Research Article

Littlewood–Paley Characterization for Musielak–Orlicz–Hardy Spaces Associated with Self-Adjoint Operators

Jiawei Shen, Shunchao Long, and Yu-long Deng

1College of Mathematics and Computer Science, Zhejiang Normal University, Jinhua 321001, China
2School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, China
3School of Mathematics Science, Changsha Normal University, Changsha 410100, China

Correspondence should be addressed to Shunchao Long; longsc84@163.com and Yu-long Deng; yuldeng@163.com

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1 Shunchao Long

1. Introduction

The metric measure space \((X, d, \mu)\) is a set \(X\) equipped with a metric \(d\) and a non-negative Borel doubling measure \(\mu\) on \(X\). Let \(f \in L^2(X)\) and \(L\) be a densely defined operator on \(L^2(X)\) which satisfies the following two conditions:

(i) \((H1)\) \(L\) is a non-negative self-adjoint operator on \(L^2(X)\).

(ii) \((H2)\) the kernel of \(e^{-tL}\), denoted by \(p_t(x, y)\), is a measurable function on \(X \times X\) satisfying the Gaussian estimates, i.e., there exist \(C_1, C_2 > 0\) such that

\[
|p_t(x, y)| \leq \frac{C_1}{V(x, \sqrt{t})} e^{-\left(\frac{d(x, y)^2}{C_2 t}\right)},
\]

holds for all \(t > 0\) and \(x, y \in X\), where \(V(x, \sqrt{t}) = \mu(B(x, \sqrt{t}))\).

The Littlewood–Paley function \(G_L(f)\) and Lusin-area function \(S_L(f)\) associated with the heat semigroup generated by \(L\) are given by

\[
G_L(f)(x) = \left(\int_{0}^{\infty} |t^{1/2} e^{-tL} f(x)|^2 \frac{dt}{t}\right)^{1/2},
\]

\[
S_L(f)(x) = \left(\int_{0}^{\infty} \int_{d(x, y) < t} |t^{1/2} e^{-tL} f(x)|^2 \frac{d\mu(y)}{\mu(B(x, t))} \frac{dt}{t}\right)^{1/2}.
\]

In this paper, we focus on the characterization of the Musielak–Orlicz–Hardy spaces \(H_{\varphi, L}\) and \(H_{L,G, \varphi}\), where the operator \(L\) satisfies \((H1)\) and \((H2)\) and \(\varphi\) is a growth function (cf. Definition 6 below).

Definition 1. Suppose that the operator \(L\) satisfies \((H1)\) and \((H2)\) and \(\varphi\) is a growth function. A function \(f \in H^2(X)\) is said to be in \(H_{\varphi, L}(X)\) if \(S_L(f) \in L^\varphi(X)\) (cf. Definition 7 below). Moreover, we define

\[
\|f\|_{H_{\varphi, L}(X)} = \left\|S_L(f)\right\|_{L^\varphi} = \inf \left\{ \lambda \in (0, \infty) : \frac{1}{\lambda} \int_X \varphi \left( x, \frac{S_L(f)(x)}{\lambda} \right) d\mu(x) \leq 1 \right\}.
\]
Definition 2. Suppose that the operator \( L \) satisfies (H1) and (H2) and \( \varphi \) is a growth function. A function \( f \in H^2(X) \) is said to be in \( H_{L,G,\varphi}(X) \) if \( G_L(f) \in L^p(X) \). Moreover, we define
\[
\| f \|_{H_{L,G,\varphi}(X)} = \| G_L(f) \|_{L^p} \quad = \inf \left\{ \lambda \in (0, \infty) : \int_X \varphi \left( x, \frac{G_L(f)(x)}{\lambda} \right) \, d\mu(x) \leq 1 \right\}.
\]
(4)

The Musielak–Orlicz–Hardy space \( H_{L,G,\varphi}(X) \) is defined to be the complement space of \( H_{L,G,\varphi}^\perp(X) \).

Theorem 1. Suppose that the operator \( L \) satisfies (H1) and (H2) and \( \varphi \) is a growth function of uniformly lower type \( p_1 \).

Then, the spaces \( H_{\varphi,L}(X) \) and \( H_{L,G,\varphi}(X) \) coincide and their norms are equivalent.

Theorem 1 obtains the behavior of Littlewood–Paley \( \varphi \)-function \( G_L \) on \( H_{\varphi,L} \) and partly improves the result in [9]. To make it clear, we first establish the discrete characterization of the Musielak–Orlicz spaces \( H_{\varphi,L}(X) \) and \( H_{L,G,\varphi}(X) \) and state these results as follows.

Theorem 2. Suppose that the operator \( L \) satisfies (H1) and (H2) and \( \varphi \) is a growth function of uniformly lower type \( p_1 \). Let \( f \in H_{\varphi,L}(X) \cap L^2(X) \). Then, for all \( M \in \mathbb{N} \) with \( M > (nq(\varphi)/2p_1) \), \( f \) has an \( AT_{x,M} \)-expansion such that
\[
\| f \|_{H_{\varphi,L}(X)} \equiv \| W_f \|_{L^1(x)}.
\]
(5)

Theorem 3. Suppose that the operator \( L \) satisfies (H1) and (H2) and \( \varphi \) is a growth function of uniformly lower type \( p_1 \). Let \( f \in H_{L,G,\varphi}(X) \cap L^2(X) \). Then, for all \( M \in \mathbb{N} \) with \( M > (nq(\varphi)/2p_1) \), \( f \) has an \( AT_{x,M} \)-expansion such that
\[
\| f \|_{H_{L,G,\varphi}(X)} \equiv \| W_f \|_{L^1(x)}.
\]
(6)

Theorems 2 and 3 extend the results in [6, 7], respectively. Also, we extend the results in [9] by removing the assumption of uniformly reverse Hölder condition. As a consequence of Theorems 2 and 3, we immediately get Theorem 1.

The paper is organized as follows. Section 2 contains some basic definitions and lemmas concerning metric measure spaces, growth functions, Musielak–Orlicz space, and \( AT_{x,M} \)-family. The aim of Section 3 is to prove Theorem 2 and establish the characterization of Musielak–Orlicz–Hardy space \( H_{\varphi,L} \). We develop a method to unify the different control terms of inner integral. The aim of Section 4 is to prove Theorem 3 and set up the characterization of Musielak–Orlicz–Hardy space \( H_{L,G,\varphi} \). We borrow the ideas from [6,10]. Consequently, we get that the characterization of Musielak–Orlicz–Hardy space by means of \( H_{\varphi,L} \) and \( H_{L,G,\varphi} \) is equivalent.

Most of the notations we use are standard. \( C \) denotes a positive constant that may change from line to line and we use the subscript for the sake of eliminating confusion. We write \( A \equiv B \) if there exist constants \( C_1 \) and \( C_2 \) which are independent of \( A \) and \( B \) such that \( C_1B \leq A \leq C_2B \). For a measurable set \( A \), \( |A| \) denotes the Lebesgue measure of \( A \) and \( \chi_A \) is the characteristic function.

2. Basic Concepts and Lemmas

2.1. Metric Measure Spaces. A metric measure space \((X, d, \mu)\) is a set \( X \) equipped with a metric \( d \) and a non-negative Borel doubling measure \( \mu \) on \( X \). Fix \( x \in X \) and let \( r \in (0, \infty) \), and we denote the open ball centered at \( x \) with radius \( r \) by
\[
B(x, r) = \{ y \in X : d(x, y) < r \},
\]
and set \( V(x, r) = \mu(B(x, r)) \).
Definition 3. A space of homogeneous type \((X,d,\mu)\) is a set \(X\) with a metric \(d\) and a non-negative measure \(\mu\) on \(X\), so that there exists a constant \(C_D \in [1,\infty)\) such that for all \(x \in X\) and \(r > 0\),

\[ V(x, 2r) \leq C_D V(x, r) < \infty. \]  \tag{8}

Definition 3 was introduced by Coifman and Weiss [11]. The property of \(\mu\) in (8) is the doubling condition and it implies the strong \(n\) homogeneity property, i.e., for some constant \(C > 0\) and homogeneity \(n\),

\[ V(x, \lambda r) \leq C \lambda^n V(x, r) \]  \tag{9}

holds uniformly for all \(\lambda \in [1,\infty), x \in X\), and \(r > 0\).

