Abstract. The primordial density fluctuation inevitably couples to all forms of matter via loop corrections and depends on the ambient conditions while inflation was ongoing. This gives us an opportunity to observe processes which were in progress while the universe was inflating, provided they were sufficiently dramatic to overcome suppression by powers of \((H/M_P)^2 \approx 10^{-9}\), where \(H\) is the Hubble scale during inflation and \(M_P\) is the Planck mass. As an example, if a primordial magnetic field was synthesized during inflation, as suggested by some interpretations of the apparently universal \(10^{-6}\) gauss field observed on galactic scales, then this could leave traces in inflationary observables. In this paper, I compute corrections to the spectrum and bispectrum generated by a varying electromagnetic coupling during inflation, assuming that the variation in this coupling is mediated by interaction with a collection of light scalar fields. If the mass scale associated with this interaction is too far below the Planck scale then the stability of perturbation theory can be upset. For the mass-scale which is relevant in the standard magnetogenesis scenario, however, the theory is stable and the model is apparently consistent with observational constraints.

Keywords: Inflation, Cosmological perturbation theory, Physics of the early universe, Quantum field theory in curved spacetime.
1. Introduction

Our understanding of the very early universe has settled down over the last several years to give, on balance, a broadly consistent picture. In this *concordance* model, an adiabatic perturbation is synthesized on superhorizon scales during an early epoch of inflation and is converted by gravitational collapse into the observed distribution of baryons and cold dark matter in the universe. It is generally accepted that this simple timeline is sufficient to explain the vast majority of observational data.

A zoo of optional components can be added to improve the fit to certain small anomalies in the data, or to provide an origin for features of our macroscopic world which are presently without a theoretical rationale. Examples of these optional features are scalar isocurvature modes (which may source non-gaussianities in the temperature anisotropy \[1, 2\]), vector modes (which may source statistical anisotropy \[3, 4, 5\]), or tensor gravitational waves. All of these must be subdominant to the adiabatic perturbation. We usually assume that these optional extras can be added or taken away without penalty. However, if they mediate sufficiently dramatic processes then we must remember that the observable adiabatic fluctuation will couple to all of them via loop corrections. In this paper the loop corrections arising from an especially interesting optional addition are studied, namely a varying electromagnetic† coupling which is mediated by interaction with a collection of light scalar fields. Among other effects, a varying coupling of this sort could be responsible for generating a primordial magnetic field. Such magnetic fields are required by observation \[7\] and although a number of possible formation mechanisms are known—see, for example, Ref. \[8\] for a non-inflationary example—their large-scale coherence and homogeneity hints at an inflationary origin parallel with the origin of the primordial density fluctuation.

What can we hope to learn from the study of such loop effects? One motivation is that, despite increasing high-quality experimental data which suggests the simplest predictions of inflation are a good fit for observation, it is still unclear whether inflation is the right model of microphysics or simply a good parametrization of an approximately gaussian, scale-invariant and adiabatic spectrum. One way to approach this question

† In fact, the coupling in question cannot be the electromagnetic coupling \(\alpha\), the fine structure “constant”, because inflation is usually supposed to take place at an energy scale far above the scale of electroweak symmetry breaking, and at such energies the electromagnetic field has not yet obtained a separate identity.

It is known that non-Abelian gauge fields are shielded from obtaining a perturbation during inflation by their strong self-interactions \[6\], and therefore the gauge coupling in question must be the \(U(1)\) Standard Model hypercharge field. A fluctuation in this field will communicate some perturbation to the electromagnetic field at electroweak symmetry breaking. However, the detailed identity of the Abelian gauge field in question is irrelevant for the question posed in the present paper. For simplicity, I will refer to this field as the “electromagnetic” field and its coupling as the “electromagnetic coupling” throughout, with the understanding that for applications to magnetogenesis the hypercharge field must be substituted and the resulting fluctuation projected onto the physical electromagnetic field. Similarly, this analysis will apply to any \(U(1)\) fields in the low energy theory of inflation whatever their microscopic origin, although not to any non-Abelian fields.
is to study the leading departures from gaussianity, as measured by the bispectrum and trispectrum. Non-gaussian statistics have received considerable attention over the last several years. Indeed, recent experimental results hint that it may be possible to discriminate among different microphysical models using this technique [9, 10]. Such studies fall into the general framework of non-linear perturbation theory. However, departures of the observable adiabatic fluctuation from exact gaussianity are not the only effect we can hope to probe using this tool. In a quantum mechanical world which includes gravity we must expect the adiabatic fluctuation to couple to all other degrees of freedom which were light enough to be excited during the inflationary era. This point has been emphasized in recent work by Weinberg [11]. If inflation does more for us than merely generate the observed density fluctuation—and if we are lucky—it may be possible to probe whatever other processes were in progress during inflation, provided they were sufficiently dramatic to leave traces in the inflationary observables. The analysis given in this paper can be thought of as an exploratory step in this direction.

There are other motivations. If we are to have confidence in our predictions, it is necessary to maintain control over the theoretical tools which we use. This will become of increasing significance as the data improve and we pass from a qualitative to a quantitative description of the earliest times, excluding some models as possible theories of the early universe while promoting others as a better match for observation. The key criterion here is stability of the perturbative series which is used to extract observables from the Lagrangian, a question which has already attracted attention in the literature [12, 13]. Several potential sources of instability exist. Increasing orders of perturbation theory are typically suppressed by powers of the ratio \((H/M_P)^2 \approx 10^{-9}\), where \(H\) is the energy scale of inflation and \(M_P\) is the Planck mass. However, perturbation theory can in principle contain instabilities which scale like a positive power of the scale factor \(a(t) \approx \exp(Ht)\) during inflation.

If such “fast” divergences are present then they will rapidly overwhelm any powers of \((H/M_P)^2\) and render perturbation theory unstable after a short time—generally too short to be of any use in extracting predictions for relics of the early universe which are visible at the present day. Alternatively there may be large corrections which come from a sensitivity to physics in the ultra-violet, or from some other hierarchy which exists in the theory. The first possibility was studied by Armendariz-Picon et al. [13], whereas an example of the latter, which was studied by Leblond & Shandera [12], is the relative hierarchy \(c_s^{-2} \geq 1\) between the speed of sound and the speed of light. Whatever the source of large hierarchies which overwhelms the smallness of \((H/M_P)^2\), it is important to emphasize that an instability in perturbation theory does not necessarily imply that anything untoward is taking place. It may simply mean that we need to find a better description of the process in question.

The possibility of fast instabilities in perturbation theory was considered by Weinberg [14, 15] (see also Chaicherdsakul [16]), who gave a criterion according to which it is possible to decide whether such instabilities are forbidden. Even where this is the case, it does not necessarily follow that perturbation theory is convergent because
Weinberg's theorem does not exclude the possibility of much slower divergences: for any fluctuation which is outside the horizon, these divergences scale with the number of e-folds of expansion since the time of horizon exit.

Whether divergences are fast or slow, however, the interpretation is the same. When we expand an expectation value of some operators as a series in a loop-counting parameter or the slow-roll parameters, we are developing a series expansion based on the cut-off associated with the theory. The role of the cut-off is played by the time at which we wish to evaluate the expectation value—which is usually chosen to be at the end of inflation, or some similar time where we wish to use the expectation value as an initial condition for classical cosmological perturbation theory in the later universe. The behaviour of any expectation value as a function of this cut-off is merely its time dependence. The problem arises because truncating the series at any finite order gives the appearance of divergences. If we compute an answer which is superficially divergent in this way, then we must find some other method to compute the time dependence of the expectation value before growing secular terms take perturbation theory out of our control.

This point of view leads to a technique of computation in which we can separate calculations into a quantum initial condition [17], for which we need all the machinery of the so-called in–in (or Schwinger–Keldysh) formalism, and a subsequent classical evolution for which to a good approximation we need only the classical, homogeneous evolution equations. The details of this approach have been developed by many authors [18, 17, 19, 20, 11, 21]. There is a possible difficulty if we allow the inflating volume to become too large, because we may then encounter a source of quantum divergences which could invalidate the use of classical evolution equations even after horizon exit [22, 23, 24, 25, 26, 27, 28], but provided we work within some patch of spacetime not much larger than the size of the present horizon such effects are likely to be negligible.

Most recent work on studying non-gaussianities from inflation has centred on the evolution subsequent to horizon exit [29, 30, 31, 32], whereas the quantum initial condition has received comparatively less attention [33, 34, 17, 35]. There is a good reason for this imbalance: although there are known to be controlled examples where significant non-linearity can be generated outside the horizon [32, 36, 30], it is very hard (with canonical kinetic terms) to construct a controlled calculation in which a significant effect arises from the initial condition. Indeed, the most useful tool for extracting predictions from the underlying quantum field theory—that is, the slow-roll expansion—generally has the effect of forcing correlations to be very small. A second interpretation of the analysis given in this paper is an example in which the initial condition is modified, by including a coupling to high-energy virtual quanta which belong to the electromagnetic field. As can be expected, it will not be possible to control the calculation in the regime where this modification dominates the initial condition. However, we will be able to obtain a bound on the characteristics of the interaction among scalar and electromagnetic quanta which guarantees that the calculation is not taken beyond our control.
Throughout this paper, we use units in which $\hbar = c = 1$ and the reduced Planck mass is set equal to unity, giving $M_P \equiv (8\pi G)^{-1/2} = 1$, where $G$ is Newton’s gravitational constant. The background space time is de Sitter space with flat spatial slices and metric
\begin{equation}
    ds^2 = -dt^2 + a(t)^2 dx \cdot dx,
\end{equation}
in the $(-, +, +, +)$ sign convention. Some formulae are more conveniently written in terms of conformal time, defined locally by the rule $dt = a(t) d\eta$ and given explicitly by the quadrature $\eta = \int_\infty^t dt'/a(t')$. Spacetime indices are labelled with Latin indices $\{a, b, c, \cdots\}$; purely spatial indices are labelled with indices $\{i, j, k, \cdots\}$; and indices in the space of scalar fields are given Greek labels $\{\alpha, \beta, \gamma, \cdots\}$.

Purely spatial vectors such as $x$ or $k$ are written in bold face and an infix dot denotes index contraction with the flat background spatial metric, so that $x \cdot k \equiv \sum_i x_i k_i \equiv x_i k_i$, where the summation symbol will usually be omitted. Note that both indices are lowered. This convention is used to interpret exponentiation of any square spatial matrix $\gamma_{ij}$, giving the rule $\exp(\gamma)_{ij} \equiv \sum_{n=0}^\infty (\gamma^n)_{ij}/n!$. Repeated spacetime indices in complementary raised and lowered positions are summed using the full spacetime metric $g_{ab}$ according to the Einstein convention, as usual; this convention also applies to raised and lowered spatial indices with the substitution of the full spatial metric in contractions.

The model used as an example in this paper is Einstein gravity coupled to some collection of light scalar fields $\phi^\alpha$ and a single $U(1)$ gauge field $A_a$. The $U(1)$ gauge field is taken to have a kinetic term of the form $\lambda(\phi) F_{ab} F_{ab}$, where $F_{ab} \equiv \partial_a A_b - \partial_b A_a$ is the gauge-invariant field strength. The coupling, $\lambda(\phi)$, is determined by the vacuum expectation values of some or all of the light scalars. This model is studied in §2 where the interactions among the scalar, tensor and gauge field perturbations are derived. In §3 the magnetogenesis mechanism $[37, 38, 6, 39]$ is briefly reviewed, following an analysis by Bamba & Sasaki. In §§4–5 I compute the leading loop correction—for the spectrum and bispectrum—which comes from scalar fluctuations mixing with virtual quanta of the gauge field. The paper concludes with a discussion in §6. An auxiliary calculation of a simple pure scalar loop correction is given in Appendix A to aid comparison of the methods used in the present paper with those of other authors.

2. Scalar–magnetic couplings in the inflationary Lagrangian

Our starting point is Einstein gravity coupled to a collection of scalar fields $\phi^\alpha$, some of which are responsible for inflation, with the addition of a $U(1)$ gauge field $A_a$.

2.1. The gauge-fixed Einstein–scalar–vector action

As discussed in §1 the normalization of the gauge field is taken to be controlled by some non-canonical coupling $\lambda(\phi)$ which is determined by the vacuum expectation values of
some subset of the scalar fields. The action is
\[ S = \frac{1}{2} \int d^4x \sqrt{-g} \left( R - \nabla_a \phi^\alpha \nabla^a \phi_\alpha - 2V - \frac{1}{2} \lambda(\phi) F^{ab} F_{ab} \right), \]  
(2.1)
where the $U(1)$ field strength is defined by $F_{ab} \equiv \partial_a A_b - \partial_b A_a$ and $R$ is the spacetime Ricci scalar. The potential $V(\phi)$ is any reasonably smooth function which will generically depend on all the $\phi^\alpha$ and is arbitrary except that in order for the analysis which follows to apply it must support an epoch of inflation, at least for some range of values of the scalar fields.

The background spacetime is taken to be homogeneous and isotropic, so the gauge field has no expectation value, up to configurations which are pure gauge. One can therefore assume that $A_a$ is a perturbation, which will generically couple to the scalar and gravitational degrees of freedom in Eq. (2.1). To study this system of coupled perturbations it is especially convenient to write the spacetime metric (including the effect of gravitational fluctuations) in the so-called Arnowitt–Deser–Misner (ADM) form,
\[ ds^2 = -N^2 dt^2 + h_{ij} (dx^i + N^i dt)(dx^j + N^j dt), \]  
(2.2)
where $N$ (the “lapse function”) and $N^i$ (the “shift vector”) are not dynamical degrees of freedom, but instead are determined by constraint equations. The spatial metric $h_{ij}$ contains propagating tensor modes, and depending on the gauge it may also contain propagating scalar modes. After gauge-fixing $h_{ij}$, and solving the constraints, $N$ and $N^i$ can be written in terms of the propagating degrees of freedom in $h_{ij}$ and the $\phi^\alpha$. This is a great simplification in concrete calculations.

Consider first the gauge sector. The ADM decomposition is based on a so-called “3+1” split of spacetime into three spatial dimensions and one timelike dimension. Many formulae are simplified by making an analogous decomposition of the vector potential, writing $A_a \equiv (\rho, \omega_i)$ where $\rho$ is the timelike component, and $\omega_i$ is a spatial 3-vector. It will transpire that $\rho$ is not a dynamical field, but is removed by a constraint associated with the gauge invariance of $A_a$. Once $\rho$ has been removed a further gauge fixing condition must be applied to $\omega_i$, which leaves the expected two physical polarizations of a massless gauge boson.

