A generalization of the inequality of Audenaert et al.

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Abstract

We extend the inequality of Audenaert et al.\cite{ACMMABV} to general von Neumann algebras.

1 Introduction

Let $A, B$ be positive matrices and $0 \leq s \leq 1$. Then an inequality

$$2 \text{Tr} A^s B^{1-s} \geq \text{Tr}(A + B - |A - B|)$$

holds. This is a key inequality to prove the upper bound of Chernoff bound, in quantum hypothesis testing theory. This inequality was first proven in \cite{ACMMABV}, using an integral representation of the function $t^s$. Recently, N.Ozawa gave a much simpler proof for the same inequality. In this note, based on his proof, we extend the inequality to general von Neumann algebras. More precisely, we prove the following: Let $\{\mathcal{M}, \mathcal{H}, J, \mathcal{P}\}$ be a standard form associated with a von Neumann algebra $\mathcal{M}$, i.e., $\mathcal{H}$ is a Hilbert space where $\mathcal{M}$ acts on, $J$ is the modular conjugation, and $\mathcal{P}$ is the natural positive cone.\cite{1} Let $\mathcal{M}_{++}$ be the set of all positive normal linear functionals over $\mathcal{M}$. For each $\varphi \in \mathcal{M}_{++}$, $\xi_\varphi$ is the unique element in the natural positive cone $\mathcal{P}$ which satisfies $\varphi(x) = (x \xi_\varphi, \xi_\varphi)$ for all $x \in \mathcal{M}$. We denote the relative modular operator associated with $\varphi, \psi \in \mathcal{M}_{++}$ by $\Delta_{\varphi,\psi}$.\cite{Appendix} The main result in this note is the following:

**Proposition 1.1** Let $\varphi, \eta$ be positive normal linear functionals on a von Neumann algebra $\mathcal{M}$. Then, for any $0 \leq s \leq 1$,

$$\eta(1) - (\eta - \varphi)_+(1) \leq \left\| \Delta_{\eta,\varphi}^{s} \xi_\varphi \right\|^2.$$ \hspace{1cm} (2)

The equality holds iff $\eta = (\eta - \varphi)_+ + \psi$ and $\varphi = (\eta - \varphi)_- + \psi$ for some $\psi \in \mathcal{M}_{++}$ whose support is orthogonal to the support of $|\eta - \varphi|$.

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As a corollary of this proposition, we obtain a generalization of the inequality of [ACMABV]:

**Corollary 1.1** Let \( \varphi, \eta \) be positive normal linear functionals on a von Neumann algebra \( \mathcal{M} \). Then, for any \( 0 \leq s \leq 1 \),

\[
2 \left\| \Delta_{\varphi_\psi}^{-s} \xi \right\|^2 \geq \varphi(1) + \eta(1) - |\varphi - \eta| (1). \tag{3}
\]

The equality holds iff \( \eta = (\eta - \varphi)_+ + \psi \) and \( \varphi = (\eta - \varphi)_- + \psi \) for some \( \psi \in \mathcal{M}_{++} \) whose support is orthogonal to the support of \( |\eta - \varphi| \).

If \( s = \frac{1}{2} \), this is the Powers-Størmer inequality. Applications of this inequality for hypothesis testing problem can be found in [JOPS].

### 2 Proof of Proposition 1.1

We first prove the following lemma which we need in the proof of Proposition 1.1

**Lemma 2.1** Let \( \varphi_1, \varphi_2, \psi, \eta \) be faithful normal positive linear functionals over a von Neumann algebra \( \mathcal{M} \). Assume that \( \varphi_1 \leq \varphi_2 \) and \( \eta \leq \psi \). Then for all \( 0 < s < 1 \),

\[
\left\| \Delta_{\varphi_2 \psi}^s \xi \right\|^2 - \left\| \Delta_{\varphi_2 \psi}^{-s} \eta \right\|^2 \leq \left\| \Delta_{\varphi_2 \psi}^s \xi \right\|^2 - \left\| \Delta_{\varphi_1 \psi}^s \xi \right\|^2.
\]