Let \(C_D\) be as in (8) and set \(m = \log_2 C_D\), and Grigor’yan et al. have shown that (8)

\[ V(x, R) \leq C_D \left[ \frac{R + d(x, y)}{r} \right]^m V(y, r) \]  \tag{10}

holds for all \(x, y \in X\) and \(0 < r \leq R < \infty\). It is easy to verify, by doubling condition (8), that for any \(N > n\), there exists a constant \(C_N\) such that for all \(x \in X\) and \(t > 0\),

\[ \int_X (1 + t^{-1} d(x, y))^{-N} d\mu(y) \leq C_N V(x, t). \]  \tag{11}

The dyadic cube decomposition on spaces of homogeneous type comes from Christ [13] as follows:

Lemma 1. Let \((X,d,\mu)\) be a space of homogeneous type. Then, there exists a collection of open subsets \(\{Q^k \subset X: k \in \mathbb{Z}, \alpha \in I_k\}\) and constants \(\delta \in (0,1)\) and \(0 < C_1, C_2 < \infty\) such that

(i) \(\mu(X \setminus \cup_n \cup_k Q^k) = 0\), for each fixed \(k\), if \(\alpha \neq \beta\), then \(Q^k_\alpha \cap Q^k_\beta = \emptyset\).

(ii) For any \(\alpha, \beta, k, l\), if \(k \leq l\), then either \(Q^k_\beta \subset Q^l_\alpha\) or \(Q^k_\beta \cap Q^l_\alpha = \emptyset\).

(iii) For each \((k, \alpha)\) and each \(l < k\), there is a unique \(\beta \in I_l\) such that \(Q^k_\beta \subset Q^l_\alpha\).

(iv) Diameter \((Q^k_\alpha) \leq C_1 \delta^k\).

(v) Each \(Q^k_\alpha\) contains some ball \(B(z^n_\alpha, C_2 \delta^k)\), where \(z^n_\alpha \in X\).

The sets \(Q^k_\alpha\) are analogues of the Euclidean dyadic cubes; it may help to think of \(Q^k_\alpha\) as being essentially a cube of ball of diameter roughly \(\delta^k\) with center \(z^n_\alpha\). We then set \(\ell(Q^k_\alpha) = C_1 \delta^k\). It is worthy pointing out that the precise value of \(C_1\) is non-essential (cf. Christ [13]). Here and in what follows, we assume \(C_1 = \delta^{-1}\).

2.2 Growth Functions. We first recall the Orlicz function. A non-decreasing function \(\Phi: [0, \infty) \rightarrow [0, \infty)\) is called an Orlicz function if \(\Phi(0) = 0, \Phi(t) > 0\) for all \(t \in (0, \infty)\) and \(\lim_{t \rightarrow \infty} \Phi(t) = \infty\) (cf. Yang [9]).

The function \(\Phi\) is said to be of upper type \(p\) (resp., lower type \(p\)) for some \(p \in (0, \infty)\), if for all \(t \in [1, \infty)\) (resp., \(t \in [0, 1]\)) and \(s \in [0, \infty)\), there is a constant \(C > 0\) such that \(\Phi(st) \leq C t^p \Phi(s)\). \(\Phi\) is said to be of type \((p_1, p_2)\) if it is of both upper type \(p_1\) and lower type \(p_2\).

Given a function \(\varphi: X \times [0, \infty) \rightarrow [0, \infty)\), for any \(x \in X\), \(\varphi(x, \cdot)\) is an Orlicz function. If there exists a constant \(C > 0\) such that for all \(x \in X\), \(t \in [1, \infty)\) (resp., \(t \in [0, 1]\)) and \(s \in [0, \infty)\),

\[ \varphi(x, st) \leq C t^p \varphi(x, s), \]  \tag{12}

then \(\varphi\) is said to be of uniformly upper type \(p\) (resp., uniformly lower type \(p\)). Moreover, \(\varphi\) is said to be of positive uniformly upper type (resp., uniformly lower type) if it is of uniformly upper type (resp., uniformly lower type) for some \(p \in (0, \infty)\).

Let \(\varphi: X \times [0, \infty) \rightarrow [0, \infty)\). If for all \(t \in [0, \infty)\), \(x \rightarrow \varphi(x, t)\) is measurable and for all bounded subsets \(K\) of \(X\),

\[ \int_K \sup_{t \in (0, \infty)} \left\{ \varphi(x, t) \left[ \int_K \Phi(y, t) d\mu(y) \right]^{-1} \right\} d\mu(x) < \infty. \]  \tag{13}

Then, \(\varphi(\cdot, t)\) is said to be uniformly locally integrable (cf. [8]).

We next recall the uniformly Muckenhoupt condition in [9, 14].

Definition 4. Let \(\varphi: X \times [0, \infty) \rightarrow [0, \infty)\) be uniformly locally integrable. The function \(\varphi(\cdot, t)\) is said to satisfy the uniformly Muckenhoupt condition for some \(q \in [1, \infty)\), denoted by \(\varphi \in \mathcal{A}_q(X)\), if

\[ \mathcal{A}_q(\varphi) = \sup_{t \in (0, \infty)} \left\{ \frac{1}{\mu(B)} \left( \int_B \varphi(x, t) d\mu(x) \right) \right\} \left\{ \frac{1}{\mu(B)} \left( \int_B \varphi(y, t) d\mu(y) \right) \right\}^{1-1/q} \left( \int_B \varphi(x, t)^{1/q} d\mu(x) \right)^{1/q} \]  \tag{14}

when \(q \in (1, \infty)\) and \((1/q) + (1/q') = 1\), or...
\[ A_1(\varphi) = \sup_{t \in (0, \infty)} \sup_{B \subset X} \frac{1}{\mu(B)} \int_B \varphi(x, t) d\mu(x) \left( \text{ess sup}_{y \in B} [\varphi(y, t)]^{-1} \right) \leq C. \]  

Here the first supremum is taken over all \( t \in (0, \infty) \) and the second one is taken over all balls \( B \subset X \).

We define \( A_\infty(X) = U \cup \cup \varphi \) and let

\[ q(\varphi) = \inf \{ q \in [1, \infty) : \varphi \in A_q(X) \} \]

be the critical indices of \( \varphi \). Moreover, we denote

\[ \varphi(E, t) = \int_E \varphi(x, t) d\mu(x), \]

for any measurable subset \( E \) of \( X \) and \( t \in [0, \infty) \). Let \( \mathcal{M} \) be the Hardy–Littlewood maximal function on \( X \), namely, for all \( x \in X \),

\[ \mathcal{M}(f)(x) = \sup_{B \ni x} \frac{1}{\mu(B)} \int_B |f(y)| d\mu(y), \]

where the supremum is taken over all balls \( B \) containing \( x \). The following lemma on the properties of \( A_\infty(X) \) is Lemma 2.8 in [9].

**Lemma 2**

(i) \( A_1(X) \subset A_p(X) \subset A_q(X) \) for \( 1 \leq p \leq q < \infty \).

(ii) If \( \varphi \in A_p(X) \) with \( p \in (1, \infty) \), then there exist some \( q \in (1, p) \) such that \( \varphi \in A_q(X) \).

(iii) If \( \varphi \in A_p(X) \) with \( p \in (1, \infty) \), then there exists a constant \( C \) such that for all measurable functions \( f \) on \( X \) and \( t \in [0, \infty) \),

\[ \int_X \mathcal{M}(f)(x)^p \varphi(x, t) d\mu(x) \leq C \int_X |f(x)|^p \varphi(x, t) d\mu(x). \]

(iv) If \( \varphi \in A_p(X) \) with \( p \in (1, \infty) \), then there exists a constant \( C \) such that for all balls \( B \subset X \) and measurable set \( E \subset B \) and \( t \in [0, \infty) \),

\[ \varphi(B, t)^p \leq C \left[ \frac{\mu(B)}{\mu(E)} \right]^p. \]

We now introduce the notion of growth functions (cf. [8, 9]).

**Definition 5.** Let \( \varphi : X \times [0, \infty) \to [0, \infty) \). Then, \( \varphi(x, t) \) is a Musielak–Orlicz function, if

\[ \int_X \varphi \left( x, \left[ \sum_j \mathcal{M}(f_j)(x)^p \right]^{1/p} \right) d\mu(x) \leq C \int_X \varphi \left( x, \left[ \sum_j |f_j(x)|^p \right]^{1/p} \right) d\mu(x). \]

2.3. Musielak–Orlicz Spaces. In this section, we study the Musielak–Orlicz spaces associated with the growth function \( \varphi \).

**Definition 7.** The Musielak–Orlicz space \( L^\varphi(X) \) denotes the set of all measurable functions \( f \) on \( X \) such that

\[ \|f\|_{L^\varphi(X)} = \inf \left\{ \lambda \in (0, \infty) : \int_X \varphi \left( x, \frac{|f(x)|}{\lambda} \right) d\mu(x) \leq 1 \right\}. \]

The space \( L^\varphi(\ell^p, X) \) is defined to be the set of all \( \{f_j\}_{j \in \mathbb{Z}} \) satisfying \( \sum_j |f_j|^p \in L^\varphi(X) \) and let

\[ \left\| \{f_j\} \right\|_{L^\varphi(\ell^p, X)} = \left\| \left[ \sum_j |f_j|^p \right]^{1/p} \right\|_{L^\varphi(X)}. \]

We have the following Fefferman–Stein vector-valued inequality of Musielak–Orlicz type (cf. [15]).

