{2i}\neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_l| - 1} \right), \]

After, using Proposition III-3 in the case \((l, b_1, b_2) = (2, 0, 0)\), we can write:

\[ N(f_1, f_2, 0, 0) = \sum_{(X_i)_{i \in I', \sigma \in \mathbb{F}_q^*}} \prod_{t=t_1}^{2} q^{|J_{l,t}| - 1} + \kappa \left( \sum_{j=1}^{m-2} \sigma_{12,j} \prod_{\tau \in J_{l,t}} X_{\tau} \right) q^{|J_{l,t}| - 1} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_{l,t}| - 1} \right), \]

the desired result follows in the case \( a = 0 \).

Now assume that \( a \neq 0 \). From (2.6a) we have:

\[ N(f, a) = \sum_{u \in \mathbb{F}_q^*} N\left(f_1, f_2, u, \frac{a}{u}\right). \]

Let \( u \in \mathbb{F}_q^* \), by Proposition III-3 in the case \((l, b_1, b_2) = (2, u, \frac{a}{u})\) we get:

\[ N\left(f_1, f_2, u, \frac{a}{u}\right) = \]
\[
\sum_{(X_t) \in I_{\nu_t} \in \mathbb{F}_q} |J_{\nu_t}|^{-1} \left( 1 + \kappa \left( a_{1,1}a + \frac{a_{1,2}}{u}a - \sum_{j=1}^{m-2} \sigma_{[1,2]} t_{j} \prod_{\tau \in J_{\nu_j}} X_{\tau} \right) \left( 1 - \left( \frac{q-1}{q} \right) |J_{\nu_t}|^{-1} \right) \right),
\]

consequently, we easily get the announced result in the case \( a \neq 0 \). \( \blacksquare \)

IV THE GENERAL PROBLEM

Here, we consider the equation (1.3), namely: \( f(X_1, \ldots, X_n) = a \), where \( f = f_1 f_2 \ldots f_k \) and \( f_t(X_1, \ldots, X_n) = \sum_{j=1}^{m} a_{ij} \prod_{\tau \in J_{\nu_j}} X_{\tau} \), and we assume that the rank of the matrix \( A = (a_{ij})_{1 \leq i, j \leq m} \) is \( k \). In the general case, the number of solutions \( N(f,a) \) of \( f(X_1, \ldots, X_n) = a \) is given by next Proposition:

PROPOSITION IV-1

With the notations (2.5a),

\[
N(f,0) = kq^{n-1} + (q-1) q^{n-1} \left[ \prod_{\{l/a_i \neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right) |J_l|^{-1} \right) \right] +
\]

\[
\sum_{l=2}^{k} (-1)^{l+1} \sum_{1 \leq i_1 < i_2 < \ldots < i_l \leq k} \prod_{t=1}^{l} q |J_{\nu_t}|^{-1} \left( 1 + \kappa \left( \sum_{j=1}^{m-l} \sigma_{[1,i_2, \ldots, i_l]} t_{j} \prod_{\tau \in J_{\nu_j}} X_{\tau} \right) \left( 1 - \left( \frac{q-1}{q} \right) |J_{\nu_t}|^{-1} \right) \right),
\]

\[
N(f,a) =
\]

\[
\sum_{(a_1, \ldots, a_k) \in \mathbb{F}_q^k \setminus \{a \}} \sum_{(X_t) \in I_{\nu_t} \in \mathbb{F}_q} |J_{\nu_t}|^{-1} \left( 1 + \kappa \left( \sum_{j=1}^{m-k} \sigma_{[1,i_2, \ldots, i_l]} t_{j} \prod_{\tau \in J_{\nu_j}} X_{\tau} \right) \left( 1 - \left( \frac{q-1}{q} \right) |J_{\nu_t}|^{-1} \right) \right),
\]

if \( a \neq 0 \).

