SPDEs with $\alpha$-stable Lévy noise: a random field approach

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Abstract

This article is dedicated to the study of an SPDE of the form

$$Lu(t, x) = \sigma(u(t, x))\dot{Z}(t, x) \quad t > 0, x \in \mathcal{O}$$

with zero initial conditions and Dirichlet boundary conditions, where $\sigma$ is a Lipschitz function, $L$ is a second-order pseudo-differential operator, $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, and $\dot{Z}$ is an $\alpha$-stable Lévy noise with $\alpha \in (0, 2)$, $\alpha \neq 1$ and possibly non-symmetric tails. To give a meaning to the concept of solution, we develop a theory of stochastic integration with respect to $Z$, by generalizing the method of [11] to higher dimensions and non-symmetric tails. The idea is to first solve the equation with “truncated” noise $\dot{Z}_K$ (obtained by removing from $Z$ the jumps which exceed a fixed value $K$), yielding a solution $u_K$, and then show that the solutions $u_L, L > K$ coincide on the event $t \leq \tau_K$, for some stopping times $\tau_K \uparrow \infty$ a.s. A similar idea was used in [22] in the setting of Hilbert-space valued processes. A major step is to show that the stochastic integral with respect to $Z_K$ satisfies a $p$-the moment inequality, for $p \in (\alpha, 1)$ if $\alpha < 1$, and $p \in (\alpha, 2)$ if $\alpha > 1$. This inequality plays the same role as the Burkholder-Davis-Gundy inequality in the theory of integration with respect to continuous martingales.

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1 Introduction

Modeling phenomena which evolve in time or space-time and are subject to random perturbations is a fundamental problem in stochastic analysis. When these perturbations are known to exhibit an extreme behavior, as seen frequently in finance or environmental studies, a model relying on the Gaussian distribution is not appropriate. A suitable alternative could be a model based on a heavy-tailed distribution, like the stable distribution. In such a model, these perturbations are allowed to have extreme values with a probability which is significantly higher than in a Gaussian-based model.

In the present article, we introduce precisely such a model, given rigorously by a stochastic partial differential equation (SPDE) driven by a noise term which has a stable distribution over any space-time region, and has independent values over disjoint space-time regions (i.e. it is a Lévy noise). More precisely, we consider the SPDE:

\[ Lu(t, x) = \sigma(u(t, x)) \dot{Z}(t, x), \quad t > 0, x \in \Omega \]

with zero initial conditions and Dirichlet boundary conditions, where \( \sigma \) is a Lipschitz function, \( L \) is a second-order pseudo-differential operator on a bounded domain \( \Omega \subset \mathbb{R}^d \), and \( \dot{Z}(t, x) = \frac{\partial^{\alpha+1}Z}{\partial x_1 \cdots \partial x_d} \) is the formal derivative of an \( \alpha \)-stable Lévy noise with \( \alpha \in (0, 2) \), \( \alpha \neq 1 \). The goal is to find sufficient conditions on the fundamental solution \( G(t, x, y) \) of the equation \( Lu = 0 \), which will ensure the existence of a mild solution of equation (1). We say that a predictable process \( u = \{u(t, x); t \geq 0, x \in \Omega\} \) is a mild solution of (1) if for any \( t > 0, x \in \Omega \),

\[ u(t, x) = \int_0^t \int_\Omega G(t, x, y)\sigma(u(s, y))Z(ds, dy) \quad \text{a.s.} \]

We assume that \( G(t, x, y) \) is a function in \( t \), which excludes from our analysis the case of the wave equation with \( d \geq 3 \).

To explain the connections with other works, we describe briefly the construction of the noise (the details are given in Section 2 below). This construction is similar to that of a classical \( \alpha \)-stable Lévy process, and is based on a Poisson random measure (PRM) \( N \) on \( \mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R}\setminus\{0\}) \) of intensity \( dt dx \nu_\alpha(dz) \), where

\[ \nu_\alpha(dz) = \left[p\alpha z^{-\alpha-1}1_{(0,\infty)}(z) + q\alpha(-z)^{-\alpha-1}1_{(-\infty,0)}(z)\right]dz \]
for some $p, q \geq 0$ with $p + q = 1$. More precisely, for any set $B \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$Z(B) = \int_{B \times \{|z| \leq 1\}} z\tilde{N}(ds, dx, dz) + \int_{B \times \{|z| > 1\}} zN(ds, dx, dz) - \mu|B|,$$  \hspace{1cm} (4)

where $\tilde{N}(B \times \cdot) = N(B \times \cdot) - |B|\nu_\alpha(\cdot)$ is the compensated process and $\mu$ is a constant (specified by Lemma 2.3 below). Here, $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R}^d)$ is the class of bounded Borel sets in $\mathbb{R}_+ \times \mathbb{R}^d$ and $|B|$ is the Lebesque measure of $B$.

As the term on the right-hand side of (2) is a stochastic integral with respect to $Z$, such an integral should be constructed first. Due to the poor integrability properties of the measure $\nu_\alpha$, this cannot be done directly from (4), using integration with respect to $N$ and $\tilde{N}$. Our construction of the integral is an extension to random fields of the construction provided by Giné and Marcus in [11] in the case of an $\alpha$-stable Lévy process $\{Z(t)\}_{t \in [0,1]}$. Unlike these authors, we do not assume that the measure $\nu_\alpha$ is symmetric.

Since any Lévy noise is related to a PRM, in a broad sense, one could say that this problem originates in Itô’s papers [12] and [13] regarding the stochastic integral with respect to a Poisson noise. SPDEs driven by a compensated PRM were considered for the first time in [14], using the approach based on Hilbert-space-valued solutions. This study was motivated by an application to neurophysiology leading to the cable equation. In the case of the heat equation, a similar problem was considered in [1], [26] and [3] using the approach based on random-field solutions. One of the results of [26] shows that the heat equation:

$$\frac{\partial u}{\partial t}(t, x) = \frac{1}{2}\Delta u(t, x) + \int_{U} f(t, x, u(t, x); z)\tilde{N}(t, x, dz) + g(t, x, u(t, x))$$

has a unique solution in the space of predictable processes $u$ satisfying $\sup_{(t,x)\in[0,T] \times \mathbb{R}^d} E|u(t, x)|^p < \infty$, for any $p \in (1 + 2/d, 2]$. In this equation, $\tilde{N}$ is the compensated process corresponding to a PRM $N$ on $\mathbb{R}_+ \times \mathbb{R}^d \times U$ of intensity $dtdx\nu(dz)$, for an arbitrary $\sigma$-finite measure space $(U, \mathcal{B}(U), \nu)$ with measure $\nu$ satisfying $\int_U |z|^p\nu(dz) < \infty$. Because of this later condition, this result cannot be used in our case with $U = \mathbb{R}\setminus\{0\}$ and $\nu = \nu_\alpha$. For similar reasons, the results of [3] also do not cover the case of an $\alpha$-stable noise. However, in the case $\alpha > 1$, we will be able to exploit successfully some ideas of [26] for treating the equation with “truncated” noise $Z_K$, obtained by removing from $Z$ the jumps exceeding a value $K$ (see Section 5.2 below).
The heat equation with the same type of noise as in the present article was examined in [16] and [18] in the cases $\alpha < 1$, respectively $\alpha > 1$, assuming that the noise has only positive jumps (i.e. $q = 0$). The methods used by these authors are different from those presented here, since they investigate the more difficult case of a non-Lipschitz function $\sigma(u) = u^\delta$ with $\delta > 0$. In [16], Mueller removes the atoms of $Z$ of mass smaller than $2^{-n}$ and solves the equation driven by the noise obtained in this way; here we remove the atoms of $Z$ of mass larger than $K$ and solve the resulting equation. In [18], Mytnik uses a martingale problem approach and gives the existence of a pair $(u, Z)$ which satisfies the equation (the so-called “weak solution”), whereas in the present article we obtain the existence of a solution $u$ for a given noise $Z$ (the so-called “strong solution”). In particular, when $\alpha > 1$ and $\delta = 1/\alpha$, the existence of a “weak solution” of the heat equation with $\alpha$-stable Lévy noise is obtained in [18] under the condition

$$\alpha < 1 + \frac{2}{d}$$

which we encounter here as well. It is interesting to note that (5) is the necessary and sufficient condition for the existence of the density of the super-Brownian motion with “$\alpha - 1$”-stable branching (see [7]). Reference [17] examines the heat equation with multiplicative noise (i.e. $\sigma(u) = u$), driven by an $\alpha$-stable Lévy noise $Z$ which does not depend on time.

To conclude the literature review, we should point out that there are many references related to stochastic differential equations with $\alpha$-stable Lévy noise, using the approach based on Hilbert-space valued solutions. We refer the reader to Section 12.5 of the monograph [22], and to [21], [2], [15], [23] for a sample of relevant references. See also the survey article [20] for an approach based on the white noise theory for Lévy processes.

This article is organized as follows.

- In Section 2 we review the construction of the $\alpha$-stable Lévy noise $Z$, and we show that this can be viewed as an independently scattered random measure with jointly $\alpha$-stable distributions.

- In Section 3 we consider the linear equation (11) (with $\sigma(u) = 1$) and we identify the necessary and sufficient condition for the existence of the solution. This condition is verified in the case of some examples.
• Section 4 contains the construction of the stochastic integral with respect to the $\alpha$-stable noise $Z$, for $\alpha \in (0, 2)$. The main effort is dedicated to proving a maximal inequality for the tail of the integral process, when the integrand is a simple process. This extends the construction of [11] to the case random fields and non-symmetric measure $\nu_{\alpha}$.

• In Section 5, we introduce the process $Z_K$ obtained by removing from $Z$ the jumps exceeding a fixed value $K$, and we develop a theory of integration with respect to this process. For this, we need to treat separately the cases $\alpha < 1$ and $\alpha > 1$. In both cases, we obtain a $p$-th moment inequality for the integral process for $p \in (\alpha, 1)$ if $\alpha < 1$, and $p \in (\alpha, 2)$ if $\alpha > 1$. This inequality plays the same role as the Burkholder-Davis-Gundy inequality in the theory of integration with respect to continuous martingales.

• In Section 6, we prove the main result about the existence of the mild solution of equation (1). For this, we first solve the equation with “truncated” noise $Z_K$ using a Picard iteration scheme, yielding a solution $u_K$. We then introduce a sequence $(\tau_K)_{K \geq 1}$ of stopping times with $\tau_K \uparrow \infty$ a.s. and we show that the solutions $u_L, L > K$ coincide on the event $t \leq \tau_K$. For the definition of the stopping times $\tau_K$, we need again to consider separately the cases $\alpha < 1$ and $\alpha > 1$.

• Appendix A contains some results about the tail of a non-symmetric stable random variable, and the tail of an infinite sum of random variables. Appendix B gives an estimate for the Green function associated to the fractional power of the Laplacian. Appendix C gives a local property of the stochastic integral with respect to $Z$ (or $Z_K$).

2 Definition of the noise

In this section we review the construction of the $\alpha$-stable Lévy noise on $\mathbb{R}_+ \times \mathbb{R}^d$ and investigate some of its properties.

Let $N = \sum_{i \geq 1} \delta_{(t_i, X_i, z_i)}$ be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$, defined on a probability space $(\Omega, \mathcal{F}, P)$, with intensity measure $dtdx\nu_{\alpha}(dz)$, where $\nu_{\alpha}$ is given by (3). Let $(\varepsilon_j)_{j \geq 0}$ be a sequence of positive real numbers such that $\varepsilon_j \to 0$ as $j \to \infty$ and $1 = \varepsilon_0 > \varepsilon_1 > \varepsilon_2 > \ldots$. Let

$$\Gamma_j = \{z \in \mathbb{R}; \varepsilon_j < |z| \leq \varepsilon_{j-1}\}, \quad j \geq 1 \quad \text{and} \quad \Gamma_0 = \{z \in \mathbb{R}; |z| > 1\}.$$
For any set $B \in \mathcal{B}_b(\mathbb{R}^+ \times \mathbb{R}^d)$, we define

$$L_j(B) = \int_{B \times \Gamma_j} zN(dt, dx, dz) = \sum_{(T_i, X_i) \in B} Z_i 1\{z_i \in \Gamma_j\}, \quad j \geq 0.$$ 

**Remark 2.1** The variable $L_0(B)$ is finite since the sum above contains finitely many terms. To see this, we use the “one-point uncompactification” as explained in Section 6.1.3 of [25], i.e. we view $N$ as a point process on the space $\mathbb{R}^+ \times \mathbb{R}^d \times (\mathbb{R}\backslash\{0\})$. Since the set $\Gamma_0$ is relatively compact in $\mathbb{R}\backslash\{0\}$, and the point process $N$ has finitely many points in any relatively compact set, $N(B \times \Gamma_0) = \text{card}\{i \geq 1; (T_i, X_i, J_i) \in B \times \Gamma_0\} < \infty$.

For any $j \geq 0$, the variable $L_j(B)$ has a compound Poisson distribution with jump intensity measure $|B| \cdot \nu_\alpha|_{\Gamma_j}$, i.e.

$$E[e^{iuL_j(B)}] = \exp\left\{|B| \int_{\Gamma_j} (e^{iu - 1})\nu_\alpha(dz)\right\}, \quad u \in \mathbb{R}. \quad (6)$$

It follows that $E(L_j(B)) = |B| \int_{\Gamma_j} z\nu_\alpha(dz)$ and $\text{Var}(L_j(B)) = |B| \int_{\Gamma_j} z^2\nu_\alpha(dz)$ for any $j \geq 0$. Hence $\text{Var}(L_j(B)) < \infty$ for any $j \geq 1$ and $\text{Var}(L_0(B)) = \infty$.

If $\alpha > 1$, then $E(L_0(B))$ is finite. Define

$$Y(B) = \sum_{j \geq 1} [L_j(B) - E(L_j(B))] + L_0(B). \quad (7)$$

This sum converges a.s. by Kolmogorov’s criterion since $\{L_j(B) - E(L_j(B))\}_{j \geq 1}$ are independent zero-mean random variables with $\sum_{j \geq 1} \text{Var}(L_j(B)) < \infty$.

From (6) and (7), it follows that $Y(B)$ is an infinitely divisible random variable with characteristic function:

$$E(e^{iuY(B)}) = \exp\left\{|B| \int_{\mathbb{R}} (e^{iu - 1} - iuz 1\{|z| \leq 1\})\nu_\alpha(dz)\right\}, \quad u \in \mathbb{R}. \quad (8)$$

Hence $E(Y(B)) = |B| \int_{\mathbb{R}} z 1\{|z| > 1\}\nu_\alpha(dz)$ and $\text{Var}(Y(B)) = |B| \int_{\mathbb{R}} z^2\nu_\alpha(dz)$.

**Lemma 2.2** The family $\{Y(B); B \in \mathcal{B}_b(\mathbb{R}^+ \times \mathbb{R}^d)\}$ defined by (7) is an independently scattered random measure, i.e.

(a) for any disjoint sets $B_1, \ldots, B_n$ in $\mathcal{B}_b(\mathbb{R}^+ \times \mathbb{R}^d)$, $Y(B_1), \ldots, Y(B_n)$ are independent;

(b) for any sequence $(B_n)_{n \geq 1}$ of disjoint sets in $\mathcal{B}_b(\mathbb{R}^+ \times \mathbb{R}^d)$ such that $\bigcup_{n \geq 1} B_n$ is bounded, $Y(\bigcup_{n \geq 1} B_n) = \sum_{n \geq 1} Y(B_n)$ a.s.
Proof: (a) Note that for any function $\varphi \in L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ with compact support $K$, we can define the random variable $Y(\varphi) = \sum_{j \geq 1} [L_j(\varphi) - E(L_j(\varphi))] + L_0(\varphi)$ where $L_j(\varphi) = \int_{K \times \Gamma_j} \varphi(t, x)zN(dt, dx, dz)$. For any $u \in \mathbb{R}$, we have:

$$E(e^{iuY(\varphi)}) = \exp \left\{ \int_{\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}} (e^{iuz\varphi(t, x)} - 1 - iuz\varphi(t, x)1_{\{|z| \leq 1\}})dt dx \nu_\alpha(dz) \right\}.$$  

(9)

For any disjoint sets $B_1, \ldots, B_n$ and for any $u_1, \ldots, u_n \in \mathbb{R}$, we have:

$$E[\exp(i \sum_{k=1}^n u_k Y(B_k))] = E[\exp(iY(\sum_{k=1}^n u_k 1_{B_k}))]$$

$$= \exp \left\{ \int_{\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}} (e^{i\sum_{k=1}^n u_k 1_{B_k}(t,x)} - 1 - iz\sum_{k=1}^n u_k 1_{B_k}(t,x))dt dx \nu_\alpha(dz) \right\}$$

$$= \exp \left\{ \sum_{k=1}^n |B_k| \int_{\mathbb{R}} (e^{izu_k} - 1 - iu_k z1_{\{|z| \leq 1\}})\nu_\alpha(dz) \right\}$$

(10)

using (9) with $\varphi = \sum_{k=1}^n u_k 1_{B_k}$ for the second equality, and (6) for the last equality. This proves that $Y(B_1), \ldots, Y(B_n)$ are independent.

