Zeros of Dirichlet series with periodic coefficients

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Abstract

Let $a = (a_n)_{n \geq 1}$ be a periodic sequence, $F_a(s)$ the meromorphic continuation of $\sum_{n \geq 1} a_n/n^s$, and $N_a(\sigma_1, \sigma_2, T)$ the number of zeros of $F_a(s)$, counted with their multiplicities, in the rectangle $\sigma_1 < \Re s < \sigma_2$, $|\Im s| \leq T$. We extend previous results of Laurinčikas, Kaczorowski, Kučas, and Steuding, by showing that if $F_a(s)$ is not of the form $P(s)L_\chi(s)$, where $P(s)$ is a Dirichlet polynomial and $L_\chi(s)$ a Dirichlet L-function, then there exists an $\eta = \eta(a) > 0$ such that for all $1/2 < \sigma_1 < \sigma_2 < 1 + \eta$, we have $c_1T \leq N_a(\sigma_1, \sigma_2, T) \leq c_2T$ for sufficiently large $T$, and suitable positive constants $c_1$ and $c_2$ depending on $a$, $\sigma_1$, and $\sigma_2$.

1 Introduction

One of the most important open problems in mathematics is the

Conjecture. (Generalized Riemann Hypothesis) Every $L_\chi(s)$ function, associated with a Dirichlet character $\chi$, is zero-free in the open half-plane $\Re(s) > 1/2$.

In this paper, we enlarge the set of $L_\chi(s)$ to Dirichlet series with periodic coefficients, and investigate what can be shown in the opposite direction about zeros in $\Re(s) > 1/2$. For the distribution of zeros of these meromorphic functions in the whole complex plane, we refer to Chapter 11 of the book by Steuding [6].

Let $q$ be a positive integer. Let $H_q$ be the $q$-dimensional Hilbert space of Dirichlet series $\sum_{n \geq 1} a_n/n^s$, where $a = (a_n)_{n \geq 1}$ is a $q$-periodic sequence of complex numbers, and with the scalar product given by

$$\langle \sum_{n \geq 1} a_n/n^s, \sum_{n \geq 1} b_n/n^s \rangle = \sum_{n=1}^q a_n \overline{b_n}.$$  

It is well known that these Dirichlet series $\sum_{n \geq 1} a_n/n^s$ have a meromorphic continuation to the entire complex plane with at most one simple pole at $s = 1$. We shall denote this meromorphic continuation by $F_a(s)$. 



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Let $D^\text{pr}_q$ be the set of primitive characters that induce the Dirichlet characters modulo $q$. For $\psi$ in $D^\text{pr}_q$, we denote by $E_{q,\psi}$ the subspace of $H_q$ generated by the functions $L_{\psi}(s)/d^s$ where $d$ divides $q/\text{conductor}(\psi)$.

We denote by $N_F(\sigma_1, \sigma_2, T)$ (respectively $N'_F(\sigma_1, \sigma_2, T)$) the number of zeros of the function $F(s)$ in the rectangle $\sigma_1 < \text{Re} \ s < \sigma_2$, $|\text{Im} \ s| \leq T$, counted with their multiplicities (resp. without their multiplicities).

We begin with a structural theorem for $H_q$.

**Theorem 1.** Let $q$ be a positive integer.

(i) The functions $L_\chi(s)/d^s$, where $d$ runs through the divisors of $q$, and $\chi$ is a Dirichlet character modulo $q/d$, form an orthogonal basis of $H_q$.

(ii) We have the orthogonal decomposition

$$H_q = \bigoplus_{\psi \in D^\text{pr}_q} E_{q,\psi}$$

Thus every function $F_a(s)$ can be written in a unique way as

$$F_a(s) = \sum_{\psi \in D^\text{pr}_q} P_{\psi}(s)L_{\psi}(s)$$

where the $P_{\psi}(s)$ are Dirichlet polynomials that satisfy certain specific conditions. Ignoring these conditions for the moment, we get a much larger set of functions, for which we have the following result.

