Vacuum polarisation of Dirac fermions in the cosmological de Sitter-global monopole spacetime

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Abstract

We study the vacuum polarisation effects of the Dirac fermionic field induced by a pointlike global monopole located in the cosmological de Sitter spacetime. First we derive the four orthonormal Dirac modes in this background in a closed form. Quantising the field using these modes, we then compute the fermionic condensate, $\langle 0 | \Psi \Psi | 0 \rangle$, as well as the vacuum expectation value of the energy-momentum tensor for a massive Dirac field, regularised in a particular way. We have used the Abel-Plana summation formula in order to extract the global monopole contribution to these quantities and have investigated their variations numerically with respect to relevant parameters.

Keywords : de Sitter spacetime, topological defects, fermions, vacuum polarisation

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1 Introduction

The early inflationary period of our universe is a phase of very rapid and nearly exponential accelerated expansion. This phase was proposed in order to resolve the three puzzles of the standard Big Bang cosmology – the horizon problem, the flatness problem and the problem of the hitherto unobserved magnetic monopoles [1, 2] (also references therein). The inflationary paradigm also satisfactorily explains how the primordial perturbations, which were quantum field theoretic in nature initially, grew, became large and classical and eventually developed into the large scale cosmic structures as we observe them today. After the end of the inflation, our universe entered respectively the phases of radiation and matter domination. These phases were also expanding but not with acceleration. Interestingly, the astrophysical data from high redshift supernovae, the galaxy cluster and the cosmic microwave background suggest that our current universe is also undergoing a phase of accelerated expansion (see [3] and references therein). In order to drive such accelerated expansions, the universe should be endowed with some exotic matter with negative isotropic pressure, called the dark energy. The simplest and phenomenologically one of the most successful model of the dark energy is simply the positive cosmological constant, and the corresponding solution of the Einstein equation is known as the de Sitter spacetime, having an exponential scale factor and hence a constant Hubble rate.

Quantum field theory in the de Sitter background might thus make interesting physical predictions whose imprint can be found for example, in the cosmic microwave background. A very important topic in this area is the non-perturbative infrared or secular effect at late times, which is likely to have connection to the cosmological constant and the cosmic coincidence problem, e.g. [4] and references therein.

In this paper, we shall however be interested in the field theoretic effects in the de Sitter universe endowed with topological defects. Such defects might have created via some symmetry breaking phase transitions in
the early universe after the Big Bang. They may be in the form of cosmic strings, the global monopoles, domain walls and textures \[5, 6, 7, 8\]. Our interest in this paper will be the global monopoles, first proposed in \[9\] (see also \[10, 11\]). It is a spherically symmetric topologically stable gravitational defect with a deficit in the solid angle \(4\pi\). This may result from a global-symmetry breaking \((O(3) \rightarrow U(1))\) of a self interacting scalar triplet. Such defects might have singular structure at the centre (like the pointlike global monopole). However, such singularity can be relaxed by taking a finite core and allowing the core to inflate \[12\]. The topological defect that has perhaps received more attention than the others is the cosmic string \[6\], and references therein. It is a cylindrically symmetric spacetime with a \(\delta\)-function like line singularity along the symmetry axis. Note that although such defects introduce inhomogeneity, their role in the primordial structure formation is highly suppressed \[13\]. Nevertheless, such defects may also create other physical effects like the emission of gravitational waves and lensing \[14, 15\]. In particular, for a realistic situation like a network of cosmic strings, the characteristic caustics and the cusps in the lensing phenomena is largely expected. Besides the structure formation, the topological defects may be relevant to the baryon asymmetry of the universe \[16\]. The superconducting cosmic strings \[17\] or the stable loops of current-carrying string called the vortons \[18\], may be responsible for the production of the high energy cosmic rays \[19\] or even the gamma ray bursts \[20\].

The vacuum polarisation effects in the presence of a cosmic string is a much cultivated topic, both in flat and (anti)-de Sitter backgrounds. The study of the vacuum expectation values of the field squared, \(\langle 0|\phi^2(x)|0\rangle\) of a scalar field and of the fermionic condensate, \(\langle 0|\psi(x)\bar{\psi}(x)|0\rangle\), the vacuum expectation values of their energy-momentum tensors as well as of the conserved currents and various anomalies have extensively been investigated in \[21]-\[61\]. Apart from the usual Minkowski, de Sitter/anti-de Sitter backgrounds, these studies also include spacetimes with compactified dimensions, and even the presence of a magnetic flux. Several of such studies have been made in higher dimensions as well.

In this paper we wish to compute the vacuum polarisation effects of the Dirac fermions induced by a pointlike global monopole located in the cosmological de Sitter spacetime. For example, the renormalised Euclidean Green function and the vacuum expectation value of the energy-momentum tensor for a fermionic field in a global monopole located in flat spacetime was computed in \[62\] in the context of the braneworld scenario. The vacuum expectation value of the energy-momentum tensor for massless as well as massive fermions were computed in \[63, 64\]. Similar investigations were carried out by considering nontrivial core structure instead of a pointlike global monopole in \[65\]. The ground state energy of a massive scalar field in this background using the \(\zeta\)-function regularisation can be seen in \[66\]. Finite temperature effects on the vacuum expectation values of the energy-momentum tensor of the massless spin-1/2 fermions can be seen in \[67, 68\]. We further refer our reader to various quantum field theoretic computations in such backgrounds including the higher dimensions and the Kaluza-Klein theory in \[69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79\]. While all the above computations are done in flat backgrounds, the computation of renormalised vacuum expectation value of the two point function for a scalar field as well the energy-momentum tensor in a de Sitter global monopole background can be seen in \[80\].

