THE TWO WEIGHT $T^1$ THEOREM FOR FRACTIONAL RIESZ TRANSFORMS WHEN ONE MEASURE IS SUPPORTED ON A CURVE

By

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Abstract. Let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$. We assume that at least one of the two measures $\sigma$ and $\omega$ is supported on a regular $C^1,\delta$ curve in $\mathbb{R}^n$. Let $R^{\alpha,n}_\sigma$ be the $\alpha$-fractional Riesz transform vector on $\mathbb{R}^n$. We prove the $T^1$ theorem for $R^{\alpha,n}_\sigma$: namely that $R^{\alpha,n}_\sigma$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the $A^{\alpha}_2$ conditions with holes hold, the punctured $A^{\alpha}_2$ conditions hold, and the cube testing condition for $R^{\alpha,n}_\sigma$ and its dual both hold. The special case of the Cauchy transform, $n=2$ and $\alpha=1$, when the curve is a line or circle, was established by Lacey, Sawyer, Shen, Uriarte-Tuero and Wick in [LaSaShUrWi]. This $T^1$ theorem represents essentially the most general $T^1$ theorem obtainable by methods of energy reversal. More precisely, for the pushforwards of the measures $\sigma$ and $\omega$, under a change of variable to straighten out the curve to a line, we use reversal of energy to prove that the quasienergy conditions in [SaShUr7] are implied by the $A^{\alpha}_2$ with holes, punctured $A^{\alpha}_2$, and quasicube testing conditions for $R^{\alpha,n}_\sigma$. Then we apply the main theorem in [SaShUr7] to deduce the $T^1$ theorem above.

Contents

1 Introduction 454
  1.1 A brief history of the $T^1$ theorem. 454
  1.2 Statement of results. 459
  1.3 Techniques. 463

2 Definitions 468
  2.1 Quasicubes. 469
  2.2 The $A^{\alpha}_2$ conditions. 471
    2.2.1 Punctured Muckenhoupt conditions. 472

*E. T. Sawyer is supported in part by NSERC.
†C.-Y. Shen is supported in part by MOST, through grant 104-2628-M-002-015-MY4.
‡I. Uriarte-Tuero has been partially supported by grants DMS-1056965 (US NSF), MTM2010-16232, MTM2009-14694-C02-01 (Spain), and a Sloan Foundation Fellowship.
2.3 Quasicube testing and quasiweak boundedness property. 472
2.4 Quasienergy conditions. 473

3 One measure supported in a line
3.1 Backward quasienergy condition. 477
3.2 Forward quasienergy condition. 487
3.3 Backward triple testing and quasiweak boundedness property. 496

4 One measure compactly supported on a $C^{1,\delta}$ curve
4.1 Changes of variable. 501
4.2 A preliminary $T1$ theorem. 512
4.3 The $T1$ theorem for a measure supported on a regular $C^{1,\delta}$ curve. 514

1 Introduction

1.1 A brief history of the $T1$ theorem. The celebrated $T1$ theorem of David and Journé [DaJo] characterizes those singular integral operators $T$ on $\mathbb{R}^n$ that are bounded on $L^2(\mathbb{R}^n)$, and does so in terms of a weak boundedness property, and the membership of the two functions $T1$ and $T^*1$ in the space of bounded mean oscillation,

$$\|T1\|_{BMO(\mathbb{R}^n)} \lesssim \|1\|_{L^\infty(\mathbb{R}^n)} = 1,$$

$$\|T^*1\|_{BMO(\mathbb{R}^n)} \lesssim \|1\|_{L^\infty(\mathbb{R}^n)} = 1.$$ 

These latter conditions are actually the following testing conditions in disguise,

$$\|T1_Q\|_{L^2(\mathbb{R}^n)} \lesssim \|1_Q\|_{L^2(\mathbb{R}^n)} = \sqrt{|Q|},$$

$$\|T^*1_Q\|_{L^2(\mathbb{R}^n)} \lesssim \|1_Q\|_{L^2(\mathbb{R}^n)} = \sqrt{|Q|},$$

tested over all indicators of cubes $Q$ in $\mathbb{R}^n$ for both $T$ and its dual operator $T^*$. This theorem was the culmination of decades of investigation into the nature of cancellation conditions required for boundedness of singular integrals.\(^1\)

A parallel thread of investigation culminated in the theorem of Coifman and Fefferman\(^2\) that characterizes those nonnegative weights $w$ on $\mathbb{R}^n$ for which all of the ‘nicest’ of the $L^2(\mathbb{R}^n)$ bounded singular integrals $T$ above are bounded on weighted spaces $L^2(\mathbb{R}^n; w)$, and does so in terms of the $A_2$ condition of Muckenhoupt,

$$\left(\frac{1}{|Q|} \int_Q w(x)dx\right) \left(\frac{1}{|Q|} \int_Q \frac{1}{w(x)}dx\right) \lesssim 1,$$

\(^1\)See, e.g., Chapter VII of Stein [Ste] and the references given there for a historical background.
\(^2\)See, e.g., Chapter V of [Ste] and the references given there for the long history of this investigation, in which the celebrated theorem of Hunt, Muckenhoupt and Wheeden played a critical role.
taken over all cubes \( Q \) in \( \mathbb{R}^n \). This condition is also a testing condition in disguise, namely it is a consequence of
\[
\left\| T \left( s^Q \frac{1}{|Q|} \right) \right\|_{L^2(\mathbb{R}^n; \omega)} \lesssim \| s^Q \|_{L^2(\mathbb{R}^n; \frac{1}{|Q|})},
\]
tested over all ‘indicators with tails’
\[
s^Q(x) = \frac{\ell(Q)}{\ell(Q) + |x - c_Q|}
\]
of cubes \( Q \) in \( \mathbb{R}^n \), provided \( T \) satisfies an appropriate nondegeneracy condition, e.g., the strongly elliptic condition in [SaShUr7].

A natural synthesis of these two results leads to the ‘two weight’ question of which pairs of weights \((\sigma, \omega)\) have the property that nice singular integrals are bounded from \( L^2(\mathbb{R}^n; \sigma) \) to \( L^2(\mathbb{R}^n; \omega) \). The simplest (nontrivial) singular integral of all is the Hilbert transform
\[
Hf(x) = \int_{\mathbb{R}} \frac{f(y)}{y - x} \, dy
\]
on the real line, and Nazarov, Treil and Volberg formulated the two weight question for the Hilbert transform [Vol], that in turn led to the NTV conjecture:

**Conjecture 1** ([Vol]). The Hilbert transform is bounded from \( L^2(\mathbb{R}^n; \sigma) \) to \( L^2(\mathbb{R}^n; \omega) \), i.e.,
\[
\|Hf\|_{L^2(\mathbb{R}^n; \omega)} \lesssim \|f\|_{L^2(\mathbb{R}^n; \sigma)}, \quad f \in L^2(\mathbb{R}^n; \sigma),
\]
if and only if the two weight \( A_2 \) condition with tails holds,
\[
\left( \frac{1}{|Q|} \int_Q s^Q_2 \, d\omega(x) \right) \left( \frac{1}{|Q|} \int_Q s^Q_2 \, d\sigma(x) \right) \lesssim 1,
\]
for all cubes \( Q \), and the two testing conditions hold,
\[
\|H1^Q\sigma\|_{L^2(\mathbb{R}^n; \omega)} \lesssim \|1^Q\|_{L^2(\mathbb{R}^n; \sigma)} = \sqrt{|Q|}_\sigma,
\]
\[
\|H^*1^Q\omega\|_{L^2(\mathbb{R}^n; \sigma)} \lesssim \|1^Q\|_{L^2(\mathbb{R}^n; \omega)} = \sqrt{|Q|}_\omega,
\]
for all cubes \( Q \).

In a groundbreaking series of papers including [NTV1], [NTV2] and [NTV4], Nazarov, Treil and Volberg used weighted Haar decompositions with random grids, introduced their ‘pivotal’ condition, and proved the above conjecture under the assumption that the pivotal condition held. Subsequently, in joint work of two of the authors, Sawyer and Uriarte-Tuero, with Lacey [LaSaUr2], it was shown that the pivotal condition was not necessary in general, a necessary ‘energy’ condition was introduced as a substitute, and a hybrid merging of these two conditions was shown to be sufficient for use as a side condition. Eventually, these three authors
with Shen established the NTV conjecture in a two part paper: Lacey, Sawyer, Shen and Uriarte-Tuero [LaSaShUr2] and Lacey [Lac]. A key ingredient in the proof was an ‘energy reversal’ phenomenon enabled by the Hilbert transform kernel equality
\[
\frac{1}{y - x} - \frac{1}{y - x'} = \frac{x - x'}{(y - x)(y - x')},
\]
having the remarkable property that the denominator on the right hand side remains positive for all \(y\) outside the smallest interval containing both \(x\) and \(x'\). This proof of the NTV conjecture was given in the special case that the weights \(\sigma\) and \(\omega\) had no point masses in common, largely to avoid what were then thought to be technical issues. However, these issues turned out to be considerably more interesting, and this final assumption of no common point masses was removed shortly after by Hytönen [Hyt2], who also simplified some aspects of the proof.

At this juncture, attention naturally turned to the analogous two weight inequalities for higher dimensional singular integrals, as well as \(\alpha\)-fractional singular integrals such as the Cauchy transform in the plane. In a long paper [SaShUr4], begun on the arXiv in 2013, and culminating in our publication in Revista [SaShUr], the authors introduced the appropriate notions of Poisson kernel to deal with the \(A^2_\alpha\) condition on the one hand, and the \(\alpha\)-energy condition on the other hand (unlike for the Hilbert transform, these two Poisson kernels differ in general). The main result of that paper established the \(T1\) theorem for ‘elliptic’ vectors of singular integrals under the side assumption that an energy condition and its dual held, thus identifying the culprit in higher dimensions as the energy conditions (see also [SaShUr7] where the restriction to no common point masses was removed, and the extension to quasicubes was obtained3). A general \(T1\) conjecture is this.

**Conjecture 2.** Let \(T^{a,n}\) denote an elliptic vector of standard \(\alpha\)-fractional singular integrals in \(\mathbb{R}^n\). Then \(T^{a,n}\) is bounded from \(L^2(\mathbb{R}^n; \sigma)\) to \(L^2(\mathbb{R}^n; \omega)\), i.e.,
\[
\|T^{a,n}(f\sigma)\|_{L^2(\mathbb{R}^n; \omega)} \lesssim \|f\|_{L^2(\mathbb{R}^n; \sigma)}, \quad f \in L^2(\mathbb{R}^n; \sigma),
\]
if and only if the two one-tailed \(A^2_\alpha\) conditions with holes hold, the punctured \(A^2_\alpha\) conditions hold, and the two testing conditions hold,
\[
\|T^{a,n}1_Q\sigma\|_{L^2(\mathbb{R}^n; \omega)} \lesssim \|1_Q\|_{L^2(\mathbb{R}^n; \sigma)} = \sqrt{|Q|_\sigma},
\|
T^{a,n,\text{dual}}1_Q\omega\|_{L^2(\mathbb{R}^n; \sigma)} \lesssim \|1_Q\|_{L^2(\mathbb{R}^n; \omega)} = \sqrt{|Q|_\omega},
\]
for all cubes \(Q\) in \(\mathbb{R}^n\) (whose sides need not be parallel to the coordinate axes).

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3All of the relevant material from references [SaShUr] and [SaShUr7] can be found in a single long file [SaShUr5] on the arXiv.
A positive resolution to this conjecture could have implications for a number of problems that are higher dimensional analogues of those connected to the Hilbert transform,\(^4\)

(1) when a rank one perturbation of a unitary operator is similar to a unitary operator (see, e.g., [Vol] and [NiTr]): this could extend to an analogous question for a rank one perturbation of a normal operator \(T\) and lead to a two weight inequality for the Cauchy transform with one measure being the spectral measure of \(T\),

(2) when a product of two densely defined Toeplitz operators \(T_aT_b\) is a bounded operator, which is equivalent to the Birkhoff–Wiener–Hopf factorization for a given function \(c = ab\); the same questions for the Bergman space could lead to a two weight problem for the Beurling transform,

(3) questions regarding subspaces of the Hardy space invariant under the inverse shift operator (see, e.g., [Vol] and [NaVo]),

(4) questions concerning orthogonal polynomials (see, e.g., [VoYu], [PeVoYu] and [PeVoYu1]),

and also to a variety of questions in quasiconformal theory (due to the relevance of the Beurling transform in that context) such as

(1) the conjecture of Iwaniec and Martin, at the level of Hausdorff dimension distortion (see [IwMa]) and, at the level of Hausdorff measures, higher dimensional analogues of the Astala conjecture (see e.g., [LaSaUr]), for which proof of an (essentially) two weight inequality for the Beurling transform was crucial, and in general, similar questions pertaining to the higher dimensional analogues of the Beurling transform,

(2) the problem of characterizing which Beltrami coefficients give rise to biLipschitz maps (see, e.g., [AsGo]),

(3) and to the well known problem of the connectivity of the manifold of planar chord-arc curves (see, e.g., [AsGo] and [AsZi]).

The theoretical relevance and potential applications of such a general \(T1\) theorem has led to repeated attempts to prove it, and its substantial difficulty has caused progress to happen in small increments at the cost of great effort, as can already be seen from the history summarized above. In essentially all of the efforts made towards proving the general \(T1\) theorem, the energy conditions in Definition 24 below appear as a key step in the general proof strategy (explicitly in the various efforts by the authors, and implicitly in [LaWi2]). In view of the aforementioned

\(^4\)We thank Professor N. Nikolski for providing information on some of these topics.
main result in [SaShUr7], the general $T_1$ theorem would hold if it were the case that the energy conditions were implied by boundedness of $T^{a,n}$ from $L^2(\mathbb{R}^n; \sigma)$ to $L^2(\mathbb{R}^n; \omega)$. When the first version of this paper was written, it was not in fact known if this was true, and since that time, the energy conditions have been shown in [SaShUr11] to be not necessary in dimension $n = 1$ for elliptic operators, and later in [Saw4] to be not necessary in dimension $n \geq 2$ for the Riesz transforms.

While the energy conditions are not in general necessary, there are some cases in which they have been proved to hold, and in fact without exception the technique used to prove them is that of energy reversal (see [SaShUr3] where energy reversal is shown to fail spectacularly in higher dimensions—the initial failure was noted by M. Lacey). Of course, the energy conditions hold for the Hilbert transform on the line [LaSaUr2]. A natural case in which they hold is when both measures are doubling (since then even the pivotal conditions hold), or somewhat more generally, Ahlfors–David regular. A further generalization of this case is the case considered by Lacey and Wick in [LaWi2], namely when the measures are uniformly not concentrated on hyperplanes, at all locations and scales. Their paper overlaps [SaShUr] to some extent and has been further generalized in [SaShUr9] to conditions on $k$-dimensional planes for $k < n$.

Another special case in which the energy conditions hold (proved by the authors in recent joint work [LaSaShUrWi] with M. Lacey and B. Wick) is that of the Cauchy transform in the plane in the special case when one of the measures is supported on either a straight line or a circle, thus proving the $T_1$ theorem in this case. This case has relevant applications to model spaces. The key to this result was an extension of the energy reversal phenomenon for the Hilbert transform to the setting of the Cauchy transform, and here the one-dimensional nature of the line and circle played a critical role. In particular, a special decomposition of a 2-dimensional measure into ‘end’ and ‘side’ pieces played a crucial role, and was in fact discovered independently in both the initial version of this paper and in [LaWi].

The one-dimensional cases mentioned above are reminiscent of a famous problem of Calderón, namely the boundedness of the Cauchy transform on Lipschitz curves, which can indeed be recast in terms of a $T_1$ theorem. The current technology in the two weight $T_1$ theorems requires $1 + \delta$ degrees of differentiability, so the corresponding natural question is the one solved in this paper. While the extra differentiability required is a drawback, the possibility of using two different weights adds strength and flexibility to the original Calderón problem (which by now has many proofs, see, e.g., E. Stein [Ste]).
In this paper, we extend the $T_1$ theorem to the setting where one of the measures is supported on a Hölder continuously differentiable curve (in higher dimensions). This result seems to represent the best possible $T_1$ theorem that can be obtained from the methods of energy reversal, and requires a number of new ideas, especially of a geometric nature, as opposed to the more algebraic ‘corona’ ideas developed for the solution to the NTV conjecture. In particular, changes of variable are made to straighten sufficiently small pieces of the curve to a line, and the resulting operator norms, $A_2^p$ conditions, and testing condition constants are tracked under these changes of variable. This tracking presents significant subtleties, especially for the testing constants, which require appropriate tangent plane approximations to the phase function of the testing kernel. Further effort is then needed to control the testing conditions associated with these pieces by the testing conditions we assume for the curve in the first place. In particular, a ‘localized triple testing’ condition is derived that enables the reduction of testing conditions for small pieces of transformed measure on a line to the testing conditions for the global measures. Yet another complication arises here in the use of quasicubes in the proof—dictated by pushforwards of ordinary cubes—and this requires the new notion of ‘$L$-transverse’ and its properties in order to control the intersections of quasicubes with lines.

We now give a more precise description of what is in this paper and its relation to the literature.

1.2 Statement of results. In [SaShUr7] (see also [SaShUr] for special cases), under a side assumption that certain energy conditions hold, the authors show in particular that the two weight inequality

$$\|R^{\alpha,n}(f\sigma)\|_{L^2(\omega)} \lesssim \|f\|_{L^2(\sigma)}$$

for the vector of Riesz transforms $R^{\alpha,n}$ in $\mathbb{R}^n$ (with $0 \leq \alpha < n$) holds if and only if the $A_2^p$ conditions with holes hold, the punctured $A_2^{p,\text{punct}}$ conditions hold, the quasicube testing conditions hold, and the quasiweak boundedness property holds. Here, a quasicube is a globally biLipschitz image of a usual cube. Precise definitions of all terms used here are given in the next section. It is not known at the time of this writing whether or not these or any other energy conditions are necessary for any vector $T^{\alpha,n}$ of fractional singular integrals in $\mathbb{R}^n$ with $n \geq 2$, apart from the trivial case of positive operators. In particular, there are no known counterexamples. We also showed in [SaShUr2] and [SaShUr3] that the technique of reversing energy, typically used to prove energy conditions, fails spectacularly in higher dimension (and we thank M. Lacey for showing us this failure for the
Cauchy transform with the circle measure). See also the counterexamples for the fractional Riesz transforms in [LaWi2].

The purpose of this paper is to show that if \( \sigma \) and \( \omega \) are locally finite positive Borel measures (possibly having common point masses), and at least one of the two measures \( \sigma \) and \( \omega \) is supported on a line in \( \mathbb{R}^n \), or on a regular \( C^{1,\delta} \) curve in \( \mathbb{R}^n \), then the energy conditions are indeed necessary for boundedness of the fractional Riesz transform \( R^{\alpha,n} \), and hence that a \( T_1 \) theorem holds for \( R^{\alpha,n} \) (see Theorem 8 below). Just after the first version of this paper appeared on the arXiv (see v.1 of [SaShUr6]), M. Lacey and B. Wick [LaWi] independently posted a similar result for the special case of the Cauchy transform in the plane where one measure is supported on a line or a circle, and the five authors have combined on the paper [LaSaShUrWi] in this setting.

The vector of \( \alpha \)-fractional Riesz transforms is given by

\[
R^{\alpha,n} = \{ R^{\alpha,n}_\ell : 1 \leq \ell \leq n \}, \quad 0 \leq \alpha < n,
\]

where the component Riesz transforms \( R^{\alpha,n}_\ell \) are the convolution fractional singular integrals \( R^{\alpha,n}_\ell f \equiv K^{\alpha,n}_\ell \ast f \) with odd kernel defined by

\[
K^{\alpha,n}_\ell (w) \equiv c^{\alpha,n}_\ell \frac{w^\ell}{|w|^{n+1-\alpha}}, \quad w = (w^1, w^2, \ldots, w^n).
\]

Finally, we remark that the \( T_1 \) theorem under this geometric condition has application to the weighted discrete Hilbert transform \( H_{(\Gamma,\nu)} \) when the sequence \( \Gamma \) is supported on an appropriate \( C^{1,\delta} \) curve in the complex plane. See [BeMeSe] where \( H_{(\Gamma,\nu)} \) is essentially the Cauchy transform with \( n = 2 \) and \( \alpha = 1 \).

We now recall a special case of our main two weight theorem from [SaShUr7] which plays a key role here—see also [SaShUr] and [SaShUr4] for earlier versions. Let \( Q^n_{\text{par}} \) denote the collection of all cubes in \( \mathbb{R}^n \) with sides parallel to the coordinate axes, and denote by \( D^n \subset Q^n_{\text{par}} \) a dyadic grid in \( \mathbb{R}^n \). The side conditions \( A^{\alpha}_2, A^{\alpha,\text{dual}}_2, A^{\alpha,\text{punct}}_2, A^{\alpha,\text{punct,\dual}}_2, E_\alpha \) and \( c^{\alpha,\text{dual}}_2 \) depend only on the measure pair \( (\sigma, \omega) \), while the necessary conditions \( \mathfrak{S}_{R^{\alpha,n}}^{\text{dual}}, \mathfrak{T}_{R^{\alpha,n}}^{\text{dual}} \) and \( \mathfrak{W}B^p_{R^{\alpha,n}} \) depend on the measure pair \( (\sigma, \omega) \) as well as the singular operator \( R^{\alpha,n}_2 \). These conditions will be explained below in the more general setting of quasicubes; see Definitions 22 and 24 for \( A^{\alpha}_2, A^{\alpha,\text{dual}}_2 \) and \( E_\alpha, c^{\alpha,\text{dual}}_2 \), and see displays (2.3) and (2.4) and (2.5) for \( A^{\alpha,\text{punct}}_2, A^{\alpha,\text{punct,\dual}}_2 \) and \( \mathfrak{S}_{R^{\alpha,n}}^{\text{dual}}, \mathfrak{T}_{R^{\alpha,n}}^{\text{dual}} \) and \( \mathfrak{W}B^p_{R^{\alpha,n}} \). For convenience in notation, we use Fraktur font for \( \mathcal{A} \) to denote

\[
\mathfrak{A}^{\alpha}_2 = A^{\alpha}_2 + A^{\alpha,\text{dual}}_2 + A^{\alpha,\text{punct}}_2 + A^{\alpha,\text{punct,\dual}}_2,
\]

or when the measure pair is important,

\[
\mathfrak{A}^{\alpha}_2(\sigma, \omega) = A^{\alpha}_2(\sigma, \omega) + A^{\alpha,\text{dual}}_2(\sigma, \omega) + A^{\alpha,\text{punct}}_2(\sigma, \omega) + A^{\alpha,\text{punct,\dual}}_2(\sigma, \omega).
\]
Notation 3. In order to avoid confusion with the use of $\ast$ for pullbacks and pushforwards of maps, we will use the superscript dual in place of $\ast$ to denote ‘dual conditions’ throughout this paper.

Theorem 4. Suppose that $R^{\alpha,n}_R$ is the vector of $\alpha$-fractional Riesz transforms in $\mathbb{R}^n$, and that $\omega$ and $\sigma$ are positive locally finite Borel measures on $\mathbb{R}^n$ (possibly having common point masses). Set $R^{\alpha,n}_Rf = R^{\alpha,n}_R(f\sigma)$ for any smooth truncation of $R^{\alpha,n}_R$. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be a globally biLipschitz map.

(1) Suppose $0 \leq \alpha < n$ and that $\gamma \geq 2$ is given. Then the vector Riesz transform $R^{\alpha,n}_R$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.,

$$\|R^{\alpha,n}_Rf\|_{L^2(\omega)} \leq N_{R^{\alpha,n}_R} \|f\|_{L^2(\sigma)},$$

uniformly in smooth truncations of $R^{\alpha,n}_R$, and moreover,

$$N_{R^{\alpha,n}_R} \leq C(\sqrt{\mathcal{A}_2^{\alpha} + \mathcal{E}_\alpha + \mathcal{E}_\alpha^{\text{dual}} + \mathcal{I}_{R^{\alpha,n}_R} + \mathcal{I}_{R^{\alpha,n}_R}^{\text{dual}} + \mathcal{WBP}_{R^{\alpha,n}_R}},$$

provided that the two dual $A^2_\alpha$ conditions with holes hold, the punctured dual $A^{\text{punct}}_2$ conditions hold, and the two dual quasicube testing conditions for $R^{\alpha,n}_R$ hold, the quasiweak boundedness property for $R^{\alpha,n}_R$ holds for a sufficiently large constant $C$ depending on the goodness parameter $r$, and provided that the two dual quasienergy conditions $\mathcal{E}_\alpha + \mathcal{E}_\alpha^{\text{dual}} < \infty$ hold uniformly over all dyadic grids $\mathcal{D}^n$, and where the goodness parameters $r$ and $\varepsilon$ implicit in the definition of $M_{r,\text{deep}}$ below are fixed sufficiently large and small respectively depending on $n$, $\alpha$ and $\gamma$. Here $N_{R^{\alpha,n}_R} = N_{R^{\alpha,n}_R}(\sigma, \omega)$ is the least constant in (1.3).

(2) Conversely, suppose $0 \leq \alpha < n$ and that the Riesz transform vector $R^{\alpha,n}_R$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$,

$$\|R^{\alpha,n}_Rf\|_{L^2(\omega)} \leq N_{R^{\alpha,n}_R} \|f\|_{L^2(\sigma)},$$

Then the testing conditions and weak boundedness property hold for $R^{\alpha,n}_R$, the fractional $A^2_\alpha$ conditions with holes hold, and the punctured dual $A^{\text{punct}}_2$ conditions hold, and moreover,

$$\sqrt{\mathcal{A}_2^{\alpha} + \mathcal{I}_{R^{\alpha,n}_R} + \mathcal{I}_{R^{\alpha,n}_R}^{\text{dual}} + \mathcal{WBP}_{R^{\alpha,n}_R}} \leq C N_{R^{\alpha,n}_R}.$$

Problem 5. As mentioned above, the energy conditions are not in general necessary for boundedness of singular integral operators ([SaShUr11] and [Saw4]), and it is an open question whether or not some weaker version of the energy conditions might be necessary and sufficient for boundedness of $R^{\alpha,n}_R$ in the presence of testing and Muckenhoupt conditions. See also [SaShUr3] for a failure of energy reversal in higher dimensions—such an energy reversal was used in dimension $n = 1$ to prove the necessity of the energy condition for the Hilbert transform.
Remark 6. In [LaWi2], M. Lacey and B. Wick use the NTV technique of surgery to show that an expectation over grids of an analogue of the weak boundedness property for the Riesz transform vector $R^{a,n}$ is controlled by the $A^a_2$ and cube testing conditions, together with a small multiple of the operator norm. They then obtain a $T1$ theorem with a side condition of uniformly full dimensional measures, using independent grids corresponding to each measure, resulting in an elimination of the weak boundedness property as a condition. See also the subsequent paper [SaShUr10] for an elimination of the weak boundedness property in Theorem 4 above. In any event, the weak boundedness property is always necessary for the norm inequality, and as such can be viewed as a weak close cousin of the testing conditions.

