A Codicil To Massless Gauge Superfields of Higher Half-Odd Integer Superspins

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ABSTRACT

We study theories of 4D, $\mathcal{N} = 1$ supersymmetric massless, arbitrary higher half odd-integer superspins. A new series of such theories is found to exist for arbitrary superspin $Y$ ($Y = s + 1/2$ for any integer $s$). The lowest member ($s = 1$) of the series is the original off-shell formulation of 4D, $\mathcal{N} = 1$ supergravity first presented by Breitenlohner in 1977.

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1 Introduction

The state-of-the-art understanding on the subject of higher spin supersymmetric multiplets was established in a work by Kuzenko, Postnikov, and Sibiryakov [1]. In fact, they established two such formulations for each and every possible value of the superspin $Y$. These formulations are based on the introduction of constrained compensating superfields. The goal of this work is to re-examine these schemes in order to be able to reproduce their results and, if possible, to discover new formulations in the case of half odd superspins. This is exactly what will happen in the following. Their results will emerge naturally from our algorithm as a possible way a theory of higher, half odd massless superspins can be formulated.

In an accompany paper [2] devoted to the study of massless 4D, $\mathcal{N} = 1$ higher integer superspins, we developed an algorithm that was able to do two things:

- generate all known results for massless 4D, $\mathcal{N} = 1$ higher integer superspins up to that point, and
- introduce a new formulation of the theory.

After the success of this algorithm in the investigation of higher, integer superspins we would like to apply a similar way of thinking in the case of half odd superspins.

The conceptual backbone of the method followed, can be summarized as following:

Step 1) find the main physical superfield that will be used to construct the theory,

Step 2) find the most general free action which is quadratic to this superfield,

Step 3) find the gauge transformation of the main superfield,

Step 4) find the type of superfield(s) we have to introduce as compensators,

Step 5) find the possible gauge transformations of the compensators which on-shell give just the degrees of freedom needed, and

Step 6) check invariances of the action with respect to all transformations.

This fixes the index structure, mass dimensions and says something about reality of this main physical superfield. Within the context of supergravity, this main physical superfield is known as the ‘superconformal submultiplet.’
2 General Action and Gauge Transformations

The goal is to develop a theory for massless half odd superspin \( Y = s + \frac{1}{2} \), for integers \( s \). This means the highest superspin projection operator acting on the ‘main superfield’ used to develop the theory must generate an object with an odd number of indices \( (2s + 1) \). As suggested by supergravity theory, the fundamental superfield for this theory should be a bosonic superfield with an even number of indices, \( H \) of a real bosonic superfield theory) it also needs to be real. Therefore our theory must be constructed in terms of the Fronsdal action of massless integer spins (which must be the bosonic piece of our theory) it also needs to be real. Therefore our theory must be constructed in terms of a real bosonic superfield \( H_{\alpha(s)\dot{\alpha}(s)} \) with zero mass dimensions \( (|H| = 0) \). The most general action that can be written for such an object has the form:

\[
S = \int d^8z \left\{ c_1 H^{\alpha(s)\dot{\alpha}(s)} D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\dot{\alpha}(s)} + c_2 H^{\alpha(s)\dot{\alpha}(s)} \Box H_{\alpha(s)\dot{\alpha}(s)} + c_3 H^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha(s)\dot{\alpha}(s)} \bar{D}^\gamma H_{\gamma\alpha(s-1)\dot{\gamma}(s-1)} + c_4 H^{\alpha(s)\dot{\alpha}(s)} \left[ D_{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\gamma(s-1)\dot{\gamma}(s-1)} \right] \right\} \tag{1}
\]

In writing this action, we have also made an assumption that parity violating terms should be excluded. If this assumption is not used then an additional term of the form

\[
S_{P-\text{violation}} = \int d^8z \left\{ H^{\alpha(s)\dot{\alpha}(s)} \partial_{\alpha(s)\dot{\alpha}(s)} [\bar{D}^\gamma, \bar{D}^\gamma] H_{\gamma\alpha(s-1)\dot{\gamma}(s-1)} \right\} \quad \tag{2}
\]

may be considered.

The massless property of the theory suggest there must be an underlying gauge symmetry. This symmetry, of course, must respect the highest superspin projection operator. Taking this into account there is only one option. The gauge transformation of \( H_{\alpha(s)\dot{\alpha}(s)} \) must be of the form:

\[
\delta H_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{s!} \bar{D}_{\dot{\alpha}(s-1)} L_{\alpha(s)\dot{\alpha}(s-1)} - \frac{1}{s!} D_{\alpha(s-1)\dot{\alpha}(s)} \tag{3}
\]

written in terms of some complex gauge parameter superfield \( L_{\alpha(s)\dot{\alpha}(s-1)} \).

The change of the above action under this transformation is:

\[
\delta S = \int d^8z \left\{ -2c_1 + 2c_2 + \frac{2}{s} c_3 + 2\frac{2s+1}{s} c_4 \right\} H^{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}(s-1)} D^2 \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} \]

\[\text{5}\]In principle we could repeat the whole analysis including this term and show that it’s coefficient will vanish. But just knowing that the final results are Frondal’s actions for bosons and fermions and they preserves parity, allows us to set this term to zero from the very beginning.
\[ + 2c_2 H^\alpha(\dot{a}(s)) \bar{D}^2 \bar{D}_{\dot{a}_s} L_{\alpha(s)\dot{a}(s-1)} \]
\[ + \left[ -\frac{2}{s} c_3 + 2\frac{s+1}{s} c_4 \right] H^\alpha(\dot{a}(s)) D_{\alpha_s} \bar{D}^2 \bar{D}_{\dot{a}_s} L_{\gamma\alpha(s-1)\dot{a}(s-1)} \]
\[ + \left[ 2c_3 - 2c_4 \right] H^\alpha(\dot{a}(s)) \bar{D}_{\dot{a}_s} D_{\alpha_s} \bar{D}^2 \bar{D}^\gamma L_{\gamma\alpha(s-1)\dot{a}(s-1)} \]
\[ - \left[ \frac{s-1}{s} \right] \left[ 2c_3 - 2c_4 \right] H^\alpha(\dot{a}(s)) \bar{D}_{\dot{a}_s} D_{\alpha_s} \bar{D}_{\dot{a}_{s-1}} \bar{D}^\gamma \bar{D}^\gamma L_{\gamma\alpha(s-1)\dot{a}(s-2)} \]
\[ + c.c. \right] . \]

It is obvious that the above action is not invariant under the proposed gauge transformation. There are two ways around this. One way is to impose differential constraints (using either \( D \) or \( \bar{D} \)) on the gauge parameter superfield \( L_{\alpha(s)\dot{a}(s-1)} \). In general this procedure leads to the ‘ghost-for-ghost’ phenomenon in a quantum theory. We wish to avoid this.

The other way, is to introduce a set of compensators. In order to keep the propagating degrees of freedom down to the minimal number and in order to have on-shell an irreducible representation of the Super-Poincare group, we need exactly one ‘propagating’ compensator \(^6\) and some arbitrary number of auxiliary compensators \(^7\).

The propagating compensator must satisfy several constraints. It must provide the extra degrees of freedom in order to complete the irreducible representation and the rest of its components must vanish on shell. In principle there are two options, it can be either bosonic or fermionic. In the first case, the gauge transformation of a bosonic compensator has to be \(^8\) of the form \( D_{\alpha} L_{\alpha(s)\dot{a}(s-1)} + \bar{D}^{\dot{a}_s} L_{\alpha(s-1)\dot{a}(s)} \). But with a transformation like that we can not gauge away all the degrees of freedom, besides the ones needed for the irreducible representation. So this option can not lead to the desired result. Therefore the propagating compensator must be a fermionic superfield \( \Upsilon_{\alpha(s)\dot{a}(s-1)} \). This is in accord with the feature of all previous studied theories, where the statistics of the main superfield and the compensator opposite to one another.

Also the gauge transformation of the fermionic compensator \( \Upsilon \) must be such that, it satisfies the following:

- the component \( \Upsilon^{(1,0)}_{\alpha(s+1)\dot{a}(s-1)} \) must be gauged away. This can be done if

\[ D_{(\alpha_{s+1})} \Upsilon_{\alpha(s)\dot{a}(s-1)} \sim \text{some component of the gauge parameter (algebraically)} \]

\(^6\)This is a superfield of mass dimensions 0 or 1/2.
\(^7\)These are superfields of mass dimensions 1
\(^8\)This is fixed just by considering the index structure and mass dimensions.
• the component $\Upsilon^{(0,1)S}_{\alpha(s)\hat{\alpha}(s)}$ must be gauged away. This can be done if

$$\bar{D}_{(\hat{\alpha}s} \Upsilon_{\alpha(s)\hat{\alpha}(s-1))} \sim \text{some component of the gauge parameter (algebraically)}$$

• the component $\Upsilon^{(0,1)A}_{\alpha(s)\hat{\alpha}(s-2)}$ must be gauged away. This can be done if

$$\bar{D}^\gamma \Upsilon_{\alpha(s)\hat{\gamma}(s-2)} \sim \text{some component of the gauge parameter (algebraically)}$$

• the component $\Upsilon^{(1,1)(A,A)}_{\alpha(s-1)\hat{\alpha}(s-2)}$ must propagate on shell and have a specific gauge transformation. This can be done if

$$[D^\gamma, \bar{D}^\gamma] \Upsilon_{\gamma\alpha(s-1)\hat{\gamma}\hat{\alpha}(s-2)} \sim \partial^\gamma \bar{\Upsilon} \text{ some component of the gauge parameter}$$

The above constraints and equation (4) will be our guideline. Based on equation (4), we must find all possible ways that we can introduce a fermionic compensator with mass dimensions 1/2 and with the specific index structure $\Upsilon_{\alpha(s)\hat{\alpha}(s-1)}$, which has a gauge transformation that satisfies all the above constraints. This can be done only by two ways.

