Complete monotonicity of some entropies

Ioan Raşa
Department of Mathematics, Technical University of Cluj-Napoca,
Memorandumului Street 28,
400114 Cluj-Napoca,
Romania, ioan.rasa@math.utcluj.ro

Abstract

It is well-known that the Shannon entropies of some parameterized probability distributions are concave functions with respect to the parameter. In this paper we consider a family of such distributions (including the binomial, Poisson, and negative binomial distributions) and investigate the Shannon, Rényi, and Tsallis entropies of them with respect to the complete monotonicity.

keywords: entropies; concavity; complete monotonicity; inequalities

subject class: 94A17; 60E15; 26A51

1 Introduction

Let $c \in \mathbb{R}$, $I_c := [0, -\frac{1}{c}]$ if $c < 0$, and $I_c := [0, +\infty)$ if $c \geq 0$.
For $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}_0$ the binomial coefficients are defined as usual by

$$\binom{\alpha}{k} := \frac{\alpha(\alpha - 1) \ldots (\alpha - k + 1)}{k!} \quad \text{if } k \in \mathbb{N}, \text{ and } \binom{\alpha}{0} := 1.$$ 

Let $n > 0$ be a real number such that $n > c$ if $c \geq 0$, or $n = -cl$ with some $l \in \mathbb{N}$ if $c < 0$.
For $k \in \mathbb{N}_0$ and $x \in I_c$ define

$$p_n^{[c]}(x) := (-1)^k \binom{-\frac{n}{c}}{k}(cx)^k(1 + cx)^{-\frac{n}{c} - k}, \quad \text{if } c \neq 0,$$
\[
p_{n,k}^{[0]}(x) := \lim_{c \to 0} p_{n,k}^{[c]}(x) = \frac{(nx)^k}{k!} e^{-nx}.
\]

Details and historical notes concerning these functions can be found in [3], [7], [21] and the references therein. In particular,

\[
d \frac{d}{dx} p_{n,k}^{[c]}(x) = n \left( p_{n+c,k-1}^{[c]}(x) - p_{n+c,k}^{[c]}(x) \right).
\]  \tag{1}

Moreover,

\[
\sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) = 1; \quad \sum_{k=0}^{\infty} kp_{n,k}^{[c]}(x) = nx,
\]  \tag{2} \tag{3}

so that \( \left( p_{n,k}^{[c]}(x) \right)_{k \geq 0} \) is a parameterized probability distribution. Its associated Shannon entropy is

\[
H_{n,c}(x) := - \sum_{k=0}^{\infty} p_{n,k}^{[c]}(x) \log p_{n,k}^{[c]}(x),
\]

while the Rényi entropy of order 2 and the Tsallis entropy of order 2 are given, respectively, by (see [18], [20])

\[
R_{n,c}(x) := - \log S_{n,c}(x); \quad T_{n,c}(x) := 1 - S_{n,c}(x),
\]

where

\[
S_{n,c}(x) := \sum_{k=0}^{\infty} \left( p_{n,k}^{[c]}(x) \right)^2, \quad x \in I_c.
\]

The cases \( c = -1, \ c = 0, \ c = 1 \) correspond, respectively, to the binomial, Poisson, and negative binomial distributions. For other details see also [15], [16].

In this paper we investigate the above entropies with respect to the complete monotonicity.
2 Shannon entropy

A. Let’s start with the case $c < 0$.

$H_{n-1}$ is a concave function; this is a special case of the results of [19]; see also [6], [8], [9] and the references therein.

Here we shall determine the signs of all the derivatives of $H_{n,c}$.

**Theorem 1** Let $c < 0$. Then, for all $k \geq 0$,

$$H_{n,c}^{(2k+2)}(x) \leq 0, \quad x \in \left(0, -\frac{1}{c}\right), \quad (4)$$

$$H_{n,c}^{(2k+1)}(x) = \begin{cases} 
\geq 0 & x \in (0, -\frac{1}{2c}], \\
\leq 0 & x \in [-\frac{1}{2c}, -\frac{1}{c}). 
\end{cases} \quad (5)$$

**Proof** We have $n = -cl$ with $l \in \mathbb{N}$. As in [10], let us represent log ($l!$) by integrals:

$$\log (l!) = \int_{0}^{\infty} \left( l - \frac{1 - e^{-st}}{1 - e^{-s}} \right) \frac{e^{-s}}{s} ds = \int_{0}^{1} \left( \frac{1 - (1 - t)^{l}}{t} - l \right) \frac{dt}{\log (1 - t)}. \quad (6)$$

