PROJECTIONS OF FOUR CORNER CANTOR SET: TOTAL SELF-SIMILARITY, SPECTRUM AND UNIQUE CODINGS

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Abstract. Given $\rho \in (0, 1/4]$, the four corner Cantor set $E \subset \mathbb{R}^2$ is a self-similar set generated by the iterated function system
\[
\{(\rho x, \rho y), (\rho x, \rho y + 1 - \rho), (\rho x + 1 - \rho, \rho y), (\rho x + 1 - \rho, \rho y + 1 - \rho)\}.
\]
For $\theta \in [0, \pi)$ let $E_\theta$ be the orthogonal projection of $E$ onto a line with an angle $\theta$ to the $x$-axis. In this paper we give a complete characterization on which the projection $E_\theta$ is totally self-similar. We also study the spectrum of $E_\theta$, which turns out that the spectrum of $E_\theta$ achieves its maximum value if and only if $E_\theta$ is totally self-similar. Furthermore, when $E_\theta$ is totally self-similar, we calculate its Hausdorff dimension and study the subset $U_\theta$ which consists of all $x \in E_\theta$ having a unique coding. In particular, we show that $\dim H U_\theta = \dim H E_\theta$ for Lebesgue almost every $\theta \in [0, \pi)$. Finally, for $\rho = 1/4$ we describe the distribution of $\theta$ in which $E_\theta$ contains an interval. It turns out that the possibility for $E_\theta$ to contain an interval is smaller than that for $E_\theta$ to have an exact overlap.

1. Introduction

The study of linear projections of a planar set has a long history, which can be dated back to Besicovitch [3] and Marstrand [15]: for a Borel or analytic set $E \subset \mathbb{R}^2$, let $E_\theta = \text{proj}_\theta(E)$ denote its orthogonal projection of $E$ onto a line at an angle $\theta$ to the $x$-axis. Then for Lebesgue almost every $\theta \in [0, \pi)$ we have $\dim_H E_\theta = \min \{\dim_H E, 1\}$, and in particular, if $\dim_H E > 1$ then $\text{Leb}(E_\theta) > 0$. Here $\dim_H$ denotes the Hausdorff dimension. In this paper we study projections of the four corner Cantor set $E$ (cf. [16 Ch. 10]), and give a complete characterization for which $E_\theta$ is totally self-similar (see Definition 1.1). Moreover, we study the spectrum of $E_\theta$ (see Definition 1.3) and show that $E_\theta$ is totally self-similar if and only if its spectrum achieves its maximum value. Assuming $E_\theta$ is totally self-similar, we calculate its Hausdorff dimension and study its subset $U_\theta$ which consists of all $x \in E_\theta$ having a unique coding. We show that $\dim_H U_\theta = \dim_H E_\theta$ for Lebesgue almost every $\theta \in [0, \pi)$. Furthermore, when $\rho = 1/4$ we give the distribution of $\theta$ in which $E_\theta$ contains an interval.

Given $\rho \in (0, 1/4]$, let $E \subset \mathbb{R}^2$ be the four corner Cantor set, which is a self-similar set generated by the iterated function system (IFS)
\[
\{(\rho x, \rho y), (\rho x, \rho y + 1 - \rho), (\rho x + 1 - \rho, \rho y), (\rho x + 1 - \rho, \rho y + 1 - \rho)\}.
\]
It is well known that $\dim_H E = \frac{2 \log 2}{\log \rho}$ and its Hausdorff measure $\mathcal{H}^{\frac{2 \log 2}{\log \rho}}(E) \in (0, \infty)$ (cf. [11]). Furthermore, the self-similar set $E$ can be written algebraically as
For $t \in \mathbb{R}$ let $E(t)$ be its orthogonal projection onto a line with slope $t$. Then

$$E(t) = \left\{ \frac{x + ty}{\sqrt{1 + t^2}} : (x, y) \in E \right\},$$

and it is also a self-similar set. By symmetry and scaling, we may reduce the projection $E(t)$ to the self-similar set $E_\lambda$ generated by the IFS

$$F_\lambda := \{ f_d(x) = \rho x + d : d \in \Omega_\lambda \} \quad \text{with} \quad \Omega_\lambda := \{ 0, \lambda, 1 - \rho - \lambda, 1 - \rho \},$$

where $\lambda \in [0, 1 - \rho]$. In other word, it suffices to consider

$$E_\lambda = \bigcup_{d \in \Omega_\lambda} f_d(E_\lambda) = \left\{ \sum_{i=1}^{\infty} \rho^{i-1}d_i \mid d_i \in \Omega_\lambda \forall i \in \mathbb{N} \right\}, \quad \lambda \in [0, 1 - \rho].$$

Note that $\rho \in (0, 1/4]$ is fixed, and it is always suppressed in our notation. In fact, we can restrict our parameter $\lambda$ to the interval $(0, \rho) \cup \left(\frac{1-2\rho}{2}, \frac{1-\rho}{2}\right)$. Note by the symmetry that $E_\lambda$ has the same geometrical structure as $E_{\lambda - \rho}$ for any $\lambda \in [0, 1 - \rho)$. Then we only need to consider $\lambda \in [0, \frac{1-2\rho}{2}]$. Moreover, for $\lambda \in [\rho, \frac{1-2\rho}{2}]$ the self-similar set $E_\lambda$ satisfies the open set condition, and for $\lambda = 0$ or $\lambda = \frac{1-\rho}{2}$ the self-similar set $E_\lambda$ can be degenerated to a self-similar set satisfies the strong separation condition (SSC). So we only need to consider $\lambda \in (0, \rho) \cup \left(\frac{1-2\rho}{2}, \frac{1-\rho}{2}\right)$, and in this case the self-similar set $E_\lambda$ has non-trivial overlaps (see Figure 1 for the two types of overlapping structure).

\begin{figure}[h]
\centering
\begin{tabular}{c c c}
0 & $\lambda$ & 1 \\
\hline
$\lambda$ & $f_\lambda$ & $\frac{1 - \rho - \lambda}{1 - \rho}$ \\
0 & $\frac{1 - \rho - \lambda}{1 - \rho}$ & 1
\end{tabular}
\hspace{1cm}
\begin{tabular}{c c c}
0 & $\lambda$ & 1 \\
\hline
$\lambda$ & $f_\lambda$ & $\frac{1 - \rho - \lambda}{1 - \rho}$ \\
0 & $\frac{1 - \rho - \lambda}{1 - \rho}$ & 1
\end{tabular}
\caption{The first two levels for the geometric construction of $E_\lambda$ with $\lambda \in (0, \rho)$ (left) and $\lambda \in \left(\frac{1-2\rho}{2}, \frac{1-\rho}{2}\right)$ (right).}
\end{figure}

In 2004 Broomhead, Montaldi and Sidorov \cite{4} introduced the following finer family of self-similar sets with overlaps. For $n \in \mathbb{N} \cup \{0\}$ let $\Omega_\lambda^n := \{ i_1i_2 \cdots i_n \mid i_k \in \Omega_\lambda, 1 \leq k \leq n \}$, where for $n = 0$ we set $\Omega_\lambda^0 := \{ \epsilon \}$ with $\epsilon$ the empty word. Let $\Omega_\lambda^*$ be the set of all finite words over the alphabet $\Omega_\lambda$, i.e., $\Omega_\lambda^* = \bigcup_{n=0}^{\infty} \Omega_\lambda^n$. Furthermore, let $\Omega_\lambda^\infty$ be the set of all infinite sequences over the alphabet $\Omega_\lambda$. For $i = i_1i_2 \cdots i_n \in \Omega_\lambda^*$ we write

$$f_i(x) := f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(x) = \rho^n x + \sum_{k=1}^{n} \rho^{k-1}i_k$$

as compositions of maps. In particular, for $i = \epsilon$ we set $f_\epsilon$ as the identity map.
**Theorem 1.4.** Overlapping self-similar sets (cf. [6]) we consider the spectrum of \( E \).

**Definition 1.3.**

Our first result characterizes when \( E \) is totally self-similar. For \( k \in \mathbb{N} \cup \{0\} \) we define

\[
\lambda_k := \frac{\rho(1-\rho^k)}{1+\rho^k}, \quad \gamma_k := \rho(1-\rho^k) \quad \text{and} \quad \eta_k := \frac{1-2\rho+\rho^{k+1}}{2}.
\]

Then it is clear that

\[
0 = \lambda_0 = \gamma_0 < \lambda_1 < \gamma_1 < \ldots < \lambda_k < \gamma_k < \ldots < \rho, \quad \frac{1}{2} = \eta_0 > \eta_1 > \eta_2 > \ldots > \eta_k > \eta_{k+1} > \cdots > \frac{1-2\rho}{2}.
\]

Furthermore, \( \lambda_k, \gamma_k \not> \rho \) and \( \eta_k \not< \frac{1-2\rho}{2} \) as \( k \to \infty \).

**Theorem 1.2.** Let \( \lambda \in (0, \rho) \cup \left( \frac{1-2\rho}{2}, \frac{1-\rho}{2} \right) \).

(i) If \( \lambda \in (0, \rho) \), then \( E \) is totally self-similar if and only if \( \lambda = \lambda_k \) or \( \gamma_k \) for some \( k \in \mathbb{N} \).

(ii) If \( \lambda \in \left( \frac{1-2\rho}{2}, \frac{1-\rho}{2} \right) \), then \( E \) is totally self-similar if and only if \( \lambda = \eta_k \) for some \( k \in \mathbb{N} \).

Given two words \( i, j \in \Omega_\lambda^n \), by (1.4) it is clear that \( f_i = f_j \) if and only if \( f_i(0) = f_j(0) \).

So, the scaled distance \( \frac{|f_i(0) - f_j(0)|}{\rho^n} \) describes the closeness of the two maps \( f_i \) and \( f_j \), which reveals the complexity of the overlapping structure of \( E_\lambda \). Let

\[
A_\lambda := \left\{ \frac{|f_i(0) - f_j(0)|}{\rho^n} : i, j \in \Omega_\lambda^n \text{ with } f_i \neq f_j; \ n \in \mathbb{N} \right\}
\]

\[
= \left\{ \left| \sum_{i=1}^{n} \frac{d_i}{\rho^i} \right| \neq 0 : d_i \in \Omega_\lambda^n; \ n \in \mathbb{N} \right\},
\]

where \( \Omega_\lambda^\pm := \Omega_\lambda - \Omega_\lambda = \{0, \pm \lambda, \pm (1-\rho-2\lambda), \pm (1-\rho-\lambda), \pm (1-\rho)\} \).

Motivated by the spectrum from non-integer base expansions (cf. [7]) and the spectrum for overlapping self-similar sets (cf. [8]) we consider the spectrum of \( E_\lambda \).

**Definition 1.3.** For \( \lambda \in (0, \rho) \cup \left( \frac{1-2\rho}{2}, \frac{1-\rho}{2} \right) \), the spectrum of \( E_\lambda \) is defined by

\[
l_\lambda := \inf A_\lambda = \inf \left\{ \left| \sum_{i=1}^{n} \frac{d_i}{\rho^i} \right| \neq 0 : d_i \in \Omega_\lambda^\pm; \ n \in \mathbb{N} \right\}.
\]

Our second result describes the spectrum \( l_\lambda \) of \( E_\lambda \).

**Theorem 1.4.**

(i) For any \( k \in \mathbb{N} \) we have

\[
l_{\lambda_k} = 1 - \rho - \lambda_k, \quad l_{\gamma_k} = 1 - \rho \quad \text{and} \quad l_{\eta_k} = 1 - \rho.
\]

(ii) If \( \lambda \in (0, \rho) \), then \( E_\lambda \) is not totally self-similar if and only if \( l_k < 1 - \rho - \lambda \).

(iii) If \( \lambda \in \left( \frac{1-2\rho}{2}, \frac{1-\rho}{2} \right) \), then \( E_\lambda \) is not totally self-similar if and only if \( l_k < 1 - \rho \).
Remark 1.5.  (i) By Theorem 1.4 it follows that the spectrum \( l_\lambda \) attains its maximum value if and only if \( E_\lambda \) is totally self-similar.

(ii) If \( 0 < \rho < 1/9 \), then by \[1.1\] we have \( \dim_H(E - E) < 1 \). By \[18\] Lemma 2.7 it follows that \( \mathcal{H}^s(E_\lambda) > 0 \) for Lebesgue almost every \( \lambda \in (0, \rho) \cup \left( \frac{1-2\rho}{2}, \frac{1-\rho}{2} \right) \), where \( s = \frac{\log 4}{-\log \rho} \).

Hence, by \[3\] Corollary 3.2 we can deduce that \( E_\lambda \) satisfies the weak separate condition for Lebesgue almost every \( \lambda \). Note that \( l_\lambda > 0 \) is equivalent to that \( E_\lambda \) satisfies the weak separation condition (cf. \[21\] \[8\]). So, if \( \rho \in (0, 1/9) \) then \( l_\lambda > 0 \) for Lebesgue almost every \( \lambda \).

In view of \[1.3\] and \[1.4\], for each \( \lambda \in E_\lambda \) we can find a sequence \( (d_i) = d_1d_2\ldots \in \Omega_\lambda^N \) such that

\[
(1.6) \quad x = \lim_{n \to \infty} f_{d_1\ldots d_n}(0) = \sum_{i=1}^{\infty} \rho^{i-1}d_i =: \pi_\lambda((d_i)).
\]

The infinite sequence \( (d_i) \) is called a coding of \( x \) with respect to the digit set \( \Omega_\lambda \). Since \( E_\lambda \) has overlaps, \( x \in E_\lambda \) might have multiple codings. In this paper, we are also interested in the subset

\[
U_\lambda := \{ x \in E_\lambda : \#\pi_\lambda^{-1}(x) = 1 \}.
\]

Then each \( x \in U_\lambda \) has a unique coding.

Our third result shows that if \( E_\lambda \) is totally self-similar, then \( \dim_H U_\lambda < \dim_H E_\lambda \). Furthermore, we give the analytic formula for the dimension of \( U_\lambda \).

Theorem 1.6. If \( E_\lambda \) is totally self-similar, i.e., \( \lambda \in \bigcup_{k=1}^{\infty} \{ \lambda_k, \gamma_k, \eta_k \} \), then

\[
\dim_H U_\lambda < \dim_H E_\lambda.
\]

(i) If \( \lambda = \lambda_k \) for some \( k \in \mathbb{N} \), then \( \dim_H U_\lambda = s \), where \( s \in (0, 1) \) is an appropriate root of

\[
4\rho^s - 2\rho^{ks} = 1.
\]

(ii) If \( \lambda = \gamma_k \) for some \( k \in \mathbb{N} \), then \( \dim_H U_\lambda = s, \dim_H E_\lambda = t \), where \( s, t \in (0, 1) \) are respectively appropriate roots of

\[
4\rho^s - \rho^{ks} = 1 \quad \text{and} \quad 4\rho^t - 2\rho^{(k+1)t} = 1.
\]

(iii) If \( \lambda = \eta_k \) for some \( k \in \mathbb{N} \), then \( \dim_H U_\lambda = s, \dim_H E_\lambda = t \), where \( s, t \in (0, 1) \) are respectively appropriate roots of

\[
4\rho^s - 2\rho^{(k+1)s} = 1 \quad \text{and} \quad 4\rho^t - \rho^{(k+1)t} = 1.
\]

Remark 1.7. When \( \lambda = \lambda_k \), although the self-similar set \( E_\lambda \) can be represented as a graph-directed set, the directed graph does not satisfy the open set condition. So we don’t know how to calculate the Hausdorff dimension of \( E_\lambda \) in this case.