**Lemma 4.** Let \( p \in (1, \infty) \), \( \varphi \) be a Musielak–Orlicz function with uniformly lower type \( p_1 \) and upper type \( p_2 \), \( q \in (1, \infty) \), and \( \varphi \in A_q(X) \). If \( q(\varphi) < p_1 \leq p_2 < \infty \), then there exists a constant \( C > 0 \) such that, for all \( \{f_j\}_{j \in \mathbb{Z}} \in L^\varphi(\ell^p, X) \),

\[ \int_X \varphi \left( x, \left[ \sum_j \mathcal{M}(f_j)(x)^p \right]^{1/p} \right) d\mu(x) \leq C \int_X \varphi \left( x, \left[ \sum_j |f_j(x)|^p \right]^{1/p} \right) d\mu(x). \]
\textbf{Corollary 1.} Let \( p, \varphi \) be as in Lemma 4. Then, for all \( r \in (0, (p_1/q(\varphi))) \) and \( \{f_j\}_{j \in \mathbb{Z}} \in L^p(\ell^q, X) \), there exists a constant \( C > 0 \) such that

\[
\int_X \varphi \left( x, \left[ \sum_j \mathcal{M}(f_j)(x) \right]^{1/pr} \right) \mu(x) \leq C \int_X \varphi \left( x, \left[ \sum_j \left| f_j(x) \right| \right]^{1/pr} \right) \mu(x).
\]

\textit{Proof.} Fix \( r \in (0, (p_1/q(\varphi))) \) and let \( \varphi(x,t) = \varphi(x,t^{1/r}) \). We claim that \( \varphi \) is of uniformly lower type \( p_1/r \) and upper type \( p_2/r \). In fact, there exist constants \( C_1, C_2 > 0 \) such that

\[
\varphi(x, \alpha) \leq C_1 t^{p_1/r} \varphi(x, s^{1/r}),
\]

\[
\varphi(x, \alpha) \leq C_2 t^{p_2/r} \varphi(x, s^{1/r}).
\]

Hence, Lemma 4 yields

\[
\int_X \varphi \left( x, \left[ \sum_j \mathcal{M}(f_j)(x) \right]^{1/pr} \right) \mu(x) \leq C \int_X \varphi \left( x, \left[ \sum_j \left| f_j(x) \right| \right]^{1/pr} \right) \mu(x).
\]

It finishes the proof of Corollary 1. \( \square \)

\section*{2.4. AT_{X,M}-Family and Decomposition Theorem.} In this section, we assume that the space \( X \) satisfies the strong homogeneity property \( (9) \) with homogeneous dimension \( n \). In view of Lemma 1, the space \( X \) possesses a dyadic decomposition analogous to the Euclidean dyadic cubes, i.e., there exists a collection of open subsets \( \{Q_k \in X: k \in \mathbb{Z}, \alpha \in \mathbb{I}_k \} \) such that for every \( k \in \mathbb{Z} \),

\[
X = \bigcup_{\alpha \in \mathbb{I}_k} Q_k^\alpha,
\]

where \( I_k \) is some index set and \( Q_k^\alpha \) has the properties as in Lemma 1. Such open subsets \( \{Q_k^\alpha \in X: k \in \mathbb{Z}, \alpha \in \mathbb{I}_k \} \) are said to be a family of dyadic cubes of \( X \) (cf. [61]).

\textbf{Definition 8.} Suppose that the operator \( L \) satisfies \( (H1) \) and \( (H2) \) and \( M \in \mathbb{N} \). Then, a collection of functions \( \{a_Q\}_{Q \ 	ext{dyadic}} \) in \( L^2(X) \) is said to be an \( \text{AT}_{X,M} \)-family associated with an operator \( L \), if for every dyadic \( Q \), there exists a function \( \mathcal{D}(L^2M) \) such that

(i) \( a_Q = L^M(b_Q)(x) \),

(ii) \( \text{supp}(L^k(b_Q)) \subset 3Q, k = 0, 1, \ldots, 2M \),

(iii) \( |\ell(Q)^2L^k(b_Q)(x)| \leq \ell(Q)^2V(Q)^{-1/2}, k = 0, 1, \ldots, 2M \).

\textbf{Proposition 1.} Suppose that the operator \( L \) satisfies \( (H1) \) and \( (H2) \). Let \( f \in L^2(X) \). Then, for \( M \in \mathbb{N} \), \( f \) has an \( \text{AT}_{X,M} \)-expansion

\[
f = \sum_{Q \ 	ext{dyadic}} s_Q a_Q.
\]

Moreover, let \( Q_k^\alpha \) and \( \delta \) be as in Lemma 1. Then,
\[
S_{Q_0}^d = \left( \int_{\delta - n}^{\delta} \int_{Q_0^d} r^2 L e^{-r^2 L} f(y) d\mu(y) \frac{dr}{t} \right)^{2}.
\]

**Proof.** The proof of Proposition 1 can be found in [6, Theorem 3.2]. \(\Box\)

### 3. The Proof of Theorem 2

In this section, we establish a characterization of the Musielak–Orlicz–Hardy space \(H_{p,L}^1\), where the operator \(L\) satisfies (H1) and (H2) and \(\varphi\) is a growth function.

For every \(v \in (0, \infty)\) and \(x \in X\), let \(\Gamma_v(x) = \{(y, t) \in X \times (0, \infty) : d(x, y) < vt\}\) be the cone of aperture \(v\) and vertex \(x \in X\). For any closed subset \(F\) of \(X\), we denote the union of all cones with vertices in \(F\) by

\[
\mathcal{R}_v(F) = \bigcup_{x \in F} \Gamma_v(x).
\]

When \(v = 1\), \(\Gamma(x)\) and \(\mathcal{R}(F)\) stand for \(\Gamma_1(x)\) and \(\mathcal{R}_1(x)\), respectively. Given an open subset \(O\) of \(X\), we establish Lemma 5 of \(\mathcal{R}(O^c)\) on the geometric properties. We also remark that Aguilera and Segovia [16] obtained the same result in the case of Euclidean space.

**Lemma 5.** Suppose that \((X, d, \mu)\) is a space of homogeneous type and there exists a constant \(C_D > 1\) such that (8) holds. Let \(O\) be an open subset of \(X\) and \(F = O^c\). For \(v > 1\), we denote \(O^*\) by

\[
O^* = \{ x \in X : \mathcal{M}(\chi_O)(x) > (4v)^{-2} \log C_D \},
\]

and write \(F^* = (O^*)^c\). Then,

(i) \(\mathcal{R}_v(F^*) \subset \mathcal{R}(F)\).

(ii) There exists a constant \(C_v\) such that

\[
V(x, t) < C_v \mu(B(z, t \cap F)
\]

holds for \((z, t) \in \mathcal{R}_v(F^*)\).

**Proof.** It suffices to show that the lemma holds when \(\mathcal{R}_v(F^*) \neq \emptyset\) since it is trivial if \(\mathcal{R}_v(F^*) = \emptyset\). We first prove (i) on the condition that \(\mathcal{R}_v(F^*) \neq \emptyset\), which implies \(O \neq X\).

Let \((z, t) \in \mathcal{R}_v(F^*)\). We thus have \(z \in F\) or \(z \in O\). It is easy to see that \((z, t) \in \mathcal{R}_v(F)\) since \(d(z, z) = 0 < t\) in the case \(z \in F\) and then \(\mathcal{R}_v(F^*) \subset \mathcal{R}(F)\) holds. The proof of (i) is reduced to the verification in the case \(z \in O\).

Suppose \(z \in O\) and let \(\delta = \text{dist}(z, F)\). Then, \(0 < \delta < \infty\) and \(B(z, \delta) \subset O\) since \(F\) is closed and non-empty. For every \((z, t) \in \mathcal{R}_v(F^*)\), we have \(y \in F^*\) such that \(d(z, y) < vt\). Thus, writing \(r = \delta + d(z, y)\), we get \(B(z, \delta) \subset B(y, r)\) and

\[
B(z, \delta) \subset B(z, \delta \cap O \subset B(y, r) \cap O.
\]

Hence,

\[
V(z, t) < \mu(B(y, r) \cap O) \leq (4v)^{-2} \log C_D V(y, t).
\]

By using (10) twice, we have

\[
V(y, r) \leq C_D r e^{r^2 L} \mu(B(z, \delta) \cap O)
\]

and then

\[
\delta \leq \frac{r}{2v} = \frac{\delta + d(z, y) < vt}{2v}.\]

It follows that \(\delta < t\) since \(v > 1\). Recalling the definition of \(\delta\), we get \(x \in F\) such that \(d(x, z) < t\), which implies \((z, t) \in \mathcal{R}(F)\). It completes the proof of (i).

