ÉTALE TRIVIALITY OF FINITE VECTOR BUNDLES OVER COMPACT COMPLEX MANIFOLDS

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Abstract. A vector bundle $E$ over a projective variety $M$ is called finite if it satisfies a nontrivial polynomial equation with nonnegative integral coefficients. Introducing finite bundles, Nori proved that $E$ is finite if and only if the pullback of $E$ to some finite étale covering of $M$ is trivializable [No1]. The definition of finite bundles extends naturally to holomorphic vector bundles over compact complex manifolds. We prove that a holomorphic vector bundle over a compact complex manifold $M$ is finite if and only if the pullback of $E$ to some finite étale covering of $M$ is holomorphically trivializable. Therefore, $E$ is finite if and only if it admits a flat holomorphic connection with finite monodromy. In [BP] this result was proved under the extra assumption that the compact complex manifold $M$ admits a Gauduchon astheno-Kähler metric.

1. Introduction

Given a vector bundle $E$ on a projective variety and a polynomial $P(x) = \sum_{i=0}^{N} a_i x^i$, where $a_i$ are nonnegative integers, define

$$P(E) := \bigoplus_{i=0}^{N} (E^\otimes i)^{\oplus a_i},$$

where $E^\otimes 0$ is the trivial line bundle. A vector bundle $E$ is called finite if there are two distinct polynomials $P_1$ and $P_2$ of the above type such that the two vector bundles $P_1(E)$ and $P_2(E)$ are isomorphic; finite bundles were introduced by Nori in [No1]. The above definition clearly makes sense if $E$ is a holomorphic vector bundle on a compact complex manifold.

There are several equivalent definitions of finiteness [No1, p. 35, Lemma 3.1]. The one that is most useful to work with is the following: A vector bundle $E$ is finite if and only if there are finitely many vector bundles $V_1, \cdots, V_p$ such that

$$E^\otimes j = \bigoplus_{i=1}^{p} V_i^\oplus j_i$$

for all $j \geq 1$, where $j_i$ are nonnegative integers.

Nori proved that a vector bundle $E$ on a complex projective variety $Z$ is finite if and only if the pullback of $E$ to some finite étale covering of $Z$ is trivializable [No1, No2]. In [No1, No2] the base field is allowed to be any algebraically closed field, although we have stated here for complex numbers. When $Z$ is smooth, a vector bundle $E$ pulls back

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to the trivial bundle under some finite étale covering of $Z$ if and only if $E$ admits a flat holomorphic connection with finite monodromy.

It is natural to ask whether the above characterization of finite vector bundles remains valid for holomorphic vector bundles over compact complex manifolds.

In [BHS] this characterization of finite bundles was proved for holomorphic vector bundles on compact Kähler manifolds. In [BP] this characterization was proved for holomorphic vector bundles on compact complex manifolds that admit a Gauduchon astheno-Kähler metric.

The following is proved here (Theorem 4.3 and Corollary 4.4):

**Theorem 1.1.** Let $E$ be a holomorphic vector bundle over a compact complex manifold $M$. Then $E$ is finite if and only if the pullback of $E$ to some finite étale covering of $M$ is holomorphically trivializable. Also, $E$ is finite if and only if it admits a flat holomorphic connection with finite monodromy.

Both [BHS] and [BP] used, very crucially, the numerically flat bundles introduced by J.-P. Demailly, T. Peternell and M. Schneider in [DPS]. Finite bundles are numerically flat. Theorem 1.18 of [DPS, p. 311] proves a characterization of numerically flat bundles on compact Kähler manifolds. This theorem of [DPS] was directly used in [BHS], and it was extended in [BP, p. 5, Theorem 3.2] to numerically flat bundles on compact complex manifolds admitting a Gauduchon astheno-Kähler metric, which was then used in an essential way. The proof of Theorem 1.1 avoids use of numerically flat bundles.