The main result of this paper is the $T1$ theorem for a measure supported on a regular $C^{1,\delta}$ curve. We point out that the cubes occurring in the testing conditions in the following theorem are of course restricted to those that intersect the curve, otherwise the integrals vanish, and include not only those in $Q^n_{\text{par}}$ with sides parallel to the axes, but also those in $Q^n$ consisting of all rotations of the cubes in $Q^n_{\text{par}}$:

\begin{align}
\|T_{\sigma}^{\alpha,n}\|_{L^2(\sigma)} &\equiv \sup_{Q \in \Omega^n} \frac{1}{|Q|} \int_Q |R^{a,n}(1_Q \sigma)|^2 \omega < \infty, \\
(\|T^{\text{dual}}_{\sigma}^{\alpha,n}\|_{L^2(\omega)})^2 &\equiv \sup_{Q \in \Omega^n} \frac{1}{|Q|} \int_Q |(R^{a,n})^{\text{dual}}(1_Q \omega)|^2 \sigma < \infty.
\end{align}

(1.5)

In the special case considered in [LaSaShUrWi] of the Cauchy transform in the plane with $\omega$ supported on the unit circle $\mathbb{T}$ or the real line $\mathbb{R}$, the testing is taken over the smaller collection of all Carleson squares.

We consider regular $C^{1,\delta}$ curves in $\mathbb{R}^n$ defined as follows.

Definition 7. Suppose $\delta > 0$, $I = [a, b]$ is a closed interval on the real line with $-\infty < a < b < \infty$, and that $\Phi : I \rightarrow \mathbb{R}^n$ is a $C^{1,\delta}$ curve parameterized by arc length. The curve is one-to-one with the possible exception that $\Phi(a) = \Phi(b)$. We refer to any curve as above as a regular $C^{1,\delta}$ curve.

Theorem 8. Let $0 \leq \alpha < n$ and suppose $\Phi$ is a regular $C^{1,\delta}$ curve. Suppose further that

1. $\sigma$ and $\omega$ are positive locally finite Borel measures on $\mathbb{R}^n$ (possibly having common point masses), and $\omega$ is supported in $\mathcal{L} \equiv \text{range} \Phi$, and
2. $R^{a,n}$ is the vector of $\alpha$-fractional Riesz transforms in $\mathbb{R}^n$, and $R^{a,n}_\sigma f = R^{a,n}_\sigma(f \sigma)$ for any smooth truncation of $R^{a,n}$.

Then $R^{a,n}_\sigma$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.,

$$
\|R^{a,n}_\sigma f\|_{L^2(\omega)} \leq N^{\alpha,n}_{R^{a,n}} \|f\|_{L^2(\sigma)},
$$

where

$$
N^{\alpha,n}_{R^{a,n}} \approx \frac{\alpha}{\delta} \left( \frac{|I|}{\delta} \right) \frac{1}{\alpha^{\alpha/2}} \left( \frac{|I|}{\delta} \right)^{\alpha/2}.
$$

In the special case considered in [LaSaShUrWi] of the Cauchy transform in the plane with $\omega$ supported on the unit circle $\mathbb{T}$ or the real line $\mathbb{R}$, the testing is taken over the smaller collection of all Carleson squares.
uniformly in smooth truncations of $R^{a,n}$, if and only if the two dual $A_2^a$ conditions with holes hold, the punctured dual $A_2^{a\text{ punct}}$ conditions hold, and the two dual cube testing conditions (1.5) for $R^{a,n}$ hold. Moreover, we have the equivalence
\[ \mathcal{N}_{R^{a,n}} \approx \sqrt{A_2^a + T_{R^{a,n}}} + T_{\text{dual}} R^{a,n}. \]

1.3 Techniques. The remainder of the introduction is devoted to giving an overview of the techniques and arguments needed to obtain Theorem 8 from Theorem 4. For this we need $\Omega$-quasicubes and conformal $\alpha$-fractional Riesz transforms $R_{\psi}^{a,n}$ where $\Omega$ is a globally biLipschitz map and $\Psi$ is a $C^{1,\delta}$ diffeomorphism of $\mathbb{R}^n$. We now describe these issues in more detail. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be a globally biLipschitz map as defined in Definition 20 below, and refer to the images $\Omega Q$ of cubes in $Q^n$ under the map $\Omega$ as $\Omega$-quasicubes or simply quasicubes. These $\Omega$-quasicubes will often be used in place of cubes in the testing conditions, energy conditions and weak boundedness property, and we will use $\Omega Q^n$ as a superscript to indicate this. For example, the quasitesting analogue of the usual testing conditions (1.5) is
\begin{align*}
(\mathcal{T}_{R^{a,n}})^2 &\equiv \sup_{Q \in \Omega Q^n} \frac{1}{|Q|_\sigma} \int_Q |R^{a,n}(1_Q \sigma)|^2 \omega < \infty, \\
(\mathcal{T}_{R^{a,n}}^{\text{dual}})^2 &\equiv \sup_{Q \in \Omega Q^n} \frac{1}{|Q|_\sigma} \int_Q |(R^{a,n})^{\text{dual}}(1_Q \omega)|^2 \sigma < \infty.
\end{align*}
We alert the reader to the fact that different collections of quasicubes will be considered in the course of proving our theorem. The definitions of these terms, and the remaining terms used below, will be given precisely in the next section. When the superscript $\Omega Q^n$ is omitted, it is understood that the quasicubes are the usual cubes $Q^n$.

The next result shows that the quasienergy conditions are in fact necessary for boundedness of the Riesz transform vector $\mathbf{R}^{a,n}$ when one of the measures is supported on a line. In that case, the quasienergy conditions are even implied by the Muckenhoupt $A_2^a$ conditions with holes and the quasitesting conditions. Moreover, the backward tripled quasitesting condition and the quasiweak boundedness property are implied by the Muckenhoupt with holes and quasitesting conditions as well, but provided the quasicubes come from a $C^1$ diffeomorphism and are rotated in an appropriate way.

Finally, in order to obtain the $T1$ theorem when one measure is supported on a curve, we will need to generalize the fractional Riesz transforms $\mathbf{R}^{a,n}$ that we can consider in this theorem. Consider $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ given by
\[ \Psi(x) = (x^1, x^2 - \psi^2(x^1), x^3 - \psi^3(x^1), \ldots, x^n - \psi^n(x^1)) = x - (0, \psi(x_1)), \]
where \( x = (x^1, x') \) and

\[
\psi(t) = (\psi^2(t), \psi^3(t), \ldots, \psi^n(t)) \in \mathbb{R}^{n-1}
\]
is a \( C^{1,\delta} \) function \( \psi : \mathbb{R} \to \mathbb{R}^{n-1} \). Let \( K_{\psi}^{a,n}(x, y) \) denote the vector Riesz kernel and define

\[
K_{\psi}^{a,n}(x, y) = \frac{|y - x|^{n+1-\alpha}}{|\psi(y) - \psi(x)|^{n+1-\alpha}} K^{a,n}(x, y) = c_{a,n} \frac{y - x}{|\psi(y) - \psi(x)|^{n+1-\alpha}}.
\]

**Definition 9.** We refer to the operator \( R_{\psi}^{a,n} \) with kernel \( K_{\psi}^{a,n} \) as a *conformal \( \alpha \)-fractional Riesz transform*. We also define the factor

\[
\Gamma_{\psi}(x, y) \equiv \frac{|y - x|^{n+1-\alpha}}{|\psi(y) - \psi(x)|^{n+1-\alpha}}
\]
to be the conformal factor associated with \( \psi \) and \( K_{\psi}^{a,n} \).

**Notation 10.** We emphasize that the \( C^{1,\delta} \) diffeomorphism \( \psi \) that appears in the definition of the conformal \( \alpha \)-fractional Riesz transform \( R_{\psi}^{a,n} \) need not have any relation to the biLipschitz map \( \Omega : \mathbb{R}^n \to \mathbb{R}^n \) that is used to define the quasicubes under consideration. On the other hand, we will have reason to consider \( \psi \)-quasicubes as well in connection with changes of variable.

We use the tangent line approximations to \( R_{\psi}^{a,n} \) having kernels \( K_{\psi}^{a,n}(x, y) \rho_{R, \eta}^{a} \) where \( \rho_{R, \eta}^{a} \) is defined in the next section. These approximations are dictated by our use of energy conditions, which are in turn applied to first order Taylor expansions of the kernel truncations in order to obtain the Quasienergy Lemma 25 below. It is shown in [SaShUr7] (see also [LaSaShUr2] for the one-dimensional case without holes) that once one has obtained the theorem for tangent line approximations, these approximations can be replaced with any reasonable notion of truncation, including the usual cutoff truncations.

We now introduce a condition on \( \Omega \)-quasicubes that plays a role in deriving the necessity of the tripled testing and weak boundedness conditions. See Lemma 27 below for the relevant consequences of this condition. We begin with a collection of ‘good’ rotations \( R \) that take the standard basis \( \{e_i\}_{i=1}^n \) to a basis \( \{Re_j\}_{j=1}^n \) in which no unit vector \( Re_j \) is too close to any unit vector \( e_i \).

Let \( \mathcal{R}^n \) denote the group of rotations in \( \mathbb{R}^n \). For \( e \in S^{n-1} \) and \( 0 < \eta < 1 \) let

\[
F_{e, \eta} = \{ R \in \mathcal{R}^n : |\langle Re, e_k \rangle| \leq \eta \text{ for } 1 \leq k \leq n \}.
\]

Note that the condition \( F_{e, \eta} \neq \emptyset \) is independent of the unit vector \( e \), and depends only on \( \eta \), by transitivity of rotations. Fix \( \eta = \eta_n \in (0, 1) \) so that \( F_{e, \eta} \neq \emptyset \) for all \( e \in S^{n-1} \) (this requires \( \eta_n \geq \frac{1}{\sqrt{n}} \)).
Definition 11. Let $L$ be a line in $\mathbb{R}^n$. A $C^1$ diffeomorphism $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is $L$-transverse if

$$\|D\Omega^{-1} - R\|_{\infty} < \frac{1 - \eta}{4}$$

for some $R \in F_{e_L, \eta}$ where $e_L$ is a unit vector in the direction of $L$.

Theorem 12. Fix a collection of $\Omega$-quasicubes. Let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$ (possibly having common point masses). Suppose that $R^{a,n}_\psi$ is a conformal $a$-fractional Riesz transform with $0 \leq a < n$, and where $\Psi$ is a $C^{1,\delta}$ diffeomorphism with $\Psi(x) = x - (0, \psi(x_1))$ where

$$\|D\psi\|_{\infty} < \frac{1}{8n} \left(1 - \frac{a}{n}\right).$$

Impose the tangent line truncations for $R^{a,n}_\psi$ in the $\Omega$-quasitesting conditions. If the measure $\omega$ is supported on a line $L$, then

$$E_{\alpha}^{\Omega, \Omega^n} \lesssim \sqrt{A^\alpha_2 + \Sigma_{R^{a,n}_\psi}^{\Omega^n}} \quad \text{and} \quad E_{\alpha}^{\Omega, \Omega^n, \text{dual}} \lesssim \sqrt{A^\alpha_2, \text{dual} + \Sigma_{R^{a,n}_\psi}^{\Omega^n, \text{dual}}}.$$

If in addition $\Omega$ is a $C^1$ diffeomorphism and $L$-transverse, then

$$W_{\text{par}}^{\Omega, \Omega^n} \lesssim \Sigma_{R^{a,n}_\psi}^{\Omega^\text{par}, \text{dual}} \lesssim \Sigma_{R^{a,n}_\psi}^{\Omega^\text{par}, \text{dual}} + \sqrt{A^\alpha_2} + \sqrt{A^\alpha_2, \text{dual}}.$$

Remark 13. We restrict Theorem 12 to conformal Riesz transforms $R^{a,n}_\psi$ in order to exploit the special property that for $j \geq 2$, the scalar transforms $R^{a,n}_j$ and $(R^{a,n}_\psi)$ behave like a Poisson operator when acting on a measure supported on the $x_1$-axis. This property is not shared by higher order Riesz transforms, such as the Beurling transform in the plane, and this accounts for our failure to treat such singular integrals at this time. The restriction to conformal Riesz transforms $R^{a,n}_\psi$ is dictated by the reversal of energy that is possible for these special transforms when the phase of the singular integral is $y - x$ and one of the measures is supported on a line.

Since the conformal factor $\Gamma(x, y)$ in Definition 9 satisfies the estimates

$$\frac{1}{C} \leq \Gamma(x, y) \leq C,$$

$$|\nabla \Gamma(x, y)| \leq C|x - y|^{-1},$$

$$|\nabla \Gamma(x, y) - \nabla \Gamma(x', y)| \leq C\left(\frac{|x - x'|}{|x - y|}\right)\delta |x - y|^{-1}, \quad \frac{|x - x'|}{|x - y|} \leq \frac{1}{2},$$

$$|\nabla \Gamma(x, y) - \nabla \Gamma(x, y')| \leq C\left(\frac{|y - y'|}{|x - y|}\right)\delta |x - y|^{-1}, \quad \frac{|y - y'|}{|x - y|} \leq \frac{1}{2},$$

$$|\nabla \Gamma(x, y) - \nabla \Gamma(x', y')| \leq C\left(\frac{|x - x'|}{|x - y|}\right)\delta |x - y|^{-1}, \quad \frac{|x - x'|}{|x - y|} \leq \frac{1}{2},$$

$$|\nabla \Gamma(x, y) - \nabla \Gamma(x, y')| \leq C\left(\frac{|y - y'|}{|x - y|}\right)\delta |x - y|^{-1}, \quad \frac{|y - y'|}{|x - y|} \leq \frac{1}{2}.$$
it is easy to see from the product rule that the conformal fractional Riesz transforms are standard fractional singular integrals in the sense used in [SaShUr7]. Since they are also strongly elliptic as in [SaShUr7], it follows from the main theorem in [SaShUr7] that

**Conclusion 14.** Theorem 4 above holds with $R_{\Psi}^{a,n}$ in place of $R^{a,n}$ provided $\Psi$ is a $C^{1,\delta}$ diffeomorphism.

If we combine Theorem 12 with this extension of Theorem 4, we immediately obtain the following $T1$ theorem as a corollary.

**Remark 15.** The following theorem generalizes the $T1$ theorem for the Hilbert transform ([Lac], [LaSaShUr2] and [Hyt2]) both in that the supports of measures are more general, and in that the kernels treated are more general. See also related work in the references given at the end of the paper.

**Theorem 16.** Fix a line $L$ and a collection of $\Omega$-quasicubes and suppose that $\Omega$ is a $C^{1}$ diffeomorphism and $L$-transverse. Let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^{n}$ (possibly having common point masses). Suppose that $R_{\Psi}^{a,n}$ is a conformal fractional Riesz transform with $0 \leq \alpha < n$, where $\Psi$ is a $C^{1,\delta}$ diffeomorphism given by $\Psi(x) = x - (0, \psi(x_{1}))$ where $\psi$ satisfies (1.7), i.e.,

$$||D\psi||_{\infty} < \frac{1}{8n} \left(1 - \frac{\alpha}{n}\right).$$

Set $(R_{\Psi}^{a,n})_{\sigma} = R_{\Psi}^{a,n}(\sigma)$ for any smooth truncation of $R_{\Psi}^{a,n}$. If at least one of the measures $\sigma$ and $\omega$ is supported on the line $L$, then the operator norm $\mathcal{N}_{R_{\Psi}^{a,n}}$ of $(R_{\Psi}^{a,n})_{\sigma}$ as an operator from $L^{2}(\sigma)$ to $L^{2}(\omega)$, uniformly in smooth truncations, satisfies

$$\mathcal{N}_{R_{\Psi}^{a,n}} \approx C_{\alpha} \left(\sqrt{\mathcal{Q}_{\alpha}^{n} + \mathcal{Q}_{R_{\Psi}^{a,n}}^{n} + \mathcal{Q}_{R_{\Psi}^{a,n,\text{dual}}}^{n}}\right).$$

Our extension of Theorem 16 to the case when one measure is compactly supported on a $C^{1,\delta}$ curve $L$ presented as a graph requires additional work. More precisely, we suppose that $L$ is presented as the graph of a $C^{1,\delta}$ function $\psi : \mathbb{R} \to \mathbb{R}^{n-1}$ given by

$$\psi(t) = (\psi^{2}(t), \psi^{3}(t), \ldots, \psi^{n}(t)) \in \mathbb{R}^{n-1}, \quad t \in \mathbb{R}.$$ 

Define $\Psi : \mathbb{R}^{n} \to \mathbb{R}^{n}$ by

$$\Psi(x) = (x^{1}, x^{2} - \psi^{2}(x^{1}), x^{3} - \psi^{3}(x^{1}), \ldots, x^{n} - \psi^{n}(x^{1})) = x - (0, \psi(x^{1})),$$

where $x = (x^{1}, x')$. Then $\Psi$ is globally invertible with inverse map

$$\Psi^{-1}(\zeta) = (\zeta^{1}, \zeta^{2} + \psi^{2}(\zeta^{1}), \zeta^{3} + \psi^{3}(\zeta^{1}), \ldots, \zeta^{n} + \psi^{n}(\zeta^{1})) = \zeta + (0, \psi(\zeta^{1})).$$
Both $\Psi$ and its inverse $\Psi^{-1}$ are $C^{1,\delta}$ maps, and $\Psi_{\mid L}$ is a $C^{1,\delta}$ diffeomorphism from the curve $L$ to the $x_1$-axis. Set $\Psi, Q^n = (\Psi^{-1})^* Q^n = \{ \Psi Q : Q \in Q^n \}$. The images $\Psi Q$ of cubes $Q$ under the map $\Psi$ are $\Psi$-quasicubes.

The next theorem is a preliminary version of the main Theorem 8 that requires only the change of variable estimates in Propositions 29 and 30 below. The ‘defects’ in this preliminary version are that the quasitesting conditions are related to the map $\Psi$ defining the curve $L$, and that the smallness condition (1.7) is imposed on the derivative of $\psi$.

**Theorem 17.** Let $n \geq 2$ and $0 \leq \alpha < n$. Suppose that $L$ is a $C^{1,\delta}$ curve in $\mathbb{R}^n$ presented as the graph of a $C^{1,\delta}$ function $\psi : \mathbb{R} \rightarrow \mathbb{R}^{n-1}$ as above, and assume that (1.7) holds, i.e.,

$$\|D\psi\|_{\infty} < \frac{1}{8n} \left(1 - \frac{\alpha}{n}\right).$$

Let $\omega$ and $\sigma$ be positive Borel measures (possibly having common point masses), and assume that $\omega$ is compactly supported in $L$. Let $\Psi$ be associated to $\psi$ as above. Finally, set $R^n = RQ^n_{\text{par}}$ where $R$ is a rotation that is $L$-transverse when $L$ is the $x_1$-axis. Then the $\alpha$-fractional Riesz transform $R_{\alpha}^n$ is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the Muckenhoupt conditions hold,

$$A_2^\alpha + A_2^{\alpha,\text{dual}} + A_2^{\alpha,\text{punct}} + A_2^{\alpha,\text{punct,dual}} < \infty,$$

and the quasitesting conditions hold,

$$\Sigma_{R^n_{\psi}}^{\Psi R^n_{\psi}} + \Sigma_{R^n_{\psi}}^{\Psi R^n_{\psi,\text{dual}}} < \infty,$$

where $\Sigma_{R^n_{\psi}}^{\Psi R^n_{\psi}}$ and $\Sigma_{R^n_{\psi}}^{\Psi R^n_{\psi,\text{dual}}}$ are the best constants in

$$\int_{\Psi Q} |R_{\alpha,n}^n(1_{\Psi Q}\sigma)|^2 d\omega \leq (\Sigma_{R^n_{\psi}}^{\Psi R^n_{\psi}})^2 |\Psi Q|_\sigma,$$

$$\int_{\Psi Q} |R_{\alpha,n,\text{dual}}(1_{\Psi Q}\omega)|^2 d\sigma \leq (\Sigma_{R^n_{\psi}}^{\Psi R^n_{\psi,\text{dual}}})^2 |\Psi Q|_\omega,$$

for all cubes $Q \in R^n = RQ^n_{\text{par}}$.

Moreover, we have the equivalence

$$\mathcal{N}_{R^n_{\psi}}(\sigma, \omega) \approx \sqrt{\mathcal{A}_2^\alpha(\sigma, \omega) + \Sigma_{R^n_{\psi}}^{\Psi R^n_{\psi}}(\sigma, \omega) + \Sigma_{R^n_{\psi}}^{\Psi R^n_{\psi,\text{dual}}}(\sigma, \omega)}.$$

The bound on $\|D\psi\|_{\infty}$ can be relaxed, but we will not pursue this here. To obtain Theorem 8, we instead remove the Lipschitz assumption by cutting the support $L$ of $\omega$ into sufficiently small pieces $L_i$ where the oscillation of the tangents to $L_i$ is small. Here the necessity of the tripled testing condition $\Sigma_{R^n_{\psi}}^{\Omega Q^n_{\psi,\text{triple,dual}}}$ in Theorem 12 plays a key role in permitting our testing conditions to be taken with
respect to the entire measure $\omega$, rather than with respect to the corresponding pieces $1 L_i \omega$. Then the restriction to quasitesting conditions is removed using the fact that Theorem 4 holds for conformal Riesz transforms with general quasicubes; see Conclusion 14.

Finally, we mention another direction in which Theorem 16 can be generalized, namely to the setting where $\sigma$ and $\omega$ are locally finite positive Borel measures supported on orthogonal subspaces intersecting in a line. We refer the reader to the arXiv version [SaShUr6] of this paper for a discussion of the proof of the following theorem.

**Theorem 18.** Let

$$S = \{ (x_1, x', 0) \in \mathbb{R} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} : (x_1, x') \in \mathbb{R} \times \mathbb{R}^{k_1} \},$$

$$W = \{ (x_1, 0, x'') \in \mathbb{R} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} : (x_1, x'') \in \mathbb{R} \times \mathbb{R}^{k_2} \},$$

$$L = S \cap W = \{ (x_1, 0, 0) \in \mathbb{R} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} : x_1 \in \mathbb{R} \},$$

be $(k_1+1)$-, $(k_2+1)$- and 1-dimensional subspaces respectively of $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2}$. Let $\sigma$ and $\omega$ be locally finite positive Borel measures supported on $S$ and $W$ respectively (possibly having common point masses in the intersection $L$ of their supports). Suppose that $\Omega$ is a $C^1$ diffeomorphism and $L$-transverse. Suppose also that $R^{a,n}_\Psi$ is a conformal fractional Riesz transform with $0 \leq \alpha < n$, where $\Psi$ is a $C^{1,\delta}$ diffeomorphism given by $\Psi(x) = x - (0, \psi(x_1))$ where $\psi$ satisfies (1.7). Set $(R^{a,n}_\Psi)_\sigma f = R^{a,n}_\Psi(f\sigma)$ for any smooth truncation of $R^{a,n}_\Psi$. Then the operator norm $\mathcal{N}_{R^{a,n}_\Psi}$ of $(R^{a,n}_\Psi)_\sigma$ as an operator from $L^2(\sigma)$ to $L^2(\omega)$, uniformly in smooth truncations, satisfies

$$\mathcal{N}_{R^{a,n}_\Psi} \approx C_\alpha \left( \sqrt{\mathcal{Q}^{\Omega}_2} + \Sigma^{\Omega}_{R^{a,n}_\Psi} + \tau^{\Omega,\text{dual}}_{R^{a,n}_\Psi} \right).$$

**Remark 19.** The above theorem generalizes Theorem 16 by permitting the support of the measure $\omega$ to extend into an orthogonal subspace in a higher dimension. There is an analogous theorem that generalizes Theorem 8 in this way, but we will not pursue this here.

### 2 Definitions

The $\alpha$-fractional Riesz vector $R^{a,n} = \{ R^{a,n}_\ell : 1 \leq \ell \leq n \}$ has as components the Riesz transforms $R^{n,\alpha}_\ell$ with odd kernel

$$K^{a,n}_\ell(w) = \frac{\Omega_\ell(w)}{|w|^{n-\alpha}}.$$
where
\[ \Omega_\ell(w) = c_{n,\alpha} \frac{w^\ell}{|w|} \]
is homogeneous of degree 0. The tangent line truncation of the Riesz transform \( R_{\alpha,\ell}^{n,a} \) has kernel \( \Omega_\ell(w) \rho_{\eta,R}^{a}(|w|) \) where \( \rho_{\eta,R}^{a} \) is continuously differentiable on an interval \((0, S)\) with \( 0 < \eta < R < S \), and where \( \rho_{\eta,R}^{a}(r) = r^{a-n} \) if \( \eta \leq r \leq R \), and has constant derivative on both \((0, \eta)\) and \((R, S)\) where \( \rho_{\eta,R}^{a}(S) = 0 \). As shown in [SaShUr7] (see [LaSaShUr2] for the one-dimensional case without holes), boundedness of \( R_{\alpha,\ell}^{n,a} \) with one set of appropriate truncations together with the offset \( A_{2}^{a} \) condition (see (2.2) below) is equivalent to boundedness of \( R_{\alpha,\ell}^{n,a} \) with all truncations. In particular, this includes the smooth truncations with kernels
\[ \varphi_{\eta,R}(|w|) K_{\ell,n}^{n,a}(w) = \varphi_{\eta,R}(|w|) \frac{\Omega_\ell(w)}{|w|^{a-n}} \]
where \( \varphi_{\eta,R} \) is infinitely differentiable and compactly supported on the interval \((0, \infty)\) with \( 0 < \eta < R < \infty \), and where \( \varphi_{\eta,R}(r) = 1 \) if \( \eta \leq r \leq R \).

### 2.1 Quasicubes

Our general notion of quasicube will be derived from the following definition.

**Definition 20.** We say that a map \( \Omega : \mathbb{R}^n \to \mathbb{R}^n \) is a globally biLipschitz map if
\[ \|\Omega\|_{Lip} \equiv \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{\|\Omega(x) - \Omega(y)\|}{\|x - y\|} < \infty, \]
and \( \|\Omega^{-1}\|_{Lip} < \infty \). We say that a map \( \Psi : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^{1,\delta} \) diffeomorphism if
\[ \|\Psi\|_{C^{1,\delta}} \equiv \sup_{x \in \mathbb{R}^n} \|\nabla \Psi(x)\| + \sup_{x,y \in \mathbb{R}^n, x \neq y} \frac{\|\nabla \Psi(x) - \nabla \Psi(y)\|}{\|x - y\|^\delta} < \infty, \]
and \( \|\Psi^{-1}\|_{C^{1,\delta}} < \infty \). When \( \delta = 0 \), we write
\[ C^1 = C^{1,0} \quad \text{and} \quad \|\Psi\|_{C^1} \equiv \sup_{x \in \mathbb{R}^n} \|\nabla \Psi(x)\|. \]

Note that if \( \Omega \) is a globally biLipschitz map, then there are constants \( c, C > 0 \) such that
\[ c \leq J_{\Omega}(x) \equiv |\det D\Omega(x)| \leq C, \quad x \in \mathbb{R}^n. \]

Special cases of globally biLipschitz maps are given by \( C^{1,\delta} \) diffeomorphisms
\[ \Psi : \mathbb{R}^n \to \mathbb{R}^n, \]
and these include those used in the definition of conformal Riesz transforms above, and defined by

$$\Psi(x) = x - (0, \psi(x^1)),$$

where \( x = (x^1, x') \in \mathbb{R}^n \) and \( \psi : \mathbb{R} \to \mathbb{R}^{n-1} \) is a \( C^{1,\delta} \) function. We denote by \( Q^n \) the collection of all cubes in \( \mathbb{R}^n \), and by \( Q^n_{\text{par}} \) the subcollection of cubes in \( \mathbb{R}^n \) with sides parallel to the coordinate axes, and by \( D^n \) (contained in \( Q^n_{\text{par}} \)) a dyadic grid in \( \mathbb{R}^n \).

