SIMILAR FILLINGS AND ISOLATION OF CUSPS OF HYPERBOLIC 3-MANIFOLDS

ROBERTO FRIGERIO

Abstract. In this paper we deepen the analysis of certain classes $M_{g,k}$ of hyperbolic 3-manifolds that were introduced in a previous work by B. Martelli, C. Petronio and the author. Each element of $M_{g,k}$ is an oriented complete finite-volume hyperbolic 3-manifold with compact connected geodesic boundary of genus $g$ and $k$ cusps. We study small deformations of the complete hyperbolic structure of manifolds in $M_{g,k}$ via a close analysis of their geodesic triangulations. We prove that several elements in $M_{g,k}$ admit non-homeomorphic hyperbolic Dehn fillings sharing the same volume, homology, cusp volume, cusp shape, Heegaard genus, complex length of the shortest geodesic, length of the shortest return path, and Turaev-Viro invariants. Manifolds which share all these invariants are called geometrically similar, and were first studied by C. D. Hodgson, R. G. Meyerhoff and J. R. Weeks. The examples of geometrically similar manifolds they described are commensurable with each other. We show here that many elements in $M_{g,k}$ admit non-commensurable geometrically similar Dehn fillings.

The notion of geometric isolation for cusps in a hyperbolic 3-manifold was introduced by W. D. Neumann and A. W. Reid and studied by D. Calegari, who provided explanations for all the previously known examples of isolation phenomena. We show here that the cusps of any manifold $M \in M_{g,k}$ are geometrically isolated from each other. Apparently, isolation of cusps in our examples arises for different reasons from those described by Calegari.

We also show that any element in $M_{g,k}$ admits an infinite family of hyperbolic Dehn fillings inducing non-trivial deformations of the hyperbolic structure on the geodesic boundary.

Let $N$ be an oriented complete finite-volume hyperbolic 3-manifold with compact geodesic boundary. Mostow-Prasad’s rigidity Theorem implies that the space (of homotopy classes) of complete finite-volume structures supported by $N$ reduces to a single point, so non-trivial deformations of the complete structure can give rise only to incomplete metrics. It is a well-known fact that such deformations are closely related to the geometry of manifolds which can be obtained from $N$ via Dehn filling, as we are now going to explain.

A slope on a torus is an isotopy class of simple unoriented closed curves. Let $X$ be an oriented 3-manifold with boundary tori $T_1, \ldots, T_k$ and let $V_1, \ldots, V_h$ be solid tori, $h \leq k$. Let $s_i$ be a slope on $T_i$ for $i = 1, \ldots, h$ and choose an attaching homeomorphism $\varphi_i : \partial V_i \to T_i$ taking a meridian of $V_i$ onto a loop representing $s_i$. Set $\Phi = (\varphi_1, \ldots, \varphi_h)$ and $X(s_1, \ldots, s_h) = X \bigcup_\Phi (V_1 \cup \ldots \cup V_h)$. We say that $X(s_1, \ldots, s_h)$ is obtained by Dehn filling $X$ along the $s_i$’s. It is easily seen that $X(s_1, \ldots, s_h)$ is a 3-manifold whose homeomorphism type depends solely on the $s_i$’s. Also observe that the orientation of $X$ naturally induces an orientation also on $X(s_1, \ldots, s_h)$.

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A complete finite-volume hyperbolic $N$ admits a natural compactification obtained by adding some boundary tori. Thus, up to identifying $N$ with its compactification, it does make sense to consider the Dehn fillings of $N$. A crucial fact is that the metric completions of many small deformations of the complete metric of $N$ actually define complete hyperbolic structures on manifolds obtained by Dehn filling $N$. This phenomenon is at the heart of the proof of Thurston’s hyperbolic Dehn filling Theorem [Thu79], which states that “almost all” the Dehn fillings of a cusped 3-manifold support a complete finite-volume hyperbolic metric.

This paper is devoted to the description of certain classes $\mathcal{M}_{g,k}$ of cusped hyperbolic 3-manifolds with geodesic boundary. Such classes were first introduced in [FMP03]. We concentrate here on describing how partially truncated triangulations can be employed in order to study the Dehn fillings of manifolds in $\mathcal{M}_{g,k}$.

1. Preliminaries and statements

All the manifolds considered in this paper will be connected and oriented. Let $\Delta$ denote the standard tetrahedron, and let $\hat{\Delta}$ be $\Delta$ with its vertices removed. An ideal triangulation of a compact 3-manifold $M$ with boundary is a realization of the interior of $M$ as a gluing of a finite number of copies of $\hat{\Delta}$, induced by a simplicial face-pairing of the corresponding $\Delta$’s. Let $\Sigma_g$ be the closed orientable surface of genus $g$. The following result [FMP03] motivates the definition of $\mathcal{M}_{g,k}$.

**Proposition 1.1.** An ideal triangulation of a manifold whose boundary is the union of $\Sigma_g$ and $k$ tori contains at least $g+k$ tetrahedra.

For all $g > k \geq 1$ we then define $\mathcal{M}_{g,k}$ as follows:

$$\mathcal{M}_{g,k} = \{ \text{compact oriented manifolds } M \text{ having an ideal triangulation}$$

\[\text{with } g+k \text{ tetrahedra, and } \partial M = \Sigma_g \cup \left( \bigsqcup_{i=1}^k T_i \right) \text{ with } T_i \cong \Sigma_1 \}.\]

Let $N$ be a compact manifold with boundary. When this does not create ambiguities, we will denote by $N$ also the manifold obtained by removing the boundary tori from the original $N$. Thus the natural compactification of a hyperbolic manifold will be usually denoted by the same symbol denoting the manifold itself. We say that a numerical sequence $(a_n)_{n=1}^\infty$ has growth type $n^g$ if there exist constants $C > c > 0$ such that $n^{c-n} < a_n < n^{C-n}$ for $n \gg 0$. The following results are taken from [FMP03].

**Theorem 1.2.** Any element in $\mathcal{M}_{g,k}$ admits a complete finite-volume hyperbolic structure with geodesic boundary.

**Theorem 1.3.** For all $g > k \geq 1$ we have $\mathcal{M}_{g,k} \neq \emptyset$. Moreover, for any fixed $k$ the sequence $(\# \mathcal{M}_{g,k})_{g=2}^\infty$ has growth type $g^9$.

1.1. Isolation of cusps. Recall that if $N$ is a complete finite-volume hyperbolic 3-manifold, then every boundary torus of $N$ is naturally endowed with a Euclidean structure, defined up to similarity. Neumann and Reid introduced in [NR93] the notion of geometric isolation for cusps in a hyperbolic manifold:

**Definition 1.4.** Let $N$ be a complete finite-volume hyperbolic 3-manifold with (possibly empty) geodesic boundary and cusps $C_1, \ldots, C_h, C_{h+1}, \ldots, C_k$. We say that $C_1, \ldots, C_h$ are geometrically isolated from $C_{h+1}, \ldots, C_k$ if any small deformation of the hyperbolic structure on $N$ induced by Dehn filling $C_{h+1}, \ldots, C_k$ while keeping $C_1, \ldots, C_h$ complete does not affect the Euclidean structure at $C_1, \ldots, C_h$. 
Calegary described in [Cal01] different strategies for constructing manifolds with isolated cusps, also providing explanations for all the previously known examples of isolation phenomena. In Section 3 we show that the cusps of any manifold \( M \in \mathcal{M}_{g,k} \) are geometrically isolated from each other:

**Theorem 1.5.** Let \( M \in \mathcal{M}_{g,k} \) with cusps \( C_1, \ldots, C_k \) and let \( h \leq k \). Then \( C_1, \ldots, C_h \) are geometrically isolated from \( C_{h+1}, \ldots, C_k \).

Apparently, isolation of cusps in our examples arises for different reasons from those described in [Cal01].

1.2. **Non-isolation of the boundary.** The natural question if the geodesic boundary of an element in \( \mathcal{M}_{g,k} \) is isolated from the cusps is also answered. Examples of isolation of the geodesic boundary from cusps of hyperbolic 3-manifolds were provided in [NR93, Fuj93]. On the other hand, non-isolation phenomena were described in [Fuj92, FK97]. In Section 4 we prove the following:

**Theorem 1.6.** Let \( M \in \mathcal{M}_{g,k} \). Then there exists an infinite set \( \{N_i\}_{i \in \mathbb{N}} \) of complete finite-volume hyperbolic 3-manifolds with the following property: each \( N_i \) is obtained by Dehn filling \( M \), and the hyperbolic surfaces \( \partial M, \partial N_1, \ldots, \partial N_i \) are pairwise non-isometric.

1.3. **Some invariants of hyperbolic 3-manifolds.** Let \( N \) be a complete hyperbolic 3-manifold and take a closed geodesic \( \ell \subset N \). Then a well-defined complex length \( \mathbb{C}L(\ell) \in \mathbb{C}/2\pi i\mathbb{Z} \) exists which can be described as follows. The universal covering \( \tilde{N} \) of \( N \) is isometric to a convex polyhedron in \( \mathbb{H}^3 \) bounded by a countable number of hyperbolic planes [Koj90]. Choose an orientation on \( \ell \), and realize \( \tilde{N} \) in \( \mathbb{H}^3 \cong \mathbb{C} \times (0, \infty) \) in such a way that \( \ell \) lifts in \( \tilde{N} \) to the oriented geodesic \( \tilde{\ell} \) with endpoints 0 and \( \infty \). Let \( \gamma \in \text{Aut}(\tilde{N}) \subset \text{Isom}^+(\mathbb{H}^3) \) be the element corresponding to the oriented curve \( \ell \) which leaves \( \tilde{\ell} \) invariant. A complex number \( a \) exists such that

\[
\gamma(z,t) = (a \cdot z, |a| \cdot t), \quad (z,t) \in \mathbb{C} \times (0, \infty).
\]

We set \( \mathbb{C}L(\ell) = \ln a \in \mathbb{C}/2\pi i\mathbb{Z} \). It is easily seen that this is a good definition, i.e. that \( a \) only depends on the unoriented curve \( \ell \), and that the usual length of \( \ell \) is equal to \( \Re(\mathbb{C}L(\ell)) \).

If \( N \) is complete finite-volume with compact geodesic boundary and \( k \) cusps, the **cusp shape** of \( N \) is the set of Euclidean structures (up to a scale factor) induced on the boundary tori of \( N \). A **regular horocusp neighbourhood** for \( N \) is a set \( O_1 \sqcup \ldots \sqcup O_k \subset N \), where \( O_i \) is an open embedded horospherical neighbourhood of the \( i \)-th cusp of \( N \), \( O_i \cap O_j = \emptyset \) for \( i \neq j \) and \( \text{vol}(O_i) = \text{vol}(O_j) \) for \( i, j \in \{1, \ldots, k\} \). The **cusp volume** of \( N \) is the volume of a maximal regular horocusp neighbourhood for \( N \) (where this volume is intended to be 0 if \( N \) is compact). A **return path** in \( N \) is a geodesic segment in \( N \) intersecting \( \partial N \) perpendicularly in its endpoints. Since the boundary of \( N \) is compact, it is easily seen that there exists a (not necessarily unique) shortest return path in \( N \).

If \( N \) is a compact 3-manifold with \( \partial N = \partial_0 N \sqcup \partial_1 N \), one can define the **Heegaard genus** of \( (N, \partial_0 N, \partial_1 N) \) as the minimal genus of a surface that splits \( N \) as \( C_0 \sqcup C_1 \), where \( C_i \) is obtained by attaching 1-handles on one side of a collar of \( \partial_i N \). Moreover, for any integer \( r \geq 2 \), after fixing in \( \mathbb{C} \) a primitive \( 2r \)-th root of unity, a real-valued invariant \( TV_r(N) \) was defined by Turaev and Viro in [TV92].
1.4. Similar fillings. We are now ready to give the following:

**Definition 1.7.** Let $N, N'$ be complete finite-volume hyperbolic 3-manifolds with geodesic boundary and the same number of cusps. We say that $N$ and $N'$ are *geometrically similar* if the following conditions hold:

- $N$ and $N'$ share the same volume, the same cusp volume and the same cusp shape;
- The shortest return paths of $N$ and $N'$ have the same length;
- The shortest closed geodesics of $N$ and $N'$ have the same complex length;
- $H_1(N; \mathbb{Z}) \cong H_1(N'; \mathbb{Z})$;
- if $\Sigma$ (resp. $\Sigma'$) is the geodesic boundary of $N$ (resp. of $N'$) and $T_1, \ldots, T_k$ (resp. $T_1', \ldots, T_k'$) are the boundary tori of $N$ (resp. of $N'$), then the Heegaard genus of $(N, \Sigma, T_1 \sqcup \cdots \sqcup T_k)$ is equal to the Heegaard genus of $(N', \Sigma', T_1' \sqcup \cdots \sqcup T_k')$;
- $N$ and $N'$ have the same Turaev-Viro invariants;
- Manifolds obtained by sufficiently complicated Dehn fillings on $N$ can be paired to manifolds obtained by sufficiently complicated Dehn fillings on $N'$ in such a way that the elements in each pair share the same volume, first homology group, cusp volume, cusp shape, length of the shortest return path, complex length of the shortest geodesic, Heegaard genus and Turaev-Viro invariants.

Geometrically similar hyperbolic 3-manifolds were first studied in [HMW92], where it was shown that the Whitehead link complement admits an infinite sequence of pairs of non-homeomorphic geometrically similar Dehn fillings (the definition of geometric similarity introduced in [HMW92] is actually a bit different from ours, and regards cusped manifolds without geodesic boundary). The elements of any pair of geometrically similar manifolds described in [HMW92] are both obtained by filling one cusp of the Whitehead link complement, and they are commensurable with each other. We show here that if $M \in M_{g,k}$ is generic, i.e., if it does not admit too many isometries, then we can construct different geometrically similar manifolds by filling $M$ along slopes on any chosen set of cusps of $M$. This allows us to prove the following:

**Theorem 1.8.** For any $k > 0$ there exist $g > k$ and an element $X_k \in M_{g,k}$ with boundary tori $T_1, \ldots, T_k$ having the following property. For each $i = 1, \ldots, k$ there exists a finite set $S_i$ of slopes on $T_i$ such that if $h \leq k$ and $s_i \notin S_i$ is a slope on $T_i$, then $X_k(s_1, \ldots, s_h)$ is hyperbolic and at least $((k! \cdot 3^h)/(h! \cdot (k-h)!))$ pairwise non-homeomorphic hyperbolic Dehn fillings of $X_k$ are geometrically similar to $X_k(s_1, \ldots, s_h)$.

Moreover, the geometrically similar manifolds we obtain are typically non-commensurable with each other (however, examples are also provided of non-homeomorphic geometrically similar commensurable Dehn fillings of a specific element of $M_{g,k}$).

2. Triangulations and deformation space

In order to construct a hyperbolic structure on a manifold $M \in M_{g,k}$, we choose a suitable triangulation of $M$ and we solve the corresponding hyperbolicity equations. We recall that the *valence* of an edge in a triangulation is the number of tetrahedra incident to it (with multiplicity). The following result is proved in [EM13].

**Proposition 2.1.** Let $M \in M_{g,k}$ and suppose that $T$ is an ideal triangulation of $M$ with $g+k$ tetrahedra. Then the following holds:
For any $i = 1, \ldots, k$ there are exactly two tetrahedra of $\mathcal{T}$ with 3 vertices on $\Sigma_g$ and one on $T_i$; the remaining $g - k$ tetrahedra have all 4 vertices on $\Sigma_g$.

$\mathcal{T}$ has $k + 1$ edges $e_0, \ldots, e_k$ such that $e_0$ has both its endpoints on $\Sigma_g$ and valence 6, while $e_i$ connects $\Sigma_g$ to $T_i$ and has valence 6 for $i = 1, \ldots, k$.

2.1. Geometric tetrahedra. In order to construct a hyperbolic structure on our manifold $M \in \mathcal{M}_{g,k}$ we realize the tetrahedra of an ideal triangulation of $M$ as special geometric blocks in $\mathbb{H}^3$ and then we require that the structures match under the gluings. To describe the blocks to be used we need some definitions.

A partially truncated tetrahedron is a pair $(\Delta, I)$, where $\Delta$ is a tetrahedron and either $I = \emptyset$ or $I = \{v\}$, where $v$ is a vertex of $\Delta$. In the latter case we say that $v$ is the ideal vertex of $\Delta$. In the sequel we will always refer to $\Delta$ itself as a partially truncated tetrahedron, tacitly implying that $I$ is also fixed. The topological realization $\Delta^*$ of $\Delta$ is obtained by removing from $\Delta$ the ideal vertex, if $I \neq \emptyset$, and small open stars of the non-ideal vertices. We call lateral hexagon and truncation triangle the intersection of $\Delta^*$ respectively with a face of $\Delta$ and with the link in $\Delta$ of a non-ideal vertex. The edges of the truncation triangles, which also belong to the lateral hexagons, are called boundary edges, and the other edges of $\Delta^*$ are called internal edges. If $\Delta$ has an ideal vertex, three lateral hexagons of $\Delta^*$ are in fact pentagons with a vertex removed, and they are called exceptional lateral hexagons.

A geometric realization of $\Delta$ is an embedding of $\Delta^*$ in $\mathbb{H}^3$ such that the truncation triangles are geodesic triangles, the lateral hexagons are geodesic polygons with ideal vertices corresponding to missing edges, and truncation triangles and lateral hexagons lie at right angles to each other. The following theorem [Fuj90, FP04] classifies isometry classes of geometric partially truncated tetrahedra.