2. Holomorphic bundles on compact complex manifolds

2.1. Finite vector bundles. Let $M$ be a compact connected complex manifold. A holomorphic vector bundle $E$ over $M$ is called *decomposable* if there are holomorphic vector bundles $V$ and $W$ over $M$ of positive ranks such that $V \oplus W$ is holomorphically isomorphic to $E$. A holomorphic vector bundle is called *indecomposable* if it is not decomposable. Clearly, any holomorphic vector bundle can be expressed as a direct sum of indecomposable vector bundles. Atiyah proved the following uniqueness theorem for such decompositions.

**Theorem 2.1** ([At, p. 315, Theorem 3]). Let $E$ be a holomorphic vector bundle over $M$ holomorphically isomorphic to both $\bigoplus_{i=1}^{m} V_i$ and $\bigoplus_{j=1}^{n} W_j$, where $V_i$, $1 \leq i \leq m$, and $W_j$, $1 \leq j \leq n$, are all indecomposable vector bundles. Then $m = n$, and there is a permutation $\sigma$ of $\{1, \cdots, m\}$ such that $V_i$ is holomorphically isomorphic to $W_{\sigma(i)}$ for all $1 \leq i \leq m$.

**Definition 2.2.** A holomorphic vector bundle $V$ will be called a **component** of $E$ if there is another holomorphic bundle $W$ such that $V \oplus W$ is isomorphic to $E$. If a component $V$ is also indecomposable, then it would be called an **indecomposable component**.
A holomorphic vector bundle $E$ on $M$ is called finite if there are finitely many holomorphic vector bundles $V_1, \ldots, V_p$ such that

$$E^\otimes j = \bigoplus_{i=1}^p V_i^\otimes j_i$$

for all $j \geq 1$, where $j_i$ are nonnegative integers ([No1, p. 35, Definition], [No1, p. 35, Lemma 3.1(d)], [No2, p. 80, Definition]). As mentioned in the introduction, $E$ is finite if and only if there are two distinct polynomials $P_1$ and $P_2$, whose coefficients are nonnegative integers, such that $P_1(E)$ is holomorphically isomorphic to $P_2(E)$ [No1, p. 35, Definition], [No1, p. 35, Lemma 3.1(a)].

Any vector bundle of the form $\bigoplus_{i=1}^N W_i^\otimes n_i$, where $n_i$ are nonnegative integers, will be called a direct sum of copies of $W_1, \ldots, W_N$.

We can assume that each $V_i$ in the above definition is a component (see Definition 2.2) of some $E^\otimes m$. If $V_i$ is a component of $E^\otimes m$, then $V^\otimes k$ is a component of $E^\otimes km$ for all $k \geq 1$. Hence using Theorem 2.1 it follows that $V^\otimes k$ is a direct sum of copies of the indecomposable components (see Definition 2.2) of $\bigoplus_{i=1}^p V_i$. So $V_1, \ldots, V_p$ are finite bundles. Similarly, it follows from Theorem 2.1 that any component of a finite bundle is also a finite bundle [No1, p. 36, Lemma 3.2(2)], [No2, p. 80, Lemma 3.2(2)].

Hence we may assume that each $V_i$ in (2.1) is indecomposable.

If $E_1$ and $E_2$ are finite vector bundles, then both $E_1 \oplus E_2$ and $E_1 \otimes E_2$ are also finite [No1, p. 36, Lemma 3.2(1)], [No2, p. 80, Lemma 3.2(1)]. A holomorphic line bundle is finite if and only if it is of finite order [No1, p. 36, Lemma 3.2(3)], [No2, p. 80, Lemma 3.2(3)]. If $E$ is finite then clearly the dual bundle $E^*$ is also finite.

