Elements of Torelli topology:
II. The extension problem

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1. Introduction

**Torelli groups and diffeomorphisms.** The Teichmüller modular group $\text{Mod}(S)$ of an orientable surface $S$ is defined as the group of isotopy classes of orientation-preserving diffeomorphisms of $S$. Both diffeomorphisms and isotopies are required to preserve the boundary $\partial S$ only set-wise. If $S$ is closed, the subgroup of elements $\text{Mod}(S)$ acting trivially on $H_1(S, \mathbb{Z})$ is called the **Torelli group** of $S$ and is denoted by $\mathcal{T}(S)$. We will denote the homology group $H_1(S, \mathbb{Z})$ simply by $H_1(S)$. A diffeomorphism of a closed surface $S$ is called a **Torelli diffeomorphism** if it acts trivially on $H_1(S)$.

There are good reasons to hesitate with defining Torelli groups and Torelli diffeomorphisms for surfaces with boundary. Requiring diffeomorphisms to act trivially on $H_1(S)$ in the case when $\partial S \neq \emptyset$ seems to be fairly naive.

**The extension problem.** This paper is devoted to the extension problem in Torelli topology. By this we understand the following circle of questions. Let $Q$ be a subsurface of a connected closed orientable surface $S$. We are interested in extensions of diffeomorphisms of $Q$ to diffeomorphisms of $S$ acting trivially on $H_1(S)$, i.e. in extensions which are Torelli diffeomorphisms. As usual, the case of connected subsurfaces $Q$ is the most important one.

The theory of Teichmüller modular groups suggests to consider only diffeomorphisms of $Q$ equal to the identity on $\partial Q$, and that the appropriate isotopies of such diffeomorphisms are the isotopies fixed on $\partial Q$. At the same times this allows to simultaneously address the question of when the extension $\varphi \setminus S$ of a diffeomorphism $\varphi$ of $Q$ by the identity to a diffeomorphism of $S$ is a Torelli diffeomorphism. The latter question was already addressed by A. Putman, whose paper [P] was one of the main sources of inspiration for the present paper.

Closely related is the question of when there exists a multi-twist diffeomorphism $\tau$ about $\partial Q$ such that $\tau \circ (\varphi \setminus S)$ is a Torelli diffeomorphism. For some applications this is more important than $\varphi \setminus S$ itself being a Torelli diffeomorphism.

**The ambient surface $S$ as a structure on $Q$.** It is hardly surprising that the answers to such questions depend on the embedding of $Q$ into the ambient closed surface $S$. But they depend this embedding only in a mild and controlled way discovered by A. Putman [P]. In contrast with [P], we prefer not to spell out in advance what structure on the surface $Q$ is induced by its the embedding in $S$. We treat the embedding of $Q$ in $S$ as a part of the structure of $Q$, keeping the mind open to using whatever part of this structure may be needed.

Let $c = \partial Q$ and let $cQ$ be the closure of the complement $S \setminus Q$. Clearly, $cQ$ is a subsurface of $S$ with the boundary $c$. If $Q$ is connected, then the answers to these questions depend only on the partition of $c$ into the boundaries $\partial P$ of components $P$ of $cQ$. 
Weakly Torelli diffeomorphisms. Our answers are stated in terms of homology groups related to $Q$ and the ambient surface $S$. We will use the singular homology theory and work with singular chains when this is either necessary or more natural than working with homology classes.

An obvious necessary condition for the existence of a Torelli diffeomorphism of $S$ extending a diffeomorphism $\varphi$ of $Q$ is the following: for every cycle $\sigma$ in $Q$ the cycle $\varphi^*(\sigma)$ should be homologous to $\sigma$ in $S$. We call diffeomorphisms $\varphi$ of $Q$ fixed on $c$ and having this property weakly Torelli diffeomorphisms of $Q$.

Homology theory. We use the standard notations $C_*(\bullet)$ and $\partial$ for the groups of singular chains and boundary maps. Given a topological space $X$ and a subspace $A$ of $X$, we denote by $Z_1(X, A)$ the group of relative cycles of $(X, A)$, i.e. the group of chains $\alpha \in C_*(X)$ such that $\partial \alpha \in C_*(A)$. By $[\sigma]$ we denote the homology class of a cycle $\sigma$.

The action on relative cycles. Let $Z_1(Q, c)$ be the group of relative cycles $\alpha \in Z_1(Q, c)$ such that the boundary $\partial \alpha$ is a boundary in $cQ$ also.

Let $\varphi$ be a weakly Torelli diffeomorphism of $Q$. It turns out that understanding the action of $\varphi$ on relative cycles $\alpha \in Z_1(Q, c)$ is the key to the extension problem for $\varphi$. If $\alpha$ is an arbitrary relative cycle of $(Q, c)$, then $\varphi^*(\alpha) - \alpha$ is a cycle in $Q$ because $\varphi$ is fixed on $c$. The following theorem is the first main result of this paper.

Theorem. If $\varphi$ is a weakly Torelli diffeomorphism, then for every $\alpha \in Z_1(Q, c)$ the cycle $\varphi^*(\alpha) - \alpha$ is homologous in $Q$ to a cycle in $c$.

See Theorem 5.3. It turns out that the homology classes of cycles in $c$ homologous to the cycle $\varphi^*(\alpha) - \alpha$ in $Q$ to a big extent depend only on $\partial \alpha$ and even only on the homology class $[\partial \alpha] \in H_0(c)$. In order to state this in a precise and efficient form, one need to introduce two homology groups reflecting the way the manifold $c$ is situated in $S$.

Let $K_0(c)$ be the intersection of the kernels of the inclusion homomorphisms

$$H_0(c) \rightarrow H_0(Q) \quad \text{and} \quad H_0(c) \rightarrow H_0(cQ).$$

A relative cycle $\alpha \in Z_1(Q, c)$ belongs to $Z_1(Q, c)$ if and only if $[\partial \alpha] \in K_0(c)$. Let $H_1(c)$ be the quotient of $H_1(c)$ by the kernel of the inclusion homomorphism

$$\iota: H_1(c) \rightarrow H_1(S).$$

The group $H_1(c)$ is canonically isomorphic to the image of $\iota$. The advantage of the group $H_1(c)$ is that it lives on $c$ instead of $S$. If $Q$ is connected, then the groups $K_0(c)$ and $H_1(c)$ depend only on the partition of $c$ into the boundaries $\partial P$ of components $P$ of $cQ$. 


Observation. Suppose that $\varphi$ is a weakly Torelli diffeomorphism and $\alpha \in \mathbb{Z}_1(Q, c)$. Let $\gamma \in C_1(c)$ be a cycle homologous in $Q$ to the cycle $\varphi_*(\alpha) - \alpha$. Then the image of the homology class $[\gamma] \in H_1(c)$ in the group $H_1(c)$ depends only on $[\partial \alpha] \in H_0(c)$.

See Section 5. The image of $[\gamma]$ in the group $H_1(c)$ is called the difference class of $\alpha$ and is denoted by $\Delta \varphi(\alpha)$. The difference classes $\Delta \varphi(\alpha)$ lead to a homomorphism

$$\delta_{\varphi} : K_0(c) \rightarrow H_1(c)$$

called the $\delta$-difference map of $\varphi$. The $\delta$-difference map of $\varphi$ turns out to be the obstruction for the extension $\varphi \setminus S$ of $\varphi$ by the identity to be a Torelli diffeomorphism of $S$.

Theorem. The extension $\varphi \setminus S$ of $\varphi$ to a diffeomorphism of $S$ by the identity is a Torelli diffeomorphism if and only if $\varphi$ is a weakly Torelli diffeomorphism and $\delta_{\varphi} = 0$.

See Theorem 6.2. This theorem provides a characterization of diffeomorphisms $\varphi$ such that $\varphi \setminus S$ is a Torelli diffeomorphism in terms of the action of $\varphi$ on $Q$ and some minor and independent of $\varphi$ information about the embedding of $Q$ into $S$. For connected subsurfaces $Q$ it is closely related to a theorem of A. Putman. See A. Putman [P], Theorem 3.3.

Symmetric homomorphisms. The intersection pairing $H_0(c) \times H_1(c) \rightarrow \mathbb{Z}$ leads to a canonical pairing $K_0(c) \times H_1(c) \rightarrow \mathbb{Z}$, denoted by $\langle \cdot, \cdot \rangle_c$. A map $\delta : K_0(c) \rightarrow H_1(c)$ is called symmetric if

$$\langle a, \delta(b) \rangle_c = \langle b, \delta(a) \rangle_c$$

for all $a, b \in K_0(c)$. It turns out that the $\delta$-difference map $\delta_{\varphi}$ is symmetric for every weakly Torelli diffeomorphism $\varphi$. See Theorem 6.3.

Completely reducible homomorphisms. Suppose now that the subsurface $Q$ is connected. Then the decomposition of $c = \partial Q$ into the disjoint union of the boundaries $\partial P$ of the components $P$ of $cQ$ leads to direct sum decompositions of the groups $K_0(c)$ and $H_1(c)$. A homomorphism $K_0(c) \rightarrow H_1(c)$ is said to be completely reducible if it respects these direct sum decompositions.

Theorem. The diffeomorphism $\varphi$ can be extended to a Torelli diffeomorphism of $S$ if and only if $\varphi$ is peripheral and the $\delta$-difference map $\delta_{\varphi}$ is completely reducible.

See Theorem 7.5. This theorem is complemented by the following theorem (which is, in fact, used in its proof) about the realization of completely reducible symmetric homomorphisms $K_0(c) \rightarrow H_1(c)$ as $\delta$-difference maps.
Theorem. If \( \delta : K_0(c) \to H_1(c) \) is a completely reducible and symmetric, then \( \delta = \delta_\varphi \) for some diffeomorphism \( \varphi \) of \( Q \) which is equal to the identity on \( \partial Q \) and can be extended to a Torelli diffeomorphism of \( S \).

See Theorem 7.4. The above characterizations of diffeomorphisms \( \varphi \) such that \( \varphi \backslash S \) is a Torelli diffeomorphisms and such that \( \varphi \) can be extended to a Torelli diffeomorphism of \( S \) imply, together with the last theorem, that neither of these properties can be characterized entirely in terms of action of \( \varphi \) on \( Q \). These characterizations do depend on the embedding of \( Q \) in \( S \), although in a minor and controlled way. For diffeomorphisms \( \varphi \) such that \( \varphi \backslash S \) is a Torelli diffeomorphism this was observed by A. Putman [P].

Diagonal maps. A homomorphism \( H_0(c) \to H_1(c) \) is called diagonal if for every component \( C \) of \( c \) it maps the homology class of any 0-simplex in \( C \) to the image in \( H_1(c) \) of an integer multiple of the fundamental class \( [C] \).

Theorem. There exists a multi-twist diffeomorphism \( \tau \) about \( \partial Q \) such that \( \tau \circ (\varphi \backslash S) \) is a Torelli diffeomorphism if and only if \( \varphi \) a weakly Torelli diffeomorphism and its difference map \( \delta_\varphi \) is equal to the restriction to \( K_0(c) \) of a diagonal map \( H_0(c) \to H_1(c) \).

See Theorem 8.2. It turns out that if each component of \( cQ \) has \( \leqslant 3 \) boundary components, then every completely reducible symmetric map is equal to the restriction of a diagonal map. See Theorem 8.3. This leads to the following theorem.

Theorem. Suppose that every component of \( cQ \) has \( \leqslant 3 \) boundary components. If \( \varphi \) can be extended to a Torelli diffeomorphism of \( S \), then there exists a multi-twist diffeomorphism \( \tau \) about \( \partial Q \) such that \( \tau \circ (\varphi \backslash S) \) is a Torelli diffeomorphism.

See Theorem 8.4. As the examples in Section 4 show, the assumption that every component of \( cQ \) has \( \leqslant 3 \) boundary components cannot be relaxed even to \( \leqslant 4 \).

Outline of the paper. Section 2 is devoted mostly to fixing the meaning of such common terms as a subsurface, a circle, a twist. Section 3 is devoted to a review of reduction systems and of their basic properties. This material is used only in Section 4. Section 4 is devoted partly to the motivation behind this paper, partly to the dangers one may encounter while venturing in the Torelli sea after gaining experience in the Teichmüller modular groups waters. The next four sections are devoted to the detailed statements and proofs of the results outlined above. Section 9 is devoted to speculations about how one should define Torelli groups for surfaces with non-empty boundary. While Section 9 depends on the previous sections, it is more of a continuation of Section 4 than of Sections 5–8. The Appendix is devoted to a converse of the first theorem above.
2. Surfaces, circles, and twists

**Surfaces.** By a *surface* we understand a compact orientable 2-manifold with a possibly empty boundary. We denote by $S$ a fixed connected oriented surface. By a *subsurface* of $S$ we understand a codimension 0 submanifold $Q$ of $S$ such that each component of $\partial Q$ is either equal to a component of $\partial S$, or disjoint from $\partial S$. For a subsurface $Q$ of $S$ we will denote by $cQ$ the closure of its set-theoretic complement $S \setminus Q$. Clearly, $cQ$ is also a subsurface of $S$.

