A complete solution of the partition of a number into arithmetic progressions†

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Abstract

We solve the enumeration of the set \( AP(n) \) of partitions of a positive integer \( n \) in which the nondecreasing sequence of parts forms an arithmetic progression. In particular, we establish a formula for the number of nondecreasing arithmetic progressions of positive integers with sum \( n \). We also present an explicit method to calculate all the partitions of \( AP(n) \).

2020 Mathematics Subject Classification: 11P81, 11A51.

Keywords: partition, arithmetic progression, arithmetic generated by a sequence.

1 Introduction

A partition of a positive integer \( n \) is a nondecreasing sequence of positive integers whose sum is \( n \). The summands are called parts of the partition. We consider the problem of enumerating the set \( AP(n) \) of partitions of \( n \) in which the nondecreasing sequence of parts forms an arithmetic progression (AP), that is, the nondecreasing arithmetic progressions of positive integers with sum \( n \).

A related work was made by Mason [4] who studied the representation of an integer as the sum of consecutive integers. Bush [1] extended Mason’s results to integers in arithmetic progressions. An analytical approach was made by Leveque [3]. A few authors have considered a combinatorial perspective of the problem of enumerating the set \( AP(n) \) (see [2, 5, 6, 7]). The sequence \((|AP(n)|)_{n>0}\) occurs as sequence number A049988 in the Online Encyclopedia of Integer Sequences [8].

Our paper proposes a novel way to study this problem based on [9]. The main idea is as follows: the usual divisors trivially solve the problem of the partition of a number into equal parts. Now, for each \( k \in \mathbb{Z} \), we will consider a new product mapping \((\circ_k)\) that will generate an arithmetic \((k\text{-arithmetic})\) similar to the usual one. In this new arithmetic, the divisors of an integer \( n \) will trivially solve the problem of the representation of \( n \) as the sum of arithmetic progressions whose difference is \( k \). We will prove the following theorem:

† JP J. Algebra Number Theory Appl. 53(2) (2022), 109-122. DOI: 10.17654/09725555220006
Theorem 1. Given a positive integer \( n \), let \( \tau(n) \) denote the number of positive divisors of \( n \), \( D_E(n) \) denote the set of divisors of \( n \) and \( D_O(n) \) denote the set of divisors of \( 2n \) except the even divisors of \( n \). Then the cardinality of the set \( AP(n) \), denoted by \( |AP(n)| \), is equal to

\[
\tau(n) + \sum_{d \in D_E(n)} \left[ \frac{1}{2} \left( \frac{2n}{d(d-1)} \right) - 1 \right] + \sum_{d \in D_O(n)} \left[ \frac{1}{2} \left( \frac{2n}{d(d-1)} \right) \right].
\]

We also present an explicit method to calculate all the partitions of \( AP(n) \).

In the following section, we present a brief introduction to the methods used in [9].

2 Arithmetic progressions and the usual arithmetic

The usual product \( m \cdot n \) on \( \mathbb{Z} \) can be viewed as the sum of \( n \) terms of an arithmetic progression \( (a_n) \) whose first term is \( a_1 = m - n + 1 \) and whose difference is \( d = 2 \).

Example 1. \( 6 \cdot 3 = (6 - 3 + 1) + 6 + 8 = 3 \cdot 6 = (3 - 6 + 1) + 0 + 2 + 4 + 6 + 8. \)

The previous example motivates the following definition.

Definition 1 (**k-arithmetic** product \( \odot_k \)). Given \( m, k \in \mathbb{Z} \), for all positive integers \( n \), we define the following expression

\[
m \odot_k n = (m - n + 1) + (m - n + 1 + k) + \ldots + (m - n + 1 + k + \ldots (n-1)) + k
\]
as the **k-arithmetic** product.

This arithmetic progression can be added to obtain the following formula:

\[
m \odot_k n = (m - n + 1) \cdot n + \frac{n \cdot (n-1) \cdot k}{2}. \tag{1}
\]

We take (1) as Definition 1 and consider \( n \in \mathbb{Z} \).

In connection with the above result, for each \( k \in \mathbb{Z} \), the expression “given a **k-arithmetic**” refers to the fact that we are going to work with integers, the sum, the new product and the usual order. This means that we are going to work on \( \mathbb{Z}_k = \{\mathbb{Z}, +, \odot_k, <\} \). Clearly, \( \mathbb{Z}_2 \), the **2-arithmetic**, will be the usual arithmetic.

