Gravitational Collapse of an Imperfect Non Adiabatic Fluid

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Abstract

We study the evolution of an anisotropic shear-free fluid with heat flux and kinematic self-similarity of the second kind. We found a class of solution to the Einstein field equations by assuming that the part of the tangential pressure which is explicitly time dependent of the fluid is zero and that the fluid moves along time-like geodesics. The energy conditions, geometrical and physical properties of the solutions are studied. The energy conditions are all satisfied at the beginning of the collapse but when the system approaches the singularity the energy conditions are violated, allowing for the appearance of an attractive phantom energy. We have found that, depending on the self-similar parameter $\alpha$ and the geometrical radius, they may represent a naked singularity. We speculate that the apparent horizon disappears due to the emergence of exotic energy at the end of the collapse, or due to the characteristics of null acceleration systems as shown by recent work.

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I. INTRODUCTION

There are two exciting subjects in General Relativity (GR) nowadays. One of them is related to the Cosmic Censorship Conjecture and the other one is Critical Phenomena in the context of the Gravitational Collapse.

One of the most important problems in gravitation theory is the final state of a collapsing massive star, after it has exhausted its nuclear fuel. Despite many efforts over the last four decades, our understanding is still limited to several conjectures, such as, the cosmic censorship conjecture [1], and the hoop conjecture [2]. To the former, many counter-examples have been found [3]-[6], although it is still not clear whether those particular examples are stable and generic. To the latter, no counter-examples have been found so far in four-dimensional Einstein’s Theory of gravity, although it has been shown that this is no longer the case in five dimensions [7, 8]. On the other hand, Choptuik’s discovery on critical phenomena in gravitational collapse near the threshold of black holes formation gave us deep insight to the non-linearity of the Einstein field equations [9]. Now critical phenomena in gravitational collapse already become a well-established subarea in GR [10, 11].

Many works have been done so far in this area, it can be seen that the critical solutions, which separate the collapse that form black holes from the one that does not form black holes, can have discrete self-similarity (DSS), continuous self-similarity (CSS), or none of them, depending on both matter fields and regions of initial data space. The collapse can be type II if the black holes start to form with zero mass. In this case it is found that the critical solutions have either DSS or CSS. The collapse can also be type I, that is, the formation of black holes starts with a finite non-zero mass. It is found that in the latter case the critical solutions have no self-similarities, neither DSS nor CSS [10, 11]. It is interesting that even the collapse is not critical, but if it has self-similarity, it is found that the formation of black holes can still start with zero-mass [12–16]. For details, we would like to refer readers to [12-21] and references therein.

We and other authors have studied gravitational collapse of anisotropic fluids with kinematic self-similarities in four-dimensional spacetimes [36][22][23][24] and references therein. For example, Brandt et al. (2003) [23] have analyzed the collapse of an anisotropic fluid with self similarity of the first kind, with no heat flux. They have shown that the system formed a black holes at the end. Besides, Brandt et al. (2006) [22] have studied the collapse
of the second type for values $\alpha = 1$ and $3/2$, with equation of state with radial pressure proportional to the energy density, tangential pressure is zero and no heat flux. They have shown that there is formation of black hole ($\alpha = 3/2$) and naked singularity ($\alpha = 1$). Another naked singularity appears at the end of the collapse for an anisotropic fluid with heat flow and with self-similarity of the second kind [24].

In this work, we have studied general solutions of the Einstein’s equations for a second kind self-similar anisotropic shear-free fluid with heat flux. We have analyzed some particular cases, which gives us completely different final states, including a naked singularity, representing a new counter-example to the cosmic censorship. The paper is organized as follows. In Section 2 we present the Einstein field equations. In Section 3 we present a class of exact solutions that represents an anisotropic fluid moving along time-like geodesics. The ingoing, outgoing null congruence scalar expansions and the energy conditions are analyzed [37]. The energy conditions are all satisfied at the beginning of the collapse but when the system approaches the singularity the energy conditions are violated, allowing for the appearance of an attractive phantom energy [30]. Finally, in Section 4 we present the conclusions.