(b) Let $S_n = \sum_{k=1}^n Y(B_k)$ and $S = Y(B)$, where $B = \bigcup_{n \geq 1} B_n$. By Lévy’s equivalence theorem, $(S_n)_{n \geq 1}$ converges a.s. if and only if it converges in distribution. By (10), with $u_k = u$ for all $i = 1, \ldots, k$, we have:

$$E(e^{iuS_n}) = \exp \left\{ \left| \bigcup_{k=1}^n B_k \right| \int_{\mathbb{R}} (e^{izu} - 1 - iz1_{\{|z| \leq 1\}})\nu_\alpha(dz) \right\}.$$  

This clearly converges to $E(e^{iuS}) = \exp \left\{ |B| \int_{\mathbb{R}} (e^{izu} - 1 - iz1_{\{|z| \leq 1\}})\nu_\alpha(dz) \right\}$, and hence $(S_n)_{n \geq 1}$ converges in distribution to $S$. □

Recall that a random variable $X$ has an $\alpha$-stable distribution with parameters $\alpha \in (0, 2), \sigma \in [0, \infty), \beta \in [-1, 1], \mu \in \mathbb{R}$ if for any $u \in \mathbb{R},$

$$E(e^{iuX}) = \exp \left\{ -|u|^\alpha \sigma^\alpha \left( 1 - i\text{sgn}(u)\beta \tan \frac{\pi \alpha}{2} \right) + iu\mu \right\} \quad \text{if } \alpha \neq 1, \text{ or}$$

$$E(e^{iuX}) = \exp \left\{ -|u|\sigma \left( 1 + i\text{sgn}(u)\frac{2}{\pi} \ln |u| \right) + iu\mu \right\} \quad \text{if } \alpha = 1$$

(see Definition 1.1.6 of [27]). We denote this distribution by $S_\alpha(\sigma, \beta, \mu).$
Lemma 2.3 \( Y(B) \) has a \( S_\alpha(\sigma|B|^{1/\alpha}, \beta, \mu|B|) \) distribution with \( \beta = p - q \),

\[
\sigma^\alpha = \int_0^\infty \frac{\sin x}{x^\alpha} dx = \left\{ \begin{array}{ll}
\frac{\Gamma(2-\alpha)}{2} \cos \frac{\pi \alpha}{2} & \text{if } \alpha \neq 1 \\
\frac{\pi}{2} & \text{if } \alpha = 1
\end{array} \right., \quad \mu = \left\{ \begin{array}{ll}
\frac{\beta}{\alpha-1} & \text{if } \alpha \neq 1 \\
\beta c_0 & \text{if } \alpha = 1
\end{array} \right.,
\]

and \( c_0 = \int_0^\infty (\sin z - z 1_{(z \leq 1)}) z^{-2} dz \). If \( \alpha > 1 \), then \( E(Y(B)) = \mu|B| \).

Proof: We first express the characteristic function \( \mathcal{E}(\alpha) \) of \( Y(B) \) in Feller's canonical form (see Section XVII.2 of [9]):

\[
E(e^{i\alpha Y(B)}) = \exp \left\{ ib|B| + |B| \int_\mathbb{R} \frac{e^{iz} - 1 - iz}{z^2} M_\alpha(dz) \right\}
\]

with \( M_\alpha(dz) = z^2 \nu_\alpha(dz) \) and \( b = \int_\mathbb{R} (\sin z - z 1_{(|z| \leq 1)}) \nu_\alpha(dz) \). Then the result follows from the calculations done in Example XVII.3.(g) of [9]. □

From Lemma 2.2 and Lemma 2.3 it follows that

\[
Z = \{ Z(B) = Y(B) - \mu|B|; B \in \mathcal{B}_0(\mathbb{R}_+ \times \mathbb{R}^d) \}
\]

is an \( \alpha \)-stable random measure, in the sense of Definition 3.3.1 of [27], with control measure \( m(B) = \sigma^\alpha|B| \) and constant skewness intensity \( \beta \). In particular, \( Z(B) \) has a \( S_\alpha(\sigma|B|^{1/\alpha}, \beta, 0) \) distribution.

We say that \( Z \) is an \( \alpha \)-stable Lévy noise. Coming back to the original construction \( \mathcal{T} \) of \( Y(B) \) and noticing that

\[
\mu|B| = -|B| \int_\mathbb{R} z 1_{(|z| \leq 1)} \nu_\alpha(dz) = -\sum_{j \geq 1} E(L_j(B)) \quad \text{if } \alpha < 1,
\]

\[
\mu|B| = |B| \int_\mathbb{R} z 1_{(|z| > 1)} \nu_\alpha(dz) = E(L_0(B)) \quad \text{if } \alpha > 1,
\]

it follows that \( Z(B) \) can be represented as:

\[
Z(B) = \sum_{j \geq 0} L_j(B) = \int_{B \times (\mathbb{R}\setminus\{0\})} z N(dt, dx, dz) \quad \text{if } \alpha < 1,
\]

\[
Z(B) = \sum_{j \geq 0} [L_j(B) - E(L_j(B))] = \int_{B \times (\mathbb{R}\setminus\{0\})} z \tilde{N}(dt, dx, dz) \quad \text{if } \alpha > 1.
\]

Here \( \tilde{N} \) is the compensated Poisson measure associated to \( N \), i.e., \( \tilde{N}(A) = N(A) - E(N(A)) \) for any relatively compact set \( A \) in \( \mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R}\setminus\{0\}) \).

In the case \( \alpha = 1 \), we will assume that \( p = q \) so that \( \nu_\alpha \) is symmetric around 0, \( E(L_j(B)) = 0 \) for all \( j \geq 1 \), and \( Z(B) \) admits the same representation as in the case \( \alpha < 1 \).
3 The linear equation

As a preliminary investigation, we consider first equation (11) with \( \sigma = 1 \):

\[
Lu(t, x) = \dot{Z}(t, x), \quad t > 0, x \in \mathcal{O}
\]  

(13)

with zero initial conditions and Dirichlet boundary conditions. In this section \( \mathcal{O} \) is a bounded domain in \( \mathbb{R}^d \) or \( \mathcal{O} = \mathbb{R}^d \).

By definition, the process \( \{u(t, x); t \geq 0, x \in \mathcal{O}\} \) given by:

\[
u(t, x) = \int_0^t \int_{\mathcal{O}} G(t - s, x, y)Z(ds, dy)
\]  

(14)

is a mild solution of (13), provided that the stochastic integral on the right-hand side of (14) is well-defined.

We define now the stochastic integral of a deterministic function \( \varphi \):

\[
Z(\varphi) = \int_0^\infty \int_{\mathbb{R}^d} \varphi(t, x)Z(dt, dx).
\]

If \( \varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d) \), this can be defined by approximation with simple functions, as explained in Section 3.4 of [27]. The process \( \{Z(\varphi); \varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d)\} \) has jointly \( \alpha \)-stable finite dimensional distributions. In particular, each \( Z(\varphi) \) has a \( S_\alpha(\sigma_\varphi, \beta, 0) \)-distribution with scale parameter:

\[
\sigma_\varphi = \sigma \left( \int_0^\infty \int_{\mathbb{R}^d} |\varphi(t, x)|^\alpha dx dt \right)^{1/\alpha}.
\]

More generally, a measurable function \( \varphi : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R} \) is integrable with respect to \( Z \) if there exists a sequence \( \{\varphi_n\}_{n \geq 1} \) of simple functions such that \( \varphi_n \to \varphi \) a.e., and for any \( B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d) \), the sequence \( \{Z(\varphi_n1_B)\} \) converges in probability (see [24]).

The next result shows that condition \( \varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d) \) is also necessary for the integrability of \( \varphi \) with respect to \( Z \). Due to Lemma [22] this follows immediately from the general theory of stochastic integration with respect to independently scattered random measures developed in [24].

**Lemma 3.1** A deterministic function \( \varphi \) is integrable with respect to \( Z \) if and only if \( \varphi \in L^\alpha(\mathbb{R}_+ \times \mathbb{R}^d) \).
Proof: We write the characteristic function of $Z(B)$ in the form used in [24]:

$$E(e^{iuZ(B)}) = \exp\left\{ \int_B [iua + \int_{\mathbb{R}} (e^{izu} - 1 - iu\tau(z))\nu_\alpha(dz)]\,dtdx \right\}$$

with $a = \beta - \mu$, $\tau(z) = z$ if $|z| \leq 1$ and $\tau(z) = \text{sgn}(z)$ if $|z| > 1$. By Theorem 2.7 of [24], $\phi$ is integrable with respect to $Z$ if and only if

$$\int_{\mathbb{R}^+ \times \mathbb{R}^d} |U(\phi(t,x))|dtdx < \infty \quad \text{and} \quad \int_{\mathbb{R}^+ \times \mathbb{R}^d} V(\phi(t,x))dtdx < \infty$$

where $U(y) = ay + \int_{\mathbb{R}} (\tau(yz) - y\tau(z))\nu_\alpha(dz)$ and $V(y) = \int_{\mathbb{R}} (1 \wedge |yz|^2)\nu_\alpha(dz)$. Direct calculations show that in our case, $U(y) = -\beta \alpha^{-1}y^\alpha$ if $\alpha \neq 1$, $U(y) = 0$ if $\alpha = 1$, and $V(y) = \frac{2\alpha}{2^\alpha - \alpha}y^\alpha$. □

The following result follows immediately from (14) and Lemma 3.1.

**Proposition 3.2** Equation (13) has a mild solution if and only if for any $t > 0, x \in \mathcal{O}$

$$I_\alpha(t) = \int_0^t \int_\mathcal{O} G(s,x,y)^\alpha dy\,ds < \infty. \quad (15)$$

In this case, $\{u(t,x); t \geq 0, x \in \mathcal{O}\}$ has jointly $\alpha$-stable finite-dimensional distributions. In particular, $u(t,x)$ has a $S_\alpha(\sigma I_\alpha(t)^{1/\alpha}, \beta, 0)$ distribution.

Condition (15) can be easily verified in the case of several examples.

**Example 3.3** (Heat equation) Let $L = \frac{\partial}{\partial t} - \frac{1}{2}\Delta$. Assume first that $\mathcal{O} = \mathbb{R}^d$. Then $G(t,x,y) = G(t,x-y)$, where

$$G(t,x) = \frac{1}{(2\pi t)^{d/2}} \exp\left( -\frac{|x|^2}{2t} \right), \quad (16)$$

and condition (15) is equivalent to (5). In this case, $I_\alpha(t) = c_{d_0}t^{d(1-\alpha)/2+1}$. If $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, then $G(t,x,y) \leq \overline{G}(t,x-y)$ (see p. 74 of [17]) and condition (15) is implied by (5).

**Example 3.4** (Parabolic equations) Let $L = \frac{\partial}{\partial t} - \mathcal{L}$ where

$$\mathcal{L}f(x) = \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x) \frac{\partial f}{\partial x_i}(x) \quad (17)$$
is the generator of a Markov process with values in $\mathbb{R}^d$, without jumps (a diffusion). Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ or $\mathcal{O} = \mathbb{R}^d$. By Aronson estimate (see e.g. Theorem 2.6 of [22]), under some assumptions on the coefficients $a_{ij}, b_i$, there exist some constants $c_1, c_2 > 0$ such that

$$G(t, x, y) \leq c_1 t^{-d/2} \exp \left( -\frac{|x - y|^2}{c_2 t} \right)$$

for all $t > 0$ and $x, y \in \mathcal{O}$. In this case, condition (15) is implied by (15).

**Example 3.5 (Heat equation with fractional power of the Laplacian)** Let $L = \frac{\partial}{\partial t} + (-\Delta)_{\gamma}$ for some $\gamma > 0$. Assume that $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$ or $\mathcal{O} = \mathbb{R}^d$. Then (see e.g. Appendix B.5 of [22])

$$G(t, x, y) = \int_0^\infty G(s, x, y) g_{t,\gamma}(s) ds = \int_0^\infty G(t^{1/\gamma}, x, y) g_{1,\gamma}(s) ds,$$

where $G(t, x, y)$ is the fundamental solution of $\frac{\partial u}{\partial t} - \Delta u = 0$ on $\mathcal{O}$ and $g_{t,\gamma}$ is the density of the measure $\mu_{t,\gamma}$, $(\mu_{t,\gamma})_{t \geq 0}$ being a convolution semigroup of measures on $[0, \infty)$ whose Laplace transform is given by:

$$\int_0^\infty e^{-us} g_{t,\gamma}(s) ds = \exp (-tu^\gamma), \quad \forall u > 0.$$

Note that if $\gamma < 1$, $g_{t,\gamma}$ is the density of $S_t$, where $(S_t)_{t \geq 0}$ is a $\gamma$-stable subordinator with Lévy measure $\rho_\gamma(dx) = \frac{1}{\Gamma(2 - \gamma)} x^{-\gamma - 1} 1_{(0,\infty)}(x) dx$.

Assume first that $\mathcal{O} = \mathbb{R}^d$. Then $G(t, x, y) = \overline{G}(t, x - y)$, where

$$\overline{G}(t, x) = \int_{\mathbb{R}^d} e^{\xi x} e^{-t|\xi|^{2\gamma}} d\xi.$$

If $\gamma < 1$, then $\overline{G}(t, \cdot)$ is the density of $X_t$, $(X_t)_{t \geq 0}$ being a symmetric $(2\gamma)$-stable Lévy process with values in $\mathbb{R}^d$ defined by $X_t = W_{S_t}$, with $(W_t)_{t \geq 0}$ a Brownian motion in $\mathbb{R}^d$ with variance 2. By Lemma B.1 (Appendix B), if $\alpha > 1$, then (15) holds if and only if

$$\alpha < 1 + \frac{d}{2\gamma}.$$

If $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$, then $G(t, x, y) \leq \overline{G}(t, x - y)$ (by Lemma 2.1 of [16]). In this case, if $\alpha > 1$, then (15) is implied by (21).
Example 3.6 (Cable equation in $\mathbb{R}$) Let $Lu = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u$ and $\mathcal{O} = \mathbb{R}$. Then $G(t, x, y) = \overline{G}(t, x - y)$, where

$$G(t, x) = \frac{1}{\sqrt{4\pi t}} \exp \left( -\frac{|x|^2}{4t} - t \right),$$

and condition (15) holds for any $\alpha \in (0, 2)$.

Example 3.7 (Wave equation in $\mathbb{R}^d$ with $d = 1, 2$) Let $L = \frac{\partial^2}{\partial t^2} - \Delta$ and $\mathcal{O} = \mathbb{R}^d$ with $d = 1$ or $d = 2$. Then $G(t, x, y) = \overline{G}(t, x - y)$, where

$$G(t, x) = \begin{cases} 
1 & \text{if } d = 1 \\
\frac{1}{2\pi} \cdot \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x| < t\}} & \text{if } d = 2.
\end{cases}$$

Condition (15) holds for any $\alpha \in (0, 2)$. In this case, $I_\alpha(t) = 2^{-\alpha}t^2$ if $d = 1$ and $I_\alpha(t) = \frac{(2\pi)^{1-\alpha}}{(2-\alpha)(3-\alpha)} t^{3-\alpha}$ if $d = 2$.

4 Stochastic integration

In this section we construct a stochastic integral with respect $Z$ by generalizing the ideas of [11] to the case of random fields. Unlike these authors, we do not assume that $Z(B)$ has a symmetric distribution, unless $\alpha = 1$.

Let $\mathcal{F}_t = \mathcal{F}_t^N \vee \mathcal{N}$ where $\mathcal{N}$ is the $\sigma$-field of negligible sets in $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_t^N$ is the $\sigma$-field generated by $N([0, s] \times A \times \Gamma)$ for all $s \in [0, t], A \in \mathcal{B}_b(\mathbb{R}^d)$ and for all Borel sets $\Gamma \subset \mathbb{R}\setminus\{0\}$ bounded away from 0. Note that $\mathcal{F}_t^Z \subset \mathcal{F}_t^N$ where $\mathcal{F}_t^Z$ is the $\sigma$-field generated by $Z([0, s] \times A), s \in [0, t], A \in \mathcal{B}_b(\mathbb{R}^d)$.

A process $X = \{X(t, x)\}_{t \geq 0, x \in \mathbb{R}^d}$ is called elementary if it of the form

$$X(t, x) = 1_{(a,b)]}(t)1_A(x)Y$$

(22)

where $0 \leq a < b$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and $Y$ is $\mathcal{F}_a$-measurable. A simple process is a linear combination of elementary processes. Note that any simple process $X$ can be written as:

$$X(t, x) = 1_{[0]}(t)Y_0(x) + \sum_{i=0}^{N-1} 1_{(t_i, t_{i+1}]}(t)Y_i(x)$$

(23)
with \( 0 = t_0 < t_1 < \ldots < t_N < \infty \) and \( Y_i(x) = \sum_{j=1}^{m_i} 1_{A_{ij}}(x)Y_{ij} \), where \((Y_{ij})_{j=1,\ldots,m_i}\) are \( \mathcal{F}_{t_i} \)-measurable and \((A_{ij})_{j=1,\ldots,m_i}\) are disjoint sets in \( \mathcal{B}_b(\mathbb{R}^d) \).

Without loss of generality, we assume that \( Y_0 = 0 \).

We denote by \( \mathcal{P} \) the predictable \( \sigma \)-field on \( \Omega \times \mathbb{R}_+ \times \mathbb{R}^d \), i.e., the \( \sigma \)-field generated by all simple processes. We say that a process \( X = \{X(t, x)\}_{t \geq 0, x \in \mathbb{R}^d} \) is predictable if the map \((\omega, t, x) \mapsto X(\omega, t, x)\) is \( \mathcal{P} \)-measurable.

