PRESENTING AFFINE $q$-SCHUR ALGEBRAS

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Abstract. We obtain a presentation of certain affine $q$-Schur algebras in terms of generators and relations. The presentation is obtained by adding more relations to the usual presentation of the quantized enveloping algebra of type affine $\hat{gl}_n$. Our results extend and rely on the corresponding result for the $q$-Schur algebra of the symmetric group, which were proved by the first author and Giaquinto.

Introduction

Let $V'$ be a vector space of finite dimension $n$. On the tensor space $(V')^\otimes r$ we have natural commuting actions of the general linear group $GL(V')$ and the symmetric group $S_r$. Schur observed that the centralizer algebra of each action equals the image of the other action in $\text{End}((V')^\otimes r)$, in characteristic zero, and Schur and Weyl used this observation to transfer information about the representations of $S_r$ to information about the representations of $GL(V')$. That this Schur–Weyl duality holds in arbitrary characteristic was first observed in [4], although a special case was already used in [2]. In recent years, there have appeared various applications of the Schur–Weyl duality viewpoint to modular representations. The Schur algebras $S(n, r)$ first defined in [9] play a fundamental role in such interactions.

Jimbo [13] and (independently) Dipper and James [6] observed that the tensor space $(V')^\otimes r$ has a $q$-analogue in which the mutually centralizing actions of $GL(V')$ and $S_r$ become mutually centralizing actions of a quantized enveloping algebra $U(gl_n)$ and of the Iwahori-Hecke algebra $H(S_r)$ corresponding to $S_r$. In this context, the ordinary Schur algebra $S(n, r)$ is replaced by the $q$-Schur algebra $S_q(n, r)$. Dipper and James also showed that the $q$-Schur algebras determine the representations of finite general linear groups in non-defining characteristic.

An affine version of Schur–Weyl duality was first described in [3]. A different version, in which the vector space $V'$ is replaced by an infinite dimensional vector space $V$, is given in [11], and we follow the latter approach here. In the affine (type $A$) setting, the mutually commuting actions are of an affine quantized enveloping algebra $U(\hat{gl}_n)$ and an extended affine Hecke algebra $H(\hat{W})$ corresponding to an extended affine Weyl group $\hat{W}$ containing the affine Weyl group $W$ of type $\hat{A}_{r-1}$. The affine $q$-Schur algebra $\hat{S}_q(n, r)$ in this context, which is also infinite dimensional, was first studied in [11], [17], and [19].

Recently, a new approach to Schur algebras or their $q$-analogues was given in [7], where it was shown that they may be defined by generators and relations in a

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manner compatible with the usual defining presentation of the enveloping algebra or its corresponding quantized enveloping algebra. The purpose of this paper is to extend that result to the affine case — that is, to describe the affine $q$-Schur algebra $\hat{S}_q(n,r)$ by generators and relations compatible with the defining presentation of $U(\hat{\mathfrak{gl}}_n)$. This result is formulated in Theorem 1.6.1, under the assumption that $n > r$. An equivalent result, which describes the affine $q$-Schur algebra as a quotient of Lusztig’s modified form of the quantized enveloping algebra, is given in Theorem 2.6.1. These results depend on a different presentation, also valid for $n > r$, of the $q$-Schur algebra given in [11, Proposition 2.5.1]. A different approach to the results of this paper seems to be indicated for the case $n \leq r$.

The organization of the paper is as follows. In Section 1 we give necessary background information, and formulate our main result. In Section 2 we give the proof of Theorem 1.6.1, and we also give, in Section 2.6, the alternative presentation mentioned above. Finally, in Section 3 we outline the analogous results in the classical case, when the quantum parameter is specialized to 1.

1. Preliminaries and statement of main results

Our main result, stated in §1.6, is a presentation by generators and relations of the affine $q$-Schur algebra. In order to put this result in context, we review some of the definitions of the algebra that have been given in the literature.

1.1. Affine Weyl groups of type $A$. The affine Weyl group will play a key role, both in our definitions and our methods of proof, so we define it first.

The Weyl group we consider in this paper is that of type $\hat{A}_{r-1}$, where we intend $r \geq 3$. This corresponds to the Dynkin diagram in Figure 1.1.1.

![Dynkin diagram of type $\hat{A}_{r-1}$](image)

**Figure 1.1.1.** Dynkin diagram of type $\hat{A}_{r-1}$

The number of vertices in the graph in Figure 1.1.1 is $r$, as the top vertex (numbered $r$) is regarded as an extra relative to the remainder of the graph, which is a Coxeter graph of type $A_{r-1}$.

We associate a Weyl group, $W = W(\hat{A}_{r-1})$, to this Dynkin diagram in the usual way (as in [12, §2.1]). This associates to node $i$ of the graph a generating involution $s_i$ of $W$, where $s_is_j = s_js_i$ if $i$ and $j$ are not connected in the graph, and

$$s_is_js_i = s_js_is_j$$

if $i$ and $j$ are connected in the graph. For $t \in \mathbb{Z}$, it is convenient to denote by $\bar{t}$ the congruence class of $t$ modulo $r$, taking values in the set $\{1,2,\ldots,r\}$. For the purposes of this paper, it is helpful to think of the group $W$ as follows, based on
a result of Lusztig [15]. (Note that we write maps on the right when dealing with permutations.)

**Proposition 1.1.2.** There exists a group isomorphism from $W$ to the set of permutations of $\mathbb{Z}$ satisfying the following conditions:

(a) $(i + r)w = (i)w + r$

(b) $\sum_{t=1}^{r}(t)w = \sum_{t=1}^{r}t$

such that $s_i$ is mapped to the permutation

$$t \mapsto \begin{cases} 
  t & \text{if } \overline{t} \neq \overline{i}, \overline{i+1}, \\
  t - 1 & \text{if } \overline{t} = \overline{i+1}, \\
  t + 1 & \text{if } \overline{t} = \overline{i},
\end{cases}$$

for all $t \in \mathbb{Z}$.

For reasons relating to weight spaces which will become clear later, we consider a larger group $\hat{W}$ of permutations of $\mathbb{Z}$.

**Definition 1.1.3.** Let $\rho$ be the permutation of $\mathbb{Z}$ taking $t$ to $t + 1$ for all $t$. Then the group $\hat{W}$ is defined to be the subgroup of permutations of $\mathbb{Z}$ generated by the group $W$ and $\rho$.

As will become clear later, the point of $\rho$ is that conjugation by $\rho$ will correspond to a graph automorphism of the Dynkin diagram given by rotation by one place.

**Proposition 1.1.4.** (i) There exists a group isomorphism from $\hat{W}$ to the set of permutations of $\mathbb{Z}$ satisfying the following conditions:

(a) $(i + r)w = (i)w + r$

(b) $\sum_{t=1}^{r}(t)w \equiv \sum_{t=1}^{r}t \mod r$.

(ii) Any element of $\hat{W}$ is uniquely expressible in the form $\rho^z w$ for $z \in \mathbb{Z}$ and $w \in W$. Conversely, any element of this form is an element of $\hat{W}$.

(iii) Let $S \cong S_r$ be the subgroup of $\hat{W}$ generated by

$$\{s_1, s_2, \ldots, s_{r-1}\}.$$ Let $Z$ be the subgroup of $\hat{W}$ consisting of all permutations $z$ satisfying

$$(t)z \equiv t \mod r$$

for all $t$. Then $Z \hat{W} \cong Z \circ \hat{W}$ and $\hat{W}$ is the semidirect product of $S$ and $Z$.

**Proof.** The three parts are proved in [11, Proposition 1.1.3, Corollary 1.1.4, Proposition 1.1.5] respectively. □
It is convenient to extend the usual notion of the length of an element of a Coxeter group to the group $\hat{W}$ in the following way.

**Definition 1.1.5.** For $w \in W$ the length $\ell(w)$ of $w$ is the length of a word of minimal length in the group generators $s_i$ of $W$ which is equal to $w$. The length, $\ell(w')$, of a typical element $w' = \rho^zw$ of $\hat{W}$ (where $z \in \mathbb{Z}$ and $w \in W$) is defined to be $\ell(w)$.

When the affine Weyl group is thought of in the above way, the familiar notions of length and distinguished coset representatives may be adapted from the corresponding notions for Coxeter groups.

**Definition 1.1.6.** Let $\Pi$ be the set of subsets of $S = \{s_1, s_2, \ldots, s_r\}$, excluding $S$ itself. For each $\pi \in \Pi$, we define the subgroup $\hat{W}_\pi$ of $\hat{W}$ to be that generated by $\{s_i \in \pi\}$. (Such a subgroup is called a parabolic subgroup.) We will sometimes write $W_\pi$ for $\hat{W}_\pi$ to emphasize that it is a subgroup of $W$. Let $\Pi'$ be the set of elements of $\Pi$ that omit the generator $s_r$.

All the subgroups $\hat{W}_\pi$ are subgroups of $\hat{W}$, and are parabolic subgroups in the usual sense of Coxeter groups. Furthermore, each such $\hat{W}_\pi$ is isomorphic to a direct product of Coxeter groups of type $A$ (i.e., finite symmetric groups) corresponding to the connected components of the Dynkin diagram obtained after omitting the elements $s_i$ which do not occur in $\pi$. We will appeal to these facts freely in the sequel.

**Definition 1.1.7.** Let $\pi \in \Pi$. The subset $D_\pi$ of $\hat{W}$ is the set of those elements such that for any $w \in \hat{W}_\pi$ and $d \in D_\pi$,

$$\ell(wd) = \ell(w) + \ell(d).$$

We call $D_\pi$ the set of distinguished right coset representatives of $\hat{W}_\pi$ in $\hat{W}$.

The subset $D_\pi^{-1}$ is called the set of distinguished left coset representatives of $\hat{W}_\pi$ in $\hat{W}$; elements $d \in D_\pi^{-1}$ have the property that $\ell(dw) = \ell(d) + \ell(w)$ for any $w \in \hat{W}_\pi$.

**Proposition 1.1.8.** (i) Let $\pi \in \Pi$ and $w \in \hat{W}$. Then $w = w_\pi w^\pi$ for a unique $w_\pi \in \hat{W}_\pi$ and $w^\pi \in D_\pi$.

(ii) Let $\pi' \in \Pi$ and $w \in \hat{W}$. Then $w = w^{\pi'} w_\pi$ for a unique $w_\pi \in \hat{W}_{\pi'}$ and $w^{\pi'} \in D_{\pi'}$.

(iii) Let $\pi_1, \pi_2 \in \Pi$. The set $D_{\pi_1, \pi_2} := D_{\pi_1} \cap D_{\pi_2}^{-1}$ is an irredundantly described set of double $\hat{W}_{\pi_1}$-$\hat{W}_{\pi_2}$-coset representatives, each of minimal length in its double coset.

**Proof.** See [11, Propositions 1.4.4, 1.4.5].

1.2. **Affine Hecke algebras of type $A$.** We now define the extended affine Hecke algebra $\mathcal{H} = \mathcal{H}(\hat{W})$ of type $A$. The Hecke algebra is a $q$-analogue of the group algebra of $\hat{W}$, and is related to $\hat{W}$ in the same way as the Hecke algebra $\mathcal{H}(S_\pi)$ of type $A$ is related to the symmetric group $S_\pi$. In particular, one can recover the group
algebra of $\hat{W}$ by replacing the parameter $q$ occurring in the definition of $\mathcal{H}(\hat{W})$ by $1$.

**Definition 1.2.1.** The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\hat{W})$ over $\mathbb{Z}[q, q^{-1}]$ is the associative, unital algebra with algebra generators
\[
\{T_{s_1}, \ldots, T_{s_r}\} \cup \{T_{\rho}, T_{\rho}^{-1}\}
\]
and relations
\begin{align*}
(1) \quad T_{s_i}^2 &= qT_{s_i} + (q - 1), \\
(2) \quad T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} \text{ if } s \text{ and } t \text{ are not adjacent in the Dynkin diagram,} \\
(3) \quad T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} \text{ if } s \text{ and } t \text{ are adjacent in the Dynkin diagram,} \\
(4) \quad T_{\rho}T_{s_{i+1}}T_{\rho}^{-1} &= T_{s_i}.
\end{align*}
In relation (4), we interpret $s_{r+1}$ to mean $s_1$.

The algebra $\mathcal{H}$ has a better known presentation, known as the Bernstein presentation, but this is not convenient for our purposes. The equivalence of the two presentations is well known, and a proof may be found, for example, in [11, Theorem 4.2.5]. However, it will be convenient to have the following modified version of the presentation in Definition 1.2.1.

**Lemma 1.2.2.** The affine Hecke algebra $\mathcal{H} = \mathcal{H}(\hat{W})$ over $\mathbb{Z}[q, q^{-1}]$ is the associative, unital algebra with algebra generators
\[
\{T_{s_1}, \ldots, T_{s_{r-1}}\} \cup \{T_{\rho}, T_{\rho}^{-1}\}
\]
and relations
\begin{align*}
(1') \quad T_{s_i}^2 &= qT_{s_i} + (q - 1), \\
(2') \quad T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} \text{ if } |i - j| > 1, \\
(3') \quad T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} \text{ if } |i - j| = 1, \\
(4') \quad T_{\rho}T_{s_{i+1}}T_{\rho}^{-1} &= T_{s_i} \text{ if } 1 \leq i < r - 1, \\
(5') \quad T_{\rho}T_{s_i}T_{\rho}^{-1} &= T_{s_i} \text{ if } 1 \leq i \leq r - 1.
\end{align*}

*Proof.* It is clear that relations (1'–5') are consequences of relations (1)–(4). For the converse direction, we define $T_{s_r} := T_{\rho}T_{s_1}T_{\rho}^{-1}$; the remaining cases of relations (1)–(4) may then be obtained from relations (1')–(5') by conjugating by $T_{\rho}$ or by $T_{\rho}^{-1}$. \qed

**Definition 1.2.3.** Let $w \in W$. The element $T_w$ of $\mathcal{H}(W)$ is defined as
\[
T_{s_{i_1}} \cdots T_{s_{i_m}},
\]
where $s_{i_1} \cdots s_{i_m}$ is a reduced expression for $w$ (i.e., one with $m$ minimal). (This is well-defined by standard properties of Coxeter groups.)

