HILBERT’S IRREDUCIBILITY THEOREM FOR PRODUCTS OF ELLIPTIC CURVES

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Abstract. Corvaja-Zannier conjectured that an abelian variety \( A \) over a number field satisfies a modified version of the Hilbert property. We investigate their conjecture for products of elliptic curves using Kawamata’s structure result for ramified covers of abelian varieties, and Faltings’s finiteness theorem for rational points on higher genus curves.

1. Introduction

Recall that a normal integral variety \( X \) over a field \( k \) satisfies the Hilbert property over \( k \) (as defined in [Ser08, §4]) if, for every positive integer \( n \) and every collection of finite surjective morphisms \( \pi_i : Y_i \to X \), \( 1 \leq i \leq n \), with \( Y_i \) geometrically integral over \( k \) and \( \deg \pi_i \geq 2 \), the set \( X(k) \setminus \cup_{i=1}^n \pi_i(Y_i(k)) \) is dense in \( X \). In particular, if \( X \) satisfies the Hilbert property over \( k \), then \( X(k) \) is dense. The Hilbert property is closely related to the inverse Galois problem for \( \mathbb{Q} \); see [Ser08, §4]. In this paper we study a modified version of the Hilbert property, motivated by conjectures of Campana and Corvaja-Zannier on rational points for varieties over number fields.

By Hilbert’s Irreducibility Theorem [Ser08, Theorem 3.4.1], a rational variety over a number field satisfies the Hilbert property. On the other hand, an abelian variety over a number field does not satisfy the Hilbert property. Nonetheless, despite the failure of the Hilbert property for abelian varieties, Lang’s conjecture on rational points of pseudo-hyperbolic varieties (see [Lan74]) predicts that abelian varieties should satisfy a modified version of the Hilbert property.

The aim of this paper is to investigate such modified Hilbert properties for products of elliptic curves.

We were first led to investigate the weak-Hilbert property for abelian varieties by the work of Corvaja-Zannier on the Hilbert property for the Fermat K3 surface [CZ17], the work of Coccia on the “affine” Hilbert property [Coc19], Demeio’s extensions of Corvaja-Zannier’s work [Dema, Demb], Streeter’s verification of the Hilbert property for certain del Pezzo surfaces [Str], and Zannier’s seminal work on Hilbert’s irreducibility theorem for powers of elliptic curves [Zan10].

Let us recall that in [CZ17] Corvaja-Zannier introduced the following modified version of the Hilbert property.

Theorem 1.2. Let \( k \) be a field. A normal projective geometrically connected variety \( X \) over \( k \) satisfies the weak-Hilbert property over \( k \) if, for every finite surjective non-unramified morphism \( \pi : Y \to X \) with \( Y \) geometrically integral and normal, the set \( X(k) \setminus \pi(Y(k)) \) is dense in \( X \).

If \( k \) is a finitely generated field of characteristic zero and \( A \) is an abelian variety over \( k \), then the Mordell-Weil and Lang-Néron theorem imply that \( A(k) \) is a finitely generated abelian group; see [Con06, Corollary 7.2]. We prove that the product \( \prod_{i=1}^n E_i \) of elliptic curves \( E_1, \ldots, E_n \) over a number field satisfies the weak-Hilbert property under the (necessary) assumption that the rank of each \( E_i(k) \) is positive.

Theorem 1.2. Let \( k \) be a finitely generated field of characteristic zero, and let \( E_1, \ldots, E_n \) be elliptic curves over \( k \) with positive rank over \( k \). Then \( \prod_{i=1}^n E_i \) satisfies the weak-Hilbert property over \( k \).

We were first led to investigate the weak-Hilbert property for abelian varieties by the work of Corvaja-Zannier on the Hilbert property for the Fermat K3 surface [CZ17], the work of Coccia on the “affine” Hilbert property [Coc19], Demeio’s extensions of Corvaja-Zannier’s work [Dema, Demb], Streeter’s verification of the Hilbert property for certain del Pezzo surfaces [Str], and Zannier’s seminal work on Hilbert’s irreducibility theorem for powers of elliptic curves [Zan10].

