DISTANCE SETS OF WELL-DISTRIBUTED PLANAR SETS FOR POLYGONAL NORMS

SERGEI KONYAGIN AND IZABELLA LABA

Abstract. Let $X$ be a 2-dimensional normed space, and let $BX$ be the unit ball in $X$. We discuss the question of how large the set of extremal points of $BX$ may be if $X$ contains a well-distributed set whose distance set $\Delta$ satisfies the estimate $|\Delta \cap [0, N]| \leq C N^{3/2 - \epsilon}$. We also give a necessary and sufficient condition for the existence of a well-distributed set with $|\Delta \cap [0, N]| \leq C N$.

§0. INTRODUCTION

The classical Erdős Distance Problem asks for the smallest possible cardinality of

$$\Delta(A) = \Delta_{l_2^2}(A) = \left\{ \|a - a'\|_{l_2^2} : a, a' \in A \right\}$$

if $A \subset \mathbb{R}^2$ has cardinality $N < \infty$ and

$$\|x\|_{l_2^\infty} = \sqrt{x_1^2 + x_2^2}$$

is the Euclidean distance between the points $a$ and $a'$. Erdős conjectured that $|\Delta(A)| \gg N/\sqrt{\log N}$ for $N \geq 2$. (We write $U \ll V$, or $V \gg U$, if the functions $U, V$ satisfy the inequality $|U| \leq CV$, where $C$ is a constant which may depend on some specified parameters). The best known result to date in two dimensions is due to Katz and Tardos who prove in [KT04] that $|\Delta(A)| \gg N^{0.864}$ improving an earlier breakthrough by Solymosi and Tóth [ST01].

More generally, one can examine an arbitrary two-dimensional space $X$ with the unit ball

$$BX = \{ x \in \mathbb{R}^2 : \|x\|_X \leq 1 \}$$

and define the distance set

$$\Delta_X(A) = \{ \|a - a'\|_X : a, a' \in A \}.$$ 

For example, let

$$\|x\|_{l_\infty^2} = \max(|x_1|, |x_2|)$$

then for $N \geq 1$, $A = \{ m \in \mathbb{Z}^2 : 0 \leq m_1 \leq N^{1/2}, 0 \leq m_2 \leq N^{1/2} \}$ we have $|A| \gg N$, $|\Delta_{l_2^2}(A)| \ll N^{1/2}$. This simple example shows that the Erdős Distance Conjecture can not be directly extended for arbitrary two-dimensional spaces. Erdős [E46] (see also [I01]) proved the estimate $|\Delta_X(A)| \gg N^{1/2}$ for any space $X$. 

Typeset by $\LaTeX$
Also, for a positive integer \(N\) we denote

\[
\Delta_{X,N}(A) = \{\|a - a'\|_X \leq N : a, a' \in A\}.
\]

We say that a set \(S \subset X\) is well-distributed if there is a constant \(K\) such that every closed ball of radius \(K\) in \(X\) contains a point from \(S\). In other words, for every point \(x \in X\) there is a point \(y \in S\) such that \(\|x - y\|_X \leq K\). Sometimes it is said that \(S\) is a \(K\)-net for \(X\). Clearly, for any well-distributed set \(S\) and \(N \geq 2K\) we have

\[
(1) \quad |\{x \in S : \|x\|_X \leq N/2\}| \gg N^2
\]

where the constant in \(\gg\) depends only on \(K\). Therefore, for any well-distributed set \(S \in l^2\) we have, by [T02],

\[
|\Delta_{l^2,N}(S)| \gg N^{1.728},
\]

and the Erdős Distance Conjecture implies for large \(N\)

\[
|\Delta_{l^2,N}(S)| \gg N^2 / \sqrt{\log N}.
\]

On the other hand, for a well-distributed set \(S = \mathbb{Z}^2 \subset l^\infty\) we have

\[
|\Delta_{l^\infty,N}(S)| = 2N + 1.
\]

Iosevich and the second author [IL03] have recently established that a slow growth of \(\Delta_{X,N}(S)\) for a well-distributed set \(S \subset X\) is possible only in the case if \(BX\) is a polygon with finitely or infinitely many sides. Let us discuss possible definitions of polygons with infinitely many sides. For a convex set \(A \subset X\) by \(\text{Ext}(A)\) we denote the set of extremal points of \(A\). Namely, \(x \in \text{Ext}(A)\) if and only if \(x \in A\) and for any segment \([y, z]\) the conditions \(x \in [y, z] \subset A\) imply \(x = y\) or \(x = z\). Clearly, \(\text{Ext}(BX)\) is a closed subset of the unit circle

\[
\partial BX = \{x \in X : \|x\|_X = 1\}.
\]

Also, it is easy to see that \(\text{Ext}(BX)\) is finite if and only if \(BX\) is a polygon with finitely many sides, and it is natural to consider \(BX\) as a polygon with infinitely many sides if \(\text{Ext}(BX)\) is small. There are different ways to define smallness of \(\text{Ext}(BX)\) and, thus, polygons with infinitely many sides:

1) in category: \(\text{Ext}(BX)\) is nowhere dense in \(\partial BX\);
2) in measure: \(\text{Ext}(BX)\) has a zero linear measure (or a small Hausdorff dimension);
3) in cardinality: \(\text{Ext}(BX)\) is at most countable.

Clearly, 3) implies 2) and 2) implies 1).

It has been proved in [IL03] that the condition

\[
(0.1) \quad \lim_{N \to \infty} |\Delta_{X,N}(S)|N^{-3/2} = 0
\]

for a well-distributed set \(S\) implies that \(BX\) is a polygon in a category sense. Following [IL03], we prove that, moreover, \(BX\) is a polygon in a measure sense.
Theorem 1. Let $S$ be a well-distributed set.
(i) Assume that (0.1) holds. Then the one-dimensional Hausdorff measure of $Ext(BX)$ is 0;
(ii) If moreover
\[(0.2) \quad |\Delta_{X,N}(S)| = O(N^{1+\alpha})\]
for some $\alpha \in (0, 1/2)$ then the Hausdorff dimension of $Ext(BX)$ is at most $2\alpha$.

If $|\Delta_{X,N}(S)|$ has an extremally slow rate of growth for some well-distributed set $S$, namely,
\[(0.3) \quad |\Delta_{X,N}(S)| = O(N)\]
then, as it has been proved in [IL03], $BX$ is a polygon with finitely many sides. However, if we weaken (0.3) we cannot claim that $BX$ is a polygon in a cardinality sense.

Theorem 2. Let $\psi(u)$ be a function $(0, \infty) \to (0, \infty)$ such that $\lim_{u \to \infty} \psi(u) = \infty$. Then there exists a 2-dimensional space $X$ and a well-distributed set $S \subset X$ such that
\[(0.4) \quad |\Delta_{X,N}(S)| = o(N\psi(N)) \quad (N \to \infty)\]
but $Ext(BX)$ is a perfect set (and therefore is uncountable).