### 2.2. Semistable sheaves

Let $d$ be the complex dimension of $M$. A Gauduchon metric on $M$ is a Hermitian structure $g$ on $M$ such that the associated positive $(1, 1)$–form $\omega_g$ on $M$ satisfies the equation

$$\partial \bar{\partial} \omega_g^{d-1} = 0.$$  

Gauduchon metric exists [Ga, p. 502, Théorème]. Fix a Gauduchon metric $g$ on $M$. This enables us to define the degree of a torsionfree coherent analytic sheaf $F$ on $M$ as follows: Fix a Hermitian structure on the determinant line bundle $\det(F) \to M$ (see [Ko2, Ch. V, § 6] for determinant bundle), and denote by $\mathcal{K}$ the curvature of the corresponding Chern connection; now define

$$\deg(F) = \frac{1}{2\pi \sqrt{-1}} \int_M \mathcal{K} \wedge \omega_g^{d-1} \in \mathbb{R}$$

[LT, p. 43, Definition 1.4.1]. Define the slope of $F$

$$\mu(F) := \frac{\deg(F)}{\text{rank}(F)} \in \mathbb{R}.$$  

This allows us to define stability and semistability [LT, p. 44, Definition 1.4.3]. We recall that a torsionfree coherent analytic sheaf $F$ on $M$ is called polystable if it is a direct sum of stable sheaves of same slope; in particular, polystable sheaves are semistable.
For a torsionfree coherent analytic sheaf $F$ on $M$, define $\mu_{\max}(F)$ to be the slope of the maximal semistable subsheaf of $F$. In other words, $\mu_{\max}(F)$ is the slope of the first term in the Harder–Narasimhan filtration of $F$.

A holomorphic vector bundle on $M$ is polystable if it admits a Hermitian–Einstein metric [LT, p. 55, Theorem 2.3.2] (see also [Lu, p. 12, Proposition 4], [Ko2, p. 177–178, Theorem 8.3] and [Ko1]). Moreover, A stable holomorphic vector bundle on $M$ admits a Hermitian–Einstein metric [LY, p. 563, Theorem 1], [LT, p. 61, Theorem 3.0.1]. If $H_1$ and $H_2$ are Hermitian–Einstein metrics on $E$ and $F$ respectively, then $H_1 \otimes H_2$ is a Hermitian–Einstein metric on $E \otimes F$. This immediately implies that the tensor product of two stable vector bundles is polystable. Now it is straightforward to deduce the following:

Corollary 2.3. If $E$ and $F$ are polystable vector bundles, then $E \otimes F$ is also polystable.

For the following proposition see [HL, p. 22, Proposition 1.5.2] (in [HL, Proposition 1.5.2] Gieseker (semi)stability is used, but the proof works for $\mu$–(semi)stability; see [HL, p. 25, Theorem 1.6.6] for it), [Ko2, p. 175–176, Theorem 7.18], [BDL, p. 1034, Proposition 3.1], [BTT, p. 998, Proposition 2.1].

Proposition 2.4. Let $E$ be a semistable holomorphic vector bundle on $M$, and let

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{b-1} \subset F_b = E$$

be a filtration of reflexive subsheaves such that $F_i/F_{i-1}$ is stable with $\mu(F_i/F_{i-1}) = \mu(E)$ for all $1 \leq i \leq b$. Then the holomorphic isomorphism class of the graded object

$$\bigoplus_{i=1}^{b} F_i/F_{i-1}$$

is independent of the choice of the filtration.

Definition 2.5. The stable sheaves $F_i/F_{i-1}$, $1 \leq i \leq b$, in Proposition 2.4 will be called the stable graded factors of $E$.

3. Homomorphisms of finite bundles

Lemma 3.1. Let $E$ be a finite vector bundle on $M$ and

$$\phi \in H^0(M, E) \setminus \{0\}$$

a nonzero holomorphic section. Then $\phi$ does not vanish at any point of $M$.

Proof. Assume that $\phi$ vanishes at a point $x_0 \in M$. Then

$$\phi^{\otimes j} \in H^0(M, E^{\otimes j})$$

vanishes at $x_0$ of order at least $j$.

