ON CLASSICAL GLOBAL SOLUTIONS OF NONLINEAR WAVE EQUATIONS
WITH LARGE DATA

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Abstract. This paper studies the Cauchy problem for systems of semi-linear wave equations on \( \mathbb{R}^{3+1} \)
with nonlinear terms satisfying the null conditions. We construct future global-in-time classical solutions with
arbitrarily large initial energy. The proof adapts the short pulse method introduced by Christodoulou in his works
on the black-hole formations in general relativity. He studied the characteristic data prescribed on the past null
infinity so that the decay rate of the solutions may be regarded as given. The main difficulty of the current work
is to derive the decay estimates for the Cauchy problem. It requires a more delicate analysis on a relaxed
propagation estimate both on the decay and on the largeness of the solutions.

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1. Introduction

We consider the Cauchy problem of the following system of wave equations on \( \mathbb{R}^{3+1} \):

\[ \square \phi = Q(\nabla \phi, \nabla \phi). \]  

Here, \( \square = -\partial_t^2 + \Delta \) is the standard wave operator. The function \( \phi \) is vector valued. In fact, \( \phi \) stands for
a vector of \( N \) unknown functions \( \phi^I, I = 1, \ldots, N \). The symbol \( \nabla \phi \) denotes all possible partial derivatives
\( \partial_\gamma \phi^I \)'s for \( \gamma = 0, 1, 2, 3 \) and \( I = 1, 2, \ldots, N \). The nonlinearity \( Q(\nabla \phi, \nabla \phi) \) is a quadratic
form in \( \nabla \phi \) satisfying the null condition. The problem of constructing global-in-time solutions for small
initial data has been studied intensively

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in the literature. The purpose of the current paper is to propose a large data regime for (1.1) which also leads to global classical solutions.

We discuss briefly the small data theory for (1.1) on $\mathbb{R}^{n+1}$. The idea is to use the decay mechanism of linear waves, i.e. solutions of $\Box \phi = 0$, and treat the nonlinear problem as a perturbation of the linear waves. In dimensions greater than or equal to 4, i.e. $n \geq 4$, the pointwise decay rate of linear waves is at least $t^{-3/2}$, which is integrable on for $t \geq 1$. This fast decay rate can be used to prove the small-data-global-existence results; see the pioneering works of Klainerman [5] and [6]. However, in $\mathbb{R}^{3+1}$, the pointwise decay of the linear wave is merely at the rate $t^{-1}$ which is not integrable. This weak decay rate is not enough to control the nonlinear interaction: F. John [4] showed that there were quadratic forms (which are not null forms) such that for arbitrarily small non-zero smooth data, solutions to (1.1) blow up in finite time.

The importance of the null condition was first observed in the breakthrough work [7] by Klainerman, where he proved that small data lead to global-in-time classical solutions if the non-linearity $Q$ is a null form, or equivalently, satisfies the null condition. In [1] Christodoulou obtained a similar result based on the conformal compactification of the Minkowski spacetime. Although the approaches in [7] and [1] are very different, both proofs rely on the cancellation structure of null condition, which is absent for general quadratic nonlinearities.

The idea of exploiting the cancellation structure of the null conditions can also be used to handle certain large data problems. In a recent breakthrough in general relativity, Christodoulou [2] rigorously proved for the first time that black holes can form dynamically from arbitrary dispersed initial data. The key to this work was the new idea of the “short pulse method”. Roughly speaking, this is a choice of special large initial data, called short pulse data, so that these large profiles can be propagated along the flow of Einstein vacuum equations. One of the key observations in the proof is still tightly related to the cancellation of the null structure: the profile is only large in certain components and these large components are always coupled with some small components so that their contributions are still manageable. Christodoulou’s work has been generalized in [8] by Klainerman and Rodnianski. A key ingredient in their work is the relaxed propagation estimates which allows profiles with more large components.

The ideas used in [2] and [8] have been adapted to the main equation (1.1) by Wang and Yu to construct future-in-time global solutions with large initial data; see [10] and [11]. Their approach is indirect. The authors essentially impose the characteristic data on the past null infinity and solve the inverse scattering problem all the way up to a finite time to construct the initial Cauchy data. Very recently, Yang [12] has obtained a global existence theorem for semi-linear wave equations with large Cauchy initial energy. The largeness in [12] is from a slower decay of the initial data at spatial infinity, but not from the short pulse method.

The aim of the current work is to study the behavior of the solutions of (1.1) with short pulse data. We give short pulse Cauchy data directly (one should compare with the indirect approach of [10]) and prove that the data lead to future-global-in-time classical solutions for (1.1). We remark that compared to the characteristic data approach in [10], one of the main difficulties is to prove quantitative decay of the solution. This difficulty does not appear in [10], because the data there are radiation fields given on the past null infinity, so that the decay rate is already explicitly given.

1.1. The short pulse data and main results. We use $(x_0, x_1, x_2, x_3)$ to denote the standard Cartesian coordinates $(t, x, y, z)$ on $\mathbb{R}^{3+1}$. In particular, $\partial_0$ stands for $\partial_t$. Let $\phi : \mathbb{R}^{3+1} \rightarrow \mathbb{R}^N$ be a vector valued function, and we use $\phi^I$ to denote its components. The Cauchy problem we study is the following system
of nonlinear wave equations

\[ \Box \phi^I = Q^I(\nabla \phi, \nabla \phi), \text{ for } I = 1, 2, \cdots, N, \]

and \( (\phi, \partial_t \phi)|_{t=1} = (\phi_0, \phi_1) \). \hspace{1cm} (1.2)

We remark that in order to simplify some of the expressions appearing in the proof of the main theorem, we give the initial data at \( t = 1 \) rather than \( t = 0 \). Because of the invariance of the equation under time translations, this is the same as giving data on \( t = 0 \). The nonlinearities \( Q^I \) are null forms, i.e. we can write

\[ Q^I(\nabla \phi, \nabla \phi) = \sum_{0 \leq \alpha, \beta \leq 3, 1 \leq J, K \leq N} A^{\alpha \beta, I}_{JK} \partial_\alpha \phi^J \partial_\beta \phi^K, \]

and for all null vector \( \xi \in \mathbb{R}^{3+1} \), i.e. \( \xi = (\xi_0, \cdots, \xi_3) \) satisfying \( -\xi_0^2 + \sum_{i=1}^{3} \xi_i^2 = 0 \), the coefficient matrices \( A^{\alpha \beta, I}_{JK} \) satisfy

\[ A^{\alpha \beta, I}_{JK} \xi_\alpha \xi_\beta = 0. \]

For the sake of simplicity, instead of writing all the components, we shall always use \( \phi \) to denote the \( \phi^I \)'s and use \( Q(\nabla \phi, \nabla \phi) \) to denote the nonlinearity. In particular, we always write the main equation (1.2) as (1.1).

Before describing the short pulse data, we introduce some notations: \( r \) and \( \theta \) are used to denoted the usual radial and angular coordinates on \( \mathbb{R}^3 \). Let \( \delta \) be a small positive constant which will be determined later. We choose once and forever two smooth functions \( \psi_0(r, \theta) \) and \( \psi_1(r, \theta) \) and we call them the seed functions. The seed functions \( \psi_0 \) and \( \psi_1 \) are compactly supported smooth functions defined on \((-1, 0) \times S^2 \), i.e. \( (y, \theta) \in (-1, 0) \times S^2 \). We also identify the \( t = 1 \) hypersurface with \( \mathbb{R}^3 \) and divide it into three parts:

\[ \{ t = 1 \} = B_{1-2\delta} \cup (B_1 - B_{1-2\delta}) \cup (\mathbb{R}^3 - B_1), \]

where \( B_r \) is the ball centered at the origin with radius \( r \).

The initial data \((\phi_0, \phi_1)\) on \( \{ t = 1 \} \) of (1.2) are given in the following way:

- On \( B_{1-2\delta} \), we set \((\phi_0, \phi_1) \equiv (0, 0)\).
- On \( B_1 - B_{1-2\delta} \), we first set

\[ \phi_0(r, \theta) = \delta^{1/2} \psi_0(\frac{r-1}{2\delta}, \theta), \]

then define \( \phi_1 \) as follows

\[ \phi_1(r, \theta) = -\partial_r \phi_0(r, \theta) + \delta^{1/2} \psi_1(\frac{r-1}{2\delta}, \theta). \]

- On \( \mathbb{R}^3 - B_1 \), we set \((\phi_0, \phi_1) \equiv (0, 0)\).

It is instructive to draw a picture of the data. Ignoring the rotational argument \( \theta \), the graph of \( \phi_0 \) is as follows:

\[ \begin{array}{c}
\text{\( \phi_0 \)} \\
\downarrow^\delta^1 \\
2\delta \\
\downarrow \\
r=0 \hspace{2cm} r = 1 - 2\delta \hspace{2cm} r = 1
\end{array} \]
The pulse-like shape of the graph explains the name “short pulse” used for this data. The width of the pulse is $2\delta$ and its amplitude is $\delta^2$, which is very large relative to the width.

The choice of $\phi_1(r,\theta)$ is obscure and artificial in the above form. In fact, we have a natural geometric explanation of this choice, which can also serve as heuristics to understand why one expects a global solution.

**Remark 1.1 (Geometric / Physical interpretation).** In terms of the solution $\phi$, we should rewrite the definition of $\phi_1$ in the following equivalent form:

$$(\partial_t + \partial_r)\phi \bigg|_{t=1} = \delta^{1/2} \psi_1 \left( \frac{r-1}{2\delta}, \theta \right).$$

Recall that $L = \partial_t + \partial_r$ is the normal (with respect to the Minkowski metric!) of the outgoing light cones $t-r = \text{constant}$ in $\mathbb{R}^{3+1}$. If we integrate $|(\partial_t + \partial_r)\phi|^2$ on such an outgoing light cone $C$, the quantity

$$\int_C |(\partial_t + \partial_r)\phi|^2 d\mu_C$$

measures the incoming energy through this light cone. Therefore, since $\delta$ will be eventually very small, the choice of $\phi_1$ is to keep the incoming energy as small as possible. Intuitively, we expect all the energy will be emanated in the outgoing direction so that the solution $\phi$ disperses.

We now explain in what sense the short pulse data are large. It appears that the short pulse data is at least small in the $L^\infty$ sense due to the presence of the factor $\delta$ in front of the seed functions. First of all, we notice that the $L^\infty$ norm is irrelevant since we may always add a constant to get a new solution for (1.2). The size of the data should be measured at least on the level of first derivatives. Secondly, we notice that, if we take derivatives in the $\partial_r$ direction many times, the data can be extremely large in the $L^\infty$ sense, because each $\partial_r$ derivative will bring out a $\delta^{-1}$ factor from the first argument of $\phi_0$ or $\phi_1$.

A more natural way to see the largeness of the data is to consider the energy spaces, i.e. the Sobolev spaces $H^k(\mathbb{R}^3)$. The critical $H^s$-exponent (with respect to scaling) of (1.2) is $\frac{3}{2}$. Therefore, the $0^{\text{th}}$ order energy $E_0 = \int_{\mathbb{R}^3} |\nabla \psi_0|^2 + |\psi_1|^2 dx$ is subcritical and the $1^{\text{st}}$ order energy $E_1 = \sum_{i=1}^{\frac{3}{2}} \int_{\mathbb{R}^3} |\nabla \partial_i \phi_0|^2 + |\partial_i \phi_1|^2 dx$ is supercritical. Here we use $\nabla$ to denote spatial gradient.

**Remark 1.2 (Largeness of short pulse data).** We can compute the $0^{\text{th}}$ order energy $E_0$ and the $1^{\text{st}}$ order energy $E_1$ as follows:

$$E_0 \sim \|\nabla \psi_0\|_{L^2}^2 + \|\psi_1\|_{L^2}^2 \sim 1,$$

$$E_1 \sim \|\nabla^2 \psi_0\|_{L^2}^2 + \|\nabla \psi_1\|_{L^2}^2 \sim \delta^{-2}.$$  

Since $E_0$ and $E_1$ are subcritical and supercritical, respectively, we cannot make both of them small by the scaling invariance of the equation. It is in this sense that the data are large on the level of energy.

Moreover, for all $k \geq 0$, we can show that

$$E_k = \int_{\mathbb{R}^3} |\nabla^{k+1} \phi_0|^2 + |\nabla^k \phi_1|^2 dx \sim \delta^{-2k}.$$  

We note in passing that the higher order energies can be extremely large. Also, we remark that the symbol $\sim$ depends only on the size of the seed functions $\psi_0$ and $\psi_1$.

We are now ready to state the main theorem of the paper:
Main Theorem. For any given pair of seed functions \((\psi_0, \psi_1)\) as above, we give the corresponding short pulse data for the following system of wave equations

\[
\begin{align*}
\Box \phi^I &= Q^I(\nabla \phi, \nabla \phi), \quad \text{for } I = 1, 2, \cdots, N, \\
(\phi, \partial_t \phi) \big|_{t=1} &= (\phi_0, \phi_1),
\end{align*}
\]

where the \(Q^I\)'s are null forms.

Then there exists a positive number \(\delta_0\) depending only on the seed functions \((\psi_0, \psi_1)\), so that for all \(\delta < \delta_0\), the above Cauchy problem admits a unique smooth solution \(\phi\) with lifespan \([1, +\infty)\). Moreover, when \(t \to \infty\), the nonlinear wave \(\phi\) scatters.

1.2. Notations. We review the basic geometry of Minkowski space \(\mathbb{R}^{3+1}\). In particular, we discuss the standard double null (cone) foliations on \(\mathbb{R}^{3+1}\) which will play a central role for the energy estimates.

Let \(r = \sqrt{x_1^2 + x_2^2 + x_3^2}\). We define two optical functions \(u\) and \(\bar{u}\) as follows

\[
u = \frac{1}{2}(t - r), \quad \bar{u} = \frac{1}{2}(t + r).
\]

For a given constant \(c\), we use \(C_c\) to denote the level surface \(u = c\) with an extra constraint that \(t \geq 1\) (since we will construct a future-global-in-time solution starting from the initial hypersurface \(\{t = 1\}\)). According to the different value of \(u\), we use also \(C_u\) to denote these hypersurfaces. These are called outgoing light cones. Thus, \(\{C_u | u \in \mathbb{R}\}\) defines a foliation of \(\mathbb{R}^{3+1}_{t \geq 1}\). We also call this foliation null because each leaf \(C_u\) is a null hypersurface with respect to the Minkowski metric.

Similarly, using the level sets of the optical function \(\bar{u}\), we define another null foliation of \(\mathbb{R}^{3+1}_{t \geq 1}\), denoted by \(\{\bar{C}_u | u \in \mathbb{R}\}\). Each \(\bar{C}_u\) is a truncated incoming light cone. The intersection \(C_u \cap \bar{C}_{\bar{u}}\) is a round 2-sphere with radius \(u - \bar{u}\), denoted by \(S_{u, \bar{u}}\). We say that the two foliations \(\{C_u | u \in \mathbb{R}\}\) and \(\{\bar{C}_u | u \in \mathbb{R}\}\) form a double null foliation of \(\mathbb{R}^{3+1}_{t \geq 1}\).

We recall that the following two null vector fields,

\[
L = \partial_t + \partial_r, \quad \text{and} \quad \bar{L} = \partial_t - \partial_r
\]

are the normals of (also parallel to) \(C_u\) and \(\bar{C}_{\bar{u}}\) respectively.

The following picture depicts the outgoing null foliation \(C_u\) of \(\mathbb{R}^{3+1}_{t \geq 1}\).

Since the foliation is spherically symmetric, we ignore the spherical part of \(\mathbb{R}^{3+1}\) and only draw the \(t\) and \(r\) components in the schematic diagram. The other pictures in the paper should also be understood in this way. In the above picture, a 45° line denotes an outgoing cone \(C_u\). Two outgoing cones \(C_0\) and \(C_\delta\) divide \(\mathbb{R}^{3+1}\) into three regions: the small data region, i.e. region I in the picture, the short pulse region,
i.e. the region with light grey color, region II in the picture, and the region III in the picture, i.e. the region with dark grey color.

**Remark 1.3** (Vanishing Property on $C_0$). Recall that the short pulse data prescribed on $\{t = 1\}$ in the last subsection are identically zero for $r \geq 1$, therefore, according to the weak Huygen’s principle, the solution of the main equation (1.2) vanishes identically in the region III (dark grey). In particular, the solution $\phi$ (if it exists) vanishes to infinity order on $C_0$.

We now pay more attention to the short pulse region (region II with light grey color). We use $D_{\underline{u},u}$ to denote the spacetime region enclosed by the hypersurfaces $\{t = 1\}, C_0, C_u$ and $C_{\underline{u}}$, where $u \in [0, \delta]$ and $\underline{u} \geq 1 - u$. The following picture is the schematic diagram for all the notations introduced in this section for the short pulse region.

A dashed 45° segment denotes an incoming cone $C_{\underline{u}}$. A thickened black point denotes a 2-sphere $S_{\underline{u},u}$. An orthogonal pair of arrows denotes the null vector pair $(L, \underline{L})$. A typical picture (if $\underline{u} \geq 1$) of $D_{\underline{u},u}$ is the grey region. If $\underline{u} < 1$, the picture of $D_{\underline{u},u}$ looks like a triangle:

We remark that, for both cases, both $\{C_u | 0 \leq u' \leq u\}$ and $\{C_{\underline{u}} | 1 - u \leq u' \leq u\}$ foliate $D_{\underline{u},u}$.

In view of Remark 1.1 we also remark that the choice of the short pulse data is also adapted to the double null foliation in the short pulse region: the data is chosen in a way that very little energy propagates in the incoming direction through $C_u$’s. We expect most of the energy will radiate through the $C_{\underline{u}}$’s to the future null infinity.
We construct a solution \( \phi \) to denote the induced metric from the Minkowski metric on \( S_{\alpha,\mu} \). The intrinsic covariant derivative on \( S_{\alpha,\mu} \) is denoted by \( \nabla \). This covariant derivative is closely related to the rotational symmetry of \( \mathbb{R}^{3+1} \). Recall that the infinitesimal rotations are represented by the following three vector fields:

\[
\Omega_{ij} = x_i \partial_j - x_j \partial_i, \quad 1 \leq i < j \leq 3.
\]

We use \( \Omega \) as a short hand notation for an arbitrary choice from the above vector fields. We also use \( \Omega^2 \) to denote an operator of the form \( \Omega_{ij} \Omega_{ij} \); similarly for \( \Omega^n \). For a given function \( \phi \), we use \( |\Omega \phi| \) to denote \( \sum |\Omega_{ij}\phi| \) and use \( |\Omega^2 \phi| \) to denote \( \sum |\Omega_{ij} \Omega_{ij'} \phi| \), and so on. Therefore, by direct computation, we obtain

\[
|\Omega \phi| \sim r |\nabla \phi|,
\]

where the size \( \nabla \phi \) is measured with respect to \( \mathcal{g} \). Moreover, for all \( n \), we have

\[
|\Omega^n \phi| \sim r^n |\nabla^n \phi|.
\]

In the rest of the paper, the number of derivatives that we impose on the solution is a fixed number which does not exceed, say, 100. Therefore, we may forget the dependence on \( n \) in the above inequality.