**Proof** First we consider the case \( \varphi_2 \leq \psi \). In this case, by Lemma 1.1, \( (\Delta \varphi_1 : D \psi)_i, (\Delta \varphi_2 : D \psi)_1, (\Delta \eta : D \psi)_i, (\Delta \eta : D \psi)_1 \) have continuations \( (\Delta \varphi_1 : D \psi)_z, (\Delta \varphi_2 : D \psi)_z, (\Delta \eta : D \psi)_z \in \mathcal{M} \), analytic on \( I_{-\frac{1}{2}} := \{ z \in \mathbb{C} : -\frac{1}{2} < \Im z < 0 \} \) and bounded continuous on \( I_{-\frac{1}{2}} \), with norm less than or equal to 1.

We define a positive operator

\[
T := (\Delta \varphi_2 : D \psi)_i - (\Delta \varphi_2 : D \psi)_- - (\Delta \varphi_1 : D \psi)_i + (\Delta \varphi_1 : D \psi)_- \in \mathcal{M}.
\tag{4}
\]

To see that \( T \) is positive, recall from Lemma 1.1 that for any \( \xi \in D(\Delta_{\varphi_\psi}^{-s}) \), we have \( \Delta_{\psi \psi}^{-s} \xi \in D(\Delta_{\varphi_\psi}^{-s}) \), and

\[
\Delta_{\varphi_\psi}^{-s} \Delta_{\psi \psi}^{-s} \xi = (\Delta \varphi : D \psi)_- - (\Delta \varphi : D \psi)_i \xi, \quad k = 1, 2.
\]

From this, we obtain

\[
(T \xi, \xi) = \left\| (\Delta \varphi : D \psi)_- - (\Delta \varphi : D \psi)_i \xi \right\|^2 - \left\| (\Delta \varphi_1 : D \psi)_i - (\Delta \varphi_1 : D \psi)_- \xi \right\|^2 = \left\| \Delta_{\varphi_\psi}^{-s} \Delta_{\psi \psi}^{-s} \xi \right\|^2 - \left\| \Delta_{\varphi_1 \psi}^{-s} \Delta_{\psi \psi}^{-s} \xi \right\|^2. \tag{5}
\]

As \( \Delta_{\psi \psi}^{-s} \xi \) is in \( D(\Delta_{\varphi_\psi}^{-s}) \), the last term is positive from Lemma 1.2. This proves \( T \geq 0 \).
Next we define \( x' := J((D\eta : D\psi)_{-1}\Delta_{\psi}) J \in \mathcal{M}' \). From \( \|x'\| \leq 1 \) and \( 0 \leq T \in \mathcal{M} \), we have
\[
(x''T x' \xi_\psi, \xi_\psi) = \left( T^{\frac{1}{2}} x'' x T^{\frac{1}{2}} \xi_\psi, \xi_\psi \right) \leq (T \xi_\psi, \xi_\psi).
\]
(6)

As \( \xi_\psi \in D \left( \Delta_{\psi \psi}^{\frac{1}{2}} \right) \), from Lemma A.1, we have
\[
x' \xi_\psi = J(D\eta : D\psi)_{-1}\Delta_{\psi \psi} \xi_\psi = J \Delta_{\psi \psi}^{\frac{1}{2}} \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi = J \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi = \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi = \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi \in D(\Delta_{\psi \psi}^{\frac{1}{2}}).
\]
(7)

By this and Lemma A.1, we have \( \Delta_{\psi \psi}^{\frac{1}{2}} x' \xi_\psi \in D(\Delta_{\psi \psi}^{\frac{1}{2}}) \) and
\[
(D\varphi_k : D\psi)_{-1}\Delta_{\psi \psi} \xi_\psi = \Delta_{\psi \psi}^{\frac{1}{2}} \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi = \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi
\]
for \( k = 1, 2 \). Hence we obtain
\[
(x''T x' \xi_\psi, \xi_\psi) = \left( \left\| \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi \right\|^2 - \left\| \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi \right\|^2 \right).
\]
(9)

On the other hand, substituting \( \xi = \xi_\psi \in D(\Delta_{\psi \psi}^{\frac{1}{2}}) \) to (6), we have
\[
(T \xi_\psi, \xi_\psi) = \left( \left\| \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi \right\|^2 - \left\| \Delta_{\psi \psi}^{\frac{1}{2}} \xi_\psi \right\|^2 \right).
\]
(10)

From (6), (9), (10), we obtain the result for the \( \varphi_2 \leq \psi \) case.

