Thermal Green Functions at Zero Energy

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Abstract

The thermal expectation values of all possible bosonic generalised retarded functions evaluated at zero energy are studied. The relationship of such functions to calculational schemes, technical problems and physical applications is outlined. It is then shown that all generalised retarded functions constructed from any one set of bosonic fields are equal at zero energy. This is done completely generally and is not limited to any approximation scheme such as perturbation theory.

1 Introduction

There are many types of Green function which appear in thermal field theory. The retarded and advanced Green functions [1, 2] are merely a subset of the much more numerous GRF (generalised retarded functions) [3, 4, 5]. The GRF as a whole are most easily calculated in ITF (the Imaginary-Time Formalism or Matsubara method) [2, 6, 7, 11], and they are the only ones calculated directly in ITF [4, 5]. Conversely calculations in ITF invariably produce GRF. If a real-time formalism is used [1, 6, 7] of all the GRF it is only known how to extract the retarded and advanced functions [1, 4, 5]. However the precise method used to calculate the GRF is of no relevance to the manipulations performed in later sections. Calculational schemes will only be discussed with respect to the motivation and background to the problem of bosonic GRF at zero energy, to which we now turn.

The zero energy Green functions have proved to have a much more complicated behaviour than their zero temperature counterparts. This has been seen when specific
examples are calculated \([8]\). This is due to the many cuts running across the zero energy point \([9]\). By looking at simple perturbative examples \([9]\), one can quickly see that for a thermal Green function there are always cuts running across the zero energy point (though not always at the lowest order of perturbation theory). Physically this is due to Landau damping processes.

The zero energy Green functions are of great physical importance. At zero temperature they are generated by the lowest term in a time derivative expansion of the effective action. However, how can one make derivative expansions of the effective action in thermal field theories when zero-energy thermal Green functions show a marked dependence \([8]\) on the order in which spatial and time derivative expansions are made? Conversely, precisely what sort of thermal Green functions (retarded, advanced, GRF, time ordered, thermal Wightman, etc.) are generated by the effective action derivative expansions used in the literature?

A related issue is that the zero energy Green functions ought by default to be pure real if one is thinking in terms of effective action expansions around a simple stable vacuum configuration. However, from a mathematical point of view, why should these thermal Green functions be real when they are generally complex near zero energy due to cuts which are present in the thermal case?

Another question involves direct calculations of Green functions at zero energy in ITF. In this case all the external energies are set directly equal to the zero discrete Euclidean energy value. Why though do we pay no heed to the large number of cuts which are known to pass across the zero energy point? In turn, how does this static ITF calculation relate to the zero energy limits of the different GRF \([8]\). Looking at it from another point of view, why, when using contour methods to evaluate ITF energy sums, does it not matter which side of the cuts the discrete energy point at zero energy is put?

In the following study of the zero energy limit of the GRF at finite temperature, we will use the approach used in \([4, 5]\) to study general thermal Green functions. This uses the definitions of the Green functions as expectation values of fields and the fundamental relations between them such as the KMS condition. These relations only involve factors of \(e^{-\beta p}\) where \(p\) is an external energy. Thus looking at the low energy limit is equivalent to looking at the infinite temperature limit and the results are valid for either viewpoint. No other scales are relevant to the results of this analysis! A big advantage of such analysis is that the results are completely general, one can examine any type of field, and they apply to both the full Green functions and to any sensible approximation to the Green functions. All that is required is that the fundamental relations between Green functions are satisfied in whatever approximation scheme is used. As the KMS condition is equivalent to the definition of what thermal temperature field theory is, there is no problem with any useful approximation.

In section two, we recall some of the basic results of \([4]\) to be used in the subsequent proofs. This will also serve to establish the notation used. The most general proof given here involves a hideous shuffling and partitioning of subscripts and subsubscripts. For this reason in the following sections, simpler cases, which illustrate the basic principles involved, are considered in order of increasing complexity, before the final most general and undignified proof is presented.
2 Thermal Generalised Retarded Functions

Throughout this paper all quantities are being measured from the rest-frame of the heat bath. Any dependence on spatial coordinates or three-momentum is not written explicitly as this does not effect the arguments given here. This is because the thermal boundary conditions only involve the external energies. Variables such as \( p, k \) etc. refer to real Minkowskii energies and not energy-momentum four-vectors. When we have continued to the complex energy plane, we will denote general complex energies by \( z \).

We will consider a set of \( N \) bosonic fields labeled \( \phi_1, \ldots, \phi_N \) as in principle they may be distinct types of field (e.g. different components of scalar or gauge fields) as well as carrying their own time arguments. These appear in many different orders so it is convenient to use subscripts \( \phi_{b_j} \) where the \( N \) subscripts \( b_j \) (\( j = 1 \ldots N \)) are some permutation of the integers from 1 to \( N \). We will then need to shuffle these subscripts round further to enable different terms to be compared. This means that the subscripts are now manipulated. Any reference to subsubscripts outside the range \( j = 1 \ldots N \) are to be understood to be modulo \( N \) i.e. \( b_0 = b_N \) etc.

