Centers for the restricted category $\mathcal{O}$ at the critical level over affine Kac–Moody algebras

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Abstract The restricted category $\mathcal{O}$ at the critical level over an affine Kac–Moody algebra is a certain subcategory of the ordinary BGG-category $\mathcal{O}$. We study a deformed version introduced by Arakawa and Fiebig and calculate the center of the deformed restricted category $\mathcal{O}$. This complements a result of Fiebig which describes the center of the non-restricted category $\mathcal{O}$ outside the critical hyperplanes over a symmetrizable Kac–Moody algebra.

Keywords Affine Kac–Moody algebras · Critical level · Restricted category $\mathcal{O}$ · Representation theory

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1 Introduction

The first step of Fiebig’s proof of the Kazhdan–Lusztig conjecture for symmetrizable Kac–Moody algebras outside the critical hyperplanes was to calculate the center of a non-critical block of the deformed category $\mathcal{O}$ (cf. [3]). It turned out that it is isomorphic to the structure algebra of the moment graph associated to this block. Since the moment graph picture gives a description of the equivariant cohomology of the flag manifold corresponding to the Langlands dual Lie algebra, one gets a connection to geometry to prove the Kazhdan–Lusztig conjecture.

In the present paper we use analogous arguments as in [3] to describe centers of an appropriate categorical framework for the critical level representations over an affine Kac–Moody algebra. For this, let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a simple Lie algebra with a Borel and a Cartan
where $x$ to get the following: The center of the category $O$ as a transformation from the functor $(\otimes L(\delta))^n$ to the identity functor, where $L(\delta)$ is the one-dimensional simple $\widehat{g}$-module with highest weight the smallest positive imaginary root. If $n \neq 0$, this action is only non-trivial on the subcategory $O_c \subset O$ of modules with critical level. In [2] the authors introduce the restricted category $\widehat{O}_c$ which consists of those modules $M \in O_c$ on which $Z_n$ acts trivially for all $n \in \mathbb{Z} \setminus \{0\}$. Furthermore, they describe the categorical structure of the subgeneric blocks of $\widehat{O}_c$.

Now let $\Lambda \subset \widehat{\mathfrak{h}}^*$ be a subset that parameterises the highest weights of simple modules in a block of $O_c$. Denote this block by $\widehat{O}_c^{\Lambda}$ and let $R(\Lambda)$ be the subset of roots corresponding to $(\mathfrak{g}, \mathfrak{h})$ which are integral with respect to $\Lambda$. In this paper we work with a relative version of $\widehat{O}_c^{\Lambda}$. For this, let $\widehat{S}$ be the localization of the symmetric algebra over $\mathfrak{h}$. In chapter 2, we define the deformed block $\widehat{O}_{\Lambda}^{\mathfrak{S}, \Lambda}$ as a relative version of $\widehat{O}_c^{\Lambda}$. We use the main results of [2] to get the following: The center of the category $\widehat{O}_{\Lambda}^{\mathfrak{S}, \Lambda}$ is isomorphic to the $\widehat{S}$-module

$$\left\{ (z_\mu)_{\mu \in \Lambda} \in \prod_{\mu \in \Lambda} \mathfrak{S} \middle| z_\mu \equiv z_{\alpha \downarrow \mu} \pmod{\alpha^\vee} \forall \alpha \in R(\Lambda) \right\}$$

where $\alpha \downarrow : \Lambda \to \Lambda$ is a certain bijection on the set $\Lambda$, which we define in chapter 3.5, and $\alpha^\vee \in \mathfrak{h}$ denotes the coroot of the root $\alpha$.

In this sense, this paper should be seen as an addendum to the first part of Fiebig’s article [3]. But unfortunately our result does not yet have a geometric counterpart as the non-critical situation of [3] does.

## 2 Preliminaries

### 2.1 Affine Kac–Moody algebras

Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be a finite dimensional, complex, simple Lie algebra with a Borel and a Cartan subalgebra and denote by $\mathcal{L}(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ the loop algebra. The Killing form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ allows us to take a central extension $\widehat{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathcal{K}$ of the loop algebra, where $\mathcal{K}$ is a central element. Adding an outer derivation operator $D = t \frac{\partial}{\partial t}$ to $\widehat{\mathfrak{g}}$ we get the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$ corresponding to $\mathfrak{g}$. As a vector space we have $\widehat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]) \oplus \mathcal{K} \oplus \mathcal{D}$ and the Lie bracket is given by:

$$[x \otimes t^n, y \otimes t^m] = [x, y] + n \delta_{m, -n} \kappa(x, y) \mathcal{K},$$

$$[K, \widehat{\mathfrak{g}}] = \{0\},$$

$$[D, x \otimes t^n] = nx \otimes t^n,$$

where $x, y \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$. The corresponding affine Cartan subalgebra is given by $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathcal{K} \oplus \mathcal{D}$ and the affine Borel subalgebra by $\widehat{\mathfrak{b}} = (\mathfrak{g} \otimes \mathbb{C}[t] + \mathfrak{b} \otimes \mathbb{C}[t]) \oplus \mathcal{K} \oplus \mathcal{D}$.

Denote by $R \supset R^+$ the root system with positive roots corresponding to the triple $(\mathfrak{g}, \mathfrak{b}, \mathfrak{h})$. The projection $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathcal{K} \oplus \mathcal{D} \to \mathfrak{h}$ induces an embedding $\mathfrak{h}^* \hookrightarrow \widehat{\mathfrak{h}}^*$. Let $\delta \in \widehat{\mathfrak{h}}^*$ be the imaginary root defined by $\delta(\mathfrak{h} \oplus \mathcal{K}) = \{0\}$ and $\delta(\mathcal{D}) = 1$. The inclusion $\mathfrak{h}^* \hookrightarrow \widehat{\mathfrak{h}}^*$ yields an embedding of $R$ into the affine root system of $(\widehat{\mathfrak{g}}, \widehat{\mathfrak{h}})$ which we denote by $\widehat{R}$. It is given by $\widehat{R} = \widehat{R}^{re} \cup \widehat{R}^{im}$ where

\[ \square \]
\[ \hat{R}^{\text{re}} = \{ \alpha + n\delta \mid \alpha \in R \subset \hat{R}, n \in \mathbb{Z} \}, \]
\[ \hat{R}^{\text{im}} = \{ n\delta \mid n \in \mathbb{Z}, n \neq 0 \}. \]

Let \( \theta \in R^+ \) be the highest root and denote by \( \Pi \subset R^+ \) the set of simple roots. The set of affine simple roots \( \hat{\Pi} \subset \hat{R} \) is then given by \( \hat{\Pi} = \Pi \cup \{-\theta + \delta\} \) and the set of positive affine roots by \( \hat{R}^+ = R^+ \cup \{ \alpha + n\delta \mid \alpha \in R, n > 0 \} \cup \{ n\delta | n > 0 \} \). For \( \lambda, \mu \in \hat{h}^* \) we set \( \mu \leq \lambda \) if \( \lambda - \mu \) can be written as a sum of positive affine roots. This defines a partial ordering on \( \hat{h}^* \).

We denote by \( \mathcal{W} \subset \text{Gl}(\hat{h}^*) \) the Weyl group of the finite dimensional Lie algebra \( \mathfrak{g} \) and by \( \hat{\mathcal{W}} \subset \text{Gl}(\hat{h}^*) \) the affine Weyl group of \( \mathfrak{g} \). For any real root \( \alpha \in \hat{R}^{\text{re}} \) the root space \( \mathfrak{g}_\alpha \) is one dimensional and we can uniquely define the coroot \( \check{\alpha}^\vee \in \check{h} \) by the properties \( \check{\alpha}^\vee \in [\hat{g}_\alpha, \hat{g}_{-\alpha}] \) and \( \langle \alpha, \check{\alpha}^\vee \rangle = 2 \), where \( \langle \cdot, \cdot \rangle : \hat{h}^* \times \hat{h} \rightarrow \mathbb{C} \) is the natural pairing. Let \( \rho \in \hat{h}^* \) be an element with \( \rho(\alpha^\vee) = 1 \) for all coroots \( \alpha^\vee \) associated to a simple affine root \( \alpha \in \hat{\Pi} \).

The \( \rho \)-shifted dot-action of \( \hat{\mathcal{W}} \) on \( \hat{h}^* \) is now given by
\[ w \cdot \lambda = w(\lambda + \rho) - \rho \]
where \( w \in \hat{\mathcal{W}} \) and \( \lambda \in \hat{h}^* \). Although \( \rho \) is not uniquely defined, the dot-action is independent of the choice of \( \rho \).

### 2.2 The deformed category \( \mathcal{O} \)

Denote by \( S = S(\mathfrak{h}) \) the symmetric algebra over the complex vector space \( \mathfrak{h} \) and by \( \hat{S} = S(\hat{h}) \) the symmetric algebra over \( \hat{h} \). Every \( S \)-algebra \( A \) with structure map \( \tau : S \rightarrow A \) carries an \( \hat{S} \)-structure by composing \( \tau \) with the map \( \hat{S} \rightarrow S \) induced by \( \mathfrak{h}^* \subset \hat{h}^* \) from above. We call a commutative, associative, noetherian and unital \( S \)-algebra \( A \) a deformation algebra.