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Theorems 1.2 and 1.5 provide evidence for Corvaja-Zannier’s conjecture.

Corvaja-Zannier’s conjecture (Conjecture 1.4) predicts that a smooth projective geometrically connected variety over \( k \) admits a finite extension \( L/k \) such that the set \( X(k) \setminus \bigcup_{i=1}^{n} \pi_i(Y_i(k)) \) is dense in \( X \).

Note that, if \( X \) satisfies the modified-Hilbert property over \( k \), then \( X \) satisfies the weak-Hilbert property over \( k \). However, the weak-Hilbert property defined above differs a priori from Corvaja-Zannier’s definition. Nonetheless, it seems reasonable to suspect that these notions are equivalent.

Clearly, a normal projective geometrically connected variety \( X \) over a field \( k \) with the Hilbert property (as defined in [Ser08, §3]) satisfies the modified-Hilbert property over \( k \). Thus, in particular, by Hilbert’s irreducibility theorem, any rational variety over a number field \( k \) satisfies the modified-Hilbert property over \( k \) and, in particular, the weak-Hilbert property over \( k \).

By [CZ17, Theorem 1.6], if \( X \) is a smooth projective geometrically connected variety over a number field \( k \) with the modified-Hilbert property, then \( X \) satisfies the Hilbert property over \( k \) if and only if it is geometrically simply-connected (i.e., \( \pi_1^X(X_{\overline{k}}) = 1 \)). Indeed, by loc. cit., a smooth projective geometrically connected variety \( X \) over a number field \( k \) with the Hilbert property is geometrically simply-connected. In particular, since abelian varieties over number fields are not geometrically simply connected, they do not have the Hilbert property.

Corvaja-Zannier conjectured that a smooth projective geometrically connected variety \( X \) over a number field \( k \) for which the set \( X(k) \) is dense satisfies the modified-Hilbert property over a finite field extension of \( k \). We state Corvaja-Zannier’s conjecture in the slightly more general context of varieties over finitely generated fields of characteristic zero, and also include the implied (currently not known) equivalence between the modified-Hilbert property and the weak-Hilbert property (up to a finite field extension).

**Conjecture 1.4** (Corvaja-Zannier). Let \( X \) be a smooth projective geometrically connected variety over a finitely generated field \( k \) of characteristic zero. Then the following statements are equivalent.

1. There is a finite extension \( M/k \) such that \( X_M \) satisfies the modified-Hilbert property over \( M \).
2. There is a finite extension \( N/k \) such that \( X_N \) satisfies the weak-Hilbert property over \( N \).
3. There is a finite extension \( L/k \) such that \( X(L) \) is Zariski-dense in \( X \).

Note that (1) \( \implies \) (2) and that (2) \( \implies \) (3). It is not known whether (3) \( \implies \) (2) nor whether (2) \( \implies \) (1).

Campana’s conjectures on “special” varieties provide another perspective on Conjecture 1.4. Indeed, Campana conjectured that (3) (and thus also (1) and (2)) should be equivalent to \( X_{\overline{k}} \) being special; see [Cam04, Conjecture 9.20] (and also [Cam11, Cam]). Examples of special varieties are abelian varieties, K3 surfaces, and rationally connected smooth projective varieties. Such varieties are thus expected (guided by the above conjectures) to satisfy the modified-Hilbert property over some finite extension of the finitely generated base field \( k \) of characteristic zero. Proving that such varieties satisfy the modified-Hilbert property seems very difficult, as it is currently not even known whether all K3 surfaces or Fano varieties have a potentially dense of rational points. We will comment a bit more on Campana’s conjectures below.

Our second result is that the product of two elliptic curves with positive rank satisfies the modified-Hilbert property. This modest contribution requires the input of Kawamata’s extension of Ueno’s fibration theorem for closed subvarieties of abelian varieties to the case of ramified covers of products of two elliptic curves (see Theorem 4.1), and uses Faltings’s finiteness theorem for higher genus curves in several ways.

