NAKAYAMA AUTOMORPHISMS OF GRADED ORE EXTENSIONS OF KOSZUL ARTIN-SCHELTER REGULAR ALGEBRAS

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Abstract. Let \( A \) be a Koszul Artin-Schelter regular algebra, \( \sigma \) a graded automorphism of \( A \) and \( \delta \) a degree-one \( \sigma \)-derivation of \( A \). We introduce an invariant for \( \delta \) called the \( \sigma \)-divergence of \( \delta \). We describe the Nakayama automorphism of the graded Ore extension \( B = A[z; \sigma, \delta] \) explicitly using the \( \sigma \)-divergence of \( \delta \), and construct a twisted superpotential \( \hat{\omega} \) for \( B \) so that it is a derivation quotient algebra defined by \( \hat{\omega} \). We also determine all graded Ore extensions of noetherian Artin-Schelter regular algebras of dimension 2 and compute their Nakayama automorphisms.

Introduction

To understand Artin-Schelter regular algebras has been a main topic in the study of noncommutative algebras and noncommutative projective geometry since the late 1980s. These algebras are exactly connected graded skew Calabi-Yau algebras ([16]), and possess a kind of automorphisms called Nakayama automorphisms. Such automorphisms are an important tool to study Hopf actions, noncommutative invariant theory and so on ([2, 12, 17, 18]). However, the computation of Nakayama automorphisms is always hard. There is a plenty of work to provide different methods to solve this problem (see [6, 9, 10, 12, 13, 16, 17, 18, 20, 23, 26, 27]).

Obtaining new Artin-Schelter regular algebras from known ones is a common approach, and the methods include Ore extensions, double Ore extensions and regular normal extensions. How the Nakayama automorphisms behave under those extensions is an interesting and concerned problem. The behavior of Nakayama automorphisms under regular normal extensions is studied in [16, 26]. The Nakayama automorphisms of trimmed double Ore extensions of Koszul Artin-Schelter regular algebras are described in [27]. Earlier, Liu, Wang and Wu consider the case of Ore extensions in a general setting ([10]). They prove that if \( A \) is (not necessarily connected graded) skew Calabi-Yau with a Nakayama automorphism \( \mu_A \), then the Ore extension \( B = A[z; \sigma, \delta] \) has a Nakayama automorphism \( \mu_B \) satisfying \( \mu_{BA} = \sigma^{-1} \mu_A \) and \( \mu_B(z) = \lambda z + b \) for some \( \lambda, b \in A \). It is natural to ask what \( \lambda \) and \( b \) are. Restricted on graded Ore extensions, Zhu, Van Oystaeyen and Zhang show \( \lambda = \text{hdet}(\sigma) \) if \( A \) is Koszul Artin-Schelter regular in [27], and Zhou, Lu and the first named author show the same equality if \( A \) is noetherian Artin-Schelter regular generated in degree 1 in [20], where \( \text{hdet} \) is the homological determinant introduced by Jørgensen and Zhang (see [8]).

In fact, \( \lambda \) can be determined by trimmed Ore extensions, namely, the \( \sigma \)-derivation \( \delta \) is trivial, by the filtered-graded technique. However, \( b \) is related to \( \delta \), and seems mysterious for us. Recently, Liu and Ma describe \( \lambda \) and \( b \) explicitly for all (ungraded) Ore extensions if \( A \) is a polynomial algebra in [9]. It inspires

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us to make a progress in the description of the parameter $b$. The main goal of this paper is to describe the Nakayama automorphisms of graded Ore extensions of Koszul Artin-Schelter regular algebras specifically.

Let $A = T(V)/(R)$ be a Koszul Artin-Schelter regular algebra of dimension $d$, $\sigma$ a graded automorphism of $A$ and $\delta$ a degree-one $\sigma$-derivation of $A$. Our idea is to construct a pair $((\delta_{i,j}), [\delta_{i,j}])$ of two sequences of linear maps associated with $\delta$:

$$
\delta_{i,j} : W_i \to W_i \otimes V, \quad \delta_{i,j} : W_i \to V \otimes W_i,
$$

where $W_i = V$ and $W_j = \bigcap_{s=1}^i V^\otimes_s \otimes R / V^{\otimes_{i-2}}$ for any $i \geq 1, j \geq 2$, by the Koszul complex of the graded trivial module $k_A$ (see Lemma 2.2 and Lemma 2.3). In general, sequence pair $((\delta_{i,j}), [\delta_{i,j}])$ is not unique for $\delta$. For each sequence pair $((\delta_{i,j}), [\delta_{i,j}])$ associated with $\delta$, there is a unique pair $(\delta, \delta)$ of elements in $V$ coming from the action of $\delta_{i,j}$ and $\delta_{i,j}$ on $W_i$ which is 1-dimensional. We find that $\delta_{i} + \mu_A^{-1}(\delta_{i})$ is independent on the choices of sequence pairs for $\delta$, where $\mu_A$ is the Nakayama automorphism of $A$ (see Corollary 3.1). We call this invariant the $\sigma$-divergence of $\delta$, denoted by $\nabla_{\sigma} \cdot \delta$. If $A$ is a polynomial algebra and $\sigma$ is the identity map $\text{id}_A$, then $\nabla_{\text{id}_A} \cdot \delta$ is precisely the usual divergence of $\delta$.

It is well known that the graded Ore extension $B = A[z; \sigma, \delta]$ is also a Koszul Artin-Schelter regular algebra of dimension $d + 1$. Using a sequence pair $((\delta_{i,j}), [\delta_{i,j}])$ for $\delta$, we compute the Yoneda product of the Ext-algebra $E(B)$ of $B$. Since $E(B)$ is a graded Frobenius algebra and its Nakayama automorphism is dual to the one of $B$ (see [12, 23]), then we obtain the main result of this paper.

**Theorem 0.1.** (Theorem 3.9) Let $B = A[z; \sigma, \delta]$ be a graded Ore extension of a Koszul Artin-Schelter regular algebra $A$, where $\sigma$ is a graded automorphism of $A$ and $\delta$ is a degree-one $\sigma$-derivation. Then the Nakayama automorphism $\mu_B$ of $B$ satisfies

$$
\mu_B = \sigma^{-1} \mu_A, \quad \mu_B(z) = \text{hdet}(\sigma) z + \nabla_{\sigma} \cdot \delta,
$$

where $\mu_A$ is the Nakayama automorphism of $A$.

It is also well-known that a Koszul Artin-Schelter regular algebra $A$ is determined by a twisted superpotential $\omega$, that is, $A$ is isomorphic to a derivation quotient algebra defined by $\omega$ (see [1, 3]). He, Van Oystaeyen and Zhang constructed a new twisted superpotential for the graded Ore extension $B = A[z; \sigma, \delta]$ in two special cases ([6, 7]). In fact, any sequence pair $((\delta_{i,j}), [\delta_{i,j}])$ for $\delta$ provides us assistance to construct a new twisted superpotential $\hat{\omega}$ for the graded Ore extension $B$ from $\omega$, and it is independent on the choices of sequence pairs. Hence, $B$ is isomorphic to a derivation quotient algebra defined by $\hat{\omega}$ (see Theorem 3.1). As an application of the main result, we determine all graded Ore extensions of noetherian Koszul Artin-Schelter regular algebras of dimension 2 and compute their Nakayama automorphisms. There is an interesting observation about Calabi-Yau property.

**Theorem 0.2.** (Theorem 4.4) Assume the base field $k$ is of characteristic 0. Let $A$ be a noetherian Artin-Schelter regular algebra of dimension 2, and $B$ a graded Ore extension $A[z; \sigma, \delta]$. Write $\mu_A$ for the Nakayama automorphism of $A$.

(a) Suppose $A$ is commutative, then $B$ is Calabi-Yau if and only if $\sigma = \mu_A$, $\delta(x_1) = l_1 x_1^2 - 2l_2 x_2 x_1 + l_3 x_2^2$, and $\delta(x_2) = l_3 x_1^2 - 2l_1 x_2 x_1 + l_4 x_2^2$, for some $l_1, l_2, l_3, l_4 \in k$.

(b) Suppose $A$ is noncommutative, then $B$ is Calabi-Yau if and only if $\sigma = \mu_A$.

The graded automorphism group for a commutative noetherian Artin-Schelter regular algebra of dimension 2 is much bigger than a noncommutative one, as well as the set of $\sigma$-derivations. So there are more
conditions for graded Ore extensions of the commutative one being Calabi-Yau. But it is still surprised that there is no any restriction on $\delta$ for $A[z; \sigma, \delta]$ being a Calabi-Yau algebra in case $A$ is a noncommutative non-nétherian Artin-Schelter regular algebra of dimension 2. It is natural to ask whether this result holds without the nonétherian assumption, or even for all noncommutative Koszul Artin-Schelter regular algebras.

This paper is organized as follows. In Section 1, we recall some definitions and properties, especially for Koszul Artin-Schelter regular algebras. In Section 2, we construct a sequence pair for any $\sigma$-derivation of a Koszul algebra where $\sigma$ is a graded automorphism, and discuss the relations between different sequence pairs. In Section 3, we focus on the Koszul Artin-Schelter regular algebras. We prove there is an invariant for graded Ore extensions of the commutative one being Calabi-Yau. But it is still surprised that there is no any restriction on $\delta$ for $A[z; \sigma, \delta]$ being a Calabi-Yau algebra in case $A$ is a noncommutative non-nétherian Artin-Schelter regular algebra of dimension 2. It is natural to ask whether this result holds without the nonétherian assumption, or even for all noncommutative Koszul Artin-Schelter regular algebras.

Throughout the paper, $k$ is a fixed field. All vector spaces and algebras are over $k$. Unless otherwise stated, the tensor product $\otimes$ means $\otimes_k$.

1. Preliminaries

A graded algebra $A = \oplus_{i \in \mathbb{Z}} A_i$ is called locally finite if $\dim A_i < \infty$ for any $i \in \mathbb{Z}$. A locally finite graded algebra $A$ is called connected if $A_i = 0$ for any $i < 0$ and $A_0 = 1$. In this case, write $e_A$ for the augmentation from $A$ to $k$. Let $M = \oplus_{i \in \mathbb{Z}} M_i$ be a right graded $A$-module. Then $n$-th shift of $M$ is a right graded $A$-module $M(n)$ with the homogenous space $M(n)_i = M_{i+n}$ for all $i \in \mathbb{Z}$. Let $N$ be a graded $A$-bimodule and $\mu$ a graded automorphism of $A$. The graded twisted $A$-bimodule $N^\mu$ is a graded $A$-bimodule with the $A$-action $a \cdot x \cdot b = ax\mu(b)$ for any $a, b \in A, x \in N$.

For a connected graded algebra $A$, there exists a minimal graded free resolution of the right graded trivial module $k_A$ of $A$,

\[
\cdots \to P_2 \overset{\partial_2}{\to} P_1 \overset{\partial_1}{\to} P_0 \overset{\partial_0}{\to} k_A \to 0,
\]

namely, $\ker \partial_i \subseteq P_i A_{i+1}$ for any $i \geq 0$. Then the graded vector space $E(A) = \bigoplus_{i \geq 0} \text{Ext}^i_A(k_A, k_A) = \bigoplus_{i \geq 0} \text{Hom}_A(P_i, k)$ equipped with the Yoneda product is a connected graded algebra, called the Ext-algebra of $A$. In the sequel, We denote the Yoneda product by “$*$”.

Let $V$ be a finite dimensional vector space. Write $\tau : V \otimes V \to V \otimes V$ for the usual twisting map. We adopt the notation in \cite{4} of a sequence of linear endomorphisms of $V^{\otimes d}$ for any $d \geq 2$:

\[
t^\tau_0 = \text{id}_{V^{\otimes d}}, \quad t^\tau_i = (\text{id}_V^{\otimes i-1} \otimes \tau \otimes \text{id}_V^{\otimes d-i-1}) t^{\tau}_{d-1}, \quad \text{for any } 1 \leq i \leq d-1.
\]

**Definition 1.1.** Let $V$ be a finite dimensional vector space and $\nu : V \to V$ an isomorphism of vector spaces. If an element $\omega \in V^{\otimes d}$ for some $d \geq 2$ such that

\[
\omega = (-1)^{d-1} t^{\nu}_{d-1} (\nu \otimes \text{id}_V^{\otimes d-1})(\omega),
\]

then $\omega$ is called a $\nu$-twisted superpotential. In particular, $\omega$ is called a superpotential if $\nu = \text{id}_V$.

Let $\nu$ be a linear automorphism of $V$ and integer $d \geq 2$. Define the partial derivation of a $\nu$-twisted superpotential $\omega \in V^{\otimes d}$ with respect to $\psi \in (V^*)^{\otimes i}$, where $(-)^*$ is the $k$-dual of a space and $1 \leq i \leq d$, to be

\[
\partial_{\psi} \omega = (\text{id}_V^{\otimes d-i} \otimes \psi)(\omega).
\]
Definition 1.2. Let $V$ be a finite dimensional vector space, $\omega \in V^{\otimes d}$ a $\nu$-twisted superpotential for some linear automorphism $\nu$ of $V$ and $d \geq 2$. The $i$-th derivation quotient algebra $R(\omega, i)$ of $\omega$ is
\[
R(\omega, i) = T(V)/(\partial_0(\omega), \psi \in (V^*)^{\otimes i}),
\]
where $T(V)$ is the tensor algebra over $V$.

Let $T(V)$ be the tensor algebra over $V$ with the usual grading and $R$ a subspace of $V^{\otimes 2}$. Then $A = T(V)/(R)$ is a connected graded algebra, called a quadratic algebra. Write $\pi_A$ for the canonical projection from $T(V)$ to $A$.

Definition 1.3. A quadratic algebra $A$ is called Koszul, if the right graded trivial module $k_A$ admits a minimal graded free resolution \([\mathcal{A}]\) such that the right graded $A$-module $P_i$ is generated by degree $i$ for any $i \geq 0$.

Let $A = T(V)/(R)$ be a Koszul algebra. Write $W_0 = k$, $W_1 = V$, $W_2 = R$ and for $i \geq 3$,
\[
W_i = \bigcap_{0 \leq s \leq i-2} V^{\otimes s} \otimes R \otimes V^{\otimes s-2}.
\]

Then the following Koszul complex is a minimal graded free resolution of $k_A$,
\[
\cdots \xrightarrow{\partial_{i+1}^A} W_{d-i} \otimes A \xrightarrow{\partial_i^A} W_{d-i} \otimes A \xrightarrow{\partial_{i+1}^A} \cdots \xrightarrow{\partial_1^A} W_2 \otimes A \xrightarrow{\partial_1^A} W_1 \otimes A \xrightarrow{\partial_1^A} A \xrightarrow{\epsilon_A} k_A \rightarrow 0,
\]
where $\partial_i^A = (id_V^{\otimes i-1} \otimes m_A)(id_V^{\otimes i-1} \otimes \pi_A | V \otimes \text{id}_A) = id_V^{\otimes i-1} \otimes m_A$ and $m_A$ is the multiplication of $A$ for $i \geq 1$.

