On the Euler-Maruyama scheme for spectrally one-sided Lévy driven SDEs with Hölder continuous coefficients

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Abstract
We study in this article the strong rate of convergence of the Euler-Maruyama scheme and associated with the jump-type equation introduced in Li and Mytnik [13]. We obtain the strong rate of convergence under similar assumptions for strong existence and pathwise uniqueness. Models of this type can be considered as a generalization of the CIR (Cox-Ingersoll-Ross) process with jumps.

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1 Introduction
In mathematical finance, a popular model for short term interest rates is the Cox-Ingersoll-Ross (CIR) model, which is the solution to the one-dimensional stochastic differential equation (SDE)

\[ X_t = x_0 + \int_0^t a(c - X_s)ds + \int_0^t \sqrt{X_s}dW_s, \quad x_0 \in \mathbb{R}, \ t \in [0, T], \]

where \( a, c > 0 \) and \( W = (W_t)_{0 \leq t \leq T} \) is a standard one-dimensional Brownian motion. There has been a push in the financial mathematics literature to generalize the CIR models to include jumps. The most notable works in this direction are the affine jump-diffusion models proposed in Duffie et al. [2, 3].

Motivated by the recent developments of continuous-state branching processes. It was shown in Fu and Li [5], and later extended in Li and Mytnik [13] to more general jump type equations, that is if \( b, \sigma \) and \( h \) are Hölder continuous and \( h \) is non-decreasing then existence and pathwise uniqueness of solution holds for SDEs of the form

\[ X_t = x_0 + \int_0^t b(X_s-)ds + \int_0^t \sigma(X_s-)dW_s + \int_0^t h(X_s-)dL_s, \quad x_0 \in \mathbb{R}, \ t \in [0, T]. \]

\[ L_t = \int_0^t \int_0^{\infty} z \tilde{N}(ds, dz). \]

where the process \( W = (W_t)_{0 \leq t \leq T} \) is a standard one-dimensional Brownian motion and \( \tilde{N} \) is a compensated Poisson random measure with intensity or Lévy measure \( \nu \) satisfying the condition \( \int_0^{\infty} \{z^2 \wedge z\} \nu(dz) < \infty \). In the recent paper of Jiao et al. [8, 9], in order to capture the persistency of low interest rate, self-exciting and large jump behaviours exhibited by sovereign interest rates and power markets, a version of the model considered in [5, 13] was introduced to the financial mathematics literature as the α-CIR process.

In practice, the solution to equation (2) is rarely analytically tractable, the goal of this article is to study under similar assumptions to those of [13], the strong rate of convergence for Euler-Maruyama scheme associated with the
SDE [2]. From the point of view of strong existence and pathwise uniqueness of a solution, the fact that the Lévy measure \( \nu \) is stable plays (as chosen in Jiao et al. [3]) very little role (see Theorem 2.3 in [13]). One can consider any spectrally positive Lévy process of the form given in [3] and produce a wide range of generalized CIR processes with different jump structures.

Given \( n \in \mathbb{N} \) and a time grid \( 0 = t_0 < t_1 \cdots < t_n = T \), the Euler-Maruyama scheme associated with equation (3) is given by \( X_0 := x_0 \) and

\[
X^{(n)}_{t_i} := x_0 + \sum_{j=0}^{n-1} b(X^{(n)}_{t_j}) \mathbf{1}_{(t_j, t_{j+1})}(s) ds + \sum_{j=0}^{n-1} \sigma(X^{(n)}_{t_j}) \mathbf{1}_{(t_j, t_{j+1})}(s) dW_s + \sum_{j=0}^{n-1} h(X^{(n)}_{t_j}) \mathbf{1}_{(t_j, t_{j+1})}(s) dL_s
\]

and one can extend the definition of the Euler-Maruyama scheme to continuous time by setting

\[
X^{(n)}_t = x_0 + \int_{[0,t]} b(X^{(n)}_{\eta_n(s)}) ds + \int_{[0,t]} \sigma(X^{(n)}_{\eta_n(s)}) dW_s + \int_{[0,t]} h(X^{(n)}_{\eta_n(s)}) dL_s
\]

where \( \eta_n(s) := t_j \) if \( s \in (t_j, t_{j+1}] \). The process \( (X^{(n)}_{\eta_n(t)})_{0 \leq t \leq T} \) is left continuous and for the purpose of this paper, we take equally spaced time grid of size \( T/n \).

Using techniques from Yamada and Watanabe [21], Gyöngy and Rásonyi [11] proved that if the drift coefficient \( b \) is the sum of a Lipschitz and a non-increasing \( \gamma \)-Hölder continuous, the diffusion coefficient \( \sigma \) is \( \gamma \)-Hölder continuous with \( \gamma \in [1/2, 1] \) and the jump coefficient \( h = 0 \), then

\[
\mathbb{E} ||X_T - X^{(n)}_T|| \leq \begin{cases} Cn^{-\alpha} (\gamma - \frac{1}{2}), & \text{if } \gamma \in (1/2, 1], \\ C(\log n)^{-1}, & \text{if } \gamma = 1/2. \end{cases}
\]

In [22], Yan proved similar results when \( \gamma > 1/2 \) by using Tanaka’s formula. These results are later extended, in for example [14, 17], to SDEs with irregular drift and diffusion coefficients. In the case where \( h \neq 0 \), \( L \) is a symmetric \( \alpha \)-stable process with \( \alpha \in (1, 2) \) and \( b = \sigma = 0 \), Hashimoto and Tsuchiya [7] shown using the method of Komatsu [10], if the coefficient jump \( h \) is bounded \( \gamma \)-Hölder continuous with \( \gamma \in [1/\alpha, 1] \), then

\[
\mathbb{E} ||X_T - X^{(n)}_T||^{\alpha - 1} \leq \begin{cases} Cn^{-(\gamma - \frac{1}{2})}, & \text{if } \gamma \in (1/\alpha, 1], \\ C(\log n)^{-(\alpha - 1)}, & \text{if } \gamma = 1/\alpha. \end{cases}
\]

We mention here also the works of Hashimoto [6], Mikulevičius and Xu [10], Qiao [19] for strong convergence and Mikulevičius and Zhang [15] for weak convergence. However there is little in the current literature on the Euler-Maruyama scheme for jump-type equation with Hölder continuous coefficients and drift. To the best of our knowledge, there is no result on the strong rate of convergence for equation of the form (2).

The structure of the current work is as follows. In section 1.1 we introduce the necessary notations and our standing assumptions. In section 1.2 we introduce the Yamada-Watanabe approximation technique and give two auxiliary results in Lemma 1.3 and Lemma 1.4, which are used in controlling the jump part of the approximation. In section 2.1, under boundedness assumption on the coefficients \( \sigma \) and \( h \), we obtain in Theorem 2.2 the strong rate of convergence of the Euler-Maruyama scheme for driving Lévy processes which are non-square integrable. In section 2.2 we consider the case of square integrable Lévy processes and obtain in Theorem 2.3 the strong rate of convergence without any boundedness assumption on the coefficients.

1.1 Notations and Assumptions

We work on the usual filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\) endowed with a filtration \( \mathcal{F} := (\mathcal{F}_t)_{t \geq 0} \) which satisfies the usual conditions and \( \mathcal{F}_\infty \subset \mathcal{F} \). We denote the sup-norm by \( || \cdot ||_{\infty} \) and set

\[
\alpha_\nu := \inf\{ \tilde{\alpha} > 1; \lim_{x \to 0^+} x^{\tilde{\alpha} - 1} \int_x^\infty z^\nu(z) dz = 0 \}.
\]

**Assumption 1.1.** We assume that the Lévy measure \( \nu \), and the coefficients \( b, \sigma \) and \( h \) satisfies the following conditions:

(i) There exist \( \zeta \in [1/2, 1] \) and \( K_0 > 0 \) such that

\[
\sup_{t, s \in [0, T]} \mathbb{E} ||L_t - L_s|| \leq K_0 |t - s|^\zeta.
\]

Note that examples of \( L \) include compensated \( \alpha \)-stable Lévy process for \( \alpha \in [1, 2] \), compensated square integrable Lévy processes and compensated compound Poisson process with integrable jump size.
(ii) The Lévy measure $\nu$ is such that $\nu((-\infty,0)) = 0$ and $\int_0^\infty \{z \wedge z^2\} \nu(dz) < \infty$.