If $w' \in \hat{W}$ is of form $\rho^z w$ for $w \in W$, we denote by $T_{w'}$ the element $T_{\rho}^z T_w$. (This is well-defined by Proposition 1.1.4 (ii).)
Proposition 1.2.4. (i) A free $\mathbb{Z}[q, q^{-1}]$-basis for $\mathcal{H}$ is given by the set $\{T_w : w \in \hat{W}\}$.
(ii) As a $\mathbb{Z}[q, q^{-1}]$-algebra, $\mathcal{H}$ is generated by $T_{s_1}, T_\rho$ and $T_\rho^{-1}$.

Proof. See [11, Proposition 1.2.3, Lemma 1.2.4]. \qed

1.3. The affine $q$-Schur algebra as an endomorphism algebra. We first present the definition of the affine $q$-Schur algebra as given in [11, §2].

Definition 1.3.1. A weight is a composition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of $r$ into $n$ pieces, that is, a finite sequence of nonnegative integers whose sum is $r$. (There is no monotonicity assumption on the sequence.) We denote the set of weights by $\Lambda(n, r)$.

The $r$-tuple $\ell(\lambda)$ of a weight $\lambda$ is the weakly increasing sequence of integers where there are $\lambda_i$ occurrences of the entry $i$.

The Young subgroup $S_\lambda \subseteq S_r \subseteq W \subseteq \hat{W}$ is the subgroup of permutations of the set $\{1, 2, \ldots, r\}$ that leaves invariant the following sets of integers:

$$\{1, 2, \ldots, \lambda_1\}, \{\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2\}, \{\lambda_1 + \lambda_2 + 1, \ldots\}.$$

The weight $\omega$ is given by the $n$-tuple

$$\ell(\lambda) = (1, 1, \ldots, 1, 0, 0, \ldots, 0).$$

Remark 1.3.2. The Young subgroup $S_\lambda \subseteq S_r$ can be thought of as a group $\hat{W}_\lambda$ for some $\lambda \in \Pi'$. Note, however, that different compositions $\lambda$ can give rise to canonically isomorphic groups. Also note that we require $n \geq r$ for $\omega$ to exist.

Definition 1.3.3. Let $\lambda \in \Pi$. For $t \in \mathbb{Z}$, the parabolic subgroup $\hat{W}_{\lambda+t}$ is the one generated by those elements $s_{i+t}$ where $i$ is such that $s_i$ lies in $\hat{W}_\lambda$. We also use the notation $D_{\lambda+t}$ with the obvious meaning.

The element $x_{\lambda+t} \in \mathcal{H}$ is defined as

$$x_{\lambda+t} := \sum_{w \in \hat{W}_{\lambda+t}} T_w.$$

We will write $x_\lambda$ for $x_{\lambda+0}$.

Definition 1.3.4. The affine $q$-Schur algebra $\hat{S}_q(n, r)$ over $\mathbb{Z}[q, q^{-1}]$ is defined by

$$\hat{S}_q(n, r) := \text{End}_\mathcal{H} \left( \bigoplus_{\lambda \in \Lambda(n, r)} x_\lambda \mathcal{H} \right),$$

where $\mathcal{H} = \mathcal{H}(\hat{W})$.

There is a basis for $\hat{S}_q(n, r)$ similar to Dipper and James’ basis for the ordinary $q$-Schur algebra.
Definition 1.3.5. Let $d \in \hat{W}$ be an element of $D_{\lambda,\mu}$. Write $d = \rho^c$ (as in Proposition 1.1.4 (ii)) with $c \in W$. Then the element

$$\phi^d_{\lambda,\mu} \in \text{Hom}(x_\mu \mathcal{H}(\hat{W}), x_\lambda \mathcal{H}(\hat{W}))$$

is defined as

$$\phi^d_{\lambda,\mu}(x_\mu) := \sum_{d' \in D_{\nu} \cap W_\mu} x_\lambda T^z_{\rho} T_{cd'}$$

$$= \sum_{d' \in D_{\nu} \cap W_\mu} T^z_{\rho} x_{\lambda + z} T_{cd'} = \sum_{w \in W_{\lambda + z} \cap W_\mu} T^z_{\rho} T_w = \sum_{w \in W_{\lambda} \cap dW_\mu} T_w$$

where $\nu$ is the composition of $n$ corresponding to the standard Young subgroup $d^{-1}W_{\lambda + z} \cap W_\mu$ of $W$.

Theorem 1.3.6. (i) A free $\mathbb{Z}[q, q^{-1}]$-basis for $\hat{S}_q(n, r)$ is given by the set

$$\{\phi^d_{\lambda,\mu} : \lambda, \mu \in \Lambda(n, r), d \in D_{\lambda,\mu}\}.$$

(ii) The set of basis elements

$$\{\phi^d_{\lambda,\mu} : \lambda, \mu \in \Lambda(n, r), d \in S_r \cap D_{\lambda,\mu}\}$$

spans a subalgebra of $\hat{S}_q(n, r)$ canonically isomorphic to the $q$-Schur algebra $S_q(n, r)$.

(iii) The set of basis elements

$$\{\phi^d_{\lambda,\mu} : d \in \hat{W}\}$$

spans a subalgebra canonically isomorphic to the Hecke algebra $\mathcal{H}(\hat{W})$, where $\phi^d_{\lambda,\mu}$ is identified with $T_d$.

Proof. See [11, Theorem 2.2.4] for part (i), and [11, Proposition 2.2.5] for parts (ii) and (iii). \qed

Note again that parts (ii) and (iii) of Theorem 1.3.6 only apply if $n \geq r$.

1.4. Quantum groups and tensor space. The affine $q$-Schur algebras are closely related to certain quantum groups (Hopf algebras). The following Hopf algebra is crucial for our purposes.

Definition 1.4.1. The associative, unital algebra $U(\mathfrak{gl}_n)$ over $\mathbb{Q}(v)$ is given by generators

$$E_i, F_i \ (1 \leq i \leq n - 1); \quad K_i, K_i^{-1} \ (1 \leq i \leq n)$$

subject to the following relations:

(Q1) \quad $K_i K_j = K_j K_i$,

(Q2) \quad $K_i K_i^{-1} = K_i^{-1} K_i = 1$,

(Q3) \quad $K_i E_j = v^{\epsilon^+(i,j)} E_j K_i$, 

\[ K_i F_j = v^{\epsilon^+(i,j)} F_j K_i, \]
\[ E_i F_j - F_j E_i = \delta_{ij} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{v - v^{-1}}, \]
\[ E_i E_j = E_j E_i \quad \text{if } i \text{ and } j \text{ are not adjacent}, \]
\[ F_i F_j = F_j F_i \quad \text{if } i \text{ and } j \text{ are not adjacent}, \]
\[ E_i^2 E_j - (v + v^{-1}) E_i E_j E_i + E_j E_i^2 = 0 \quad \text{if } i \text{ and } j \text{ are adjacent}, \]
\[ F_i^2 F_j - (v + v^{-1}) F_i F_j F_i + F_j F_i^2 = 0 \quad \text{if } i \text{ and } j \text{ are adjacent}. \]

Here, we regard \( i \) and \( j \) as “adjacent” if \( i \) and \( j \) index adjacent nodes in the Dynkin diagram of type \( \tilde{A}_{n-1} \). In the relations, \( i \) and \( j \) vary over all values of the indices for which the relation is defined. Also,

\[ \epsilon^+(i,j) := \begin{cases} 1 & \text{if } j = i; \\ -1 & \text{if } j = i - 1; \\ 0 & \text{otherwise}; \end{cases} \]

and

\[ \epsilon^-(i,j) := \begin{cases} 1 & \text{if } j = i - 1; \\ -1 & \text{if } j = i; \\ 0 & \text{otherwise}. \end{cases} \]

where we write \( \overline{a} \) for \( a \in \mathbb{Z} \) to denote the residue class of \( a \) in the residue class ring \( \mathbb{Z}/n\mathbb{Z} \). The residue class notation has no effect in the above definition, where indices are restricted to the range \( 1, \ldots, n - 1 \). However, the notation is important in the next two definitions.

The following Hopf algebra is a quantized affine enveloping algebra associated with the affine Lie algebra \( \tilde{\mathfrak{gl}}_n \).

**Definition 1.4.2.** The associative, unital algebra \( \mathbb{U}(\tilde{\mathfrak{gl}}_n) \) over \( \mathbb{Q}(v) \) is given by generators

\[ E_i, F_i, K_i, K_i^{-1} \]

(where \( 1 \leq i \leq n \)) subject to relations (Q1) to (Q9) of Definition 1.4.1 (reading indices modulo \( n \)).

In this definition, the notion of “adjacent” in relations (Q1)–(Q9) must now be interpreted in the Dynkin diagram of type \( \tilde{A}_{n-1} \). More precisely, \( i \) and \( j \) are to be regarded as “adjacent” if \( i \) and \( j \) index adjacent nodes in the Dynkin diagram of type \( \tilde{A}_{n-1} \). Note that \( i, j \) index adjacent nodes if and only if \( i - j \equiv \pm 1 \) (mod \( n \)).

In [11], a larger Hopf algebra is considered. It is an extended version of the quantized affine algebra \( \mathbb{U}(\tilde{\mathfrak{gl}}_n) \) considered in Definition 1.4.2.

**Definition 1.4.3.** The associative, unital algebra \( \hat{\mathbb{U}}(\tilde{\mathfrak{gl}}_n) \) over \( \mathbb{Q}(v) \) is given by generators

\[ E_i, F_i, K_i, K_i^{-1}, R, R^{-1} \]

(where \( 1 \leq i \leq n \)) subject to relations (Q1) to (Q9) of Definition 1.4.1 (reading indices modulo \( n \)).

In this definition, the notion of “adjacent” in relations (Q1)–(Q9) must now be interpreted in the Dynkin diagram of type \( \tilde{A}_{n-1} \). More precisely, \( i \) and \( j \) are to be regarded as “adjacent” if \( i \) and \( j \) index adjacent nodes in the Dynkin diagram of type \( \tilde{A}_{n-1} \). Note that \( i, j \) index adjacent nodes if and only if \( i - j \equiv \pm 1 \) (mod \( n \)).
(where $1 \leq i \leq n$) subject to relations (Q1) to (Q9) of Definition 1.4.1 (reading indices modulo $n$), together with the relations

(Q10) \[ RR^{-1} = R^{-1}R = 1, \]
(Q11) \[ R^{-1}K_{i+1}R = K_i, \]
(Q12) \[ R^{-1}K_{i+1}^{-1}R = K_i^{-1}, \]
(Q13) \[ R^{-1}E_{i+1}R = E_i, \]
(Q14) \[ R^{-1}F_{i+1}R = F_i. \]

The following result was proved in [11, Theorem 3.1.10].

**Theorem 1.4.4.** The algebra \( \hat{U}(\hat{gl}_n) \) is a Hopf algebra with multiplication \( \mu \), unit \( \eta \), comultiplication \( \Delta \), counit \( \varepsilon \) and antipode \( S \). The comultiplication is defined by

\[
\Delta(1) = 1 \otimes 1, \\
\Delta(E_i) = E_i \otimes K_i K_i^{-1} + 1 \otimes E_i, \\
\Delta(F_i) = K_i^{-1} K_{i+1} \otimes F_i + F_i \otimes 1, \\
\Delta(X) = X \otimes X \quad \text{for} \quad X \in \{K_i, K_i^{-1}, R, R^{-1}\}.
\]

The counit is defined by

\[
\varepsilon(E_i) = \varepsilon(F_i) = 0, \\
\varepsilon(K_i) = \varepsilon(K_i^{-1}) = \varepsilon(R) = \varepsilon(R^{-1}) = 1.
\]

The antipode is defined by

\[
S(E_i) = -E_i K_i^{-1} K_{i+1}, \\
S(F_i) = -K_i K_i^{-1} F_i, \\
S(K_i) = K_i^{-1}, \\
S(K_i^{-1}) = K_i, \\
S(R) = R^{-1}, \\
S(R^{-1}) = R.
\]

The unit satisfies \( \eta(1) = 1_U \).

Note that the usual Hopf algebra structure on \( U(gl_n) \) and \( U(\hat{gl}_n) \) is obtained by restricting the operations of Theorem 1.4.4 above.

Let \( V \) be the \( \mathbb{Q}(v) \)-vector space with basis \( \{e_t : t \in \mathbb{Z}\} \). This has a natural \( \hat{U}(\hat{gl}_n) \)-module structure as follows.

**Lemma 1.4.5.** There is a left action of \( \hat{U}(\hat{gl}_n) \) on \( V \) defined by the conditions

\[
E_i e_{t+1} = e_t \quad \text{if} \quad i = t \mod n, \\
E_i e_{t+1} = 0 \quad \text{if} \quad i \neq t \mod n,
\]
$F_i e_t = e_{t+1}$ if $i = t \mod n$,  \\
$F_i e_t = 0$ if $i \neq t \mod n$,  \\
$K_i e_t = v e_t$ if $i = t \mod n$,  \\
$K_i e_t = e_t$ if $i \neq t \mod n$,  \\
$R e_t = e_{t+1}$.

**Proof.** See [11, Lemma 3.2.1]. \hfill \Box

Since \( \widehat{\mathfrak{gl}}_n \) is a Hopf algebra, the tensor product of two \( \widehat{\mathfrak{gl}}_n \)-modules has a natural \( \widehat{\mathfrak{gl}}_n \)-module structure via the comultiplication \( \Delta \).

**Definition 1.4.6.** The vector space \( V \otimes r \) has a natural \( \widehat{\mathfrak{gl}}_n \)-module structure given by
\[
 u.x = \Delta(u)(r-1).x
\]
We call this module tensor space. The weight \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda(n, r) \) of a basis element
\[
eq_1 \otimes e_2 \otimes \cdots \otimes e_r
\]
of \( V \otimes r \) is given by the condition
\[
\lambda_i := |\{ j : t_j \equiv i \mod n \}|
\]
for \( i = 1, \ldots, n \). The \( \lambda \)-weight space, \( V_\lambda \), of \( V \otimes r \) is the span of all the basis vectors of weight \( \lambda \).

Henceforth, we will always assume that \( q = v^2 \), and regard \( \mathbb{Q}(v) \) as an algebra over \( A = \mathbb{Z}[q, q^{-1}] \) by means of the ring homomorphism \( A \to \mathbb{Q}(v) \) such that \( q \to v^2 \), \( q^{-1} \to v^{-2} \).

The following result about the affine \( q \)-Schur algebra, which will be used frequently in the sequel, was proved in [11, Theorem 3.4.8].