**Remark 4.1** One can show that the predictable \( \sigma \)-field \( \mathcal{P} \) is the \( \sigma \)-field generated by the class \( \mathcal{C} \) of processes \( X = \{X(t, x)\}_{t \geq 0, x \in \mathbb{R}^d} \) such that \( t \mapsto X(\omega, t, x) \) is left-continuous for any \( \omega \in \Omega, x \in \mathbb{R}^d \) and \((\omega, x) \mapsto X(\omega, t, x)\) is \( \mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d) \)-measurable for any \( t > 0 \).

Let \( L_\alpha \) be the class of all predictable processes \( X \) such that

\[
\|X\|_{\alpha, T, B} := E \int_0^T \int_B |X(t, x)|^\alpha \, dx \, dt < \infty,
\]

for all \( T > 0 \) and \( B \in \mathcal{B}_b(\mathbb{R}^d) \). Note that \( L_\alpha \) is a linear space.

Let \((E_k)_{k \geq 1}\) be an increasing sequence of sets in \( \mathcal{B}_b(\mathbb{R}^d) \) such that \( \bigcup_k E_k = \mathbb{R}^d \). We define

\[
\|X\|_\alpha = \sum_{k \geq 1} \frac{1 \wedge \|X\|_{\alpha, k, E_k}}{2^k} \quad \text{if} \quad \alpha > 1,
\]

\[
\|X\|_\alpha = \sum_{k \geq 1} \frac{1 \wedge \|X\|_{\alpha, k, E_k}}{2^k} \quad \text{if} \quad \alpha \leq 1.
\]

We identify two processes \( X \) and \( Y \) for which \( \|X - Y\|_\alpha = 0 \), i.e. \( X = Y \) \( \nu \)-a.e., where \( \nu = P \, dt \, dx \). In particular, we identify two processes \( X \) and \( Y \) if \( X \) is a modification of \( Y \), i.e. \( X(t, x) = Y(t, x) \) a.s. for all \( (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \).

The space \( L_\alpha \) becomes a metric space endowed with the metric \( d_\alpha \):

\[
d_\alpha(X, Y) = \|X - Y\|_\alpha \quad \text{if} \quad \alpha > 1, \quad d_\alpha(X, Y) = \|X - Y\|_\alpha^\alpha \quad \text{if} \quad \alpha \leq 1.
\]

This follows using Minkowski’s inequality if \( \alpha > 1 \), and the inequality \( |a + b|^\alpha \leq |a|^\alpha + |b|^\alpha \) if \( \alpha \leq 1 \).

The following result can be proved similarly to Proposition 2.3 of [29].

**Proposition 4.2** For any \( X \in L_\alpha \) there exists a sequence \( (X_n)_{n \geq 1} \) of bounded simple processes such that \( \|X_n - X\|_\alpha \to 0 \) as \( n \to \infty \).
By Proposition 5.7 of [25], the process \( \{Z(t, B) = Z([0, t] \times B); t \geq 0\} \) has a càdlàg modification, for any \( B \in \mathcal{B}_b(\mathbb{R}^d) \). We work with these modifications. If \( X \) is a simple process given by (23), we define

\[
I(X)(t, B) = \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} Y_{ij} Z((t_i \land t, t_{i+1} \land t] \times (A_{ij} \cap B)).
\]  

(24)

Note that for any \( B \in \mathcal{B}_b(\mathbb{R}^d) \), \( I(X)(t, B) \) is \( \mathcal{F}_t \)-measurable for any \( t \geq 0 \), and \( \{I(X)(t, B)\}_{t \geq 0} \) is càdlàg. We write

\[
I(X)(t, B) = \int_0^t \int_B X(s, x)Z(ds, dx).
\]

The following result will be used for the construction of the integral. This result generalizes Lemma 3.3 of [11] to the case of random fields and non-symmetric measures \( \nu_\alpha \).

**Theorem 4.3** If \( X \) is a bounded simple process then

\[
\sup_{\lambda > 0} \lambda^\alpha P(\sup_{t \in [0, T]} |I(X)(t, B)| > \lambda) \leq c_\alpha E \int_0^T \int_B |X(t, x)|^\alpha dxdt,
\]

(25)

for any \( T > 0 \) and \( B \in \mathcal{B}_b(\mathbb{R}^d) \), where \( c_\alpha \) is a constant depending only on \( \alpha \).

**Proof:** Suppose that \( X \) is of the form (23). Since \( \{I(X)(t, B)\}_{t \in [0, T]} \) is càdlàg, it is separable. Without loss of generality, we assume that its separating set \( D \) can be written as \( D = \bigcup_n F_n \) where \( (F_n)_n \) is an increasing sequence of finite sets containing the points \( (t_k)_{k=0,\ldots,N} \). Hence,

\[
P(\sup_{t \in [0, T]} |I(X)(t, B)| > \lambda) = \lim_{n \to \infty} P(\max_{t \in F_n} |I(X)(t, B)| > \lambda).
\]

(26)

Fix \( n \geq 1 \). Denote by \( 0 = s_0 < s_1 < \ldots < s_m = T \) the points of the set \( F_n \). Say \( t_k = s_{i_k} \) for some \( 0 = i_0 < i_1 < \ldots < i_N \). Then each interval \( (t_k, t_{k+1}] \) can be written as the union of some intervals of the form \( (s_i, s_{i+1}] \):

\[
(t_k, t_{k+1}] = \bigcup_{i \in t_k} (s_i, s_{i+1}]
\]

(27)
where \( I_k = \{i; i_k \leq i < i_{k+1}\} \). By (24), for any \( k = 0, \ldots, N - 1 \) and \( i \in I_k \),

\[
I(X)(s_{i+1}, B) - I(X)(s_i, B) = \sum_{j=1}^{m_k} Y_{kj} Z((s_i, s_{i+1}] \times (A_{kj} \cap B)).
\]

For any \( i \in I_k \), let \( N_i = m_k \), and for any \( j = 1, \ldots, N_i \), define \( \beta_{ij} = Y_{kj} \), \( H_{ij} = A_{kj} \) and \( Z_{ij} = Z((s_i, s_{i+1}] \times (H_{ij} \cap B)) \). With this notation, we have:

\[
I(X)(s_{i+1}, B) - I(X)(s_i, B) = \sum_{j=1}^{N_i} \beta_{ij} Z_{ij} \quad \text{for all } i = 0, \ldots, m.
\]

Consequently, for any \( l = 1, \ldots, m \)

\[
I(X)(s_l, B) = \sum_{i=0}^{l-1} (I(X)(s_{i+1}, B) - I(X)(s_i, B)) = \sum_{i=0}^{l-1} \sum_{j=1}^{N_i} \beta_{ij} Z_{ij}.
\] (28)

Using (26) and (28), it is enough to prove that for any \( \lambda > 0 \),

\[
P\left( \max_{l = 0, \ldots, m-1} \left| \sum_{i=0}^{l} \sum_{j=1}^{N_i} \beta_{ij} Z_{ij} \right| > \lambda \right) \leq c \alpha \lambda^{-\alpha} E \int_0^T \int_B |X(s, x)|^\alpha dx ds.
\] (29)

First, note that

\[
E \int_0^T \int_B |X(s, x)|^\alpha dx ds = \sum_{i=0}^{m-1} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B|.
\]

This follows from the definition (23) of \( X \) and (27), since \( X(t, x) = \sum_{i=0}^{N-1} \sum_{i \in I_k} 1(s_i, s_{i+1}[t) \sum_{j=1}^{N_i} \beta_{ij} 1_{H_{ij}}(x) \).

We now prove (29). Let \( W_i = \sum_{j=1}^{N_i} \beta_{ij} Z_{ij} \). For the event on the left-hand side, we consider its intersection with the event \( \{\max_{0 \leq i \leq m-1} |W_i| > \lambda\} \) and its complement. Hence the probability of this event can be bounded by

\[
\sum_{i=0}^{m-1} P(|W_i| > \lambda) + P\left( \max_{0 \leq i \leq m-1} |W_i| > \lambda \right) =: I + II.
\]

We treat separately the two terms.
For the first term, we note that \( \bar{\beta}_i = (\beta_{ij})_{1 \leq j \leq N_i} \) is \( \mathcal{F}_{s_i} \)-measurable and \( \bar{Z}_i = (Z_{ij})_{1 \leq j \leq N_i} \) is independent of \( \mathcal{F}_{s_i} \). By Fubini’s theorem

\[
I = \sum_{i=0}^{m-1} \int_{\mathbb{R}^{N_i}} P\left( \left| \sum_{j=1}^{N_i} x_j Z_{ij} \right| > \lambda \right) P_{\beta_i}(d\bar{x}),
\]

where \( \bar{x} = (x_j)_{1 \leq j \leq N_i} \) and \( P_{\beta_i} \) is the law of \( \bar{\beta}_i \).

We examine the tail of \( U_i = \sum_{j=1}^{N_i} x_j Z_{ij} \) for a fixed \( \bar{x} \in \mathbb{R}^{N_i} \). By Lemma 2.3, \( Z_{ij} \) has a \( S_\alpha(\sigma(s_{i+1} - s_i)^{1/\alpha}|H_{ij} \cap B|^{1/\alpha}, \beta, 0) \) distribution. Since the sets \( (H_{ij})_{1 \leq j \leq N_i} \) are disjoint, the variables \( (Z_{ij})_{1 \leq j \leq N_i} \) are independent. Using elementary properties of the stable distribution (Properties 1.2.1 and 1.2.3 of [27]), it follows that \( U_i \) has a \( S_\alpha(\sigma_i, \beta^*_i, 0) \) distribution with parameters:

\[
\sigma^*_i = \sigma^*(s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j| \sigma_i \alpha |H_{ij} \cap B|,
\]

\[
\beta^*_i = \frac{\beta}{\sum_{j=1}^{N_i} |x_j| \sigma_i \alpha |H_{ij} \cap B|} \sum_{j=1}^{N_i} \text{sgn}(x_j) |x_j| \sigma_i \alpha |H_{ij} \cap B|.
\]

By Lemma A.1 (Appendix A), there exists a constant \( c^*_\alpha > 0 \) such that

\[
P(|U_i| > \lambda) \leq c^*_\alpha \lambda^{-\alpha} \sigma^*(s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j| \alpha |H_{ij} \cap B| \tag{30}
\]

for any \( \lambda > 0 \). Hence

\[
I \leq c^*_\alpha \lambda^{-\alpha} \sum_{i=0}^{m-1} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E|\beta_{ij}| \alpha |H_{ij} \cap B| = c^*_\alpha \lambda^{-\alpha} \sigma^*_i \int_0^T \int_B |X(s, x)| \alpha dx ds.
\]

We now treat \( II \). We consider three cases. For the first two cases we deviate from the original argument of [11] since we do not require that \( \beta = 0 \).

**Case 1.** \( \alpha < 1 \). Note that

\[
II \leq P\left( \max_{0 \leq l \leq m-1} M_l > \lambda \right) \tag{31}
\]

where \( \{M_l = \sum_{l=0}^l |W_i|1\{|W_i| \leq \lambda\}, \mathcal{F}_{s_{l+1}}; 0 \leq l \leq m - 1\} \) is a submartingale. By the submartingale maximal inequality (Theorem 35.3 of [3]),

\[
P\left( \max_{0 \leq l \leq m-1} M_l > \lambda \right) \leq \frac{1}{\lambda} E(M_{m-1}) = \frac{1}{\lambda} \sum_{i=0}^{m-1} E(|W_i|1\{|W_i| \leq \lambda\}). \tag{32}
\]
Using the independence between \( \beta_i \) and \( Z_i \) it follows that

\[
E[|W_i|1_{|W_i|\leq \lambda}] = \int_{\mathbb{R}^{N_i}} E[|\sum_{j=1}^{N_i} x_j Z_{ij}|1_{|\sum_{j=1}^{N_i} x_j Z_{ij}|\leq \lambda}] P_{\mathcal{F}}(d\tau)
\]

Let \( U_i = \sum_{j=1}^{N_i} x_j Z_{ij} \). Using (30) and Remark A.2 (Appendix A), we get:

\[
E[|U_i|1_{|U_i|\leq \lambda}] \leq c^* \sigma^\alpha \frac{1}{1-\alpha} \lambda^{1-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|.
\]

Hence

\[
E[|W_i|1_{|W_i|\leq \lambda}] \leq c^* \sigma^\alpha \frac{1}{1-\alpha} \lambda^{1-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E[|\beta_{ij}|^\alpha |H_{ij} \cap B|].
\]

From (31), (32) and (33), it follows that:

\[
II \leq c^* \sigma^\alpha \frac{1}{1-\alpha} \lambda^{1-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} E[|\beta_{ij}|^\alpha |H_{ij} \cap B|].
\]

Case 2. \( \alpha > 1 \). We have

\[
II \leq P(\max_{0 \leq i \leq m-1} |\sum_{i=0}^{l} X_i| > \lambda/2) + P(\max_{0 \leq i \leq m-1} Y_i > \lambda/2) =: II' + II'',
\]

where \( X_i = W_i 1_{|W_i|\leq \lambda} - E[W_i 1_{|W_i|\leq \lambda}|\mathcal{F}_{s_i}] \) and \( Y_i = |E[W_i 1_{|W_i|\leq \lambda}|\mathcal{F}_{s_i}]| \).

We first treat the term \( II' \). Note that \( \{M_l = \sum_{i=0}^{l} X_i, \mathcal{F}_{s_{l+1}}; 0 \leq l \leq m-1\} \) is a zero-mean square integrable martingale, and

\[
II' = P(\max_{0 \leq i \leq m-1} |M_i| > \lambda/2) \leq \frac{4}{\lambda^2} \sum_{i=0}^{m-1} E(X_i^2) \leq \frac{4}{\lambda^2} \sum_{i=0}^{m-1} E[W_i^2 1_{|W_i|\leq \lambda}].
\]

Let \( U_i = \sum_{j=1}^{N_i} x_j Z_{ij} \). Using (30) and Remark A.2 (Appendix A), we get:

\[
E[U_i^2 1_{|U_i|\leq \lambda}] \leq 2c^* \sigma^\alpha \frac{1}{2-\alpha} \lambda^{2-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|.
\]

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As in Case 1, we obtain that:

\[ E[W_i^2 1_{|W_i| \leq \lambda}] \leq c_\alpha^* \sigma^\alpha \frac{2}{2-\alpha} \lambda^{2-\alpha}(s_i+1-s_i) \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B|, \quad (34) \]

and hence

\[ II' \leq 8c_\alpha^* \sigma^\alpha \frac{1}{2-\alpha} \lambda^{-\alpha} E \int_0^T \int_B |X(s, x)|^\alpha dx ds. \]

We now treat \( II'' \). Note that \( \{N_i = \sum_{i=0}^t Y_i; 0 \leq l \leq m - 1\} \) is a semimartingale and hence, by the submartingale inequality,

\[ II'' \leq \frac{2}{\lambda} E(N_{m-1}) = \frac{2}{\lambda} \sum_{i=0}^{m-1} E(Y_i). \]

To evaluate \( E(Y_i) \), we note that for almost all \( \omega \in \Omega \),

\[ E[W_i 1_{|W_i| \leq \lambda}] |\mathcal{F}_{s_i}](\omega) = E[\sum_{j=1}^{N_i} \beta_{ij}(\omega)Z_{ij} 1_{\{\sum_{j=1}^{N_i} \beta_{ij}(\omega)Z_{ij} \leq \lambda\}}], \quad (35) \]

due to the independence between \( \beta_i \) and \( Z_i \). We let \( U_i = \sum_{j=1}^{N_i} x_j Z_{ij} \) with \( x_j = \beta_{ij}(\omega) \). Since \( \alpha > 1 \), \( E(U_i) = 0 \). Using (30) and Remark A.2 we obtain

\[ |E[U_1 1_{|U_1| \leq \lambda}]| = |E[U_1 1_{|U_1| > \lambda}]| \leq E[|U_1| 1_{|U_1| > \lambda}] \leq c_\alpha^* \sigma^\alpha \frac{\alpha}{\alpha-1} \lambda^{1-\alpha}(s_i+1-s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|. \]

Hence, \( E(Y_i) \leq c_\alpha^* \sigma^\alpha \frac{\alpha}{\alpha-1} \lambda^{1-\alpha}(s_i+1-s_i) \sum_{j=1}^{N_i} E|\beta_{ij}|^\alpha |H_{ij} \cap B| \) and

\[ II'' \leq c_\alpha^* \sigma^\alpha \frac{2\alpha}{\alpha-1} \lambda^{-\alpha} E \int_0^T \int_B |X(t, x)|^\alpha dx dt. \]

**Case 3.** \( \alpha = 1 \). In this case we assume that \( \beta = 0 \). Hence \( U_i = \sum_{j=1}^{N_i} x_j Z_{ij} \) has a symmetric distribution for any \( \pi \in \mathbb{R}^N \). Using (35), it follows that \( E[W_i 1_{|W_i| \leq \lambda}] |\mathcal{F}_{s_i}] = 0 \) a.s. for all \( i = 0, \ldots, m - 1 \). Hence, \( \{M_i = \sum_{i=0}^t W_i 1_{|W_i| \leq \lambda}, \mathcal{F}_{s_{i+1}}; 0 \leq l \leq m - 1\} \) is a zero-mean square integrable martingale. By the martingale maximal inequality,

\[ II \leq \frac{1}{\lambda^2} E[M_{m-1}^2] = \frac{1}{\lambda^2} \sum_{i=0}^{m-1} E[W_i^2 1_{|W_i| \leq \lambda}]. \]
The result follows using (34). □

