Finite Groups With Minimal CSS-Subgroups

Abd El-Rahman Heliel¹,²,∗, Rola Hijazi¹, Shorouq Al-Shammari¹,³

¹ Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
² Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt
³ Department of Science and Technology, The University College in Al Khafji, University of Hafir Al Batin, Al Khafji, Saudi Arabia

Abstract. Let G be a finite group. A subgroup H of G is called SS-quasinormal in G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B. A subgroup H of G is called CSS-subgroup in G if there exists a normal subgroup K of G such that G = HK and H ∩ K is SS-quasinormal in G. In this paper, we investigate the influence of minimal CSS-subgroups of G on its structure. Our results improve and generalize several recent results in the literature.

2020 Mathematics Subject Classifications: 20D10, 20D15, 20D20
Key Words and Phrases: CSS-subgroup, c-normal subgroup, SS-quasinormal subgroup, p-nilpotent group, saturated formation.

1. Introduction

All groups considered in this paper are finite. The terminology and notions employed agree with standard usage, as in [2, 5], and G always denotes a finite group.

Following Kegel [9], a subgroup H of G is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G, i.e., HP = PH for any Sylow subgroup P of G. A subgroup H of G is said to be c-normal in G if H has a normal subgroup K such that G = HK and H ∩ K ≤ HG, where HG = CoreG(H) is the largest normal subgroup of G contained in H (see Wang [18]). Recently, in 2008, Li et al. [12] extended S-quasinormal subgroups of a group G to SS-quasinormal subgroups and they gave the following definition: A subgroup H of G is said to be SS-quasinormal in G if there is a supplement B of H to G such that H permutes with every Sylow subgroup of B.

Obviously, every S-quasinormal subgroup is SS-quasinormal. The converse is not true in general. For instance, S3 is SS-quasinormal subgroup of the symmetric group S4 but...
not $S$-quasinormal. More recently, in 2019, Zhao et al. [26] introduced a new subgroup embedding property of a finite group, called CSS-subgroup, which generalize and unify both of $c$-normality and $SS$-quasinormality as follows: A subgroup $H$ of $G$ is called CSS-subgroup of $G$ if there exists a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K$ is $SS$-quasinormal in $G$. It is clear that each of $c$-normality and $SS$-quasinormality concepts implies CSS-subgroup. The converse does not hold in general (see [26, Examples 1 and 2]).

Over years, many authors studied the influence of minimal subgroups of a finite group on its structure (a subgroup of prime order is called a minimal subgroup). In this context, Buckley [3] got the supersolvability of a group of odd order when all its minimal subgroups are normal. In [17], Shaalan proved that a group $G$ is supersolvable if all subgroups of prime order $p$ or of order $4$ (if $p = 2$) of $G$ are $S$-quasinormal in $G$. Later on, Wang [18] got the same result of Shaalan [17] just he replaced $S$-quasinormality by $c$-normality. By using the $SS$-quasinormality concept, Li et al. [11] extended these results through the theory of formations and proved that: Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and let $G$ be a group. Then $G \in \mathfrak{F}$ if and only if $G$ has a normal subgroup $H$ such that $G/H \in \mathfrak{F}$ and every subgroup of $F^s(H)$ of prime order $p$ or of order $4$ (if $p = 2$) is $SS$-quasinormal in $G$, where $F^s(H)$ is the generalized Fitting subgroup of $H$. Also, Wei et al. in [21] used the $c$-normality concept and obtained the same previous result. For more results in this direction (see [1, 11, 12, 16–18, 20, 21, 24]).

The main purpose of this paper is to improve and extend the above mentioned results by using the recent concept CSS-subgroup. More precisely, we investigate the structure of a finite group $G$ when every subgroup of $G$ of prime order $p$ or of order $4$ (if $p = 2$) is CSS-subgroup in $G$.

2. Basic Definitions and Preliminaries

In this section, we list some definitions and state some known results from the literature which will be used in proving our results.

A class of groups $\mathfrak{F}$ is said to be a formation if $\mathfrak{F}$ is closed under taking epimorphic images and every group $G$ has a smallest normal subgroup with quotient in $\mathfrak{F}$. This subgroup is called the $\mathfrak{F}$-residual of $G$ and it is denoted by $G^\mathfrak{F}$. A formation $\mathfrak{F}$ is called saturated if it is closed under taking Frattini extensions. Throughout this paper, $\mathfrak{U}$ and $\mathfrak{N}$ will denote the classes of supersolvable groups and nilpotent groups, respectively. It is known that $\mathfrak{U}$ and $\mathfrak{N}$ are saturated formations (see [7, Satz 8.6, p. 713 and Satz 3.7, p. 270]).

A normal subgroup $N$ of a group $G$ is an $\mathfrak{F}$-hypercentral subgroup of $G$ provided $N$ possesses a chain of subgroups $1 = N_0 \unlhd N_1 \unlhd \ldots \unlhd N_s = N$ such that $N_{i+1}/N_i$ is an $\mathfrak{F}$-central chief factor of $G$ (see [5, p. 387]). The product of all $\mathfrak{F}$-hypercentral subgroups of $G$ is again an $\mathfrak{F}$-hypercentral subgroup, denoted by $Z_\mathfrak{F}(G)$, and it is called the $\mathfrak{F}$-hypercentral of $G$ (see [5, IV 6.8]). For the formation $\mathfrak{U}$, the $\mathfrak{U}$-hypercentral of a group $G$ will be denoted by $Z_\mathfrak{U}(G)$, that is, $Z_\mathfrak{U}(G)$ is the product of all normal subgroups $N$ of $G$ such that each chief factor of $G$ below $N$ has prime order and for the formation $\mathfrak{N}$, the $\mathfrak{N}$-hypercentral of
a group $G$ is simply the terminal member $Z_\infty(G)$ of the ascending central series of $G$. For more details about saturated formations, see \[5, \text{IV}\].