II. THE FIELD EQUATIONS

The general metric of spacetimes with spherical symmetry can be cast in the form,

$$ds^2 = r_1^2 \left[ e^{2\Phi(t,r)} dt^2 - e^{2\Psi(t,r)} dr^2 - r^2 S^2(t,r) d\Omega^2 \right], \quad (1)$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$, and $r_1$ is a constant with the dimension of length. Then, we can see that the coordinates $t, r, \theta$ and $\phi$, as well as the functions $\Phi, \Psi$ and $S$ are all dimensionless.

Self-similar solutions of the second kind are given by

$$\Phi(t,r) = \Phi(x), \quad \Psi(t,r) = \Psi(x), \quad S(t,r) = S(x), \quad (2)$$

where

$$x \equiv \ln \left[ \frac{r}{(-t)^{1/\alpha}} \right], \quad (3)$$

and $\alpha$ is a dimensionless constant. The general energy-momentum tensor of an anisotropic fluid can be cast in the form

$$T_{\mu\nu} = \rho u_{\mu} u_{\nu} + p_t (\theta_\mu \theta_\nu + \phi_\mu \phi_\nu) + p_r n_\mu n_\nu + q (u_\mu n_\nu + u_\nu n_\mu), \quad (4)$$
where $u^\mu$ denotes the four-velocity of the fluid and $n^a$ is a unit spacelike vector orthogonal to $u^a$, while $\theta_\nu$ and $\phi_\mu$ denote the unit vectors in the tangential directions. Then, we can see that $\rho$ is the energy density of the fluid measured by observers comoving with the fluid, $p_t$ and $p_r$ are respectively the tangential and radial pressures and $q$ is the radial heat flux. In the comoving coordinates, we have

\begin{align}
  u_\mu &= e^{\Phi(x)} \delta_t^\mu, \quad n_\mu = e^{\Psi(x)} \delta^r_\mu, \\
  \theta_\mu &= rS(x) \delta^\theta_\mu, \quad \phi_\mu = rS(x) \sin \theta \delta^\phi_\mu. \quad (5)
\end{align}

Defining

\[ y \equiv \frac{\dot{S}}{S}, \quad (6) \]

where the symbol dot over the variable denotes differentiation with respect to $x$. We find that the non-null components of the Einstein tensor in the coordinates $\{t, r, \theta, \phi\}$ can be written as

\begin{align}
  G_{tt} &= -\frac{1}{r^2} e^{2(\Phi - \Psi)} \left[ 2\dot{y} + y(3y + 4) + 1 - 2(1 + y)\dot{\Psi} - S^{-2} e^{2\Psi} \right] \\
  &\quad + \frac{1}{\alpha^2t^2} (2\ddot{\Psi} + y), \quad (7) \\
  G_{tr} &= \frac{2}{\alpha tr} \left[ \dot{y} + (1 + y)(y - \dot{\Psi}) - y\dot{\Phi} \right], \quad (8) \\
  G_{rr} &= \frac{1}{r^2} \left[ 2(1 + y)^2 \dot{\Phi} + (1 + y)^2 - S^{-2} e^{2\Psi} \right] \\
  &\quad - \frac{1}{\alpha^2t^2} e^{2(\Psi - \Phi)} \left[ 2\dot{y} + y \left( 3y - 2\dot{\Phi} + 2\alpha \right) \right], \quad (9) \\
  G_{\theta\theta} &= S^2 e^{-2\Psi} \left[ \ddot{\Phi} + \dot{y} + \dot{\Phi} \left( \dot{\Phi} - \dot{\Psi} + y \right) + (1 + y) \left( y - \dot{\Psi} \right) \right] \\
  &\quad - \frac{r^2S^2}{\alpha^2t^2} e^{-2\Psi} \left[ \ddot{\Psi} + \dot{y} + y^2 - \left( \dot{\Psi} + y \right) \left( \dot{\Phi} - \dot{\Psi} - \alpha \right) \right], \quad (10) \\
  G_{\phi\phi} &= G_{\theta\theta} \sin^2 \theta \quad (11)
\end{align}