Remark 1.4. In the sequel, we treat $V$ as the vector space $V$ or the homogeneous space $A_1$ of a Koszul algebra $A$ freely. So we write $m_A$ shortily for the liner map $m_A(\pi_A | V \otimes \text{id}_A) : V \otimes A \rightarrow A$ for convenience.

As vector spaces, the Ext-algebra of $A$ is
\[
E(A) = \bigoplus_{i \geq 0} \text{Hom}_A(W_i \otimes A, k_A) \cong \bigoplus_{i \geq 0} W_i^*.
\]

Remark 1.5. For Koszul algebras, researchers always use their Koszul dual as a common tool, which are also isomorphic to their Ext-algebras. In this paper, we use the language of Ext-algebras to study the Nakayama automorphisms of graded Ore extensions, since it is convenient to obtain some induced maps.

Definition 1.6. A connected graded algebra $C$ is Artin-Schelter regular (AS-regular, for short) of dimension $d$, if it has finite global dimension $d$, $\dim \left( \text{Ext}_C^d(k_C, C_C) \right) = 1$ and $\dim \left( \text{Ext}_C^i(k_C, C_C) \right) = 0$ for $i \neq d$.

AS-regular algebras have an important homological invariant.

Theorem 1.7. \cite{Y4} Proposition 4.5 (b) If $C$ is AS-regular of dimension $d$, then there exists a unique graded automorphism $\mu_C$ of $C$ such that
\[
\text{Ext}_C^i(C, C') \cong \begin{cases} 
0 & \text{if } i \neq d, \\
C^{\mu_C}(l) & \text{if } i = d,
\end{cases}
\]
as graded $C$-bimodules,

where $C' = C \otimes C^{op}$ is the enveloping algebra of $C$ for some $l \geq 0$.

The automorphism $\mu_C$ is called the Nakayama automorphism of $C$. In particular, if the Nakayama automorphism of an AS-regular algebra is the identity map, then it is a Calabi-Yau (CY, for short) algebra (see \cite{Y4} [16]).
The Ext-algebras of AS-regular algebras also carry important information. Let \( E = \oplus_{i \in \mathbb{Z}} E^i \) be a finite dimensional graded algebra. We say \( E \) is a graded Frobenius algebra, if there exists a nondegenerate associative graded bilinear form \( \langle -, - \rangle : E \otimes E \to k(d) \) for some \( d \in \mathbb{Z} \). In particular, there exists a graded automorphism \( \mu_E \) of \( E \) such that
\[
\langle \alpha, \beta \rangle = (-1)^{d/2} \beta \mu_E(\alpha),
\]
for any \( \alpha \in E^i, \beta \in E^j \). We say the automorphism \( \mu_E \) is the (classical) Nakayama automorphism of the graded Frobenius algebra \( E \). (see \cite{21} for details).

In this paper, we mainly consider Koszul AS-regular algebras. We list some important results about Koszul AS-regular algebras below.

**Theorem 1.8.** Let \( A \) be a Koszul AS-regular algebra of dimension \( d \) and \( \mu_A \) the Nakayama automorphism of \( A \).

1. (a) \cite{21} Proposition 5.10] The Ext-algebra \( E(A) \) of \( A \) is a graded Frobenius algebra.
2. (b) \cite{23} Proposition 3] Let \( \mu_E \) be the (classical) Nakayama automorphism of \( E(A) \), then
\[
\mu_{E^1(A)} = (\mu_{A^1})^*,
\]
where \( E^1(A) \) is identified with \( V^* = A_1^\ast \).
3. (c) \cite{6} Lemma 4.3] Any nonzero element \( \omega \in W_d \) is a \( \mu_{A^1V} \)-twisted superpotential.
4. (d) \cite{3} Theorem 11 [\cite{6} Theorem 4.4(i)] For any nonzero element \( \omega \in W_d \), \( A \cong \mathcal{A}(\omega, d - 2) \).

**Remark 1.9.** If \( A \) is a Koszul AS-regular algebra of dimension \( d \), then \( \dim W_d = 1 \). So an element \( \omega \in W_d \) is nonzero is equivalent to it is a basis of \( W_d \).

2. A SEQUENCE PAIR

Let \( V \) be a vector space with a basis \( \{x_1, x_2, \ldots, x_n\} \), \( T(V) \) the tensor algebra over \( V \), and \( A = T(V)/(R) \) a Koszul algebra, where \( R \) is a subspace of \( V^\otimes 2 \). Recall the Koszul complex as follows
\[
\cdots \to W_I \otimes A \xrightarrow{\partial^1} W_{I-1} \otimes A \xrightarrow{\partial^1} \cdots \to W_2 \otimes A \xrightarrow{\partial^1} W_1 \otimes A \xrightarrow{\partial^1} A \xrightarrow{\epsilon} k_A \to 0,
\]
where \( W_0 = k, W_1 = V, W_2 = R \) and \( W_I = \bigcap_{|I| \geq 2} V^\otimes i \otimes R \otimes V^\otimes -i-2 \) for \( i \geq 3 \).

Let \( \sigma \) be a graded automorphism of \( A \) and \( \sigma \) a degree-one \( \sigma \)-derivation of \( A \). Then we have a graded Ore extension \( B = [z; \sigma, \delta] \) with \( \deg z = 1 \). Clearly, \( B \) is a quotient algebra of the tensor algebra \( T(V \otimes k[z]) \) and a Koszul algebra. Write \( \pi_B \) for the canonical projection from \( T(V \otimes k[z]) \) to \( B \).

Write \( \sigma = \sigma_{[V]} \in GL(V) \). It is easy to know that \( \sigma_{[V]}(W_i) = W_i \) for each \( i \geq 0 \) and \( \sigma_T := \oplus_{i \geq 0} \sigma_{[V]}^i \) is a graded automorphism of \( T(V) \) such that \( \sigma_T \) is induced by \( \sigma_T \).

Choose a linear map \( \delta : V \to V \otimes V \) such that \( \pi_B \delta = \delta_{[V]} \). In fact, \( \delta \) extends to a degree-one \( \sigma_T \)-derivation of \( T(V) \) (also denoted by \( \delta \)) in a unique way, and
\[
\delta(R) \subseteq R \otimes V + V \otimes R.
\]
So \( \delta \) can be induced by \( \delta \). All \( \sigma \)-derivations can be obtained in this way.

The condition \cite{22} provides an approach to decomposing \( \delta \) into two parts, that is, there exist two linear maps \( \delta_{2,1} : R \to R \otimes V, \delta_{2,2} : R \to V \otimes R \) such that
\[
\delta_{R} = \delta_{2,1} + \delta_{2,2}.
\]
The decomposition can be realized as follows. One can choose a basis \( \{ r_1, r_2, \cdots, r_t \} \) of \( R \), obtain that
\[
(2.4) \quad \delta(r_i) = \sum_{j=1}^{t} r_j \otimes \alpha_j + \sum_{j=1}^{t} \beta_j \otimes r_j \in R \otimes V + V \otimes R,
\]
for some \( \alpha_j, \beta_j \in V \) and \( j, i = 1, \cdots, t \), and then define
\[
\delta_{2,r}(r_i) = \sum_{j=1}^{t} r_j \otimes \alpha_j, \quad \delta_{2,l}(r_i) = \sum_{j=1}^{t} \beta_j \otimes r_j, \quad \forall i = 1, \cdots, t.
\]

**Remark 2.1.** It is clear that the choice of \( \delta_{2,r} \) and \( \delta_{2,l} \) for \( \delta \) is not unique, which depends on the decomposition \((2.4)\).

2.1. **Minimal free resolutions.** Now we begin to construct a minimal free resolution of \( k_B \). In the sequel, write \( \delta_{1,r} = \delta_{1,l} = \delta_V \), and \( \lambda_c \) (resp. \( \rho_c \)) for the left (resp. right) multiplication of \( z \) on \( B \).

Applying \(- \otimes_A B\) to \((2.1)\), one obtains an exact sequence,
\[
\cdots \xrightarrow{\partial_{d+1}} W_d \otimes B \xrightarrow{\partial_d} W_{d-1} \otimes B \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_{2}} W_2 \otimes B \xrightarrow{\partial_1} W_1 \otimes B \xrightarrow{\partial_0} B \xrightarrow{\epsilon_{B} \otimes_A B} B/A_{\geq 1}B \rightarrow 0,
\]
where \( \partial_1 = \partial_1^B \otimes_A B = (\text{id}_V)_{B} \otimes m_B)(\text{id}_V)_{B} \otimes \pi_B \otimes \text{id}_B) = \text{id}_V \otimes m_B \) and \( m_B \) is the multiplication of \( B \) for \( i \geq 1 \).

**Lemma 2.2.** There exist graded linear maps \( \delta_{i,r} : W_i \rightarrow W_i \otimes V \) for \( i \geq 2 \) such that the following diagram is commutative
\[
\begin{array}{c}
\cdots \xrightarrow{\partial_{d+1}} W_d \otimes B(1) \xrightarrow{\partial_d} W_{d-1} \otimes B(1) \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_{2}} W_2 \otimes B(1) \xrightarrow{\partial_1} W_1 \otimes B(1) \xrightarrow{\partial_0} B(1) \xrightarrow{\epsilon_{B} \otimes_A B} B(1)/A_{\geq 1}B(1) \rightarrow 0 \\
\cdots \xrightarrow{\partial_{d+1}^B} W_d \otimes B \xrightarrow{\partial_d^B} W_{d-1} \otimes B \xrightarrow{\partial_{d-1}^B} \cdots \xrightarrow{\partial_{2}^B} W_2 \otimes B \xrightarrow{\partial_1^B} W_1 \otimes B \xrightarrow{\partial_0^B} B \xrightarrow{\epsilon_{B} \otimes_A B} B/A_{\geq 1}B \rightarrow 0,
\end{array}
\]
where graded right \( B \)-module homomorphisms
\[
\phi_0 = \lambda_c, \quad (id_V \otimes m_B)\delta_{i,r} = (id_V \otimes m_B)(\delta_{i,r} \otimes \text{id}_B), \quad i \geq 1.
\]
Moreover, as linear maps \( W_i \rightarrow W_{i-1} \otimes B \),
\[
(2.5) \quad (\text{id}_V \otimes m_B)\delta_{i,r} = (\text{id}_V \otimes m_B)(\sigma \otimes \delta + \delta_{i-1,r} \otimes \text{id}_V), \quad i \geq 2.
\]

**Proof.** It’s easy to check that \( \lambda_c(\text{id}_A \otimes \text{id}_B) = (\text{id}_A \otimes \text{id}_B)\phi_0 \) and \( \phi_0 \partial_1 = \partial_1 \phi_1 \), since
\[
\phi_0 \partial_1 = \lambda_c m_B = m_B(\sigma \otimes \lambda_c + (\pi_B \otimes \text{id}_B)(\delta \otimes \text{id}_B)) = m_B(\sigma \otimes \lambda_c) + m_B(\delta \otimes \text{id}_B),
\]
\[
\partial_1 \phi_1 = m_B(\sigma \otimes \lambda_c) + m_B(id_V \otimes m_B)(\delta \otimes \text{id}_B) = m_B(\sigma \otimes \lambda_c) + m_B(\delta \otimes \text{id}_B).
\]
By the construction of \( \delta_{2,r} \) and \( \delta_{2,l} \), we have
\[
(id_V \otimes m_B)(\sigma \otimes \delta + \delta \otimes id_V) = (id_V \otimes m_B)(\delta_{2,r} + \delta_{2,l}) = (id_V \otimes m_B)\delta_{2,r},
\]
in case restricted on \( W_2 = R \). One obtains
\[
\partial_2 \phi_2 = (id_V \otimes m_B)(\sigma \otimes \lambda_c) + (id_V \otimes m_B)(id_V \otimes m_B)(\delta_{2,r} \otimes \text{id}_B)
\]
\[
= (id_V \otimes m_B)(\sigma \otimes \lambda_c) + (id_V \otimes m_B)(id_V \otimes m_B)(\delta_{2,r} \otimes \text{id}_B + \delta_{2,l} \otimes \text{id}_B)
\]
\[
= (id_V \otimes m_B)(\sigma \otimes \lambda_c) + (id_V \otimes m_B)(id_V \otimes m_B)(\delta \otimes \text{id}_B),
\]
\[
\phi_i\partial_2 = (\sigma \otimes \lambda_i)(\text{id}_V \otimes m_B) + (\text{id}_V \otimes m_B)(\delta \otimes \text{id}_B)\text{id}_V \otimes m_B) \\
= (\text{id}_V \otimes m_B)(\sigma^{\otimes 2} \otimes \lambda_i) + (\text{id}_V \otimes m_B)(\text{id}_V^{\otimes 2} \otimes m_B)\left((\sigma \otimes \delta) \otimes \text{id}_B + (\delta \otimes \text{id}_V) \otimes \text{id}_B\right) \\
= (\text{id}_V \otimes m_B)(\sigma^{\otimes 2} \otimes \lambda_i) + (\text{id}_V \otimes m_B)(\text{id}_V^{\otimes 2} \otimes m_B)(\delta \otimes \text{id}_B).
\]

So \(\partial_2\phi_2 = \phi_1\partial_2\). By the Comparison Theorem, there exists a graded \(B\)-module homomorphism \(\phi_i : W_i \otimes B(-1) \to W_i \otimes B\) such that \(\phi_i\partial_{i+1} = \partial_i\phi_{i+1}\) for any \(i \geq 3\).

It is clear that \((\phi_i - \sigma^{\otimes i} \otimes \lambda_i)(W_i \otimes B_0) \subseteq W_i \otimes V\), so write \(\delta_{i,r}\) for the linear map \(\phi_i - \sigma^{\otimes i} \otimes \lambda_i\) restricted on \(W_i\) for any \(i \geq 3\). One obtains that
\[
\phi_i = \sigma^{\otimes i} \otimes \lambda_i + (\text{id}_V^{\otimes i} \otimes m_B)(\delta_{i,r} \otimes \text{id}_B), \quad i \geq 3.
\]

Then \(\partial_i\phi_i = (\text{id}_V^{\otimes i-1} \otimes m_B)(\sigma^{\otimes i} \otimes \lambda_i) + (\text{id}_V^{\otimes i-1} \otimes m_B)(\text{id}_V^{\otimes i} \otimes m_B)(\delta_{i,r} \otimes \text{id}_B)\), and
\[
\phi_{i-1}\partial_i = (\sigma^{\otimes i-1} \otimes \lambda_i)(\text{id}_V^{\otimes i-1} \otimes m_B) + (\text{id}_V^{\otimes i-1} \otimes m_B)(\delta_{i-1,r} \otimes \text{id}_B)
\]
\[
= (\text{id}_V^{\otimes i-1} \otimes m_B)(\delta^{\otimes i} \otimes \lambda_i) + (\text{id}_V^{\otimes i-1} \otimes m_B)(\text{id}_V^{\otimes i} \otimes m_B)(\sigma^{\otimes i-1} \otimes \delta \otimes \text{id}_B)
\]
\[
+ (\text{id}_V^{\otimes i-1} \otimes m_B)(\text{id}_V^{\otimes i} \otimes m_B)(\delta_{i-1,r} \otimes \text{id}_V \otimes \text{id}_B).
\]

Since \(\partial_i\phi_i = \phi_{i-1}\partial_i\), as linear maps \(W_i \to W_{i-1} \otimes B\),
\[
(\text{id}_V^{\otimes i-1} \otimes m_B)\delta_{i,r} = (\text{id}_V^{\otimes i-1} \otimes m_B)(\sigma^{\otimes i-1} \otimes \delta + \delta_{i-1,r} \otimes \text{id}_V), \quad \forall i \geq 3.
\]

**Remark 2.3.** By the proof of Lemma 2.2, any linear map from \(W_i\) to \(W_i \otimes V\) satisfying (2.5) can be chosen to be \(\delta_{i,r}\) for \(i \geq 2\). So the sequence \(\{\delta_{i,r} \mid i \geq 1\}\) of linear maps is not unique for the map \(\delta\), even when \(\delta_{2,r}\) is fixed.

