Physical Foundation of the Mathematical Concepts in the Nonstandard Analysis Theory of Turbulence

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Abstract

The physical foundation of the main mathematical concepts in the nonstandard analysis theory of turbulence are presented and discussed. The basic fact is that there does not exist the absolute zero fluid-volume. Therefore, the corresponding physical object to the absolute point is just the uniform fluid-particle. The fluid-particle, in general, corresponds to the monad. The uniform fluid-particle corresponds to the uniform monad, the nonuniform fluid-particle to the nonuniform monad. There are two kinds of the differentiations, one based on the absolute point, another based on the monad. The former is adopted in the Navier-Stokes equations, the latter in the fundamental equations, the closed forms are the equations (7)-(11) in this article, in the nonstandard analysis theory of turbulence. The continuity of fluid is shown by virtue of the concepts of the fluid-particle and fluid-particle in lower level. The character of the continuity in two cases, the standard and nonstandard analysis, is presented in the paper. And the difference in the discretization between the Navier-Stokes equations and the equations (7)-(11) is pointed out too.

Key words: the monad, the point, the uniform and nonuniform point, the continuity of fluid, the fluid-particle, the fluid-particle in lower level

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In papers [1] [2] [3], one new description of turbulence is presented and the preliminary discussion on some important and key concepts about the description is given too. The new
description of turbulence is called as Nonstandard Analysis Theory of Turbulence (NATT). In this article, we will have a discussion on the reasonability of the basic mathematical concepts in the NATT. Some of these concepts are in being in the standard analysis, but they have new meanings in the NATT. And the others are not in being in the standard analysis at all.

Obviously, the reasonability of these mathematical concepts originates in the physical foundation of them. The mathematical logic is important, but should not be first. In the mathematical logic, the introduction of these concepts should only meet a demand for being not contradictory.

These mathematical concepts are: the point, the structure of point, the point-average, the infinitesimal, the monad, the new derivative, the continuity of fluid, etc. We will discuss their physical meanings in order to show the reasonability of these mathematical concepts as follows.

1 Point and continuity of fluid

A fluid is composed of numerous molecule, but could be thought of as continuous provided that the scales of any physical process in the fluid are much greater than the free path of molecule. Then the flow of the fluid usually is called as a flow field, i.e., the fluid in motion is modeled as a flow field. The flow field is continuously composed of the real number points, and is called as real number space mathematically. Yet after contemplation it is obvious that there does not exist the “point” of real number space in a real fluid. Therefore, the physical meaning of the point of the real number space (i.e., flow field), or in other words, the physical reality from which the concept of point is drawn, should be indicated. In fact, the logic of the process, that the real number space (i.e., flow field) is drawn from the flow fluid composed of molecule, should be as follows.

It is well known that the motion of the molecule contained in a very small volume will reach, by collision with each other, the equilibrium in a very small time interval. By virtue of this fact, the concept of the fluid-particle has been presented. The fluid, in fact, should be thought to be physically composed of the fluid-particles. The fluid-particle is a very small volume of fluid and is real physical object. Hence, the concept of the mathematical point is drawn from the fluid-particle, i.e., the abstraction of the fluid-particle is the point. After the abstraction from
the fluid-particles, the flow field composed of the points is obtained. So the point forming flow field is an abstracted concept and is drawn from the fluid-particle, that is to say, the concept of point is based on the fluid-particle.

Obviously, it is necessary to make this abstraction for knowing the properties of flow. The mathematical derivation could hardly be put on the base of the fluid-particle. Otherwise, on the base of real number space composed of the real number points the mathematical derivation, by which the law and properties of fluid flow can be obtained, is made reasonably.

The fluid-particle is a very small volume of fluid. However small the scale of the fluid-particle is, it is still finite from the angle of physical practice. We would like to point out some features about the scale of the fluid-particle.

The physical world is possessed of the hierarchical structure. The scales of the fluid-particles in different levels are different. Surely, the scale of the fluid-particle in higher level should be much greater than that in lower level. The scale of the fluid-particle in the lowest level in a physical problem should not be less than a certain scale \( \xi \). The certain scale \( \xi \) is the lowest scale in the scales, any one of which can form the volume which contains enough number of moleculae for stable statistical average value of physical quantities. The fluid volume, the linear scale of which is less than the scale \( \xi \), does not have certain average value of physical quantities. Therefore, that fluid volume could not be called as a fluid-particle.

