Large versus bounded solutions to sublinear elliptic problems
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Summary. Let $L$ be a second order elliptic operator with smooth coefficients defined on a domain $\Omega \subset \mathbb{R}^d$ (possibly unbounded), $d \geq 3$. We study nonnegative continuous solutions $u$ to the equation $Lu(x) - \varphi(x, u(x)) = 0$ on $\Omega$, where $\varphi$ is in the Kato class with respect to the first variable and it grows sublinearly with respect to the second variable. Under fairly general assumptions we prove that if there is a bounded nonzero solution then there is no large solution.

1. Introduction. Let $L$ be a second order elliptic operator

$$L = \sum_{i,j=1}^{d} a_{ij}(x) \partial_{x_i} \partial_{x_j} + \sum_{i=1}^{d} b_i(x) \partial_{x_i}$$

with smooth coefficients $a_{ij}, b_i$ defined on a domain $\Omega \subset \mathbb{R}^d$, $d \geq 3$ \(\text{(1)}\). No conditions are put on the behavior of $a_{ij}, b_j$ near the boundary of $\partial \Omega$. We study nonnegative continuous functions $u$ such that

$$Lu(x) - \varphi(x, u(x)) = 0 \quad \text{on} \quad \Omega,$$

in the sense of distributions, where $\varphi : \Omega \times [0, \infty) \to [0, \infty)$ grows sublinearly with respect to the second variable. Such $u$ will be later called solutions. A solution $u$ to \(\text{(1.2)}\) is called large if $u(x) \to \infty$ when $x \to \partial \Omega$ or $\|x\| \to \infty$.

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\(\text{(1)}\) By a domain we always mean a set that is open and connected.
Large solutions, i.e. the boundary blow-up problems, are of considerable interest due to their applications in different fields. Such problems arise in the study of Riemannian geometry \[3\], non-Newtonian fluids \[1\], subsonic motion of a gas \[24\] and electric potentials in some bodies \[22\].

We prove that under fairly general conditions bounded and large solutions cannot exist at the same time. Classical examples the reader may have in mind are

\[(1.3) \quad \Delta u - p(x)u^\gamma = 0 \quad \text{with} \quad 0 < \gamma \leq 1 \quad \text{and} \quad p \in L^\infty_{\text{loc}},\]

where \(\Delta\) is the Laplace operator on \(\mathbb{R}^d\), but we go far beyond that. Not only the operator may be more general but the special form of the nonlinearity in (1.3) may be replaced by \(\varphi(x,t)\) satisfying

\[
(\text{SH}_1) \quad \text{There exists a function } p \in K^\text{loc}_d(\Omega) \quad (p \text{ is locally in the Kato class)} \quad \text{such that } \varphi(x,t) \leq p(x)(t+1) \quad \text{for all } t \geq 0 \quad \text{and} \quad x \in \Omega.
\]

\[
(\text{H}_2) \quad \text{For every } x \in \Omega, \ t \mapsto \varphi(x,t) \text{ is continuous nondecreasing on } [0,\infty).
\]

\[
(\text{H}_3) \quad \varphi(x,t) = 0 \quad \text{for every } x \in \Omega \quad \text{and} \quad t \leq 0.
\]

We recall that a Borel measurable function \(\psi\) on \(\Omega\) is locally in the Kato class in \(\Omega\) if

\[
\lim_{\alpha \to 0} \sup_{x \in D} \int_{D \cap \{|x-y| \leq \alpha\}} \frac{|\psi(y)|}{|x-y|^{d-2}} \, dy = 0
\]

for every open bounded set \(D\) with \(\bar{D} \subset \Omega\). Hypothesis (H\(_1\)) makes \(\varphi\) locally integrable against against the Green function \((2)\) for \(L\), which plays an important role in our approach. (H\(_3\)) is a technical extension of \(\varphi\) to \((-\infty,0)\) needed as a tool. For a part of our results we replace (SH\(_1\)) by a weaker condition:

\[
(\text{H}\_1) \quad \text{For every } t \in [0,\infty), \ x \mapsto \varphi(x,t) \in K^\text{loc}_d(\Omega).
\]

Applying methods of potential theory we obtain the following result.

**THEOREM 1.** Assume that \(\Omega\) is Greenian for \(L\)(\(^3\)). Suppose that \(\varphi(x,t) = p(x)\psi(t)\) satisfies (SH\(_1\)), (H\(_2\)), (H\(_3\)) and there exists a nonnegative nontrivial bounded solution to (1.2). Then there is no large solution to (1.2).