and the components of the energy-momentum tensor are

\begin{align}
  T_{tt} &= \rho e^{2\Phi}, \quad (12) \\
  T_{tr} &= q e^{\Phi + \Psi}, \quad (13) \\
  T_{rr} &= p_r e^{2\Psi}, \quad (14) \\
  T_{\theta\theta} &= p_r r^2 S^2, \quad (15) \\
  T_{\phi\phi} &= p_r r^2 S^2 \sin^2 \theta, \quad (16)
\end{align}
where in writing the above expressions we have set \( r_1 = 1 \). From these expressions we find that the Einstein field equations \( G_{\mu\nu} = T_{\mu\nu} \) can be written as

\[
\begin{align*}
\rho &= \frac{V^{(1)}(x)}{r^2} + \frac{V^{(2)}(x)}{t^2}, \\
p_r &= \frac{P^{(1)}_r(x)}{r^2} + \frac{P^{(2)}_r(x)}{t^2}, \\
p_t &= \frac{P^{(1)}_t(x)}{r^2} + \frac{P^{(2)}_t(x)}{t^2}, \\
q &= \frac{Q(x)}{tr},
\end{align*}
\]

where

\[
\begin{align*}
V^{(1)}(x) &= \frac{1}{r^2} e^{-2\Psi} \left[ 2y + y(3y + 4) + 1 - 2(1 + y) \dot{\Psi} - S^{-2}e^{2\Psi} \right], \\
V^{(2)}(x) &= \frac{1}{\alpha^2} e^{-2\Phi} [y + 2\dot{\Psi}], \\
P^{(1)}_r(x) &= -\frac{1}{S^2} + e^{-2\Psi} (1 + y)[1 + y + 2\dot{\Phi}], \\
P^{(2)}_r(x) &= -\frac{1}{\alpha^2} e^{-2\Phi} [2\dot{y} + 2\alpha y + 3y^2 - 2y\dot{\Phi}], \\
P^{(1)}_t(x) &= e^{-2\Psi} \left[ \dot{\Phi} + \dot{y} + \dot{\Phi} (\dot{\Phi} - \dot{\Psi} + y) + (1 + y) (y - \dot{\Psi}) \right], \\
P^{(2)}_t(x) &= \frac{1}{\alpha^2} e^{-2\Phi} \left[ \dot{\Psi} + \dot{y} + y^2 - (\dot{\Psi} + y) (\dot{\Phi} - \dot{\Psi} - \alpha) \right], \\
Q(x) &= \frac{2}{\alpha} e^{-(\Phi + \Psi)} \left[ y - (1 + y)(\dot{\Psi} - y) - y\dot{\Phi} \right].
\end{align*}
\]

In the next section we have solved the Einstein’s equations.

### III. GEODESIC SHEAR-FREE MODEL WITH AN EQUATION OF STATE

We study now the solutions of anisotropic fluid with self-similarity in a geodesic model, that is, a situation in which the acceleration \( \ddot{\Phi} = 0 \), and in particular we made \( \Phi = 0 \). Thus, we can write equations (18) as

\[
\begin{align*}
V^{(1)}(x) &= \frac{1}{S^2} - e^{-2\Psi} (1 + y)^2, \\
V^{(2)}(x) &= \frac{1}{\alpha^2} y[y + 2\dot{\Psi}], \\
P^{(1)}_r(x) &= -\frac{1}{S^2} + e^{-2\Psi} (1 + y)^2, \\
P^{(2)}_r(x) &= -\frac{1}{\alpha^2} [2\dot{y} + 2\alpha y + 3y^2],
\end{align*}
\]
\[ P_t^{(1)}(x) = e^{-2\Psi} \left[ \dot{y} + (1 + y) \left( y - \dot{\Psi} \right) \right], \]  
\[ P_t^{(2)}(x) = \frac{1}{\alpha^2} \left[ \ddot{\Psi} + \dot{y} + y^2 + (\dot{\Psi} + y) \left( \dot{\Psi} + \alpha \right) \right], \]  
\[ Q(x) = \frac{2e^{-\Psi}}{\alpha} \left[ \dot{y} + (1 + y)(y - \dot{\Psi}) \right]. \]

Then, in principle, we can have solutions with geometrical, but not physical self-similarity.

In order to obtain a particular solution of the Einstein’s equations, let us assume the shear-free condition

\[ y = \dot{\Psi}, \]  

and the equation of state

\[ P_t^{(2)}(x) = 0. \]  

This equation corresponds to the part of the tangential pressure which is explicitly time dependent, keeping the self-similar variable \( x \) as an independent variable.

