Scale invariant cosmology I: the vacuum and the cosmological constant

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ABSTRACT

Context. The source of the acceleration of the expansion of the Universe is still unknown.

Aims. We examine some consequences of the possible scale invariance of the empty space at large scales.

Methods. The central hypothesis of this work is that, at macroscopic and large scales where General Relativity may be applied, the empty space in the sense it is used in the Minkowski metric, is also scale invariant.

Results. It is shown that if this applies, the Einstein cosmological constant \( \Lambda \) and the scale factor \( \lambda \) of the scale invariant framework are related by two differential equations.

Key words. Cosmology: theory - Cosmology: dark energy - Cosmology: cosmological parameters

1. Introduction

The cause of the accelerating expansion of the Universe (Riess et al. 1998, Perlmutter et al. 1999) is one of the major scientific problems at present. It leads to many fundamental studies. These concern the observational evidences of the acceleration (Shapiro & Turner 2006, Frieman et al. 2008, Cervantes-Cota & Smoot 2011), the possibilities related to the cosmological constant (Carroll et al. 1992, Frieman et al. 2008), the possibilties of a modified gravity (Milgrom 1983, 2009, Wesson 1983, 2015), the huge discrepancy between the astronomical estimates of \( \Lambda \) and the values derived from the vacuum energy in particle physics (Weinberg 1989, Solà 2013), the nature of the so-called dark energy and the search of possible dark matter candidates in astroparticle physics (Feng 2010, Porter et al. 2011). Among the about 50’000 papers with “cosmological constant” in their title according to the NASA ADS data basis, there is an impressive variety of suggestions and tentative explanations of various natures, as also illustrated by some of the above reviews. We apologize for adding some pages to the above mentioned paper avalanche.

The laws of physics generally are not unchanged under a change of scale, a fact originally discovered by Galileo Galilei as recalled by Feynman (1963), who mentions that Galileo realized “that the strengths of materials were not in exactly the right proportion to their sizes”. The scale references appear closely related to the material content of the medium. Even the vacuum at the quantum level is not scale invariant, since quantum effects produce some units of time and length. This point is as a matter of fact a historical argument, already forwarded long time ago against the theory of Weyl (1923), who built a geometry that added the scale invariance to the general covariance properties of the gravitational field equations.

Let us consider the space at macroscopic and large scales, for example such as the scales considered in cosmology. There, the empty space, in the sense it is used for example in the case of the Minkowski metric, does not appear to have preferred scales of length or time, even if a particular velocity is considered in Special Relativity. We now make the hypothesis that the empty space at large scales is scale invariant. The purpose of this work is to explore some consequences of this particular hypothesis. We note that the cosmological scales differ by an enormous factor, up to \( 10^{19} \), from the nuclear scales where quantum effects intervene. Thus, in the same way as we may use Newton or Einstein theory at macroscopic and large scales, even if we do not have a quantum theory of gravitation, we may consider that the large scale empty space is scale invariant, even if this is not true at scales where quantum effects intervene. We remark that it is not uncommon for a physical law to be valid under certain scales or under some conditions.

Scale invariance of a system of equations means that the equations do not change by a transformation of the space and time coordinates of the form \( dx^i \to \lambda(x^i) \, ds \). There \( \lambda(x^i) \) is the scale factor, where \( x^i \) represents space-time coordinates (\( \lambda \) may only have a time dependence as discussed below). We shall explore some implications of a possible scale invariance of the empty space at large scales.

A strong reason for doing that has been emphasized by Dirac (1973): It appears as one of the fundamental principles of Nature that the equations expressing basic laws should be invariant under the widest possible group of transformations. It is well known for example that the Maxwell equations of electrodynamics in absence of charges and currents show the property of scale invariance. While scale invariance has often been studied in relation with possible variations of the gravitational constant \( G \), no such hypothesis of variable \( G \) is considered here. We do not know whether the above hypothesis of scale invariance applies. However, it is by carefully examining the implications of such a hypothesis that we will find whether it corresponds to Nature or not.

