The Post-Decoherence Density Matrix Propagator for Quantum Brownian Motion

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ABSTRACT: Using the path integral representation of the density matrix propagator of quantum Brownian motion, we derive its asymptotic form for times greater than the so-called localization time, \((\hbar/\gamma kT)^{1/2}\), where \(\gamma\) is the dissipation and \(T\) the temperature of the thermal environment. The localization time is typically greater than the decoherence time, but much shorter than the relaxation time, \(\gamma^{-1}\). We use this result to show that the reduced density operator rapidly evolves into a state which is approximately diagonal in a set of generalized coherent states. We thus reproduce, using a completely different method, a result we previously obtained using the quantum state diffusion picture \((\text{Phys.Rev.D} 52, 7294 \text{ (1995)})\). We also go beyond this earlier result, in that we derive an explicit expression for the weighting of each phase space localized state in the approximately diagonal density matrix, as a function of the initial state. For sufficiently long times it is equal to the Wigner function, and we confirm that the Wigner function is positive for times greater than the localization time (multiplied by a number of order 1).
One of the simplest open systems that is amenable to straightforward analysis is the quantum Brownian motion model. This model consists of a non-relativistic point particle, possibly in a potential, coupled to a bath of harmonic oscillators in a thermal state. The quantum Brownian motion model has been used very extensively in studies of decoherence and emergent classicality (see for example, Refs.[1,2,3,4,5,6,7,8]).

In the simplest case of a free particle of mass $m$ in a high temperature bath, with negligible dissipation the master equation for the reduced density matrix $\rho(x,y)$ of the point particle is,

$$\frac{\partial \rho}{\partial t} = \frac{i\hbar}{2m} \left( \frac{\partial^2 \rho}{\partial x^2} \right) \left( \frac{\partial^2 \rho}{\partial y^2} \right) - \frac{1}{2} a^2 (x - y)^2 \rho$$

(1.1)

where $a^2 = 4m\gamma kT/\hbar^2$. (More general forms of this equation, together with the derivations of it may be found in many places. See, for example, Refs.[2,9,5]).

One of the most important properties of (1.1) (and also its more general forms) is that the density operator tends to become approximately diagonal in both position and momentum after a short time. This has been seen in numerical solutions and in the evolution of particular types of initial states for which analytic solution is possible [10,6,11,7,12,8,13,14,15].

A more precise demonstration of this statement was given in Ref.[16] by appealing to an alternative description of open systems known as the quantum state diffusion picture [17,18,19,20,21]. In that picture, the density operator $\rho$ satisfying (1.1) is regarded as a mean over a distribution of pure state density operators,

$$\rho = M|\psi\rangle\langle\psi|$$

(1.2)

where $M$ denotes the mean (defined below), with the pure states evolving according to a non-linear stochastic Langevin-Ito equation, which for the model of this paper is,

$$|d\psi\rangle = -\frac{i}{\hbar} H|\psi\rangle dt - \frac{1}{2} (L - \langle L \rangle)^2 |\psi\rangle dt + (L - \langle L \rangle) |\psi\rangle d\xi(t)$$

(1.3)
for the normalized state vector $|\psi\rangle$, where $H = p^2/2m$ and $L = a\dot{x}$. Here, $d\xi$ is a complex differential random variable representing a complex Wiener process. The linear and quadratic means are,

$$M[d\xi d\xi^*] = dt, \quad M[d\xi] = 0, \quad M[d\xi^*] = 0 \quad (1.4)$$

The appeal of this picture is that the solutions to the stochastic equation (1.3) appear to describe the expected behaviour of an individual history of the system, and have been seen to correspond to single runs of laboratory experiments. For example, for the quantum Brownian motion model, the solutions tend to phase space localized states of constant width whose centres undergo classical Brownian motion [20,16,22,23,24]. The timescale of this process, the localization time, is at slowest of order $(\hbar/\gamma kT)^{1/2}$, which is the timescale on which the thermal fluctuations overtake the quantum fluctuations [25,26,27]. For an initial superposition of localized states a distance $\ell$ apart, localization initially proceeds on a much shorter timescale, of order $\hbar^2/(\ell^2 m\gamma kT)$ (which is often called the decoherence time [8,14]), thereafter going over to the slower timescale above.