**Definition 21.** Suppose that \( \Omega : \mathbb{R}^n \to \mathbb{R}^n \) is a globally biLipschitz map.

1. If \( E \) is a measurable subset of \( \mathbb{R}^n \), we define

   $$\Omega E \equiv \{ \Omega(x) : x \in E \}$$

   to be the image of \( E \) under the homeomorphism \( \Omega \).

   (a) In the special case that \( E = Q \in Q^n \) is a cube in \( \mathbb{R}^n \), we will refer to \( \Omega Q \) as a quasicube (or \( \Omega \)-quasicube if \( \Omega \) is not clear from the context).

   (b) We define the center of the quasicube \( \Omega Q \) to be \( \Omega x_Q \) where \( x_Q \) is the center of \( Q \).

   (c) We define the side length \( \ell(\Omega Q) \) of the quasicube \( \Omega Q \) to be the sidelenth \( \ell(Q) \) of the cube \( Q \).

   (d) For \( r > 0 \) we define the ‘dilation’ \( r\Omega Q \) of a quasicube \( \Omega Q \) to be \( \Omega rQ \) where \( rQ \) is the usual ‘dilation’ of a cube in \( \mathbb{R}^n \) that is concentric with \( Q \) and having side length \( r\ell(\Omega Q) \).

2. If \( \mathcal{K} \) is a collection of cubes in \( \mathbb{R}^n \), we define

   $$\Omega \mathcal{K} \equiv \{ \Omega Q : Q \in \mathcal{K} \}$$

   to be the collection of quasicubes \( \Omega Q \) as \( Q \) ranges over \( \mathcal{K} \).

3. If \( \mathcal{F} \) is a grid of cubes in \( \mathbb{R}^n \), we define the inherited grid structure on \( \Omega \mathcal{F} \) by declaring that \( \Omega Q \) is a child of \( \Omega Q' \) in \( \Omega \mathcal{F} \) if \( Q \) is a child of \( Q' \) in the grid \( \mathcal{F} \).

Note that if \( \Omega Q \) is a quasicube, then

$$|\Omega Q|^{\frac{1}{n}} \approx |Q|^{\frac{1}{n}} = \ell(Q) = \ell(\Omega Q)$$

shows that the measure of \( \Omega Q \) is approximately its sidelenth to the power \( n \). Moreover, there is a positive constant \( R_{\text{big}} \) such that we have the comparability containments

$$Q + \Omega x_Q \subset R_{\text{big}} \Omega Q \quad \text{and} \quad \Omega Q \subset R_{\text{big}}(Q + \Omega x_Q).$$
2.2 The $A^a_2$ conditions. Recall that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map. Now let $\mu$ be a locally finite positive Borel measure on $\mathbb{R}^n$, and suppose $Q$ is a $\Omega$-quasicube in $\mathbb{R}^n$. Recall that $|Q|^{1/2} = \ell(Q)$ for a quasicube $Q$. The two $\alpha$-fractional Poisson integrals of $\mu$ on a quasicube $Q$ are given by

$$P^\alpha(Q, \mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{1/2}}{(|Q|^{1/2} + |x-x_Q|)^{n+1-\alpha}} d\mu(x),$$

$$P^\alpha(Q, \mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{1/2}}{(|Q|^{1/2} + |x-x_Q|^2)^{n-\alpha}} d\mu(x),$$

where we emphasize that $|x-x_Q|$ denotes Euclidean distance between $x$ and $x_Q$ and $|Q|$ denotes the Lebesgue measure of the quasicube $Q$. We refer to $P^\alpha$ as the standard Poisson integral and to $P^\alpha$ as the reproducing Poisson integral. Let $\sigma$ and $\omega$ be locally finite positive Borel measures on $\mathbb{R}^n$, possibly having common point masses, and suppose $0 \leq \alpha < n$.

We say that the pair $(K, K')$ in $Q^n \times Q^n$ are neighbours if $K$ and $K'$ live in a common dyadic grid and both $K \subset 3K' \setminus K'$ and $K' \subset 3K \setminus K$, and we denote by $N^n$ the set of pairs $(K, K')$ in $Q^n \times Q^n$ that are neighbours. Let $\Omega N^n \subset \Omega Q^n \times \Omega Q^n$ be the corresponding collection of neighbour pairs $(\Omega K, \Omega K')$ of quasicubes. Then we define the classical offset $A^a_2$ constants by

$$A^a_2(\sigma, \omega) \equiv \sup_{(Q, Q') \in \Omega N^n} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q'|_\omega}{|Q'|^{1-\frac{\alpha}{n}}}.$$  

Since the cubes in $\Omega^n_{\text{par}}$ are products of half open, half closed intervals $[a, b)$, the neighbouring quasicubes $(Q, Q') \in \Omega N^n$ are disjoint, and the common point masses of $\sigma$ and $\omega$ do not simultaneously appear in each factor.

We now define the one-tailed $A^a_2$ constant using $P^\alpha$. The energy constants $E_\alpha$ introduced in the next subsection will use the standard Poisson integral $P^\alpha$.

**Definition 22.** The one-sided constants $A^a_2$ and $A^{a,\text{dual}}_2$ for the weight pair $(\sigma, \omega)$ are given by

$$A^a_2(\sigma, \omega) \equiv \sup_{Q \in \Omega Q^a} P^\alpha(Q, 1_Q \sigma) \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} < \infty,$$

$$A^{a,\text{dual}}_2(\sigma, \omega) \equiv \sup_{Q \in \Omega Q^a} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}} P^\alpha(Q, 1_Q \omega) < \infty.$$  

Note that these definitions are the analogues of the corresponding conditions with ‘holes’ introduced by Hytönen [Hyt] in dimension $n = 1$—the supports of the measures $1_Q \sigma$ and $1_Q \omega$ in the definition of $A^a_2$ are disjoint, and so the common point masses of $\sigma$ and $\omega$ do not appear simultaneously in each factor.
2.2.1 Punctured Muckenhoupt conditions. Given an at most countable set
\[ \mathcal{P} = \{p_k\}_{k=1}^{\infty} \]
in \( \mathbb{R}^n \), a quasicube \( Q \in \Omega^Q \), and a positive locally finite Borel measure \( \mu \), define
\[ \mu(Q, \mathcal{P}) \equiv |Q|_\mu - \sup \{ \mu(p_k) : p_k \in Q \cap \mathcal{P} \}, \]
where we note that the sup above is achieved at some point \( p_k \) since \( \mu \) is locally finite. The quantity \( \mu(Q, \mathcal{P}) \) is simply the \( \tilde{\mu} \) measure of \( Q \) where \( \tilde{\mu} \) is the measure \( \mu \) with its largest point mass in \( Q \) removed. Given a locally finite measure pair \( (\sigma, \omega) \), let
\[ \mathcal{P}_{(\sigma, \omega)} = \{p_k\}_{k=1}^{\infty} \]
be the at most countable set of common point masses of \( \sigma \) and \( \omega \). Then as shown in [SaShUr7] (as pointed out by Hytönen [Hyt2], the one-dimensional case follows from the proof of Proposition 2.1 in [LaSaUr2]), the weighted norm inequality (1.1) implies finiteness of the following punctured Muckenhoupt conditions:
\[
A^{\alpha, \text{punct}}_2(\sigma, \omega) \equiv \sup_{Q \in \Omega^Q} \frac{\sigma(Q, \mathcal{P}_{(\sigma, \omega)})}{|Q|^{1-\frac{\alpha}{n}}} |Q|^{1-\frac{\alpha}{n}}, \\
A^{\alpha, \text{punct}, \text{dual}}_2(\sigma, \omega) \equiv \sup_{Q \in \Omega^Q} \frac{\omega(Q, \mathcal{P}_{(\sigma, \omega)})}{|Q|^{1-\frac{\alpha}{n}}} |Q|^{1-\frac{\alpha}{n}}.
\]

Finally, we point out that the intersection of these conditions, namely \( A^\alpha_2 + A^{\alpha, \text{dual}}_2 + A^{\alpha, \text{punct}}_2 + A^{\alpha, \text{punct, dual}}_2 < \infty \), is independent of the biLipschitz map \( \Omega \) as follows from taking \( \Psi = \Omega^{-1} \) in Proposition 29 below.

2.3 Quasicube testing and quasiweak boundedness property. The following ‘dual’ quasicube testing conditions are necessary for the boundedness of \( R^{a,n} \) from \( L^2(\sigma) \) to \( L^2(\omega) \):
\[
T^{a,n}_{R^a} \equiv \sup_{Q \in \Omega^Q} \frac{1}{|Q|_\sigma} \int_Q |R^{a,n}(1_Q \sigma)|^2 \omega < \infty, \\
(T^{\text{dual}}_{R^a})^2 \equiv \sup_{Q \in \Omega^Q} \frac{1}{|Q|_\omega} \int_Q |(R^{a,n})^{\text{dual}}(1_Q \omega)|^2 \sigma < \infty.
\]
Note that these conditions are required to hold uniformly over tangent line truncations of \( R^{a,n} \), and where again we point out that in the presence of the \( A^a_2 \) conditions, we can equivalently replace the tangent line truncations with any other admissible truncations.
The quasiweak boundedness property for $R^{α,n}$ is another necessary condition for (1.1) given by
\[
\left| \int_Q R^{α,n}(1_Q σ) dω \right| \leq WBP_{R^{α,n}} \sqrt{|Q| |Q'|_σ},
\]
for all dyadic quasicubes $Q, Q' ∈ Ω^D$ with $1 ≤ \frac{|Q|^{\frac{1}{n}}}{|Q'|^{\frac{1}{n}}} ≤ C$,
and either $Q ⊂ 3Q' \setminus Q'$ or $Q' ⊂ 3Q \setminus Q$,
and all dyadic quasigrids $Ω^D$.

2.4 Quasienergy conditions. Suppose $Ω : R^n → R^n$ is a $C^{1,δ}$ diffeomorphism. We begin by briefly recalling some of the notation used in [SaShUr7]. Given a dyadic $Ω$-quasicube $K ∈ D$ and a positive measure $μ$ we define the $Ω$-quasiHaar projection $P^K_μ ≡ ∑_{J ∈ D: J ⊂ K} △^μ_J$ where the projections $△^μ_J$ are the usual orthogonal projections onto the space of mean value zero functions that are constant on the children of $J$—see, e.g., [SaShUr7]. Now we recall the definition of a good dyadic quasicube—see [NTV4], [LaSaUr2] and [SaShUr] for more detail—and the definition of a quasicube that is deeply embedded in another quasicube. We say that a dyadic quasicube $J$ is $(r, ε)$-deeply embedded in a dyadic quasicube $K$, or simply $r$-deeply embedded in $K$, which we write as $J ⊂ r K$, when $J ⊂ K$ and both
\[
ℓ(J) ≤ 2^{−r} ℓ(K),
\]
\[
\text{quasidist}(J, \partial K) ≥ \frac{1}{2} ℓ(J)^ε ℓ(K)^{1−ε},
\]
where we define the quasidistance $\text{quasidist}(E, F)$ between two sets $E$ and $F$ to be the Euclidean distance $\text{dist}(Ω^{-1} E, Ω^{-1} F)$ between the preimages $Ω^{-1} E$ and $Ω^{-1} F$ of $E$ and $F$ under the map $Ω$, and where we recall that $ℓ(J) ≈ |J|^\frac{1}{n}$. A quasicube $I$ is a superquasicube of a quasicube $J$ if $I ⊂ J$.

Definition 23. Let $r ∈ N$ and $0 < ε < 1$. A dyadic quasicube $J$ is $(r, ε)$-good, or simply good, if for every dyadic superquasicube $I$, it is the case that either $J$ has side length at least $2^{−r}$ times that of $I$, or $J ⊂ r I$ is $(r, ε)$-deeply embedded in $I$.

The parameters $r, ε$ will be fixed sufficiently large and small respectively later on, and we denote the set of such good dyadic quasicubes by $Ω^D_{\text{good}}$.

We note that
\[
\|P^K_μ x\|_{L^2(μ)}^2 = \int_I \|x - E^K_μ x\|^2 dμ(x) = \int_I \left| x - \left( \frac{1}{|I|_μ} \int_I x dx \right) \right|^2 dμ(x), \quad x = (x_1, \ldots, x_n),
\]
where $P^\mu_I x$ is the orthogonal projection of the identity function $x : \mathbb{R}^n \to \mathbb{R}^n$ onto the vector-valued subspace of $\bigoplus_{k=1}^n L^2(\mu)$ consisting of functions supported in $I$ with $\mu$-mean value zero, and where $E^\mu_I x$ is the expectation ($\mu$-average) of $x$ on the cube $I$. At this point we emphasize that in the setting of quasicubes we continue to use the linear function $x$ and not the pushforward of $x$ by $\Omega$. The reason of course is that the quasienergy defined below is used to capture the first order information in the Taylor expansion of a singular kernel.

We use the collection $\mathcal{M}_{r-deep}(K)$ of maximal $r$-deeply embedded dyadic subquasicubes of a dyadic quasicube $K$. We let $J^* = \gamma J$ where $\gamma \geq 2$. The goodness parameter $r$ is chosen sufficiently large, depending on $\varepsilon$ and $\gamma$, that the bounded overlap property

\begin{equation}
\sum_{J \in \mathcal{M}_{r-deep}(K)} 1_{J^*} \leq \beta 1_K
\end{equation}

holds for some positive constant $\beta$ depending only on $n$, $\gamma$, $r$ and $\varepsilon$ (see [SaShUr4]). We will also need the following refinements of $\mathcal{M}_{r-deep}(K)$ for each $\ell \geq 0$:

$$\mathcal{M}_{r-deep}^\ell(K) \equiv \{ J \in \mathcal{M}_{r-deep}(\pi^\ell K) : J \subset L \text{ for some } L \in \mathcal{M}_{r-deep}(K) \},$$

where $\pi^\ell K$ denotes the $\ell$th parent above $K$ in the dyadic grid. Since $J \in \mathcal{M}_{r-deep}^\ell(K)$ implies $\gamma J \subset K$, we also have from (2.7) that

\begin{equation}
\sum_{J \in \mathcal{M}_{r-deep}^\ell(K)} 1_{J^*} \leq \beta 1_K, \quad \text{for each } \ell \geq 0.
\end{equation}

Of course $\mathcal{M}_{r-deep}^0(K) = \mathcal{M}_{r-deep}(K)$, but $\mathcal{M}_{r-deep}^\ell(K)$ is in general a finer subdecomposition of $K$ the larger $\ell$ is, and may in fact be empty.

There is one final generalization we need. We say that a quasicube $J \in \Omega^n_{par}$ is $(r, \varepsilon)$-deeply embedded in a quasicube $K \in \Omega^n_{par}$, or simply $r$-deeply embedded in $K$, which we write as $J \Subset_{r} K$, when $J \subset K$ and both

$$\ell(J) \leq 2^{-r} \ell(K),$$

$$\text{quasidist}(J, \partial K) \geq \frac{1}{2} \ell(J)^\varepsilon \ell(K)^{1-\varepsilon}.$$

This is the same definition as we gave earlier for dyadic quasicubes, but is now extended to arbitrary quasicubes $J, K \in \Omega^n_{par}$. Now given $K \in \Omega^n_{D}$ and $F \in \Omega^n_{par}$ with $\ell(F) \geq \ell(K)$, define

$$\mathcal{M}_{r-deep}^F(K) \equiv \{ \text{maximal } J \in \Omega^n_{D} : J \Subset_{r} K \text{ and } J \Subset_{r} F \},$$
and
\[(E^{\text{refined}}_\alpha)^2 \equiv \sup_{I} \sup_{F \in \Omega^n_{\text{par}} : \ell(F) \geq \ell(I)} \frac{1}{|I|} \sum_{J \in M_{F - \text{deep}}(I)} \left( \frac{P^\alpha(J, 1_{I \setminus J} \sigma)}{|J|^\frac{1}{n}} \right)^2 \|P_{J}^{\text{subgood}, \omega} x \|^2_{L^2(\omega)}.
\]

The important difference here is that the quasicube \(F \in \Omega^n_{\text{par}}\) is permitted to lie outside the quasigrid \(\Omega^n_D\) containing \(K\). Similarly we have a dual version of \(E^{\text{refined}}_\alpha\).

**Definition 24.** Suppose \(\sigma\) and \(\omega\) are positive Borel measures on \(\mathbb{R}^n\). Then the quasienergy condition constant \(E^{\Omega^n_\alpha}\) is given by
\[(E^{\Omega^n_\alpha})^2 \equiv \sup_{F \in \Omega^n_{\text{par}} : \ell(F) \geq \ell(I)} \frac{1}{|I|} \sum_{r=1}^{\infty} \sum_{J \in M_{F - \text{deep}}(I_r)} \left( \frac{P^\alpha(J, 1_{I \setminus J} \sigma)}{|J|^\frac{1}{n}} \right)^2 \|P_{J}^{\omega} x \|^2_{L^2(\omega)} ,
\]
where \(\sup_{I=r \cup \hat{I}_r}\) above is taken over

1. all dyadic quasigrids \(\Omega^n_D\),
2. all \(\Omega^n_D\)-dyadic quasicubes \(I\),
3. and all subpartitions \(\{I_r\}_{r=1}^{N} \) or \(\infty\) of the quasicube \(I\) into \(\Omega^n_D\)-dyadic subquasicubes \(I_r\).

This definition of the quasienergy constant \(E^{\Omega^n_\alpha}\) is larger than that used in [SaShUr7]. There is a similar definition for the dual (backward) quasienergy condition that simply interchanges \(\sigma\) and \(\omega\) everywhere. These definitions of the quasienergy condition depend on the choice of goodness parameters \(r\) and \(\varepsilon\).

Finally, we record the following elementary special case of the Energy Lemma (see, e.g., [SaShUr7] or [SaShUr] or [LaWi]) that we will need here. Recall that our quasicubes come from a fixed globally biLipschitz map \(\Omega\) in \(\mathbb{R}^n\). Our singular integrals below will be conformal fractional Riesz transforms associated with an unrelated \(C^{1, \delta}\) diffeomorphism \(\Psi\) of \(\mathbb{R}^n\) that is presented as a graph.

**Lemma 25 (Quasienergy Lemma).** Suppose that \(\Omega\) is a globally biLipschitz map, and that \(\Psi\) is a \(C^{1, \delta}\) diffeomorphism of \(\mathbb{R}^n\). Let \(J\) be a quasicube in \(\Omega^n_D\). Let \(\psi_J\) be an \(L^2(\omega)\) function supported in \(J\) and with \(\omega\)-integral zero. Let \(\nu\) be a positive measure supported in \(\mathbb{R}^n \setminus \gamma J\) with \(\gamma \geq 2\). Then for \(R^{\alpha,n}_\Psi\) a conformal \(\alpha\)-fractional Riesz transform, we have
\[
|\langle R^{\alpha,n}_\Psi(\nu), \psi_J \rangle_\omega| \lesssim \|\psi_J\|_{L^2(\omega)} \left( \frac{P^\alpha(J, \nu)}{|J|^\frac{1}{n}} \right) \|P_{J}^{\nu} x \|^2_{L^2(\omega)},
\]
3 One measure supported in a line

In this section we prove Theorem 12, i.e., we prove that the \( \Omega \)-quasienergy conditions, the backward tripled \( \Omega \)-quasitesting conditions (for appropriately rotated quasicubes), and the \( \Omega \)-quasiweak boundedness property (for appropriately rotated quasicubes) are implied by the Muckenhoupt \( A_2^\alpha \) conditions and the \( \Omega \)-quasitesting conditions \( T^{\Omega \Omega^n}_{R_\psi} \) and \( T^{\Omega \Omega^n,\text{dual}}_{R_\psi} \) associated to the tangent line truncations of a conformal \( \alpha \)-fractional Riesz transform \( R^{\alpha,n}_\psi \), when one of the measures \( \omega \) is supported in a certain line, and the other measure \( \sigma \) is arbitrary. The one-dimensional character of just one of the measures is enough to circumvent the failure of strong reversal of energy as described in [SaShUr2] and [SaShUr3].

**Notation 26.** We emphasize again that the \( C^{1,\delta} \) diffeomorphism \( \Psi \) that appears in the definition of the conformal \( \alpha \)-fractional Riesz transform \( R^{\alpha,n}_\psi \) need not have any relation to the globally biLipschitz map \( \Omega \) that is used to define the quasicubes under consideration.

Recall that the conformal Riesz transforms \( R^{\alpha,n}_\psi \) considered here have vector kernel \( K^{\alpha,n}_\psi \) defined by

\[
K^{\alpha,n}_\psi(y, x) \equiv \frac{y - x}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}},
\]

where we suppose that \( \Psi \) is given as the graph of \( \psi : \mathbb{R} \to \mathbb{R}^{n-1} \):

\[
\Psi(x) = (x^1, x^2 + \psi(x^1), \ldots, x^n + \psi^n(x^1)),
\]

where \( \psi \in C^{1,\delta} \).

Fix a collection of \( \Omega \)-quasicubes where \( \Omega : \mathbb{R}^n \to \mathbb{R}^n \) is a globally biLipschitz map unrelated to \( \Psi \). Fix a dyadic quasigrid \( \Omega D \), and suppose that \( \omega \) is supported in the \( x_1 \)-axis, which we denote by \( L \). We will show that both quasienergy conditions hold relative to \( \Omega D \). Furthermore, when \( \Omega \) is a \( C^1 \) diffeomorphism and \( L \)-transverse, we will show that the backward tripled quasienergy condition, and hence also the quasiweak boundedness property, is controlled by \( A_2^{\alpha,\text{dual}} \) and dual quasitesting.

Let \( 3 < \gamma = \gamma(n, \alpha) \) where \( \gamma \) will be taken sufficiently large depending on \( n \) and \( \alpha \) for the arguments below to be valid—see, in particular, (3.7), (3.8), (3.11), (3.17) and (3.23) below—and where we also need \( \|D\psi\|_{\infty} \) sufficiently small depending on \( n \) and \( \alpha \) as in (1.7) above, i.e.,

\[
\|D\psi\|_{\infty} < \frac{1}{8n^2}(n - \alpha).
\]
3.1 **Backward quasienergy condition.** The dual (backward) quasienergy condition $\frac{\mathcal{E}_{\alpha}^{\Omega^\alpha,\text{dual}}}{\mathcal{R}_{R}^{1,\text{dual}}} \lesssim \frac{\mathcal{E}_{\Omega^\alpha,\text{dual}}}{\mathcal{R}_{R}^{1,\text{dual}}} + \sqrt{A_{2}^{\alpha,\text{dual}}}$ is the more straightforward of the two to verify, and so we turn to it first. We will show

$$\sup_{\ell \geq 0} \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\text{deep}}(I_{r})} \left( \frac{P^{\alpha}(J, 1_{\Gamma})}{|J|^2} \right)^{2} \|P_{f}^{\alpha}x\|_{L^{2}(\sigma)}^{2} \leq \left( \sum_{r=1}^{\infty} \left( \frac{\mathcal{E}_{\alpha}^{\Omega^\alpha,\text{dual}}}{\mathcal{R}_{R}^{1,\text{dual}}} \right)^{2} |I|_{\omega},
$$

for all partitions of a dyadic quasicube $I = \bigcup_{r=1}^{\infty} I_{r}$ into dyadic subquasicubes $I_{r}$.

We fix $\ell \geq 0$ and suppress both $\ell$ and $r$ in the notation $\mathcal{M}_{\text{deep}}(I_{r}) = \mathcal{M}_{\text{r-deep}}(I_{r})$. Recall that $J^{*} = \gamma J$, and that the bounded overlap property (2.8) holds. We may of course assume that $I$ intersects the $x_{1}$-axis $L$. Now we set $\mathcal{M}_{\text{deep}} \equiv \bigcup_{r=1}^{\infty} \mathcal{M}_{\text{deep}}(I_{r})$ and write

$$\sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\text{deep}}(I_{r})} \left( \frac{P^{\alpha}(J, 1_{\Gamma})}{|J|^2} \right)^{2} \|P_{f}^{\alpha}x\|_{L^{2}(\sigma)}^{2} = \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^{\alpha}(J, 1_{\Gamma})}{|J|^2} \right)^{2} \|P_{f}^{\alpha}x\|_{L^{2}(\sigma)}^{2}.$$

We will consider the cases $3J \cap L = \emptyset$ and $3J \cap L \neq \emptyset$ separately.

Suppose $3J \cap L = \emptyset$. There is $c > 0$ and a finite sequence $\{\xi_{k}\}_{k=1}^{N}$ in $S^{n-1}$ (actually of the form $\xi_{k} = (0, \xi_{k}^{2}, \ldots, \xi_{k}^{n})$) with the following property. For each $J \in \mathcal{M}_{\text{deep}}$ with $3J \cap L = \emptyset$, there is $1 \leq k = k(J) \leq N$ such that for $y \in J$ and $x \in I \cap L$, the linear combination $\xi_{k} \cdot K_{\psi}^{\alpha,n}(y, x)$ is positive and satisfies

$$\xi_{k} \cdot K_{\psi}^{\alpha,n}(y, x) = \frac{\xi_{k} \cdot (y-x)}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}} \gtrsim \frac{\ell(J)}{|y-x|^{n+1-\alpha}}.$$

For example, if $J$ lies above the horizontal hyperplane

$$H \equiv \{(x_{1}, \ldots, x_{n-1}, 0) : (x_{1}, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\}$$

containing $L$, then for $y \in J$ and $x \in L$ we have $y_{n} \gtrsim (3-1)\ell(J) > \ell(J)$ and $x_{n} = 0$, hence the estimate

$$(0, \ldots, 0, 1) \cdot K_{\psi}^{\alpha,n}(y, x) = \frac{y_{n} - x_{n}}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}} \gtrsim \frac{\ell(J)}{|y-x|^{n+1-\alpha}}.$$

For $J$ below $H$ we take the unit vector $(0, \ldots, 0, -1)$ in place of $(0, 1)$. Finally, the general case is handled by choosing a hyperplane $H_{k} \equiv \{x \in \mathbb{R}^{n} : x \cdot \xi_{k} = 0\}$ normal to $\xi_{k}$ that contains $L$ and is disjoint from $3J$, and where a finite number of normal vectors $\{\xi_{k}\}_{k=1}^{N} \in S^{n-1}$ clearly suffice for this. Thus for $y \in J \in \mathcal{M}_{\text{deep}}$ and $k = k(J)$ we have the following ‘weak reversal’ of quasienergy for the conformal Riesz transform $R_{\psi}^{\alpha,n}$ with kernel $K_{\psi}^{\alpha,n}(y, x)$:

$$|R_{\psi}^{\alpha,n}(1_{I \cap L}\omega)(y)| = \left| \int_{I \cap L} K_{\psi}^{\alpha,n}(y, x) d\omega(x) \right| \geq \left| \int_{I \cap L} \xi_{k} \cdot K_{\psi}^{\alpha,n}(y, x) d\omega(x) \right| \gtrsim \int_{I \cap L} \frac{\ell(J)}{|y-x|^{n+1-\alpha}} d\omega(x) \approx P^{\alpha}(J, 1_{I\omega}).$$
Thus from (3.3) and the pairwise disjointedness of $J \in \mathcal{M}_{\text{deep}}$, we have
\[
\sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^\sigma(J, \mathbf{1}_I\omega)}{|J|^{\frac{1}{2}}} \right)^2 \|P^\sigma x\|^2_{L^2(\sigma)} \lesssim \sum_{J \in \mathcal{M}_{\text{deep}}} P^\sigma(J, \mathbf{1}_I\omega)^2 |J|_{\sigma}
\lesssim \sum_{J \in \mathcal{M}_{\text{deep}}} \int_J |\mathcal{R}_\Psi^\sigma,\omega(\mathbf{1}_{I\cap L}\omega)(y)|^2 d\sigma(y)
\leq \int_J |\mathcal{R}_\Psi^\sigma,\omega(\mathbf{1}_I\omega)(y)|^2 d\sigma(y) \leq (\mathfrak{T}_\mathcal{R}_\Psi^{\sigma,\text{dual}})^2 |J|_{\omega}.
\]

Now we turn to estimating the sum over those quasicubes $J \in \mathcal{M}_{\text{deep}}$ for which $3J \cap L \neq \emptyset$. In this case we use the one-dimensional nature of the support of $\omega$ to obtain a strong reversal of one of the partial quasienergies. Recall the Hilbert transform inequality for intervals $J$ and $I$ with $2J \subset I$ and $\sup \mu \subset \mathbb{R} \setminus I$:
\[
\sup_{y,z \in J} \frac{H_\mu(y) - H_\mu(z)}{y - z} = \int_{\mathbb{R} \setminus I} \left\{ \frac{1}{x} - \frac{1}{y} \right\} d\mu(x)
= \int_{\mathbb{R} \setminus I} \frac{1}{(x-y)(x-z)} d\mu(x) \approx \frac{P(J, \mu)}{|J|}.
\]

We wish to obtain a similar control in the situation at hand, but the matter is now complicated by the extra dimensions. Fix $y = (y^1, y')$, $z = (z^1, z') \in J$ and $x = (x^1, 0) \in L \setminus yJ$.