**Theorem 2.2.** Let $\Delta$ be a partially truncated tetrahedron and let $\Delta^{(1)}$ be the set of edges of $\Delta$. The geometric realizations of $\Delta$ are parameterized up to isometry by the dihedral angle assignements $\theta : \Delta^{(1)} \to (0, \pi)$ such that for each vertex $v$ of $\Delta$, if $e_1, e_2, e_3$ are the edges that emanate from $v$, then $\theta(e_1) + \theta(e_2) + \theta(e_3)$ is equal to $\pi$ for ideal $v$ and less than $\pi$ for non-ideal $v$.

The following well-known hyperbolic trigonometry formulae will prove useful later:
Lemma 2.3. With notation as in Fig. 4 we have

\begin{align*}
(1) & \quad \cosh a_1 = \frac{(\cos \alpha_2 \cdot \cos \alpha_3 + \cos \alpha_1)}{(\sin \alpha_2 \cdot \sin \alpha_3)}, \\
(2) & \quad \sinh a_1 / \sin \alpha_1 = \sinh a_2 / \sin \alpha_2 = \sinh a_3 / \sin \alpha_3, \\
(3) & \quad \cosh b_1 = \frac{(\cosh c_2 \cdot \cosh c_3 + \cosh c_1)}{(\sinh c_2 \cdot \sinh c_3)}. 
\end{align*}

2.2. Hyperbolicity equations. Let \( M \) be an element of \( \mathcal{M}_{g,k} \) and \( \mathcal{T} \) be an ideal triangulation of \( M \) with \( g + k \) tetrahedra. We try to give \( M \) a hyperbolic structure with geodesic boundary by looking for a geometric realization \( \theta \) of \( \mathcal{T} \) such that the structures of the tetrahedra match under the gluings. In order to define a global hyperbolic structure on \( M \), the tetrahedra of \( \mathcal{T} \) must satisfy two obvious necessary conditions, which in fact are also sufficient. Namely, we should be able to glue the lateral hexagons by isometries, and we should have a total dihedral angle of \( 2\pi \) around each edge of the manifold. The first condition ensures that the hyperbolic structure defined by \( \theta \) on the complement of the 2-skeleton of \( \mathcal{T} \) extends to the complement of the 1-skeleton. Since any tetrahedron of \( \mathcal{T} \) contains at most one ideal vertex, the second one ensures that such a structure glues up without singularities also along the edges.

By Proposition 2.1 if we suppose \( M \) to be hyperbolic and \( \mathcal{T} \) to be geometric (i.e., to define a hyperbolic structure on the whole of \( M \)), than the edges of the tetrahedra with all the vertices on \( \Sigma_g \) should have all the same length. This would force the realizations of the compact tetrahedra in \( \mathcal{T} \) to be regular and isometric to each other. Moreover, all the finite internal edges of the tetrahedra with one vertex on the boundary tori should also have the same length.

On each tetrahedron of \( \mathcal{T} \) we fix the orientation compatible with the global orientation of \( M \). As a result also the lateral hexagons have a fixed orientation, which is reversed by the gluing maps. Let us now fix some notation we will use extensively later on. Let \( T_1, \ldots, T_k \) be the boundary tori of \( M \). We denote by \( \Delta_{2i-1}, \Delta_{2i} \) the tetrahedra of \( \mathcal{T} \) incident to \( T_i \) and by \( F_{2i-1}^j, F_{2i-1}^j, F_{2i-1}^j, F_{2i-1}^j, F_{2i-1}^j, F_{2i-1}^j, F_{2i-1}^j, F_{2i-1}^j \) the exceptional hexagons of \( \Delta_{2i-1}, \Delta_{2i} \), in such a way that \( F_{2i-1}^j \) is glued to \( F_{2i}^j \) for \( j = 1, 2, 3 \). For \( l = 1, \ldots, 2k \) we also suppose that \( F_{1}^l, F_{2}^l, F_{3}^l \) are positively arranged around the ideal vertex of \( \Delta_l \), and we call \( e_{l}^1 \) the only finite internal edge of \( F_{i}^l \), and \( f_{l}^j \) the edge of \( \Delta_l \) opposite to \( e_{l}^j \). We now consider a geometric realization \( \theta \) of the tetrahedra of \( \mathcal{T} \) such that compact tetrahedra are regular and isometric to each other, and for \( l = 1, \ldots, 2k \), \( j = 1, 2, 3 \) we set \( \alpha_{l}^j = \theta(e_{l}^j) \), and \( \gamma_{l}^j = \theta(f_{l}^j) \) (see Fig. 2). We set \( \beta \) to be the dihedral angle along the edges of the \( g - k \) compact tetrahedra of \( \mathcal{T} \). We denote by \( L^\theta \) the length with respect to the realization \( \theta \).

2.3. Consistency along the faces. We first determine the conditions on dihedral angles under which all the compact lateral hexagons of the tetrahedra in \( \mathcal{T} \) are regular and isometric to each other. By equation (3), this is equivalent to asking that the lengths of all the boundary edges of all the compact lateral hexagons are equal to each other, and by equation (1), this condition translates into the following set of equations:

\begin{equation}
\frac{\cos \alpha_{l}^j \cdot \cos \alpha_{l}^{j+1} + \cos \gamma_{l}^{j+2}}{\sin \alpha_{l}^j \cdot \sin \alpha_{l}^{j+1}} = \frac{\cos^2 \beta + \cos \beta}{\sin^2 \beta}, \quad l = 1, \ldots, 2k, \; j = 1, 2, 3.
\end{equation}

2.4. Exceptional hexagons. Let us consider an exceptional hexagon \( F_{126}^* \) as in Fig. 8 and recall that the hexagon is embedded in \( \mathbb{H}^3 \) by \( \theta \). We consider the horospheres \( O_1 \) and \( O_2 \) centred at \( v_{123} \) and passing through the non-ideal ends of \( e_1 \) and \( e_2 \) respectively. We define \( \sigma^\theta(F_{126}) \) to be \( \pm \text{dist}(O_1, O_2) \), the sign being positive if \( e_2, v_{123}, e_1 \) are arranged
positively on \( \partial F_{126}^* \) and \( O_1 \) is contained in the horoball bounded by \( O_2 \), or if \( e_2, v_{123}, e_1 \) are arranged negatively on \( \partial F_{126}^* \) and \( O_2 \) is contained in the horoball bounded by \( O_1 \), and negative otherwise. Let us denote by \( e_{ij} \) the boundary edge joining \( e_i \) with \( e_j \). From [Frib, Proposition 1.8] and equation (2) we deduce:

\[
\sigma^\theta(F_{126}) = \ln \frac{\sinh L^\theta(e_{56})}{\sinh L^\theta(e_{46})} + \ln \frac{\sin \theta(e_2) \cdot \sin \theta(e_5)}{\sin \theta(e_1) \cdot \sin \theta(e_4)}.
\]

We now set \( \ell^\theta(F_{126}) = L^\theta(e_6) \). The next proposition [Frib] shows that the functions \( \sigma \) and \( \ell \) provide a parameterization of isometry classes of exceptional hexagons.

**Proposition 2.4.** Let \( F \) and \( F' \) be paired exceptional lateral hexagons. Their pairing can be realized by an isometry if and only if \( \sigma^\theta(F) + \sigma^\theta(F') = 0 \) and \( \ell^\theta(F) = \ell^\theta(F') \).

Recall now that we are considering the geometric realization of the triangulation \( \mathcal{T} \) of \( M \) parameterized by the dihedral angles \( \alpha^i_j, \gamma^i_j, \beta, i = 1, \ldots, 2k, j = 1, 2, 3 \). Under the assumption that equations (11) are in force (i.e., that the boundary edges of the compact faces of \( \mathcal{T} \) have all the same length), Proposition 2.4 and equation (5) imply that the
Theorem 2.5. We have

matching exceptional hexagons can be glued by isometries if and only if for \( i = 1, \ldots, k \) we have:

\[
\sin \alpha_i^1 \sin \alpha_{i-1}^3 \sin \gamma_{i-1}^1 \sin \gamma_i^1 = \sin \alpha_{2i-1}^3 \sin \alpha_{2i-1}^2 \sin \gamma_{2i-1}^3 \sin \gamma_{2i-1}^1.
\]

2.5. Consistency around the edges. Since \( \gamma_i^1 + \gamma_i^2 + \gamma_i^3 = \pi \) for \( i = 1, \ldots, 2k \), the total angle along any half-infinite edge of \( \mathcal{T} \) is automatically forced to be equal to \( 2\pi \), so consistency around the edges translate into the following equation only:

\[
6 \cdot (g - k) \cdot \beta + \sum_{i=1}^{2k} (\alpha_i^1 + \alpha_i^2 + \alpha_i^3) = 2\pi.
\]

Any solution of consistency equations (4), (5), (7) defines a non-singular hyperbolic structure with geodesic boundary on \( M \).

2.6. Completeness equations. For \( i = 1, \ldots, k \) let now \( \mu_i, \lambda_i \) be the basis of \( H_1(T_i; \mathbb{Z}) \) which is defined as follows: \( \mu_i \) is the projection on \( T_i \) of the edge in the link of the ideal vertex of \( \Delta_{2i-1} \) that joins \( f_{2i-1}^1 \) to \( f_{2i-1}^2 \); \( \lambda_i \) is the projection on \( T_i \) of the edge in the link of the ideal vertex of \( \Delta_{2i} \) that joins \( f_{2i}^3 \) to \( f_{2i}^2 \). A solution

\[
x = (\alpha_1^1, \alpha_1^2, \alpha_1^3, \gamma_1^1, \gamma_1^2, \gamma_1^3, \ldots, \alpha_{2k}^1, \alpha_{2k}^2, \alpha_{2k}^3, \gamma_{2k}^1, \gamma_{2k}^2, \gamma_{2k}^3, \beta) \in \mathbb{R}^{12k+1}
\]

of consistency equations naturally defines an \( \text{Aff}(\mathbb{C}) \)-structure on \( T_i \) (see e. g. [BP92, Fri03]). We denote by \( a_i(x) \in \mathbb{C} \) (resp. by \( b_i(x) \in \mathbb{C} \)) the dilation component of the holonomy of \( \mu_i \) (resp. of \( \lambda_i \)) corresponding to the \( \text{Aff}(\mathbb{C}) \)-structure defined by \( x \) on \( T_i \). It is well-known that the hyperbolic structure defined by \( x \) on \( M \) induces a complete metric on the \( i \)-th end of \( M \) if and only if \( a_i(x) = b_i(x) = 1 \). Moreover, one can explicitly compute \( a_i \) and \( b_i \) in terms of the dihedral angles:

**Theorem 2.5.** We have

\[
a_i(x) = \frac{(\sin \gamma_{2i-1}^1 \sin \gamma_{2i}^2)}{(\sin \gamma_{2i-1}^2 \sin \gamma_{2i}^1)} \exp(i(\gamma_{2i-1}^3 - \gamma_{2i}^3)),
b_i(x) = \frac{(\sin \gamma_{2i-1}^2 \sin \gamma_{2i}^3)}{(\sin \gamma_{2i-1}^3 \sin \gamma_{2i}^2)} \exp(i(\gamma_{2i-1}^1 - \gamma_{2i}^1)).
\]
2.7. The complete solution. The following theorem [FMP03] shows that a solution of consistency and completeness equations always exists, and is as symmetric as possible.

**Theorem 2.6.** There exist constants \( \alpha_{g,k}, \beta_{g,k} \in (0, \pi/3) \) such that the point
\[
x_0 = (\alpha_{g,k}, \alpha_{g,k}, \alpha_{g,k}, \pi/3, \pi/3, \pi/3, \ldots, \alpha_{g,k}, \alpha_{g,k}, \alpha_{g,k}, \pi/3, \pi/3, \pi/3, \beta_{g,k}) \in \mathbb{R}^{12k+1}
\]
provides the unique solution of consistency and completeness equations for \( \mathcal{T} \).

Thus the complete hyperbolic structure of \( M \) induces on each boundary torus the regular hexagonal Euclidean structure which is obtained by gluing two Euclidean equilateral triangles.

**Lemma 2.7.** We have \( \alpha_{g,k} < \beta_{g,k} < 2\alpha_{g,k} \leq \pi/3 \).

**Proof:** From equation (1) we easily get \( \cos \beta_{g,k} = (2 + \cos 2\alpha_{g,k})/3 > \cos 2\alpha_{g,k} \), whence \( \beta_{g,k} < 2\alpha_{g,k} \). Moreover, since \( \alpha_{g,k} < \pi/3 \) we have \( 2\cos^2 \alpha_{g,k} - 3\cos \alpha_{g,k} + 1 > 0 \), whence \( \cos \beta_{g,k} = (2\cos^2 \alpha_{g,k} + 1)/3 < \cos \alpha_{g,k} \) and \( \alpha_{g,k} < \beta_{g,k} \). This inequality also implies \( 2\pi = 6(g-k) \cdot \beta_{g,k} + 6k \cdot \alpha_{g,k} \geq 6g \cdot \alpha_{g,k} \), whence \( \alpha_{g,k} \leq \pi/6 \).

**Lemma 2.8.** Let \( \Delta^* \subset \mathbb{H}^3 \) be a non-compact geometric tetrahedron of the realization of \( \mathcal{T} \) parameterized by \( x_0 \) and let \( v \) be the ideal vertex of \( \Delta^* \). Then the horosphere centred at \( v \) and tangent to the truncation triangles of \( \Delta^* \) does not intersect the lateral hexagon opposite to \( v \).

**Proof:** For \( \epsilon > 0 \) let \( \Delta^*_\epsilon \) be the small deformation of \( \Delta^* \) having angle \( \pi/3 - \epsilon \) along the edges emanating from \( v \), and \( \alpha_{g,k} \) along the other internal edges. Let \( l_\epsilon \) be the length of an internal edge emanating from \( v \) and \( d_\epsilon \) be the distance between the truncation triangle corresponding to \( v \) and the opposite lateral hexagon. By [FPT04 Proposition 5.6], Lemma 2.3 and Lemma 2.7 we have
\[
\lim_{\epsilon \to 0} (\cosh d_\epsilon / \cosh l_\epsilon) = \sqrt{4\cos^2 \alpha_{g,k} - 1} > 1,
\]
whence the conclusion.

2.8. Smoothness at the complete structure. From now on we denote by \( \Omega_{g,k} \subset \mathbb{R}^{12k+1} \) the set of solutions of consistency equations for \( \mathcal{T} \) (it is clear that this set indeed depends only on \( g \) and \( k \), and not on \( \mathcal{T} \)). If \( x \in \Omega_{g,k} \) is as in equation (3), we set
\[
\begin{align*}
u_i(x) &= \ln a_i(x) &= \ln((\sin \gamma_{2i-1} \sin \gamma_{2i})/(\sin \gamma_{2i-1} \sin \gamma_{2i}^3)) + i(\gamma_{2i-1}^3 - \gamma_{2i}^3), \\
v_i(x) &= \ln b_i(x) &= \ln((\sin \gamma_{2i-1} \sin \gamma_{2i}^3)/(\sin \gamma_{2i-1} \sin \gamma_{2i}^3)) + i(\gamma_{2i-1} - \gamma_{2i}^3).
\end{align*}
\]
The following theorem is proved in [Frib] (see also [NZSS5]).

**Theorem 2.9.** Near \( x_0 \), the space \( \Omega_{g,k} \subset \mathbb{R}^{12k+1} \) is a smooth manifold of real dimension \( 2k \), whose tangent space \( T_{x_0} \Omega_{g,k} \) at \( x_0 \) is given by the solutions of the linearization of consistency equations (7), (8), (9). Moreover, there exists a small neighbourhood \( U \) of \( x_0 \) in \( \Omega_{g,k} \) with the following properties:

1. For \( x \in U \), we have \( u_i(x) = 0 \iff v_i(x) = 0 \iff \) the metric structure defined by \( x \) is complete at the \( i \)-th end of \( M \);
2. The map \((u_1, \ldots, u_k) : U \to \mathbb{C}^k\) is a diffeomorphism between \( U \) and an open neighbourhood of \( 0 \) in \( \mathbb{C}^k \).
Due to our choice of $\mu_i, \lambda_i$ we also have the following:

**Lemma 2.10.** If $j \in \{1, \ldots, k\}$ and $\{y_n\}_{n \in \mathbb{N}} \subset \Omega_{g,k}$ is a sequence with $\lim_{n \to \infty} y_n = x_0$ and $u_j(y_n) \neq 0$ for every $n \in \mathbb{N}$, then $\lim_{n \to \infty} v_j(y_n)/u_j(y_n) = -1/2 + i\sqrt{3}/2$.