Next, we prove (ii). Given \((z, t) \in \mathcal{R}_v(F^*)\), we get \(y \in F^*\) such that \(d(z, y) < vt\). Thus, \(B(z, t) \subset B(y, (1 + vt)t)\) and

\[
\mu(B(z, t) \cap O) \leq \mu(B(y, (1 + vt)t) \cap O)
\]

and then

\[
\mu(B(z, t) \cap O) \leq (4v)^{-2} \log C_D V(y, (1 + vt)t).
\]

Therefore,

\[
\mu(B(z, t) \cap O) \leq (4v)^{-2} \log C_D V(y, t)
\]

and

\[
\mu(B(z, t) \cap O) \leq (4v)^{-2} \log C_D V(y, t)
\]

\[
(1 + r^{-1} d(y, z)) \log C_D V(z, t)
\]

\[
\leq \frac{1 + v}{2v}^{2} \log C_D V(z, t).
\]

We obtain

\[
\left[ 1 - \frac{1 + v}{2v}^{2} \log C_D \right] V(z, t) < \mu(B(z, t) \cap F),
\]

since \(V(z, t) = \mu(B(z, t) \cap O) + \mu(B(z, t) \cap F)\), and complete the proof of (ii). It finishes the proof of Lemma 5. \(\Box\)

For all \(v \in (0, \infty), f \in L^2(X)\) and \(x \in X\), the variant Lusin-area function associated with \(L\) is given by

\[
S_{L,v}(f)(x) = \left( \int_{0}^{\infty} \int_{d(x,y)<vt} |r^2 L e^{-r^2 L}(f)(y)| \frac{d\mu(y)}{V(x, t)} \frac{dr}{t} \right)^{1/2}.
\]

**Lemma 6.** Suppose that the operator \(L\) satisfies (H1) and (H2). Let \(\varphi \in A_p(X)\) for \(1 \leq p < \infty\) and \(O, O^*\), \(F, F^*\) be as in Lemma 5. Then, there exists a finite constant \(C\), which is independent of \(O\), such that for all \(x \in (0, \infty)\) and \(f \in L^2(X)\),

\[
\int_{F} |S_{L,v}(f)(x)|^2 \varphi(x, \lambda) d\mu(x) \leq C \int_{F} |S_{L}(f)(x)|^2 \varphi(x, \lambda) d\mu(x),
\]

where \(S_L\) is the short hand of \(S_{L,L}^1\).
\[ V(x, t)^{-1} \leq C_D \left( 1 + r^{-1}d(x, y) \right)^{\log_{C_D} V(y, t)^{-1}} \]
\[ < C_D (1 + r)^{\log_{C_D} V(y, t)^{-1}}. \]

\[ \quad \int_{F, \rho} |S_{L,v}(f)(x)|^2 \varphi(x, \lambda) d\mu(x) \leq C_D (1 + r)^{\log_{C_D} \int_{F, \rho} \left( \int_{\gamma, (x)} \left| t^2 \cdot \frac{\log_{\lambda} (y)}{V(y, t)} \right|^2 \varphi(x, \lambda) d\mu(x) \right) dt} \]
\[ = C_{v, D} \int_{\gamma, (x)} \left| t^2 \cdot \frac{\log_{\lambda} (y)}{V(y, t)} \right|^2 \varphi(B(y, v, t) \cap F, \lambda) \frac{d\mu(y) dt}{t}. \]

Then, applying Lemma 2 to the sets \( B(y, t) \) and \( B(y, v, t) \), respectively, we get
\[ \varphi(B(y, v, t), \lambda) \leq C(2v)^{\log_{C_D} \varphi(B(y, t), \lambda)}, \]
\[ \varphi(B(y, t), \lambda) \leq C \left( \frac{V(y, t)}{\mu(B(y, t) \cap F)} \right)^{\log_{C_D} \varphi(B(y, t) \cap F, \lambda)}. \]

Therefore, (47), (48), and Lemma 5 yield
\[ \varphi(B(y, v, t), \lambda) \leq C \varphi(B(y, t) \cap F, \lambda). \]

Thus, \( \int_{F, \rho} |S_{L,v}(f)(x)|^2 \varphi(x, \lambda) d\mu(x) \) is bounded by
\[ C \int_{F, \rho} \left| t^2 \cdot \frac{\log_{\lambda} (y)}{V(y, t)} \right|^2 V(y, t)^{-1} \varphi(B(y, t) \cap F, \lambda) \frac{d\mu(y) dt}{t}. \]

Finally, in view of \( \gamma, (F^*) \subset \gamma, (F) \) (see Lemma 5), it follows immediately that (50) is bounded by
\[ C \int_{\gamma, (F)} \left| t^2 \cdot \frac{\log_{\lambda} (y)}{V(y, t)} \right|^2 V(y, t)^{-1} \varphi(B(y, t) \cap F, \lambda) \frac{d\mu(y) dt}{t} \]
\[ = C \int_{F} \left( \int_{\gamma, (x)} \left| t^2 \cdot \frac{\log_{\lambda} (y)}{V(y, t)} \right|^2 \varphi(x, \lambda) d\mu(x) \right) dt \]
\[ \leq C \int_{F} |S_{L}(f)(x)|^2 \varphi(x, \lambda) d\mu(x), \]

where we use the fact that
\[ V(y, t)^{-1} \leq C_D \left( 1 + r^{-1}d(x, y) \right)^{\log_{C_D} V(x, t)^{-1}} < C_D \frac{V(y, t)^{-1}}{V(x, t)^{-1}}, \]
\[ \quad \text{for } (y, t) \in \Gamma, (x) \text{ in the last line. It finishes the proof of Lemma 6.} \]

**Lemma 7.** Suppose that the operator \( L \) satisfies (H1) and (H2). Let \( \varphi \) be a growth function and \( \varphi \in A_q(X) \) for \( 1 < q < \infty \). Then, there exists a constant \( C, > 0 \) such that
\[ \int_X \varphi(x, S_{L,v}(f)(x)) d\mu(x) \leq C \int_X \varphi(x, S_{L}(f)(x)) d\mu(x) \]
holds for all \( v \in (0, \infty) \) and all measurable functions \( f \).

**Proof.** It suffices to show that Lemma 7 holds in the case \( v \in (1, \infty) \) since the conclusion is trivial if \( v \in (0, 1] \). Given \( \lambda \in (0, \infty) \), we introduce the notations
\[ O_1 = \left\{ x \in X : S_{L,v}(f)(x) > \lambda \right\}, \]
\[ O_1^* = \left\{ x \in X : \mathcal{M}(\lambda, (O_1))(x) > (4v)^{-\log_{C_D} \varphi} \right\}, \]
where \( \mathcal{M} \) is the Hardy--Littlewood maximal function. Noting that \( \varphi \in A_q(X) \), Lemma 2 yields
\[ \varphi(O_1, \lambda) = \varphi \left( \left\{ x \in X : \mathcal{M}(\lambda, (O_1))(x) > (4v)^{-\log_{C_D} \varphi} \right\} \right) \]
\[ \leq \int_X (4v)^{\log_{C_D} \varphi} \varphi(x, \lambda) d\mu(x) \]
\[ \leq C \varphi(O_1, \lambda). \]

Writing \( F_1 = O_1^* \), \( F_1^* = O_1^* \) and applying Lemma 6, we obtain
\[ \int_{F_1} |S_{L,v}(f)(x)|^2 \varphi(x, \lambda) d\mu(x) \leq \int_{F_1} |S_{L}(f)(x)|^2 \varphi(x, \lambda) d\mu(x). \]

Thus, by using (55) and (56), we have
\[
\varphi \left( \left\{ x \in X : S_{L,v}(f)(x) > \lambda \right\} \right) \\
\leq \varphi(O^*, \lambda) + \varphi \left( \left\{ x \in F_1^* : S_{L,v}(f)(x) > \lambda \right\} \right) \\
\leq C \varphi(O_1, \lambda) + \frac{1}{\lambda^2} \int_{F_1} |S_{L,v}(f)(x)|^2 \varphi(x, \lambda) d\mu(x) \\
\leq C \left[ \varphi(O_1, \lambda) + \frac{1}{\lambda^2} \int_{F_1} |S_{L}(f)(x)|^2 \varphi(x, \lambda) d\mu(x) \right] \\
\leq C \left[ \varphi(O_1, \lambda) + \frac{1}{\lambda^2} \int_0^1 t \varphi \left( \left\{ x \in X : S_{L}(f)(x) > t \right\} \right) dt \right]. \\
\tag{57}
\]

\[
\int_X \varphi(x, S_{L,v}(f)(x)) d\mu(x) \leq C \int_0^\infty \frac{1}{\lambda} \varphi \left( \left\{ x \in X : S_{L,v}(f)(x) > \lambda \right\} \right) d\lambda \\
\leq C \int_0^\infty \frac{1}{\lambda} \varphi(O_1, \lambda) d\lambda + C \int_0^\infty \frac{1}{\lambda^2} \int_0^t \varphi \left( \left\{ x \in X : S_{L}(f)(x) > t \right\} \right) dt d\lambda \\
\leq C \int_0^\infty \frac{1}{\lambda} \varphi \left( \left\{ x \in X : S_{L}(f)(x) > \lambda \right\} \right) d\lambda \\
\quad + C \int_0^\infty \frac{1}{\lambda^2} \int_0^t \varphi \left( \left\{ x \in X : S_{L}(f)(x) > t \right\} \right) dt d\lambda \\
\leq C \int_X \varphi(x, S_L(f)(x)) d\mu(x) + C \int_0^\infty \varphi \left( \left\{ x \in X : S_L(f)(x) > t \right\} \right) d\lambda \\
\quad \int_0^\infty \frac{\lambda^2}{\lambda^2} dt \\
\leq C \int_X \varphi(x, S_L(f)(x)) d\mu(x). \\
\tag{58}
\]