We now proceed to the construction of the stochastic integral. If $Y = \{Y(t)\}_{t \geq 0}$ is a jointly measurable random process, we define:

$$\|Y\|_{\alpha, T} = \sup_{\lambda > 0} \lambda^{\alpha} P(\sup_{t \in [0,T]} |Y(t)| > \lambda).$$

Let $X \in L_\alpha$ be arbitrary. By Proposition 4.2, there exists a sequence $(X_n)_{n \geq 1}$ of simple functions such that $\|X_n - X\|_\alpha \to 0$ as $n \to \infty$. Let $T > 0$ and $B \in \mathcal{B}_b(\mathbb{R}^d)$ be fixed. By linearity of the integral and Theorem 4.3,

$$\|I(X_n)(\cdot, B) - I(X_m)(\cdot, B)\|_{\alpha, T} \leq c\|X_n - X_m\|_{\alpha, T, B} \to 0 \tag{36}$$

as $n, m \to \infty$. In particular, the sequence $\{I(X_n)(\cdot, B)\}_n$ is Cauchy in probability in the space $D[0, T]$ equipped with the sup-norm. Therefore, there exists a random element $Y(\cdot, B)$ in $D[0, T]$ such that for any $\lambda > 0$,

$$P(\sup_{t \in [0,T]} |I(X_n)(t, B) - Y(t, B)| > \lambda) \to 0.$$

Moreover, there exists a subsequence $(n_k)_k$ such that

$$\sup_{t \in [0,T]} |I(X_{n_k})(t, B) - Y(t, B)| \to 0 \ a.s.$$

as $k \to \infty$. Hence $Y(t, B)$ is $\mathcal{F}_t$-measurable for any $t \in [0, T]$. The process $Y(\cdot, B)$ does not depend on the sequence $(X_n)_n$ and can be extended to a càdlàg process on $[0, \infty)$, which is unique up to indistinguishability. We denote this extension by $I(X)(\cdot, B)$ and we write

$$I(X)(t, B) = \int_0^t \int_B X(s, x)Z(ds, dx).$$

If $A$ and $B$ are disjoint sets in $\mathcal{B}_b(\mathbb{R}^d)$, then

$$I(X)(t, A \cup B) = I(X)(t, A) + I(X)(t, B) \ a.s. \tag{37}$$

**Lemma 4.4** Inequality (25) holds for any $X \in \mathcal{L}_\alpha$.

**Proof:** Let $(X_n)_n$ be a sequence of simple functions such that $\|X_n - X\|_\alpha \to 0$. For fixed $B$, we denote $I(X) = I(X)(\cdot, B)$. We let $\|\cdot\|_\infty$ be the sup norm on $D[0, T]$. For any $\varepsilon > 0$, we have:

$$P(\|I(X)\|_\infty > \lambda) \leq P(\|I(X) - I(X_n)\|_\infty > \lambda \varepsilon) + P(\|I(X_n)\|_\infty > \lambda(1 - \varepsilon)).$$
Multiplying by $\lambda^\alpha$, and using Theorem 4.3, we obtain:
\[
\sup_{\lambda > 0} \lambda^\alpha P(\|I(X)\|_\infty > \lambda) \leq \varepsilon^{-\alpha} \sup_{\lambda > 0} \lambda^\alpha P(\|I(X) - I(X_n)\|_\infty > \lambda) + (1 - \varepsilon)^{-\alpha} c_\alpha \|X_n\|_{\alpha,T,B}^\alpha.
\]

Let $n \to \infty$. Using (36) one can prove that $\sup_{\lambda > 0} \lambda^\alpha P(\|I(X_n) - I(X)\|_\infty > \lambda) \to 0$. We obtain that $\sup_{\lambda > 0} \lambda^\alpha P(\|I(X)\|_\infty > \lambda) \leq (1 - \varepsilon)^{-\alpha} c_\alpha \|X\|_{\alpha,T,B}^\alpha$.

The conclusion follows letting $\varepsilon \to 0$. □

For an arbitrary Borel set $\mathcal{O} \subset \mathbb{R}^d$ (possibly $\mathcal{O} = \mathbb{R}^d$), we assume in addition, that $X \in L_\alpha$ satisfies the condition:
\[
E \int_0^T \int_\mathcal{O} |X(t,x)|^\alpha dx dt < \infty \quad \text{for all } T > 0. \tag{38}
\]

Then we can define $I(X)(\cdot, \mathcal{O})$ as follows. Let $\mathcal{O}_k = \mathcal{O} \cap E_k$ where $(E_k)_k$ is an increasing sequence of sets in $\mathcal{B}_b(\mathbb{R}^d)$ such that $\bigcup_k E_k = \mathbb{R}^d$. By (37), Lemma 4.4 and (38),
\[
\sup_{\lambda > 0} \lambda^\alpha P(\sup_{t \leq T} |I(X)(t, \mathcal{O}_k) - I(X)(t, \mathcal{O}_l)| > \lambda) \leq c_\alpha E \int_0^T \int_{\mathcal{O}_k \setminus \mathcal{O}_l} |X(t,x)|^\alpha dx dt \to 0
\]
as $k, l \to \infty$. This shows that $\{I(X)(\cdot, \mathcal{O}_k)\}_k$ is a Cauchy sequence in probability in the space $D[0,T]$ equipped with the sup norm. We denote by $I(X)(\cdot, \mathcal{O})$ its limit. As above, this process can be extended to $[0, \infty)$ and $I(X)(t, \mathcal{O})$ is $\mathcal{F}_t$-measurable for any $t > 0$. We denote
\[
I(X)(t, \mathcal{O}) = \int_0^t \int_\mathcal{O} X(s,x)Z(ds,dx).
\]

Similarly to Lemma 4.4, one can prove that for any $X \in L_\alpha$ satisfying (38),
\[
\sup_{\lambda > 0} \lambda^\alpha P(\sup_{t \leq T} |I(X)(t, \mathcal{O})| > \lambda) \leq c_\alpha E \int_0^T \int_\mathcal{O} |X(t,x)|^\alpha dx dt.
\]

5 The truncated noise

For the study of non-linear equations, we need to develop a theory of stochastic integration with respect to another process $Z_K$ which is defined by removing from $Z$ the jumps whose modulus exceed a fixed value $K > 0$. More precisely, for any $B \in \mathcal{B}_b(\mathbb{R}_+ \times \mathbb{R}^d)$, we define
\[
Z_K(B) = \int_{B \times \{0 < |z| \leq K\}} zN(ds,dx,dz) \quad \text{if } \alpha \leq 1 \tag{39}
\]
\[ Z_K(B) = \int_{B \times \{0 < |z| \leq K\}} z \tilde{N}(ds, dx, dz) \quad \text{if} \quad \alpha > 1. \]  

We treat separately the cases \( \alpha \leq 1 \) and \( \alpha > 1 \).

5.1 The case \( \alpha \leq 1 \)

Note that \( \{Z_K(B); B \in \mathcal{B}^b(\mathbb{R}_+ \times \mathbb{R}^d)\} \) is an independently scattered random measure on \( \mathbb{R}_+ \times \mathbb{R}^d \) with characteristic function given by:

\[ E(e^{iuZ_K(B)}) = \exp \left\{ |B| \int_{|z| \leq K} (e^{izu} - 1) \nu_\alpha(dz) \right\} \quad \forall u \in \mathbb{R}. \]

We first examine the tail of \( Z_K(B) \).

**Lemma 5.1** For any set \( B \in \mathcal{B}^b(\mathbb{R}_+ \times \mathbb{R}^d) \),

\[ \sup_{\lambda > 0} \lambda^\alpha P(|Z_K(B)| > \lambda) \leq r_\alpha |B| \]  

(41)

where \( r_\alpha > 0 \) a constant depending only on \( \alpha \) (given by Lemma A.3).

**Proof:** This follows from Example 3.7 of [11]. We denote by \( \nu_{\alpha,K} \) the restriction of \( \nu_\alpha \) to \( \{z \in \mathbb{R}; 0 < |z| \leq K\} \). Note that

\[ \nu_{\alpha,K}(\{z \in \mathbb{R}; |z| > t\}) = \begin{cases} 
  t^{-\alpha} - K^{-\alpha} & \text{if } 0 < t \leq K \\
  0 & \text{if } t > K
\end{cases} \]

and hence \( \sup_{t > 0} t^\alpha \nu_{\alpha,K}(\{z \in \mathbb{R}; |z| > t\}) = 1 \). Next we observe that we do not need to assume that the measure \( \nu_{\alpha,K} \) is symmetric since we use a modified version of Lemma 2.1 of [10] given by Lemma A.3 (Appendix A).

\[ \square \]

In fact, since the tail of \( \nu_{\alpha,K} \) vanishes if \( t > K \), we can obtain another estimate for the tail of \( Z_K(B) \) which, together with (41), will allow us to control its \( p \)-th moment for \( p \in (\alpha, 1) \). This new estimate is given below.

**Lemma 5.2** If \( \alpha < 1 \), then

\[ P(|Z_K(B)| > u) \leq \frac{\alpha}{1 - \alpha} K^{1-\alpha} |B| u^{-1} \quad \text{for all } u > K. \]

If \( \alpha = 1 \), then \( P(|Z_K(B)| > u) \leq K |B| u^{-2} \quad \text{for all } u > K. \)
Proof: We use the same idea as in Example 3.7 of [11]. For each $k \geq 1$, let $Z_{k,K}(B)$ be a random variable with characteristic function:

$$E(e^{iuZ_{k,K}(B)}) = \exp \left\{ |B| \int_{\{k^{-1} < |z| \leq K\}} (e^{izu} - 1) \nu_{\alpha}(dz) \right\}.$$ 

Since $\{Z_{k,K}(B)\}_{k}$ converges in distribution to $Z_K(B)$, it suffices to prove the lemma for $Z_{k,K}(B)$. Let $\mu_k$ be the restriction of $\nu_{\alpha}$ to $\{z; |z| = K\}$. Since $\mu_k$ is finite, $Z_{k,K}(B)$ has a compound Poisson distribution with

$$P(|Z_{k,K}(B)| > u) = e^{-|B|\mu_k(\mathbb{R})} \sum_{n \geq 0} \frac{|B|^n}{n!} \mu_k^*n(\{z; |z| > u\}). \quad (42)$$

where $\mu_k^*n$ denotes the $n$-fold convolution. Note that

$$\mu_k^*n(\{z; |z| > u\}) = [\mu_k(\mathbb{R})]^n P(\sum_{i=1}^n |\eta_i| > u),$$

where $(\eta_i)_{i \geq 1}$ are i.i.d. random variables with law $\mu_k/\mu_k(\mathbb{R})$.

Assume first that $\alpha < 1$. To compute $P(\sum_{i=1}^n |\eta_i| > u)$ we consider the intersection with the event $\{\max_{1 \leq i \leq n} |\eta_i| > u\}$ and its complement. Note that $P(|\eta_i| > u) = 0$ for any $u > K$. Using this fact and Markov’s inequality, we obtain that for any $u > K$,

$$P(\sum_{i=1}^n |\eta_i| > u) \leq P(\sum_{i=1}^n |\eta_i|1_{\{|\eta_i| \leq u\}} > u) \leq \frac{1}{u} \sum_{i=1}^n E(\eta_i 1_{\{|\eta_i| \leq u\}}).$$

Note that $P(|\eta_i| > s) \leq (s^{-\alpha} - K^{-\alpha})/\mu_k(\mathbb{R})$ if $s \leq K$. Hence, for any $u > K$

$$E(\eta_i 1_{\{|\eta_i| \leq u\}}) \leq \int_0^u P(|\eta_i| > s) ds = \int_0^K P(|\eta_i| > s) ds \leq \frac{1}{\mu_k(\mathbb{R})} \frac{\alpha}{1 - \alpha} K^{1-\alpha}.$$ 

Combining all these facts, we get: for any $u > K$

$$\mu_k^*n(\{z; |z| > u\}) \leq [\mu_k(\mathbb{R})]^{n-1} \frac{\alpha}{1 - \alpha} K^{1-\alpha} nu^{-1},$$

and the conclusion follows from (42).

Assume now that $\alpha = 1$. In this case, $E(\eta_i 1_{\{|\eta_i| \leq u\}}) = 0$ since $\eta_i$ has a symmetric distribution. Using Chebyshev’s inequality this time, we obtain:

$$P(\sum_{i=1}^n |\eta_i| > u) \leq P(\sum_{i=1}^n |\eta_i|1_{\{|\eta_i| \leq u\}} > u) \leq \frac{1}{u^2} \sum_{i=1}^n E(\eta_i^2 1_{\{|\eta_i| \leq u\}}).$$

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The result follows as above using the fact that for any \( u > K \),

\[
E(\eta^2 1_{(|\eta| \leq u)}) \leq 2 \int_0^u sP(|\eta| > s)ds = 2 \int_0^K sP(|\eta| > s)ds \leq \frac{1}{\mu_k(\mathbb{R})}K.
\]

□

Lemma 5.3  If \( \alpha < 1 \) then

\[
E|Z_K(B)|^p \leq C_{\alpha,p}K^{p-\alpha}|B| \quad \text{for any } p \in (\alpha, 1),
\]

where \( C_{\alpha,p} \) is a constant depending on \( \alpha \) and \( p \). If \( \alpha = 1 \), then

\[
E|Z_K(B)|^p \leq C_p K^{p-1}|B| \quad \text{for any } p \in (1, 2),
\]

where \( C_p \) is a constant depending on \( p \).

Proof: Note that

\[
E|Z_K(B)|^p = \int_0^\infty P(|Z_K(B)|^p > t)dt = p \int_0^\infty P(|Z_K(B)| > u)u^{p-1}du.
\]

We consider separately the integrals for \( u \leq K \) and \( u > K \). For the first integral we use (41):

\[
\int_0^K P(|Z_K(B)| > u)u^{p-1}du \leq r_\alpha |B| \int_0^K u^{-\alpha+p-1}du = r_\alpha |B| \frac{1}{p-\alpha} K^{p-\alpha}.
\]

For the second one we use Lemma 5.2 if \( \alpha < 1 \) then

\[
\int_K^\infty P(|Z_K(B)| > u)u^{p-1}du \leq \frac{\alpha}{1-\alpha} K^{1-\alpha}|B| \int_K^\infty u^{p-2}du = \frac{\alpha}{(1-\alpha)(1-p)} |B| K^{p-\alpha},
\]

and if \( \alpha = 1 \), then

\[
\int_K^\infty P(|Z_K(B)| > u)u^{p-1}du \leq K |B| \int_K^\infty u^{p-3}du = |B| \frac{1}{2-p} K^{p-1}.
\]

□

We now proceed to the construction of the stochastic integral with respect to \( Z_K \). For this, we use the same method as for \( Z \). Note that \( \mathcal{F}_t^{Z_K} \subset \mathcal{F}_t \), where \( \mathcal{F}_t^{Z_K} \) is the \( \sigma \)-field generated by \( Z_K([0, s] \times A) \) for all \( s \in [0, t] \) and
A ∈ \mathcal{B}_b(\mathbb{R}^d). For any B ∈ \mathcal{B}_b(\mathbb{R}^d), we will work with a càdlàg modification of the Lévy process \{Z_K(t, B) = Z_K([0, t] \times B); t \geq 0\}.

If X is a simple process given by (23), we define

\[ I_K(X)(t, B) = \int_0^t \int_B X(s, x)Z(ds, dx) \]

by the same formula (24) with Z replaced by Z_K. The following result shows that I_K(X)(t, B) has the same tail behavior as I(X)(t, B).

**Proposition 5.4** If X is a bounded simple process then

\[ \sup_{\lambda > 0} \lambda^\alpha P(\sup_{t \in [0, T]} |I_K(X)(t, B)| > \lambda) \leq d_\alpha E\int_0^T \int_B |X(t, x)|^\alpha dx dt, \quad (43) \]

for any T > 0 and B ∈ \mathcal{B}_b(\mathbb{R}^d), where d_\alpha is a constant depending only on \alpha.