**Proof:** See [NZ85, Fri05].

### 2.9. Dehn filling equations

Let $U$ be a sufficiently small neighbourhood of $x_0$ in $\Omega_{g,k}$ and let $x \in U$. For $j = 1, \ldots, k$, we define the $j$-Dehn filling coefficient $(p_j(x), q_j(x)) \in \mathbb{R}^2 \cup \{-\infty\}$ as follows: if $u_j(x) = 0$, then $(p_j(x), q_j(x)) = \infty$; otherwise, $p_j(x), q_j(x)$ are the unique real solutions of the equation

$$p_j(x)u_j(x) + q_j(x)v_j(x) = 2\pi i.$$ 

(Existence and uniqueness of such solutions near $x_0$ can be easily deduced from Theorem 2.9 and Lemma 2.10.) Let us set

$$d = (d_1, \ldots, d_k): U \to \prod_{i=1}^k S^2, \quad d_j(x) = (p_j(x), q_j(x)) \in S^2 = \mathbb{R}^2 \cup \{-\infty\}.$$ 

As a consequence of Theorem 2.9 and Lemma 2.10 we have the following:

**Theorem 2.11.** If $U$ is small enough, the map $d$ defines a diffeomorphism onto an open neighbourhood of $(\infty, \ldots, \infty)$ in $S^2 \times \cdots \times S^2$.

For $x \in \Omega_{g,k}$ we denote by $M(x)$ the hyperbolic structure induced on $M$ by $x$, and by $\hat{M}(x)$ the metric completion of $M(x)$. We also set

$$\Omega_{g,k} = \{ x \in U \subset \Omega_{g,k} : \text{for } i = 1, \ldots, k \text{ the } i \text{-th Dehn filling coefficient associated to } x \text{ is equal either to } \infty \text{ or to a pair of coprime integers} \}.$$ 

**Theorem 2.12.** If $U$ is sufficiently small and $x$ belongs to $\Omega_{g,k} \cap U$, then $\hat{M}(x)$ admits a complete finite-volume smooth hyperbolic structure which is obtained by adding to $M(x)$ a closed geodesic at any cusp with non-infinite Dehn filling coefficient. From a topological point of view, $\hat{M}(x)$ is obtained by filling the $i$-th cusp of $M$ along the slope $p_i(x)\mu_i + q_i(x)\lambda_i$ if $(p_i(x), q_i(x)) \neq \infty$, and by leaving the $i$-th cusp of $M$ unfilled if $(p_i(x), q_i(x)) = \infty$, $i = 1, \ldots, k$.

**Proof:** See e. g. [Thu79, NZ85, BP92, Fri05].

**Proposition 2.13.** Let $U$ be sufficiently small, take $x \in \Omega_{g,k} \cap U$ and suppose $(p_j(x), q_j(x)) \neq \infty$. Let $\ell_j \subset \hat{M}(x)$ be the added geodesic at the $j$-th cusp of $M$ and let $CL^x(\ell_j)$ be its complex length. Choose integers $r_j(x), s_j(x)$ with $p_j(x)s_j(x) - q_j(x)r_j(x) = -1$. Then we have

$$CL^x(\ell_j) = r_j(x)u_j(x) + s_j(x)v_j(x).$$ 

**Proof:** See [NZ85].

### 3. Isolation of cusps

We now study small deformations of the complete hyperbolic structure of $M$ by analyzing deformations of the shapes of the geometric tetrahedra of $\mathcal{T}$. For $x \in \Omega_{g,k}$ let $\ell(x) \in \mathbb{R}$ be the length of any boundary edge of any compact lateral hexagon in the geometric realization of $\mathcal{T}$ parameterized by $x$. The following results are proved in [Fri13, Section 3].
Lemma 3.1. The map $\ell : \Omega_{g,k} \to \mathbb{R}$ is smooth, and $d\ell_{x_0} = 0$.

Lemma 3.2. For $x \in \Omega_{g,k}$ let $\beta(x) = x^{12k+1}$ be the dihedral angle along the edges of any compact tetrahedron in the geometric realization of $T$ parameterized by $x$. Then $d\beta_{x_0} = 0$.

3.1. Infinitesimal deformations. We begin by looking for explicit equations for the tangent space $T_{x_0}\Omega_{g,k}$. So fix a smooth arc $\varphi : (-\varepsilon, \varepsilon) \to \Omega_{g,k}$ and for any smooth $f : \Omega_{g,k} \to \mathbb{R}$ let us denote by $\tilde{f}$ the derivative of $f \circ \varphi$ at $t = 0$. With notation as in Subsection 2.2 if $\theta(t)$ is the geometric realization of $T$ parameterized by $\varphi(t) \in \Omega_{g,k}$, we set $\alpha_i^j(t) = \theta(t)(\alpha_i^j)$, $\gamma_i^j(t) = \theta(t)(\gamma_i^j)$.

Recall that for every $l = 1, \ldots, 2k; j = 1, 2, 3$ we have

$$\ell(\varphi(t)) = \frac{\cos \alpha_i^{j+1}(t) \cdot \cos \alpha_i^{j+2}(t) + \cos \gamma_i^j(t)}{\sin \alpha_i^{j+1}(t) \cdot \sin \alpha_i^{j+2}(t)},$$

where the primes are considered mod 3. Moreover, by Lemma 3.1 we have $\dot{\ell} = 0$, so differentiating at $0$ the equation above we easily get

$$\sqrt{3}(\dot{\alpha}_i^j + \dot{\alpha}_i^{j+2}) \cos \alpha_{g,k} + \dot{\gamma}_i^j \sin \alpha_{g,k} = 0.$$

Summing up these equations for $j = 1, 2, 3$ and observing that $\dot{\gamma}_1^1 + \dot{\gamma}_1^2 + \dot{\gamma}_1^3 = 0$ we obtain

$$(10) \quad \dot{\alpha}_1^1 + \dot{\alpha}_1^2 + \dot{\alpha}_1^3 = 0,$$

whence

$$(11) \quad \sqrt{3} \dot{\alpha}_i^j \cos \alpha_{g,k} = \dot{\gamma}_i^j \sin \alpha_{g,k}.$$

Let $i \in \{1, \ldots, k\}$. Evaluating equations (8) along $\varphi$ and differentiating at $0$ we get

$$\sqrt{3}(\dot{\alpha}_2^1 + \dot{\alpha}_2^2) \cos \alpha_{g,k} + (\dot{\gamma}_2^1 + \dot{\gamma}_2^2) \sin \alpha_{g,k} = \sqrt{3}(\dot{\alpha}_2^1 + \dot{\alpha}_2^2) \cos \alpha_{g,k} + (\dot{\gamma}_2^1 + \dot{\gamma}_2^2) \sin \alpha_{g,k}$$

$$= \sqrt{3}(\dot{\alpha}_2^3 + \dot{\alpha}_2^3) \cos \alpha_{g,k} + (\dot{\gamma}_2^3 + \dot{\gamma}_2^3) \sin \alpha_{g,k}.$$

Together with equations (10) and (11), this implies

$$(12) \quad \dot{\alpha}_2^1 = -\dot{\alpha}_2^1, \quad \dot{\alpha}_2^2 = -\dot{\alpha}_2^2, \quad \dot{\alpha}_2^3 = -\dot{\alpha}_2^3.$$

We can now summarize all these computations giving explicit equations for $T_{x_0}\Omega_{g,k}$. Let $Z$ be the linear subspace of $\mathbb{R}^{12}$ defined by the following equations:

$$\begin{align*}
(\sqrt{3} \cos \alpha_{g,k})x_1 &= (\sin \alpha_{g,k})x_4 \\
(\sqrt{3} \cos \alpha_{g,k})x_2 &= (\sin \alpha_{g,k})x_5 \\
(\sqrt{3} \cos \alpha_{g,k})x_3 &= (\sin \alpha_{g,k})x_6 \\
x_1 + x_7 = x_2 + x_8 = x_3 + x_9 = x_4 + x_{10} = x_5 + x_{11} = x_6 + x_{12} = 0 \\
x_1 + x_2 + x_3 = 0.
\end{align*}$$

Observe that $\dim_{\mathbb{R}} Z = 2$. For $i = 1, \ldots, k$ let $r_i : \mathbb{R}^{12k+1} \to \mathbb{R}^{12}$ be the map defined by $r_i(x) = (x_{12i-11}, x_{12i-10}, \ldots, x_{12i-1}, x_{12i})$. Let

$$\mathcal{Z} = \{x \in \mathbb{R}^{12k+1} : x_{12k+1} = 0, r_i(x) \in Z \text{ for } i = 1, \ldots, k\}$$

be the product of one copy of $Z$ for each cusp (so $\dim_{\mathbb{R}} Z = 2k$).

Proposition 3.3. We have $T_{x_0}\Omega_{g,k} = \mathcal{Z}$.

Proof: Lemma 3.2 and equations (10), (11), (12) imply that $T_{x_0}\Omega_{g,k} \subseteq \mathcal{Z}$. But $\dim_{\mathbb{R}} \mathcal{Z} = 2k = \dim_{\mathbb{R}} T_{x_0}\Omega_{g,k}$, whence the conclusion.
3.2. Isolation of cusps. We now go into the proof of Theorem 1.5. So let \( C_1, \ldots, C_k \) be the cusps of our fixed manifold \( M \in \mathcal{M}_{g,k} \) corresponding to the boundary tori \( T_1, \ldots, T_k \). We look for equations defining the set of structures in \( \Omega_{g,k} \) which are complete at \( C_1, \ldots, C_k \). To this aim we set:

\[
J_h = \{ x \in \mathbb{R}^{12k+1} : x_{12i+1} = x_{12i+2} = x_{12i+3} \text{ for all } i = 0, \ldots, h-1 \}.
\]

**Lemma 3.4.** Near \( x_0 \), the set \( J_h \cap \Omega_{g,k} \) is a smooth submanifold of \( \Omega_{g,k} \) of real dimension \( 2(k-h) \).

**Proof:** It is easily seen that \( T_{x_0}J_h + T_{x_0}\Omega_{g,k} = J_h + \mathbb{R} = \mathbb{R}^{12k+1} = T_{x_0}\mathbb{R}^{12k+1} \), so the conclusion follows from basic results about transverse intersections of submanifolds. □

Let \( \Delta \) be a topological partially truncated tetrahedron with ideal vertex \( v_0 \), and take \( \vartheta \in (0, \pi/3) \). Then there exists, up to isometry, exactly one geometric realization of \( \Delta \) with dihedral angles \( \pi/3 \) along the internal edges emanating from \( v_0 \), and angle \( \vartheta \) along the other internal edges. We denote this geometric tetrahedron by \( \Delta^\vartheta \).

**Proposition 3.5.** For each \( l = 1, \ldots, 2h \) let \( \Delta^\vartheta_l(p) \) be the geometric realization of \( \Delta^\vartheta \) parameterized by \( p \in J_h \cap \Omega_{g,k} \). Then a real number \( \vartheta(p) \in (0, \pi/3) \) exists such that \( \Delta^\vartheta_l(p) \) is isometric to \( \Delta^\vartheta(p) \) for \( l = 1, \ldots, 2h \).

**Proof:** For \( l = 1, \ldots, 2h \), \( j = 1, 2, 3 \) let \( T^j_l(p) \) be the truncation triangle of \( \Delta^\vartheta_l(p) \) having a vertex on the edge \( f^j_l \). Fix \( i \in \{ 0, \ldots, h-1 \} \) and consider the tetrahedron \( \Delta_{2i+1}^\vartheta(p) \). The compact face of such tetrahedron is a regular right-angled hexagon, so condition \( x_{12i+3}(p) = x_{12i+2}(p) = x_{12i+1}(p) \) implies that \( T^1_{2i+1}(p), T^2_{2i+1}(p) \) and \( T^3_{2i+1}(p) \) are isometric to each other. This gives \( x_{12i+4}(p) = x_{12i+5}(p) = x_{12i+6}(p) \), so \( \Delta^\vartheta_{2i+1} \) is isometric to \( \Delta^\vartheta_{2i+1} \) for some \( \xi_{2i+1} = x_{12i+1} \). Moreover, since the non-compact faces of \( \Delta^\vartheta_{2i+1} \) are isometrically glued to the non-compact faces of \( \Delta^\vartheta_{2i+2} \) we easily see that the truncation triangles \( T^1_{2i+2}(p), T^2_{2i+2}(p) \) and \( T^3_{2i+2}(p) \) are isoceles and isometric to each other. This forces \( x_{12i+7}(p) = x_{12i+8}(p) = x_{12i+9}(p) = \xi_{2i+2} \) and \( x_{12i+10}(p) = x_{12i+11}(p) = x_{12i+12}(p) = \pi/3 \), so \( \Delta^\vartheta_{2i+2} \) is isometric to \( \Delta^\vartheta_{2i+2} \) for some \( \xi_{2i+2} \in (0, \pi/3) \).

Finally, since the length of the compact internal edges of the \( \Delta^\vartheta_l \)'s does not depend on \( l \), we have \( \xi_1 = \ldots = \xi_{2h} \), whence the conclusion. □

**Corollary 3.6.** Let \( p \) be a point in \( J_h \cap \Omega_{g,k} \) and denote by \( M(p) \) the hyperbolic structure defined by \( p \) on \( M \). Then for all \( i = 1, \ldots, h \) the following holds:

- \( M(p) \) induces a complete metric on the cusp \( C_i \);
- The Euclidean structure induced on \( T_i \) by \( M(p) \) is isometric to the regular hexagonal structure induced on \( T_i \) by the complete hyperbolic structure \( M(x_0) \).

The corollary just stated says that the Euclidean structures on \( T_1, \ldots, T_h \) are not affected by the deformations of the hyperbolic metric on \( M \) which correspond to points in \( J_h \cap \Omega_{g,k} \). Therefore to conclude the proof of Theorem 1.5 we only need the following:

**Proposition 3.7.** Let \( K_h \) be the subset of \( \Omega_{g,k} \) corresponding to the structures inducing complete metrics on \( C_1, \ldots, C_h \). Then there exists a neighbourhood \( V \) of \( x_0 \) in \( \Omega_{g,k} \) with \( K_h \cap V = J_h \cap V \).

**Proof:** By Theorem 2.9 and Lemma 3.4 there exists a neighbourhood \( W \) of \( x_0 \) in \( \Omega_{g,k} \) such that both \( K_h \cap W \) and \( J_h \cap W \) are smooth submanifolds of \( \Omega_{g,k} \) of real dimension \( 2(k-h) \). Moreover Corollary 3.6 shows that \( J_h \cap W \subset K_h \cap W \), whence \( J_h \cap V = K_h \cap V \) for some (maybe smaller) neighbourhood of \( x_0 \) in \( \Omega_{g,k} \). □
Remark 3.8. Theorem 1.3 shows that Dehn fillings along sufficiently complicated slopes on some boundary tori of \( M \) do not affect the Euclidean structure on the non-filled boundary tori. Using SnapPea we have checked in a number of cases that the same isolation phenomenon still holds when filling along short non-exceptional slopes. It is conjectured in [HK98] that the space of Dehn filling deformations of complete finite-volume hyperbolic 3-manifolds is connected and smooth. If this were true, then the Euclidean structure on the non-filled tori would remain unchanged under all the partial hyperbolic Dehn fillings of \( M \).

4. Non-isolation of the boundary

Let \( \text{Teich}(\partial M) \) be the Teichmüller space of hyperbolic structures on \( \partial M \), i.e. the space of equivalence classes of hyperbolic metrics on \( \partial M \), where two such metrics are considered equivalent if they are isometric through a diffeomorphism homotopic to the identity of \( \partial M \). Since \( \partial M \) is compact, it is well-known that for any \( \gamma \in \pi_1(\partial M) \) and any metric \( h \in \text{Teich}(\partial M) \) there exists a unique closed \( h \)-geodesic in the free homotopy class of \( \gamma \). We denote the \( h \)-length of this geodesic by \( L_\gamma(h) \). An easy computation shows that if \( \bar{\rho}_h : \pi_1(\partial M) \to \text{PSL}(2, \mathbb{R}) \cong \text{Isom}^+(\mathbb{H}^2) \) is a holonomy representation for the hyperbolic structure \( h \) then we have

\[
\text{tr} \rho_h(\gamma) = \pm 2 \cosh(L_\gamma(h)/2), \quad \gamma \in \pi_1(\partial M),
\]

where \( \rho_h(\gamma) \) is a lift in \( \text{SL}(2, \mathbb{R}) \) of \( \bar{\rho}_h(\gamma) \in \text{PSL}(2, \mathbb{R}) \).

For \( x \in \Omega_{g,k} \) we denote by \( M(x) \) the hyperbolic structure defined on \( M \) by \( x \), and by \( B(x) \in \text{Teich}(\partial M) \) the equivalence class of the hyperbolic structure induced by \( M(x) \) on \( \partial M \). It is well-known that \( \text{Teich}(\partial M) \) admits a structure of differentiable manifold such that:

- \( \text{Teich}(\partial M) \) is diffeomorphic to the Euclidean space \( \mathbb{R}^{6g-6} \);
- For any \( \gamma \in \pi_1(M) \) the map \( L_\gamma : \text{Teich}(\partial M) \to \mathbb{R} \) defined above is smooth;
- The map \( B : \Omega_{g,k} \to \text{Teich}(\partial M) \) is smooth.

The following proposition is proved in [Frib].

Proposition 4.1. We have \( dB_{x_0} = 0 \).

Thus in order to understand how deformations of the complete structure affect the geodesic boundary we need to analyze the map \( B \) up to the second order in a neighbourhood of \( x_0 \). We begin with the following:

Definition 4.2. Let \( y_0 \) be a point of a smooth \( n \)-manifold \( Y \) and let \( \varphi : U \to \mathbb{R}^n \) be a diffeomorphism with \( \varphi(y_0) = 0 \), where \( U \subset Y \) is a small open neighbourhood of \( y_0 \). Let \( 0 \neq v \in T_{y_0}Y \), and consider a sequence \( \{y_j\}_{j \in \mathbb{N}} \subset U \setminus \{y_0\} \). We say that \( y_n \) converges to \( y_0 \) along \( v \) if

\[
\lim_{j \to \infty} y_j = y_0, \quad \lim_{j \to \infty} \varphi(y_j)/||\varphi(y_j)|| = d\varphi_{y_0}(v)/||d\varphi_{y_0}(v)||,
\]

where we are identifying \( T_0\mathbb{R}^n \) with \( \mathbb{R}^n \), endowed with the Euclidean norm \( || \cdot || \).