For a Lie algebra \( \mathfrak{a} \) and a deformation algebra \( A \) we define \( \mathfrak{a}_A := \mathfrak{a} \otimes_\mathbb{C} A \). We can compose the structure morphism \( \tau : S \rightarrow A \) to get a map \( \hat{\mathfrak{g}}_A \hookrightarrow S(\hat{\mathfrak{h}}) \rightarrow S(\mathfrak{h}) \rightarrow A \). We can extend this map to an \( A \)-linear map \( \tau : \hat{\mathfrak{g}}_A \rightarrow A \) which we denote by \( \tau \) again and call it the canonical weight corresponding to \( A \). We can identify the \( A \)-dual space of \( \hat{\mathfrak{g}}_A \) with \( \hat{\mathfrak{h}}^* \otimes_\mathbb{C} A \) and use this identification to view every weight \( \lambda \in \hat{h}^* \) as an element in \( \hat{\mathfrak{g}}_A^\vee \). This implies \( \tau(D \otimes 1) = \tau(K \otimes 1) = 0 \).

For a \( \hat{\mathfrak{g}}_A \)-module \( M \) and a weight \( \lambda \in \hat{h}^* \) we define the deformed \( \lambda \)-weight space
\[ M_\lambda := \{ v \in M \mid Hv = (\lambda(H) + \tau(H))v \forall H \in \hat{h} \}. \]

We call \( M \) a weight module if it decomposes into the direct sum of its deformed weight spaces.

**Definition 1** The deformed category \( \mathcal{O}_A \) is the full subcategory of \( \hat{\mathfrak{g}}_A \)-mod consisting of all modules \( M \) which are
1. weight modules and
2. locally \( \mathfrak{b}_A \)-finite, i.e., for any \( m \in M \) the submodule \( \mathfrak{b}_Am \) is finitely generated over \( A \).

The deformed Verma module with highest weight \( \lambda \in \hat{h}^* \) is an element of \( \mathcal{O}_A \) and is defined by
\[ \Delta_A(\lambda) := U(\hat{\mathfrak{g}}_A) \otimes_{U(\hat{\mathfrak{h}}_A)} A_\lambda \]
where \( A_\lambda \) is \( A \) as an \( A \)-module and the left action of \( \hat{\mathfrak{h}} \) is given by the composition \( \hat{\mathfrak{h}} \rightarrow \hat{\mathfrak{h}}^{\check{\lambda} + \tau} \rightarrow A \). For a homomorphism \( A \rightarrow A' \) of \( S \)-algebras we get a functor
\[ \cdot \otimes_A A' : \mathcal{O}_A \rightarrow \mathcal{O}_{A'} \]
which maps \( \Delta_A(\lambda) \) to \( \Delta_{A'}(\lambda) \).
Lemma 1 ([3], Proposition 2.1) Let $A$ be a local deformation algebra with residue field $\mathbb{K}$. Then the base change functor $\cdot \otimes_A \mathbb{K}$ induces a bijection between isomorphism classes of simple objects in $\mathcal{O}_A$ and the ones in $\mathcal{O}_\mathbb{K}$.

This lemma implies that the simple modules of $\mathcal{O}_A$ and $\mathcal{O}_\mathbb{K}$ are parameterized by $\hat{\mathfrak{h}}^*$ (cf. [2], section 2.8). We denote the simple module in $\mathcal{O}_A$ (resp.) with highest weight $\lambda \in \hat{\mathfrak{h}}^*$ by $L_A(\lambda)$ (resp.).

In contrast to the category $\mathcal{O}$ over the simple Lie algebra $\mathfrak{g}$ we do not have projective covers of simple objects in the category $\mathcal{O}$ over $\hat{\mathfrak{g}}$ in general. However, in certain truncated subcategories projective covers do exist.

Definition 2
1. A subset $J \subset \hat{\mathfrak{h}}^*$ is called open, if for all $\lambda \in J$ and $\mu \in \hat{\mathfrak{h}}^*$ with $\mu \leq \lambda$ we have $\mu \in J$.
2. We call a subset $J \subset \hat{\mathfrak{h}}^*$ bounded from above if for any $\lambda \in J$ the set $J \cap \{ \geq \lambda \} = \{ \mu \in J \mid \mu \geq \lambda \}$ is finite.

Definition 3 Let $J \subset \hat{\mathfrak{h}}^*$ be open and $M \in \mathcal{O}_A$.

1. We define $M^J := M/(U(\hat{\mathfrak{g}}_A) \oplus_{\lambda \in \hat{\mathfrak{h}}^* \setminus J} M_\lambda)$.
2. We define $\mathcal{O}_A^J \subset \mathcal{O}_A$ to be the full subcategory of objects $M$ with $M = M^J$.

It is easy to see that $M \mapsto M^J$ induces a functor $\mathcal{O}_A \rightarrow \mathcal{O}_A^J$ which is left adjoint to the inclusion functor $\mathcal{O}_A^J \subset \mathcal{O}_A$. Furthermore, $\Delta_A(\lambda)$ and $L_A(\lambda)$ are contained in $\mathcal{O}_A^J$ if and only if $\lambda \in J$.

Theorem 1 ([2], Theorem 2.8) Let $J$ be a bounded, open subset of $\hat{\mathfrak{h}}^*$ and $A$ a local deformation algebra with residue field $\mathbb{K}$.

1. For each $\lambda \in J$ there exists a projective cover $P_A^J(\lambda)$ of $L_A(\lambda)$ in $\mathcal{O}_A^J$. It admits a Verma flag, i.e., a filtration with layers isomorphic to deformed Verma modules, and we have

$$
(P_A^J(\lambda) : \Delta_A(\mu)) = \begin{cases} 
[\Delta_\mathbb{K}(\mu) : L_\mathbb{K}(\lambda)] & \text{if } \mu \in J, \\
0 & \text{otherwise}
\end{cases}
$$

where the left hand side denotes the multiplicity of $\Delta_A(\mu)$ in a Verma flag and the right hand side the multiplicity of $L_\mathbb{K}(\lambda)$ as a simple subquotient of $\Delta_\mathbb{K}(\mu)$.
2. If $J' \subset J$ is open as well, then $P_A^J(\lambda) \cong P_A^{J'}(\lambda)$.
3. If $A \rightarrow A'$ is a homomorphism of local deformation algebras and $P \in \mathcal{O}_A^J$ is projective, then $P \otimes_A A'$ is projective in $\mathcal{O}_{A'}^J$.
4. We have $P_A^J(\lambda) \otimes_A \mathbb{K} \cong P_{A'}^J(\lambda)$.
5. Let $P$ be a finitely generated projective object in $\mathcal{O}_A^J$ and $A \rightarrow A'$ be a homomorphism of local deformation algebras. For any $M \in \mathcal{O}_A^J$ the natural map

$$
\text{Hom}_{\mathcal{O}_A}(P, M) \otimes_A A' \rightarrow \text{Hom}_{\mathcal{O}_{A'}}(P \otimes_A A', M \otimes_A A')
$$

is an isomorphism.
For $A$ a local deformation algebra with residue field $\mathbb{K}$, we introduce an equivalence relation on $\widehat{h}^*$. Let $\lambda, \mu \in \widehat{h}^*$. We call $\mu A$-larger than $\lambda$ if $[\Delta_{\mathbb{K}}(\mu) : L_{\mathbb{K}}(\lambda)] \neq 0$. Then $\sim_A$ is defined to be the equivalence relation on $\widehat{h}^*$ generated by pairs $(\lambda, \mu)$, s.t. $\mu$ is $A$-larger than $\lambda$. Let $\Lambda \in \widehat{h}^*/\sim_A$. $\mathcal{O}_{A,A}$ is defined to be the full subcategory of $\mathcal{O}_A$ that consists of all objects $M$ with the property that any simple subquotient of $M$ has highest weight in $\Lambda$.

**Theorem 2** ([3], Proposition 2.8) *The functor*

$$\bigoplus_{\Lambda \in \widehat{h}^*/\sim_A} \mathcal{O}_{A,A} \rightarrow \mathcal{O}_A$$

(\(M_A\) \mapsto \bigoplus M_A)

*is an equivalence of categories.*

**3 Restricted representations**

The main purpose of this chapter is to introduce the restricted deformed category $\mathcal{O}$. It is studied in [2] and is a natural categorical framework for highest weight modules at the critical level.