Definition 2 (**k-arithmetic** divisor). Given a **k-arithmetic**, an integer \( d > 0 \) is called a divisor of \( a \) (**arith k**) if there exists some integer \( b \) such that \( a = b \odot_k d \). We can write: \( d \mid a \) (**arith k**) if there exists some integer \( b \) such that \( b \odot_k d = a \).

In other words, \( d \) is the number of terms of the summation that represents the **k-arithmetic** product.
Example 2. Consider the following expression:

$$9 \odot_3 8 = 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23 = 100.$$  

The number of terms is 8; hence, we can say that 8 is a divisor of 100 in 3-\textit{arithmetic}, that is, 8 is a divisor of 100 (\textit{arith} 3). Notably, a divisor is always a positive number, and the number 9 indicates where we should start the summation. However, we cannot be sure that 9 is a divisor of 100 (\textit{arith} 3).

To characterize the set of divisors, we define the \textit{k-arithmetic} quotient:

\textbf{Definition 3} \textit{(k-arithmetic quotient \(\odot_k\)).} Let \(a, b \in \mathbb{Z}\), \(b \neq 0\). Given a \(k\)-arithmetic, an integer \(c\) is called a quotient of \(a\) divided by \(b\) (\textit{arith} \(k\)) if and only if \(c \odot_k b = a\). We write:

\[ a \odot_k b = c \iff c \odot_k b = a. \]

Also, we can express the \(k\)-\textit{arithmetic} quotient with the usual one:

\[ a \odot_k b = \frac{a}{b} + (b - 1) \cdot (1 - \frac{k}{2}). \]  \tag{2}

We must consider \(\odot_k\) in the following manner. If we want to write \(a\) as the sum of \(b\) terms of an arithmetic progression, then the quotient will give us the place to start the summation. For instance, if we want to express 57 as the sum of 6 terms of an arithmetic progression whose difference is 3, we can do: \(57 \odot_3 6 = 57/6 + 5 \cdot (1 - 3/2) = 7\). Hence, \(57 = 7 \odot_3 6\). The first term is \(7 - 6 + 1 = 2\), and the solution is \(2 + 5 + 8 + 11 + 14 + 17 = 57\).

\textbf{Corollary 1.} Let \(b > 0\), \(b\) is a divisor of \(a\) (\textit{arith} \(k\)) \(\iff a \odot_k b\) is an integer.

Consider Example 2: \(100 = 9 \odot_3 8\) but 9 is not a divisor of 100 (\textit{arith} 3) because \(100 \odot_3 9 = 100/9 + 8 \cdot (1 - 3/2) = 64/9 \notin \mathbb{Z}\). That is, 100 is not the sum of 9 terms of an arithmetic progression of integers whose difference is 3.

If we have the divisors of \(n\) (\textit{arith} \(k\)), then we have the arithmetic progressions whose difference is \(k\) with sum \(n\): if \(d\) is a divisor of \(n\) (\textit{arith} \(k\)), there must exist an integer \(a\) such that \(n = a \odot_k d\). Then, \(n = (a - d + 1) + (a - d + 1 + k) + \ldots + (a - d + 1 + (d - 1)k)\); hence \(n\) is the sum of an arithmetic progression of integers whose difference is \(k\). On the other hand, if \(n\) is the sum of an arithmetic progression of integers whose difference is \(k\), there must exist \(a, d \in \mathbb{Z}\), \(d > 0\) such that \(n = a + (a + k) + (a + 2k) + \ldots + (a + (d - 1)k)\). Then, \(n = (a + d - 1) \odot_k d\) and \(d\) is a divisor of \(n\) (\textit{arith} \(k\)).

Now it is clear that the problem to calculate the arithmetic progressions of integers whose difference is \(k\) with sum \(n\) is equivalent to calculate the set of divisors of \(n\) (\textit{arith} \(k\)).

For the upcoming lemma and the rest of this paper, we use the following notation for even and odd numbers.

\textbf{Notation 1.} We write the set of even and odd numbers as follows:

- \(E = \{\ldots, -4, -2, 0, 2, 4, 6, \ldots\}\).
$O = \{\ldots, -3, -1, 1, 3, 5, 7, \ldots \}$.

The following lemma characterizes the divisors of $n$ (arith $k$). The proof appears in [9].