To extend the result to a general case, we use Lemma A.3. For any \( \varepsilon > 0 \), we have
\[
\varepsilon \varphi_1 \leq \varepsilon \varphi_2 \leq \psi + \varepsilon \varphi_2, \quad \eta \leq \psi + \varepsilon \varphi_2.
\]
(11)

Therefore, for any \( 0 < s < 1 \), we have
\[
\left\| \Delta_{\psi \psi}^{\frac{s}{2}} \xi_\psi \right\|^2 - \left\| \Delta_{\psi \psi}^{\frac{s}{2}} \xi_\psi \right\|^2 \leq \left\| \Delta_{\psi \psi}^{\frac{s}{2}} \psi + \varepsilon \varphi_2 \xi_\psi + \varepsilon \varphi_2 \right\|^2 - \left\| \Delta_{\psi \psi}^{\frac{s}{2}} \psi + \varepsilon \varphi_2 \xi_\psi + \varepsilon \varphi_2 \right\|^2.
\]

Using relations \( \Delta_{\varphi \varphi}^{\frac{s}{2}} = \varepsilon \varphi \Delta_{\varphi \varphi}^{\frac{s}{2}} \eta \) etc, we have
\[
\left\| \Delta_{\psi \psi}^{\frac{s}{2}} \xi_\psi \right\|^2 - \left\| \Delta_{\psi \psi}^{\frac{s}{2}} \xi_\psi \right\|^2 \leq \left\| \Delta_{\psi \psi}^{\frac{s}{2}} \psi + \varepsilon \varphi_2 \xi_\psi + \varepsilon \varphi_2 \right\|^2 - \left\| \Delta_{\psi \psi}^{\frac{s}{2}} \psi + \varepsilon \varphi_2 \xi_\psi + \varepsilon \varphi_2 \right\|^2
\]
\[
= \left\| \frac{\Delta_{\psi \psi}^{\frac{s}{2}}}{\psi + \varepsilon \varphi_2} \xi_\psi \right\|^2 - \left\| \frac{\Delta_{\psi \psi}^{\frac{s}{2}}}{\psi + \varepsilon \varphi_2} \xi_\psi \right\|^2.
\]

In the second line we used Lemma A.5. Taking \( \varepsilon \to 0 \) and applying Lemma A.3 and Lemma A.5, we obtain the result. □

*Proof of Proposition 1.1*
It is trivial for $s = 0, 1$. We prove the claim for $0 < s < 1$. We first consider faithful $\varphi, \eta$. From $\varphi \leq \varphi + (\eta - \varphi)_+$ and Lemma \[\text{A.2}\] we have

$$
\left\| \Delta_{\varphi, \varphi}^s \xi_\varphi \right\|^2 - \left\| \Delta_{\varphi + (\eta - \varphi)_+}^s \xi_\varphi \right\|^2 \leq \left\| \Delta_{\varphi + (\eta - \varphi)_+}^s \xi_\varphi \right\|^2 - \left\| \Delta_{\varphi, \varphi}^s \xi_\varphi \right\|^2. \tag{12}
$$

By Lemma \[\text{2.1}\] and inequalities $\eta \leq \varphi + (\eta - \varphi)_+$, $\varphi \leq \varphi + (\eta - \varphi)_+$, the last term is bounded as

$$
\leq \left\| \Delta_{\varphi + (\eta - \varphi)_+}^s \xi_\varphi \right\|^2 - \left\| \Delta_{\varphi + (\eta - \varphi)_+}^s \xi_\varphi \right\|^2 = \left\| \xi_{\varphi + (\eta - \varphi)_+} \right\|^2 - \left\| \Delta_{\varphi + (\eta - \varphi)_+}^s \xi_\varphi \right\|^2.
$$