By [5] Theorem 1] or [15] Lemma 2.4, one obtains a minimal free resolution of \(k_B\).

**Lemma 2.4.** The following complex is exact:
\[
\cdots \to W_d \otimes B(-1) \otimes W_{d+1} \otimes B \overset{\phi_{d+1}}{\to} W_{d-1} \otimes B(-1) \otimes W_d \otimes B \to \cdots
\]
\[
\overset{\phi_{d}}{\to} W_1 \otimes B(-1) \otimes W_2 \otimes B \overset{\phi_2}{\to} B(-1) \otimes W_1 \otimes B \overset{\phi_1}{\to} B \overset{e_B}{\to} k_B \to 0.
\]

2.2. A sequence pair. We have constructed a sequence \(\{\delta_{i,r} \mid i \geq 2\}\) of linear maps from \(\delta\) to obtain a minimal free resolution of \(k_B\). It seems a right version of linear maps arose from \(\delta\). Symmetrically, we construct a left version of linear maps. Write \(\delta_{0,r} = \delta_{0,i} = 0\).

**Lemma 2.5.** Let \(\{\delta_{i,r} : W_i \to W_i \otimes V \mid i \geq 1\}\) be a sequence of linear maps as in Lemma 2.2. Then there exists a unique set \(\{\delta_{i,j} : W_i \to V \otimes W_j \mid i \geq 1\}\) of linear maps with respect to \(\{\delta_{i,r}\}\) such that
\[
\delta_{i,r} + (-1)^{i} \delta_{i-1,r} = \sigma \otimes \delta_{i-1,r} + (-1)^{i} \delta_{i-1,j} \otimes \text{id}_V, \quad \forall i \geq 1.
\]

**Proof.** Since \(\delta_{1,r} = \delta_{1,j}\), the result holds if \(i = 1\). By the definition (2.3) of \(\delta_{2,r}\), it is a linear map from \(W_2\) to \(V \otimes W_2\). Clearly,
\[
\delta_{2,r} - \delta_{1,j} \otimes \text{id}_V - \sigma \otimes \delta_{1,r} = \delta_{2,r} - \delta \otimes \text{id}_V - \sigma \otimes \delta = \delta_{2,r} - \delta = -\delta_{2,l}.
\]
Suppose $i \geq 3$ and there exist linear maps $\delta_{ij} : W_j \to V \otimes W_j$ for $j < i$, such that

$$\delta_{ij} + (-1)^i \delta_{ij} = \sigma \otimes \delta_{i-1,j} + (-1)^i \delta_{i-1,j} \otimes \text{id}_V.$$  

Then we have the following commutative diagram

$$\begin{array}{ccc}
W_i \otimes B(-1) & \xrightarrow{\partial_i} & V \otimes W_i \otimes B \\
\downarrow{\pi_i} & & \downarrow{\theta_i} \\
V \otimes W_{i-1} \otimes B & \xrightarrow{(\text{id}_V \otimes \delta_{i-1})B} & V \otimes W_{i-2} \otimes B,
\end{array}$$

where $\theta_i = (\text{id}_V^{\otimes i} \otimes m_B)\left((-1)^{i-1}(\delta_{i-1} - \sigma \otimes \delta_{i-1} - \sigma + (-1)^{i-1}(-1)^{i-1}i \otimes \text{id}_V) \otimes \text{id}_B\right)$. In fact,

\begin{align*}
(-1)^{i-1}(\text{id}_V \otimes \delta_{i-1})\theta_i &= (\text{id}_V^{\otimes i} \otimes m_B)(\text{id}_V^{\otimes i} \otimes m_B)\left((-1)^{i-1}(\delta_{i-1} - \sigma \otimes \delta_{i-1} + (-1)^{i-1}(-1)^{i-1}i \otimes \text{id}_V) \otimes \text{id}_B\right) \\
&= (\text{id}_V^{\otimes i} \otimes m_B)(\text{id}_V^{\otimes i} \otimes m_B)\left((-1)^{i-1}(\delta_{i-1} + (-1)^{i-1}(-1)^{i-1}i \otimes \text{id}_V) \otimes \text{id}_B\right) \\
&= (\text{id}_V^{\otimes i} \otimes m_B)(\text{id}_V^{\otimes i} \otimes m_B)\left((-1)^{i-1}(\delta_{i-1} + (-1)^{i-1}(-1)^{i-1}i \otimes \text{id}_V) \otimes \text{id}_B\right) + (-1)^{i-1}(\text{id}_V^{\otimes i} \otimes m_B)(\text{id}_V^{\otimes i} \otimes m_B)(\delta_{i-1} + (-1)^{i-1}(-1)^{i-1}i \otimes \text{id}_V) \otimes \text{id}_B \\
&= (\text{id}_V^{\otimes i} \otimes m_B)(\text{id}_V^{\otimes i} \otimes m_B)\left((-1)^{i-1}(\delta_{i-1} + (-1)^{i-1}(-1)^{i-1}i \otimes \text{id}_V) \otimes \text{id}_B\right) + (-1)^{i-1}(\text{id}_V^{\otimes i} \otimes m_B)(\text{id}_V^{\otimes i} \otimes m_B)(\delta_{i-1} + (-1)^{i-1}(-1)^{i-1}i \otimes \text{id}_V) \otimes \text{id}_B \\
&= 0,
\end{align*}

the second and sixth equalities hold by Lemma 2.2, the fourth equality holds by the assumption and the fifth equality holds by $W_i \subseteq V^{\otimes i} \otimes R$. So $\text{Im} \theta_i \subseteq \text{Ker}(\text{id}_V \otimes \delta_{i-1}) = \text{Im}(\text{id}_V \otimes \delta_i)$, and there exists a graded $B$-module homomorphism $\overline{\theta}_i : W_i \otimes B(-1) \to V \otimes W_i \otimes B$ such that $(\text{id}_V \otimes \delta_i)\overline{\theta}_i = \theta_i$, since $W_i \otimes B$ is free.

It is easy to know that $\overline{\theta}_i|_{W_i}$ is a map from $W_i$ to $V \otimes W_i$ by an argument on degree, denoted this map by $\delta_{i,j}$. Hence,

$$\delta_{i,j} + (-1)^j \delta_{i,j} = \sigma \otimes \delta_{i-1,j} + (-1)^j \delta_{i-1,j} \otimes \text{id}_V.$$  

The uniqueness can be obtained by the construction easily. \hfill \Box

**Definition 2.6.** Let $[\delta_{i,j}] : W_i \to W_i \otimes V$ for $i \geq 1$ be a sequence of linear maps constructed in Lemma 2.2 and $[\delta_{i,j}] : W_i \to V \otimes W_i$ for $i \geq 1$ a sequence of linear maps constructed in Lemma 2.5. Then $([\delta_{i,j}],[\delta_{i,j}])$ is called a sequence pair for the $\sigma$-derivation $\delta$.

We give a relation between $[\delta_{i,j}]$ and $[\delta_{i,j}]$, which will be useful in the construction of twisted superpotentials for graded Ore extensions.

**Proposition 2.7.** Let $([\delta_{i,j}],[\delta_{i,j}])$ be a sequence pair for $\delta$. Then

(a) For any integer $d \geq 1$,

$$\sum_{i=1}^{d} (-1)^i \delta_{i,j} \otimes \text{id}_V^{\otimes i} = (-1)^d i \sum_{i=1}^{d} (-1)^i (\sigma^{\otimes d - i} \otimes \delta_{i,j}).$$

(b) For any integer $d \geq 2$,

$$\sum_{i=1}^{d-1} (-1)^i (\sigma^{\otimes d - i - 1} \otimes \delta_{i,j} \otimes \text{id}_V) = (-1)^d i \sum_{i=1}^{d-1} (-1)^i (\delta_{i,j} \otimes \text{id}_V^{\otimes d-i}).$$
Proposition 2.8. Let $\delta$, $\delta'$ be two linear maps from $V$ to $V \otimes V$ such that $\pi_A \delta = \pi_A \delta' = \delta_V$, and $\left(\delta_{\iota_{\sigma_1}^i}, \delta_{\iota_{\sigma_1}^i}\right)$ and $\left(\delta_{\iota_{\sigma_1}^i}, \delta'_{\iota_{\sigma_1}^i}\right)$ two sequence pairs for $\delta$ constructed from $\delta$ and $\delta'$ respectively. Then for any $i \geq 1$,

(a) $\text{Im} \left(\sum_{j=0}^{i-1} (-1)^j \left(\delta_{\iota_{\sigma_1}^j} - \delta'_{\iota_{\sigma_1}^j} \otimes \mathfrak{id}_V\right)\right) \subseteq W_{i+1}$. 

Proof: (a) If $d = 1$, the equality is clear. Assume $d \geq 2$. By Lemma 2.6,

\[
\begin{align*}
\sum_{j=0}^{d-2} \sum_{i=1}^{d} (-1)^i & \left(\alpha_{\delta^j} \otimes \delta_{\iota_{\sigma_1}^i} \otimes \mathfrak{id}_V\right) \\
= \sum_{j=1}^{d} (-1)^j & \left(\alpha_{\delta^j} \otimes \delta_{\iota_{\sigma_1}^j} \otimes \mathfrak{id}_V\right) + \sum_{i=2}^{d} (-1)^i \sum_{j=0}^{d-2} \sum_{i=1}^{d} \alpha_{\delta^j} \otimes \delta_{\iota_{\sigma_1}^i} \otimes \mathfrak{id}_V \\
= \sum_{j=1}^{d} (-1)^j & \left(\alpha_{\delta^{j+1}} \otimes \delta_{\iota_{\sigma_1}^{j+1}} \otimes \mathfrak{id}_V\right) - \sum_{j=0}^{d-2} \sum_{i=1}^{d} \alpha_{\delta^j} \otimes \delta_{\iota_{\sigma_1}^i} \otimes \mathfrak{id}_V \\
= \sum_{j=0}^{d-2} \sum_{i=1}^{d} (-1)^i & \left(\delta_{\iota_{\sigma_1}^i} \otimes \mathfrak{id}_V\right) \\
\text{The result follows.} \\
\end{align*}
\]

(b) By Lemma 2.8,

\[
\sum_{j=0}^{d-2} \sum_{i=1}^{d} (-1)^i \left(\delta_{\iota_{\sigma_1}^i} \otimes \mathfrak{id}_V\right) + \sum_{i=1}^{d-1} \alpha_{\delta^{i-1}} \otimes \delta_{\iota_{\sigma_1}^i} = \sum_{i=1}^{d-1} \alpha_{\delta^{i-1}} \otimes \delta_{\iota_{\sigma_1}^i} + \sum_{i=1}^{d-1} (-1)^i \left(\alpha_{\delta^{i+1}} \otimes \delta_{\iota_{\sigma_1}^{i+1}}\right).
\]

Then

\[
\begin{align*}
\sum_{i=1}^{d-1} (-1)^i & \left(\delta_{\iota_{\sigma_1}^i} \otimes \mathfrak{id}_V\right) = \delta_{\iota_{\sigma_1}^{d+1}} - \sum_{i=1}^{d-1} \alpha_{\delta^{i-1}} \otimes \delta_{\iota_{\sigma_1}^i} + \sum_{i=1}^{d-1} (-1)^i \left(\alpha_{\delta^{i+1}} \otimes \delta_{\iota_{\sigma_1}^{i+1}}\right) \\
\end{align*}
\]

where the second equality holds by (a). \(\square\)

As shown above, a sequence pair $\left(\left(\delta_{\iota_{\sigma_1}^i}, \left(\delta_{\iota_{\sigma_1}^i}\right)\right)\right)$ is constructed from a linear map $\delta : V \to V \otimes V$ which induces the $\sigma$-derivation $\sigma$ in the graded Ore extensions $B$. It is also shown that such sequence pairs vary according to the decomposition $\left(\sigma_1\right)$ and the choices of $\left(\delta_{\iota_{\sigma_1}^i}\right)$ in Lemma 2.2. On the other hand, the $\sigma$-derivation $\delta$ can be induced from different linear maps $\delta, \delta' : V \to V \otimes V$, which also arise different sequence pairs for $\sigma$. Here, we give a relation between different sequence pairs.
(b) \( \text{Im} \left( (\delta_{i,j} - \delta'_{i,j}) + (-1)^j \sum_{j=0}^{i-1} (-1)^j (\sigma \otimes (\delta_{j-1,j} - \delta'_{j-1,j}) \otimes \text{id}_V^{\otimes j-1}) \right) \subseteq W_{i+1} \).

