Minimal period problems for brake orbits of nonlinear autonomous reversible semipositive Hamiltonian systems

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Abstract

In this paper, for any positive integer \( n \), we study the Maslov-type index theory of \( i_{L_0}, i_{L_1} \) and \( i\sqrt{-1} \) with \( L_0 = \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{2n} \) and \( L_1 = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n} \). As applications we study the minimal period problems for brake orbits of nonlinear autonomous reversible Hamiltonian systems. For first order nonlinear autonomous reversible Hamiltonian systems in \( \mathbb{R}^{2n} \), which are semipositive, and superquadratic at zero and infinity we prove that for any \( T > 0 \), the considered Hamiltonian systems possesses a nonconstant \( T \) periodic brake orbit \( X_T \) with minimal period no less than \( \frac{T}{2n+2} \). Furthermore if \( \int_0^T H_{22}'(x_T(t)) dt \) is positive definite, then the minimal period of \( x_T \) belongs to \( \{T, \frac{T}{2}\} \). Moreover, if the Hamiltonian system is even, we prove that for any \( T > 0 \), the considered even semipositive Hamiltonian systems possesses a nonconstant symmetric brake orbit with minimal period belonging to \( \{T, \frac{T}{4}\} \).

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1 Introduction and main results

In this paper, let \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \) and \( N = \begin{pmatrix} -I_n & 0 \\ 0 & I_n \end{pmatrix} \), where \( I_n \) is the identity in \( \mathbb{R}^n \) and \( n \in \mathbb{N} \). We suppose the following condition

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(H1) \( H \in C^2(\mathbb{R}^{2n}, \mathbb{R}) \) and satisfies the following reversible condition
\[
H(Nx) = H(x), \quad \forall x \in \mathbb{R}^{2n}.
\]

We consider the following problem:
\[
\dot{x} = JH'(x), \quad x \in \mathbb{R}^{2n}, \quad (1.1)
\]
\[
x(-t) = Nx(t), \quad x(T + t) = x(t), \quad \forall t \in \mathbb{R}. \quad (1.2)
\]

A solution \((T, x)\) of \((1.1)-(1.2)\) is a special periodic solution of the Hamiltonian system \((1.1)\). We call it a *brake orbit* and \(T\) the period of \(x\). Moreover, if \(x(R) = -x(R)\), we call it a *symmetric brake orbit*. It is easy to check that if \(\tau\) is the minimal period of \(x\), there must holds \(x(t + \frac{\tau}{2}) = -x(t)\) for all \(t \in \mathbb{R}\).

Since 1948, when H. Seifert in [47] proposed his famous conjecture of the existence of \(n\) geometrically different brake orbits in the potential well in \(\mathbb{R}^n\) under certain conditions, many people began to study this conjecture and related problems. Let \(\#\tilde{O}(\Omega)\) and \(\#\tilde{J}_b(\Sigma)\) the number of geometrically distinct brake orbits in \(\Omega\) for the second order case and on \(\Sigma\) for the first order case respectively. S. Bolotin proved first in [7] (also see [8]) of 1978 the existence of brake orbits in general setting. K. Hayashi in [27], H. Gluck and W. Ziller in [25], and V. Benci in [5] in 1983-1984 proved \(\#\tilde{O}(\Omega) \geq 1\) if \(V\) is \(C^1\), \(\tilde{\Omega} = \{V \leq h\}\) is compact, and \(V'(q) \neq 0\) for all \(q \in \partial\Omega\). In 1987, P. Rabinowitz in [45] proved that if \(H\) is \(C^1\) and satisfies the reversible condition, \(\Sigma \equiv H^{-1}(h)\) is star-shaped, and \(x \cdot H'(x) \neq 0\) for all \(x \in \Sigma\), then \(\#\tilde{J}_b(\Sigma) \geq 1\). In 1987, V. Benci and F. Giannoni gave a different proof of the existence of one brake orbit in [6].

In 1989, A. Szulkin in [49] proved that \(\#\tilde{J}_b(H^{-1}(h)) \geq n\), if \(H\) satisfies conditions in [43] of Rabinowitz and the energy hypersurface \(H^{-1}(h)\) is \(\sqrt{2}\)-pinched. E. van Groesen in [26] of 1985 and A. Ambrosetti, V. Benci, Y. Long in [1] of 1993 also proved \(\#\tilde{O}(\Omega) \geq n\) under different pinching conditions.

In [42] of 2006, Long, Zhu and the author of this paper proved that there exist at least 2 geometrically distinct brake orbits on any central symmetric strictly convex hypersurface \(\Sigma\) in \(\mathbb{R}^{2n}\) for \(n \geq 2\). Recently, in [35], Liu and the author of this paper proved that there exist at least \(\lceil n/2 \rceil + 1\) geometrically distinct brake orbits on any central symmetric strictly convex hypersurface \(\Sigma\) in \(\mathbb{R}^{2n}\) for \(n \geq 2\), if all brake orbits on \(\Sigma\) are nondegenerate then there are at least \(n\) geometrically distinct brake orbits on \(\Sigma\). For more details one can refer to [42], [35] and the reference there in.

In his pioneering paper [43] of 1978, P. Rabinowitz proved the following famous result via the variational method. Suppose \(H\) satisfies the following conditions:
(H1’) $H \in C^1(\mathbb{R}^{2n}, \mathbb{R})$.

(H2) There exist constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu H(x) \leq H'(x) \cdot x, \quad \forall |x| \leq r_0.$$ 

(H3) $H(x) = o(|x|^2)$ at $x = 0$.

(H4) $H(x) \geq 0$ for all $x \in \mathbb{R}^{2n}$.

Then for any $T > 0$, the system (1.1) possesses a non-constant $T$-periodic solution. Because a $T/k$ periodic function is also a $T$-periodic function, in [33] Rabinowitz proposed a conjecture that under conditions (H1’) and (H2)-(H4), there is a non-constant solution possessing any prescribed minimal period. Since 1978, this conjecture has been deeply studied by many mathematicians. A significant progress was made by Ekeland and Hofer in their celebrated paper [16] of 1985, where they proved Rabinowitz’s conjecture for the strictly convex Hamiltonian system. For Hamiltonian systems with convex or weak convex assumptions, we refer to [2]-[3], [12]-[17], [11], [20]-[23], and references therein for more details. For the case without convex condition we refer to [37]-[39] and Chapter 13 of [11] and references therein. A interesting result is for the semipositive first order Hamiltonian system, in [18] G. Fei, S.-T. Kim, and T. Wang proved the existence of a $T$ periodic solution of system (1.1) with minimal period no less than $T/2n$ for any given $T > 0$.

Note that in the second order Hamiltonian systems there are many results on the minimal problem of brake orbits such us [37]-[39] and [50]. For the even first order Hamiltonian system, in [51], the author of this paper studied the minimal period problem of semipositive even Hamiltonian system and gave a positive answer to Rabinowitz’s conjecture in that case. In [19], G. Fei, S.-T. Kim, and T. Wang proved the same result for second order Hamiltonian systems.

So it is natural to consider the minimal period problem of brake orbits in reversible first order nonlinear Hamiltonian systems. In [32], Liu have considered the strictly convex reversible Hamiltonian systems case and proved the existence of nonconstant brake orbit of (1.1) with minimal period belonging to $\{T, T/2\}$ for any given $T > 0$.

Since [51], we also hope to obtain some interesting results in the even Hamiltonian system for the minimal period problem of brake orbits.

It can be found in many papers mentioned above that the Maslov-type index theory and its iteration theory play a important role in the study of minimal period problems in Hamiltonian systems. In this paper we study some monotonicity properties of Maslov-type index and apply it to prove our main results.
In this paper we denote by $\mathcal{L}(\mathbb{R}^{2n})$ and $\mathcal{L}_s(\mathbb{R}^{2n})$ the set of all real $2n \times 2n$ matrices and symmetric matrices respectively. And we denote by $y_1 \cdot y_2$ the usual inner product for all $y_1, y_2 \in \mathbb{R}^k$ with $k$ being any positive integer. Also we denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of positive integers and integers respectively.

Let $\text{Sp}(2n) = \{ M \in \mathcal{L}(\mathbb{R}^{2n}) | M^T J M = J \}$ be the $2n \times 2n$ real symplectic group. For any $\tau > 0$, set $\mathcal{P}_\tau = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) | \gamma(0) = I_{2n} \}$ and $S_\tau = \mathbb{R}/(\tau \mathbb{Z})$.

For any $\gamma \in \mathcal{P}_\tau$ and $\omega \in \mathbb{U}$, where $\mathbb{U}$ is the unit circle of the complex plane $\mathbb{C}$, the Maslov-type index $(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}$ was defined by Long in [40]. We have a brief review in Appendix of Section 6.

For convenience to introduce our results, we define the following (B1) condition, since the Hamiltonian systems considered here are reversible, this condition is natural.

**(B1) Condition.** For any $\tau > 0$ and $B \in C([0, \tau], \mathcal{L}_s(\mathbb{R}^{2n}))$ with the $n \times n$ matrix square block form $B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}$ satisfying $B_{12}(0) = B_{21}(0) = 0 = B_{12}(\tau) = B_{21}(\tau)$, We will call $B$ satisfies the condition (B1).

Throughout this paper, we denote by

$$L_0 = \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{2n}, \quad L_1 = \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{2n}. \quad (1.3)$$

The definitions of Maslov-type indices $(i_{L_{\sqrt{-1}}}(\gamma), \nu_{L_{\sqrt{-1}}}(\gamma))$ and $(i_{L_j}(\gamma), \nu_{L_j}(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, n\}$ for $j = 0, 1$ and $\gamma \in \mathcal{P}_\tau(2n)$ with $\tau > 0$ can be found in [42] and Section 2 below. Also for $B \in C([0, \tau], \mathcal{L}_s(\mathbb{R}^{2n}))$ satisfies condition (B1), the definitions of $(i_{L_{\sqrt{-1}}}(B), \nu_{L_{\sqrt{-1}}}(B))$ and $(i_{L_j}(B), \nu_{L_j}(B)) \in \mathbb{Z} \times \{0, 1, \ldots, n\}$ for $j = 0, 1$ and $\gamma \in \mathcal{P}_\tau(2n)$ can be found in Section 2 and references therein.

For any $B \in C([0, \tau], \mathcal{L}_s(\mathbb{R}^{2n}))$, denote by $\gamma_B$ the fundamental solution of the following problem:

$$\dot{\gamma}_B(t) = JB(t)\gamma_B(t), \quad (1.4)$$

$$\gamma_B(0) = I_{2n}. \quad (1.5)$$

Then $\gamma_B \in \mathcal{P}_\tau$. We call $\gamma_B$ the *symplectic path associated to $B$*.

**Definition 1.1.** If $H \in C^2(\mathbb{R}^n, \mathbb{R})$ is a reversible function, for any $x_\tau$ be a $\tau$-periodic brake orbit solution of (1.1), let $B(t) = H''(x(t))$, we define $\gamma_{x_\tau} = \gamma_B|_{[0, \tau]}$ and call it the symplectic path associated to $x_\tau$. We define

$$i_{L_0}(x_\tau) = i_{L_0}(\gamma_{x_\tau}), \quad \nu_{L_0}(x_\tau) = i_{L_0}(\gamma_{x_\tau}). \quad (1.6)$$
Moreover, if $H$ is even and $x_\tau$ is a $\tau$-periodic symmetric brake orbit solution of (1.1), let $B(t) = H''(x(t))$, we define $\gamma_{x_\tau} = \gamma_B|_{[0,\tau]}$ and call it the symplectic path associated to $x_\tau$. We define

$$i^{L_0}_{\sqrt{-1}}(x_\tau) = i^{L_0}_{\sqrt{-1}}(\gamma_{x_\tau}), \quad \nu^{L_0}_{\sqrt{-1}}(x_\tau) = i^{L_0}_{\sqrt{-1}}(\gamma_{x_\tau}).$$

(1.7)

**Definition 1.2.** For any $\tau$-period and $k \in \mathbb{N} \equiv \{1, 2, \ldots\}$, we define the $k$ times iteration $x^k$ of $x$ by

$$x^k(t) = x(t - j\tau), \quad j\tau \leq t \leq (j + 1)\tau, \quad 0 \leq j \leq k.$$ (1.8)

As in [35], for any $\gamma \in \mathcal{P}_\tau$ and $k \in \mathbb{N} \equiv \{1, 2, \ldots\}$, in this paper the $k$-time iteration $\gamma^k$ of $\gamma \in \mathcal{P}_\tau(2n)$ in brake orbit boundary sense is defined by $\tilde{\gamma}|_{[0,k\tau]}$ with

$$\tilde{\gamma}(t) = \begin{cases} \gamma(t - 2j\tau)(N\gamma(\tau)^{-1}N\gamma(\tau))^j, & t \in [2j\tau, (2j + 1)\tau], j = 0, 1, 2, \ldots \\ N\gamma(2j\tau + 2\tau - t)N(N\gamma(\tau)^{-1}N\gamma(\tau))^j+1, & t \in [(2j + 1)\tau, (2j + 2)\tau], j = 0, 1, 2, \ldots \end{cases}$$

The followings are our main results of this paper.

**Theorem 1.1.** Suppose that $H$ satisfies conditions (H1)-(H4) and (H5) $H''(x)$ is semipositive definite for all $x \in \mathbb{R}^{2n}$.

Then for any $T > 0$, the system (1.1)-(1.2) possesses a nonconstant $T$ periodic brake orbit solution $x_T$ with minimal period no less that $\frac{T}{2n+2}$. Moreover, for $x = (x_1, x_2)$ with $x_1, x_2 \in \mathbb{R}^n$, denote by $H''_{22}(x)$ the second order differential of $H$ with respect to $x_2$, if

$$\int_0^T H''_{22}(x_T(t)) \, dt > 0,$$ (1.9)

then the minimal period of $x_T$ belongs to $\{T, \frac{T}{2}\}$.

**Remark 1.1.** (Theorem 1.1 of [32]) Suppose that $H$ satisfies conditions (H1)-(H4) and if $x_T$ satisfies

$$(H'') \int_0^T H''(X_T(t)) \, dt > 0.$$ (H'')

Then the minimal period of $x_T$ belongs to $\{T, \frac{T}{2}\}$.

In the case $n = 1$, the result can be better, i.e., the following

**Theorem 1.2.** For $n = 1$, suppose that $H$ satisfies conditions (H1)-(H4).

Then for any $T > 0$, the system (1.1)-(1.2) possesses a nonconstant $T$ periodic brake orbit solution with minimal period belong to $\{T, \frac{T}{2}\}$.

Consider the minimal period problem for $H(x) = \frac{1}{2}B_0x \cdot x + \dot{H}(x)$, where $B_0 \in \mathcal{L}_s(\mathbb{R}^{2n})$. This is motivated by [18], [22], and [43], where in [18] $B_0$ was considered to be semipositive, in [22] and [43] $B_0$ was considered to be positive.

We have the following general result.
Theorem 1.3. Let $2n \times 2n$ be real semipositive matrix $B_0 = \text{diag}(B_{11}, B_{22})$ with $B_{11}$ and $B_{22}$ being $n \times n$ matrix. Assume $H(x) = \frac{1}{2}B_0 x \cdot x + \hat{H}(x)$ for all $x \in \mathbb{R}^{2n}$, and $\hat{H}$ satisfies conditions (H1)-(H5).

Then for any $T > 0$, (1.1) possesses a nonconstant $T$-periodic brake orbit $x_T$ with minimal period no less than $\frac{T}{2L_0(B_0) + 2\nu(B_0)+2n+2}$, where we see $B_0$ as an element in $C([0,T/2], L_\nu(\mathbb{R}^{2n}))$ satisfies condition (B1).

Remark 1.2. In section 3, we will show that $i\lambda_0(B_0) + \nu\lambda_0(B_0) \geq 0$.

As a direct consequence of Theorem 1.3, we have the following Corollary 1.1.

Corollary 1.1. For $T > 0$ such that $i\lambda_0(B_0) + \nu\lambda_0(B_0) = 0$, where we see $B_0$ as an element in $C([0,T/2], L_\nu(\mathbb{R}^{2n}))$ satisfies condition (B1), under the same assumptions of Theorem 1.2, the system (1.1) possesses a nonconstant $T$-periodic brake orbit with minimal period no less that $\frac{T}{2n+2}$.

We can also prove the following Corollary 1.2 of Theorem 1.3.

Corollary 1.2. If $B_0 \neq 0$, then for $0 < T < \frac{T}{||B_0||}$ with $||B_0||$ being the operator norm of $B_0$, under the same condition of Theorem 1.2, possesses a nonconstant $T$-periodic brake orbit $x_T$ with minimal period no less than $\frac{T}{2n+2}$. Moreover, if

$$\int_0^T H''_{22}(x_T(t)) \, dt > 0,$$

then the minimal period of $x_T$ belongs to $\{T, \frac{T}{2}\}$.

Theorem 1.4. Suppose that $H$ satisfies conditions (H1)-(H5) and (H6) $H(-x) = H(x)$ for all $x \in \mathbb{R}^{2n}$.

Then for any $T > 0$, the system (1.1)-(1.2) possesses a nonconstant symmetric brake orbit with minimal period belonging to $\{T, T/3\}$.

Theorem 1.5. Let $2n \times 2n$ be real semipositive matrix $B_0 = \text{diag}(B_{11}, B_{22})$ with $B_{11}$ and $B_{22}$ being $n \times n$ matrix, assume $H(x) = \frac{1}{2}B_0 x \cdot x + \hat{H}(x)$ for all $x \in \mathbb{R}^{2n}$, and $\hat{H}$ satisfies conditions (H1)-(H6). Then for any $T > 0$, the system (1.1)-(1.2) possesses a nonconstant symmetric brake orbit $x_T$ with minimal period no less than $\frac{T}{4(\hat{\nu}\sqrt{\nu}(B_0)+\nu\hat{\nu}(B_0))+7}$. Moreover, if $i\lambda_0\sqrt{-1}(B_0) + \nu\lambda_0\sqrt{-1}(B_0)$ is even, then the minimal period of $x_T$ is no less than $\frac{T}{4(\hat{\nu}\sqrt{\nu}(B_0)+\nu\hat{\nu}(B_0))+3}$ where we see $B_0$ as an element in $C([0,T/4], L_\nu(\mathbb{R}^{2n}))$ satisfies condition (B1).

Remark 1.3. In section 3, we will show that $i\lambda_0\sqrt{-1}(B_0) \geq 0$, hence $i\lambda_0\sqrt{-1}(B_0) + \nu\lambda_0\sqrt{-1}(B_0) \geq 0$.

As a direct consequence of Theorem 1.5, we have the following Corollary 1.3.

Corollary 1.3. For $T > 0$ such that $i\lambda_0\sqrt{-1}(B_0) + \nu\lambda_0\sqrt{-1}(B_0) = 0$, under the same assumptions of Theorem 1.4, the system (1.1) possesses a nonconstant symmetric brake orbit with minimal period belonging to $\{T, T/3\}$. 
We can also prove the following Corollary 1.4 of Theorem 1.5.

**Corollary 1.4.** If $B_0 \neq 0$, then for $0 < T < \frac{\pi}{||B_0||}$ with $||B_0||$ being the operator norm of $B_0$, under the same condition of Theorem 1.5, the system (1.1) possesses a nonconstant symmetric brake orbit with minimal period belonging to $\{T, T/3\}$.

This paper is organized as follows. In section 2, we study the Maslov-type index theory of $i_{L_0}, i_{L_1}$ and $i_{\sqrt{-1}}^{L_0}$. We compute the difference between $i_{L_0}(\gamma)$ and $i_{L_1}(\gamma)$. In Section 3, we study the relation between the Maslov-type index $(i_{\sqrt{-1}}^{L_0}(B), \nu_{\sqrt{-1}}^{L_0}(B))$ for $B \in C([0, \tau], L_s(\mathbb{R}^{2n}))$ satisfies condition (B1) and the Morse indices of the corresponding Galerkin approximation. As applications we get some monotonicity properties of $i_{L_0}(B), i_{L_1}(B)$ and $i_{\sqrt{-1}}^{L_0}(B)$ and we prove Theorem 3.2 which is very important in the proof of Theorems 1.4-1.5. In Section 4, based on the preparations in Sections 2 and 3 we prove Theorems 1.1-1.3 and Corollary 1.2. In Section 5, we prove Theorems 1.4-1.5 and corollary 1.4. In Section 6, we give a briefly review of $(i_\omega, \nu_\omega)$ index theory with $\omega \in U$ for symplectic paths starting with identity as appendix.

2 Maslov-type index theory associated with Lagrangian subspaces

2.1 A brief review of index function $(i_{L_j}, \nu_{L_j})$ with $j = 0, 1$ and $(i_{\sqrt{-1}}^{L_0}, \nu_{\sqrt{-1}}^{L_0})$

Let

$$F = \mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$$

possess the standard inner product. We define the symplectic structure of $F$ by

$$\{v, w\} = (Jv, w), \forall v, w \in F, \text{ where } J = (-J) \oplus J = \begin{pmatrix} -J & 0 \\ 0 & J \end{pmatrix}.$$  \hspace{1cm} (2.2)

We denote by $\text{Lag}(F)$ the set of Lagrangian subspaces of $F$, and equip it with the topology as a subspace of the Grassmannian of all $2n$-dimensional subspaces of $F$.

It is easy to check that, for any $M \in \text{Sp}(2n)$ its graph

$$\text{Gr}(M) \equiv \left\{ \begin{pmatrix} x \\ Mx \end{pmatrix} \mid x \in \mathbb{R}^{2n} \right\}$$

is a Lagrangian subspace of $F$.

Let

$$V_1 = \{0\} \times \mathbb{R}^n \times \{0\} \times \mathbb{R}^n \subset \mathbb{R}^{4n}, \quad V_2 = \mathbb{R}^n \times \{0\} \times \mathbb{R}^n \times \{0\} \subset \mathbb{R}^{4n}.$$  \hspace{1cm} (2.3)
By Proposition 6.1 of [35] and Lemma 2.8 and Definition 2.5 of [42], we give the following definition.

**Definition 2.1.** For any continuous path \( \gamma \in \mathcal{P}_\tau(2n) \), we define the following Maslov-type indices:

\[
i_{L_0}(\gamma) = \mu_{FCLM}^CLM(V_1, \text{Gr}(\gamma), [0, \tau]) - n, \quad (2.4)
\]
\[
i_{L_1}(\gamma) = \mu_{FCLM}^CLM(V_2, \text{Gr}(\gamma), [0, \tau]) - n, \quad (2.5)
\]
\[
\nu_{L_j}(\gamma) = \dim(\gamma(\tau)L_j \cap L_j), \quad j = 0, 1, \quad (2.6)
\]

where we denote by \( i_{CLM}^F(V, W, [a, b]) \) the Maslov index for Lagrangian subspace path pair \((V, W)\) in \( F \) on \([a, b]\) defined by Cappell, Lee, and Miller in [11].