Proof

First consider the case \( a = 0 \). From Proposition III-1 and for all \( i \in \{1, \ldots, k\} \):

\[
N(f_i,0) = q^{n-1} + (q-1) q^{n-1} \left[ \prod_{\{l/a_i \neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right) |J_l|^{-1} \right) \right],
\]

thus:

\[
\sum_{i=1}^{k} N(f_i,0) = kq^{n-1} + (q-1) q^{n-1} \sum_{i=1}^{k} \left[ \prod_{\{l/a_i \neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right) |J_l|^{-1} \right) \right].
\]

Now let \( 1 \leq i_1 < i_2 < \ldots < i_l \leq k \), Proposition III-3 gives:

\[
N(f_{i_1}, f_{i_2}, \ldots, f_{i_l}, 0, 0, \ldots, 0) =
\]

\[
\sum_{(X_t) \in I_{\nu_t} \in \mathbb{F}_q} \prod_{t=1}^{l} q |J_{\nu_t}|^{-1} + \kappa \left( \sum_{j=1}^{m-l} \sigma_{[1,i_2, \ldots, i_l]} t_{j} \prod_{\tau \in J_{\nu_j}} X_{\tau} \right) \left( 1 - \left( \frac{q-1}{q} \right) |J_{\nu_t}|^{-1} \right) \right) \right).
\]

8
After, we get the desired result for the case $a = 0$ from (2.6b).

Now assume that $a \neq 0$ and let $(a_1, \ldots, a_k) \in \mathbb{F}_q^k$ such that $a_1 a_2 \ldots a_k = a$, then from Proposition III-3, we can write:

$$N(f_1, f_2, \ldots, f_k, a_1, \ldots, a_k) =$$

$$\sum_{(X_i) \in I_{_F} \in \mathbb{F}_q^I_{t-1}} \prod_{t=1}^k (q^{|J_{\nu_t}|-1} + \kappa \left( \sum_{j=1}^{m-t} \sigma_{i_j} a_j - \sum_{j=1}^{m-k} \sigma_{i_j}^{k_2} X_1 \prod_{t \in J_{\nu_j}} X_2 \right) q^{|J_{\nu_t}|-1} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_{\nu_t}|^{-1}} \right) \right),$$

it follows from (2.6a) that:

$$N(f, a) =$$

$$\sum_{(a_1, \ldots, a_k) \in \mathbb{F}_q^k, (a_1, \ldots, a_k = a)} \sum_{(X_i) \in I_{_F} \in \mathbb{F}_q^I_{t-1}} \prod_{t=1}^k (q^{|J_{\nu_t}|-1} \left( 1 + \kappa \left( \sum_{j=1}^{m-t} \sigma_{i_j} a_j - \sum_{j=1}^{m-k} \sigma_{i_j}^{k_2} X_1 \prod_{t \in J_{\nu_j}} X_2 \right) \left( 1 - \left( \frac{q-1}{q} \right)^{|J_{\nu_t}|^{-1}} \right) \right),$$

which completes the proof.■

The formula given in previous Proposition is generally a semi-explicit formula because of terms

$$\kappa \left( \sum_{j=1}^{m-t} \sigma_{i_j} a_j - \sum_{j=1}^{m-k} \sigma_{i_j}^{k_2} X_1 \prod_{t \in J_{\nu_j}} X_2 \right) .$$

In the next Proposition, particular cases related to the matrix $A$ are considered.

**PROPOSITION IV-2**

If $m = k$ and $a \neq 0$ then:

$$N(f, a) =$$

$$\sum_{(a_1, \ldots, a_{m-1}) \in (\mathbb{F}_q^*)^{m-1}} \left[ \prod_{i=1}^m \left( q^{|J_i|^{-1}} + \kappa \left( \sum_{j=1}^{m-1} a_{i_j} a_j + \frac{a}{a_{i_1} a_{i_2} \ldots a_{i_{m-1}}} \right) q^{|J_i|^{-1}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_i|^{-1}} \right) \right) \right].$$

If

| $m = 2k$ |
|---|
| $A = (D_1 D_2)$ |

where $D_1$ and $D_2$ are two invertible diagonale $k \times k$ matrices, then:

$$N(f, a) =$$

$$(q - 1)^{k-1} \prod_{j=1}^k \left[ q^{|J_i|+|J_{k+j}|^{-1}} - q^{|J_i|+|J_{k+j}|^{-1}} \left( 1 - \left( \frac{q-1}{q} \right)^{|J_i|^{-1}} \right) \right],$$