By $\varphi + (\eta - \varphi)_+ \geq \eta$ and Lemma \[\text{A.2}\] we have

$$
\left\| \xi_{\varphi + (\eta - \varphi)_+} \right\|^2 - \left\| \Delta_{\eta, \eta}^s \xi_\eta \right\|^2 = \varphi(1) + (\eta - \varphi)_+(1) - \eta(1). \tag{14}
$$

Hence we obtain

$$
\varphi(1) - \left\| \Delta_{\eta, \eta}^s \xi_\varphi \right\|^2 \leq \varphi(1) + (\eta - \varphi)_+(1) - \eta(1), \tag{15}
$$

which is equal to

$$
\eta(1) - (\eta - \varphi)_+(1) \leq \left\| \Delta_{\eta, \eta}^s \xi_\varphi \right\|^2. \tag{16}
$$

We now prove the inequality for general $\varphi, \eta$. By considering a von Neumann algebra $\mathcal{M}_e := e\mathcal{M}e$ with $e := s(\eta) \vee s(\varphi)$ instead of $\mathcal{M}$ if it is necessary, we may assume $\varphi + \varepsilon \eta, \eta + \delta \varphi$ are faithful on $\mathcal{M}$ for any $\varepsilon, \delta > 0$. We then have

$$
(\eta + \delta \varphi)(1) - (\eta + \delta \varphi - (\varphi + \varepsilon \eta))_+ \leq \left\| \Delta_{\eta, \eta}^s \xi_{\varphi + \varepsilon \eta} \right\|^2. \tag{17}
$$

Taking the limit $\varepsilon \to 0$ and then the limit $\delta \to 0$, and using Lemma \[\text{A.3}\] and Lemma \[\text{A.5}\] we obtain the inequality \[2\] for general $\varphi, \eta$.

To check the condition for the equality, by approximating $\varphi$ and $\eta$ by $\varphi + \varepsilon \eta, \eta + \delta \varphi$ in \[12, 13\], and \[14\], just as in \[17\], and taking the limit $\varepsilon \to 0$ and $\delta \to 0$, we obtain

$$
\left\| \Delta_{\varphi, \varphi}^s \xi_\varphi \right\|^2 - \left\| \Delta_{\varphi + (\eta - \varphi)_+}^s \xi_\varphi \right\|^2 \leq \left\| \Delta_{\varphi + (\eta - \varphi)_+}^s \xi_\varphi \right\|^2 - \left\| \Delta_{\varphi, \varphi}^s \xi_\varphi \right\|^2
$$

$$
\leq \left\| \Delta_{\varphi + (\eta - \varphi)_+}^s \xi_\varphi \right\|^2 - \left\| \Delta_{\varphi + (\eta - \varphi)_+}^s \xi_\varphi \right\|^2 = \left\| \xi_{\varphi + (\eta - \varphi)_+} \right\|^2 - \left\| \Delta_{\eta, \eta}^s \xi_\eta \right\|^2
$$

$$
= \varphi(1) + (\eta - \varphi)_+(1) - \eta(1). \tag{18}
$$
By Lemma A.4, the first inequality is an equality iff the support of \((\eta - \varphi)_+\) is orthogonal to \(\varphi\) and the third inequality is an equality iff the support of \((\eta - \varphi)_-\) is orthogonal to \(\eta\). Therefore, if the equality in (18) holds, then \((\eta - \varphi)_+\) is orthogonal to \(\varphi\) and \((\eta - \varphi)_-\) is orthogonal to \(\eta\). Conversely, if \((\eta - \varphi)_+\) is orthogonal to \(\varphi\) and \((\eta - \varphi)_-\) is orthogonal to \(\eta\). Then we have \(\varphi + (\eta - \varphi)_+ = \eta + (\eta - \varphi)_-\), where both sides of the equality are sum of orthogonal elements. Therefore, we have

\[
\|\Delta_{\varphi + (\eta - \varphi)_+}^2 \xi_{\varphi + (\eta - \varphi)_+} \|^2 - \|\Delta_{\eta,\varphi + (\eta - \varphi)_+}^2 \xi_{\varphi + (\eta - \varphi)_+} \|^2 = \|\Delta_{\eta - (\eta - \varphi)_+}^2 \varphi + (\eta - \varphi)_+ \xi_{\varphi + (\eta - \varphi)_+} \|^2
\]

Furthermore, we have

\[
\|\Delta_{\varphi + (\eta - \varphi)_+}^2 \xi_{\varphi} \|^2 - \|\Delta_{\eta,\varphi}^2 \xi_{\varphi} \|^2 = \|\Delta_{\eta - (\eta - \varphi)_+}^2 \varphi \xi_{\varphi} \|^2.
\]

Hence, the second inequality is an equality in this case. As the first and third inequalities are equalities from the orthogonality of \((\eta - \varphi)_+\) with \(\varphi\) and \((\eta - \varphi)_-\) with \(\eta\) respectively, the equality holds in (18).