**Theorem 1.4.7.** The quotient of \( \widehat{\mathfrak{gl}}_n \) by the kernel of its action on tensor space is isomorphic as a \( \mathbb{Q}(v) \)-algebra to the algebra \( \mathbb{Q}(v) \otimes_A \widehat{S}_q(n, r) \).

There is a corresponding result for the finite \( q \)-Schur algebra. This was introduced in [1]; see [8] or [10] for more details.

**Theorem 1.4.8.** Let \( V' \) be the submodule of \( V \) spanned by the \( e_j \) for \( 1 \leq j \leq n \). Then the quotient of \( \mathfrak{gl}_n \) by the kernel of its action on \( (V')^\otimes r \) is isomorphic as a \( \mathbb{Q}(v) \)-algebra to the algebra \( \mathbb{Q}(v) \otimes_A S_q(n, r) \). We denote the corresponding epimorphism from \( \mathfrak{gl}_n \) to \( \mathbb{Q}(v) \otimes_A S_q(n, r) \) by \( \alpha \).

**Definition 1.4.9.** For convenience of notation, we shall henceforth denote by \( \widehat{S}_v(n, r) \) the algebra \( \mathbb{Q}(v) \otimes_A \widehat{S}_q(n, r) \) and by \( S_v(n, r) \) its finite analogue \( \mathbb{Q}(v) \otimes_A S_q(n, r) \). We may refer to these algebras as the affine \( v \)-Schur algebra and \( v \)-Schur algebra, respectively.

It will be useful in the sequel to consider the weight spaces of \( S_v(n, r) \) as right \( \mathbb{Q}(v) \otimes_A \mathcal{H}(S_v) \) modules. The following result is useful in such a context.
Lemma 1.4.10. Let \(1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n\), and let \(\lambda \in \Lambda(n, r)\) be such that \(\lambda_j\) is the number of occurrences of \(j\) in the sequence \((i_1, i_2, \ldots, i_r)\). Then the \(\lambda\)-weight space of \(V^{\otimes r}\) is generated as a right \(\mathbb{Q}(v) \otimes_A \mathcal{H}(S_r)\)-module by the element
\[
e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}.
\]

Proof. This is a well known result, which can be seen for example by using the definition of \(S_v(n, r)\) together with the isomorphism, given in [8], between tensor-space and Dipper and James’ “\(q\)-tensor space” (see [6]).

Although \((V')^{\otimes r}\) is not a \(\widehat{\mathcal{U}(\mathfrak{gl}_n)}\)-module, we have the following

Lemma 1.4.11. The action of \(u \in \widehat{\mathcal{U}(\mathfrak{gl}_n)}\) on \(V^{\otimes r}\) is determined by its action on the subspace \((V')^{\otimes r}\).

Proof. This is part of [11, Proposition 3.2.5].

1.5. Lusztig’s approach. In §1.5 we review the approach to the affine \(q\)-Schur algebra used by Lusztig [17], McGerty [18, §2] and others.

Let \(V\) be a free rank \(r\) module over \(k[\epsilon, \epsilon^{-1}]\), where \(k\) is a finite field of \(q\) elements, and \(\epsilon\) is an indeterminate.

Let \(F^n\) be the space of \(n\)-step periodic lattices, i.e. sequences \(L = (L_i)_{i \in \mathbb{Z}}\) of lattices in our free module \(V\) such that \(L_i \subseteq L_{i+1}\), and \(L_{i-n} = \epsilon L_i\). The group \(G = \text{Aut}(V)\) acts on \(F^n\) in the natural way. Let \(\mathcal{G}_{r,n}\) be the set of nonnegative integer sequences \((a_i)_{i \in \mathbb{Z}}\), such that \(a_i = a_{i+n}\) and \(\sum_{i=1}^{n} a_i = r\), and let \(\mathcal{G}_{r,n,n}\) be the set of \(\mathbb{Z} \times \mathbb{Z}\) matrices \(A = (a_{i,j})_{i,j \in \mathbb{Z}}\) with nonnegative entries such that \(a_{i,j} = a_{i+n,j+n}\) and \(\sum_{i \in [1,n]} a_{i,j} = r\). The orbits of \(G\) on \(F^n\) are indexed by \(\mathcal{G}_{r,n,n}\), where \(L\) is in the orbit \(F_a\) corresponding to \(a\) if \(a_i = \dim_k(L_i/L_{i-1})\). The orbits of \(G\) on \(F^n \times F^n\) are indexed by the matrices \(\mathcal{G}_{r,n,n}\), where a pair \((L, L')\) is in the orbit \(O_A\) corresponding to \(A\) if
\[
a_{i,j} = \dim \left( \frac{L_i \cap L'_j}{(L_{i-1} \cap L'_j) + (L_i \cap L'_{j-1})} \right).
\]

For \(A \in \mathcal{G}_{r,n,n}\) let \(r(A), c(A) \in \mathcal{G}_{r,n}\) be given by \(r(A)_i = \sum_{j \in \mathbb{Z}} a_{i,j}\) and \(r(A)_j = \sum_{i \in \mathbb{Z}} a_{i,j}\).

Similarly let \(B^r\) be the space of complete periodic lattices, that is, sequences of lattices \(L = (L_i)\) such that \(L_i \subseteq L_{i+1}\), \(L_{i-r} = \epsilon L_i\), and \(\dim_k(L_i/L_{i-1}) = 1\) for all \(i \in \mathbb{Z}\). Let \(b_0 = (\ldots, 1, 1, \ldots)\). The orbits of \(G\) on \(B^r \times B^r\) are indexed by matrices \(A \in \mathcal{G}_{r,n,n}\) where the matrix \(A\) must have \(r(A) = c(A) = b_0\).

Let \(\mathfrak{A}_{r,q}, \mathfrak{S}_{r,q}\) and \(\mathfrak{T}_{r,q}\) be the span of the characteristic functions of the \(G\) orbits on \(F^n \times F^n, B^r \times B^r\) and \(F^n \times B^r\) respectively. Convolution makes \(\mathfrak{A}_{r,q}\) and \(\mathfrak{S}_{r,q}\) into algebras and \(\mathfrak{T}_{r}\) into a \(\mathfrak{A}_{r,q} \otimes \mathfrak{S}_{r,q}\) bimodule. For \(A \in \mathcal{G}_{r,n,n}\) set
\[
d_A = \sum_{i \geq k, j < l, 1 \leq i \leq n} a_{ij} a_{kl}.
\]

Let \(\{e_A : A \in \mathcal{G}_{r,n,n}\}\) be the basis of \(\mathfrak{A}_{r,q}\) given by the characteristic function of the orbit corresponding to \(A\), and let \(\{[A] : A \in \mathcal{G}_{r,n,n}\}\) be the basis of \(\mathfrak{A}_{r,q}\) given by
When \( n = r \), the subset of either basis spanned by all monomial matrices \( A \) spans \( \mathfrak{H}_{r,q} \).

All of these spaces of functions are the specialization at \( v = \sqrt{q} \) of modules over \( A = \mathbb{Z}[v,v^{-1}] \), which we denote by \( \mathfrak{A}_r \), \( \mathfrak{H}_r \) and \( \mathfrak{T}_r \) respectively; here \( v \) is an indeterminate.

**Proposition 1.5.1** (Varagnolo–Vasserot). The \( A \)-algebra \( \mathfrak{A}_r \) is naturally isomorphic to the affine \( q \)-Schur algebra \( \hat{\mathcal{S}}_v(n,r) \) of Definition 1.3.4. Furthermore, the isomorphism may be chosen to identify the basis of Definition 1.3.5 with the basis \( \{e_A : A \in \mathfrak{S}_{r,n,n}\} \).

**Proof.** The necessary isomorphism is the map \( \Phi \) given in [19, Proposition 7.4 (a)]. \( \Box \)

We will also need the canonical basis, \( \{[A] : A \in \mathfrak{S}_{r,n,n}\} \), for \( \hat{\mathfrak{S}}_v(n,r) \). This is related to the basis \( \{[A] : A \in \mathfrak{S}_{r,n,n}\} \) in a unitriangular way: we have

\[
\{A\} = \sum_{A_1 : A_1 \leq A} \Pi_{A_1,A}[A_1],
\]

where \( \leq \) is a certain natural partial order and the \( \Pi_{A_1,A} \) are certain Laurent polynomials (similar to the famous Kazhdan–Lusztig polynomials \( P_{y,w} \) of [14]) satisfying \( \Pi_{A,A} = 1 \). The reader is referred to [17, §4] for full details, or to [11, §2.4] for a more elementary construction.

An element \( A \in \mathfrak{S}_{r,n,n} \) is said to be aperiodic if for any \( p \in \mathbb{Z} \setminus \{0\} \) there exists \( k \in \mathbb{Z} \) such that \( a_{k,k+p} = 0 \). Let \( \mathfrak{S}_{r,n,n}^{ap} \) be the set of aperiodic elements in \( \mathfrak{S}_{r,n,n} \).

**Theorem 1.5.2** (Lusztig). Under the identifications of Theorem 1.4.7, the subalgebra \( \hat{U}(\mathfrak{gl}_n) \) of \( \hat{\mathfrak{U}}(\mathfrak{gl}_n) \) projects to the \( \mathbb{Q}(v) \)-span of the elements

\[
\{[A] : A \in \mathfrak{S}_{r,n,n}^{ap}\}.
\]

**Proof.** This is [17, Theorem 8.2]. \( \Box \)

**Remark 1.5.3.** Theorem 1.5.2 is not true if we replace the canonical basis by one of the other two bases so far discussed.

If we have \( n > r \), elementary considerations show that every element of \( \mathfrak{S}_{r,n,n} \) is aperiodic. This means that the subalgebra of \( \hat{\mathfrak{S}}_v(n,r) \) described in Theorem 1.5.2 is in fact the whole of \( \hat{\mathfrak{S}}_v(n,r) \), so that we may refer to the algebra of Theorem 1.5.2 as “the affine \( q \)-Schur algebra” without confusion. We will concentrate on the case \( n > r \) in this paper.

**1.6. Main results.** Our main aim is to prove the following

**Theorem 1.6.1.** Let \( n > r \), and identify \( \hat{\mathfrak{S}}_v(n,r) \) with the quotient of \( \hat{U}(\mathfrak{gl}_n) \) described in Theorem 1.5.2 (see Remark 1.5.3). Over \( \mathbb{Q}(v) \), the affine \( v \)-Schur algebra \( \hat{\mathfrak{S}}_v(n,r) \) is given by generators \( E_i, F_i, K_i, K_i^{-1} \) \((1 \leq i \leq n)\) subject to relations (Q1) to (Q9) of Definition 1.4.2 (reading indices modulo \( n \)), together with the relations

\[
K_1 K_2 \cdots K_n = v^r
\]
Theorem 1.6.2. Identify $S_v(n, r)$ with the quotient of $U(\mathfrak{gl}_n)$ described in Theorem 1.4.8. Over $\mathbb{Q}(v)$, the $v$-Schur algebra $S_v(n, r)$ is given by generators $E_i, F_i$ ($1 \leq i \leq n - 1$) and $K_i, K_i^{-1}$ ($1 \leq i \leq n$) subject to relations (Q1) to (Q9) of Definition 1.4.1, together with the relations

$$K_1K_2\cdots K_n = v^r$$

$$(K_i - 1)(K_i - v)(K_i - v^2)\cdots(K_i - v^r) = 0.$$  

Proof. This is [7, Theorem 2.1].

Definition 1.6.3. For now, we will denote by $T$ the $\mathbb{Q}(v)$-algebra given by the generators and relations of Theorem 1.6.1, and we will denote the corresponding epimorphism from $U(\hat{\mathfrak{gl}}_n)$ to $T$ by $\beta$. The main aim is thus to show that $T$ is isomorphic to $\hat{S}_v(n, r)$.

Remark 1.6.4. There is an obvious isomorphism between the algebra given by the generators and relations of Theorem 1.6.2 and the subalgebra of $T$ generated by the images of the the $E_i, F_i, K_j$ and $K_j^{-1}$, where $1 \leq i < n$ and $1 \leq j \leq n$. This means that if a relation in $\hat{S}_v(n, r)$ involving the $E_i, F_i$ and $K_j$ avoids all occurrences of $E_a$ and $F_a$ for some $1 \leq a \leq n$, then by Theorem 1.6.2 and symmetry, the relation is a consequence of relations (Q15) and (Q16).

The following result establishes a natural surjection from $T$ to $\hat{S}_v(n, r)$, and our main task in proving Theorem 1.6.1 will be to show that this map is an isomorphism, in other words, that relations (Q15) and (Q16) are sufficient.

Proposition 1.6.5. Relations (Q15) and (Q16) of Theorem 1.6.1 hold in $\hat{S}_v(n, r)$, and therefore $\hat{S}_v(n, r)$ is a quotient of the algebra $T$. (We denote the corresponding epimorphism by $\gamma : T \longrightarrow \hat{S}_v(n, r)$.)

Proof. Using the comultiplication on $U(\hat{\mathfrak{gl}}_n)$, it may be easily checked that

$$K_1K_2\cdots K_n - v^r$$

and

$$(K_i - 1)(K_i - v)(K_i - v^2)\cdots(K_i - v^r)$$

act as zero on the tensor space $(V')^\otimes r$ given in Theorem 1.4.8. The result now follows from Lemma 1.4.11. 

Remark 1.6.6. For later reference, we note that the maps $\alpha, \beta, \gamma$ respectively from Theorem 1.4.8, Remark 1.6.4, and Proposition 1.6.5 fit together into the following
commutative diagram

\[
\begin{array}{c}
\hat{U}(\hat{\mathfrak{gl}}_n) \\
\downarrow \\
U(\hat{\mathfrak{gl}}_n) \\
\downarrow \beta \\
T \rightarrow \hat{S}_v(n, r) \\
\downarrow \gamma \\
\hat{S}_v(n, r) \\
\downarrow \\
U(\mathfrak{gl}_n) \\
\downarrow \alpha \\
S_v(n, r)
\end{array}
\]

in which all horizontal maps and the diagonal one are surjections, and all vertical maps are injections.

2. PROOF OF THE MAIN RESULTS

Most of this section is devoted to proving Theorem 1.6.1. The final result of this section, Theorem 2.6.1, is an equivalent formulation of Theorem 1.6.1, compatible with Lusztig’s modified form of the quantized enveloping algebra.