The rest of the paper is organised as follows. In the next Section and Appendix A, we present the derivation of the Dirac modes in the de Sitter global monopole background in a closed form. Using these mode functions, we compute the fermionic vacuum condensate, \(\langle 0|\nabla^2\psi|0\rangle\), and the vacuum expectation value of the energy-momentum tensor for the fermionic field respectively in Section 3, Section 4 and Section 5. Appendix B and Appendix C contain some detail for these Sections. Finally we conclude in Section 6. We shall work in 3 + 1-dimensions with mostly negative signature of the metric, and will set \(\hbar = 1\) throughout. We shall chiefly follow the method involving the Abel-Plana summation formula, as described in e.g. \[28, 53\] in the context of the de Sitter cosmic string background.
2 The Dirac modes

We shall present below the derivation of an orthonormal, complete set of Dirac modes in the de Sitter background with a pointlike global monopole defect. The metric in the cosmological time reads,

\[ ds^2 = dt^2 - e^{2t/\alpha} \left[ dr^2 + \beta^2 r^2 d\Omega^2 \right], \]

(1)

where \( \alpha^{-2} = \Lambda/3 \), \( \Lambda \) being the cosmological constant. The parameter \( \beta \) is conventionally expressed as \( \beta^2 = (1 - 8\pi G \xi^2) \) \cite{10}, where \( \xi \) stands for the symmetry breaking scale, which lies well below the Planck energy, \( G^{-1/2} \). Thus \( 0 < \beta^2 \leq 1 \), and we have a deficit in the solid angle, \( 4\pi(1 - \beta^2) \) in Eq. (1).

In terms of the conformal time, \( \eta = -\alpha e^{-t/\alpha} \) \((\infty < \eta < 0^-)\), the above metric reads,

\[ ds^2 = \frac{\alpha^2}{\eta^2} \left[ d\eta^2 - dr^2 - \beta^2 r^2 d\Omega^2 \right] \]

(2)

The pointlike global monopole breaks the maximal symmetry of the de Sitter spacetime and introduces a curvature singularity at \( r = 0 \) \cite{80},

\[ R = \frac{12}{\alpha^2} + \frac{2(1 - \beta^2)}{\alpha^2 \beta^2 r^2} \eta^2 \]

Setting \( \eta/\alpha \to 1 \) in Eq. (1) recovers the metric of the flat spacetime with a global monopole, which may interestingly also describe an effective metric produced in the superfluid \(^3\)He-A, but with a negative deficit in the solid angle \cite{65}.

The four orthonormal Dirac modes in the above background are given by (see Appendix A for detail),

\[
\Psi_{\sigma jlm}^{(+)}(\eta, r, \theta, \phi) = \frac{\lambda \sqrt{\pi} e^{\pi \alpha/2}}{2\beta \alpha^{3/2} \sqrt{r}} \left( \eta^2 H_{1/2 - im\alpha}^{(1)}(\lambda|\eta|) J_{\nu_s}(\lambda r) \Omega_{jl,m} \right) \\
\Psi_{\sigma jlm}^{(-)}(\eta, r, \theta, \phi) = \frac{\lambda \sqrt{\pi} e^{-\pi \alpha/2}}{2\beta \alpha^{3/2} \sqrt{r}} \left( i(-1)^\sigma (\hat{\sigma} \cdot \hat{\sigma}) \eta^2 H_{-1/2 - im\alpha}^{(2)}(\lambda|\eta|) J_{\nu_s}(\lambda r) \Omega_{jl,m} \right)
\]

(3)

where \( \sigma = 0, 1 \) correspond to two different solutions and the \((\pm)\)-sign in the superscripts represent respectively positive and negative frequency solutions. The \( \hat{\sigma} \)'s are the Pauli matrices. The \( \Omega \)'s are the spin-1/2 spherical harmonics and \( l_\sigma, \nu_\sigma \) are respectively given in Eq. (74) and below Eq. (82). \( \lambda \) is a real, positive parameter.

It is easy to check that the above modes satisfy

\[
(\Psi_{\sigma jlm}^{(+)}(r), \Psi_{\sigma' j'l'm'}^{(+)}) = (\Psi_{\sigma jlm}^{(-)}(r), \Psi_{\sigma' j'l'm'}^{(-}) = \delta_{jj'} \delta_{\nu \nu'} \delta_{ll'} \delta_{mm'} \delta_{\sigma \sigma'}
\]

with rest of the inner products vanishing. Thus they form a complete and orthonormal set.

