A NEW APPROACH TO STEINER SYMMETRIZATION OF COERCIVE CONVEX FUNCTIONS

YOUJIANG LIN, GANGSONG LENG, AND LUJUN GUO

Abstract. In this paper, a new approach of defining Steiner symmetrization of coercive convex functions is proposed and some fundamental properties of the new Steiner symmetrization are proved. Further, using the new Steiner symmetrization, we give a different approach to prove a functional version of the Blaschke-Santaló inequality due to Ball [2].

1. Introduction

The purpose of this paper is to introduce a new way of defining Steiner symmetrization for coercive convex functions, and to explore its applications. Our new definition is motivated by and can be regarded as an improvement of a functional Steiner symmetrization of [1]. In particular, our new definition has a key property: the invariance of integral, which is not true for the definition of [1]. Moreover, our definition provides a new approach to the familiar functional Steiner symmetrization (see [7, 8]), but we do not use geometric Steiner symmetrization and our approach is more suitable for certain functional problems.

Steiner symmetrization was invented by Steiner [32] to prove the isoperimetric inequality. For over 160 years Steiner symmetrization has

Key words and phrases. Rearrangements of functions; Steiner symmetrizations; Coercive convex functions; Blaschke-Santaló inequality.

2010 Mathematics Subject Classification. 46E30, 52A40.

The authors would like to acknowledge the support from China Postdoctoral Science Foundation Grant 2013M540806, National Natural Science Foundation of China under grant 11271244 and National Natural Science Foundation of China under grant 11271282 and the 973 Program 2013CB834201.
been a fundamental tool for attacking problems regarding isoperimetry and related geometric inequalities [17, 18, 32, 33]. Steiner symmetrization appears in the titles of dozens of papers (see e.g. [4, 5, 6, 8, 10, 13, 16, 20, 21, 24, 26, 27, 31]) and plays a key role in recent work such as [19, 25, 34, 35].

Steiner symmetrization is a type of rearrangement. In the 1970s, interest in rearrangements was renewed, as mathematicians began to look for geometric proofs of functional inequalities. Rearrangements were generalized from smooth or convex bodies to measurable sets and to functions in Sobolev spaces. Functional Steiner symmetrization, as a kind of important rearrangement of functions, has been studied in [1, 7, 8, 9, 11, 12, 14]. In [7], Brascamp, Lieb, and Luttinger established that the spherical symmetrization of a nonnegative function can be approximated in $L^p(\mathbb{R}^{n+1})$ by a sequence of Steiner symmetrizations and rotations. In [8], Burchard proved that Steiner symmetrization is continuous in $W^{1,p}(\mathbb{R}^{n+1})$, $1 \leq p < \infty$, for every dimension $n \geq 1$, in the sense that $f_k \to f$ in $W^{1,p}$ implies $Sf_k \to Sf$ in $W^{1,p}$. In [14], Fortier gave a thorough review and exposition of results regarding approximating the symmetric decreasing rearrangement by polarizations and Steiner symmetrizations.

For a nonnegative measurable function $f$, the familiar definition of its Steiner symmetrization (see [7, 8, 9, 14]) is defined as following:

**Definition 1.** For a measurable function $f : \mathbb{R}^n \to \mathbb{R}^+$, let $m$ denote the Lebesgue measure, if $m([f > t]) < +\infty$ for all $t > 0$, then its Steiner symmetrization is defined as

$$
S_u f(x) = \int_0^\infty \chi_{S_u E(t)}(x) dt,
$$

where $S_u E(t)$ is the Steiner symmetrization of the level set $E(t) := \{x \in \mathbb{R}^n : f(x) > t\}$ about $u^\perp$ and $\chi_A$ denotes the characteristic function of set $A$. 

During the study of the analogy between convex bodies and log-concave functions, Artstein-Klartag-Milman in [1] defined another functional Steiner transformation as follows:

**Definition 2.** For a coercive convex function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) and a hyperplane \( H = u^\perp \ (u \in S^{n-1}) \) in \( \mathbb{R}^n \), for any \( x = x' + tu \), where \( x' \in H \) and \( t \in \mathbb{R} \), we define the Steiner symmetrization \( \tilde{S}_u f \) of \( f \) about \( H \) by

\[
(\tilde{S}_u f)(x) = \inf_{t_1 + t_2 = t} \left[ \frac{1}{2} f(x' + 2t_1 u) + \frac{1}{2} f(x' - 2t_2 u) \right]. 
\]

(1.2)

In this paper, we introduce a new way of defining the functional Steiner symmetrization for coercive convex functions. A function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \), not identically \(+\infty\), is called convex if

\[
f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)
\]

for all \( x, y \in \mathbb{R}^n \) and for \( 0 \leq \lambda \leq 1 \). A convex function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is called coercive if \( \lim_{|x| \to +\infty} f(x) = +\infty \).

**Definition 3.** For a coercive convex function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) and a hyperplane \( H = u^\perp \ (u \in S^{n-1}) \) in \( \mathbb{R}^n \), for any \( x = x' + tu \), where \( x' \in H \) and \( t \in \mathbb{R} \), we define the Steiner symmetrization \( S_u f \) of \( f \) about \( H \) by

\[
(S_u f)(x) = \sup_{\lambda \in [0,1]} \inf_{t_1 + t_2 = t} \left[ \lambda f(x' + 2t_1 u) + (1 - \lambda)f(x' - 2t_2 u) \right]. 
\]

(1.3)

Our definition \( S_u f \) is motivated by and can be regarded as an improvement of \( \tilde{S}_u f \) in Definition 2. When compared with \( \tilde{S}_u f \) in Definition 1, our definition symmetrizes a parabola-like (one-dimension) cure once at a time instead of symmetrizing the level set as in \( \tilde{S}_u f \).

The rest of the paper is organized as follows. In Section 2, we explore the analogy between convex bodies and coercive convex functions using our new definition (see Table 1). In Section 3, we will elaborate on the relations between Definition 3 and Definitions 1, 2. In Section 4, we give a completely different approach to prove a functional version of the Blaschke-Santaló inequality due to Ball [2].
Table 1. A contrast between convex bodies and coercive convex functions on Steiner symmetrization

|   | Convex bodies                                                                 | Coercive convex functions                                           |
|---|------------------------------------------------------------------------------|---------------------------------------------------------------------|
| 1 | For a convex body $K$, $S_uK$ is still a convex body and symmetric about $u^\perp$. | For a coercive convex function $f$, $S_u f$ is still a coercive convex function and symmetric about $u^\perp$. |
| 2 | $Vol_\mathbb{R}^n(S_uK) = Vol_\mathbb{R}^n(K)$.                              | $\int_{\mathbb{R}^n} \exp(-S_u f) = \int_{\mathbb{R}^n} \exp(-f)$. |
| 3 | $K$ can be transformed into an unconditional body using $n$ Steiner symmetrizations. | $f$ can be transformed into an unconditional function using $n$ Steiner symmetrizations. |
| 4 | For any convex bodies $K_1 \subset K_2$, then $S_u K_1 \subset S_u K_2$.      | For any coercive convex functions $f_1 \leq f_2$, then $S_u f_1 \leq S_u f_2$. |
| 5 | If $K$ is a symmetric about $z$, then $S_u K$ is symmetric about $z | u^\perp$. | If $f$ is even about $z$, then $S_u f$ is even about $z | u^\perp$. |
| 6 | If the sequence $\{K_i\}$ converges in the Hausdorff metric to $K$, then the sequence $\{S_u K_i\}$ will converge to $S_u K$. | If the sequence $\{\exp(-f_i)\}$ converges in the $L^p$ distance to $\exp(-f)$, then the sequence $\{\exp(-S_u f_i)\}$ will converge to $\exp(-S_u f)$. |
| 7 | There is a sequence of directions $\{u_i\}$ so that the sequence of convex bodies $K_i = S_{u_i} \ldots S_{u_1} K$ converges to the ball with the same volume as $K$. | There is a sequence of directions $\{u_i\}$ so that the sequence of log-concave functions $\exp(-f_i)$, where $f_i = S_{u_i} \ldots S_{u_1} f$, converges to a radial function with the same integral as $\exp(-f)$. |

2. The functional Steiner symmetrization

We first study the one-dimensional case. In Definition 3, when $n = 1$, $S^0 = \{-1, 1\}$ and $H = \{0\}$, it is clear that $(S_1 f)(x) = (S_{-1} f)(x)$ for any $x \in \mathbb{R}$. Let $S f$ denote Steiner symmetrization of one-dimensional function, then

$$Sf(x) = \sup_{\lambda \in [0, 1]} \inf_{x_1 + x_2 = x} [\lambda f(2x_1) + (1 - \lambda) f(2x_2)].$$
Theorem 1. If $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a coercive convex function, then $Sf(x)$ is a coercive even convex function and for any $s \in \mathbb{R}$,

$$Vol_1([f \leq s]) = Vol_1([Sf \leq s]),$$

(2.2)

where $[f \leq s] = \{x \in \mathbb{R} : f(x) \leq s\}$ denotes the sublevel set of $f$.

The following lemma is straightforward, and we omit its proof.

Lemma 1. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a coercive convex function, then we have

(i) If $a = \inf f(t)$, then $a \in (-\infty, +\infty)$ and $f^{-1}(a) = \{x \in \mathbb{R} : f(x) = a\}$ is a nonempty finite closed interval $[\mu, \nu]$, where $\mu$ may equal to $\nu$.

(ii) $f(t)$ is strictly decreasing on the interval $(-\infty, \mu]$ and strictly increasing on the interval $[\nu, +\infty)$.

(iii) If $f(c) = f(d)$ and $c < d$, then $\mu < d$ and $c < \nu$.

(iv) For $c$ and $d$ given in (iii), we have the right derivative $f'_r(d) \geq 0$ for $f$ is increasing on $[\mu, +\infty)$, we also have $f'_r(c) \leq 0$ for $f$ is decreasing on $(-\infty, \nu]$.

(v) For two intervals $[a, a + t_0]$ and $[b, b + t_0]$ with the same length $t_0 > 0$, if $f(a) = f(a + t_0)$, then either $f(b) \geq f(a)$ or $f(b + t_0) \geq f(a + t_0)$.

Proof of Theorem 1. First, we show that $Sf$ is even. For any $x \in \mathbb{R}$, by (2.1), we have

$$Sf(-x) = \sup_{\lambda \in [0,1]} \inf_{x_2 \in \mathbb{R}} [\lambda f(-2x_2 - 2x) + (1 - \lambda)f(-2x_2)]$$

$$= \sup_{\lambda \in [0,1]} \inf_{x_2 \in \mathbb{R}} [\lambda f(2x_2 - 2x) + (1 - \lambda)f(2x_2)]$$

$$= \sup_{\lambda' \in [0,1]} \inf_{x_2 \in \mathbb{R}} [\lambda' f(2x_2) + (1 - \lambda')f(2x_2 - 2x)]$$

$$= Sf(x),$$

(2.3)

which implies that $Sf$ is even.