In ITF one calculates expectation values of Euclidean time ordered fields whose time arguments lie between 0 and \(-i\beta\). The result of a calculation of an \( N \)-point function in ITF is given by some function, say \( \Phi \), evaluated at discrete Euclidean energy points, i.e. \( \Phi(\{z_j = 2\pi\nu_j/\beta\}) \) (\( \nu_j \) are integers for bosonic fields). To look at the region around the zero energy point we need to make an analytic continuation. Analytic continuation to general complex energies, \( \{z\} \), can be performed uniquely if one chooses certain behaviour at large \( |z| \) \[4, 10\]. In many practical perturbative calculations this is usually trivial. The resulting function, \( \Phi(\{z\}) \), is found to represent the analytic continuation from real to complex energies of what are called GRF (Generalised Retarded Functions) \[3, 5\]. The form found is \[4, 5\]

\[
\Phi(\{z\}) = \left(\frac{-1}{2\pi}\right)^{N-1} \int dk_1 \ldots dk_j \delta(\sum_{j=1}^{N} k_j) \sum_{\text{perm.}\{b\}} \mathcal{W}(b_1, b_2, \ldots, b_N; \{k\}) \prod_{j=2}^{N} \frac{i}{B_j^N} \tag{2.1}
\]

where the first sum takes \( \{b_j\} \) through all permutations of the numbers \((1, 2, \ldots, N)\) and

\[
B_i^j = \sum_{l=0}^{\lfloor j-i\rfloor N} (z_{b_{i+l}} - k_{b_{i+l}}), \tag{2.2}
\]

To write \( \Phi \) in this way, an \( N \)-th redundant complex energy variable has been introduced defined through the constraint

\[
\sum_{j=1}^{N} z_j = 0. \tag{2.3}
\]

The \( \mathcal{W}(b_1, b_2, \ldots, b_N; \{k\}) \) are the thermal Wightman functions in energy space defined for pure bosonic fields to be

\[
\tilde{\mathcal{W}}(b_1, b_2, \ldots, b_N; \tau_1, \tau_2, \ldots, \tau_N) = \]

3
\begin{equation}
Tr\{e^{-\beta H} \phi_{b_1}(\tau_{b_1})\phi_{b_2}(\tau_{b_2})...\phi_{b_N}(\tau_{b_N})\}/Tr\{e^{-\beta H}\}. \tag{2.4}
\end{equation}

\begin{equation}
\tilde{\mathcal{W}}(b_1, b_2, ..., b_N; \tau_1, \tau_2, ..., \tau_N) = (2\pi)^{-N} \left( \prod_{j=1}^{N} \int_{-\infty}^{\infty} dp_j e^{-ip_j\tau_j} \right) \mathcal{W}(b_1, b_2, ..., b_N; p_1, p_2, ..., p_N) \tag{2.5}
\end{equation}

The \{\tau\} are complex times. From this definition in terms of the trace and Boltzmann factor we find a fundamental property of these Green functions, a generalisation to \(N\)-point functions of the well known KMS (Kubo-Martin-Schwinger) condition for two-point functions \[6, 7, 10, 11\], namely for pure bosonic functions

\begin{equation}
\exp(-\beta p_{b_1})\mathcal{W}(b_1, b_2, ..., b_N; p_1, p_2, ..., p_N) = \mathcal{W}(b_1, b_2, ..., b_N; p_1, p_2, ..., p_N). \tag{2.6}
\end{equation}

Immediately we can see that there are poles in the integrand of (2.1) for real external energies. This leads to discontinuities in \(\Phi(\{z\})\) at real energies (as \(\mathcal{W} \neq 0\) in general for the thermal case) so we must specify how we approach the real energy axes. In fact we find that there are many different results depending on how the axes are approached (many more than \(N!\) in general \[4, 5\]). We will specify which sides of the cuts along the real energy axes are being considered by setting the complex energies to

\begin{equation}
z_i = p_1 + \epsilon_i, \quad p_i, \epsilon_i \in \mathbb{R}e, \quad |\epsilon_i| \ll 1 \tag{2.7}
\end{equation}

To ensure that one is not sitting on any cuts, and from (2.3), the epsilons must satisfy:-

1. All sums of subsets of the epsilons must be non zero

\begin{equation}
\sum_{j=1}^{N} c_j \epsilon_j \neq 0 \quad \forall \{c_j\} = \{0, 1|0 < \sum_{j=1}^{N} c_j < N\} \tag{2.8}
\end{equation}

2. The sum of all epsilons must equal zero

\begin{equation}
\sum_{j=1}^{N} \epsilon_j = 0 \tag{2.9}
\end{equation}

It is the sign of the epsilons and the sign of all possible sums of the epsilons which completely specify which side of the real energy cuts of \(\Phi\) function is being studied. Complex energy space is split into a large number of regions, each bounded by real energy cuts and each region corresponds to a unique GRF \[4, 5\]. We therefore define two sets of epsilons to be equivalent if they have selected the same region of complex energy space and so are giving precisely the same GRF.

\begin{equation}
\{\epsilon\} \equiv \{\epsilon'\} \quad \text{iff} \quad \text{SGN}(\sum_{j=1}^{j=N} c_j \epsilon_j) = \text{SGN}(\sum_{j=1}^{j=N} c_j \epsilon'_j) \quad \forall \{c_j\} = \{0, 1|0 < \sum_{j=1}^{N} c_j < N\} \tag{2.10}
\end{equation}

where

\begin{equation}
\text{SGN}(x) = \begin{cases} +1, & x > 0 \\ -1, & x < 0 \end{cases} \tag{2.11}
\end{equation}
An important subset of the generalised retarded functions comes from just $2N$ different ways of choosing these epsilons. Suppose in (2.7) we pick out one epsilon, and set it positive, and then set all the other epsilons negative, say $\epsilon_a = (N-1)\epsilon$, $\epsilon_{\text{other}} = -\epsilon$ where $\epsilon$ is an infinitesimal positive quantity. In this case, the ITF result after analytic continuation, is found to be, in terms of real times,