Moreover, if \( W_{d+1} = 0 \) for some \( d \geq 1 \), then
\[
\sum_{j=0}^{d-1} (-1)^j (\delta_{d-j,j} \otimes \text{id}_V^j) = \sum_{j=0}^{d-1} (-1)^j (\delta'_{d-j,j} \otimes \text{id}_V^j),
\]
\[
\sum_{j=0}^{d-1} (-1)^j (\sigma \otimes \delta_{d-j,j}) = \sum_{j=0}^{d-1} (-1)^j (\sigma \otimes \delta'_{d-j,j}),
\]
\[
(-1)^d \delta_{d,j} + \sum_{j=1}^{d-1} (-1)^j (\sigma \otimes \delta_{d-j,j} \otimes \text{id}_V^{\otimes j-1}) = (-1)^d \delta'_{d,j} + \sum_{j=1}^{d-1} (-1)^j (\sigma \otimes \delta'_{d-j,j} \otimes \text{id}_V^{\otimes j-1}).
\]

**Proof.** (a) Let \( \{ \phi_i \} \) and \( \{ \phi'_i \} \) be two lifts of \( A : B/A_{\geq 1}B(-1) \to B/A_{\geq 1}B \) as in Lemma 2.2 associated with \( \{ \delta_{i,j} \} \) and \( \{ \delta'_{i,j} \} \) respectively. Clearly, \( \phi_0 = \phi'_0 \), and \( \phi_i - \phi'_i = (\text{id}_V^0 \otimes \text{sm}_B) \left( (\delta_{i,j} - \delta'_{i,j}) \otimes \text{id}_V \right) \) for any \( i \geq 1 \). There exist graded \( B \)-module homomorphisms \( s_i : W_i \otimes B(-1) \to W_{i+1} \otimes B \) for \( i \geq 1 \) such that the following diagram is commutative
\[
\begin{array}{cccccccc}
\cdots & W_i \otimes B(-1) & \downarrow \delta_{i-1} & W_{i-1} \otimes B(-1) & \downarrow \delta_{i-2} & \cdots & W_1 \otimes B(-1) & \downarrow \delta_{1} & B(-1) \\
& \cdots & \downarrow s_{i} & \cdots & \downarrow s_{i-1} & \cdots & \downarrow s_{1} & \downarrow s_{1} & 0 \\
\cdots & W_i \otimes B & \downarrow \sigma \downarrow & \cdots & \downarrow \sigma \downarrow & \cdots & \downarrow \sigma \downarrow & \downarrow \sigma & 0 \\
& \cdots & \downarrow \delta_{i-1} & \cdots & \downarrow \delta_{i-2} & \cdots & \downarrow \delta_{1} & \downarrow \delta_{1} & B
\end{array}
\]

that is, \( \phi_i - \phi'_i = s_{i-1} \delta_{i} + \delta_{i-1} s_i \), for any \( i \geq 1 \), where \( s_0 = 0 \). By an easy argument on degree, one obtains that for any \( i \geq 1 \), \( \text{Im} s_{i|W_i} \subseteq W_{i+1} \) and
\[
s_{i|W_i} = \delta_{i,j} - \delta'_{i,j} - s_{i-1|W_{i-1}} \otimes \text{id}_V.
\]

Since \( s_{1|W_1} = \delta_{1,j} - \delta'_{1,j} \), we have
\[
s_{i|W_i} = \sum_{j=0}^{i-1} (-1)^j \left( (\delta_{i,j} - \delta'_{i,j}) \otimes \text{id}_V^{\otimes j} \right).
\]

Then the result follows.

(b) By Lemma 2.3, we have
\[
\delta_{j,r} + (-1)^j \delta_{j,0} = \sigma \otimes \delta_{j-1,r} + (-1)^j \delta_{j-1,j} \otimes \text{id}_V, \quad \delta'_{j,r} + (-1)^j \delta'_{j,0} = \sigma \otimes \delta'_{j-1,r} + (-1)^j \delta'_{j-1,j} \otimes \text{id}_V.
\]

for any \( j \geq 1 \). So
\[
(\delta_{j,r} - \delta'_{j,r}) \otimes \text{id}_V^{\otimes j} - \sigma \otimes (\delta_{j-1,r} - \delta'_{j-1,r}) \otimes \text{id}_V^{\otimes j} = (-1)^j (\delta_{j-1,j} - \delta'_{j-1,j}) \otimes \text{id}_V^{\otimes j-1} - (-1)^j \delta_{j,r} - \delta'_{j,0} \otimes \text{id}_V^{\otimes j-1},
\]

for any \( 1 \leq j \leq i \), where \( \delta_{0,r} = \delta_{0,0} = \delta'_{0,0} = 0 \). Then
\[
\sum_{j=1}^{i} (-1)^{i-j} (\delta_{j,r} - \delta'_{j,r}) \otimes \text{id}_V^{\otimes j} - \sum_{j=1}^{i} (-1)^{i-j} \sigma \otimes (\delta_{j-1,r} - \delta'_{j-1,r}) \otimes \text{id}_V^{\otimes j-1}
\]
\[
= \sum_{j=1}^{i} (-1)^{i-j} (\delta_{j-1,j} - \delta'_{j-1,j}) \otimes \text{id}_V^{\otimes j-1} - \sum_{j=1}^{i} (-1)^{j} (\delta_{j,r} - \delta'_{j,0}) \otimes \text{id}_V^{\otimes j-1}
\]
\[
= (-1)^{i+1} (\delta_{i,i} - \delta'_{i,i}).
\]
Equivalently,
\[
\sum_{j=0}^{i-1} (-1)^j (\delta_{d-jr} - \delta'_{d-jr}) \otimes \id_V^j = (-1)^i (\delta_{d,i} - \delta'_{d,i}) - \sum_{j=1}^{i-1} (-1)^j \sigma \otimes (\delta_{d-jr} - \delta'_{d-jr}) \otimes \id_V^{j-1}.
\]
Then the results holds by (a).

The last consequence is an immediate consequence of (a, b) and Proposition 2.7(a).

3. Nakayama automorphisms of graded Ore extensions of Koszul AS-regular algebras

Keep the notations in the last section. In this section, we always assume \( A = T(V)/(R) \) is a Koszul AS-regular algebra of dimension \( d \), where \( V \) is a vector space with a basis \( \{x_1, x_2, \ldots, x_n\} \). Write \( \mu_A \) for the Nakayama automorphism of \( A \). In this case, by [22, Proposition 3.1.4], one obtains
\[
\dim W_j = \begin{cases} 
\dim W_{d-i} & \text{if } 0 \leq i \leq d. \\
0 & \text{if } i > d.
\end{cases}
\]
Write \( \{\eta_1, \eta_2, \ldots, \eta_n\} \) for a basis of \( W_{d-1} \) and \( \omega \) for a basis of \( W_d \).

It is well known that the graded Ore extension \( B = A[z; \overline{\sigma}, \overline{\delta}] \) is a Koszul AS-regular algebra of dimension \( d + 1 \) provided \( \overline{\delta} \) is a graded automorphism of \( A \). Write \( \sigma = \overline{\sigma} \). The minimal free resolution (2.6) of \( k_B \) becomes

\[
\begin{align*}
0 & \to W_d \otimes B(-1) \to W_{d-1} \otimes B(-1) \oplus W_d \otimes B \to \\
& \to W_1 \otimes B(-1) \oplus W_2 \otimes B \to B(-1) \oplus W_1 \otimes B \to B \to k_B \to 0.
\end{align*}
\]

3.1. An invariant. Let \( ([\delta_{ij}], [\delta_{ij}]) \) be a sequence pair for \( \overline{\delta} \). Since \( \dim W_d = 1 \), there exists a unique pair \( (\delta_r, \delta_l) \) of elements in \( V \) with respect to \( ([\delta_{ij}], [\delta_{ij}]) \) such that
\[
\delta_{d,i}(\omega) = \omega \otimes \delta_r, \quad \delta_{d,i}(\omega) = \delta_l \otimes \omega.
\]

Corollary 3.1. Let \( (\delta_r, \delta_l) \) and \( (\delta'_r, \delta'_l) \) be two pairs with respect to two sequence pairs \( ([\delta_{ij}], [\delta_{ij}]) \) and \( ([\delta'_{ij}], [\delta'_{ij}]) \) for \( \overline{\delta} \), respectively. Then
\[
\delta_r + \mu_A \sigma^{-1}(\delta_l) = \delta'_r + \mu_A \sigma^{-1}(\delta'_l).
\]

Proof. Firstly, one obtains that
\[
\tau^{d-1}_d(\mu_A \sigma^{-1} \otimes \id_V^d)(\delta_{d,l} - \delta'_l)(\omega) = \omega \otimes \left( \mu_A \sigma^{-1}(\delta_l - \delta'_l) \right).
\]
On the other hand,
\[
\tau^{d-1}_d(\mu_A \sigma^{-1} \otimes \id_V^d)(-1)^{d-1} \sum_{j=1}^{d-1} (-1)^j \left( \sigma \otimes (\delta_{d-jr} - \delta'_{d-jr}) \otimes \id_V^{j-1} \right)(\omega)
\]
\[
= \tau^{d-1}_d \left( -1 \right)^{d-1} \sum_{j=1}^{d-1} (-1)^j \left( \mu_A \otimes (\delta_{d-jr} - \delta'_{d-jr}) \otimes \id_V^{j-1} \right)(\omega)
\]
where the third equality holds by $W_{d+1} = 0$, Proposition 2.8 and Theorem 1.8(e). Then the result follows by Proposition 2.8 again.

**Definition 3.2.** Let $A = T(V)/(R)$ be a Koszul AS-regular algebra, $\sigma$ a graded automorphism of $A$ and $\delta$ a degree-one $\sigma$-derivation of $A$. Let $(\delta_i, \delta_j)$ be the pair of elements in $V$ with respect to some sequence pair $((\delta_{i,j}), (\delta_{i,j}))$ for the $\sigma$. Then the element $\omega + \mu_i \sigma^{-1}(\delta_i)$ is called the $\sigma$-divergence of $\delta$, denoted by $\nabla_{\sigma} \cdot \delta$.

**Remark 3.3.** If $A$ is a commutative graded polynomial algebra generated in degree 1 and $\sigma$ is the identity map, $\nabla_{\sigma} \cdot \delta$ is the usual divergence $\nabla \cdot \delta$ of $\delta$ (see Theorem 4.1 or [9, Theorem 1.1(1)]). It motivates the name “$\sigma$-divergence” of a $\sigma$-derivation $\delta$.

### 3.2. Ext-algebras

In this subsection, we compute the Yoneda product of Ext-algebras $E(B)$ partially. We refer [8] for the definition of homological determinant $hdet(\sigma)$ of a graded automorphism $\sigma$.

Following the minimal free resolution \(\mathbb{L}^1\) of $k_B$, one obtains that

\[
\begin{align*}
E^1(B) &= \text{Hom}_k(B(-1) \oplus W_1 \otimes B, k) \cong k(1) \oplus W_1^*, \\
E^d(B) &= \text{Hom}_k(W_{d-1} \oplus B(-1) \oplus W_d \otimes B, k) \cong W_{d-1}^*(1) \oplus W_d^*, \\
E^{d+1}(B) &= \text{Hom}_k(W_d \otimes B(-1), k) \cong W_d^*(1).
\end{align*}
\]

Then $E^1(B)$ has a basis $\xi, x_1, x_2, \ldots, x_n$, where $\xi$ corresponds to the identity of $k$ (or the augmentation map $e_B : B(-1) \to k(1)$), $E^1(B)$ has a basis $\eta_1, \eta_2, \ldots, \eta_n, \omega^*$, and $E^{d+1}(B)$ has a basis $\omega^*$ corresponding to the element $\omega^*$ in $W_d$.

**Remark 3.4.** In order to make notations succinct, we use the basis of $W_1, W_{d-1}, W_d$ to represent the basis of $E^1(B), E^d(B), E^{d+1}(B)$. However, it should keep in mind that each element in such basis also has a corresponding homomorphism through the minimal free resolution. To be specific, each $x_i^*$ corresponds to $x_i^* \otimes e_B : W_1 \otimes B \to k(-1)$, $\eta_i^*$ corresponds to $\eta_i^* \otimes e_B : W_{d-1} \otimes B(-1) \to k(-d)$, $\omega^*$ corresponds to $\omega^* \otimes e_B : W_d \otimes B \to k(-d)$ and $\omega^*$ corresponds to $\omega^* \otimes e_B : W_d \otimes B(-1) \to k(-d-1)$ for $i = 1, \ldots, n$. In the sequel, we use such correspondence freely.

Define a graded algebra homomorphism

\[
p_\ast : B \to k[z], \quad \sum_{i=1}^m a_i z^i \mapsto \sum_{i=1}^m e_A(a_i) z^i.
\]

By [19] Theorem 1, $E(p_\ast)$ is a graded algebra homomorphism from $E(k[z])$ to $E(B)$. Notice that,

\[
0 \to k[z] \otimes k[z] \xrightarrow{h} k[z] \xrightarrow{e_{k[z]}} k[k[z]] \to 0,
\]

where $k[z]$ is the vector space spanned by $z$ and the right graded $k[z]$-module homomorphism $h$ mapping $z \otimes 1$ to $z$, is a minimal free resolution of the graded trivial module $k[k[z]]$. So

\[
E^1(k[z]) = \text{Hom}_{k[z]}(k[z] \otimes k[z], k) = (k[z])^*.
\]

and $z^\ast$ (or $z^\ast \otimes e_{k[z]}$) is a basis of $E^1(k[z])$. 

Lemma 3.5. $E(p_2)(z^*) = \xi$.

Proof. Clearly, we have the following commutative diagram

\[
\begin{array}{ccccccccc}
W_1 \otimes B(-1) \oplus W_2 \otimes B & \longrightarrow & B(-1) \oplus W_1 \otimes B & \longrightarrow & B & \longrightarrow & k_B & \longrightarrow & 0 \\
0 & \downarrow f & 0 & \longrightarrow & k[z] \otimes k[z] & \longrightarrow & k[z] & \longrightarrow & 0,
\end{array}
\]

where $f(b, w_1 \otimes b') = z \otimes p_1(b)$ for any $b, b' \in B$ and $w_1 \in W_1$. By [13] Theorem 1, one obtains that

\[ E(p_2)(z^*)(b, w_1 \otimes b') = (z^* \otimes \varepsilon_{k[z]})(f(b, w_1 \otimes b')) = \varepsilon_{k[z]} p_1(b) = \varepsilon_B(b), \]

as elements in $\text{Hom}_B(B(-1) \oplus W_1 \otimes B, k)$. Hence, $E(p_2)(z^*) = \xi$. \qed

Now it turns to compute the Yoneda product of $E^1(B)$ and $E^d(B)$. We fix a sequence pair $((\delta_i), (\delta_{ij}))$ for $\delta$, and $(\delta_i, \delta_i)$ is the pair of elements in $V$ with respect to $((\delta_{ij}), (\delta_{ij}))$.

Lemma 3.6. In $E(B)$, for any $i, j = 1, \cdots, n$,

\[
\begin{align*}
\xi \ast \omega^* &= (-1)^d \text{hdet}(\overline{\partial} \partial) \omega^*, & \xi \ast \eta_j^* &= 0, \\
x_i^* \ast \omega^* &= (-1)^d x_i^*(\delta_i) \ast \omega^*, & x_i^* \ast \eta_j^* &= (-1)^d x_i^*(\eta_j^* \otimes \omega^* \delta_i^*). 
\end{align*}
\]

Proof. We claim that the following diagram is commutative.

\[
\begin{array}{ccccccccc}
W_j \otimes B(-1) & \longrightarrow & W_{d-1} \otimes B(-1) \oplus W_d \otimes B & \longrightarrow & B(-d) & \longrightarrow & k_B(-d),
\end{array}
\]

where $\varphi_1 = (\eta_j^* \otimes \text{id}_B, \omega^* \otimes \text{id}_B)$, and

\[
\varphi_2 = (-1)^d \begin{pmatrix}
-\delta_i \\
\phi_i
\end{pmatrix} + (-1)^d \begin{pmatrix}
0 \\
(\eta_j^* \otimes \text{id}_B) \otimes \text{id}_B \\
\text{hdet}(\overline{\partial} \partial)(\omega^* \otimes \text{id}_B) \\
(\omega^* \otimes \text{id}_B) \delta_{ij} \otimes \text{id}_B
\end{pmatrix},
\]

and the first summand of $\varphi_2$ is induced by $\eta_j^* \otimes \varepsilon_B$ and the other one by $\omega^* \otimes \varepsilon_B$.