**3.1 The critical level**

Let $M \in \mathcal{O}_A$ and $\lambda \in \widehat{h}^*$. The central element $K \in \widehat{g}$ acts on the weight space $M_k$ by $\lambda(K) \in \mathbb{C}$. We denote by $M_k$ the eigenspace of the action of $K$ with eigenvalue $k \in \mathbb{C}$. If $M = M_k$ we call $M$ a module of level $k$. The category $\mathcal{O}_A$ decomposes into the direct sum of subcategories of modules with equal level, i.e., $\mathcal{O}_A = \bigoplus_{k \in \mathbb{C}} \mathcal{O}_{A,k}$, where $\mathcal{O}_{A,k}$ consists of those modules which are of level $k$.

Let $(\cdot|\cdot) : \widehat{g} \times \widehat{g} \rightarrow \mathbb{C}$ be the non-degenerate, symmetric and invariant bilinear form on $\widehat{g}$ which is induced by the Killing form on $g$. Its restriction to $\widehat{h} \times \widehat{h}$ is non-degenerate as well and induces a non-degenerate bilinear form $(\cdot|\cdot)$ on $\widehat{h}^*$. For $A$ a local deformation algebra with canonical weight $\tau : \widehat{h}_A \rightarrow A$, we can extend it to $(\cdot|\cdot)_A : \widehat{h}^*_A \times \widehat{h}^*_A \rightarrow A$, the $A$-linear continuation of $(\cdot|\cdot)$. Recall the element $\rho \in \widehat{h}^*$ with $\rho(\alpha^\vee) = 1$ for any simple root $\alpha \in \widehat{h}$. Note that for the imaginary root $\delta \in \widehat{R}$ we have $(\delta|\delta) = 0$.

**Definition 4** For $\lambda \in \widehat{h}^*$ we define the set of **integral roots** (with respect to $\lambda$ and $A$) by

$$\widehat{R}_A(\lambda) := \{ \alpha \in \widehat{R} \mid 2(\lambda + \rho + \tau|\alpha)_A \in \mathbb{Z}(\alpha|\alpha)_A \}.$$ 

The corresponding **integral Weyl group** is defined by

$$\widehat{W}_A(\lambda) := \langle \sigma_\alpha \mid \alpha \in \widehat{R}_A(\lambda) \cap \widehat{R}_{\text{rc}} \rangle \subset \widehat{W}.$$ 

For an equivalence class $\Lambda \in \widehat{h}^*/\sim_A$ and $\mu, \lambda \in \Lambda$, we have $\lambda(K) = \mu(K)$ and also $\widehat{R}_A(\lambda) = \widehat{R}_A(\mu)$. The complex number $\text{crit} := -\rho(K)$ is called **critical level**.

**Lemma 2** ([1], Lemma 4.2) *For $\Lambda \in \widehat{h}^*/\sim_A$ the following are equivalent.*

1. $\Lambda$ is critical, i.e., $\lambda(K) = \text{crit}$ for all $\lambda \in \Lambda$.
2. We have $\lambda + \delta \in \Lambda$ for all $\lambda \in \Lambda$.
3. We have $n\delta \in \widehat{R}_A(\lambda)$ for all $n \neq 0$ and all $\lambda \in \Lambda$. 

\(\text{Springer}\)
3.2 Restricted category $\mathcal{O}$

We recall the action of the Feigin–Frenkel center on $\mathcal{O}_A$ as explained in [1] and [2]. Let $A$ be a local deformation algebra with residue field $\mathbb{K}$. In chapter 3.1 of [2] the authors introduce an equivalence

$$T : \mathcal{O}_A \rightarrow \mathcal{O}_A$$

defined by $T := \cdot \otimes_{C} L(\delta)$ where $L(\delta)$ is the one dimensional simple $\hat{g}$-module with highest weight $\delta$. The inverse functor of $T$ is given by $T^{-1} = \cdot \otimes_{C} L(-\delta)$. Denote by $T^n$ the $n$-fold composition of $T$.

**Lemma 3** ([2], Chapter 3.1) A block $\mathcal{O}_{A,A}$ with $A \in \hat{h}^*/_{\sim} A$ is preserved by the functor $T$ if and only if $A$ is of level $k(A) = \text{crit}$.

Let $V^{\text{crit}}(g)$ be the universal affine vertex algebra associated with $g$ at the critical level and denote by $z$ its center. On any $M \in \mathcal{O}_{A,\text{crit}}$, $z$ induces the structure of a graded module over the polynomial ring (of infinite rank)

$$Z := \bigoplus_{n \in \mathbb{Z}} Z_n = \mathbb{C}\left[ p_s^{(i)} \mid i = 1, \ldots, \text{rk}_g, s \in \mathbb{Z} \right].$$

Any $z \in Z_n$ for $n \in \mathbb{Z}$ acts on $\mathcal{O}_A$ as a natural transformation from $T^n$ to the identity functor on $\mathcal{O}_A$. Thus, the action of $z$ on $M$ is given by a homomorphism $z^M : T^n M \rightarrow M$. This action respects base change, i.e., for $A \rightarrow A'$ a homomorphism of deformation algebras, the base change functor $\cdot \otimes_A A' : \mathcal{O}_{A,\text{crit}} \rightarrow \mathcal{O}_{A',\text{crit}}$ induces a natural isomorphism $z^M \otimes_A A' \cong z^M \otimes_A \text{id}_{A'}$.

**Definition 5** Let $M \in \mathcal{O}_{A,\text{crit}}$. We call $M$ restricted if for all $n \neq 0$ and all $z \in Z_n$, the homomorphism $z^M : T^n M \rightarrow M$ is zero. Then the restricted deformed category $\overline{\mathcal{O}}_{A,\text{crit}}$ is the full subcategory of $\mathcal{O}_{A,\text{crit}}$ of all restricted modules.

For $\mathcal{J} \subset \hat{h}^*$ open and bounded let $\overline{\mathcal{O}}_{A,\text{crit}}^\mathcal{J} := \overline{\mathcal{O}}_{A,\text{crit}} \cap \mathcal{O}_{A^\mathcal{J}}^\mathcal{J}$. Denote by $Z_n M$ the submodule of $M$ generated by the subset $\{ z^M(m) \mid m \in T^n M, z \in Z_n \} \subset M$. The functor $(\cdot)^{\text{res}} : \mathcal{O}_{A,\text{crit}} \rightarrow \overline{\mathcal{O}}_{A,\text{crit}}$ given by

$$M \rightarrow M^{\text{res}} := M \left/ \sum_{n \in \mathbb{Z}\{0\}} Z_n M \right.$$ 

is well defined and left adjoint to the inclusion functor $\overline{\mathcal{O}}_{A,\text{crit}} \subset \mathcal{O}_{A,\text{crit}}$.

**Definition 6** Let $\lambda \in \hat{h}^*$ be of critical level. The restricted Verma module is defined by

$$\Delta_A(\lambda) := \Delta^\text{res}_A(\lambda).$$

We collect some results which can be found in [2] and which we need for the calculation of the center of a critical restricted block.

**Lemma 4** Restricted Verma modules respect base change, i.e., for any homomorphism $A \rightarrow A'$ of deformation algebras we have $\Delta_A(\lambda) \otimes_A A' \cong \Delta_{A'}(\lambda)$.

**Proof** The claim is obvious for non-restricted Verma modules. Since the action of the center respects base change it is also true for restricted Verma modules. \(\square\)
Lemma 5 ([2], Lemmas 3.8, 3.3, 3.4) Let \( \lambda \in \hat{h}^* \) be critical, \( \mathcal{J} \subset \hat{h}^* \) open, \( A \to A' \) a homomorphism of local deformation algebras and \( M \in \mathcal{O}_A \). Then
1. for any \( \mu \in \hat{h}^* \) and \( \kappa \) the residue field of \( A \) the weight space \( \Delta_A(\lambda)_\mu \) is a free \( A \)-module with
\[
\text{rk}_A \Delta_A(\lambda)_\mu = \dim \kappa \Delta_\kappa(\lambda)_\mu,
\]
2. \( (M^\text{res})^{\mathcal{J}} \cong (M^{\mathcal{J}})^\text{res} \),
3. the canonical map
\[
(M \otimes_A A')^\text{res} \to M^\text{res} \otimes_A A'
\]
is an isomorphism.

Remark 1 Note that in [2], Lemma 3.4, the isomorphism in (3) of the above lemma is formulated as
\[
(M \otimes_A A')^\text{res} \sim (M^\text{res} \otimes_A A')^\text{res}.
\]
But since \( M^\text{res} \otimes_A A' \) is an object of \( \mathcal{O}_{A'} \), we do not have to restrict it anymore.

3.3 Restricted projective objects

We recall the construction of projective covers from [2]. Let \( A \) be a local deformation algebra with residue field \( \kappa \), \( \mathcal{J} \subset \hat{h}^* \) an open, bounded subset and let \( \lambda \in \mathcal{J} \) be of critical level. Recall the projective cover \( P^\mathcal{J}_A(\lambda) \to L_A(\lambda) \). Define \( P^\mathcal{J}_A(\lambda) := P^\mathcal{J}_A(\lambda)^\text{res} \). We say a module \( M \in \mathcal{O}_{A,\text{crit}} \) has a restricted Verma flag, if it has a finite filtration with subquotients isomorphic to restricted Verma modules. As in the non-restricted case, we denote by \( (M : \Delta_A(\lambda)) \) the multiplicity of \( \Delta_A(\lambda) \) as a subquotient in a restricted Verma flag of \( M \).