3 The fermionic condensate, \( \langle 0 | \Psi \Psi | 0 \rangle \)

Using the complete, orthonormal mode functions of Eq. (3), we can quantise the fermionic field in the de Sitter-global monopole background. Using this field quatisation, we wish to compute below the fermionic
condensate, i.e. the vacuum expectation value of the operator $\overline{\Psi} \Psi$. By the mode-sum formula (e.g. [55] and references therein), we have

$$\langle 0 \vert \overline{\Psi} \Psi \vert 0 \rangle = \sum_{\sigma=0,1} \sum_{l,j,m} \int_0^\infty d\lambda \overline{\Psi}_{\sigma jlm}^{(-)}(x) \Psi_{\sigma jlm}^{(-)}(x)$$

(5)

where $\overline{\Psi}^{(\pm)} = \Psi^\dagger (\pm \gamma^{(0)})$ is the adjoint spinor. Using the second of Eq. (3) and the completeness relationship for the spherical harmonics, Eq. (5) can be expanded as

$$\langle 0 \vert \overline{\Psi} \Psi \vert 0 \rangle = \frac{\eta^4 e^{-\eta \sigma}}{16 \beta^2 \alpha^3 r} \sum_{j=1/2}^\infty (2j + 1) \int_0^\infty d\lambda \lambda^2 (J_{\nu_1}^2(\lambda r) + J_{\nu_0}^2(\lambda r)) \left( |H_{1/2-na}^{(2)}(\lambda \eta)|^2 - |H_{-1/2-na}^{(2)}(\lambda \eta)|^2 \right)$$

(6)

where $\nu_\sigma (\sigma = 0, 1)$ is given below Eq. (82). In order to evaluate the above integral, we use the following identities e.g. [53],

$$|H_{\pm 1/2-na}^{(2)}(\lambda \eta)|^2 = \frac{4}{\pi} \eta e^{\eta \sigma} |K_{1/2-na}(i\lambda \eta)|^2, \quad K_{1/2-na}(x) = K_{-1/2-na}(x)$$

$$K_{1/2-na}(i\lambda \eta)^2 - K_{1/2+na}(i\lambda \eta)^2 = -\frac{i}{\lambda} \left( \partial_{\eta} + \frac{1 - 2i\alpha}{\eta} \right) K_{1/2-na}(i\lambda \eta) K_{1/2-na}(-i\lambda \eta)$$

(7)

where $K$ is the modified Bessel function of the second kind. Then Eq. (6) takes the form

$$\langle 0 \vert \overline{\Psi} \Psi \vert 0 \rangle = -\frac{i \eta^3}{4 \pi^2 \beta^2 \alpha^3 r} \sum_{j=1/2}^\infty (2j + 1) \int_0^\infty d\lambda \lambda^2 (J_{\nu_1}^2(\lambda r) + J_{\nu_0}^2(\lambda r)) (\eta \partial_{\eta} + 1 - 2i\alpha) K_{1/2-na}(i\lambda \eta) K_{1/2-na}(-i\lambda \eta)$$

(8)

In order to evaluate the above integral we need to simplify the product $K_{1/2-na}(i\lambda \eta) K_{1/2-na}(-i\lambda \eta)$. We write it as an integral representation [81, 82, 83, 84]

$$K_{1/2-na}(i\lambda \eta) K_{1/2-na}(-i\lambda \eta) = \int_0^\infty \frac{du}{u} \int_0^\infty dy' \cosh 2\mu y' e^{-2u \eta^2 \sinh^2 y' - 1/2u}, \mu = 1/2 - i\alpha$$

(9)

Substituting Eq. (9) into Eq. (8), we have

$$\langle 0 \vert \overline{\Psi} \Psi \vert 0 \rangle = -\frac{i \eta^3}{4 \pi^2 \beta^2 \alpha^3 r} (\eta \partial_{\eta} + 1 - 2i\alpha) \sum_{j=1/2}^\infty (2j + 1) \int_0^\infty \frac{du}{u} e^{-1/2u} \int_0^\infty dy' \cosh 2\mu y'$$

$$\times \int_0^\infty d\lambda \left( J_{\nu_1}^2(\lambda r) + J_{\nu_0}^2(\lambda r) \right) e^{-2\lambda^2 u r^2 \sinh^2 y'}$$

$$= -\frac{i \eta^3}{4 \pi^2 \beta^2 \alpha^3 r} (\eta \partial_{\eta} + 1 - 2i\alpha) \sum_{j=1/2}^\infty (2j + 1) \int_0^\infty dy' \cosh 2\mu y' \int_0^\infty \frac{du}{u} e^{-1/2u} \int_0^\infty \frac{du}{u} e^{-u/4\eta^2 \sinh^2 y'} e^{-r^2/(4\eta^2 \sinh^2 y')}$$

$$\times \left[ I_{\nu_1}(r^2/4\eta^2 \sinh^2 y') + I_{\nu_0}(r^2/4\eta^2 \sinh^2 y') \right],$$

(10)

where $I_{\nu}$ is the modified Bessel function of the first kind. We introduce a new variable $x = r^2/(4\eta^2 \sinh^2 y')$ and perform the $y'$-integral in Eq. (10) to have

$$\langle 0 \vert \overline{\Psi} \Psi \vert 0 \rangle = -\frac{i \eta^3}{8 \pi^2 \beta^2 \alpha^3 r^3} (\eta \partial_{\eta} + 1 - 2i\alpha) \sum_{j=1/2}^\infty (2j + 1) \int_0^\infty dx e^{-x(1-\eta^2/r^2)} (I_{\nu_1}(x) + I_{\nu_0}(x)) K_{1/2-na}(x \eta^2/r^2)$$