**Circles.** A *circle* on $S$ is defined as a submanifold of $S$ diffeomorphic to the standard circle $S^1$ and disjoint from $\partial S$. A circle in $S$ is said to be *non-peripheral* if it does not bound an annulus together with a component of the boundary $\partial S$, and *non-trivial* if, in addition, it does not bound a disc in $S$.

A circle $D$ in $S$ is called *separating* if $S \setminus D$ is not connected. Then $S \setminus D$ consist of two components. The closures of these components are subsurfaces of $S$ having $D$ as a boundary component. Their other boundary components are included in $\partial S$.

A *bounding pair of circles* on a connected closed surface $S$ is as an unordered pair $C, C'$ of disjoint non-isotopic circles in $S$ such that both $C$ and $C'$ are non-separating, but $S \setminus (C \cup C')$ is not connected. In this case $S \setminus (C \cup C')$ consist of two components.

**Twist diffeomorphisms and Dehn twists.** Let $A$ be an annulus, i.e. a surface diffeomorphic to $S^1 \times [0, 1]$. As is well known, the group of diffeomorphisms of $A$, fixed in a neighborhood of $\partial A$ and considered up to isotopies fixed in a neighborhood $\partial A$, is an infinite cyclic group. A diffeomorphism of $A$ is called a *twist diffeomorphism* of $A$ if it is fixed in a neighborhood of $\partial A$ and its isotopy class is a generator of this group.

If an annulus $A$ is a subsurface of a surface $S$, then any twist diffeomorphism of $A$ can be extended by the identity to a diffeomorphism of $S$. Such extensions are called *twist diffeomorphisms* of $S$, and their isotopy classes are called *Dehn twists*. The Dehn twist resulting from a twist diffeomorphism of an annulus $A$ in $S$ is said to be a *Dehn twist about* a circle $C$ in $S$ if $C$ is contained in $A$ as a deformation retract.

**Left and right twists.** Since $S$ is oriented, every annulus $A$ contained in $S$ is also oriented. The orientation of an annulus $A$ allows to choose a preferred isotopy class of twist diffeomorphisms of $A$. The twist diffeomorphisms in this isotopy class and their extensions to twist diffeomorphisms of $S$ are called the *left twist diffeomorphisms*, and other twist diffeomorphisms are called the *right twist diffeomorphisms*.

The isotopy classes of the left or right twist diffeomorphisms about $C$ are called, respectively, *left* or *right Dehn twists* about $C$. They are uniquely determined by $C$. The left Dehn twist...
about $C$ is denoted by $t_C$. The right Dehn twist about $C$ is the inverse of the left one and hence is equal to $t_C^{-1}$. Let $G$ be a diffeomorphism of $S$ and let $g \in \text{Mod}(S)$ be its isotopy class. Then

$$g t_C g^{-1} = t_{G(C)}.$$

In particular, if $G(C) = C$, or if $G(C)$ is isotopic to $C$, then $t_C$ and $g$ commute.

**The action of Dehn twists on homology.** Since $S$ is oriented, there is a canonical skew-symmetric pairing on $H_1(S)$, known as the *intersection pairing*. We denote it by

$$(a, b) \mapsto \langle a, b \rangle.$$

Let $C$ be a circle on $S$. Let us choose an orientation of $C$. Let $[C]$ be the image in $H_1(S)$ of the fundamental class of $C$ with this orientation. Then $t_C$ acts on $H_1(S)$ by the formula

$$(t_C)_*(a) = a + \langle a, [C] \rangle [C],$$

and the powers of $t_C$ act by the formula

$$(t_C^m)_*(a) = a + m \langle a, [C] \rangle [C].$$

Changing the orientation of $C$ replaces $[C]$ by $-[C]$ and hence does not change the right hand sides of these formulas. The formula for the action of $t_C$ implies that $t_C$ belongs to the Torelli group $\mathcal{I}(S)$ if and only if $[C] = 0$, i.e. if and only if $C$ is a separating circle.

**Multi-twists.** Let $c$ be a one-dimensional closed submanifold of $S$. A *Dehn multi-twist about* $c$ is defined as a product $t$ of the form

$$t = \prod_O t_O^{m_O},$$

where $O$ runs over all components of $c$ and $m_O$ are integers. Every Dehn multi-twist about $c$ can be represented by a diffeomorphism of $S$ equal to the identity outside of a subsurface of $S$ containing $c$ as a deformation retract. Such diffeomorphisms are called *multi-twist diffeomorphisms* about $c$.

Suppose that $C, D$ is a bounding pair of circles. Then $C$ and $D$ can be oriented in such a way that $[C] = [D]$ and hence $t_C$ and $t_D$ induce the same maps on $H_1(S)$. Therefore the multi-twists $t_C t_D^{-1}$, $t_D t_C^{-1}$ belong to $\mathcal{I}(S)$. Both of them are called the *Dehn–Johnson twists* about the bounding pair $C, D$. 
3. Pure diffeomorphisms and reduction systems

This section is devoted to an overview of the main notions related to reduction systems of diffeomorphisms of a surface $S$ and elements of $\text{Mod}(S)$. This material is needed only for the next section, which is devoted to the motivation behind the main results of this paper, but is not needed for the proofs of these results.

**Cutting surfaces and diffeomorphisms.** Let $c$ be one-dimensional closed submanifold of a surface $S$. We denote by $S/c$ the result of cutting $S$ along $c$. The components of $S/c$ are called parts into which $c$ divides $S$. The canonical map

$$p/c : S/c \rightarrow S$$

induces a diffeomorphism $(p/c)^{-1}(S-c) \rightarrow S-c$, treated as an identification. If $p/c$ is injective on a component $Q$ of $S/c$, then the induced map $Q \rightarrow p/c(Q)$ is also treated as an identification and $Q$ is treated as a subsurface of $S$.

Any diffeomorphism $\psi : S \rightarrow S$ such that $\psi(c) = c$ induces a diffeomorphism

$$\psi/c : S/c \rightarrow S/c.$$  

If $\psi/c$ leaves a component $Q$ of $S/c$ invariant, then $\psi/c$ induces a diffeomorphism

$$\psi_Q : Q \rightarrow Q,$$

called the restriction of $\psi$ to $Q$.

**Systems of circles and reduction systems.** A one-dimensional closed submanifold $c$ of a surface $S$ is called a system of circles on $S$ if the components of $c$ are all non-trivial circles on $S$ and are pair-wise non-isotopic.

A system of circles $c$ on $S$ is called a reduction system for a diffeomorphism $\psi$ of $S$ if $\psi(c) = c$. A system of circles $c$ on $S$ is called a reduction system for an element $f \in \text{Mod}(S)$ if $c$ is a reduction system for some diffeomorphism $\psi$ in the isotopy class $f$, i.e. if $f$ can be represented by a diffeomorphism $\psi$ of $S$ such that $\psi(c) = c$.

**Reducible and pseudo-Anosov elements.** A non-trivial element $f \in \text{Mod}(S)$ is said to be reducible if there exists a non-empty reduction system for $f$, and irreducible otherwise. An irreducible element of infinite order is called a pseudo-Anosov element. Thurston's theory [T] provides a lot of information about pseudo-Anosov elements.
**Pure diffeomorphisms and elements.** Let $\psi$ be a diffeomorphism of $S$. A system of circles $c$ is said to be a *pure reduction system* for $\psi$ if $c$ is a reduction system for $\psi$ and the following four conditions hold.

(a) $\psi$ is orientation-preserving.

(b) Every component of $S/c$ is invariant under $\psi/c$.

(c) $\psi$ is equal to the identity in a neighborhood of $c \cup \partial S$.

(d) For each component $Q$ of $S/c$ the isotopy class of the restriction $\psi_Q : Q \to Q$ is either pseudo-Anosov, or contains $\text{id}_Q$.

A diffeomorphism $\psi$ of $S$ is said to be *pure* if $\psi$ admits a pure reduction system. An isotopy class $f \in \text{Mod}(S)$ is said to be *pure* if $f$ contains a pure diffeomorphism. A system of circles $c$ is said to be a *pure reduction system* for an element $f \in \text{Mod}(S)$ if $c$ is a pure reduction system for some diffeomorphism $\psi$ in the isotopy class $f$. The *isotopy extension theorem* implies that the property of being a pure reduction system for $f$ depends only on the isotopy class of the submanifold $c$ in $S$.

The subgroups $\mathcal{I}_m(S)$. Given an integer $m$, the group $\mathcal{I}_m(S)$ is defined as the subgroup of $\text{Mod}(S)$ consisting of the isotopy classes of diffeomorphisms acting trivially on $H_1(S, \mathbb{Z}/m\mathbb{Z})$. Clearly, $\mathcal{I}(S)$ is contained in $\mathcal{I}_m(S)$ for every $m \in \mathbb{Z}$. In the present paper the groups $\mathcal{I}_m(S)$ play only a technical role and only in Section 4 devoted to the motivation.

**3.1. Theorem.** If $m \geq 3$, then all elements of $\mathcal{I}_m(S)$ are pure.

**3.2. Lemma.** Suppose that $c$ is a reduction system for a diffeomorphism $\psi$ representing an element of $\mathcal{I}_m(S)$, where $m \geq 3$. Then $\psi$ leaves every component of $c$ invariant, and $\psi/c$ leaves every component of $S/c$ invariant.

**Proofs.** See Theorem 1.7 and Theorem 1.2 of [I2] respectively.

**Minimal pure reduction systems.** A system of circles $c$ is said to be a *minimal pure reduction system* for an element $f \in \text{Mod}(S)$ if $c$ is a pure reduction system for $f$, but no proper subsystem of $c$ is. Any pure reduction system for $f$ contains a minimal pure reduction system, and, in fact, it is unique. Moreover, up to isotopy such a minimal pure reduction system depends only on $f$.

In fact, the set of the isotopy classes of components of a minimal pure reduction system for $f$ is nothing else but the canonical reduction system of $f$ in the sense of [I2]. This easily follows from the results of [I2], Chapter 7. See also [IM], Section 3. Since the canonical reduction system of $f$ is defined invariantly in terms of $f$, it depends only on $f$ and hence the same is true for the minimal pure reduction systems for $f$. 

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4. The sirens of Torelli topology

Extensions by the identity. Suppose that $Q$ is a subsurface of $S$. If $\varphi$ is a diffeomorphism of $Q$ equal to the identity in a neighborhood of the boundary $\partial Q$, then $\varphi$ canonically extends to a diffeomorphism of $S$ equal to the identity on the complementary surface $cQ$. We will denote this extension by $\varphi \backslash S$.

More generally, let $c$ be a one-dimensional closed submanifold of $S$ and let $Q$ be a component of $S/c$. The canonical map $p/c : S/c \to S$ is injective on $Q \setminus \partial Q$, but may map two different components of $\partial Q$ onto the same component of $c$. The image $p/c(Q)$ is always a subsurface of $S$. If $\varphi$ is a diffeomorphism of $Q$ equal to the identity in a neighborhood of $\partial Q$, then $\varphi$ induces a diffeomorphism of the image $p/c(Q)$. The latter, in turn, canonically extends to a diffeomorphism of $S$ equal to the identity on $c(p/c(Q))$. We will denote this extension also by $\varphi \backslash S$.

A construction of abelian subgroups of Teichmüller modular groups. While the extensions by the identity are rarely mentioned explicitly, they play a crucial role in the background of the theory of Teichmüller modular groups. As an example, let us outline their role in the classification of abelian subgroups of Teichmüller modular groups.

Let $c$ be a one-dimensional closed submanifold of $S$. Suppose that for each component $Q$ of $S/c$ we are given a diffeomorphism $\varphi(Q) : Q \to Q$ equal to identity in a neighborhood of the boundary $\partial Q$. Extensions $\varphi(Q) \backslash S$ of these diffeomorphisms by the identity to diffeomorphisms $S \to S$ obviously commute. Therefore, the isotopy classes of these extensions generate an abelian subgroup of $\text{Mod}(S)$.

It turns out that every abelian subgroup of $\text{Mod}(S)$ consisting of pure elements is contained in an abelian subgroup of this form. Together with Theorem 3.1 this implies that for every subgroup $G$ of $\text{Mod}(S)$ the intersection $G \cap \mathcal{I}_m(S)$, which is a subgroup of finite index in $G$, is contained in a subgroup of this form.

A better construction of abelian subgroups. It takes into account the special role of Dehn multi-twists. The extensions $\varphi(Q) \backslash S$ are equal to the identity in a neighborhood of $c$. Therefore, these extensions commute with twist diffeomorphisms representing Dehn twists about components of $c$, and hence the isotopy classes of these extensions together with multi-twists about $c$ generate an abelian subgroup $\mathfrak{A}$ of $\text{Mod}(S)$.

