In this paper we introduce a nonlinear integral equation such that the system of global solution to this equation represents a class of a very narrow beam at $T \to \infty$ (an analogue to the laser beam) and this sheaf of solutions leads to an almost-exact representation of the Hardy-Littlewood integral. The accuracy of our result is essentially better than the accuracy of related results of Balasubramanian, Heath-Brown and Ivic.

1. Introduction

Let us remind that Hardy and Littlewood started to study the following integral in 1918:

\[
\int_0^T \left| \zeta \left( \frac{1}{2} + it \right) \right|^2 dt = \int_0^T Z^2(t) dt,
\]

where

\[
Z(t) = e^{i\vartheta(t)} \zeta \left( \frac{1}{2} + it \right), \quad \vartheta(t) = -\frac{T}{2} \ln(\pi) + \text{Im} \ln \left[ \Gamma \left( \frac{1}{4} + it \right) \right],
\]

and they have derived the following formula (see [4], page 122, 151-156)

\[
\int_0^T Z^2(t) dt \sim T \ln(T), \quad T \to \infty.
\]

In this paper we show that except the asymptotic formula (1.2) that posses an unbounded error there is an infinite family of other asymptotic representations of the Hardy-Littlewood integral (1.1). Each member of this family is an almost-exact representation of the given integral. The proof of this will be based on properties of a kind of functions having some canonical properties on the set of zeroes of the function $\zeta(1/2 + it)$.

(A) Let us remind further that in 1928 Titchmarsh has discovered a new treatment to the integral (1.1) by which the Titchmarsh-Kober-Atkinson (TKA) formula:

\[
\int_0^\infty Z^2(t)e^{-2\delta t} dt = \frac{c - \ln(4\pi\delta)}{2\sin(\delta)} + \sum_{n=0}^{N} c_n\delta^n + O(\delta^{N+1}),
\]

where $\delta \to 0$, $C$ is the Euler constant, $c_n$ are constant depending upon $N$, was derived.

**Key words and phrases.** Riemann zeta-function.
The TKA formula has been published for the first time in 1951 in the fundamental monograph by Titchmarsh (see [8], [1], [6], [7]). It was thought for about 56 years that the TKA formula is a kind of curiosity (see [5], page 139). However, in this paper we show that the TKA formula itself contains new kinds of principles.

(B) Namely, in this work we introduce a new class of curves, which are the solutions to the following nonlinear integral equation:

\[ \int_0^T \mu[x(T)] Z^2(t) e^{-\frac{2}{\pi \mu t}} dt = \int_0^T Z^2(t) dt, \]

where the class of functions \( \{ \mu \} \) is specified as: \( \mu \in C^\infty((y_0, \infty)) \) is a monotonically increasing (to \( +\infty \)) function and it obeys \( \mu(y) \geq 7y \ln(y) \). The following holds true: for any \( \mu \in \{ \mu \} \) it exists just one solution to the equation (1.4):

\[ \varphi(T) = \varphi_\mu(T), \quad T \in [T_0, \infty), \quad T_0 = T_0[\varphi], \quad \varphi(T) \to \infty \text{ as } T \to \infty. \]

Let us denote by the symbol \( \{ \varphi \} \) the system of these solutions. The function \( \varphi(T) \) is related to the zeroes of the Riemann zeta-function on the critical line by the following way. Let \( t = \gamma \) be a zero of the function \( \zeta(1/2 + it) \) of the order \( n(\gamma) \), where \( n(\gamma) = O(\ln(\gamma)) \), (see [3], page 178). Then the points \( [\gamma, \varphi(\gamma)], \gamma > T_0 \) (and only these points) are the inflection points with the horizontal tangent. In more details, it holds true the following system of equations:

\[ \varphi'(\gamma) = \varphi''(\gamma) = \cdots = \varphi^{(2n)}(\gamma) = 0, \quad \varphi^{(2n+1)}(\gamma) \neq 0, \]

where \( n = n(\gamma) \).