From equation (25) we can see that if we have chosen \( y = -1 \), we would not have heat flux. This particular case has been studied in a previous paper [22].

Using equation (26) and (27) and substituting into (24) we get

\[ 2 \ddot{\Psi} + \dot{\Psi}(3\dot{\Psi} + 2\alpha) = 0, \]  

which can be solved giving

\[ \Psi = \frac{2}{3} \ln \left| 1 - 3e^{\alpha(x_0-x)} \right| + \Psi_0, \]  

where \( x_0 \) is an arbitrary integration constant.

From equations (26) and (6) we obtain that

\[ \Psi = \ln \left( \frac{S}{S_0} \right), \]  

which furnishes

\[ S = S_0 e^\Psi, \]  

where \( S_0 \) is another arbitrary integration constant. Substituting equation (29) into (31) we get

\[ S = S_0 e^{\Psi_0} \left[ 1 - 3e^{\alpha(x_0-x)} \right]^{2/3}. \]

We can always assume \( S_0 = 1 \) and \( \Psi_0 = 0 \) without any loss of generality.
Thus the metric (1) takes the form

\[ ds^2 = dt^2 - \left[ 1 - 3e^{\alpha x_0}(-t) \right]^{4/3} dr^2 - r^2 \left[ 1 - 3e^{\alpha x_0}(-t) \right]^{4/3} d\Omega^2. \]  

(33)

The geometric radius is given by

\[ R = rS = r \left[ 1 + 3e^{\alpha x_0}tr^{-\alpha} \right]^{2/3}. \]  

(34)

Before looking for solutions of the Einstein field equations, we must take into account that for the metric to represent spherical symmetry some physical and geometrical conditions must be imposed [25]-[35]. We impose the regularity condition for the gravitational collapse at the center, i.e., \( \lim_{r \to 0} R = 0 \) at least at the initial time. Thus, we must have that \( \alpha \leq 0 \).

Using equation (34) and the expression for outgoing and ingoing null geodesics [38–41]

\[ \theta_l = \frac{f}{R} (R_t + e^{-\Psi} R_r), \]  

(35)

and

\[ \theta_n = \frac{g}{R} (R_t - e^{-\Psi} R_r), \]  

(36)

we obtain that

\[ \theta_l = \frac{f}{r \left[ 1 + 3e^{\alpha x_0}tr^{-\alpha} \right]^{2/3}} \left\{ \frac{2e^{\alpha x_0}r^{1-\alpha}}{\left[ 1 + 3e^{\alpha x_0}tr^{-\alpha} \right]^{1/3}} + 1 - \frac{2\alpha e^{\alpha x_0}tr^{-\alpha}}{1 + 3e^{\alpha x_0}tr^{-\alpha}} \right\}, \]  

(37)

and

\[ \theta_n = \frac{g}{r \left[ 1 + 3e^{\alpha x_0}tr^{-\alpha} \right]^{2/3}} \left\{ \frac{2e^{\alpha x_0}r^{1-\alpha}}{\left[ 1 + 3e^{\alpha x_0}tr^{-\alpha} \right]^{1/3}} - 1 + \frac{2\alpha e^{\alpha x_0}tr^{-\alpha}}{1 + 3e^{\alpha x_0}tr^{-\alpha}} \right\}, \]  

(38)

where \( f \) and \( g \) are positive functions and the comma means partial differentiation. Hereinafter we will analyze the expansions without the factors \( f \) and \( g \) since these quantities are always positive for reasonable physical system [23].

The inequalities that are all common to the energy conditions:

\[ C_1 = |\rho + p| - 2 |q| > 0, \]  

(39)

\[ C_2 = \rho - p + 2 p_t + \Delta > 0, \]  

(40)

the weak energy conditions are given by

\[ C_3 = \rho - p + \Delta > 0, \]  

(41)
the dominant energy conditions are written as

\[ C_4 = \rho - p > 0 \]  \hspace{1cm} (42)

\[ C_5 = \rho - p - 2pt + \Delta > 0 \]  \hspace{1cm} (43)

and the strong energy conditions can be written as

\[ C_6 = 2pt + \Delta > 0 \]  \hspace{1cm} (44)

where \( \Delta = \sqrt{\rho^2 + 2\rho p + p^2 - 4q^2} \).

