Equilibration in the Kac Model Using the GTW Metric $d_2$

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Abstract We use the Fourier based Gabetta–Toscani–Wennberg metric $d_2$ to study the rate of convergence to equilibrium for the Kac model in 1 dimension. We take the initial velocity distribution of the particles to be a Borel probability measure $\mu$ on $\mathbb{R}^n$ that is symmetric in all its variables, has mean $\vec{0}$ and finite second moment. Let $\mu_t(dv)$ denote the Kac-evolved distribution at time $t$, and let $R_\mu$ be the angular average of $\mu$. We give an upper bound to $d_2(\mu_t, R_\mu)$ of the form $\min \left\{ B e^{-\frac{\lambda_1 t}{n+1}}, d_2(\mu, R_\mu) \right\}$, where $\lambda_1 = \frac{n+2}{2(n-1)}$ is the gap of the Kac model in $L^2$ and $B$ depends only on the second moment of $\mu$. We also construct a family of Schwartz probability densities $\{ f_0^{(n)} : \mathbb{R}^n \to \mathbb{R} \}$ with finite second moments that shows practically no decrease in $d_2( f_0(t), R_{f_0} )$ for time at least $\frac{1}{2\lambda}$ with $\lambda$ the rate of the Kac operator. We also present a propagation of chaos result for the partially thermostated Kac model in Tossounian and Vaidyanathan (J Math Phys 56(8):083301, 2015).

Keywords Kac model · Kac master equation · Kinetic theory · Gabetta–Toscani–Wennberg metric $d_2$ · Rate of approach to equilibrium · Non-equilibrium statistical mechanics · Propagation of chaos · Cercignani type conjecture

1 Introduction

In [12] Kac introduced a linear $n$ particle model with the goal of deriving the Boltzmann equation with Maxwellian molecules. He derived a space homogeneous Boltzmann-type equation using the notion of propagation of chaos, which he called the “propagation of the Boltzmann property”. A sequence of densities $\{ f_n : L^1(S^{n-1}(\sqrt{nE}), \sigma) \to \mathbb{R} \}$ on the spheres $S^{n-1}$ where each $f_n$ invariant under the exchange of the variables is called chaotic with limit $h$ if
\[ \lim_{n \to \infty} \int_{S^{n-1}(\sqrt{nE})} f_n(v_1, \ldots, v_n) \phi(v_1, \ldots, v_k) \sigma(dv) = \int_{\mathbb{R}^k} \prod_{i=1}^k h(v_i) \phi(v) dv^k. \]

for all \( k \) and all \( \phi \in L^\infty \) that depends only on \( v_1, \ldots, v_k \). Here \( E \) is the average energy per particle and is independent of \( n \), and \( \sigma \) is the uniform probability measure on \( S^{n-1}(\sqrt{nE}) \).

Kac showed for his model that if \( \{f_n(t = 0, \cdot)\}_n \) is a chaotic sequence with limit \( f_0 \), then so is the time evolved \( \{f_n(t, \cdot)\}_n \) for any time \( t \geq 0 \) and the chaotic limit \( h(t, v) \) of the sequence \( \{f_n(t, \cdot)\}_n \) satisfies the Kac–Boltzmann equation

\[ \frac{\partial h}{\partial t}(t, v) = 2 \int_\mathbb{R} \int_0^{2\pi} \left( h(t, v^* h(t, w^*) - h(t, v) h(t, w) \right) d\theta dw \]

with initial condition \( h(0, v) = f_0(v) \). Here \( v^*(\theta) \) and \( w^*(\theta) \) are given by the equation:

\[ (v^*(\theta), w^*(\theta)) = (v \cos \theta - w \sin \theta, v \sin \theta + w \cos \theta). \] (2)

The dynamical variables in Kac’s model are the 1 dimensional velocities of \( n \) identical particles. The particles are assumed to be uniformly distributed in space and only their velocities evolve. Let \( \vec{v} = (v_1, \ldots, v_n) \) denote the velocities of the particles, and \( f(t, \vec{v}) \) denote the distribution of the velocities. A binary collision takes place at a sequence of random times \( \{t_i\} \) with \( \{t_{i+1} - t_i\} \) i.i.d. with law \( \exp(n\lambda) \), for some parameter \( \lambda \) independent of \( n \), as follows. At \( t_i \), a pair of particles \( k, l \) is chosen randomly and uniformly among the \( \binom{n}{2} \) pairs to collide. Let \( v_k \) and \( v_l \) be their velocities prior to the collision. After the collision their velocities become \( v^*_k(\theta) \) and \( v^*_l(\theta) \) given by Eq. (2) with \( v \) and \( w \) replaced by \( v_k \) and \( v_l \), and where \( \theta \) is chosen randomly and uniformly in \( [0, 2\pi] \). These collisions preserve energy. We represent the effect of rotating particles \( k \) and \( l \) on a probability density \( f \) by the operator \( Q_{k,l} \). \( Q_{k,l} \) is given by:

\[ Q_{k,l} f = \int_0^{2\pi} f(v_1, \ldots, v_{k-1}, v^*_k(\theta), \ldots, v^*_l(\theta), \ldots, v_n) d\theta, \] (3)

and the collision operator \( Q \) is given by \( Q = \sum_{i \neq j} Q_{i,j} \). The Fokker–Planck equation of this process is known as the Kac master equation and is given by

\[ \frac{\partial f(t, \vec{v})}{\partial t} = n\lambda(Q - I)f := -Lf. \] (4)

Here \( -L \) is the generator of the Kac process. Kac worked on the sphere \( \sum_{i=1}^n v_i^2 = nE \) and took the initial distributions to be a symmetric under the exchange of its variables. \( L^2 \left( S^{n-1}(\sqrt{nE}) \right) \). This symmetry, which is preserved by the Kac evolution, is the physically interesting case. The restriction to a sphere is possible because Kac’s evolution preserves the energy \( v_1^2 + \cdots + v_n^2 \) and therefore preserves the property of being supported on a sphere too. It is well known (see the introduction of [5]) that on each sphere the only stationary solutions are the constants and that the Kac process is ergodic. On \( \mathbb{R}^n \), i.e. when the energy at \( t = 0 \) is not fixed, the equilibria are the radial functions.

In the following, let

- \( \sigma^r \) (or \( \sigma \) if \( r \) is clear from the context) denote the normalized uniform probability measure on \( S^{n-1}(r) \) for any \( r > 0 \),
- \( |h|_{L^p(r)} \) be \( \int_{S^{n-1}(r)} |h(w)|^p \sigma^r(dw) \) for \( 1 \leq p < \infty \),
- \( |h|_{L^\infty(r)} = \text{ess sup} \{|h(w)| : |w| = r\} \).
\begin{itemize}
  \item $R_h$ denote the angular average of $h$: $R_h(v) = \int_{\mathbb{S}^{n-1}(|v|)} h(w)\sigma^r(dw)$. $R_\mu$ can be defined similarly for Borel probability measures $\mu$ ($R_h$ was called the radial average of $h$ in [3] and [14]), and
  \item $Q_{i,j}(\theta)$ map $(v_1, \ldots, v_n)$ to $(v_1, \ldots, v_i \cos \theta - v_j \sin \theta, v_{i+1}, \ldots, v_i \sin \theta - v_j \cos \theta, v_{j+1}, \ldots, v_n)$.
\end{itemize}

The aim of this paper is to study the Gabetta–Toscani–Wennberg (GTW) metric $d_2$ in relation with the Kac evolution, and to give a propagation of chaos result for the partially thermostated Kac model in [14]. The speed of approach to equilibrium is one of the central questions in this field. Kac in [12] conjectured that there is a spectral gap for the generator of the master equation on $L^2(S^{n-1}(r))$ that is independent of the number of particles. Kac’s conjecture was proved by Janvresse in [11] and the gap was computed explicitly in [5], where the authors show if $f : L^2(S^{n-1}(r)) \to \mathbb{R}$ is symmetric in its variables with integral 1, then the following inequality holds:

$$||e^{-L^2} f - 1||_{L^2(r)} \leq e^{-\lambda n^2/(n-1)} ||f - 1||_{L^2(r)}. \tag{5}$$

The $L^2$ gap requires time of order $n$ to show fast convergence to equilibrium because the initial norm $||f - 1||_{L^2(r)}$ can grow exponentially in $n$ if $f = \prod f_i(v_i)/Z$ is a normalized product on $S^{n-1}(r)$.

The (negative) of the relative entropy $S(f(t)|1) = \int f \ln \left( \frac{f}{1} \right) d\sigma^r$ was studied as a distance to equilibrium because it is an extensive quantity. We have $S(f|1) \geq 0$ and $S(f|1) = 0$ if and only if $f = 1$ a.e. Villani showed in [15] that

$$S(f(t)|1) \leq e^{-\frac{2}{n+1}t} S(f|1), \tag{6}$$

using entropy production techniques. The initial entropy production is defined by $-\frac{1}{S(f(t)|1)} \frac{d}{dt} S(f(t)|1)_{|t=0^+}$. Einav showed in [8] that the rate in (6) is essentially sharp in the $n$ behavior at $t = 0$, disproving Cercignani’s conjecture in the context of the Kac model which states that there is a positive lower bound on the entropy production that is independent of $n$ for the class of $L^1$ functions with finite entropy and finite second moment (see [2] and Sect. 6 in [15]).

Exponentially fast decay with rate independent of the number of particles was established in [3] for the Kac model coupled to a thermostat. In this model, the particles in addition to colliding among themselves, collide at a rate $\eta$ with particles from a Maxwellian thermostat at a fixed temperature $\beta^{-1}$. The energy of the system is no longer conserved since the thermostats can pump in or drain out energy from the system. So, in this model, the solution $f(t, \cdot)$ is supported on all of $\mathbb{R}^n$. Equilibrium is reached when all the (non-thermostat) particles are independent and have the Gaussian distribution at the same temperature as the thermostat.