**Theorem 2.** Let $C$ be a finite set of at least two primitive Dirichlet characters, and let $(P_{\psi})_{\psi \in C}$ be a family of non-zero Dirichlet polynomials. Define

$$F(s) := \sum_{\psi \in C} P_{\psi}(s)L_{\psi}(s).$$

Then there exists a number $\eta = \eta(F) > 0$ such that, for all real numbers $\sigma_1$ and $\sigma_2$ with $1/2 \leq \sigma_1 < \sigma_2 \leq 1 + \eta$ and all sufficiently large $T$, we have

$$N'_F(\sigma_1, \sigma_2, T) \gg_{F,\sigma_1,\sigma_2} T.$$

For the upper bound for the number of zeros, we come back to the smaller set of Dirichlet series with periodic coefficients.

**Theorem 3.** Let $a = (a_n)_{n \geq 1}$ be a non-zero periodic sequence. Then

$$N_{F_a} \left( \frac{1}{2} + u, +\infty, T \right) \ll_a T \frac{\log(1/u)}{u}$$

for $0 < u \leq 1/2$ and $T \geq 1$.

Combining these three results, we finally get the result that motivated this paper.
Theorem 4. Let $q \geq 1$. Let $a = (a_n)_{n \geq 1}$ be a $q$-periodic sequence such that $\sum_{n \geq 1} a_n/n^s$ does not belong to one of the subspaces $E_{q,\psi}$, $\psi \in \mathcal{D}_{pr}^q$. Then there exists a number $\eta = \eta(a) > 0$ such that, for all real numbers $\sigma_1$ and $\sigma_2$ with $1/2 < \sigma_1 < \sigma_2 \leq 1 + \eta$ and all sufficiently large $T$, we have

$$N_{F_a}(\sigma_1, \sigma_2, T) \asymp_{a, \sigma_1, \sigma_2} N'_{F_a}(\sigma_1, \sigma_2, T) \asymp_{a, \sigma_1, \sigma_2} T.$$ 

Remark. It follows that, if $F_a(s)$ does not vanish in $\Re s > 1/2$, then

$$F_a(s) = P(s)L_\psi(s)$$

for some primitive Dirichlet character $\psi$ and some Dirichlet polynomial $P(s)$. Thus the functions $L_\psi(s)$ with $\psi \in \mathcal{D}_{pr}^q$ turn out to be a kind of “primitive function” for all those $F_a(s)$ with $q$-periodic $a = (a_n)_{n \geq 1}$, which conjecturally do not vanish in $\Re s > 1/2$.

We recall that the Dirichlet characters are exactly the arithmetic functions which are both periodic and completely multiplicative. What are the roles of these two properties for the Generalized Riemann Hypothesis (GRH)? What we find here about the zeros of Dirichlet series with periodic coefficients, confirms the commonly held idea that in any proof of GRH, the Euler Product, which comes from complete multiplicativity, must play a significant role.

Theorem 4 follows easily from the orthogonal basis of $q$-periodic sequences canonically associated with Dirichlet characters modulo $q$, which has been used, and perhaps discovered, by Codecà, Dvornicich, and Zannier [1], Lemma 1. The case $\eta = 0$ in Theorem 2 follows from [3], Theorem 2, of Kaczorowski and Kulas. They use the classical way to get zeros off the critical line, which is to apply a strong joint universal property for the Dirichlet L-functions. But, as far as we know, that method requires that one works within the strip $1/2 < \Re s < 1$. To get zeros in the half-plane $\Re s > 1$, we use here a kind of weak joint universal property for the Dirichlet L-functions. This leads us to add a new tool into the picture: the Brouwer fixed point theorem (see Lemma 2).