In Sect. 2, we briefly recall the basic scale invariant field equations necessary for the present study. In Sect. 3, we apply
these equations to the macroscopic space where the Minkowski metric applies and obtain some fundamental relations between the scale parameter \( \lambda \) (with its derivatives) and Einstein cosmological constant. Section 4 provides the conclusions.

2. The basic scale invariant field equations

We make some recalls about scale invariance, limiting them to the necessary minimum. More developments can be found in works on the scale covariant theory by giants of Physics like Eddington [1923], Dirac [1973] and Canuto et al. [1977]. Their works are based on some particular case of Weyl’s geometry (Weyl 1923). We recall that General Relativity is not scale invariant. In its 4-dimensional space, the element interval coordinates \( ds^2 \) writes \( ds^2 = g_{\mu
u}dx^\mu dx^\nu \), (in this work the symbols with a prime refer to the space of General Relativity). A scale (or gauge) transformation is considered to a new coordinate system \( x' \) with the following relation between the two systems,

\[
d s' = A(x')dx \, ,
\]

where \( ds^2 = g_{\mu\nu}dx^\mu dx^\nu \) is the line element in the new more general framework where scale invariance is supposed to be a fundamental property in addition to the general covariance of General Relativity. Parameter \( A(x') \) is the scale factor connecting the two line elements. The Cosmological Principle of space homogeneity and isotropy in cosmology demands that the scale factor only depends on time. From the above definitions, we have a conformal transformation between the metrics of the two coordinate systems,

\[
g'_{\mu\nu} = A^2 g_{\mu\nu} \, .
\]

In this framework, scalars, vectors or tensors that transform like

\[
Y'_\mu = A^n Y^\mu \, ,
\]

are respectively called coscalars, covectors or cotensors of power \( n \). If \( n = 0 \), one has an inscalar, invector or intensor, such objects are invariant to the scale transformation \([1]\). The term scale covariance (called co-covariance by Dirac) refers to the general case of transformations \([3]\) with possibly different powers \( n \), while we reserve the term scale invariance more specifically to the case \( n = 0 \).

An extensive cotensor analysis has been developed by the above mentioned authors, see also [Rouvier & Maeder 1978]. The derivative of a scale invariant object is not in general scale invariant. Thus, co-covariant derivatives of the first and second order have been developed preserving scale covariance. For example, the co-covariant derivatives \( A_{\mu\nu} \) and \( A'_\mu \) of a co-vector \( A_\mu \) become

\[
A_{\mu\nu} = \partial_\nu A_\mu - \Gamma^\alpha_{\mu\nu} A_\nu - n \kappa_{\nu} A_\mu \, ,
\]

\[
A'_{\mu} = \partial_\mu A^\nu + \Gamma^\mu_{\nu\alpha} A^\nu - n \kappa_{\mu} A^\nu \, ,
\]

with \( \Gamma^\mu_{\nu\alpha} = \Gamma^\mu_{\alpha\nu} - g'_{\mu\nu} \kappa_{\nu} - g'_{\mu\alpha} \kappa_{\nu} + g_{\mu\nu} \kappa'_{\nu} \).

There, \( \Gamma^\mu_{\nu\alpha} \) is a modified Christoffel symbol, and \( \Gamma^\mu_{\nu\alpha} \) is the usual Christoffel symbol. The term \( \kappa_{\nu} \) is called the coefficient of metrical connection, it is

\[
\kappa_{\nu} = -\frac{\partial}{\partial x^\nu} \ln \lambda \, .
\]

In the scale covariant theory, it is as a fundamental quantity as are the \( g_{\mu\nu} \). A modified Riemann-Cristoffel tensor \( R'_{\mu\nu\lambda\rho} \) its contracted form \( R'_{\mu\nu} \) and the total curvature \( R \) also have their corresponding terms [Eddington 1923, Dirac 1973, Canuto et al. 1977]. The last two are

\[
R'_{\mu\nu} = R'_{\mu\nu} - k_{\mu} - k_{\nu} - g'_{\mu\nu} \kappa_{\rho} - 2 \kappa_{\mu} \kappa_{\nu} + 2 g'_{\mu\nu} \kappa_{\rho} \, ,
\]

\[
R = R' - 6 \kappa_{\alpha} + 6 \kappa_{\alpha} \kappa_{\beta} \, .
\]

