LATTICE POLYTOPES, FINITE ABELIAN SUBGROUPS IN
SL(n, C) AND CODING THEORY

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Abstract. We consider \( d \)-dimensional lattice polytopes \( \Delta \) with \( h^* \)-polynomial \( h^*_\Delta = 1 + h^*_1 t^k \) for \( 1 < k < (d + 1)/2 \) and relate them to some abelian subgroups of \( \SL_{d+1}(\mathbb{C}) \) of order \( 1 + h^*_p = p^r \) where \( p \) is a prime number. These subgroups can be investigate by means of coding theory as special linear constant weight codes in \( \mathbb{F}_p^{d+1} \). If \( p = 2 \), then the classification of these codes and corresponding lattice polytopes can be obtained using a theorem of Bonisoli. If \( p > 2 \), the main technical tool in the classification of these linear codes is the non-vanishing theorem for generalized Bernoulli numbers \( B_{p^r, \chi} \) associated with odd characters \( \chi: \mathbb{F}_p^* \to \mathbb{C}^* \) where \( q = p^r \). Our result implies a complete classification of all lattice polytopes whose \( h^* \)-polynomial is a binomial.

Introduction

Let \( M \cong \mathbb{Z}^d \) be a \( d \)-dimensional lattice and \( \Delta \subset M_R := M \otimes \mathbb{R} \cong \mathbb{R}^d \) a \( d \)-dimensional lattice polytope, i.e. the vertices of \( \Delta \) are contained in the lattice \( M \). It is well-known (see e. g. [BR07, Section 3]) that the Ehrhart series

\[
\text{Ehr}_\Delta(t) = 1 + \sum_{k \geq 1} |k \Delta \cap M| t^k
\]

is a rational function of the form

\[
\text{Ehr}_\Delta(t) = \frac{1 + h^*_1 t + \ldots + h^*_d t^d}{(1-t)^{d+1}} = \frac{h^*_\Delta(t)}{(1-t)^{d+1}}
\]

where the coefficients \( h^*_1, \ldots, h^*_d \) of the polynomial \( h^*_\Delta(t) \) are nonnegative integers. We call the polynomial \( h^*_\Delta(t) \) the \( h^* \)-polynomial of \( \Delta \). Recall some basic facts about \( h^* \)-polynomials.

Two lattice polytopes \( \Delta \subset M_R \) and \( \Delta' \subset M'_R \) are called isomorphic if there exists a bijective affine linear map \( \varphi: M_R \to M'_R \) such that \( \varphi(\Delta) = \Delta' \) and \( \varphi(M) = M' \). If \( \Delta \) and \( \Delta' \) are isomorphic lattice polytopes, then \( |k \Delta \cap M| = |k \Delta' \cap M'| \) for all \( k \geq 1 \) and we have \( h^*_\Delta(t) = h^*_{\Delta'}(t) \).

For any \( d \)-dimensional lattice polytope \( \Delta \subset M_R \), we construct a new \((d+1)\)-dimensional lattice polytope

\[
\Delta' := \text{conv}(\Delta \times \{0\}, (0,1)) \subset M_R \oplus \mathbb{R}
\]

which is called the pyramid over \( \Delta \). It is easy to show that \((1-t)\text{Ehr}_\Delta(t) = \text{Ehr}_{\Delta'}(t) \) (see e. g. [BR07, Theorem 2.4]). So we obtain again \( h^*_\Delta(t) = h^*_{\Delta'}(t) \).

Let \( e_1, \ldots, e_d \) be a \( \mathbb{Z} \)-basis of \( M \). A \( d \)-dimensional simplex \( \Delta \) is called a unimodular simplex if it is isomorphic to the convex hull of \( 0, e_1, \ldots, e_d \). For any lattice polytope \( \Delta \) we denote by \( \text{vol}(\Delta) \) the integral volume of \( \Delta \), i.e. \( \text{vol}(\cdot) \) is the \( d \)-multiple of the standard Lebesgue volume on \( \mathbb{R}^d \) such that for any unimodular simplex \( \Delta \) holds \( \text{vol}(\Delta) = 1 \). By [BR07, Corollary 3.21], we have \( h^*_\Delta(1) = 1 + \sum_{k=1}^d h^*_k = \text{vol}(\Delta) \).

Date: May 22, 2014.

2000 Mathematics Subject Classification. Primary 52B20; Secondary 14B05, 11B68.
The knowledge of the $h^*$-polynomial of $\Delta$ imposes strong conditions on the polytope $\Delta$. For example, the equalities $h^*_1 = \ldots = h^*_d = 0$ hold if and only if $\Delta$ is a unimodular simplex.

The purpose of this paper is to classify (up to isomorphism) lattice polytopes $\Delta$ whose $h^*$-polynomial has exactly one nonzero coefficient $h^*_k \neq 0$ for $k \in \{1, \ldots, d\}$, i.e. $h^*_\Delta = 1 + h^*_k t^k$. Such an $h^*$-polynomial will be called an $h^*$-binomial.

Simplest examples of $h^*$-binomials are the ones of degree $k = 1$, i.e. linear $h^*$-polynomials. Lattice polytopes $\Delta$ such that $h^*_\Delta$ is linear have been classified by the first author and Benjamin Nill in [BN07] (we will give a summary of these results in section 1). Thus, for our purpose, it remains to investigate $d$-dimensional lattice polytopes having an $h^*$-binomial of degree $k > 1$. One can show that the condition $k > 1$ implies that $\Delta$ is a simplex and the degree $k$ of $h^*_\Delta$ is at most $(d + 1)/2$. In particular, 3-dimensional lattice simplices $\Delta \subseteq \mathbb{R}^3$ having nonlinear $h^*$-binomial are precisely empty lattice tetrahedra with $h^*$-polynomial $h^*_\Delta = 1 + h^*_2 t^2$.

These tetrahedra have been classified by G.K. White in [Whi64]. In [BH10], we generalized the classification of White to arbitrary odd dimension $d \geq 3$ using the so called “Terminal Lemma” of M. Reid [Rei87]. This implies the classification of $d$-dimensional lattice simplices having an $h^*$-binomial of degree $k = (d + 1)/2 > 1$ in (see also section 1). Therefore, this paper is devoted mainly to lattice polytopes having an $h^*$-binomial with degree $1 < k < (d + 1)/2$ which are not pyramids over lower-dimensional lattice simplices.

Let us give an overview of our approach. For a $d$-dimensional lattice simplex $\Delta \subseteq \mathbb{M}_\mathbb{R}$ with vertices $v_0, \ldots, v_d$, we define a subgroup $G_\Delta \subseteq SL_{d+1}(\mathbb{C})$ which consists of all diagonal matrices

$$g(\lambda) := \text{diag}(e^{2\pi i \lambda_0}, \ldots, e^{2\pi i \lambda_d})$$

such that $\lambda_i \in [0, 1]$ for all $i \in \{0, \ldots, d\}$, $\sum_{i=0}^{d} \lambda_i v_i \in M$ and $\sum_{i=0}^{d} \lambda_i \in \mathbb{Z}$. The simplex $\Delta$ is determined up to isomorphism by the subgroup $G_\Delta \subseteq SL_{d+1}(\mathbb{C})$.

Moreover, the coefficients of its $h^*$-polynomial $h^*_\Delta = 1 + \sum_{k=1}^{d} h^*_k t^k$ can be computed as follows

$$h^*_k = |\{\text{diag}(e^{2\pi i \lambda_0}, \ldots, e^{2\pi i \lambda_d}) \in G_\Delta : \sum_{i=0}^{d} \lambda_i = k\}|, \quad 0 \leq k \leq d.$$

It is easy to show that a lattice simplex $\Delta$ is a pyramid over a lower-dimensional simplex if and only if there exists $i \in \{1, \ldots, d + 1\}$ such that $\lambda_i = 0$ for all $g(\lambda) \in G_\Delta$.

For a prime number $p$, we denote the group of $p$-th roots of unity in $\mathbb{C}$ by $\mu_p$.

**Theorem.** Let $\Delta$ be a $d$-dimensional lattice simplex which is not a pyramid over a lower-dimensional simplex. If the degree $k$ of the $h^*$-polynomial $h^*_\Delta = 1 + h^*_k t^k$ satisfies $1 < k < (d + 1)/2$, then there exists a prime number $p$ such that all nontrivial elements $g(\lambda) \in G_\Delta$ have order $p$. In particular, $G_\Delta$ can be considered as a subgroup of $\mu_p^{d+1}$ and one obtains $\text{vol}(\Delta) = 1 + h^*_k = |G_\Delta| = p^r$ for a positive integer $r$.

For a prime number $p$, we denote by $\mathbb{F}_p$ the finite field of order $p$. By the above theorem, the subgroup $G_\Delta \subseteq \mu_p^{d+1}$ can be identified with an $r$-dimensional linear subspace $L_\Delta \subseteq \mathbb{F}_p^{d+1}$.

