BRST Symmetry: Boundary Conditions and Edge States in QED

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Abstract

In manifolds with spatial boundary, BRST formalism can be used to quantize gauge theories. We show that, in a $U(1)$ gauge theory, only a subset of all the boundary conditions allowed by the self-adjointness of the Hamiltonian preserves BRST symmetry. Hence, the theory can be quantized using BRST formalism only when that subset of boundary conditions is considered. We also show that for such boundary conditions, there exist fermionic states which are localized near the boundary.

1 Introduction

Topological insulators and their surface modes are subjects of emerging interest (for example, see [1–5]). Especially, understanding a two-dimensional topological insulator in the light of fractional Hall effect has been a priority in the subject for the last decade [6,7]. In this context, theories of gauge fields interacting with matter, especially in two and three spatial dimensions, have gained importance. We investigate the quantization of $U(1)$ gauge theories with Dirac fermions from this perspective.

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Quantization of gauge theories using BRST formalism is conventional. It is elegant, yet simple. One introduces a ghost field for every constraint of the system. This breaks the gauge symmetry, but introduces a new global symmetry (called BRST symmetry) generated by appropriate combinations of the ghosts and the constraints. The generators of this new global symmetry are fermionic and hence nilpotent and the physical Hilbert space is identified by its cohomology.

We are interested in systems like topological insulators. All such real systems available for experiment are of finite size and hence have spatial boundaries. The presence of boundaries, in general, can reduce the symmetry of the system. As a reflection of this, all boundary conditions might not preserve the symmetry (as shown in [8]). Therefore, boundary conditions naturally assume significance in the discussion of gauge theories in manifolds with boundaries and their quantization using the BRST formalism.

The boundary conditions cannot be chosen arbitrarily. Rather we need to only impose those boundary conditions which define domains of self-adjointness of the Hamiltonian [9–11]. However, all such domains might not be preserved under BRST transformations. Therefore, in order to quantize the system by BRST formalism, we must choose only those boundary conditions which not only define a self-adjoint Hamiltonian, but are also consistent with the BRST symmetry.

The presence of boundaries also naturally leads to the discussion of edge states, which, if extant, play an important role in the physics of the boundary in systems like topological insulators [5, 12].

In this paper, we consider a $U(1)$ gauge theory with Dirac fermions on a $(d+1)$-dimensional manifold $M \times \mathbb{R}$ with spatial boundary $\partial M$ of codimension one. In section 2, we review the usual discussion of a $U(1)$ gauge theory. In section 3, we introduce the ghosts and invoke BRST symmetry. We obtain the set of all allowed boundary conditions by demanding the self-adjointness of the gauge fixed Hamiltonian. We show that, out of the set of boundary conditions on the gauge fields consistent with the self-adjointness of the Hamiltonian, only some of them are invariant under BRST transformations. This subset of boundary conditions is the same as that obtained by quantization of the system using the canonical formalism [9, 13].

However, we show that there is no such constraint on the boundary conditions of the Dirac fermions. Hence, any domain of self-adjointness of the Dirac Hamiltonian is compatible with the BRST symmetry.

For a system like a topological insulator, we are further required to use physical conditions to choose the suitable boundary conditions from this set of allowed BRST-preserving boundary conditions.

Finally, we discuss the possibility of fermionic edge states in the system. In a simple $(2+1)$-dimensional geometry, we solve for the eigensates of the Hamiltonian in the limit of small coupling constant, with boundary conditions that ensure the self-adjointness of the Hamiltonian and preserve the BRST symmetry. We show that there exist fermionic edge states (protected by a mass gap), which interact with soft photons and do not break BRST symmetry. These states should be experimentally detectable.
2 The Maxwell-Dirac System

Consider a gauge theory of Dirac fields (which we call \(U(1)\) Maxwell-Dirac system) on a \((d+1)\)-dimensional flat manifold \(M \times \mathbb{R}\) with spatial boundary \(\partial M\) of codimension one. We choose the metric \(g^{\mu\nu} = \text{diag}(1, -1, \ldots, -1)\). We use the convention that Greek alphabets \((\mu, \nu, \ldots)\) range from 0 to \(d\) and indices with Latin alphabets \((i, j, \ldots)\) range from 1 to \(d\).

The \(U(1)\) gauge fields \(A_\mu\) are Hermitian
\[
A_\mu^* = A_\mu
\]
and the field strength is given by
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu.
\]
The covariant derivative is
\[
D_\mu = \partial_\mu - ieA_\mu,
\]
with \(e\) the gauge coupling constant.