Also, we find a necessary and sufficient condition for a space $X$ to make (0.3) possible for some well-distributed set $S \subset X$. Take two non-collinear vectors $e_1, e_2$ in $X$. They determine coordinates for any $x \in X$, namely, $x = x_1 e_1 + x_2 e_2$. Then, for any non-degenerate segment $I \subset X$, we can define its slope $SL(I)$: if the line containing $I$ is given by an equation $u_1 x_1 + u_2 x_2 + u_0 = 0$, then we set $SL(I) = -u_1/u_2$. We write $SL(I) = \infty$ if $u_2 = 0$; it will be convenient for us to consider $\infty$ as an algebraic number.

Theorem 3. The following conditions on $X$ are equivalent:
(i) $BX$ is a polygon with finitely many sides, and there is a coordinate system in $X$ such that the slopes of all sides of $BX$ are algebraic;
(ii) there is a well-distributed set $S \subset X$ such that (0.3) holds.

Corollary 1. If a norm $\|\cdot\|_X$ on $\mathbb{R}^2$ is so that $BX$ is a polygon with finitely many sides and all angles between its sides are rational multiples of $\pi$ then there is a well-distributed set $S \subset X$ such that (0.3) holds.

Corollary 2. If a norm $\|\cdot\|_X$ on $\mathbb{R}^2$ is defined by a regular polygon $BX$ then there is a well-distributed set $S \subset X$ such that (0.3) holds.

The Falconer conjecture (for the plane) says that if the Hausdorff dimension of a compact $A \subset \mathbb{R}^2$ is greater than 1 then $\Delta(A)$ has positive Lebesgue measure. The best known result is due to Wolff who proved in [W99] that the distance set has positive Lebesgue measure if the Hausdorff dimension of $A$ is greater than $4/3$. One can ask a similar question for an arbitrary two-dimensional normed space $X$. It turns out that this question is related to distance sets for well-distributed and separated sets. By Theorem 4 from [IL04], Theorem 3 and Proposition 1 we get the following.
Corollary 3. If a norm $\| \cdot \|_X$ on $\mathbb{R}^2$ is defined by a polygon $BX$ with finitely many sides all of which have algebraic slopes then there is a compact $A \subset X$ such that the Hausdorff dimension of $A$ is 2 and Lebesgue measure of $\Delta_X(A)$ is 0.

It would be interesting to know if the result is true without supposition on the slopes of the sides.

Recall that, by [I/suppress L03], it is enough to prove the implication (ii) $\rightarrow$ (i) in Theorem 3 assuming that $BX$ is a polygon. In that case we prove a stronger result.

Theorem 4. Let $BX$ be a polygon with finitely many sides which does not satisfy the condition (i) of Theorem 3. Then for any well-distributed set $S$ we have

$$|\Delta_{X,N}(S)| \gg N \log N / \log \log N \quad (N \geq 3).$$

Comparison of Theorem 4 with Theorem 2 shows that the growth of $|\Delta_{X,N}(S)|$ for well-distributed sets and $N \to \infty$ does not distinguish the spaces $X$ with small and big cardinality of $\text{Ext}(BX)$.

§1. PROOF OF THEOREMS 1 AND 2

Proof of (i). Without loss of generality we may assume that $BX \subset B\mathbb{L}^2$ and the set $S$ is well-distributed in $X$ with the constant $K = 1/2$. Also, choose $\delta > 0$ so that

$$\delta B\mathbb{L}^2 \subset BX.$$ 

By (0.1), for any $\varepsilon > 0$ there are arbitrary large $N_0$ such that

$$|\Delta_{X,N_0}(S)| \leq \varepsilon N_0^{3/2}.$$ 

If $N_0 \geq 8$ then the number of integers $j \geq 0$ with $N_0/2 + 4j \leq N_0 - 2$ is

$$\geq (N_0/2 - 2)/4 \geq N_0/8.$$ 

Thus, there is at least one $j$ such that $N = N_0/2 + 4j$ satisfies the condition

$$|(\Delta_X(S) \cap (N-2,N+2))| \leq 8\varepsilon N_0^{3/2}/N_0 \leq 12\varepsilon N^{1/2}.$$ 

So, (1.2) holds for arbitrary large $N$.

We take any $N$ satisfying (1.2) and an arbitrary $P \in S$. Let $Q$ be the closest point to $P$ in the space $X$ (observe that it exists since $S$ is closed due to (0.1)). Then, by well-distribution of $S$ (recall that $K = 1/2$) we have

$$\|P - Q\|_X \leq 1.$$ 

Without loss of generality, $P = 0$. Denote $M = [2N\delta]$ and consider the rays

$$L_j = \{(r,\theta) : \theta = \theta_j = 2\pi j/M\},$$ 

where $(r,\theta)$ are the polar coordinates in $l^2$. Consider a point $R_j$, $1 \leq j \leq M$, with the polar coordinates $(r_j, (\theta_{j-1} + \theta_j)/2)$ such that $\|R_j\|_X = N$. By (1.1) we have

$$r_j \geq \delta N.$$
Therefore, the Euclidean distance from $R_j$ to the rays $L_{j-1}$ and $L_j$ is
\begin{equation}
(1.4) \quad r_j \sin(\pi/M) \geq N\delta \sin(\pi/(2N\delta)) > 1.
\end{equation}
provided that $N$ is large enough. Therefore, the distance from $R_j$ to these rays in $X$ is also greater than 1. Also, the distance from $R_j$ to the circles
\[ \Gamma_1 = \{ R : \|R\|_X = N - 1 \}, \quad \Gamma_2 = \{ R : \|R\|_X = N + 1 \} \]
in $X$ is equal to 1. Thus, the $X$-disc of radius $1/2$ with the center at $R_j$ is contained in the open region $U_j$ bounded by $L_{j-1}$, $L_j$, $\Gamma_1$, and $\Gamma_2$. By the supposition on $S$ there is a point $P_j \in U_j \cap S$.

Observe that for any $j$ we have
\[ N - 1 < \|P - P_j\|_X < N + 1, \quad N - 2 < \|Q - P_j\|_X < N + 2. \]
Let $U = \{ (\|P - P_j\|_X, \|Q - P_j\|_X) \}$. By (1.2),
\begin{equation}
(1.5) \quad |U| \leq 144\varepsilon^2 N.
\end{equation}
For any $(n_1, n_2) \in U$ we denote
\[ J_{n_1, n_2} = \{ j : \|P - P_j\|_X = n_1, \quad \|Q - P_j\|_X = n_2 \}. \]
By [IL03, Lemma 1.4, (i)], if $j_1, j_2, j_3 \in J_{n_1, n_2}$ then one of the points $P_{j_1}, P_{j_2}, P_{j_3}$ must lie on the segment connecting two other points and contained in the circle $\{ R : \|P - R\|_X = n_1 \}$. This implies that for all $j \in J_{n_1, n_2}$ but at most two indices the intersection of $\partial BX$ with the sector $S_j$ bounded by $L_{j-1}$ and $L_j$ is inside some line segment contained in $\partial BX$. Therefore, by (1.5), the number of sectors $S_j$ containing an extremal point of $BX$ is at most $288\varepsilon^2 N$. For $R \in \partial BX$ with the polar coordinates $(r, \theta)$ denote $\Theta(R) = \theta$. Define the measure on $\partial BX$ in such a way that for any Borel set $V \subset \partial BX$ the measure $\mu_p(V)$ is defined as the Lebesgue measure of $\Theta(V)$. In particular,
\[ \mu_p(\partial BX \cap S_j) = \frac{2\pi}{M}. \]
Clearly, $\mu_p$ is equivalent to the standard Lebesgue measure on $\partial BX$. We have proved that
\[ \mu_p(Ext(BX)) \leq 288\varepsilon^2 N \frac{2\pi}{M}. \]
But $1/M \leq 1/(N\delta)$. Hence,
\[ \mu_p(Ext(BX)) \leq 2\pi \times 288\varepsilon^2 /\delta. \]
As $\varepsilon$ can be chosen arbitrarily small, we get $\mu_p(Ext(BX)) = 0$, and this completes the proof of (i).