We consider first the case
\[
|y' - z'| \leq |y^1 - z^1|.
\]

We pause to recall the main assumption in (3.2) regarding the size of the graphing function:
\[
\|D\psi\|_\infty < \frac{1}{8n^2}(n - \alpha).
\]

Now the first component $(\mathcal{R}_\Psi^\sigma,\omega)_1$ is ‘positive’ in the direction of the $x^1$-axis $L$, and so for $(y^1, y'), (z^1, z') \in J$, we write
\[
\frac{(\mathcal{R}_\Psi^\sigma,\omega)_1(\mathbf{1}_{I\cap J}\omega)(y^1, y') - (\mathcal{R}_\Psi^\sigma,\omega)_1(\mathbf{1}_{I\cap J}\omega)(z^1, z')}{{y^1} - {z^1}}
= \int_{I \cap J} \left\{ \frac{(K_{\Psi,\omega}^\sigma)_1((y^1, y'), x) - (K_{\Psi,\omega}^\sigma)_1((z^1, z'), x)}{{y^1} - {z^1}} \right\} d\omega(x)
= \int_{I \cap J} \left\{ \frac{\Psi(y) - \Psi(y')^{y^1 - x^1}}{y^1 - z^1} - \frac{\Psi(z) - \Psi(z')^{z^1 - x^1}}{z^1 - x^1} \right\} d\omega(x).
\]
For $0 \leq t \leq 1$ define

$$w_t \equiv ty + (1 - t)z = z + t(y - z),$$

so that $w_t - x = t(y - x) + (1 - t)(z - x)$,

and

$$\Phi(t) \equiv \frac{w_t^1 - x^1}{|\Psi(w_t) - \Psi(x)|^{n+1-\alpha}},$$

so that

$$\frac{y^1 - x^1}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}} - \frac{z^1 - x^1}{|\Psi(z) - \Psi(x)|^{n+1-\alpha}} = \Phi(1) - \Phi(0) = \int_0^1 \Phi'(t) dt.$$

Now we will use (3.1) and $\nabla|\xi|^r \equiv r|\xi|^{r-2}\xi$ to compute that

$$\frac{d}{dt} \Phi(t) = \frac{y^1 - z^1}{|\Psi(w_t) - \Psi(x)|^{n+1-\alpha}} - (n + 1 - \alpha)(w_t^1 - x^1) \frac{(w_t^1 - x^1)(y^1 - z^1)}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}}$$

$$- (n + 1 - \alpha)(w_t^1 - x^1) \frac{(w_t^1 - x^1 + \psi(w_t^1) - \psi(x^1)) \cdot (y^1 - z^1)}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}}$$

$$- (n + 1 - \alpha)(w_t^1 - x^1) \frac{(w_t^1 - x^1 + \psi(w_t^1) - \psi(x^1)) \cdot D\psi(w_t^1)(y^1 - z^1)}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}}$$

$$= (y^1 - z^1) \left\{ \frac{|\Psi(w_t) - \Psi(x)|^2}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}} - (n + 1 - \alpha) \frac{|w_t^1 - x^1|^2}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}} \right\}$$

$$+ (y^1 - z^1)$$

$$\times \left\{ - (n + 1 - \alpha)(w_t^1 - x^1) \frac{(w_t^1 - x^1 + \psi(w_t^1) - \psi(x^1)) \cdot (\frac{y^1 - z^1}{|y^1 - z^1|})}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}} \right\}$$

$$+ (y^1 - z^1)$$

$$\times \left\{ - (n + 1 - \alpha)(w_t^1 - x^1) \frac{(w_t^1 - x^1 + \psi(w_t^1) - \psi(x^1)) \cdot D\psi(w_t^1)}{|\Psi(w_t) - \Psi(x)|^{n+3-\alpha}} \right\}$$

$$= (y^1 - z^1)(A(t) + B(t) + C(t)).$$
From (3.6) we have \( \|D\psi\|_\infty < \frac{1}{8n^2}(n - \alpha) \). Now
\[
\left| w_i^1 - x^1 \right| \approx |y - x| \quad \text{and} \quad \left| w_i' - x' \right| = \left| w_i' \right| \lesssim \frac{|y - x|}{\gamma}
\]
because \( x \in L \setminus \gamma J \) and \( y, z \in J \) and \( 3J \cap L \neq \emptyset \), and so we obtain from (3.2), with \( \gamma = \gamma(n, \alpha) \) sufficiently large, that both
\[
|w_i' - x'| \leq \frac{1}{4} \sqrt{n - \alpha} |w_i^1 - x^1|,
\]
(3.7)
\[
|\psi(w_i^1) - \psi(x^1)| \leq \|D\psi\|_\infty |w_i^1 - x^1| \leq \frac{1}{4} \sqrt{n - \alpha} |w_i^1 - x^1|.
\]
Hence we have
\[
-A(t) = -\frac{|\Psi(w_i') - \Psi(x)|^2}{|\Psi(w_i) - \Psi(x)|^{n+3-\alpha}} + (n + 1 - \alpha) \frac{|w_i^1 - x^1|^2}{|\Psi(w_i') - \Psi(x)|^{n+3-\alpha}}
\]
\[
= -\frac{|\Psi(w_i') - \Psi(x)|^2 + (n + 1 - \alpha)|w_i^1 - x^1|^2}{|\Psi(w_i') - \Psi(x)|^{n+3-\alpha}}
\]
\[
\geq \frac{3}{4} (n - \alpha) \frac{(w_i^1 - x^1)^2}{|\Psi(w_i') - \Psi(x)|^{n+3-\alpha}},
\]
where the inequality in the final line holds because of (3.7). Note that we are able to control the sign of \( A(t) \) above by using the hypothesis that \( \|D\psi\|_\infty \) is small to keep \( |\psi(w_i^1) - \psi(x^1)| \) sufficiently small, and then using the hypothesis that \( \gamma \) is large to keep \( |w_i' - x'| \) sufficiently small, so that altogether \( (n - \alpha)(w_i^1 - x^1)^2 \) is the dominant term in the numerator.

Now from our assumption (3.5) and (3.6), i.e., \( \|D\psi\|_\infty < \frac{1}{8n^2}(n - \alpha) \), we have
\[
|B(t)| = \left| (n + 1 - \alpha)(w_i^1 - x^1) \right| \frac{(w_i' - x' + \psi(w_i^1) - \psi(x^1)) \cdot (\frac{w_i' - x'}{\psi(x)})}{|\Psi(w_i') - \Psi(x)|^{n+3-\alpha}}
\]
\[
\leq (n + 1 - \alpha) |w_i^1 - x^1| \left( |w_i' - x'| + \|D\psi\|_\infty |w_i^1 - x^1| \right) \frac{|w_i' - x'|}{|\Psi(w_i') - \Psi(x)|^{n+3-\alpha}}
\]
\[
\leq (n + 1 - \alpha) \frac{|w_i^1 - x^1|(|w_i' - x'| + \|D\psi\|_\infty |w_i^1 - x^1|)}{|\Psi(w_i') - \Psi(x)|^{n+3-\alpha}}
\]
\[
\leq (n + 1 - \alpha) \left( \frac{a_{n,\alpha}}{\gamma} + \frac{1}{8n^2(n - \alpha)} \right) |w_i^1 - x^1|^2 \frac{|w_i^1 - x^1|^2}{|\Psi(w_i') - \Psi(x)|^{n+3-\alpha}}
\]
\[
\leq \frac{1}{4} (n - \alpha) \frac{(w_i^1 - x^1)^2}{|\Psi(w_i') - \Psi(x)|^{n+3-\alpha}},
\]
with a constant \( a_{n,\alpha} \) independent of \( \gamma \), provided

\[
\frac{a_{n,\alpha}}{\gamma} + \frac{1}{8n^2}(n - \alpha) \leq \frac{1}{4} \frac{n - \alpha}{n + 1 - \alpha},
\]

which holds for \( \gamma = \gamma(n, \alpha) \) sufficiently large. We also have from the same calculation that

\[
|C(t)| = \left| (n + 1 - \alpha)(w_1^1 - x^1)(w_1' - x' + \psi(w_1^1) - \psi(x^1)) \cdot D\psi(w_1^1) \right| \left| \Psi(w_1) - \Psi(x)^{n+3-\alpha} \right|
\]

\[
\leq (n + 1 - \alpha)|w_1^1 - x^1|^1 \left( |w_1' - x'| + \|D\psi\|_\infty |w_1^1 - x^1| \right) \|D\psi\|_\infty |\Psi(w_1) - \Psi(x)^{n+3-\alpha}|
\]

\[
\leq \frac{1}{4}(n - \alpha) \left( \frac{w_1^1 - x^1}{|\Psi(w_1) - \Psi(x)|^{n+3-\alpha}} \right)^2 \|D\psi\|_\infty
\]

\[
\leq \frac{1}{4}(n - \alpha) \left( \frac{w_1^1 - x^1}{|\Psi(w_1) - \Psi(x)|^{n+3-\alpha}} \right)^2 \frac{1}{8n^2}(n - \alpha)
\]

\[
< \frac{1}{4}(n - \alpha) \left( \frac{w_1^1 - x^1}{|\Psi(w_1) - \Psi(x)|^{n+3-\alpha}} \right)^2 .
\]

Thus altogether in case (3.5) we have

\[
|\langle R^{\alpha,\gamma}_{\Psi} \rangle_1(1_{I_{\gamma,j}} \omega)(y^1, y') - \langle R^{\alpha,\gamma}_{\Psi} \rangle_1(1_{I_{\gamma,j}} \omega)(z^1, z')| \n\]

\[
= |y^1 - z^1| \int_{I_{\gamma,j}} \int_0^1 \left[ \frac{d}{dt} \Phi(t) dt \right] \frac{y^1 - z^1}{y^1 - z^1 - d\omega(x)}
\]

\[
= |y^1 - z^1| \int_{I_{\gamma,j}} \int_0^1 \left\{ A(t) + B(t) + C(t) \right\} dt \omega(dx)
\]

\[
\geq |y^1 - z^1| \int_{I_{\gamma,j}} \int_0^1 \left\{ (n - \alpha) \frac{w_1^1 - x^1}{|\Psi(w_1) - \Psi(x)|^{n+3-\alpha}} \right\} dt \omega(dx)
\]

\[
\approx |y^1 - z^1| \int_{I_{\gamma,j}} \left\{ (n - \alpha) \frac{w_1^1 - x^1}{|\Psi(w_1) - \Psi(x)|^{n+3-\alpha}} \right\} d\omega(x)
\]

\[
\approx |y^1 - z^1| \left( c_1^1 - x^1 \right)^2 \frac{1}{|J|^2} ,
\]

where the constants implicit in \( \approx \) depend only on \( n \) and \( \alpha \).

On the other hand, in the case that

\[
|y' - z'| > |y^1 - z^1| ,
\]
we write

\[(R^{a,n})' = (R_2^{a,n}, \ldots, R_n^{a,n}),\]

\[\Phi(t) = \frac{w'_t - x'}{|\Psi(w_t) - \Psi(x)|^{n+1-a}},\]

with \(w_t = ty + (1 - t)z\) as before. Then as above we obtain

\[
\frac{y' - x'}{|\Psi(y) - \Psi(x)|^{n+1-a}} - \frac{z' - x'}{|\Psi(z) - \Psi(x)|^{n+1-a}} = \Phi(1) - \Phi(0) = \int_0^1 \frac{d}{dt} \Phi(t) dt,
\]

where if we write \(\hat{y}^k \equiv (0, y^2, \ldots, y^{k-1}, 0, y^{k+1}, \ldots, y^n)\), we have, similarly to the computation of \(\frac{d}{dt} \Phi(t)\) above,

\[
\frac{d}{dt} \Phi(t) \equiv \left\{ \frac{d}{dt} \Phi_k(t) \right\}_{k=2}^n
\]

\[
\equiv \left\{ (y^k - z^k) \left[ \frac{|\Psi(w_t) - \Psi(x)|^2}{|\Psi(w_t) - \Psi(x)|^{n+1-a}} \right] \right\}_{k=2}^n
\]

\[
- \left\{ (n + 1 - \alpha)(w_t^k - x^k) \right\}_{k=2}^n
\]

\[
\times \left( (y^k - z^k)\right)^n \left[ \frac{|\Psi(w_t) - \Psi(x)|^{n+1-a}}{|\Psi(w_t) - \Psi(x)|^{n+1-a}} \right]
\]

\[
\equiv \left\{ (y^k - z^k)A_k(t) \right\}_{k=2}^n + \left\{ V_k(t) \right\}_{k=2}^n + \left\{ (y^k - z^k)C_k(t) \right\}_{k=2}^n
\]

Now for \(2 \leq k \leq n\) we have \(x^k = 0\) and so

\[
A_k(t) = \frac{|\Psi(w_t) - \Psi(x)|^2}{|\Psi(w_t) - \Psi(x)|^{n+1-a}} - (n + 1 - \alpha)(w_t^k - x^k) \cdot \left( \Psi(w_t^1 - \Psi(x^1)) \right)_{k=2}^n
\]

\[
= \frac{|\Psi(w_t) - \Psi(x)|^2 - (n + 1 - \alpha)w_t^k + \Psi(w_t^1 - \Psi(x^1))_{k=2}^n}{|\Psi(w_t) - \Psi(x)|^{n+1-a}}
\]

\[
= \frac{|w_t^1 - x^1|^2}{|\Psi(w_t) - \Psi(x)|^{n+1-a}} - \sum_{j=2}^n \frac{|w_t^j + \Psi(w_t^1) - \Psi(x^1)|^2}{|\Psi(w_t) - \Psi(x)|^{n+1-a}}
\]

\[
- (n + 1 - \alpha)w_t^k(x^k - \Psi(x^1))_{k=2}^n
\]
Then using
\[ |w^k| \leq \frac{1}{\gamma} |w^1 - x^1| \quad \text{and} \quad |\psi^k(w^1) - \psi^k(x^1)| \leq \|D\psi\|_{\infty} |w^1 - x^1|, \]
we claim
\begin{equation}
A_k(t) \geq \frac{1}{2} \frac{|w^1 - x^1|^2}{|\Psi(w^1) - \Psi(x)|^{n+3-\alpha}},
\end{equation}
where \(\|D\psi\|_{\infty}\) satisfies (3.2) and \(\gamma = \gamma(n, \alpha)\) is sufficiently large. Indeed, use
\[ |w^k| \leq \frac{b_{n,a}}{\gamma} |w^1 - x^1|, \]
where the constant \(b_{n,a}\) is independent of \(\gamma\), to obtain
\[ |\Psi(w^1) - \Psi(x)|^{n+3-\alpha} \frac{A_k(t)}{A_k(t)} \]
\[ \geq |w^1 - x^1|^2 - (n + 1 - \alpha)|w^k||w^k + \psi^k(w^1) - \psi^k(x^1)| \]
\[ \geq |w^1 - x^1|^2 - (n + 1 - \alpha)|w^k|^2 |D\psi|_{\infty} |w^1 - x^1| \]
\[ \geq |w^1 - x^1|^2 - (n + 1 - \alpha) \left( \left( \frac{b_{n,a}}{\gamma} \right)^2 + \frac{b_{n,a}}{\gamma} |D\psi|_{\infty} \right) |w^1 - x^1|^2 \]
\[ \geq \frac{1}{2} |w^1 - x^1|^2, \]
for \(\gamma = \gamma(n, \alpha)\) sufficiently large since \(\|D\psi\|_{\infty} < \frac{n-\alpha}{8n^2}\) by (3.2).

Thus we have
\[ \int_{f_{\gamma,J}^1} A_k(t) d\omega(x) \geq \frac{1}{2} c_{n,a} \int_{f_{\gamma,J}^1} \frac{1}{c_{J} - x^{n+1-\alpha}} d\omega(x) \geq c'_{n,a} \frac{P^a(J, 1_{f_{\gamma,J}^1})}{|J|^1}, \]
where \(c'_{n,a}\) is independent of the choice of \(\gamma = \gamma(n, \alpha)\), and hence
\[ \left| \int_{f_{\gamma,J}^1} U(t) d\omega(x) \right|^2 = \left| \int_{f_{\gamma,J}^1} \int_0^1 \{ (y^k - z^k) A_k(t) \}_{k=1}^n dtd\omega(x) \right|^2 \]
\[ = \sum_{k=2}^n (y^k - z^k)^2 \left| \int_{f_{\gamma,J}^1} \int_0^1 A_k(t) dtd\omega(x) \right|^2 \]
\[ \geq (c'_{n,a})^2 \sum_{k=2}^n (y^k - z^k)^2 \left( \frac{P^a(J, 1_{f_{\gamma,J}^1})}{|J|^1} \right)^2 \]
\[ = (c'_{n,a})^2 |y' - z'|^2 \left( \frac{P^a(J, 1_{f_{\gamma,J}^1})}{|J|^1} \right)^2. \]
For $2 \leq k \leq n$ we also have using $x^k = 0$ and (3.9) that

\[
\frac{1}{n + 1 - \alpha} |V_k(t)| = \left| (w^k_i - x^k_i)(w^i_1 - x^1_i)(y^1_i - z^1_i) + (\hat{w}^k_i - \hat{x}^k_i + \hat{\psi}^k(w^1_i) - \hat{\psi}^k(x^1_i)) \cdot (\hat{y}^k_i - \hat{z}^k_i) \right|
\]

\[
\leq |w^k_i| \left| (w^i_1 - x^1_i)|y^1_i - z^1_i| + \sum_{j \neq 1, k} |w^j_i + \psi'(w^1_i) - \psi'(x^1_i)||y^j_i - z^j_i| \right|
\]

\[
\leq \left\{ \frac{|w^k_i||y^1_i - z^1_i|}{|\Psi(w_i) - \Psi(x)|^{n+2-\alpha}} + \sum_{j \neq 1, k} \frac{|w^j_i|(|w^1_i| + \|D\psi\|_{\infty}|w^1_i - x^1_i||y^j_i - z^j_i|)}{|\Psi(w_i) - \Psi(x)|^{n+3-\alpha}} \right\}
\]

\[
\lesssim \left\{ \frac{b_{n,a}}{\gamma |\Psi(w_i) - \Psi(x)|^{n+1-\alpha}} \right\} + \left( \sqrt{n\ell(J)}(\sqrt{n\ell(J)} + \|D\psi\|_{\infty}|w^1_i - x^1_i|) \right) \frac{|y^j_i - z^j_i|}{|\Psi(w_i) - \Psi(x)|^{n+3-\alpha}} \right\}
\]

as well as

\[
\frac{1}{n + 1 - \alpha} |W_k(t)| = \frac{1}{n + 1 - \alpha} |y^1_i - z^1_i| |C_k(t)|
\]

\[
= |y^1_i - z^1_i||w^k_i - x^k_i| \left| (w^i_1 - x^1_i + \psi(w^1_i) - \psi(x^1_i)) \cdot D\psi(w^1_i) \right|
\]

\[
\leq |y^1_i - z^1_i||w^k_i| \left| (|w^1_i| + \|\psi(w^1_i) - \psi(x^1_i)||D\psi(w^1_i)| \right|
\]

\[
\leq |y^1_i - z^1_i| \left( \frac{b_{n,a}}{\gamma^2} + \frac{b_{n,a}\|D\psi\|_{\infty}}{\gamma} \right) \|D\psi\|_{\infty} \frac{|w^1_i - x^1_i|^2}{|\Psi(w_i) - \Psi(x)|^{n+3-\alpha}}
\]

\[
\lesssim |y^1_i - z^1_i| \left( \frac{1}{\gamma^2} + \frac{\|D\psi\|_{\infty}}{\gamma} \right) \|D\psi\|_{\infty} \frac{|w^1_i - x^1_i|}{|\psi - x|^{n+1-\alpha}}
\]

Thus

\[
\left| \int_{I \cap J} \int_0^1 V(t)dtd\omega(x) \right|
\]

\[
\leq d_{n,a}(n + 1 - \alpha) \left\{ \frac{1}{\gamma} + \left( \frac{1}{\gamma} \right) \left( \frac{1}{\gamma} + \|D\psi\|_{\infty} \right) \right\} \int_{I \cap J} \frac{|y^j_i - z^j_i|}{|\psi - x|^{n+1-\alpha}} d\omega(x)
\]

\[
\leq d'_{n,a}(n + 1 - \alpha) \left\{ \frac{1}{\gamma} + \left( \frac{1}{\gamma} \right) \left( \frac{1}{\gamma} + \frac{n - \alpha}{8\gamma} \right) \right\} |J|^\frac{1}{n}
\]
and

\[ \left| \int_{I_{\gamma J}} \int_0^1 W(t)dt d\omega(x) \right| \leq \frac{d'_{n,a}}{\gamma} \frac{|y' - z'|}{|c_J - x|^{n+1-\alpha}}, \]

where the constants \( d'_{n,a} \) and \( d''_{n,a} \) are independent of \( \gamma \), and so we conclude that

\[ \left| \int_{I_{\gamma J}} \int_0^1 V(t)dt d\omega(x) \right| + \left| \int_{I_{\gamma J}} \int_0^1 W(t)dt d\omega(x) \right| \leq \frac{1}{2} \left| \int_{I_{\gamma J}} \int_0^1 U(t)dt d\omega(x) \right|, \]

provided \( \gamma = \gamma(n, \alpha) \) is sufficiently large. Then if (3.9) holds with \( \gamma \) sufficiently large, we have

\[ |(R_{\Psi}^{a,n})_1 I_{\gamma J} \omega(y^1, y') - (R_{\Psi}^{a,n})_1 I_{\gamma J} \omega(z^1, z')| \]

\[ = \left| \int_{I_{\gamma J}} \left\{ \frac{y^k - x^k}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}} - \frac{z^k - x^k}{|\Psi(z) - \Psi(x)|^{n+1-\alpha}} \right\} d\omega(x) \right| \]

\[ = \left| \int_{I_{\gamma J}} \int_0^1 \Phi'(t)dt d\omega(x) \right| \]

\[ \geq \left| \int_{I_{\gamma J}} \int_0^1 U(t)dt d\omega(x) \right| - \left| \int_{I_{\gamma J}} \int_0^1 V(t)dt d\omega(x) \right| - \left| \int_{I_{\gamma J}} \int_0^1 W(t)dt d\omega(x) \right| \]

\[ \geq \frac{1}{2} \left| \int_{I_{\gamma J}} \int_0^1 U(t)dt d\omega(x) \right| \]

\[ \geq \frac{1}{2} \left| \frac{P^\alpha(J, 1_{I_{\gamma J} \omega})}{|J|^\frac{3}{2}} \right| |y' - z'| \]

\[ \geq \frac{1}{2} \left| \frac{P^\alpha(J, 1_{I_{\gamma J} \omega})}{|J|^\frac{3}{2}} \right| |y^1 - z^1| \]

Combining the inequalities from each case (3.5) and (3.9) above, and assuming \( \gamma \) sufficiently large, we conclude that for all \( y, z \in J \) we have the following ‘strong reversal’ of the 1-partial quasienergy:

\[ |y^1 - z^1|^2 \left( \frac{P^\alpha(J, 1_{I_{\gamma J} \omega})}{|J|^\frac{3}{2}} \right)^2 \lesssim |R_{\Psi}^{a,n} 1_{I_{\gamma J} \omega}(y^1, y') - R_{\Psi}^{a,n} 1_{I_{\gamma J} \omega}(z^1, z')|^2. \]
Thus we have

\[
\sum_{J \in M_{\text{deep}}} \left( \frac{P^\alpha(J, 1_{I \setminus j} \omega)}{|J|^\frac{1}{n}} \right)^2 \int_J |y^1 - E^\alpha_j y^1|^2 \, d\sigma(y)
\]

for \( 2 \leq j \leq n \), we have the following ‘weak reversal’ of energy:

\[
\sum_{J \in M_{\text{deep}}} \frac{1}{|J|^\frac{1}{n}} \int_J \int_J (y^1 - z^1)^2 d\sigma(y) d\sigma(z)
\]

and now we obtain in the usual way that this is bounded by

\[
\int_J |R^\alpha_{\psi'}(1_{I \setminus j} \omega)(y^1, y')|^2 \, d\sigma(y) + \sum_{J \in M} (\Sigma^\text{dual}_{R^\alpha_{\psi'}})^2 |y^1|_{I \setminus j} \omega
\]

\[
\leq (\Sigma^\text{dual}_{R^\alpha_{\psi'}})^2 |I|_{I \setminus j} \omega + \beta (\Sigma^\text{dual}_{R^\alpha_{\psi'}})^2 |I|_{I \setminus j} \omega \lesssim (\Sigma^\text{dual}_{R^\alpha_{\psi'}})^2 |I|_{I \setminus j} \omega.
\]

Now we turn to the other partial quasienergies and begin with the estimate that for \( 2 \leq j \leq n \), we have the following ‘weak reversal’ of energy:

\[
| (R^\alpha_{\psi'} J (1_{I \setminus j} \omega))(y) | = \left| \int_{I \setminus j} \frac{y^j - 0}{|\Psi(y) - \Psi(x)|_{n+1-\alpha}} \, d\omega(x_1, 0, \ldots, 0) \right|
\]

\[
(3.13)
\]

Thus for \( 2 \leq j \leq n \), we use

\[
\int_J |y^j - E^\alpha_j y^j|^2 d\sigma(y) \leq \int_J |y^j|^2 d\sigma(y)
\]
to obtain in the usual way
\[
\sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^\alpha(J, 1_{\cdot \setminus J \emptyset})}{|J|^1} \right)^2 \int_J |y|^2 d\sigma(y)
\]
\[
\leq \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^\alpha(J, 1_{\cdot \setminus J \emptyset})}{|J|^1} \right)^2 \int_J |y|^2 d\sigma(y)
\]
\[
= \sum_{J \in \mathcal{M}_{\text{deep}}} \int_J \left( \frac{P^\alpha(J, 1_{\cdot \setminus J \emptyset})}{|J|^1} \right)^2 |y|^2 d\sigma(y)
\]
(3.14)
\[
\lesssim \sum_{J \in \mathcal{M}_{\text{deep}}} \int_J \left| (R_{\psi}^\alpha)_{\gamma} (1_{\cdot \setminus J \emptyset})(y) \right|^2 d\sigma(y)
\]
\[
\lesssim (\mathcal{T}^{\Omega Q^n, \text{dual}}_{R_{\psi}})^2 |I|_\sigma + \sum_{J \in \mathcal{M}} (\mathcal{T}^{\Omega Q^n, \text{dual}}_{R_{\psi}})^2 |yJ|_\sigma
\]
\[
\leq (\mathcal{T}^{\Omega Q^n, \text{dual}}_{R_{\psi}})^2 |I|_\sigma + \beta (\mathcal{T}^{\Omega Q^n, \text{dual}}_{R_{\psi}})^2 |I|_\sigma
\]
\[
\lesssim (\mathcal{T}^{\Omega Q^n, \text{dual}}_{R_{\psi}})^2 |I|_\sigma.
\]
Summing these estimates for \( j = 1 \) and \( 2 \leq j \leq n \) completes the proof of the backward quasienergy condition
\[
E^{\Omega Q^n, \text{dual}}_{\alpha} \lesssim \mathcal{T}^{\Omega Q^n, \text{dual}}_{R_{\psi}}.
\]