With respect to this property an element \( \varphi \in \{ \varphi \} \) is to be named as the Jacob’s ladder leading to \([+\infty, +\infty]\) (the rungs of the Jacob’s ladder are the segments of the curve \( \varphi \) lying in the neighborhoods of the points \( [\gamma, \varphi(\gamma)], \gamma > T_0[\varphi] \)). Finally, also the composition of the functions \( G[\varphi(T)] \) is to be named the Jacob’s ladder if the following conditions are fulfilled: \( G \in C^\infty((y_0, +\infty)), \) \( G \) grows to \( +\infty \) and \( G \) has a positive derivative everywhere.

Let us mention that the mapping (the operator)

\[ \hat{H} : \{ \mu \} \to \{ \varphi \} \]

can be named the \( Z^2 \)-mapping of the functions of the class \( \{ \mu \} \).

(C) Jacob’s ladder implies the following results:

(a) An almost exact asymptotic formula for the Hardy-Littlewood integral. Let us mention that our new formula makes more exact also the leading term in (1.2):

\[ \int_0^T Z^2(t) dt \sim \frac{\varphi(T)}{2} \ln \left( \frac{\varphi(T)}{2} \right), \quad T \to +\infty, \quad \forall \varphi \in \{ \varphi \}, \]

i.e. the leading term in this formula is also the Jacob’s ladder, and therefore we can say that the leading term has a very fine structure.

(b) The system \( \{ \varphi \} \) has the property of an ”infinitely close approach” of any two Jacob’s ladders at \( T \to +\infty \).
(c) Our new formula for the Hardy-Littlewood integral is stable with respect to the choice of the elements from some subset \( \{ \varphi \}^* \) in \( \{ \varphi \} \).

2. Results

The following theorem holds true:

**Theorem 1.** The TKA formula implies:

(A) \[
\int_0^T Z^2(t)dt = F[\varphi(T)] + r[\varphi(T)], \quad T \geq T_0[\varphi],
\]

where \( F[y] = \frac{y}{2} \ln \left( \frac{y}{2} \right) + (c - \ln(2\pi))\frac{y}{2} + c_0, \quad r[\varphi(T)] = O \left( \frac{\ln(\varphi(T))}{\varphi(T)} \right) = O \left( \frac{\ln(T)}{T} \right). \) \( (2.2) \)

(B) For all \( \varphi_1(T), \varphi_2(T) \in \{ \varphi \} \) we have

\[
\varphi_1(T) - \varphi_2(T) = O \left( \frac{1}{T} \right), \quad T \geq T_0 = \max\{T_0[\varphi_1], T_0[\varphi_2]\}. \]

(C) If the set \( \{ \mu(y_0) \} \) is bounded then for all \( \varphi(T) \) in \( \{ \varphi \} \) and for any fixed \( \varphi_0(T) \in \{ \varphi \} \):

\[
\varphi_0(T) - \frac{A}{T} < \varphi(T) < \varphi_0(T) + \frac{A}{T}, \quad T \geq T_0 = \sup\{\mu(y_0)\}. \]

Let us mention that the constants in the \( O \)-symbols do not depend upon the choice of \( \varphi(T) \).

Let us remind the Balasubramanian’s formula (see [2]):

\[
\int_0^T Z^2(t)dt = T \ln(T) + (2c - 1 - \ln(2\pi))T + O \left( T^{1/3+\epsilon} \right), \]

and \( \Omega \)-theorem of Good (see [3]):

\[
\int_0^T Z^2(t)dt - T \ln(T) - (2c - 1 - \ln(2\pi))T = \Omega \left( T^{1/4} \right). \]

**Remark 2.** Combining the formulae \( 2.1 \), \( 2.2 \) and \( 2.5 \) one obtains that:

- Formula \( 2.3 \) possesses quite large uncertainty since the deviation from the value of \( 1.1 \) is given by \( R(T) = O(T^{1/3+\epsilon}) \), and (see \( 2.6 \)) since

\[
\lim_{T \to \infty} |R(T)| = +\infty,
\]

this cannot be removed.