Let $L \subseteq \mathbb{F}_p^n$ be an arbitrary $r$-dimensional linear subspace in $\mathbb{F}_p^n$. Choose a basis $a_1, \ldots, a_r$ of $L$. For $i = 1, \ldots, r$ we consider the vector $a_i = (a_{i1}, \ldots, a_{in})$ as the $i$-th row of a $(r \times n)$-matrix $A$ which we call a generator matrix of $L$. The linear subspace $L$ is uniquely determined by the matrix $A$. Let us write an arbitrary vector
Consider the following two functions on $F_p^n$ with values in $\mathbb{Z}_{\geq 0}$:

- the weight-function
  \[
  \omega(v) := |\{i \in \{1, \ldots, n\} : v_i \neq 0\}|
  \]
- and the age-function
  \[
  \alpha(v) := \sum_{i=0}^d v_i.
  \]

We say that a linear subspace $L \subset F_p^n$ has constant weight (resp. constant age), if the weight-function $\omega$ (resp. age-function $\alpha$) is constant on the set of all nonzero vectors in $L$. It is easy to show that if a linear subspace $L \subseteq F_p^n$ has constant age then $L$ has also constant weight and one has

\[
p\omega(v) = 2\alpha(v) \ \forall v \in L \setminus \{0\}.
\]

We will see below some examples of linear subspaces $L$ in $F_p^n$ of constant weight that do not have constant age.

In coding theory linear subspaces of $F_p^n$ are called linear codes. It is important to remark that the linear code $L_{\Delta} \subseteq F_p^{d+1}$ associated to a $d$-dimensional lattice simplex $\Delta$ with $h^*$-binomial of degree $1 < k < (d + 1)/2$ has constant age $kp$. There are special linear codes of constant weight which are called simplex codes. These linear codes are constructed as follows. Fix a positive integer $l \in \mathbb{N}$ and put $m := (p^l - 1)/(p - 1)$ to be the number of points in $(l - 1)$-dimensional projective space $\mathbb{P}^{l-1}$ over $F_p$. For any point $x \in \mathbb{P}^{l-1}$ one chooses a nonzero vector $A(x) \in F_p^m$ in the corresponding $1$-dimensional linear subspace in $F_p^n$. We consider the vectors $A(x_1), \ldots, A(x_m)$ as columns of a $l \times m$-matrix $A$ and define the $l$-dimensional linear simplex code in $F_p^n$ by the generator matrix $A$. We remark that the above matrix $A$ is determined uniquley up to permutations of its columns and multiplications by nonzero elements in $F_p$.

If $p = 2$, then the weight-function $\omega$ and the age-function $\alpha$ coincide. We observe that in this case the generator matrix of a simplex code is unique up to permutation of the columns and the classification of linear codes of constant age can be obtained using a theorem of Bonisoli (see e.g. [Bon84] or [WW96]).

**Main Theorem** (Case $p = 2$). Let $\Delta$ be a $d$-dimensional lattice simplex with $h^*$-binomial of degree $1 < k < (d + 1)/2$. Assume that $\text{vol}(\Delta) = 2^r$ for a positive integer $r$ and $\Delta$ is not a pyramid over a lower-dimensional simplex. Then the numbers $k, d, r$ are related by the equation:

\[
2^{r-2}(d + 1) = k(2^r - 1)
\]

and the generating $(r \times (d + 1))$-matrix of the linear code $L_{\Delta}$ can be written up to a permutation of the columns in the form $(S_r, \ldots, S_r)$ where $S_r$ is repeated $(k/2^{r-2})$-times.

**Example.** Let $\Delta$ be a $d$-dimensional lattice simplex with $h^*$-binomial of degree $1 < k < (d + 1)/2$. Assume that $\text{vol}(\Delta) = 2$ (i.e. $r = 1$, or $h^*_\Delta = 1 + t^k$) and $\Delta$ is not a pyramid over a lower-dimensional simplex. Then the above theorem implies that $d + 1 = 2k$ and $G_\Delta = \{\pm E\}$.

The case $p > 2$ is more involved.
Example. If $p = 3$ then the matrix
\[ A_1 = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \]
generates a 2-dimensional simplex code $L_1 \subset \mathbb{F}_4^2$ of constant weight 3. But the rows of $A_1$ have different age. So the linear code $L_1$ does not have constant age.

Example. We can modify the previous example as follows. Consider the 2-dimensional linear code $L$ in $\mathbb{F}_8^3$ generated by the matrix:
\[ A = (A_1, -A_1) = \begin{pmatrix} 1 & 0 & 1 & 2 & 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 0 & 2 & 2 & 2 \end{pmatrix} \]
Then the linear code $L$ has not only the constant weight 6, but it has the constant age 9, because for any vector $(v_1, \ldots, v_8) \in L$ one has $v_i + v_{i+4} \in \{0, 3\}$ for all $1 \leq i \leq 4$.

Our main theorem in the case $p > 2$ claims that the last example is in some sense typical for constructing any linear code of constant age:

**Main Theorem.** Let $\Delta$ be a $d$-dimensional lattice simplex having an $h^*$-binomial of degree $1 < k < (d+1)/2$. Assume that $\text{vol}(\Delta) = p^r$ for a prime number $p$ and a positive integer $r$. Assume that $\Delta$ is not a pyramid over a lower-dimensional simplex. Then the number $p, d, k, r$ are related by the equation
\[
(p^r - p^{r-1})(d + 1) = 2k(p^r - 1)
\]
and the generating $(r \times (d + 1))$-matrix of the linear code $L_\Delta$ can be written up to permutation of the columns in the form
\[
(A_1, -A_1, A_2, -A_2, \ldots, A_s, -A_s),
\]
where $s = k/p^r - 1$ and $A_1, \ldots, A_s$ are generator matrices of $r$-dimensional simplex codes.

In the proof of this theorem we will use some number theoretic results which have probably independent interest. Fix an odd prime $p$ and a prime power $q = p^r$.

Let $B_1$ be the 1st (periodic) Bernoulli function which maps a real number $x$ to
\[
B_1(x) = \begin{cases} 
\{x\} - \frac{1}{2} & x \notin \mathbb{Z} \\
0 & x \in \mathbb{Z}
\end{cases}
\]
where $\{x\}$ denotes the fractional part of $x$, i.e. $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the biggest integer which is smaller than or equal to $x$. For any $a \in \mathbb{F}_q$ we denote by $\text{Tr}(a) \in \{0, 1, \ldots, p - 1\}$ the representative of the trace of $a$ in $\mathbb{F}_p$. We define generalized Bernoulli numbers associated to characters $\chi : \mathbb{F}_q^* \to \mathbb{C}^*$ as
\[
B_1^{(r)}(\chi) := \sum_{a \in \mathbb{F}_q^*} \chi(a) B_1\left(\frac{\text{Tr}(a)}{p}\right).
\]
The number $B_1^{(1)}$ coincides with the classical generalized Bernoulli number $B_1, \chi$ (see e.g. [Was97], Chapter 4). The important technical tool in the proof of the main theorem is the following non-vanishing theorem similar to the classical case (see e.g. [Was97]).

**Theorem.** Let $\chi : \mathbb{F}_q^* \to \mathbb{C}^*$ be an arbitrary odd character, i.e. $\chi(-1) = -1$. Then $B_1^{(r)}(\chi) \neq 0$. 

This paper is organized as follows. In Section 1, we recall some already known classification results of lattice polytopes having an $h^*$-binomial of degree $k$. In Section 2, we define the group $G$ associated to a lattice simplex $\Delta$ and make some basic observations. In Section 3, we investigate the group $G\Delta$ of a lattice simplex $\Delta$ having an $h^*$-binomial of degree $1 < k < (d+1)/2$ and show how these groups are related to coding theory. In Section 4, we recall the notions and results from coding theory which we will need later on. In Section 5, we classify linear codes of constant age and prove our main classification result. Finally, we will prove the non-vanishing theorem for the generalized Bernoulli numbers $B^{(1)}_{1,\chi}$ in section 6.

1. The cases $k = 1$ and $k = (d+1)/2$

Let $\Delta$ be a $d$-dimensional lattice polytope with $h^*$-binomial of degree $k$. The case $k \leq 1$ has been studied by the first author and Benjamin Nill in [BN07]. We shortly recall their results.

Definition 1.1. Let $M$ be a lattice. Consider $r+1$ lattice polytopes $\Delta_0, \ldots, \Delta_r \subset M_R$ and the cone

$$\sigma := \{(\lambda_0, \ldots, \lambda_r, \sum \lambda_i \Delta_i) \subset \mathbb{R}^{r+1} \oplus M_R : \lambda_i \geq 0\}.$$ 

The polytope which arises by intersecting the cone $\sigma$ with the hyperplane $H := \{x \in \mathbb{R}^{r+1} \oplus M'_R : \sum_i x_i = 1\}$ is called the Cayley polytope of $\Delta_0, \ldots, \Delta_r$ and we denote it by $\Delta_0 \ast \ldots \ast \Delta_r$.