Take any holomorphic vector bundles $A$ on $M$. For any nonzero holomorphic section $h \in H^0(M, A) \setminus \{0\}$, let $\text{Ord}_A(h, x_0)$ denote the order of vanishing of $h$ at $x_0$. It can be shown that the set of all such integers

$$\{\text{Ord}_A(h, x_0)\}_{h \in H^0(M, A) \setminus \{0\}}$$
is bounded above. Indeed, the space of holomorphic sections of $A$ vanishing at $x_0$ of order at least $k$ is a subspace of the finite dimensional vector space $H^0(M, A)$, and $H^0(M, A)$ is filtered by such subspaces.

Consider the decomposition of $E^\otimes j$ in (2.1); fix an isomorphism of $E^\otimes j$ with the direct sum. Take any $1 \leq k \leq p$ such that

1. $j_k > 0$, and
2. the projection of the section $\phi^\otimes j$ to some component $V_k$ of $E^\otimes j$ (of the $j_k$ components) is nonzero.

Since $\phi^\otimes j$ vanishes at $x_0$ of order at least $j$, the projection of $\phi^\otimes j$ to any component of $E^\otimes j$ vanishes at $x_0$ of order at least $j$ (if the projection does not vanish identically). But this contradicts the above observation that the orders of vanishing, at $x_0$, of the nonzero sections of a given holomorphic vector bundle are bounded above. This proves that $\phi$ does not vanish at any point of $M$. □

**Proposition 3.2.** Let $E$ and $F$ be finite holomorphic vector bundles on $M$, and let $\Phi : E \rightarrow F$ be an $\mathcal{O}_M$–linear homomorphism. Then the image $\Phi(E)$ is a subbundle of $F$.

**Proof.** Let $r$ be the rank of $\Phi(E)$. We assume that $r > 0$, because the proposition is obvious for $r = 0$. Consider the holomorphic vector bundle

$$\text{Hom}(\bigwedge^r E, \bigwedge^r F) = (\bigwedge^r F) \otimes (\bigwedge^r E)^*.$$  

Since $F$ is a finite vector bundle, we know that $\bigotimes^r F$ is also finite. Therefore, the direct summand $\bigwedge^r F$ of $\bigotimes^r F$ is also finite. Similarly, $(\bigwedge^r E)^*$ is also finite. Consequently, $\text{Hom}(\bigwedge^r E, \bigwedge^r F)$ is a finite bundle.

Let

$$\tilde{\Phi} := \bigwedge^r \Phi \in H^0(M, \text{Hom}(\bigwedge^r E, \bigwedge^r F))$$

be the homomorphism of $r$–th exterior products corresponding to $\Phi$. Since the rank of $\Phi(E)$ is $r$, the rank of the subsheaf $\tilde{\Phi}(\bigwedge^r E) \subset \bigwedge^r F$ is one, and the section $\tilde{\Phi}$ vanishes exactly on the closed subset of $M$ over which $\Phi(E)$ fails to be a subbundle of $F$. But $\tilde{\Phi}$ does not vanish anywhere by Lemma 3.1. Consequently, $\Phi(E)$ is a subbundle of $F$. □

**Lemma 3.3.** Let $E$ be a finite vector bundle on $M$. Then $E$ is semistable, and $\text{degree}(E) = 0$.

**Proof.** Consider the decomposition in (2.1). We have

$$\mu(\bigoplus_{i=1}^p V_i^\otimes j_i) = \sum_{i=1}^p j_i \cdot \text{rank}(V_i) / \sum_{i=1}^p j_i \cdot \text{rank}(V_i) \mu(V_i),$$

in particular, $\min\{\mu(V_i)\}_{i=1}^p \leq \mu(\bigoplus_{i=1}^p V_i^\otimes j_i) \leq \max\{\mu(V_i)\}_{i=1}^p$. Hence from (2.1) it follows that $\{\mu(E^\otimes j)\}_{j=1}^\infty$ is bounded. On the other hand,

$$\mu(E^\otimes j) = j \cdot \mu(E).$$
Therefore, we have \( \mu(E) = 0 \). This implies that \( \text{deg}(E) = 0 \).