Adding Dehn multi-twists generators can be replaced by adding to $c$ components isotopic to the already present ones. But if we allow adding Dehn multi-twists, we may assume that $c$ is a system of circles and that for each $Q$ the diffeomorphism $\varphi(Q)$ is either isotopic to $\text{id}_Q$, or belongs to a pseudo-Anosov isotopy class.
The resulting class of the abelian subgroups is the same as the original one. But now the constructed subgroup admits a more detailed description. Without changing the subgroup, we can put aside diffeomorphisms $\varphi(Q)$ which are isotopic to the identity. The isotopy classes of extensions of remaining diffeomorphisms together with Dehn twists about the components of $c$ turn out to be the free generators of $\mathfrak{A}$.

**Pasting.** This construction of abelian subgroups is a *pasting argument* par excellence. We start with a system of circles $c$ on our surface $S$. It subdivides $S$ into several pieces. For each piece we consider an object located on it (a diffeomorphism of the piece in our example) and paste them together into a new object on the whole surface $S$ (an abelian group in our example). Usually one needs to add something located, at least morally, on the cutting submanifold $c$ (Dehn multi-twists about $c$).

**Commutants and bicommutants.** The *commutant* $X'$ of a subset $X$ of a group $\mathfrak{G}$ is the subgroup of $\mathfrak{G}$ consisting of elements commuting with all $g \in X$. The *bicommutant* of $X$ is the commutant of the commutant of $X$, i.e. the subgroup $X''$. The *commutant* $g'$ and the *bicommutant* $g''$ of an element $g \in \mathfrak{G}$ are the defined as the commutant and the bicommutant of the one-element subset $\{g\}$.

**Commutants and bicommutants in $\text{Mod}(S)$.** Let us fix a subgroup $\Gamma$ of finite index in $\text{Mod}(S)$ consisting of pure elements. For example, by Theorem 3.1 we can take as $\Gamma$ the subgroup $\mathfrak{H}_m(S)$ for any $m \geq 3$.

Suppose that $f \in \Gamma$ and $c$ is a pure reduction system for $f$. Then $f$ can be represented by a diffeomorphism $\varphi: S \to S$ such that $c$ is a pure reduction system for $\varphi$. Let us consider for every component $Q$ of $S/c$ the restriction

$$\varphi_Q: Q \to Q,$$

i.e. the diffeomorphism $Q \to Q$ induced by $\varphi/c$. The restriction $\varphi_Q$ is equal to the identity in a neighborhood of $\partial Q$, and hence the extension $\varphi_Q\backslash S$ of $\varphi_Q$ to a diffeomorphism of $S$ is defined.

By applying the above construction to $c$ and diffeomorphisms $\varphi_Q\backslash S$, we get an abelian subgroup of $\text{Mod}(S)$. Let $\mathfrak{B}(f)$ be the intersection of this subgroup with $\Gamma$. It turns out that $\mathfrak{B}(f)$ is a subgroup of finite index in the bicommutant $f''$ of $f$ in $\Gamma$. The bicommutant $f''$ itself is equal to $\mathfrak{B}(g)$ for an element $g$ closely related to $f$.

The (somewhat disguised) commutants, bicommutants, and groups $\mathfrak{B}(f)$ are the key tool for algebraic characterizations of Dehn twists and related elements of $\text{Mod}(S)$ and for the classification of automorphisms of $\text{Mod}(S)$. See [I1], [I2]. Ad hoc analogues of these construction for Torelli groups were used by B. Farb and the author [FI] in order to give an algebraic characterization of Dehn and Dehn–Johnson twists in $\mathfrak{H}(S)$.  

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**Cut-and-paste.** The construction of the groups \( \mathcal{B}(f) \) is a *cut-and-paste argument* par excellence. We started with a pure isotopy class \( f \). It has a representative \( \varphi \) which can be cut into sufficiently well understood pieces \( \varphi_Q \).

Namely, the isotopy class \( f_Q \) of each piece \( \varphi_Q \) is either pseudo-Anosov, or contains the identity. In both cases the commutant and the bicommutant of \( f_Q \) in \( \text{Mod}(Q) \) are well understood. The case when \( f_Q \) contains identity is trivial. If \( f_Q \) is pseudo-Anosov, then both \( f'_Q \) and \( f''_Q \) contain an infinite cyclic group generated by a pseudo-Anosov element as a subgroup of finite index. See [M] or [I\text{I}2].

After cutting \( \varphi \) into pieces \( \varphi_Q \), we considered the extensions \( \varphi_Q \backslash S \) of these pieces to \( S \) and pasted them together into the group \( \mathcal{B}(f) \).

**A siren song.** The sirens sing that one can cut and paste in Torelli groups also. A. Putman begins his paper [P] as follows.

We introduce machinery to allow “cut-and-paste”-style inductive arguments in the Torelli subgroup …. In the past these arguments have been problematic because restricting the Torelli group to subsurfaces gives different groups depending on how the subsurfaces are embedded. We define a category ...

While Putman's machinery allows to use “cut-and-paste” arguments in some situations, they remain problematic in other ones. The main obstacle is the lack of a suitable definition of Torelli groups for surfaces with boundary. By this reason after cutting \( \varphi \) into restrictions \( \varphi_Q \) we are not in the theory of Torelli groups anymore.

In fact, the heart of Putman's paper [P] is a definition of Torelli groups for surfaces with boundary. But he does not address the question if the isotopy classes of restrictions \( \varphi_Q \) belong to his version of Torelli groups. The siren sings that they do belong, and one only has to read [P] carefully …

**A bicommutant siren song.** Suppose that \( S \) is a closed surface and \( f \in \mathcal{I}(S) \). It is tempting to think that one construct a subgroup of finite index in the bicommutant \( f'' \) of \( f \) in \( \mathcal{I}(S) \) by adapting the construction of \( \mathcal{B}(f) \).

The element \( f \) is a pure element by Theorem 3.1. Let \( c \) be a pure reduction system for a diffeomorphism \( \varphi \) of \( S \) representing \( f \). Let us consider, for each component \( Q \) of \( S/c \), the restriction \( \varphi_Q \) and the extension \( \varphi_Q \backslash S \) of \( \varphi_Q \).

The sirens sing that the group generated by the Dehn multi-twists about \( c \) belonging to \( \mathcal{I}(S) \) together with the isotopy classes of extensions \( \varphi_Q \backslash S \) is a subgroup of finite index in the bicommutant \( f'' \) of \( f \) in \( \mathcal{I}(S) \).
An extension siren song. The sirens sing that for each component $Q$ of $S/c$ the isotopy class $f_Q \setminus S$ of the extension $\varphi_Q \setminus S$ belongs to $\mathcal{J}(S)$, at least up to Dehn multi-twists about $c$. Otherwise the bicommutant song does not make much sense.

Since only the component $Q$ of $S/c$ affects $f_Q \setminus S$, one may forget about other components. This leads to a plain extension version of the bicommutant song.

Let $f \in \mathcal{J}(S)$, and let $\varphi$ be a representative of $f$ leaving a subsurface $Q$ of $S$ invariant. Without any loss of generality one can assume that $\varphi$ is equal to the identity in a neighborhood of $\partial Q$ in $S$. Consider the restriction $\varphi_Q$ and its extension $\varphi_Q \setminus S$.

Let $f_Q \setminus S$ be the isotopy class of $\varphi_Q \setminus S$. The sirens sing that $f_Q \setminus S$ belongs to $\mathcal{J}(S)$ up to Dehn multi-twist about $\partial Q$. In other words,

$$t \cdot (f_Q \setminus S) \in \mathcal{J}(S)$$

for some Dehn multi-twist $t$ about $\partial Q$. Let us return to the reality.

**Example.** Suppose that $S$ is a closed surface and $Q$ is a subsurface of $S$ such that both $Q$ and its complementary surface $cQ$ are connected. Suppose that the boundary $\partial Q$ consists of four components $C_1, C_2, C_3, C_4$. Let $C$ be a circle bounding in $Q$ a disc with two holes together with $C_1$ and $C_2$, and $D$ be a circle bounding in $cQ$ a disc with two holes together with the same circles $C_1$ and $C_2$. The union of these two discs with two holes is a genus 1 surface with the boundary $C \cup D$, and hence

$$f = t_C t_D^{-1}$$

is a Dehn-Johnson twist. In particular, $f$ belongs to the Torelli group $\mathcal{J}(S)$.

Let $\tau_C, \tau_D$ be twist diffeomorphisms of $S$ representing $t_C, t_D$ and equal to the identity on $cQ, Q$ respectively. By the definition, $f = t_C t_D^{-1}$ is the isotopy class of

$$\varphi = \tau_C \circ \tau_D^{-1}.$$ 

Since $\tau_D$ is equal to identity on $Q$, the restriction $\varphi_Q$ is equal to the restriction $(\tau_C)_Q$. Since $\tau_C$ is equal to the identity on $cQ$, the extension $\tau_C \setminus S$ is equal to $\tau_C$. Hence

$$\varphi_Q \setminus S = \tau_C$$

and the isotopy class $f_Q \setminus S$ of $\varphi_Q \setminus S$ is equal to $t_C$. Since $C$ is non-separating in $S$,

$$f_Q \setminus S = t_C \not\in \mathcal{J}(S).$$
This ruins the most optimistic extensions hopes. The rest is ruined by the fact that
\[ t \cdot (f_Q \setminus S) = t \cdot t_C \not\in \mathcal{J}(S) \]
for any Dehn multi-twist \( t \) about \( \partial Q \). Let us prove this.

To begin with, let us orient the circles \( C_1, \ldots, C_4 \) as components of \( \partial Q \). The only relation between the homology classes \([C_1], \ldots, [C_4]\) is
\[ [C_1] + [C_2] + [C_3] + [C_4] = 0. \]
In particular, any three of the classes \([C_1], \ldots, [C_4]\) are linearly independent. Let us orient \( C \) and \( D \) in such a way that \([C] = [D] = [C_1] + [C_2]\).

There is a circle \( A \) in \( S \) such that \( A \) is disjoint from \( C_2 \) and \( C_4 \) and intersects each of three circles \( C, C_1, \) and \( C_3 \) transversely at one point. One can orient \( A \) in such a way that the intersection numbers of the homology class \( a = [A] \) are
\[ \langle a, [C] \rangle = \langle a, [C_1] \rangle = 1, \quad \langle a, [C_3] \rangle = -1, \]
\[ \langle a, [C_2] \rangle = \langle a, [C_4] \rangle = 0. \]

Every Dehn multi-twist \( t \) about \( \partial Q \) has the form
\[ t = \prod_{i=1}^{4} (t_{C_i})^{m_i}, \]
where \( m_1, \ldots, m_4 \) are integers. The product \( t \cdot t_C \) acts on \( a \) as follows:
\[ (t \cdot t_C)_* (a) = a + [C] + m_1[C_1] - m_3[C_3] \]
\[ = a + ([C_1] + [C_2]) + m_1[C_1] - m_3[C_3] \]
\[ = a + (m_1 + 1)[C_1] + [C_2] - m_3[C_3] \]
\[ \neq a, \]
because the classes \([C_1], [C_2], [C_3]\) are linearly independent. It follows that
\[ t \cdot (f_Q \setminus S) = t \cdot t_C \not\in \mathcal{J}(S) \]
for any Dehn multi-twist \( t \) about \( \partial Q \).
**Example: a modification.** In the above example \( \partial Q \) is not a pure reduction system for \( f \). Let us modify the example in such a way that it will be. Let \( m \in \mathbb{Z} \) and let

\[
f^m = t^m_C \cdot t^{-m}_D \in \mathcal{S}(S)
\]

be the isotopy class of \( \varphi^m = \tau^m_C \circ \tau^{-m}_D \). Then the isotopy class \( f^m_Q \setminus S \) of \( \varphi^m_Q \setminus S \) is equal to \( t^m_C \). As above,

\[
t \cdot (f^m_Q \setminus S) = t \cdot t^m_C \notin \mathcal{S}(S)
\]

for any Dehn multi-twist \( t \) about \( \partial Q \).

Let \( \psi \) be a diffeomorphism of \( S \) equal to the identity in a neighborhood of \( cQ \). Let

\[
\kappa = \varphi^m \circ \psi.
\]

Then \( \kappa_Q = T^m_C \circ \psi_Q \) and \( \kappa_Q \setminus S = T^m_C \circ \psi \). Let

\[
g, \quad g_Q, \quad k, \quad k_Q, \quad k_Q \setminus S
\]

be the isotopy classes of, respectively,

\[
\psi, \quad \psi_Q, \quad \kappa, \quad k_Q, \quad k_Q \setminus S.
\]

Then \( k = f^m \cdot g, \quad k_Q = t^m_C \cdot g_Q, \quad k_Q \setminus S = t^m_C \cdot g, \) and if \( g \in \mathcal{S}(S) \), then

\[
k = f^m \cdot g \in \mathcal{S}(S) \quad \text{and} \quad t \cdot (k_Q \setminus S) = t \cdot t^m_C \cdot g \notin \mathcal{S}(S)
\]

for any Dehn multi-twist \( t \) about \( \partial Q \). If, in addition, \( t^m_C \cdot g_Q \in \text{Mod}(Q) \) is a pseudo-Anosov element, then \( k_Q \) is irreducible.