If $\dim \Delta_i = 0$ for $i = 1, \ldots, r$, we call $\Delta_0 \ast \ldots \ast \Delta_r$ the $r$-fold pyramid over $\Delta_0$.

Definition 1.2. An $n$-dimensional lattice polytope $\Delta$ ($n \geq 1$) is called Lawrence prism with heights $h_1, \ldots, h_n$, if there exist non-negative integers $h_1, \ldots, h_n$ such that $\Delta$ is the Cayley polytope of $n$ segments $[0, h_1], \ldots, [0, h_n] \subset \mathbb{R}$.

Definition 1.3. We call an $n$-dimensional lattice polytope $\Delta$ ($n \geq 2$) exceptional, if it is a simplex which is the $(n-2)$-fold pyramid over the 2-dimensional basic simplex multiplied by 2.

Theorem 1.4 ([BN07, Theorem 2.5]). Let $\Delta$ be a $d$-dimensional lattice polytope. Then $\deg(h^*_\Delta) \leq 1$ if and only if $\Delta$ is an exceptional simplex or a Lawrence prism.

It remains to consider the case $k > 1$, which we will assume from now on. By [BR07, Corollary 3.16], $h^*_1 = |\Delta \cap M| - d - 1$, so $\Delta$ must be a simplex. For $d$-dimensional lattice simplices the coefficients $h^*_k$ of the $h^*$-polynomial $h^*_\Delta(t) =$
\[
\sum_{k=0}^{d} h_k^* t^k \text{ have a simple combinatorial description: Let } v_0, \ldots, v_d \text{ be the vertices of } \Delta. \text{ Then }
\]
\[
\Pi_\Delta := \left\{ \sum_{i=0}^{d} \lambda_i (v_i, 1) : 0 \leq \lambda_i < 1 \right\} \subset M_\mathbb{R} \oplus \mathbb{R}
\]
is called the fundamental parallelepiped of \( \Delta \). By [BR07 Corollary 3.11], \( h_k^* \) equals the number of lattice points in the fundamental parallelepiped \( \Pi_\Delta \) with last coordinate equal to \( k \).

**Proposition 1.5.** Let \( M \) be a lattice of rank \( d \) and \( \Delta \subseteq M_\mathbb{R} \) a \( d \)-dimensional lattice polytope with \( h^* \)-binomial of degree \( k > 1 \). Then \( k \leq (d + 1)/2 \).

**Proof.** Recall that \( \Delta \) is a simplex. Let \( v_0, \ldots, v_d \) be the vertices of \( \Delta \). Assume that \( k > (d + 1)/2 \). Then there exist \( 0 \leq \lambda_i < 1 \) for \( i = 0, \ldots, d \) such that \( \sum_{i=0}^{d} \lambda_i (v_i, 1) \in M \oplus \mathbb{Z} \) has last coordinate strictly bigger than \( (d + 1)/2 \). The last coordinate of
\[
\sum_{i=0}^{d} \{1 - \lambda_i\} (v_i, 1) \in M \oplus \mathbb{Z}
\]
is strictly smaller than \( (d + 1)/2 \), so \( h_\Delta^* \) is not a binomial. Contradiction. \( \square \)

We have studied the case \( k = (d + 1)/2 \) in [BH10]. In this case, the dimension of the lattice simplex \( d = 2k - 1 \) is odd. We give a summary of the classification result.

**Theorem 1.6.** Let \( \Delta \) be a \((2k-1)\)-dimensional lattice simplex which is not a lattice pyramid. The following two statements are equivalent:

1. The \( h^* \)-binomial of \( \Delta \) has degree \( k \).
2. \( \Delta \) is isomorphic to the Cayley polytope \( \Delta_1^* \ast \ldots \ast \Delta_k^* \) of \( 1 \)-dimensional empty lattice simplices \( \Delta_i \subset M'_\mathbb{R} \) where \( M' \) is a \( k \)-dimensional lattice.

2. **The Group \( G_\Delta \) Associated to a Simplex \( \Delta \)**

Let \( M \cong \mathbb{Z}^d \) be a \( d \)-dimensional lattice. To a \( d \)-dimensional lattice simplex \( \Delta \subseteq M_\mathbb{R} \) with vertices \( v_0, \ldots, v_d \in M \), we associate a group
\[
G_\Delta = \{ \text{diag}(e^{2\pi ix_0}, \ldots, e^{2\pi ix_d}) : x_i \in [0, 1], \sum_{i=0}^{d} x_i (v_i, 1) \in M_\mathbb{R} \oplus \mathbb{Z} \} \subset \text{SL}_{d+1}(\mathbb{C})
\]

The composition of this group is component wise multiplication. Below, we will mostly need an additive structure. That is why, we introduce
\[
\Lambda_\Delta = \{ x = (x_0, \ldots, x_d) \in (\mathbb{R}/\mathbb{Z})^{d+1} : \{x_i\} \in [0, 1], \sum_{i=0}^{d} \{x_i\} (v_i, 1) \in M_\mathbb{R} \oplus \mathbb{Z} \}
\]
where \((\mathbb{R}/\mathbb{Z})^{d+1}\) denotes the \((d+1)\)-dimensional real torus and where \( \{y\} \) denotes the unique representative in \([0, 1]\) of \( y \in \mathbb{R}/\mathbb{Z} \). To be more precise, the fractional part is the function \( \{\cdot\} : \mathbb{R} \to [0, 1] \) which associates to \( x \in \mathbb{R} \) the unique representative in \([0, 1]\) of \( x + \mathbb{Z} \in \mathbb{R}/\mathbb{Z} \). From this we get a well-defined function \( \{\cdot\} : \mathbb{R}/\mathbb{Z} \to [0, 1] \).

**Remark 2.1.** It is clear that \( G_\Delta \) and \( \Lambda_\Delta \) are uniquely determined by each other. Indeed, they give the same group but \( G_\Delta \) has a multiplicative structure while \( \Lambda_\Delta \) has an additive structure.
Remark 2.2. There is a bijection between lattice points in the fundamental parallelepiped $\Pi_{\Delta}$ and elements in $\Lambda_{\Delta}$

$$\Lambda_{\Delta} \rightarrow \Pi_{\Delta} \cap (M \oplus \mathbb{Z}); x \mapsto \sum_{i=0}^{d} \{x_i\} (v_i, 1).$$

One might think of $\Lambda_{\Delta}$ as the lattice points in the fundamental parallelepiped. But instead of remembering the lattice points $\sum_{i=0}^{d} \{x_i\} (v_i, 1)$, we keep track of the coefficients $x_0, \ldots, x_d$ in the linear combination.

Let $\mathcal{F}$ be the set of isomorphism classes of $d$-dimensional lattice simplices with a chosen order of the vertices. Then

$$\mathcal{F} \rightarrow \{\text{finite subgroups } \Lambda \subset (\mathbb{R}/\mathbb{Z})^{d+1}; \Delta \mapsto \Lambda_{\Delta}\}$$

is a bijection. We define the inverse map. Let $\Lambda \subseteq (\mathbb{R}/\mathbb{Z})^{d+1}$ be a finite group. Consider the natural projection map $\pi : \mathbb{R}^{d+1} \rightarrow (\mathbb{R}/\mathbb{Z})^{d+1}$. The preimage $M := \pi^{-1}(\Lambda) \subseteq \mathbb{R}^{d+1}$ is a $(d+1)$-dimensional lattice such that $Z^{d+1} \subseteq M$ has finite index. We denote the standard basis of $Z^{d+1}$ by $e_1, \ldots, e_{d+1}$. Then $\Delta_{\Lambda} := \text{conv}(e_1, \ldots, e_{d+1}) \subseteq M_{R}$ is a $d$-dimensional lattice simplex with respect to the affine lattice $\text{aff}(e_1, \ldots, e_{d+1}) \cap M$.

The following theorem is easy to prove.

**Theorem 2.3.** The maps $\Delta \mapsto \Lambda_{\Delta}$ and $\Lambda \mapsto \Delta_{\Lambda}$ are inverse to each other. In particular, a lattice simplex is uniquely determined up to isomorphism by its group $\Lambda_{\Delta}$.