If \( E \) is not semistable, take any subsheaf \( W \subset E \) such that \( \mu(W) > \mu(E) = 0 \). So \( \text{det}(W) \) is a subsheaf of \( \bigwedge^s E \subset \bigotimes^s E \), where \( s \) is the rank of \( W \). Hence the line bundle \( (\text{det}(W))^{\otimes j} \) is a subsheaf of \( \bigotimes^{rj} E \). Now from \((2.1)\) it follows that \( (\text{det}(W))^{\otimes j} \) is a subsheaf of some \( V_i \) for some \( i \in \{1, \ldots, p\} \) (the projection of \( (\text{det}(W))^{\otimes j} \) to one of the direct summands in \((2.1)\) has to be nonzero). This implies that 
\[
\mu((\text{det}(W))^{\otimes j}) \leq \max\{\mu_{\max}(V_1), \ldots, \mu_{\max}(V_p)\}.
\]
Consequently, the collection of real numbers \( \{\mu((\text{det}(W))^{\otimes j})\}_{j=1}^{\infty} \) is bounded above.

On the other hand, we have \( \mu((\text{det}(W))^{\otimes j}) = j\mu(W) \). Since \( \mu(W) > 0 \), this contradicts the above observation that \( \{\mu((\text{det}(W))^{\otimes j})\}_{j=1}^{\infty} \) is bounded above. Therefore, \( E \) is semistable.

**Proposition 3.4.** Let \( E \) be a finite vector bundle on \( M \), and let \( Q \) be a torsionfree quotient of \( E \) of degree zero. Then \( Q \) is locally free.

**Proof.** The open subset of \( M \) over which \( Q \) is locally free will be denoted by \( U \). The complement \( M \setminus U \) is a complex analytic subspace of \( M \) of codimension at least two; this is because \( Q \) is torsionfree.

Let \( r \) be the rank of \( Q \). The quotient map \( E \to Q \) produces an \( \mathcal{O}_M \)-linear homomorphism 
\[
\varphi : \bigwedge^r E \to \text{det}(Q).
\]
(3.1)

Note that the restriction \( \varphi|_U : (\bigwedge^r E)|_U \to (\text{det}(Q))|_U \) is surjective, because we have \( (\text{det}(Q))|_U = \bigwedge^r (Q|_U) \).

We shall first show that the holomorphic line bundle \( \text{det}(Q) \) is of finite order.

The image \( \varphi(\bigwedge^r E) \subset \text{det}(Q) \) will be denoted by \( L \), where \( \varphi \) is the homomorphism in \((3.1)\). Hence \( L^j \) is a quotient of \( (\bigwedge^r E)^{\otimes j} \). Since \( (\bigwedge^r E)^{\otimes j} \) is a component of \( E^{\otimes rj} \) (see Definition \((2.2)\)), from Theorem \((2.1)\) and \((2.1)\) we conclude that \( (\bigwedge^r E)^{\otimes j} \) is a direct sum of copies of \( V_1, \ldots, V_p \) (recall that \( V_1, \ldots, V_p \) are assumed to be indecomposable). The vector bundles \( V_1, \ldots, V_p \) are semistable of degree zero by Lemma \((3.1)\) and hence using Proposition \((2.4)\) it follows that the stable quotients of degree zero of a direct sum of copies of \( V_1, \ldots, V_p \) are just the stable quotients of degree zero of some \( V_i \) occurring in the direct sum \( \bigoplus_{i=1}^p V_i \).

Since \( \deg(L^j/\text{Torsion}) = j \cdot \deg(\text{det}(Q)) = 0 \), and \( L^j/\text{Torsion} \) is a quotient of a direct sum of copies of \( V_1, \ldots, V_p \), we conclude that \( L^j/\text{Torsion} \) is a quotient of some \( V_i, 1 \leq i \leq p \). Each \( V_i \) has only finitely many torsionfree quotients of rank one and degree zero (see Proposition \((2.4)\)); this can also be deduced from \([BDL\text{, p. 1034, Proposition 3.1}]\) (this proposition says that the semistable vector bundle \( V_i \) admits only finitely many isomorphisms classes of reflexive stable subsheaves of degree zero). Consequently, we have \( L^{a|U} = L^{b|U} \) for some \( a \neq b \). This implies that \( \text{det}(Q)^{\otimes a} = \text{det}(Q)^{\otimes b} \), because the complement \( M \setminus U \) is a complex analytic subspace of \( M \) of codimension at least two, and \( L|_U = \text{det}(Q)|_U \). Therefore, the line bundle \( \text{det}(Q) \) is of finite order.
Since det($Q$) is a finite bundle, from Proposition 3.2 we conclude that the homomorphism $\varphi$ in (3.1) is surjective.