The Maxwell-Dirac action is given by
\[
S = \int_{M \times \mathbb{R}} d^{d+1}x \ L, \quad \text{(2.4)}
\]
\[
L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i \bar{\psi} \gamma^\mu D_\mu \psi - m \bar{\psi} \psi,
\]
where \(m\) is the mass of the fermions and \(\bar{\psi} = \psi^\dagger \gamma^0\). The Gamma matrices generate the Clifford algebra:
\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \quad \gamma^0 = \gamma^0, \quad \gamma^i = -\gamma^i.
\]

The conjugate momenta to the gauge fields \(A_\mu\) and the fermions \(\bar{\psi}, \psi\) are given by
\[
\Pi^i_{\text{gauge}} \equiv \frac{\partial L}{\partial \dot{A}_i} = F^{i0}, \quad \Pi^0_{\text{gauge}} \equiv \frac{\partial L}{\partial \dot{A}_0} = 0, \quad \Pi_{\bar{\psi}} \equiv \frac{\partial L}{\partial \dot{\bar{\psi}}} = 0, \quad \Pi_\psi \equiv \frac{\partial L}{\partial \dot{\psi}} = 0,
\]
where the dot denotes derivation with respect to time. Notice that the field \(A_0\) is not dynamical. In other words, the Lagrangian (2.5) does not depend on \(\dot{A}_0\). As a consequence, the momentum \(\Pi^0_{\text{gauge}}\) conjugate to \(A_0\) vanishes and thus \(A_0\) is arbitrary and plays the role of a Lagrange multiplier. In fact, \(\Pi^0_{\text{gauge}} = 0\) is a primary constraint and as such it is part of the gauge symmetry generator.\(^1\) The Hamiltonian is
\[
H = \int_M d^d x \left( \Pi_{\text{gauge}}^i \dot{A}_i + \Pi_\psi \dot{\bar{\psi}} - L \right),
\]
\[
= \int_M d^d x \left[ \frac{1}{2} (\Pi_{\text{gauge}}^i)^2 + \frac{1}{4} F_{ij} F^{ij} - \Pi_\psi \gamma^0 (\gamma^i D_i + im) \psi + G A_0 \right],
\]
\(^1\)A detailed review of primary constraints and their relation to gauge transformations can be found in [14].
where $G$, the Gauss law operator, is

$$G = \partial_i \Pi^i_{\text{gauge}} - ie \Pi_\psi \psi. \quad (2.11)$$

In order for this operator to generate gauge transformations infinitesimally, the correct expression for the Gauss law is not (2.11), but rather

$$G(h) \equiv \int_M d^d x \left[ \Pi^i_{\text{gauge}} \partial_i - ie \Pi_\psi \psi \right] h(x^0, \vec{x}) = 0, \quad (2.12)$$

with $h(x^0, \vec{x})$ a test function that vanishes at the spatial boundary of our manifold:

$$h(x^0, \vec{x}) \big|_{\partial M} = 0. \quad (2.13)$$

The operator $G(h)$ vanishes on quantum state vectors in the physical subspace.

This analysis must followed by a suitable choice of boundary conditions on $A_i$ and $\psi$ invoking the self-adjointness of the Hamiltonian and subsequent canonical quantization, as in [9].

### 3 BRST Symmetry

In this section, we explore the quantization of the Maxwell-Dirac theory using BRST formalism. The BRST formalism deals with the quantization of gauge fields in a rigorous mathematical framework. This approach amounts to replacing the gauge symmetry of the theory by a global BRST symmetry, which enlarges the number of degrees of freedom in the original theory. In this enlarged Hilbert space, the usual canonical quantization can be performed. Then, restricting attention to BRST-invariant states, one recovers the Hilbert space of physical states of the original theory.

The gauge symmetry of the above Maxwell-Dirac system can be replaced by the BRST global symmetry by introducing three additional fields: an auxiliary field $B$, a ghost field $G$ and an anti-ghost field $\bar{G}$. This new action is given by

$$S_{\text{BRST}} = \int_M d^{d+1} x \mathcal{L}_{\text{BRST}}, \quad (3.1)$$

$$\mathcal{L}_{\text{BRST}} = \mathcal{L} + B (\partial^\mu A_\mu - \frac{\zeta}{2} B) + (\partial^\mu \bar{G})(\partial_\mu G), \quad (3.2)$$

where $\zeta$ is a real parameter and $\mathcal{L}$ is given in (2.5).