$$\Phi^{(N)}(\{t\}; \epsilon_a > 0, \epsilon_{\text{other}} < 0) = R_a(\{t\}),$$

where $R_a$ is one of the $N$ retarded $N$-point functions. For pure bosonic fields they can be written as

$$R_a(t_1, t_2, ..., t_N) = \sum_{\text{perm}(a)|a_N=a} \theta(t_a - t_{aN-1})\theta(t_{aN-1} - t_{aN-2})\theta(t_{aN-2} - t_{a1}).$$

where $\phi_a = \phi_a(t_a)$ and the $\phi_a$ can be different bosonic fields. The sum takes the $\{a_j\}$, $j = 1$ to $N-1$, through all permutations of the numbers 1 to $N$ less the number $a$. The number $a$, which is the subscript on the $R$, indicates that the $a$'th field has the largest time and we set $a_N = a$.

The $N$ advanced functions, $A_a$, are obtained in an identical manner except that the theta functions are reversed in time and an overall factor of $(-1)^{N-1}$ is added. This corresponds to switching the signs of all the epsilon in (2.1) terms so we have

$$\Phi^N(\{t\}; \epsilon_a < 0, \epsilon_{\text{other}} > 0) = A_a(\{t\}).$$

The retarded and advanced functions are merely a subset of the generalised retarded functions \cite{3, 4, 5}. In this case the various different analytic continuations of $\Phi$ to the real energy axes form a definition of the generalised retarded functions in energy space. This in turn then gives a definition of the GRF in terms of fields and theta functions in real time.

### 3 Two-point functions

The method to be used here to study zero energy N-point Green functions is easily illustrated for two-point functions. Some well known results are reproduced in a slightly more cumbersome notation \cite{7, 10, 11}. For $N = 2$ we have

$$\Phi^{(2)}(z_1) = \frac{1}{2\pi} \int dk_1 dk_2 \delta(k_1 + k_2) \left[ W(12; k) \frac{i}{z_1 - k_1} + W(21; k) \frac{i}{z_2 - k_2} \right]$$

$$R^{(2)}(p_1) = \Phi(p_1 + i\epsilon), \quad A^{(2)}(p_1) = \Phi(p_1 - i\epsilon)$$

where $\epsilon$ without a subscript is an infinitesimal positive real number, $p_1 + p_2 = 0$, and $\epsilon_1 + \epsilon_2 = 0$. We use the usual representation

$$\frac{i}{p_j - k_j + i\epsilon_j} = \frac{i\text{PP}}{p_j - k_j} + \theta_j \pi \delta(p_j - k_j)$$

where

$$\Phi^{(N)}(\{t\}; \epsilon_a > 0, \epsilon_{\text{other}} < 0) = R_a(\{t\}),$$
where $\theta_j = +1 \ (-1)$ when $\epsilon_j > 0 \ (<0)$. The PP indicates that the principal part is to be taken.

We now use (3.3) to split (3.1) into two pieces

$$\Phi^{(2)}(z_1) = \Phi^{(2)}_0(z_1) + \Phi^{(2)}_1(z_1)$$

(3.4)

From this we find for the two-point functions

$$\Phi^{(2)}_0(p_1 + i\epsilon_1) = \frac{1}{2\pi} \int dk_1dk_2 \delta(k_1 + k_2) \left[ W(12; k) \frac{i\text{PP}}{p_1 - k_1} + W(21; k) \frac{i\text{PP}}{p_2 - k_2} \right]$$

$$\Phi^{(2)}_1(p_1 + i\epsilon_1) = \frac{1}{2} \int dk_1dk_2 \delta(k_1 + k_2) \left[ W(12; k)\theta_1\delta(p_1 - k_1) + W(21; k)\theta_2\delta(p_2 - k_2) \right]$$

(3.5)

The KMS condition (2.6) tells us that $e^{-\beta p}W(12; p) = W(21; p)$. If we now use this in the zero energy limit, and use the various relations between the energy variables such as

$$\theta_1 + \theta_2 = 0$$

(3.6)

we find

$$\Phi^{(2)}_0(p_1 + i\epsilon_1) = \frac{1}{2\pi} \int dk (W(12; k) - W(21; k)) \frac{i\text{PP}}{-k} = R^{(2)}(p = 0) = A^{(2)}(p = 0)$$

(3.7)

$$\Phi^{(2)}_1(p_1 + i\epsilon_1) = 0$$

(3.8)

This is a well known result [10, 7, 11] that the imaginary part of two-point bosonic functions is zero at zero energy and arbitrary three-momenta. Physically this result is important for the study of electric and magnetic static field screening in a gauge theory.

