Arithmetic surfaces and adelic quotient groups

D. V. Osipov

Abstract. We explicitly calculate an arithmetic adelic quotient group for a locally free sheaf on an arithmetic surface when the fibre over the infinite point of the base is taken into account. The result is stated in the form of a short exact sequence. We relate the last term of this sequence to the projective limit of groups which are finite direct products of copies of the one-dimensional real torus and are connected with the first cohomology groups of locally free sheaves on the arithmetic surface.

Keywords: arithmetic surface, Parshin–Beilinson adeles, arithmetic adeles.

Dedicated to A. N. Parshin on the occasion of his 75th birthday

§ 1. Introduction

The ring of adeles for number fields and algebraic curves was introduced by Chevalley and Weil.

Higher adeles (or the ring of adeles in higher dimensions) were introduced by Parshin and Beilinson. Parshin used adeles for smooth algebraic surfaces over a field in [1]. Beilinson defined adeles for arbitrary Noetherian schemes in a short note [2]. Proofs of Beilinson’s results concerning adelic resolutions of quasicoherent sheaves appeared later in the paper [3] by Huber.

The ultimate goal of the higher adeles programme is the generalization of the Tate–Iwasawa method from the one-dimensional case to the case of higher dimensions; see [4], [5]. The Tate–Iwasawa method enables one to obtain a meromorphic continuation (to the whole complex plane \( \mathbb{C} \)) and functional equations for the zeta functions and \( L \)-functions of number fields and fields of rational functions on curves defined over finite fields. This method works simultaneously in the number-theoretic case and in the geometric case; see [6].

A well-known approach in arithmetic algebraic geometry is to solve the problem first for function fields by a method that can be transferred to the scheme part of an arithmetic surface related to a number field. The next and final step is to include the Archimedean fibres of the surface. This approach was used successfully in Faltings’ proof of the Mordell conjecture; see [7]. Archimedean fibres have been studied using Arakelov’s theory of arithmetic surfaces; see [8].

The author is partially supported by Laboratory of Mirror Symmetry NRU HSE, RF Government grant, ag. no. 14.641.31.0001.

AMS 2010 Mathematics Subject Classification. 11R56, 14G40, 11G99.

© 2018 Russian Academy of Sciences (DoM), London Mathematical Society, Turpion Ltd.
Parshin has recently developed a new version of the Tate–Iwasawa method (see [9]), in which he removed the well-known manipulations with formulae and replaced them by considerations of functoriality and duality. Parts of these constructions can be made in the case of algebraic surfaces over finite fields. In this paper we extend the adelic results known for algebraic surfaces to the case of arithmetic surfaces.

Let \( X \) be an arithmetic surface, that is, a two-dimensional normal integral scheme which is surjectively fibred over \( \text{Spec} \mathbb{Z} \). The Parshin–Beilinson ring of adeles \( \mathbb{A}_X \) of \( X \) does not take into account the fibre \( X \times_{\mathbb{Z}} \mathbb{R} \) over the ‘infinite point’ of \( \text{Spec} \mathbb{Z} \), that is, over the Archimedean valuation of the ring \( \mathbb{Z} \). In [10], Parshin and the author defined a ring of arithmetic adeles \( \mathbb{A}_{\text{ar}}^X \) of an arithmetic surface \( X \), which takes into account the fibre of \( X \) over the ‘infinite point’ of \( \text{Spec} \mathbb{Z} \). Informally speaking, the ring \( \mathbb{A}_X \) is a complicated restricted product of two-dimensional local fields over all pairs consisting of a closed point in \( X \) and a formal branch at \( x \) of an integral one-dimensional subscheme of \( X \), where the two-dimensional local field is a finite extension of either \( \mathbb{Q}_p((t)) \) or \( \mathbb{Q}_p\{(t)\} \) (for more details, see, for example, [11], §2.1). Now, to obtain the ring \( \mathbb{A}_{\text{ar}}^X \) from the Parshin–Beilinson ring of adeles \( \mathbb{A}_X \), we add the fields \( \mathbb{R}((t)) \) or \( \mathbb{C}((t)) \) associated with ‘horizontal curves’ on \( X \) and ‘infinite points’ of the ‘horizontal curves’. In other words, we add the restricted (adelic) product of fields which are completions of local fields at points with non-transcendental coordinates on \( X \times_{\mathbb{Z}} \mathbb{R} \). As in the case of Parshin–Beilinson adeles, it is important to define various subgroups in the additive group of \( \mathbb{A}_{\text{ar}}^X \) to preserve the analogy of the results obtained with the classical one-dimensional case of number fields as well as with the case of projective surfaces over a field.

The goal of this paper is to calculate explicitly the adelic quotient group

\[
\mathbb{A}_{\text{ar}}^X(\mathcal{F})/(\mathbb{A}_{\text{ar},01}^X(\mathcal{F}) + \mathbb{A}_{\text{ar},02}^X(\mathcal{F})),
\]

where \( \mathcal{F} \) is a locally free sheaf on the arithmetic surface \( X \) and \( \mathbb{A}_{\text{ar}}^X(\mathcal{F}) \), \( \mathbb{A}_{\text{ar},01}^X(\mathcal{F}) \), \( \mathbb{A}_{\text{ar},02}^X(\mathcal{F}) \) stand for the group of arithmetic higher adeles of \( X \) and two subgroups of it; for more details, see §2 below. The subgroups \( \mathbb{A}_{\text{ar},01}^X(\mathcal{F}) \) and \( \mathbb{A}_{\text{ar},02}^X(\mathcal{F}) \) were introduced in [12], [13]. (Similar subgroups were also considered in [14].)

The subgroup \( \mathbb{A}_{\text{ar},01}^X(\mathcal{F}) + \mathbb{A}_{\text{ar},02}^X(\mathcal{F}) \) is an analogue of the subgroup \( K \) in the additive group of the adelic ring \( \mathbb{A}_K \), where \( K \) is a global field, that is, \( K \) is either a number field or the field of rational functions of an algebraic curve over a finite field. (For example, if \( K \) is the field of rational functions of a projective algebraic curve \( Z \) over a finite field, then \( K = \mathbb{A}_0(\mathcal{O}_Z) \) and \( \mathbb{A}_K = \mathbb{A}_{01}(\mathcal{O}_Z) \) in the adelic complex for the structure sheaf \( \mathcal{O}_Z \) of the curve \( Z \); see, for example, [11], §3.) It is well known that the group \( \mathbb{A}_K/K \) is compact for every global field \( K \), and this fact has many applications. By explicit calculations we obtain a similar result in the case of arithmetic surfaces \( X \) and the group \( \mathbb{A}_{\text{ar}}^X(\mathcal{F}) \).

We recall an explicit calculation in the case when \( K \) is a number field with \([K : \mathbb{Q}] = l\), and \( E \) is the ring of integers in \( K \). The strong approximation theorem

\[\text{This was shown in his talks at the Steklov Mathematical Institute in November 2017.}\]
immediately gives the exact sequence

$$0 \to \prod_{\sigma} \hat{E}_\sigma \to \mathbb{A}_K/K \to \left(\prod_{\nu} K_\nu\right)/E \to 0,$$

(2)

where $\mathbb{A}_K$ is the ring of adeles of $K$, $\sigma$ runs over the set of maximal ideals of $E$, $\hat{E}_\sigma$ is the corresponding completion, $\nu$ runs over the (finite) set of Archimedean places of $K$, and $K_\nu$ is the completion. We note that the last non-zero term in the exact sequence (2) is isomorphic to $T_l$, where $T = \mathbb{R}/\mathbb{Z}$ is the torus.

On the other hand, given a normal irreducible projective algebraic surface $Y$ over a field $k$, we fix an ample divisor $\tilde{C}$ on $Y$. Then the complement $U = Y \setminus \text{supp} \tilde{C}$ is affine. This situation may be regarded as an analogue of the one-dimensional case, where instead of fixing finitely many points of a global field (Archimedean places in the number-theoretic case) we fix a divisor with affine complement. For a locally free sheaf $F$ on $Y$, the corresponding adelic quotient group was calculated in [15] and [10], § 14, in the following form:

$$0 \to G_1 \to \mathbb{A}_Y(F)/\left(\mathbb{A}_Y,0_1(F) + \mathbb{A}_Y,0_2(F)\right) \to G_2 \to 0$$

(3)

with a linearly compact $k$-vector space

$$G_1 \simeq \prod_{D \subset Y, D \not\subset \text{supp} \tilde{C}} \left(\prod_{x \in D} \frac{\mathcal{O}_{K_x,D}}{\mathcal{O}_D} \hat{F}_D\right) \otimes \prod_{x \in U} \hat{F}_x,$$

where $x$ runs over the set of (closed) points in the subscheme $U$, $D$ runs over the set of irreducible curves on $Y$ such that $D \not\subset \text{supp} \tilde{C}$, $\prod'$ means an adelic (on the algebraic surface) product, the ring $\mathcal{O}_{K_x,D}$ is the product of the discrete valuation rings belonging to the finite set of two-dimensional local fields constructed from the pair $x \in D$ (for example, if $K_x,D = k((u))((t))$ is a two-dimensional local field, then $\mathcal{O}_{K_x,D} = k((u))[[t]]$), $\hat{F}_x$ is the completion of the stalk of $\mathcal{F}$ at $x$, and $\hat{F}_D$ is the completion of the stalk of $\mathcal{F}$ at the generic point of $D$.