When expressed in terms of $\rho$ and $\omega_i$, the part of the action involving the gauge field can be written
\[ S \supset \frac{1}{2} \int d^3x dt \sqrt{h} \left\{ -\frac{\lambda}{2} h^{mn} h^{ij} f_{ij} f_{mn} + \frac{\lambda}{N^2} h^{ij} \partial_i \omega_j \partial_j \omega \right\}, \]  
(2.3)
where we have defined a useful quantity $\partial_i \omega$ by the rule
\[ \partial_i \omega \equiv \partial_i \omega_i - \partial_\rho \omega_i + f_{ij} N^j \]  
(2.4)
(denoting a time derivative with respect to $t$ by an overdot) and $f_{ij}$ is the spatially gauge-invariant 3-vector field strength, $f_{ij} \equiv \partial_i \omega_j - \partial_j \omega_i$. The original gauge invariance associated with $A_a$ corresponding to $U(1)$ gauge transformations was $A_a \mapsto A_a + \partial_a \Lambda$.
Magnetogenesis and the primordial non-gaussianity

for some arbitrary spacetime-dependent function \( \Lambda \). Under such a transformation, \( \rho \) and \( \omega_i \) undergo separate transformations determined by

\[
\rho \mapsto \rho + \dot{\Lambda},
\]

\[
\omega_i \mapsto \omega_i + \partial_i \Lambda.
\]

It follows that \( \bar{\partial}_i \omega \) is gauge invariant, and therefore so is any action built out of \( f_{ij} \) and \( \bar{\partial}_i \omega \) alone.

The ADM action for \( \rho \) and \( \omega_i \) is singular, because the lagrangian is degenerate along pure gauge directions. It therefore cannot be quantized as it stands, but must be put into a form suitable for quantum mechanical calculations by adding a term of the form \( s(c^* f) \), where \( s \) is a so-called Slavnov operator,

\[
s \equiv b(\delta \omega_i) \left( \frac{\delta}{\delta c^*} - h \frac{\delta}{\delta c} \right),
\]

\( h \) is an auxiliary field, and \( b, c^* \) are a pair of anti-commuting ghost fields which nevertheless obey Bose-Einstein statistics. The function \( f \) is chosen to remove the degeneracy along gauge directions, but is otherwise essentially arbitrary provided that it is not invariant under gauge transformations. It is usually described as a gauge-fixing function. For the purposes of the present paper the most appropriate choice is an analogue of the Lorentz–Coulomb gauge, specified by \( f = \partial_i \omega_i \). It might have been thought that the covariant condition \( f' = \partial_i \omega_i \) would be more appropriate in order to maintain manifest spatial covariance. Ultimately, however, we will be performing calculations in a version of the interaction picture in which it is simple to compute with \( f \), but more complicated to compute with \( f' \). It is immaterial whether we choose \( f \) or \( f' \), and since there are no serious consequences associated with losing manifest spatial covariance we will stick with \( f \). (If desired, the spatially covariant action and Feynman rules can be obtained from those given here by the replacement \( \delta^{ij} \mapsto h^{ij} \) in the gauge-fixing terms written in Eqs. (2.8)–(2.25) below.) The total action is invariant under a quantized form of the original gauge symmetry, usually known as a BRST symmetry, which is generated by \( s \). In the classical theory, the gauge condition is enforced by solving the constraint \( f = 0 \) for one linear combination of the components of \( \omega_i \). This removes one degree of freedom from the theory.

After inserting the action into a path integral and appropriately integrating out the auxiliary field \( h \), one finds that the ghosts \( b \) and \( c^* \) decouple and contribute only to an irrelevant overall normalization. The result can be written as a path integral over the action

\[
S = \frac{1}{2} \int d^3 x \, dt \, \sqrt{h} \left( NB_1 + \frac{1}{N} B_{-1} \right),
\]

where the quantities \( B_1 \) and \( B_{-1} \) are defined by

\[
B_1 \equiv R - h^{ij} \partial_i \phi^a \partial_j \phi_a - 2V - \frac{\lambda}{2} h^{im} h^{jn} f_{ij} f_{mn} - \frac{1}{\xi} \delta^{ij} \delta^{mn} \partial_i \omega_j \partial_m \omega_n
\]

\[
B_{-1} \equiv E^{ij} E_{ij} - E^2 + \pi^a \pi_a + \lambda h^{ij} \bar{\partial}_i \omega \bar{\partial}_j \omega.
\]

† Recall that in the summation convention which is being used here, \( \partial_i \omega_i \equiv \sum \partial_i \omega_i \).
In these formulae, $R$ is the spatial Ricci curvature associated with $h_{ij}$; the quantities $\pi^\alpha \equiv \dot{\phi}^\alpha - N^j \partial_j \phi^\alpha$ are a collection of momenta associated with the scalar fields; and $E_{ij} = \dot{h}_{ij}/2 - \nabla_i (N^j)$ is the gravitational momentum, where $\nabla_i$ is the covariant derivative compatible with $h_{ij}$.

### 2.2. The constraint equations

The physical degrees of freedom in the action (2.8) are a collection of propagating modes associated with the scalars, $\phi^\alpha$, together with modes arising from the components of $h_{ij}$. These are the fields whose time derivatives appear in the action. On the other hand, the fields $N$, $N^i$ and $\rho$ do not appear in the action with any time derivatives, and therefore are associated with constraints. These constraints can be satisfied and the unwanted degrees of freedom eliminated by simply solving for $N$, $N^i$ and $\rho$ in terms of the other fields in the system, and substituting the result back into the action.

The constraints associated with $N$ and $N^i$ are well-known, and are modified here only by the presence of a component in the action associated with a gauge boson. The $N$ constraint is

$$R - h^{ij} \partial_i \phi^\alpha \partial_j \phi^\alpha - 2V - \frac{\lambda}{2} h^{ij} h^{mn} f_{ij} f_{mn} - \frac{1}{\xi} \delta^{ij} \delta^{mn} \partial_i \omega_j \partial_m \omega_n$$

$$- \frac{1}{N^2} \left( E_{ij} E^{ij} - E^2 + \pi^\alpha \pi_\alpha + \lambda h^{ij} \bar{\partial}_i \omega \bar{\partial}_j \omega \right) = 0,$$

and the $N^i$ constraint is

$$\nabla^j \left( \frac{1}{N} (E_{ij} - h_{ij} E) \right) = \frac{\pi^\alpha}{N} \partial_j \phi^\alpha - \frac{\lambda}{N} h^{jk} \bar{\partial}_i \omega f_{kj}.$$  \hspace{1cm} (2.12)

On the other hand, from integrating out $\rho$ we obtain a very simple constraint

$$\nabla_j \left( \frac{\lambda}{N} h^{ij} \bar{\partial}_i \omega \right) = 0.$$  \hspace{1cm} (2.13)

Consider first the $N^i$ constraint. We take the background field configuration to be spatially homogeneous and isotropic, so that the $\phi^\alpha$ are functions of $t$ only, with small perturbations $\delta \phi^\alpha$ which satisfy the smallness condition $|\delta \phi^\alpha| \ll |\phi^\alpha|$. If inflation has been ongoing for an exponentially large number of e-folds then this may be a poor approximation over the whole inflating volume, owing to back reaction effects which can cause large fluctuations in the scalar expectation values between widely separated regions. However, in any region of spacetime in the neighbourhood of exit from inflation this field theory is likely to be an accurate effective description.

To parametrize the degrees of freedom associated with the spatial metric, we write $h_{ij} = a^2(t)(e^\gamma)_{ij}$, where $\gamma_{ij}$ is a spatial $3 \times 3$ matrix whose components are taken to be of the same magnitude as $\delta \phi^\alpha$. One then aims for a perturbative solution in $\delta \phi^\alpha$ and $\gamma_{ij}$, with the gauge field $\omega_i$ also taken to be perturbative and of the same formal magnitude. This implies that the gauge field does not modify the background evolution of the scalar
fields. We define quantities \( \alpha_n, \vartheta_n \) and \( \beta_{nj} \) by writing

\[
N = 1 + \sum_{n=1}^{\infty} \alpha_n \quad (2.14)
\]

\[
N_i = \sum_{n=1}^{\infty} (\partial_i \vartheta_n + \beta_{nj}) \quad ,
\]

where each of \( \alpha_n, \vartheta_n \) and \( \beta_{nj} \) is taken to contain exactly \( n \) powers of the perturbations, and \( \partial_j \beta_{nj} = 0 \) for all \( n \).

At \( \mathcal{O}(1) \), the \( N \) constraint (2.11) gives the Friedmann equation for the background,

\[
3H^2 = \frac{1}{2} \dot{\phi}^\alpha \dot{\phi}_\alpha + V.
\]

At \( \mathcal{O}(\delta \phi) \), one obtains an equation for the scalar component of the shift vector, \( \vartheta_1 \),

\[
\frac{4H}{a^2} \partial^2 \vartheta_1 = -2V_{\alpha \delta} \phi^\alpha - 2\dot{\phi} \delta \phi_\alpha + 2\alpha_1 (-6H^2 + \dot{\phi}^\alpha \dot{\phi}_\alpha). \quad (2.17)
\]

To obtain equations for \( \alpha_1 \) and the vector component \( \beta_{1j} \) one must return to the shift constraint, Eq. (2.12). One finds \( \beta_{1j} = 0 \) and

\[
\alpha_1 = \frac{\dot{\phi}^\alpha \delta \phi_\alpha}{2H}. \quad (2.18)
\]

For the purpose of computing interactions between the \( \delta \phi^\alpha, \gamma_{ij} \) and \( \omega_i \), together with their interactions. Any terms in the action which are linear in the perturbations vanish as a consequence of the background equations of motion. The leading non-trivial terms are therefore quadratic. Any theory defined by purely quadratic terms is free and gives rise to gaussian statistics. Therefore, the leading correction to gaussian statistics comes from interaction terms at cubic order or above. The details of these interactions can be calculated by treating them as small perturbations to the quadratic pieces, which we suppose still supply the dominant
evolution. This formulation is equivalent to the interaction picture in the canonical formalism.

At quadratic order, the perturbations $\delta \phi^\alpha$, $\gamma_{ij}$ and $\omega_i$ decouple and do not communicate with each other. The action therefore breaks into a sum of terms for each fluctuation which can be treated separately.

Consider first the fluctuations in the light scalar fields $\delta \phi^\alpha$, which are described by the action

\[
S_2 \supseteq \frac{1}{2} \int \mathrm{d}^3 x \mathrm{d} t \ a^3 \left\{ \delta \dot{\phi}^\alpha \delta \dot{\phi}^\alpha - \frac{1}{a^2} \partial_i \gamma^\alpha \partial_i \gamma^\alpha - V_{\alpha\beta} \delta \phi^\alpha \delta \phi^\beta - \frac{2 \dot{\phi}^\alpha}{a^2} \partial_j \gamma^\alpha \partial_j \gamma^\alpha \\
+ \alpha_1 \left[ - \frac{4H}{a^2} \partial^2 \gamma_1 - 2 V_{\alpha} \delta \phi^\alpha - 2 \dot{\phi}^\alpha \delta \phi^\alpha + \alpha_1 (-6H^2 + \dot{\phi}^\alpha \dot{\phi}^\alpha) \right] \right\}.
\] (2.20)

In comparison the tensor modes $\gamma_{ij}$ obey a very simple action, containing only the minimal kinetic term

\[
S_2 \supseteq \frac{1}{8} \int \mathrm{d}^3 x \mathrm{d} t \ a^3 \left\{ \dot{\gamma}_{ij} \dot{\gamma}_{ij} - \frac{1}{a^2} \partial_k \gamma_{ij} \partial_k \gamma_{ij} \right\}.
\] (2.21)

The action controlling the evolution of the gauge field satisfies

\[
S_2 \supseteq \frac{1}{2} \int \mathrm{d}^3 x \mathrm{d} t \ a \left\{ \lambda (\dot{\omega}_i - \partial_i \rho_1) (\dot{\omega}_i - \partial_i \rho_1) - \frac{\lambda}{2a^2} f_{ij} f_{ij} - \frac{1}{\xi a^2} \partial_i \omega_i \partial_j \omega_j \right\}.
\] (2.22)

### 2.4. Interactions: Cubic terms

The leading interactions among the $\gamma_{ij}$ and a single scalar degree of freedom were obtained by Maldacena, and are not affected by the presence of a gauge field. However, in addition to the terms found by Maldacena there are now cubic interactions which involve gauge bosons together with the $\delta \phi^\alpha$ or $\gamma_{ij}$. Because the spacetime gauge connexion $A_a$ appeared in the original action Eq. (2.1) quadratically, all these cubic interactions involve two gauge bosons and only a single other field. There is therefore a term which describes the interaction of two gauge fields with a scalar,

\[
S_3 \supseteq \frac{1}{2} \int \mathrm{d}^3 x \mathrm{d} t \ a^3 \left\{ \frac{\lambda_{\alpha}}{a^2} \dot{\phi}^\alpha \left( \dot{\omega}_i \dot{\omega}_i - 2 \dot{\omega}_i \partial_i \rho_1 + \partial_i \rho_1 \partial_i \rho_1 \right) + \frac{2\lambda}{a^4} \dot{\omega}_i f_{ij} \dot{\omega}_j - \frac{1}{\xi a^4} \partial_i \omega_i \partial_j \omega_j - \frac{\lambda_{\alpha}}{a^2} \dot{\phi}^\alpha f_{ij} f_{ij} \\
+ \alpha_1 \left[ - \frac{\lambda}{2a^4} f_{ij} f_{ij} - \frac{1}{\xi a^4} \partial_i \omega_i \partial_j \omega_j - \frac{\lambda}{a^4} \left( \dot{\omega}_i \dot{\omega}_i - 2 \dot{\omega}_i \partial_i \rho_1 + \partial_i \rho_1 \partial_i \rho_1 \right) \right] \right\} (2.23)
\]

There is also an interaction between two gauge fields and a single $\gamma_{ij}$,

\[
S_3 \supseteq \frac{1}{2} \int \mathrm{d}^3 x \mathrm{d} t \ a^3 \left\{ - \frac{\lambda}{a^2} \dot{\gamma}_{ij} \left( \dot{\omega}_i \dot{\omega}_j - 2 \dot{\omega}_i \partial_j \rho_1 + \partial_i \rho_1 \partial_j \rho_1 \right) + \frac{\lambda}{a^4} \dot{\gamma}_{ij} f_{mi} f_{mj} \right\}.
\] (2.24)

In principle there is also a self-interaction between three gauge bosons, described by the term

\[
S_3 \supseteq \frac{1}{2} \int \mathrm{d}^3 x \mathrm{d} t \ a^3 \left\{ - \frac{2\lambda}{a^2} \omega_i \partial_i \rho_2 \right\}.
\] (2.25)

We will see in [3] below that it is possible to make a choice of gauge in which this interaction only involves the unphysical polarization of $\omega_i$. Therefore, in this gauge it decouples from all physical amplitudes, although it may remain present in more general gauges.
3. Magnetogenesis

In this section the process of magnetogenesis is briefly reviewed, with the aim of establishing the relevant notation and formulae which will be required for a calculation of loop corrections in §§4–5. Our starting point is the assumption that the magnetic field arises from the coupling $\lambda F^{ab} F_{ab}$ between the gauge field and the scalar sector. This calculation has been given in some generality by Bamba & Sasaki [40], whose methods we follow. (See also Refs. [41, 42, 43].)