(iii) The drift coefficient $b$ is of the form $b = b_1 + b_2$ where $b_1$ is a Lipschitz continuous function, and $b_2$ is a non-increasing $\rho$-Hölder continuous function with $\rho \in (0,1)$, that is,
\[ K_1 := \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|b_1(x) - b_1(y)|}{|x - y|} + \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|b_2(x) - b_2(y)|}{|x - y|^\rho} < \infty. \]

(iv) The diffusion coefficient $\sigma$ is an $\gamma$-Hölder continuous function with $\gamma \in [1/2,1)$ and the coefficient $h$ is an $\beta$-Hölder continuous function with $\beta \in (1-1/\alpha_\nu,1)$, that is,
\[ K_2 := \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|\sigma(x) - \sigma(y)|}{|x - y|^\gamma} + \sup_{x,y \in \mathbb{R}, x \neq y} \frac{|h(x) - h(y)|}{|x - y|^\beta} < \infty. \]

(v) The coefficient $h$ is a non-decreasing function.

By Assumption (iii) and (iv), there exists $K_3$ such that for any $x \in \mathbb{R}$, $|b(x)| + |\sigma(x)| + |h(x)| \leq K_3(1 + |x|)$ and we denote $K := \max\{K_0, K_1, K_2, K_3\}$.

**Remark 1.2.** We list now some consequences of Assumption (iii).

(i) From Lemma 2.1 of Li and Mytnik [13], if $\int_0^\infty \{z \wedge z^2\} \nu(dz) < \infty$ then $\alpha_\nu = 1$, and for any $\alpha > \alpha_\nu$, $\lim_{\varepsilon \to 0^+} x^{\alpha - 2} \int_0^\varepsilon z^2 \nu(dz) = 0$.

(ii) From Theorem 25.3 and Theorem 25.18 of Sato [20], we know that for any $p > 0$, $\mathbb{E}[|L_t|^p]$ and $\mathbb{E}[\sup_{s \leq t}|L_s|^p]$ are finite for all $t \geq 0$ and only if $\int_1^\infty z^p \nu(dz) < \infty$.

### 1.2 Yamada and Watanabe Approximation Technique

To deal with the Hölder continuity of the coefficients $\sigma$ and $h$, we introduce below the Yamada and Watanabe approximation technique (see for example [11, 13, 21]). For each $\varepsilon \in (1,\infty)$ and $\delta \in (0,1)$, we select a continuous function $\psi_{\delta,\varepsilon} : \mathbb{R} \to \mathbb{R}^+$ with support of $\psi_{\delta,\varepsilon}$ belongs to $[\varepsilon/\delta,\varepsilon]$ and is such that
\[ \int_{\varepsilon/\delta}^\varepsilon \psi_{\delta,\varepsilon}(z)dz = 1 \quad \text{and} \quad 0 \leq \psi_{\delta,\varepsilon}(z) \leq \frac{2}{z \log \delta}, \quad z > 0. \]

We define a function $\phi_{\delta,\varepsilon} \in C^2(\mathbb{R};\mathbb{R})$ by setting
\[ \phi_{\delta,\varepsilon}(x) := \int_0^{(|x|)} \int_0^{|y|} \psi_{\delta,\varepsilon}(z)dzdy. \]

It is straight forward to verify that $\phi_{\delta,\varepsilon}$ has the following useful properties:

\begin{align*}
|\psi_{\delta,\varepsilon}(x)| &\leq \varepsilon + \phi_{\delta,\varepsilon}(x), \quad \text{for any } x \in \mathbb{R}, \quad (4) \\
0 &\leq |\phi_{\delta,\varepsilon}(x)| \leq 1, \quad \text{for any } x \in \mathbb{R}, \quad (5) \\
\phi_{\delta,\varepsilon}(x) &\geq 0, \quad \text{for } x \geq 0 \text{ and } \phi_{\delta,\varepsilon}(x) < 0, \quad \text{for } x < 0, \quad (6) \\
\phi_{\delta,\varepsilon}'(\pm|x|) = \psi_{\delta,\varepsilon}(|x|) &\leq \frac{2}{|x| \log \delta} 1_{[\varepsilon/\delta,\varepsilon]}(|x|) \leq \frac{2\delta}{\varepsilon \log \delta}, \quad \text{for any } x \in \mathbb{R} \setminus \{0\}. \quad (7)
\end{align*}

We present below two auxiliary lemmas, which are used to control the jumps in the estimation of the strong error. Lemma 1.3 below is analogous to Lemma 3.2 given in [13].

**Lemma 1.3.** Suppose that the Lévy measure $\nu$ satisfies $\int_0^\infty \{z \wedge z^2\} \nu(dz) < \infty$. Let $\varepsilon \in (0,1)$ and $\delta \in (1,\infty)$. Then for any $x \in \mathbb{R}, y \in \mathbb{R} \setminus \{0\}$ with $xy \geq 0$ and $u > 0$, it holds that
\[ \int_0^\infty \{\phi_{\delta,\varepsilon}(y + xz) - \phi_{\delta,\varepsilon}(y) - xz\phi_{\delta,\varepsilon}'(y)\nu(dz) \leq 2 \cdot \mathbf{1}_{(0,\varepsilon]}(|y|) \left\{ \frac{|x|^2}{\log \delta} \left( \frac{1}{|y|} \wedge \frac{\delta}{\varepsilon} \right) \int_0^u z^2 \nu(dz) + |x| \int_0^\infty z \nu(dz) \right\}. \]
Proof. Let $x \in \mathbb{R}$, $y \in \mathbb{R} \setminus \{0\}$ with $xy \geq 0$ and $z > 0$. By the second order Taylor’s expansion for $\phi_{\delta, \varepsilon}$, it follows from (11) that

$$\phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y) - xz\phi'_{\delta, \varepsilon}(y) = |xz|^2 \int_0^1 \theta \phi''_{\delta, \varepsilon}(y + \theta xz) d\theta \leq \frac{2|xz|^2}{\log \delta} \int_0^1 \frac{\theta 1_{[x/\delta, \varepsilon]}(|y + \theta xz|)}{|y + \theta xz|} d\theta.$$  

Since $xy \geq 0$, we have $|y| \leq |y + \theta xz|$ and $1_{[x/\delta, \varepsilon]}(|y + \theta xz|) \leq 1_{(0, \varepsilon]}(|y|)$. Hence we obtain

$$\phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y) - xz\phi'_{\delta, \varepsilon}(y) \leq \frac{2|xz|^2 1_{(0, \varepsilon]}(|y|)}{\log \delta} \left( \frac{1}{|y|} \wedge \frac{\delta}{\varepsilon} \right).$$  

(8)

Moreover, since $xy \geq 0$, by (6) we have $x \phi'_{\delta, \varepsilon}(y) \geq 0$. This together with the fact that the right hand side of (8) has $1_{(0, \varepsilon]}(|y|)$, we obtain

$$\phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y) - xz\phi'_{\delta, \varepsilon}(y) \leq 1_{(0, \varepsilon]}(|y|) \{ \phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y) \}$$

$$= 1_{(0, \varepsilon]}(|y|) xz \int_0^1 \phi'_{\delta, \varepsilon}(y + \theta xz) d\theta \leq 1_{(0, \varepsilon]}(|y|) xz.$$  

(9)

The result then follows from (8) and (9).