Theorem \(^1\) considerably improves a similar result of El Mabrouk and Hansen \(^9\) for \(L\) being the Laplace operator \(\Delta\) on \(\mathbb{R}^d\), \(\varphi(x,t) = p(x)\psi(t)\), \(p \in L^\infty_{\text{loc}}(\mathbb{R}^d)\) and \(\psi(t) = t^\gamma, \ 0 < \gamma < 1\). It is proved in Section 4.

In fact, we prove a few statements more general than Theorem \(^1\) but a little more technical to formulate (see Theorem \(^3\) in Section 2). Generally, we do not assume that \(\varphi\) has product form, and in particular we characterize

\(^2\) See \((1.1)-(4.3)\) for the definition of \(G_\Omega\).

\(^3\) See Section 4 for the definition, more precisely, \((4.2), (4.3)\).
a class of functions $p(x)$ in (SH$_1$) for which there are bounded solutions but no large solutions to (1.2) (see Theorem 9 in Section 4).

Besides the theorem due to El Mabrouk and Hansen [9] there are other results indicating that the equation $\Delta u - p(x)u^\gamma = 0$, or more generally $\Delta u - p(x)\psi(u) = 0$, cannot have bounded and large solutions at the same time [10], [17], [21]. We prove such a statement in considerable generality:

- $L$ is an elliptic operator (1.1);
- $\Omega$ is Greenian for $L$, generally unbounded;
- the nonlinearity is assumed to have only sublinear growth; no concavity with respect to the second variable and no product form of $\varphi$ is required.

Our main strategy adopted from [7] and [9] is to relate solutions of (1.2) to $L$-harmonic functions and to make extensive use of potential theory. We rely on the results of [11] and [12] where this approach was developed.

Existence of large solutions for the equation

$$\Delta u = p(x)f(u)$$

was studied under more regularity: $p$ Hölder continuous and $f$ Lipschitz (not necessarily monotone) [19] (4) or on the whole of $\mathbb{R}^d$ [29]. In our approach very little regularity is involved but monotonicity of $\varphi$ with respect of $t$ is essential. Suppose $\varphi$ is not of the product form but the following condition is satisfied:

(H$_4$) For every $x \in \Omega$, $t \mapsto \varphi(x,t)$ is concave on $[0, \infty)$.

Then we have

**Theorem 2.** Suppose that (H$_1$)–(H$_4$) hold and that there is a bounded solution to

$$Lu(x) - \varphi(x, u(x)) = 0.$$  

Then there is no large solution.

Theorem 2 follows directly from Theorem 3. Our strategy for the proof of Theorem 1 is to construct a function $\varphi_1 \geq \varphi$ satisfying (SH$_1$), (H$_2$)–(H$_4$) and to apply Theorem 3 to $\varphi$ and $\varphi_1$ (5). To make use of both equations, for $\varphi$ and $\varphi_1$, we need a criterion for existence of bounded solutions to (1.2) (see Theorem 8). The latter, proved in this generality, is itself interesting.

Semilinear problems $\Delta u + g(x, u) = 0$ have been extensively studied under a variety of hypotheses on $g$, and various questions have been asked. The function $g$ is not necessarily monotone or negative but there are often other restrictive assumptions like more regularity of $g$ or the product form. The problem is usually considered either in bounded domains or in $\Omega = \mathbb{R}^d$.

(4) More generally, $\Delta u = p(x)f(u) + q(x)g(u)$, $p, q$ Hölder continuous [18].

(5) The main difficulty is to guarantee that $\varphi_1(x, 0) = 0$ (see Section 3).
Finally, there are not many results for general elliptic operators, and they mostly have the same restrictions \[4\], \[15\], \[25\], \[27\]. Clearly, stronger regularity of \(g\) or \(\Omega\) is used to obtain conclusions other than the one we are interested in.

2. Large solutions to \(Lu - \varphi(\cdot, u) = 0\) under (H\(_1\))–(H\(_3\)). In this section we replace (SH\(_1\)) by (H\(_1\)) which is weaker. Our aim is to prove that under fairly general assumptions, bounded and large solutions to (1.2) cannot occur at the same time \((6)\).

**Theorem 3.** Let \(\Omega\) be a domain and suppose \(\varphi, \varphi_1\) satisfy (H\(_1\))–(H\(_3\)). Assume that \(\varphi \leq \varphi_1\) and \(\varphi_1\) is concave with respect to the second variable. If the equation \(Lu = \varphi_1(\cdot, u)\) has a nontrivial nonnegative bounded solution in \(\Omega\) then \(Lu = \varphi(\cdot, u)\) does not have a large solution in \(\Omega\).