Proof of (ii) follows the same scheme. Inequality (1.2) should be replaced by
\[ |\Delta_X(S) \cap (N - 2, N + 2)| \leq \Delta N^\alpha, \]
where ...
where $\Delta$ may depend only on $X$, $S$, and $\alpha$. We define the distance $d_p$ on $\partial BX$ as the distance between the polar coordinates. This metric is equivalent to the $X$-metric. The set $\text{Ext}(BX)$ can be covered by at most $2\Delta^2N^{2\alpha}$ arcs $\partial BX \cap S_j$ each of them has the $d_p$-diameter at most $2\pi/(\Delta\delta)$. This implies the required estimate for the Hausdorff dimension of $\text{Ext}(BX)$.

**Proof of Theorem 2.** We select an increasing sequence $\{N_j\}$ of positive integers such that

$$
\psi(N) \geq 5^j \quad (N \geq N_j).
$$

By $\Lambda_j$ we denote the set of numbers $a/q$ with $a \in \mathbb{Z}$, $q \in \mathbb{N}$, $q \leq N_j$. We will construct a ball $BX$ on the Euclidean plane. Moreover, it will be symmetric with respect to the lines $x_1 = x_2$ and $x_1 = -x_2$, and thus it suffices to construct $BX$ in the quadrant $Q = \{x : x_2 \geq |x_1|\}$.

Let $D_0$ be the square

$$
D_0 = \{x : 0 \leq x_2 + x_1 \leq 1, 0 \leq x_2 - x_1 \leq 1\}.
$$

We will construct a decreasing sequence of polygons $D_j$; each one will be defined as a result of cutting some angles from the previous one. The sides $V_1, V_2$ of $D_0$ with an endpoint at the origin will not be changed. The intersection of the sequence $D_j$ will define the part of our $BX$ in $Q$. In particular, the points $(\pm 1/2, 1/2)$ will be vertices of all polygons $D_j$. Therefore, these points as well as the symmetrical points $(\pm 1/2, -1/2)$ will be in $\partial BX$.

First, we construct $D_1$ as a result of cutting $D_0$ by a line $x_2 = u$ for some $u \in (1/2, 1)$. We choose $u$ such that for intersection points $x^1$ and $x^2$ of this line with the boundary of $D_0$ the ratios $x^1_j/x^2_j (j = 1, 2)$ differs from all numbers $\lambda \in \Lambda_1$. Moreover, we take neighborhoods $U_j$ of the points $x^j (j = 1, 2)$ such that

$$
\forall y \in U_j \quad y_2/y_1 \notin \Lambda_1 \quad (j = 1, 2).
$$

In the sequel we shall make other cuts only inside the sets $U_1$ and $U_2$. This means that all points $x$ on the boundary of $D_1$ with $x_1/x_2 \notin \Lambda_1$ not belonging to the sides $V_1, V_2$ as well as their neighborhoods in the boundary of $D_1$ will remain in all polygons $D_2, D_3, \ldots$, and eventually they will be interior points of some segments in the boundary of $BX$ with a slope $-1, 0, \text{or } 1$.

On the second step, we construct $D_2$ as a result of cutting $D_1$ by lines with slopes $-1/2$ and $1/2$ such that for any new vertex $x$ of a polygon $D_2$ we have $x_2/x_1 \notin \Lambda_2$. Moreover, we take neighborhoods $U(x)$ of all these points $x$ (each is contained in $U_1$ or in $U_2$) such that

$$
\forall y \in U(x) \quad y_2/y_1 \notin \Lambda_2.
$$

Again, we shall make other cuts only inside the sets $U(x)$. This means that all points $x$ on the boundary of $D_2$ with $x_1/x_2 \notin \Lambda_2$ not belonging to the sides $V_1, V_2$ as well as their neighborhoods in the boundary of $D_2$ will remain in all polygons $D_3, D_4, \ldots$, and eventually they will be interior points of some segments in the boundary of $BX$ with a slope $a/2$, $a \in \mathbb{Z}$, $|a| \leq 2$.

Proceeding in the same way, we shall get a ball $BX$ with the following property: if $x \in \partial BX$ and $x_1/x_2 \in \Lambda_j+1$ for some $j$ then $x$ is an interior point of some segment.
contained in \( \partial BX \) with a slope \( a/2^j \), \( a \in \mathbb{Z}, |a| \leq 2^j \). This segment is a part of a line \( 2^j x_2 - ax_1 = b(a, j) \) or a symmetrical line \( 2^j x_2 - ax_1 = -b(a, j) \). Also, by symmetry, if \( x \in \partial BX \) and \( x_2/x_1 = \Lambda_{j+1} \) for some \( j \) then \( 2^j x_1 - ax_2 = b(a, j) \) or \( 2^j x_1 - ax_2 = -b(a, j) \). In terms of the norm \( \| \cdot \|_X \) we conclude that if \( x \in X \) and \( x_1/x_2 \in \Lambda_{j+1} \) or \( x_2/x_1 \in \Lambda_{j+1} \) then \( \|x\|_X \) is equal to one of the numbers \( |2^j x_1 - ax_2|/|b(a, j)| \) or \( |2^j x_2 - ax_1|/|b(a, j)| \), \( a \in \mathbb{Z}, |a| \leq 2^j \). Also, observe that, by our construction, \( BX \) is contained in the square \([-1, 1]^2 \). Therefore,

\[
(1.7) \quad \|x\|_X \geq \max(|x_1|, |x_2|).
\]

Now let us take the lattice \( S = \mathbb{Z}^2 \) and estimate \( |\Delta_{X,N}(S)| \) for \( N_j < N \leq N_{j+1} \). If \( x, y \in S \) and \( \|x-y\|_X \leq N \), then we have \( \|x-y\|_X = \|(z_1, z_2)\|_X \) where \( z_1, z_2 \in \mathbb{Z} \) and, by (1.7), \( \max(|z_1|, |z_2|) \leq N \). Hence, \( (z_1, z_2) = (0,0) \), or \( x_1/x_2 \in \Lambda_{j+1} \), or \( x_2/x_1 \in \Lambda_{j+1} \). Therefore, \( \|x-y\|_X = 0 \) or \( \|x-y\|_X \) is equal to one of the numbers \( |2^j x_1 - ax_2|/|b(a, j)| \) or \( |2^j x_2 - ax_1|/|b(a, j)| \), \( a \in \mathbb{Z}, |a| \leq 2^j \). For every \( a \) we have

\[
|2^j x_1 - ax_2| \leq 2^j |x_1| + |a| \times |x_2| \leq 2^{j+1} N.
\]

Taking the sum over all \( a \) we get

\[
(1.8) \quad |\Delta_{X,N}(S)| \leq (2^{j+1} + 1)2^{j+1} N + 1 \leq 2^{2j+3} N.
\]

On the other hand, by (1.6),

\[
(1.9) \quad \psi(N) \geq 5^j.
\]

Comparing (1.8) and (1.9), we get (0.4) and thus complete the proof of the theorem.

