Intertwining connectivities in representable matroids

Tony Huynh† Stefan H. M. van Zwam‡

November 8, 2017

Abstract

Let $M$ be a representable matroid, and $Q, R, S, T$ subsets of the ground set such that the smallest separation that separates $Q$ from $R$ has order $k$, and the smallest separation that separates $S$ from $T$ has order $l$. We prove that, if $M$ is sufficiently large, then there is an element $e$ such that in one of $M \setminus e$ and $M/ e$ both connectivities are preserved.

For matroids representable over a finite field we prove a stronger result: we show that we can remove $e$ such that both a connectivity and a minor of $M$ are preserved.

1 Introduction

For a matroid $M$ on ground set $E$ we define, as usual, the connectivity function $\lambda_M$ by

$$\lambda_M(X) = \text{rk}_M(X) + \text{rk}_M(E - X) - \text{rk}(M).$$

For disjoint sets $S, T \subseteq E$, the connectivity between $S$ and $T$ is

$$\kappa_M(S, T) := \min\{\lambda_M(X) : S \subseteq X \subseteq E - T\}. \quad (1)$$

Geelen, in private communication, conjectured the following.

Conjecture 1.1. There exists a function $c : \mathbb{N}^2 \to \mathbb{N}$ with the following property. Let $M$ be a matroid, and let $Q, R, S, T \subseteq E(M)$ be sets of elements such that $Q \cap R = S \cap T = \emptyset$. Let $k := \kappa_M(Q, R)$ and $l := \kappa_M(S, T)$. If $|E(M) - (Q \cup R \cup S \cup T)| \geq c(k, l)$, then there exists an element $e \in E(M) - (Q \cup R \cup S \cup T)$ such that one of the following holds:

(i) $\kappa_{M \setminus e}(Q, R) = k$ and $\kappa_{M \setminus e}(S, T) = l$;
(ii) $\kappa_{M/e}(Q, R) = k$ and $\kappa_{M/e}(S, T) = l$.

∗Research supported by the NWO (The Netherlands Organization for Scientific Research) free competition project "Matroid Structure – for Efficiency" led by Bert Gerards, and by the National Science Foundation, Grant No. 1161650.
†Department of Mathematics, Simon Fraser University, Burnaby, B.C., Canada. Email: tony.bourbaki@gmail.com
‡Department of Mathematics, Princeton University, Princeton, NJ, United States. Email: svanzwam@math.princeton.edu
In other words, for fixed \( Q, R, S, T \), there is a finite number of minor-minimal matroids with the prescribed connectivities. This formulation is reminiscent of the definition of an intertwine, which is a minor-minimal matroid containing two prescribed minors. For that reason we speak of the intertwining of connectivities.

For graphs the result follows readily from Robertson and Seymour’s Graph Minors Theorem [11]. In this paper we prove the conjecture for all representable matroids.

For matroids representable over a finite field we prove a stronger result:

**Theorem 1.2.** There exists a function \( c : \mathbb{N}^3 \to \mathbb{N} \) with the following property. Let \( q \) be a prime power, let \( M \) be a \( \text{GF}(q) \)-representable matroid, let \( N \) be a minor of \( M \), let \( S, T \subseteq E(M) \) be disjoint, and let \( k := \kappa_M(S, T) \). If \( |E(M) - (S \cup T \cup E(N))| > c(q, |E(N)|, k) \), then there exists an element \( e \in E(M) - (S \cup T \cup E(N)) \) such that at least one of the following holds:

(i) \( \kappa_{M\setminus e}(S, T) = k \) and \( N \) is a minor of \( M \setminus e \);
(ii) \( \kappa_{M/e}(S, T) = k \) and \( N \) is a minor of \( M/e \).

By repeated use of this theorem, it is possible to bound the size of an intertwine of any fixed number of connectivities. This gives a (highly unsatisfying) answer to the following problem:

**Problem 1.3.** Let \( M = (S, \mathcal{I}) \) be a matroid that is a gammoid. Give an upper bound, in terms of \( |S| \), on the size of the graph needed to represent \( M \) as a gammoid.

Good upper bounds can potentially be useful in the study of parametrized complexity (c.f. [8]).

Our proof technique for Theorem 1.2 has been used previously in, for instance, [4, 6, 7]. For graphs it dates back at least to the work of Robertson and Seymour on graph minors (cf. [12]). In fact, Theorem 1.2 is a generalization of [6, Theorem 1.1] and [13, Theorem 13.3].

Theorem 1.2 becomes false when the dependence on \( q \) is removed. A counterexample is readily obtained from a construction of arbitrarily long blocking sequences in [6, Proposition 6.1]. It follows that different techniques are needed to prove Conjecture 1.1.

Our proof of Conjecture 1.1 for representable matroids uses a different approach, based on a suggestion by Geelen (private communication). Unfortunately, the proof uses a property of representable matroids that does not hold for general matroids.

The paper is organized as follows. In Section 2 we fix some terminology and state some easy lemmas. Section 3 contains results related to Tutte’s Linking Theorem. The main result in that section shows that, if Conjecture 1.1 is false, there exist matroids with arbitrarily long sequences of nested separations. In Section 4 we prove Theorem 1.2, and in Section 5 we prove Conjecture 1.1 for all representable matroids.