**Theorem 1.5.** Let \( k \) be a finitely generated field of characteristic zero, and let \( E_1 \) and \( E_2 \) be elliptic curves over \( k \). If \( E_1(k) \) and \( E_2(k) \) have positive rank, then \( E_1 \times E_2 \) has the modified-Hilbert property over \( k \).

Hassett-Tschinkel proved that an abelian variety \( A \) over a finitely generated field \( k \) of characteristic zero admits a finite extension \( L/k \) such that \( A(L) \) is Zariski-dense in \( X \); see [HT00, §3] (or [Jav18, §3]). Thus, Corvaja-Zannier’s conjecture (Conjecture 1.4) predicts that an abelian variety \( A \) over a finitely generated field \( k \) of characteristic zero satisfies the modified-Hilbert property over some finite field extension of \( k \). Theorems 1.2 and 1.3 provide evidence for Corvaja-Zannier’s conjecture.
The fact that an elliptic curve of positive rank over \( k \) satisfies the modified-Hilbert-property is already known and, as noted in \( \text{[CZ17]} \), a consequence of Faltings’s theorem (\textit{quodam} Mordell’s conjecture) \( \text{[Fal83, Fal84]} \).

Our results (Theorem 1.2 and Theorem 1.5) generalize earlier work of Zannier in which evidence for Conjecture 1.4 was provided for abelian varieties \( A \) which are isogenous to \( E^n \) with \( E \) a non-CM elliptic curve \( \text{[Zan10, Zan09]} \). Note that Zannier’s arguments are very different from ours and rely on Hilbertian properties of cyclotomic fields (see \( \text{[DZ07, Zan00]} \)). Theorem 1.2 also provides a non-linear analogue of Corvaja’s theorem for linear algebraic groups \( \text{[Cor04]} \).

Since elliptic curves of positive rank over a number field satisfy the modified-Hilbert property, the most natural approach to proving that the product of elliptic curves satisfies the modified-Hilbert property would be to show that the product of two varieties satisfying the modified-Hilbert property over \( k \) satisfies the modified-Hilbert property. This product property seems however difficult to establish. Instead, to prove Theorem 1.2 we verify a “weaker” expectation.

**Theorem 1.6.** Let \( k \) be a field and \( X_1, \ldots, X_n \) be integral normal projective varieties over \( k \). Assume that, for every \( i = 1, \ldots, n \), the variety \( X_i \) satisfies the modified-Hilbert property over \( k \). Then \( X_1 \times \cdots \times X_n \) satisfies the weak-Hilbert property over \( k \).

Our approach to Theorem 1.6 is inspired greatly by the arguments of Bary-Soroker–Fehm–Petersen \( \text{[BSPP14]} \). Indeed, in \textit{loc. cit.} it is shown that, if \( X \) and \( Y \) satisfy the Hilbert property over \( k \), then \( X \times Y \) satisfies the Hilbert property over \( k \). Their result answers an old question of Serre in the positive (see the Problem stated in \( \text{[Ser08, §3.1]} \)). We mention that Bary-Soroker–Fehm–Petersen’s product theorem for varieties with the Hilbert property can also be deduced from \( \text{[HW16, Lemma 8.12]} \) (which builds on Wittenberg’s thesis \( \text{[Wit07]} \) Lemma 3.12).

The most general criterion we prove for verifying the Weak-Hilbert property for a variety is Theorem 2.3. It is precisely this result which was inspired by Bary-Soroker–Fehm–Petersen’s work \( \text{[BSPP14]} \).

Let us briefly mention that Theorem 1.6 has further consequences. For example, if \( E \) is an elliptic curve over a finitely generated field \( k \) of characteristic zero with \( E(k) \) of positive rank, then the variety \( E^n \times \mathbb{P}^m_k \) satisfies the weak-Hilbert property over \( k \). Moreover, if \( X \) is the K3 surface defined by \( x^4 + y^4 = z^4 + w^4 \) in \( \mathbb{P}^3_k \), then \( E^n \times X \) also satisfies the weak-Hilbert property over \( k \), as Corvaja-Zannier proved that \( X \) satisfies the Hilbert property over \( k \) (see \( \text{[CZ17, Theorem 1.4]} \)).