if $a \neq 0.$
First consider the case \( m = k \). Then \( A \) is an invertible \( m \times m \) matrix. Let \((a_1, a_2, \ldots, a_{m-1}) \in (\mathbb{F}_q^*)^{m-1}\). In (2.4a), (2.5a) we can take:
\[
B_{[i_1, i_2, \ldots, i_l]} = A, \quad C_{[i_1, i_2, \ldots, i_l]} = 0, \quad (b_1, \ldots, b_m) = \left( a_1, a_2, \ldots, a_{m-1}, \frac{a}{a_{1}a_{2}a_{m-1}} \right),
\]
\[
l = m, \quad [i_1, i_2, \ldots, i_l] = [1, 2, \ldots, m], \quad I_{\nu'} = \emptyset,
\]
then from (3.2), we can write:
\[
N \left( f_1, f_2, \ldots, f_m, a_1, \ldots, a_{m-1}, \frac{a}{a_{1}a_{2}\ldots a_{m-1}} \right) = \prod_{i=1}^{m} \left( q^{|J_i|-1} \sum_{j=1}^{m} \left( a'_{i,j}a_j + a'_{i,m}a_{a_{1}a_{2}\ldots a_{m-1}} \right) q^{|J_i|-1} \left( 1 - \left( \frac{q^2}{q^3} \right)^{|J_i|-1} \right) \right)
\]
and the result (4.1) follows from (2.6a).

Now consider the case (4.2) and let \((a_1, a_2, \ldots, a_{k-1}) \in (\mathbb{F}_q^*)^{k-1}\). Recalling that \( A = (a_{ij}) \), it’s easy to see that: \( D_1 = \text{Diag} (a_{11}, \ldots, a_{kk}) \), \( D_2 = \text{Diag} (a_{1,k+1}, a_{2,k+2}, \ldots, a_{k,k+k}) \), \( a_{ij} = 0 \) for all \((i, j)\) such that \( j - i \notin \{0, i\} \), \( a_{ii}a_{i,k+i} \neq 0 \) for all \( i \in \{1, \ldots, k\} \). This means that we can make the following choices in (2.4a):
\[
B_{[i_1, i_2, \ldots, i_l]} = D_1, \quad C_{[i_1, i_2, \ldots, i_l]} = D_2, \quad (b_1, \ldots, b_k)^T = \left( a_1, a_2, \ldots, a_{k-1}, \frac{a}{a_{1}a_{2}\ldots a_{k-1}} \right)^T,
\]
\[
l = k, \quad [i_1, i_2, \ldots, i_k] = [1, 2, \ldots, k], \quad (\nu_1, \ldots, \nu_k) = (1, 2, \ldots, k),
\]
\[
(\nu'_{1}, \ldots, \nu'_{m-k}) = (k + 1, \ldots, m - k),
\]
thus:
\[
\left( \prod_{\tau \in J_1} X_{\tau}, \prod_{\tau \in J_2} X_{\tau}, \ldots, \prod_{\tau \in J_k} X_{\tau} \right)^T = D_1^{-1} \left( a_1, \ldots, a_{k-1}, \frac{a}{a_{1}a_{2}\ldots a_{k-1}} \right)^T - D_1^{-1}D_2 \left( \prod_{\tau \in J_{k+1}} X_{\tau}, \prod_{\tau \in J_{k+2}} X_{\tau}, \ldots, \prod_{\tau \in J_m} X_{\tau} \right)^T
\]
and for all \( i \in \{1, \ldots, k\} \),
Thus:

\[
\prod_{\tau \in J_i} X_\tau = a_{i_1}^{-1} a_i - a_{i_2}^{-1} a_{i,k+i} \prod_{\tau \in J_{k+i}} X_\tau,
\]

with \(a_k = \frac{a}{a_1 \ldots a_{k-1}}\).

We deduce from Proposition III-1 and the hypothesis \((a_{i_1}, a_i, a_{i,k+i}) \in (\mathbb{F}_q^*)^3\), that the number of solutions in \(\mathbb{F}_q^{|J_i|+|J_{k+i}|}\) of (4.5) is:

\[
q^{ |J_i|+|J_{k+i}| - 1 } - q^{ |J_i|+|J_{k+i}| - 1 } \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_i| - 1 } \right) \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_{k+i}| - 1 } \right).
\]

Then, it’s easy to see that:

\[
N \left( f_1, f_2, \ldots, f_k, a_1, \ldots, a_{k-1}, \frac{a}{a_1 a_2 \ldots a_{k-1}} \right) = \\
\prod_{i=1}^{k} \left[ q^{ |J_i|+|J_{k+i}| - 1 } - q^{ |J_i|+|J_{k+i}| - 1 } \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_i| - 1 } \right) \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_{k+i}| - 1 } \right) \right]
\]

and from (2.6a) the result (4.3) follows.

