MILNOR NUMBERS AND EULER OBSTRUCTION

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ABSTRACT. Using a geometric approach, we determine the relations between the local Euler obstruction \( Eu_f \) of a holomorphic function \( f \) and several generalizations of the Milnor number for functions on singular spaces.

1. Introduction

In the case of a nonsingular germ \((X, x_0)\) and a function \( f \) with an isolated critical point at \( x_0 \), the following three invariants coincide (for (c), up to sign):

(a) the Milnor number of \( f \) at \( x_0 \), denoted \( \mu(f) \);
(b) the number of Morse points in a Morsification of \( f \);
(c) the Poincaré-Hopf index of \( \text{grad} f \) at \( x_0 \);

This fact is essentially due to Milnor’s work in the late sixties [Mi]. There exist extensions of all these invariants to the case when \((X, x_0)\) is a singular germ, but they do not coincide in general. One of the extensions of (c) is the Euler obstruction of \( f \) at \( x_0 \), denoted \( Eu_f(X, x_0) \). This was introduced in [BMPS]; roughly, it is the obstruction to extending the conjugate of the gradient of the function \( f \) as a section of the Nash bundle of \((X, x_0)\). It measures how far the local Euler obstruction is from satisfying the local Euler condition with respect to \( f \) in bivariant theory. It is then natural to compare \( Eu_f(X, x_0) \) to the Milnor number of \( f \) in the case of a singular germ \((X, x_0)\). This has been also a question raised in [BMPS].

The main idea of this paper is that, for singular \( X \), the Euler obstruction \( Eu_f(X, x_0) \) is most closely related to (b). We use the homological version of the bouquet theorem for the Milnor fiber given in [Ti], which relates the contributions in the bouquet to the number of Morse points. Through this relation, one may compare \( Eu_f(X, x_0) \) to the highest Betti number of the Milnor fiber of \( f \). In case \( X \) has Milnor’s property, the comparison is optimal and yields a general inequality, see §3.1. We further compare \( Eu_f(X, x_0) \) with two different generalizations of the Milnor number for functions with isolated singularity on singular spaces, one due to [Le3], the other to [Go, MS] for curve singularities and to [IS] for functions on isolated complete intersection germs in general. In case when the
germ \((X, x_0)\) is an isolated complete intersection singularity, we use in addition the GSV-index of vector fields \[\text{GSV}\] to completely determine the relations between \(\text{Eu}_f(X, x_0)\), the Milnor number of \(f\) and the GSV-index attached to \(f\).

2. Euler obstruction and Morsification of functions

Let \((X, x_0)\) denote the germ at some point \(x_0\) of a reduced pure dimensional complex analytic space embedded in \(\mathbb{C}^N\), for some \(N\). Consider a Whitney stratification \(W\) of some representative of \(X\). Let \(W_0\) be the stratum containing \(x_0\) and let \(W_i\ldots W_q\) be the finitely many strata of \(X\) having \(x_0\) in their closure, other than \(W_0\). Let also \(F : (\mathbb{C}^N, x_0) \to (\mathbb{C}, 0)\) denote some extension of \(f\).

**Definition 2.1.** (Lazzeri '73, Benedetti '77, Pignoni '79, Goresky-MacPherson '83 \[GM, p.52\].) One says that \(f : (X, x_0) \to \mathbb{C}\) is a general function germ if \(dF_{x_0}\) does not vanish on any limit of tangent spaces to \(W_i\), \(\forall i \neq 0\), and to \(W_0 \setminus \{x_0\}\). One says that \(f : (X, x_0) \to \mathbb{C}\) is a stratified Morse function germ if: \(\dim W_0 \geq 1\), \(f\) is general with respect to the strata \(W_i\), \(i \neq 0\) and the restriction \(f_{|W_0} : W_0 \to \mathbb{C}\) has a Morse point at \(x_0\).

Let us recall some definitions and notations from \[BMPS\]. The complex conjugate of \(X\), \(\bar{X}\) is the usual tangent bundle of \(X\), and \(X\) is a holomorphic function with isolated singularity at \(x_0\). If \(\nu : \bar{X} \to X\) is the Nash blow-up of \(X\) and \(\bar{T}\) is the Nash bundle over \(\bar{X}\), then \(\nu^{-1}(x) \cap S_{\epsilon}\) is a small enough sphere around \(x_0\), given by Milnor’s result \[Mi, \text{Cor. 2.8}\]. Following \[BMPS\], the obstruction to extend \(\nu^{-1}(X \cap S_{\epsilon})\) is denoted by \(\text{Eu}_f(X, x_0)\) and is called the local Euler obstruction of \(f\).

