ZEROS OF FUNCTIONS IN HILBERT SPACES OF DIRICHLET SERIES

KRISTIAN SEIP

ABSTRACT. The Dirichlet–Hardy space $H^2$ consists of those Dirichlet series $\sum_n a_n n^{-s}$ for which $\sum_n |a_n|^2 < \infty$. It is shown that the Blaschke condition in the half-plane $\text{Re } s > 1/2$ is a necessary and sufficient condition for the existence of a nontrivial function $f$ in $H^2$ vanishing on a given bounded sequence. The proof implies in fact a stronger result: every function in the Hardy space $H^2$ of the half-plane $\text{Re } s > 1/2$ can be interpolated by a function in $H^2$ on such a Blaschke sequence. Analogous results are proved for the Hilbert space $D_\alpha$ of Dirichlet series $\sum_n a_n n^{-s}$ for which $\sum_n |a_n|^2 [d(n)]^\alpha < \infty$; here $d(n)$ is the divisor function and $\alpha$ a positive parameter. In this case, the zero sets are related locally to the zeros of functions in weighted Dirichlet spaces of the half-plane $\text{Re } s > 1/2$. Partial results are then obtained for the zeros of functions in $H^p$ ($L^p$ analogues of $H^2$) for $2 < p < \infty$, based on certain contractive embeddings of $D_\alpha$ in $H^p$.

1. INTRODUCTION

This paper studies the zeros of functions in certain Hilbert spaces of Dirichlet series. The prototypical case is that of the Dirichlet–Hardy space $H^2 = \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \|f\|_{H^2}^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}$, which we view as a space of analytic functions in the half-plane $\mathbb{C}_{1/2}^+ = \{ s = \sigma + it : \sigma > 1/2 \}$. Our result regarding $H^2$ is as follows.

**Theorem 1.** Suppose $S = (\sigma_j + it_j)$ is a bounded sequence of points in $\mathbb{C}_{1/2}^+$. Then there is a nontrivial function in $H^2$ vanishing on $S$ if and only if $\sum_j (\sigma_j - 1/2) < \infty$. Here multiple zeros are accounted for in the usual way: $n$ occurrences of some point in the sequence $S$ correspond to a zero of order at least $n$ at that point. Note that the summability condition on $S$ is just the Blaschke condition for bounded sequences in $\mathbb{C}_{1/2}^+$.

The “only if” part of Theorem 1 is well-known [6]; it is a consequence of the fact that if $f$ is in $H^2$, then $f(s)/s$ is in $H^2(\mathbb{C}_{1/2}^+)$. The novelty of Theorem 1 is thus the positive direction, i.e., the sufficiency of the local Blaschke condition. This result may come as no surprise to readers familiar with recent developments such as [11] and [10], which quantify the local resemblance between $H^2$ and the Hardy space $H^2(\mathbb{C}_{1/2}^+)$. It should be kept in mind, however, that in the present setting the usual one-variable tool of dividing out zeros must be abandoned, and no 2000 Mathematics Subject Classification. 30B50, 30C15, 30H10.
The author is supported by the Research Council of Norway grant 160192/V30.
direct use of Blaschke products is possible. Moreover, as shown in [6], the multiplier algebra of \( H^2 \) is the much smaller space \( H^\infty \) consisting of those functions in \( H^2 \) that extend to bounded holomorphic functions in the larger half-plane \( \mathbb{C}^+ = \{ s = \sigma + it : \sigma > 0 \} \). These facts point to a certain rigidity of our problem.

A simple argument using almost periodicity of the function \( t \mapsto f(\sigma + it) \) along with Rouché’s theorem [11] shows that a function in \( H^2 \) has either none or an infinite number of zeros. It appears to be an inaccessible problem to describe the pattern of “repetitions” of zeros in the vertical direction, and therefore a full description of the zero sets of functions in \( H^2 \) does not seem to be within reach.

The main ingredient in our construction of a nontrivial function in \( H^2 \) vanishing on a bounded Blaschke sequence is an iteration involving approximations by Dirichlet series on compact sets and solutions to certain \( \partial \) equations. We will present this proof in the next section, where we also notice that our argument implies a stronger result: The values of an arbitrary function in \( H^2(\mathbb{C}_1^+) \) can be interpolated by a function in \( H^2 \) on a bounded Blaschke sequence.

Suitably elaborated, the same technique leads to analogous results for most of the Hilbert spaces studied in [9]. In Section 3, we choose to present these results for the Hilbert spaces

\[
\mathcal{D}_\alpha = \left\{ f(s) = \sum_{n=1}^\infty a_n n^{-s} : \| f \|_{\mathcal{D}_\alpha} = \sum_{n=1}^\infty |a_n|^2 [d(n)]^\alpha < \infty \right\},
\]

where \( d(n) \) is the divisor function and \( \alpha \) a real parameter; we define \( \mathcal{D}_\infty \) as the subspace of \( H^2 \) consisting of those \( \sum_n a_n n^{-s} \) for which \( a_n = 0 \) unless \( n = 1 \) or \( n \) is a prime. We will restrict to the case \( 0 < \alpha \leq \infty \). Our reason for doing so is that this leads to nontrivial results for the zeros of functions in the Dirichlet–Hardy spaces \( H^p \) for \( p > 2 \), to be considered in Section 4.