Now, let \(1 \leq i_1 < i_2 < \ldots < i_l \leq k\). The system of equations (2.3) in the case (4.2) and \((b_1, \ldots, b_l) = (0, \ldots, 0)\), namely:

\[
\begin{align*}
&f_{i_1} (X_1, \ldots, X_n) = 0, \\
&f_{i_2} (X_1, \ldots, X_n) = 0, \\
&\quad \vdots \\
&f_{i_l} (X_1, \ldots, X_n) = 0,
\end{align*}
\]

is equivalent to

\[
\begin{align*}
&\prod_{\tau \in J_{i_1}} X_\tau + a_{i_1}^{-1} a_{i_1,k+i_1} \prod_{\tau \in J_{k+i_1}} X_\tau = 0, \\
&\prod_{\tau \in J_{i_2}} X_\tau + a_{i_2}^{-1} a_{i_2,k+i_2} \prod_{\tau \in J_{k+i_2}} X_\tau = 0, \\
&\quad \vdots \\
&\prod_{\tau \in J_{i_l}} X_\tau + a_{i_l}^{-1} a_{i_l,k+i_l} \prod_{\tau \in J_{k+i_l}} X_\tau = 0,
\end{align*}
\]

thus:

\[
N \left( f_{i_1}, f_{i_2}, \ldots, f_{i_l}, 0, 0, \ldots, 0 \right) = \\
q^{n - \sum_{j=1}^{l} (|J_{i_j}| + |J_{k+i_j}|)} \prod_{j=1}^{l} \left( q^{ |J_{i_j}|+|J_{k+i_j}| - 1 } + \kappa (0) q^{ |J_{i_j}|+|J_{k+i_j}| - 1 } \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_{i_j}| - 1 } \right) \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_{k+i_j}| - 1 } \right) \right).
\]

After by Proposition III-1, we have for all \(j \in \{1, \ldots, l\}\):

\[
N (f_i, 0) = q^{n-1} + \kappa (0) q^{n-1} \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_i| - 1 } \right) \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_{k+i}| - 1 } \right),
\]

consequently from (2.6b), we have:

\[
N (f, 0) = kq^{n-1} + \kappa (0) q^{n-1} \sum_{i=1}^{k} \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_i| - 1 } \right) \left( 1 - \left( \frac{q-1}{q} \right) ^{ |J_{k+i}| - 1 } \right)
\]

11
\[
\sum_{1 \leq i_1 < i_2 \leq k} q^{n-\sum_{j=1}^{l} (|J_{i_1}| + |J_{i_2}|)} \left[ \prod_{j=1}^{l} q^{(|J_{i_1}| + |J_{i_2}|)^{-1}} \left( 1 + \kappa(0) \left( 1 - \left( \frac{q-1}{q} \right)^{|J_{i_1}|^{-1}} \right) \left( 1 - \left( \frac{q-1}{q} \right)^{|J_{i_2}|^{-1}} \right) \right) \right] \] 

\[ \cdots \]

\[ \sum_{1 \leq i_1 < \ldots < i_l \leq k} q^{n-\sum_{j=1}^{l} (|J_{i_1}| + |J_{i_2}|)} \left( 1 + \kappa(0) \left( 1 - \left( \frac{q-1}{q} \right)^{|J_{i_1}|^{-1}} \right) \left( 1 - \left( \frac{q-1}{q} \right)^{|J_{i_2}|^{-1}} \right) \right) \] 

\[ \cdots + (-1)^{k+1} \sum_{j=1}^{l} q^{(|J_j|)} \prod_{j=1}^{l} q^{(|J_{i_1}| + |J_{i_2}|)^{-1}} \left( 1 + \kappa(0) \left( 1 - \left( \frac{q-1}{q} \right)^{|J_j|^{-1}} \right) \left( 1 - \left( \frac{q-1}{q} \right)^{|J_{i_2}|^{-1}} \right) \right) \]

which is the formula (4.4). \[ \blacksquare \]

This Proposition indicates for instance that for particular interesting cases related to the matrix \( A \), we can get an explicit formula for the number of solutions of (1.3).