**Remark 1.4.** In the short pulse region II, if \( \delta \) is sufficiently small, then \(|u| \sim r\). Therefore, for all \( n \), we have

\[
|\Omega^n \phi| \sim |u|^n |\nabla^n \phi|.
\]

In particular, for all \( p \geq 1 \) and \( n \), we have

\[
||\Omega^n \phi||_{L^p(S_{\alpha,\mu})} \sim |u|^n ||\nabla^n \phi||_{L^p(S_{\alpha,\mu})}.
\]

### 1.3. Comments on the proof.

We construct a solution \( \phi \) in three steps:

- **Step 1** We construct \( \phi \) in the short pulse region II.

  The initial data for this region are given on initial hypersurface \( \Sigma_1 \). We expect to see the largeness of the data in the proof. In particular, the \( L \) derivative of the solution causes a loss of \( \delta^{-1} \). This makes the proof difficult and also different from the classical small data problem. The decay of \( \phi \) is another difficulty.

- **Step 2** Smallness of \( \phi \) on \( C_\delta \).

  Although \( \phi \) constructed in Step 1 is large in many respects, we show that \( \phi \) is indeed small on the inner boundary \( C_\delta \). This is a key intermediate step: since in next step, the \( \phi \) restricted on \( C_\delta \) gives initial characteristic data, this step allows one to reduce the problem to a small data problem in region I.

- **Step 3** We construct \( \phi \) in the small data region I.

  In region II, the problem is reduced to a small data problem. We can then use the classical approach to construct \( \phi \).

#### 1.3.1. Vector field method.

We will derive energy estimates for the main equation (1.2). Our approach is based on the classical vector field method and we briefly recall the main structure of the method as follows.

Let \( \phi \) be a (scalar) solution for a non-homogenous wave equation \( \Box \phi = \Phi \) on \( \mathbb{R}^{3+1} \). The energy-stress tensor associated to \( \phi \) is \( T_{\alpha\beta}[\phi] = \nabla_\alpha \phi \nabla_\beta \phi - \frac{1}{2} g_{\alpha\beta} \nabla^\mu \phi \nabla_\mu \phi \) where \( g_{\alpha\beta} \) is the Minkowski metric. In particular, in terms of null pair \((L, L)\), we have \( T[\phi](L, L) = (L \phi)^2 \), \( T[\phi](L, L) = (L \phi)^2 \) and \( T[\phi](L, L) = |\nabla \phi|^2 \). Given a vector field \( X \), we use \( (X) \pi_{\mu\nu} = \frac{1}{2} \mathcal{L}_X g_{\mu\nu} \) to denote its deformation tensor. The energy currents associated to \( \phi \) are defined by \( J^X_\alpha[\phi] = T_{\alpha\mu}[\phi] X^\mu \) and \( K^X[\phi] = T^{\mu\nu}[\phi] (X) \pi_{\mu\nu} \). The following divergence identity is the key to the energy estimates:

\[
\nabla^\alpha J^X_\alpha[\phi] = L^{\alpha\beta}[\phi] X^\alpha \Phi + \Phi \cdot X \phi.
\] (1.3)
In applications, we integrate this identity on the spacetime region. This is equivalent to multiplying □φ = Φ by Xφ and then integrating by parts. This is the reason that we call X a multiplier vector field.

In the short pulse region II, we integrate (1.3) on $D_{z,u}$. Since φ vanishes on $C_0$ up to infinite order, this yields

$$
\int_{C_u} T[φ](X,L) + \int_{C_u} T[φ](X,L) = \frac{1}{2} \int_{\Sigma_1} T[φ](X,L+L) + \int_{D_{z,u}} K^X[φ] + Φ \cdot Xφ. \tag{1.4}
$$

where $Σ_1$ is the initial Cauchy hypersurface $\{t = 1\}$.

In the short pulse region II, we will use two multiplier vector fields: $X = L$ and $X = u^α L$, where the power $α = 1 - ε_0$ and $ε_0 \in (0, \frac{1}{2})$ is a given constant. The first plays a similar role to the time vector field $\partial_t$; the second plays a similar role to the Morawetz vector field $K = (t^2 + r^2)\partial_t + 2tr\partial_r$.

For $X = L$ and $X = u^α L$, the corresponding deformation tensors and energy currents are

- For $X = L$, $π_{AB} = -\frac{1}{2}g_{AB}$ and $K = -\frac{1}{2}Lφ \cdot Lφ$.
- For $X = u^α L$, $π_{AL} = -αu^{α-1}$, $π_{AB} = \frac{1}{2}u^α g_{AB}$ and $K = -\frac{α}{2} u^{α-1} |∇φ|^2 + \frac{1}{r} u^α Lφ \cdot Lφ$.

respectively. We remark that indices $A$ and $B$ are used to denote a frame on $S_{z,u}$ and we only listed the nonzero components of the deformation tensors.

We will also need estimates for higher order derivatives for φ. To achieve this, we will construct the main equation (1.2) with certain vector fields, i.e. the commutator vector fields. These vector fields are essentially the Lie algebras of the conformal isometries of $\mathbb{R}^{3+1}$. We list all of them as follows:

$$
Z = \{Ω_{ij}, Ω_{0i}, Ω_{i}, Ω_{j}, S|i, j = 1, 2, 3, i \neq j\},
$$

where $Ω_{0i} = x_i \partial_t + t \partial_i$ and $S = t\partial_t + r\partial_r = uL + uL$. We also define the good and bad commutator vector fields:

$$
Z = Z_g \cup Z_b, \quad Z_b = \{Ω_i, Ω_0 | i = 1, 2, 3\}.
$$

As shorthand notations, we use $Z$ to denote an arbitrary vector field from $Z$; similarly, we use $Z_g$ and $Z_b$ to denote vectors from $Z_g$ and $Z_b$ respectively. Geometrically, a good vector field $Z_g$ is tangential to the outgoing light cone $C_0$, but a bad vector field $Z_b$ is transversal to $C_0$.

1.3.2. A word on null forms. Recall that a quadratic form $Q$ over $\mathbb{R}^{3+1}$ is a null form if $Q(ξ, ξ) = 0$ for all null vector $ξ \in \mathbb{R}^{3+1}$. The space of null forms are spanned by the following seven forms: $Q_0(ξ, η) = g(ξ, η)$ and $Q_{αβ}(ξ, η) = ξ_α η_β - η_α ξ_β (0 \leq α, β \leq 3)$. Given scalar functions $φ, ψ$ and a null form $Q(ξ, η) = Q^{αβ} ξ_α η_β$, we use $Q(∇ φ, ∇ ψ)$ as a shorthand for $Q(∇ φ, ∇ ψ) = Q^{αβ} ∇_α φ ∇_β ψ$.

For a (conformal) Killing vector field $Z \in Z$, we have

$$
ZQ(∇ φ, ∇ ψ) = Q(∇Z φ, ∇ ψ) + Q(∇ φ, ∇Z ψ) + Q(∇ φ, ∇ ψ), \tag{1.5}
$$

where $Q$ is a null form, which may or may not be $Q$.

In terms of the null pair $(L, L)$, a null form $Q$ satisfies the following pointwise estimates

$$
|Q(∇ φ, ∇ ψ)| \lesssim |Lφ| |Lψ| + |Lφ| |Lψ| + |∇ φ| |∇ ψ| + |Lφ| |Lφ| |Lψ| + |∇ φ| |Lψ| + |Lφ| |Lψ|. \tag{1.6}
$$

In particular, on the right hand side of the inequality, the term $|Lφ|^2$ does not appear.
1.3.3. **Main features of the proof.** We discuss some main difficulties of the problem and also the ideas to get around them.

- **Largeness/Loss of $\delta^{-1}$ in the short pulse region.**

  In the short pulse region, if one differentiates $\phi$ in the $L$ direction, then the resulting function will be approximately $\delta^{-1}$ times as large as the initial functions. Schematically, we can regard $L$ as $L \sim \delta^{-1}$. Similarly, $L \sim 1$ and $\nabla \sim 1$.

  The large factor $\delta^{-1}$ maybe fatal to the energy estimates for nonlinear terms. The resolution of this difficulty is exactly the basic philosophy of null conditions: if one term behaves badly, say $|\mathcal{L}\phi| \sim \delta^{-\frac{1}{2}}$ in the nonlinearities, it must be coupled with the a good term, say $L\phi$ or $\nabla\phi$, which are both of size $\delta^{\frac{1}{2}}$. Their product will then be a term of size 1 which will be manageable in the proof.

- **Relaxation in $\delta$ for the propagation estimates.**

  On the initial hypersurface $\Sigma_1$, it is easy to see that the data satisfy $\|L\phi\|_{L^\infty_{\Sigma_1}} \sim \delta^{-\frac{1}{2}}$ and $\|\nabla\phi\|_{L^\infty_{\Sigma_1}} \sim \delta^{\frac{1}{2}}$. Up to a correct decay factor in $t$, we hope the size of $L\phi$ and $\nabla\phi$ measured in $\delta$ can be propagated for later times, i.e. $\|L\phi\|_{L^\infty_{\Sigma_t}} \sim \delta^{-\frac{1}{2}}$ and $\|\nabla\phi\|_{L^\infty_{\Sigma_t}} \sim \delta^{\frac{1}{2}}$ should be always true. Recall that the proof will be based on energy estimates. If we use $L$ as a multiplier vector field and integrate in $D_{\phi,u}$ in the short pulse region, the energy on the left hand side of (1.4) is $\int_{C_{\Sigma_1}} |L\phi|^2 + \int_{C_{\Sigma_1}} |\nabla\phi|^2$. Therefore, the expected propagation estimates suggest that $\int_{C_{\Sigma_1}} |L\phi|^2 \lesssim 1$ and $\int_{C_{\Sigma_1}} |\nabla\phi|^2 \lesssim \delta$. Therefore, in view of the form of the energy, the disparity of the $\delta$ power for these two quantities only gives the desired bound for $L\phi$, but not for $\nabla\phi$. This may lead to the failure of closing the bootstrap argument.

  To get around this difficulty, we pretend that the amplitude of $\nabla\phi$ was worse than that suggested by the initial data. The purpose of this relaxation is to make the two terms in $\int_{C_{\Sigma_1}} |L\phi|^2 + \int_{C_{\Sigma_1}} |\nabla\phi|^2$ comparable: we compromise at the moment and we want to show at least $\|\nabla\phi\|_{L^\infty(\Sigma_1)} \lesssim 1$ can be propagated. In this way, we can close the argument, which in turn allows us to gain $\delta^{\frac{1}{2}}$ for the estimate of $\nabla\phi$ at the cost of losing one derivative.

- **Relaxation in the decay factor in the short pulse region.**

  According to the decay rate of linear waves, one may expect the decay of $\phi$ or more precisely the derivatives of $\phi$ should be $\frac{1}{t}$ or $\frac{1}{2}$ in the short pulse region. This expected decay will cause a loss of $\log t$ in the energy estimates since we may need to integrate a factor of size $\frac{1}{t}$ coming from the nonlinear term.

  The idea to get around this point is also to relax the decay rate a little bit. This is why we choose $X = |u|^{1-\epsilon_0} L$ as a multiplier vector field instead of using the standard $S$ vector field. The $|u|^{-\epsilon_0}$ will be amplified to $|u|^{-2\epsilon_0}$ in the energy estimates due to the nonlinearity. Therefore, we can gain a little more decay relative to the relaxed decay. This is just enough to close the argument for the energy estimates.

- **Smallness of the solution on $C_\delta$.**

  This is precisely the question that we will answer in Step 2 of the proof. As we discussed, in the short pulse region, we expect $L \sim \delta^{-1}$. In particular, we expect that, for all the bad vector fields $Z_b$, we also have $Z_b \sim \delta^{-1}$. Therefore, for a given $n$, the restriction of $Z_b^n\phi$ on $C_\delta$ may be of size $\delta^{\frac{1}{2}-n}$. This is by no means small.
The key point of the proof is the following observation: on the 2-sphere $S_{1-\delta,\delta}$, i.e. the initial sphere of $C_{\delta}$, the data vanish completely since they are compactly supported on $\Sigma_1$ between $S_{1-\delta,\delta}$ and $S_{1,0}$. Therefore, even the bad derivatives of $\phi$ are small initially. To get the smallness of $\phi$, we will integrate along null geodesics on $C_0$ to trace all the information back to the data. In this way, we can show that up to an error of size $\delta^{\frac{1}{2}}$, all derivatives of $\phi$ are comparable to their initial values.

1.3.4. Outline of the paper. The rest of the paper is organized as follows:

In Section 2, we establish a priori energy estimates for higher order derivatives of the solution in the short pulse region. As consequences, first of all, we can construct the solution in the short pulse regions; Secondly, we can obtain a smallness estimate for the solution on $C_{\delta}$, i.e. the inner boundary of the short pulse data region.

In Section 3, with a modified Klainerman-Sobolev inequality, we construct global solutions in the small data region.

2. Short pulse region

The goal of the current section is to construct the solution $\phi$ in the short pulse region. The construction relies on a priori energy estimates. We assume that the solution $\phi$ exists on spacetime domain $D_{u^*,u^*}$. This domain is inside the short pulse region, i.e. $u^* \in (0, \delta)$ and $u^* \in (1 - u^*, +\infty)$.

We first introduce the energy norms. Let $u, u' \in (0, u^*)$ and $u, u' \in (1 - u^*, u^*)$. Let $C_{u'}^u$ be the part of the cone $C_u$ so that $1 - u^* \leq u \leq u'$ and let $C_{u^*}^{u'}$ be the part of the cone $C_{u^*}$ so that $0 \leq u \leq u'$. Whenever there is no confusion, we will use $C_u$ and $C_{u^*}^{u'}$ instead of $C_{u'}^u$ and $C_{u^*}^{u'}$. We use $\Sigma_1$ to denote the initial hypersurface $\Sigma_1$, we introduce the following homogeneous norms:

$$E_k(u, u) = \sum_{Z_g \in Z_1, l \leq k} \left( \delta^l \|\nabla'_g Z^l_g \phi\|_{L^2(C^{u^*}_{u^*})} + \delta^{\frac{l}{2}} \|u\|_{Z^l_g} \|\nabla'_g Z^l_g \phi\|_{L^2(C^{u^*}_{u^*})} \right).$$

$$E_k(u, u) = \sum_{Z_g \in Z_1, l \leq k} \left( \delta^l \|LZ^l_g \phi\|_{L^2(C_{u^*}^{u'})} + \delta^{\frac{l}{2}} \|u\|_{Z^l_g} \|\nabla'_g L Z^l_g \phi\|_{L^2(C_{u^*}^{u'})} \right).$$

We also introduce the inhomogeneous norms:

$$E_{\leq k}(u, u) = \sum_{0 \leq j \leq k} E_j(u, u), \quad E_{\leq k}(u, u) = \sum_{0 \leq j \leq k} E_j(u, u).$$

On the initial hypersurface $\Sigma_1$, we introduce the following initial energy norms:

$$E_{\leq n}(\Sigma_1) = \sum_{Z_g \in Z_1, l \leq k} \delta^l \|LZ^l_g \phi\|_{L^2(\Sigma_1)} + \delta^{\frac{l}{2}} \|\nabla'_g L Z^l_g \phi\|_{L^2(\Sigma_1)} + \delta^{\frac{l}{2}} \|LZ^l_g \phi\|_{L^2(\Sigma_1)}.$$

Recall that the short pulse data are determined by the seed functions $\psi_0$ and $\psi_1$. We expect that the initial energy norm $E_{\leq n}(\Sigma_1)$ can be bounded by the seed functions.

Lemma 2.1. Given the seed functions $\psi_0$ and $\psi_1$, we define

$$I_n(\psi_0, \psi_1) = \sum_{k=1}^{n} \left( \|\nabla^k \psi_0\|_{L^2(\mathbb{R}^2 \times (-1,0))} + \|\nabla^k \psi_1\|_{L^2(\mathbb{R}^2 \times (-1,0))} \right).$$

Then for all $\delta$, we have

$$E_{\leq n-1}(\Sigma_1) \lesssim I_n(\psi_0, \psi_1). \quad (2.1)$$

Here $\nabla$ denotes the spatial gradient.
The proof is straightforward: we rescale the $\delta$ factors appearing in the definitions of $\phi|_{t=0}$ and $\partial_t \phi|_{t=1}$ to 1 so that we can write everything in terms of $\psi_0$ and $\psi_1$.

2.1. **Main a priori estimates.** This subsection is the central part of the paper. The goal is to bound $E_{\leq 3}(u, u)$ on $D_{u^*, u^*}$, where the solution $\phi$ is assumed to exist.

**Proposition 2.2.** Given seed functions $\psi_0$ and $\psi_1$, there exists $\delta_0 > 0$, so that for all $\delta < \delta_0$, for all $u \in (0, u^*)$ and $\bar{u} \in (1 - u^*, \bar{u}^*)$, we have

$$E_{\leq 3}(u, \bar{u}) + E_{\leq 3}(u, \bar{u}) \leq C(I_4(\psi_0, \psi_1)), \quad (2.2)$$

where $C(I_4(\psi_0, \psi_1))$ is a constant depending only on $I_4(\psi_0, \psi_1)$.

The proof of the proposition is based on a standard bootstrap argument. On $D_{u^*, u^*}$, since we assume that $\phi$ exists, there is a large constant $M$, so that

$$E_{\leq 3}(u, \bar{u}) + E_{\leq 3}(u, \bar{u}) \lesssim M, \quad (2.3)$$

for all $u \in (0, u^*)$ and $\bar{u} \in (1 - u^*, \bar{u}^*)$. The large constant $M$ may depend on $\phi$ itself. The purpose of the bootstrap argument is to show that, if $\delta$ is sufficiently small, then one can choose $M$ in such a way that it depends only on $I_4(\psi_0, \psi_1)$. Hence, we obtain the proof of (2.2).

2.1.1. **Preliminary estimates.** The goal of this subsection is to use the bootstrap assumption (2.3) to get estimates on lower order derivatives of $\phi$ (up to second derivatives).

We first recall the Sobolev inequalities on $S_{\Sigma_1}$, $C_{\Sigma_1}$, and $C_0$ in the short pulse region. Recall that in the short pulse region, we have $|u| \sim r$ provided $\delta$ is sufficiently small. Let $\phi$ be a smooth function.

On $S_{\Sigma_1}$, we have

$$\|\phi\|_{L^\infty(S_{\Sigma_1})} \lesssim |u|^{-1/2}(\|\phi\|_{L^4(S_{\Sigma_1})} + \|\Omega \phi\|_{L^4(S_{\Sigma_1})}), \quad (2.4)$$

$$\|\phi\|_{L^4(S_{\Sigma_1})} \lesssim |u|^{-1/2}(\|\phi\|_{L^2(S_{\Sigma_1})} + \|\Omega \phi\|_{L^2(S_{\Sigma_1})}). \quad (2.5)$$

On $S_{\Sigma_1}$, if in addition we assume that $\phi \equiv 0$ on $C_0$, we have

$$\|\phi\|_{L^2(S_{\Sigma_1})} \lesssim \|L \phi\|_{L^2(C_0)}^{1/2} \|\phi\|_{L^2(C_0)}^{1/2},$$

$$\|\phi\|_{L^4(S_{\Sigma_1})} \lesssim \|L \phi\|_{L^2(C_0)}^{1/2} (\|\phi\|_{L^2(C_0)} + \|\Omega \phi\|_{L^2(C_0)}). \quad (2.6)$$

We remark that the assumption $\phi \equiv 0$ on $C_0$ will be always true when we apply the above inequalities in the short pulse region in the rest of the paper, since the solution $\phi$ of the main equations (1.2) (if it exists) vanishes to infinite order on $C_0$.