The vector space $G_2$ is also linearly compact (over the field $k$). We recall (see [10], § 14.3) that in the simplest case, when $Y = C \times_k C$, $C = \mathbb{P}^1_k$, $\tilde{C} = C \times y + y \times C$ ($y$ is a fixed $k$-rational point on the projective line $C$) and $\mathcal{F} = \mathcal{O}_Y$, we have

$$G_2 \simeq \frac{k((u))((t))}{k[u^{-1}](t) + k((u))[t^{-1}]}.$$

Now, if we replace the locally linearly compact field $k((u))$ by the locally compact field $\mathbb{R}$, and the discrete subspace $k[u^{-1}]$ by the discrete subgroup $\mathbb{Z}$, then $G_2$ becomes the group $t\mathbb{R}[[t]]/t\mathbb{Z}[[t]] \simeq t\mathbb{T}[[t]].$

In this paper we carry out an explicit calculation of the adelic quotient group (1) on an arithmetic surface $X$ for a locally free sheaf $\mathcal{F}$ on $X$. This is the content

\footnote{We shall sometimes use the ‘fractional’ notation $\frac{N_1}{N_2}$ for the quotient group $N_1/N_2$.}
of Theorem 2.1 in §2. We do not state this theorem in the introduction, but note that this result generalizes the result (2) in the number field case and is analogous to the result (3) in the case of a projective algebraic surface $Y$.

More precisely, the answer obtained in Theorem 2.1 is a description of the group (1) by means of a short exact sequence whose first term is written in the same way as the first term in the exact sequence (3) (but we must take into account additional terms like $\mathbb{R}((t))$ or $\mathbb{C}((t))$) and the group in this term is compact while $G_1$ is a linearly compact $k$-vector space. Moreover, the last term in this exact sequence is equal to $t\mathbb{R}[[t]]/t\mathbb{Z}[[t]] \simeq t\mathbb{T}[[t]]$ in the simplest case, when $X = \mathbb{P}^1_\mathbb{Z}$ and $F = \mathcal{O}_X$ (this follows from the explicit calculations for $\mathbb{P}^1_\mathbb{Z}$ in Example 2.3).

We make another interesting remark. In Proposition 4.1 we prove that the last term in the short exact sequence for the group (1) is isomorphic to the projective limit of the groups $H^1(X, \mathcal{F}(nC)) \otimes_\mathbb{Z} \mathbb{T}$ over the set of all $n \leq 0$, where $C$ is an ample Cartier divisor (fixed before the calculation) on $X$. Each of these groups is a finite direct product of several copies of $\mathbb{T}$, and the projective limit is isomorphic to a countable product of groups isomorphic to $\mathbb{T}$. (This becomes clear if we pass to Pontryagin-dual groups.) On the other hand, returning to the exact sequence (2), we note that the one-dimensional Arakelov geometry contains the Riemann–Roch theorem in the form of the Poisson summation formula. It uses theta functions constructed from the right-hand side of (2), that is, from the cocompact lattice $E$ in the finite-dimensional $\mathbb{R}$-vector space $\prod_{v} \mathbb{K}_v$; for more details, see Remark 4.3.

The paper is organized as follows. In §2 we recall the basic notions of the Parshin–Beilinson adelic theory and give the necessary definitions for the groups of arithmetic adeles on an arithmetic surface. We also state the main result of the paper, Theorem 2.1. In §3 we prove this theorem. Firstly, in §3.1 we calculate the first term of the corresponding short exact sequence. Secondly, in §3.2 we calculate the last term of this sequence. Finally, in §4 we prove that the last term is isomorphic to the projective limit of groups that are finite direct products of copies of $\mathbb{T}$ and are related to the first cohomology groups of the sheaves $\mathcal{F}(nC)$.

§2. Statement of the result

Let $X$ be a two-dimensional integral normal scheme. We assume that the natural morphism $f: X \rightarrow \text{Spec} \mathbb{Z}$ is projective and surjective. Then it follows that $f$ is a flat morphism. We will call such a scheme $X$ an arithmetic surface.

Let $C = \bigcup_{1 \leq i \leq w} C_i$ be a reduced one-dimensional subscheme of $X$, where each of the one-dimensional subschemes $C_i$, $1 \leq i \leq w$, is integral and $f(C_i) = \text{Spec} \mathbb{Z}$.

Suppose that the open subscheme $U = X \setminus C$ is an affine scheme. We note that this condition on $U$ holds, for example, when the coherent sheaf $\mathcal{O}_X(nC)$ is an ample invertible sheaf on $X$ for some integer $n \geq 1$. By Corollary 3.24 in Ch. 5 of [16], an invertible sheaf $\mathcal{L}$ on $X$ is ample if and only if so is the invertible sheaf $(i_p)^* \mathcal{L}$ on the subscheme $f^{-1}(p)$ for every closed point $p$ of $\text{Spec} \mathbb{Z}$.

Let $\mathcal{F}$ be a locally free sheaf of $\mathcal{O}_X$-modules on $X$. For every point $q$ in $X$ let $\widehat{\mathcal{F}}_q$ be the completion of the stalk $\mathcal{F}_q$ of $\mathcal{F}$ at $q$. 
We write $\mathbb{A}_X(F)$ for the Parshin–Beilinson adelic group of $F$ on $X$:

$$\mathbb{A}_X(F) = \mathbb{A}_{X,01}(F) = \prod_{x \in D} K_{x,D}(F) \subset \prod_{x \in D} K_{x,D}(F),$$

where $x \in D$ is a pair consisting of a closed point $x \in X$ and an integral closed one-dimensional subscheme $D \subset X$, the ring $K_{x,D}$ is equal to $\prod_{1 \leq i \leq l} K_i$, where the two-dimensional local field $K_i$ is the completion of the field $\text{Frac} \hat{O}_x$ with respect to the prime ideal of height 1 in the ring $\hat{O}_x$ associated with the ideal of the subscheme $D$ restricted to $\text{Spec} \hat{O}_x$, and the group $K_{x,D}(F)$ is equal to $\hat{F}_x \otimes_{\hat{O}_x} K_{x,D}$ (for more details, see [3], [11], [15], §3.1).

We put $\mathcal{O}_{K_{x,D}}(F) = \hat{F}_x \otimes_{\hat{O}_x} \mathcal{O}_{K_{x,D}}$, where the ring $\mathcal{O}_{K_{x,D}}$ is equal to $\prod_{1 \leq i \leq l} \mathcal{O}_{K_i}$ and each $\mathcal{O}_{K_i}$ is the discrete valuation ring in the two-dimensional local field $K_i$ associated with the pair $x \in D$.

We let $\prod'$ denote the adelic product on $X$, that is, the intersection with the subgroup $\mathbb{A}_X(F)$ inside the product $\Pi$.

We recall that there are natural subgroups $\mathbb{A}_{X,01}(F)$ and $\mathbb{A}_{X,02}(F)$ of the adelic group $\mathbb{A}_X(F)$:

$$\mathbb{A}_{X,01}(F) = \prod_D' K_D(F) = \mathbb{A}_X(F) \cap \prod_D K_D(F),$$

$$\mathbb{A}_{X,02}(F) = \prod_x' K_x(F) = \mathbb{A}_X(F) \cap \prod_x K_x(F),$$

where the intersections are taken inside the group $\prod_{x \in D} K_{x,D}(F)$. Here we use the group

$$K_D(F) = \hat{F}_D \otimes_{\hat{O}_D} K_D,$$

the field $K_D$ is the completion of the field $\mathbb{Q}(X)$ of rational functions on $X$ with respect to the discrete valuation given by $D$, the group $K_x(F)$ is equal to $\hat{F}_x \otimes_{\hat{O}_x} K_x$, and the ring $K_x$ is the localization of the ring $\hat{O}_x$ with respect to the multiplicative system $\mathcal{O}_x \setminus 0$. The products $\prod_D$ and $\prod_x$ are diagonally embedded in the product $\prod_{x \in D}$ by means of the natural maps

$$p_{x,D,F} : K_x(F) \hookrightarrow K_{x,D}(F) \quad \text{and} \quad q_{x,D,F} : K_D(F) \hookrightarrow K_{x,D}(F).$$

Let $V$ be a locally linearly compact vector space over a field $k$. Let $k'/k$ be a field extension. We put

$$V \otimes_k k' \overset{\text{def}}{=} \lim_{W} \left( (V/W) \otimes_k k' \right) = \lim_{W'} \lim_{W' \subseteq W'} \left( (W'/W) \otimes_k k' \right),$$

where $W$ and $W'$ run over the set of open linearly compact $k$-vector subspaces of $V$. For example, $\mathbb{Q}(t) \otimes_{\mathbb{Q}} \mathbb{R} = \mathbb{R}(t)$.

\footnote{Following the notation used for higher adeles on algebraic surfaces, we write ‘∈’ when $x$ is a closed point in the corresponding scheme, and we write ‘⊂’ when the one-dimensional integral scheme $D$ is a closed subscheme of $X$.}

\footnote{By $\hat{F}_D$ we mean the completion of the stalk of $F$ at the generic point of $D$, that is, at the non-closed point on $X$ whose closure coincides with $D$.}
Define a curve \( X_\mathbb{Q} = X \times_{\text{Spec} \mathbb{Z}} \text{Spec} \mathbb{Q} \). Let \( \mathbb{A}_X(\mathcal{F}_\mathbb{Q}) \) be the adelic group of the locally free sheaf \( \mathcal{F}_\mathbb{Q} = (i_\mathbb{Q})^* \mathcal{F} \) on the generic fibre \( X_\mathbb{Q} \xrightarrow{i_\mathbb{Q}} X \). Then \( \mathbb{A}_X(\mathcal{F}_\mathbb{Q}) \) is a locally linearly compact \( \mathbb{Q} \)-vector space whose base of open subspaces is given by the divisors on the curve \( X_\mathbb{Q} \).