**1.1. Campana’s conjectures.** Campana’s aforementioned notion of special variety forms an important guiding principle in our study of varieties with the modified-Hilbert property. In fact, Campana’s conjectures reach much further and also predict a precise interplay between density of rational points and dense entire curves (much like Lang’s conjectures \( \text{[Lan74]} \)); this is also hinted at by Corvaja-Zannier (see \( \text{[CZ17, §2.4]} \)).

To explain this, let us say that a variety \( X \) over \( \mathbb{C} \) satisfies the \textit{Brody-modified-Hilbert property} if, for every integer \( n \geq 1 \) and finite surjective non-unramified morphisms \( \pi_i : Y_i \rightarrow X \) with \( Y_i \) integral and normal \((i = 1, \ldots, n)\), there is a holomorphic map \( C \rightarrow X^n \) with Zariski-dense image which does not lift to any of the covers \( \pi_i^n : Y_i^n \rightarrow X^n \). A special smooth projective connected variety over \( \mathbb{C} \) is conjectured to satisfy the Brody-modified-Hilbert property; see \( \text{[Cam]} \). In this direction it was shown recently by Campana-Winkelmann that a rationally connected variety over \( \mathbb{C} \) satisfies the Brody-modified-Hilbert property; see \( \text{[CW]} \). We also mention that the work of Noguchi-Winkelmann-Yamanoi can be used to prove that a non-zero abelian variety over \( \mathbb{C} \) satisfies the Brody-modified-Hilbert property (see \( \text{[NWY07, NWY08, NWY13, Yam13]} \)). On the other hand, it is not known whether every K3 surface satisfies the Brody-modified-Hilbert property, as we do not know whether such surfaces admit a dense entire curve.

This being said, our motivation for writing this short note is to call some attention to the beautiful string of new ideas surrounding the modified-Hilbert property, potential density of rational points on varieties over number fields, the existence of dense entire curves, and Campana’s special varieties. In fact, we were naturally led to investigating these problems by our work on Lang’s conjectures \( \text{[Lan74]} \) (see \( \text{[BJK, Jav, JK, JLa, JLb, JX]} \)).
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Conventions. If $k$ is a field, then a variety over $k$ is a finite type separated scheme over $k$. If $X$ and $Y$ are varieties over $k$, then we let $X \times Y$ denote the fiber product $X \times_{\text{Spec} k} Y$. A field $k$ is said to be finitely generated if it is finitely generated over its prime field.

2. The weak-Hilbert property

Throughout this section, let $k$ be a field. Moreover, let $f : X \to S$ be a morphism of smooth projective integral varieties over $k$. Furthermore, let $\pi : Y \to X$ be a finite surjective non-unramified morphism and let

$$
\begin{array}{ccc}
Y & \to & T, \\
\psi & \to & X
\end{array}
$$

be the Stein factorization of the composed morphism $Y \to X \to S$; see [Har77, §II.11]. Note that the variety $T$ is projective normal integral over $k$ and that the geometric fibers of $Y \to T$ are connected.

Proposition 2.1. Let $U \subset S$ be a dense open subset. Assume that $S$ satisfies the weak-Hilbert property over $k$ and that, for every $s$ in $U(k) \setminus \psi(T(k))$, the set $X_s(k)$ is dense in $X_s$. If the morphism $\psi : T \to S$ is non-unramified, then $X(k) \setminus \pi(Y(k))$ is dense.

Proof. Since $S$ satisfies the weak-Hilbert property over $k$ and $T \to S$ is a non-unramified finite surjective morphism with $T$ a normal integral variety over $k$, the set $S(k) \setminus \psi(T(k))$ is dense in $S$. In particular, the set $U(k) \setminus \psi(T(k))$ is dense in $S$. Now, note that the set

$$
\bigcup_{s \in U(k) \setminus \psi(T(k))} X_s(k).
$$

is dense in $X$. Indeed, since $X_s(k)$ is dense in $X_s$, the closure of $\bigcup_{s \in U(k) \setminus \psi(T(k))} X_s(k)$ in $X$ contains the dense set $\bigcup_{s \in U(k) \setminus \psi(T(k))} X_s(k)$. Now, note that $X(k) \setminus \pi(Y(k))$ contains the (dense) set

$$
\bigcup_{s \in U(k) \setminus \psi(T(k))} X_s(k).
$$

This concludes the proof. \qed

Lemma 2.2. Assume that the branch locus $D$ of $\pi : Y \to X$ dominates $S$ (i.e., $f(D) = S$). Then, for every point $s$ in $S$, the morphism $Y_s \to X_s$ is finite surjective non-unramified.