### 4 Three-point functions

For three-point functions there are three retarded, $R_a$, and three advanced, $A_a$, functions. They illustrate why the general $N$-point case is not nearly as simple as the two-point example. From (2.11), the three-point GRF is given by

$$\Phi^{(3)}(z_1, z_2, z_3 = -z_1 - z_2) = \frac{1}{4\pi^2} \int dk_1dk_2dk_3 \delta(k_1 + k_2 + k_3) \sum_{\text{perm.123}} W(123; k_1, k_2, k_3) \frac{i}{z_2 + k_3 - k_2} \frac{i}{z_3 - k_3}$$

(4.1)

where the sum is over all permutations of the indices. The retarded and advanced products are obtained by considering the 6 regions of complex energy space separated by the cuts along the real energy axes. We set $z_a = p_a + i\epsilon_a$ where $p_a, \epsilon_a \in \mathbb{R}$, $\epsilon_a$ is a non-zero
infinitesimal and \( \theta_a = +1 \) \((-1)\) if \( \epsilon_a > 0 \) \(< 0\) etc. We then split \( \Phi^{(3)} \) into three pieces using (3.3) and find

\[
\Phi^{(3)} = \Phi^{(3)}_0 + \Phi^{(3)}_1 + \Phi^{(3)}_2 \tag{4.2}
\]

where

\[
\Phi^{(3)}_0(p_1 + i \epsilon_1, p_2 + i \epsilon_2) = \frac{1}{4\pi^2} \int dk_1dk_2dk_3 \delta(k_1 + k_2 + k_3) \sum_{\text{perm.123}} W(123; k_1, k_2, k_3) \frac{iPP}{(p_2 + p_3 - k_2 - k_3)(p_3 - k_3)} \tag{4.3}
\]

\[
\Phi^{(3)}_1(p_1 + i \epsilon_1, p_2 + i \epsilon_2) = \frac{1}{4\pi} \int dk_1dk_2dk_3 \delta(k_1 + k_2 + k_3) \sum_{\text{perm.123}} W(123; k_1, k_2, k_3) \left[ \frac{iPP}{p_2 + p_3 - k_2 - k_3} \theta_3 \delta(p_3 - k_3) + \theta_{23} \delta(p_2 + p_3 - k_2 - k_3) \frac{iPP}{p_3 - k_3} \right] \tag{4.4}
\]

\[
\Phi^{(3)}_2(p_1 + i \epsilon_1, p_2 + i \epsilon_2) = \frac{1}{4} \sum_{\text{perm.123}} W(123; p_1, p_2, p_3) \theta_{23} \theta_3 \tag{4.5}
\]

where \( \theta_{23} = +1 \) \((-1)\) if \( \epsilon_2 + \epsilon_3 > 0 \) \(< 0\) etc. Clearly \( \Phi^{(3)}_0 \) is independent of the which argument we choose to be retarded or advanced, while the remaining two are not obviously independent of which retarded or advanced function we are looking at.

Looking at the \( \Phi^{(3)}_2 \) term we can show that it is independent of the index \( a \) at zero external energy by using the KMS condition for bosonic functions (2.6). For three-point functions at zero energy this is

\[
W(312; 0) = W(123; 0) = W(231; 0), \quad W(321; 0) = W(213; 0) = W(132; 0). \tag{4.6}
\]

We also have that

\[
\theta_{23} \theta_3 + \theta_{31} \theta_1 + \theta_{12} \theta_2 = -1 \tag{4.7}
\]

whatever \( a \) is chosen. This then gives

\[
\frac{1}{2} \left( R^{(3)}_a(0, 0) + A^{(3)}_a(0, 0) \right) = \Phi^{(3)}_0 - \frac{1}{4} (W(123; 0) + W(321; 0)) \text{ independent of } a. \tag{4.8}
\]

The terms with neither all delta functions nor all principal parts are the difficult ones in the general case, and the \( \Phi^{(3)}_1 \) term illustrates how such terms are manipulated. The principal part integrands have to be integrated over and it is difficult to say what these give in general. It also means we can not put all the integration variables to zero in any one term, as we could with \( \Phi^{(3)}_0 \) and so the equality (4.6) between bosonic \( N \)-point functions at zero energy can not be immediately used. One proceeds by collecting terms with the same delta functions. Consider terms containing \( \delta(p_3 - k_3) \). This comes from two places, the term shown explicitly in (3.4) and one from the second delta function
term where we have cycled the indices $123 \rightarrow 231$. Thus we can rewrite the second term in the square bracket in (4.4) as

$$\Phi_1^{(3)}(p_1 + i\epsilon_1, p_2 + i\epsilon_2) = \frac{1}{4\pi} \int dk_1 dk_2 dk_3 \delta(k_1 + k_2 + k_3)$$

$$\sum_{\text{perm.} 123} \left[ \theta_3 \delta(p_3 - k_3) W(123; k_1, k_2, k_3) \frac{i\text{PP}}{p_2 + p_3 - k_2 - k_3} + \theta_{12} \delta(p_1 + p_2 - k_1 - k_2) W(312; k_1, k_2, k_3) \frac{i\text{PP}}{p_2 - k_2} \right]$$

(4.9)

Now at zero external energy, the delta function does at least set one of the integration variables to zero. Together with the KMS condition (2.6) for pure bosonic functions in this case, we find

$$W(312; k_1, -k_1, 0) = W(123; k_1, -k_1, 0)$$

(4.10)

$$\frac{i\text{PP}}{p_2 - k_2} = -\frac{i\text{PP}}{p_2 + p_3 - k_2 - k_3}$$

(4.11)

$$\theta_3 + \theta_{12} = 0$$

(4.12)

and hence we see that

$$\frac{1}{2} \left( R_a^{(3)}(p = 0) - A_a^{(3)}(p = 0) \right) = \Phi_1^{(3)} = 0.$$ 

(4.13)

This leaves us with

$$R_a^{(3)}(0, 0) = A_a^{(3)}(0, 0) = \Phi_0^{(3)} - \frac{1}{4} (W(123; 0) + W(321; 0))$$ independent of $a.$

(4.14)

So all the three-point GRF made out of the same set of bosonic fields are equal at zero energy.