4.1. An alternative formulation. First of all we show how Theorem 1.6 can be deduced from the following:

Theorem 4.3. There exist a smooth path \( \varphi : (-\varepsilon, \varepsilon) \to \Omega_{g,k} \) and an element \( \varphi \in \pi_1(\partial M) \) such that \( \varphi(0) = x_0 \) and the map \( t \mapsto L^{B(\varphi(t))}(\varphi) \) has non-zero second derivative at 0.
4.2. Proving Theorem 4.3 Let \( \gamma : (-\varepsilon, \varepsilon) \to \Omega_{g,k} \) be a fixed smooth path with \( \gamma(0) = x_0 \) and for any smooth function \( f : \Omega_{g,k} \to \mathbb{R} \) let \( \dot{f}(t) \) (resp. \( \ddot{f}(t) \)) denote the first (resp. second) derivative of \( f \circ \gamma \) in \( t \). We will denote by \( \ddot{f} \) (resp. \( \dddot{f} \)) the value of \( \ddot{f}(0) \) (resp. \( \dddot{f}(0) \)). We recall that \( \dot{\gamma} \in MCG(\partial M) \), and for any smooth function \( f \) recall that \( \dot{f}(0) \) (resp. \( \dddot{f}(0) \)) gives a \( \partial K \) of \( \partial M \).

We have \( \dddot{f}(0) = \frac{\sin x}{\sin x} \) and evaluating at 0 we get

\[
(2 \sin^4 \alpha_{g,k}) \dddot{f} = (-3 \cos \alpha_{g,k} \sin \alpha_{g,k})(\dddot{x}_1 + \dddot{x}_2) - \sqrt{3} \sin^2 \alpha_{g,k} \dddot{x}_6 + (5 \cos^2 \alpha_{g,k} + 1) (\dddot{x}_1^2 + \dddot{x}_2^2) - \sin^2 \alpha_{g,k} \dddot{x}_6 + (2 \cos^2 \alpha_{g,k} + 4) \dddot{x}_1 \dddot{x}_2 + 2 \sqrt{3} \cos \alpha_{g,k} \sin \alpha_{g,k} (\dddot{x}_1 \dddot{x}_6 + \dddot{x}_2 \dddot{x}_6).
\]

Observe now that on \( \Omega_{g,k} \) we also have

\[
\dddot{f}(0) = \frac{\sin x \cos x + \cos x}{\sin x \sin x} (\dddot{x}_1 \sin x_3 - \cos x_3) / (\sin x_1 \sin x_3) = (\cos x_2 \cos x_3 + \cos x_4) / (\sin x_2 \sin x_3).
\]

Differentiating two times these equalities along \( \gamma \), evaluating at 0 as above and summing up with equality \([15]\) we get

\[
(6 \sin^4 \alpha_{g,k}) \dddot{f} = -6 \cos \alpha_{g,k} \sin \alpha_{g,k} \dddot{x}_1 + \dddot{x}_2 + \dddot{x}_3 - \sqrt{3} \sin^2 \alpha_{g,k} (\dddot{x}_4 + \dddot{x}_5 + \dddot{x}_6) + (10 \cos^2 \alpha_{g,k} + 2)(\dddot{x}_1^2 + \dddot{x}_2^2 + \dddot{x}_3^2) - \sin^2 \alpha_{g,k} (\dddot{x}_4^2 + \dddot{x}_5^2 + \dddot{x}_6^2) + 2 \sqrt{3} \cos \alpha_{g,k} \sin \alpha_{g,k} (\dddot{x}_1 \dddot{x}_5 + \dddot{x}_1 \dddot{x}_6 + \dddot{x}_2 \dddot{x}_4 + \dddot{x}_2 \dddot{x}_6 + \dddot{x}_3 \dddot{x}_4 + \dddot{x}_3 \dddot{x}_5).
\]

Since \( \dddot{x} \) belongs to \( T_{x_0} \Omega_{g,k} \) we have \( (\sin \alpha_{g,k}) \dddot{x}_{l+i} = (\sqrt{3} \cos \alpha_{g,k}) \dddot{x}_i \) for \( i = 1, 2, 3 \) and \( \dddot{x}_1 + \dddot{x}_2 + \dddot{x}_3 = 0 \). Substituting these relations in \([16]\) we (rather strikingly) get \( \dddot{f} = -(\cos \alpha_{g,k} / \sin^3 \alpha_{g,k})(\dddot{x}_1 + \dddot{x}_2 + \dddot{x}_3) \). By the very same argument it follows that

\[
(17) \quad \dddot{f} = -(\cos \alpha_{g,k} / \sin^3 \alpha_{g,k})(\dddot{x}_{6l+1} + \dddot{x}_{6l+2} + \dddot{x}_{6l+3}), \quad l = 0, \ldots, 2k - 1.
\]

Thus the condition forcing the dihedral angle along the compact edge of \( \mathcal{T} \) to be constant is \( 2\pi \) given by \( -2k \sin^3 \alpha_{g,k} / \cos \alpha_{g,k} + 6(g-k) \dddot{x}_{12k+1} = 0 \), and implies that \( \dddot{f} \) has the same sign as \( \dddot{x}_{12k+1} \). On the other hand, condition \( \ell(x) = \cos x_{12k+1} / (1 - \cos x_{12k+1}) \) implies \( \dddot{f} = -(\sin \beta_{g,k} / (1 - \cos \beta_{g,k})^2) \dddot{x}_{12k+1} \), so \( \dddot{f} \) and \( \dddot{x}_{12k+1} \) should have opposite signs. This forces \( \dddot{f} = \dddot{x}_{12k+1} = 0 \), whence the conclusion by equation \([17]\). \( \square \)
4.3. The chosen curve. By Proposition 3.5 the subspace of $\mathbb{R}^{12k+1}$ having equations 
$x \in \mathbb{R}^{12k+1} : \ x_2 = x_3, \ x_{12i+1} = x_{12i+2} = x_{12i+3}, \ i = 1, \ldots, k - 1$ \} intersects $\Omega_{g,k}$ \transversely near $x_0$ in the support of a smooth curve $\zeta: (-\epsilon, \epsilon) \to \Omega_{g,k}$ \with \( \zeta(0) = x_0 \) and 
(18) 
\[
\ddot{x}(0) = (2\sin \alpha_{g,k}, -\sin \alpha_{g,k}, -\sin \alpha_{g,k}, 2\sqrt{3} \cos \alpha_{g,k}, -\sqrt{3} \cos \alpha_{g,k}, -\sqrt{3} \cos \alpha_{g,k}, \\
-2\sin \alpha_{g,k}, \sin \alpha_{g,k}, \sin \alpha_{g,k}, -2\sqrt{3} \cos \alpha_{g,k}, \sqrt{3} \cos \alpha_{g,k}, \sqrt{3} \cos \alpha_{g,k}, \sqrt{3} \cos \alpha_{g,k}, \\
0, \ldots, 0).
\]
As before, for any smooth $f : \Omega_{g,k} \to \mathbb{R}$ we set $f(t) := f(\zeta(t))$ and we denote by $\dot{f}(t)$ (resp. $\ddot{f}(t)$) the first (resp. second) derivative of $f \circ \zeta$ in $t$. We also denote simply by $\dot{f}$, $\ddot{f}$ the values $\dot{f}(0)$, $\ddot{f}(0)$ respectively.

If $\psi : (-\epsilon', \epsilon') \to (-\epsilon, \epsilon)$ is a local diffeomorphism with $\dot{\psi}(0) = 1$, then $(\zeta \circ \psi)(0) = \zeta(0) + \ddot{\psi}(0) \cdot \zeta(0)$. Thus, up to reparameterizing $\zeta$ without changing its tangent vector at 0 we can assume that the following condition holds:
(19) 
\[
\ddot{x}_1 = \ddot{x}_7.
\]
A very similar argument to the proof of Proposition 3.5 gives the following:

Proposition 4.5. For any $t \in (-\epsilon, \epsilon)$ we have 
\[
x_2(t) = x_3(t), \quad x_5(t) = x_6(t), \quad x_8(t) = x_9(t), \quad x_{11}(t) = x_{12}(t),
\]
and 
\[
x_{12i+1}(t) = x_{12i+2}(t) = x_{12i+3}(t), \quad x_{12i+4}(t) = x_{12i+5}(t) = x_{12i+6}(t) = \pi/3, \\
x_{12i+7}(t) = x_{12i+8}(t) = x_{12i+9}(t), \quad x_{12i+10}(t) = x_{12i+11}(t) = x_{12i+12}(t) = \pi/3
\]
for $i = 1, \ldots, k - 1$.

We are now ready to prove the following:

Proposition 4.6. We have 
\[
\ddot{x}(0) = (8 \cos \alpha_{g,k} \sin \alpha_{g,k} - 4 \cos \alpha_{g,k} \sin \alpha_{g,k}, -4 \cos \alpha_{g,k} \sin \alpha_{g,k}, 2\sqrt{3}, -\sqrt{3}, -\sqrt{3}, \\
8 \cos \alpha_{g,k} \sin \alpha_{g,k}, -4 \cos \alpha_{g,k} \sin \alpha_{g,k}, -4 \cos \alpha_{g,k} \sin \alpha_{g,k}, 2\sqrt{3}, \sqrt{3}, -\sqrt{3}, \\
0, \ldots, 0).
\]

Proof: Since $x_4(t) + x_5(t) + x_6(t) = x_{10}(t) + x_{11}(t) + x_{12}(t) = \pi$, by Proposition 4.5 we have $\ddot{x}_5 = \ddot{x}_6 = -(1/2)\ddot{x}_4$ and $\ddot{x}_{11} = \ddot{x}_{12} = -(1/2)\ddot{x}_{10}$, while by Lemma 4.4 and Proposition 4.5 we get $\ddot{x}_2 = \ddot{x}_3 = -(1/2)\ddot{x}_1$ and $\ddot{x}_8 = \ddot{x}_9 = -(1/2)\ddot{x}_7$. Substituting these relations in equation 15 and recalling that $\dot{\ell} = 0$ we have after some computations 
(20) 
\[
\sqrt{3} \cos \alpha_{g,k} \dddot{x}_1 - \sin \alpha_{g,k} \dddot{x}_4 = 2\sqrt{3} \sin \alpha_{g,k}(4 \cos^2 \alpha_{g,k} - 1).
\]

The very same argument also gives 
(21) 
\[
\sqrt{3} \cos \alpha_{g,k} \dddot{x}_7 - \sin \alpha_{g,k} \dddot{x}_{10} = 2\sqrt{3} \sin \alpha_{g,k}(4 \cos^2 \alpha_{g,k} - 1).
\]

In the same way, differentiating two times the equality 
\[
\sin x_1(t) \sin x_4(t) \sin x_7(t) \sin x_{10}(t) = \sin x_2(t) \sin x_5(t) \sin x_8(t) \sin x_{11}(t),
\]
evaluating at 0 and substituting in the result the relations $\ddot{x}_2 = -(1/2)\ddot{x}_1$, $\ddot{x}_5 = -(1/2)\ddot{x}_4$, $\ddot{x}_8 = -(1/2)\ddot{x}_7$ and $\ddot{x}_{11} = -(1/2)\ddot{x}_{10}$ we get 
(22) 
\[
\sqrt{3} \cos \alpha_{g,k}(\dddot{x}_1 + \dddot{x}_7) + \sin \alpha_{g,k}(\dddot{x}_4 + \dddot{x}_{10}) = 4\sqrt{3} \sin \alpha_{g,k}(4 \cos^2 \alpha_{g,k} + 1).
\]
Solving equations 19, 20, 21 and 22 we get the desired result for $\dddot{x}_1, \ldots, \dddot{x}_{12}$. 
Let now \( i = 1, \ldots, k - 1 \). By Proposition 4.6 and Lemma 4.4 we get

\[
\ddot{x}_{12i+1} = \ddot{x}_{12i+2} = \ddot{x}_{12i+3} = (\ddot{x}_{12i+1} + \ddot{x}_{12i+2} + \ddot{x}_{12i+3})/3 = 0,
\]
\[
\ddot{x}_{12i+7} = \ddot{x}_{12i+8} = \ddot{x}_{12i+9} = (\ddot{x}_{12i+7} + \ddot{x}_{12i+8} + \ddot{x}_{12i+9})/3 = 0.
\]
Moreover, Proposition 4.6 forces

\[
\ddot{x}_{12i+4} = \ddot{x}_{12i+5} = \ddot{x}_{12i+6} = \ddot{x}_{12i+10} = \ddot{x}_{12i+11} = \ddot{x}_{12i+12} = 0,
\]
and Lemma 4.4 also gives \( \ddot{x}_{12i+1} = 0 \).

**Remark 4.7.** For \( i = 1, \ldots, k \) recall that the map \( r_i : \mathbb{R}^{12k+1} \to \mathbb{R}^{12} \) is defined by \( r_i(x) = (x_{12i-11}, x_{12i-10}, \ldots, x_{12i-1}, x_{12i}) \). If \( h \leq k \), a slight modification of the strategy adopted to construct \( \tilde{\xi} \) yields a curve \( \tilde{\xi}_h : (-\varepsilon, \varepsilon) \to \Omega_{g,k} \) with the following properties:

\( \tilde{\xi}_h(0) = x_0 \), the structure defined by \( \xi(t) \) on \( M \) is complete at the last \( k - h \) cusps, and \( r_i(\tilde{\xi}_h(0)) = \pi_1(\tilde{\xi}(0)) \) for all \( i = 1, \ldots, h \).

4.4. **The final step.** The smooth path \( \xi : (-\varepsilon, \varepsilon) \to \Omega_{g,k} \) determines a smooth family of developing maps \( D_t : \partial M \to \mathbb{H}^2 \), which gives in turn a smooth path of holonomy representations \( \pi_t : \pi_1(\partial M) \to \text{PSL}(2, \mathbb{R}) \) lifting to a smooth path of representations \( \rho_t : \pi_1(\partial M) \to \text{SL}(2, \mathbb{R}) \). For any \( \gamma \in \pi_1(\partial M) \), \( t \in (-\varepsilon, \varepsilon) \) we set \( \text{tr}_\gamma(t) = \text{trace} \rho_t(\gamma) \). Of course \( \text{tr}_\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R} \) is smooth for any \( \gamma \in \pi_1(\partial M) \). Moreover, from Proposition 4.1 and equation (11) we easily deduce \( \text{tr}_\gamma(0) = 0 \). The following result readily implies Theorem 4.3 and whence Theorem 1.6.

**Proposition 4.8.** There exists an element \( \xi \in \pi_1(\partial M) \) such that \( \text{tr}_\xi(0) \neq 0 \).

In order to find the element \( \xi \in \pi_1(\partial M) \) mentioned in Proposition 4.8 we have to describe in some detail a particular portion of the triangulation induced on \( \partial M \) by the canonical decomposition \( \mathcal{T} \) of \( M \). So we fix our attention on the geodesic hexagon (with identifications on edges and vertices) which results from the gluing of the truncation triangles of \( \Delta_1, \Delta_2 \in \mathcal{T} \). Let \( l_1, l_2 \) be the oriented edges of this hexagon described in Fig. 5 and observe that the starting point of \( l_1 \) and the endpoint of \( l_2 \) both coincide with the same point \( b \in \partial M \). Thus the loop \( l_2 * l_1 \) defines an element \( \eta \) of \( \pi_1(\partial M, b) \). Our next aim is to give an explicit description of the isometry \( \tilde{\pi}_t(\xi) \in \text{PSL}(2, \mathbb{R}) \) in terms of angles and lengths of edges of the triangulation of \( \partial M \). Let us fix two consecutive lifts \( \tilde{l}_1, \tilde{l}_2 \) of \( l_1, l_2 \) in \( \partial M \) and let \( \tilde{l}_1 = \gamma(\tilde{l}_1) \) be the lift of \( l_1 \) starting at the endpoint of \( \tilde{l}_2 \). Then \( \tilde{\pi}_t(\xi) \) is the unique orientation-preserving isometry of \( \mathbb{H}^2 \) taking the oriented geodesic segment \( D_t(l_1) \) onto the oriented geodesic segment \( D_t(l_1') \).

Let \( \eta(t) \) be the angle formed by \( D_t(l_1) \) and \( D_t(l_2) \) at the endpoint of \( D_t(l_1) \) and \( \zeta(t) \) the angle formed by \( D_t(l_2) \) and \( D_t(l_1') \) at the endpoint of \( D_t(l_2) \) (see Fig. 4). Of course we have \( \eta(t) = x_3(t) + x_9(t) \), while

\[
\zeta(t) = (x_2(t) + x_8(t)) + (x_1(t) + x_7(t)) + \delta(x_3(t) + x_9(t)) + r(t),
\]
where \( \delta \in \{0, 1\} \) is determined by the combinatorics of \( \mathcal{T} \) and \( r(t) \in (0, 2\pi) \) is given by the sum (with multiplicity) of some of the \( x_i(t) 's \) with \( i \geq 13 \). Equation (18) and Proposition 4.6 give the following:

**Lemma 4.9.** We have \( \dot{\eta}(0) = \dot{\zeta}(0) = 0 \) and \( \ddot{\eta}(0) \neq 0 \). Moreover, if \( \delta = 0 \) then \( \ddot{\zeta}(0) = -\ddot{\eta}(0) \); if \( \delta = 1 \) then \( \ddot{\zeta}(0) = 0 \).