For us, the interesting feature of the quantum state diffusion picture is that it gives some useful information about the form of the density operator on time scales greater than the localization time. Given a set of localized phase space solutions $|\Psi_{pq}\rangle$, the density operator may be reconstructed via (1.2). This, it may be shown [16], may be written explicitly as

$$\rho = \int dpdq \ f(p,q,t)|\Psi_{pq}\rangle\langle\Psi_{pq}| \quad (1.5)$$

Here, $f(p,q,t)$ is a non-negative, normalized solution to the Fokker-Planck equation describing the classical Brownian motion undergone by the centres of the stationary solutions. This is therefore an explicit, albeit indirect, demonstration of the approach to approximately phase space diagonal form on short time scales.

The above demonstration was described by us in detail in Ref. [16]. However, we were not able to deduce an explicit form for the function $f(p,q,t)$ using the quantum state diffusion picture. That is, we know that it is a solution to the Fokker-Planck equation, but it was not clear how to pick out the particular solution corresponding to a particular
initial density operator. Intuitively, it is clear that \( f(p, q, t) \) is something like the Wigner function of the initial state, coarse-grained sufficiently to make it positive, evolved forwards in time, and with the interference terms thrown away. We would like to be able to show this explicitly.

The aim of the present paper is to derive the form (1.5) for times greater than the localization time directly from the path integral representation of the density matrix propagator corresponding to (1.1), without using the quantum state diffusion picture. As we shall see, this derivation has the advantage that it gives an explicit expression for \( f(p, q, t) \). In particular, we shall show that \( f(p, q, t) \) coincides with the Wigner function \( W_t(p, q) \) of the density operator at time \( t \), for sufficiently large times.

## 2. THE DENSITY MATRIX PROPAGATOR

The solution to the master equation (1.1) may be written in terms of the propagator, \( J \),

\[
\rho_t(x, y) = \int dx_0 dy_0 \, J(x, y, t|x_0, y_0, 0) \, \rho_0(x_0, y_0) \tag{2.1}
\]

(see, for example, Refs.[2,27] for further details of the quantum Brownian motion model). The propagator may be given in general by a path integral expression, which for the particular case considered here is

\[
J(x_f, y_f, t|x_0, y_0, 0) = \int Dx Dy \, \exp \left( \frac{im}{2\hbar} \int dt (\dot{x}^2 - \dot{y}^2) - \frac{a^2}{2} \int dt (x - y)^2 \right) \tag{2.2}
\]

This is readily evaluated, with the result,

\[
J(x_f, y_f, t|x_0, y_0, 0) = \exp \left( \frac{im}{2\hbar t} \left[ (x_f - x_0)^2 - (y_f - y_0)^2 \right] \right.

- \left. \frac{a^2 t}{6} \left[ (x_f - y_f)^2 + (x_f - y_f)(x_0 - y_0) + (x_0 - y_0)^2 \right] \right) \tag{2.3}
\]

(For convenience we will ignore prefactors in what follows. They may be recovered where required by appropriate normalizations.)
The main result of the present paper comes from the simple observation that the real part of the exponent in the path integral (2.2) may be written

$$\exp\left(-\frac{a^2}{2} \int dt \ (x-y)^2\right) = \int D\bar{x} \ \exp\left(-a^2 \int dt \ (x-\bar{x})^2 - a^2 \int dt \ (y-\bar{x})^2\right)$$

(2.4)

The path integral representation of the propagator may therefore be written,

$$J(x_f, y_f, t|x_0, y_0, 0) = \int D\bar{x} \ K_{\bar{x}}(x_f, t|x_0, 0) \ K^*_{\bar{x}}(y_f, t|y_0, 0)$$

(2.5)

where

$$K_{\bar{x}}(x_f, t|x_0, 0) = \int Dx \ \exp\left(\frac{im}{2\hbar} \int dt \ x^2 - a^2 \int dt \ (x-\bar{x})^2\right)$$

(2.6)

For a pure initial state, $\rho_0(x, y) = \Psi_0(x)\Psi_0^*(y)$, the density operator at time $t$ may therefore be written,

$$\rho_t(x, y) = \int D\bar{x} \ \Psi_{\bar{x}}(x, t)\Psi_{\bar{x}}^*(y, t)$$

(2.7)

where the (unnormalized) wave function $\Psi_{\bar{x}}$ is given by

$$\Psi_{\bar{x}}(x_f, t) = \int dx_0 \ K_{\bar{x}}(x_f, t|x_0, 0) \ \Psi_0(x_0)$$

(2.8)

(Wave functions of this type often appear in discussions of systems undergoing continuous measurement [28,29,30,31].)