§2. PROOF OF THEOREM 3, PART I

In this section we prove that the condition (i) of Theorem 3 implies (ii).

Assume that \( \partial BX \) consists of a finite number of line segments with slopes \( \beta_1, \beta_2, \ldots, \beta_r \), all real and algebraic. Let \( \mathbb{Q} \beta_1, \ldots, \beta_r \) be the field extension of \( \mathbb{Q} \) generated by \( \beta_1, \ldots, \beta_r \), and let \( \alpha_0 \) be its primitive element, i.e. an algebraic number such that \( \mathbb{Q} \beta_1, \ldots, \beta_r = \mathbb{Q}[\alpha_0] \). We may assume that \( \alpha_0 \) is an algebraic integer: indeed, if \( \alpha_0 \) is a root of \( P(x) = a_d x^d + \cdots + a_0 \), then \( \alpha_0' = a_d \alpha_0 \) is a root of \( a_d^{d-1}P(x/a_d) = x^d + a_{d-1}x^{d-1} + a_{d-2}a_dx^{d-2} + \cdots + a_0a_d^{d-1} \), hence an algebraic integer, and generates the same extension.

It suffices to prove that there is a well-distributed set \( S \subset \mathbb{R}^2 \) such that

\[
(2.1) \quad |\{ (x + \beta y : (x, y) \in S - S, |x| + |y| \leq R \}| \leq R,
\]

for each \( \beta \in \mathbb{Q}[\alpha] \).

Since \( \mathbb{Q} \beta_1, \ldots, \beta_r \subset \mathbb{R} \), we have \( \alpha_0 \in \mathbb{R} \). Let \( \alpha_1, \ldots, \alpha_{d-1} \) be the algebraic conjugates of \( \alpha_0 \) in \( \mathbb{C} \) (of course they need not belong to \( \mathbb{Q}[\alpha_0] \)). Define for \( C > 0 \)

\[
T(C) = \{ \sum_{j=0}^{d-1} a_j \alpha_j^k : a_j \in \mathbb{Z}, |\sum_{j=0}^{d-1} a_j \alpha_j^k| \leq C, k = 1, \ldots, d - 1 \},
\]

where

\[
|\sum_{j=0}^{d-1} a_j \alpha_j^k| = \left| \sum_{j=0}^{d-1} a_j \alpha_j^k \right| = \left| \sum_{j=0}^{d-1} a_j \alpha_j^k \right|.
\]
and

\[ S = T(C) \times T(C) , \]

where \( C \) will be fixed later.

We first claim that \( T(C) \) is well distributed in \( \mathbb{R} \) (with the implicit constant dependent on \( C \)), and that

\[ \tag{2.2} \left| T(C) \cap [-R, R] \right| \ll R. \]

Indeed, let \( x = (x_0, x_1, \ldots, x_{d-1})^T \) solve

\[
\begin{align*}
\sum_{j=0}^{d-1} \alpha_0^j x_j &= 1, \\
\sum_{j=0}^{d-1} \alpha_k^j x_j &= 0, \quad k = 1, \ldots, d - 1.
\end{align*}
\]

Since the Vandermonde matrix \( A = (\alpha_k^j) \) is nonsingular, \( x \) is unique. In particular, \( x \) solves the system of equations.

To prove the first part of the claim, it suffices to show that there is a constant \( K_1 \) such that for any \( y \in \mathbb{R} \) there is a \( v \in T(C) \) with \( |y - v| \leq K_1 \). Fix \( y \), then we have

\[ y = \sum_{j=0}^{d-1} \alpha_0^j y x_j . \]

Let \( v_j \) be an integer such that \( |v_j - y x_j| \leq 1/2 \), and let \( v = \sum_{j=0}^{d-1} \alpha_0^j v_j . \) Then

\[ |y - v| = \left| \sum_{j=0}^{d-1} \alpha_0^j (y x_j - v_j) \right| \leq \frac{1}{2} \sum_{j=0}^{d-1} |\alpha_0^j| =: K_1 , \]

and, for \( k = 1, \ldots, d - 1 \),

\[ \left| \sum_{j=0}^{d-1} \alpha_k^j v_j \right| \leq \left| \sum_{j=0}^{d-1} \alpha_k^j (y x_j - v_j) \right| + |y| \sum_{j=0}^{d-1} \alpha_k^j x_j \leq \frac{1}{2} \sum_{j=0}^{d-1} |\alpha_k^j| . \]

The claim follows if we let \( C \geq \frac{1}{2} \sum_{j=0}^{d-1} |\alpha_k^j| . \)

We now prove (2.2). It suffices to verify that there is a constant \( K_2 \) such that for any \( y \in \mathbb{R} \) there are at most \( K_2 \) elements of \( T(C) \) in \( [y - C, y + C] \). Let \( a = \sum_{j=0}^{d-1} \alpha_0^j a_j \), then the conditions that \( a \in T(C) \) and \( |y - a| \leq C \) imply that

\[ A\tilde{a} - \tilde{y} \in CQ , \]

where \( \tilde{a} = (a_0, \ldots, a_{d-1})^T , \) \( \tilde{y} = (y, 0, \ldots, 0)^T , \) and \( Q = [-1, 1]^d \). In other words, \( \tilde{a} \in A^{-1} \tilde{y} + CA^{-1}Q \). But it is clear that the number of integer lattice points contained in any translate of \( CA^{-1}Q \) is bounded by a constant.
It remains to prove (2.1). Observe first that if $x, x' \in T(C)$, then $x - x' \in T(2C)$. Thus, in view of (2.2), it is enough to prove that for any two algebraic integers $\beta, \gamma \in \mathbb{Z}[\alpha]$ there is a $C_1 = C_1(\beta, \gamma)$ such that if $x, y \in T(2C)$, then $x\beta + y\gamma \in T(C)$. By the triangle inequality, it suffices to prove this with $y = 0$. Let $x \in T(C)$, then $x = \sum_{j=0}^{d-1} \alpha_i^j x_j$ for some $x_j \in \mathbb{Z}$. We also write $\beta = \sum_{j=0}^{d-1} \alpha_i^j b_j$, with $b_j \in \mathbb{Z}$. Then $\beta y = \sum_{i,j=0}^{d-1} \alpha_i^j x_i b_j$. We thus need to verify that

$$|\sum_{i,j=0}^{d-1} \alpha_i^j x_i b_j| \leq C_1$$

for $k = 1, \ldots, d - 1$. But the left side is equal to

$$\left|\sum_{i=0}^{d-1} \alpha_i^k x_i\right| \cdot \left|\sum_{j=0}^{d-1} \alpha_i^j b_j\right|,$$

which is bounded by $C_1(\beta) = C \max_k |\sum_{j=0}^{d-1} \alpha_i^j b_j|$. 