For any group $G$, the generalized Fitting subgroup $F^*(G)$ is the set of all elements $x$ of $G$ which induce an inner automorphism on every chief factor of $G$.

**Lemma 1.** (See [26, Lemma 2.3]) Let $H$ be CSS-subgroup of $G$.

1. If $H \leq M \leq G$, then $H$ is CSS-subgroup of $M$.

2. Let $N \leq G$ and $N \leq H$. Then $H$ is CSS-subgroup of $G$ if and only if $H/N$ is CSS-subgroup of $G/N$.

3. Let $\pi$ be a set of some primes and $N$ a normal $\pi'$-subgroup of $G$. If $H$ is a $\pi$-subgroup of $G$, then $HN/N$ is CSS-subgroup of $G/N$.

**Lemma 2.** (See [7, Satz 5.4, p. 434 and Satz 5.2, p. 281]) Let $G$ be a minimal non-$p$-nilpotent group (a non-$p$-nilpotent group all of its proper subgroups are $p$-nilpotent), where $p$ is a prime.

1. $G$ is a minimal non-nilpotent group.

2. $G = PQ$, where $P$ is a normal Sylow $p$-subgroup of $G$ and $Q$ is a non normal cyclic Sylow $q$-subgroup of $G$.

3. $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

4. If $p > 2$, then the exponent of $P$ is $p$ and when $p = 2$, the exponent of $P$ is at most 4.

**Lemma 3.** (See [17, Theorem 3.2]) Let $p$ be the smallest prime dividing $|G|$ and $P$ a Sylow $p$-subgroup of $G$. If every subgroup of $P$ of order $p$ or of order 4 (if $p = 2$) is $S$-quasinormal in $G$, then $G$ is $p$-nilpotent.

**Lemma 4.** (See [23]) Let $H$ be a subnormal subgroup of $G$.

1. If $H$ is a Hall-subgroup of $G$, then $H$ is normal in $G$.

2. If $H$ is a $\pi$-subgroup of $G$, then $H \leq O_\pi(G)$.

**Lemma 5.** (See [11, Lemma 2.2]) Suppose that $P$ is a $p$-subgroup of $G$. Then $P$ is $S$-quasinormal in $G$ if and only if $P \leq O_p(G)$ and $P$ is SS-quasinormal in $G$.

**Lemma 6.** (See [13, Theorem 3.3]) Suppose that $P$ is a normal $p$-subgroup of $G$, where $p > 2$. If every subgroup of $P$ of order $p$ is $S$-quasinormal in $G$, then $P \leq Z_\Delta(G)$.

**Lemma 7.** (See [22, Theorem 7.7, p. 31]) Let $N$ be a normal subgroup of $G$ such that $N \leq Z_\Delta(G)$. Then $Z_\Delta(G/N) = Z_\Delta(G)/N$.

**Lemma 8.** (See [22, Theorem 6.3, p. 220 and Corollary 7.8, p. 33]) Let $P$ be a normal $p$-subgroup of $G$ such that $|G/C_G(P)|$ is a power of $p$. Then $P \leq Z_\Delta(G)$.
Lemma 9. (See [5, Proposition 3.11, p. 362]) If \( \mathfrak{F}_1 \) and \( \mathfrak{F}_2 \) are two saturated formations such that \( \mathfrak{F}_1 \subseteq \mathfrak{F}_2 \), then \( Z_{\mathfrak{F}_1}(G) \subseteq Z_{\mathfrak{F}_2}(G) \).

Lemma 10. (See [4]) Let \( K \) be a normal subgroup of \( G \) such that \( G/K \in \mathfrak{F} \), where \( \mathfrak{F} \) is a saturated formation. If \( \Omega(P) \leq Z_{\mathfrak{F}}(G) \), where \( P \) is a Sylow \( p \)-subgroup of \( K \), then \( G/O_{p'}(K) \in \mathfrak{F} \).

Lemma 11. (See [8, X 13] and [14, Lemma 2.3(4)]) Let \( M \) be a subgroup of \( G \).

1. If \( M \) is normal in \( G \), then \( F^*(M) \leq F^*(G) \).
2. \( F^*(G) \neq 1 \) if \( G \neq 1 \).
3. If \( F^*(G) \) is solvable, then \( F^*(G) = F(G) \).
4. Suppose \( K \) is a subgroup of \( G \) contained in \( Z(G) \). Then \( F^*(G/K) = F^*(G)/K \).

Lemma 12. (See [10, Corollary 3]) Let \( \mathfrak{F} \) be a saturated formation and \( G \) a group. Suppose that \( C_G(N) \leq N \unlhd G \). Then \( G \in \mathfrak{F} \) if every cyclic subgroup of \( N \) of prime order or of order 4 (if \( p = 2 \)) is contained in \( Z_{\mathfrak{F}}(G) \).

