Quasi-exactly solvable quartic: elementary integrals and asymptotics

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Abstract
We study elementary eigenfunctions \( y = pe^h \) of operators \( L(y) = y'' + Py \), where \( p, h \) and \( P \) are polynomials in one variable. For the case when \( h \) is an odd cubic polynomial, we investigate the real level crossing points and asymptotics of eigenvalues. This study leads to an interesting identity with elementary integrals.

Littlewood, when he makes use of an algebraic identity, always saves himself the trouble of proving it: he maintains that an identity, if true, can be verified in few lines by anybody obtuse enough to feel the need of verification.

Freeman Dyson 1949 Some guesses in theory of partitions Eureka 8 10–15

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1. Introduction
Following Bender and Boettcher [3], we consider the eigenvalue problem
\[
\frac{d^2 w}{d\xi^2} + (\xi^4 + 2b\xi^2 + 2iJ\xi + \lambda)w = 0, \quad w(t e^{-i(\pi/2 \pm \pi/3)}) \to 0, \quad t \to +\infty,
\]
with real parameters \( b \) and \( J \).

When \( J \) is a positive integer, this problem is quasi-exactly solvable (QES) which means that for every \( b \) there are \( J \) eigenvalues \( \lambda_{J,k} \), \( k = 1, \ldots, J \), with elementary eigenfunctions
\[
y_{J,k}(\xi) = p_{J,k}(\xi) \exp(-i\xi^3/3 - ib\xi),
\]
where \( p_{J,k} \) are polynomials of degree \( J - 1 \). These \( J \) eigenvalues are found from algebraic equations
\[
Q_J(b, \lambda) = 0, \quad J = 1, 2, \ldots,
\]
where \( Q_J \) are real polynomials of degree \( J \) in \( \lambda \).
For every $J$ and $b$, problem (1) has infinitely many eigenvalues $\lambda$ tending to infinity, which satisfy an equation

$$F_J(b, \lambda) = 0,$$

(3)

where $F_J$ is a real entire function of two variables which is called the spectral determinant.

We denote by $Z_J(R)$ and $Z_QES_J(R)$ the sets of real solutions $(b, \lambda)$ of equations (3) and (2), respectively. These are certain analytic curves in $R^2$. A computer-generated picture of these curves is given in [3]. The symbols $Z_J$ and $Z_QES_J$ denote the sets of all complex solutions of these equations.

The most conspicuous feature seen in the picture in [3] is a set of real crossing points where the curves of $Z_QES_J(R)$ intersect the rest of $Z_J(R)$.

In this communication, we prove that infinitely many such points exist for each odd $J$ (proposition 1). We characterize real and complex crossing points as intersection points of $Z_QES_J$ with $Z_{-J}$ (theorem 1). We prove that all non-QES eigenvalues are real when $b$ is real and $J$ is a positive integer (theorem 2).

In sections 8 and 9, we study the asymptotic behavior of polynomials $Q_J$ as $b \to \pm \infty$, using singular perturbation methods from [10]. Appendix A with a description of the relevant results of [10] is included.

Our proof of theorem 1 is related to a challenging conjecture on certain elementary integrals. We rigorously verified this conjecture for integers $J \leq 5$ using symbolic computation. The conjecture is discussed in sections 3 and 5.

2. Elementary solutions

Let $h$ and $p$ be two polynomials in one variable. When $y = p(z) e^{h(z)}$ satisfies a second order differential equation

$$y'' + Py = \lambda y,$$

(4)

where $P$ is a polynomial? Substitution gives

$$\frac{p''}{p} + 2\frac{p'}{p} h' + h'' + h'^2 + P - \lambda = 0.$$  

(5)

Such $P$ exists if and only if

$$p'' + 2p'h'$$

is divisible by $p$.  

(6)

Another criterion is obtained if we consider the second solution $y_1$ of (4) which is linearly independent of $y$. This second solution can be found from the condition

$$yy'_1 - y'y_1 = 1.$$  

(7)

Solving (7) with respect to $y_1$, we obtain

$$y_1 = pe^h \int p^{-2} e^{-2h}.$$  

(8)

As all solutions of (4) must be entire functions, we conclude that all residues of $p^{-2} e^{-2h}$ vanish.  

(9)

This condition is necessary and sufficient for $y = pe^h$ to satisfy equation (4) with some $P$. Indeed, if (9) holds, then $y_1$ defined by (8) is an entire function, so $(y, y_1)$ is a pair of entire functions whose Wronskian determinant equals 1, so this pair must satisfy a differential equation (4) with entire $P$, and asymptotics at infinity show that $P$ must be a polynomial.
Thus, conditions (6) and (9) are equivalent. One can give another equivalent condition in terms of zeros of $p$, as in [19]. Let

$$p(z) = \prod_{j=1}^{n}(z - z_j), \quad p_k(z) = p(z)/(z - z_k).$$

Then, (6) is equivalent to

$$p''(z_k) + 2p'(z_k)h'(z_k) = 0,$$

for all $k = 1, \ldots, n$. We have $p'(z_k) = p_k(z_k)$ and $p''(z_k) = 2p'_k(z_k)$, so the condition

$$\sum_{j \neq k} \frac{1}{z_k - z_j} = -h'(z_k), \quad 1 \leq k \leq n,$$

(10)

is equivalent to (6) and (9). Equation (10) is the equilibrium condition for $n$ unit charges at the points $z_k$ in the plane, repelling each other with the force inversely proportional to the distance, and in the presence of the external field $h'(z)$. Equations (10) express the fact that $(z_1, \ldots, z_n)$ is a critical point of the master function

$$\Psi(z_1, \ldots, z_n) = \prod_{(j,k) : k<j}(z_k - z_j) \prod_{k} e^{h(z_k)}.$$