It finishes the proof of Lemma 7. \hfill \square

Lemma 8 says that the sequence \( \{s_{Q_x^k}\}_{x \in I_k} \) can be majorized by the Hardy–Littlewood maximal operator \( M \) on \((X, d, \mu)\) (cf. [17], pp.147, where we take \( r = 1 \)).

**Lemma 8.** Suppose \( 0 < q \leq 1 \) and \( N > (n/q) \). Fix \( k \in \mathbb{Z} \) and let \( \{s_{Q_x^k}\}_{x \in I_k} \) be as in Proposition 1. Then, for any subsequence \( I_{k'} \subseteq I_k \) and for every \( x \in X \),

\[
\sum_{a \in I_{k'}} \left[ \frac{1}{1 + \varepsilon \left( Q_x^k \right)^{-1}} d(x, y_a^k) \right]^{1/q} \leq C \left[ \mathcal{M} \left( \sum_{a \in I_{k'}} \left| s_{Q_x^k} \right|^q \right) \chi_{Q_x^k} \right]^{1/q},
\]

where \( y_a^k \) is the center of \( Q_x^k \) and \( C \) depends only on \( n \) and \( N - (n/q) \).

Therefore, we employ (57) together with the assumption \( v \in (1, \infty) \), Lemma 3, and the uniformly upper type 1 of \( \varphi \) to get

\[
\int_X \varphi \left( x, \frac{S_L(f)(x)}{\lambda} \right) d\mu(x) \leq C \int_X \varphi \left( x, \frac{|W_f(x)|}{\lambda} \right) d\mu(x). \\
\tag{60}
\]

In fact, since (60) holds for all \( \lambda \in (0, \infty) \), there exists a constant \( C_0 \) such that

\[
\int_X \varphi \left( x, \frac{S_L(f)(x)}{\lambda} \right) d\mu(x) \leq C_0 \int_X \varphi \left( x, \frac{|W_f(x)|}{\lambda} \right) d\mu(x) \leq C_0. \\
\tag{61}
\]

Using (12) and (61), we have
\[
\int_X \varphi \left( x, \frac{G_t(f)(x)}{C_t \delta_1} \right) d\mu(x) \leq 1, \quad (62)
\]
for some constant \( C_1 \), which implies \( \lambda_0 \leq C_1 \lambda_1 \). Analogously, there exists a constant \( C_2 \) such that \( \lambda_1 \leq C_2 \lambda_0 \) and we get the desired result.

We now turn to prove (60). Given \((x, k) \in X \times \mathbb{Z}_+\), by Lemma 1, there exists a unique \( \alpha \in I_k \) such that \( x \in Q^k_{\alpha} \). We denote such \( Q^k_{\alpha} \) by \( Q^k_{x} \) and write

\[
W_f(x) = \left\{ \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} \left[ \mu(Q^k_{\alpha})^{-1} \right] \left| s_{Q^k_{\alpha}}(x) \right|^2 \right\}^{1/2} \tag{63}
\]

where constants \( \delta \in (0, 1) \) satisfy Lemma 1 and the last line is obtained by using Proposition 1.

Moreover, for any fixed \((x, k) \in X \times \mathbb{Z}_+\), Lemma 1 also tells us that there are \( \tilde{x}_k \in Q^k_{x} \) and constants \( C_1 \in (0, 1), C_2 = \delta^{-1} \) such that

\[
B(\tilde{x}_k, C_1 \delta^k) \subset Q^k_{x} \subset B(x, C_2 \delta^k) \subset B(x, C_2 \delta^{k-1}), \quad (64)
\]

for all \( t \in (\delta^{k+1}, \delta^k) \). Consequently,

\[
W_f(x) \leq C \left\{ \sum_{k \in \mathbb{Z}} \int_{\delta^{k+1}}^{\delta^k} V(x, t)^{-1} \int_{B(x, C_2 \delta^{k-1})} \left| t^2 L e^{-tI} f(y) \right|^2 \frac{d\mu(y)}{t} \right\}^{1/2} \tag{66}
\]

Thus, by using Lemma 7, we deduce that

\[
\int_X \varphi \left( x, \frac{S_{\alpha}(f)(x)}{\lambda} \right) d\mu(x) \geq C \int_X \varphi \left( x, \frac{W_f(x)}{\lambda} \right) d\mu(x). \tag{67}
\]

It remains to establish the reverse inequality of (67). Let \( \delta \) be as in Lemma 1. In view of Proposition 1, we write

\[
f = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} s_{Q^k_{\alpha}} a_{Q^k_{\alpha}}, \tag{68}
\]

and get
\[ S_L(f)(x) = \left( \int_0^\infty \int_{d(x,y)<t} \left| t^2 L e^{-t L} f(y) \right|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2} \]

\[ = \left( \int_0^\infty \int_{d(x,y)<t} \left| t^2 L e^{-t L} \left( \sum_{k \in Z \alpha \in I_k} s_{Q_k} a_{Q_k} \right) \right|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2} \]

\[ = \left( \sum_{j \in Z} \int_{\delta} \int_{d(x,y)<t} \left| t^2 L e^{-t L} \left( \sum_{k \in Z \alpha \in I_k} s_{Q_k} a_{Q_k} \right) \right|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2} \]

\[ \leq \left( \sum_{j \in Z} \int_{\delta} \int_{d(x,y)<t} \left| t^2 L e^{-t L} \left( \sum_{k \in Z \alpha \in I_k} s_{Q_k} a_{Q_k} \right) \right|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2} \]

\[ + \left( \sum_{j \in Z} \int_{\delta} \int_{d(x,y)<t} \left| t^2 L e^{-t L} \left( \sum_{k \in Z \alpha \in I_k} s_{Q_k} a_{Q_k} \right) \right|^2 \frac{d\mu(y)}{V(x,t)} \frac{dt}{t} \right)^{1/2} \]

\[ = I_1 + I_2. \]

We firstly estimate the inner integral of \( I_1 \). For any \( k > j \) and \( \alpha \in I_k \), noting \( a_{Q_k} = L^M b_{Q_k} \), we have

\[ \left| t^2 L e^{-t L} (a_{Q_k})(y) \right| = \left| t^2 L^{M+1} e^{-t L} (b_{Q_k})(y) \right| = t^{-2M} \left| \left( t^2 L \right)^{M+1} e^{-t L} (b_{Q_k})(y) \right|. \]  
\[ (70) \]

Since \( M > (nq(q)/2p_1) \) with \( n \) given as in (9), we can choose some \( q \) satisfying Corollary 1 such that \( 2M > n/q \). Thus, there is some \( N > 0 \) such that \( 2M > N > (n/q) \). Then, applying Definition 8, the upper bound of the kernel \( (t^2 L)^{M+1} e^{-t L} \) (cf. [18], Proposition 3.1), and (11), we get

\[ \left( \int_{d(x,y)<t} \left| t^2 L e^{-t L} (a_{Q_k})(y) \right|^2 \frac{d\mu(y)}{V(x,t)} \right)^{1/2} \]

\[ \leq C t^{-2M} \ell(Q_k^{t})^{2M} \mu(Q_k^{t})^{-1/2} \left( \int_{d(x,y)<t} t + d(x,y) \right)^{2N} \frac{d\mu(y)}{V(x,t)} \]

\[ \leq C t^{-2M} \ell(Q_k^{t})^{2M} \mu(Q_k^{t})^{-1/2} (1 + t^{-1} d(x,z_k^t))^{-N}, \]  
\[ (72) \]