Proof. A morphism of varieties $V \to W$ over $k$ is unramified if and only if, for every $w$ in $W$, the morphism $U_w \to \text{Spec} k(w)$ is unramified (i.e., étale); see [Sta15, Tag 00UV]. Now, let $s$ be a point of $S$. To show that the finite surjective morphism $Y_s \to X_s$ is non-unramified, let $d \in D$ be a point lying over $s$. Then, by the definition of the branch locus, $Y_d \to \text{Spec} k(d)$ is non-unramified. Note that $Y_d = Y_s \times_{X_s} d$ as schemes over $d = \text{Spec} k(d)$. As the fibre of $Y_s \to X_s$ over $d$ is non-unramified, it follows that $Y_s \to X_s$ is non-unramified. \qed

Theorem 2.3. Let $U \subset S$ be a dense open subscheme of $S$. Assume that the following statements hold.

1. The variety $S$ satisfies the weak-Hilbert property over $k$.
2. For every $s$ in $U(k)$, the projective variety $X_s$ is normal integral and satisfies the modified-Hilbert property over $k$.
3. The branch locus $D$ of $\pi : Y \to X$ dominates $S$, i.e., $f(D) = S$.

Then $X(k) \setminus \pi(Y(k))$ is dense in $X$.

Proof. If $\psi : T \to S$ is non-unramified, then it follows from Proposition 2.1 that $X(k) \setminus \pi(Y(k))$ is dense in $X$. (We do not need here the assumption that $f(D) = S$.) Thus, to prove the theorem, we may and do assume that $\psi : T \to S$ is unramified. Since $S$ is smooth and $\psi : T \to S$ is a finite surjective unramified morphism, it follows that $T$ is smooth, so that $\psi : T \to S$ is in fact flat, hence étale.
Note that we have a commutative diagram of morphisms

\[
\begin{array}{ccc}
D_T & \xrightarrow{f_T} & T \\
\downarrow{\pi_T} & & \downarrow{\psi} \\
X_T & \xrightarrow{f} & S \\
\end{array}
\]

As the branch locus \( D \) of \( \pi \) dominates \( S \), it follows that the branch locus \( D_T \) of \( \pi_T : Y_T \to X_T \) dominates \( T \). This implies that, for all \( t \) in \( T \), the morphism \( Y_t \to X_t \) is non-unramified (Lemma 2.2). We now use this observation.

For \( s \in U(k) \), consider the finite surjective morphism \( Y_s \to X_s \). Let \( \{t_1, \ldots, t_r\} = \psi^{-1}\{s\} \). Then \( Y_s = Y_{t_1} \sqcup \ldots \sqcup Y_{t_r} \) and, as explained above, every induced finite surjective morphism \( \pi_{s,j} : Y_{t_j} \to X_s \) is non-unramified. Since every \( Y_{t_j} \) is integral and normal and \( X_s \) satisfies the modified-Hilbert property over \( k \), it follows that

\[ X_s(k) \setminus \bigcup_{j=1}^r \pi_{s,j}(Y_{t_j}(k)) = X_s(k) \setminus \pi_s(Y_s(k)) \]

is dense in \( X_s \). Since, for every \( s \) in \( U(k) \), the latter set is dense in \( X_s \), we conclude that \( X(k) \setminus \pi(Y(k)) \) is dense in \( X \), as required.

\[ \square \]

3. PRODUCTS OF VARIETIES

To study products of varieties \( X_1, \ldots, X_n \), we will exploit the many projections such a product is equipped with.

**Definition 3.1.** Let \( X_1, \ldots, X_n \) be varieties over \( k \) and let \( X := X_1 \times \ldots \times X_n \). Define \( \tilde{X}_i \) to be the product of \( X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n \). We let \( p_i : X \to \tilde{X}_i \) be the natural projection.