If $\phi$ is supported in the annular region $\{ (r, \theta) | 1 - \delta < r \leq 1 \}$ on the initial Cauchy hypersurface $\Sigma_1$, we have

$$\|\phi\|_{L^2(S_{\Sigma_1})} \lesssim \delta^{1/2} (\|\phi\|_{L^2(\Sigma_1)} + \|L \phi\|_{L^2(\Sigma_1)}). \quad (2.7)$$

For the proof of the above inequalities, we refer the reader to [2].

We also recall the Gronwall’s inequality. Let $f(t)$ be a non-negative function defined on an interval $I$ with initial point $t_0$. If $f$ satisfies

$$\frac{d}{dt} f \leq a \cdot f + b$$

with two non-negative functions $a, b \in L^1(I)$, then for all $t \in I$, we have

$$f(t) \leq e^{A(t)} (f(t_0) + \int_{t_0}^t e^{-A(\tau)} b(\tau) d\tau)$$

where $A(t) = \int_{t_0}^t a(\tau) d\tau$. 

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We start to derive estimates and we treat $u$ as a fixed constant. By virtue of null pair $(L, L)$, we rewrite the main system of equations (1.2) as

$$-LL\phi + \dot{\phi} + \frac{1}{r}(L\phi - L_\phi) = Q(\nabla \phi, \nabla \phi).$$  \hspace{1cm} (2.8)

We remark that, $\phi$ is now a $\mathbb{R}^N$-valued function and the norms used in the rest of the paper are with respect to a fixed inner product in $\mathbb{R}^N$. For example, the symbol $|L\phi|$ denotes $\sqrt{\sum_{l \leq N} (L\phi^l)^2}$.

We also need to commute derivatives with (2.8). Recall that, for all $Z \in \mathcal{Z}$ except for $Z = S$, we have $[\Box, Z] = 0$. Indeed, we have $[\Box, S] = 2\Box$. Combining this remark with (1.5), for all $k \geq 0$, we can commute $k$ vectors $Z_1, Z_2, \ldots, Z_k \in \mathcal{Z}$ with (1.2) to obtain a semilinear wave equation for $Z_1 Z_2 \cdots Z_k \phi$.

We use the shorthand notation $Z^k \phi$ to denote $Z_1 Z_2 \cdots Z_k \phi$, therefore, we have

$$\Box Z^k \phi = \sum_{p+q \leq k} Q(\nabla Z^p \phi, \nabla Z^q \phi).$$  \hspace{1cm} (2.9)

We combine (2.4), (2.5), (2.6) and bootstrap assumption (2.3). We first have

$$\|\nabla \phi\|_{L^4(S_{u}\omega)} \lesssim |u|^{-\frac{1}{2}} \|L \nabla \phi\|_{L^4(S_{u}\omega)}^\frac{1}{2} \left( \|\nabla \phi\|_{L^4(S_{u}\omega)}^\frac{1}{2} + |u|^{\frac{1}{2}} \|\nabla^2 \phi\|_{L^4(S_{u}\omega)} \right)$$

$$\lesssim |u|^{-\frac{1}{2}} (|u|^{-1} M)^\frac{1}{2} \left( (\delta^\frac{1}{2} |u|^{-\frac{1}{2}} M)^{\frac{1}{2}} + |u|^{\frac{1}{2}} (\delta^\frac{1}{2} |u|^{-\frac{1}{2}} M)^{\frac{1}{2}} \right).$$

Hence,

$$\|\nabla \phi\|_{L^4(S_{u}\omega)} \lesssim \delta^\frac{1}{2} |u|^{-1 - \frac{1}{2}} M.$$

Similarly, since in the bootstrap assumption (2.3), we have assumed bounds on four derivatives on $\phi$, we can repeat the above argument to derive

$$\|\nabla \Omega \phi\|_{L^4(S_{u}\omega)} \lesssim \delta^\frac{1}{2} |u|^{-1 - \frac{1}{2}} M,$$  \hspace{1cm} (2.10)

and

$$\|\nabla \Omega^2 \phi\|_{L^4(S_{u}\omega)} \lesssim \delta^\frac{1}{2} |u|^{-1 - \frac{1}{2}} M.$$  \hspace{1cm} (2.11)

Combining (2.11) and (2.10), the Sobolev inequality implies

$$\|\nabla \phi\|_{L^{\infty}(S_{u}\omega)} \lesssim |u|^{-\frac{1}{2}} \left( \|\nabla \phi\|_{L^4(S_{u}\omega)} + \|\nabla \Omega \phi\|_{L^4(S_{u}\omega)} \right)$$

$$\lesssim \delta^\frac{1}{2} |u|^{-\frac{1}{2} - \frac{1}{4}} M.$$  \hspace{1cm} (2.12)

Similarly, we have

$$\|\nabla \Omega \phi\|_{L^{\infty}(S_{u}\omega)} \lesssim \delta^\frac{1}{2} |u|^{-\frac{1}{2} - \frac{1}{4}} M.$$  \hspace{1cm} (2.13)

By repeating the above argument, for $0 \leq l \leq k \leq 2$, we can also easily obtain

$$\|\nabla Z^l Z^k \phi\|_{L^4(S_{u}\omega)} \lesssim \delta^{\frac{1}{2} - l} |u|^{-\frac{1}{2} - \frac{1}{4}} M.$$  \hspace{1cm} (2.14)

We turn to the bound of $L\phi$ in $L^{\infty}(S_{u}\omega)$. Let $a = \frac{1}{r} + |L\phi| + |\nabla \phi|$ and $b = |\ddot{\phi} + \frac{1}{r} |L\phi| + |L\phi \nabla \phi| + |\nabla \phi|^2$, in view of (1.6), (2.8) yields

$$\|L\phi\| \lesssim a |L\phi| + b.$$

We would like to integrate this equation directly along $L$ to derive the pointwise bound on $L\phi$. Since $L\phi$ vanishes along $C_0$, in view of Gronwall’s inequality, it suffices to control $\|a\|_{L^1 L^{\infty}(S_{u}\omega)}$ and $\|b\|_{L^1 L^{\infty}(S_{u}\omega)}$. 
We only give the estimates on $|L\phi|$ appearing in $a$ and $b$. The others can be estimated directly from (2.14). According to Sobolev inequality, we have
\[
\|L\phi\|_{L^1 L^\infty(S_{u,w})} \lesssim |u|^{-1} \sum_{0 \leq j \leq 2} \|\Omega^j L\phi\|_{L^1 L^\infty(S_{u,w})} \lesssim |u|^{-1} \delta^{\frac{1}{2}} \sum_{0 \leq j \leq 2} \|\Omega^j L\phi\|_{L^1 L^\infty(S_{u,w})} \lesssim |u|^{-1} \delta^{\frac{1}{2}} M.
\]
Finally, we can prove
\[
\|a\|_{L^1 L^\infty(S_{u,w})} \lesssim |u|^{-1} \delta^{-\frac{1}{2}} M,
\]
\[
\|b\|_{L^1 L^\infty(S_{u,w})} \lesssim |u|^{-2} \delta^{-\frac{1}{2}} M.
\]
Therefore, the Gronwall’s inequality provide us the following estimates for $L\phi$:
\[
\|L\phi\|_{L^1 L^\infty(S_{u,w})} \lesssim \delta^{1/2} |u|^{-2} M. \tag{2.15}
\]

By virtue of (2.9) (where $k \leq 2$), we can also bound $LZ^k_b Z^{k-l}\phi$ in $L^2(S_{u,w})$ in a similar way. Therefore, for $0 \leq l \leq k \leq 2$, we have
\[
\|LZ^k_b Z^{k-l}\phi\|_{L^2(S_{u,w})} \lesssim \delta^{1/2-l} |u|^{-1} M. \tag{2.16}
\]

We turn to the $L^\infty(S_{u,w})$ estimates on $L\phi$. We start with a computation of $L(u^2 (L\phi)^2)$:
\[
L(u^2 (L\phi)^2) = 2u(L\phi)^2 + 2u^2 (L\phi)(LL\phi) = 2u(L\phi)^2 + 2u^2 (L\phi) [\Delta \phi + \frac{1}{r} (L\phi - L\phi) + Q(\nabla \phi, \nabla \phi)] \lesssim |u|^2 |L\phi|^2 \left(\frac{2}{r} - \frac{2}{u} + |L\phi| + |\nabla \phi| + |u|^2 |L\phi| (|\Delta \phi| + \frac{1}{r} |L\phi| + |L\phi| |\nabla \phi| + |\nabla \phi|^2)\right).
\]
We make the following important observation: in the short pulse region, $|\frac{2}{r} - \frac{2}{u}| \lesssim \frac{\delta}{|u|^2}$. Therefore, if we define $y = |u||L\phi|$, according to the estimate obtained so far, the previous computation yields
\[
Ly^2 \lesssim \left(\frac{\delta}{|u|^2} + \frac{\delta^{\frac{1}{2}}}{|u|^2} M\right) y^2 + \frac{\delta^{\frac{1}{2}}}{|u|^2} M y.
\]
We divide both sides of the equation by $y$, thus, we have
\[
L(|u||L\phi|) \lesssim \left(\frac{\delta}{|u|^2} + \frac{\delta^{\frac{1}{2}}}{|u|^2} M\right) (|u||L\phi|) + \frac{\delta^{\frac{1}{2}}}{|u|^2} M.
\]
By integrating directly this equation, if $\delta$ is sufficiently small, we obtain
\[
|u||L\phi|(u, u, \theta) - C|1 - u||L\phi|(1 - u, u, \theta) | \lesssim \delta^{\frac{1}{2}} M. \tag{2.17}
\]
where the absolute constant $C$ comes from the use of Gronwall’s inequality. Therefore, according to (2.17), we finally obtain
\[
\|L\phi\|_{L^\infty(S_{u,w})} \lesssim \frac{\delta^{-\frac{1}{2}}}{|u|} I_3(\psi_0, \psi_1) + \delta^{\frac{1}{2}} |u|^{-1} M. \tag{2.18}
\]
We remark that the derivation of (2.15) and (2.18) depends on not only on the bootstrap assumption (2.3) but also the main equation (1.2). We summarize the estimates derived so far as follows:

\[
\|\nabla Z_l^k \|_{L^4(S_{\lambda_0})} \lesssim \delta^{\frac{1}{2} - l} |\lambda|^{-\frac{3}{4} + \frac{3}{2}} M, \quad 0 \leq l \leq k \leq 2, \\
\|\nabla \phi\|_{L^\infty(S_{\lambda_0})} \lesssim \delta^{\frac{1}{2}} |\lambda|^{-\frac{3}{4} + \frac{3}{2}} M, \\
\|L \phi\|_{L^\infty(S_{\lambda_0})} \lesssim \delta^{\frac{1}{2}} |\lambda|^{-2} M, \\
\|LZ_l^k \|_{L^2(S_{\lambda_0})} \lesssim \delta^{1/2 - l} |\lambda|^{-1} M, \quad 0 \leq l \leq k \leq 2, \\
\|L \phi\|_{L^\infty(S_{\lambda_0})} \lesssim \delta^{\frac{1}{4}} |\lambda|^{-1} M.
\]

(2.19)

**Remark 2.3.** The bootstrap assumptions (2.3) are adapted to the relaxed estimates. Roughly speaking, we expect the behaviors of \(\nabla \phi\) with respect to \(\delta\) and \(\lambda\) are approximately \(|\lambda|^{-\frac{3}{4} + \frac{3}{2}} M\), i.e.

\[
\|\nabla \phi\|_{L^\infty(S_{\lambda_0})} \sim |\lambda|^{-\frac{3}{4} + \frac{3}{2}} \cdot \delta^{\frac{1}{4}} M.
\]

However, the estimates on \(\nabla \phi\) in (2.19) shows that, by affording two more derivatives (via Sobolev inequalities), we can improve the bound on \(\nabla \phi\): we get an extra \(\delta^{\frac{1}{2}}\) factor and an extra \(\lambda^{-\frac{3}{4} + \frac{3}{2}}\) decay factor, i.e.

\[
\|\nabla \phi\|_{L^\infty(S_{\lambda_0})} \sim |\lambda|^{-\frac{3}{4} + \frac{3}{2}} \cdot \delta^{\frac{1}{2}} M.
\]

**2.1.2. Estimates on \(E_{\leq 2}\) and \(E_{\leq 2}\).** Recall that for \(Z \in \mathcal{Z}\) and \(k \geq 0\), we have

\[
\Box Z^k \phi = \sum_{p + q \leq k} Q(\nabla Z^p \phi, \nabla Z^q \phi).
\]

(2.20)

In Section 2.1.2, we fix \(k \leq 2\). Let \(l \leq k\) be the number of \(Z^l\)'s appearing in \(Z^k\), i.e. \(Z^k = Z^l Z^{k-l}\).

We use the vector field method outlined in the introduction to estimate \(E_{\leq 2}\) and \(E_{\leq 2}\).

In the fundamental energy identity (1.4) and (2.20), we replace \(\phi\) by \(Z^k \phi\) and take \(X = \nabla\) to obtain

\[
\int_{C_{\lambda_0}} |\nabla Z^k \phi|^2 + \int_{E_{\lambda_0}} |LZ^k \phi|^2 = \int_{\Sigma_1} |\nabla Z^k \phi|^2 + |LZ^k \phi|^2 + \int_{D_{\lambda_0}} Q(\nabla Z^k \phi, \nabla \phi)LZ^k \phi \\
+ \sum_{p \leq k < q} \int_{D_{\lambda_0}} Q(\nabla Z^p \phi, \nabla Z^q \phi)LZ^k \phi - \int_{D_{\lambda_0}} \frac{1}{r} LZ^k \cdot LZ^k \phi.
\]

We multiply both sides of the equation by \(\delta^{2l}\) to renormalize the contribution from the initial data to be close to 1, therefore, we obtain

\[
\delta^{2l} \int_{C_{\lambda_0}} |\nabla Z^k \phi|^2 + \delta^{2l} |LZ^k \phi|^2 \lesssim I_3(\psi_0, \psi_1)^2 + \delta^{2l} |\int_{D_{\lambda_0}} Q(\nabla Z^k \phi, \nabla \phi)LZ^k \phi| \\
+ \sum_{p + q \leq k, \ p < k < q} \delta^{2l} |\int_{D_{\lambda_0}} Q(\nabla Z^p \phi, \nabla Z^q \phi)LZ^k \phi| + \delta^{2l} |\int_{D_{\lambda_0}} \frac{1}{r} LZ^k \cdot LZ^k \phi|.
\]

(2.21)

We rewrite the right-hand side of the above inequality as

\[
I_3(\psi_0, \psi_1)^2 + S + T + W.
\]

where \(S\), \(T\) and \(W\) denote the three bulk integral terms in (2.21). We will bound \(S\), \(T\) and \(W\) one by one.
We begin with $S$, by definition, $S$ is bounded by the sum of the following integrals:

$$S_1 = \delta^2 \int_{D_{u,u}} \left( |L\phi| + |\nabla\phi| \right) |LZ^k\phi|^2,$$

$$S_2 = \delta^2 \int_{D_{u,u}} |L\phi|LZ^k\phi|LZ^k\phi|,$$

$$S_3 = \delta^2 \int_{D_{u,u}} |L\phi|\nabla Z^k\phi|LZ^k\phi|,$$

$$S_4 = \delta^2 \int_{D_{u,u}} |\nabla\phi|LZ^k\phi|LZ^k\phi|,$$

$$S_5 = \delta^2 \int_{D_{u,u}} \left( |L\phi| + |\nabla\phi| \right) |\nabla Z^k\phi|LZ^k\phi|.$$

It suffices to bound the $S_i$'s one by one.