Theorem 3 ([2], Theorem 4.9, and [4], Theorem 4.3) Let \( A \) be a local deformation algebra with residue field \( \kappa \), \( \mathcal{J} \) an open and bounded subset of \( \hat{h}^* \) and let \( \mu \in \mathcal{J} \) be of critical level. Then \( P^\mathcal{J}_A(\mu) \) admits a restricted Verma flag and we have
\[
(P^\mathcal{J}_A(\mu) : \Delta_A(\lambda)) = \begin{cases} 
[\Delta_\kappa(\lambda) : L_\kappa(\mu)] & \text{if } \lambda \in \mathcal{J} \\
0 & \text{otherwise}
\end{cases}
\]
for all \( \lambda \in \hat{h}^* \). Furthermore, \( P^\mathcal{J}_A(\mu) \) is a projective cover of \( L_A(\mu) \) in \( \mathcal{O}^\mathcal{J}_A \).

For the method of proof we are using for calculating the center of a restricted critical block, the following lemma will be quite important.

Lemma 6 Let \( A \to A' \) be a homomorphism of local deformation algebras. Let \( M, \overline{P} \in \mathcal{O}^\mathcal{J}_A \) and \( \overline{P} \) be a finitely generated projective object. Then the canonical map
\[
\text{Hom}_{\mathcal{O}_A}(\overline{P}, M) \otimes_A A' \longrightarrow \text{Hom}_{\mathcal{O}_{A'}}(\overline{P} \otimes_A A', M \otimes_A A')
\]
is an isomorphism of \( A' \)-modules.

Proof By Theorem 3 we find a finitely generated projective object \( P \in \mathcal{O}^\mathcal{J}_A \), s.t. \( P^\text{res} \cong \overline{P} \). Since \((\cdot)^\text{res}\) is left adjoint to the inclusion functor \( \mathcal{O}^\mathcal{J}_A \hookrightarrow \mathcal{O}^\mathcal{J}_A \) and since \( M \otimes_A A' \) is an object of \( \mathcal{O}_{A'} \), we get
\[
\text{Hom}_{\mathcal{O}_A}(P^\text{res}, M) \otimes_A A' \sim \text{Hom}_{\mathcal{O}_A}(P, M) \otimes_A A'
\]
and

\[ \text{Hom}_{\mathcal{O}_A'} \left( (P \otimes_A A')^\text{res}, M \otimes_A A' \right) \xrightarrow{\sim} \text{Hom}_{\mathcal{O}_A'} \left( P \otimes_A A', M \otimes_A A' \right) \]

By Lemma 5, we have \((P \otimes_A A')^\text{res} \cong P^\text{res} \otimes_A A'\). Thus, the claim follows from Theorem 1 (5).

**Remark 2** The Feigin–Frenkel conjecture claims that for \(\lambda, \mu \in J\) the multiplicities \((P_{J A}(\mu) : \Delta_{A}(\lambda))\) are given by certain periodic Kazhdan–Lusztig polynomials evaluated at one. These periodic polynomials depend on the relative position between \(\lambda\) and \(\mu\), and if \(\lambda\) and \(\mu\) are “far away” from each other these polynomials are zero. Thus, if the Feigin–Frenkel conjecture was true, the projective cover \(P_{J A}(\mu)\) would stabilize for \(J\) big enough and \(P_{J A}(\mu)\) would also be a projective cover of \(L_{A}(\mu)\) in the bigger non-truncated category \(\mathcal{O}_A\).

### 3.4 The restricted block decomposition

We set

\[
\hat{h}^*_{\text{crit}} := \{ \lambda \in \hat{h}^* | \lambda(K) = \text{crit} \}.
\]

We define an equivalence relation \(\sim_{\text{res}}^{\mathcal{A}}\) on the critical hyperplane \(\hat{h}^*_\text{crit}\). Let \(\lambda, \mu \in \hat{h}^*_\text{crit}\). We say, \(\mu\) is restricted \(\mathcal{A}\)-larger than \(\lambda\) if \(L_{A}(\lambda)\) appears as a subquotient of \(P_{\mathcal{J} A}(\mu)\) for \(\mathcal{J} \subset \hat{h}^*\) big enough. \(\sim_{\text{res}}^{\mathcal{A}}\) is then defined to be the equivalence relation on \(\hat{h}^*_\text{crit}\) that is generated by pairs \((\lambda, \mu)\), s.t. \(\mu\) is restricted \(\mathcal{A}\)-larger than \(\lambda\). For an equivalence class \(\Lambda \in \hat{h}^*_\text{crit} \sim_{\text{res}}^{\mathcal{A}}\) let \(\mathcal{O}_{A, \Lambda} \subset \mathcal{O}_{A, \text{crit}}\) be the full subcategory of objects \(M\), such that \([M : L_{A}(\lambda)] \neq 0\) implies \(\lambda \in \Lambda\).

**Theorem 4** ([2], Theorem 5.2) The functor

\[
\bigoplus_{\Lambda \in \hat{h}^*_\text{crit} \sim_{\text{res}}^{\mathcal{A}}} \mathcal{O}_{A, \Lambda} \longrightarrow \mathcal{O}_{A, \text{crit}}
\]

\((M_{\Lambda}) \mapsto \bigoplus M_{\Lambda}\)

is an equivalence of categories.

We want to recall a more detailed description of the restricted critical blocks. Denote by \(\gamma : \hat{h}^* \rightarrow h^*, \lambda \mapsto \overline{\lambda}\) the projection with respect to the decomposition \(\hat{h} = h \oplus CD \oplus CK\) and denote by \(\overline{A} \subset h^*\) the image of a subset \(A \subset \hat{h}^*\).

For \(A \in \hat{h}^*_\text{crit} \sim_{\text{res}}^{\mathcal{A}}\) we define the **finite integral root system** (with respect to \(A\) and \(A\)) by

\[
R_{A}(A) := \{ \alpha \in R | 2(\lambda + \rho + \tau | \alpha)_{A} \in \mathbb{Z}(\alpha | \alpha)_{A} \text{ for all } \lambda \in \Lambda \}
\]

and the **finite integral Weyl group** by

\[
\mathcal{W}_{A}(A) := \{ s_{\alpha} | \alpha \in R_{A}(A) \} \subset \mathcal{W}.
\]

**Lemma 7** ([2], Lemma 5.3) Let \(\Lambda \in \hat{h}^*_\text{crit} \sim_{\text{res}}^{\mathcal{A}}\) be a critical restricted equivalence class. Then, for all \(\lambda \in \Lambda\)

\[
\overline{A} = \mathcal{W}_{A}(A) \cdot \overline{\lambda}.
\]
3.5 The generic and subgeneric cases

For the calculation of the center we will need the description of the generic and subgeneric equivalence classes (cf. [1]). In the rest of this paper, $\tilde{S}$ will be the localization of $S = S(\mathfrak{h})$ at the maximal ideal generated by $\mathfrak{h}$. If $p \subset \tilde{S}$ is a prime ideal, we denote by $S_p$ the localization of $\tilde{S}$ at $p$.

Now let $A$ be such a localization of $\tilde{S}$. We need some more notation: Let $\Lambda \in \hat{h}^\ast_{\text{crit}}/\sim_{\text{res}}^A$. For a root $\alpha \in R_A(\Lambda)$ and $\lambda \in \Lambda$ we define $\alpha \downarrow \lambda$ (resp. $\alpha \uparrow \lambda$) to be the element in the set $\{s_\alpha \cdot \lambda, s_{-\alpha} + \delta \cdot \lambda\}$ which is smaller (resp. larger) than or equal to $\lambda$. We then define inductively $\alpha \downarrow^n \lambda := \alpha \downarrow (\alpha \downarrow^{n-1} \lambda)$ and $\alpha \uparrow^n \lambda := \alpha \uparrow (\alpha \uparrow^{n-1} \lambda)$.

**Definition 7** Let $\Lambda \in \hat{h}^\ast_{\text{crit}}/\sim_{\text{res}}^A$ be a critical restricted equivalence class. We call $\Lambda$

1. **generic** if $\Lambda \subset h^\ast$ consists of one element,
2. **subgeneric** if $\Lambda \subset h^\ast$ consists of two elements.

**Lemma 8** ([2], Lemma 5.5) Let $p \subset \tilde{S}$ be a prime ideal of height one and let $\Lambda \subset \hat{h}^\ast_{\text{crit}}$ be an equivalence class for $\sim_{\text{res}}^S$.

1. If $\alpha \not\in p$ for all $\alpha \in R$, then $\Lambda$ is generic.
2. If $\alpha \not\in p$ for some $\alpha \in R$, then $R_{p}(\Lambda) \subset \{\alpha, -\alpha\}$ and $\Lambda$ is either generic or subgeneric.