If we assume that \( q = 0 \) then we get the following inequalities:

\[ C_1 = \rho + p > 0 \]  \hspace{1cm} (45)

\[ C_2 = \rho + pt > 0 \]  \hspace{1cm} (46)

\[ C_3 = \rho > 0 \]  \hspace{1cm} (47)

\[ C_4 = \rho - p > 0 \]  \hspace{1cm} (48)

\[ C_5 = \rho - pt > 0 \]  \hspace{1cm} (49)

\[ C_6 = 2pt + \rho + p > 0 \]  \hspace{1cm} (50)

Hereinafter, we will use the classification of types of matter given by Chan, da Silva and Villas da Rocha (2009) \[30\]. If, for example, \( C_1 \) is not satisfied, but \( C_2, C_3 \) and \( C_6 \) are satisfied, then we have a physical system with attractive phantom energy.

In the following we will study two particular cases, with suitable choices for the parameter \( \alpha \), in order to obtain analytical and manageable solutions for the proposed problem.

IV. CASE \( \alpha = -1 \)

Let us now analyze a particular case where \( \alpha = -1 \) and \( x_0 = 0 \), giving

\[ \theta_l = \frac{2r^2(1 + 3tr)^{\frac{4}{3}} + 1 + 5tr}{r(1 + 3tr)^{\frac{4}{3}}} \]  \hspace{1cm} (51)

\[ \theta_n = \frac{2r^2(1 + 3tr)^{\frac{4}{3}} - 1 - 5tr}{r(1 + 3tr)^{\frac{4}{3}}} \]  \hspace{1cm} (52)
Solving $\theta_l = 0$ we get

$$t_{ah} = \frac{1}{r} \left\{ \frac{2}{125} \left[ 3600r^{12} - 1250r^6 - 1728r^{18} + 250\sqrt{-16r^{18} + 25r^{12}} \right]^{\frac{1}{3}} - \right. $$

$$\left. \frac{125}{2} \left( \frac{32}{625}r^6 - \frac{576}{15625}r^{12} \right) \right\} \left[ 3600r^{12} - 1250r^6 - 1728r^{18} + 250\sqrt{-16r^{18} + 25r^{12}} \right]^{\frac{1}{3}} - \frac{24}{125}r^6 - \frac{1}{5} \right\} \right\} \right\} \right. $$

(53)

and solving $\theta_n = 0$ we get

$$t_n = \frac{1}{r} \left\{ \frac{2}{125} \left( 3600r^{12} + 1250r^6 + 1728r^{18} + 250\sqrt{16r^{18} + 25r^{12}} \right)^{\frac{1}{3}} - \right. $$

$$\left. \frac{125}{2} \left( -\frac{32}{625}r^6 - \frac{576}{15625}r^{12} \right) \right\} \left[ 3600r^{12} + 1250r^6 + 1728r^{18} + 250\sqrt{16r^{18} + 25r^{12}} \right]^{\frac{1}{3}} + \frac{24}{125}r^6 - \frac{1}{5} \right\} \right\} \right. $$

(54)

In order to have a black hole we should have $\theta_n$ horizon being interior to the $\theta_l$ horizon, which is the case (see Figure 6). However, another condition to have black hole is to have $\theta_n < 0$, while $\theta_l > 0$, in the exterior to the $\theta_n$ horizon. Since this horizon is located inside the singularity, we do not have to analyze it. The Figure 6 shows that the apparent horizon does
FIG. 2: Zoom of the outgoing null geodesic expansion \( \theta_l \), for \( \alpha = -1 \) and \( x_0 = 0 \).

not cover the singularity for a radius greater than \( r_c \), then the structure characterizes a naked singularity formation. For the radius \( r < r_c \), there is a region of exotic matter, localized before the apparent horizon. The apparent horizon disappears for the radius \( r > r_c \), but the singularity remains, then it seems to exist a naked singularity there. The disappearance of the apparent horizon may be related to the presence of exotic matter.