3. Remarkable identity

From now on we suppose that $h$ is an odd polynomial of degree 3, which we write in the form

$$h(z) = z^3/3 \cdot b.$$

Suppose that all residues of $p^{-2} e^{-2h}$ vanish. Then, the integral $\int p^{-2} e^{-2h}$ is a meromorphic function in the plane. Surprisingly, the integral of some linear combination

$$\int (p^2(-z) e^{-2h(z)} - C p^2(z) e^{-2h(z)})$$

is not only meromorphic but is an elementary function! Here $C$ is a constant depending on $b$ and $p$.

**Conjecture 11.** Let $h$ be given by (11). Let $p$ be a polynomial. All residues of $p^{-2} e^{-2h}$ vanish if and only if there exist a constant $C$ and a polynomial $q$ such that

$$\left( p^2(-z) - \frac{C}{p^2(z)} \right) e^{-2h(z)} = \frac{d}{dz} \left( \frac{q(z)}{p(z)} e^{-2h(z)} \right).$$

(12)

In other words,

$$p^2(z)p^2(-z) - C = q'(z)p(z) - q(z)p'(z) - 2q(z)p(z)h'(z).$$

It is known [3] that for given $h$ of the form (11), there exist polynomials $p$ of any given degree such that all residues of $p^{-2} e^{-2h}$ vanish. These polynomials $p$ have simple roots. We verified the conjecture for $deg \ p \leq 4$ by symbolic computation with Maple. We do not know whether there is any analog of the conjecture for other polynomials $h$.

Substituting $p_k(z) = z^n + az^{n-1} + \cdots$ into (5) and using (11), we conclude that

$$P(z) - \lambda = -h''(z) - h''(z) - 2nz + 2a = -z^4 + 2z^2b - 2(n + 1)z - b^2 + 2a.$$

(13)

1 After this paper was submitted, Evgene Mukhin and Vitaly Tarasov informed us that they proved this conjecture.
We choose $\lambda = b^2 - 2a$ so that $P(0) = 0$. Equation (5) now becomes

$$p''_n + 2(z^2 - b)p'_n - (2nz - 2a)p_n = 0.$$  \hfill (14)

Coefficients of $p_n$ can be now determined by a linear recurrence formula. Putting

$$p_n(z) = \sum_{j=0}^{n} a_j z^{n-j}, \quad a_{-1} = 0, \quad a_0 = 1, \quad a_1 = a,$$

we obtain the recurrence

$$ja_j = aa_{j-1} - b(n - j + 2)a_{j-2} + \frac{(n - j + 2)(n - j + 3)}{2} a_{j-3}.$$  \hfill (15)

Coefficients $a_j$ are found from this formula one by one beginning from $a_1 = a$. Vanishing of the constant term in (14) gives a polynomial equation $Q^{*}_{n+1}(b, a) = 0$ in which we can substitute $a = (b^2 - \lambda)/2$ and write it as

$$Q_{n+1}(b, \lambda) = 0.$$  \hfill (16)

Polynomials $p_n, Q^{*}_{n+1}$ for small $n$ are given in appendix B.

We have $\deg_\lambda Q_{n+1} = n + 1$ [3]. For every $b$ and every $\lambda$ satisfying this equation, the differential equation (4), with $P$ as in (13), has a unique solution $y = p_ne^h$ where $p_n$ is a monic polynomial of degree $n$. Coefficients of $p_n$ are polynomials in $b$ and $\lambda$.

Polynomials $Q_{n+1}$ are fundamental for our subject, but little is known about them. It seems difficult to investigate them algebraically. In section 8, we will use analytic tools to establish some properties of these polynomials; in particular, we will find the terms of top weight and asymptotics of $\lambda$ as $b \to \infty$.

Functions $y = p_ne^h$ are eigenfunctions of the operator

$$L_J(y) = y'' - (z^4 - 2bz^2 + 2Jz)y, \quad J = n + 1,$$  \hfill (17)

which is equivalent to the operator in (4) by the change of the independent variable $z = i\zeta$. This operator maps the space $\{pe^h : \deg p \leq n\}$ of dimension $n + 1$ into itself. For each non-negative integer $n$, and generic $b$, the operator (17) has $n + 1$ eigenfunctions of the form $p_ne^h$ with eigenvalues $\lambda$ which are solutions of (16).

We assume without loss of generality that $Q_{n+1}$ is monic as a polynomial in $\lambda$, and $p_n$ is a monic polynomial in $z$. Then, the constant $C$ in the conjecture turns out to be

$$C(b, \lambda) = a_n \frac{\partial}{\partial \lambda} Q_{n+1}.$$  \hfill (18)

Symbolic computation for small $n$ shows that $a_n = (-1)^n 2^{-2n}$.