We finish this chapter with the main results of [2].

**Theorem 5** ([2], Theorem 5.6) Let $\Lambda \in \hat{h}^\ast_{\text{crit}}/\sim_{\text{res}}^A$, $\lambda \in \Lambda$ and $\mathcal{J} \subset \hat{h}^\ast_{\text{crit}}$ an open and bounded subset.

1. Suppose that $\Lambda$ is generic. Then $\Lambda$ consists of one element and

$$P^\mathcal{J}_A(\lambda) \cong \Delta_A(\lambda)$$

if $\lambda \in \mathcal{J}$.

2. Suppose that $\Lambda$ is subgeneric and that $\Lambda = \{\lambda, s_\alpha \cdot \lambda\}$ for some $\alpha \in R$. Then there is a non-split short exact sequence

$$0 \to \Delta_A(\alpha \uparrow \lambda) \to P^\mathcal{J}_A(\lambda) \to \Delta_A(\lambda) \to 0$$

and a short exact sequence

$$0 \to L_A(\alpha \downarrow \lambda) \to \Delta_A(\lambda) \to L_A(\lambda) \to 0$$

if $\mathcal{J}$ contains $\lambda$ and $\alpha \uparrow \lambda$.

3. If $\Lambda$ is subgeneric with $R_A(\Lambda) = \{\pm \alpha\}$ and $\lambda \in \Lambda$, we have

$$\Lambda = \{\ldots, \alpha \downarrow^2 \lambda, \alpha \downarrow \lambda, \lambda, \alpha \uparrow \lambda, \alpha \uparrow^2 \lambda, \ldots\}.$$

The second short exact sequence in (2) follows from the first one and BGGH-reciprocity.

4 The center of $\mathcal{O}$ for $\Lambda$ critical

Recall that the center of a category is the ring of endo-transformations of the identity functor. Let $A$ be a deformation algebra and fix a critical equivalence class $\Lambda \in \hat{h}^\ast_{\text{crit}}/\sim_{\text{res}}^A$. For an
open, bounded subset \( \mathcal{J} \subset \mathfrak{h}^* \) denote by \( Z_A(\Lambda, \mathcal{J}) \) the center of \( \mathcal{O}_A^{\mathcal{J}} \) and by \( Z_A(\Lambda) \) the center of \( \mathcal{O}_{A, \Lambda} \). We first consider the case \( \Lambda = \tilde{S} \), the localization of \( S \) at \( S_{\mathfrak{h}} \). Thus, we have

\[
Z_{\tilde{S}}(\Lambda, \mathcal{J}) := Z \left( \mathcal{O}_A^{\mathcal{J}} \right) = \text{End} \left( \text{id}_{\mathcal{O}_A^{\mathcal{J}}} \right).
\]

We want to describe the center by observing its action on projective objects. Since in general we only have enough projective objects in the truncated categories, we have to express the center of \( \mathcal{O}_{A, \Lambda} \) as a limit of the centers \( Z_{\mathcal{A}}(\Lambda, \mathcal{J}) \) which runs over open and bounded subsets \( \mathcal{J} \subset \mathfrak{h}^* \). The main result is then

**Theorem 6** Let \( \tilde{S} \) be the localization of \( S \) at the maximal ideal generated by \( \mathfrak{h} \) and \( \Lambda \in \mathfrak{h}^*/\sim_{\mathfrak{h}} \). Then we have an isomorphism of \( \tilde{S} \)-modules

\[
Z_{\tilde{S}}(\Lambda) \cong \left\{ (z_\mu)_{\mu \in \Lambda} \in \prod_{\mu \in \Lambda} \tilde{S} : z_\mu \equiv z_{\alpha \uparrow \mu} \pmod{\alpha \downarrow} \forall \alpha \in R_{\tilde{S}}(\Lambda) \right\}.
\]

We use Lemma 6 and a localization process to split the problem into generic and subgeneric situations. Following [3], we first relate the center to the endomorphism rings of a generating set of restricted projective objects.

**Remark 3** Let \( A \) be a localization of \( \tilde{S} \) at a prime ideal \( \mathfrak{p} \subset \tilde{S} \) and \( \Gamma \) an equivalence class under \( \sim_{\mathfrak{p}} \). The block \( \mathcal{O}_{A, \Gamma} \) is generated by the set of projective covers \( \{ P_{A}^{\mathcal{J}}(\lambda) \}_{\lambda \in \Gamma \cap \mathcal{J}} \). By the same arguments as given in [3], chapter 3.1, we get that evaluating on indecomposable projective objects induces an injective map

\[
Z_A(\Gamma, \mathcal{J}) \hookrightarrow \prod_{\mu \in \Gamma \cap \mathcal{J}} \text{End}_{\mathcal{O}_A} \left( P_{A}^{\mathcal{J}}(\mu) \right)
\]

and the image of this map is given by the subset

\[
\left\{ (z_\mu)_{\mu \in \Gamma \cap \mathcal{J}} \in \prod_{\mu \in \Gamma \cap \mathcal{J}} \text{End}_{\mathcal{O}_A} \left( P_{A}^{\mathcal{J}}(\mu) \right) : z_\mu \circ f = f \circ z_\lambda \forall \lambda, \mu \in \Gamma \cap \mathcal{J}, f \in \text{Hom}_{\mathcal{O}_A} \left( P_{A}^{\mathcal{J}}(\lambda), P_{A}^{\mathcal{J}}(\mu) \right) \right\}.
\]

Let \( A \rightarrow A' \) be a homomorphism of local deformation algebras. The base change, Lemma 6, for the endomorphism rings of restricted projective objects induces a map

\[
Z_A(\Gamma, \mathcal{J}) \rightarrow Z_{A'}(\Gamma, \mathcal{J}).
\]

For another open, bounded subset \( \mathcal{J}' \subset \mathfrak{h}^* \) with \( \mathcal{J} \subset \mathcal{J}' \) the restriction of the identity functor to the subcategory \( \mathcal{O}_{A, \Gamma}^{\mathcal{J}} \subset \mathcal{O}_{A, \Gamma}' \) induces a map \( Z_A(\Gamma, \mathcal{J}') \rightarrow Z_A(\Gamma, \mathcal{J}) \). This defines a directed system. Since for every finitely generated module \( M \) of \( \mathcal{O}_{A, \Gamma}^{\mathcal{J}} \) there exists an open and bounded subset such that \( M \) lies in \( \mathcal{O}_{A, \Gamma}^{\mathcal{J}} \), and since the center is already uniquely defined by its action on the finitely generated objects, we have

\[
Z_A(\Gamma) \cong \lim_{\leftarrow} Z_A(\Gamma, \mathcal{J}).
\]

The base change maps \( Z_A(\Gamma, \mathcal{J}) \rightarrow Z_{A'}(\Gamma, \mathcal{J}) \) then induce a base change map \( Z_A(\Gamma) \rightarrow Z_{A'}(\Gamma) \), by the universal property of inverse limits.
Lemma 9 Let $A$ be a localization of $\tilde{S}$ at a prime ideal $p \subset \tilde{S}$. The evaluation on restricted Verma modules induces an injective map

$$Z_A(\Gamma) \hookrightarrow \prod_{\mu \in \Gamma} \text{End}_{\Sigma_A}(\overline{\Delta}_A(\mu)) \cong \prod_{\mu \in \Gamma} A.$$ 

Proof Let $Q := Q(A)$ be the quotient field of $A$. Then by Theorem 5 (1), all Verma modules $\overline{\Delta}_Q(\lambda)$ are projective and we have $\text{Hom}_{\Sigma_Q}(\overline{\Delta}_Q(\mu), \overline{\Delta}_Q(\lambda)) = 0$ for $\mu \neq \lambda$. But by the description of the center from above and by base change for the center we get a commutative diagram

$$Z_A(\Gamma) \longrightarrow \prod_{\mu \in \Gamma} \text{End}_{\Sigma_A}(\overline{\Delta}_A(\mu)) \cong \prod_{\mu \in \Gamma} A$$

$$\downarrow$$

$$Z_Q(\Gamma) \sim \longrightarrow \prod_{\mu \in \Gamma} \text{End}_{\Sigma_Q}(\overline{\Delta}_Q(\mu)) \cong \prod_{\mu \in \Gamma} Q$$

where the lower horizontal is an isomorphism and the verticals are injective. But then the upper horizontal is injective as well. 

Remark 4 We want to describe the image of the map

$$Z_{\tilde{S}}(\Gamma) \hookrightarrow \prod_{\mu \in \Gamma} \tilde{S}.$$ 

The strategy to do this is to localize all appearing modules at prime ideals $p$ of height one. So let $p \subset \tilde{S}$ be such an ideal. If $\alpha^\lor \notin p$ for all $\alpha \in R_{\tilde{S}}(\Gamma)$, all $\overline{S}_p(\mu)$ are projective and we get $Z_{\tilde{S}}(\Gamma) \sim \prod_{\mu \in \Gamma} S_p$ which is the generic situation. We will deal with the subgeneric case in the next chapter.