Our strategy is to first evaluate the quantity $K_{\bar{x}}$, examine its asymptotic form for times greater than the localization time, and then use it to reconstruct the density matrix propagator, $J$. The reason we expect this to yield the desired result is that up to normalization factors and ignoring the fact that $\bar{x}$ is real not complex, Eq.(2.8) is essentially the solution to the Langevin–Ito equation, (1.3), so the phase space localization effect should be visible in its long time limit. Moreover, Eq.(2.7) is the analogue of (1.2) or (1.5), so by reorganizing the functional integral over $\bar{x}(t)$, we might reasonably expect to derive (1.5).

The path integral (2.6) is essentially the same as that for a harmonic oscillator coupled to an external source, with the complication that the frequency is complex. The path integral is therefore readily carried out (see Ref.[32], for example), with the result,

$$K_{\bar{x}}(x_f, t|x_0, 0) = N \exp\left(\frac{i}{\hbar}c_1(x_f^2 + x_0^2) + \frac{i}{\hbar}c_2x_0x_f + c_3x_f + c_4x_0 + c_5 - a^2 \int_0^t ds \ \bar{x}^2(s)\right)$$

(2.9)
where,
\[ c_1 = \frac{m \omega \cos \omega t}{2 \sin \omega t} \]  \hspace{1cm} (2.10)
\[ c_2 = -\frac{m \omega}{\sin \omega t} \]  \hspace{1cm} (2.11)
\[ c_3 = \frac{2a^2}{\sin \omega t} \int_0^t ds \, \bar{x}(s) \sin \omega s \]  \hspace{1cm} (2.12)
\[ c_4 = \frac{2a^2}{\sin \omega t} \int_0^t ds \, \bar{x}(s) \sin \omega(t - s) \]  \hspace{1cm} (2.13)
\[ c_5 = \frac{4iha^2}{m \omega \sin \omega t} \int_0^t ds \int_0^s ds' \bar{x}(s) \bar{x}(s') \sin \omega(t - s) \sin \omega s' \]  \hspace{1cm} (2.14)

Here \( \omega = \alpha(1 - i) \) and
\[ \alpha = \left( \frac{ha^2}{4m} \right)^{\frac{1}{2}} = \left( \frac{\gamma kT}{\hbar} \right)^{\frac{1}{2}} \]  \hspace{1cm} (2.15)

The timescale of evolution according to (2.9) is therefore \( \alpha^{-1} \), which coincides with the localization time discussed in Ref. [16]. The asymptotic properties of \( K_{\bar{x}} \) are now easily seen. As \( t \to \infty \), \( c_2 \to 0 \) and \( c_1 \to \frac{1}{2} m \alpha(1+i) \) like \( e^{-\alpha t} \). Since \( c_2 \to 0 \), the propagator \( K_{\bar{x}} \) factors into a product of functions of \( x_0 \) and \( x_f \). The wave function (2.8) therefore “forgets” its initial conditions and becomes proportional to a Gaussian of the form
\[ \exp \left( \frac{i}{\hbar} c_1 x_f^2 + c_3 x_f \right) \]  \hspace{1cm} (2.16)
on a timescale \( \alpha^{-1} \). This is in complete agreement with the quantum state diffusion picture analysis of Refs. [20,16].

Now introduce
\[ \bar{q} = \frac{\hbar}{m \alpha} \text{Re} \, c_3, \quad \bar{p} = \hbar (\text{Re} \, c_3 + \text{Im} \, c_3) \]  \hspace{1cm} (2.17)

Then the Gaussian may be written
\[ \exp \left( \frac{i}{\hbar} c_1 x_f^2 + c_3 x_f \right) = \exp \left( -\frac{ma}{2\hbar} (1 - i)(x - \bar{q})^2 + \frac{i}{\hbar} px + \frac{ma}{2\hbar} (1 - i)\bar{q}^2 \right) \]
\[ \equiv \langle x | \Psi_{\bar{p}\bar{q}} \rangle e^{\frac{ma}{2\hbar} (1-i)\bar{q}^2} \]  \hspace{1cm} (2.18)