(11)
We next use \[53\]

\[(\eta \partial_\eta + 1 - 2i m \alpha) e^{x \eta^2 / r^2} K_{1/2 - im \alpha}(x \eta^2 / r^2) = \frac{2x \eta^2}{r^2} e^{x \eta^2 / r^2} \left( K_{1/2 - im \alpha}(x \eta^2 / r^2) - K_{-1/2 - im \alpha}(x \eta^2 / r^2) \right),\]

in order to further simplify Eq. (11) as

\[
\langle 0|\overline{\Psi}\Psi|0 \rangle = \frac{\eta}{2\pi^2 \beta^2 \alpha^2} \sum_{j=1/2}^{\infty} (2j + 1) \int_0^\infty dy \, y e^{y(1 - r^2 / \eta^2)} \text{Im}[K_{1/2 - im \alpha}(y)] (I_{\nu_1}(yr^2 / \eta^2) + I_{\nu_0}(yr^2 / \eta^2)) \quad (12)
\]

where following [53], we also have defined a new variable, \( y = x \eta^2 / r^2 \). We wish to split now \( \langle 0|\overline{\Psi}\Psi|0 \rangle \) into two parts – one corresponding to the pure de Sitter spacetime without any monopole (\( \beta = 1 \)) and the other corresponding to the global monopole defect, so that we define

\[
\langle 0|\overline{\Psi}\Psi|0 \rangle_{gm} := \langle 0|\overline{\Psi}\Psi|0 \rangle - \langle 0|\overline{\Psi}\Psi|0 \rangle_{dS} \quad (13)
\]

Similar things can be seen in e.g. [53] and references therein in the context of cosmic strings. For the pure de Sitter part, we have the renormalised expression,

\[
\langle 0|\overline{\Psi}\Psi|0 \rangle_{dS, \text{Ren.}} = \frac{m}{2\pi^2 \alpha^2} \left( 1 + m^2 \alpha^2 \right) \left[ \ln(m \alpha) - \text{Re}(m \alpha) + \frac{1}{12} \right], \quad (14)
\]

where \( \psi \) is the digamma function [84]. Although the above expression has been found earlier in various references (e.g. [53] in the context of de Sitter cosmic strings), we have outlined its derivation in Appendix B, as a check of consistency of our modes, Eq. (3). A couple of comments regarding the definition of Eq. (13) are in order here. It will turn out below that the quantity \( \langle 0|\overline{\Psi}\Psi|0 \rangle_{gm} \) is ultraviolet finite. In other words, the curvature induced by the defect \( \beta < 1 \) in Eq. (1) does not induce any divergence. The divergence only comes from the pure de Sitter part and can be renormalised via a cosmological constant counterterm. This seems to have some qualitative similarity (but not the same) with the Hadamard subtraction cum regularisation of the Feynman propagator in a general curved spacetime using the Riemann normal coordinates, e.g. [85]. One can try regularisation techniques different from that of the present one, as well. Apart from the left hand side being regular, the decomposition of Eq. (13) has the obvious advantage regarding the application of the Abel-Plana summation formula, eventually helping us to do many computations analytically, we will see below. Despite this advantage, however, this regularisation scheme might have some caveat as well, as we shall point out in Section 6.

After subtracting the integral expression of \( \langle 0|\overline{\Psi}\Psi|0 \rangle_{dS} \), Eq. (96), from Eq. (12), it turns out from Eq. (13) that we need to evaluate the expression,

\[
\sum_j (j + 1/2) \left[ \left( \frac{1}{\beta^2} I_{j+1/2} + 1/2 - I_{j+1/2} + 1/2 \right) + \left( \frac{1}{\beta^2} I_{j+1/2} - 1/2 - I_{j+1/2} - 1/2 \right) \right],
\]

which can be done conveniently by using the Abel-Plana summation formula [86],

\[
\sum_{j=0}^\infty f(j) = \frac{f(0)}{2} + \int_0^\infty dv f(v) - i \int_0^\infty du \frac{(f(iu) - f(-iu))}{e^{2\pi u} + 1} \quad (15)
\]
Thus it follows from the above equation that

$$\sum_j [f(j/\beta)/\beta - f(j)] = -i \int_0^\infty du (f(iu) - f(-iu)) \left( \frac{1}{e^{2\pi u/\beta} + 1} - \frac{1}{e^{2\pi u} + 1} \right)$$  \hspace{1cm} (16)$$

When we apply this formula in our case we have the relevant integral to be

$$\sum_{j=1/2}^\infty (j + 1/2) \left[ \frac{1}{\beta^2} I_{1+1/2+1/2}(y^{2}/\eta^{2}) - I_{j+1/2+1/2}(y^{2}/\eta^{2}) + \frac{1}{\beta^2} I_{1+1/2-1/2}(y^{2}/\eta^{2}) - I_{j+1/2-1/2}(y^{2}/\eta^{2}) \right]$$

$$= -\frac{4}{\pi} \int du g(\beta, u) \text{Im} \left[ K_{1/2+iu}(y^{2}/\eta^{2}) \right]$$  \hspace{1cm} (17)$$

where we have defined,

$$g(\beta, u) = \cosh \pi u \left( \frac{1}{e^{2\pi \beta u} + 1} - \frac{1}{e^{2\pi u} + 1} \right)$$  \hspace{1cm} (18)$$

Using this, we have after some algebra

$$\langle 0 | \overline{\Psi} \Psi | 0 \rangle_{\text{gm}} = \langle 0 | \overline{\Psi} \Psi | 0 \rangle - \langle 0 | \overline{\Psi} \Psi | 0 \rangle_{\text{dS}}$$

$$= -\frac{4\eta}{\pi^3 \alpha^{3/2}} \int_0^\infty du g(\beta, u) \int_0^\infty dy y e^{y(1-r^{2}/\eta^{2})} \text{Im}[K_{1/2-im\alpha}(y)] \text{Im}[K_{1/2+iu}(y^{2}/\eta^{2})]$$  \hspace{1cm} (19)$$