Now using (2), (3) and (6) we get

$$H_{n,c}(x) = H_{l-1}(-cx) = -l \left[ (-cx) \log (-cx) + (1 + cx) \log (1 + cx) \right] + \int_{0}^{1} \frac{-t}{\log (1 - t)} \left( 1 + cx \right)^{l} + (1 - t - cx)^{l} - 1 - (1 - t)^{l} \frac{dt}{t^{2}}.$$

It is a matter of calculus to prove that

$$H_{n,c}''(x) = cl \left( \frac{1}{x} - \frac{c}{1 + cx} \right) + c^{2} l(l-1) \int_{0}^{1} \frac{-t}{\log (1 - t)} \left[ (1 + cx)^{l-2} + (1 - t - cx)^{l-2} \right] dt,$$

and for $k \geq 0$
\[ H_{n,c}^{(2k+2)}(x) = cl(2k)! \left( \frac{1}{x^{2k+1}} - \left( \frac{c}{1+cx} \right)^{2k+1} \right) \]
\[ + \ l(l-1) \ldots (l-2k-1)c^{2k+2} \int_0^1 \frac{-t}{\log(1-t)} \left[ (1+cx)^{l-2k-2} + (1-t-cxt)^{l-2k-2} \right] t^{2k} dt. \]

For \( 0 < t < 1 \) we have
\[ 0 < \frac{-t}{\log(1-t)} < 1, \]
so that
\[ H_{n,c}^{(2k+2)}(x) \leq cl(2k)! \left( \frac{1}{x^{2k+1}} - \left( \frac{c}{1+cx} \right)^{2k+1} \right) + \]
\[ + l(l-1) \ldots (l-2k-1)c^{2k+2} \int_0^1 \left[ (1+cx)^{l-2k-2} + (1-t-cxt)^{l-2k-2} \right] t^{2k} dt. \]

Repeated integration by parts yields
\[ \int_0^1 (1+cx)^{l-2k-2} t^{2k} dt \leq \frac{(2k)!}{(l-2)(l-3) \ldots (l-2k-1)(cx)^2} \int_0^1 (1+cx)^{l-2} dt, \]
and so
\[ \int_0^1 (1+cx)^{l-2k-2} t^{2k} dt \leq \frac{(2k)!}{(l-2)(l-3) \ldots (l-2k-1)(cx)^2} \int_0^1 (1+cx)^{l-2} dt, \]
\[ \leq \frac{(2k)!}{(l-1)(l-2) \ldots (l-2k-1)(cx)^2} \int_0^1 (1+cx)^{l-2} dt. \]

Replacing \( x \) by \(-\frac{1}{c} - x\) we obtain
\[ \int_0^1 (1-t-cxt)^{l-2k-2} t^{2k} dt \leq \frac{(2k)!}{(l-1)(l-2) \ldots (l-2k-1)(1+cx)^2} \int_0^1 (1-cx)^{l-2} dt. \]

From (8), (9) and (10) it follows that
\[ H_{n,c}^{(2k+2)}(x) \leq cl(2k)! \left[ \frac{(1+cx)^{l-1}}{x^{2k+1}} - \frac{c^{2k+1}(-cx)^{l-1}}{(1+cx)^{2k+1}} \right] \leq 0, \]
and this proves (4).

It is easy to verify that \( H_{n,c}^{(2k+1)} \left( -\frac{1}{2c} \right) = 0. \) Since \( H_{n,c}^{(2k+2)} \leq 0, \) it follows that \( H_{n,c}^{(2k+1)} \) is decreasing, and this implies (5).
B. Consider the case \( c = 0 \).

\( H_{n,0} \) is the Shannon entropy of the Poisson distribution. The derivative of this function is completely monotonic: see, e.g., [2, p. 2305]. For the sake of completeness we insert here a short proof.

**Theorem 2** \( H'_{n,0} \) is completely monotonic, i.e.,

\[
(-1)^k H^{(k+1)}_{n,0}(x) \geq 0, \quad k \geq 0, \quad x > 0.
\]  

**Proof** Let us remark that \( H_{n,0}(y) = H_{1,0}(ny) \); so it suffices to investigate the derivatives of \( H_{1,0}(x) \).