But the scale of a fluid-particle in any level could not be too large to show the variation of physical quantities with time and space.

It is important to indicate that the scales of the fluid-particles in any level are all objective. The scale of the fluid-particle is not determined by people at will, but by the nature of the specific physical problem under study. In other words, the scale of the fluid-particle should be put on the base of physics and be corresponding to the nature of the specific physical problem under discussion. As an example, the scale of the fluid-particle in atmospheric flow is much greater than that in the flow around the wing of an aircraft.

In practical activities, people usually can not give how large, in the concrete, the fluid-particle is, yet we believe that the scale of the fluid-particle is not set by people’s will, but is objective.

By the concept of the fluid-particle, the clearer physical meaning of the continuity of a fluid
can be given, that is: If there is a space scale of $\xi$, enough number of moleculae are contained in the fluid volume with linear scale $\xi$, so that the statistical averaging over the motion of these moleculae can give the stable value, i.e., the fluid-volume with linear scale $\xi$ just is a fluid-particle in certain level. And, the space scales of any physical process in the fluid are all greater than the scale $\xi$. Further, the motion of the moleculae contained in the volume with linear scale $\xi$, by collision with each other, has already reached thermodynamic equilibriums in the time interval less than or equal to the least time scale in the various time scales of the various physical processes in the fluid. Then the fluid can be thought of as continuous. In this case, the effect of fluid composed of moleculae can be ignored, and the real flow of fluid will be modeled after the flow field.

2 Uniform and non-uniform points

The fundamental physical idea in the NATT is that the fluid-particle is wholly uniform in a laminar flow, but not uniform, i.e., there is internal structure in the fluid-particle in the case of turbulence. By that mentioned above, though the fluid-particle in the turbulence can be divided at will into smaller and uniform fluid-particles in mathematical abstract, we could not reject this idea, because the fluid-particle can not be divided at will in physical practice. The scale of the fluid-particle has the objective physical base in practice.

What is the mathematical meaning of this fundamental physical idea in the NATT?

As is stated above, a point (the absolute geometric point) should be mathematically drawn from only the uniform fluid-particle. If the non-uniform fluid-particle was, in mathematical abstraction, thought of as an absolute geometric point too, the internal structure of the non-uniform fluid-particle can not be described. So this will go against the physical reality. Thus, the non-uniform fluid-particle should be, in mathematical abstraction, taken as the monad. The linear scale of the monad is infinitesimal $\varepsilon, \varepsilon \neq 0$, but $\varepsilon > 0$. The internal structure can be permitted in the monad. From the nonstandard analysis, the infinitesimal $\varepsilon$ is also number, i.e., the nonstandard number. The real number is called as the standard number.

Even in the case of a laminar flow, in fact, the monad is also the mathematical abstraction
of the uniform fluid-particle, but is uniform, no internal structure in the monad. The monad no
having internal structure essentially is identified with the absolute point. This concept can be
shown clearly by the figure 1-2.

Laminar flow :

![Diagram of laminar flow]

The wholly flow field is just a mathematically real number
space composed of the absolute real number points.

Figure 1: Mathematical-abstraction of the fluid-particle in laminar flow

Turbulence :

![Diagram of turbulence]

The wholly flow field is just a mathematically hyperreal number
space composed of the monads(i.e., standard points).

Figure 2: Mathematical-abstraction of the fluid-particle in turbulence

Therefore, the physical meaning of mathematical monad is the fluid-particle, i.e., the monad
is the mathematical abstraction of the fluid-particle. In the case of a laminar flow the fluid-
particle is wholly uniform, so the monad(mathematical abstraction of the fluid-particle) is also
uniform. And yet, in turbulence, the fluid-particle is not wholly uniform, so the abstract monad
is non-uniform. In paper [1], the monad is also called as the standard point being different from
the absolute point. Thus, correspondingly there are also uniform and non-uniform standard
points (usually called as uniform and non-uniform points for short). In a laminar flow, the uniform standard point drawn from the uniform fluid-particle is identified essentially with the absolute point. So the physical meaning of the absolute point is also the uniform fluid-particle.

The structure of the point in NATT represents just the internal structure of the standard point, rather than the absolute point, which has no the internal structure. Obviously, the internal structure of the standard point is the mathematical abstraction just from the internal structure of the fluid-particle. So the physical meaning of the point-structure is just the internal structure of the fluid-particle.