Let \( \eta \) be a diffeomorphism of \( S \) equal to the identity in a neighborhood of \( Q \). Let

\[
\lambda = \kappa \circ \eta.
\]

Then \( \lambda_Q = \kappa_Q, \quad \lambda_Q \setminus S = \kappa_Q \setminus S, \) and \( \lambda_cQ = \tau^m_D \circ \eta_cQ \). Let

\[
e, \quad e_cQ, \quad l, \quad l_Q, \quad l_cQ
\]

be the isotopy classes of, respectively,

\[
\eta, \quad \eta_cQ, \quad \lambda, \quad \lambda_Q, \quad \lambda_cQ.
\]
Then \( l = k \cdot e \), \( l_Q = k_Q \), \( l_{cQ} = t_D^{-m} \cdot e_{cQ} \), and if \( g, e \in \mathcal{I}(S) \), then

\[
l = k \cdot e \in \mathcal{I}(S) \quad \text{and} \quad t \cdot (l_Q \backslash S) = t \cdot (k_Q \backslash S) \notin \mathcal{I}(S)
\]

for any Dehn multi-twist \( t \) about \( \partial Q \). If, in addition, \( t_D^{-m} \cdot e_{cQ} \in \text{Mod}(cQ) \) is a pseudo-Anosov element, then \( l_{cQ} \) is irreducible.

To sum up, if \( g, e \in \mathcal{I}(S) \) and if the isotopy classes \( t_D^m \cdot g_Q, t_D^{-m} \cdot e_{cQ} \) are pseudo-Anosov, then

\[
l \in \mathcal{I}(S), \quad t \cdot (l_Q \backslash S) \notin \mathcal{I}(S),
\]

and by Lemma 3.2 \( \partial Q \) is contained in any pure reduction system for \( l \). Since the isotopy classes of \( \lambda_Q \) and \( \lambda_{cQ} \) are pseudo-Anosov, \( \partial Q \) is actually a pure reduction system for \( l \). It remains to construct diffeomorphisms \( \psi, \eta \) with such properties.

**Construction of** \( \psi, \eta \). We will construct \( \psi, \eta \) under the assumption that the genus of both \( Q \) and \( cQ \) is at least 1. Then \( Q \) contains circles bounding in \( Q \) a surface of genus \( \geq 1 \) with one boundary component. Such circles are bounding circles in both \( Q \) and \( S \). A well known construction of Thurston leads to pseudo-Anosov elements of \( \text{Mod}(Q) \) of the form \( t_1^a \cdot t_2^b \), where both \( t_1 \) and \( t_2 \) are Dehn twists about such bounding circles.

Such an element \( t_1^a \cdot t_2^b \) can be represented by a composition of the form \( \tau_1^a \circ \tau_2^b \), where \( \tau_1, \tau_2 \) are twist diffeomorphisms of \( Q \) about bounding circles. Consider now \( \tau_1, \tau_2 \) as twist diffeomorphisms of \( S \) about the same circles. Then

\[
\psi = \tau_1^a \circ \tau_2^b
\]

is a diffeomorphism of \( S \) equal to the identity on \( cQ \). Since Dehn twists about bounding circles belong to \( \mathcal{I}(S) \), the isotopy class \( g \) of \( \psi \) belongs to \( \mathcal{I}(S) \). The isotopy class \( g_Q \) of the restriction \( \psi_Q \) is equal to \( t_1^a \cdot t_2^b \) and hence \( g_Q \) is pseudo-Anosov. It follows that \( t_1^m \cdot (g_Q)^n \) is pseudo-Anosov for all sufficiently large \( m, n \). See [I2], the proof of Lemma 5.8 in the case when \( g \) from this Lemma is reducible (not our \( g \)).

By the same arguments, there is a diffeomorphism \( \eta \) of \( S \) such that \( \eta \) is equal to the identity on \( Q \), its isotopy class \( e \) belongs to \( \mathcal{I}(S) \), and the isotopy class \( e_{cQ} \) of its restriction \( \eta_{cQ} \) is a pseudo-Anosov element of \( \text{Mod}(cQ) \). By the same reasons as above, \( t_D^{-m} \cdot (e_{cQ})^n \) is pseudo-Anosov for all sufficiently large \( m, n \).

Let \( m, n \) be two sufficiently large integers and replace \( \psi, \eta \) by their \( n \)-th powers. Then \( m \) and the new \( \psi, \eta \) have all the required properties.
5. Weakly Torelli diffeomorphisms

Notations related to the singular homology theory. For a topological space \( X \) we denote by \( C_n(X) \) the group of singular \( n \)-chains in \( X \), and by \( \partial \) the boundary map of the singular chain complex \( C_*(X) \). For a continuous map \( F: X \to Y \) we denote by \( F_* \) the induced maps \( C_*(X) \to C_*(Y) \) and \( H_*(X) \to H_*(Y) \).

For a cycle \( \alpha \) we denote by \([\alpha]\) the homology class. The space in question usually is clear from the context, but if it is important to distinguish between homology classes of \( \alpha \) in different spaces, we denote by \([\alpha]_X \in H_*(X)\) the homology class of \( \alpha \) in \( X \).

By \( Z_n(X) \) we denote the group of \( n \)-cycles in \( X \). For subspace \( A \) of \( X \) we denote by \( Z_n(X, A) \) the group of relative \( n \)-cycles of the pair \( (X, A) \), i.e. the group of singular chains \( \alpha \in C_n(X) \) in \( X \) such that \( \partial \alpha \in C_{n-1}(A) \). Then

\[
H_n(X, A) = Z_n(X, A) / (\partial C_{n+1}(X) + C_n(A)).
\]

Let \( i_*: H_n(A) \to H_n(X) \) and \( \partial_*: H_{n+1}(X, A) \to H_n(A) \) be the maps induced by the inclusion \( i: A \to X \) and the boundary map of the homology sequence of the pair \( (X, A) \) respectively. The exactness of the homology sequence of the pair \( (X, A) \) implies that the kernel of \( i_* \) is equal to the image of \( \partial_* \). Let

\[
K_n(A \to X) = \text{kernel of } i_*: H_n(A) \to H_n(X),
\]

\[
H_n(A: X) = H_n(A) / \partial_* H_{n+1}(X, A) = H_n(A) / K_n(A \to X).
\]

Obviously, the canonical homomorphism \( H_n(A: X) \to H_n(X) \) is an isomorphism onto the image of \( i_* \). We will often identify \( H_n(A: X) \) with this image.

The framework. Let us fix for the rest of the paper a closed connected surface \( S \) and a subsurface \( Q \) of \( S \). Let \( c = \partial Q \). The subsurface \( Q \) is not required to be connected, although the case of a connected subsurface is the most important one. Let

\[
K_0(c) = K_0(c \to Q) \cap K_0(c \to cQ),
\]

\[
Z_1(Q, c) = \{ \alpha \in Z_1(Q, c) \mid [\partial \alpha] \in K_0(c) \},
\]

\[
H_1(c) = H_1(c: S) = H_1(c) / K_1(c \to S).
\]

Since \( \alpha \in Z_1(Q, c) \) implies that \([\partial \alpha] \in K_0(c \to Q)\), the condition \([\partial \alpha] \in K_0(c)\) in the definition of \( Z_1(Q, c) \) can be replaced by \([\partial \alpha] \in K_0(c \to cQ)\).
Weakly Torelli diffeomorphisms. Let \( \varphi : Q \to Q \) be a diffeomorphism fixed in a neighborhood of \( c \). If \( \alpha \in \mathbb{Z}_1(Q, c) \), then

\[
\partial \varphi_*(\alpha) = \varphi_*(\partial \alpha) = \partial \alpha
\]

and hence \( \varphi_*(\alpha) - \alpha \) is a cycle in \( Q \).

The diffeomorphism \( \varphi \) is said to be weakly Torelli if for every cycle \( \sigma \) in \( Q \) the cycle \( \varphi_*(\sigma) - \sigma \) is homologous to 0 in \( S \) or, what is the same, \( [\varphi_*(\sigma) - \sigma]_S = 0 \). If \( \varphi \) admits an extension to a Torelli diffeomorphism \( \psi : S \to S \), then

\[
[\varphi_*(\sigma) - \sigma]_S = [\psi_*(\sigma) - \sigma]_S = \psi_*[\sigma]_S - [\sigma]_S = 0
\]

for every \( \sigma \in \mathbb{Z}_1(Q) \) and hence \( \varphi \) is a weakly Torelli diffeomorphism. It turns out that the property of being a weakly Torelli diffeomorphism has strong implications for the cycles \( \varphi_*(\alpha) - \alpha \) for all relative cycles \( \alpha \in \mathbb{Z}_1(Q, c) \).

The Mayer–Vietoris equivalence. The surface \( S \) is equal to the union of subsurfaces \( Q \) and \( cQ \), with the intersection \( Q \cap cQ = \partial Q = \partial cQ = c \). By a well known result, usually hidden in the proof of the Mayer–Vietoris theorem, the inclusion

\[
C_*(Q) + C_*(cQ) \to C_*(S)
\]

is a homology equivalence, i.e. it induces an isomorphism in homology.

The Mayer–Vietoris boundary map. The Mayer–Vietoris boundary map

\[
\mathcal{B} : H_1(S) \to H_0(c)
\]

is defined as follows. By the Mayer–Vietoris equivalence, every homology class \( \alpha \in H_1(S) \) had the form \( \alpha = [\alpha + \beta] \) for some \( \alpha \in C_1(Q) \) and \( \beta \in C_1(cQ) \). It is well known that \([\partial \alpha] \in H_0(c)\) depends only on \( \alpha \). By the definition, \( \mathcal{B}(\alpha) = [\partial \alpha] \).

The image of the Mayer–Vietoris boundary map \( \mathcal{B} \) is equal to \( K_0(c) \). Indeed, the group \( K_0(c) \) is the group of homology classes of \( 0 \)-chains in \( c \) which bound in both \( Q \) and \( cQ \). Obviously, the same is true for the image of \( \mathcal{B} \).

The intersection pairing. The orientation \( S \) induces an orientation of \( Q \), which, in turn, induces an orientation of \( c = \partial Q \). The Poincaré duality leads to a bilinear map

\[
(\bullet, \bullet)_c : H_0(c) \times H_1(c) \to \mathbb{Z}
\]

called the intersection pairing. By the Poincaré duality or by the following description this pair-
ing is non-degenerate. For each component $C$ of $c$, let $o(C) \in H_0(c)$ be the homology class of any singular $o$-simplex in $C$, and let $[C]$ be the fundamental class of $C$. Then

\[(5.2) \quad \langle o(C), [D] \rangle_c = 1 \text{ if } C = D, \quad \text{and} \quad \langle o(C), [D] \rangle_c = 0 \quad \text{if } C \neq D.\]

Let $\iota : H_1(c) \to H_1(S)$ be the inclusion homomorphism. The pairing $\langle \bullet, \bullet \rangle_c$ is related to the standard intersection pairing

\[\langle \bullet, \bullet \rangle : H_1(S) \times H_1(S) \to Z\]

and the Mayer–Vietoris boundary map $\partial$ by the formula

\[(5.3) \quad \langle a, \iota(b) \rangle = \langle \partial(a), b \rangle_c,\]

where $a \in H_1(S)$ and $b \in H_1(c)$.

5.1. Lemma. The groups $K_0(c)$ and $K_1(c \to S)$ are the orthogonal complements of each other with respect to the pairing $\langle \bullet, \bullet \rangle_c$.

Proof. Let $\theta \in H_0(c)$. Then $\theta$ is a linear combination of the form

\[\theta = \sum_C n(C) o(C),\]

where $C$ runs over all components of $c$ and $n(C) \in \mathbb{Z}$. If $P$ is a component of $Q$, then

\[\langle \theta, [\partial P] \rangle_c = \sum_C n(C)\]

where $C$ runs over all components of $\partial P$. On the other hand, $\theta \in K_0(c \to Q)$ if and only if all such sums are equal to $o$. Since $K_1(c \to Q)$ is generated by the fundamental classes $[\partial P]$ for components $P$ of $Q$, it follows that $K_0(c \to Q)$ and $K_1(c \to Q)$ are the orthogonal complements of each other. A completely similar argument proves that $K_0(c \to cQ)$ and $K_1(c \to cQ)$ are the orthogonal complements of each other.

By the Mayer–Vietoris equivalence the kernel $K_1(c \to S)$ is equal to the sum of the kernels $K_1(c \to Q)$ and $K_1(c \to cQ)$. By the definition $K_0(c)$ is equal to the intersection of the kernels $K_0(c \to Q)$ and $K_1(c \to cQ)$. By combining these facts with the previous paragraph, we see that $K_0(c)$ and $K_1(c \to S)$ are the orthogonal complements of each other. $\blacksquare$

5.2. Lemma. A cycle $\delta$ in $Q$ is homologous to a cycle in $c$ if and only if $\langle \delta, \tau \rangle = 0$ for all cycles $\tau$ in $Q$. 

