Von Neumann’s Uniqueness Theorem in Theories
with Nonphysical Particles

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Abstract—Von Neumann’s uniqueness theorem is extended to the class of anti-Fock representations of canonical commutation relations.

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INTRODUCTION

Canonical commutation relations (CCR), or Heisenberg algebra representations, lie at the heart of quantum theory. In the case of a single degree of freedom, the momentum and position are represented by self-adjoint operators \( P \) and \( Q \) in a Hilbert space \( \mathcal{H} \), which satisfy the relation

\[
[P, Q] = -iI. \tag{1}
\]

Transition to an arbitrary number of degrees of freedom, namely, to the relations

\[
[P_n, Q] = -i\delta_{nj}I, \quad [Q_n, Q_j] = 0, \quad [P_n, P_j] = 0, \quad 1 \leq i, j \leq n, \tag{2}
\]

does not entail any significant difficulties, and all the main results for CCR admit a straightforward generalization [1].

It is known that no Heisenberg algebra representations of CCR by bounded operators [2] exist. In connection with this, a representation of canonical commutation relations in the Weyl form turns out to be more convenient:

\[
e^{itP}e^{isQ} = e^{ist}e^{isQ}e^{itP}, \quad s, t \in \mathbb{R}. \tag{3}
\]

Indeed, according to the Stone theorem [3], the operators \( e^{itP} \) and \( e^{itQ} \) are bounded.

The relation in (3) is however satisfied only for a certain class of CCR representations, which includes a Schrodinger one [4, 5]. The operators of the latter \( p \) and \( q \) are defined on a space \( L^2(-\infty, \infty) \) by the following relations:

\[
(pf)(x) = -i\frac{\partial}{\partial x}f(x), \quad (qf)(x) = xf(x). \tag{4}
\]

CCR representations unitarily equivalent to the Schrodinger one are commonly referred to as regular ones [4, 5].

Note that, even in the case of a single degree of freedom, there are CCR representations which are not unitarily equivalent to the Schrodinger one; i.e., they are irregular. For example, if the operators in (4) are defined on \( L^2(a, b), a, b \in \mathbb{R} \), then they will form an irregular CCR representation.

Thus, to describe regular representations it is necessary to impose some extra conditions on \( P \) and \( Q \) operators. An important result in this direction is the Rellich [6] and Dixmier [7] theorem.

Theorem 1. In order for a pair of closed symmetric operators \( (P, Q) \) acting upon a Hilbert space \( \mathcal{H} \) to form a regular CCR representation, it is necessary and sufficient that there be a linear manifold \( \mathcal{D} \) contained in \( \mathcal{D}(P) \cap \mathcal{D}(Q) \), dense on \( \mathcal{H} \) and such that

1. \( \mathcal{D} \) is stable under the action of \( P \) and \( Q \) operators,
2. operator contraction \( (P^2 + Q^2) \) on \( \mathcal{D} \) is essentially self-adjoint,
3. equality \( PQ - QP = -iI \) holds on \( \mathcal{D} \).

If the commutation relations can be defined in the Weyl form (3), then the following uniqueness theorem proven by von Neumann [8] is valid.

Theorem 2 (von Neumann’s theorem). Let \( \mathcal{H} \) be a separable Hilbert space, \( \{U(t)|t \in \mathbb{R}\} \) and \( \{V(s)|s \in \mathbb{R}\} \) are two weakly continuous one-parameter groups of unitary operators acting on \( \mathcal{H} \) and such that

\[
U(t)V(s) = e^{ist}V(s)U(t) \quad \forall t, s \in \mathbb{R};
\]

then, representation \( \{U(t), V(s)|t, s \in \mathbb{R}\} \) is said to be unitarily equivalent to the direct sum of Schrodinger representations.

