Modeling and Analysis of Micro-cantilever Plate Piezoelectric Energy Harvester with a Tip Mass

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Abstract. With the development of MEMS (Micro-electro-mechanical Systems) techniques, research interest has deepened in micro piezoelectric harvesting device, which can supply electrical power for wireless sensors from ambient vibration. This paper proposes a nonlinear analysis of a micro-scale PEH (piezoelectric energy harvester), which is modeled by a micro-cantilever plate with two layers and a tip mass attached to the free end. Considering size effect and nonlinear curvature, based on strain gradient and inextensible plate theory, nonlinear dynamic equations of the micro cantilever piezoelectric energy harvester with tip mass are established by virtue of the Hamilton's principle. The maximum output voltage of piezoelectric energy harvester with different tip mass is calculated respectively using MATLAB software. Comparative analysis confirms that an increase of $K$ (the mass ratio of the tip mass to piezoelectric plate) reduces the natural frequency of energy harvester and significantly enhances the nonlinear phenomenon. In addition, when the mass ratio $K$ exceeds 1, obvious superharmonic resonance phenomena generates, which considerably improves the output voltage of the proposed PEH in a lower frequency band. This study proposes an approach to enhance the output performance of micro-cantilever plate energy harvesters and broaden the bandwidth of PEH to adapt low-frequency ambient vibration.

Keywords: Micro-cantilever plate, Piezoelectric energy harvester, Strain gradient, Nonlinear

1. Introduction

In recent years, MEMS technique has developed rapidly. Many researchers have focused on micro-size energy harvester, which has the advantages of long service life and low cost [1]. The additional tip mass is an effective way to improve the performance of the energy harvester, which can significantly increase the amplitude and generate more electricity through the electro-mechanical coupling effect of piezoelectric materials. In addition, the tip mass has the function of adjusting the natural frequency of the structure to adapt to a wide range of external excitation environments [2].

Although many researchers have studied Piezoelectric energy harvesters with cantilever plate structure [3,4], there is still little work to analyze the effects of size and nonlinear dynamic behaviors for micro-plate piezoelectric energy harvester based on strain gradient theory. Therefore, this paper proposes the model of the micro-cantilever plate PEH with tip mass. Then the natural frequencies, phase figures, bifurcation diagrams and frequency-output voltage curves are given to discuss the
2. Dynamic model of the harvester

$\mathbf{x, y, z}$ position on the coordinate axis

$u, v, w$ displacement in the x, y, and z directions

$\kappa_{ij}$ nonlinear curvature of mid-plane

$\varepsilon_{xx}, \varepsilon_{yy}, \gamma_{xy}, \gamma_{yx}$ strain in plate

$\sigma_{xx}', \sigma_{yy}', \tau_{xy}', \tau_{yx}'$ stress in the base layer

$\sigma_{xx}^p, \sigma_{yy}^p, \tau_{xy}^p, \tau_{yx}^p$ stress in the piezoelectric layer

$h', h^p$ thicknesses of the base layer and piezoelectric layer

$a', a^p$ lengths of the base layer and the piezoelectric layer

$b$ width of the plate

$m_t$ mass of the tip mass

$R$ total resistance in the circuit

$w_0$ amplitude of base excitation

$\Omega$ frequency of base excitation

$l$ material scale parameter according to Ramezani [5]

$\rho'$ density of base layer

$\rho^p$ density of piezoelectric material

Structure diagram of the piezoelectric energy harvester is shown in figure.1, which consists of two layers, the lower is the base cantilever plate and the upper is the PVDF piezoelectric material.

Figure 1. schematic diagram of PEH.

2.1. Potential energy

According to the nonlinear curvature is given by Dowell [6], The strain field is represented as:

$$
\varepsilon_{xx} = -z \frac{\partial^2 w}{\partial x^2} \left[1 + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2\right]
$$

$$
\varepsilon_{yy} = -z \frac{\partial^2 w}{\partial y^2} \left[1 + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2\right]
$$

(1)

$$
\gamma_{xy} = \gamma_{yx} = -2z \frac{\partial^2 w}{\partial x \partial y} \left[1 + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2\right]
$$