Theorem 3 is an explicit form of the upper bound of Steuding

$$(3) \quad N_{F_a}(1/2 + u, +\infty, T) \ll_{a, u} T.$$ 

More precisely, the proof of the slightly weaker $N_{F_a}(1/2 + u, +\infty, T) \ll_{a, u} T \log T$ appears in [5]. In [6], the upper bound (3) is stated in Theorem 11.3, but the proof is given only in the analog situation of the extended Selberg class. For the sake of completeness, we give here the details of the proof in our situation of Dirichlet series with periodic coefficients, and take the opportunity to make the dependence on $u$ explicit.

For Theorem 3, the lower bound $N_{F_a}(\sigma_1, \sigma_2, T) \gg T$ appears in the paper of Laurinčikas [4] with the condition $1/2 < \sigma_1 < \sigma_2 < 1$, and the restriction that the sequence $a$ be a linear combination of at least two Dirichlet characters modulo $q$. 

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2 Proof of Theorem 1

(i) For a Dirichlet character \( \chi \) modulo \( q/d \), we denote by \( \tilde{\chi} \) the arithmetic function defined by

\[
\tilde{\chi} = \chi \left( \frac{n}{d} \right)
\]

with the usual convention that \( \chi(t) = 0 \) if \( t \) is not a positive integer. By Lemma 1 of \([\Pi]\), the functions \( \tilde{\chi} \) form an orthogonal basis for the \( q \)-periodic sequences \((a_n)_{n \geq 1}\) with scalar product \( \langle a, b \rangle = \sum_{n=1}^{q} a_n b_n \). The result now follows from

\[
\sum_{n=1}^{+\infty} \frac{\tilde{\chi}(n)}{n^s} = \frac{L_\chi(s)}{d^s}, \quad \text{Re } s > 1.
\]

(ii) We are going to apply part (i) and a change of basis in each \( E_{q,\psi} \). Let \( \psi \) be a primitive Dirichlet character whose conductor \( m \) is a divisor of \( q \). For a Dirichlet character \( \chi \) modulo \( q/d \) induced by \( \psi \) we have

\[
\frac{L_\chi(s)}{d^s} \psi(s) = \frac{1}{d^s} \prod_{p|q} \left( 1 - \frac{\psi(p)}{p^s} \right) = \frac{1}{d^s} \prod_{p|m} \left( 1 - \frac{\psi(p)}{p^s} \right) = \frac{1}{q'} b^s \prod_{p|b} \left( 1 - \frac{\psi(p)}{p^s} \right)
\]

where \( q' := \frac{q}{m} \) and \( b := \frac{q'}{d} \).

By part (i) of the theorem, we thus have the orthogonal sum

\[
H_q = \bigoplus_{\psi \in D^*_q} \frac{L_\psi(s)}{q'^s} \cdot V_{q,\psi},
\]

where

\[
V_{q,\psi} = \text{Vect} \left\{ b^s \prod_{p|b} \left( 1 - \frac{\psi(p)}{p^s} \right) : b|q' \right\}
\]

\[
= \text{Vect} \left\{ \left( \prod_{i=1}^{r} x_i^{\beta_i} \right) \left( \prod_{\beta_i \geq 1} \left( 1 - \frac{\psi(p_i)}{x_i} \right) \right) : 0 \leq \beta_i \leq \alpha_i \right\}
\]

with

\[
q' = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \quad \text{and} \quad x_i := p_i^r.
\]

We order the two families

1. the free family of \( \mathbb{C}[x_1, \ldots, x_r] : \left( \prod_{i=1}^{r} x_i^{\beta_i} \right)_{0 \leq \beta_i \leq \alpha_i} \)
2. the family \( \prod_{i=1}^{r} x_i^{\beta_i} \left( \prod_{\beta_i \geq 1} \left( 1 - \frac{\psi(p_i)}{x_i} \right) \right) \) according to the order on the \( \beta = (\beta_i) \) given by

\[
\beta < \beta' \iff \sum_{i=1}^{r} \beta_i < \sum_{i=1}^{r} \beta'_i \quad \text{or} \quad \sum_{i=1}^{r} \beta_i = \sum_{i=1}^{r} \beta'_i \quad \text{and there is a } j \text{ with} \quad \beta_j < \beta'_j
\]