Let $\text{dom} f := \{x \in \mathbb{R}^n : f(x) < +\infty\}$ denote the effective domain of $f$. To prove the remaining part of the theorem, we shall consider two cases: $\text{dom} f = \mathbb{R}$ and $\text{dom} f \neq \mathbb{R}$. 
Case (1) $\text{dom } f = \mathbb{R}$. There are two steps.

First Step. We shall prove that $Sf(0) = \inf f$ and for any $x > 0$, there exists some $x' \in \mathbb{R}$ such that $$Sf(x) = f(x') = f(x' - 2x).$$ (2.4)

Let $x = 0$, by (2.1), we have $$Sf(0) = \sup_{\lambda \in [0,1]} \inf_{x_1 + x_2 = 0} [\lambda f(2x_1) + (1 - \lambda)f(-2x_2)]$$ $$= \inf_{x_1 \in \mathbb{R}} f(2x_1) = \inf_{x \in \mathbb{R}} f(x).$$ (2.5)

For $x > 0$, since $f$ is coercive and convex, there exists some $x' \in \mathbb{R}$ satisfying $$f(x') = f(x' - 2x).$$ (2.6)

Indeed, let $f_x(x_1) := f(x_1) - f(x_1 - 2x)$, $a = \inf f$ and $f^{-1}(a) = [\mu, \nu]$, by Lemma 1(ii), $f_x(x_1) < 0$ if $x_1 < \mu$ and $f_x(x_1) > 0$ if $x_1 > \nu$. Since $f(x_1)$ and $f(x_1 - 2x)$ are convex functions about $x_1 \in \mathbb{R}$ and any convex function is continuous on the interior of its effective domain, thus $f_x(x_1)$ is continuous in $\mathbb{R}$. Therefore, there exists some $x'$ such that $f_x(x') = 0$.

Now we prove $Sf(x) = f(x')$, where $x > 0$ and $x'$ satisfies equality (2.6). Let $G_x(\lambda)$ be a function about $\lambda \in [0,1]$ defined as $$G_x(\lambda) := \inf_{x_1 \in \mathbb{R}} [\lambda f(2x_1) + (1 - \lambda)f(2x_1 - 2x)].$$ (2.7)

For any $\lambda \in [0,1]$, choose $x_1 = \frac{x'}{2}$, we have $$G_x(\lambda) \leq \lambda f(x') + (1 - \lambda)f(x' - 2x) = f(x').$$ (2.8)

Thus, $Sf(x) = \sup_{\lambda \in [0,1]} G_x(\lambda) \leq f(x')$.

On the other hand, we prove that there exists some $\lambda_0 \in [0,1]$ such that $G_x(\lambda_0) = f(x')$. Since $f$ is a convex function defined in $\mathbb{R}$ and by Theorem 1.5.2 in [30], both the right derivative $f'_r$ and the left derivative $f'_l$ exist and $f'_l \leq f'_r$. 

Claim 1. There exists some \( \lambda_0 \in [0, 1] \) satisfying
\[
\lambda_0 f_r'(x') + (1 - \lambda_0) f_r'(x' - 2x) = 0. \tag{2.9}
\]

Proof of Claim 1. Since \( f(x') = f(x' - 2x) \) and \( x > 0 \), by Lemma 1(iv),
we have \( f_r'(x') \geq 0 \) and \( f_r'(x' - 2x) \leq 0 \), thus \( f_r'(x') - f_r'(x' - 2x) \geq 0 \).

(i) If \( f_r'(x') - f_r'(x' - 2x) > 0 \), then (2.9) can be obtained by choosing
\[
\lambda_0 = -\frac{f_r'(x' - 2x)}{f_r'(x') - f_r'(x' - 2x)}. \tag{2.10}
\]

(ii) If \( f_r'(x') - f_r'(x' - 2x) = 0 \), then \( f_r'(x') = f_r'(x' - 2x) = 0 \), thus,
for any \( \lambda_0 \in [0, 1] \), we can get (2.9). \( \square \)

Choose a \( \lambda_0 \) satisfying (2.9), we define
\[
\Phi_{\lambda_0}(x_1) = \lambda_0 f(2x_1) + (1 - \lambda_0) f(2x_1 - 2x). \tag{2.11}
\]

Since \( f \) is a convex function, then \( \Phi_{\lambda_0} \) is a convex function about \( x_1 \).
By (2.9), we have that the right derivative and the left derivative of
\( \Phi_{\lambda_0} \) at \( x_1 = \frac{x'}{2} \) satisfy
\[
\Phi'_{\lambda_0 r}(x_1)|_{x_1 = \frac{x'}{2}} = 2\lambda_0 f_r'(x') + 2(1 - \lambda_0) f_r'(x' - 2x) = 0, \tag{2.12}
\]
and \( \Phi'_{\lambda_0 l}(x_1)|_{x_1 = \frac{x'}{2}} \leq \Phi'_{\lambda_0 r}(x_1)|_{x_1 = \frac{x'}{2}} = 0. \)

By (2.6), (2.11) and the fact that if a convex function \( f : \mathbb{R} \to \mathbb{R} \)
satisfies \( f_r'(x_0) \geq 0 \) and \( f_l'(x_0) \leq 0 \) then \( f(x_0) = \min \{ f(x) : x \in \mathbb{R} \} \),
we have
\[
\inf_{x_1 \in \mathbb{R}} \Phi_{\lambda_0}(x_1) = \Phi_{\lambda_0}(\frac{x'}{2}) = f(x'). \tag{2.13}
\]

By (2.11) and (2.13), we have
\[
Sf(x) = \sup_{\lambda \in [0, 1]} G_x(\lambda) \geq G_x(\lambda_0) = \inf_{x_1 \in \mathbb{R}} \Phi_{\lambda_0}(x_1) = f(x'). \tag{2.14}
\]

Thus, we have \( Sf(x) = f(x') = f(x' - 2x) \).

Second Step. We shall prove that \( Sf \) is coercive and convex, and
for any \( s \in \mathbb{R} \), \( \text{Vol}_1([Sf \leq s]) = \text{Vol}_1([f \leq s]) \).
First, we prove that $Sf$ is coercive. Suppose that there exists $M_0 > 0$ and a sequence $\{x_n\}$ satisfying $|x_n| > n$ and $Sf(x_n) < M_0$ for any positive integer $n$, then by (2.4), there exists $x'_n$ such that
\[
Sf(x_n) = f(x'_n) = f(x'_n - 2x_n) < M_0.
\]
(2.15)
Since $2 \max\{|x'_n|, |x'_n - 2x_n|\} \geq |x'_n| + |x'_n - 2x_n| \geq 2|x_n| > 2n$, there is a sequence $\{y_n\}$, where $y_n = x'_n$ if $|x'_n| \geq |x'_n - 2x_n|$ and $y_n = x'_n - 2x_n$ if $|x'_n| \leq |x'_n - 2x_n|$, satisfying $\lim_{n \to +\infty} |y_n| = +\infty$ and $f(y_n) < M_0$, which is contradictory with $f$ is coercive.

Next, we prove that $Sf$ is a convex function on $\mathbb{R}$. First, we prove that $Sf(x)$ is increasing on $[0, +\infty)$. In fact, by (2.4), for any $0 < x_1 < x_2$, there exist $x'_1$ and $x'_2$ such that $Sf(x_i) = f(x'_i) = f(x'_i - 2x_i)$ ($i = 1, 2$). By Lemma 1(iii), for $\mu$ and $\nu$ given in Lemma 1, we have $x'_i > \mu$ ($i = 1, 2$) and $x'_i - 2x_i < \nu$ ($i = 1, 2$). If $f(x'_i) > f(x'_2)$, since $f$ is increasing on the interval $[\mu, +\infty)$, then $x'_1 > x'_2$. By $0 < x_1 < x_2$, we have $x'_1 - 2x_1 > x'_2 - 2x_2$. Since $f$ is decreasing on the interval $(-\infty, \nu]$, we have $f(x'_1 - 2x_1) \leq f(x'_2 - 2x_2)$, which is a contradiction. The contradiction means that $f(x'_1) \leq f(x'_2)$, thus $Sf$ is increasing on $[0, +\infty)$. Since $Sf$ is even, to prove $Sf$ is convex on $\mathbb{R}$, it suffices to prove that $Sf$ is convex on $[0, +\infty)$.

For any $0 \leq x_1 < x_2$ and $0 < \alpha < 1$, by (2.4), let $x'_1$, $x'_2$ and $x_0 \triangleq (\alpha x_1 + (1 - \alpha) x_2)'$ be three real numbers satisfying
\[
Sf(x_1) = f(x'_1) = f(x'_1 - 2x_1),
\]
(2.16)
\[
Sf(x_2) = f(x'_2) = f(x'_2 - 2x_2),
\]
(2.17)
\[
Sf(\alpha x_1 + (1 - \alpha) x_2) = f(x_0) = f(x_0 - 2(\alpha x_1 + (1 - \alpha) x_2)).
\]
(2.18)
Since $f$ is a convex function, we have
\[
\alpha f(x'_1) + (1 - \alpha) f(x'_2) \geq f(\alpha x'_1 + (1 - \alpha) x'_2),
\]
(2.19)
\[
\alpha f(x'_1 - 2x_1) + (1 - \alpha) f(x'_2 - 2x_2) \\
\geq f(\alpha x'_1 + (1 - \alpha) x'_2 - 2(\alpha x_1 + (1 - \alpha) x_2)).
\]
(2.20)
Since \( f(x_0) = f(x_0 - 2(\alpha x_1 + (1 - \alpha)x_2)) \) and both \([x_0 - 2(\alpha x_1 + (1 - \alpha)x_2), x_0] \) and \([\alpha x_1' + (1 - \alpha)x_2' - 2(\alpha x_1 + (1 - \alpha)x_2), \alpha x_1' + (1 - \alpha)x_2']\) have the same length \(2(\alpha x_1 + (1 - \alpha)x_2) > 0\), by Lemma 1(v), either
\[
f(\alpha x_1' + (1 - \alpha)x_2') \geq f(x_0)
\]
(2.21)
or
\[
f(\alpha x_1' + (1 - \alpha)x_2' - 2(\alpha x_1 + (1 - \alpha)x_2)) \geq f(x_0 - 2(\alpha x_1 + (1 - \alpha)x_2)).
\]
(2.22)
If (2.21) holds, then we use (2.19) and if (2.22) holds, then we use (2.20), by (2.16)-(2.18), \( Sf \) is a convex function.

Finally, we prove that \( Vol_1([f \leq s]) = Vol_1([Sf \leq s]) \) for any \( s \in \mathbb{R} \). Since \( Sf(x) \) is an even convex function, \( Sf(0) = \inf Sf \). Since \( Sf(0) = \inf f \) by (2.5), thus \( \inf Sf = \inf f \). Let \( a = \inf Sf = \inf f \), \( (Sf)^{-1}(a) = [-\delta, \delta] \), and \( f^{-1}(a) = [\mu, \nu] \).