Theorem 3 gives, in particular, the most general conditions for \(\Delta\) implying nonexistence of a bounded and a large solution at the same time. Compare with Theorem 3.1 in \[9\], where the statement was proved for \(\varphi(x, u) = p(x)u^\gamma, p \in L^\infty_{\text{loc}}(\Omega)\).

Applying Theorem 3 to \(\varphi\) being concave with respect to the second variable we obtain Theorem 2. In the next section, we will prove that under (SH\(_1\)) such a \(\varphi_1\) always exists, which makes Theorem 3 widely applicable.

For the proof we need to recall a number of properties satisfied by solutions to (1.2). For \(L = \Delta\) they were proved in \[7\], and the general case is similar (see \[12\]).

Let \(C^+(\Omega)\) and \(C^+(\partial \Omega)\) be the sets of nonnegative continuous functions on \(\Omega\) and \(\partial \Omega\) respectively.

**Lemma 4 (\[12\] Lemma 5).** Suppose that \(\varphi\) satisfies (H\(_2\)). Let \(u, v \in C^+(\Omega)\) be such that \(Lu, Lv \in L^1_{\text{loc}}(\Omega)\). If

\[Lu - \varphi(\cdot, u) \leq Lv - \varphi(\cdot, v)\]

in the sense of distributions and

\[\lim_{x \to y} \inf_{y \in \partial \Omega} (u - v)(x) \geq 0,\]

then

\[u - v \geq 0\quad \text{in } \Omega.\]

For a bounded regular domain \(D \subset \mathbb{R}^d\) and a nonnegative function \(f\) continuous on \(\partial D\), we define \(U_D^\varphi f\) to be the function such that

\(U_D^\varphi f = f\)
on $\mathbb{R}^d \setminus D$ and $U_D^\varphi f \mid_D$ is the unique solution of

$$
\begin{cases}
Lu - \varphi(\cdot, u) = 0 & \text{in } D \text{ in the sense of distributions,} \\
u \geq 0 & \text{in } D, \\
u = f & \text{on } \partial D.
\end{cases}
$$

Existence of $U_D^\varphi f$ was proved in [12, Theorem 4]. Let $G_D$ be the Green function for $D$. Then

$$
H_D f = U_D^\varphi f + G_D \varphi(\cdot, U_D^\varphi f) \quad \text{in } D,
$$

where $H_D f$ is an $L$-harmonic function in $D$ with boundary values $f$, and for a function $u$ we set

$$
G_D(\varphi(\cdot, u))(x) = \int_D G_D(x, y) \varphi(y, u(y)) dy.
$$

In particular $U_D^\varphi f$ is not identically $0$ in $D$ if $f$ is not identically $0$ on $\partial D$.

Now we focus on the properties of $U_D^\varphi f$. We say that $u$ is a supersolution to (1.2) if $Lu - \varphi(\cdot, u) \leq 0$, and a subsolution if $Lu - \varphi(\cdot, u) \geq 0$. In the following lemma we shall apply $U_D^\varphi$ to $f, g, u, v \in C^+(\Omega)$, that is, to their restrictions to $\partial D$. The lemma is a direct consequence of Lemma 4 and existence of solutions to (2.1). For $L = \Delta$ it was proved in [7].

**Lemma 5.** Suppose that $\varphi$ satisfies (H$_1$)–(H$_3$) and let $D$ be a bounded regular domain such that $\bar{D} \subset \Omega$. Then $U_D^\varphi$ is nondecreasing in the following sense:

$$
U_D^\varphi f \leq U_D^\varphi g \quad \text{if } f \leq g \text{ in } \Omega.
$$

Let $u$ be a continuous supersolution and $v$ a continuous subsolution of (1.2) in $\Omega$. Suppose further that $D$ and $D'$ are regular bounded domains such that $D' \subset D \subset \Omega$. Then

$$
\begin{align*}
U_D^\varphi u &\leq u \quad \text{and} \quad U_D^\varphi v \geq v, \\
U_{D'}^\varphi u &\geq U_D^\varphi u \quad \text{and} \quad U_{D'}^\varphi v \leq U_D^\varphi v.
\end{align*}
$$

If in addition (H$_4$) holds (7) then $U_D^\varphi$ is a convex function on $C^+(\partial D)$, i.e. for every $\lambda \in [0, 1]$,

$$
U_D^\varphi(\lambda f + (1 - \lambda)g) \leq \lambda U_D^\varphi f + (1 - \lambda)U_D^\varphi g.
$$

In particular, for every $\alpha \geq 1$,

$$
U_D^\varphi(\alpha f) \geq \alpha U_D^\varphi f.
$$

Now, let $(D_n)$ be a sequence of bounded regular domains such that for every $n \in \mathbb{N}$, $D_n \subset D_{n+1} \subset \Omega$ and $\bigcup_{n=1}^\infty D_n = \Omega$. Such a sequence will be called a regular exhaustion of $\Omega$ and it is used to generate solutions to (1.2).