**Proof:** As in the proof of Theorem 4.3, it is enough to prove that

\[ P(\max_{t=0, \ldots, m-1} \sum_{i=0}^{t} \sum_{j=1}^{N_i} \beta_{ij}^* Z_{ij}^* > \lambda) \leq d_\alpha \lambda^{-\alpha} \sum_{i=0}^{m-1} (s_{i+1} - s_i) \sum_{j=1}^{N_i} |E|\beta_{ij}|^\alpha |H_{ij} \cap B|, \]

where \( Z_{ij}^* = Z_K((s_i, s_{i+1}] \times (H_{ij} \cap B)) \). This reduces to showing that \( U_i^* = \sum_{j=1}^{N_i} x_j Z_{ij}^* \) satisfies an inequality similar to (30) for any \( \tau \in \mathbb{R}^{N_i} \), i.e.

\[ P(|U_i^*| > \lambda) \leq d_\alpha^* \lambda^{-\alpha} (s_{i+1} - s_i) \sum_{j=1}^{N_i} |x_j|^\alpha |H_{ij} \cap B|, \quad (44) \]

for any \( \lambda > 0 \), for some \( d_\alpha^* > 0 \). We first examine the tail of \( Z_{ij}^* \). By (11),

\[ P(|Z_{ij}^*| > \lambda) \leq r_\alpha (s_{i+1} - s_i) K_{ij} \lambda^{-\alpha}. \]

where \( K_{ij} = |H_{ij} \cap B| \). Letting \( \eta_{ij} = K_{ij}^{-1/\alpha} Z_{ij}^* \), we obtain that for any \( u > 0 \),

\[ P(|\eta_{ij}| > u) \leq r_\alpha (s_{i+1} - s_i) u^{-\alpha} \quad \forall j = 1, \ldots, N_i. \]

By Lemma A.3 (Appendix A), it follows that for any \( \lambda > 0 \),

\[ P(|\sum_{j=1}^{N_i} b_j \eta_{ij}| > \lambda) \leq r_\alpha^2 (s_{i+1} - s_i) \sum_{j=1}^{N_i} |b_j|^\alpha \lambda^{-\alpha}, \]

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for any sequence \((b_j)_{j=1,...,N}\) of real numbers. Inequality (44) (with \(d^*_\alpha = r^2_\alpha\)) follows by applying this to \(b_j = x_j K_{ij}^{1/\alpha}\). □

In view of the previous result and Proposition 4.2, for any process \(X \in \mathcal{L}_\alpha\) we can construct the integral

\[
I_K(X)(t, B) = \int_0^t \int_B X(s, x)Z_K(ds, dx)
\]

in the same manner as \(I(X)(t, B)\), and this integral satisfies (13). If in addition the process \(X \in \mathcal{L}_\alpha\) satisfies (38), then we can define the integral \(I_K(X)(t, O)\) for an arbitrary Borel set \(O \subset \mathbb{R}^d\) (possibly \(O = \mathbb{R}^d\)). This integral will satisfy an inequality similar to (43) with \(B\) replaced by \(O\).

The appealing feature of \(I_K(X)(t, B)\) is that we can control its moments, as shown by the next result.

**Theorem 5.5** If \(\alpha < 1\), then for any \(p \in (\alpha, 1)\) and for any \(X \in \mathcal{L}_p\),

\[
E|I_K(X)(t, B)|^p \leq C_{\alpha, p} K^{p-\alpha} E \int_0^t \int_B |X(s, x)|^p dx ds,
\]

for any \(t > 0\) and \(B \in \mathcal{B}_0(\mathbb{R}^d)\), where \(C_{\alpha, p}\) is a constant depending on \(\alpha, p\). If \(O \subset \mathbb{R}^d\) is an arbitrary Borel set, and we assume in addition, that the process \(X \in \mathcal{L}_p\) satisfies:

\[
E \int_0^T \int_O |X(s, x)|^p dx ds < \infty \quad \forall \ T > 0,
\]

then inequality (45) holds with \(B\) replaced by \(O\).

**Proof:** Step 1. Suppose that \(X\) is an elementary process of the form (22). Then \(I_K(X)(t, B) = Y Z_K(H)\) where \(H = (t \land \alpha, t \land b] \times (A \cap B)\). Note that \(Z_K(H)\) is independent of \(F_\alpha\). Hence, \(Z_K(H)\) is independent of \(Y\). Let \(P_Y\) denote the law of \(Y\). By Fubini’s theorem,

\[
E|YZ_K(H)|^p = p \int_0^\infty P(|YZ_K(H)| > u) u^{p-1} du = p \int_{\mathbb{R}} \left( \int_0^\infty P(|yZ_K(H)| > u) u^{p-1} du \right) P_Y(dy).
\]
We evaluate the inner integral. We split this integral into two parts, for \( u \leq K|y| \), respectively \( u > K|y| \). For the first integral, we use (41). For the second one, we use Lemma 5.2. Therefore, the inner integral is bounded by:

\[
\int_{0}^{K|y|} u^{-\alpha-1} du + \frac{\alpha}{1-\alpha} |y| K^{1-\alpha} |H| \int_{K|y|}^{\infty} u^{p-2} du = C'_{\alpha,p} K^{p-\alpha} |y|^p |H|
\]

and

\[
E[YZ_K(H)]^p \leq pC'_{\alpha,p} K^{p-\alpha} |H| E[Y]^p = C_{\alpha,p} K^{p-\alpha} E \int_{0}^{t} \int_{B} |X(s,x)|^p dx ds.
\]

**Step 2.** Suppose now that \( X \) is a simple process of the form (23). Then \( X(t,x) = \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} X_{ij}(t,x) \) where \( X_{ij}(t,x) = 1_{(t_{i+1},t_i)}(t) A_{ij}(x) Y_{ij} \).

Using the linearity of the integral, the inequality \(|a + b|^p \leq |a|^p + |b|^p\), and the result obtained in Step 1 for the elementary processes \( X_{ij} \), we get:

\[
E[I_K(X)(t,B)]^p \leq E \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} |I_K(X_{ij})(t,B)|^p \leq C_{\alpha,p} K^{p-\alpha} E \sum_{i=0}^{N-1} \sum_{j=1}^{m_i} \int_{0}^{t} \int_{B} |X_{ij}(s,x)|^p dx ds = C_{\alpha,p} K^{p-\alpha} E \int_{0}^{t} \int_{B} |X(s,x)|^p dx ds.
\]

**Step 3.** Let \( X \in \mathcal{L}_p \) be arbitrary. By Proposition 4.2, there exists a sequence \( (X_n)_n \) of bounded simple processes such that \( \|X_n - X\|_{p} \to 0 \). Since \( \alpha < p \), it follows that \( \|X_n - X\|_{\alpha} \to 0 \). By the definition of \( I_K(X)(t,B) \) there exists a subsequence \( \{n_k\}_k \) such that \( \{I_K(X_{n_k})(t,B)\}_k \) converges to \( I_K(X)(t,B) \) a.s. Using Fatou’s lemma and the result obtained in Step 2 (for the simple processes \( X_{n_k} \)), we get:

\[
E[I_K(X)(t,B)]^p \leq \liminf_{k \to \infty} E[I_K(X_{n_k})(t,B)]^p \\
\leq C_{\alpha,p} K^{p-\alpha} \liminf_{k \to \infty} \int_{0}^{t} \int_{B} |X_{n_k}(s,x)|^p dx ds \\
= C_{\alpha,p} K^{p-\alpha} \int_{0}^{t} \int_{B} |X(s,x)|^p dx ds.
\]

**Step 4.** Suppose that \( X \in \mathcal{L}_p \) satisfies (40). Let \( \mathcal{O}_k = \mathcal{O} \cap E_k \) where \( (E_k)_k \) is an increasing sequence of sets in \( \mathcal{B}_0(\mathbb{R}^d) \) such that \( \bigcup_{k \geq 1} E_k = \mathbb{R}^d \).
By the definition of $I_K(X)(t, \mathcal{O})$, there exists a subsequence $(k_i)$, such that
\{I_K(X)(t, \mathcal{O}_{k_i})\}_i converges to $I_K(X)(t, \mathcal{O})$ a.s. Using Fatou's lemma, the result obtained in Step 3 (for $B = \mathcal{O}_{k_i}$) and the monotone convergence theorem, we get:

\[
E|I_K(X)(t, \mathcal{O})|^p \leq \liminf_{i \to \infty} E|I_K(X)(t, \mathcal{O}_{k_i})|^p \\
\leq C_{\alpha,p}K^{p-\alpha} \liminf_{i \to \infty} E \int_0^t \int_{\mathcal{O}_{k_i}} |X(s, x)|^p dx ds \\
= C_{\alpha,p}K^{p-\alpha} E \int_0^t \int_{\mathcal{O}} |X(s, x)|^p dx ds.
\]

□

**Remark 5.6** Finding a similar moment inequality for the case $\alpha = 1$ and $p \in (1, 2)$ remains an open problem. The argument used in Step 2 above relies on the fact that $p < 1$. Unfortunately, we could not find another argument to cover the case $p > 1$.

### 5.2 The case $\alpha > 1$

In this case, the construction of the integral with respect to $Z_K$ relies on an integral with respect to $\hat{N}$ which exists in the literature. We recall briefly the definition of this integral. For more details, see Section 1.2.2 of [26], Section 24.2 of [28] or Section 8.7 of [22].

Let $\mathbb{E} = \mathbb{R}^d \times (\mathbb{R}\setminus\{0\})$ endowed with the measure $\mu(dx, dz) = dx\nu_\alpha(dz)$ and $\mathcal{B}_b(\mathbb{E})$ be the class of bounded Borel sets in $\mathbb{E}$. For a simple process $Y = \{Y(t, x, z); t \geq 0, (x, z) \in \mathbb{E}\}$, the integral $I_{\hat{N}}(Y)(t, B)$ is defined in the usual way, for any $t > 0, B \in \mathcal{B}_b(\mathbb{E})$. The process $I_{\hat{N}}(Y)(\cdot, B)$ is a (càdlàg) zero-mean square-integrable martingale with quadratic variation

\[
[I_{\hat{N}}(Y)(\cdot, B)]_t = \int_0^t \int_B |Y(s, x, z)|^2 N(ds, dx, dz)
\]

and predictable quadratic variation

\[
\langle I_{\hat{N}}(Y)(\cdot, B) \rangle_t = \int_0^t \int_B |Y(s, x, z)|^2 \nu_\alpha(dz) dx ds.
\]
By approximation, this integral can be extended to the class of all $\mathcal{P} \times \mathcal{B}(\mathbb{R}\backslash\{0\})$-measurable processes $Y$ such that, for any $T > 0$ and $B \in \mathcal{B}_b(\mathbb{E})$

$$\|Y\|^2_{2,T,B} := E\int_0^T \int_B |Y(s, x, z)|^2 \nu_\alpha(dz)dxds < \infty.$$  

The integral is a martingale with the same quadratic variations as above, and has the isometry property: $E|\hat{I}^N(Y)(t, B)|^2 = \|Y\|^2_{2,T,B}$. If in addition, $\|Y\|_{2,T,\mathbb{E}} < \infty$, then the integral can be extended to $\mathbb{E}$. By the Burkholder-Davis-Gundy inequality for discontinuous martingales, for any $p \geq 1$

$$E\sup_{t \leq T} |\hat{I}^N(Y)(t, \mathbb{E})|^p \leq C_p E[\hat{I}^N(Y)(\cdot, \mathbb{E})]^p_T.$$

The previous inequality is not suitable for our purposes. A more convenient inequality can be obtained for another stochastic integral, constructed for $p \in [1, 2]$ fixed, as suggested on page 293 of [26]. More precisely, one can show that for any bounded simple process $Y$,

$$E\sup_{t \leq T} |\hat{I}^N(Y)(t, \mathbb{E})|^p \leq C_p E[\hat{I}^N(Y)(\cdot, \mathbb{E})]^p_T,$$

where $C_p$ is the constant appearing in (47) (see Lemma 8.22 of [22]).

By the usual procedure, the integral can be extended to the class of all $\mathcal{P} \times \mathcal{B}(\mathbb{R}\backslash\{0\})$-measurable processes $Y$ such that $[Y]_{p,T,\mathbb{E}} < \infty$. The integral is defined as an element in the space $L^p(\Omega; D[0, T])$ and will be denoted by

$$\hat{I}^N(Y)(t, \mathbb{E}) = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}\backslash\{0\}} Y(s, x, z) \hat{N}(ds, dx, dz).$$

Its appealing feature is that it satisfies inequality (48).

From now on, we fix $p \in [1, 2]$. Based on (40), for any $B \in \mathcal{B}_b(\mathbb{R}^d)$, we let

$$I_K(X)(t, B) = \int_0^t \int_B X(s, x)Z_K(ds, dx) = \int_0^t \int_B \int_{\{|z| \leq K\}} X(s, x)z \hat{N}(ds, dx, dz),$$

for any predictable process $X = \{X(t, x); t \geq 0, x \in \mathbb{R}^d\}$ for which the rightmost integral is well-defined. Letting $Y(t, x, z) = X(t, x)z1_{\{0 < |z| \leq K\}}$, we see that this is equivalent to saying that $p > \alpha$ and $X \in \mathcal{L}_p$. By (48),

$$E\sup_{t \leq T} |I_K(X)(t, B)|^p \leq C_{\alpha,p}K^{p-\alpha}E\int_0^T \int_B |X(s, x)|^p dxds.$$

(49)
where $C_{\alpha,p} = C_p \alpha / (p - \alpha)$. If in addition, the process $X \in \mathcal{L}_p$ satisfies (46) then (49) holds with $B$ replaced by $\mathcal{O}$, for an arbitrary Borel set $\mathcal{O} \subset \mathbb{R}^d$.

Note that (49) is the counterpart of (45) for the case $\alpha > 1$. Together, these two inequalities will play a crucial role in Section 6.

The following table summarizes all the conditions:

| $\alpha < 1$ | $\alpha > 1$ |
|---------------|---------------|
| $B$ is bounded | $X \in \mathcal{L}_\alpha$ | $X \in \mathcal{L}_p$ for some $p \in (\alpha, 2]$ |
| $B = \mathcal{O}$ is unbounded | $X \in \mathcal{L}_\alpha$ and $X$ satisfies (38) | $X \in \mathcal{L}_p$ and $X$ satisfies (46) for some $p \in (\alpha, 2]$ |

Table 1: Conditions for $I_K(X)(t, B)$ to be well-defined

6 The main result

In this section, we state and prove the main result regarding the existence of a mild solution of equation (1). For this result, $\mathcal{O}$ is a bounded domain in $\mathbb{R}^d$. For any $t > 0$, we denote

$$J_p(t) = \sup_{x \in \mathcal{O}} \int_0^T G(t, x, y)^p dy.$$  

**Theorem 6.1** Let $\alpha \in (0, 2)$, $\alpha \neq 1$. Assume that for any $T > 0$,

$$\lim_{h \to 0} \int_0^T \int_{\mathcal{O}} |G(t, x, y) - G(t + h, x, y)|^p dy dt = 0 \quad \forall x \in \mathcal{O},$$  

(50)

$$\lim_{|h| \to 0} \int_0^T \int_{\mathcal{O}} |G(t, x, y) - G(t, x + h, y)|^p dy dt = 0 \quad \forall x \in \mathcal{O},$$  

(51)

$$\int_0^T J_p(t) dt < \infty,$$  

(52)

for some $p \in (\alpha, 1)$ if $\alpha < 1$, or for some $p \in (\alpha, 2]$ if $\alpha > 1$. Then equation (1) has a mild solution. Moreover, there exists a sequence $(\tau_K)_{K \geq 1}$ of stopping times with $\tau_K \uparrow \infty$ a.s. such that for any $T > 0$,

$$\sup_{(t, x) \in [0, T] \times \mathcal{O}} E(|u(t, x)|^p 1_{\{t \leq \tau_K\}}) < \infty.$$  

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Example 6.2 *(Heat equation)* Let $L = \frac{\partial}{\partial t} - \frac{1}{2} \Delta$. Then $G(t, x, y) \leq \overline{G}(t, x-y)$ where $\overline{G}(t, x)$ is the fundamental solution of $Lu = 0$ on $\mathbb{R}^d$. Condition (52) holds if $p < 1 + 2/d$. If $\alpha < 1$, this condition holds for any $p \in (\alpha, 1)$. If $\alpha > 1$, this condition holds for any $p \in (\alpha, 1 + 2/d]$, as long as $\alpha$ satisfies (5). Conditions (50) and (51) hold by the continuity of the function $G$ in $t$ and $x$, by applying the dominated convergence theorem. To justify the application of this theorem, we use the trivial bound $(2\pi t)^{-d/2}$ for both $G(t+h, x, y)^p$ and $G(t, x+h, y)^p$, which introduces the extra condition $dp < 2$. Unfortunately, we could not find another argument for proving these two conditions. (In the case of the heat equation on $\mathbb{R}^d$, Lemmas A.2 and A.3 of [26] estimate the integrals appearing in (51) and (50), with $p = 1$ in (50). These arguments rely on the structure of $G$ and cannot be used when $\mathcal{O}$ is a bounded domain.)

Example 6.3 *(Parabolic equations)* Let $L = \frac{\partial}{\partial t} - \mathcal{L}$ where $\mathcal{L}$ is given by (17). Assuming (18), we see that (52) holds if $p < 1 + 2/d$. The same comments as for the heat equation apply here as well. (Although in a different framework, a condition similar to (50) was probably used in the proof of Theorem 12.11 of [22] (page 217) for the claim $\lim_{s \to t} \mathbb{E}|J_3(X)(s) - J_3(X)(t)|_{L^p(\mathcal{O})} = 0$. We could not see how to justify this claim, unless $dp < 2$.)

Example 6.4 *(Heat equation with fractional power of the Laplacian)* Let $L = \frac{\partial}{\partial t} + (-\Delta)_{\gamma}$ for some $\gamma > 0$. By Lemma B.23 of [22], if $\alpha > 1$, then condition (52) holds for any $p \in (\alpha, 1 + 2\gamma/d)$, provided that $\alpha$ satisfies (21). (This condition is the same as in Theorem 12.19 of [22], which examines the same equation using the approach based on Hilbert-space valued solution.) To verify condition (50) and (51), we use the continuity of $G$ in $t$ and $x$ and apply the dominated convergence theorem. To justify the application of this theorem, we use the trivial bound $C_{d, \gamma}t^{-d\gamma/(2\gamma)}$ for both $G(t+h, x, y)^p$ and $G(t, x+h, y)^p$, which introduces the extra condition $dp < 2\gamma$. This bound can be seen from (19), using the fact that $G(t, x, y) \leq \overline{G}(t, x-y)$ where $G$ and $\overline{G}$ are the fundamental solutions of $\frac{\partial u}{\partial t} - \Delta u = 0$ on $\mathcal{O}$, respectively $\mathbb{R}^d$. (In the case of the same equation on $\mathbb{R}^d$, elementary estimates for the time and space increments of $\overline{G}$ can be obtained directly from (20), as on p. 196 of [4]. These arguments cannot be used when $\mathcal{O}$ is a bounded domain.)