**Example.** Let $BX$ be a symmetric convex octagon whose sides have slopes $0, -1, \infty, \sqrt{2}$. Let also $T(C) = \{i + j\sqrt{2} : |i - j\sqrt{2}| \leq C\}$, and $S = T(10) \times T(10)$. It is easy to see that $T(C)$ is well distributed and that (2.2) holds. Let $x, y \in S$, then $x - y = (i + j\sqrt{2}, k + l\sqrt{2})$, where $i + j\sqrt{2}, k + l\sqrt{2} \in T(20)$. Depending on where $x - y$ is located, the distance from $x$ to $y$ will be one of the following numbers:

$$c_1|i + j\sqrt{2}|,$$
$$c_2|k + l\sqrt{2}|,$$
$$c_3|(i + k) + (j + l)\sqrt{2}|,$$
$$c_4|(i + j\sqrt{2})\sqrt{2} - (k + l\sqrt{2})| = c_4(2j - k) + (i - l)\sqrt{2}|.$$

Clearly, the first three belong to $T(20 \max(c_1, c_2, c_3))$. For the fourth one, we have

$$c_4(2j - k) - (i - l)\sqrt{2}| = c_4 - (i - j\sqrt{2})\sqrt{2} - (k - l\sqrt{2})|$$
$$\leq 20c_4(1 + \sqrt{2}).$$

Hence all distances between points in $S$ belong to $T(C)$ for some $C$ large enough, and in particular satisfy the cardinality estimate (2.2).

§3. ADDITIVE PROPERTIES OF MULTIDIMENSIONAL SETS
AND SETS WITH SPECIFIC ADDITIVE RESTRICTIONS

Let $Y$ be a linear space over $\mathbb{R}$ or over $\mathbb{Q}$. For $A, B \subset Y$ and $\alpha \in \mathbb{R}$ or $\mathbb{Q}$ we denote

$$A + B = \{a + b : a \in A, b \in B\}, \quad \alpha A = \{\alpha a : a \in A\}.$$ 

We say that a set $A \subset Y$ is a $d$-dimensional if $A$ is contained in some $d$-dimensional affine subspace of $Y$, but in no $d - 1$-dimensional affine subspace of $Y$. We will denote the dimension of a set $A$ by $d_A$.

The following result is due to Ruzsa [Ru94, Corollary 1.1].
Lemma 3.1. Let \( A, B \subset \mathbb{R}^d \), \(|A| \leq |B|\), and assume that \( A + B \) is \( d \)-dimensional. Then

\[
|A + B| \geq |B| + d|A| - d(d + 1)/2.
\]

The special case of Lemma 3.1 with \( A = B \) was proved earlier by Freiman [F73, p. 24]. In this case we also have the following corollary.

Corollary 3.1. Let \( A \subset \mathbb{R}^d \), and assume that \(|A + A| \leq K|A|\), \( K \leq |A|^{1/2} \). Then the dimension of \( A \) does not exceed \( K \).

Proof. Let \(|A| = N \geq 1\), then \( d_A \leq N - 1 \). Suppose that \( d_A > K \). The function \( f(x) = (x + 1)N - x(x + 1)/2 \) is increasing for \( x \leq N - 1/2 \), hence by (3.1) we have

\[
KN \geq f(d_A) > f(K) = (K + 1)N - \frac{K(K + 1)}{2},
\]
i.e. \( K(K + 1) > 2N \), which is not possible if \( K^2 \leq N \).

We observe that Lemma 3.1, and hence also Corollary 3.1, extends to the case when \( A, B \) are subsets of a linear space \( Y \) over \( \mathbb{Q} \). Assume that \( Y \) is \( d \)-dimensional, and take a basis \( \{e_1, \ldots, e_d\} \) in \( Y \). Consider the space \( \mathbb{R}^d \) with a basis \( \{e'_1, \ldots, e'_d\} \).

We can arrange a mapping \( \Phi : Y \rightarrow Y' \) by

\[
\Phi(\sum_{j=1}^d \alpha_j e_j) = \sum_{j=1}^d \alpha_j e'_j.
\]

It is easy to see that \( \Phi \) is Freiman’s isomorphism of any order and, in particular, of order 2: this means that for any \( y_1, y_2, z_1, z_2 \) from \( Y \) the condition

\[
y_1 + y_2 \neq z_1 + z_2
\]
implies

\[
\Phi(y_1) + \Phi(y_2) \neq \Phi(z_1) + \Phi(z_2).
\]

Therefore, if \( A, B \) are finite subsets of \( Y \) and \( A' = \Phi(A) \), \( B' = \Phi(B) \), then \(|A + B| = |A' + B'|\), and we get the required inequality for \(|A + B|\).

The following is a special case of [N96, Theorem 7.8].

Lemma 3.2. If \( N \in \mathbb{N} \), \( K > 1 \), \( A \subset Y \), and \( B \subset Y \) satisfy

\[
\min(|A|, |B|) \geq N, \quad |A + B| \leq KN,
\]

we have

\[
|A + A| \leq K^2|A|.
\]
Corollary 3.2. If $N \in \mathbb{N}$, $K > 1$, and if $A, B \subset Y$ satisfy (3.2) for some $K$ with $K^2(2K^2 + 1) < N$, then $d_{A+B} \leq K$. In particular, $d_A \leq K$ and $d_B \leq K$.

Proof. By Lemma 3.2, we have $|A + A| \leq K^2N$, hence Corollary 3.1 implies that $d_A \leq K^2$, and similarly for $B$. Hence $d_{A+B} \leq d_A + d_B \leq 2K^2$. By Lemma 3.1, we have

$$ KN \geq |A + B| \geq (1 + d_{A+B})N - \frac{d_{A+B}(d_{A+B} + 1)}{2} $$

$$ \geq d_{A+B}N + N - K^2(2K^2 + 1) \geq d_{A+B}N, $$

which proves the first inequality. To complete the proof, observe that $d_{A+B} \geq \max(d_A, d_B)$.

Lemma 3.3. Let $K > 0$, $A$ and $B$ be finite nonempty subsets of $\mathbb{R}$, $\alpha \in \mathbb{R} \setminus \{0\}$. Also, suppose that the following conditions are satisfied

$$(3.3) \quad |A - \alpha B| \leq K|B|. $$

Then there is a set $B' \subset B$ such that

$$(3.4) \quad |A - \alpha B'| \leq K|B'|, $$

$$(3.5) \quad |B'| \geq |A|/K, $$

and for any $b_1, b_2 \in B'$ the number $\alpha (b_1 - b_2)$ is a linear combination of differences $a_1 - a_2$, $a_1, a_2 \in A$, with integer coefficients.