For $S_1$, in view of the $L^\infty(S_{u,u})$ estimates on $L\phi$ and $\nabla\phi$, we have

$$S_1 \leq \delta^2 \int_{1-u}^u \int_0^u (|L\phi|_{L^\infty(S_{u,u}')} + ||\nabla\phi||_{L^\infty(S_{u,u}')}||LZ^k\phi||^2_{L^2(S_{u,u}')} du' du''$$

$$\leq \delta^2 \int_{1-u}^u \int_0^u \delta^\frac{3}{4} |u'|^{-\frac{2+\alpha}{4}} M||LZ^k\phi||^2_{L^2(S_{u,u}')} du' du''$$

$$\lesssim \int_{1-u}^u \delta^\frac{3}{4} |u'|^{-\frac{2+\alpha}{4}} M \left( \delta^2 ||LZ^k\phi||^2_{L^2(C_{\alpha/2})} \right) du'.$$

In view of the bootstrap assumption, we bound $\delta^2 ||LZ^k\phi||^2_{L^2(C_{\alpha/2})}$ by $M^2$. After an integration over $u'$ on $[1-u,u]$, we have

$$S_1 \lesssim \delta^\frac{3}{4} (1-u)^{-\frac{2+\alpha}{4}} - |u|^{-\frac{2+\alpha}{4}} M^3.$$

Because of $u \in [0,\delta]$, for sufficiently small $\delta$, we have

$$S_1 \lesssim \delta^\frac{3}{4} M^3. \tag{22.22}$$

For $S_2$, we have

$$S_2 \lesssim \delta^2 \int_{1-u}^u ||LZ^k\phi||_{L^\infty L^2(S_{u,u}')} ||L\phi||_{L^2 L^\infty(S_{u,u}')} ||LZ^k\phi||_{L^2(C_{\alpha/2})} du'.$$

According to the bootstrap assumption \(\square\) and the estimates \(\square\), we have $||LZ^k\phi||_{L^\infty L^2(S_{u,u}')} \lesssim \delta^\frac{3}{4} |u|^{-1} M$, $||L\phi||_{L^2 L^\infty(S_{u,u}')} \lesssim I_3(\psi_0,\psi_1)|u|^{-1}$ and $||LZ^k\phi||_{L^2(C_{\alpha/2})} \lesssim \delta^{-1} M$, therefore, we can conclude that

$$S_2 \lesssim \delta^\frac{3}{4} M^2. \tag{22.23}$$

For $S_3$, we have

$$S_3 \lesssim \delta^2 \int_{1-u}^u ||\nabla Z^k\phi||_{L^\infty L^2(S_{u,u}')} ||L\phi||_{L^2 L^\infty(S_{u,u}')} ||LZ^k\phi||_{L^2(C_{\alpha/2})} du'.$$

$$\lesssim I_3(\psi_0,\psi_1) \cdot M \cdot \delta^3 \int_{1-u}^u ||\nabla Z^k\phi||_{L^\infty L^2(S_{u,u}')} |u|^{-1} du'.$$

According to the $L^4$ estimates on $\nabla Z^k\phi$ on $S_{u,u}$ in \(\square\), we have $||\nabla Z^k\phi||_{L^2(S_{u,u})} \lesssim \delta^\frac{3}{4} |u|^{-\frac{3}{4} - \alpha} M$, this leads to

$$S_3 \lesssim \delta^\frac{3}{4} M^2. \tag{22.24}$$
For $S_4$, we can proceed exactly as for $S_2$ (we just replace the factor $L\phi$ in $S_2$ by $\nabla \phi$), this gives

$$S_4 \lesssim \delta^4 M^3. \quad (2.25)$$

For $S_5$, we can proceed exactly as for $S_3$ (we just replace the factor $|L\phi|$ in $S_2$ by $|\nabla \phi| + |L\phi|$), this gives

$$S_5 \lesssim \delta M^3. \quad (2.26)$$

We now estimate the second term in (2.21), i.e., the estimates on $T$. According to the structure of null forms, we have

$$T = \sum_{p+q \leq k, \ p<k, q<k} \delta^{2l} \left| \iint_{\mathbb{D}_{\omega}} Q(\nabla Z^p \phi, \nabla Z^q \phi) LZ^k \phi \right|$$

$$\lesssim \sum_{p+q \leq k, \ p<k, q<k} \delta^{2l} \left| \partial Z^p \phi \| \partial Z^q \phi \| LZ^k \phi \right|,$$

where $\partial \in \{\nabla, L\}$ and $\partial_g \in \{\nabla, L\}$. For each given term in the above sum, let $l'$ and $l''$ be total numbers of bad commutator $Z_0$'s appearing in $Z^p$ and $Z^q$ respectively. We remark that $l' + l'' \leq l$. Since $q \leq 1$, we have

$$T \lesssim \delta^2 \sum_{p+q \leq k, \ p<k, q<k} \int_{1-u}^u \int_0^u |\partial Z^p \phi \| L^4(S_{\omega'}, \omega') \| \partial Z^q \phi \| L^4(S_{\omega'}, \omega') \| LZ^k \phi \| L^2(S_{\omega'}, \omega') du' du''$$

By the second of (2.6), we have:

$$\| \partial Z^q \phi \| L^4(S_{\omega'}, \omega') \lesssim u'^{-1/2} \| L Z^q \phi \| L^4(C_{\omega'}) \left( \| \partial Z^q \phi \| L^2(C_{\omega'}) + \| \Omega \partial Z^q \phi \| L^2(C_{\omega'}) \right)^{1/2} \quad (2.27)$$

Now by (2.2), we have schematically:

$$L \sim \frac{1}{u} S + \frac{1}{u} x^i \Omega_{ij}, \quad \nabla \sim \frac{1}{u} \Omega_{ij}$$

which imply, for any smooth function $f$:

$$L \partial_g f \sim \frac{1}{u} L Z_g f, \quad \nabla f \sim u \nabla f, \quad \partial_g f \sim \frac{1}{u} Z_g f$$

If $\partial_g = \nabla$, then the second factor on the right hand side of (2.27) is bounded through the bootstrap assumption (2.3) by:

$$\left( \| \nabla Z^q \phi \| L^2(C_{\omega'}) + \| \nabla Z_g Z^q \phi \| L^2(C_{\omega'}) \right)^{1/2} \lesssim \delta^{1/4 - l'/2} M^{1/2}$$

If $\partial_g = L$, by virtue of (2.16) the first term in the parenthesis is bounded by:

$$\| L Z^q \phi \| L^2(C_{\omega'}) \lesssim \delta^{1 - l''} u^{-1} M$$

while for the second term we have:

$$\| \Omega \partial Z^g \phi \| L^2(C_{\omega'}) \lesssim \| \Omega Z_g Z^q \phi \| L^2(C_{\omega'}) \lesssim \delta^{1/2 - l''} M^{1/2}$$

These together with (2.27) imply:

$$\| \partial Z^q \phi \| L^4(S_{\omega'}, \omega') \lesssim \delta^{1/4 - l''} M u^{l'-1} \quad (2.28)$$
Therefore (2.5) implies:
\[
T \lesssim \delta^{1/4} M \sum_{p + q \leq k \atop p < k, q < k} \int_{1 - u}^{u} u'^{-3/2} \delta^{l''} \left( \| \partial Z^q Z_g^q \phi \|_{L^2(S_u^l)} \right) \| Z^{k} \phi \|_{L^2(S_u^l)} \, du \, du'
\]
\[
\lesssim \delta^{1/4} M \sum_{p + q \leq k \atop p < k, q < k} \int_{1 - u}^{u} u'^{-3/2} \delta^{l''} \left( \| \partial Z^q Z_g^q \phi \|_{L^2(C_u^l)} \right) \| Z^{k} \phi \|_{L^2(C_u^l)} \, du'
\]
\[
\lesssim \delta^{1/4} M^3 \int_{1 - u}^{u} u'^{-3/2} \, du'
\]
where \(|q'| \leq 1\). This eventually yields
\[
T \lesssim \delta^{1/4} M^3.
\] (2.29)

It remains to bound the third term \(W\) in (2.21). It is similar to \(T_2\). We simply bound \(LZ^k \phi\) and \(LZ^k \phi\) on \(\mathbb{C}_u^k\). Although the bound of \(|LZ^p \phi|\) on \(\mathbb{C}_u^k\) is not directly from the bootstrap assumption, in view of (2.19) and the fact that \(k \leq 2\), we can bound \(LZ^k \phi\) first on \(L^2(S_u^l)\) and then on \(L^2(C_u^l)\). This leads to
\[
W \lesssim \delta M^2.
\] (2.30)

By combining (2.21) with (2.22), (2.23), (2.24), (2.25), (2.26), (2.29) and (2.30), for sufficiently small \(\delta\), we obtain
\[
\delta^{2l} \int_{\mathbb{C}_u} |\nabla Z^k \phi|^2 + \delta^{2l} \int_{\mathbb{C}_u} |Z^k \phi|^2 \lesssim I_3(\psi_0, \psi_1)^2 + \delta^{1/2} M^3.
\]
In other words, for all \(0 \leq l \leq k \leq 2\), we have
\[
\delta^{1/2} |\nabla Z^k \phi|^2 + \delta^{1/2} |LZ^k \phi|^2 \lesssim I_3(\psi_0, \psi_1) + \delta^{1/4} M^3.
\] (2.31)

In the fundamental energy identity (1.4) and (2.20), we replace \(\phi\) by \(Z^k \phi\) and take \(X = u^\alpha L\) to obtain
\[
\int_{\mathbb{C}_u} |u|^{\alpha} |LZ^k \phi|^2 + \int_{\mathbb{C}_u} |u|^{\alpha} |\nabla Z^k \phi|^2 = \int_{\mathbb{C}_u} |u|^{\alpha} \left( |\nabla Z^k \phi|^2 + |LZ^k \phi|^2 \right) + \int_{D_{\mathbb{C}_u}} |u|^{\alpha} Q(\nabla Z^k \phi, \nabla \phi) L Z^k \phi
\]
\[+ \sum_{p + q \leq k \atop p < k, q < k} \int_{D_{\mathbb{C}_u}} |u|^{\alpha} Q(\nabla Z^p \phi, \nabla Z^q \phi) L Z^k \phi + \int_{D_{\mathbb{C}_u}} \frac{|u|^{\alpha}}{r} L Z^k \phi \cdot L Z^k \phi
\]
\[- 2\alpha \int_{D_{\mathbb{C}_u}} |u|^{\alpha - 2} |\nabla Z^k \phi|^2.
\]
We multiply both sides of the equation by \(\delta^{2l-1}\) to renormalize the contribution from the initial data to be close to 1. We remark that this normalization is respect to the relaxed estimates on \(\nabla \phi\). By dropping of the last negative term in the above equation, we obtain
\[
\delta^{2l-1} \int_{\mathbb{C}_u} |u|^{\alpha} |LZ^k \phi|^2 + \delta^{2l-1} \int_{\mathbb{C}_u} |u|^{\alpha} |\nabla Z^k \phi|^2 \lesssim I_3(\psi_0, \psi_1)^2 + \delta^{2l-1} \int_{D_{\mathbb{C}_u}} |u|^{\alpha} Q(\nabla Z^k \phi, \nabla \phi) L Z^k \phi
\]
\[+ \sum_{p + q \leq k \atop p < k, q < k} \delta^{2l-1} \int_{D_{\mathbb{C}_u}} |u|^{\alpha} Q(\nabla Z^p \phi, \nabla Z^q \phi) L Z^k \phi + \delta^{2l-1} \int_{D_{\mathbb{C}_u}} \frac{|u|^{\alpha}}{r} L Z^k \phi \cdot L Z^k \phi,
\] (2.32)

We rewrite the right-hand side of the above inequality as
\[
I_3(\psi_0, \psi_1)^2 + S + T + W.
\]
where $S$, $T$ and $W$ denote the three bulk integral terms in (2.32). We now bound $S$, $T$ and $W$ one by one.

We begin with $S$. According to the definition of $S$ and the structure (1.6) for null forms, $S$ is bounded by the sum of the following terms:

\[
S_1 = \delta^{2l-1} \int_{D_{u_0}}^u |u|^{\alpha} (|L\phi| + |\nabla \phi|) |LZ^k \phi|, \\
S_2 = \delta^{2l-1} \int_{D_{u_0}}^u |u|^{\alpha} (|L\phi| + |LZ^k \phi|), \\
S_3 = \delta^{2l-1} \int_{D_{u_0}}^u |u|^{\alpha} (|L\phi| + |\nabla Z^k \phi|), \\
S_4 = \delta^{2l-1} \int_{D_{u_0}}^u |u|^{\alpha} (|L\phi| |\nabla Z^k \phi| |LZ^k \phi|).
\]

The idea to bound the $S_i$'s are exactly the same as before. Roughly speaking, we bound all the first order derivative components of $\nabla \phi$ in $L^\infty(S_{u_0})$.

For $S_1$, we have

\[
S_1 \leq \delta^{2l-1} \int_{D_{u_0}}^u |u|^{\alpha} |\delta^{-\frac{l}{2}} M| |LZ^k \phi|^2, \\
\lesssim \delta^{-\frac{1}{2}} M \int_{0}^{u} \left( \delta^{2l-1} \int_{C_{u'}} |u|^{\alpha} |LZ^k \phi|^2 \right) du'.
\]

According to the bootstrap assumption on $\delta^{l-\frac{1}{2}} |LZ^k \phi|_{L^2(C_{u'})}$, we obtain

\[
S_1 \lesssim \delta^{l} M^3. \tag{2.33}
\]

For $S_2$, since $k \leq 2$, we use the bound on $LZ^k \phi$ on $S_{u_0}$ to derive $\delta^{l} |LZ^k \phi|_{L^2(C_{u'})} \lesssim |u|^{-1} M$. Therefore, we can proceed as follows:

\[
S_2 \leq \delta^{2l-1} \int_{1-u}^u |u|^{\alpha} |\delta^{-\frac{l}{2}} \delta^{\frac{l}{2}} M||LZ^k \phi|_{L^2(C_{u'})} |LZ^k \phi|_{L^2(C_{u'})} du.
\]

According to the bootstrap assumptions, we finally obtain

\[
S_2 \lesssim \delta^{l} M^3. \tag{2.34}
\]

The estimates on $S_3$ can be obtained in a similar way as $S_2$: we simply replace $LZ^k \phi$ by $\nabla Z^k \phi$ and proceed exactly the same as before. This gives

\[
S_3 \lesssim \delta^{l} M^3. \tag{2.35}
\]

For $S_4$, we first make the following remark:

**Remark 2.4.** It seems to be natural to derive the estimates by putting $\nabla Z^k \phi$ in the $L^2(C_{u'})$ norm. In fact, this does not work due to the fact that we have relaxed the estimates on the rotational directions. To illustrate the idea, we may proceed as follows:

\[
S_4 \leq \delta^{2l-1} \int_{1-u}^u |u|^{\alpha} (|u|^{-1} \delta^{-\frac{l}{2}} I_3(\psi_0, \psi_1)) ||\nabla Z^k \phi||_{L^2(C_{u'})} |LZ^k \phi||_{L^2(C_{u'})} du, \\
\leq \delta^{-1} \int_{1-u}^u |u|^{\alpha} (|u|^{-1} \delta^{-\frac{l}{2}} I_3(\psi_0, \psi_1)) (\delta^{l} M) (\delta |u|^{-1} M) du', \\
\lesssim M^2.
\]
This estimate is certainly not good since we do not have a $\delta$ (to some positive power) factor in front of the possibly large constant $M$.

At this point, we have to use the bootstrap assumptions on the fourth order derivatives of $\phi$ to improve the relaxed estimates on $\nabla$-direction.

The above remark suggests to put $\nabla Z^k \phi$ in $L^4(S_{u,L})$ norm to get an extra $\delta^2$ factor. In fact, we have

$$S_4 \lesssim \delta^{2l-1} \int_{1-u} u |\nabla^{\alpha} (L\phi)_{L^2 L^4(S_{u,L})}\|LZ^k \phi\|_{L^2 S_{u,L}}\|LZ^k \phi\|_{L^2 (S_{u,L})} du'.$$

Since we have already derived estimates on $\|L\phi\|_{L^4(S_{u,L})}$ and $\|LZ^k \phi\|_{L^2 (S_{u,L})}$ ($k \leq 2$), a direct computation yields

$$S_4 \lesssim \delta^4 M^2.$$ \hfill (2.36)

We turn to the estimates on $T$. According to the structure of null forms, we have

$$T \lesssim \delta^{2l-1} \sum_{p+q \leq k} \int_{D_{u,L}} |\nabla^{\alpha} (LZ^k \phi)\|LZ^k \phi\|,$$

where $\partial \in \{\nabla, L\}$ and $\partial \phi \in \{\nabla, L\}$.

Here we postpone the estimate for $T$ until we estimate the top order energy $E_{\leq 3}(u,u)$, because the estimates for $T$ corresponding to $E_{\leq 2}$ and $E_{\leq 3}$ are identical. Instead, we just state the result:

$$T \lesssim \delta^{-1} \sum_{p+q \leq k} \int_0^u \delta^{2l-1} \|u^{\alpha/2} LZ^k \phi\|^2_{L^2 (S_{u,L})} du' + I_3(\psi_0, \psi_1)^4 \hfill (2.37)$$

It remains to control $W = \delta^{2l-1} \int_{D_{u,L}} |\nabla Z^k \phi\|LZ^k \phi|$. We proceed as follows

$$W \lesssim \delta^{2l-1} \int_{D_{u,L}} (\delta^2 \|u\| \|LZ^k \phi\|) \|LZ^k \phi\| \frac{1}{u^{\alpha/2}} \|LZ^k \phi\| \frac{1}{u^{\alpha/2}} \|LZ^k \phi\|$$

$$\lesssim \delta^{2l-1} \left( \int_{D_{u,L}} \|u\|^2 \|LZ^k \phi\|^2 + \int_{D_{u,L}} \frac{1}{\delta} \|u\| \|LZ^k \phi\|^2 \right)$$

$$= \delta^{2l} \int_{D_{u,L}} \frac{1}{u^{\alpha/2}} \|LZ^k \phi\|^2_{L^2 (S_{u,L})} du' + \delta^{2l-2} \int_0^u \|u\| \|LZ^k \phi\|^2_{L^2 (S_{u,L})} du'.$$

The first term in the last line has already been controlled in (2.31). In view of the fact that $\alpha < 1$ (this is crucial to make the first factor integrable in $u!$), for sufficiently small $\delta$, we obtain

$$W \lesssim I_3(\psi_0, \psi_1)^2 + \delta^{-1} \int_0^u \delta^{2l-1} \|u\| \|LZ^k \phi\|^2_{L^2 (S_{u,L})} du'. \hfill (2.38)$$

By combining (2.32) with (2.33), (2.34), (2.35), (2.36), (2.37) and (2.38), for sufficiently small $\delta$, we obtain

$$\delta^{2l-1} \int_{S_{u,L}} |\nabla^{\alpha} (LZ^k \phi)|^2 + \delta^{2l-1} \int_{S_{u,L}} |\nabla^{\alpha} \nabla Z^k \phi|^2 \lesssim I_3(\psi_0, \psi_1)^2 + \delta^2 M^3 + \delta^{-1} \int_0^u \delta^{2l-1} \|u\| \|LZ^k \phi\|^2_{L^2 (S_{u,L})} du'.$$

The last term on the right-hand side can be removed by the Gronwall’s inequality. This finally proves that, for all $0 \leq l \leq k \leq 2$, we have

$$\delta^{-\frac{l}{2}} \|u\| \|LZ^k \phi\|^2_{L^2 (S_{u,L})} + \delta^{-\frac{l}{2}} \|\nabla Z^k \phi\|_{L^2 (S_{u,L})} \lesssim C(I_3(\psi_0, \psi_1)) + \delta^2 M^3. \hfill (2.39)$$
The estimates \([2.31]\) and \([2.39]\) together implies
\[
E_{<2}(u, \overline{u}) + E_{<2}(u, \overline{u}) \leq C(I_3(\psi_0, \psi_1)) + \delta^k M^{\frac{3}{2}}.
\] (2.40)

2.1.3. Estimates on \(E_3, E_3\). We take \(k = 3\) in \([2.20]\). Let \(l \leq k\) be the number of \(Z_i\)'s appearing in \(Z^3\), i.e. \(Z^3 = Z_1 Z_2 l^{-3}\). We take \(Z^3 \phi\) in the place of \(\phi\) in \([1.4]\) and take the multiplier \(X = L\), this yields
\[
\int_{C_u} |\nabla Z^3 \phi|^2 + \int_{C_u} |LZ^3 \phi|^2 = \int_{\Sigma_1} |\nabla Z^3 \phi|^2 + |LZ^3 \phi|^2 + \int_{D_{\overline{u}}} Q(\nabla Z^3 \phi, \nabla \phi)LZ^3 \phi
+ \sum_{p+q=3, p < 3} \int_{D_{\overline{u}}} Q(\nabla Z^p \phi, \nabla Z^q \phi)LZ^3 \phi - \int_{D_{\overline{u}}} \frac{1}{r}LZ^3 \phi \cdot LZ^3 \phi.
\]

After a renormalization in \(\delta\), we obtain
\[
\delta^{2l} \int_{C_u} |\nabla Z^3 \phi|^2 + \delta^{2l} \int_{C_u} |LZ^3 \phi|^2 \lesssim I_4(\psi_0, \psi_1)^2 + \delta^{2l} \int_{D_{\overline{u}}} Q(\nabla Z^3 \phi, \nabla \phi)LZ^3 \phi
+ \sum_{p+q=3, p < 3} \delta^{2l} \int_{D_{\overline{u}}} Q(\nabla Z^p \phi, \nabla Z^q \phi)LZ^3 \phi + \delta^{2l} \int_{D_{\overline{u}}} \frac{1}{r}LZ^3 \phi \cdot LZ^3 \phi
\]
\[
= I_4(\psi_0, \psi_1)^2 + S + T + W,
\]
where \(S, T\) and \(W\) denote the three bulk integral terms. We will bound \(S, T\) and \(W\) one by one.