## 2 Preliminaries

We will use the following elementary observation (c.f. [10, 3]):
Lemma 2.1. Let $M$ be a matroid and let $(A, \{e\}, B)$ be a partition of $E(M)$. Then $e \in \text{cl}_M(A)$ if and only if $e \notin \text{cl}_M(B)$.

It is well-known that the connectivity function is submodular:

Lemma 2.2. Let $M$ be a matroid, and let $X, Y \subseteq E(M)$. Then

$$\lambda_M(X) + \lambda_M(Y) \geq \lambda_M(X \cap Y) + \lambda_M(X \cup Y).$$

The following lemmas are easily verified:

Lemma 2.3. Let $M$ be a matroid, let $X \subseteq E(M)$, and let $N$ be a minor of $M$ with $X \subseteq E(N)$. Then $\lambda_N(X) \leq \lambda_M(X)$.

Lemma 2.4. Let $M$ be a matroid, let $S \subseteq E(M)$, and let $N$ be a minor of $M$ with $S \subseteq E(N)$. Then $\lambda_N(S \cup T) \leq \lambda_M(S \cup T)$.

We introduce some terminology.

Definition 2.5. Let $M$ be a matroid and let $S, T$ be disjoint subsets of $E(M)$. A partition $(A, B)$ of $E(M)$ is $S-T$-separating of order $k+1$ if $S \subseteq A$, $T \subseteq B$, and $\lambda_M(A) = k$. If $B$ is implicit, we also say that $A$ is $S-T$-separating of order $k+1$.

If, moreover, $|A|, |B| \geq k+1$ then $(A, B)$ is an (exact) $(k+1)$-separation of $M$. Sometimes we will be sloppy and say that $(A, B)$ is $S-T$-separating if $S \subseteq B$ and $T \subseteq A$.

Lemma 2.6. Let $M$ be a matroid, let $S, T \subseteq E(M)$ be disjoint subsets, and let $k := \kappa_M(S, T)$. If $(A_1, B_1)$ and $(A_2, B_2)$ are $S-T$-separating with $\lambda_M(A_1) = \lambda_M(A_2) = k$, then $(A_1 \cap A_2, B_1 \cup B_2)$ is $S-T$-separating of order $k+1$.

Proof. Clearly, $(A_1 \cap A_2, B_1 \cup B_2)$ and $(A_1 \cup A_2, B_1 \cap B_2)$ are $S-T$-separating. Since $\kappa_M(S, T) = k$, we must have $\lambda_M(A_1 \cap A_2) \geq k$ and $\lambda_M(A_1 \cup A_2) \geq k$. It follows from Lemma 2.2 that equality must hold.

Finally, we will frequently use the following well-known result and its dual.

Lemma 2.7. Let $M$ be a matroid, let $S, T \subseteq E(M)$ be disjoint subsets, let $k := \kappa_M(S, T)$, and let $e \in E(M) - (S \cup T)$. A partition $(A, B)$ of $E(M) - e$ is $S-T$-separating of order $k$ in $M/e$ if and only if $(A \cup e, B)$ is $S-T$-separating of order $k+1$ in $M$ with $e \in \text{cl}_M(A) \cap \text{cl}_M(B)$.

3 Tutte’s Linking Theorem

In [14], Tutte proved the following result, which can be seen to be a generalization of Menger’s theorem to matroids (see [9, Section 8.5]):

Theorem 3.1. Let $M$ be a matroid and let $S, T$ be disjoint subsets of $E(M)$. Then

$$\kappa_M(S, T) = \max\{\lambda_N(S) : N \text{ minor of } M \text{ such that } E(N) = S \cup T\}.$$ (2)
Theorem 3.2. Let $M$ be a matroid and let $S, T$ be disjoint subsets of $E(M)$. For each $e \in E(M) - (S \cup T)$, at least one of the following holds:

(i) $\kappa_{M/e}(S, T) = \kappa_M(S, T)$, or
(ii) $\kappa_{M/e}(S, T) = \kappa_M(S, T)$.

Definition 3.3. Let $M$ be a matroid, let $S$ be disjoint subsets of $E(M)$, and let $e \in E(M) - (S \cup T)$.

(i) If $\kappa_{M/e}(S, T) = \kappa_M(S, T)$ then we say $e$ is deletable with respect to $(S, T)$.
(ii) If $\kappa_{M/e}(S, T) = \kappa_M(S, T)$ then we say $e$ is contractible with respect to $(S, T)$.
(iii) If $e$ is both deletable and contractible then we say $e$ is flexible with respect to $(S, T)$.

We may omit the phrase “with respect to $(S, T)$” if it can be deduced from the context. We will mainly be concerned with non-flexible elements. The following theorem is the main result of this section:

Theorem 3.4. Let $M$ be a matroid, let $S, T$ be disjoint subsets of $E(M)$, let $k := \kappa_M(S, T)$, and let $F \subseteq E(M) - (S \cup T)$ be a set of non-flexible elements. There exist an ordering $(f_1, f_2, \ldots, f_t)$ of $F$ and a sequence $(A_1, A_2, \ldots, A_t)$ of subsets of $E(M)$, such that

(i) $A_i$ is $S - T$-separating of order $k + 1$ for each $i \in \{1, \ldots, t\}$;
(ii) $A_i \subseteq A_{i+1}$ for each $i \in \{1, \ldots, t-1\}$;
(iii) $A_i \cap F = \{f_1, \ldots, f_i\}$ for each $i \in \{1, \ldots, t\}$;
(iv) $f_i \in cl_M(A_i) - f_i \cap cl_M(E(M) - A_i)$ or $f_i \in cl_M^*(A_i - f_i) \cap cl_M^*(E(M) - A_i)$.