### 5 N-point functions

Using (3.3) split up $\Phi$,

$$\Phi(\{p + i\epsilon\}) = \sum_{l=0}^{N-1} \Phi_l(\{p + i\epsilon\})$$

(5.1)

where $\Phi_l$ has $l$ of the $B$ terms of $\phi$ in (2.1) replaced by delta functions and the other $N - l - 1$ factors of $B$ are replaced by principal parts. Thus

$$\Phi_l(\{z_a = p_a + i\epsilon_a\}) = \left( \frac{-1}{2} \right)^N \left( \frac{1}{\pi} \right)^{N-1} \int dk_1 \ldots dk_N \delta(k_1 + k_2 + \ldots + k_N).$$

$$\sum_{\text{perm.} \{b\}} \sum_{\{x\} \in X_0} W(b_1 b_2 \ldots b_N; \{k\}) \frac{i\text{PP}}{B_N^2} \ldots \frac{i\text{PP}}{B_{x_1 - 1}^N} \psi_{x_1}^N \delta(B_{x_1}^N) \frac{i\text{PP}}{B_{x_1 + 1}^N} \ldots \psi_{x_l}^N \delta(B_{x_l}^N)$$

(5.2)
where we are replacing the \(x_i\)-th \(B\) factor, \(B_{x_i}^N\), by a delta function and the other \(B\) factors are replaced by their principal parts. The effective definition of \(B\) is now

\[
B^j_i = B(i,j; \{p-k\}) = \sum_{l=0}^{\lfloor j-i \rfloor_N} (p_{b_{l+i}} - k_{b_{l+i}}),
\]

as the infinitesimal imaginary parts of the external energies are dealt with explicitly. This replacement of the \(B\) terms by principal parts and delta functions is done in all possible ways, and this performed by the \(\sum_{\{x\}}\). This sum takes the \(x_j\) variables through all values between 1 and \(N\) keeping the \(x\)'s in ascending order i.e. \(\{x\} \in X_0\) where

\[
X_j = \{\{x\} | 1 + j = x_0 < x_1 < \ldots < x_l \leq N + j \equiv x_0 - 1\}.
\]

It is implicit throughout this work that whenever needed, indices, such as the \(x\)'s and \(b\)'s, are periodic so that \(x_j \equiv x_j + N\). The signs of the small epsilon regulating terms are encoded by the \(\psi = \pm 1\) factors which are defined to be

\[
\psi^m_j = \begin{cases} +1 & \text{if } \sum_{l=0}^{m-j} \epsilon_{b_{l+i}} > 0 \\ -1 & \text{if } \sum_{l=0}^{m-j} \epsilon_{b_{l+i}} < 0 \end{cases}
\]

The idea is to exploit the delta functions in (5.2). The indices are cycled as a subset of the sum over all permutations of indices. This means that terms with similar delta functions and principal parts appear \(l + 1\) times. In fact one can quickly see that by using the delta functions, the KMS condition and the fact that we will put \(\{p\}\) to zero, the only complicated part comes from the \(\psi\) terms.

It is more useful, in view of the cyclic properties of the thermal Wightman functions, to write out this part of the permutation of \(b\) indices explicitly. It is possible to write \(\Phi_l\) in a slightly different form, namely

\[
\Phi_l(\{z_a = p_a + i\epsilon_a\}) = \left(\frac{-1}{2}\right)^{N-1} \left(\frac{1}{\pi}\right)^{N-l-1} \int dk_1 \ldots dk_j \delta(k_1 + k_2 + \ldots + k_N) \sum_{\text{perm.\{b\}\mid NC}} \sum_{j=0}^{N-1} \sum_{\{x\} \in X_j} \mathcal{W}(b_{x_0}b_{x_0+1} \ldots b_{x_0-1}; \{k\}).
\]

\[
\frac{iPP}{B_{x_0+1}} \ldots \psi^{N+j}_{x_1} \delta(B_{x_1}^{N+j}) \ldots \psi^{N+j}_{x_l} \delta(B_{x_l}^{N+j}) \ldots \frac{iPP}{B_{N+j}}
\]

Here \(\sum_{\text{perm.\{b\}\mid NC}}\) indicates that no cyclic permutations are included when summing over the permutations of the \(b\) indices, the sum over different cycles of a given permutation being performed by the \(j\) sum. \(X_j\) was defined in (5.4).

The next step is to try and sum over all cycles of the partitions, \(\{x\}\), of the \(b\) indices, and then to vary the partitions instead of summing over all cycles of the partitions and then partitioning up the indices into \(l + 1\) pieces. Now define the label \(k\) such that \(x_k - N\) is the smallest positive integer for a given set \(\{x\}\) i.e.

\[
1 \leq x_k - N < x_{k+1} < \ldots < x_l - N < x_0 < \ldots < x_{k-1} \leq N.
\]
We can relabel these \( \{x\} \) partitions using \( \{y\} \), \( i = 0, \ldots, l \), such that \( y_0 \) (\( y_l \)) is the smallest (biggest) in this series i.e. \( \{y\} \in Y \) where

\[
Y := \{ \{y\} | 1 \leq y_0 < y_1 < \ldots < y_l \leq N \} \tag{5.8}
\]

The definition is therefore

\[
y_i = x_{i+k} - N \quad \text{if} \quad 0 \leq i \leq l - k
\]
\[
y_i = x_{i+k-l-1} \quad \text{if} \quad l - k < i \leq l, \tag{5.9}
\]

so \( y_{l-k+1} = x_0 = j + 1, y_{l-k} = x_l \leq j + 1 + N, y_0 = x_k - N \) etc. For each partition \( \{x\} \) and cycle \( j \) we have a unique \( \{y\} \) and \( k \) defined, (5.9) is a 1:1 mapping from \( \{x\} \) to \( \{y\} \).