If \( E_\lambda \) is totally self-similar, then \( E_\lambda \) has exact overlaps, i.e., \( f_i = f_j \) for some \( i \neq j \) (see Remark 4.1 and Lemma 4.2). This implies that \( \dim_H E_\lambda < \dim_S E_\lambda = \frac{\log 4}{-\log \rho} \), where \( \dim_S \) denotes the similarity dimension. However, by \[20\] Theorem 2.1 we know that \( \dim_H E_\lambda = \frac{\log 4}{-\log \rho} \) for Lebesgue almost every \( \lambda \in (0, \rho) \cup \left( \frac{1-2\rho}{2}, \frac{1-\rho}{2} \right) \). Our fourth result states that for typical \( \lambda \) the univoque set \( U_\lambda \) has the same Hausdorff dimension as \( E_\lambda \).
Theorem 1.8. If \( \rho \in (0, 1/4) \), then for Lebesgue almost every \( \lambda \in (0, \rho) \cup \left( \frac{1-2\rho}{2}, \frac{1-\rho}{2} \right) \) we have
\[
\dim_H U_\lambda = \dim_H E_\lambda = \frac{\log 4}{-\log \rho}.
\]
In particular, if \( \rho \in (0, 1/16) \), then \( U_\lambda = E_\lambda \) for Lebesgue almost every \( \lambda \in (0, \rho) \cup \left( \frac{1-2\rho}{2}, \frac{1-\rho}{2} \right) \).

Remark 1.9. (i) Theorem 1.8 can be easily extended to all \( \lambda \in \mathbb{R} \);
(ii) If \( \rho \in (0, 1/16) \), then Theorem 1.8 suggests that for Lebesgue almost every \( \lambda \in (0, \rho) \cup \left( \frac{1-2\rho}{2}, \frac{1-\rho}{2} \right) \) the self-similar set \( E_\lambda \) satisfies the SSC, i.e., \( f_i(E_\lambda) \cap f_j(E_\lambda) = \emptyset \) for any \( i \neq j \in \Omega_\lambda \). An extension to a larger class of self-similar sets with overlaps can be found in [2].

When \( \rho = 1/4 \), projections of the four corner Cantor set \( E \) defined in (1.1) are extensively studied (cf. [12] [13] [14] [16]). Note by (1.2) that the scaled projection \( E_\lambda \) is a self-similar set generated by the IFS
\[
\left\{ f_d(x) = \frac{x + d}{4} : d \in \{0, 4\lambda, 3 - 4\lambda, 3\} \right\}.
\]

It is known that \( E_\lambda \) has zero Lebesgue measure for Lebesgue almost every \( \lambda \in \mathbb{R} \). Indeed, there are only countably many \( \lambda \) for which \( E_\lambda \) has positive Lebesgue measure. More precisely, if \( \lambda \notin \mathbb{Q} \), then \( E_\lambda \) has zero Lebesgue measure but full Hausdorff dimension; if \( \lambda \in \mathbb{Q} \) and \( E_\lambda \) has exact overlaps (see its definition below), then \( \dim_H E_\lambda < 1 \); if \( \lambda \in \mathbb{Q} \) but \( E_\lambda \) does not have an exact overlap, then \( E_\lambda \) is a perfect set containing a non-degenerate interval.

Definition 1.10. \( E_\lambda \) is said to have an exact overlap if there exist two blocks \( i = i_1 \ldots i_n, j = j_1 \ldots j_n \in \{0, 4\lambda, 3 - 4\lambda, 3\}^n \) such that \( \hat{f}_i = \hat{f}_j \).

The following complete characterization of exact overlaps of \( E_\lambda \) can be essentially deduced from [16] Theorem 10.5 (see also, [12] [13] [14]). For \( n \in \mathbb{N} \) let \( \text{ord}_2(n) \) be the highest power of 2 that divides \( n \) (cf. [13] P. 2).

Theorem 1.11 ([16]). Let \( \rho = 1/4 \) and \( \lambda \in (0, \frac{3}{8}) \). Then \( E_\lambda \) has an exact overlap if and only if
\[
\lambda = \frac{3p}{4(p+q)} \in \mathbb{Q} \quad \text{with} \quad (p, q) \in W,
\]
where
\[
W := \{(p, q) \in \mathbb{N}^2 : \text{both ord}_2(p) \text{ and ord}_2(q) \text{ are even}, \ p < q \text{ and } p, q \text{ are coprime}\}.
\]

Furthermore, the following statements hold true.

(i) If \( \lambda = \frac{3p}{4(p+q)} \in \mathbb{Q} \) in reduced form with \( (p, q) \in W \), then \( \dim_H E_\lambda < 1 \).
(ii) If \( \lambda = \frac{3p}{4(p+q)} \in \mathbb{Q} \) in reduced form with \( (p, q) \notin W \), then \( E_\lambda \) contains an interval.
(iii) If \( \lambda \notin \mathbb{Q} \), then \( C_\lambda \) has zero Lebesgue measure and \( \dim_H E_\lambda = 1 \).

Remark 1.12. Note that in [16] Theorem 10.5 the result was stated using the following notation. For \( n \in \mathbb{N} \) let \( n^* \in \{1, 2, 3\} \) be defined by
\[
n^* = \frac{n}{4^p} \mod 4,
\]
where \( j_0 \) is the largest integer \( j \) such that \( 4^j \) divides \( n \). One can easily verify that \( n^* \) is odd if and only if \( \text{ord}_2(n) \) is even. So, Theorem 1.11 is the same as [16, Theorem 10.5].

Our final result describes the density of \( W \) in \( \mathbb{N}^2 \), which reveals the possibility in which the projection \( E_\lambda \) has an exact overlap. We also consider the density of

\[
W := \{(p, q) \in \mathbb{N}^2 : \text{ord}_2(p) \text{ odd or ord}_2(q) \text{ odd, and } p < q \text{ with } p, q \text{ coprimes}\},
\]

which describes the possibility in which \( E_\lambda \) contains a non-degenerate interval. For a set \( A \) let \( \#A \) denote its cardinality.

**Theorem 1.13.** Let \( W \) and \( \hat{W} \) be defined as in (1.7) and (1.8) respectively. Then

\[
\lim_{N \to \infty} \frac{(W \cap [1, N]^2)}{N^2} = \frac{5}{3\pi^2} \quad \text{and} \quad \lim_{N \to \infty} \frac{(\hat{W} \cap [1, N]^2)}{N^2} = \frac{4}{3\pi^2}.
\]

Theorem 1.13 indicates that the possibility for \( E_\lambda \) to contain a non-degenerate interval is smaller than that for \( E_\lambda \) to have an exact overlap.

The rest of the paper is organized as follows. In the next section we give a complete characterization when \( E_\lambda \) is totally self-similar and prove Theorem 1.2. In Section 3 we study the spectrum of \( E_\lambda \) and prove Theorem 1.4. In particular, we show that \( E_\lambda \) is totally self-similar if and only if its spectrum achieves its maximum value. In Section 4 we consider the subset \( U_\lambda \) which consists of all \( x \in E_\lambda \) having a unique coding, and calculate its Hausdorff dimension (Theorem 1.6). In Section 5 we show that \( \dim_H U_\lambda = \dim_H E_\lambda \) for typical \( \lambda \), and prove Theorem 1.8. Finally, we consider the four corner Cantor set \( E \) with dimension one, i.e., \( \rho = 1/4 \). Although the projection \( E_\lambda \) was extensively studied in the literature (see, e.g., [16, Ch. 10]), we add a new result on the distribution of \( \lambda \) in which \( E_\lambda \) contains an interval.

2. Total self-similarity of \( E_\lambda \)

In this section we will characterize the total self-similarity of \( E_\lambda \), and prove Theorem 1.2. Given \( \rho \in (0, 1/4] \) and \( \lambda \in (0, \rho) \cup \left(\frac{1-2\rho}{2}, \frac{1-\rho}{2}\right) \), we recall from Definition 1.1 that \( I = [0, 1] \) is the convex hull of \( E_\lambda \). Set \( I_0 = I \), and for \( n \geq 1 \) let

\[
I_n := \bigcup_{i \in \Omega^n_\lambda} f_i(I),
\]

where \( \Omega^n_\lambda \) consists of all length \( n \) words over \( \Omega_\lambda = \{0, \lambda, 1-\rho-\lambda, 1-\rho\} \). Let

\[
H := I \setminus I_1 = I \setminus \bigcup_{d \in \Omega_\lambda} f_d(I)
\]

be a hole of \( E_\lambda \). The following characterization of total self-similarity of \( E_\lambda \) can be found in [6, Proposition 2.1].

**Proposition 2.1.** The set \( E_\lambda \) is totally self-similar if and only if for any two words \( i, j \in \Omega^n_\lambda \) with \( n \in \mathbb{N} \),

\[
\text{either } f_i = f_j \quad \text{or} \quad f_i(I) \cap f_j(H) = \emptyset.
\]

Recall from (1.5) the definitions of \( \lambda_k, \gamma_k \) and \( \eta_k \) for \( k \in \mathbb{N} \). Let

\[
\ell_k := \frac{\rho(1 - \rho^{k+1})}{1 + \rho^k + \rho^{k+1}}, \quad k \in \mathbb{N}.
\]
Then by using $\rho \in (0, 1/4]$ it follows that
$$\lambda_k < \ell_k < \gamma_k \quad \text{for all } k \in \mathbb{N}.$$ To prove Theorem 1.2 we need the following two lemmas.

**Lemma 2.2.** Let $\rho \in (0, 1/4]$ and $k \in \mathbb{N}$.

(i) If $\lambda \geq \lambda_k$, then $f_{\lambda_0^k}(1 - \rho - \lambda) > f_0(1 - \rho)^{k-1}(1 - \rho - \lambda)(0)$.

(ii) If $\lambda \geq \ell_k$, then $f_{\lambda_0^k}(1 - \rho - \lambda) > f_0(1 - \rho)^k(0)$.

(iii) If $\lambda < \eta_k$, then $f_{\lambda(1 - \rho)^{k-1}}(0) < f_{(1 - \rho - \lambda)^{k-1}}(0)$.

**Proof.** Since the proofs of the three items are similar, we only prove (i). Note by (1.4) that
$$f_{\lambda_0^k}(1 - \rho - \lambda) = \lambda + \rho^{k+1}(1 - \rho - \lambda), \quad f_0(1 - \rho)^{k-1}(1 - \rho - \lambda)(0) = \rho - \rho^{k+1} - \rho^k \lambda.$$ Then $f_{\lambda_0^k}(1 - \rho - \lambda) > f_0(1 - \rho)^{k-1}(1 - \rho - \lambda)(0)$ is equivalent to
$$\lambda > \frac{\rho - 2\rho^{k+1} + \rho^{k+3}}{1 + \rho^k - \rho^{k+1}}.$$ Since $\lambda \geq \lambda_k = \frac{(1 - \rho)^k}{1 + \rho^k}$, it suffices to prove
$$\frac{\rho(1 - \rho^k)}{1 + \rho^k} > \frac{\rho - 2\rho^{k+1} + \rho^{k+3}}{1 + \rho^k - \rho^{k+1}},$$ which holds by using $0 < \rho \leq 1/4$. \qed

For $\lambda \in (0, \rho)$ we recall that $\Omega_{\lambda_1}^\pm = \{0, \pm \lambda, \pm (1 - \rho - 2\lambda), \pm (1 - \rho - \lambda), \pm (1 - \rho)\}$.

**Lemma 2.3.** Let $\lambda = \lambda_1 = \frac{\rho(1 - \rho)}{1 + \rho}$. If $d \in \Omega_{\lambda_1}^\pm$ and $d \leq 0$, then
$$d + \frac{\lambda_1}{\rho} \in \Omega_{\lambda_1}^\pm.$$ **Proof.** Note that
$$\frac{\lambda_1}{\rho} = \frac{1 - \rho}{1 + \rho} = (1 - \rho) \left(1 - \frac{\rho}{1 + \rho}\right) = 1 - \rho - \frac{\rho(1 - \rho)}{1 + \rho} = 1 - \rho - \lambda_1.$$ This implies that
$$0 + \frac{\lambda_1}{\rho} = 1 - \rho - \lambda_1 \in \Omega_{\lambda_1}^\pm, \quad -\lambda_1 + \frac{\lambda_1}{\rho} = 1 - \rho - 2\lambda_1 \in \Omega_{\lambda_1}^\pm,$$ $$-(1 - \rho - 2\lambda_1) + \frac{\lambda_1}{\rho} = \lambda_1 \in \Omega_{\lambda_1}^\pm, \quad -(1 - \rho - \lambda_1) + \frac{\lambda_1}{\rho} = 0 \in \Omega_{\lambda_1}^\pm,$$ $$-(1 - \rho) + \frac{\lambda_1}{\rho} = -\lambda_1 \in \Omega_{\lambda_1}^\pm$$ as desired. \qed

In the following we split our proof of Theorem 1.2 into two subsections for $\lambda \in (0, \rho)$ and $\lambda \in \left(\frac{1 - 2\rho}{2}, \frac{1 - \rho}{2}\right)$, separately.
2.1. Total self-similarity of $E_\lambda$ for $\lambda \in (0, \rho)$. Since $\lambda \in (0, \rho)$, the hole $H$ is given by (see the left graph of Figure 1)

$$H = I \setminus I_1 = (\rho + \lambda, 1 - \rho - \lambda).$$

First we show that $\lambda \in \bigcup_{k=1}^\infty \{\lambda_k, \gamma_k\}$ is necessary for the total self-similarity of $E_\lambda$.

**Proposition 2.4.** If $\lambda \in (0, \rho) \setminus \bigcup_{k=1}^\infty \{\lambda_k, \gamma_k\}$, then $E_\lambda$ is not totally self-similar.

**Proof.** Note by (1.5) and (2.1) that $0 < \lambda_k < \ell_k < \gamma_k < \lambda_{k+1}$ for any $k \in \mathbb{N}$, and $\lambda_k \not> \rho$ as $k \to \infty$. So, we only need to prove that $E_\lambda$ is not totally self-similar for any (I) $\lambda \in (0, \lambda_1)$; (II) $\lambda \in \bigcup_{k=1}^\infty \{\lambda_k, \ell_k\}$; and (III) $\lambda \in \bigcup_{k=1}^\infty \{\ell_k, \gamma_k\} \cup \{\gamma_k, \lambda_{k+1}\}$. By Proposition 2.1 it suffices to prove in the three different cases that there exist two words $i, j \in \Omega_\lambda^n$ for some $n \in \mathbb{N}$ such that

(2.2) \quad $f_i \neq f_j$ and \quad $f_i(I) \cap f_j(H) \neq \emptyset$,

where $I = [0, 1]$ and $H = (\rho + \lambda, 1 - \rho - \lambda)$.

Case (I) $\lambda \in (0, \lambda_1)$. Take $i = 0$, $j = \lambda \in \Omega_\lambda$. Since $f_0(0) = 0 < \lambda = f_\lambda(0)$, by (1.4) we have $f_0 \neq f_\lambda$. Furthermore, note that

(2.3) \quad $f_0(I) = [0, \rho]$ \quad and \quad $f_\lambda(H) = (\rho(\rho + \lambda) + (1 - \rho - \lambda) + \lambda)$.

Then by using $0 < \lambda < \rho < 1/4$ it is clear that $0 < \rho(\rho + \lambda) + (1 - \rho - \lambda)$. Since $\lambda < \lambda_1 = \frac{\rho(1 - \rho)}{1 + \rho}$, we have $\rho > \rho(\rho + \lambda) + \lambda$. This together with (2.3) implies that $f_0(I) \cap f_\lambda(H) \neq \emptyset$, proving (2.2).