- Following \( 2.2 \) we have

\[
\lim_{T \to \infty} r[\varphi(T)] = 0,
\]

and this means that formula \( 2.1 \) seems to be almost exact.

**Remark 3.** We have found a new fact that the leading term in the Hardy-Littlewood integral is a ladder (see \( 2.1 \)), i.e. it has a fine structure. There is no analogue of this in the formulae \( 1.2 \) or \( 2.5 \).

**Remark 4.** We say explicitly that:
• Formula (2.3) contains a new effect, namely, any two Jacob’s ladders approach each other at $T \to \infty$.

• (2.4) implies that the representations (2.1) and (2.2) of the integral of Hardy-Littlewood (1.1) are stable under the choice of $\varphi(T) \in \{\varphi\}$ in the case of a bounded set $\{\mu(y_0)\}$.

**Remark 5.** Following the second part of Remark 3 the representations (2.1) and (2.2) of the Hardy-Littlewood integral (1.1) is microscopically unique in sense that any two Jacob’s ladders $\varphi_1, \varphi_2$, $T \geq T_0$ (see (2.4)) cannot be distinguished at $T \to \infty$.

In the fifth part of this work we establish the relation between the Jacob’s ladder and the prime-counting function $\pi(T)$:

$$\pi(T) \sim \frac{1}{1 - c} \left\{ T - \frac{\varphi(T)}{2} \right\}, \; T \to \infty.$$

### 3. Existence of the Jacob’s Ladder

The following lemma holds true:

**Lemma 1.** Let $\mu(y) \in \{\mu\}$ be fixed. Then there exists an unique solution $\varphi(T), \; T \geq T_0$ to the integral equation (1.4) that obeys the property (1.5).

**Proof.** By the Bonnet’s mean-value theorem in the case $y > 0$ and $\mu > 0$ we have

$$\int_0^\mu Z^2(t)e^{-\frac{2}{y}t}dt = \int_0^M Z^2(t)dt, \; M > 0,$$

where $e^{-\frac{2}{y}t}, \; t \in [0, \mu]$ is decreasing and equals 1 at $t = 0$.

First of all we will show that the formula (3.1) maps to any fixed $\mu > 0$ just one $M > 0$, since the case $M_1 \neq M_2$ is impossible because it would mean that

$$\int_{M_1}^{M_2} Z^2(t)dt = 0.$$

Let $\mu(y) \in \{\mu\}$. Then the following formula

$$\int_0^{\mu(y)} Z^2(t)e^{-\frac{2}{y}t}dt = \int_0^{M(y)} Z^2(t)dt$$

defines a function $M(y) = M_\mu(y), \; y \geq y_0$. Let the symbol $\{M\}$ denote the class of the images of the elements $\mu(y) \in \{\mu\}$.

The function $M(y)$ is positive and increases to $+\infty$. In fact, let

$$\Phi(y) = \int_0^{\mu(y)} Z^2(t)e^{-\frac{2}{y}t}dt.$$

Then

$$\Phi'(y) = \frac{2}{y^2} \int_0^{\mu(y)} tZ^2(t)e^{-\frac{2}{y}t}dt + Z^2[\mu(y)]e^{-\frac{2}{y}\mu(y)}\frac{d\mu(y)}{dy} > 0$$
These properties of the M have been shown by analogy. And with respect to the continuity of the left-hand side of eq. (3.3) we obtain (see (3.9)): 

\[ \int_{M(y)}^{\hat{M}} Z^2(t)dt = 0 \quad M_1(\hat{y}) = M_2(\hat{y}) = M(\hat{y} + 0). \]

The existence of M(\hat{y} - 0) and the identities: M(\hat{y}) = M(\hat{y} - 0) = M(\hat{y} + 0) can be shown by analogy.