**Example 2.4.** Let $G \subseteq \text{SL}_3(\mathbb{C})$ be the subgroup of order 4 consisting of diagonal matrices with entries 1 and $-1$. The singular locus of the quotient $\mathbb{C}^3/G$ consists of 3 irreducible curves having one common point which comes from 1-dimensional fixed point sets $(\mathbb{C}^3)^g$ for 3 non-unit elements $g \in G$. Using methods of toric geometry one can identify all minimal resolutions of the non-isolated Gorenstein quotient singularities of $\mathbb{C}^3/G$ with all triangulations of the lattice triangle

$$\Delta = \{(x_1, x_2) \in \mathbb{R}^2 : x_1, x_2 \geq 0, x_1 + x_2 \leq 2\}$$

whose $h^*$-polynomial has the form $1 + 3t$ and $G_{\Delta} = G$.

We collect some basic observations concerning the group $\Lambda_{\Delta}$ of a lattice simplex $\Delta$. Recall from the introduction that the $h^*$-polynomial stays unchanged under pyramid constructions. In contrast, the group $\Lambda_{\Delta}$ behaves as follows.

**Proposition 2.5.** Let $\Delta$ be a $d$-dimensional lattice simplex. The following statements are equivalent:

1. $\Delta$ is a pyramid;
2. there exists $i = 0, \ldots, d$ such $x_i = 0 + \mathbb{Z}$ for all $x \in \Lambda_{\Delta}$.

**Proof.** The implication (1) $\Rightarrow$ (2) is obvious.

For the other direction, let $x_d = 0 + \mathbb{Z}$ for all $x \in \Lambda_{\Delta}$. It suffices to show that $\Delta_{\Lambda_{\Delta}}$ is a pyramid. Let $\pi : \mathbb{R}^{d+1} \rightarrow (\mathbb{R}/\mathbb{Z})^{d+1}$ be the natural projection map. We have $M := \pi^{-1}(\Lambda_{\Delta}) = M' \oplus \mathbb{Z}$ where $M' = \text{pr}(M)$ for $\text{pr} : \mathbb{R}^{d+1} \rightarrow \mathbb{R}^d$ the projection onto the first $d$ coordinates. From this it follows that $\Delta_{\Lambda_{\Delta}}$ is a pyramid with apex $e_{d+1}$. $\square$

By [BR07, Corollary 3.11], we have
Proposition 2.6. Let $\Delta$ be a $d$-dimensional lattice simplex. Then the coefficients of the $h^*$-polynomial $h^*_\Delta = \sum_{k=0}^d h_k^* t^k$ can be computed as follows

$$h_k^* = |\{(x_0, \ldots, x_d) \in \Lambda_\Delta : \sum_{i=0}^d x_i = k\}|.$$

3. The case $1 < k < (d + 1)/2$

Consider a $d$-dimension lattice simplex $\Delta$ with $h^*$-binomial of degree $1 < k < (d + 1)/2$ which is not a pyramid over a lower-dimensional simplex. Let $v_0, \ldots, v_d$ be the vertices of $\Delta$. Let $0 \neq x = (x_0, \ldots, x_d) \in \Lambda_\Delta$. We can uniquely express $\{x_i\} = a_i/n$ for a positive integer $n$ and non-negative integers $a_i < n$ subject to the condition $\gcd(a_0, \ldots, a_d, n) = 1$. By rearranging the coordinates, we may assume that

$$a_0 \cdots a_l \neq 0 \quad \text{and} \quad a_{l+1} = \ldots = a_d = 0$$

for $l = 0, \ldots, d$.

Proposition 3.1. $l = 2k$; in other words: the number of nonzero coordinates $a_i/n$ is a constant.

Proof. We have $0 \neq -x = ((n-a_0)/n+\mathbb{Z}, \ldots, (n-a_d)/n+\mathbb{Z}) \in \Lambda_\Delta$. By Proposition 2.6, we obtain

$$2k = \sum_{i=0}^d \left\{\frac{n-a_i}{n}\right\} + \sum_{i=0}^d \frac{a_i}{n} = \sum_{i=0}^l \frac{n-a_i}{n} + \sum_{i=0}^l \frac{a_i}{n} = l.$$

□

Corollary 3.2. $\gcd(a_i, n) = 1$ for $i = 0, \ldots, d$.

Proof. Assume by contradiction that $\gcd(a_0, n) = q$. Then

$$0 \neq \frac{n}{q} \cdot x = \left(\frac{a_0}{q} + \mathbb{Z}, \ldots, \frac{a_d}{q} + \mathbb{Z}\right) \in \Lambda_\Delta.$$

But $a_0/q + \mathbb{Z} = 0 + \mathbb{Z}$ and there are at most $l-1$ nonzero coefficients contradicting the previous proposition. □

Let $0 \neq y = (y_0, \ldots, y_d) \in \Lambda_\Delta$. Like above, we can write $y_i = b_i/m$ for a positive integer $m$ and non-negative integers $b_i < m$ coprime to $m$.

Proposition 3.3. $m = n$.

To be able to easily formulate the proof, we introduce the following notion.

Definition 3.4. Let $z = (z_0, \ldots, z_d) \in \Lambda_\Delta$. The support of $z$ is given by

$$\text{supp}(z) := \{i = 0, \ldots, d : y_i \neq 0 + \mathbb{Z}\}.$$

Proof of Proposition 3.3. We distinguish two cases.

Assume that $\text{supp}(x) \neq \text{supp}(y)$. Consider

$$0 \neq x + y = \left(\frac{a_0}{n} + \frac{b_0}{m} + \mathbb{Z}, \ldots, \frac{a_d}{n} + \frac{b_d}{m} + \mathbb{Z}\right) \in \Lambda_\Delta.$$

By Proposition 3.1, there exists $i = 0, \ldots, d$ such that $(a_i/n + b_i/m) + \mathbb{Z} = 0 + \mathbb{Z}$, i.e.

$$\frac{a_i}{n} + \frac{b_i}{m} = 1.$$

The assertion follows by the fact that $(a_i, n) = 1 = (b_i, m)$.
Assume that \( \text{supp}(x) = \text{supp}(y) \). Since for all \( 0 \neq z \in \Lambda_\Delta \), \( |\text{supp}(z)| = 2k < d + 1 \), there exists \( i = 0, \ldots, d \) such that \( z_i = 0 \). Since \( \Delta \) is not a pyramid there exists \( 0 \neq z \in \Lambda_\Delta \) with \( \text{supp}(x) \neq \text{supp}(z) \). We may write \( z_i = c_i/n \) for a positive integer \( l \) and non-negative integers \( c_i < l \) coprime to \( l \). By the same argument as above applied to \( x \) and \( z \), it follows that \( \ell = n \). Applying this argument once again to \( z \) and \( y \) yields \( \ell = m \). Hence \( n = l = m \). 

\[ \square \]

**Corollary 3.5.** \( n = p \) is a prime number.

**Proof.** We may assume \( n \neq 2 \). Assume by contradiction that \( n \) is not prime, i.e. there is a nontrivial divisor \( d \mid n \). Since \( \Delta \) is not a pyramid there exists \( 0 \neq z \in \Lambda_\Delta \) with \( \text{supp}(x) \neq \text{supp}(z) \). We can write \( z_i = c_i/n \) for non-negative integers \( c_i < n \) coprime to \( n \). By Proposition 3.1 there exists \( i \in \text{supp}(x) \cap \text{supp}(z) \) such that \( a_i/n + c_i/n = 1 \). Let \( a_i' \) be an integer such that \( a_i'a_i = 1 \in \mathbb{Z}/n\mathbb{Z} \). Consider \( w = \sum (d - c_i)a_i' \cdot x + z \in \Lambda_\Delta \). The \( i \)-th coordinate of \( w \) is equal to \( d/n + \mathbb{Z} \). Contradiction to Corollary 3.2.

Our purpose is to relate the classification of \( d \)-dimensional lattice polytopes \( \Delta \) with an \( h^* \)-binomial of degree \( 1 < k < (d+1)/2 \) to coding theory using linear codes \( L_\Delta \subseteq \mathbb{F}_p^{d+1} \).

**Definition 3.6.** A linear subspace \( L \subseteq \mathbb{F}_p^d \) of dimension \( k \) will be called a linear code of dimension \( k \) and block length \( d \). Let \( x \) be a vector in \( \mathbb{F}_p^{d+1} \). The (Hamming) weight of \( x \) is the number of nonzero coordinates in \( x \), i.e. \( \omega(x) = |\{ i = 1, \ldots, d+1 : x_i \neq 0 \}| \). A linear subspace \( L \subseteq \mathbb{F}_p^{d+1} \) has constant weight if every nonzero vector has the same weight.

**Definition 3.7.** Let \( L \subseteq \mathbb{F}_p^d \) be a linear code. For every \( x \in L \), let \( x_i \) be the unique integer representative between 0 and \( p-1 \) of the \( i \)-th coordinate of \( x \). Then the age of \( x \) is given by

\[
\alpha(x) = \sum_{i=1}^{d} x_i \in \mathbb{N}.
\]

Let \( L \subseteq \mathbb{F}_p^d \) be a linear code. We say that \( L \) has constant age \( l \), if for all \( 0 \neq x \in L \): \( \alpha(x) = l \).