We start with \(S\). It can be bounded by the sum of the following terms:
\[
S_1 = \delta^{2l} \int_{D_{\overline{u}}} |L\phi|(|LZ^3 \phi| + |\nabla Z^3 \phi|)|LZ^3 \phi|,
S_2 = \delta^{2l} \int_{D_{\overline{u}}} |L\phi|(|\nabla Z^3 \phi| + |LZ^3 \phi|)|LZ^3 \phi|,
S_3 = \delta^{2l} \int_{D_{\overline{u}}} |\nabla \phi|(|LZ^3 \phi| + |LZ^3 \phi| + |LZ^3 \phi|)|LZ^3 \phi|.
\]

For \(S_1\), according to the \(L^\infty(S_{\overline{u}})\) estimates on \(L\phi\), we have
\[
S_1 \lesssim \delta^{2l-\frac{1}{2}} \int_{D_{\overline{u}}} |u|^{-1} |\nabla Z^3 \phi||LZ^3 \phi| + \delta^{2l-\frac{1}{2}} \int_{D_{\overline{u}}} |u|^{-\frac{1}{2}} \cdot (|u|^{\frac{1}{2}} L Z^3 \phi) \cdot (|u|^{-\frac{1}{2}} |LZ^3 \phi|)
= S_{11} + S_{12}.
\]

For \(S_{11}\), according to Cauchy-Schwarz inequality, we have
\[
S_{11} \lesssim \delta^{2l-1} \int_{D_{\overline{u}}} |\nabla Z^3 \phi|^2 + \delta^{2l} \int_{D_{\overline{u}}} |u|^{-2} |LZ^3 \phi|^2
\]
\[
\lesssim \int_0^1 \frac{1}{2} \cdot \delta^{2l} \int_{C_{u'}} |\nabla Z^3 \phi|^2 du' + \int_{1-u}^u |u'|^{-2} \delta^{2l} \int_{\overline{C}_u} |LZ^3 \phi|^2 du'.
\] (2.42)
For $S_{12}$, we still use Cauchy-Schwarz inequality to derive

$$S_{12} \lesssim \delta^{2l} \int_{D_{\alpha}^-} (|u|^\frac{2l}{2} |LZ^3 \phi|)^2 + \delta^{2l} \int_{D_{\alpha}^-} |u|^{-2-\alpha} |LZ^3 \phi|^2$$

$$\lesssim \int_0^u \frac{1}{\delta} \delta^{2l} \int_{C_{u'}} |LZ^3 \phi|^2 du' + \int_{1-u}^u |u'|^{-2-\alpha} \delta^{2l} \int_{C_{u'}} |LZ^3 \phi|^2 du'$$

$$\lesssim \delta M^2 + \int_{1-u}^u |u'|^{-2-\alpha} \delta^{2l} \int_{C_{u'}} |LZ^3 \phi|^2 du'.$$

Therefore, we obtain

$$S_1 \lesssim \delta M^2 + \int_0^u \frac{1}{\delta} \delta^{2l} \int_{C_{u'}} |\nabla Z^3 \phi|^2 du' + \int_{1-u}^u |u'|^{-2} \delta^{2l} \int_{C_{u'}} |LZ^3 \phi|^2 du'. \quad (2.43)$$

For $S_2$, according to the $L^\infty(S_{L,u})$ estimates on $L\phi$, we have

$$S_2 \lesssim \delta^{2l+\frac{1}{2}} M \int_{D_{\alpha}^-} |u|^2 |\nabla Z^3 \phi| |LZ^3 \phi| + \delta^{2l+\frac{1}{2}} M \int_{D_{\alpha}^-} |u|^{-2} |LZ^3 \phi|^2 = S_{21} + S_{22}.$$  

For $S_{21}$, since $1 \lesssim |u|$, we have

$$S_{21} \lesssim \delta^{2l} M \int_{D_{\alpha}^-} |\nabla Z^3 \phi|^2 + \delta^{2l+1} M \int_{D_{\alpha}^-} |u|^{-2} |LZ^3 \phi|^2$$

$$\lesssim \int_0^u \delta^{2l} M \int_{C_{u'}} |\nabla Z^3 \phi|^2 du' + \delta M \int_{1-u}^u |u'|^{-2} \delta^{2l} \int_{C_{u'}} |LZ^3 \phi|^2 du'$$

$$\lesssim \delta M^3.$$  

For $S_{22}$, we have

$$S_{22} \lesssim \delta^{\frac{1}{2}} M \int_{1-u}^u |u'|^{-2} \delta^{2l} \int_{C_{u'}} |LZ^3 \phi|^2 du'$$

$$\lesssim \delta^{\frac{1}{2}} M^3. \quad (2.44)$$

For $S_3$, according to the $L^\infty(S_{L,u})$ estimates on $\nabla \phi$, it is bounded by the following three terms:

$$S_{31} = \delta^{2l+\frac{1}{2}} M \int_{D_{\alpha}^-} |u|^{-\frac{3}{2}-\frac{1}{2}} |LZ^3 \phi| |LZ^3 \phi|,$$

$$S_{32} = \delta^{2l+\frac{1}{2}} M \int_{D_{\alpha}^-} |u|^{-\frac{3}{2}-\frac{1}{2}} |LZ^3 \phi|^2,$$

$$S_{33} = \delta^{2l+\frac{1}{2}} M \int_{D_{\alpha}^-} |u|^{-\frac{3}{2}-\frac{1}{2}} |\nabla Z^3 \phi| |LZ^3 \phi|.$$  

To bound $S_{31}$, we follow exactly the same way for $S_{11}$, this yields

$$S_{31} \lesssim \delta^{\frac{1}{2}} M^3.$$  

To bound $S_{32}$, we follow exactly the same way for $S_{22}$, this yields

$$S_{32} \lesssim \delta^{\frac{1}{2}} M^3.$$
To bound $S_{33}$, we follow exactly the same way for $S_{21}$, this yields

$$S_{33} \lesssim \delta^\frac{3}{2} M^3.$$  

Therefore, we obtain

$$S_3 \lesssim \delta^\frac{3}{2} M^3. \quad (2.45)$$

We turn to the estimates on $T$. According to the structure of null forms, we have

$$T \lesssim \sum_{p+q \leq 3, \ p \leq q} \delta^{2l} \int_{D_{u^n}} \left| \partial Z^p \phi \right| \partial_y Z^q \phi \|LZ^3 \phi\|.$$  

where $\partial \in \{\nabla, L\}$ and $\partial_y \in \{\nabla, L\}$. By using exactly the same method as we derive (2.29), we obtain

$$T \lesssim \delta^\frac{3}{2} M^3.$$  

(2.46)

It remains to bound $W = \delta^{2l} \int_{D_{u^n}} \frac{1}{2} |LZ^3 \phi| \|LZ^3 \phi\|$. According to Cauchy-Schwarz inequality, we have

$$W \lesssim \delta^{2l} \left( \int_{D_{u^n}} \frac{\delta}{|u|^{2-\alpha}} |LZ^3 \phi|^2 + \int_{D_{u^n}} \frac{1}{\delta} \|u\|^2 \|LZ^3 \phi\|^2 \right)$$

$$= \int_{1-u}^u \frac{1}{|u|^{2-\alpha}} \cdot \delta^{2l+1} \|LZ^3 \phi\|^2_{L^2(C_{u'})} du' + \delta^{2l-1} \int_0^u \|u\|^2 \|LZ^3 \phi\|^2_{L^2(C_{u'})} du'.$$

This yields $W \lesssim \delta M^2$. Combining this estimate with (2.43), (2.44), (2.45) and (2.46), we obtain

$$\delta^{2l} \int_{C_u} |\nabla Z^3 \phi|^2 + \delta^{2l} \int_{C_u} \|LZ^3 \phi\|^2 \lesssim I_4(\psi_0, \psi_1)^2 + \delta^\frac{3}{2} M^3.$$

In other words, for all $0 \leq l \leq 3$, we have

$$\delta^l \|\nabla Z^3 \phi\|_{L^2(C_u)} + \delta^l \|LZ^3 \phi\|_{L^2(C_u)} \lesssim I_4(\psi_0, \psi_1) + \delta^\frac{3}{2} M^\frac{3}{2}. \quad (2.47)$$

Similar to the derivation for (2.32) (by taking $k = 3$), we have

$$\delta^{2l-1} \int_{C_u} |u|^3 |LZ^3 \phi|^2 + \delta^{2l-1} \int_{C_u} |u|^3 |\nabla Z^3 \phi|^2 \lesssim I_4(\psi_0, \psi_1)^2 + \delta^{2l-1} \int_{D_{u^n}} |u|^3 Q(\nabla Z^3 \phi, \nabla \phi)LZ^3 \phi|$$

$$+ \sum_{p+q \leq 3, \ p \leq q \leq 2} \delta^{2l-1} \int_{D_{u^n}} |u|^3 Q(\nabla Z^p \phi, \nabla Z^q \phi)LZ^3 \phi| + \delta^{2l-2} \int_{D_{u^n}} |u|^3 |LZ^3 \phi| \cdot \|LZ^3 \phi\|,$$

We rewrite the above inequality as

$$\delta^{2l-1} \int_{C_u} |u|^3 |LZ^3 \phi|^2 + \delta^{2l-1} \int_{C_u} |u|^3 |\nabla Z^3 \phi|^2 \lesssim I_4(\psi_0, \psi_1)^2 + S + T + W.$$  

(2.48)

where $S$, $T$ and $W$ denote the three bulk integral terms in an obvious way. We now bound $S$, $T$ and $W$ one by one.
We begin with $S$ which is bounded by the sum of the the following terms:

$$
S_1 = \delta^{2l-1} \iint_{\mathbb{R}^n_{\omega,u}} |u|^{\alpha} (|L\phi| + |\nabla \phi|)|LZ^3\phi|^2,
$$

$$
S_2 = \delta^{2l-1} \iint_{\mathbb{R}^n_{\omega,u}} |u|^{\alpha} (|\nabla \phi| + |L\phi|)|LZ^3\phi||LZ^3\phi|,
$$

$$
S_3 = \delta^{2l-1} \iint_{\mathbb{R}^n_{\omega,u}} |u|^{\alpha} (|L\phi| + |\nabla \phi|)|\nabla Z^3\phi||LZ^3\phi|,
$$

$$
S_4 = \delta^{2l-1} \iint_{\mathbb{R}^n_{\omega,u}} |u|^{\alpha} |L\phi||\nabla Z^3\phi||LZ^3\phi|.
$$

For $S_1$, we have

$$
S_1 \leq \delta^{2l-1} \iint_{\mathbb{R}^n_{\omega,u}} |u|^{\alpha} (|u|^{-1} \delta^{-\frac{1}{2}} M)|LZ^3\phi|^2
\leq \delta^{-\frac{1}{2}} M \int_{0}^{u} \left( \delta^{2l-1} \int_{C_{\omega}} |u|^\alpha |LZ^k\phi|^2 \right) du'
\leq \delta^{\frac{1}{2}} M^3.
$$

For $S_2$, we use $L^\infty$ bound on $\nabla \phi$ and $L\phi$. Since we have already derived estimates on $E_{\leq 2}(u, u)$ and $E_{\leq 2}(u, u)$, for sufficiently small $\delta$, we indeed have

$$
|L\phi| + |\nabla \phi| \lesssim |u|^{-\frac{1}{2}} \delta^{\frac{1}{2}} C(I_3(\psi_0, \psi_1)).
$$

We remark that this estimate is better than those in (2.19) since we have improved the big bootstrap constant $M$ to be a constant depending only on the size of the seed data $(\psi_0, \psi_1)$. Therefore, according to Cauchy-Schwarz inequality, we have

$$
S_2 \leq C(I_3(\psi_0, \psi_1)) \delta^{2l-1} \iint_{\mathbb{R}^n_{\omega,u}} (|u|^{-\frac{1}{2}} \delta^{\frac{1}{2}} M)|LZ^3\phi||LZ^3\phi|
\leq \delta^{\frac{1}{2}} \int_{1-u}^{u} \frac{1}{|u'|^{3+\frac{1}{2}\alpha}} \cdot \delta^{\frac{1}{2}} |LZ^3\phi|_{L^2(C_{\omega})} du' + \delta^{\frac{1}{2}} \int_{0}^{u} \frac{1}{\delta} \cdot \delta^{2l-1} |u|^\frac{1}{2} |LZ^3\phi|_{L^2(C_{\omega})} du'
\leq \delta^{\frac{1}{2}} M^2.
$$

For $S_3$, we have

$$
S_3 \leq \delta^{2l-1} \iint_{\mathbb{R}^n_{\omega,u}} (|u|^{-\frac{1}{2}} \delta^{\frac{1}{2}} M)|\nabla Z^3\phi||LZ^3\phi|
\leq M \delta^{-\frac{1}{2}} \int_{0}^{u} \left( \delta^l ||\nabla Z^3\phi||_{L^2(C_{\omega})} \right) \left( \delta^{-\frac{1}{2}} ||u|^\frac{1}{2} \delta^{\frac{1}{2}} M^3 \right) du'
\leq \delta^2 M^3.
$$

For $S_4$, we have

$$
S_4 \leq \delta^{2l-1} \iint_{\mathbb{R}^n_{\omega,u}} (|u|^{-1+\alpha} \delta^{-\frac{1}{2}} I_3(\psi_0, \psi_1))|\nabla Z^3\phi||LZ^3\phi|
\leq \delta^{-1} \int_{0}^{u} \left( \delta^l ||\nabla Z^3\phi||_{L^2(C_{\omega})} \right) \left( \delta^{-\frac{1}{2}} ||u|^\frac{1}{2} \delta^{\frac{1}{2}} M^3 \right) du'.
$$

By virtue of (2.47), we can bound $\delta^l ||\nabla Z^3\phi||_{L^2(C_{\omega})}$ to derive

$$
S_4 \leq C(I_4(\psi_0, \psi_1)) M.
$$
The estimates on $S_1$, $S_2$, $S_3$ and $S_4$ together yield
\begin{equation}
S \lesssim \delta^2 M^3 + C(I_4(\psi_0, \psi_1)) M. \tag{2.50}
\end{equation}

For $T$, according to (1.6), we have
\begin{equation}
T \lesssim \sum_{p+q \leq 3, \ p \geq 2, q \leq 2} \delta^{2l-1} \int_{D_{2,u}} |u|^l |\partial Z^p \phi| |\partial_y Z^q \phi| |LZ^3 \phi|.
\end{equation}
where $\partial \in \{\nabla, L\}$ and $\partial_y \in \{\nabla, L\}$.

We first consider $\partial_y = \nabla, \partial = L$, and denote its contribution by $T_1$, then we see all the other cases are lower order compared to this case. By (2.5) we have:
\begin{equation}
T_1 \lesssim \delta^{2l-1} \sum_{p+q \leq 3, \ p \geq 2, q \leq 2} \int_{1-u}^u \int_0^1 u'^{\alpha/2} |LZ^p \phi| L^4(S_{y', u'}) |\nabla Z^q \phi| L^4(S_{y', u'}) \|u'^{\alpha/2} LZ^3 \phi\|_{L^2(S_{y', u'})} du' du.
\end{equation}

By the second of (2.6),
\begin{equation}
\delta'^{l'} |LZ^p \phi| L^4(S_{y', u'}) \lesssim \delta'^{l''} \delta'^{l'-1/2} |LZ_0 Z^p \phi| L^{1/2}(C_{y'}) \left( |LZ^p \phi| L^2(C_{y'}) + |LZ_0 Z^p \phi| L^2(C_{y'}) \right)^{1/2} \lesssim \delta'^{l'-1/2} \delta'^{l''} \left( I_4(\psi_0, \psi_1) + \delta'^{3/8} M^{3/2} \right) \lesssim \delta'^{l'-1/2} I_4(\psi_0, \psi_1)
\end{equation}
provided that $\delta$ is sufficiently small.

On the other hand, by (2.5),
\begin{equation}
\delta'^{l''} |\nabla Z^q \phi| L^2(S_{y', u'}) \lesssim \delta'^{l''} \left( \delta'' |\nabla Z^q \phi| L^2(S_{y', u'}) + \delta'' |\nabla Z_0 Z^q \phi| L^2(S_{y', u'}) \right)
\end{equation}

Therefore we have:
\begin{equation}
T_1 \lesssim \delta^{l-1} \sum_{p+q \leq 3, \ p \geq 2, q \leq 2} \int_{-u}^u \delta'^{l'-1/2} I_4(\psi_0, \psi_1) \left( \delta'' |\nabla Z^q \phi| L^2(C_{y'}) + \delta'' |\nabla Z_0 Z^q \phi| L^2(C_{y'}) \right) \cdot \|u'^{\alpha/2} LZ^3 \phi\|_{L^2(C_{y'})} du' \lesssim \delta^{l-1} \sum_{p+q \leq 3, \ p \geq 2, q \leq 2} \int_{-u}^u \delta'^{l'-1/2} I_4(\psi_0, \psi_1)^2 \|u'^{\alpha/2} LZ^3 \phi\|_{L^2(C_{y'})} du'.
\end{equation}

By Cauchy-Schwarz, this implies:
\begin{equation}
T_1 \lesssim \delta^{l-1} \sum_{p+q \leq 3, \ p \geq 2, q \leq 2} \int_{0}^{u} \delta^{2l-1} \|u'^{\alpha/2} LZ^3 \phi\|_{L^2(C_{y'})}^2 du' + I_4(\psi_0, \psi_1)^4
\end{equation}

If $\partial = L, \partial_y = L$, we denote its contribution by $T_2$, then by the estimates we have derived for $\|u'^{\alpha/2} LZ^3 \phi\|_{L^2(C_{y'})}$, a similar argument leads to the estimate on $T_2$:
\begin{equation}
T_2 \lesssim \delta'^{l'-1/2} \sum_{p+q \leq 3, \ p \geq 2, q \leq 2} \int_{0}^{u} \delta^{2l-1} \|u'^{\alpha/2} LZ^3 \phi\|_{L^2(C_{y'})}^2 du' + \delta^{1/2} I_4(\psi_0, \psi_1)^4
\end{equation}
If $\partial = \nabla$, the estimates for $\partial_s Z^g \phi$ are the same as before. While for $\nabla Z^p \phi$, we have, if $\delta$ is sufficiently small:

$$
\delta^{\alpha} \| u^{\alpha/2} \nabla Z^p \phi \|_{L^1(S,u)} \lesssim \delta^{\alpha} \| u^{\alpha-1+\alpha/4} \| L^2 Z^p Z^g \phi \|_{L^2(C_u)}^{1/2} \cdot \left( \| u^{\alpha/2} \nabla Z^p \phi \|_{L^2(C_u)} + \| u^{\alpha/2} \nabla Z^p \phi \|_{L^2(C_u)} \right)^{1/2} 
\lesssim \delta^{1/4} I_4(\psi_0, \psi_1)
$$