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The "only if" part is obvious. Let us prove the "if" part. The exactness of the part
\[ H_1(c) \rightarrow H_1(Q) \rightarrow H_1(Q, c) \]
of the homology sequence of the pair \( (Q, c) \) implies that \( \delta \) is homologous to a cycle in \( c \) if and only if the homology class of \( \delta \) in the relative homology group \( H_1(Q, c) \) is equal to 0. By the Poincaré–Lefschetz duality a \( a \in H_1(Q, c) \) is equal to 0 if and only if \( \langle a, b \rangle = 0 \) for all \( b \in H_1(Q) \). It remains to note that
\[ \langle a, b \rangle = \langle \delta, \tau \rangle \]
if \( a \) is the homology class of \( \delta \) and \( b = [\tau] \) for some \( \tau \in Z_1(Q) \).

**5.3. Theorem.** If \( \varphi \) is a weakly Torelli diffeomorphism, then for every \( \alpha \in Z_1(Q, c) \) the cycle \( \varphi_*(\alpha) - \alpha \) is homologous in \( Q \) to a cycle in \( c \).

**Proof.** Suppose that \( \alpha \in Z_1(Q, c) \). Let \( \tau \in Z_1(Q) \) be an arbitrary cycle. Let
\[ \delta = \varphi_*(\alpha) - \alpha \quad \text{and} \quad \sigma = \varphi_*(\tau) - \tau. \]
Since \( \alpha \in Z_1(Q, c) \), there exists \( \beta \in Z_1(cQ, c) \) such that \( \partial \alpha = \partial \beta \) and hence
\[ \gamma = \alpha - \beta \]
is a cycle.

1. As the first step of the proof, let us prove that
\[ \langle \delta, \tau \rangle + \langle \gamma, \sigma \rangle + \langle \delta, \sigma \rangle = 0. \]

Let \( \psi: S \rightarrow S \) is the extension of \( \varphi \) by the identity. Then
\[ \psi_*(\gamma) - \gamma = \psi_*(\alpha + \beta) - (\alpha + \beta) = \varphi_*(\alpha) - \alpha = \delta. \]
Since \( \psi \) is a diffeomorphism, \( \psi \) preserves the intersection pairing and hence
\[ \langle \gamma, \tau \rangle = \langle \psi_*(\gamma), \psi_*(\tau) \rangle = \langle \psi_*(\gamma), \varphi_*(\tau) \rangle. \]
Obviously, \( \psi_*(\gamma) = \gamma + \delta \) and \( \varphi_*(\tau) = \tau + \sigma \) and hence
\[ \langle \psi_*(\gamma), \varphi_*(\tau) \rangle = \langle \gamma + \delta, \tau + \sigma \rangle \]
\[ = \langle \gamma, \tau \rangle + \langle \delta, \tau \rangle + \langle \gamma, \sigma \rangle + \langle \delta, \sigma \rangle. \]
By combining the last two chains of equalities we see that
\[ \langle \gamma, \tau \rangle = \langle \gamma, \tau \rangle + \langle \delta, \tau \rangle + \langle \gamma, \sigma \rangle + \langle \delta, \sigma \rangle. \]
Cancelling the terms \( \langle \gamma, \tau \rangle \) completes the proof of (5.4).

2. The next step is to prove that
\[ (5.5) \quad \langle \delta, \sigma \rangle = 0. \]
Since \( \delta \) is a cycle in \( Q \) and \( \rho \) is a cycle in \( c \), the intersection number \( \langle \delta, \rho \rangle = 0 \). On the other hand, \( \langle \delta, \sigma \rangle = \langle \delta, \rho \rangle \) because \( \sigma \) and \( \rho \) are homologous. This implies (5.5).

3. Now we will use the weak Torelli property in order to prove that
\[ (5.6) \quad \langle \gamma, \sigma \rangle = 0. \]
Since \( \varphi \) is a weak Torelli diffeomorphism, the cycle \( \sigma \) is homologous to \( 0 \) in \( S \). By the Mayer–Vietoris equivalence this implies that \( \sigma \) is homologous in \( Q \) to a cycle \( \rho \in \mathbb{Z}_1(c) \) such that \( \rho \) is homologous to \( 0 \) in \( cQ \). Let \( r \in H_1(c) \) be the homology class of \( \rho \) in \( c \). Then \( r \in K_1(c \to cQ) \) and hence \( r \in K_1(c \to S) \).

The fact that \( \sigma \) and \( \rho \) are homologous also implies that
\[ \langle \gamma, \sigma \rangle = \langle \gamma, \rho \rangle = \langle [\gamma], [\rho]_S \rangle. \]
Obviously, the inclusion \( \iota \) maps \( r = [\rho]_c \) to \( [\rho]_S \) and hence (5.3) implies that
\[ \langle [\gamma], [\rho]_S \rangle = \langle D[\gamma], r \rangle_c \]
But \( D[\gamma] = [\partial \alpha] \) by the definition of \( D \) and hence
\[ \langle D[\gamma], r \rangle_c = \langle [\partial \alpha], r \rangle_c \]
Since \( [\partial \alpha] \in K_0(c) \) and \( r \in K_1(c \to S) \), Lemma 5.1 implies that
\[ \langle [\partial \alpha], r \rangle_c = 0. \]
The last four displayed equalities together immediately imply (5.6).

4. Finally, the equalities (5.4), (5.5), and (5.6) together imply that \( \langle \delta, \tau \rangle = 0 \). Since \( \tau \) is an arbitrary cycle in \( Q \), Lemma 5.2 implies that \( \delta = \varphi_*(\alpha) - \alpha \) is homologous to a cycle in \( c \). The theorem follows. ■
**The difference classes.** Suppose that \( \varphi \) is a weakly Torelli diffeomorphism. Let us identify the group \( H_1(c) = H_1(c)/K_1(c \to S) \) with the image of the map \( \iota: H_1(c) \to H_1(S) \).

Let \( \alpha \in Z_1(Q, c) \). By Theorem 5.3 the cycle \( \varphi_*(\alpha) - \alpha \) is homologous in \( Q \) to a cycle \( \gamma \) in \( c \). The identification of \( H_1(c) \) with the image of \( \iota \) turns the image of \( [\gamma] \in H_1(c) \) in the group \( H_1(c) \) into the homology class \( [\gamma]_S \) of the cycle \( \gamma \) in \( S \).

Since \( \gamma \) is homologous to \( \varphi_*(\alpha) - \alpha \), the homology class \( [\gamma]_S \) is equal to the homology class \( [\varphi_*(\alpha) - \alpha]_S \) of the cycle \( \varphi_*(\alpha) - \alpha \) in \( S \). In particular, the image of \( [\gamma] \in H_1(c) \) in the group \( H_1(c) \) depends only on \( \alpha \), and if \( \alpha \) is a cycle in \( Q \), then this image is equal to \( \circ \) by the definition of weakly Torelli maps.

The image of \( [\gamma] \in H_1(c) \) in the group \( H_1(c) \) is called the *difference class* of \( \alpha \) and is denoted by \( \Delta_\varphi(\alpha) \). Obviously, the map \( \alpha \mapsto \Delta_\varphi(\alpha) \) is a homomorphism

\[
\Delta_\varphi : Z_1(Q, c) \to H_1(c).
\]

It is called the \( \Delta \)-*difference map* of \( \varphi \). The difference class \( \Delta_\varphi(\alpha) \) depends only on \( \partial \alpha \).

Indeed, if \( \alpha, \beta \in Z_1(Q, c) \) and \( \partial \alpha = \partial \beta \), then \( \alpha - \beta \) is a cycle in \( Q \) and hence

\[
\Delta_\varphi(\alpha) - \Delta_\varphi(\beta) = \Delta_\varphi(\alpha - \beta) = \circ.
\]

If \( \delta \in C_1(c) \), then \( \varphi_*(\delta) = \delta \) and hence \( \varphi_*(\delta) - \delta = \circ \in Z_1(c) \) because the diffeomorphism \( \varphi \) is fixed on \( c \). It follows that \( \Delta_\varphi(\delta) = \circ \) for every \( \delta \in C_1(c) \). By combining this observation with the fact that \( \Delta_\varphi(\alpha) \) depends only on \( \partial \alpha \), we see that, moreover, \( \Delta_\varphi(\alpha) \) depends only on the homology class \( [\partial \alpha] \in H_0(c) \).

**The \( \delta \)-difference map.** Let \( b : Z_1(Q, c) \to H_0(c) \) be the homomorphism \( \alpha \mapsto [\partial \alpha] \).

By the definition the group \( Z_1(Q, c) \) is the preimage of \( K_0(c) \) under the map \( b \). It follows that \( b \) induces a surjective homomorphism \( b : Z_1(Q, c) \to K_0(c) \).

If \( \varphi \) is a weakly Torelli diffeomorphism, then the difference classes \( \Delta_\varphi(\alpha) \) depend only on \( [\partial \alpha] \in H_0(c) \) and hence the surjectivity of \( b \) implies that there exists a unique homomorphism \( \delta_\varphi \) such that the triangle

\[
\begin{array}{ccc}
Z_1(Q, c) & \xrightarrow{b} & K_0(c) \\
\Delta_\varphi & \downarrow & \\
H_1(c) & \xleftarrow{\delta_\varphi} & \\
\end{array}
\]

is commutative. The map \( \delta_\varphi \) is called the \( \delta \)-*difference map* of \( \varphi \).
5.4. **Lemma.** If $\varphi, \psi$ are weakly Torelli diffeomorphisms, then $\psi \circ \varphi$ is also a weakly Torelli diffeomorphism, and $\delta_{\psi \circ \varphi} = \delta_\psi + \delta_\varphi$.

**Proof.** Let $\alpha \in Z_1(Q, c)$. Then $\varphi_*(\alpha) \in Z_1(Q, c)$ because $\varphi$ is fixed on $c$. Obviously,

\[(5.7) \quad \psi_*(\varphi_*(\alpha)) - \alpha = (\psi_*(\varphi_*(\alpha)) - \varphi_*(\alpha)) + (\varphi_*(\alpha) - \alpha).\]

If, moreover, $\alpha \in Z_1(Q)$, the $\varphi_*(\alpha) \in Z_1(Q)$ also and hence both $\varphi_*(\alpha) - \alpha$ and $\psi_*(\varphi_*(\alpha)) - \varphi_*(\alpha)$ are homologous to $0$ in $S$. Together with (5.7) this implies that the cycle $\psi_*(\varphi_*(\alpha)) - \alpha$ is homologous to $0$ in $S$. It follows that $\psi \circ \varphi$ is weakly Torelli.

In general, Theorem 5.3 implies that the cycles $\varphi_*(\alpha) - \alpha$ and $\psi_*(\varphi_*(\alpha)) - \varphi_*(\alpha)$ are homologous in $Q$ to some cycles $\delta, \varepsilon \in C_1(c)$ respectively. In view of (5.7) this implies that $\psi_*(\varphi_*(\alpha)) - \alpha$ is homologous in $Q$ to $\varepsilon + \delta \in Z_1(c)$.

Let $\theta = [\partial \alpha] = [\partial \varphi_*(\alpha)]$. The images in $H_1(c)$ of $[\delta], [\varepsilon]$ and $[\varepsilon + \delta]$ are equal to

$$\delta_\varphi(\theta), \quad \delta_\psi(\theta), \quad \text{and} \quad \delta_{\psi \circ \varphi}(\theta)$$

respectively. Since the last one is equal to the sum of the first two,

$$\delta_{\psi \circ \varphi}(\theta) = \delta_\psi(\theta) + \delta_\varphi(\theta).$$

Since every chain $\theta \in K_1(c)$ is equal to $[\partial \alpha]$ for some $\alpha \in Z_1(Q, c)$, the last displayed equality implies that $\delta_{\psi \circ \varphi} = \delta_\psi + \delta_\varphi$. ■

5.5. **Lemma.** If $\varphi$ is a weakly Torelli diffeomorphism, then $\varphi^{-1}$ is also a weakly Torelli diffeomorphism and $\delta_{\varphi^{-1}} = -\delta_\varphi$.

**Proof.** Let $\alpha \in Z_1(Q, c)$. Then $\beta = \varphi^{-1}_*(\alpha) \in Z_1(Q, c)$ and

\[(5.8) \quad \varphi^{-1}_*(\alpha) - \alpha = \beta - \varphi_*(\beta)\]

If, moreover, $\alpha \in Z_1(Q)$, then $\beta \in Z_1(Q)$ also and hence $\varphi_*(\beta) - \beta$ is homologous to $0$ in $S$. In view of (5.8) this implies that the cycle $\varphi^{-1}_*(\alpha) - \alpha$ is homologous to $0$ in $S$. It follows that $\varphi^{-1}$ is a weakly Torelli diffeomorphism.

In general, Theorem 5.3 implies that the cycle $\varphi_*(\beta) - \beta$ is homologous in $Q$ to a cycle $\delta \in Z_1(c)$. In view of (5.8) $\varphi^{-1}_*(\alpha) - \alpha$ is homologous in $Q$ to $-\delta$.