**Remark 3.8.** The name “age” is inspired by a definition of Ito and Reid (see [IR96 Theorem 1.3]).

We have the following connection between age and weight:

**Proposition 3.9.** If a linear code \( L \subseteq \mathbb{F}_p^d \) has constant age, then it has constant weight and for all \( 0 \neq x \in L \): \( 2\alpha(x) = p\omega(x) \).

**Proof.** Let \( 0 \neq x \in L \). Then

\[
\alpha(x) = \alpha(-x) = \sum_{i=1}^{d} (p - x_i) = p\omega(x) - \alpha(x).
\]

\[ \square \]

**Remark 3.10.** Constant weight does not imply constant age. Indeed, a linear subspace \( L \subseteq \mathbb{F}_p^d \) of dimension 1 has constant weight but not constant age in general.

**Theorem 3.11.** Let \( \Delta \subseteq \mathbb{R}^d \) be a \( d \)-dimensional lattice simplex with \( h^*-\)binomial \( h_\Delta^{k} = 1 + h_k' t^k \) of degree \( 1 < k < (d+1)/2 \). Assume that \( \Delta \) is not a pyramid over a lower-dimensional simplex. Then there exists a prime number \( p \) such that \( \Lambda_\Delta \) can be identified with a linear code \( L_\Delta \subseteq \mathbb{F}_p^{d+1} \) of constant age \( kp \). In particular, \( L_\Delta \) has constant weight \( 2k \) and \( \text{vol}(\Delta) = 1 + h_k' = |L_\Delta| = p^r \) where \( r = \dim L_\Delta \).
Proof. By the statements 3.4, 3.5 it follows that there is a prime number $p$ such that $A_\Delta$ can be identified with a linear code $L_\Delta \subseteq \mathbb{F}_p^{d+1}$. By Proposition 2.6 it follows that $L_\Delta$ has constant age $kp$ and so, by Proposition 3.9, $L_\Delta$ has constant weight $2k$.

By [BR07, Corollary 3.21], $\text{vol}(\Delta) = 1 + h_k^*$. By Proposition 2.6 $|L_\Delta| = 1 + h_k^*$. Since $L_\Delta$ is a linear subspace over $\mathbb{F}_p$, it follows that $|L_\Delta| = p^r$ where $r = \dim L_\Delta$.

We will use Theorem 3.11 to characterize $d$-dimensional lattice simplices $\Delta$ with $h^*$-binomial of degree $1 < k < (d+1)/2$ by investigating their corresponding linear codes $L_\Delta$ more closely.

4. Elements of Coding Theory

In this section, we will recall the notions and statements from coding theory used later on. Fix a prime number $p$.

A linear map $f : \mathbb{F}_p^d \rightarrow \mathbb{F}_p^d$ is a monomial transform if there exists a permutation $\sigma \in S_d$ and elements $\tau_1, \ldots, \tau_d \in \mathbb{F}_p^*$ such that

$$f(x_1, \ldots, x_d) = (\tau_1 x_{\sigma(1)}, \ldots, \tau_d x_{\sigma(d)}).$$

In other words $f$ is the composition of a permutation of the coordinates followed by a coordinate wise dilation. Two linear codes $L_1, L_2 \subseteq \mathbb{F}_p^d$ are called equivalent if there exists a monomial transform $f : \mathbb{F}_p^d \rightarrow \mathbb{F}_p^d$ such that $f(L_1) = L_2$. This gives an equivalence relation on the set of linear codes in $\mathbb{F}_p^d$.

Definition 4.1. Let $L \subseteq \mathbb{F}_p^d$ be a linear code of dimension $k$ and block length $d$. A matrix $A \in \mathbb{F}_p^{k \times d}$ is called a generator matrix of $L$ if the row space is the given code, i.e. $L = \{xA : x \in \mathbb{F}_p^k\}$. In other words: To get a generator matrix of $L$ just take a basis of $L$ and arrange these vectors as the rows of a matrix $A$.

Next we introduce the class of linear codes which we are interested in.

Definition 4.2. Fix a natural number $r$. Let $m = (p^r - 1)/(p - 1)$ be the number of points in $(r-1)$-dimensional projective space over $\mathbb{F}_p$. Consider $A$ an $r \times m$-matrix over $\mathbb{F}_p$ whose columns consist of one nonzero vector from each 1-dimensional subspace of $\mathbb{F}_p^r$. Then $A$ forms the generator matrix of the simplex code $L(A)$ of dimension $r$ and block length $m$. It is evident from the definition that all the linear codes which are produced by this procedure are equivalent. Furthermore, the simplex codes are the dual codes to the well-known Hamming codes.

Definition 4.3. Let $L \subseteq \mathbb{F}_p^d$ be a linear code of dimension $r$ and block length $d$. Let $A \in \mathbb{F}_p^{r \times d}$ be a generator matrix for $L$. If there is a linear code $L' \subseteq \mathbb{F}_p^{d'}$ of dimension $r$ and block length $d' < d$ with generator matrix $A' \in \mathbb{F}_p^{r \times d'}$ such that $A$ is equal to the $d/d'$-replication of the matrix $A'$, i.e.

$$A = (A', A', \ldots, A')$$

then $L$ will be called the $d/d'$-replication of the linear code $L'$.

Theorem 4.4 ([Mac61], [WW95]). Two linear codes $L_1, L_2 \subseteq \mathbb{F}_p^d$ are equivalent if and only if there exists a linear isomorphism $f : L_1 \rightarrow L_2$ that preserves weights.

Proof. If $L_1$ and $L_2$ are equivalent, then there is a monomial transform $f : \mathbb{F}_p^d \rightarrow \mathbb{F}_p^d$ which maps $L_1$ onto $L_2$. Obviously $f$ preserves weights.

Next assume that there is a linear isomorphism $f : L_1 \rightarrow L_2$ which preserves weights. First, we need to choose a non-trivial character $\chi : \mathbb{F}_p \rightarrow \mathbb{C}^\times$ of $(\mathbb{F}_p, +)$. This is done by choosing a complex $p$-th root of unity $\zeta$ and defining $\chi(a) = \zeta^a$ for $a \in \{0, 1, \ldots, p - 1\}$. 
Denote by $\mathcal{X}(F_p)$ the group of all characters of $(F_p,+).$ For any $a \in F_p$ we consider the multiplication map $\mu_a : F_p \to F_p; x \mapsto ax.$ Then

$$F_p \to \mathcal{X}(F_p), a \mapsto \chi \circ \mu_a$$

is an isomorphism of finite abelian groups. This is a special instance of the isomorphism

$$\text{Hom}(F^d_p, F_p) \to \mathcal{X}(F^d_p); \lambda \mapsto \chi \circ \lambda$$

where $\mathcal{X}(F^d_p)$ denotes the group of characters of the group $(F^d_p,+).$

Let $x \in L_1$ and set $y := f(x).$ We can express the weights of $x$ and $y$ using the character $\chi$:

$$\omega(x) = \sum_{i=1}^{d} \left( 1 - \frac{1}{q} \sum_{a \in F_p} \chi(ax_i) \right)$$

and

$$\omega(y) = \sum_{i=1}^{d} \left( 1 - \frac{1}{q} \sum_{b \in F_p} \chi(by_i) \right).$$

Since $f$ preserves weights, we get the equality

$$\sum_{i=1}^{d} \sum_{a \in F_p} \chi(ax_i) = \sum_{i=1}^{d} \sum_{b \in F_p} \chi(by_i)$$

We denote the projection onto the $i$th coordinate by $p_i : L_1 \to F_p.$ Then the previous equation can be written as an equation of characters:

$$\sum_{i=1}^{d} \sum_{a \in F_p} \chi(a \cdot p_i) = \sum_{i=1}^{d} \sum_{b \in F_p} \chi(b \cdot p_i \circ f)$$

By subtracting $d$ times the constant $1$ function on both sides yields:

$$\sum_{i=1}^{d} \sum_{a \in F_p} \chi(a \cdot p_i) = \sum_{i=1}^{d} \sum_{b \in F_p} \chi(b \cdot p_i \circ f)$$

Let $i = 1$ and $a = 1.$ Since the characters of $F^d_p$ are linearly equivalent, there exists $\sigma(1) \in \{1, \ldots, d\}$ and $\tau_1 \in F^*_p$ such that $\chi \circ p_1 = \chi(\tau_1 \cdot p_{\sigma(1)} \circ f).$ By the isomorphism $\text{Hom}(F^d_p, F_p) \to \mathcal{X}(F^d_p)$ from above, we get that $p_1 \equiv \tau_1 \cdot p_{\sigma(1)} \circ f.$ Hence

$$\sum_{a \in F^*_p} \chi(a \cdot p_1) = \sum_{b \in F^*_p} \chi((b\tau_1) \cdot p_{\sigma(1)} \circ f).$$

By subtracting this equation from the equation above, we get

$$\sum_{i=2}^{d} \sum_{a \in F_p} \chi(a \cdot p_i) = \sum_{i=1}^{d} \sum_{b \in F^*_p} \chi(b \cdot p_i \circ f)$$

Inductively, we obtain a permutation $\sigma \in S_d$ and nonzero scalars $\tau_1, \ldots, \tau_d \in F^*_p$ such that $p_i = \tau_i \cdot p_{\sigma(i)} \circ f.$ We define a monomial transform

$$F : F^d_p \to F^d_p; (x_1, \ldots, x_d) \mapsto (\tau_{\sigma(1)}^{-1}(x_{\sigma^{-1}(1)}), \ldots, \tau_{\sigma^{-1}(d)}^{-1}(x_{\sigma^{-1}(d)})).$$

Then, by the choice of $\tau_1$ and $\sigma,$ we obtain $F|_{L_1} = f.$ In particular, $F$ induces an equivalence of codes.