If \( s = a \), then \( Vol_1([f \leq s]) = \nu - \mu \) and \( Vol_1([Sf \leq s]) = 2\delta \).

Next, we prove \( \nu - \mu = 2\delta \). By Lemma 1, \( Sf \) is strictly decreasing on \((-\infty, -\delta)\) and strictly increasing on \((\delta, +\infty)\), and \( f \) is strictly decreasing on \((-\infty, \mu)\) and strictly increasing on \((\nu, +\infty)\). For \( \delta \geq 0 \), if \( \nu - \mu > 2\delta \), then \( x_0 := \delta + \frac{\nu - \mu - 2\delta}{2} > \delta \), thus \( Sf(x_0) > Sf(\delta) \), which is contradictory with
\[
Sf(x_0) = \sup_{\lambda \in [0, 1]} \inf_{x_1 \in \mathbb{R}} [\lambda f(2x_1) + (1 - \lambda)f(2x_1 - 2x_0)] 
\leq \sup_{\lambda \in [0, 1]} [\lambda f(\nu) + (1 - \lambda)f(\nu - 2x_0)] = a,
\]
(2.23)
where inequality is by choosing \( x_1 = \frac{\nu}{2} \) and last equality is by \( \nu - 2x_0 = \mu \). Thus, \( \nu - \mu \leq 2\delta \). Thus if \( \delta = 0 \), then \( \mu = \nu \). For \( \delta > 0 \), by (2.4), there exists \( \delta' \) such that \( Sf(\delta) = f(\delta') = f(\delta' - 2\delta) = a \), which implies that \( \nu - \mu \geq 2\delta \). Thus, \( \nu - \mu = 2\delta \).

If \( s > a \), by Lemma 1, equality (2.4), and \( Sf \) is even, there is a unique \( x > 0 \) and a unique \( x' \in \mathbb{R} \) such that \( Sf(-x) = Sf(x) = s = f(x') = f(x' - 2x) \), thus we have \( Vol_1([f \leq s]) = Vol_1([Sf \leq s]) = 2x \).
If \( s < a \), then \([Sf \leq s] = [f \leq s] = \emptyset\), thus \( Vol_1([f \leq s]) = Vol_1([Sf \leq s]) = 0\).

Case (2) \( \text{dom} f \neq \mathbb{R} \). There exist eight cases for \( \text{dom} f \): 1) \([\alpha, \beta]\); 2) \((\alpha, \beta)\); 3) \((\alpha, \beta]\); 4) \([\alpha, \beta]\); 5) \((-\infty, \beta]\); 6) \((-\infty, \beta)\); 7) \([\alpha, +\infty)\); 8) \((\alpha, +\infty)\). We need only prove our conclusion for \( \text{dom} f = (\alpha, \beta) \). By the same method we can prove our conclusion for other cases. For \( \text{dom} f = (\alpha, \beta) \), there exist three cases: (i) \( f \) is decreasing on \((\alpha, \beta)\); (ii) \( f \) is increasing on \((\alpha, \beta)\); (iii) \( f \) is decreasing on \((\alpha, \gamma)\] and increasing on \([\gamma, \beta)\) for some \( \gamma \in (\alpha, \beta) \). Cases (i) and (ii) are corresponding to the cases of \( \lim_{\gamma \to \beta, \gamma < \beta} \) and \( \lim_{\gamma \to \alpha, \gamma > \alpha} \gamma \) in case (iii), respectively, thus we need only prove our conclusion for case (iii).

If \( \lim_{x \to \alpha, x > \alpha} f(x) = \lim_{x \to \beta, x < \beta} f(x) \), following the proof of Case (1) (i.e., \( \text{dom} f = \mathbb{R} \)), we have that \( Sf \) is convex on \((-\frac{\beta - \alpha}{2}, \frac{\beta - \alpha}{2})\) and \( Vol_1([Sf \leq s]) = Vol_1([f \leq s]) \) for any \( s < \lim_{x \to \alpha, x > \alpha} f(x) \).

If \( \lim_{x \to \alpha, x > \alpha} f(x) \neq \lim_{x \to \beta, x < \beta} f(x) \), we may assume that

\[
\lim_{x \to \alpha, x > \alpha} f(x) = b > \lim_{x \to \beta, x < \beta} f(x) = c. \quad (2.24)
\]

If \( c = a = \inf f \), then \( f \) is decreasing on \((\alpha, \beta)\). Thus we may suppose that \( c > a \). Let \( \gamma \in (\alpha, \beta) \) satisfy \( f(\gamma) = c \). If \( |x| < \frac{\beta - \gamma}{2} \), by the proof of Case (1), there exists \( x' \in (\gamma, \beta) \) such that \( Sf(x) = f(x') = f(x' - 2x) \).

**Step 1.** We shall prove that for \( |x| \geq \frac{\beta - \gamma}{2} \) and \( |x| < \frac{\beta - \alpha}{2} \),

\[
Sf(x) = f(\beta - 2|x|). \quad (2.25)
\]

Since \( Sf \) is even, we may assume \( \frac{\beta - \gamma}{2} \leq x < \frac{\beta - \alpha}{2} \). For any \( \lambda \in [0, 1] \),

\[
\inf_{x_1 \in \mathbb{R}} [\lambda f(2x_1) + (1 - \lambda)f(2x_1 - 2x)] \\
\leq \lambda \lim_{t \to \beta, t < \beta} f(t) + (1 - \lambda)f(\beta - 2x) \\
= \lambda c + (1 - \lambda)f(\beta - 2x). \quad (2.26)
\]
Since $\frac{\beta - \gamma}{2} \leq x < \frac{\beta + \alpha}{2}$, then $\alpha < \beta - 2x \leq \gamma$. Since $f$ is decreasing on $(\alpha, \gamma]$, thus $f(\beta - 2x) \geq f(\gamma) = c$. Thus, by (2.26), we have

$$
S_f(x) = \sup_{\lambda \in [0, 1]} \inf_{x_1 \in \mathbb{R}^n} [\lambda f(2x_1) + (1 - \lambda)f(2x_1 - 2x)] \\
\leq \sup_{\lambda \in [0, 1]} [\lambda c + (1 - \lambda)f(\beta - 2x)] = f(\beta - 2x). \tag{2.27}
$$

On the other hand, we prove that $S_f(x) \geq f(\beta - 2x)$. Since $\text{dom } f = (\alpha, \beta)$ and $\inf_{x_1 \in \mathbb{R}} [\lambda f(x_1) + (1 - \lambda)f(x_1 - 2x)] = \inf f$ for $\lambda = 0$ or $\lambda = 1$, we have

$$
S_f(x) = \sup_{\lambda \in (0, 1)} \inf_{x_1 \in (x + 2x, \beta)} [\lambda f(x_1) + (1 - \lambda)f(x_1 - 2x)]. \tag{2.28}
$$

By $b > c > a$, if $f^{-1}(a) = [\mu, \nu]$, then $\alpha < \mu \leq \nu < \beta$, thus $f$ is strictly decreasing on $(\alpha, \mu]$ and strictly increasing on $[\nu, \beta)$.

**Claim 2.** For a fixed $\beta' \in (\nu, \beta) \cap (\alpha + 2x, \beta)$, there exists $\delta > 0$ such that function

$$
G_x(x_1) := \lambda f(x_1) + (1 - \lambda)f(x_1 - 2x) \tag{2.29}
$$

is decreasing on $(\alpha + 2x, \beta']$ for any $0 < \lambda < \delta$.

**Proof of Claim 2.** For $x_1 \in (\alpha + 2x, \beta']$, the right derivative of $G_x(x_1)$

$$
G'_{xr}(x_1) = \lambda f'_r(x_1) + (1 - \lambda)f'_r(x_1 - 2x) \\
\leq \lambda f'_r(\beta') + (1 - \lambda)f'_r(\beta' - 2x), \tag{2.30}
$$

where the inequality is by the right derivative of a convex function is increasing on the interior of its effective domain. Since $\beta' \in (\nu, \beta) \cap (\alpha + 2x, \beta)$ and $x \in [\frac{\beta - \gamma}{2}, \frac{\beta - \alpha}{2}]$, then $\beta' - 2x \in (\alpha, \gamma + \beta' - \beta)$, thus $f'_r(\beta') > 0$ and $f'_r(\beta' - 2x) < 0$ for $f$ is strictly increasing on $(\nu, \beta)$ and strictly decreasing on $(\alpha, \gamma]$. Thus, by (2.30), we choose

$$
\delta = \frac{-f'_r(\beta' - 2x)}{f'_r(\beta') - f'_r(\beta' - 2x)}. \tag{2.31}
$$

then $G'_{xr}(x_1) < 0$ on $(\alpha + 2x, \beta']$ for any $\lambda \in (0, \delta)$. Therefore, $G_x(x_1)$ is decreasing on $(\alpha + 2x, \beta']$ for any $\lambda \in (0, \delta)$. \qed
By (2.28) and Claim 2, we have that

\[
Sf(x) = \sup_{\lambda \in (0, 1)} \inf_{x_1 \in (\alpha + 2x, \beta)} [\lambda f(x_1) + (1 - \lambda)f(x_1 - 2x)] \\
\geq \sup_{\lambda \in (0, \delta)} \inf_{x_1 \in (\alpha + 2x, \beta)} [\lambda f(x_1) + (1 - \lambda)f(x_1 - 2x)] \\
= \sup_{\lambda \in (0, \delta)} \inf_{x_1 \in [\beta', \beta]} [\lambda f(x_1) + (1 - \lambda)f(x_1 - 2x)] \\
\geq \sup_{\lambda \in (0, \delta)} [\lambda f(\beta') + (1 - \lambda)f(\beta - 2x)] \\
= f(\beta - 2x),
\]

(2.32)

where the second inequality is by \(x_1 \in [\beta', \beta) \subset (\nu, \beta)\) and \(\beta' - 2x \leq x_1 - 2x < \beta - 2x \leq \gamma\) and \(f\) is strictly increasing on \((\nu, \beta)\) and strictly decreasing on \((\alpha, \gamma]\), and the last equality is by \(f(\beta - 2x) \geq f(\beta')\).