Let $\theta = [\partial \alpha] = [\partial \beta]$. Then $\delta_\varphi(\theta)$ is equal to the image of $[\delta]$ in $H_1(c)$ and $\delta_{\varphi^{-1}}(\theta)$ is equal to the image of $[-\delta]$ in $H_1(c)$. Therefore $\delta_\varphi(\theta) = -\delta_{\varphi^{-1}}(\theta)$. Since every chain $\theta \in K_1(c)$ is equal to $[\partial \alpha]$ for some $\alpha \in Z_1(Q, c)$, the lemma follows. ■
6. Extensions by the identity and the symmetry property

6.1. Theorem. Let $\varphi : Q \to Q$ be a weakly Torelli diffeomorphism and let $\psi : S \to S$ be the extension of $\varphi$ by the identity map of cQ. Then

$$\psi_*(a) - a = \delta_\varphi \circ \varnothing(a)$$

for all $a \in H_1(S)$, where the target $H_1(c)$ of $\delta_\varphi$ is identified with the image of the inclusion homomorphism $\iota : H_1(c) \to H_1(S)$.

Proof. Let $a \in H_1(S)$. By the Mayer–Vietoris equivalence, $a = [\alpha - \beta]$ for some chains $\alpha \in C_1(Q)$ and $\beta \in C_1(cQ)$ such that $\alpha - \beta$ is a cycle. Then $\varnothing(a) = [\partial \alpha]$.

Since $\alpha - \beta$ is a cycle, $\partial \alpha = \partial \beta \in C_0(c)$ and hence $\partial \alpha = \partial \beta$ bounds both in $Q$ and in $cQ$. It follows that $[\partial \alpha] \in K_0(c)$ and $\alpha \in Z_1(Q, c)$. Therefore $\Delta_\varphi(\alpha)$ is defined.

After the identification of $H_1(c)$ with the image of $\iota : H_1(c) \to H_1(S)$ the difference class $\Delta_\varphi(\alpha)$ turns into the homology class $[\varphi_*(\alpha) - \alpha]_S \in H_1(S)$ of $\varphi_*(\alpha) - \alpha$ in $S$.

On the other hand, $\psi_*(a) - a$ is equal to the homology class in $S$ of the cycle

$$\psi_*(\alpha - \beta) - (\alpha - \beta) = \psi_*(\alpha) - \psi_*(\beta) - \alpha + \beta = \varphi_*(\alpha) - \beta - \alpha + \beta = \varphi_*(\alpha) - \alpha.$$

It follows that $\psi_*(a) - a = \Delta_\varphi(\alpha)$. But $\Delta_\varphi(\alpha) = \delta_\varphi([\partial \alpha])$ by the definition of $\delta_\varphi$. By combining the last two equalities with $\varnothing(a) = [\partial \alpha]$ we see that

$$\psi_*(a) - a = \Delta_\varphi(\alpha) = \delta_\varphi([\partial \alpha]) = \delta_\varphi \circ \varnothing(a).$$

The theorem follows. ■

6.2. Theorem. Let $\varphi : Q \to Q$ be a weakly Torelli diffeomorphism and let $\psi : S \to S$ be the extension of $\varphi$ by the identity map of cQ. Then $\psi$ is a Torelli diffeomorphism if and only if $\delta_\varphi = 0$.

Proof. If $\delta_\varphi = 0$, then Theorem 6.1 implies that the induced map $\psi_* : H_1(S) \to H_1(S)$ is equal to the identity and hence $\psi$ is a Torelli diffeomorphism. Conversely, if $\psi$ is a Torelli diffeomorphism, then $\varphi$ is a weakly Torelli diffeomorphism by the remarks after the definition of the latter. ■
The induced pairing. Lemma 5.1 implies that (5.1) induces a non-degenerate pairing

\[ \langle \bullet, \bullet \rangle_c : K_0(c) \times H_1(c) \rightarrow \mathbb{Z}. \]

The identification of \( H_1(c) \) with the image of \( \iota : H_1(c) \rightarrow H_1(S) \) turns (5.3) into

\[ (6.1) \quad \langle a, b \rangle = \langle D(a), b \rangle_c, \]

where \( b \in H_1(c), \quad a \in H_1(S) \) and hence \( D(a) \in K_0(c) \) by the remarks after the definition of the Mayer–Vietoris boundary map \( D \). A homomorphism

\[ \delta : K_0(c) \rightarrow H_1(c) \]

is said to be symmetric if \( \langle a, \delta(b) \rangle_c = \langle b, \delta(a) \rangle_c \) for all \( a, b \in K_0(c) \).

6.3. Theorem. If \( \varphi \) is a weakly Torelli diffeomorphism, then \( \delta_\varphi \) is symmetric.

Proof. Let \( \psi : S \rightarrow S \) be the extension of \( \varphi \) by the identity. By Theorem 6.1

\[ \psi_*(a) - a = \delta_\varphi \circ D(a) \]

for all \( a \in H_1(S) \). Let \( a, b \in H_1(S) \). Since \( \psi_\ast \) preserves the intersection pairing,

\[ \langle a, b \rangle = \langle a + \delta_\varphi \circ D(a), b + \delta_\varphi \circ D(b) \rangle. \]

The intersection pairing \( \langle \bullet, \bullet \rangle \) is equal to \( \circ \) on the image of \( H_1(c) \rightarrow H_1(S) \), and hence

\[ \langle \delta_\varphi \circ D(a), \delta_\varphi \circ D(b) \rangle = \circ. \]

It follows that \( \langle a, b \rangle = \langle a, b \rangle + \langle a, \delta_\varphi \circ D(b) \rangle + \langle \delta_\varphi \circ D(a), b \rangle \), and hence

\[ \langle a, \delta_\varphi \circ D(b) \rangle + \langle \delta_\varphi \circ D(a), b \rangle = \circ. \]

Since the intersection pairing is skew-symmetric, this, in turn, implies that

\[ \langle a, \delta_\varphi \circ D(b) \rangle = \langle b, \delta_\varphi \circ D(a) \rangle. \]

By combining this identity with (6.1), we get

\[ \langle D(a), \delta_\varphi \circ D(b) \rangle_c = \langle D(b), \delta_\varphi \circ D(a) \rangle_c. \]

By the remarks after the definition of \( D \) the image of \( D \) is equal to \( K_0(c) \). It follows that \( \delta_\varphi \) is symmetric. ■
7. General extensions

**The connectivity assumption.** For the rest of the paper we will assume that the subsurface \( Q \) is connected. Of course, this is the most important case, and the general case, at least in principle, can be approached by dealing with each component separately. But one should keep in mind that in Torelli topology glueing the information about different components is not automatic. The glueing problem is one of the topics of the sequel \([I_3]\) of this paper.

The main features of the case of connected \( Q \) are the following. The kernel \( K_0(c \to cQ) \) is contained in the kernel \( K_0(c \to Q) \). The kernel \( K_1(c \to Q) \) is generated by the fundamental class \( [\partial Q] \) and hence is contained in \( K_1(c \to cQ) \). It follows that

\[
K_0(c) = K_0(c \to cQ) \quad \text{and} \quad K_1(c \to S) = K_1(c \to cQ).
\]

Hence \( H_1(c) = H_1(c)/K_1(c \to cQ) \).

**Completely reducible homomorphisms.** There is a canonical direct sum decomposition

\[
H_\ast(c) = \bigoplus_P H_\ast(\partial P),
\]

where \( P \) runs over all components of \( cQ \). Since the subsurface \( Q \) is connected and hence \( K_0(c) = K_0(c \to cQ) \), this decomposition leads to the direct sum decomposition

\[
K_0(c) = \bigoplus_P K_0(\partial P),
\]

where \( K_0(\partial P) = K_0(\partial P \to P) \), and to the direct sum decomposition

\[
H_1(c) = \bigoplus_P H_1(\partial P),
\]

where \( H_1(\partial P) = H_1(\partial P; P) = H_1(\partial P)/K_1(\partial P \to P) \). A homomorphism

\[
\delta : K_0(c) \longrightarrow H_1(c)
\]

is called completely reducible if \( \delta \) respects the above direct sum decompositions, i.e. if \( \delta \) maps \( K_0(\partial P) \) into \( H_1(\partial P) \) for every component \( P \) of \( cQ \). If \( \delta \) is completely reducible, then \( \delta \) admits a direct sum decomposition

\[
\delta = \bigoplus_P \delta_P
\]

into induced homomorphisms \( \delta_P : K_0(\partial P) \to H_1(\partial P) \). 

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7.1. **Theorem.** Of $\varphi : Q \rightarrow Q$ can be extended to a Torelli diffeomorphism $S \rightarrow S$, then $\varphi$ is a weakly Torelli diffeomorphism and $\delta_{\varphi}$ is completely reducible.

**Proof.** By the remarks after the definition of weakly Torelli diffeomorphisms $\varphi$ is a weakly Torelli diffeomorphism. Let $R$ be an arbitrary component of $cQ$, and let $\alpha \in Z_1(Q, c)$. It is sufficient to prove that $[\partial \alpha] \in K_0(\partial R)$ implies $\Delta_{\varphi}(\alpha) \in H_1(\partial R)$.

Let us present $\partial \alpha$ as the sum $\theta + \rho$, where $\theta \in C_0(\partial R)$ and $\rho \in C_0(c \sim \partial R)$. Then $[\partial \alpha] = [\theta]$ and $[\rho] = 0$. It follows that $\theta$ bounds in $R$. Since $Q$ is connected, $\theta$ bounds in $Q$ also, i.e. $\theta = \partial \alpha'$ for some $\alpha' \in Z_1(Q, c)$. Then $[\partial \alpha'] = [\partial \alpha]$ and $\partial \alpha' \in C_0(\partial R)$. On the other hand, $\Delta_{\varphi}(\alpha)$ depends only on $[\alpha]$ and hence $\Delta_{\varphi}(\alpha') = \Delta_{\varphi}(\alpha)$.

Therefore we may assume that $\partial \alpha \in C_0(\partial R)$. Then the assumption $[\partial \alpha] \in K_0(\partial R)$ implies that there exists $\beta \in C_1(R)$ such that $\partial \alpha = \partial \beta$ and hence

$$\gamma = \alpha - \beta$$

is a cycle. Let $\psi$ be a Torelli diffeomorphism of $S$ extending $\varphi$. Then $\psi_*(\gamma)$ is homologous to $\gamma$ in $S$. By the Mayer–Vietoris equivalence

$$\psi_*(\gamma) - \gamma = \partial(A + B)$$

for some chains $A \in C_2(Q)$, $B \in C_2(cQ)$. Since $\psi$ is an extension of $\varphi$, it follows that

$$(7.1) \quad \psi_*(\alpha) - \alpha - \partial A = \psi_*(\beta) - \beta - \partial B \in C_1(c).$$

Let us represent $B$ as a sum $B = C + D$ with $C \in C_2(R)$ and $D \in C_2(cQ \sim R)$. Then

$$\psi_*(\alpha) - \alpha - \partial(A + D) = \psi_*(\beta) - \beta - \partial C \in C_1(\partial R).$$

Let $\delta = \psi_*(\beta) - \beta - \partial C$. Then $\delta$ is a cycle in $\partial R$ and the cycle $\psi_*(\alpha) - \alpha$ is homologous in $Q$ to $\delta + \partial D$. By the definition, $\Delta_{\varphi}(\alpha)$ is equal to the image of the homology class $[\delta + \partial D] \in H_1(c)$ in the group $H_1(c)$.

Since, obviously, $[\partial D] \in K_1(c \rightarrow S)$, the image of $[\delta + \partial D]$ in $H_1(c)$ is equal to the image of $[\delta]$ and hence belongs to $H_1(\partial R)$. It follows that $\Delta_{\varphi}(\alpha) \in H_1(\partial R)$.

**Matrix presentations.** Let $P$ be a component of $cQ$. Let $C_0$, $C_1$, ..., $C_n$ be the components of $\partial P$ oriented as components of $\partial Q$. For $i = 1, 2, \ldots, n$ let

$$o_i = o_{C_i} - o_{C_0}$$
and let \( c_i \) be the image of \([C_i]\) in \( H_1(\partial P)\). Then

\[
o_1, \ o_2, \ldots, \ o_n
\]

is a basis of \( K_0(\partial P) \) and

\[
c_1, \ c_2, \ldots, \ c_n
\]

is a basis of \( H_1(\partial P) \).

Suppose that \( \delta \) is completely reducible. Every \( \delta(o_i) \) belongs to \( H_1(\partial P) \), and hence

\[
\delta(o_i) = \sum_{j=1}^{n} m_{ij} c_j
\]

for some integers \( m_{ij} \). The matrix \((m_{ij})_{1 \leq i,j \leq n}\) is the matrix of the component \( \delta_P \).

The symmetry condition holds for \( \delta_P \) if and only if it holds for every pair \( o_i, o_j \) of elements of our basis of \( K_0(\partial P) \), i.e. if and only if

\[
\langle o_i, \delta(o_j) \rangle = \langle o_j, \delta(o_i) \rangle
\]

for all \( i, j \) between 1 and \( n \). But \( \langle o_i, \delta(o_j) \rangle = m_{ji} \) and \( \langle o_j, \delta(o_i) \rangle = m_{ij} \). Therefore, \( \delta_P \) is symmetric if and only if \( m_{ij} = m_{ji} \) for all \( i, j \) between 1 and \( n \), i.e. if and only if the corresponding matrix \((m_{ij})_{1 \leq i,j \leq n}\) is symmetric.