Since $L_{\alpha(s)\hat{\alpha}(s-1)}$ and $\bar{L}_{\alpha(s-1)\hat{\alpha}(s)}$ are the only gauge parameters available, the gauge transformation of $\Upsilon$ must include at least one of them. Their mass dimensions are -1/2 ($[L] = [\bar{L}] = -1/2$), so we need 2 $D(\bar{D})$’s to build something with mass dimensions 1/2. Therefore, the transformation of $\Upsilon_{\alpha(s)\hat{\alpha}(s-1)}$ must include at least one of the following terms.

- $A)$ $\bar{D}^2 L_{\alpha(s)\hat{\alpha}(s-1)}$,
- $B)$ $\frac{1}{s!} \bar{D}^{\alpha\hat{\alpha}} D_{(\alpha s} \bar{L}_{\alpha(s-1))\hat{\alpha}(s)}$,
- $C)$ $D^2 L_{\alpha(s)\hat{\alpha}(s-1)}$,
- $D)$ $\frac{1}{s!} D_{(\alpha s} \bar{D}^{\alpha\hat{\alpha}} \bar{L}_{\alpha(s-1)\hat{\alpha}(s)}$.

In order, these types of possible transformations to have a hope to give something desirable, they need to be completed appropriately so the above constraints are satisfied. The minimal way to do that is the following:

- $A)$ $\bar{D}^2 L_{\alpha(s)\hat{\alpha}(s-1)} + D^{\alpha s+1} \Lambda_{\alpha(s+1)\hat{\alpha}(s)}$,
- $B)$ $\frac{1}{s!} \bar{D}^{\alpha\hat{\alpha}} D_{(\alpha s} \bar{L}_{\alpha(s-1))\hat{\alpha}(s)} + D^{\alpha s+1} \Lambda_{\alpha(s+1)\hat{\alpha}(s)}$,
- $C)$ $D^2 L_{\alpha(s)\hat{\alpha}(s-1)} + D^{\alpha\hat{\alpha}} V_{\alpha(s)\hat{\alpha}(s)}$.

the last case was eliminated through this last requirement.
Now it is really straightforward to check if any of the above transformations can be used together with (4) in order to introduce the fermionic compensator. Just by observing (4) we see that case C can not happen and we are left with two possibilities. Case (A) can arise from the first term of (4) and Case (B) can arise from the third term. Next we will study this two cases.

### 3 The Higher Superspin KPS-Series

For Case (A) consider, we impose

\[ c_2 = c_3 = c_4 = 0 \tag{5} \]

so then equation (4) becomes:

\[
\delta S = \int d^8z \left\{ -2c_1 H^\alpha(s)\dot{\alpha}(s) \bar{D}_\alpha D^2 L_{\alpha(s)}\dot{\alpha}(s-1) + c.c. \right\} \\
= \int d^8z \left\{ -2c_1 H^\alpha(s)\dot{\alpha}(s) \bar{D}_\alpha D^2 \left[ \bar{D}^2 L_{\alpha(s)}\dot{\alpha}(s-1) + D^{\alpha+1} \Lambda_{\alpha(s+1)}\dot{\alpha}(s-1) \right] + c.c. \right\} . \tag{6} \]

At this point we can introduce a fermionic compensator \( \Upsilon_{\alpha(s)}\dot{\alpha}(s-1) \) with the following gauge transformation:

\[
\delta \Upsilon_{\alpha(s)}\dot{\alpha}(s-1) = \bar{D}^2 L_{\alpha(s)}\dot{\alpha}(s-1) + D^{\alpha+1} \Lambda_{\alpha(s+1)}\dot{\alpha}(s) , \tag{7} \]

and in order to construct a fully invariant action and give to the compensator some dynamics we have to add to the initial action some more terms

- Add a counter term, which cancels the change of the initial action:

\[
S_c = \int d^8z \ 2c_1 H^\alpha(s)\dot{\alpha}(s) \left( \bar{D}_\alpha D^2 \Upsilon_{\alpha(s)}\dot{\alpha}(s-1) - D_\alpha \bar{D}^2 \Upsilon_{\alpha(s-1)}\dot{\alpha}(s) \right) \tag{8} \]

- Add a kinetic energy term for the compensator (the most general free action quadratic to \( \Upsilon_{\alpha(s)}\dot{\alpha}(s-1) \))

\[
S_{k.e} = \int d^8z \left\{ \hbar_1 \Upsilon^\alpha(s)\dot{\alpha}(s-1) D^2 \Upsilon_{\alpha(s)}\dot{\alpha}(s-1) + c.c. \right\} \\
+ \hbar_2 \Upsilon^\alpha(s)\dot{\alpha}(s-1) \bar{D}^2 \Upsilon_{\alpha(s)}\dot{\alpha}(s-1) + c.c. \\
+ \hbar_3 \Upsilon^\alpha(s)\dot{\alpha}(s-1) \bar{D}_{\alpha} D_{\alpha} \Upsilon_{\alpha(s-1)}\dot{\alpha}(s) \\
+ \hbar_4 \Upsilon^\alpha(s)\dot{\alpha}(s-1) D_{\alpha} \bar{D}^2 \Upsilon_{\alpha(s-1)}\dot{\alpha}(s) \right\} \tag{9} \]
The full action is thus given by

\[
S_{\text{Full}} = \int d^8 z \left\{ c_1 H^{\alpha(s)\dot{\alpha}(s)} \bar{D}^7 D_7 H_{\alpha(s)\dot{\alpha}(s)} \\
+ 2c_1 H^{\alpha(s)\dot{\alpha}(s)} \left( \bar{D}_{\dot{\alpha}} D^2 Y_{\alpha(s)\dot{\alpha}(s-1)} - D_{\alpha} \bar{D}^2 \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} \right) \\
+ h_1 Y^{\alpha(s)\dot{\alpha}(s-1)} D^2 \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
+ h_2 Y^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 Y_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
+ h_3 Y^{\alpha(s)\dot{\alpha}(s-1)} \bar{D} \bar{D} \bar{D}_{\alpha} \bar{Y}_{\alpha(s)\dot{\alpha}(s)} \\
+ h_4 Y^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha} \bar{D} \bar{D} \bar{D}_{\alpha} \bar{Y}_{\alpha(s)\dot{\alpha}(s)} \right\}.
\]

Now we can define the superfields \( \mathcal{G}_{\alpha(s)\dot{\alpha}(s)} \) and \( \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} \) as the variations of the full action with respect to the superfields \( H_{\alpha(s)\dot{\alpha}(s)} \) and \( \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} \). These variations yield respectively,

\[
\mathcal{G}_{\alpha(s)\dot{\alpha}(s)} = 2c_1 D^7 \bar{D} D_7 H_{\alpha(s)\dot{\alpha}(s)} + \frac{2c_1}{s!} \left( \bar{D}_{\dot{\alpha}} D^2 Y_{\alpha(s)\dot{\alpha}(s-1)} - D_{\alpha} \bar{D}^2 \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} \right),
\]

and

\[
\mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} = 2c_1 D^2 \bar{D} \bar{D}_{\alpha} H_{\alpha(s)\dot{\alpha}(s)} + 2h_1 D^2 Y_{\alpha(s)\dot{\alpha}(s-1)} + 2h_2 D^2 \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} \\
+ \frac{h_3}{s!} \bar{D} \bar{D} \bar{D}_{\alpha} \bar{Y}_{\alpha(s-1)\dot{\alpha}(s)} + \frac{h_4}{s!} D_{\alpha} \bar{D} \bar{D} \bar{D}_{\alpha} \bar{Y}_{\alpha(s-1)\dot{\alpha}(s)}.
\]

The invariance of the full action under the above gauge transformations, forces a set of constraints that must be satisfied. These are the Bianchi Identities which are going to determine all the free parameters.

\[
\bar{D} \mathcal{G}_{\alpha(s)\dot{\alpha}(s)} + \bar{D}^2 \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} = 0,
\]

\[
D_{(\alpha+1)} \mathcal{T}_{\alpha(s)} = 0.
\]

The solution of the first one is:

\[
h_1 = -\frac{s+1}{s} c_1, \quad h_4 = 2c_1,
\]

and the solution of the second one is:

\[
h_2 = 0, \quad h_3 = 0.
\]

Therefore the final action is:

\[
S = \int d^8 z \left\{ c_1 H^{\alpha(s)\dot{\alpha}(s)} \bar{D}^7 D_7 H_{\alpha(s)\dot{\alpha}(s)} \\
+ 2c_1 H^{\alpha(s)\dot{\alpha}(s)} \left( \bar{D}_{\dot{\alpha}} D^2 Y_{\alpha(s)\dot{\alpha}(s-1)} - D_{\alpha} \bar{D}^2 \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} \right) \\
- \left[ \frac{s+1}{s} \right] c_1 Y^{\alpha(s)\dot{\alpha}(s-1)} D^2 Y_{\alpha(s)\dot{\alpha}(s-1)} + c.c. \\
+ 2c_1 Y^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha} \bar{D} \bar{D} \bar{D}_{\alpha} \bar{Y}_{\alpha(s-1)\dot{\alpha}(s)} \right\}.
\]
and it is invariant under the gauge transformations

\[ \delta H_{\alpha(s)\dot{\alpha}(s)} = \frac{1}{s!} \bar{D}_{(\dot{\alpha}_s L_{\alpha(s)\dot{\alpha}(s-1)})} - \frac{1}{s!} D_{(\alpha_s \bar{L}_{\alpha(s-1)\dot{\alpha}(s)})} \]

\[ \delta \Upsilon_{\alpha(s)\dot{\alpha}(s-1)} = \bar{D}^2 L_{\alpha(s)\dot{\alpha}(s-1)} + D^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s)} \].