**Step 2.** We shall prove that \(Sf\) is convex in \(\mathbb{R}\). Since \(Sf\) is increasing on \([0, \frac{\beta - \alpha}{2})\) and \(Sf\) is even on \((-\frac{\beta - \alpha}{2}, \frac{\beta - \alpha}{2})\). Thus, it suffices to prove \(Sf\) is convex in \([0, \frac{\beta - \alpha}{2})\). For any \(x_1, x_2 \in [\frac{\beta - \gamma}{2}, \frac{\beta - \alpha}{2})\) and \(\lambda \in (0, 1)\), by (2.25) and \(f\) is convex function, then

\[
\lambda Sf(x_1) + (1 - \lambda) Sf(x_2) = \lambda f(\beta - 2x_1) + (1 - \lambda)f(\beta - 2x_2) \\
\geq f(\beta - 2(\lambda x_1 + (1 - \lambda)x_2)) \\
= Sf(\lambda x_1 + (1 - \lambda)x_2),
\]

(2.33)

where the last equality is by \(\lambda x_1 + (1 - \lambda)x_2 \in [\frac{\beta - \gamma}{2}, \frac{\beta - \alpha}{2})\). By (2.33), \(Sf\) is convex on \([\frac{\beta - \gamma}{2}, \frac{\beta - \alpha}{2})\). Because that \(Sf\) is convex in \([0, \frac{\beta - \alpha}{2}]\) by the proof in Case (1), it suffices to prove that the left derivative of \(Sf\) at \(x = \frac{\beta - \gamma}{2}\) is less than its right derivative at \(x = \frac{\beta - \gamma}{2}\).

By (2.25), we have

\[
Sf'(\frac{\beta - \gamma}{2}) = \lim_{t \to 0, \delta > 0} \frac{Sf(\frac{\beta - \gamma}{2} + t) - Sf(\frac{\beta - \gamma}{2})}{t} \\
= \lim_{t \to 0, \delta > 0} \frac{f(\gamma - 2t) - f(\gamma)}{t} = -2f'(\gamma).
\]

(2.34)

For any \(t \in (-\frac{\beta - \gamma}{2}, 0)\), we have \(\frac{\beta - \gamma}{2} + t \in (0, \frac{\beta - \gamma}{2})\). Thus there exist \(x', x'' \in (\gamma, \beta)\) such that \(x'' - x' = 2(\frac{\beta - \gamma}{2} + t)\) and \(Sf(\frac{\beta - \gamma}{2} + t) = \)
\[ f(x') = f(x''). \] Since

\[ (x' - \gamma) + 2\left(\frac{\beta - \gamma}{2} + t\right) = (x' - \gamma) + (x'' - x') = x'' - \gamma < \beta - \gamma, \]

\( x' < \gamma - 2t. \) Let \(|t|\) be sufficiently small such that \( \gamma + 2|t| < \mu, \) where \( \mu \) satisfies \( f^{-1}(a) = [\mu, \nu], \) then \( f(x') > f(\gamma - 2t) \) for \( f \) is strictly decreasing on \( (\gamma, \mu). \) Then

\[
Sf'(\frac{\beta - \gamma}{2}) = \lim_{t \to 0, t < 0} \frac{Sf(\frac{\beta - x}{2} + t) - Sf(\frac{\beta - x}{2})}{t} = \lim_{t \to 0, t < 0} \frac{f(x') - f(\gamma)}{t} \leq \lim_{t \to 0, t < 0} \frac{f(\gamma - 2t) - f(\gamma)}{t} = -2f'(\gamma). \tag{2.35}
\]

Since \( f \) is a convex function, then \( f'_1(\gamma) \leq f'_r(\gamma), \) by (2.34) and (2.35), we have \( Sf'(\frac{\beta - \gamma}{2}) \leq Sf'_r(\frac{\beta - \gamma}{2}). \)

**Step 3.** Proof of \( \text{Vol}_1([Sf \leq a]) = \text{Vol}([f \leq s]) \) for any \( s \in \mathbb{R}. \)

If \( s < c, \) the proof is the same as in Case (1).

If \( c \leq s < b, \) since \( f \) is strictly decreasing on \( (\alpha, \gamma), \) there is a unique \( x' \in (\alpha, \gamma) \) such that \( f(x') = s, \) thus \([f < s] = [x', \beta]. \) By (2.25), we have \( Sf(\frac{\beta - x'}{2}) = f(x') = s, \) thus \([Sf < s] = [-\frac{\beta - x'}{2}, \frac{\beta - x'}{2}]. \) Therefore, \( \text{Vol}_1([Sf < s]) = \text{Vol}_1([f < s]) = \beta - x'. \)

If \( s \geq b, \) then \( b < +\infty \) for \( s \in \mathbb{R}, \) we have \( \text{Vol}_1([Sf < s]) = \text{Vol}_1([f < s]) = \beta - \alpha. \quad \square \)

**Remark. 1)** By Theorem 1, for any \( x \in \mathbb{R}, \) if \( x = 0, \) then \( Sf(0) = \inf f; \) if \( x \neq 0, \) then there exist three cases:

i) \( Sf(x) = f(x') = f(x' - 2|x|) \) for some \( x' \in \mathbb{R}; \)

ii) \( Sf(x) = f(x_0 - 2|x|) \) for some \( x_0 \in \mathbb{R}; \)

iii) \( Suf(x) = f(x_0 + 2|x|) \) for some \( x_0 \in \mathbb{R}. \)

2) In Theorem 1, there exist three cases for \( \text{dom} Sf: \) i) \( \text{dom} Sf = (-\delta, \delta); \) ii) \( \text{dom} Sf = [-\delta, \delta]; \) iii) \( \text{dom} Sf = \mathbb{R}. \) \( \text{dom} Sf = (-\delta, \delta) \) is corresponding to \( \text{dom} f = (\alpha, \beta), \) \( \text{dom} f = (\alpha, \beta) \) or \( \text{dom} f = [\alpha, \beta], \) where \( \delta = \frac{\beta - \alpha}{2}. \) \( \text{dom} Sf = [-\delta, \delta] \) is corresponding to \( \text{dom} f = [\alpha, \beta]. \) \( \text{dom} Sf = \mathbb{R} \) is corresponding to \( \text{dom} f = (-\infty, \beta), \) \( \text{dom} f = (-\infty, \beta], \) \( \text{dom} f = (\alpha, +\infty), \) \( \text{dom} f = [\alpha, +\infty) \) or \( \text{dom} f = \mathbb{R}. \) For a non-empty
convex set $K \subset \mathbb{R}^n$ and a hyperplane $H = u^\perp$, where $u \in S^{n-1}$, the Steiner symmetrization $S_H K$ of $K$ about $H$ is defined as

$$S_H K = \{x' + \frac{1}{2}(t_1 - t_2)u : x' \in P_H(K), t_i \in I_K(x') \text{ for } i = 1, 2\},$$

where $P_H(K) = \{x' \in H : x' + tu \in K \text{ for some } t \in \mathbb{R}\}$ is the projection of $K$ onto $H$ and $I_K(x') = \{t \in \mathbb{R} : x' + tu \in K\}$. By the above definition and Definition 3, for coercive convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and its Steiner symmetrization $S_{u^\perp} f$, we have

$$\text{dom}(S_{u^\perp} f) = S_{u^\perp}(\text{dom} f).$$

(2.36)

We know that $\text{dom} f$ is convex if $f$ is convex and the Steiner symmetrization of a non-empty convex set is still a convex set, thus by (2.36), $\text{dom}(S_{u^\perp} f)$ is a convex set.

3) For a convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the epigraph of $f$ is defined as $\text{epi} f := \{(x, y) \in \mathbb{R}^{n+1} : x \in \text{dom} f, y \geq f(x)\}$. By the definition of epigraph and Theorem 1, for one-dimensional coercive convex function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$, we have $\text{cl}(\text{epi} Sf) = S_{e^\perp}(\text{cl}(\text{epi} f))$, where $e$ is a unit vector along the $x$-axis and $\text{cl} A$ denotes the closure of a subset $A \subset \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a coercive and convex function and $u \in S^{n-1}$. For any $x' \in u^\perp$ and $t \in \mathbb{R}$, if $\tilde{f}(t) = f(x' + tu)$ is considered as a one-dimensional function about $t$, then $S\tilde{f}(t) = S_uf(x' + tu)$. By Theorem 1, $\text{cl}(\text{epi}(S\tilde{f})) = S_{e^\perp}(\text{cl}(\text{epi}\tilde{f})).$

Since $x' \in u^\perp$ is arbitrary, thus we have

$$\text{cl}(\text{epi}(S_{u^\perp} f)) = S_{\tilde{u}^\perp}(\text{cl}(\text{epi} f)),
(2.37)$$

where $\tilde{u}^\perp \subset \mathbb{R}^{n+1}$ denotes the hyperplane through the origin and orthogonal to the unit vector $\tilde{u} = (u, 0) \in \mathbb{R}^{n+1}$.

Next, by Definition 3 and Theorem 1, we shall prove five propositions which are corresponding to properties 1-5 in Table 1.

The following lemma is an obvious fact, and we omit its proof.

Lemma 2. For $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, let $u \in S^{n-1}$ and $H = u^\perp$, if
i) \( f \) is symmetric with respect to hyperplane \( H \), i.e., for any \( x' \in H \) and \( t \in \mathbb{R} \), \( f(x' + tu) = f(x' - tu) \);

ii) for any \( x' \in H \) and \( t_1, t_2 \in \mathbb{R} \), if \( |t_1| \leq |t_2| \), then \( f(x' + t_1 u) \leq f(x' + t_2 u) \);

iii) \( f \) is convex on half-space \( H^+ := \{x' + tu : x' \in u^\perp, t \geq 0\} \).

Then \( f \) is a convex function on \( \mathbb{R}^n \).

**Proposition 2.1.** If \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is a coercive convex function and \( u \in S^{n-1} \), then \( S_u f \) is a coercive convex function and symmetric about \( u^\perp \).

**Proof.** It is clear that \( S_u f \) is symmetric about \( u^\perp \). Indeed, for any \( x' \in u^\perp \) and \( t \in \mathbb{R} \), if we consider \( S_u f(x' + tu) \) as a one-dimensional function about \( t \), then by Theorem 1 and Definition 3, we have \( S_u f(x' + tu) = S_u f(x' - tu) \).

**Step 1.** We shall prove that \( S_u f \) is coercive.

Suppose that there exist \( M_0 > 0 \) and a sequence \( \{x_n\}^{\infty}_{n=1} \subset \mathbb{R}^n \) satisfying that \( |x_n| > n \) and \( S_u f(x_n) < M_0 \). Next, we shall construct a sequence \( \{y_n\} \) satisfying \( |y_n| > n \) and \( f(y_n) < M_0 \), which is contradictory with \( f \) is coercive.

For any positive integer \( n \geq 1 \), let \( x_n = x_n' + t_n u \) and \( x_n' \in u^\perp \). There exist two cases of \( t_n \neq 0 \) and \( t_n = 0 \).

1. If \( t_n \neq 0 \), then by Theorem 1, there exist three cases:
   i) \( S_u f(x_n) = f(x_n' + t_n u) = f(x_n' + (t_n - 2t_n)u) \) for some \( t_n' \in \mathbb{R} \);
   ii) \( S_u f(x_n) = f(x_n' + (t_0 - 2t_n)u) \) for some \( t_0 \in \mathbb{R} \);
   iii) \( S_u f(x_n) = f(x_n' + (t_0 + 2t_n)u) \) for some \( t_0 \in \mathbb{R} \).