The function M(y), y \geq y_0 obeys the following properties:

(a) It has a continuous derivative at any point y such that M(y) \neq \gamma, where \gamma is a zero of the function \zeta(1/2 + it).

(b) It has a derivative equal to +\infty at any point y such that M(y) = \gamma, \gamma mentioned above.

These properties of the M function can be proved as follows. By (3.3) and (3.4) we have

\[ \frac{\Phi(y + \Delta y) - \Phi(y)}{\Delta y} = \frac{M(y + \Delta y) - M(y)}{\Delta y} Z^2 \{M(y) + \theta \cdot [M(y + \Delta y) - M(y)]\}, \]

where \theta \in (0, 1). Using the fact that \Phi'(y) > 0 (see (3.5)) we can deduce from (3.6) that:

(i) for any values of y such that Z^2[M(y)] > 0 there exists a continuous derivative, and

(ii) for any values of y such that Z^2[M(y)] = 0 we have

\[ \lim_{\Delta y \to 0} \frac{M(y + \Delta y) - M(y)}{\Delta y} = +\infty. \]

Since our function T = M(y), y \geq y_0 is continuous and increasing (to +\infty) there exists unique inverse function that is also continuous and increasing (to +\infty):

\[ y = \varphi(T) = \varphi_M(T), \quad T \geq T_0[\varphi] = M_\mu(y_0). \]

As a consequence of eqs. (3.3) and (3.4) we have

\[ \frac{d\varphi(T)}{dT} = \frac{Z^2(T)}{\Phi'[\varphi(T)]}, \quad \Phi' = \Phi'_y[\varphi(T)] > 0, \quad T \geq T_0[\varphi], \]

(\varphi'(\gamma) = 0, see (3.7)). Eq. (3.9) implies that \varphi(T) \in C^\infty([T_0[\varphi], +\infty)) and also that the property (1.5) holds true. Inserting (3.8) into (3.3) one obtains that X(T) = \varphi_\mu(T), T \geq T_0[\varphi] is a solution to the integral equation (1.4). □
4. CONSEQUENCES FROM THE TKA FORMULA

The following Lemma holds true:

**Lemma 2.** Let \( M(y) \in \{ M \} \) be arbitrary, then

\[
\int_0^{M(y)} Z^2(t) dt = F(y) + r(y),
\]

where

\[
F(y) = \frac{y}{2} \ln \left( \frac{y}{2} \right) + E \frac{y}{2} + c_0, \quad r(y) = \mathcal{O} \left( \frac{\ln(y)}{y} \right), \quad E = c - \ln(2\pi),
\]

and the constant within the \( \mathcal{O} \)-symbol is an absolute constant.

**Proof.** We start with the formula (1.3) and let \( N = 1 \):

\[
\int_0^\infty Z^2(t)e^{-2\delta t} dt = c - \frac{\ln(4\pi\delta)}{2\sin(\delta)} + c_0 + c_1\delta + \mathcal{O}(\delta^2).
\]

As long as \( |Z(t)| < At^{1/4}, \ t \geq t_0 \), we have

\[
f(t,\delta) = t^{1/2}e^{-\delta t} \leq f \left( \frac{1}{2\delta}, \delta \right) = \frac{1}{\sqrt{2e\delta}}
\]

\[
\int_U^\infty Z^2(t)e^{-2\delta t} dt < B\frac{e^{-\delta U}}{\delta^{3/2}}, \quad B = \frac{A^2}{\sqrt{2e}}, \ U \geq t_0.
\]

The value \( U = \mu(1/\delta) \) is to be chosen by the following rule:

\[
B\delta^{-3/2}e^{-\delta U} \leq \delta^2 \Rightarrow \mu \left( \frac{1}{\delta} \right) \geq \frac{7}{\delta} \ln \left( \frac{1}{\delta} \right) > \frac{1}{\delta} \ln \left( \frac{B}{\delta^{3/2}} \right).
\]