Lemma 1.4. Suppose that the Lévy measure $\nu$ satisfies $\int_0^\infty \{ z \wedge z^2 \} \nu(dz) < \infty$. Let $\varepsilon \in (0, 1)$ and $\delta \in (1, \infty)$. Then for any $x, x' \in \mathbb{R}$, $y \in \mathbb{R}$ and $u \in (0, \infty)$, it holds that

$$\int_0^\infty \left| \phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y) - xz\phi'_{\delta, \varepsilon}(y) \right| \nu(dz) \leq 2 \left\{ \frac{\delta |x|^2}{\varepsilon \log \delta} \int_0^u z^2 \nu(dz) + |x - x'| \int_u^\infty z \nu(dz) \right\}.$$  

(10)

In particular, if $x' = 0$, then

$$\int_0^\infty \{ \phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y) - xz\phi'_{\delta, \varepsilon}(y) \} \nu(dz) \leq 2 \left\{ \frac{\delta |x|^2}{\varepsilon \log \delta} \int_0^u z^2 \nu(dz) + |x| \int_u^\infty z \nu(dz) \right\}.$$  

(11)

Proof. For $z \in (0, u)$, from the second order Taylor’s expansion for $\phi_{\delta, \varepsilon}$ and mean value theorem applied to $\phi'_{\delta, \varepsilon}$, we obtain from (10),

$$\left| \phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y + x'z) - (x - x')z\phi'_{\delta, \varepsilon}(y) \right| \leq \left| \phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y + x'z) - (x - x')z\phi'_{\delta, \varepsilon}(y + x'z) \right|$$

$$\leq |x - x'|^2 |z|^2 \int_0^1 \theta \phi''_{\delta, \varepsilon}(y + \theta xz + (1 - \theta)x'z) d\theta + |x' - x' ||z|^2 \int_0^1 \phi'_{\delta, \varepsilon}(y + \theta x'z) d\theta$$

$$\leq \left\{ |x - x'|^2 |z|^2 + |x' - x'|^2 |z|^2 \right\} \frac{2\delta}{\varepsilon \log \delta}.$$  

For the $z \in [u, \infty)$, apply mean value theorem to $\phi_{\delta, \varepsilon}$,

$$\left| \phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y + x'z) - (x - x')z\phi'_{\delta, \varepsilon}(y) \right|$$

$$= |x - x'| |z| \int_0^1 \left| \phi'_{\delta, \varepsilon}(y + \theta xz + (1 - \theta)x'z) - \phi'_{\delta, \varepsilon}(y) \right| d\theta \leq 2|x - x'| |z|.$$  

This concludes the proof of (11). In the case where $x' = 0$, then since $\phi''_{\delta, \varepsilon} \geq 0$, we have

$$\phi_{\delta, \varepsilon}(y + xz) - \phi_{\delta, \varepsilon}(y) - xz\phi'_{\delta, \varepsilon}(y) = |xz|^2 \int_0^1 \theta \phi''_{\delta, \varepsilon}(y + \theta xz) d\theta \geq 0,$$

which concludes the proof of (11).

Remark 1.5. Suppose that the Lévy measure $\nu$ satisfies the condition $\int_1^\infty z^2 \nu(dz) < \infty$ then one can take $u = \infty$ in Lemma 1.4 and the right hand side of (10) and (11) are still finite.
2 Strong Rate of Convergence

2.1 The Non-Square Integrable Case

In this subsection, we compute the strong rate of convergence in the case where \( L \) is a non-square integrable. The typical example one should keep in mind is when the Lévy measure \( \nu \) is spectrally positive \( \alpha \)-stable with \( \alpha \in [1, 2] \).

**Lemma 2.1.** Suppose that Assumption [12] holds and \( h \) is bounded.

(i) There exists \( C_1 > 0 \) depend on \( x_0, K, T \) and \( \|h\|_\infty \) such that for any \( t \in [0, T] \),

\[
\mathbb{E} \left[ \sup_{t \leq T} |X^{(n)}_t| \right] \leq C_1.
\]  

(ii) There exists \( C_2 > 0 \) depend on \( x_0, C_1, K, T \) and \( \|h\|_\infty \) such that for any \( t \in [0, T] \),

\[
\mathbb{E} |X^{(n)}_t - X^{(n)}_{\eta_n(t)}| \leq C_2 \left( \frac{1}{n} \right)^{1/2}.
\]

**Proof.** To prove (i), we aim to apply Lemma 3.2 of Gyöngy and Rásonyi [11]. To bound the stochastic integral against \( L \), we note that by Theorem 7.30 of He et al. [12], there exists a localizing sequence of stopping times \( (T_m)_{m \in \mathbb{N}} \) with \( T_m \uparrow \infty \) such that \( \int_0^T h(X^{(n)}_{\eta_n(s)})dL^T_m \in \mathcal{H}^1 \), where \( \mathcal{H}^1 \) is the martingale Hardy space. By applying the Burkholder-Davis-Gundy inequality, linear growth condition on \( b \), and the fact that for each \( m \), using the Burkholder-Davis-Gundy inequality, linear growth condition on \( \sigma \) and Jensen’s inequality, we obtain

\[
\mathbb{E} \left[ \sup_{t \leq T} \left| \int_0^T h(X^{(n)}_{\eta_n(s)})dL^T_m \right| \right] \leq c_1 \mathbb{E} \left[ \left\{ \int_0^T (h(X^{(n)}_{\eta_n(s)}))^2 d[L^T_m] \right\}^{1/2} \right] 
\]

for some \( c_1 > 0 \). The right hand side above is bounded by \( \lambda := c_2^2 \|h\|_\infty \mathbb{E} \left[ \sup_{s \leq T} |L_s| \right] < \infty \) for all \( m \in \mathbb{N} \). To take the limit as \( m \to \infty \) in the above inequalities we note that

\[
\mathbb{E} \left[ \sup_{t \leq T} \left| \int_0^T h(X^{(n)}_{\eta_n(s)})dL^T_s \right| \right] = \mathbb{E} \left[ \sup_{t \leq T \wedge T_m} \left| \int_0^t h(X^{(n)}_{\eta_n(s)})dL_s \right| \right]
\]

and monotone convergence theorem can be applied.

To estimate the time integral and the Brownian integral we proceed similarly to Remark 3.2 of [11], however we have to pay extra attention as \( X^{(n)} \) is not continuous. Using left continuity of \( X^{(n)}_{\eta_n} \), there exists a localizing sequence \( (T_m)_{m \in \mathbb{N}} \) such that \( |X^{(n)}_{\eta_n}| \) when stopped at \( T_m \) is bounded and the Brownian integral is a martingale. Using \( \mathbb{E} \left[ \sup_{t \leq T \wedge T_m} |X^{(n)}_{\eta_n(t)}| \right] \leq c_0 \mathbb{E} \left[ \int_0^{T \wedge T_m} (1 + \sup_{u \leq s} |X^{(n)}_{\eta_n(u)}|^2 )ds \right] \),