2. PROOF OF THEOREM 1

The Hardy space \( H^2(\mathbb{C}_1^+) \) is defined as the set of functions \( f \) analytic in \( \mathbb{C}_1^+ \) for which

\[
\| f \|^2_{H^2(\mathbb{C}_1^+)} = \sup_{\sigma > 1/2} \int_{-\infty}^{\infty} |f(\sigma + it)|^2 dt < \infty.
\]

Every \( f \) in \( H^2(\mathbb{C}_1^+) \) has a nontangential boundary limit at almost every point of the vertical line \( \sigma = 1/2 \), and the corresponding limit function \( t \mapsto f(1/2 + it) \) is in \( L^2(\mathbb{R}) \); the \( L^2 \) norm of this function coincides with the \( H^2 \) norm introduced above. We may represent \( f \) as

\[
f(s) = \int_0^\infty \varphi(\xi) e^{-(s-1/2)\xi} d\xi
\]

so that by the Plancherel identity \( \| f \|_{H^2(\mathbb{C}_1^+)} = \sqrt{2\pi} \| \varphi \|_2 \). By an appropriate discretization of the integral in this representation, we obtain an approximation on compact sets of functions in \( H^2(\mathbb{C}_1^+) \) by functions in \( H^2 \):
Lemma 1. Let $N$ be a positive integer. Then for every $\varphi$ in $L^2(\log N, \infty)$, there is a function $F(s) = \sum_{n=N}^{\infty} a_n n^{-s}$ in $H^2$ such that $\|F\|_{H^2} \leq \|\varphi\|_2$ and the function
\[
\Phi(s) = \int_{\log N}^{\infty} \varphi(\xi) e^{-(s-1/2)\xi} d\xi - F(s)
\]

enjoys the estimate
\[
|\Phi(s)| \leq 2|s - 1/2| N^{-\sigma-1/2} \|\varphi\|_2
\]
for $s$ in $\mathbb{C}_{1/2}^+$. 

Proof. We set
\[
a_n = \sqrt{n} \int_{\log n}^{\log(n+1)} \varphi(\xi) d\xi
\]
and perform straightforward estimates. 

Note that a version of Lemma 1 was used iteratively by J.-F. Olsen in [8, pp. 109–112], in a similar way as will be done below, to find a new proof of the interpolation theorem for $H^2$ first proved in [11]. The desired Dirichlet series will in our case be an infinite series of functions in $H^2$, where each term is determined by an application of Lemma 1 and convergence is ensured when $N$ is sufficiently large.

We turn to the second essential ingredient in our iteration. Set
\[
\Omega = \Omega(R, \tau) = \{s = \sigma + it : 1/2 \leq \sigma \leq 1/2 + \tau, -R \leq t \leq R\},
\]
for positive numbers $R$ and $\tau$. Lebesgue area measure on $\mathbb{C}$ is denoted by $m$. The following lemma has an obvious proof which is omitted.

Lemma 2. Suppose $g$ is a continuous function on $\mathbb{C}_{1/2}^+$ supported on $\Omega = \Omega(R, 2)$ and satisfying $|g(s)| \leq \epsilon$. Then
\[
u(s) = \frac{1}{\pi} \int_{\Omega} \frac{g(w)}{s - w} dm(w)
\]
solves $\partial u = g$ in $\mathbb{C}_{1/2}^+$ with bounds $\|u\|_{\infty} \leq c \epsilon \log R$ for an absolute constant $c$ (independent of $R$) and
\[
|u(s)| \leq \frac{Re \epsilon}{\pi \text{dist}(s, \Omega)}.
\]

It follows that
\[
\sup_{\sigma \geq 1/2} \left(\int_{-\infty}^{\infty} |u(\sigma + it)|^2 dt\right)^{1/2} \leq c' \epsilon \sqrt{R \log R}
\]
for an absolute constant $c'$.

Let now $S = (\sigma_j + it_j)$ be a bounded Blaschke sequence in $\mathbb{C}_{1/2}^+$ with an associated Blaschke product $B$. We may assume that $S$ is in $\Omega(R - 2, 1/2)$ for some $R > 2$. We fix once and for all a smooth function $\Theta$ on the closed half-plane $\sigma \geq 1/2$ with the following properties: $\Theta$ is
supported on $\Omega(R, 2)$ such that $\Theta(s) = 1$ for $s$ in $\Omega(R - 1, 1)$ and $|\nabla \Theta| \leq 2$. For a given positive integer $N$, we set $E_N(s) = N^{-s+1/2}$ and define an operator $T_N$ on $E_N H^2(C_1^+)$ as follows. Set

$$f(s) = \int_{\log N}^\infty \varphi(\xi)e^{-(s-1/2)\xi}d\xi$$

and $\Phi = f - F$, where $F$ is as in Lemma 1 and let $u$ denote the solution from Lemma 2 to the equation

$$\overline{\partial}u = \frac{\overline{\partial}(\Theta \Phi)}{BE_N}.$$ 

Then set

$$T_N f = \Theta \Phi - B E_N u.$$ 

It is clear that $T_N f$ is in $E_N H^2(C_1^+)$ since $\Theta$ has compact support. The virtue of $T_N$ is that $T_N f(s) = \Phi(s)$ for $s$ in $S$, i.e., $T_N f - \Phi$ is divisible by $B$.

The following estimate is crucial.

**Lemma 3.** The operator $T_N$ on $E_N H^2(C_1^+)$ enjoys the estimate $\|T_N\| \leq CN^{-1}$, where $C$ is a constant depending only on $R$ and $S$.