This theory is equivalent to that developed by Kuzenko, Postnikov, and Sibiryakov (KPS), once one solves the constraints that appear in their description (as done in [4]). Therefore without any further examination we can conclude that this action, with that set of transformations describes a massless half odd superspin \((Y = s+1/2)\).

### 4 The Higher Superspin B-Series

For Case (B) we impose

\[ c_2 = 0 \]

\[ -2c_1 + \frac{2}{s} c_3 + 2 \frac{2s + 1}{s} c_4 = 0 \Rightarrow c_1 = \frac{1}{s} [c_3 + (2s + 1)c_4] \]

so that equation (4) becomes:

\[ \delta S = \int d^8 z \left\{ \left[ \frac{2}{s} \right] (c_3 - (2s + 1)c_4) H^{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}_s} D^2 \left[ \frac{1}{s!} \bar{D}^\gamma D_{(\alpha_s \bar{L}_{\alpha(s-1)\dot{\alpha}(s-1)})} \right] \right. \]

\[ + \left. \left[ \frac{2}{s} \right] (c_3 - (2s + 1)c_4) H^{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}_s} D^2 \left[ D^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} \right] \right. \]

\[ + c.c. \]

\[ -2(c_3 - c_4) H^{\alpha(s)\dot{\alpha}(s)} [D_{\alpha_s}, \bar{D}_{\dot{\alpha}_s}] \left[ \bar{D}^2 D^\gamma L_{\gamma\alpha(s-1)\dot{\alpha}(s-1)} \right] \]

\[ -2(c_3 - c_4) H^{\alpha(s)\dot{\alpha}(s)} [D_{\alpha_s}, \bar{D}_{\dot{\alpha}_s}] \left[ D^2 \bar{D}^\gamma \bar{L}_{\alpha(s-1)\dot{\alpha}(s-1)} \right] \]

\[ -2(c_3 - c_4) H^{\alpha(s)\dot{\alpha}(s)} [D_{\alpha_s}, \bar{D}_{\dot{\alpha}_s}] \left[ - \frac{s - 1}{s!} \bar{D}_{(\alpha_{s-1} D^\gamma \bar{D}^\gamma \bar{L}_{\gamma\alpha(s-2)\dot{\gamma}(s-2)})} \right] \]

\[ -2(c_3 - c_4) H^{\alpha(s)\dot{\alpha}(s)} [D_{\alpha_s}, \bar{D}_{\dot{\alpha}_s}] \left[ - \frac{s - 1}{s!} D_{(\alpha_{s-1} D^\gamma \bar{D}^\gamma \bar{L}_{\gamma\alpha(s-2)\dot{\gamma}(s-2)})} \right] \}. \]

We introduce two compensators:

1) A fermionic propagating compensator \( \Upsilon_{\alpha(s)\dot{\alpha}(s-1)} \) with mass dimensions 1/2

and the following gauge transformation

\[ \delta \Upsilon_{\alpha(s)\dot{\alpha}(s-1)} = \frac{1}{s!} \bar{D}^\gamma D_{(\alpha_s \bar{L}_{\alpha(s-1)\dot{\alpha}(s-1)})} \gamma_{\dot{\alpha}(s-1)} + D^{\alpha_{s+1}} \Lambda_{\alpha(s+1)\dot{\alpha}(s-1)} \]
2) A real auxiliary bosonic compensator $B_{\alpha(s-1)\dot{\alpha}(s-1)}$, with mass dimensions 1 which transforms as

$$
\delta B_{\alpha(s-1)\dot{\alpha}(s-1)} = \bar{D}^2 D_\alpha^s L_{\alpha(s)\dot{\alpha}(s-1)} + D^2 \bar{D}^{\dot{\alpha}_s} \bar{L}_{\alpha(s-1)\dot{\alpha}(s)} - \left[ \frac{s-1}{s!} \right] D_{(\dot{\alpha}_{s-1} D^\gamma \bar{D}^{\dot{\gamma}} \bar{L}_{\alpha(s-1)\dot{\gamma}(s-2)})}
- \left[ \frac{s-1}{s!} \right] D_{(\dot{\alpha}_{s-1} \bar{D}^{\dot{\gamma}} \bar{D}^\gamma \bar{L}_{\alpha(s-2)\dot{\gamma}(s-1)})}.
$$

(21)

To create an invariant action and give dynamics to the compensators we have to add the following terms:

- A counter term which will cancel the change of the initial action

$$
S_c = \int d^8 z \left\{ - \left[ \frac{2}{s} \right] (c_3 - (2s + 1)c_4) H^{\alpha(s)\dot{\alpha}(s)} \bar{D}_{\dot{\alpha}_s} D^2 \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} + c.c. + 2(c_3 - c_4) H^{\alpha(s)\dot{\alpha}(s)} [D_{\alpha_s} \bar{D}_{\dot{\gamma}_s}] B_{\alpha(s-1)\dot{\alpha}(s-1)} \right\}.
$$

(22)

- A kinetic energy term for both the compensators (the most general action for $\bar{Y}$ and $B$)

$$
S_{k.e} = \int d^8 z \left\{ e B^{\alpha(s-1)\dot{\alpha}(s-1)} B_{\alpha(s-1)\dot{\alpha}(s-1)}
+ h_1 \bar{Y}^{\alpha(s)\dot{\alpha}(s-1)} D^2 \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} + c.c.
+ h_2 \bar{Y}^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^2 \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} + c.c.
+ h_3 \bar{Y}^{\alpha(s)\dot{\alpha}(s-1)} \bar{D}^{\dot{\alpha}_s} D_{\alpha_s} \bar{Y}_{\alpha(s-1)\dot{\alpha}(s)}
+ h_4 \bar{Y}^{\alpha(s)\dot{\alpha}(s-1)} D_{\alpha_s} \bar{D}^{\dot{\alpha}_s} \bar{Y}_{\alpha(s-1)\dot{\alpha}(s)} \right\}.
$$

(23)

- An interaction term among compensators (as in principle, such a term can exist)

$$
S_{int} = \int d^8 z \left\{ b B^{\alpha(s-1)\dot{\alpha}(s-1)} [D_{\alpha_s} \bar{Y}_{\alpha(s)\dot{\alpha}(s-1)} + \bar{D}^{\dot{\alpha}_s} \bar{Y}_{\alpha(s-1)\dot{\alpha}(s)}] \right\}.
$$

(24)
Therefore the full action is

\[
S = \int d^8z \left\{ \frac{1}{s} \left[ (c_3 + (2s + 1)c_4) H^{\alpha(s)\bar{\alpha}(s)} H^\gamma \bar{D}_\gamma D_{\alpha(s)\bar{\alpha}(s)} \\
+ c_3 H^{\alpha(s)\bar{\alpha}(s)} \partial_{\alpha(s)} \partial_{\bar{\alpha}(s)} \frac{\partial^{\gamma\bar{\gamma}}}{\gamma} H_{\gamma\alpha(s-1)\bar{\gamma}\bar{\alpha}(s-1)} \\
+ c_4 H^{\alpha(s)\bar{\alpha}(s)} [D_{\alpha(s)} \bar{D}_{\bar{\alpha}(s)}] [D^\gamma \bar{D}^{\bar{\gamma}}] H_{\gamma\alpha(s-1)\bar{\gamma}\bar{\alpha}(s-1)} \\
- \frac{2}{s} \left( c_3 - (2s + 1)c_4 \right) H^{\alpha(s)\bar{\alpha}(s)} \bar{D}_{\bar{\alpha}(s)} D^2 \Upsilon_{\alpha(s-1)\bar{\alpha}(s-1)} \\
+ \frac{2}{s} \left( c_3 - (2s + 1)c_4 \right) H^{\alpha(s)\bar{\alpha}(s)} D_{\alpha(s)} \bar{D}^2 \Upsilon_{\alpha(s-1)\bar{\alpha}(s-1)} \\
+ 2\left( c_3 - c_4 \right) H^{\alpha(s)\bar{\alpha}(s)} [D_{\alpha(s)} \bar{D}_{\bar{\alpha}(s)}] B_{\alpha(s-1)\bar{\alpha}(s-1)} \\
+ e B^{\alpha(s-1)\bar{\alpha}(s-1)} B_{\alpha(s-1)\bar{\alpha}(s-1)} \\
+ h_1 \Upsilon^{\alpha(s)\bar{\alpha}(s-1)} D^2 \Upsilon_{\alpha(s)\bar{\alpha}(s-1)} + c.c. \\
+ h_2 \Upsilon^{\alpha(s)\bar{\alpha}(s-1)} \bar{D}^2 \Upsilon_{\alpha(s)\bar{\alpha}(s-1)} + c.c. \\
+ h_3 \Upsilon^{\alpha(s)\bar{\alpha}(s-1)} \bar{D}^{\alpha(s)} D_{\alpha(s)} \Upsilon_{\alpha(s-1)\bar{\alpha}(s-1)} \\
+ h_4 \Upsilon^{\alpha(s)\bar{\alpha}(s-1)} D_{\alpha(s)} \bar{D}^{\bar{\alpha}(s)} \Upsilon_{\alpha(s-1)\bar{\alpha}(s-1)} \\
+ b B^{\alpha(s-1)\bar{\alpha}(s-1)} \left( D^{\alpha(s)} \Upsilon_{\alpha(s)\bar{\alpha}(s-1)} + \bar{D}^{\bar{\alpha}(s)} \Upsilon_{\alpha(s-1)\bar{\alpha}(s-1)} \right) \right\} .
\]