Therefore, the equality in (18) holds iff \((\eta - \varphi)_+\) is orthogonal to \(\varphi\) and \((\eta - \varphi)_-\) is orthogonal to \(\eta\). However, the latter condition means \(\eta = (\eta - \varphi)_+ + \psi\) and \(\varphi = (\eta - \varphi)_- + \psi\) for some \(\psi \in M_{++}\) whose support is orthogonal to the support of \(\eta - \varphi\). \(\square\)

**Proof of Corollary 1.4**

Replacing \(\eta, \varphi, s\) in (2) with \(\varphi, \eta, 1 - s\) respectively, we obtain

\[
\varphi(1) - (\varphi - \eta)_+ (1) \leq \|\Delta_{\eta,\varphi}^2 \xi_{\varphi} \|^2 = \|\Delta_{\eta,\varphi}^2 \xi_{\varphi} \|^2.
\]

Summing (2) and (20), we obtain (3). \(\square\)

### A Appendix

Let \(\{M, \mathcal{H}, J, \mathcal{P}\}\) be a standard form associated with a von Neumann algebra \(M\), i.e., \(\mathcal{H}\) is a Hilbert space where \(M\) acts on, \(J\) is the modular conjugation, and \(\mathcal{P}\) is the natural positive cone. Let \(M_{++}\) be the set of all positive normal linear functionals over \(M\). For each \(\varphi \in M_{++}\), \(\xi_{\varphi}\) is the unique element in the natural positive cone \(\mathcal{P}\) which satisfies \(\varphi(x) = (x\xi_{\varphi}, \xi_{\varphi})\) for all \(x \in M\). For \(\varphi, \psi \in M_{++}\), we define an operator \(S_{\varphi\psi}\) as the closure of the operator

\[
S_{\varphi\psi}(x\xi_{\psi} + (1 - j(s(\psi))))\zeta) := s(\psi)x^*\xi_{\varphi}, \quad x \in M, \ \zeta \in \mathcal{H},
\]

where \(s(\psi) \in M\) is the support projection of \(\psi\) and \(j(y) := JyJ\). The polar decomposition of \(S_{\varphi\psi}\) is given by \(S_{\varphi\psi} = J\Delta_{\varphi\psi}^2\) where \(\Delta_{\varphi\psi}\) is the relative modular operator associated with \(\varphi, \psi \in M_{++}\). The subspace \(M\xi_{\psi} + (1 - j(s(\psi)))\mathcal{H}\)
of $\mathcal{H}$ is a core of $\Delta^\frac{1}{2}_{\varphi,\psi}$. The support projection of the positive operator $\Delta_{\varphi,\psi}$ is $s(\varphi)j(s(\psi))$. For a complex number $z \in \mathbb{C}$, we define a closed operator $\Delta^z_{\varphi,\psi}$ by

$$\Delta^z_{\varphi,\psi} := \exp(z(\log \Delta_{\varphi,\psi})s(\varphi)j(s(\psi))))s(\varphi)j(s(\psi)).$$

For an operator $A$ on a Hilbert space $\mathcal{H}$, we denote by $D(A)$ its domain.