For \( \omega = e^{\sqrt{-1} \theta} \) with \( \theta \in \mathbb{R} \), we define a Hilbert space \( E^\omega = E_{L_0}^\omega \) consisting of those \( x(t) \) in \( L^2([0, \tau], C^{2n}) \) such that \( e^{-\theta J_0}x(t) \) has Fourier expending

\[
e^{-\frac{\theta}{\tau} J_0}x(t) = \sum_{j \in \mathbb{Z}} e^{\frac{j \pi}{\tau} J} a_j, \quad a_j \in \mathbb{C}^n
\]

with

\[
\|x\|^2 := \sum_{j \in \mathbb{Z}} \tau(1 + |j|)|a_j|^2 < \infty.
\]

For \( \omega = e^{\sqrt{-1} \theta}, \theta \in (0, \pi) \), we define two self-adjoint operators \( A^\omega, B^\omega \in \mathcal{L}(E^\omega) \) by

\[
(A^\omega x, y) = \int_0^1 \langle -J_0 \dot{x}(t), y(t) \rangle dt, \quad (B^\omega x, y) = \int_0^1 \langle B(t)x(t), y(t) \rangle dt
\]
on \( E^\omega \). Then \( B^\omega \) is also compact.

**Definition 2.2.** We define the index function

\[
i_{L_0}^\omega(B) = I(A^\omega, A^\omega - B^\omega) \equiv -\text{sf}\{A^\omega - sB^\omega, 0 \leq s \leq 1\},
\]
\[
\nu_{L_0}^\omega(B) = m^0(A^\omega - B^\omega), \quad \forall \omega = e^{\sqrt{-1} \theta}, \theta \in (0, \pi),
\]

where the definition of sf of spectral flow for the path of bounded self-adjoint linear operators one can refer to [53] and references their in.

By (3.21) of [35], we have

\[
i_{L_0}(B) \leq i_{L_0}^\omega(B) \leq i_{L_0}(B) + n. \quad (2.7)
\]

**Lemma 2.1.** For \( \omega = e^{\sqrt{-1} \theta} \) with \( \theta \in (0, \pi) \), let \( V_\omega = L_0 \times (e^{\theta J} L_0) \subset \mathbb{R}^{4n} \equiv F \). There holds

\[
i_{L_0}^\omega(B) = \mu_{FCLM}^CLM(V_\omega, \text{Gr}(\gamma_B), [0, \tau]). \quad (2.8)
\]
**Proof.** Without loss of generality we can suppose the $C^1$ path $\text{Gr}(\gamma_B)$ of Lagrangian subspaces intersects $V_\omega$ regularly (otherwise we can perturb it slightly with fixed endow points such that they intersects regularly and the index does not change by the homotopy invariant property $\mu_{CLM}^F$), where the definition of intersection form can be found in [16]. We denote by $\mu_{BF}$ the maslov index defined by Booss and Furutani in [9].

By the spectral flow formula of Theorem 5.1 in [9] or Theorem 1.5 of [10] (cf. also proof of Proposition 2.3 of [52]), we have

$$\text{sf}\{A^sB^\omega, 0 \leq s \leq 1\} = \mu_{BF}(\text{Gr}(\gamma_B), V_\omega, [0, \tau])$$

$$= \mu_{BF}((I \oplus e^{-\sqrt{-1}\theta J})\text{Gr}(\gamma_B), (I \oplus e^{-\sqrt{-1}\theta J})V_\omega, [0, \tau])$$

$$= -m^-((-\Gamma((I \oplus e^{-\sqrt{-1}\theta J})\text{Gr}(\gamma_B), V_1, 0)) + \sum_{0 < t < \tau} \text{sign}(-\Gamma((I \oplus e^{-\sqrt{-1}\theta J})\text{Gr}(\gamma_B), V_1, t))$$

$$+ m^+(-\Gamma((I \oplus e^{-\sqrt{-1}\theta J})\text{Gr}(\gamma_B), V_1, \tau))$$

$$=-\mu_{CLM}^F(V_1, (I \oplus e^{-\sqrt{-1}\theta J})\text{Gr}(\gamma_B), [0, \tau])$$

$$=-\mu_{CLM}^F((I \oplus e^{-\sqrt{-1}\theta J})V_1, \text{Gr}(\gamma_B), [0, \tau])$$

$$=-\mu_{CLM}^F(V_\omega, \text{Gr}(\gamma_B), [0, \tau]),$$

(2.9)

where in the fourth equality we have used Theorem 2.1 in [9] and the property of index $\mu_{RS}$ for symplectic paths defined in [16] (cf also (2.6)-(2.8) of [52]), in the sixth equality we have used Lemma 2.6 of [42], in the second and seventh equalities we used the symplectic invariance property of index $\mu_{BF}$ and $\mu_{CLM}^F$ respectively.

**Definition 2.3.** Let $B \in C([0, \tau], \mathcal{L}_s(\mathbb{R}^{2n})$ and $\gamma_B$ be the symplectic path associated to $B$. We define

$$i_{\omega}^{L_0}(\gamma_B) = i_{\omega}^{L_0}(B),$$

(2.10)

$$\nu_{\omega}^{L_0}(\gamma_B) = i_{\omega}^{L_0}(B).$$

(2.11)

By Lemma 2.1, in general we give the following definition.

**Definition 2.4.** For any $\gamma \in \mathcal{P}_\tau(2n)$ and $\omega = e^{\sqrt{-1}\theta}$ with $\theta \in (0, \pi)$, we define

$$i_{\omega}^{L_0}(\gamma) = \mu_{CLM}^F(V_\omega, \text{Gr}(\gamma_B), [0, \tau]),$$

(2.9)
\[ \nu^L_\omega(\gamma) = \dim \left( \gamma(\tau)L_0 \cap e^{\sqrt{-1}tJ}L_0 \right). \] 

(2.12)

For any \( \gamma \in \mathcal{P}_r(2n) \), we define a new symplectic path \( \tilde{\gamma} \in \mathcal{P}_r(2n) \) by

\[
\tilde{\gamma}(t) = \begin{cases} I_{2n}, & t \in [0, \frac{\tau}{3}], \\ \gamma(3t - \tau), & t \in [\frac{\tau}{3}, \frac{2\tau}{3}], \\ \gamma(\tau), & t \in [\frac{2\tau}{3}, \tau]. 
\end{cases}
\] 

(2.13)

So we can perturb \( \tilde{\gamma} \) slightly to a \( C^1 \) path \( \hat{\gamma} \) such that \( \hat{\gamma} \) is homotopic to \( \tilde{\gamma} \) with fixed end points and \( \hat{\gamma}(t) = I_{2n} \) for \( t \in [0, \frac{\tau}{6}] \) and \( \hat{\gamma}(t) = \gamma(\tau) \) for \( t \in [\frac{5\tau}{6}, \tau] \). Set \( \hat{B}(t) = -J\hat{\gamma}(t)(\hat{\gamma}(t))^{-1} \). So we have

\[ \hat{B}(0) = \hat{B}(\tau) = 0. \] 

(2.14)

Then this \( \hat{B} \in C([0, \tau], \mathcal{L}_s(\mathbb{R}^{2n})) \) and satisfies condition (B1). Also we have \( \hat{\gamma} \) is is homotopic to \( \gamma \) with fixed end points. So we have

\[
i_1(\hat{\gamma}^k) = i_1(\gamma^k) = i_1(\gamma^k_B), \quad \forall k \in \mathbb{N},
\] 

(2.15)

\[
\nu_1(\hat{\gamma}^k) = \nu_1(\gamma^k) = \nu_1(\gamma^k_B), \quad \forall k \in \mathbb{N}
\] 

(2.16)

and

\[
i_{L_0}(\hat{\gamma}^k) = i_{L_0}(\gamma^k) = i_{L_0}(\gamma^k_B), \quad \forall k \in \mathbb{N},
\] 

(2.17)

\[
\nu_{L_0}(\hat{\gamma}^k) = \nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^k_B), \quad \forall k \in \mathbb{N}.
\] 

(2.18)

Also by the property of index \( \mu^{CLM}_F \) and Definition 2.4 have

\[
i_{L_0}^{\sqrt{-1}}(\hat{\gamma}^k) = i_{L_0}^{\sqrt{-1}}(\gamma^k) = i_{L_0}^{\sqrt{-1}}(\gamma^k_B), \quad \forall k \in \mathbb{N},
\] 

\[
\nu_{L_0}^{\sqrt{-1}}(\hat{\gamma}^k) = \nu_{L_0}^{\sqrt{-1}}(\gamma^k) = \nu_{L_0}^{\sqrt{-1}}(\gamma^k_B), \quad \forall k \in \mathbb{N}.
\]

Hence, in \cite{35} the authors essentially proved the following Bott-type iteration formula.

**Theorem 2.1.** (Theorem 4.1 of \cite{35}) Let \( \gamma \in \mathcal{P}_r(2n) \) and \( \omega_k = e^{\pi \sqrt{-1}/k} \). For odd \( k \) we have

\[
i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + \sum_{i=1}^{(k-1)/2} i_{\omega_k^i}(\gamma^2),
\] 

\[
\nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \sum_{i=1}^{(k-1)/2} \nu_{\omega_k^i}(\gamma^2),
\]

and for even \( k \), we have

\[
i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + i_{L_0}^{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{k/2-1} i_{\omega_k^i}(\gamma^2),
\] 

\[
\nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \nu_{L_0}^{\sqrt{-1}}(\gamma^1) + \sum_{i=1}^{k/2-1} \nu_{\omega_k^i}(\gamma^2).
\]
Obviously we also have
\[ i_{L_0}(\gamma) \leq i_{L_0}^0(\gamma) \leq i_{L_0}(\gamma) + n. \] (2.19)

### 2.2 The Bott-type iteration formula for \((i_{L_0}^0, \nu_{L_0}^0)\)

In order to study the minimal period problem for Even reversible Hamiltonian systems, we need the iteration formula of the Maslov-type index of \((i_{L_0}^0, \nu_{L_0}^0)\) for symplectic paths starting with identity. We use Theorem 2.1 to obtain it.

Precisely we have the following Theorem.

**Theorem 2.2.** Let \(\gamma \in \mathcal{P}_\tau(2n)\) and \(\omega_k = e^{\pi \sqrt{-1}/k}\). For odd \(k\) we have
\[
i_{L_0}^0(\gamma^k) = i_{L_0}^0(\gamma^1) + \sum_{i=1}^{(k-1)/2} i\omega_k^{2i-1}(\gamma^2),
\] (2.20)
\[
\nu_{L_0}^0(\gamma^k) = \nu_{L_0}^0(\gamma^1) + \sum_{i=1}^{(k-1)/2} \nu\omega_k^{2i-1}(\gamma^2),
\] (2.21)

and for even \(k\), we have
\[
i_{L_0}^0(\gamma^k) = \sum_{i=1}^{k/2} i\omega_k^{2i-1}(\gamma^2),
\] (2.22)
\[
\nu_{L_0}^0(\gamma^k) = \sum_{i=1}^{k/2} \nu\omega_k^{2i-1}(\gamma^2).
\] (2.23)

**Proof.** For odd \(k\), since \(\gamma^{2k} = (\gamma^k)^2\), by Theorem 2.1 we have
\[
i_{L_0}(\gamma^{2k}) = i_{L_0}(\gamma^k) + i_{L_0}^0(\gamma^k),
\] (2.24)
\[
\nu_{L_0}(\gamma^{2k}) = \nu_{L_0}(\gamma^k) + \nu_{L_0}^0(\gamma^k).
\] (2.25)

Also by Theorem 2.1 we have
\[
i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + \sum_{i=1}^{(k-1)/2} i\omega_k^{2i}(\gamma^2),
\] (2.26)
\[
\nu_{L_0}(\gamma^k) = \nu_{L_0}(\gamma^1) + \sum_{i=1}^{(k-1)/2} \nu\omega_k^{2i}(\gamma^2),
\] (2.27)
\[
i_{L_0}(\gamma^{2k}) = i_{L_0}(\gamma^1) + i_{L_0}^0(\gamma^k) + \sum_{i=1}^{k-1} i\omega_k^{2i}(\gamma^2),
\] (2.28)
\[
\nu_{L_0}(\gamma^{2k}) = \nu_{L_0}(\gamma^1) + \nu_{L_0}^0(\gamma^k) + \sum_{i=1}^{k-1} \nu\omega_k^{2i}(\gamma^2).
\] (2.29)
Since $\omega_k = \omega_{2k}^2$, by (2.24), (2.28) minus (2.26) yields (2.20). By (2.25), (2.29) minus (2.27) yields (2.21).

For even $k$, by similar argument we obtain (2.22) and (2.23). The proof of Theorem 2.2 is complete.

2.3 The difference of $i_{L_0}(\gamma)$ and $i_{L_1}(\gamma)$.

The precise difference of $i_{L_0}(\gamma)$ and $i_{L_1}(\gamma)$ for $\gamma \in P_\tau$ with $\tau > 0$ is very important in the proof of the main results of this paper. In this subsection we use the Hörmander index (cf. [14]) to compute it. Note that in [42], in fact we have already proved that $|i_{L_0}(\gamma) - i_{L_1}(\gamma)| \leq n$.

For any $P \in \text{Sp}(2n)$ and $\varepsilon \in \mathbb{R}$, we set

$$M_\varepsilon(P) = P^T \begin{pmatrix} \sin 2\varepsilon I_n & -\cos 2\varepsilon I_n \\ -\cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix} P + \begin{pmatrix} \sin 2\varepsilon I_n & \cos 2\varepsilon I_n \\ \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix}.$$ (2.30)

Then we have the following theorem.

**Theorem 2.3.** For $\gamma \in P_\tau$ with $\tau > 0$, we have

$$i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2} \text{sgn} M_\varepsilon(\gamma(\tau)),$$ (2.31)

where $\text{sgn} M_\varepsilon(\gamma(\tau))$ is the signature of the symmetric matrix $M_\varepsilon(\gamma(\tau))$ and $\varepsilon > 0$ is sufficiently small.

We also have,

$$(i_{L_0}(\gamma) + \nu_{L_0}(\gamma)) - (i_{L_1}(\gamma) + \nu_{L_1}(\gamma)) = \frac{1}{2} \text{sgn} M_\varepsilon(\gamma(\tau)),$$ (2.32)

where $\varepsilon < 0$ and $|\varepsilon|$ is sufficiently small.

**Proof.** By the first geometrical definition of the Maslov-type index in Section 4 of [11], there exists an $\varepsilon > 0$ small enough such that

$$V_1 \cap e^{-\varepsilon J} \text{Gr}(\gamma(0)) = \{0\}, \quad V_2 \cap e^{-\varepsilon J} \text{Gr}(\gamma(\tau)) = \{0\}.$$ (2.33)

By definition 2.1, we have

$$i_{L_0}(\gamma) = \mu_{CLM}^F(V_1, e^{-\varepsilon J} \text{Gr}(\gamma), [0, \tau]) - n,$$ (2.34)

$$i_{L_1}(\gamma) = \mu_{CLM}^F(V_2, e^{-\varepsilon J} \text{Gr}(\gamma), [0, \tau]) - n.$$ (2.35)

Define $\gamma_1(t) = e^{-\varepsilon J} \text{Gr}(\gamma(t))$ and $\gamma_2(t) = e^{-\varepsilon J} \text{Gr}(\gamma(\tau - t))$ for $t \in [0, \tau]$. Then $\gamma_1$ and $\gamma_2$ are two paths of Lagrangian subspaces of the symplectic space $(\mathcal{F}, \mathcal{J})$ defined in (2.1) and (2.2). $\gamma_1$ connects
\(e^{-\varepsilon J}\text{Gr}(\gamma(0))\) and \(e^{-\varepsilon J}\text{Gr}(\gamma(\tau))\) and is transversal to \(V_1\) and \(V_2\). \(\gamma_2\) connects \(e^{-\varepsilon J}\text{Gr}(\gamma(\tau))\) and \(e^{-\varepsilon J}\text{Gr}(\gamma(0))\) and is transversal to \(V_1\) and \(V_2\). Denote by \(\gamma\) the catenation of the paths \(\gamma_1\) and \(\gamma_2\). By Definition 3.4.2 of the \(\text{Hörmander index}\) \(s(M_1, M_2; L_1, L_2)\) on p. 66 of [14] and (2.34)-(2.35), we have

\[
s(V_1, V_2; e^{-\varepsilon J}\text{Gr}(\gamma(0)), e^{-\varepsilon J}\text{Gr}(\gamma(\tau))) = \langle \gamma, \alpha \rangle
\]

\[
= \mu_{CLM}(V_1, \gamma_1) + \mu_{CLM}(V_2, \gamma_2)
\tag{2.36}
\]

\[
= \mu_{CLM}(V_1, e^{-\varepsilon J}\text{Gr}(\gamma)) - \mu_{CLM}(V_2, e^{-\varepsilon J}\text{Gr}(\gamma))
\tag{2.37}
\]

\[
= i_{L_0}(\gamma) - i_{L_1}(\gamma),
\tag{2.38}
\]

where \(\alpha\) is the Maslov-Arnold index defined in Theorem 3.4.9 on p. 64 of [14]. Since \(\gamma_1\) and \(\gamma_2\) are transversal to \(V_1\) and \(V_2\) (2.36) holds, (2.37) holds from the definition of \(\gamma_1\) and \(\gamma_2\).

In the proof of Theorem 3.3 of [42], we have proved that for \(\varepsilon > 0\) small enough, there holds

\[
\text{sgn}(V_1, e^{-\varepsilon J}\text{Gr}(I_{2n}); V_2) = 0,
\tag{2.39}
\]

where \(\text{sgn}(W_1, W_3; W_2)\) for 3 Lagrangian spaces with \(W_3\) transverses to \(W_1\) and \(W_2\) is introduced in Definition 3.2.3 on p. 67 of [14]. Note that by Claim 1 below, we can prove (2.39) at once.

**Claim 1.** For \(\varepsilon > 0\), small enough, there holds

\[
\text{sign}(V_1, e^{-\varepsilon J}\text{Gr}(\gamma(\tau)); V_2) = \text{sgn}(M_\varepsilon(\gamma(\tau))).
\tag{2.40}
\]

**Proof of Claim 1.** In fact,

\[
e^{-\varepsilon J}\text{Gr}(\gamma(\tau)) = \begin{pmatrix} e^{\varepsilon J/2} & 0 \\ 0 & e^{-\varepsilon J/2}\gamma(\tau) \end{pmatrix}
\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} cp - sq \\ sp + cq \\ (c, s)\gamma(\tau)(p, q)^T \\ (-s, c)\gamma(\tau)(p, q)^T \end{pmatrix}, \quad p, q \in \mathbb{R}^n
\tag{2.41}
\]

where we denote by \(c = \cos \varepsilon I_n\) and \(s = \sin \varepsilon I_n\). Hence the transformation \(A : V_1 \mapsto e^{-\varepsilon J}\text{Gr}(I_{2n})\) satisfies

\[
A(0, -sp - cq, 0, -(s, c)\gamma(\tau)(p, q)^T)
\]

\[
= (cp - sq, sp + cq, (c, s)\gamma(\tau)(p, q)^T, (-s, c)\gamma(\tau)(p, q)^T), \quad \forall p, q \in \mathbb{R}^n,
\tag{2.42}
\]
where \( A \) is introduced in Definition 3.4.3 of sign\((M_1, M_2; L)\) on p. 67 of [14]. For the convenience of our computation, we rewrite (2.42) as follows.

\[
A = \begin{pmatrix}
0 & 0 \\
s & c
\end{pmatrix}
\begin{pmatrix}
p \\
quation{2.43}
\end{pmatrix},
\]

Then for \( p_1, p_2, q_1, q_2 \in \mathbb{R}^n \), the symmetric bilinear form \( Q(V_2) : (x, y) \mapsto J(Ax, y) \) on \( V_1 \) defined in Definition 3.4.3 on p. 67 of [14] satisfies:

\[
Q(V_2) = \begin{pmatrix}
0 & 0 \\
s & c
\end{pmatrix}
\begin{pmatrix}
p \\
quation{2.44}
\end{pmatrix},
\]

Let \( \tilde{M}_\varepsilon(\gamma(\tau)) = \begin{pmatrix}
sc & -s^2 \\
c^2 & -sc
\end{pmatrix} + \gamma(\tau)^T \begin{pmatrix}
sc & s^2 \\
-c^2 & -sc
\end{pmatrix} \gamma(\tau) \). Then by definition of the symmetric bilinear form \( Q(V_2) \), \( \tilde{M}_\varepsilon(\gamma(\tau)) \) is an invertible symmetric \( 2n \times 2n \) matrix. We define

\[
M_\varepsilon(\gamma(\tau)) = 2\tilde{M}_\varepsilon(\gamma(\tau)) = \tilde{M}_\varepsilon(\gamma(\tau)) + \tilde{M}_\varepsilon^T(\gamma(\tau)).
\]

Then we have

\[
M_\varepsilon(\gamma(\tau)) = \gamma(\tau)^T \begin{pmatrix}
\sin 2\varepsilon I_n & -\cos 2\varepsilon I_n \\
-\cos 2\varepsilon I_n & \sin 2\varepsilon I_n
\end{pmatrix} \gamma(\tau) + \begin{pmatrix}
\sin 2\varepsilon I_n & \cos 2\varepsilon I_n \\
\cos 2\varepsilon I_n & -\sin 2\varepsilon I_n
\end{pmatrix}.
\]

It is clear that

\[
\text{sgn}Q(V_2) = \text{sgn}\tilde{M}_\varepsilon(\gamma(\tau)) = \text{sgn}M_\varepsilon(\gamma(\tau)).
\]

By the definition of \( \text{sgn}(V_1, e^{-\varepsilon J}\text{Gr}(\gamma(\tau)); V_2) \), we have

\[
\text{sgn}(V_1, e^{-\varepsilon J}\text{Gr}(\gamma(\tau)); V_2) = \text{sgn}Q(V_2).
\]
Then (2.40) holds from (2.47) and (2.48), and the proof of Claim 1 is complete.

Thus by (2.38), (2.39) and Claim 1, we have

\[
i_{L_0}(\gamma) - i_{L_1}(\gamma) = s(V_1, V_2; e^{-\varepsilon J} \text{Gr}(\gamma(0)), e^{-\varepsilon J} \text{Gr}(\gamma(\tau)))
\]

\[
= \frac{1}{2} \text{sgn}(V_1, e^{-\varepsilon J} \text{Gr}(\gamma(\tau)); V_2) - \frac{1}{2} \text{sgn}(V_1, e^{-\varepsilon J} \text{Gr}(\gamma(0)); V_2)
\]

\[
= \frac{1}{2} \text{sgn}(V_1, e^{-\varepsilon J} \text{Gr}(\gamma(\tau)); V_2) - \frac{1}{2} \text{sgn}(V_1, e^{-\varepsilon J} \text{Gr}(I_{2n}); V_2)
\]

\[
= \frac{1}{2} \text{sgn}(V_1, e^{-\varepsilon J} \text{Gr}(\gamma(\tau)); V_2)
\]

\[
= \frac{1}{2} \text{sgn}M_{\varepsilon}(\gamma(\tau)).
\]

Here in the second equality, we have used Theorem 3.4.12 of on p. 68 of [14]. Thus (2.31) holds.