Now, (4.3) implies:

\[
\int_0^{\mu(1/\delta)} Z^2(t)e^{-2\delta t} dt = c - \frac{\ln(4\pi\delta)}{2\sin(\delta)} + c_0 + c_1\delta + \mathcal{O}(\delta^2), \quad \mu \left( \frac{1}{\delta} \right) \geq \frac{1}{\delta} \ln \left( \frac{1}{\delta} \right).
\]

(see the introduction, part (B) - the condition for \( \mu(y) \)), and for the remainder term we have:

\[
-\int_{\mu(1/\delta)}^\infty Z^2(t)e^{-2\delta t} dt = \mathcal{O}(\delta^2).
\]

Let \( \delta \in (0, \delta_0) \) with \( \delta_0 \) being sufficiently small, then

\[
\frac{1}{\sin(\delta)} = \frac{1}{\delta} \left\{ 1 + \frac{\delta^2}{6} + \mathcal{O}(\delta^4) \right\},
\]

and

\[
\frac{c - \ln(4\pi\delta)}{2\sin(\delta)} = \frac{1}{2\delta} \ln \left( \frac{1}{\delta} \right) + \frac{D}{2\delta} + \mathcal{O} \left[ \delta \ln \left( \frac{1}{\delta} \right) \right],
\]

and (see \( 4.4 \))

\[
\int_0^{\mu(1/\delta)} Z^2(t)e^{-2\delta t} dt = \frac{1}{2\delta} \ln \left( \frac{1}{\delta} \right) + \frac{D}{2\delta} + c_0 + \mathcal{O} \left[ \delta \ln \left( \frac{1}{\delta} \right) \right], \quad D = c - \ln(4\pi).
\]

Putting \( \delta = 1/y, \ y_0 = 1/\delta_0 \) into eq. (4.6) and using eq. (5.3) we obtain the formulae (4.1) and (4.2), respectively. Since the constants in the eqs. (4.3), (4.5) are absolute, the constant entering the \( \mathcal{O} \)-symbol in (4.2) is absolute, too. □
5. Proof of the theorem

Putting $T = y/2$ into eq. (2.5) and comparing with the formula (4.1) we obtain

$$\frac{y}{2} < M(y), \ y \to \infty.$$ 

Furthermore, putting into eq. (2.5)

$$T = \frac{y}{2} \left(1 + \frac{A}{\ln \left(\frac{y}{2}\right)}\right), \ A > 1 - c,$$

and comparing with eq. (4.1) we have

$$M(y) < \frac{y}{2} \left(1 + \frac{A}{\ln \left(\frac{y}{2}\right)}\right), \ y \to \infty.$$ 

Subsequently (see (3.8)),

(5.1) \quad 0 < M(y) \ - \ \frac{y}{2} < \frac{A}{2} \frac{y}{\ln \left(\frac{y}{2}\right)} \Rightarrow 0 < 2T - \varphi(T) < B \frac{\varphi(T)}{\ln[\varphi(T)]},$$

i.e. the following equation holds true

(5.2) \quad 1.9T < \varphi(T) < 2T.

Inserting $y = \varphi(T) \in \{\varphi\}$ into eq. (4.1) (see (3.8)) we obtain the formula (2.1) and the estimate (2.2), (see (5.2)).

The relation (2.3) follows from $F'(y) = 1/2 \ln(y/2) + E + 1$ and from the eq. (2.1) written for $\varphi_1$ and $\varphi_2$, respectively, with help of (5.2).

Since $\mu(y) > M(y), \ y \geq y_0$ (see (3.3)) and in the case of boundedness of the set $\{\mu(y_0)\}$ the choice of values (see (2.4)):

$$T_0 = \sup_{\mu} \{\mu(y_0)\} \geq \mu(y_0) > M_\mu(y_0), \ \forall \mu(y) \in \{\mu\},$$

is regular, i.e. the interval $[T_0, +\infty)$ is the common domain of the functions $\varphi_\mu(T), \ \mu(T) \in \{\mu\}$.