Case (II) $\lambda \in (\lambda_k, \ell_k)$ for some $k \in \mathbb{N}$. Take $i = \lambda 0^k$ and $j = 0(1 - \rho)^{k-1}(1 - \rho - \lambda)$. Since $\lambda > \lambda_k$, by (1.4) and (1.5) it follows that

$$f_{\lambda 0^k}(0) = \lambda > \rho - \rho^k\lambda - \rho^{k+1} = f_{0(1 - \rho)^{k-1}(1 - \rho - \lambda)}(0),$$

which implies $f_{\lambda 0^k} \neq f_{0(1 - \rho)^{k-1}(1 - \rho - \lambda)}$. Moreover, by (1.4) it follows that

(2.4) \quad $f_{\lambda 0^k}(H) = \left(\lambda + \rho^{k+1}(\rho + \lambda), \lambda + \rho^{k+1}(1 - \rho - \lambda)\right)$,

$\quad f_{0(1 - \rho)^{k-1}(1 - \rho - \lambda)}(I) = \left[\rho - \rho^k\lambda - \rho^{k+1}, \rho - \rho^k\lambda\right]$.

Since $\lambda \in (\lambda_k, \ell_k)$, by (2.1) we have $\lambda + \rho^{k+1}(\rho + \lambda) < \rho - \rho^k\lambda$. Furthermore, by Lemma 2.2 (i) it follows that

$$\rho - \rho^k\lambda - \rho^{k+1} = f_{0(1 - \rho)^{k-1}(1 - \rho - \lambda)}(0) < f_{\lambda 0^k}(1 - \rho - \lambda) = \lambda + \rho^{k+1}(1 - \rho - \lambda).$$

So, by (2.4) it follows that $f_{\lambda 0^k}(H) \cap f_{0(1 - \rho)^{k-1}(1 - \rho - \lambda)}(I) \neq \emptyset$, establishing (2.2).

Case (III) $\lambda \in [\ell_k, \gamma_k) \cup (\gamma_k, \lambda_{k+1})$ for some $k \in \mathbb{N}$. Take $i = \lambda 0^k$ and $j = 0(1 - \rho)^k$. Note by (1.4) and (1.5) that

$$f_{0(1 - \rho)^k}(0) = \rho(1 - \rho^k) = \gamma_k \neq \lambda = f_{\lambda 0^k}(0).$$

Then $f_{0(1 - \rho)^k} \neq f_{\lambda 0^k}$. Furthermore, note by (1.4) that

(2.5) \quad $f_{\lambda 0^k}(H) = \left(\lambda + \rho^{k+1}(\rho + \lambda), \lambda + \rho^{k+1}(1 - \rho - \lambda)\right)$,

$\quad f_{0(1 - \rho)^k}(I) = \left[\rho(1 - \rho^k), \rho\right].$
Since $\lambda \in [\ell, \gamma_k) \cup (\gamma_k, \lambda_{k+1})$, by (1.5) we have $\lambda + \rho^{k+1}(\rho + \lambda) < \rho$, and by Lemma 2.2 (ii) it follows that

$$\rho(1 - \rho^k) = f_0(1 - \rho)^k(0) < f_M(1 - \rho - \lambda) = \lambda + \rho^{k+1}(1 - \rho - \lambda).$$

Therefore, by using (2.5) we obtain $f_0(1 - \rho)^k(I) \cap f_M^k(H) \neq \emptyset$, completing the proof. □

The proof for $\lambda \in \bigcup_{k=1}^{\infty} \{\lambda_k, \gamma_k\}$ to be sufficient is more involved. First we consider $\lambda = \lambda_k$ for some $k \in \mathbb{N}$.

**Proposition 2.5.** If $\lambda = \lambda_k$ for some $k \in \mathbb{N}$, then $E_\lambda$ is totally self-similar.

**Proof.** Let $\lambda = \lambda_k = \frac{\rho(1 - \rho^k)}{1 + \rho^k}$. By Proposition 2.1 it suffices to prove that for any $n \in \mathbb{N}$ and for any $i, j \in \Omega_\lambda^n$ with $f_i \neq f_j$ we have

$$f_i(H) \cap f_j(I) = (f_i(0) + \rho^n(\rho + \lambda), f_i(0) + \rho^n(1 - \rho - \lambda)) \cap [f_j(0), f_j(0) + \rho^n] = \emptyset,$$

which is equivalent to

$$f_i(0) + \rho^n(\rho + \lambda) \geq f_j(0) + \rho^n \quad \text{or} \quad f_i(0) + \rho^n(1 - \rho - \lambda) \leq f_j(0).$$

In other words,

$$|f_i(0) - f_j(0)| \geq \rho^n(1 - \rho - \lambda).$$

So, we only need to prove that for any $n \in \mathbb{N}$ and for any $i = i_1 \ldots i_n, j = j_1 \ldots j_n \in \Omega_\lambda^n$ with $f_i \neq f_j$,

$$1 - \rho - \lambda \leq \left| \sum_{m=1}^{n} \frac{\rho^{n-1} i_m}{\rho^n} - \sum_{m=1}^{n} \frac{\rho^{n-1} j_m}{\rho^n} \right| = \left| \sum_{m=1}^{n} i_m - j_m \rho^{m-1} \right| = \left| \sum_{m=1}^{n} c_m \right|,$$

where each $c_m \in \Omega_\lambda^\pm = \{0, \pm \lambda, \pm (1 - \rho - 2\lambda), \pm (1 - \rho - \lambda), \pm (1 - \rho)\}$.

Suppose on the contrary there exists a block $c_1 c_2 \cdots c_n \in (\Omega_\lambda^\pm)^n$ for some $n \in \mathbb{N}$ such that

$$0 < \left| \sum_{m=1}^{n} c_m \rho^{-m} \right| < 1 - \rho - \lambda.$$  

Furthermore, we can choose the block $c_1 c_2 \cdots c_n \in (\Omega_\lambda^\pm)^n$ satisfying (2.7) such that the finite sequence $|c_n|, |c_{n-1}|, \cdots, |c_1|$ is lexicographically minimal, and $c_n \neq 0$. Without loss of generality we can assume that $c_n > 0$. Note that for any $\lambda \in (0, \rho)$,

$$1 - \rho > 1 - \rho - \lambda > 1 - \rho - 2\lambda > \lambda > 0.$$  

If $n = 1$, then by using $\lambda = \lambda_k \geq \lambda_1 = \frac{\rho(1 - \rho)}{1 + \rho}$ and (2.8) it follows that

$$\left| \frac{c_1}{\rho} \right| \geq \frac{\lambda}{\rho} \geq 1 - \rho - \lambda,$$

leading to a contradiction with (2.7). So we must have $n \geq 2$.  

Note that \( c_n \in \Omega^\pm \) and \( c_n > 0 \). Then \( c_n \in \{ \lambda, 1 - \rho - 2\lambda, 1 - \rho - \lambda, 1 - \rho \} \). If \( c_n \in \{ 1 - \rho - 2\lambda, 1 - \rho - \lambda, 1 - \rho \} \), then by using \( 0 < \lambda < \rho \leq 1/4 \) and \( n \geq 2 \) it follows that

\[
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{1 - \rho - 2\lambda}{\rho^n} - \sum_{m=1}^{n-1} \frac{1 - \rho}{\rho^m} = \frac{1 - \rho - 2\lambda}{\rho^n} - \frac{1 - \rho^{n-1}}{\rho^{n-1}} = 1 + \frac{1 - 2\rho - 2\lambda}{\rho^n} > 1,
\]

contradicting to (2.7). So, in the following it suffices to consider \( c_n = \lambda \), which will be split into the following three cases: (I) \( \lambda = \lambda_1 \); (II) \( \lambda = \lambda_k \) with \( k \geq 2 \) and \( n \leq k \); (III) \( \lambda = \lambda_k \) with \( k \geq 2 \) and \( n > k \).

Case (I) \( c_n = \lambda = \lambda_1 \). If \( c_{n-1} \leq 0 \), then by using \( c_{n-1} \in \Omega^\pm_{\lambda_1} \) and Lemma 2.3 we have \( c_{n-1} + \frac{\lambda}{\rho} \in \Omega^\pm_{\lambda_1} \). Thus

\[
\sum_{m=1}^{n} \frac{c_m}{\rho^m} = \frac{1}{\rho^{n-1}} \left( c_{n-1} + \frac{\lambda}{\rho} \right) + \sum_{m=1}^{n-2} \frac{c_m}{\rho^m},
\]

which contradicts to our assumption that \( |c_n|, |c_{n-1}|, \ldots, |c_1| \) is lexicographically minimal. So, \( c_{n-1} > 0 \), and thus by using \( 0 < \rho \leq 1/4 \) and (2.8) it follows that

\[
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{\lambda}{\rho^n} - \sum_{m=1}^{n-2} \frac{1 - \rho}{\rho^m} = \frac{\lambda}{\rho^n-1} \left( \frac{1}{\rho} + 1 \right) - \frac{1 - \rho^{n-2}}{\rho^{n-2}} = 1 + \frac{\lambda(1+\rho) - \rho^2}{\rho^n} > 1,
\]

where the last inequality follows by \( \lambda = \lambda_1 = \frac{\rho(1-\rho)}{1+\rho} \). This leads to a contradiction with (2.7).

Case (II) \( c_n = \lambda = \lambda_k \) with \( k \geq 2 \) and \( n \leq k \). Then by (1.5) it follows that

\[
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{\lambda}{\rho^n} - \sum_{m=1}^{n-1} \frac{1 - \rho^k}{\rho^m} = \frac{1 - \rho^k}{\rho^{n-1}(1+\rho^k)} - \frac{1 - \rho^{n-1}}{\rho^{n-1}} = 1 - \frac{2\rho^{k-n+1}}{1+\rho^k} \geq 1 - \frac{2\rho}{1+\rho} = 1 - \rho - \lambda,
\]

leading to a contradiction with (2.7).

Case (III) \( c_n = \lambda = \lambda_k \) with \( k \geq 2 \) and \( n > k \). We consider two subcases.
(III A) $c_{n-1}c_{n-2}\cdots c_{n-k+1}$ has a digit $-(1 - \rho - 2\lambda)$. Then by (1.5) we have

$$\left| \sum_{m=1}^{n} c_m \rho^m \right| \geq \frac{\lambda}{\rho^n} - \sum_{m=n-k+2}^{n-1} \frac{1 - \rho}{\rho^m} - \frac{1 - \rho - 2\lambda}{\rho^{n-k+1}} - \sum_{m=1}^{n-k} \frac{1 - \rho}{\rho^m}$$

$$= \frac{\lambda}{\rho^n} + \frac{2\lambda}{\rho^{n-k+1}} - \sum_{m=1}^{n-1} \frac{1 - \rho}{\rho^m}$$

$$= \frac{1 - \rho^k}{\rho^{n-1}(1 + \rho^k)} + \frac{2(1 - \rho^k)}{\rho^{n-k}(1 + \rho^k)} - \frac{1 - \rho^{n-1}}{\rho^{n-1}}$$

$$= 1 + \frac{2(1 - \rho - \rho^k)}{\rho^{n-k}(1 + \rho^k)} > 1,$$

contradicting to (2.7).

(III B) The digit $-(1 - \rho - 2\lambda)$ does not appear in $c_{n-1}c_{n-2}\cdots c_{n-k+1}$. Then we claim that at least one digit of $c_{n-1}c_{n-2}\cdots c_{n-k}$ should be positive. Otherwise, by using $c_n = \lambda = \sum_{m=1}^{n} \rho^m (1 - \rho) + \rho^k (1 - \rho - \lambda)$ it follows that

$$\sum_{m=1}^{n} c_m \rho^m = \sum_{m=n-k+1}^{n-1} \frac{c_m + 1 - \rho}{\rho^m} + \frac{c_{n-k} + 1 - \rho - \lambda}{\rho^{n-k}} + \sum_{m=1}^{n-k-1} \frac{c_m}{\rho^m}.$$  

(2.9)

Note that $c_m \in \Omega_\lambda^\pm$, $c_m \leq 0$ and $c_m \neq -(1 - \rho - 2\lambda)$ for all $n - k + 1 \leq m \leq n - 1$. Then $c_m + 1 - \rho \in \Omega_\lambda^\pm$. Furthermore, since $c_{n-k} \in \Omega_\lambda^\pm$ and $c_{n-k} \leq 0$, we also have $c_{n-k} + 1 - \rho - \lambda \in \Omega_\lambda^\pm$. Therefore, (2.9) gives another representation of $\sum_{m=1}^{n} c_m \rho^m$, which is lexicographically smaller than $|c_n|, |c_{n-1}|, \ldots, |c_1|$, leading to a contradiction with our assumption. By the claim, (2.8) and using $0 < \rho \leq 1/4$ it follows that

$$\left| \sum_{m=1}^{n} c_m \rho^m \right| \geq \frac{\lambda}{\rho^n} - \sum_{m=n-k+1}^{n-1} \frac{1 - \rho}{\rho^m} + \frac{\lambda}{\rho^{n-k}} - \sum_{m=1}^{n-k-1} \frac{1 - \rho}{\rho^m}$$

$$= \frac{\lambda}{\rho^n} + \frac{\lambda + 1 - \rho}{\rho^{n-k}} - \sum_{m=1}^{n-1} \frac{1 - \rho}{\rho^m}$$

$$= \frac{1 - \rho^k}{\rho^{n-1}(1 + \rho^k)} + \frac{\rho(1 - \rho^k) + (1 - \rho)(1 + \rho^k)}{\rho^{n-k}(1 + \rho^k)} - \frac{1 - \rho^{n-1}}{\rho^{n-1}}$$

$$= 1 + \frac{(1 - 2\rho)(1 + \rho^k)}{\rho^{n-k}(1 + \rho^k)} > 1,$$

contradicting to (2.7).

Hence, by Cases (I)–(III) it follows that (2.7) fails, and then proves (2.6) as required. \(\square\)

Proof of Theorem 1.2 (i). Let $0 < \lambda < \rho \leq 1/4$. By Propositions 2.4 and 2.5 we only need to prove that if $\lambda = \gamma_k = \rho(1 - \rho^k)$ for some $k \in \mathbb{N}$, then $E_\lambda$ is totally self-similar. Take $\lambda = \gamma_k$ with $k \in \mathbb{N}$. By the same argument as in the proof of Proposition 2.5 it suffices to prove that for any $n \in \mathbb{N}$ and for any $c_m \in \Omega_\lambda^\pm = \{0, \pm \lambda, \pm (1 - \rho - 2\lambda), \pm (1 - \rho - \lambda), \pm (1 - \rho)\}$ with
1 \leq m \leq n \text{ we have } \left| \sum_{m=1}^{\infty} \frac{c_m}{\rho^m} \right| \geq 1 - \rho - \lambda. \text{ Indeed, we can prove }
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq 1 - \rho.

Suppose on the contrary there exists a block \( c_1 c_2 \cdots c_n \in (\Omega_\lambda^\pm)^n \) with \( n \in \mathbb{N} \) such that
\begin{equation}
0 < \left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| < 1 - \rho. \tag{2.10}
\end{equation}
Furthermore, we can choose the block \( c_1 c_2 \cdots c_n \in (\Omega_\lambda^\pm)^n \) satisfying (2.10) such that the finite sequence \( |c_n|, |c_{n-1}|, \ldots, |c_1| \) is lexicographically minimal and \( c_n \neq 0 \). Without loss of generality we can assume that \( c_n > 0 \). Note that \( \lambda = \gamma_k \geq \rho(1 - \rho) \). Then by (2.10) and the same argument as in the proof of Proposition 2.5 we have \( n \geq 2 \).

Note that \( c_n \in \Omega_\lambda^\pm \) and \( c_n > 0 \). Then \( c_n \in \{\lambda, 1 - \rho - 2\lambda, 1 - \rho - \lambda, 1 - \rho\} \). If \( c_n \in \{1 - \rho - 2\lambda, 1 - \rho - \lambda, 1 - \rho\} \), then by using \( 0 < \lambda < \rho \leq 1/4 \) and (2.8) it follows that
\[ \left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{1 - \rho - 2\lambda}{\rho^n} - \sum_{m=1}^{n-1} \frac{1 - \rho}{\rho^m} = 1 + \frac{1 - 2\rho - 2\lambda}{\rho^n} > 1, \]
leading to a contradiction with (2.10). So, in the following it suffices to consider \( c_n = \lambda \), which will be split into the following two cases: (I) \( n \leq k \); (II) \( n > k \).