This bound is better than that of $\delta^{\alpha} \| L^2 Z^p \phi \|_{L^1(S,u)}$. Therefore finally we obtain:

$$
T \lesssim \delta^{-1} \sum_{\rho + \rho' \leq 2} \int_0^u \delta^{2l-1} \| u^{\alpha/2} L^2 Z^3 \phi \|_{L^2(C_u)}^2 du' + I_4(\psi_0, \psi_1)^4
$$

(2.51)

It remains to control $W = \delta^{2l-1} \int \frac{|u|^2}{r} |LZ^3 \phi| \|LZ^3 \phi\|$. By Cauchy-Schwarz inequality, we have

$$
W \lesssim \delta^{2l-1} \left( \int \frac{\delta}{|u|^2} |LZ^3 \phi|^2 + \int \frac{1}{\delta} |LZ^3 \phi|^2 \right) 
= \delta^{2l} \int_{1-u}^u \frac{1}{|u|^2} |LZ^3 \phi|^2_{L^2(C_u)} du' + \delta^{2l-2} \int_0^u \| u \|^2 \|LZ^3 \phi\|_{L^2(C_u)}^2 du'.
$$

We use (2.47) to bound the first term in the last line. Since $\alpha < 1$, for sufficiently small $\delta$, we obtain

$$
W \lesssim I_4(\psi_0, \psi_1)^2 + \delta^{-1} \int_0^u \delta^{2l-1} \| u \|^2 \|LZ^3 \phi\|_{L^2(C_u)}^2 du'.
$$

(2.52)

By combining (2.50), (2.51) and (2.52), we obtain

$$
\delta^{2l-1} \int_{C_u} \| u \|^\alpha \|LZ^3 \phi\|^2 + \delta^{2l-1} \int_{C_u} \| u \|^\alpha \|\nabla Z^3 \phi\|^2 
\lesssim I_4(\psi_0, \psi_1)^4 + \delta^{1/4} M^3 + C(I_3(\psi_0, \psi_1)) M + \delta^{-1} \int_0^u \delta^{2l-1} \| u \|^2 \|LZ^3 \phi\|_{L^2(C_u)}^2 du'.
$$

The last term on the right-hand side can be removed by the Gronwall’s inequality so that

$$
\delta^{2l-1} \int_{C_u} \| u \|^\alpha \|LZ^3 \phi\|^2 + \delta^{2l-1} \int_{C_u} \| u \|^\alpha \|\nabla Z^3 \phi\|^2 
\lesssim I_4(\psi_0, \psi_1)^4 + \delta^{1/4} M^3 + C(I_3(\psi_0, \psi_1)) M.
$$

This finally proves that, for all $0 \leq l \leq 3$, we have

$$
\delta^{l-\frac{1}{2}} \| u \|^2 \|LZ^3 \phi\|_{L^2(C_u)} + \delta^{l-\frac{1}{2}} \|\nabla Z^3 \phi\|_{L^2(C_u)} \lesssim C(I_4(\psi_0, \psi_1)) + C(I_3(\psi_0, \psi_1)) M^{\frac{1}{4}} + \delta^{\frac{1}{4}} M^{\frac{3}{4}}.
$$

(2.53)

By combining this estimates with (2.53) and (2.40), for sufficiently small $\delta$, this finally proves

$$
E_{\leq 3}(u, u) + E_{\leq 3}(u, u) \leq C(I_4(\psi_0, \psi_1)).
$$

(2.54)

This is the end of the bootstrap argument and the Proposition 2.2 has been proved.
2.2. Higher Order Estimates. This subsection is devoted to prove a higher order analogue of Proposition 2.2.

Proposition 2.5. Given seed functions $\psi_0, \psi_1$ and a positive integer $n \geq 3$, there exists $\delta_0 > 0$, so that for all $\delta < \delta_0$, for all $u \in (0, u^*)$ and $\bar{u} \in (1 - u^*, u^*)$, we have

$$E_{\leq n}(u, \bar{u}) + E_{\leq n}(u, \bar{u}) \leq C(I_{n+1}(\psi_0, \psi_1)),$$

(2.55)

and

$$||\nabla Z_l Z_g^{-l}||_{L^\infty(S_{\pm\omega})} \lesssim \delta^{\frac{1}{2}-l} |u|^{-\frac{3}{2}-\frac{3}{2}l} C(I_{n+1}(\psi_0, \psi_1)), \quad 0 \leq l \leq k \leq n-3,$$

(2.56)

where $C(I_{n+1}(\psi_0, \psi_1))$ is a constant depending only on $I_{n+1}(\psi_0, \psi_1)$.

Remark 2.6. Although $C(I_n(\psi_0, \psi_1))$ and $\delta_0$ in the proposition may depend on the integer $n$, in the rest of the paper, we only need the result for $n = 12$.

We prove (2.55) and (2.56) together by induction on $n$. For $n = 3$, the proposition has been achieved in the previous subsection. For $n \geq 4$, we assume that the proposition holds for all $n'$ so that $n' \leq n - 1$. To prove for $n$, we first make the following bootstrap assumption: We choose a large constant $M$, so that

$$E_n(u, \bar{u}) + E_n(u, \bar{u}) \lesssim M,$$

(2.57)

for all $u \in (0, u^*)$ and $\bar{u} \in (1 - u^*, u^*)$. We remark that $M$ may depend on $\phi$ at the moment. We will show that, if $\delta$ is sufficiently small, then we can make $M$ depend only on $I_{n+1}(\psi_0, \psi_1)$. We also remark that the induction hypothesis is

$$E_{\leq n-1}(u, \bar{u}) + E_{\leq n-1}(u, \bar{u}) \lesssim C(I_{\leq n}(\psi_0, \psi_1)),$$

(2.58)

and

$$||\nabla Z_l Z_g^{-l}||_{L^\infty(S_{\pm\omega})} \lesssim \delta^{\frac{1}{2}-l} |u|^{-\frac{3}{2}-\frac{3}{2}l} C(I_{n}(\psi_0, \psi_1)), \quad 0 \leq l \leq k \leq n-4,$$

(2.59)

for all $u \in (0, u^*)$ and $\bar{u} \in (1 - u^*, u^*)$.

We claim that, together with the induction hypothesis (2.58) and (2.59), the bootstrap assumption (2.57) implies

$$||\nabla Z_l Z_g^{-2l}||_{L^\infty(S_{\pm\omega})} \lesssim \delta^{\frac{1}{2}-l} |u|^{-\frac{3}{2}-\frac{3}{2}l} M, \quad 0 \leq l \leq n-3,$$

(2.60)

for all $u \in (0, u^*)$ and $\bar{u} \in (1 - u^*, u^*)$.

The bound on $||\nabla Z_l Z_g^{-n-3l}||_{L^\infty(S_{\pm\omega})}$ is straightforward: we simply use Sobolev inequalities by affording two more $\Omega_{ij}$ derivatives. The derivation is exactly the same as for \ref{2.12}.

The bound on $||Z_l Z_g^{-n-3l}||_{L^\infty(S_{\pm\omega})}$ relies on the \ref{2.9}, i.e.

$$\square Z^n \phi = \sum_{p+q \geq n-3} Q(\nabla Z^p \phi, \nabla Z^q \phi).$$

(2.61)

According to the structure of null forms, we can rewrite it as the following inequality:

$$L|LZ^{n-3} \phi| \lesssim a|LZ^{n-3} \phi| + b,$$

(2.62)
where
\[ a = \frac{1}{r} + (|L\phi| + |\nabla \phi|), \]
and
\[ b = |\Delta Z^{n-3}\phi| + \frac{1}{r}|LZ^{n-3}\phi| + \sum_{p+q \leq n-3} (|LZ^p\phi||\nabla Z^q\phi| + |\nabla Z^p\phi||\nabla Z^q\phi|). \]

We claim that
\[ ||a||_{L^1_tL^\infty(S_{2,u})} \lesssim |u|^{-1} \delta^{-\frac{1}{4}-l} M, \]
\[ ||b||_{L^1_tL^\infty(S_{2,u})} \lesssim |u|^{-2} \delta^{-\frac{1}{4}-l} M. \tag{2.63} \]

To prove this claim, we first notice that all the terms have already been bounded by the induction hypothesis except for the top order terms, i.e. \(|\Delta Z^{n-3}\phi|, \frac{1}{r}|LZ^{n-3}\phi|, |LZ^{n-3}\phi||\nabla \phi|, |L\phi||\nabla Z^{n-3}\phi|\) and \(|\nabla \phi||\nabla Z^{n-3}\phi|\) appeared in \(b\). In view of the bound on \(||\nabla Z^p\phi||\nabla Z^q\phi||L^\infty(S_{2,u})||\) derived above, it suffices to bound \(||LZ^{n-3}\phi||\).

According to Sobolev inequality, we have
\[ ||LZ^{n-3}\phi||_{L^1_tL^\infty(S_{2,u})} \lesssim |u|^{-1} \sum_{0 \leq j \leq 2} ||\Omega^j LZ^{n-3}\phi||_{L^1_tL^2(S_{2,u})} \]
\[ \lesssim |u|^{-1} \delta^{\frac{j}{2}} \sum_{0 \leq j \leq 2} ||L\Omega^j Z^{n-3}\phi||_{L^1_tL^2(S_{2,u})} \]
\[ \lesssim |u|^{-1} \delta^{\frac{j}{2}} M. \]

We thus proved (2.63). By virtue of Gronwall’s inequality, (2.62) yields the desired estimates for \(LZ^l \phi\) in (2.60).

The estimates on \(||LZ^l Z^{n-3-l}\phi||_{L^\infty(S_{2,u})}\) relies on the use of equation (2.20). In fact, we have
\[ \Box Z^{n-3}\phi = \sum_{p+q \leq n-3} Q(\nabla Z^p\phi, \nabla Z^q\phi). \]

Let \(y = |u| ||LZ^l Z^{n-3-l}\phi||\). By computing \(L(u^2 ||LZ^l Z^{n-3-l}\phi||^2)\), we have
\[ Ly^2 \lesssim \left( \frac{\delta}{|u|^2} + \frac{\delta^{\frac{1}{2}}}{|u|^2} M \right) y^2 + \frac{\delta^{\frac{1}{2}-l}}{|u|^2} M. \]

By integrating this equation, we obtain
\[ ||u|| ||LZ^l Z^{n-3-l}\phi||(1 - u, \theta) \lesssim \delta^{-\frac{1}{4}-l} M. \tag{2.64} \]

Therefore, according to (2.7), we finally obtain
\[ ||LZ^l Z^{n-3-l}\phi||_{L^\infty(S_{2,u})} \lesssim \delta^{-\frac{1}{4}-l} \frac{I_{n+1}(\psi_0, \psi_1) + \delta^{-\frac{1}{4}-l} |u|^{-1} M.} \]

To finish the proof of Proposition 2.55, it remains to improve the constant \(M\) in (2.57). The procedure is exactly the same as for the proof of \(E_3(u, u)\) and \(E_3(u, u)\) in previous subsection.

We replace \(\phi\) by \(Z^n\phi\) and take \(X = L\) in (1.4) and (2.20), this yields
\[ \delta^{2l} \int_{C_u} |\nabla Z^n\phi|^2 + \delta^{2l} \int_{C_{2,u}} |LZ^n\phi|^2 \lesssim I_{n+1}(\psi_0, \psi_1) + \delta^{2l} \int_{D_{2,u}} Q(\nabla Z^n\phi, \nabla Z^n\phi) \]
\[ + \sum_{p+q \leq n, p+q \leq n} \delta^{2l} \int_{D_{2,u}} Q(\nabla Z^p\phi, \nabla Z^q\phi) |LZ^n\phi| + \delta^{2l} \int_{D_{2,u}} \frac{1}{r} |LZ^n\phi| \cdot |LZ^n\phi|. \tag{2.65} \]
We rewrite the right-hand side \( I_{n+1}(\psi_0, \psi_1)^2 + S + T + W \), where \( S, T \) and \( W \) denote the three bulk integral terms in (2.65). We will bound \( S, T \) and \( W \) one by one.

We start with \( S \) which bounded by the sum of the following terms:

\[
S_1 = \delta^{2l} \int_{D_{u,u}} |L\phi| (|LZ^n\phi| + |\nabla Z^n\phi|) |LZ^n\phi|, \\
S_2 = \delta^{2l} \int_{D_{u,u}} |L\phi| (|\nabla Z^n\phi| + |LZ^n\phi|) |LZ^n\phi|, \\
S_3 = \delta^{2l} \int_{D_{u,u}} |\nabla\phi| (|LZ^n\phi| + |LZ^n\phi| + |\nabla Z^n\phi|) |LZ^n\phi|.
\]

In view of the forms of \( S_1, S_2 \) and \( S_3 \) appeared in the subsection for the estimates on \( E_3(u, u) \) and \( E_3(u, u) \), i.e. the derivation of the inequalities (2.43), (2.44) and (2.45), we can proceed exactly in the same way (replace all the \( Z^3\phi \) by \( Z^n\phi \)). We take \( S_1 \) as an example to illustrate the process: by the bound on \( L\phi \) in \( L^\infty(S_{u,u}) \), we have

\[
S_1 \lesssim \delta^{2l-\frac{1}{2}} \int_{D_{u,u}} |u|^{-1} |\nabla Z^n\phi| |LZ^n\phi| + \delta^{2l-\frac{1}{2}} \int_{D_{u,u}} |u|^{-\frac{1+\alpha}{2}} \cdot (|u|^\frac{3}{2} LZ^n\phi) \cdot (|u|^{-\frac{1}{2}} |LZ^n\phi|) \\
= S_{11} + S_{12}.
\]

We bound \( S_{11} \) exactly as the derivation for (2.42):

\[
S_{11} \lesssim \delta^{2l-1} \int_{D_{u,u}} |\nabla Z^n\phi|^2 + \delta^{2l} \int_{D_{u,u}} |u|^{-2} |L\phi|^2 \\
\lesssim \int_0^u \frac{1}{\delta} \cdot \delta^{2l} \int_{C_{u'}} |\nabla Z^n\phi|^2 \, du' + \int_1^u |u'|^{-2} \delta^{2l} \int_{C_{u'}} |LZ^n\phi|^2 \, du'.
\]

Similarly, we have

\[
S_{12} \lesssim \delta^{\frac{1}{2}} M^2.
\]

We give the final result on as follows:

\[
S \lesssim \delta^{\frac{1}{2}} M^2 + \int_0^u \frac{1}{\delta} \cdot \delta^{2l} \int_{C_{u'}} |\nabla Z^n\phi|^2 \, du' + \int_1^u |u'|^{-2} \delta^{2l} \int_{C_{u'}} |LZ^n\phi|^2 \, du'.
\]

(2.66)

For \( T \), we have

\[
T \lesssim \sum_{p \leq n-1, q \leq n-1} \delta^{2l} \int_{D_{u,u}} |\partial Z^p\phi||\partial Z^q\phi||LZ^n\phi|.
\]

where \( \partial \in \{\nabla, L\} \) and \( \partial_g \in \{\nabla, L\} \). The estimate for \( T \) follows exactly the same as we derive (2.29). we have:

\[
T \lesssim \delta^{1/2} M^3
\]

(2.67)

For \( W = \delta^{2l} \int_{D_{u,u}} \frac{1}{r} |LZ^n\phi| |LZ^n\phi| \), we have

\[
W \lesssim \int_{1-u}^u \frac{1}{|u'|^{2-\alpha}} \cdot \delta^{2l+1} \cdot \|LZ^n\phi\|^2_{L^2(C_{u'})} \, du' + \delta^{2l-1} \int_0^u \|u|^\frac{3}{2} LZ^n\phi\|_{L^2(C_{u'})}^2 \, du' \\
\lesssim \delta M^2.
\]
The estimates on $S$, $T$ and $W$, together with (2.65), imply that
\[
\delta^{2l} \int_{C_u} |\nabla Z^n \phi|^2 + \delta^{2l} \int_{C_u} |LZ^n \phi|^2 \lesssim I_{n+1}(\psi_0, \psi_1)^2 + \delta^{\frac{3}{2}} M^3
\]
\[
+ \int_0^u \frac{1}{\delta} \cdot \delta^{2l} \int_{C_u'} |\nabla Z^n \phi|^2 du' + \int_{1-u}^u |u'|^{-2} \delta^{2l} \int_{C_u} |LZ^n \phi|^2 du'.
\]
The last two terms can be removed by Gronwall’s inequality. Therefore, we obtain
\[
\delta^l \|\nabla Z^n_{g} Z^{n-1} f\|_{L^2(C_u')} + \delta^l \|L Z^n_{g} Z^{n-1} f\|_{L^2(C_u)} \lesssim I_{n+1}(\psi_0, \psi_1) + \delta^{\frac{3}{2}} M^{\frac{3}{2}}.
\] (2.68)

We now change the multiplier vector field to $\nu^a L$ to derive
\[
\delta^{2l-1} \int_{C_u} |u|^a |LZ^n \phi|^2 + \delta^{2l-1} \int_{C_u} |u|^a |\nabla Z^n \phi|^2 \lesssim I_{n+1}(\psi_0, \psi_1)^2 + \delta^{2l-1} \int_{D_{z=\phi}} |u|^a \phi(\nabla Z^n \phi, \nabla \phi) LZ^n \phi|
\]
\[
+ \sum_{p+q \leq n-1} \delta^{2l-1} \int_{D_{z=\phi}} |u|^a \phi(\nabla Z^p \phi, \nabla Z^q \phi) LZ^n \phi + \delta^{2l-1} \int_{D_{z=\phi}} |u|^a \phi LZ^n \phi \cdot LZ^n \phi|
\]
We rewrite the above inequality as
\[
\delta^{2l-1} \int_{C_u} |u|^a |LZ^n \phi|^2 + \delta^{2l-1} \int_{C_u} |u|^a |\nabla Z^n \phi|^2 \lesssim I_{n+1}(\psi_0, \psi_1)^2 + S + T + W.
\] (2.69)
where $S$, $T$ and $W$ denote the three bulk integral terms. We bound $S$, $T$ and $W$ one by one.