Using the linear growth condition on \( b \) and the fact that for each \( m \), there exists a constant \( C_m \) such that \( \sup_{u \leq s \wedge T_m} |X^{(n)}_{\eta_n(u)}| \leq \sup_{u \leq s \wedge T_m} |X^{(n)}_u| \leq C_m \), we obtain

\[
\mathbb{E} \left[ \sup_{t \leq T \wedge T_m} |X^{(n)}_{t \wedge T_m}| \right] \leq |x_0| + \lambda + KT + K \mathbb{E} \left[ \int_0^{T \wedge T_m} \sup_{u \leq s} |X^{(n)}_{\eta_n(u)}| ds \right] + c_0 \left\{ \mathbb{E} \left[ \int_0^{T \wedge T_m} (1 + \sup_{u \leq s} |X^{(n)}_{\eta_n(u)}|^2 )ds \right] \right\}^{1/2}
\]

\[
\leq C_{T, x_0} + K \mathbb{E} \left[ \int_0^{T \wedge T_m} \sup_{u \leq s} |X^{(n)}_u| ds \right] + c_0 \left\{ \mathbb{E} \left[ \int_0^{T \wedge T_m} \sup_{u \leq s} |X^{(n)}_u|^2 ds \right] \right\}^{1/2} < \infty
\]

where \( C_{T, x_0} := |x_0| + \lambda + KT + c_0 \sqrt{T} \). Using the fact that \( X^{(n)} \) is a càdlàg process and we replace \( \sup_u |X^{(n)}_u| \) by \( \sup_{u \leq s} |X^{(n)}_u| \) in the Lebesgue integral, equation (13) can be estimated by

\[
\mathbb{E} \left[ \sup_{t \leq T \wedge T_m} |X^{(n)}_{t \wedge T_m}| \right] \leq C_{T, x_0} + K \mathbb{E} \left[ \int_0^T \sup_{u \leq s} |X^{(n)}_{u \wedge T_m}| ds \right] + c_0 \left\{ \mathbb{E} \left[ \int_0^T \sup_{u \leq s} |X^{(n)}_{u \wedge T_m}|^2 ds \right] \right\}^{1/2}.
\]
Then it follows from Lemma 3.2 (i) of [11] with $p = 1$, $q = 2$ and $V(t) = Z(t) = \sup_{u \leq t} |X_{u,T,m}^{(n)}|$ that there exists $C_T$ such that

$$
\mathbb{E}\left[\sup_{t \leq T} |X_{t,T,m}^{(n)}| \right] \leq C_T, \gamma_C T.
$$

Hence the result follows from an application of the monotone convergence theorem.

To prove (ii), we note that the coefficients $\theta$, $\sigma$ satisfies the linear growth condition and $h$ is bounded, then

$$
|X_t^{(n)} - X_{\eta_n(t)}^{(n)}| \leq K (1 + |X_{\eta_n(t)}^{(n)}|) \left( |t - \eta_n(t)| + |W_t - W_{\eta_n(t)}| \right) + \|h\|_\infty L_t - L_{\eta_n(t)}.
$$

From (12) and Assumption (1.1)-(i), we have

$$
\mathbb{E}[X_t^{(n)} - X_{\eta_n(t)}^{(n)}] \leq M_1 (|t - \eta_n(t)| + |t - \eta_n(t)|^{1/2} + |t - \eta_n(t)|^\gamma),
$$

where the constant $M_1$ is given by

$$
M_1 := \max \left\{ K(1 + C_1)(1 \lor \sqrt{2\pi^{-1}}), \|h\|_\infty K_0 \right\}.
$$

This concludes the proof. \hfill \Box

From Theorem 2.2 in [13], under Assumption (1.1) (and the assumption that $\sigma$ and $h$ are bounded) there exists a unique strong solution to the SDE (2). We now present our first result on the rate of convergence for the Euler-Maruyama scheme.

**Theorem 2.2.** Suppose that Assumption (1.1) holds and $\sigma$, $h$ are bounded. Then there exists $C_3 > 0$ depending on $x_0$, $K$, $T$, $\rho$, $\gamma$, $\beta$, $\|\sigma\|_{\infty}$ and $\|h\|_{\infty}$ such that for any $\varepsilon \in (0, \frac{1}{1 - \beta} - \alpha_\nu)$,

$$
\sup_{0 \leq t \leq T} \mathbb{E}[|X_t - X_t^{(n)}|] \leq C_3 \begin{cases} 
   n^{-\rho/2} + n^{-\frac{\gamma}{2}(1 - \frac{1}{\beta})} & \gamma \in (1/2, 1], \alpha_\nu < \frac{2(1 - \gamma)}{1 - \beta}, \\
   n^{-\rho/2} + n^{-\frac{\gamma}{2}(1 - (\alpha_\nu + \frac{1}{\beta})(1 - \gamma))} & \gamma \in (1/2, 1], \alpha_\nu \geq \frac{2(1 - \gamma)}{1 - \beta}, \\
   (\log n)^{-1} & \gamma = 1/2.
\end{cases}
$$

Moreover, if $\nu(dz)$ is defined by

$$
\nu(dz) = \frac{1_{(0,\infty)}(z)\mu(z)}{z^{1+\alpha}} dz,
$$

(14)

for some $\alpha \in (1, 2)$ and bounded measurable function $\mu$ then the above $\varepsilon$ can be chosen as zero and $\alpha_\nu = \alpha$.

**Remark 2.3.** We set $\alpha_\nu := \sup\{\alpha > 1; \int_1^\infty z^\alpha \nu(dz) < \infty\}$. We point out that if $\gamma \in [\frac{1}{2}, \frac{3}{2}]$ then the boundedness assumption on $\sigma$ can be removed. The rate of convergence can be retrieve by performing similar computations as in Theorem 2.2 and we leave this to reader.

**Proof.** Define $Z_t^{(n)} := X_t - X_t^{(n)}$ and let $\varepsilon \in (0, 1)$ and $\delta \in (1, \infty)$. By using (2) and Itô's formula,

$$
|Z_t^{(n)}| \leq \varepsilon + \phi_\delta,\varepsilon(Z_t^{(n)}) = \varepsilon + M_t^{n,\delta,\varepsilon} + I_t^{n,\delta,\varepsilon} + J_t^{n,\delta,\varepsilon} + K_t^{n,\delta,\varepsilon},
$$

where we set

$$
M_t^{n,\delta,\varepsilon} := \int_0^t \phi_\delta,\varepsilon(Z_s^{(n)})\{\sigma(X_s) - \sigma(X_{\eta_n(s)})\} dW_s
$$

$$
+ \int_0^t \int_0^\infty \left\{ \phi_\delta,\varepsilon(Z_s^{(n)})\{b(X_{s-}) - b(X_{\eta_n(s)})\}\right\} \tilde{N}(ds, dz),
$$

$$
I_t^{n,\delta,\varepsilon} := \int_0^t \phi_\delta,\varepsilon(Z_s^{(n)})\{b(X_s) - b(X_{\eta_n(s)})\} ds,
$$

$$
J_t^{n,\delta,\varepsilon} := \frac{1}{2} \int_0^t \phi_\delta,\varepsilon''(Z_s^{(n)})|\sigma(X_s) - \sigma(X_{\eta_n(s)})|^2 ds,
$$