When calculating processes involving exchange of virtual gauge bosons in Minkowski space it is usually a useful calculational check to leave the constant $\xi$ arbitrary. Indeed, since physical quantities do not depend on $\xi$ it must cancel out in any correct computation. Unfortunately, when attempting to compute the gauge field propagator in a time-dependent background, the presence of the $\xi$ term is an obstruction to solving the propagator equation. To simplify the process, it is helpful to take the limit $\xi \to 0$. For a gauge-fixing functional $f$ this directly enforces the constraint $f = 0$ in order that the action remain non-singular. Hence, instead of integrating over the three components of $\omega_i$, we must make the decomposition

$$\omega_i(x, t) = \sum_{\sigma \in \pm} \int \frac{d^3k}{(2\pi)^3} e^\sigma_i(k) \omega^\sigma(k, t) e^{ikx},$$

(3.1)

where the $e^\sigma$ are so-called polarization vectors, labelled by a two-valued discrete index $\sigma$. The path integral should include only the two physical polarizations $\omega_{\pm}$. The polarization vectors are chosen in such a way that $k \cdot e^\sigma(k) = 0$ for each $\sigma$, and are normalized so that

$$e^\sigma_i(k) \cdot e^{\sigma'}_j(k)^* = \delta^{\sigma\sigma'}$$

(3.2)

$$\sum_{\sigma \in \pm} e^\sigma_i(k) e^{\sigma'}_j(k)^* = \delta_{ij} - \frac{k_i k_j}{k^2}.$$  

(3.3)

After carrying out this reduction, note that the apparent three-boson interaction (2.25) couples only to the unphysical degree of freedom in $\omega_i$. It can formally be removed after integrating by parts. It follows that this interaction is not physical. Note also that $\rho_1 = 0$ in this gauge, so all $\rho$ terms drop out of the action for $\omega_i$ up to cubic order.

With these choices, the propagator for the gauge field is obtained by inverting the differential operator which appears under the integral in Eq. (2.22). Suppose we write

$$\langle \omega_\sigma(k_1, t) \omega_\sigma'(k_2, t') \rangle = (2\pi) \delta(k_1 + k_2) \delta_{\sigma\sigma'} G_k(t, t').$$

(3.4)

The equation which determines $G_k$ is

$$\ddot{G}_k + \left( H + \frac{\dot{\lambda}}{\lambda} \right) \dot{G}_k + \frac{k^2}{a^2} G_k = -\frac{i}{a \lambda} \delta(t - t'),$$

(3.5)

together with the boundary condition that in the limit $k/aH \to \infty$, where the fluctuation corresponding to this wavenumber cannot feel the curvature of spacetime, $G$ should approach the corresponding mode function from Minkowski space. This
boundary condition plays an important role in the calculation. It corresponds to the stipulation that we begin with conventional Minkowski space quantum field theory in the ultra-violet, and then attempt to determine what this implies for fluctuations deep in the infra-red. Although it is possible to make more general choices of field theory in the ultra-violet, the assumption of flat space field theory is minimal and any admixture of different ultra-violet physics is usually subdominant to the Minkowski result. When we come to define what we mean by loop integrals in de Sitter space, which are also part of the specification of the ultra-violet behaviour of the theory, it will be necessary to take care that our definition does not destroy the property that we begin with flat space field theory at very high energies.

Eq. (3.5) cannot be solved exactly for a general choice of $\lambda(\phi)$. Instead, one can obtain a solution of Wentzel–Kramers–Brillouin (WKB) type which is valid inside the horizon, and can be matched onto a long wavelength solution which is valid outside the horizon. This is equivalent to using the flat space boundary condition deep inside the horizon to determine the size of fluctuations at horizon exit, and then using this as an initial condition for a classical superhorizon calculation.

The solutions can be written most simply in terms of the conformal time coordinate $\eta$. In this variable, the relevant WKB solution is

$$G_k(\eta, \eta') = \frac{1}{2k} \frac{1}{\sqrt{\lambda(\eta)\lambda(\eta')}} \times \begin{cases} e^{ik(\eta - \eta')} & \eta < \eta' \\ e^{ik(\eta' - \eta)} & \eta' < \eta \end{cases}. \quad (3.6)$$

It follows that the power spectrum of each polarization at horizon exit satisfies

$$P_* = \frac{k^2}{4\pi^2} \frac{1}{\lambda_*}. \quad (3.7)$$

where ‘$*$’ denotes evaluation at the time the mode with wavenumber $k$ exited the horizon. The $k$ dependence gives the spectrum a steep blue tilt, which means that if it remains unprocessed by new physics in the superhorizon regime it must have an essentially negligible magnitude on observable scales.

It is sometimes said that a canonically normalized vector field does not receive a perturbation from inflation, based on the observation that positive and negative frequencies of the gauge field are not mixed as the universe expands [44, 45]. It follows that if asymptotic in and out vacua can be defined there is no particle creation in the transition between the in- and out-vacuum. In the computation of inflationary observables, however, there is usually no natural out region where a notion of particles make sense, nor any need to invoke such a region. Instead, we are interested primarily in whether expectation values of operators behave coherently over many e-folds of expansion, and in this sense a canonically normalized gauge field receives a fluctuation in exactly the same way as any light bosonic field. As we have already observed, however, the fluctuation which is imprinted in the precisely canonical case is very blue and entirely negligible on cosmologically relevant scales. One can think of this as a consequence of the fact that canonically coupled vector fluctuations must redshift like radiation, giving an extremely strong suppression for scales which exited the horizon early and have been
redshifting for longer. Once modes re-enter the horizon they begin to oscillate and expectation values of their associated operators lose their coherence, which is consistent with Parker’s observation that there is no asymptotic particle creation in this model \[44\].

The situation changes in the presence of the non-canonical coupling $\lambda(\phi)$. In this case Eq. (3.7) suggests that if $\lambda_* < 1$ the power in fluctuations of the gauge field has been amplified in comparison with a vanilla model where $\lambda = 1$ for all time. However, this is misleading. Since the energy–momentum tensor associated with the gauge field is proportional to $\lambda$, when we compute the energy density stored in $\omega_i$ at horizon exit we obtain an answer which is independent of $\lambda_*$. It follows that there is a subtle distinction between this method of generating perturbations and the familiar case of scalar perturbations. In the simplest model of scalar perturbations, we generate fluctuations of the correct magnitude at horizon exit which are then conserved until horizon re-entry. In the case of $\omega_i$ we do not make the physical fluctuations at horizon exit any larger whether or not we include a coupling $\lambda$. It is evolution of $\lambda$ after horizon crossing which amplifies fluctuations in the gauge field and prevents them redshifting to zero; in this respect, the mechanism is similar to the curvaton example for purely scalar perturbations. We should therefore expect physical quantities to involve the ratio, $\lambda_2/\lambda_1$, of $\lambda$ at different times $\eta_1$ and $\eta_2$, which can be made large only if $\lambda$ evolves sufficiently strongly that there is a large hierarchy between its values at these times. In the limit of an observation made instantaneously at a moment in time, we can conclude that the physical effect must be proportional to $\lambda'/\lambda$, or a higher derivative, where $'$ denotes differentiation with respect to the conformal time. The power spectrum and bispectrum measured at horizon exit are examples of observations made instantaneously, and we shall see in §§4–5 below that they come proportional to powers of $\lambda'/\lambda$. An alternative means of breaking conformal invariance was considered in Ref. \[46\].

Once a mode has left the horizon, its evolution is governed by Eq. (3.5) in the limit $k/aH \to 0$. Discarding powers of gradients, the homogeneous solution for each polarization mode takes the form

$$\omega(\eta, x) \equiv \omega_*(x) + \lambda_* \omega'_*(x) \int_\eta^\eta d\tau \frac{d\tau}{\lambda(\tau)},$$

which depends on the value of $\omega$ and its derivative $\omega'$ (where a prime $'$ denotes differentiation with respect to conformal time) on any initial hypersurface, $\eta = \eta_*$, provided that all relevant modes have left the horizon at that time. For a mode corresponding to a single wavenumber $k$ we can take this hypersurface to be the time of horizon exit, and the power spectra of $\omega$ and $\omega'$ can be extracted from Eq. (3.6). If $\lambda$ is increasing or decreasing less fast than $\eta$ then the integral in Eq. (3.8) converges and each polarization is constant outside the horizon up to terms which decay exponentially fast in cosmic time. On the other hand, if $\lambda$ decreases faster than $\eta$ then the integral diverges and each polarization grows rapidly. These possibilities correspond to a decreasing gauge

† I would like to thank David Lyth for helpful correspondence on this question.
Magnetogenesis and the primordial non-gaussianity

coupling (or one which increases less fast than $a^{-1}$), or an increasing gauge coupling, respectively.

At any time $\eta$ at which Eq. (3.8) applies, the proper electric and magnetic energy densities were computed by Bamba & Sasaki and are given by [40, 47]

$$
B_i \equiv \sqrt{\lambda(\eta)} \frac{\epsilon_{ijk} \partial_ j \omega_k(\eta)}{a(\eta)^2},
$$

(3.9)

$$
E_i \equiv \sqrt{\lambda(\eta)} \frac{a(\eta)^2}{\lambda} \omega'_i = \sqrt{\lambda(\eta)} \frac{d}{d\eta} \omega_i,
$$

(3.10)

where $\epsilon_{ijk}$ is the alternating tensor in three dimensions and the normalization has been chosen so that the electromagnetic energy density is given by $\rho_{EM} = (B^2 + E^2)/2$, as usual. It follows from Eqs. (3.9)–(3.10), Eq. (3.8) and Eq. (3.6) that the power spectrum of the proper magnetic energy density is

$$
\mathcal{P}_B = \frac{1}{4\pi^2} \left( \frac{k}{a} \right)^4 \left| 1 + \left( i k - \frac{1}{2} \frac{\lambda'}{\lambda} \right) \int^\eta_\tau \frac{d\tau}{\lambda(\tau)} \right|^2,
$$

(3.11)

and the power spectrum of the proper electric energy density is

$$
\mathcal{P}_E = \frac{1}{4\pi^2} \left( \frac{\lambda}{a^4} \right) \left| ik - \frac{1}{2} \frac{\lambda'}{\lambda} \right|^2.
$$

(3.12)

Bamba [47] (see also Giovannini [42] and Martin & Yokoyama [48]) observed that if $\lambda$ is increasing (or decreasing less fast than $a^{-1}$), then the integral is dominated by its lower limit and one finds that $\mathcal{P}_E/\mathcal{P}_B \sim (\lambda_*/\lambda) \rightarrow 0$ as $\eta \rightarrow 0$. On the other hand, if $\lambda$ is decreasing sufficiently fast to cause the integral to diverge then it is dominated by its upper limit, giving instead $\mathcal{P}_E/\mathcal{P}_B \sim (aH/k)$, which grows exponentially with the number of e-folds since horizon exit. One can conclude that in the first case one has a predominantly magnetic field at late times, with only an exponentially small admixture of electric field, whereas in the second case the situation is reversed.

Let us focus on the case where a magnetic field is synthesized at late times, preserving the opposite case for the discussion in §6. During radiation domination the electromagnetic field redshifts like the dominant constituent of the universe and therefore its relative density is conserved. If we assume prompt reheating after inflation, then the root mean square fluctuation in $\rho_B$ on a scale corresponding to comoving wavenumber $k$ has magnitude

$$
|B|_{\text{rms}} \sim \left( \frac{k}{a} \right)^2 \left( \frac{\lambda}{\lambda_*} \right)^{1/2}
$$

(3.13)

in Planck units. The proper wavenumber today on cluster scales is of order $k/a \sim e^{-140}$, and to seed a galactic dynamo it may be sufficient to produce fluctuations with magnitude [49] $|B|_{\text{rms}} \sim e^{-60}$ T [8]. Therefore to obtain a cosmologically interesting magnetic field, we require roughly $\lambda \sim e^{200} \lambda_*$ [48]. Although the mechanism of magnetogenesis is quite insensitive to the dependence of $\lambda$ on the scalar fields which

‡ Our conventions for magnetic field strengths are measured in tesla, where 1 T \( \sim e^{-120} \) in Planck units.
determine its value, a large class of models which invoke couplings of this form can be written in the “dilaton-like” form \( \lambda(\phi) = \exp(\phi/M_\phi) \) \[50\], where \( M_\phi \) is some characteristic mass scale. Such couplings have also been invoked in the context of models of dark energy \[51\], where they may be subject to additional constraints \[52\, 53\]. In order for a coupling of this form to yield the requisite hierarchy, \( M_\phi \) must be chosen to satisfy \( M_\phi \approx \Delta \phi/200 \) where \( \Delta \phi \) is the excursion of the field \( \phi \) between horizon crossing and the end of inflation. If \( \phi \) is a field driving a stage of chaotic inflation it can be supposed to move a distance perhaps of order 10 in fundamental units, in which case

\[
M_\phi \sim \frac{1}{20}.
\]  

(3.14)

We will adopt this value of \( M_\phi \) as the canonical one for magnetogenesis, although in practice \( M_\phi \) will vary from model to model, and may be closer to the Planck scale.

A variety of bounds are known on the energy density which can be present in magnetic fields at early times, either from direct detection \[54\] or indirect effects \[55\]. These limits typically arise from constraints at the short wavelength end of the spectrum and imply that the extremely blue raw spectrum, Eq. (3.7), must be processed into an approximately scale-invariant form. Exact scale invariance occurs for \( M_\phi = \sqrt{2\epsilon}/4 \), which gives \( M_\phi^{-1} \sim 10^3 \) and it follows that for \( M_\phi \) close to Eq. (3.14) approximate scale invariance will apply.

4. Electric loop corrections to the scalar spectrum

In this section we return to the interaction between gauge bosons and the other fluctuation modes which are relevant during inflation. If we wish to make predictions for the anisotropy seen in the cosmic microwave background (CMB) then we are principally concerned with the power in scalar perturbations around the time of last scattering, because these dominate the density fluctuation in the primordial plasma. Such fluctuations are connected to observation by making predictions for the properties of the comoving curvature perturbation, \( R \), which on superhorizon scales is equivalent to the curvature perturbation on uniform density hypersurfaces, \( \zeta \). In a model with many degrees of freedom there is a considerable simplification afforded by using \( \zeta \), which can be computed using the so-called non-linear \( \delta N \) formalism.

As an initial condition, calculations using the \( \delta N \) formalism require predictions for the correlations among the \( \delta \phi^\alpha \) around the time of horizon crossing. In this section we will compute a prediction for the two-point correlation of the \( \delta \phi^\alpha \), taking into account the leading loop correction from exchange of virtual gauge bosons. This is presumably sufficient to make predictions for the power spectrum of the CMB, although it will transpire that there may be UV-divergent terms associated with the gauge transformation between \( \zeta \) and the \( \delta \phi^\alpha \) which are not captured by the \( \delta N \) formula. Therefore the final answer can be treated as an order of magnitude estimate only.

\[ Compare, for example, with Eq. (42) of Ref. \[48\].\]
In the following section we will compute the analogous correction for the three-point correlation function, which is necessary if we wish to study higher-order statistics.