We observe that the second family is then the image of the first family under an upper triangular matrix with ones on the diagonal. Thus

\[
\frac{V_{q,\psi}}{q^s} = \frac{1}{q^s} \text{Vect}\{d^s : d|q'\} = \text{Vect}\left\{ \frac{1}{d^s} : d|q' \right\},
\]

and the result follows from (4).

### 3 Preparation for Theorem 2

In the following two lemmas, we use the notation

\[
D_n(R) := \{ z = (z_j)_{1 \leq j \leq n} \in \mathbb{C}^n : |z_j| \leq R \text{ for all } 1 \leq j \leq n \}.
\]

**Lemma 1.** Let \( q \) be a positive integer, and \( y \) and \( R \) be positive real numbers. Let \( \chi_1, \ldots, \chi_n \) be pairwise distinct Dirichlet characters modulo \( q \). Then there exists a real \( \eta > 0 \) such that for all fixed \( \sigma \) with \( 1 < \sigma \leq 1 + \eta \), and for all prime numbers \( p > y \), there exists a continuous function \( t_p : D_n(R) \to \mathbb{R} \), such that for all \( z \in D_n(R) \)

\[
z = \left( \sum_{p > y} \frac{\chi_j(p)}{p^{\sigma + it_p(z)}} \right)_{1 \leq j \leq n}.
\]

**Remark.** We can interpret this lemma as a linear system to be solved. There are \( n \) equations. The unknowns are the infinite family of \( (p^{-it_p})_{p > y} \) that must be chosen in the unit circle. The \( z \in \mathbb{C}^n \) is a parameter. Moreover, the solution must be chosen continuously in the parameter \( z \).

**Proof.** If \( n < \varphi(q) \), we extend \( (\chi_j)_{1 \leq j \leq n} \) to \( (\chi_j)_{1 \leq j \leq \varphi(q)} \), using all the Dirichlet characters modulo \( q \). This allows us to restrict the proof to the case \( n = \varphi(q) \).

We denote by \( C \) the unitary matrix of the characters modulo \( q \). That is,

\[
C := (\chi_j(a))_{1 \leq a \leq q, (a,q)=1}^{1 \leq j \leq \varphi(q)}
\]
We have
\[ \sum_{p>y} \frac{\chi_j(p)}{p^{\sigma+it_p}} = \sum_{1 \leq a \leq q} \chi_j(a) \sum_{p>y \atop p \equiv a(q)} \frac{1}{p^{\sigma+it_p}}. \]

To change variables we write
\[ z = Cw, \]
where
\[ z = (z_j)_{1 \leq j \leq \varphi(q)} \quad \text{and} \quad w = (w_a)_{1 \leq a \leq q, \atop (a,q) = 1}, \]
and
\[ \theta_p = - (\log p)(t_p \circ C). \]

To prove the lemma, it is sufficient to solve the system
\[ \sum_{p>y \atop p \equiv a(q)} e^{i \theta_p} p^{\sigma} = w_a, \quad 1 \leq a \leq q, \ (a,q) = 1, \]
in the real unknowns \((\theta_p)_{p>y},\) continuously in \(w \in D_{\varphi(q)}(\|C^{-1}\|_\infty R).\)

We put
\[ S_a = S_a(q,y,\sigma) := \sum_{p>y \atop p \equiv a(q)} \frac{1}{p^{\sigma}}. \]