The remaining part of this section is dedicated to the proof of Theorem 6.1. The idea is to solve first the equation with the truncated noise $Z_K$ (yielding a mild solution $u_K$), and then identify a sequence $(\tau_K)_{K \geq 1}$ of stopping times with $\tau_K \uparrow \infty$ a.s. such that for any $t > 0$, $x \in \mathcal{O}$ and $L > K$, 

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$u_K(t, x) = u_L(t, x)$ a.s. on the event $\{t \leq \tau_K\}$. The final step is to show that process $u$ defined by $u(t, x) = u_K(t, x)$ on $\{t \leq \tau_K\}$ is a mild solution of (1). A similar method can be found in Section 9.7 of [22] using an approach based on stochastic integration of operator-valued processes, with respect to Hilbert-space-valued processes, which is different from our approach.

Since $\sigma$ is a Lipschitz function, there exists a constant $C_\sigma > 0$ such that:

$$|\sigma(u) - \sigma(v)| \leq C_\sigma|u - v|, \quad \forall u, v \in \mathbb{R}. \quad (53)$$

In particular, letting $D_\sigma = C_\sigma \vee |\sigma(0)|$, we have:

$$|\sigma(u)| \leq D_\sigma(1 + |u|), \quad \forall u \in \mathbb{R}. \quad (54)$$

For the proof of Theorem 6.1, we need a specific construction of the Poisson random measure $N$, taken from [21]. We review briefly this construction.

Let $(O_k)_{k \geq 1}$ be a partition of $\mathbb{R}^d$ with sets in $B_b(\mathbb{R}^d)$ and $(U_j)_{j \geq 1}$ be a partition of $\mathbb{R} \setminus \{0\}$ such that $\nu_\alpha(U_j) < \infty$ for all $j \geq 1$. We may take $U_j = \Gamma_{j-1}$ for all $j \geq 1$. Let $(E_{i,j,k}^j, X_{i,j,k}^j, Z_{i,j,k}^j)_{i,j,k \geq 1}$ be independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, such that

$$P(E_{i,j,k}^j > t) = e^{-\lambda_{j,k}t}, \quad P(X_{i,j,k}^j \in B) = \frac{|B \cap O_k|}{|O_k|}, \quad P(Z_{i,j,k}^j \in \Gamma) = \frac{|\Gamma \cap U_j|}{|U_j|},$$

where $\lambda_{j,k} = |O_k|\nu_\alpha(U_j)$. Let $T_{i,j,k}^j = \sum_{l=1}^i E_{i,l,k}^j$ for all $i \geq 1$. Then

$$N = \sum_{i,j,k \geq 1} \delta_{(T_{i,j,k}^j, X_{i,j,k}^j, Z_{i,j,k}^j)} \quad (55)$$

is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times (\mathbb{R} \setminus \{0\})$ with intensity $dtdx\nu_\alpha(dz)$.

This section is organized as follows. In Section 6.1 we prove the existence of the solution of the equation with truncated noise $Z_K$. Sections 6.2 and 6.3 contain the proof of Theorem 6.1 when $\alpha < 1$, respectively $\alpha > 1$.

### 6.1 The equation with truncated noise

In this section, we fix $K > 0$ and we consider the equation:

$$Lu(t, x) = \sigma(u(t, x))Z_K(t, x) \quad t > 0, x \in \mathcal{O} \quad (56)$$

with zero initial conditions and Dirichlet boundary conditions. A mild solution of (56) is a predictable process $u$ which satisfies (2) with $Z$ replaced by $Z_K$. For the next result, $\mathcal{O}$ can be a bounded domain in $\mathbb{R}^d$ or $\mathcal{O} = \mathbb{R}^d$ (with no boundary conditions).
Theorem 6.5 Under the assumptions of Theorem 6.1, equation (56) has a unique mild solution \( u = \{ u(t, x); t \geq 0, x \in \mathcal{O} \} \). For any \( T > 0 \),

\[
\sup_{(t, x) \in [0, T] \times \mathcal{O}} E|u(t, x)|^p < \infty,
\]

and the map \((t, x) \mapsto u(t, x)\) is continuous from \([0, T] \times \mathcal{O}\) into \( L^p(\Omega)\).

Proof: We use the same argument as in the proof of Theorem 13 of [6], based on a Picard iteration scheme. We define \( u_0(t, x) = 0 \) and

\[
u_{n+1}(t, x) = \int_0^t \int_{\mathcal{O}} G(t - s, x, y) \sigma(u_n(s, y)) Z_K(ds, dy)
\]

for any \( n \geq 0 \). We prove by induction on \( n \geq 0 \) that: (i) \( u_n(t, x) \) is well-defined; (ii) \( K_n(t) := \sup_{(t, x) \in [0, T] \times \mathcal{O}} E|u_n(t, x)|^p < \infty \) for any \( T > 0 \); (iii) \( u_n(t, x) \) is \( \mathcal{F}_t \)-measurable for any \( t > 0 \) and \( x \in \mathcal{O} \); (iv) the map \((t, x) \mapsto u_n(t, x)\) is continuous from \([0, T] \times \mathcal{O}\) into \( L^p(\Omega)\) for any \( T > 0 \).

The statement is trivial for \( n = 0 \). For the induction step, assume that the statement is true for \( n \). By an extension to random fields of Theorem 30, Chapter IV of [8], \( u_n \) has a jointly measurable modification. Since this modification is \((\mathcal{F}_t)_t\)-adapted, (in the sense of (iii)), it has a predictable modification (using an extension of Proposition 3.21 of [22] to random fields). We work with this modification, that we call also \( u_n \).

We prove that (i)-(iv) hold for \( u_{n+1} \). To show (i), it suffices to prove that \( X_n \in L_p \), where \( X_n(s, y) = 1_{[0, t]}(s) G(t - s, x, y) \sigma(u_n(s, y)) \). By (51) and (52),

\[
E \int_0^t \int_{\mathcal{O}} |X_n(s, y)|^p dy ds \leq D_0^{p/2} 2^{p-1}(1 + K_n(t)) \int_0^t J_p(t - s) ds < \infty.
\]

In addition, if \( \mathcal{O} = \mathbb{R}^d \), we have to prove that \( X_n \) satisfies (58) if \( \alpha < 1 \), or (46) if \( \alpha > 1 \) (see Table 1). If \( \alpha < 1 \), this follows as above, since \( \alpha < p \) and hence \( \sup_{(t, x) \in [0, T] \times \mathcal{O}} E|u(t, x)|^\alpha < \infty \); the argument for \( \alpha > 1 \) is similar. Combined with the moment inequality (45) (or (49)), this proves (ii), since:

\[
E|u_{n+1}(t, x)|^p \leq C_{\alpha, p} K^{p-\alpha} D_0^{p/2} 2^{p-1}(1 + K_n(t)) \int_0^t J_p(t - s) ds,
\]

for any \( x \in \mathcal{O} \). Property (iii) follows by the construction of the integral \( I_K \).
To prove (iv), we first show the right continuity in \( t \). Let \( h > 0 \). Writing the interval \([0, t + h]\) as the union of \([0, t]\) and \((t, t + h]\), we obtain that 
\[
E|u_{n+1}(t + h, x) - u_{n+1}(t, x)|^p \leq 2^{p-1}(I_1(h) + I_2(h)),
\]
where
\[
I_1(h) = E \left| \int_0^t \int_O (G(t + h - s, x, y) - G(t - s, x, y))\sigma(u_n(s, y))Z_K(ds, dy) \right|^p,
\]
\[
I_2(h) = E \left| \int_t^{t+h} \int_O G(t + h - s, x, y)\sigma(u_n(s, y))Z_K(ds, dy) \right|^p.
\]
Using again (54) and the moment inequality (45) (or (49)), we have:
\[
I_1(h) \leq D_\sigma^p 2^{p-1}(1 + K_n(t)) \int_0^t \int_O |G(s + h, x, y) - G(s, x, y)|^p dy ds
\]
\[
I_2(h) \leq D_\sigma^p 2^{p-1}(1 + K_n(t)) \int_t^{t+h} \int_O G(s, x, y)^p dy ds.
\]
It follows that both \( I_1(h) \) and \( I_2(h) \) converge to 0 as \( h \to 0 \), using (50) for \( I_1(h) \), respectively the Dominated Convergence Theorem and (52) for \( I_2(h) \).

The left continuity in \( t \) is similar, by writing the interval \([0, t - h]\) as the difference between \([0, t]\) and \((t - h, t]\) for \( h > 0 \). For the continuity in \( x \), similarly as above, we see that \( E|u_{n+1}(t, x + h) - u_{n+1}(t, x)|^p \) is bounded by:
\[
D_\sigma^p 2^{p-1}(1 + K_n(t)) \int_0^t \int_O |G(s, x + h, y) - G(s, x, y)|^p dy ds,
\]
which converges to 0 as \(|h| \to 0\) due to (51). This finishes the proof of (iv).

We denote \( M_n(t) = \sup_{x \in O} E|u_n(t, x)|^p \). Similarly to (58), we have:
\[
M_n(t) \leq C_1 \int_0^t (1 + M_{n-1}(s)) J_p(t - s) ds, \quad \forall n \geq 1,
\]
where \( C_1 = C_{\alpha, p} K^{p-\alpha} D_\sigma^p 2^{p-1} \). By applying Lemma 15 of Erratum to [6] with \( f_n = M_n, k_1 = 0, k_2 = 1 \) and \( g(s) = C J_p(s) \), we obtain that:
\[
\sup \sup_{n \geq 0, t \in [0, T]} M_n(t) < \infty \quad \text{for all } T > 0. \tag{59}
\]

We now prove that \( \{u_n(t, x)\}_n \) converges in \( L^p(\Omega) \), uniformly in \((t, x) \in [0, T] \times \mathcal{O} \). To see this, let \( U_n(t) = \sup_{x \in \mathcal{O}} E|u_{n+1}(t, x) - u_n(t, x)|^p \) for \( n \geq 0 \). Using the moment inequality (15) (or (19)) and (53), we have:
\[
U_n(t) \leq C_2 \int_0^t U_{n-1}(s) J_p(t - s) ds
\]
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where \( C_2 = C_{\alpha,p}K^{p-\alpha}C_{\sigma}^p \). By Lemma 15 of Erratum to [6], \( \sum_{n\geq 0} U_n(t)^{1/p} \) converges uniformly on \([0,T]\). (Note that this lemma is valid for all \( p > 0 \).)

We denote by \( u(t,x) \) the limit of \( u_n(t,x) \) in \( L^p(\Omega) \). One can show that \( u \) satisfies properties (ii)-(iv) listed above. So \( u \) has a predictable modification. This modification is a solution of (56). To prove uniqueness, let \( v \) be another solution and denote \( H(t) = \sup_{x \in \Omega} |u(t,x) - v(t,x)|^p \). Then

\[
H(t) \leq C_2 \int_0^t H(s) J_p(t-s) ds.
\]

Using (52), it follows that \( H(t) = 0 \) for all \( t > 0 \). \( \square \)

### 6.2 Proof of Theorem 6.1: case \( \alpha < 1 \)

In this case, for any \( t > 0 \) and \( B \in \mathcal{B}_b(\mathbb{R}^d) \), we have: (see (11))

\[
Z(t,B) = \int_{[0,t] \times B \times (\mathbb{R} \setminus \{0\})} zN(ds,dx,dz).
\]

The characteristic function of \( Z(t,B) \) is given by:

\[
E(e^{iuZ(t,B)}) = \exp \left\{ t|B| \int_{\mathbb{R} \setminus \{0\}} (e^{iaz} - 1) \nu_\alpha(dz) \right\}, \quad \forall u \in \mathbb{R}.
\]

Note that \( \{Z(t,B)\}_{t \geq 0} \) is not a compound Poisson process since \( \nu_\alpha \) is infinite.

We introduce the stopping times \( (\tau_K)_{K \geq 1} \), as on page 239 of [21]:

\[
\tau_K(B) = \inf \{ t > 0; |Z(t,B) - Z(t-,B)| > K \},
\]

where \( Z(t-,B) = \lim_{s \uparrow t} Z(s,B) \). Clearly, \( \tau_L(B) \geq \tau_K(B) \) for all \( L > K \).

We first investigate the relationship between \( Z \) and \( Z_K \) and the properties of \( \tau_K \). Using construction (55) of \( N \) and definition (39) of \( Z_K \), we have:

\[
Z(t,B) = \sum_{i,j,k \geq 1} Z_i^{j,k} 1_{\{T_i^{j,k} \leq t\}} 1_{\{X_i^{j,k} \in B\}} =: \sum_{j,k \geq 1} Z_i^{j,k}(t,B)
\]

\[
Z_K(t,B) = \sum_{i,j,k \geq 1} Z_i^{j,k} 1_{\{|Z_i^{j,k}| \leq K\}} 1_{\{T_i^{j,k} \leq t\}} 1_{\{X_i^{j,k} \in B\}}.
\]

We observe that \( \{Z_i^{j,k}(t,B)\}_{t \geq 0} \) is a compound Poisson process with

\[
E(e^{iuZ_i^{j,k}(t,B)}) = \exp \left\{ t|\mathcal{O}_k \cap B| \int_{U_j} (e^{iaz} - 1) \nu_\alpha(dz) \right\}, \quad \forall u \in \mathbb{R}.
\]
Note that \( \tau_K(B) > T \) means that all the jumps of \( \{Z(t, B)\}_{t \geq 0} \) in \([0, T]\) are smaller than \( K \) in modulus, i.e. \( \{\tau_K(B) > T\} = \{\omega; |Z^{j,k}_i(\omega)| \leq K \text{ for all } i, j, k \geq 1 \text{ for which } T^{j,k}_i(\omega) \leq T \text{ and } X^{j,k}_i(\omega) \in B\}. \) Hence, on \( \{\tau_K(B) > T\}, \)

\[
Z([0, t] \times A) = Z_K([0, t] \times A) = Z_L([0, t] \times A),
\]

for any \( L > K, t \in [0, T], A \in \mathcal{B}_0(\mathbb{R}^d) \) with \( A \subset B \). Using an approximation argument and the construction of the integrals \( I(X) \) and \( I_K(X) \), it follows that for any \( X \in \mathcal{L}_\alpha \) and for any \( L > K, \) a.s. on \( \{\tau_K(B) > T\}, \) we have:

\[
I(X)(T, B) = I_K(X)(T, B) = I_L(X)(T, B). \tag{60}
\]

The next result gives the probability of the event \( \{\tau_K(B) > T\} \).

**Lemma 6.6** For any \( T > 0 \) and \( B \in \mathcal{B}_0(\mathbb{R}^d), \)

\[
P(\tau_K(B) > T) = \exp(-T|B|K^{-\alpha}).
\]

Consequently, \( \lim_{K \to \infty} P(\tau_K(B) > T) = 1 \) and \( \lim_{K \to \infty} \tau_K(B) = \infty \) a.s.

**Proof:** Note that \( \{\tau_K(B) > T\} = \bigcap_{j, k \geq 1} \{\tau^{j,k}_K(B) > T\}, \) where

\[
\tau^{j,k}_K(B) = \inf\{t > 0; |Z^{j,k}_i(t, B) - Z^{j,k}_i(t-, B)| > K\}
\]

Since \( \nu_\alpha(\{z; |z| > K\}) = K^{-\alpha} \) and \( \{\tau^{j,k}_K(B)\}_{j, k \geq 1} \) are independent, it is enough to prove that for any \( j, k \geq 1, \)

\[
P(\tau^{j,k}_K(B) > T) = \exp\{-T|B \cap O_k|\nu_\alpha(\{z; |z| > K\} \cap U_j)\}. \tag{61}
\]

Note that \( \{\tau^{j,k}_K(B) > T\} = \{\omega; |Z^{j,k}_i(\omega)| \leq K \text{ for all } i \text{ for which } T^{j,k}_i \leq T \text{ and } X^{j,k}_i \in B\} \) and \( (T^{j,k}_n)_{n \geq 1} \) are the jump times of a Poisson process with intensity \( \lambda_{j,k}. \) Hence

\[
P(\tau^{j,k}_K(B) > T) = \sum_{n \geq 0} \sum_{m=0}^n \sum_{I \subset \{1, \ldots, n\}, \text{card}(I) = m} P(T^{j,k}_n \leq T < T^{j,k}_{n+1}) P(\bigcap_{i \in I} \{X^{j,k}_i \in B\})
\]

\[
P(\bigcap_{i \in I} \{|Z^{j,k}_i| \leq K\}) P(\bigcap_{i \in I^c} \{X^{j,k}_i \notin B\})
\]

\[
= \sum_{n \geq 0} e^{-\lambda_{j,k} T} \frac{(\lambda_{j,k} T)^n}{n!} \left[1 - P(X^{j,k}_1 \in B) P(|Z^{j,k}_1| > K)\right]^n
\]

\[
= \exp \left\{ -\lambda_{j,k} T P(X^{j,k}_1 \in B) P(|Z^{j,k}_1| > K) \right\},
\]

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which yields (61).