To bound $S$, we can follow exactly the same way as the derivation for (2.50) (we simply replace all the $Z^3 \phi$’s by $Z^n \phi$), this gives
\[
S \lesssim \delta^{\frac{3}{2}} M^3 + C(I_{n+1}(\psi_0, \psi_1)) M.
\]

To bound $T$, we can follow exactly the same way as the derivation for (2.51) (we simply replace all the $Z^3 \phi$’s by $Z^n \phi$ and $Z^2 \phi$’s by $Z^{n-1} \phi$). We obtain
\[
T \lesssim \delta^{-1} \sum_{p+q \leq n-1} \int_0^u \delta^{2l-1} |u|^a \phi LZ^n \phi|^2_{L^2(C_{z=\phi})} du' + I_{n+1}(\psi_0, \psi_1)^4
\]
To bound $W$, we can follow exactly the same way as the derivation for (2.52) by replacing the $Z^3 \phi$’s by $Z^n \phi$, this gives
\[
W \lesssim I_{n+1}(\psi_0, \psi_1)^2 + \delta^{-1} \int_0^u \delta^{2l-1} |u|^a \phi LZ^n \phi|^2_{L^2(C_{z=\phi})} du'.
\]

The estimates on $S$, $T$ and $W$, together with (2.65), imply that
\[
\delta^{2l} \int_{C_u} |u|^a |LZ^n \phi|^2 + \delta^{2l} \int_{C_u} |u|^a |\nabla Z^n \phi|^2 \lesssim I_{n+1}(\psi_0, \psi_1)^2 + \delta^{\frac{3}{2}} M^3 + C(I_{n}(\psi_0, \psi_1)) M + \delta^{-1} \int_0^u \delta^{2l-1} |u|^a \phi LZ^n \phi|^2_{L^2(C_{z=\phi})} du'
\]
By the Gronwall’s inequality again, for $l \leq n$, we finally obtain
\[
\delta^{l-\frac{3}{2}} |u|^a \phi LZ^n_{g} Z^{n-1} f\|_{L^2(C_u)} + \delta^{l-\frac{3}{2}} |\nabla Z^n_{g} Z^{n-1} f\|_{L^2(C_u)} \lesssim C(I_{n+1}(\psi_0, \psi_1)) + C(I_{n}(\psi_0, \psi_1)) M^{\frac{3}{2}} + \delta^{\frac{3}{2}} M^{\frac{3}{2}}.
\] (2.70)

For sufficiently small $\delta$, the estimate (2.68) and (2.70) show that
\[
E_n(u, u) + E_n(u, u) \lesssim C(I_{n+1}(\psi_0, \psi_1)).
\]
This completes the bootstrap argument and the Proposition 2.5 has been proved.

2.3. **Existence based on a priori estimates.** The existence of solutions of (1.2) follows immediately from the a priori energy estimates derived previously. Since the procedure is standard, we only give a sketch of the proof in this subsection.

We start with solving local solution for Cauchy problem with data prescribed on \( \Sigma_1 \) with \( 1 - \delta < r \leq 1 \). Therefore, we obtain a local solution confined in the region bounded by \( C_\delta \) and \( C_1 \). In particular, on a neighborhood of \( S_{1,0} \) on the incoming cone \( C_1 \), the solution has been constructed.

We then use \( C_0 \) and \( C_1 \) as initial hypersurfaces. The classical local existence result [9] of Rendall can be applied in this situation. Therefore, we know that there exists a solution in the entire spacetime neighborhood (which lies in the domain of dependence of \( C_0 \) and \( C_1 \)) of \( S_{1,0} \). Combined with the local solution of the Cauchy problem, we have constructed a local solution for \( t \in [1, 1 + \epsilon] \) for some small \( \epsilon \).

Since the a priori energy estimates (as well as the companying \( L^\infty \) estimates) depends only on the size of the seed data on \( \Sigma_1 \), this solution is well behaved on \( \Sigma_1 \). Therefore, we can use this as initial surface (instead of \( \Sigma_1 \)) to repeat the above argument. Eventually, we obtain an solution in the entire short pulse region.

2.4. **Improved Estimates on \( C_\delta \).** Recall that, given \( n \geq 12 \), in the short pulse region, we have derived the following a priori \( L^\infty \) estimates on the solution \( \phi \):

\[
\begin{align*}
\| \nabla Z_0^l Z_g^{k-l} \phi \|_{L^\infty(S_{\infty,0})} & \lesssim \delta^l |u|^{-\frac{n}{2} - \frac{3}{2}} C(I_{n+1}(\psi_0, \psi_1)), \quad 0 \leq l \leq k \leq n - 3, \\
\| L Z_0^l Z_g^{k-l} \phi \|_{L^\infty(S_{\infty,0})} & \lesssim \delta^l |u|^{-2} C(I_{n+1}(\psi_0, \psi_1)), \quad 0 \leq l \leq k \leq n - 3, \\
\| L Z_0^l Z_g^{k-l} \phi \|_{L^\infty(S_{\infty,0})} & \lesssim \delta^l |u|^{-1} C(I_{n+1}(\psi_0, \psi_1)), \quad 0 \leq l \leq k \leq n - 3.
\end{align*}
\]

The goal of this section is to improve these bounds for the solution on \( C_\delta \). More precisely, we will prove that

**Proposition 2.7.** On \( C_\delta \), for sufficiently small \( \delta \), we have

\[
\begin{align*}
\| \nabla Z_0^l Z_g^{k-l} \phi \|_{L^\infty(S_{\infty,\delta})} & \lesssim \delta^l |u|^{-\frac{n}{2} - \frac{3}{2}} C(I_{n+1}(\psi_0, \psi_1)), \quad 0 \leq l \leq k \leq n - 3, \\
\| L Z_0^l Z_g^{k-l} \phi \|_{L^\infty(S_{\infty,\delta})} & \lesssim \delta^l |u|^{-2} C(I_{n+1}(\psi_0, \psi_1)), \quad 0 \leq l \leq k \leq n - 3, \\
\| L Z_0^l Z_g^{k-l} \phi \|_{L^\infty(S_{\infty,\delta})} & \lesssim \delta^l |u|^{-1} C(I_{n+1}(\psi_0, \psi_1)), \quad 0 \leq l \leq k \leq n - 3.
\end{align*}
\]

Notice the power of \( \delta \) for \( L Z_0^l Z_g^{k-l} \phi \) has been modified to \( \frac{1}{2} \). Since \( C_0 \) is also the outer boundary of the small data region (region I), the smallness (in terms of \( \delta \)) of the solution stated in the proposition is indispensable for the construction of a global solution in the small data region. As we mentioned in the introduction, the proof relies on the following observation: on the \( S_{1-\delta, \delta} \) or equivalently the lower boundary of \( C_\delta \), the data are identically zero. This is because that the data are compactly supported on \( \Sigma_1 \) between \( S_{1-\delta, \delta} \) and \( S_{1,0} \). Therefore, even for bad derivatives of \( \phi \), it is small at least initially. The idea of the proof is to integrate along \( L \) direction to show that the smallness indeed propagates.

We use an induction argument on the pair \((l, k)\) \((0 \leq k \leq n - 3, 0 \leq l \leq k)\) to prove (2.71). First of all, we give an order on the set of such pairs: we say that \((l', k') < (l, k)\) if one of the following holds: (1) \(k' < k\) or (2) \(l' < l, k = k'\). We do the induction with respect to this order.

For \((l, k) = (0, 0)\), the bounds on \( L \phi \) and \( \nabla \phi \) are clear. It remains to prove that

\[
\| L \phi \|_{L^\infty(S_{\infty, \delta})} \lesssim \delta^l |u|^{-1} C(I_{n}(\psi_0, \psi_1)).
\]

Recall that, in (2.17), we have obtained

\[
\| |u| L \phi |(u, u, \theta) - C |u| L \phi |(1 - u, u, \theta) | \lesssim \delta^l M.
\]
In view of the higher order energy estimates derived in the previous subsection, the constant $M$ should be replaced by $C(I_n(\psi_0, \psi_1))$. Let $u = \delta$, then the second term vanishes on the initial sphere $S_{1 - \delta, \delta}$. This gives the desired estimates on $\|L^2g\|_{L^\infty(S_{\delta, \delta})}$.

For $(l, k) = (0, k)$, we can use (2.64) to obtain the desired estimates in a similar way.

We assume that for all $(l', k') < (l, k)$, we have

$$|
abla \Omega^{l'}_{k'} Z^l_g \phi| \lesssim \frac{1}{4} |\Omega^{l'}_{k'} Z^l_g \phi| \lesssim \frac{1}{4} |\Omega^{l'}_{k'} Z^l_g \phi| \lesssim \frac{1}{4} \delta \frac{|u|}{|\Omega|}^{-1} C(I_{n+1}(\psi_0, \psi_1)).$$

Since $\Omega \in Z_g$, we can use induction hypothesis (since we can reduce $l$), the first term is bounded by

$$\sum_{\partial \in \{L, L, \psi\}} \frac{1}{4} |\partial Z^{l-1}_{k} Z^{k-1}_g \phi| \lesssim \frac{1}{4} |\partial Z^{l-1}_{k} Z^{k-1}_g \phi| \lesssim \frac{1}{4} \delta \frac{|u|}{|\Omega|}^{-1} C(I_{n+1}(\psi_0, \psi_1)).$$

For the second term, notice that $\Omega Z = \delta, \delta$), we now reduce the estimates to the above induction hypothesis. Because we have already proved the case for $(l, k) = (0, k)$, so we can assume in addition that $l \geq 1$.

We first bound $\nabla Z^{l-1}_{k} Z^{k-1}_g \phi$. In fact, we have

$$\|\nabla \Omega^{l'}_{k'} Z^l_g \phi\|_{L^\infty(S_{\delta, \delta})} \lesssim \delta \frac{|u|}{|\Omega|}^{-1} C(I_{n+1}(\psi_0, \psi_1)).$$

For $(l, k)$, we now reduce the estimates to the above induction hypothesis. Because we have already proved the case for $(l, k) = (0, k)$, so we can assume in addition that $l \geq 1$.

We turn to the bound on $\nabla Z^{l-1}_{k} Z^{k-1}_g \phi$. Evidently, it is bounded by $\sum_{\partial \in \{L, L, \psi\}} |\partial Z^{l-1}_{k} Z^{k-1}_g \phi|$. When $\partial = \nabla$ in the sum, it can be bounded directly by the bound on $\nabla Z^{l-1}_{k} Z^{k-1}_g \phi$ derived above. Therefore, it suffices to bound $LL Z^{l-1}_{k} Z^{k-1}_g \phi$ and $LL Z^{l-1}_{k} Z^{k-1}_g \phi$.

For $LL Z^{l-1}_{k} Z^{k-1}_g \phi$, according to (2.20) (where we use $Z^{l-1}_{k} Z^{k-1}_g$ as commutator vector field), we have

$$-LL Z^{l-1}_{k} Z^{k-1}_g \phi + \Delta Z^{l-1}_{k} Z^{k-1}_g \phi = \frac{1}{r} (LL Z^{l-1}_{k} Z^{k-1}_g \phi - LL Z^{l-1}_{k} Z^{k-1}_g \phi) + \sum_{p+q \geq k-1} Q(\nabla Z^p \phi, \nabla Z^q \phi).$$

The second term on the left-hand side can be bounded by $\nabla Z^{l-1}_{k} Z^{k-1}_g \phi$. The terms on the right-hand side are all of lower degrees ($< k$) so that they are bounded by the induction hypothesis. Therefore, we have

$$\|LL Z^{l-1}_{k} Z^{k-1}_g \phi\|_{L^\infty(S_{\delta, \delta})} \lesssim \delta \frac{|u|}{|\Omega|}^{-1} C(I_{n+1}(\psi_0, \psi_1)).$$

For $LL Z^{l-1}_{k} Z^{k-1}_g \phi$, we use the following identity:

$$LL Z^{l-1}_{k} Z^{k-1}_g \phi = L \left( \frac{1}{4} (S Z^{l-1}_{k} Z^{k-1}_g \phi - u LL Z^{l-1}_{k} Z^{k-1}_g \phi) \right).$$
The second term on the right-hand side can be bounded directly by the bound on $LLZ_{b}^{l-1}Z_{g}^{k-l}\phi$ just derived. Therefore, it suffices to control the contribution from the first term, i.e.

\[
L\left(\frac{1}{u}SZ_{b}^{l-1}Z_{g}^{k-l}\phi\right) = -\frac{1}{u^{2}}SZ_{b}^{l-1}Z_{g}^{k-l}\phi + \frac{1}{u}LSZ_{b}^{l-1}Z_{g}^{k-l}\phi
\]

\[
= \frac{1}{|g|^{2}}SZ_{b}^{l-1}Z_{g}^{k-l}\phi + \frac{1}{u}LLZ_{b}^{l-1}Z_{g}^{k-l+1}\phi.
\]

For $A$, by rewriting $S$ as $uL + u\bar{L}$, we can use induction hypothesis for $(l,k-1)$; for $B$, we can use induction hypothesis for $(l-1,k)$.

Hence, we have obtained the desired estimates for $LLZ_{b}^{l}Z_{g}^{k-l}\phi$.

Finally, to bound $LLZ_{b}^{l}Z_{g}^{k-l}\phi$, we use the equation

\[
-LLZ_{b}^{l}Z_{g}^{k-l}\phi + \Delta Z_{b}Z_{g}^{k-l}\phi = -\frac{1}{r}(LLZ_{b}^{l}Z_{g}^{k-l}\phi - LLZ_{b}^{l}Z_{g}^{k-l}\phi) + \sum_{p+q\leq k}Q(\nabla Z^{p}\phi, \nabla Z^{q}\phi).
\]

We rewrite this as

\[
-LLZ_{b}^{l}Z_{g}^{k-l}\phi - \frac{1}{r}LLZ_{b}^{l}Z_{g}^{k-l}\phi + Q(\nabla \phi, \nabla Z_{b}^{l}Z_{g}^{k-l}\phi)
\]

\[
= -\Delta Z_{b}Z_{g}^{k-l}\phi - \frac{1}{r}LLZ_{b}^{l}Z_{g}^{k-l}\phi + \sum_{p+q\leq k}Q(\nabla Z^{p}\phi, \nabla Z^{q}\phi).
\]

All the terms on the right-side have been controlled in previous steps. Therefore, it is straightforward to see that the right-hand side is bounded by $C(I_{n+1}(\psi_{0}, \psi_{1}))|u|^{-2}\delta^{4}$. We now can mimic the proof for (2.72) by defining $y = uLLZ_{b}^{l}Z_{g}^{k-l}\phi$, this leads to

\[
|LLZ_{b}^{l}Z_{g}^{k-l}\phi(u, \delta, \theta) - \frac{C(1-\delta)}{u}LLZ_{b}^{l}Z_{g}^{k-l}\phi(1-\delta, \delta, \theta)| \lesssim \delta^{4}|u|^{-1}C(I_{n+1}(\psi_{0}, \psi_{1})).
\]

Taking into account of the vanishing property of $LLZ_{b}^{l}Z_{g}^{k-l}\phi$ on $S_{1-\delta, \delta}$, we complete the proof of Proposition 2.71.

**Remark 2.8.** For applications in the next section, we only need a slightly weakened (in decay) version of the estimates from Proposition 2.71:

\[
||\nabla Z_{b}^{l}Z_{g}^{k-l}\phi||_{L^{\infty}(S_{\delta, \delta})} + ||LLZ_{b}^{l}Z_{g}^{k-l}\phi||_{L^{\infty}(S_{\delta, \delta})} \lesssim \delta^{4}|u|^{-2}C(I_{n+1}(\psi_{0}, \psi_{1})),
\]

\[
||LLZ_{b}^{l}Z_{g}^{k-l}\phi||_{L^{\infty}(S_{\delta, \delta})} \lesssim \delta^{4}|u|^{-1}C(I_{n+1}(\psi_{0}, \psi_{1})),
\]

where $0 \leq l \leq k \leq n - 3$.

3. Small data region

In this section, we construct solutions in the entire small data region, i.e. region I. The approach is a modification of the classical approach with additional difficulties arising from the boundary $C_{\delta}$.

3.1. Klainerman-Sobolev inequality revisited. We first introduce notations needed for the statement of the Klainerman-Sobolev inequality. We use $\Sigma_{t}$ to denote the constant time slices in the small data region, i.e. for a fixed $t \in (1, +\infty)$,

\[
\Sigma_{t} := \{(x,t)|t - r \geq \delta\}.
\]

This is a ball of radius $t - \delta$. We recall that we use $\Sigma_{t}$ to denote the entire $t = 1$ hyperplane. Given a point $(t, x) \in \Sigma_{t}$ (assuming that $x \neq 0$), we use the $(t, B(t, x))$ to denote its corresponding boundary.
point, i.e. \((t, B(t, x))\) is the unique point on the boundary of \(\Sigma_t\) (also on \(C_\delta\)) which is the intersection of the boundary of \(\Sigma_t\) with the ray emanated from \((t, 0)\) and passing from \((t, x)\). We now state the Klainerman-Sobolev inequality:

**Proposition 3.1.** For all \(f \in C^\infty(\mathbb{R}^{3+1})\), \(t > 1\) and a point \((t, x)\) in the small data region, we have

\[
|f(t, x)| \lesssim \frac{1}{(1 + |u|)^{1/2}} |f(t, B(t, x))| + \frac{1}{(1 + |u|)^{1/2}} \sum_{Z \in Z, k \leq 3} \|Z^k f\|_{L^2(\Sigma_t)}.
\]  

We recall the following identities on \(\mathbb{R}^{3+1}\):

\[
\begin{align*}
\partial_t &= \frac{1}{t - r} \left( \frac{t}{t + r} S - \sum_{i=1}^3 \frac{x_i}{t + r} \Omega_{0i} \right), \\
\partial_i &= -\frac{1}{t - r} \left( \frac{x_i}{t + r} S - \frac{t}{t + r} \Omega_{0i} - \sum_{j=1}^3 \frac{x_j}{t + r} \Omega_{ij} \right), \\
\partial_r &= \frac{1}{t - r} \left( -\frac{r}{t + r} S + \sum_{i=1}^3 \frac{tx_i}{(t + r)r} \Omega_{0i} \right).
\end{align*}
\]  

Therefore, schematically, in terms of \(Z_g \in Z_g\), we write the above identities as

\[
\partial = \frac{1}{|t - r|} Z_g
\]

Near light cone \(C_0\), i.e. the hypersurface \(t = r\), we can take \(Z\) to be \(\partial_i\) or \(\partial_t\), therefore, schematically we have

\[
\partial = \left( 1 + \frac{1}{u} \right) Z.
\]

We remark that this schematic expression means, for any function \(f\), we have the following pointwise estimates:

\[
|\partial f| \lesssim \left( 1 + \frac{1}{u} \right) |Z f|.
\]

We start the proof of (3.1). Let \(\chi\) be a non-negative smooth cut-off function on \(\mathbb{R}_{\geq 0}\) so that \(\chi\) is supported in \([0, \frac{1}{2}]\) and \(\chi \equiv 1\) on \([0, \frac{1}{4}]\). We decompose \(f(t, x)\) as

\[
f(t, x) = f_1(t, x) + f_2(t, x) = \chi \left( \frac{x}{t} \right) f(t, x) + (1 - \chi \left( \frac{x}{t} \right)) f(t, x).
\]

Therefore, the function \(f_1(t, x)\) is supported in region

\[
D_1 = \{(t, x) \mid 2r \leq t, t \geq 1\},
\]

which is far away from the cone \(C_\delta\); the function \(\phi_2(t, x)\) is supported in region

\[
D_2 = \{(t, x) \mid t - r \geq \delta, 4r \geq t, t \geq 1\}
\]

which is close to the cone \(C_\delta\).