and
By localization arguments, we can take $M_t^{n, \delta, \varepsilon}$ to be a martingale and can be removed after taking the expectation. Therefore we only estimate the terms $I_t^{n, \delta, \varepsilon}$, $J_t^{n, \delta, \varepsilon}$ and $K_t^{n, \delta, \varepsilon}$. The coefficient $b_1$ is Lipschitz continuous and $b_2$ is non-increasing, we have for $x, y \in \mathbb{R}$ with $x \neq y$,
\[
\phi'_{\delta, \varepsilon}(x - y)(b(x) - b(y)) = \frac{\phi'_{\delta, \varepsilon}(x - y)}{x - y} (x - y)(b(x) - b(y)) \leq K_1 \frac{|\phi'_{\delta, \varepsilon}(x - y)|}{|x - y|} |x - y|^2 \leq K |x - y|,
\]
where in the first inequality, we used (5) and the fact that $(x - y)(b(x) - b(y)) \leq 0$ and in the last inequality, we used (8) and Lipschitz continuity of $b_1$. From the above we have
\[
I_t^{n, \delta, \varepsilon} \leq \int_0^t \phi'_{\delta, \varepsilon}(Z_s^{(n)})(b(X_s) - b(X_s^{(n)})) ds + \int_0^t \phi'_{\delta, \varepsilon}(Z_s^{(n)})(b(X_s^{(n)})) - b(\bar{X}_s^{(n)}) ds
\leq K \int_0^t |Z_s^{(n)}| ds + K \int_0^t |X_s^{(n)} - \bar{X}_s^{(n)}| + |X_s^{(n)} - X_s^{(n)}| |d\nu|.
\]
Using the fact that $\sigma$ is bounded and (7), we have
\[
J_t^{n, \delta, \varepsilon} \leq \int_0^t \phi''_{\delta, \varepsilon}(Z_s^{(n)})(\sigma(X_s) - \sigma(X_s^{(n)})) ds + \int_0^t \phi''_{\delta, \varepsilon}(Z_s^{(n)})(\sigma(X_s) - \sigma(X_s^{(n)})) \|d\nu\|_{\infty} 2^{1/\gamma} ds
\leq K_2^2 \int_0^t \frac{1}{|Z_s^{(n)}| \log \delta} ds + 2K_1^2 \gamma \int_0^t \frac{1}{|Z_s^{(n)}| \log \delta} ds
\leq 2TK_2^2 \gamma 2^{1/\gamma} ds.
\]
Finally, to estimate $K_t^{n, \delta, \varepsilon}$, we write it into two terms
\[
K_t^{n, \delta, \varepsilon} = K_t^{n, \delta, \varepsilon, 1} + K_t^{n, \delta, \varepsilon, 2},
\]
where $K_t^{n, \delta, \varepsilon, 1}$ and $K_t^{n, \delta, \varepsilon, 2}$ are given by
\[
K_t^{n, \delta, \varepsilon, 1} := \int_0^t \int_0^\infty \left\{ \phi_{\delta, \varepsilon}(Z_s^{(n)} + h(X_s) - h(X_s^{(n)})) z \phi'_{\delta, \varepsilon}(Z_s^{(n)}) \right\} \nu(dz) ds
\]
\[
K_t^{n, \delta, \varepsilon, 2} := \int_0^t \int_0^\infty \left\{ \phi_{\delta, \varepsilon}(Z_s^{(n)} + h(X_s) - h(X_s^{(n)})) z \phi'_{\delta, \varepsilon}(Z_s^{(n)}) \right\} \nu(dz) ds.
\]
We observe that for each $s \in [0, t]$, if $Z_s^{(n)} = 0$ then $h(X_s) = h(X_s^{(n)}) = 0$. Therefore we can apply Lemma 13 with $y = Z_s^{(n)}$ and $x = h(X_s) - h(X_s^{(n)})$ since $h$ is non-decreasing. That is for any $u > 0$,
\[
\int_0^\infty \left\{ \phi_{\delta, \varepsilon}(Z_s^{(n)} + h(X_s) - h(X_s^{(n)})) z \phi'_{\delta, \varepsilon}(Z_s^{(n)}) \right\} \nu(dz)
\leq \frac{2|h(X_s) - h(X_s^{(n)})|^2 1_{[0, u]}(|Z_s^{(n)}|)}{|Z_s^{(n)}| \log \delta} \int_u^\infty z^2 \nu(dz) + 2K_1^2 \gamma \frac{|Z_s^{(n)}|^2 1_{[0, u]}(|Z_s^{(n)}|)}{|Z_s^{(n)}| \log \delta} \int_u^\infty z \nu(dz)
\leq \frac{2K^2 \gamma 1_{[0, u]}(|Z_s^{(n)}|)}{|Z_s^{(n)}| \log \delta} \int_u^\infty z^2 \nu(dz) + 2K_1^2 \gamma 1_{[0, u]}(|Z_s^{(n)}|) \int_u^\infty z \nu(dz)
\leq \frac{2K^2 \gamma 2^{3/2} 1_{[0, u]}(|Z_s^{(n)}|)}{|Z_s^{(n)}| \log \delta} \int_u^\infty z^2 \nu(dz) + 2K_1^2 \gamma 1_{[0, u]}(|Z_s^{(n)}|) \int_u^\infty z \nu(dz),
\]
where in the second last inequality, we used the fact that $h$ is a $\beta$-Hölder continuous function with $\beta \in (1 - 1/\alpha, 1)$. 

We recall that $\alpha_\nu = \inf \{ \tilde{\alpha} > 1; \lim_{x \to 0^+} x^{\tilde{\alpha}-1} \int_x^\infty z \nu(dz) = 0 \}$. From Lemma 2.1 in [13], we know that $\alpha_\nu \in [1, 2]$ and for any $\tilde{\alpha} > \alpha_\nu$, $\lim_{x \to 0^+} x^{\tilde{\alpha}-1} \int_x^\infty z \nu(dz) = 0$. Also by the definition of $\alpha_{\nu}$, $\lim_{x \to 0^+} x^{\tilde{\alpha}-1} \int_x^\infty z \nu(dz) = 0$. Let $u = \varepsilon^q$ for some $q > 0$, which we will choose later. Since $\beta \in (1 - 1/\alpha_\nu, 1)$, we can take $\tilde{\alpha}$ such that $\alpha_\nu < \tilde{\alpha} < \frac{1}{1 - \beta}$. Then for sufficiently small $\varepsilon$, equation (17) can be further bounded as follows

$$
\frac{2K^2}{\log \delta} \varepsilon^{2\beta-1} \int_0^{\varepsilon^q} z^2 \nu(dz) + 2K \varepsilon^\beta \int_{\varepsilon^q}^\infty z \nu(dz)
= \frac{K^2}{\log \delta} \varepsilon^{2\beta-1-q(\tilde{\alpha}-2)} \varepsilon^{q(\tilde{\alpha}-2)} \int_0^{\varepsilon^q} z^2 \nu(dz) + 2K \varepsilon^{\beta-q(\tilde{\alpha}-1)} \varepsilon^{q(\tilde{\alpha}-1)} \int_{\varepsilon^q}^\infty z \nu(dz)
\leq \frac{2K^2}{\log \delta} \varepsilon^{2\beta-1-q(\tilde{\alpha}-2)} + 2K \varepsilon^{\beta-q(\tilde{\alpha}-1)} = 2 \left( \frac{K^2}{\log \delta} + K \right) \varepsilon^{1-\tilde{\alpha}(1-\beta)},
$$

where in the last equality, we have chosen $q > 0$ such that $2\beta - 1 - q(\tilde{\alpha} - 2) = \beta - q(\tilde{\alpha} - 1)$, that is, $q = 1 - \beta$. From the above computation we have

$$
K_t^{1, n, \delta, \varepsilon, 1} \leq 2T \left( \frac{K^2}{\log \delta} + K \right) \varepsilon^{1-\tilde{\alpha}(1-\beta)}.
$$

By applying (19) in Lemma 14 with $u = 1, y = Z_s^{(n)}$, $x = h(X_s) - h(X_{\eta_{n(s)}})$, $x' = h(X_s) - h(X_{\eta_{n(s)}})$ and using the fact that $h$ is bounded, $K_t^{n, \delta, \varepsilon, 2}$ can be bounded above by

$$
K_t^{n, \delta, \varepsilon, 2} \leq |K_t^{n, \delta, \varepsilon, 2}|
\leq 2 \int_0^1 z^2 \nu(dz) \int_0^t \frac{\varepsilon}{\log \delta} \left( |h(X_s^{(n)}) - h(X_{\eta_{n(s)}}^{(n)})| + |h(X_s) - h(X_{\eta_{n(s)}})| \right) ds
+ 2 \int_1^\infty z \nu(dz) \int_0^t |h(X_s^{(n)}) - h(X_{\eta_{n(s)}}^{(n)})| ds
\leq 2 \left\{ \frac{4 \|h\|_\infty}{\varepsilon} \int_0^1 z^2 \nu(dz) ds + \int_1^\infty z \nu(dz) \right\} \int_0^t |h(X_s^{(n)}) - h(X_{\eta_{n(s)}}^{(n)})| ds
\leq 2K \left\{ \left( \frac{4 \|h\|_\infty}{\varepsilon} \int_0^1 z^2 \nu(dz) \right) \vee \int_1^\infty z \nu(dz) \right\} \left( \frac{\varepsilon}{\log \delta} + 1 \right) \int_0^t |X_s^{(n)} - X_{\eta_{n(s)}}^{(n)}| \beta ds.
$$