In the presence of such new fields, the conjugate momentum $\Pi^0_{\text{gauge}}$ becomes non-zero:

$$\Pi^0_{\text{gauge}} = B. \quad (3.3)$$

On the other hand, the conjugate momenta to the auxiliary, ghost and anti-ghost fields are given by

$$\Pi_B \equiv \frac{\partial \mathcal{L}_{\text{BRST}}}{\partial B} = 0, \quad \Pi_G \equiv \frac{\partial \mathcal{L}_{\text{BRST}}}{\partial \bar{G}} = \dot{G}, \quad \Pi_{\bar{G}} \equiv \frac{\partial \mathcal{L}_{\text{BRST}}}{\partial G} = \dot{\bar{G}}. \quad (3.4)$$
The Hamiltonian is

\[ H = \int_M d^d x \left( \Pi^\mu_{\text{gauge}} \dot{A}_\mu + \Pi_\psi \dot{\psi} + \Pi_\varphi \dot{\varphi} - \mathcal{L}_{\text{BRST}} \right) \]

\[ = \int_M d^d x \left[ \frac{\zeta - 1}{2} (\Pi^0_{\text{gauge}})^2 + \frac{1}{2} (\Pi_{\text{gauge}} - \partial^i A_i)^2 - \frac{1}{2} (\partial^i A_i)^2 \right. \]

\[ - \frac{1}{2} (\partial_i A_0)^2 + \frac{1}{4} F^i_j F_{ij} - \Pi_\psi \gamma^0 (\gamma^i D_i + im - i e \gamma^0 A_0) \psi + \Pi_\varphi \Pi_\varphi - (\partial^i \varphi) (\partial_i \varphi) \] (3.6)

Defining

\[ \mathcal{P}^0 \equiv \Pi^0_{\text{gauge}} - \partial^i A_i, \quad \mathcal{P}^i \equiv \Pi^i_{\text{gauge}} - \partial^i A_0, \] (3.7)

we can rewrite the Hamiltonian as

\[ H = \int_M d^d x \left[ \frac{\zeta - 1}{2} (\Pi^0_{\text{gauge}})^2 + \frac{1}{2} (\mathcal{P}^0)^2 + \frac{1}{2} (\mathcal{P}^i)^2 - \Pi_\psi \gamma^0 (\gamma^i D_i + im - i e \gamma^0 A_0) \psi \right. \]

\[ \left. + \Pi_\varphi \Pi_\varphi - \bar{\varphi} (\partial^i \varphi) + \frac{1}{2} A_0 (\partial^i A_0) + A_i (\partial_i \partial_j A_j) - \frac{1}{2} A_i (\partial^2 A_i) \right] \]

\[ + \int_{\partial M} d^{d-1} x \left[ D_n \varphi - \frac{1}{2} A_0 \partial_n A_0 + \frac{1}{2} A_i \partial_n A_i - \frac{1}{2} A_n \partial_i A_i - \frac{1}{2} A_i \partial_i A_n \right], \] (3.8)

where \( n \) denotes the outward pointing unit vector of the boundary \( \partial M \). Removing the boundary terms, the Hamiltonian is

\[ H = \int_M d^d x \left[ \frac{\zeta - 1}{2} (\Pi^0_{\text{gauge}})^2 + \frac{1}{2} (\mathcal{P}^0)^2 + \frac{1}{2} (\mathcal{P}^i)^2 - \Pi_\psi \gamma^0 (\gamma^i D_i + im - i e \gamma^0 A_0) \psi \right. \]

\[ \left. + \Pi_\varphi \Pi_\varphi - \bar{\varphi} (\partial^i \varphi) + \frac{1}{2} A_0 (\partial^i A_0) + \frac{1}{2} A_i (\partial^2 A_i) \right]. \] (3.9)

The fields can be expanded in the basis of the eigenfunctions of the following operators:

\[ -\partial^2 A_i + 2 \partial_i \partial_j A_j = \omega^2 A_i, \quad \partial_i A_0 = \omega_0^2 A_0, \quad \partial^2 \varphi = \omega_\varphi^2 \varphi, \quad H_D \psi = E_D \psi, \] (3.10)

where \( H_D \) is the Dirac Hamiltonian given by

\[ H_D = i \gamma^0 \gamma^\mu \partial_\mu - m \gamma^0 \] (3.11)

and \( \omega^2, \omega_0^2, \omega_\varphi^2, E_D \geq 0 \), by the requirement of positivity of the Hamiltonian.

As we show in detail in the appendix, this requirement leads to the following most general boundary conditions on the fields:

\[ (\vec{A}_\perp + i \vec{F}_{n, \perp})(x) \bigg|_{\partial M} = U_\perp(x) (\vec{A}_\perp - i \vec{F}_{n, \perp})(x) \bigg|_{\partial M}, \] (3.12)

\[ (A_n + i \partial_n A_i)(x) \bigg|_{\partial M} = U_n(x) (A_n - i \partial_n A_i)(x) \bigg|_{\partial M}, \] (3.13)

\[ (A_0 + i \partial_0 A_0)(x) \bigg|_{\partial M} = U_0(x) (A_0 - i \partial_0 A_0)(x) \bigg|_{\partial M}, \] (3.14)

\[ (\varphi + i \partial_\varphi \varphi)(x) \bigg|_{\partial M} = U_\varphi(x) (\varphi - i \partial_\varphi \varphi)(x) \bigg|_{\partial M}, \] (3.15)