4.1 The subgeneric case

Let $S_\alpha$ be the localization of $\tilde{S}$ at the prime ideal generated by $\alpha^\lor$. We fix an equivalence class $\Gamma \subset \hat{h}^*_\text{crit}$ under $\sim_{\text{res}}$ which is not generic. For $\lambda \in \Gamma$ we get, by Lemma 8 and Theorem 5 (3), $\Gamma = \hat{W}_\alpha \cdot \lambda = \{\ldots, \alpha \downarrow \lambda, \lambda, \alpha \uparrow \lambda, \alpha \uparrow^2 \lambda, \ldots\}$, where $\hat{W}_\alpha \subset \hat{W}$ is the affine subgroup generated by the reflections $s_{\alpha + n \delta}$ with $n \in \mathbb{Z}$. Recall that by Theorem 5 we have a short exact sequence

$$\overline{\Delta}_{S_\alpha}(\alpha \uparrow \lambda) \hookrightarrow \overline{P}_{S_\alpha}(\lambda) \twoheadrightarrow \overline{\Delta}_{S_\alpha}(\lambda)$$

if $\alpha \uparrow \lambda$, $\lambda \in \mathcal{J}$. This implies that for $\mathcal{J}' \supset \mathcal{J}$ we have $\overline{P}_{S_\alpha}(\lambda) \cong \overline{P}_{S_\alpha}(\mathcal{J}')$ in $\overline{S}_\alpha$. Thus, for any $\mu \in \Gamma$ we will always assume that the corresponding open, bounded subset $\mathcal{J}_\mu$ is big enough, such that we can write $\overline{P}_{S_\alpha}(\mu) = \overline{P}_{S_\alpha}(\mathcal{J}_\mu)$.

Lemma 10 Restriction to the restricted Verma module $\overline{\Delta}_{S_\alpha}(\alpha \uparrow \mu) \rightarrow \overline{P}_{S_\alpha}(\mu)$ for every $\mu \in \Gamma$ induces a surjective map

$$\prod_{\mu \in \Gamma} \text{End}_{\Sigma_{S_\alpha}}(\overline{P}_{S_\alpha}(\mu)) \twoheadrightarrow \prod_{\mu \in \Gamma} \text{End}_{\Sigma_{S_\alpha}}(\overline{\Delta}_{S_\alpha}(\mu))$$

$$(f_\mu) \mapsto (f_\mu|_{\overline{\Delta}_{S_\alpha}(\alpha \uparrow \mu)}).$$

Proof Since $\text{End}_{\Sigma_{S_\alpha}}(\overline{\Delta}_{S_\alpha}(\mu)) = S_\alpha \cdot \text{id}_{\overline{\Delta}_{S_\alpha}(\mu)}$, every endomorphism of the restricted Verma module lifts to an endomorphism of $\overline{P}_{S_\alpha}(\alpha \downarrow \mu)$. 

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Identifying $\prod_{\mu \in \Gamma} \text{End}_{\mathcal{G}_{S_a}}(\overline{\Delta}_{S_a}(\mu))$ with $\prod_{\mu \in \Gamma} S_a$ and using the naturality of the action of the center, we get a commutative diagram

$$
\mathcal{Z}_{S_a}(\Gamma) \xrightarrow{\cdot} \prod_{\mu \in \Gamma} S_a
\downarrow \quad \downarrow
\prod_{\mu \in \Gamma} \text{End}_{\mathcal{G}_{S_a}}(\overline{P}_{S_a}(\mu)).
$$

The aim is now to describe the image of the composition of the down- with the up-going arrow in this diagram.

**Proposition 1** Let $\lambda, \mu \in \Gamma$, then $\text{Hom}_{\mathcal{G}_{S_a}}(\overline{P}_{S_a}(\mu), \overline{P}_{S_a}(\lambda))$ is a free $S_a$-module and we have

$$
\text{rk}_{S_a} \text{Hom}_{\mathcal{G}_{S_a}}(\overline{P}_{S_a}(\mu), \overline{P}_{S_a}(\lambda)) = \begin{cases} 
1 & \text{if } \mu = \alpha \downarrow \lambda \text{ or } \mu = \alpha \uparrow \lambda, \\
2 & \text{if } \mu = \lambda, \\
0 & \text{otherwise}.
\end{cases}
$$

**Proof** By our assumption, $\Gamma$ is subgeneric, i.e., it contains exactly two elements. $\text{Hom}_{\mathcal{G}_{S_a}}(\overline{P}_{S_a}(\mu), \overline{P}_{S_a}(\lambda))$ is a finitely generated $S_a$-module. Let $\mathbb{K}$ be the residue field and $Q$ the quotient field of $S_a$. By Lemma 6 we get

$$
\text{Hom}_{\mathcal{G}_{S_a}}(\overline{P}_{S_a}(\mu), \overline{P}_{S_a}(\lambda)) \otimes_{S_a} \mathbb{K} \cong \text{Hom}_{\mathcal{G}_{\mathbb{K}}}(\overline{P}_{\mathbb{K}}(\mu), \overline{P}_{\mathbb{K}}(\lambda))
$$

and

$$
\text{Hom}_{\mathcal{G}_{S_a}}(\overline{P}_{S_a}(\mu), \overline{P}_{S_a}(\lambda)) \otimes_{S_a} Q \cong \text{Hom}_{\mathcal{G}_{Q}}(\overline{P}_{Q}(\mu), \overline{P}_{Q}(\lambda)).
$$

But by [5], chapter 6.2, the dimension of the right hand side of the first equality is given by

$$
\text{dim}_{\mathbb{K}} \text{Hom}_{\mathcal{G}_{\mathbb{K}}}(\overline{P}_{\mathbb{K}}(\mu), \overline{P}_{\mathbb{K}}(\lambda)) = \begin{cases} 
1 & \text{if } \mu = \alpha \downarrow \lambda \text{ or } \mu = \alpha \uparrow \lambda, \\
2 & \text{if } \mu = \lambda, \\
0 & \text{otherwise}.
\end{cases}
$$

Since we have a decomposition $\overline{P}_{S_a}(\mu) \otimes_{S_a} Q \cong \overline{\Delta}_Q(\alpha \uparrow \mu) \oplus \overline{\Delta}_Q(\mu)$ and $\overline{P}_{S_a}(\lambda) \otimes_{S_a} Q \cong \overline{\Delta}_Q(\alpha \downarrow \lambda) \oplus \overline{\Delta}_Q(\lambda)$ into simple Verma modules we also get

$$
\text{dim}_{Q} \left( \text{Hom}_{\mathcal{G}_{S_a}}(\overline{P}_{S_a}(\mu), \overline{P}_{S_a}(\lambda)) \otimes_{S_a} Q \right) = \begin{cases} 
1 & \text{if } \mu = \alpha \downarrow \lambda \text{ or } \mu = \alpha \uparrow \lambda, \\
2 & \text{if } \mu = \lambda, \\
0 & \text{otherwise}.
\end{cases}
$$

But then by [2], Lemma 3.7 (involving Nakayama’s Lemma), we get that $\text{Hom}_{\mathcal{G}_{S_a}}(\overline{P}_{S_a}(\mu), \overline{P}_{S_a}(\lambda))$ is a free $S_a$-module with

$$
\text{rk}_{S_a} \text{Hom}_{\mathcal{G}_{S_a}}(\overline{P}_{S_a}(\mu), \overline{P}_{S_a}(\lambda)) = \begin{cases} 
1 & \text{if } \mu = \alpha \downarrow \lambda \text{ or } \mu = \alpha \uparrow \lambda, \\
2 & \text{if } \mu = \lambda, \\
0 & \text{otherwise}.
\end{cases}
$$

This finishes the proof. \(\Box\)
For fixed $\lambda \in \Gamma$ we want to give bases for the following $S_\alpha$-modules:

\begin{align*}
\text{End}_{\mathcal{O}_S}(P_{S_\alpha}(\lambda)), \\
\text{Hom}_{\mathcal{O}_S}(P_{S_\alpha}(\lambda), P_{S_\alpha}(\alpha \uparrow \lambda)), \\
\text{Hom}_{\mathcal{O}_S}(P_{S_\alpha}(\lambda), P_{S_\alpha}(\alpha \downarrow \lambda)).
\end{align*}

We follow the notation of [5]. Clearly, we can take the identity in $\text{End}_{\mathcal{O}_S}(P_{S_\alpha}(\lambda))$ as the first basis element. Over the residue field $K$ we have a composition $P_K(\lambda) \rightarrow \Delta_K(\lambda) \rightarrow \Delta_K(\alpha \uparrow \lambda)$.