Using the prime number theorem for arithmetic progressions, we readily find that there exists an \(\eta > 0,\) such that for each \(1 < \sigma \leq 1 + \eta \) and \(1 \leq a \leq q, \ (a,q) = 1,\) we have
\[ S_a \geq 10\|C^{-1}\|_\infty R, \]
and there exist prime numbers \(p_{1,a}\) and \(p_{2,a}\), such that
\[ \frac{1}{3} \leq \lambda_0 := \frac{1}{S_a} \sum_{y<p \leq p_{1,a} \atop p \equiv a(q)} \frac{1}{p^{\sigma}} \leq \frac{1}{3} + \frac{1}{100} \]
and
\[ \frac{1}{3} \leq \lambda_1 := \frac{1}{S_a} \sum_{p_{1,a} < p \leq p_{2,a} \atop p \equiv a(q)} \frac{1}{p^{\sigma}} \leq \frac{1}{3} + \frac{1}{100} \]

We also write
\[ \lambda_2 := \frac{1}{S_a} \sum_{p>p_{2,a} \atop p \equiv a(q)} \frac{1}{p^{\sigma}}, \]
such that
\[ \lambda_0 + \lambda_1 + \lambda_2 = 1. \]
We choose
\[
\theta_p = \begin{cases} 
0 & \text{if } y < p \leq p_{1,a} \\
\pi + u_1 & \text{if } p_{1,a} < p \leq p_{2,a} \\
\pi - u_2 & \text{if } p_{2,a} < p
\end{cases}
\]
with $u_1$ and $u_2$ to be fixed later. In view of (5) it is sufficient to solve, for each $a$, the equation
\[
(7) \quad \lambda_1 e^{iu_1} + \lambda_2 e^{-iu_2} = \lambda_0 - \frac{w_a}{S_a}
\]
in the real unknowns $u_1$ and $u_2$, continuously in $w_a$ for $|w_a| \leq \|C^{-1}\|_\infty R$. We define the function $F$ by
\[
F : \left[0, \frac{\pi}{2}\right]^2 \to \mathbb{C} \\
(u_1, u_2) \mapsto \lambda_1 e^{iu_1} + \lambda_2 e^{-iu_2}.
\]
$F$ is a diffeomorphism onto its image. Moreover, since $\frac{1}{3} \leq \lambda_0, \lambda_1 \leq \frac{1}{3} + \frac{1}{100}$, and $\frac{1}{3} - \frac{1}{50} \leq \lambda_2 \leq \frac{1}{3}$, we have
\[
\left\{ s \in \mathbb{C} : |s - \lambda_0| \leq \frac{1}{10} \right\} \subset \text{Im } F,
\]
as illustrated in the following figure.

Figure 1: The image of $F$, depicted by the region with the dotted boundary, contains the disk with center $\lambda_0$ and radius $\frac{1}{10}$.

Thus by (6) we can solve (7) continuously in $w_a$. This concludes the proof of Lemma 1. \hfill \square
Lemma 2. Let $q$ and $L$ be positive integers, and $R \geq 1$ be real. Let $\chi_1, \ldots, \chi_n$ be pairwise distinct Dirichlet characters modulo $q$. For all $1 \leq j \leq n$, let $h_j$ be a non-zero rational function in $L$ complex variables. Then there exists a real $\eta > 0$ such that, for all $\sigma$ with $1 < \sigma \leq 1 + \eta$, we have

$$\{ z \in \mathbb{C}^n : \frac{1}{R} \leq |z_j| \leq R \} \subset \left\{ h_j \left( \frac{1}{p_1^{\sigma+it_p}}, \ldots, \frac{1}{p_L^{\sigma+it_p}} \right) \prod_{p > p_L} \left( 1 - \frac{\chi_j(p)}{p^{\sigma+it_p}} \right)^{-1} : t_p \in \mathbb{R} \right\}$$

Proof. We first consider the particular case where all the $h_j$ are 1. We put $y = p_L$ and $R' = \pi + \log R$. Applying Lemma 1 (and changing the letter $z$ to $w$) we have continuous functions $t_p$ such that

$$w_j = \sum_{p > y} \frac{\chi_j(p)}{p^{\sigma+it_p(w)}}, \quad w \in D_n(1 + R'), \quad 1 \leq j \leq n.$$