(7) Notice that concavity together with (H$_1$) and (H$_2$) implies (SH$_1$).
Proposition 6 ([12, Proposition 10]). Let \( g \in C^+(\Omega) \) be an \( L \)-superharmonic function. Then the sequence \( (U^{\varphi}_{D_n}g) \) is decreasing to a solution \( u \in C^+(\Omega) \) of (1.2) satisfying \( u \leq g \).

Now we are ready to prove the main result of this section.

Proof of Theorem 3. Suppose that \( Lu - \varphi_1(\cdot, u) = 0 \) has a nontrivial nonnegative bounded solution \( \tilde{u} \) in \( \Omega \). Let \( (D_n) \) be an increasing sequence of bounded regular domains exhausting \( \Omega \). Then by Proposition 6 for every \( \lambda \geq \lambda_1 = \|\tilde{u}\|_{L^\infty} > 0 \), \( v_\lambda = \lim_{n \to \infty} U^{\varphi_1}_{D_n}\lambda \) is a nontrivial nonnegative bounded solution of \( Lu - \varphi_1(\cdot, u) = 0 \) in \( \Omega \) too.

Let \( \lambda \geq \lambda_1 \). Then by Lemma 5, \( U^{\varphi_1}_{D_n}\lambda \geq \frac{\lambda}{\lambda_1} U^{\varphi_1}_{D_n}\lambda_1 \). Therefore, letting \( n \to \infty \) we obtain
\[
v_\lambda \geq \frac{\lambda}{\lambda_1} v_{\lambda_1}, \quad \text{where} \quad v_{\lambda_1} = \lim_{n \to \infty} U^{\varphi_1}_{D_n}\lambda_1.
\]

Furthermore, \( \varphi \leq \varphi_1 \) implies, by Lemma 4, that \( U^{\varphi}_{D_n}\lambda \geq U^{\varphi_1}_{D_n}\lambda \), because \( U^{\varphi}_{D_n}\lambda \) is a supersolution to \( Lu - \varphi_1(\cdot, u) = 0 \). Hence
\[
u_\lambda = \lim_{n \to \infty} U^{\varphi}_{D_n}\lambda \geq v_\lambda.
\]

Suppose now that there is a large solution \( u \) to (1.2). Then it satisfies \( \liminf_{x \to \partial \Omega} u(x) = \infty \). Hence for sufficiently large \( n \), \( u \geq U^{\varphi}_{D_n}\lambda \) on \( \partial D_n \), and so by Lemma 4
\[
u \geq u_\lambda \geq v_\lambda.
\]

Consequently, \( u \geq \frac{\lambda}{\lambda_1} v_{\lambda_1} \) and so \( \frac{v}{\lambda} \geq \frac{1}{\lambda_1} v_{\lambda_1} \) for every \( \lambda \geq \lambda_1 \). When \( \lambda \) tends to infinity, we get \( v_{\lambda_1} = 0 \), which gives a contradiction.

3. Domination by a concave function. The aim of this section is to show that (SH\(_1\)), (H\(_2\)), (H\(_3\)) imply existence of a function \( \varphi_1 \) concave with respect to the second variable and such that
\[
\varphi(x, t) \leq \varphi_1(x, t), \quad \varphi_1(x, 0) = 0.
\]

Clearly, a nonnegative function \( \psi \) concave on \([0, \infty)\), continuous at zero, and with \( \psi(0) = 0 \) is dominated by an affine function. Indeed, given \( \beta > 0 \), we have
\[
\psi(t) \leq \frac{t}{\beta} \psi(\beta), \quad t \geq \beta,
\]
and so
\[
\psi(t) \leq \frac{t}{\beta} \psi(\beta) + \sup_{0 \leq s \leq \beta} \psi(s).
\]

(\(^8\)) Note here that \( u \) may be zero and usually an extra argument is needed to ensure it is not.
The idea behind (SH\(_1\)) is to formulate a condition as weak as possible to go beyond concavity in Theorem 1. It turns out that (SH\(_1\)) together with Theorem 7 below does the job. Clearly, the most delicate part is to guarantee that \(\varphi_1(x,0) = 0\).