**Lemma 1.** Given a $k$-arithmetic and $a \in \mathbb{Z}$, the divisors of $a$ (arith $k$) are:

1. The usual divisors of $a$ if $k \in E$.
2. The usual divisors of $2a$ except the even usual divisors of $a$ if $k \in O$.

**Example 3.** Express the number 20 in all possible ways as a sum of an arithmetic progression whose difference is 3.

**Solution.** The divisors of 20 (arith 3) are the usual divisors of 40 except the even usual divisors of 20: $\{1, 2, 4, 5, 8, 10, 20, 40\}$. We obtain:

- $d = 1 \Rightarrow 20 \odot 3 \ 1 = 20 \Rightarrow 20 = 20 \odot 3 \ 1 \Rightarrow 20 = 20$.
- $d = 5 \Rightarrow 20 \odot 3 \ 5 = 2 \Rightarrow 20 = 2 \odot 3 \ 5 \Rightarrow 20 = -2 + 1 + 4 + 7 + 10$.
- $d = 8 \Rightarrow 20 \odot 3 \ 8 = -1 \Rightarrow 20 = -1 \odot 3 \ 8 \Rightarrow 20 = -8 - 5 - 2 + 1 + 4 + 7 + 10 + 13$.
- $d = 40 \Rightarrow 20 \odot 3 \ 40 = -19 \Rightarrow 20 = -19 \odot 3 \ 40 \Rightarrow 20 = -58 - 55 - \ldots + 56 + 59$.

**Definition 4.** Given a positive integer $n$ and $k \in \mathbb{Z}$, let $D_k(n)$ denote the set of divisors of $n$ (arith $k$).

By Lemma 1, we have two options:

- If $k \in E$, $D_k(n)$ is the set of the usual divisors of $n$. $D_E(n)$ denote this case.
- If $k \in O$, $D_k(n)$ is the set of the usual divisors of $2n$ except the even usual divisors of $n$. $D_O(n)$ denote this case.

Lemma 1 clarifies the problem of the representation of a number as the sum of an arithmetic series. We can easily obtain the results previously studied by other authors. For instance, the following corollary appears in [1].

**Corollary 2.** Let $n = 2^e p_1^{e_1} \cdots p_r^{e_r}$ be any positive integer, where $p_1, \ldots, p_r$ are distinct odd primes. The number of different ways in which $n$ can be expressed as the sum of an arithmetic series of integers with a specified odd common difference, $k$, is twice the number of distinct positive odd divisors of $n$.

**Proof.** Let $\tau_O(n) = (e_1 + 1) \cdot \ldots \cdot (e_r + 1)$ denote the number of odd usual divisors of $n$. Let $\tau_E(n) = e \cdot (e_1 + 1) \cdot \ldots \cdot (e_r + 1)$ denote the number of even usual divisors of $n$. We have to calculate the number of elements of $D_k(n)$, $k \in O$, denoted by $|D_k(n)|$. By Lemma 1,

$$|D_k(n)| = \tau(2n) - \tau_E(n) = (e + 2) \prod_{i=1}^{r} (e_i + 1) - e \prod_{i=1}^{r} (e_i + 1) = 2 \tau_O(n).$$

Also, if $k \in O$, $|D_k(n)| = (\tau_E(2n) - \tau_O(n)) + \tau_O(2n) = (\tau_O(n)) + \tau_O(n)$. Hence, exactly half of the elements of $D_k(n)$ are even and the other half are odd. \hfill \Box

Let us now study the main result of this paper.
3 Remarks and examples

If we think about Example 3, then we have a constructive method to solve the main problem of this paper. We are interested in the partitions of a positive integer \( n \) in which the nondecreasing sequence of parts forms an arithmetic progression. Therefore, the first term must be a positive integer. Let us consider some remarks.

**Remark 1.** The case \( k = 0 \) produces the trivial partitions. There are \( \tau(n) \) trivial partitions in this case.

**Remark 2.** The divisor \( d = 1 \in D_k(n) \) always produces the trivial partition \( n = n \) for all \( k \).

**Remark 3.** If \( k \in E, k > 0 \), then we have only to study the divisors \( d \in D_E(n) \) such that \( 1 < d \leq \sqrt{n} \).

**Proof.** We are interested in the partitions whose first term is greater than 0. If \( d \in D_k(n) \), then we can calculate \( a \in \mathbb{Z} \) such that \( a \odot_k d = n \). By (2), \( a = n \odot_k d = n/d + (d-1)(1-k/2) \).