**Proof.** By our assumption on $S$, $|B(s)|$ is bounded below on the set where $\nabla \Theta \neq 0$. We therefore obtain the result by combining the estimates of Lemma 1 and Lemma 2. \hfill \Box

**Final part of the proof of Theorem 1.** Set $f_0(s) = B(s)E_N(s)/(s + 1/2)$, where $N$ is a sufficiently large integer. Then $f_0$ is in $E_N H^2(C_1^+)$ and its $H^2(C_1^+)$ norm is $\sqrt{2\pi}$. Set $f_{j+1} = T_N f_j$, and let $F_j$ be the Dirichlet series in $\mathcal{H}^2$ obtained by applying Lemma 1 to $f_j$. Then $F_j$ vanishes on $S$ since $f_1 = f_0 - F_0$ on $S$. Iterating, we get that $F_0 + \cdots + F_{j+1}$ also vanishes on $S$. Thus the function

$$F = \sum_{j=0}^\infty F_j$$

is a nontrivial function in $\mathcal{H}^2$ vanishing on $S$ if we choose $N$ so large that $\|T_N\| < 1$ and, say,

$$|F_0(3/2)| > \sum_{j=1}^\infty |F_j(3/2)|.$$ 

Both inequalities can be achieved in view of respectively Lemma 3 and Lemma 1. \hfill \Box

It is of interest to note that our algorithm can be applied in the following way. Let $f$ be an arbitrary function in $H^2(C_1^+)$ that is not divisible by the Blaschke product $B$. We may set $f_0 = f$ and apply $T_N$ to $f$ as before, now viewing $T_N$ as an operator from $H^2(C_1^+)$ to $E_N H^2(C_1^+)$. In the first step of the iteration, we get a poorer bound (depending on $N$), but Lemma 3 applies when $T_N$ acts on $f_j$ for $j \geq 1$. The proof shows that $F$ agrees with $f$ on $S$, which we again take to mean that $f - F$ is divisible by $B$. What we have proved, can be summarized as follows.

**Theorem 1'.** Suppose $S = (\sigma_j + it_j)$ is a bounded Blaschke sequence in $C_1^+$. Then for every function $f$ in $H^2(C_1^+)$ there is a Dirichlet series $F$ in $\mathcal{H}^2$ that agrees with $f$ on $S$. 

This theorem can be viewed as a precise statement about the local resemblance between $H^2(\mathbb{C}_{1/2}^+)$ and $H^2$ but of a different nature than the approximation theorem in [10]. Namely, since the function $f - F$ is divisible by a Blaschke product, it can not be continued analytically across any segment containing an accumulation point of the Blaschke sequence.

On the other hand, we may relate our discussion to the results of [11] and [10]. First, note that Theorem 1’ implies the local interpolation theorem for $H^2$ presented in [11]. Second, if we replace $B$ by the constant $1$ in the proof of Theorem 1’, we get that the function $f - F$ does indeed extend to an analytic function in $\mathbb{C}\setminus(1/2 + i(-R + 1, R - 1))$; here $f$ is an arbitrary function in $H^2(\mathbb{C}_{1/2}^+)$ and $F$ is the function in (1) obtained by starting the iteration with $f_0 = f$. This result may be compared to Theorem 1 in [10].

3. Zeros of Functions in $D_\alpha$

In [9], Olsen studied the relation between $D_\alpha$ and the ordinary weighted Dirichlet spaces in the half-plane $\mathbb{C}_{1/2}^+$. We will use the link found by Olsen to obtain local results on the zeros of functions in $D_\alpha$.

For $0 \leq \beta \leq 1$, we let $D_\beta(\mathbb{C}_{1/2}^+)$ consist of those functions $f$ in $H^2(\mathbb{C}_{1/2}^+)$ for which

$$\int_{\mathbb{C}_{1/2}^+} |f'(s)|^2 (\sigma - 1/2)^{1-\beta} dm(s) < \infty.$$  

Since $f$ belongs to $H^2(\mathbb{C}_{1/2}^+)$, we may write

$$f(s) = \int_0^\infty \varphi(\xi) e^{-(s-1/2)\xi} d\xi$$

for some function $\varphi$ in $L^2(0, \infty)$. A computation involving the Plancherel identity shows that

$$\int_{\mathbb{C}_{1/2}^+} |f'(s)|^2 (\sigma - 1/2)^{1-\beta} dm(s) = c \int_0^\infty |\varphi(\xi)|^2 \xi^\beta d\xi$$

for an absolute constant $c$. We may therefore equip $D_\beta(\mathbb{C}_{1/2}^+)$ with the norm

$$\|f\|_{D_\beta(\mathbb{C}_{1/2}^+)} = \left( \int_0^\infty |\varphi(\xi)|^2 (1 + \xi^\beta) d\xi \right)^{1/2}.$$

It will be convenient to use the following terminology. Given a class of analytic functions $\mathcal{C}$ on some domain $\Omega$ in $\mathbb{C}$, we say that a sequence $\Lambda$ of not necessarily distinct points in $\Omega$ belongs to $Z(\mathcal{C})$ if there is a nontrivial function in $\mathcal{C}$ having $\Lambda$ as its zero set. We will also a few times resort to the notation $U \lesssim V$ which means that there is a positive constant $C$ such that $U \leq CV$ holds for whatever arguments the quantities $U$ and $V$ depend on.

**Theorem 2.** Suppose $0 < \alpha \leq \infty$ and let $S = (\sigma_j + it_j)$ be a bounded sequence in $\mathbb{C}_{1/2}^+$. Then there is a nontrivial function in $D_\alpha$ vanishing on $S$ if and only if $S$ belongs to $Z(D_{1-2-\alpha}(\mathbb{C}_{1/2}^+))$.