The arithmetic adelic group of an arithmetic surface \( X \) was introduced in [10], Example 11 (see also the applications in [17], § 4). By definition,

\[
\mathbb{A}_X^\text{ar}(\mathcal{F}) \overset{\text{def}}{=} \mathbb{A}_X(\mathcal{F}) \oplus (\mathbb{A}_{X_\mathbb{Q}}(\mathcal{F}_\mathbb{Q}) \hat{\otimes} \mathbb{Q} \mathbb{R}).
\]

We note that

\[
\mathbb{A}_{X_\mathbb{Q}}(\mathcal{F}_\mathbb{Q}) \hat{\otimes} \mathbb{Q} \mathbb{R} \cong \prod_{D, f(D) = \text{Spec} \mathbb{Z}}' (K_D(\mathcal{F}) \hat{\otimes} \mathbb{Q} \mathbb{R}),
\]

where \( \prod' \) means the restricted product (as in adeles on a curve) with respect to the subgroups \( \hat{\mathcal{F}}_D \hat{\otimes} \mathbb{Q} \mathbb{R} \).

The subgroups \( \mathbb{A}_{X_\mathbb{Q},01}^\text{ar}(\mathcal{F}) \) and \( \mathbb{A}_{X_\mathbb{Q},02}^\text{ar}(\mathcal{F}) \) of the group \( \mathbb{A}_X^\text{ar}(\mathcal{F}) \) were introduced in [12], [13]. The subgroup \( \mathbb{A}_{X_\mathbb{Q},01}^\text{ar}(\mathcal{F}) \) coincides with \( \mathbb{A}_{X_\mathbb{Q}}(\mathcal{F}) \) but is mapped into both summands of the group \( \mathbb{A}_X^\text{ar}(\mathcal{F}) \) in (6), where the first map is the natural embedding in the definition (4) and the second is the composite of the natural maps

\[
\mathbb{A}_{X_\mathbb{Q},01}(\mathcal{F}) \rightarrow \prod_{D, f(D) = \text{Spec} \mathbb{Z}}' K_D(\mathcal{F}) \hookrightarrow \prod_{D, f(D) = \text{Spec} \mathbb{Z}}' (K_D(\mathcal{F}) \hat{\otimes} \mathbb{Q} \mathbb{R}) \cong \mathbb{A}_{X_\mathbb{Q}}(\mathcal{F}_\mathbb{Q}) \hat{\otimes} \mathbb{Q} \mathbb{R}.
\]

The subgroup \( \mathbb{A}_{X_\mathbb{Q},02}^\text{ar}(\mathcal{F}) \) is defined as

\[
\mathbb{A}_{X_\mathbb{Q},02}^\text{ar}(\mathcal{F}) \overset{\text{def}}{=} \mathbb{A}_{X,02}(\mathcal{F}) \oplus (\mathcal{F}_\mathbb{Q} \otimes \mathbb{Q} \mathbb{R}),
\]

where \( \eta \) is the generic point of \( X \), the first summand in (8) is naturally embedded in the first summand of (6) by the formula (5), and the second summand in (8) is naturally diagonally embedded in the second summand of (6).

We recall the definition of the subgroup \( A_C(\mathcal{F}) \subset \prod_{1 \leq i \leq w} K_C_i(\mathcal{F}) \) in [15], § 3.2.2. To do this, we first define a subgroup \( B_{x,C}(\mathcal{F}) \subset K_x(\mathcal{F}) \) putting

\[
B_{x,C}(\mathcal{F}) \overset{\text{def}}{=} \bigcap_{D \ni x, D \not\subset C} p_{x,D,\mathcal{F}}^{-1}(p_{x,D,\mathcal{F}}(K_x(\mathcal{F})) \cap \mathcal{O}_{K_x(D)}(\mathcal{F})).
\]

By the remark after Lemma 1 in § 3.2.1 of [15], we also have

\[
B_{x,C}(\mathcal{F}) \cong \hat{\mathcal{F}}_x \otimes_{\mathcal{O}_x} (j_x j^* \mathcal{O}_X)_x.
\]

Now the subgroup \( A_C(\mathcal{F}) \) is defined as the image of the projection of the group \( \text{Ker} \Xi \) to the group \( \prod_{1 \leq i \leq w} K_{C_i}(\mathcal{F}) \), where the map

\[
\Xi: \prod_{1 \leq i \leq w} K_{C_i}(\mathcal{F}) \oplus \prod_{x \in C} B_{x,C}(\mathcal{F}) \rightarrow \prod_{1 \leq i \leq w} \prod_{x \in C_i} K_{x,C}(\mathcal{F})
\]

is given by

\[
\Xi(z \oplus v) = \prod_{1 \leq i \leq w} \prod_{x \in C_i} q_{x,C_i,\mathcal{F}}(z) - \prod_{1 \leq i \leq w} \prod_{x \in C_i} p_{x,C_i,\mathcal{F}}(v),
\]
where \( z \in \prod_{1 \leq i \leq w} K_{C_i}(\mathcal{F}) \) and \( v \in \prod_{x \in C} B_{x,C}(\mathcal{F}) \). By Proposition 1 in [15], we also have

\[
A_C(\mathcal{F}) \simeq \lim_{n \to \infty} \lim_{m < n} H^0(X, \mathcal{F} \otimes \mathcal{O}_X (\mathcal{O}_X(nC)/\mathcal{O}_X(mC))).
\]

We define an affine curve \( U_\mathbb{Q} = U \times_{\text{Spec} \mathbb{Z}} \text{Spec} \mathbb{Q} \) and a \( \mathbb{Q} \)-vector space

\[
A_{U_\mathbb{Q}}(\mathcal{F}) \overset{\text{def}}{=} H^0(U_\mathbb{Q}, \mathcal{F}_\mathbb{Q} |_{U_\mathbb{Q}}).
\]

The main result of this paper is the following theorem.

**Theorem 2.1.** Consider the group \( \Phi = \mathbb{A}_X^\text{ar} (\mathcal{F})/(\mathbb{A}_X^\text{ar,01}(\mathcal{F}) + \mathbb{A}_X^\text{ar,02}(\mathcal{F})) \). Then there is an exact sequence

\[
0 \to \left( \prod_{x \in D, D \subseteq C} \mathcal{O}_{K_{x,D}}(\mathcal{F}) \right) \oplus \left( \prod_{D \subseteq C, f(D) = \text{Spec} \mathbb{Z}} (\hat{\mathcal{F}}_D \otimes \mathbb{Q} \mathbb{R}) \right) \to \Phi
\]

\[
\prod_{D \subseteq C, D \subset X} \hat{\mathcal{F}}_D + \prod_{x \in U} \hat{\mathcal{F}}_x
\]

\[
\to \frac{\prod_{1 \leq i \leq w} (K_{C_i}(\mathcal{F}) \otimes \mathbb{Q} \mathbb{R})}{A_C(\mathcal{F}) + (A_{U_\mathbb{Q}}(\mathcal{F}) \otimes \mathbb{Q} \mathbb{R})} \to 0,
\]

(11)

where the group \( \prod_{D \subseteq C, D \subseteq C} \hat{\mathcal{F}}_D \) is mapped canonically to both summands of the ‘numerator’ (compare with (7)), and the group \( \prod_{x \in U} \hat{\mathcal{F}}_x \) is mapped only to the first summand of the ‘numerator’.

**Remark 2.2.** It was shown in [10], §14.3, Remark 27, that the first non-zero term of the exact sequence (11) is a compact group. The last non-zero term of this sequence will be analyzed in §4 below.

**Example 2.3.** In the simplest case, when \( X = \mathbb{P}_\mathbb{Z}^1 \), \( C \) is a hyperplane section, \( U = \mathbb{A}_\mathbb{Z}^1 \) and \( \mathcal{F} = \mathcal{O}_X \), we have the following subgroups which occur in the last non-zero term of the exact sequence (11):

\[
\prod_{1 \leq i \leq w} (K_{C_i}(\mathcal{F}) \otimes \mathbb{Q} \mathbb{R}) \simeq \mathbb{R}((t)), \quad A_C(\mathcal{F}) \simeq \mathbb{Z}((t)), \quad A_{U_\mathbb{Q}}(\mathcal{F}) \otimes \mathbb{Q} \mathbb{R} \simeq \mathbb{R}[t^{-1}].
\]

§ 3. Proof of the theorem

In this section we prove Theorem 2.1. In §3.1 we calculate the first non-zero term of the exact sequence (11). In §3.2 we calculate the last non-zero term of the exact sequence (11).

We also note that we will use the adelic complex \( \mathcal{A}_Y(\mathcal{G}) \) on a two-dimensional integral normal scheme \( Y \) of finite type over \( \text{Spec} \mathbb{Z} \) for a locally free sheaf \( \mathcal{G} \) of \( \mathcal{O}_Y \)-modules:

\[
\mathbb{A}_{Y,0}(\mathcal{G}) \oplus \mathbb{A}_{Y,1}(\mathcal{G}) \oplus \mathbb{A}_{Y,2}(\mathcal{G}) \to \mathbb{A}_{Y,01}(\mathcal{G}) \oplus \mathbb{A}_{Y,02}(\mathcal{G}) \oplus \mathbb{A}_{Y,12}(\mathcal{G}) \to \mathbb{A}_{Y,012}(\mathcal{G})
\]

\[(a_0, a_1, a_2) \mapsto (a_1 - a_0, a_2 - a_0, a_2 - a_1)
\]

\[(a_{01}, a_{02}, a_{12}) \mapsto a_{01} - a_{02} + a_{12}.
\]
Here $\mathbb{A}_{Y,1}(\mathcal{G})$ and $\mathbb{A}_{Y,2}(\mathcal{G})$ are the subgroups of the group $\mathbb{A}_{Y,02}(\mathcal{G}) = \mathbb{A}_Y(\mathcal{G})$ that were defined above; see (4), (5). By definition, the other subgroups of $\mathbb{A}_Y(\mathcal{G})$ that occur in the adelic complex are of the following form:

$$\mathbb{A}_{Y,1}(\mathcal{G}) = \prod_{D \subset Y} \hat{\mathcal{G}}_D, \quad \mathbb{A}_{Y,2}(\mathcal{G}) = \prod_{x \in Y} \hat{\mathcal{G}}_x,$$

(12)

where $\eta$ is the generic point of $Y$, and the pairs $x \in D$ on $Y$ are as in §2.