Motivated by the result in [3], Vaidyanathan and I worked in [14] with the Kac model where we thermostated $m$ of the particles, $m < n$ using a stronger thermostat at temperature $\beta^{-1}$. Let $P_i$ be the operator representing the action of the strong thermostat on the $i^{th}$ particle. $P_i$ is given by

$$P_i[f](v_1, \ldots, v_n) = g_\beta(v_i) \int_{\mathbb{R}} f(v_1, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_n) dw, \tag{7}$$

where $g_\beta(v)$ the Gaussian at temperature $\frac{1}{\beta}$:

$$g_\beta(x) = \sqrt{\frac{\beta}{2}} e^{-\frac{\beta}{2} x^2}. \tag{8}$$
The generator of the partially thermostated Kac model in [14] is given by

\[- L_{n,m} = n\lambda(Q - I) + \eta \sum_{i=1}^{m} (P_i - I). \tag{9}\]

The minus sign is there to make \( L_{n,m} \) positive definite in \( L^2(\mathbb{R}^n) \).

A propagation of chaos result for the partially thermostated Kac model will be presented below, where the \( f_n \)'s are supported on all of \( \mathbb{R}^n \) instead of the only on the spheres \( S^{n-1}(\sqrt{nE}) \).

The Fourier based GTW metric \( d_2 \) was used in [4] to show that the infinite thermostat model in [3] can be approximated uniformly in time by the Kac model with a finite reservoir having \( n + N \) particles. Here \( N \gg n \) and the initial conditions are taken to have the special form \( f(\nu) = l_0(v_1, \ldots, v_n) \prod_{j=n+1}^{n+N} g_j(v_j) \). The last \( N \) particles are the reservoir particles. This approximation was proven under a technical finite fourth moment assumption.

Let \( \mu \) and \( \nu \) be Borel probability measures on \( \mathbb{R}^n \). The GTW metrics \( d_\alpha \) are given by

\[ d_\alpha(\mu, \nu) = \sup_{|\xi| = a} |\hat{\mu}(\xi) - \hat{\nu}(\xi)|. \tag{10}\]

Here we use the convention that the Fourier transform of \( \phi \) is \( \hat{\phi}(\xi) = \int_{\mathbb{R}^n} \phi(v)e^{-2\pi i\xi \cdot v} \, dv \). We will use only \( d_2 \) even though analogs of Theorems 1 and 2 are valid for any \( d_\alpha \) with \( \alpha > 0 \).

The GTW metrics \( \{d_\alpha\}_{\alpha > 0} \) were introduced in [10] in the context of the space homogeneous Kac–Boltzmann Eq. (1) where they helped in showing exponentially fast convergence to equilibrium for the initial data with finite \( 2 + \epsilon \) moment for some \( \epsilon > 0 \). \( d_1 \) and \( d_2 \) were used in [7] to show exponential convergence to steady states for the Kac–Boltzmann system coupled to multiple Maxwellian thermostats at different temperatures. Similarly, \( d_1 \) and \( d_2 \) were used by J. Evans in [9] to show existence and ergodicity of non-equilibrium steady states in the Kac model coupled to multiple thermostats.

An interesting feature of the \( d_2 \) metric that we will elaborate in Sects. 2 and 3 is its intensivity property given in [4]: Let \( f_1, \ldots, f_n \) and \( g_1, \ldots, g_n \) be probability densities on \( \mathbb{R} \) with finite second moments and 0 first moment. Then

\[ d_2 \left( \prod_{i=1}^{n} f_i(v_i), \prod_{j=1}^{n} g_j(v_j) \right) = \max_{i \leq n} d_2(f_i, g_i). \tag{11}\]

We take our initial distribution \( \mu \) to be a Borel probability measure \( \mu \) on \( \mathbb{R}^n \). A special case is a density on \( S^{n-1}(\sqrt{nE}) \). We adapt Eq. (4) to measures and study the Kac-evolved \( \mu, e^{-tL}\mu \) using the GTW distance \( d_2 \). In Sect. 3 we give the “almost” intensivity properties of the \( d_2 \) metric. Proposition 1 shows that, after time of \( O(\ln n) \), a good quantity to compare \( d_2(\mu, R_\mu) \) with is

\[ \int \frac{|v|^2}{n} \mu(dv). \]

While at \( t = 0 \), there are states for which \( d_2(\mu, R_\mu) \) is as big as \( \int |v|^2 \mu(dv) \) which is of order \( n \). The function \( d_2(e^{-tL}\mu, R_\mu) \) is not guaranteed to be differentiable with respect to \( t \) due to the supremum taken in the definition of \( d_2 \). So Cercignani’s conjecture cannot be formulated in the same way as in the relative entropy. But one could formulate the following conditional statement:

“(C) Let \( \mu \) be a Borel probability measure on \( \mathbb{R}^n \) with finite second moment and zero first moment. If \( d_2(\mu, R_\mu) > 0 \) and \( d_2(e^{-tL}\mu, R_\mu) \) is differentiable at \( t = 0 \) then

\[ \frac{d}{dt}(d_2(e^{-tL}\mu, R_\mu)) \bigg|_{t=0} \geq a \]

for some \( a > 0 \) independent of \( \mu \) or \( n \).” We will disprove this conjecture in Theorem 2.
In Sect. 4 we give the first main theorem: Theorem 1, a convergence result that provides an upper bound for \(d^2(e^{-tL}\mu, R_\mu)\) when \(\mu\) has zero mean and finite second moment, and is symmetric under the exchange of its variables. This upper bound has the form

\[
\min\{Be^{\frac{-d_1^2}{n-1}}\lambda, d_2(\mu, R_\mu)\}
\]

with \(B\) depending only on the second moment of \(\mu\). This shows that \(d^2(e^{-tL}\mu, R_\mu)\) goes to zero. It is curious that the proof uses the \(L^2\) gap of the Kac evolution in Eq. (5) in an unexpected context. We show in Proposition 1 that our bound has the correct order of magnitude at \(t = 0\). This upper bound gives decay after time of order \(n\), in agreement with the upper bounds using the \(L^2\) and relative entropy metrics. Next, in Sect. 5, we use the \(L^\infty\) nature of the \(d^2\) metric to construct a family of functions \(f_n \in L^1(\mathbb{R}^n)\) having \(O(t^{n-1})\) decay in \(d^2_2\) when \(0 \leq t \leq 1/(2\lambda)\). This disproves the Cercignani-type conjecture (C) for the Kac evolution in the \(d^2_2\) metric. We give the construction in Theorem 2. In Sect. 6, we give a propagation of chaos result for the partially thermostated Kac model in [14] by adapting McKean’s proof of propagation of chaos for the regular Kac model in [13]. In Sect. 7 we give some concluding remarks. The results are given in Sect. 2.

2 Results

We first give Proposition 1 that generalizes Eq. (11). It says that \(\int \frac{|u|^2}{n} \mu(dv)\) essentially gives the order of magnitude of \(d^2(e^{-tL}\mu, R_\mu)\), the distance between a measure and its angular average.

**Proposition 1** \((d^2\text{-energy comparison})\) Let \(\mu\) and \(v\) be Borel probability measures on \(\mathbb{R}^n\) with \(n \geq 2\). Let \(\int \tilde{\nu}\mu(dv) = \tilde{\nu}, \int \tilde{\nu}v(dv) = \tilde{\nu},\) and \(\int |v|^2(\mu(dv) + \nu(dv)) < \infty.\) Let \(-L = n(I - Q)\) \((\lambda = 1)\) be the generator of the Kac evolution \((\lambda = 1).\) Then

\[
d^2(e^{-tL}\mu, R_\mu) \leq \frac{(2\pi)^2}{2} \left(2 - e^{-\frac{n}{n-1}t}\right) \int \frac{|v|^2}{n} \mu(dv)
+ e^{-\frac{n}{n-1}t} \max_i \int v_i^2 \mu(dv) + (n-1)e^{-\frac{n}{n-1}t} \max_{i \neq j} \left| \int v_i v_j \mu(dv) \right|
\]

(12)

\[
d^2(e^{-tL}\mu, e^{-tL}v) \leq \frac{(2\pi)^2}{2} ((n-1)e^{-t} + 1) \int_{\mathbb{R}^n} |\mu(dv) - v(dv)| \frac{|v|^2}{n}.
\]

(13)

**Remark 1** If \(\mu\) has mean \(\tilde{\mu} \neq \tilde{\nu}. Then d^2(\mu, R_\mu) = \infty because the angular average \(R_\mu\) has mean \(\tilde{\nu}. One way around this is to use a centered GTW distance \(d^2_2\) as in [7] and [9]. This handles the \(\frac{1}{|\xi|}\) divergence as \(\xi \to \tilde{\nu}\) in the definition of \(d^2_2\). We will omit this case.

With the help of this proposition, the statement of Theorem 1 becomes more natural.

**Theorem 1** Let \(\mu\) be a Borel probability measure on \(\mathbb{R}^n\) that is invariant under permutation of coordinates. Let \(\int |v|^2 \mu(dv) < \infty and \int \tilde{\nu}\mu(dv) = \tilde{\nu}. Set \lambda in (4) be 1. Then

\[
d^2(e^{-tL}\mu, R_\mu) \leq \min \left\{ K \left( e^{-\frac{d_1^2}{n-1}t} \right) \left[ 2 \int v_i^2|\mu|(dv) + (n-1)e^{-\frac{n}{n-1}t} \int_{\mathbb{R}^n} v_1 v_2 \mu(dv) \right], \right.
\]

(14)

\[
K = 6.64(2\pi)^2 and \lambda_1 is the gap in (5).
\]
Theorem 1 implies that \(d_2(e^{-tL} \mu, R_\mu) \leq K(ne^{-t} + 1) \int \frac{|v|^2}{n} \mu(dv) \left( e^{-\frac{4t}{n+3}} \right)\) for all \(t\), and that if \(\mu\) has zero correlations between the \(v_i\) (e.g. \(\mu = \prod_i \mu_0(dv_i)\) and \(\mu_0\) centered at 0), then \((ne^{-t} + 1)\) can be replaced by 1. The important information in this theorem is the exponential rate of decay \(\frac{4t}{n+3}\) for large time. The constant \(K\) is not optimal at \(t = 0\). It would be desirable to have a bound of the form \(d_2(e^{-tL} \mu, R_\mu) \leq 1e^{-ct/n} d_2(\mu, R_\mu)\). But Theorem 2 implies that no such bound exists at least on \([0, 1/2]\) even if \(\mu\) has a Schwartz density with respect to the Lebesgue measure. Theorem 2 also implies that, for some Schwartz densities \(f_0 \times \frac{d}{dt} d_2(e^{-tL} f, R_f)\) exists and equals 0. The conjecture that “the best constant \(K_{\text{best}}\) in Eq. (14) satisfies

\[K_{\text{best}}(n) \geq H \left( \int v_1^2 \mu(dv), \int v_1 v_2 \mu(dv) \right) \left( 1 + \frac{c}{n} \right)\] (15)

for some optimal \(H\) that is at most linear in its arguments.” is consistent with Proposition 1 and Theorem 2 because there is decay in Eq. (15) only after time of order 1 (since \((1 + \frac{c}{n})e^{-\frac{t}{n}} \leq 1\) when \(t \geq n \ln(1 + \frac{c}{n}) \approx c\).