Choose \(\varepsilon < 0\) such that \(|\varepsilon|\) is sufficiently small, by the discussion of \(\mu^{CLM} F\) index we have

\[
i_{L_0}(\gamma) = \mu^{CLM}_F(V_1, e^{-\varepsilon J} \text{Gr}(\gamma), [0, \tau]) - \nu_{L_0}(\gamma),
\]

\[
i_{L_1}(\gamma) = \mu^{CLM}_F(V_2, e^{-\varepsilon J} \text{Gr}(\gamma), [0, \tau]) - \nu_{L_1}(\gamma).
\]

Then by the same proof as above, we have

\[
i_{L_0}(\gamma) + \nu_{L_0}(\gamma) - i_{L_1}(\gamma) - \nu_{L_1}(\gamma) = \frac{1}{2} \text{sgn}M_{\varepsilon}(\gamma(\tau)),
\]

where \(\varepsilon < 0\) is small enough. Hence (2.32) holds. The proof of Theorem 2.3 is complete.

We have the following consequence.

**Corollary 2.1.** (Theorem 2.3 of [35]) For \(\gamma \in \mathcal{P}_+(2n)\) with \(\tau > 0\), there hold

\[
|i_{L_0}(\gamma)) - i_{L_1}(\gamma)| \leq n, \quad |i_{L_0}(\gamma) + \nu_{L_0}(\gamma) - i_{L_1}(\gamma) - \nu_{L_1}(\gamma)| \leq n.
\]

Moreover if \(\gamma(1)\) is a orthogonal matrix then there holds

\[
i_{L_0}(\gamma) = i_{L_1}(\gamma).
\]

**Proof.** (2.52) holds directly from Theorem 2.3, so we only need to prove (2.53). Since \(\gamma(\tau)\) is an orthogonal and symplectic matrix, we have

\[
\gamma^T(\tau)J\gamma(\tau) = J, \quad \gamma^T(\tau)\gamma(\tau) = I_{2n}.
\]

So we have

\[
\gamma(\tau)J = J\gamma(\tau), \quad \gamma(\tau)^TJ = J\gamma(\tau)^T.
\]
It is easy to check that for any \( \varepsilon \in \mathbb{R} \), there holds
\[
J \begin{pmatrix} \sin 2\varepsilon I_n & \pm \cos 2\varepsilon I_n \\ \pm \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix} J = \begin{pmatrix} \sin 2\varepsilon I_n & \pm \cos 2\varepsilon I_n \\ \pm \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix}.
\] (2.56)

Hence by (2.55) and (2.56), we have
\[
JM_\varepsilon(\gamma(\tau))J = J \begin{pmatrix} \gamma(\tau)^T \sin 2\varepsilon I_n & -\cos 2\varepsilon I_n \\ -\cos 2\varepsilon I_n & \sin 2\varepsilon I_n \end{pmatrix} \gamma(\tau) + \begin{pmatrix} \sin 2\varepsilon I_n & \cos 2\varepsilon I_n \\ \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix} J
\]
\[
= \begin{pmatrix} \gamma(\tau)^T \sin 2\varepsilon I_n & -\cos 2\varepsilon I_n \\ -\cos 2\varepsilon I_n & \sin 2\varepsilon I_n \end{pmatrix} \gamma(\tau) + \begin{pmatrix} \sin 2\varepsilon I_n & \cos 2\varepsilon I_n \\ \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix} J
\]
\[
= \begin{pmatrix} \gamma(\tau)^T \sin 2\varepsilon I_n & -\cos 2\varepsilon I_n \\ -\cos 2\varepsilon I_n & \sin 2\varepsilon I_n \end{pmatrix} \gamma(\tau) + \begin{pmatrix} \sin 2\varepsilon I_n & \cos 2\varepsilon I_n \\ \cos 2\varepsilon I_n & -\sin 2\varepsilon I_n \end{pmatrix} J
\]
\[
= M_\varepsilon(\gamma(\tau)).
\] (2.57)

So we have
\[
M_\varepsilon(\gamma(\tau))J = -JM_\varepsilon(\gamma(\tau)).
\] (2.58)

Thus for any \( x \in \mathbb{R}^{2n} \) and \( \lambda \in \mathbb{R} \) satisfying
\[
M_\varepsilon(\gamma(\tau))x = \lambda x.
\] (2.59)

By (2.58) we have
\[
M_\varepsilon(\gamma(\tau))(Jx) = -JM_\varepsilon(\gamma(\tau))x = -\lambda(Jx).
\] (2.60)

Since for \( \varepsilon > 0 \) small enough \( M_\varepsilon(\gamma(\tau)) \) is an invertible symmetric matrix, by (2.60) we have
\[
m^+(M_\varepsilon(\gamma(\tau))) = m^-(M_\varepsilon(\gamma(\tau))) = n
\] (2.61)

which yields
\[
\text{sgn}M_\varepsilon(\gamma(\tau)) = m^+(M_\varepsilon(\gamma(\tau))) - m^-(M_\varepsilon(\gamma(\tau))) = 0.
\] (2.62)

Then (2.53) holds from Theorem 2.3.

**Lemma 2.2.** For a symplectic path \( P : [0, \tau] \to \text{Sp}(2n) \) with \( \tau > 0 \), if for \( j = 0, 1 \) there holds \( \nu_{L_j}(P(t)) = \text{constant} \) for all \( t \in [0, \tau] \), then for \( \varepsilon > 0 \) small enough we have
\[
\text{sgn}M_\varepsilon(P(0)) = \text{sgn}M_\varepsilon(P(\tau)).
\] (2.63)
Proof. Since $\text{Sp}(2n)$ is path connected, we can choose a path $\gamma \in \mathcal{P}_\tau$ with $\gamma(\tau) = P(0)$. By Proposition 2.11 of [42] and the definition of $\mu_j$ for $j = 1, 2$ in [42], we have

$$\mu_{CM}^F(V_j, \text{Gr}(P), [0, \tau]) = 0, \quad j = 0, 1. \quad (2.64)$$

So by the Path Additivity and Reparametrization Invariance properties of $\mu_{CM}^F$ in [11], we have

$$i_{L_j}(P \ast \gamma) = \mu_{CM}^F(V_j, \text{Gr}(P \ast \gamma), [0, \tau]) - n$$

$$= \mu_{CM}^F(V_j, \text{Gr}(\gamma), [0, \tau]) + \mu_{CM}^F(V_j, \text{Gr}(P), [0, \tau]) - n$$

$$= i_{L_j}(\gamma), \quad (2.65)$$

where the definition of joint path $\eta \ast \xi$ is given by (6.1) in Section 6 below. Then by Theorem 2.3 we have

$$i_{L_0}(\gamma) - i_{L_1}(\gamma) = \frac{1}{2} \text{sgn}(M_\varepsilon(P(0))), \quad (2.66)$$

$$i_{L_0}(P \ast \gamma) - i_{L_1}(P \ast \gamma) = \frac{1}{2} \text{sgn}(M_\varepsilon(P(\tau))). \quad (2.67)$$

Then (2.63) holds from (2.65)-(2.67). The proof of Lemma 2.2 is complete.

Remark 2.1. It is easy to check that for $n_j \times n_j$ symplectic matrix $P_j$ with $j = 1, 2$ and $n_j \in \mathbb{N}$, we have

$$M_\varepsilon(P_1 \diamond P_2) = M_\varepsilon(P_1) \diamond M_\varepsilon(P_2),$$

$$\text{sgn} M_\varepsilon(P_1 \diamond P_2) = \text{sgn} M_\varepsilon(P_1) + \text{sgn} M_\varepsilon(P_2).$$

By direct computation according to Theorem 2.3 and Corollary 2.1, for $\gamma \in \mathcal{P}_\tau(2)$, $b > 0$, and $\varepsilon > 0$ small enough we have

$$\text{sgn} M_\varepsilon(R(\theta)) = 0, \quad \text{for } \theta \in \mathbb{R}, \quad (2.68)$$

$$\text{sgn} M_\varepsilon(P) = 0, \quad \text{if } P = \pm \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}, \quad (2.69)$$

$$\text{sgn} M_\varepsilon(P) = 2, \quad \text{if } P = \pm \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}, \quad (2.70)$$

$$\text{sgn} M_\varepsilon(P) = -2, \quad \text{if } P = \pm \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}. \quad (2.71)$$
Also we give an example as follows to finish this section

\[ \text{sgn} M_\varepsilon(P) = 2, \quad \text{if } P = \pm \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}. \]  

(2.72)

3  Relation between $i_{L_0}$, $i_{L_1}$, $i_{L_0}^{\sqrt{-1}}$ and the corresponding Morse indices, and their monotonicity properties.

In [31], Liu studied the relation between the $L$-index of solutions of Hamiltonian systems with $L$-boundary conditions and the Morse index of the corresponding functional defined via the Galerkin approximation method on the finite dimensional truncated space at its corresponding critical points. In order to prove the main results of this paper, in this section we use the results of [31] to study some monotonicity properties of $i_{L_0}$ and $i_{L_1}$. We also study the index $i_{L_0}^{\sqrt{-1}}(B)$ with $B$ being a continuous symmetric matrices path satisfying condition (B1) defined in Section 1 and the Morse index of the corresponding functional defined via the Galerkin approximation method. Then as applications we study some monotonicity properties of $i_{L_0}^{\sqrt{-1}}(B)$ which will be important in the proof of Theorems 1.4-1.5 in Section 5 below.

For any $\tau > 0$ and $B \in C([0, \tau/4], L_s(\mathbb{R}^{2n}))$ (in order to apply the results in this section conveniently Section 5, we always assume $B \in C([0, \tau/4], L_s(\mathbb{R}^{2n}))$ satisfying condition (B1). We extend $B$ to $[0, \frac{\tau}{2}]$ by

\[ B\left(\frac{\tau}{4} + t\right) = NB\left(\frac{\tau}{4} - t\right)N, \quad \forall t \in [0, \frac{\tau}{4}]. \]  

(3.1)

Then since $B(\frac{\tau}{2}) = B(0)$, we can extend it $\frac{\tau}{2}$-periodically to $\mathbb{R}$, so we can see $B$ as an element in $C(S_{\tau/2}, L_s(\mathbb{R}^{2n}))$.

Let $E_\tau = \{ x \in W^{1/2,2}(S_\tau, \mathbb{R}^{2n}) | x(-t) = N x(t) a.e. t \in \mathbb{R} \}$ with the usual norm and inner product denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ respectively.

By the Sobolev embedding theorem, for any $s \in [1, +\infty)$, there is a constant $C_s > 0$ such that

\[ \|z\|_{L^s} \leq C_s \|z\|, \quad \forall z \in E_{2\tau}. \]  

(3.2)

Note that $B$ can also be seen as an element in $C(S_\tau, L_s(\mathbb{R}^{2n}))$. We define two selfadjoint operators $A_\tau$ and $B_\tau$ on $E_\tau$ by the following bilinear forms

\[ \langle A_\tau x, y \rangle = \int_0^\tau -J \dot{x} \cdot y \, dt, \quad \langle B_\tau x, y \rangle = \int_0^\tau B(t)x \cdot y \, dt. \]  

(3.3)
Then $A_\tau$ is a bounded operator on $E_\tau$ and $\dim \ker A_\tau = n$, the Fredholm index of $A_\tau$ is zero, and $B_\tau$ is a compact operator on $E_\tau$.

Set
\[
E_\tau(j) = \left\{ z \in E_\tau \left| z(t) = \exp\left( \frac{2j\pi t}{\tau} \right) a + \exp\left( -\frac{2j\pi t}{\tau} \right) b, \forall t \in \mathbb{R}; \forall a, b \in L_0 \right. \right\}.
\]
and
\[
E_{\tau,m} = E_\tau(0) + E_\tau(1) + \cdots + E_\tau(m).
\]

Let $\Gamma_\tau = \{ P_{\tau,m} : m = 0, 1, 2, \ldots \}$ be the usual Galerkin approximation scheme w.r.t. $A_\tau$, just as in [31], i.e., $\Gamma_\tau$ is a sequence of orthogonal projections satisfies:

1. $E_{\tau,0} = P_{\tau,0} E_\tau = \ker A_\tau$, $E_{\tau,m} = P_{\tau,m} E_\tau$ is finite dimension for $m \geq 0$;
2. $P_{\tau,m} \to x$ as $m \to \infty$ for any $x \in E_\tau$;
3. $P_{\tau,m} A_\tau = A_\tau P_{\tau,m}, \forall m \geq 0$.

For $d > 0$, we denote by $M^+_d(\cdot)$, $M^-_d(\cdot)$ and $M^0_d(\cdot)$ the eigenspace corresponding to the eigenvalue $\lambda$ belong to $[d, +\infty)$, $(-\infty, -d]$ and $(-d, d)$ respectively, and $M^+(\cdot)$, $M^-(\cdot)$ and $M^0(\cdot)$ the positive, negative and null subspace of of the selfadjoint operator defining it respectively. For any bounded selfadjoint linear operator on $E$, We denote $L^\# = (L|_{ImL})^{-1}$, and we also denote by $P_{\tau,m} L P_{\tau,m} = (P_{\tau,m} L P_{\tau,m})|_{E_{\tau,m}} : E_{\tau,m} \to E_{\tau,m}$.

Similarly we define two subspaces of $E_\tau$ by $\hat{E} = \{ x \in E \left| x(t + \frac{\tau}{2}) = -x(t), a.e. t \in \mathbb{R} \right. \}$ and $\tilde{E} = \{ x \in E \left| x(t + \frac{\tau}{2}) = x(t), a.e. t \in \mathbb{R} \right. \}$ be the symmetric ones and $\frac{\tau}{2}$-periodic ones of $E_\tau$ respectively.

We define two selfadjoint operators $\hat{A}$ and $\hat{B}$ on $\hat{E}$ by the following bilinear forms

\[
\langle \hat{A}x, y \rangle = \int_0^\tau -J \hat{x} \cdot y \, dt, \quad \langle \hat{B}x, y \rangle = \int_0^\tau B(t)x(t) \cdot y(t) \, dt.
\]

Then $\hat{A}$ is a bounded Fredholm operator on $\hat{E}$ and $\dim \ker \hat{A} = 0$, the Fredholm index of $\hat{A}$ is zero. $\hat{B}$ is a compact operator on $\hat{E}$.

For any positive integer $m$, we define
\[
\hat{E}_m = \sum_{j=1}^m E_\tau(2j - 1).
\]
For $m \geq 1$, let $\hat{P}_m$ be the orthogonal projection from $\hat{E}$ to $\hat{E}_m$. Then $\{ \hat{P}_m \}$ is a Galerkin approximation scheme w.r.t. $\hat{A}$. 
Theorem 3.1. For any \( B(t) \in C([0, \frac{T}{4}], \mathcal{L}_s(\mathbb{R}^{2n})) \) satisfying condition (B1) and \( 0 < d \leq \frac{1}{4}||(A_r - B_r)^\#||^{-1} \), there exists \( m^* > 0 \) such that for \( m \geq m^* \) there hold

\[
\dim M_d^+(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) = mn - i\sqrt{L_0}(B) - \nu\sqrt{L_0}(B),
\]

\[
\dim M_d^-(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) = mn + i\sqrt{L_0}(B),
\]

\[
\dim M_d^0(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) = \nu\sqrt{L_0}(B).
\]

Proof. The method of the proof here is similar as that of Theorem 2.1 in [51].

For any positive integer \( m \), we define

\[
\hat{E}_m = \sum_{j=0}^{m} E_r(2j).
\]

For \( m \geq 1 \), let \( \hat{P}_m \) be the orthogonal projection from \( \hat{E} \) to \( \hat{E}_m \). Then \( \{\hat{P}_m\} \) is a Galerkin approximation scheme w.r.t. \( \hat{A} \).

For any \( y \in \hat{E}_m \) and \( z \in \hat{E}_m \), it is easy to check that

\[
\langle (P_{\tau,m}(A_r - B_r)P_{\tau,m}y, z) \rangle = 0.
\]

So we have the following \( P_{\tau,m}(A_r - B_r)P_{\tau,m} \) orthogonal decomposition

\[
E_{\tau,2m} = \hat{E}_m \oplus \hat{E}_m.
\]

Similarly, we have the following \( A_r - B_r \) orthogonal decomposition

\[
E_{\tau} = \hat{E} \oplus \hat{E}.
\]

Hence, under above decomposition we have

\[
(A_r - B_r) = (\hat{A} - \hat{B}) \oplus (\hat{A} - \hat{B}).
\]

Thus

\[
||(A_r - B_r)^\#||^{-1} \leq ||(\hat{A} - \hat{B})^\#||^{-1}
\]

\[
||(A_r - B_r)^\#||^{-1} \leq ||(\hat{A} - \hat{B})^\#||^{-1}
\]

By the definitions of \( M_d^*(\cdot) \) for \( P_{\tau,2m}(A_r - B_r)P_{\tau,2m}, \hat{P}_m(\hat{A} - \hat{B})\hat{P}_m, \) and \( \hat{P}_m(\hat{A} - \hat{B})\hat{P}_m \) with \( * = +, -, 0 \). So for \( * \in \{+,-,0\} \) we have

\[
\dim M_d^*(P_{\tau,2m}(A_r - B_r)P_{\tau,2m}) = \dim M_d^*(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) + \dim M_d^*(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m).
\]
Note that, the space $E_{\tau}$ and the operators $A_{\tau}$, $B_{\tau}$ and $P_{\tau,m}$ are also defined in the same way. So by the definition we see that $\tilde{E}$ is the $\tau$-periodic extending of $E_{\tau}$ from $S_{\tau}$ to $S_{2\tau}$, and $\tilde{E}_{m}$ is the $\tau$-periodic extending of $E_{\tau,2m}$ from $S_{\tau}$ to $S_{2\tau}$ too.

Thus we have
\[ \| (A_{\tau} - B_{\tau})^\# \|^{-1} = \| (\tilde{A} - \tilde{B})^\# \|^{-1}. \] (3.15)

By (3.13) and (3.15) we have
\[ \| (A_{2\tau} - B_{2\tau})^\# \|^{-1} \leq \| (A_{\tau} - B_{\tau})^\# \|^{-1}. \] (3.16)

For $* \in \{+,-,0\}$ we have
\[ \dim M_d^*(P_{\tau,m}(A_{\tau} - B_{\tau})P_{\tau,m}) = M_d^*(\tilde{P}_m(\tilde{A} - \tilde{B})\tilde{P}_m). \] (3.17)

Then for $0 < d \leq \frac{1}{4} \| (A_{\tau} - B_{\tau})^\# \|^{-1}$, by Theorem 2.1 in [31] there exists $m_1 > 0$ such that for $m \geq m_1$ we have
\[ \dim M_d^+(P_{\tau,2m}(A_{\tau} - B_{\tau})P_{\tau,2m}) = 2mn - i_{L_0}(\gamma_B^2) - \nu_{L_0}(\gamma_B^2), \] (3.18)
\[ \dim M_d^-(P_{\tau,2m}(A_{\tau} - B_{\tau})P_{\tau,2m}) = 2mn + n + i_{L_0}(\gamma_B^2), \] (3.19)
\[ \dim M_d^0(P_{\tau,2m}(A_{\tau} - B_{\tau})P_{\tau,2m}) = \nu_{L_0}(\gamma_B^2). \] (3.20)

By (3.16), we have $0 < d \leq \frac{1}{4} \| (A_{\tau} - B_{\tau})^\# \|^{-1}$. By Theorem 2.1 in [31] again there exists $m_2 > 0$, such that for $m \geq m_2$ we have
\[ \dim M_d^+(P_{\tau,m}(A_{\tau} - B_{\tau})P_{\tau,m}) = mn - i_{L_0}(\gamma_B) - \nu_{L_0}(\gamma_B)), \] (3.21)
\[ \dim M_d^-(P_{\tau,m}(A_{\tau} - B_{\tau})P_{\tau,m}) = mn + n + i_{L_0}(\gamma_B)), \] (3.22)
\[ \dim M_d^0(P_{\tau,m}(A_{\tau} - B_{\tau})P_{\tau,m}) = \nu_{L_0}(\gamma_B)), \] (3.23)

Let $m^* = \max \{m_1, m_2\}$. Then for $m \geq m^*$, all of (3.18)-(3.23) hold.

So by (3.14), (3.17), and (3.18)-(3.23) we have
\[ \dim M_d^+(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) = mn - (i_{L_0}(\gamma_B^2) - i_{L_0}(\gamma_B)) - (\nu_{L_0}(\gamma_B^2) - \nu_{L_0}(\gamma_B)), \] (3.24)
\[ \dim M_d^-(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) = mn + i_{L_0}(\gamma_B^2) - i_{L_0}(\gamma_B), \] (3.25)
\[ \dim M_d^0(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) = \nu_{L_0}(\gamma_B^2) - \nu_{L_0}(\gamma_B). \] (3.26)

Thus (3.5)-(3.7) hold from (3.24)-(3.26), Definition 2.3, and Theorem 2.2. The proof of Theorem 3.1 is complete.

\[ \blacksquare \]
Remark 3.1. Let any \( B \in C([0, \frac{3}{2}], \mathcal{L}_s(\mathbb{R}^{2n})) \) be a constant matrix path satisfying condition (B1). By Theorem 5.1 of [42], for \( d = 0 \) the same conclusions of Theorem 2.1 of [31] still holds. Hence for \( d = 0 \) the same conclusions of Theorem 3.1 still hold, i.e., there exists \( m^* > 0 \) such that for \( m \geq m^* \) there hold

\[
\dim M^+(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) = mn - i\frac{L_0}{\sqrt{-1}}(B) - \nu\frac{L_0}{\sqrt{-1}}(B),
\]
\[
\dim M^-(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) = mn + i\frac{L_0}{\sqrt{-1}}(B),
\]
\[
\dim M^0(\hat{P}_m(\hat{A} - \hat{B})\hat{P}_m) = \nu\frac{L_0}{\sqrt{-1}}(B).
\]

In the following, we study some monotonicity of the the Maslov-type \( i\frac{L_0}{\sqrt{-1}} \) index. In this paper, for any two symmetric matrices \( B_1 \) and \( B_2 \), we say \( B_1 > B_2 \) if \( B_1 - B_2 \) is positive definite and we say \( B_1 \geq B_2 \) if \( B_1 - B_2 \) is semipositive. Similarly for two symmetric matrix paths \( B_1, B_2 \in C([0, \tau], \mathcal{L}_s(\mathbb{R}^{2n})) \), we say \( B_1 > B_2 \) if \( B_1(t) - B_2(t) \) is positive definite for all \( t \in [0, \tau] \) and we say \( B_1 \geq B_2 \) if \( B_1(t) - B_2(t) \) is semipositive definite for all \( t \in [0, \tau] \).