We will need two lemmas to prove this theorem.

Lemma 3.5. Let $M$ be a matroid, let $S, T$ be disjoint subsets of $E(M)$, let $k := \kappa_M(S, T)$, and let $e \in E(M) - (S \cup T)$ be non-contractible. If $(A, B)$ is an $S - T$-separating partition of order $k + 1$ such that $e \in A$ and $|A|$ is minimum, then $e \in cl_M(A - e) \cap cl_M(B)$.

Proof. Suppose not. By Lemma 2.7, there is an $S - T$-separating partition $(A', B')$ of order $k + 1$ such that $e \in A'$ and $e \in cl_M(A' - e) \cap cl_M(B')$. By Lemma 2.6, $A \cap A'$ is $S - T$-separating of order $k + 1$. By minimality of $A$, it then follows that $A \subseteq A'$, and therefore $B \supseteq B'$. But then $e \in cl_M(B)$. By Lemma 2.1, then, $e \notin cl_M^*(A - e)$. If also $e \notin cl_M^*(A - e)$ then $\lambda_M(A - e) = k + 1$, contradicting $\kappa_M(S, T) = k$. The result follows.

Lemma 3.6. Let $M$ be a matroid, let $S, T$ be disjoint subsets of $E(M)$, let $k := \kappa_M(S, T)$, and let $U$ be an $S - T$-separating set of order $k + 1$. If $e \in E(M) - (T \cup U)$ is non-contractible with respect to $(S, T)$, then $e$ is non-contractible with respect to $(U, T)$.
Proof. First, observe that $\kappa_M(U, T) = k$. If the lemma is false, then there is an $S - T$-separating partition $(A, B)$ of order $k$ in $M/e$, yet $\kappa_{M/e}(U, T) = k$. In particular, $\lambda_{M/e}(U) = k$. By submodularity,

$$2k - 1 = \lambda_{M/e}(A) + \lambda_{M/e}(U) \geq \lambda_{M/e}(U \cap A) + \lambda_{M/e}(U \cup A). \quad (3)$$

Since $U \cup A$ is $U - T$-separating, we have $\lambda_{M/e}(U \cup A) \geq k$. Hence $\lambda_{M/e}(U \cap A) \leq k - 1$. But $\lambda_M(U \cap A) = k$ since $U \cap A$ is $S - T$-separating. It follows that $e \in \text{cl}_M(U \cap A)$, and in particular $e \in \text{cl}_M(U)$. By Lemma 2.7, we cannot have $e \in \text{cl}_M(E(M) - (U \cup e))$. But then $\lambda_M(U \cup e) = k - 1$, contradicting the fact that $\kappa_M(U, T) = k$. \hfill \square

Proof of Theorem 3.4. We prove the result by induction on $|F|$, the case $|F| = 0$ being trivial. Suppose the result fails for a matroid $M$ with subsets $S, T, F$ as in the theorem. Let $k := \kappa_M(S, T)$ and $t := |F|$. For each $e \in F$, let $(A_e, B_e)$ be $S - T$-separating of order $k + 1$ with $e \in A_e$ and $|A_e|$ as small as possible. Let $f$ be such that $|A_f| \leq |A_e|$ for all $e \in F$.

Claim 3.6.1. $A_f \cap F = \{ f \}$.

Proof. Suppose $g \in A_f \cap F$ with $g \neq f$. By our choice of $f$, we must have that $A_g = A_f$ (using Lemma 2.6). Since $g$ is not flexible, Lemma 3.5 implies that $(A_f - g, B_f \cup g)$ is $S - T$-separating of order $k + 1$, contradicting minimality of $|A_f|$. \hfill \square

By Lemma 3.6 we can apply the theorem inductively, replacing $S$ by $A_f$ and $F$ by $F - f$, thus finding a sequence $(A_2, \ldots, A_t)$ of nested $A_f - T$-separating sets of order $k + 1$. But now the sequence $(A_f, A_2, \ldots, A_t)$ satisfies all conditions of the theorem. \hfill \square

We will use the following two facts:

Lemma 3.7. Let $M$ be a matroid, let $S, T$ be disjoint subsets of $E(M)$, let $k := \kappa_M(S, T)$, and let $(A_1, \ldots, A_t)$ be a sequence of nested $S - T$-separating sets of order $k + 1$. Let $(C, D)$ be a partition of $E(M) - (S \cup T)$ such that $C$ is independent, $D$ is coindependent, and $\lambda_{M_\cap D}(S) = k$. Let $i, j \in \{1, \ldots, t\}$ with $i < j$. Let $C' := C \cap (A_j - A_i)$, let $D' := D \cap (A_j - A_i)$, and let $M' := M/C' \setminus D'$. Then $(A_i, B_j)$ is $S - T$-separating of order $k + 1$ in $M'$. Moreover, $M'|A_i = M|A_i$ and $M'|B_j = M|B_j$.