The main property of the adelic complex is that $H^i(\mathbb{A}_Y(\mathcal{G})) = H^i(Y, \mathcal{G})$ for integers $i$, $0 \leq i \leq 2$. In particular, if $H^i(Y, \mathcal{G}) = 0$ for $i = 1$ or $i = 2$, then the condition for the $i$th cocycle to be a coboundary in the adelic complex gives rise to analogues of the approximation theorems for two-dimensional schemes (compare with [15], §2).

3.1. The first non-zero term of the exact sequence (11). Consider the natural embedding

$$\left( \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_{x,D}}(\mathcal{F}) \right) \oplus \left( \prod_{D \not\subset C, f(D) = \text{Spec} \mathbb{Z}} (\hat{\mathcal{F}}_D \otimes_{\mathbb{Q}} \mathbb{R}) \right) \to \mathbb{A}^{ar}_X(\mathcal{F}),$$

where the first and second summands are mapped to the corresponding summands of (6). We put

$$E = \left( \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_{x,D}}(\mathcal{F}) \right) \oplus \left( \prod_{D \not\subset C, f(D) = \text{Spec} \mathbb{Z}} (\hat{\mathcal{F}}_D \otimes_{\mathbb{Q}} \mathbb{R}) \right) \cap (\mathbb{A}^{ar}_{X,01}(\mathcal{F}) + \mathbb{A}^{ar}_{X,02}(\mathcal{F})).$$

(13)

Recall the calculations for the quotient group of the group of Beilinson–Parshin adeles on two-dimensional schemes. Namely, it follows from Theorem 1 in [15] that

$$G = \left( \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_{x,D}}(\mathcal{F}) \right) \cap (\mathbb{A}_{X,01}(\mathcal{F}) + \mathbb{A}_{X,02}(\mathcal{F}))$$

$$= \prod_{D \subset X, D \not\subset C} \hat{\mathcal{F}}_D + \prod_{x \in U} \hat{\mathcal{F}}_x + (\tau - \gamma)(A_C(\mathcal{F})),$$

(14)

where the intersection is taken inside the group $\mathbb{A}_X(\mathcal{F})$, the maps $\gamma$ and $\tau$ are the natural maps

$$\gamma: A_C(\mathcal{F}) \hookrightarrow \prod_{1 \leq i \leq w} K_{C_i}(\mathcal{F}) \hookrightarrow \mathbb{A}_X(\mathcal{F}),$$

$$\tau: A_C(\mathcal{F}) \hookrightarrow \prod_{x \in C} B_{x,C}(\mathcal{F}) \hookrightarrow \prod_{x \in X} K_x(\mathcal{F}) \hookrightarrow \mathbb{A}_X(\mathcal{F}),$$

(15)
and the first arrow in (15) is the composite of the maps

\[ A_C(F) \simeq \text{Ker}\, \Xi \to \prod_{x \in C} B_{x,C}(F) \]

and can also be described as the map \( z \mapsto v \) (see the definition of \( \Xi \) in (10)).

The natural projection map \( \mathbb{A}^n_X(F) \to \mathbb{A}_X(F) \) induces a well-defined map

\[ \theta: E \to G. \]

We first prove that \( \theta \) is injective. We shall need the following simple lemma.

**Lemma 3.1.** Let \( G \) be a locally free sheaf on an irreducible algebraic curve \( S \) over a number field \( F \). Then we have

\[ (\hat{G}_s \otimes_{O_s} F(S)) \cap (G_{\eta} \otimes_Q \mathbb{R}) = G_{\eta}, \]

where \( s \) is a (closed) point on \( S \), \( O_s \) is the local ring of the point \( s \), \( \eta \) is the generic point, and the intersection is taken inside the group \( (\hat{G}_s \otimes_{O_s} F(S)) \otimes_Q \mathbb{R} \).

**Proof.** Clearly, we can replace the sheaf \( G \) by the structure sheaf \( O_S \). When \( S = \mathbb{P}^1_F \) and \( s = [0 : 1] \), the lemma becomes obvious since

\[ F((t)) \cap (F[t] \otimes_Q \mathbb{R}) = F[t] \]

and we can replace \( F[t] \) by \( F(t) \) in the last formula. For an arbitrary curve \( S \) and any \( s \in S \) we consider a morphism \( r: S \to \mathbb{P}^1_F \) such that \( r(s) = [0 : 1] \) and \( r(s') \neq [0 : 1] \) for all other points \( s' \) on \( S \). (For example, \( r = g^{-1} \), where \( g \in H^0(S \setminus s, O_{S \setminus s}) \) and \( g \) is non-constant: we use the fact that \( S \setminus s \) is an affine curve.) We have an embedding of fields \( r^*: F(t) \hookrightarrow F(S) \). The lemma now follows by taking the tensor product (over the field \( F(t) \)) of the field \( F(S) \) and the sequence

\[ 0 \to F(t) \to F((t)) \oplus (F(t) \otimes_Q \mathbb{R}) \to F((t)) \otimes_Q \mathbb{R}. \]

We now suppose that there is an element \((0 \oplus y) \in \text{Ker}\, \theta\), where

\[ y \in \prod_{D \subset C, f(D) = \text{Spec} \, \mathbb{Z}} (\hat{F}_D \otimes_Q \mathbb{R}). \quad (16) \]

We will prove that \( y = 0 \). This yields that the map \( \theta \) is injective.

We have a decomposition

\[ (0 \oplus y) = y_{01} + y_{02} + y_{\infty}, \quad (17) \]

where

\[ y_{01} = \prod_D y_{01,D} \in \prod_{D \subset X} K_D(\mathcal{F}), \quad y_{02} = \prod_x y_{02,x} \in \prod_{x \in X} K_x(\mathcal{F}), \quad y_{\infty} \in \mathcal{F}_{\eta} \otimes_Q \mathbb{R}. \]

Consider two cases.
Case 1. If \( y_\infty = 0 \), then \( y_{01,i} = 0 \) for \( 1 \leq i \leq w \) since the image of an element \( y_{01,C_i} = 0 \) in \( K_{C_i}(\mathcal{F}) \otimes_{\mathbf{Q}} \mathbf{R} \) will be equal to zero by the condition (16) on \( y \). Hence we obtain that \( y_{02,x} = 0 \) for every closed point \( x \in C_i \) because the projection of the element \((0 \oplus y)\) to the group \( K_{x,C_i}(\mathcal{F}) \) is equal to zero. By considering the fibre of \( f \) that contains a previous point \( x \), we obtain that \( y_{01,D} = 0 \) when \( f(D) \neq \text{Spec} \mathbf{Z} \) (because the projection of \((0 \oplus y)\) to \( K_{x,D}(\mathcal{F}) \) is equal to zero). Now, for every closed point \( x' \) on a previous subscheme \( D \), we obtain that \( y_{02,x'} = 0 \). By considering an integral one-dimensional closed subscheme \( D' \) with \( f(D') = \mathbf{Z} \) and \( x' \in D' \), we obtain that \( y_{01,D'} = 0 \). Hence, by (17), \( y = 0 \). Case 1 is covered.

Case 2. If \( y_\infty \neq 0 \), then the image of \( y_\infty \) in \( K_{C_i}(\mathcal{F}) \otimes_{\mathbf{Q}} \mathbf{R} \) (for \( 1 \leq i \leq w \)) is equal to the image of \( -y_{01,C_i} \) because of the condition (16) on \( y \). Therefore, by Lemma 3.1 applied to the curve \( X_\eta \), we obtain that \( y_\infty = -y_{01,C_i} \in \mathcal{F}_\eta \). Arguing as in Case 1, we deduce from this that \( y_{02,x} = -y_{01,C_i} \) for every closed point \( x \in C_i \). Therefore \( y_{01,D} = -y_{02,x} \) for every one-dimensional integral closed subscheme \( D \) such that \( f(D) \neq \text{Spec} \mathbf{Z} \) and \( D \) contains a previous point \( x \). This yields that \( y_{02,x'} = -y_{01,D} \) for every closed point \( x' \) on a previous subscheme \( D \). Finally, for every integral one-dimensional closed subscheme \( D' \) with \( f(D') = \mathbf{Z} \) and \( x' \in D' \), we have \( y_{01,D'} = -y_{02,x'} \). Thus \( y_{01,D'} = -y_\infty \in \mathcal{F}_\eta \). Therefore, by (17), we obtain that \( y = 0 \). Case 2 is covered.

Thus we have proved that the map \( \theta \) is injective. We now want to prove that

\[
E = \prod_{D \subset X, D \not\subset C} \tilde{\mathcal{F}}_D + \prod_{x \in U} \tilde{\mathcal{F}}_x. \tag{18}
\]

This will give us the first non-zero term of the exact sequence (11) in Theorem 2.1.

We have a commutative diagram

\[
\begin{array}{ccc}
\prod_{D \subset X, D \not\subset C} \tilde{\mathcal{F}}_D & + & \prod_{x \in U} \tilde{\mathcal{F}}_x \\
\uparrow \alpha & & \downarrow \beta \\
E & & A^\text{ar}_X(\mathcal{F}),
\end{array}
\]

Recall that the summand \( \prod_{D \subset X, D \not\subset C} \tilde{\mathcal{F}}_D \) is mapped to both summands of the group \( A^\text{ar}_X(\mathcal{F}) \); see (6).