**Lemma A.1** Let $\varphi, \psi$ be faithful normal positive linear functionals over a von Neumann algebra $\mathcal{M}$. Suppose that there exists a constant $\lambda > 0$ such that $\lambda \varphi \leq \psi$. Then the cocyle $\mathbb{R} \ni t \mapsto (D\varphi : D\psi)_t \in \mathcal{M}$ has an extension $(D\varphi : D\psi)_z \in \mathcal{M}$ analytic on $I_{-\frac{1}{2}} := \{z \in \mathbb{C} : -\frac{1}{2} < \Re z < 0\}$ and bounded continuous on $\overline{I_{-\frac{1}{2}}}$ with the bound $\|(D\varphi : D\psi)_z\| \leq \lambda^{|z|}$ for all $z \in \overline{I_{-\frac{1}{2}}}$. Furthermore, for any faithful $\zeta \in \mathcal{M}_{+\star}$, $0 < s < \frac{1}{2}$, and any element $\xi$ in $D(\Delta^{-s}_{\varphi,\psi})$, $\Delta^{-s}_{\varphi,\psi}\xi$ is in the domain of $\Delta^s_{\varphi,\psi}$, and

$$\Delta^s_{\varphi,\psi}\Delta^{-s}_{\varphi,\psi}\xi = (D\varphi : D\psi)_{-is}\xi. \quad (21)$$

**Proof** The existence and boundedness of $(D\varphi : D\psi)_z$ is proven in [AM]. To show the latter part of the Lemma, let $\zeta \in \mathcal{M}_{+\star}$ be faithful. We define the region $I_s$ in the complex plane by $I_s := \{z \in \mathbb{C} : -s < \Re z < 0\}$ for each $0 < s < \frac{1}{2}$. For any $\xi \in D(\Delta^{-s}_{\varphi,\psi})$ and $\xi_1 \in D(\Delta^{s}_{\varphi,\psi})$, we consider two functions on $\overline{I_{-s}}$ by $F(z) := \left(\Delta^{is}_{\varphi,\psi}\xi, \Delta^{-is}_{\varphi,\psi}\xi_1\right)$, and $G(z) := ((D\varphi : D\psi)_z\xi, \xi_1)$. Both of these functions are bounded continuous on $\overline{I_{-s}}$ and analytic on $I_{-s}$. Furthermore, they are equal on $\mathbb{R}$:

$$F(t) = \left(\Delta^{it}_{\varphi,\psi}\Delta^{-it}_{\varphi,\psi}\xi_1\right) = ((D\varphi : D\psi)_t\xi, \xi_1) = G(t), \quad \forall t \in \mathbb{R}.$$ 

This means $F(z) = G(z)$ for all $z \in \overline{I_{-s}}$. In particular, we have $F(-is) = G(-is)$, i.e.,

$$\left(\Delta^{-is}_{\varphi,\psi}\xi, \Delta^{is}_{\varphi,\psi}\xi_1\right) = ((D\varphi : D\psi)_{-is}\xi, \xi_1).$$

As this holds for all $\xi_1 \in D(\Delta^{s}_{\varphi,\psi})$, $\Delta^{-s}_{\varphi,\psi}\xi$ is in the domain of $\Delta^s_{\varphi,\psi}$, and (21) holds.

**Lemma A.2** Let $\varphi, \eta, \psi$ be normal positive linear functionals over a von Neumann algebra $\mathcal{M}$ such that $\varphi \leq \eta$. Then for any $0 \leq s \leq 1$, we have $D(\Delta^s_{\varphi, \psi}) \subset D(\Delta^s_{\eta, \psi})$ and

$$\|\Delta^s_{\varphi, \psi}\xi\| \leq \|\Delta^s_{\eta, \psi}\xi\|, \quad \forall \xi \in D(\Delta^s_{\eta, \psi}). \quad (22)$$

**Proof** This is proven in [AM].

**Lemma A.3** Let $\varphi$ and $\eta$ be elements in $\mathcal{M}_{+\star}$ and $\varphi_n$ a sequence in $\mathcal{M}_{+\star}$ such that $\lim_{n \to \infty} \|\varphi_n - \varphi\| = 0$. Then for any and $0 < s < 1$,

$$\lim_{n \to \infty} \|\Delta^s_{\varphi_n, \eta}\xi_\eta\| = \|\Delta^s_{\varphi, \eta}\xi_\eta\|. \quad (23)$$
Proof By the integral representation of $t^s$, we have
\[
\frac{\sin s\pi}{\pi} \int_0^\infty d\lambda \lambda^{-1} \left( \left( \Delta_{\varphi, \eta} (\Delta_{\varphi, \eta} + \lambda)^{-1} - \Delta_{\varphi, \eta} (\Delta_{\varphi, \eta} + \lambda)^{-1} \right) \xi_\eta, \xi_\eta \right).
\]