Case (I) \( c_n = \lambda \) with \( n \leq k \). Then by using \( \lambda = \gamma_k = \rho(1 - \rho^k) \) it follows that
\[ \left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{\lambda}{\rho^n} - \sum_{m=1}^{n-1} \frac{1 - \rho}{\rho^m} = \frac{1 - \rho^k}{\rho^{n-1}} - \frac{1 - \rho^{n-1}}{\rho^{n-1}} = 1 - \rho^{k-n+1} \geq 1 - \rho, \]
leading to a contradiction with (2.10).

Case (II) \( c_n = \lambda \) with \( n > k \). We consider two subcases.

(II A) \( c_{n-1} c_{n-2} \cdots c_{n-k} \) contains a digit \(-(1 - \rho - 2\lambda)\). Then by (2.8) and \( \lambda = \gamma_k = \rho(1 - \rho^k) \) it follows that
\begin{align*}
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| &\geq \frac{\lambda}{\rho^n} - \sum_{m=n-k+1}^{n-1} \frac{1 - \rho}{\rho^m} - \sum_{m=1}^{n-k-1} \frac{1 - \rho}{\rho^m} \\
&= \frac{\lambda}{\rho^n} + \frac{2\lambda}{\rho^{n-k}} - \sum_{m=1}^{n-1} \frac{1 - \rho}{\rho^m} \\
&= 1 + \frac{1 - 2\rho^k}{\rho^{n-k-1}} \geq 2 - 2\rho^k > 1 - \rho, \tag{2.11}
\end{align*}
where the last inequality follows by \( 0 < \rho \leq 1/4 \). This leads to a contradiction with (2.10).

(II B) The digit \( -(1 - \rho - 2\lambda) \) never occurs in the block \( c_{n-1} c_{n-2} \cdots c_{n-k} \). Then we claim that at least one digit in \( c_{n-1} c_{n-2} \cdots c_{n-k} \) is positive. Otherwise, by using \( c_n = \lambda = \rho - \rho^{k+1} = \)}
\[
\sum_{m=1}^{k} \rho^m (1 - \rho) \text{ we have}
\]
\[
(2.12) \quad \sum_{m=1}^{n} \frac{c_m}{\rho^m} = \sum_{m=n-k}^{n-1} \frac{c_m + 1 - \rho}{\rho^m} + \sum_{m=1}^{n-k-1} \frac{c_m}{\rho^m}.
\]
Note that \(c_m \in \Omega^{\pm}_{\lambda}\), \(c_m \leq 0\) and \(c_m \neq -(1 - \rho - 2\lambda)\) for any \(n - k \leq m \leq n - 1\). Then \(c_m + 1 - \rho \in \Omega^{\pm}_{\lambda}\) for all \(n - k \leq m \leq n - 1\). Therefore, (2.12) contradicts to the minimality of \(|c_n|, |c_{n-1}|, \cdots, |c_1|\). This proves the claim. So by (2.8) it follows that
\[
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{\lambda}{\rho^n} - \sum_{m=n-k+1}^{n-1} \frac{1 - \rho}{\rho^m} + \frac{\lambda}{\rho^{n-k}} - \sum_{m=1}^{n-k-1} \frac{1 - \rho}{\rho^m}
\]
\[
> \frac{\lambda}{\rho^n} - \sum_{m=n-k+1}^{n-1} \frac{1 - \rho}{\rho^m} - \frac{1 - \rho - 2\lambda}{\rho^{n-k}} - \sum_{m=1}^{n-k-1} \frac{1 - \rho}{\rho^m} > 1 - \rho,
\]
where the last inequality holds by the same argument as in (2.11). Again this leads to a contradiction with (2.10).

Therefore, (2.10) fails by Cases (I) and (II). This completes the proof.

\[\square\]

2.2. Total self-similarity of \(E_\lambda\) for \(\lambda \in \left(\frac{1 - 2\rho}{2}, \frac{1 - \rho}{2}\right)\). The proof of Theorem 1.2 (ii) is similar to that for Theorem 1.2 (i). Take \(\lambda \in \left(\frac{1 - 2\rho}{2}, \frac{1 - \rho}{2}\right)\). Then the hole \(H\) is given by (see the right graph of Figure 1)
\[
H = \Omega \setminus \bigcup_{d \in \Omega} f_d(I) = (\rho, \lambda) \cup (1 - \lambda, 1 - \rho).
\]

Let \(H_1 := (\rho, \lambda)\) and \(H_2 := (1 - \lambda, 1 - \rho)\). Then \(H = H_1 \cup H_2\) with the union disjoint. Recall from (1.5) that \(\eta_k = \frac{1 - 2\rho + \rho^{k+1}}{2} \in \left(\frac{1 - 2\rho}{2}, \frac{1 - \rho}{2}\right)\) for all \(k \in \mathbb{N}\), and \(\eta_k \searrow \frac{1 - 2\rho}{2}\) as \(k \to \infty\).

**Proposition 2.6.** If \(\lambda = \eta_k\) for some \(k \in \mathbb{N}\), then \(E_\lambda\) is totally self-similar.

**Proof.** Take \(\lambda = \eta_k\). By Proposition 2.1 it suffices to prove that for any \(n \in \mathbb{N}\) and for any \(i, j \in \Omega^n\) with \(f_i \neq f_j\) we have
\[
f_i(H) \cap f_j(I) = f_i(H_1 \cup H_2) \cap f_j(I) = (f_i(H_1) \cap f_j(I)) \cup (f_i(H_2) \cap f_j(I))
\]
\[
= \left((f_i(0) + \rho^{n+1}, f_i(0) + \rho^n \lambda) \cap [f_j(0), f_j(0) + \rho^n]\right)
\]
\[
\cup \left((f_i(0) + \rho^n(1 - \lambda), f_i(0) + \rho^n(1 - \rho)) \cap [f_j(0), f_j(0) + \rho^n]\right)
\]
\[
= \emptyset,
\]
which is equivalent to
\[
|f_i(0) - f_j(0)| \geq \max \{\rho^n \lambda, \rho^n(1 - \rho)\} = \rho^n(1 - \rho),
\]
where the equality holds since \(\lambda < \frac{1 - 2\rho}{2}\). So, we only need to prove that for any \(n \in \mathbb{N}\) and for any \(i = i_1 \ldots i_n, j = j_1 \ldots j_n \in \Omega^n\) with \(f_i \neq f_j\),
\[
1 - \rho \leq \left| \frac{f_i(0) - f_j(0)}{\rho^n} \right| = \left| \sum_{m=1}^{n} \frac{j_m - i_m}{\rho^m} \right| = \left| \sum_{m=1}^{n} \frac{i_m - j_m}{\rho^{n-m+1}} \right| = \left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right|,
\]
where each \(c_m \in \Omega^{\pm}_{\lambda} = \{0, \pm \lambda, \pm (1 - \rho - 2\lambda), \pm (1 - \rho - \lambda), \pm (1 - \rho)\}.\]
Suppose on the contrary there exists a block $c_1c_2 \cdots c_n \in (\Omega^\pm_\lambda)^n$ for some $n \in \mathbb{N}$ such that

$$
0 < \left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| < 1 - \rho. \tag{2.13}
$$

By the same argument as in the proof of Proposition 2.5, we can choose the block $c_1c_2 \cdots c_n \in (\Omega^\pm_\lambda)^n$ satisfying (2.13) such that the finite sequence $|c_n|, |c_{n-1}|, \ldots, |c_1|$ is lexicographically minimal and $c_n > 0$. Note that $\lambda = \eta_k \leq \eta_1 = \frac{1-2\rho+\rho^2}{2}$ and $\rho \in (0, 1/4]$. Then by (2.13) it follows that $n \geq 2$. Furthermore, by using $0 < \rho \leq 1/4$ and $1 - \frac{2\rho}{2} < \lambda < 1 - \rho$ we have

$$
0 < 1 - \rho - 2\lambda < \rho \leq \lambda < 1 - \rho - \lambda < 1 - \rho. \tag{2.14}
$$

Note that $c_n \in \Omega^\pm_\lambda$ and $c_n > 0$. Then $c_n \in \{1 - \rho - 2\lambda, \lambda, 1 - \rho - \lambda, 1 - \rho\}$. If $c_n \in \{\lambda, 1 - \rho - \lambda, 1 - \rho\}$, then by (2.14) we have

$$
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{\lambda}{\rho^n} - \sum_{m=1}^{n-1} \frac{1 - \rho}{\rho^m} = \frac{\lambda}{\rho^n} - \frac{1 - \rho^{n-1}}{\rho^{n-1}} = 1 + \frac{\lambda - \rho}{\rho^n} \geq 1,
$$

leading to a contradiction with (2.13). So, in the following it suffices to consider $c_n = 1 - \rho - 2\lambda$, which will be split into the following two cases: (I) $n \leq k$; (II) $n > k$.

Case (I) $c_n = 1 - \rho - 2\lambda$ with $n \leq k$. Then by (2.14) and using $\lambda = \eta_k = \frac{1-2\rho+\rho^k+1}{2}$ it follows that

$$
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{1 - \rho - 2\lambda}{\rho^n} - \sum_{m=1}^{n-1} \frac{1 - \rho}{\rho^m} = \frac{1 - \rho^k}{\rho^{n-1}} - \frac{1 - \rho^{n-1}}{\rho^{n-1}} = 1 - \rho^{k-n+1} \geq 1 - \rho,
$$

leading to a contradiction with (2.13).

Case (II) $c_n = 1 - \rho - 2\lambda$ with $n > k$. We consider two subcases.

(II A) $c_{n-1}c_{n-2} \cdots c_{n-k}$ contains a digit $-(1 - \rho - 2\lambda)$. Then by (2.14) and $\lambda = \eta_k = \frac{1-2\rho+\rho^k+1}{2}$ it follows that

$$
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{1 - \rho - 2\lambda}{\rho^n} - \sum_{m=n-k+1}^{n-1} \frac{1 - \rho}{\rho^m} - \sum_{m=1}^{n-k-1} \frac{1 - \rho}{\rho^m} = \frac{1 - \rho^k}{\rho^{n-k}} + \frac{2\lambda}{\rho^{n-k}} - \frac{1 - \rho}{\rho^m} \geq 1 + \frac{1 - 3\rho + \rho^k+1}{\rho^{n-k}} > 1,
$$

where the last inequality follows by $0 < \rho \leq 1/4$. This leads to a contradiction with (2.13).

(II B) The digit $-(1 - \rho - 2\lambda)$ never occurs in $c_{n-1}c_{n-2} \cdots c_{n-k}$. Then we claim that at least one digit of $c_{n-1}c_{n-2} \cdots c_{n-k}$ is positive. Otherwise, by using $c_n = 1 - \rho - 2\lambda = \rho - \rho^k+1 = \sum_{m=1}^{k} \rho^m(1 - \rho)$ we have

$$
\sum_{m=1}^{n} \frac{c_m}{\rho^m} = \sum_{m=n-k}^{n-1} \frac{c_m + 1 - \rho}{\rho^m} + \sum_{m=1}^{n-k-1} \frac{c_m}{\rho^m}. \tag{2.15}
$$
Note that \( c_m \in \Omega^+_\lambda \), \( c_m \leq 0 \) and \( c_m \neq -(1 - \rho - 2\lambda) \) for any \( n - k \leq m \leq n - 1 \). Then \( c_m + 1 - \rho \in \Omega^\pm_\lambda \) for all \( n - k \leq m \leq n - 1 \). Therefore, (2.16) contradicts to the minimality of \(|c_n|, |c_{n-1}|, \ldots, |c_1|\). This proves the claim, and then by (2.14) it follows that

\[
\left| \sum_{m=1}^{n} \frac{c_m}{\rho^m} \right| \geq \frac{1 - \rho - 2\lambda}{\rho^n} \geq 1 - \rho - 2\lambda - \frac{1 - \rho}{\rho^n} \geq \frac{1 - \rho}{\rho^n} > 1,
\]

where the last inequality holds by the same argument as in (2.15). This again contradicts to (2.13).

By Cases (I) and (II) it follows that (2.13) does not hold, and thus \( f_i(H) \cap f_j(I) = \emptyset \) for any \( i, j \in \Omega^n_\lambda \) with \( n \in \mathbb{N} \). This completes the proof. \( \Box \)

**Proof of Theorem 1.2 (ii).** By Proposition 2.6 we only need to prove that if \( \lambda \in (\frac{1-2\rho}{2}, \frac{1-\rho}{2}) \setminus \bigcup_{k=1}^\infty \{\eta_k\} \), then \( E_\lambda \) is not totally self-similar. Note that \( \eta_0 = \frac{1-\rho}{2} \) and \( \eta_k \setminus \frac{1-2\rho}{2} \) as \( k \to \infty \).

So by Proposition 2.7 it suffices to show that for any \( k \in \mathbb{N} \) and for any \( \lambda \in (\eta_k, \eta_{k-1}) \) we can find two words \( i, j \in \Omega^n_\lambda \) with \( n \in \mathbb{N} \) such that

(2.17) \( f_1 \neq f_j \) and \( f_i(I) \cap f_j(H) \neq \emptyset \),

where \( I = [0,1] \) and \( H = (\rho, \lambda) \cup (1 - \lambda, 1 - \rho) \).

Let \( \lambda \in (\eta_k, \eta_{k-1}) \) for some \( k \in \mathbb{N} \), and take \( i = \lambda(1 - \rho)^{k-1}, j = (1 - \rho - \lambda)0^{k-1} \in \Omega^k_\lambda \). Since \( \lambda < \eta_{k-1} = \frac{1-2\rho+\rho^k}{2} \), by Lemma 2.2 (iii) we have

\[
f_{\lambda(1-\rho)^{k-1}}(0) < f_{(1-\rho-\lambda)0^{k-1}}(0).
\]

Then \( f_{\lambda(1-\rho)^{k-1}} \neq f_{(1-\rho-\lambda)0^{k-1}} \). Furthermore, since \( H_1 = (\rho, \lambda) \subset H \), by (1.4) it follows that

(2.18) \[
f_{\lambda(1-\rho)^{k-1}}(I) = \left[ \rho - \rho^k + \lambda, \rho + \lambda \right], \quad f_{(1-\rho-\lambda)0^{k-1}}(H_1) = \left( 1 - \rho - \lambda + \rho^{k+1}, 1 - \rho - \lambda + \rho^k \right).
\]

Note that \( \lambda \in (\eta_k, \eta_{k-1}) \). Then \( 1 - \rho - \lambda + \rho^{k+1} < \rho + \lambda \). Furthermore, by Lemma 2.2 (iii) we obtain

\[
\rho - \rho^k + \lambda = f_{\lambda(1-\rho)^{k-1}}(0) < f_{(1-\rho-\lambda)0^{k-1}}(0) < f_{(1-\rho-\lambda)0^{k-1}}(\lambda) = 1 - \rho - \lambda + \rho^k \lambda.
\]

So, by (2.18) it follows that \( f_{\lambda(1-\rho)^{k-1}}(I) \cap f_{(1-\rho-\lambda)0^{k-1}}(H_1) \neq \emptyset \), proving (2.17). \( \Box \)

### 3. The Spectrum of \( E_\lambda \)

Recall by Definition 1.3 that the spectrum \( \lambda_\lambda \) of \( E_\lambda \) is given by

\[
\lambda_\lambda = \inf \left\{ \frac{|f_i(0) - f_j(0)|}{\rho^n} : i, j \in \Omega^n_\lambda \text{ with } f_i \neq f_j; \ n \in \mathbb{N} \right\}.
\]

In this section we will characterize the total self-similarity of \( E_\lambda \) by using the spectrum, and prove Theorem 1.4. First we consider the spectrum \( \lambda_\lambda \) when \( E_\lambda \) is totally self-similar.