**Lemma 3.1.** For any \( \tau > 0 \) and \( B_1, B_2 \in C([0, \frac{3}{2}], \mathcal{L}_s(\mathbb{R}^{2n})) \) satisfying condition (B1). If \( B_1 \geq B_2 \), then there hold

\[
i\frac{L_0}{\sqrt{-1}}(B_1) \geq i\frac{L_0}{\sqrt{-1}}(B_2) \tag{3.27}
\]

and

\[
i\frac{L_0}{\sqrt{-1}}(B_1) + \nu\frac{L_0}{\sqrt{-1}}(B_1) \geq i\frac{L_0}{\sqrt{-1}}(B_2) + \nu\frac{L_0}{\sqrt{-1}}(B_2). \tag{3.28}
\]

Moreover, if

\[
\int_0^\tau (B_1(t) - B_2(t))dt > 0, \tag{3.29}
\]

then there holds

\[
i\frac{L_0}{\sqrt{-1}}(B_1) \geq i\frac{L_0}{\sqrt{-1}}(B_2) + \nu\frac{L_0}{\sqrt{-1}}(B_2). \tag{3.30}
\]

**Proof.** Let the space \( \hat{E} \) and the orthogonal projection operator \( \hat{P}_m \) be the ones defined in Section 2. Correspondingly we define the compact operators \( \hat{B}_1 \) and \( \hat{B}_2 \). By Theorem 3.1, for \( d > 0 \) small enough, there exists \( m^* > 0 \) such that

\[
\dim M^+(\hat{P}_m(\hat{A} - \hat{B}_1)\hat{P}_m) = mn - i\frac{L_0}{\sqrt{-1}}(B_1) - \nu\frac{L_0}{\sqrt{-1}}(B_1), \tag{3.31}
\]
\[
\dim M^-(\hat{P}_m(\hat{A} - \hat{B}_1)\hat{P}_m) = mn + i\frac{L_0}{\sqrt{-1}}(B_1), \tag{3.32}
\]
\[
\dim M^0(\hat{P}_m(\hat{A} - \hat{B}_1)\hat{P}_m) = \nu\frac{L_0}{\sqrt{-1}}(B_1). \tag{3.33}
\]
and
\[
\dim M^+_d(\hat{P}_m(\hat{A} - \hat{B}_2)\hat{P}_m) = mn - \frac{iL_0}{\sqrt{-1}}(B_2) - \nu \frac{L_0}{\sqrt{-1}}(B_2),
\]
\[
\dim M^-_d(\hat{P}_m(\hat{A} - \hat{B}_2)\hat{P}_m) = mn + \frac{iL_0}{\sqrt{-1}}(B_2),
\]
\[
\dim M^0_d(\hat{P}_m(\hat{A} - \hat{B}_2)\hat{P}_m) = \nu \frac{L_0}{\sqrt{-1}}(B_2).
\]

If \( B_1 \geq B_2 \), we have \( \hat{P}_m(\hat{A} - \hat{B}_1)\hat{P}_m \leq \hat{P}_m(\hat{A} - \hat{B}_2)\hat{P}_m \), so
\[
\dim M^-_d(\hat{P}_m(\hat{A} - \hat{B}_1)\hat{P}_m) \geq \dim M^-_d(\hat{P}_m(\hat{A} - \hat{B}_2)\hat{P}_m).
\]

Then by (3.32) and (3.35), (3.37) holds. Also we have
\[
\dim M^+_d(\hat{P}_m(\hat{A} - \hat{B}_1)\hat{P}_m) \leq \dim M^+_d(\hat{P}_m(\hat{A} - \hat{B}_2)\hat{P}_m).
\]

Then by (3.31) and (3.34), (3.28) holds.

**Corollary 3.1.** For any \( \tau > 0 \) and \( B \in C([0, \frac{\tau}{4}], L_s(\mathbb{R}^{2n})) \) satisfying condition (B1) and \( B \geq 0 \), there holds
\[
i\frac{L_0}{\sqrt{-1}}(B) \geq 0.
\]

**Proof.** By Lemma 3.1, we have
\[
i\frac{L_0}{\sqrt{-1}}(B) \geq i\frac{L_0}{\sqrt{-1}}(0).
\]

Then the conclusion holds from the fact that
\[
i\frac{L_0}{\sqrt{-1}}(0) = i\frac{L_0}{\sqrt{-1}}(\gamma_0) = 0,
\]
Where \( \gamma_0 \) is the identity symplectic path.

By Theorem 2.1 of [31] and the Remark below Theorem 2.1 in [31] and the similar proof of Lemma 3.1 we have the following lemma.

**Lemma 3.2.** If \( \tau > 0 \) and \( B_1, B_2 \in C([0, \frac{\tau}{4}], L_s(\mathbb{R}^{2n})) \) satisfying condition (B1) and \( B_1 \geq B_2 \), then for \( j = 0, 1 \) there hold
\[
i_{L_j}(B_1) \geq i_{L_j}(B_2)
\]
and

\[ i_{L_j}(B_1) + \nu_{L_j}(B_1) \geq i_{L_j}(B_2) + \nu_{L_j}(B_2). \]  

(3.45)

Moreover, if \( \int_0^\tau (B_1(t) - B_2(t)) dt > 0 \), then there holds

\[ i_{L_j}(B_1) \geq i_{L_j}(B_2) + \nu_{L_j}(B_2). \]  

(3.46)

Since \( i_{L_j}(0) = -n \) and \( \nu_{L_j}(0) = n \) for \( j = 0, 1 \), a direct consequence of Lemma 3.2 is the following

**Corollary 3.2.** If \( \tau > 0 \) and \( B \in C([0, \tau], L_s(\mathbb{R}^{2n})) \) satisfying condition (B1) and \( B \geq 0 \), then for \( j = 0, 1 \) there hold

\[ i_{L_j}(B) + \nu_{L_j}(B) \geq 0, \quad i_{L_j}(B) \geq -n. \]  

(3.47)

Moreover if \( \int_0^\tau B(t) dt > 0 \), there holds

\[ i_{L_j}(B) \geq 0. \]  

(3.48)

Moreover we can give a stronger version of Corollary 3.2, i.e., the following Lemma 3.3.

**Lemma 3.3.** Let \( \tau > 0 \) and \( B \in C([0, \tau], L_s(\mathbb{R}^{2n})) \) with the \( n \times n \) matrix square block form

\[ B(t) = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} \]

satisfying condition (B1) and \( B \geq 0 \).

If \( \int_0^\tau B_{22}(t) dt > 0 \), there holds

\[ i_{L_0}(B) \geq 0. \]  

(3.49)

If \( \int_0^\tau B_{11}(t) dt > 0 \), there holds

\[ i_{L_1}(B) \geq 0. \]  

(3.50)

**Proof.** Without loss of generality, assume \( \lambda > 0 \) such that

\[ \int_0^\tau B_{22}(t) \geq \lambda I_n. \]  

(3.51)

Also we can extend \( B \) to \([0, \tau]\) by

\[ B(\frac{\tau}{2} + t) = NB(\frac{\tau}{2} - t)N, \forall t \in [0, \frac{\tau}{2}]. \]  

(3.52)

Then since \( B(\tau) = B(0) \), we can extend it \( \tau \)-periodically to \( \mathbb{R} \), so we can see \( B \) as an element in \( C(S_\tau, L_s(\mathbb{R}^{2n})) \). Then we have

\[ \int_0^\tau B_{22}(t) \geq 2\lambda I_n. \]  

(3.53)
For any \( m \in \mathbb{N} \), we define two subspaces of \( E \) as follows

\[
E_{\tau,m} = \left\{ z \in E_\tau \left| z(t) = \sum_{j=1}^{m} \exp(-\frac{2j\pi t}{\tau})b_j, \forall t \in \mathbb{R}; \forall b_j \in L_0 \right. \right\},
\]

\[
E_\tau(0) = \{ z \in E_\tau \left| z(t) \equiv b, b \in L_0 \right. \}.
\]

Then for any \( z = \alpha x + \beta y \in E_\tau(0) \oplus E_{\tau,m} \) with \( \alpha^2 + \beta^2 = 1 \) and \( ||x|| = ||y|| = 1 \), we have

\[
\langle (A_\tau - B_\tau)z, z \rangle = \langle (A_\tau - B_\tau)(\alpha x + \beta y), \alpha x + \beta y \rangle \\
= -\beta^2 \langle A_\tau y, y \rangle - \langle B_\tau (\alpha x + \beta y), \alpha x + \beta y \rangle \\
\leq -||A_\tau^#||^{-1} \beta^2 - \langle B_\tau (\alpha x + \beta y), \alpha x + \beta y \rangle.
\]

(3.54)

Since \( B \geq 0 \), note that \( x(t) \equiv b = (0, b_1) \in L_0 \) for all \( t \in S_\tau \) with \( \tau|b_1|^2 = 1 \), we have

\[
\langle B_\tau (\alpha x + \beta y), \alpha x + \beta y \rangle \\
= \int_0^\tau (\alpha^2 Bx \cdot x + \beta^2 By \cdot y + 2\alpha\beta Bx \cdot y) dt \\
\geq \alpha^2 \int_0^\tau Bx \cdot x dt + \beta^2 \int_0^\tau By \cdot y dt - 2\alpha |\beta| (\int_0^\tau Bx \cdot x dt)^{1/2} (\int_0^\tau By \cdot y dt)^{1/2} \\
\geq \alpha^2 \int_0^\tau Bx \cdot x dt + \beta^2 \int_0^\tau By \cdot y dt - \frac{1}{1 + \varepsilon} \alpha^2 \int_0^\tau Bx \cdot x dt - (1 + \varepsilon) \beta^2 \int_0^\tau By \cdot y dt \\
= \frac{\varepsilon \alpha^2}{1 + \varepsilon} \int_0^\tau Bx \cdot x dt - \varepsilon \beta^2 \int_0^\tau By \cdot y dt \\
= \frac{\varepsilon \alpha^2}{1 + \varepsilon} \left( \int_0^\tau B(t) dt \right) b \cdot b - \varepsilon \beta^2 \int_0^\tau By \cdot y dt \\
= \frac{\varepsilon \alpha^2}{1 + \varepsilon} \left( \int_0^\tau B_{22}(t) dt \right) b_1 \cdot b_1 - \varepsilon \beta^2 \int_0^\tau By \cdot y dt \\
\geq \frac{\varepsilon \alpha^2}{1 + \varepsilon} 2\lambda |b_1|^2 - \varepsilon \beta^2 ||B_\tau|| ||y||^2 \\
= \frac{2\varepsilon \lambda \alpha^2}{(1 + \varepsilon) \tau} - \varepsilon \beta^2 ||B_\tau||
\]

(3.55)

for any \( \varepsilon > 0 \).

Let \( \varepsilon = \min\{1, \frac{||A_\tau^#||^{-1} ||B_\tau||^{-1}}{2} \} \). By (3.54) and (3.55), we have

\[
\langle (A_\tau - B_\tau)z, z \rangle \leq -||A_\tau^#||^{-1} \beta^2 - \frac{2\varepsilon \lambda \alpha^2}{(1 + \varepsilon) \tau} + \varepsilon \beta^2 ||B_\tau|| \\
\leq \frac{||A_\tau^#||^{-1} \beta^2}{2} - \frac{\varepsilon \lambda \alpha^2}{\tau} \\
\leq -d_0(\alpha^2 + \beta^2) \\
= -d_0.
\]

(3.56)
where \( d_0 = \min \{ \frac{||A^\#||^{-1}}{2}, \frac{\lambda}{\tau}, \frac{\lambda||A^\#||^{-1}||B^\#||^{-1}}{2\tau} \} \). Note that \( d_0 \) is independent of \( m \), so for \( 0 < d \leq \min \{ d_0, \frac{||(A^\#_\tau - B^\#_\tau)^\#||^{-1}}{4} \} \), by Theorem 2.1 of [31] there exists \( m^* > 0 \) such that, for \( m \geq m^* \), we have

\[
\dim M_d^- (P_{\tau,m}(A^\#_{\tau} - B^\#_{\tau})P_{\tau,m}) = mn + n + i_{L_0}(B). \tag{3.57}
\]

By (3.56) we have

\[
\dim M_d^- (P_{\tau,m}(A^\#_{\tau} - B^\#_{\tau})P_{\tau,m}) \geq \dim(E^\tau_\tau(0) \oplus E^-_{\tau,m}) = mn + n. \tag{3.58}
\]

Then by (3.57) and (3.58) we have

\[
i_{L_0}^L(\gamma_p^B) \geq i_{L_0}^L(\gamma_q^B). \tag{3.59}
\]

Proof. Extend \( \gamma_B(t) \) to \([0, \frac{\tau}{4}]\) as \( \gamma_p^B \), we still denote it by \( \gamma_B \). By definition of \( i_{L_0}^L \) and the Path additivity and Symplectic invariance property of \( \mu_{CLM}^F \) in [11], we have

\[
i_{L_0}^L(\gamma_p^B) - i_{L_0}^L(\gamma_q^B) = \mu_{CLM}^F(L_0 \times JL_0, \text{Gr}(\gamma_B), [0, \frac{\tau}{4}]) - \mu_{CLM}^F(L_0 \times JL_0, \text{Gr}(\gamma_B), [0, \frac{\tau}{4}])
\]

\[
= \mu_{CLM}^F(L_0 \times JL_0, \text{Gr}(\gamma_B), [\frac{\tau}{4}, \frac{\tau}{4}]) - \mu_{CLM}^F(L_0 \times JL_0, \text{Gr}(\gamma_B), [\frac{\tau}{4}, \frac{\tau}{4}])
\]

\[
= \mu_{CLM}^F(L_0 \times L_0, \text{Gr}(-J\gamma_B), [\frac{\tau}{4}, \frac{\tau}{4}]). \tag{3.60}
\]

By the first geometrical definition of the index \( \mu_{CLM}^F \) in section 4 of [11], there is a \( \varepsilon > 0 \) small enough such that

\[
(e^{-\varepsilon J}\text{Gr}(-J\gamma_B(\frac{\tau}{4}))) \cap (L_0 \times L_0) = \{0\} = (e^{-\varepsilon J}\text{Gr}(\gamma_B(\frac{\tau}{4}))) \cap (L_0 \times L_0) \tag{3.61}
\]
It is clear that

\[ \text{By the Homotopy invariance with respect to end points and Path additivity properties of} \] 
\[ \text{and} \]

\[
\mu_{CLM}^F(L_0 \times L_0, \text{Gr}(-J\gamma_B), \left[ \frac{qT}{4}, \frac{pT}{4} \right])
\]

\[ = \mu_{CLM}^F(L_0 \times L_0, e^{-\varepsilon J} \text{Gr}(-J\gamma_B), \left[ \frac{qT}{4}, \frac{pT}{4} \right])
\]

\[ = \mu_{CLM}^F(L_0 \times L_0, \text{Gr}(-e^{-\varepsilon J} J\gamma_B e^{-\varepsilon J}), \left[ \frac{qT}{4}, \frac{pT}{4} \right]), \] (3.62)

where in the second equality we have used Symplectic invariance property of \( \mu_{CLM}^F \) index in [11].

Choose a \( C^1 \) path \( \gamma \in \mathcal{P}_{\mathcal{F}_\varepsilon} \) such that \( \gamma(t) = -e^{-\varepsilon J} J\gamma_B e^{-\varepsilon J} \) for all \( t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right] \). Denote by \( D(t) = -J\dot{\gamma}(t)\gamma(t)^{-1} \) for \( t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right] \). For \( t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right] \), by direct computation we have

\[
D(t) = -J \frac{d}{dt} \left( -e^{-\varepsilon J} J\gamma e^{-\varepsilon J} \right) \left( -e^{-\varepsilon J} J\gamma e^{-\varepsilon J} \right)^{-1} = -Je^{-\varepsilon J} B(t)e^{\varepsilon J} J. \] (3.63)

Since \( B \geq 0 \) we have \( D(t) \geq 0 \) for \( t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right] \) and \( D \in C\left( \left[ \frac{qT}{4}, \frac{pT}{4} \right], \mathcal{L}_s(\mathbb{R}^{2n}) \right) \). For \( s \geq 0 \), we define \( D_s(t) = D(t) + sI_{2n} \) and symplectic path \( \gamma_s(t) \) by

\[
\frac{d}{dt} \gamma_s(t) = JD_s(t)\gamma_s(t), \quad t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right] 
\]

\[
\gamma_s(0) = I_{2n}. \]

It is clear that

\[
\gamma_0 = \gamma. \] (3.64)

By the same argument of step2 of the proof of Theorem 5.1 in [12], we have

\[
-J \frac{d}{ds} \gamma_s(t)(\gamma_s(t))^{-1} > 0, \quad \text{for} \quad t = \frac{pT}{4}, \frac{qT}{4}. \] (3.65)

By (3.64) and definition of \( \gamma_s \) we have

\[
\nu_{L_0}(\gamma_0(\frac{pT}{4})) = 0 = \nu_{L_0}(\gamma_0(\frac{qT}{4})). \] (3.66)

So by (3.65), there is a \( \sigma > 0 \) small enough such that

\[
\nu_{L_0}(\gamma_s(\frac{pT}{4})) = 0 = \nu_{L_0}(\gamma_s(\frac{qT}{4})), \quad \forall s \in [0, \sigma]. \] (3.67)

So we have

\[
\mu_{CLM}^F(L_0 \times L_0, \text{Gr}(\gamma_s(\frac{pT}{4})), s \in [0, \sigma]) = 0, 
\]

\[
\mu_{CLM}^F(L_0 \times L_0, \text{Gr}(\gamma_s(\frac{qT}{4})), s \in [0, \sigma]) = 0. \] (3.68)

By the Homotopy invariance with respect to end points and Path additivity properties of \( \mu_{CLM}^F \) index in [11], we have

\[
\mu_{CLM}^F(L_0 \times L_0, \text{Gr}(\gamma_s(\frac{pT}{4})), s \in [0, \sigma]) + \mu_{CLM}^F(L_0 \times L_0, \text{Gr}(\gamma_0(t)), t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right])
\]

\[ = \mu_{CLM}^F(L_0 \times L_0, \text{Gr}(\gamma_0(t)), t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right]) + \mu_{CLM}^F(L_0 \times L_0, \text{Gr}(\gamma_s(\frac{pT}{4})), s \in [0, \sigma]). \] (3.69)
So by (3.60), (3.62), (3.64), (3.68) and (3.69), we have
\[ iL_0 \sqrt{\gamma_B} (\gamma_B^p) - iL_0 \sqrt{\gamma_B} (\gamma_B^q) = \mu^{CLM}_F (L_0 \times L_0, Gr(\gamma_\sigma(t)), t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right]). \] (3.70)

Since \( D(t) \geq 0 \) for \( t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right] \), we have
\[ D_\sigma(t) > 0, \quad \forall t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right]. \] (3.71)

So by the proof of Lemma 3.1 of [42] and Lemma 2.6 of [42], we have
\[ \mu^{CLM}_F (L_0 \times L_0, Gr(\gamma_\sigma(t)), t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right]) = \sum_{t \in \left[ \frac{qT}{4}, \frac{pT}{4} \right]} \nu_{L_0} (\gamma_\sigma(t)) \geq 0. \] (3.72)

Thus by (3.70) and (3.72), (3.59) holds. The proof of Theorem 3.1 is complete. \[ \square \]

By similar proof of Theorem 3.2 we have the following Theorem 3.3.

**Theorem 3.3.** If \( \tau > 0 \) and \( B \in C([0, \frac{T}{4}], L_4(R^{2n})) \) satisfying condition (B1) and \( B \geq 0 \), then for \( j = 0, 1 \) and any two positive integers \( p \geq q \) there holds
\[ iL_j (\gamma_B^p) \geq iL_j (\gamma_B^q). \] (3.73)

### 4 Proof of Theorems 1.1-1.3 and Corollary 1.2

In this section we study the minimal period problem for brake orbits of the reversible Hamiltonian system (1.1) and complete the proof of Theorems 1.1-1.3 and Corollary 1.2.

For \( T > 0 \), we set \( E = W^{1,2,2}(S_T, R^{2n}) \) with the usual norm and inner product denoted by \( || \cdot || \) and \( \langle \cdot, \cdot \rangle \) respectively, and two subspaces of \( E \) by \( E_T = \{ x \in W^{1,2,2}(S_T, R^{2n}) | x(-t) = Nx(t) \ a.e. \ t \in R \} \) and \( \tilde{E}_T = \{ x \in W^{1,2,2}(S_T, R^{2n}) | x(-t) = -Nx(t) \ a.e. \ t \in R \} \). Then we have
\[ E = E_T \oplus \tilde{E}_T. \] (4.1)

As in Section 3, we define two selfadjoint operators \( A_T \) on \( E_T \) by the same way as (3.3). We also define two selfadjoint operators \( \tilde{A}_T \) on \( \tilde{E}_T \) by the following bilinear form:
\[ \langle \tilde{A}_T x, y \rangle = \int_0^T -J\dot{x} \cdot y \, dt. \] (4.2)

Then \( A_T \) is a bounded operator on \( E_T \) and \( \text{dim ker } A_T = n \), the Fredholm index of \( A_T \) is zero, and \( \tilde{A}_T \) is a bounded operator on \( \tilde{E}_T \) and \( \text{dim ker } \tilde{A}_T = n \), the Fredholm index of \( \tilde{A}_T \) is zero.
Set
\[ E_T(j) = \left\{ z \in E \mid z(t) = \exp\left( 2j\pi T J \right) a + \exp\left( -2j\pi T J \right) b, \forall t \in \mathbb{R}; \forall a, b \in L_0 \right\}, \]
\[ E_{T,m} = E_T(0) + E_T(1) + \cdots + E_T(m) \]
and
\[ \tilde{E}_T(j) = \left\{ z \in E_T \mid z(t) = \exp\left( 2j\pi t T J \right) a + \exp\left( -2j\pi t T J \right) b, \forall t \in \mathbb{R}; \forall a, b \in L_1 \right\}, \]
\[ \tilde{E}_{T,m} = \tilde{E}_T(0) + \tilde{E}_T(1) + \cdots + \tilde{E}_T(m). \]

Let \( P_{T,m} \) be the orthogonal projection from \( E_T \) to \( E_{T,m} \) and \( \tilde{P}_{T,m} \) be the orthogonal projection from \( \tilde{E}_T \) to \( \tilde{E}_{T,m} \) for \( m = 0, 1, 2, \ldots \), then \( \Gamma_T = \{ P_{T,m} : m = 0, 1, 2, \ldots \} \) and \( \tilde{\Gamma}_T = \{ \tilde{P}_{T,m} : m = 0, 1, 2, \ldots \} \) are the usual Galerkin approximation schemes w.r.t. \( A_T \) and \( \tilde{A}_T \) respectively.

For \( z \in E_T \), we define
\[ f(z) = \frac{1}{2} \langle A_T z, z \rangle - \int_0^T H(z)dt. \] (4.3)

It is well known that \( f \in C^2(E_T, \mathbb{R}) \) whenever,
\[ H \in C^2(\mathbb{R}^{2n}) \quad \text{and} \quad |H''(x)| \leq a_1|x|^s + a_2 \] (4.4)
for some \( s \in (1, +\infty) \) and all \( x \in \mathbb{R}^{2n} \).

By similar argument of Lemma 4.1 of [51], looking for \( T \)-periodic brake orbit solutions of (1.1) is equivalent to look for critical points of \( f \).

In order to get the information about the Maslov-type indices, we need the following theorem which was proved in [24, 28, 48].

**Theorem 4.1.** Let \( W \) be a real Hilbert space with orthogonal decomposition \( E = X \oplus Y \), where \( \dim X < +\infty \). Suppose \( f \in C^2(W, \mathbb{R}) \) satisfies (PS) condition and the following conditions:

(i) There exist \( \rho, \delta > 0 \) such that \( f(w) \geq \delta \) for any \( w \in W \);

(ii) There exist \( e \in \partial B_1(0) \cap Y \) and \( r_0 > \rho > 0 \) such that for any \( w \in \partial Q \), \( f(w) < \delta \) where \( Q = (B_{r_0}(0) \cap X) \oplus \{ re : 0 \leq r \leq r_0 \} \), \( B_r(0) = \{ w \in W : ||w|| \leq r \} \).