4. Boundary value problem

Eigenfunctions $pe^h$ do not belong to $L^2(\mathbb{R})$, but they satisfy the boundary conditions

$$y(t e^{\pm \pi i/3}) \to 0 \quad \text{as} \quad t \to \infty.$$  \hfill (19)

With these boundary conditions, operator (17) is not Hermitian but PT-symmetric [3, 14, 18].

Physicists write the boundary value problem for the operator $L_J$ with the boundary conditions (19) in the equivalent form (1), which corresponds to the rotation of the independent variable $i\zeta = z, w(\zeta) = y(i\zeta)$. We also find this form convenient in certain arguments, and will use it in sections 7 and 8. We keep the notation $y(z)$ for an eigenfunction of (17) and (19), while $w(\zeta)$ stands for an eigenfunction of (1).
It is known that the boundary value problems (17) and (19) have an infinite sequence of eigenvalues tending to infinity [16]. Eigenvalues $\lambda$ are solutions of the equation
\[ F_{n+1}(b, \lambda) = 0, \] (20)
where $F_{n+1}$ is a real entire function on $\mathbb{C}^2$ which is called the spectral determinant [14]. The set of all solutions of (20) in $\mathbb{C}^2$ is called the spectral locus and we denote it by $Z_{n+1}$. As $F_{n+1}$ is real, the set of eigenvalues is symmetric with respect to the real line when $b$ is real. For each real $b$, all sufficiently large eigenvalues (how large, depends on $b$) are real [14].

Equation (20) is reducible: $F_{n+1}$ is evidently divisible by $Q_{n+1}$. On the other hand, equation (16) is irreducible, as follows from [9] or [2], and it defines a smooth algebraic curve in $\mathbb{C}^2$. This algebraic curve will be denoted by $Z_{QES}^{n+1}$.

5. Theorem 1

Now we discuss a corollary of our conjecture that we can prove. Let us fix a simple curve $\gamma$ in $\mathbb{C}$ parametrized by the real line, with the properties $\gamma(t) \to \infty$, $\arg \gamma(t) \to \pm \pi/3$, $t \to \pm \infty$. Then (12) implies
\[ \int_\gamma p(z) e^{2h(z)} \, dz = C \int_\gamma p^{-2}(-z) e^{2h(z)} \, dz. \] (21)
To obtain this, we replace $z \mapsto -z$ in (12) and then integrate along $\gamma$; the integral on the right-hand side of (12) vanishes because $\Re h(z) \to -\infty$ as $z \to \infty$ on $\gamma$. Let $\gamma_0$ be a curve consisting of the piece $\{\gamma(t) : -\infty < t \leq 0\}$ followed by a curve from $\gamma(0)$ to $z$. Put
\[ g(z) = p(-z) e^{-h(z)} \int_{\gamma_0} p^{-2}(-\zeta) e^{2h(\zeta)} \, d\zeta. \]
Then, $g^-(z) := g(-z)$ satisfies $L_{n+1}(g^-) = \lambda g^-$, so $g$ satisfies $L_{-n-1}(g) = \lambda g$. To check this, we make a substitution $z \mapsto -z$ in (17). Moreover, if the integral on the right-hand side of (21) is zero, then $g$ also satisfies boundary condition (19). Therefore, our conjecture (12) and (18) has the following corollary.

Theorem 1. The points $(b, \lambda) \in Z_{n+1}^{QES}$ where the eigenfunction $y = p e^{h}$ satisfies
\[ \int_\gamma y^2(z) \, dz = 0 \] (22)
are either zeros of $C(b, \lambda)$ or points of intersection of $Z_{n+1}^{QES}$ with $Z_{-n-1}$. We will prove theorem 1 in the next section.

Equation (22) is the well-known condition of level crossing, which we discuss in section 7.

Thus, the corollary says that the eigenvalues at the points on $Z_{n+1}^{QES}$ which are singular points of $Z_{n+1}$ are eigenvalues of two spectral problems, one for $L_{n+1}$ and the other for $L_{-n-1}$.

6. Darboux transform

Proof of theorem 1. Assuming that all residues of $p^{-2} e^{-2h}$ vanish, we will prove that the right and left sides of (21) with $C$ as in (18) have the same zeros on $Z_{n+1}^{QES}$. Fix the integer $n \geq 0$. Let $\psi_k = p_k e^{h}$, $k = 0, \ldots, n$, be all elementary eigenfunctions of $L_{n+1}$. They are linearly independent, and they span a space $V$ invariant under $L_{n+1}$. As $V$ is a
subspace of \( U = \{ p e^{h} : \deg p \leq n \} \), we conclude that \( V = U \). So the Wronski determinant 
\( W = W(\psi_0, \ldots, \psi_n) \) is proportional to the Wronski determinant 
\[
W(e^{h}, z e^{h}, \ldots, z^{n} e^{h}) = \left( \prod_{k=0}^{n} k! \right) e^{(n+1)h}.
\]

Now let us perform the Darboux transform of \( L_{n+1} \) killing these \( n+1 \) eigenfunctions. We recall that the Darboux transform \([5, 6, 8, 13]\) applies to any operator \(-D^2 + V\) with eigenfunctions \( \phi_0, \ldots, \phi_n \) and corresponding eigenvalues \( \lambda_0, \ldots, \lambda_n \). The transformed operator is 
\[
-D^2 + V - 2 \frac{d^2}{dz^2} \log W(\phi_0, \ldots, \phi_n),
\]
and its eigenvalues are those eigenvalues of \(-D^2 + V\) which are distinct from \( \lambda_0, \ldots, \lambda_n \). As in our case \( 2(\log W)' = 2(n + 1)h' = 4(n + 1)z \), the result of application of the Darboux transform to \( L_{n+1} \) and eigenfunctions \( \psi_k, k = 0, \ldots, n \), is \( L_{-n-1} \).