4.1. Dominant contributions to the interaction vertex

Consider the vertex for interaction of two gauge bosons with a scalar particle, given by Eq. (2.23) in the limit $\xi \to 0$ with $\omega_i$ replaced by Eq. (3.1). If we are only computing around the time of horizon crossing, then we can obtain a good approximation by truncating all quantities to leading order in the slow-roll expansion. In the interaction vertex, this leaves us with terms of the form

$$S_3 \supset \frac{1}{2} \int \! d^3x \, dt \, a \lambda_\alpha \delta \phi^\alpha \left\{ \dot{\omega}_i \dot{\omega}_i - \frac{1}{a^2} (\partial_i \omega_j \partial_i \omega_j - \partial_i \omega_j \partial_j \omega_i) \right\}. \quad (4.1)$$

The first term in Eq. (4.1) involves an interaction with $\dot{\omega}^2$, which according to Eq. (3.10) can be thought of as a measure of the electric field intensity. The second term involves interactions with terms of the form $(\partial \omega)^2$, which according to Eq. (3.9) are a measure of the magnetic field intensity. We can likewise imagine interactions which are dominated by the first or second term to represent interaction with the electric or magnetic field, respectively. This distinction is useful because if $\lambda$ is carrying a strong time dependence at horizon exit we expect the ambient electric field to be enhanced in comparison with the magnetic field, which depends only on spatial gradients. Indeed, this is true irrespective of whether $\lambda$ is increasing or decreasing, provided that it has a strong time dependence in either direction, because the properties of the fluctuations at horizon crossing do not determine whether the final configuration will be an electric or magnetic field. The initial condition involves a strong electric field in either case, but the final character of the field is only determined by the long-term evolution of $\lambda$ after horizon crossing.

To see this in detail, it is simplest to use Eqs. (3.1) and (3.6) to rewrite Eq. (4.1) as an effective vertex which takes the form

$$S_3 \supset \int \! d\eta \int \frac{d^3k_1 \, d^3k_2 \, d^3k_3}{(2\pi)^9} (2\pi)^3 \delta(k_1 + k_2 + k_3) \times \frac{\lambda_\alpha}{2} \delta \phi^\alpha (k_1, \eta) \alpha_{ij}^{\pm \pm} (\eta) \omega_i (k_2, \eta) \omega_j (k_3, \eta), \quad (4.2)$$

where the vertex function $\alpha_{ij}^{\pm \pm}$ is defined by

$$\alpha_{ij}^{\pm \pm} = \delta_{ij} \left\{ k_2 \cdot k_3 + \left( i \epsilon_2 k_2 + \frac{\Omega(\eta)}{\eta} \right) \left( i \epsilon_3 k_3 + \frac{\Omega(\eta)}{\eta} \right) \right\} - k_2 j k_3 i. \quad (4.3)$$

In this equation, $\Omega$ is an abbreviation for the dimensionless combination

$$\Omega \equiv \frac{1}{2} \frac{\lambda_\alpha \delta \phi^\alpha}{\lambda \hat{H}} \quad (4.4)$$

and the $\pm$ symbols are chosen according the details of time ordering and the assignment of '+' or '-' vertices (to be described in §4.2 below) among the gauge fields which participate in the vertex, and are fixed by the structure of the diagram in which Eq. (4.2) holds.
is inserted. (In Eq. (4.3), we have temporarily abandoned our summation convention—for this equation only—in the interests of notational clarity: this choice of $\alpha_{ij}^{\omega_2}$ should be inserted directly in Eq. (4.2) without concern for the position of the spatial indices $i$ and $j$.) The parameter $\Omega$ measures the hierarchy between the wavefunction of the gauge field and its time derivative, and arises from the interaction with the electric field. In the limit $|\Omega| \gg |k\eta|$, the total interaction is dominated by this electric piece and the details of the time ordering become irrelevant. This is true whenever the rate of change of the coupling $\lambda$ with the fields is tuned to satisfy

$$ \frac{1}{2} \left( \frac{\dot{\phi}_\alpha}{H} \right) \left( \frac{\lambda_\alpha}{\lambda} \right) \gg e^{-N_*}. $$ (4.5)

In Eq. (4.5), ‘$*$’ denotes evaluation at the time when the mode with wavenumber $k$ exited the horizon, or more precisely a small but non-zero number of e-folds $N_* \sim 1$ afterwards. This time should be chosen so that the fluctuations in scalar modes are close to their asymptotic superhorizon values, and the canonical commutation relation $[\delta \dot{\phi}, \delta \phi] \sim e^{-N_*}$ allows the $\delta \phi_\alpha$ to be treated as approximately classical quantities, but it should not be so late after horizon exit that appreciable evolution might have occurred, which would spoil the accuracy of the slow-roll approximation. Whenever Eq. (4.5) applies we can ignore the purely magnetic part of the interaction. It is important to note, however, that the electric interaction can only become dominant if at least one of the scalars which couple to $\omega_i$ is rolling on cosmological timescales, so that $\dot{\phi}_\alpha/H$ is not totally negligible for some $\phi_\alpha$. This is not really a restriction if one wishes to use this interaction for the purposes of magnetogenesis, because one is then dependent on $\lambda$ developing a large hierarchy between its value at horizon exit and its value at some much later time, such as the end of inflation, and this can occur only if some of the scalar fields are in motion.

Alternatively, the scalar fields whose vacuum expectation values determine the magnitude of $\lambda$ may not be rolling during inflation, or the dependence of $\lambda$ on these fields may be so weak that $(\ln \lambda)_\alpha$ is never large enough for Eq. (4.5) to apply. In this limit the interaction is still interesting, but it is magnetically dominated and the momentum integral which describes the loop is somewhat more complicated to compute. For the remainder of this paper, we focus on the electric part of the interaction only.

### 4.2. The in–in formalism

The appropriate formalism in which to compute expectation values of cosmological fluctuations is the so-called in–in formalism introduced by Schwinger. In the cosmological case we wish to compute expectation values of the form $\langle 0 | O | 0 \rangle$ for some collection of operators $O$, in the state $| 0 \rangle$ which following the discussion of ultraviolet physics given in 3 should be taken to be the Minkowski vacuum deep inside the horizon. We know from scattering calculations in Minkowski space that one can compute $\langle \text{out} | O | \text{in} \rangle$ using a conventional Feynman path integral for any ‘in-state’ $| \text{in} \rangle$ and ‘out-state’ $| \text{out} \rangle$. It follows that after integrating over all possible out-states we can
compute $\langle \text{in}|O|\text{in} \rangle$ using two path integrals, which gives the so-called Schwinger–Keldysh path integral formula

$$\langle \text{in}|O|\text{in} \rangle = \int [d\phi_+ d\phi_-] O \exp \left( iS[\phi_+] - iS[\phi_-] \right), \quad (4.6)$$

where the $\phi$ label the elementary fields in the theory and $O$ is taken to be built out of either ‘+’ or ‘−’ fields but not both. The integral is defined by prescribing that only those + and − field configurations which begin in the appropriate vacuum $|0\rangle$ and end with coincident values at some late time are to be included in the integration. The precise choice of this late time is immaterial, provided it is chosen to be later than the time of evaluation of any field which appears in $O$. We will conventionally choose $O$ to be composed only from + fields and carry the integral from past infinity to the time of observation, $\eta_*$. 

When coupling to gravity is taken into account, there can be a subtlety in the construction of the path integrals which appear in Eq. (4.6). From a given formula for the lagrangian, one would ordinarily obtain the canonical momenta and construct the hamiltonian. Integrating over the coordinates and canonical momenta, with time evolution supplied by the hamiltonian, one arrives at the standard path integral. If the lagrangian is not quadratic in the momenta, however, one cannot explicitly integrate them out [56]. Instead, one must include their degrees of freedom in the path integral as “derivative ghosts” which compensate for the fact that the fields are not canonically normalized. They are associated with interactions which are cubic or higher in derivatives such as $\delta \dot{\phi}$. In our example, the only such interactions are associated with scalar fluctuations and for this reason we ignore derivative ghosts. Instead, they should instead included with scalar loop corrections, which presumably do not lead to large effects [56, 57, 58].

The doubling of degrees of freedom in Eq. (4.6) implies that when we expand the path integral into diagrams we encounter extra graphs, which mix vertices constructed from + and − fields. These vertices collectively ensure that all expectation values are real. If we apply Eq. (4.6) to the calculation of the scalar two-point function $\langle \delta \phi^\alpha(\mathbf{k}_1)\delta \phi^\beta(\mathbf{k}_2) \rangle_*$, one obtains the diagrams shown in Fig. 1.

The diagrams in Fig. 1 divide into two pairs of complex conjugates, corresponding to the $(+, +)$ and $(-, -)$ diagrams in one pair and the $(+, -)$ and $(-, +)$ diagrams in another. Consider first the $(+, +)$ diagram. This makes a contribution to the scalar two-point function of the form

$$\langle \delta \phi^\alpha(\mathbf{k}_1)\delta \phi^\beta(\mathbf{k}_2) \rangle_* \gtrsim \frac{H_*^4}{32} \prod_i k_i^3 \left( \frac{\lambda_{\alpha} \lambda_{\beta}}{\lambda^2} \right)_* \Omega_4^4(1 + ik_1 \eta_*)(1 + ik_2 \eta_*) e^{-\eta_*(k_1 + k_2)}$$

† In writing these and all subsequent expressions, I have chosen to nest the integrals which result from mixed $(+, -)$ and $(-, +)$-type diagrams. Alternatively, it is possible to factorize these contributions, obtaining expressions which are manifestly non-singular for all momenta $k_i$ [59, 60, 61]. Whichever method is chosen, the answer is always the same. I would like to thank Peter Adshead, Richard Easther, Eugene Lim, Martin Sloth and Filippo Vernizzi for correspondence on this issue.
**Figure 1.** Schwinger-formalism diagrams for the loop correction to the scalar two-point function which arise from mixing with gauge bosons. Straight lines indicate scalar quanta, which only appear on the external legs. The interior loop, composed of wavy lines, indicates mixing with virtual quanta borrowed from the ambient electric and magnetic fields. The fields associated with external legs are always of + type, whereas the vertices can be of + or − type. The (+, +) and (−, −) diagrams form one complex conjugate pair, and the (+, −) and (−, +) diagrams form another.

\[
\times \int \frac{d^3q}{q^3} \frac{d^3r}{r^3} P_2(q, r) \delta(-k_1 - r + q) \delta(-k_2 - q + r) \\
\times \left( \begin{bmatrix} k_1 & k_2 \\ k_1 + r + q & k_2 - r - q \end{bmatrix} + \begin{bmatrix} k_2 & k_1 \\ k_2 + r + q & k_1 - r - q \end{bmatrix} \right), \tag{4.7}
\]

where \( i \in \{1, 2\} \), which depends on a four-parameter integral, defined by

\[
\left[ \begin{array}{cc} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{array} \right] \equiv -\int_{-\infty}^{\eta} \frac{d\tau}{\tau^2} \int_{-\infty}^{\tau} \frac{d\eta}{\eta^2} (1 - i\alpha_1 \eta)(1 - i\alpha_2 \tau)e^{i\beta_1 \eta}e^{i\beta_2 \tau}. \tag{4.8}
\]

To make sense of this for real \( \beta_1 \) and \( \beta_2 \), the contours of integration for \( \eta \) and \( \tau \) must be deformed so that \( e^{i\beta_1 \eta} \) and \( e^{i\beta_2 \tau} \) are decaying for large \( |\eta| \) and \( |\tau| \), respectively. Indeed, this contour prescription follows automatically from the choice of vacuum \( |0\rangle \) in Eq. (4.6). In addition, \( P_2(q, r) \) is a polynomial in the vector momenta which depends on the transverse structure of the gauge field propagator and the detailed momentum dependence at the vertex. It satisfies

\[
P_2(q, r) \equiv q^2 r^2 + (q \cdot r)^2. \tag{4.9}
\]

Eq. (4.7) has been written in a form which emphasizes the symmetry between \( k_1 \) and \( k_2 \). In order to simplify it further, one can integrate out either \( q \) or \( r \), which leaves behind a single momentum-conservation delta-function, \( \delta(k_1 + k_2) \), together with an integral over the remaining momentum. Since local field theories usually exhibit bad behaviour at high energies, we may expect this integral to receive a significant contribution from the region where the 3-momentum which circulates in the loop becomes large. Indeed, neglecting any powers of momentum which are introduced by the time integrals in Eq. (4.8), it easy to see that the loop diverges at least as fast as \( \int dq \) in the ultra-violet, whereas it converges at least as fast as \( \int q \, dq \) in the infra-red. The time integral has dimension \( q^2 \), so it can only make this divergence worse at high energy, and improve convergence at low energy. To make sense of such an integral it must first be regularized, removing the contribution of arbitrarily high-energy modes, and made finite by applying a renormalization prescription. Once this is done, the ultra-violet sensitivity of the original integral implies that we can expect the loop to be dominated by contributions near some UV scale. In what follows, we use this to ignore effects associated with the infra-red cutoff.
Magnetogenesis and the primordial non-gaussianity

How are we to choose a cut-off in an expanding, inflationary spacetime? In flat space quantum field theory we are familiar with the use of a variety of regulators. The method of the renormalization group tells us that these are all equivalent in the continuum limit, after the introduction of counter-terms and a renormalization prescription. The simplest choice is a hard cutoff, $M$, on the invariant momenta associated with particles which circulate within the loops. To apply this cutoff in practice one must Wick rotate loop integrals to Euclidean spacetime where $M$ becomes a $SO(4)$-invariant cut-off on the Euclidean momentum. When we revert to Lorentzian signature, this procedure gives a $SO(3,1)$-invariant result and therefore preserves Lorentz invariance. As a result, the hard momentum cutoff in Minkowski space provides a popular means of estimating the magnitude of loops in an effective theory. The problem at hand is to find a way to perform similar estimates in de Sitter space.

In any curved spacetime we do not have global Lorentz invariance, although according to the equivalence principle we must recover approximate local $SO(3,1)$ invariance in a small neighbourhood of any point. On the basis of the discussion in §3 it follows that this restoration of Lorentz invariance in the ultra-violet corresponds to choosing conventional flat space quantum field theory deep inside the horizon. Whatever regulator we pick, it is necessary to be sure that it does not conflict with local Lorentz invariance in the ultra-violet, or we shall obtain nonsensical results. Recently, van der Meulen and Smit [62] have pointed out that that a momentum cut-off at some characteristic scale $M$ should be taken to apply to proper momenta, rather than comoving momenta, since only the former have physical significance. However, such a cut-off violates local Lorentz invariance for any finite value of the cut-off. This is because the cut-off grows between adjacent spatial slices which are taken at later and later times. If we pick some particular point in spacetime and ask whether Lorentz invariance is restored in a local neighbourhood of that point we find that no matter how small a patch of spacetime we choose, it necessarily intersects a sheaf of spatial slices. The growth in the cut-off transverse to the sheaf picks out a preferred direction, and the properties of the theory do not become invariant under local spacetime rotations within the patch. It follows that Lorentz invariance is broken.

Breakdown of Lorentz invariance in the ultra-violet would have dramatic consequences for the stability of the theory. As time increases, the cut-off grows. This means that between two adjacent spatial slices, one taken at a slightly later time than the other, new quanta are introduced to our universe [66]. These new quanta destabilize the theory, because although they are at very high energies when they are added to the description, they can scatter with each other to produce other quanta which are much softer. These scattered soft quanta, whose correlations are essentially random, impinge on superhorizon correlation functions and scramble their values. Thus, in the presence of the Lorentz-violating cut-off, correlation functions do not retain coherent values outside the horizon but instead behave as if they are coupled to a quantum noise.