Proof. Let us construct a graph $H$ on $B$. We join $b_1, b_2 \in B$ (not necessarily distinct) by an edge if there are $a_1, a_2 \in A$ such that $a_1 - \alpha b_1 = a_2 - \alpha b_2$. Let $B_1, \ldots, B_s$ be the components of connectedness of the graph $H$. Thus, for any $j = 1, \ldots, s$ and for any $b_1, b_2 \in B_j$ there is a path connecting $b_1$ and $b_2$ consisting of edges of $H$ (a one-point path for $b_1 = b_2$ is allowed). This implies that $\alpha (b_1 - b_2)$ is a sum of differences $a_1 - a_2$ for some pairs $(a_1, a_2) \in A \times A$. Also, denoting

$$ S = A - \alpha B, \quad S_j = A - \alpha B_j, $$

we see that, by the choice of $B_1, \ldots, B_s$, the sets $S_j (j = 1, \ldots, s)$ are disjoint.

Since

$$ |B| = \sum_{j=1}^s |B_j|, \quad |S| = \sum_{j=1}^s |S_j|, $$

there is some $j$ such that

$$ |S_j|/|B_j| \leq |S|/|B|, $$

and, by (3.3),

$$ |S_j| \leq K|B_j|, $$

On the other hand,

$$ |S_j| = |A - \alpha B_j| \geq |A|. $$

Hence,

$$ |B_j| \geq |S_j|/K \geq |A|/K. $$

So, the set $B' = B_j$ satisfies (3.4) and (3.5), and Lemma 3.3 follows.
Lemma 3.4. Let $K > 0$, $A$ and $B$ be finite nonempty subsets of $\mathbb{R}$, $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$. Also, suppose that the conditions

\begin{align*}
(A) & \quad |A - \alpha_1 B| \leq K|B|,
(B) & \quad |A - \alpha_2 B| \leq K|A|,
\end{align*}

are satisfied. Then there are nonempty sets $A' \subset A$ and $B' \subset B$ such that

\begin{align*}
(A') & \quad |A_0 - \alpha_1 B_0| \leq K|B_0|,
(B') & \quad |A_0 - \alpha_2 B_0| \leq K|A_0|,
\end{align*}

\begin{align*}
(A'') & \quad |A' \setminus A_0| \geq \frac{|A|}{K^2},
\end{align*}

and for any $a_1', a_2' \in A'$ the difference $a_1' - a_2'$ is a linear combination of numbers $\frac{\alpha_2}{\alpha_1}(a_1 - a_2)$, $a_1, a_2 \in A$, with integer coefficients.

Proof. By (A), we can use Lemma 3.3 for $\alpha = \alpha_1$, and we get (3.8) and (3.5). Further, we use Lemma 3.3 again for $B', A$ (thus, in the reverse order), and we get (3.9) and also

\begin{align*}
|A'| \geq \frac{|B'|}{K}.
\end{align*}

Combining the last inequality with (3.5) we obtain (3.10). The proof of the lemma is complete.

Replacing (3.8) by a weaker inequality

\begin{align*}
|A' - \alpha_1 B'| \leq K|B'|
\end{align*}

and iterating Lemma 3.4, we get the following.

Lemma 3.5. Let $K > 0$, $A$ and $B$ be finite nonempty subsets of $\mathbb{R}$, $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$. Also, suppose that the conditions (3.6) and (3.7) are satisfied. Then there are nonempty sets $A_j \subset A$ and $B_j \subset B$ ($j = 0, 1, \ldots$) such that $A_0 = A$, $B_0 = B$, $A_j \subset A_{j-1}$, $B_j \subset B_{j-1}$ for $j \geq 1$,

\begin{align*}
|A_j - \alpha_2 B_j| \leq K|A_j| & \quad (j \geq 1),
|A_j| \geq \frac{|A|}{K^{2j}},
\end{align*}

and for any $a_1, a_2 \in A_j$ the difference $a_1 - a_2$ is a linear combination of numbers $\frac{\alpha_2}{\alpha_1}(a_1 - a_2)$, $a_1, a_2 \in A$, with integer coefficients.

Now we are in position to come to the main object of our constructions: to show that under the assumptions of Lemma 3.5, providing that the number $\alpha_1/\alpha_2$ is transcendental, we can conclude that the dimension of the set $A$ over $\mathbb{Q}$ cannot be too small.
Corollary 3.6. Let $K > 0$, $A$ and $B$ be finite nonempty subsets of $\mathbb{R}$, $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \{0\}$ such that $\alpha_1/\alpha_2$ is transcendental. Also, suppose that the conditions (3.6) and (3.7) are satisfied. Then, if for some $d \in \mathbb{N}$ the inequality
\begin{equation}
|A| > K^{2d}
\end{equation}
holds, then the dimension of $A$ over $\mathbb{Q}$ is greater than $d$.

Proof. By Lemma 3.5 and (3.11), we have $|A_d| \geq 2$. Take distinct $a_1, a_2 \in A_d$. Then also $a_1, a_2 \in A_j$ for $j = 0, 1, \ldots, d$, and, by Lemma 3.6, the difference $a_1 - a_2$ is a linear combination of numbers $\frac{\alpha_2}{\alpha_1}(a'_1 - a'_2)$, $a'_1, a'_2 \in A$, with integer coefficients. Therefore, all numbers $b_j = \frac{\alpha_2}{\alpha_1}(a_1 - a_2)$ belong to the linear span of $a'_1 - a'_2$, $a'_1, a'_2 \in A$, over $\mathbb{Q}$. But, since $\alpha_1/\alpha_2$ is transcendental, the numbers $b_j (j = 0, \ldots, d)$ are linearly independent over $\mathbb{Q}$. Therefore, the dimension of the linear span of $a'_1 - a'_2$, $a'_1, a'_2 \in A$, over $\mathbb{Q}$ is at least $d + 1$, as required.

Corollary 3.7. If $A$ is a subset of $\mathbb{R}$, $2 \leq |A| < \infty$, $\alpha$ is a transcendental real number, then
\begin{equation}
|A - \alpha A| \gg |A| \log |A|/\log \log |A|.
\end{equation}

Proof. Suppose that the conclusion fails, then for any $\epsilon > 0$ we may find arbitrarily large $N$ and $A \subset \mathbb{R}$ with $|A| = N$ such that
\begin{equation}
|A - \alpha A| \leq KN, \quad K = \epsilon \frac{\log N}{\log \log N}.
\end{equation}

By Corollary 3.2, we have $d_A \leq K$. On the other hand, (3.6) holds with $B = A$, $\alpha_1 = \alpha$, and, since $A - \alpha^{-1} A = -\alpha^{-1}(A - \alpha A)$, (3.7) holds with $B = A$ and $\alpha_2 = \alpha^{-1}$. Corollary 3.7 then implies that
\begin{equation}
N \leq K^{2K}.
\end{equation}

Taking logarithms of both sides, and assuming that $2\epsilon < 1$, we obtain
\begin{equation}
\log N \leq 2\epsilon \frac{\log N}{\log \log N} (\log(2\epsilon) + \log \log N + \log \log \log N) \leq 2\epsilon \log N,
\end{equation}
which is not possible if $N$ was chosen large enough.

Remark. On the other hand, if $\alpha \in \mathbb{R}$ is an algebraic number, then one can use our construction from §2 to show that for any $N \in \mathbb{N}$ there is a set $A \subset \mathbb{R}$, $|A| = N$, such that
\begin{equation}
|A - \alpha A| \leq C|A|,
\end{equation}
where $C$ depends only on $\alpha$.