We shall evaluate the above integral numerically. However, two special cases are of interest, for which analytic expressions can be found. The first is when the proper distance from the monopole ($r = 0$) is small, $r\alpha/|\eta| \to 0$. Recalling $y = x\eta^{2}/r^{2}$ (cf., the discussion below Eq. (12)), we have for large $y$-values \[84\]

$$\text{Im}[K_{1/2-im\alpha}(y)] \approx -\frac{m\sqrt{\pi \alpha}}{2y^{3/2}} e^{-y},$$

so that Eq. (19) becomes

$$\langle 0 | \overline{\Psi} \Psi | 0 \rangle_{\text{gm}} = \frac{2^{3/2} \eta m}{\pi^3 \alpha^{3/2} r} \int_0^\infty du g(\beta, u) \int_0^\infty dy y^{-1/2} e^{-y^{2}/\eta^{2}} \text{Im}[K_{1/2+iu}(y^{2}/\eta^{2})]$$  \hspace{1cm} (20)$$

Using now the integral \[82\]

$$\int_0^\infty d\zeta \zeta^{\beta-1} e^{-\zeta} K_\nu(\zeta) = \frac{\sqrt{\pi} \Gamma(\beta + \nu) \Gamma(\beta - \nu)}{2^\beta \Gamma(\beta + 1/2)}$$  \hspace{1cm} (21)$$

Eq. (20) becomes

$$\langle 0 | \overline{\Psi} \Psi | 0 \rangle_{\text{gm}} \bigg|_{r/|\eta| \to 0} = \frac{m \eta^{2}}{\pi^3 \alpha^{3/2} r^{2}} \text{Im} \int_0^\infty du g(\beta, u) \Gamma(1 + iu) \Gamma(-iu)$$

$$= \frac{m}{24\pi (r\alpha/\eta)^{2}} \left( \frac{1}{\beta^2} - 1 \right) + \frac{m}{2\pi^3 (r\alpha/\eta)^{2}} \sum_{n=0}^{\infty} (-1)^n \left( \frac{\zeta(2, 1 + (n + 1)\beta)}{\zeta(2, 1 + (n + 1))} - \zeta(2, 1 + (n + 1)) \right)$$  \hspace{1cm} (22)$$
where we have used $\Gamma(1 - iu)\Gamma(iu) = i\pi/\sin\pi u$ [84] and the expansion,

$$g(\beta, u) = \frac{1}{\sinh \pi u} \left( \frac{1}{\pi^{2\beta u} + 1} - \frac{1}{\pi^{2\beta u} + 1} \right) + \frac{2}{\pi^{2\beta u} - 1} \sum_{n=0}^{\infty} (-1)^n \left( e^{-2\pi(n+1)\beta u} - e^{-2\pi(n+1)u} \right)$$

(23)

In order to obtain Eq. (22), we also have used the formula given in [82] involving the multiple $\zeta$-function

$$\zeta(2, 1 + (n + 1)\beta) = \sum_{n_1=1}^{\infty} \frac{1}{(n_1 + 1)^2} \sum_{n_2=1}^{n_1} \frac{1}{(n_1 + 1)^{1+n(n+1)\beta}}$$

The above series is obviously convergent, as $\beta > 0$. For example, for $\beta = 0.5$, its numerical value is close to 0.9, as can be easily checked using e.g., Mathematica.

Thus, the fermionic condensate Eq. (22) diverges as the square of the proper radial distance $r\alpha/|\eta|$, as we move towards the monopole, expected due to the curvature singularity at $r = 0$. Note also that the condensate vanishes for $m = 0$.

As the second special case, let us consider the opposite scenario, i.e. large proper distance from the monopole, $r\alpha/|\eta| \gg 1$. Since $y = x\eta^2/r^2$ (cf., the discussion below Eq. (12)), we make an expansion for small $y$-values [82]

$$K_{1/2 - ima}(y) \approx \frac{1}{2} \Gamma\left( \frac{1}{2} - ima \right) \left( \frac{y}{2} \right)^{ima - 1/2}$$

Substituting the above into Eq. (19) and using Eq. (21), we find

$$\langle 0 | \bar{\Psi} \Psi | 0 \rangle_{gm} \bigg|_{r/|\eta| \to \infty} \approx -\frac{4\eta}{\pi^4 \alpha^3 r} \int_0^\infty du \, u \, g(\beta, u) \int_0^\infty dy \, e^{-yr^2/\eta^2} \text{Im} \left[ \frac{1}{2} \Gamma(1/2 - ima) \left( \frac{y}{2} \right)^{ima - 1/2} \text{Im}[K_{1/2 + im}(yr^2/\eta^2)] \right] = \frac{1}{2^{1/2} \pi^{7/2} \alpha^3 (r/\eta)^4} \int_0^\infty du \, u^2 \, g(\beta, u) \left[ \frac{\Gamma(1/2 - ima)}{\Gamma(2 + ima)} \left( \frac{\eta}{2r} \right)^{2ima} \Gamma(1 + ima + iu)\Gamma(1 + ima - iu) \right] = \frac{2^{1/2} \alpha f(\beta, ma)}{\pi^{7/2} (r/\eta)^4} \sin (2ma \ln (2r/|\eta|) - \phi_0)$$