**Theorem 7.** Suppose that \(\varphi(x,t)\) satisfies (SH\(_1\)), (H\(_2\)), (H\(_3\)). Then there is \(\varphi_1(x,t)\) satisfying (SH\(_1\)), (H\(_2\))–(H\(_4\)) such that

\[
\varphi(x,t) \leq \varphi_1(x,t).
\]

Moreover, there exists a constant \(C > 0\) such that

\[
\varphi_1(x,t) \leq Cp(x)(t + 1).
\]

**Proof.** For \(t \geq 1\),

\[
\varphi(x,t) \leq 2p(x)t.
\]

We need to dominate \(\varphi\) for \(t \leq 1\). Let \(\eta \in C^\infty(\mathbb{R})\), \(\eta \geq 0\), \(\text{supp}\ \eta \subset (-1,1)\), \(\eta(-t) = \eta(t)\) and \(\int_{\mathbb{R}} \eta(s) \, ds = 1\). Given \(0 < \delta \leq 1\), let \(\eta_\delta(t) = \frac{1}{\delta} \eta(\frac{1}{\delta} t)\), \(t \in \mathbb{R}\). Let \(x \in \Omega\). We write \(\varphi_x(t) = \varphi(x,t)\), \(t \in \mathbb{R}\). Then

\[
(3.1) \quad \varphi_x \ast \eta_\delta(0) = \int_{-\delta}^{\delta} \varphi_x(-t)\eta_\delta(t) \, dt = \int_{-1}^{1} \varphi(x, \delta s) \eta(s) \, ds.
\]

Hence

\[
(3.2) \quad 0 \leq \inf_{\delta} \varphi_x \ast \eta_\delta(0) = \lim_{\delta \to 0} \varphi_x \ast \eta_\delta(0) = \varphi_x(0) = 0.
\]

Secondly, \((\varphi_x \ast \eta_\delta)' = \varphi_x \ast (\eta_\delta)'\) and

\[
(3.3) \quad (\eta_\delta)'(t) = \frac{1}{\delta^2} \eta' \left( \frac{1}{\delta} t \right).
\]

Moreover,

\[
\int_{\mathbb{R}} |(\eta_\delta)'(t)| \, dt \leq \frac{1}{\delta^2} \int_{\mathbb{R}} \left| \eta' \left( \frac{1}{\delta} t \right) \right| \, dt = \frac{1}{\delta} \int_{\mathbb{R}} |\eta'(s)| \, ds.
\]

Therefore, if \(0 \leq t \leq 2\) then

\[
|(\varphi_x \ast \eta_\delta)'(t)| \leq \int_{\mathbb{R}} \varphi_x(t-s) |(\eta_\delta)'(s)| \, ds \leq p(x) \frac{4}{\delta} \int_{\mathbb{R}} |\eta'(s)| \, ds.
\]

Consequently, there exists a constant \(c_1\) such that for \(0 \leq t \leq 2\) we have