**Symmetric matrices.** Symmetric integer \( n \times n \) matrices form a free abelian group \( SM(n) \) of rank \( n(n + 1)/2 \). A standard basis of \( SM(n) \) consists of the following matrices. For a pair \( k, l \in \{1, 2, \ldots, n\} \), let \( e(k,l) \) be \( n \times n \) matrix with entries

\[
e(k,l)_{ij} = 1 \quad \text{if} \quad \{i, j\} = \{k, l\} \quad \text{and} \quad e(k,l)_{ij} = 0 \quad \text{otherwise}.
\]

The matrices \( e(k,l) \) with \( k \leq l \) obviously form a basis of \( SM(n) \).

For a subset \( a \) of \( \{1, 2, \ldots, n\} \), let \( m(a) \) be \( n \times n \) matrix with the entries

\[
m(a)_{ij} = 1 \quad \text{if} \quad i, j \in a, \quad \text{and} \quad m(a)_{ij} = 0 \quad \text{otherwise}.
\]

For \( k \leq l \), let \( a_{k,1} = \{k, k+1, \ldots, l\} \), and let \( m(k,l) = m(a_{k,1}) \).

**7.2. Lemma.** The matrices \( m(k,l) \) with \( k \leq l \) form a basis of \( SM(n) \).

**Proof.** Since \( SM(n) \) is a free abelian group of rank \( n(n + 1)/2 \) and there are \( n(n + 1)/1 \) matrices \( m(k,l) \) with \( k \leq l \), it is sufficient to prove that these matrices generate the abelian group \( SM(n) \).
Therefore it is sufficient to express the matrices $e(k,l)$ with $k \leq l$ in terms of the matrices $m(k,l)$ with $k \leq l$. But

$$e(k,k) = m(k, k),$$

$$e(k, k + 1) = m(k, k + 1) - m(k, k) - m(k + 1, k + 1),$$

and if $k + 1 < l$, then

$$e(k,l) = m(k, l) - m(k + 1, l) - m(k, l - 1) + m(k + 1, l - 1).$$

It follows that the matrices $m(k,l)$ with $k \leq l$ generate $SM(n)$, and hence form a basis of $SM(n)$. ■

**Peripheral twists and bi-twists.** Let $P$ be a component of $cQ$, and let $U$ be the union of several components of $\partial P$. Let us orient $U$ as a part of the boundary of $Q$.

There are circles $C$ in $Q$ bounding together with $U$ a subsurface of $Q$. We will call twist diffeomorphisms about such circles the *peripheral twists diffeomorphisms about* $U$.

Let $C$ be a circle in $Q$ bounding together with $U$ a subsurface of $Q$, and let $D$ be a circle in $cQ$ bounding together with $U$ a subsurface of $cQ$. The union of these two subsurfaces is a subsurface of $S$ with boundary $C \cup D$. Let $T_C, T_D$ be twists diffeomorphisms of $S$ about $C, D$ fixed on $cQ, cP$ respectively.

The diffeomorphism $\psi = T_C \circ T_D^{-1}$ leaves $Q$ invariant and is equal to the identity on $c$. The isotopy class $t_C \circ t_D^{-1}$ of $\psi$ is a Dehn-Johnson twist and hence belongs to $\mathcal{F}(S)$. Therefore $\psi$ is a Torelli diffeomorphism.

We will call such diffeomorphisms $\psi$ *peripheral bi-twist diffeomorphisms* about $U$.

The diffeomorphism $\psi = T_C \circ T_D^{-1}$ induces a weak Torelli diffeomorphism $\varphi$ of $Q$. It is a twist diffeomorphism of $Q$ about $C$, and its extension $\chi$ by the identity to a diffeomorphism of $S$ is equal to $T_C$.

**7.3. Lemma.** Let $[U] \in H_1(c)$ be the image of the fundamental class of $U$. Then

$$\delta_\varphi(a) = \langle a, [U] \rangle_c [U]$$

for all $a \in K_0(c)$.

**Proof.** The orientation of $U$ is the same as the orientation of $U$ as a part of the boundary of the subsurface of $S$ bounded by $C \cup U$. Let us consider $C$ with the orientation opposite
to its orientation as a part of this boundary. Then
\[ [C] = [U] \]

after the identification of \( H_1(c) \) with the image of \( H_1(c) \) in \( H_1(S) \).

Let \( a \in H_1(S) \). By Theorem 6.1
\[ \delta \phi \circ \mathcal{D}(a) = \chi_*(a) - a. \]

Since \( \chi \) is a twist diffeomorphism of \( S \) about \( C \),
\[ \chi_*(a) - a = \langle a, [C] \rangle [C] = \langle a, [U] \rangle [U]. \]

Since \( \langle a, [U] \rangle = \langle \mathcal{D}(a), [U] \rangle \) by the formula (6.1), it follows that
\[ \delta \phi \circ \mathcal{D}(a) = \langle \mathcal{D}(a), [U] \rangle [U]. \]

Since \( \mathcal{D} \) is a map onto \( K_0(c) \), the lemma follows. ■

7.4. Theorem. For every symmetric completely reducible map
\[ \delta: K_0(c) \rightarrow H_1(c) \]

there exist a weakly Torelli diffeomorphism \( \phi \) of \( Q \) admitting an extension to a Torelli diffeomorphism of \( S \) and such that \( \delta \phi = \delta \).

Proof. Suppose that \( \delta \) is equal to \( \phi \) on all summands of \( K_0(c) \) except the summand \( K_0(\partial P) \) for some component \( P \) of \( cQ \). Then \( \delta \) maps \( K_0(\partial P) \) into \( H_1(\partial P) \).

We will use the notations from the discussion of matrix presentations and symmetric matrices above. For a subset
\[ \mathcal{A} \subset \{1, 2, \ldots, n\} \]

we will denote by \( U(\mathcal{A}) \) be the union of circles \( C_i \) with \( i \in \mathcal{A} \). Let us choose for each such subset \( \mathcal{A} \) a peripheral bi-twist diffeomorphism \( \psi(\mathcal{A}) \) about \( U(\mathcal{A}) \). Let \( \phi(\mathcal{A}) \) be the diffeomorphism of \( Q \) induced by \( \psi(\mathcal{A}) \). It is a peripheral twist diffeomorphism about \( U(\mathcal{A}) \).

Lemma 7.3 implies that the \( \delta \)-difference map of \( \phi(\mathcal{A}) \) is equal to \( \phi \) on all summands of
$K_0(c)$ except of $K_0(\partial P)$, and the matrix of the map

$$K_0(\partial P) \longrightarrow H_1(\partial P)$$

induced by this $\delta$-difference map is equal to $\mathfrak{m}(\mathfrak{A})$. Since every symmetric matrix is an integer linear combination of matrices of this form, Lemmas 5.4 and 5.5 imply that $\delta$ is equal to the $\delta$-difference map of a product $\varphi$ of diffeomorphisms of the form $\varphi(\mathfrak{A})$. The corresponding product $\psi$ of diffeomorphisms $\psi(\mathfrak{A})$ is a Torelli diffeomorphism extending $\varphi$. This proves the theorem when $\delta$ is equal to $\circ$ on all summands of $K_0(c)$ except one.

Any completely reducible map $\delta$ is equal to the sum over the components $P$ of maps equal to $\circ$ on all summands of $K_0(c)$ except of $K_0(\partial P)$. Moreover, if $\delta$ is symmetric, then all summands of this sum are also symmetric. Therefore, the general case of the theorem follows from the already proved case and Lemmas 5.4 and 5.5. ■

7.5. Theorem. For a diffeomorphism $\varphi: Q \to Q$ fixed on $c$ the following two conditions are equivalent.

(i) $\varphi$ can be extended to a Torelli diffeomorphism of $S$.

(ii) $\varphi$ a weakly Torelli diffeomorphism and its $\delta$-difference map $\delta_\varphi$ is completely reducible.

Proof. By remarks after the definition of weakly Torelli diffeomorphisms and Theorem 7.1 the condition (i) implies (ii).

Conversely, suppose that $\varphi$ satisfies (ii). Then by Theorem 6.3 the $\delta$-difference map $\delta_\varphi$ is symmetric, and hence by Theorem 7.4 there exists a peripheral diffeomorphism $\varphi'$ of $Q$ admitting an extension $\psi'$ to a Torelli diffeomorphism of $S$ and such that

$$\delta_{\varphi'} = \delta_{\varphi}.$$ 

Lemmas 5.4 and 5.5 imply that the map

$$\varphi'' = (\varphi')^{-1} \circ \varphi$$

is peripheral and its $\delta$-difference map is $\circ$. Let $\chi'' = \varphi''\setminus S$ be the extension of $\varphi''$ to $S$ by the identity. By Theorem 6.2 $\chi''$ is a Torelli diffeomorphism. Let

$$\psi = \psi' \circ \chi''.$$ 

By Lemma 5.4 $\psi$ is a Torelli diffeomorphism. Since $\psi'$ is an extension of $\varphi'$ and $\chi''$ is an extension of $(\varphi')^{-1} \circ \varphi$, the diffeomorphism $\psi$ is an extension of $\varphi$. Therefore $\varphi$ satisfies (i). ■
8. Multi-twists about the boundary

Multi-twist diffeomorphism about the boundary. Suppose that for each a component $C$ of $c$ an annulus $A_C$ in $Q$ having $C$ as one of its boundary components is fixed, and suppose that these annuli are pair-wise disjoint. Let

$$\tau_C : Q \to Q$$

be the extension by the identity of a left twist diffeomorphism of $A_C$. A multi-twist diffeomorphism of $Q$ about the boundary is a diffeomorphism of $Q$ of the form

$$(8.1) \quad \tau = \prod_C \tau_C^{m_C},$$

where $C$ runs over the components of $c = \partial Q$ and $m_C$ are integers.

The extension $\tau \backslash S$ of such a diffeomorphism $\tau$ by the identity is a multi-twist diffeomorphism of $S$ about $c$ and its isotopy class is equal to the Dehn multi-twist

$$(8.2) \quad t = \prod_C t_C^{m_C} \in \text{Mod}(S),$$

where each $t_C \in \text{Mod}(S)$ is the left Dehn twist of $S$ about $C$.

Diagonal maps. A homomorphism

$$D : H_0(c) \to H_1(c)$$

is said to be diagonal if for every component $C$ of $c$ it maps the generator $o_C$ of $H_0(c)$ from Section 7 to an integer multiple of $[C]$. Every multi-twist diffeomorphism $\tau$ about the boundary tautologically defines a diagonal map $D_\tau$. Namely, if $\tau$ has the form (8.1), then $D_\tau$ is given the rule

$$D_\tau : o_C \mapsto m_C[C].$$

8.1. Lemma. Suppose that $\tau$ is a multi-twist diffeomorphism of $Q$ about the boundary. Then $\tau$ can be extended to a Torelli diffeomorphism of $S$ and hence is a weakly Torelli diffeomorphism. If $\tau$ is given by the formula (8.1), then its difference map

$$\delta_\tau : K_0(c) \to H_1(c)$$

is equal to the restriction to $K_0(c)$ of the diagonal map $D_\tau : H_0(c) \to H_1(c)$. 

Proof. It is sufficient to consider the diffeomorphism (8.1). For each component $C$ of $c$ let $B_C$ be an annulus in $cQ$ having $C$ as one of its boundary components, and let $T_C$ be the extension by the identity to $S$ of a left twist diffeomorphism of $B_C$. Let

$$T = \prod C T^m C.$$

Then $T$ is equal to the identity on $Q$, and hence $T^{-1} \circ \tau$ extends $\tau$. On the other hand, $T^{-1} \circ \tau$ is isotopic to the identity, and hence is a Torelli diffeomorphism. Therefore, $\tau$ can be extended to a Torelli diffeomorphism of $S$ and hence is a weakly Torelli diffeomorphism.

The action of the Dehn multi-twist (8.2) on $H_1(S)$ is given by the formula

$$t_*(a) = a + \sum C m_C \langle a, [C]\rangle [C],$$

where $a \in H_1(S)$. Theorem 6.1 implies that $t_*(a) - a = \delta_\tau \circ \mathcal{D}(a)$, and hence

$$\delta_\tau \circ \mathcal{D}(a) = \sum C m_C \langle a, [C]\rangle [C].$$

Let us represent $\mathcal{D}(a)$ as a linear combination

$$\mathcal{D}(a) = \sum C n_C o_C$$

of generators $o_C$ with integer coefficients $n_C$. Then (5.2) implies that

$$n_D = \sum C \langle n_C o_C, [D]\rangle_c = \langle \mathcal{D}(a), [D]\rangle_c$$

for every component $D$ of $c$. On the other hand, $\langle \mathcal{D}(a), [D]\rangle_c = \langle a, [D]\rangle$ by (6.1) and hence $n_D = \langle a, [D]\rangle$ for every component $D$ of $c$ and

$$\mathcal{D}(a) = \sum_D \langle a, [D]\rangle o_D = \sum C \langle a, [C]\rangle o_C.$$

After applying $D_\tau$ to the last equality we get

$$D_\tau \circ \mathcal{D}(a) = \sum C m_C \langle a, [C]\rangle [C].$$

By comparing this formula with the above formula for $\delta_\tau \circ \mathcal{D}(a)$, we see that

$$\delta_\tau \circ \mathcal{D} = D_\tau \circ \mathcal{D}.$$

Since $\mathcal{D}$ is a map onto $K_o(c)$, the lemma follows. $\blacksquare$
Theorem. For a diffeomorphism \( \varphi : Q \to Q \) fixed on \( c \) the following two conditions are equivalent.