Finally, it should be pointed out that the fluid-particle is the physical reality and has finite scale. Because the scale of the fluid-particle is very small, it is, in the mathematical abstraction, thought of as monad, the scale of which is infinitesimal, even thought of as the absolute point, the scale of which is zero.

In paper [1], it was shown that the fluid-particle, whether or not is uniform, is composed of numerous fluid-particles in lower level (Fig. 3), and a fluid-particle in lower level is uniform and contains still enough number of fluid moleculae for the stable statistic-average.

![Fluid-particle composed of fluid-particles in lower level](image)

**Figure 3:** Fluid-particle composed of fluid-particles in lower level

The mathematical abstract of the fluid-particle in lower level is the nonstandard point (i.e., the internal point of a monad). Therefore, the nonstandard point (i.e., the internal point of a monad) is uniform point.

So a turbulent field is possessed of hierarchical structure: the global turbulent field, composed
of the monads, i.e., the abstraction from the fluid-particles, is in a level. And the monad field (i.e., the abstraction from the flow in the fluid-particle), composed of the nonstandard points (the mathematical abstraction from the fluid-particles in lower level), is in another lower level.

3 Two kinds of differentiations

Having a function \( f(x) \), the derivative of which is

\[
\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
\]

Here the mathematical meaning of \( \Delta x \to 0 \) is that \( \Delta x \) tends to the absolute zero. But, in many cases, people can not give the clearly physical meaning of the absolute zero, for example, the physical meaning of “a fluid volume tends to the absolute zero”. In many cases, in fact, there is not “absolute zero” in physics. The physical meaning of the statement of “a fluid volume tends to the absolute zero” is ambiguous. Therefore, the statement of that \( \Delta x \) tends to absolute zero (i.e., \( \Delta x \to 0 \)) in mathematics does not correspond to the fact that any physical quantity tends to absolute zero, but corresponds to, for example in the case of fluid flow, the fact that physical quantity \( \Delta x \) tends to the scale of the uniform fluid-particle. In other words, the mathematical abstraction of the fact that \( \Delta x \) tends to the scale of the uniform fluid-particle in physics is that \( \Delta x \to 0 \). The physical meaning of \( \Delta x \to 0 \) is that \( \Delta x \) tends to the scale of the uniform fluid-particle.

In NATT, there is other derivative:

\[
\frac{\partial f}{\partial x} = \frac{f(x + \varepsilon) - f(x)}{\varepsilon}
\]

Here \( \varepsilon \) is nonstandard number infinitesimal. The \( \varepsilon \) is not arbitrary infinitesimal, but the linear scale of a monad. The monad is the abstraction from the fluid-particle. So the infinitesimal \( \varepsilon \), the linear scale of the monad, is logically the abstraction from the linear scale of the fluid-particle, namely, the physical meaning of the infinitesimal \( \varepsilon \) is the linear scale of the fluid-particle.

The definition (2) has already been presented in the nonstandard analysis. But there the infinitesimal is arbitrary, here in NATT the \( \varepsilon \) is certain infinitesimal, the linear scale of a monad.
The definition (1), in general, denotes mathematically that the \( \frac{\partial f}{\partial x} \) is the rate of increment of function \( f \) at some point (absolute geometric point), i.e., the slope of tangent to the curve of function \( f \). But, in many cases, the absolute point does not exist in physics, let alone the rate of increment of function at this point. In the case of fluid, it can be understood that the physical meaning, which the definition (1) corresponds to, is the increment (i.e., the increment of the physical quantity \( f \) between two neighbour uniform fluid-particles in some level) divided by the scale of the fluid-particle in the same level. Let \( A,B \) be two neighbour uniform fluid-particles, the scale of the fluid-particle be \( L \), then the physical meaning of \( \frac{\partial f}{\partial x} \) in definition (1) is that

\[
\frac{f(B) - f(A)}{L}
\]

Therefore, from the angle of physics, the definition (1) is applicable to only the uniform fluid-particle in some level. Correspondingly, in mathematics, the definition (1) is applicable to only the uniform point in the same level.