\[ \psi_+(x) \bigg|_{\partial M} = U_F(x) \gamma^0 \psi_-(x) \bigg|_{\partial M}. \] (3.16)
Here, \( \forall x \in \partial M \), we have defined
\[
\psi_\pm \equiv \frac{1}{2} (\pm \gamma^0 \vec{\gamma} \cdot \hat{n}) \psi,
\]
\[
F_{in}^{(A)} \equiv \partial_i A_n - \partial_n A_i
\]
(3.17)
and the operators \( U_\perp, U_n, U_0, U_g \) and \( U_F \) satisfy
\[
U_\perp U_\perp = I, \quad U_n U_n = I, \quad U_0 U_0 = I,
\]
\[
U_g U_g = I, \quad U_F U_F = I, \quad [U_F, \gamma^0 \vec{\gamma} \cdot \hat{n}] = 0.
\]
(3.18)
The ghost field can be expanded in a complete orthonormal set of functions \( \{ H_k(x^0, x^i) \} \) as
\[
G(x^0, x^i) = \sum_k C_k H_k(x^0, x^i).
\]
(3.19)
Using the Gauss law (2.12), the momenta (2.7)-(2.8) and (3.3)-(3.4) and the above ghost field expansion, the BRST charge can be written as
\[
\hat{\Omega} \equiv G(\sum_k C_k H_k) - i \int_M d^d x \ \Pi_\psi \Pi_{\text{gauge}},
\]
(3.20)
where
\[
G(\sum_k C_k H_k) = \int_M [\Pi_{\text{gauge}}(\partial_i - i e \Pi_\psi \psi)] \sum_k C_k H_m(x^0, x^i),
\]
(3.21)
This BRST charge generates the variation of the fields under which the action (3.1) remains invariant. In this work we are only interested in the BRST variation of the gauge fields \( A^i \) and fermions \( \psi \). Upon imposing the following canonical commutation relations:
\[
[\Pi_{\text{gauge}}(x^0, \vec{x}), A^j(x^0, \vec{y})] = -i \delta^{ij} \delta^d(\vec{x} - \vec{y}),
\]
(3.22)
\[
\{\Pi_\psi(x^0, \vec{x}), \psi(x^0, \vec{y})\} = \delta^d(\vec{x} - \vec{y}),
\]
(3.23)
the BRST variations of our interest are
\[
\delta A^0 = i \epsilon [\hat{\Omega}, A^0] = \epsilon \partial^0 G,
\]
(3.24)
\[
\delta A^i = i \epsilon [\hat{\Omega}, A^i] = \epsilon \partial^i G,
\]
(3.25)
\[
\delta \psi = i \epsilon [\hat{\Omega}, \psi] = -\epsilon G \psi,
\]
(3.26)
\[
\delta G = i \epsilon [\hat{\Omega}, G] = 0,
\]
(3.27)
where \( \epsilon \) is a Grassmannian number.
4 The Boundary Conditions

The boundary conditions (3.12)-(3.16) which preserve the self-adjointness of the Hamiltonian are not consistent with BRST symmetry. In the following we show that only a smaller subset of these boundary conditions preserve BRST symmetry.

As we mentioned, the BRST charge $\hat{\Omega}$ in (3.20) generates a global BRST symmetry in the action (3.1). However, in order for $\hat{\Omega}$ to generate the BRST symmetry infinitesimally, all $H_k(x^0, x^i)$ in (3.19) must vanish on $\partial M$:

$$G\big|_{\partial M} = \sum_k C_k H_k(x^0, x^i) = 0.$$  \hspace{1cm} (4.1)

This requirement implies that

$$\hat{\nabla}_\perp G(x)\big|_{\partial M} = 0, \quad \partial_0 G(x)\big|_{\partial M} = 0. \hspace{1cm} (4.2)$$

Thus, the BRST transformation (3.27) enforces $U_g = -\mathbb{I}$ in (3.15).

4.1 Allowed boundary conditions on $A_\mu$

From (3.24) and (4.2) it follows that

$$\delta A_0(x)\big|_{\partial M} = 0. \hspace{1cm} (4.3)$$

Using the above in (3.14), we get

$$[1 + U_0(x)]\delta(\partial_\mu A_0)(x)\big|_{\partial M} = \epsilon[1 + U_0(x)\partial_0 \partial_\mu G(x)\big|_{\partial M} = 0, \hspace{1cm} (4.4)$$

As $\partial_\mu G(x)\big|_{\partial M} \neq 0$ in general, the above implies that BRST symmetry enforces $U_0 = -\mathbb{I}$ and hence, the BRST-preserving boundary condition on $A_0$ is

$$A_0(x)\big|_{\partial M} = 0. \hspace{1cm} (4.5)$$

For any other boundary condition on $A_0$, the BRST symmetry will be broken.