\[
(3.4) \quad \varphi_x \ast \eta_\delta(t) \leq \frac{c_1}{\delta} p(x)t + \varphi_x \ast \eta_\delta(0).
\]
Moreover,
\[
\varphi_x * \eta_\delta(t) = \int_{\mathbb{R}} \varphi_x(t - s) \eta_\delta(s) \, ds \geq \int_{-\delta}^{0} \varphi_x(t - s) \eta_\delta(s) \, ds
\geq \varphi_x(t) \int_{-\delta}^{0} \eta_\delta(s) \, ds = \frac{1}{2} \varphi_x(t).
\]
Hence
\[
\varphi_x(t) \leq 2 \varphi_x * \eta_\delta(t)
\]
and so for \( t \in [0, 2] \),
\[
\varphi_x(t) \leq \frac{2c_1}{\delta} p(x)t + 2 \varphi_x * \eta_\delta(0).
\]
Let
\[
\psi_\delta(x, t) = \frac{2c_1}{\delta} p(x)t + 2 \varphi_x * \eta_\delta(0), \quad \psi(x, t) = \inf_{0<\delta<1} \psi_\delta(x, t).
\]
First we prove that for every fixed \( x \in \Omega \), \( \psi(x, t) \) is concave on \([0, 2]\). For \( t, s \in [0, 2] \) and \( \alpha \in [0, 1] \), we have
\[
\psi(x, \alpha t + (1 - \alpha)s) = \inf_{\delta} \psi_\delta(x, \alpha t + (1 - \alpha)s)
= \inf_{\delta} \left( \alpha \psi_\delta(x, t) + (1 - \alpha) \psi_\delta(x, s) \right)
\]
and
\[
\inf_{\delta} \left( \alpha \psi_\delta(x, t) + (1 - \alpha) \psi_\delta(x, s) \right) \geq \inf_{\delta} \alpha \psi_\delta(x, t) + \inf_{\delta} (1 - \alpha) \psi_\delta(x, s).
\]
Hence
\[
\psi(x, \alpha t + (1 - \alpha)s) \geq \alpha \psi(x, t) + (1 - \alpha) \psi(x, s)
\]
and so \( \psi(x, t) \) is continuous on \((0, 2)\) in \( t \). Secondly,
\[
\psi(x, 0) = \inf_{\delta} 2 \varphi_x * \eta_\delta(0) = 2 \varphi(x, 0) = 0,
\]
and for every \( \delta \),
\[
\limsup_{t \to 0} \psi(x, t) \leq \limsup_{t \to 0} \left( \frac{2c_1 c(x)}{\delta} \frac{1}{2} t + 2 \varphi_x * \eta_\delta(0) \right)
\leq 2 \varphi_x * \eta_\delta(0) \leq 2 \varphi_x(\delta).
\]
Hence \( \lim_{t \to 0^+} \psi(x, t) = 0 \) and so \( \psi(x, t) \) is continuous on \([0, 2]\). Moreover, \( \psi(x, \cdot) \) is nondecreasing and
\[
\psi(x, t) \leq \psi_1(x, t) \leq 2c_1 p(x)t + 2 \varphi(x, 1)
\leq 2c_1 p(x)t + 4 p(x) \leq 4c_1 p(x)(t + 1).
\]
Finally, we define
\[
\varphi_1(x, t) = \begin{cases} 
2p(x)t + \psi(x, t) & \text{if } 0 \leq t \leq 1, \\
2p(x)t + \psi(x, 1) & \text{if } t > 1,
\end{cases}
\]
and we set \( \varphi_1(x, t) = 0 \) if \( t \leq 0 \).

4. Large solutions to \( Lu - \varphi(\cdot, u) = 0 \) under (SH\(_1\)), (H\(_2\)), (H\(_3\)). In this section we prove Theorem \( \boxed{1} \). The argument is based on a very convenient characterization of existence of bounded solutions to \( \boxed{1.2} \). It is formulated in terms of thinness at infinity.

Let \( \Omega \subset \mathbb{R}^d, d \geq 3 \), be a domain. A subset \( A \subset \Omega \) is called \textit{thin at infinity} if there is a continuous nonnegative \( L \)-superharmonic function \( s \) on \( \Omega \) such that
\[
\begin{cases}
s \geq 1 & \text{on } A, \\
 s(x_0) < 1 & \text{for some } x_0 \in \Omega.
\end{cases}
\]

We say that \( \Omega \) is \textit{Greenian} if there is a function \( G_\Omega \) called the \textit{Green function for} \( L \) satisfying
\[
G_\Omega(x, y) \in C^\infty(\Omega \times \Omega \setminus \{(x, x) : x \in \Omega\}),
\]
for every \( y \in \Omega \) we have
\[
LG_\Omega(\cdot, y) = -\delta_y \quad \text{in the sense of distributions,}
\]
and
\[
G_\Omega(\cdot, y) \quad \text{is a potential,}
\]
i.e. every nonnegative \( L \)-harmonic function \( h \) such that \( h(x) \leq G_\Omega(x, y) \) is identically zero. For a given domain \( \Omega \), the Green function \( G_\Omega \) may or may not exist, but existence of \( s \) as above implies that it does.

**Theorem 8 (\cite[Theorem 19]{12}).** Suppose that \( \Omega \) is Greenian and \( \varphi \) is a measurable function satisfying (H\(_1\))–(H\(_3\)). Equation \( \boxed{1.2} \) has a nonnegative nontrivial bounded solution in \( \Omega \) if and only if there exists a Borel set \( A \subset \Omega \) which is thin at infinity and \( c_0 > 0 \) such that
\[
\int_{\Omega \setminus A} G_\Omega(\cdot, y) \varphi(y, c_0) \, dy \neq \infty.
\]

In the case of \( L = \Delta \) and \( \varphi(x, t) = p(x)t^\gamma, 0 < \gamma < 1, p \in L^\infty_{\text{loc}} \), Theorem 8 was proved in \cite{7}. Notice that no concavity (H\(_4\)) is required.