Finally, we state a lemma due to J. Bourgain[B99, Lemma 2.1]. For our purposes, we need a slightly more precise formulation than that given in [B99]; the required modifications are described below.
**Lemma 3.8.** Let $N \geq 2$, $A, B$ be finite subsets of $\mathbb{R}$ and $G \subset A \times B$ such that

\[(3.12)\] $|A|, |B| \leq N,$

\[(3.13)\] $|S| \leq N$ where $S = \{a + b : (a, b) \in G\},$

\[(3.14)\] $|G| \geq \delta N^2.$

Then there exist $A' \subset A$, $B' \subset B$ satisfying the conditions

\[(3.15)\] $|(A' \times B') \cap G| \gg \delta^5 N^2 (\log N)^{-C_1},$

\[(3.16)\] $|A' - B'| \ll N^{-1} (\log N)^{C_2 \delta^{-13}} |(A' \times B') \cap G|.$

In [B], the bounds (3.15) and (3.16) involved factors of the form $N^{\gamma^+}$ and $N^{\gamma^-}$, where $N^{\gamma^+}$ ($N^{\gamma^-}$) means $\leq C(\varepsilon) N^{\gamma^+ \varepsilon}$ for all $\varepsilon > 0$ and some $C(\varepsilon) > 0$ (resp., $\geq c(\varepsilon) N^{\gamma^- \varepsilon}$ for all $\varepsilon > 0$, $c(\varepsilon) > 0$). We need a slightly stronger statement, namely that the same bounds hold with the factors in question obeying the inequalities

$\ll N^{\gamma} (\log N)^C$ or $\gg N^{\gamma} (\log N)^{-C}$, respectively, for some appropriate choice of a constant $C$. A careful examination of the proof in [B99] shows that it remains valid with this new meaning of the notation $N^{\gamma^+}$ and $N^{\gamma^-}$, and that one may in fact take $C_1 = 5$, $C_2 = 10$. We further note that although Bourgain states his lemma for $A, B \subset \mathbb{Z}^d$, the same proof works for $A, B \subset \mathbb{R}$ if the exponential sum inequality [B99,(2.7)] is replaced by

$|G| < \int_S \chi_A \ast \chi_B \leq |S|^{1/2} \|\chi_A \ast \chi_B\|_2;$

we then observe that

$$\|\chi_A \ast \chi_B\|_2^2 = \left| \{ (a, a', b, b') \in A \times A \times B \times B : a + b = a' + b' \} \right|$$

$$= \left| \{ (a, a', b, b') \in A \times A \times B \times B : a - b = a' - b \} \right| = \|\chi_A \ast \chi_B\|_2^2,$$

and proceed further as in [B99]. A similar modification should be made in [B99,(2.36)].

**§4. PROOF OF THEOREM 4**

In this section we prove Theorem 4; note that this also proves the implication (ii)$\Rightarrow$(i) of Theorem 3.

Suppose that $B\chi$ is a polygon with finitely many sides for which the conclusion of the theorem fails, i.e. that there is a well distributed set $S$ such that for any $\varepsilon > 0$ there is an increasing sequence of positive integers $N_1, N_2, \cdots \to \infty$ with

\[(4.1)\] $|\Delta_{X, N_j}(S)| < \varepsilon N_j \psi(N_j),$

where $\psi(n)$ is a slowly increasing function such that $\sum \psi(N_j)$ diverges. Then, for each $j$ such that $\varepsilon N_j \psi(N_j) \leq 1$, we have

$$|\Delta_{X, N_j}(S)| < \varepsilon N_j \psi(N_j).$$

This implies that for each $j$, there exists a subset $S_j \subset S$ such that

$$|\Delta_{X, N_j}(S_j)| < \varepsilon N_j \psi(N_j).$$

Then, we can take $S = \bigcup_{j=1}^\infty S_j$ and $S_j \cap S_{j+1} = \emptyset$, and we have

$$|\Delta_{X, N_j}(S)| < \varepsilon N_j \psi(N_j).$$

Finally, we have

$$|\Delta_{X, N_j}(S)| < \varepsilon N_j \psi(N_j).$$

This completes the proof of Theorem 4.
where

\[ \psi(N) = \log \frac{N}{\log \log N}. \]

Without loss of generality we may assume that \( \partial BX \) contains a vertical line segment and a horizontal line segment, and that \( c_1 BL^2 \subset BX \subset BL^2 \). Let also \( c_2 \in (0, 1/10) \) be a small constant such that all sides of \( BX \) have length at least \( 8c_2 \).

Let \( M \) be a sufficiently large number which may depend on \( \epsilon \); all other constants in the proof will be independent of \( \epsilon \). Let \( T = N \cdot j_0 \) for some \( j_0 \) large enough so that \( T > M \), and let \( N = c_2 T \). Let also \( Q = \text{Int} (N \cdot BX), v = (v_1, v_2) \), and

\[ A = \{ x_1 : (x_1, x_2) \in S \cap Q \text{ for some } x_2 \}, \]

\[ Q' = Q + (T - 2N)v. \]

Observe that both \( Q \) and \( Q' \) have Euclidean diameter \( \leq 2N \), and that

\[ Q' \subset \{(x_1, x_2) : (T - 3N)v_1 < x_1 < (T - N)v_1 \}, \]

so that

\[ \|x - x'\|_X \geq (1 - 4c_2)T > T/2, \ x \in Q, x' \in Q'. \]

By our choice of \( c_2 \) we have \( c_2 \leq r/4 \), so that

\[ T/2 \cdot r \geq 2N. \]

Hence all \( X \)-distances between points in \( Q \) and \( Q' \) are measured using the vertical segments of \( \partial BX \), i.e.

\[ \|x - x'\|_X = |x_1 - x'_1|/v_1, \ x = (x_1, x_2) \in Q, x' = (x'_1, x'_2) \in Q_1. \]

Next, we claim that

\[ |\{\|x - x'\|_X : x \in S \cap Q, x' \in S \cap Q'\}| < K_0 \epsilon N \psi(N), \]

where \( K_0 \) is a constant depending only on \( c_2 \). Indeed, we have

\[ \{\|x - x'\|_X : x \in Q, x' \in Q'\} \subset [0, T], \]

hence the failure of (4.2) would imply that

\[ |\Delta_{X,T}(S)| \geq K_0 \epsilon N \psi(N) \geq \epsilon T \psi(T), \]

if \( K_0 \) is large enough (at the last step we used that \( \psi(N) \gg \psi(c_2^{-1}N) = \psi(T) \)). But this contradicts (4.1).