Form (3.25) and (4.2) it is easy to check that

$$\delta \vec{A}_\perp(x)\big|_{\partial M} = 0, \quad \delta \vec{F}_{n\perp}(x)\big|_{\partial M} = 0. \hspace{1cm} (4.6)$$

Consequently, the BRST variation of (3.12) becomes trivial and the boundary conditions (3.12) are allowed by BRST symmetry for all $U_\perp$. In a similar fashion, the boundary conditions (3.13) are also not constrained by the BRST symmetry and any $U_n(x)$ is allowed.

These are the same set of boundary conditions (4.5) and (3.12) that one obtains if the theory is quantized using Dirac constraints in canonical formalism [9, 13].

In a system like a topological insulator where boundaries play a vital role, we can further use physical conditions to constrain this allowed set of BRST-preserving boundary conditions. The surface of a topological insulator, unlike the bulk (which is an insulator), behaves like
a conductor. Therefore, the tangential component of the electric field must vanish on the boundary of the topological insulator. Then, recalling that $A_0$ vanishes on the boundary, we need to choose

$$\vec{A}_\perp(x)\bigg|_{\partial M} = 0.$$  \hspace{1cm} (4.7)

This is one of the allowed boundary conditions from the set (3.12) (for this case, $U_\perp = -\mathbb{I}$) and this ensures that the tangential component of the electric field $\vec{E}_\perp = \partial_0 \vec{A}_\perp - \nabla_\perp A_0$ vanishes on the boundary. Also, this is one of the boundary conditions obtained in [13] using canonical formalism.

### 4.2 Fermionic Boundary Conditions

From (3.26) and (4.1) it follows that

$$\delta \psi(x)\bigg|_{\partial M} = 0.$$  \hspace{1cm} (4.8)

Using the above in (3.17), it is easy check that

$$\delta \psi_\pm(x)\bigg|_{\partial M} = 0.$$  \hspace{1cm} (4.9)

A BRST variation of the boundary condition (3.16) is thus trivial and hence, the boundary condition (3.16) is compatible with the BRST symmetry for any choice of $U_F$. Thus, the BRST symmetry constrains the boundary conditions on the gauge fields $A_\mu$, but it does not constrain the fermionic boundary conditions.

### 5 A (2 + 1)-Dimensional Example

In the following, we consider the (2 + 1)-dimensional case, which is particularly relevant in the context of topological insulators. Consider the (2 + 1)-dimensional manifold

$$\tilde{M} \equiv \{x^0, x^1, x^2 : \ x^1 \leq 0\}$$  \hspace{1cm} (5.1)

with spatial boundary

$$\partial \tilde{M} = \{x^0, x^1, x^2 : \ x^1 = 0\}.$$  \hspace{1cm} (5.2)

We choose the following representation of the Gamma matrices:

$$\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^3,$$  \hspace{1cm} (5.3)

with $\sigma^i$'s the Pauli matrices. It follows then that $\psi_\pm$ are given by

$$\psi_+ = \begin{pmatrix} \psi_1 \\ 0 \end{pmatrix}, \quad \psi_- = \begin{pmatrix} 0 \\ \psi_2 \end{pmatrix}.$$  \hspace{1cm} (5.4)

It is easy to check that the matrix $U_F$ must then take the form

$$U_F = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta, \tilde{\theta} \in \mathbb{R}.$$  \hspace{1cm} (5.5)
The boundary conditions in (3.16) in this case are simply
\[ \psi_1 \bigg|_{x_1=0} = -ie^{i\theta} \psi_2 \bigg|_{x_1=0} \] (5.6)
and the gauge fields satisfy the following boundary conditions:
\[ A_0 \bigg|_{x_1=0} = 0, \quad A_2 \bigg|_{x_1=0} = 0. \] (5.7)
The vanishing of \( A_0 \) on the boundary \( x_1 = 0 \) is required by BRST symmetry. However, the condition \( A_2 \big|_{x_1=0} = 0 \) is one of the many boundary conditions (3.12) that preserves BRST symmetry. We choose this particular boundary condition because it leads to the vanishing of the tangential component of the electric field on the boundary, as it should in a topological insulator.