In the second case, \( (b, \lambda) \) belongs to the spectral locus of the Darboux transform \( L_{-n-1} \). This means that the equation 
\[
L_{-n-1}(y^*) = \lambda y^*
\]
with these parameters \( (b, \lambda) \) has a solution \( y^* \) that tends to 0 on both ends of \( \gamma \). Then, \( y_1 = y^*(-z) \) tends to 0 on both ends of \(-\gamma\) and satisfies \( L_{n+1}(y_1) = \lambda y_1 \). So \( y_1 \) satisfies the same differential equation \( L_{n+1}(y) = \lambda y \) as \( y \) does and is linearly independent of \( y \). So \( y_1 = y \int y^{-2} e^{-2h} \) tends to 0 on both ends of \(-\gamma\). So \( y^*(z) = y_1(-z) \) tends to 0 on both ends of \( \gamma \), and this means that the right-hand side of \((21)\) is 0.

The argument is evidently reversible. This proves that the right- and the left-hand sides of \((21)\) have the same zeros on \( Z_{n+1}^{\text{QES}} \), that is, \((21)\) with \( C = \alpha_n \partial Q_n / \partial \lambda \), where \( \alpha_n(b, \lambda) \neq 0 \) on \( Z_{n+1}^{\text{QES}} \). □

According to the theorem of Shin \([15]\), all eigenvalues of \( L_J \) for \( J \leq 0 \) are real. Shin’s proof of this uses the ODE-IM correspondence discovered by Dorey, Dunning and Tateo \([7]\).

Combining the Darboux transform used in the proof of theorem 1 with the result of Shin \([15]\), we obtain

**Theorem 2.** For every positive integer \( J \), all non-QES eigenvalues of \( L_J \) with boundary conditions \((19)\) are real.

**Proof.** These eigenvalues are also eigenvalues of \( L_{-J} \) with boundary conditions \((19)\), and the eigenvalues of \( L_{-J} \) are all real by Shin’s theorem \([15]\). □

For \( J > 1 \), there are always some non-real eigenvalues when \( b \) is sufficiently negative. For integer \( J > 1 \), these non-real eigenvalues belong to the QES part of the spectrum.

7. Level crossing

As \( Q_{n+1} \) and \( F_{n+1} \) are real functions, it is reasonable to consider real solutions of equations \((20)\) and \((16)\). Eigenfunctions \( y(z) \) corresponding to these real solutions are real,
while eigenfunctions \( w(\zeta) \) (see (1)) are PT-symmetric, that is, \( w(-\zeta) = \overline{w(\zeta)} \). These real solutions \( (b, \lambda) \) form curves in \( \mathbb{R}^2 \) which we call the real spectral locus \( Z_{n+1}^{\text{R}}(\mathbb{R}) \) and the QES real spectral locus \( Z_{n+1}^{\text{QES}}(\mathbb{R}) \), respectively.

Now we discuss (22). First we state a result which describes \( Z_{n+1}^{\text{QES}}(\mathbb{R}) \).

**Theorem 3.** For \( n \geq 0 \), the spectral locus \( Z_{n+1}^{\text{QES}}(\mathbb{R}) \) consists of \( \lfloor n/2 \rfloor + 1 \) disjoint analytic curves \( \Gamma_{1n,m} \), \( 0 \leq m \leq \lfloor n/2 \rfloor \) (analytic embeddings of \( \mathbb{R} \) to \( \mathbb{R}^2 \)).

For \( (b, \lambda) \in \Gamma_{1n,m} \), the eigenfunction has \( n \) zeros, \( n-2m \) of them real.

If \( n \) is odd then \( b \to +\infty \) on both ends of each curve \( \Gamma_{n,m} \). If \( n \) is even, then the same holds for \( 0 \leq m < n/2 \), but on the ends of \( \Gamma_{n,n/2} \) we have \( b \to \pm \infty \).

If \( (b, \lambda) \in \Gamma_{1n,m} \), \( (b, \mu) \in \Gamma_{1n,m} + 1 \) and \( b \) is sufficiently large, then \( \mu > \lambda \).

The proof of this theorem can be found in [11]. It follows the method of [10] where similar results were established for real spectral loci of other families of cubic and quartic potentials. The method is based on singular perturbation and Nevanlinna parametrization of the spectral locus.

Computer-generated pictures of \( Z_{n+1}(\mathbb{R}) \) show an interesting phenomenon: when \( n \) is even, the curve \( \Gamma_{n,n/2} \) crosses the non-QES part of the spectral locus [3, figure A1]. We will prove that infinitely many such crossings exist for even \( n \) and negative \( b \).