It follows from this diagram and (14) that to establish (18), it suffices to prove that the group \( \text{Im} \ \alpha \) contains the subgroup

\[
H = (\tau - \gamma)(A_C(\mathcal{F})) \cap \text{Im} \ \theta. \tag{20}
\]

We now proceed to prove this. Suppose that

\[
H \ni a = \theta(b). \tag{21}
\]

We have \( b \in A^\text{ar}_{X,01}(\mathcal{F}) + A^\text{ar}_{X,02}(\mathcal{F}) \). Therefore,

\[
b = b_{01} + b_{02} + b_\infty,
\]
where $b_{01} = \prod_D b_{01,D} \in \prod'_{D \subset X} K_D(\mathcal{F})$, $b_{02} = \prod_x b_{02,x} \in \prod'_{x \in X} K_x(\mathcal{F})$ and $b_\infty \in \mathcal{F}_\eta \otimes \mathbb{Q} \mathbb{R}$. We again consider two cases.

Case 1. Suppose that $b_{01,C_i} = 0$ for some integer $i$, $1 \leq i \leq w$. Since the image of $b$ in $K_{C_i}(\mathcal{F}) \otimes \mathbb{Q} \mathbb{R}$ is equal to zero by (13), we have $b_\infty = 0$. This implies (again by (13)) that $b_{01,C_j} = 0$ for $1 \leq j \leq w$. By (13), the image of $b$ in $K_{x,C_j}(\mathcal{F})$ is equal to zero for every closed point $x \in C_j$. Hence it follows from $b_{01,C_j} = 0$ that $b_{02,x} = 0$. Thus we obtain that

$$b = \left( \prod_{D \not\subset C} b_{01,D} \right) + \left( \prod_{x \in U} b_{02,x} \right).$$

Let $D \subset X$ be a one-dimensional integral closed subscheme not contained in a fibre of $f$. Then the image of $b_{01,D}$ in the group $K_D(\mathcal{F}) \otimes \mathbb{Q} \mathbb{R}$ belongs to the subgroup $\hat{\mathcal{F}}_D \otimes \mathbb{Q} \mathbb{R}$ by (13) and the equality $b_\infty = 0$. Hence $b_{01,D} \in \hat{\mathcal{F}}_D$. Now let $D \subset X$ be a one-dimensional integral closed subscheme contained in a fibre of $f$. Then there is a closed point $x \in D \cap C$. By (13), the image of $b$ in the group $K_{x,D}(\mathcal{F})$ belongs to the subgroup $\mathcal{O}_{K_{x,D}}(\mathcal{F})$. Since $b_{02,x} = 0$, we obtain that $b_{01,D} \in \hat{\mathcal{F}}_D$. Hence, for every closed point $x \in U$, we have $b_{02,x} \in \hat{\mathcal{F}}_x$ because for every one-dimensional integral closed subscheme $D \ni x$ it follows from the fact that $\theta(b) \in H$ that $b_{02,x} + b_{01,D} = 0$ inside the group $K_{x,D}(\mathcal{F})$ (that is, the image of $b_{02,x}$ belongs to the subgroup $\mathcal{O}_{K_{x,D}}(\mathcal{F})$). Thus we have proved that

$$b \in \prod_{D \subset X, D \not\subset C} \hat{\mathcal{F}}_D + \prod_{x \in U} \hat{\mathcal{F}}_x.$$

It follows that $a = \theta(b) = \alpha(b)$ (see (21)). Case 1 is covered.

Case 2. We now suppose that $b_{01,C_i} \neq 0$ for every integer $i$, $1 \leq i \leq w$. Since the image of $b$ in $K_{C_i}(\mathcal{F}) \otimes \mathbb{Q} \mathbb{R}$ is equal to zero for $1 \leq i \leq w$ (see (13)), we have $b_{01,C_i} + b_\infty = 0$ inside the group $K_{C_i}(\mathcal{F}) \otimes \mathbb{Q} \mathbb{R}$. Therefore, by Lemma 3.1 applied to the curve $X_\mathbb{Q}$, we obtain that

$$b_{01,C_i} = -b_\infty \in \mathcal{F}_\eta.$$

We have $b_{02,x} = -b_{01,C_i}$ for every closed point $x \in C_i$ because, by (13), the image of $b$ in $K_{x,C_i}(\mathcal{F})$ is equal to zero. Thus we have

$$\theta\left( \prod_D b_{01,D} + \prod_x b_{02,x} + b_\infty \right) \in (\tau - \gamma)(\mathbb{A}_C(\mathcal{F})), \quad (22)$$

where $b_{01,C_i} = -b_{02,x} = -b_\infty \in \mathcal{F}_\eta$ for every integer $i$, $1 \leq i \leq w$, and every pair $x \in C_i$. The formula (22) yields the following equality in the group $\mathbb{A}_U(\mathcal{F})$, which is the Parshin–Beilinson adelic group of $\mathcal{F}$ on $U$:

$$\prod_{D \not\subset C} b_{01,D} + \prod_{x \in U} b_{02,x} = 0. \quad (23)$$

Since the scheme $U$ is affine, we have $H^1(U, \mathcal{O}_U) = 0$. Combining this with (23), we obtain that the 1-cocycle $(\prod_{D \not\subset C} b_{01,D} - \prod_{x \in U} b_{02,x}, 0)$ in the adelic complex
on $U$ is a 1-coboundary. This means that there are elements $g \in \mathcal{F}_\eta$, $e_{2,x} \in \widehat{\mathcal{F}}_x$ (for every closed point $x \in U$) and elements $e_{1,D} \in \widehat{\mathcal{F}}_D$ (for every integral 1-dimensional closed subscheme $D$ on $U$) such that

$$b_{01,D} = e_{1,D} - g, \quad b_{02,x} = e_{2,x} + g, \quad e_{1,D} + e_{2,x} = 0,$$

where the first equality holds in $K_D(\mathcal{F})$, the second in $K_x(\mathcal{F})$, and the third in $\mathcal{O}_{x,D}(\mathcal{F})$. Thus we obtain that

$$b = \left( \prod_{D \subset U} b_{01,D} \right) + \left( \prod_{x \in U} b_{02,x} \right) + b_{\infty} = \left( \prod_{D \subset U} e_{01,D} \right) + \left( \prod_{x \in U} e_{02,x} \right) + e_{\infty},$$

where $e_{01,D} = e_{1,D} = b_{01,D} + g \in \widehat{\mathcal{F}}_D$ for $D \not\subset C$, $e_{02,x} = e_{2,x} = b_{02,x} - g \in \widehat{\mathcal{F}}_x$ for $x \in U$, and

$$e_{\infty} = -e_{01,C_i} = e_{02,x} = b_{\infty} - g = -b_{01,C_i} - g = b_{02,x} - g \in \mathcal{F}_\eta$$

for $x \in C_i$ and $1 \leq i \leq w$.

**Remark 3.2.** We did not use the last equality in (24). This equality means that

$$\prod_{D \subset U} e_{01,D} = - \prod_{x \in U} e_{02,x} \in A_{1,U}(\mathcal{F}) \cap A_{2,U}(\mathcal{F}),$$

where $A_{1,U}(\mathcal{F})$ and $A_{2,U}(\mathcal{F})$ are the subgroups defined in (12) and the intersection is taken in the group $\mathcal{A}_U(\mathcal{F})$. This intersection contains the subgroup $H^0(U, \mathcal{F}|_U)$ but (as shown in [18], §6) is not equal to $H^0(U, \mathcal{F}|_U)$. (More precisely, the example in [18], §6 corresponds to the case when $U$ is an affine regular surface over a countable field, but the same argument works in our case.)

We now obtain from (13) and (25) that $e_{\infty} \in A_{U_\eta}(\mathcal{F}) = H^0(U, \mathcal{F}_\eta | U_\eta)$. Since $e_{01,D} \in \widehat{\mathcal{F}}_D$ for every $D \not\subset C$, (13) yields that the image of $e_{02,x}$ in the group $K_{x,D}(\mathcal{F})$ belongs to the subgroup $\mathcal{O}_{K_{x,D}}(\mathcal{F})$ for every pair $x \in C_i$, where $1 \leq i \leq w$. Using the fact that $e_{\infty} = e_{02,x}$, we obtain

$$e_{\infty} = -e_{01,C_i} = e_{02,x} \in H^0(U, \mathcal{F}|_U).$$

Thus we have

$$b = \left( \prod_{D \not\subset C} e_{01,D} \right) + \left( \prod_{x \in U} e_{02,x} \right) + e_{\infty} = \left( \prod_{D \not\subset C} f_{01,D} \right) + \left( \prod_{x \in U} f_{02,x} \right),$$

where $f_{02,x} = e_{02,x} - e_{\infty} \in \widehat{\mathcal{F}}_x$ for $x \in U$ and $f_{01,D} = e_{01,D} + e_{\infty} \in \widehat{\mathcal{F}}_D$ for $D \not\subset C$. This yields that $a = \theta(b) = \alpha(b)$ (see (21)). Case 2 is covered.

Thus we have proved (18).