The “only if” part of this theorem follows from Theorem 1 of [9] and will therefore not be considered in what follows.
A number of partial results regarding $Z(D^\beta(C_{1/2}^+))$ for $0 < \beta \leq 1$ are known thanks to work of L. Carleson, H. Shapiro and A. Shields, and others [4, 5, 14, 1, 7, 12]. These papers deal with $D^\beta(\mathbb{D})$, i.e., the space of analytic functions $f$ on the unit disk $\mathbb{D}$ for which
\[
\int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{1-\beta} dm(z) < \infty;
\]
the results regarding $D^\beta(\mathbb{D})$ apply because of the following fact: If $\phi$ is a conformal map of $C_{1/2}^+$ onto the unit disk $\mathbb{D}$, then $S$ is in $Z(D^\beta(C_{1/2}^+))$ if and only if $\phi(S)$ is in $Z(D^\beta(\mathbb{D}))$. Indeed, if $f$ is in $D^\beta(\mathbb{D})$ and we choose
\[
\phi(s) = \frac{s + 1/2}{s - 3/2},
\]
then a calculation shows that the function $F(s) = f(\phi(s))(s + 1/2)^{\beta-2}$ is in $D^\beta(C_{1/2}^+)$; a similar transformation can be found for the reverse inclusion.

One of Carleson’s theorems [4] implies that
\[
\sum_j (\sigma_j - 1/2)^{1-\beta} < \infty
\]
is a sufficient condition for a bounded sequence $S = (\sigma_j + it_j)$ to be in $Z(D^\beta(C_{1/2}^+))$ for $0 < \beta < 1$; a theorem of Shapiro and Shields [14] shows that
\[
\sum_j |\log(\sigma_j - 1/2)|^{-1} < \infty
\]
suffices for $S$ to be in $Z(D_1(C_{1/2}^+))$. If $S = (\sigma_j + it_j)$ is restricted to a cone $|t - t_0| \leq c(\sigma - 1/2)$, then the Blaschke condition is sufficient for the sequence to belong to $Z(D_1(C_{1/2}^+))$, as follows from an observation in [5]. For information about further developments, we refer to [12] and [7].

Of particular interest to us is the following result.

**Lemma 4.** Assume $0 < \beta \leq 1$. If $S = (\sigma_j + it_j)$ belongs to $Z(D^\beta(C_{1/2}^+))$, then there exists a function $G$ in $D^\beta(C_{1/2}^+) \cap H^\infty(C_{1/2}^+)$ that has $S$ as its zero set and that can be continued analytically to the domain $\mathbb{C} \setminus \bigcup_j (\{\infty\}, 1 - \sigma_j) + it_j$.

**Proof.** The lemma is a combination of Proposition 9.33 and Proposition 9.37 in [11, p. 137] and Theorem 2 in [2]. Here we use the fact that the multiplier algebra of $D^\beta(C_{1/2}^+)$ is a subset of $H^\infty(C_{1/2}^+)$. \qed

Note that the only property that we will need regarding the analytic continuation of the function $G$ in Lemma [4] is that $G$ is uniformly bounded with bounded derivative on any subset of the closed half-plane $\sigma \geq 1/2$ which is at a positive distance from $S$. 

We set
\[
\|\varphi\|_{2,\beta}^2 = \int_R^\infty |\varphi(\xi)|^2 \xi^\beta d\xi,
\]
assuming \(\varphi\) is a function defined for \(\xi \geq R > 0\), and let \(L_2^\beta(R, \infty)\) be the set of measurable functions \(\varphi\) on the half-line \([R, \infty)\) for which \(\|\varphi\|_{2,\beta} < \infty\). The following lemma is analogous to Lemma \([1]\)

**Lemma 5.** Let \(N\) be a positive integer and \(K\) a compact subset of the closed half-plane \(\mathbb{C}_1^+\), and assume that \(0 < \alpha \leq \infty\). Then there exist constants \(c, C, C_K\) and \(\eta > 1/2, \nu > 1\) such that, for every \(\varphi \in L_2^\alpha(\log N, \infty)\), there is a function \(F(s) = \sum_{n=N}^\infty a_n n^{-s}\) in \(\mathcal{D}_\alpha\) satisfying \(\|F\|_{\mathcal{D}_\alpha} \leq c \|\varphi\|_{2,1-2^{-\alpha}}\) and such that the function

\[
\Phi(s) = \int_{\log N}^\infty \varphi(\xi)e^{-(s-1/2)}d\xi - F(s)
\]

enjoys the estimates
\[
(2) \quad |\Phi(s)| \leq C|s - 1/2|N^{-\sigma+1/2}(\log N)^{-\eta}\|\varphi\|_{2,1-2^{-\alpha}}
\]
for \(s\) in \(\mathbb{C}_1^+\) and

\[
(3) \quad \int_K |\Phi'(s)|^2(\sigma - 1/2)^{2-\alpha} dm(s) \leq C_K(\log N)^{-\nu}\|\varphi\|_{2,1-2^{-\alpha}}^2.
\]

We get a poorer bound in (2) than in Lemma \([1]\) because asymptotic estimates for the divisor function are involved in the construction of the Dirichlet series \(F\).

**Lemma 6.** If \(0 < \alpha < \infty\), then we have
\[
\sum_{n \leq M} [d(n)]^{-\alpha} = AM(\log M)^{2^{-\alpha} - 1}(1 + O((\log M)^{-1}))
\]
for some absolute constant \(A > 0\) when \(M \to \infty\).