Note that relation (1) can be written in the form

\[
[a, a^*] = I, \tag{5}
\]
where
\[ a = \frac{1}{\sqrt{2}}(Q + iP), \quad a^* = \frac{1}{\sqrt{2}}(Q - iP). \quad (6) \]

It is clear that in the Schrodinger representation, there is a vacuum operator \( \psi_0 = C \exp(-x^2/2) \) that satisfies the condition
\[ a \psi_0 = 0. \quad (7) \]

CCR representations, which Eq. (7) holds for, are referred to as the Fock representations. It can be shown that all Fock representations are unitarily equivalent. In particular, each Fock representation is unitarily equivalent to the Schrodinger one and is therefore regular.

The properties of CCR representation in the Weyl form in a Hilbert space are studied in detail in [9]. However, modern gauge quantum theories, where nonphysical particles appear in the covariant gauge, have to deal with a space with an indefinite metric [10-12], namely, the Krein space. To clarify this point, we note that a scalar product which defines probability to observe a particle cannot obviously be positive for nonphysical particles. In addition, scalar products for all nonphysical particles cannot be zero simultaneously. Indeed, in this case all scalar products containing the vector, which describes nonphysical particles, are zero. It can easily be seen by using the Schwarz inequality
\[ |(x, y)|^2 \leq (x, x)(y, y), \]
which is valid in an arbitrary space with nonnegative metric. Therefore, at least for one of the vectors describing nonphysical particles, a scalar product on itself must be negative. This means that a respective space possesses indefinite metric. We emphasize that in a space with indefinite metric there are always positive, negative, and neutral vectors [14]. Note that a set of neutral vectors may include a subset of isotropic vectors, which are orthogonal to all vectors of a space under consideration. It is known that an arbitrary space with indefinite metric is a sum of isotropic space consisting of isotropic vectors and nondegenerate space that does not contain them. Since isotropic vectors do not affect in any way the theory under study, only nondegenerate spaces are considered in quantum field theory. The simplest nondegenerate space with an indefinite metric is the Krein one, whose basic properties are described in the next section. Note that the present paper is a step toward deriving the von Neumann’s theorem in gauge theory, in which the number of coordinates and momenta is infinite. It is shown in [13] that, besides the Fock representation, two other types of representations arise in a Krein space: one with a negative spectrum of the particle number operator \( N = a^*a \) (an anti-Fock case) and another with a reciprocal discrete spectrum. The purpose of this article is to generalize the von Neumann’s theorem to the case of anti-Fock representation in a Krein space.

KREIN SPACE

In this section we sketch the basic properties of a Krein space (for more detail see [14-16]).

If a linear space \( \mathcal{H} \) with a scalar product (in general indefinite) admits decomposition into a direct sum of the form
\[ \mathcal{H} = \mathcal{H}^+ + \mathcal{H}^-, \quad \mathcal{H}^+ \perp \mathcal{H}^-, \quad (8) \]
where \( \mathcal{H}^+ \) and \( \mathcal{H}^- \) are complete subspaces with positively and negatively defined metrics, respectively, then \( \mathcal{H} \) is a Krein space.

According to this definition, the class of Krein spaces includes both Hilbert (\( \mathcal{H}^+ = 0 \)) and “anti-Hilbert” (\( \mathcal{H}^- = 0 \)) spaces.

Since each vector in a Krein space satisfies the following decomposition
\[ x = x_+ + x_-, \quad x = \mathcal{H}_ \quad (9) \]
a scalar product of elements \( x \) and \( y \) can be written in the form
\[ (x, y) = (x_+ , y_+) + (y_-, x_-). \quad (10) \]

The operator \( J \) is called a fundamental symmetry operator if
\[ J(x_+ + x_-) = x_+ - x_- \quad (11) \]

The following properties can easily be checked:

1. \( J \) is an invertible operator and \( J^{-1} = J \),
2. \( J \) is a self-adjoint operator with respect to both indefinite and positively defined scalar products, which will be introduced later.