For case i), since \( |t_n'| + |t_n' - 2t_n| \geq 2|t_n| \), then either \( |t_n'| \geq |t_n| \) or \( |t_n' - 2t_n| \geq |t_n| \).

If \( |t_n'| \geq |t_n| \), let \( y_n = x_n' + t_n' u \), then \( S_u f(x_n) = f(y_n) \) and \( |y_n| = |x_n'| + |t_n'| \geq |x_n'| + |t_n| = |x_n| \).

If \( |t_n' - 2t_n| \geq |t_n| \), let \( y_n = x_n' + (t_n - 2t_n)u \), then \( S_u f(x_n) = f(y_n) \) and \( |y_n| = |x_n'| + |t_n - 2t_n| \geq |x_n'| + |t_n| = |x_n| \). Since \( |x_n| > n \), we have \( |y_n| > n \) and \( f(y_n) = S_u f(x_n) < M_0 \).
For case ii), since $|t_0| + |t_0 - 2t_n| \geq 2|t_n|$, we have either $|t_0| \geq |t_n|$
or $|t_0 - 2t_n| \geq |t_n|$. If $|t_0 - 2t_n| \geq |t_n|$, let $y_n = x_n' + (t_0 - 2t_n)u$, then $S_u f(x_n) = f(y_n)$ and $|y_n| \geq |x_n|$. If $|t_0| \geq |t_n|$, let $y_n = x_n' + t_0u$ if $x_n' + t_0u \in \text{dom } f$, otherwise let $y_n = x_n' + t_0'u$, where $t_0' \in \text{dom } f$. Since $x_n' + t_0'u \in \text{dom } f$, $|x_n' + t_0'u| > n$ and $f(x_n' + t_0'u) < f(x_n' + (t_0 - 2t_n)u)$, which can be satisfied for $\lim_{t \to t_0} t < t_0 f(x_n' + tu) \leq f(x_n' + (t_0 - 2t_n)u)$
by Theorem 1. Thus, we have $|y_n| > n$ and $f(y_n) < M_0$.

For case iii), we can construct $\{y_n\}$ with the same method as in case (ii).

(2) If $t_n = 0$, by Definition 3, we have $S f(x_n) = \inf_{t \in \mathbb{R}} f(x_n' + tu)$. Since $S_u f(x_n) < M_0$, there exists $y_n = x_n' + t'u$ such that $f(y_n) < M_0$. Since $|y_n| = |x_n'| + |t'| \geq |x_n'| = |x_n|$, we have $|y_n| > n$ and $f(y_n) < M_0$.

Step 2. We shall prove that $S_u f$ is convex.

**Claim 3.** $S_u f$ is proper, i.e., $[S_u f = +\infty] \neq \mathbb{R}^n$ and $[S_u f = -\infty] = \emptyset$.

**Proof of Claim 3.** For any $x \in \mathbb{R}^n$, let $x = x' + tu$, where $x' \in u^\perp$.

Since $f$ is a coercive convex function defined on $\mathbb{R}^n$, one dimensional function $f(x' + tu)$ about $t \in \mathbb{R}$ either is a coercive convex function or is identically $+\infty$. If $f(x' + tu)$ is a coercive convex function, then there exists $s \in \mathbb{R}$ such that $s = \inf \{f(x' + tu) : t \in \mathbb{R}\}$. Thus, we have

$$S f(x) = \sup_{\lambda \in [0, 1]} \inf_{t_1 + t_2 = t} [\lambda f(x' + 2t_1 u) + (1 - \lambda) f(x' - 2t_2 u)] \geq s,$$

which implies that $S f(x) > -\infty$. If $f(x' + tu)$ is identically $+\infty$, then $S_u f(x) = +\infty > -\infty$. By the definition of convex functions, $f$ is not identically $+\infty$, there exists $x \in \mathbb{R}^n$ such that $f(x) < +\infty$. Let $x = x_0 + tu$, where $x_0 \in u^\perp$, then

$$S_u f(x_0) = \inf_{t_1 \in \mathbb{R}} f(x_0 + t_1 u) \leq f(x) < +\infty,$$

which implies that $S_u f$ is not identically $+\infty$. \hfill \Box

By Definition 3 and Theorem 1, for any $x' \in u^\perp$, one-dimensional function $S_u f(x' + tu)$ is either an even and coercive convex function about $t \in \mathbb{R}$ or identically $+\infty$. Thus, $S_u f$ satisfies conditions i) and ii)
Lemma 2. Therefore, to prove that $S_u f$ is convex, it suffices to prove that $S_u f$ satisfies condition iii) of Lemma 2. For any $x, y \in \{x' + tu : x' \in u^\perp, t \geq 0\}$ and $\lambda \in (0, 1)$, if $x \notin \text{dom}(S_u f)$ or $y \notin \text{dom}(S_u f)$, then $S_u f(x) = +\infty$ or $S_u f(y) = +\infty$, thus

$$S_u f(\lambda x + (1 - \lambda)y) \leq \lambda S_u f(x) + (1 - \lambda)S_u f(y). \quad (2.38)$$

By Remark 2), $\text{dom}(S_u f)$ is convex. Therefore, if $x \in \text{dom}(S_u f)$ and $y \in \text{dom}(S_u f)$, then $\lambda x + (1 - \lambda)y \in \text{dom}(S_u f)$. Let $x = x' + tu$ and $y = y' + su$, where $x', y' \in u^\perp$ and $t \geq 0$ and $s \geq 0$, then $\lambda x + (1 - \lambda)y = [\lambda x' + (1 - \lambda)y'] + [\lambda t + (1 - \lambda)s]u$.

**Case 3.1.** The case of $t = 0$ and $s = 0$. For the case we have $x, y \in u^\perp$, thus $\lambda x + (1 - \lambda)y \in u^\perp$. By Definition 3 and $f$ is convex, we have

$$\lambda S_u f(x) + (1 - \lambda)S_u f(y) = \lambda \inf_{t \in \mathbb{R}} f(x + tu) + (1 - \lambda) \inf_{s \in \mathbb{R}} f(y + su)$$

$$= \inf_{(t,s) \in \mathbb{R}^2} \left[ \lambda f(x + tu) + (1 - \lambda) f(y + su) \right]$$

$$\geq \inf_{(t,s) \in \mathbb{R}^2} f(\lambda x + (1 - \lambda)y + (\lambda t + (1 - \lambda)s)u)$$

$$= S_u f(\lambda x + (1 - \lambda)y). \quad (2.39)$$

**Case 3.2.** The case of $t > 0$ and $s > 0$.

For $x = x' + tu \in \text{dom}(S_u f)$, by Theorem 1, there exist three cases:

$a_1$) There exists some $t' \in \mathbb{R}$ such that

$$S_u f(x) = f(x' + t'u) = f(x' + (t' - 2t)u); \quad (2.40)$$

$a_2$) There exists some $t_0 \in \mathbb{R}$ such that

$$S_u f(x) = f(x' + (t_0 - 2t)u) \geq \lim_{t_0 \to t_0, t_0 < t_0} f(x' + t_0'u); \quad (2.41)$$

$a_3$) There exists some $t_0 \in \mathbb{R}$ such that

$$S_u f(x) = f(x' + (t_0 + 2t)u) \geq \lim_{t_0 \to t_0, t_0 > t_0} f(x' + t_0'u). \quad (2.42)$$

For $y = y' + su \in \text{dom}(S_u f)$, by Theorem 1, there exist three cases:
There exists some $s' \in \mathbb{R}$ such that

$$S_u f(y) = f(y' + s'u) = f(y' + (s' - 2s)u); \quad (2.43)$$

There exists some $s_0 \in \mathbb{R}$ such that

$$S_u f(y) = f(y' + (s_0 - 2s)u) \geq \lim_{s_0' \to s_0, s_0' < s_0} f(y' + s'u); \quad (2.44)$$

There exists some $s_0 \in \mathbb{R}$ such that

$$S_u f(y) = f(y' + (s_0 + 2s)u) \geq \lim_{s_0' \to s_0, s_0' > s_0} f(y' + s'u). \quad (2.45)$$

We may assume that

$$f(x' + t_0 u) = \lim_{t_0' \to t_0, t_0' < t_0} f(x' + t_0'u) \text{ for case } a_2,$$

$$f(x' + t_0 u) = \lim_{t_0' \to t_0, t_0' > t_0} f(x' + t_0'u) \text{ for case } a_3,$$

$$f(y' + s_0 u) = \lim_{s_0' \to s_0, s_0' < s_0} f(y' + s_0'u) \text{ for case } b_2,$$

$$f(y' + s_0 u) = \lim_{s_0' \to s_0, s_0' > s_0} f(y' + s_0'u) \text{ for case } b_3. \quad (2.46)$$

Let $(\tilde{t}_1, \tilde{t}_2)$ be a pair of real numbers satisfying

$$(\tilde{t}_1, \tilde{t}_2) = \begin{cases} 
(t' - 2t, t') & \text{for case } a_1 \\
(t_0 - 2t, t_0) & \text{for case } a_2 \\
(t_0, t_0 + 2t) & \text{for case } a_3.
\end{cases} \quad (2.47)$$

Let $(\tilde{s}_1, \tilde{s}_2)$ be a pair of real numbers satisfying

$$(\tilde{s}_1, \tilde{s}_2) = \begin{cases} 
(s' - 2s, s') & \text{for case } b_1 \\
(s_0 - 2s, s_0) & \text{for case } b_2 \\
(s_0, s_0 + 2s) & \text{for case } b_3.
\end{cases} \quad (2.48)$$

Since $f$ is convex and by (2.40-2.45), for $i = 1, 2$, we have

$$\lambda S_u f(x) + (1 - \lambda) S_u f(y) \geq \lambda f(x' + \tilde{t}_i u) + (1 - \lambda) f(y' + \tilde{s}_i u) \geq f(\lambda x' + (1 - \lambda)y' + (\lambda \tilde{t}_i + (1 - \lambda)\tilde{s}_i)u). \quad (2.49)$$
By (2.47) and (2.48), we have

\[
\begin{align*}
[\lambda \tilde{t}_2 + (1 - \lambda)\tilde{s}_2] - [\lambda \tilde{t}_1 + (1 - \lambda)\tilde{s}_1] \\
= \lambda (\tilde{t}_2 - \tilde{t}_1) + (1 - \lambda)(\tilde{s}_2 - \tilde{s}_1) = 2[\lambda t + (1 - \lambda)s].
\end{align*}
\]

(2.50)

By \(\lambda x + (1 - \lambda)y = \lambda x' + (1 - \lambda)y' + (\lambda t + (1 - \lambda)s)u\) and Definition 3, we have

\[
S_u f(\lambda x + (1 - \lambda)y) = \sup_{\delta \in [0, 1]} \inf_{\omega \in \mathbb{R}} [\delta f(\lambda x' + (1 - \lambda)y' + \omega u) + (1 - \delta)f(\lambda x' + (1 - \lambda)y' + (\omega - 2(\lambda t + (1 - \lambda)s))u)] \\
\leq \sup_{\delta \in [0, 1]} [\delta f(\lambda x' + (1 - \lambda)y' + (\lambda \tilde{t}_2 + (1 - \lambda)\tilde{s}_2)u) + (1 - \delta)f(\lambda x' + (1 - \lambda)y' + (\lambda \tilde{t}_1 + (1 - \lambda)\tilde{s}_1)u)] \\
\leq \max_{i=1,2} f(\lambda x' + (1 - \lambda)y' + (\lambda \tilde{t}_i + (1 - \lambda)\tilde{s}_i)u) \\
\leq \lambda S_u f(x) + (1 - \lambda)S_u f(y),
\]

(2.51)

where the first inequality is by choosing \(\omega = \lambda \tilde{t}_2 + (1 - \lambda)\tilde{s}_2\) and (2.50), and the last inequality is by (2.49).