According to [10, (2.5)],

\[
H_{1,0}(x) = x - x \log x + \int_0^\infty \frac{e^{-t}}{t} \left( x - \frac{1 - \exp (x(e^{-t} - 1))}{1 - e^{-t}} \right) dt
\]

\[
= x - x \log x - \int_0^1 \left( x - \frac{1 - e^{-sx}}{s} \right) \frac{ds}{\log (1 - s)}.
\]

It follows that

\[
H'_{1,0}(x) = - \log x - \int_0^1 (1 - e^{-sx}) \frac{ds}{\log (1 - s)}
\]

and for \( k \geq 1 \),

\[
H^{(k+1)}_{1,0}(x) = (-1)^k \left( \frac{(k-1)!}{x^k} + \int_0^1 s^k e^{-sx} \frac{ds}{\log (1 - s)} \right). \tag{12}
\]

By using (7) we get

\[
\int_0^1 \frac{s^k e^{-sx}}{\log (1 - s)} ds \geq - \int_0^1 s^{k-1} e^{-sx} ds =
\]

\[
= - \int_0^x \frac{t^{k-1}}{x^k} e^{-t} dt \geq - \int_0^\infty \frac{1}{x^k} t^{k-1} e^{-t} dt = - \frac{(k-1)!}{x^k}.
\]

Combined with (12), this proves (11) for \( k \geq 1 \). In particular, we see that \( H_{n,0} \) is concave and non-negative on \([0, +\infty)\); it follows that \( H'_{n,0} \geq 0 \) and so (11) is completely proved.
C. Let now $c > 0$.

**Theorem 3** For $c > 0$, $H'_{n,c}$ is completely monotonic.

**Proof** Since $H_{m,c}(y) = H_{m,1}(cy)$, it suffices to study the derivatives of $H_{n,1}(x)$.

By using (2), (3) and

$$\log A = \int_0^\infty \frac{e^{-x} - e^{-Ax}}{x} dx, \quad A > 0,$$

we get

$$H_{n,1}(x) = n ((1 + x) \log (1 + x) - x \log x) + \int_0^\infty \frac{e^{-ns} - e^{-s}}{s(1 - e^{-s})} (1 - (1 + x - xe^{-s})^{-n}) ds$$

$$= n ((1 + x) \log (1 + x) - x \log x) + \int_0^1 \frac{1 - (1 - t)^{n-1}}{t \log (1 - t)} (1 - (1 + tx)^{-n}) dt.$$

It follows that, for $j \geq 1$,

$$\frac{1}{n} H_{n,1}^{(j+1)}(x) = (-1)^{j-1} (j - 1)! \left( (x + 1)^{-j} - x^{-j} \right) +$$

$$+ (-1)^{j-1} (n+1)(n+2) \ldots (n+j) \int_0^1 \frac{-t}{\log (1 - t)} \left[ 1 - (1 - t)^{n-1} \right] (1 + xt)^{-n-j-1} t^{j-1} dt.$$

Using again (7), we get

$$(-1)^{j-1} \frac{1}{n} H_{n,1}^{(j+1)}(x) \leq (j - 1)! \left( (x + 1)^{-j} - x^{-j} \right) +$$

$$+ (n + 1)(n + 2) \ldots (n + j) \int_0^1 \left[ 1 - (1 - t)^{n-1} \right] (1 + xt)^{-n-j-1} t^{j-1} dt$$

$$= u(x) + v(x),$$

where

$$u(x) := \frac{(j - 1)!}{(x + 1)^j} - (n+1)(n+2) \ldots (n+j) \int_0^1 t^{j-1} (1 - t)^{n-1} (1 + xt)^{-n-j-1} dt,$$

$$v(x) := (n + 1)(n + 2) \ldots (n + j) \int_0^1 t^{j-1} (1 + xt)^{-n-j-1} dt - \frac{(j - 1)!}{x^j}.$$
We shall prove that $u(x) \leq 0$ and $v(x) \leq 0$, $x > 0$. Let us remark that
\[
\int_0^1 t^{j-1}(1-t)^{n-1}(1+xt)^{-n-j-1}dt \geq \int_0^1 t^{j-1}(1-t)^n(1+xt)^{-n-j-1}dt,
\]
and integration by parts yields
\[
\int_0^1 \frac{t^{j-1}(1-t)^n}{(1+xt)^{n+j+1}}dt = \frac{j-1}{(n+1)(x+1)} \int_0^1 \frac{t^{j-2}(1-t)^{n+1}}{(1+xt)^{n+j+1}}dt.
\]
Applying repeatedly this formula we obtain
\[
\int_0^1 \frac{t^{j-1}(1-t)^n}{(1+xt)^{n+j+1}}dt = \frac{(j-1)!}{(n+1)(n+2)\ldots(n+j)(x+1)^j}.
\]
Now (13) and (14) imply $u(x) \leq 0$.