In view of Theorems 8 and 7, the proof of Theorem \( \boxed{1} \) is straightforward:

**Proof of Theorem \( \boxed{1} \)** If \( Lu - p(x)\psi(u) = 0 \) has a nonnegative nontrivial bounded solution then by Theorem 8 there is a set \( A \subset \Omega \) thin at infinity
such that
\[
\int_{\Omega \setminus A} G_\Omega(\cdot, y)p(y) \, dy \neq \infty.
\]

Let \(\varphi_1\) be the function constructed in Theorem 7. Then \(\varphi_1\) can be taken such that
\[
\varphi_1(x,t) \leq C p(x)(t + 1),
\]
and so again by Theorem 8, \(L u - \varphi_1(\cdot, u) = 0\) has a nonnegative nontrivial bounded solution. Hence the conclusion follows by Theorem 3.

Now we are going to apply Theorem 3 to \(\varphi\) that satisfies (SH1).

**Theorem 9.** Let \(\Omega\) be a Greenian domain. Assume that \(\varphi\) satisfies (SH1), (H2), (H3) and there exists a set \(A \subset \Omega\) thin at infinity such that the function \(p(x)\) in (SH1) satisfies
\[
\int_{\Omega \setminus A} G_\Omega(\cdot, y)p(y) \, dy \neq \infty.
\]
Then (1.2) has a nonnegative nontrivial bounded solution and it has no large solution.

**Proof.** By Theorem 8 there is a nonnegative nontrivial bounded solution to (1.2). Let \(\varphi_1(x,t)\) be the function constructed in Theorem 7. Then
\[
\varphi_1(x,t) \leq C p(x)(t + 1).
\]
Hence there is a nonnegative nontrivial bounded solution to \(L u - \varphi_1(\cdot, u) = 0\), and so by Theorem 3 there is no large solution to (1.2).

Suppose now that for every \(t_0 > 0\) there is a constant \(C_{t_0} > 0\) such that for every \(t \geq 0\) and \(x \in \Omega\), \(\varphi(x,t) \leq C_{t_0} \varphi(x,t_0)(t + 1)\). We do not assume any integrability of \(\varphi(x,t_0)\) in the spirit of (4.6). Then

**Theorem 10.** Let \(\Omega\) be a Greenian domain. Assume that \(\varphi\) satisfies (H1)–(H3). Suppose further that for every \(t_0 > 0\) there is \(C_{t_0} > 0\) such that
\[
\varphi(x,t) \leq C_{t_0} \varphi(x,t_0)(t + 1).
\]
If (1.2) has a nonnegative nontrivial bounded solution, then (1.2) has no large solution.

**Proof.** By Theorem 8 there exists a set \(A \subset \Omega\) thin at infinity and \(t_0 > 0\) such that
\[
\int_{\Omega \setminus A} G_\Omega(\cdot, y)\varphi(y,t_0) \, dy \neq \infty.
\]
Let $\varphi_1(x, t)$ be the function constructed in Theorem 7. We can take $\varphi_1$ such that $\varphi_1(x, t) \leq C(t_0)\varphi(x, t_0)(t + 1)$. Then

\[(4.8) \int_{\Omega \setminus A} G_\Omega(\cdot, y)\varphi_1(y, t_0) dy \neq \infty.\]

Hence there is a nonnegative nontrivial bounded solution to $Lu - \varphi(\cdot, u) = 0$, and so by Theorem 3 there is no large solution to (1.2).

5. Bounded solutions to $Lu - \varphi(\cdot, u) = 0$. Theorems 7 and 8 allow us to remove concavity and get the following characterization of bounded solutions.

**Proposition 11.** Let $\Omega$ be a Greenian domain. Suppose that $\varphi(x, t) = p(x)\psi(t)$ satisfies (SH1), (H2) and (H3). Let $(D_n)$ be an increasing sequence of regular bounded domains exhausting $\Omega$. The following statements are equivalent:

1. Equation (1.2) has a nonnegative nontrivial bounded solution.
2. For every $c > 0$, $v_c = \inf_{n \in \mathbb{N}} U_{D_n}^\varphi c$ is a nonnegative nontrivial bounded solution of (1.2).
3. There exists $c > 0$ such that $v_c = \inf_{n \in \mathbb{N}} U_{D_n}^\varphi c$ is a nonnegative nontrivial bounded solution of (1.2).

Furthermore if any of the above conditions holds then

\[(5.1) \sup_{x \in \Omega} v_c(x) = c.\]

The proof of Proposition 11 is given at the end of this section. We proceed as before: first we obtain the result for a concave nonlinear term, i.e. under (H1)–(H4), and then we apply Theorem 7.