The invariance of this action under the corresponding gauge transformations is guaranteed by the satisfaction of the following two Bianchi identities

\[
0 = D^{\alpha(s)} \mathcal{G}_{\alpha(s)\bar{\alpha}(s)} = \frac{1}{s} D^{\alpha(s)} \bar{D}(\bar{\alpha}(s) \mathcal{T}_{\alpha(s)\bar{\alpha}(s-1)}) ,
\]

\[
0 = D_{\alpha(s+1)} \mathcal{T}_{\alpha(s)} = \frac{1}{s!} D_{\alpha(s+1)} \bar{D} \mathcal{Y}_{\alpha(s-1)\bar{\alpha}(s-1)} - \frac{s - 1}{s!} D_{\alpha(s-1)} \bar{D} \mathcal{Y}_{\alpha(s-2)\bar{\alpha}(s-1)}) = 0 ,
\]

where \( \mathcal{G}_{\alpha(s)\bar{\alpha}(s)} \), \( \mathcal{T}_{\alpha(s)\bar{\alpha}(s-1)} \), \( \mathcal{Y}_{\alpha(s-1)\bar{\alpha}(s-1)} \) are the variations of the action with respect the corresponding superfields \( H_{\alpha(s)\bar{\alpha}(s)} \), \( \Upsilon_{\alpha(s-1)\bar{\alpha}(s-1)} \), and \( B_{\alpha(s-1)\bar{\alpha}(s-1)} \).

The solution of the first bianchi identity gives:

\[
h_1 = - \left[ \frac{1}{s} \right] (c_3 - (2s + 1)c_4) , \quad h_3 = 0 ,
\]

\[
h_4 = \left[ \frac{2(s + 1)}{s^2} \right] (c_3 - (2s + 1)c_4) ,
\]

10
\[ e = \frac{1}{2} b - (c_3 - c_4) \text{ ,} \]
\[ b = -h_4 = - \left[ \frac{2(s + 1)}{s^2} \right] (c_3 - (2s + 1)c_4) \text{ ,} \]
\[ b = \left[ \frac{2(2s + 1)}{s} \right] (c_3 - c_4) \text{ .} \]

The last two equations will give a relationship among \( c_3 \) and \( c_4 \)

\[ c_4 = \left[ \frac{2s^2 + 2s + 1}{(2s + 1)^2} \right] c_3 \text{ .} \]

The second Binachi identity has as a solution:

\[ h_2 = 0 \text{ ,} \quad h_3 = 0 \text{ .} \]

So the full action takes the form

\[
S = \int d^8 z \left\{ \frac{2(s + 1)^2}{s(2s + 1)} c_3 H^{\alpha(s)\bar{\alpha}(s)} [D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\bar{\alpha}(s)}] 
+ c_3 H^{\alpha(s)\bar{\alpha}(s)} \partial_{\alpha_s \bar{\alpha}_s} \partial^{\gamma \bar{\gamma}} H_{\gamma \alpha(s-1)\bar{\gamma} \bar{\alpha}(s-1)} 
+ \left[ \frac{2s^2 + 2s + 1}{(2s + 1)^2} \right] c_3 H^{\alpha(s)\bar{\alpha}(s)} [D_{\alpha_s}, \bar{D}_{\bar{\alpha}_s}] [D^\gamma, \bar{D}^{\bar{\gamma}}] H_{\gamma \alpha(s-1)\bar{\gamma} \bar{\alpha}(s-1)} 
+ \left[ \frac{4s}{2s + 1} \right] c_3 H^{\alpha(s)\bar{\alpha}(s)} D_{\bar{\alpha}_s} D^2 Y_{\alpha(s)\bar{\alpha}(s-1)} 
- \left[ \frac{4s}{2s + 1} \right] c_3 H^{\alpha(s)\bar{\alpha}(s)} D_{\alpha_s} \bar{D}^2 \bar{Y}_{\alpha(s)\bar{\alpha}(s-1)} 
+ \left[ \frac{4s(s + 1)}{(2s + 1)^2} \right] c_3 H^{\alpha(s)\bar{\alpha}(s)} [D_{\alpha_s}, \bar{D}_{\bar{\alpha}_s}] B_{\alpha(s-1)\bar{\alpha}(s-1)} 
+ \left[ \frac{2(s + 1)^2}{(2s + 1)^2} \right] c_3 B^{\alpha(s-1)\bar{\alpha}(s-1)} B_{\alpha(s-1)\bar{\alpha}(s-1)} 
+ \left[ \frac{2s}{2s + 1} \right] c_3 Y^{\alpha(s)\bar{\alpha}(s-1)} D^2 Y_{\alpha(s)\bar{\alpha}(s-1)} + c.c. 
- \left[ \frac{4(s + 1)}{2s + 1} \right] c_3 Y^{\alpha(s)\bar{\alpha}(s-1)} D_{\alpha_s} \bar{D}^{\bar{\alpha}_s} \bar{Y}_{\alpha(s)\bar{\alpha}(s-1)} 
+ \left[ \frac{4(s + 1)}{2s + 1} \right] c_3 B^{\alpha(s-1)\bar{\alpha}(s-1)} (D^{\alpha_s} Y_{\alpha(s)\bar{\alpha}(s-1)} + \bar{D}^{\bar{\alpha}_s} \bar{Y}_{\alpha(s-1)\bar{\alpha}(s)}) \right\} \text{ .} \]
At this point we can use the equation of motion of the auxiliary superfield $B_{\alpha(s-1)\hat{a}(s-1)}$

\[
\mathcal{Y}_{\alpha(s-1)\hat{a}(s-1)} = 0 \Rightarrow
\]

\[
B_{\alpha(s-1)\hat{a}(s-1)} = -\left[ \frac{s}{s+1} \right] [D\alpha, \bar{D}\hat{a}] H_{\alpha(s)\hat{a}(s)}
- \left[ \frac{2s+1}{s} \right] (D\alpha \mathcal{Y}_{\alpha(s)\hat{a}(s-1)} + \bar{D}\hat{a} \bar{\mathcal{Y}}_{\alpha(s-1)\hat{a}(s)})
\]

in order to integrate it out and simplify the action. So our final action is:

\[
S = \int d\hat{s}_2 \left\{ \left[ \frac{2(s+1)^2}{s(2s+1)} \right] c_3 H^{\alpha(s)\hat{a}(s)} D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\hat{a}(s)}
+ c_3 H^{\alpha(s)\hat{a}(s)} [D_{\alpha}, \bar{D}_{\hat{a}}][D^\gamma, \bar{D}^\gamma] H_{\gamma\alpha(s-1)\hat{a}(s-1)}
+ c_3 \frac{1}{(2s+1)} \left[ \frac{1}{2s+1} \right] c_3 H^{\alpha(s)\hat{a}(s)} [D_{\alpha}, \bar{D}_{\hat{a}}][D^\gamma, \bar{D}^\gamma] H_{\gamma\alpha(s-1)\hat{a}(s-1)}
+ \left[ \frac{4s}{2s+1} \right] c_3 H^{\alpha(s)\hat{a}(s)} D_{\alpha} \bar{D}_{\hat{a}} D^\gamma \bar{D}^\gamma H_{\alpha(s-1)\hat{a}(s-1)}
+ \left[ \frac{4s}{2s+1} \right] c_3 H^{\alpha(s)\hat{a}(s)} D_{\alpha} \bar{D}_{\hat{a}} D^\gamma \bar{D}^\gamma H_{\alpha(s-1)\hat{a}(s-1)}
+ \left[ \frac{4s}{2s+1} \right] c_3 \bar{H}^{\alpha(s)\hat{a}(s-1)} D_{\alpha} \bar{D}^\gamma \bar{D}^\gamma H_{\alpha(s-1)\hat{a}(s-1)} + c.c.
\right\}
\]

and it is invariant under the following gauge transformations

\[
\delta H_{\alpha(s)\hat{a}(s)} = \frac{1}{s!} \bar{D}_{\hat{a}} (\bar{L}_{\alpha(s)\hat{a}(s-1)} - \frac{1}{s!} D_{\alpha} \bar{L}_{\alpha(s-1)\hat{a}(s)})
\]

\[
\delta \mathcal{Y}_{\alpha(s)\hat{a}(s-1)} = \frac{1}{s!} \bar{D}^\gamma \bar{D}_{\hat{a}} (\bar{L}_{\alpha(s-1)\hat{a}(s-1)} + D^\alpha \Lambda_{\alpha(s+1)\hat{a}(s-1)}).
\]