Let us show that the expression
\[ (x, y)_J = (x_+ , y_+) - (x_-, y_-) \quad (12) \]
gives a positively defined scalar product in the Krein space called \( J \) product. Indeed,
\[ (x, x)_J = (x_+ , x_+) - (x_-, x_-) \geq 0. \]

The norm on \( \mathcal{H} \) can be introduced by the expression
\[ \|x\|_J = \sqrt{(x, x)_J}, \quad x \in \mathcal{H}. \quad (13) \]

In this case it is called \( J \) norm. It is easy to see that the fundamental symmetry operator relates indefinite and positively defined scalar products as follows:
\[ (x, y) = (x, Jy)_J, \quad (x, y)_J = (x, Jy). \quad (14) \]

It is not hard to obtain a relationship between operators \( A^* \) and \( A^+ \), where symbols “*” and “+” denote self-adjoint operators relative to the \( J \) product and inner product, respectively. Indeed, from the equalities
\[ (Ax, y) = (Ax, Jy)_J = (x, A^* Jy)_J = (x, J A^* Jy) \quad (15) \]

\[ (Ax, y) = (Ax, Jy)_J = (x, A^* Jy)_J = (x, J A^* Jy) \quad (15) \]
it follows that

\[ A^+ = JA^+J. \]  

### REGULAR REPRESENTATIONS OF HEISENBERG ALGEBRA IN KREIN SPACE

Assume that the particle number operator \( N = a^+a \) possesses an eigenvector \( \psi_0 \) in a Hilbert space and

\[ N\psi_\alpha = \alpha \psi_\alpha. \]  

Let us show that \( \alpha \geq 0 \) in a Hilbert space. Indeed, if \( \alpha = -\beta, \beta > 0 \), then

\[ (N\psi_\beta, \psi_\beta) = -\beta(\psi_\beta, \psi_\beta) \leq 0. \]  

On the other hand,

\[ (N\psi_\beta, \psi_\beta) = (a\psi_\beta, a\psi_\beta) \geq 0. \]  

From these two latter inequalities, it follows that \( \psi_\beta = 0 \). Since \( a\psi_\alpha = \psi_{\alpha-1} \), requiring \( \alpha \) be nonnegative imposes condition (7); i.e., the representation considered is a Fock one. The condition (17) can be used as a definition of a regular CCR representation in a Hilbert space.

By definition, condition (17) introduces regular representations also in a space with an indefinite metric. The following theorem describes regular representations in a Krein space [13]:

**Theorem 3.** Regular representations of Heisenberg algebra in a Krein space belong to one of three classes:

1. **Fock representations** are characterized by the presence of \( \psi_0 \) vector (a Fock vector) such that

\[ a\psi_0 = 0. \]

Particle number operator \( N \) has a complete set of eigenvectors \( \psi_n, n = 0, 1, 2, \ldots \), with nonnegative eigenvalues;

2. **Anti-Fock representations** are characterized by the presence of \( \psi_-1 \) vector (an anti-Fock vector) such that

\[ a^+\psi_-1 = 0. \]

Particle number operator \( N \) has a complete set of eigenvectors \( \psi_n, n = -1, -2, \ldots \), with negative eigenvalues;

3. **\( \lambda \)-representations:** particle number operator \( N \) has a complete set of eigenvectors such that

\[ \text{Sp}N = 1 + \mathbb{Z}, \quad -1 < \lambda < 0. \]

Fock representations in a Krein space are in fact those in a Hilbert space. We will be interested in a second (anti-Fock) case when

\[ a^+\psi_-1 = 0. \]  

Note that no neutral eigenvectors of an \( \lambda \) operator exist in a Krein space. Recall that, in a space with indefinite metric, vectors are called neutral if their scalar product on themselves is zero. Indeed, a neutral eigenvector of \( \lambda \) operator must be orthogonal to all vectors of the space under consideration, because a Krein space is spanned by the eigenvectors of the \( \lambda \) operator. However, no such nonzero vectors exist in a Krein space. It is easy to see that, for a set of eigenvectors of \( \lambda \) operator

\[ \psi_{-n} = \frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\psi_1, \quad n \in \mathbb{N} \]

the following relation holds

\[ (\psi_{-m}, \psi_{-n}) = (-1)^{n-1}. \]  

Indeed,

\[ (\psi_{-m}, \psi_{-n}) = \frac{1}{n-1}(a\psi_{-m+1}, a\psi_{-n+1}) = \frac{1}{n-1}(\psi_{-m+1}, a^2\psi_{-n+1}) = \psi_{-m+1}(\psi_{-1}, \psi_{-1}), \]

where one can always take \( (\psi_{-1}, \psi_{-1}) = 1 \).