‡ I would like to thank Andrew Tolley for emphasizing this property of the proper momentum cut-off. A similar observation has been made in several places in the literature; see especially Refs. [63, 64, 65].
bath and are swamped exponentially quickly by quantum noise. In this cut-off theory there is no quantum-to-classical transition, and presumably there are no conserved quantities outside the horizon. On the other hand, if we find a way to preserve Lorentz invariance then none of these undesirable features become manifest. One can think of this stabilizing property of Lorentz invariance as analogous to the conservation of the velocity of the centre of energy of any isolated system. This is a consequence of invariance under Lorentz boosts and implies, for example, that quantum effects associated with virtual ultra-violet quanta do not disturb the motion of isolated particles. A form of this analogy was earlier used by Lyth [67].

Let us return to Eqs. (4.7)–(4.8) and study the time and loop integrals in more detail. Along the way, we will encounter the features described in the previous paragraph. In Appendix A an analogous analysis is given for the scalar loop which arises from a $V''''$ interaction, which has previously appeared in the literature [62][25]. The $(+, +)$ diagram described by Eq. (4.7) must be supplemented by a $(+, -, -)$ diagram, which gives a contribution corresponding to

$$\langle \delta \phi^\alpha(k_1) \delta \phi^\beta(k_2) \rangle \geq - \frac{H^4}{32 \prod_i k_i^3} \left( \frac{\lambda \alpha \lambda \beta}{\lambda^2} \right) \Omega^4_s (1 + i k_1 \eta_s)(1 - i k_2 \eta_s) e^{-i \eta_s(k_1 - k_2)}$$

$$\times \int \frac{d^3q d^3r}{q^3 r^3} P_2(q, r) \delta(-k_1 - r + q) \delta(-k_2 - q + r)$$

$$\times \left( \left[ \frac{k_1}{k_1 + r + q} \right] - \left[ \frac{-k_2}{-k_2 - r - q} \right] + \left[ \frac{-k_2}{-k_2 - r - q} \right] \right). \quad (4.10)$$

Once we have aggregated the contribution of Eq. (4.7) and Eq. (4.10) we must remember to add in their complex conjugates in order to account for the $(-, -)$ and $(-, +)$ diagrams which have not been written explicitly.

After integration by parts in $\eta$ and $\tau$, the fundamental time integral given in Eq. (4.8) can be re-expressed in the form

$$\left[ \frac{\alpha_1 \alpha_2}{\beta_1 \beta_2} \right] = \left[ -\frac{1}{2 \eta_s^2} + i \eta_s \left( \frac{\alpha_2 - \alpha_1 + \beta_1 - \beta_2}{2} \right) \right] e^{i(\beta_1 + \beta_2) \eta_s} + i \eta_s (\beta_1 - \alpha_1) e^{i \beta_2 \eta_s} \int_{-\infty}^{\eta_s} \frac{d\tau}{\tau} e^{i \beta_1 \tau}$$

$$+ (\beta_1 + \beta_2) \left( \frac{\alpha_2 - \alpha_1 + \beta_1 - \beta_2}{2} \right) \int_{-\infty}^{\eta_s} \frac{d\tau}{\tau} e^{i (\beta_1 + \beta_2) \tau}$$

$$+ (\beta_1 - \alpha_1)(\beta_2 - \alpha_2) \int_{-\infty}^{\eta_s} \frac{d\tau}{\eta_s} e^{i \beta_2 \tau} \int_{-\infty}^{\eta_s} \frac{d\tau}{\eta} e^{i \beta_1 \eta}. \quad (4.11)$$

This equation shows primitive fast divergences but some of these will cancel and others are purely imaginary, which implies that they disappear when the contribution of all four diagrams is accounted for. Write $k_1 = k_2 = k$ and consider the deep ultra-violet region, where $q \approx r \gg k$. In this region, the $\beta_i$ terms are individually very large, approximately satisfying $\beta_i \sim \pm 2q \gg k$, so that the integrals are almost all very small—except for the integral with $\beta_1 + \beta_2$ in the exponent, since this combination is independent of $q$.

§ Conversely, these integrals diverge like powers of $\ln |\eta_s|$ near the infra-red cutoff, where $q$ is a measure of the 3-momentum circulating in the loop. However, in this region such logarithms are suppressed by positive powers of $q$ and therefore the loop integral will be well-behaved. Indeed, for the purposes
Collecting all necessary terms, expanding for small $|k\eta|$ and keeping only contributions which are relevant in this limit, the expectation value can be written in the form
\[
\langle \delta \phi^\alpha(k_1) \delta \phi^\beta(k_2) \rangle_s = (2\pi)^3 \delta\left(\sum_i k_i\right) \frac{H_s^4}{32 \prod_i k_i} \left(\frac{\lambda^\alpha \lambda^\beta}{\lambda^2}\right)_s \Omega_s^4 I_2,
\]
(4.12)
with the left-over time and momentum dependence consolidated into an integral of the form
\[
I_2 = 4 \int \frac{d^3 q}{q^3 |k - q|^3} P_2(q, k - q) \left\{ k^2 - k(q + |k - q|)(N_s + \ln 2 - \gamma_E - 1) \right\},
\]
(4.13)
where $\gamma_E \approx 0.57722$ is the Euler–Mascheroni constant and $N_s$ measures by how many e-folds the mode with wavenumber $k$ is outside the horizon at time $\eta_s$.

Eq. (4.13) is quadratically divergent. Introducing a cut-off $\Lambda$ (as yet unspecified) on the momentum circulating in the loop, this integral takes the form
\[
I_2 \approx -\frac{8k}{\pi^2} \Lambda^2 (N_s + \ln 2 + \gamma_E - 1) + \frac{4k^2}{\pi^2} \Lambda + \frac{16k^3}{3\pi^2} \ln \frac{\Lambda}{\mu} (N_s + \ln 2 + \gamma_E - 1) + \cdots,
\]
(4.14)
where ‘…’ denotes terms which are subdominant in the limit $\Lambda \to \infty$, and $\mu$ is an arbitrary scale, of the same dimensions as $\Lambda$, which has been introduced to make sense of the logarithm. It is clear that if we take $\Lambda$ to correspond to a proper momentum cut-off, which would take the form $\Lambda = a_s \tilde{\Lambda}$ for some constant $\tilde{\Lambda}$, then we introduce fast divergences as $a_s \to \infty$. These divergences correspond to hard quanta which are redshifted into the effective theory as the universe expands but subsequently scatter to produce soft quanta and contaminate the spectrum, as described above Eq. (4.10).

Indeed, since the accumulation of such quanta is presumably highly incoherent, it seems unlikely that one can ascribe any definite value to the spectrum after horizon crossing.

The key lesson I wish to draw from this example is that such divergent effects are fictional. If we begin with a Lorentz invariant theory valid at high scales and integrate out modes above a proper momentum cut-off, then cancelling divergences would automatically appear in the coefficients of the resulting effective Lagrangian \[68, 69\]. For this reason, power law divergences such as those appearing in Eq. (4.14) are devoid of physical significance; only the coefficient of the logarithmic divergence can have meaning. If we begin with a Lorentz invariant effective low energy theory such as Eq. (2.1) it is not possible to see these cancellations taking place. Thus, taken literally, our analysis up to this point is not compatible with a Lorentz violating cut-off; instead, sensible answers can be obtained only by using a Lorentz-invariant regulator such as dimensional reduction \[14, 15, 16, 58, 60\]. However, we can equally well make use of our knowledge that the power law terms in Eq. (4.14) must ultimately cancel, leaving behind an unfixed finite term or threshold correction \[70, 71\].

The threshold correction can be determined by matching to a more complete theory which resolves the details of ultraviolet physics, or by specifying a renormalization of the present calculation, we are assuming that the loop integral is dominated by exchange of virtual quanta near the ultra-violet cutoff, so these integrals (and other similar integrals to be encountered in \cite{55} while studying the loop-corrected three-point function) actually play no role in the analysis.
prescription which allows us to make contact with measurement. However, unlike simple theories such as quantum electrodynamics it is difficult to find an appropriate renormalization prescription in cosmology. This is because it is not possible to directly measure the expectation values we have computed: they are only important as an initial condition for the purpose of computing the structure in the late universe which is visible to us. A similar problem afflicts calculations in quantum chromodynamics, where interactions among hadrons are handled by first studying the predominantly electromagnetic interactions among their constituent partons.\[ Indeed, there is an interesting analogy between quantum chromodynamics and the calculation of cosmological expectation values using the $\delta N$ formalism. In QCD, one calculates correlation functions among partons at high energies, where the QCD coupling is small and perturbation theory applies. One then chooses a “factorization scale” which determines the energy below which partons are summed up into hadrons according to certain “parton distribution functions.” In doing so, it is possible to encounter large logarithmic singularities at soft or collinear momenta which enhance the phase space for interacting partons to dress themselves into jets, or which manifest as initial state radiation. One can find analogues for all these effects in cosmology: high energy corresponds to early times during inflation, where the slow-roll approximation applies and perturbation theory in slow-roll quantities makes sense; the time of horizon crossing plays the role of the factorization scale; the coefficients of the $\delta N$ expansion correspond to the parton distribution functions; and logarithms such as $N_\ast = \ln |k\eta|$ play the role of the singularities at soft or collinear momenta. The details of this analogy have been explored in more detail in Ref. [72].

\[ P^{\alpha\beta}(k) = P_0^\ast(k) \left\{ \delta^{\alpha\beta} + \frac{4}{3} P_0^\ast \Omega_\ast^4 \left( \frac{\Lambda^\ast}{\lambda^2} \right) \left( N_\ast + \ln 2 + \gamma_E - 1 \right) \ln \frac{\Lambda'}{H_\ast} \right\}, \quad (4.15) \]

where $P_0^\ast(k) = H^2/2k^2$ is the tree-level power spectrum, and $P_0(k) = H^2/4\pi^2$ is its dimensionless equivalent. Note that since a flat metric is being assumed on field space, the placement of the $\alpha$ and $\beta$ field indices is immaterial. Eq. (4.15) is the first principal result of this paper. It is interesting to observe that the logarithm $\ln(\Lambda'/\mu_\ast)$ cannot be too large; although the precise value of $H_\ast$ during inflation is model dependent, in chaotic models it is reasonable to assume that $H_\ast \approx 10^{-5}$ in fundamental units. It follows that $\ln(\Lambda'/H_\ast)$ cannot be more than of order $1 - 10$.

The two-point function $\langle \delta\phi^\alpha(k_1)\delta\phi^\beta(k_2) \rangle$ is one contribution to the power spectrum $\zeta$, but in practice it would be accompanied by other contributions arising from non-
linear terms in the gauge transformation between \( \zeta \) and the \( \delta \phi^\alpha \). These may themselves carry ultra-violet divergences which should be accounted for in an accurate calculation. However, there is no reason to believe that these contributions would be any larger than Eq. (4.15), and we can therefore suppose that this expression suffices for the purpose of obtaining order-of-magnitude estimates.

How large is the loop correction? Specializing for simplicity to the case of a single field, and adopting the parametrization \( \lambda(\phi) = \exp(\phi/M_\phi) \), it follows that this loop correction does not overwhelm the tree-level provided \( M_\phi^{-1} \lesssim 120 \). This compares favourably with the value \( M_\phi^{-1} \lesssim 20 \), given in Eq. (3.14), which was suggested as appropriate for magnetogenesis (see [3]); although \( M_\phi^{-1} \) varies from model to model, it is unlikely to be as large as 120. Note that although these values may seem closer than desirable, the tuning here is in an exponential. A mass scale \( M_\phi = 120 \) corresponds to a hierarchy \( \lambda/\lambda_* = e^{1200} \) which is enormously larger than is required or desirable to produce a primordial magnetic field: the large energy density of an electromagnetic field amplified by such an enormous factor would swamp any other constituents of the universe and lead to an entirely unacceptable late time phenomenology.

5. Electric loop corrections to the scalar bispectrum

Eq. (4.15) and the associated bound on \( \lambda \) or \( M_\phi \) which guarantees that the loop correction does not overpower the tree-level are interesting in their own right. However, in this section we return to the question of non-gaussianity in the cosmic microwave background. One can obtain a marginally tighter bound on \( M_\phi \) by demanding that the loop expansion is stable for this expectation value as well as for the spectrum, and we shall see that increasingly stringent bounds are possible for higher correlation functions.

The Schwinger formalism diagrams contributing to the three-point function are shown in Fig. [2]. These diagrams make contributions to the three-point function which can be put into a form similar to Eqs. (4.7) and (4.10) for the two-point function. For example, for the \((+,+,+)\) diagram we obtain

\[
\langle \delta \phi^\alpha(k_1) \delta \phi^\beta(k_2) \delta \phi^\gamma(k_3) \rangle_s \supseteq \frac{H^6_s}{64 \prod_i k_i^3} \left( \frac{\lambda_\alpha \lambda_\beta \lambda_\gamma}{\lambda^3} \right)_s \Omega^6_\phi \prod_i (1 + ik_i \eta_s) e^{-ik_i \eta_i},
\]

where now \( i \in \{1, 2, 3\} \) and \( Q_+ \) is a function of the scalar momenta which is defined by

\[
Q_+ \equiv \left[ \frac{k_1}{k_1 + q + r} \right] \left[ \frac{k_2}{k_2 - r + s} \right] \left[ \frac{k_3}{k_3 - q + s} \right] + \left[ \frac{k_1}{k_1 + q + e} \right] \left[ \frac{k_3}{k_3 - q + s} \right] \left[ \frac{k_2}{k_2 - r + s} \right] + \left[ \frac{k_2}{k_2 + r + s} \right] \left[ \frac{k_1}{k_1 + q + r} \right] \left[ \frac{k_3}{k_3 - q - s} \right] + \left[ \frac{k_3}{k_3 + q + s} \right] \left[ \frac{k_1}{k_1 - q + r} \right] \left[ \frac{k_2}{k_2 - r - s} \right] + \left[ \frac{k_3}{k_3 + q + s} \right] \left[ \frac{k_2}{k_2 + r - s} \right] \left[ \frac{k_1}{k_1 - q - r} \right].
\]

The momentum polynomial \( P_3(q, r, s) \) is the analogue of Eq. (4.9) for the bispectrum,
and depends symmetrically on each of the vector momenta \( q, r \) and \( s \). Specifically, it obeys

\[
P_3(q, r, s) \equiv q^2(r \cdot s)^2 + r^2(q \cdot s)^2 + s^2(q \cdot r)^2 - (q \cdot r)(q \cdot s)(r \cdot s),
\]

whereas the time integrals can now be cast in the form of a six-parameter exponential integral,

\[
\left[ \alpha_1 \alpha_2 \alpha_3 \beta_1 \beta_2 \beta_3 \right] \equiv -i \int_{-\infty}^{\eta^*} d\zeta \int_{-\infty}^{\tau} d\tau \int_{-\infty}^{\eta} d\eta \frac{1}{\eta^2} (1 - i\alpha_1 \eta)(1 - i\alpha_2 \tau)(1 - i\alpha_3 \zeta)e^{i\beta_1 \eta}e^{i\beta_2 \tau}e^{i\beta_3 \zeta},
\]

with the necessary deformations of the contour of integrations—or, more accurately, the inclusion of suppression factors in the exponentials—to cause convergence at large \( |\eta|, |\tau| \) or \( |\zeta| \). We expect the same general considerations to govern the result of the integrals in Eq. (5.1) which applied for the two-point function: after integrating by parts, Eq. (5.4) can be expressed as an asymptotic series in inverse powers of \( \eta^* \), together with nested integrals of the form \( \int d\zeta \zeta^{-1} e^{i\mu \zeta} \) for some \( \mu \).