**Theorem 2** Let \(n \geq 2\) and let \(\Lambda\) be as in Eq. (4) with \(\lambda = 1\). There is a Schwartz probability density \(f_n\) on \(\mathbb{R}^n\) that satisfies

\[d_2(e^{-tL} f_0, R_{f_0}) \geq \max \left\{ d_2(f_0, R_{f_0}) \left( 1 - \frac{e}{n} (2t)^n \right), 0 \right\} \quad \text{for all } t \geq 0.\] (16)

The lower bound in Theorem 2 endures for \(t \in [0, 1/2]\). We will give the \(f_n\) explicitly in Lemma 4 up to two parameters \(\widetilde{A}(n)\) and \(B(n)\) that are shown to be finite but are not computed. The functions \(f_n\) will be perturbations of the Gaussians \(\prod_{i=1}^n \Gamma_{\alpha(n)}(v_i)\) at high temperature by Schwartz functions that have small \(L^1\) norms.

Finally, we give the propagation of chaos result for the partially thermostated Kac model in [14]. This result is independent of the previous Theorems. As mentioned in the introduction, the energy of the system of particles is no longer conserved. Thus our functions will be in \(L^1(\mathbb{R}^n)\) for various \(n\) instead of \(L^1(S^{n-1}(\sqrt{n}E))\). Let \(n_0, m_0 < n_0\) be such that \(\alpha = \frac{n_0}{m_0}\) is the fraction of particles that are thermostated. Let \(L_{m,n}\) be given by (9). Then we have the following theorem.

**Theorem 3** Propagation of Chaos for the Partially thermostated Kac Model

Let \(A = \{ i : i \geq 1, \text{ and } (i \text{ mod } n_0) \in \{1, 2, \ldots, m_0\} \}\) and \(B = \mathbb{N} - A\). Let \(\{f_k \in L^1(\mathbb{R}^{kn_0})\}_{k=1}^\infty\) be a family of probability distributions that are symmetric under the exchange of particles with indices in \(A\) and under the exchange of particles with indices in \(B\). If

\[
\lim_{k \to \infty} \int_{\mathbb{R}^{kn_0}} f_k(v_1, \ldots, v_{kn_0}) \phi(v_1, \ldots, v_l) dv = \int_{\mathbb{R}^{kn_0}} \prod_{i \in A, j \leq l} \tilde{f}_0(v_i) \times \prod_{j \in B, j \leq l} \tilde{f}_0(v_j) \phi(v_1, \ldots, v_l) dv,
\]

for every \(\phi\) in \(L^\infty(\mathbb{R}^l)\), then

\[
\lim_{k \to \infty} \int_{\mathbb{R}^{kn_0}} e^{-tL_{kn_0, kn_0}} [f_k](v_1, \ldots, v_{kn_0}) \phi(v_1, \ldots, v_l) dv = \int_{\mathbb{R}^{kn_0}} \prod_{i \in A, j \leq l} \tilde{f}(t, v_i) \times \prod_{j \in B, j \leq l} \tilde{f}(t, v_j) \phi(v_1, \ldots, v_l) dv,
\]
for every \( \phi \) in \( L^\infty(\mathbb{R}^l) \) where \((\tilde{f}, \tilde{f})\) satisfy the following system of Boltzmann–Kac equations:
\[
\begin{align}
\frac{\partial \tilde{f}}{\partial t}(t, v) &= 2\lambda \left[ \int_{\mathbb{R}} \int_{0}^{2\pi} \tilde{f}(t, v^*)(\alpha \tilde{f}(t, w^*) + (1 - \alpha) \tilde{f}(t, w^*)) \, d\theta \, dw - \tilde{f}(t, v) \right] \\
&\quad + \eta(P_1 - 1) \tilde{f}
\end{align}
\]
\[
\frac{\partial \tilde{b}}{\partial t}(t, v) = 2\lambda \left[ \int_{\mathbb{R}} \int_{0}^{2\pi} \tilde{f}(t, v^*)(\alpha \tilde{f}(t, w^*) + (1 - \alpha) \tilde{f}(t, w^*)) \, d\theta \, dw - \tilde{b}(t, v) \right],
\]
(17)

\[
\text{together with the initial conditions } (\tilde{f}(t = 0), \tilde{b}(t = 0)) = (\tilde{f}_0, \tilde{b}_0).
\]

This roughly says that a given particle collides with a thermostated particle a fraction \( \alpha \) of the time, and with non-thermostated particles: the fraction \( 1 - \alpha \) of the time.

### 3 Proof of Proposition 1

The proof of Proposition 1 relies on the action of the Kac evolution on quadratic polynomials. The following lemma says that after time of order \( \ln(n) \), \((v, \xi)^2\) is effectively \( \frac{|v|^2}{n} |\xi|^2 \).

**Lemma 1** (Kac action on Quadratic Polynomials) \textit{Let } \( n \geq 2 \) \textit{and let } \( L \) \textit{be as the operator in (4) with } \( \lambda = 1 \). \textit{For any } \( v, \xi \in \mathbb{R}^n \), \textit{we have}
\[
e^{-tL}(v, \xi)^2 = \left(1 - e^{-\frac{n}{n-1}t}\right) \frac{|v|^2 |\xi|^2}{n} + e^{-\frac{n}{n-1}t} \sum_{i=1}^{n} \xi_i^2 v_i^2 + e^{-\frac{4n-6}{n-1}t} \sum_{i \neq j} \xi_i \xi_j v_i v_j \quad (18)
\]

\textit{It follows that for all } \( n \geq 2 \) \textit{and } \( t \geq 0 \) \textit{we have}
\[
\left|e^{-tL}(v, \xi)^2 - \frac{|v|^2 |\xi|^2}{n}\right| \leq e^{-t} \left(1 - \frac{1}{n}\right) |v|^2 |\xi|^2. \quad (19)
\]

**Proof** (of Lemma 1) \textit{We will look at the action of } \( Q \) \textit{on } \( v_1 v_2 \) \textit{and on } \( v_1^2 \) \textit{separately. First,}
\[
Q_{i,j} v_1 v_2 = \begin{cases}
0, & \{i, j\} \cap \{1, 2\} \neq \emptyset \\
v_1 v_2, & \text{otherwise}
\end{cases}
\]

\textit{It follows that } \( e^{-tL} v_1 v_2 = e^{-n(1 - \frac{n-2}{2})t} v_1 v_2 \). \textit{Similarly, } \( Q v_1^2 = (1 - \frac{1}{n-1}) v_1^2 + \frac{1}{n-1} \frac{|v|^2}{n} \).

Thus
\[
n(Q - I) \left( \frac{v_1^2}{|v|^2} \right) = \left( \begin{array}{cc}
-\frac{n}{n-1} & \frac{n}{n-1} \\
0 & 0
\end{array} \right) \left( \begin{array}{c}
\frac{v_1^2}{|v|^2}
\end{array} \right)
\]

\text{And since } \exp \left( t \left( \begin{array}{cc}
-\frac{n}{n-1} & \frac{n}{n-1} \\
0 & 0
\end{array} \right) \right) = \left( \begin{array}{cc}
e^{-\frac{n}{n-1}t} & 1 - e^{-\frac{n}{n-1}t} \\
0 & 1
\end{array} \right), \text{we obtain:}
\[
e^{-tL} v_1^2 = e^{-\frac{n}{n-1}t} v_1^2 + \left(1 - e^{-\frac{n}{n-1}t}\right) \frac{|v|^2}{n} \cdot
\]

\text{From these two identities Eq. (18) follows.}

\text{Next we prove Eq. (19). Let } \( a = n \left(1 - \frac{n-2}{2}\right) = \frac{4n-6}{n-1} \text{ and let } b = \frac{n}{n-1} \). \text{ We have}
\[a \geq 2b \text{ when } n \geq 2\]. \text{ The right-hand side of (19) can be written as}
\[e^{-at} (v, \xi)^2 + \left(e^{-bt} - e^{-at}\right) \sum_{i=1}^{n} \xi_i^2 v_i^2 - \frac{|\xi|^2 |v|^2}{n} e^{-bt}\]
This is bounded above by \( e^{-bt} \left( 1 - \frac{1}{n} \right) |v|^2 |\xi|^2 \) because \( \xi_i^2 \leq |\xi|^2 \). Similarly, \( e^{-tL} (v, \xi)^2 - \frac{|v|^2 |\xi|^2}{n} \geq -\frac{|v|^2 |\xi|^2}{n} \). Taking absolute values and using the observation that \( 1 - \frac{1}{n} \geq \frac{1}{n} \) completes the proof. \(\square\)

We are now ready to prove Proposition 1.