We say that a level crossing occurs at a point \( (b, \lambda) \) of the spectral locus if \( \partial F_{n+1}/\partial \lambda = 0 \) at this point. If \( y \) is the eigenfunction corresponding to a point \( (b, \lambda) \), then the level crossing occurs if and only if (22) is satisfied [17, II.7], [18, theorem 8]. There are two types of level crossing points.

(a) Critical points of the function \( \lambda \) at non-singular points of \( Z_{n+1} \).

If such a critical point \( (b_0, \lambda_0) \) is simple and belongs to \( Z_{n+1}(\mathbb{R}) \), then the two eigenvalues that meet at this point are both real for \( b \) on one side of \( b_0 \) and complex conjugate on the other side.

(b) Singular points of \( Z_{n+1} \).

If two eigenvalues collide at a simple self-intersection point of \( Z_{n+1}(\mathbb{R}) \) with two distinct non-vertical tangents, then these eigenvalues both remain real in a neighborhood of \( b_0 \). The operator \( L_{n+1} \) with \( b = b_0 \) contains a Jordan cell in this case.

We recall that \( Z_{n+1}^{\text{QES}} \) is a smooth curve. Thus, the crossing points on \( Z_{n+1}^{\text{QES}} \) where only QES eigenvalues collide are all of type (a), and they satisfy

\[
Q_{n+1}(b, \lambda) = 0, \quad \frac{\partial}{\partial \lambda} Q_{n+1}(b, \lambda) = 0.
\]

For each \( n \), there are finitely many such points on \( Z_{n+1}^{\text{QES}} \).

We will show that there are always infinitely many crossing points of type (b) where QES eigenvalues collide with non-QES eigenvalues. So the curve defined by (20) is not smooth: it has infinitely many self-intersections.

We do not know whether more complicated singularities than (a) and (b) exist; numerical experiments only show singularities of types (a) and (b).

**Proposition 1.** The function

\[
\Phi_n(b, \lambda) = \int y^2(z) \, dz, \quad Z_{n+1}^{\text{QES}} \to C,
\]

where \( y \) is the eigenfunction corresponding to \( (b, \lambda) \), has infinitely many zeros \( (b_k, \lambda_k) \), \( b_k \to \infty \). When \( n \) is even, \( \Phi_n \) has infinitely many zeros with negative \( b_k \) and real \( \lambda_k \).
Proof. We have
\[ \Phi_n(b) = \int p_n^2(z) e^{2b(z)} \, dz. \]
We recall that coefficients of \( p_n \) and \( h \) are algebraic functions of \( b \). When \( n = 0 \), we can take \( p_0 = 1 \), and then
\[ \Phi_0(b) = \int e^{(2/3)z^3 - 2bc} \, dz = 2^{2/3} i \pi \text{Ai}(2^{2/3} b), \]
where \( \text{Ai} \) is the Airy function [1]. The Airy function is a real entire function of order 3/2 with infinitely many negative simple zeros.

To generalize this to other values of \( n \), we express \( \Phi_n \) as a linear combination of \( \Phi_0 \) and \( \Phi_0' \) with coefficients depending on \( b \) algebraically. Differentiating \( \Phi_0(b) \) with respect to \( b \), we obtain
\[ \int z^k e^{(2/3)z^3 - 2bc} \, dz = (-2)^{-k} \Phi_0^{(k)}(b), \]
and thus
\[ \Phi_n(b) = p_n^2(-D/2) \Phi_0(b), \]
where \( D = d/db \). Now all \( \Phi_0^{(k)} \) are linear combinations of \( \Phi_0 \) and \( \Phi_0' \) with polynomial coefficients because \( \text{Ai} \) satisfies the differential equation \( \text{Ai}'(s) = s \text{Ai}(s) \). So \( \Phi_n \) is of the form
\[ \Phi_n(b) = A_n(b) \Phi_0(b) + B_n(b) \Phi_0'(b), \]
(23)
where \( A_n \) and \( B_n \) are algebraic functions.

We claim that every linear combination \( \phi \) of \( \Phi_0 \) and \( \Phi_0' \) with algebraic coefficients has infinitely many zeros. We prove this claim by contradiction. Suppose that such a linear combination
\[ \phi = a_0 \Phi_0 + a_1 \Phi_0', \]
(24)
has finitely many zeros. Let \( F \) be a compact Riemann surface spread on the Riemann sphere on which \( a_0 \) and \( a_1 \) are meromorphic. Then, \( \phi \) is meromorphic on \( F \backslash E \), where \( E \) is the finite set of points of \( F \) lying over \( \infty \). At the points of \( E \), \( \phi \) has isolated essential singularities. As \( \phi \) has finitely many zeros and poles on \( F \backslash E \), we conclude that \( \phi'/\phi \) is meromorphic on \( F \backslash E \). The growth estimate \( \log|\phi(b)| \lesssim O(|b|^{3/2}) \), \( b \to \infty \), implies that the points of \( E \) are removable singularities of \( \phi'/\phi \). Thus, \( \phi \) is the exponent of an Abelian integral. Now consider (24) as a linear differential equation of first order with respect to \( \Phi_0 \), whose coefficients belong to the minimal field \( K \) that contains \( C(b) \), is algebraically closed, and contains a primitive of every element, and the exponent of a primitive of every element. As every first order linear differential equation can be solved by integration, we conclude that \( \Phi_0 \in K \) which implies that \( \text{Ai} \in K \). But this is not so by a well-known classical theorem of Picard and Vessiot [12, theorem 6.6]. This proves our claim.