Proof. Let $M' := M/C' \setminus D'$. By definition of $C$ and $D$, $\kappa_M(S, T) = k$. By monotonicity of $\alpha$, $\lambda_M(A_i) = k$. It follows from Lemma 2.7 that for all $e \in C'$, $e \notin \text{cl}_M(A_i \cup (C' - \{e\}))$ and $e \notin \text{cl}_M(B_j \cup (C' - \{e\}))$. From this the second claim follows. \hfill \square

Lemma 3.8 (Geelen, Gerards, and Whittle [5, Lemma 4.7]). Let $M$ be a matroid, let $S, T$ be disjoint subsets of $E(M)$, and let $k := \kappa_M(S, T)$. There exist sets $S_1 \subseteq S$ and $T_1 \subseteq T$ such that $|S_1| = |T_1| = \kappa_M(S_1, T_1) = k$. 

5
4 Proof of the result for finite fields

Let $M$ be a rank-$r$ matroid on ground set $E$. Write $M = M[D]$ if the $r \times E$ matrix $D$ (over field $F$) represents $M$. For $S \subseteq E$, denote by $D[S]$ the submatrix of $D$ induced by the columns labeled by $S$, and denote by $\langle D[S] \rangle$ the vector space spanned by the columns of $D[S]$. To clean up notation we will write $\langle S \rangle$ for $\langle D[S] \rangle$ if $D$ is clear from the context.

Recall that, if $(A, B)$ is such that $\lambda_{M[D]}(A) = k$, then $\langle A \rangle \cap \langle B \rangle$ is a $k$-dimensional subspace of $\mathbb{F}^r$. Assume $\mathbb{F} = \text{GF}(q)$. Denote by $M^+_{(A,B)}$ the matroid obtained from $M$ by adding a copy of $\text{PG}(k-1,q)$ to $M$, such that in the representation it is contained in $\langle A \rangle \cap \langle B \rangle$. Furthermore, $M^+_A := M^+_{(A,B)} \setminus B$ and $M^+_B := M^+_{(A,B)} \setminus A$. Now we can carry out row operations to get $M^+_{(A,B)} = M[D']$, with

$$D' = \begin{bmatrix}
A & X & B \\
D_1 & 0 & 0 \\
0 & P & D_2
\end{bmatrix},$$

where $P$ is a $k \times X$ matrix representing $\text{PG}(k-1,q)$ (with elements labeled by $X$). We remark that $M^+_{(A,B)}$ is the generalized parallel connection of $M^+_A$ and $M^+_B$ along $X$ (cf. [9, Section 11.4]). The following lemma follows easily from Lemma 3.7.

Lemma 4.1. Let $M$ be a $\text{GF}(q)$-representable matroid, let $S$ and $T$ be disjoint subsets of $E(M)$ with $\kappa_M(S,T) = k$, and let $(A, B)$ be $S - T$-separating of order $k + 1$. Let $(C, D)$ be a partition of $E(M) - (S \cup T)$ such that $\lambda_{M[C \cup D]}(S) = k$. Then $(M^+_{(A,B)} / C \setminus D)[X] = M^+_{(A,B)}[X].$

We repeat the main result, filling in an explicit value for the constant:

Theorem 4.2. Let $q$ be a prime power, let $M$ be a $\text{GF}(q)$-representable matroid on ground set $E$, let $N$ be a minor of $M$ on $n$ elements, let $S, T \subseteq E$, and let $k := \kappa_M(S, T)$. If $|E - (S \cup T)| > n + 2(n + 1)q^r$, then there exists an element $e \in E$ such that at least one of the following holds:

(i) $\kappa_{M/e}(S, T) = k$ and $N$ is a minor of $M \setminus e$;
(ii) $\kappa_{M/e}(S, T) = k$ and $N$ is a minor of $M/e$.

The proof is not hard, but unfortunately we could not avoid using rather involved notation. For that reason we give a rough sketch of the idea. Let $M$ be a counterexample. First we construct a long sequence $(A_1, B_1), \ldots, (A_i, B_i)$ of nested $S - T$-separating partitions of order $k + 1$. For each $i$ we define the matroid $M_i$, obtained from $M^+_B$ by deleting or contracting the elements of $B_i - E(N)$ so that the minor $N$ is preserved. Since each $M_i$ will have the same number of elements, only a finite number of distinct represented matroids can arise. Since our matroid is sufficiently large it follows that, after suitably relabeling the new elements, $M_i = M_j$ for some $i < j$. This shows that the elements in $A_j - A_i$
can be removed in such a way that both $N$ and the $S - T$-connectivity are
preserved, which contradicts our choice of $M$.

Proof. Let $q, M, N, n, S, T$, and $k$ be as stated, and assume $|E - (S \cup T)| >
n + 2(n + 1)q^n$, yet no element can be removed keeping both the $S - T$-
connectivity and the minor $N$. Let $(C, D)$ be a partition of $E - (S \cup T)$
such that $\lambda_{M/C, D}(S) = \kappa_M(S, T)$ and such that $C$ is independent and
$D$ coindendent. Let $(C_N, D_N)$ be a partition of $E - E(N)$ such that
$N = M/C_N \setminus D_N$ and such that $C_N$ is independent and $D_N$
coindendent. By our assumption, $C \cap C_N = \emptyset$ and $D \cap D_N = \emptyset$.