2.1. A subalgebra of \( \hat{S}_v(n, r) \) isomorphic to \( \mathcal{H}(W) \). A presentation for \( \hat{S}_v(n, r) \) in the case \( n > r \) was given in [11, Proposition 2.5.1], and our main strategy for proving Theorem 1.6.1 will be to adapt this presentation.

Proposition 2.1.1. The algebra \( \hat{S}_v(n, r) \) is generated by elements

\[
\{ \phi^d_{\omega, \omega} : d \in \hat{W} \} \cup \{ \phi^1_{\lambda, \omega} : \lambda \in \Lambda(n, r) \} \cup \{ \phi^1_{\omega, \lambda} : \lambda \in \Lambda(n, r) \}.
\]

The elements \( \phi^d_{\omega, \omega} \) are subject to the relations of the affine Hecke algebra of Definition 1.2.1 under the identification given by Theorem 1.3.6 (iii). The generators are also subject to the following defining relations, where \( s \) denotes a generator \( s_i \in \hat{W}_\lambda \).

\[
\begin{align*}
(\text{Q17}) & \quad \phi^1_{\omega, \lambda} \phi^1_{\mu, \omega} = \delta_{\lambda, \mu} \sum_{d \in \hat{W}_\lambda} \phi^d_{\omega, \omega}, \\
(\text{Q18}) & \quad \phi^1_{\omega, \omega} \phi^1_{\omega, \lambda} = q \phi^1_{\omega, \lambda}, \\
(\text{Q19}) & \quad \phi^1_{\lambda, \omega} \phi^1_{\omega, \omega} = q \phi^1_{\lambda, \omega}.
\end{align*}
\]

A key step in understanding the structure of the algebra \( T \) of Definition 1.6.3 is locating within it a subalgebra isomorphic to the affine Hecke algebra \( \mathcal{H}(\hat{W}) \). Theorem 1.3.6 (iii) shows that this can be done for the algebra \( \hat{S}_v(n, r) \), and we now review how this works in terms of endomorphisms of tensor space. Recall the definition of weight space from Definition 1.4.6, and the definition of the weight \( \omega \) from Definition 1.3.1.

Definition 2.1.2. For each \( 1 \leq i < r \), let \( \tau(T_{s_i}) : V_\omega \rightarrow V_\omega \) be the endomorphism corresponding to the action of \( vF_iE_i - 1 \in \hat{U}(\mathfrak{gl}_n) \). Similarly, let \( \tau(T_{\rho^{-1}}) \) be the endomorphism corresponding to \( F_nF_{n-1}\cdots F_{r+1}R \),
and let $\tau(T_\rho)$ be the endomorphism corresponding to 

$$E_rE_{r+1}\cdots E_{n-1}R^{-1}.$$  

**Lemma 2.1.3.** The endomorphisms $\tau(T_w)$ defined above (for $w \in \{s_i : 1 \leq i < r\} \cup \{\rho, \rho^{-1}\}$) satisfy the relations of Lemma 1.2.2 (after replacing $T_w$ by $\tau(T_w)$).

**Proof.** Using the epimorphism $\alpha' : U(\hat{gl}_n) \twoheadrightarrow S_v(n, r)$ studied in [1], [8], [10], one finds that the action of $\tau(T_{s_i})$ on $V_\omega$ in the case where $i \neq r$ corresponds to the action of $\phi_{\omega, \omega}^s \in S_v(n, r)$. (Recall from Theorem 1.4.8 that $S_v(n, r)$ is the quotient of $U(\hat{gl}_n)$ by the annihilator of $V_n^{\otimes r}$.) This proves relations (1'), (2') and (3') of Lemma 1.2.2.

The effect of $\tau(T_\rho)$ on $V_\omega$ is

$$\tau(T_\rho)(e_{i_1} \otimes \cdots \otimes e_{i_r}) = e_{j_1} \otimes \cdots \otimes e_{j_1},$$

where $j_i = i_i - 1 \mod r$. The effect of $\tau(T_{\rho^{-1}})$ on $V_\omega$ is the inverse of this action. The proof of relations (4') and (5') now follow by calculation of the action of $vF_iE_{i-1}$ on $V_\omega$ using the comultiplication. \qed

**Remark 2.1.4.** Definition 2.1.2 and Lemma 2.1.3 are very similar to [11, Definition 3.3.1] and [11, Lemma 3.3.2], respectively. They are included here because [11, Definition 3.3.1] contains an incorrect definition for $\tau(T_{s_r})$.

**Lemma 2.1.5.** Define $\tau(T_{s_i}) := \tau(T_\rho)\tau(T_{s_i})\tau(T_{\rho^{-1}})$. Then the map taking $\tau(T_w)$ to $T_w$ (where $w \in \{s_i : 1 \leq i \leq r\} \cup \{\rho, \rho^{-1}\}$) extends uniquely to an isomorphism of algebras between $\mathcal{H}(\hat{W})$ and the algebra $\tau(\mathcal{H})$ generated by the endomorphisms $\tau(T_w)$.

**Proof.** This follows from Lemma 1.2.2 and the argument given in [11], namely [11, Lemma 3.3.3, Lemma 3.3.4]. \qed

For later purposes, it will be convenient to have versions of the above results that do not make reference to the grouplike elements $R$ and $R^{-1}$. The following lemma is the key to the necessary modifications. (Recall that $n \geq r + 1$ by assumption.)

**Lemma 2.1.6.** Let $e \in V_\omega$. Then we have

$$R.e = (F_1F_2\cdots F_r).e$$

and

$$R^{-1}.e = (E_{r-1}E_{r-2}\cdots E_1)E_n.e.$$

**Proof.** It is enough to consider the case where

$$e = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r}$$

is a basis element, and this turns out to be a straightforward exercise using the comultiplication in $\hat{U}(\hat{gl}_n)$. \qed
Proposition 2.1.7. For each $1 \leq i < r$, let $\tau'(T_{s_i}) : V_w \to V_w$ be the endomorphism corresponding to the action of $vF_iE_i - 1 \in U(\mathfrak{gl}_n)$. Similarly, let $\tau'(T_{\rho})$ be the endomorphism corresponding to

$$(F_nF_{n-1} \cdots F_{r+1})(F_1F_2 \cdots F_r),$$

and let $\tau'(T_{\rho})$ be the endomorphism corresponding to

$$(E_rE_{r+1} \cdots E_{n-1})(E_{r-1}E_{r-2} \cdots E_1)E_n.$$

Then, after replacing $T_w$ by $\tau'(T_w)$, these endomorphisms satisfy the relations of Lemma 1.2.2.

Proof. Combine Lemma 2.1.6 with Lemma 2.1.3. □

2.2. Weight space decomposition of $T$. An important property of the algebra $T$ is that it possesses a decomposition into left and right weight spaces, similar to that enjoyed by the ordinary and affine $q$-Schur algebras.

Definition 2.2.1. An element $t \in T$ is said to be of left weight $\lambda \in \Lambda(n, r)$ if for each $i$ with $1 \leq i \leq n$ we have

$$\beta(K_i)t = v^\lambda t,$$

where $\beta$ is the map defined in Definition 1.6.3. There is an analogous definition for elements of right weight $\lambda$. The left (respectively, right) $\lambda$-weight space of $T$ is the $\mathbb{Q}(v)$-submodule spanned by all elements of left (respectively, right) weight $\lambda$.

Definition 2.2.2. For each $\lambda \in \Lambda(n, r)$, define the idempotent element $1_\lambda \in T$ by the image of $1_\lambda \in S_v(n, r)$ under the canonical inclusion map from Remark 1.6.4. Here the $1_\lambda$ are the idempotents which were defined in [7, (3.4)]. The sum of the $1_\lambda$, as $\lambda$ varies over $\Lambda(n, r)$, is $1$. Moreover, $1_\lambda 1_\mu = 0$ for $\lambda \neq \mu$, i.e. the idempotents are pairwise orthogonal.

Proposition 2.2.3. The algebra $T$ is the direct sum of its left $\lambda$-weight spaces, and the nonzero $\lambda$-weight spaces are indexed by the elements of $\Lambda(n, r)$.

Proof. Thanks to the above orthogonal decomposition of the identity in $T$, there is a direct sum decomposition

$$T = \bigoplus_{\lambda \in \Lambda(n, r)} 1_\lambda T.$$

Moreover, in $S_v(n, r)$ we have the identity

$$\alpha(K_i)1_\lambda = \lambda_i 1_\lambda \quad (i = 1, \ldots, n)$$

from [7, Proposition 8.3(a)], where $\alpha$ is the quotient map $U(\mathfrak{gl}_n) \to S_v(n, r)$ of Theorem 1.4.8. Now it follows from the embedding of Remark 1.6.4, or more precisely from the commutativity of the diagram in Remark 1.6.6, that

$$\beta(K_i)1_\lambda = \lambda_i 1_\lambda \quad (i = 1, \ldots, n)$$

holds in the algebra $T$. Thus it follows that $\beta(K_i)v = \lambda_i v$ for all $i = 1, \ldots, n$ and all $v \in 1_\lambda T$. This proves that $1_\lambda T$ is the $\lambda$-weight space in $T$. □
For simplicity’s sake, we will write $E_i$ in place of $\beta(E_i)$ and $F_i$ in place of $\beta(F_i)$ for the remainder of §2.2.

**Lemma 2.2.4.** (i) In $T$ we have $K_i^{\pm 1}1_\lambda = v^{\pm\lambda_i}1_\lambda$.
(ii) The idempotent $1_\lambda$ lies within the subalgebra of $T$ generated by the $K_i$.
(iii) In $T$, the idempotent $1_\lambda$ coincides with the image of $\phi_1^1_{\lambda_\lambda}$ under $\beta$.

**Proof.** Part (i) is already contained in the proof of the preceding proposition, and part (ii) is due to the definition of $1_\lambda$ in [7] as $1_\lambda = \left[ \frac{K_1}{\lambda_1} \right] \cdots \left[ \frac{K_n}{\lambda_n} \right]$.

Part (iii) is a consequence of the remarks preceding [10, Lemma 2.9] combined with [10, Corollary 2.10].

**Definition 2.2.5.** For each $i$ with $1 \leq i \leq n$, let $\alpha_i = ((\alpha_i)_1, \ldots, (\alpha_i)_n)$ be the $n$-tuple of integers given by

$$(\alpha_i)_j = \begin{cases} 1 & \text{if } j \equiv i \mod n, \\ -1 & \text{if } j \equiv i+1 \mod n, \\ 0 & \text{otherwise}. \end{cases}$$

The following identities will be used frequently in the sequel, often without explicit reference. In these identities, it will be convenient to regard a weight $\lambda = (\lambda_1, \ldots, \lambda_n)$ as an infinite periodic sequence of integers, indexed by $\mathbb{Z}$, by setting $\lambda_j$ for any $j \in \mathbb{Z}$ to the corresponding value $\lambda_i$ such that $1 \leq i \leq n$ and $j \equiv i \mod n$.

**Lemma 2.2.6.** Let $\lambda \in \Lambda(n, r)$, extended to an infinite periodic sequence as above. The following identities hold in $T$:
(i) For any $1 \leq i \leq n$, we have

$$E_i1_\lambda = \begin{cases} 1_{\lambda+i},E_i & \text{if } \lambda_{i+1} > 0; \\ 0 & \text{otherwise}. \end{cases}$$

(ii) For any $1 \leq i \leq n$, we have

$$F_i1_\lambda = \begin{cases} 1_{\lambda-i},F_i & \text{if } \lambda_i > 0; \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** By Remark 1.6.4 and Lemma 2.2.4, it is enough to check that both sides of each identity agree after projection to $\hat{S}_v(n, r)$. By Theorem 1.4.7, it is enough to check that both sides of each identity agree in their action on tensor space, which is a routine calculation.

The following lemma will be used extensively in the sequel. We will sometimes refer to it as the *cancellation principle* for $T$. 

Lemma 2.2.7. Let $c \geq 1$. Let $\lambda \in \Lambda(n, r)$, extended to an infinite periodic sequence as above. The following identities hold in $T$:

(i) For each $1 \leq i \leq n$ with $\lambda_i = 0$, there exists a nonzero element $z \in A$ such that

$$F_i^c E_i^c 1_\lambda = \begin{cases} z 1_\lambda & \text{if } \lambda_{i+1} \geq c; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $c = \lambda_{i+1} = 1$ then $z = 1$.

(ii) For each $1 \leq i \leq n$ with $\lambda_{i+1} = 0$, there exists a nonzero $z' \in A$ such that

$$E_i^c F_i^c 1_\lambda = \begin{cases} z' 1_\lambda & \text{if } \lambda_i \geq c; \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, if $c = \lambda_i = 1$, then $z' = 1$.

Proof. By the formulas in [16, 3.1.9] we have the following identities in $U(\mathfrak{g} / n)$:

$$E_i^{(c)} F_i^{(c)} = \sum_{t \geq 0} F_i^{(c-t)} \prod_{s=1}^t \frac{v^{2t-2c-s+1} K_i - v^{-2t+2c+s-1} \bar{K}_i^{-1}}{v^s - v^{-s}} E_i^{(c-t)}$$

$$F_i^{(c)} E_i^{(c)} = \sum_{t \geq 0} E_i^{(c-t)} \prod_{s=1}^t \frac{v^{2t-2c-s+1} \bar{K}_i^{-1} - v^{-2t+2c+s-1} K_i}{v^s - v^{-s}} F_i^{(c-t)}$$

where $\bar{K}_i = K_i K_i^{-1}$ and $X^{(m)} = X^m/[m]!$ for $X = E_i, F_i$. Here $[m]$ is the quantum integer $[m] = (v^m - v^{-m})/(v - v^{-1})$ and $[m]! = [1] \cdots [m - 1][m]$ for any $m \in \mathbb{N}$.