We denote the term inside of the integral by $f_n(\lambda)$. It is easy to see
\[
|f_n(\lambda)| \leq \lambda^{s-1} \eta(1),
\]
\[
|f_n(\lambda)| \leq \lambda^{s-2} \left( \left| \Delta_{\varphi, \eta}^\frac{1}{n} \xi_\eta \right|^2 + \left| \Delta_{\varphi, \eta}^\frac{1}{n} \xi_\eta \right|^2 \right) \leq \lambda^{s-2} \left( \varphi(1) + \sup_n \varphi_n(1) \right).
\]

Hence $|f_n(\lambda)|$ is bounded from above by an integrable function independent of $n$.

Next we show $\lim_{n \to \infty} f_n(\lambda) = 0$ for all $\lambda > 0$. To do so, we first observe that $\Delta_{\varphi, \eta}^\frac{1}{n}$ converges to $\Delta_{\varphi, \eta}^\frac{1}{s}$ in the strong resolvent sense: For all $x\xi_\eta + (1 - j(s(\eta)))\zeta \in M\xi_\eta + (1 - j(s(\eta)))\mathcal{H}$, using Powers-Stormer inequality, we have
\[
\left\| \Delta_{\varphi, \eta}^\frac{1}{n} \left( x\xi_\eta + (1 - j(s(\eta)))\zeta - \Delta_{\varphi, \eta}^\frac{1}{s} (x\xi_\eta + (1 - j(s(\eta)))\zeta) \right) \right\|^2 = \left\| s(\eta)x^*\xi_\varphi - s(\eta)x^*\xi_\varphi \right\|^2
\leq \left\| x^* \right\|^2 \left\| \xi_\varphi - \xi_\eta \right\|^2 \leq \left\| x^* \right\|^2 \left\| \varphi_n - \varphi \right\| \rightarrow 0, \text{ as } n \rightarrow \infty.
\]

As $M\xi_\eta + (1 - j(s(\eta)))\mathcal{H}$ is a common core for all $\Delta_{\varphi, \eta}^\frac{1}{n}$ and $\Delta_{\varphi, \eta}^\frac{1}{s}$, this means $\Delta_{\varphi, \eta}^\frac{1}{n}$ converges to $\Delta_{\varphi, \eta}^\frac{1}{s}$ in the strong resolvent sense. Therefore, for a bounded continuous function $g(t) = t^2(t^2 + \lambda)^{-1}$, $g(\Delta_{\varphi, \eta}^\frac{1}{n})$ converges to $g(\Delta_{\varphi, \eta}^\frac{1}{s})$ strongly. Hence we have $\lim_{n \to \infty} f_n(\lambda) = 0$.

By the Lebesgue’s theorem, we obtain the result. □

Lemma A.4 For any $\varphi, \eta \in M_{++}$ with $\varphi \leq \eta$ and $0 < s < 1$,
\[
\left\| \Delta_{\eta \varphi}^\frac{2}{n} \xi_\varphi \right\| = \left\| \xi_\varphi \right\| (24)
\]
if and only if $\eta - \varphi$ is orthogonal to $\varphi$.