Let $F := C \cup D - E(N)$, and let $t^\prime := |F|$. Then $t^\prime > 2(n + 1)q^n$. By
Theorem 3.4, there is a nested sequence $(A'_i, \ldots, A''_i)$ of $S - T$-separating
sets of order $k + 1$ such that $A'_i \subseteq A''_{i+1}$ for $i \in \{1, \ldots, t^\prime - 1\}$. Let
$(f'_i, \ldots, f''_i)$ be the corresponding ordering of $F$. Consider the sequence
$(A'_1 \cap E(N), \ldots, A''_1 \cap E(N))$. This sequence contains at most $n + 1$
different elements. It follows that $(A'_1, \ldots, A''_1)$ has a subsequence $(A'_i, \ldots, A''_i)$
such that $A'_i \cap E(N) = A''_j \cap E(N)$ for all $i, j \in \{1, \ldots, t^\prime\}$, and such that
$t^\prime \geq t^\prime/(n + 1) > 2q^n$.

Let $(f'_1, \ldots, f''_1)$ be the corresponding subsequence of $F$. Using duality
if necessary we may assume that $|\{f'_1, \ldots, f''_1\} \cap C| \leq |\{f'_1, \ldots, f''_1\} \cap D|$. Let
$(A_1, \ldots, A_t)$ be a subsequence of $(A'_1, \ldots, A''_1)$ such that $A_{i+1} - A_i$
contains an element of $C$ for all $i \in \{1, \ldots, t - 1\}$, and such that $t \geq t^\prime/2 > q^n$.
For each $i \in \{1, \ldots, t\}$, define $B_i := E - A_i$.

Let $H$ be an $r \times E$ matrix over $GF(q)$ representing $M$. Let $s := (q^k -
1)/(q - 1)$. For each $i$, let $W_i := (A_i) \cap (B_i)$, and let $X_i := \{x'_i, \ldots, x''_i\}$
be a set of labels disjoint from $E$ and disjoint from $X_j$ for all $j \in \{1, \ldots, t\} - \{i\}$. Let
the $k \times X_1$ matrix $P_1$ be an arbitrary representation of $PG(k - 1, q)$
having ground set $X_1$.

For each $i \in \{1, \ldots, t\}$, let $M^+_i$ be the matroid $M^{+}_{(A_i, B_i)}$ with
the set $X$ relabeled by $X_i$. Moreover, we assume this labeling was chosen such that,
in $(M^+_i)^{\prime}/C \setminus D_i$, $x''_i$ is parallel to $x'_i$ for all $j \in \{1, \ldots, s\}$ (where $(M^+_i)^{\prime}$
is defined in the obvious way). This can be done because of Lemma 4.1.

Now we define, for each $i$, a matroid $N_i$ as follows: first set $N'_i :=
(M^+_i \setminus A_i) \cap (C_N \cap B_i) \setminus (D_N \cap B_i)$. Now $N_i$ is obtained from
$N'_i$ by relabeling $x'_i$ by $x''_i$. Let $H_i$ be the corresponding representation matrix. Note
that, for $i, j \in \{1, \ldots, t\}$, $E(N_i) = E(N_j) \subseteq E(N) \cup X_1$. Hence $|E(N_i)| \leq n + s$. Since
$X_i \subseteq B_i$, we find that $rk(N_i) \leq n$. Furthermore, for all $x \in X_i, H_i[x] =
H_i[x]$. Hence there are at most $((q^n - 1)/(q - 1) + 1)^{n} \leq q^n - q^n$ distinct
representation matrices $H_i$. Since $t > q^n$, there exist $i, j \in \{1, \ldots, t\}$ with
$i < j$ such that $H_i = H_j$. But then $M/(B_i \cap C_N) \setminus (B_j \cap D_N)$ is equal to

$$(M/(\{A_j - A_i\} \cap C) \setminus (\{A_j - A_i\} \cap D) \setminus (B_j \cap C_N) \setminus (B_j \cap D_N),$$

using Lemmas 3.7 and 4.1. In particular, since $(A_j - A_i) \cap C \neq \emptyset$, there
exists an $e \in C$ such that $\kappa_{M/e}^2(S, T) = k$ and $M/e$ has $N$ as minor,
a contradiction.

\end{proof}

For completeness we show that Conjecture 1.1 follows from Theorem 4.2 when $M$
is $GF(q)$-representable.
Proof of Conjecture 1.1 for GF(q)-representable matroids. Let \( n := |Q \cup R| \) and set \( c := n + 2(n + 1)q^n \). Let \( (C, D) \) be a partition of \( E - (Q \cup R) \) such that \( \lambda_{M \setminus Q, D}(Q) = k \). By Theorem 3.1, \( C \) and \( D \) exist. Now apply Theorem 4.2 with \( N = M \setminus C \setminus D, S, \) and \( T \). The result follows.