By taking the expectation in [15], [16], [18] and [19], we obtain for any $t \in [0, T]$,

$$
\mathbb{E}[Z_t^{(n)}] \leq \varepsilon + \mathbb{E}[J_t^{n, \delta, \varepsilon}] + \mathbb{E}[J_t^{1, n, \delta, \varepsilon}] + \mathbb{E}[K_t^{n, \delta, \varepsilon}]
\leq \varepsilon + K \int_0^t \mathbb{E}[Z_s^{(n)}] ds + \frac{2TK^2 \varepsilon^{2\gamma-1}}{\log \delta} + 2T \left( \frac{K^2}{\log \delta} + K \right) \varepsilon^{1-\tilde{\alpha}(1-\beta)}
+ K \int_0^t \mathbb{E}[|X_{\eta_{n(s)}}^{(n)} - X_{\eta_{n(s)}}^{(n)}|] ds + \mathbb{E}[|X_{\eta_{n(s)}}^{(n)} - X_{\eta_{n(s)}}^{(n)}|] |\sigma|_\infty \right) ds
+ 2K \left\{ \left( \frac{4 \|h\|_\infty}{\varepsilon} \int_0^1 z^2 \nu(dz) \right) \vee \int_1^\infty z \nu(dz) \right\} \left( \frac{\varepsilon}{\log \delta} + 1 \right) \int_0^t \mathbb{E}[|X_s^{(n)} - X_{\eta_{n(s)}}^{(n)}| \beta] ds.
$$

Using (ii) of Lemma 2.1 we have

$$
\mathbb{E}[Z_t^{(n)}] \leq \varepsilon + K \int_0^t \mathbb{E}[|Z_s^{(n)}|] ds + \frac{2TK^2 \varepsilon^{2\gamma-1}}{\log \delta} + 2T \left( \frac{K^2}{\log \delta} + K \right) \varepsilon^{1-\tilde{\alpha}(1-\beta)}
+ KT \left\{ \frac{C_2}{n^{1/2}} + \frac{C_2}{n^{\sigma/2}} \right\} + 2K \frac{1}{T} T(2|\sigma|_\infty)^{2-1/\gamma} \frac{\varepsilon}{\log \delta} \frac{C_2}{n^{1/2}}
+ 2K \left\{ \left( \frac{4 \|h\|_\infty}{\varepsilon} \int_0^1 z^2 \nu(dz) \right) \vee \int_1^\infty z \nu(dz) \right\} \left( \frac{\varepsilon}{\log \delta} + 1 \right) \frac{C_2}{n^{\sigma/2}}.
$$
By using Gronwall’s inequality, we have
\[
e^{-KT}E[|Z_t^{(n)}|] \leq \varepsilon + \frac{2TK^2\varepsilon^{2\gamma-1}}{\log \delta} + 2T \left\{ \frac{K^2}{\log \delta} + K \right\} \varepsilon^{1-\hat{\alpha}(1-\beta)}
\]
\[+ KT \left\{ \frac{C_2}{n^{1/2}} + \frac{C_2^p}{n^{p/2}} \right\} + 2K^{1/\gamma}T(2\|\sigma\|_\infty)^{2-1/\gamma} \frac{\delta}{\varepsilon \log \delta \cdot n^{1/2}}
\]
\[+ 2KT \left\{ \left( 4\|h\|_\infty \int_0^1 z^2 \nu(dz) \right) \vee \int_1^\infty z \nu(dz) \right\} \left( \frac{\delta}{\varepsilon \log \delta} + 1 \right) \frac{C_2^3}{n^{3/2}}.
\]
To optimize the above bound, if \( \gamma \in (1/2, 1] \), then we choose \( \delta = 2 \) and obtain
\[
E[|Z_t^{(n)}|] \leq M_2 \left\{ \varepsilon + \varepsilon^{2\gamma-1} + \varepsilon^{1-\hat{\alpha}(1-\beta)} + \frac{1}{n^{p/2}} + \frac{1}{\varepsilon n^{1/2}} + \left( \frac{1}{\varepsilon} + 1 \right) \frac{1}{n^{3/2}} \right\},
\]
where the constant \( M_2 \) given by
\[
M_2 := e^{KT} \max \left\{ 1, \frac{2TK^2}{\log 2}, T \left\{ \frac{K^2}{\log 2} + K \right\}, 2KT \{C_2 + C_2^p\}, \frac{4K^{1/\gamma}T(2\|\sigma\|_\infty)^{2-1/\gamma}}{\log 2}, \right\}
\]
\[2KT \left\{ \left( 2\|h\|_\infty \int_0^1 z^2 \nu(dz) \right) \vee \int_1^\infty z \nu(dz) \right\} \frac{2C_2^3}{\log 2}.
\]
We let \( \varepsilon = n^{-q} \), where the optimal \( q > 0 \) is chosen later. There are two cases to consider. If \( \alpha_\nu < \frac{2(1-\gamma)}{1-\beta} \), then we choose \( \hat{\alpha} = \frac{2(1-\gamma)}{1-\beta} \) and we have \( 2\gamma - 1 = 1 - \hat{\alpha}(1-\beta) \). Hence by choosing \( q \) such that \( q(2\gamma - 1) = \beta/2 - q \), that is \( q = \frac{\beta}{2(\gamma - 1)} \), we have
\[
E[|Z_t^{(n)}|] \leq 6M_2 \left\{ n^{-p/2} + n^{-\frac{q}{4}(1-\frac{1}{p})} \right\}.
\]
If \( \alpha_\nu \geq \frac{2(1-\gamma)}{1-\beta} \), then we choose \( \hat{\alpha} = \alpha_\nu + \varepsilon \) for any \( \varepsilon \in (0, \frac{1}{\beta} - \alpha_\nu) \) and then \( 2\gamma - 1 > 1 - (\alpha_\nu + \varepsilon)(1-\beta) \). Hence by choosing \( q \) such that \( q(1 - (\alpha_\nu + \varepsilon)(1-\beta)) = \beta/2 - q \), that is \( q = \frac{\beta}{2 - (\alpha_\nu + \varepsilon)(1-\beta)} \), we have
\[
E[|Z_t^{(n)}|] \leq 6M_2 \left\{ n^{-p/2} + n^{-\frac{q}{4}(1-\frac{1}{p})} \right\}.
\]
This concludes the proof for \( \gamma \in (1/2, 1] \).

If \( \gamma = 1/2 \), then we choose \( \varepsilon = n^{-q} \) and \( \delta = n^p \) with \( p, q > 0 \) and \( p + q < \beta/2 \), we have
\[
e^{-KT}E[|Z_t^{(n)}|] \leq \frac{1}{n^{q}} + \frac{2TK^2}{p \log n} + 2T \left\{ \frac{K^2}{p \log n} + K \right\} \frac{1}{n^{q-\hat{\alpha}(1-\beta)}}
\]
\[+ KT \left\{ \frac{C_2}{n^{1/2}} + \frac{C_2^p}{n^{p/2}} \right\} + K^2T \frac{n^{p+q}}{p \log n \cdot n^{1/2}}
\]
\[+ 2KT \left\{ \left( 4\|h\|_\infty \int_0^1 z^2 \nu(dz) \right) \vee \int_1^\infty z \nu(dz) \right\} \frac{1}{p \log n} \frac{1}{p^{1/2}} \frac{C_2^3}{n^{3/2}}.
\]
Hence we can conclude that
\[
E[|Z_t^{(n)}|] \leq \frac{M_3}{\log n},
\]
where the constant \( M_3 \) is given by
\[
M_3 := e^{KT} \max \left\{ 1, \frac{2TK^2}{p}, T \left\{ \frac{K^2}{p} + K \right\}, \frac{2KT \{C_2 + C_2^p\}}{p^{-1} K^2 C_2}, \right\}
\]
\[2KT \left\{ \left( 4\|h\|_\infty \int_0^1 z^2 \nu(dz) \right) \vee \int_1^\infty z \nu(dz) \right\} \left( p^{-1} + 1 \right) \frac{C_2^3}{n^{3/2}}.
\]
This concludes the proof for \( \gamma = 1/2 \).