**Proof** (of Proposition 1) We start with the definition of \( d_2(e^{-tL} \mu, R_\mu) \).

\[
d_2(e^{-tL} \mu, R_\mu) = \sup_{\xi \neq 0} \frac{1}{|\xi|^2} \left| \int_{\mathbb{R}^n} \left( e^{-tL} \mu(dv) - R_\mu \right) e^{-2\pi i v \cdot \xi} \right|
\]

\[
= \sup_{\xi \neq 0} \frac{1}{|\xi|^2} \left| \int_{\mathbb{R}^n} e^{-tL} \mu(dv) \left( e^{-2\pi i v \cdot \xi} - R_{e^{-2\pi i v \cdot \xi}}(v) \right) \right|
\]

Here we used the self-adjointness of taking the angular average and the fact that \( e^{-tL} \mu \) and \( \mu \) have the same angular average. Here \( R_{e^{-2\pi i v \cdot \xi}} \) is the angular average of \( \exp(-2\pi i v \cdot \xi) \) which we study next. For brevity, let \( R \) denote \( R_{e^{-2\pi i v \cdot \xi}} \). Then \( R \) is also the angular average of \( \cos(2\pi v \cdot \xi) \) and we have

\[
R(v) = \int_{|y|=|v|} \cos(2\pi y_n |\xi|) \, dy = |S_n|-1 \int_{S_n-1} \cos(2\pi |v| |\xi| \cos \theta_1) \, d\sigma^n
\]

\[
= \frac{|S_n|-2}{|S_n|-1} \int_{\theta_1=0}^{\pi} \cos(2\pi |v| |\xi| \cos \theta_1) \sin(\theta_1)^{n-2} \, d\theta_1
\]

\[
= \frac{\int_{\theta_1=0}^{\pi} \cos(2\pi |v| |\xi| \cos \theta_1) \sin(\theta_1)^{n-2} \, d\theta_1}{\int_{\theta_1=0}^{\pi} \sin(\theta_1)^{n-2} \, d\theta_1}.
\]

Note that \( R(v) = F_1 \left( \frac{n}{2}, \frac{1}{4} (2\pi |v| |\xi|)^2 \right) \). Here \( F_1 \) is the hypergeometric function given by \( F_1(a, x) = 1 + \sum_{k=1}^\infty \frac{1}{a+1+2a} \cdots (k-1+a) \frac{x^k}{k!} \). Going back to \( d_2(e^{-tL} \mu, R_\mu) \), we can use the fact that using \( \int v \mu(dv) = 0 \) we have the following.

\[
d_2(e^{-tL} \mu, R_\mu) = \sup_{\xi \neq 0} \frac{1}{|\xi|^2} \left| \int_{\mathbb{R}^n} e^{-tL} \mu(dv) \left( e^{-2\pi i v \cdot \xi} - 1 + 2\pi i v \cdot \xi + R_{e^{-2\pi i v \cdot \xi}}(v) \right) \right|
\]

thus \( d_2(e^{-tL} \mu, R_\mu) \leq \sup_{\xi \neq 0} \int_{\mathbb{R}^n} e^{-tL} \mu(dv) \frac{|e^{-2\pi i v \cdot \xi} - 1 + 2\pi i v \cdot \xi|}{|\xi|^2} + \int_{\mathbb{R}^n} e^{-tL} \mu(dv) \frac{1-R(v)}{|\xi|^2} \, dv. \)

Taylor’s theorem gives \( |e^{ix} - 1 - ix| \leq \frac{1}{2} x^2 \) for all \( x \in \mathbb{R} \). We thus have

\[
\sup_{\xi \neq 0} \int_{\mathbb{R}^n} e^{-tL} \mu(dv) \frac{|e^{-2\pi i v \cdot \xi} - 1 + 2\pi i v \cdot \xi|}{|\xi|^2} \leq \frac{(2\pi)^2}{2} \int_{\mathbb{R}^n} e^{-tL} \mu(dv) \frac{(v, \xi)^2}{|\xi|^2} \, dv. \quad (20)
\]

This is of order \( 1 \) after time of \( O(\ln(n)) \) by Lemma 1. We now study the term containing \( (1-R(v))/|\xi|^2 \). It equals
This, together with Lemma 1, proves inequality (12).

To prove inequality (13), we need a way to “liberate” $e^{-tL}$ from inside the absolute values, so that Lemma 1 can be used. For $d_2(e^{-tL} \mu, e^{-tL} v)$, we can write

$$
|\xi|^{-2} \left| e^{-tL} \hat{\mu}(\xi) - e^{-tL} \hat{v}(\xi) \right| = \int_{\mathbb{R}^n} e^{2\pi i \xi \cdot v} \left( e^{-tL} \mu(dv) - e^{-tL} v(dv) \right)
$$

$$= \int_{\mathbb{R}^n} \left| e^{2\pi i \xi \cdot v} - 1 + 2\pi i \xi \cdot v \right| \left( e^{-tL} \mu(dv) - e^{-tL} v(dv) \right)
$$

$$\leq \frac{(2\pi)^2}{2} |\xi|^{-2} \int_{\mathbb{R}^n} (v, \xi)^2 \left| e^{-tL} [\mu(dv) - v(dv)] \right|,
$$
as in inequality (20). We now look at the term $|e^{-tL} [\mu(dv) - v(dv)]|$. Let $A$ be a measurable set. Recall that $Q_{i,j}(\theta)[A] = \{v : Q_{i,j}(\theta)[v] \in A\}; Q_{i,j}(\theta)$ can act on measures by the adjoint action $[Q_{i,j}(\theta)\mu](A) := \mu[Q_{i,j}(\theta)[A]]$. We have

$$
\left| [Q_{i,j}\mu](A) - [Q_{i,j}v](A) \right| = \int_{\mathbb{R}} \left| (Q_{i,j}(\theta)\mu)(A) - (Q_{i,j}(\theta)v)(A) \right| d\theta
$$

$$= \int_{\mathbb{R}} \left| (\mu[Q_{i,j}(\theta)(A)] - v[Q_{i,j}(\theta)(A)]) \right| d\theta
$$

$$\leq \int_{\mathbb{R}} |\mu - v| (Q_{i,j}(\theta)[A]) d\theta
$$

$$= Q_{i,j}|\mu - v|[A].
$$

From the convexity of $s \mapsto |s|$ it follows that $|e^{-tL}[\mu(dv) - v(dv)]| \leq e^{-tL}|\mu(dv) - v(dv)|$. Thus, we can use the self-adjointness of $L$ and let $e^{-tL}$ act on $(v, \xi)^2$. This allows us to apply Lemma 1 and obtain the desired upper bounds related to the second moment as follows.

$$
|\xi|^{-2} \left| e^{-tL} \hat{\mu}(\xi) - e^{-tL} \hat{v}(\xi) \right| \leq \frac{(2\pi)^2}{2} |\xi|^{-2} \int_{\mathbb{R}^n} e^{-tL} (v, \xi)^2 |\mu(dv) - v(dv)|
$$

$$\leq \frac{(2\pi)^2}{2} ((n - 1)e^{-t} + 1) \int_{\mathbb{R}^n} |v|^2 \mu(dv) - v(dv)|
$$

We now show that we can have $d_2(\mu, R \mu)$ can be of order $n$ at $t = 0$. If $\mu$ is a measure which is even, symmetric in its variables, and satisfies $\int \hat{v} \mu(dv) = 0$, $\int |v|^2 \mu(dv) < \infty$, but $\int_{\mathbb{R}^n} v_1 v_2 \mu(dv) = 0$, then we have

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\[ d_2(\mu, R_\mu) \geq \lim_{s \to 0} \lim_{\xi \to s(1,1,\ldots,1)} |\xi|^{-2} \left| \int \cos(2\pi v.\xi) [\mu(dv) - R_\mu(dv)] \right| \]

\[ = \lim_{s \to 0} \lim_{\xi \to s(1,1,\ldots,1)} |\xi|^{-2} \left| \int (\cos(2\pi v.\xi) - 1)[\mu(dv) - R_\mu(dv)] \right| \]

\[ = \frac{(2\pi)^2}{2} \lim_{s \to 0} \lim_{\xi \to s(1,1,\ldots,1)} |\xi|^{-2} \left| \int (v.\xi)^2 [\mu(dv) - R_\mu(dv)] \right| \]

\[ = \frac{(2\pi)^2}{2} \lim_{s \to 0} \lim_{\xi \to s(1,1,\ldots,1)} |\xi|^{-2} \left[ \int \left( (v.\xi)^2 - \frac{|v|^2|\xi|^2}{n} \right) \mu(dv) \right] \]

\[ = \frac{(2\pi)^2}{2} \left| \int v_1 v_2 \mu(dv) \right| \lim_{i \neq j} |\xi_i - \xi_j| = \frac{(n - 1)(2\pi^2)}{2} \left| \int v_1 v_2 \mu(dv) \right| . \]

and if \( \mu \) is an even measure concentrated on the line \( v_1 = v_2 = \ldots = v_n \), then \( \int v_1 v_2 \mu(dv) = \int v_1^2 \mu(dv) \) and \( d_2(\mu, R_\mu) \) is a multiple of the total energy. Proposition 1 says that this condition won’t last for time longer than \( O(\ln(n)) \). Also, if \( \mu \) has mean zero and has all correlations zero (as in Eq. (11)) then \( d_2(e^{-tL} \mu, R_\mu) \) never becomes of order \( n \) as shown by Eq. (12).

\section*{4 Proof of Theorem 1}

Let \( \mu \) be a probability measure with mean zero and finite second moment and let \( -L = n(Q - I) \). We use the fact that the Fourier transform commutes with the Kac evolution to take the problem into Fourier space. Because the second moment of \( \mu \) is finite, \( \hat{\mu} \) has bounded second derivatives. This will allow us to control \( |\hat{\mu} - R_\mu|_{L^\infty(r)} \) by \( |\hat{\mu} - R_\mu|_{L^2(Q)} \) on each sphere. The fact that the \( L^2 \) gap of the Kac operator in [5] gives an exponential decay in \( L^2(r) \) for each \( r \) leads to a decay in \( d_2(e^{-tL} \mu, R_\mu) \) after carefully obtaining order \( r^2 \) decay in \( e^{-tL} \hat{\mu} - \hat{R}_\mu \) as \( r \to 0^+ \) and combining the decay results on each sphere.