Consider the projective bundle $\mathbb{P}(\bigwedge^r E) \longrightarrow M$ parametrizing the hyperplanes in the fibers of $\bigwedge^r E$. Let Gr($E$) be the Grassmann bundle over $M$ parametrizing the $r$-dimensional quotients of the fibers of $E$. Let

$$\beta : \text{Gr}(E) \longrightarrow \mathbb{P}(\bigwedge^r E) \quad (3.2)$$

be the Plücker embedding that sends a quotient $\hat{q}_x : E_x \longrightarrow Q_x$ of dimension $r$ to the kernel of the homomorphism $\bigwedge^r \hat{q}_x : \bigwedge^r E_x \longrightarrow \bigwedge^r Q_x$.

Let

$$\sigma : M \longrightarrow \mathbb{P}(\bigwedge^r E)$$

be the holomorphic section defined by the surjective homomorphism $\varphi$ in (3.1). The image of the restriction $\sigma|_U$ lies in $\beta(\text{Gr}(E)|_U) \subset \mathbb{P}(\bigwedge^r E)$, where $\beta$ is the map in (3.2). Consequently, $\sigma(M)$ lies in $\beta(\text{Gr}(E))$, because $\beta(\text{Gr}(E))$ is a closed submanifold of $\mathbb{P}(\bigwedge^r E)$.

Since $\sigma(M)$ lies in $\beta(\text{Gr}(E))$, it is evident that $Q$ is a quotient bundle of $E$. $\square$

4. Flat connection on finite bundles

A holomorphic vector bundle $E$ over $M$ would be called étale trivial if there is a finite connected étale covering

$$\varpi : Y \longrightarrow M$$

such that the vector bundle $\varpi^*E$ is holomorphically trivializable.

**Proposition 4.1.** Let $E$ be a finite stable vector bundle on $M$. Then $E$ is étale trivial.

*Proof.* Consider the vector bundles $V_1, \ldots, V_p$ in (2.1). As before, we assume that each $V_i$ is indecomposable and it is a component of some $E \otimes j$. From the given condition that $E$ is stable it follows that $E \otimes j$ is polystable (see Corollary 2.3). Using Lemma 3.3 we conclude that the degree of $E \otimes j$ is zero. Hence any component (see Definition 2.2) of $E \otimes j$ is also polystable of degree zero. In particular, all $V_i$ are polystable of degree zero. Since each $V_i$ is also indecomposable, it is stable of degree zero.

If $V_i$ is a component (see Definition 2.2) of $E \otimes j$, then $V_i \otimes n$ is a component of $E \otimes nj$, so $V_i \otimes n$ is also a direct sum of copies of $V_1, \ldots, V_p$.

Any nonzero homomorphism between two stable vector bundles of degree zero is an isomorphism. Take any reflexive subsheaf $S$ of degree zero of a direct sum $U$ of copies of $V_1, \ldots, V_p$. Then $S$ is a subbundle of $U$ by Proposition 3.4. Moreover, $S$ is a component of $U$, because $U$ is polystable of degree zero, and hence $S$ is a direct sum of copies of $V_1, \ldots, V_p$. Furthermore, any torsionfree quotient of degree zero of such a subsheaf $S$ is again a direct sum of copies of $V_1, \ldots, V_p$, because $S$ is polystable of degree zero. In
particular, for any any homomorphism

\[ f : \bigoplus_{i=1}^{p} V_i^{\oplus c_i} \to \bigoplus_{i=1}^{p} V_i^{\oplus d_i}, \]

both kernel\((f)\) and cokernel\((f)\) are direct sums of copies of \(V, \cdots, V_p\).