**Corollary 4.5.** Any two $r$-dimensional codes $L_1, L_2 \subset F^d_p$ of the same constant weight are equivalent. □
Proof. Choose bases $v_1, \ldots, v_r$ of $L_1$ and $w_1, \ldots, w_r$ of $L_2$. Let $f : L_1 \rightarrow L_2$ be the linear isomorphism which maps $v_i$ onto $w_i$. Since $L_1$ and $L_2$ are two constant weight codes of the same weight, the linear function preserves weights. Then by the previous Theorem, $L_1$ and $L_2$ are equivalent. □

Proposition 4.6. An $r$-dimensional simplex code has constant weight $p^{r-1}$.

Proof. Consider the first row of a generating matrix $A$ for a simplex code. Then it contains exactly $p^{r-1}$ non-zero entries, because there exist exactly $p^{r-1}$ points in the projective space $\mathbb{P}^{r-1}(\mathbb{F}_p)$ having non-zero first homogeneous coordinate. □

Theorem 4.7 ([Bon84], [WW96]). Every $r$-dimensional code $L \subseteq \mathbb{F}_p^d$ of constant weight such that no coordinate is 0 for all vectors in $L$ is equivalent to an $m$-fold replication of $r$-dimensional simplex codes.

Proof. Let $A$ be the generating $(r \times n)$-matrix for the code $L \subseteq \mathbb{F}_p^d$. Without loss of generality we assume that $A$ has no zero-columns. Let $B \in GL(r, \mathbb{F}_p)$ be an arbitrary invertible $(r \times r)$-matrix. Then $BA$ is another generating matrix for $L$ and $B$ defines a natural bijective linear map $L \rightarrow L$. Since $L$ has constant weight, by Theorem 4.4 the columns of $BA$ are obtained by a monomial transformation of columns of $A$. On the other hand, we can choose $B$ in such a way that the first column of $BA$ will be any prescribed nonzero vector in $\mathbb{F}_p^r$. Therefore, every nonzero vector in $\mathbb{F}_p^r$ is proportional to some column in $A$, i.e., $A$ contains a $(r \times (p^r - 1)/(p-1))$-submatrix which determines a $r$-dimensional simplex code. Since any simplex code has constant weight, the remaining $(d - (p^r - 1)/(p-1))$ columns of $A$ also generate an $r$-dimensional linear code $L' \subseteq \mathbb{F}_p^{d-(p^r-1)/(p-1)}$ of constant weight. We can apply to $L'$ the same arguments. By this procedure, we decompose $L$ into a sequence of $m$ $r$-dimensional simplex codes. □

Corollary 4.8. Let $\Delta \subseteq \mathbb{R}^d$ be a $d$-dimensional lattice simplex with $h^*$-binomial of degree $1 < k < (d+1)/2$. Assume that $\Delta = p^r$ for a prime number $p$ and a positive integer $r$. The number of vertices of $\Delta$ is a multiple of $(p^r - 1)/(p-1)$.

Proof. By Theorem 4.11, the linear code $L_\Delta$ is an $r$-dimensional linear subspace of $\mathbb{F}_p^{d+1}$. By Theorem 4.7, $d+1$ is an integer multiple of $(p^r - 1)/(p-1)$. The number of vertices of $\Delta$ equals to $d+1$. □

Corollary 4.9. With the assumptions of the previous corollary we have
\[ 2k(p^r - 1) = (d+1)(p-1)p^{r-1}. \]

Proof. By Theorem 4.7, the linear code $L_\Delta$ is equivalent to a replicated simplex code $H \subseteq \mathbb{F}_p^{(p^r-1)/(p-1)}$, i.e. $d+1 = m(p^r - 1)/(p-1)$ for a positive integer $m$. In particular, we have
\[ m = \frac{(d+1)(p-1)}{p^r - 1}. \]

By Proposition 4.6, the weight of the simplex code $H$ is equal to $p^{r-1}$. Thus, the weight of $L_\Delta$ is $mp^{r-1}$. By Theorem 5.11 the weight of $L_\Delta$ equals to $2k$. □

5. Proof of the Main theorem

By Theorem 3.11 a $d$-dimensional lattice simplex $\Delta$ with $h^*$-binomial of degree $1 < k < (d+1)/2$ and a chosen order of its vertices is determined up to isomorphism by its associated linear code $L_\Delta \subseteq \mathbb{F}_p^{d+1}$ of constant age $kp$. So it is enough to classify linear codes in $\mathbb{F}_p^{d+1}$ of constant age.
If \( p = 2 \), then, by Proposition 3.9, the weight-function and the age-function coincide. Hence, our classification result follows by the Theorem of Bonisoli (see Theorem 4.7).

**Theorem 5.1.** The \( d \)-dimensional lattice simplices \( \Delta \) with \( h^* \)-binomial of degree \( 1 < k < (d + 1)/2 \) and \( \text{vol}(\Delta) = 2^r \) for a positive integer \( r \) which are not a pyramid over a lower-dimensional simplex are in correspondence with linear constant weight codes in \( F_p^{d+1} \) of weight \( 2k \). These codes are classified by the Theorem of Bonisoli. Furthermore, the numbers \( k, d, r \) are related by the equation:

\[
2^{r-2(d + 1)} = k(2^r - 1)
\]

For \( p > 2 \), the age-function and the weight-function differ in general. We consider a construction to get linear codes of constant age.

**Proposition 5.2.** Fix a prime number \( p > 2 \) and a positive integer \( r \). Set \( m = (p^r - 1)/(p - 1) \). Let \( A_1, \ldots, A_s \) be \( (r \times m) \)-generator matrices of \( r \)-dimensional simplex codes. Then the linear code with generator matrix:

\[
A := (A_1, -A_1, \ldots, A_s, -A_s)
\]

has constant age \( sp^r \).

**Theorem 5.3.** The \( d \)-dimensional lattice simplices \( \Delta \) with \( h^* \)-binomial of degree \( 1 < k < (d + 1)/2 \) and \( \text{vol}(\Delta) = p^r \) for a prime number \( p \) and a positive integer \( r \) which are not a pyramid over a lower-dimensional simplex are in correspondence with linear codes \( L_\Delta \subseteq F_p^{d+1} \) of constant age \( kp \). The numbers \( p, d, k, r \) are related by the equation

\[
(p^r - p^{r-1})(d + 1) = 2k(p^r - 1)
\]

and the generator matrix of those codes can be written up to permutation of the columns in the form

\[
(A_1, -A_1, A_2, -A_2, \ldots, A_s, -A_s),
\]

where \( s = k/p^{r-1} \) and \( A_1, \ldots, A_s \) are generator matrices of \( r \)-dimensional simplex codes.

**Proof.** The result is a direct consequence of Theorem 3.11, Corollary 4.9, Proposition 5.2, and the following theorem.

**Theorem 5.4.** Let \( L \subseteq F_p^n \) be an \( r \)-dimensional linear code of constant age such that no coordinate is 0 for all vectors in \( L \). Then \( n = m(p^r - 1)/(p - 1) \) for some even number \( m = 2s \) and up to a permutation of the columns the generating \( (r \times n) \)-matrix of \( L \) can be written in the form

\[
(A_1, -A_1, A_2, -A_2, \ldots, A_s, -A_s),
\]

where \( A_1, \ldots, A_s \) are generating matrices for \( r \)-dimensional simplex codes.

**Proof.** Let \( A \) be a generating \( (r \times n) \)-matrix of the \( r \)-dimensional linear code \( L \subseteq F_p^n \). By Proposition 3.9, \( L \) has constant weight. Since \( A \) does not contain zero-columns, by Theorem 4.7, it is a sequence of \( m \) matrices of \( r \)-dimensional simplex codes. In particular, there exist exactly \( m \) columns in \( A \) representing any given point in \( \mathbb{P}^r(F_p) \), we have

\[
n = m\frac{p^r - 1}{p - 1}.
\]

and \( L \) has constant weight \( mp^{r-1} \) (see Proposition 4.6).