The first term of the partition is \( a - d + 1 = n/d + (d-1)(1-k/2) - d + 1 \). If the first term is greater than 0, then we have the following expression:

\[
\frac{n}{d} + (d-1)(1 - \frac{k}{2}) - d + 1 > 0 \iff k < \frac{2n}{d(d-1)}.
\]  (3)

By (3), if \( k \in E, k > 0 \) and \( \frac{2n}{d(d-1)} \leq 2 \), then there will be no partition with a positive first term. Hence, if \( d > \sqrt{n} \), then we will not have an element of \( \text{AP}(n) \).

**Remark 4.** If \( k \in O, k > 0 \), then we have only to study the divisors \( d \in D_O(n) \) such that \( 1 < d < \sqrt{2n} \).

**Proof.** By (3), if \( k \in O, k > 0 \) and \( \frac{2n}{d(d-1)} \leq 1 \), then there will be no partition with a positive first term. Hence, if \( d > \sqrt{2n} \), then we will not have an element of \( \text{AP}(n) \).

If \( d = \sqrt{2n} \), then \( \sqrt{2n} \) is even and \( \sqrt{2n} \mid n \). Thus, by Lemma 1, we do not have to consider this case.

**Notation 2.** Let \( n \) a positive integer. Let \( d \in D_k(n), d > 1 \). Then we denote by \( k_d \) the critical value \( \frac{2n}{d(d-1)} \).

Let us look at all these remarks with an example:

**Example 4.** Calculate \( \text{AP}(6) \).

**Solution.** We are going to consider three cases:

- \( k = 0 \Rightarrow D_0(6) = \{1, 2, 3, 6\} \). We have the following possibilities:
  \* \( d = 1 \Rightarrow 6 \odot_0 1 = 6 \Rightarrow 6 \odot_0 1 = 6 \Rightarrow 6 = 6 \).
  \* \( d = 2 \Rightarrow 6 \odot_0 2 = 4 \Rightarrow 4 \odot_0 2 = 6 \Rightarrow 3 + 3 = 6 \).
\[ d = 3 \Rightarrow 6 \div 3 = 4 \Rightarrow 4 \div 3 = 6 \Rightarrow 2 + 2 + 2 = 6. \]
\[ d = 6 \Rightarrow 6 \div 6 = 6 \Rightarrow 6 \div 6 = 6 \Rightarrow 1 + 1 + 1 + 1 + 1 + 1 = 6. \]

We have \( \tau(6) = 4 \) trivial partitions in this case. We do not need to repeat this trivial case anymore. Note that the case \( k = 0 \) includes the trivial partition produced by the divisor \( d = 1 \). In the following cases, we will consider the divisors of \( D_k(6) \) greater than 1.

- \( k > 0, k \in E \Rightarrow D_E(6) = \{1, 2, 3, 6\} \). By Remark 3, we have only to study the divisors \( d \in D_E(n) \) such that \( 1 < d \leq \sqrt{6} \), hence we need to study the divisor \( d = 2 \).
  - \( d = 2 \Rightarrow k_2 = \frac{2d}{d(d-1)} = 6 \). By (3), the divisor \( d = 2 \) produces partitions of \( \text{AP}(6) \) in cases such that \( k \in E, 0 < k < 6 \). Hence, \( d = 2 \) produces 2 partitions \( (k = 2, k = 4) \).
  - \( d = 2, k = 2 \Rightarrow 6 \div 2 \cdot 2 = 3 \Rightarrow 3 \div 2 = 6 \Rightarrow 2 + 4 = 6. \)
  - \( d = 2, k = 4, \Rightarrow 6 \div 4 \cdot 2 = 2 \Rightarrow 2 \div 4 = 6 \Rightarrow 1 + 5 = 6. \)

- \( k > 0, k \in O \Rightarrow D_O(6) = \{1, 2, 3, 4, 6, 12\} \). By Remark 4, we have only to study the divisors \( d \in D_O(n) \) such that \( 1 < d < \sqrt{12} \), hence we need to study the divisor \( d = 3 \).
  - \( d = 3 \Rightarrow k_3 = \frac{2d}{3(d-1)} = 2 \). By (3), the divisor \( d = 3 \) produces partitions of \( \text{AP}(6) \) in cases such that \( k \in O, 0 < k < 2 \). Hence \( d = 3 \) produces 1 partition \( (k = 1) \).
  - \( d = 3, k = 1 \Rightarrow 6 \div 1 \cdot 3 = 3 \Rightarrow 3 \div 1 = 3 \Rightarrow 1 + 2 + 3 = 6. \)

Hence \( |\text{AP}(6)| = 4 + 2 + 1 = 7. \)

Let us do a slightly more complicated example.