From this action we can calculate the following superfields

\[
\mathcal{G}_{\alpha(s)\hat{a}(s)} = \left[ \frac{4(s+1)^2}{s(2s+1)} \right] c_3 D^\gamma \bar{D}^2 D_\gamma H_{\alpha(s)\hat{a}(s)}
+ \frac{2c_3}{s! s!} \partial_{\alpha(s)} \partial^\gamma \bar{D} H_{\gamma\alpha(s-1)\hat{a}(s-1)}
+ \left[ \frac{4}{2s+1} \right] c_3 \frac{1}{s! s!} [D_{\alpha}, \bar{D}_{\hat{a}}][D^\gamma, \bar{D}^\gamma] H_{\gamma\alpha(s-1)\hat{a}(s-1)}
- \left[ \frac{4s}{2s+1} \right] c_3 \frac{1}{s! s!} D_{\alpha} \bar{D}_{\hat{a}} D^\gamma \bar{D}^\gamma H_{\alpha(s-1)\hat{a}(s-1)}
+ \left[ \frac{4s}{2s+1} \right] c_3 \frac{1}{s! s!} \bar{D}_{\hat{a}} D_{\alpha} \bar{D}^\gamma \bar{D}^\gamma H_{\alpha(s-1)\hat{a}(s-1)}
\]

,
\[ \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} = - \left[ \frac{4s}{2s+1} \right] \frac{c_3}{s!} D_{\alpha} D_{\dot{\alpha}}^* D_{\gamma} H_{\gamma\alpha(s-1)\dot{\alpha}(s-1)} \\
- \left[ \frac{4(s+1)}{2s+1} \right] \frac{c_3}{s!} D^{\dot{\alpha}} Y_{\alpha(s)\dot{\alpha}(s-1)} \\
+ \left[ \frac{4s}{2s+1} \right] \frac{c_3}{s!} D_{\alpha} D_{\dot{\alpha}}^* Y_{\alpha(s-1)\dot{\alpha}(s)} \] 

which satisfy the Bianchi identities for this action

\[
0 = D^{\alpha(s)} g_{\alpha(s)\dot{\alpha}(s)} - \frac{1}{s!} D^{\alpha(s)} D_{\dot{\alpha}(s)} \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} \\
0 = D_{(\alpha s+1)} \mathcal{T}_{\alpha(s)\dot{\alpha}(s-1)} \] 

It is also straightforward to prove that they satisfy another identity

\[
D^{\alpha(s+1)} \mathcal{W}_{\alpha(2s+1)} = \left[ \frac{s(2s+1)}{4(s+1)^2} \right] \frac{1}{c_3} \partial_{(\alpha s+1)} \mathcal{W}_{\alpha(s)\dot{\alpha}(s)} \\
+ \frac{i}{4(s+1)^2} \frac{s^2}{c_3} \partial_{(\alpha s+1)} \mathcal{W}_{\alpha(s)\dot{\alpha}(s)} \\
+ \left[ \frac{s^2}{4(s+1)^2} \right] \frac{1}{c_3} \partial_{(\alpha s+1)} \mathcal{W}_{\alpha(s)\dot{\alpha}(s)} \\
+ \left[ \frac{s^2}{4(s+1)^2} \right] \frac{1}{c_3} \partial_{(\alpha s+1)} \mathcal{W}_{\alpha(s)\dot{\alpha}(s)} \\
+ \left[ \frac{s^2}{4(s+1)^2} \right] \frac{1}{c_3} \partial_{(\alpha s+1)} \mathcal{W}_{\alpha(s)\dot{\alpha}(s)} \\
+ \left[ \frac{s^2}{4(s+1)^2} \right] \frac{1}{c_3} \partial_{(\alpha s+1)} \mathcal{W}_{\alpha(s)\dot{\alpha}(s)} \\
+ \left[ \frac{s^2}{4(s+1)^2} \right] \frac{1}{c_3} \partial_{(\alpha s+1)} \mathcal{W}_{\alpha(s)\dot{\alpha}(s)} \\
+ \left[ \frac{s^2}{4(s+1)^2} \right] \frac{1}{c_3} \partial_{(\alpha s+1)} \mathcal{W}_{\alpha(s)\dot{\alpha}(s)} \\
+ \left[ \frac{s^2}{4(s+1)^2} \right] \frac{1}{c_3} \partial_{(\alpha s+1)} \mathcal{W}_{\alpha(s)\dot{\alpha}(s)} 
\] 

That means that on-shell \((\mathcal{T} = \mathcal{G} = 0)\) the object \(\mathcal{W}_{\alpha(2s+1)}\) satisfies the equations

\[
D^{\alpha(s+1)} \mathcal{W}_{\alpha(2s+1)} = 0 \quad ; \quad \bar{D}_{\dot{\alpha}} \mathcal{W}_{\alpha(2s+1)} = 0 \]

therefore it describes a massless half odd superspin. Now we know that this theory, on-shell has an irreducible representation propagating. The last thing we need to check is whether these are the only degrees of freedom propagating or if there are more. The easiest way to do that is to go to components notation and calculate the action in the Wess-Zumino gauge. If the only thing propagating is this half odd supermultiplet, the components action must be the Fronsdal action for bosons and fermions respectively.

Because of the gauge transformation, we have the freedom to gauge away some of the components. Specifically,\(^9\)

\[
\Phi_{\gamma\alpha(s-1)} = \phi^{(S)}_{\gamma\alpha(s-1)} + \frac{s-1}{s!} C_{\gamma \alpha(s-1) \dot{\alpha}(s-2)} \phi_{\dot{\alpha}(s-2)} \quad ; \quad \phi_{\gamma\alpha(s-1)} = \frac{1}{s!} \Phi_{\gamma\alpha(s-1)} \quad ; \quad \phi_{\dot{\alpha}(s-2)} = C^{\gamma\alpha(s-1)} \Phi_{\dot{\alpha}(s-1)}
\]

Furthermore the notation \(\phi^{(m,n)}\) represents the \(\theta^m \bar{\theta}^n\) component in the taylor series of the superfield \(\Phi\)
So in the Wess-Zumino gauge for the two superfields are:

\[ H_{\alpha(s)\bar{\alpha}(s)} = \theta^{\alpha(s+1)} \bar{\theta}^{\alpha(s+1)} h_{\alpha(s+1)\bar{\alpha}(s+1)} - \frac{s}{s!s!} \theta_{(s)} \bar{\theta}_{(s)} \bar{h}_{\alpha(s+1)\bar{\alpha}(s+1)} \]

\[
\begin{align*}
&+ \frac{1}{\sqrt{2}} \bar{\theta}^{\alpha(s+1)} \bar{\psi}_{\alpha(s+1)\bar{\alpha}(s)} + \frac{1}{\sqrt{2}} \theta^{\alpha(s+1)} \psi_{\alpha(s+1)\bar{\alpha}(s)} \\
&+ \frac{1}{\sqrt{2s!}} \theta(\alpha(s)\bar{\alpha}(s+1)) - \frac{1}{\sqrt{2s!}} \theta(\alpha(s)\bar{\alpha}(s+1)) \\
&+ \theta^2 \bar{\theta}^2 A_{\alpha(s)\bar{\alpha}(s)} ,
\end{align*}
\]

and

\[
\begin{align*}
\Gamma_{\alpha(s)\bar{\alpha}(s-1)} &= \theta^2 \left[ \rho_{(s)}(s-1) + \frac{1}{\sqrt{2}} \psi_{(s)}(s-1) \right] \\
&+ \frac{1}{\sqrt{2s!}} \theta_{(s)} \bar{\theta}_{(s-1)} \bar{\psi}_{\alpha(s+1)\bar{\alpha}(s+2)} \\
&+ \theta^2 \bar{\theta}^2 \left[ \bar{\nu}_{(s)}(s) + \frac{1}{\sqrt{2}} \bar{\psi}_{(s)}(s) \right] - \frac{1}{2s+1} \bar{A}_{(s)}(s) \\
&- \frac{1}{2s}(s+1) \theta_{(s)} \bar{\theta}_{(s-1)} \bar{h}_{\alpha(s+1)\bar{\alpha}(s+1)} \\
&\quad \cdot \left[ \frac{1}{s(s+1)} \theta_{(s)} \bar{\theta}_{(s-1)} \left[ U_{\alpha(s)\bar{\alpha}(s-2)} + \frac{1}{s} \bar{h}_{\alpha(s-1)\bar{\alpha}(s-2)} \right] \\
&+ \frac{s}{(s+1)!} \theta^2 \theta_{(s)} \left[ S_{\alpha(s+1)\bar{\alpha}(s+2)} + \frac{s}{2s} \bar{h}_{\alpha(s+1)\bar{\alpha}(s+1)} \bar{h}_{\alpha(s+2)\bar{\alpha}(s+2)} \right] \\
&+ \frac{s}{(s+1)!} \theta^2 \theta_{(s)} \left[ \beta_{\alpha(s)\bar{\alpha}(s-1)} + \frac{1}{2s!} \theta_{(s)} \bar{\theta}_{(s-1)} \bar{h}_{\alpha(s+1)\bar{\alpha}(s+1)} \bar{h}_{\alpha(s+2)\bar{\alpha}(s+2)} \right] \\
&- \frac{i}{\sqrt{2}(s+1)!} \theta_{(s)} \bar{\theta}_{(s)} \bar{\psi}_{\alpha(s+1)\bar{\alpha}(s+1)} + \frac{i}{2(s+1)!} \theta_{(s)} \bar{\theta}_{(s)} \bar{h}_{\alpha(s+1)\bar{\alpha}(s+1)} ,
\end{align*}
\]
From mass dimensions arguments we can tell immediately that the components \( A, U, S, P, \rho, \beta \) are auxiliary fields, so they cannot appear with derivatives in the component action. The rest of the degrees of freedom left are exactly those that compose the half odd superspin supermultiplet and therefore, the action in components has to be the Fronsdal action.