The formula in the lemma (in a more precise form) was stated by Ramanujan \([13]\) Formula (9)) and later proved by Wilson in \([15]\) Formula (2.39)). For \(\alpha = \infty\), the estimate follows from the prime number theorem.

**Proof of Lemma 5** Let \(n_j\) be the smallest integer \(n\) such that \(e^{j^\gamma} \leq n\). We set \(\xi_n = j^\gamma\); when \(e^{j^\gamma} < n \leq e^{(j+1)^\gamma}\), we choose \(\xi_n\) inductively such that

\[
(4) \quad \xi_{n+1} - \xi_n = A_j[d(n)]^{-\alpha}e^{-j^\gamma j^{(1-2^{-\alpha})}},
\]

where \(A_j\) is chosen such that \(\xi_{n+1} = (j + 1)^\gamma\). Note if \(1/2 < \gamma < 1\), then Lemma 6 implies that

\[
(5) \quad \sum_{j^\gamma \leq \log n \leq (j+1)^\gamma} [d(n)]^{-\alpha} = Ae^{j^\gamma j^{-1} j^{(2^{-\alpha} - 1)}}(1 + o(1))
\]
for some absolute constant \(A > 0\) when \(j \to \infty\). We will therefore assume that \(1/2 < \gamma < 1\).

We set
\[
a_n = \sqrt{n} \int_{\xi_n}^{\xi_{n+1}} \varphi(\xi) d\xi
\]
and note first that
\[ |a_n|^2 \leq n(\xi_{n+1} - \xi_n) \int_{\xi_n}^{\xi_{n+1}} |\varphi(\xi)|^2 d\xi \]
by the Cauchy–Schwarz inequality. Taking into account (4), we obtain the norm estimate \( \|F\|_{\mathcal{D}^\alpha} \leq c\|\varphi\|_{2,1-2^{-\alpha}} \) from this inequality.

To obtain the pointwise estimate (2), we consider first the finite sum
\[ \Sigma_j = \sum_{n=n_j}^{n_{j+1}-1} \left( \int_{\xi_n}^{\xi_{n+1}} \varphi(\xi)e^{-(s-1/2)\xi}d\xi - a_n n^{-s} \right). \]
We observe that we have
\[ |n^{-s+1/2} - e^{-(s-1/2)\xi}| \leq N^{-\sigma+1/2}|s - 1/2|j^{-\gamma-1} \]
as long as \( \xi \) and \( \xi_n \) both are in the interval \([\xi_{n_j}, \xi_{n_{j+1}}]\). Applying again the Cauchy–Schwarz inequality and taking into account (4), we therefore obtain
\[ \left| \int_{\xi_n}^{\xi_{n+1}} \varphi(\xi)e^{-(s-1/2)\xi}d\xi - a_n n^{-s} \right|^2 \leq A_j N^{-2\sigma+1}|s - 1/2|^2j^{2(\gamma-1)}[d(n)]^{-\alpha}n^{-1}(\log n)^{1-2^{-\alpha}} \int_{\xi_n}^{\xi_{n+1}} |\varphi(\xi)|^2 d\xi. \]
Now applying the Cauchy–Schwarz inequality to the sum, we get
\[ |\Sigma_j|^2 \leq A_j N^{-2\sigma+1}|s - 1/2|^2j^{2(\gamma-1)}\sum_{n=n_j}^{n_{j+1}}[d(n)]^{-\alpha}n^{-1}(\log n)^{1-2^{-\alpha}} \int_{\xi_n}^{\xi_{n+1}} |\varphi(\xi)|^2 d\xi. \]
By (5), we therefore get
\[ |\Sigma_j|^2 \lesssim N^{-2\sigma+1}|s - 1/2|^2j^{3(\gamma-1)-\gamma(1-2^{-\alpha})} \int_{\xi_{n_j}}^{\xi_{n_{j+1}}} \xi^{1-2^{-\alpha}} |\varphi(\xi)|^2 d\xi. \]
If we now choose a sufficiently small \( \gamma \) in the interval \((1/2, 1)\), then the series
\[ \sum_{j=1}^{\infty} j^{-3+\gamma(2+2^{-\alpha})} \]
is summable and we get (2) with
\[ \eta = (2 - \gamma(2 + 2^{-\alpha}))/\gamma. \]

To deal with (3), we begin by assuming that \( K \subset \Omega(R, \tau) \) for suitable \( R \) and \( \tau \). By duality, we have
\[ \left( \int_{-R}^{R} |\Phi'(\sigma + it)|^2 dt \right)^{1/2} = \sup_{\|g\|_2 = 1} \left| \int_{-R}^{R} \Phi'(\sigma + it)g(t) dt \right|. \]
Lemma 7. The operator $T_N$ acts boundedly on $E_N D_{1-2-\alpha}(\mathbb{C}^+_{1/2})$ with $\|T_N\| \leq C(\log N)^{-\delta}$, where $\delta > 0$ and $C$ is a constant depending only on $R$ and $S$. 

where the supremum is taken over all $g$ of norm 1 in $L^2(-R, R)$. We obtain

$$
\int_{-R}^{R} \Phi'(\sigma + it)g(t)dt = \sum_{n=N}^{\infty} a_n(\log n)n^{-\sigma} \hat{g}(\log n) - \int_{\xi_n}^{\xi_{n+1}} \varphi(\xi)e^{-\sigma(1/2)\xi} \hat{g}(\xi)d\xi
$$

$$
= \sum_{n=N}^{\infty} \int_{\xi_n}^{\xi_{n+1}} \varphi(\xi)((\log n)n^{-1/2} \hat{g}(\log n) - \xi e^{-\sigma(1/2)\xi} \hat{g}(\xi))d\xi.
$$