In fact, $\varepsilon_B(\varphi_1) = (\eta_j^* \otimes \varepsilon_B, \omega^* \otimes \varepsilon_B)$ is obvious, and

\[
\begin{align*}
\varphi_1(-\delta_i \phi_i)^T &= -((\eta_j^* \otimes \text{id}_B) \phi_i) + (\omega^* \otimes \text{id}_B) \phi_i \\
&= -\eta_j^* \odot m_B + (\omega^* \otimes \text{id}_B)(\sigma^\text{ordl} \odot \lambda_2) + (\omega^* \otimes m_B)(\delta_{d, i} \otimes \text{id}_B), \\
(-1)^d \phi_i \varphi_2 &= -\eta_j^* \odot m_B + \text{hdet}(\overline{\partial} \partial)(\omega^* \odot \lambda_2) + (\omega^* \otimes m_B)(\delta_{d, i} \otimes \text{id}_B).
\end{align*}
\]

By [14] Theorem 1.2, $\sigma^\text{ordl} = \text{hdet}(\overline{\partial} \partial) : W_d \to W_d$. So the diagram is commutative. Then

\[
\begin{align*}
\xi \ast (\eta_j^*, \omega^*) &= (\varepsilon_B, 0) \varphi_2 = (-1)^d \text{hdet}(\overline{\partial} \partial) \omega^* \otimes \varepsilon_B, \\
x_i^* \ast (\eta_j^*, \omega^*) &= (0, x_i^* \otimes \varepsilon_B) \varphi_2 = (-1)^d x_i^*(\eta_j^* \otimes \omega^* \delta_i^*) + (-1)^d x_i^*(\delta_i) \omega^* \otimes \varepsilon_B.
\end{align*}
\]

The proof is completed. \qed
Lemma 3.7. In $E(B)$, for any $i, j = 1, \cdots, n$,
\[ \omega^* \ast \xi = \bar{\omega}^*, \quad \eta_j^* \ast \xi = 0, \]
\[ \omega^* \ast x_i^* = -x_i^*(\delta_{ij})\bar{\omega}^*, \quad \eta_j^* \ast x_i^* = (-1)^d(x_i^* \sigma \otimes \eta_j^*)(\omega)\bar{\omega}^*. \]

Proof. Firstly, we check the following diagram is commutative.

\[
\begin{array}{ccc}
W_d \otimes B(-1) & \xrightarrow{\phi_d} & W_{d-1} \otimes B(-1) \oplus W_d \otimes B \\
\downarrow & & \downarrow \\
W_{d-1} \otimes B(-2) \oplus W_d \otimes B(-1) & \xrightarrow{\phi_{d-1}} & W_{d-2} \otimes B(-2) \oplus W_{d-1} \otimes B(-1)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-2} \otimes B(-3) \oplus W_{d-1} \otimes B(-2) & \xrightarrow{\phi_{d-2}} & W_{d-3} \otimes B(-3) \oplus W_{d-2} \otimes B(-2)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-3} \otimes B(-4) \oplus W_{d-2} \otimes B(-3) & \xrightarrow{\phi_{d-3}} & W_{d-4} \otimes B(-4) \oplus W_{d-3} \otimes B(-3)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-4} \otimes B(-5) \oplus W_{d-3} \otimes B(-4) & \xrightarrow{\phi_{d-4}} & W_{d-5} \otimes B(-5) \oplus W_{d-4} \otimes B(-4)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-5} \otimes B(-6) \oplus W_{d-4} \otimes B(-5) & \xrightarrow{\phi_{d-5}} & W_{d-6} \otimes B(-6) \oplus W_{d-5} \otimes B(-5)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-6} \otimes B(-7) \oplus W_{d-5} \otimes B(-6) & \xrightarrow{\phi_{d-6}} & W_{d-7} \otimes B(-7) \oplus W_{d-6} \otimes B(-6)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-7} \otimes B(-8) \oplus W_{d-6} \otimes B(-7) & \xrightarrow{\phi_{d-7}} & W_{d-8} \otimes B(-8) \oplus W_{d-7} \otimes B(-7)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-8} \otimes B(-9) \oplus W_{d-7} \otimes B(-8) & \xrightarrow{\phi_{d-8}} & W_{d-9} \otimes B(-9) \oplus W_{d-8} \otimes B(-9)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-9} \otimes B(-10) \oplus W_{d-8} \otimes B(-9) & \xrightarrow{\phi_{d-9}} & W_{d-10} \otimes B(-10) \oplus W_{d-9} \otimes B(-10)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-10} \otimes B(-11) \oplus W_{d-9} \otimes B(-10) & \xrightarrow{\phi_{d-10}} & W_{d-11} \otimes B(-11) \oplus W_{d-10} \otimes B(-10)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-11} \otimes B(-12) \oplus W_{d-10} \otimes B(-11) & \xrightarrow{\phi_{d-11}} & W_{d-12} \otimes B(-12) \oplus W_{d-11} \otimes B(-11)
\end{array}
\]

\[
\begin{array}{ccc}
W_{d-12} \otimes B(-13) \oplus W_{d-11} \otimes B(-12) & \xrightarrow{\phi_{d-12}} & W_{d-13} \otimes B(-13) \oplus W_{d-12} \otimes B(-12)
\end{array}
\]

where

\[
\psi_0 = (\mathbf{id}_B, x_i^* \otimes \mathbf{id}_B),
\]
\[
\psi_s = \begin{pmatrix}
0 & 0 \\
\mathbf{id}_{W_S \otimes B(-1)} & 0
\end{pmatrix} + (-1)^s \begin{pmatrix}
(x_i^* \sigma \otimes \mathbf{id}_B^{d-1}) \otimes \mathbf{id}_B & 0 \\
(\mathbf{id}_B \otimes \mathbf{id}_B^{d-1}) \delta_{ij} & (x_i^* \sigma \otimes \mathbf{id}_B^{d-1}) \otimes \mathbf{id}_B
\end{pmatrix},
\]
\[
1 \leq s \leq d - 1,
\]
\[
\psi_d = \begin{pmatrix}
0 & 0 \\
\mathbf{id}_{W_S \otimes B(-1)} & 0
\end{pmatrix} + (-1)^d \begin{pmatrix}
(x_i^* \sigma \otimes \mathbf{id}_B^{d-1}) \otimes \mathbf{id}_B \\
(\mathbf{id}_B \otimes \mathbf{id}_B^{d-1}) \delta_{ij} & (x_i^* \sigma \otimes \mathbf{id}_B^{d-1}) \otimes \mathbf{id}_B
\end{pmatrix},
\]

and the first summand of those maps is induced by $\xi$ and the other one by $x_i^* \otimes \mathbf{e}_B$.

Write $\partial^B_i$ for the $i$-th differential in the minimal resolution of $k_B$ for $i \geq 1$. Obviously, $\mathbf{e}_B \psi_0 = (\epsilon_B, x_i^* \otimes \mathbf{e}_B)$. Also, $\psi_0 \partial^B_i = -\partial^B_i \psi_1$, since $\delta = \delta_1, \ldots, \delta_{d-1}$ and

\[
\psi_0 \partial^B_i = (-\partial_1 + (x_i^* \otimes \mathbf{id}_B)\phi_1, (x_i^* \otimes \mathbf{id}_B)\partial_2)
\]
\[
= (-\partial_1 + x_i^* \sigma \otimes \lambda_2 + (x_i^* \otimes m_B)(\mathbf{id}_B \otimes \mathbf{id}_B), x_i^* \otimes m_B),
\]
\[
-\partial^B_1 \psi_1 = (-\partial_1 + \phi_B (x_i^* \sigma \otimes \mathbf{id}_B) + \partial_1 [(x_i^* \otimes \mathbf{id}_B)\delta_{ij} \otimes \mathbf{id}_B], \partial_1 ((x_i^* \otimes \mathbf{id}_B) \otimes \mathbf{id}_B))
\]
\[
= (-\partial_1 + x_i^* \sigma \otimes \lambda_2 + (x_i^* \otimes m_B)(\delta_{ij} \otimes \mathbf{id}_B), x_i^* \otimes m_B).
\]

For $1 \leq s \leq d - 2$,

\[
\psi_s \partial^B_s = \begin{pmatrix}
0 & 0 \\
-\partial_{s+1} & 0
\end{pmatrix} + (-1)^s \begin{pmatrix}
(x_i^* \sigma \otimes \mathbf{id}_B^{d-1}) \otimes m_B & 0 \\
\zeta_s & (x_i^* \otimes \mathbf{id}_B^{d-1}) \otimes m_B
\end{pmatrix},
\]
\[
-\partial^B_{s+1} \psi_s = \begin{pmatrix}
0 & 0 \\
-\partial_{s+1} & 0
\end{pmatrix} + (-1)^s \begin{pmatrix}
(x_i^* \sigma \otimes \mathbf{id}_B^{d-1}) \otimes m_B & 0 \\
\zeta_s & (x_i^* \otimes \mathbf{id}_B^{d-1}) \otimes m_B
\end{pmatrix},
\]

where

\[
\zeta_s = (-1)^s \left( [(x_i^* \otimes \mathbf{id}_B^{d-1}) \delta_{ij}] \otimes \mathbf{id}_B \right) \phi_{s+1}
\]
\[
= (-1)^s \left( (x_i^* \otimes \mathbf{id}_B^{d-1}) \delta_{ij} \otimes m_B + (x_i^* \sigma \otimes \mathbf{id}_B^{d-1}) \otimes \lambda_2 + (x_i^* \otimes \mathbf{id}_B^{d-1}) \otimes m_B \right) \phi_{s+1}
\]
\[
= \left( x_i^* \sigma \otimes \mathbf{id}_B^{d-1} \right) \otimes \lambda_2 + (-1)^s \left( (x_i^* \otimes \mathbf{id}_B^{d-1}) \otimes \lambda_2 + (x_i^* \otimes \mathbf{id}_B^{d-1}) \otimes m_B \right) \phi_{s+1},
\]
By Lemma 2.5, of a Koszul AS-regular algebra, which includes [27, Proposition 3.15] partially. For the automorphism \( \eta^* \). Since \( \eta^* \) is the Yoneda product in \( \mu^* \otimes \eta^* \otimes \varepsilon_B \). Similarly, one obtains that the Yoneda products in \( \mu^* \otimes \eta^* \otimes \varepsilon_B \).

The result follows. \( \square \)

3.3. Nakayama automorphisms. This subsection devotes to proving the main result of this paper. For the completeness, we give a whole computation of the Nakayama automorphism of a graded Ore extension of a Koszul AS-regular algebra, which includes [27] Proposition 3.15] partially. For the automorphism \( \sigma \in GL(V) \) and the Nakayama automorphism \( \mu_A \) of \( A \), there exist two invertible \( n \times n \) matrices \( M = (m_{ij}) \), \( P = (p_{ij}) \) over \( k \), such that

\[
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = M \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mu_A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = P \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.
\]

**Lemma 3.8.** For any \( j = 1, \cdots, n \),

\[
\begin{pmatrix} \eta^*_j \otimes x^*_n(\omega) \\ \eta^*_j \otimes x^*_2(\omega) \\ \vdots \\ \eta^*_j \otimes x^*_1(\omega) \end{pmatrix} = (-1)^{d-1} P^T \begin{pmatrix} x^*_1 \otimes \eta^*_j(\omega) \\ x^*_2 \otimes \eta^*_j(\omega) \\ \vdots \\ x^*_n \otimes \eta^*_j(\omega) \end{pmatrix}.
\]

**Proof.** Since \( A \) is a Koszul AS-regular algebra of dimension \( d \), the minimal free resolution \( \mathfrak{R} \) of trivial module \( k_A \) becomes

\[
0 \rightarrow W_d \otimes A \xrightarrow{\partial^1} W_{d-1} \otimes A \xrightarrow{\partial^2} \cdots \xrightarrow{\partial^{d-1}} W_1 \otimes A \xrightarrow{\partial^d} A \xrightarrow{\varepsilon} k_A \rightarrow 0.
\]

So \( E^1(A) = W_1^* \), \( E^{d-1}(A) = W_{d-1}^* \), \( E^d(A) = W_d^* \). By a similar argument in the proof of Lemma 3.6 and Lemma 3.7, one obtains that the Yoneda products in \( E(A) \) of \( \{x^*_i\}^n_{i=1} \) and \( \{\eta^*_j\}^n_{j=1} \), which are basis of \( E^1(A) \) and \( E^{d-1}(A) \) respectively, are as follows:

\[
x^*_i \otimes \eta^*_j = (-1)^{d-1}(\eta^*_j \otimes x^*_i)(\omega)\omega^* \quad \eta^*_j \otimes x^*_i = (-1)^{d-1}(x^*_i \otimes \eta^*_j)(\omega)\omega^*.
\]

By Theorem 1.8(a,b), \( \mu_{E(A)}(x^*_i) = \sum_{j=1}^n p_{ij} x^*_j \), and \( E(A) \) is graded Frobenius with the bilinear form as follows

\[
\langle x^*_i, \eta^*_j \rangle = \langle x^*_i \otimes \eta^*_j \rangle(\omega) = (-1)^{d-1}(\eta^*_j \otimes x^*_i)(\omega),
\]

\[
\langle \eta^*_j, \mu_{E(A)}(x^*_i) \rangle = \sum_{i=1}^n p_{ij} (\eta^*_j \otimes x^*_i)(\omega) = \sum_{i=1}^n p_{ij} \eta^*_j(\omega) x^*_i = (-1)^{d-1} \sum_{i=1}^n p_{ij} (\eta^*_j \otimes x^*_i)(\omega),
\]

for any \( i, j = 1, \cdots, n \). The result follows. \( \square \)

Now we prove the main result of this paper.
Theorem 3.9. Let $B = A[z; ς, δ]$ be a graded Ore extension of a Koszul AS-regular algebra $A$, where $ς$ is a graded automorphism of $A$ and $δ$ is a degree-one $ς$-derivation. Then the Nakayama automorphism $μ_B$ of $B$ satisfies

$$μ_B|_A = ς^{-1}μ_A, \quad μ_B(z) = hdet(ς)z + ∇ς · δ,$$

where $μ_A$ is the Nakayama automorphism of $A$ and $∇ς · δ$ is the $ς$-derivative of $δ$.

Proof. Let $(δ_i, δ_i)$ be the pair of elements in $V$ with respect to some sequence pair $((δ_i), (δ_i))$ for $ς$.