Since the above identities hold in $U(\mathfrak{g} / n)$, they hold in the quotient $T$. Multiply the second identity on the right by $1_\lambda$. By Lemma 2.2.6 and the hypothesis $\lambda_i = 0$ all terms on the right hand side will then vanish, excepting the term corresponding to $t = c$. So we obtain the identity

$$F_i^{(c)} E_i^{(c)} 1_\lambda = \prod_{s=1}^c \frac{v^{-s+1} \bar{K}_i^{-1} - v^{s-1} \bar{K}_i}{v^s - v^{-s}} 1_\lambda$$

and a similar argument with the first identity above in light of the hypothesis $\lambda_{i+1} = 0$ yields the identity

$$E_i^{(c)} F_i^{(c)} 1_\lambda = \prod_{s=1}^c \frac{v^{-s+1} K_i - v^{s-1} K_i}{v^s - v^{-s}} 1_\lambda.$$ 

These are identities in the quotient $T$. In fact, they hold in the subalgebra $S_{v^n}(n, r)$ under the embedding of Remark 1.6.4. By Lemma 2.2.4(i) the above identities in $T$ take the form

$$F_i^{(c)} E_i^{(c)} 1_\lambda = \prod_{s=1}^c \frac{v^{\lambda_i+1 - \lambda_i - s+1} - v^{\lambda_i - \lambda_i+1 + s-1}}{v^s - v^{-s}} 1_\lambda$$

$$E_i^{(c)} F_i^{(c)} 1_\lambda = \prod_{s=1}^c \frac{v^{\lambda_i - \lambda_i+1 - s+1} - v^{\lambda_i+1 - \lambda_i, s-1}}{v^s - v^{-s}} 1_\lambda.$$
Remembering that $\lambda_i = 0$ in the first formula and $\lambda_{i+1} = 0$ in the second, by multiplying through by $([c]!)^2$ we obtain the desired result, where
\[
z = ([c]!)^2 \left[ \frac{\lambda_{i+1}}{c} \right], \quad z' = ([c]!)^2 \left[ \frac{\lambda_i}{c} \right].
\]
in terms of the standard Gaussian binomial coefficients (see e.g. [16, §1.3]). The proof is complete. \qed

**Definition 2.2.8.** Maintain the notation of Lemma 2.2.7. Let $M$ be a monomial in the various elements $E_i, F_i$ and $1_\lambda$ of $T$. We call a monomial $M'$ a reduction of $M$ if it (a) represents the same element of $T$ as $M$ and (b) $M'$ can be obtained from $M$ by omitting zero or more generators of $M$ of the form $1_\mu$.

A distinguished term in the algebra $T$ is an element of $T$ of one of the following two forms:
(i) $E_i^c 1_\lambda$, where $c \geq 0$ and $\lambda_i = 0$;
(ii) $F_i^c 1_\lambda$, where $c \geq 0$ and $\lambda_{i+1} = 0$.

A strictly distinguished monomial in the algebra $T$ is a monomial in the elements $F_i, E_i$ and $1_\lambda$ that can be parsed as a word in the distinguished terms. A reduction of a strictly distinguished monomial is called a distinguished monomial.

**Example 2.2.9.** The idempotents $1_\lambda$ are both distinguished terms and distinguished monomials in $T$: here, we take $c = 0$.

If $\lambda_i = 0$ and $\lambda_{i+1} = c$ then the element $M' = F_i^c E_i^c 1_\lambda$ of Lemma 2.2.7 (i) is a distinguished monomial. Indeed, it can be seen by repeated applications of Lemma 2.2.6 (i) and the fact that $1_\lambda$ is idempotent, that $M'$ is a reduction of the strictly distinguished monomial $M = (F_i^c 1_{\lambda+c\alpha_i})(E_i^c 1_\lambda)$. (To verify that $M$ is strictly distinguished, one must note that $(\lambda+c\alpha_i)_{i+1} = 0$.) Furthermore, by Lemma 2.2.7 and the fact that $\lambda_{i+1} \geq c$, we see that $M = M'$ is a nonzero element of $T$.

Similarly, if $\lambda_i = c$ and $\lambda_{i+1} = 0$ then the element $E_i^c F_i^c 1_\lambda$ of Lemma 2.2.7 (ii) is a distinguished monomial.

In the sequel, we will make use of various automorphisms of $T$; part (i) below may be used without explicit comment.

**Proposition 2.2.10.** (i) There is a unique automorphism $\nu$ of $T$ of order $n$ satisfying
\[
\nu(E_i) = E_{i+1}, \quad \nu(F_i) = F_{i+1} \quad \text{and} \quad \nu(K_i^{\pm 1}) = K_{i+1}^{\pm 1},
\]
for all $1 \leq i \leq n$, and reading subscripts modulo $n$. Let $\lambda \in \Lambda(n,r)$ and define
\[
\lambda_+ = (\lambda_n, \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_{n-1}).
\]
Then $\nu(1_\lambda) = 1_{\lambda_+}$. 

(ii) There is a unique anti-automorphism $\sigma$ of $T$ satisfying

$$
\sigma(E_i) = F_i, \\
\sigma(F_i) = E_i \text{ and} \\
\sigma(K_i^{\pm1}) = K_i^{\pm1},
$$

for all $1 \leq i \leq n$. The anti-automorphism $\sigma$ fixes all elements $1_\lambda \in T$.

Proof. For (i), we note that there is an automorphism of $U(\hat{\mathfrak{gl}}_n)$ corresponding to $\nu$, that in addition fixes the elements $R^{\pm1}$; this can be verified by checking the defining relations for $U(\hat{\mathfrak{gl}}_n)$. Since this automorphism preserves setwise the set of relations (Q15) and (Q16) in $T$, we obtain an automorphism of $T$ as claimed; it is unique because we have given its effect on a generating set (see Theorem 1.6.1). The last claim of (i) follows from the relationship between the $K_i$ and $1_\lambda$; see for example [10, Corollary 2.10].

The same line of argument can be used to prove (ii). □

Lemma 2.2.11. Let $M = t_1t_2\cdots t_k$ be a strictly distinguished monomial with distinguished terms $t_i$. Then $M \neq 0$ if and only if the following two conditions hold:

(i) each term $t_i$ is nonzero;

(ii) for each $1 \leq i < k$, there exists $\lambda \in \Lambda(n, r)$ such that $t_i = t_i1_\lambda$ and $t_{i+1} = 1_\lambda t_{i+1}$.

Proof. Condition (i) is clearly necessary for $M$ to be nonzero. To see the necessity of condition (ii), recall from Lemma 2.2.6 that for each term $t_i$, there exist $\lambda, \mu \in \Lambda(n, r)$ such that $t_i = t_i1_\lambda t_i = t_i1_\mu$.

We now check sufficiency. It will be enough to show that $\sigma(M)M \neq 0$, where $\sigma$ is as in Proposition 2.2.10. This follows from Lemma 2.2.7. Indeed, the hypotheses $\lambda_{i+1} \geq c$ or $\lambda_i \geq c$ follow from condition (i) above, and condition (ii) above implies that if $t_i1_\lambda = t_i$ then we have

$$
\sigma(t_{i+1})\sigma(t_i) = z''\sigma(t_{i+1})1_\lambda t_{i+1} = z''\sigma(t_{i+1})t_{i+1},
$$

where $z''$ is equal either to $z$ or to $z'$ as in Lemma 2.2.7. There is a unique $\mu \in \Lambda(n, r)$ such that $M = M1_\mu$, and an induction now shows that $\sigma(M)M$ is a nonzero scalar multiple of $1_\mu$, completing the proof. □

2.3. A subalgebra of $T$ isomorphic to $\hat{S}_\nu(n, r)$. The aim of §2.3 is to show that the relations satisfied by the endomorphisms of Proposition 2.1.7 are in fact consequences of the defining relations (Q15) and (Q16) of the algebra $T$. In this section, we may abuse notation by identifying elements $u$ of $U(\hat{\mathfrak{gl}}_n)$ with their images $\beta(u)$ in $T$ (see Definition 1.6.3).

Recall from Remark 1.6.4 that there is a natural subalgebra of $T$ isomorphic to the ordinary $\nu$-Schur algebra, $S_\nu(n, r)$. Using this fact, we can make the following

Definition 2.3.1. For each $1 \leq i < r$, define elements of $T$ by

$$
\zeta(T_n) = (vF_iE_i - 1)1_\omega,
$$

Lemma 2.3.3. The expressions given for \( \zeta(T_{\rho^{-1}}) \) and \( \zeta(T_{\rho}) \) are distinguished monomials.

**Proof.** This is a routine exercise, in which the hypothesis that \( n > r \) plays an important part. \( \square \)

Lemma 2.3.4. The following identities hold in \( T \):

(i) \( \zeta(T_{\rho^{-1}}) = (F_n(F_1F_2 \cdots F_{n-1}F_{n+1})F_{n-2}F_{n+2}) \) 1\( \omega \);

(ii) \( \zeta(T_{\rho}) = ((E_rE_{r-1}E_{n-1})(E_{r-1}E_{r-2} \cdots E_1E_n)) 1\omega. \)

**Proof.** Equation (i) (respectively, (ii)) follows by applying repeated commutations between the generators \( F_i \) (respectively, \( E_i \)). \( \square \)

Lemma 2.3.5. The following identities hold in \( T \):

(i) \( \zeta(T_{\rho^{-1}})\zeta(T_{\rho}) = 1\omega; \)

(ii) \( \zeta(T_{\rho})\zeta(T_{\rho^{-1}}) = 1\omega. \)

**Proof.** Let \( \omega' \in \Lambda(n, r) \) be the weight \( \omega' = (0, 1, \ldots, 1, 0, \ldots, 0) \), where the occurrences of 1 appear in positions 2, 3, 4, \ldots, \( r + 1 \leq n \). Then it follows from Lemma 2.3.4 that

\[
\zeta(T_{\rho}) = 1\omega ((E_rE_{r-1} \cdots E_2E_1) 1\omega' (E_{r+1}E_{n-1}E_n)) 1\omega
\]

and it follows from Definition 2.3.1 that

\[
\zeta(T_{\rho^{-1}}) = 1\omega ((F_nF_{n-1} \cdots F_{n+1}) 1\omega') (F_1F_2 \cdots F_r) 1\omega.
\]

We will prove (i), and (ii) follows by a similar argument.

To prove (i), we first show that

\[
1\omega' (F_1F_2 \cdots F_r) 1\omega (E_rE_{r-1} \cdots E_2E_1) 1\omega' = 1\omega'.
\]

The left hand side of the equation is readily checked to be a good monomial, and then the equation follows by repeated applications of the \( c = 1 \) case of the cancellation principle (Lemma 2.2.7), starting in the middle of the equation (i.e., with \( F_1 1\omega E_r \)).

A similar argument shows that

\[
1\omega (F_nF_{n-1} \cdots F_{n+1}) 1\omega' (E_{r+1}E_{n-1}E_n) 1\omega = 1\omega.
\]

Part (i) follows by combining these last two identities. \( \square \)

Lemma 2.3.6. Let \( 1 < i < r \), and let

\[
M = (E_{r-1}E_{r-2} \cdots E_1)(E_{r+1}E_{r+2} \cdots E_n) 1\omega.
\]

Then the identity \((vF_{i-1} - 1)M = M(vF_i - 1)\) holds in \( T \).

**Note.** Notice that both sides of the identity have right weight \( \omega. \)
**Proof.** Since the identity involves no occurrences of $E_r$ or $F_r$, Remark 1.6.4 applies. More precisely, after applying a suitable symmetry of the Dynkin diagram, we see that it suffices to prove the identity

$$(vF_{n-r+i-1}E_{n-r+i-1} - 1)M' = M'(vF_{n-r+i}E_{n-r+i} - 1)$$

in the ordinary $v$-Schur algebra, where indices are read modulo $n$, and we have

$$M' = (E_{n-1}E_{n-2} \cdots E_{n-r+1})(E_1E_2 \cdots E_{n-r})1_\omega,$$

and

$$\omega' = (0, 0, \ldots, 0, 1, \ldots, 1),$$

where 1 occurs $r$ times in $\omega'$.

By Theorem 1.4.8, it suffices to show that both sides of the identity act in the same way on tensor space $V^\otimes r$, and because both sides of the identity have right weight $\omega'$, it is enough to check this on the $\omega'$-weight space. By Lemma 1.4.10, it is enough to check that each side of the identity acts the same on the element

$$e_{\omega'} = e_{n-r+1} \otimes e_{n-r+2} \otimes \cdots \otimes e_n.$$

Fix $j$ with $1 \leq j < r$, and let $e_{j,\omega'}$ be the tensor obtained by exchanging the occurrences of $e_{n-r+j}$ and $e_{n-r+j+1}$ in $e_{\omega'}$. Using the comultiplication, it is a routine calculation to show that

$$(F_{n-r+j}E_{n-r+j})e_{\omega} = e_{j,\omega'} + v^{-1}e_{\omega'},$$

and it is immediate from this that

$$(vF_{n-r+j}E_{n-r+j} - 1)e_{\omega} = ve_{j,\omega'}.$$

Another calculation with the comultiplication shows that

$$(E_{n-1}E_{n-2} \cdots E_{n-r+1})(E_1E_2 \cdots E_{n-r})1_\omega e_{\omega'} = e'_{\omega'},$$

where

$$e'_{\omega'} = e_1 \otimes (e_{n-r+1} \otimes e_{n-r+2} \otimes \cdots \otimes e_{n-1}).$$

Let $j$ be such that $1 \leq j < r$. Acting $(vF_{n-r+j}E_{n-r+j} - 1)$ on the left, we deduce that

$$(vF_{n-r+j}E_{n-r+j} - 1)(E_{n-1}E_{n-2} \cdots E_{n-r+1})(E_1E_2 \cdots E_{n-r})1_\omega e'_{\omega'} = ve'_{j,\omega'},$$

where $e'_{j,\omega'}$ is obtained from $e_{j,\omega}$ by exchanging the occurrences of $e_{n-r+j}$ and $e_{n-r+j+1}$.

The result now follows after we observe that

$$(E_{n-1}E_{n-2} \cdots E_{n-r+1})(E_1E_2 \cdots E_{n-r})1_\omega e_{j+1,\omega'} = e'_{j,\omega'}.
\square$$

**Corollary 2.3.7.** If $i$ is such that $1 < i < r$, then the relation

$$\zeta(T_{s_{i-1}})\zeta(T_i) = \zeta(T_i)\zeta(T_{s_i})$$

holds in $T$.

**Proof.** Observe that $E_r$ commutes with $F_i$, $F_{i-1}$, $E_i$ and $E_{i-1}$. The assertion now follows by left-multiplying the identity of Lemma 2.3.6 by $E_r$. \qed
The techniques of proof of Lemma 2.3.6 play an important part in the next brace of results.