It follows that

$$
\left| \int_{-R}^{R} \Phi'(\sigma + it)g(t)dt \right| \lesssim (\log N)^{1-\gamma} \sum_{j} \sum_{n=n_j}^{n_{j+1}-1} \left| a_n \right|((\log n)n^{-\sigma}(|\hat{g}(\xi_j^*)| + |\hat{g}'(\xi_j^*)|)),
$$

where $|\hat{g}(\xi_j^*)|$ and $|\hat{g}'(\xi_j^*)|$ are the maxima of the respective functions $|\hat{g}(\xi)|$ and $|\hat{g}'(\xi)|$ on $[\xi_{n_j}, \xi_{n_{j+1}}]$. We apply the Cauchy–Schwarz inequality and employ again Lemma 6, we also use the Plancherel–Pólya inequality (cf. the proof on pp. 2674–2675 in [9]) and get

$$
\int_{-R}^{R} |\Phi'(\sigma + it)|^2 dt \lesssim (\log N)^{2-\gamma} \sum_{n \geq N} \left| a_n \right|^2[d(n)]^\alpha(\log n)^{2\alpha + 1}n^{-2\sigma + 1},
$$

where the implicit constant depends on $R$. Since

$$
\int_{1/2}^{\infty} (\log n)^{2\alpha + 1}n^{-2\sigma + 1}(\sigma - 1/2)^{2\alpha} d\sigma = 2^{-1-2\alpha} \Gamma(1+2\alpha),
$$

(3) follows with $\nu = 2/\gamma - 2$, where as in the preceding case we get the desired result by choosing a sufficiently small $\gamma$ in the interval $(1/2, 1)$. \hfill \Box

Let now $S = (\sigma_j + it_j)$ be a bounded sequence in $Z(D_{1-2-\alpha}(\mathbb{C}^+_{1/2}))$, and let $G$ be the function from Lemma 4 vanishing on $S$. We may again assume that $S$ is in $\Omega(R-2, 1/2)$ for some $R > 2$ and let $\Theta$ be as above. We view $E_N D_{1-2-\alpha}(\mathbb{C}^+_{1/2})$ as a Hilbert subspace of $D_{1-2-\alpha}(\mathbb{C}^+_{1/2})$ and define $T_N$ on $E_N D_{1-2-\alpha}(\mathbb{C}^+_{1/2})$ similarly as in the preceding section: Set

$$
f(s) = \int_{\log N}^{\infty} \varphi(\xi)e^{-(s-1/2)\xi}d\xi
$$

and $\Phi = f - F$, where $F$ is as in Lemma 5 and let $u$ denote the solution from Lemma 2 to the equation

$$
\partial u = \frac{\overline{\partial}(\Theta \Phi)}{GE_N}.
$$

Then set

$$
T_N f = \Theta \Phi - GE_N u.
$$

We will again use that $T_N f(s) = \Phi(s)$ for $s$ in $S$. 

Lemma 7. The operator $T_N$ acts boundedly on $E_N D_{1-2-\alpha}(\mathbb{C}^+_{1/2})$ with $\|T_N\| \leq C(\log N)^{-\delta}$, where $\delta > 0$ and $C$ is a constant depending only on $R$ and $S$. 

We observe that (3) implies that
\[ \int_{C_{1/2}^+} |\partial(T_N f)(s)|^2 (\sigma - 1/2)^{1-\beta} dm(z) \leq C(\log N)^{-\delta} \|f\|_{D_\beta(C_{1/2}^+)}^2. \]
We treat each of the terms on the right-hand side of (6) separately. We get
\[ \int_{C_{1/2}^+} |\partial(\Theta \Phi)(s)|^2 (\sigma - 1/2)^{1-\beta} dm(z) \lesssim (\log N)^{-\nu} \|f\|_{D_\beta(C_{1/2}^+)}^2, \]
and so it remains to consider the weighted area integral for the function
\[ (6) \quad \partial(G_E u) = \partial G E N u + G \partial E N u + G E N \partial u. \]
We set for convenience
\[ w = \Theta(N) \int_{\partial u} \Psi. \]
It remains to estimate the contribution from the third term on the right-hand side of (6). Set
\[ \Delta = \Omega(R, 2) \setminus \Omega(R - 1, 1) \supseteq \text{supp}(\nabla \Theta). \]
Then trivially
\[ |\partial u(s)| \lesssim (\log N)^{-\eta} (1 + |s|)^{-2} \]
if \( \text{dist}(s, \Delta) \geq 1/8. \) On the other hand, if \( \text{dist}(s, \Delta) < 1/8, \) we argue as follows. Set
\[ \Delta_+ = \Omega(R + 1, 3) \setminus \Omega(R - 3/2, 3/4) \]
and apply the Cauchy–Pompeiu formula to \( \Theta \Phi/(G E N) \) in \( \Delta_+. \) Hence we get
\[ \partial u(s) = \partial \left( \frac{\Theta \Phi}{G E N} \right)(s) - \frac{1}{2\pi i} \int_{\partial \Delta_+} \frac{\Theta(w) \Phi(w)}{G(w) E N(w)} \frac{1}{(s - w)^2} dw. \]
We write \( \Theta(w) = \Theta(s) + \Theta(w) - \Theta(s). \) Then, by the smoothness of \( \Theta \) and analyticity of \( \Phi/(G E N), \) we get
\[ \left| \int_{\partial \Delta_+} \frac{\Theta(w) \Phi(w)}{G(w) E N(w)} \frac{1}{(s - w)^2} dw \right| \lesssim |\Theta(s)| \left| \partial \left( \frac{\Phi}{G E N} \right) \right| (s) + \int_{\partial \Delta_+} \left| \frac{\Phi(w)}{|G(w) E N(w)| s - w} \big| dw \right|. \]
Using Lemma 4 and (2), we therefore deduce that
\[ |\partial u(s)| \lesssim \left| \partial \left( \frac{\Theta \Phi}{G E N} \right) \right| (s) + (\log N)^{-\eta} |\log(\sigma - 1/2)| \|f\|_{D_\beta(C_{1/2}^+)} \].
when ever $\text{dist}(s, \Delta) < 1/8$. The remaining part of the estimation is essentially a repetition of the preceding calculations, and the details are therefore omitted. \hfill \Box