**Proof** (of Eq. (14)) Let \( u(t, \xi) \) be \( \hat{u}(t, \xi) = \hat{R}_\mu(\xi) \). From Eq. (13) we have that

\[
\left| \sum_{i,j} \eta_i \eta_j \partial_i \partial_j u(t, \xi) \right| = \left| -2\pi^2 \int (\vec{\eta}, \vec{v})^2 e^{-2\pi i v.\xi} e^{-tL} (\mu - R_\mu) \right| 
\]

\[
= (2\pi)^2 \left| \int_{R^2} (\vec{\eta}, \vec{v})^2 e^{-2\pi i v.\xi} e^{-tL} \{ (I - R)[\mu] \} (dv) \right| 
\]

\[
= (2\pi)^2 \left| \int_{R^2} (\vec{\eta}, \vec{v})^2 e^{-2\pi i v.\xi} (I - R)[e^{-tL}(\mu)](dv) \right| 
\]

\[
= (2\pi)^2 \left| \int_{R^2} (I - R) \left( (\vec{\eta}, \vec{v})^2 e^{-2\pi i v.\xi} \right) [e^{-tL}(\mu)](dv) \right| 
\]

\[
\leq (2\pi)^2 \left| \int_{R^2} (I + R)(\vec{\eta}, \vec{v})^2 e^{-tL}[\mu](dv) \right| 
\]

\[
= (2\pi)^2 \left| \int e^{-tL}(\vec{\eta}, \vec{v})^2 \mu(dv) + (2\pi)^2 \frac{|v|^2|\eta|^2}{n} \mu(dv) \right| 
\]
\[
= (2\pi)^2|\eta|^2 \left\{ 2 \int v_2^2 \mu(dv) + (n - 1)e^{-\frac{4\pi - \theta_1}{R}} \int v_1 v_2 \mu(dv) \right\}
=: L_p(t)|\eta|^2
\]
for all \( \xi, \eta \) and all \( t \geq 0 \). Fix \( t \) and \( r > 0 \). Let \( S = S^{n-1}(r) \) and choose \( \xi_0 \in S \) and \( \theta_0 \) so that \( e^{-i\theta_0}u(\xi_0) = |u|_{L^\infty(S)} \). Let
\[
B = S \cap \left\{ |\xi - \xi_0| \leq \frac{|u(\xi_0)|}{\sqrt{3Lp(t)}} \right\}.
\]
All of \( u, \xi_0, \) and \( B \) depend on \( t \), but we will suppress this dependence in many places. Our next task is to show that \( |u|_{L^\infty(r)} \) is of order \( r^2 \) as \( r \to 0 \), for \( d_2 \) to be bounded. We will accomplish this in Eq. (25) which shows that \( |u(\xi_0)| - |u(\xi)| \) is actually quadratic in \( |\xi - \xi_0| \) for \( \xi \in B \).

Let us first show that \( |u(\eta)| \geq |u(\xi_0)|/2 \) on \( B \). Let \( \eta \) be any point in \( \mathbb{R}^n \). By Taylor’s theorem we have:
\[
u(\eta) = u(\xi_0) + (\nabla u)(\xi_0).(|\eta - \xi_0|) + \frac{1}{2} \sum_{i,j} \partial_i \partial_j u(\xi_0^*)(|\eta - \xi_0|, |\eta - \xi_0|) j.
\]
Equation (21) bounds the term \( \frac{1}{2} \sum_{i,j} \partial_i \partial_j u(\xi_0^*) \) in absolute value by \( \frac{1}{2} Lp(t)|\eta - \xi_0|^2 \). We next study the linear term in Eq. (22) when \( \eta = \xi \in B \).

Since \( |u(\xi)|^2 \) has a maximum on \( S \) at \( \xi_0 \), we have either \( u(\xi_0) = 0 \) or \( \nabla |u(\xi_0)| \) is perpendicular to \( S \) at \( \xi_0 \), and thus \( \nabla u(\xi_0) \) is parallel to \( \xi_0 \). Without loss of generality we can take \( u(\xi_0) \neq 0 \) for otherwise \( u \equiv 0 \) on \( S \) and \( S \) does not contribute to \( d_2 \).

If follows from our assumptions, including the assumption that \( u(\xi_0) \neq 0 \), that we have
\[
|\nabla u(\xi_0)| \leq Lp(t)|\xi_0|, \tag{23}
\]
which might be false at other points on \( B \).

Equation (23) follows from the following observations. First, the fact that \( \nabla u(\xi_0) \) is parallel to \( \xi_0 \), thus \( |\xi_0, \nabla u(\xi_0)| = |\xi_0||\nabla(\xi_0)| \). Second,
\[
|\xi_0, \nabla(\xi_0)| = \sum (\xi_0) i \int_0^1 \partial_s (\partial_i u)(s\xi_0)ds = \left| \sum \sum (\xi_0) i \int_0^1 \partial_j \partial_i u(s\xi_0)ds \right|
\leq |\xi_0|^2 Lp(t)
\]
by (21).

Note that Eq. (22) with \( \xi_0 \) replaced by zero implies that
\[
|u(\eta, t)| \leq \frac{Lp(t)}{2} |\eta|^2 \quad \text{for any } \eta \text{ and } t,
\]
(24)
since \( u(t, 0) = \int \mu(dv) - \int R_\mu(dv) = 0 \) and \( \nabla u(t, 0) = -2\pi i \int \tilde{v} e^{-itL_{\mu}}(dv) = 0 \) for all \( t \).

In particular, we have \( \sqrt{|u(\xi_0)|} \leq \sqrt{Lp(t)/2} |\xi_0| \) and thus, for all \( \xi \in B \) we have \( |\xi - \xi_0| \leq \frac{1}{\sqrt{6}} |\xi_0| \). Hence \( \xi, \xi_0 \geq 0 \) on \( B \).

We now find an upper bound for \( |\nabla u(\xi_0).(|\xi - \xi_0|)| \) on \( S \). We choose a coordinate system in which \( \xi_0 = (0, \ldots, 0, 0, r) \) and \( \xi = (0, \ldots, w, \sqrt{r^2 - w^2}) \). Here we’re using the fact that \( \xi, \xi_0 > 0 \) on \( B \). Set the \( n \)th coordinate direction \( \tilde{e}_n \) to \( \xi_0/r \). Then \( |(\xi - \xi_0).e_n| = \)
At the same time we have the following lower bound.

\[ |r - \sqrt{r^2 - w^2}| = \frac{w^2}{r + \sqrt{r^2 - w^2}} \leq \frac{w^2}{r}. \]

Similarly, \(|\xi - \xi_0|^2 = w^2 + (r - \sqrt{r^2 - w^2})^2 = 2r^2(1 - \frac{w^2}{r^2}) \geq w^2\), which together with Eq. (23), gives the inequality

\[ |(\nabla u)(\xi_0). (\xi - \xi_0)| \leq Lp(t) r \times \frac{w^2}{r} \leq Lp(t) |\xi - \xi_0|^2. \]

In summary, we have shown that the for all \(\xi \in B\) the following inequality holds.

\[ |u(\xi_0) - u(\xi)| \leq \frac{3}{2} Lp(t) |\xi - \xi_0|^2. \tag{25} \]

This implies that we have \(|u(\xi)| \geq |u(\xi_0)| - \frac{3}{2} Lp(r) |\xi - \xi_0|^2 \geq \frac{|u(\xi_0)|}{2}\) on \(B\).

We complete the proof of Eq. (14) by a simple computation. We choose a coordinate system in which \(\xi_0\) points towards the North Pole and we denote by \(\theta\) the angle from the \(\xi_0\) axis. The largest value \(\theta_{\text{max}}\) of \(\theta\) on \(B\) satisfies the equation

\[ |\xi - \xi_0|_{\text{max}} = 2 r \sin \left( \frac{1}{2} \theta_{\text{max}} \right). \]

By integrating out the rest of the angular variables in \(\sigma^r\), we obtain

\[ \sigma^r (B) = \frac{2 \sin^{-1} \left( \frac{|u(\xi_0)|}{2r Lp(t)} \right)}{2^{\pi} \sin(\theta)^n} \theta \sin(\theta)^n \, d\theta \]

\[ = \frac{1}{2} \int_0^{r} \sin(\theta)^n \, d\theta \times \frac{2}{\sin(\theta)^n} \theta \sin(\theta)^n \, d\theta \times \frac{2}{\sin(\theta)^n} \theta \sin(\theta)^n \, d\theta \]

This gives us the lower bound \(||u(t, \xi)||_{L^2(r)}^2 \geq \frac{|u(t, \xi_0)|^2}{4} \sigma^r (B)\). Letting \(b(t, r) = \frac{|u(t, \xi_0)|}{r^2 Lp(t)}\), we obtain \(b \leq \frac{1}{2}\) for all \(t\) and have the following upper bound.

\[ \frac{||u(t, .)||_{L^2(r)}^2}{(Lp(t)r^2)^2} \leq e^{-2\lambda_1 t} \frac{||u(0, .)||_{L^2(r)}^2}{(Lp(t)r^2)^2} \leq e^{-2\lambda_1 t} \frac{||u(0, .)||_{L^\infty(r)}^2}{(Lp(t)r^2)^2} \leq \frac{(Lp(0))}{4} \frac{2^e}{4}. \tag{26} \]

At the same time we have the following lower bound.

\[ \frac{||u(t, .)||_{L^2(r)}^2}{(Lp(t)r^2)^2} \geq \frac{|u(t, \xi_0)|^2}{4(Lp(t)r^2)^2} \sigma^r (B) \geq \frac{b(t, r)^2}{4} \frac{2^e}{4} \frac{|\theta_{\text{max}}|^2}{2} \tag{27} \]

Equations (26) and (27) give the following inequality:

\[ b(t, r) \leq \frac{72}{23} e^{-\frac{4\lambda_1}{2\pi} t} \left( n - 1 \right) \left( \frac{n + 1}{n e^{-t} + 1} \right)^2 \int_0^{\pi} \sin(\theta)^n \, d\theta \times \left( \frac{72}{23} \right)^{\frac{n-1}{2}} \times 2^{(n+3)} \]

Finally, since \(n \geq 2\) and we have

\[ \sup_{k \geq 2} \left( \frac{23^2 (k - 1)(k + 1)^2}{72^2 (ke^{-t} + 1)^2} \int_0^{\pi} \sin(\theta)^n \, d\theta \right) \leq 2.1207 \]

\(\text{its the value when } k = 6 \text{ and } t = \infty\), we have \(b(t, r)\) is less than or equal to \(\frac{72}{23} \times 2.1207 e^{-\frac{4\lambda_1}{2\pi} t}\) and \(d_2(e^{-tL}\mu, R_\mu)\) is at most \(6.64Lp(t) e^{-\frac{4\lambda_1}{2\pi} t}\). 