Hence, Lemma 8 yields the inner integral of \( I_1 \) which is bounded by
\[
\left( \int_{d(x,y)<t} t^2 e^{-t^2 L} \left( \sum_{k>j} \sum_{a \in I_k} s_{Q_a} \alpha Q_a \right)(y) \right)^{1/2} \frac{d\mu(y)}{V(x,t)} \leq C \sum_{k>j} \sum_{a \in I_k} t^{-2M} \ell(Q_a) \mu(Q_a)^{-1/2} \left| s_{Q_a} \right| \left[ \frac{1 + t^{-1} d(x, z_a)}{1 + \ell(Q_a)^{-1} d(x, z_a)} \right]^N
\]
\[
\leq C \sum_{k>j} \delta(2M-N) (k-j) \sum_{a \in I_k} \mu(Q_a)^{-1/2} \left| s_{Q_a} \right| \left[ \frac{1 + \ell(Q_a)^{-1} d(x, z_a)}{1 + \ell(Q_a)^{-1} d(x, z_a)} \right]^N
\]
\[
\leq C \sum_{k>j} \delta(2M-N) (k-j) \left[ \mathcal{M} \left( \sum_{a \in I_k} |Q_a| \mu(Q_a)^{-2} \chi_Q(x) \right)(x) \right]^{1/q}.
\]

Secondly, we estimate the inner integral of \( I_2 \). For any \( k \leq j \) and \( \alpha \in I_k \), we write
\[
\left| t^2 e^{-t^2 L} \left( \sum_{k>j} \sum_{a \in I_k} s_{Q_a} \alpha Q_a \right)(y) \right|^2 = t^2 e^{-t^2 L} \left( \sum_{k>j} \sum_{a \in I_k} s_{Q_a} \alpha Q_a \right)(y).
\]

Then, using Definition 8, Gaussian estimate (1), and inequality (11), we obtain
\[
\left| t^2 e^{-t^2 L} \left( \sum_{k>j} \sum_{a \in I_k} s_{Q_a} \alpha Q_a \right)(y) \right| \leq C t^2 \ell(Q_a)^{-2} \mu(Q_a)^{-1/2} \left( 1 + \ell(Q_a)^{-1} d(y, z_a) \right)^{-N}.
\]

Since \( d(x, y) < t \leq \ell(Q_a) \), we further have
\[
\left( \int_{d(x,y)<t} \left| t^2 e^{-t^2 L} \left( \sum_{k>j} \sum_{a \in I_k} s_{Q_a} \alpha Q_a \right)(y) \right|^2 \frac{d\mu(y)}{V(x,t)} \right)^{1/2} \leq C t^2 \ell(Q_a)^{-2} \mu(Q_a)^{-1/2} \left( \int_{d(x,y)<t} \left( \ell(Q_a) + d(x, y) \right)^{2N} \frac{d\mu(y)}{V(x,t)} \right)^{1/2}
\]
\[
\leq C t^2 \ell(Q_a)^{-2} \mu(Q_a)^{-1/2} \left( 1 + \ell(Q_a)^{-1} d(x, z_a) \right)^{-N}.
\]

Hence, Lemma 8 yields the inner integral of \( I_2 \) which is bounded by
\[
\begin{aligned}
&\left( \int_{d(x,y) \neq 0} t^2 L e^{-\frac{t^2 L}{2}} \left( \sum_{k \leq j} \sum_{n \in I_k} s_{Q_n} a_{Q_n}^j \right)(y) \right)^{1/2} \frac{d\mu(y)}{V(x,t)} \\
\leq C \sum_{k \leq j} \sum_{n \in I_k} t^2 e^{Q_n^j} \mu(Q_n^j)^{(1/2)} \left( s_{Q_n}^j \right) \left[ 1 + \ell(Q_n^j)^{-1} d(x,z_n^j) \right]^N \\
\leq C \sum_{k \leq j} \delta_2^{(j-k)} \sum_{n \in I_k} \mu(Q_n^j)^{(1/2)} \left( s_{Q_n}^j \right) \left[ 1 + \ell(Q_n^j)^{-1} d(x,z_n^j) \right]^N \\
\leq C \sum_{k \leq j} \delta_2^{(j-k)} \left[ \mathcal{M} \left( \sum_{n \in I_k} s_{Q_n}^j \mu(Q_n^j)^{\frac{2}{q}} \alpha \chi_{Q_n^j} \right)(x) \right]^{1/2}.
\end{aligned}
\] (77)

Fix \( j \in \mathbb{Z} \), and we let \( \beta > 0 \) and
\[
\tau = \begin{cases} 
1, & k > j \\
-1, & k \leq j.
\end{cases}
\] (78)

Writing \( F_k(x) = \mathcal{M} \left( \sum_{n \in I_k} s_{Q_n}^j \mu(Q_n^j)^{\frac{2}{q}} \alpha \chi_{Q_n^j} \right)(x) \), we now turn to estimate
\[
J = \left( \sum_k \delta_2^{\tau(k-j)} F_k(x)^{(1/2)} \right)^2.
\] (79)

Since
\[
\begin{aligned}
S_j^\tau (f)(x) \leq C \left( \sum_k \int_{j \in \mathbb{Z}} \delta_2^{\tau(k-j)} F_k(x) \left( \sum_{k \geq j} \delta_2^{(2M-N)(k-j)} \right) \right)^{1/2} \\
+ \sum_{j \in \mathbb{Z}} \int_{j \in \mathbb{Z}} \delta_2^{(2M-N)(k-j)} \left( \sum_{k \geq j} \delta_2^{(j-k)} F_k(x) \right) \left( \sum_{k \geq j} \delta_2^{(2M-N)(k-j)} \right) \left( \sum_{k \geq j} \delta_2^{(j-k)} \right) ^{1/2} \\
\leq C \left( \sum_k \int_{j \in \mathbb{Z}} \delta_2^{\tau(k-j)} F_k(x) \left( \sum_{k \geq j} \delta_2^{(2M-N)(k-j)} \right) \right)^{1/2} \\
+ \sum_{j \in \mathbb{Z}} \int_{j \in \mathbb{Z}} \delta_2^{(2M-N)(k-j)} \left( \sum_{k \geq j} \delta_2^{(j-k)} F_k(x) \right) \left( \sum_{k \geq j} \delta_2^{(2M-N)(k-j)} \right) \left( \sum_{k \geq j} \delta_2^{(j-k)} \right)^{1/2} \\
\leq C \left( \sum_{k \in \mathbb{Z}} F_k(x) \right)^{2/2}.
\end{aligned}
\] (82)

Therefore, (82) and Corollary 1 yield

\[
\delta_2^{\tau(k-j)} = \frac{\beta \delta_2^{\beta}}{1 - \delta_2^{\beta}} \int_{\delta_2^{\tau(k-j)}} s^{\beta-1} \, ds,
\] (80)

we have
\[
J = C_\beta \left( \sum_k \int_{\delta_2^{\tau(k-j)}} F_k(x)^{(1/2)} s^{\beta-1} \, ds \right)^2
\]
\[
= C_\beta \left( \int_0^1 \left( \sum_k \delta_2^{\tau(k-j)} \right)^2 \right)^{1/2}
\]
\[
\leq C_\beta \left( \int_0^1 s^{\beta-1} \, ds \right) \left( \sum_k \delta_2^{\tau(k-j)} \right)^2
\]
\[
\leq C_\beta \int_0^1 \left( \sum_k \delta_2^{\tau(k-j)} \right)^2 s^{\beta-1} \, ds
\]
\[
= C_\beta \sum_k \frac{\delta_2^{\tau(k-j)} F_k(x)^{2/2}}{\delta_2^{\tau(k-j)}}
\]
\[
= C_\beta \sum_k \delta_2^{\tau(k-j)} F_k(x)^{2/2},
\] (81)

where \( E_k = [\delta_2^{\tau(k-j)}, \delta_2^{\tau(k-j)-1}] \). In view of inequalities (73)–(81), taking \( \beta = 2M - N, \tau = 1 \), and \( \beta = 2, \tau = -1 \) respectively, we get
\[
\int_X \varphi \left( x, \frac{S_k f(x)}{\lambda} \right) d\mu(x) \leq C \int_X \varphi \left( x, \lambda^{-1} \left( \sum_{k \in Z} F_k(x)^{2/q} \right)^{1/2} \right) d\mu(x)
\]

\[
\leq C \int_X \varphi \left( x, \lambda^{-1} \left( \sum_{k \in Z} \sum_{n \in I_k} \| Q^k_n \|^{\theta(q)} \mu(Q^k_n(x)) \right)^{1/2} \right) d\mu(x)
\]

\[
= C \int_X \varphi \left( x, \frac{W_f(x)}{\lambda} \right) d\mu(x),
\]

which gives the reverse inequality of (67). It finishes the proof of Theorem 2. \qed

4. The Proof of Theorem 3

In this section, we establish a characterization of the Musielak–Orlicz–Hardy space \( H_{\text{LGO}} \), where the operator \( L \) satisfies (H1) and (H2) and \( \varphi \) is a growth function. Our proof will borrow some ideals from Duong et al. [6].