We must also take account of the \((+, +, +)\)- and \((- +, +)\)-type diagrams. Since these are related by complex conjugation it suffices to consider only diagrams of one particular signature, for which we will choose the \((+, +, -)\)-type, and then add in their complex conjugates. The \(-\) vertex can be attached to any of the external legs, corresponding to any one of the momenta \( k_i \). The contribution from any such diagram, for example where the \(-\) vertex is associated with \( k_3 \), can be written in a form similar to Eq. (5.1), taking account of the fact that, irrespective of its numerical value, the time at the \(-\) vertex is taken to be later than the time of evaluation \( \eta^* \), and with \( Q_+ \) replaced by a different function \( Q_- \) which accounts for the necessary sign interchanges. We therefore arrive at

\[
\langle \delta \phi^\alpha(k_1)\delta \phi^{\beta}(k_2)\delta \phi^{\gamma}(k_3) \rangle \supseteq \frac{H^6}{64 \Pi_i k^3_i} \left( \frac{\lambda_i \lambda_j \lambda_k}{\lambda^3} \right) \Omega^6.
\]
Magnetogenesis and the primordial non-gaussianity

\begin{align}
&\times (1 + i k_1 \eta_*) (1 + i k_2 \eta_*) (1 - i k_3 \eta_*) e^{-i \eta_*(k_1 + k_2 - k_3)} \\
&\times \int \frac{d^3 q}{q^3} \frac{d^3 r}{r^3} \frac{d^3 s}{s^3} P_3(p, q, r) \delta(-k_1 - r + q) \delta(-k_2 - s + r) \delta(-k_3 - q + s) \\
&\times Q_- (k_1, k_2, k_3, q, r, s),
\end{align}

(5.5)

together with the equivalent expressions obtained by exchanging \(k_1\) with \(k_2\) and \(k_3\). In this expression, \(Q_-\) is defined by

\begin{align}
Q_- &\equiv \left[ \begin{array}{ccc}
-k_1 & k_2 & -k_3 \\
k_1 + q + r & k_2 - r + s & -k_3 - q - s \\
-k_1 + q + r & -k_2 - r + s & k_3 - q - s \\
0 & 0 & 0
\end{array} \right] + \left[ \begin{array}{ccc}
-k_1 & k_2 & -k_3 \\
k_1 + q - r & k_2 + r - s & -k_3 + q - s \\
-k_1 + q - r & -k_2 + r - s & k_3 + q - s \\
0 & 0 & 0
\end{array} \right] \\
&\quad + \left[ \begin{array}{ccc}
-k_1 & k_2 & -k_3 \\
k_1 + q - r & -k_2 - r + s & k_3 - q - s \\
-k_1 + q - r & k_2 - r + s & -k_3 + q - s \\
0 & 0 & 0
\end{array} \right] + \left[ \begin{array}{ccc}
-k_1 & k_2 & -k_3 \\
-k_1 + q + r & k_2 + r + s & k_3 + q + s \\
-k_1 + q + r & -k_2 + r + s & k_3 + q + s \\
0 & 0 & 0
\end{array} \right].
\end{align}

(5.6)

To see how this works in detail, it follows after integrating by parts that we can express Eq. (5.4) in the form

\begin{align}
\left[ \begin{array}{c}
\alpha_1 \\
\beta_1 \\
\alpha_2 \\
\beta_2 \\
\alpha_3 \\
\beta_3
\end{array} \right] &= \left( \frac{i}{6 \eta_*^3} + \frac{\xi_1}{\eta_*^2} + \frac{\xi_2}{\eta_*} \right) e^{i(\beta_1 + \beta_2 + \beta_3) \eta_*} + (\beta_1 + \beta_2 + \beta_3) \xi_3 \int_{-\infty}^{\eta_*} \frac{d\zeta}{\zeta} e^{i(\beta_1 + \beta_2 + \beta_3) \zeta} \\
&- \frac{i \xi_3}{\eta_*} e^{i \beta_1 \eta_*} \int_{-\infty}^{\eta_*} \frac{d\zeta}{\zeta} e^{i(\beta_1 + \beta_2) \zeta} + \left( \frac{\xi_4}{2 \eta_*^2} - \frac{\xi_5}{\eta_*} \right) e^{i(\beta_2 + \beta_3) \eta_*} \int_{-\infty}^{\eta_*} \frac{d\zeta}{\zeta} e^{i \beta_1 \zeta} \\
&- \xi_6 \int_{-\infty}^{\eta_*} \frac{d\zeta}{\zeta} e^{i \beta_2 \zeta} \int_{-\infty}^{\zeta} \frac{d\tau}{\tau} e^{i(\beta_1 + \beta_2) \tau} - \xi_7 \int_{-\infty}^{\eta_*} \frac{d\zeta}{\zeta} e^{i \beta_3 \zeta} \int_{-\infty}^{\zeta} \frac{d\tau}{\tau} e^{i(\beta_2 + \beta_3) \tau} \\
&- \xi_8 \int_{-\infty}^{\eta_*} \frac{d\zeta}{\zeta} e^{i \beta_2 \zeta} \int_{-\infty}^{\zeta} \frac{d\tau}{\tau} e^{i \beta_3 \tau} - \xi_9 \int_{-\infty}^{\eta_*} \frac{d\zeta}{\zeta} e^{i \beta_2 \zeta} \int_{-\infty}^{\zeta} \frac{d\tau}{\tau} e^{i \beta_3 \tau} \int_{-\infty}^{\tau} \frac{d\eta}{\eta} e^{i \beta_1 \eta},
\end{align}

(5.7)

where the coefficients \(\xi_1\) to \(\xi_9\) are defined by

\begin{align}
\xi_1 &= \frac{1}{2} \left\{ 9(\alpha_2 - \alpha_1) + 3 \alpha_3 + 2(\beta_1 - 2 \beta_2) - \beta_3 \right\}
\end{align}

(5.8)

\begin{align}
\xi_2 &= \frac{1}{2} \left\{ 9 \alpha_3 \beta_2 + 2 \beta_2^2 - 3 \alpha_3 \beta_1 - 4 \beta_1^2 - 2 \beta_1 \beta_2 - \beta_3^2 + 3 \alpha_1 (8 \alpha_3 + \beta_1 - \beta_2 - 5 \beta_3) \\
&\quad + \beta_3 (3 \alpha_3 + \beta_1 - 5 \beta_2) + 3 \alpha_2 (\beta_2 - 8 \alpha_3 - \beta_1 + 5 \beta_3) \right\}
\end{align}

(5.9)

\begin{align}
\xi_3 &= (\beta_1 + \beta_2) \left( \alpha_2 - \alpha_1 - \frac{\beta_2 - \beta_1}{2} \right)
\end{align}

(5.10)

\begin{align}
\xi_4 &= \beta_1 - \alpha_1
\end{align}

(5.11)

\begin{align}
\xi_5 &= \frac{1}{2} (\beta_1 - \alpha_1) (2 \alpha_2 + 2 \alpha_3 - 3 \beta_2 - \beta_3)
\end{align}

(5.12)

\begin{align}
\xi_6 &= (\beta_3 - \alpha_3) (\beta_1 + \beta_2) \left( \alpha_2 - \alpha_1 - \frac{\beta_2 - \beta_1}{2} \right)
\end{align}

(5.13)

\begin{align}
\xi_7 &= (\beta_2 + \beta_3) \left\{ \alpha_3 + (\beta_1 - \alpha_1) (\beta_2 - \alpha_1) - \frac{\beta_2 - \beta_3}{2} \right\}
\end{align}

(5.14)

\begin{align}
\xi_8 &= (\beta_1 - \alpha_1) (\beta_2 - \alpha_2)
\end{align}

(5.15)

\begin{align}
\xi_9 &= (\beta_1 - \alpha_1) (\beta_2 - \alpha_2) (\beta_3 - \alpha_3)
\end{align}

(5.16)
This function contains primitive fast divergences, but as in the case of the spectrum these will conspire to cancel among themselves in the final answer, leaving a result which contains only slow divergences which scale as powers of \(N_s\). As before, the combination \(\beta_1 + \beta_2 + \beta_3\) is always independent of the loop momentum, whereas most of the integrals involve decaying exponentials of \(|\beta_1|, |\beta_2|\) or \(|\beta_3|\) and will play no role when the loop integral is dominated by its ultra-violet region. In \(Q_+\), there is no combination other than \(\beta_1 + \beta_2 + \beta_3\) which can contribute, whereas in \(Q_-\)—for four of the terms—the combination \(\beta_1 + \beta_2\) (and therefore \(\beta_3\) on its own) also remains finite in the extreme ultra-violet limit. It follows that the terms involving \(\xi_3\) and \(\xi_6\) can also contribute, although it will turn out that in Eq. (5.7) all these pieces cancel out of the final answer.

In order to simplify the calculation from this point onwards, it is convenient to specialize to the equilateral limit in which all the \(k_i\) are equal, giving \(k_1 = k_2 = k_3\). In practice this does not entail much loss of generality, since the approximations we are obliged to make in evaluating the time integrals in Eqs. (4.8) and (5.4) mean that we cannot take the \(k_i\) to exit the horizon with a separation of more than a few e-folds. (Exactly similar remarks apply to the standard calculation of the tree-level three-point function.) One may now proceed by analogy with the scalar two-point function, as described in \([4]\). The three-point function is written in the form

\[
\langle \delta \phi^a(k_1) \delta \phi^b(k_2) \delta \phi^c(k_3) \rangle_{*,k_i=k} = (2\pi)^3 \delta(\sum_i k_i) \frac{H^6}{64 \prod_i k_i^3} \left( \frac{\lambda^a \lambda^b \lambda^c}{\lambda^3} \right)_\ast \Omega^6 I_3, \tag{5.17}
\]

and the time- and momentum-dependence is absorbed into \(I_3\). This integral is somewhat more complicated than its counterpart for the two-point function. It is defined by

\[
I_3 \equiv 4k \int \frac{d^3q}{q^3 r^3 s^3} P_3(q,r,s) \left\{ 4k(q + r + s) + (q^2 + r^2 + s^2) \ln 27 \right. \\
+ (qr + qs + rs)(4N_s + 4\gamma_E + 9 \ln 3 - 4 - 2k^2) \left. \right\}, \tag{5.18}
\]

with the specific assignments \(r \equiv q - k_1\) and \(s \equiv q + k_3\). In addition, \(N_s\) satisfies the usual definition \(N_s \equiv \ln |k\eta_s|\), which is unambiguous here because of our assumption that the momentum triangle formed by the \(k_i\) is equilateral.

To proceed, we pick \(k_1\) to point along the \(\hat{z}\) axis of a spherical polar coordinate system. (We could also pick \(k_3\), but the result is independent of our choice.) In this system of coordinates the only non-trivial inner product we require is that of \(q\) with \(k_3\), which takes the form

\[
q \cdot k_3 = kq \left\{ \cos \theta \cos \theta_{13} + \sin \theta \sin \theta_{13} \cos(\phi - \phi_{13}) \right\}, \tag{5.19}
\]

where \(\{\theta, \phi\}\) are the zenith and azimuth of \(q\) relative to \(k_1\), and \(\{\theta_{13}, \phi_{13}\}\) are the corresponding zenith and azimuth of \(k_{13}\). Note that \(\phi_{13}\) is devoid of significance, being only a coordinate choice, and cannot appear in any physical quantity, whereas \(\theta\) and \(\phi\) are simply variables of integration. The integral is quadratically divergent. With a cut-off \(\Lambda\), we find

\[
I_3 = -\frac{24k}{\pi^2} \Lambda^2 \left( \tilde{N}_s - \frac{1}{4} \ln 3 \right) - \frac{48k^2}{\pi^2} \Lambda + \frac{8k^3}{\pi^2} \ln \frac{\Lambda}{\mu} \tilde{N}_s \left( 4 + 2 \cos \theta_{13} + \frac{1}{N_s} \right), \tag{5.20}
\]
where $\tilde{N}_\ast$ is a slightly modified count of the number of e-folds since horizon exit, defined by

$$\tilde{N}_\ast \equiv N_\ast + \gamma_E + \frac{11}{4}\ln 3 - 1 \approx N_\ast + 2.60.$$  

(5.21)

One might worry that the appearance of $\theta_{13}$ in Eq. (5.20) represents a failure of symmetry between $k_1, k_2,$ and $k_3$. However, it should be remembered that once we specify any two of the momenta the third is determined by the triangle law $k_1 + k_2 + k_3 = 0$. Only the relative orientations of these vectors can enter physical quantities, and the absolute orientation of one vector and the azimuth of a second are simply coordinate choices. For an equilateral configuration, this leaves only one cosine which can play a role in physical quantities.

The tree-level, single-field equilateral bispectrum is known to take the form \cite{17, 20}

$$B_{00} = \frac{11}{2}\sqrt{2}\epsilon_\ast H_\ast^4 \frac{H_\ast^4}{2k^6}.$$  

(5.22)

We drop power law divergences and neglect a possible threshold correction. This implies that, if we specialize to single-field inflation and return to the parametrization $\lambda(\phi) = \exp(\phi/M_\phi)$, the one-loop corrected bispectrum can be written as

$$B_\ast \approx B_{00}\left(1 - \frac{4\epsilon_\ast^{7/2}}{11\sqrt{2}M_\phi^3} P_R \tilde{N}_\ast \ln \frac{\Lambda'}{H_\ast}\right),$$  

(5.23)

where $\Lambda'$ is again the proper scale of new physics, which can have a hierarchy with respect to the Planck scale of no more than around $\ln(\Lambda'/H_\ast) \sim 10$. The power spectrum of the curvature perturbation, $P_R$, is computed using the tree-level scalar spectrum, which we assume is not destabilized by the loop correction from electric quanta. It follows that if, likewise, we do not wish the loop correction to be larger than the tree-level, we must have $M_\phi^{-1} \lesssim 50$.

This bound is somewhat stronger than is required to guarantee stability of the spectrum. Indeed, it is clear that successively higher-order expectation values will contain increasingly large negative powers of $M_\phi$, whereas the leading correction to an $n$-point scalar expectation value comes from diagrams such as those in Fig. 2 where the electromagnetic quanta circulate in a ring between the external scalar legs. Such a diagram is always suppressed with respect to the tree-level by only one power of the loop-counting parameter $(H/M_P)^2$, and therefore for very large $n$ we will require $M_\phi$ to be closer and closer to unity. Does this imply that only $M_\phi = 1$ makes sense for an effective field theory? In fact, this conclusion would be too strong. For $M_\phi$ too far below the Planck scale the present analysis does not show that the two- and three-point scalar expectation values are inconsistent with observation, but rather only that perturbation theory is insufficient to compute them.