Lemma 13. (See [15, Lemma 2.8]) Suppose that \( G \) is a group and \( P \) is a normal \( p \)-subgroup of \( G \) contained in \( Z_{\infty}(G) \). Then \( C_G(P) \geq O^p(G) \).

Lemma 14. (See [7, Satz 2.8, p. 420]) If \( P \) is a cyclic Sylow \( p \)-subgroup of \( G \), where \( p \) is the smallest prime dividing \( |G| \), then \( G \) is \( p \)-nilpotent.

Lemma 15. (See [6, Theorem 3.10, p. 184]) If \( H \) is a \( p' \)-group of automorphisms of the \( p \)-group \( P \) with \( p \) odd which acts trivially on \( \Omega_1(P) \), then \( H = 1 \).

Lemma 16. (See [6, Theorem 2.4, p. 178]) If \( H \) is a \( p' \)-group of automorphisms of the abelian \( p \)-group \( P \) which acts trivially on \( \Omega_1(P) \), then \( H = 1 \).

3. Main Results

First we prove:

Theorem 1. Let \( p \) be the smallest prime dividing \( |G| \) and \( P \) a Sylow \( p \)-subgroup of \( G \). If every subgroup of \( P \) of prime order \( p \) or of order 4 (if \( p = 2 \)) is CSS-subgroup of \( G \), then \( G \) is \( p \)-nilpotent.

Proof. Assume that the result is false and let \( G \) be a counterexample of minimal order. Let \( L \) be an arbitrary proper subgroup of \( G \). Then every subgroup of \( L \) of prime order \( p \) or of order 4 (if \( p = 2 \)) is CSS-subgroup of \( G \) by the hypothesis. Thus, by Lemma 11, every subgroup of \( L \) of prime order \( p \) or of order 4 (if \( p = 2 \)) is CSS-subgroup of \( L \).

That means \( L \) satisfies the hypothesis of the theorem and so \( L \) is \( p \)-nilpotent by the minimal choice of \( G \). Hence, \( G \) is not \( p \)-nilpotent but all of its proper subgroups are \( p \)-nilpotent.
By Lemma 2, $G$ is a minimal non-nilpotent group and so $G = PQ$, where $P$ is a normal Sylow $p$-subgroup of $G$ and $Q$ is a non-normal cyclic Sylow $q$-subgroup of $G$, for some prime $q \neq p$. Furthermore, if $p > 2$, then $P$ is of exponent $p$ and if $p = 2$, $P$ is of exponent at most 4. If every subgroup of $P$ with order $p$ or 4 (if $p = 2$) is $S$-quasinormal in $G$, then, by Lemma 3, we get the $p$-nilpotency of $G$, a contradiction. Therefore, there exists a subgroup $S$ of $P$ of prime order $p$ or of order 4 (if $p = 2$) such that $S$ is not $S$-quasinormal in $G$. By hypothesis, $S$ is CSS-subgroup of $G$. Then there exists a normal subgroup $K$ of $G$ such that $G = SK$ and $S \cap K$ is $SS$-quasinormal in $G$. Assume that $K = G$. It follows that $S$ is $SS$-quasinormal in $G$. Since $P$ is normal in $G$, then $S$ is subnormal in $G$. Thus, by Lemma 4, $S \leq O_p(G)$. Applying Lemma 5, we get $S$ is $S$-quasinormal in $G$, a contradiction. Hence, $K$ is a proper normal nilpotent subgroup of $G$ which implies that $Q$ is characteristic in $K$. Therefore $Q$ is a normal subgroup in $G$, a final contradiction completing the proof.

**Lemma 17.** Let $P$ be a non-trivial normal $p$-subgroup of $G$ (where $p > 2$). If every minimal subgroup of $P$ is CSS-subgroup of $G$, then $P \leq Z_G(G)$.

**Proof.** We prove the theorem by induction on $|G| + |P|$. If every minimal subgroup of $P$ is $S$-quasinormal in $G$, then by Lemma 6, we get $P \leq Z_G(G)$ and we are done. Thus, we may assume that $P$ has a minimal subgroup $L$ such that $L$ is not $S$-quasinormal in $G$. By the hypothesis of the lemma, $L$ is CSS-subgroup of $G$, i.e., $G$ has a normal subgroup $K$ such that $G = LK$ and $L \cap K$ is $SS$-quasinormal in $G$. If $L \cap K \neq 1$, we have $L \cap K = L$. Hence, $L$ is $SS$-quasinormal in $G$. Since $P$ is normal in $G$, then $L$ is subnormal in $G$. Lemma 4 implies that $L \leq O_p(G)$. Applying Lemma 5, $L$ is $S$-quasinormal in $G$, a contradiction. Therefore, we may assume $L \cap K = 1$. Then, $P = P \cap G = P \cap LK = L(P \cap K)$ and $P \cap K \leq G$. By the hypothesis, every minimal subgroup of the non-trivial normal $p$-subgroup $P \cap K$ is CSS-subgroup of $G$. This leads to $P \cap K \leq Z_G(G)$ by induction on $|G| + |P|$. Hence, $P/(P \cap K) \leq Z_G(G)/(P \cap K)$ as $P/(P \cap K)$ is a normal subgroup of $G/(P \cap K)$ of order $p$. But $P \cap K \leq Z_G(G)$, then $Z_G(G)/(P \cap K) = Z_G(G)/(P \cap K)$ by Lemma 7. Thus, $P/(P \cap K) \leq Z_G(G)/(P \cap K)$. Now it follows easily that $P \leq Z_G(G)$.