We can lift this composition via base change to a map $P_{S_\alpha}(\lambda) \rightarrow \Delta_{S_{\alpha}}(\alpha \uparrow \lambda)$ which is unequal to 0. Composing this map with the inclusion $\Delta_{S_{\alpha}}(\alpha \uparrow \lambda) \hookrightarrow P_{S_\alpha}(\lambda)$ yields an endomorphism $n_{\lambda} : P_{S_\alpha}(\lambda) \rightarrow P_{S_\alpha}(\lambda)$ which is unequal to 0 and after applying $\cdot \otimes S_{\alpha} \mathbb{K}$ corresponds to the composition $n^K_{\lambda} : P_K(\lambda) \rightarrow \Delta_K(\lambda) \rightarrow \Delta_K(\alpha \uparrow \lambda) \rightarrow \Delta_{S_{\alpha}}(\alpha \uparrow \lambda).$

Since the identity and $n_{\lambda}$ are linearly independent, $\{\text{id}, n_{\lambda}\} \subset \text{End}_{\mathcal{O}_S}(P_{S_\alpha}(\lambda))$ is a basis.

Taking the map $P_{S_\alpha}(\lambda) \rightarrow \Delta_{S_{\alpha}}(\alpha \uparrow \lambda)$ from above and the projectivity of $P_{S_\alpha}(\lambda)$ we get a morphism $b_{\lambda} : P_{S_\alpha}(\lambda) \rightarrow P_{S_\alpha}(\alpha \uparrow \lambda)$ as the dotted arrow from the following diagram

\[ P_{S_\alpha}(\lambda) \rightarrow \Delta_{S_{\alpha}}(\alpha \uparrow \lambda). \]

This morphism is unequal to 0 after applying $\cdot \otimes S_{\alpha} \mathbb{K}$. We conclude that $b_{\lambda}$ is a basis of $\text{Hom}_{\mathcal{O}_S}(P_{S_\alpha}(\lambda), P_{S_\alpha}(\alpha \uparrow \lambda))$.

Finally, we have the composition $a_{\lambda} : P_{S_\alpha}(\lambda) \rightarrow \Delta_{S_{\alpha}}(\lambda) \hookrightarrow P_{S_\alpha}(\alpha \downarrow \lambda)$ which is unequal to 0 after base change $\cdot \otimes S_{\alpha} \mathbb{K}$ with the residue field. Thus it is a basis of $\text{Hom}_{\mathcal{O}_S}(P_{S_\alpha}(\lambda), P_{S_\alpha}(\alpha \downarrow \lambda))$.

**Remark 5** Since $\alpha \uparrow \lambda \geq \lambda$, the endomorphism $n_{\lambda} : P_{S_\alpha}(\lambda) \rightarrow P_{S_\alpha}(\lambda)$ restricts to an endomorphism $\Delta_{S_{\alpha}}(\alpha \uparrow \lambda) \rightarrow \Delta_{S_{\alpha}}(\alpha \uparrow \lambda)$ and thus induces an endomorphism on the cokernel $\Delta_{S_{\alpha}}(\alpha \uparrow \lambda)$ of the embedding $\Delta_{S_{\alpha}}(\alpha \uparrow \lambda) \hookrightarrow P_{S_\alpha}(\lambda)$.

**Proposition 2** Up to an invertible element of $S_\alpha$, $n_{\lambda}$ induces the map $\alpha \vee \cdot \text{id}$ on $\Delta_{S_{\alpha}}(\alpha \uparrow \lambda)$ and the zero map on $\Delta_{S_{\alpha}}(\lambda)$.

Before we prove this proposition we repeat a result of [8] about the Jantzen filtration on $\Delta_{K}(\lambda)$.

The Shapovalov form for Kac–Moody algebras introduced in [6] induces a contravariant form $(\cdot, \cdot)_{S_{\alpha}}$ on $\Delta_{S_{\alpha}}(\mu)$ for any $\mu \in \Gamma$. We define a filtration on $\Delta_{S_{\alpha}}(\mu)$ by setting

\[ \overline{\Delta}_{S_{\alpha}}(\mu)^i := \left\{ m \in \Delta_{S_{\alpha}}(\mu) \mid (m, \Delta_{S_{\alpha}}(\mu))_{S_{\alpha}} \in (\alpha \vee)^i \cdot S_{\alpha} \right\} . \]
We then get the Jantzen filtration on $\Delta_\mathbb{K}(\mu)$ by

$$\Delta_\mathbb{K}(\mu)^i := \text{im} \left( \Delta_{S_\alpha}(\mu)^i \hookrightarrow \Delta_{S_\mu}(\mu) \twoheadrightarrow \Delta_\mathbb{K}(\mu) \right)$$

where $\Delta_{S_\alpha}(\mu) \twoheadrightarrow \Delta_\mathbb{K}(\mu)$ is the map induced by $\cdot \otimes S_\alpha \mathbb{K}$.

**Lemma 11** ([8], Proposition 5.6) The Jantzen filtration on $\Delta_\mathbb{K}(\mu)$ is given by

$$\Delta_\mathbb{K}(\mu) \supset \mathcal{L}_{(\alpha \downarrow \mu)} \supset 0.$$ 

**Proof** We give a proof which is adapted from [7], chapter 5.14, and slightly different to the one given in [8].

Since, by construction, $\Delta_\mathbb{K}(\mu)^1$ coincides with the maximal submodule of $\Delta_\mathbb{K}(\mu)$, we get, by Theorem 5, that $\Delta_\mathbb{K}(\mu)^1 \cong \mathcal{L}_\mu(\alpha \downarrow \mu)$. We have to prove $\Delta_\mathbb{K}(\mu)^2 = 0$. Let $m \in \Delta_\mathbb{K}(\mu)^1$ be a generator of highest weight $\alpha \downarrow \mu$. We have to prove $m \not\in \Delta_\mathbb{K}(\mu)^2$.

Let us assume $m \in \Delta_\mathbb{K}(\mu)^2$. Thus, there is an element $m' \in \Delta_{S_\mu}(\mu)^2$ such that $m' \mapsto m$ under specialization $\Delta_{S_\mu}(\mu) \twoheadrightarrow \Delta_\mathbb{K}(\mu)$. Since for $\nu > \alpha \downarrow \mu$ we have $\Delta_\mathbb{K}(\mu)^1 \nu = 0$, we conclude

$$\Delta_{S_\mu}(\mu)^1 \subset (\alpha \lor) \cdot \Delta_{S_\mu}(\mu) \nu$$ 

and

$$\Delta_{S_\mu}(\mu)^2 \subset (\alpha \lor)^2 \cdot \Delta_{S_\mu}(\mu) \nu.$$ 

The generalized Casimir operator $C$ can be split into a sum $C = C_1 + C_2$, where $C_1(\Delta_{S_\mu}(\mu) \nu) \subset \bigoplus_{\eta \lor \nu} \Delta_{S_\mu}(\mu) \eta$, $C_2(\Delta_{S_\mu}(\mu) \nu) \subset \Delta_{S_\mu}(\mu) \nu$ and $C_2$ acts on $\Delta_{S_\mu}(\mu) \nu$ by multiplication with $(\nu + \tau + \rho | \mu + \tau + \rho)_{S_\mu} - (\rho | \rho)_{S_\mu} \in S_\alpha$. Since $C$ commutes with the $\hat{\mathcal{W}}$-action, we get that $C$ acts on $\Delta_{S_\mu}(\mu)$ by multiplication with $(\mu + \tau + \rho | \mu + \tau + \rho)_{S_\mu} - (\rho | \rho)_{S_\mu}$. Thus, applying $C$ to $m'$ yields

$$C m' = ((\mu + \tau + \rho | \mu + \tau + \rho)_{S_\mu} - (\rho | \rho)_{S_\mu}) m'$$

on the one hand, but also

$$C m' \in ((\alpha \downarrow \mu + \tau + \rho | \alpha \downarrow \mu + \tau + \rho)_{S_\mu} - (\rho | \rho)_{S_\mu}) m' + (\alpha \lor)^2 \cdot \Delta_{S_\mu}(\mu).$$

Therefore, by $\hat{\mathcal{W}}$-invariance of $(\cdot | \cdot)_{S_\mu}$ and a little calculation we get

$$(\mu - \alpha \downarrow \mu | \tau) m' \in (\alpha \lor)^2 \cdot \Delta_{S_\mu}(\mu).$$

Since $\mu - \alpha \downarrow \mu$ is either equal to $n\alpha$ or $n(-\alpha + \delta)$ for $n > 0$ an integer and since $(\delta | \tau)_{S_\mu} = 0$, we get $(\mu - \alpha \downarrow \mu | \tau)_{S_\mu} = k\alpha \lor$ with $k \in \mathbb{C}\setminus\{0\}$. But then specializing $\alpha \lor \mapsto 0$ yields $km = 0$ which is a contradiction. Thus $m \not\in \Delta_\mathbb{K}(\mu)^2$. 