**Case 3.3.** The case of \(t = 0\) and \(s > 0\) (or \(t > 0\) and \(s = 0\)). In this case, there exists \(t_0\) such that

\[
S_u f(x) = \lim_{t \to t_0, x+tu \in \text{dom } f} f(x + tu).
\]

(2.52)

We may assume that

\[
f(x + t_0u) = \lim_{t \to t_0, x+tu \in \text{dom } f} f(x + tu).
\]

(2.53)

In the proof of Case 3.2, let \(\tilde{t}_1 = \tilde{t}_2 = t_0\), we can get the required inequality. \(\square\)

**Proposition 2.2.** Let \(f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}\) be a coercive convex function and \(u \in S^{n-1}\), then

\[
\int_{\mathbb{R}^n} e^{-(S_u f)(x)} dx = \int_{\mathbb{R}^n} e^{-f(x)} dx.
\]

(2.54)
Proof. By (2.37), for any $t \in \mathbb{R}$, we have $\text{cl}[S_u f < t] = S_u(\text{cl}[f < t])$. Since Steiner symmetrization of convex sets preserves volume, $\text{Vol}([S_u f < t]) = \text{Vol}([f < t])$. By Fubini’s theorem, we have

$$
\int_{\mathbb{R}^n} e^{-(S_u f)(x)} \, dx = \int_{\mathbb{R}} \text{Vol}([S_u f < t]) e^{-t} \, dt = \int_{\mathbb{R}} \text{Vol}([f < t]) e^{-t} \, dt = \int_{\mathbb{R}^n} e^{-f(x)} \, dx. \quad (2.55)
$$

Lemma 3. Let $u_1, u_2 \in S^{n-1}$ and $(u_1, u_2) = 0$. If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is a coercive convex function and $f$ is symmetric about $u_1^\perp$, then $S_{u_2} f$ is symmetric about both $u_1^\perp$ and $u_2^\perp$.

Proof. By Proposition 2.1, $S_{u_2} f$ is symmetric about $u_2^\perp$. Next, we prove that $S_{u_2} f$ is symmetric about $u_1^\perp$. Since $(u_1, u_2) = 0$, then $u_1 \in u_2^\perp$ and $u_2 \in u_1^\perp$. For any $x' \in u_1^\perp$, let $x'' = x' + t_{x'} u_2$, where $x'' = x'|u_2^\perp$. Then $x'' = x' - t_{x'} u_2 \in u_1^\perp$, thus $x'' + tu_2 \in u_1^\perp$. Because that $x'' \in u_2^\perp$ and $u_1 \in u_2^\perp$, thus $x'' + tu_1 \in u_2^\perp$. Thus, for any $x' \in u_1^\perp$ and $t \in \mathbb{R}$, we have

$$
(S_{u_2} f)(x' + tu_1) = (S_{u_2} f)(x'' + tu_1 + t_{x'} u_2)
$$

$$
= \sup_{\lambda \in [0, 1]} \inf_{t_1 + t_2 = x'} \left[ \lambda f(x'' + tu_1 + 2t_1 u_2) + (1 - \lambda) f(x'' + tu_1 - 2t_2 u_2) \right]
$$

$$
= \sup_{\lambda \in [0, 1]} \inf_{t_1 + t_2 = x'} \left[ \lambda f(x'' - tu_1 + 2t_1 u_2) + (1 - \lambda) f(x'' - tu_1 - 2t_2 u_2) \right]
$$

$$
= (S_{u_2} f)(x'' - tu_1 + t_{x'} u_2)
$$

$$
= (S_{u_2} f)(x' - tu_1), \quad (2.56)
$$

where the second equality is by $f$ is symmetric about $u_1^\perp$ and $x'' + tu_2 \in u_1^\perp$. This completes the proof. \qed

We say that a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is unconditional if $f(x_1, \ldots, x_n) = f(|x_1|, \ldots, |x_n|)$ for every $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

Proposition 2.3. Any coercive convex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ can be transformed into an unconditional function $\tilde{f}$ using $n$ Steiner symmetrizations.
Proof. Let \( \{u_1, \ldots, u_m\} \) be an orthonormal basis of \( \mathbb{R}^n \). By Proposition 2.1 and Lemma 3, \( S_{u_1} \cdots S_{u_m} f \) is symmetric about \( u_i \), \( i = 1, \cdots, m \), which implies that \( f \) can be transformed into an unconditional function \( \tilde{f} = S_{u_1} \cdots S_{u_m} f \) using \( n \) Steiner symmetrizations. \( \square \)

**Proposition 2.4.** Let \( f_1 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) and \( f_2 : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be coercive convex functions and \( u \in S^{n-1} \). If \( f_1 \leq f_2 \) (which implies that \( f_1(x) \leq f_2(x) \) for any \( x \in \mathbb{R}^n \)), then \( S_u f_1 \leq S_u f_2 \).

Proof. By Definition 3 and \( f_1 \leq f_2 \), for \( x = x' + tu \), where \( x' \in u^\perp \), we have

\[
S_u f_1(x) = \sup_{\lambda \in [0,1]} \inf_{t_1+t_2=t} [\lambda f_1(x' + 2t_1 u) + (1-\lambda) f_1(x' - 2t_2 u)] \\
\leq \sup_{\lambda \in [0,1]} \inf_{t_1+t_2=t} [\lambda f_2(x' + 2t_1 u) + (1-\lambda) f_2(x' - 2t_2 u)] \\
= S_u f_2(x). \tag{2.57}
\]

\( \square \)

We say a function \( f \) is even about point \( z \in \mathbb{R}^n \) if \( f(z+x) = f(z-x) \) for any \( x \in \mathbb{R}^n \). Let \( z|H \) denote the projection of \( z \) onto hyperplane \( H \).

**Proposition 2.5.** Let \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) be a coercive convex function and \( u \in S^{n-1} \), if \( f \) is even about \( z \), then \( S_u f \) is even about \( z|u^\perp \).

Proof. For any \( x \in \mathbb{R}^n \), let \( x = x' + tu \), where \( x' = x|u^\perp \). Let \( z = z' - t_0 u \), where \( z' = z|u^\perp \). By Definition 3, we have

\[
(S_u f)(z' + x) = (S_u f)(z' + x' + tu) = (S_u f)(z' + x' - tu) \\
= \sup_{\lambda \in [0,1]} \inf_{t_1+t_2=-t} [\lambda f(z' + x' + 2t_1 u) + (1-\lambda) f(z' + x' - 2t_2 u)] \\
= \sup_{\lambda \in [0,1]} \inf_{t_2 \in \mathbb{R}} [\lambda f(z + t_0 u + x' - 2t_2 u - 2tu) + (1-\lambda) f(z + t_0 u + x' - 2t_2 u)] \\
= \sup_{\lambda \in [0,1]} \inf_{t_2 \in \mathbb{R}} [\lambda f(z + x' - 2t_2 u - 2tu) + (1-\lambda) f(z + x' - 2t_2 u)] \\
= \sup_{\lambda' \in [0,1]} \inf_{t_2 \in \mathbb{R}} [\lambda' f(z + x' - 2t_2 u) + (1-\lambda') f(z + x' - 2t_2 u - 2tu)], \tag{2.58}
\]
where the second equality is by \( S_u f \) is symmetric about \( u^\perp \) and the fifth equality is by replacing \( t_0 - 2t_2 \) by \(-2t_2\).

On the other hand, since \( f \) is even about \( z \), we have

\[
(S_u f)(z' - x) = (S_u f)(z' - x' - tu)
\]

\[
= \sup_{\lambda \in [0,1]} \inf_{t_1 \in \mathbb{R}} \left[ \lambda f(z' - x' + 2t_1u) + (1 - \lambda) f(z' - x' - 2t_2u) \right]
\]

\[
= \sup_{\lambda \in [0,1]} \inf_{t_1 \in \mathbb{R}} \left[ \lambda f(z + t_0u - x' + 2t_1u) + (1 - \lambda) f(z + t_0u - x' + 2t_1u + 2tu) \right]
\]

\[
= \sup_{\lambda \in [0,1]} \inf_{t_1 \in \mathbb{R}} \left[ \lambda f(z - x' + 2t_1u) + (1 - \lambda) f(z - x' + 2t_1u + 2tu) \right]
\]

\[
= \sup_{\lambda \in [0,1]} \inf_{t_1 \in \mathbb{R}} \left[ \lambda f(z + x' - 2t_1u) + (1 - \lambda) f(z + x' - 2t_1u - 2tu) \right]
\]

where the last equality is by \( f \) is even about \( z \). By (2.58) and (2.59), we have \((S_u f)(z' + x) = (S_u f)(z' - x)\) for any \( x \in \mathbb{R}^n \). \( \square \)

3. The relation between new definition and former definitions

3.1. The relation between Definition 3 and Definition 2.

The relation can be generalized as follows:

(i) \( S_u f \) is in general larger than \( \tilde{S}_u f \) (look at Example 1).

(ii) For one-dimensional coercive convex function \( f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\} \), if \( f \) is symmetric about an axes \( x = x_0 \), i.e., \( f(x_0 - x) = f(x_0 + x) \) for any \( x \in \mathbb{R} \), then \( S f = \tilde{S} f \).

(iii) For \( n \)-dimensional coercive convex function \( f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) and \( u \in S^{n-1} \), if for any \( x' \in u^\perp \), one-dimensional function \( f(x' + tu) \) about \( t \in \mathbb{R} \) is symmetric about an axes \( t = t_0 \), then \( S_u f = \tilde{S}_u f \).