Immediate consequence of Lemma 17 and Theorem 1, we have the following corollary:

**Corollary 1.** Let $P$ be a normal $p$-subgroup of $G$. If every subgroup of $P$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$, then $P \leq Z_G(G)$.

**Proof.** Assume that $p > 2$. Then, by Lemma 17, $P \leq Z_G(G)$ and we are done. Hence, consider $p = 2$. Let $Q$ be any Sylow $q$-subgroup of $G$, where $q \neq 2$. It is clear that $PQ$ is a subgroup of $G$. Since every subgroup of $P$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$, then by Lemma 11, every subgroup of $P$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $PQ$.

By applying Theorem 1, we have $PQ$ is 2-nilpotent. This implies that $PQ = P \times Q$ and so $Q$ centralizes $P$. Thus, $O_p(G) \leq C_G(P)$ and it follows that $|G/C_G(P)|$ is a power of 2. By Lemma 8, we conclude $P \leq Z_G(G)$. 

We now prove:

**Theorem 2.** Let $\mathcal{F}$ be a saturated formation containing $\Omega$ and $G$ a group. Then $G \in \mathcal{F}$ if and only if there exists a normal subgroup $H$ in $G$ such that $G/H \in \mathcal{F}$ and every subgroup of $H$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$.

Proof. If $G \in \mathcal{F}$, then we set $H = 1$ and the result follows. Conversely, assume that the result is false and let $G$ be a counterexample of minimal order. By using Lemma 11 and repeated applications of Theorem 1, the group $H$ has a Sylow tower of supersolvable type which means that $H$ has a normal Sylow $p$-subgroup $P$, where $p$ is the largest prime dividing $|H|$. Clearly, $P$ is normal in $G$ and hence $(G/P)/(H/P) \cong G/H \in \mathcal{F}$. By Lemma 12, every subgroup of $H/P$ of prime order or of order 4 (if $p = 2$) is CSS-subgroup of $G/P$. Then, by the minimal choice of $G$, we have $G/P \in \mathcal{F}$ and so $1 \neq G/\mathcal{F} \leq P$. By the hypothesis, every subgroup of $G/\mathcal{F}$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$. Then, by Corollary 1, $G/\mathcal{F} \leq Z_{\Omega}(G)$. Since $\Omega(G/\mathcal{F}) \leq G/\mathcal{F}$ and $Z_{\Omega}(G) \leq Z_{\mathcal{F}}(G)$, by Lemma 9, we have $\Omega(G/\mathcal{F}) \leq G/\mathcal{F} \leq Z_{\mathcal{F}}(G)$. Hence, $\Omega(G/\mathcal{F}) \leq Z_{\mathcal{F}}(G)$. Therefore, by Lemma 10, $G \in \mathcal{F}$, a contradiction.

The following corollaries are immediate consequences of Theorem 2:

**Corollary 2.** Let $H$ be a normal subgroup of $G$ such that $G/H$ is supersolvable. If every subgroup of $H$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$, then $G$ is supersolvable.

**Corollary 3.** Let $H$ be a normal subgroup of $G$ such that $(G/H)′$ is nilpotent. If every subgroup of $H$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$, then $G′$ is nilpotent.

**Corollary 4.** Let $G$ be a group such that every subgroup of $G$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$, then $G$ is supersolvable.

Now we can prove:

**Theorem 3.** Let $\mathcal{F}$ be a saturated formation containing $\Omega$ and $G$ a group. Then $G \in \mathcal{F}$ if and only if $G$ has a normal subgroup $H$ such that $G/H \in \mathcal{F}$ and every subgroup of $F^*(H)$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$.

Proof. If $G \in \mathcal{F}$, then we set $H = 1$ and the theorem follows. Now we prove the converse. By the hypothesis and Lemma 11, every subgroup of $F^*(H)$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $F^*(H)$. Corollary 4 implies that $F^*(H)$ is supersolvable. Hence, by Lemma 11, $F^*(H) = F(H)$. Then, by Corollary 1, $O_p(H) \leq Z_{\Omega}(G)$. Since $Z_{\Omega}(G) \leq Z_{\mathcal{F}}(G)$, by Lemma 9, it follows that $O_p(H) \leq Z_{\mathcal{F}}(G)$ and so $F^*(H) = F(H) \leq Z_{\mathcal{F}}(G)$. Applying Lemma 12, we get $G \in \mathcal{F}$.

Immediately from Theorem 3, we have the following corollaries:

**Corollary 5.** Let $H$ be a normal subgroup $G$ such that $G/H$ is supersolvable. If every subgroup of $F^*(H)$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$, then $G$ is supersolvable.
Corollary 6. If every subgroup of $F^*(G)$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$, then $G$ is supersolvable.

Corollary 7. Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and $G$ a group. Then $G \in \mathcal{F}$ if and only if $G$ has a solvable normal subgroup $H$ such that $G/H \in \mathcal{F}$ and every subgroup of $F(H)$ of prime order $p$ or of order 4 (if $p = 2$) is CSS-subgroup of $G$.