To prove the last statement, let \(A_k^{(n)} = \{ \tau_K(B) > n \} \). Then \( P(\lim_k A_k^{(n)}) \geq \lim_k P(A_k^{(n)}) = 1 \) for any \( n \geq 1 \), and hence \( P(\bigcap_{n \geq 1} \lim_k A_k^{(n)}) = 1 \). Hence, with probability 1, for any \( n \), there exists some \( K_n \) such that \( \tau_{K_n} > n \). Since \((\tau_K)_K\) is non-decreasing, this proves that \( \tau_K \to \infty \) with probability 1. □

**Remark 6.7** The construction of \( \tau_K(B) \) given above is due to [21] (in the case of a symmetric measure \( \nu_\alpha \)). This construction relies on the fact that \( B \) is a bounded set. Since \( Z(t, \mathbb{R}^d) \) (and consequently \( \tau_K(\mathbb{R}^d) ) \) is not well-defined, I could not see why this construction can also be used when \( B = \mathbb{R}^d \), as it is claimed in [21]. To avoid this difficulty, one could try to use an increasing sequence \( (E_n)_n \) of sets in \( B_b(\mathbb{R}^d) \) with \( \bigcup_n E_n = \mathbb{R}^d \). Using (60) with \( B = E_n \) and letting \( n \to \infty \), we obtain that \( I(X)(t, \mathbb{R}^d) = I_K(t, \mathbb{R}^d) \) a.s. on \( \{ t \leq \tau_K \} \), where \( \tau_K = \inf_{n \geq 1} \tau_K(E_n) \). But \( P(\tau_K > t) \leq P(\lim, \{ \tau_K(E_n) > t \}) \leq \lim_n P(\tau_K(E_n) > t) = \lim_n \exp(-t|E_n|^K^{-\alpha}) = 0 \) for any \( t > 0 \), which means that \( \tau_K = 0 \) a.s. Finding a suitable sequence \( (\tau_K)_K \) of stopping times which could be used in the case \( O = \mathbb{R}^d \) remains an open problem.

In what follows, we denote \( \tau_K = \tau_K(O) \). Let \( u_K \) be the solution of equation (56), whose existence is guaranteed by Theorem 6.5.

**Lemma 6.8** Under the assumptions of Theorem 6.1, for any \( t > 0 \), \( x \in O \) and \( L > K \),

\[
u_K(t, x) = u_L(t, x) \quad \text{a.s. on } \{ t \leq \tau_K \}.
\]

**Proof:** By the definition of \( u_L \) and (60),

\[
u_L(t, x) = \int_0^t \int_O G(t - s, x, y) \sigma(u_L(s, y)) Z_L(ds, dy)
\]

\[
= \int_0^t \int_O G(t - s, x, y) \sigma(u_L(s, y)) Z_K(ds, dy)
\]

a.s. on the event \( \{ t \leq \tau_K \} \). Using the definition of \( u_K \) and Proposition C.1 (Appendix C), we obtain that, with probability 1,

\[
u(u_K(t, x) - u_L(t, x))1_{\{ t \leq \tau_K \}} = 1_{\{ t \leq \tau_K \}} \int_0^t \int_O G(t - s, x, y) (\sigma(u_K(s, y)) - \sigma(u_L(s, y))) 1_{\{ s \leq \tau_K \}} Z_K(ds, dy).
\]
Let $M(t) = \sup_{x \in \mathcal{O}} E(|u_K(t, x) - u_L(t, x)|^p 1_{\{t \leq \tau_K\}})$. Using the moment inequality (45) and the Lipschitz condition (53), we get:

$$M(t) \leq C \int_0^t J_p(t-s)M(s)ds,$$

where $C = C_{\alpha,p}K^{-\alpha}C_\sigma^p$. Using (52), it follows that $M(t) = 0$ for all $t > 0$. □

For any $t > 0$, $x \in \mathcal{O}$, let $\Omega_{t,x} = \bigcap_{L \geq K} \{ t \leq \tau_K(t), u_K(t, x) \neq u_L(t, x) \}$, where $L$ and $K$ are positive integers. Let $\Omega^*_t = \Omega_{t,x} \cap \{ \lim_{K \to \infty} \tau_K = \infty \}$. By Lemma 6.6 and Lemma 6.8, $P(\Omega^*_t) = 1$.

The next result concludes the proof of Theorem 6.1.

**Proposition 6.9** Under the assumptions of Theorem 6.1, the process $u = \{u(t, x); t \geq 0, x \in \mathcal{O}\}$ defined by:

$$u(\omega, t, x) = u_K(\omega, t, x) \text{ if } \omega \in \Omega^*_t \text{ and } t \leq \tau_K(\omega)$$
$$u(\omega, t, x) = 0 \text{ if } \omega \not\in \Omega^*_t$$

is a mild solution of equation (1).

**Proof:** We first prove that $u$ is predictable. Note that

$$u(t, x) = \lim_{K \to \infty} (u_K(t, x)1_{\{t \leq \tau_K\}})1_{\Omega^*_t \mathcal{O}}.$$

The process $X(\omega, t, x) = 1_{\{t \leq \tau_K\}}(\omega)\text{ is clearly predictable, being in the class } \mathcal{C} \text{ defined in Remark 4.1. By the definition of } \Omega_{t,x}, \text{ since } u_K, u_L \text{ are predictable, it follows that } (\omega, t, x) \mapsto 1_{\Omega^*_t}(\omega) \text{ is } \mathcal{P}-\text{measurable. Hence, } u \text{ is predictable.}$

We now prove that $u$ satisfies (2). Let $t > 0$ and $x \in \mathcal{O}$ be arbitrary. Using (60) and Proposition C.1 (Appendix C), with probability 1, we have:

$$1_{\{t \leq \tau_K\}}u(t, x) = 1_{\{t \leq \tau_K\}}u_K(t, x)$$
$$= 1_{\{t \leq \tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s, x, y)\sigma(u_K(s, y))Z_K(ds, dy)$$
$$= 1_{\{t \leq \tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s, x, y)\sigma(u_K(s, y))Z(ds, dy)$$
$$= 1_{\{t \leq \tau_K\}} \int_0^t \int_{\mathcal{O}} G(t-s, x, y)\sigma(u_K(s, y))1_{\{s \leq \tau_K\}}Z(ds, dy)$$
\[
1\{t \leq \tau_K\} \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u(s, y)) 1\{s \leq \tau_K\} Z(ds, dy)
\]
\[
= 1\{t \leq \tau_K\} \int_0^t \int_{\mathcal{O}} G(t-s, x, y) \sigma(u(s, y)) Z(ds, dy).
\]

For the second last equality, we used the fact that processes \(X(s, y) = 1_{[0,t]}(s) G(t-s, x, y) \sigma(u_K(s, y)) 1\{s \leq \tau_K\}\) and \(Y(s, y) = 1_{[0,t]}(s) G(t-s, x, y) \sigma(u(s, y)) 1\{s \leq \tau_K\}\) are modifications of each other (i.e. \(X(s, y) = Y(s, y) \text{ a.s. for all } s > 0, y \in \mathcal{O}\)), and hence, \([X - Y]_{\alpha,t,\mathcal{O}} = 0\) and \(I(X)(t, \mathcal{O}) = I(Y)(t, \mathcal{O}) \text{ a.s.}\)
The conclusion follows letting \(K \to \infty\), since \(\tau_K \to \infty\) a.s. \(\square\)

6.3 Proof of Theorem 6.1: case \(\alpha > 1\)

In this case, for any \(t > 0\) and \(B \in \mathcal{B}_b(\mathbb{R}^d)\), we have: (see (12))
\[
Z(t, B) = \int_{[0,t] \times B \times (\mathbb{R} \setminus \{0\})} z \tilde{N}(ds, dx, dz).
\]

To introduce the stopping times \((\tau_K)_{K \geq 1}\) we use the same idea as in Section 9.7 of [22].

Let \(M(t, B) = \sum_{j \geq 1} (L_j(t, B) - EL_j(t, B))\) and \(P(t, B) = L_0(t, B)\), where \(L_j(t, B) = \sum_{[0,t] \times B} z \tilde{N}(ds, dx, dz)\)

\[
1\{1 < |z| \leq K\} N(ds, dx, dz)
\]

and \(\mu_K = \int_{1 < |z| \leq K} z \nu_\alpha(dz)\). Recalling definition (40) of \(Z_K\), it follows that:
\[
Z_K(t, B) = M(t, B) + P_K(t, B) - t|B| \mu_K.
\]

For any \(K > 0\), we let
\[
\tau_K(B) = \inf\{t > 0; |P(t, B) - P(t-, B)| > K\},
\]
where $P(t, B) = \lim_{s \to t} P(s, B)$.

Lemma 6.6 holds again, but its proof is simpler than in the case $\alpha < 1$, since $\{P(t, B)\}_{t \geq 0}$ is a compound Poisson process. By (55),

$$P(t, B) = \sum_{i,j,k \geq 1} Z_{i,j,k}^{1} 1_{\{|Z_{i,j,k}^{1}| > 1\}} 1_{\{T_{i,j,k}^{1} \leq t\}} 1_{\{X_{i,j,k}^{1} \in B\}}$$

$$P_{K}(t, B) = \sum_{i,j,k \geq 1} Z_{i,j,k}^{1} 1_{\{|Z_{i,j,k}^{1}| \leq K\}} 1_{\{T_{i,j,k}^{1} \leq t\}} 1_{\{X_{i,j,k}^{1} \in B\}}.$$

Hence, on $\{\tau_{K}(B) > T\}$, for any $L > K$, $t \in [0, T]$, $A \in \mathcal{B}_{b}(\mathbb{R}^{d})$ with $A \subseteq B$,

$$P([0, t] \times A) = P_{K}([0, t] \times A) = P_{L}([0, t] \times A).$$

Let $b_{K} = \mu - \mu_{K} = \int_{|z| > K} z \nu_{\alpha}(dz)$. Using (62) and (63), it follows that:

$$Z([0, t] \times A) = Z_{K}([0, t] \times A) - t|A|b_{K} = Z_{L}([0, t] \times A) - t|A|b_{L}$$

for any $L > K$, $t \in [0, T]$, $A \in \mathcal{B}_{b}(\mathbb{R}^{d})$ with $A \subseteq B$. Let $p \in (\alpha, 2]$ be fixed. Using an approximation argument and the construction of the integrals $I(X)$ and $I_{K}(X)$, it follows that for any $X \in L_{\alpha}$ and for any $L > K$, a.s. on $\{\tau_{K}(B) > T\}$, we have:

$$I_{K}(X)(T, B) = I_{K}(X)(T, B) - b_{K} \int_{0}^{T} \int_{\mathcal{O}} X(s, y)dyds \quad (64)$$

$$= I_{L}(X)(T, B) - b_{L} \int_{0}^{T} \int_{\mathcal{O}} X(s, y)dyds.$$

We denote $\tau_{K} = \tau_{K}(\mathcal{O})$. We consider the following equation:

$$Lu(t, x) = \sigma(u(t, x)) \dot{Z}_{K}(t, x) - b_{K}\sigma(u(t, x)), \quad t > 0, x \in \mathcal{O} \quad (65)$$

with zero initial conditions and Dirichlet boundary conditions. A mild solution of (65) is a predictable process $u$ which satisfies:

$$u(t, x) = \int_{0}^{t} \int_{\mathcal{O}} G(t - s, x, y)\sigma(u(s, y))Z_{K}(ds, dy) - b_{K} \int_{0}^{t} \int_{\mathcal{O}} G(t - s, x, y)\sigma(u(s, y))dyds \quad \text{a.s.}$$

for any $t > 0, x \in \mathcal{O}$. The existence and uniqueness of a mild solution of (65) can be proved similarly to Theorem 6.5. We omit these details. We denote this solution by $v_{K}$. 39
Lemma 6.10 Under the assumptions of Theorem 6.1, for any $t > 0, x \in \mathcal{O}$ and $L > K$,

$$v_K(t, x) = v_L(t, x) \quad \text{a.s. on } \{t \leq \tau_K\}.$$  

Proof: By the definition of $v_L$ and (64), a.s. on the event $\{t \leq \tau_K\}$, $v_L(t, x)$ is equal to

$$\int_0^t \int_{\mathcal{O}} G(t-s, x,y)\sigma(v_L(s, y))Z_L(ds, dy) - b_L \int_0^t \int_{\mathcal{O}} G(t-s, x,y)\sigma(v_L(s, y))dyds =$$

$$\int_0^t \int_{\mathcal{O}} G(t-s, x,y)\sigma(v_L(s, y))Z_K(ds, dy) - b_K \int_0^t \int_{\mathcal{O}} G(t-s, x,y)\sigma(v_L(s, y))dyds.$$  

Using the definition of $v_K$ and Proposition C.1 (Appendix C), we obtain that, with probability 1,

$$(v_K(t, x) - v_L(t, x))1_{\{t \leq \tau_K\}} = 1_{\{t \leq \tau_K\}} \left( \int_0^t \int_{\mathcal{O}} G(t-s, x,y)(\sigma(v_K(s, y)) - \sigma(v_L(s, y))) \right.$$

$$
\left. 1_{\{s \leq \tau_K\}}Z_K(ds, dy) - \int_0^t \int_{\mathcal{O}} G(t-s, x,y)(\sigma(v_K(s, y)) - \sigma(v_L(s, y)))1_{\{s \leq \tau_K\}}dyds \right).$$

Letting $M(t) = \sup_{x \in \mathcal{O}} E(|v_K(t, x) - v_L(t, x)|^p1_{\{t \leq \tau_K\}})$, we see that $M(t) \leq 2^{p-1}(E|A(t, x)|^p + E|B(t, x)|^p)$ where

$$A(t, x) = \int_0^t \int_{\mathcal{O}} G(t-s, x,y)(\sigma(v_K(s, y)) - \sigma(v_L(s, y)))1_{\{s \leq \tau_K\}}Z_K(ds, dy)$$

$$B(t, x) = \int_0^t \int_{\mathcal{O}} G(t-s, x,y)(\sigma(v_K(s, y)) - \sigma(v_L(s, y)))1_{\{s \leq \tau_K\}}dyds.$$  

We estimate separately the two terms. For the first term, we use the moment inequality (69) and the Lipschitz condition (53). We get:

$$\sup_{x \in \mathcal{O}} E|A(t, x)|^p \leq C \int_0^t J_p(t-s)M(s)ds,$$

(66)

where $C = C_{\alpha, p}K^{p-\alpha}C_p$. For the second term, we use Hölder’s inequality $|\int fg \, d\mu| \leq (\int |f|^p \, d\mu)^{1/p}(\int |g|^q \, d\mu)^{1/q}$ with $f(s, y) = G(t-s, x,y)^{1/p}(\sigma(v_K(s, y)) - \sigma(v_L(s, y)))1_{\{s \leq \tau_K\}}$ and $g(s, y) = G(t-s, x,y)^{1/q}$, where $p^{-1} + q^{-1} = 1$. Hence,

$$|B(t, x)|^p \leq C_p K_{\tau}^{p/q} \int_0^t G(t-s, x,y)|\sigma_v(s, y) - \sigma(v_L(s, y))|^p1_{\{s \leq \tau_K\}}dyds,$$

40
where \( K_t = \int_0^t J_1(s)ds < \infty \). (Since \( \mathcal{O} \) is a bounded set, \( J_1(s) \leq C J_p(s)^{1/p} \) where \( C \) is a constant depending on \( |\mathcal{O}| \) and \( p \). Since \( p > 1 \), \( \int_0^t J_p(s)^{1/p}ds \leq c_t (\int_0^t J_p(s)ds)^{1/p} < \infty \) by (52). This shows that \( K_t < \infty \).) Therefore,

\[
\sup_{x \in \mathcal{O}} E[|B(t, x)|^p] \leq C_t \int_0^t J_1(t-s)M(s)ds,
\]

where \( C_t = C_p K_t^{p/q} \). From (66) and (67), we obtain that:

\[
M(t) \leq C'_t \int_0^t (J_p(t-s) + J_1(t-s))M(s)ds
\]

where \( C'_t = 2^{p-1}(C \vee C_t) \). This implies that \( M(t) = 0 \) for all \( t > 0 \). \( \square \)

For any \( t > 0 \) and \( x \in \mathcal{O} \), we let \( \Omega_{t,x} = \bigcap_{K>L} \{ t \leq \tau_K, v_K(t, x) \neq v_L(t, x) \} \) where \( K \) and \( L \) are positive integers, and \( \Omega^*_{t,x} = \Omega_{t,x} \cap \{ \lim_{K \to \infty} \tau_K = \infty \} \). By Lemma 6.10 \( P(\Omega^*_{t,x}) = 1 \).

**Proposition 6.11** Under the assumptions of Theorem 6.1, the process \( u = \{u(t, x); t \geq 0, x \in \mathcal{O}\} \) defined by:

\[
\begin{align*}
u(\omega, t, x) &= v_K(\omega, t, x) \quad \text{if } \omega \in \Omega^*_{t,x} \text{ and } t \leq \tau_K(\omega) \\
u(\omega, t, x) &= 0 \quad \text{if } \omega \notin \Omega^*_{t,x}
\end{align*}
\]

is a mild solution of equation (1).