Example 1. For one-dimensional coercive convex function

\[
f(x) = \begin{cases} 
x^3 & \text{if } x \geq 0, \\
x^2 & \text{if } x \leq 0.
\end{cases}
\]

We compare \( S f \) with \( \tilde{S} f \), where

\[
S f(x) = \sup_{\lambda \in [0,1]} \inf_{x_1 + x_2 = x} \left[ \lambda f(2x_1) + (1 - \lambda) f(-2x_2) \right]
\]
\[ \tilde{S}f(x) = \inf_{x_1 + x_2 = x} \left[ \frac{1}{2}f(2x_1) + \frac{1}{2}f(-2x_2) \right]. \]

By calculation, we can get that
\[ \tilde{S}f(x) = \begin{cases} 
\frac{(-12x - 1)\sqrt{1 + 12x + 18x + 1}}{27} + 2x^2 & \text{if } x \geq 0, \\
\frac{(12x - 1)\sqrt{1 - 12x - 18x + 1}}{27} + 2x^2 & \text{if } x \leq 0.
\end{cases} \quad (3.2) \]

and
\[ Sf(x) = g^{-1}(|x|), \quad (3.3) \]
where \( g^{-1} \) is the inverse function of
\[ g(x) = \frac{1}{2}(\sqrt{x} + \sqrt{x}), \quad x \in [0, \infty). \quad (3.4) \]

By Matlab, we can draw their figures (see Figure 1). In the figure, we can find that the level sets of \( Sf \) and \( f \) have the same size and \( Sf > \tilde{S}f \).

3.2. The relation between Definition 3 and Definition 1.

In this section, we show that the two definitions are same for log-concave functions (Theorem 3.2).

**Lemma 3.1.** Let \( F = e^{-f} \) be a log-concave function, where \( f : \mathbb{R}^n \to \mathbb{R} \) is a coercive convex function, then \([S_uF > t] = S_u([F > t])\).
Proof. By Definition 1, if \( \bar{S}_u F(x) > t \), then \( x \in S_u([F > t]) \). On the other hand, if \( x \in S_u([F > t]) \), since \( S_u([F > t]) \) is an open set and \( F \) is continuous, then there exists \( t' > t \) such that \( x \in S_u([F > t']) \), by (1.1), we have \( \bar{S}_u F(x) > t \). \( \square \)

**Theorem 3.2.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a coercive convex function and \( u \in S^{n-1} \), then \( e^{-S_u f} = \bar{S}_u(e^{-f}) \), where \( S_u f \) and \( \bar{S}_u(e^{-f}) \) are given in (1.3) and (1.1), respectively.

Proof. For \( t > 0 \), we have
\[
[e^{-S_u f}] > t = [S_u f < -\ln t] = S_u([f < -\ln t]) = S_u([e^{-f} > t]),
\]
where the second equality holds by (2.37).

By Lemma 3.1, we have \([\bar{S}_u(e^{-f})] > t = S_u([e^{-f} > t])\), thus \([e^{-S_u f}] > t = [\bar{S}_u(e^{-f})] > t\). Using the “layer-cake representation”, we have
\[
e^{-S_u f} = \int_0^\infty \chi_{[e^{-S_u f}] > t}(x)\,dt = \int_0^\infty \chi_{[\bar{S}_u(e^{-f})] > t}(x)\,dt = \bar{S}_u(e^{-f}).
\]

The continuity and convergence of Steiner symmetrization in \( L^p \) space have been proved in many papers [7, 8, 9, 14], especially Proposition 3 and Theorem 2 in [14] are corresponding to the properties 6-7 in Table 1.

**4. Application to functional Blaschke-Santaló inequality**

We can use the new definition to prove some important inequalities, such as functional Blaschke-Santaló inequality, Prékopa-Leindler inequality for log-concave functions, Hardy-Littlewood inequality for log-concave functions, etc. As an illustration, here we only use it to prove the functional Blaschke-Santaló inequality for even convex functions.

For a convex body \( K \subset \mathbb{R}^n \), its polar about \( z \) is defined by \( K^z = \{ x \in \mathbb{R}^n : \sup_{y \in K} \langle x - z, y - z \rangle \leq 1 \} \). For a log-concave function
$f : \mathbb{R}^n \to [0, \infty)$, its polar about $z$ is defined by

$$f^z(x) = \inf_{y \in \mathbb{R}^n} \frac{e^{-\langle x-z, y-z \rangle}}{f(y)}. \quad (4.1)$$

To better understand this definition recall the classical Legendre transform: For a function $\phi : \mathbb{R}^n \to \mathbb{R}$, its Legendre transform about $z$ is defined by

$$L_z\phi(x) = \sup_{y \in \mathbb{R}^n} [\langle x-z, y-z \rangle - \phi(y)].$$

From above definition of polarity, if $f(x) = e^{-\phi(x)}$, where $\phi(x)$ is a convex function, then $f^z(x) = e^{-L_z\phi(x)}$. Since $L_z(L_z\phi) = \phi$ for a convex function $\phi$, $(f^z)^z = f$. For $z = 0$, we denote $L^0\phi = L\phi$.

For a convex body $K$, its Santaló point $s(K)$ satisfies $Vol(K^{s(K)}) = \min_z Vol(K^z)$. The Blaschke-Santaló inequality [3, 29] states that $Vol(K)Vol(K^{s(K)}) \leq Vol(B_2^n)^2$, where $B_2^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is the Euclidean ball ($| \cdot |$ denote the Euclidean norm). The functional Blaschke-Santaló inequality of log-concave functions is the analogue of Blaschke-Santaló inequality of convex bodies. If $f$ is a nonnegative integrable function on $\mathbb{R}^n$ such that $f^0$ has its barycenter at 0, then

$$\int_{\mathbb{R}^n} f(x)dx \int_{\mathbb{R}^n} f^0(y)dy \leq \left( \int_{\mathbb{R}^n} e^{-\frac{1}{2}|x|^2} dx \right)^2 = (2\pi)^n.$$

In the special case where the function $f$ is even, this result follows from an earlier inequality of Ball [2]; and in [15], Fradelizi and Meyer prove something more general (see also [22]). Recently, Lehec [23] gave a direct proof of the functional Blaschke-Santaló inequality.

In this paper, inspired by the proof of K. Ball [2] for Santaló inequality for centrally symmetric convex bodies, we prove functional Blaschke-Santaló inequality for even convex functions. For the non-even case, we can prove the inequality by the similar method, but we don’t prove it here.

**Theorem 4.1.** (K. Ball, [2]) Let $f : \mathbb{R}^n \to [0, \infty)$ be an even convex function. Assume that $0 < \int e^{-f} < \infty$. Then

$$\int e^{-f} \int e^{-L_f} \leq (2\pi)^n. \quad (4.2)$$
First, we give the following lemmas.

**Lemma 4.2.** Let \( f : \mathbb{R}^n \to [0, \infty) \) be an even convex function and \( u \in S^{n-1} \). Assume that \( 0 < \int e^{-f} < \infty \). Then

\[
\int e^{-\mathcal{L}f} \leq \int e^{-\mathcal{L}(S_u f)}.
\]  

(4.3)

**Proof.** After a linear transformation, it may be supposed that \( H = u^\perp = \{(x_i)_{i=1}^n : x_n = 0 \} \). For \( f \) and \( t \in \mathbb{R} \), we define a new function \( f(t)(x') := f(x' + tu) \), where \( x' \in H \).

By the definition of Steiner symmetrization, for \( x' = x'_1 + x'_2 \), where \( x', x'_1 \) and \( x'_2 \in H \), let \((x', t)\) denote \( x' + tu \), we have

\[
(\mathcal{L}(S_u f))(t)(x') = (\mathcal{L}(S_u f))(x' + tu)
\]

\[
= \sup_{(y', s) \in H \times \mathbb{R}} \left[ ((x', t), (y', s)) - (S_u f)(y' + su) \right]
\]

\[
= \sup_{(y', s) \in H \times \mathbb{R}} \left[ ((x', t), (y', s)) - \sup_{\lambda \in [0, 1]} \inf_{s_1 + s_2 = s} (\lambda f(y' + 2s_1u) + (1 - \lambda)f(y' - 2s_2u)) \right]
\]

\[
= \sup_{(y', s) \in H \times \mathbb{R}} \inf_{\lambda \in [0, 1]} \sup_{s_1 + s_2 = s} \left[ ((x', t), (y', s)) - (\lambda f(y' + 2s_1u) + (1 - \lambda)f(y' - 2s_2u)) \right]
\]

\[
\leq \sup_{(y', s) \in H \times \mathbb{R}} \left[ ((x', t), (y', s)) - \left( \frac{1}{2} f(y' + 2s_1u) + \frac{1}{2} f(y' - 2s_2u) \right) \right]
\]

\[
= \sup_{(y', s) \in H \times \mathbb{R}} \sup_{s_1 + s_2 = s} \left[ ((x', t), (y', s)) - \left( \frac{1}{2} f(y' + 2s_1u) + \frac{1}{2} f(y' + 2(s_1 - s)u) \right) \right]
\]

\[
\leq \frac{1}{2} \sup_{(y', s) \in H \times \mathbb{R}} \sup_{s_1 + s_2 = s} \left[ ((2x'_1, t), (y', 2s_1)) - f(y' + 2s_1u) \right]
\]

\[
+ \frac{1}{2} \sup_{(y', s) \in H \times \mathbb{R}} \sup_{s_1 + s_2 = s} \left[ ((2x'_2, -t), (y', 2s_1 - 2s_2)) - f(y' + 2(s_1 - s)u) \right]
\]

\[
= \frac{1}{2} \left[ (\mathcal{L}f)(2x'_1 + tu) + (\mathcal{L}f)(2x'_2 - tu) \right],
\]  

(4.4)

where the first inequality is by choosing \( \lambda = \frac{1}{2} \) and the second inequality is by \( \sup \sup (A + B) \leq \sup A + \sup B \).