The right side of the definition (2), from the angle of mathematics, denotes the increment (i.e., the increment of function \( f \) between two corresponding internal points of two infinite close monads) divided by the linear scale of the monad. And the physical meaning of the right side of definition (2) is the increment (i.e., the increment of physical quantity function \( f \) between two corresponding fluid-particles in lower level, which are uniform in NATT, of two neighbour fluid-particles) divided by the scale of the fluid-particle. Let \( L \) be the linear scale of the fluid-particle, \( b,a \) be the two corresponding fluid-particles in lower level of two neighbour fluid-particles \( B,A \), i.e., \( b,a \) are located at the same place in respective fluid-particles (Fig.4).

So the meaning of the definition (2) is that \( \frac{\partial f}{\partial x} \) in (2) is the abstraction from the physical quantity \( \frac{f(b) - f(a)}{L} \). The reasonability of the mathematical abstract \( \frac{\partial f}{\partial x} \) in (2) is based on the real physical quantity \( \frac{f(b) - f(a)}{L} \). The definition (2) can be applicable to both the uniform and nonuniform fluid-particle in physics, or to both the uniform and nonuniform point in mathematics.

If, in fluid mechanics, the conservation laws of physics (the conservation of mass, momentum and energy) are applied to uniform and nonuniform fluid-particle in some level respectively,
the different results will be obtained. In the case of uniform fluid-particle, the mathematical abstraction of the results obtained is the Navier-Stokes equations (for the incompressible fluid)

\[
\frac{\partial U_i}{\partial x_i} = 0 \quad (3)
\]

\[
\frac{\partial U_i}{\partial t} + \frac{\partial (U_i U_j)}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i \quad (4)
\]

The equations (3)(4) hold for only the uniform point. In the case of the nonuniform fluid-particle, the mathematical abstraction of the results obtained is the equations (for the incompressible fluid)

\[
\frac{\partial \tilde{U}_i}{\partial x_i} = 0 \quad (5)
\]

\[
\frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial \tilde{U}_i \tilde{U}_j}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i \quad (6)
\]

Here the sign “∼” denotes the average over all the nonstandard points contained in a monad.

The equations (5)(6) hold for both the uniform and nonuniform point. The equations (5)(6) will degenerate into the Navier-Stokes equations (3)(4) in the case of the uniform point.

The differentiation (1) is based on the limit, expressed as the statement of \( \delta - \varepsilon \), and in the frame of \( \delta - \varepsilon \). The differentiation (2) is not based on the limit, and out of the frame of \( \delta - \varepsilon \).

The statement of \( \delta - \varepsilon \) shows mathematically the endless process of infinite tending, and makes the expression of the process stricter mathematically. But it, in the stating the process
of infinite tending, for example infinite tending to zero, does not give any room to physics, i.e.,
it shows the process of tending to the absolute zero or to the absolute nil. However, the real
physics should be: the process of infinite tending to zero is just a mathematical abstraction from
the real process of gradually tending to the scale of the uniform fluid-particle in some level.

The statement of $\delta - \varepsilon$ obliterates this real uniform fluid-particle. In many cases, this oblit-
eration does not have an effect on the results. Yet the obliteration will make some problems
sometimes. How if the fluid-particle is not uniform? The structure in the fluid-particle can not
be shown by the absolute point. Turbulence just is the case. Here the statement of $\delta - \varepsilon$ should
be abandoned and the differentiation (2) is adopted. The differentiation (2) is mathematically
based on the infinitesimal. In the sixties of the last century, the nonstandard analysis was pre-
sented. It is proved strictly in the nonstandard analysis that there exists the infinitesimal. This
gives the mathematical foundation to the new description of turbulence.

4 continuity

In section 1, we showed the physical meaning of the fluid continuity. From the physical reality
the mathematical model of fluid, i.e., continuous flow field, is obtained. In the continuous flow
field, the physical properties(i.e., the various physical quantities on all points of the field) vary
continuously with time and space. Yet the mathematical description of the continuously varying
of the physical quantities is different in two cases, the standard and nonstandard analysis.

In the case of standard analysis, when the function representing the physical quantity is
continuous mathematically, the physical quantity is thought of as continuous. In other words,
the physical quantity varies continuously with time and space, the function $f$ representing the
physical quantity is, in mathematics, continuous function. The definition of continuous function
is that(Fig.5)

$$\lim_{\Delta x \to 0} [f(x + \Delta x) - f(x)] = 0$$

Note that the function $f(x)$ is characterized by only the function value.