We first recall some basic definitions and facts about Fefferman–Stein type maximal function, referring to [7] for a complete account.

Given \( f \in L^2(X), a > 0, \) and \((x, t) \in X \times (0, \infty), \) the Fefferman–Stein type maximal function is defined as

\[
\mathcal{M}^\ast_{aL}(f)(x, t) = \text{ess sup}_{y \in X} \frac{|t^2 L e^{-t^2} f(y)|}{1 + t^2 d(x, y)}.
\]

Lemma 9 is useful (cf. [7]).

**Lemma 9.** Suppose the operator \( L \) satisfies (H1) and (H2). Let \( m \) be as in (10). Then, for any \( \beta, r > 0 \) and \( \alpha > (m/2), \) there exists a constant \( C > 0 \) such that

\[
\left| \mathcal{M}^\ast_{aL}(f)(x, 2^{-j}t) \right| \leq C \sum_{j=1}^{\infty} 2^{-(j-1)a} \int_{X \times (z, 2^{-j}t)} \frac{|2^{-j}t^2 L e^{-t^2} f(x)|^r}{1 + t^2 d(x, z)} \, d\mu(z)
\]

holds for all \( f \in L^2(X), l \in \mathbb{Z}, x \in X, \) and \( t \in [1, 2). \)

We also need Lemma 10, and its proof is standard, which we omit here.

**Lemma 10.** Let \( n \) and \( m \) be as in (9) and (10), and \( N > n + m. \) Then, there exists a constant \( C > 0 \) such that

\[
\int_X \frac{|f(x)|}{xV(x, t)[1 + t^{-1} d(x, y)]^N} \, d\mu(x) \leq C \mathcal{M}(f)(y)
\]

holds for all measurable functions \( f \) on \((X, d, \mu), t > 0, \) and each \( y \in X. \)

**Proof.** of Theorem 3. Fix \( f \in H_{\text{LGO}}(X) \cap L^2(X), \) and we let \( \lambda_0 = \| f \|_{H_{\text{LGO}}(X)} \) and \( \lambda_1 = \| W_f \|_{L^p(X)}. \) It suffices to show that for all \( \lambda \in (0, \infty), \) we have

\[
\int_X \varphi \left( x, \frac{G_L f(x)}{\lambda} \right) d\mu(x) \equiv \int_X \varphi \left( x, \frac{W_f(x)}{\lambda} \right) d\mu(x).
\]

In fact, since (87) holds for all \( \lambda \in (0, \infty), \) there exists a constant \( C_0 \) such that

\[
\int_X \varphi \left( x, \frac{G_L f(x)}{\lambda} \right) d\mu(x) \leq C_0 \int_X \varphi \left( x, \frac{W_f(x)}{\lambda} \right) d\mu(x) \leq C_0.
\]

Using (12) and (88), we have

\[
\int_X \varphi \left( x, \frac{G_L f(x)}{\lambda_1} \right) d\mu(x) \leq 1,
\]

for some constant \( C_1, \) which implies \( \lambda_0 \leq C_1 \lambda_1. \) Analogously, there exists a constant \( C_2 \) such that \( \lambda_1 \leq C_2 \lambda_0 \) and we get the desired result.
We now fix arbitrary \( \lambda \in (0, \infty) \) and turn to prove (88). Given \((x, k) \in X \times Z\), by Lemma 1, there exists a unique \( a \in I_k\) such that \( x \in Q^k_a\). We denote such \( Q^k_a\) by \( Q^k\) and write

\[
W_f(x) = \sum_{k \in Z} \sum_{a \in I_k} \left[ \mu(Q^k)^{-1} \left| 3c^2 \right| \right]^{1/2}.
\]

We have

\[
W_f(x) = \sum_{k \in Z} \sum_{a \in I_k} \left[ \mu(Q^k)^{-1} \left| 3c^2 \right| \right]^{1/2}.
\]

where constants \( \delta \in (0, 1) \) satisfy Lemma 1 and the last line is obtained by using Proposition 1.

Moreover, for any fixed \((x, k) \in X \times Z\), Lemma 1 also tells us that there are \( z^k \in Q^k_a\) and constants \( C_3 \in (0, 1), C_4 > 0 \) such that

\[
B(z^k, C_4 \delta) \subset B(C, C_3 \delta) \subset B(C, C_4 \delta^{-1}) = B_x,
\]

for all \( t \in (\delta^{k+1}, \delta^k)\). Consequently, by inequalities (9) and (10), we have

\[
\text{esssup}_{y \in B_x} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \leq \text{esssup}_{y \in B_x} \left| t^2 L e^{-t^2 L} f(y) \right|^2 \left[ 1 + t^{-1} d(x, y) \right]^{2a}.
\]

Hence, (90) and (92) yield

\[
W_f(x) \leq C \left( \int_0^\infty \left[ M_{a,(a)}(f)(x, t) \right] \frac{dt}{t} \right)^{1/2}
\]

and

\[
= C \left( \sum_{k \in Z} \int_{2^k}^{2^{k+1}} \left[ M_{a,(a)}(f)(x, t) \right] \frac{dt}{t} \right)^{1/2},
\]

Thus, by using Lemma 9, we deduce that for any \( \beta, r > 0 \) and \( a > (m/2)\), there exists a constant \( C \) such that

\[
\left[ M_{a,(a)}(f)(x, 2^{-k} t) \right] \leq C \sum_{j=k}^{\infty} 2^{(j-k)\beta} \int_{xV(z, 2^{-k} t)} \left[ 2^{-j} \right]^{1/2} |L e^{-(2^{-j})^2} f(z)| \frac{d\mu(z)}{[1 + 2^k d(x, z)]^{ap}}.
\]
\[
\left( \int_1^2 \left| \mathcal{M}_{a,L}^* (f)(x, 2^{-k}y) \right|^2 \frac{dt}{t} \right)^{\frac{r}{2}} \leq C \left( \sum_{j=k}^{\infty} 2^{-(j-k)r} \int_1^2 \left| f(z) \right|^2 \frac{d\mu(z)}{V(z, 2^{-k})} \left[ 1 + 2^k d(x,z) \right]^{2r} \frac{dt}{t} \right)^{\frac{r}{2}}
\]

\[
= C \sum_{j=k}^{\infty} 2^{-(j-k)r} \left( \int_1^2 \left| \frac{(2^{-i}t)^2 \mathcal{L} \mathcal{L} f(z) \right|^r}{V(z, 2^{-k})} \left[ 1 + 2^k d(x,z) \right]^{2r} \frac{d\mu(z)}{V(z, 2^{-k})} \frac{dt}{t} \right)^{\frac{r}{2}}
\]

Thus, by using Corollary 1, we have

\[
\int_\mathbb{R} \varphi \left( x, \frac{|W_f(x)|}{\lambda} \right) d\mu(x) \leq \int_\mathbb{R} \varphi \left( x, \frac{C \left( \sum_{k \in \mathbb{Z}} \left[ \mathcal{M}_{a,L}^* (f)(x, 2^{-k}y) \right]^2 (dt/t) \right)^{\frac{1}{2}}}{\lambda} \right) d\mu(x)
\]

\[
= \int_\mathbb{R} \varphi \left( x, \frac{C \left( \sum_{k \in \mathbb{Z}} \left[ \left( \int_1^2 \left| \mathcal{M}_{a,L}^* (f)(x, 2^{-k}y) \right|^2 (dt/t) \right)^{\frac{r}{2}} \right] \frac{1}{t} \right)^{\frac{1}{2}}}{\lambda} \right) d\mu(x)
\]

\[
\leq \int_\mathbb{R} \varphi \left( x, \frac{C \left( \sum_{k \in \mathbb{Z}} \left( \mathcal{M} (G_k)(x) \right)^2 \right)^{\frac{r}{2}}}{\lambda} \right) d\mu(x)
\]

\[
\leq C \int_\mathbb{R} \varphi \left( x, \frac{\left( \sum_{k \in \mathbb{Z}} G_k(x) \right)^{\frac{r}{2}}}{\lambda} \right) d\mu(x).
\]

We now turn to estimate \( G_k^{2r} (x) \). Since
\[ 2^{-(j-k)\beta r} = \frac{\beta r}{1 - 2^{-\beta r}} \int_{2^{j-k}}^{2^{j-k+1}} s^{-\beta r-1} ds, \quad \text{for any } k \in \mathbb{Z}, \text{ we have} \]

\[
G_k(x) = C \sum_{j=k}^{\infty} \int_{2^{j-k}}^{2^{j-k+1}} \left[ \int_{1}^{2} \left| (2^{-j}t)^2 L e^{-(2^{-j}t)^2} f(x) \right|^2 dr \right]^{r/2} ds \left( s^{-\beta r-1} \right)^{1/r} \]

\[
= C \sum_{j=k}^{\infty} \int_{1}^{2} \left[ \int_{1}^{2} \left| (2^{-j}t)^2 L e^{-(2^{-j}t)^2} f(x) \right|^2 dr \right]^{r/2} ds \left( s^{-\beta r-1} \right)^{1/r} \chi_{E_j}(s) \frac{ds}{s^{\beta r+1}}, \quad \text{where } E_j = [2^{j-k}, 2^{j-k+1}]. \]