**Example 2.2.** If the germ \((X, x_0)\) is nonsingular, then its Nash blow-up can be identified to \(X\) itself, the Nash bundle is the usual tangent bundle of \(X\) and \(\text{Eu}_f(X, x_0)\) is, by definition, the Poincaré-Hopf index of \(\text{grad}_X f\) at \(x_0\). From \[Mi, \text{Th.7.2}\] one deduces: \(\text{Eu}_f(X, x_0) = (-1)^{\dim X} \mu\), where \(\mu\) is the Milnor number of \(f\). It is also easy to prove (see \[BLS \text{ and BMPS}\]) that if \((X, x_0)\) is any singular space but \(f\) is a general function germ at \(x_0\), then the obstruction \(\text{Eu}_f(X, x_0)\) is zero.

We claim that a natural way to study \(\text{Eu}_f\) is to split it according to a Morsification of \(f\). We prove the following general formula for holomorphic germs with isolated singularity:

**Proposition 2.3.** Let \(f : (X, x_0) \to (\mathbb{C}, 0)\) be a holomorphic function with isolated singularity at \(x_0\). Then

\[\text{Eu}_f(X, x_0) = (-1)^{\dim C} \alpha_q,\]

where \(\alpha_q\) is the number of Morse points on \(W_q = X_{\text{reg}}\) in a generic deformation of \(f\).
Proof. We Morsify the function \( f \), i.e. we consider a small analytic deformation \( f_\lambda \) of \( f \) such that \( f_\lambda \) only has stratified Morse points within the ball \( B \) and it is general in a small neighborhood of \( x_0 \). (See, for instance, the Morsification Theorem 2.2 in \[Lé2\].)

Since \( f_\lambda \) is a deformation of \( f \), it follows that \( \text{grad}_X f \) is homotopic to \( \text{grad}_X f_\lambda \) over the sphere \( X \cap \partial B \), so the obstructions to extend their lifts to \( \nu^{-1}(X \cap B) \) without zeros are equal.

On the other hand, the obstruction corresponding to \( \text{grad}_X f_\lambda \) is also equal to the sum of local obstructions due to the Morse points of \( f_\lambda \). Lemma 4.1 of \[STV\] shows that the local obstruction at a stratified Morse point is zero if the point lies in a lower dimensional stratum. So the points that only count are the Morse points on the stratum \( X_{\text{reg}} \) and, at such a point, the obstruction is \((-1)^{\dim C_X} \), as explain above in Example 2.2. □

Remark 2.4. The Euler obstruction is defined via the Nash blow-up and the latter only takes into account the closure of the tangent bundle over the regular part \( X_{\text{reg}} \). Since the other strata are not counting in the Nash blow-up, it is natural that they do not count for \( \text{Eu}_f(X,x_0) \) neither. The number \( \alpha_q \) does not depend on the chosen Morsification, by a trivial connectedness argument. We refer to \[STV\] for more about \( \alpha_q \) and other invariants of this type, which enter in a formula for the global Euler obstruction of an affine variety \( Y \subset \mathbb{C}^N \).

Remark 2.5. The number \( \alpha_q \) may be interpreted as the intersection number within \( T^*\mathbb{C}^N \) between \( dF \) and the conormal \( T^{*X}_{X_{\text{reg}}} \). Therefore our Proposition 2.3 may be compared to \[BMPS, Corollary 5.4\], which is proved by using different methods. J. Schürmann informed us that such a result can also be obtained using the techniques of \[Sch\].