But, in nonstandard analysis, this is not the case. Here the function representing the physical
quantity should be written as \( f(x, x') \), the sign \( x \) denotes the \( x \)-monad in some hyper real number space, \( x' \) is the internal coordinates in the monad, i.e., \( x' \) is the coordinates of the internal point of the \( x \)-monad(Fig.6). The \( f(x, x') \) expresses the value of the physical quantity at the internal point \( x' \) of the \( x \)-monad. Then the continuously varying, with time and space, of the physical quantity expressed by the function \( f(x, x') \) means that not only the value but also the structure of the function \( f(x, x') \) vary continuously. In other words, what is continuous function of \( f(x, x') \) means that not only the increment of \( f(x + \triangle x, x') - f(x, x') \) tends to infinitesimal, but also the shape of the function \( f(x + \triangle x, x') \) tends infinitely to the shape of the function \( f(x, x') \), when \( \triangle x \) tends to infinitesimal \( \varepsilon \)(the linear scale of a monad), i.e., the two monads(the monad \( (x + \triangle x) \) and the monad \( (x) \) ) are infinitely close to each other(Fig.7).

Here the function \( f(x, x') \) is possessed of both the value and the shape(i.e., the variation of the function \( f(x, x') \) with \( x' \) in \( x \)-monad). So the variation of not only the function value but also
the function shape should be continuous, if the physical quantity function is continuous. This means that the continuity of the fluid should be the continuity of the function shape as well as the function value. There are two sides in the continuity of fluid in the case of the nonstandard analysis.

In the standard analysis, the physical meaning of the mathematically continuity of the function expresses the relationship between only two neighbour uniform fluid-particles. When people say that the function representing the physical quantity is continuous, the physical meaning of the continuity is that the difference of the value of the physical quantity between two neighbour uniform fluid-particles is very small.

Similarly, in the nonstandard analysis, the mathematical concept of continuity depicts, in physics, just the relationship between two neighbour fluid-particles(both uniform and nonuniform). The difference of the physical quantity, its value and structure(i.e., the structure of the physical quantity function), between the two neighbour fluid-particles is very small. In the NATT, this is called as assumption of close property between two neighbour fluid-particles, in abstraction, between two infinitely close monads. The assumption is just the continuity(physically corresponding to the case of both uniform and nonuniform fluid-particle) in the nonstandard analysis, the expansion of the continuity(physically corresponding to the case of only uniform
fluid-particle) in normally the standard analysis. The reasonability of this assumption should be checked up by the reasonability of the results from the assumption.

There is discontinuity in opposition to the continuity. In order to show these concepts clearly, the concept of the fluctuation in a fluid will be discussed on further as follows.

If the precision of the measurement in turbulence is very high, are the data obtained from the measurement continuous (no fluctuation)?

As is stated above, the concept of the monad is drawn from the fluid-particle. Here, the monad and fluid-particle is in the space of four dimensions, time and space. What is the fluid-particle in space (three dimensions) is clear. Yet the meaning of the fluid-particle in time-direction needs to be indicated. Let the time scale of the fluid-particle be $\delta_t$, the time scale of the fluid-particle in lower level be $\tau$. Surely, the scale of $\delta_t$ and $\tau$ are all objective, and determined by the physical nature of the problem under discussion. The scale $\tau$ needs to meet the requirements of the conditions as follows: The motion of the fluid molecules contained in the space fluid-particle in lower level, by collision with each other, has already reached thermodynamic equilibrium in the time period of $\tau$. The time-direction scale, which is infinitesimal, of the monad in four dimensions and the time-direction scale, which is high order infinitesimal, of the monad-interior point (nonstandard point) in four dimensions are the mathematical abstraction of $\delta_t$ and $\tau$ respectively.

It is known that if the measurement in a fluid is carried out in the level of atom and molecule, i.e., the measurement of the motion of atom and molecule, the results of the measurement will show the random motion, controlled by the quantum law, of the atom and molecule. This measurement is out of the studying at present. Now the objective of the measurement is the mean quantity of the motion of numerous molecules.

Assume that the motion of numerous molecules contained in the space fluid-particle in lower level, by collision with each other, has already reached the thermodynamic equilibrium in only the time interval greater than or equal to $\tau$. The sampling time in the measurement is $t_0$. If $t_0 < \tau$, the data of the measurement can not indicate the physical properties of this fluid particle in lower level, but the average value, which is instable, of the motion properties of the molecules contained in the fluid volume being less than the fluid-particle in lower level. So the data are
fluctuant. If $t_0 > \tau$, but $\ll \delta_t$ (the time scale of the fluid-particle), the data of measurement should be continuous (not fluctuation) and obey the Navier-Stokes equations. The measurement, in fact, is carried out in the range of time scale of a fluid-particle. But the usual measurement in turbulence are not the cases.