When \( n \) is even, according to theorem 3, we have a real analytic branch \( \lambda(b) \) defined for all real \( b \) with sufficiently large absolute value. The graph of this branch is a part of \( \Gamma_{n,n/2} \). Using this branch, we rewrite the equation \( \Phi_n(b) = 0 \) as
\[ \Phi_0'(b)/\Phi_0(b) = A(b), \]
where \( A \) is a real branch of an algebraic function on \( (-\infty, B) \) with some \( B \in \mathbb{R} \). This last equation has infinitely many negative solutions because \( \Phi_0 \) has infinitely many negative zeros and they are interlaced with zeros of \( \Phi_0 \). This completes the proof of the proposition. □

Using the asymptotics of the zeros of the Airy function [1] we obtain that the crossing points satisfy \( b_k \sim -((3/4)\pi k)^{2/3}, k \to \infty \).
8. Asymptotics as $b \to +\infty$

Now we study asymptotics of the eigenvalues $\lambda$ as $b \to +\infty$ and make conclusions about polynomials $Q_{n+1}$. Our main result here is the explicit formula (30) for the top quasi-homogeneous part of $Q_{n+1}$.

First we obtain a preliminary estimate of solutions $\lambda(b)$ of equation (16) for large $b$:

$$\lambda(b) \sim b^2 + O(\sqrt{b}), \quad b \to \infty.$$  \hfill (25)

To prove this, consider the recurrence (15). For a monomial $a^mb$, we define the weight as $m+2k$. Then, (15) implies that

$$j!a_j = a^j + \sum_{m=1}^{[j/2]} c_{m,j} b^m a^{j-2m} + \text{terms of lower weight}.$$  

Vanishing of the constant term in (14) gives

$$Q_{n+1}^*(a, b) = a a_n + b a_{n-1} + 2 a_{n-2} = 0.$$  

so $Q_{n+1}^*$ is a sum of a quasi-homogeneous polynomial in $a$ and $b$ of weight $2(n+1)$ and a polynomial of lower weight. This means that $a = O(\sqrt{b})$ and $\lambda(b) = b^2 - 2a$ satisfies (25).

To obtain more precise asymptotics, we use singular perturbation arguments from [10], which we state in appendix A for the reader’s convenience.

Suppose that $b$ is real and $b \to +\infty$. In equation (1), we set $\epsilon = \epsilon u = -ie^{-2}$, $b = \epsilon^{-4}$, $W(u) = w(\epsilon u - ie^{-2})$.

The result is

$$W'' + (\epsilon^6 u^4 - 4i \epsilon^3 u^3 - 4 \epsilon^2 - 2i J \epsilon^3 u) W + (2 J + \epsilon^2 \lambda - \epsilon^6) W = 0,$$  \hfill (26)

or

$$-W'' - (a^2 (b^{-3/4} u - 2i)^2 - 2i J b^{-3/4} u) W = (2 J + b^{-1/2} \lambda - b^{3/2}) W.$$  \hfill (27)

When $\epsilon \to 0$, we obtain the limit eigenvalue problem

$$-W'' + 4 a^2 W = \mu W,$$  \hfill (28)

which is a harmonic oscillator with eigenvalues $\mu_k = 2(2k+1)$, $k = 0, 1, 2, \ldots$. By a general result from [10] (see appendix), (27) implies that for each $k$, there must be a unique eigenvalue $\lambda_k(b)$ which satisfies

$$\lambda_k = b^2 + (\mu_k - 2 J + o(1)) \sqrt{b},$$  \hfill (29)

Moreover, for each compact set $K$ in the $\lambda$-plane, there exists $b_0 > 0$ such that for $b > b_0$ there are no other eigenvalues $\lambda(b) \in K$, except those satisfying (29).

We conclude from (25) that QES eigenvalues must satisfy (29). That is, for each QES eigenvalue $\lambda$ there exists $k$ such that (29) holds. Now we have to find out what are the values of $k$ for the QES eigenvalues.

To do this, we consider zeros of eigenfunctions. We know that the $k$th eigenfunction of (28) has $\lfloor k/2 \rfloor$ zeros in the right half-plane, the same number of zeros in the left half-plane, and one zero on $i\mathbb{R}$ if $k$ is odd. (In fact, all these last zeros belong to the real line but this is irrelevant for our argument.) So for every $m = 0, 1, \ldots$ there are two eigenfunctions of the harmonic oscillator (with $k = 2m$ and $k = 2m + 1$) which have $m$ zeros in the right half-plane, and one of them ($k = 2m + 1$) has a zero on $i\mathbb{R}$.

Theorem 3 implies that for each given $n$ and for each $m \leq \lfloor n/2 \rfloor$ and $b$ sufficiently large positive, there is exactly one curve $\Gamma_{n,m}$, such that the corresponding eigenfunctions have $m$
zeros in the right half-plane\textsuperscript{2}. We refer to [10] for the argument showing that the zeros of eigenfunctions \( w \) in the right half-plane do not escape to infinity as \( b \to +\infty \). Zeros of \( w \) on \( i\mathbb{R} \) do escape to infinity, except possibly one of them. Thus, the branches of QES eigenvalues must be \( \lambda_0, \ldots , \lambda_n \) satisfying (29).