Using again integration by parts we get
\[
\int_0^1 t^{j-1}(1+xt)^{-n-j-1}dt \leq \frac{j-1}{(n+j)x} \int_0^1 t^{j-2}(1+xt)^{-n-j}dt \leq \cdots \leq \frac{(j-1)!}{(n+1)(n+2)\ldots(n+j)} \frac{1}{x^j},
\]
which shows that $v(x) \leq 0$.

We conclude that
\[(-1)^{j-1}H_{n,1}^{(j+1)}(x) \leq 0, \quad j \geq 1, x > 0.\]  
(15)

In particular, (15) shows that $H_{n,1}$ is concave on $[0, +\infty)$; it is also non-negative, which means that $H'_{n,1} \geq 0$. Combined with (15), this shows that $H''_{n,1}$ is completely monotonic, and the proof is finished.

**Remark 3.1** (14) can be obtained alternatively by using the change of variables $y = (1-t)/(1+xt)$ and the properties of the Beta function. An alternative proof of the inequality $v(x) \leq 0$ follows from
\[
\int_0^1 t^{j-1}(1+xt)^{-n-j-1}dt \leq \frac{1}{x^{j-1}} \int_0^\infty \frac{(xt)^{j-1}}{(1+xt)^{n+j+1}}dt = \frac{1}{x^{j}} \int_0^\infty \frac{s^{j-1}}{(1+s)^{j+n+1}}ds = \frac{1}{x^j} B(j, n+1) = \frac{1}{x^j} \frac{(j-1)!n!}{(n+j)!}.
\]
Corollary 3.1 The following inequalities are valid for \( x > 0 \) and \( c \geq 0 \):

\[
\log \frac{x}{cx + 1} \leq \sum_{k=0}^{\infty} p_n^{[c]}(x) \log \frac{k + 1}{ck + n} \leq \log \frac{nx + 1}{ncx + n}.
\]  
(16)

In particular, for \( c = 0 \) and \( n = 1 \),

\[
\log x \leq \sum_{k=0}^{\infty} e^{-x} \frac{x^k}{k!} \log (k + 1) \leq \log (x + 1).
\]

Proof We have seen that \( H'_n,c(x) \geq 0 \). An application of (1) yields

\[
H'_n,c(x) = n \left( \log \frac{1 + cx}{x} + \sum_{k=0}^{\infty} p_n^{[c]}(x) \log \frac{k + 1}{n + ck} \right).
\]

This proves the first inequality in (16); the second is a consequence of Jensen’s inequality applied to the concave function \( \log t \).

## 3 Rényi entropy and Tsallis entropy

The following conjecture was formulated in [13]:

Conjecture 3.1 \( S_{n,-1} \) is convex on \([0, 1]\).

Th. Neuschel [11] proved that \( S_{n,-1} \) is decreasing on \([0, \frac{1}{2}]\) and increasing on \([\frac{1}{2}, 1]\). The conjecture and the result of Neuschel can be found also in [3].

A proof of the conjecture was given by G. Nikolov [12], who related it with some new inequalities involving Legendre polynomials. Another proof can be found in [4].

Using the important results of Elena Berdysheva [3], the following extension was obtained in [17]:

Theorem 4 ([17, Theorem 9]). For \( c < 0 \), \( S_{n,c} \) is convex on \([0, -\frac{1}{c}]\).

A stronger conjecture was formulated in [13] and [17]:

Conjecture 4.1 For \( c \in \mathbb{R} \), \( S_{n,c} \) is logarithmically convex, i.e., \( \log S_{n,c} \) is convex.
It was validated for $c \geq 0$ by U. Abel, W. Gawronski and Th. Neuschel \[1\], who proved a stronger result:

**Theorem 5** ([1]). For $c \geq 0$, the function $S_{n,c}$ is completely monotonic, i.e.,

$$(-1)^m \left(\frac{d}{dx}\right)^m S_{n,c}(x) > 0, \quad x \geq 0, \ m \geq 0.$$  

Consequently, for $c \geq 0$, $S_{n,c}$ is logarithmically convex, and hence convex.