However we can now think of starting with a set of \( l + 1 \) variables \( \{y\} \) which satisfy \( 1 \leq y_0 < \ldots < y_l \leq N \) and a variable \( 0 \leq k < N \). The \( k \) represents reverse cycling of the \( l \ y \)'s that define a partition of \( 1, \ldots, N \) (whereas \( j \) represented cycling the \( N \ b \) indices). We can then define a unique partition \( \{x\} \) and variable \( j \) through (5.9) and this is a 1:1 map from \( \{\{y\}, k\} \rightarrow \{\{x\}, j\} \). Thus this \( \{x\} \) to \( \{y\} \) map is 1:1 and onto and we can rewrite (5.8) as

\[
\Phi_1(\{z_a = p_a + i \epsilon_a\}) = \left( \frac{-1}{2} \right)^{N-1} (\frac{1}{\pi})^{N-l-1} \int dk_1 \ldots dk_j \delta(k_1 + k_2 + \ldots + k_N) \times
\]
\[
\sum_{\text{perm.}\{b\} | NC} \sum_{Y \in k=0} \sum_{l} W(b_{x_0} b_{x_0+1} \ldots b_{x_0-l}; \{k\}) \frac{i \text{PP}}{B_{x_0-1}^{x_0-1}} \ldots \psi_{x_1}^{x_0-1} \delta(B_{x_1}^{x_0-1}) \ldots \]
\[
\psi_{x_1}^{x_0-1} \delta(B_{x_1}^{x_0-1}) \ldots \frac{i \text{PP}}{B_{x_0-1}^{x_0-1}} \tag{5.10}
\]

where the \( \{x\} \)'s and \( j \) are defined in terms of the \( \{y\} \)'s and \( k \) using (5.9).

Now the \( \sum_k \) shifts the indices \( l + 1 \) times, shifting by \( x_{k+1} - x_k \). The reason for doing this rearrangement is in the delta functions. If we now set the external energies \( \{p\} \) to zero, the integration variables associated with each block sum to zero because the delta functions in (5.11) can be written as

\[
\delta(k_1 + k_2 + \ldots + k_N) \prod_{j=1}^l \delta(B_{x_j}^{x_0-1}) = \prod_{j=1}^l \delta(\sum_{i=x_{j-1}}^{x_j} k_{b_i}) \tag{5.12}
\]

Together with the property (2.6), this shows that the same thermal Wightman function factor appears, for a given partition \( \{y\} \), however often the partition has been cycled. In a similar way the delta functions ensure that the principal part denominators can be written in a way that is independent of the \( k \) sum. This leaves us with

\[
\Phi_1(\{z_a = 0 + i \epsilon_a\}) = \left( \frac{-1}{2} \right)^{N-1} (\frac{1}{\pi})^{N-l-1} \int dk_1 \ldots dk_j \delta(k_1 + k_2 + \ldots + k_N). \tag{5.11}
\]
where the sum of cycles and all the signs of the epsilon’s are encoded in the

\[ \Psi(\psi) = \sum_{k=0}^{l} (-\theta_a)^{k}\theta_a^{l-k} = \left\{ \begin{array}{ll} 0, & l \text{ odd} \\ -1, & l \text{ even} \end{array} \right. \]  

(5.14)

This \( \Psi \) is a generalisation of the expression (5.6) encountered in the case of two-point functions, and (4.12) and (4.7) seen with three-point functions. All dependence on the precise analytic continuation, and hence the specification of which GRF is being studied, is contained in this factor. We are now using the second definition of \( B \) from (5.3) for convenience.

### 5.1 The special case of the retarded and advanced functions

For the case of retarded and advanced functions, the sum in \( \Psi \) of (5.13) is easy to perform. Suppose we are looking at \( R_a \) or \( A_a \) so that \( \epsilon_a \) is of a different sign from all the others.

From (2.9) \( \epsilon_a \) is then as large as the sum of the other epsilons. Then any sum of epsilons containing \( \epsilon_a \) has the same sign as \( \epsilon_a \). The remaining sums must have the opposite sign. Thus the product of \( \psi \)'s is equal to

\[ \Psi = \sum_{k=0}^{l} (-\theta_a)^{k}\theta_a^{l-k} = \left\{ \begin{array}{ll} 0, & l \text{ odd} \\ -1, & l \text{ even} \end{array} \right. \]  

(5.15)

where \( \theta_a = \pm 1 \) depending on the sign of \( \epsilon_a \). Looking at the terms with an even \( l \), an even number of delta functions, we see that they are independent of the \( a \) label and equal

\[ \Phi_l(\{ z_a = 0 + i\epsilon_a \}) \]

\[ = -\left( \frac{1}{2} \right)^{N-1} \left( \frac{1}{\pi} \right)^{N-l-1} \int dk_1 \ldots dk_j \delta(k_1 + k_2 + \ldots + k_N) \cdot \sum_{\text{perm.} \{b\} NC \{y\} \in Y} \mathcal{W}(b_{y_0}b_{y_0+1} \ldots b_{y_0-1}; \{k\}). \]

(5.16)
and this is independent of the label $a$ chosen. Hence

$$R_a(0) = A_a(0) = \sum_{l=0,2,...}^{N} \Phi_l = \text{independent of } a \quad (5.17)$$

with the same number found for all values of the index $a$. This shows that all the zero energy retarded and advanced $N$-point thermal Green functions made out of the same set of bosonic fields are equal at zero energy.