(24)

showing quartic fall off along with oscillatory behaviour. The phase $\phi_0$ and the function $f(\beta, ma)$ above are determined by the complex relationship

$$f(\beta, ma)e^{i\phi_0} = \frac{\Gamma(1/2 - ima)}{\Gamma(2 + ima)} \int_0^\infty du \, u^2 \, g(\beta, u)\Gamma(1 + ima + iu)\Gamma(1 + ima - iu)$$

(25)

Note that if we set $m = 0$ above, the integral becomes real and hence $\phi_0$ becomes vanishing. This makes Eq. (24) vanishing too, in the massless limit. We also note that the radial dependence of the divergence in Eq. (22) or the fall off in Eq. (24), are qualitatively similar to that of the de Sitter cosmic string [53].

Finally, we have plotted the variation of the condensate $\langle 0 | \bar{\Psi} \Psi | 0 \rangle_{gm}$, Eq. (19), with respect to the dimensionless mass parameter $ma$ and the dimensionless proper distance squared $r^2/\eta^2$, in Fig. 1, for two different values of the defect parameter $\beta$. For sufficiently high values of either of these variables, the curves tend to merge. The first plot shows that the condensate is vanishing for $m = 0$, in agreement with our previous results found for the special cases.
4 Vacuum expectation value of the energy-momentum tensor

We next wish to compute the vacuum expectation value of the fermionic energy-momentum tensor in the background Eq. (2). We have for the Dirac field,

\[ T_{\mu\nu} = i \frac{1}{2} [\Psi (\gamma_{\mu} \nabla_{\nu}) \Psi - (\nabla_{\mu} \Psi)(\gamma_{\nu}) \Psi] \]  

(26)

Expanding the spinor and its adjoint in terms of mode functions, and taking the expectation value with respect to the vacuum, we have

\[ \langle 0 | T_{\mu\nu} | 0 \rangle = i \frac{1}{2} \int_{0}^{\infty} d\lambda \sum_{\sigma,j,l,m} [\Psi^{(-)}(x) \gamma_{\mu} \nabla_{\nu} \Psi^{(-)}(x) - (\nabla_{\mu} \Psi^{(-)})(\gamma_{\nu}) \Psi^{(-)}(x)] \]  

(27)

In the above expression we encounter anti-commutators between the \( \gamma \)'s and the spin connection matrices, \( [\gamma_{\mu}, \Gamma_{\nu}]_{+} \). However, using Eq. (64) and Eq. (65), it is easy to see that such anti-commutators vanish for all \( \mu, \nu \), leaving us only with the partial derivatives to deal with,

\[ \langle 0 | T_{\mu\nu} | 0 \rangle = i \frac{1}{2} \int_{0}^{\infty} d\lambda \sum_{\sigma,j,l,m} [\Psi_{\sigma;jlm}^{(-)}(x) \gamma_{\nu} \partial_{\mu} \Psi_{\sigma;jlm}^{(-)}(x) - (\partial_{\mu} \Psi_{\sigma;jlm}^{(-)})(\gamma_{\nu}) \Psi_{\sigma;jlm}^{(-)}(x)] \]  

(28)

Using now the second of Eq. (3), we shall explicitly compute below the above expression component-wise.

4.1 Energy density

Using Eq. (64) in order to express the curved space \( \gamma \)-matrices in terms of the flat space ones, we have from Eq. (28)

\[ \langle 0 | T_{\mu\nu}^{0} | 0 \rangle = i \frac{1}{2} \int_{0}^{\infty} d\lambda \sum_{\sigma,j,l,m} [\Psi_{\sigma;jlm}^{(-)}(x) \partial_{\nu} \Psi_{\sigma;jlm}^{(-)}(x) - (\partial_{\nu} \Psi_{\sigma;jlm}^{(-)})(x) \Psi_{\sigma;jlm}^{(-)}(x)] \]  

(29)
Using now the second of Eq. (3) and the formula for the spherical harmonics,

\[ \sum_{m_j = -j}^{m_j = +j} |\Omega_{jlm}|^2 = \frac{2j + 1}{4\pi}, \]  

Eq. (29) takes the form after some algebra,

\[ \langle 0| T_0^0 | 0 \rangle = \frac{\eta^5}{4\pi^2 \beta^2 \lambda^4 r^4} \sum_{j=1/2}^{\infty} (2j + 1) \int d\lambda \left( J_{2j+1/2+1/2}^2 (\lambda r) + J_{2j+1/2-1/2}^2 (\lambda r) \right) \left[ \left( \partial_n^2 + \frac{2}{\eta} \partial_n + \frac{4i\alpha (1/2 - i\alpha)}{\eta^2} \right) K_{1/2-\alpha m} (i\lambda |\eta|) K_{1/2-\alpha m} (-i\lambda |\eta|) \right] \]  