So, in order to prove the theorem, it is sufficient to show that the columns of the matrix \( A \) can be reordered in pairs of the type \( \{X, -X\} \).
Remark 5.5. Observe that if $p = 2$, then this is obvious since the only nonzero element in $\mathbb{F}_2$ is 1 and $1 + 1 = 0$. In the following, we will assume that $p$ is an odd prime.

The standard Terminal Lemma [Rei87], which was used in the generalized theorem of White [BH10], claims that up to some zero entries the last statement holds true for every row of $A$. However, different rows of $A$ may define different decompositions of coordinates in pairs. The main difficulty is to guarantee that this decomposition into pairs holds true for all rows of $A$ simultaneously.

We want to reformulate the last statement using the finite field $\mathbb{F}_q$ with $q = p^r$ elements. For this, we choose an $\mathbb{F}_p$-basis $v_1, \ldots, v_n$ of $\mathbb{F}_q$ and identify the $r$-dimensional columns of $A$ with some elements $\alpha_i$ of $\mathbb{F}_q^*$.

Proposition 5.6. Our goal is to show that $n = 2s(p^r - 1)/(p - 1)$ is an even integer (for $s$ an integer) and up to reordering the coordinates, we can write the matrix $A$ in the form

$$(\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots, \alpha_n/2, -\alpha_n/2).$$

Denote by $T$ the matrix of the $\mathbb{F}_p$-bilinear form

$$\text{Tr} : \mathbb{F}_q \times \mathbb{F}_q \to \mathbb{F}_p, \quad (x, y) \mapsto \text{Tr}(xy)$$

in the basis $v_1, \ldots, v_n$. Since $\mathbb{F}_q$ is separable over $\mathbb{F}_p$, the matrix $T$ is non-degenerate and the $r \times n$-matrix $TA$ is also a generating matrix of the $r$-dimensional linear code $L$. Since $TA$ can be obtained from $A$ by row operations, the statement of equation (2) for $A$ is equivalent to the same statement for $TA$. Any nonzero element $v \in L$ can be obtained as product of a nonzero $(1 \times r)$-matrix $\Lambda = (\lambda_1, \ldots, \lambda_r)$ with the $r \times n$-matrix $TA$. If we consider the matrix $\Lambda \in \mathbb{F}_q^* \setminus \{0\}$ as an element $\lambda$ of the multiplicative group $\mathbb{F}_q^*$, then any nonzero vector $v \in L$ can be written as

$$v = (\text{Tr}(\lambda \alpha_1), \text{Tr}(\lambda \alpha_2), \ldots, \text{Tr}(\lambda \alpha_n)) \in \mathbb{F}_q^n$$

for some $\lambda \in \mathbb{F}_q^*$.

Since $L$ has constant age, we get $2 \alpha(v) = \omega(v)$ for all $0 \neq v \in L$ (see Proposition 3.9). In particular, we have

$$\sum_{i=1}^n B_1 \left( \frac{\text{Tr}(y \alpha_i)}{p} \right) = 0, \quad \forall y \in \mathbb{F}_q^*$$

where $B_1$ denotes the 1st (periodic) Bernoulli function which maps a real number $x$ to

$$B_1(x) = \begin{cases} \{x\} - \frac{1}{2} & x \notin \mathbb{Z} \\ 0 & x \in \mathbb{Z} \end{cases}$$

where $\{x\}$ denotes the fractional part of $x$, i.e. $\{x\} = x - \lfloor x \rfloor$ where $\lfloor x \rfloor$ is the biggest integer which is smaller than or equal to $x$.

Let $G := \mathbb{F}_q^*$ be the cyclic group of even order $p^r - 1$ ($p$ is an odd prime by Remark 5.5). We consider $V := \mathbb{C}[G]$ the regular representation space of $G$ over $\mathbb{C}$ together with the canonical basis $\sigma_g$ ($g \in G$). The $\mathbb{C}$-space $V$ splits into a direct sum of 1-dimensional $G$-invariant subspaces $V_i$ ($i = 1, \ldots, |G|$) corresponding to $p^r - 1$ different complex characters of $G$. The 1-dimensional $G$-invariant subspace $V_i$ corresponding to a character $\chi_i$ is generated by the vector

$$e_i := \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1})\sigma_g \in \mathbb{C}[G].$$
To any character $\chi : G \to \mathbb{C}^*$ one associates a complex number

$$B_{1,\chi}^{(r)} := \sum_{g \in G} \chi(g)B_1\left(\frac{\text{Tr}(g)}{p}\right).$$

**Remark 5.7.** Observe that $B_{1,\chi}^{(1)}$ coincides with the classical generalized Bernoulli number (see [Was97, p. 31]).

A character $\chi : G \to \mathbb{C}^*$ is called odd if $\chi(-1) = -1$. There exist exactly $(p^r - 1)/2$ odd characters of $G$. We need the following statement, which will be proved in section 6.

**Theorem 5.8.** Let $\chi : \mathbb{F}_q^* \to \mathbb{C}^*$ be a an odd character. Then $B_{1,\chi}^{(r)} \neq 0$.

Let $\{\sigma_g\}_{g \in G}$ be the dual basis of the dual space $V^*$. We consider in $V$ the subspace $U$ generated by the elements $\{S(x)\}_{x \in G}$

$$S(x) := \sum_{g \in G} B_1\left(\frac{\text{Tr}(xg)}{p}\right)\sigma_g$$

Denote by $U^\perp$ the orthogonal complement in $V^*$ with respect to the canonical pairing

$$(\cdot, \cdot) : V \times V^* \to \mathbb{C}.$$ 

Then $(p^r - 1)/2$ linearly independent elements belong to $U^\perp$, because

$$\langle S(x), e_g^* + e_{-g}^* \rangle = B_1\left(\frac{\text{Tr}(xg)}{p}\right) + B_1\left(\frac{\text{Tr}(-xg)}{p}\right) = 0.$$ 

In order to show that $\{e_g^* + e_{-g}^* : g \in G\}$ forms a basis of $U^\perp$, we need to show that $\dim U^\perp = (p^r - 1) - \dim U \leq (p^r - 1)/2$, or, equivalently, that $\dim U \geq (p^r - 1)/2$.

We construct explicitly $(p^r - 1)/2$ linearly independent vectors $u_i$ corresponding to odd characters $\chi_i$ of $G$:

$$u_i := |G|B_{1,\chi_i}^{(r)}e_i = B_{1,\chi_i}^{(r)} \sum_{g \in G} \chi_i(g^{-1})\sigma_g = \sum_{g \in G} \chi_i(g^{-1})B_{1,\chi_i}^{(r)}\sigma_g$$

$$= \sum_{g \in G} \sum_{h \in G} \chi_i(g^{-1}h)B_1\left(\frac{\text{Tr}(h)}{p}\right)\sigma_g$$

and show that they are contained in $U$. Using the substitution $g' = g^{-1}h$, this follows from

$$u_i = \sum_{g \in G} \sum_{g' \in G} \chi_i(g')B_1\left(\frac{\text{Tr}(gg')}{p}\right)\sigma_g = \sum_{g' \in G} \chi_i(g')S(g') \in U.$$ 

Now we can prove Proposition 5.6. The sum $\sum_{i=1}^n e_{\alpha_i}^*$ belongs to $U^\perp$, because

$$\langle S(x), \sum_{i=1}^n e_{\alpha_i}^* \rangle = \sum_{i=1}^n B_1\left(\frac{\text{Tr}(x\alpha_i)}{p}\right) = 0 \quad \text{for all } x \in G.$$ 

This means that $e_g^*$ and $e_{-g}^*$ appear in the sum $\sum_{i=1}^n e_{\alpha_i}^*$ with the same multiplicity and we can divide the sequence $\alpha_1, \ldots, \alpha_n$ into pairs of elements of $G = \mathbb{F}_q^*$ with opposite sign. □
6. Nonvanishing of $B_{1,\chi}^{(r)}$

In this section we will prove Theorem 5.8. Fix a prime power $q = p^r$ and an odd character $\chi : \mathbb{F}_q^* \to \mathbb{C}^*$. Let us consider the square of the absolute value of $B_{1,\chi}^{(r)}$:

$$|B_{1,\chi}^{(r)}|^2 = B_{1,\chi}^{(r)} \cdot B_{1,\chi}^{(r)} = \sum_{a \in \mathbb{F}_q^*} \sum_{b \in \mathbb{F}_q^*} \chi(ab^{-1}) B_1 \left( \frac{\text{Tr}(a)}{p} \right) B_1 \left( \frac{\text{Tr}(b)}{p} \right).$$

Using the substitution $c = ab^{-1}$, we get

$$|B_{1,\chi}^{(r)}|^2 = \sum_{c \in \mathbb{F}_q^*} \chi(c) \sum_{a \in \mathbb{F}_q^*} B_1 \left( \frac{\text{Tr}(a)}{p} \right) B_1 \left( \frac{\text{Tr}(ac^{-1})}{p} \right).$$

We can extend the inner sum over $a$ to contain 0 $\in \mathbb{F}_q$, since $\text{Tr}(0) = 0$. Thus, we have to understand the sum $\sum_{a \in \mathbb{F}_q^*} B_1(\text{Tr}(a)/p) B_1(\text{Tr}(ac^{-1})/p)$ for all $c \in \mathbb{F}_q^*$.