Let now \( \mathbb{H} = \{ z \in \mathbb{C} : \Im(z) > 0 \} \) be the upper half-plane model of \( \mathbb{H}^2 \). Without loss of generality we may assume that for any \( t \in (-\varepsilon, \varepsilon) \) the developing map \( D_t : \partial M \to \mathbb{H} \)
Figure 5: Up to cyclic reorderings of the internal edges $e_1, e_2, e_3$ of $\Delta_1$, we show here the only possible identifications of the vertices of the hexagon tessellated by the truncation triangles of $\Delta_1$ and $\Delta_2$.

sends $\tilde{l}_1$ onto the geodesic segment starting at $i \in \mathbb{H}$ and ending at $\lambda(t) \cdot i$, where $\lambda(t) = \exp \ell(t)$. For $1 < \lambda \in \mathbb{R}$, $\theta \in (0, 2\pi)$ we set

$$A(\lambda, \theta) = \begin{pmatrix} \sqrt{\lambda} \sin(\theta/2) & -\sqrt{\lambda} \cos(\theta/2) \\ (1/\sqrt{\lambda}) \cos(\theta/2) & (1/\sqrt{\lambda}) \sin(\theta/2) \end{pmatrix} \in \text{PSL}(2, \mathbb{R}).$$

It is easily seen that $A(\lambda, \theta)$ sends the half-geodesic $s$ starting at $i \in \mathbb{H}$ and ending at $\infty$ onto the half-geodesic $s'$ starting at $\lambda \cdot i$ such that $s$ and $s'$ define at $\lambda \cdot i$ an angle equal to $\theta$. Now both $D_t(\tilde{l}_2)$ and $D_t(\tilde{l}_1)$ have length $\ell(t)$, and the isometry $\tilde{\rho}_t(\tilde{\gamma})$ takes the oriented geodesic segment $D_t(\tilde{l}_1)$ onto the oriented geodesic segment $D_t(\tilde{l}_1')$, so

$$\tilde{\rho}_t(\tilde{\gamma}) = A(\lambda(t), \eta(t)) \cdot A(\lambda(t), \zeta(t)).$$

By Lemmas 3.1, 4.4 we have $\dot{\lambda}(0) = \ddot{\lambda}(0) = 0$. Also recall that $\dot{\eta}(0) = \dot{\zeta}(0) = 0$, so differentiating two times equality (23), evaluating at 0 and taking the trace we obtain

$$2t\dddot{\tau}(0) = \dot{\eta}(0) \cdot ((\lambda(0) + \lambda(0)^{-1}) \cos(\eta(0)/2) \sin(\zeta(0)/2)
+ 2\sin(\eta(0)/2) \cos(\zeta(0)/2))$$

$$+ \dot{\zeta}(0) \cdot ((\lambda(0) + \lambda(0)^{-1}) \sin(\eta(0)/2) \cos(\zeta(0)/2)
+ 2\cos(\eta(0)/2) \sin(\zeta(0)/2)).$$

Let us suppose $\ddot{\zeta}(0) = -\dot{\eta}(0) \neq 0$. In this case, from equation (24) we obtain

$$t\dddot{\tau}(0) = (\dot{\eta}(0)/2)(\lambda(0) + \lambda(0)^{-1} - 2)(\sin((\zeta(0)/2) - (\eta(0)/2))) \neq 0.$$
When $\ddot{\zeta}(0) = 0$ computations are more involved, and in order to prove that $\dot{t} \gamma(0) \neq 0$ one can show that

\begin{equation}
(\lambda(0) + \lambda(0)^{-1}) \cos(\eta(0)/2) \sin(\zeta(0)/2) + 2 \sin(\eta(0)/2) \cos(\zeta(0)/2) > 0.
\end{equation}

We skip this computation here, addressing the reader to [Fri05] for the details. The proof of Proposition 4.8 is now concluded.

5. Similar fillings

Kojima proved in [Koj90] that every complete finite-volume hyperbolic manifold $N$ with non-empty geodesic boundary admits a canonical decomposition into geometric polyhedra. For later reference we record the following:

Proposition 5.1. Any shortest return path in $N$ is an edge of the Kojima decomposition of $N$. Moreover, any compact regular partially truncated tetrahedron isometrically immersed in $N$ whose internal edges are shortest return paths is a piece of the canonical decomposition of $N$. 

Figure 6: Notation for the proof of equation (24).
The following result is proved in [FMP03].

**Theorem 5.2.** Let \( M \in \mathcal{M}_{g,k} \) with \( \partial M = \Sigma_g \sqcup ( \bigsqcup_{i=1}^k T_i ) \). Then the following holds:

1. \( M \) has a unique triangulation with \( g + k \) tetrahedra, which gives the canonical Kojima decomposition of \( M \);
2. The volume of the complete hyperbolic structure of \( M \) depends only on \( g \) and \( k \);
3. The Heegaard genus of \( (M, \Sigma_g, \bigsqcup_{i=1}^k T_i) \) is \( g + 1 \);
4. \( H_1(M; \mathbb{Z}) = \mathbb{Z}^{g+k} \);
5. The Turaev-Viro invariant \( TV_r(M) \) depends only on \( r, g \) and \( k \).

Manifolds in \( \mathcal{M}_{g,k} \) also share other geometric invariants:

**Theorem 5.3.** Let \( M \in \mathcal{M}_{g,k} \) be endowed with its complete hyperbolic structure. Then the following holds:

1. The cusp volume of \( M \) depends only on \( g \) and \( k \);
2. The Euclidean structures on the boundary tori of \( M \) are all isometric to the regular hexagonal one;
3. The length of the shortest return path of \( M \) depends only on \( g \) and \( k \).

**Proof:** By Lemma 2.8, a maximal regular horocusp neighbourhood for \( M \) is obtained by gluing the maximal horocusp neighbourhoods of the ideal vertices of the non-compact tetrahedra of \( T \), whence point (i). Point (ii) has already been established and point (iii) is a consequence of Proposition 5.1 and Theorem 5.2 (ii). \( \square \)

### 5.1. Spines and homology.

We now prove a refinement of Theorem 5.2 (ii) that will be useful later. To this aim we switch from the viewpoint of ideal triangulations to the dual viewpoint of special spines, suggested in Fig. 7. Recall that a spine of a manifold is a subpolyhedron onto which the manifold collapses. A polyhedron is special if it is locally homeomorphic to that of Fig. 7 right and its natural stratification consists of 0-, 1-, and 2-cells.

**Proposition 5.4.** Let \( M \in \mathcal{M}_{g,k} \). Then we have the exact sequence

\[
0 \rightarrow H_1 \left( \bigsqcup_{i=1}^k T_i; \mathbb{Z} \right) \xrightarrow{i_*} H_1(M; \mathbb{Z}) \rightarrow \mathbb{Z}^{g-k} \rightarrow 0,
\]

where \( i_* \) is the map induced by the inclusion \( i : \bigsqcup_{i=1}^k T_i \rightarrow M \).
Proof: Let $P$ be the spine dual to the triangulation of $M$ with $g + k$ tetrahedra. Note that $P$ has a cellularization into vertices, edges, and faces corresponding to tetrahedra, faces, and edges of the triangulation. We denote in particular by $S(P)$ the 1-skeleton of $P$ (a 4-valent graph). By Proposition 2.1 the spine $P$ contains $k$ (open) hexagonal faces $E_1, \ldots, E_k$ and one big face $G$ with $6g$ vertices (with multiplicity). For $i = 1, \ldots, k$ the closure $\overline{E_i}$ of $E_i$ is a torus which bounds a collar of the $i$-th toric component $T_i$ of $\partial M$, and the rest of $P$ lies outside this collar. Since $M$ collapses onto $P$, we have $H_1(M; \mathbb{Z}) \cong H_1(P; \mathbb{Z})$, and we can use cellular homology to compute $H_1(P; \mathbb{Z})$.

Since $g > k$, a vertex $v$ of $P$ exists which corresponds to a partially truncated tetrahedron without ideal vertices. Notice that $G$ is the only face incident to $v$. Moreover, an easy analysis of the local structure of $P$ near $v$ shows that the number of edges emanating from $v$ on which $G$ passes three times with the same orientation is at most two. Let us also observe that any finite graph has an even number of vertices with odd valence, so the number of connected components of $S(P) \setminus \{v\}$ is at most two. These facts easily imply that a maximal tree $Y$ in $S(P)$ exists with the following properties: $S(P) \setminus Y$ consists of $g + k + 1$ edges $e_1, \ldots, e_{g+k+1}$, where $e_{2i-1}$ and $e_{2i}$ represent generators of $H_1(T_i; \mathbb{Z})$ for $i = 1, \ldots, k$, and $G$ passes three times on $e_{g+k+1}$ with different orientations. Therefore we get a presentation for $H_1(P; \mathbb{Z})$ with generators $e_1, \ldots, e_{g+k+1}$ and one relator $w$ containing $\pm e_{g+k+1}$ once. This implies in turn that the classes of $e_1, \ldots, e_{g+k}$ give a free basis of $\mathbb{Z}^{g+k+1}/\langle w \rangle \cong \mathbb{Z}^{g+k} \cong H_1(P; \mathbb{Z})$, whence the conclusion. \qed

5.2. Boundary slopes and Dehn filling. Let $T_i$ be the $i$-th boundary torus of a manifold $M \in \mathcal{M}_{g,k}$, and recall that the unique complete finite-volume hyperbolic structure on $M$ induces on $T_i$ a Euclidean structure defined up to similarity. For the sake of simplicity, we endow $T_i$ with a fixed Euclidean structure choosing the scale factor in such a way that $\text{Area}(T_i) = \sqrt{3}/2$. This easily implies that $T_i$ is isometric to $\mathbb{C}/\Gamma$, where $\mathbb{C}$ is endowed with the standard Euclidean metric, and $\Gamma$ is the discrete additive subgroup of $\mathbb{C}$ with generators $1, -1/2 + i\sqrt{3}/2$. We denote by $\mathcal{M}(T_i)$ the group of isotopy classes of isometries of $T_i$. Of course $\mathcal{M}(T_i)$ acts on the set of slopes on $T_i$.

Let $D_6$ be the dihedral group with 12 elements, i.e., the group of isometries of $\mathbb{C}$ generated by the rotation $r : \mathbb{C} \to \mathbb{C}$, $r(z) = e^{i\pi/3}z$ and the reflection $s : \mathbb{C} \to \mathbb{C}$, $s(z) = \overline{z}$. Any element of $D_6$ induces an isometry of $T_i$, and any isometry of $T_i$ lifts up to isotopy to an element of $D_6$. Thus $\mathcal{M}(T_i)$ is canonically isomorphic to $D_6$.

Let $\mu_i, \lambda_i$ be the preferred basis of $H_1(T_i; \mathbb{Z})$ chosen in Subsection 2.6. In what follows we will often represent slopes as indivisible elements in $H_1(T_i; \mathbb{Z})$ without emphasizing the fact that each slope corresponds in fact to two such elements. Any slope $s$ on $T_i$ determines a well-defined isotopy class of geodesics on $T_i$, and we denote by $L(s)$ the Euclidean length of such geodesics. An elementary calculation gives the following:

Lemma 5.5. Let $s = p \cdot \mu_i + q \cdot \lambda_i$ be a slope on $T_i$. Then $L(s) = \sqrt{p^2 + q^2 - pq}$.

Let $\{\kappa_1 < \kappa_2 < \ldots < \kappa_n < \ldots\}$ be the set of lengths of slopes on $T_i$. The following result is easily deduced from Lemma 5.5.

Proposition 5.6. The following holds:

- There are exactly three slopes of length $\kappa_1 = 1$. They are represented by $\mu_i, \lambda_i$ and $\mu_i + \lambda_i$, and they are $\mathcal{M}(T_i)$-equivalent to each other.
- There are exactly three slopes of length $\kappa_2 = \sqrt{3}$. They are represented by $\mu_i - \lambda_i, \mu_i + 2 \cdot \lambda_i$ and $2 \cdot \mu_i + \lambda_i$, and they are $\mathcal{M}(T_i)$-equivalent to each other.
• If $s$ is a slope with $L(s) \geq \kappa_3 = \sqrt{7}$, then there exist exactly six slopes $\mathcal{M}(T_i)$-equivalent to $s$.

**Remark 5.7.** Let $s$ and $s'$ be slopes on $T_i$. Of course if $s'$ is $\mathcal{M}(T_i)$-equivalent to $s$ then $L(s') = L(s)$, but the converse is not true. For example, the slopes $s = 19 \mu_i + 11 \lambda_i$ and $s' = 16 \mu_i - \lambda_i$ have the same length $L(s) = L(s') = \sqrt{273}$, even if they are not $\mathcal{M}(T_i)$-equivalent.

### 5.3. Dehn fillings

The following result, which is proved in [FMP03], completely classifies the Dehn fillings of elements in $\mathcal{M}_{g,k}$.

**Theorem 5.8.** Let $M \in \mathcal{M}_{g,k}$ with $\partial M = \Sigma_g \cup \left( \bigsqcup_{i=1}^{k} T_i \right)$, let $h \leq k$, let $s_i$ be a slope on $T_i$ for $i = 1, \ldots, h$ and $N = M(s_1, \ldots, s_h)$. Then $N$ is hyperbolic if and only if $L(s_i) \geq \kappa_3$ for all $i = 1, \ldots, h$. Moreover, when $N$ is hyperbolic the Heegaard genus of $(N, \Sigma_g, T_{h+1} \cup \ldots \cup T_k)$ is $g+1$.

Actually, it is proved in [FMP03] that each manifold in $\mathcal{M}_{g,k}$ is a link complement in the handlebody of genus $g$. The following result is an easy consequence of Proposition 5.4.

**Proposition 5.9.** If $N$ is as in the statement of Theorem 5.8 then $H_1(N; \mathbb{Z}) \cong \mathbb{Z}^{g+k-h}$.

### 5.4. Symmetries of $\Omega_{g,k}$

We now describe the symmetries of the deformation space $\Omega_{g,k}$, and explain how these symmetries act on the space of Dehn filling coefficients. In order to clarify our arguments it is convenient to denote the coordinates of $\mathbb{R}^{12k+1}$ as in equation (25):

$$
\beta(x) = x_{12k+1}, \quad \alpha^l_i(x) = x_{6(l-1)+j}, \quad \gamma^l_i(x) = x_{6(l-1)+3+j}, \quad l = 1, \ldots, 2k, \quad j = 1, 2, 3.
$$

### 5.5. Symmetries of $\Omega_{g,k}$

Let $\text{Aut}(\Omega_{g,k})$ denote the set of diffeomorphisms of $\Omega_{g,k}$ onto itself, fix $i \in \{1, \ldots, k\}$ and take an element $\sigma$ of the symmetric group $\mathfrak{S}_3$. We can make $\sigma$ act on the apices of the dihedral angles of $\Delta_{2i-1}$ and of $\Delta_{2i}$, thus obtaining an automorphism $\tilde{\sigma}_i \in \text{Aut}(\Omega_{g,k})$ which leaves the angles of all the other tetrahedra unchanged:

\begin{equation}
\begin{aligned}
\alpha^l_i(\tilde{\sigma}_i(x)) &= \alpha^{\sigma^{-1}(l)}(x), & \gamma^l_i(\tilde{\sigma}_i(x)) &= \gamma^{\sigma^{-1}(l)}(x) & \text{if} \ l = 2i - 1, 2i; \\
\beta(\tilde{\sigma}_i(x)) &= \beta(x).
\end{aligned}
\end{equation}

The fact that $\tilde{\sigma}_i$ takes indeed $\Omega_{g,k}$ into itself is a consequence of the invariance of consistency equations (4), (6), (7) under the permutation of apices described in (26).

Another symmetry $\zeta_i : \Omega_{g,k} \rightarrow \Omega_{g,k}$ exists which corresponds to interchanging the roles of the tetrahedra $\Delta_{2i-1}$ and $\Delta_{2i}$:

\begin{equation}
\begin{aligned}
\beta(\zeta_i(x)) &= \beta(x); & \alpha^{l}_{2i-1}(\zeta_i(x)) &= \alpha^l_{2i}(x), & \gamma^{l}_{2i-1}(\zeta_i(x)) &= \gamma^l_{2i}(x); \\
\alpha^{l}_i(\zeta_i(x)) &= \alpha^l_i(x), & \gamma^{l}_i(\zeta_i(x)) &= \gamma^l_i(x) & \text{if} \ l \neq 2i - 1, 2i.
\end{aligned}
\end{equation}

Also in this case the fact that $\zeta_i (\Omega_{g,k}) = \Omega_{g,k}$ easily follows from a straightforward analysis of the consistency equations. We can now define a map

\begin{equation}
\varphi_i : \mathfrak{S}_3 \times \mathbb{Z}/2 \rightarrow \text{Aut}(\Omega_{g,k}), \quad \varphi_i(\sigma, \epsilon) = \tilde{\sigma}_i \circ \zeta^\epsilon_i, \quad \sigma \in \mathfrak{S}_3, \quad \epsilon = 0, 1.
\end{equation}

Using that $\zeta_i$ commutes with $\tilde{\sigma}_i$ for all $\sigma \in \mathfrak{S}_3$ it is easily seen that $\varphi_i$ is an injective homomorphism with image a certain subgroup $\text{Sym}_i(\Omega_{g,k})$ of $\text{Aut}(\Omega_{g,k})$. 