In the turbulence measurement at present, the time-samples are taken from the numerous time scales of the numerous fluid-particles in lower level, which are contained in the various fluid-particles of four dimensions. The results for one sample are just the statistic average values of the motion properties, which are reached equilibrium by molecule-collision with each other in the interval $\tau$, of the numerous molecule in contained in the space fluid-particle in lower level. In other words, in the mathematical abstract, the samples are taken from the numerous time-scales of the numerous internal points of various monads (Fig.8). The essential character of this measurement is that the measurer can not know which internal point of a monad is measured, but know only which monad contains the measured internal point. So the measurer know only the range over a monad, but can not know the strict position of the measured objective. This is the uncertainty of turbulence measurement. Therefore, obtained data will show the fluctuation provided that the motion properties of different internal points of a monad are obviously different.

Figure 8: The scheme of the sampling in turbulence measurement

Though it is possible in theory that the measurements are carried out continuously on the different internal points of one monad, this precision can be reached by modern technology, the
continuous data can be obtained in the monad. And the consecutive measurement is in the next monad. And so on and so forth, there are continuous measurements in various monads one by one. However, the concrete operation of the measurement is borne hardly at all, because it is nearly the process in which infinite samples are taken. This process of the measurement as well as the process of solving infinite equations is not possible. Therefore, this measurement exists only in theory and imagining, but not in reality.

Thus the practical measurement of turbulence will give, as mentioned above, the fluctuant results. This fluctuation results from the measurement of the regular flow (the turbulence is also regular). Note that the fluid flow itself is one thing, the result from the measurement of the fluid flow is another. They have relation closely with each other, but are not the same thing. The irregular measurement results does not denote the irregularity and un-continuity of the fluid flow itself, conversely, the continuity and regularity of the fluid flow does not exclude the possibility of the irregular results (or called as the fluctuation) from the measurement of the fluid flow. When the turbulence measurement is carried out, what is met with is the fact that the irregular and fluctuant data are obtained from the measurement of regular and continuous fluid flow. It is maybe said that the turbulence-fluctuation, in fact, just is the fluctuation of the data from the turbulence measurement.

About the fluctuation, in a word, except the random results of measurement in molecule level, there are the fluctuation results from not reaching the stable state of statistic average, i.e., from the measurement in the range of the scale less than that of the fluid-particle in the least level, and the fluctuation results especially from the uncertainty of turbulence measurement (in the level of global field). Only the turbulence-measurement in a monad, the mathematical concept drawn from the fluid-particle, possibly gives the continuous results.

In NATT, there is a mathematical concept of the monad-average (point-average). The physical meaning of the point-average is the average over the motion of numerous lower level fluid-particles in a fluid-particle. The point-average is drawn from the average over the fluid-particle. Here the point-average is different from the normal average in turbulence researching, and it is not over measurement data. Surely, we easy connect the point-average in NATT with the normal average in turbulence researching. The value of the point-average can be obtained by
the theoretical computation (after discretization of the controlling equations the point-average is, in fact, the grid-average), then the average value, being comparable with the average-value from the measurement, is easy obtained from the grid-average.

The standard and nonstandard analysis are mentioned in NATT. In a laminar flow, the standard analysis is employed; in turbulence, the nonstandard analysis is employed. Either laminar or turbulent flow, in fact, either the standard or nonstandard analysis all can be employed. The standard analysis means that there is description of one level, the nonstandard analysis means that there is description of multi-levels. In the standard analysis, the description of a fluid flow is based on the uniform fluid-particle (i.e., the fluid-particle in a laminar flow, the fluid-particle in lower level in a turbulent flow). The fluid-particle as well as the fluid-particle in lower level is wholly uniform in the case of a laminar flow, so the fluid-particle can be thought of as basic element in a laminar flow. And just the fluid-particle corresponds to the physical practice. Therefore, the standard analysis, the description of one level, is employed in a laminar flow. However, in turbulence, the fluid-particle is not wholly uniform and the fluid-particles in lower level contained in the fluid-particle are possessed of respective different motion properties. Obviously, there are multi-levels in turbulence. This is why the nonstandard analysis is employed in turbulence. The standard analysis, in theory, can also be employed, i.e., the description of one level and still the Navier-Stokes equations controlling the flow, in turbulence, but the fluid-particle in lower level must be the basic element. Yet the fluid-particle in lower level does not correspond to the physical practice. There does exist some unsuitability in practice. The evident example of this unsuitability is that the difficulty of the very enormous amount of calculation work will be met, as the Navier-Stokes equations are solved by numerical calculation in turbulence.