**Lemma 2.3.8.** The following identities hold in $T$, where $1 < i < r$:

(i) $(F_i E_i - v^{-1}) 1_\omega = (E_i F_i - v) 1_\omega$;
(ii) $1_\omega(F_n F_{n-1} \cdots F_r E_r E_{r+1} \cdots E_n - v^{-1}) = 1_\omega(E_r E_{r+1} \cdots E_n F_n F_{n-1} \cdots F_r - v)$;
(iii) $(E_n F_n - v) E_1 E_n 1_\omega = E_1 E_n (E_1 F_1 - v) 1_\omega$;
(iv) $1_\omega(F_{r-1} E_{r-1} - v^{-1}) E_r E_{r-1} \cdots E_2 E_1 = 1_\omega E_r E_{r-1} \cdots E_2 E_1 (F_r E_r - v^{-1})$.

**Note.** The expressions appearing in (i) and (ii) above have both left and right weight equal to $\omega$.

**Proof.** We omit the proof of (i), because it is similar to, but easier than, the proof of (ii).

To prove (ii), it is enough, by symmetry of the defining relations of $T$, to prove the identity

$$1_\omega^{-1}(F_n F_{n-1} \cdots F_r E_{r-1} \cdots E_n - v^{-1}) = 1_\omega^{-1}(E_{r-1} \cdots E_n F_n F_{n-1} \cdots F_r - v),$$

where

$$\omega^{-1} = (1, 1, \ldots, 1, 0, 0, \ldots, 0).$$

This can be regarded as an identity in $S_n(n, r)$. By Lemma 1.4.10, it is enough to check that each side of the identity acts in the same way on the element

$$e_{\omega^{-1}} = e_1 \otimes e_2 \otimes \cdots \otimes e_{r-2} \otimes e_{r-1} \otimes e_n.$$

A calculation shows that each side of the identity acts on $e_{\omega^{-1}}$ to give

$$e_1 \otimes e_2 \otimes \cdots \otimes e_{r-2} \otimes e_n \otimes e_{r-1}.$$

To prove (iii), it is enough by symmetry of the defining relations to prove

$$(E_1 F_1 - v) E_2 E_1 1_{\omega^+} = E_2 E_1 (E_2 F_2 - v) 1_{\omega^+},$$

where

$$\omega^+ = (0, 1, 1, \ldots, 1, 0, 0, \ldots, 0).$$

By Lemma 1.4.10, it is enough to show that both sides of the identity act in the same way on

$$e_{\omega^+} = e_2 \otimes e_3 \otimes \cdots \otimes e_{r+1}.$$

A calculation shows that each side sends $e_{\omega^+}$ to

$$e_2 \otimes e_1 \otimes e_4 \otimes e_5 \otimes \cdots \otimes e_{r+1}.$$

For (iv), observe that both sides of the identity have right weight $\omega^+$, as defined above. Since (iv) can be regarded as an identity in $S_n(n, r)$, Lemma 1.4.10 applies and it is enough to check that both sides of the identity have the same effect on $e_{\omega^+}$. A calculation shows that both sides of the identity send $e_{\omega^+}$ to

$$e_1 \otimes e_2 \otimes \cdots \otimes e_{r-3} \otimes e_{r-2} \otimes e_r \otimes e_{r-1}.$$
Definition 2.3.9. We define $\zeta(T_{s_r})$ to be the element of $T$ given by
\[ 1_\omega(F_n F_{n-1} \cdots F_r E_r E_{r+1} \cdots E_n - v^{-1}). \]

Lemma 2.3.10. The following identities hold in $T$:
(i) $\zeta(T_p)\zeta(T_{s_r}) = \zeta(T_{s_{r-1}})\zeta(T_p)$;
(ii) $\zeta(T_p)\zeta(T_{s_1}) = \zeta(T_{s_r})\zeta(T_p)$.

Proof. We prove (i) first. Using Lemma 2.3.4(ii), it is enough to prove that the expressions
\[ M_1 = 1_\omega(E_r E_{r-1} \cdots E_1)(E_{r+1} E_{r+2} \cdots E_n)(F_n F_{n-1} \cdots F_{r+1} F_r)(E_r E_{r+1} \cdots E_n) \]
and
\[ M_2 = 1_\omega(F_{r-1} E_{r-1})(E_r E_{r-1} \cdots E_1) F_r E_{r+1} \cdots E_n \]
are equal.

Using Lemma 2.2.6 repeatedly, and the notation of the proof of Lemma 2.3.6, we find that
\[ M_1 = 1_\omega E_r E_{r-1} \cdots E_1(1_\omega+ E_{r+1} E_{r+2} \cdots E_n F_n F_{n-1} \cdots F_{r+1} 1_{\omega^+}) F_r E_{r+1} \cdots E_n, \]
and repeated applications of the cancellation principle (Lemma 2.2.7) show that the parenthetic expression shown is simply equal to $1_{\omega^+}$. By Lemma 2.2.6 again, we have
\[ M_1 = 1_\omega E_r E_{r-1} \cdots E_1(1_\omega+ E_r E_{r+1} \cdots E_n) = 1_\omega (E_r E_{r-1} \cdots E_1) F_r E_r (E_{r+1} E_{r+2} \cdots E_n). \]
Applying Lemma 2.3.8 (iv) gives
\[ M_1 = 1_\omega F_{r-1} E_{r-1}(E_r E_{r-1} \cdots E_1)(E_{r+1} E_{r+2} \cdots E_n), \]
which is $M_2$, as required.

We now turn to (ii). By Lemma 2.3.8, parts (i) and (ii), it is enough to show that the monomials
\[ M_3 = 1_\omega(E_r E_{r+1} \cdots E_n)(F_n F_{n-1} \cdots F_r)(E_r E_{r+1} \cdots E_{n-1})(E_{r-1} E_{r-2} \cdots E_1)E_n \]
and
\[ M_4 = 1_\omega(E_r E_{r+1} \cdots E_{n-1})(E_{r-1} E_{r-2} \cdots E_1)E_n(E_1 F_1) \]
are equal. Using Lemma 2.2.6 and the notation of the proof of Lemma 2.3.6 again, we find that $M_3$ is equal to
\[ 1_\omega E_r E_{r+1} \cdots E_n F_n (1_\omega- F_{n-1} F_{n-2} \cdots F_r E_r E_{r+1} \cdots E_{n-1} 1_{\omega^-}) E_{r-1} E_{r-2} \cdots E_1 E_n. \]
By the cancellation principle, this simplifies to
\[ M_3 = 1_\omega E_r E_{r+1} \cdots E_n F_n 1_\omega- E_{r-1} E_{r-2} \cdots E_1 E_n \]
\[ = 1_\omega (E_r E_{r+1} \cdots E_{n-1})(E_n F_n)(E_{r-1} E_{r-2} \cdots E_1)E_n. \]
Applying commutations yields
\[ M_3 = 1_\omega(E_r E_{r+1} \cdots E_{n-1})(E_{r-1} E_{r-2} \cdots E_2)(E_n F_n E_1 E_n). \]
Using Lemma 2.3.8 (iii), we have
\[ M_3 = 1_\omega(E_r E_{r+1} \cdots E_{n-1})(E_{r-1} E_{r-2} \cdots E_2)(E_1 E_n E_1 F_1), \]
which is equal to \( M_4 \), as desired. □

**Proposition 2.3.11.** Let \( \gamma : T \twoheadrightarrow \hat{S}_\nu(n, r) \) be the epimorphism of Proposition 1.6.5. Then \( \gamma \) admits a right inverse: there is an injective homomorphism \( \iota : \hat{S}_\nu(n, r) \longrightarrow T \) such that \( \gamma \circ \iota \) is the identity homomorphism on \( \hat{S}_\nu(n, r) \).

**Proof.** We start by specifying \( \iota \) on the subalgebra of \( \hat{S}_\nu(n, r) \) spanned by the elements \( \phi_{\omega, \omega}^d \), as in §2.1. We define
\[ \iota(\tau(T_{s_i})) := \zeta(T_{s_i}) \]
for \( 1 \leq i < r \), and define
\[ \iota(\tau(T_{\rho^\pm 1})) := \zeta(T_{\rho^\pm 1}). \]
In these cases, \( \gamma \circ \iota \) is the identity map by Proposition 2.1.7 and the definition of \( \zeta \), so it is enough to check that the relations of Lemma 1.2.2 are satisfied in the image of \( \iota \). The difficult cases, (4′) and (5′), follow from Lemma 2.3.5, Corollary 2.3.7 and Lemma 2.3.10. Since those cases hold, it is enough to check cases (1′), (2′) and (3′) assuming that neither \( s \) nor \( t \) is equal to \( r \); this follows from Lemma 2.1.3 and Remark 1.6.4.

It remains to check relations (Q17), (Q18) and (Q19) of Proposition 2.1.1. Since, by Remark 1.6.4, there is a canonically embedded copy of \( S_\nu(n, r) \) in the algebra \( T \) (namely, the subalgebra generated by all \( E_i, F_i, K_j^{\pm 1} \) with \( i \neq n \)), we may send the elements \( \phi_{\omega, \lambda}^1, \phi_{\mu, \omega}^1 \) and \( \phi_{\omega, \omega}^d \) (where \( d \) lies in the finite symmetric group) to the corresponding elements of \( T \). (Observe that this construction is compatible with the definitions in the previous paragraph.) More explicitly, if \( M \) is a polynomial in the generators \( E_i, F_i, K_j, K_j^{-1} \) and there are no occurrences of \( E_n \) or \( F_n \), then \( \alpha(M) \in S_\nu(n, r) \) and \( \iota(\alpha(M)) = \beta(M) \) by construction. It then follows that
\[ (\gamma \circ \iota)(\alpha(M)) = \gamma(\beta(M)) = \alpha(M), \]
as required. Relations (Q17), (Q18) and (Q19) can be seen to hold in \( T \) (and thus in the image of \( \iota \)) by Theorem 1.6.2 and Remark 1.6.4. □

2.4. **Surjectivity of \( \iota \).** So far we have shown that there is a monomorphism \( \iota : \hat{S}_\nu(n, r) \longrightarrow T \). From Definition 1.6.3 we have a surjective map \( \beta : U(\mathfrak{gl}_n) \longrightarrow T \). We aim in §2.4 to show that the image of \( \beta \) is contained in the image of \( \iota \), which will complete the proof of Theorem 1.6.1.

**Lemma 2.4.1.** The algebra \( T \) is generated by the all elements of the form \( E_i 1_\lambda, 1_\lambda F_i \) and \( 1_\lambda \), for \( 1 \leq i \leq n \) and \( \lambda \in \Lambda(n, r) \).
Proof. The images of the $K_i$ and $K_i^{-1}$ in $T$ are linear combinations of the elements $1_\lambda$, as they are in the ordinary $\nu$-Schur algebra (see [10, Corollary 2.10]). The image of $E_i$ in $T$ is a linear combination of elements $E_i1_\lambda$, because the $1_\lambda$ form an orthogonal decomposition of the identity (see Lemma 2.2.4). Similarly, the image of $F_i$ in $T$ is a linear combination of elements $1_\lambda F_i$. This shows that the usual algebra generators of $T$ lie in the span of the elements listed in the statement. Conversely, it follows from the definitions (see Definition 2.2.2) that the elements listed lie in $T$, completing the proof. \hfill $\Box$

**Lemma 2.4.2.** If $i \neq n$ and $\lambda \in \Lambda(n,r)$, the elements $E_i1_\lambda$, $1_\lambda F_i$ and $1_\lambda$ lie in the image of $\iota$. \hfill $\Box$

**Proof.** The elements listed in the statement lie in the canonically embedded copy of $S_\nu(n,r)$ in $T$. By the construction of $\iota$ (see the proof of Proposition 2.3.11), such elements lie in the image of $\iota$. \hfill $\Box$

**Lemma 2.4.3.** (i) The element $E_n1_\omega$ of $T$ lies in $\iota(\tilde{S}_\nu(n,r))$.
(ii) The element $1_\omega F_n$ of $T$ lies in $\iota(\tilde{S}_\nu(n,r))$.

**Proof.** By construction of $\iota$, the element $\zeta(T_\rho)$ lies in the image of $\iota$. Since $T$ contains a canonical copy of $S_\nu(n,r)$ (see the proof of Proposition 2.3.11), the element

$$(F_1F_2 \cdots F_{r-1})(F_{n-1}F_{n-2} \cdots F_r)1_\omega$$

lies in the image of $\iota$. By multiplying these two elements we see that

$$(F_1F_2 \cdots F_{r-1})(F_{n-1}F_{n-2} \cdots F_r)(E_rE_{r+1} \cdots E_{n-1})(E_{r-1}E_{r-2} \cdots E_1)E_n1_\omega$$

lies in the image of $\iota$. Applying the cancellation principle, this latter expression simplifies to $E_n1_\omega$, completing the proof of (i).

A similar argument using $\zeta(T_{\rho^{-1}})$ in place of $\zeta(T_\rho)$ can be used to prove (ii). \hfill $\Box$

Our main effort will be directed towards proving that the elements $E_n1_\lambda$ lie in the image of $\iota$. More precisely, we will prove that $E_n1_\lambda$ lies in the ideal of $\text{Im}(\iota)$ generated by $E_n1_\omega$. Our argument will rely on the following technical lemma whose proof will be deferred to §2.5.

**Lemma 2.4.4.** Fix $\lambda \in \Lambda(n,r)$ such that $\lambda_1 > 0$. There exists a distinguished monomial $M$ in the generators $E_i$, $F_i$ and $1_\mu$ satisfying the following conditions:
(i) $M = 1_\omega M1_\lambda \neq 0$;
(ii) $M$ contains no occurrences of $E_n$, $F_n$, $E_1$ or $F_{n-1}$;
(iii) all the occurrences of $F_1$ occur consecutively, as do all the occurrences of $E_{n-1}$;
(iv) there are at most $\lambda_1 - 1$ occurrences of $F_1$.

**Lemma 2.4.5.** Let $\sigma$ be the antiautomorphism of $T$ given in Proposition 2.2.10 (ii), and let $M$ and $\lambda$ be as in Lemma 2.4.4. Then there is a nonzero scalar $z \in \mathbb{Q}(\nu)$ such that

$$E_n1_\lambda = z\sigma(M)(E_n1_\omega)M.$$ 

In particular, $E_n1_\lambda$ lies in the ideal of $\text{Im}(\iota)$ generated by $E_n1_\omega$. 