The density, radial and tangential pressure, and the heat flow are given by

\begin{align}
\rho &= 4 \frac{9r^4 \sqrt{\delta t} t + 3r^3 \sqrt{\delta t} - 4t^2 r - t}{\delta t^{10/3} r}, \\
p_r &= 4 \frac{t (1 + 4tr)}{\delta t^{10/3} r}, \\
p_t &= 2 \frac{t}{\delta t^{10/3} r}, \\
q &= -4 \delta t^{-8/3},
\end{align}

and where \( \delta_1 = 1 + 3tr \).
The energy conditions are given by

\[ C_1 = 4 \left( 3 \left| \frac{t^2 (1 + 3tr)}{\delta_I^{10/3}} \right| (|\delta_I|)^{8/3} - 2 \right) (|\delta_I|)^{-8/3} > 0, \]  

\[ C_2 = 4 \frac{9r^4 \sqrt{\delta_I} t + 3r^3 \sqrt{\delta_I} - 4t^2 r - t}{\delta_I^{10/3} r} - 4 \frac{t (1 + 4tr)}{\delta_I^{10/3} r} + 4 \frac{t}{\delta_I^{10/3} r} + \Delta_1 > 0, \]  

\[ C_3 = 4 \frac{9r^4 \sqrt{\delta_I} t + 3r^3 \sqrt{\delta_I} - 4t^2 r - t}{\delta_I^{10/3} r} - 4 \frac{t (1 + 4tr)}{\delta_I^{10/3} r} + \Delta_1 > 0 \]  

\[ C_4 = 4 \frac{9r^4 \sqrt{\delta_I} t + 3r^3 \sqrt{\delta_I} - 4t^2 r - t}{\delta_I^{10/3} r} - 4 \frac{t (1 + 4tr)}{\delta_I^{10/3} r} > 0 \]  

\[ C_5 = 4 \frac{9r^4 \sqrt{\delta_I} t + 3r^3 \sqrt{\delta_I} - 4t^2 r - t}{\delta_I^{10/3} r} - 4 \frac{t (1 + 4tr)}{\delta_I^{10/3} r} - 4 \frac{t}{\delta_I^{10/3} r} + \Delta_1 > 0 \]  

\[ C_6 = 4 \frac{t}{\delta_I^{10/3} r} + \Delta_1 > 0 \]  

where

\[ \Delta_1 = \Delta = 4 \sqrt{-\frac{81 r^6 t^2 - 54 r^5 t - 9 r^4 + 4 \delta_I^{2/3}}{\delta_I^6}} \]  

We can see in Figure 3 that all the energy conditions are satisfied except the Condition 1 \((C_1)\) in the neighborhood of the curves where \(C_1 = 0\) (see also Figure 8).
FIG. 4: Zoom of the ingoing null geodesics expansion $\theta_n$, for $\alpha = -1$ and $x_0 = 0$.

V. CASE $\alpha = -2$

Let us now analyze a particular case where $\alpha = -2$ and $x_0 = 0$, giving

\[
\theta_l = \frac{2r^3(1 + 3tr^2)^{\frac{2}{3}} + 1 + 7tr^2}{r(1 + 3tr^2)^{\frac{2}{3}}},
\]

(66)

\[
\theta_n = \frac{2r^2(1 + 3tr^2)^{\frac{2}{3}} - 1 - 7tr^2}{r(1 + 3tr^2)^{\frac{2}{3}}},
\]

(67)

Solving $\theta_l = 0$ we get

\[
t_{ah} = \frac{1}{r^2} \left\{ \frac{4}{343} \left[ 1764r^{18} - 2401r^9 - 216r^{27} + 343\sqrt{-8r^{27} + 49r^{18}} \right]^\frac{1}{3} - \frac{343}{4} \left( \frac{64}{2401}r^9 - \frac{576}{17649}r^{18} \right) \right\} - \frac{24}{343}r^9 - \frac{1}{7}
\]

(68)

and solving $\theta_n = 0$ we get

\[
t_n = \frac{1}{r^2} \left\{ \frac{4}{343} \left[ 1764r^{18} + 2401r^9 - 216r^{27} + 343\sqrt{8r^{27} + 49r^{18}} \right]^\frac{1}{3} - \frac{343}{4} \left( \frac{64}{2401}r^9 - \frac{576}{17649}r^{18} \right) \right\} - \frac{24}{343}r^9 - \frac{1}{7}
\]
FIG. 5: Apparent horizon $\theta_l = 0$ (solid and dashed curves), for $\alpha = -1$ and $x_0 = 0$, $\theta_n = 0$ curve (dotted curve), singularity curve (dot-dashed curve), $C_1$ energy condition curves (long-dashed and space-dashed curves) where the energy condition $C_1$ is not fulfilled.