Putting \( \lambda_k = b^2 - 2a(k) \), and \( J = n + 1 \) in (29), we obtain

\[
\lambda_k \sim \sqrt{b(n - 2k)}, \quad \quad 0 \leqslant k \leqslant n.
\]

We conclude that the top weight term of the polynomial \( Q_{n+1}^* \) is

\[
\prod_{k=0}^{n}(a - (n - 2k)\sqrt{b}) = \begin{cases} (a^2 - b)(a^2 - 3b) \ldots (a^2 - nb), & n \text{ is odd,} \\ a(a^2 - 2b) \ldots (a^2 - nb), & n \text{ is even.} \end{cases} \tag{30}
\]

This implies that the degree of the discriminant of \( Q_{n+1}^* \) is \( n(n + 1)/2 \), and the genus of the QES spectral locus is \( n(n - 2)/4 \) when \( n \) is even and \( (n - 1)^2/4 \) when \( n \) is odd.

9. Asymptotics as \( b \to -\infty \)

When \( b \to -\infty \), our operator (1) also degenerates to a harmonic oscillator. However, none of the QES eigenvalues of (1) tend to the eigenvalues of this harmonic oscillator as \( b \to -\infty \).

To study this limit, we set \( z = \epsilon u \), \( b = -\epsilon^{-4} \) and \( W(u) = w(\epsilon u) \) in (1). The result is

\[
W'' + (\epsilon^6 u^4 - 2a^2 + 2iJ \epsilon^3 u + \epsilon^2 \lambda)W = 0. \tag{31}
\]

As \( \epsilon \to 0 \), this tends to the harmonic oscillator

\[
-W'' + 2u^2 W = \mu W,
\]

whose eigenvalues are \( \mu_k = \sqrt{2}(2k + 1) \), \( k = 0, 1, 2, \ldots \). So by the results in [10] (see appendix A) for every \( k \) and for \( b < -b_k \), there is an eigenvalue \( \lambda_k(b) \) which satisfies

\[
\lim_{\epsilon \to 0} \epsilon^2 \lambda_k(b) = \sqrt{2}(2k + 1),
\]

or \( \lambda_k(b) \sim \sqrt{-b} \). Comparison with (25) shows that these eigenvalues \( \lambda_k \) cannot come from the QES spectrum.

A similar method can be used to study asymptotics of the spectrum as \( J \to \infty \), which we plan to consider elsewhere. The results in this direction for the quasi-exactly solvable sextic potentials can be found in [4].

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\textsuperscript{2} Remember that we are working here with eigenfunctions \( w(\xi) = y(i\xi) \), where \( y \) is an eigenfunction from theorem 3.
Appendix A. Singular perturbation of polynomial potentials

Here we state the main singular perturbation result of [10] and verify that the eigenvalue problems (26) and (31) satisfy all conditions that imply continuity of the discrete spectrum at \( \epsilon = 0 \).

Consider the eigenvalue problem
\[
-\psi'' + P_\epsilon(z, b)\psi = \lambda \psi, \quad \psi(z) \to 0, \quad z \in R_1 \cup R_2.
\] (A.1)

Here, \( z \) is the independent variable, \( P \) is a polynomial in \( z \) whose coefficients depend on parameters \( \epsilon > 0 \) and \( b \in \mathbb{C} \), dependence on \( b \) is holomorphic, and \( R_1, R_2 \) are two rays in the complex plane defined by \( R_k = \{ t e^{i\theta_k} \in \mathbb{C} : t > 0 \}, k = 1, 2 \).

Suppose that
\[
P_\epsilon(z, b) = \sum_{j=0}^{d} a_j(b, \epsilon) z^j,
\]
where \( a_d(\epsilon) > 0 \) does not depend on \( b \), \( P_0(z, b) = a_m(b, 0)z^m + \cdots \), where \( m < d \), and the dots stand for the terms of smaller degree in \( z \).

Let
\[
P^*_\epsilon(z, b) = \sum_{j=m}^{d} a_j(b, \epsilon) z^j.
\]

For every polynomial potential \( P(z) = a_n z^n + \cdots \) of degree \( n \), the separation rays are defined by
\[
\{ z \in \mathbb{C} : a_n z^{n+2} < 0 \}.
\]

Turning points are just zeros of the potential \( P \) in the complex plane. The vertical line at a point \( z \) is the line defined by \( P(z) \) \( dz^2 < 0 \). If \( P \) depends on parameters, then the separation rays, turning points and the vertical line field depend on the same parameters.

We assume that there exist \( \delta > 0 \) and \( \epsilon_0 > 0 \) and a compact \( K \subset \mathbb{C} \) such that for all \( \epsilon \in (0, \epsilon_0) \) and for all \( b \in K \) and \( k \in \{1, 2\} \), the following conditions are satisfied.

(i) \( | \arg z - \theta_k | \geq \delta \) for all turning points \( z \in \mathbb{C} \setminus \{0\} \) of \( P^*_\epsilon \).

(ii) For every point \( z \in R_k \), the smallest angle between \( R_k \) and the vertical line with respect to \( P^*_\epsilon \) at this point is at least \( \delta \).