5 Discretization of the equations

There are, for the incompressible fluid, three sets of the closed equations for a turbulent flow in the NATT.

A. Choice one:

\[
\frac{\partial \bar{U}_i}{\partial x_i} = 0, \quad \frac{\partial \bar{U}_i}{\partial t} + \frac{\partial (\bar{U}_i \bar{U}_j)}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{P}}{\partial x_i} + \nu \nabla^2 \bar{U}_i + o(\varepsilon^2)
\]
B. Choice two:

\[
\frac{\partial U_i}{\partial x_i} = 0, \quad \frac{\partial U_i}{\partial t} + \frac{\partial (U_i U_j)}{\partial x_j} = \frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \nabla^2 U_i
\]  

\[
\frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial u_i}{\partial t} + U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} - 2 \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + o(\varepsilon^3)
\]  

C. Choice three:

\[
\frac{\partial \tilde{U}_i}{\partial x_i} = 0, \quad \frac{\partial \tilde{U}_i}{\partial t} + \frac{\partial (\tilde{U}_i \tilde{U}_j)}{\partial x_j} + \frac{\partial (u_i u_j)}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \tilde{P}}{\partial x_i} + \nu \nabla^2 \tilde{U}_i + o(\varepsilon^3)
\]  

\[
\frac{\partial u_i}{\partial x_i} = 0, \quad \frac{\partial u_i}{\partial t} + \tilde{u}_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial \tilde{U}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} + \nu \nabla^2 u_i + o(\varepsilon^3)
\]

Here the \( \varepsilon \) is the linear scale of a monad and is infinitesimal (nonstandard number). The \( U_i, P \) are the instantaneous velocity and pressure. The \( u_i, p \) are the fluctuant velocity and pressure. They are the functions of \((x_1, x_2, x_3, t, x'_1, x'_2, x'_3, t')\). The \( \tilde{U}_i, \tilde{P} \) are the point-averages (monad-averages) of the instantaneous velocity and pressure. The independent variables of the \( \tilde{U}_i, \tilde{P} \) are only \((x_1, x_2, x_3, t)\). Note that here the fluctuant quantities are different from the normal fluctuations in a turbulent measurement. The definition of the fluctuations \( u_i, p \) is that \( u_i = U_i - \tilde{U}_i \) and \( p = P - \tilde{P} \). Obviously, the \( u_i, p \) are defined in a monad. Therefore, the \( u_i, p \) would be called as the fluctuations in a monad, or the internal fluctuations.

The differentiation (2) has been applied to \( \frac{\partial}{\partial t}, \frac{\partial}{\partial x} \) in these equations. So the equations can be applied to both the uniform and nonuniform monads. What is the physical meaning of the discretization of the equations (7)-(11)? By virtue of the fact that the infinitesimal \( \varepsilon \) in the definition (2) is the linear scale of a monad, the discretization of the equations (7)-(11) means the coarsening of the monad. But the discretization of the Navier-Stokes equations means, in the case of turbulence, the coarsening of the uniform internal point, essentially identified with the absolute point, of a monad. There are different meanings between two. The monad is abstract from the fluid-particle, but the scale of grid of discretization of (7)-(11) is, in general, not equal to the scale of fluid-particle in practice. If the physical relation between the neighbour grids is very close to the real physical relation between the neighbour fluid-particles, the results of computation will be approximately correct, and the discretization is reasonable. Conversely, if
the physical relation between the neighbour grids is not close to that between the neighbour fluid-particles, the correct results can not be obtained and the discretization is not reasonable.

We know by that mentioned above that the Navier-Stokes equations are based on the limit($\Delta x \to 0, \Delta t \to 0$), and the Navier-Stokes equations can be applied to only the uniform point(i.e., the abstract from the uniform fluid-particle). Yet the equations of (7)-(11) can be permitted to apply to the nonuniform point(i.e., the abstract from the nonuniform fluid-particle).