This determines the first non-zero term of the exact sequence (11) in Theorem 2.1.
3.2. The last non-zero term of the exact sequence (11). To obtain this term, we make the following calculations:

\[
\Phi \left( \prod'_{x \in D, D \notin \mathbb{C}} \mathcal{O}_{K_{x,D}}(\mathcal{F}) \right) \oplus \left( \prod_{D \notin \mathbb{C}, f(D) = \text{Spec} \mathbb{Z}} (\hat{\mathcal{F}}_D \otimes_{\mathbb{Q}} \mathbb{R}) \right)
\]

\[
\cong \frac{\mathbb{A}_{X,01}(\mathcal{F}) + \mathbb{A}_{X,02}(\mathcal{F})}{\mathbb{A}_{X}(\mathcal{F})}
\]

\[
\cong \frac{\Phi_1 + \Phi_2}{\beta(\Phi_3)},
\]

where

\[
\Phi_1 = \frac{\mathbb{A}_{X,02}(\mathcal{F}) + \prod'_{D, f(D) \neq \text{Spec} \mathbb{Z}} K_D(\mathcal{F}) + \prod'_{x \in D, D \notin \mathbb{C}} \mathcal{O}_{K_{x,D}}(\mathcal{F})}{\mathbb{A}_{X}(\mathcal{F})},
\]

\[
\Phi_2 = \frac{\mathbb{A}_{X,02}(\mathcal{F}) \otimes_{\mathbb{Q}} \mathbb{R}}{(\mathcal{F}_\eta \otimes_{\mathbb{Q}} \mathbb{R}) + \prod_{D \notin \mathbb{C}, f(D) = \text{Spec} \mathbb{Z}} (\hat{\mathcal{F}}_D \otimes_{\mathbb{Q}} \mathbb{R})},
\]

\[
\Phi_3 = \prod'_{D, f(D) = \text{Spec} \mathbb{Z}} K_D(\mathcal{F}),
\]

and \(\beta\) is the natural map from the group \(\Phi_3\) to the group \(\Phi_1 \oplus \Phi_2\) (to both parts of the direct sum).

We claim that

\[
\frac{\mathbb{A}_{X,02}(\mathcal{F}) + \prod'_{D, f(D) \neq \text{Spec} \mathbb{Z}} K_D(\mathcal{F}) + \prod'_{x \in D, D \notin \mathbb{C}} \mathcal{O}_{K_{x,D}}(\mathcal{F}) + \Phi_3}{\mathbb{A}_{X}(\mathcal{F})} \cong 0.
\]

(26)

Indeed, by Theorem 1 in [15], we immediately obtain that the expression on the left-hand side of (26) is equal to

\[
\prod_{1 \leq i \leq w} \prod'_{x \in C_i} K_{x,C_i}(\mathcal{F}) + \prod'_{x \in C} B_{x,C}(\mathcal{F}),
\]

(27)

which vanishes because this group is isomorphic to the group \(\lim_{n \to \infty} \lim_{m \to 0} H^1(X, \mathcal{F}_{nm})\) by means of the inductive and projective limits of the coherent complexes of the coherent sheaves \(\mathcal{F}_{nm} = \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X(nC)/\mathcal{O}_X(mC))\) on the scheme \(Y_{n-m} = (C, \mathcal{O}_X/J_C^{n-m})\) with topological space \(C\) and structure sheaf \(\mathcal{O}_X/J_C^{n-m}\), where \(J_C\) is the ideal sheaf of the subscheme \(C\) on \(X\), and we have \(H^1(X, \mathcal{F}_{nm}) = H^1(Y_{n-m}, \mathcal{F}_{nm}) = 0\) because \(Y_{n-m}\) is an affine scheme (see the similar arguments in Lemma 7 of [10]).

5We recall that \(\mathcal{O}_X(nC)\) is a reflexive torsion-free coherent subsheaf of the constant sheaf of the field of rational functions on \(X\). This subsheaf consists of elements of \(j_\ast \mathcal{O}_U\) whose discrete valuations given by the subschemes \(C_i\) (where \(1 \leq i \leq w\)) are greater than or equal to \(-n\); see also the beginning of §3.2.1 in [15].
Remark 3.3. We note that the reduction of \((26)\) to the vanishing of \((27)\) also follows easily from the equality \(H^2(X, j_* j^* \mathcal{F}) = 0\) and the adelic complex for the sheaf \(j_* j^* \mathcal{F}\) on \(X\).

We now claim that

\[
\Phi_1 \oplus \Phi_2 / \beta(\Phi_3) \sim \lambda \left( \Phi_3 \cap \left( \mathbb{A}_X, 0_2(\mathcal{F}) + \prod_{D, f(D) \neq \text{Spec } \mathbb{Z}}' K_D(\mathcal{F}) + \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_x,D}(\mathcal{F}) \right) \right),
\]

(28)

where the intersection is taken inside the group \(\mathbb{A}_X(\mathcal{F})\) and \(\lambda\) is the natural map from \(\Phi_3\) to \(\Phi_2\).

To prove this, we construct a map \(\psi\) from the group on the left-hand side of \((28)\) to the group on the right-hand side in the following way. Take any element \(w\) in the group \((\Phi_1 \oplus \Phi_2) / \beta(\Phi_3)\). To define \(\psi(w)\), let \(x + y\) be a lift of \(w\), where \(x \in \Phi_1\) and \(y \in \Phi_2\), and let \(\tilde{x} \in \mathbb{A}_X(\mathcal{F})\) be a lift of \(x\). By \((26)\) there is an element \(g \in \Phi_3\) such that

\[
g + \tilde{x} \in \mathbb{A}_X, 0_2(\mathcal{F}) + \prod_{D, f(D) \neq \text{Spec } \mathbb{Z}}' K_D(\mathcal{F}) + \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_x,D}(\mathcal{F}).
\]

(29)

Now, by definition, \(\psi(w) = y + \lambda(g)\). Clearly, the map \(\psi\) is well defined since \(\psi(w)\) is independent of the choice of the lifts of \(w\) and \(x\) and for two elements \(g_1, g_2 \in \Phi_3\) with the condition as in \((29)\) we have

\[
g_1 - g_2 \in \Phi_3 \cap \left( \mathbb{A}_X, 0_2(\mathcal{F}) + \prod_{D, f(D) \neq \text{Spec } \mathbb{Z}}' K_D(\mathcal{F}) + \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_x,D}(\mathcal{F}) \right).
\]

It is also clear that \(\psi\) is surjective. Moreover, we easily see that \(\psi\) is an injective map. Thus \(\psi\) is an isomorphism.

We will now prove that

\[
\lambda \left( \Phi_3 \cap \left( \mathbb{A}_X, 0_2(\mathcal{F}) + \prod_{D, f(D) \neq \text{Spec } \mathbb{Z}}' K_D(\mathcal{F}) + \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_x,D}(\mathcal{F}) \right) \right) = \lambda(\mathbb{A}_C(\mathcal{F})).
\]

(30)

Indeed, suppose that

\[
c_0' + c_1' + c_2 \in \Phi_3 \cap \left( \mathbb{A}_X, 0_2(\mathcal{F}) + \prod_{D, f(D) \neq \text{Spec } \mathbb{Z}}' K_D(\mathcal{F}) + \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_x,D}(\mathcal{F}) \right),
\]

(31)

where

\[
c_0' = \prod_D c_{0,1,D} \in \Phi_3, \quad c_1' = \prod_D c_{01,D} \in \prod_{D, f(D) \neq \text{Spec } \mathbb{Z}}' K_D(\mathcal{F}),
\]

\[
c_0 = \prod_x c_{0,2,x} \in \mathbb{A}_X, 0_2(\mathcal{F}), \quad c_2 = \prod_{x \in D} c_{12,x,D} \in \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_x,D}(\mathcal{F}).
\]
Restricting the equality
\[ c_{02} + c_{01}' - c_{01}'' + c_{12} = 0 \] (32)
to the open affine subset \( U \), we define a 1-cocycle \((c_{01}' - c_{01}'' - c_{02}, c_{12})\) in the adelic complex of the sheaf \( \mathcal{F} \) on \( U \). Since \( H^1(U, \mathcal{O}_U) = 0 \), this 1-cocycle is a 1-coboundary. This implies the existence of elements \( d_{1,D} \in \hat{\mathcal{F}}_D \) (for every one-dimensional integral closed subscheme \( D \subset X \) with \( D \not\subset C \)), elements \( d_{2,x} \in \hat{\mathcal{F}}_x \) (for every closed point \( x \in U \)) and an element \( h \in \mathcal{F}_\eta \) such that
\[
\begin{align*}
d_{1,D} + h &= c_{01,D} \quad \text{for all } D \text{ with } f(D) \neq \text{Spec } \mathbb{Z}, \\
d_{1,D} - h &= c_{01,\bar{D}} \quad \text{for all } D \not\subset C \text{ and } f(D) = \text{Spec } \mathbb{Z}, \\
d_{2,x} - h &= c_{02,x} \quad \text{for all } x \in U.
\end{align*}
\]
We also define the elements \( d_{01,D} = d_{1,D} \) and \( d_{02,x} = d_{2,x} \) (for \( D \) and \( x \) as above) and the elements
\[
\begin{align*}
d_{01,C_i} &= c_{01,C_i} + h \quad \text{for } 1 \leq i \leq w, \\
d_{02,x} &= c_{02,x} + h \quad \text{for all } x \in C, \\
d_{12,x,D} &= c_{12,x,D} \in \mathcal{O}_{K_{x,D}}(\mathcal{F}) \quad \text{for all } x \in D \text{ with } D \not\subset C.
\end{align*}
\]
Then we have the same equation for the elements \('d'\) as for the elements \('c'\) in (32). Using this and the conditions above on the elements \('d'\), we obtain that
\[
\prod_{1 \leq i \leq w} d_{01,C_i} \in A_C(\mathcal{F}).
\]
Therefore the intersection of groups in (31) is contained in the group
\[
A_C(\mathcal{F}) + \prod_{D \not\subset C, f(D) = \text{Spec } \mathbb{Z}} \hat{\mathcal{F}}_D + \text{(the image of } \mathcal{F}_\eta \text{ in } \Phi_3).\]
Moreover, the group \( A_C(\mathcal{F}) \) is contained in the intersection of groups in (31). Indeed, it is clear that \( A_C(\mathcal{F}) \subset \Phi_3 \), and we also have an embedding
\[
A_C(\mathcal{F}) \subset \mathbb{A}_{X,02}(\mathcal{F}) + \prod_{x \in D, D \not\subset C} \mathcal{O}_{K_{x,D}}(\mathcal{F})
\]
by means of the map \( \tau \) (see (15) and (9)). Hence, bearing in mind the definition of \( \Phi_2 \), we obtain (30).
Thus we find from (28) and (30) that
\[
\frac{\Phi_1 \oplus \Phi_2}{\beta(\Phi_3)} \simeq \frac{\Phi_2}{\lambda(A_C(\mathcal{F}))}.
\]
Using the calculation of the adelic quotient group on the algebraic curve \( X_\mathbb{Q} \) (namely, applying Remark 1 in [15] to the algebraic curve \( X_\mathbb{Q} \) and the open affine subset \( U_\mathbb{Q} \)), we obtain
\[
\Phi_2 \simeq \frac{\prod_{1 \leq i \leq w} (K_{C_i}(\mathcal{F}) \otimes_\mathbb{Q} \mathbb{R})}{A_{U_\mathbb{Q}}(\mathcal{F}) \otimes_\mathbb{Q} \mathbb{R}}.
\]
Hence we obtain that
\[
\frac{\Phi_1 \oplus \Phi_2}{\beta(\Phi_3)} \cong \prod_{1 \leq i \leq w} (KC_i(F) \otimes_{\mathbb{Q}} \mathbb{R}) \cong \frac{\prod_{1 \leq i \leq w} \mathcal{O}_{K,C_i}(F)}{A_C(F) + (A_{Uq}(F) \otimes_{\mathbb{Q}} \mathbb{R})}.
\]