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Remark 2 The proof of Theorem 1 relies on Eqs. (26) and (27) which can be seen as the norm \( L^\infty \) being interpolated between \((L^2)^{1/2}\) and \(W^{2,\infty} \). \(L^p(t)\) got through intact which potentially saves a factor \( n \) compared to \( L^p(0) \). It would be interesting function-analytically to see if more information can be incorporated in this interpolation inequality using the exact form of \( u \).

5 Construction of \( f_0 \)

In this section, for each \( n \geq 2 \) we construct a probability density \( f_\alpha \) on \( \mathbb{R}^n \) that is symmetric in its variables and has the property that

\[
\frac{d_2(e^{-tL} f_n, R f_n)}{d_2(f_n, R f_n)} \geq \max \left\{ 1 - \frac{e}{n} (2\lambda t)^{n-1}, 0 \right\}.
\]

This says that no matter how large \( n \) is, \( d_2(e^{-tL} f_n, R f_n) \) is practically unchanged for time at least \( \frac{1}{2\lambda} \). Although this result provides no information about the decay after time of order 1, it does rule out bounds of the form \( d_2(f(t), R f) \leq e^{-ct} d_2(f(0), R f) \) for any \( c \). Let us rescale the time so that \( \lambda = 1 \).

In Lemmas 1–3 we will construct a Schwartz function \( \psi(u) \) for which

\[
d_2(Q^k \psi, R \psi) = d_2(\psi, R \psi) \quad \text{for } k = 0, 1, \ldots, n - 2.
\]

We will scale \( \psi \) and add to it a positive Gaussian at large enough temperature to obtain a non-negative function \( f_\alpha \). The existence of \( \psi \) satisfying Eq. (28) is not very surprising and follows from the \( L^\infty \) nature of the \( d_2 \) metric and the fact that it takes \( n - 1 \) Kac rotations \( Q \) of a vector \( \vec{v} \) to cover the whole sphere \( |\vec{w}| = |\vec{v}| \). This is analogous to the result in [1] where it is shown that the total the variation distance between an initial permutation of a deck of cards and the uniform distribution is not affected by \( O(\ln(n)) \) riffle-shuffles. The reason for this invariance is because there are permutations that cannot be reached in less than \( O(\ln(n)) \) riffle-shuffles.

Since \( d_2 \) deals with the Fourier transforms, we will use the fact that the Fourier transform commutes with rotations, and thus with the Kac rotations \( Q_{i,j} \). We will directly construct the Fourier transform of \( f_n \)-s and only afterwards ensure that the inverse Fourier transform is non-negative and in \( L^1 \). As a first step we will construct a one parameter family of functions \( \phi(\xi; \alpha) \geq 0 \) such that \( Q^k \phi((z, 0, 0, \ldots, 0); \alpha) = 0 \) for all \( z, \alpha \) and all \( k \leq n - 2 \).

Let \( h(x; \alpha) = (1 - e^{-\alpha x^2}) \) and set \( \phi(\xi; \alpha) = \prod_{i=1}^n h(\xi; \alpha) \) (We will drop the parameter \( \alpha \) in \( \phi \) below.). Then we have the following lemmas.

Lemma 2 Properties of \( \phi \) Fix \( |\xi| = r \), and let \( z_1 = (r, 0, 0, \ldots, 0) \). Then for all \( l \leq n - 2 \) we have

1. \( |Q^l \phi|(z_1) = \phi(z_1) = 0 \).
2. \( R \phi(z_1) > \frac{1}{2} |\phi|_{L^\infty(r)} \); provided \( \alpha \geq \alpha(r) \) is large enough.
3. \[\left( \frac{m}{n-1} \right)^{n-1} Q^{n-1} \phi \] \( (z_1) \leq \frac{1}{n} (2\lambda)^{n-1} |\phi|_{L^\infty(r)} \).
4. \( |\phi|_{L^\infty(r)} = (1 - e^{-ar^2/n})^n \).

Remark 3 Properties (1) and (3) are easier to prove for the function \( \prod_{i=1}^n \xi_i^2 \). We use \( h(x; \alpha) \) instead of \( x^2 \) in \( \phi \) so that property (2) is satisfied. Properties (2) and (1) tell us that the maximum of

\[ |\phi(\xi) - R \phi| \]
on $S^{n-1}(r)$ is at $(\pm z_1, 0, \ldots, 0)$ because we know that

$$R_\phi(r) - |\phi|_{L^\infty(r)} \leq R_\phi(\xi) - \phi(\xi) \leq R_\phi(r),$$

and thus, on $S^{n-1}(r)$, we have

$$|\phi(\xi) - R_\phi| \leq \max \{ R_\phi(r), |R_\phi(r) - |\phi|_{L^\infty(r)}| \} = R_\phi(r)$$

by property (2).

**Remark 4** The coefficient of $Q^{n-1}[\phi](z_1)$ in property (3) comes from the Taylor expansion of $e^{-nt}Q\phi$.

**Proof** 1. Given a sequence of Kac rotations $Q_{i_1, j_1}(\theta_1), \ldots, Q_{i_k, j_k}(\theta_k)$, we can define a sequence of trigonometric polynomials $\{P_{1}^{(k)}, \ldots, P_{n}^{(k)}\}_{k=1}^\infty$ as follows. Let

$$\begin{pmatrix}
P_1^{(0)} \\
P_2^{(0)} \\
\vdots \\
P_n^{(0)}
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
\vdots \\
0
\end{pmatrix}.
$$

Once $\{P_i^{(s)}\}_{i=1}^n$ are defined, define $P_i^{s+1}(\theta_1, \ldots, \theta_k)$ using the equality

$$P_i^{s+1} = \begin{cases}
P_i^{(s)}(\theta_1, \ldots, \theta_k), & i \notin \{i_{s+1}, j_{s+1}\} \\
P_i^{(s)}(\theta_1, \ldots, \theta_k) \cos(\theta_{s+1}) - P_{j_{s+1}}^{(s)}(\theta_1, \ldots, \theta_k) \sin(\theta_{s+1}), & i = i_{s+1} \\
P_i^{(s)}(\theta_1, \ldots, \theta_k) \sin(\theta_{s+1}) + P_{j_{s+1}}^{(s)}(\theta_1, \ldots, \theta_k) \cos(\theta_{s+1}), & j = j_{s+1}
\end{cases}.$$

We are interested in these polynomials since they determine the velocity of particle 1 after the $k$ Kac collisions above in the relation:

$$v_l(\text{after}) = r P_i^{(k)}(\theta_1, \ldots, \theta_k)$$

We now show that if $i \geq 2$ is an index for which the “edges” $\{(i_1, j_1), \ldots, (i_k, j_k)\}$ do not connect “vertex” $i$ to vertex 1, then $P_l(\theta_1, \ldots, \theta_k) = 0$. Let $G$ denote the graph on $(v_1, \ldots, v_n)$ with edges $\{(i_1, j_1), \ldots, (i_k, j_k)\}$. Let $C$ be the connected component of $v_i$. An easy inductive argument shows that $\{P_j^{(l)} : j \in C\}$ depends only on $\{P_j^{(0)} : j \in C\}$, for $l = 0, 1, \ldots, k$. In particular, $P_i^{(k)}$ is obtained from $\{P_j^{(0)}(\theta_1, \ldots, \theta_k) : j \in C\}$ after possibly multiplying them by $\cos \theta$-s and $\sin \theta$-s, and adding them up. Since $P_j^{(0)} \equiv 0$ for $j \in C$, we have $P_i^{(k)}(\theta_1, \ldots, \theta_k) \equiv 0$.

As a conclusion, it follows that if $[Q_{i_k, j_k} \cdots Q_{i_1, j_1}]G(z_1; \alpha) \neq 0$, then we have

$$Q_{i_k, j_k} \cdots Q_{i_1, j_1} \prod_i \left(1 - e^{-\alpha x_i^2}\right)|z_1| = \frac{1}{(2\pi)^k} \int \prod_{i=1}^n \left(1 - e^{-\alpha x_i^2}(P_i^{(k)}(\cos(\theta_i), \sin(\theta_i))^2)\right) \times \prod_{j=1}^k d\theta_j \neq 0.$$

Thus the connected component $C$ of $i$ must contain 1 for each $i$. So $G$ is a connected graph which means that $k \geq n - 1$. Property (1) follows from the hypothesis that $k \leq n - 2$.  

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2. For \( r > 0 \) and \( n \geq 2 \) fixed,
\[
\frac{\phi}{|\phi|_{L^\infty(r)}} = \frac{\prod (1 - e^{-a\xi^2})}{(1 - e^{-ar^2/n})^n} \to 1
\]
almost everywhere on \( S^{n-1}(r) \) as \( ar^2 \to \infty \). Thus, by the dominated convergence theorem, there exists an \( \hat{A}(n) < \infty \) such that if \( ar^2 \geq \hat{A}(n) \) then \( \int_{S^{n-1}(r)} \phi(w)\sigma^r(dw) \geq \frac{1}{2} |\phi|_{L^\infty(r)} \). Let
\[
\alpha(r, n) = \frac{\hat{A}(n)}{r^2}. \tag{29}
\]

Note that the property of having an \( L^1(r) \) norm greater than or equal to \( \frac{1}{2} \) of the \( L^\infty(r) \) norm is preserved in time under the Kac evolution \( e^{-tL} \). This is because for positive functions, the Kac evolution does not change the \( L^1 \) norm, but it can only decrease the \( L^\infty \) norm. This observation is also true when we replace \( e^{-tL} \) by \( Q^k \).