3. Milnor numbers

3.1. Lé’s Milnor number. Lé D.T. \[Lé3\] proved that for a function \( f \) with an isolated singularity at \( x_0 \in X \) (in the stratified sense) one has a Milnor fibration. He pointed out that, under certain conditions, the space \( X \) has “Milnor’s property” in homology (which means that the reduced homology of the Milnor fiber of \( f \) is concentrated in dimension \( \dim X - 1 \)). Then the Milnor number \( \mu(f) \) is well defined as the rank of this homology group. By Lé’s results \[Lé3\], Milnor’s property is satisfied for instance if \( (X,x_0) \) is a complete intersection (not necessarily isolated!) or, more generally, if \( \text{rHd} (X,x_0) \geq \dim (X,x_0) \), where \( \text{rHd} (X,x_0) \) denotes the rectified homology depth of \( (X,x_0) \), see \[Lé3\] for its definition originating in Grothendieck’s work.

To compare \( \mu(f) \) with \( \text{Eu}_f(X,x_0) \) we use the general bouquet theorem for the Milnor fiber in its homological version. Let \( M_f \) and \( M_l \) denote the Milnor fiber of \( f \) and of a general function \( l \). Let \( f : (X,x_0) \to (\mathbb{C},0) \) be a function with stratified isolated singularity and let \( \Lambda \) be the set of stratified Morse points in some chosen Morsification of \( f \) (by convention \( x_0 \notin \Lambda \)). Then by \[11\], pp.228-229 and Bouquet Theorem] we have:

\[
\tilde{H}_*(M_f) \cong \tilde{H}_*(M_l) \oplus \oplus_{i \in \Lambda} H_{*+k_i+1}(C(F_i), F_i)
\]

where, for \( a_i \in \Lambda \), \( F_i \) denotes the complex link of the stratum to which \( a_i \) belongs, \( k_i \) is the dimension of this stratum and \( C(F_i) \) denotes the cone over \( F_i \).
In particular, if the germ \((X, x_0)\) is a complete intersection (more generally, if \(r\text{Hd} (X, x_0) \geq \dim(X, x_0)\)), then:
\[
\mu(f) = \mu(l) + \sum_{i \in \Lambda} \mu_i,
\]
where \(\mu_i := \text{rank } H_{\dim X - k_i}(C(F_i), F_i)\). This result shows that the Milnor number \(\mu(f)\) gathers information from all stratified Morse points, whereas \(\text{Eu}_f(X, x_0)\) is, up to sign, the number \(\alpha_0 = \# \Lambda_0\), where \(\Lambda_0\) denotes the set of Morse points occurring on \(X_{\text{reg}}\) (see Proposition \(\text{[28]}\) above). Notice that we have \(\Lambda_0 \subset \Lambda\), \(\mu(l) \geq 0\), \(\mu_i = 1\) if \(i \in \Lambda_0\) and \(\mu_i \geq 0\) if \(i \in \Lambda \setminus \Lambda_0\). We therefore get the general inequality, whenever the space \(X\) has Milnor’s property (e.g. when \((X, x_0)\) is a complete intersection, not necessarily with isolated singularities), and therefore the Milnor-Lê number is well defined:
\[
\mu(f) \geq (-1)^{\dim X} \text{Eu}_f(X, x_0).
\]

In case \((X, x_0)\) is an isolated complete intersection singularity (ICIS for short), from the above discussion on \([\text{1}]\) we get the equality:
\[
\text{Eu}_f(X, x_0) = (-1)^{\dim X} [\mu(f) - \mu(l)].
\]

In the ICIS case, \([\text{3}]\) also shows that the inequality \([\text{2}]\) is strict whenever \(X\) is actually singular. This is so since \(\mu(l) > 0\), which can be proved inductively using Looijenga’s results \([\text{10}]\).

3.2. Another Milnor number. A different generalization of the Milnor number is due to V. Goryunov \([\text{Go}]\), D. Mond and D. van Straten \([\text{MS}]\). This is originally defined for functions on curve singularities \(X \subset \mathbb{C}^N\), and we refer to \([\text{MS}, \text{p.178}]\) for the precise definition. This number is preserved under simultaneous deformations of both the space \(X\) and the function \(f\). Thus, if the curve singularity \((X, x_0)\) is an ICIS, defined by some application \(g: (\mathbb{C}^N, x_0) \to (\mathbb{C}^p, 0)\) on an open set in \(\mathbb{C}^N\), and \(F\) is an extension of \(f\) to the ambient space, then \(\mu_G(f)\) counts the number of critical points (with their multiplicities) of the restriction of \(F\) to a Milnor fiber of \(g\), say \(X_t = g^{-1}(t)\) for some regular value \(t\) of \(g\). This is equivalent to saying that \(\mu_G(f)\) is the Poincaré-Hopf index of the gradient of the restriction \(F|_{X_t}\). In other words, this is saying that \(\mu_G(f)\) is the \(GSV\)-index of the gradient vector field of \(f\) on \(X\). We recall that the \(GSV\)-index of a vector field \(v\) on \((X, x_0)\), defined in \([\text{GSV, SS}]\), equals the Poincaré-Hopf index of an extension of \(v\) to the Milnor fiber \(X_t\).