3.2 Forward quasienergy condition. Now we turn to proving the (forward) quasienergy condition
\[
E^{\Omega Q^n}_{\alpha} \lesssim \mathcal{T}^{\Omega Q^n}_{R_{\psi}} + \sqrt{A^\alpha_{2}},
\]
where \( A^\alpha_{2} \) is the Muckenhoupt condition with holes. We must show
\[
\sup_{\ell \geq 0} \sum_{r=1}^\infty \sum_{J \in \mathcal{M}_{\text{deep}}(I_r)} \left( \frac{P^\alpha(J, 1_{\cdot \setminus J \cdot \sigma})}{|J|^1} \right)^2 \|P^\alpha_{J} x\|_{L^2(\omega)}^2 \leq \left( (\mathcal{T}^{\Omega Q^n}_{R_{\psi}})^2 + A^\alpha_{2} \right) |I|_\sigma,
\]
for all partitions of a dyadic quasicube
\[
I = \bigcup_{r \geq 1} I_r
\]
into dyadic subquasicubes \( I_r \). We again fix \( \ell \geq 0 \) and suppress both \( \ell \) and \( r \) in the notation
\[
\mathcal{M}_{\text{deep}}(I_r) = \mathcal{M}_{r-\text{deep}}(I_r).
\]
We may assume that all the quasicubes \( J \) intersect \( \text{supp} \omega \), hence that all the quasicubes \( I_r \) and \( J \) intersect \( L \), which contains \( \text{supp} \omega \). We must show

\[
\sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{\text{deep}}(I_r)} \left( \frac{P^\alpha(J, 1_{E(J^*) \sigma})}{|J|^\frac{1}{2}} \right)^2 \|P^\alpha_J x\|_{L^2(\omega)}^2 \leq ((\sum_{\mathcal{R}_n^\omega}^\alpha)^2 + A_2^\alpha)|I|_\sigma.
\]

Let

\[
\mathcal{M}_{\text{deep}} = \bigcup_{r=1}^{\infty} \mathcal{M}_{\text{deep}}(I_r)
\]
as above and, with \( J^* = \gamma J \) for each \( J \in \mathcal{M}_{\text{deep}} \), make the decomposition

\[
I \setminus J^* = E(J^*) \cup S(J^*)
\]
of \( I \setminus J^* \) into \textbf{end} \( E(J^*) \) and \textbf{side} \( S(J^*) \) disjoint pieces defined by

\[
E(J^*) \equiv (I \setminus J^*) \cap \left\{(y^1, y^2) : |y^1 - c^1_J| \leq \frac{10}{\gamma} |y^1 - c^1_J| \right\};
\]

\[
S(J^*) \equiv (I \setminus J^*) \setminus E(J^*).
\]

Then it suffices to show both

\[
A \equiv \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^\alpha(J, 1_{E(J^*) \sigma})}{|J|^\frac{1}{2}} \right)^2 \|P^\alpha_J x\|_{L^2(\omega)}^2 \leq ((\sum_{\mathcal{R}_n^\omega}^\alpha)^2 + A_2^\alpha)|I|_\sigma,
\]

\[
B \equiv \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^\alpha(J, 1_{S(J^*) \sigma})}{|J|^\frac{1}{2}} \right)^2 \|P^\alpha_J x\|_{L^2(\omega)}^2 \leq ((\sum_{\mathcal{R}_n^\omega}^\alpha)^2 + A_2^\alpha)|I|_\sigma.
\]

Term \( A \) will be estimated in analogy with the Hilbert transform estimate (3.4), while term \( B \) will be estimated by summing Poisson tails. Both estimates rely heavily on the one-dimensional nature of the support of \( \omega \), for example

\[
\|P^\alpha_J x\|_{L^2(\omega)}^2 = \|P^\alpha_J x^1\|_{L^2(\omega)}^2.
\]

Thus in this quasienergy condition, there is only one nonvanishing partial quasienergy, namely the 1-partial quasienergy measured along the \( x_1 \)-axis.

For \( (x^1, 0^1), (z^1, 0^1) \in J \) in term \( A \) we first claim the following ‘strong reversal’ of quasienergy,

\[
\left| (R_n^\omega)_{1}(1_{E(J^*) \sigma})(x^1, 0^1) - (R_n^\omega)_{1}(1_{E(J^*) \sigma})(z^1, 0^1) \right|
\]

\[
\approx \left| \int_{E(J^*)} \left\{ \frac{(K_n^\omega)_{1}(x^1, 0^1), y) - (K_n^\omega)_{1}(z^1, 0^1), y)}{x^1 - z^1} \right\} d\sigma(y) \right|
\]

\[
\approx \left| \int_{E(J^*)} \left\{ \frac{(x^1 - y^1)^2 + |\psi(x^1) - \psi(y^2)|^2}{x^1 - z^1} - \frac{(z^1 - y^1)^2 + |\psi(z^1) - \psi(y^2)|^2}{x^1 - z^1} \right\} d\sigma(y) \right|
\]

\[
\approx \frac{P^\alpha(J, 1_{E(J^*) \sigma})}{|J|^\frac{1}{2}}.
\]
Indeed, if we set
\[ a(u) = |\psi(u + y) - \psi(y_1) - y'| \]
and \( s = x^1 - y^1 \) and \( t = z^1 - y^1 \), then the term in braces in (3.15) is
\[
\frac{s - y^1}{(s^1 - y^1)^2 + (\psi(x^1) - \psi(y^1))^2} \frac{\alpha s}{2} - \frac{t - y^1}{(t^1 - y^1)^2 + (\psi(z^1) - \psi(y^1))^2} \frac{\alpha t}{2} = \frac{\phi(s) - \phi(t)}{s - t},
\]
where with \( y \) fixed for the moment,
\[
\phi(u) = u(u^2 + |\psi(u + y^1) - \psi(y^1)|^2)^{\frac{\alpha + 1}{2}} = u(u^2 + a(u)^2)^{\frac{\alpha + 1}{2}}; \\
a(u)^2 \equiv |\psi(u + y^1) - \psi(y^1) - y'|^2.
\]
Now the derivative of \( \phi(u) \) is
\[
\frac{d\phi(u)}{du} = (u^2 + a(u)^2)^{-\frac{\alpha + 1}{2}} \left[ u^2 + \frac{1}{2} a(u)^2 \right] - \frac{n + 1 - \alpha}{2} (u^2 + a(u)^2)^{-\frac{\alpha + 1}{2}} - \frac{1}{2} a(u)^2 \\
= (u^2 + a(u)^2)^{-\frac{\alpha + 1}{2}} - \frac{1}{2} \left\{ (u^2 + a(u)^2) - (n + 1 - \alpha) u^2 \right\} \\
= (u^2 + a(u)^2)^{-\frac{\alpha + 1}{2}} - \frac{1}{2} \left\{ a(u)^2 - (n + 1 - \alpha) u \frac{1}{2} a(u)^2 \right\} - (n - \alpha)u^2,
\]
and the derivative of \( a(u)^2 \) is
\[
\frac{d}{du} \frac{1}{2} a(u)^2 = \frac{1}{2} \frac{d}{du} |\psi(u + y_1) - \psi(y_1) - y'|^2 \\
= \left[ \psi(u + y_1) - \psi(y_1) - y' \right] \cdot D\psi(u + y_1).
\]
We now want to conclude that
\[
(3.16) \quad \left| a(u)^2 - (n + 1 - \alpha)u \frac{1}{2} a(u)^2 \right| \leq \frac{1}{2} (n - \alpha)u^2, \quad \text{for} \ u \text{ between} \ s \text{ and} \ t,
\]
so that \(-(n - \alpha)u^2\) is the dominant term inside the braces in the formula for \( \frac{d}{du} \phi(u) \).

For this we note that
\[
|y'| \leq C \frac{1}{y} |u|,
\]
and so using
\[
2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2
\]
twice, the left side of (3.16) is at most
\[
(||D\psi||_\infty |u| + |y'|)^2 + (n + 1 - \alpha)|u| (||D\psi||_\infty |u| + |y'|)||D\psi||_\infty \\
\leq 2 ||D\psi||^2_\infty u^2 + 2 |y'|^2 + (n + 1 - \alpha)||D\psi||^2_\infty u^2 + (n + 1 - \alpha)||D\psi||_\infty |u||y'| \\
\leq (n + 4 - \alpha)||D\psi||^2_\infty u^2 + C_1 \left( C \frac{1}{y} |u| \right)^2,
\]
which gives (3.16) if

\[ (n + 4 - \alpha) \| D\psi \|_\infty^2 + C_1 C^2 \frac{1}{\gamma^2} \leq \frac{1}{2} (n - \alpha), \]

which in turn holds for \( \| D\psi \|_\infty \) and \( \gamma = \gamma(n, \alpha) \) satisfying

\[
\| D\psi \|_\infty < \frac{1}{2} \sqrt{\frac{n - \alpha}{n + 4 - \alpha}} \quad \text{and} \quad \gamma \gg \frac{1}{\sqrt{n - \alpha}},
\]

where the first inequality in (3.17) follows from (3.2), i.e.,

\[
\| D\psi \|_\infty < \frac{1}{8n^2} (n - \alpha) < \frac{1}{2} \sqrt{\frac{n - \alpha}{n + 4 - \alpha}},
\]

and the second inequality holds for \( \gamma = \gamma(n, \alpha) \) sufficiently large.

Thus we get

\[
-\frac{d}{du} \varphi(u) \approx t^2 (t^2 + a(t)^2)^{-\frac{n+\alpha}{2}-1}, \quad \text{for } u \text{ between } s \text{ and } t,
\]

where for \((x^1, 0'), (z^1, 0') \in J \) with \( J \in \mathcal{M}_{\text{deep}} \), the implied constants of comparability are independent of \( y \in E(J^*) \). Finally, since

\[
|s - t| \lesssim \frac{1}{\gamma} |t| \ll |t|,
\]

the derivative \( \frac{d\varphi}{du} \) is essentially constant on the small interval \((s, t)\), and we can apply the tangent line approximation to \( \varphi \) to obtain

\[
\varphi(s) - \varphi(t) \approx \frac{d\varphi}{dt}(t)(s - t),
\]

and conclude that for \((x^1, 0'), (z^1, 0') \in J\),

\[
\left| \int_{E(J^*)} \left\{ \frac{x^1 - y^1}{(|x^1 - y^1|^2 + |\varphi(x^1) - \varphi(y^1)|^2)^{\frac{n+\alpha}{2}} - \frac{z^1 - y^1}{(|z^1 - y^1|^2 + |\varphi(z^1) - \varphi(y^1)|^2)^{\frac{n+\alpha}{2}}} \right\} d\sigma(y) \right|
\]

\[
\approx \int_{E(J^*)} \frac{|z^1 - y^1|^2}{(|z^1 - y^1|^2 + |\varphi(z^1) - \psi(z^1) - y^1|^2)^{\frac{n+\alpha}{2}} + 1} d\sigma(y)
\]

\[
\approx \int_{E(J^*)} \frac{|z^1 - y^1|^2}{(|z^1 - y^1|^2 + |y^1|^2)^{\frac{n+\alpha}{2}} + 1} d\sigma(y) \approx \frac{P^\alpha(J, 1_{E(J^*)}\sigma)}{|J|^{\frac{3}{2}}},
\]

which proves (3.15).
Thus we have

\begin{equation}
\sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^a(J, 1_{E(J^*)})}{|J|^n} \right)^2 \int_{J \cap L} |x^1 - \mathbb{E}_J x^1|^2 d\omega(y)
\end{equation}

\begin{align*}
&= \frac{1}{2} \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^a(J, 1_{E(J^*)})}{|J|^n} \right)^2 \int_{J \cap L} \int_{J \cap L} (x^1 - z^1)^2 d\omega(x) d\omega(z) \\
&\approx \sum_{J \in \mathcal{M}_{\text{deep}}} \frac{1}{|J|_\omega} \\
&\quad \times \int_{J \cap L} \int_{J \cap L} \{(\mathbf{R}_\Psi^{\alpha,n})_1(1_{E(J^*)})(x^1, 0') - (\mathbf{R}_\Psi^{\alpha,n})_1(1_{E(J^*)})(z^1, 0')\}^2 d\omega(x) d\omega(z) \\
&\quad + \sum_{J \in \mathcal{M}_{\text{deep}}} \frac{1}{|J|_\omega} \\
&\quad \times \int_{J \cap L} \int_{J \cap L} \{(\mathbf{R}_\Psi^{\alpha,n})_1(1_{J^*})(x^1, 0') - (\mathbf{R}_\Psi^{\alpha,n})_1(1_{J^*})(z^1, 0')\}^2 d\omega(x) d\omega(z) \\
&\quad + \sum_{J \in \mathcal{M}_{\text{deep}}} \frac{1}{|J|_\omega} \\
&\quad \times \int_{J \cap L} \int_{J \cap L} \{(\mathbf{R}_\Psi^{\alpha,n})_1(1_{S(J^*)})(x^1, 0') - (\mathbf{R}_\Psi^{\alpha,n})_1(1_{S(J^*)})(z^1, 0')\}^2 d\omega(x) d\omega(z) \\
&\equiv A_1 + A_2 + A_3,
\end{align*}

since

\[ I = J^* \cup (I \setminus J^*) = J^* \cup E(J^*) \cup S(J^*) \]

where $\cup$ denotes disjoint union. Now we can discard the difference in term $A_1$ by writing

\[ |(\mathbf{R}_\Psi^{\alpha,n})_1(1_{J^*})(x^1, 0') - (\mathbf{R}_\Psi^{\alpha,n})_1(1_{J^*})(z^1, 0')| \leq |(\mathbf{R}_\Psi^{\alpha,n})_1(1_{J^*})(x^1, 0')| + |(\mathbf{R}_\Psi^{\alpha,n})_1(1_{J^*})(z^1, 0')| \]

to obtain from pairwise disjointedness of $J \in \mathcal{M}_{\text{deep}},$

\begin{equation}
A_1 \lesssim \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{J \cap L} |(\mathbf{R}_\Psi^{\alpha,n})_1(1_{J^*})(x^1, 0')|^2 d\omega(x) \leq \int_I |(\mathbf{R}_\Psi^{\alpha,n})_1(1_{J^*})|^2 d\omega \\
\leq \left( \sum_{\mathbf{R}_\Psi^{\alpha,n}} \right)^2 |I|_\sigma,
\end{equation}

and similarly we can discard the difference in term $A_2$, and use the bounded overlap property (2.8) to obtain

$$A_2 \lesssim \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{J \cap L} |(\mathbf{R}_{\Psi}^{a,n})_1(1_{S(J^*)}\sigma)(x^1, 0')|^2 \, d\omega(x) \leq \sum_{J \in \mathcal{M}_{\text{deep}}} (\sum_{\mathcal{O}^n(J^*)_I}^\infty)^2 |J^*|_\sigma \quad (3.20)$$

$$\leq \beta (\sum_{\mathcal{O}^n(J^*)_I}^\infty)^2 |I|_\sigma.$$

This leaves us to consider the term

$$A_3 = \sum_{J \in \mathcal{M}_{\text{deep}}} \frac{1}{|J|_{L^2(\omega)}} \times \int \int (\mathbf{R}_{\Psi}^{a,n})_1(1_{S(J^*)}\sigma)(x^1, 0') - (\mathbf{R}_{\Psi}^{a,n})_1(1_{S(J^*)}\sigma)(z^1, 0')|^2 \, d\omega(x) \, d\omega(z)$$

$$= 2 \sum_{J \in \mathcal{M}_{\text{deep}}} \int (\mathbf{R}_{\Psi}^{a,n})_1(1_{S(J^*)}\sigma)(x^1, 0') - \mathbb{E}_{J \cap L}[(\mathbf{R}_{\Psi}^{a,n})_1(1_{S(J^*)}\sigma)(z^1, 0')]|^2 \, d\omega(x),$$

in which we do not discard the difference. However, because the average is subtracted off, we can apply the Quasienergy Lemma 25, together with duality

$$\|\mathbf{R}_{\Psi}^{a,n}(v) - \mathbb{E}_J^{a,\gamma} \mathbf{R}_{\Psi}^{a,n}(v)\|_{L^2(\omega)} = \sup_{\|\mathbf{P}_J\|_{L^2(\omega)} = 1} |\langle \mathbf{R}_{\Psi}^{a,n}(v) - \mathbb{E}_J^{a,\gamma} \mathbf{R}_{\Psi}^{a,n}(v), \mathbf{P}_J \rangle|$$

$$= \sup_{\mathbf{P}_J \|_{L^2(\omega)} = 1 \text{ and } \int \mathbf{P}_J d\omega = 0} |\langle \mathbf{R}_{\Psi}^{a,n}(v), \mathbf{P}_J \rangle|,$$

to each term in this sum to dominate it by

$$B = \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{P^a(J, 1_{S(J^*)}\sigma)}{|J|^\frac{1}{n}} \right)^2 \|\mathbb{E}_J^{a,\gamma}\|_{L^2(\omega)}^2 \quad (3.21)$$

To estimate $B$, we first assume that $n - 1 \leq \alpha < n$ so that

$$P^a(J, 1_{S(J^*)}\sigma) \leq \mathbb{E}_J^{a,\gamma}(J, 1_{S(J^*)}\sigma),$$

and then use

$$\|\mathbb{E}_J^{a,\gamma}\|^2_{L^2(\omega)} \lesssim |J|^\frac{1}{2} |J|_{L^2(\omega)}$$

and apply the $A_2^\alpha$ condition with holes to obtain the following ‘pivotal reversal’ of
quasienergy,

\[ B \lesssim \sum_{J \in \mathcal{M}_{\text{deep}}} P^a(J, 1_{S(J^*)})^2 |J|_\alpha \leq \sum_{J \in \mathcal{M}_{\text{deep}}} P^a(J, 1_{S(J^*)}) \{ P^a(J, 1_{S(J^*)}) |J|_\alpha \} \]

\[ \leq A_2^a \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{S(J^*)} \frac{|J|^\frac{1}{2} |J|^{1-\frac{\alpha}{2}}}{|J|^{\frac{n}{2}} + |y - c_J|} \, d\sigma(y) \]

\[ = A_2^a \sum_{J \in \mathcal{M}_{\text{deep}}} \int_{S(J^*)} \left( \frac{|J|^\frac{1}{2}}{|J|^{\frac{n}{2}} + |y - c_J|} \right)^{n+1-\alpha} \, d\sigma(y) \]

\[ = A_2^a \int_I \left\{ \sum_{J \in \mathcal{M}_{\text{deep}}} \left( \frac{|J|^\frac{1}{2}}{|J|^{\frac{n}{2}} + |y - c_J|} \right)^{n+1-\alpha} 1_{S(J^*)}(y) \right\} d\sigma(y) \]

\[ = A_2^a \int_I F(y) d\sigma(y). \]

At this point we claim that \( F(y) \leq C \) with a constant \( C \) independent of the decomposition \( \mathcal{M}_{\text{deep}} = \bigcup_{r \geq 1} \mathcal{M}_{\text{deep}}(J_r) \). Indeed, if \( y \) is fixed, then the quasicubes \( J \in \mathcal{M}_{\text{deep}} \) for which \( y \in S(J^*) \) satisfy

\[ (3.22) \quad J \cap \text{Sh}(y; \gamma) \neq \emptyset, \]

where \( \text{Sh}(y; \gamma) \) is the Carleson shadow of the point \( y \) onto the \( x_1 \)-axis \( L \), defined as the interval on \( L \) with length \( \frac{1}{2} \gamma \text{dist}(y, L) \) and center equal to the point on \( L \) that is closest to \( y \). If a quasicube \( J \) intersects \( \text{Sh}(y; \gamma) \), and \( y \in S(J^*) \), we must have

\[ \ell(J) \leq C_0 \text{dist}(y, L), \]

where \( C_0 = C_0(\gamma, R_{\text{big}}) \) is a positive constant depending on \( \gamma \) and the comparability constant \( R_{\text{big}} \) for \( \Omega \) appearing in (2.1). We have thus shown that \( J \in \mathcal{M}_{\text{deep}} \) and \( y \in S(J^*) \) imply \( J \cap L \subseteq C \text{Sh}(y; \gamma) \) with \( C = 2C_0 + 1 \).

Let \( \mathcal{J} = J \cap L \) be the intersection of the quasicube \( J \) with \( L \), and note that the linear measure of \( \mathcal{J} \) satisfies \( |\mathcal{J}| \lesssim \ell(J) \). Moreover, \( \mathcal{J} \) need not be an interval if the quasicube’s edge is close to being parallel to \( L \). Fix a point \( y \). Then for a quasicube \( J \) satisfying both (3.22) and \( y \in S(J^*) \), the set \( \mathcal{J} \) is contained inside the multiple \( C \text{Sh}(y; \gamma) \) of the shadow, and

\[ |y - c_J| \gtrsim \text{dist}(y, L). \]

Now we face two difficulties that do not arise for usual cubes with a side parallel to \( L \). First, as already mentioned, \( \mathcal{J} \) need not be an interval, and in fact may be a quite complicated set, and second that a quasicube may intersect the line \( L \) in a set having linear measure far less than its side length, for example when a tilted cube
intersects $L$ near a vertex. Both of these difficulties are surmounted using the fact that the quasicubes $J$ belong to a collection $M_{r-deep}(I_r)$ for some $r$ (we are still suppressing the index $\ell$). Indeed, we first show that there is a positive constant $C'$ such that for each $r$ we have

$$
\sum_{J \in M_{r-deep}(I_r) \atop \emptyset \neq J \subset \text{Sh}(y; \gamma), y \in S(J^*)} \ell(J) \leq \beta |I_r \cap C' \text{Sh}(y; \gamma)|,
$$

where $\beta$ is the constant appearing in the bounded overlap condition (2.8). To see this, we note that if $\emptyset \neq J \subset L$, then $J^* = \gamma J$ satisfies $|J^* \cap L| \geq \ell(J)$ provided $\gamma$ is large enough depending on the constant $R_{\text{big}}$ in (2.1), and we also have $J^* \subset C' \text{Sh}(y; \gamma)$ where $C' = C'(C, \gamma, R_{\text{big}})$ is a positive constant depending on $C, \gamma$ and $R_{\text{big}}$. Altogether we thus have

$$
\sum_{J \in M_{r-deep}(I_r) \atop \emptyset \neq J \subset \text{Sh}(y; \gamma), y \in S(J^*)} \ell(J) \leq \sum_{J \in M_{r-deep}(I_r) \atop \emptyset \neq J \subset \text{Sh}(y; \gamma), y \in S(J^*)} |J^* \cap L| \leq \beta \int_{L \cap I_r} 1_{C' \text{Sh}(y; \gamma)} dx \leq \beta \int_{C' \text{Sh}(y; \gamma) \cap I_r} dx = \beta |C' \text{Sh}(y; \gamma) \cap I_r|,
$$

which proves (3.23).

Now we continue with the estimate

$$
\sum_{J \in M_{\text{deep}} \atop \emptyset \neq J \subset \text{Sh}(y; \gamma), y \in S(J^*)} \left( \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|} \right)^{n+1-a} \leq \sum_{r=1}^{\infty} \sum_{J \in M_{r-deep}(I_r) \atop \emptyset \neq J \subset \text{Sh}(y; \gamma), y \in S(J^*)} \left( \frac{|J|^\frac{1}{n}}{|y - c_J|} \right)^{n+1-a} \frac{|J|^\frac{1}{n}}{\text{dist}(y, L)} \leq \frac{1}{\text{dist}(y, L)} \sum_{r=1}^{\infty} \beta |J^r \cap C' \text{Sh}(y; \gamma)| \leq \beta \frac{1}{\text{dist}(y, L)} |C' \text{Sh}(y; \gamma)| \leq \beta \gamma,
$$

(3.24)
because $n-\alpha > 0$ and the sets $\{ I_r \cap C' \text{ Sh}(y; \gamma) \}_{r=1}^\infty$ are pairwise disjoint in $C' \text{ Sh}(y; \gamma)$. It is here that the one-dimensional nature of $\omega$ permits the summing of the side lengths of the quasicubes. Thus we have

$$B \leq A_2^n \int_I F(y) d\sigma(y) \leq CA_2^n |J|_\omega,$$

which is the desired estimate in the case that $n - 1 \leq \alpha < n$.

Now we suppose that $0 \leq \alpha < n - 1$ and use Cauchy–Schwarz to obtain

$$P^\alpha(J, 1_{S(J^*)\sigma}) = \int_{S(J^*)} \frac{|J|^{\frac{1}{n}}}{(|J|^\frac{1}{n} + |y - c_J|)^{n+1-\alpha}} d\sigma(y) \leq \left\{ \int_{S(J^*)} \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|)^{n+1-\alpha}} \left( \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|} \right)^{n-\alpha} d\sigma(y) \right\}^{\frac{1}{2}} \times \left\{ \int_{S(J^*)} \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|)^{n+1-\alpha}} \left( \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|} \right)^{n-\alpha} d\sigma(y) \right\}^{\frac{1}{2}} = P^\alpha(J, 1_{S(J^*)\sigma}) \leq P^\alpha(J^*, 1_{S(J^*)\sigma}) \leq A_2^n \sum_{J \in M_{\text{deep}}} |J|^{1-\frac{\alpha}{n}} \int_{S(J^*)} \frac{(|J|^\frac{1}{n})^{a+2-n}}{(|J|^\frac{1}{n} + |y - c_J|)^2} d\sigma(y) \leq A_2^n \sum_{J \in M_{\text{deep}}} |J|^{1-\frac{\alpha}{n}} \int_{S(J^*)} \frac{(|J|^\frac{1}{n})^{a+2-n}}{(|J|^\frac{1}{n} + |y - c_J|)^2} d\sigma(y) \leq A_2^n \sum_{J \in M_{\text{deep}}} |J|^{1-\frac{\alpha}{n}} \int_{S(J^*)} \frac{(|J|^\frac{1}{n})^{a+2-n}}{(|J|^\frac{1}{n} + |y - c_J|)^2} d\sigma(y) \leq A_2^n \int_I \left\{ \sum_{J \in M_{\text{deep}}} \left( \frac{|J|^\frac{1}{n}}{|J|^\frac{1}{n} + |y - c_J|} \right)^2 1_{S(J^*)}(y) \right\} d\sigma(y) \equiv A_2^n \int_I F(y) d\sigma(y),$$

Then arguing as above we have

$$B \leq \sum_{J \in M_{\text{deep}}} P^\alpha(J^*, 1_{S(J^*)\sigma})^2 |J|_\omega$$

(3.25)
and again $F(y) \leq C$ is the calculation above when $n + 1 - \alpha$ is replaced by 2:

$$\sum_{J \in \mathcal{M}_{deep}} \left( \frac{|J|^{\frac{1}{2}}}{|J|^{\frac{\alpha}{2}} + |y - c_J|} \right)^2 \leq \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{r-deep}(I_r)} \left( \frac{|J|^{\frac{1}{2}}}{|J|^{\frac{\alpha}{2}} + |y - c_J|} \right)^2 \leq \sum_{r=1}^{\infty} \sum_{J \in \mathcal{M}_{r-deep}(I_r)} \frac{|J|^{\frac{1}{2}}}{|y - c_J|} \frac{|J|^{\frac{1}{2}}}{\text{dist}(y, L)} \leq \frac{1}{\text{dist}(y, L)} \sum_{r=1}^{\infty} \beta |I_r \cap C' \text{Sh}(y; \gamma)| \lesssim \beta \frac{1}{\text{dist}(y, L)} |C' \text{Sh}(y; \gamma)| \lesssim \beta \gamma.$$

Thus we again have

$$B \leq A_2^\alpha \int F(y) d\sigma(y) \leq C A_2^\alpha |I|_\sigma,$$

and this completes the proof of necessity of the quasienergy conditions when one of the measures is supported on a line.