3. By Cayley’s theorem there are \( n^{n-2} \) distinct trees on \( n \) vertices, and for each tree we can order its edges in \((n - 1)! \) ways. Each order of presentation of the edges in the tree comes with a weight \( (\frac{n}{2})^{-(n-1)} \). The terms \( Q_{i_1, j_1} \ldots Q_{i_n, j_n} |\phi(z_1) \) where the edges \( \{(i_1, j_1), \ldots, (i_n, j_n)\} \) do not connect all the vertices \( (v_1, \ldots, v_n) \) evaluate to zero. The rest of the terms are non-negative and bounded above by \( |\phi|_{L^\infty(r)} \). Thus,
\[
\frac{(nt)^{n-1}}{(n-1)!} (Q^{n-1}\phi)(z_1) \leq \frac{(nt)^{n-1}}{(n-1)!} (n-1)! n^{n-2} \leq \frac{e}{n} (2t)^{n-1} |\phi|_{L^\infty(r)}, \tag{30}
\]
proving property (3).

4. This property follows from an application of the method of Lagrange multipliers.

Since \( \alpha(r, n) \) in the above lemma is proportional to \( r^{-2} \), we need a way of keeping \( r = |\xi| \) strictly away from zero when \( d_2 \) is being evaluated. We do this in Lemma 3 by multiplying.

Let \( \psi(\xi) = \phi(\xi) A(\xi) \), where \( A(\xi) = |\xi|^d e^{-|\xi|^2} \). Then we have the following Lemma.

Lemma 3 Let \( A(\xi) = |\xi|^d e^{-|\xi|^2} \) and let \( b \) be smallest solution to \( (xe^{-x} = \frac{1}{2} e^{-1}) \) \((b \approx 0.23196)\). Let \( \alpha = \alpha(\sqrt{b}, n) \) be as in Eq. (29). If \( \psi = A(\xi)\phi(\xi) \). Then \( \frac{|\psi - R_\psi|}{|\xi|^2} \) has a maximum on \( \mathbb{R}^n - \{0\} \) at a point \((x, 0, 0, \ldots, 0) \) with \( x^2 \geq b \).

Proof Choose \( \alpha \) as in the hypothesis. Then \( R_\phi(\xi) \geq \frac{1}{2} \) when \( |\xi| \geq \sqrt{b} \) by property 2 of Lemma 2. In particular:
\[
\frac{|\psi(1,0,\ldots,0)-R_\psi(1)|}{|\xi|^2} = e^{-b} R_\phi(1,0,0,\ldots,0) \geq \frac{1}{2} e^{-b} \geq \frac{1}{2} e^{-1}. \]
So if \( |\xi|^2 < b \), then \( \frac{|\psi(\xi) - R_\psi(\xi)|}{|\xi|^2} < be^{-b} < \frac{1}{2} e^{-1} \). So we know that the maximum \( \frac{|\psi - R_\psi|}{|\xi|^2} \) is attained at a point \( \xi \) with norm at least \( \sqrt{b} \). So, for our choice of \( \alpha \), we have \( R_\phi \geq \frac{1}{2} |\phi|_{L^\infty(r)} \) and property 1 in Lemma 2 shows that \( \xi \) can be taken to be \((x, 0, \ldots, 0) \) for some \( x \geq \sqrt{b} \).

We now give an explicit formula for \( f_0 \).

Lemma 4 Let \( b, \alpha = \alpha(\sqrt{b}, n) \) be as in Lemma 3 and Eq. (29). Set
\[
f_0(v) = \left( \frac{0.9\pi}{1 + \alpha} \right)^{\frac{n}{2}} e^{-\left(\frac{0.9\pi^2}{1 + \alpha}\right)v^2} + \frac{1}{B(2\pi)^4} \Delta^2 \prod_{i=1}^n \left( \sqrt{\pi} e^{-\pi^2 v^2} - \sqrt{\frac{\pi}{1 + \alpha}} e^{-\frac{\pi^2 v^2}{1 + \alpha}} \right).
\]
If \( B > 0 \) is large enough, then \( f_0 \) is a probability density and Eq. (16) holds for \( f_0 \).
Proof Notice that \( f_0(v) \) is the sum of a Gaussian and \( \frac{1}{B} \hat{\psi} \). The Gaussian is radial at a high temperature since \( \alpha \) is large. For large \( |v| \), \( \hat{\psi} \) is bounded by polynomial of degree 4 times \( \exp(-\frac{\pi^2}{1+\sigma}|v|^2) \), so we can find a \( B = B(n) \) that makes \( |\hat{\psi}| \leq B \left( \frac{0.9\pi}{1+\sigma} \right) \frac{2}{e^{-\left( \frac{0.9\pi^2}{1+\sigma} \right)|v|^2}} \). This shows that when \( B \geq B(n) \) we have \( f_0 \geq 0 \). Since \( \psi \) is a Schwartz function, its Fourier transform is in \( L^1 \) and we have \( \int \hat{\psi}(v) \, dv = \psi(0) = 0 \). This shows that \( f_0 \) integrates to 1.

We now prove Eq. (16) for \( f_0 \). Note that \( \frac{e^{-tL}f_0(\xi) - Rf_0(\xi)}{|\xi|^2} = \frac{|e^{-tL}\psi(\xi) - R\psi|}{B|\xi|^2} \). We showed in Proposition 3 that when \( t = 0 \), this term is maximized at a point \( z_1 = (z_0, 0, 0, \ldots, 0) \) for some \( z_0 \geq \sqrt{B} \). Fix \( k \leq n - 2 \). Then

\[
0 \geq \frac{d_2(e^{-tL}f_0, Rf_0) - d_2(f_0, Rf_0)}{t^k} = \frac{d_2(e^{-tL}f_0, Rf_0) - R\psi (z_1)}{t^k}
\]

\[
\geq \frac{1}{Bt^k} \left( R\psi (z_1) - e^{-tL}\psi(z_1) \right) - R\psi(z_1) \geq -\frac{e^{-tL}\psi(z_1)}{Bt^k z_0^2} = -\frac{z_0^2 e^{-z_0^2}}{Bt^k} e^{-tL}\phi(z_1)
\]

(31)

Here we used the fact that \( e^{-tL}\psi \) and \( \psi \) have the same radial parts.

Recall from Lemma 2 that \( Q^l\phi(z_1) = \phi(z_1) = 0 \) for \( l = 0, 1, 2, \ldots, n - 2 \). Hence, the same is true for their linear combinations \( \left[ n^k(I - Q)^k\phi \right](z_1) \). Thus, by Taylor’s theorem, the right hand side in Eq. (31) converges to \( \frac{1}{2} \left( \frac{n^k}{n!} (I - Q)^k \phi(z_1) \right) \) as \( t \to 0^+ \), which is zero if \( k \leq n - 2 \). So \( e^{ntQ}(\phi) = e^{ntQ}Q^{n-1}(\phi)(z_1) \) for some \( t^* \) in \( (0, t) \) and we have:

\[
0 \geq \frac{d_2(e^{-tL}f, Rf) - d_2(f, Rf)}{t^{n-1}} = \frac{d_2(e^{-tL}f, Rf) - R\psi(z_1)}{Bz_0^2} \geq -\frac{z_0^2 e^{-z_0^2}}{Bt^{n-1}} e^{-tL}\phi(z_1) = e^{nt}Q^{n-1}(\phi)(z_1) = e^{ntQ}Q^{n-1}(\phi)(z_1)
\]

Since \( e^{ntQ}Q^{n-1}(\phi)(z_1) \) is less than \( |Q^{n-1}\phi|_{L^\infty(\mathbb{R}^n)} e^{tn} \), we conclude that

\[
\frac{d_2(e^{-tL}f, Rf) - d_2(f, Rf)}{t^{n-1}} \geq -\frac{n^{n-1}}{(n-1)!} e^{-\frac{z_0^2}{B}} e^{nt}Q^{n-1}(\phi)|_{L^\infty(\mathbb{R}^n)} e^{tn}.
\]

Combining this with property (3) in Lemma 2 gives Eq. (16).

6 Proof of the Propagation of Chaos

McKean gave in [13] a short algebraic proof of propagation of chaos for Kac’s original model on \( S^{n-1} \). This proof was adapted in [3] to give a propagation of chaos result for the fully thermostated Kac model. This section describes how McKean’s proof can be further modified to give a propagation of chaos result for the partially thermostated Kac model in [14].

Let \( Z = Z(\mathbb{R}^\infty, \text{symm}) \) be the space of bounded and continuous functions depending on an arbitrary but finite number of variables, endowed with the product

\[
f \otimes g (v_1, \ldots, v_a, v_{a+1}, \ldots, v_{a+b})\]

\[
e\frac{1}{(a+b)!} \sum_\sigma f(v_{\sigma(1)}, \ldots, v_{\sigma(a)}) g(v_{\sigma(a+1)}, \ldots, v_{\sigma(a+b)})
\]
and identify functions which have the same symmetrization: \( \int_{\mathbb{R}^\infty} f \, \phi \, dv = \int_{\mathbb{R}^\infty} g \, \phi \, dv \) for all \( \phi \in L^1(\mathbb{R}^\infty) \) that is symmetric in its variables. McKean observed that \( n\lambda(Q-I) \) can be approximated by \( 2\lambda \Gamma \). Here \( \Gamma \) is the operator given by

\[
\Gamma[\phi(v_1, \ldots, v_k)] = \sum_{i \leq k} \int_0^{2\pi} \phi(v_1, \ldots, v_i \cos \theta - v_{k+1} \sin \theta, v_{i+1}, \ldots, v_k) \, d\theta,
\]

that takes functions depending on \( k \) variables to functions depending on \( k+1 \) variables. Note that \( \Gamma \) is a derivation. That is, \( \Gamma[f \otimes g] = \Gamma[f] \otimes g + f \otimes \Gamma[g] \). McKean demonstrated that propagation of chaos holds for \( \{e^{tD} f_n\}_n \) whenever \( D \) is a derivation. McKean then showed the terms in the Taylor expansion of \( \int_{S^{n-1}} e^{\lambda n(Q-I)} f_n \phi d\sigma \) converged to the corresponding terms in \( \int_{S^{n-1}} e^{2\lambda t \Gamma} f_n \phi d\sigma \). Since both series converge absolutely when \( t \propto \frac{1}{k} \) is small enough, propagation of chaos follows.