Then (1) \( f \) possesses a critical value \( c \geq \delta \), which is given by
\[ c = \inf_{h \in \Gamma} \max_{w \in Q} f(h(w)), \]
where \( \Gamma = \{ h \in C(Q, E) : h = \text{id} \text{ on } \partial Q \} \);

(2) There exists \( w_0 \in K_c \equiv \{ w \in E : f'(w) = 0, f(w) = c \} \) such that the Morse index \( m^-(w_0) \) of \( f \) at \( w_0 \) satisfies \( m^-(w_0) \leq \dim X + 1 \).
Proof of Theorem 1.3. For any given $T > 0$, we prove the existence of $T$-periodic brake solution of (1.1) whose minimal period satisfies the inequalities in the conclusion of Theorem 1.2. We divide the proof into five steps.

Step 1. We truncate the function $\hat{H}$ suitably and evenly such that it satisfies the growth condition (4.4). Hence corresponding new reversible function $H$ satisfies condition (4.4).

We follow the method in Rabinowitz’s pioneering work \[43\] (cf. also [18], [44] and [51]). Let $K > 0$ and $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi \equiv 1$ if $y \leq K$, $\chi \equiv 0$ if $y \geq K$ and $\chi'(y) < 0$ if $y \in (K, K+1)$, where $K$ will be determined later. Set

$$\hat{H}_K(z) = \chi(|z|) \hat{H}(z) + (1 - \chi(|z|)) R_K |z|^4$$

and

$$H_K(z) = \frac{1}{2} B_0 x \cdot x + \hat{H}_K(z),$$

where the constant $R_K$ satisfies

$$R_K \geq \max_{K \leq |z| \leq K+1} \frac{H(z)}{|z|^4}.$$  

Then $H_K \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. Since $\hat{H}$ satisfies (H3), $\forall \varepsilon > 0$, there is a $\delta_1 > 0$ such that $\hat{H}_K(z) \leq \varepsilon |z|^2$ for $|z| \leq \delta_1$. It is easy to see that $H_K(z)|z|^4$ is uniformly bounded as $|z| \to +\infty$, there is an $M_1 = M_1(\varepsilon, K)$ such that $\hat{H}_K(z) \leq M_1 |z|^4$ for $|z| \geq \delta_1$. So

$$\hat{H}_K(z) \leq \varepsilon |z|^2 + M_1 |z|^4, \; \forall z \in \mathbb{R}^{2n}.$$  

Set

$$f_K(z) = \frac{1}{2} \langle A_T z, z \rangle - \int_0^T H_K(z) dt, \; \forall z \in \hat{E}.$$  

Then $f_K \in C^2(E_T, \mathbb{R})$ and

$$f_K(z) = \frac{1}{2} \langle (A_T - B_0T) z, z \rangle - \int_0^T \hat{H}_K(z) dt, \; \forall z \in \hat{E},$$

where $B_{0T}$ is the selfadjoint linear compact operator on $E_T$ defined by

$$\langle B_{0T} z, z \rangle = \int_0^T B_{0T}(t) \cdot z(t) dt.$$  

Step 2. For $m > 0$, let $f_{Km} = f|E_{T,m}$. We show $f_{Km}$ satisfies the hypotheses of Theorem 4.1. We set

$$X_m = M^-(P_{T,m}(A_T - B_{0T})P_{T,m}) \oplus M^0(P_{T,m}(A_T - B_{0T})P_{T,m}),$$

$$Y_m = M^+(P_{T,m}(A_T - B_{0T})P_{T,m}).$$
For $z \in Y_m$, by \[4.8\] and \[3.2\], and the fact that $P_{T,j}B_0T = P_{T,j}B_0T$ for $j > 0$, we have

$$f_{Km}(z) = \frac{1}{2}((A_T - B_0T)z, z) - \int_0^T \dot{H}_K(z)dt$$

$$\geq \frac{1}{2}||((A_T - B_0T)\#||^{-1}||z||^2 - (\varepsilon ||z||_{L^2}^2 + M_1||z||_{L^4}^4)$$

$$\geq \frac{1}{2}||((A_T - B_0T)\#||^{-1}||z||^2 - (\varepsilon C_2^2 + M_1 C_4^2||z||^2)||z||^2, \tag{4.9}$$

where $C_2$ and $C_4$ are constants for $s = 2, 4$ for the Sobolev embedding of inequality \[3.2\], and they are independent of $m$ and $K$.

So if choose $\varepsilon > 0$ small enough such that $\varepsilon C_2^2 < \frac{1}{4}||((A_T - B_0T)\#||^{-1}$, then there exists $\rho = \rho(K) > 0$ small enough and $\delta = \delta(K) > 0$, which are independent of $m$, such that

$$f_m(z) \geq \delta, \quad \forall z \in \partial B_\rho(0) \cap Y_m. \tag{4.10}$$

Let $e \in B_1(0) \cap Y_m$ and set

$$Q_m = \{re : 0 \leq r \leq r_1\} \oplus (B_{r_1}(0) \cap X_m),$$

where $r_1$ will be determined later. Let $z = z_0 + z_0 \in B_{r_1}(0) \cap X_m$, we have

$$f_{Km}(z + re) = \frac{1}{2}((A_T - B_0T)z, z) + \frac{1}{2}r^2((A_T - B_0T)e, e) - \int_0^T \dot{H}_K(z + re)dt$$

$$\leq \frac{1}{2}||A_T - B_0T||r^2 - \frac{1}{2}||(A_T - B_0T)\#||^{-1}||z_0||^2 - \int_0^T \dot{H}_K(z + re)dt. \tag{4.11}$$

Since $\dot{H}$ satisfies (H2) we have

$$\dot{H}_K(x) \geq a_1|x|^\alpha - a_2, \quad \forall x \in \mathbb{R}^{2n},$$

where $\alpha = \min\{\mu, 4\}$, $a_1 > 0, a_2$ are two constants independent of $K$ and $m$. Then there holds

$$\int_0^T \dot{H}_K(z + re)dt \geq a_1 \int_0^T |z + re|^\alpha - Ta_2 \geq a_3(||z_0||_{L^\alpha}^\alpha + r^\alpha) - a_4, \tag{4.12}$$

where $a_3$ and $a_4$ are constants independent of $K$ and $m$. By \[4.11\] and \[4.12\] we have

$$f_{Km}(z + re) \leq \frac{1}{2}||A_T - B_0||r^2 - \frac{1}{2}||(A - B_0)\#||^{-1}||z_0||^2 - a_5(||z_0||_{L^\alpha}^\alpha + r^\alpha) + a_4.$$  

Since $\alpha > 2$ there exists a constant $r_1 > \rho > 0$, which are independent of $K$ and $m$, such that

$$f_{Km} \leq 0, \quad \forall z \in \partial Q_m. \tag{4.13}$$
Then by Theorem 4.1, $f_{K_m}$ has a critical value $c_{K_m}$, which is given by

$$c_{K_m} = \inf_{g \in \Gamma_m} \max_{z \in \mathcal{Q}_m} f_{K_m}(g(z)), \quad (4.14)$$

where $\Gamma_m = \{g \in C(\mathcal{Q}_m, \hat{E}_m | g = id; on \partial \mathcal{Q}_m}\}$. Moreover there is a critical point $x_{K_m}$ of $f_{K_m}$ which satisfies

$$m^-(x_{K_m}) \leq \text{dim } X_m + 1. \quad (4.15)$$

**Step 3.** We prove that there exists a $T$-periodic brake orbit solution $x_T$ of (1.1) which satisfies $i_{L_0}(x_T) \leq i_{L_0}(B_0) + \nu_{L_0}(B_0) + 1$.

Note that $id \in \Gamma_m$, by (4.1) and condition (H4), we have

$$c_{K_m} \leq \sup_{z \in \mathcal{Q}_m} f_{K_m}(z) \leq \frac{1}{2}|A_T - B_{0T}|^2. \quad (4.16)$$

Then $\{c_{K_m}\}$ possesses a convergent subsequence, we still denote it by $\{c_{K_m}\}$ for convenience. So there is a $c_K \in [\delta, \overline{\delta}]$ such that $c_{K_m} \to c_K$.

By the same arguments as in section 6 of [44] we have $f_K$ satisfies $\text{(PS)}_c$ condition for $c \in \mathbb{R}$, i.e., any sequence $z_m$ such that $z_m \in E_{T,m}$, $f_{K_m}(z_m) \to 0$ and $f_{K_m}(z_m) \to c$ possesses a convergent subsequence in $E_T$. Hence in the sense of subsequence we have

$$x_{K_m} \to x_K, \quad f_K(x_K) = c_K, \quad f'_K(x_K) = 0. \quad (4.16)$$

By similar argument in [44], $x_K$ is a classical nonconstant symmetric $T$-periodic solution of

$$\dot{x} = JH'_K(x), \quad x \in \mathbb{R}^{2n}.$$

Set $B_K(t) = H''_K(x_K(t))$, Then $B_K \in C([0,T/2], \mathcal{L}_s(\mathbb{R}^{2n}))$ and satisfies condition (B1). Let $B_{K_T}$ be the operator defined by the same way of the definition of $B_{0T}$. It is easy to show that

$$||f''(z) - (A_T - B_{K_T})|| \to 0 \quad \text{as } ||z - x_K|| \to 0.$$

So for $0 < d \leq \frac{1}{4}|||A_T - B_{K_T})||^{-1}$, there exists $r_2 > 0$ such that

$$||f''_{K_m}(z) - P_{T,m}(A_T - B_{K_T})P_{T,m}|| \leq ||f''(z) - (A_T - B_{K_T})|| \leq \frac{1}{2}d, \forall z \in \{z \in E_T : ||z - x_K|| \leq r_2\}.$$

Then for $z \in \{z \in E_T : ||z - x_K|| \leq r_2\} \cap E_{T,m}$, $\forall u \in M_d(P_{T,m}(A_T - B_{K_T})P_{T,m}) \setminus \{0\}$, we have

$$\langle f''_{K_m}(z)u, u \rangle \leq \langle P_{T,m}(A_T - B_{K_T})P_{T,m}u, u \rangle + ||f''_{K_m}(z) - P_{T,m}(A_T - B_{K_T})P_{T,m}|| ||u||^2$$

$$\leq \frac{1}{2}d||u||^2.$$
So we have

$$m^-(f''_{km}(z)) \geq \dim M^-_d(PT,m(AT - BKT)PT,m).$$

(4.17)

By Theorem 2.1 of [31] and Remark 3.1, there is $m^* > 0$ such that for $m \geq m^*$ we have

$$\dim X_m = mn + n + iL_0(B_0) + \nu L_0(B_0),$$

(4.18)

$$\dim M^-_d(PT,m(AT - BKT)PT,m) = mn + n + iL_0(B_K).$$

(4.19)

Then by (4.15), (4.16), and (4.17)-(4.19), we have

$$iL_0(B_K) \leq iL_0(B_0) + \nu L_0(B_0) + 1.$$  

By the similar argument as in the section 6 of [44], there is a constant $M_2$ independent of $K$ such that $\|x_K\|_\infty \leq M_2$. Choose $K > M_2$. Then $x_K$ is a non-constant symmetric $T$-periodic solution of the problem (1.1). From now on in the proof of Theorem 1.3, we write $B = B_K$ and $x_T = x_K$. Then $x_T$ is a non-constant symmetric $T$-periodic solution of the problem (1.1), and $B$ satisfies

$$iL_0(x_T) = iL_0(B) \leq iL_0(B_0) + \nu L_0(B_0) + 1.$$

(4.20)

Since $x_T$ obtained in Step 3 is a nonconstant and symmetric $T$-period solution, its minimal period $\tau = \frac{T}{k}$ for some $k \in \mathbb{N}$.

We denote by $x_\tau = x_T|_{[0,\tau]}$, then it is a brake orbit solution of (1.1) with the minimal $\tau$ and $X_T = x_K^k$ being the $k$ times iteration of $x_\tau$. As in Section 1, let $\gamma_{x_T}$ and $\gamma_{x_\tau}$ be the symplectic path associated to $(\tau, x)$ and $(T, x_T)$ respectively. Then $\gamma_{x_\tau} \in C([0, \frac{T}{k}], \text{Sp}(2n))$ and $\gamma_{x_T} \in C([0, \frac{T}{k}], \text{Sp}(2n))$. Also we have $\gamma_{x_T} = \gamma_{x_\tau}^k$.

**Step 4.** We prove that

$$iL_1(\gamma_{x_\tau}) + \nu L_1(\gamma_{x_\tau}) \geq 1.$$ 

We follow the way of the proof of Theorem 1.2 of [18]. By the same way as $\tilde{E}_T$ and $\tilde{A}_T$ we can define the space $\tilde{E}_\tau$ and the operator $\tilde{A}_\tau$ on it. Also we can define the orthogonal projection $\tilde{P}_\tau,m$ and the subspaces $\tilde{E}_{\tau,m}$ for $m = 0, 1, 2, \ldots$. Let $\tilde{B}_\tau$ be the selfadjoint linear compact operator on $\tilde{E}_T$ defined by:

$$\langle \tilde{B}_\tau z, z \rangle = \int_0^\tau B(t)z(t) \cdot z(t) \, dt, \quad \forall z \in \tilde{E}_\tau.$$ 

For $z \in \tilde{E}_\tau$, set

$$f_\tau(z) = \frac{1}{2}\langle (\tilde{A}_\tau - \tilde{B}_\tau)z, z \rangle = \frac{1}{2}\langle \tilde{A}_\tau z, z \rangle - \frac{1}{2} \int_0^\tau H''(x_\tau(t))z \cdot z \, dt$$
and
\[ f_{\tau m}(w) = f_{\tau}(w), \quad \forall w \in \tilde{E}_{\tau, m}. \]

Let
\[ X = \{ z \in L_1 | B_0 z = 0 \text{ and } \hat{H}''(x_\tau(t))z = 0, \forall t \in \mathbb{R} \} \]
and \( Y \) be the orthogonal complement of \( X \) in \( L_1 \), i.e., \( L_1 = X \oplus Y \). Since \( H''(x_\tau(t)) = B_0 + \hat{H}''(x_\tau(t)) \), by (H4) it is easy to see that there exists \( \lambda_0 > 0 \) such that
\[ \int_0^\tau H''(x_\tau(t))z_0 \cdot z_0 \, dt \geq \lambda_0 ||z_0||, \quad \forall z_0 \in Y. \]

Thus for any \( z = z_- + z_0 \in \tilde{P}_{\tau, m} M^- (\tilde{A}_\tau) \oplus Y \) with \( ||z|| = 1 \), we have
\begin{align*}
  f_{\tau m}(z) &= \frac{1}{2} \langle (\tilde{A}_\tau - \tilde{B}_\tau)z, z \rangle - \frac{1}{2} \int_0^\tau H''(x_\tau(t))z \cdot z \, dt \\
  &\leq -\frac{1}{2} ||\tilde{A}_\tau^#||^{-1} ||z_-||^2 - \frac{1}{2} \int_0^\tau H''(x_\tau(t))z_0 \cdot z_0 \, dt - \int_0^\tau H''(x_\tau(t))z_- \cdot z_0 \, dt \\
  &\leq -\frac{1}{2} ||\tilde{A}_\tau^#||^{-1} ||z_-||^2 - \frac{\lambda_0}{2} ||z_0||^2 + \max_{t \in [0, \tau]} ||H''(x_\tau(t))|| ||z_-|| ||z_0||. \tag{4.21}
\end{align*}

Since
\[ ||z_-|| ||z_0|| \leq \frac{\varepsilon}{4} ||z_-||^2 + \frac{1}{\varepsilon} ||z_0||^2, \quad \forall \varepsilon > 0. \]

By choosing \( \varepsilon \) suitably one can see that there exists \( 0 < c_0 < 1 \) with \( |1 - c_0| \) small enough such that if \( ||z_0|| \leq c_0 \),
\[ f_{\tau m}(z) \leq -\frac{\lambda_0}{4} c_0^2. \tag{4.23} \]

When \( ||z_0|| \leq c_0 \), we have \( ||z_-||^2 \geq 1 - c_0^2 \). By (4.21) and (H4)
\[ f_{\tau m}(z) \leq -\frac{1}{2} ||\tilde{A}_\tau^#||^{-1} ||z_-||^2 \leq -\frac{1}{2} ||\tilde{A}_\tau^#||^{-1} (1 - c_0^2). \]

Hence we always have
\[ f_{\tau m}(z) \leq -c ||z||^2, \quad \forall z \in \tilde{P}_{\tau, m} M^- (\tilde{A}_\tau) \oplus Y, \tag{4.24} \]
where \( c = \max\{\frac{\lambda_0}{4} c_0^2, \frac{1}{2} ||\tilde{A}_\tau^#||^{-1} (1 - c_0^2)\} \) is independent of \( m \). Let
\[ d = \min\{\frac{1}{4} ||(\tilde{A}_\tau - \tilde{B}_\tau)^#||^{-1}, \frac{c}{2} \}. \]
By (4.24) and Theorem 2.1 of [31] and Remark 3.1 and the definition of $i_{L_1}(\gamma(x_\tau))$, for $m$ large enough, we have

$$mn + n + i_{L_1}(\gamma(x_\tau)) = \dim M_d^- (\hat{P}_{\tau,m}(\hat{A}_\tau - \hat{B}_\tau)\hat{P}_{\tau,m})$$

$$\geq \dim(\hat{P}_{\tau,m}M^- (\hat{A}_\tau) \oplus Y)$$

$$= mn + n - \dim X,$$

which implies that

$$i_{L_1}(\gamma(x_\tau)) \geq - \dim X. \quad (4.25)$$

Since $x_\tau$ is a nonconstant brake solution of (1.1), by the definition of $X$ we have

$$\nu_{L_1}(\gamma(x_\tau)) \geq \dim X + 1. \quad (4.26)$$

Hence by (4.26) and (4.27) we have

$$i_{L_1}(\gamma(x_\tau)) + \nu_{L_1}(\gamma(x_\tau)) \geq 1. \quad (4.28)$$

**Step 5. Finish the proof of Theorem 1.3.**

By Theorem 2.1 and Theorem 6.2 below (also Theorem 2.6 of [32]) we have

$$i_{L_0}(\gamma^k_{x_\tau}) \geq i_{L_0}(\gamma_{x_\tau}) + \frac{k - 1}{2}(i_1(\gamma^2) + \nu_1(\gamma^2) - n), \quad \text{if } k \in 2\mathbb{N} - 1, \quad (4.29)$$

$$i_{L_0}(\gamma^k_{x_\tau}) \geq i_{L_0}(\gamma_{x_\tau}) + \frac{i_{L_0}}{\sqrt{2}}(\gamma_{x_\tau}) + (\frac{k}{2} - 1)(i_1(\gamma^2) + \nu_1(\gamma^2) - n), \quad \text{if } k \in 2\mathbb{N}. \quad (4.30)$$

Since $B_0$ is semipositive and $\hat{H}$ satisfies (H4), by Corollary 3.2, we have

$$i_{L_0}(\gamma_{x_\tau}) + \nu_{L_0}(\gamma_{x_\tau}) \geq 0. \quad (4.31)$$

By Proposition C of [42] and the definitions of $i_{L_0}$ and $i_{L_1}$ we have

$$i_1(\gamma^2) = i_{L_0}(\gamma) + i_{L_1}(\gamma) + n,$$

$$\nu_1(\gamma^2) = \nu_{L_0}(\gamma) + \nu_{L_1}(\gamma).$$

So by (4.28) and (4.31) we have

$$i_1(\gamma^2) + \nu_1(\gamma^2) - n \geq 1. \quad (4.32)$$

So by (4.29), (4.30) and (4.32) we have
By (4.20) and the definition of $\gamma_{x\tau}$ we have

$$i_{L_0}(\gamma_{x\tau}) \leq i_{L_0}(B_0) + \nu_{L_0}(B_0) + 1.$$  

(4.35)

By Corollary 3.2, we have

$$i_{L_0}(\gamma_{x\tau}) \geq -n.$$  

(4.36)

So by (4.33)-(4.36) we have

$$k \leq 2(i_{L_0}(B_0) + \nu_{L_0}(B_0)) + 2n + 2.$$  

(4.37)

Claim 2. $k$ can not be $2(i_{L_0}(B_0) + \nu_{L_0}(B_0)) + 2n + 3$. Otherwise, we have

$$k = 2(i_{L_0}(B_0) + \nu_{L_0}(B_0)) + 2n + 3.$$  

(4.38)

The equality in (4.29) holds, then by (4.32), in this case there must hold that

$$i_1(\gamma^2) + \nu_1(\gamma^2) - n = 1.$$  

(4.39)

and

$$i_{L_0}(\gamma_{x\tau}) = -n.$$  

(4.40)

By Corollary 3.2 again we have that

$$\nu_{L_0}(\gamma_{x\tau}) = n.$$  

(4.41)

Also by (4.39) we have

$$i_{L_1}(\gamma(x\tau)) + \nu_{L_1}(\gamma(x\tau)) = 1.$$  

Denote by $\nu_{L_1}(\gamma(x\tau)) = r$. Then we have

$$i_{L_1}(\gamma(x\tau)) = 1 - r,$$  

(4.42)

$$\nu_{L_1}(\gamma(x\tau)) \geq 1.$$  

(4.43)
By (4.40) and (4.42) we have
\[ i_{L_0}(\gamma x) - i_{L_1}(\gamma x) = r - n - 1. \]

(4.44)

So we can write \( \gamma x(\tau) = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \) with \( A, C, D \) to be \( n \times n \) real matrices. Hence by (4.2) of [35] we have
\[ \gamma^2 x(\tau) = N \gamma x(\frac{\tau}{2})^{-1} N \gamma x(\frac{\tau}{2}) = \begin{pmatrix} D^T A & 0 \\ C^T A & A^T D \end{pmatrix}. \]

Since \( \gamma x(\frac{\tau}{2}) \) is a symplectic matrix we have
\[ A^T D = D^T A = I_n, \quad C^T A = A^T C. \]

So we have
\[ \gamma^2 x(\tau) = \begin{pmatrix} I_n & 0 \\ C^T A & I_n \end{pmatrix}. \]

Note that here \( C^T A \) is a symmetric matrix and \( A \) is invertible. So by (4.43) there exists an orthogonal matrix \( Q \) such that
\[ Q(C^T A)Q^T = \text{diag}(0, 0, ..., 0, \lambda_1, \lambda_2, ..., \lambda_{n-r}, \lambda_{n-r+1}, ..., \lambda_{n-p-r}) \]

with \( \lambda_j > 0 \) for \( j = 1, 2, ..., p \) and \( \lambda_j < 0 \), for \( j = p + 1, p + 2, ..., n - p - r \), where \( 1 \leq p \leq n - r \). Then it is easy to check that \( (I_2)^{\circ \phi} \circ N_1(1, -1) custom N_1(1, 1) \circ \delta^{(n-p-r)} \in \Omega^0(\gamma x) \) with \( \Omega^0(\gamma x) \) to be defined in Section 6 below. Then by Theorem 6.2 below or Theorem 2.6 of [32], when the equality in (4.29) holds, there must hold \( p = n - r \). Hence we have
\[ Q(C^T A)Q^T = \text{diag}(0, 0, ..., 0, \lambda_1, \lambda_2, ..., \lambda_{n-r}), \]

(4.46)

\[ \lambda_j > 0, \quad \text{for } j = 1, 2, ..., n - r. \]

(4.47)

**Case 1.** If \( \det A > 0 \), then there exists an invertible matrix path \( \rho(s) \) for \( s \in [0, \frac{\tau}{2}] \) connecting \( I_n \) and \( A \) such that \( \rho(0) = I_n \) and \( \rho(1) = A \).