Summing up, for the Rényi entropy $R_{n,c} = -\log S_{n,c}$ and Tsallis entropy $T_{n,c} = 1 - S_{n,c}$, we can state

**Corollary 5.1**

i) Let $c \geq 0$. Then $R_{n,c}$ is increasing and concave, while $T'_{n,c}$ is completely monotonic on $[0, +\infty)$.

ii) $T_{n,c}$ is concave for all $c \in \mathbb{R}$.

**Proof**

i) Apply Theorem 5.

ii) For $c < 0$, apply Theorem 4. For $c \geq 0$, Theorem 5 shows that $S_{n,c}$ is convex, so that $T_{n,c}$ is concave.

**Remark 5.1** As far as we know, Conjecture 4.1 is still open for $c < 0$, so that the concavity of $R_{n,c}, c < 0$, remains to be investigated.

**Acknowledgement**

The author is grateful to the referee for valuable comments and very constructive suggestions. In particular, the elegant alternative proofs presented in Remark 1 were kindly suggested by the referee.

**References**

[1] U. Abel, W. Gawronski, Th. Neuschel, Complete monotonicity and zeros of sums of squared Baskakov functions, Appl. Math. Comput., 258, 130-137 (2015)
[2] J.A. Adell, A. Lekuona and Y. Yu, Sharp bounds on the entropy of the Poisson Law and related quantities, IEEE Trans. Information Theory, 56, 2299-2306 (2010)

[3] E. Berdysheva, Studying Baskakov-Durrmeyer operators and quasi-interpolants via special functions, J. Approx. Theory, 149, 131-150 (2007)

[4] I. Gavrea, M. Ivan, On a conjecture concerning the sum of the squared Bernstein polynomials, Appl. Math. Comput., 241, 70-74 (2014)

[5] H. Gonska, I. Raşa, M.-D. Rusu, Chebyshev-Grüss-type inequalities via discrete oscillations, Bul. Acad. Stiinte Repub. Mold. Mat., 1, (74), 63-89; arxiv 1401.7908 [math.CA] (2014)

[6] P. Harremoës, Binomial and Poisson distributions as maximum entropy distributions, IEEE Trans. Information Theory, 47, 2039 - 2041 (2001)

[7] M. Heilmann, Erhöhung der Konvergenzgeschwindigkeit bei der Approximation von Funktionen mit Hilfe von Linearkombinationen spezieller positiver linearer Operatoren, Habilitationsschrift, Universität Dortmund (1992)

[8] E. Hillion, Concavity of entropy along binomial convolutions, Electron. Commun. Probab., 17, 1-9 (2012)

[9] E. Hillion, O. Johnson, A proof of the Shepp-Olkin entropy concavity conjecture, arXiv: 1503.01570v1, (2015)

[10] C. Knessl, Integral representations and asymptotic expansions for Shannon and Rényi entropies, Appl. Math. Lett., 11, (1998), 69-74.

[11] Th. Neuschel, Unpublished manuscript (2012)

[12] G. Nikolov, Inequalities for ultraspherical polynomials. Proof of a conjecture of I. Raşa, J. Math. Anal. Appl., 418, (2014), 852-860.

[13] I. Raşa, Unpublished manuscripts (2012)

[14] I. Raşa, Special functions associated with positive linear operators, arxiv: 1409.1015v2, (2014)
[15] I. Raşa, Rényi entropy and Tsallis entropy associated with positive linear operators, arxiv: 1412.4971v1, (2014)

[16] I. Raşa, Entropies and the derivatives of some Heun functions, arxiv: 1502.05570v1, (2015)

[17] I. Raşa, Entropies and Heun functions associated with positive linear operators, Appl. Math. Comput., 268, 422-431 (2015)

[18] A. Rényi, On measures of entropy and information, Proc. Fourth Berkeley Symp. Math. Statist. Prob., Vol. 1, Univ. of California Press, pp. 547-561 (1961)

[19] L.A. Shepp, I. Olkin, Entropy of the sum of independent Bernoulli random variables and of the multinomial distribution, Proc. Contributions to Probability, New York, pp. 201-206 (1981)

[20] C. Tsallis, Possible generalization of Boltzmann-Gibbs statistics, J. Stat. Phys., 52, 479-487 (1988)

[21] M. Wagner, Quasi-Interpolaten zu genuinen Baskakov-Durrmeyer-Typ Operatoren, Shaker Verlag, Aachen (2013)