**Proposition 3.1.**
Proof. For (i) let $\lambda = \lambda_k$ for some $k \in \mathbb{N}$. Then by the proof of Proposition 2.5 it follows that

$$l_{\lambda} = \inf \left\{ \frac{|f_i(0) - f_j(0)|}{\rho^n} : i, j \in \Omega_{\lambda}^n, f_i \neq f_j, n \in \mathbb{N} \right\} \geq 1 - \rho - \lambda.$$ 

On the other hand, let $i = 0(1 - \rho)^{k-1}$ and $j = \lambda 0^{k-1}$. By using $\rho \in (0, 1/4]$ and $\lambda = \frac{\rho(1 - \rho^k)}{1 + \rho^k}$ it follows by (1.4) that

$$f_{0(1-\rho)^{k-1}}(0) = \rho(1 - \rho^{k-1}) < \lambda = f_{\lambda 0^{k-1}}(0),$$

which implies $f_{0(1-\rho)^{k-1}}(0) \neq f_{\lambda 0^{k-1}}$. Then

$$\frac{|f_{0(1-\rho)^{k-1}}(0) - f_{\lambda 0^{k-1}}(0)|}{\rho^k} = \frac{|\rho(1 - \rho^{k-1}) - \lambda|}{\rho^k} = \frac{1 + \rho^k - 2\rho}{1 + \rho^k} = 1 - \rho - \lambda.$$

This proves $l_{\lambda} = 1 - \rho - \lambda$.

Next we consider (ii). Let $\lambda = \gamma_k = \rho(1 - \rho^k)$ for some $k \in \mathbb{N}$. Then by the proof of Theorem 1.2 (i) it follows that

$$l_{\lambda} = \inf \left\{ \frac{|f_i(0) - f_j(0)|}{\rho^n} : i, j \in \Omega_{\lambda}^n, f_i \neq f_j, n \in \mathbb{N} \right\} \geq 1 - \rho.$$ 

On the other hand, take $i = \lambda 0^{k-1}$ and $j = 0(1 - \rho)^{k-1}$. Then by using $\lambda = \rho(1 - \rho^k)$ we have

$$f_{\lambda 0^{k-1}}(0) = \lambda > \rho(1 - \rho^{k-1}) = f_{0(1-\rho)^{k-1}}(0),$$

which yields $f_{\lambda 0^{k-1}}(0) \neq f_{0(1-\rho)^{k-1}}$. Furthermore,

$$\frac{|f_{\lambda 0^{k-1}}(0) - f_{0(1-\rho)^{k-1}}(0)|}{\rho^k} = \frac{|\lambda - \rho + \rho^k|}{\rho^k} = \frac{|\rho - \rho^{k+1} - \rho + \rho^k|}{\rho^k} = 1 - \rho.$$ 

Thus, $l_{\lambda} = 1 - \rho$.

Finally we prove (iii). Let $\lambda = \eta_k = \frac{1 - 2\rho + \rho^{k+1}}{2}$ for some $k \in \mathbb{N}$. Then by the proof of Proposition 2.6 it follows that

$$l_{\lambda} = \inf \left\{ \frac{|f_i(0) - f_j(0)|}{\rho^n} : i, j \in \Omega_{\lambda}^n, f_i \neq f_j, n \in \mathbb{N} \right\} \geq 1 - \rho.$$ 

On the other hand, let $i = (1 - \rho - \lambda) 0^{k-1}$ and $j = \lambda(1 - \rho)^{k-1}$. By using $\lambda = \frac{1 - 2\rho + \rho^{k+1}}{2}$ it follows that

$$f_{(1-\rho-\lambda) 0^{k-1}}(0) = 1 - \rho - \lambda > \lambda + \rho - \rho^k = f_{\lambda(1-\rho)^{k-1}}(0),$$

which implies $f_{(1-\rho-\lambda) 0^{k-1}}(0) \neq f_{\lambda(1-\rho)^{k-1}}$. Furthermore,

$$\frac{|f_{(1-\rho-\lambda) 0^{k-1}}(0) - f_{\lambda(1-\rho)^{k-1}}(0)|}{\rho^k} = \frac{\left|\lambda + \rho - \rho^k - 1 + \rho + \lambda\right|}{\rho^k} = \frac{|2\rho - \rho^k - 1 + 1 - 2\rho + \rho^{k+1}|}{\rho^k} = 1 - \rho.$$
So, \( l_\lambda = 1 - \rho. \)

**Proof of Theorem 4.4.** By Proposition 3.1 it suffices to prove (ii) and (iii). First we prove (ii).

By Theorem 1.2 and Proposition 3.1 it suffices to prove that for any \( \lambda \in (0, \rho) \setminus \bigcup_{k=1}^{\infty} \{ \lambda_k, \gamma_k \} \) we have \( l_\lambda < 1 - \rho - \lambda. \) Note by (1.5) and (2.1) that \( \lambda_k = \frac{\rho(1 - \rho^k)}{1 + \rho^k}, \ell_k = \frac{\rho(1 - \rho^{k+1})}{1 + \rho^k + \rho^{k+1}}, \gamma_k = \rho(1 - \rho^k). \) Then \( 0 < \lambda_k < \ell_k < \gamma_k < \lambda_{k+1} \) for any \( k \in \mathbb{N}, \) and \( \lambda_k \uparrow \rho \) as \( k \to \infty. \) So, it suffices to prove \( l_\lambda < 1 - \rho - \lambda \) in the following four cases: (I) \( \lambda \in (0, \lambda_1); \) (II) \( \lambda \in \bigcup_{k=1}^{\infty} (\ell_k, \gamma_k); \) (III) \( \lambda \in \bigcup_{k=1}^{\infty} (\ell_k, \gamma_k); \) (IV) \( \lambda \in \bigcup_{k=1}^{\infty} (\gamma_k, \lambda_{k+1}). \)

Case (I) \( \lambda \in (0, \lambda_1). \) Take \( i = 0 \) and \( j = \lambda. \) Since \( \lambda < \lambda_1 = \frac{\rho(1 - \rho)}{1 + \rho}, \) we have \( \frac{\lambda}{\rho} < 1 - \rho - \lambda, \) and then

\[
l_\lambda \leq \frac{|f_0(0) - f_\lambda(0)|}{\rho} = \frac{\lambda}{\rho} < 1 - \rho - \lambda.
\]

Case (II) \( \lambda \in (\lambda_k,\ell_k) \) for some \( k \in \mathbb{N}. \) Take \( i = \lambda 0^k \) and \( j = 0(1 - \rho)^{k-1}(1 - \rho - \lambda). \) Since

\[
l_\lambda \leq \frac{|f_0(0) - f_{\lambda 0^k}(0)|}{\rho^{k+1}} = \frac{\lambda + \rho^k \lambda - \rho + \rho^{k+1}}{\rho^{k+1}} < 1 - \rho - \lambda.
\]

Case (III) \( \lambda \in (\ell_k,\gamma_k) \) for some \( k \in \mathbb{N}. \) Let \( i = \lambda 0^k \) and \( j = 0(1 - \rho)^k. \) Then

\[
f_0(1 - \rho)^k(0) = \rho(1 - \rho^k) = \gamma_k > \lambda = f_{\lambda 0^k}(0),
\]

which implies \( f_{\lambda 0^k} \neq f_0(1 - \rho)^{k-1}(1 - \rho - \lambda). \) Note that \( \lambda < \ell_k = \frac{\rho(1 - \rho^{k+1})}{1 + \rho^k + \rho^{k+1}}. \) Then by (3.1) it follows that

\[
l_\lambda \leq \frac{|f_{\lambda 0^k}(0) - f_0(1 - \rho)^{k-1}(1 - \rho - \lambda)(0)|}{\rho^{k+1}} = \frac{\lambda + \rho^k \lambda - \rho + \rho^{k+1}}{\rho^{k+1}} < 1 - \rho - \lambda.
\]

Case (IV) \( \lambda \in (\gamma_k,\lambda_{k+1}) \) for some \( k \in \mathbb{N}. \) Take \( i = \lambda 0^k \) and \( j = 0(1 - \rho)^k. \) Similar to Case (III) we have \( f_0(1 - \rho)^k \neq f_{\lambda 0^k}. \) Since \( \gamma_k < \lambda < \lambda_{k+1}, \) by (1.5) it follows that

\[
0 < l_\lambda \leq \frac{|f_{\lambda 0^k}(0) - f_0(1 - \rho)^k(0)|}{\rho^{k+1}} = \frac{\lambda - \rho(1 - \rho^k)}{\rho^{k+1}} < 1 - \rho - \lambda.
\]

By Cases (I)–(IV) we prove (ii).

Next we prove (iii). By Theorem 1.2 and Proposition 3.1 it suffices to prove that for any \( \lambda \in \left( \frac{1 - 2\rho}{2}, \frac{1 - \rho}{2} \right) \setminus \bigcup_{k=1}^{\infty} \{ \eta_k \} \) we have \( l_\lambda < 1 - \rho. \) Note that \( \eta_0 = \frac{1 - \rho}{2} \) and \( \eta_k \searrow \frac{1 - 2\rho}{2} \) as \( k \to \infty. \) Then we only need to prove \( l_\lambda < 1 - \rho \) for any \( \lambda \in (\eta_k, \eta_{k-1}) \) with \( k \in \mathbb{N}. \) Now take \( \lambda \in (\eta_k, \eta_{k-1}) \) for some \( k \in \mathbb{N}. \) Let \( i = (1 - \rho - \lambda)0^{k-1} \) and \( j = \lambda(1 - \rho)^{k-1}. \)
Since $\lambda < \eta_{k-1}$, by Lemma 2.2 (iii) we have $f_{\lambda(1-\rho)^{k-1}}(0) < f_{(1-\rho-\lambda)^{0k-1}}(0)$, which gives $f_{\lambda(1-\rho)^{k-1}} \neq f_{(1-\rho-\lambda)^{0k-1}}$. Furthermore, by using $\lambda > \eta_k = \frac{1-2\rho+\rho^{k+1}}{2}$ it follows that

$$l_\lambda \leq \frac{|f_{\lambda(1-\rho)^{k-1}}(0) - f_{(1-\rho-\lambda)^{0k-1}}(0)|}{\rho^k} = \frac{1 - \rho - \lambda - \rho^k - \lambda}{\rho^k} < 1 - \rho$$

as desired. This completes the proof.

4. Unique codings of $E_\lambda$ when $E_\lambda$ is totally self-similar

Recall that $U_\lambda$ consists of all $x \in E_\lambda$ having a unique coding with respect to the IFS $F_\lambda$ defined in [1,2]. In this section we will study the set $U_\lambda$ when $E_\lambda$ is totally self-similar, and prove Theorem 1.6. Let $\lambda \in (0, \rho) \cup \left(\frac{1-2\rho}{2}, \frac{1-2\rho}{2}\right)$. Note by Theorem 1.2 that $E_\lambda$ is totally self-similar if and only if $\lambda \in \bigcup_{k=1}^\infty \{\lambda_k, \gamma_k, \eta_k\}$. So we will calculate the Hausdorff dimension of $U_\lambda$ for $\lambda = \lambda_k, \gamma_k$ and $\eta_k$.

First we prove Theorem 1.6 (ii) and (iii) for $\lambda = \gamma_k$ and $\lambda = \eta_k$ respectively, which can be essentially deduced from [5, Theorem 2].

Proof of Theorem 1.6 (ii) and (iii). For (ii) let $\lambda = \gamma_k = \rho(1-\rho^k)$ with $k \in \mathbb{N}$. Then, in view of Figure 1 (left), we have $f_\lambda(I) \cap f_{1-\rho-\lambda}(I) = \emptyset$, where $I = [0, 1]$. Furthermore,

$$f_\lambda(I) \cap f_{\lambda}(I) = [\lambda, \rho] = [\rho(1-\rho^k), \rho] = f_{\lambda^0k}(I) = f_{0(1-\rho)^k}(I),$$

and symmetrically,

$$f_{1-\rho-\lambda}(I) \cap f_{1-\rho}(I) = [1-\rho, 1-\lambda] = f_{(1-\rho-\lambda)(1-\rho)^k}(I) = f_{(1-\rho)^0}(I).$$

Thus, the IFS $(E_\lambda, F_\lambda)$ belongs to the class $\mathcal{E}$ studied in [5]. So, by [5, Theorem 2] it follows that $\dim_H E_\lambda = t \in (0, 1)$ satisfies

$$\rho^t(1-\rho^{kt}) + \rho^t(1-\rho^{kt}) + \rho^t = 1,$$

which can be simplified as $4\rho^t - 2\rho^{(k+1)t} = 1$. Furthermore, $\dim_H U_\lambda = s \in (0, 1)$ satisfies

$$2\rho^s \left(1 - \frac{\rho^{ks}(2 - \rho^k - \rho^{ks})}{1 - \rho^{2ks}}\right) + 2\rho^s = 1,$$

which can be deduced as $4\rho^s - \rho^{ks} = 1$. This establishes (ii).

Next we prove (iii). Let $\lambda = \eta_k = \frac{1-2\rho+\rho^{k+1}}{2}$ for some $k \in \mathbb{N}$. Then, in view of Figure 1 (right), we have $f_0(I) \cap f_\lambda(I) = \emptyset$ and $f_{1-\rho-\lambda}(I) \cap f_{1-\rho}(I) = \emptyset$. Furthermore,

$$f_\lambda(I) \cap f_{1-\rho-\lambda}(I) = [1-\rho - \lambda, \rho + \lambda] = \left[1 - \frac{\rho^{k+1} + 1 + \rho^{k+1}}{2}, \frac{1 + \rho^{k+1}}{2}\right] = f_{\lambda(1-\rho)^k}(I) = f_{(1-\rho^-\lambda)^0k}(I).$$

So, $(E_\lambda, F_\lambda)$ also belongs to the class $\mathcal{E}$ in [5]. By [5, Theorem 2] it follows that $\dim_H E_\lambda = t \in (0, 1)$ satisfies

$$\rho^t + \rho^t(1-\rho^{kt}) + \rho^t + \rho^t = 1,$$

which can be deduced as $4\rho^t - \rho^{(k+1)t} = 1$. Moreover, $\dim_H U_\lambda = s \in (0, 1)$ satisfies

$$\rho^s + \rho^s(1 - 2\rho^k) + \rho^s + \rho^s = 1,$$

which can be simplified as $4\rho^s - 2\rho^{(k+1)s} = 1$. This proves (iii).

By (i) and (ii) it is easy to verify that $\dim_H U_\lambda < \dim_H E_\lambda$, completing the proof.
Remark 4.1. By the above proof it follows that $E_\lambda$ has an exact overlap when $\lambda \in \bigcup_{k=1}^{\infty} \{\gamma_k, \eta_k\}$.

Note that when $\lambda \in \bigcup_{k=1}^{\infty} \{\gamma_k, \eta_k\}$, the IFS $(E_\lambda, \mathcal{F}_\lambda)$ belongs to the class $\mathcal{E}$ studied in [5]. Then by [5, Theorem 1] it follows that if $\lambda = \gamma_k$ for some $k \in \mathbb{N}$, then $U_\lambda$ is not closed, and there are infinitely many $x \in E_\lambda$ having countably infinitely many codings. On the other hand, if $\lambda = \eta_k$ for some $k \in \mathbb{N}$, then $U_\lambda$ is closed, and there is no $x \in E_\lambda$ having countably infinitely many codings. Furthermore, by [5, Theorem 2] it follows that for any $\lambda \in \bigcup_{k=1}^{\infty} \{\gamma_k, \eta_k\}$ we have

$$\mathcal{H}^t(E_\lambda) \in (0, +\infty) \quad \text{and} \quad \mathcal{H}^s(U_\lambda) \in (0, +\infty),$$

where $t = \dim_H E_\lambda$ and $s = \dim_H U_\lambda$.