(i) There exists a multi-twist diffeomorphism \( \tau : S \to S \) about \( c \) such that \( \tau \circ (\varphi \setminus S) \) is a Torelli diffeomorphism.

(ii) \( \varphi \) a weakly Torelli diffeomorphism and \( \delta \varphi \) is the restriction of a diagonal map.

Proof. Suppose that (i) holds. We may assume that \( \tau \) is the extension by the identity \( \rho \setminus S \) of a multi-twist diffeomorphism \( \rho \) of \( Q \) about the boundary. Then

\[
(\rho \circ \varphi) \setminus S = \tau \circ (\varphi \setminus S)
\]

is a Torelli diffeomorphism and hence Theorem 6.2 implies that \( \rho \circ \varphi \) is a weakly Torelli diffeomorphism and its difference map is equal to \( \phi \). Since \( \rho \) is a weakly Torelli diffeomorphism, Lemmas 5.4 and 5.5 imply that \( \varphi \) is also a weakly Torelli diffeomorphism and \( \delta \varphi = -\delta \rho \). Since \( \delta \rho \) is equal to the restriction of a diagonal map by Lemma 8.1, it follows that the same is true for \( \delta \varphi \). This proves that (ii) holds.

Suppose that (ii) holds. Then there is a multi-twist diffeomorphism \( \tau \) of \( Q \) about the boundary such that \( \delta \tau = -\delta \varphi \). In this case \( \tau \circ \varphi \) is a weakly Torelli diffeomorphism and its difference map is equal to \( \phi \). By Theorem 6.2 the extension by the identity \( (\tau \circ \varphi) \setminus S \) is a Torelli diffeomorphism of \( S \). But

\[
(\tau \circ \varphi) \setminus S = (\tau \setminus S) \circ (\varphi \setminus S)
\]

and \( \tau \setminus S \) is a multi-twist about \( \partial Q \). This proves that (i) holds.

Theorem. If \( \partial P \) consists of \( \leq 3 \) circles for every component \( P \) of \( cQ \), then every completely reducible symmetric map \( \delta : K_0(c) \to H_1(c) \) is the restriction of a diagonal map.

Proof. Suppose that \( \delta \) is equal to \( \phi \) on all summands of \( K_0(c) \) except the summand \( K_0(\partial P) \) for some component \( P \) of \( cQ \). We will use the notations from the discussion of matrix presentations and symmetric matrices from Section 7.

If \( \partial P \) is connected, then \( K_0(\partial P) = H_1(\partial P) = \phi \) and the theorem is trivial.

If \( \partial P \) consists of 2 components \( C_0, C_1 \), then \( K_0(\partial P) \) is an infinite cyclic group with the generator \( o_1 = o_{C_1} - o_{C_0} \), and \( H_1(\partial P) \) is an infinite cyclic group with the generator \( [C_0] = [C_1] \). Let \( \delta(o_1) = m[C_1] \). If \( n_o, n_1 \in \mathbb{Z} \) and \( D \) is the (unique) diagonal map such that \( D(o_C) = \phi \) for \( C \neq C_0, C_1 \) and

\[
D(o_{C_0}) = n_o[C_0], \quad D(o_{C_1}) = n[C_1],
\]

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then $\delta$ is equal to the restriction of $D$ if and only if $m = n_1 - n_0$. It follows that if $\partial P$ consists of 2 components, then $\delta$ is equal to the restriction of a diagonal map.

If $\partial P$ consists of 3 components $C_0, C_1, C_2$, then $K_0(\partial P)$ is free abelian and

$$o_1 = o_{C_1} - o_{C_0}, \quad o_2 = o_{C_2} - o_{C_0}$$

is a basis of $K_0(\partial P)$. The group $H_1(\partial P)$ is free abelian having $[C_1], [C_2]$ as a basis. The induced map $\delta_P : K_0(\partial P) \to H_1(\partial P)$ is symmetric and hence has the form

$$o_1 \mapsto m_1 [C_1] + m_2 [C_2].$$

Let $D$ be the diagonal map such that $D(o_C) = o$ if with $C \neq C_0, C_1, C_2$ and

$$D(o_{C_0}) = n_0 [C_0], \quad D(o_{C_1}) = n_1 [C_1], \quad D(o_{C_2}) = n_2 [C_2].$$

As in Section 7, we assume that circles $C_0, C_1, C_2$ are oriented as components of the boundary of $Q$. Then $[C_0] + [C_1] + [C_2] = 0$, and hence

$$D(o_1) = n_1 [C_1] - n_0 [C_0] = (n_1 + n_0) [C_1] + n_0 [C_2],$$

$$D(o_2) = n_2 [C_2] - n_0 [C_0] = n_0 [C_1] + (n_2 + n_0) [C_2].$$

It follows that $\delta$ is equal to the restriction of $D$ if and only if

$$m_1 = n_1 + n_0, \quad m = n_0, \quad m_2 = n_2 + n_0.$$

For any given $m, m_1, m_2 \in \mathbb{Z}$ this system of equation for $n_0, n_1, n_2 \in \mathbb{Z}$ has a unique solution. It follows that $\delta$ is the restriction of a diagonal map in this case also.

This completes the proof of the theorem when $\delta$ is equal to $o$ on all summands of $K_0(c)$ except one. In order to deduce from this case the general one, one needs only to repeat word by word the last paragraph of the proof of Theorem 7.4. ■

\textbf{8.4. Theorem.} Suppose that $\partial P$ consists of $\leq 3$ circles for every component $P$ of $cQ$. If $\varphi$ can be extended to a Torelli diffeomorphism of $S$, then $\tau \circ (\varphi \setminus S)$ is a Torelli diffeomorphism for some multi-twist diffeomorphism $\tau$ of $S$ about $c$.

\textbf{Proof.} By Theorem 7.5 $\varphi$ is a weakly Torelli diffeomorphism and $\delta_\varphi$ is completely reducible. By Theorem 6.3 $\delta_\varphi$ is symmetric. Theorem 8.3 implies that $\delta_\varphi$ is equal to the restriction of a diagonal map. It remains to apply Theorem 8.2. ■
9. Torelli groups of surfaces with boundary?

What is the Torelli group of a connected subsurface $Q$ of a closed orientable surface $S$? The first real insight into this question is due to A. Putman [P]. He suggested that a good candidate is the group $P(Q)$ of isotopy classes of diffeomorphisms of $Q$ fixed on the boundary $\partial Q$ and such that their extensions by the identity to $S$ act trivially on $H_1(S)$. The isotopies are also supposed to be fixed on $\partial Q$. Putman showed that $P(Q)$ depends only on the partition of $\partial Q$ into the boundaries $\partial P$ of components $P$ of the complementary subsurface $cQ$.

The present paper suggests that another good candidate is the group $I(Q)$ of isotopy classes of diffeomorphisms of $Q$ fixed on the boundary $\partial Q$ and admitting an extension to a diffeomorphism of $S$ acting trivially on $H_1(S)$. The isotopies are again supposed to be fixed on $\partial Q$. Like $P(Q)$, the group $I(Q)$ depends only on the partition of $\partial Q$ into the boundaries $\partial P$ of components $P$ of the complementary subsurface $cQ$.

By the very definition, the groups $I(Q)$ have better restriction properties (being closed under restrictions is one of two properties put forward by Putman as the motivation of his definition). Suppose that $\psi$ is a diffeomorphism of $S$ leaving $Q$ invariant and fixed on $\partial Q$, and let $\varphi$ be the induced diffeomorphism $Q \rightarrow Q$. If the isotopy class of $\psi$ belongs to $I(S)$, then the isotopy class of $\varphi$ belongs to $I(Q)$, but not necessarily belongs to $P(Q)$.

The groups $P(Q)$ and $I(Q)$ are related by the short exact sequence

$$1 \rightarrow P(Q) \rightarrow I(Q) \xrightarrow{\delta} D(c) \rightarrow 1$$

where $D(c)$ is the group of completely reducible symmetric map $K_0(c) \rightarrow H_1(c)$, and the difference homomorphism $\delta$ maps the isotopy class of a diffeomorphism $\varphi$ into its difference map $\delta_\varphi$. It seems that $I(Q)$ should be considered together with the homomorphism $\delta$ and the pair $(I(Q), \delta)$ is an even better candidate for the title of the Torelli group of $Q$.

Suppose that $S$ is partitioned into the union of several subsurfaces such that their interiors are disjoint and every two of them intersect by several common boundary components. The elements of groups $P(Q)$ for subsurfaces $Q$ of this partition can be pasted together into an element of the Torelli group $I(S)$, but many other elements can be also pasted. If the elements of groups $\text{Mod}(Q)$ can be pasted, then they belong to the groups $I(Q)$, but not all collections of elements of $I(Q)$ can be pasted, as shown in the sequel [I3] of this paper. Still, the groups $I(Q)$ together with the $\delta$-difference maps provide a good framework for analyzing the pasting problem and the classification of abelian subgroups of $I(S)$.

* The use of the same notation $I(\bullet)$ as the one used by D. Johnson for the Torelli groups of closed surfaces should not be construed as the desire to dismiss other candidates.
Appendix. A converse to Theorem 5.3

A. 1. Theorem. If for every $\alpha \in Z_1(Q, c)$ the cycle $\varphi_*(\alpha) - \alpha$ is homologous in $Q$ to a cycle in $c$, then $\varphi$ is a weakly Torelli diffeomorphism.

Proof. Suppose that $\tau$ is a cycle in $Q$. Let $\alpha \in Z_1(Q, c)$. Let
\[
\sigma = \varphi_*(\tau) - \tau \quad \text{and} \quad \delta = \varphi_*(\alpha) - \alpha.
\]
Suppose that $\beta \in Z_1(cQ, c)$ and $\partial \alpha = \partial \beta$. Then
\[
\gamma = \alpha - \beta
\]
is a cycle. The first step of the proof of Theorem 5.3 does not use the assumption that $\varphi$ is a weakly Torelli diffeomorphism and implies that
\[
\langle \delta, \tau \rangle + \langle \gamma, \sigma \rangle + \langle \delta, \sigma \rangle = 0.
\]
By the assumption $\delta$ is homologous to a cycle in $c$. Since $\tau, \sigma$ are cycles in $Q$, it follows that $\langle \delta, \tau \rangle = \langle \delta, \sigma \rangle = 0$. In view of the last displayed equality, this implies that $\langle \gamma, \sigma \rangle = 0$. Let $\alpha$ be an arbitrary element of $H_1(S)$. By the Mayer–Vietoris equivalence the homology class $\alpha$ can be represented by a cycle of the form $\gamma = \alpha - \beta$ and hence
\[
\langle \alpha, [\sigma]_S \rangle = \langle \gamma, \sigma \rangle = 0.
\]
Since $\alpha \in H_1(S)$ is arbitrary, it follows that $[\sigma]_S = 0$. In other words, the cycle $\varphi_*(\tau) - \tau$ is a boundary in $S$ for every $\tau \in Z_1(Q)$ and hence $\varphi$ is weakly Torelli. 

Remark. Let us assume only that $\varphi_*(\alpha) - \alpha$ is homologous in $Q$ to a cycle in $c$ for every cycle $\alpha$ in $S$. As in Section 5, one can define the difference class $\Delta_\varphi(\alpha)$ of a cycle $\alpha$ in $Q$ as the image in $H_1(c)$ of the homology class $[\gamma] \in H_1(c)$ of any cycle $\gamma \in Z_1(c)$ homologous to $\varphi_*(\alpha) - \alpha$ in $Q$. Clearly, the difference class $\Delta_\varphi(\alpha)$ depends only on the homology class $[\alpha] \in H_1(Q)$, and hence these difference classes define a homomorphism
\[
d_\varphi : H_1(Q) \longrightarrow H_1(c)
\]
It turns out that every homomorphism $H_1(Q) \longrightarrow H_1(c)$ equal to zero on the image of $H_1(c)$ in $H_1(Q)$ can be realized as $d_\varphi$ for some diffeomorphism $\varphi$ of $Q$ fixed on $c$ and satisfying the above assumption. This follows from a construction of D. Johnson. See [J], Lemma 5 and Appendix I. In particular, such a diffeomorphism $\varphi$ does not need to be weakly Torelli.
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