It is easy to check that
\[ A_0^{(k)} = 0, \quad A_1^{(k)} = a_k k_2 \cos(k_1 x_1) \cos(k_2 x_2), \quad A_2^{(k)} = a_k k_1 \sin(k_1 x_1) \sin(k_2 x_2), \] (5.8)
with \( a_k \in \mathbb{C} \), satisfy the above boundary conditions and are solutions of the eigenvalue equations
\[ -\partial_j^2 A_i^{(k)} + 2\partial_i \partial_j A_j^{(k)} = \omega_k^2 A_i^{(k)}, \quad \partial_i^2 A_0^{(k)} = \omega_0^2 A_0^{(k)}, \] (5.9)
with
\[ \omega_k^2 = k_1^2 + k_2^2, \quad \omega_0 = 0. \] (5.10)
Thus, the gauge field can be expressed as
\[ A_0 = 0, \quad A_1 = \sum_{k_1, k_2} a_k k_2 \cos(k_1 x_1) \cos(k_2 x_2), \quad A_2 = \sum_{k_1, k_2} a_k k_1 \sin(k_1 x_1) \sin(k_2 x_2). \] (5.11)
Demanding reality of the gauge fields yields
\[ a_k^* = a_k \implies a_k \in \mathbb{R}. \] (5.12)
The ghost field can be expanded in the eigenfunctions of the scalar Laplacian
\[ H_{\tilde{k}} = e^{i\tilde{k}_2 x_2} \sin(\tilde{k}_1 x_1) \] (5.13)
with eigenvalues
\[ \omega_g^2 = \tilde{k}_1^2 + \tilde{k}_2^2. \] (5.14)
Hence, the ghost field can be expressed as
\[ G = \sum_{\tilde{k}_1, \tilde{k}_2} C_{\tilde{k}} H_{\tilde{k}}, \quad C_{\tilde{k}} \in \mathbb{C}. \] (5.15)
5.1 Eigenstates of the Dirac Operator

In this section, we solve for the fermionic edge states in $\tilde{M}$ when the coupling constant $g$ is small. We want to consider the interaction of the fermions with photons of very small energies. For such soft-photons, we can terminate the sums in (5.11) at small values of $k_1, k_2$, which in turn imply a small $\omega_k$.

For simplicity, we will assume that $\tilde{\theta} = \pi/2$ in (5.6). With this choice, the fermionic boundary condition (5.6) reduces to

$$\psi_1|_{x_1=0} = \psi_2|_{x_1=0}. \quad (5.16)$$

However, it is not difficult to generalize the analysis to arbitrary $\tilde{\theta}$.

For small gauge coupling constant $e$, we expand the field $\psi$ in $e$ as

$$\psi = \chi + e\xi + \ldots \quad (5.17)$$

The eigenvalue equation for the Dirac fermions

$$H_D\psi \equiv [i\gamma^0\gamma^i(\partial_i - ieA_i) + eA_0 + m\gamma_0]\psi = E\psi, \quad E \in \mathbb{R}, \quad (5.18)$$

at order 1, leads to

$$i\gamma^0(\gamma^i\partial_i - im)\chi = E\chi, \quad (5.19)$$

subject to the boundary condition (5.16). It is easy to see that the above has solution

$$\chi = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{mx_1+iEx_2}. \quad (5.20)$$

At order $e$, the eigenvalue equation (5.18) gives

$$i\gamma^0(\gamma^i\partial_i - im)\xi + \gamma^0\gamma^iA_i\chi = E\xi. \quad (5.21)$$

To solve this, we start by rewriting $A_i$ as

$$A_1 = \sum_{k_1, k_2} \frac{ak_1}{4} k_2 (e^{ik_1 x_1} + e^{-ik_1 x_1})(e^{ik_2 x_2} + e^{-ik_2 x_2}), \quad (5.22)$$

$$A_2 = -\sum_{k_1, k_2} \frac{ak_1}{4} k_1 (e^{ik_1 x_1} - e^{-ik_1 x_1})(e^{ik_2 x_2} - e^{-ik_2 x_2}). \quad (5.23)$$

Inserting the ansatz

$$\xi = \sum_{k_1, k_2} \left( \xi_k^{(1)} e^{(ik_1+m)x_1+i(k_2+E)x_2} + \xi_k^{(2)} e^{-(ik_1+m)x_1+i(k_2+E)x_2} ight.$$

$$+ \xi_k^{(3)} e^{(ik_1+m)x_1+i(-k_2+E)x_2} + \xi_k^{(4)} e^{-(ik_1+m)x_1+i(-k_2+E)x_2} \right) \quad (5.24)$$
in \((5.21)\), we obtain

\[
\xi_k^{(1)} = -\frac{a_k}{4} (2E_k - 2imk_1 + \omega_k^2)^{-1} \left( \frac{2E_k + 2imk_2 - \omega_k^2}{2E_k + 2imk_2 + \omega_k^2} \right),
\]

\[
\xi_k^{(2)} = \frac{a_k}{4} (2E_k + 2imk_1 + \omega_k^2)^{-1} \left( \frac{2E_k - 2imk_2 + \omega_k^2}{2E_k - 2imk_2 - \omega_k^2} \right),
\]

\[
\xi_k^{(3)} = -\frac{a_k}{4} (2E_k + 2imk_1 - \omega_k^2)^{-1} \left( \frac{2E_k - 2imk_2 - \omega_k^2}{2E_k - 2imk_2 + \omega_k^2} \right),
\]

\[
\xi_k^{(4)} = \frac{a_k}{4} (2E_k - 2imk_1 - \omega_k^2)^{-1} \left( \frac{2E_k + 2imk_2 + \omega_k^2}{2E_k + 2imk_2 - \omega_k^2} \right).
\]