We define a symplectic path \( \phi_1 \) by
\[ \phi_1(s) = \begin{pmatrix} \rho(s)^{-1} & 0 \\ 0 & \rho(s)^T \end{pmatrix} \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}, \quad \forall s \in [0, \frac{\tau}{2}]. \]
Then \( \nu_{L_j}(\phi_1(s)) = \text{constant} \) for \( j = 0, 1 \) and \( s \in [0, \frac{\tau}{2}] \). So by Definition 2.5 and Lemma 2.8 and Proposition 2.11 of [42], for \( j = 1, 2 \) we have

\[
\mu_{F}^{CLM}(V_j, \text{Gr}(\phi_1), [0, \frac{\tau}{2}]) = 0. \tag{4.48}
\]

Also we have

\[
\phi_1(0) = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \quad \text{and} \quad \phi_1(0) = \begin{pmatrix} I_n & 0 \\ A^T C & I_n \end{pmatrix}.
\]

Note that we can always choose the orthogonal matrix \( Q \) in (4.46) such that \( \det Q = 1 \) (otherwise we replace it by \( \text{diag}(-1, 1, ..., 1)Q \)). Then there exists an invertible matrix path \( \rho_2(s) \) for \( s \in [0, \frac{\tau}{2}] \) connecting it and \( I_n \) such that \( \rho_2(0) = I_n \) and \( \rho_2(\frac{\tau}{2}) = Q \). We define a symplectic path \( \phi_2 \) by

\[
\phi_2(s) = \begin{pmatrix} I_n & 0 \\ \rho_2(s)A^T C \rho_2(s)^T & I_n \end{pmatrix}, \quad \forall s \in [0, \frac{\tau}{2}].
\]

Then \( \nu_{L_j}(\phi_2(s)) = \text{constant} \) and for \( j = 0, 1 \) and \( s \in [0, \frac{\tau}{2}] \). So by Definition 2.5 and Lemma 2.8 and Proposition 2.11 of [42] again, for \( j = 1, 2 \) we have

\[
\mu_{F}^{CLM}(V_j, \text{Gr}(\phi_2), [0, \frac{\tau}{2}]) = 0. \tag{4.49}
\]

Also we have

\[
\phi_2(0) = \begin{pmatrix} I_n & 0 \\ A^T C & I_n \end{pmatrix}, \\
\phi_2(\frac{\tau}{2}) = \begin{pmatrix} I_n & 0 \\ QA^T C Q^T & I_n \end{pmatrix} = (I_2)^{\circ r} \circ N_1(1, \lambda_1) \circ \cdots \circ N_1(1, \lambda_{n-r}). \tag{4.50}
\]

By the Reparametrization invariance and Path additivity of the Maslov index \( \mu_{F}^{CLM} \) in [11] and (4.48) and (4.49), for \( j = 1, 2 \) we have

\[
\mu_{F}^{CLM}(V_j, \text{Gr}(\gamma_{x_r}), [0, \frac{\tau}{2}]) = \mu_{F}^{CLM}(V_j, \text{Gr}(\phi_2 \ast (\phi_1 \ast \gamma_{x_r})), [0, \frac{\tau}{2}]),
\]

where the joint path \( \phi_2 \ast (\phi_1 \ast \gamma_{x_r}) \) is defined by (6.1). So by definition for \( j = 0, 1 \) we have

\[
i_{L_j}(\gamma_{x_r}) = i_{L_j}(\phi_2 \ast (\phi_1 \ast \gamma_{x_r})). \tag{4.51}
\]

Then by Theorem 2.3 and (4.50) we have

\[
i_{L_0}(\gamma_{x_r}) - i_{L_1}(\gamma_{x_r}) = \frac{1}{2} \text{sgn} M_{\varepsilon}((I_2)^{\circ r} \circ N_1(1, \lambda_1)^T \circ \cdots \circ N_1(1, \lambda_{n-r})^T). \tag{4.52}
\]

By Remark 2.1 and the computations (2.68)-(2.71) at the end of Section 2, for \( \varepsilon > 0 \) small enough we have

\[
\text{sgn} M_{\varepsilon}((I_2)^{\circ r} \circ N_1(1, \lambda_1)^T \circ \cdots \circ N_1(1, \lambda_{n-r})^T) = 2(r - n). \tag{4.53}
\]
So we have
\[ iL_0(\gamma x_\tau) - iL_1(\gamma x_\tau) = r - n, \tag{4.54} \]
which contradicts to (4.44).

**Case 2.** If \( \det A < 0 \), then there exists a invertible matrix path \( \rho(s) \) for \( s \in [0, \frac{\tau}{2}] \) such that \( \rho(0) = \text{diag}(-1, 1, 1, ..., 1) \) and \( \rho(1) = A \). by similar arguments we can show that
\[ iL_0(\gamma x_\tau) - iL_1(\gamma x_\tau) = \frac{1}{2} \text{sgn} M_\varepsilon((-I_2) \triangle (I_2)^{\circ(r-1)} \triangle N_1(1, \lambda_1) \triangle ... \triangle N_1(1, \lambda_{n-r})) = r - n, \tag{4.55} \]
which still contradicts to (4.44).

Hence we have proved that \( k \) can not be \( 2(iL_0(B_0) + \nu L_0(B_0)) + 2n + 3 \). By the same argument we can prove that \( k \) can not be \( 2(iL_0(B_0) + \nu L_0(B_0)) + 2n + 4 \). Thus Claim 2 is proved and the proof of Theorem 1.3 is complete.

**Proof of Theorem 1.1.** Note that this is the case \( B_0 = 0 \) of Theorem 1.3. Then by Theorem 1.3 and the fact that \( iL_0(0) = -n \) and \( \nu L_0(0) = n \), the minimal period of \( x_\tau \) is no less than \( \frac{T}{2n+2} \). In the following we prove that if (1.9) holds then the minimal period of \( x_\tau \) belongs to \( \{T, \frac{T}{2}\} \).

Let \( x_\tau \) is the \( k \)-time iteration of \( x_\tau \) with \( \tau \) being the minimal period of \( x_\tau \) and \( \tau = \frac{T}{k} \). Then by the proof of Theorem 1.3 with \( B_0 = 0 \) we have (4.28), (4.29) and (4.30) hold. Since (1.9) holds, by Lemma 3.3 we have
\[ iL_0(\gamma x_\tau) \geq 0. \tag{4.56} \]
So by (4.29) if \( k \) is odd, we have
\[ 1 \geq 0 + \frac{k-1}{2}. \tag{4.57} \]
Hence \( k \leq 3 \). Now we prove that \( k \) can not be 3, other wise we have
\[ iL_0(\gamma x_\tau) = 0, \tag{4.58} \]
\[ \nu L_0(\gamma x_\tau) = 0, \tag{4.59} \]
\[ iL_1(\gamma x_\tau) + \nu L_1(\gamma x_\tau) = 1. \tag{4.60} \]
And by Theorem 2.1 and Theorem 6.2 we have
\[ 1 \geq iL_0(\gamma^3 x_\tau) = iL_0(\gamma x_\tau) + i_\varepsilon x_\tau/3(\gamma x_\tau) \geq (i_1(\gamma^2 x_\tau) - \nu_1(\gamma x_\tau) - n) \geq 1. \tag{4.61} \]
Then all the equalities of (4.61) hold. By Lemma 6.2 and 2 of Theorem 6.2 again, there exist \( p \geq 0 \), \( q \geq 0 \) with \( p + q \leq n \) and \( 0 < \theta_1 \leq \theta_2 \leq ... \leq \theta_{n-(p+q)} \leq 2\pi/3 \) such that
\[ (I_2)^{\circ p} \circ N_1(1, -1)^{\circ q} \circ R(\theta_1) \circ R(\theta_2) \circ ... \circ R(\theta_{n-p-q}) \in \Omega^0((\gamma x_\tau)(\tau)), \tag{4.62} \]
where $\Omega^0(M)$ for a symplectic matrix $M$ is defined in Section 6. By (4.62) we have

$$-1 \notin \sigma((\gamma_2)(\tau)).$$

Now we denote by $\gamma_{x,}(\frac{\tau}{2}) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A, B, C, D$ are all $n \times n$ matrices.

Claim 1. Both $D$ and $A$ are invertible.

We first prove $D$ is invertible. Otherwise, there exists a $n \times n$ invertible matrix $P$ such that

$$P^{-1}DP = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}$$

and $R$ is a $(n-r) \times (n-r)$ matrix with $r \geq 1$. So we have

$$\begin{pmatrix} P^T & 0 \\ 0 & P^{-1} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} (P^{-1})^T & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$$

with $\tilde{D} = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}$. Since $\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}$ is a symplectic matrix, we have

$$\tilde{A}^T\tilde{D} - \tilde{C}^T\tilde{B} = I_n. \quad (4.64)$$

Since $\tilde{D} = \begin{pmatrix} 0 & 0 \\ 0 & R \end{pmatrix}$, $\tilde{B}^T\tilde{D}$ and $\tilde{A}^T\tilde{D}$ both have form

$$\tilde{B}^T\tilde{D} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}, \quad \tilde{A}^T\tilde{D} = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}. \quad (4.65)$$

So by (4.64) and (4.65) we have

$$\tilde{A}^T\tilde{D} + \tilde{C}^T\tilde{B} = 2\tilde{A}^T\tilde{D} - I_n = \begin{pmatrix} -I_r & * \\ 0 & * \end{pmatrix}. \quad (4.66)$$

By direct computation and (4.65) and (4.66) we have

$$N \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix}^{-1} N \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \quad (4.67)$$

$$= \begin{pmatrix} P^T & 0 \\ 0 & P^{-1} \end{pmatrix} N \begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} N \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} (P^{-1})^T & 0 \\ 0 & P \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{D}^T\tilde{A} + \tilde{B}^T\tilde{C} & 2\tilde{B}^T\tilde{D} \\ 2\tilde{A}^T\tilde{C} & \tilde{A}^T\tilde{D} + \tilde{C}^T\tilde{B} \end{pmatrix}$$

$$= \begin{pmatrix} * & * & 0 & * \\ * & * & 0 & * \\ * & * & -I_r & * \\ * & * & 0 & * \end{pmatrix} \quad (4.68)$$
Since by (4.2) of [35] we have
\[ \gamma_{x, r}^2(\tau) = N \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)^{-1} N \left( \begin{array}{cc} A & B \\ C & D \end{array} \right), \]
by (4.67) and (4.68) we have
\[-1 \in \sigma(\gamma_{x, r}^2(\tau)),\]
which contradicts to (4.63). Thus we have proved that \( D \) is invertible. Similarly we can prove \( A \) is invertible, and Claim 1 is proved.

**Claim 2.** There exists an invertible \( n \times n \) real matrix \( Q \) with \( \det Q > 0 \) such that
\[ Q^{-1} (B^T C) Q = \text{diag}(0, 0, ..., 0, \lambda_1, \lambda_2, ..., \lambda_{n-r}) \] (4.69)
with \( r = \nu_{L_1}(\gamma_{x, r}) \) and \( \lambda_i \in (-1, 0) \) for \( i = 1, 2, ..., n - r \).

In fact
\[ \gamma_{x, r}^2(\tau) = N \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)^{-1} N \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \]
\[ = \left( \begin{array}{cc} D^T & B^T \\ C^T & A^T \end{array} \right) \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \]
\[ = \left( \begin{array}{cc} I + 2B^T C & 2B^T D \\ 2A^T C & I + 2C^T B \end{array} \right). \] (4.70)

Since \( B \) and \( D \) are both invertible, for any \( \omega \in \mathbb{C} \), we have
\[ \begin{pmatrix} I_n & 0 \\ -\frac{1}{2}(I_n + 2C^T B - \omega I_n)D^{-1}(B^T)^{-1} & I_n \end{pmatrix} \left( \begin{array}{cc} I + 2B^T C - \omega I_n & 2B^T D \\ 2A^T C & I + 2C^T B - \omega I_n \end{array} \right) \]
\[ = \begin{pmatrix} I + 2B^T C - \omega I_n & 2B^T D \\ -\frac{1}{2}(I_n + 2C^T B - \omega I_n)D^{-1}(B^T)^{-1}(I + 2B^T C - \omega I_n) + 2A^T C & 0 \end{pmatrix}. \]

So we have
\[ \det(\gamma_{x, r}^2(\tau) - \omega I_{2n}) = \det(B^T D) \det((I_n + 2C^T B - \omega I_n)D^{-1}(B^T)^{-1}(I + 2B^T C - \omega I_n) - 4A^T C) \]
\[ = \det(D(I_n + 2C^T B - \omega I_n)D^{-1}(B^T)^{-1}(I + 2B^T C - \omega I_n) - 4A^T C)B^T) \]
\[ = \det(D[I_n + 2C^T B - \omega I_n]D^{-1}(B^T)^{-1}(I + 2B^T C - \omega I_n)B^T - 4DA^T CB^T) \]
\[ = \det((I + 2B^T C - \omega I_n)^2 - 4(1 + CB^T)CB^T) \]
\[ = \det(\omega^2 I_n - 2\omega(I + 2CB^T) + I). \] (4.71)
By \([4.62]\) we have

$$\sigma(\gamma^2_{x_\tau}(\tau)) \subset U.$$  \hspace{1cm} (4.72)

So for \(\omega \in U\) by \([4.71]\) we have

$$\det(\gamma^2_{x_\tau}(\tau) - \omega I_{2n}) = (-4)^n \omega^n \det(CB^T - \frac{1}{2}(\Re \omega - 1)).$$  \hspace{1cm} (4.73)

Hence by \([4.62]\) again we have \(\sigma(CB^T) \subset (-1, 0]\), moreover there exists a invertible \(n \times n\) matrix \(S\) such that

$$S^{-1}CB^T S = \text{diag}(0, 0, \ldots, 0, \lambda_1, \lambda_2, \ldots, \lambda_{n-r}).$$  \hspace{1cm} (4.74)

with \(r = \nu_{L_1}(\gamma_{x_\tau})\) and \(\lambda_i \in (-1, 0)\) for \(i = 1, 2, \ldots, n - r\). Since \(S^{-1}CB^T S = (B^T S)^{-1}B^T C(B^T S)\), let \(Q = B^T S\), if \(\det Q < 0\) we replace it by \(B^T S \text{diag}(-1, 1, \ldots, 1)\), Claim 2 is proved.

**Continue the proof of Theorem 1.1.**

If \(\det B > 0\), there is a continuous symplectic matrix path joint \(\begin{pmatrix} \begin{array}{cc} B^{-1} & 0 \\ 0 & B^T \end{array} \end{pmatrix}\) and \(I_{2n}\). Since

$$\begin{pmatrix} \begin{array}{cc} B^{-1} & 0 \\ 0 & B^T \end{array} \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} B^{-1}A & I_n \\ B^T C & B^T D \end{pmatrix}.$$  

By Lemma 2.2, for \(\varepsilon > 0\) small enough, we have

$$\text{sgn} M_{\varepsilon} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \text{sgn} M_{\varepsilon} \left( \begin{array}{cc} B^{-1}A & I_n \\ B^T C & B^T D \end{array} \right).$$  \hspace{1cm} (4.75)

If \(\det B < 0\), there is a continuous symplectic matrix path joint \(\begin{pmatrix} \begin{array}{cc} B^{-1} & 0 \\ 0 & B^T \end{array} \end{pmatrix}\) and \((-I_2) \circ I_{2(n-1)}\). By direct computation we have

$$\text{sgn} M_{\varepsilon} \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \text{sgn} M_{\varepsilon} \left( (-I_2) \circ I_{2(n-1)} \right) \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$  

So by Lemma 2.2 again we have \([4.75]\) holds. So whenever \(\det(B) > 0\) or not, \([4.75]\) always holds.

Denote by \(\begin{pmatrix} \begin{array}{cc} P^T & 0 \\ 0 & P^{-1} \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{cc} B^{-1}A & I_n \\ B^T C & B^T D \end{array} \end{pmatrix} \begin{pmatrix} \begin{array}{cc} P & 0 \\ 0 & (P^{-1})^T \end{array} \end{pmatrix} = \begin{pmatrix} \tilde{A} & I_n \\ \tilde{C} & \tilde{D} \end{pmatrix}.$$  

By Claim 2, we have

$$\tilde{C} = \text{diag}(0, 0, \ldots, 0, \lambda_1, \lambda_2, \ldots, \lambda_{n-r}).$$  \hspace{1cm} (4.76)

Since \(\begin{pmatrix} \begin{array}{cc} \tilde{A} & I_n \\ \tilde{C} & \tilde{D} \end{array} \end{pmatrix}\) is a symplectic matrix, we have \(\tilde{A}\) and \(\tilde{D}\) are both symmetric and have the following forms:

$$\tilde{A} = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \quad \tilde{D} = \begin{pmatrix} D_{11} & 0 \\ 0 & D_{22} \end{pmatrix}.$$  

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where $A_{11}$ and $D_{11}$ are $r \times r$ invertible matrices, $A_{22}$ and $D_{22}$ are $(n - r) \times (n - r)$ invertible matrices. So we have

$$
\begin{pmatrix}
\tilde{A} & I_n \\
\tilde{C} & \tilde{D}
\end{pmatrix} = \begin{pmatrix} A_{11} & I_r \\ 0 & D_{11} \end{pmatrix} \circ \begin{pmatrix} A_{22} & I_{n-r} \\ \Lambda & D_{22} \end{pmatrix},
$$

(4.77)

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_{n-r})$.

Since $N \begin{pmatrix} A_{11} & I_r \\ 0 & D_{11} \end{pmatrix}^{-1} N \begin{pmatrix} A_{11} & I_r \\ 0 & D_{11} \end{pmatrix} = \begin{pmatrix} I_r & 2D_{11} \\ 0 & I_r \end{pmatrix}$, by (4.62) $D_{11}$ is negative definite. So we can joint it to $-I_r$ by a invertible symmetric matrix path. Then by Lemma 2.2, Remark 2.1, and computations below Remark 2.1 in Section 2, we have

$$
\text{sgn} M_\varepsilon \begin{pmatrix} A_{11} & I_r \\ 0 & D_{11} \end{pmatrix} = \text{sgn} M_\varepsilon \begin{pmatrix} -I_r & I_r \\ 0 & -I_r \end{pmatrix} = r \text{sgn} M_\varepsilon (N_1(-1, 1)) = 2r.
$$

(4.78)

Since $\begin{pmatrix} A_{22} & I_{n-r} \\ \Lambda & D_{22} \end{pmatrix}$ is invertible for $\varepsilon = 0$, for $\varepsilon > 0$ small enough, we have

$$
\text{sgn} M_\varepsilon \begin{pmatrix} A_{22} & I_{n-r} \\ \Lambda & D_{22} \end{pmatrix} = \text{sgn} M_0 \begin{pmatrix} A_{22} & I_{n-r} \\ \Lambda & D_{22} \end{pmatrix} = \text{sgn} \begin{pmatrix} A_{22} & I_{n-r} \\ \Lambda & D_{22} \end{pmatrix} \circ \begin{pmatrix} 0 & -I_{n-r} \\ -I_{n-r} & 0 \end{pmatrix} \begin{pmatrix} A_{22} & I_{n-r} \\ \Lambda & D_{22} \end{pmatrix} + \begin{pmatrix} 0 & I_{n-r} \\ I_{n-r} & 0 \end{pmatrix}
$$

$$
= \text{sgn} \begin{pmatrix} 2 \begin{pmatrix} -A_{22} \Lambda & -\Lambda \\ -\Lambda & -D_{22} \end{pmatrix} \\ \begin{pmatrix} -A_{22} \Lambda & -\Lambda \\ -\Lambda & -D_{22} \end{pmatrix} \end{pmatrix}.
$$

(4.79)

Since $\begin{pmatrix} A_{22} & I_{n-r} \\ \Lambda & D_{22} \end{pmatrix}$ is a symplectic matrix, we have

$$
A_{22}D_{22} - \Lambda = I_{n-r},
$$

$$
A_{22}\Lambda = \Lambda A_{22}.
$$

(4.80)

Hence

$$
A_{22}^{-1}\Lambda - D_{22} = A_{22}^{-1}(\Lambda - A_{22}D_{22}) = -A_{22}^{-1}.
$$
So we have
\[
\begin{pmatrix}
I_{n-r} & 0 \\
-A_{22}^{-1} & I_{n-r}
\end{pmatrix}
\begin{pmatrix}
-A_{22} & -\Lambda \\
-\Lambda & -D_{22}
\end{pmatrix}
\begin{pmatrix}
I_{n-r} & -A_{22}^{-1} \\
0 & I_{n-r}
\end{pmatrix}
= \begin{pmatrix}
-A_{22} & 0 \\
0 & A_{22}^{-1} \Lambda - D_{22}
\end{pmatrix}
= \begin{pmatrix}
-A_{22} & 0 \\
0 & -A_{22}^{-1}
\end{pmatrix}.
\]
(4.81)

By (4.80), there exist invertible matrix \( R \) such that
\[
R^{-1} A_{22} R = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_{n-r}), \quad \alpha_i \in \mathbb{R} \setminus \{0\}, \ i = 1, 2, \ldots, n-r,
\]
(4.82)
\[
R^{-1} \Lambda R = \text{diag}(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_{n-r}}), \quad \{i_1, i_2, \ldots, i_{n-r}\} = \{1, 2, \ldots, n-r\}.
\]
(4.83)

So we have
\[
R^{-1}(-A_{22} \Lambda) R = \text{diag}(-\lambda_{i_1} \alpha_1, -\lambda_{i_2} \alpha_2, \ldots, -\lambda_{i_{n-r}} \alpha_{n-r}),
\]
(4.84)
\[
R^{-1}(-A_{22}^{-1}) R = \text{diag}(-\frac{1}{\alpha_1}, -\frac{1}{\alpha_2}, \ldots, -\frac{1}{\alpha_{n-r}}).
\]
(4.85)

Since \( \lambda_i \in (-1, 0) \) for \( i = 1, 2, \ldots, n-r \), by (4.82)-(4.85) we have
\[
\text{sgn}(-A_{22} \Lambda) + \text{sgn}(-A_{22}^{-1}) = 0.
\]
(4.86)

Hence by (4.79), (4.81) and (4.86) we have
\[
\text{sgn} M_\epsilon \begin{pmatrix}
A_{22} & I_{n-r} \\
\Lambda & D_{22}
\end{pmatrix} = \text{sgn}(-A_{22} \Lambda) + \text{sgn}(-A_{22}^{-1}) = 0.
\]
(4.87)

Since \( \det Q > 0 \) we can joint it to \( I_n \) by a invertible matrix path. Hence by Lemma 2.2 and Remark 2.1, (4.77), (4.78) and (4.87), we have
\[
\text{sgn} M_\epsilon \begin{pmatrix}
B^{-1} A & I_n \\
B^T C & B^T D
\end{pmatrix} = \text{sgn} M_\epsilon \begin{pmatrix}
A_{11} & I_r \\
0 & D_{11}
\end{pmatrix} + \text{sgn} M_\epsilon \begin{pmatrix}
A_{22} & I_{n-r} \\
\Lambda & D_{22}
\end{pmatrix}
= 2r + 0
= 2r.
\]
(4.88)

Then by Theorem 2.3, (4.75) and (4.88) we have
\[
i_L_0(\gamma_{x_r}) - i_{L_1}(\gamma_{x_r}) = r.
\]
(4.89)
However by (4.58), (4.60) and \( \nu_{L_1}(\gamma_{x_T}) = r \) we have

\[ i_{L_0}(\gamma_{x_T}) - i_{L_1}(\gamma_{x_T}) = r - 1, \]

which contradicts to (4.89).