5 Intertwining two connectivities

In this section we prove Conjecture 1.1 for all representable matroids. The key property we need for our proof is that we can add a point to the intersection of two non-skew flats. Formally:

**Definition 5.1.** A matroid \( M \) has the **intersection property** if for all flats \( S, T \in E(M) \) such that \( \cap(S, T) > 0 \), there exist a matroid \( N \) and a non-loop element \( e \in E(N) \) such that \( N \setminus e = M \) and \( e \in \text{cl}_N(S) \cap \text{cl}_N(T) \). In this case, we say that \( N \) is a **good extension** of \( M \) (with respect to \( S, T \)). A class of matroids \( \mathcal{M} \) is **intersection-closed** if every \( M \in \mathcal{M} \) has the intersection property, and \( \mathcal{M} \) is closed under minors, duality, and good extensions.

Note that the class of representable matroids is evidently intersection-closed. The Vámos matroid shows that not all matroids have the intersection property. See [1] for more on matroids with the intersection property.

The restriction we use is reminiscent of the double-circuit property from the min-max theorem for matroid matching (see [2]). However, whereas the min-max theorem is false even for affine spaces, in our case the condition appears to be just an artifact of our proof. We remain hopeful that Conjecture 1.1 can be proven without this condition. We will now state and prove the main result.

**Theorem 5.2.** There exists a function \( c : \mathbb{N}^2 \rightarrow \mathbb{N} \) with the following property. Let \( M \) be a matroid in an intersection-closed family, and let \( Q, R, S, T, F \subseteq E(M) \) be sets of elements such that \( Q \cap R = S \cap T = \emptyset \) and \( F \subseteq E(M) - (Q \cup R \cup S \cup T) \). Let \( k := \kappa_M(Q, R) \) and \( l := \kappa_M(S, T) \). If \( |F| \geq c(k, l) \), then there exists an element \( e \in F \) such that one of the following holds:

(i) \( \kappa_M(Q, R) = k \) and \( \kappa_M(S, T) = l \);
(ii) \( \kappa_M(Q, R) = k \) and \( \kappa_M(S, T) = l \).

**Proof.** We prove that the result holds for \( c(k, l) := 4^{k+l} \). We proceed by induction on \( k + l \), noting that the base case where \( k = 0 \) or \( l = 0 \) is straightforward. Assume that the result holds for all \( k', l' \) with \( k' + l' < k + l \), but that \( M, Q, R, S, T, F \) form a counterexample. Possibly after relabeling we may assume \( k \leq l \). By Lemma 3.8 we can assume that \( |S| = |T| = l \), and that \( S \) and \( T \) are independent sets. Furthermore, we can assume that for each \( e \in F \) either \( \kappa_M(Q, R) < k \) or \( \kappa_M(Q, R) < k \).

**Claim 5.2.1.** There exists a \( Q-R \) separating partition \( (A, B) \) with \( \lambda(A) = k \), such that \( A \cap S \neq \emptyset, A \cap T \neq \emptyset, |A \cap (S \cup T)| \geq l \), and \( |B \cap F| \geq \frac{1}{2} c(k, l) \).
3.4

follows. If of $A$ and $B$, meets both of $S$ and $T$. Indeed: otherwise we have (possibly after swapping $S$ and $T$) that $S \subseteq A_i$ and $T \subseteq B_i$. In that case $(A_i, B_i)$ is $S - T$ separating with $\lambda_M(A_i) = k$. It follows that $k = l$. Assume $f_i$ is non-contractible with respect to $(Q, R)$. Then $\lambda_{M/F}(B_i) = k - 1$, and therefore $f_i$ is also non-contractible with respect to $(S, T)$, so the theorem holds with $e = f_i$.

Hence, possibly after exchanging the sequences $(A_1, \ldots, A_r)$ and $(B_1, \ldots, B_s)$, we can assume $A_1 \cap S \neq \emptyset$ and $A_r \cap T \neq \emptyset$. If $|A_i \cap (S \cup T)| < l$ then $|B_i \cap (S \cup T)| > l$, and therefore $(A, B) = (B_i, A_i)$ is a partition as desired; otherwise we simply take $(A, B) = (A_i, B_i)$.

If necessary, we relabel $Q$ and $R$ so that $Q \subseteq A$ and $R \subseteq B$. Define

$$S_1 := A \cap S \quad T_1 := A \cap T$$

$$S_2 := B \cap S \quad T_2 := B \cap T.$$

Also define $F_2 := B \cap F$. We try to remove the elements from $A$ while preserving the $S - T$ connectivity. Let $N_0 := M$, and order the elements of $A - (S_1 \cup T_1)$ arbitrarily as $a_1, \ldots, a_u$. For $i = 1, 2, \ldots, u$ define $N_i$ as follows. If $\kappa_{N_{i-1}:a_i}(S, T) = l$ and $a_i \not\in \text{cl}_{N_{i-1}}(B)$, then $N_i := N_{i-1}/a_i$. Else, if $\kappa_{N_{i-1}:a_i}(S, T) = l$ and $a_i \not\in \text{cl}_{N_{i-1}}(B)$, then $N_i := N_{i-1}/a_i$. Otherwise $N_i := N_{i-1}$. Observe that $\kappa_{N_i}(S, T) = l$ and $\kappa_{N_i}(A \cap E(N_u), R) = \lambda_{N_i}(B) = k$. We distinguish two cases.