$$\left\{ \begin{array}{c} \frac{343}{4} \left( \frac{64}{2401} r^9 - \frac{576}{17649} r^{18} \right) \\
\left[ 1764 r^{18} + 2401 r^9 + 216 r^{27} + 343 \sqrt{8 r^{27} + 49 r^{18}} \right]^2 - \frac{24}{343} r^9 - \frac{1}{7} \end{array} \right\} = (69)$$

As in the case $\alpha = -1$, in order to have a black hole we should have that $\theta_n$ horizon being interior to the $\theta_l$ horizon, which is the case (see Figure [14]). However, another condition to have a black hole is to have $\theta_n < 0$ exterior to the $\theta_n$ horizon. Again, since this horizon is located inside the singularity, we do not have to analyze it. Again, the Figure [14] shows that the apparent horizon covers the singularity for a radius greater than $r_c$, then the structure characterizes a naked singularity formation. For the radius $r < r_c$, there is a region of exotic matter, localized before the apparent horizon. The apparent horizon disappears for the radius $r > r_c$, but the singularity remains, then it seems that there is a naked singularity there.
FIG. 6: Apparent horizon (solid curve) $\theta_l = 0$, for $\alpha = -1$ and $x_0 = 0$, singularity curve (dotted curve) and $C_1$ energy condition curve (dashed curve). The time $t = t_c \approx -0.14$ (where $r_c \approx 2.35$) denotes the moment when the apparent horizon curve disappears. The dotted curve represents also $R = 0$ and the dot-dashed curve denotes $R = 1.2$.

The density, radial and tangential pressure, and the heat flow are given by

$$
\rho = 4 \frac{9 r^6 \sqrt{\delta_2} t + 3 r^4 \sqrt{\delta_2} - 10 t^2 r^2 - 2 t}{\delta_2^{10/3}}, \quad (70)
$$

$$
p_r = 8 \frac{t (1 + 5 r^2 t)}{\delta_2^{10/3}}, \quad (71)
$$

$$
p_t = 8 \frac{t}{\delta_2^{10/3}}, \quad (72)
$$

$$
q = -8 \frac{r}{\delta_2^{8/3}}, \quad (73)
$$

where $\delta_2 = 1 + 3tr^2$.

The energy conditions are given by

$$
C_1 = 12 \left| \frac{r^4}{\delta_2^2} \right| - 16 \left| \frac{r}{\delta_2^{8/3}} \right| > 0 \quad (74)
$$
FIG. 7: The energy conditions for $\alpha = -1$ and $x_0 = 0$. 
FIG. 8: The positive and negative part of the energy condition $C1$ for $\alpha = -1$ and $x_0 = 0$.

FIG. 9: The outgoing null geodesic expansion $\theta_t$, for $\alpha = -2$ and $x_0 = 0$. 
FIG. 10: Zoom of the outgoing null geodesic expansion $\theta_l$, for $\alpha = -2$ and $x_0 = 0$.

\[ C_2 = 4 \frac{9 r^6 \sqrt{\delta_2} t + 3 r^4 \sqrt{\delta_2} - 10 t^2 r^2 - 2 t}{\delta_2^{10/3}} - 8 \frac{t (1 + 5 r^2 t)}{\delta_2^{10/3}} + 16 \frac{t}{\delta_2^{10/3}} + \Delta_2 > 0 \] (75)

\[ C_3 = 4 \frac{9 r^6 \sqrt{\delta_2} t + 3 r^4 \sqrt{\delta_2} - 10 t^2 r^2 - 2 t}{\delta_2^{10/3}} - 8 \frac{t (1 + 5 r^2 t)}{\delta_2^{10/3}} + \Delta_2 > 0 \] (76)