Proof First we prove if $\left\| \Delta_{\eta \varphi}^\frac{2}{n} \xi_\varphi \right\| = \left\| \xi_\varphi \right\|$, then $\eta - \varphi$ is orthogonal to $\varphi$. From Lemma A.2, for any $\zeta \in D(\Delta_{\eta \varphi}^\frac{2}{s})$, $\Delta_{\eta \varphi}^\frac{2}{s} \zeta$ is in $D(\Delta_{\eta \varphi}^\frac{2}{n})$ and
\[
\left\| \Delta_{\eta \varphi}^\frac{2}{n} \Delta_{\eta \varphi}^\frac{2}{s} \zeta \right\| \leq \left\| \Delta_{\eta \varphi}^\frac{2}{n} \Delta_{\eta \varphi}^\frac{2}{s} \zeta \right\| \leq \left\| \zeta \right\|.
\]
Therefore, $\Delta_{\eta \varphi}^\frac{2}{n} \Delta_{\eta \varphi}^\frac{2}{s}$ defined on $D(\Delta_{\eta \varphi}^\frac{2}{n})$ can be uniquely extended to a bounded operator $A$ on $\mathcal{H}$, with norm $\| A \| \leq 1$. We define an operator $0 \leq T \leq 1$ by $T := A^* A$. Note that
\[
A \Delta_{\eta \varphi}^\frac{2}{n} \xi_\varphi = \Delta_{\eta \varphi}^\frac{2}{n} \Delta_{\eta \varphi}^\frac{2}{s} \xi_\varphi = \Delta_{\eta \varphi}^\frac{2}{n} (s(\eta)) \xi_\varphi = \Delta_{\eta \varphi}^\frac{2}{n} \xi_\varphi = \xi_\varphi.
\]
From this, and the assumption, we have
\[
(T \Delta_{\eta \varphi}^2 \xi \varphi, \Delta_{\eta \varphi}^2 \xi \varphi) = \| A \Delta_{\eta \varphi}^2 \xi \varphi \|^2 = \| \xi \varphi \|^2 = \| \Delta_{\eta \varphi}^2 \xi \varphi \|^2.
\]
As the spectrum of \( T \) is included in \([0, 1]\), this equality means
\[
T \Delta_{\eta \varphi}^2 \xi \varphi = \Delta_{\eta \varphi}^2 \xi \varphi.
\]
(25)

For any \( \zeta \in D(\Delta_{\eta \varphi}^2) \), we have
\[
(\Delta_{\eta \varphi}^2 \xi \varphi, \Delta_{\eta \varphi}^2 \zeta) = (T \Delta_{\eta \varphi}^2 \xi \varphi, \Delta_{\eta \varphi}^2 \zeta) = (A \Delta_{\eta \varphi}^2 \xi \varphi, A \Delta_{\eta \varphi}^2 \zeta) = (\xi \varphi, \zeta),
\]
from (25). Therefore, \( \xi \varphi \in D(\Delta_{\eta \varphi}^2) \) and \( \Delta_{\eta \varphi}^2 \xi \varphi = \xi \varphi \). Hence we obtain
\[
\Delta_{\eta \varphi}^2 \xi \varphi = \xi \varphi.
\]
(26)

We then obtain
\[
(\eta - \varphi)(s(\varphi)) = 0,
\]
i.e., the support of \( \eta - \varphi \) is orthogonal to the support of \( \varphi \).

Conversely, if the support of \( \eta - \varphi \) is orthogonal to \( \varphi \), then we have
\[
\| \Delta_{\eta \varphi}^2 \xi \varphi \|^2 = \| \Delta_{\eta \varphi}^2 \xi \varphi \|^2 + \| \Delta_{\varphi \varphi}^2 \xi \varphi \|^2 = \| \xi \varphi \|^2.
\]
(27)

\[\square\]

**Lemma A.5** For all normal positive linear functionals \( \psi_1, \psi_2 \) over a von Neumann algebra \( \mathcal{M} \), and \( 0 \leq s \leq 1 \),
\[
\left\| \Delta_{\psi_1, \psi_2}^2 \xi \psi_2 \right\| = \left\| \Delta_{\psi_2, \psi_1}^2 \xi \psi_1 \right\|.
\]
(28)

**Proof** Functions \( F(z) := \left( \Delta_{\psi_1, \psi_2}^2 \xi \psi_2, \Delta_{\psi_2, \psi_1}^2 \xi \psi_1 \right) \) and \( G(z) := \left( \Delta_{\psi_1, \psi_1}^2 \xi \psi_1, \Delta_{\psi_2, \psi_2}^2 \xi \psi_2 \right) \) are bounded continuous on \( 0 \leq \Re z \leq 1 \) and analytic on \( 0 < \Re z < 1 \). It is easy to check \( F(it) = G(it) \) for \( t \in \mathbb{R} \). Hence we obtain \( F(z) = G(z) \) on \( 0 \leq \Re z \leq 1. \)

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