The propagator \( K_{\bar{x}} \) therefore has the form
\[ K_{\bar{x}}(x_f, t|x_0, 0) = N \langle x_f | \Psi_{\bar{p}\bar{q}} \rangle e^{\frac{ma}{2\hbar} (1-i)\bar{q}^2} \]
\[ \times \exp \left( \frac{i}{\hbar} c_1 x_0^2 + c_4 x_0 + c_5 - a^2 \int_0^t ds \, \bar{x}^2(s) \right) \]  \hspace{1cm} (2.19)
The generalized coherent states $|\Psi_{pq}\rangle$ depend on $\bar{x}(t)$ only through $\bar{p}$ and $\bar{q}$, which are functionals of $\bar{x}(t)$. They are close to minimal uncertainty states, satisfying, $\Delta p\Delta q = \hbar/\sqrt{2}$ [20,16].

The desired form of the propagator is now obtained by inserting (2.19) in (2.5), but reorganizing the functional integral over $\bar{x}(t)$ into ordinary integrations over $\bar{p}$ and $\bar{q}$ and functional integrations over remaining parts of $\bar{x}(t)$. This may be achieved by writing the functional integral over $\bar{x}(t)$ as

$$\int D\bar{x} = \int dp dq \int D\bar{x} \delta(p - \bar{p}) \delta(q - \bar{q})$$

(2.20)

with $\bar{p}$ and $\bar{q}$ given in terms of $\bar{x}$ by (2.17). We thus obtain,

$$J(x_f, y_f, t|x_0, y_0, 0) = \int dp dq \int D\bar{x} \delta(p - \bar{p}) \delta(q - \bar{q}) \langle x_f|\Psi_{pq}\rangle \langle \Psi_{pq}|y_f\rangle e^{\frac{m\alpha}{\hbar} q^2} \times \exp\left(\frac{i}{\hbar} c_1 x_0^2 - \frac{i}{\hbar} c_1^* y_0^2 + c_4 x_0 + c_4^* y_0\right) \times \exp\left(c_5 + c_5^* - 2a^2 \int_0^t ds \bar{x}^2(s)\right)$$

(2.21)

This may be written,

$$J(x_f, y_f, t|x_0, y_0, 0) = \int dp dq f(p, q, t|x_0, y_0) \langle x_f|\Psi_{pq}\rangle \langle \Psi_{pq}|y_f\rangle$$

(2.22)

where

$$f(p, q, t|x_0, y_0) = \int D\bar{x} \delta(p - \bar{p}) \delta(q - \bar{q}) e^{\frac{m\alpha}{\hbar} q^2} \times \exp\left(\frac{i}{\hbar} c_1 x_0^2 - \frac{i}{\hbar} c_1^* y_0^2 + c_4 x_0 + c_4^* y_0\right) \times \exp\left(c_5 + c_5^* - 2a^2 \int_0^t ds \bar{x}^2(s)\right)$$

(2.23)

We have clearly cast the result in the desired form. Folding an arbitrary initial state into the expression for the density matrix propagator (2.22), we obtain an expression of the desired form (1.5), where $f(p, q, t)$ is given explicitly by,

$$f(p, q, t) = \int dx_0 dy_0 f(p, q, t|x_0, y_0) \rho_0(x_0, y_0)$$

(2.24)

This is our first main result.
3. THE PHASE SPACE DISTRIBUTION FUNCTION

It remains to evaluate the path integral expression (2.23). To do this first notice that (2.23) may be written

\[
 f(p,q,t|x_0,y_0) = \exp \left( i \frac{\hbar}{\ell} c_1 x_0^2 - \frac{m \alpha}{\hbar} q^2 \right) \int dk dk' e^{i \hbar k p + i \hbar k' q} \times \int D\bar{x} \exp \left( -i \hbar k \bar{p} - i \hbar k' \bar{q} + c_4 x_0 + c_4^* y_0 \right) \times \exp \left( c_5 + c_5^* - 2a^2 \frac{t}{ds} \bar{x}^2(s) \right) \quad (3.1)
\]