Using Hölder's inequality, we obtain

\[
G_k(x)^{2r} = \left[ C \int_{1}^{2} \left[ \int_{1}^{2} \left| (2^{-j}t)^2 L e^{-(2^{-j}t)^2} f(x) \right|^2 dr \right]^{r/2} ds \right]^{2/r} \times \int_{1}^{2} \left( \sum_{j=k}^{\infty} \left[ \int_{1}^{2} \left| (2^{-j}t)^2 L e^{-(2^{-j}t)^2} f(x) \right|^2 dr \right]^{r/2} ds \chi_{E_j}(s) \right)^{2/r} \frac{ds}{s^{\beta r+1}} \]

\[
= C \int_{1}^{2} \left( \sum_{j=k}^{\infty} \left( \int_{1}^{2} \left| (2^{-j}t)^2 L e^{-(2^{-j}t)^2} f(x) \right|^2 dr \right) \chi_{E_j}(s) \right) ds \frac{ds}{s^{\beta r+1}} \]

\[
= C \sum_{j=k}^{\infty} \left( \int_{2^{j-k}}^{2^{j-k+1}} ds \frac{ds}{s^{\beta r+1}} \int_{1}^{2} \left| (2^{-j}t)^2 L e^{-(2^{-j}t)^2} f(x) \right|^2 dr \right).
\]

Summarizing all \( k \in \mathbb{Z}, \text{ we have} \)
\[
\sum_{k \in \mathbb{Z}} G_k(x) x^{2r} \leq C \sum_{k \in \mathbb{Z}} \sum_{j < k} \left( 2^{-(j-k)\beta r} \int_x^\infty \left( (2^{-j})^2 e^{-\gamma j^2} f(x) \right) \frac{dt}{t} \right)
\]
\[
= C \sum_{j \in \mathbb{Z}} \sum_{k < j} \left( 2^{-(j-k)\beta r} \int_x^\infty \left( (2^{-j})^2 e^{-\gamma j^2} f(x) \right) \frac{dt}{t} \right)
\]
\[
= C \sum_{j \in \mathbb{Z}} \left( 2^{-(j)\beta r} \int_x^\infty \left( (2^{-j})^2 e^{-\gamma j^2} f(x) \right) \frac{dt}{t} \right)
\]
\[
= C \left( 2^{-\beta r} \right)^{-1} \sum_{j \in \mathbb{Z}} \left( 2^{-(j)\beta r} \int_x^\infty \left( (2^{-j})^2 e^{-\gamma j^2} f(x) \right) \frac{dt}{t} \right)
\]
\[
= C \int_0^\infty t^2 e^{-\gamma t} f(x) \frac{dt}{t} = C (G_2(f)(x))^2.
\]

Therefore, (11), (97), and (101) yield
\[
\int x \phi(x, \frac{G_2(f)(x)}{\lambda}) \, d\mu(x) \geq C \int x \phi(x, \frac{W_f(x)}{\lambda}) \, d\mu(x).
\]

(102)

It reduces to show the reverse inequality of (102). Let \( \delta \) be as in Lemma 1. In view of Proposition 1, we write
\[
f = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k},
\]

and get
\[
G_L(f)(x) = \left( \int_0^\infty t^2 e^{-\gamma t} f(x) \frac{dt}{t} \right)^{1/2}
\]
\[
= \left( \int_0^\infty t^2 e^{-\gamma t} \left( \sum_{k \in \mathbb{Z}} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k} \right) (x) \right) \frac{dt}{t} \right)^{1/2}
\]
\[
\leq \left( \sum_{\alpha \in I_k} \left( \int_0^\infty t^2 e^{-\gamma t} \left( \sum_{k \in \mathbb{Z}} s_{Q_k} a_{Q_k} \right) (x) \right) \frac{dt}{t} \right)^{1/2}
\]
\[
+ \left( \sum_{\alpha \in I_k} \left( \int_0^\infty t^2 e^{-\gamma t} \left( \sum_{k \leq j} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k} \right) (x) \right) \frac{dt}{t} \right)^{1/2}
\]
\[
= I_3 + I_4.
\]

We now estimate the integrand function of \( I_3 \). For any \( k > j \) and \( \alpha \in I_k \), noting \( a_{Q_k} = L^M b_{Q_k} \), we have
\[
\left| t^2 e^{-\gamma t} \left( \sum_{k \geq j} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k} \right) (x) \right| \leq C \sum_{k \geq j} \sum_{\alpha \in I_k} t^{-2M} \ell(Q_k)^{2M} \mu(Q_k)^{-1} \int_0^\infty d(x, y_k^\alpha)^N \right| \frac{s_{Q_k}}{1 + t^{-1} d(x, y_k^\alpha)^N}
\]
\[
\leq C \sum_{k \geq j} \delta^{(2M-N)(k-j)} \sum_{\alpha \in I_k} \mu(Q_k)^{-1} \int_0^\infty \left| \frac{s_{Q_k}}{1 + t^{-1} d(x, y_k^\alpha)^N} \right|^N \]
\[
\leq C \sum_{k \geq j} \delta^{(2M-N)(k-j)} \left[ \mathcal{M} \left( \sum_{\alpha \in I_k} \left| s_{Q_k} \right|^q \mu(Q_k)^{-q/2} \chi_{Q_k} \right) (x) \right]^{1/q}.
\]

For the integrand function of \( I_4 \) in the case \( k \leq j \) and \( \alpha \in I_k \), we write
\[
\left| t^2 e^{-\gamma t} \left( \sum_{k \leq j} \sum_{\alpha \in I_k} s_{Q_k} a_{Q_k} \right) (x) \right| \leq C \sum_{k \leq j} \sum_{\alpha \in I_k} t^{-2M} \ell(Q_k)^{2M} \mu(Q_k)^{-1} \int_0^\infty d(x, y_k^\alpha)^N \right| \frac{s_{Q_k}}{1 + t^{-1} d(x, y_k^\alpha)^N}
\]
Then, using Definition 8, Gaussian estimate (1), and inequality (11), we obtain

\[ t^2 e^{-t^2L} (a_{Q_k})(x) \leq C \sum_{k \leq j} \sum_{n \in I_k} s_{Q_n} \alpha_{Q_n} (x) \leq C t^2 \ell(Q_k)^{-2} \mu(Q_k)^{-(1/2)} \frac{\|s_{Q_n}\|}{\left[ 1 + \ell(Q_k)^{-1} d(x, y_n) \right]^N} \]  

(109)

Consequently,

\[ t^2 e^{-t^2L} \left( \sum_{k \leq j} \sum_{n \in I_k} s_{Q_n} \alpha_{Q_n} (x) \right) \leq C \sum_{k \leq j} \sum_{n \in I_k} t^2 \ell(Q_k)^{-2} \mu(Q_k)^{-(1/2)} \frac{\|s_{Q_n}\|}{\left[ 1 + \ell(Q_k)^{-1} d(x, y_n) \right]^N} \]  

(110)

Similar to the discussion in (82) and combining (104)–(110), we get

\[ G_L(f)(x) \leq C \left( \sum_{j \in \mathbb{Z}} \int_{k > j} \left| \sum_{k \leq j} \delta^{(2M-N)(k-j)} G_k(x) \right|^{1/q} \frac{dt}{t} \right)^{1/2} \]  

(111)
Therefore, (111) and Corollary 1 yield

\[
\int_X \phi\left(x, \frac{G_T(f)(x)}{\lambda}\right) d\mu(x) \leq C \int_X \phi\left(x, \lambda^{-1} \left( \sum_{k \in Z} G_k(x)^{2q} \right)^{1/2}\right) d\mu(x)
\]

\[
\leq C \int_X \phi\left(x, \lambda^{-1} \left( \sum_{k \in Z} \left( \sum_{a \in I_k} |s_{Q_k}^a|^q \mu(Q_k^{-1} x) \right)^{1/2}\right) d\mu(x)
\]

\[
= C \int_X \phi\left(x, \lambda^{-1} \left( \sum_{k \in Z} \sum_{a \in I_k} |s_{Q_k}^a|^q \mu(Q_k^{-1} x) \right)^{1/2}\right) d\mu(x)
\]

\[
= C \int_X \phi\left(x, \frac{W_f(x)}{\lambda}\right) d\mu(x),
\]

which gives the reverse inequality of (102). It finishes the proof of Theorem 2. □

Data Availability

No data were used to support this study.

Disclosure

This work is a renewed version of [19].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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