There, \( R'_{\mu\nu} \) and \( R' \) are the usual expressions in General Relativity. The symbol \( \kappa_{\alpha} \) indicates a derivative. The above forms allow us to express the first member of a scale invariant field equation, which is thus a generalization of the first member of the field equation of the General Relativity, including also scale invariance as a fundamental property. This scale invariant first member is

\[
R'_{\mu\nu} = \frac{1}{2} R g'_{\mu\nu} \, ,
\]

with \( R'_{\mu\nu} \) and \( R \) given \([8]\) and \([9]\). This first member depends on the \( g'_{\mu\nu} \), \( \kappa_{\nu} \) and their derivatives. In General Relativity, the second member of the field equation writes

\[
-8 \pi G T^E_{\mu\nu} - \Lambda_{E} g'_{\mu\nu} \, .
\]

The velocity of light \( c \) is taken as unity, \( G \) is the gravitational constant, taken as an inscalar. \( T^E_{\mu\nu} \) is the energy-momentum tensor for a perfect fluid in the system of General Relativity and \( \Lambda_{E} \) is the cosmological constant of General Relativity, (we do not put a prime to \( \Lambda_{E} \), since the index “E” is explicit enough). The second member of the scale invariant field equation must be an intensor, as is the first one given by \([10]\), it is scale invariant [Canuto et al. 1977]. Thus, we have

\[
T'^{\mu\nu} = T^{\mu\nu} \, ,
\]

where the right-hand term is the energy-momentum tensor in the new more general coordinate system. This has further implications, which are easily examined in the case of a perfect fluid [Canuto et al. 1977]. We may write \([12]\) as,

\[
(p + \varrho)u_{\mu}u_{\nu} - g_{\nu\rho}p = (p' + \varrho' \lambda)u'_{\mu}u'_{\nu} - g'_{\nu\rho}p' \, .
\]

The velocities \( u^{\mu} \) and \( u'_{\mu} \) transform like

\[
u^{\mu} = \frac{dx^{\mu}}{ds} = \lambda^{-1} dx^{\mu}/ds = \lambda^{-1} u^{\mu} \, ,
\]

\[
u'_{\mu} = g'_{\mu\nu}u'^{\nu} = \lambda^2 g_{\mu\nu} \lambda^{-1} u^{\nu} = \lambda u_{\nu} \, .
\]

Thus, relation \([13]\) becomes with \([15]\)

\[
(p + \varrho)u_{\mu}u_{\nu} - g_{\nu\rho}p = (p' + \varrho' \lambda^2)u_{\mu}u_{\nu} - \lambda^2 g_{\nu\rho}p' \, .
\]

This implies the following scaling of pressure and density in the new general coordinate system

\[
p = p' \lambda^2 \quad \text{and} \quad \varrho = \varrho' \lambda^2 \, .
\]
Pressure and density are therefore not scalars, but coscalars of power -2. For the empty space studied in this work, these relations do not intervene (since $p$ and $\rho$ are zero), but they do in the scale invariant cosmological equations where matter is present in the Universe.

Let us now consider the last term in (11), which contains $\Lambda_E$ and is also globally scale invariant. Expression (2) shows that the Einstein metric tensor $g_{\mu\nu}$, behaves like $\lambda^2$. It is thus a cotensor of power $+2$. We can write

$$\Lambda_E g^\mu\nu = \Lambda_E \lambda^2 g^\mu\nu .$$  \hspace{1cm} (18)

We could possibly define a new $\Lambda$, by

$$\Lambda = \Lambda_E \lambda^2 .$$  \hspace{1cm} (19)