Fix a point \(x_0 \in M\). Assign the vector space \(F_{x_0}\) to any holomorphic vector bundle \(F\) on \(M\). In view of the above observations, \(V_1, \cdots, V_p\) produce a Tannakian category (defined in \([\mathrm{No1}, \text{p. 30}], [\mathrm{No2}, \text{p. 76}]\)). Now using \([\mathrm{No1}, \text{p. 31, Theorem 1.1}], [\mathrm{No2}, \text{p. 77, Theorem 1.1}])\), this Tannakian category produces a complex affine group-scheme. The isomorphism classes of indecomposable objects of this Tannakian category are contained in the union of the following three:

(1) \(\{V_1, \cdots, V_p\}\),

(2) \(\{V_1^*, \cdots, V_p^*\}\), and

(3) all indecomposable components (see Definition 2.2) of all \(V_i \otimes V_j^*\), \(1 \leq i, j \leq p\).

Note that \(V_i \otimes V_j^*\) is polystable by Corollary 2.3; also, degree\((V_i \otimes V_j^*)\) = 0, because degree\((V_i)\) = 0 for all \(i\). So all indecomposable components of \(V_i \otimes V_j^*\) are stable of degree zero.

Since the above union is a finite collection, from \([\mathrm{No1}, \text{p. 31, Theorem 1.2}], [\mathrm{No2}, \text{p. 77, Theorem 1.2}])\) it follows that the group-scheme defined by the above Tannakian category is finite; this finite group will be denoted by \(\Gamma\).

Fix a basis of \(E_{x_0}\). Now the structure group \(\text{GL}(r_E, \mathbb{C})\) of \(E\), where \(r_E\) denotes the rank of \(E\), has a reduction of structure group to this group \(\Gamma\) \([\mathrm{No2}, \text{p. 79, Theorem 2.9}])\) (see also \([\mathrm{No2}, \text{p. 34, Proposition 2.9}])\). Since our group \(\Gamma\) is finite, the proof of \([\mathrm{No2}, \text{Theorem 2.9}])\) remains valid without any modification.

Therefore, there is a finite étale Galois covering \(\varpi : \tilde{Y} \to M\) with Galois group \(\Gamma\) such that the vector bundle \(\varpi^*E\) is holomorphically trivializable. Now taking \(Y\) to be a connected component of \(\tilde{Y}\) we conclude that \(E\) is étale trivial. \(\square\)

Let \(E\) be a finite vector bundle over \(M\). As in Proposition 2.4 let

\[ 0 = F_0 \subset F_1 \subset \cdots \subset F_{m-1} \subset F_m = E \quad (4.1) \]

be a filtration of reflexive subsheaves of \(E\) such that \(F_k/F_{k-1}\) is stable of degree zero for all \(1 \leq k \leq m\). From Proposition 3.4 we know that each \(F_k\) is a subbundle of \(E\).

**Lemma 4.2.** For every \(1 \leq k \leq m\), the holomorphic vector bundle \(F_k/F_{k-1}\) in \((4.1)\) is finite.

**Proof.** Consider the vector bundles \(V_1, \cdots, V_p\) in \((2.1)\). Since they are all finite, just as in \((4.1)\), for every \(1 \leq i \leq p\) there is a filtration of holomorphic subbundles

\[ 0 = U^i_0 \subset U^i_1 \subset \cdots \subset U^i_{n_i-1} \subset U^i_{n_i} = V_i \quad (4.2) \]
such that $U_i^j/U^j_{i-1}$ is a stable vector bundle of degree zero for all $1 \leq \nu \leq n_i$. Now consider the collection of stable vector bundles
\[ \{U_i^j/U^j_{i-1}\}_{i=1}^{n_i} \] (4.3)
From Proposition 2.4 it follows that this collection does not depend on the choice of the filtrations in (4.2).