**Proof** (Proof of Proposition 2) Over the residue field $\mathbb{K}$, we get the following diagram of short exact sequences in the horizontals.
where the composition \( n^\mathbb{K}_\lambda \) in the middle is induced by \( n_\lambda \) and \( x, y \in \mathbb{K} \). Since the composition \( \overline{\Delta}_\mathbb{K}(\alpha \uparrow \lambda) \hookrightarrow \overline{P}_\mathbb{K}(\lambda) \rightarrow \overline{\Delta}_\mathbb{K}(\lambda) \) is zero, we get that both scalars, \( x \) and \( y \), are zero.

Over \( S_\alpha \) the composition

\[
\overline{P}_S(\lambda) \rightarrow \overline{\Delta}_S(\alpha \uparrow \lambda) \hookrightarrow \overline{P}_S(\lambda) \rightarrow \overline{\Delta}_S(\lambda)
\]

is zero, so \( n_\lambda \) induces the zero map on \( \overline{\Delta}_S(\lambda) \). But the composition \( \overline{\Delta}_S(\alpha \uparrow \lambda) \hookrightarrow \overline{P}_S(\lambda) \rightarrow \overline{\Delta}_S(\alpha \uparrow \lambda) \) is unequal to zero since otherwise we would have a factorization over the cokernel, in formulas

\[
\overline{\Delta}_S(\alpha \uparrow \lambda) \rightarrow \overline{P}_S(\lambda) \rightarrow \overline{\Delta}_S(\alpha \uparrow \lambda)
\]

But \( \overline{\Delta}_S(\lambda) \rightarrow \overline{\Delta}_S(\alpha \uparrow \lambda) \) is the zero map, because it is zero after applying \( \cdot \otimes_{S_\alpha} \mathbb{Q} \), while \( \overline{P}_S(\lambda) \rightarrow \overline{\Delta}_S(\alpha \uparrow \lambda) \) is unequal to zero.

Now, after multiplying \( n_\lambda \) with an appropriate invertible element of \( S_\alpha \), \( n_\lambda \) induces \( (\alpha \lor \cdot \mathrm{id}) \) on \( \overline{\Delta}_S(\alpha \uparrow \lambda) \) and 0 on \( \overline{\Delta}_S(\lambda) \).

**Lemma 12**

1. We have \( n_\lambda \circ a_{\alpha \uparrow \lambda} = a_{\alpha \uparrow \lambda} \circ (\alpha \lor \cdot \mathrm{id} - n_{\alpha \uparrow \lambda}) \).
2. We have \( (\alpha \lor \cdot \mathrm{id} - n_{\alpha \uparrow \lambda}) \circ b_\lambda = b_\lambda \circ n_\lambda \).

**Proof**

1. This part of the lemma is clear by definition of the map \( n_\lambda \) and the effect it has on restricted Verma modules.
2. Applying $\otimes_{S_\alpha} Q$ to the diagram

\[
\begin{array}{ccc}
\overline{P}_{S_\alpha}(\lambda) & \xrightarrow{b_\lambda} & \overline{P}_{S_\alpha}(\alpha \uparrow \lambda) \\
\downarrow n_\lambda & & \downarrow \alpha^\vee \cdot \text{id} - n_{\alpha \uparrow \lambda} \\
\overline{P}_{S_\alpha}(\lambda) & \xrightarrow{b_\lambda} & \overline{P}_{S_\alpha}(\alpha \uparrow \lambda)
\end{array}
\]

the resulting diagram identifies with

\[
\begin{array}{ccc}
\overline{\Delta}_Q(\lambda) \oplus \overline{\Delta}_Q(\alpha \uparrow \lambda) & \xrightarrow{g_1} & \overline{\Delta}_Q(\alpha \uparrow 2 \lambda) \oplus \overline{\Delta}_Q(\alpha \uparrow \lambda) \\
\downarrow g_2 & & \downarrow g_2 \\
\overline{\Delta}_Q(\lambda) \oplus \overline{\Delta}_Q(\alpha \uparrow \lambda) & \xrightarrow{g_1} & \overline{\Delta}_Q(\alpha \uparrow 2 \lambda) \oplus \overline{\Delta}_Q(\alpha \uparrow \lambda)
\end{array}
\]

where $g_1, g_2$ are both given by the matrix

\[
\begin{pmatrix}
0 & 0 \\
0 & (\alpha^\vee)
\end{pmatrix}
\]

But this diagram certainly commutes. Then the diagram above over $S_\alpha$ commutes as well.

By the description of the center in Remark 3, and by what we have discovered above, the image of the inclusion

\[
\mathcal{Z}_{S_\alpha}(\Gamma) \hookrightarrow \prod_{\lambda \in \Gamma} \text{End}_{\overline{S}_\alpha}(\overline{P}_{S_\alpha}(\lambda))
\]

is generated by the tuples $(\text{id}_{\overline{P}_{S_\alpha}(\mu)})_{\mu \in \Gamma}$ and the set of tuples $\{((\delta^\mu_\lambda))_{\mu \in \Gamma})_{\lambda \in \Gamma}$ where

\[
\delta^\mu_\lambda = \begin{cases}
  n_\lambda & \text{if } \mu = \lambda, \\
  \alpha^\vee \cdot \text{id} - n_{\alpha \uparrow \lambda} & \text{if } \mu = \alpha \uparrow \lambda, \\
  0 & \text{otherwise}.
\end{cases}
\]

But the images of these generators under the map

\[
\phi : \prod_{\mu \in \Gamma} \text{End}_{\overline{S}_\alpha}(\overline{P}_{S_\alpha}(\mu)) \rightarrow \prod_{\mu \in \Gamma} \text{End}_{\overline{S}_\alpha}(\overline{\Delta}_S(\mu)) \cong \prod_{\mu \in \Gamma} S_\alpha
\]

are $\phi((\text{id}_{\overline{P}_{S_\alpha}(\mu)})_{\mu \in \Gamma}) = (1)_{\mu \in \Gamma}$ and $\phi((\delta^\mu_\lambda))_{\mu \in \Gamma}) = (\kappa^\mu_\lambda)_{\mu \in \Gamma}$ with

\[
\kappa^\mu_\lambda = \begin{cases}
  \alpha^\vee & \text{if } \mu = \alpha \uparrow \lambda , \\
  0 & \text{else}
\end{cases}
\]

As a conclusion we have

**Proposition 3**

\[
\mathcal{Z}_{S_\alpha}(\Gamma) \cong \left\{(z_{\mu})_{\mu \in \Gamma} \in \prod_{\mu \in \Gamma} S_\alpha \mid z_{\mu} \equiv z_{\alpha \uparrow \mu} \quad (\text{mod } \alpha^\vee) \right\}.
\]
4.2 The general case

In this chapter we want to collect our results in the subgeneric and generic cases to prove the main theorem. Let $\Lambda$ be an equivalence class under $\sim^{\text{res}}_{\tilde{S}}$.

**Theorem 7**

$$Z_{\tilde{S}}(\Lambda) \cong \left\{ (z_{\mu})_{\mu \in \Lambda} \in \prod_{\mu \in \Lambda} \tilde{S} \mid z_{\mu} \equiv z_{\alpha \downarrow \mu} \pmod{\alpha} \forall \alpha \in R_{\tilde{S}}(\Lambda) \right\}.$$  

**Proof** For $p \subset \tilde{S}$ a prime ideal of height one and for $Q = \text{Quot}(\tilde{S})$, we have a base change map

$$Z_{\tilde{S}}(\Lambda) \hookrightarrow Z_{\tilde{S}}(\Lambda) \otimes_{\tilde{S}} S_p \subset Z_Q(\Lambda) \cong \prod_{\mu \in \Lambda} Q.$$  

Note that $\Lambda$ splits into a disjoint union $\bigcup \Lambda_i$ of generic and subgeneric $S_p$-blocks and we get a decomposition

$$Z_{\tilde{S}}(\Lambda) \otimes_{\tilde{S}} S_p \cong \prod_{i} Z_{S_p}(\Lambda_i).$$  

We also have

$$Z_{\tilde{S}}(\Lambda) = \bigcap_{p \in \mathfrak{P}} \prod_{i} Z_{S_p}(\Lambda_i)$$

in the $Q$-vector space $\prod_{\mu \in \Lambda} Q$. Here, $\mathfrak{P}$ denotes the set of all prime ideals of $\tilde{S}$ of height one. If $\alpha^\vee \not\in p$ for all $\alpha \in R_{\tilde{S}}(\Lambda)$, then all restricted deformed Verma modules are projective and we have $Z_{\tilde{S}}(\Lambda) \otimes_{\tilde{S}} S_p \cong \prod_{\mu \in \Lambda} S_p$.

If $p$ is generated by $\alpha^\vee$ for $\alpha \in R_{\tilde{S}}(\Lambda)$, $Z_{\tilde{S}}(\Lambda) \otimes_{\tilde{S}} S_p$ decomposes into the product of modules of the form described in Proposition 3 and in modules of the form $\prod_{\mu \in \Lambda_i} S_p$, if $\Lambda_i$ is generic. Putting these together in the above intersection proves the claim.  

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