In the following we only need to consider $\lambda = \lambda_k = \frac{\rho(1-\rho^k)}{1+\rho^k}$ for some $k \in \mathbb{N}$. The overlapping structure is completely different from that for $\lambda \in \bigcup_{k=1}^{\infty} \{\gamma_k, \eta_k\}$ (see Figure 2). In particular, the IFS $(E_\lambda, \mathcal{F}_\lambda)$ does not belong to the class $\mathcal{E}$ studied in [5]. Let $\lambda = \lambda_k$ for some $k \in \mathbb{N}$, and set

$$
\begin{align*}
i_1 &= 0(1-\rho)^{k-1}(1-\rho-\lambda), \\
i_2 &= 0(1-\rho)^k, \\
i_3 &= (1-\rho-\lambda)(1-\rho)^k, \\
i_4 &= (1-\rho-\lambda)(1-\rho)^{k-1}(1-\rho-\lambda), \\
j_1 &= \lambda 0^k; \\
j_2 &= \lambda 0^{k-1} \lambda; \\
j_3 &= (1-\rho) 0^{k-1} \lambda; \\
j_4 &= (1-\rho) 0^k.
\end{align*}
$$

Figure 2. The overlapping structure of $E_\lambda$ with $\lambda = \lambda_k$ for some $k \in \mathbb{N}$. Then $f_\lambda(E_\lambda) \cap f_\lambda(E_\lambda) = f_1(E_\lambda) \cup f_2(E_\lambda)$ and $f_{1-\rho-\lambda}(E_\lambda) \cap f_{1-\rho}(E_\lambda) = f_3(E_\lambda) \cup f_4(E_\lambda)$ with $f_\ell = f_\ell'$ for all $\ell \in \{1, 2, 3, 4\}$.

Lemma 4.2. Let $\lambda = \lambda_k$ for some $k \in \mathbb{N}$. Then

$$f_\ell = f_\ell' \quad \text{for all} \quad \ell \in \{1, 2, 3, 4\}. $$

Furthermore,

$$f_0(E_\lambda) \cap f_\lambda(E_\lambda) = f_1(E_\lambda) \cup f_2(E_\lambda), \quad f_{1-\rho-\lambda}(E_\lambda) \cap f_{1-\rho}(E_\lambda) = f_3(E_\lambda) \cup f_4(E_\lambda).$$

Before proving Lemma 4.2 we point out that the unions in the lemma are NOT disjoint: $f_1(E_\lambda) \cap f_2(E_\lambda) \neq \emptyset$ and $f_3(E_\lambda) \cap f_4(E_\lambda) \neq \emptyset$. 
Proof. Note by the symmetry that $E_\lambda$ has the same structure as $E_{1-\rho-\lambda}$. Then it suffices to prove
\[ f_1 = f_{j_1}, \quad f_2 = f_{j_2} \quad \text{and} \quad f_0(E_\lambda) \cap f_1(E_\lambda) = f_1(E_\lambda) \cup f_{j_1}E_\lambda). \]
Since $\lambda = \lambda_k = \frac{\rho(1-\rho^k)}{1+\rho^k}$, by (4.1) it follows that
\[ f_1(0) = \rho(1-\rho^k) - \rho^k \lambda = \lambda = f_{j_1}(0), \quad f_{j_2}(0) = \rho(1-\rho^k) = \lambda(1+\rho^k) = f_{j_2}(0). \]
This implies that $f_{j_\ell} = f_{j_k}$ for $\ell = 1, 2$. Furthermore, by (4.2) it follows that
\[ f_{j_1}(I) \cup f_{j_2}(I) = [\lambda, \rho-\rho^k]\cup[\rho-\rho^{k+1}, \rho] = [\lambda, \rho] = f_0(I) \cap f_\lambda(I), \]
where the second equality follows by $\lambda < \rho$. Then by Definition 4.1 and Theorem 4.2 we obtain
\[ f_1(E_\lambda) \cap f_{j_2}(E_\lambda) = (f_{j_1}(I) \cap E_\lambda) \cup (f_{j_2}(I) \cap E_\lambda) = (f_{j_1}(I) \cup f_{j_2}(I)) \cap E_\lambda = (f_0(I) \cap f_{j_\lambda}(I)) \cap E_\lambda = f_0(E_\lambda) \cap f_{j_\lambda}(E_\lambda) \]
as desired. \qed

Our next result shows that for $\lambda = \lambda_k$, $U_\lambda$ can be represented as a strongly connected graph-directed set satisfying the SSC. Let $X_\lambda \subset \Omega^N_\lambda$ be a subshift of finite type with the set of forbidden blocks given by
\[ F = \bigcup_{\ell=1}^4 \{i_\ell, j_\ell\}, \]
where $i_\ell, j_\ell$ are defined in (4.1). Let $\sigma$ be the left-shift map on $\Omega^N_\lambda$.

**Lemma 4.3.** Let $\lambda = \lambda_k$ for some $k \in \mathbb{N}$. Then $(X_\lambda, \sigma)$ is a transitive subshift of finite type.

**Proof.** Note that each forbidden block in $F$ has length $k+1$. Then $(X_\lambda, \sigma)$ is a $k$-step subshift of finite type. By [14] Theorem 2.1.8 it suffices to prove that for any two admissible words $c = c_1 \ldots c_k, d = d_1 \ldots d_k \in B_*(X_\lambda)$, we can find a word $w$ such that $c\cdot d \in B_*(X_\lambda)$. Here $B_*(X_\lambda)$ denotes the set of all admissible words appearing in some sequence of $X_\lambda$. Take $c = c_1 \ldots c_k, d = d_1 \ldots d_k \in B_*(X_\lambda)$. We will prove in the following two cases the existence of $w$ so that $c\cdot d \in B_*(X_\lambda)$.

Case I. $c_k \in \{0, 1-\rho-\lambda\}$. If $d_1 \in \{0, \lambda\}$, then by taking $w = (1-\rho-\lambda)^{k-1}\lambda(1-\rho-\lambda)$ one can verify that the longer word $c\cdot d$ does not contain any block from $F$, i.e., $c\cdot d \in B_*(X_\lambda)$. If $d_1 \in \{1-\rho-\lambda, 1-\rho\}$, then by taking $w = (1-\rho-\lambda)^{k-1}\lambda(1-\rho-\lambda)$ we have $c\cdot d \in B_*(X_\lambda)$.

Case II. $c_k \in \{\lambda, 1-\rho\}$. If $d_1 \in \{0, \lambda\}$, then by taking $w = \lambda^{k-1}(1-\rho-\lambda)\lambda(1-\rho-\lambda)$ we have $c\cdot d \in B_*(X_\lambda)$. If $d_1 \in \{1-\rho-\lambda, 1-\rho\}$, then by taking $w = \lambda^{k-1}(1-\rho-\lambda)\lambda$ one can verify that $c\cdot d \in B_*(X_\lambda)$. \qed

**Lemma 4.4.** Let $\lambda = \lambda_k$ for some $k \in \mathbb{N}$. Then $U_\lambda = \pi_\lambda(X_\lambda)$.

**Proof.** Take $\lambda = \lambda_k$ and let $x \in E_\lambda \setminus \pi_\lambda(X_\lambda)$. Note that $E_\lambda = \pi_\lambda(\Omega^N_\lambda)$. Then $x$ has a coding $(x_i) \in \Omega^N_\lambda$ which contains a block from $F = \bigcup_{\ell=1}^4 \{i_\ell, j_\ell\}$. So by Lemma 4.2 it follows that $x$ has at least two different codings with the substitution: $i_\ell \sim j_\ell$ for $\ell \in \{1, 2, 3, 4\}$. Thus, $x \notin U_\lambda$. 


On the other hand, take \( x \in E_\lambda \setminus U_\lambda \). Then \( x \) has two different codings, say \((c_i), (d_i) \in \Omega_\lambda^N\). Without loss of generality we may assume \( c_1 < d_1 \). By the overlapping structure of \( E_\lambda \) we have \( x \in [\lambda, \rho] \cup [1 - \rho, 1 - \lambda] \), and by symmetry we may assume \( x \in [\lambda, \rho] \). Then \( c_1 = 0 \) and \( d_1 = \lambda \). Note that \( \sum_{i=1}^{\infty} \rho^{i-1}c_i = x = \sum_{i=1}^{\infty} \rho^{i-1}d_i \). Then

\[
\sum_{i=2}^{\infty} \rho^{i-1}c_i = \sum_{i=2}^{\infty} \rho^{i-1}d_i + \lambda. \tag{4.3}
\]

**Claim.** If \( k \geq 2 \), then \( c_2 \ldots c_k = (1 - \rho)^{k-1} \) and \( d_2 \ldots d_k = 0^{k-1} \).

Suppose the claim does not hold, and let \( \tau \in \{2, 3, \ldots, k\} \) be the smallest integer in which \( c_\tau \neq 1 - \rho \) or \( d_\tau \neq 0 \). Then \( c_2 \ldots c_{\tau-1} = (1 - \rho)^{\tau-2} \) and \( d_2 \ldots d_{\tau-1} = 0^{\tau-2} \). So, by (4.3) and \( \lambda = \lambda_k = \frac{\rho(1 - \rho^k)}{1 + \rho^k} \) it follows that

\[
\sum_{i=\tau}^{\infty} \rho^{-\tau}c_i - \sum_{i=\tau}^{\infty} \rho^{-\tau}d_i = \frac{\lambda}{\rho^{\tau-1}} - \sum_{i=1}^{\tau-2} \frac{1 - \rho}{\rho^i} = \frac{1 - \rho^k}{\rho^{\tau-2}(1 + \rho^k)} - \frac{1 - \rho^{\tau-2}}{1 + \rho^k} = 1 - \frac{2\rho^{\tau+2-\tau}}{1 + \rho^k} \geq 1 - \frac{2\rho^2}{1 + \rho^k}.
\]

Since \( \lambda = \frac{\rho(1 - \rho^k)}{1 + \rho^k} > \frac{2\rho^2}{1 + \rho^k} \), by (4.4) we have

\[
\sum_{i=\tau}^{\infty} \rho^{-\tau}c_i - \sum_{i=\tau}^{\infty} \rho^{-\tau}d_i > 1 - \lambda = f_{1-\rho}(1) - f_\lambda(0),
\]

which implies that \( c_\tau = 1 - \rho \) and \( d_\tau = 0 \). This leads to a contradiction with the definition of \( \tau \), and thus proves the claim.

By (4.3) and the claim it follows that for \( k \in \mathbb{N} \),

\[
\sum_{i=k+1}^{\infty} \rho^{-k-1}c_i - \sum_{i=k+1}^{\infty} \rho^{-k-1}d_i = \frac{\lambda}{\rho^k} - \sum_{i=1}^{k-1} \frac{1 - \rho}{\rho^i} = 1 - \frac{2\rho}{1 + \rho^k} > \rho + \lambda = f_\lambda(1) - f_0(0),
\]

where the inequality holds by using \( \lambda = \frac{\rho(1 - \rho^k)}{1 + \rho^k} \) and \( 0 < \rho \leq 1/4 \). This implies that \( c_{k+1} \in \{1 - \rho - \lambda, 1 - \rho\} \) and \( d_{k+1} \in \{0, \lambda\} \). Thus, by (4.1) it follows that \( c_1c_2 \ldots c_{k+1} \in \{i_1, i_2\} \) and \( d_1d_2 \ldots d_{k+1} \in \{j_1, j_2\} \), which yields

\[(c_i), (d_i) \notin X_\lambda.
\]

Since \((c_i), (d_i)\) are two arbitrary codings of \( x \), we conclude that \( x \notin \pi_\lambda(X_\lambda) \). \( \square \)

**Lemma 4.5.** Let \( \lambda = \lambda_k \) for some \( k \in \mathbb{N} \). Then \( U_\lambda \) can be represented as a strongly connected graph-directed set satisfying the SSC.

**Proof.** Note that \( X_\lambda \) is a subshift of finite type with the set \( \mathcal{F} = \bigcup_{\ell=1}^{k} \{i_\ell, j_\ell\} \) of forbidden blocks. Since each block in \( \mathcal{F} \) has length \( k + 1 \), \( X_\lambda \) is a \( k \)-step subshift of finite type which can be represented as a directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) constructed in the following way. Let \( \mathcal{V} = B_k(X_\lambda) \) be the set of all length \( k \) admissible blocks appearing in some sequence of \( X_\lambda \). For two vertices \( \mathbf{c} = c_1 \ldots c_k, \mathbf{d} = d_1 \ldots d_k \in \mathcal{V} \) we draw a directed edge from \( \mathbf{c} \) to \( \mathbf{d} \), denoted by \( \mathbf{cd} \), if

\[c_2 \ldots c_k = d_1 \ldots d_{k-1} \text{ and } c_1 \ldots c_kd_k \in B_{k+1}(X_\lambda).\]
Proposition 4.6. Let \( \lambda = \lambda_k \) for some \( k \in \mathbb{N} \). Then

\[
\dim_H U_\lambda = s,
\]

where \( s \in (0, 1) \) satisfies \( 4\rho^s - 2\rho^{ks} = 1 \).

Proof. Take \( \lambda = \lambda_k \). For a word \( w = w_1 \ldots w_n \in B_n(X_\lambda) \) let \( U_\lambda(w) := U_\lambda \cap f_w(E_\lambda) \). Then \( U_\lambda(w) \) consists of all \( x \in U_\lambda \) whose unique coding beginning with the word \( w \). So,

\[
U_\lambda = \bigcup_{d \in \Omega_\lambda} U_\lambda(d)
\]

with the union pairwise disjoint. Note by Lemma 4.5 that \( U_\lambda \) is a strongly connected graph-directed set satisfying the SSC. Then by \([17]\) it follows that for \( s = \dim_H U_\lambda \) we have \( H^s(U_\lambda) \in (0, \infty) \). So by (4.7) it suffices to prove that for each \( d \in \Omega_\lambda = \{0, \lambda, 1 - \rho - \lambda, 1 - \rho\} \),

\[
H^s(U_\lambda(d)) = \left( \rho^s - \frac{2\rho^{(k+1)s}}{1+2\rho^{ks}} \right) H^s(U_\lambda).
\]

Since the proofs of (4.8) for different \( d \in \Omega_\lambda \) are similar, we only prove (4.8) for \( d = 0 \).
Note by Lemma 4.4 that $U_\lambda = \pi_\lambda(X_\lambda)$. This means that for a given $x \in E_\lambda$, if all of its codings do not contain any block from $F = \bigcup_{\ell=1}^{4} \{i_\ell, j_\ell\}$ then $x \in U_\lambda$. So by the definition of $U_\lambda(w)$ we obtain that

$$U_\lambda(0) = f_0(U_\lambda) \setminus \bigcup_{\ell=1}^{2} f_0(U_\lambda(\hat{i}_\ell)),$$

where for $\ell = 1, 2, 3, 4$ we set $\hat{i}_\ell := \sigma(i_\ell)$ and $\hat{j}_\ell := \sigma(j_\ell)$. Note by (4.1) that $\hat{i}_{5-\ell} = \hat{i}_\ell$ and $\hat{j}_{5-\ell} = \hat{j}_\ell$ for any $\ell \in \{1, 2, 3, 4\}$. Then by the definition of $U_\lambda(w)$ and using $U_\lambda = \pi_\lambda(X_\lambda)$ it follows that

$$U_\lambda(\hat{i}_1) = f_{i_1}(U_\lambda) \setminus \bigcup_{\ell=1}^{2} f_{i_1}(U_\lambda(\hat{i}_\ell)),$$

$$U_\lambda(\hat{i}_2) = f_{i_2}(U_\lambda) \setminus \bigcup_{\ell=1}^{2} f_{i_2}(U_\lambda(\hat{i}_\ell)),$$

$$U_\lambda(\hat{j}_1) = f_{j_1}(U_\lambda) \setminus \bigcup_{\ell=1}^{2} f_{j_1}(U_\lambda(\hat{i}_\ell)),$$

$$U_\lambda(\hat{j}_2) = f_{j_2}(U_\lambda) \setminus \bigcup_{\ell=1}^{2} f_{j_2}(U_\lambda(\hat{j}_\ell)).$$

Since the unions in (4.10) are pairwise disjoint, by taking the $s$-dimensional Hausdorff measure on both sides of (4.10) we obtain that

$$\mathcal{H}^s(U_\lambda(\hat{i}_\ell)) = \mathcal{H}^s(U_\lambda(\hat{j}_\ell)) = \frac{\rho^{ks}}{1 + 2\rho^{ks}} \mathcal{H}^s(U_\lambda) \quad \forall \ell = 1, 2.$$

Thus, by (4.9) it follows that

$$\mathcal{H}^s(U_\lambda(0)) = \rho^s \left[ \mathcal{H}^s(U_\lambda) - \mathcal{H}^s(U_\lambda(\hat{i}_1)) - \mathcal{H}^s(U_\lambda(\hat{i}_2)) \right] = \left( \rho^s - \frac{2\rho^{(k+1)s}}{1 + 2\rho^{ks}} \right) \mathcal{H}^s(U_\lambda)$$

proving (4.8) for $d = 0$. This completes the proof.