The same proof was used in [3] to show that there is propagation of chaos for the fully thermostated Kac model. The observation there is that the generator \( -L = \eta \sum_{i=1}^n (M_i - I) + n\lambda(Q-I) \) can be approximated by \( \eta \sum_{i=1}^\infty (M_i - I) + 2\lambda \Gamma \) which is a derivation. Here \( M_i \) is the weaker Maxwellian thermostat acting on the \( i^{th} \) particle:

\[
M_i[f] = \int_{\mathbb{R}} \int_0^{2\pi} f(v_1, \ldots, v_i \cos \theta - w \sin \theta, v_{i+1}, \ldots, v_n) g(v_i \sin \theta + w \cos \theta) \, d\theta \, dw.
\]

We will tweak this proof, which works on both \( \{L(S^{n-1})\}_n \) and \( \{L^1(\mathbb{R}^n)\}_n \), for the partially thermostated Kac model. Suppose \( \alpha = \frac{m_0}{n_0} \) is the fraction of thermostated particles. Thermostating part of the particles divides the indices \( 1, \ldots, n \) into two groups \( A_n \) (the thermostated) and \( B_n \) (the rest). Our initial condition \( f_n(0, \ldots) \) should be symmetric under the exchange of particles in \( A_n \) and under the exchange of particles in \( B_n \). We want to have a space similar to \( Z \) and a derivation similar to \( \Gamma \) that adapt to the fact that a new particle introduced in the system is not always thermostated.

One approach is to let the underlying space be \( \bar{Z} = \bar{Z}((\mathbb{R}^{n_0})^\infty) \) and to let \( f, g \) all depend on \( kn_0, ln_0 \) variables. We can let every particle with index \( i \equiv 1, 2, \ldots, m_0(\text{mod} \ n_0) \) to be thermostated. We can define \( f \otimes g \) analogously by

\[
f \otimes g(v_1, \ldots, v_{kn_0}, v_{kn_0+1}, \ldots, v_{(k+1)n_0}) = \frac{1}{((k+1)m_0)!(k+1)(n_0 - m_0)!!} \times \sum_{\sigma} f(v_{\sigma(1)}, \ldots, v_{\sigma(kn_0)}) g(v_{\sigma(kn_0+1)}, \ldots, v_{\sigma((k+1)n_0)}).
\]

Here \( \sigma \) runs over all permutations leaving \( A_n \) (and also \( B_n \)) invariant. Our generator becomes \( -\mathcal{L}_k \) given by the equation

\[
-\mathcal{L}_k = kn_0\lambda(Q-I) + \eta \sum_{i=1}^{kn_0} 1_{[1, \ldots, m_0]}(i \text{ mod } n_0)(P_i - I).
\]

We replace \( \Gamma \) by \( \bar{\Gamma} : \bar{Z} \mapsto \bar{Z} \) that takes functions depending on \( kn_0 \) variables to functions on \( (k+1)n_0 \) variables. \( \bar{\Gamma} \) is given by

\[
\bar{\Gamma}[\phi](v_1, \ldots, v_{(k+1)n_0}) = \sum_{i \leq kn_0} \sum_{l=kn_0+1}^{(k+1)n_0} \int_0^{2\pi} \phi(v_1, \ldots, v_i \cos \theta - v_{k+1} \sin \theta, v_{i+1}, \ldots, v_k) \, d\theta.
\]
We see that \( \tilde{\Gamma} = 2\lambda \Gamma + \eta \sum_{i=1}^{kn_0} i [1, \ldots, m_0](i \mod n_0)(P_i - I) \). Hence \( \tilde{\Gamma} \) is a derivation. Note that we have the inequality

\[
\left\| \mathcal{L}_k \phi - 2\lambda \tilde{\Gamma} \phi - \eta \sum_{i=1}^{kn_0} i [1, \ldots, m_0](i \mod n_0)(P_i - I) \phi \right\| \leq \frac{t^2 n_0}{k^2} \left( \frac{ln_0}{2} \right) ||\phi|| + 2\lambda \frac{ln_0}{kn_0 + 1} ||\tilde{\Gamma} \phi||.
\]

whenever \( \phi \) depends only on \( ln_0 \) variables with \( l < k \). This goes to 0 when \( l \) is fixed and \( k \to \infty \).

Finally, for every \( k, l \geq 0 \), we have the following bound

\[
||\mathcal{L}_{k+l} \circ \mathcal{L}_{k+l-1} \circ \cdots \circ \mathcal{L}_{k+1} \circ \mathcal{L}_k f||_\infty \leq (4\lambda + 2\eta)/(l+1) k(k+1) \ldots (k+l-1)||f||_\infty.
\]

This makes \( \sum_{j=1}^{l^2} ||\mathcal{L}_{k+l-1} \circ \cdots \circ \mathcal{L}_{k+1} \circ \mathcal{L}_k f||_\infty \) converge for all \( k \) when \( t < \frac{1}{4\lambda + 2\eta} \). McKean’s proof can be used step by step from this point on (see also Lemma 19 in [3]) to give propagation of chaos for time \( t = \frac{0.9}{4\pi + 2\eta} \). Iterating this process \( j \)-times shows propagation of chaos for time up to \( \frac{0.9 j}{4\lambda + 2\eta} \), and hence for all \( t > 0 \) since \( j \) is arbitrary.

7 Conclusion

We saw in Theorem 1 that under the Kac evolution a Borel measure \( \mu \) approaches its angular average \( R_\mu \) in the GTW metric \( d_2 \) exponentially with rate at least \( O \left( \frac{1}{n} \right) \) and saw in Theorem 2 that the initial decay in \( d_2 \) can be very slow at least for time \( 1/(2\lambda) \) which is a macroscopic quantity. We also saw that the average energy per particle also controls \( d_2(\mu, R_\mu) \) after time of order \( ln(n) \). Proposition 1 suggests that the constant \( K \) in Theorem 1 is not optimal. This raises the question of what is the optimal \( K(n) \)? And whether our conjecture in (15) is correct. The proof of Theorem 1 gives an application of the \( L^2 \) gap to initial states that are not necessarily in \( L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \) and can be generalized to other evolutions which have gaps in \( L^2 \) provided their generators commute with the Fourier transform. For example: the Kac model in 1 dimension with an initial state not symmetric in its variables; the Kac model in 1 dimension with symmetric collision rules for which \( \theta \) in (3) has weight \( \rho(\theta) \) where \( \rho \) is not necessarily constant but satisfies \( \rho(2\pi - \theta) = \rho(\theta) \). It would be interesting to check if decay rates for Fourier based metrics can be obtained for non-Maxwellian molecules, where the collision rate between particles \( i \) and \( j \) is proportional to \( |v_i^2 + v_j^2|^\gamma \) for some \( \gamma \in (0, 2] \); and for the momentum conserving Kac model in 3 dimensions with Maxwellian molecules whose gap was computed in [6]. The functions \( \{f_n\} \) suggest a set of questions such as: can there be a sequence of distributions \( \mu_n \) similar to the \( \{f_n\} \)-s except that they are supported on the sphere? and, since the \( f_n \) are small \( L^1 \)-perturbations of Gaussians by Schwartz functions with a very particular algebraic structure, is there a physical interpretation to these structures? can we find functions \( f_n \) similar to the \( f_n \) for which there is a physical interpretation? Our lower bound in Theorem 2 is effective only when \( t \leq \frac{1}{\Delta} \). It should be possible to make this bound effective for a longer time interval by improving the upper bound in property (3) of Lemma 2. If we improve the bound \( |Q_{l_1, j_1} \cdots Q_{l_2, j_2} Q_{l_1, j_1} \phi(\gamma(z_1))| \leq |\phi||_{L^\infty(\gamma)} \) in Eq. (30), we will have a larger lower bound for \( d_2(e^{-itL_\mu}, R_\mu) \).
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References

1. Bayer, D., Diaconis, P.: Trailing the dovetail shuffle to its lair. Ann. Appl. Probab. 2(2), 294–313 (1992)
2. Bobylev, A.V., Cercignani, C.: On the rate of entropy production for the Boltzmann equation. J. Stat. Phys. 94, 3–4 (1999)
3. Bonetto, F., Loss, M., Vaidyanathan, R.: The Kac model coupled to a thermostat. J. Stat. Phys. 156(4), 647–667 (2014)
4. Bonetto, F., Loss, M., Tossounian, H., Vaidyanathan, R.: Uniform approximation of a Maxwellian thermostat by finite reservoirs. Commun. Math. Phys. 351(1), 311339 (2017)
5. Carlen, E.A., Carvalho, M.C., Loss, M.: Many-body aspects of approach to equilibrium. In: Journées Equations aux Dérivées Partielles (La Chapelle sur Erdre, 2000), pages Exp. No. XI, 12. Univ. Nantes, Nantes (2000)
6. Carlen, E.A., Geronimo, J., Loss, M.: On the Markov sequence problem for Jacobi polynomials. Adv. Math. 226(4), 3426–3466 (2011)
7. Carlen, E.A., Lebowitz, J.L., Mouhot, C.: Exponential approach to, and properties of, a non-equilibrium steady state in a dilute gas. Braz. J. Probab. Stat. 29(2), 372–386 (2015)
8. Einav, A.: On Villani’s conjecture concerning entropy production for the Kac master equation. Kinet. Relat. Models 4(2), 479–497 (2011)
9. Evans, J.: Non-equilibrium steady states in Kac’s model coupled to a thermostat. J. Stat. Phys. 164(5), 1103–1121 (2016)
10. Gabetta, E., Toscani, G., Wennberg, B.: Metrics for probability distributions and the trend to equilibrium for solutions of the Boltzmann equation. J. Stat. Phys. 81(5–6), 901–934 (1995)
11. Janvresse, E.: Spectral gap for Kac’s model of Boltzmann equation. Ann. Probab. 29, 288–304 (2001)
12. Kac, M.: Foundations of kinetic theory. In: Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. III, pp. 171–197. University of California Press, Berkeley (1956)
13. McKean Jr., H.P.: Speed of approach to equilibrium for Kac’s caricature of a Maxwellian gas. Arch. Ration. Mech. Anal. 21, 343–367 (1966)
14. Tossounian, H., Vaidyanathan, R.: Partially thermostat Kac model. J. Math. Phys. 56(8), 083301 (2015)
15. Villani, C.: Cercignani’s conjecture is sometimes true and always almost true. Commun. Math. Phys. 234(3), 455–490 (2003)