It is known that the tree-level power spectrum is an excellent match for observation, and that the temperature anisotropy is gaussian to high accuracy. It follows that there is a reasonably secure foundation to demand that the two-point function can be computed perturbatively. The situation for the bispectrum is less clear. It is known
that the non-linearity of the temperature anisotropy is not too large, although it may perhaps be of order $f_{NL} \sim 60$ in the squeezed limit. This limit is the opposite of the configuration studied here, where one of the momenta is going to zero while the other two become approximately equal and opposite, but even in the equilateral case where estimation is more difficult there is a relatively stringent limit, $|f_{NL}| \lesssim 300$. There is not yet any hint from the data that the three-point function requires anything beyond conventional perturbation theory. Even less is known concerning the properties of the $n$-point expectation values for $n$ larger than three. For this reason, it seems most conservative to adopt the approximate bound $M_{\phi^{-1}} \lesssim 120$ which corresponds to the applicability of perturbation theory for the power spectrum.

6. Discussion

In this paper I have computed the corrections to the spectrum and bispectrum of a set of light scalar fields which arise from coupling to an electromagnetic field during inflation. The calculation of inflationary observables is conventionally split into a quantum-mechanical initial condition, which specifies the momentum-dependence of expectation values at the time of horizon crossing, and a subsequent classical evolution. The interaction studied in this paper constitutes a correction to the initial condition, and can be thought of as arising from interactions with virtual quanta of the electromagnetic field. In a scenario where the gauge coupling is rolling rapidly during inflation, these virtual electromagnetic quanta experience strong amplification as they are drawn across the horizon and “freeze in” to the correlations among the scalar modes at that time.

To which scenarios would this correction apply? The most direct application is to theories of so-called magnetogenesis, where one aims to produce a late-time magnetic field by breaking the conformal invariance of the $U(1)$ gauge field. In this scenario the scalar fields which determine the expectation value $\lambda(\phi)$ must be rolling on cosmological timescales, meaning that these scalars will typically also contribute to the curvature perturbation, $\zeta$. It is $\zeta$ which is visible as the adiabatic temperature perturbation in the Cosmic Microwave Background. Therefore, in order that the conventional perturbative predictions for the power spectrum and bispectrum are not destabilized we must demand that the correction from electromagnetic interactions does not dominate the tree-level. As shown in \[\text{§4}\], this requires that $M_{\phi^{-1}} \lesssim 120$ in order that the spectrum is not destabilized, and the stronger condition (\[\text{§5}\]) $M_{\phi^{-1}} \lesssim 50$ for the bispectrum. Both of these conditions are approximately satisfied for conventional pictures of magnetogenesis, which, as shown in \[\text{§3}\] typically requires $M_{\phi^{-1}} \approx 20$. According to the discussion in \[\text{§5}\] it is probably most conservative to adopt the limit $M_{\phi^{-1}} \lesssim 120$, because the evidence that current observation matches perturbative calculations of the bispectrum is rather less strong than for the spectrum.

There is another mechanism by which synthesis of an electric or magnetic field could destabilize conventional perturbation theory. If the energy density in electromagnetic radiation begins to compete with the potential energy associated with the scalar
fields, then this would deform the comoving hypersurface on which inflation ends. This deformation would imply that $\zeta$ was not conserved, but rather would acquire a contribution from this final hypersurface \[73\]. On the other hand, if the energy density in the electromagnetic field remains small, then we can be confident that there is no back-reaction on the metric. In this case, it follows that the hypersurface on which inflation ends suffers a negligible deformation and $\zeta$ is given by the usual procedure, because the electromagnetic energy density never contributes to the expansion rate or the integrated number of e-foldings, $N$.

When does energy in the electromagnetic field remain small? This question was addressed by Bamba \[40\] and Martin & Yokoyama \[48\], who gave a detailed discussion of the back-reaction problem. They concluded that when a magnetic field is synthesized there is never a problem with back reaction, whereas if the outcome is an electric field then the scenario is only free of instabilities associated with back reaction if the scale of inflation is taken to be very low. One way to understand this phenomenon is to note that the magnetic energy density $\rho_B \sim (\partial \omega)^2$ is proportional to spatial gradients, whereas the electric energy density $\rho_E \sim \dot{\omega}^2$ involves a time derivative. We must therefore include $\rho_E$ at leading order in a gradient expansion of the scalar Klein–Gordon equation. If $\rho_E$ becomes too large it begins to act as a source and causes an unwanted growing instability in the scalar fluctuations. In practice this “instability” may entail nothing more dramatic than a macroscopic flow of charge which shorts out the electric field \[40\], but our ability to calculate is impaired and it is difficult to draw conclusions with any confidence. Therefore, unless the scale of inflation is taken to be very low, it seems that one should restrict attention to scenarios where $\lambda(\phi)$ grows during inflation.

Couplings between scalars and $U(1)$ gauge fields are invoked in certain theories of dark energy, such as the so-called “chameleon” \[74\] model. Such couplings were proposed in Ref. \[51\], following earlier work \[75\], \[76\], and studied using astrophysical means in Refs. \[52\], \[53\], \[77\]. (For laboratory limits, see Refs. \[78\], \[79\].) However, the chameleon is essentially inert during inflation: it rolls rapidly to its minimum, and is fixed there for the duration of inflation \[80\]. Since it is not in motion while observable scales are leaving the horizon, the parameter $\Omega$ which controls the magnitude of the correction tends to zero. Moreover, because the energy density carried by the chameleon is cosmologically negligible, its expectation values do not contribute to those of the curvature perturbation. Therefore, although perturbation theory may fail for expectation values of the chameleon field if the associated mass scale is too small, this has no observational consequences. We conclude that no limit on the chameleon coupling $\beta$ analogous to those obtained in Refs. \[51\], \[52\], \[53\] can be extracted from the present analysis.

Although fast instabilities were encountered at intermediate points in the calculation, the final expectation values were found to be free of fast divergences. Their time dependence was instead described by powers of $N_*$, which measures by how many e-folds the fluctuation in question has passed outside the horizon. One can understand this as a consequence of a theorem due to Weinberg, which guarantees that
interactions involving gauge fields are well behaved in this sense \cite{15}. In the present case, although the electromagnetic part of the interaction would be “dangerous” (in Weinberg’s sense) as part of a scalar interaction because it involves time derivatives, it does not produce fast divergences. Indeed, the only effect of differentiation with respect to time when applied to a mode of the gauge field is to introduce factors of the slowly varying parameter $\Omega$. Such factors merely set the scale of the interaction, rather than changing the character of the time dependence, as would be the case for a scalar mode.

**Acknowledgments**

I would like to thank Clare Burrage, Anne Davis, Jim Lidsey, David Lyth, Karim Malik, Amanda Weltman and Daniel Wesley, who helped me with a large number of details concerning cosmological magnetic fields. Peter Adshead, Emanuela Dimastrogiovanni, Richard Easther, Eugene Lim, Meindert van der Meulen, Sarah Shandera, Jan Smit, Andrew Tolley and Daniel Wesley made many important suggestions concerning the treatment of loop diagrams in de Sitter space.

I would like to thank the Astronomy Unit at Queen Mary, University of London, for their hospitality.

**Appendix A. Scalar loop corrections from the $V'''$ vertex**

In this Appendix, I compute the scalar loop correction coming from a simple $V'''$ vertex. This loop correction is common to any inflationary model, and arises simply from the self-interactions implied by the scalar potential, unless $V'''$ is somehow tuned to be zero. This loop correction corresponds to the diagram in Fig. A1 with two external scalar legs which are connected by a loop of circulating virtual scalar quanta. Just like the processes considered in the main text, this diagram comes in four different types labelled by the $+$ or $-$ flavours at each vertex, giving complex conjugate pairs $(+,+)$, $(-,-)$ and $(+, -), (-, +)$. The loop process described by Fig. A1 has already appeared in the literature, and has been the subject of detailed study by several previous authors. van der Meulen & Smit \cite{62} studied this loop for both large and small internal momenta, finding corrections to the classical time dependence. These corrections are one possible source of the “quantum logarithms” described in the Introduction (§I). Bartolo et al. \cite{25} considered
the same loop process, but focused on the infra-red region and found results in this regime which were in agreement with those of van der Meulen & Smit. In this Appendix the same loop is studied using the methodology applied in the main text, principally for the purpose of aiding comparison of the methods of the present paper with those of either van der Meulen & Smit or Bartolo et al.

We work with a single-field model of inflation. The generalization to multiple fields is easy to obtain by identical methods, and keeping track of the necessary indices in the space of scalar fields only introduces needless notational clutter. It follows that the vertex is simple, and takes the form

\[ S_3 \equiv - \int \frac{d^3k_1 d^3k_2 d^3k_3}{(2\pi)^9} (2\pi)^3 \delta(\sum_i k_i) \frac{V'''}{3} \delta\phi(k_1) \delta\phi(k_2) \delta\phi(k_3). \] (A.1)

As with the two-point loop correction studied in \[§§\] 4–5, although Eq. (A.4) contains more negative powers of \(\tau\) and \(\eta\) than (for example) Eq. (4.11) and therefore requires more integrations by parts to separate its asymptotic dependence on \(\eta\), and also because the integrands contain more terms anyway. Fewer terms are present when dealing with the gauge field because of its conformal invariance, which gives the
In terms of these combinations, the ρδW we also define a combination βγW. We now have all the ingredients necessary to define the σk. Let us focus first on the infra-red region, where the momentum circulating in the loop is small compared to the momenta k1 = k2 = |k| on the external legs. In this limit, we can take q ≈ 0 and r ≈ k. Although Eq. (A.4) contains a large number of primitive fast divergences (up to and including η−6 in the pure power-law sector, and up to and including η−2 when multiplied by additional logarithmic divergences, bearing in mind that purely imaginary divergences cancel out of the final answer), these all cancel in both the ultra-violet and infra-red. The leading time dependence instead comes from the double exponential integral. This can be written

$$\int_{-\infty}^{\eta} \frac{d\eta}{\eta} e^{i\alpha_{1}\eta} \int_{-\infty}^{\eta} \frac{d\tau}{\tau} e^{i\beta_{1}\tau} = \frac{1}{2} \ln^{2} |\alpha_{1}\eta_{s}| + \left( \gamma_{E} + \ln \left| \frac{\alpha_{1}}{\beta_{1}} \right| \right) \ln |\alpha_{1}\eta_{s}| + f(\alpha_{1}/\beta_{1}),$$

where f(x) is a complicated function of its argument which can be given in closed form in terms of the Euler–Mascheroni constant when x = 1. We will focus only on the

$$\left[ \begin{array}{ccc} \alpha_{1} & \alpha_{2} & \alpha_{3} \\ \beta_{1} & \beta_{2} & \beta_{3} \end{array} \right] = \left( \begin{array}{ccc} \sigma_{1} & \sigma_{2} & \sigma_{3} \\ \eta_{s}^{6} & \eta_{s}^{5} & \eta_{s}^{4} \end{array} \right) + i \left( \begin{array}{ccc} \sigma_{4} & \sigma_{5} & \sigma_{6} \\ \eta_{s}^{3} & \eta_{s}^{2} & \eta_{s}^{1} \end{array} \right) e^{i\eta_{s}},$$

$$\delta \sigma_{6} \int_{-\infty}^{\eta} \frac{d\eta}{\eta} e^{i\alpha_{1}\eta} \int_{-\infty}^{\eta} \frac{d\tau}{\tau} e^{i\beta_{1}\tau} - (\alpha_{1}\rho_{3} + \alpha_{3}\lambda_{4}) \int_{-\infty}^{\eta} \frac{d\eta}{\eta} e^{i\alpha_{1}\eta} \int_{-\infty}^{\eta} \frac{d\tau}{\tau} e^{i\beta_{1}\tau}.$$

The parameters appearing in this expression involve certain combinations of the αi and βj, which are described by the choices

$$\lambda_{1} \equiv \frac{1}{3}; \quad \lambda_{2} \equiv \beta_{1}\lambda_{1}; \quad \lambda_{3} \equiv \beta_{2} + \beta_{1}\lambda_{2}; \quad \text{and} \quad \lambda_{4} \equiv \beta_{3} + \beta_{1}\lambda_{3};$$

$$\gamma_{1} \equiv \lambda_{1}; \quad \gamma_{2} \equiv \lambda_{2} + \alpha_{1}\lambda_{1}; \quad \gamma_{3} \equiv \lambda_{3} + \alpha_{2}\lambda_{1} - \alpha_{1}\lambda_{2};$$

$$\gamma_{4} \equiv \alpha_{1}\lambda_{3} + \alpha_{2}\lambda_{2} - \alpha_{3}\lambda_{1}; \quad \gamma_{5} \equiv \alpha_{3}\lambda_{2} + \alpha_{2}\lambda_{3}; \quad \text{and} \quad \gamma_{6} \equiv \alpha_{3}\lambda_{3}.$$

We also define a combination δ,

$$\delta \equiv \alpha_{1} + \beta_{1}.$$
leading divergence as $\eta \rightarrow 0$. Keeping only this term, one finds that the loop corrected power spectrum takes the form

$$P(k) = P_0(k) \left( 1 + \frac{2}{9} \left( \frac{V'''}{2\pi} \right)^2 \frac{N^2}{H^2_*} \ln k\ell \right), \quad (A.15)$$

where $P_0(k)$ is the tree-level power spectrum and $\ell$ is an infra-red regulator which has been introduced to prevent a divergence in the momentum integral. As described in Refs. [81, 82, 83, 56, 24, 25], this regulator can be understood as the size of a comoving “box” in which we perform the quantum field theory calculation. This box must be chosen to be sufficiently large that the region of space for which we wish to obtain a prediction can fit comfortably inside it, but small enough that all fluctuations remain perturbatively small. Eq. (A.15) is identical with Eq. (1) of Ref. [25].