We include a brief proof of the following simple observation.

**Lemma 3.2.** Let \( X_1, \ldots, X_n \) be smooth projective geometrically integral varieties over \( k \), and let \( D \subset \prod_{i=1}^n X_i \) be a non-empty closed subscheme of codimension one. Then, there is an integer \( j \in \{1, \ldots, n\} \) such that \( p_j(D) = \tilde{X}_j \).

**Proof.** We argue by induction on \( n \). We may and do assume that \( D \) is integral. Write \( X = \prod_{i=1}^n X_i \). Note that

\[ D \subseteq X_1 \times p_1(D) \subseteq X. \]

If \( X_1 \times p_1(D) = X \), then \( p_1(D) = \tilde{X}_1 \), as required. Thus, we may assume that \( X_1 \times p_1(D) \neq X \). Then, as \( D \) is of codimension one, it follows that \( D = X_1 \times p_1(D) \). In this case, as \( p_1(D) \) is integral and of codimension one in \( \tilde{X}_1 \), after relabeling if necessary, it follows from the induction hypothesis that \( p_1(D) \) surjects onto \( X_3 \times \ldots \times X_n \). This implies that \( D = X_1 \times p_1(D) \) surjects onto \( \tilde{X}_3 = X_1 \times X_3 \times \ldots \times X_n \), as required.

\[ \square \]

**Lemma 3.3.** Let \( X_1, \ldots, X_n \) be smooth projective geometrically integral varieties over \( k \), and let \( \pi : Y \to X_1 \times \ldots \times X_n \) be a finite surjective non-unramified morphism with \( Y \) an integral normal projective variety. Let \( D \) be the branch locus of \( \pi \). Then there is an integer \( j \in \{1, \ldots, n\} \) such that \( p_j(D) = \tilde{X}_j \).

**Proof.** Note that \( D \) is non-empty, as \( \pi \) is non-unramified. Then, by Zariski-Nagata purity [Gro63, Theorem X.3.1], the branch locus \( D \) is a closed subscheme pure of codimension one, so that the lemma follows from Lemma 3.2.

\[ \square \]

We are now ready to prove that a product of varieties satisfying the modified-Hilbert property over \( k \) satisfies the weak-Hilbert property over \( k \).
Proof of Theorem 1.6. We argue by induction on \( n \). If \( n = 1 \), the statement is obvious. Thus, we may and do assume that \( n > 1 \). Write \( X = \prod_{i=1}^{n} X_i \) and let \( \pi : Y \to X \) be a finite surjective non-unramified morphism. It suffices to show that \( X(k) \setminus \pi(Y(k)) \) is dense in \( X \). By Lemma 3.3 there is an integer \( j \in \{1, \ldots, n\} \) such that the branch locus of \( \pi : Y \to X \) dominates \( X_j \) (Definition 3.1). Define \( S := X_j \) and consider the natural morphism \( j : X \to S \). Note that, by the induction hypothesis, the smooth projective integral variety \( S \) satisfies the weak-Hilbert property over \( k \). Moreover, for every \( s \in S(k) \), the projective variety \( X_s \) is naturally isomorphic to \( X_j \), and is therefore a smooth projective integral variety over \( k \) satisfying the modified-Hilbert property over \( k \). Thus, conditions (1), (2), (3) of Theorem 2.3 are satisfied. We conclude that \( X(k) \setminus \pi(Y(k)) \) is dense in \( X \).

Remark 3.4. Let \( X \) and \( Y \) be smooth projective connected varieties over a finitely generated field \( k \) of characteristic zero. If \( X \) and \( Y \) are special in the sense of Campana [Cam04, Cam11, Cam], then \( X \times Y \) is special. Moreover, the conjectures of Campana and Corvaja-Zannier predict that \( X \) is special if and only if there is a finite field extension \( L/k \) such that \( X_L \) has the modified-Hilbert property over \( L \). In particular, Theorem 1.6 is in accordance with the conjectures of Campana and Corvaja-Zannier as it verifies that a product of varieties with the modified-Hilbert property satisfies the weak-Hilbert property.