We define the error term $E$ by

$$E = (\sum_{p > y} \log \left( 1 - \frac{\chi_j(p)}{p^{\sigma+it_p}} \right))_{1 \leq j \leq n} = \left( -\sum_{p > y} \frac{\chi_j(p)}{p^{\sigma+it_p}} \right)_{1 \leq j \leq n} + E ((t_p)_{p > y}).$$

The real number $\sigma > 1$ being fixed, the function $E$ is continuous for the weak-convergence topology. Moreover, for all $j$ and all $(t_p)_{p > y}$, we have

$$|E_j ((t_p)_{p > y})| \leq \sum_p \frac{1}{p^2} < 1.$$

Let $z \in D_n(R')$ be fixed. From (10) we see that, for all $j$ and all $(t_p)_{p > y}$,

$$|z_j + E_j ((t_p)_{p > y})| \leq 1 + R'.$$

Thus we have the following continuous function.

$$F : D_n(1 + R') \rightarrow D_n(1 + R')$$

$$w \mapsto z + E ((t_p(w))_{p > y})$$

The Brouwer fixed point theorem shows that there exists a $w \in D_n(1 + R')$ such that $F(w) = w$. Together with (8) and (9) this yields

$$\left( -\sum_{p > y} \log \left( 1 - \frac{\chi_j(p)}{p^{\sigma+it_p(w)}} \right) \right)_{1 \leq j \leq n} = z.$$

Taking exponentials allows us to conclude the case when $h_j \equiv 1$. 
We now consider the case with a general \( h \). Let us choose \((t_{p_1}, \ldots, t_{p_L})\) such that for all \( j \), \( h_j \left( \frac{1}{p_1^{\sigma + it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma + it_{p_L}}} \right) \) has neither zeros nor poles for \( 1 \leq \sigma \leq 2 \). We put

\[
\begin{align*}
   c := \min_{1 \leq j \leq n} \min_{1 \leq \sigma \leq 2} |h_j \left( \frac{1}{p_1^{\sigma + it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma + it_{p_L}}} \right)|,
   
   C := \max_{1 \leq j \leq n} \max_{1 \leq \sigma \leq 2} |h_j \left( \frac{1}{p_1^{\sigma + it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma + it_{p_L}}} \right)|.
\end{align*}
\]

Applying the particular case where \( h_j \equiv 1 \) with \( \tilde{R} = \max \left( \frac{C}{\pi}, \frac{R}{c} \right) \) allows us to conclude the general case.

4 Proof of Theorem 2

If \( \sigma_1 < 1 \) then \( N_p(\sigma_1, \sigma_2, T) \gg_{\sigma_1, \sigma_2} T \) by Theorem 2 of [M]. We may thus restrict our attention to the case \( \sigma_1 \geq 1 \).

We choose \( q \) to be the least common multiple of the conductors of the \( \psi \) in \( C \), and we write \( C = \{\psi_1, \ldots, \psi_n\} \) with \( 2 \leq n \leq \varphi(q) \). We use the notation

\[
F_j(s) = P_{\psi_j}(s) L_{\psi_j}(s)
\]

and

\[
P_{\psi_j}(s) = \sum_{k \geq 1} \frac{c_{j,k}}{k^s}
\]

We choose \( y = p_L \) such that if \( p \) divides a \( k \) for which there is a \( j \) such that \( c_{j,k} \neq 0 \), then \( p \leq y \). Denoting by \( \chi_j \) the Dirichlet character modulo \( q \) that is induced by \( \psi_j \) we can thus write

\[
F_j(s) = h_j \left( \frac{1}{p_1^{\sigma + it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma + it_{p_L}}} \right) \prod_{p > p_L} \left( 1 - \frac{\chi_j(p)}{p^s} \right)^{-1}
\]

where \( h_j \) is a nonzero rational function such that

\[
(11) \quad h_j \quad \text{has no poles in} \quad \{(z_1, \ldots, z_L) \in \mathbb{C}^L : |z_i| < 1\}.
\]