Proof. The monomial $M$ is equal to a strictly distinguished monomial

$$M' = t'_m t'_{m-1} \cdots t'_1.$$  

After moving unnecessary idempotents in $M$ to the right using Lemma 2.2.7, and omitting the corresponding idempotents from the terms $t'_i$, we may assume that $M$ is of the form

$$M = t_m t_{m-1} \cdots t_1 1_\lambda.$$  

We will prove by induction on $k \leq m$ that

$$(\sigma(t_1)\sigma(t_2)\cdots\sigma(t_k))E_n(t_k t_{k-1} \cdots t_1 1_\lambda)$$

is a nonzero multiple of $E_n 1_\lambda$; the case $k = m$ is the assertion of the Lemma. The base case, $k = 0$, is trivial.

There are two cases to consider for the inductive step. The first case, which is easier to deal with, is that $t_k$ is of the form $E^c_i$ for some $c > 0$. In this case, we have

$$\sigma(t_k)E_n t_k = F^c_i E^c_i E_n,$$

which can be rewritten as

$$E_n F^c_i E^c_i$$

using the relations of $U(\widehat{\mathfrak{gl}}_n)$. (Note that we do not have $i = n$, because $M$ does not contain occurrences of $E_n$.) We now have

$$F^c_i E_n t_k t_{k-1} \cdots t_1 1_\lambda = E_n F^c_i t_k 1_\mu t_{k-1} t_{k-2} \cdots t_1 1_\lambda$$

for a suitable $\mu \in \Lambda(n, r)$. The hypothesis $M \neq 0$ means that we have $\mu_{i+1} \geq c$, so we may apply Lemma 2.2.7 (i) to replace $F^c_i t_k 1_\mu$ by $z 1_\mu$ with $z$ nonzero. The proof is now completed in this case by the inductive hypothesis.

The second case is that $t_k$ is of the form $F^c_i$ for some $c > 0$. In this case, we cannot have $i = n$ or $i = n - 1$ because of condition (ii) of Lemma 2.4.4. Suppose for the moment that $i \neq 1$. Then the relations in $U(\widehat{\mathfrak{gl}}_n)$ show that

$$E^c_i E_n F^c_i = E_n E^c_i F^c_i.$$  

We can then proceed as in the first case to show that

$$E^c_i E_n t_k t_{k-1} \cdots t_1 1_\lambda = E_n E^c_i t_k 1_\mu t_{k-1} t_{k-2} \cdots t_1 1_\lambda$$

where $z'$ is nonzero. Here we have used Lemma 2.2.7 (ii), which is applicable because $M$ is nonzero and $\mu_i \geq c$. Again, we are done by induction in this case.

The remaining case is the possibility that $t_k = F^c_1$ for some $c > 0$. The relations in $U(\widehat{\mathfrak{gl}}_n)$ show that

$$E^c_1 E_n F^c_1 = E^c_1 F^c_1 E_n,$$

and so we have

$$E^c_1 E_n t_k t_{k-1} \cdots t_1 1_\lambda = E^c_1 t_k E_n 1_\mu t_{k-1} t_{k-2} \cdots t_1 1_\lambda.$$
Because $F_1^c$ arises from a distinguished term, we have $\mu_2 = 0$. By Lemma 2.2.6 (i), we have
\[ E_1^c F_1^c E_n 1_\mu = E_1^c F_1^c 1_{\mu'} E_n, \]
where again $\mu'_2 = 0$ (recall that $n \geq 3$). Hence
\[ E_1^c E_n t_k t_{k-1} \cdots t_1 1_\lambda = z' 1_{\mu'} E_n t_k t_{k-2} \cdots t_1 1_\lambda. \]
Furthermore, $z'$ is nonzero. To see why, we recall that by condition (ii) of Lemma 2.4.4, there are no occurrences of $E_1$ or $E_n$ or $F_n$ in $M$ and that by condition (iii), all the occurrences of $F_1$ occur consecutively. Repeated applications of Lemma 2.2.6 then show that $\mu'_1 = \lambda_1$. Lemma 2.2.7 (ii) then applies again to yield
\[ E_1^c 1_{\mu'} = z'' 1_{\mu'}, \]
and $z''$ is nonzero because by condition (iv) of Lemma 2.4.4, $c \leq \lambda_1 - 1 < \lambda_1 = \mu'_1$. Once again, the assertion follows by induction in this case.

Finally, we observe that since both $M$ and $\sigma(M)$ avoid occurrences of $E_n$ and $F_n$, they lie in the subalgebra of $T$ corresponding to $S_v(n, r)$. This means that $M$ and $\sigma(M)$ lie in $\text{Im}(\iota)$, and the proof follows.

**Corollary 2.4.6.** If $\lambda \in \Lambda(n, r)$, the elements $E_n 1_\lambda$, $1_\lambda F_n$ and $1_\lambda$ lie in the image of $\iota$.

**Proof.** If $\lambda_1 = 0$, then $E_n 1_\lambda = 1_\lambda F_n = 0$ and the assertion is trivial. Otherwise, the assertion follows by combining lemmas 2.4.3 and 2.4.5. \qed

**Proof of Theorem 1.6.1 (modulo Lemma 2.4.4).** By Lemma 2.4.2 and Corollary 2.4.6, the generators of $T$ listed in Lemma 2.4.1 all lie in $\text{Im}(\iota)$. This proves that $\iota$ is surjective, and taken in conjunction with Proposition 2.3.11, we see that $\iota$ is an isomorphism. This completes the proof of Theorem 1.6.1 (modulo Lemma 2.4.4). \qed

2.5. **Proof of Lemma 2.4.4.** The only other ingredient needed to prove Theorem 1.6.1 is Lemma 2.4.4.

**Definition 2.5.1.** Let $\mu, \nu \in \Lambda(n, r)$. We say that $\mu$ and $\nu$ are **Z-equivalent** if they become equal after their zero parts have been deleted. (In other words, $\mu$ and $\nu$ correspond to the same parabolic subgroup of the symmetric group.)

**Example 2.5.2.** Let $n = 6$, $r = 5$, $\lambda = (0, 2, 1, 0, 2, 0)$ and $\mu = (2, 0, 1, 2, 0, 0)$. After deletion of zero parts, each of $\lambda$ and $\mu$ reduces to $(2, 1, 2)$, so $\lambda$ and $\mu$ are Z-equivalent.

**Lemma 2.5.3.** Let $\lambda \in \Lambda(n, r)$ with $\lambda_1 > 0$.

(i) There exists a nonzero distinguished monomial $M_1$ in the generators $E_2, E_3, \ldots, E_{n-1}$, such that the occurrences of $E_{n-1}$ occur consecutively, satisfying
\[ M_1 = 1_\mu M_1 1_\lambda. \]
Here, $\mu = \mu(\lambda)$ is such that (a) $\mu$ and $\lambda$ are $\mathbb{Z}$-equivalent and (b) $\ell(\mu)$ (see Definition 1.3.1) is of the form

$$(1, 1, \ldots, 1, 2, 2, \ldots, 2, \ldots, k, k, \ldots, k).$$

Furthermore, $\mu_1 = \lambda_1$, $k \leq r$ and thus $\mu_{k+1} = \mu_{k+2} = \cdots = \mu_{r+1} = \cdots = \mu_n = 0$.

(ii) Let $\mu$ be as in part (i) above, and let $\nu$ be the unique element of $\Lambda(n, r)$ such that (a) $\nu_a = \mu_{i+1}$ whenever

$$a = 1 + \sum_{j=1}^{i} \mu_j$$

for any $0 \leq i < r$, and (b) $\nu_a = 0$ for other values of $a$. (In particular, $\nu_1 = \mu_1 = \lambda_1$, and $\nu$ and $\mu$ are $\mathbb{Z}$-equivalent.) Then there exists a nonzero distinguished monomial $M_2$ in the generators $F_2, F_3, \ldots, F_{n-2}$ satisfying

$$M_2 = 1_{\nu}M_21_{\mu}.$$ (iii) Let $\mu$ and $\nu$ be as in (ii) above. Then for each $1 \leq i \leq r$ we have

$$\sum_{j=1}^{\mu_i} \nu_{b(i)+j} = \mu_i = \nu_{b(i)+1},$$

where $b(i) = \sum_{k<i} \mu_k$. Summing over all $i$, this yields

$$\sum_{j=1}^{r} \nu_j = r,$$

and hence $\nu_{r+1} = \nu_{r+2} = \cdots = \nu_n = 0$.

(iv) Let $\nu$ be as in part (ii) above. Then there exists a nonzero distinguished monomial $M_3$ in the generators $F_1, F_2, F_3, \ldots, F_{n-2}$ satisfying

$$M_3 = 1_{\omega}M_31_{\nu}.$$ Furthermore, the occurrences of $F_1$ occur consecutively, and there are $\lambda_1 - 1$ occurrences of $F_1$.

Note Note that in the above situation, $\mu$ and $\nu$ are uniquely determined by $\lambda$. For example, if $n = 9$, $r = 7$ and

$$\lambda = (2, 0, 0, 3, 0, 0, 0, 0, 2),$$

then we have

$$\mu = (2, 3, 2, 0, 0, 0, 0, 0, 0),$$

and

$$\nu = (2, 0, 3, 0, 0, 2, 0, 0, 0).$$

In this case, we could take

$$M_1 = 1_{\mu}E_3^2E_4^2E_5^2E_6^2E_7^2E_8^2E_9^2E_1^3\lambda.$$
\[ M_2 = 1_\nu F_2^3 F_5^2 F_4^2 F_3^2 1_\mu, \]
\[ M_3 = 1_\omega F_1 F_4 F_3^2 F_6 1_\nu. \]

**Proof.** To prove (i), let \( \lambda \in \Lambda(n, r) \) be such that \( \lambda_i = 0 \) and \( \lambda_{i+1} = c > 0 \). Lemma 2.2.7 (i) implies that \( E_i^c 1_\lambda \) is a nonzero element of \( T \), and iterated applications of Lemma 2.2.6(i) show that \( E_i^c 1_\lambda = 1_\xi E_i^c \), where \( \xi_j = \begin{cases} 
\lambda_{i+1} & \text{if } j = i, \\
\lambda_i & \text{if } j = i + 1, \\
\lambda_j & \text{otherwise.} 
\end{cases} \)

Repeated applications of this procedure can be used to move all the zero parts of \( \lambda \) to the right. Since \( \lambda_1 > 0 \) by assumption, we have \( \mu_1 = \lambda_1 \), and it is never necessary to use an application of \( E_1 \). It is necessary to use applications of \( E_{n-1} \) if and only if \( \lambda_n > 0 \), but in this case they may all be applied consecutively. Since there are at most \( r \) nonzero parts in \( \lambda_1 \) and since \( n > r \), we have \( k \leq r \). We have \( M_1 \neq 0 \) based on an inductive argument using Lemma 2.2.11. The other assertions of (i) follow.

The claims in (ii) concerning \( \nu \) are routine apart from the assertion regarding the distinguished monomial. The proof of (iii) then follows from the characterization of \( \nu \) given in (ii). (The reader may find it helpful here to look at the note preceding this proof.) The entries \( \nu_{r+1}, \nu_{r+2}, \ldots, \nu_n \) can effectively be ignored for the rest of the proof.

The proof of the last assertion of (ii) follows similar lines to the proof of (i). The main difference is that the aim is to move certain of the zero components of \( \lambda \) to the left. The basic step involves \( \mu \in \Lambda(n, r) \) such that \( \mu_i = c > 0 \) and \( \mu_{i+1} = 0 \), in which case Lemma 2.2.7 (ii) shows that \( F_i^c 1_\mu \) is a nonzero element of \( T \), and iterated applications of Lemma 2.2.6 (ii) show that \( F_i^c 1_\mu = 1_\xi F_i^c \), where \( \xi_j = \begin{cases} 
\mu_{i+1} & \text{if } j = i, \\
\mu_i & \text{if } j = i + 1, \\
\mu_j & \text{otherwise.} 
\end{cases} \)

Since \( \nu_1 = \mu_1 = \lambda_1 > 0 \) by the definition of \( \nu \), no applications of \( F_1 \) are necessary. The fact that \( \nu_n = 0 \) shows why no applications of \( F_{n-1} \) are necessary. As before, Lemma 2.2.11 shows why \( M_2 \neq 0 \).

To prove (iv), observe that \( \nu \) can be written as the concatenation of maximal segments of the form

\[ (\nu_i, 0, 0, \ldots, 0). \]

Let us first deal with the case where \( \nu_1 = r \). Here, the monomial may be given explicitly as

\[ M_3 = 1_\omega F_{\nu_1-1}^{1} F_{\nu_2-2}^{2} \cdots F_{\nu_{r-1}-2}^{\nu_1-2} F_1^{\nu_1-1} 1_\nu. \]

This monomial is easily checked to be distinguished. In the general case, there is a monomial in the \( F_i \) corresponding to each of the maximal segments mentioned.
above, and monomials corresponding to distinct segments commute. When these monomials are concatenated and flanked by $1_\omega$ and $1_\nu$, we obtain $M_3$. The occurrences of $F_1$ all correspond to the segment containing $\nu_1$, and the explicit formula above shows that these occurrences are consecutive and that there are $\lambda_1 - 1$ of them. (This number may be zero.) As in the proof of (ii) above, the fact that $\omega_n = 0$ explains why there are no occurrences of $F_{n-1}$. Also as above, Lemma 2.2.11 shows why $M_3 \neq 0$. \hfill \Box

**Proof of Lemma 2.4.4.** In the notation of Lemma 2.5.3, the required monomial is

$$M = 1_\omega M_3 1_\nu M_2 1_\mu M_1 1_\lambda.$$  

Properties (i)–(iv) of Lemma 2.4.4 follow from the various parts of Lemma 2.5.3. Since $M_1$, $M_2$ and $M_3$ are nonzero, $M$ is nonzero by Lemma 2.2.11.

This completes the proof of Lemma 2.4.4, and therefore of Theorem 1.6.1. \hfill \Box

### 2.6. An alternative presentation of $\hat{S}_v(n, r)$. Lusztig [16, Part IV] has defined a modified form of a quantized enveloping algebra, by replacing the zero part of the algebra with an infinite system of pairwise orthogonal idempotents, acting on modules as weight space projectors. The modified form has a canonical basis with remarkable properties, similar to properties of the canonical basis of the positive part of the original quantized algebra.

The following presentation of the algebra $\hat{S}_v(n, r)$, compatible with Lusztig’s modified form of $U(\hat{\mathfrak{gl}}_n)$, is equivalent with the presentation given in Theorem 1.6.1.