### 3.3 Backward triple testing and quasiweak boundedness property.

In this subsection we show that for a measure supported on a line, the backward triple quasitesting condition,

$$\int_{Q'} |R_{\psi}^{\mu,\nu}(1_Q \omega)|^2 d\sigma \leq \mathcal{S}_{R_{\psi}^{\mu,\nu},\text{triple, dual}} |Q'|_\omega,$$

is controlled by the $A_2^\alpha$ conditions with holes and the two quasitesting conditions, namely

$$\mathcal{S}_{R_{\psi}^{\mu,\nu},\text{triple, dual}} \lesssim \mathcal{S}_{R_{\psi}^{\mu,\nu},\text{dual}} + \sqrt{A_2^{\alpha}} + \sqrt{A_2^{\alpha,\text{dual}}},$$

provided that $\Omega$ is a $C^1$ diffeomorphism and $L$-transverse, where $L$ is the support of $\omega$. It is then an easy consequence of the Cauchy–Schwarz inequality that the weak boundedness property is also controlled by the $A_2$ conditions and the two
quasitesting conditions,

\[\int_Q \mathbf{R}_{\eta}^n(1_Q^*\omega) d\sigma \lesssim \left(4\mathbf{r}_{\Omega_1}^\alpha,\text{dual} + \sqrt{A_2^\alpha} + \sqrt{A_2^\alpha,\text{dual}}\right) \sqrt{|Q|_\sigma |Q'|_\omega},\]

for all pairs of quasicubes \(Q\) and \(Q'\) of size comparable to their distance apart.

We will use the following two properties of an \(L\)-transverse \(C^1\) diffeomorphism \(\Omega\) as defined in Definition 11 above.

**Lemma 27.** Suppose that \(\Omega : \mathbb{R}^n \to \mathbb{R}^n\) is a \(C^1\) diffeomorphism and \(L\)-transverse. Then if \(e_L\) is a unit vector in the direction of \(L\), we have

\[(3.27) \quad \left| \langle \frac{D\Omega^{-1}(x)e_L}{|D\Omega^{-1}(x)e_L|}, e_k \rangle \right| \leq \frac{1 + \eta}{2}, \quad \text{for } x \in \mathbb{R}^n \text{ and } 1 \leq k \leq n,

and

\[(3.28) \quad Q \cap L \text{ is connected whenever } Q \in \Omega_{\Omega_1}^n\par.

**Proof.** Choose a rotation \(R \in \mathcal{G}_{e_L,\eta}\). Then we have

\[\left| \langle \frac{D\Omega^{-1}(x)e_L}{|D\Omega^{-1}(x)e_L|}, e_k \rangle - \Omega^{-1}(x)e_L, e_k \rangle \right| \leq \left| \frac{D\Omega^{-1}(x)e_L}{|D\Omega^{-1}(x)e_L|} - D\Omega^{-1}(x)e_L \right|

\leq |Re_L - D\Omega^{-1}(x)e_L|

since

\[\left| \frac{v}{|v|} - v \right| = \text{dist}(v, S^{n-1}) \leq |v - Re_L|.

Thus using \(\|D\Omega^{-1} - R\|_\infty < \frac{1 - \eta}{4}\) from the definition of \(L\)-transverse, we obtain

\[\left| \langle \frac{D\Omega^{-1}(x)e_L}{|D\Omega^{-1}(x)e_L|}, e_k \rangle \right| = \left| \langle \frac{D\Omega^{-1}(x)e_L}{|D\Omega^{-1}(x)e_L|} - D\Omega^{-1}(x)e_L, e_k \rangle + \langle Re_L, e_k \rangle + \langle (D\Omega^{-1}(x) - R)e_L, e_k \rangle \right|

\leq \|D\Omega^{-1} - R\|_\infty + \eta + \|D\Omega^{-1} - R\|_\infty < \frac{1 + \eta}{2},

which proves (3.27).

Now suppose that \(Q \in \Omega_{\Omega_1}^n\par\) satisfies \(Q \cap L \neq \emptyset\). Then \(\Omega^{-1}L\) is the image of a differentiable curve. Let \(\varphi : \mathbb{R} \to \mathbb{R}^n\) be a parameterization of \(\Omega^{-1}L\). The tangent directions \(\frac{D\varphi(t)}{D\varphi(t)}\) of the curve \(\Omega^{-1}L\) are given by \(\frac{D\Omega^{-1}(\varphi(t))e_L}{|D\Omega^{-1}(\varphi(t))e_L|}\), which satisfy (3.27).

Set

\[K \equiv \Omega^{-1}Q\]
and note that $K$ is an ordinary half-open half-closed cube in $\Omega_{\text{par}}^n$, which without loss of generality we may take to be $K = [-1, 1]^n$. Let

$$\alpha \equiv \inf \varphi^{-1} K \quad \text{and} \quad \beta \equiv \sup \varphi^{-1} K.$$ 

Now assume in order to derive a contradiction that $\Omega^{-1} L \cap \Omega^{-1} Q$ is not connected. It follows from (3.27) that if $\varphi(t) \in \partial K$, then the tangent line at $\varphi(t)$ intersects the complement of $\overline{K}$ in any neighbourhood of $\varphi(t)$, and hence there is $t_0 \in (\alpha, \beta)$ such that $\varphi(t_0) = (\varphi_1(t_0), \ldots, \varphi_n(t_0)) \notin \overline{K}$. Thus there is $k_2$ such that $|\varphi_{k_2}(t_0)| > 1$. Let $k_1$ be any index other than $k_2$.

Let $P$ be the orthogonal projection of $\mathbb{R}^n$ onto the 2-plane $\Pi$ spanned by $e_{k_1}$ and $e_{k_2}$. Then $P\varphi$ is a differentiable curve whose image lies in $\Pi$ and satisfies the following analogue of (3.27):

$$(3.29) \quad \left\langle \frac{D P \varphi(t)}{|D P \varphi(t)|}, e_k \right\rangle = \left\langle \frac{D P \Omega^{-1}(\varphi(t))e_L}{|D P \Omega^{-1}(\varphi(t))e_L|}, e_k \right\rangle \leq \frac{1 + \eta}{2} < 1,$$

for $t \in \mathbb{R}$ and $k = k_1, k_2$.

Thus the image $P \Omega^{-1} L$ of the curve $P \varphi$ may be written as the graph of a continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ whose domain is identified with the $x_{k_1}$-axis and whose range is identified with the $x_{k_2}$-axis. Then we have

$$|f(x_{k_1})| = |\varphi_{k_2}(t_0)| > 1$$

for $x_{k_1} = \varphi_{k_1}(t_0)$. The map $g(t) = f^{-1}(\varphi_{k_2}(t))$ is differentiable and one-to-one, hence monotone and $g(\alpha), g(\beta) \in [-1, 1]$. We may suppose $g$ is strictly increasing. We also have $f(g(\alpha)) = \varphi_{k_1}(\alpha), f(g(\beta)) = \varphi_{k_1}(\beta) \in [-1, 1]$ since $\varphi(\alpha), \varphi(\beta) \in \overline{K}$. It follows that $x_{k_1} \in (g(\alpha), g(\beta))$, and hence $f$ must have a relative extreme value at some point $z \in (g(\alpha), g(\beta))$. But then $f'(z) = 0$ which implies $D P \varphi(g^{-1}(z))$ is parallel to $e_{k_1}$, and so contradicts

$$\left| \left\langle \frac{D P \varphi(g^{-1}(z))}{|D P \varphi(g^{-1}(z))|}, e_{k_1} \right\rangle \right| < 1$$

from (3.29).

To prove the backward triple quasitesting condition in (3.26),

$$\int_{3Q'} |R_{\varphi}^{\alpha,n}(1_Q |\omega)|^2 d\sigma \leq \xi_{R_{\varphi}^{\alpha,n,\text{triple, dual}}} |Q' |\omega,$$

it suffices to decompose the triple $3Q'$ into $3^n$ dyadic quasicubes $Q$ of side length that of $Q'$, and then apply backward testing to $Q = Q'$ which gives $\xi_{R_{\varphi}^{\alpha,n,\text{dual}}} |Q' |\omega$, and then to prove

$$\int_Q |R_{\varphi}^{\alpha,n}(1_Q |\omega)|^2 d\sigma \leq \left( \sqrt{A_{2d}^a} + \sqrt{A_{2d}^{\alpha,\text{dual}}} \right) |Q' |\omega,$$
where \( Q \) and \( Q' \) are distinct quasicubes of equal side length in a common dyadic quasigrid that share an \((n - 1)\)-dimensional quasiface \( \mathbb{F} \) in common. We also assume that \( \omega \) is supported on a line \( L \) that is parallel to a coordinate axis. The cases when the quasicubes \( Q \) and \( Q' \) meet in an ‘edge’ of smaller dimension is handled in similar fashion.

From Lemma 27 we see that the line \( L \) meets the quasihyperplane \( \mathbb{H} \) containing the quasiface \( \mathbb{F} \) at an angle at least \( \varepsilon > 0 \) depending on the constant \( \eta \) in Definition 11, and that the intersection \( Q' \cap L = Q'' \cap L \) is an interval. Now select the smallest possible dyadic subquasicube \( Q'' \) of \( Q' \) such that \( Q' \cap L = Q'' \cap L \). It follows that the length of the interval \( Q'' \cap L \) is comparable to \( \ell(Q'') \), with comparability constants depending on \( \varepsilon > 0 \). Then since \( Q \setminus 3Q'' \) is well separated from \( Q'' \) we have

\[
\int_{Q \setminus 3Q''} |\mathcal{R}_\Psi^{a,n}(1_{Q'\omega})|^2 \, d\sigma \leq \left( \sqrt{A_2^{a_d}} + \sqrt{A_2^{a_d}} \right) |Q''|_{\omega},
\]

and it remains to consider the integrals \( \int_{Q''} |\mathcal{R}_\Psi^{a,n}(1_{Q'\omega})|^2 \, d\sigma \) as \( Q''' \) ranges over all dyadic quasicubes in \( 3Q'' \cap Q \) with side length \( \ell(Q'') \). Without loss of generality, we may ignore the cases where \( Q''' \) and \( Q'' \) fail to share a face, since in the proof below, the only relevant property is that \( 1_{Q'\omega} \) is supported on a segment \( Q'' \cap L \) that is ‘transverse’ to the quasicube \( Q'' \) that supports \( 1_{Q'\sigma} \), and meets it in a single point.

So we assume now that, with \( R \) denoting an appropriate rotation, we have

1. \( Q''' = \Psi K''' \) where \( K''' = R([-1, 0] \times [-\frac{1}{2}, \frac{1}{2}])^{n-1} \),
2. \( Q'' = \Psi K' \) where \( K'' = R([0, 1] \times [-\frac{1}{2}, \frac{1}{2}])^{n-1} \),
3. \( \text{supp } \omega \subset L = (-\infty, \infty) \times \{(0, \ldots, 0)\} \), the \( x_1 \)-axis,
4. \( Q'' \cap L \) is an interval of length comparable to \( \ell(Q'') \).

Then the restriction \( \omega_{Q''} \) of \( \omega \) to the quasicube \( Q'' \) has support \( \text{supp } \omega_{Q''} \) contained in the line segment \( S \equiv Q'' \cap L \), while the restriction \( \sigma_{Q''} \) of \( \sigma \) to the quasicube \( Q''' \) has support \( \text{supp } \sigma_{Q''} \) contained in the quasicube \( Q''' \). We exploit the distinguished role played by the unique point in \( \partial Q''' \cap \partial Q'' \cap L \), which we relabel as the origin, by writing \( y = t \xi \in Q''' \) where \( t = |y| \) and \( \xi \in \mathbb{S}^{n-1} \), and by writing \( x = (s, 0) \in Q'' \cap L \), so that for an appropriate \( a \approx 1 \), we have

\[
\int_{Q''} |\mathcal{R}_\Psi^{a,n}(1_{Q''\omega})|^2 \, d\sigma \lesssim \int_{Q''} \left\{ \int_{Q''} |y - x|^{a-n} \, d\sigma(y) \right\}^2 \, d\sigma(x) \approx \int_{Q''} \left\{ \int_{Q''} \left| \psi - (s, 0) \right|^{a-n} \, d\sigma(y) \right\}^2 \, d\sigma(x)
\]

\[
= \int_{Q''} \left\{ \int_0^a |t \xi - (s, 0)|^{a-n} \, d\sigma(t \xi) \right\}^2 \, d\sigma(t) \approx \int_{Q''} \left\{ \int_0^a (t + s)^{a-n} \, d\sigma(s) \right\}^2 \, d\sigma(t)
\]

\[
\approx \int_{Q''} \left\{ \int_0^\infty \left( t + s \right)^{a-n} \, d\sigma(t) \right\}^2 \, d\sigma(s) \approx \left\{ \int_0^\infty t^{a-n} \, d\sigma(t) \right\}^2 \, d\sigma(s) \equiv I + II,
\]
where the one-dimensional measures $\tilde{\omega}$ and $\tilde{\sigma}$ are uniquely determined by $\omega$, $Q''$ and $\sigma, Q'''$ respectively by the passage from the second line to the third line above. Note also that the approximation in the second line above follows from (3.27). Now as in [LaSaShUr], we apply Muckenhoupt’s two-weight Hardy inequality for general measures (see Hytönen [Hyt2] for a proof), to obtain

$$
\int_0^\infty \left\{ \int_{(0,r]} f(s)d\mu(s) \right\}^2 dv(t) \leq \left\{ \sup_{0<r<\infty} \left( \int_{[r,\infty)} dv \right) \left( \int_{(0,r]} d\mu \right) \right\} \int_0^\infty f(s)^2 d\mu(s),
$$

with $\mu = \tilde{\omega}$, $dv(t) = t^{2\alpha-2n}d\tilde{\sigma}(t)$ and $f = 1$ to obtain that

$$
I = \int_0^\infty \left\{ \int_{(0,r]} \tilde{d}\tilde{\omega}(s) \right\}^2 t^{2\alpha-2n}d\tilde{\sigma}(t)
$$

$$
\lesssim \left\{ \sup_{0<r<\infty} \left( \int_{[r,\infty)} t^{2\alpha-2n}d\tilde{\sigma}(t) \right) \left( \int_{(0,r]} d\tilde{\omega}(s) \right) \right\} \int_0^\infty d\tilde{\omega}(s) \lesssim A_2^a |Q''|_\omega
$$

since $\int_0^\infty d\tilde{\omega}(s) = |Q''|_\omega$ and

$$
\left( \int_{[r,\infty)} t^{2\alpha-2n}d\tilde{\sigma}(t) \right) \left( \int_{(0,r]} d\tilde{\omega}(s) \right)
$$

$$
\approx \int_{Q'' \cap \{ |y| \geq r \}} |y-x|^{2\alpha-2n}d\sigma(y) \int_{Q'' \cap \{ |x| \leq r \}} d\omega(x)
$$

$$
\approx \int_{Q'' \cap \{ |y| \geq r \}} \left( \frac{r}{|y|+r} \right)^{n-\alpha}d\sigma(y) r^{\alpha-\eta} |Q'' \cap \{ |x| \leq r \}|_\omega \lesssim A_2^a r.
$$

Then we apply the two-weight dual Hardy inequality

$$
\int_0^\infty \left\{ \int_{(0,r]} f(s)d\mu(s) \right\}^2 dv(t) \leq \left\{ \sup_{0<r<\infty} \left( \int_{(0,r]} d\mu \right) \left( \int_{[r,\infty)} dv \right) \right\} \int_0^\infty f(s)^2 d\mu(s),
$$

with $d\mu(s) = s^{2\alpha-2n}d\tilde{\omega}(s)$, $dv(t) = d\tilde{\sigma}(t)$ and $f(s) = s^{\alpha-\eta}$ to obtain that

$$
II = \int_0^\infty \left\{ \int_{(0,r]} s^{\alpha-\eta} d\tilde{\omega}(s) \right\}^2 d\tilde{\sigma}(t) = \int_0^\infty \left\{ \int_{[r,\infty)} s^{\alpha-\eta} d\mu(s) \right\}^2 d\tilde{\sigma}(t)
$$

$$
\lesssim \left\{ \sup_{0<r<\infty} \left( \int_{(0,r]} d\tilde{\sigma}(t) \right) \left( \int_{[r,\infty)} s^{2\alpha-2n}d\tilde{\omega}(s) \right) \right\} \int_0^\infty s^{2n-2\alpha}d\mu(s) \lesssim A_2^a |Q'|_\omega
$$

since

$$
\int_0^\infty s^{2n-2\alpha}d\mu(s) = \int_0^\infty d\tilde{\omega}(s) = |Q''|_\omega
$$

and

$$
\left( \int_{(0,r]} d\tilde{\sigma}(t) \right) \left( \int_{[r,\infty)} s^{2\alpha-2n}d\tilde{\omega}(s) \right) \lesssim A_2^{a,\text{dual}}
$$

just as above.
Remark 28. In the case where one measure is supported on the $x$-axis, we need only test over cubes with sides parallel to the coordinate axes. For the Cauchy operator this is in [LaSaShUrWi] and for the higher-dimensional case see the earlier versions of the current paper on the arXiv.

4 One measure compactly supported on a $C^{1,\delta}$ curve

Suppose that $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^{1,\delta}$ diffeomorphism. Recall the associated class of conformal Riesz vector transforms $R_{\Psi}^{a,n}$ whose kernels $K_{\Psi}^{a,n}$ are given by

$$K_{\Psi}^{a,n}(x, y) = \frac{y - x}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}}.$$ 

First we investigate the effect of a change of variable on the statement of the $T1$ theorem.

4.1 Changes of variable. Suppose that $\Psi : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^{1,\delta}$ diffeomorphism, i.e., that both $\Psi$ and its inverse $\Psi^{-1}$ are globally $C^{1,\delta}$ maps. In particular, we have that

$$|\Psi(y) - \Psi(x)| \leq \|\Psi\|_{C^1} |y - x|,$$

$$|\Psi^{-1}(w) - \Psi^{-1}(z)| \leq \|\Psi^{-1}\|_{C^1} |w - z|,$$

(4.1) $$\Rightarrow \frac{1}{\|\Psi^{-1}\|_{C^1}} \leq \frac{|\Psi(y) - \Psi(x)|}{|y - x|} = \frac{|w - z|}{|\Psi^{-1}(w) - \Psi^{-1}(z)|} \leq \|\Psi\|_{C^1},$$

$$\Rightarrow \frac{1}{\|\Psi^{-1}\|_{C^1}} \leq \|D\Psi\|_{\infty} \leq \|\Psi\|_{C^1}.$$ 

Let

(4.2) $$\Psi^*K^{a,n}(x, y) \equiv K^{a,n}(\Psi(x), \Psi(y)) = \frac{\Psi(y) - \Psi(x)}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}}$$

be the pullback of the kernel $K^{a,n}$ under $\Psi$, and define the corresponding operator

$$(\Psi^*R^{a,n})(f\mu)(x) = \int \Psi^*K^{a,n}(x, y)f(y)d\mu(y).$$

We claim the equality

(4.3) $$\mathcal{N}_{R^{a,n}}(\sigma, \omega) = \mathcal{N}_{\Psi^*R^{a,n}}(\Psi^*\sigma, \Psi^*\omega),$$

where $\Psi^*\sigma = (\Psi^{-1})_*\sigma$ denotes the pushforward of $\sigma$ under $\Psi^{-1}$, but as $\Psi$ is a homeomorphism we abuse notation by writing $\Psi^*$ for $(\Psi^{-1})_*$, and where $\mathcal{N}_{R^{a,n}}(\sigma, \omega)$ is the best constant in the inequality

(4.4) $$\int |R^{a,n}f\sigma|^2 d\omega \leq \mathcal{N}_{R^{a,n}}(\sigma, \omega) \int |f|^2 d\sigma,$$
and similarly for $\mathcal{N}_y\mathcal{R}^{\alpha,n}(\Psi^*\sigma, \Psi^*\omega)$. Indeed, with the change of variable $x' = \Psi(x)$, $y' = \Psi(y)$, and setting $\Psi^*f = f \circ \Psi$, etc., we have

$$\int |R^{\alpha,n}f\sigma(x')|^2 d\omega(x') = \int |R^{\alpha,n}f\sigma(\Psi(x))|^2 d\Psi^*\omega(x);$$

$$R^{\alpha,n}f\sigma(\Psi(x)) = \int K^{\alpha,n}(\Psi(x), y')f(y')d\sigma(y')$$
$$= \int K^{\alpha,n}(\Psi(x), \Psi(y))f(\Psi(y))d\Psi^*\sigma(y)$$
$$= (\Psi^*R^{\alpha,n})(\Psi^*f\Psi^*\sigma)(x);$$

$$\int |f(y')|^2 d\sigma(y') = \int |\Psi^*f(y)|^2 d\Psi^*\sigma(y),$$

which shows that (4.4) becomes

$$\int |(\Psi^*R^{\alpha,n})(\Psi^*f\Psi^*\sigma)(x)|^2 d\Psi^*\omega(x) \leq \mathcal{N}_{\mathcal{R}^{\alpha,n}} \int |\Psi^*f(y)|^2 d\Psi^*\sigma(y),$$

and hence that (4.3) holds.

Now the operator $\Psi^*R^{\alpha,n}$ is easily seen to be a standard fractional singular integral, but it fails to be a conformal Riesz transform in general because the phase $\Psi(y) - \Psi(x)$ in the kernel in (4.2) is not $y-x$. We will rectify this drawback by showing that the boundedness of the conformal Riesz transform $\mathcal{R}^{\alpha,n}_\Psi$ is equivalent to that of $\Psi^*R^{\alpha,n}$, and that the appropriate testing conditions are equivalent as well. So consider the two inequalities (4.4) and

$$(4.5) \int |R^{\alpha,n}_\Psi(f^2\Psi^*\sigma)|^2 d\Psi^*\omega \leq \mathcal{N}_{R^{\alpha,n}_\Psi}(\Psi^*\sigma, \Psi^*\omega) \int |f|^2 d\Psi^*\sigma,$$

where we recall that the measures $\Psi^*\omega$ and $\Psi^*\sigma$ are the pushforwards under $\Psi^{-1}$ of the measures $\omega$ and $\sigma$ respectively. Here the constants $\mathcal{N}_{\mathcal{R}^{\alpha,n}_\Psi}(\sigma, \omega)$ and $\mathcal{N}_{R^{\alpha,n}_\Psi}(\Psi^*\sigma, \Psi^*\omega)$ are the smallest constants in their respective inequalities.

At this point we fix a collection of quasicubes $\Omega\mathcal{Q}^n$ with $\Omega$ biLipschitz, and recall the Muckenhoupt and energy constants

$$A^2_2(\sigma, \omega), A^2_{2,\text{dual}}(\sigma, \omega), A^2_{2,\text{punct}}(\sigma, \omega), A^2_{2,\text{punct,dual}}(\sigma, \omega),$$

$$\mathcal{E}_\alpha^{\Omega\mathcal{Q}^n}(\sigma, \omega), \mathcal{E}^{\Omega\mathcal{Q}^n,\text{dual}}(\sigma, \omega);$$

$$A^2_2(\Psi^*\sigma, \Psi^*\omega), A^2_{2,\text{dual}}(\Psi^*\sigma, \Psi^*\omega), A^2_{2,\text{punct}}(\Psi^*\sigma, \Psi^*\omega), A^2_{2,\text{punct,dual}}(\Psi^*\sigma, \Psi^*\omega),$$

and $\mathcal{E}_\alpha^{\Omega\mathcal{Q}^n}(\Psi^*\sigma, \Psi^*\omega), \mathcal{E}^{\Omega\mathcal{Q}^n,\text{dual}}(\Psi^*\sigma, \Psi^*\omega),$
that depend only on the measures and the quasicubes, and the testing constants
\[ T_{\Omega_1}(\sigma, \omega), \quad T_{\Omega_1}^{\text{dual}}(\sigma, \omega); \]
\[ T_{\Psi_1}^{\text{dual}}(\Psi^*\sigma, \Psi^*\omega), \quad T_{\Psi_1}^{\text{dual}}(\Psi^*\sigma, \Psi^*\omega), \]
that depend on the measures and the quasicubes as well as the fractional singular integral and its tangent line truncations. We sometimes suppress the dependence \((\sigma, \omega)\) on the measures when they are understood from the context, or when they do not play a significant role.

Finally, we define an even more general testing condition. Let \( F \) be a collection of bounded Borel sets, and let \( T^{\alpha, n} \) be an \( \alpha \)-fractional singular integral. Then define \( \mathcal{T}^{\mathcal{F}}_{\mathcal{T}^{\alpha, n}}(\sigma, \omega) \) to be the smallest constant in the inequality
\[
\int |T^{\alpha, n}1_F \sigma|^2 \, d\omega \leq \mathcal{T}^{\mathcal{F}}_{\mathcal{T}^{\alpha, n}}|F|_\sigma, \quad F \in \mathcal{F},
\]
and similarly for the dual \( \mathcal{T}^{\mathcal{F}, \text{dual}}_{\mathcal{T}^{\alpha, n, \text{dual}}}(\sigma, \omega) \). Note that our testing conditions above are with \( \mathcal{F} = \Omega^{\mathcal{Q}^n} \) and \( T^{\alpha, n} = R^{\alpha, n} \) or \( R^{\alpha, n}_\psi \). Given \( \Psi \) and \( \mathcal{F} \) as above, denote by
\[
\Psi^*\mathcal{F} \equiv \{ \Psi^{-1}(F) : F \in \mathcal{F} \}
\]
the pullback of \( \mathcal{F} \) under the map \( \Psi \), i.e., \( \Psi^*1_F = 1_{\Psi^{-1}(F)} \). Of particular interest for us is the set of quasicubes \( \mathcal{Q} = \Omega^{\mathcal{Q}^n} \) which is used in the versions given above of the testing conditions. Then we have \( \Psi^*\Omega = \{ \Psi^{-1}(Q) : Q \in \Omega \} \), and the sets \( \Psi^{-1}(Q) \) form a new family of quasicubes since \( \Psi^{-1} \circ \Omega \) is a globally biLipschitz map, which if necessary we will refer to as \( \Psi^{-1} \circ \Omega \)-quasicubes.

Our first proposition concerns the equivalence of the Muckenhoupt conditions under a biLipschitz change of variable, and the next proposition considers norm inequalities and testing conditions.