By Lemma 3.6 and Lemma 3.7

$$⟨ξ, ω⟩ = (1)^d hdet(ς) = (1)^d(ω, hdet(ς)ξ),$$

$$⟨ξ, η_j⟩ = 0 = (1)^d(η_j, hdet(ς)ξ),$$

for any $j = 1, \cdots, n$. Hence,

$$μ_E(B)(ξ) = hdet(ς)ξ.$$

Write $ω = x_1 ⊗ v_1 + x_2 ⊗ v_2 + \cdots + x_n ⊗ v_n$, where $v_1, v_2, \cdots, v_n ∈ W_{d−1}$. Then $(x_i^* ⊗ η_j)(ω) = η_j^j(v_i)$, and

$$(x_i^* ς ⊗ η_j)(ω) = (x_i^* ⊗ η_j) \left( \sum_{j=1}^n m_{ij}(x_i ⊗ v_j) \right) = \sum_{j=1}^n m_{ij} η_j^j(v_j).$$

By Lemma 3.6 one obtains

$$\begin{pmatrix}
(x_1^* ς ⊗ η_j)^j(ω) \\
(x_2^* ς ⊗ η_j)^j(ω) \\
\vdots \\
(x_n^* ς ⊗ η_j)^j(ω)
\end{pmatrix}
= \left( \begin{array}{c}
η_j^j(v_1) \\
η_j^j(v_2) \\
\vdots \\
η_j^j(v_n)
\end{array} \right) = \left( \begin{array}{c}
(x_1^* ⊗ η_j)^j(ω) \\
(x_2^* ⊗ η_j)^j(ω) \\
\vdots \\
(x_n^* ⊗ η_j)^j(ω)
\end{array} \right) = (1)^d (P^1 M)^T \begin{pmatrix}
(η_j^j ⊗ x_1^j)(ω) \\
(η_j^j ⊗ x_2^j)(ω) \\
\vdots \\
(η_j^j ⊗ x_n^j)(ω)
\end{pmatrix}.$$

By Lemma 3.6 and Lemma 3.7

$$\begin{pmatrix}
⟨x_1^j, η_j⟩ \\
⟨x_2^j, η_j⟩ \\
\vdots \\
⟨x_n^j, η_j⟩
\end{pmatrix}
= (1)^d + 1 \begin{pmatrix}
⟨(η_j^j ⊗ x_1^j)(ω) \\
(η_j^j ⊗ x_2^j)(ω) \\
\vdots \\
(η_j^j ⊗ x_n^j)(ω)
\end{pmatrix}
= (M^1 P)^T \begin{pmatrix}
⟨x_1^j ς ⊗ η_j⟩ \\
⟨x_2^j ς ⊗ η_j⟩ \\
\vdots \\
⟨x_n^j ς ⊗ η_j⟩
\end{pmatrix}
= (1)^d (M^1 P)^T \begin{pmatrix}
⟨η_j^j⟩, ⟨x_1^j⟩ \\
⟨η_j^j⟩, ⟨x_2^j⟩ \\
\vdots \\
⟨η_j^j⟩, ⟨x_n^j⟩
\end{pmatrix}.$$

Write $c_i = x_i^j(∇ς · δ)$ for any $i = 1, \cdots, n$, and it is easy to see that

$$\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n
\end{pmatrix}
= \begin{pmatrix}
x_1^j(δ_j) \\
x_2^j(δ_j) \\
\vdots \\
x_n^j(δ_j)
\end{pmatrix} + (M^1 P)^T \begin{pmatrix}
x_1^j(δ_j) \\
x_2^j(δ_j) \\
\vdots \\
x_n^j(δ_j)
\end{pmatrix}.$$

Then

$$\begin{pmatrix}
⟨x_1^j, η_j⟩ \\
⟨x_2^j, η_j⟩ \\
\vdots \\
⟨x_n^j, η_j⟩
\end{pmatrix}
= (1)^d (M^1 P)^T \begin{pmatrix}
⟨η_j^j⟩, ⟨x_1^j⟩ \\
⟨η_j^j⟩, ⟨x_2^j⟩ \\
\vdots \\
⟨η_j^j⟩, ⟨x_n^j⟩
\end{pmatrix} + \begin{pmatrix}
⟨η_j^j⟩, ⟨c_1⟩ \\
⟨η_j^j⟩, ⟨c_2⟩ \\
\vdots \\
⟨η_j^j⟩, ⟨c_n⟩
\end{pmatrix}.$$
Suppose $B$ be a Koszul AS-regular algebra of dimension $d$ and $\omega$ a basis of $W_d$. Suppose $B = A[z; \overline{\tau}, \overline{\sigma}]$ is graded Ore extension of $A$, where $\overline{\tau}$ is a graded automorphism of $A$ and $\overline{\sigma}$ is a degree-one $\overline{\tau}$-derivation of $A$. Let $((\delta_{ij}), ([\delta_{ij}])$ be a sequence pair for $\overline{\sigma}$, then

$$\tilde{\sigma} = \sum_{i=0}^{d} (-1)^i \tau_{d+1}^{i} (\omega^i \otimes \mathbf{i}_{V}^{(d-i)}) (z \otimes \omega) + \sum_{i=1}^{d} (-1)^i (\delta_{ij} \otimes \mathbf{i}_{V}^{(d-i)}) (\omega)$$

$$\tilde{\tau} = \sum_{i=0}^{d} (-1)^i \tau_{d+1}^{i} (\sigma^i \otimes \mathbf{i}_{V}^{(d-i)}) (z \otimes \omega) + \sum_{i=1}^{d} (-1)^i (\sigma^i \otimes \delta_{ij}) (\omega).$$

is a $\mu_B$-twisted superpotential, where $\mu_B$ is the Nakayama automorphism of $B$ and $\sigma = \overline{\tau}_{V}$. Moreover, $B \cong \mathcal{A}(\tilde{\sigma}, d-1)$.

Proof. By Proposition 2.7(a), one obtains that (3.2) and (3.3) are equal. By Theorem 1.8(c),

$$\tau_{d+1}^{d-1} (\mu_B \otimes \mathbf{i}_{V}^{(d)}) (\omega) = (-1)^{d-1} \omega.$$
where the third equation holds by [14, Theorem 1.2] and Proposition 2.7(b).

Let $\delta : V \to V \otimes V$ be a linear map such that the map $\overline{\delta}$ and the sequence pair $\{\delta_{i,r}\}$ are induced by $\delta$, and $\{x_1, \ldots, x_n\}$ a basis of $V$. Write

$\tilde{V} = V \oplus k[z], \quad \tilde{R} = R \oplus k[z \otimes x_i - \sigma(x_i) \otimes z - \delta(x_i) | \ i = 1, \ldots, n].$

Then $B \cong T(\tilde{V})/\langle \tilde{R} \rangle$ is a Koszul regular algebra of dimension $d + 1$. Write

$\tilde{W}_i = \tilde{V}, \quad \tilde{W}_i = \bigcap_{0 \leq s \leq i-2} \tilde{V}^{\otimes s} \otimes \tilde{R} \otimes \tilde{V}^{\otimes d-s-2}, \quad \forall i \geq 2.$

Clearly, $W_i \subseteq \tilde{W}_i$ for any $i \geq 1$ and $\dim \tilde{W}_{d+1} = 1$.

It is easy to know that $\hat{\omega} \neq 0$. Since $B$ is Koszul AS-regular, it suffices to prove $\hat{\omega} \in \tilde{W}_{d+1}$ by Theorem 1.8(d). Write $\omega = \sum v_1 \otimes v_2 \otimes \cdots \otimes v_d$. By (3.2), one obtains that

$\hat{\omega} = \sum (z \otimes v_1 - \sigma(v_1) \otimes z - \delta(v_1)) \otimes v_2 \otimes \cdots \otimes v_d$

$\quad + \sum_{i=2}^{d}(-1)^i(\sigma(v_1) \otimes \sigma(v_2) \otimes \cdots \otimes \sigma(v_i) \otimes z \otimes v_{i+1} \otimes \cdots \otimes v_d) + \sum_{i=2}^{d}(-1)^i(\delta(x_i) \otimes \sigma(v_1) \otimes \cdots \otimes \sigma(v_d)) \otimes \cdots \otimes v_d) \in \tilde{R} \otimes \tilde{V}^{\otimes d-1}.$

Since $(\mu^{\hat{\eta}_{d+1}})_i \cdots (\mu^{\hat{\eta}_{d+1}})_{i-1}((\tilde{R} \otimes \tilde{V}^{\otimes d-1}) \subseteq \tilde{V}^{\otimes i} \otimes \tilde{R} \otimes \tilde{V}^{\otimes d-1}$ and $\hat{\omega}$ is a $B_{\tilde{V}}$-twisted superpotential, we have

$\hat{\omega} = (-1)^{d_i}((\mu^{\hat{\eta}_{d+1}})_{i-1} \cdots (\mu^{\hat{\eta}_{d+1}}))_1((\sigma(v_1) \otimes \cdots \otimes (\mu^{\hat{\eta}_{d+1}})_{i-1}((\tilde{R} \otimes \tilde{V}^{\otimes d-1}) \subseteq \tilde{V}^{\otimes i} \otimes \tilde{R} \otimes \tilde{V}^{\otimes d-1},$

for any $1 \leq i \leq d - 1$. It implies that $\hat{\omega} \in \tilde{W}_{d+1}$. \(\Box\)

**Remark 3.12.** (a) By Proposition 2.8, the twisted superpotential $\hat{\omega}$ constructed in the last theorem is independent on the choices of sequence pairs for $\overline{\delta}$. 
There exists a sequence pair
Lemma 4.3. By the extension of 1
for any (not necessarily distinguished)
Firstly, we determine the vector spaces
Degree
Theorem 4.1. [9, Theorem 1.1(1)]
To prove this theorem, we need some preparation. Let
(a) The results in [6, Theorem 4.4] and [7, Theorem 0.1(ii)] are both the special case of \( \delta_{ij} = \delta_{i,j} = 0 \) for \( i \geq 2 \).

4. Applications

4.1. Graded polynomial algebras. In this part, we assume \( A = k[x_1, x_2, \cdots, x_n] \) is a graded polynomial algebra generated in degree 1. The Nakayama automorphism of a graded Ore extension of \( A \) is just a graded version of [9, Theorem 1.1]. We use our method to prove the differential case as an example.

Theorem 4.1. [9, Theorem 1.1(1)] Let \( A = k[x_1, x_2, \cdots, x_n] \) be a graded polynomial algebra generated in degree 1. Then the Nakayama automorphism \( \mu_B \) of the graded Ore extension \( B = A[z; \delta] \) is

\[
\mu_{B/A} = id_A \quad \mu_B(z) = z + \nabla \cdot \delta;
\]

where \( \nabla \cdot \delta \) is the divergence of \( \delta \), that is \( \nabla \cdot \delta = \sum_{i=1}^{n} \partial \delta(x_i)/\partial x_i \).

To prove this theorem, we need some preparation. Let \( V \) be the vector space spanned by \( \{x_1, x_2, \cdots, x_n\} \).

Firstly, we determine the vector spaces \( \{W_i \mid i \geq 2\} \) for \( A \). Write \( r_{i_1, i_2} = x_{i_1} \otimes x_{i_2} - x_{i_2} \otimes x_{i_1} \), for any (not necessarily distinguished) \( i_1, i_2 \in \{1, 2, \cdots, n\} \). For any integer \( m \geq 3 \), we write inductively,

\[
r_{i_1, i_2, \cdots, i_m} = \sum_{j=1}^{m} (-1)^{j+1} r_{i_1, i_2, \cdots, \hat{i}_j, \cdots, i_m} \otimes x_{i_j} \in V^\otimes m,
\]

for any (not necessarily distinguished) \( i_1, i_2, \cdots, i_m \in \{1, 2, \cdots, n\} \). The following result is clear.

Lemma 4.2. Let an integer \( m \geq 2 \), (not necessarily distinguished) \( i_1, i_2, \cdots, i_m \in \{1, 2, \cdots, n\} \). Then

(a) \( r_{i_1, i_2, \cdots, i_m} = \sum_{j=1}^{m} (-1)^{j+1} x_{i_j} \otimes r_{i_1, \cdots, \hat{i}_j, \cdots, i_m} \),
(b) \( r_{i_1, \cdots, i_m} \in W_m \).
(c) \( r_{i_1, \cdots, i_m} = 0 \), if \( i_s = i_t \) for some \( s \neq t \).
(d) \( r_{i_1, \cdots, i_m} = (-1)^{sgn(r_{i_1, \cdots, i_m})} \), for any \( \sigma \in S_m \).
(e) the set \( \{r_{i_1, \cdots, i_m} \mid i_1 < i_2 < \cdots < i_m\} \) is a basis of \( W_m \).

Let \( \delta : V \to V \otimes V \) be a linear map such that the map \( \overline{\delta} \) in Theorem 4.1 is induced by it. Write

\[
\delta(x_i) = \sum_{s,t=1}^{n} k^{(s,t)}_{i,s} x_s \otimes x_t,
\]

where \( k^{(s,t)}_{i,s} \in k \) for \( s, t, i = 1, \cdots, n \). Then we construct a sequence pair for \( \overline{\delta} \).

Lemma 4.3. There exists a sequence pair \((\{\delta_{ij}\}, \{\delta_{ij}\})\) for \( \overline{\delta} \) such that

\[
\delta_{ij}(r_{i_1, i_2, \cdots, i_m}) = \sum_{j=1}^{m} \sum_{s,t=1}^{n} k^{(j)}_{i_s} x_s \otimes r_{i_1, \cdots, \hat{i}_j, \cdots, i_m} \otimes x_t, \quad \delta_{ij}(r_{i_1, i_2, \cdots, i_m}) = \sum_{j=1}^{m} \sum_{s,t=1}^{n} k^{(j)}_{i_s} x_s \otimes r_{i_1, \cdots, \hat{i}_j, \cdots, i_m} \otimes x_t,
\]

for any \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n \) and \( m \geq 2 \).