Thus we have prove that \( k \) can not be 3. So if \( k \) is odd, it must be 1. By the same proof we have if \( k \) is even, it must be 2. Then \( \tau \in \{ T, \frac{T}{2} \} \). The proof of Theorem 1.1 is complete.

**Proof of Corollary 1.2.** Since \( 0 < T < \frac{\pi}{||B_0||} \), there is \( \varepsilon > 0 \) small enough such that

\[ 0 \leq B_0 \leq ||B_0||I_{2n} < \left( \frac{\pi}{T} - \varepsilon \right)I_{2n}. \]

It is easy to see that

\[ \gamma(\frac{\pi}{T} - \varepsilon)I_{2n}(t) = \exp\left( (\frac{\pi}{T} - \varepsilon) tJ \right) \quad \forall t \in [0, \frac{T}{2}]. \]

So we have

\[ \nu_{L_0}(\gamma(\frac{\pi}{T} - \varepsilon)I_{2n}) = 0, \]
\[ i_{L_0}(\frac{\pi}{T} - \varepsilon)I_{2n}) = 0. \]

Then by (5.40) and Lemma 3.1 and Corollary 3.1 we have

\[ 0 \leq i_{-1}(B_0) + \nu_{-1}(B_0) \leq i_{-1}(\frac{\pi}{T} - \varepsilon)I_{2n}) = 0. \]

So we have

\[ i_{-1}(B_0) + \nu_{-1}(B_0) = 0. \]

Hence by the same proof of Theorem 1.1, the conclusions of Corollary 1.2 holds.

**Remark 4.1.** Under the same conditions of Theorem 1.3, if \( \int_0^T H_{22}^n(x_T(t)) \, dt > 0 \), by the same proof of Theorem 1.1, we have

\[ \tau \geq \frac{T}{2(i_{L_0}(B_0) + \nu_{L_0}(B_0)) + 2}. \]

Moreover, if \( 0 < T < \frac{\pi}{||B_0||} \) or \( i_{L_0}(B_0) + \nu_{L_0}(B_0) = 0 \), we have \( \tau \in \{ T, \frac{T}{2} \} \).

**Proof of Theorem 1.2.** This is the case \( n = 1 \) and \( B_0 = 0 \) of Theorem 1.3, by the proof Theorem 1.3, for any \( T > 0 \) we obtain an \( T \)-periodic brake solution \( x_T \) satisfies

\[ i_{L_0}(\gamma_{x_T}) \leq 1. \]
If it’s minimal period is $\tau = T/k$ for some $k \in \mathbb{N}$, we denote $x_\tau = x_{T[0,\tau]}$. Then by the proof of Theorem 1.3 we have

$$i_1(\gamma^2_\tau) + \nu_1(\gamma^2_\tau) \geq 2.$$  \hfill (4.91)

In the following we prove Theorem 1.2 in 2 steps.

**Step 1.** For $k = 2p + 1$ for some $p \geq 0$, we prove that $p = 0$.

Firstly by the proof of Theorem 1.3 we have

$$1 \geq i_{L_0}(\gamma^{2p+1}_x) \geq p(i_1(\gamma^2_x) + \nu_1(\gamma^2_x) - 1) + i_{L_0}(\gamma).$$  \hfill (4.92)

We divide the argument into three cases.

**Case 1.** $i_1(\gamma^2_x) + \nu_1(\gamma^2_x) = 2$. If $\nu_1(\gamma^2_x) = 1$, then $i_1(\gamma^2_x) = 1 \in 2\mathbb{Z} + 1$. By Lemma 6.3, we have $N_1(1,1) \in \Omega^0(\gamma^2_x(\tau))$. Since

$$1 = i_1(\gamma^2_x) = i_{L_0}(\gamma_x) + i_{L_1}(\gamma_x) + 1.$$  \hfill (4.93)

By Corollary 2.1 we have

$$|i_{L_0}(\gamma_x) - i_{L_1}(\gamma_x)| \leq 1.$$  \hfill (4.94)

Then by (4.93) and (4.94) we have

$$i_{L_0}(\gamma_x) = i_{L_1}(\gamma_x) = 0.$$  \hfill (4.95)

So by Theorem 2.1, Lemma 6.2, and (6.13), we have

$$i_{L_0}(\gamma^3_x) = i_{L_0}(\gamma_x) + i_{2\pi\sqrt{-1/3}}(\gamma^2_x)$$
$$= i_{L_0}(\gamma_x) + (\gamma^2_x) + S_{N_1(1,1)}(1)$$
$$= 0 + 1 + 1$$
$$= 2 > 1 \geq i_{L_0}(\gamma^{2p+1}_x).$$  \hfill (4.96)

Then by Theorem 3.3 we have

$$2p + 1 < 3.$$

Hence $p = 0$.

If $\nu_1(\gamma^2_x) = 2$, then $i_1(\gamma^2_x) = 0$. But now $\gamma^2_x(\tau) = I_2$, by Lemma 6.3 $i_1(\gamma^2_x) \in 2\mathbb{Z} + 1$, which yields a contradiction. So this case can not happen. So in Case 1, we have proved $p = 0$.

**Case 2.** $i_1(\gamma^2_x) + \nu_1(\gamma^2_x) = 3$. 


If $\nu_1(\gamma^2_{x\tau}) = 1$, then

$$i_1(\gamma^2_{x\tau}) = 2 \in 2\mathbb{Z}. \quad (4.97)$$

By Lemma 6.3 we have $N_1(1, -1) \in \Omega^0((\gamma^2_{x\tau})(\tau))$. So if $p \geq 1$, by Theorem 3.3, Theorem 2.1, Lemma 6.2 and (6.13), we have we have

$$1 \geq i_{L_0}(\gamma^{2p+1}_{x\tau}) \geq i_{L_0}(\gamma^{3}_{x\tau})$$

$$= i_{L_0}(\gamma_{x\tau}) + i_{e^{2\pi \sqrt{-1/3}}(\gamma^2_{x\tau})}$$

$$= i_{L_0}(\gamma_{x\tau}) + i_1(\gamma^2_{x\tau}) + S_{N_1}(1, -1)(1)$$

$$\geq -1 + 2 + 0$$

$$= 1. \quad (4.98)$$

So there must hold

$$i_{L_0}(\gamma_{x\tau}) = -1.$$ 

Then by Corollary 2.1 we have

$$i_{L_1}(\gamma_{x\tau}) \leq 0.$$ 

So we have

$$i_1(\gamma^2_{x\tau}) = i_{L_0}(\gamma_{x\tau}) + i_{L_1}(\gamma_{x\tau}) + 1 \leq -1 + 0 + 1 = 0,$$

which contradicts (4.97). Thus we have $p = 0$.

If $\nu_1(\gamma^2_{x\tau}) = 2$, then

$$i_1(\gamma^2_{x\tau}) = 1, \quad \gamma^2_{x\tau}(\tau) = I_2. \quad (4.99)$$

If $p \geq 1$, by Theorem 3.3, Theorem 2.1, Corollary 3.2, Lemma 6.2 and (6.13), we have we have

$$1 \geq i_{L_0}(\gamma^{2p+1}_{x\tau}) \geq i_{L_0}(\gamma^{2p+1}_{x\tau})$$

$$= i_{L_0}(\gamma_{x\tau}) + i_{e^{2\pi \sqrt{-1/3}}(\gamma^2_{x\tau})}$$

$$= i_{L_0}(\gamma_{x\tau}) + i_1(\gamma^2_{x\tau}) + S_{I_2}(1)$$

$$\geq -1 + 1 + 1$$

$$= 1.$$ 

So there must hold

$$i_{L_0}(\gamma_{x\tau}) = -1.$$
Then by Corollary 2.1 we have

\[ i_{L_1}(\gamma_{x_T}) \leq 0. \]

So we have

\[ i_1(\gamma^2_{x_T}) = i_{L_0}(\gamma_{x_T}) + i_{L_1}(\gamma_{x_T}) + 1 \leq -1 + 0 + 1 = 0, \]

which contradicts (4.99). Thus we have \( p = 0. \)

**Case 3.** \( i_1(\gamma^2_{x_T}) + \nu_1(\gamma^2_{x_T}) \geq 4. \)

In this case \( i_1(\gamma^2_{x_T}) + \nu_1(\gamma^2_{x_T}) - 1 \geq 3. \) By Corollary 3.2 we have \( i_{L_0} \geq -1. \) So by (4.92) we have

\[ p \leq 2/3, \quad (4.100) \]

which yields \( p = 0. \) So we finish Step 1.

**Step 2.** For \( k = 2p + 2 \) for some \( p \geq 0, \) we prove that \( p = 0. \)

In fact, apply Bott-type iteration formula of Theorem 2.1 to the the case of the iteration time equals to 4 and note that by Corollary 3.1 \( i_{\sqrt{-1}}(\gamma_{x_T}) \geq 0. \) Then by the same argument of Step 1, we can prove that \( p = 0. \)

Thus by Steps 1 and 2, Theorem 1.2 is proved.

A natural question is that can we prove the minimal period is \( T \) in this way? We have the following remark.

**Remark 4.2.** Only use the Maslov-type index theory to estimate the iteration time of the \( T \)-periodic brake solution \( x_T \) obtained by the first 4 steps in the proof of Theorem 1.3 with \( B_0 = 0, \) we can not hope to prove \( T \) is the minimal period of \( x_T. \) Even \( H''(z) > 0 \) for all \( z \in \mathbb{R}^{2n} \setminus \{0\}. \)

For \( n = 1 \) and \( T = 4\pi, \) we can not exclude the following case:

\[ x_T(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}, \]

\[ H'(x_T(t)) = x_T(t), \]

\[ H''(x_T(t)) \equiv I_{2n}. \]

It is easy to check that \( \gamma_{x_T}(t) = R(t) \) for \( t \in [0, 2\pi]. \) Hence by Lemma 5.1 of [30] or the proof of Lemma 3.1 of [42] we have

\[ i_{L_0}(\gamma_{x_T}) = \sum_{0<s<2\pi} \nu_{L_0}(\gamma_{x_T})(s) = 1. \]

In this case the minimal period of \( x_T \) is \( T \). Similarly for \( n > 1 \) we can construct examples to support this remark.
5 Proof of Theorems 1.4-1.5 and Corollary 1.4

In this section we study the minimal period problem for symmetric brake orbit solutions of the even reversible Hamiltonian system (1.1) and complete the proof of Theorems 1.4-1.5 and Corollary 1.4.

For $T > 0$, let $E_T = \{ x \in W^{1/2,2}(S_T, \mathbb{R}^{2m}) | x(-t) = Nx(t) \ a.e. \ t \in \mathbb{R} \}$ with the usual $W^{1/2,2}$ norm and inner product. Correspondingly $\hat{E}$ and $\tilde{E}$ are defined to be the symmetric ones and the $\frac{T}{2}$-periodic ones in $E_T$ respectively. Also $\{ P_{T,m} \}$ and $\{ \hat{P}_m \}$ are the Galerkin approximation scheme w.r.t. $A_T$ and $\hat{A}$ respectively, where $\{ P_{T,m} \}$, $\{ \hat{P}_m \}$, $A_T$, and $\hat{A}$ are defined by the same way as in Section 2, we only need to replace $\tau$ by $T$.

For $z \in E_T$, we define
\[
\hat{f}(z) = \frac{1}{2} \langle A_T z, z \rangle - \int_0^T H(z) dt.
\] (5.1)

For $z \in \hat{E}$, we define
\[
\hat{f}(z) = \frac{1}{2} \langle \hat{A} z, z \rangle - \int_0^T H(z) dt.
\] (5.2)

We have the following lemma.

**Lemma 5.1.** Let $z \in \hat{E}$. If $\hat{f}'(z) = 0$, then $f'(z) = 0$.

**Proof.** Let $z \in \hat{E}$ and $\hat{f}'(z) = 0$. So for any $y \in \hat{E}$ we have
\[
\langle \hat{f}'(z), y \rangle = \int_0^T J \dot{z}(t) \cdot y(t) dt - \int_0^T H'(z(t)) \cdot y(t) dt = 0, \ \forall y \in \hat{E}.
\] (5.3)

Since $H$ is even and $z \in \hat{E}$, we have
\[
H'(z(t + \frac{T}{2})) = H'(-z(t)) = -H'(z(t)).
\] (5.4)

So $H'(z) \in \hat{E}$ and
\[
\langle f'(z), y \rangle = \int_0^T J \dot{z}(t) \cdot y(t) dt - \int_0^T H'(z(t)) \cdot y(t) dt = 0, \ \forall y \in \hat{E}.
\] (5.5)

By (5.4) and (5.5), we have
\[
\langle f'(z), y \rangle = \int_0^T J \dot{z}(t) \cdot y(t) dt - \int_0^T H'(z(t)) \cdot y(t) dt = 0, \ \forall y \in E_T.
\] (5.6)

Hence $f'(z) = 0$.

By Lemma 5.1 and arguments in the proof of Theorem 1.3 in Section 4, to look for the $T$-period symmetric solutions of (1.1) is equivalent to look for critical points of $\hat{f}$.

**Proof of Theorem 1.5.** For any given $T > 0$, we prove the existence of $T$-periodic symmetric brake orbit solution of (1.1) whose minimal period satisfies the inequalities in the conclusion of
Theorem 1.5. Since the proof of existence of $T$-periodic symmetric brake orbit solution $x_T$ of (1.1) is similar to that of the proof of Theorem 1.3, we will only give the sketch. We divide the proof into several steps.

**Step 1.** Similarly as Step 1 in the proof of Theorem 1.3, for any $K > 0$ we can truncate the function $\hat{H}$ suitably and evenly to $\hat{H}_K$ such that it satisfies the growth condition (4.4). Correspondingly we obtain a new even and reversible function $H_K$ satisfies condition (4.4).

Set

$$\hat{f}_K(z) = \frac{1}{2} \langle \hat{A}z, z \rangle - \int_0^T H_K(z) dt, \quad \forall z \in \hat{E}.$$ (5.7)

Then $\hat{f}_K \in C^2(\hat{E}, \mathbb{R})$ and

$$\hat{f}_K(z) = \frac{1}{2} \langle (\hat{A} - \hat{B}_0)z, z \rangle - \int_0^T \hat{H}_K(z) dt, \quad \forall z \in \hat{E},$$ (5.8)

where $\hat{B}_0$ is the selfadjoint linear compact operator on $\hat{E}$ defined by

$$\langle \hat{B}_0 z, z \rangle = \int_0^T B_0 z(t) \cdot z(t) dt.$$ (5.9)

**Step 2.** For $m > 0$, let $\hat{f}_{Km} = \hat{f}|\hat{E}_m$, where $\hat{E}_m = \hat{P}_m \hat{E}$. Set

$$X_m = M^- (\hat{P}_m (\hat{A} - \hat{B}_0) \hat{P}_m) \oplus M^0 (\hat{P}_m (\hat{A} - \hat{B}_0) \hat{P}_m),$$

$$Y_m = M^+ (\hat{P}_m (\hat{A} - \hat{B}_0) \hat{P}_m).$$

By the same argument of Step 2 in the proof of Theorem 1.3, we can show that $\hat{f}_{Km}$ satisfies the hypotheses of Theorem 4.1. Moreover, we obtain a critical point $x_{Km}$ of $\hat{f}_{Km}$ with critical value $C_{Km}$ which satisfies

$$m^-(x_{Km}) \leq \dim X_m + 1.$$ (5.10)

and

$$\delta \leq C_{Km} \leq \frac{1}{2} ||\hat{A} - \hat{B}_0|| r_1^2,$$ (5.11)

where $\delta$ is a positive number depending on $K$ and $r_1 > 0$ is independent of $K$ and $m$.

**Step 3.** We prove that there exists a symmetric $T$-periodic brake orbit solution $x_T$ of (1.1) which satisfies

$$i_{\sqrt{-1}} L_0 (\gamma x_T) \leq i_{\sqrt{-1}} L_0 (B_0) + \nu L_0 (B_0) + 1.$$ (5.12)

From the proof of Theorem 1.3 we have $f_K$ satisfies $(PS)_c^*$ condition for $c \in \mathbb{R}$, by the same proof of Lemma 5.1, we have $\hat{f}_K$ satisfies $(PS)_c^*$ condition for $c \in \mathbb{R}$, i.e., any sequence $z_m$ such
that $z_m \in \hat{E}_m$, $\hat{f}'_{Km}(z_m) \to 0$ and $\hat{f}_{Km}(z_m) \to c$ possesses a convergent subsequence in $\hat{E}$. Hence in the sense of subsequence we have

$$x_{Km} \to x_K, \quad \hat{f}_K(x_K) = c_K, \quad \hat{f}'_K(x_K) = 0. \quad (5.13)$$

By similar argument as in [44], $x_K$ is a classical nonconstant symmetric $T$-periodic solution of

$$\dot{x} = JH'_K(x), \quad x \in \mathbb{R}^{2n}. \quad (5.14)$$

Set $B_K(t) = H'_K(x_K(t))$, then $B_K \in C(S_{T/2}, \mathcal{L}_s(\mathbb{R}^{2n}))$. Let $\hat{B}_K$ be the operator defined by the same way of the definition of $\hat{B}_0$. It is easy to show that

$$||\hat{f}''(z) - (\hat{A} - \hat{B}_K)|| \to 0 \quad \text{as} \quad ||z - x_K|| \to 0. \quad (5.15)$$

So for $0 < d < \frac{1}{2}||(A_T - B_{K_T})^\#||^{-1}$, there exists $r_2 > 0$ such that

$$||\hat{f}''_{Km}(z) - \hat{P}_m(\hat{A} - \hat{B}_K)\hat{P}_m|| \leq ||\hat{f}''(z) - (\hat{A} - \hat{B}_K)|| < \frac{1}{2}d, \quad \forall z \in \{z \in \hat{E} : ||z - x_K|| \leq r_2\}. \quad (5.16)$$

Then for $z \in \{z \in \hat{E} : ||z - x_K|| \leq r_2\} \cap \hat{E}_m$, $\forall u \in M^{-}_d(\hat{P}_m(\hat{A} - \hat{B}_T)\hat{P}_m) \setminus \{0\}$, we have

$$\langle \hat{f}''_{Km}(z)u, u \rangle \leq \langle \hat{P}_m(\hat{A} - \hat{B}_K)\hat{P}_mu, u \rangle + ||\hat{f}''_{Km}(z) - \hat{P}_m(\hat{A} - \hat{B}_K)\hat{P}_m||u||^2 \leq -\frac{1}{2}d||u||^2.$$  

So we have

$$m^{-}(\hat{f}''_{Km}(z)) \geq \dim M^{-}_d(\hat{P}_m(\hat{A} - \hat{B}_K)\hat{P}_m). \quad (5.17)$$

By Theorem 3.1, Remark 3.1, there is $m^* > 0$ such that for $m \geq m^*$ we have

$$\dim X_m = mn + i\frac{L_0}{\sqrt{-1}}(B_0) + \nu\frac{L_0}{\sqrt{-1}}(B_0), \quad (5.18)$$

$$\dim M^{-}_d(\hat{P}_m(\hat{A} - \hat{B}_K)\hat{P}_m) = mn + i\frac{L_0}{\sqrt{-1}}(B_0). \quad (5.19)$$

Then by (5.10), (5.13), and (5.17)-(5.19), we have

$$i\frac{L_0}{\sqrt{-1}}(B_K) \leq i\frac{L_0}{\sqrt{-1}}(B_0) + \nu\frac{L_0}{\sqrt{-1}}(B_0) + 1. \quad (5.20)$$

By the similar argument as in the section 6 of [44], there is a constant $M_3$ independent of $K$ such that $||x_K||_\infty \leq M_3$. Choose $K > M_3$. Then $x_K$ is a non-constant symmetric $T$-periodic brake orbit solution of the problem (1.1). From now on in the proof of Theorem 1.2, we write $B = B_K$ and $x_T = x_K$. Then $x_T$ is a non-constant symmetric $T$-periodic solution of the problem (1.1), and $B$ satisfies

$$i\frac{L_0}{\sqrt{-1}}(\gamma x_T) = i\frac{L_0}{\sqrt{-1}}(B) \leq i\frac{L_0}{\sqrt{-1}}(B_0) + \nu\frac{L_0}{\sqrt{-1}}(B_0) + 1. \quad (5.21)$$
Step 4. Finish the proof of Theorem 1.5.

Since $x_T$ obtained in Step 3 is a nonconstant and symmetric $T$-period brake orbit solution, its minimal period $\tau = \frac{T}{4r+s}$ for some nonnegative integer $r$ and $s = 1$ or $s = 3$. We now estimate $r$.

We denote by $x_T = x_T[0,\tau]$, then it is a symmetric period solution of (1.1) with the minimal $\tau$ and $X_T = x_T^{4r+s}$ being the $4r+s$ times iteration of $x_T$. As in Section 1, let $\gamma_{x_T}$ and $\gamma_{x_T}$ the symplectic path associated to $(\tau,x)$ and $(T,x_T)$ respectively. Then $\gamma_{x_T} \in C([0,\frac{T}{4}],\text{Sp}(2n))$ and $\gamma_{x_T} \in C([0,\frac{T}{4}],\text{Sp}(2n))$. Also we have $\gamma_{x_T} = \gamma_{x_T}^{4r+s}$, which is the $4r+s$ times iteration of $\gamma_{x_T}$.

By (5.21) we have
\[ i_L^0 \sqrt{-1} (\gamma_{x_T}^{4r+s}) \leq i_L^0 \sqrt{-1} (B_0) + \nu L^0 \sqrt{-1} (B_0) + 1. \] (5.22)

Since $x_T$ is also a nonconstant symmetric periodic solution of (1.1). It is clear that $\nu_{-1}(x_T^2) \geq 1$. (5.23)

Since $\hat{H}$ satisfies condition (H5) and $B_0$ is semipositive, by Corollary 3.1 of [51] (also by Theorem 6.2) we have
\[ i_{-1}(\gamma_{x_T}^2) \geq 0. \] (5.24)

By Corollary 3.2 of [51] (cf. also [29]), we have
\[ i_1(\gamma_{x_T}^2) + \nu_1(\gamma_{x_T}^2) \geq n. \] (5.25)

It is easy to see that
\[ \gamma_{x_T}^4(\frac{T}{2} + t) = \gamma_{x_T}(t) \gamma_{x_T}(\frac{T}{2}), \quad \forall t \in [0,\frac{T}{2}]. \] (5.26)

So by Theorem 6.1 of Bott-type iteration formula we have
\[ i_1(\gamma_{x_T}^4) + \nu_1(\gamma_{x_T}^4) = i_1(\gamma_{x_T}^2) + \nu_1(\gamma_{x_T}^2) + i_{-1}(\gamma_{x_T}^2) + \nu_{-1}(\gamma_{x_T}^2) \geq n + 0 + 1 \]
\[ = n + 1. \] (5.27)

If $r \geq 1$, then by Theorems 2.2 and 6.2 and (5.27) we have
\[ i_{-1}(\gamma_{x_T}^{4r}) = i_{-1}((\gamma_{x_T}^2)^{2r}) \]
\[ = \sum_{j=1}^{r} i_{\omega^2 j - 1}(\gamma_{x_T}^4) \]
\[ \geq \sum_{j=1}^{r} (i_1(\gamma_{x_T}^4) + \nu_1(\gamma_{x_T}^4) - n) \] (5.28)
\[ = r(i_1(\gamma_{x_T}^4) + \nu_1(\gamma_{x_T}^4) - n) \]
\[ \geq r, \] (5.29)
where $\omega_{2r} = e^{\pi\sqrt{-1}/(2r)}$ as defined in Theorem 2.2.