And the stress field of the basic layer and piezoelectric layer can be obtained on the basis of the constitutive equation respectively:
\[
\begin{align*}
\sigma_{xx}^\epsilon &= \frac{E_s}{1-v^2} (\varepsilon_{xx} + v \varepsilon_{yy}) \\
\sigma_{yy}^\epsilon &= \frac{E_s}{1-v^2} (\varepsilon_{yy} + v \varepsilon_{xx}) \\
\tau_{xy}^\epsilon &= \frac{E_s}{2(1+v)} \cdot \gamma_{xy} = \frac{E_s}{2(1+v)} \cdot \gamma_{xy} \\
\sigma_{xx}^p &= \frac{E_p}{1-v_p^2} (\varepsilon_{xx} + v_p \varepsilon_{yy}) + d_{31} \cdot D_3 \\
\sigma_{yy}^p &= \frac{E_p}{1-v_p^2} (\varepsilon_{yy} + v_p \varepsilon_{xx}) + d_{32} \cdot D_3 \\
\tau_{xy}^p &= \frac{E_p}{2(1+v_p)} \cdot \gamma_{xy} = \frac{E_p}{2(1+v_p)} \cdot \gamma_{xy} \\
\end{align*}
\]

where \(D\) represents the tensor of electric displacement and \(d\) is the electrical polarization coupling coefficient.

The strain energy of the plate can be expressed as according to continuum mechanics theory:

\[
\mathcal{U}^s = \iiint \left( \sigma_{ij}^\epsilon E_{ij} + \sigma_{ij}^p E_{ij} + \tau_{ij}^\epsilon \gamma_{ij} \right) dx dy dz \\
\tag{4}
\]

Considering the size effect of micro-size plate, the strain gradient \(\eta\) can be expressed as:

\[
\eta_{ijk} = \eta_{jik} = \epsilon_{jik} \quad (i = x, y, z \quad j, k = x, y) \\
\tag{5}
\]

The expressions of strain energy density and corresponding strain energy based on strain gradient are given by Mindlin [6]:

\[
\begin{align*}
\dot{W}^s &= L_1^s \eta_{xxx} \eta_{xxx} + L_2^s \eta_{xyy} \eta_{xyy} + L_3^s \eta_{zzz} \eta_{zzz} + L_4^s \eta_{xx} \eta_{yy} + L_5^s \eta_{yy} \eta_{yy} + L_6^s \eta_{zz} \eta_{zz} \\
\dot{U}^s &= \iiint \dot{W}^s dx dy dz \\
\tag{6}
\end{align*}
\]

where \(L_i (i = 1, 2, 3, 4, 5)\) are material parameters given by Ramezani [5]:

\[
\begin{align*}
L_1^s &= \frac{1}{2} \frac{E_s v_s}{(1-2v_s)(1+v_s)} \\
L_2^s &= \frac{1}{2} \frac{E_s}{2(1+v_s)} \quad (\zeta = s, p) \\
L_4^s &= L_5^s = L_6^s = 0 \\
\end{align*}
\]

According to strain gradient theory, one obtains the expression of Potential energy of micro cantilever plate PEH:

\[
U = \mathcal{U}^s + \mathcal{U}^p + \dot{U}^s + \dot{U}^p + \frac{1}{2} E_s \cdot D_3 \\
\tag{8}
\]

where:

\[
E_s = a_{33} \cdot D_3 + d_{31} \cdot \varepsilon_{xx} + d_{32} \cdot \varepsilon_{yy} = \frac{V_1}{h_p^3} \\
\tag{9}
\]

\(\alpha_{33}\) refer to the dielectric constant; \(V_1\) is the out-put voltage.

2.2. Kinetic energy

Kinetic energy of the system consists of three parts:

\[
T = T^s + T^p + T_0 \\
\tag{10}
\]
where $T'$ and $T''$ represent the kinetic energy of base layer and piezoelectric layer respectively, and $T_2$ represents kinetic energy of the additional mass.

\[
T' = \iiint_V \frac{\rho'}{2} \frac{\partial w}{\partial t} - w_0 \Omega \cos(\Omega t) \, dV'
\]

\[
T'' = \iiint_V \frac{\rho''}{2} \left( \frac{\partial w}{\partial t} - w_0 \Omega \cos(\Omega t) \right)^2 \, dV''
\]

\[
T_o = \frac{\rho}{2} \left( \frac{\partial w}{\partial t} \bigg|_{t=0} - w_0 \Omega \cos(\Omega t) \right)^2
\]

(11)

2.3. Hamilton principle

Virtual work done by damping can be written as:

\[
\partial W_r = -\int_v r' \cdot \frac{\partial w}{\partial t} \cdot \delta wdV' - \int_v r'' \cdot \frac{\partial w}{\partial t} \cdot \delta wdV''
\]

(13)