However, to avoid any ambiguity, we always keep all expressions with $\Lambda_E$, the true Einstein cosmological constant. Thus, the second member of the scale invariant field equation becomes

$$-8\pi GT^\mu\nu - \lambda^2 \Lambda_E g^\mu\nu .$$  \hspace{1cm} (20)

With (8), (9), (10) and (20), the scale invariant field equation becomes [Dirac 1937; Canuto et al. 1977].

$$R^\mu\nu - \frac{1}{2} \eta^\mu\nu R = -2k_0 \kappa^\nu - 2g^\mu\nu \kappa^\rho - g^\mu\nu \kappa^\rho \kappa^\sigma = -8\pi GT^\mu\nu - \lambda^2 \Lambda_E g^\mu\nu .$$  \hspace{1cm} (21)

The first member only depends on $g^\mu\nu$ and $\kappa$ (or $\lambda$). This equation can be applied to various physical systems, characterized by their line element $ds^2$ and their energy-momentum tensor $T^\mu\nu$. Interestingly enough, we shall see below that this equation, when applied to the empty space, leads to useful relations between the $\kappa$ terms and $\Lambda_E$.

3. Consequences of scale invariance of the empty space at large scales

3.1. Relations between the scale factor $\lambda$ and the cosmological constant $\Lambda_E$

We consider the case of the empty space, with thus an energy-momentum tensor $T^\mu\nu$ equal to zero. The line-element is given by the Minkowski metric,

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) .$$  \hspace{1cm} (22)

In General Relativity, the above metric implies that the first member of Einstein equation is equal to zero,

$$R^\mu\nu = \frac{1}{2} g^\mu\nu R = 0 .$$  \hspace{1cm} (23)

Thus, in the scale invariant field equation (21), only the following terms are remaining,

$$\kappa^\mu + \kappa^\nu + 2\kappa^\rho - 2g^\mu\rho \kappa^\sigma + g^\mu\rho \kappa^\sigma = \Lambda^2 \Lambda_E g^\mu\nu .$$  \hspace{1cm} (24)

All other terms have disappeared and we are left only with a relation between some functions of the scale factor $\lambda$ (through the $\kappa$-terms), the $g^\mu\nu$, and the Einstein cosmological constant. The term $\kappa^\nu$ is related to $\lambda$ and on its first two derivatives according to relation (7). It is interesting to remark that the cosmological constant which can be interpreted as the energy density of the vacuum is also related, in the present context, to the properties of scale invariance in the empty space.

At this stage, it may be opportune to recall that the problem of the cosmological constant in the empty space is not a new one. Bertotti et al. (1991) are quoting the following remark they got from Professor Bondi, who stated that: ”Einstein’s disenchantment with the cosmological constant was partially motivated by a desire to preserve scale invariance of the empty space Einstein equations”. This remark is in agreement with the fact that $\Lambda_E$ is not scale invariant as are the $T^\mu\nu$. The above developments show that the scale invariant theory may offer a possibility to reconcile the existence of $\Lambda_E$ with the scale invariance of the empty space. This reconciliation takes the form of relations (24), which are now further analyzed.

As stated above, the scale factor $\lambda$ can only be a function of time, therefore only the zero component of $\kappa^\nu$ is non-vanishing. Thus, the coefficient of metrical connection $\kappa^\nu$ is

$$\kappa^\nu = \kappa^\rho \delta^\nu_\rho = \kappa^0 0^0 = \partial_0 \kappa_0 = \frac{d\kappa_0}{dt} \equiv k_0 .$$  \hspace{1cm} (25)

The 0 and the 1, 2, 3 components of what remains from the field equation (24) become respectively

$$3\kappa_0^2 = \lambda^2 \Lambda_E ,$$  \hspace{1cm} (26)

$$2(3-\kappa_0^2) = -\lambda^2 \Lambda_E .$$  \hspace{1cm} (27)

From the definition (7), one has $\kappa_0 = -\lambda/\lambda$ (with c=1 at the denominator) and expressions (26) and (27) lead to the two following differential relations for $\lambda$,

$$3 \frac{\dot{\lambda}^2}{\lambda^2} = \lambda^2 \Lambda_E ,$$  \hspace{1cm} (28)

$$2 \frac{\dddot{\lambda}}{\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} = \lambda^2 \Lambda_E .$$  \hspace{1cm} (29)