We now prove:

Theorem 4. Let $G$ be a group. If every subgroup of $G$ of prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of $G$ is CSS-subgroup of $G$ or lies in $Z_\infty(G)$, then $G$ is nilpotent.

Proof. Assume that the result is false and let $G$ be a counterexample of minimal order. Let $L$ be an arbitrary proper subgroup of $G$ and $K$ a cyclic subgroup of $L$ of prime order or of order 4. Then $K \leq Z_\infty(G) \cap L \leq Z_\infty(L)$. By hypotheses and Lemma 11, $K$ is CSS-subgroup of $L$. The minimal choice of $G$ implies that $L$ is nilpotent. Since $L$ is an arbitrary proper subgroup of $G$, we have that $G$ is a minimal non-nilpotent group. Hence, by Lemma 2, $G = PQ$, where $P$ is a normal Sylow $p$-subgroup of $G$ and $Q$ is a non normal cyclic Sylow $q$-subgroup of $G$, $p \neq q$. Moreover, $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Now we have:

1. $p = 2$ and every element of order 4 is CSS-subgroup of $G$.

   Assume that $p > 2$. By Lemma 2, the exponent of $P$ is $p$. Then, by the hypotheses, $P \leq Z_\infty(G)$. Applying Lemma 13, $O^p(G) \leq C_G(P)$ which means that $G = PQ = P \times Q$ is nilpotent, a contradiction. If every element of order 4 of $G$ lies in $Z_\infty(G)$, then $P \leq Z_\infty(G)$ which means that $G = PQ = P \times Q$ is nilpotent, again contradiction.

2. For every $x \in P \setminus \Phi(P)$, $|x| = 4$.

   Assume that $|x| \neq 4$. Then there exists $x \in P \setminus \Phi(P)$ and $|x| = 2$. Since $P \leq G$, we have that $<x^G> \leq P$. Then $<x^G>/\Phi(P) \leq G/\Phi(P)$. But as we mentioned above $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$. Then $P = <x^G>/\Phi(P) = <x^G> \leq Z_\infty(G)$. In particular, $G$ is nilpotent, a contradiction.

3. Finishing the proof.

   From 2, every element $x$ in $P \setminus \Phi(P)$ is of order 4. From 1, $<x>$ is CSS-subgroup of $G$. Then there exists a normal subgroup $S$ of $G$ such that $G = <x> S$ and $<x> \cap S$ is $SS$-quasinormal in $G$. Clearly, $P \cap S \leq G$. Hence, $(P \cap S)/\Phi(P) \leq G/\Phi(P)$. Since $P/\Phi(P)$ is a minimal normal subgroup $G/\Phi(P)$, it follows that either $P \cap S \leq \Phi(P)$ or $P \cap S = P$. Assume first that $P \cap S \leq \Phi(P)$. Then $P = P \cap G = P \cap (<x> \cap S) = <x> (P \cap S) = <x> \Phi(P)$. Therefore, $P = <x>$ and this means that $P$ is a cyclic normal Sylow 2-subgroup of $G$ of order 4. By Lemma 14, $G$ is 2-nilpotent and so $G = PQ = P \times Q$ is nilpotent, a contradiction. Thus, assume that $P \cap S = P$. Then $<x> = <x> \cap P = <x> \cap (P \cap S) = (<x> \cap P) \cap S = <x> \cap S$. Hence, $<x>$ is $SS$-quasinormal in $G$. $<x> \leq P \leq G$
implies that $<x>$ is subnormal in $G$. By Lemma 4, $<x> \leqslant O_2(G)$. Applying Lemma 5, $<x>$ is S-quasinormal in $G$. Thus, $<x>Q \leqslant G$. If $<x>Q = G$, then $<x> = P$ which implies $G$ is nilpotent, a contradiction. Therefore, $<x>Q < G$ and it follows that $<x>Q$ is nilpotent. Then $<x>Q = <x> \times Q$. Thus, $<x>Q \leqslant N_G(Q)$ implies that $P \leqslant N_G(Q)$ and so $G = PQ = P \times Q$ is nilpotent, a final contradiction completing the proof.

**Theorem 5.** Let $H$ be a normal subgroup of $G$ such that $G/H$ is nilpotent. If every subgroup of $H$ of prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of $H$ is CSS-subgroup of $G$ or lies in $Z_\infty(G)$, then $G$ is nilpotent.

**Proof.** Assume that the result is false and let $G$ be a counterexample of minimal order. Let $L$ be an arbitrary proper subgroup of $G$. Since $G/H$ is nilpotent, we have $L/L \cap H \cong LH/H$ is nilpotent. The element of prime order or of order 4 of $L \cap H$ is contained in $Z_\infty(G) \cap L \leqslant Z_\infty(L)$. By hypotheses and Lemma 11, every cyclic subgroup of order 4 of $L \cap H$ is CSS-subgroup in $L$. Thus the pair $(L, L \cap H)$ satisfies the hypotheses of the theorem in any case. Then $L$ is nilpotent, that is, $G$ is a minimal non-nilpotent group. Applying Lemma 2, $G = PQ$, where $P$ is normal Sylow $p$-subgroup of $G$ and $Q$ is non normal cyclic Sylow $q$-subgroup of $G$, $p \neq q$. Since $G/H$ and $G/P$ are nilpotent, then $G/P \cap H \leqslant G/P \times G/H$ is nilpotent. Now we deal with:

(1) $P \leqslant H$.