Since \( x'_1 \) and \( x'_2 \) are arbitrary, by (4.4), we can get

\[
\left( e^{-\mathcal{L}(S_u f)(t)} \right)(x') \geq \sup_{x'_1 + x'_2 = x'} \left( e^{-\frac{1}{2}(\mathcal{L}f)(2x'_1)} \times e^{-\frac{1}{2}(\mathcal{L}f)(-2x'_2)} \right).
\]  

(4.5)
By (4.5) and Prékopa-Leindler inequality, we have
\[\int_{H} e^{-(\mathcal{L}(S_{u}f))(x')}dx' \geq \left( \int_{H} e^{-(\mathcal{L}f)(x')}dx' \right)^{\frac{1}{2}} \left( \int_{H} e^{-(\mathcal{L}f)(-t)(x')}dx' \right)^{\frac{1}{2}} \]
\[= \int_{H} e^{-(\mathcal{L}f)(t)(x')}dx', \tag{4.6}\]
where the last equality is by \(\mathcal{L}f\) is even (since \(f\) is even). Thus, by Fubini’s theorem, we can get the desired inequality. □

**Lemma 4.3.** Let \(h(t)\) be an increasing convex function defined on \([0, +\infty)\) and \(\int_{0}^{+\infty} e^{-h(t)}dt < \infty\). Let \(\mathcal{L}(h(\cdot \mid \cdot))\) denote the Legendre transform of function \(h(|x|)\) defined on \(\mathbb{R}^{n}\). Then
\[\int_{\mathbb{R}^{n}} e^{-h(|x|)}dx \int_{\mathbb{R}^{n}} e^{-(\mathcal{L}(h(\cdot \mid \cdot)))(x)}dx \leq (2\pi)^{n}. \tag{4.7}\]

**Proof.** By spherical coordinate transformation, we have
\[\int_{\mathbb{R}^{n}} e^{-h(|x|)}dx = \int_{S^{n-1}} \left[ \int_{0}^{+\infty} e^{-h(r)} r^{n-1}dr \right] d\omega. \tag{4.8}\]
For any \(x \in \mathbb{R}^{n}\), let \(x = t_{x}\theta_{x}\), where \(\theta_{x} = \frac{x}{|x|} \in S^{n-1}\) for \(|x| \neq 0\) and \(\theta_{x}\) is any unit vector for \(|x| = 0\), and \(t_{x} = |x|\). Then, we have
\[\mathcal{L}(h(\cdot \mid \cdot))(x) = \sup_{y \in \mathbb{R}^{n}} (\langle x, y \rangle - h(|y|)) \]
\[= \sup_{\theta_{y} \in S^{n-1}, t_{y} \geq 0} (t_{x}\theta_{x}, t_{y}\theta_{y}) - h(t_{y}) = \sup_{t_{y} \geq 0} (t_{x}t_{y} - h(t_{y})). \]
Thus, we have
\[\int_{\mathbb{R}^{n}} e^{-(\mathcal{L}(h(\cdot \mid \cdot)))(x)}dx = \int_{S^{n-1}} \left[ \int_{0}^{+\infty} e^{-\sup_{t \geq 0}(rt-h(t))} r^{n-1}dr \right] d\omega. \tag{4.9}\]
For \(r \in [0, +\infty)\), let \(f_{1}(r) = (e^{-h(r)}) r^{n-1}\), \(f_{2}(r) = (e^{-\sup_{t \geq 0}(rt-h(t))}) r^{n-1}\) and \(f_{3}(r) = \left(\frac{2}{r^{2}}\right) r^{n-1}\). Next, we shall prove that
\[\int_{0}^{+\infty} f_{1}(r)dr \int_{0}^{+\infty} f_{2}(r)dr \leq \left( \int_{0}^{+\infty} f_{3}(r)dr \right)^{2}. \tag{4.10}\]
Let \(g_{i}(t) = f_{i}(e^{t})e^{t}\) for \(i = 1, 2, 3\), then \(\int_{0}^{+\infty} f_{i}(r)dr = \int_{\mathbb{R}} g_{i}(t)dt\) and for every \(s, t \in \mathbb{R}\), \(g_{1}(s)g_{2}(t) \leq \left( g_{3}(\frac{s+t}{2}) \right)^{2}\). Hence inequality (4.10) follows from Prékopa-Leindler inequality.
By (4.8), (4.9) and (4.10), we have
\[
\int_{\mathbb{R}^n} e^{-h(|x|)} dx \int_{\mathbb{R}^n} e^{-L(h(||))(x)} dx = \omega_n^2 \int_0^{+\infty} f_1(r) dr \int_0^{+\infty} f_2(r) dr \leq \omega_n^2 \left( \int_0^{+\infty} e^{-\frac{r^2}{2}} r^{n-1} dr \right)^2 = (2\pi)^n,
\]
where \(\omega_n = n\pi^{n/2}/\Gamma(1 + \frac{n}{2})\) is the surface area of Euclidean unit ball.

**Proof of Theorem 4.1.** By the integral invariance under Steiner symmetrization (Proposition 2.2), for any \(u \in S^{n-1}\), we have
\[
\int_{\mathbb{R}^n} e^{-(Su_f)(x)} dx = \int_{\mathbb{R}^n} e^{-f(x)} dx. \quad (4.11)
\]
By (4.11) and Lemma 4.2, we have
\[
\int e^{-f} \int e^{-Lf} \leq \int e^{-Su_f} \int e^{-L(Su_f)}. \quad (4.12)
\]
By property 7 in Table 1, for log-concave function \(e^{-f} \in L^1(\mathbb{R}^n)\), there exists a sequence of directions \(\{u_i\}_{i=1}^{\infty} \subset S^{n-1}\) such that \(e^{-Su_1,...,u_if}\) converges to a radial function \(e^{-h(||)}\), where \(h(t)\) is a one-dimensional increasing convex function defined on \([0, +\infty)\). By (4.12) and Lemma 4.3 and the continuity of integral in \(L^1(\mathbb{R}^n)\), we have
\[
\int e^{-f} \int e^{-Lf} \leq \lim_{i \to +\infty} \int e^{-Su_1,...,u_if} \int e^{-L(Su_1,...,u_if)} = \int e^{-h(||)} \int e^{-L(h(||))} \leq (2\pi)^n. \quad (4.13)
\]
This completes the proof. 

**References**

[1] S. Artstein, B. Klartag, V. D. Milman, *On the Santaló point of a function and a functional Santaló inequality*, Mathematika 54 (2004), 33-48.

[2] K. Ball, *Isometric problems in \(l_p\) and sections of convex sets*, Doctoral thesis, University of Cambridge, 1986.
[3] W. Blaschke, *über affine Geometrie 7: Neue Extremeigenschaften von Ellipse und Ellipsoid*. Wilhelm Blaschke Gesammelte Werke 3. Thales Verlag, Essen (1985)

[4] G. Bianchi, P. Gronchi, *Steiner symmetrals and their distance from a ball*, Israel J. Math. 135 (2003), 181-192.

[5] G. Bianchi, D. A. Klain, E. Lutwak, D. Yang, G. Zhang, *A countable set of directions is sufficient for Steiner symmetrization*, Adv. in Appl. Math. in press.

[6] J. Bourgain, J. Lindenstrauss, V. D. Milman, *Estimates related to Steiner symmetrizations*, in: Geometric Aspects of Functional Analysis (1987-88), Lecture Notes in Math., vol. 1376, Springer, Berlin, 1989, pp. 264-273.

[7] H. J. Brascamp, E. H. Lieb, J. M. Luttinger, *A General Rearrangement Inequality for Multiple Integrals*, J. Funct. Anal. 17 (1974), 227-237.

[8] A. Burchard, *Steiner symmetrization is continuous in $W^{1,p}$*, Geom. Funct. Anal. 7 (1997), 823-860.

[9] A. Burchard, *A short course on rearrangement inequalities*, available at http://www.math.utoronto.ca/almut/rearrange.pdf, 2009.

[10] A. Cianchi, M. Chlebík, N. Fusco, *The perimeter inequality under Steiner symmetrization: cases of equality*, Ann. of Math. 162 (2005), 525-555.

[11] A. Cianchi, N. Fusco, *Strict monotonicity of functionals under Steiner symmetrization*, in: Calculus of Variations: Topics from the Mathematical Heritage of E. De Giorgi, Quad. Mat., vol. 14, Dept. Math., Seconda Univ. Napoli, Caserta, 2004, pp. 187-220.

[12] A. Cianchi, N. Fusco, *Steiner symmetric extremals in Pólya-Szegő type inequalities*, Adv. Math. 203 (2006), 673-728.

[13] K. J. Falconer, *A result on the Steiner symmetrization of a compact set*, J. London Math. Soc. 14 (1976), 385-386.

[14] M. Fortier, *Convergence results for rearrangements: Old and new*, M.S. Thesis, University of Toronto, December 2010.

[15] M. Fradelizi, M. Meyer, *Some functional forms of Blaschke-Santaló inequality*, Math. Z. 256 (2007), 379-395.

[16] R. J. Gardner, *Symmetrals and X-rays of planar convex bodies*, Arch. Math. (Basel) 41 (1983), 183-189.

[17] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. 39 (2002), 355-405.

[18] R. J. Gardner, Geometric Tomography (2nd Ed.), Cambridge University Press, New York, 2006.
[19] C. Haberl, F. Schuster, *General $L_p$ affine isoperimetric inequalities*, J. Differential Geom. 83 (2009), 1-26.

[20] B. Klartag, V. Milman, *Isomorphic Steiner symmetrization*, Invent. Math. 153 (2003), 463-485.

[21] B. Klartag, *An isomorphic version of the slicing problem*, J. Funct. Anal. 218 (2005), 372-394.

[22] J. Lehec, *Partitions and functional Santaló inequality*, Arch. Math. 92 (2009), 89-94.

[23] J. Lehec, *A direct proof of the functional Santaló inequality*, C. R. Math. Acad. Sci. Paris, 347 (2009), 55-58.

[24] M. Longinetti, *An isoperimetric inequality for convex polygons and convex sets with the same symmetrals*, Geom. Dedicata 20 (1986), 27-41.

[25] E. Lutwak, D. Yang, G. Zhang, *Orlicz projection bodies*, Adv. Math. 223 (2010), 220-242.

[26] P. Mani-Levitska, *Random Steiner symmetrizations*, Studia Sci. Math Hungar. 21 (1986), 373-378.

[27] A. McNabb, *Partial Steiner symmetrization and some conduction problems*, J. Math. Anal. Appl. 17 (1967), 221-227.

[28] M. Meyer, A. Pajor, *On the Blaschke Santaló inequality*, Arch. Math. 55 (1990), 82-93.

[29] L. A. Santaló, *An affine invariant for convex bodies of n-dimensional space*. Port. Math. 8 (1949), 155-161.

[30] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory*, Encyclopedia Math. Appl., vol. 44, Cambridge University Press, Cambridge, 1993.

[31] P. R. Scott, *Planar rectangular sets and Steiner symmetrization*, Elem. Math. 53 (1998), 36-39.

[32] J. Steiner, *Einfacher Beweis der isoperimetrische Hauptsätze*, J. Reine Angew. Math. 18 (1838), 281-296.

[33] G. Talenti, The standard isoperimetric theorem, in: P. Gruber and J. M.Wills, (Eds.), Handbook of Convex Geometry, North-Holland, Amsterdam, 1993, pp. 73-124.

[34] J. Van Schaftingen, *Universal approximation of symmetrizations by polarizations*, Proc. Amer. Math. Soc. 134 (2005), 177-186.

[35] J. Van Schaftingen, *Approximation of symmetrizations and symmetry of critical points*, Topol. Methods Nonlinear Anal. 28 (2006), 61-85.
School of Mathematical Sciences, Peking University, Beijing, 100871, China; Department of Mathematics, Department of Mathematics, Shanghai University, Shanghai, 200444, China

E-mail address: lxyoujiang@126.com

Department of Mathematics, Shanghai University, Shanghai, 200444, China

E-mail address: gieng@staff.shu.edu.cn

Department of Mathematics, Shanghai University, Shanghai, People’s Republic of China 200444

E-mail address: lujunguo0301@163.com