As noted in the introduction to \([\text{BMPS}]\), this definition of \(\mu_G(f)\) makes sense in all dimensions and one may generalize \(\mu_G\) as follows. Given an ICIS \((X, x_0)\) and a function \(f\) on it with an isolated singularity at \(x_0\), we denote by \(\nabla_X f\) the gradient vector field of \(f\) (not the conjugate of the gradient as we did for defining \(\text{Eu}_f(X, x_0)\)). Thus we may define \(\mu_G(f)\) as the \(GSV\)-index of \(\nabla_X f\) at \(x_0\). We notice that this invariant is precisely the \emph{virtual multiplicity} at \(x_0\) of the function \(f\) on \(X\) introduced by Izawa and Suwa in \([\text{IS}]\) and denoted \(\overline{m}(f; x_0)\). This multiplicity is by definition the localization at \(x_0\) of the top Chern class of the virtual cotangent bundle \(T^*(X)\) of \(X\) defined by the differential of \(f\), which is non-zero on \(X \setminus \{x_0\}\) by hypothesis. This invariant has the advantage of being defined even if the singular set of \(X\) is non-isolated and it is related to global properties of the variety (we refer to \([\text{IS}]\) for details). This coincides with the index of the 1-form
\( dq \) defined in [EG], and it is similar to the interpretation of the GSV index of vector fields given in [LSS] as a localization of the top Chern class of the virtual tangent bundle.

One can easily find the relation between \( \mu_G(f) \) and \( \mu(f) \) in case \( X \) is an ICIS. The proof can be found for instance in [Ld]. Let \( \mu(X, x_0) \) be the Milnor number of the ICIS \( (X, x_0) \) and let \( f \) be some function with isolated singularity on \( (X, x_0) \). Then:

\[
\mu_G(f) = \mu(f) + \mu(X, x_0).
\]

Using [3], we get:

\[
(4) \quad \text{Eu}_f(X, x_0) = (-1)^{\text{dim}X} [\mu_G(f) - \mu_G(l)].
\]

These equalities completely determine the relation between \( \text{Eu}_f(X, x_0) \), the GSV-index and the Milnor number of \( f \), in terms of the Milnor number of the ICIS \( (X, x_0) \).

4. Further remarks

It is proved in [BMPS], using [BLS], that one has:

\[
(5) \quad \text{Eu}_f(X, x_0) = \sum_{i=0}^{q} [\chi(M(l, x_0) \cap W_i) - \chi(M(f, x_0) \cap W_i)] \cdot \text{Eu}_X(W_i),
\]

where \( M(f, x_0) \) and \( M(l, x_0) \) denote representatives of the Milnor fibers of \( f \) and of the generic linear function \( l \), respectively. Combining this relation with Proposition 2.3 one gets:

\[
\sum_{i=0}^{q} [\chi(M(l, x_0) \cap W_i) - \chi(M(f, x_0) \cap W_i)] \cdot \text{Eu}_X(W_i) = (-1)^{\text{dim}X} \alpha_q.
\]

**Example 4.1.** Let \( X = \{x^2 - y^2 = 0\} \times \mathbb{C} \subset \mathbb{C}^3 \) and \( f \) be the restriction to \( X \) of the function \( (x, y, z) \mapsto x + 2y + z^2 \). Take \( x_0 := (0, 0, 0) \) and take as general linear function \( l \) the restriction to \( X \) of the projection \( (x, y, z) \mapsto z \). Then \( X \) has two strata: \( W_0 = \text{the z-axis, } W_1 = X \setminus \{x = y = 0\} \). We compute \( \text{Eu}_f(X, x_0) \) from the relation [5].