Assume that $p > 2$. Then, by Lemma 2, the exponent of $P$ is $p$ and so $P = P \cap H \leqslant Z_\infty(G)$. Applying Lemma 13, we have $O^p(G) \leqslant C_G(P)$. This implies $G = PQ = P \times Q$ is nilpotent, a contradiction. Thus, we may assume that $p = 2$. Since $P \leqslant G$, it follows that every element of order 2 or 4 of $G$ is contained in $P$; in particular in $H$. Thus, every element of order 2 of $G$ lies in $Z_\infty(G)$ and, by hypotheses, every cyclic subgroup of order 4 is CSS-subgroup of $G$ or lies also in $Z_\infty(G)$. Applying similar arguments to those in (2) and (3) of the proof of Theorem 4, we have that $G$ is nilpotent, a contradiction.

(2) $P \not< H$.

Then $P \cap H < P$ and hence $Q(P \cap H) < G$. Therefore, $Q(P \cap H)$ is nilpotent which implies that $Q(P \cap H) = Q \times (P \cap H)$. Moreover, $Q$ is characteristic in $Q(P \cap H)$. Clearly, as $G/P \cap H = (P/P \cap H)(Q(P \cap H)/P \cap H)$ is nilpotent, then $Q(P \cap H)/P \cap H \leqslant G/P \cap H$. Thus $Q(P \cap H) \leqslant G$. Hence $Q \leqslant G$, a contradiction.

**Theorem 6.** Let $H$ be a normal subgroup of $G$ such that $G/H$ is nilpotent and every cyclic subgroup of order 4 of $F^*(H)$ is CSS-subgroup of $G$. Then $G$ is nilpotent if and only if every subgroup of prime order of $F^*(H)$ is contained in $Z_\infty(G)$.

**Proof.** If $G$ is nilpotent, then we set $H = 1$ and the result follows. Conversely, assume that the result is false and let $G$ be a counterexample of minimal order. With the same arguments to those in steps (1) and (2) of the proof of Theorem 4.4 in [11], we have:
(1) Every proper normal subgroup of $G$ is nilpotent, and $F(G)$ is the unique maximal normal subgroup of $G$.

(2) $H = G$, $G' = G$ and $F^*(G) = F(G) < G$.

(3) Let $q$ be a minimal prime divisor of $|F(G)|$ and $Q$ a Sylow $q$-subgroup of $F(G)$. Then $G/C_G(Q)$ is a $q$-group.

Since $F^*(G) \neq 1$, then we may assume that $q$ is a minimal prime divisor of $|F(G)|$ and $Q$ is a Sylow $q$-subgroup of $F(G)$ which is a non-trivial normal subgroup of $G$. Clearly, from hypotheses, $\Omega_1(Q) \leq Z_\infty(G)$. Thus, by Lemma 13, $C_G(\Omega_1(Q)) \supseteq O^q(G)$. If $q > 2$, then, by Lemma 15, $C_G(Q) \supseteq O^q(G)$. This implies that $G/C_G(Q)$ is a $q$-group. If $q = 2$, let $< x >$ be an arbitrary cyclic subgroup of $Q$ of order 4. By hypotheses, $< x >$ is CSS-subgroup of $G$. Then, there exists a normal subgroup $L$ of $G$ such that $G = < x > L$ and $< x > \cap L$ is $SS$-quasinormal in $G$. If $< x > \cap L = 1$, then $L$ is a proper normal subgroup of $G$ and, by (1), $L$ is nilpotent. It follows that any Sylow $p$-subgroup of $L$ is normal in $G$, where $p$ is any prime number such that $p \neq 2$. Therefore, $G$ is nilpotent, a contradiction. Hence we may assume that $< x > \leq L$ and $< x >$ is $SS$-quasinormal in $G$. Since $Q$ is a Sylow $q$-subgroup of $G$, it follows that $< x >$ is subnormal in $G$. Hence, by Lemma 4, $< x > \leq O_2(G)$. Applying Lemma 5, $< x >$ is $S$-quasinormal in $G$. Now, let $P$ be any Sylow $p$-subgroup of $G$, where $p \neq 2$. Therefore $< x > P \leq G$. Clearly, as $< x >$ is subnormal in $< x > P$ and $< x >$ is a Sylow 2-subgroup of $< x > P$, we have $< x >$ is normal in $< x > P$. Hence, by Lemma 16, $< x > P$ is nilpotent. It follows that $P \leq C_G(< x >)$ and so $O^2(G) \leq C_G(< x >)$. This implies that $O^2(G) \leq C_G(Q)$ and so $G/C_G(Q)$ is a 2-group.

(4) We have a contradiction.