Choosing \( R = 1 \) we get by Lemma 2 a real \( \eta > 0 \), which will be the one we use for Theorem 2. Let \( \sigma_1 \) and \( \sigma_2 \) be real numbers such that \( 1 \leq \sigma_1 < \sigma_2 \leq 1 + \eta \). We choose

\[
\sigma = \frac{\sigma_1 + \sigma_2}{2}.
\]

By Lemma 2 there is a sequence \((t_p)_p\) of real numbers such that for all \( j \), \( 1 \leq j \leq n \),

\[
\begin{align*}
   h_j \left( \frac{1}{p_1^{\sigma + it_{p_1}}}, \ldots, \frac{1}{p_L^{\sigma + it_{p_L}}} \right) \prod_{p > p_L} \left( 1 - \frac{\chi_j(p)}{p^{\sigma + it_{p}}} \right)^{-1} = e^{2i\pi j/n}
\end{align*}
\]
We write

\[ G_j(s) := h_j \left( \frac{1}{p_1^{s+it_p}}, \ldots, \frac{1}{p_L^{s+it_p}} \right) \prod_{p \leq p_L} \left( 1 - \frac{\chi_j(p)}{p^{s+it_p}} \right)^{-1}. \]

As \( n \geq 2 \), we have

\[ (12) \quad \sum_{j=1}^{n} G_j(\sigma) = 0. \]

We now choose a circle \( C = C(\sigma, r) \) centered at \( \sigma = \frac{\sigma_1 + \sigma_2}{2} \) and with a radius \( r \)
with \( 0 < r < \frac{2\sigma_2 - \sigma_1}{2} \), such that \( \sum_{j=1}^{n} G_j(s) \) does not vanish on \( C \). We write

\[ \gamma := \min_{s \in C} \left| \sum_{j=1}^{n} G_j(s) \right| > 0. \]

Because of \( (11) \) and the uniform convergence of the infinite products, we can choose a prime number \( p_M \geq p_L \) such that for all \( j, 1 \leq j \leq n, \)

\[ \left| F_j(z) - h_j \left( \frac{1}{p_1^{s+it_p}}, \ldots, \frac{1}{p_L^{s+it_p}} \right) \prod_{p_L < p \leq p_M} \left( 1 - \frac{\chi_j(p)}{p^{s+it_p}} \right)^{-1} \right| < \frac{\gamma}{3n}, \quad \text{Re} \ z \geq \sigma - r, \]

and

\[ \left| G_j(s) - h_j \left( \frac{1}{p_1^{s+it_p}}, \ldots, \frac{1}{p_L^{s+it_p}} \right) \prod_{p_L < p \leq p_M} \left( 1 - \frac{\chi_j(p)}{p^{s+it_p}} \right)^{-1} \right| < \frac{\gamma}{3n}, \quad \text{Re} \ s \geq \sigma - r. \]

By Weyl’s criterion, we know that the set \( \{p_1^t, \ldots, p_M^t\} \) is uniformly distributed in \( \{z : |z| = 1\}^M \). Using \( (11) \) once more it follows that the set of \( t \in \mathbb{R} \), such that for all \( s \) with \( |s - \sigma| \leq r \) and all \( j, 1 \leq j \leq n, \)

\[ \left| h_j \left( \frac{1}{p_1^{s+it}}, \ldots, \frac{1}{p_L^{s+it}} \right) \prod_{p_L < p \leq p_M} \left( 1 - \frac{\chi_j(p)}{p^{s+it}} \right)^{-1} \right| < \frac{\gamma}{3n}. \]

has positive lower density. For these real \( t \), we have thus

\[ \max_{s \in C} \left| \sum_{j=1}^{n} F_j(s + it) - G_j(s) \right| < \gamma = \min_{s \in C} \left| \sum_{j=1}^{n} G_j(s) \right|. \]