It follows that if we define

\[ A' = \{ x'_1 : (x'_1, x'_2) \in S \cap Q' \text{ for some } x'_2 \}, \]

then we can estimate the cardinality of the difference set \( A - A' \) using (4.2):

\[ |A - A'| < K_0 \epsilon N \psi(N). \]
On the other hand, since $S$ is well distributed, we must have

\[(4.4) \quad |A|, |A'| \gg N.\]

Hence by Corollary 3.2 we have

\[(4.5) \quad d_A \ll \varepsilon \psi(N).\]

We may now repeat the same argument with the vertical side of $\partial BX$ replaced by its other sides. In particular, using the horizontal segment in $\partial BX$ instead, we obtain the following. Let

\[B = \{x_2: (x_1, x_2) \in S \cap Q \text{ for some } x_1\},\]

then there is a set $B' \subset \mathbb{R}$ such that

\[(4.6) \quad |B|, |B'| \gg N,\]

\[(4.7) \quad |B - B'| < K_0 \varepsilon N \psi(N),\]

\[(4.8) \quad d_B \ll \varepsilon \psi(N).\]

Furthermore, assume that $\partial BX$ contains a segment of a line $x_1 + \alpha x_2 = \beta$, then

\[(4.9) \quad |\{x_1 + \alpha x_2: (x_1, x_2) \in S \cap Q\}| \leq K_0 \varepsilon N \psi(N);\]

this estimate is an easier analogue of (4.3) obtained by counting distances between points in $Q$ and just one point in the appropriate analogue of $Q'$.

Suppose that $\partial BX$ contains segments of lines $x_1 + \alpha_1 x_2 = C_1$, $x_2 + \alpha_2 x_2 = C_2$ (i.e. with slopes $-1/\alpha_1$, $-1/\alpha_2$), where $\alpha_1, \alpha_2$ are neither 0 nor $\infty$, and that the ratio $\alpha_1/\alpha_2$ is transcendental. Let $G = (A \times B) \cap S$, then $|G| \geq c_4 N^2$ since $S$ is well distributed. By (4.4), (4.6), and (4.9) with $\alpha = \alpha_1$, the assumptions of Lemma 3.8 are satisfied with $N$ replaced by $K_0 \varepsilon N \psi(N)$ and $\delta = c_4 (K_0 \varepsilon \psi(N))^{-2}$. We conclude that there are subsets $A_1 \subset A$ and $B_1 \subset B$ such that

\[(4.10) \quad |(A_1 \times B_1) \cap G| \gg N^2 \varepsilon \psi(\log N)^{-2},\]

\[(4.11) \quad |A_1 - \alpha_1 B_1| \ll N^{-1} \varepsilon^{-2} \psi(\log N)^{\epsilon} |(A_1 \times B_1) \cap G|.\]

Here and below, $c$ denotes a constant which may change from line to line but is always independent of $N$. We also simplified the right sides of (4.10) and (4.11) by noting that $\psi(N) \leq \log N$.

Similarly, applying Lemma 3.8 with $G$ replaced by $(A_1 \times B_1) \cap G$ and $\alpha_1$ replaced by $\alpha_2$, we find subsets $A_2 \subset A_1$ and $B_2 \subset B_1$ such that

\[(4.12) \quad |(A_2 \times B_2) \cap G| \gg N^2 \varepsilon \psi(\log N)^{-2},\]
(4.13) \[ |A_2 - \alpha_2 B_2| \ll N^{-1} \epsilon^{-c}(\log N)^{\epsilon} |(A_2 \times B_2) \cap G|. \]

Clearly, (4.11) also holds with \( A_1, B_1 \) replaced by \( A_2, B_2 \).

Thus \( A_2, B_2 \) satisfy the assumptions (3.14), (3.15) of Corollary 3.7, with \( K = \epsilon^{-c}(\log N)^{\epsilon} \). By (4.4), (4.5) and Corollary 3.7, we must have for some constants \( c, K_2 \),

\[ cN \leq |A_2| < (\epsilon^{-1} \log N)^{K_2 \epsilon \log N / \log \log N}, \]

hence

\[ \log c + \log N \leq \frac{K_2 \epsilon \log N}{\log \log N} (\log \log N - \log \epsilon) \leq 2K_2 \epsilon \log N, \]

a contradiction if \( \epsilon \) was chosen small enough. This proves that if (0.5) fails, then the ratio between any two slopes, other than 0 or \( \infty \), of sides of \( BX \) is algebraic.

To conclude the proof of the theorem, we first observe that if \( BX \) is a rectangle, then there is nothing to prove. If \( BX \) is a hexagon with slopes 0, \( \infty \), \( \alpha \), we may always find a coordinate system as in Theorem 3 (i); namely, if we let

\[ x_1' = x_1, \quad x_2' = \alpha x_2, \]

then the slopes 0 and \( \infty \) remain unchanged, and lines \( \alpha x_1 - x_2 = C \) with slope \( \alpha \) are mapped to lines \( x_1' - x_2' = C/\alpha \) with slope 1. Finally, suppose that \( BX \) is a polygon with slopes 0, \( \infty \), \( \alpha_1, \alpha_2, \ldots, \alpha_l \), and apply the linear transformation (4.14) with \( \alpha = \alpha_1 \). Then the sides of \( \partial BX \) with slope \( \alpha_1 \) is mapped to line segments with slope 1; moreover, since the ratios \( \alpha_j/\alpha_1, j = 2, 3, \ldots, l \), remain unchanged in the new coordinates, and since we have proved that these ratios are algebraic, all remaining sides of \( \partial BX \) are mapped to line segments with algebraic slopes.

**Acknowledgements.** This work was completed while the first author was a PIMS Distinguished Chair at the University of British Columbia, and was partially supported by NSERC grant 22R80520. We are indebted to Ben Green for pointing out to us the reference [Ru].

**REFERENCES**

[B99] J. Bourgain, On the dimension of Kakeya sets and related maximal inequalities, Geom. Funct. Anal. 9 (1999), 256–282.

[E46] P. Erdős, On sets of distances of \( n \) points, Amer. Math. Monthly 53 (1946), 248–250.

[F73] G. Freiman, Foundations of a structural theory of set addition (translation from Russian), Translations of Mathematical Monographs, vol. 37, American Mathematical Society, Providence, RI, 1973.

[I01] A. Iosevich, Curvature, combinatorics and the Fourier transform, Notices Amer. Math. Soc. 46 (2001), 577–583.

[IL03] A. Iosevich and I. Laba, Distance sets of well-distributed planar point sets, Discrete Comput. Geometry 31 (2004), 243–250.

[IL04] A. Iosevich and I. Laba, \( K \)-distance sets, Falconer conjecture and discrete analogs, preprint, 2003.

[KT04] N.H. Katz and G. Tardos, A new entropy inequality for the Erdős distance problem. in: Towards a Theory of Geometric Graphs.(ed.J Pach) Contemporary Mathematics, vol. 342, Amer.Math Soc. 2004
[N96] M. Nathanson, Additive Number Theory, II: Inverse Problems and the Geometry of Sumsets, Springer-Verlag, New York, 1996.
[Ru94] I. Ruzsa, Sum of sets in several dimensions, Combinatorica 14 (1994), 485–490.
[ST01] J. Solymosi and Cs. Tóth, Distinct distances in the plane, Discrete Comput. Geometry 25 (2001), 629–634.
[W99] T. Wolff, Decay of circular means of Fourier transforms of measures, Int. Math. Res. Notices 10 (1999), 547–567.

Department of Mechanics and Mathematics, Moscow State University, Moscow, 119992, Russia, e-mail: konyagin@ok.ru

Department of Mathematics, University of British Columbia, Vancouver, B.C. V6T 1Z2, Canada, e-mail: ilaba@math.ubc.ca