We now prove Theorem 1.2. Note that the proof is a straightforward application of Theorem 1.6 and Faltings’s finiteness theorems for higher genus curves.

Proof of Theorem 1.2. As in the statement of the theorem, we let \( k \) be a finitely generated field of characteristic zero. Moreover, let \( E_1, \ldots, E_n \) be elliptic curves over \( k \) of positive rank over \( k \). Then, for every \( i = 1, \ldots, n \), the elliptic curve \( E_i \) satisfies the modified-Hilbert property over \( k \) by Faltings’s theorem [Fal83, Fal84]. Indeed, it suffices to note that, if \( E \) is an elliptic curve over \( k \) and \( \pi : Y \to E \) is a non-unramified finite surjective morphism, then the set \( Y(k) \) is finite.) Thus, it follows from Theorem 1.6 that \( E_1 \times \cdots \times E_n \) satisfies the weak-Hilbert property over \( k \).

4. Kawamata’s Theorem

To prove that the product of two elliptic curves satisfies the modified-Hilbert property, we will use Kawamata’s theorem on finite covers of abelian varieties. Note that Kawamata’s theorem is a generalization of Ueno’s fiberation theorem for closed subvarieties of abelian varieties.

Theorem 4.1 (Kawamata). Let \( K \) be an algebraically closed field of characteristic zero. Let \( X \) be a normal algebraic variety over \( K \) and let \( X \to A \) be a finite morphism. Then the following data exists.

(1) An abelian subvariety \( B \) of \( A \);
(2) finite étale Galois covers \( X' \to X \) and \( B' \to B \);
(3) a normal projective variety \( Y \) of general type over \( K \);
(4) a finite morphism \( Y \to A/B \) with \( A/B \) the quotient of \( A \) by \( B \) such that \( X' \) is a fiber bundle over \( Y \) with fibers \( B' \) and with translations by \( B' \) as structure group.

Proof. See [Kaw81, Theorem 23].

Lemma 4.2. Let \( k \) be a finitely generated field of characteristic zero, let \( A \) be an abelian surface over \( k \), and let \( Y \to A \) be a finite surjective non-unramified morphism with \( Y \) integral normal. If the Kodaira dimension of \( Y \) is not two, then \( Y(k) \) is not dense.

Proof. Note that the Kodaira dimension of \( Y \) is non-negative, as \( Y \) admits a finite surjective morphism to an abelian variety. If the Kodaira dimension of \( Y \) is zero, then \( Y \to A \) is étale by Kawamata’s theorem (Theorem 4.1). This contradicts our assumption that \( Y \to A \) is non-unramified. Thus, we may and do assume that the Kodaira dimension of \( Y \) equals one. Then, by Kawamata’s theorem (Theorem 4.1), there is a finite field extension \( L/k \) and a finite étale cover \( Y' \to Y_L \) of the surface \( Y_L \) such that \( Y' \) dominates a curve \( C \) over \( L \) of genus at least two. By Chevalley-Weil [Hir86, §8], if \( Y(k) \) is dense, then there is a finite field extension \( M/k \) such that \( Y'(M) \) is dense. As \( Y' \to C \) is surjective, it follows that \( C(M) \) is dense. However, this contradicts Faltings’s theorem [Fal84] that \( C(M) \) is finite. We conclude that the set \( Y(k) \) is not dense in \( Y \).

□
Lemma 4.3. Let $A$ and $B$ be elliptic curves over $k$, and let $\pi : Y \to A \times B$ be a finite surjective morphism with $Y$ of general type. Then the branch locus of $\pi$ dominates $A$ and $B$.