### 5.2 Generalised Retarded Functions

The generalised retarded functions are the Green functions obtained from any valid choice of epsilons, as specified by (2.8) and (2.9). Given that the retarded and advanced are subsets of the generalised functions we therefore have to show that (5.15) holds for all possible epsilon choices. This can be done as follows.

We first note that in (5.14) that the epsilons only appear in blocks running from $y_j \rightarrow (y_{j+1} - 1)$. It helps to simplify the notation if one works in terms of such blocks. So define

$$\bar{\epsilon}_j = \sum_{m=y_j}^{(y_{j+1})-1} \epsilon_{bm} \quad (5.18)$$

$$\bar{\sigma}_c^a = \psi_{yc}^{-1} = \sum_{l=0}^{a-\epsilon} \bar{\epsilon}_{c+l} \quad \tilde{\psi}_c^a = \text{SGN}(\bar{\sigma}_c^a) = \psi_{yc}^{-1} \quad (5.19)$$

$$\Lambda_k = \prod_{j=0}^{l-1} \tilde{\psi}_{k-j}^j \quad (5.20)$$

$$\Psi = \sum_{k=0}^{l} \Lambda_k \quad (5.21)$$

It is clear from the properties of the epsilons that the $\{\bar{\epsilon}\}$ also satisfy the same properties though with $N \rightarrow (l + 1)$. The idea of equivalent classes of $\bar{\epsilon}$’s as defined in (2.10) is also of use, so we have

$$\{\bar{\epsilon}\} \equiv \{\bar{\epsilon}'\} \quad \text{iff} \quad \text{SGN} \left( \sum_{j=1}^{j=N} c_j \bar{\epsilon}_j \right) = \text{SGN} \left( \sum_{j=1}^{j=N} c_j \bar{\epsilon}'_j \right)$$

$$\forall \{c_j\} = \{0,1 \mid 0 < \sum c_j < N\} \quad (5.22)$$

In this case though each equivalence class of $\bar{\epsilon}$’s does not pick out a unique $N$-point GRF (unless $l + 1 = N$).

A useful lemma to prove is as follows:-

**Lemma** Each equivalence class of $\bar{\epsilon}_j$ ($j = 0, \ldots, l$), as defined by (5.22), $\Lambda_k$ is a product of a unique number of $+1$ and a unique number of $-1$.

**Proof**

Consider two different terms in $\Psi$, $\Lambda_{k_1}$ and $\Lambda_{k_2}$. From the definition of the $\bar{\sigma}$’s we have that

$$\bar{\sigma}_{k_1-j_1}^{k_1} - \bar{\sigma}_{k_1-j_2}^{k_1} = \bar{\sigma}_{k_1-j_1}^{k_1-j_2-1} \quad (5.23)$$
where we will now take $j = 0 \ldots l$. For the case of $j_2 = k_1 - k_2 - 1$ we find a relation between terms appearing in $\Lambda_{k_1}$ and $\Lambda_{k_2}$ namely

$$\tilde{\sigma}_{k_1-j_1}^{k_1} - \tilde{\sigma}_{k_2+1}^{k_1} = \tilde{\sigma}_{k_1-j_1}^{k_2}$$

(5.24)

Now three different cases need to be considered, namely

$$\begin{align*}
\tilde{\psi}_{k_1-j_1}^{k_1} & \in \Lambda_{k_1} \quad \text{if} \quad j_1 \neq l - 1 & (5.25) \\
\tilde{\psi}_{k_1-j_1}^{k_2} & \in \Lambda_{k_2} \quad \text{if} \quad j_1 \neq k_1 - k_2 - 1 & (5.26) \\
\tilde{\psi}_{k_2+1}^{k_1} & \in \Lambda_{k_1} \quad \text{as} \quad k_1 \neq k_2 & (5.27)
\end{align*}$$

where the indices are now periodic in $l + 1$, $j_1 \equiv j_1 + (l + 1)$ etc. The set $\Lambda$ is just the set of all the terms $\tilde{\psi}$ appearing in the product in the definition of $\Lambda$ (5.20).

The indices $k_1$ and $k_2$ are fixed so that $\tilde{\psi}_{k_2+1}^{k_1}$ has a definite value. Suppose $\tilde{\psi}_{k_2+1}^{k_1} > 0(<0)$. Then

1. For the $(l - 1)$ normal cases where $j_1 \neq l - 1$ and $j_1 \neq k_1 - k_2 - 1$ we see that

$$\forall \ j_1 \neq l - 1, j_1 \neq k_1 - k_2 - 1, \quad \tilde{\psi}_{k_1-j_1}^{k_1} < 0(>0) \quad \Rightarrow \quad \tilde{\psi}_{k_1-j_1}^{k_2} < 0(>0).$$

(5.28)

Note that where $\tilde{\psi}_{k_1-j_1}^{k_1} > 0(<0)$ nothing is learnt from (5.24). Only one $\tilde{\psi}$ term from each of $\Lambda_{k_1}$ and $\Lambda_{k_2}$ has not considered in the above $j_1$ range. It follows from (5.28) that apart from these last terms, the number of negative (positive) terms in the $\Lambda_{k_2}$ product is greater than or equal to the number of negative (positive) terms in the $\Lambda_{k_1}$ product.