**Proposition 29.** Suppose \( \Omega : \mathbb{R}^n \to \mathbb{R}^n \) is a globally biLipschitz map and that the Muckenhoupt conditions are defined by taking supremums over the collection \( \Omega^{\mathcal{Q}^n} \) of \( \Omega \)-quasicubes. Let \( \Psi \) be another globally biLipschitz map, and let \( \sigma \) and \( \omega \) be positive Borel measures possibly having common point masses. Then we have the following three equivalences:
\[
A_2^{\alpha, \text{punct}}(\sigma, \omega) \approx A_2^{\alpha, \text{punct}}(\Psi^*\sigma, \Psi^*\omega),
\]
\[
A_2^{\alpha, \text{punct, dual}}(\sigma, \omega) \approx A_2^{\alpha, \text{punct, dual}}(\Psi^*\sigma, \Psi^*\omega),
\]
\[
\mathcal{A}_2^d(\sigma, \omega) = \mathcal{A}_2^d(\Psi^*\sigma, \Psi^*\omega).
\]

Note the absence of any statement regarding the one-tailed Muckenhoupt conditions with holes, \( A_2^a \) and \( A_2^{a, \text{dual}} \), where it is not obvious that an equivalence is possible.
Proof. For convenience we set \( \tilde{\sigma} = \Psi^* \sigma \), \( \tilde{\omega} = \Psi^* \omega \) and \( \tilde{Q} = \Psi(Q) \). In order to show that

\[ A_2^{a, \text{punct}}(\Psi^* \sigma, \Psi^* \omega) \lesssim A_2^{a, \text{punct}}(\sigma, \omega), \]

it suffices to show that

\[ \frac{\tilde{\omega}(K, \Psi_1(\tilde{\sigma}, \tilde{\omega}))}{|K|^{1 - \frac{d}{p}}} \lesssim \sup_{Q \in \Omega^u} \frac{\omega(Q, \Psi_1(\sigma, \omega))}{|Q|^{1 - \frac{d}{p}}} \]

for all \( \Omega \)-quasicubes \( K \in \Omega \Omega^u \). Now a change of variable shows that

\[ \frac{\tilde{\omega}(K, \Psi_1(\tilde{\sigma}, \tilde{\omega}))}{|K|^{1 - \frac{d}{p}}} = \frac{\omega(\tilde{K}, \Psi_1(\sigma, \omega))}{|\tilde{K}|^{1 - \frac{d}{p}}} \]

where of course \( |K| \approx |\tilde{K}| \). Now choose a quasicube \( Q \) containing \( \tilde{K} \) with \( \ell(Q) \leq C_\Omega \ell(K) \)

so that we have

\[ |K| \approx |Q|. \]

If a largest common point mass for \( \omega \) in \( \tilde{K} \) (respectively \( Q \)) occurs at \( x \) (respectively \( y \)), then \( \omega([x]) \leq \omega([y]) \) and so we have

\[ \omega(\tilde{K}, \Psi_1(\sigma, \omega)) = \omega([\tilde{K}]) \delta_x \leq |Q|_\omega - \omega([y]) \delta_y = \omega(Q, \Psi_1(\sigma, \omega)), \]

since if \( x = y \) we use \( |\tilde{K}|_\omega \leq |Q|_\omega \), while if \( x \neq y \), then \( y \notin \tilde{K} \) and we use \( |Q|_\omega - \omega([y]) \delta_y \geq |\tilde{K}|_\omega \). Thus

\[ \frac{\tilde{\omega}(K, \Psi_1(\tilde{\sigma}, \tilde{\omega}))}{|K|^{1 - \frac{d}{p}}} \lesssim \frac{\omega(\tilde{K}, \Psi_1(\sigma, \omega))}{|\tilde{K}|^{1 - \frac{d}{p}}} \]

\[ \lesssim \frac{\omega(Q, \Psi_1(\sigma, \omega))}{|Q|^{1 - \frac{d}{p}}} \]

\[ \lesssim A_2^{a, \text{punct}}(\sigma, \omega), \]

which completes the proof of the first assertion in Proposition 29. The second assertion is proved in similar fashion.

Now we turn to the third assertion in Proposition 29, where in view of what we have just shown, it suffices to show that

\[ A_2^a(\tilde{\sigma}, \tilde{\omega}) + A_2^{a, \text{dual}}(\tilde{\sigma}, \tilde{\omega}) \lesssim A_2^a(\sigma, \omega). \]

By symmetry it is then enough to show

\[ \mathcal{P}(K, 1_{K^c} \tilde{\sigma}) \frac{|K|_{\tilde{\omega}}}{|K|^{1 - \frac{d}{p}}} \lesssim A_2^a(\sigma, \omega), \]
for all $\Omega$-quasicubes $K \in \Omega \mathcal{Q}^n$. Now a change of variable shows that
\[
\frac{|K|_{\tilde{\sigma}}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{K^c \tilde{\sigma}}) = \frac{|K|_{\tilde{\sigma}}}{|K|^{1-\frac{\alpha}{n}}} \int_{\mathbb{R}^n \setminus K} \left( \frac{|K|_{\tilde{\sigma}}^\frac{1}{n}}{(|K|_{\tilde{\sigma}}^\frac{1}{n} + |x - c_K|)^2} \right)^{n-\alpha} d\tilde{\sigma}(x)
\approx \frac{|K|_{\tilde{\sigma}}}{|K|^{1-\frac{\alpha}{n}}} \int_{\mathbb{R}^n \setminus \tilde{K}} \left( \frac{|K|_{\tilde{\sigma}}^\frac{1}{n}}{(|K|_{\tilde{\sigma}}^\frac{1}{n} + |x' - c_{\tilde{K}}|)^2} \right)^{n-\alpha} d\tilde{\sigma}(x')
\approx \frac{|K|_{\tilde{\sigma}}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(\tilde{K}, 1_{\tilde{K}^c \tilde{\sigma}}) \approx \frac{|K|_{\tilde{\sigma}}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}^c \tilde{\sigma}}),
\]
where of course $|K| \approx |\tilde{K}|$ and $\mathcal{P}(K, \mu) \approx \mathcal{P}(\tilde{K}, \mu)$. We have written $\tilde{K}$ in place of $K$ in the final equivalence only when it matters. Now choose quasicubes $Q, P \in \Omega \mathcal{Q}^n$ such that
\[
Q \subset \tilde{K} \subset P \quad \text{and} \quad \ell(P) \leq C_{\Omega} \ell(Q),
\]
so that $|Q| \approx |\tilde{K}| \approx |P|$, and $\mathcal{P}(Q, \mu) \approx \mathcal{P}(\tilde{K}, \mu) \approx \mathcal{P}(P, \mu)$ for any positive measure $\mu$. Let $y \in P$ be a point where the largest common point mass of $\sigma$ occurs, and let $z \in P$ be a point where the largest common point mass of $\omega$ occurs. Define
\[
\dot{\sigma} = \sigma - \sigma(\{y\}) \quad \text{and} \quad \dot{\omega} = \omega - \omega(\{z\}).
\]
Now we have the two ‘punctured’ inequalities,
\[
\frac{|\tilde{K}|_{\dot{\sigma}}}{|\tilde{K}|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}^c \dot{\sigma}}) \leq \frac{|P|_{\dot{\sigma}}}{|P|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{P^c \dot{\sigma}}) + \frac{|P|_{\dot{\sigma}}}{|P|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{P^c \dot{\sigma}})
\lesssim A_2^{\dot{\alpha}, \text{punct}}(\sigma, \omega) + A_2^{\dot{\alpha}}(\sigma, \omega) \leq \mathcal{Q}_2^{\dot{\alpha}}(\sigma, \omega)
\]
and
\[
\frac{|\tilde{K}|_{\dot{\sigma}}}{|\tilde{K}|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}^c \dot{\sigma}}) \leq \frac{|P|_{\dot{\sigma}}}{|P|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{P^c \dot{\sigma}}) + \frac{|P|_{\dot{\sigma}}}{|P|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{P^c \dot{\sigma}})
\lesssim A_2^{\dot{\alpha}, \text{punct,dual}}(\sigma, \omega) + A_2^{\dot{\alpha}}(\sigma, \omega) \leq \mathcal{Q}_2^{\dot{\alpha}}(\sigma, \omega).
\]
Next, we claim that if $y \neq z$, then
\[
\frac{\sigma(\{y\}) \omega(\{z\})}{|P|^{1-\frac{\alpha}{n}}} \leq A_2^{\dot{\alpha}}(\sigma, \omega) + A_2^{\dot{\alpha}, \text{dual}}(\sigma, \omega) \leq \mathcal{Q}_2^{\dot{\alpha}}(\sigma, \omega).
\]
Indeed, it is easy to find a quasicube $R \subset P$ with half the side length of $P$ (but not necessarily a quasicubich of $P$) such that exactly one of $y$ and $z$ lies in $R$. For if $y$ and $z$ lie on opposite sides of a quasihyperplane $H$ that is ‘parallel’ to the coordinate axes, then $P \setminus H$ consists of two disjoint quasirectangles, each containing one of the points $y$ and $z$. Clearly, the larger of the two quasirectangles contains a quasicube $R$ with side length at least $\frac{1}{2} \ell(P)$ and containing one of $y$ and $z$. 
With such a quasicube \( R \) in hand, say with \( y \in R \) and \( z \in R^c \), then
\[
\frac{\sigma(\{y\}) \omega(\{z\})}{|P|^{1-\frac{\alpha}{n}} |P|^{1-\frac{\delta}{n}}} \leq \frac{|R|_o}{|R|^{1-\frac{\alpha}{n}}} \mathcal{P}(R, 1_R; \omega) \leq A_2^{\alpha, \text{dual}}(\sigma, \omega) \leq 2 A_2^0(\sigma, \omega).
\]

Now we write
\[
\frac{|\tilde{K}|_{\omega}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma) = \frac{|\tilde{K}|_{\omega+\sigma(\{y\})}}{|\tilde{K}|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma + \sigma(\{y\}))
\]
\[
\leq \frac{|\tilde{K}|_{\omega}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma) + \frac{|\tilde{K}|_{\omega}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma + \sigma(\{y\}) \omega(\{z\}))
\]
\[
\frac{|\tilde{K}|_{\omega}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma) \lesssim 2 A_2^0(\sigma, \omega)
\]
If \( y \neq z \), then we have
\[
\frac{|\tilde{K}|_{\omega}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma) \lesssim 2 A_2^0(\sigma, \omega)
\]
by the three inequalities (4.6), (4.7) and (4.8) proved above. On the other hand, if \( y = z \), then either \( y \in \tilde{K} \) or \( z \in P \setminus \tilde{K} \). If \( y \in \tilde{K} \) we have
\[
\frac{|\tilde{K}|_{\omega}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma) \leq \frac{|\tilde{K}|_{\omega}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma) \lesssim 2 A_2^0(\sigma, \omega)
\]
by (4.6), and if \( z \in P \setminus \tilde{K} \) we have
\[
\frac{|\tilde{K}|_{\omega}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma) \leq \frac{|\tilde{K}|_{\omega}}{|K|^{1-\frac{\alpha}{n}}} \mathcal{P}(K, 1_{\tilde{K}}; \sigma) \lesssim 2 A_2^0(\sigma, \omega)
\]
by (4.7). This completes the proof of Proposition 29. \( \square \)

**Proposition 30.** Suppose \( \Psi : \mathbb{R}^n \to \mathbb{R}^n \) is a \( C^{1,\delta} \) diffeomorphism, i.e., both \( \Psi \) and its inverse \( \Psi^{-1} \) are globally \( C^{1,\delta} \) maps, let \( \alpha \) and \( \omega \) be positive Borel measures (possibly having common point masses) with one of the measures supported in a compact subset \( K \) of \( \mathbb{R}^n \), and let \( F \) be a collection of bounded Borel sets. Then with the fractional Riesz transform \( R^{\alpha,n} \) and the conformal fractional Riesz transform \( R_{\Phi,\alpha}^{\alpha,n} \) as above, we have the following three equivalences:

1. \( \mathcal{M}_{R^{\alpha,n}}(\sigma, \omega) + \sqrt{2 A_2^0(\sigma, \omega)} \approx \mathcal{M}_{R_{\Phi,\alpha}^{\alpha,n}}(\Psi^*\sigma, \Psi^*\omega) + \sqrt{2 A_2^0(\sigma, \omega)} \)
2. \( \frac{\mathcal{F}_R^{\alpha,n}(\sigma, \omega) + \sqrt{A_2^0(\sigma, \omega)}}{\sigma(\{y\}) } \approx \frac{\mathcal{F}_{R_{\Phi,\alpha}^{\alpha,n}}(\Psi^*\sigma, \Psi^*\omega) + \sqrt{A_2^0(\sigma, \omega)}}{\sigma(\{y\}) } \)
3. \( \frac{\mathcal{F}_{R_{\Phi,\alpha}^{\alpha,n}}(\sigma, \omega) + \sqrt{2 A_2^0(\sigma, \omega)}}{\sigma(\{y\}) } \approx \frac{\mathcal{F}_{R_{\Phi,\alpha}^{\alpha,n}}(\Psi^*\sigma, \Psi^*\omega) + \sqrt{2 A_2^0(\sigma, \omega)}}{\sigma(\{y\}) } \)

where the implied constants depend only on \( n, \alpha \), \( \text{diam}(K) \), \( ||\Gamma||_{1,\delta} \) and \( ||\Psi||_{C^{1,\delta}} + ||\Psi^{-1}||_{C^{1,\delta}} \).

In particular, we see that in the presence of the Muckenhoupt conditions \( A_2^0 \), and when one of the measures is supported in a compact set, the testing conditions
Indeed, if supported in a compact ball $B$ below are usual cubes since we invoke no testing or energy in the proof of $\Psi_1^{-1}(Q)$. Thus in order to deform the sets over which we test from $Q$ to $\Psi_1^{-1}(Q)$, we need only push the measures forward and alter the conformal factor $\Gamma$ to the associated conformal factor $\Gamma_\psi$ in the operator, keeping the critical phase $y - x$ in the numerator of the kernel unchanged. The same results hold for the inverse $C^{1,\alpha}$ diffeomorphism $\Psi^{-1}$ in place of $\Psi$.

Before beginning the proof it is convenient to introduce two auxiliary operators $\Psi^{*,\text{tan},1}R^{a,n}$ and $\Psi^{*,\text{tan},2}R^{a,n}$ with kernels related to the pullback kernel $\Psi^*K^{a,n}$ defined above by

$$\Psi^*K^{a,n}(x, y) = \frac{\Psi(y) - \Psi(x)}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}}.$$  

We define the kernels of $\Psi^{*,\text{tan},1}R^{a,n}$ and $\Psi^{*,\text{tan},2}R^{a,n}$ by

$$\Psi^{*,\text{tan},1}K^{a,n}(x, y) \equiv \Psi'(x) \frac{y - x}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}},$$

$$\Psi^{*,\text{tan},2}K^{a,n}(x, y) \equiv \Psi'(y) \frac{y - x}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}}.$$  

The superscript $\text{tan}$, 1 (respectively $\text{tan}$, 2) indicates that we are replacing the phase function $\Psi(y) - \Psi(x)$ with its tangent line approximation at $x$ (respectively $y$). Now we prove Proposition 30.

**Proof.** We begin with the first statement, where we may assume that $\omega$ is supported in a compact ball $B$. Moreover, by Proposition 29, we may assume the cubes below are usual cubes since we invoke no testing or energy in the proof of this first statement. Then we may also assume that $\sigma$ is supported in the double $2B$. Indeed, if $\omega$ is supported in a ball $B$ and $\sigma$ is supported outside the double $2B$ of the ball $B$, then the associated norm inequality for a fractional singular integral operator $T^{a,n}$ is easily seen to be controlled solely in terms of the Muckenhoupt constant $\mathcal{A}_2^q$ with holes:

$$\int_{\mathbb{R}^n \setminus 2B} |T^{a,n}1_B g(x-\omega)|^2 d\sigma \lesssim \int_{\mathbb{R}^n \setminus 2B} \int_B |x-y|^{a-n} g(x) d\omega(x) d\sigma(y)$$

$$(4.9) \lesssim \left( \int_B |g|^2 d\omega \right) \int_{\mathbb{R}^n \setminus 2B} \left( \int_B |x-y|^{2a-2n} d\omega(x) \right) d\sigma(y)$$

$$\lesssim \|g\|_{L^2(\omega)}^2 \frac{|B|_{\omega}}{|B|^{1-\beta}} \mathcal{A}_2^q (B, \sigma) \lesssim \mathcal{A}_2^q \|g\|_{L^2(\omega)}^2,$$

where we have used that $|x-y|^{2a-2n} \approx |c_B - y|^{2a-2n}$ for $y \in \mathbb{R}^n \setminus 2B$ and $x \in B$.

We write the pullback $\Psi^*K^{a,n}(x, y)$ of the vector kernel

$$K^{a,n}(x, y) = \frac{y - x}{|y - x|^{n+1-\alpha}}.$$
given in formula (4.2), in the form

\[ \Psi^* K^{a,n}(x,y) = \frac{\Psi(y) - \Psi(x)}{|\Psi(y) - \Psi(x)|^{n+1-a}} \equiv \frac{\Psi'(x)(y-x)}{|\Psi(y) - \Psi(x)|^{n+1-a}} + E_1^a(x,y), \]

(4.10)

where we write \( \Psi' \) for the derivative \( D\Psi \), and where \( \Psi^{*,\tan,1} K^{a,n} \) is the first of our auxiliary kernels defined above. We claim that the error kernel \( E_1^a(x,y) \) satisfies the improved local estimate

\[ E_1^a(x,y) = O(|y-x|^{a-n+\delta}; M) \]

for some constant \( M \) depending on \( \|\Gamma\|_{1,\delta}, \|\Psi\|_{C^{1,\delta}} \) and \( \|(\Psi')^{-1}\|_{\infty} \). Indeed, we have

\[ \frac{\Psi(y) - \Psi(x)}{|\Psi(y) - \Psi(x)|^{n+1-a}} - \frac{\Psi'(x)(y-x)}{|\Psi(y) - \Psi(x)|^{n+1-a}} = \frac{1}{|\Psi(y) - \Psi(x)|^{n+1-a}} \{ \Psi(y) - \Psi(x) - \Psi'(x)(y-x) \}. \]

Now we use the estimate

\[ |\Psi(y) - \Psi(x) - \Psi'(x)(y-x)| \leq \|\Psi\|_{C^{1,\delta}} |y-x|^{1+\delta} = O(|y-x|^{1+\delta}; \|\Psi\|_{C^{1,\delta}}), \]

(4.12)

together with the bound \( |\Gamma(\Psi(x), \Psi(y))| \leq \|\Gamma\|_{\infty} \), to obtain (4.11):

\[ |E_1^a(x,y)| \lesssim \frac{1}{|y-x|^{n+1-a}} |y-x|^{1+\delta} = |y-x|^{a-n+\delta}, \]

for \( |y-x| \leq C \).

Recall that both \( \sigma \) and \( \omega \) are supported in a fixed compact set \( K \). Let \( C_{2B} \) be a sufficiently large constant exceeding the diameter of \( 2B \). Now we bound the operator norm of the error term. For this we write \( \sigma = \Psi^* \sigma \) and \( \omega = \Psi^* \omega \) for convenience, and then observe that the norm of the error operator

\[ E_1^a f \sigma(x) \equiv \int E_1^a(x,y)f(y)d\sigma(y) \]

as a map from \( L^2(\sigma) \) to \( L^2(\omega) \) is controlled by the offset \( A_2^\delta \) constant. Indeed, from the definition of \( E_1^a(x,y) \) in (4.10), we see that the kernel \( E_1^a \) vanishes on the
diagonal, and so for $a \sim \log_2 \frac{1}{c_k}$,

$$
|\mathcal{E}_a f \sigma(x)| \leq \sum_{k=a}^{\infty} \int_{B(x, 2^{-k}) \setminus B(x, 2^{-k-1})} M|y - x|^n |f(y)|d\sigma(y)
$$

$$
\lesssim \sum_{k=a}^{\infty} 2^{-k|B(x, 2^{-k})|} \int_{B(x, 2^{-k}) \setminus B(x, 2^{-k-1})} |f(y)|d\sigma(y)
$$

$$
\lesssim \sum_{k=a}^{\infty} 2^{-k|A^k_{\alpha}(f)\sigma)(x)|},
$$

where $A^k_{\alpha, c_k}$ is the annular $\alpha$-averaging operator given by

$$
A^k_{\alpha}(f)\sigma)(x) \equiv |B(x, 2^{-k})| \int_{B(x, 2^{-k}) \setminus B(x, 2^{-k-1})} |f|d\sigma.
$$

We now claim that the boundedness of $A^k_{\alpha, c_k}$, and hence also that of $\mathcal{E}_a f$, is controlled by the offset $A^k_{\alpha, c_k}$ constant. Indeed, for a sufficiently small positive constant $c$, we have

$$
\|A^k_{\alpha}(f)\sigma)\|_{L^\infty(\omega)}^2 \leq \int_{\mathbb{R}^n} (2^{-k(a-n)} \int_{B(x, 2^{-k}) \setminus B(x, 2^{-k-1})} |f|^2 d\sigma) d\omega(x)
$$

$$
\leq 2^{-2k(a-n)} \int_{\mathbb{R}^n} \left( \int_{B(x, 2^{-k})} |f|^2 d\sigma \right) |B(x, 2^{-k}) \setminus B(x, 2^{-k-1})| d\omega(x)
$$

$$
= 2^{-2k(a-n)} \sum_{z \in \mathbb{Z}^n} \int_{B(c^{-k} z, \sqrt{nc} 2^{-k})} \left( \int_{B(c^{-k} z, 2^{-k})} |f|^2 d\sigma \right) |B(x, 2^{-k}) \setminus B(x, 2^{-k-1})| d\omega(x)
$$

$$
\leq 2^{-2k(a-n)} \sum_{z \in \mathbb{Z}^n} \int_{B(c^{-k} z, \sqrt{nc} 2^{-k})} \left( \int_{B(c^{-k} z, (\sqrt{nc} + 1) 2^{-k})} |f|^2 d\sigma \right)
$$

$$
\times |B(c^{-k} z, (\sqrt{nc} + 1) 2^{-k}) \setminus B(c^{-k} z, nc 2^{-k})| d\omega(x),
$$

$$
= \sum_{z \in \mathbb{Z}^n} \frac{|B(c^{-k} z, \sqrt{nc} 2^{-k})| d\omega|B(c^{-k} z, (\sqrt{nc} + 1) 2^{-k}) \setminus B(cn 2^{-k} z, nc 2^{-k})| d\omega(x)}{2^{2k(a-n)}}
$$

$$
\times \int_{B(\sqrt{nc} + 1) 2^{-k})} |f|^2 d\sigma.
$$

Using the separation between $B(c^{-k} z, (\sqrt{nc} + 1) 2^{-k}) \setminus B(cn 2^{-k} z, nc 2^{-k})$ and $B(c^{-k} z, \sqrt{nc} 2^{-k})$, it is easy to see that

$$
\frac{|B(c^{-k} z, \sqrt{nc} 2^{-k})| d\omega|B(c^{-k} z, (\sqrt{nc} + 1) 2^{-k}) \setminus B(cn 2^{-k} z, nc 2^{-k})| d\omega(x)}{2^{2k(a-n)}} \lesssim A^k_{\alpha, c_k}.
$$
Combining inequalities we then obtain
\[ \|A_\alpha^k(f, \sigma)\|_{L^2(\tilde{\sigma})} \lesssim \sum_{\sigma \in \mathbb{Z}^n} A_2^{\alpha, k} \int_{B(2^{-k}, (\sqrt{n}+1)^{-2^{-k}})} |f|^2 \, d\sigma \]
\[ = A_2^{\alpha, k} \int_{\mathbb{R}^n} \sum_{\sigma \in \mathbb{Z}^n} 1_{B(2^{-k}, (\sqrt{n}+1)^{-2^{-k}})} |f|^2 \, d\sigma \]
\[ \lesssim A_2^{\alpha, k} \int |f|^2 \, d\tilde{\sigma} = A_2^{\alpha} \|f\|_{L^2(\tilde{\sigma})}^2, \]
and hence
\[ \|E_1^\alpha f, \tilde{\sigma}\|_{L^2(\tilde{\sigma})} \lesssim \sum_{k=\alpha}^{\infty} 2^{-k\delta} \|A_\alpha^k(f, \sigma)\|_{L^2(\tilde{\sigma})} \lesssim \sum_{k=\alpha}^{\infty} 2^{-k\delta} A_2^{\alpha} \|f\|_{L^2(\tilde{\sigma})} \lesssim C_2B^\alpha A_2^{\alpha} \|f\|_{L^2(\tilde{\sigma})}. \]

This completes the proof that the norm of the error operator $E_1^\alpha$ as a map from $L^2(\tilde{\sigma})$ to $L^2(\tilde{\omega})$ is controlled by the offset $A_2^\alpha$ constant. For reference in proving statements (2) and (3) below, we record that in similar fashion, using the reduction that $\tilde{\sigma}$ also has compact support, the norm of the dual error operator
\[ E_2^\alpha f, \tilde{\sigma}(x) \equiv \int E_2^\alpha(x, y)f(y) \, d\sigma(y);
\]
$\Psi^*K^{\alpha, n}(x, y) = \Psi^*K^{\alpha, n}(x, y) + E_2^\alpha(x, y)$,
as a map from $L^2(\tilde{\omega})$ to $L^2(\tilde{\sigma})$ is controlled by the offset $A_2^\alpha$ constant.

Now we further analyze the first kernel on the right-hand side of (4.10), namely
\[ \Psi^{*, \tan, 1}K^{\alpha, n}(x, y) = \frac{\Psi'(x)(y-x)}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}}, \]
by writing
\[ \Psi^{*, \tan, 1}K^{\alpha, n}(x, y) = \Psi'(x)\frac{y-x}{|\Psi(y) - \Psi(x)|^{n+1-\alpha}} = \Psi'(x)K^{\alpha, n}_\Psi(x, y). \]

We now compute that
\[ \int_{\mathbb{R}^n} |\Psi^{*, \tan, 1}R^{\alpha, n}(f, \sigma)|^2 \, d\tilde{\sigma} = \int_{\mathbb{R}^n} \left( \int \Psi^{*, \tan, 1}K^{\alpha, n}_\Psi(x, y)f(y) \, d\sigma(y) \right)^2 \, d\tilde{\sigma}(x) \]
\[ = \int_{\mathbb{R}^n} \left( \int \Psi'(x)K^{\alpha, n}_\Psi(x, y)f(y) \, d\sigma(y) \right)^2 \, d\tilde{\sigma}(x) \]
\[ = \int_{\mathbb{R}^n} \Psi'(x) \left( \int K^{\alpha, n}_\Psi(x, y)f(y) \, d\sigma(y) \right)^2 \, d\tilde{\sigma}(x) \]
\[ = \int_{\mathbb{R}^n} \Psi'(x) \int R^{\alpha, n}_\Psi(f, \sigma)(x) \, d\tilde{\sigma}(x), \]
(4.13)
where the matrix $\Psi'(x)$ is acting on the vector

$$T_{\Gamma_{\Psi}}^{a,n}f\tilde{\sigma}(x) = \int K_{\Gamma_{\Psi}}^{a,n}(x, y)f(y)d\tilde{\sigma}(y).$$

Using the inequality

$$\int_{\mathbb{R}^n} |\Psi'(x)v|d\tilde{\omega} \approx |v|,$$

we conclude that

$$\int_{\mathbb{R}^n} |\Psi^{\ast,\text{tan},1} R^{a,n}f\tilde{\sigma}|^2 d\tilde{\omega} \approx \int_{\mathbb{R}^n} |R_{\Psi}^{a,n}f\tilde{\sigma}|^2 d\tilde{\omega},$$

which shows that

$$\mathcal{M}_{R^{a,n}}(\tilde{\sigma}, \tilde{\omega}) \approx \mathcal{M}_{\Psi^{\ast,\text{tan},1} R^{a,n}}(\tilde{\sigma}, \tilde{\omega}).$$

Similarly we have

$$\mathcal{M}_{R^{a,n}}(\tilde{\sigma}, \tilde{\omega}) \approx \mathcal{M}_{\Psi^{\ast,\text{tan},2} R^{a,n}}(\tilde{\sigma}, \tilde{\omega}).$$

Reverting to the notation with $\Psi^\ast$ it now follows from this, and then the boundedness of the error operator $\mathcal{E}^a$, that

$$\mathcal{M}_{R^a}^{\Psi^\ast}(\sigma, \omega) \approx \mathcal{M}_{\Psi^{\ast,\text{tan},1} R^{a,n}}(\Psi^\ast \sigma, \Psi^\ast \omega) \lesssim \mathcal{M}_{\Psi^{\ast,\text{tan},1} R^{a,n}}(\Psi^\ast \sigma, \Psi^\ast \omega) + [A_2 + A_2^{\text{dual}}]$$

$$= \mathcal{M}_{R^{\ast}}(\sigma, \omega) + [A_2 + A_2^{\text{dual}}],$$

where the final equality is (4.3). The reverse inequality in the second statement of Proposition 30 is proved in similar fashion, or simply by replacing $\Psi$ with $\Psi^{-1}$.