\textbf{Example 5.} Calculate \( |\text{AP}(100)|. \)
\textbf{Solution.} We will write the divisors by pairs. By Remarks 3 and 4, we will have to study the first row of divisors only.

- \( k = 0 \). There are \( \tau(100) = 9 \) trivial partitions.

- \( k > 0, k \in E: D_E(100) = \{1, 2, 4, 5, 10, 100, 50, 25, 20\} \).

\[
\begin{array}{|c|c|c|c|}
\hline
\text{d} & \frac{\text{d(100)}}{\text{d(d-1)}} & \text{d \in E} & \text{Number of AP partitions} \\
\hline
2 & 100 & k = 2, k = 4, \ldots, k = 98 & 49 \\
4 & 16.6 & k = 2, k = 4, \ldots, k = 16 & 8 \\
5 & 10 & k = 2, k = 4, k = 6, k = 8 & 4 \\
10 & 2.2 & k = 2 & 1 \\
\hline
\end{array}
\]
• \( k > 0, \ k \in O: \ D_O(100) = \{ 1, 2, 4, 5, 8, 100 \} \).

| \( d \) | \( k_d = \frac{2 \cdot 100}{d(d-1)} \) | \( k \in O \) and \( 0 < k < k_d \) | Number of AP partitions |
|-----|-----------------|-----------------|-----------------|
| 5   | 10              | \( k = 1, k = 3, \ldots, k = 9 \) | 5               |
| 8   | 3.57 \ldots    | \( k = 1, k = 3 \) | 2               |

Hence \( |\text{AP}(100)| = 9 + 49 + 8 + 4 + 1 + 5 + 2 = 78 \).

If we want to calculate a concrete partition, for instance \( d = 5, \ k = 7 \), then we can do:
\[
100 \odot_7 5 = 100/5 + 4(1 - 7/2) = 10 \Rightarrow 100 = 10 \odot_7 5 = 6 + 13 + 20 + 27 + 34.
\]

We can use the floor and the ceiling functions to count the even and odd numbers in each case.

**Remark 5.** The number of positive even numbers less than a real \( x > 0 \) is given by the expression \( \left\lfloor \frac{1}{2} (\lceil x \rceil - 1) \right\rfloor \), where \( \lceil x \rceil \) is the greatest integer \( \leq x \) and \( \lfloor x \rfloor \) is the smallest integer \( \geq x \).

**Remark 6.** The number of positive odd numbers less than a real \( x > 0 \) is given by the expression \( \left\lfloor \frac{1}{2} \lceil x \rceil \right\rfloor \).

With all of the above, we can prove Theorem 1.

## 4 Proof of Theorem 1

Let us summarize the method explained in the previous section. Then we have to study three cases to calculate \( |\text{AP}(n)| \):

- **Case \( k = 0 \):** there are \( \tau(n) \) trivial partitions (Remark 1). The divisor \( d = 1 \) always produces the trivial partition \( n = n \) (Remark 2). This partition is counted in this case only. In the following cases, we will consider the divisors of \( D_k(n) \) greater than 1.

- **Case \( k > 0, \ k \in E \):** we have only to study the divisors \( d \in D_E(n) \) such that \( 1 < d \leq \sqrt{n} \) (Remark 3). By (3), each divisor produces partitions in the cases such that \( k \in E, \ 0 < k < k_d \). By Remark 5, there are \( \left\lfloor \frac{1}{2} (\lceil k_d \rceil - 1) \right\rfloor \) partitions of \( \text{AP}(n) \) in this case.

- **Case \( k > 0, \ k \in O \):** we have only to study the divisors \( d \in D_O(n) \) such that \( 1 < d < \sqrt{2n} \) (Remark 4). By (3), each divisor produces partitions in the cases such that \( k \in O, \ 0 < k < k_d \). By Remark 6, there are \( \left\lfloor \frac{1}{2} \lceil k_d \rceil \right\rfloor \) partitions of \( \text{AP}(n) \) in this case.