In the opposite limit, where very hard virtual quanta are circulating in the loop, $q$ and $r$ are both much larger than $k$. In this limit, one finds that the loop correction has the form

$$P(k) = P_0(k) \left\{ 1 - \frac{16 \ln(\Lambda'/H_*)}{27} \left( \frac{V'''}{2\pi} \right)^2 \right\}. \quad (A.16)$$

References

[1] F. Vernizzi and D. Wands, Non-Gaussianities in two-field inflation, JCAP 0605 (2006) 019, arXiv:astro-ph/0603799.
[2] T. Battefeld and R. Easther, Non-gaussianities in multi-field inflation, JCAP 0703 (2007) 020, arXiv:astro-ph/0610296.
[3] S. Yokoyama and J. Soda, Primordial statistical anisotropy generated at the end of inflation, JCAP 0808 (2008) 005, arXiv:0805.4265.
[4] S. Kanno, M. Kimura, J. Soda, and S. Yokoyama, Anisotropic Inflation from Vector Impurity, JCAP 0808 (2008) 034, arXiv:0806.2422.
[5] K. Dimopoulos, D. H. Lyth, and Y. Rodriguez, Statistical anisotropy of the curvature perturbation from vector field perturbations, arXiv:0809.1055.
[6] A.-C. Davis and K. Dimopoulos, Primordial magnetic fields in false vacuum inflation, Phys. Rev. D55 (1997) 7398–7414, arXiv:astro-ph/9506132.
[7] J.-L. Han and R. Wielebinski, Milestones in the Observations of Cosmic Magnetic Fields, arXiv:astro-ph/0209090.
[8] D. Battefeld, T. Battefeld, D. H. Wesley, and M. Wyman, Magnetogenesis from Cosmic String Loops, JCAP 0802 (2008) 001, arXiv:0708.2901.
[9] A. P. S. Yadav and B. D. Wandelt, Evidence of Primordial Non-Gaussianity ($f_{NL}$) in the Wilkinson Microwave Anisotropy Probe 3-Year Data at 2.8$\sigma$, Phys. Rev. Lett. 100 (2008) 181301, arXiv:0712.1148.
[10] WMAP Collaboration, E. Komatsu et al., Five-Year Wilkinson Microwave Anisotropy Probe Observations: Cosmological Interpretation, arXiv:0803.0547.
[11] S. Weinberg, Non-Gaussian Correlations Outside the Horizon, Phys. Rev. D78 (2008) 123521, arXiv:0808.2909.
[12] L. Leblond and S. Shandera, Simple Bounds from the Perturbative Regime of Inflation, JCAP 0808 (2008) 007, arXiv:0802.2290.
[13] C. Armendariz-Picon, M. Fontanini, R. Penco, and M. Trodden, Where does Cosmological Perturbation Theory Break Down?, arXiv:0805.0114.
[14] S. Weinberg, Quantum contributions to cosmological correlations, Phys. Rev. D72 (2005) 043514, \texttt{arXiv:hep-th/0506236}.
[15] S. Weinberg, Quantum contributions to cosmological correlations. II: Can these corrections become large?, Phys. Rev. D74 (2006) 023508, \texttt{arXiv:hep-th/0605244}.
[16] K. Chaicherdsakul, Quantum cosmological correlations in an inflating universe: Can fermion and gauge fields loops give a scale free spectrum?, Phys. Rev. D75 (2007) 063522, \texttt{arXiv:hep-th/0611352}.
[17] D. Seery and J. E. Lidsey, Primordial non-gaussianities from multiple-field inflation, JCAP 0509 (2005) 011, \texttt{arXiv:astro-ph/0506056}.
[18] D. H. Lyth and Y. Rodriguez, The inflationary prediction for primordial non-gaussianity, Phys. Rev. Lett. 95 (2005) 121302, \texttt{arXiv:astro-ph/0504045}.
[19] D. H. Lyth and D. Seery, Classicality of the primordial perturbations, Phys. Lett. B662 (2008) 309–313, \texttt{arXiv:astro-ph/0607647}.
[20] D. Seery, K. A. Malik, and D. H. Lyth, Non-gaussianity of inflationary field perturbations from the field equation, JCAP 0803 (2008) 014, \texttt{arXiv:0802.0588}.
[21] S. Weinberg, Non-Gaussian Correlations Outside the Horizon II: The General Case, Phys. Rev. D79 (2009) 043504, \texttt{arXiv:0810.2831}.
[22] S.-P. Miao and R. P. Woodard, Leading log solution for inflationary Yukawa, Phys. Rev. D74 (2006) 044019, \texttt{arXiv:gr-qc/0602110}.
[23] T. Prokopec, N. C. Tsamis, and R. P. Woodard, Stochastic Inflationary Scalar Electrodynamics, Annals Phys. 323 (2008) 1324–1360, \texttt{arXiv:0707.0847}.
[24] D. Seery, One-loop corrections to the curvature perturbation from inflation, JCAP 0802 (2008) 006, \texttt{arXiv:0707.3378}.
[25] N. Bartolo, S. Matarrese, M. Pietroni, A. Riotto, and D. Seery, On the Physical Significance of Infra-red Corrections to Inflationary Observables, JCAP 0801 (2008) 015, \texttt{arXiv:0711.4263}.
[26] K. Enqvist, S. Nurmi, D. Podolsky, and G. I. Rigopoulos, On the divergences of inflationary superhorizon perturbations, JCAP 0804 (2008) 025, \texttt{arXiv:0802.0395}.
[27] H. R. S. Cogollo, Y. Rodríguez, and C. A. Valenzuela-Toledo, On the Issue of the $\zeta$ Series Convergence and Loop Corrections in the Generation of Observable Primordial Non-Gaussianity in Slow-Roll Inflation. Part I: the Bispectrum, JCAP 0808 (2008) 029, \texttt{arXiv:0806.1546}.
[28] Y. Rodríguez and C. A. Valenzuela-Toledo, On the Issue of the $\zeta$ Series Convergence and Loop Corrections in the Generation of Observable Primordial Non-Gaussianity in Slow-Roll Inflation. Part II: the Trispectrum, \texttt{arXiv:0811.4092}.
[29] K.-Y. Choi, L. M. H. Hall, and C. van de Bruck, Spectral running and non-Gaussianity from slow-roll inflation in generalised two-field models, JCAP 0702 (2007) 029, \texttt{arXiv:0701.0247}.
[30] M. Sasaki, Multi-brid inflation and non-Gaussianity, Prog. Theor. Phys. 120 (2008) 159–174, \texttt{arXiv:0805.0974}.
[31] A. Naruko and M. Sasaki, Large non-Gaussianity from multi-brid inflation, \texttt{arXiv:0807.0180}.
[32] C. T. Byrnes, K.-Y. Choi, and L. M. H. Hall, Conditions for large non-Gaussianity in two-field slow-roll inflation, \texttt{arXiv:0807.1101}.
[33] J. M. Maldacena, Non-Gaussian features of primordial fluctuations in single field inflationary models, JHEP 05 (2003) 013, \texttt{arXiv:astro-ph/0210603}.
[34] M. Alishahiha, E. Silverstein, and D. Tong, DBI in the sky, Phys. Rev. D70 (2004) 123505, \texttt{arXiv:hep-th/0404084}.
[35] X. Chen, M.-x. Huang, S. Kachru, and G. Shiu, Observational signatures and non-Gaussianities of general single field inflation, JCAP 0701 (2007) 002, \texttt{arXiv:hep-th/0605045}.
[36] C. T. Byrnes, K.-Y. Choi, and L. M. H. Hall, Large non-Gaussianity from two-component hybrid inflation, JCAP 0902 (2009) 017, \texttt{arXiv:0812.0807}.
[37] M. S. Turner and L. M. Widrow, Inflation Produced, Large Scale Magnetic Fields, Phys. Rev.
D37 (1988) 2743.

[38] B. Ratra, *Cosmological ‘seed’ magnetic field from inflation*, Astrophys. J. **391** (1992) L1–L4.

[39] A.-C. Davis, K. Dimopoulos, T. Prokopec, and O. Tornkvist, *Primordial spectrum of gauge fields from inflation*, Phys. Lett. **B501** (2001) 165–172, [arXiv:astro-ph/0007214](http://arxiv.org/abs/astro-ph/0007214).

[40] K. Bamba and M. Sasaki, *Large-scale magnetic fields in the inflationary universe*, JCAP **0702** (2007) 030, [arXiv:astro-ph/0611701](http://arxiv.org/abs/astro-ph/0611701).

[41] M. Giovannini, *Magnetogenesis, variation of gauge couplings and inflation*, arXiv:astro-ph/0212346.

[42] M. Giovannini, *The magnetized universe*, Int. J. Mod. Phys. **D13** (2004) 391–502, [arXiv:astro-ph/0312614](http://arxiv.org/abs/astro-ph/0312614).

[43] O. Bertolami and R. Monteiro, *Varying electromagnetic coupling and primordial magnetic fields*, Phys. Rev. **D71** (2005) 123525, [arXiv:astro-ph/0504211](http://arxiv.org/abs/astro-ph/0504211).

[44] L. Parker, *Particle creation in expanding universes*, Phys. Rev. Lett. **21** (1968) 562–564.

[45] L. Parker, *Quantized fields and particle creation in expanding universes: I*, Phys. Rev. **183** (1969) 1057–1068.

[46] A. Ashoorioon and R. B. Mann, *Generation of cosmological seed magnetic fields from inflation with cutoff*, Phys. Rev. **D71** (2005) 103509, [arXiv:gr-qc/0410053](http://arxiv.org/abs/gr-qc/0410053).

[47] K. Bamba, *The interrelation between the generation of large-scale electric fields and that of large-scale magnetic fields during inflation*, JCAP **0710** (2007) 015, [arXiv:0710.1906](http://arxiv.org/abs/0710.1906).

[48] J. Martin and J. Yokoyama, *Generation of Large-Scale Magnetic Fields in Single-Field Inflation*, JCAP **0801** (2008) 025, [arXiv:0711.4307](http://arxiv.org/abs/0711.4307).

[49] A.-C. Davis, M. Lilley, and O. Tornkvist, *Relaxing the Bounds on Primordial Magnetic Seed Fields*, Phys. Rev. **D60** (1999) 021301, [arXiv:astro-ph/9904022](http://arxiv.org/abs/astro-ph/9904022).

[50] K. Bamba and J. Yokoyama, *Large-scale magnetic fields from inflation in dilaton electromagnetism*, Phys. Rev. **D69** (2004) 043507, [arXiv:astro-ph/0310824](http://arxiv.org/abs/astro-ph/0310824).

[51] P. Brax, C. van de Bruck, A.-C. Davis, D. F. Mota, and D. J. Shaw, *Testing Chameleon Theories with Light Propagating through a Magnetic Field*, Phys. Rev. **D76** (2007) 085010, [arXiv:0707.2801](http://arxiv.org/abs/0707.2801).

[52] C. Burrage, *Supernova Brightening from Chameleon–Photon Mixing*, Phys. Rev. **D77** (2008) 043009, [arXiv:0711.2966](http://arxiv.org/abs/0711.2966).

[53] C. Burrage, A.-C. Davis, and D. J. Shaw, *Detecting Chameleons: The Astronomical Polarization Produced by Chameleon-like Scalar Fields*, arXiv:0809.1763.

[54] T. Kahniashvili, Y. Maravin, and A. Kosowsky, *Primordial Magnetic Field Limits from WMAP Five-Year Data*, arXiv:0806.1876.

[55] C. Caprini and R. Durrer, *Gravitational wave production: A strong constraint on primordial magnetic fields*, Phys. Rev. **D65** (2001) 023517, [arXiv:astro-ph/0007214](http://arxiv.org/abs/astro-ph/0007214).

[56] D. Seery, *One-loop corrections to a scalar field during inflation*, JCAP **0711** (2007) 025, [arXiv:0707.3377](http://arxiv.org/abs/0707.3377).

[57] E. Dimastrogiovanni and N. Bartolo, *One-loop graviton corrections to the curvature perturbation from inflation*, arXiv:0807.2790.

[58] P. Adshead, R. Easther, and E. A. Lim, *Cosmology With Many Light Scalar Fields: Stochastic Inflation and Loop Corrections*, arXiv:0809.4008.

[59] D. Seery, *M. S. Sloth, and F. Vernizzi, Inflationary trispectrum from graviton exchange*, JCAP **0903** (2009) 018, [arXiv:0811.3934](http://arxiv.org/abs/0811.3934).

[60] P. Adshead, R. Easther, and E. A. Lim, *The ‘in–in’ Formalism and Cosmological Perturbations*, arXiv:0904.4207.

[61] X. Chen, B. Hu, M.-x. Huang, G. Shiu, and Y. Wang, *Large Primordial Trispectra in General Single Field Inflation*, arXiv:0905.3494.

[62] M. van der Meulen and J. Smit, *Classical approximation to quantum cosmological correlations*, JCAP **0711** (2007) 023, [arXiv:0707.0842](http://arxiv.org/abs/0707.0842).

[63] J. C. Niemeyer, *Inflation with a high frequency cutoff*, Phys. Rev. **D63** (2001) 123502,
Magnetogenesis and the primordial non-gaussianity

[64] A. Kempf and J. C. Niemeyer, Perturbation spectrum in inflation with cutoff, Phys. Rev. D64 (2001) 103501, arXiv:astro-ph/0103225.

[65] A. Ashoorioon, A. Kempf, and R. B. Mann, Minimum length cutoff in inflation and uniqueness of the action, Phys. Rev. D71 (2005) 023503, arXiv:astro-ph/0410139.

[66] R. H. Brandenberger, Inflationary cosmology: Progress and problems, arXiv:hep-th/9910410.

[67] D. H. Lyth, Large Scale Energy Density Perturbations and Inflation, Phys. Rev. D31 (1985) 1792–1798.

[68] C. P. Burgess and D. London, Uses and abuses of effective Lagrangians, Phys. Rev. D48 (1993) 4337–4351, arXiv:hep-ph/9203216.

[69] C. Arzt, M. B. Einhorn, and J. Wudka, Effective Lagrangian approach to precision measurements: The Anomalous magnetic moment of the muon, Phys. Rev. D49 (1994) 1370–1377, arXiv:hep-ph/9304206.

[70] H. Georgi, H. R. Quinn, and S. Weinberg, Hierarchy of Interactions in Unified Gauge Theories, Phys. Rev. Lett. 33 (1974) 451–454.

[71] S. Weinberg, Gauge Hierarchies, Phys. Lett. B82 (1979) 387.

[72] D. Seery, A parton picture of de Sitter space during slow-roll inflation, JCAP 0905 (2009) 021, arXiv:0903.2788.

[73] D. H. Lyth, Generating the curvature perturbation at the end of inflation, JCAP 0511 (2005) 006, arXiv:astro-ph/0510443.

[74] J. Khoury and A. Weltman, Chameleon fields: Awaiting surprises for tests of gravity in space, Phys. Rev. Lett. 93 (2004) 171104, arXiv:astro-ph/0309300.

[75] D. F. Mota and J. D. Barrow, Local and Global Variations of The Fine Structure Constant, Mon. Not. Roy. Astron. Soc. 349 (2004) 291, arXiv:astro-ph/0309273.

[76] T. Clifton, D. F. Mota, and J. D. Barrow, Inhomogeneous gravity, Mon. Not. Roy. Astron. Soc. 358 (2005) 601, arXiv:gr-qc/0406001.

[77] C. Burrage, A.-C. Davis, and D. J. Shaw, Active Galactic Nuclei Shed Light on Axion-like-Particles, arXiv:0902.2320.

[78] A. S. Chou et al., A search for chameleon particles using a photon regeneration technique, arXiv:0806.2438.

[79] A. Afanasev et al., New Experimental limit on Optical Photon Coupling to Neutral, Scalar Bosons, arXiv:0806.2631.

[80] P. Brax, C. van de Bruck, A.-C. Davis, J. Khoury, and A. Weltman, Detecting dark energy in orbit: The cosmological chameleon, Phys. Rev. D70 (2004) 123518, arXiv:astro-ph/0408415.

[81] M. S. Sloth, On the one loop corrections to inflation and the CMB anisotropies, Nucl. Phys. B748 (2006) 149–169, arXiv:astro-ph/0604488.

[82] M. S. Sloth, On the one loop corrections to inflation. II: The consistency relation, Nucl. Phys. B775 (2007) 78–94, arXiv:hep-th/0612138.

[83] D. H. Lyth, The curvature perturbation in a box, JCAP 0712 (2007) 016, arXiv:0707.0361.