V NUMERICAL EXAMPLES

V-1 A FIRST EXAMPLE

Consider the following particular situation of (1.3):

(5.1) \( (X_1X_2 + X_5X_6X_7) (X_3X_4 + X_5X_6X_7) = a \).

Then:

\[
\begin{align*}
(k, m, n) &= (2, 3, 7), \\
A &= \left( \begin{array}{ccc}
1 & 0 & 1 \\
0 & 1 & 1
\end{array} \right), \\
J_1 &= \{1, 2\}, \quad J_2 = \{3, 4\}, \quad J_3 = \{5, 6, 7\}.
\end{align*}
\]

We have the following Proposition.

PROPOSITION V-1

Let \( \alpha \) be a primitive element of \( \mathbb{F}_q \), \( 1 \leq \tau \leq \frac{q-1}{2} \). Then the number of solutions of (5.1) is:

(5.2) \( N(f, 0) = 2q^6 + 3q^5 - 13q^4 + 16q^3 - 9q^2 + 2q, \)

(5.3) \( N(f, \alpha^{2\tau}) = (q - 1)^2 q (q^3 + q^2 - 2q + 2), \)

(5.4) \( N(f, \alpha^{2\tau - 1}) = (q - 1)^3 q (q^2 + 2q - 2). \)

Proof

In (2.4a) we can take:

\[ l = 2, \quad [i_1, i_2] = [1, 2], \quad (\nu_1, \nu_2) = (1, 2), \quad \nu' = 3, \quad B_{[1,2]} = \left( \begin{array}{cc}
1 & 0 \\
0 & 1
\end{array} \right), \quad C_{[1,2]} = \left( \begin{array}{c}
1 \\
1
\end{array} \right), \]

thus:
\[ B_{[1,2]}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}, \quad B_{[1,2]}^{-1} C_{[1,2]} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sigma^{1,1}_{[1,2]} \\ \sigma^{2,1}_{[1,2]} \end{pmatrix}. \]

From the above assumptions and Proposition III-4, we first can establish easily (5.2) and secondly we can deduce that for all \( 1 < t < q - 1 \):

\[(5.5) \quad N(f, \alpha^t) = \sum_{u \in \mathbb{F}_q^3} \sum_{(X_5, X_6, X_7) \in \mathbb{F}_q^3} \left[ (q + \kappa (u - X_5 X_6 X_7)) \left( q + \kappa \left( \frac{\alpha^t}{u} - X_5 X_6 X_7 \right) \right) \right],\]

it follows that:

\[(5.6) \quad T_1 = \sum_{u \in \{-\alpha^t, \alpha^t\}} \sum_{(X_5, X_6, X_7) \in \mathbb{F}_q^3} \left[ (q + \kappa (u - X_5 X_6 X_7)) \left( q + \kappa \left( \frac{\alpha^t}{u} - X_5 X_6 X_7 \right) \right) \right],\]

\[(5.7) \quad T_2 = \sum_{u \notin \{-\alpha^t, 0, \alpha^t\}} \sum_{(X_5, X_6, X_7) \in \mathbb{F}_q^3} \left[ (q + \kappa (u - X_5 X_6 X_7)) \left( q + \kappa \left( \frac{\alpha^t}{u} - X_5 X_6 X_7 \right) \right) \right],\]

thus:

\[(5.8) \quad T_1 = 2(q - 1)^2 (2q - 1)^2 + 2 \left( q^2 - (q - 1)^2 \right) (q - 1)^2 = 2(q - 1)^2 q \left[ q^2 + 3q - 2 \right].\]

Now, (5.7) implies that:

\[ T_2 = \sum_{u \notin \{-\alpha^t, 0, \alpha^t\}} \left[ \sum_{(X_5, X_6, X_7) \in \mathbb{F}_q^3} \left[ (q + \kappa (u - X_5 X_6 X_7)) \left( q + \kappa \left( \frac{\alpha^t}{u} - X_5 X_6 X_7 \right) \right) \right] + \right. \]

\[ \left. \sum_{(X_5, X_6, X_7) \in \mathbb{F}_q^3} \sum_{X_5 X_6 X_7 = \frac{\alpha^t}{u}} \sum_{X_5 X_6 X_7 \neq \frac{\alpha^t}{u}} \left[ (q + \kappa (u - X_5 X_6 X_7)) \left( q + \kappa \left( \frac{\alpha^t}{u} - X_5 X_6 X_7 \right) \right) \right] \right].\]
It follows from Corollary III-2 that:

\[
\begin{align*}
T_2 &= \sum_{u \in \{-\alpha^r, 0, \alpha^r\}} \left[ 2(q-1)^3(2q-1) + \left( q^3 - 2(q-1)^2 \right)(q-1)^2 \right] \\
&= (q-3) \left[ 2(q-1)^3(2q-1) + \left( q^3 - 2(q-1)^2 \right)(q-1)^2 \right] \\
&= q(q-3) \left( q-1 \right)^2 \left( q^2 + 2q - 2 \right),
\end{align*}
\]

and from (5.8) and (5.9), the formula (5.3) follows.

Now a similar analysis leads to the following:

\[
N(f, \alpha^{2r-1}) = \sum_{u \in \mathbb{F}_q^*} \left[ \sum_{(X_5,X_6,X_7) \in \mathbb{F}_q^3/ \langle X_5,X_6,X_7=0 \rangle} \left[ (q + \kappa(u)(X_5,X_6,X_7)) \right] + \sum_{(X_5,X_6,X_7) \in \mathbb{F}_q^3/ \langle X_5,X_6,X_7=0 \rangle} \left[ (q + \kappa(u)(X_5,X_6,X_7)) \right] \right] + (q-1) \left[ 2(q-1)^3(2q-1) + \left( q^3 - 2(q-1)^2 \right)(q-1)^2 \right].
\]

It follows that \( N(f, \alpha^{2r-1}) = (q-1)^3 q(q^2 + 2q - 2) \), which is the formula (5.4). \( \blacksquare \)

The result given in this Proposition indicates for particular cases of (1.3) that the obtained solution is a simple explicit formula.

V-2 A SECOND EXAMPLE

The example that we consider here shows us that the obtained formula for \( N(f,a) \) in Proposition IV-1 is more efficient than a trivial systematic computation of all the solutions. In the semi-explicit formula (5.12), the use of a classical machine is certainly required, but lead to easier computation involving sums and the obvious function \( \kappa \). This example is the following:

\[
(5.10) \quad (X_1 + X_6X_7X_8)(X_1 + X_2X_3 + X_6X_7X_8) (X_2X_3 + X_4X_5) = a
\]

We have:

\[
\begin{align*}
(k,m,n) &= (3,4,8), \\
A &= \begin{pmatrix} 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 \end{pmatrix}, \\
J_1 &= \{1\}, \quad J_2 = \{2,3\}, \quad J_3 = \{4,5\}, \quad J_4 = \{6,7,8\}
\end{align*}
\]

and the following result:
PROPOSITION V-2

The number $N(f,a)$ of solutions of (5.10) is:

(5.11) $N(f,0) = 3q^7 - 3q^6 + q^3(2q^2 - 2q + 1)$,

(5.12) $N(f,a) = q^3 \sum_{a_1,a_2 \in \mathbb{F}_q^*} (q + \kappa (a_2 - a_1)) \left( q + \kappa \left( a_1 - a_2 + \frac{a}{a_1a_2} \right) \right)$, 

if $a \neq 0$.

Proof

With obvious notations, Proposition IV-1 gives:

(5.13) $N(f,0) = \xi - \sum_{1 \leq i_1 < i_2 \leq 3} C_{i_1,i_2} + C_{1,2,3}$.

It is easily seen that $\xi = 3q^7 + (q - 1)q^5$. To compute $C_{1,2}$, we consider in (2.4a):

$$B_{[1,2]} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \quad C_{[1,2]} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$ 

Hence $B_{[1,2]}^{-1} C_{[1,2]} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \left( \sigma_{[1,2]}^{i,j} \right)_{i,j}$ and $C_{1,2} = q^5(2q - 1)$, as is easy to check. Likewise we obtain:

$C_{1,3} = C_{2,3} = q^6 + q^3 \sum_{(x_1,x_5) \in \mathbb{F}_q^2} \kappa (X_4X_5) = q^6 + q^4(q - 1)$ and $C_{1,2,3} = (2q - 1)^2 q^3$.

Then it is easy to see that (5.13) is equivalent to (5.11).