In turbulence, the fluid-particle, the abstraction of which is the monad, is not uniform, but the fluid-particle in lower level, the abstraction of which is the nonstandard point(i.e., the internal point of the monad), is uniform. Therefore, in turbulence, the Navier-Stokes equations can be applied to only the internal points of a monad rather than to the monad itself. But the equations (7)-(11) can be applied to the monad itself. When the computation of these equations is carried out, the discretization of the Navier-Stokes equations means the coarsening of the internal point of a monad, physically corresponding to the fluid-particle in lower level, and the discretization of equations (7)-(11) means the coarsening of the monad itself, physically corresponding to the fluid-particle. Therefore, in the turbulent computation, the grid of discretization corresponds to the fluid-particle in lower level in the case of computation of the Navier-Stokes equations, but corresponds to the fluid-particle itself in the case of computation of the equations (7)-(11). In other words, only the very fine grids, which introduces the very enormous amount of calculation work, can meet the needs for obtaining the reasonable solution of the Navier-Stokes equations, but the coarse grids can meet the needs for the reasonable solution of the equations (7)-(11) in the case of the turbulence computation. It does not need to be fine grids, so there is not the difficulty of enormous amount of calculation work in the turbulent computation of the equations (7)-(11).

First, the mathematical concepts are drawn from the physical reality. And by virtue of the mathematical concepts a series of the mathematical derivations and calculations are performed in order to obtain the various results about the physical properties, for example, the controlling equations. Then, after the discretization of the controlling equations the computation of the discretization-equations is carried out and the various physical properties of the concrete problem are obtained. There are two procedures, from the physical reality to the mathematical abstract
and from the mathematical abstract to the physical reality, the latter is a converse procedure of the former.

6 Conclusions

In the sections above, the discussion on some important mathematical concepts and the corresponding physical meanings to these concepts is given. What are called as the corresponding physical meanings are the physical objects from which the mathematical concepts are drawn. The results are, in sum, as follows.

**MATHEMATICAL CONCEPTS** **CORRESPONDING PHYSICAL OBJECTS**

* The absolute point the uniform fluid-particle

* The uniform point (uniform monad) the uniform fluid-particle

* The nonuniform point (nonuniform monad) the nonuniform fluid-particle

* The nonstandard point (internal point of a monad) the fluid-particle in lower level

* $\triangle x \to 0$ $\triangle x$ tends to the scale of uniform fluid-particle

* $\frac{\partial f}{\partial x} = \lim_{\triangle x \to 0} \frac{f(x + \triangle x) - f(x)}{\triangle x}$ the difference of the $f$-values between only two neighbour uniform fluid-particles divided by the linear scale of the fluid-particle

* The infinitesimal $\varepsilon$ the linear scale of the fluid-particle

* $\frac{\partial f}{\partial \varepsilon} = \frac{f(x + \varepsilon) - f(x)}{\varepsilon}$ the difference of $f$-values between the corresponding two lower level fluid-particles of the neighbour fluid-particles divided by the linear scale of the fluid-particle

* The N-S equations the conservation equations (i.e., the mass, momentum and energy equations) about only the uniform fluid-particle

* The fundamental equations in the NATT the conservation equations about both the (the closed forms are the equations uniform and nonuniform fluid-particle (7)-(11) in this article)

* The standard analysis the flow fluid composed of only uniform fluid-particles, only one level
Finally, what is stated in this paper is that the reasonability of the mathematical concepts, especially in physics, is based just on their corresponding physical reality. Every reasonable mathematical concept, even mathematical derivation method, must be possessed of its physical foundation, i.e., is the abstract from the physical object (physical reality). In physics, there does not exist the mathematical concept broken away from the physical reality. People, sometimes, ignore carelessly this fact, and are limited to and only stay on the stage of the pure mathematics. Because getting accustomed, for example, to the absolute point in mathematical abstraction, some researchers usually do not pay attention to the effect of the fact that there does not exist the absolute point in physical reality on the description of physical problem such as the turbulence. And the physical problem becomes very difficult to comprehend due to this carelessness. But after the contemplation over the physical meanings of these mathematical concepts, in other words, over what are the real physical objects from which these mathematical concepts are drawn, the difficult problem may be suddenly enlightened.

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