The second and third statements are proved in the same way as the first statement just proved above, but with the following difference. The functions $f$ under consideration are restricted to indicators $f = 1_E$ with $E \in \Psi^\ast \mathcal{F}$, and as a result we have from (4.13), and its dual version, the two identities

$$\int_{\mathbb{R}^n} |\Psi^{\ast,\text{tan},1} R^{a,n} 1_E \tilde{\sigma}|^2 d\tilde{\omega} = \int_{\mathbb{R}^n} |\Psi'(x)R_{\Psi}^{a,n}(1_E\tilde{\sigma}(x))|^2 d\tilde{\omega}(x)$$

and

$$\int_{\mathbb{R}^n} |\Psi^{\ast,\text{tan},2} R^{a,n} 1_E \tilde{\sigma}|^2 d\tilde{\sigma} = \int_{\mathbb{R}^n} |\Psi'(y)R_{\Psi}^{a,n}(1_E\tilde{\sigma}(y))|^2 d\tilde{\sigma}(y),$$

since the kernels $K^{a,n}(x, y)$ and $K_{\Gamma_{\Psi}}^{a,n}(x, y)$ are antisymmetric. Just as for the norm estimate above, we use (4.14) to obtain both

$$\int_{\mathbb{R}^n} |\Psi^{\ast,\text{tan},1} R^{a,n} 1_E \tilde{\sigma}|^2 d\tilde{\omega} \approx \int_{\mathbb{R}^n} |R_{\Psi}^{a,n}(x, y)1_E d\tilde{\sigma}|^2 d\tilde{\omega};$$

$$\mathcal{M}_{\Psi^{\ast,\text{tan},1} R^{a,n}}(\Psi^\ast \sigma, \Psi^\ast \omega) \approx \mathcal{M}_{R^{a,n}}(\Psi^\ast \sigma, \Psi^\ast \omega),$$

where the matrix $\Psi'(x)$ is acting on the vector

$$T_{\Gamma_{\Psi}}^{a,n}f\tilde{\sigma}(x) = \int K_{\Gamma_{\Psi}}^{a,n}(x, y)f(y)d\tilde{\sigma}(y).$$
and
\[ \int_{\mathbb{R}^n} |\Psi_{*}^{*, \lambda, 2} R_{\alpha, \mu}^* \alpha \tilde{\omega}|^2 d\tilde{\sigma} \approx \int_{\mathbb{R}^n} |R_{\mu, \alpha}^{*, n}(x, y) 1_E d\tilde{\omega}|^2 d\tilde{\sigma}; \]
\[ \mathcal{E}_{\Psi_*^{*, \lambda, 2} R_{\alpha, \mu}^*}^{\Psi^*, \sigma, \Psi^* \omega} \approx \mathcal{E}_{R_{\mu, \alpha}^{*, n}}^{\Psi^*, \sigma, \Psi^* \omega}. \]

Now, noting that both of the measures \( \tilde{\sigma} \) and \( \tilde{\omega} \) are compactly supported, we use that both of the error operators
\[ \tilde{E}_1^\alpha f \tilde{\sigma}(x) \equiv \int E_1^\alpha(x, y) f(y) d\tilde{\sigma}(y) \quad \text{and} \quad \tilde{E}_2^\alpha f \tilde{\omega}(x) \equiv \int E_2^\alpha(x, y) f(y) d\tilde{\omega}(y) \]
have norms controlled by the offset \( A_2^\alpha \) condition to obtain
\[ \mathcal{E}_{\Psi_*^{*, \lambda, 2} R_{\alpha, \mu}^*}^{\Psi^*, \sigma, \Psi^* \omega} + [A_2^\alpha + A_2^{\alpha, \text{dual}}] \approx \mathcal{E}_{R_{\mu, \alpha}^{*, n}}^{\Psi^*, \sigma, \Psi^* \omega} + [A_2^\alpha + A_2^{\alpha, \text{dual}}] \]
and
\[ \mathcal{E}_{\Psi_*^{*, \lambda, 2} R_{\alpha, \mu}^*}^{\Psi^*, \sigma, \Psi^* \omega} + [A_2^\alpha + A_2^{\alpha, \text{dual}}] \approx \mathcal{E}_{R_{\mu, \alpha}^{*, n}}^{\Psi^*, \sigma, \Psi^* \omega} + [A_2^\alpha + A_2^{\alpha, \text{dual}}]. \]

Note that once again we need the one-tailed Muckenhoupt conditions to reduce to the case where both measures have common compact support. Combining inequalities we have
\[ \mathcal{E}_{\Psi_*^{*, \lambda, 2} R_{\alpha, \mu}^*}^{\Psi^*, \sigma, \Psi^* \omega} + [A_2^\alpha + A_2^{\alpha, \text{dual}}] \approx \mathcal{E}_{R_{\mu, \alpha}^{*, n}}^{\Psi^*, \sigma, \Psi^* \omega} + [A_2^\alpha + A_2^{\alpha, \text{dual}}], \]
and this completes the proof of Proposition 30. \( \square \)

### 4.2 A preliminary T1 theorem.
We can use just the change of variable Proposition 30 and Theorem 16 to prove the preliminary Theorem 17. Recall that \( \mathcal{L} \) is presented as the graph of a \( C^{1, \delta} \) function \( \psi : \mathbb{R} \to \mathbb{R}^n \) given by
\[ \psi(t) = (\psi^2(t), \psi^3(t), \ldots, \psi^n(t)), \]
and that both
\[ \Psi(x) = (x^1, x^2 - \psi^2(x^1), x^3 - \psi^3(x^1), \ldots, x^n - \psi^n(x^1)) = x - (0, \psi(x^1)), \]
\[ \Psi^{-1}(\xi) = (\xi^1, \xi^2 + \psi^2(\xi^1), \xi^3 + \psi^3(\xi^1), \ldots, \xi^n + \psi^n(\xi^1)) = \xi + \psi(\xi^1), \]
are \( C^{1, \delta} \) maps, and that \( \Psi \) is a \( C^{1, \delta} \) homeomorphism from the curve \( \mathcal{L} \) to the \( x_1 \)-axis. Recall \( \Psi_* \Omega = (\Psi^{-1})^* \Omega = \{ \Psi \Omega : Q \in \Omega \} \). In the next subsection, the small Lipschitz assumption (1.7) will be removed, and the testing conditions below will be permitted to be taken over usual cubes.
Finally recall that in Theorem 17, we assume the small Lipschitz condition (1.7), i.e.,
\[ \|D\psi\|_\infty < \frac{1}{8n}(1 - \frac{\alpha}{n}), \]
and that \( \omega \) and \( \sigma \) are positive Borel measures (possibly having common point masses) with \( \omega \) compactly supported in \( \mathcal{L} \), and that \( \mathcal{R}^n = RQ^n_{\text{par}} \) where \( R \) is a fixed rotation that is \( L \)-transverse when \( L \) is the \( x_1 \)-axis. The conclusion of Theorem 17 is then that
\[
\int_{\psi \mathcal{Q}} |\mathcal{R}^{n}(1_{\psi \mathcal{Q} \sigma})|^2 d\omega \leq \mathcal{F}_{\mathcal{R}^{n}}^{\psi,\mathcal{Q}} |\psi \mathcal{Q}|_\sigma,
\]
\[
\int_{\psi \mathcal{Q}} |\mathcal{R}^{n,\text{dual}}(1_{\psi \mathcal{Q} \sigma})|^2 d\sigma \leq \mathcal{F}_{\mathcal{R}^{n,\text{dual}}}^{\psi,\mathcal{Q}} |\psi \mathcal{Q}|_\omega,
\]
for all cubes \( Q \in \mathcal{R}^n \).

**Proof of Theorem 17.** By the testing equivalences (2) and (3) of Proposition 30 with \( \mathcal{F} = \Psi \mathcal{Q} \), and using \( \Psi^{*}\mathcal{F} = \Psi^{*}\Psi \mathcal{Q} = \Omega \), we have
\[
\sqrt{\mathcal{A}_2^{\psi}(\sigma, \omega) + \mathcal{H}_{\mathcal{R}^{n}}^{\psi,\mathcal{Q}}(\sigma, \omega) + \mathcal{H}_{\mathcal{R}^{n,\text{dual}}}^{\psi,\mathcal{Q}}(\sigma, \omega)} \\
\approx \sqrt{\mathcal{A}_2^{\psi}(\sigma, \omega) + \mathcal{H}_{\mathcal{R}^{n}}^{\psi,\mathcal{Q}}(\Psi^{*}\sigma, \Psi^{*}\omega) + \mathcal{H}_{\mathcal{R}^{n,\text{dual}}}^{\psi,\mathcal{Q}}(\Psi^{*}\sigma, \Psi^{*}\omega)} \\
\approx \sqrt{\mathcal{A}_2^{\psi}(\sigma, \omega) + \mathcal{H}_{\mathcal{R}^{n}}^{\psi,\mathcal{Q}}(\Psi^{*}\sigma, \Psi^{*}\omega)}
\]
where the final line follows from Theorem 16 because \( \Psi^{*}\omega \) is supported on a line. Then we continue with equivalence (1) of Proposition 30 to obtain
\[
\sqrt{\mathcal{A}_2^{\psi}(\sigma, \omega) + \mathcal{H}_{\mathcal{R}^{n}}^{\psi,\mathcal{Q}}(\Psi^{*}\sigma, \Psi^{*}\omega)} \approx \sqrt{\mathcal{A}_2^{\psi}(\sigma, \omega) + \mathcal{H}_{\mathcal{R}^{n}}^{\psi,\mathcal{Q}}(\sigma, \omega)}.
\]
Altogether we now obtain from this that
\[
\mathcal{H}_{\mathcal{R}^{n}}^{\psi,\mathcal{Q}}(\sigma, \omega) \approx \sqrt{\mathcal{A}_2^{\psi}(\sigma, \omega) + \mathcal{H}_{\mathcal{R}^{n}}^{\psi,\mathcal{Q}}(\sigma, \omega) + \mathcal{H}_{\mathcal{R}^{n,\text{dual}}}^{\psi,\mathcal{Q}}(\sigma, \omega)},
\]
and this completes the proof of Theorem 17. \( \square \)

**Remark 31.** At this point, one can obtain a ‘\( T1 \) type’ theorem when \( \omega \) is compactly supported on a \( C^{1,\delta} \) curve \( \mathcal{L} \), without any additional restriction on the Lipschitz constant of the curve, by decomposing \( \omega = \sum_{i=1}^{N} \omega_i \) into finitely many measures \( \omega_i \) with support so small that the supporting curve is presented as a graph of a \( C^{1,\delta} \) function \( \psi_i \) relative to a rotated axis, and such that \( \|D\psi_i\|_\infty < \frac{1}{8n}(1 - \frac{\alpha}{n}) \) (this requires some work). Then Theorem 17 applies to each measure pair \( (\omega_i, \sigma) \) (appropriately rotated), and the corresponding rotated quasitesting conditions must now be taken over the finitely many measure pairs \( (\omega_i, \sigma) \). In the next subsection we will improve on this observation by eliminating the small Lipschitz assumption, and by taking the testing conditions over the single measure pair \( (\omega, \sigma) \).
4.3 The $T^1$ theorem for a measure supported on a regular $C^{1,\delta}$ curve.

Now we prove our main result, Theorem 8.

**Proof of Theorem 8.** Step 1. Given $0 \leq \alpha < n$, we define
\[
\varepsilon \equiv \frac{1}{8n}(1 - \frac{\alpha}{n}),
\]
where the right-hand side is the constant appearing in (1.7) in Theorem 12. Now let $0 < \varepsilon' < \varepsilon$ and choose a finite collection of points \( \{\xi_j\}_{j=1}^J \subset \mathbb{S}^{n-1} \) in the unit sphere such that the spherical balls \( \mathcal{B}(\xi_j, \frac{\varepsilon}{4}) \) cover \( \mathbb{S}^{n-1} \). Observe that our curve $\Phi$ and its derivative $\Phi'$ are uniformly continuous. We now claim that we can decompose the curve $\mathcal{L}$ into finitely many consecutive pieces $\{\mathcal{L}_i\}_{i=0}^N$ such that with $\tilde{\mathcal{L}}_i$ defined to be $\bigcup_{k=-C_n}^{C_n} \mathcal{L}_{i+k}$, the union of $\mathcal{L}_i$ and the previous and subsequent $C_n$ pieces, and $\mathcal{L}_{i}^\ast$ defined to be $\bigcup_{k=-2C_n}^{2C_n} \mathcal{L}_{i+k}$, where $C_n$ is a dimensional constant defined in (4.16) in step 2 below (without circularity), the following three properties hold:

1. There is $\eta > 0$ such that $\mathcal{L}_i = \operatorname{range} \Phi_i$ where $\Phi_i = \Phi\mid_{[\eta i, \eta(i+1))]}$ is the restriction of $\Phi$ to the interval $[\eta i, \eta(i+1))]$ for all $i$, and

2. For each $i$, there is $j = j(i)$ depending on $i$ such that, after a rotation $R_i$ that takes the point $\xi_j$ to the point $(0, \ldots, 0, 1)$, followed by an appropriate translation $T_i$, the curve $T_iR_i\tilde{\mathcal{L}}_i$ is the graph of the restriction $\psi_i\mid_{(-\varepsilon, \varepsilon)}$ of a globally defined $C^{1,\delta}$ function $\psi_i : \mathbb{R} \to \mathbb{R}^{n-1}$ with $\|D\psi_i\|_\infty < \varepsilon$, and

3. $\psi_i(t) = (0, \ldots, 0) \in \mathbb{R}^{n-1}$ for all $|t| > 4\varepsilon$.

Thus we are claiming that we can locally rotate and translate the curve so that it is given locally as part of the graph of a globally defined $C^{1,\delta}$ function $\psi_i$ with $\|D\psi_i\|_\infty < \varepsilon$, where in view of the definition of $\varepsilon$, this latter inequality is what is required in (1.7) of Theorem 17. Note that $\varepsilon_i \approx C_n \eta$.

To see that these three properties can be obtained, we use uniform continuity of $\Phi'$ to take a small piece $\mathcal{L}_i^\ast$ of the curve, such that the oscillation of tangent lines across the piece $\mathcal{L}_i^\ast$ is less than $\varepsilon'$, and then translate and rotate the chord joining its endpoints so as to lie on the $x_1$-axis. Note that with this done, the resulting curve is the graph of a function $\psi_i^\ast(t)$ defined for $t \in I_i^\ast$, which satisfies
\[
|\psi_i^\ast(t)| \leq C_{\varepsilon'}\frac{\varepsilon'}{2}\eta, \quad t \in I_i^\ast,
\]
since $\psi_i^\ast = T_iR_i\Phi\mid_{I_i^\ast}$. Here we are using the convention that $I_i$ is the parameter interval of $\Phi$ corresponding to the image $\mathcal{L}_i$, and similarly for $\tilde{I}_i$ and $I_i^\ast$.

Then we construct the extended function $\psi_i(t)$ so that its graph includes the translated and rotated piece $\tilde{\mathcal{L}}_i$, and so that away from $\tilde{I}_i$ the function $\psi_i$ smoothly straightens out from $\psi_i^\ast$ so as to vanish on the remaining $x_1$-axis, and in such a way
that $\|D\psi_i\|_{\infty} < \varepsilon$. This is most easily seen by taking $\psi_i(t) = \psi^*_i(t)\rho(t)$, where $\rho(t)$ is an appropriate smooth bump function that is identically 1 on $\hat{I}_i$ and vanishes outside $I^*_i$. Note then that

$$D\psi_i(t) = D\psi^*_i(t)\rho(t) + \psi^*_i(t)D\rho(t)$$

satisfies $|D\psi_i(t)| \leq C\varepsilon'$ since $|\psi^*_i(t)| \leq C\varepsilon\eta$ and $|D\rho(t)| \leq C\frac{1}{\eta}$. Consequently we have

$$\|D\psi_i\|_{\infty} \leq C\varepsilon' < \varepsilon.$$ Of course the function $\psi_i = \psi^*_i(t)\rho(t)$ is $C^{1,\delta}$ since $\text{supp} \rho$ is contained in the interior of the interval $I^*_i$. This completes the verification of properties (1), (2) and (3) above.

In the next step, we will restrict $\omega$ to the small piece $\sim_i$ and it will be important that we can straighten out the larger piece $\sim_i$ via a global $C^{1,\delta}$ diffeomorphism $\Psi_i^{-1}$ of $\mathbb{R}^n$ (defined using $\psi_i$), so that we can derive a tripled quasitesting condition for intervals in $I_i$ whose triples are contained in $\hat{I}_i$, the straightened out portion of $\sim_i$.

**Step 2.** We now apply Theorem 12 to the pullbacks $\hat{I}_i$ under $\psi_i$ of the localized pieces $\sim_i$ as follows. Fix $i$ and denote by $\omega_i$ the restriction of $\omega$ to $\sim_i$. We are assuming the usual cube testing conditions $\mathcal{T}^{Q_n}_{\mathbb{R}^n}(\sigma, \omega)$ and $\mathcal{T}^{Q_n,\text{dual}}_{\mathbb{R}^n}(\sigma, \omega)$ on the weight pair $(\sigma, \omega)$ over all cubes $Q \in Q^n$. Under the change of variable given by the $C^{1,\delta}$ map

$$\Psi^{-1}_i : \mathbb{R}^n \to \mathbb{R}^n,$$

corresponding to $\Psi_i(x) = (x_1, x' + \psi_i(x_1))$, the pair of measures $(\sigma, \omega)$ is transformed to the pullback pair $(\tilde{\sigma}, \tilde{\omega})$ (since $i$ is fixed we suppress the dependence of the change of variable on $i$ and simply write $\tilde{\sigma}$ and $\tilde{\omega}$, but we will use the subscript $i$ to emphasize restrictions of $\tilde{\omega}$). Now define $\tilde{\omega}_i$ to be the transform of the small piece of measure $\omega_i$, and note that it is supported on the $x_1$-axis, and moreover that the transform $\hat{I}_i$ of the larger piece $\sim_i$ is also supported on the $x_1$-axis. By Proposition 30, and in the presence of $\mathcal{A}_2$, the testing conditions $\mathcal{T}^{Q_n}_{\mathbb{R}^n}(\sigma, \omega)$ and $\mathcal{T}^{Q_n,\text{dual}}_{\mathbb{R}^n}(\sigma, \omega)$ for the measure pair $(\sigma, \omega)$ over cubes in $Q^n$ for the $\alpha$-fractional Riesz transform $\mathbf{R}^{\alpha,n}$ are transformed into the testing conditions $\mathcal{T}^{\Psi^*_i Q_n}_{\mathbf{R}^{\alpha,n}}(\sigma, \omega)$ and $\mathcal{T}^{\Psi^*_i Q_n,\text{dual}}_{\mathbf{R}^{\alpha,n}}(\sigma, \omega)$ for the measure pair $(\tilde{\sigma}, \tilde{\omega})$ over quasicubes in $\Psi^*_i Q^n$ for the conformal $\alpha$-fractional Riesz transform $\mathbf{R}^{\alpha,n}_{\Psi^*_i}$. Now in order to apply Theorem 12 to the conformal $\alpha$-fractional Riesz transform $\mathbf{R}^{\alpha,n}_{\Psi^*_i}$, we will choose below a specific rotation $\mathbb{R}^n = RQ^n_{\text{par}}$ of the collection of cubes $Q^n_{\text{par}}$. Then we consider the testing conditions for the pair $(\tilde{\sigma}, \tilde{\omega})$ over these quasicubes $\Psi^*_i \mathbb{R}^n$ that form a subset of the quasicubes $\Psi^*_i Q^n$. Provided we choose $\varepsilon > 0$ small enough in Step 1, the map $\Psi^*_i$ will have its derivative $D\Psi^*_i$ close to the
identity \( I \). Choose a rotation \( R \) that is \( L \)-transverse when \( L \) is the \( x_1 \)-axis. From Lemma 27 applied with \( \Omega = \Psi_i^{-1} \circ R \), we then obtain (3.27) and the key geometric property:

\[
\text{(4.15) The intersection of any } Q \in \Psi_i^* \mathcal{R}^n \text{ with the } x_1 \text{-axis is an interval in the } x_1 \text{-axis.}
\]

Using (3.27) and this geometric property we will now deduce, for the special fractional Riesz transform \( R_{\Psi}^{a,n} \), that in the presence of the \( A^a_2 \) conditions, the \( \Psi_i^* \mathcal{R}^n \)-quasicube testing conditions for the pair \((\tilde{\sigma}, \tilde{\omega})\) follow from the \( \Psi_i^* \mathcal{R}^n \)-quasicube testing conditions for the pair \((\tilde{\sigma}, \tilde{\omega})\).

Indeed, fix a quasicube \( Q \in \Psi_i^* \mathcal{R}^n \) and consider the left hand sides of the two dual testing conditions, namely

\[
\int_Q |R_{\Psi}^{a,n} 1_Q \tilde{\sigma}|^2 d\tilde{\omega}_i \quad \text{and} \quad \int_Q |R_{\Psi}^{a,n} 1_Q \tilde{\omega}_i|^2 d\tilde{\sigma}.
\]

Now the first integral is trivially dominated by

\[
\int_Q |R_{\Psi}^{a,n} 1_Q \tilde{\sigma}|^2 d\tilde{\omega} \leq (\Psi_i^{\mathcal{R}^n,dual})^2 |Q|_{\tilde{\sigma}}
\]

as required. To estimate the second integral, we first use (4.15) to choose a quasi cube \( Q' \in \Psi_i^* \mathcal{R}^n \) such that \( Q' \subset Q \) and \( 1_{Q'} \tilde{\omega} = 1_Q \tilde{\omega}_i \), and in addition that

\[
\text{(4.16) } \ell(Q') \leq \tilde{C}_n \ell(I)
\]

where \( I = Q \cap \{x_1 \text{-axis}\} \). This latter condition (4.16) simply means that \( Q' \) is taken essentially as small as possible so that \( 1_{Q'} \tilde{\omega} = 1_Q \tilde{\omega}_i \). Then we write

\[
\int_Q |R_{\Psi}^{a,n} 1_Q \tilde{\omega}_i|^2 d\tilde{\sigma} = \int_{Q' \cup Q' \cap I} |R_{\Psi}^{a,n} 1_Q \tilde{\omega}_i|^2 d\tilde{\sigma} = \int_{Q' \setminus Q} |R_{\Psi}^{a,n} 1_Q \tilde{\omega}_i|^2 d\tilde{\sigma} + \int_{Q' \cap I} |R_{\Psi}^{a,n} 1_Q \tilde{\omega}_i|^2 d\tilde{\sigma} = I + II.
\]

Now

\[
I \lesssim A^a_2,dual |Q'|_{\tilde{\omega}} = A^a_2,dual |Q|_{\tilde{\omega}_i}
\]

by a standard calculation similar to that in (4.9), and

\[
II \leq \int_{\partial Q'} |R_{\Psi}^{a,n} 1_Q \tilde{\omega}_i|^2 d\tilde{\sigma} \leq (\Psi_i^{\mathcal{R}^n,\text{triple},dual})^2 |Q'|_{\tilde{\omega}} = (\Psi_i^{\mathcal{R}^n,\text{triple},dual})^2 |Q|_{\tilde{\omega}_i},
\]

by the local backward triple quasicube testing condition for the pair \((\tilde{\sigma}, \tilde{\omega})\), whose necessity was proved in Theorem 12 above with the measure \( \tilde{\omega}_i \), the restriction
of \( \tilde{\omega} \) to \( \hat{I}_i \), which is compactly supported on the real axis. By local backward triple quasicube testing here we mean that we are restricting attention to those triples \( 3Q' \) such that \( 3Q' \cap \{x_{1\text{-axis}}\} \subset [-\zeta_i, \zeta_i) \), the image of the larger piece \( \tilde{\mathcal{L}}_i \) under \( \psi_i^{-1} \). This restriction is necessary since our arguments for necessity of triple testing require support on a line. Now we use condition (4.16) in our choice of quasicube \( Q' \). Indeed, with this choice we then have \( 3Q' \cap \{x_{1\text{-axis}}\} \subset [-\zeta_i, \zeta_i) \), and so have the backward tripled quasicube testing condition at our disposal.

**Step 3.** Now we use Theorem 12 again to obtain the quasienenergy conditions for the conformal fractional Riesz transform \( R_{\alpha,n} \) for each pair \( (\tilde{\sigma}, \tilde{\omega}_i) \). Thus, assuming only the \( \mathcal{A}_2^{\alpha} \) conditions, and \( \Omega^n \) cube testing conditions for the weight pair \( (\sigma, \omega) \) for the \( \alpha \)-fractional Riesz transform \( R_{\alpha,n} \), we have established that the weight pair \( (\tilde{\sigma}, \tilde{\omega}_i) \) satisfies the \( \mathcal{A}_2^{\alpha} \) conditions, the quasienenergy conditions, the quasitesting conditions and the quasiboundedness property (which follows from the backward triple quasitesting condition) all for the conformal \( \alpha \)-fractional Riesz transform \( R_{\alpha,n} \). Now Theorem 4 for conformal Riesz transforms (Conclusion 14) and parts (2) and (3) of Proposition 30 apply to show that

\[
\mathcal{N}_{R_{\alpha,n}^n}(\tilde{\sigma}, \tilde{\omega}_i) \lesssim \sqrt{\mathcal{A}_2^{\alpha}(\sigma, \omega) + \mathcal{S}_{R_{\alpha,n}^n}(\sigma, \omega) + \mathcal{S}_{\text{dual}}^{R_{\alpha,n}^n}(\sigma, \omega)}
\]

for each \( i \). Then by part (1) of Proposition 30 we have

\[
\mathcal{N}_{R_{\alpha,n}^n}(\sigma, \omega_i) \lesssim \mathcal{N}_{R_{\alpha,n}^n}^{\alpha}(\tilde{\sigma}, \tilde{\omega}_i),
\]

and we have

\[
\mathcal{N}_{R_{\alpha,n}^n}(\sigma, \omega) \leq \sum_{i=1}^{N} \mathcal{N}_{R_{\alpha,n}^n}(\sigma, \omega_i).
\]

This completes the proof of Theorem 8.

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