If $\kappa$ is an element of the symmetric group $\mathcal{S}_k$, then there exists a symmetry $\tilde{\kappa} \in \text{Aut}(\Omega_{g,k})$ which induces the corresponding permutation of the shape of the cusps:

\[
\alpha^j_{2i-1}(\tilde{\kappa}(x)) = \alpha^j_{2i-1(i)-1}(x), \quad \alpha^j_{2i}(\tilde{\kappa}(x)) = \alpha^j_{2i-1(i)}(x),
\]

\[
\gamma^i_{2j-1}(\tilde{\kappa}(x)) = \gamma^i_{2j-1(i)-1}(x), \quad \gamma^i_{2j}(\tilde{\kappa}(x)) = \gamma^i_{2j-1(i)}(x), \quad i = 1, \ldots, k, \quad j = 1, 2, 3; \quad \beta(\tilde{\kappa}(x)) = \beta(x).
\]

The map $\kappa \mapsto \tilde{\kappa}$ defines an injective homomorphism $\nu: \mathcal{S}_k \to \text{Aut}(\Omega_{g,k})$.

Let us denote by $\text{Sym}(\Omega_{g,k})$ the subgroup of $\text{Aut}(\Omega_{g,k})$ generated by $\nu(\mathcal{S}_k) \cup \left( \bigcup_{i=1}^k \text{Sym}_i(\Omega_{g,k}) \right)$. Elements in $\text{Sym}_i(\Omega_{g,k})$ commute with elements in $\text{Sym}_j(\Omega_{g,k})$ whenever $i \neq j$, thus the group generated by the $\text{Sym}_i(\Omega_{g,k})$’s is actually isomorphic to the product $\prod_{i=1}^k \text{Sym}_i(\Omega_{g,k})$. Moreover if $\pi: \text{Sym}(\Omega_{g,k}) \to \mathcal{S}_k$ is the natural homomorphism which maps each symmetry to the corresponding permutation of cusps we have $\text{Ker } \pi = \prod_{i=1}^k \text{Sym}_i(\Omega_{g,k})$ and $\pi \circ \nu = \text{Id } \mathcal{S}_k \to \mathcal{S}_k$. Thus

\[
\text{Sym}(\Omega_{g,k}) = \nu(\mathcal{S}_k) \ltimes \left( \prod_{i=1}^k \text{Sym}_i(\Omega_{g,k}) \right) \cong \mathcal{S}_k \ltimes \left( \prod_{i=1}^k \mathcal{S}_3 \times \mathbb{Z}/2 \right).
\]

Our next task is to investigate how symmetries in $\text{Sym}(\Omega_{g,k})$ act on the space of Dehn filling coefficients parameterizing a small neighbourhood of $x_0$ in $\Omega_{g,k}$.

5.6. Action on Dehn filling coefficients. Let us denote by $\mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ the group of isotopy classes of isometries of $T_1 \sqcup \ldots \sqcup T_k$. For $\sigma \in \mathcal{S}_k$ we define an element $\eta(\sigma) \in \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ which permutes the marked tori $T_1, \ldots, T_k$ according to $\sigma$. Namely, $\eta(\sigma)$ is the isotopy class of any element $\sigma' \in \text{Isom}(T_1 \sqcup \ldots \sqcup T_k)$ with the following properties: $\sigma'(T_j) = T_{\sigma(j)}$, $\sigma'(|\mu_j|) = |\sigma_j|$, $\sigma'(\lambda_j) = \lambda_j$ for $j = 1, \ldots, k$. It is easily seen that the map $\eta: \mathcal{S}_k \to \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ is a well-defined injective homomorphism. Let now $\pi': \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k) \to \mathcal{S}_k$ be the natural projection which associates to any element in $\mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ the induced permutation of the $T_j$’s. Of course we have $\pi' \circ \eta = \text{Id } \mathcal{S}_k \to \mathcal{S}_k$, and the kernel of $\pi'$ is canonically isomorphic to $\mathcal{M}(T_1) \times \ldots \times \mathcal{M}(T_k)$. Therefore we have

\[
\mathcal{M}(T_1 \sqcup \ldots \sqcup T_k) = \eta(\mathcal{S}_k) \ltimes \left( \prod_{i=1}^k \mathcal{M}(T_i) \right) \cong \mathcal{S}_k \ltimes \left( \prod_{i=1}^k D_6 \right),
\]

where $D_6 \cong \mathcal{S}_3 \times \mathbb{Z}/2$ is the dihedral group with 12 elements.

By Theorems 2.9 and 2.11 from now on we can fix a small neighbourhood $V$ of $x_0$ in $\Omega_{g,k}$ such that for all $x \in V$ the Dehn filling coefficient $(p_j(x), q_j(x)) \in S^2 = \mathbb{R}^2 \cup \{\infty\}$ is well-defined, and the map

\[
d = (d_1, \ldots, d_k): V \to \prod_{i=1}^k S^2, \quad d_j(x) = (p_j(x), q_j(x)) \in S^2
\]

is a diffeomorphism onto an open neighbourhood of $\{\infty\} \times \ldots \times \{\infty\}$ in $S^2 \times \ldots \times S^2$. It is easily seen that we can also assume $\psi(V) = V$ for all $\psi \in \text{Sym}(\Omega_{g,k})$.

We now observe that any element $h \in \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k)$ induces an automorphism of $H_1(T_i; \mathbb{R}) \oplus \ldots \oplus H_1(T_k; \mathbb{R})$. The basis $\mu_i, \lambda_i$ defines a canonical isomorphism $H_1(T_i; \mathbb{R}) \cong \mathbb{R}^2$, so $h$ induces an automorphism $h_* \in \prod_{i=1}^k \mathbb{R}^2$ that preserves $\{\infty\} \times \ldots \times \{\infty\}$. 
Proposition 5.10. For any \( \psi \in \text{Sym}(\Omega_{g,k}) \) there exists a unique \( h(\psi) \in \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k) \) such that \( d(\psi(x)) = h(\psi)_*(d(x)) \) for all \( x \in V \). Moreover the map

\[
\text{Sym}(\Omega_{g,k}) \to \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k), \quad \psi \mapsto h(\psi)
\]

is a group isomorphism which preserves decompositions \([\text{B}], \text{C}]\).

**Proof:** We need to describe as explicitly as possible how the maps \( d_j : V \to S^2, \ i = 1, \ldots, k \) change under precompositions with elements in \( \text{Sym}(\Omega_{g,k}) \). Let us consider the action of \( \mathfrak{S}_3 \times \mathbb{Z}/2 \) on \( \Omega_{g,k} \) via the representation \( \varphi_j : \mathfrak{S}_3 \times \mathbb{Z}/2 \to \text{Sym}(\Omega_{g,k}) \) defined in equation \([28]\). By definition the element \( (\sigma, 0) \in \mathfrak{S}_3 \times \mathbb{Z}/2 \) acts on the \( \gamma_{2j-1} \)'s and the \( \gamma_{2j} \)'s just by applying \( \sigma^{-1} \) to the apices, while the action of \( (\text{Id}, 1) \in \mathfrak{S}_3 \times \mathbb{Z}/2 \) interchanges the indices \( 2j-1, 2j \). Let \( r = ((132), 1), \ s = ((12), 0) \) be fixed elements of \( \mathfrak{S}_3 \times \mathbb{Z}/2 \). Together with equations \([33]\), the description of the action of \( \varphi_j(r), \varphi_j(s) \) given above implies (after some computations) that

\[
\begin{align*}
\text{equation (33)}: \quad & u_j(\varphi_j(r)(x)) = -v_j(x), \quad v_j(\varphi_j(r)(x)) = u_j(x) + v_j(x), \\
\text{equation (34)}: \quad & u_j(\varphi_j(s)(x)) = -u_j(x), \quad v_j(\varphi_j(s)(x)) = \underline{u_j(x)} + \underline{v_j(x)}.
\end{align*}
\]

We can now compute the action of \( h(\varphi_j(r)) \) and \( h(\varphi_j(s)) \) on Dehn filling coefficients. Using equations \([33], \ [34]\) and the very definition of Dehn filling coefficients we get

\[
2\pi i = p_j(\varphi_j(r)(x))u_j(\varphi_j(r)(x)) + q_j(\varphi_j(r)(x))v_j(\varphi_j(r)(x))
\]

\[
= p_j(\varphi_j(r)(x))(-v_j(x)) + q_j(\varphi_j(r)(x))(u_j(x) + v_j(x))
\]

\[
= q_j(\varphi_j(r)(x))(u_j(x) + (q_j(\varphi_j(r)(x)) - p_j(\varphi_j(r)(x)))v_j(x),
\]

whence \( p_j(x) = q_j(\varphi_j(r)(x)), \ q_j(x) = q_j(\varphi_j(r)(x)) - p_j(\varphi_j(r)(x)) \) and

\[
\text{equation (35)}: \quad p_j(\varphi_j(r)(x)) = p_j(x) - q_j(x), \quad q_j(\varphi_j(r)(x)) = p_j(x).
\]

A similar computation also gives

\[
\text{equation (36)}: \quad p_j(\varphi_j(s)(x)) = p_j(x) - q_j(x), \quad q_j(\varphi_j(s)(x)) = -q_j(x).
\]

This easily implies that \( h(\varphi_j(r)) \) and \( h(\varphi_j(s)) \) act on Dehn filling coefficients at the \( j \)-th end of \( M \) respectively as a rotation of angle \( \pi/3 \) and a reflection with respect to the line \( \mathbb{R} \cdot \mu_j \). This gives in turn that \( h \) restricts to an isomorphism

\[
\text{Sym}(\Omega_{g,k}) \supset \text{Sym}_j(\Omega_{g,k}) \cong \mathcal{M}(T_j) \subset \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k).
\]

Moreover, with notation as in formulae \([\text{B}], \ [\text{C}]\), any permutation of cusps in \( \nu(\mathfrak{S}_k) \subset \text{Sym}(\Omega_{g,k}) \) is taken by \( h \) into the corresponding permutation in \( \eta(\mathfrak{S}_k) \subset \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k) \), and this concludes the proof.

5.7. **Return paths.** Recall that for \( x \in \Omega_{g,k} \) we denote by \( M(x) \) the hyperbolic structure induced on \( M \) by \( x \), and by \( \hat{M}(x) \) the metric completion of \( M(x) \). Moreover, if \( x \in \Omega_{g,k} \) then \( \hat{M}(x) \) is a complete finite-volume hyperbolic manifold with geodesic boundary. In this case the unique compact edge in the geometric triangulation of \( M(x) \) defines a return path \( l_x \) in \( \hat{M}(x) \). For \( y \in \Omega_{g,k} \) we denote by \( L^y \) the length with respect to the hyperbolic metric on \( \hat{M}(y) \). Of course we have \( \lim_{x \to \Omega_{g,k} \leftarrow y} L^y(l_x) = L^{x_0}(l_{x_0}) \). Moreover, a positive number \( \delta \) exists such that any return path in \( M \) different from \( l_{x_0} \) has length at least \( L^{x_0}(l_{x_0}) + 2\delta \).
Lemma 5.11. There exists a neighbourhood $V' \subset V$ of $x_0$ in $\Omega_{g,k}$ such that if $x \in \Omega_{g,k} \cap V'$, then $l_x$ is the only return path in $\hat{M}(x)$ having length less than $L^{x_0}(l_{x_0}) + \delta$. In particular, if $x \in \Omega_{g,k} \cap V'$ then $l_x$ is the unique shortest return path in $\hat{M}(x)$.

Proof: We suppose by contradiction that there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset \Omega_{g,k}$ converging to $x_0$ such that $\hat{M}(y_n)$ contains a return path $l_n \neq l_{y_n}$ with $L^{y_n}(l_n) < L^{x_0}(l_{x_0}) + \delta$ for every $n \in \mathbb{N}$. Since the distance between the added geodesics $\hat{M}(y) \setminus M(y)$ and the geodesic boundary of $\hat{M}(y)$ approaches $\infty$ as $y$ tends to $x_0$, we can suppose that the compact set $K_y \subset \hat{M}(y)$ of points whose distance from $\partial \hat{M}(y)$ is less than or equal to $2L^{x_0}(l_{x_0})$ is entirely contained in $M(y)$. Moreover, up to passing to a subsequence we can suppose that there exists an $\epsilon_n$-bilipschitz homeomorphism $f_n : K_{y_n} \to K_{x_0}$ taking $\partial \hat{M}(y_n)$ onto $\partial \hat{M}(x_0) = \partial M(x_0)$, where $\epsilon_n$ tends to 1 as $n$ tends to $\infty$. Thus $L^{x_0}(f_N(l_N)) < L^{x_0}(l_{x_0}) + 2\delta$ for some $N \gg 0$. Since $l_N$ is not boundary-parallel in $\hat{M}(y_n)$, the path $f_N(l_N)$ is not boundary-parallel in $M(x_0)$. Since return paths minimize length in their relative homotopy class, this implies that $f_N(l_N)$ is homotopic to $l_{x_0} = f_N(l_{y_n})$ relatively to the boundary in $M(x_0)$, so $l_N$ is homotopic to $l_{y_N}$ relatively to the boundary in $\hat{M}(y_n)$. Since $l_N$ and $l_{y_n}$ are both return paths, this implies in turn $l_N = l_{y_N}$, a contradiction. 

Let $V'$ be a neighbourhood of $x_0$ in $\Omega_{g,k}$ as in the statement of Lemma 5.11 and for $x \in \Omega_{g,k} \cap V$ let $U(x)$ be the universal covering of the hyperbolic manifold $\hat{M}(x)$. We recall that $U(x)$ is isometric to a convex polyhedron of $\mathbb{H}^3$ bounded by a countable number of hyperbolic planes. Lemma 5.11 readily implies the following:

Corollary 5.12. The minimal distance between distinct connected components of $\partial U(x)$ is equal to $L^x(l_x)$ for all $x \in \Omega_{g,k} \cap V'$. Moreover, if $S_1, S_2$ are distinct connected components of $\partial U(x)$, then the distance between $S_1$ and $S_2$ equals $L^x(l_x)$ if and only if the shortest path joining $S_1$ with $S_2$ projects onto $l_x$ in $\hat{M}(x)$.

The following proposition relates the intrinsic geometry of $U(x)$ to properties of our geometric triangulation of $M(x)$, when $x \in \Omega_{g,k}$.

Proposition 5.13. There exists a neighbourhood $V'' \subset V'$ of $x_0$ in $\Omega_{g,k}$ with the following property. Let $x \in \Omega_{g,k} \cap V''$ and $S_1, \ldots, S_4$ be pairwise distinct connected components of $\partial U(x)$. Then the distance between $S_i$ and $S_j$ equals $L^x(l_x)$ for all $i \neq j$ if and only if there exists a lift of a compact tetrahedron in the geometric triangulation parameterized by $x$ whose truncation triangles lie on $S_1, \ldots, S_4$.

Proof: We concentrate on the “only if” part of the statement, the “if” part being obvious. Let $\delta$ be as in the statement of Lemma 5.11. Then there exist $\epsilon > 0$ and a small neighbourhood $V'' \subset V'$ of $x_0$ in $\Omega_{g,k}$ such that for every $y \in \Omega_{g,k} \cap V''$ we have

$$(1 + \epsilon)L^y(l_y) < L^{x_0}(l_{x_0}) + \delta.$$ 

Let $K_y$ be the set of points of $\hat{M}(y)$ whose distance from $\partial \hat{M}(y)$ is at most twice the diameter of the regular truncated tetrahedron with edge-length equal to $L^y(l_y)$. Up to resizing $V''$ we can suppose that for all $y \in V''$ the set $K_y$ is contained in $M(y) \subset \hat{M}(y)$, and there exists a $(1 + \epsilon)$-bilipschitz homeomorphism $p_y : K_y \to K_{x_0}$.