Substitute the equation (8), and equation (10)- equation (13) into the equation of Hamilton principle and rearrange:

\[
\int_v (\sigma_e \delta e_x + \tau_{\theta} \delta \eta_{\theta}) \, dV' - \frac{1}{4} \int m \left( \frac{\partial^2}{\partial t^2} w \bigg|_{t=0} - w_0 \Omega^2 \sin(\Omega t) \right)
\]

\[
-\int_v \rho' \left( \frac{\partial^2 w}{\partial t^2} - w_0 \Omega^2 \sin(\Omega t) \right) \, dV'' + \int_v (\sigma_e \delta e_x + \tau_{\theta} \delta \eta_{\theta}) \, dV''
\]

\[
-\int_v r' \cdot \frac{\partial w}{\partial t} \, \delta wdV' - \int_v r'' \left( \frac{\partial^2 w}{\partial t^2} - w_0 \Omega^2 \sin(\Omega t) \right) \, \delta wdV''
\]

\[
-\int_v r' \cdot \frac{\partial w}{\partial t} \, \delta wdV'' = 0
\]

(14)

2.4. Electrical equations

Electric displacement can be expressed by equation (9):

\[
D_i = \frac{E_3 - (d_{31} \cdot \varepsilon_x + d_{32} \cdot \varepsilon_y)}{a_{33}}
\]

(15)

According to Kirchhoff’s law, the electrical governing equation can be obtained as:

\[
\frac{S_i}{a_{33}h^2} \frac{V_i}{R} + \frac{1}{a_{33}h^2} \int_v (d_{31} \cdot \varepsilon_{11} + d_{32} \cdot \varepsilon_{22}) \, dV'' = 0
\]

(16)

2.5. Establishment of model

Modal function of plate is regarded as a combination of fixed-free beam function $X_i(x)$ in $x$ direction and free-free beam function $Y_j(y)$ in $y$ direction, and the third order dispersion of deflection is described as:

\[
w(x, y, t) = w_1(t) \cdot X_i(x) \cdot Y_j(y) + w_2(t) \cdot X_i(x) \cdot Y_j(y) + w_3(t) \cdot X_2(x) \cdot Y_j(y)
\]

(17)

where the first two order of $X_i(x)$ and $Y_j(y)$ is given as follows:

\[
X_i = \sin \frac{1.9x}{a} - \sinh \frac{1.9x}{a} - 0.7(\cosh \frac{1.9x}{a} - \cos \frac{1.9x}{a})
\]

\[
X_j = \sin \frac{4.7x}{a} - \sinh \frac{4.7x}{a} - (\cosh \frac{4.7x}{a} - \cos \frac{4.7x}{a})
\]

\[
Y_j = 1
\]

\[
Y_j = (3) \left( \frac{1-2y}{b} \right)
\]

(18)

(19)
The expansion expression of the deflection $w$ (equation (17)) is applied to the equations of the theoretical dynamics model. Dynamic ordinary differential equations can be derived by equation (14) and electrical ordinary differential equation can be obtained by equation (16). Then the generalized coordinate $w_1, w_2, w_3$ and $V_1$ can be solved using equation (20).

$$
\ddot{w}_1 = a_1 w_1 + a_2 w_1^3 + a_3 w_2 + a_4 w_1 w_2 + a_5 w_1 w_3 + a_6 w_2 w_3 + a_7 w_1 + a_8 V_1 \\
+ a_9 w_1^2 V_1 + a_{10} w_2^2 V_1 + a_{11} w_3^2 V_1 + a_{12} w_1 \sin (\Omega t) \\
\ddot{w}_2 = b_1 w_2 + b_2 w_1 + b_3 w_1^3 + b_4 w_2 + b_5 w_1 w_2 + b_6 w_1 w_3 + b_7 w_2 w_3 + b_8 w_1 + b_9 V_1 \\
+ b_{10} w_1^2 V_1 + b_{11} w_2^2 V_1 + b_{12} w_3^2 V_1 + b_{13} w_1 \sin (\Omega t) \\
\ddot{w}_3 = c_1 w_3 + c_2 w_1^3 + c_3 w_2 + c_4 w_1 w_2 + c_5 w_1 w_3 + c_6 w_2 w_3 + c_7 w_1 + c_8 V_1 \\
+ c_{19} w_1^2 V_1 + c_{20} w_2^2 V_1 + c_{21} w_3^2 V_1 + c_{22} w_1 \sin (\Omega t) \\
\ddot{V}_1 = d_1 w_1 + d_2 w_1^3 + d_3 w_2 + d_4 w_1 w_2 + d_5 w_1 w_3 + d_6 w_2 w_3 + d_7 w_1 + d_8 V_1 \\
+ d_{10} w_1^2 V_1 + d_{11} w_2^2 V_1 + d_{12} w_3^2 V_1 + d_{13} \dot{w}_1 + d_{14} \dot{V}_1
$$

where parameters $a_i, b_i, c_i, d_i$ $(i = 0, 1, ..., 17)$ are the coefficients about $w_1, w_2, w_3$ and $V_1$ after integrating equation (14) and equation (16), which relate to geometric dimensions and material parameters. Because the expanded formula is complex and space is limited, letters are used instead of specific expansion expressions to make the form of the final expression more intuitive.