Then,
\[
|\text{AP}(n)| = \tau(n) + \sum_{d \in D_E(n), \ 1 < d \leq \sqrt{n}} \left\lfloor \frac{1}{2} (\lceil k_d \rceil - 1) \right\rfloor + \sum_{d \in D_O(n), \ 1 < d < \sqrt{2n}} \left\lfloor \frac{1}{2} k_d \right\rfloor.
\]
Once the problem is understood and solved, the only difficulty in calculating $AP(n)$ is to obtain the set of divisors of $2n$.

Figure 1 looks like the famous Goldbach’s comet.

![Figure 1: $|AP(n)|$, $n = 1, 2, \ldots, 100000$](image)

5 On the lengths of the partitions of $AP(n)$

A question proposed in [5, 6] deals with the different lengths of the partitions of $AP(n)$. If we consider a partition of $AP(n)$ as a $d$-tuple $(n_1, n_2, \ldots, n_d)$, then we can define the set $AP\text{div}(n)$ as the different lengths $d$ of the partitions of $AP(n)$. Since the trivial partitions have lengths equal to the divisors of $n$, $D_E(n) \subseteq AP\text{div}(n)$. By Theorem 1, the different lengths will be the elements of $D_E(n)$ (usual divisors) and the even elements of $D_O(n)$ that produce partitions of $AP(n)$.

**Corollary 3.** $|AP\text{div}(n)| = \tau(n) + \sum_{d \in E \cap D_O(n) \atop 1 < d < \sqrt{2n}} 1$.

**Proof.** By Theorem 1,

$$AP\text{div}(n) = D_E(n) \cup \{d \in E \cap D_O(n) : 1 < d < \sqrt{2n}\},$$

and the result follows.
**Example 6.** Calculate $|\text{APdiv}(500)|$.

*Solution.* Since $500 = 2^2 \cdot 5^3$, $\tau(500) = 3 \cdot 4 = 12$.

$$D_O(500) = \{ 1, \frac{2}{500}, \frac{4}{250}, \frac{5}{200}, \frac{8}{125}, \frac{16}{100}, \frac{20}{50}, \frac{25}{40} \}.$$Then, $\text{APdiv}(500) = D_E(500) \cup \{ 8 \}$ and $|\text{APdiv}(500)| = 12 + 1 = 13$.

The sequence $(|\text{APdiv}(n)|)_{n>0}$ occurs as sequence number A175239 in the Online Encyclopedia of Integer Sequences [8].

6 Conclusion

The novel way of studying a partition problem by calculating the divisors of a number in an arithmetic similar to the usual one is the main contribution of this paper. The study of the arithmetic generated by $\odot_k$ ($k$-arithmetic) is interesting by itself. An improvement of [9] proposes to study the arithmetic generated by any integer sequence $(a_n)_{n>0}$. Try to convert a partition problem to a divisors problem in an arithmetic generated by an integer sequence is a topic that needs more work.

Acknowledgements

This work was supported by King Juan Carlos University under grant C2PREDOC2020.

References

[1] L. E. Bush, *On the expression of an integer as the sum of an arithmetic series*, Amer. Math. Monthly 37(7) (1930), 353–357. DOI: 10.1080/00029890.1930.11987091.

[2] R. Cook and D. Sharp, *Sums of arithmetic progressions*, Fib. Quart. 33(3) (1995), 218–221.

[3] W. Leveque, *On representations as a sum of consecutive integers*, Canad. J. Math. 2 (1950), 399–405. DOI: 10.4153/CJM-1950-036-3.

[4] T. E. Mason, *On the representation of an integer as the sum of consecutive integers*, Amer. Math. Monthly 19(3) (1912), 46–50. DOI: 10.1080/00029890.1912.11997664.

[5] A. O. Munagi and T. Shonhiwa, *On the partitions of a number into arithmetic progressions*, J. Integer Seq. 11(5) (2008), #08.5.4.

[6] A. O. Munagi, *Combinatorics of integer partitions in arithmetic progression*, Integers 10 (2010), 73–82. DOI: 10.1515/INTEG.2010.007.

[7] M. A. Nyblom and C. Evans, *On the enumeration of partitions with summands in arithmetic progression*, Australas. J. Combin. 28 (2003), 149–159.
[8] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, 2021. https://oeis.org.

[9] F. J. de Vega, An extension of Furstenberg’s theorem of the infinitude of primes, JP J. Algebra Number Theory Appl. 53(1) (2022), 21-43. DOI: 10.17654/0972555522002.