To see in details how all this takes place, we substitute the component field expansions from the above expression for the superfields to the action\((32)\). The bosonic piece is:

\[
S_{\text{Bosons}} = \int d^4x \left[ \frac{2(s + 1)^2}{s(2s + 1)} \right] c_3 h^{\alpha(s+1)\tilde{\alpha}(s+1)} \square h_{\alpha(s+1)\tilde{\alpha}(s+1)}
\]

\[
- \left[ \frac{(s + 1)^3}{s(2s + 1)} \right] c_3 h^{\alpha(s+1)\tilde{\alpha}(s+1)} \partial_{\alpha s+1} \partial_{\alpha s+1} \partial_{\gamma \gamma} h_{\gamma}(s) \gamma \alpha(s) \gamma \tilde{\alpha}(s)
\]

\[
+ \left[ \frac{2(s + 1)^3}{2s + 1} \right] c_3 h^{\alpha(s+1)\tilde{\alpha}(s+1)} \partial_{\alpha s+1} \partial_{\alpha s} h_{\alpha(s-1)\tilde{\alpha}(s-1)}
\]

\[
- \left[ \frac{2(s + 1)^3}{s} \right] c_3 h^{\alpha(s-1)\tilde{\alpha}(s-1)} \square h_{\alpha(s-1)\tilde{\alpha}(s-1)}
\]

\[
- \left[ \frac{(s + 1)^3(s - 1)^2}{s(2s + 1)} \right] c_3 h^{\alpha(s-1)\tilde{\alpha}(s-1)} \partial_{\alpha s-1} \partial_{\alpha s-1} \partial_{\gamma \gamma} h_{\gamma}(s-2) \gamma \alpha(s-2)
\]

\[
+ \left[ \frac{4s}{2s + 1} \right] c_3 s^{\alpha(s-1)\tilde{\alpha}(s-1)} s^{\alpha(s-1)\tilde{\alpha}(s-1)}
\]

\[
+ \left[ \frac{4s}{2s + 1} \right] c_3 p^{\alpha(s-1)\tilde{\alpha}(s-1)} p^{\alpha(s-1)\tilde{\alpha}(s-1)}
\]

\[
+ \left[ \frac{4(s + 1)^3 - 16s^4}{s(2s + 1)} \right] c_3 a^{\alpha(s)\tilde{\alpha}(s)} A_{\alpha(s)\tilde{\alpha}(s)}
\]

\[
+ 4c_3 v^{\alpha(s)\tilde{\alpha}(s)} v^{\alpha(s)\tilde{\alpha}(s)}
\]

\[
+ \left[ \frac{4}{2s + 1} \right] c_3 w^{\alpha(s)\tilde{\alpha}(s)} w^{\alpha(s)\tilde{\alpha}(s)}
\]

\[
+ \left[ \frac{2(s + 1)(s - 1)}{s(2s + 1)} \right] c_3 u^{\alpha(s)\tilde{\alpha}(s-2)} U_{\alpha(s)\tilde{\alpha}(s-2)} + c.c.
\]

The component fields above all correspond to the zero-\( \theta \) limit of a corresponding
superfield. The equations of motions for the auxiliary superfields are:

\[ A_{\alpha(s)} \dot{\alpha}(s) = 0 , \]
\[ S_{\alpha(s-1)} \dot{\alpha}(s-1) = 0 , \]
\[ P_{\alpha(s-1)} \dot{\alpha}(s-1) = 0 , \]
\[ v_{\alpha(s)} \dot{\alpha}(s) = 0 , \]
\[ U_{\alpha(s)} \dot{\alpha}(s-2) = 0 , \]
\[ w_{\alpha(s)} \dot{\alpha}(s) = 0 . \]  

(42)

and the final action for the propagating bosonic components is

\[
S_{\text{Bosons}} = \int d^4x \left[ \frac{2(s + 1)^2}{s(2s + 1)} c_3 h^{\alpha(s+1)\dot{\alpha}(s+1)} \Box h_{\alpha(s+1)\dot{\alpha}(s+1)} - \left[ \frac{(s + 1)^3}{s(2s + 1)} \right] c_3 h^{\alpha(s+1)\dot{\alpha}(s+1)} \partial_{\alpha s+1} \partial_{\dot{\alpha} s+1} \partial^{\gamma \dot{\gamma}} h_{\gamma \alpha(s)\dot{\gamma}(s)} + \left[ \frac{2(s + 1)^3}{2s + 1} \right] c_3 h^{\alpha(s+1)\dot{\alpha}(s+1)} \partial_{\alpha s+1} \partial_{\dot{\alpha} s+1} \partial_{\alpha \dot{\alpha}} h_{\alpha(s-1)\dot{\alpha}(s-1)} - \frac{2(s + 1)^3}{s} c_3 h^{\alpha(s-1)\dot{\alpha}(s-1)} \Box h_{\alpha(s-1)\dot{\alpha}(s-1)} - \left[ \frac{(s + 1)^3(s - 1)^2}{s(2s + 1)} \right] c_3 h^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha s-1} \partial_{\dot{\alpha} s-1} \partial^{\gamma \dot{\gamma}} h_{\gamma \alpha(s-2)\dot{\gamma}(s-2)} \right].
\]

(43)

By setting \( c_3 = \frac{s(2s+1)}{2(s+1)^2} \) we obtain:

\[
S_{\text{Bosons}} = \int d^4x \left[ h^{\alpha(s+1)\dot{\alpha}(s+1)} \Box h_{\alpha(s+1)\dot{\alpha}(s+1)} - \frac{s + 1}{2} h^{\alpha(s+1)\dot{\alpha}(s+1)} \partial_{\alpha s+1} \partial_{\dot{\alpha} s+1} \partial^{\gamma \dot{\gamma}} h_{\gamma \alpha(s)\dot{\gamma}(s)} + (s + 1) s h^{\alpha(s+1)\dot{\alpha}(s+1)} \partial_{\alpha s+1} \partial_{\dot{\alpha} s+1} \partial_{\alpha \dot{\alpha}} h_{\alpha(s-1)\dot{\alpha}(s-1)} - (s + 1)(2s + 1) h^{\alpha(s-1)\dot{\alpha}(s-1)} \Box h_{\alpha(s-1)\dot{\alpha}(s-1)} - \frac{(s + 1)(s - 1)^2}{2} h^{\alpha(s-1)\dot{\alpha}(s-1)} \partial_{\alpha s-1} \partial_{\dot{\alpha} s-1} \partial^{\gamma \dot{\gamma}} h_{\gamma \alpha(s-2)\dot{\gamma}(s-2)} \right].
\]

(44)

which is the Fronsdal action for a propagating spin-(s + 1) bosonic field. For the limiting value of \( s = 1 \), this is the linearized Einstein-Hilbert action.

The fermionic piece of the action is:

\[
S_{\text{Fermions}} = \int d^4x \left[ \frac{4(s + 1)^2}{s(2s + 1)} i \bar{\psi}^{\alpha(s)} \dot{\alpha}(s+1) \partial^{\alpha s+1} \bar{\dot{\alpha} s+1} \psi^{\alpha(s+1)\dot{\alpha}(s)} \right].
\]

16
The equation of motion for the fermionic auxiliary fields are

\[ \rho_\alpha(s) \dot{\dot{a}}(s-1) = 0 \quad , \quad \beta_\alpha(s) \dot{a}(s-1) = 0 \]  

and the action for propagating fermions takes it’s final form:

\[ S_{\text{Fermions}} = \int d^4x \left[ \frac{2(s+1)^2}{s(2s+1)} i c_3 \bar{\psi}^{\alpha(s)}(s) \dot{\dot{a}}(s(s+1)) \gamma^{s+1} \hat{a}_{s+1} \psi^{\alpha(s(s+1))} \right. 
\]

\[ - \left. \frac{2}{s} ic_3 \bar{\psi}^{\alpha(s-1)}(s) \gamma^s \hat{a}_{\hat{s}} \hat{a}_{s(s+1)} \psi^{\alpha(s(s+1))} \right] 
\]

\[ + \left. \frac{2(s+1)}{2s+1} ic_3 \bar{\psi}^{\alpha(s)}(s) \dot{\dot{a}}(s) \hat{a}_{s+1} \hat{a}_{s(s+1)} \psi^{\alpha(s(s+1))} \right] + \text{c.c.} 
\]

\[ - \left. \frac{2(s+1)^2}{s(2s+1)} ic_3 \bar{\psi}^{\alpha(s-1)}(s) \dot{\dot{a}}(s-1) \hat{a}_{s(s+1)} \psi^{\alpha(s(s+1))} \right] + \text{c.c.} 
\]

\[ - \left. \frac{2(s+1)^2}{s(2s+1)} ic_3 \bar{\psi}^{\alpha(s-2)}(s) \dot{\dot{a}}(s-1) \hat{a}_{s(s+1)} \psi^{\alpha(s(s+1))} \right] + \text{c.c.} 
\]