**Proof of Theorem 1.6 (i).** By Proposition 4.6 it suffices to prove that for $\lambda = \lambda_k$ we have $\dim_H U_\lambda < \dim_H E_\lambda$. Let $F' = F \setminus \{i_1\} = \{i_2, i_3, i_4, j_1, j_2, j_3, j_4\}$, and let $X'_\lambda$ be the subshift of finite type over $\Omega_\lambda$ with the set $F'$ of forbidden blocks. Then

$$X'_\lambda = \left\{ (d_i) \in \Omega^\mathbb{N}_\lambda : d_{i+1} \ldots d_{i+k+1} \notin F' \forall i \geq 0 \right\}.$$

By a similar argument as in the proof of Lemmas 4.3 and 4.5 one can show that $\pi_\lambda(X'_\lambda)$ is a strongly connected graph-directed set satisfying the SSC. Note that $X_\lambda$ is a proper subset of $X'_\lambda$. Then by [13] Corollary 4.4.9 it follows that $h_{top}(X_\lambda) < h_{top}(X'_\lambda)$. This implies that

$$\dim_H \pi_\lambda(X_\lambda) = \frac{h_{top}(X_\lambda)}{-\log \rho} < \frac{h_{top}(X'_\lambda)}{-\log \rho} = \dim_H \pi_\lambda(X'_\lambda).$$

Since $U_\lambda = \pi_\lambda(X_\lambda)$ by Lemma 4.4 and $\pi_\lambda(X'_\lambda) \subset E_\lambda$, we conclude that

$$\dim_H U_\lambda < \dim_H E_\lambda.$$
completing the proof. □

5. Typical result for the Hausdorff dimension of $U_{\lambda}$

When $E_{\lambda}$ is totally self-similar, we determine the Hausdorff dimension of $U_{\lambda}$ in the previous section. In this section we show that $U_{\lambda}$ has the same Hausdorff dimension as $E_{\lambda}$ for typical $\lambda$, and prove Theorem 4.8. Note that

\[ E_{\lambda} \setminus U_{\lambda} = \bigcup_{i \in \Omega_{\lambda}^i} f_i(M_{\lambda}), \]

where

\[ M_{\lambda} := \bigcup_{c,d \in \Omega_{\lambda}, c \neq d} f_c(E_{\lambda}) \cap f_d(E_{\lambda}). \]

Then $E_{\lambda} \setminus U_{\lambda}$ is a countable union of scaling copies of $M_{\lambda}$. By the countable stability of Hausdorff dimension, to prove Theorem 1.8 (i) it suffices to prove that for $\lambda \in (0, 1)$ there exist $\lambda$ such that $\dim_H M_{\lambda} < \dim_H E_{\lambda}$ for Lebesgue almost every $\lambda \in (0, \rho) \cup (1 - \rho, 1)$. Moreover, to prove Theorem 1.8 (ii) we only need to prove that for $\lambda \in (0, 1)$ we have $M_{\lambda} = \emptyset$ for Lebesgue almost every $\lambda \in (0, \rho) \cup (1 - \rho, 1)$. Since the proof for $\lambda \in (1 - \rho, 1)$ is similar, we only prove it for $\lambda \in (0, \rho)$.

Let $J := [a, b] \subset (0, \rho)$, and take $\lambda \in J$. Then (see the left graph of Figure 1)

\[ M_{\lambda} = (f_0(E_{\lambda}) \cap f_{\lambda}(E_{\lambda})) \cup (f_{1/\lambda}(E_{\lambda}) \cap f_{1/\lambda}(E_{\lambda})). \]

Note by symmetry that $f_{1/\lambda}(E_{\lambda}) \cap f_{1/\lambda}(E_{\lambda}) = 1 - f_0(E_{\lambda}) \cap f_{\lambda}(E_{\lambda})$. So in the following it suffices to prove that for $\rho \in (0, 1/4)$ we have $\dim_H (f_0(E_{\lambda}) \cap f_{\lambda}(E_{\lambda})) < \dim_H E_{\lambda}$ for Lebesgue almost every $\lambda \in J$. And for $\rho \in (0, 1/4)$ we have $f_0(E_{\lambda}) \cap f_{\lambda}(E_{\lambda}) = \emptyset$ for Lebesgue almost every $\lambda \in J$. Suppose $f_0(E_{\lambda}) \cap f_{\lambda}(E_{\lambda}) = \emptyset$ for some $\lambda \in J$. Otherwise, we are done. Observe that each $x \in E_{\lambda}$ has a coding $i \in \Omega_{\lambda}^N$ satisfying $x = \pi_{\lambda}(i)$. Set

\[ D_J := \left\{ (i, j) \in \Omega_{\lambda}^N \times \Omega_{\lambda}^N : \exists \lambda \in J \text{ such that } f_0(\pi_{\lambda}(i)) = f_{\lambda}(\pi_{\lambda}(j)) \right\}. \]

**Lemma 5.1.** Each pair $(i, j) \in D_J$ determines a unique $\lambda \in J$.

**Proof.** Take $(i, j) \in D_J$ with $i = i_1 i_2 \ldots, j = j_1 j_2 \ldots$. Then $f_0(\pi_{\lambda}(i)) = f_{\lambda}(\pi_{\lambda}(j))$. By (1.6) it follows that

\[ \sum_{n=1}^{\infty} \rho^n i_n = \sum_{n=1}^{\infty} \rho^n j_n + \lambda. \]

Note that the digits $i_n, j_n \in \Omega_{\lambda} = \{0, \lambda, 1 - \rho - \lambda, 1 - \rho\}$ might contain the parameter $\lambda$. In order to separate the parameter $\lambda$ we partition the set $\mathbb{N}$ into $\mathcal{N}_{j}^{3} := \{n : i_n = 0\}, \mathcal{N}_{j}^{2} := \{n : i_n = \lambda\}, \mathcal{N}_{j}^{1} := \{n : i_n = 1 - \rho - \lambda\}$ and $\mathcal{N}_{j}^{0} := \{n : i_n = 1 - \rho\}$. Similarly, for $s \in \{1, 2, 3, 4\}$ we define $\mathcal{N}_{j}^{s}$ by replacing the $i_n$ in $\mathcal{N}_{j}^{s}$ by $j_n$. Thus (5.2) can be rewritten as

\[ \lambda \sum_{n \in \mathcal{N}_{j}^{3}} \rho^n + (1 - \rho - \lambda) \sum_{n \in \mathcal{N}_{j}^{2}} \rho^n + (1 - \rho) \sum_{n \in \mathcal{N}_{j}^{1}} \rho^n + (1 - \rho - \lambda) \sum_{n \in \mathcal{N}_{j}^{0}} \rho^n + (1 - \rho) \sum_{n \in \mathcal{N}_{j}^{4}} \rho^n + \lambda, \]
which can be reorganized as
\[
\lambda \left( 1 + \sum_{n \in \mathcal{N}_j^2} \rho^n + \sum_{n \in \mathcal{N}_i^3} \rho^n - \sum_{n \in \mathcal{N}_j^3} \rho^n - \sum_{n \in \mathcal{N}_i^2} \rho^n \right) = (1 - \rho) \left( \sum_{n \in \mathcal{N}_j^3 \cup \mathcal{N}_i^4} \rho^n - \sum_{n \in \mathcal{N}_j^2 \cup \mathcal{N}_i^4} \rho^n \right).
\]

Since \(0 < \rho < 1/4\), we have
\[(5.3)\]
\[
1 + \sum_{n \in \mathcal{N}_j^2} \rho^n + \sum_{n \in \mathcal{N}_i^3} \rho^n - \sum_{n \in \mathcal{N}_j^3} \rho^n - \sum_{n \in \mathcal{N}_i^2} \rho^n \geq 1 - 2 \sum_{n=1}^{\infty} \rho^n = \frac{1 - 3\rho}{1 - \rho} > 0.
\]

This gives
\[
\lambda = \frac{(1 - \rho) \left( \sum_{n \in \mathcal{N}_j^3 \cup \mathcal{N}_i^4} \rho^n - \sum_{n \in \mathcal{N}_j^2 \cup \mathcal{N}_i^4} \rho^n \right)}{1 + \sum_{n \in \mathcal{N}_j^2} \rho^n + \sum_{n \in \mathcal{N}_i^3} \rho^n - \sum_{n \in \mathcal{N}_j^3} \rho^n - \sum_{n \in \mathcal{N}_i^2} \rho^n}
\]
as desired.

In terms of Lemma 5.1, let \(\lambda_{ij}\) be the unique \(\lambda \in J\) determined by the pair \((i,j) \in D_J\). Then
\[
(5.4)\]
\[
\lambda_{ij} = \frac{(1 - \rho)p_{ij}}{1 + q_{ij}},
\]
where
\[
p_{ij} := \sum_{n \in \mathcal{N}_j^3 \cup \mathcal{N}_i^4} \rho^n - \sum_{n \in \mathcal{N}_j^2 \cup \mathcal{N}_i^4} \rho^n, \quad q_{ij} := \sum_{n \in \mathcal{N}_j^3} \rho^n + \sum_{n \in \mathcal{N}_i^4} \rho^n - \sum_{n \in \mathcal{N}_j^3} \rho^n - \sum_{n \in \mathcal{N}_i^4} \rho^n.
\]

Equipped with the metric \(d\) on \(\Omega^N_{\lambda}\) given by
\[
d(i,j) := \rho^{\inf\{n : i_n \neq j_n\}},
\]
we define a metric \(\| \cdot \|\) on the product space \(\Omega^N_{\lambda} \times \Omega^N_{\lambda}\) by
\[
\|(i,j), (u,v)\| = \max\{d(i,u), d(j,v)\}.
\]

Lemma 5.2. The map \(\Phi : D_J \to J \times [0,1]\) defined by
\[
\Phi((i,j)) = (\lambda_{ij}, f_0(\pi_{\lambda_{ij}}(i)))
\]
is Lipschitz continuous with respect to the metric \(\| \cdot \|\) on \(D_J\).

Proof. Take two pairs \((i,j), (u,v) \in D_J\). It suffices to prove that
\[
(5.5)\]
\[
|\lambda_{ij} - \lambda_{u,v}| \leq C_1\|(i,j), (u,v)\|
\]
and
\[
(5.6)\]
\[
|f_0(\pi_{\lambda_{ij}}(i)) - f_0(\pi_{\lambda_{u,v}}(u))| \leq C_2\|(i,j), (u,v)\|
\]
for some constants $C_1, C_2 > 0$. Note by (5.4) that
\[
|\lambda_{i,j} - \lambda_{u,v}| \leq \frac{|p_{i,j} - p_{u,v}|}{1 + q_{i,j}} + \frac{|p_{u,v}|}{(1 + q_{i,j})(1 + q_{u,v})} |q_{i,j} - q_{u,v}|
\]
(5.7)
\[
\leq \frac{1}{1 + q_{i,j}} |p_{i,j} - p_{u,v}| + \frac{|p_{u,v}|}{(1 + q_{i,j})(1 + q_{u,v})} |q_{i,j} - q_{u,v}|
\]
where the last inequality follows by (5.5). This proves (5.6), completing the proof. □

In particular, if $q_{i,j}, 1 + q_{u,v} \geq \frac{1 - 3\rho}{1 - \rho} > 0$ and $|p_{u,v}| \leq \sum_{n=1}^{\infty} \rho^n = \frac{\rho}{1 - \rho}$. Observe that
\[
|p_{i,j} - p_{u,v}| \leq \left| \sum_{n \in N_i^1 \cup N_i^4} \rho^n - \sum_{n \in N_u^1 \cup N_u^4} \rho^n \right| + \left| \sum_{n \in N_i^2 \cup N_i^4} \rho^n - \sum_{n \in N_u^2 \cup N_u^4} \rho^n \right|
\]
\[
\leq \tilde{C}_1 |d(i, u) + d(j, v)| \leq \tilde{C}_2 ||(i, j), (u, v)||
\]
for some constants $\tilde{C}_1, \tilde{C}_2 > 0$, and similarly, $|q_{i,j} - q_{u,v}| \leq \tilde{C}_3 ||(i, j), (u, v)||$ for some constant $\tilde{C}_3 > 0$. So, by (5.7) we prove (5.5).

On the other hand, observe by $\lambda_{i,j} \in J = [a, b]$ that
\[
|f_0(\pi_{\lambda_{i,j}}(1)) - f_0(\pi_{\lambda_{u,v}}(u))| \leq (1 - \rho) \left| \sum_{n \in N_i^1 \cup N_i^4} \rho^n - \sum_{n \in N_u^1 \cup N_u^4} \rho^n \right|
\]
\[
+ |\lambda_{i,j} \left( \sum_{n \in N_i^2} \rho^n - \sum_{n \in N_u^2} \rho^n \right) - \lambda_{u,v} \left( \sum_{n \in N_u^2} \rho^n - \sum_{n \in N_u^3} \rho^n \right) |
\]
\[
\leq \tilde{C}_4 \cdot d(i, u) + |\lambda_{i,j}| \cdot \left| \sum_{n \in N_i^1} \rho^n + \sum_{n \in N_u^1} \rho^n - \sum_{n \in N_i^3} \rho^n - \sum_{n \in N_u^3} \rho^n \right|
\]
\[
+ \left| \sum_{n \in N_u^2} \rho^n - \sum_{n \in N_u^3} \rho^n \right| \cdot |\lambda_{i,j} - \lambda_{u,v}|
\]
\[
\leq \tilde{C}_4 \cdot d(i, u) + \tilde{C}_5 \cdot d(i, u) + \frac{\rho}{1 - \rho} |\lambda_{i,j} - \lambda_{u,v}|
\]
\[
\leq C_2 ||(i, j), (u, v)||
\]
for some constant $C_2 > 0$, where the last inequality follows by (5.5). This proves (5.6), completing the proof.

The following famous slicing theorem is due to Marstrand [15].

**Lemma 5.3.** Let $K \subset \mathbb{R}^2$ be a Borel set. Then for Lebesgue almost every $x \in \mathbb{R}$ we have
\[
\dim_H K \cap \{(x, y) : y \in \mathbb{R}\} \leq \max \{0, \dim_H K - 1\}.
\]
In particular, if $\dim_H K < 1$, then for Lebesgue almost every $x \in \mathbb{R}$ the intersection
\[
K \cap \{(x, y) : y \in \mathbb{R}\} = \emptyset.
\]
Note that $E_\lambda$ is an affine image of the projection of the four corner Cantor set $E$ generated by the IFS $\{(px, py), (px, py + 1 - \rho), (px + 1 - \rho, py), (px + 1 - \rho, py + 1 - \rho)\}$. Another useful result was essentially due to Hochman [10] (see also [20, Theorem 2.1]).