3. Results and discussion

Parameters of the device are given in Table 1.

| Table 1. Parameters of the structure. |
|--------------------------------------|
| Physical parameter | Value | Physical parameter | Value |
|---------------------|-------|---------------------|-------|
| $a_s^s$             | 450 μm | $E_s^s$             | 185 Gpa |
| $a_p^s$             | 225 μm | $E_p^s$             | 2 Gpa  |
| $b$                 | 2225 μm | $\nu^s$            | 0.29   |
| $h_s^s$             | 1.2 μm | $\nu^p$            | 0.36   |
| $h_p^s$             | 0.6 μm | $R$                | $10^6 \Omega$ |

Based on Rayleigh-Ritz method, the natural frequencies of structures with different mass ratios are obtained as follows and $K$ refer to mass ratio of the additional tip mass to piezoelectric plate.

| Table 2. Natural frequencies of structure. |
|-------------------------------------------|
| Natural frequency (KHz) | $K = 0$ | $K = 1$ | $K = 2$ | $K = 3$ |
|--------------------------|---------|---------|---------|---------|
| First order              | 9.928   | 4.501   | 3.344   | 2.762   |
| Second order             | 41.72   | 21.82   | 16.52   | 13.84   |
| Third order              | 57.54   | 43.31   | 42.53   | 42.27   |

From Table 2, it can be seen that the natural frequency of the structure will decrease accordingly with the tip mass of the cantilever plate increasing, and the reduction of the first order natural frequency is more obvious than higher order ones. When $K = 3$, the first order natural frequency decreases by 72%, and the third order natural frequency decreases by 27%.
Figure 2 shows the comparison of the maximum output voltage of PEH between different mass ratios, and the abscissa represents the range of external excitation frequency. Figure 3 illustrates the relationship between nonlinear behavior and linear behavior. Figure 4 presents the nonlinear phenomenon of the proposed structure through phase diagram and bifurcation.

**Figure 2.** Maximum output voltage for different mass ratio.

**Figure 3.** Natural frequency and maximum voltage for superharmonic and primary resonance.

**Figure 4.** For $\kappa = 3$: (A) phase diagram at 1040 Hz; (B) bifurcation diagram.

When the ambient vibration frequency $\Omega$ is equal to the first order natural frequency of the PEH $p_1$, primary resonance occurs. When $3\Omega = p_1$, three times superharmonic resonance with nonlinear characteristics occurs. As is presented in figure 3, when the mass ratio $\kappa < 1$, only the primary resonance occurs and there is no superharmonic resonance. The result can be interpreted as the stiffness of the structure is relatively large, which accordingly makes the deflection is small and the nonlinear phenomenon is not obvious. When the mass ratio $\kappa \geq 2$, the phenomenon of three times superharmonic resonance becomes more obvious with the increasing of additional mass, which
enables the proposed PEH to output high voltage in a lower frequency \((p/3)\) and expands the frequency bandwidth of the energy harvester.

Through the analysis of simulation results above, we can find that the improvement of the performance of the micro-size PEH is significant both in the increasing of output voltage and broadening the bandwidth of low frequency, which is due to the superharmonic resonance caused by the nonlinearity of the structure. Therefore, the bifurcation diagrams of amplitude with frequency at \(K = 3\) and the phase diagrams at \(p/3\) are given in figures.4 to further analyze this nonlinear phenomenon.

Figures.4 show that, with the increase of \(K\), the system appears bifurcation and superharmonic resonance, which means that vibration amplitude and output voltage are also significantly enhanced in superharmonic resonance region except for the primary resonance. At this time, the system can capture energy from the ambient vibration in the low frequency band through the super harmonic resonance with nonlinear characteristics, which explains the improvement of two key performances of the proposed micro-size PEH in detail, namely, the output voltage and the working bandwidth.

4. Conclusion
This paper proposes a micro-size piezoelectric energy harvester consisting of a cantilever plate with a tip mass on the free end. The dynamic model of the PEH is established considering electromechanical coupling effect and size effect. Based on the nonlinear dynamic equations, the performance of PEH is discussed with different additional mass ratios. Simulation results indicate that bifurcation and superharmonic resonance occurs with the increase of mass ratio \(K\). This nonlinear phenomenon enables the device to work in a lower bandwidth and capture more power from ambient vibration, which observably improves harvesting efficiency. The research direction of this paper is to proposes an approach to enhance micro-size PEH performance in a lower bandwidth, and how to obtain an optimal mass ratio to meet the requirement of the structural characteristics in practical applications remains to be further studied.

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