**Final part of the proof of Theorem 2.** We act in the same way as in the proof of Theorem 1. Set $f_0 = E_N G$, where $N$ is a sufficiently large integer, and then $f_j = T_N f_0$. Let $F_j$ be the Dirichlet series in $\mathcal{D}_\alpha$ obtained from $f_j$. Then, by the same argument as in the proof of Theorem 1, $F = \sum_{j=0}^\infty F_j$ is a nontrivial function in $\mathcal{D}_\alpha$ vanishing on $S$ if $N$ is chosen so large that $\|T_N\| < 1$ and, say,

$$|F_0(3/2)| > \sum_{j=1}^\infty |F_j(3/2)|.$$  

This time we use respectively Lemma 7 and Lemma 5 to conclude that these inequalities are fulfilled when $N$ is sufficiently large. \hfill \Box

In accordance with the observation leading to Theorem 1' in the preceding section, we get that our algorithm in fact yields the following stronger result.

**Theorem 2'.** Suppose $S = (\sigma_j + it_j)$ is a bounded sequence in $Z(D_{1-2-\alpha}(\mathbb{C}^+_1/2))$. Then for every function $f$ in $D_{1-2-\alpha}(\mathbb{C}^+_1/2)$ there is a Dirichlet series in $\mathcal{D}_\alpha$ that agrees with $f$ on $S$.

4. **The relation between $\mathcal{D}_\alpha$ and $\mathcal{H}^p$**

For $1 \leq p < \infty$, we define $\mathcal{H}^p$ as the closure of the set of finite Dirichlet polynomials in the norm

$$\lim_{T \to \infty} \left( \frac{1}{T} \sum_n n^{-it} \right)^p.$$  

Alternatively, we may express this limit as an $L^p$ norm over the infinite-dimensional torus $\mathbb{T}^\infty$. We refer to F. Bayart’s paper [3], where these spaces were first studied.

We will only consider the case $2 \leq p < \infty$. Since then $\mathcal{H}^p$ is a subset of $\mathcal{H}^2$, the Blaschke condition for bounded sequences in $\mathbb{C}^+_1/2$ is a necessary condition for the existence of a nontrivial Dirichlet series $f$ in $\mathcal{H}^p$ vanishing on a bounded sequence $S$ in $\mathbb{C}^+_1/2$. We are not able to determine whether, in general, this condition is sufficient as well, but the following special case can be settled at once. Note first that Khinchin’s inequality implies that $\mathcal{D}_\infty$ is included in $\mathcal{H}^p$ for every finite $p$. (This fact will also follow from the sharper results to be established below.) By Theorem 2 and Carleson’s observation from [5] mentioned above, this inclusion leads to the result that for $1 \leq p < \infty$ the Blaschke condition is necessary and sufficient for there to be a nontrivial function in $\mathcal{H}^p$ vanishing on a bounded sequence contained in a cone $|t - t_0| \leq c|\sigma - 1/2|$. It is of interest to note that this result fails spectacularly when $p = \infty$ because $\mathcal{H}^\infty$ is a space of functions analytic in the larger half-plane $\mathbb{C}^+_1$.

We will now establish a more precise relation between $\mathcal{D}_\alpha$ and $\mathcal{H}^p$.

---

\(^1\)The Blaschke condition $\sum_j \sigma_j < \infty$ is trivially a necessary and sufficient condition for the existence of a nontrivial function in $\mathcal{H}^\infty$ vanishing on a bounded sequence $(\sigma_j + it_j)$ in $\mathbb{C}^+$; it suffices to observe that $(2^{-\sigma_j - it_j})$ is a Blaschke sequence in $\mathcal{D}$, and so we may pick an associated Blaschke product $B$ and use the function $B(2^{-s})$. 

Lemma 8. For $\alpha$ a nonnegative integer, the space $\mathcal{D}_\alpha$ is contractively embedded in $\mathcal{H}^{2\alpha+1}$.

Proof. We prove the lemma by induction on $\alpha$. The statement is a tautology when $\alpha = 0$. Assume it holds for $\alpha = k$. This means that if we write $g = f^2$ with $g(s) = \sum b_n n^{-s}$, then we have

$$\|f\|_{2^{k+1}}^4 = \|g\|_{2^k}^2 \leq \sum |b_n|^2 |d(n)|^k.$$

Writing $f(s) = \sum a_n n^{-s}$ and using the Cauchy–Schwarz inequality, we obtain

$$|b_n|^2 \leq d(n) \sum_{k|n} |a_k|^2 |a_{n/k}|^2.$$

The result follows since $d(lk) \leq d(l)d(k)$. \[ \square \]

We denote by $[\alpha]$ the integer part of $\alpha$ and obtain from Lemma 8 the following result for general $\alpha > 0$.