By Theorem 3.2, we have
\begin{equation}
 i L_0 \sqrt{-1} (\gamma_{x_r}^{4s}) \geq i L_0 \sqrt{-1} (\gamma_{x_r}^{4r}).
\end{equation}

Then (5.22), (5.29) and (5.30) yield
\begin{equation}
r \leq i L_0 \sqrt{-1} (B_0) + \nu L_0 \sqrt{-1} (B_0) + 1.
\end{equation}

Thus for $i L_0 \sqrt{-1} (B_0) + \nu L_0 \sqrt{-1} (B_0)$ is odd, by (5.31) we have
\begin{equation}
4r + s \leq 4r + 3 \leq 4(i L_0 \sqrt{-1} (B_0) + \nu L_0 \sqrt{-1} (B_0)) + 7.
\end{equation}

**Claim 3.** For $i L_0 \sqrt{-1} (B_0) + \nu L_0 \sqrt{-1} (B_0)$ is even, the equality in (5.31) can not hold.

Otherwise, $r \geq 1$ and the equality in (5.28) holds i.e.,
\begin{equation}
i \omega_{2r}^{2j-1} (\gamma_{x_r}^4) = i_1 (\gamma_{x_r}^4) + \nu_1 (\gamma_{x_r}^4) - n = 1, \quad j = 1, 2, ..., r.
\end{equation}

By the definition of $\omega_{2r}$, we have $\omega_{2r}^{2j-1} \neq -1$ for $j = 1, 2, ..., r$. So by (5.33) and 2 of Theorem 6.2, we have $I_2^p \circ N_1 (1, -1)^{eq} \circ K \in \Omega^0 (\gamma_{x_r}^4 (\tau))$ for some non-negative integers $p$ and $q$ satisfying $0 \leq p + q \leq n$ and $K \in \text{Sp}(2(n - p - q))$ with $\sigma(K) \in U \setminus \{1\}$ satisfying the condition that all eigenvalues of $K$ located with the arc between 1 and $\omega_{2r}$ in $U^+ \setminus \{-1\}$ possess total multiplicity $n - p - q$. So there are no eigenvalues of $K$ on the arc between $\omega_{2r}^{2j-1}$ and $-1$ except $\omega_{2r}^{2r-1}$ with $r = 1$. However, whether $\omega_{2r}^{2r-1} \in \sigma(\gamma_{x_r}^4 (\tau))$ or not, we always have
\begin{equation}
S_{\gamma_{x_r}^4 (\tau)}^+ (\omega_{2r}^{2r-1}) = 0,
\end{equation}
\begin{equation}
i \omega_{2r}^{2r-1} (\gamma_{x_r}^4) = 1.
\end{equation}

So (6.13) and Lemma 6.2, we have
\begin{align*}
i_{-1} (\gamma_{x_r}^4) & = i \omega_{2r}^{2r-1} (\gamma_{x_r}^4) + S_{\gamma_{x_r}^4 (\tau)}^+ (\omega_{2r}^{2r-1}) \\
& = 1 + 0 = 1.
\end{align*}

But by (5.26), Lemma 6.1, and Theorem 6.1, we have
\begin{align*}
i_{-1} (\gamma_{x_r}^4) & = i_{-1} (\gamma_{x_r}^{2r})^2 \\
& = i \sqrt{-1} (\gamma_{x_r}^{2r}) + i_{-} \sqrt{-1} (\gamma_{x_r}^{2r}) \\
& = 2i \sqrt{-1} (\gamma_{x_r}^{2r}).
\end{align*}
Then \( i_{-1}(r_{x_t}) \) is an even integer, which yields a contradiction to (5.36). So Claim 3 holds, and we have
\[
r \leq i_{\sqrt{1-T}}(B_0) + \nu_{\sqrt{1-T}}(B_0).
\] (5.37)
Hence
\[
4r + s \leq 4r + 3 \leq 4(i_{\sqrt{1-T}}(B_0) + \nu_{\sqrt{1-T}}(B_0)) + 3.
\] (5.38)
Then by (5.32) and (5.38), Theorem 1.5 holds from (5.32) and (5.38).

**Proof of Theorem 1.4.** This is the case \( B_0 \equiv 0 \) of Theorem 1.2. From Theorem 3.1 it is easy to see that
\[
i_{\sqrt{1-T}}(0) = 0, \quad \nu_{\sqrt{1-T}}(0) = 0.
\] (5.39)
Then \( i_{\sqrt{1-T}}(0) + \nu_{\sqrt{1-T}}(0) = 0 \) and is also even. So Theorem 1.4 holds from Theorem 1.5.

**Proof of Corollary 1.2.** Since \( 0 < T < \frac{\pi}{||B_0||} \), there is \( \varepsilon > 0 \) small enough such that
\[
0 \leq B_0 \leq ||B_0||I_{2n} < \left(\frac{\pi}{T} - \varepsilon\right)I_{2n}.
\] (5.40)
It is easy to see that
\[
\gamma\left(\frac{\pi}{T} - \varepsilon\right)I_{2n}(t) = \exp\left(\frac{\pi}{T} - \varepsilon\right)tJ \quad \forall t \in \left[0, \frac{T}{4}\right].
\] (5.41)
Since
\[
\nu_{L_0}(\exp\left(\frac{\pi}{T} - \varepsilon\right)tJ)) = 0, \quad \forall t \in \left[0, \frac{T}{T-1}\right].
\] (5.42)
We have
\[
i_{\sqrt{1-T}}\left(\gamma\left(\frac{\pi}{T} - \varepsilon\right)I_{2n}\right) = 0, \quad i_{\sqrt{1-T}}\left(\gamma\left(\frac{\pi}{T} - \varepsilon\right)I_{2n}\right) = 0.
\] (5.43)
So by Theorem 2.1 we have
\[
i_{\sqrt{1-T}}\left(\frac{\pi}{T} - \varepsilon\right)I_{2n} = i_{\sqrt{1-T}}\left(\gamma\left(\frac{\pi}{T} - \varepsilon\right)I_{2n}\right) - i_{\sqrt{1-T}}\left(\gamma\left(\frac{\pi}{T} - \varepsilon\right)I_{2n}\right) = 0.
\] (5.44)
Then by (5.40) and Lemma 3.1 and Corollary 3.1 we have
\[
0 \leq i_{-1}(B_0, \frac{T}{2}) + \nu_{-1}(B_0, \frac{T}{2}) \leq i_{-1}\left(\frac{\pi}{T} - \varepsilon\right)I_{2n}, \frac{T}{2} = 0.
\] (5.45)
So we have
\[
i_{-1}(B_0, \frac{T}{2}) + \nu_{-1}(B_0, \frac{T}{2}) = 0.
\] (5.46)
Hence by Theorem 1.1 or Corollary 1.1, the conclusion of Corollary 1.2 holds.

Also a natural question is that can we prove the minimal period is \( T \) in this way? We have the following remark.
Remark 5.1. Only use the Maslov-type index theory to estimate the iteration time of the symmetric $T$-periodic brake solution $x_T$ obtained in the proof of Theorem 1.5 with $B_0 = 0$, we can not hope to prove $T$ is the minimal period of $x_T$. Even $H''(z) > 0$ for all $z \in \mathbb{R}^{2n} \setminus \{0\}$. For $n = 1$ and $T = 6\pi$, we can not exclude the following case:

$$x_T(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix},$$

$$H'(x_T(t)) = x_T(t),$$

$$H''(x_T(t)) \equiv I_{2n}.$$  \hspace{1cm} (5.47)

It is easy to check that $\gamma_{x_T}(t) = R(t)$ for $t \in [0, 3\pi]$. Hence by Theorem 2.1 and Lemma 5.1 of [30] or the proof of Lemma 3.1 of [42] we have

$$i \sqrt{\nu_{L_0}(\gamma_{x_T}) = \sum_{3\pi/4 \leq s < 3\pi} \nu_{L_0}(\gamma_{x_T})(s) = 1.}$$  \hspace{1cm} (5.48)

In this case the minimal period of $x_T$ is $T/3$. Similarly for $n > 1$ we can construct examples to support this remark.

6 Appendix on Maslov-type indices $(i_\omega, \nu_\omega)$

We first recall briefly the Maslov-type index theory of $(i_\omega, \nu_\omega)$. All the details can be found in [11].

For any $\omega \in U$, the following codimension 1 hypersuface in Sp$(2n)$ is defined by:

$$\text{Sp}(2n)_\omega^0 = \{M \in \text{Sp}(2n) | \text{det}(M - \omega I_{2n}) = 0\}.$$  

For any two continuous path $\xi$ and $\eta$: $[0, \tau] \to \text{Sp}(2n)$ with $\xi(\tau) = \eta(0)$, their joint path is defined by

$$\eta \ast \xi(t) = \begin{cases} 
\xi(2t) & \text{if } 0 \leq t \leq \frac{\tau}{2}; \\
\eta(2t - \tau) & \text{if } \frac{\tau}{2} \leq t \leq \tau.
\end{cases}$$  \hspace{1cm} (6.1)

Given any two $(2m_k \times 2m_k)$- matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\
C_k & D_k \end{pmatrix}$ for $k = 1, 2$, as in [11], the $\odot$-product of $M_1$ and $M_2$ is defined by the following $(2(m_1 + m_2) \times 2(m_1 + m_2))$-matrix $M_1 \odot M_2$:

$$M_1 \odot M_2 = \begin{pmatrix}
A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2
\end{pmatrix}.$$
A special path $\xi_n$ is defined by

$$
\xi_n(t) = \begin{pmatrix}
2 - \frac{t}{\tau} & 0 \\
0 & (2 - \frac{t}{\tau})^{-1}
\end{pmatrix}^n, \quad \forall t \in [0, \tau].
$$

**Definition 6.1.** For any $\omega \in U$ and $M \in \text{Sp}(2n)$, define

$$
\nu_\omega(M) = \dim \ker(M - \omega I_{2n}).
$$

(6.2)

For any $\gamma \in \mathcal{P}_\tau(2n)$, define

$$
\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)).
$$

(6.3)

If $\gamma(\tau) \notin \text{Sp}(2n)_0^\omega$, we define

$$
i_\omega(\gamma) = [\text{Sp}(2n)_0^\omega : \gamma \ast \xi_n],
$$

(6.4)

where the right-hand side of (6.4) is the usual homotopy intersection number and the orientation of $\gamma \ast \xi_n$ is its positive time direction under homotopy with fixed endpoints.

If $\gamma(\tau) \in \text{Sp}(2n)_0^\omega$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of $\gamma$ in $\mathcal{P}_\tau(2n)$, and define

$$
i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{i_\omega(\beta) | \beta(\tau) \in U \text{ and } \beta(\tau) \notin \text{Sp}(2n)_0^\omega\}.
$$

(6.5)

Then $(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}$, is called the index function of $\gamma$ at $\omega$.

**Lemma 6.1.** (Lemma 5.3.1 of [41]) For any $\gamma \in \mathcal{P}_\tau(2n)$ and $\omega \in U$, there hold

$$
i_\omega(\gamma) = \bar{i}_\omega(\gamma), \quad \nu_\omega(\gamma) = \bar{\nu}_\omega(\gamma).
$$

(6.6)

As in [38], for any $M \in \text{Sp}(2n)$ we define

$$
\Omega(M) = \{P \in \text{Sp}(2n) \mid \sigma(P) \cap U = \sigma(M) \cap U \\
\text{and } \nu_\lambda(P) = \nu_\lambda(M), \quad \forall \lambda \in \sigma(M) \cap U\}.
$$

(6.7)

We denote by $\Omega^0(M)$ the path connected component of $\Omega(M)$ containing $M$, and call it the homotopy component of $M$ in $\text{Sp}(2n)$.

The following symplectic matrices were introduced as basic normal forms in [41]:

$$
D(\lambda) = \begin{pmatrix}
\lambda & 0 \\
0 & \lambda^{-1}
\end{pmatrix}, \quad \lambda = \pm 2,
$$

(6.8)

$$
N_1(\lambda, b) = \begin{pmatrix}
\lambda & b \\
0 & \lambda
\end{pmatrix}, \quad \lambda = \pm 1, \ b = \pm 1, \ 0,
$$

(6.9)
\[ R(\theta) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (6.10) \]

\[ N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (6.11) \]

where \( b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \) with \( b_i \in \mathbb{R} \) and \( b_2 \neq b_3 \).

For any \( M \in \text{Sp}(2n) \) and \( \omega \in U \), splitting number of \( M \) at \( \omega \) is defined by

\[ S^\pm_M = \lim_{\epsilon \to 0^+} i^\epsilon \exp(\pm \sqrt{-t^2} \omega) \gamma - i^\omega(\gamma) \quad (6.12) \]

for any path \( \gamma \in \mathcal{P}_r(2n) \) satisfying \( \gamma(\tau) = M \).

Splitting numbers possesses the following properties.

**Lemma 6.2.** (cf. [40], Lemma 9.1.5 and List 9.1.12 of [41]) Splitting number \( S^\pm_M(\omega) \) are well defined; that is they are independent of the choice of the path \( \gamma \in \mathcal{P}_r(2n) \) satisfying \( \gamma(\tau) = M \). For \( \omega \in U \) and \( M \in \text{Sp}(2n) \), \( S^\pm_M(\omega) \) are constant for all \( N \in \Omega^0(M) \). Moreover we have

1. \((S^+_M(\pm 1), S^-_M(\pm 1)) = (1, 1) \) for \( M = \pm N_1(1, b) \) with \( b = 1 \) or 0;
2. \((S^+_M(\pm 1), S^-_M(\pm 1)) = (0, 0) \) for \( M = \pm N_1(1, b) \) with \( b = -1 \);
3. \((S^+_M(e^{\sqrt{-t^2} \theta}), S^-_M(e^{\sqrt{-t^2} \theta})) = (0, 1) \) for \( M = R(\theta) \) with \( \theta \in (0, \pi) \cup (\pi, 2\pi) \);
4. \((S^+_M(\omega), S^-_M(\omega)) = (0, 0) \) for \( \omega \in U \setminus R \) and \( M = N_2(\omega, b) \) is trivial i.e., for sufficiently small \( \alpha > 0 \), \( MR((t-1)\alpha)^\alpha \) possesses no eigenvalues on \( U \) for \( t \in [0, 1) \).
5. \((S^+_M(\omega), S^-_M(\omega)) = (1, 1) \) for \( \omega \in U \setminus R \) and \( M = N_2(\omega, b) \) is non-trivial.
6. \((S^+_M(\omega), S^-_M(\omega)) = (0, 0) \) for any \( \omega \in U \) and \( M \in \text{Sp}(2n) \) with \( \sigma(M) \cap U = \emptyset \).
7. \( S^+_M(\omega) = S^+_M(\omega) + S^+_M(\omega) \), for any \( M_j \in \text{Sp}(2n_j) \) with \( j = 1, 2 \) and \( \omega \in U \).

By the definition of splitting numbers and Lemma 6.2, for \( 0 \leq \theta_1 < \theta_2 < 2\pi \) and \( \gamma \in \mathcal{P}_r(2n) \) with \( \tau > 0 \), we have

\[ i^\epsilon \exp(\sqrt{-t^2} \theta) \gamma = i^\epsilon \exp(\sqrt{-t^2} \theta_1) + S^+_\gamma(\tau) (e^{\sqrt{-t^2} \theta_1}) + \sum_{\theta \in (\theta_1, \theta_2)} \left( S^+_\gamma(\tau) (e^{\sqrt{-t^2} \theta}) - S^-_\gamma(\tau) (e^{\sqrt{-t^2} \theta}) \right) - S^-_\gamma(\tau) (e^{\sqrt{-t^2} \theta_2}). \quad (6.13) \]

For any symplectic path \( \gamma \in \mathcal{P}_r(2n) \) and \( m \in \mathbb{N} \), we define its \( m \)th iteration in the periodic boundary sense \( \gamma(m) : [0, m\tau] \to \text{Sp}(2n) \) by

\[ \gamma(m)(t) = \gamma(t - j\tau) \gamma(\tau)^j \quad \text{for} \quad j\tau \leq t \leq (j + 1)\tau, \quad j = 0, 1, ..., m - 1. \quad (6.14) \]
**Definition 6.2.** (cf. [40], [41]) For any $\gamma \in P_{\tau}(2n)$ and $\omega \in U$, we define
\[
(i_\omega(\gamma, m), \nu_\omega(\gamma, m)) = (i_\omega(\gamma(m)), \nu_\omega(\gamma(m))), \quad \forall m \in \mathbb{N}.
\]

We have the following Bott-type iteration formula.

**Theorem 6.1.** (cf. [40], Theorem 9.2.1 of [41]) For any $\tau > 0$, $\gamma \in P_{\tau}(2n)$, $z \in U$, and $m \in \mathbb{N}$,
\[
i_z(\gamma, m) = \sum_{\omega^k = z} i_\omega(\gamma), \quad \nu_z(\gamma, m) = \sum_{\omega^m = z} \nu_\omega(\gamma).
\]

By Theorem 8.1.4 of [41], we have the following Lemma.

**Lemma 6.3.** For $\gamma \in P_\tau(2)$ with $\tau > 0$, the following results hold.
1. If $N_1(1, 1) \in \Omega^0(\gamma(\tau))$, then
\[
i_1(\gamma, m) = m(i_1(\gamma) + 1) - 1, \quad \nu_1(\gamma, m) = 1, \quad \forall m \in \mathbb{N},
\]
\[
i_1(\gamma) \in 2\mathbb{Z} + 1.
\]
2. If $N_1(1, 1) \in \Omega^0(\gamma(\tau))$, then
\[
i_1(\gamma, m) = m(i_1(\gamma) + 1) - 1, \quad \nu_1(\gamma, m) = 2, \quad \forall m \in \mathbb{N},
\]
\[
i_1(\gamma) \in 2\mathbb{Z} + 1.
\]
3. If $N_1(1, -1) \in \Omega^0(\gamma(\tau))$, then
\[
i_1(\gamma, m) = m(i_1(\gamma) + 1), \quad \nu_1(\gamma, m) = 1, \quad \forall m \in \mathbb{N},
\]
\[
i_1(\gamma) \in 2\mathbb{Z}.
\]

Denote by $U^+ = \{\omega \in U | Im \omega \geq 0\}$ and $U^- = \{\omega \in U | Im \omega \leq 0\}$. The following theorem was proved by Liu and Long in [33, 34], which plays a important role in the proof of our main results in Sections 4-5.

**Theorem 6.2.** (Theorem 10.1.1 of [41])
1. For any $\gamma \in P_\tau(2n)$ and $\omega \in U \setminus \{1\}$, it always holds that
\[
i_1(\gamma) + \nu_1(\gamma) - n \leq i_\omega(\gamma) \leq i_1(\gamma) + n - \nu_\omega(\gamma).
\]
2. The left equality in (6.23) holds for some $\omega \in U^+ \setminus \{1\}$ (or $U^- \setminus \{1\}$) if and only if $I_{2p} \circ N_1(1, -1)^{cq} \circ K \in \Omega^0(\gamma(\tau))$ for some non-negative integers $p$ and $q$ satisfying $0 \leq p + q \leq n$ and $K \in Sp(2(n - p - q))$ with $\sigma(K) \in U \setminus \{1\}$ satisfying the condition that all eigenvalues of $K$ located
with the arc between 1 and \( \omega \) including \( U^+ \setminus \{1\} \) (or \( U^- \setminus \{1\} \)) possess total multiplicity \( n - p - q \). If \( \omega \neq -1 \), all eigenvalues of \( K \) are in \( U \setminus R \) and those in \( U^+ \setminus R \) (or \( U^- \setminus R \)) are all Krein-negative (or Krein-positive) definite. If \( \omega = -1 \), it holds that \( (-I_{2s}) \circ N_1(-1,1)^{ot} \circ H \in O^0(\gamma(\tau)) \) for some non-negative integers \( s \) and \( t \) satisfying \( 0 \leq s + t \leq n - p - q \), and some \( H \in Sp(2(n-p-q-s-t)) \) satisfying \( \sigma(H) \subset U \setminus R \) and that all elements in \( \sigma(H) \cap U^+ \) (or \( \sigma(H) \cap U^- \)) are all Krein-negative (or Krein-positive) definite.

3. The left equality of (6.23) holds for all \( \omega \in U \setminus \{1\} \) if and only if \( I_{2p} \circ N_1(1,-1)^{(n-p)} \in O^0(\gamma(\tau)) \) for some integer \( p \in [0,n] \). Especially in this case, all the eigenvalues of \( \gamma(\tau) \) are equal to 1 and \( \nu_\gamma = n + p \geq n \).

4. The right equality in (6.23) holds for some \( \omega \in U^+ \setminus \{1\} \) (or \( U^- \setminus \{1\} \)) if and only if \( I_{2p} \circ N_1(1,1)^{or} \circ K \in O^0(\gamma(\tau)) \) for some non-negative integers \( p \) and \( r \) satisfying \( 0 \leq p + r \leq n \) and \( K \in Sp(2(n-p-r)) \) with \( \sigma(K) \in U \setminus \{1\} \) satisfying the condition that all eigenvalues of \( K \) located with the arc between 1 and \( \omega \) including \( U^+ \setminus \{1\} \) (or \( U^- \setminus \{1\} \)) possess total multiplicity \( n - p - r \). If \( \omega \neq -1 \), all eigenvalues of \( K \) are in \( U \setminus R \) and those in \( U^+ \setminus R \) (or \( U^- \setminus R \)) are all Krein-positive (or Krein-negative) definite. If \( \omega = -1 \), it holds that \( (-I_{2s}) \circ N_1(-1,1)^{ot} \circ H \in O^0(\gamma(\tau)) \) for some non-negative integers \( s \) and \( t \) satisfying \( 0 \leq s + t \leq n - p - r \), and some \( H \in Sp(2(n-p-q-r-t)) \) satisfying \( \sigma(H) \subset U \setminus R \) and that all elements in \( \sigma(H) \cap U^+ \) (or \( \sigma(H) \cap U^- \)) are all Krein-positive (or Krein-negative) definite.

5. The right equality of (6.23) holds for all \( \omega \in U \setminus \{1\} \) if and only if \( I_{2p} \circ N_1(1,1)^{(n-p)} \in O^0(\gamma(\tau)) \) for some integer \( p \in [0,n] \). Especially in this case, all the eigenvalues of \( \gamma(\tau) \) are equal to 1 and \( \nu_\gamma = n + p \geq n \).

6. Both equalities of (6.23) holds for all \( \omega \in U \setminus \{1\} \) if and only if \( \gamma(\tau) = I_{2n} \).

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