Now assume that $a \neq 0$. According to proposition IV-1 and (2.4a), we have:

$$N(f,a) = \sum_{a_1,a_2 \in \mathbb{F}_q^*} \sum_{X_6,X_7,X_8} \prod_{t=1}^3 q^{|J_{tr}| - 1} \left( 1 + \kappa \left( \sum_{j=1}^3 a_{t,j} - \sigma_{[1,2,3]}^{i,j}X_6X_7X_8 \right) \left( 1 - \left( \frac{q-1}{q} \right)^{|J_{tr}| - 1} \right) \right),$$

$B_{[1,2,3]}^{-1} a_{t,j} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}, \quad B_{[1,2,3]}^{-1} C_{[1,2,3]} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \left( \sigma_{[1,2,3]}^{i,j} \right)_{i,j}$.

A straightforward computation gives us formula (5.12).■

VI WEIGHT HIERARCHY OF A CLASS OF CODES

Let us denote by $J$ the partition $\{J_1, J_2, ..., J_m\}$ of $\{1, ..., n\}$. Following Cherdieu and Rolland in [2], we can introduce the map

$$c : E(q,n,J) \to \mathbb{F}_q^n$$

where $E(q,n,J)$ is the set of all multilinear polynomials with separated variables of the form $f(X_1, ..., X_n) = \sum_{j=1}^m a_j \prod_{\tau \in J_j} X_\tau$ where $a_1, ..., a_m \in \mathbb{F}_q$. The map $c$ is injective and its image $\text{Im} g$ is known as a multilinear
code with separated variables. We write $\text{Im} \ g = C (q, n, \mathcal{J})$. Proposition III.1 gives:

$$N (f, a) = q^{n-1} + (q-1) q^{n-1} \left( \prod_{\{j/a \neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|j|} \right) \right).$$

and we easily deduce:

**PROPOSITION VI-1**

The length of the code $C (q, n, \mathcal{J})$ is $q^n$, and its dimension is $m$. The weight $\omega (c (f))$ of the code-word $c (f)$, where $f (X_1, \ldots, X_n) = \sum_{j=1}^m a_j \prod_{\tau \in J_j} X_\tau$, is

$$\omega (c (f)) = q^{n-1} (q-1) \left( 1 - \prod_{\{j/a \neq 0\}} \left( 1 - \left( \frac{q-1}{q} \right)^{|j|} \right) \right),$$

and the minimum distance of $C (q, n, \mathcal{J})$ is

$$\text{dist} (C (q, n, \mathcal{J})) = q^{n-s} (q-1)^s$$

where $s = \max_{1 \leq j \leq m} (|J_j|)$.

If $C$ denotes a code with parameters $[n, r, d]$, and if $D$ denotes one of its subcodes, we recall that $\chi (D) = \{i \in \{1, \ldots, n\} / \exists (x_1, \ldots, x_n) \in D \text{ with } x_i \neq 0 \}$ is the support of $D$, and that the number $\omega (D) = |\chi (D)|$ of elements in $\chi (D)$ is the Wei weight of $D$ ([6] p. 1412). The $h^{\text{th}}$ minimum weight of $C$ is given by

$$d_h = d_h (C) = \min \{ \omega (D) / D \text{ subcode of } C \text{ with dim}(D) = h \}.$$

The minimum distance of $C$ is $d_1$, and the weight hierarchy of the code $C$ is the sequence $\{d_1, \ldots, d_r\}$.

If $D = \text{Vect} (f_1, \ldots, f_h)$ denotes the subcode of $C (q, n, \mathcal{J})$ generated by the polynomials $f_1, \ldots, f_h$ in $E (q, n, \mathcal{J})$, and if we write $\mathbb{F}_q^n = \{P_1, \ldots, P_{q^n}\}$, the Wei weight of $D$ is

$$\omega (D) = \left| \{i \in \{1, \ldots, q^n\} / \exists g \in D \text{ with } g (P_i) \neq 0 \} \right|$$

$$= \left| \left\{ i \in \{1, \ldots, q^n\} / \exists \lambda_1, \ldots, \lambda_h \in \mathbb{F}_q \sum_{j=1}^h \lambda_j f_j (P_i) \neq 0 \right\} \right|$$

$$= \left| \{i \in \{1, \ldots, q^n\} / \exists j \in \{1, \ldots, h\} \text{ and } f_j (P_i) \neq 0 \} \right|,$$

thus

$$\omega (D) = q^n - N (f_1, \ldots, f_h, 0, \ldots, 0).$$

Each polynomial $f_i$ ($1 \leq i \leq h$) can be written in the form $f_i (X_1, \ldots, X_n) = \sum_{j=1}^m a_{ij} \prod_{\tau \in J_j} X_\tau$ and the rank of the matrix $A = (a_{ij})$ is $k$ as soon as we assume that $D$ is of dimension $k$. We can therefore apply Proposition IV.1 to compute $\omega (D)$ and $d_h$.