**Proposition 12.** Suppose that $\varphi$ satisfies (H1)–(H4). Then the conclusion of Proposition 11 holds true.

Proposition 12 was proved in [7] for $L = \Delta$ and $\varphi(x, t) = p(x)t^\gamma$ where $0 < \gamma < 1$ and $p \in L^\infty_{\text{loc}}$. Generalization to elliptic operators and $\varphi$ satisfying (H1)–(H4) is straightforward and $\varphi$ need not to be of the product form.

**Proof of Proposition 12.** The proof is the same as in [7] Lemmas 3 and 4], but we include the argument here for the reader’s convenience. Let $u_n = U_{D_n}^\varphi c$ and $u_c = \inf_{n \in \mathbb{N}} u_n$. Under hypotheses (H1)–(H4), $\sup_{x \in \Omega} u_c(x)$ is either zero or $c$. Indeed, by Proposition 6 $u_c$ is a nonnegative solution of (1.2) bounded above by $c$. Suppose now that there exists $0 < c_0 \leq c$ such that $\sup_{x \in \Omega} u_c = c_0$. By Lemma 4

\[U_{D_n}^\varphi \left( \frac{c}{c_0} u_c \right) \leq U_{D_n}^\varphi c = u_n.\]
Also by Lemma 5, 
\[
\frac{c}{c_0} U_{D_n}^\varphi u_c \leq U_{D_n}^\varphi \left( \frac{c}{c_0} u_c \right).
\]
Hence 
\[
U_{D_n}^\varphi u_c = u_c \leq \frac{c_0}{c} u_n,
\]
and letting \(n\) tend to infinity we obtain 
\[
u_c \leq \frac{c_0}{c} u_c,
\]
which implies \(c = c_0\).

Therefore, under (H4), if any of conditions (1)–(3) is satisfied then (5.1) follows. It is clear that (2)⇒(3)⇒(1). So it is enough to prove that (1) implies (2). Let \(w\) be a nonnegative nontrivial bounded solution of (1.2).

Suppose first that \(r \geq \sup_\Omega w\). Then \(v = \lim_{n \to \infty} U_{D_n}^\varphi r\) is a nonnegative nontrivial bounded solution satisfying \(w \leq v \leq r\) in \(\Omega\). Hence
\[
\sup_{x \in \Omega} v(x) = r.
\]
Secondly, we take \(0 < c < \sup_\Omega w\).

By Lemma 5, \(u_n = U_{D_n}^\varphi c \leq U_{D_n}^\varphi r = v_n\) in \(D_n\). Hence 
\[
G_{D_n}(\varphi(\cdot, u_n)) \leq G_{D_n}(\varphi(\cdot, v_n)) \quad \text{in} \quad D_n.
\]
Furthermore by (2.2),
\[
v_n + G_{D_n}(\varphi(\cdot, v_n)) = r \quad \text{in} \quad D_n,
\]
and 
\[
u_n + G_{D_n}(\varphi(\cdot, u_n)) = c \quad \text{in} \quad D_n.
\]
We can deduce 
\[0 \leq c - u_n \leq r - v_n \quad \text{in} \quad D_n.
\]
When \(n\) tends to infinity, we get 
\[c - u \leq r - v \quad \text{in} \quad \Omega.
\]
Suppose now that \(u\) is trivial. Then 
\[v \leq r - c \quad \text{in} \quad \Omega.
\]
But \(\sup_\Omega v = r\), which gives a contradiction. 

Proof of Proposition 11. As before, it is enough to prove that (1) implies (2). By Theorem 8, there is a set \(A \subset \Omega\) thin at infinity such that
\[
\int_{\Omega \setminus A} G_\Omega(\cdot, y)p(y) \, dy \neq \infty.
\]
Let \(\varphi_1(x, t)\) be the function constructed in Theorem 7. We can take \(\varphi_1\) such that \(\varphi_1(x, t) \leq Cp(x)(t + 1)\), so again by Theorem 8, \(Lu - \varphi_1(\cdot, u) = 0\) has
a nonnegative nontrivial bounded solution. Let $c > 0$. By Proposition 12,
\[ v_1^c = \lim_{n \to \infty} U^\varphi_{D_n} c \]
is a nonnegative nontrivial bounded solution of $Lu - \varphi_1(\cdot, u) = 0$ and
\[ \sup_{x \in \Omega} v_1^c(x) = c. \]  
(5.4)
But in view of Lemma 4,
\[ c \geq v_c = \lim_{n \to \infty} U^\varphi_{D_n} c \geq \lim_{n \to \infty} U^\varphi_{D_n} c = v_1^c. \]
Thus $v_c$ is a nonnegative nontrivial solution to (1.2) satisfying
\[ \sup_{x \in \Omega} v_c(x) = c. \]

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