6. Relation between the Jacob’s ladders and prime-counting function $\pi(T)$

Comparing the formulae (2.1) and (2.5) we obtain

$$\omega \left(\frac{\varphi}{2}\right) - \omega(T) = (1 - c)T + O \left(T^{1/3+\epsilon}\right), \ \omega(t) = t \ln(t) + (c - \ln(2\pi))T.$$ 

Let us consider the power series expansion in the variable $\varphi/2 - T$ of the previous formula. We obtain the following nonlinear equation

(6.1) \quad x(\ln(T) - a) - \sum_{k=2}^{\infty} \frac{x^k}{k(k - 1)} = 1 - c + O \left(T^{-2/3+\epsilon}\right), \ x = \frac{T - \varphi}{T},$$

where $a = \ln(2\pi) - 1 - c$. Because of (see (5.1) and (5.2))

$$x = O \left(\frac{1}{\ln(T)}\right),$$

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we obtain from (6.1):
\[ x = \frac{1 - c}{\ln(T) - a} + O \left( \frac{1}{\ln^3(T)} \right) \Rightarrow T - \frac{\varphi(T)}{2} = \frac{T}{\ln(T)} \left\{ 1 + O \left( \frac{1}{\ln(T)} \right) \right\}, \]
and furthermore, by using the Selberg-Erdős theorem, we obtain the formula:
\[ (6.2) \quad \pi(T) \sim \frac{1}{1 - c} \left\{ T - \frac{\varphi(T)}{2} \right\}, \quad T \to \infty, \quad \forall \varphi \in \{ \varphi \}. \]

**Remark 6.** As a consequence of the above written we have that the Jacob’s ladders are connected (along to the zeroes of the function \( \zeta(1/2 + it) \)) also to the prime-counting function \( \pi(T) \), see (6.2).

More interesting information can be deduced from the nonlinear equation (6.1). Namely, inserting
\[ (6.3) \quad T e^{-a} = \tau, \quad e^{-a} \varphi(e^{a} \tau) = \psi(\tau) \]
into (6.1) we obtain:
\[ (6.4) \quad x \ln(\tau) - \sum_{k=2}^{\infty} \frac{x^k}{k(k-1)} = 1 - c + O \left( \tau^{-2/3+\varepsilon} \right), \quad x = \frac{\tau - \psi(\tau)}{\tau}. \]

The following statement holds true: if in the equation:
\[ (6.5) \quad x = \frac{A_1}{\ln(\tau)} + \frac{A_3}{\ln^3(\tau)} + \ldots + \frac{A_n}{\ln^n(\tau)} + \frac{A_{n+1}}{\ln^{n+1}(\tau)} + \ldots, \]
the coefficients \( A_1, A_3, \ldots, A_n \) are already known, then the coefficient \( A_{n+1} \) is determined by (6.4). We obtain:
\[ A_1 = 1 - c, \quad A_3 = \frac{1}{2}(1 - c)^2, \quad A_4 = \frac{1}{6}(1 - c)^3, \]
\[ A_5 = \frac{1}{2}(1 - c)^3 + \frac{1}{12}(1 - c)^4, \ldots . \]

Changing the variables in (6.5) into the initial ones (see (6.3)) we obtain the following asymptotic formula
\[ (6.6) \quad \frac{1}{T} \left\{ T - \frac{\varphi(T)}{2} \right\} \sim \frac{A_1}{\ln(T) - a} + \frac{A_3}{(\ln(T) - a)^3} + \ldots , \quad T \to \infty, \]
where \( B_2 = a A_1, \quad B_3 = a^2 A_1 + A_3, \ldots . \)

**Remark 7.** Let us remark that the asymptotic formula (6.6) is an analogue to the following asymptotic formula
\[ \frac{1}{T} \int_2^T \frac{dt}{\ln(t)} \sim \frac{1}{\ln(T)} + \frac{1}{\ln^2(T)} + \frac{2!}{\ln^3(T)} + \ldots , \]
for the Gauss logarithmic integral.
7. Fundamental properties of $Z^2$-transformation

Let us mention explicitly that the key idea of the proof of the theorem was to introduce the new integral transformation: $Z^2$-transformation.