The functional integral over \(\bar{x}\) is a Gaussian, since \(c_5\) is quadratic in \(\bar{x}\) and \(\bar{p}, \bar{q}\) and \(c_4\) are linear in \(\bar{x}\), but it involves inverting the functional matrix contain in the last exponential in (3.1), which does not look particularly easy. However, we are saved from having to do this calculation by the following observation. From Eq.(2.5) and Eq.(2.9) (for \(\alpha t >> 1\)), we see that

\[
 J(x_f,y_f,t|x_0,y_0,0) = \exp \left( \frac{i}{\hbar} c_1 (x_f^2 + x_0^2) - \frac{i}{\hbar} c_1^* (y_f^2 + y_0^2) \right) \times \int D\bar{x} \exp \left( c_3 x_f + c_3^* y_f + c_4 x_0 + c_4^* y_0 \right) \times \exp \left( c_5 + c_5^* - 2a^2 \frac{t}{ds} \bar{x}^2(s) \right) \quad (3.2)
\]

This functional integral over \(\bar{x}\) in this expression is very similar in form to (3.1) but we already know what the answer is: it is Eq.(2.3). In particular, equating (3.2) and (2.3), we obtain

\[
 \int D\bar{x} \exp \left( c_3 x_f + c_3^* y_f + c_4 x_0 + c_4^* y_0 + c_5 + c_5^* - 2a^2 \frac{t}{ds} \bar{x}^2(s) \right) \]

\[
 = \exp \left( \frac{im}{2ht} \left[ (x_f - x_0)^2 - (y_f - y_0)^2 \right] \right) - \frac{a^2 t}{6} \left[ (x_f - y_f)^2 + (x_f - y_f)(x_0 - y_0) + (x_0 - y_0)^2 \right] \]

\[
 \times \exp \left( -\frac{i}{\hbar} c_1 (x_f^2 + x_0^2) + \frac{i}{\hbar} c_1^* (y_f^2 + y_0^2) \right) \quad (3.3)
\]

Now the point is that the formula (3.3) is true for arbitrary \(x_f, y_f\). In particular, using (2.17), we see that

\[
 c_3 x_f + c_3^* y_f = \frac{m \alpha}{\hbar} [x_f + y_f - i(x_f - y_f)] q + \frac{i}{\hbar} \bar{p}(x_f - y_f) \quad (3.4)
\]
Hence the functional integral (3.3) is exactly the same as the one appearing in (3.1) if, in (3.3), we make the substitutions

\[(x_f - y_f) \rightarrow -k, \quad \frac{ma}{\hbar} \left[ x_f + y_f - i(x_f - y_f) \right] \rightarrow -\frac{i}{\hbar} k' \]  \hspace{1cm} (3.5)

Inverting for \(x_f\) and \(y_f\), we therefore find that the functional integral over \(\bar{x}(t)\) in (3.1) is equal to the right-hand side of (3.3) with

\[x_f = -\frac{(1 + i)}{2} k - \frac{i}{2ma} k' \]  \hspace{1cm} (3.6)
\[y_f = -\frac{(1 - i)}{2} k - \frac{i}{2ma} k' \]  \hspace{1cm} (3.7)

Using this result, and changing variables from \(k'\) to \(K = k + k'/ma\) in (3.1), we obtain

\[f(p, q|t|x_0, y_0) = \exp \left( \frac{ma}{\hbar} \left( 1 - i \right) (x_f - q)^2 - \frac{i}{\hbar} px_f \right) \times \frac{J(x_f, y_f, t|x_0, y_0, 0)}{\rho_t(x_f, y_f)} \]  \hspace{1cm} (3.9)

An alternative way of writing (3.8) is to carry out the same steps, but to change variables in (3.1) from \(k, k'\) to \(x_f, y_f\), with the formal result,

\[f(p, q, t|x_0, y_0) = \int dx_f dy_f \exp \left( \frac{ma}{2\hbar} (1 - i)(x_f - q)^2 - \frac{i}{\hbar} px_f \right) \times \exp \left( \frac{ma}{2\hbar} (1 + i)(y_f - q)^2 + \frac{i}{\hbar} py_f \right) J(x_f, y_f, t|x_0, y_0, 0) \]  \hspace{1cm} (3.10)

Folding in the initial state via (2.24), we obtain,

\[f(p, q, t) = \int dx_f dy_f \exp \left( \frac{ma}{2\hbar} (1 - i)(x_f - q)^2 - \frac{i}{\hbar} px_f \right) \times \exp \left( \frac{ma}{2\hbar} (1 + i)(y_f - q)^2 + \frac{i}{\hbar} py_f \right) \rho_t(x_f, y_f) \]  \hspace{1cm} (3.10)

which has the appearance of a formal inversion of the relation (1.5).