**Theorem 2.6.1.** Assume that $n > r$. Over $\mathbb{Q}(v)$, the algebra $\hat{S}_v(n, r)$ is isomorphic with the associative algebra (with 1) given by the generators $i_\lambda$ ($\lambda \in \Lambda(n, r)$), $E_i, F_i$ ($1 \leq i \leq n$) and relations

\begin{align*}
(R1) \quad i_\lambda i_\mu &= \delta_{\lambda,\mu} i_\lambda; \quad \sum_{\lambda \in \Lambda(n, r)} i_\lambda = 1; \\
(R2) \quad E_i i_\lambda &= \begin{cases} i_{\lambda+\alpha_i} E_i & \text{if } \lambda_{i+1} > 0, \\ 0 & \text{otherwise}; \end{cases} \\
(R3) \quad F_i i_\lambda &= \begin{cases} i_{\lambda-\alpha_i} F_i & \text{if } \lambda_i > 0, \\ 0 & \text{otherwise}; \end{cases} \\
(R4) \quad E_i F_j - F_j E_i &= \delta_{i,j} \sum_{\lambda \in \Lambda(n, r)} [\lambda_j - \lambda_{j+1}] i_\lambda
\end{align*}

along with relations (Q6)–(Q9) of Definition 1.4.1. Here we regard weights as infinite periodic sequences, as in Section 2.2 above.

**Proof.** Let $A$ be the algebra defined by the presentation of the theorem, and let $T$ be the algebra defined by the presentation of Theorem 1.6.1.

By Definition 2.2.2, Lemma 2.2.4, and Lemma 2.2.6 the elements $1_\lambda$ ($\lambda \in \Lambda(n, r)$), $E_i, F_i$ ($1 \leq i \leq n$) of $T$ satisfy the relations (R1)–(R3). The $E_i, F_i$ ($1 \leq i \leq n$)
already satisfy relations (Q6)–(Q9) by assumption. By applying Remark 1.6.4 to the results in [7, Theorem 3.4] we see that (R4) holds as well for all \(1 \leq i, j < n\). If one or both of \(i, j\) is equal to \(n\), then we choose a different embedding of \(S_v(n, r)\) in \(T\), one which includes the values of \(i, j\) in question, and again apply Remark 1.6.4 to the results in [7, Theorem 3.4] to see that (R4) holds as well. In \(T\) we have by Lemma 2.2.4 the equalities

\[
(*) \quad K_i = K_i \sum_\lambda 1_\lambda = \sum_\lambda v^{\lambda_i} 1_\lambda; \quad K_i^{-1} = K_i^{-1} \sum_\lambda 1_\lambda = \sum_\lambda v^{-\lambda_i} 1_\lambda
\]

for any \(1 \leq i \leq n\), where the sums are taken over all \(\lambda \in \Lambda(n, r)\). Hence, the elements \(1_\lambda (\lambda \in \Lambda(n, r))\), \(E_i, F_i (1 \leq i \leq n)\) generate \(T\), and the map

\[i_\lambda \rightarrow 1_\lambda, \quad E_i \rightarrow E_i, \quad F_i \rightarrow F_i\]

defines a surjective quotient map from \(A\) onto \(T\).

On the other hand, in the algebra \(A\) one defines elements \(K_i = \sum_\lambda v^{\lambda_i} 1_\lambda, \quad K_i^{-1} = \sum_\lambda v^{-\lambda_i} 1_\lambda\). By following the same line of argument as in the proof of [7, Theorem 3.4], these elements, along with the elements \(E_i, F_i\) for \(1 \leq i < n\), satisfy the defining relations (Q1)–(Q9), (Q15), (Q16) of \(T\). It remains to show that the elements \(E_n, F_n\) also satisfy those relations (along with the \(K_i, K_i^{-1}\)). Only relations (Q3), (Q4), and (Q5) are in question since the other relations either hold by assumption or do not involve the elements \(E_n, F_n\).

We now verify that relation (Q3) holds for \(E_n\). By definition of \(K_i\) we have

\[K_i E_n = \sum_\lambda v^{\lambda_i} i_\lambda E_n\]

and by relation (R2) this takes the form

\[K_i E_n = \sum_\lambda v^{\lambda_i} E_n i_{\lambda - \alpha_n}\]

where, for convenience of notation, we take both sums over the set of all \(\lambda \in \mathbb{Z}^n\) satisfying \(\sum \lambda_i = r\), with the understanding that \(i_\lambda\) is interpreted to be 0 in case any part of \(\lambda\) is negative. (This makes all the sums in question finite.) Now replace \(\lambda - \alpha_n\) by \(\mu \in \mathbb{Z}^n\) and the above gives relation (Q3) for \(E_n\).

The proof that relation (Q4) holds for \(F_n\) is similar.

Finally, we verify that (Q5) holds. By the given relation (R4) we have

\[E_i F_j - F_j E_i = \delta_{ij} \sum_\lambda [\lambda_j - \lambda_{j+1}] i_\lambda\]

and this gives 0 unless \(i = j\), so (Q5) holds in case \(i \neq j\). Assuming that \(i = j\), the above sum becomes

\[E_i F_j - F_j E_i = \sum_\lambda \frac{v^{\lambda_i - \lambda_{i+1}} - v^{-\lambda_i + \lambda_{i+1}}}{v - v^{-1}} i_\lambda\]

\[= \left( \sum_\lambda v^{\lambda_i} i_\lambda v^{-\lambda_{i+1}} i_\lambda \right) - \left( \sum_\lambda v^{-\lambda_i} i_\lambda v^{\lambda_{i+1}} i_\lambda \right)\]

for any \(1 \leq i \leq n\), where the sums are taken over all \(\lambda \in \Lambda(n, r)\). Hence, the elements \(1_\lambda (\lambda \in \Lambda(n, r))\), \(E_i, F_i (1 \leq i \leq n)\) generate \(T\), and the map

\[i_\lambda \rightarrow 1_\lambda, \quad E_i \rightarrow E_i, \quad F_i \rightarrow F_i\]

defines a surjective quotient map from \(A\) onto \(T\).
using the orthogonality of the system of idempotents. By the definition of the $K_i, K_i^{-1}$ this proves (Q5) in case $i = j$.

We claim that the elements $K_i, K_i^{-1}, E_i, F_i$ (for $1 \leq i \leq n$) generate $A$. To see this, it suffices to show that the $K_i, K_i^{-1}$ generate the zero part of $A$ (the span of the $\mathbf{i}_\lambda$).

From the definition of $K_i$ and $K_i^{-1}$ it follows that

$$K_j \mathbf{i}_\lambda = v^{\lambda_j} \mathbf{i}_\lambda; \quad K_j^{-1} \mathbf{i}_\lambda = v^{-\lambda_j} \mathbf{i}_\lambda$$

and thus $K_j = K_j \sum \lambda \mathbf{i}_\lambda = \sum \lambda v^{\lambda_j} \mathbf{i}_\lambda$ and $K_j^{-1} = K_j^{-1} \sum \lambda \mathbf{i}_\lambda = \sum \lambda v^{-\lambda_j} \mathbf{i}_\lambda$, where the sums are over all $\lambda \in \Lambda(n, r)$. Hence it follows that

$$\left[ K_j \right]_t = \prod_{s=1}^t \frac{K_j v^{s+1} - K_j^{-1} v^{s-1}}{v^s - v^{-s}}$$

$$= \prod_{s=1}^t \frac{(\sum \lambda v^{\lambda_j} v^{s+1}) \mathbf{i}_\lambda - (\sum \lambda v^{-\lambda_j} v^{s-1}) \mathbf{i}_\lambda}{v^s - v^{-s}}$$

$$= \prod_{s=1}^t \sum \lambda \frac{v^{\lambda_j} - v^{-\lambda_j} v^{s-1}}{v^s - v^{-s}} \mathbf{i}_\lambda$$

$$= \sum \lambda \prod_{s=1}^t \frac{v^{\lambda_j} - v^{-\lambda_j} v^{s-1}}{v^s - v^{-s}} \mathbf{i}_\lambda$$

where we have again made use of the orthogonality of the idempotents to interchange the product and sum. From this and the orthogonality of idempotents it follows that for any $\mu \in \Lambda(n, r)$ we have

$$\left[ K_1 \right]_{\mu_1} \cdots \left[ K_n \right]_{\mu_n} = \prod_{j=1}^n \left( \sum \lambda \left[ \frac{\lambda_j}{\mu_j} \right] \mathbf{i}_\lambda \right)$$

$$= \sum \lambda \left[ \frac{\lambda_1}{\mu_1} \right] \cdots \left[ \frac{\lambda_n}{\mu_n} \right] \mathbf{i}_\lambda$$

where $\lambda$ runs over the set $\Lambda(n, r)$ in the sums. The only non-zero term in the last sum is when $\lambda = \mu$, so

$$\left[ K_1 \right]_{\mu_1} \cdots \left[ K_n \right]_{\mu_n} = \mathbf{i}_\mu.$$

This proves the claim. (The reader should refer to [16, §1.3] for definitions and basic properties of quantized binomial coefficients used here.)
We have shown that the elements $K_i, K^{-1}_i, E_i, F_i$ (for $1 \leq i \leq n$) generate the algebra $A$, and moreover satisfy all the defining relations for the algebra $T$. It follows that the map

$$K_i^{\pm 1} \to K_i^{\pm 1}, \quad E_i \to E_i, \quad F_i \to F_i$$

is a surjective quotient map from $T$ onto $A$.

Now consider the composite surjective map $T \to A \to T$. This is the clearly identity on $E_i, F_i$. Moreover, by equations ($\ast$) above the composite map takes $K_i$ to $\sum \lambda v^\lambda \lambda_i v_\lambda = K_i$. Similarly, it takes $K^{-1}_i$ to itself. Thus the composite is identity, and thus each quotient map $T \to A$ and $A \to T$ is an algebra isomorphism. \hfill \Box

3. The classical case

All of the results of this paper have analogues in the case $v = 1$. The proofs run parallel to the arguments given here, but are often easier. We will outline the main results here, and leave it to the reader to fill in the details.

3.1. The affine Schur algebra. The analogue of Definition 1.3.4 is the following

**Definition 3.1.1.** The affine Schur algebra $\hat{S}(n, r)$ over $\mathbb{Z}$ is defined by

$$\hat{S}(n, r) := \mathrm{End}_{\hat{W}} \left( \bigoplus_{\lambda \in \Lambda(n, r)} x_{\lambda} \hat{W} \right),$$

where $x_{\lambda} = \sum_{w \in \hat{W}_\lambda} w$.

There is a basis of $\hat{S}(n, r)$ similar to the basis of $\hat{S}_q(n, r)$ given in Theorem 1.3.6. The details are left to the reader.

**Definition 3.1.2.** The associative, unital algebra $U(\hat{\mathfrak{gl}}_n)$ over $\mathbb{Q}$ is given by generators

$$e_i, f_i, H_i \quad (1 \leq i \leq n)$$

subject to the relations

(q1) \quad $H_i H_j = H_j H_i$;

(q2) \quad $H_i e_j - e_j H_i = \epsilon^v(i, j) e_j$;

(q3) \quad $H_i f_j - f_j H_i = \epsilon^-(i, j) f_j$;

(q4) \quad $e_i f_j - f_j e_i = \delta_{ij} (H_j - H_{j+1})$;

(q5) \quad $e_i e_j = e_j e_i$ if $i$ and $j$ are not adjacent;

(q6) \quad $f_i f_j = f_j f_i$ if $i$ and $j$ are not adjacent;

(q7) \quad $e_i^2 e_j - 2e_i e_j e_i + e_j e_i^2 = 0$ if $i$ and $j$ are adjacent;

(q8) \quad $f_i^2 f_j - 2f_i f_j f_i + f_j f_i^2 = 0$ if $i$ and $j$ are adjacent.

As in Definition 1.4.3 the notion of adjacency takes place in the Dynkin diagram of type $\hat{A}_{n-1}$, so we read indices modulo $n$ in this definition.
This algebra is a Hopf algebra in a natural way, and the quotient of $U(\hat{\mathfrak{gl}}_n)$ by the kernel of its action on a suitably defined tensor space is isomorphic as a $\mathbb{Q}$-algebra to $\mathbb{Q} \otimes_{\mathbb{Z}} \hat{S}(n, r)$.

3.2. Main results. The analogue of Theorem 1.6.1 is the

**Theorem 3.2.1.** Let $n > r$, and identify $\hat{S}(n, r)$ with the quotient of $U(\hat{\mathfrak{gl}}_n)$ acting on tensor space. Over $\mathbb{Q}$, the affine Schur algebra $\hat{S}(n, r)$ is given by generators $e_i, f_i, H_i$ ($1 \leq i \leq n$) subject to relations (q1) to (q8) of Definition 3.1.2 (reading indices modulo $n$), together with the relations

\[(q9)\quad H_1 + \cdots + H_n = r; \]
\[(q10)\quad H_i(H_i - 1)(H_i - 2) \cdots (H_i - r) = 0.\]

There is also an equivalent version in terms of idempotents, analogous to Theorem 2.6.1, which we now state.

**Theorem 3.2.2.** Assume that $n > r$. Over $\mathbb{Q}$, the algebra $\hat{S}(n, r)$ is isomorphic with the associative algebra (with 1) given by the generators $\lambda_i$ ($\lambda \in \Lambda(n, r)$), $e_i, f_i$ ($1 \leq i \leq n$) and relations

\[(r1)\quad \lambda_i \lambda_\mu = \delta_{\lambda, \mu} \lambda_\lambda; \quad \sum_{\lambda \in \Lambda(n, r)} \lambda_i = 1;\]
\[(r2)\quad e_i \lambda = \begin{cases} \lambda_{i+\alpha_i} e_i & \text{if } \lambda_{i+1} > 0, \\ 0 & \text{otherwise}; \end{cases}\]
\[(r3)\quad f_i \lambda = \begin{cases} \lambda_{i-\alpha_i} f_i & \text{if } \lambda_i > 0, \\ 0 & \text{otherwise}; \end{cases}\]
\[(r4)\quad e_i f_j - f_j e_i = \delta_{ij} \sum_{\lambda \in \Lambda(n, r)} (\lambda_j - \lambda_{j+1}) \lambda_i\]

along with relations (q5)–(q8) of Definition 3.1.2. Here we regard weights as infinite periodic sequences, as in Section 2.2 above.

These relations are obtained from those in Theorem 2.6.1 by setting $v = 1$.

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