Let now $x \in \Omega_{g,k} \cap V''$ and $S^x_1, \ldots, S^x_4$ be pairwise distinct connected components of $\partial U(x)$ such that the distance between $S^x_i$ and $S^x_j$ equals $L^x(l_x)$ for all $i \neq j$. Let $\Delta$
be a topologically partially truncated tetrahedron with truncation triangles $B_1, B_2, B_3, B_4$ and internal edges $e_{ij}$ joining $B_i$ with $B_j$. We consider a geometric realization $\tilde{\varphi}_x : \Delta \rightarrow U(x) \subset \mathbb{H}^3$ with $\tilde{\varphi}_x(B_i) \subset S^2_i$, $i = 1, 2, 3, 4$, and we observe that $\tilde{\varphi}_x(e_{ij})$ is the shortest geodesic arc joining $S^2_i$ with $S^2_j$. Let $r_x : \Delta \rightarrow \tilde{M}(x)$ be the composition of $\tilde{\varphi}_x$ with the projection $U(x) \rightarrow \tilde{M}(x)$. Since $r_x(\Delta) \subset K_x$, we can consider the map $r_{x_0} = p_x \circ r_x : \Delta \rightarrow \tilde{M}(x_0) = M(x_0)$, which lifts in turn to $\tilde{r}_{x_0} : \Delta \rightarrow U(x_0)$. For $i = 1, 2, 3, 4$ let $S^2_{x_0}$ be the component of $\partial U(x_0)$ containing $\tilde{r}_{x_0}(B_i)$, so that $\tilde{r}_{x_0}(e_{ij})$ is a (not necessarily geodesic) arc joining $S^2_{x_0}$ with $S^2_{x_0}$. Since $r_x(e_{ij})$ is not null-homotopic relatively to the boundary in $\tilde{M}(x)$, $r_{x_0}(e_{ij})$ is not null-homotopic relatively to the boundary in $U(x_0)$, so $S^2_{x_0} \neq S^2_{x_0}$ for $i \neq j$. Moreover we have $L^2_0(r_{x_0}(e_{ij})) \leq (1 + \varepsilon)L^2(l_x) + \delta$, where we denote by $d_{x_0}$ the hyperbolic metric on $U(x_0)$. Thus by Lemma 5.11 and Proposition 5.8 the hyperbolic planes $S^2_{x_0}, S^2_{x_0}, S^2_{x_0}, S^2_{x_0}$ bound a compact geodesic regular partially truncated tetrahedron which projects onto a piece of the Kojima decomposition of $M(x_0)$. This easily implies that $\tilde{r}_x(\Delta)$ projects onto a compact partially truncated tetrahedron in the geometric triangulation of $M(x)$.  

5.8. Similar fillings. A set of slopes for a complete finite-volume hyperbolic 3-manifold $N$ is a set $S = \{s_1, \ldots, s_h\}$ of either 0 or 1 slope per boundary torus. If $S = \{s_1, \ldots, s_h\}$ is a set of slopes for $N$ we denote by $N(S)$ the manifold obtained by filling $N$ along $s_1, \ldots, s_h$.

Let $M$ and $M'$ be elements in $\mathcal{M}_{g,k}$ (we do not exclude the case $M = M'$) with boundary tori $T_1, \ldots, T_k$ and $T'_1, \ldots, T'_k$. We endow each of these tori with the Euclidean metric defined on them by the hyperbolic structures on $M, M'$ together with the requirement that $\text{Area}(T_i) = \text{Area}(T'_i) = \sqrt{3}/2$ for $i = 1, \ldots, k$. We say that a set of slopes $S$ for $M$ is equivalent to the set of slopes $S'$ for $M'$ if there exists an orientation-preserving isometry $\psi : T_1 \sqcup \ldots \sqcup T_k \rightarrow T'_1 \sqcup \ldots \sqcup T'_k$ taking $S$ onto $S'$. Of course if $S$ is equivalent to $S'$ then the lengths of the slopes in $S$ are equal to the lengths of the slopes in $S'$. We recall however that the converse is not true (see Remark 5.7).

Theorem 5.14. Let $M, M'$ be elements of $\mathcal{M}_{g,k}$ and $S$ (resp. $S'$) be a set of slopes for $M$ (resp. $M'$). Then there exists a positive constant $C$ such that the following holds: if all the slopes of $S$ are longer than $C$ and $S$ is equivalent to $S'$, then $M(S)$ is geometrically similar to $M'(S')$.

Proof: Let $V'$ be a neighbourhood of $x_0$ in $\Omega_{g,k}$ as in the statement of Lemma 5.11 and $d : V' \rightarrow S^2 \times \ldots \times S^2$ be the map defined in equation \ref{eq:definition_of_d}. We can choose a positive constant $C$ depending only on $g$ and $k$ such that the following holds: if $S = \{s_1', \ldots, s_{h'}\}$ is a set of slopes for $M$ with $L(s_{i}) > C$ for $l = 1, \ldots, h$, then any $k$-tuple of Dehn filling coefficients corresponding to $S$ lie in $d(V')$ (due to the choice of the signs, there exist exactly $2^h$ such $k$-uples).

Let now $S$ be a set of slopes for $M$ whose elements are longer than $C$ and let $S'$ be a set of slopes for $M'$ which is equivalent to $S$. Choose also points $x, x' \in V' \subset \Omega_{g,k}$ such that $d(x)$ (resp. $d(x')$) gives a $k$-uple of Dehn filling coefficients corresponding to $S$ (resp. $S'$). By Proposition 5.10 it follows that a symmetry $\varphi \in \text{Sym}(\Omega_{g,k})$ exists with $\varphi(x) = x'$. Let $M(x)$ (resp. $M'(x')$) be the hyperbolic structure defined by $x$ on $M$ (resp. by $x'$ on $M'$). Recall that $M(S)$ (resp. $M'(S')$) is isometric to the metric completion of $M(x)$ (resp. of
indeed homeomorphic to open discs because the loop $\partial D$ as above, with 

of these solid tori with $O$ we have $\text{vol}(O) = 0$. By Theorem 5.8, if $\Sigma$ (resp. $\Sigma'$) has the same length. Moreover, $M(x)$ and $M'(x')$ have the same volume, whence $\text{volume}(M(S)) = \text{volume}(M'(S'))$.

By Theorem 5.14 the bases of the cusps of $M(S)$ and $M'(S')$ are all isometric to regular hexagonal tori, so $M(S)$ and $M'(S')$ share the same cusp shape.

Consider now the shape of the geometric tetrahedra in the triangulations $\mathcal{T} = \{\Delta_1, \ldots, \Delta_{g+k}\}$, $\mathcal{T}' = \{\Delta'_1, \ldots, \Delta'_{g+k}\}$ of $M$, $M'$ respectively. Without loss of generality we can order the tetrahedra of these triangulations in such a way that $\Delta_i, \Delta'_i$ are asymptotic to the cusps of $M(S), M'(S')$ for $l = 2h + 1, \ldots, 2k$ (this is equivalent to requiring that the slopes in $S$ and $S'$ lie on $T_1, \ldots, T_h$ and $T'_1, \ldots, T'_h$). Then by Proposition 5.9 a real number $\vartheta(x) = \vartheta(x') \in (0, \pi/3]$ exists such that $\Delta_i$ and $\Delta'_i$ are isometric to $\Delta^{\vartheta(x)}$ for $l = 2h + 1, \ldots, 2k$. For $l = 2h + 1, \ldots, 2k$ let now $v_l, v'_l$ be the ideal vertices of $\Delta_i, \Delta'_i$ respectively. Due to Lemma 2.8 and the symmetric shape of $\Delta^{\vartheta(x)}$, up to increasing $C$ we can suppose that a unique horocusp neighbourhood $H_l$ of $v_l$ in $\Delta_i$ exists which is tangent to the truncation triangles of $\Delta_i$ and is entirely contained in $\Delta_i$. Moreover $H_{2i-1}$ and $H_{2i}$ glue up in $M(S)$ giving a horocusp neighbourhood $O_i$ of the $i$-th cusp for $i = h + 1, \ldots, k$. Also notice that the total horocusp neighbourhood $O_{h+1} \sqcup \ldots \sqcup O_k$ is regular (since the $H_l$'s are isometric to each other) and maximal (since each $O_i$ is tangent to the boundary of $M(S)$). The very same construction also leads to a horocusp neighbourhood $O'_{h+1} \sqcup \ldots \sqcup O'_k$ for $M'(S')$. Since $\Delta_i$ is isometric to $\Delta'_i$ for $l = 2h + 1, \ldots, 2k$, we have $\text{vol}(O_{h+1} \sqcup \ldots \sqcup O_k) = \text{vol}(O'_{h+1} \sqcup \ldots \sqcup O'_k)$, so $M(S)$ and $M'(S')$ share the same cusp volume.

Recall now that Thurston's hyperbolic Dehn filling Theorem ensures that if for all $l = 1, \ldots, h$ we have $L(s_l) > C'$ for some sufficiently large $C'$, then the shortest geodesics of $M(S)$ and $M'(S')$ are exactly the geodesics added to $M(x)$ and $M'(x')$. Thus under the hypothesis that $L(s_l) > C'$ for all $l = 1, \ldots, h$, in order to prove that the shortest geodesics of $M(S)$ and $M'(S')$ have the same length we only have to compute the complex length of these added geodesics. The desired result is then easily obtained from Proposition 2.13 and equations 33, 35.

The fact that $H_1(M(S); \mathbb{Z})$ is isomorphic to $H_1(M'(S'); \mathbb{Z})$ is an immediate consequence of Proposition 5.9. By Theorem 5.8 if $\Sigma$ (resp. $\Sigma'$) is the geodesic boundary of $M(S)$ (resp. of $M'(S')$), then both the Heegaard genus of $(M(S), \Sigma, \partial M(S) \setminus \Sigma)$ and the Heegaard genus of $(M'(S'), \Sigma', \partial M'(S') \setminus \Sigma')$ are equal to $g + 1$.

In order to prove our statement about Turaev-Viro invariants we need to construct special spines for $M(S)$ and $M'(S')$. Let $P \subset M$ be the special spine of $M$ dual to the canonical decomposition $\mathcal{T}$ of $M$, and recall that for $j = 1, \ldots, k$ a hexagonal face $E_j$ of $P$ exists which is parallel to the $j$-th boundary torus of $M$. Let $E_1, \ldots, E_h$ be the faces corresponding to the filled tori in $M(S)$ and for $l = 1, \ldots, h$ let $m_l$ be a loop on $E_l$ which represents the slope $s_l \in S$ and is in general position with respect to the singular locus $S(P)$ of $P$. The complement of $P \subset M \subset M(S)$ inside $M(S)$ consists of the disjoint union of an open collar of $\partial M(S)$ and $h$ open solid tori. Take meridional discs $D_1, \ldots, D_h$ of these solid tori with $\partial D_l = m_l$ for $l = 1, \ldots, h$. The complement of $P \cup D_1 \sqcup \ldots \sqcup D_h$ is as above, with $h$ open balls instead of the $h$ open solid tori. Fix now $l \in \{1, \ldots, h\}$. The loop $\partial D_l \subset E_l$ cuts $E_l$ into several discal open faces of $P \cup D_1 \sqcup \ldots \sqcup D_h$ (these faces are indeed homeomorphic to open discs because the loop $m_l$ is sufficiently complicated with
respect to the graph \( S(P) \cap \overline{E}_t \). Each such face separates \( \Sigma_g \subset \partial M(S) \) from the open ball corresponding to the \( l \)-th added solid torus, so, if we remove from \( P \cup D_1 \cup \ldots \cup D_h \) a face for each added solid torus we end up with a special spine \( P(S) \) of \( M(S) \). The very same procedure also provides a special spine \( P'(S') \) for \( M'(S') \).

Let \( T(S), T'(S') \) be the triangulations of \( M(S), M'(S') \) dual to \( P(S), P'(S') \) respectively. It is not difficult to show that since \( S \) is equivalent to \( S' \), the loops representing the slopes in \( S \) and in \( S' \) can be chosen so that the incidence numbers between edges and tetrahedra are the same for \( T(S) \) and for \( T'(S') \). As pointed out in [MN94], this implies that \( M(S) \) and \( M'(S') \) share the same Turaev-Viro invariants.

**Remark 5.15.** Let \( M, M' \) be elements of \( \mathcal{M}_{g,k} \) and suppose that \( s, s' \) are sufficiently long slopes on the tori \( T \subset \partial M, T' \subset \partial M' \). If there exists an orientation-reversing isometry of \( T \) onto \( T' \) taking \( s \) into \( s' \), then the complex length of the added geodesic in \( M(s) \) is equal to the conjugate of the complex length of the added geodesic in \( M'(s') \).

5.9. **Non-homeomorphic fillings.** This paragraph is devoted to the proof of Theorem 1.8. Let \( P_k \) be the special polyhedron whose 1-skeleton has the regular neighbourhood described in Fig. 8. It is easily seen that \( P_k \) is the spine of a manifold \( X_k \). Computing the boundary of \( X_k \) as explained in [BP95] one can easily prove that \( X_k \in \mathcal{M}_{k+1,k} \) if \( k \) is odd and \( X_k \in \mathcal{M}_{k+2,k} \) if \( k \) is even.

**Proposition 5.16.** For all \( k \geq 1 \), the manifold \( X_k \) admits no non-trivial isometries.

**Proof:** Let \( \mathcal{T}_k \) be the triangulation of \( X_k \) dual to \( P_k \). Since \( \mathcal{T}_k \) is the Kojima decomposition of \( X_k \), the group of isometries of \( X_k \) is canonically isomorphic to the group \( \text{Aut}(\mathcal{T}_k) \) of the combinatorial automorphisms of \( \mathcal{T}_k \). Now a straightforward analysis of the combinatorics of \( \mathcal{T}_k \) shows that \( \text{Aut}(\mathcal{T}_k) \) is trivial, whence the conclusion. \( \square \)

**Proposition 5.17.** Let \( X \in \mathcal{M}_{g,k} \) with boundary tori \( T_1, \ldots, T_k \) and suppose that \( X \) admits no non-trivial isometry. For each \( i = 1, \ldots, k \) we can choose a finite set \( S_i \) of slopes on \( T_i \) with the following property. Let \( S \) be a set of slopes for \( X \) whose elements do not belong to \( S_i \), \( i = 1, \ldots, k \) and let \( h = \#S \leq k \). Then the number of sets of slopes equivalent to \( S \) is greater than or equal to \((k! \cdot 3^h)/(h! \cdot (k-h)!))\). Moreover, if \( S' \) is a set of slopes equivalent to \( S \) and \( X(S) \) is homeomorphic to \( X(S') \), then \( S = S' \).

**Proof:** Thurston’s hyperbolic Dehn filling Theorem and Theorem 5.14 imply that we can choose the finite set \( S_i \) in such a way that if \( S \) is as in the statement and \( S' \) is a set of slopes equivalent to \( S \), then the following conditions hold: no slope in \( S' \) is contained in some \( S_i \); \( X(S), X(S') \) are geometrically similar hyperbolic 3-manifolds; the cores of the added solid tori give the \( h \) shortest geodesics both of \( X(S) \) and of \( X(S') \).

An elementary combinatorial argument shows that the number of sets of slopes equivalent to \( S \) is at least \((k! \cdot 3^h)/(h! \cdot (k-h)!))\).
Figure 8: The regular neighbourhood of the 1-skeleton $S(P_k)$ of $P_k$. Each pair of vertices joined by three edges in $S(P_k)$ gives rise to a toric cusp in $X_k$. 
Suppose now that \( S' \) is equivalent to \( S \) and let \( \psi : X(S) \to X(S') \) be a homeomorphism. By Mostow-Prasad’s rigidity Theorem, \( \psi \) is homotopic to an isometry \( \psi' \), which must take the added geodesics of \( X(S) \) to the added geodesics of \( X(S') \). This gives in turn a homeomorphism of \( X \) onto itself taking \( S \) onto \( S' \). By rigidity again, up to homotopy such a homeomorphism restricts to an isometry of \( X \), whence \( S = S' \) since \( X \) admits no non-trivial isometry. \( \square \)

6. Commensurability of similar Dehn fillings

Let \( M, M' \) be elements in \( \mathcal{M}_{g,k} \) with canonical decompositions \( T, T' \) respectively. Let \( N \) (resp. \( N' \)) be a hyperbolic manifold obtained by Dehn filling \( M \) (resp. \( M' \)) along sufficiently complicated slopes, and let \( x \in \Omega_{g,k} \) (resp. \( x' \in \Omega_{g,k} \)) be such that \( N \cong \tilde{M}(x), N' \cong \tilde{M'}(x') \). In this paragraph we describe an explicit criterion which allows us to determine if \( N \) is commensurable with \( N' \) just by looking at \( x, x' \) and at the combinatorics of \( T, T' \).

**Definition 6.1.** Two complete hyperbolic \( n \)-manifolds with geodesic boundary \( M_1, M_2 \) are commensurable if a hyperbolic manifold with geodesic boundary \( M_3 \) exists which is the total space of a finite Riemannian covering both of \( M_1 \) and of \( M_2 \).

**Proposition 6.2.** Let \( N_1, N_2 \) be complete finite-volume hyperbolic \( n \)-manifolds with non-empty geodesic boundary and denote by \( \tilde{N}_1, \tilde{N}_2 \) the universal coverings of \( N_1, N_2 \) respectively. Then \( N_1 \) is commensurable with \( N_2 \) if and only if \( \tilde{N}_1 \) is isometric to \( \tilde{N}_2 \).

**Proof:** See [Fria]. \( \square \)

From now on, let \( k \) be a fixed odd natural number and let \( X_k \) be the manifold defined in Subsection 5.9.