(iii) \( R_k \) are not separation rays, for \( P_\epsilon, \epsilon > 0 \) or \( P_0 \).

(iv) All coefficients \( a_j(b, \epsilon) \) are bounded from above and \( |a_m(b, \epsilon)| \) is bounded from below.

**Theorem A.** If the conditions (i)–(iv) are satisfied, then the spectral determinant \( F_\epsilon \) of the eigenvalue problem (A.1) converges as \( \epsilon \to 0 \) to the spectral determinant of (A.1) with \( \epsilon = 0 \):
\[
F_\epsilon \to F_0, \quad \epsilon \to 0,
\]
uniformly for \( (b, \lambda) \in K \times K_1 \) for every compact \( K_1 \subset \mathbb{C} \).

Now we verify that the family of potentials in (26) satisfies all conditions (i)–(iv) with \( d = 4, m = 2 \). We have
\[
P^*_\epsilon(z) = -\epsilon^5 z^4 + 4\epsilon^3 z^3 + 4z^2,
\]
\[
P^*_0(z) = 4z^2. \quad \text{The turning points are 0 and } 2i \epsilon^{-3}. \quad \text{The separation rays are arg } z \in [0, \pi, \pm \pi/3, \pm 2\pi/3] \text{ for } P^*_\epsilon, \epsilon > 0 \text{ (shown in thin solid lines in figure A1), and}
\]

3 This terminology is somewhat unusual but convenient here. In the standard terminology, turning points are zeros of \( P - \lambda \).
arg \( z \in \{ \pm \pi/4, \pm 3\pi/4 \} \) for \( P_0 \) (dashed lines in figure A1). The normalization rays are \( \arg z \in \{ -\pi/2 \pm \pi/3 \} \). The bold lines in figure A1 represent the Stokes complex that is the integral curves of the vertical direction field \( P^*_{\epsilon}(z) \, dz^2 < 0 \) that are adjacent to the turning points.

Thus, conditions (i), (iii) and (iv) evidently hold. It remains to verify (ii).

To do this we parametrize \( R_1 \) as \( z = t e^{-i\pi/6} : t > 0 \) and find the direction of the line field \( \arg dz \) at \( z \) by inserting this parametrization to \( \arg(P^*_{\epsilon}(z) \, dz^2) = \pi \). We obtain

\[
\arg P^*_{\epsilon}(z) \in (-\pi/2, \pi/3),
\]

\[
\pm \arg dz^2 \in (2\pi/3, 4\pi/3),
\]

so the angle between \( dz \) and \( R_1 \) is at least \( \pi/6 \). The verification for \( R_2 \) is similar.

We leave to the reader to verify that conditions of theorem A are satisfied for (31).

**Appendix B. Explicit expressions**

We recall that \( \lambda = b^2 - 2a \) where \( p_n(z) = z^n + az^{n-1} + \cdots \). Since \( Q_{n+1}(b, \lambda) \) and \( Q^*_{n+1}(b, a) \) are normalized so that they are monic polynomials in \( \lambda \) and \( a \), respectively, we have

\[
Q^*(b, a) = (-1)^{n+1} Q_{n+1}(b, b^2 - 2a)/2^{n+1}.
\]

We use the notation \( C^*(b, a) = C(b, b^2 - 2a) \), where \( C \) is the constant from (12). Then, (18) can be rewritten as

\[
C^*(b, a) = \alpha_{-n}^* \frac{\partial}{\partial a} Q^*_{n+1},
\]

(B.1)

where \( \alpha_{-n}^* = (-1)^n 2^n a_n \).

Here are the results of symbolic computations with Maple.

For \( n = 1 \):

\[
p_1(z) = z + a,
\]

\[
Q^*_2(b, a) = a^2 - b,
\]

\[
C^*(b, a) = a.
\]
For $n = 2$:

$$p_2(z) = z^2 + az + \left(\frac{a^2}{2} - b\right),$$

$$Q_2^*(b, a) = a^3 - 4ab + 2,$$

$$C^*(b, a) = \frac{3}{4}a^2 - b = \frac{1}{4} \frac{\partial Q_2^*}{\partial a}.$$

For $n = 3$:

$$p_3(z) = z^3 + az^2 + \left(\frac{1}{2}a^2 - \frac{3}{2}b\right)z - \frac{7}{6}ab + \frac{1}{6}a^3 + 1,$$

$$Q_3^*(b, a) = a^4 - 10a^2b + 12a + 9b^2,$$

$$C^*(b, a) = \frac{1}{2}a^3 - \frac{5}{2}ab + \frac{3}{2} = \frac{1}{8} \frac{\partial Q_3^*}{\partial a}.$$

For $n = 4$:

$$p_4(z) = z^4 - az^3 + \left(\frac{1}{2}a^2 - 2b\right)z^2 - \left(2 + \frac{1}{6}a^2 - \frac{5}{2}ab\right)z - \frac{2}{3}a^2b + b^2 + \frac{5}{4}a + \frac{1}{24}a^4,$$

$$Q_4^*(b, a) = 42a^2 - 96b - 20a^3b + 64ab^2 + a^5,$$

$$C^*(b, a) = \frac{21}{4} + \frac{5}{16}a^4 - \frac{15}{4}a^2b + 4b^2 = \frac{1}{16} \frac{\partial Q_4^*}{\partial a}.$$

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