Because the coordinate transformation (3.6), (3.7) is complex some attention to the integration contour is necessary. In particular, \(k\) and \(k'\) are integrated along the real axis,
therefore $x_f + y_f$ is integrated along a purely imaginary contour and $x_f - y_f$ along a real contour. More precisely, let $X = \frac{1}{2}(x_f + y_f)$ and $\xi = x_f - y_f$. Then (3.9) becomes

$$f(p, q, t|x_0, y_0) = \int_{-i\infty}^{i\infty} dX \int_{-\infty}^{+\infty} d\xi \exp\left(\frac{m\alpha}{\hbar} \left((X - q)^2 + \frac{\xi^2}{4}\right) - \frac{i}{\hbar} m\alpha \xi (X - q) - \frac{i}{\hbar} p\xi\right)$$

$$\times J(X + \frac{\xi}{2}, X - \frac{\xi}{2}, t|x_0, y_0, 0)$$

(3.11)

Explicitly, this integral reads,

$$f(p, q, t|x_0, y_0) = \int_{-i\infty}^{i\infty} dX \int_{-\infty}^{+\infty} d\xi \exp\left(\frac{m\alpha}{\hbar} \left((X - q)^2 + \frac{\xi^2}{4}\right) - \frac{i}{\hbar} m\alpha \xi (X - q) - \frac{i}{\hbar} p\xi\right)$$

$$\times \exp\left(\frac{im}{\hbar t} (X - X_0)(\xi - \xi_0) - \frac{2m\alpha^2 t}{3\hbar} (\xi^2 + \xi_0^2)\right)$$

(3.12)

where $X_0$ and $\xi_0$ defined in the same way as $X$ and $\xi$. The $X$ integral will clearly converge since the contour is along the imaginary axis, and the $\xi$ integral will converge for sufficiently large $\alpha t$.

Letting $X \to X + q$, the integral over $X$ is readily carried out, with the result

$$f(p, q, t|x_0, y_0) = \int d\xi \exp\left(-\frac{i}{\hbar} p\xi + \frac{im}{\hbar t} (q - X_0)(\xi - \xi_0) - \frac{2m\alpha^2 t}{3\hbar} (\xi^2 + \xi_0^2)\right)$$

$$\times \exp\left(\frac{m\alpha}{4\hbar} \left[\xi^2 + \left(\xi - \frac{\xi_0 - 2\xi_0}{\alpha t}\right)^2\right]\right)$$

(3.13)

The integral over $\xi$ may now be evaluated but it is not necessary to do this, since the form of the answer is now clear. For $\alpha t >> 1$, the terms in the second exponential are negligible compared to the similar terms in the first. Furthermore, the remaining terms have the form of the Wigner transform of the propagator [27,33]. We thus have the simple result,

$$f(p, q, t|x_0, y_0) \approx \int d\xi \ e^{-\frac{i}{\hbar} p\xi} \ J(q, \xi, t|X_0, \xi_0, 0)$$

(3.14)

Attaching an arbitrary initial density matrix, it then follows from (2.24) that

$$f(p, q, t) \approx \int d\xi \ e^{-\frac{i}{\hbar} p\xi} \ \rho(t(q + \frac{1}{2}\xi, q - \frac{1}{2}\xi))$$

$$= W_t(p, q)$$

(3.15)

That is, for $\alpha t >> 1$, $f(p, q, t)$ is the Wigner function of the density operator at time $t$. This is the second main result of this paper.
From any of the above representations of $f(p, q, t)$ (other than (3.14)), or from Ref. [16], it is straightforward to show that $f(p, q, t)$ obeys the Fokker-Planck equation,

$$\frac{\partial f}{\partial t} = -\frac{p}{m} \frac{\partial f}{\partial q} + 2m\gamma kT \frac{\partial^2 f}{\partial p^2} + \left(\frac{2\hbar\gamma kT}{2m}\right) \frac{\partial^2 f}{\partial p \partial q} + \frac{\hbar}{2m} \frac{\partial^2 f}{\partial q^2}$$  \hspace{1cm} (3.16)

As we have seen, $f(p, q, t)$ approaches the Wigner function $W_t(p, q)$ for $\alpha t >> 1$, which obeys the Fokker-Planck equation of classical Brownian motion:

$$\frac{\partial W}{\partial t} = -\frac{p}{m} \frac{\partial W}{\partial q} + 2m\gamma kT \frac{\partial^2 W}{\partial p^2}$$  \hspace{1cm} (3.17)

What happens is that the last two terms in Eq. (3.16) become negligible for large $\alpha t$, as may be seen by studying the Wigner function propagator (below).