\[ C_4 = 4 \frac{9 r^6 \sqrt{\delta_2} t + 3 r^4 \sqrt{\delta_2} - 10 t^2 r^2 - 2 t}{\delta_2^{10/3}} - 8 \frac{t (1 + 5 r^2 t)}{\delta_2^{10/3}} > 0 \] (77)

\[ C_5 = 4 \frac{9 r^6 \sqrt{\delta_2} t + 3 r^4 \sqrt{\delta_2} - 10 t^2 r^2 - 2 t}{\delta_2^{10/3}} - 8 \frac{t (1 + 5 r^2 t)}{\delta_2^{10/3}} - 16 \frac{t}{\delta_2^{10/3}} + \Delta_2 > 0 \] (78)

\[ C_6 = 16 \frac{t}{\delta_2^{10/3}} + \Delta_2 > 0 \] (79)

where

\[ \Delta_2 = \Delta = 4 \sqrt{\frac{r^2 \left(-81 r^{10} t^2 - 54 r^8 t - 9 r^6 + 8 \delta_2^{2/3}\right)}{\delta_2^6}}. \] (80)

Again, we can see in Figure 15 that all the energy conditions are satisfied except the Condition 1 ($C_1$) in the neighborhood of the curves where $C_1 = 0$ (see also Figure 16).
VI. CONCLUSION

In this work we have studied the evolution of an anisotropic shear-free fluid with heat flux fluid and self-similarity of the second kind, in order to build gravitational collapse model. We have found a class of solution to the Einstein field equations by assuming that the part of the tangential pressure which is explicitly time dependent of the fluid is zero and that the fluid moves along time-like geodesics. The energy conditions, geometrical and physical properties of the solutions were studied. We have also found that, depending on the parameter \( \alpha \) and the geometrical radius, they may represent a naked singularity. Besides, the energy conditions are satisfied almost everywhere.

Comparing this collapsing system of imperfect fluid with heat flow with the solution obtained by Chan, da Silva & Villas da Rocha [24], we can see an important similarity, because in both we have self-similarity, although corresponding to different kind, and shear-free configuration, with naked singularity formation. The present results reinforce our conclusions in the first paper, that is, although Joshi, Dadhich & Maartens [43] concluded that the
formation of naked singularities is due to shear of the fluid, in such mode that sufficiently strong shearing effects could delay the formation of apparent horizons, thereby exposing the strong gravitational regions to the outside world and leading to naked singularities, their study is based on a gravitational collapse of spherically symmetric dust fluid. In another work more recent, Joshi, Malafarina & Saraykar [44] showed that the causal structure of the spacetime is in fact affected by the introducing of a small amount of pressure. Our present results, again seem point out that the presence of anisotropic pressures can really modify the final structure in the gravitational collapse. In addition, Giambo & Magli, in a recent paper [45], studied the gravitational collapse of perfect fluids, more specifically for linear barotropic fluids and, as in our case, a family of geodesic fluid, and they concluded that the pressure plays a relevant role on the causal structure of the collapsing model.

Moreover, the fluid here evolves to an exotic fluid near the formation of the apparent horizon. Considering this kind of fluid can impose a repulsive gravitational effect (due to the phantom energy) , we speculate that this can be the reason to the inexistence of the formation of a horizon.
FIG. 13: Apparent horizon $\theta_l = 0$ (solid and dashed curves), for $\alpha = -2$ and $x_0 = 0$, $\theta_n = 0$ curve (dotted curve), singularity curve (dot-dashed curve), $C_1$ energy condition curves (long-dashed and space-dashed curves) where the energy condition $C_1$ is not fulfilled.

Finally, it would be very interesting verify if the naked singularity could represent a critical solution for the gravitational collapse.

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FIG. 14: Apparent horizon (solid curve) $\theta_l = 0$, for $\alpha = -2$ and $x_0 = 0$, singularity curve (dotted curve) and $C_1$ energy condition curve (dashed curve). The time $t = t_c \approx -0.17$ (where $r_c \approx 2.0$) denotes the time when the apparent horizon curve disappears. The dotted curve represents also $R = 0$ and the dot-dashed curve denotes $R = 2$.

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FIG. 15: The energy conditions for $\alpha = -2$ and $x_0 = 0$. 
FIG. 16: The positive and negative part of the energy condition $C_1$ for $\alpha = -2$ and $x_0 = 0$.

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