The last term in each of the $\Lambda_{k_1}$ and $\Lambda_{k_2}$ products, which are not covered by the above range of $j_1$, correspond to the two cases $j_1 \neq l - 1$ and $j_1 \neq k_1 - k_2 - 1$. In these cases one has trivial identities involving $\tilde{\psi}_{k_1+1}^{k_1} = -\tilde{\psi}_{k_1+1}^{k_1} > 0(<0)$. As these last members of the $\Lambda$ products have these definite signs, we therefore know that there must be more negative (positive) terms in the $\Lambda_{k_2}$ product than in the $\Lambda_{k_1}$ product.

Each term in the $\Lambda$ product is either positive or negative (never zero from (2.8)) and there are the same number of terms in all the products. So there must be fewer positive (negative) terms in the $\Lambda_{k_2}$ product than in the $\Lambda_{k_1}$ product. Thus it is clear that the number of positive and negative terms in the two $\Lambda_{k_1}$ and $\Lambda_{k_2}$ products must both be different whatever the sign of $\tilde{\psi}_{k_1-j_1}^{k_1}$ is.

Since $k_1$ and $k_2$ are arbitrary, each $\Lambda_k$ term must therefore have a unique number of positive and a unique number of negative terms in its product. Q.E.D.

**Theorem** For any allowed set of $\varepsilon_j$ ($j = 0, \ldots, l$) as defined by (2.8) and (2.9), $\Psi$ of (5.24) is 0 if $l$ is odd, -1 if $l$ is even.

**Proof**

From (5.24), $\Psi$ is the sum of $(l + 1)$ different $\Lambda_k$’s. Each $\Lambda_k$ is a product of $l$ terms, so there are $(l + 1)$ different combinations of positive and negative terms each $\Lambda_k$ can be, from 0 positive and $(l + 1)$ negative through to $(l + 1)$ positive and 0 negative terms (none of them zero by (2.9)). From the lemma proved above, each of the $(l + 1)$ $\Lambda_k$’s is a product of a unique number of positive and negative terms. Hence each of the $(l + 1)$
different combinations of $l$ positive and negative terms in the $\Lambda$'s, $l$ term products, appear once and only once in the $\Psi$ sum. Thus

$$\Psi = \sum_{k=0}^{l} (1)^{l+1-k}(-1)^{k} = \begin{cases} 0, & l \text{ odd} \\ -1, & l \text{ even} \end{cases} \text{ Q.E.D.}$$

(5.29)

Thus we have shown that the GRF at zero energy are all equal. The N-point retarded and advanced function results hold for all N-point GRF namely

$$\Phi(\{0 + i\epsilon_{a}\}) = \sum_{l=0,2,\ldots}^{N} \Phi_{l} \ \text{ independent of } \{\epsilon \} \ \text{ chosen}$$

(5.30)

$$\Phi_{l}(\{z_{a} = 0 + i\epsilon_{a}\}) = -\left(\frac{1}{2}\right)^{l} \frac{1}{(l!)} \int dk_{1} \ldots dk_{j} \delta(k_{1} + k_{2} + \ldots + k_{N}).$$

$$\sum_{\text{perm. } \{b\}|NC} \sum_{\{y\} \in \{y\}} \mathcal{W}(b_{y_{0}}b_{y_{0}+1}b_{y_{0}-1}; \{k\}).$$

$$iPP \ B(y_{0} + 1, y_{1} - 1; \{k\}) B(y_{0} + 2, y_{1} - 1; \{k\}) \ldots$$

$$iPP \ B(y_{1} - 1, y_{1} - 1; \{k\}) \delta(B(y_{1}, y_{2} - 1; \{k\}) \ldots$$

$$iPP \ B(y_{1} + 1, y_{2} - 1; \{k\}) \ldots B(y_{0} - 1, y_{0} - 1; \{k\}).$$

$$= \text{ independent of } \{\epsilon \} \ \text{ chosen}$$

(5.31)

Hence, at the point where all external energies are zero, all bosonic generalised retarded functions are equal. The discontinuities across the cuts in these functions are zero at this point.

6 Conclusions

We have shown that the discontinuities across the cuts in the $\Phi$ function of (2.1), which contains all the various possible GRF made out of a single set of bosonic fields, are zero. This answers several of the questions raised in the introduction.

For derivative expansions of effective actions, it shows that the lowest order term (which includes the free energy or effective potential) is the same which ever side of the energy cuts one starts from. However one should expect differences to appear in higher order terms of a derivative expansion as we have only shown that the discontinuities are zero precisely at the zero energy point, and not in a neighbourhood of the zero energy point. Conversely it will not matter which GRF one uses to calculate the lowest order term of the derivative expansions. This is one reason why there is never any discussion of analytic continuation in an ITF calculation of free energies. The lack of discontinuities at zero energy is also the technical reason why the free energy can be real.

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The result also means that the discrete zero energy point of the set of Euclidean ITF energy values, \( z_j = 2\pi i\nu_j/\beta \), can be thought of as lying on any side of the of any of the cuts without any inconsistency. Hence, a Green function evaluated at this discrete point can be thought of as being related by analytic continuation to any one of the GRF. It also means that one can turn a sum over discrete energies into integrals along the real energy axis in the usual way without any ambiguity arising from the position of the zero energy point relative to the cuts. This technicality is not usually noted in the discussion of the use of such methods for doing ITF energy sums.

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