Let’s set the value for \( c_3 = \frac{s(s+1)^2}{2(s+1)^2} \) as in the bosonic case, the action becomes

\[ S_{\text{Fermions}} = \int d^4x \left[ \right. 
\]

\[ - \left. \frac{2s+1}{(s+1)^2} ic_3 \bar{\psi}^{\alpha(s-1)}(s) \dot{\dot{a}}(s(s+1)) \gamma^{s+1} \hat{a}_{s+1} \psi^{\alpha(s(s+1))} \right] 
\]

\[ + \left. \frac{s}{s+1} ic_3 \bar{\psi}^{\alpha(s-1)}(s) \dot{\dot{a}}(s) \hat{a}_{s+1} \hat{a}_{s(s+1)} \psi^{\alpha(s(s+1))} \right] + \text{c.c.} 
\]

\[ - \left. ic_3 \bar{\psi}^{\alpha(s-1)}(s) \dot{\dot{a}}(s-1) \hat{a}_{s(s+1)} \psi^{\alpha(s(s+1))} \right] + \text{c.c.} 
\]

\[ - \left. ic_3 \bar{\psi}^{\alpha(s-2)}(s) \dot{\dot{a}}(s-1) \hat{a}_{s(s+1)} \psi^{\alpha(s(s+1))} \right] + \text{c.c.} 
\]

which is the Fronsdal action for spin-\((s+1/2)\). Therefore we conclude only an irreducible supermultiplet propagates on-shell and therefore the action\([32]\) describes
The results in (35), (36), and (37), taken together with the component expansions in (39) and (40), and the component results discussed thereafter are very revealing...when one considers them for the special case of the $s = 1$ theory.\footnote{When we take the limit to $s = 1$ we have to keep in mind, that fields with a negative number of undotted or dotted indices must vanish. That is true because these components do not exist in the $\theta$ expansion of the superfields. Specifically the component $\psi_{\alpha(s-1)\dot{\alpha}(s-2)}$ which is the antisymmetric component of the $\theta\bar{\theta}$ term of the $Y$ superfield doesn’t exist in the $s = 1$ limit, so it must disappear from the action.}

The first component level off-shell description of supergravity was provided in 1977 in a work by Breitenlohner [5]. Two years later and in a subsequent series of papers [6], these results were put into the context of the general superspace formalism for 4D, $\mathcal{N} = 1$ superfield supergravity. These old results and the special case of the higher spin $s = 1$ results for B-series discussed above match perfectly. This is especially clear from an examination of the auxiliary fields in the table immediately above. In $s = 1$ limit, only the $U$ auxiliary boson must be set to zero and the remaining fields are the well known ones of the non-minimal off-shell 4D, $\mathcal{N} = 1$ SG multiplet.

Another way to see this, one can initially compare the results of the current paper for the component expansions given in (39) and (40) to the similar expansions given in equation (4.9) of the first work in Ref. [6]. The (35), (36), and (37), can be compared with the linearized versions of the results found in the remaining papers.
of Ref. [6]. In other words, the implication of our present effort reveals that the non-minimal formulation of Breitenlohner is the lowest member of a class of arbitrary higher superspin, off-shell formulation of massless supermultiplets!

\[
\begin{align*}
\vdots \\
\hline \\
\hline \\
\hline \\
\hline \\
\end{align*}
\begin{align*}
\begin{array}{ccc}
s = 1 & s = 1 & s = 1 \\
\text{KPS-series} & \text{Minimal / New-Minimal} & \text{B-series}
\end{array}
\end{align*}
\]

5 Perspectives On Future Investigations

In the current work, we have been able to advance the state-of-the-art with regard to the understanding of 4D, \( \mathcal{N} = 1 \) superfields and the issue of higher spin supermultiplets. The discovery of the B-series of superfield theories, suggests that many features of off-shell 4D, \( \mathcal{N} = 1 \) supergravity may well persists in the cases of higher spin \( (s > 1) \). The gauge transformation law in (3) for \( s = 1 \) is known to define the superspace superconformal group. It thus seems reasonable for values of \( s \neq 1 \) to use this as a definition of the 4D, \( \mathcal{N} = 1 \) superspace superconformal group acting on the entire B-series of theories. Furthermore, there is no obvious reason not to use this to define a 4D, \( \mathcal{N} = 1 \) superspace superconformal group for the KPS-series also.

If it is accepted that the gauge transformation law of \( H_{\alpha(s)\dot{\alpha}(s)} \) defines a 4D, \( \mathcal{N} = 1 \) superspace superconformal group, the second equation in (33) has an obvious interpretation. In the case of \( s = 1 \) limit, the superfield \( \Upsilon_{\alpha(s)\dot{\alpha}(s-1)} \) is known to constitute a conformal compensator whose functions is to break the 4D, \( \mathcal{N} = 1 \) superspace superconformal group down to the 4D, \( \mathcal{N} = 1 \) superspace super Poincaré group. Once more it is suggestive that this interpretation can be carried over to the entirety of the B-series and as well to the KPS-series (though the governing equations for the KPS-series are given by (17)). For both cases we have verified the existence of field strength superfields \( W_{\alpha(2s+1)} \), \( G_{\alpha(s)\dot{\alpha}(s)} \) and \( T_{\alpha(s)\dot{\alpha}(s-1)} \) which occur for both
the B-series and the KPS-series.

In a future work, we will revisit all of these results in the context of a Fock-space formulation. We conjecture that all the structures we have met in this investigation will likely generalize to such a formulation. Should this be the case, then we may have a new avenue to ask questions of covariant superstring field theory. Can there exist a limit of covariant superstring field theory which recovers all the structure found in a Fock space extension of our current work?

“Never express yourself more clearly than you are able to think.”

– Niels Bohr

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Appendix: Recovering the missing $s = 1$ pieces

For the purpose of completeness we study separately the case of $s = 1$. This limit is special because in a practical level the index structure of the entire theory gets simplified and [4], which was a guideline, becomes simpler as well. Furthermore it is known that there exist two dual, well studied, theories of supergravity, the minimal and the new minimal. We would like to find if and how they emerge from our construction. For this purpose we will not bother, with the $s = 1$ limits of the two theories described above. Instead we will search for different routes that could lead to a consistent theory.

There are a couple of interesting observations that one can make. Using the table above that counts the off-shell degrees of freedom, in the $s = 1$ limit gives the answer twenty. Minimal and new minimal formulations of supergravity are known to have twelve off-shell degrees of freedom. So it can not be a limit of the above theories. This means that in our framework there must be a different mechanism that is capable of generating these theories. We would like to find this.

The second and more important observation is that the superfield $H$ in the $s=1$ limit includes all the propagating bosonic and fermionic degrees of freedom need to construct the Fronsdal action. It is very easy to verify that, just be looking equation (39) and (40). One can check that in the $s = 1$ limit all the $h$’s and $\psi$’s components are in the taylor expansion of the $H_{\alpha\dot{\alpha}}$. This means that the compensator looses one of it’s roles, to provide the extra degrees of freedom needed for the irreducible representation. It’s sole purpose now is to guarantee the gauge invariance of the action. This infers that the set of all compensators must be auxiliary superfields (so their mass dimensions must be one) and as a result their gauge transformation must be made out of 3 D’s (\(\bar{D}’s\)) acting on the only gauge parameter available $L_\alpha$. So we must look for compensators that transform like

- $\bar{D}^2 D^\alpha L_\alpha$
- $D^2 \bar{D}_\dot{\alpha} L_\alpha$
- $D^\alpha \bar{D}^2 L_\alpha$
- $\bar{D}_\dot{\alpha} D^2 L_\alpha$

The last two possibilities will introduce compensators with exactly the same index structure (and therefore fields content) as the main superfield, that is why we will not allow them. So there are two cases left that correspond at the minimal and new minimal formulations of supergravity.

The starting action is the $s = 1$ version of [1] and the change of this action under the gauge transformation [3] is
\[ \delta S = \int d^8 z \left\{ [-2c_1 + 2c_2 + 2c_3 + 6c_4] H^{\alpha \dot{\alpha}} D_\alpha D^2 \bar{D}^{2} L_\alpha \\
+ 2c_2 H^{\alpha \dot{\alpha}} \bar{D}^{2 \dot{D}^2} \bar{D}_{\dot{\alpha}} L_\alpha \\
+ [-2c_3 + 6c_4] H^{\alpha \dot{\alpha}} D_\alpha \bar{D}^{2} D^\gamma \bar{D}_{\dot{\alpha}} L_\gamma \\
+ [2c_3 - 2c_4] H^{\alpha \dot{\alpha}} \bar{D}_{\dot{\alpha}} D_\alpha \bar{D}^{2} D^\gamma L_\gamma \\
+ c.c. \right\} . \] (49)

Setting \( c_2 = 0 \) gives

\[ \delta S = \int d^8 z \left\{ H^{\alpha \dot{\alpha}} \left\{ (-2c_3 + 6c_4) D_\alpha \bar{D}_{\dot{\alpha}} - (-2c_1 + 2c_3 + 6c_4) \bar{D}_{\dot{\alpha}} D_\alpha \right\} D^\gamma \bar{D}^{2} L_\gamma \\
+ 2(c_3 - c_4) H^{\alpha \dot{\alpha}} \bar{D}_{\dot{\alpha}} D_\alpha \bar{D}^{2} D^\gamma L_\gamma \\
+ c.c. \right\} . \] (50)