By using the appropriate terminology from optics (see example: Landau & Lifshitz, Field theory, GIFML, Moscow 1962, page 167):

- the elements $\mu(y) \in \{\mu\}$ are called the rays and the set $\{\mu\}$ itself is called the beam,
- the beams crossing each other in a given point are called homocentric beams.

The fundamental property of the $Z^2$-transformation lies in the following:

- if the set $\{\mu(y_0)\}$ is bounded then the beam $\{\mu\}$ (and, at the same time, also any other homocentric beam) is transformed into the homocentric beam $\{\varphi\}$ of the Jacob’s ladders, with respect to the point $[+\infty, +\infty]$ (see (2.3), (2.4)),
- the transformed beam $\{\varphi\}$ is very narrow in sense of (2.4), i.e. an $Z^2$-optical system generates an analogue of a laser beam.

Let us consider for example the homocentric (with respect to the point $[y_0, y_0^2]$) sheaf of rays:

\[
(7.1) \quad u(y; \rho, n) = y^2[1 + \rho(y - y_0)^n], \quad \rho \in [0, 1], \quad n \in \mathbb{N}.
\]

Following the equation $u_0(y + \Delta; 0, n) = u_1(y; 1, n)$ we obtain that

\[
\Delta = \frac{y(y - y_0)^n}{\sqrt{1 + (y - y_0)^2}} \to \infty, \quad \text{at} \quad y \to \infty,
\]

i.e. (7.1) is a diverging beam. Anyway, the $Z^2$-mapping transforms (7.1) into an analogue of a laser beam.

8. On intervals that cannot be reached by estimates of Heath-Brown and Ivic

We will show the accuracy of our formula (2.1) in comparison with known estimates of Heath-Brown and Ivic (see [5], (7.20) page 178, and (7.62) page 191)

\[
(8.1) \quad \int_T^{T+G} Z^2(t)dt = O\left(G \ln^2(T)\right), \quad G = T^{1/3-c_0}, \quad c_0 = \frac{1}{108}.
\]

First of all, one can easily obtain the tangent law from both (2.1) and (8.1):

\[
(8.2) \quad \int_T^{T+U} Z^2(t)dt = U \ln \left(e^{-\alpha(T)}\right) \tan[\alpha(T, U)] + O\left(\frac{1}{T^{1/3+2c_0}}\right)
\]

for $0 < U < T^{1/3+c_0}$, where $\alpha = \alpha(U, T)$ is the angle of the chord of the curve $y = \frac{1}{2}\varphi(T)$ crossing the points $[T, \frac{1}{2}\varphi(T)]$ and $[T + U, \frac{1}{2}\varphi(T + U)]$. Further, from (2.1) and (2.5) we have

\[
(8.3) \quad \tan(\alpha_0) = \tan[\alpha(T, U_0)] = 1 + O\left(\frac{1}{\ln(T)}\right), \quad U_0 = T^{1/3+2c}.
\]

And finally, considering the set of all chords of the curve $y = \frac{1}{2}\varphi(T)$ which are parallel to our fundamental chord joining points $[T, \frac{1}{2}\varphi(T)]$ and $[T + U, \frac{1}{2}\varphi(T + U)]$,
we obtain the continuum of formulae:
\[ \int_a^b Z^2(t) \, dt = (b - a) \ln(T) + \mathcal{O}(b - a) + \mathcal{O}\left(\frac{1}{T^{1/3+2\varepsilon}}\right), \]
(8.4) \hspace{1cm} 0 < b - a < 1, (a, b) \subset \left(T, T + T^{1/3+\varepsilon_0}\right).

**Remark 8.** It is quite evident that the interval (0, 1) cannot be reached in known theories leading to estimates of Heath-Brown and Ivic.

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