**Lemma 5.4.** Let $\rho \in (0, 1/4)$. Then for Lebesgue almost every $\lambda \in (0, \frac{1}{2} \rho)$ we have

$$\dim_H E_\lambda = \frac{2 \log 2}{-\log \rho}.$$

**Proof of Theorem 1.8.** First we consider $\rho \in (0, 1/4)$. Take $J = [a, b] \subset (0, \rho)$. By Lemma 5.2 it follows that

$$(5.8) \quad \dim_H \Phi(D_J) \leq \dim_H D_J \leq 2 \dim_H \Omega^N_\lambda = 2 \frac{\log 4}{-\log \rho},$$

where the last equality holds since $\Omega^N_\lambda$ is a compact metric space under the metric $d(i, j) = \rho^{\inf\{n: n \neq j_n\}}$. Note that $M_\lambda = [f_0(E_\lambda) \cap f_\lambda(E_\lambda)] \cup [1 - f_0(E_\lambda) \cap f_\lambda(E_\lambda)]$. So, by Lemma 5.3 it follows that for Lebesgue almost every $\lambda \in J$,

$$\dim_H M_\lambda = \dim_H (f_0(E_\lambda) \cap f_\lambda(E_\lambda)) \leq \dim_H (\Phi(D_J) \cap \{(\lambda, y) : y \in \mathbb{R}\})$$

$$\leq \dim_H \Phi(D_J) - 1 \leq \frac{2 \log 4}{-\log \rho} - 1 < \frac{\log 4}{-\log \rho},$$

where the last inequality follows by $0 < \rho < 1/4$. Hence, by (5.1) and Lemma 5.4 it follows that for Lebesgue almost every $\lambda \in J$,

$$\dim_H (E_\lambda \setminus U_\lambda) \leq \frac{\log 4}{-\log \rho} = \dim_H E_\lambda,$$

which yields $\dim_H U_\lambda = \dim_H E_\lambda$. Since $J \subset (0, \rho)$ was arbitrary, it follows that $\dim_H U_\lambda = \dim_H E_\lambda = \frac{2 \log 4}{-\log \rho}$ for Lebesgue almost every $\lambda \in (0, \rho)$.

Next we assume $\rho \in (0, 1/16)$. Then by (5.8) we have $\dim_H \Phi(D_J) \leq \frac{2 \log 4}{-\log \rho} < 1$. So, by the second statement of Lemma 5.3 one can deduce that for Lebesgue almost every $\lambda \in J$ the intersection $f_0(E_\lambda) \cap f_\lambda(E_\lambda) = \emptyset$, and then $M_\lambda = \emptyset$. So, by (5.1) we conclude that $U_\lambda = E_\lambda$ for Lebesgue almost every $\lambda \in J$. Since $J \subset (0, \rho)$ was arbitrary, it follows that $U_\lambda = E_\lambda$ for Lebesgue almost every $\lambda \in (0, \rho)$.

**6. The possibility for $E_\lambda$ to contain an interval**

Recall by (1.7) that $W$ consists of all coprime pairs $(p, q) \in \mathbb{N}^2$ with $p < q$ and $\text{ord}_2(p), \text{ord}_2(q)$ even. By Theorem 1.11 it follows that for $\lambda \in (0, 3/8)$ the self-similar set $E_\lambda$ has an exact overlap if and only if $\lambda = \frac{3p}{4(p+q)}$ with $(p, q) \in W$. Similarly, we recall from (1.8) that $\tilde{W}$ consists of all coprime pairs $(p, q) \in \mathbb{N}^2$ satisfying $p < q$, $\text{ord}_2(p)$ odd or $\text{ord}_2(q)$ odd. Furthermore, by Theorem 1.11 it follows that for $\lambda \in (0, 3/8)$, $E_\lambda$ contains a non-degenerate interval if and only if $\lambda = \frac{3p}{4(p+q)}$ with $(p, q) \in \tilde{W}$. In this section we will describe the densities of $W$ and $\tilde{W}$ in $\mathbb{N}^2$, and prove Theorem 1.13.

First we recall some known results from analytic number theory (cf. [9]). Let $\phi$ be the Euler’s function such that for $n \in \mathbb{N}$, $\phi(n)$ is the number of positive integers no larger than and
prime to $n$. Then $\phi(1) = 1$, and for $n \in \mathbb{N}_{\geq 2}$, if we write it in a standard form $n = p_1^{c_1} p_2^{c_2} \cdots p_r^{c_r}$ with $p_1, p_2, \ldots, p_r$ distinct primes, then (cf. [9, Theorem 62])

$$\phi(n) = \prod_{i=1}^{r} p_i^{c_i-1}(p_i - 1).$$

Furthermore, the summation $\sum_{n=1}^{N} \phi(n)$ increases to infinity of order $N^2$. In fact, by [9, Theorem 330] we have

$$\lim_{N \to \infty} \frac{1}{N^2} \sum_{n=1}^{N} \phi(n) = \frac{3}{\pi^2}. \quad (6.2)$$

Another useful representation of $\phi$ is based on the Möbius function $\mu$ defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } n \text{ has a squared factor} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ different primes} \end{cases}$$

Then by (6.1) the function $\phi$ can be rewritten as

$$\phi(n) = n \sum_{m|n} \frac{\mu(m)}{m}, \quad (6.3)$$

where the summation is taken over all positive factors $m$ of $n$.

The following result can be easily deduced from [9, Theorem 287].

**Lemma 6.1.** For any $s > 1$ let $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ be the zeta function. Then

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \prod_{p} (1 - \frac{1}{p^s}) = \frac{1}{\zeta(s)},$$

where the product is taken over all prime numbers.

Next we prove a useful lemma which is comparable with (6.2).

**Lemma 6.2.**

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \frac{\phi(2n - 1)}{2n - 1} = \frac{8}{\pi^2}. \quad (6.4)$$

**Proof.** By (6.3) it follows that

$$\sum_{n=1}^{N} \frac{\phi(2n - 1)}{2n - 1} = \sum_{n=1}^{N} \sum_{m|(2n-1)} \frac{\mu(m)}{m} = \sum_{n=1}^{N} \sum_{(2m-1)|(2n-1)} \frac{\mu(2m-1)}{2m-1}$$

$$= \sum_{m=1}^{N} \frac{\mu(2m-1)}{2m-1} \sum_{n=1}^{N} \mathbb{I}((2n-1)(2m-1) \leq 2N-1)$$

$$= \sum_{m=1}^{N} \frac{\mu(2m-1)}{2m-1} \left[ \frac{N + m - 1}{2m - 1} \right]$$

$$= \sum_{m=1}^{N} \frac{\mu(2m-1)}{2m-1} \cdot \frac{N}{2m - 1} + \varepsilon_N,$$
where $I$ is the indicator function and
\[
|\varepsilon_N| \leq \sum_{m=1}^{N} \frac{m-1}{(2m-1)^2} + \sum_{m=1}^{N} \frac{1}{2m-1} < \sum_{m=1}^{N} \frac{2}{2m-1}.
\]
Clearly, $\frac{|\varepsilon_N|}{N} \to 0$ as $N \to \infty$. This implies that
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \phi(2n-1) = \lim_{N \to \infty} \sum_{m=1}^{N} \frac{\mu(2m-1)}{(2m-1)^2} = \sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(2m-1)^2}.
\]
Therefore, the lemma follows by Lemma 6.1 that
\[
\sum_{m=1}^{\infty} \frac{\mu(2m-1)}{(2m-1)^2} = \prod_{p \geq 3} \left(1 + \frac{\mu(p)}{p^2} + \frac{\mu(p^2)}{p^4} + \cdots\right) = \prod_{p \geq 3} \left(1 - \frac{1}{p^2}\right)
= \frac{1}{1 - \frac{1}{2^2}} \prod_{p} \left(1 - \frac{1}{p}\right) = \frac{4}{3} \frac{1}{\zeta(2)} = \frac{8}{\pi^2},
\]
where the first two products are taken over all primes at least three, the third product is taken over all primes, and the last equality follows by using $\zeta(2) = \sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. \hfill \Box

**Proposition 6.3.** Let $\hat{W}$ be defined as in (1.8). Then
\[
\lim_{N \to \infty} \frac{\#(\hat{W} \cap [1, N]^2)}{N^2} = \frac{4}{3\pi^2}.
\]

**Figure 3.** The graph of $\hat{W} \cap [1, N]^2$ with $N = 100$. 
Proof. Note that for \( n \in \mathbb{N} \), \( \text{ord}_2(n) \) is odd if and only if \( n = (2k - 1)2^{2\ell - 1} \) for some \( k, \ell \in \mathbb{N} \). In view of the definition of \( \hat{W} \), we will count the number of pairs \((p, q) \in \hat{W} \cap [1, N]^2\) in the following way (see Figure 3). First, by conditioned on \( q = (2k - 1)2^{2\ell - 1} \) the number of \( p \)'s satisfying \( (p, q) \in \hat{W} \cap [1, N]^2 \) is \( \phi((2k - 1)2^{2\ell - 1}) \). Second, by conditioned on \( p = (2k - 1)2^{2\ell - 1} \) the number of \( q \)'s satisfying \( (p, q) \in \hat{W} \cap [1, N]^2 \) is given by \( \varphi(N, (2k - 1)2^{2\ell - 1}) - \phi((2k - 1)2^{2\ell - 1}) \), where

\[
\varphi(N, (2k - 1)2^{2\ell - 1}) = \#\left\{ 1 \leq n \leq N : n \text{ and } (2k - 1)2^{2\ell - 1} \text{ are coprime} \right\}.
\]

For the range of \( k \) and \( \ell \), let \( k_1 \) be the largest \( k \in \mathbb{N} \) such that \( (2k - 1)2^{2\ell - 1} \leq N \) for some \( \ell \in \mathbb{N} \). Then

\[
k_1 = k_1(N) = \left\lfloor \frac{N + 2}{4} \right\rfloor.
\]

Furthermore, for \( k \in [1, k_1] \) let \( \ell_k \) be the largest \( \ell \in \mathbb{N} \) such that \( (2k - 1)2^{2\ell - 1} \leq N \). Then

\[
\ell_k = \ell_k(N) = \left\lfloor \log_2 \left( \frac{2N}{2k - 1} \right) \right\rfloor.
\]

Therefore,

\[
\#(\hat{W} \cap [1, N]^2) = \sum_{k=1}^{k_1} \sum_{\ell=1}^{\ell_k} \phi((2k - 1)2^{2\ell - 1})
\]

\[
= \sum_{k=1}^{k_1} \sum_{\ell=1}^{\ell_k} \left( \varphi(N, (2k - 1)2^{2\ell - 1}) - \phi((2k - 1)2^{2\ell - 1}) \right)
\]

\[
= \sum_{k=1}^{k_1} \sum_{\ell=1}^{\ell_k} \varphi(N, (2k - 1)2^{2\ell - 1}).
\]

Observe that (cf. [1, Page 47, Exercise 9])

\[
\varphi(N, (2k - 1)2^{2\ell - 1}) = \sum_{n|(2k-1)2^{2\ell - 1}} \frac{\mu(n) \left\lfloor \frac{N}{n} \right\rfloor}{n} = \sum_{n|(2k-1)2^{2\ell - 1}} \frac{\mu(n)}{n} \frac{N}{n} + \varepsilon_{k,\ell}
\]

\[
= \frac{N \phi((2k - 1)2^{2\ell - 1})}{(2k - 1)2^{2\ell - 1}} + \varepsilon_{k,\ell} = N \frac{\phi(2k - 1)}{2(2k - 1)} + \varepsilon_{k,\ell},
\]

where the last two equalities follow by (6.3) and (6.1) respectively. Here the error term \( \varepsilon_{k,\ell} \) is bounded by

\[
|\varepsilon_{k,\ell}| \leq \sum_{n=1}^{\infty} \mathbb{I}_{\{n|(2k-1)2^{2\ell - 1}\}} = d((2k - 1)2^{2\ell - 1}) = 2\ell d(2k - 1),
\]

where \( d(m) \) denotes the number of all positive factors of \( m \), and the last equality follows since \( d(m) = \prod_{i=1}^{n}(c_i + 1) \) if \( m = \prod_{i=1}^{n} p_i^{c_i} \) (cf. [9, Theorem 273]). Thus, by (6.6) and (6.7) it
follows that

$$\lim_{N \to \infty} \frac{\#(\hat{W} \cap [1, N]^2)}{N^2} = \lim_{N \to \infty} \frac{1}{N^2} \sum_{k=1}^{k_1} \sum_{\ell=1}^{\ell_k} \left( N \frac{\phi(2k-1)}{2(2k-1)} + \varepsilon_{k, \ell} \right)$$

(6.8)

$$= \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{k_1} \sum_{\ell=1}^{\ell_k} \frac{\phi(2k-1)}{2(2k-1)},$$

where the last equality follows by (6.4) and (6.5) that

$$\frac{1}{N^2} \sum_{k=1}^{k_1} \sum_{\ell=1}^{\ell_k} \varepsilon_{k, \ell} \leq \frac{1}{N^2} \sum_{k=1}^{k_1} \sum_{\ell=1}^{\ell_k} 2\ell d(2k-1) = \frac{1}{N^2} \sum_{k=1}^{k_1} \ell_k(\ell_k + 1)d(2k-1)$$

$$\leq \frac{\log_4(2N)(\log_4(2N) + 1)}{N^2} \sum_{k=1}^{N+2} d(2k-1) \to 0 \text{ as } N \to \infty.$$
where the little ‘o’ stands for the higher order indefinite small. Therefore, by (6.9) we obtain

\[
\lim_{N \to \infty} \frac{\#(\hat{W} \cap [1, N]^2)}{N^2} = \lim_{N \to \infty} \frac{1}{2N} \left( \sum_{j=1}^{\lfloor \log_2 2N \rfloor} j \left( \frac{8}{\pi^2} k_j - \frac{8}{\pi^2} k_{j+1} \right) \right)
\]

\[+ \lim_{N \to \infty} \frac{1}{2N} \sum_{j=1}^{\lfloor \log_2 2N \rfloor} j \cdot o(k_j + k_{j+1}) \]

\[= \lim_{N \to \infty} \frac{4}{\pi^2 N} \sum_{j=1}^{\lfloor \log_2 2N \rfloor} j \left( \left\lfloor \frac{1}{2} + \frac{N}{4^j} \right\rfloor - \left\lfloor \frac{1}{2} + \frac{N}{4^{j+1}} \right\rfloor \right) + \lim_{N \to \infty} \frac{o(N)}{2N} \]

\[= \lim_{N \to \infty} \frac{4}{\pi^2} \sum_{j=1}^{\infty} \frac{3j}{4j+1} = \frac{4}{3\pi^2} \]

as desired. \(\square\)

**Proof of Theorem 1.13.** By Proposition 6.3 we only need to consider the density of \(W\). Note by (1.7) and (1.8) that \(W\) and \(\hat{W}\) are disjoint, and

\[W \cup \hat{W} = \{(p, q) \in \mathbb{N}^2 : p < q \text{ and } p, q \text{ are coprime}\}.\]

Then for large \(N \in \mathbb{N}\) we have \(#(W \cup \hat{W} \cap [1, N]^2) = \sum_{n=2}^{N} \phi(n)\). So, by (6.2) and Proposition 6.3 it follows that

\[
\lim_{N \to \infty} \frac{\#(W \cap [1, N]^2)}{N^2} = \lim_{N \to \infty} \frac{\sum_{n=2}^{N} \phi(n)}{N^2} - \lim_{N \to \infty} \frac{\#(\hat{W} \cap [1, N]^2)}{N^2} = \frac{3}{\pi^2} - \frac{4}{3\pi^2} = \frac{5}{3\pi^2},
\]

completing the proof. \(\square\)

**Acknowledgements**

The first author was supported by Chongqing NSF: CQYC20220511052.

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