We consider now the Lévy measure \( \nu(dz) \) defined by

\[
\nu(dz) = \frac{1_{(0,\infty)}(z)\mu(z)}{z^{1+\alpha}}dz,
\]

for some \( \alpha \in (1, 2) \) and bounded measurable function \( \mu \). Then since

\[
\int_x^\infty z\nu(dz) \leq \|\mu\|_\infty \int_x^\infty z^{-\alpha}dz = \frac{\|\mu\|_\infty x^{1-\alpha}}{\alpha - 1},
\]

we have \( \alpha_\nu = \alpha \). To conclude the statement, it is sufficient to estimate the upper bounded of \( K_t^{n,\delta,\epsilon,1} \). From (17), with \( \epsilon = \frac{q}{\alpha} \) and \( q > 0 \), we have

\[
K_t^{n,\delta,\epsilon,1} \leq \frac{2K^2T}{\log \delta} \epsilon^{2\beta-1} \int_0^{\epsilon^q} z^2\nu(dz) + 2KT\epsilon^\beta \int_{\epsilon^q}^\infty z\nu(dz)
\]

\[
\leq \frac{2K^2\|\mu\|_\infty}{\log \delta} \epsilon^{2\beta-1} + \frac{2K\|\mu\|_\infty}{\alpha - 1} \epsilon^\beta
\]

\[
= \left( \frac{2K^2}{(2-\alpha)\log \delta} + \frac{2K}{(\alpha - 1)} \right) \|\mu\|_\infty \epsilon^{1-\alpha(1-\beta)}
\]

where in the last equality, we have chosen \( q = 1 - \beta \). This upper bound concludes the proof. \( \square \)

### 2.2 The Square Integrable Case

In this subsection we compute the strong rate of convergence in the case where \( L \) is a square integrable. In this case, the boundedness condition on the coefficients \( \sigma \) and \( h \) can be lifted. Examples of square integrable Lévy process which can be simulated include compensated Poisson process, spectrally positive tempered stable processes or spectrally positive truncated stable processes.

**Lemma 2.4.** Suppose that Assumption \( \ref{assumption} \) holds and \( \int_1^\infty z^2\nu(dz) < \infty \).

(i) Then there exists a constant \( C_3 > 0 \) such that

\[
\mathbb{E}\left[ \sup_{t \leq T} |X_t^{(n)}|^2 \right] \leq C_3,
\]

(20)

(ii) Then there exists a constant \( C_4 > 0 \) such that and for any \( t \in [0,T] \),

\[
\mathbb{E}\left[ |X_t^{(n)} - X_{0_n(t)}^{(n)}|^2 \right] \leq \frac{C_4}{n}.
\]

(21)

**Proof.** The proof is similar to Lemma 2.1. It is sufficient to apply Itô’s isometry and linear growth condition on the coefficients. \( \square \)

**Theorem 2.5.** Suppose that Assumption \( \ref{assumption} \) holds and \( \int_1^\infty z^2\nu(dz) < \infty \). Then there exists \( C_5 > 0 \) such that for any \( \epsilon \in (0, \frac{1}{1-\beta} - \alpha_\nu) \),

\[
\sup_{t \leq T} \mathbb{E}[|X_t - X_t^{(n)}|] \leq C_5 \begin{cases} 
\left( \frac{n^{-\rho/2} + n^{-\frac{\beta}{2}+\alpha(1-\frac{1}{2\delta})}}{1+1/(\log n)^{1/2}} \right) & \gamma \in (1/2, 1], \alpha_\nu < \frac{2(1-\gamma)}{1-\beta}, \\
\left( \frac{n^{-\rho/2} + n^{-\frac{\beta}{2}+\alpha(1-\frac{1}{2\delta})}}{1+1/(\log n)^{1/2}} \right) & \gamma \in (1/2, 1], \alpha_\nu \geq \frac{2(1-\gamma)}{1-\beta}, \\
\gamma = 1/2.
\end{cases}
\]

**Proof.** The proof is similar to that of Theorem 2.2. We recall that \( Z_t^{(n)} := X_t - X_t^{(n)} \) and in the proof of Theorem 2.2 the boundedness of \( \sigma \) and \( h \) were only used in the estimation of \( J_t^{n,\delta,\epsilon} \) and \( K_t^{n,\delta,\epsilon} \). Therefore, we present here only the estimates of \( J_t^{n,\delta,\epsilon} \) and \( K_t^{n,\delta,\epsilon} \).
Using the fact that $\sigma$ is $\gamma$-Hölder continuous, we have

\[
J_{t}^{n,\delta,\varepsilon} \leq \int_{0}^{t} \phi_{s,\delta,\varepsilon}''(Z_{s}^{n})|\sigma(X_{s}) - \sigma(X_{s}^{(n)})|^{2}ds + \int_{0}^{t} \phi_{s,\delta,\varepsilon}'(Z_{s}^{n})|\sigma(X_{s}) - \sigma(X_{s}^{(n)})|ds
\]

\[
\leq 2K^{2} \int_{0}^{t} \frac{1}{|Z_{s}^{n}|} \log \delta |Z_{s}^{n}|^{2\gamma}ds + 2K^{2} \int_{0}^{t} \frac{1}{|Z_{s}^{n}|} \log \delta |Z_{s}^{n}|^{2\gamma}ds
\]

\[
\leq \frac{2TK^{2}e^{2\gamma - 1}}{\log \delta} + \frac{2K^{2}\delta}{\varepsilon \log \delta} \int_{0}^{t} |X_{s}^{(n)} - X_{\eta_{s}(n)}^{(n)}|^{2\gamma}ds. \tag{22}
\]

Next, we estimate the $K_{t}^{n,\delta,\varepsilon}$ term. By applying (19) in Lemma 14 with

\[
u = +\infty, \quad y = Z_{s}^{n}, \quad x = h(X_{s}) - h(X_{\eta_{s}(n)}^{(n)}) \quad \text{and} \quad x' = h(X_{s}) - h(X_{\eta_{s}(n)}^{(n)}),
\]

the term $K_{t}^{n,\delta,\varepsilon}$ can be bounded above by (see Remark 13.5),

\[
K_{t}^{n,\delta,\varepsilon} \leq |K_{t}^{n,\delta,\varepsilon}|
\]

\[
\leq 2 \int_{0}^{t} \frac{\delta}{\varepsilon \log \delta} \left(|h(X_{s}^{(n)}) - h(X_{\eta_{s}(n)}^{(n)})|^{2} + |h(X_{s}) - h(X_{\eta_{s}(n)}^{(n)})|^{2} + |h(X_{s}) - h(X_{\eta_{s}(n)}^{(n)})| |h(X_{s}^{(n)}) - h(X_{\eta_{s}(n)}^{(n)})| \right) \int_{0}^{\infty} z^{2} \nu(dz) \, ds.
\]

Hence by taking the expectation of both hand sides and using the Hölder inequality, we have

\[
\mathbb{E}[K_{t}^{n,\delta,\varepsilon}] \leq 2 \int_{0}^{t} \frac{\delta}{\varepsilon \log \delta} \left(|h(X_{s}^{(n)}) - h(X_{\eta_{s}(n)}^{(n)})|^{2} + |h(X_{s}) - h(X_{\eta_{s}(n)}^{(n)})|^{2} + |h(X_{s}) - h(X_{\eta_{s}(n)}^{(n)})| |h(X_{s}^{(n)}) - h(X_{\eta_{s}(n)}^{(n)})| \right) \int_{0}^{\infty} z^{2} \nu(dz) \, ds.
\]