This gives us the last non-zero term of the exact sequence (11) in Theorem 2.1. Thus we have proved this theorem. □

Remark 3.4. It follows from the proof of the vanishing of the group in (27) and from the construction of the isomorphism \( \psi \) in (28) that the image of the group \( \prod_{1 \leq i \leq w} \mathcal{O}_{K,C_i}(F) \) in the group \( \Phi \) is mapped to the image of the group \( \prod_{1 \leq i \leq w} \hat{F}_{C_i} \) in the group \( A_C(F) + (A_{Uq}(F) \otimes_{\mathbb{Q}} \mathbb{R}) \) in the exact sequence (11).

§ 4. Connection with the first cohomology groups

In this section we say more about the group which is the last non-zero term of the exact sequence (11) in Theorem 2.1. In particular, we relate this group to the first cohomology groups of the sheaves \( F(nC) \) on \( X \), where \( F(nc) = F \otimes_{\mathcal{O}_X} \mathcal{O}_X(nC) \). We require in addition that \( C \) is an ample Cartier divisor on \( X \).

For every integer \( m \geq 0 \) we consider a quotient group of
\[
\prod_{1 \leq i \leq w} (KC_i(F) \otimes_{\mathbb{Q}} \mathbb{R}) \cong \frac{\prod_{1 \leq i \leq w} \mathcal{O}_{K,C_i}(F)}{A_C(F) + (A_{Uq}(F) \otimes_{\mathbb{Q}} \mathbb{R})}
\]
of the form
\[
\Theta_m(F) = \frac{\prod_{1 \leq i \leq w} (KC_i(F) \otimes_{\mathbb{Q}} \mathbb{R})}{A_C(F) + (A_{Uq}(F) \otimes_{\mathbb{Q}} \mathbb{R}) + \left( \prod_{1 \leq i \leq w} F(-mC) \otimes_{\mathbb{Q}} \mathbb{R} \right)}.
\]

Proposition 4.1. The following assertions hold.

1) For every integer \( m \geq 0 \), the group
\[
\Theta_m(F) \cong H^1(X, F(-mC)) \otimes_{\mathbb{Z}} \mathbb{T} \cong \mathbb{T}^{\text{rank}(H^1(X, F(-mC))})
\]
is a finite direct product of copies of \( \mathbb{T} \) (we recall that \( \mathbb{T} \cong \mathbb{R}/\mathbb{Z} \)).

2) There is an integer \( m_0 \geq 0 \) (depending on the sheaf \( F \)) such that for every integer \( m \geq m_0 \) we have an exact sequence
\[
0 \to \mathbb{T}^r \to \Theta_{m+1}(F) \to \Theta_m(F) \to 0,
\]
where \( r \) is the rank of the locally free sheaf \( F \) and \( l \) is the degree of the finite morphism \( f|_C : C \to \text{Spec}\mathbb{Z} \).
Proof. 1) Since $C$ is an affine one-dimensional scheme and $f|_C: C \to \text{Spec} \mathbb{Z}$ is a flat morphism, we have the following embedding of a free Abelian group in a $\mathbb{Q}$-vector space:

$$A_C(\mathcal{F})/( \lim_{j \to -m} H^0(X, \mathcal{F}(-mC)/\mathcal{F}(jC))) \cong \lim_{n \to -m} H^0(X, \mathcal{F}(nC)/\mathcal{F}(-mC))$$

$$\cong \bigoplus_{n > -m} H^0(X, \mathcal{F}(nC)/\mathcal{F}((n-1)C)) \hookrightarrow \bigoplus_{n > -m} H^0(X, \mathcal{F}(nC)/\mathcal{F}((n-1)C)) \otimes \mathbb{Z} \mathbb{Q}$$

$$\cong \left( \prod_{1 \leq i \leq w} K_{C_i}(\mathcal{F}) \right) / \left( \prod_{1 \leq i \leq w} \mathcal{F}(-mC)_{C_i} \right).$$

(Note that the direct sum decomposition in this formula is not canonical, but we fix it.) We also have another embedding of a free Abelian group in a $\mathbb{Q}$-vector space:

$$H^0(U, \mathcal{F}|_U) \cong \lim_{n} H^0(X, \mathcal{F}(nC)) \hookrightarrow \lim_{n} H^0(X, \mathcal{F}(nC)) \otimes \mathbb{Z} \mathbb{Q}$$

$$\cong H^0(U_Q, \mathcal{F}|_{U_Q}) \cong A_{U_Q}(\mathcal{F}).$$

Hence we obtain that

$$\Theta_m(\mathcal{F}) \cong \frac{A_C(\mathcal{F})}{H^0(U, \mathcal{F}|_U) + \lim_{j \to -m} H^0(X, \mathcal{F}(-mC)/\mathcal{F}(jC)) \otimes \mathbb{Z} \mathbb{R}}. \quad (34)$$

By Theorem 9 in [11], for every integer $m$ we have

$$A_C(\mathcal{F}) \cong H^0(U, \mathcal{F}|_U) + \lim_{j \to -m} H^0(X, \mathcal{F}(-mC)/\mathcal{F}(jC)) \simeq H^1(X, \mathcal{F}(-mC))$$

and we know that these groups are finitely generated Abelian groups (this also follows from the argument in the next paragraph).

Since $C$ is an ample divisor, there is an integer $n_0 > 0$ such that for every integer $n > n_0$ we have a surjective map

$$H^0(X, \mathcal{F}(nC)) \to H^0(X, \mathcal{F}(nC)/\mathcal{F}((m-1)C)). \quad (35)$$

This means that the Abelian group $H^0(U, \mathcal{F}|_U)$ is mapped surjectively onto the Abelian group $\bigoplus_{n > n_0} H^0(X, \mathcal{F}(nC)/\mathcal{F}((m-1)C))$ by means of the maps (35), and the kernel is a finitely generated Abelian group. This proves the first part of the proposition.

2) The exact sequence of locally free sheaves

$$0 \to \mathcal{F}((-m-1)C) \to \mathcal{F}(-mC) \to \mathcal{F}(-mC)/\mathcal{F}((-m-1)C) \to 0$$

induces a long exact sequence of cohomology groups

$$H^0(X, \mathcal{F}(-mC)) \to H^0(X, \mathcal{F}(-mC)/\mathcal{F}((-m-1)C))$$

$$\to H^1(X, \mathcal{F}((-m-1)C) \to H^1(X, \mathcal{F}(-mC)) \to 0.$$
Moreover, there is an integer $m_0 \geq 0$ such that for every integer $m \geq m_0$ the group $H^0(\mathcal{X}, \mathcal{F}(-mC))$ is equal to zero since the free Abelian group $H^0(\mathcal{X}, \mathcal{F})$ is finitely generated and

$$\bigcap_{j \geq 0} H^0(\mathcal{X}, \mathcal{F}(-jC)) = 0.$$ 

For every integer $n$, the finitely generated free Abelian group $H^0(\mathcal{X}, \mathcal{F}(nC)/\mathcal{F}((n-1)C))$ is a cocompact lattice in the $\mathbb{R}$-vector space $H^0(\mathcal{X}, \mathcal{F}(nC)/\mathcal{F}((n-1)C)) \otimes_{\mathbb{Z}} \mathbb{R}$ and the quotient group of this vector space by this lattice is isomorphic to $\mathbb{T}^{rl}$. This proves the second part of the proposition. \[\square\]

We obtain from the proof of Proposition 4.1 (see (34)) that

$$\prod_{1 \leq i \leq w} (K_{C_i}(\mathcal{F}) \otimes_{\mathbb{Q}} \mathbb{R}) \cong \text{lim}_{\leftarrow} M \geq 0 \Theta_m(\mathcal{F}).$$

Clearly, it follows that the group in (36) is compact.