**Proof:** We proceed as in the proof of Proposition 6.9. In this case, with probability 1, we have:

\[
1_{\{t \leq \tau_K\}} u(t, x) = 1_{\{t \leq \tau_K\}} \left( \int_0^t \int_{\mathcal{O}} G(t-s, x, y)\sigma(u(s, y))Z(ds, dy) - b_K \int_0^t \int_{\mathcal{O}} G(t-s, x, y)\sigma(u(s, y))dyds \right).
\]

The conclusion follows letting \( K \to \infty \), since \( \tau_K \to \infty \) a.s. and \( b_K \to 0 \). \( \square \)

**A Some auxiliary results**

The following result is used in the proof of Theorem 4.3
Lemma A.1 If \( X \) has a \( S_\alpha(\sigma, \beta, 0) \) distribution then
\[
\lambda^\alpha P(|X| > \lambda) \leq c_\alpha^\star \sigma^\alpha \quad \text{for all } \lambda > 0,
\]
where \( c_\alpha^\star > 0 \) is a constant depending only on \( \alpha \).

Proof: Step 1. We first prove the result for \( \sigma = 1 \). We treat only the right tail, the left tail being similar. We denote \( X \) by \( X_\beta \) to emphasize the dependence on \( \beta \). By Property 1.2.15 of [27], \( \lim_{\lambda \to \infty} \lambda^\alpha P(X_\beta > \lambda) = C_\alpha \frac{\lambda^\beta}{\sqrt{\alpha}} \), where \( C_\alpha = (\int_0^\infty x^{-\alpha} \sin x dx)^{-1} \). We use the fact that for any \( \beta \in [0, 1] \),
\[
P(X_1 < \lambda) \leq P(X_\beta < \lambda) \leq P(X_0 < \lambda)
\]
for all \( \lambda \in \mathbb{R}, |\lambda| > \lambda_\alpha \) for some \( \lambda_\alpha > 0 \) (see Section 1.5 of [19] or p.37 of [27]). If \( \beta \in [-1, 0] \), \( -X_\beta \) has the same distribution as \( X_{-\beta} \), and hence for \( \lambda > \lambda_\alpha \),
\[
P(X_\beta > \lambda) = P(X_{-\beta} < -\lambda) \leq P(X_0 < -\lambda) = P(X_0 > \lambda) \leq P(X_1 > \lambda).
\]
Since \( \lim_{\lambda \to \infty} \lambda^\alpha P(X_1 > \lambda) = C_\alpha \), there exists \( \lambda_\alpha^\star > \lambda_\alpha \) such that
\[
\lambda^\alpha P(X_1 > \lambda) < 2C_\alpha \quad \text{for all } \lambda > \lambda_\alpha^\star.
\]
It follows that \( \lambda^\alpha P(X_\beta > \lambda) < 2C_\alpha \) for all \( \lambda > \lambda_\alpha^\star \) and \( \beta \in [-1, 1] \). Clearly, for all \( \lambda \in (0, \lambda_\alpha^\star] \) and \( \beta \in [-1, 1] \), \( \lambda^\alpha P(X_\beta > \lambda) \leq \lambda^\alpha \leq (\lambda_\alpha^\star)^\alpha \).

Step 2. We now consider the general case. Since \( X/\sigma \) has a \( S_\alpha(1, \beta, 0) \) distribution, by Step 1, it follows that \( \lambda^\alpha P(|X| > \sigma\lambda) \leq c_\alpha^\star \) for any \( \lambda > 0 \). The conclusion follows multiplying by \( \sigma^\alpha \). □

In the proof of Theorem 4.3 and Lemma A.3 below, we use the following remark, due to Adam Jakubowski (personal communication).

Remark A.2 Let \( X \) be a random variable such that \( P(|X| > \lambda) \leq K\lambda^{-\alpha} \) for all \( \lambda > 0 \), for some \( K > 0 \) and \( \alpha \in (0, 2) \). Then, for any \( A > 0 \),
\[
E(|X| 1_{\{|X| \leq A\}}) \leq \int_0^A P(|X| > t) dt \leq K \frac{1}{1 - \alpha} A^{1-\alpha} \quad \text{if } \alpha < 1,
\]
\[
E(|X| 1_{\{|X| > A\}}) \leq \int_A^\infty P(|X| > t) dt + A P(|X| > A) \leq K \frac{\alpha}{\alpha - 1} A^{1-\alpha} \quad \text{if } \alpha > 1,
\]
\[
E(X^2 1_{\{|X| \leq A\}}) \leq 2 \int_0^A t P(|X| > t) dt \leq K \frac{2}{2 - \alpha} A^{2-\alpha} \quad \text{for any } \alpha \in (0, 2).
\]
The next result is a generalization of Lemma 2.1 of [10] to the case of non-symmetric random variables. This result is used in the proof of Lemma 5.1 and Proposition 5.4.

**Lemma A.3** Let \( \eta_k \) be independent random variables such that
\[
\sup_{\lambda > 0} \lambda^\alpha P(|\eta_k| > \lambda) \leq K \quad \forall k \geq 1
\]  
(68)
for some \( K > 0 \) and \( \alpha \in (0, 2) \). If \( \alpha > 1 \), we assume that \( E(\eta_k) = 0 \) for all \( k \), and if \( \alpha = 1 \), we assume that \( \eta_k \) has a symmetric distribution for all \( k \).

Then for any sequence \((a_k)_{k \geq 1}\) of real numbers, we have:
\[
\sup_{\lambda > 0} \lambda^\alpha P\left(\sum_{k \geq 1} a_k \eta_k > \lambda\right) \leq r_\alpha K \sum_{k \geq 1} |a_k|^\alpha
\]  
(69)
where \( r_\alpha > 0 \) is a constant depending only on \( \alpha \).

**Proof:** We consider the intersection of the event on the left-hand side of (69) with the event \( \{\sup_{k \geq 1} |a_k \eta_k| > \lambda\} \) and its complement. Hence,
\[
P\left(\sum_{k \geq 1} a_k \eta_k > \lambda\right) \leq \sum_{k \geq 1} P(|a_k \eta_k| > \lambda) + P\left(\left|\sum_{k \geq 1} a_k \eta_k 1_{\{|a_k \eta_k| \leq \lambda\}}\right| > \lambda\right) =: I + II.
\]
Using (68), we have \( I \leq K \lambda^{-\alpha} \sum_{k \geq 1} |a_k|^\alpha \). To treat \( II \), we consider 3 cases.

**Case 1.** \( \alpha < 1 \). By Markov’s inequality and Remark A.2, we have:
\[
II \leq \frac{1}{\lambda} \sum_{k \geq 1} |a_k| E(|\eta_k| 1_{\{|a_k \eta_k| \leq \lambda\}}) \leq K \frac{1}{1 - \alpha} \lambda^{-\alpha} \sum_{k \geq 1} |a_k|^\alpha.
\]

**Case 2.** \( \alpha > 1 \). Let \( X = \sum_{k \geq 1} a_k \eta_k 1_{\{|a_k \eta_k| \leq \lambda\}} \). Since \( E(\sum_{k \geq 1} a_k \eta_k) = 0 \),
\[
|E(X)| = |E\left(\sum_{k \geq 1} a_k \eta_k 1_{\{|a_k \eta_k| > \lambda\}}\right)| \leq \sum_{k \geq 1} |a_k| E(|\eta_k| 1_{\{|a_k \eta_k| > \lambda\}}) \leq \frac{K \alpha}{\alpha - 1} \lambda^{1-\alpha} \sum_{k \geq 1} |a_k|^\alpha,
\]
where we used Remark A.2 for the last inequality. From here, we infer that
\[
|E(X)| < \frac{\lambda}{2} \quad \text{for any } \lambda > \lambda_\alpha.
\]
where $\lambda_0^\alpha = 2K\frac{\alpha}{\alpha-1}\sum_{k\geq 1}|a_k|^\alpha$. By Chebyshev’s inequality, for any $\lambda > \lambda_0$, $II = P(|X| > \lambda) \leq P(|X - E(X)| > \lambda - |E(X)|) \leq \frac{4}{\lambda^2} E|X - E(X)|^2$

\[
\leq \frac{4}{\lambda^2} \sum_{k\geq 1} a_k^2 E(\eta_k^2 1_{|a_k\eta_k| \leq \lambda}) \leq \frac{8K}{2-\alpha} \lambda^{-\alpha} \sum_{k\geq 1} |a_k|^\alpha,
\]

using Remark [A.2] for the last inequality. On the other hand, if $\lambda \in (0, \lambda_0]$, $II = P(|X| > \lambda) \leq 1 \leq \lambda_0^\alpha \lambda^{-\alpha} = 2K\frac{\alpha}{\alpha-1}\lambda^{-\alpha} \sum_{k\geq 1} |a_k|^\alpha$.

Case 3. $\alpha = 1$. Since $\eta_k$ has a symmetric distribution, we can use the original argument of [10]. $\square$

## B Fractional power of the Laplacian

Let $G(t, x)$ be the fundamental solution of $\frac{\partial u}{\partial t} + (-\Delta)^\gamma u = 0$ on $\mathbb{R}^d$, $\gamma > 0$.

**Lemma B.1** For any $p > 1$, there exist some constants $c_1, c_2 > 0$ depending on $d, p, \gamma$ such that

\[
c_1 t^{-\frac{d}{2\gamma}(p-1)} \leq \int_{\mathbb{R}^d} G(t, x)^p dx \leq c_2 t^{-\frac{d}{2\gamma}(p-1)}.
\]

**Proof:** The upper bound is given by Lemma B.23 of [22]. For the lower bound, we use the scaling property of the functions $(g_t, \gamma)_{t>0}$. We have:

\[
G(t, x) = \int_0^\infty \frac{1}{(4\pi t^{1/\gamma})^{d/2}} \exp \left(-\frac{|x|^2}{4t^{1/\gamma}}\right) g_{1, \gamma}(r) dr
\]

\[
\geq \int_1^\infty \frac{1}{(4\pi t^{1/\gamma})^{d/2}} \exp \left(-\frac{|x|^2}{4t^{1/\gamma}}\right) g_{1, \gamma}(r) dr
\]

\[
\geq \frac{1}{(4\pi t^{1/\gamma})^{d/2}} \exp \left(-\frac{|x|^2}{4t^{1/\gamma}}\right) C_{d, \gamma} \quad \text{with} \quad C_{d, \gamma} := \int_1^\infty r^{-d/2} g_{1, \gamma}(r) dr < \infty,
\]

and hence

\[
\int_{\mathbb{R}^d} G(t, x)^p dx \geq c_{d, \gamma, p} t^{-\frac{d}{2\gamma}} \int_{\mathbb{R}^d} \exp \left(-\frac{p|x|^2}{4t^{1/\gamma}}\right) dx = c_{d, p, \gamma} t^{-\frac{d}{2\gamma}(p-1)}.
\]

$\square$
C  A local property of the integral

The following result is the analogue of Proposition 8.11 of [22].

Proposition C.1 Let $T > 0$ and $\mathcal{O} \subset \mathbb{R}^d$ be a Borel set. Let $X = \{X(t, x); t \geq 0, x \in \mathbb{R}^d\}$ be a predictable process such that $X \in \mathcal{L}_\alpha$ if $\alpha < 1$, or $X \in \mathcal{L}_p$ for some $p \in (\alpha, 2]$ if $\alpha > 1$. If $\mathcal{O}$ is unbounded, assume in addition that $X$ satisfies (38) if $\alpha < 1$, or $X$ satisfies (40) for some $p \in (\alpha, 2)$, if $\alpha > 1$. Suppose that there exists an event $A \in \mathcal{F}_T$ such that $X(\omega, t, x) = 0$ for all $\omega \in A, t \in [0, T], x \in \mathcal{O}$. Then for any $K > 0$, $I(X)(T, \mathcal{O}) = I_K(X)(T, \mathcal{O}) = 0$ a.s. on $A$.

Proof: We only prove the result for $I(X)$, the proof for $I_K(X)$ being the same. Moreover, we include only the argument for $\alpha < 1$; the case $\alpha > 1$ is similar. The idea is to reduce the argument to the case when $X$ is a simple process, as in the proof Proposition of 8.11 of [22].

Step 1. We show that the proof can be reduced to the case of a bounded set $\mathcal{O}$. Let $X_n(t, x) = X(t, x)1_{\mathcal{O}_n}(x)$ where $\mathcal{O}_n = \mathcal{O} \cap E_n$ and $(E_n)_n$ is an increasing sequence of sets in $\mathcal{B}_b(\mathbb{R}^d)$ such that $\bigcup_n E_n = \mathbb{R}^d$. Then $X_n \in \mathcal{L}_\alpha$ satisfies (70). By the dominated convergence theorem,

$$E \int_0^T \int_\mathcal{O} |X_n(t, x) - X(t, x)|^\alpha \to 0.$$ 

By the construction of the integral, $I(X_n)(T, \mathcal{O}) \to I(X)(T, \mathcal{O})$ a.s. for a subsequence $\{n_k\}$. It suffices to show that $I(X_n)(T, \mathcal{O}) = 0$ a.s. on $A$ for all $n$. But $I(X_n)(T, \mathcal{O}) = I(X_n)(T, \mathcal{O}_n)$ and $\mathcal{O}_n$ is bounded.

Step 2. We show that the proof can be reduced to the case of a bounded processes. For this, let $X_n(t, x) = X(t, x)1_{\{|X(t, x)| \leq n\}}$. Clearly, $X_n \in \mathcal{L}_\alpha$ is bounded and satisfies (70) for all $n$. By the dominated convergence theorem, $|X_n - X|_\alpha \to 0$, and hence $I(X_n)(T, \mathcal{O}) \to I(X)(T, \mathcal{O})$ a.s. for a subsequence $\{n_k\}$. It suffices to show that $I(X_n)(T, \mathcal{O}) = 0$ a.s. on $A$ for all $n$.

Step 3. We show that the proof can be reduced to the case of bounded continuous processes. Assume that $X \in \mathcal{L}_\alpha$ is bounded and satisfies (70). For any $t > 0$ and $x \in \mathbb{R}^d$, we define

$$X_n(t, x) = n^{d+1} \int_{t-1/n}\cdots\int_{(x-1/n,x)\cap \mathcal{O}} X(s, y)dyds,$$

for

$$\int_{t-1/n}\cdots\int_{(x-1/n,x)\cap \mathcal{O}} X(s, y)dyds.$$
where \((a, b] = \{y \in \mathbb{R}^d; a_i < y_i \leq b_i \text{ for all } i = 1, \ldots, d\}\). Clearly, \(X_n\) is bounded and satisfies (70). We prove that \(X_n \in \mathcal{L}_\alpha\). Since \(X_n\) is bounded, \([X_n]_\alpha < \infty\). To prove that \(X_n\) is predictable, we consider

\[
F(t, x) = \int_0^t \int_{(0, x]} X(s, y) dy ds.
\]

Since \(X\) is predictable, it is progressively measurable, i.e. for any \(t > 0\), the map \((\omega, s, x) \mapsto X(\omega, s, x)\) from \(\Omega \times [0, t] \times \mathbb{R}^d\) to \(\mathbb{R}\) is \(\mathcal{F}_t \times \mathcal{B}([0, t]) \times \mathcal{B}(\mathbb{R}^d)\)-measurable. Hence, \(F(t, \cdot)\) is \(\mathcal{F}_t \times \mathcal{B}(\mathbb{R}^d)\)-measurable for any \(t > 0\). Since the map \(t \mapsto F(\omega, t, x)\) is left-continuous for any \(\omega \in \Omega, x \in \mathbb{R}^d\), it follows that \(F\) is predictable, being in the class \(\mathcal{C}\) defined in Remark 4.1. Hence, \(X_n\) is predictable, being a sum of \(2^{d+1}\) terms involving \(F\).

Since \(F\) is continuous in \((t, x)\), \(X_n\) is continuous in \((t, x)\). By Lebesque differentiation theorem in \(\mathbb{R}^{d+1}\), \(X_n(\omega, t, x) \to X(\omega, t, x)\) for any \(\omega \in \Omega, t > 0, x \in \mathcal{O}\). By the bounded convergence theorem, \([X_n - X]_\alpha \to 0\). Hence \(I(X_{n_k})(T, \mathcal{O}) \to I(X)(T, \mathcal{O})\) a.s. for a subsequence \(\{n_k\}\). It suffices to show that \(I(X_{n_k})(T, \mathcal{O}) = 0\) a.s. on \(A\) for all \(n\).

Step 4. Assume that \(X \in \mathcal{L}_\alpha\) is bounded, continuous and satisfies (70). Let \((U_j^{(n)})_{j=1,\ldots,m_n}\) be a partition of \(\mathcal{O}\) in Borel sets with Lebesque measure smaller than \(1/n\). Let \(x_j^n \in U_j^{(n)}\) be arbitrary. Define

\[
X_n(t, x) = \sum_{k=0}^{n-1} \sum_{j=1}^{m_n} X\left(\frac{kT}{n}, x_j^n\right) 1_{(\frac{kT}{n}, \frac{(k+1)T}{n})}(t) 1_{U_j^{(n)}}(x).
\]

Since \(X\) is continuous in \((t, x)\), \(X_n(t, x) \to X(t, x)\). By the bounded convergence theorem, \([X_n - X]_\alpha \to 0\), and hence \(I(X_{n_k})(T, \mathcal{O}) \to I(X)(T, \mathcal{O})\) a.s. for a subsequence \(\{n_k\}\). Since on the event \(A\),

\[
I(X_n)(T, \mathcal{O}) = \sum_{k=0}^{n-1} \sum_{j=1}^{m_n} X\left(\frac{kT}{n}, x_j^n\right) Z\left(\left(\frac{kT}{n}, \frac{(k+1)T}{n}\right) \times U_j^{(n)}\right) = 0,
\]

it follows that \(I(X)(T, \mathcal{O}) = 0\) a.s. on \(A\). □

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