Theorem 3. For $\alpha > 0$, the space $\mathcal{D}_\alpha$ is contractively embedded in $\mathcal{H}^{2[\alpha]+2(2+|\alpha|-\alpha)^{-1}}$.

Proof. We use an argument similar to the one used in the proof of the Riesz–Thorin theorem. Suppose

$$f_\alpha(s) = \sum a_n n^{-s}$$

has norm 1 in $\mathcal{D}_\alpha$. Then the function

$$f_z(s) = \sum a_n [d(n)]^{\alpha z/2 + 1/2} n^{-s}$$

has norm bounded by 1 in $\mathcal{D}_z$, where $z = x + iy$. We take the inner product between $f_z$ and $g|g|^{a_2+b-1}$ for an arbitrary function $g$ on $T^\infty$ taking finitely many values, where the parameters $a$ and $b$ are chosen such that

$$a[\alpha] + b = 1 - 2^{-(\alpha)+1} \quad \text{and} \quad a([\alpha] + 1) + b = 1 - 2^{-(\alpha)+2}.$$

This means that

$$a = 2^{-(\alpha)+2} \quad \text{and} \quad b = 1 - 2^{-(\alpha)+2} ([\alpha] + 2).$$

We now conclude from the preceding lemma and the three lines lemma that $\|f_\alpha\|_p \leq 1$ when

$$1 - 1/p = a\alpha + b,$$

or in other words when $p = 2^{\alpha+2}(2 + [\alpha] - \alpha)^{-1}$. \[ \square \]

For $p = 2^{k+1}$ and $k$ a positive integer, we get that if $S$ is a sequence of bounded numbers belonging to $Z(D_{1-2/p}(C_{1/2}^+))$, then there is a nontrivial function in $\mathcal{H}^p$ vanishing on $S$. For general $p$, the criterion becomes a bit more cumbersome, and we state it explicitly only for $2 < p < 4$: If a bounded sequence $S$ is in $Z(D_{1-2-2+4/p}(C_{1/2}^+))$, then there is a nontrivial function in $\mathcal{H}^p$ vanishing on $S$.

It may be objected that we have found a rather indirect route to arrive at these results for $\mathcal{H}^p$. Our method of proof reflects that, in general, it remains a challenge to develop techniques that would allow a more direct approach to the study of $\mathcal{H}^p$. 
Acknowledgements. I am grateful to Jan-Fredrik Olsen, Joaquim Ortega-Cerdà, Eero Saksman, and Jordi Pau for helpful comments on the subject matter of this paper.

REFERENCES

[1] J. Agler and J. E. McCarthy, *Pick Interpolation and Hilbert Function Spaces*, Graduate Studies in Mathematics 44, Amer. Math. Soc., Providence RI, 2002.
[2] J. E. Akutowicz and L. Carleson, *The analytic continuation of interpolatory functions*, J. Analyse Math. 7 (1959/1960), 223–247.
[3] F. Bayart, *Hardy spaces of Dirichlet series and their composition operators*, Monatsh. Math. 136 (2002), 203–236.
[4] L. Carleson, *On a Class of Meromorphic Functions and Its Exceptional Sets*, Thesis, University of Uppsala, 1950.
[5] L. Carleson, *On the zeros of functions with bounded Dirichlet integrals*, Math. Z. 56 (1952), 289–295.
[6] H. Hedenmalm, P. Lindqvist, and K. Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in $L^2(0, 1)$*, Duke Math. J. 86 (1997), 1-37.
[7] J. Mashreghi and M. Shabankah, *Admissible functions for the Dirichlet space*, Studia Math. 198 (2010), 147–156.
[8] J.-F. Olsen, *Boundary Properties of Modified Zeta Functions and Function Spaces of Dirichlet series*, Doctoral Thesis, Norwegian University of Science of Technology, 2009.
[9] J.-F. Olsen, *Local properties of Hilbert spaces of Dirichlet series*, J. Funct. Anal. 261 (2011), 2669–2696.
[10] J.-F. Olsen and E. Saksman, *On the boundary behaviour of the Hardy spaces of Dirichlet series and a frame bound estimate*, J. Reine Angew. Math. 663 (2012), 33–66.
[11] J.-F. Olsen and K. Seip, *Local interpolation in Hilbert spaces of Dirichlet series*, Proc. Amer. Math. Soc. 136 (2008), 203-212.
[12] J. Pau and J. A. Peláez, *On the zeros of functions in Dirichlet-type spaces*, Trans. Amer. Math. Soc. 363 (2011), 1981-2002.
[13] S. Ramanujan, *Some formulae in the analytic theory of numbers*, Messenger Math. 45 (1916), 81–84.
[14] H. S. Shapiro and A. L. Shields, *On the zeros of functions with finite Dirichlet integral and some related function spaces*, Math. Z. 80 (1962), 217–229.
[15] B. M. Wilson, *Proofs of some formulae enunciated by Ramanujan*, Proc. London Math. Soc. 21 (1922), 235–255.

DEPARTMENT OF MATHEMATICAL SCIENCES, NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY, NO-7491 TRONDHEIM, NORWAY

E-mail address: seip@math.ntnu.no