Next, by using the fact that $h$ is of linear growth and $\beta$-Hölder continuous,

\[
\mathbb{E}[K_{t}^{n,\delta,\varepsilon}] \leq 2 \int_{0}^{t} \frac{\delta}{\varepsilon \log \delta} \left(|h(X_{s}^{(n)}) - h(X_{\eta_{s}(n)}^{(n)})|^{2} \right) \int_{0}^{\infty} z^{2} \nu(dz) \, ds
\]

\[
+ 2 \int_{0}^{t} \frac{\delta}{\varepsilon \log \delta} \left(|h(X_{s}) - h(X_{\eta_{s}(n)}^{(n)})|^{2} + |h(X_{s}) - h(X_{\eta_{s}(n)}^{(n)})| |h(X_{s}^{(n)}) - h(X_{\eta_{s}(n)}^{(n)})| \right) \int_{0}^{\infty} z^{2} \nu(dz) \, ds.
\]

Take the expectation in (15), (22), (18) and (23), we obtain from (21) and the Gronwall’s inequality, for any $t \in [0, T],$

\[
e^{-KT} \mathbb{E}[\|Z_{t}^{(n)}\|] \leq \varepsilon + \frac{2TK^{2}e^{2\gamma - 1}}{\log \delta} + 2T \left\{ \frac{K^{2}}{\log \delta} + K \right\} \varepsilon^{1-\alpha(1-\beta)}
\]

\[
+ KT \left\{ \left( \frac{C_{4}}{n} \right)^{1/2} + \left( \frac{C_{4}}{n} \right)^{\beta/2} \right\}
\]

\[
+ K^{2}TC_{4}^{\beta} \delta \frac{1}{\varepsilon \log \delta} \left( \frac{1}{n} \right)^{\gamma} + 2K^{2}TC_{4}^{\beta} \int_{0}^{\infty} z^{2} \nu(dz) \delta \frac{1}{\varepsilon \log \delta} \left( \frac{1}{n} \right)^{\beta}
\]

\[
+ 2 \cdot 3^{1/2} K^{3/2} T C_{4}^{\beta/2} \int_{0}^{\infty} z^{2} \nu(dz) \left( 4 + \sup_{t \leq T} \mathbb{E}[|X_{t}|^{2}] + C_{3} \right) + 2 \varepsilon + \frac{1}{n^{\beta/2}} + \frac{1}{\varepsilon n^{\gamma}} + \frac{1}{\varepsilon n^{\beta}} + \left( \frac{1}{\varepsilon} + 1 \right) \frac{1}{n^{\beta/2}}.
\]

To optimize the above bound, if $\gamma \in (1/2, 1]$, then we choose $\delta = 2$ and obtain

\[
\mathbb{E}[\|Z_{t}^{(n)}\|] \leq M_{4} \left\{ \varepsilon + \varepsilon^{1-\alpha(1-\beta)} + \frac{1}{n^{\beta/2}} + \frac{1}{\varepsilon n^{\gamma}} + \frac{1}{\varepsilon n^{\beta}} + \left( \frac{1}{\varepsilon} + 1 \right) \frac{1}{n^{\beta/2}} \right\}.
\]
where $M_4$ is some constant defined by

$$M_4 := e^{KT} \max \left\{ 1, \frac{2TK^2}{\log 2}, 2T \left\{ \frac{K^2}{\log 2} + K \right\}, KT \{C_4^{1/2} + C_4^{p/2}\}, \frac{2K^2TC_4^{\beta}}{\log 2}, \frac{4K^2TC_4^{\beta}}{\log 2} \int_0^\infty z^2 \nu(dz), \frac{4 \cdot 3^{1/2}K^{3/2}TC_4^{\beta/2}}{\log 2} \int_0^\infty z^2 \nu(dz) \{ 4 + \sup_{t \leq T} \mathbb{E}[|X_t|^2] + C_3 \}^{1/2} \right\}.$$  

We choose $\epsilon = n^{-q}$ and then we choose the optimal $q > 0$. There are again two cases to consider, if $\alpha_\nu < \frac{2(1-\gamma)}{1-\beta}$, then we choose $\tilde{\alpha} = \frac{2(1-\gamma)}{1-\beta}$ and then $2\gamma - 1 = 1 - \tilde{\alpha}(1 - \beta)$. Hence by choosing $q$ as $q(2\gamma - 1) = \beta/2 - q$, that is $q = \frac{\beta}{2\gamma}$, we have

$$\mathbb{E}[|Z_t(n)|] \leq 7M4 \left\{ \left( \frac{1}{n} \right)^{\rho/2} + \left( \frac{1}{n} \right)^{\frac{\beta}{2}(1 - \frac{1}{\gamma})} \right\}. $$

If $\alpha_\nu \geq \frac{2(1-\gamma)}{1-\beta}$, then we choose $\tilde{\alpha} = \alpha_\nu + \epsilon$ for any $\epsilon \in (0, \frac{1}{1-\beta} - \alpha_\nu)$ and then $2\gamma - 1 > 1 - (\alpha_\nu + \epsilon)(1 - \beta)$. Hence by choosing $q$ such that $q \{ 1 - (\alpha_\nu + \epsilon)(1 - \beta) \} = \beta/2 - q$, that is $q = \frac{\beta}{2 - (\alpha_\nu + \epsilon)(1 - \beta)}$, we have

$$\mathbb{E}[|Z_t(n)|] \leq 7M4 \left\{ \left( \frac{1}{n} \right)^{\rho/2} + \left( \frac{1}{n} \right)^{\frac{\beta}{2}(1 - \frac{1}{\gamma})} \right\}. $$

This concludes the proof for $\gamma \in (1/2, 1]$.

If $\gamma = 1/2$, then we choose $\epsilon = n^{-q}$ and $\delta = n^p$ with $p, q > 0$ and $p + q < \beta/2 < 1/2 = \gamma$. Then

$$e^{-KT} \mathbb{E}[|Z_t(n)|] \leq \frac{1}{n^q} + \frac{2TK^2}{p \log n} + 2T \left\{ \frac{K^2}{p \log n} + K \right\} \frac{1}{n^{q - \frac{\beta\gamma}{(1 - \beta)}}} + KT \left\{ \left( \frac{C_4}{n} \right)^{1/2} + \left( \frac{C_4}{n} \right)^{p/2} \right\} $$

$$+ K^2TC_4^{1/2} \frac{n^{p+q}}{p \log n} \left( \frac{1}{n} \right)^{1/2} + 2K^2TC_4^{\beta} \int_0^\infty z^2 \nu(dz) \frac{n^{p+q}}{p \log n} \left( \frac{1}{n} \right)^{\beta} $$

$$+ 2 \cdot 3^{1/2}K^{3/2}TC_4^{\beta} \int_0^\infty z^2 \nu(dz) \left\{ 4 + \sup_{s \leq t} \mathbb{E}[|X_s|^2] + C_3 \right\}^{1/2} \frac{n^{p+q}}{p \log n} \left( \frac{1}{n} \right)^{\beta/2}. $$

Hence we can conclude that

$$\mathbb{E}[|Z_t(n)|] \leq \frac{M_5}{\log n},$$

where the constant $M_5$ is given by

$$M_5 = e^{KT} \max \left\{ 1, \frac{2TK^2}{p}, T \left\{ \frac{K^2}{p} + K \right\}, 2KT \left\{ C_4^{1/2} + C_4^{p/2} \right\}, \frac{K^2TC_4^{1/2}}{p}, \frac{2K^2TC_4^{\beta}}{p} \int_0^\infty z^2 \nu(dz), \frac{2 \cdot 3^{1/2}K^{3/2}TC_4^{\beta/2}}{p} \int_0^\infty z^2 \nu(dz) \left\{ 4 + \sup_{s \leq t} \mathbb{E}[|X_s|^2] + C_3 \right\}^{1/2} \right\}. $$

This concludes the proof for $\gamma = 1/2$.  

\[\square\]

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