Rearranging the above expression a little bit, we have

\[ \langle 0| T_0^0 | 0 \rangle = \frac{\eta^5}{4\pi^2 \beta^2 \lambda^4 r^4} \sum_{j=1/2}^{\infty} (2j + 1) \left[ \left( \partial_n^2 + \frac{2}{\eta} \partial_n + \frac{4i\alpha (1/2 - i\alpha)}{\eta^2} \right) \right. \]

\[ \times \int d\lambda \left( J_{2j+1/2+1/2}^2 (\lambda r) + J_{2j+1/2-1/2}^2 (\lambda r) \right) K_{1/2-\alpha m} (i\lambda |\eta|) K_{1/2-\alpha m} (-i\lambda |\eta|) \]

\[ +4 \int d\lambda \lambda^2 \left( J_{2j+1/2+1/2}^2 (\lambda r) + J_{2j+1/2-1/2}^2 (\lambda r) \right) K_{1/2-\alpha m} (i\lambda |\eta|) K_{1/2-\alpha m} (-i\lambda |\eta|) \]

Using the integral representation for the product \( K_{1/2-\alpha m} (i\lambda |\eta|) K_{1/2-\alpha m} (-i\lambda |\eta|) \), Eq. (9), we reduce the first integral in Eq. (32) to

\[ \int d\lambda \left( J_{2j+1/2+1/2}^2 (\lambda r) + J_{2j+1/2-1/2}^2 (\lambda r) \right) K_{1/2-\alpha m} (i\lambda |\eta|) K_{1/2-\alpha m} (-i\lambda |\eta|) \]

\[ = \frac{1}{2r^2} \int_0^\infty dx e^{-x} \left( I_{2j+1/2+1/2} (x) + I_{2j+1/2-1/2} (x) \right) e^{x\eta^2/r^2} K_{1/2-\alpha m} (x\eta^2/r^2), \]

where for the variable \( x \) we used the previous definition \( x = r^2 / (4\eta^2 \sinh^2 y') \) (cf., the discussion below Eq. (10)) and have performed the integration over \( y' \).

Similarly, for the second integral in Eq. (32), we have

\[ \int d\lambda \lambda^2 \left( J_{2j+1/2+1/2}^2 (\lambda r) + J_{2j+1/2-1/2}^2 (\lambda r) \right) K_{1/2-\alpha m} (i\lambda |\eta|) K_{1/2-\alpha m} (-i\lambda |\eta|) \]

\[ = -\frac{2}{r^4} \frac{\partial}{\partial \eta^2} \int_0^\infty dx e^{-x} \left( I_{2j+1/2+1/2} (x) + I_{2j+1/2-1/2} (x) \right) e^{x\eta^2/r^2} K_{1/2-\alpha m} (x\eta^2/r^2) \]

As earlier, by defining \( x = yr^2 / \eta^2 \), we rewrite the differential operator appearing in Eq. (32) as,

\[ \partial_n^2 + \frac{2\partial_n}{\eta} + \frac{4i\alpha (1/2 - i\alpha)}{\eta^2} = \frac{4x}{r^2} \left( x\partial_x^2 + \frac{3\partial_x}{x} + \frac{i\alpha (1/2 - i\alpha)}{x} \right) \]

\[ (35) \]
Substituting now Eq. (33), Eq. (34) and Eq. (35) into Eq. (32), we have the vacuum expectation value of the energy density,

\[ \langle 0 | T_0^0 | 0 \rangle = \frac{\eta^5}{2\pi^3\beta^2\alpha^4 r^5} \sum_{j=1/2}^{\infty} (2j+1) \int_0^{\infty} dx \, x \, e^{-x} \left( I_{-1/2+1/2}(x) + I_{1/2-1/2}(x) \right) \times \left( x \partial_x^2 + \left( \frac{3}{2} - 2x \right) \partial_x - 2 + \frac{ima(1/2 - im\alpha)}{x} \right) e^{x \eta^2 / r^2} K_{1/2-ima}(x\eta^2 / r^2) \]  

(36)

Following now [53], we use the properties of \( K_{1/2-ima} \) to simplify the operator part of the above equation as

\[ \left( x \partial_x^2 + \left( \frac{3}{2} - 2x \right) \partial_x - 2 + \frac{ima(1/2 - im\alpha)}{x} \right) e^{x \eta^2 / r^2} K_{1/2-ima}(x\eta^2 / r^2) = -\frac{1}{2} e^{x \eta^2 / r^2} \left( K_{1/2-ima}(x\eta^2 / r^2) + K_{-1/2+ima}(x\eta^2 / r^2) \right) \]  

(37)

Using the above expression and also once again converting the variable to \( y = x\eta^2 / r^2 \) for our convenience, the integral of Eq. (36) takes the form,

\[ \langle 0 | T_0^0 | 0 \rangle = \frac{\eta}{4\pi^2\beta^2\alpha^4 r} \sum_{j=1/2}^{\infty} (2j+1) \int_0^{\infty} dy \, y \, e^{y(1-r^2/\eta^2)} \left( I_{-1/2+1/2}(y\eta^2 / r^2) + I_{1/2-1/2}(y\eta^2 / r^2) \right) \Re [K_{1/2-ima}(y)], \]  

(38)

### 4.2 Radial pressure

We have

\[ \langle 0 | T_{rr} | 0 \rangle = \frac{i}{2} \int d\lambda \sum_{\sigma jlm} \left[ \Psi_{\sigma jlm}^{(+)}(0) \gamma_r \partial_r \Psi_{\sigma jlm}^{(-)} - (\partial_r \Psi_{\sigma jlm}^{(+)}(0) \gamma_r \Psi_{\sigma jlm}^{(-)} \right] \]  