Proof. By the extension of \( \delta \) to \( T(V) \), one obtains that for any \( 1 \leq i_1 < i_2 \leq n \),

\[
\overline{\delta}(r_{i_1, i_2}) = \overline{\delta}(x_{i_1} \otimes x_{i_2} - x_{i_2} \otimes x_{i_1}) = x_{i_1} \otimes \overline{\delta}(x_{i_2}) + \overline{\delta}(x_{i_1}) \otimes x_{i_2} - x_{i_2} \otimes \overline{\delta}(x_{i_1}) - \overline{\delta}(x_{i_2}) \otimes x_{i_1}
\]

with the above sequence pair.
By Remark 2.3, we can define

\[ \delta_{2,r}(r_{i_1i_2}) = \sum_{s,t=1}^n (k^{(i_1)}_{st} r_{st} + k^{(i_2)}_{st} r_{st}) \otimes x_t, \quad \delta_{2,l}(r_{i_1i_2}) = \sum_{s,t=1}^n x_s \otimes (k^{(i_1)}_{st} r_{st} + k^{(i_2)}_{st} r_{st}). \]

Suppose we have obtained that for any \( u < m, \)

\[ \delta_{u,r}(r_{i_1i_2}) = \sum_{j=1}^u \sum_{s,t=1}^n k^{(i_1)}_{st} r_{i_1i_2} \otimes x_t, \quad \delta_{u,l}(r_{i_1i_2}) = \sum_{j=1}^u \sum_{s,t=1}^n k^{(i_1)}_{st} x_s \otimes r_{i_1i_2} \]

For any \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n, \) one obtains

\[
\begin{align*}
& (id_V^{\otimes m-1} \otimes m_B)(id_V^{\otimes m-1} \otimes \delta + \delta_{m-1,r} \otimes id_V)(r_{i_1i_m}) \\
& = \sum_{s,t=1}^n \sum_{p=1}^m (-1)^{m-p}(id_V^{\otimes m-1} \otimes m_B)(k^{(i_1)}_{st} r_{i_1i_2} \otimes x_s \otimes x_t) \\
& + \sum_{j=1}^m \sum_{s,t=1}^n (-1)^{m-p}(id_V^{\otimes m-1} \otimes m_B)(\sum_{p=1}^{j-1} k^{(i_1)}_{st} r_{i_1i_2} \otimes x_s \otimes x_t + \sum_{p=j}^m k^{(i_1)}_{st} r_{i_1i_2} \otimes x_s \otimes x_t) \\
& = \sum_{s,t=1}^n \sum_{j=1}^m (id_V^{\otimes m-1} \otimes m_B)(-1)^{m-j} k^{(i_1)}_{st} r_{i_1i_2} \otimes x_s \otimes x_t) \\
& + \sum_{j=1}^m \sum_{s,t=1}^n (id_V^{\otimes m-1} \otimes m_B)(\sum_{p=1}^{j-1} (-1)^{m-p} k^{(i_1)}_{st} r_{i_1i_2} \otimes x_s \otimes x_t + \sum_{p=j}^m (-1)^{m-p} k^{(i_1)}_{st} r_{i_1i_2} \otimes x_s \otimes x_t) \\
& = \sum_{s,t=1}^n \sum_{j=1}^m (id_V^{\otimes m-1} \otimes m_B)(k^{(i_1)}_{st} r_{i_1i_2} \otimes x_s \otimes x_t).
\end{align*}
\]

By Remark 2.3, we can define

\[ \delta_{u,r}(r_{i_1i_2} \otimes x_t) = \sum_{j=1}^m \sum_{s,t=1}^n k^{(i_1)}_{st} r_{i_1i_2} \otimes x_s \otimes x_t. \]

Similarly, one obtains the result for \( \delta_{u,l} \) by Lemma 4.5 and Lemma 4.2 a).

**Proof of Theorem 4.7** By Lemma 4.2, \( r_{1-n} \) is the basis of \( W_n. \) By Lemma 4.2 and Lemma 4.3, one obtains

\[ \delta_{u,r}(r_{1-n}) = r_{1-n} \otimes \left( \sum_{s,t=1}^n k^{(i)}_{st} x_t \right), \quad \delta_{u,l}(r_{1-n}) = \left( \sum_{s,t=1}^n k^{(i)}_{st} x_s \right) \otimes r_{1-n}. \]
So \( \delta_r = \sum_{i=1}^{n} n \gamma_i x_i \) and \( \delta_l = \sum_{i=1}^{n} n \gamma_i x_i \), and
\[
\delta_r + \delta_l = \sum_{i=1}^{n} n \left( k^{(i)}_{\mu} + k^{(i)}_{\nu} \right) x_i = \sum_{i=1}^{n} \frac{\partial (\delta(x_i))}{\partial x_i} = \nabla \cdot \bar{\delta}.
\]
Since \( A \) is Koszul CY, the result follows by Theorem \[3.9\]. \( \square \)

4.2. **Koszul AS-regular algebras of dimension 2.** In this subsection, we assume \( \text{char } k = 0 \). We give a formula of Nakayama automorphisms for graded Ore extensions of Koszul AS-regular algebras of dimension 2, and then compute specific Nakayama automorphisms for noetherian ones.

Now we assume \( A \) is an AS-regular algebra of dimension 2. By \[25\] Theorem 0.1, one obtains that \( A \) is always Koszul and there is an invertible matrix \( Q \in M_n(k) \) such that \( A \cong k(x_1, x_2, \cdots, x_n)/r \), where
\[
r = x^T Q x,
\]
and \( x = (x_1, x_2, \cdots, x_n)^T \). It is well known (for example, \[6\] Section 3) that the Nakayama automorphism of \( \mu_A \) satisfies
\[
\mu_A(x) = -(Q^{-1})^T Q x.
\]

Let \( \bar{\sigma} \) be a graded automorphism of \( A \) and \( \bar{\delta} \) a degree-one \( \bar{\sigma} \)-derivation of \( A \). Write \( M \in M_n(k) \) for the invertible matrix such that \( \bar{\sigma}(x) = M x \). Choose a linear map \( \delta : V \to V \otimes V \) such that \( \bar{\delta} \) can be induced by it, where \( V \) is the vector space spanned by \( \{x_1, \cdots, x_n\} \). Since \( A \) is 2-dimensional, there is a unique pair \((\delta_r, \delta_l)\) of elements in \( V \) such that
\[
\delta(r) = r \otimes \delta_r + \delta_l \otimes r.
\]
That is, there are two certain elements \( c_r = (c_{r_1}, c_{r_2}, \cdots, c_{r_m}), c_l = (c_{l_1}, c_{l_2}, \cdots, c_{l_n}) \in k^n \) such that
\[
\delta_r = c_r x, \quad \delta_l = c_l x.
\]

By Theorem \[3.9\] the Nakayama automorphism \( \mu_B \) of the graded Ore extension \( B = A[z; \bar{\tau}, \bar{\delta}] \) satisfies
\[
\mu_B \begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} -M^{-1}(Q^T)^{-1} Q & 0 \\ c_r - c_l M^{-1}(Q^T)^{-1} Q \text{hdet}(\bar{\tau}) \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix}.
\]

Now we focus on noetherian ones. There is an interesting result about CY property for noetherian cases.

**Theorem 4.4.** Let \( A \) be a noetherian AS-regular algebra of dimension 2 and \( B = A[z; \bar{\tau}, \bar{\delta}] \) is a graded Ore extension. Write \( \mu_A \) for the Nakayama automorphism of \( A \).

(a) Suppose \( A \) is commutative, then \( B \) is CY if and only if \( \bar{\tau} = \mu_A \) and
\[
\bar{\delta}(x_1) = l_1 x_1^2 - 2l_4 x_2 x_1 + l_2 x_2^2, \quad \bar{\delta}(x_2) = l_3 x_1^2 - 2l_1 x_2 x_1 + l_4 x_2^2,
\]
for some \( l_1, l_2, l_3, l_4 \in k \).

(b) Suppose \( A \) is noncommutative, then \( B \) is CY if and only if \( \bar{\tau} = \mu_A \).

To prove this result, we determine all graded Ore extensions of noetherian Koszul AS-regular algebras of dimension 2 and compute their Nakayama automorphisms. Let \( A \) be a noetherian Koszul AS-regular algebra of dimension 2. Then \( A = k(x_1, x_2)/(r) \), and there are only two classes of \( A \) up to isomorphism, that is,
\[
Q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \quad \text{or} \quad Q = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.
\]
where $q$ is a nonzero element in $k$. Let $V$ be the vector space spanned by $\{x_1, x_2\}$.

In the following, $\sigma$ is a graded automorphism of $A$, $M = (m_{ij}) \in M_2(k)$ is the invertible matrix such that $\sigma(x_1, x_2)^T = M(x_1, x_2)^T$. Write $\sigma = \sigma_T \otimes \sigma_i \otimes \sigma_l$. Let $\delta$ be a linear map from $V$ to $V \otimes V$. Then any degree-one $\sigma$-derivation $\delta$ of $A$ can be induced by $\delta$, in case $\delta$ extends to a degree-one $\sigma_T$-derivation of $T(V)$ such that

$$
\delta(r) \in r \otimes V + V \otimes r.
$$

Since the forms of $r$ (or $Q$), we assume without loss of generality,

$$
\delta(x_i) = \gamma_i x_1^2 + \gamma_{i2} x_2 x_1 + \gamma_{i3} x_2^2,
$$

where $\gamma_{ij} \in k$ for $j = 1, 2, 3, i = 1, 2$.

4.2.1. Case (i): commutative polynomial. In this case, $r = x_1 x_2 - x_2 x_1$, or equivalently

$$
Q = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
$$

$M = (m_{ij})$ is an arbitrary invertible matrix, and $\text{hdet}(\sigma) = \det(M)$ (We refer [8, 20] for the computation of homological determinant). We have

$$
\delta(r) = (m_{11} \gamma_{21} - m_{21} \gamma_{11} - \gamma_{21}) x_1^3 + (m_{12} \gamma_{23} - m_{22} \gamma_{13} + \gamma_{13}) x_2^3 + (m_{11} \gamma_{22} - m_{21} \gamma_{12}) x_1 x_2 x_1 + \gamma_{12} x_2 x_1 x_2
$$

+ $\gamma_{11} x_1^2 x_2 + (m_{11} \gamma_{23} - m_{21} \gamma_{13}) x_1 x_2^2 + (m_{12} \gamma_{21} - m_{22} \gamma_{11} - \gamma_{22}) x_2 x_1^2 + (m_{12} \gamma_{22} - m_{22} \gamma_{12} - \gamma_{23}) x_2^2 x_1$.

By a straightforward computation, one obtains that the following result.

**Lemma 4.5.** The condition (4.2) is equivalent to the following equations hold

$$
\begin{align*}
&m_{21} \gamma_{11} + (1 - m_{11}) \gamma_{21} = 0, \\
&(1 - m_{22}) \gamma_{13} + m_{12} \gamma_{23} = 0, \\
&(m_{22} - 1) \gamma_{11} + m_{21} \gamma_{12} - m_{12} \gamma_{21} + (1 - m_{11}) \gamma_{22} = 0, \\
&(m_{22} - 1) \gamma_{12} + m_{21} \gamma_{13} - m_{12} \gamma_{22} + (1 - m_{11}) \gamma_{23} = 0.
\end{align*}
$$

In this case, $\delta(r) = r \otimes \delta_r + \delta_l \otimes r$, where

$$
\delta_r = (m_{22} \gamma_{11} - m_{12} \gamma_{21} + \gamma_{22}) x_1 + (m_{11} \gamma_{23} - m_{21} \gamma_{13}) x_2, \quad \delta_l = \gamma_{11} x_1 + (m_{22} \gamma_{12} - m_{12} \gamma_{22} + \gamma_{23}) x_2.
$$

By (4.1), the Nakayama automorphisms of graded Ore extension $B = A[z; \sigma, \delta]$ of $A$ by (4.1), where $\delta$ is induced by $\delta$, satisfies

$$
\begin{align*}
\mu_B(x_1) &= \det(M)^{-1}(m_{22} x_1 - m_{12} x_2), \\
\mu_B(x_2) &= \det(M)^{-1}(-m_{21} x_1 + m_{11} x_2), \\
\mu_B(z) &= \det(M)z + (x_1, x_2)\nu^T,
\end{align*}
$$

where $\nu = (m_{22} \gamma_{11} - m_{12} \gamma_{21} + \gamma_{22}, m_{11} \gamma_{23} - m_{21} \gamma_{13}) + (\gamma_{11}, m_{22} \gamma_{12} - m_{12} \gamma_{22} + \gamma_{23}) M^{-1}$. Then we list all solutions of equations (4.3).

**Solution 4.6.** All solutions of equations (4.3) are as follows.

(a) If $M$ is the identity matrix $E_2$, that is $m_{11} = m_{22} = 1$ and $m_{12} = m_{21} = 0$, then each $\gamma_{ij}$ is free for $i = 1, 2, j = 1, 2, 3$;

(b) If $m_{21} = 0, m_{22} = 1$ and $M \neq E_2$, then $\gamma_{21} = \gamma_{22} = \gamma_{23} = 0$ and $\gamma_{11}, \gamma_{12}, \gamma_{13}$ are free variables;
In this case, by (4.1), we have the Nakayama automorphism of a graded Ore extension 
\((g)\) if 
\((e)\) if 
m 
\(m\) and 
\(m\) are free; 
(f) if 
\(m_{22}\) is not 0 and 
\((m_{22} - 1)(m_{11} - 1) = m_{22}m_{12},\) then 
\(\gamma_{11} = m_{21}(m_{11} - 1) \gamma_{21},\gamma_{12} = m_{21}(m_{11} - 1)(m_{22} - 1) \gamma_{22}, y_{23} = m_{22}(m_{12} - 1) \gamma_{13};\) 
(g) if 
\(m_{22}\) is not 0 and 
\((m_{22} - 1)(m_{11} - 1) \neq m_{22}m_{12},\) then 
\(\gamma_{11} = m_{21}(m_{11} - 1) \gamma_{21},\gamma_{12} = m_{21}(m_{11} - 1)(m_{22} - 1) \gamma_{22}, y_{23} = m_{22}(m_{12} - 1) \gamma_{13}.\)

4.2.2. Case (ii): noncommutative quantum plane. In this case, 
\(r = x_{1}x_{2} - qx_{2}x_{1},\) or equivalently,
\[ Q = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix}, \]
where nonzero element \(q \neq 1.\) There are two subcases: \(q = -1\) and \(q \neq 1.\)

(1) \(q = -1.\) The graded automorphisms of 
\(A = k(x_{1}, x_{2})/(x_{1}x_{2} + x_{2}x_{1})\) have two forms

\[ M = \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 0 & m_{12} \\ m_{21} & 0 \end{pmatrix}, \]

(1) \(M = \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix}.\) Then \(\text{hdet}(\overline{r}) = m_{11}m_{22}.\) Since 
\(r = x_{1}x_{2} + x_{2}x_{1},\) then 
\[ \delta(r) = (m_{11} + 1) \gamma_{21}x_{1}^{3} + (m_{22} + 1) \gamma_{21}x_{2}^{3} + m_{11} \gamma_{21}x_{1}x_{2}x_{1} + \gamma_{12}x_{2}x_{1}x_{2} + m_{11} \gamma_{21}x_{1}x_{2}^{2} + \gamma_{11}x_{1}^{2}x_{2} + (m_{22} \gamma_{11} + \gamma_{22})x_{2}x_{1}^{2} + (m_{22} \gamma_{12} + \gamma_{23})x_{2}^{2}x_{1}. \]

The following result is easy to get.

Lemma 4.7. The condition \((4.2)\) is equivalent to the following equations hold

\[ (m_{11} + 1) \gamma_{21} = 0, \]
\[ (m_{22} + 1) \gamma_{13} = 0, \]
\[ (m_{22} + 1) \gamma_{11} + (1 - m_{11}) \gamma_{22} = 0, \]
\[ (m_{22} - 1) \gamma_{12} + (m_{11} + 1) \gamma_{23} = 0. \]

In this case, \(\delta(r) = r \circ \delta_{r} \circ r,\) where

\[ \delta_{r} = (m_{22} \gamma_{11} + \gamma_{22})x_{1} + m_{11} \gamma_{23}x_{2}, \quad \delta_{l} = \gamma_{11}x_{1} + (m_{22} \gamma_{12} + \gamma_{23})x_{2}. \]

By \((4.1),\) we have the Nakayama automorphism of a graded Ore extension 
\(B = A[z, \overline{\gamma}, \overline{\delta}]\) satisfies that 
\[ \mu_{b}(x_{1}) = -m_{11}^{-1}x_{1}, \quad \mu_{b}(x_{2}) = -m_{22}^{-1}x_{2}, \]
\[ \mu_{b}(z) = m_{11}m_{22}z + ((m_{22} - m_{11}^{-1}) \gamma_{11} + \gamma_{22})x_{1} + ((m_{11} - m_{22}^{-1}) \gamma_{23} - \gamma_{12})x_{2}. \]

To be explicit, we give all solutions of \((4.5).\)

Solution 4.8. The solutions of \((4.5)\) are as follows.