**VII CONCLUSION**

As we have seen, Proposition IV.1 gives us a semi-explicit formula to compute the number $N (f_1, \ldots, f_h, 0, \ldots, 0)$ when the rank of the matrix $A$ is $k$, and it would be interesting to build an explicit program to do it, and compute the weight hierarchy of the code $C (q, n, \mathcal{J})$. This could be a numerical continuation of this work.
Several problems can arise from the present paper, and we gives here two ways for further investigations. First, it would be interesting to define a partition \( \{ \triangle_1, \triangle_2, \ldots, \triangle_\eta \} \) of \( \mathbb{F}_q^n \) and write the number \( N(f, a) \) in this way:

\[
N(f, a) = \sum_{j=1}^\eta N_{\triangle_j}(f, a) = \sum_{j=1}^\eta \sum_{a_1 \ldots a_k = a} N_{\triangle_j}(f_1, \ldots, f_k, a_1, \ldots, a_k),
\]

where \( N_{\triangle_j}(f, a) \) denotes the number of solutions of (1.3) in \( \triangle_j \), and \( N_{\triangle_j}(f_1, \ldots, f_k, a_1, \ldots, a_k) \) stands for the number of solutions of the system \( f_i(X_1, \ldots, X_n) = a_i \) (1 \( \leq i \leq k \)) in \( \triangle_j \). Specific partitions could lead us to less calculus. For instance,

\[
\triangle_1 = \{ (u_1, \ldots, u_n) \in \mathbb{F}_q^n / \exists (i_1, i_2, \ldots, i_m) \in J_1 \times J_2 \times \cdots \times J_m, (u_{i_1}, \ldots, u_{i_m}) = 0 \},
\]

\[
\triangle_2 = \{ (u_1, \ldots, u_n) \in (\mathbb{F}_q^*)^n / \forall \tau \in \{1, \ldots, m\} \forall (i, j) \in J_\tau \times J_\tau, u_i = u_j \text{ for all } \},
\]

lead to a trivial numbers and diagonal equations.

The second problem consists to study the more general case where:

\[
f = f_1 f_2 \cdots f_k,
\]

\[
f_i(X_1, \ldots, X_n) = \sum_{j=1}^s a_{ij} \prod_{\tau \in J^{(i)}_j} X_{\tau}, \quad i = 1, \ldots, k,
\]

\[
\{ J^{(1)}_1, J^{(1)}_2, \ldots, J^{(1)}_s \}, \quad \{ J^{(2)}_1, J^{(2)}_2, \ldots, J^{(2)}_s \}, \ldots, \quad \{ J^{(k)}_1, J^{(k)}_2, \ldots, J^{(k)}_s \}\] are partitions of \( \{1, \ldots, n\} \),

and to obtain semi-explicit formulas for the number of solutions of \( f(X_1, \ldots, X_n) = a \) in this case.

REFERENCES

[1] L. Carlitz, The number of solutions of some special equations in a finite field, Pacific J. Math. 4, 1954, pp. 207-217.

[2] J.-P. Cherdieu and R. Rolland, On hypersurfaces defined by a separated variables polynomial over a finite field, Arithmetic, geometry and coding theory (Luminy, 1993), De Gruyter, Berlin, 1996, pp. 35-43.

[3] J.-R. Joly, Equations et variétés algébriques sur un corps fini, Enseignement Math. 19, 1973, pp. 1-117.

[4] A. K. Lenstra, Factoring multivariate polynomials over finite fields, Journal of computer and system sciences 30, 1985, pp. 235-248.

[5] R.G. Van Meter, The number of solutions of certain systems of equations in a finite field, Duke Math. Journal 38, 1971, pp. 365-377.

[6] V. Wei, Generalized hamming weights for linear codes, IEEE Transactions on information theory 37, 1991, pp. 1412-1418.