When \(\omega_k\) is very small, we can set \(\omega_k^2 \approx 0\) and hence the above reduces to

\[
\xi_k^{(1)} = -\frac{a_k}{4} \frac{E_k + imk_2}{E_k - imk_1} \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \quad \xi_k^{(2)} = \frac{a_k}{4} \frac{E_k - imk_2}{E_k + imk_1} \left( \begin{array}{c} 1 \\ 1 \end{array} \right),
\]

\[
\xi_k^{(3)} = -\frac{a_k}{4} \frac{E_k - imk_2}{E_k + imk_1} \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \quad \xi_k^{(4)} = \frac{a_k}{4} \frac{E_k + imk_2}{E_k - imk_1} \left( \begin{array}{c} 1 \\ 1 \end{array} \right).
\]

Therefore, in the presence of soft photons,

\[
\psi = \left( e^{mx_1 + iEx_2} + e \sum_{k_1,k_2} a_k^{(1)} e^{(ik_1 + m)x_1 + (i(k_2 + E)x_2} + a_k^{(2)} e^{(-ik_1 + m)x_1 + (ik_2 + E)x_2} + a_k^{(3)} e^{(ik_1 + m)x_1 + (ik_2 + E)x_2} + a_k^{(4)} e^{(-ik_1 + m)x_1 + (ik_2 + E)x_2} \right) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) + O(e^2)
\]

with

\[
a_k^{(1)} = -\frac{a_k}{4} \frac{E_k + imk_2}{E_k - imk_1}, \quad a_k^{(2)} = \frac{a_k}{4} \frac{E_k - imk_2}{E_k + imk_1},
\]

\[
a_k^{(3)} = -\frac{a_k}{4} \frac{E_k - imk_2}{E_k + imk_1}, \quad a_k^{(4)} = \frac{a_k}{4} \frac{E_k + imk_2}{E_k - imk_1},
\]

are eigenmodes of \((5.18)\) and satisfy the boundary condition \((5.16)\).

For a sufficiently large mass \(m\), these eigenmodes are exponentially damped in the bulk and are localized near the edge \(x_1 = 0\). In real systems, like topological insulators, these modes are experimentally detectable.

### 6. Discussions

The BRST formalism provides a natural framework to quantize gauge theories in the presence of spatial boundaries, which are particularly important in real systems, like topological insulators. We have shown that in a \(U(1)\) gauge theory, out of the set of all local boundary conditions on the gauge fields allowed by the self-adjointness of the Hamiltonian, only some preserve
BRST symmetry. These BRST-preserving boundary conditions are, in general, consistent with observations in a topological insulator.

The presence of fermionic edge states in the theory is also very interesting from the perspective of a system like a topological insulator. These edge states are expected to assume an important role in the physics at the boundary: it is possible to experimentally verify the presence of these fermions localized at the boundary.

To demonstrate the presence of edge states, in the previous section we have considered a very simple (2+1)-dimensional system with flat boundaries. However, those results can be easily extended to any spacetime dimension and to any curved boundary of codimension one. Also, we considered the fermions to be massive so that the edge states are protected by the corresponding mass gap. However, one might also consider a gapless system with time-reversal symmetry. There also, we expect to find edge-localized fermions in a similar fashion, though the details in that case will be a bit different.

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Appendices

A Boundary Conditions of the Gauge Fields

As mentioned in section 3, the fields $A_i$ can be expanded in the basis of the eigenfunctions of the operator $\hat{\mathcal{O}} \equiv (-\partial_j^2 + 2\partial_i\partial_j)$. This operator is studied in [13]. To find the domain of self-adjointness of this operator we impose that

$$\int_M d^d x \left[ B_i^\dagger (-\partial_j^2 A_i + 2\partial_i\partial_j A_j) - (-\partial_j^2 B_i^\dagger + 2\partial_i\partial_j B_j^\dagger) A_i \right], \quad \forall A_i \in \mathcal{D}_\hat{\mathcal{O}}, B_i \in \mathcal{D}_\hat{\mathcal{O}}^\dagger \tag{A.1}$$

vanishes if and only if the same boundary conditions are imposed on both $A_i$ and $B_i$. Now (A.1) leads to the boundary term

$$\int_{\partial M} d^{d-1} x \left[ B_i^\dagger (-\partial_n A_i + \partial_i A_n) + B_n^\dagger (\partial_i A_i) - (-\partial_n B_i^\dagger + \partial_i B_n^\dagger) A_i - (\partial_i B_n^\dagger) A_n \right], \quad \tag{A.2}$$

which must vanish with the same conditions on $A_i$ and $B_i$. The most general local boundary conditions for which the above rule is satisfied are