These expressions may also be written in equivalent forms

$$\frac{\dot{\lambda}}{\lambda} = 2 \frac{\dot{\lambda}^2}{\lambda^2} ,$$  \hspace{1cm} (30)

and

$$\frac{\dddot{\lambda}}{\lambda} - \frac{\dot{\lambda}^2}{\lambda^2} = \frac{\lambda^2 \Lambda_E}{3} .$$  \hspace{1cm} (31)

These are the relations between Einstein’s cosmological constant $\Lambda_E$ and the scale factor $\lambda$. Relation (30), that does not contain $\Lambda_E$, also places a constraint on $\lambda(t)$. These various relations may intervene in the equations of cosmology and, as a matter of fact, they will reveal themselves most useful in leading to valuable simplifications.

Let us examine the solution of the differential equation (30).

We consider a solution of the form

$$\lambda = a (t - b)^n + d .$$  \hspace{1cm} (32)

Equation (30) imposes $d = 0$, meaning there is no additive constant to $\lambda$. It also implies $n = -1$. Interestingly enough, there is no constraint on the value of $b$, which means that the origin...
b of the timescale is not determined by the above equations. (The origin of the time, in the scale invariant cosmology like in other cosmologies, will be determined by the solutions of the equations of the particular cosmological model considered). The constant $a$ can be fixed for example by (31), which gives

$$a = \sqrt{\frac{3}{A_E}}.$$  

(33)

In physical units, we would have $a = \sqrt{3/(c^2 A_E)}$. Thus, we may finally write $\lambda$ as follows,

$$\lambda = \sqrt{\frac{3}{A_E}} \frac{1}{c t}.$$  

(34)

We see that the scale invariance of the empty space at macroscopic and large scales imposes a scale factor $\lambda$, such that $\lambda^2$ is related to the inverse of the energy density of the vacuum (to which $A_E$ is proportional). The factor $\lambda$ varies like the inverse of the cosmic time $t$, with no origin fixed yet at this stage of the developments.

### 3.2. Further remarks on the scale factor $\lambda$

The scale invariant equations are identical to those of General Relativity at a given fixed time, which we may choose to be the present one $t_0$. Some departures from General Relativity may appear when the evolution of a physical effect over the ages is intervening. Then, there may be different values of $\lambda$ at different epochs. At this stage, it is difficult to anticipate about the kind and size of the effects resulting from scale invariance, but we will see in future works that the main effect is a cosmic acceleration of comoving galaxies.

The fact that $A_E$ and the energy of the vacuum are related in General Relativity implies, since $\lambda$, its derivatives and $A_E$ are connected, that there may also be an energy-density associated the scale factor and its variations. The exact form of this energy-density will be studied on the basis of the appropriate cosmological equations.

Noticeably in the present framework, if the Einstein cosmological constant $A_E$ is different from zero, the scale factor $\lambda$ must necessarily vary with time. Reciprocally, if $\lambda$ is a constant, the cosmological constant should be zero. This conclusion depends on the assumption that the vacuum at large scales is scale invariant.

An estimate of $A_E$ can be obtained from (33), if we assume for example that $\lambda = 1$ at the present time $t_0$. We get in physical units

$$A_E = \frac{3}{c^2 t_0^2}.$$  

(35)

For an age of the Universe of 13.8 Gyr (Frieman et al. 2008), we obtain $A_E = 1.59 \cdot 10^{-35}$ s$^{-2}$ (with $c=1$) or in current physical units $A_E = 1.76 \cdot 10^{-58}$ cm$^{-2}$. Expression (35) is similar to classical expressions and it gives a numerical estimate in agreement with the current value (see for example (Carmeli & Kuzmenko 2001)) derived from the observations of cosmic acceleration.

### 4. Conclusions

We have made the hypothesis that the empty space is scale invariant at large scales, where General Relativity also applies.

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