By 2, $G = G'$ and so $C_G(Q) = Q \leq Z(G)$. By Lemma 11, $F^*(G/Q) = F^*(G)/Q$. Let $\overline{G} = G/Q$. Then, 3 imply that each element $\overline{y}$ of prime order $n$ in $F^*(\overline{G})$ can be viewed as an image in element $y$ of prime order $n$ in $F^*(G)$, for each $n > q$. Thus, by hypotheses, $y \leq Z_\infty(G)$. Since $Q \leq Z(G)$, then $Z_\infty(G/Q) = Z_\infty(G)/Q$. Hence $\overline{y} \leq Z_\infty(G/Q)$. Clearly, $F^*(G/Q)$ does not have an element of order 2. This means that $\overline{G}$ satisfies the hypotheses of the theorem. Then $\overline{G} = G/Q$ is nilpotent by our choice of $G$ and so $G$ is nilpotent which yields the desired contradiction.

4. Some Applications

As it was mentioned in the introduction each of $c$-normality and $SS$-quasinormality subgroups implies CSS-subgroups. Therefore the following results are direct consequences of our results.

**Corollary 8.** ([1, Lemma 3.1]) Let $p$ be the smallest prime dividing $|G|$ and $P$ a Sylow $p$-subgroup of a group $G$. If every subgroup of $P$ of prime order or of order 4 (if $p = 2$) is $c$-normal in $G$, then $G$ is $p$-nilpotent.
Corollary 9. ([18, Theorem 4.2]) Let $G$ be a group such that every subgroup of $G$ of prime order or of order 4 (if $p = 2$) is $c$-normal in $G$, then $G$ is supersolvable.

Corollary 10. ([1, Theorem 3.2] and [16, Theorem 3.9]) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $G$ a group. $G \in \mathfrak{F}$ if and only if there exists a normal subgroup $H$ in $G$ such that $G/H \in \mathfrak{F}$ and every subgroup of $H$ of prime order or of order 4 (if $p = 2$) is $c$-normal in $G$.

Corollary 11. ([1, Theorem 3.6] and [24, Theorem 3]) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $G$ a group. If $G$ has a normal subgroup $H$ such that $G/H \in \mathfrak{F}$ and every subgroup of $F(H)$ of prime order or of order 4 (if $p = 2$) is $c$-normal in $G$.

Corollary 12. ([21, Theorem 3.2]) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $G$ a group. If $G$ is nilpotent and every cyclic subgroup of order 4 of $F^*(H)$ is c-normal in $G$, then $G \in \mathfrak{F}$.

Corollary 13. ([19, Theorem 3.1]) Let $H$ be a normal subgroup of a group $G$ such that $G/H$ is nilpotent and every cyclic subgroup of order 4 of $F^*(H)$ is c-normal in $G$, then $G$ is nilpotent if and only if every subgroup of prime order of $F^*(H)$ is contained in the hypercenter $Z_\infty(G)$ of $G$.

Corollary 14. Let $p$ be the smallest prime dividing $|G|$ and $P$ a Sylow $p$-subgroup of a group $G$. If every subgroup of $P$ of prime order or of order 4 (if $p = 2$) is SS-quasinormal in $G$, then $G$ is $p$-nilpotent.

Corollary 15. ([11, Theorem 3.4]) Let $G$ be a group such that every subgroup of $G$ of prime order or of order 4 (if $p = 2$) is SS-quasinormal in $G$, then $G$ is supersolvable.

Corollary 16. Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $G$ a group. $G \in \mathfrak{F}$ if and only if there exists a normal subgroup $H$ in $G$ such that $G/H \in \mathfrak{F}$ and every subgroup of $H$ of prime order or of order 4 (if $p = 2$) is SS-quasinormal in $G$.

Corollary 17. ([11, Theorem 3.5]) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $G$ a group. $G \in \mathfrak{F}$ if and only if there exists a normal solvable subgroup $H$ in $G$ such that $G/H \in \mathfrak{F}$ and every subgroup of $F(H)$ of prime order or of order 4 (if $p = 2$) is SS-quasinormal in $G$.

Corollary 18. ([11, Theorem 3.6]) Let $G$ be a group. If $G$ has a normal subgroup $H$ such that $G/H$ is supersolvable and every subgroup of $F^*(H)$ of prime order or of order 4 is SS-quasinormal in $G$, then $G$ is supersolvable.

Corollary 19. ([11, Theorem 3.7]) Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $G$ a group. Then $G \in \mathfrak{F}$ if and only if $G$ has a normal subgroup $H$ such that $G/H \in \mathfrak{F}$ and every subgroup of $F^*(H)$ of prime order or of order 4 is SS-quasinormal in $G$.

Corollary 20. ([11, Theorem 4.1]) Let $G$ be a group. If every subgroup of $G$ of prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of $G$ is SS-quasinormal in $G$ or lies in $Z_\infty(G)$, then $G$ is nilpotent.
Corollary 21. ([11, Theorem 4.2]) Let $H$ be a normal subgroup of a group $G$ such that $G/H$ is nilpotent. If every subgroup of $H$ of prime order is contained in $Z_\infty(G)$ and every cyclic subgroup of order 4 of $H$ is SS-quasinormal in $G$ or lies in $Z_\infty(G)$, then $G$ is nilpotent.

Corollary 22. ([11, Theorem 4.4]) Let $H$ be a normal subgroup of a group $G$ such that $G/H$ is nilpotent and every cyclic subgroup of order 4 of $F^*(H)$ is SS-quasinormal in $G$, then $G$ is nilpotent if and only if every subgroup of prime order of $F^*(H)$ is contained in the hypercenter $Z_\infty(G)$ of $G$.