First, \( M(l, x_0) \cap W_0 \) is one point and \( M(f, x_0) \cap W_0 \) is two points. Next, \( M(l, x_0) \cap W_1 \) is the disjoint union of two copies of \( \mathbb{C}^* \) and \( M(f, x_0) \cap W_1 \) is the disjoint union of two copies of \( \mathbb{C}^{**} \), where \( \mathbb{C}^* \) is \( \mathbb{C} \) minus a point and \( \mathbb{C}^{**} \) is \( \mathbb{C} \) minus two points. Then formula [5] gives:

\[
\text{Eu}_f(X, x_0) = (1 - 2) \cdot \text{Eu}(X, x_0) + (0 - (-2)) \cdot 1.
\]

We have \( \text{Eu}(X, x_0) = \text{Eu}(X \cap \{l = 0\}, x_0) \). Next \( \text{Eu}(X \cap \{l = 0\}, x_0) \) is just the Euler characteristic of the complex link of the slice \( X \cap \{l = 0\} = \{x^2 - y^2 = 0\} \). This complex link is two points, so \( \text{Eu}(X \cap \{l = 0\}, x_0) = 2 \). We therefore get \( \text{Eu}_f(X, x_0) = 0 \).

**References**

[BLS] J.-P. Brasselet, Lê D.T., J. Seade, *Euler obstruction and indices of vector fields*, Topology 39, no. 6 (2000), 1193–1208.

[BMPS] J.-P. Brasselet, D.B. Massey, A.J. Parameswaran, J. Seade, *Euler obstruction and defects of functions on singular varieties*, [math.AG/0902238](http://arxiv.org/abs/math.AG/0902238) to appear in J. London Math. Soc.

[EG] W. Ebeling and S. Gusein-Zade, *Indices of 1-forms on an isolated complete intersection singularity*, Moscow. Math. J. 3, no. 2 (2003), 439–455.
[GM] M. Goresky, R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, Bd. 14. Berlin Springer-Verlag 1988.

[Go] V. Goryunov, *Functions on space curves*, J. London Math. Soc. 61 (2000), 807-822.

[GSV] X. Gómez-Mont, J. Seade and A. Verjovsky, *The index of a holomorphic flow with an isolated singularity*, Math. Ann. 291 (1991), 737-751.

[IS] T. Izawa, T. Suwa, *Multiplicity of functions on singular varieties*, Internat. J. Math. 14, 5 (2003), 541-558.

[Lé1] Lê D.T., *Some remarks on the relative monodromy*, Real and Complex Singularities Oslo 1976, Sijhoff en Nordhoff, Alphen a.d. Rijn 1977, pp. 397-403.

[Lé2] Lê D. T., *Le concept de singularité isolée de fonction analytique*, Adv. Stud. Pure Math. 8 (1986), 215-227, North Holland.

[Lé3] Lê D.T., *Complex analytic functions with isolated singularities*, J. Algebraic Geom. 1 (1992), 83–100.

[LSS] D. Lehmann, M. Soares and T. Suwa, *On the index of a holomorphic vector field tangent to a singular variety*, Bol.Soc.Bras.Mat. 26 (1995), 183–199.

[Lo] E.J.N. Looijenga, *Isolated Singular Points on Complete Intersections*, LMS Lecture Notes 77, Cambridge Univ. Press 1984.

[Mi] J. Milnor, *Singular points of complex hypersurfaces*, Ann. of Math. Studies 61, Princeton 1968.

[MS] D. Mond and D. Van Straten, *Milnor number equals Tjurina number for functions on space curves*, J. London Math. Soc. 63 (2001), 177–187.

[Sch] J. Schürmann, *Topology of singular spaces and constructible sheaves*, Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series), 63. Birkhäuser Verlag, Basel, 2003.

[SS] J. Seade, T. Suwa, *A residue formula for the index of a holomorphic flow*, Math. Annalen 304 (1996), 621–634.

[St] N. Steenrod, *The Topology of Fiber Bundles*, Princeton Univ. Press, 1951.

[STV] J. Seade, M. Tibăr, A. Verjovsky, *Global Euler obstruction and polar invariants*, math.AG/0310431

[Ti] M. Tibăr, *Bouquet decomposition of the Milnor fiber*, Topology 35 (1996), no. 1, 227-241.

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