**Remark 4.2.** It is natural to consider the following subgroup $\mathbb{A}_{X,12}^{ar}(\mathcal{F})$ of the arithmetic adelic group $\mathbb{A}^{ar}_X(\mathcal{F})$:

$$\mathbb{A}_{X,12}^{ar}(\mathcal{F}) \overset{\text{def}}{=} \mathbb{A}_{X,12}(\mathcal{F}) \oplus \mathbb{F}_D \otimes_{\mathbb{Q}} \mathbb{R} = \prod_{D, f(D) = \text{Spec } \mathbb{Z}} \mathcal{O}_{K_D, D}(\mathcal{F}) \oplus \prod_{D, f(D) = \text{Spec } \mathbb{Z}} \mathbb{F}_D \otimes_{\mathbb{Q}} \mathbb{R}.$$ 

Sugahara and Weng introduced the following groups in [12];

$$H^0_{ar}(\mathcal{X}, \mathcal{F}) = \mathbb{A}_{X,01}^{ar}(\mathcal{F}) \cap \mathbb{A}_{X,02}^{ar}(\mathcal{F}) \cap \mathbb{A}_{X,12}^{ar}(\mathcal{F}),$$

$$H^2_{ar}(\mathcal{X}, \mathcal{F}) = \mathbb{A}_{X}^{ar}(\mathcal{F})/\left(\mathbb{A}_{X,01}^{ar}(\mathcal{F}) + \mathbb{A}_{X,02}^{ar}(\mathcal{F}) + \mathbb{A}_{X,12}^{ar}(\mathcal{F})\right).$$ 

It follows immediately from Theorem 2.1, Remark 3.4 and Proposition 4.1 that

$$H^2_{ar}(\mathcal{X}, \mathcal{F}) \simeq \Theta_0(\mathcal{F}) \simeq H^1(\mathcal{X}, \mathcal{F}) \otimes_{\mathbb{Z}} \mathbb{T}.$$ 

Moreover, we easily see from Lemma 3.1 that

$$H^0_{ar}(\mathcal{X}, \mathcal{F}) \simeq H^0(\mathcal{X}, \mathcal{F}).$$

We also note that a topological duality between the topological groups $H^i_{ar}$ and $H^{2-i}_{ar}$ was defined in [12], [13] using the properties of the topology of inductive and projective limits on adelic groups (here we do not define the group $H^1_{ar}$ and do not indicate the sheaves in the cohomology groups occurring in the duality). More about this topological duality will appear in a forthcoming paper [19].
Remark 4.3. By (36) (and Proposition 4.1) we obtain that the last non-zero term of the exact sequence (11) in Theorem 2.1 is equal to the projective limit of groups which are finite direct products of copies of \( \mathbb{T} \). This gives us an interesting analogy with the values of theta functions (or, in other words, \( \theta \)-invariants) constructed from the cocompact lattice \( E \) in the finite-dimensional \( \mathbb{R} \)-vector space \( \prod_v K_v \), occurring in the exact sequence (2) in the introduction. Using these \( \theta \)-invariants, one obtains the Riemann–Roch theorem in the form of the Poisson summation formula in one-dimensional Arakelov geometry, and this was noticed by many authors; see [20].

Indeed, let \( \overline{M} \) be a Hermitian vector bundle over the ‘arithmetic curve’ \( \text{Spec} \ E \). Then, by definition,

\[
h_0^0(\overline{M}) = \log \left( \sum_{v \in M} \exp(-\pi \|v\|^2_{g^*_v \overline{M}}) \right), \quad h_1^0(\overline{M}) = h_0^0(\varpi_{E/\mathbb{Z}} \otimes \overline{M}^\vee),
\]

where \( M \) is the corresponding projective \( E \)-module, \( g: \text{Spec} \ E \to \text{Spec} \mathbb{Z} \) is the natural morphism, \( \| \cdot \|_{g^*_v \overline{M}} \) is the Euclidean norm on the \( \mathbb{R} \)-vector space \( M \otimes \mathbb{R} \) corresponding to the Hermitian vector bundle \( g^*_v \overline{M} \) over the ‘arithmetic curve’ \( \text{Spec} \mathbb{Z} \), \( \overline{M}^\vee \) is the Hermitian vector bundle dual to \( \overline{M} \), and \( \varpi_{E/\mathbb{Z}} \) is the canonical Hermitian line bundle over \( \text{Spec} \ E \). Then it follows from the Poisson summation formula that

\[
h_1^0(\overline{M}) - h_1^1(\overline{M}) = \widehat{\deg} \overline{M} - \frac{1}{2} \log |\Delta_K| \cdot \text{rk} M,
\]

where \( \widehat{\deg} \) is the Arakelov degree of a Hermitian bundle and \( \Delta_K \) is the discriminant of the number field \( K \).

Remark 4.4. More about generalizations of Remark 4.2 and the \( \theta \)-invariants in Remark 4.3 will appear in the forthcoming paper [19].

Bibliography

[1] A. N. Parshin, “On the arithmetic of two-dimensional schemes. I. Distributions and residues”, Izv. Akad. Nauk SSSR Ser. Mat. 40:4 (1976), 736–773; English transl., Math. USSR-Izv. 10:4 (1976), 695–729.
[2] A. A. Beilinson, “Residues and adeles”, Funktsional. Anal. i Prilozhen. 14:1 (1980), 44–45; English transl., Func. Anal. Appl. 14:1 (1980), 34–35.
[3] A. Huber, “On the Parshin–Beilinson adeles for schemes”, Abh. Math. Sem. Univ. Hamburg 61 (1991), 249–273.
[4] A. N. Parshin, “Higher dimensional local fields and \( L \)-functions”, Invitation to higher local fields (Münster 1999), Geom. Topol. Monogr., vol. 3, Geom. Topol. Publ., Coventry 2000, pp. 199–213.
[5] A. N. Parshin, “Representations of higher adelic groups and arithmetic”, Proceedings of the International Congress of Mathematicians, vol. 1 (Hyderabad 2010), Hindustan Book Agency, New Delhi 2011, pp. 362–392.
[6] J. T. Tate, Jr., Fourier analysis in number fields and Hecke’s zeta functions, Thesis (Ph.D.), Princeton Univ., Princeton 1950.
Yu. G. Zarkhin and A. N. Parshin, “Finiteness problems in Diophantine geometry”, Appendix in: S. Lang, Foundations of Diophantine geometry, Mir, Moscow 1986, pp. 369–438 (Russian); English transl., Amer. Math. Soc. Transl. Ser. 2, vol. 143, Amer. Math. Soc., Providence, RI 1989, pp. 35–102; arXiv:0912.4325.

S. J. Arakelov, “Theory of intersections on the arithmetic surface”, Proceedings of the International Congress of Mathematicians, vol. 1 (Vancouver, BC 1974), Canad. Math. Congress, Montreal, Que. 1975, pp. 405–408.

A. N. Parshin, “A holomorphic version of the Tate–Iwasawa method for unramified $L$-functions”, Mat. Sb. 205:10 (2014), 107–124; English transl., Sb. Math. 205:10 (2014), 1473–1491.

D. V. Osipov and A. N. Parshin, “Harmonic analysis on local fields and adelic spaces. II”, Izv. Ross. Akad. Nauk Ser. Mat. 75:4 (2011), 91–164; English transl., Izv. Math. 75:4 (2011), 749–814.

D. V. Osipov, “$n$-dimensional local fields and adeles on $n$-dimensional schemes”, Surveys in contemporary mathematics, London Math. Soc. Lecture Note Ser., vol. 347, Cambridge Univ. Press, Cambridge 2008, pp. 131–164.

K. Sugahara and L. Weng, “Arithmetic cohomology groups”, Appendix in: L. Weng, Zeta functions of reductive groups and their zeros, World Sci., Hackensack, NJ 2018 (to appear); arXiv:1507.06074.

K. Sugahara and L. Weng, “$H^1_{ar}$ for arithmetic surface is finite”, Appendix in: L. Weng, Zeta functions of reductive groups and their zeros, World Sci., Hackensack, NJ 2018 (to appear); arXiv:1603.02353.

Yongchang Zhu, “Weil representations and theta functionals on surfaces”, Perspectives in representation theory, Contemp. Math., vol. 610, Amer. Math. Soc., Providence, RI 2014, pp. 353–370.

D. V. Osipov, “On adelic quotient group for algebraic surface”, Algebra i Analiz 30:1 (2018), 151–169; English transl., St. Petersburg Math. J. 30:1 (to appear).

Qing Liu, Algebraic geometry and arithmetic curves, Transl. from the French, Oxf. Grad. Texts Math., vol. 6, Oxford Univ. Press, Oxford 2002.

D. V. Osipov, “Second Chern numbers of vector bundles and higher adeles”, Bull. Korean Math. Soc. 54:5 (2017), 1699–1718.

R. Ya. Budylin and S. O. Gorchinskiy, “Intersections of adelic groups on a surface”, Mat. Sb. 204:12 (2013), 3–14; English transl., Sb. Math. 204:12 (2013), 1701–1711.

D. V. Osipov, “Serre–Pontryagin duality on arithmetic surfaces via higher adeles” (to appear).

J. B. Bost, Theta invariants of euclidean lattices and infinite-dimensional hermitian vector bundles over arithmetic curves, arXiv:1512.08946.

Denis V. Osipov
Steklov Mathematical Institute
of Russian Academy of Sciences, Moscow
National Research University
Higher School of Economics, Moscow
National University of Science
and Technology “MISiS”, Moscow
E-mail: d.osipov@mi.ras.ru

Received 14/JAN/18
Translated by THE AUTHOR

Received 27/FEB/18