### 4. THE POSITIVITY OF THE WIGNER FUNCTION

We have shown that the density operator approaches the form (1.5), where $f(p, q, t)$ is given by the Wigner function. However, $f(p, q, t)$ is by construction positive, yet the Wigner function is not guaranteed to be positive in general [33]. What happens is that the Wigner function becomes strictly non-negative after a period of time, under evolution according to (the Wigner transform of) Eq. (1.1), as we now show.

The Wigner transform of the relation (2.1) yields,

$$W_t(p, q) = \int dp_0 dq_0 \ K(p, q, t|p_0, q_0, 0) \ W_0(p_0, q_0)$$  \hspace{1cm} (4.1)

where $K(p, q, t|p_0, q_0, 0)$ is the Wigner function propagator, and is given by [27],

$$K(p, q, t|p_0, q_0, 0) = \exp \left( -\mu(p - p_0)^2 - \nu \left( q - q_0 - \frac{p_0 t}{m} \right)^2 + \sigma(p - p_0) \left( q - q_0 - \frac{p_0 t}{m} \right) \right)$$  \hspace{1cm} (4.2)

where, introducing $D = 2m\gamma kT$,

$$\mu = \frac{1}{Dt}, \quad \nu = \frac{3m^2}{Dt^3}, \quad \sigma = \frac{3m}{Dt^2} \hspace{1cm} (4.3)$$

It is well-known that the Wigner function may take negative values only through oscillations in $\hbar$-sized regions of phase space, and that it may be rendered positive by
coarse-graining of such a region. Considered for example, the smeared Wigner function
\[ W_H(p, q) = 2 \int dp'dq' \exp \left( -\frac{2\sigma^2(p-p')^2}{\hbar^2} - \frac{(q-q')^2}{2\sigma_q^2} \right) W_\rho(p', q') \] (4.4)
This object is called the Husimi function [34]. It is equal to the expectation value of the corresponding density operator in a coherent state (of position width \( \sigma_q \)), \( \langle p, q | \rho | p, q \rangle \), so is non-negative.

Loosely speaking, what happens during time evolution according to (4.1), is that, after a certain amount of time, the propagator effectively smears the Wigner function over a region of phase space greater than \( \hbar \), and it becomes positive, in the manner of (4.4). We will now show this explicitly.

Letting \( p_0 \to p_0 + p \) and \( q_0 \to q_0 + q - p_0 t/m \) in (4.1) yields,
\[ W_t(p, q) = \int dp_0 dq_0 \exp \left( -\mu p_0^2 - \nu q_0^2 + \sigma p_0 q_0 \right) W_0(p_0 + p, q_0 + q - \frac{p_0 t}{m}) \] (4.5)
The further transformation \( p_0 \to p_0 + \frac{\sigma^2}{2\mu} q_0 \) yields,
\[ W_t(p, q) = \int dp_0 dq_0 \exp \left( -\mu p_0^2 - \beta q_0^2 \right) W_0(p_0 + \frac{\sigma}{2\mu} q_0 + p, q_0 + q - \frac{p_0 t}{m}) \] (4.6)
where \( \beta = \left( \nu - \frac{\sigma^2}{4\mu} \right) \). These two transformations are canonical, and therefore the transformed Wigner function appearing in the integrand of (4.6) is still the Wigner function of some state (unitarily related to the original one). Hence,
\[ W_t(p, q) = \int dp_0 dq_0 \exp \left( -\mu p_0^2 - \beta q_0^2 \right) \tilde{W}_{pq}(p_0, q_0) \] (4.7)
for some Wigner function \( \tilde{W}_{pq} \) depending on \( p, q \). This may now be recast as the smearing of a Husimi function:
\[ W_t(p, q) = \int dp' \exp \left( -\frac{p'^2}{(\mu^{-1} - \frac{\hbar^2}{\beta})} \right) \int dp_0 dq_0 \exp \left( -\frac{(p' - p_0)^2}{\hbar^2 \beta} - \beta q_0^2 \right) \tilde{W}_{pq}(p_0, q_0) \] (4.8)
The integral over \( p_0, q_0 \) is a Husimi function with \( \sigma_q^2 = 1/(2\beta) \). Hence \( W_t(p, q) \geq 0 \) provided the integral over \( p' \) in (4.8) exists. This will be the case if \( \mu^{-1} > \frac{\hbar^2}{\beta} \), that is, if
\[ t > \left( \frac{\sqrt{3}}{2} \right)^{\frac{1}{2}} \left( \frac{\hbar}{\gamma kT} \right)^{\frac{1}{2}} \] (4.9)
The Wigner function will therefore be non-negative for times greater than the localization time (multiplied by a number of order 1).
5. DISCUSSION