Let \( \Delta_1, \ldots, \Delta_{2k+1} \) be the partially truncated tetrahedra of the canonical decomposition \( T_k \) of \( X_k \), and suppose as usual that for \( i = 1, \ldots, k \) the tetrahedra \( \Delta_{2i-1}, \Delta_{2i} \) are non-compact and glue up to a neighbourhood of the \( i \)-th cusp of \( X_k \), while \( \Delta_{2k+1} \) is compact regular. We denote by \( F^i_{2i-1}, F^i_{2i-1}, F^i_{2i}, F^i_{2i+1}, F^i_{2i+1}, F^i_{2i+1} \) the exceptional hexagons of \( \Delta_{2i-1}, \Delta_{2i} \), in such a way that \( F^i_{2i-1} \) is glued to \( F^i_{2i} \) for \( i = 1, \ldots, k, j = 1, 2, 3 \). For \( l = 1, \ldots, 2k \) we also call \( e^i_l \) the only finite edge of \( F^i_l \), and \( f^i_l \) the edge of \( \Delta_l \) opposite to \( e^i_l \). We emphasize that here we do not require that \( F^1_l, F^2_l, F^3_l \) are positively arranged around the ideal vertex of \( \Delta_l \). Recall that a point \( x \in \Omega_{k+1,k} \) determines the geometric realization of \( T_k \) with dihedral angle \( x_{6l-6+j} \) along \( e^i_l \), angle \( x_{6l-3+j} \) along \( f^i_l \), and angle \( x_{12k+1} \) along the compact edges of the unique compact tetrahedron. It is easily seen that the exceptional lateral hexagons of the non-compact tetrahedra can be ordered around the ideal vertices in such a way that the following condition holds:

- For \( i = 1, \ldots, k-1, j = 1, 2, 3 \) the isometry which glues the compact face of \( \Delta_{2i} \) to the compact face of \( \Delta_{2i+1} \) sends \( e_{2i} \) to \( e_{2i+1} \). Moreover, if \( i \) is odd (resp. even) then \( F^1_{2i-1}, F^2_{2i-1}, F^3_{2i-1} \) and \( F^1_{2i}, F^2_{2i}, F^3_{2i} \) are positively (resp. negatively) arranged around the ideal vertices of \( \Delta_{2i-1} \) and \( \Delta_{2i} \).

(The fact that these conditions are coherent with each other depends on the combinatorial properties of \( T_k \). The second condition will be taken into account when we will explicitly consider the action of \( \text{Sym}(\Omega_{k+1,k}) \) on \( \Omega_{k+1,k} \). Let \( l \subset X_k \) be the compact edge of \( T_k \). A straight-forward analysis of the combinatorics of \( T_k \) shows that the dihedral angles of the
tetrahedra of \( T_k \) are arranged along \( l \) according to the following cyclic ordering:

- \( x_{12k+1}, x_7, x_{13}, \ldots, x_{6l+1}, \ldots, x_{12k-5}, \)
- \( x_{12k+1}, x_{12k+1}, x_8, x_{14}, \ldots, x_{6l+2}, \ldots, x_{12k-4}, \)
- \( x_{12k+1}, x_{12k+1}, x_{12k+1}, x_9, x_{15}, \ldots, x_{6l+3}, \ldots, x_{12k-3}. \)

For \( i = 1, \ldots, k \) let \( a_i, b_i, c_i : \Omega_{k+1,k} \to \mathbb{R} \) be the functions defined as follows:

\[
\begin{align*}
  a_i(x) &= x_{12(j-1)+1} + x_{12(j-1)+7}, \\
  b_i(x) &= x_{12(j-1)+2} + x_{12(j-1)+8}, \\
  c_i(x) &= x_{12(j-1)+3} + x_{12(j-1)+9},
\end{align*}
\]

and set

\[
a, b, c : \Omega_{k+1,k} \to \mathbb{R}, \quad a(x) = \sum_{i=1}^{k} a_i(x), \quad b(x) = \sum_{i=1}^{k} b_i(x), \quad c(x) = \sum_{i=1}^{k} c_i(x).
\]

For \( x \in \Omega_{k+1,k} \) we denote by \( X_k(x) \) the hyperbolic structure defined on \( X_k \) by \( x \), and by \( \hat{X}_k(x) \) the metric completion of \( X_k(x) \). Let \( V'' \) be a neighbourhood of \( x_0 \) in \( \Omega_{k+1,k} \) as in the statement of Proposition 5.13 and for \( x \in V'' \cap I \Omega_{k+1,k} \) let us denote by \( U(x) \) the universal covering of \( \hat{X}_k(x) \). We now show that the real numbers \( a(x), b(x), c(x) \) completely determine the isometry type of the universal covering \( U(x) \) of \( \hat{M}(x) \), whence the commensurability class of \( \hat{M}(x) \).

**Proposition 6.3.** Let \( x, x' \) be points in \( I \Omega_{k+1,k} \cap V'' \). Then \( \hat{X}_k(x) \) is commensurable with \( \hat{X}_k(x') \) if and only if \( a(x) = a(x') \), \( b(x) = b(x') \), \( c(x) = c(x') \).

**Proof:** Let \( L^x(l_x) \) be the minimal distance between different connected components of \( \partial U(x) \), and let \( t \subset U(x) \) be a geodesic arc of length \( L^x(l_x) \) joining two such components \( S_1, S_2 \). By Corollary 5.12 if \( t' \subset U(x) \) is any other geodesic arc of length \( L^x(l_x) \) connecting different components of \( \partial U(x) \), then there exists an isometry of \( U(x) \) taking \( t \) to \( t' \). Let us consider the set \( R \subset U(x) \) given by the union of all the compact regular truncated tetrahedra whose truncation triangles lie on \( S_1 \cup S_2 \cup S' \cup S'' \) for some connected components \( S', S'' \) of \( \partial U(x) \). Let \( N_\epsilon(t) \) be the \( \epsilon \)-neighbourhood of \( t \), and consider the sets \( A = N_\epsilon(t) \cap R \) and \( B = N_\epsilon(t) \setminus R \). Both \( A \) and \( B \) are unions of germs of dihedral sectors whose number, amplitude and cyclic order (up to the choice of a positive orientation around \( t \)) only depend on the isometry type of \( U(x) \). We will call such sectors \( A \)-sectors or \( B \)-sectors, according to the fact that they are contained in \( A \) or \( B \). Lemma 5.11 implies that \( t \) is a lift in \( U(x) \) of the unique compact edge of the geometric triangulation of \( X_k(x) \), while by Proposition 5.13 the set \( R \) is the union of the lifts containing \( t \) of the geometric tetrahedron \( \Delta_{12k+1} \subset X_k(x) \). Thus \( A \)-sectors are in number of three and have angles \( x_{12k+1}, 2x_{12k+1} \) and \( 3x_{12k+1} \). Moreover, the \( B \)-sector between the \( A \)-sectors with angles \( x_{12k+1}, 2x_{12k+1} \) has angle \( a(x) \); the \( B \)-sector between the \( A \)-sectors with angles \( 2x_{12k+1}, 3x_{12k+1} \) has angle \( b(x) \); the \( B \)-sector between the \( A \)-sectors with angles \( 3x_{12k+1}, x_{12k+1} \) has angle \( c(x) \). This shows that \( a(x), b(x), c(x) \) can be recovered solely from the isometry type of \( U(x) \), so if \( \hat{X}_k(x) \) is commensurable with \( \hat{X}_k(x') \) we have \( a(x) = a(x'), b(x) = b(x'), c(x) = c(x') \).

Suppose now that \( a(x) = a(x'), b(x) = b(x'), c(x) = c(x') \). Since \( a(x) + b(x) + c(x) + 6x_{12k+1} = a(x') + b(x') + c(x') + 6x_{12k+1} \) we have \( x_{12k+1} = x'_{12k+1} \), so the compact tetrahedron in the decomposition of \( X_k(x) \) is isometric to the compact tetrahedron in the decomposition of \( X_k(x') \). Let now \( U_k(x) \) (resp. \( U_k(x') \)) be the complement in \( U(x) \) (resp. in \( U(x') \)) of the preimage of the added geodesics \( \hat{X}_k(x) \setminus X_k(x) \) (resp. \( \hat{X}_k(x') \setminus X_k(x') \)). The
Let now \( h \) induce a complete metric on the last \( h \) smooth manifold of dimension 2.

We recall that in a neighbourhood of \( x \) and \( x \) commensurable with each other, we are now reduced to underst and when the functions \( \psi \) are commensurable with each other.

\( \psi_K \) (resp. \( \partial K \) \( \psi \)) it is easily seen that an element \( \psi \in \text{Isom}(\mathbb{H}^3) \) exists which takes \( K(x) \subset U(x) \subset \mathbb{H}^3 \) onto \( K(x') \subset U(x') \subset \mathbb{H}^3 \). Since any component of \( \partial U(x) \) (resp. \( \partial K(x') \)) in a non-empty open subset of a hyperbolic plane, this readily implies that \( \psi(\partial U(x)) = \psi(\partial U(x')) \). Now \( U(x), U(x') \) are the hyperbolic convex hulls of \( \partial U(x), \partial U(x') \) respectively, so \( \psi(U(x)) = U(x') \). By Proposition 5.22 this implies that \( \tilde{X}_k(x) \) and \( \tilde{X}_k(x') \) are commensurable with each other.

In order to determine if geometrically similar manifolds obtained by Dehn filling \( X_k \) are commensurable with each other, we are now reduced to understand when the functions \( a, b \) and \( c \) introduced above take different values on \( \text{Sym}(\Omega_{k+1,1}) \)-equivalent points in \( \Omega_{k+1,1} \).

Let us set

\[ H_h = \{ x \in \mathbb{R}^{12k+1} : x_{12i+1} = x_{12i+2} = x_{12i+3} \text{ for all } i = h - 1, \ldots, k - 1 \}. \]

We recall that in a neighbourhood of \( x_0 \) in \( \Omega_{k+1,1} \) the set \( \Omega^h_{k+1,1} := H_h \cap \Omega_{k+1,1} \) is a smooth manifold of dimension \( 2h \) whose points correspond to those structures which induce a complete metric on the last \( h \) cusps of \( X_k \) (see Lemma 3.3 and Proposition 3.7).

Let now \( \tau_h : (\epsilon, \varepsilon) \to \Omega^h_{k+1,1} \) be the curve mentioned in Remark 4.7. For a smooth \( f : \Omega_{k+1,1} \to \mathbb{R} \) let us denote by \( \tilde{f} \) (resp. \( \check{f} \)) the first (resp. second) derivative of \( f \circ \tau_h \) at 0. From Proposition 4.9 we deduce:

\[ \dot{a}_i = \dot{b}_i = \dot{c}_i = 0 \text{ for all } i = 1, \ldots, k; \]
\[ \ddot{a}_i > \ddot{b}_i > \ddot{c}_i \text{ for all } i = 1, \ldots, h; \]
\[ \dddot{a}_i = \dddot{b}_i = \dddot{c}_i = 0 \text{ for all } i = h + 1, \ldots, k. \]

We are now ready to prove the following:

**Theorem 6.4.** Fix \( 1 \leq h \leq k \), where \( k \) is odd. Then there exists a sequence \( \{W^n_h\}_{n \in \mathbb{N}} \) of pairwise non-homeomorphic complete finite-volume hyperbolic manifolds with geodesic boundary with the following properties:

- Each \( W^n_h \) is obtained by Dehn filling the first \( h \) cusps of \( X_k \);
- For any \( n \in \mathbb{N} \) there exist at least three (including \( W^n_h \) itself) pairwise non-commensurable hyperbolic Dehn fillings of \( X_k \) which are geometrically similar to \( W^n_h \).

**Proof:** We choose an infinite sequence \( \{y_n\}_{n \in \mathbb{N}} \subset \Omega^h_{k+1,1} \setminus \{x_0\} \) converging to \( x_0 \) along \( \tilde{\tau}_h(0) \) (see Definition 4.12), and we set \( W_n^h = \tilde{X}_k(y_n) \).

We first observe that \( W_n^h \) is obtained from \( X_k \) by Dehn filling the first \( h \) cusps of \( X_k \): since \( y_n \) belongs to \( \Omega^h_{k+1,1} \), the last \( k - h \) cusps of \( X_k(y_n) \) have to be complete; moreover, up to extracting a subsequence we can suppose \( a_i(y_n) > b_i(y_n) > c_i(y_n) \) for all \( i = 1, \ldots, h \), so the angles along the compact edges of \( \Delta_{2i-1}, \Delta_{2i} \) are not equal to each other, and the \( i \)-th cusp of \( X_k(y_n) \) is not complete.

Let now \( r \in \mathcal{M}(T_1 \cup \ldots \cup T_k) \) be the element acting as a positive (resp. negative) rotation by an angle of \( \pi/3 \) on \( T_i \) for \( i \in \{1, \ldots, k\} \), \( i \) odd (resp. even), and let \( \Theta : \mathcal{M}(T_1 \cup \ldots \cup T_k) \to \text{Sym}(\Omega_{k+1,1}) \) be the isomorphism described in Proposition 5.10. We set \( y'_n = \Theta(r)(y_n) \).
and \( y'_n = \Theta(r^2)(y_n) \). By construction, \( W^n_h = \tilde{X}_k(y_n), \tilde{X}_k(y'_n) \) and \( \tilde{X}_k(y''_n) \) are pairwise geometrically similar. Moreover an easy computation shows that for \( x \in \Omega_{k+1,k} \) we have
\[
\begin{align*}
a_i(\Theta(r^2)(x)) &= c_i(\Theta(r)(x)) = b_i(x), \\
b_i(\Theta(r^2)(x)) &= a_i(\Theta(r)(x)) = c_i(x), \\
c_i(\Theta(r^2)(x)) &= b_i(\Theta(r)(x)) = a_i(x),
\end{align*}
\]
whence \( a_i(y_n) > a_i(y'_n) > a_i(y''_n) \), and \( W^n_h = \tilde{X}_k(y_n), \tilde{X}_k(y'_n), \tilde{X}_k(y''_n) \) are pairwise non-commensurable by Proposition 6.3.

**Remark 6.5.** Let \( M \) be an element of \( \mathcal{M}_{g,k} \) with canonical decomposition \( \mathcal{T} \). Suppose that the arrangement of compact and non-compact tetrahedra around the compact edge of \( \mathcal{T} \) is sufficiently irregular and let \( S = \{ s_1, \ldots, s_h \} \) be a set of slopes for \( M \) such that \( s_i \) is not equivalent to \( s_m \) for \( i \neq m \). The same argument used to prove Theorem 6.4 shows that the Dehn fillings of \( M \) which are geometrically similar to \( M(S) \) are expected to be non-commensurable with each other.

We conclude with some examples of non-homeomorphic geometrically similar commensurables Dehn fillings of \( X_k \).

**Theorem 6.6.** Let \( k \geq 3 \) be odd. Then there exists an infinite sequence of pairs \( \{ Y_1^n, Y_2^n \}_{n \in \mathbb{N}} \) of complete finite-volume hyperbolic manifolds with geodesic boundary such that for every \( n \in \mathbb{N} \) the following conditions hold:
- \( Y_1^n \) is obtained by Dehn filling the first cusp of \( X_k \);
- \( Y_2^n \) is obtained by Dehn filling the third cusp of \( X_k \);
- \( Y_1^n \) is geometrically similar to \( Y_2^n \);
- \( Y_1^n \) is commensurable with \( Y_2^n \);
- \( Y_1^n \) is not homeomorphic to \( Y_2^n \).

**Proof:** We choose an infinite sequence \( \{ y_n \}_{n \in \mathbb{N}} \subseteq IO_{k+1,k}^1 \setminus \{ x_0 \} \) converging to \( x_0 \) along \( \tilde{\xi}(0) \). Let \( \tau_{13} \in \text{Sym}(\Omega_{k+1,k}) \) be the element which exchanges the first cusp of \( X_k \) with the third one according to equation (20) and let \( y'_n = \tau_{13}(y_n) \). We set \( Y_1^n = \tilde{X}_k(y_n) \) and \( Y_2^n = \tilde{X}_k(y'_n) \). It is easily seen that \( Y_1^n \) is obtained by filling the first cusp of \( X_k \), while \( Y_2^n \) is obtained by filling the third one. The element of \( \mathcal{M}(T_1 \sqcup \ldots \sqcup T_k) \) corresponding to \( \tau_{13} \) is orientation-preserving, so \( Y_1^n \) is geometrically similar to \( Y_2^n \). Moreover an easy computation shows that for every \( x \in \Omega_{k+1,k} \) we have
\[
\begin{align*}
a_1(\tau_{13}(x)) &= a_3(x), & b_1(\tau_{13}(x)) &= b_3(x), & c_1(\tau_{13}(x)) &= c_3(x), \\
a_3(\tau_{13}(x)) &= a_1(x), & b_3(\tau_{13}(x)) &= b_1(x), & c_3(\tau_{13}(x)) &= c_1(x), \\
a_j(\tau_{13}(x)) &= a_j(x), & b_j(\tau_{13}(x)) &= b_j(x), & c_j(\tau_{13}(x)) &= c_j(x), & j = 2, 4, 5, 6, \ldots, k.
\end{align*}
\]
This easily implies \( a(y_n) = a(y'_n), b(y_n) = b(y'_n), c(y_n) = c(y'_n) \), so \( Y_1^n \) is commensurable with \( Y_2^n \) by Proposition 6.3.

Let us prove that \( Y_1^n \) is not homeomorphic to \( Y_2^n \). Up to passing to a subsequence we can suppose that the added geodesic \( Y_1^n \setminus X_k \) (resp. \( Y_2^n \setminus X_k \)) is the shortest geodesic of \( Y_1^n \) (resp. \( Y_2^n \)). Let \( f_n : Y_1^n \to Y_2^n \) be a homeomorphism. By Mostow-Prasad’s rigidity theorem we may assume that \( f_n \) is an isometry, which implies \( f_n(Y_1^n \setminus X_k) = Y_2^n \setminus X_k \). Thus \( f_n \) restricts to a homeomorphism \( f'_n : X_k \to X_k \). By rigidity again we can homotope \( f'_n \) into an isometry, which by construction should take the first cusp of \( X_k \) onto the third one, against Proposition 6.6.

\qed
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Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa, Italy
E-mail address: frigerio@mail.dm.unipi.it