Case I: $\cap_{N_k}(S_1, T_1) > 0$. Since $N_u$ is a member of an intersection-closed family, we can find a matroid $N^+$ in this family with a non-loop element $s$ such that $N^+ \cap s = N_u$, and $s \in \text{cl}_{N^+}(S_1 \cap \text{cl}_{N^+}(T_1))$. We distinguish two subcases:

Case Ia: $s \not\in \text{cl}_{N^+}(B)$. Let $N := N^+ \cap s$, and define $Q' := A \cap E(N)$. Then $\kappa_N(S, T) = l - 1$ and $\kappa_N(Q', R) = k$. Since $|F_2| \geq c(k, l - 1)$, by induction we can find an element $e \in F_2$ such that either $\kappa_{N^+}(S, T) = l - 1$ and $\kappa_{N^+}(Q', R) = k$, or $\kappa_{N^+}(S, T) = l - 1$ and $\kappa_{N^+}(Q', R) = k$. We assume the former, and remark that the proof for the latter case is similar.

Claim 5.2.2. $\kappa_{M/F}(Q, R) = k$ and $\kappa_{M/F}(S, T) = l$.

Proof. Suppose $\kappa_{M/F}(Q, R) < k$, that is, $e$ is non-contractible with respect to $(Q, R)$. By Lemma 3.6, $e$ is also non-contractible with respect to $(A, R)$ in $M$. But $(A, B)$ is $Q' - R$ separating, so we must have $\lambda_{M/F}(A) = k$, a contradiction.

Next, let $C, D$ be such that $C$ is independent in $N$, $e \in C$ and, in $N_0 := N / C \setminus D$, we have $E(N_0) = S \cup T$ and $\lambda_{N_0}(S) = l - 1$. Since $C$ is independent in $N^+ / s$, it follows that $s$ is not a loop in $N^+ / C$. Let $N' := N^+ / C \setminus D$. Since $s \in \text{cl}_{N'}(S) \cap \text{cl}_{N'}(T)$, we must have that $\lambda_{N'_0}(S) = l$. It follows that $\kappa_{M/F}(S, T) = l$ as desired. □
Case II: $s \in \text{cl}_{N^+}(B)$. Again we define $Q' := A \cap E(N)$. Let $(A_1, \ldots, A_r)$ be the nested sequence of $Q' - R$ separating sets in $N^+$ from Theorem 3.4 (applied to $N, Q', R$, and $F_2$), let $(B_1, \ldots, B_r)$ be their complements, and let $(f_1, \ldots, f_r)$ be the corresponding ordering of $F_2$. Let $j := c(k-1, l-1)$. If $s \not\in \text{cl}_{N^+}(B_j)$ then we apply the arguments from Case (Ia) with $A_j \cap E(N)$ replacing $Q'$, $B_j$ replacing $B$, and $F \cap B_j$ replacing $F_2$. Otherwise, let $N := N^+ / s$, define $R' := B_j$ and $F_2' := F_2 - B_j$. We have $\kappa_N(Q', R') = k - 1$ and $\kappa_N(S, T) = l - 1$. Since $|F_2'| \geq c(k-1, l-1)$, we find by induction an element $e \in F_2'$ such that either $\kappa_{N/e}(Q', R') = k - 1$ and $\kappa_{N/e}(S, T) = l - 1$, or $\kappa_{N/e}(Q', R') = k - 1$ and $\kappa_{N/e}(S, T) = l - 1$. We assume the latter, and remark that the proof in the former case is similar.

Claim 5.2.3. $\kappa_{M/e}(Q, R) = k$ and $\kappa_{M/e}(S, T) = l$.

Proof. Suppose $e = f_j' \in F_j'$ is non-deletable with respect to $(Q, R)$. Then $e \in \text{cl}_{N/e}(B_j)$, so $\lambda_{N/e}(B_j) = k - 1$. But $s \in \text{cl}_{N/e}(B_j) \cap \text{cl}_{N/e}(A_r - e)$, so we must have $\lambda_{N/e}(B_r) = k - 1$. But then $\lambda_{N/e}(B_r) = k = 2$, contradicting our choice of $e$. Hence $e$ is deletable with respect to $(Q, R)$.

The proof that $\kappa_{M/e}(S, T) = l$ is the same as before and we omit it. □

Case II: $\cap_{N^+}(S_1, T_1) = \cap_{N^+}(S_1, T_1) = 0$. By dualizing if necessary, we may assume there is an element $e \in \text{cl}_{N}(A) \cap \text{cl}_{N}(B) \cap F$, i.e. an element that is deletable with respect to $(Q, R)$ in $M$. We assume $e \in A$ (replacing $(A, B)$ by $(A \cup e, B - e)$ otherwise).

Claim 5.2.4. $e \in \text{cl}_{N}(S_1 \cup T_1)$.

Proof. First we show that $\text{cl}_{N^+}(B) - (S_1 \cup T_1)$ spans $S_1 \cup T_1$. Suppose not, and let $X := (S_1 \cup T_1) - \text{cl}_{N^+}(B)$. By construction of $N^+$, all remaining elements are in $\text{cl}_{N^+}(B)$, so we have that $N^+ \setminus X$ has lower rank than $N^+$. Hence $X$ contains a cocircuit. But this contradicts the fact that $S_1$ and $T_1$ are coskew.