Now, let us show that the equality

\[ \{a, J\} = \{a^+, J\} = 0, \]  

where \( \{x, y\} = xy + yx \) is valid for all linear combinations of \( \psi_{-n} \) vectors. Let \( n = 2m \). Since \( a\psi_{-2m} = \psi_{-2m-1} \), from (11) and (21) it follows that

\[ J^\lambda a\psi_{-2m} = J\psi_{-2m-1} = \psi_{-2m-1}. \]

On the other hand,

\[ aJ^\lambda \psi_{-2m} = -a\psi_{-2m} = -\psi_{-2m-1}. \]

The case \( n = 2m+1 \) can be considered in a similar way. Thus,

\[ \{a, J\} \psi_{-n} = 0, \quad n \in \mathbb{N}. \]

Then, for any vector \( \psi \) such that

\[ \psi = \sum_{p=-k}^{-l} c_p\psi_p, \quad k, l \in \mathbb{N}, \quad k > l \geq 1, \]

it follows that

\[ \{a, J\} \psi = 0. \]  

This way, the equality \( \{a, J\} = 0 \) holds within dense region of a Krein space. From the property \( J^\lambda = J \), it follows that the equality \( \{a^+, J\} = 0 \) is valid as well.
WEYL REPRESENTATION ANALOGUE AND GENERALIZATION OF VON NEUMANN’S THEOREM FOR THE CASE OF ANTI-FOCK REPRESENTATION

A Weyl representation analogue for the anti-Fock case was obtained in [17].

Let \( a \) and \( a^+ \) be operators of anti-Fock representation in a Krein space \( \mathcal{H} \). Introduce new operators \( b = a^+ + a \), in the terms of which we obtain

\[
[b, b^*] = -1, \quad \tilde{N} = -N - 1, \quad \text{Sp} \tilde{N} = \mathbb{N},
\]

\[
\tilde{\psi}_n = \psi_{n-1},
\]

where “\( \sim \)” denotes all quantities that are related with new operators.

Then relations in (20) and (22) assume the form

\[
b\tilde{\psi}_n = 0, \quad \{b, J\} = \{b^+, J\} = 0.
\]

From (16) we obtain

\[
b^* = Jb^+ J = -b^*.
\]

Note that operators \( b \) and \( b^* \) satisfy standard commutation relations of the type (5):

\[
[b, b^*] = I.
\]

By using (6), \( \tilde{P} \) and \( \tilde{Q} \) operators that satisfy CCR (1) can be expressed as follows:

\[
\tilde{P} = \frac{1}{\sqrt{2i}}(b - b^*), \quad \tilde{Q} = \frac{1}{\sqrt{2}}(b + b^*).
\]

They can also be easily expressed via initial operators of anti-Fock representation: using the relations \( b^* = -b^+ = -a, b = a^+ \), we obtain

\[
\tilde{P} = \frac{1}{\sqrt{2i}}(a^+ + a), \quad \tilde{Q} = \frac{1}{\sqrt{2}}(a^+ - a).
\]

\( \tilde{P} \) and \( \tilde{Q} \) are self-adjoint operators defined on some region in a Hilbert space. Further, we can repeat arguments given in [1] while proving the relations in (3). Thus, we derive an analogue of CCR representation in the Weyl form for the anti-Fock case.

Finally, the von Neumann’s theorem can be applied to the representation obtained, because unitary groups of operators \( \{U(t) = e^{it\tilde{P}}\} \) and \( \{V(s) = e^{is\tilde{Q}}\} \) satisfy all the conditions of that theorem.

CONCLUSIONS

Von Neumann’s uniqueness theorem, according to which any two irreducible CCR representations that can be written down in the Weyl form are unitarily equivalent, is generalized to theories with nonphysical particles.

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