Consider the vector bundle $F_k/F_{k-1}$ in the statement of the proposition. Since $F_k/F_{k-1}$ is stable, by Corollary 2.3, the vector bundle $(F_k/F_{k-1})^{\otimes j}$ is polystable for every positive integer $j$. Also,
\[ \mu((F_k/F_{k-1})^{\otimes j}) = j \cdot \mu(F_k/F_{k-1}) = 0. \]
Since $F_k^{\otimes j}$ is a subbundle of $E^{\otimes j}$ of degree zero, and $E^{\otimes j}$ is semistable of degree zero (recall that $E^{\otimes j}$ is a finite bundle), it follows that $F_k^{\otimes j}$ is semistable of degree zero. Also, $(F_k/F_{k-1})^{\otimes j}$ is a polystable quotient of $F_k^{\otimes j}$ of degree zero. From these it follows that $(F_k/F_{k-1})^{\otimes j}$ is a direct sum of copies of stable graded factors of $E^{\otimes j}$ (see Definition 2.5).

Now using (2.1) and Proposition 2.4 we conclude that $(F_k/F_{k-1})^{\otimes j}$ is a direct sum of copies of the vector bundles in the collection in (4.3). Therefore, $F_k/F_{k-1}$ is a finite bundle. \[\Box\]

Theorem 4.3. A holomorphic vector bundle on $M$ is finite if and only if it is étale trivial.

Proof. Let $E$ be a finite vector bundle on $M$. Take a filtration of subbundles of it
\[ 0 = F_0 \subset F_1 \subset \cdots \subset F_m \subset E \]
as in (4.1). From Lemma 4.2 and Proposition 4.1 we know that $F_k/F_{k-1}$ is étale trivial for all $1 \leq k \leq m$. Hence there is a finite connected étale Galois covering
\[ \varpi : Y \longrightarrow M \]
such that for the pulled back filtration
\[ 0 = \varpi^*F_0 \subset \varpi^*F_1 \subset \cdots \subset \varpi^*F_m \subset \varpi^*E, \] (4.4)
the quotient bundle $(\varpi^*F_k)/(\varpi^*F_{k-1}) = \varpi^*(F_k/F_{k-1})$ is trivializable for all $1 \leq k \leq m$ (take a connected component of the fiber product of the étale Galois coverings for each $F_k/F_{k-1}$). Also, $\varpi^*E$ is finite because $E$ is so.

Lemma 4.3 of [BP] says that if a finite vector bundle $W$ admits a filtration of holomorphic subbundles such that each successive quotient is holomorphically trivializable, then $W$ is holomorphically trivializable. From this and (4.4) it follows that $\varpi^*E$ is holomorphically trivializable. Hence $E$ is étale trivial.

To prove the converse, let $E$ be an étale trivial vector bundle on $M$. Let
\[ \varpi : Y \longrightarrow M \]
be a connected étale covering such that $\varpi^*E$ is holomorphically trivializable. Consider the trivial connection $\nabla$ on $\varpi^*E$ corresponding to any holomorphic trivialization of $\varpi^*E$. This connection $\nabla$ does not depend on the choice of the holomorphic trivialization of $\varpi^*E$. 
It can be shown that $\nabla$ descends to a connection on $E$. Indeed, by taking an étale covering of $Y$ we may assume that $\varpi$ is Galois. The connection $\nabla$ is invariant under the action of the Galois group $\text{Gal}(\varpi)$ on $\varpi^*E$, and hence $\nabla$ descends to a connection on $E$. This descended connection on $E$ is holomorphic flat because $\nabla$ is holomorphic flat; also, its monodromy group is finite because $\varphi$ is a finite covering and the monodromy group of $\nabla$ is trivial. This immediately implies that $E$ is finite; see [BP, p. 7, Section 4]. □

Theorem 4.3 and its proof together have the following corollary.

**Corollary 4.4.** A holomorphic vector bundle on $M$ is finite if and only if it admits a flat holomorphic connection with finite monodromy.

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