As \( \sum_{j=1}^{n} G_j(\sigma) = 0 \) (formula \( (12) \)), it follows by Rouche’s theorem that \( F(s + it) = \sum_{j=1}^{n} F_j(s + it) \) has at least one zero in \( |s - \sigma| < r \). By the positive lower density of these \( t \), we conclude that \( N_F(\sigma_1, \sigma_2, T) \gg_{F, \sigma_1, \sigma_2} T \) for sufficiently large \( T \).
5 Proof of Theorem 3

We give here only the upper bound for the number \( N^+_a(1/2 + u, +\infty, T) \) of zeros in \( \frac{1}{2} + u < \Re s < +\infty, 0 \leq \Im s \leq T \). The proof is similar for zeros with negative real part.

Let \( \zeta(s, r) \) denote the Hurwitz zeta function. From Theorem 1 of [2] we have, for \( 1/2 < \sigma < 1 \),

\[
\int_0^T |F_a(\sigma + it)|^2 dt = \frac{T}{q^{2\sigma}} \sum_{j=1}^q |a_j|^2 \zeta(2\sigma, j/q) + O \left( \frac{q^{2-2\sigma} T^{2\sigma - 2} \sum_{j=1}^q |a_j|^2}{(2\sigma - 1)(1 - \sigma)} \right)
\]

\[
= O_a \left( \frac{T}{(2\sigma - 1)(1 - \sigma)} \right),
\]

since \( \zeta(2\sigma, r) = O_e((2\sigma - 1)^{-1}) \). By Jensen’s inequality,

\[
\int_0^T \log |F_a(\sigma + it)| dt \leq \frac{T}{2} \log \left( \frac{1}{T} \int_0^T |F_a(\sigma + it)|^2 dt \right) = O_a(T \log(1/u)),
\]

for \( \sigma = (1 + u)/2 \), according to (13).

Let \( a_{m} \) be the first nonzero term of the sequence \( (a_n)_{n\geq1} \), and let \( c \geq 2 \) be large enough such that, for \( \Re (s) \geq c \), we have \( F_a(s) = \frac{a_{m}}{m^s} (1 + \theta(s)) \) with \( |\theta(s)| \leq 1/2 \). We apply Littlewood’s lemma (see [7], Section 3.8) to the rectangle \( R \) with vertices \( c + i, c + iT, (1 + u)/2 + iT, (1 + u)/2 + i \), to get

\[
2\pi \sum_{\beta>(1+u)/2 \atop 1<\gamma\leq T} (\beta - (1 + u)/2) = \int_1^T \log |F_a((1+u)/2 + it)| dt - \int_1^T \log |F_a(c + it)| dt
\]

\[
+ \int_{(u+1)/2}^c \arg F_a(\sigma + iT) d\sigma - \int_{(u+1)/2}^c \arg F_a(\sigma + i) d\sigma.
\]

The second integral is clearly \( O_a(T) \) since \( \log |F_a(c + it)| \ll 1 \). Steuding shows on page 302 of [5] that \( |\arg(1 + \theta(s + iT))| \ll \log T \), if \( \sigma \) is from a bounded interval. Thus the third integral is \( O_a(T) \). The last integral is bounded. Together with (14) this shows that

\[
\sum_{\beta>(1+u)/2 \atop 1<\gamma\leq T} (\beta - (1 + u)/2) = O_a(T \log(1/u)).
\]

The desired bound now follows from

\[
\frac{u}{2} N^+_a(1/2 + u, +\infty, T) = \sum_{\beta>1/2+u} \frac{u}{2} \leq \sum_{\beta>(1+u)/2} (\beta - (1 + u)/2) \ll_a T \log(1/u).
\]
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