(a) If 
\(m_{11} = m_{22} = -1,\) then 
\(\gamma_{12} = \gamma_{22} = 0\) and the other variables are free;

(b) If 
\(\gamma_{11} = 1, m_{22} = -1,\) then 
\(\gamma_{21} = 0, \gamma_{23} = \gamma_{12} \) and 
\(\gamma_{11}, \gamma_{13}, \gamma_{22} \) are free;

(c) If 
\(m_{11} \neq \pm 1, m_{22} = -1,\) then 
\(\gamma_{21} = \gamma_{22} = 0, \gamma_{23} = 2(m_{11} + 1)^{-1} \gamma_{12} \) and 
\(\gamma_{11}, \gamma_{13} \) are free;
(d) If $m_{11} = -1, m_{22} = 1$, then $\gamma_{11} = 0, \gamma_{11} = -\gamma_{22}$ and $\gamma_{12}, \gamma_{21}, \gamma_{23}$ are free;
(e) If $m_{11} = -1, m_{22} \neq \pm 1$, then $\gamma_{12} = \gamma_{13} = 0, \gamma_{11} = -2(m_{22} + 1)^{-1}\gamma_{22}$ and $\gamma_{21}, \gamma_{23}$ are free;
(f) If $m_{11} \neq -1, m_{22} \neq -1$, then $\gamma_{13} = \gamma_{21} = 0, \gamma_{11} = (m_{11} - 1)(m_{22} + 1)^{-1}\gamma_{22}, \gamma_{23} = (m_{11} + 1)^{-1}(1 - m_{22})\gamma_{12}$.

\[ M = \begin{pmatrix}
0 & m_{12} \\
m_{21} & 0
\end{pmatrix}. \] Then $\text{hdet}(\mathbf{M}) = m_{12}m_{21}$. Similarly, one obtains that
\[
\delta(r) = (m_{21}\gamma_{11} + \gamma_{21})x_1^3 + (m_{12}\gamma_{23} + \gamma_{11})x_1^3 + m_{12}\gamma_{12}x_1x_2 + \gamma_{12}x_1x_2
+ m_{21}\gamma_{13}x_2^2 + \gamma_{11}x_1x_2 + (m_{12}\gamma_{21} + \gamma_{22})x_2x_1^2 + (m_{12}\gamma_{22} + \gamma_{23})x_2x_1.
\]

**Lemma 4.9.** The condition (4.2) is equivalent to the following equations hold
\[
m_{21}\gamma_{11} + \gamma_{21} = 0, \quad m_{12}\gamma_{23} + \gamma_{13} = 0,
\]

\[ (4.7) \]
\[
\gamma_{11} - m_{12}\gamma_{12} + m_{12}\gamma_{21} + \gamma_{22} = 0, \quad -\gamma_{12} + m_{12}\gamma_{13} + m_{12}\gamma_{22} + \gamma_{23} = 0.
\]
In this case, $\delta(r) = r \otimes \delta_{r} + \delta_{l} \otimes r$, where
\[
\delta_{r} = (m_{12}\gamma_{21} + \gamma_{22})x_1 + m_{21}\gamma_{13}x_2, \quad \delta_{l} = \gamma_{11}x_1 + (m_{12}\gamma_{22} + \gamma_{23})x_2.
\]

**Solution 4.10.** The solutions to equations (4.7) are as follows.

(a) If $m_{12}m_{21} = 1$, then $\gamma_{11} = -m_{12}\gamma_{21}, \gamma_{12} = m_{12}\gamma_{22}, \gamma_{13} = -m_{12}\gamma_{23};$

(b) If $m_{12}m_{21} \neq 1$, then $\gamma_{11} = -m_{21}^{-1}\gamma_{21}, \gamma_{12} = m_{12}m_{21}^{-1}\gamma_{22} + \gamma_{21}, \gamma_{13} = -m_{12}\gamma_{23}, \gamma_{22} = m_{21}^{-1}\gamma_{21} + m_{21}\gamma_{23}$.
Moreover, the Nakayama automorphism of $B = A[z; \overline{\mathbf{m}}, \overline{\mathbf{g}}]$ satisfies
\[
\mu_B(x_1) = -m_{21}^{-1}x_2, \quad \mu_B(x_2) = -m_{12}^{-1}x_1,
\]
\[
\mu_B(z) = m_{12}m_{21}z + (m_{12}\gamma_{21} - m_{21}\gamma_{21})x_1 + (m_{12}^2\gamma_{21} - m_{12}m_{21}\gamma_{23})x_2.
\]

(2) $q \neq -1$. The graded automorphism $\mathbf{g}$ of $A = k(x_1, x_2)/(x_1, x_2 - qx_2x_1)$ must be the following form
\[
M = \begin{pmatrix}
m_{11} & 0 \\
0 & m_{22}
\end{pmatrix}, \quad \text{and hdet}(\mathbf{M}) = m_{11}m_{22}. \quad \text{One obtains that}
\]
\[
\delta(r) = (m_{11} - q)\gamma_{21}x_1^3 + (1 - qm_{22})\gamma_{13}x_1^3 + m_{11}\gamma_{22}x_1x_2x_1 + \gamma_{12}x_1x_3
+ m_{11}\gamma_{23}x_1x_2 + \gamma_{11}x_1x_2 - q(m_{22}\gamma_{11} + \gamma_{22})x_2x_1x_2 - q(m_{22}\gamma_{12} + \gamma_{23})x_2x_1.
\]

**Lemma 4.11.** The condition (4.2) is equivalent to the following equations hold
\[
(m_{11} - q)\gamma_{21} = 0, \quad (1 - qm_{22})\gamma_{13} = 0,
\]

\[ (4.8) \]
\[
(m_{22} - q)\gamma_{11} + (1 - m_{11})\gamma_{22} = 0, \quad (m_{22} - 1)\gamma_{12} + (1 - qm_{11})\gamma_{23} = 0.
\]
In this case, $\delta(r) = r \otimes \delta_{r} + \delta_{l} \otimes r$, where
\[
\delta_{r} = (m_{22}\gamma_{11} + \gamma_{22})x_1 + m_{11}\gamma_{23}x_2, \quad \delta_{l} = \gamma_{11}x_1 + (m_{22}\gamma_{12} + \gamma_{23})x_2.
By \((4.11)\), we have the Nakayama automorphism of the graded Ore extension \(B = A[z; \overline{\eta}, \overline{\delta}]\) satisfies that 
\[
\mu_B(x_1) = qm_1^{-1}x_1, \quad \mu_B(x_2) = (qm_2^{-1})^{-1}x_2.
\]

\[(4.9)\]
\[
\mu_B(z) = m_1m_2z + ((m_2 + qm_1^{-1})\gamma_{11} + \gamma_{22})x_1 + ((m_1 + (qm_2^{-1})^{-1})\gamma_{23} + q^{-1}\gamma_{12})x_2.
\]

**Solution 4.12.** All solutions of \((4.9)\) are as follows.

(a) If \(m_1 = q, m_2 = q^{-1}\), then \(\gamma_{12} = -q(1 + q)\gamma_{23}, \gamma_{22} = -(q^{-1} + 1)\gamma_{11}\) and \(\gamma_{13}, \gamma_{21}\) are free;

(b) If \(m_1 \neq q, m_2 = q^{-1}\), then \(\gamma_{21} = 0, \gamma_{11} = (q^{-1} - q)^{-1}(m_1 - 1)\gamma_{22}, \gamma_{12} = (q^{-1} - 1)^{-1}(qm_1 - 1)\gamma_{23}\), and \(\gamma_{13}\) is free;

(c) If \(m_1 = q, m_2 \neq q^{-1}\), then \(\gamma_{13} = 0, \gamma_{22} = (1 - q)^{-1}(q - m_2)\gamma_{11}, \gamma_{23} = (1 - q^2)^{-1}(1 - m_2)\gamma_{12}\) and \(\gamma_{21}\) is free;

(d) If \(m_1 = 1, m_2 \neq q^{-1}\), then \(\gamma_{11} = \gamma_{13} = \gamma_{21} = 0, \gamma_{23} = (1 - q^{-1})(1 - m_2)\gamma_{12}\) and \(\gamma_{22}\) is free;

(e) If \(m_1 = q^{-1}, m_2 \neq q^{-1}\), then \(\gamma_{13} = \gamma_{21} = 0, \gamma_{22} = (1 - q^{-1})(q - m_2)\gamma_{11}, \gamma_{23}\) is free and \(\gamma_{12} = 0\) if \(m_2 \neq 1\) or \(\gamma_{12}\) is free if \(m_2 = 1\);

(f) If \(m_1 \neq q^{-1}, 1, m_2 \neq q^{-1}\), then \(\gamma_{13} = \gamma_{21} = 0, \gamma_{22} = (1 - m_1)^{-1}(q - m_2)\gamma_{11}\), and \(\gamma_{23} = (1 -qm_1^{-1})^{-1}(1 - m_2)\gamma_{12}\).

**4.2.3. Case (iii): Jordan plane.** In this case, \(r = x_1x_2 - x_2x_1 - x_2^2\), or equivalently,

\[
Q = \begin{pmatrix}
0 & 1 \\
-1 & -1 \\
\end{pmatrix}
\]

\(M\) has the form \(\begin{pmatrix}
m_{11} & m_{12} \\
0 & m_{11} \\
\end{pmatrix}\) and \(\text{hdet}(\overline{\eta}) = m_1^{-1}\). One obtains that

\[
\delta(r) = (m_{11} - 1)\gamma_{21}x_1^3 + m_{11}\gamma_{22}x_1x_2x_1 + (\gamma_{12} - \gamma_{22})x_2x_1x_2 + m_{11}\gamma_{23}x_1x_2^2 + (\gamma_{11} - \gamma_{21})x_1x_2^2 \\
+ ((m_{12} - m_{11})\gamma_{21} - m_{11}\gamma_{11} - \gamma_{22})x_2x_1^3 + ((m_{12} - m_{11})\gamma_{22} - m_{11}\gamma_{12} - \gamma_{23})x_2x_1^2 \\
+ ((1 - m_{11})\gamma_{13} + (m_{12} - m_{11} - 1)\gamma_{23})x_1^2.
\]

**Lemma 4.13.** The condition \((4.2)\) is equivalent to the following equations hold

\[
(m_{11} - 1)\gamma_{21} = 0,
\]

\[
(m_{11} - 1)\gamma_{11} + (m_{11} - m_{12} + 1)\gamma_{21} + (1 - m_{11})\gamma_{22} = 0,
\]

\[
(m_{11} + 1)\gamma_{11} + (m_{11} - 1)\gamma_{12} + (1 - m_{11} + m_{12})\gamma_{21} + (m_{11} - m_{12})\gamma_{22} + (1 - m_{11})\gamma_{23} = 0,
\]

\[
-2\gamma_{11} - \gamma_{12} + (m_{11} - 1)\gamma_{13} + 2\gamma_{21} + \gamma_{22} + (1 - m_{11} - m_{12})\gamma_{23} = 0.
\]

In this case, \(\delta(r) = r \otimes \delta_r + \delta_l \otimes r\), where

\[
\delta_r = (m_{11}\gamma_{11} + (m_{11} - m_{12})\gamma_{21} + \gamma_{22})x_1 + (\gamma_{11} - \gamma_{21} + m_{11}\gamma_{23})x_2,
\]

\[
\delta_l = (\gamma_{11} - \gamma_{21})x_1 + (\gamma_{11} + \gamma_{12} - \gamma_{21} - \gamma_{22} + m_{11}\gamma_{23})x_2.
\]

By \((4.11)\), we have the Nakayama automorphism of a graded Ore extension \(B = A[z; \overline{\eta}, \overline{\delta}]\) satisfies that 
\[
\mu_B(x_1) = m_1^{-1}x_1 + (2m_1^{-1} - m_1^{-2}m_2)z, \quad \mu_B(x_2) = m_1^{-1}x_2.
\]

\[(4.11)\]
\[
\mu_B(z) = m_1^{-1}z + ((m_{11} + m_{11}^{-1})\gamma_{11} + (m_{11} - m_{11}^{-1} - m_2)\gamma_{21} + \gamma_{22})x_1 \\
+ ((1 + 3m_1^{-1} - m_1^{-2}m_2)\gamma_{11} + m_1^{-1}\gamma_{12} + (m_1^{-1}m_2 - 3m_1^{-1})\gamma_{21} - m_1^{-1}\gamma_{22} + (1 + m_{11})\gamma_{23})x_2.
\]

**Solution 4.14.** All solutions to equations \((4.10)\) are as follows.
(a) If \( m_{11} = 1 \), then \( \gamma_{11} = (m_{12}\gamma_{21} + (1 - m_{12})\gamma_{22})/2 \), \( \gamma_{12} = m_{12}\gamma_{22} - m_{12}\gamma_{23} \), and \( \gamma_{13} \) is free, where \( \gamma_{21} = 0 \) if \( m_{12} = 2 \) or \( \gamma_{21} \) is free if \( m_{12} \neq 2 \).

(b) If \( m_{11} \neq 1 \), \( \gamma_{21} = 0 \), \( \gamma_{11} = \gamma_{22} \), \( \gamma_{12} = (m_{11} - 1)^{-1}(m_{12} + 1)\gamma_{22} + \gamma_{23}, \gamma_{13} = (m_{11} - 1)^{-1}(m_{11} + m_{12})((m_{11} - 1)^{-1}\gamma_{22} + \gamma_{23}) \).

Proof of Theorem 4.2. If \( B \) is CY, then \( \mathcal{T} = \mu_A \) follows by Theorem 4.3.

If \( A \) is commutative, then \( \mu_A \) is the identity map, that is, \( M = E_2 \). Hence, (a) is an immediate result by Solution 4.6(a) and (4.4), or Theorem 4.1.

Now assume \( A \) is noncommutative and \( \mathcal{T} = \mu_A \).

If \( A \) is a quantum plane, then \( \mu_A(x_1) = qx_1 \) and \( \mu_A(x_2) = q^{-1}x_2 \), that is \( m_{11} = q, m_{22} = q^{-1}, m_{12} = m_{21} = 0 \). Hence, \( B \) is CY by Solution 4.8 and (4.6) when \( q = -1 \), and Solution 4.12 and (4.9) when \( q \neq -1 \).

If \( A \) is the Jordan plane, then \( \mu_A(x_1) = x_1 + 2x_2 \) and \( \mu_A(x_2) = x_2 \), that is \( m_{11} = m_{22} = 1, m_{12} = 2, m_{21} = 0 \). By Solution 4.14 and (4.11), one obtains that \( B \) is CY.

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