Based on the results that have been achieved in this paper and [25, 26], the following questions arise:

Question 1. Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is the smallest prime dividing $|G|$. Assume that all maximal subgroups of $P$ are CSS-subgroups of $G$. Is $G$ $p$-nilpotent?

Question 2. Assume that all maximal subgroups of every Sylow subgroup of a group $G$ are CSS-subgroups of $G$. Is $G$ supersolvable?

Question 3. Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $H$ a normal subgroup of $G$ such that $G/H \in \mathfrak{F}$. Assume that every non-cyclic Sylow subgroup $P$ of $H$ has a subgroup $D$ with $1 < |D| < |P|$ such that every subgroup of $P$ of order $|D|$ (and 4 if $|D| = 2$) is CSS-subgroup of $G$. Is $G \in \mathfrak{F}$?

Question 4. Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and $H$ a normal subgroup of $G$ such that $G/H \in \mathfrak{F}$. Assume that every non-cyclic Sylow subgroup $P$ of $F^*(H)$ has a subgroup $D$ with $1 < |D| < |P|$ such that every subgroup of $P$ of order $|D|$ (and 4 if $|D| = 2$) is CSS-subgroup of $G$. Is $G \in \mathfrak{F}$?

References

[1] M. Asaad and M. E. Mohamed. On $c$-normality of finite groups. *J. Aust. Math. Soc.*, 78:297–304, 2005.

[2] A. Ballester-Bolinches and L.M. Ezquerro. *Classes of finite groups*, vol. 584 of Mathematics and its Applications. Springer, New York, 2006.

[3] J. Buckley. Finite groups whose minimal subgroups are normal. *Math. Z.*, 116:15–17, 1970.

[4] J. B. Derr, W. E. Deskins, and N. P. Mukherjee. The influence of minimal $p$-subgroups on the structure of finite groups. *Arch. Math.*, 45:1–4, 1985.

[5] K. Doerk and T. Hawkes. *Finite Soluble Groups*. Walter de Gruyter, Berlin, New York, 1992.
REFERENCES

[6] D. Gorenstein. *Finite Groups*. Harper and Row Publishers, New York, 1968.

[7] B. Huppert. *Endliche Gruppen I*. Springer-Verlag, Berlin, Heidelberg, New York, 1979.

[8] B. Huppert and N. Blackburn. *Finite Groups III*. Springer-Verlag, Berlin, Heidelberg, New York, 1982.

[9] O. H. Kegel. Sylow-Gruppen und Subnormalteiler endlicher Gruppen. *Math. Z.*, 78:205–221, 1962.

[10] R. Laue. Dualization of saturation for locally defined formations. *J. Algebra*, 52:347–353, 1978.

[11] S. Li, Z. Shen, and X. Kong. On SS-quasinormal subgroup of finite groups. *Comm. Algebra*, 36(12):4436–4447, 2008.

[12] S. Li, Z. Shen, J. Liu, and X. Liu. The influence of SS-quasinormality of some subgroups on the structure of finite groups. *J. Algebra*, 319:4275–4287, 2008.

[13] Y. Li and B. Li. On minimal weakly s-supplemented subgroups of finite groups. *J. Algebra Appl.*, 10(5):811–820, 2011.

[14] Y. Li and Y. Wang. The influence of minimal subgroups on the structure of a finite group. *Proc. Amer. Math. Soc.*, 131(2):337–341, 2002.

[15] Y. Li and Y. Wang. On π-quasinormally embedded subgroups of finite group. *J. Algebra*, 281:109–123, 2004.

[16] M. Ramadan, M. E. Mohamed, and A. A. Heliel. On c-normality of certain subgroups of prime power order of finite groups. *Arch. Math.*, 85:203–210, 2005.

[17] A. Shaalan. The influence of π-quasinormality of some subgroups on the structure of a finite group. *Acta Math. Hungar.*, 56:287–293, 1990.

[18] Y. Wang. C-normality of groups and its properties. *J. Algebra*, 180:954–965, 1996.

[19] Y. Wang. The influence of minimal subgroups on the structure of finite groups. *Acta Math. Sin. (Engl. Ser.)*, 16(1):63–70, 2000.

[20] H. Wei. On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups. *Comm. Algebra*, 29(5):2193–2200, 2001.

[21] H. Wei, Y. Wang, and Y. Li. On c-normal maximal and minimal subgroups of Sylow subgroups of finite groups II. *Comm. Algebra*, 31(10):4807–4816, 2003.

[22] M. Weinstein, editor. *Between Nilpotent and Solvable*. Polygonal Publishing House, Passaic, 1982.
[23] H. Wielandt. *Subnormal Subgroups and Permutation Groups*. Lectures Given at the Ohio State University, Columbus, Ohio, USA, 1971.

[24] L. Yangming. Some notes on the minimal subgroups of Fitting subgroups of finite groups. *J. Pure Appl. Algebra*, 171:289–294, 2002.

[25] X. Zhao, J. Sui, R. Chen, and Q. Huang. On the supersolvablity of finite groups. *Bull. Iranian Math. Soc.*, 46:1485–1491, 2020.

[26] X. Zhao, L. Zhou, and S. Li. On the p-nilpotence of finite groups. *Ital. J. Pure Appl. Math.*, 41:97–103, 2019.