If \(-2c_3 + 6c_4 = -2c_1 + 2c_3 + 6c_4 \Rightarrow c_1 = 2c_3\) then

\[ \delta S = \int d^8 z \left\{ (-2c_3 + 6c_4) H^{\alpha \dot{\alpha}} \left\{ D_\alpha, \bar{D}_{\dot{\alpha}} \right\} \left\{ D^\gamma \bar{D}^{2} L_\gamma + \bar{D}^{\gamma} D^2 \bar{L}_\gamma \right\} \\
+ 2(c_3 - c_4) H^{\alpha \dot{\alpha}} \bar{D}_{\dot{\alpha}} D_\alpha \left\{ \bar{D}^{2} D^\gamma L_\gamma \right\} + c.c. \right\} . \] (51)

At this point, we introduce two compensators, a real scalar \( U \) with mass dimensions \( [U] = 1 \) and a complex scalar \( \sigma \) with mass dimensions \( [\sigma] = 1 \). Their transformations are defined to be

\[ \delta U = D^\gamma \bar{D}^{2} L_\gamma + \bar{D}^\gamma D^2 \bar{L}_\gamma \]
\[ \delta \sigma = \bar{D}^{2} D^\gamma L_\gamma \] (52)

We also add to the action the following terms so the compensators have dynamics
\[ S_c = \int d^8 z \left\{ -(-2c_3 + 6c_4)H^{\dot{\alpha}\dot{\alpha}} [D_{\alpha}, \bar{D}_{\dot{\alpha}}] U \\
- 2(c_3 - c_4)H^{\alpha\dot{\alpha}} \bar{D}_{\dot{\alpha}}D_{\alpha}\sigma + c.c. \right\} \]  

\[ S_{k.e} = \int d^8 z \left\{ bU^2 + e\sigma\bar{\sigma} + f\sigma\sigma + f^*\bar{\sigma}\bar{\sigma} \right\} \]

\[ S_{\text{int.}} = \int d^8 z \left\{ gU(\sigma + \bar{\sigma}) \right\} \]

The full action is

\[ S = \int d^8 z \left\{ 2c_3H^{\alpha\dot{\alpha}}D^\gamma\bar{D}^2D_\gamma H_{\alpha\dot{\alpha}} \\
+ c_3H^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}\partial^{\gamma\dot{\gamma}}H_{\gamma\dot{\gamma}} \\
+ c_4H^{\alpha\dot{\alpha}}[D_{\alpha}, \bar{D}_{\dot{\alpha}}][D^\gamma, \bar{D}^{\dot{\gamma}}]H_{\gamma\dot{\gamma}} \\
-(-2c_3 + 6c_4)H^{\alpha\dot{\alpha}} [D_{\alpha}, \bar{D}_{\dot{\alpha}}] U \\
-2(c_3 - c_4)H^{\alpha\dot{\alpha}} \bar{D}_{\dot{\alpha}}D_{\alpha}\sigma + c.c. \\
+ bU^2 + e\sigma\bar{\sigma} + f\sigma\sigma + f^*\bar{\sigma}\bar{\sigma} \\
+ gU(\sigma + \bar{\sigma}) \right\} \]  

\[ \bar{D}^{\dot{\alpha}}G_{\alpha\dot{\alpha}} - \bar{D}^2D_{\alpha}\mathcal{E}^1 - D_{\alpha}\bar{D}^2\mathcal{E}^2 = 0 \]  

where \( G_{\alpha\dot{\alpha}}, \mathcal{E}^1, \mathcal{E}^2 \) are the variations of the full action with respect the superfields \( H_{\alpha\dot{\alpha}}, U, \sigma \)

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\( \mathcal{G}_{\alpha\dot{\alpha}} = 4c_3 D^\gamma \bar{D}^2 D_\gamma H_{\alpha\dot{\alpha}} + 2c_3 \partial_{\alpha\dot{\alpha}} \partial^{\gamma\dot{\gamma}} H_{\gamma\dot{\gamma}} \\
+ 2c_4 [D_\alpha, \bar{D}_{\dot{\alpha}}] [D^\gamma, \bar{D}^\dot{\gamma}] H_{\gamma\dot{\gamma}} - (-2c_3 + 6c_4) [D_\alpha, \bar{D}_{\dot{\alpha}}] U \\
- 2(c_3 - c_4) \bar{D}_{\dot{\alpha}} D_\alpha \sigma + (c_3 - c_4) D_\alpha \bar{D}_{\dot{\alpha}} \bar{\sigma} \)

\[ \mathcal{E}^1 = (2c_3 - 6c_4) [D^\gamma, \bar{D}^\dot{\gamma}] H_{\gamma\dot{\gamma}} + 2bU \\
+ g\sigma + g\bar{\sigma} \]

\[ \mathcal{E}^2 = 2(c_3 - c_4) D^\gamma \bar{D}^\dot{\gamma} H_{\gamma\dot{\gamma}} + c\bar{\sigma} + 2f\sigma\bar{\sigma} + gU \]

The solution of the Bianchi identity is

- \( \sigma \) is chiral
- \( c_3 = c_4 \)
- \( e = 0 \)
- \( b = 6c_4 \)
- \( g = 4c_4 \)

Hence the action takes the form

\[ S = \int d^8 z \left\{ \begin{align*}
2c_4 H^{\alpha\dot{\alpha}} D^\gamma \bar{D}^2 D_\gamma H_{\alpha\dot{\alpha}} \\
+ c_4 H^{\alpha\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \partial^{\gamma\dot{\gamma}} H_{\gamma\dot{\gamma}} \\
+ c_4 H^{\alpha\dot{\alpha}} [D_\alpha, \bar{D}_{\dot{\alpha}}] [D^\gamma, \bar{D}^\dot{\gamma}] H_{\gamma\dot{\gamma}} \\
- 4c_4 H^{\alpha\dot{\alpha}} [D_\alpha, \bar{D}_{\dot{\alpha}}] U \\
+ 6c_4 U^2 \\
+ 4c_4 U(\sigma + \bar{\sigma}) \end{align*} \right\} \]

which is invariant under the gauge transformations:

\[ \delta H_{\alpha\dot{\alpha}} = \bar{D}_{\dot{\alpha}} L_\alpha - D_\alpha \bar{L}_{\dot{\alpha}} \]

\[ \delta U = D^\gamma \bar{D}^2 L_\gamma + \bar{D}^{\dot{\gamma}} D^2 \bar{L}_{\dot{\gamma}} \]

\[ \delta \sigma = \bar{D}^2 D^\gamma L_\gamma \]

The equation of motion for the superfield \( U \) is

\[ \mathcal{E}^1 = 0 \Rightarrow U = \frac{1}{3} [D^\gamma, \bar{D}^\dot{\gamma}] H_{\gamma\dot{\gamma}} - \frac{1}{3} (\sigma + \bar{\sigma}) \]
and the action becomes:

\[
S_1 = \int d^8 z \left\{ 2c_4 H^{\alpha \dot{\alpha}} \bar{D}^\gamma \bar{D}^2 D_\gamma H_{\alpha \dot{\alpha}} \\
+ c_4 H^{\alpha \dot{\alpha}} \partial_{\alpha \dot{\alpha}} \partial^{\gamma \dot{\gamma}} H_{\gamma \dot{\gamma}} \\
+ \frac{1}{3} c_4 H^{\alpha \dot{\alpha}} [D_\alpha, \bar{D}_{\dot{\alpha}}] [D^\gamma, \bar{D}^{\dot{\gamma}}] H_{\gamma \dot{\gamma}} \\
+ i \frac{4}{3} c_4 H^{\alpha \dot{\alpha}} \partial_{\alpha \dot{\alpha}} (\bar{\sigma} - \sigma) \\
n - \frac{4}{3} c_4 \sigma \bar{\sigma} \right\}
\]

which is invariant under the transformations:

\[
\delta H_{\alpha \dot{\alpha}} = \bar{D}_{\dot{\alpha}} L_\alpha - D_\alpha \bar{L}_{\dot{\alpha}} \\
\delta \sigma = \bar{D}^2 D^\gamma L_\gamma
\]

The action \( S_1 \), up to redefinitions, is the minimal supergravity formulation.

Instead of using the equation of motion for the superfield \( U \), we can use the equation of motion for the superfield \( \sigma \) then we get:

\[
\mathcal{E}^2 = 0 \Rightarrow \bar{D}^2 U = 0
\]

therefore \( U \) is now a linear compensator and the action is

\[
S_2 = \int d^8 z \left\{ 2c_4 H^{\alpha \dot{\alpha}} \bar{D}^\gamma \bar{D}^2 D_\gamma H_{\alpha \dot{\alpha}} \\
+ c_4 H^{\alpha \dot{\alpha}} \partial_{\alpha \dot{\alpha}} \partial^{\gamma \dot{\gamma}} H_{\gamma \dot{\gamma}} \\
+ c_4 H^{\alpha \dot{\alpha}} [D_\alpha, \bar{D}_{\dot{\alpha}}] [D^\gamma, \bar{D}^{\dot{\gamma}}] H_{\gamma \dot{\gamma}} \\
- 4c_4 H^{\alpha \dot{\alpha}} [D_\alpha, \bar{D}_{\dot{\alpha}}] U \\
+ 6c_4 U^2 \right\}
\]

which is invariant under the transformations

\[
\delta H_{\alpha \dot{\alpha}} = \bar{D}_{\dot{\alpha}} L_\alpha - D_\alpha \bar{L}_{\dot{\alpha}} \\
\delta U = D^\gamma \bar{D}^2 L_\gamma + \bar{D}^{\dot{\gamma}} \bar{D}^2 \bar{L}_{\dot{\gamma}}
\]

This action is the new-minimal supergravity formulation
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