Proof. Let $\psi : \tilde{Y} \to Y$ be a resolution of singularities, and let $E = E_1 \cup \ldots \cup E_n$ be the exceptional locus. Let $R$ be the ramification divisor of $\pi : Y \to A \times B$. Then, by Riemann-Hurwitz, we have that

$$K_Y = \pi^*K_{A \times B} + R = R, \quad K_{\tilde{Y}} = \psi^*R + \sum a_iE_i.$$

As the canonical divisor $K_Y$ is big on $\tilde{Y}$ (as $\tilde{Y}$ is of general type), we see that $\pi_*R$ is big on $A \times B$. Now, assume that the branch locus of $\pi$ does not dominate $A$. Then, the big divisor $\pi_*R$ is contained in $S \times B$ with $S$ a finite closed subset of $E_1$. However, as $S \times B$ is not big, this contradicts the bigness of $\pi_*R$. We conclude that the branch locus of $\pi$ dominates $A$ (hence also $B$ by symmetry).

Proof of Theorem 7.3. Define $X := E_1 \times E_2$ and $S := E_1$. Let $f : X \to S$ be the projection map. For $i = 1, \ldots, n$, let $Y_i$ be an integral normal variety over $k$ and let $\pi_i : Y_i \to X$ be a finite surjective non-unramified morphism. It suffices to show that $X(k) \setminus \cup_{i=1}^n \pi_i(Y_i(k))$ is dense in $X$. To this end, let us first note that $X(k)$ is dense in $X$ (as $E_1(k)$ and $E_2(k)$ have positive rank). Now, if $Y_i$ has Kodaira dimension $< 2$, then $Y_i(k)$ is not dense (Lemma 4.2), so that we may discard such $Y_i$ from the collection of coverings $\pi_i : Y_i \to X$. That is, we may and do assume that, for $i = 1, \ldots, n$, the variety $Y_i$ of general type.

Moreover, if $Y_i \to T_i \to E_1$ is the Stein factorization of the composed morphism $Y_i \to E_1 \times E_2 \to E_1$ and $T_i \to E_1$ is non-unramified, then $Y_i(k)$ is not dense in $X$, as $T_i(k)$ is finite by Faltings’s finiteness theorem [Fal83, Fal84]. Therefore, we may also discard such morphisms $\pi_i : Y_i \to X$ from the collection of coverings $\pi_i : Y_i \to X$. Thus, for $i = 1, \ldots, n$, the morphism $T_i \to S$ is finite unramified, hence étale. Moreover, as $Y_i$ is of general type, by Lemma 4.3, for $i = 1, \ldots, n$, the branch locus of $\pi_i$ dominates $S := E_1$. We now argue similarly as in the end of the proof of Theorem 2.3.

Let $U \subset S$ be a dense open subset such that, for every $s$ in $U$, the scheme $Y_s$ is normal. For $s \in U(k)$, consider the finite surjective morphism $\pi_{i,s} : Y_{i,s} \to X_s$. Let $\{t_{i,1}, \ldots, t_{i,r_i}\} = \psi_i^{-1}\{s\}$. Then $Y_{i,s} = Y_{i,1} \cup \ldots \cup Y_{i,r_i}$ with $Y_{i,1}, \ldots, Y_{i,r_i}$ integer normal varieties over $k$. Moreover, for every $i = 1, \ldots, n$, and every integer $1 \leq j \leq r_i$, by Lemma 2.2, the induced finite surjective morphism $\pi_{i,s,j} : Y_{i,s,j} \to X_s$ is non-unramified (as the branch locus of $Y_i \to X$ dominates $S$, so that the branch locus of $Y_{i,T} \to X_T$ dominates $T$). Therefore, since $X_s = E_2$ satisfies the modified-Hilbert property over $k$ (by assumption), it follows that

$$X_s(k) \setminus \cup_{i=1}^n \pi_{i,s,j}(Y_{i,s,j}(k)) = X_s(k) \setminus \cup_{i=1}^n \pi_i(Y_i(k))$$

is dense in $X_s$. Note that, for every $s$ in $U(k)$, the set $X(k) \setminus \cup_{i=1}^n \pi_i(Y_i(k))$ contains the set

$$X_s(k) \setminus \cup_{i=1}^n \pi_{i,s}(Y_{i,s}(k)).$$

Since $S(k) = E_1(k)$ is dense in $E_1$, we have that $U(k)$ is dense in $E_1$, so that $X(k) \setminus \cup_{i=1}^n \pi_i(Y_i(k))$ is dense in $X$. \qed

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