Now pick $B' := \text{cl}_{N^+}(B) - (S_1 \cup T_1 \cup e)$ and $A' := A - B'$. Then $k' := \lambda_{N}(A') \leq k$. But since $S_1 \cup T_1 \cup e \subseteq A'$ and $S_1 \cup T_1 \cup e \subseteq \text{cl}_{N}(B')$, we must have that $\text{rk}_{N}(S_1 \cup T_1 \cup e) \leq k' \leq k \leq l$. Note that $|S_1 \cup T_1| \geq l$ and, since $S_1$ and $T_1$ are skew, $\text{rk}_{N}(S_1 \cup T_1) \geq l$. It follows that $k' = k = l$, and therefore $e \in \text{cl}_{N}(S_1 \cup T_1)$ as desired. □

Similar to before, we define $Q' := A \cap E(N) - \{e\}$. Let $(A_1, \ldots, A_r)$ be the nested sequence of $Q' - R$ separating sets in $N_e$ from Theorem 3.4 (applied to $Q', R$, and $F_2$), let $(B_1, \ldots, B_r)$ be their complements, and let $(f_1, \ldots, f_r)$ be the corresponding ordering of $F_2$. Let $j := c(k-1, l)$. Again we distinguish two cases.

Case IIA: $e \not\in \text{cl}_{N}(B_j)$. Let $N_0$ be obtained from $N_e$ by contracting $e$ and removing the other elements from $A_j$ according to the same rules used to obtain $N_e$. We can then apply the arguments of Case I to $N_0$ (with $A_j$ replacing $A$ and $B_j$ replacing $B$), observing that $|F \cap B_j| \geq 2(c(k-1, l) + c(k, l-1))$. 

10
Case IIb: \( e \in \text{cl}_N(B_j) \). Let \( N := N/e \), and define \( R' := B_j \). By induction we find an element \( f \in \{f_1, \ldots, f_j\} \) such that either \( \kappa_{N/f}(Q', R') = k - 1 \) and \( \kappa_{N/f}(S, T) = l \), or \( \kappa_{N/f}(Q', R') = k - 1 \) and \( \kappa_{N/f}(S, T) = l \). As before, in the former case we have \( \kappa_{M/f}(Q, R) = k \) and \( \kappa_{M/f}(S, T) = l \) and in the latter case we have \( \kappa_{M/f}(Q, R) = k \) and \( \kappa_{M/f}(S, T) = l \). This completes the proof of the theorem.

Acknowledgements We thank Jim Geelen for suggesting the problem to us, and for suggesting the proof approach of Theorem 5.2. We thank Bert Gerards for his support and several valuable insights.

References

[1] Joseph E. Bonin. A note on the sticky matroid conjecture. *Ann. Comb.*, 15(4):619–624, 2011.

[2] A. Dress and L. Lovász. On some combinatorial properties of algebraic matroids. *Combinatorica*, 7(1):39–48, 1987.

[3] J. Geelen and G. Whittle. Inequivalent representations of matroids over prime fields. Submitted. Preprint at arXiv:1101.4683, 2011.

[4] James F. Geelen, A. M. H. Gerards, and Geoff Whittle. Branch-width and well-quasi-ordering in matroids and graphs. *J. Combin. Theory Ser. B*, 84(2):270–290, 2002.

[5] Jim Geelen, Bert Gerards, and Geoff Whittle. Excluding a planar graph from GF(q)-representable matroids. *J. Combin. Theory Ser. B*, 97(6):971–998, 2007.

[6] Jim Geelen, Petr Hliněný, and Geoff Whittle. Bridging separations in matroids. *SIAM J. Discrete Math.*, 18(3):638–646 (electronic), 2004/05.

[7] Daniel Král’. Computing representations of matroids of bounded branch-width. In *Proceedings of the 24th annual conference on Theoretical aspects of computer science, STACS’07*, pages 224–235, Berlin, Heidelberg, 2007. Springer-Verlag.

[8] Dániel Marx. A parameterized view on matroid optimization problems. *Theoret. Comput. Sci.*, 410(44):4471–4479, 2009.

[9] J. Oxley. *Matroid Theory, Second Edition*. Oxford University Press, 2011.

[10] J. Oxley, C. Semple, and G. Whittle. Exposing 3-separations in 3-connected matroids. *Adv. in Appl. Math.* Accepted.

[11] Neil Robertson and P. D. Seymour. Graph minors. XX. Wagner’s conjecture. *J. Combin. Theory Ser. B*, 92(2):325–357, 2004.
[12] Neil Robertson and P. D. Seymour. Graph minors. XXI. graphs with unique linkages. *J. Combin. Theory Ser. B*, 99(3):583–616, 2009.

[13] K. Truemper. A decomposition theory for matroids. III. Decomposition conditions. *J. Combin. Theory Ser. B*, 41(3):275–305, 1986.

[14] W. T. Tutte. Menger’s theorem for matroids. *J. Res. Nat. Bur. Standards Sect. B*, 69B:49–53, 1965.