We have shown that for times greater than the localization time, \((\hbar/\gamma kT)^{\frac{1}{2}}\), the density operator satisfying (1.1) approaches the form

\[
\rho = \int dpdq \ W_t(p, q) |\Psi_{pq}\rangle \langle \Psi_{pq}|
\]

where \(W_t(p, q)\) is the Wigner function and the \(|\Psi_{pq}\rangle\) are close to minimum uncertainty generalized coherent states. The Wigner function is strictly non-negative for times greater than the localization time (times a number of order 1).

Diósi has also discussed the possibility of the phase space diagonal form (1.5) under evolution according to the master equation (1.1) [35]. His method was very different to ours, in that he used the properties of the coherent states to regard (1.5) as an expansion of the density operator. He found that such an expansion is possible for times greater than the localization time, times a number of order 1, in tune with our results.

An advantage of deriving (5.1) using path integral methods, rather than quantum state diffusion, is that it yields and explicit expression for the phase space distribution function \(f(p, q, t)\). Another advantage is that it is not obviously restricted to Markovian master equations. The quantum state diffusion picture, in its current state of development, exists only for systems described by a Markovian master equation. It may exist in the non-Markovian case, but is yet to be developed. The exact propagator for quantum Brownian motion, for quadratic potentials, can be given in terms of a path integral [5], and is (mildly) non-Markovian. Since the method described here utilizes path integrals, rather than the quantum state diffusion picture, there is a chance that our method may be valid in the non-Markovian case also, but this is still to be investigated.

We have concentrated in this paper on the simplest possible model of quantum Brownian motion: the free particle in a high temperature environment with negligible dissipation \(\gamma\). It is clear, however, that remaining in the context of a Markovian master equation, it would be straightforward (although perhaps tedious) to extend our considerations to the case of a harmonic oscillator with non-trivial dissipation. In the quantum state diffusion
picture analysis, this case was covered in Ref.[16] and we expect the path integral treatment of the present paper to yield comparable results.

It is perhaps enlightening to comment on the various timescales involved in a more general quantum Brownian motion model, and sketch the expected general physical picture part of which is described by the results of this paper.

In this paper, we have largely been concerned with the localization time, \((\hbar/\gamma kT)^{1/2}\), which is the timescale on which an arbitrary initial density operator approaches the form (1.5). The nomenclature “localization time” comes from the quantum state diffusion picture, which was the picture first used to derive some of the results described in this paper. It is so named because it is the time scale on which an arbitrary initial wave function becomes localized in phase space under evolution according to Eq.(1.3) [16,20].

Also relevant is the decoherence time, \(\hbar^2/((\ell^2 m \gamma kT))\), which is the timescale on which the off-diagonal terms of the density matrix are suppressed (in the position representation) [14]. The decoherence time necessarily involves a length scale \(\ell\), which comes from the initial state. It could, for example, be the separation of a superposition of localized wave packets, and the decoherence time is then the time scale on which the interference between these packets is suppressed.

If one is interested in emergent classicality for macroscopic systems, it is appropriate to choose values order 1 in c.g.s. units, for \(\ell\), \(T\), \(m\) and \(\gamma\). The decoherence time is then typically much shorter than the localization time. This is in turn typically much shorter than the relaxation time, \(\gamma^{-1}\), which is the time scale on which the system approaches thermal equilibrium (when this is possible).

Hence the general picture we have is as follows. Suppose the initial state of the system is a superposition of localized wave packets. Then the interference terms between these wave packets is destroyed on the decoherence timescale. After a few localization times the density matrix subsequently approaches the phase space diagonal form (1.5). After a much longer time of order the relaxation time, the system reaches thermal equilibrium. Discussions of emergent classicality usually concern times between the decoherence time
and the relaxation time, and it is this range of time which has been the primary concern of this paper.

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