**Proposition 6.1.** We distinguish two cases:

1. If $c \in \mathbb{F}_p \subseteq \mathbb{F}_q$, then

$$\sum_{a \in \mathbb{F}_q^*} B_1 \left( \frac{\text{Tr}(a)}{p} \right) B_1 \left( \frac{\text{Tr}(ac)}{p} \right) = p^{r-1} \sum_{a=1}^{p-1} B_1 \left( \frac{a}{p} \right) B_1 \left( \frac{ac^{-1}}{p} \right).$$

2. If $c \in \mathbb{F}_q \setminus \mathbb{F}_p$, then

$$\sum_{a \in \mathbb{F}_q^*} B_1 \left( \frac{\text{Tr}(a)}{p} \right) B_1 \left( \frac{\text{Tr}(ac)}{p} \right) = 0.$$

Let us grant the previous result for a moment. We get

$$|B_{1,\chi}^{(r)}|^2 = p^{r-1} \sum_{c \in \mathbb{F}_p^*} \chi(c) \sum_{a=1}^{p-1} B_1 \left( \frac{a}{p} \right) B_1 \left( \frac{ac^{-1}}{p} \right).$$

Using the substitution $b = ac^{-1}$, we obtain

$$|B_{1,\chi}^{(r)}|^2 = p^{r-1} \sum_{b \in \mathbb{F}_p^*} \chi(ab^{-1}) \sum_{a=1}^{p-1} B_1 \left( \frac{a}{p} \right) B_1 \left( \frac{b}{p} \right)$$

$$= p^{r-1} \left( \sum_{a=1}^{p-1} \chi(a) B_1 \left( \frac{a}{p} \right) \right) \cdot \left( \sum_{b=1}^{p-1} \chi(b) B_1 \left( \frac{b}{p} \right) \right) = p^{r-1} \cdot |B_{1,\chi|_{\mathbb{F}_p}}^{(1)}|^2.$$

By [Was97, Chapter 4, p. 38]), it follows that $B_{1,\chi|_{\mathbb{F}_p}}^{(1)} \neq 0$. This proves Theorem 5.8.

**Proof of Proposition 6.1.** For the first part, observe that the trace map $\text{Tr} : \mathbb{F}_q \to \mathbb{F}_p$ is a linear functional with kernel $W := \ker(\text{Tr})$ having codimension 1 in $\mathbb{F}_q$. We can find a vector $v \in \mathbb{F}_q$ such that $\mathbb{F}_q = W \oplus \mathbb{F}_p v$. By using the fact that $\text{Tr}(\lambda a) = \lambda \text{Tr}(a)$ for all $\lambda \in \mathbb{F}_p$ and all $a \in \mathbb{F}_q$, we get

$$\sum_{a \in \mathbb{F}_q^*} B_1 \left( \frac{\text{Tr}(a)}{p} \right) B_1 \left( \frac{\text{Tr}(ac)}{p} \right) = \sum_{\lambda \in \mathbb{F}_p} \sum_{w \in W} B_1 \left( \frac{\text{Tr}(w + \lambda v)}{p} \right) B_1 \left( \frac{c\text{Tr}(w + \lambda v)}{p} \right)$$

$$= \sum_{\lambda \in \mathbb{F}_p} \sum_{w \in W} B_1 \left( \frac{\lambda \text{Tr}(v)}{p} \right) B_1 \left( \frac{\lambda c \text{Tr}(v)}{p} \right).$$
The product of the Bernoulli functions does not depend on \( w \in W, |W| = p^{r-1} \) and \( \text{Tr}(v) \neq 0 \). Thus, by substituting \( \lambda' := \lambda \text{Tr}(v) \), we obtain

\[
\sum_{a \in F_q} B_1 \left( \frac{\text{Tr}(a)}{p} \right) B_1 \left( \frac{\text{Tr}(ac)}{p} \right) = p^{r-1} \sum_{\lambda' = 1}^{p-1} B_1 \left( \frac{\lambda'}{p} \right) B_1 \left( \frac{\lambda c}{p} \right).
\]

For the second part, assume that \( c \in F_q \setminus F_p \). We get two linear functionals \( \text{Tr} : F_q \to F_p; a \mapsto \text{Tr}(a) \) and \( f : F_q \to F_p; a \mapsto \text{Tr}(ac) \) with kernels \( W_1 := \ker(\text{Tr}) \) and \( W_2 := \ker(f) \). We claim that \( W_1 \neq W_2 \). Assume by contradiction that \( W_1 = W_2 \). Then, we can find \( v \in F_q \) such that \( W_i \oplus F_p v = F_q \) for \( i = 1, 2 \). Using those two decompositions of \( F_q \), we obtain for all \( a \in F_q \)

\[
\frac{f(v)}{\text{Tr}(v)} \text{Tr}(a) = f(a)
\]

where the quotient \( c' := \frac{f(v)}{\text{Tr}(v)} \) is an element of \( F_p^* \). This implies that \( f(a) = \text{Tr}(c' a) \) for all \( a \in F_q \). Since \( F_q \) is separable over \( F_p \), the \( F_p \)-bilinear form

\[
\text{Tr} : F_q \times F_q \to F_p; (a, b) \mapsto \text{Tr}(ab)
\]

is not degenerate. Hence, the equality

\[
f(a) = \text{Tr}(ac) = \text{Tr}(ac')
\]

for all \( a \in F_q \) yields \( c' = c \). Contradiction.

Thus \( W_1 \cap W_2 \) is a subspace of codimension 2 in \( F_q \). Thus we can find vectors \( w_1 \in W_1 \) and \( w_2 \in W_2 \) such that

\[
F_q = (W_1 \cap W_2) \oplus F_p w_1 \oplus F_p w_2.
\]

Using this decomposition, we can write

\[
\sum_{a \in F_q} B_1 \left( \frac{\text{Tr}(a)}{p} \right) B_1 \left( \frac{\text{Tr}(ac)}{p} \right)
= \sum_{w \in W_1 \cap W_2} \sum_{\lambda_1, \lambda_2 \in F_p} B_1 \left( \frac{\text{Tr}(w + \lambda_1 w_1 + \lambda_2 w_2)}{p} \right) B_1 \left( \frac{\text{Tr}(w + \lambda_1 w_1 + \lambda_2 w_2)c}{p} \right)
= \sum_{w \in W_1 \cap W_2} \sum_{\lambda_1, \lambda_2 \in F_p} B_1 \left( \frac{\lambda_2 \text{Tr}(w_2)}{p} \right) B_1 \left( \frac{\lambda_1 c \text{Tr}(w_1)}{p} \right)
\]

The product of the Bernoulli functions does not depend on \( w \in W_1 \cap W_2 \) and \( |W_1 \cap W_2| = p^{r-2} \). Using the substitutions \( \lambda'_1 := \lambda_1 c \text{Tr}(w_1) \) and \( \lambda'_2 := \lambda_2 \text{Tr}(w_2) \), we get

\[
\sum_{a \in F_q} B_1 \left( \frac{\text{Tr}(a)}{p} \right) B_1 \left( \frac{\text{Tr}(ac)}{p} \right) = p^{r-2} \sum_{\lambda'_1 \in F_p} \sum_{\lambda'_2 \in F_p} B_1 \left( \frac{\lambda'_2}{p} \right) B_1 \left( \frac{\lambda'_1}{p} \right)
= p^{r-2} \left( \sum_{\lambda'_1 \in F_p} B_1 \left( \frac{\lambda'_1}{p} \right) \right) \left( \sum_{\lambda'_2 \in F_p} B_1 \left( \frac{\lambda'_2}{p} \right) \right)
\]

It is straightforward to verify

\[
\sum_{\lambda \in F_p} B_1 \left( \frac{\lambda}{p} \right) = \sum_{\lambda = 1}^{p-1} \left( \frac{\lambda}{p} - \frac{1}{2} \right) = \frac{p(p-1)}{2p} - \frac{p-1}{2} = 0
\]

□
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