$$\left. (\bar{A}_\perp + i\bar{F}_{n\perp})(x) \right|_{\partial M} = U_{\perp}(x) \left( \bar{A}_\perp - i\bar{F}_{n\perp} \right)(x) \left|_{\partial M} \right., \quad \tag{A.3}$$

$$\left. (A_n + i\partial_i A_i)(x) \right|_{\partial M} = U_n(x) \left( A_n - i\partial_i A_i \right)(x) \left|_{\partial M} \right., \quad x \in \partial M, \quad \tag{A.4}$$

with

$$\bar{F}_{n\perp} = \partial_n \bar{A}_\perp - \bar{\nabla}_\perp A_n, \quad \left. U_{\perp} \right|_{\partial M} = 1 = U_n^{\dagger} U_n. \quad \tag{A.5}$$
Similarly, $A_0$ can be expanded in the eigenfunctions of $\hat{O}_0 \equiv \partial_j^2$. The domain of self-adjointness of $\hat{O}_0$ is obtained by demanding that
\[
\int_M d^d x \left[ B_0^\dagger (\partial_j^2 A_0) - (\partial_j^2 B_0^\dagger) A_0 \right], \quad \forall A_0 \in \mathcal{D}_{\hat{O}_0}, \ B_0 \in \mathcal{D}_{\hat{O}_0}^\dagger
\] (A.6)
vanishes with the same boundary conditions on $A_0$ and $B_0$. The above leads to the boundary term
\[
\int_{\partial M} d^{d-1} x \left[ B_0^\dagger (\partial_n A_0) - (\partial_n B_0^\dagger) A_0 \right],
\] (A.7)
which must vanish with the same conditions on $A_0$ and $B_0$. It is easy to check that the most general local boundary condition which satisfies the above requirement is
\[
(A_0 + i\partial_n A_0)(x) \bigg|_{\partial M} = U_0(x)(A_0 - i\partial_n A_0)(x) \bigg|_{\partial M}, \ x \in \partial M,
\] (A.8)
with $U_0^\dagger U_0 = I$.

B Fermionic Boundary Conditions

The conventional way to quantize the fermionic field is to expand it in the basis of eigenfunctions of the Dirac Hamiltonian $H_D$ given by
\[
H_D = i\gamma^0 \gamma^\mu D_\mu + m\gamma^0 = i\gamma^0 \gamma^i (\partial_i - ieA_i) + eA_0 + \gamma^0 m.
\] (B.1)
(B.2)
The domain of self-adjointness of $H_D$ can be obtained by demanding that
\[
\int_M d^d x \chi^\dagger H_D \psi - \int_M d^d x (H_D \chi)^\dagger \psi = 0, \quad \forall \psi \in \mathcal{D}_{H_D}, \ \chi \in \mathcal{D}_{H_D}^\dagger
\] (B.3)
if and only if $\psi$ and $\chi$ fulfill the same boundary conditions.

We assume that the photon fields are real:
\[
A_\mu^\dagger = A_\mu.
\] (B.4)
Then, (B.3) reduces to
\[
i \int_M d^d x \left[ \chi^\dagger \gamma^0 \gamma^i \partial_i \psi + (\partial_i \chi)^\dagger \gamma^0 \gamma^i \psi \right] = 0,
\] (B.5)
which leads to
\[
\int_{\partial M} d^{d-1} x \chi^\dagger \gamma^0 \gamma^i \hat{n} \psi = 0.
\] (B.6)
We define the operators
\[
P_\pm \equiv \frac{1}{2}(\mathbb{1} \pm \gamma^0 \gamma^i \hat{n}).
\] (B.7)
These are projectors, since they satisfy \((P_{\pm})^2 = P_{\pm}\). In terms of these projectors, the above integral can be written as

\[
\int_{\partial M} d^{d-1}x \, \chi^\dagger (P_+ - P_-)\psi = \int_{\partial M} d^{d-1}x \, \chi^\dagger (P_+^2 - P_-^2)\psi = 0, \tag{B.8}
\]

Calling \(\psi_{\pm} \equiv P_{\pm} \psi\), we can further rewrite the above as

\[
\int_{\partial M} d^{d-1}x \, \left( \chi^\dagger \psi_+ - \chi^-\psi_- \right) = 0. \tag{B.9}
\]

This requirement leads to the following domain of self-adjointness of \(H_D\):

\[
\mathcal{D}_{H_D} = \left\{ \psi : \left. \psi_+ \right|_{\partial M} = U_F \gamma^0 \psi_- \right\}, \tag{B.10}
\]

where the matrix \(U_F\) satisfies

\[
U^\dagger_F U_F = 1. \tag{B.11}
\]

Also, as \(P_+ \gamma^0 = \gamma^0 P_+\) and \(P_\pm^2 = P_\pm\), \(U_F\) must satisfy

\[
[U_F, \gamma^0 \gamma \cdot \hat{n}] = 0. \tag{B.12}
\]

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