We first bound \(f_1(t, x)\) in region \(D_1\). In the rest of the subsection, we regard \(t\) as a fixed large parameter. Let

\[
\tilde{f}_1(x) = f_1(t, tx) = f(t, tx) \chi(x),
\]
therefore, for a given positive integer \( m \), we have

\[
\| \partial^m \tilde{f}_1 \|_{L^2(\mathbb{R}^3)}^2 = \int_{\mathbb{R}^3} |\partial^m (f(t, tx)\chi(x))|^2 dx \\
\lesssim \sum_{j=0}^m \int_{\mathbb{R}^3} |\partial^j (f(t, tx))|^2 |\nabla^{m-j}\chi|^2 dx \\
\lesssim \sum_{j=0}^m \int_{\mathbb{R}^3} |\partial^j (\partial^i f)(t, tx)|^2 dx.
\]

Recall that we have \( \partial = \frac{1}{1+|t-r|} Z \). In the region \( D_1 \), we have \( t \geq 2r \), hence \( |t-r| \sim t \). Therefore, in \( D_1 \), we have

\[
|t \partial f| \lesssim |Zf|.
\]

Thus, we have

\[
\| \partial^m \tilde{f}_1 \|_{L^2(\mathbb{R}^3)}^2 \lesssim \sum_{j \leq m, \alpha, \in \mathbb{Z}} \int_{\mathbb{R}^3} |Z^j f|^2(t, tx) dx \\
= t^{-3} \sum_{j \leq m, \alpha, \in \mathbb{Z}} \int_{\mathbb{R}^3} |Z^j f(t, y)|^2 dy.
\]

Therefore, according to the classical Sobolev inequality on \( \mathbb{R}^3 \), we obtain

\[
\| f_1 \|_{L^\infty(\Sigma_r)} = \| \tilde{f}_1 \|_{L^\infty(\Sigma_r)} \lesssim \frac{1}{t^2} \sum_{k \leq 2, \alpha, \in \mathbb{Z}} \| Z^k f(t, \cdot) \|_{L^2(\Sigma_r)}. \tag{3.3}
\]

We turn to the estimates on \( f(t, x) \) in the region \( D_2 \). On the hyperplane \( \Sigma_r \), we draw a line from the origin and the point \((x, t)\). When a point moves along the radial direction on this line, it hits the characteristic boundary of \( C_\delta \) at one point \((t, B(t, x))\). By integrating \( \partial_r ((1 + |t-r|)f^2(t, x)) \) from \((t, B(t, x))\) to \((t, x)\), we obtain

\[
(1 + |t-r|)f^2(t, x) = (1 + \delta)f^2(t, B(t, x)) + \int_r^{t-\delta} \partial_r ((1 + |t-r|)f^2(t, x)) dr \\
= (1 + \delta)f^2(t, B(t, x)) + \int_r^{t-\delta} -f^2(t, x) + 2(1 + |t-r|)f(t, x)\partial_r f(t, x) dr
\]

For the integrand in the last line, we apply the classical Sobolev inequalities on spheres \( S_{t-r} \) (the sphere of radius \( r \) on \( \Sigma_r \)). Therefore, we obtain

\[
((1 + |t-r|)f^2(t, x) \lesssim f^2(t, B(t, x)) + \int_r^{t-\delta} \frac{1}{r^2} \sum_{|\alpha| \leq 2} \| \Omega^\alpha f \|_{L^2(S_{t-r})}^2 dr \\
+ \int_r^{t-\delta} \frac{1 + |t-r|}{r^2} \sum_{|\alpha|, |\beta| \leq 2} \| \Omega^\alpha f \|_{L^2(S_{t-r})} \| \Omega^\beta \partial_r f \|_{L^2(S_{t-r})} dr.
\]

Since \( |t-r|\partial_r \lesssim Z \), we have

\[
((1 + |t-r|)f^2(t, x) \lesssim f^2(t, B(t, x)) + \int_r^{t-\delta} \frac{1}{r^2} \sum_{k \leq 4, \alpha, \in \mathbb{Z}} \| Z^k f \|_{L^2(S_{t-r})}^2 dr \\
= f^2(t, B(t, x)) + \frac{1}{r^2} \sum_{k \leq 4, \alpha, \in \mathbb{Z}} \| Z^k f(t, \cdot) \|_{L^2(\Sigma_r)}^2. \tag{3.4}
\]
The estimates (3.3) and (3.4) together give the desired estimates (3.1) and we complete the proof.

3.2. A priori energy estimates. For a \( t \in (1, +\infty) \), we will use \( \Sigma_t = \{(x, t) \mid t - r \geq \delta\} \) to denote the constant time slices in the small data region. For \( k \in \mathbb{Z}_{\geq 0} \) and \( t > 1 \), we introduce the following energy norms:

\[
E_k(t) = \left( \sum_{z \in Z} \int_{\Sigma_z} |\partial_t Z^k \phi|^2 + \sum_{j=1}^{3} |\partial_j Z^k \phi|^2 \right)^{\frac{1}{2}},
\]

\[
E_{\leq k}(t) = \left( \sum_{0 \leq j \leq k} E_j(t)^2 \right)^{\frac{1}{2}}.
\]

We use \( D_{t, \delta} \) to denote the space-time region bounded by \( \Sigma_t, \Sigma_0 \) and \( C_\delta \). This region is obviously foliated by the constant time foliation \( \{ \Sigma_\tau \mid \tau \in [1, t]\} \) and this foliation is one of the foliations we use to derive energy estimates. The second foliation is the null foliation of outgoing null cones \( \{ C_u \mid u \in [\delta, t/2]\} \). This foliation is depicted as follows:

\[ t > 1 \]

\[ \Sigma_t \]

\[ C_{\delta} \]

\[ C_{\delta} \]

\[ C_{\delta} \]

\[ D_t \]

Whenever there is no confusion, we still use \( C_u \) to denote \( C_u \cap D_t \). We use \( D_{t, u} \) to denote the space-time region bounded by \( \Sigma_t, \Sigma_1 \) and \( C_u \). This is a truncated solid light cone in \( \mathbb{R}^{3+1} \). We use \( \Sigma_{1, u} \) and \( \Sigma_{t, u} \) to denote its bottom and top respectively. We remark that the bottom can be a single point.

Recall that (assuming that the solution \( \phi \) exists up to time \( t \)), for \( k \geq 0 \) and \( |\alpha| = k \), we have

\[
\Box Z^k \phi = \sum_{p+q \leq k} Q(\nabla Z^p \phi, \nabla Z^q \phi).
\]

We multiply both sides by \( \partial_t Z^k \phi \) and we then integrate over \( D_{t, u} \). This leads to the following energy identity:

\[
\int_{\Sigma_{t, u}} |\partial_t Z^k \phi|^2 + \sum_{j=1}^{3} |\partial_j Z^k \phi|^2 = \int_{\Sigma_{1, u}} |\partial_t Z^k \phi|^2 + \sum_{j=1}^{3} |\partial_j Z^k \phi|^2 + \int_{C_u} |LZ^k \phi|^2 + |\nabla Z^k \phi|^2
\]

\[ + \sum_{p+q \leq k} \int_{D_{t, u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi.\]

Recall that we use \( \partial \in \{ L, L, \nabla \} \) to denote a generic derivative and use \( \partial_g \in \{ L, \nabla \} \) to denote a good derivative. Therefore, by using \( |\partial Z^k \phi|^2 \) as a shorthand notation for \( |\partial_t Z^k \phi|^2 + \sum_{j=1}^{3} |\partial_j Z^k \phi|^2 \) and using \( |\partial_g Z^k \phi|^2 \) as a shorthand notation for \( |LZ^k \phi|^2 + |\nabla Z^k \phi|^2 \), we have

\[
\int_{\Sigma_{t, u}} |\partial Z^k \phi|^2 = \int_{\Sigma_{1, u}} |\partial Z^k \phi|^2 + \int_{C_u} |\partial Z^k \phi|^2 + \sum_{p+q \leq k} \int_{D_{t, u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi.
\]

In applications, since the data prescribed on \( \Sigma_{1, u} \) are trivial, we have

\[
\int_{\Sigma_{1, u}} |\partial Z^k \phi|^2 = \int_{C_u} |\partial Z^k \phi|^2 + \sum_{p+q \leq k} \int_{D_{t, u}} Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi. \tag{3.6}
\]
Before we state the main estimates of the section, we first compute the energy flux \( \int_{C_\delta} |\partial_y Z^k \phi|^2 \) through the outermost cone \( C_\delta \). According to (2.71), for \( k \leq n - 2 \), we have \( |\partial_y Z^k \phi| \lesssim \delta^{\frac{1}{2}} |u|^{-\frac{3}{2} - \frac{n}{2}} C(I_{n+1}(\psi_0, \psi_1)) \), therefore,

\[
\int_{C_\delta} |\partial_y Z^k \phi|^2 \lesssim \delta^{\frac{1}{2}} C(I_{n+1}(\psi_0, \psi_1)), \tag{3.7}
\]

where we still use \( C(I_{n+1}(\psi_0, \psi_1)) \) to denote \( C(I_{n+1}(\psi_0, \psi_1))^2 \).

**Proposition 3.2.** Under the same assumptions as in the previous section, for sufficiently small \( \delta \), there exists a unique global future in time solution \( \phi \) of (1.2) on the small data region, so that together with the solution constructed in the short pulse region, we have a unique future in time solution \( \phi \). Moreover, this solution \( \phi \) on the small data region enjoys the following energy estimates:

\[
\overline{E}_{\leq n}(t) \lesssim \delta^{\frac{1}{2}} C(I_{n+1}(\psi_0, \psi_1)), \tag{3.8}
\]

for all \( t > 1 \).

**Remark 3.3.** The existence of solutions in the small data region follows from the a priori estimate (3.8). Since the argument is routine, we will not pursue this point here.

We use a bootstrap argument to prove the proposition. We assume that the solution exists up to time \( t \) and for all \( 1 \leq t' \leq t \), we have

\[
\overline{E}_{\leq 7}(t') \lesssim M \delta^{\frac{1}{2}}. \tag{3.9}
\]

It suffices to show that we can indeed choose \( M \) so that it depends only on \( \psi_0 \) and \( \psi_1 \).

We first point out that we can derive \( L^\infty \) bound on \( \partial Z^p \phi \) for \( p \leq 4 \). According to Klainerman-Sobolev inequality, we have

\[
|\partial Z^p \phi(\tau, x)| \lesssim \frac{1}{(1 + |u|)^{1/2}} |\partial Z^p \phi(\tau, B(\tau, x))| + \frac{1}{(1 + |u|)^{1/2}} \sum_{Z \in Z, l \leq 3} \|Z^p \partial Z \phi\|_{L^2(\Sigma)} \lesssim \frac{C(I_{n+1}(\psi_0, \psi_1))}{(1 + |u|)^{1/2}} \delta^{\frac{1}{2}} + \frac{M}{(1 + |u|)^{1/2}} \delta^{\frac{1}{2}}.
\]

In particular, based on (3.2), it is well known that for good derivatives \( \partial_y \), we have

\[
|\partial_y Z^p \phi(\tau, x)| \lesssim \frac{M}{t^2} \delta^{\frac{1}{2}}.
\]

For all \( u \geq \delta \), according to (3.6), we have

\[
\int_{C_u} |\partial_y Z^k \phi|^2 \leq \int_{\Sigma_{t,u}} |\partial Z^k \phi|^2 + \sum_{p+q \leq k} \int_{D_{t,u}} |Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi| \leq \int_{\Sigma_t} |\partial Z^k \phi|^2 + \sum_{p+q \leq k} \int_{D_{t,i}} |Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi|.
\]

For the last step, we have enlarged the domain for integration. Therefore, according to the foliation \( \{ u \in [\delta, \frac{1}{2}] \mid C_u \} \), for the given constant \( \varepsilon_0 \in (0, \frac{1}{2}) \), we have

\[
\int_{D_{t,i}} |\partial_y Z^k \phi|^2 \leq \int_{\delta}^{t/2} \frac{1}{(1 + |u|)^{1+\varepsilon_0}} \left( \int_{C_u} |\partial_y Z^k \phi|^2 \right) du \leq \int_{\delta}^{t/2} \frac{1}{(1 + |u|)^{1+\varepsilon_0}} \left( \int_{\Sigma_\tau} |\partial Z^k \phi|^2 + \sum_{p+q \leq k} \int_{D_{t,i}} |Q(\nabla Z^p \phi, \nabla Z^q \phi) \partial_t Z^k \phi| \right) du'.
\]
Since the quantity inside the parenthesis is independent of \( u' \), we obtain
\[
\int_{D_{t,s}} \frac{\|\partial_t Z_k^p \phi\|^2}{(1 + |u|)^{1+\varepsilon_0}} \lesssim \int_{\Sigma_t} |\partial Z_k^p \phi|^2 + \sum_{p+q \leq k} \int_{D_{t,s}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| \partial_t Z_k^q \phi|.
\] (3.10)

We take \( u = \delta \) in (3.6). In view of (3.7), we obtain immediately that
\[
\int_{\Sigma_t} |\partial Z_k^p \phi|^2 \lesssim \int_{C_\delta} |\partial Z_k^p \phi|^2 + \sum_{p+q \leq k} \int_{D_{t,s}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| \partial_t Z_k^q \phi| \lesssim \delta^\frac{1}{2} C(I_{n+1}(\psi_0, \psi_1)) + \sum_{p+q \leq k} \int_{D_{t,s}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| \partial_t Z_k^q \phi|.
\] (3.11)

Together with (3.10), we have
\[
\int_{D_{t,s}} \frac{\|\partial_t Z_k^p \phi\|^2}{(1 + |u|)^{1+\varepsilon_0}} \lesssim \delta^\frac{1}{2} C(I_{n+1}(\psi_0, \psi_1)) + \sum_{p+q \leq k} \int_{D_{t,s}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| \partial_t Z_k^q \phi|.
\] (3.12)

In view of (3.11), we arrive at the following energy estimates:
\[
\int_{\Sigma_t} |\partial Z_k^p \phi|^2 + \int_{D_{t,s}} \frac{\|\partial_t Z_k^p \phi\|^2}{(1 + |u|)^{1+\varepsilon_0}} \lesssim \delta^\frac{1}{2} C(I_{n+1}(\psi_0, \psi_1)) + \sum_{p+q \leq k} \int_{D_{t,s}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| \partial_t Z_k^q \phi|.
\]

By summing over \( k \), we finally obtain that
\[
\widetilde{E}_{\leq k}(t) + \sum_{i \leq k} \int_{D_{t,s}} \frac{\|\partial_t Z_i^p \phi\|^2}{(1 + |u|)^{1+\varepsilon_0}} \lesssim \delta^\frac{1}{2} C(I_{n+1}(\psi_0, \psi_1)) + \sum_{i \leq k} \int_{D_{t,s}} |Q(\nabla Z^p \phi, \nabla Z^q \phi)| \partial_t Z_i^q \phi|.
\] (3.13)

Since we have the energy term \( \widetilde{E}_{\leq k}(t) \) on the left-hand side, to complete the bootstrap argument, it suffices to control the second term on the right-hand side. According to the structure of null forms, this term is bounded by
\[
\sum_{i \leq k, p+q \leq l} \int_{D_{t,s}} |\partial_t Z_i^p \phi| |\partial Z^q \phi| |\partial_t Z_i^l \phi|.
\]

According to whether \( p < q \) or \( p \geq q \), we break this term into two pieces (we replace \( \partial_t \) by \( \partial \)):
\[
S_1 + S_2 = \sum_{i \leq k, p+q \leq l, p < q} \int_{D_{t,s}} |\partial_t Z_i^p \phi| |\partial Z^q \phi| |\partial Z_i^l \phi| + \sum_{i \leq k, p+q \leq l, p \geq q} \int_{D_{t,s}} |\partial_t Z_i^p \phi| |\partial Z^q \phi| |\partial Z_i^l \phi|.
\]

For \( S_1 \), since \( p < q \), we have \( k - p \geq \lfloor \frac{1}{2} k \rfloor \geq 3 \). Here \( \lfloor \frac{1}{2} k \rfloor \) denotes the largest integer less or equal to \( \frac{1}{2} k \). We can apply the \( L^\infty \) estimates for good derivatives \( \partial_t Z^p \phi \). Therefore,
\[
S_1 \lesssim \sum_{i \leq k, p+q \leq l, p \leq q} \int_{\Sigma_t} M^3 \delta^\frac{1}{2} |\partial Z^q \phi|_{L^\infty(\Sigma_t)} |\partial Z_i^l \phi|_{L^2(\Sigma_t)} d\tau 
\lesssim M^3 \delta^\frac{1}{2}.
\]
For $S_2$, we apply Klainerman-Sobolev to $|\partial Z^j \phi|$ and we obtain

$$S_2 \lesssim \sum_{i \leq k, \sum_{l \in \mathbb{Z}} \sum_{p \in \mathbb{Z}} \sum_{q \in \mathbb{Z}}} M \int_{D_{t,\delta}} (1 + |u|)^{1+\varepsilon_0} \delta^{\frac{1}{2}} |\partial_p Z^p \phi| |\partial Z^j \phi|$$

$$\lesssim \varepsilon \sum_{p \geq k, \sum_{l \in \mathbb{Z}}} \int_{D_{t,\delta}} |\partial_p Z^p \phi|^2 (1 + |u|)^{1+\varepsilon_0} + \frac{1}{\varepsilon} \sum_{i \leq k, \sum_{l \in \mathbb{Z}}} \int_{D_{t,\delta}} M^2 (1 + |u|)^{\frac{1}{2}} \delta^{\frac{1}{2}} |\partial Z^j \phi|^2$$

$$\lesssim \varepsilon \sum_{p \geq k, \sum_{l \in \mathbb{Z}}} \int_{D_{t,\delta}} |\partial_p Z^p \phi|^2 (1 + |u|)^{1+\varepsilon_0} + \frac{1}{\varepsilon} M^4 \delta,$$

where the constant $\varepsilon$ will be determined later on.

Back to (3.13), the estimates on $S_1$ and $S_2$ yield

$$\tilde{E}_{\leq k}(t) + \sum_{i \leq k, \sum_{l \in \mathbb{Z}}} \sum_{p \geq k, \sum_{l \in \mathbb{Z}}} \int_{D_{t,\delta}} |\partial_p Z^p \phi|^2 (1 + |u|)^{1+\varepsilon_0} \lesssim \delta^{\frac{1}{2}} C(I_{n+1}(\psi_0, \psi_1)) + M^3 \delta^{\frac{1}{2}} + \varepsilon \sum_{p \geq k, \sum_{l \in \mathbb{Z}}} \int_{D_{t,\delta}} |\partial_p Z^p \phi|^2 (1 + |u|)^{1+\varepsilon_0} + \frac{1}{\varepsilon} M^4 \delta.$$ 

By choosing a suitable small constant $\varepsilon$, we can remove the integral term on the right-hand side and obtain

$$\tilde{E}_{\leq k}(t) \lesssim \delta^{\frac{1}{2}} C(I_{n+1}(\psi_0, \psi_1)) + M^3 \delta^{\frac{1}{2}} + \frac{1}{\varepsilon} M^4 \delta.$$ 

Hence,

$$\tilde{E}_{\leq k}(t) \lesssim \delta^{\frac{1}{2}} C(I_{n+1}(\psi_0, \psi_1)) + M^3 \delta^{\frac{1}{2}} + \frac{1}{\varepsilon} M^4 \delta.$$ 

We then can choose a sufficiently small $\delta$ and this completes the bootstrap argument.

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