(39)

which, after using Eq. (3) and Eq. (64) becomes

\[ \langle 0 | T_{rr} | 0 \rangle = \frac{\eta^5}{4\pi^2\beta^2\alpha^4 r} \sum_{j=1/2}^{\infty} (2j+1) \int d\lambda \lambda^3 \left( J_{-1/2+1/2}(\lambda r) \partial_r J_{1/2-1/2}(\lambda r) - J_{1/2+1/2}(\lambda r) \partial_r J_{-1/2-1/2}(\lambda r) \right) \times \left( K_{1/2-ima}(i\lambda|\eta|) K_{1/2-ima}(-i\lambda|\eta|) + c.c. \right) \]  

(40)

In order to evaluate the integral in Eq. (40), we further use [53]

\[ J_{-1/2+1/2}(\lambda r) \partial_r J_{1/2-1/2}(\lambda r) - J_{1/2+1/2}(\lambda r) \partial_r J_{-1/2-1/2}(\lambda r) = \frac{1}{2} \left( \frac{1}{2} \partial_r^2 + \frac{1}{r^2} \partial_r + \frac{j + 1/2}{\beta} \left( \frac{1/2 - j + 1/2}{r^2} \right) \right) J_{2-1/2-1/2}(\lambda r) + 2J_{2j+1/2-1/2}(\lambda r) \]  

(41)
Substituting this into Eq. (40), we have

\[
\langle 0 | T_r^\nu | 0 \rangle = -\frac{\eta^5}{4\pi^2\beta^2\alpha^4r^4} \sum_{j=1/2}^{\infty} (2j + 1) \int d\lambda \left( \frac{1}{2} \partial_r^2 + \frac{1}{r} \partial_r + \frac{j + 1/2}{\beta} \left( \frac{1/2 - j + 1/2}{r^2} \right) \right) J^2_{j+1/2-1/2}(\lambda r)
\]

\[
+ 2\lambda^3 J^2_{\frac{3}{2}j-1/2}(\lambda r) \left( K_{1/2-ima}(i\lambda|\eta)|K_{1/2-ima}(-i\lambda|\eta)| + c.c. \right)
\]

Now the first term in the above integral can be rewritten as, after using the integral representation Eq. (9)

\[
\int d\lambda \lambda J^2_{j+1/2-1/2}(\lambda r) \left( K_{1/2-ima}(i\lambda|\eta)|K_{1/2-ima}(-i\lambda|\eta)| + c.c. \right)
\]

\[
= \frac{1}{2r^2} \int_0^\infty dx e^{-x} I_{\frac{3}{2}j+1/2-1/2}(x) e^{\eta^2/r^2} \left( K_{1/2-ima}(x|\eta|^2/r^2) + c.c. \right)
\]

Likewise, the second integral of Eq. (42) can be recast as

\[
\int d\lambda J^2_{\frac{3}{2}j-1/2}(\lambda r) \left( K_{1/2-ima}(i\lambda|\eta)|K_{1/2-ima}(-i\lambda|\eta)| + c.c. \right)
\]

\[
= \frac{2}{r^4} \int_0^\infty dx \left( x\partial_x + 1 \right) e^{-x} I_{\frac{3}{2}j+1/2-1/2}(x) e^{\eta^2/r^2} \left( K_{1/2-ima}(x|\eta|^2/r^2) + c.c. \right)
\]

where in both the above cases we have defined the variable \( x = yr^2/\eta^2 \), as earlier.

Using now Eq. (43) Eq. (44) and once again Eq. (9) into Eq. (42), and converting the variables to \( y = x\eta^2/r^2 \), we find after a little algebra

\[
\langle 0 | T_r^\nu | 0 \rangle = \frac{\eta^5}{4\pi^2\beta^2\alpha^4r^4} \sum_{j=1/2}^{\infty} (2j + 1) \int_0^\infty dy \ y e^{y(1-r^2/\eta^2)} \text{Re}[K_{1/2-ima}(y)]
\]

\[
\times \left( I_{\frac{3}{2}j+1/2+1/2}(yr^2/\eta^2) + I_{\frac{3}{2}j+1/2-1/2}(yr^2/\eta^2) \right) = \langle 0 | T_r^\nu | 0 \rangle
\]

where the last equality follows from comparison with Eq. (38).

### 4.3 The angular stresses

We have

\[
\langle 0 | T_\theta^\theta | 0 \rangle = \frac{\eta^5}{4\pi^2\beta^2\alpha^4r^4} \sum_{j=1/2}^{\infty} (2j + 1)^2 \int d\lambda (j + 1/2) \partial_r \left( J_{j+1/2}(\lambda r) J_{j+1/2}(\lambda r) \right)
\]

\[
\times \left( K_{1/2-ima}(i\lambda|\eta)|K_{1/2-ima}(-i\lambda|\eta)| + c.c. \right)
\]

For the product of the two Bessel functions appearing above, we use the relation [28]

\[
J_{j+1/2-1/2}(\lambda r) J_{j+1/2+1/2}(\lambda r) = \frac{1}{\lambda} \left( \frac{(j + 1/2)/\beta - 1/2}{r^2} - \frac{1}{2} \partial_r \right) J^2_{j+1/2-1/2}(\lambda r)
\]
We then have after using Eq. (9),

$$
\sum_{j=1/2}^{\infty} \frac{(2j+1)^2}{2} \int d\lambda \left( \frac{(j+1/2)/\beta - 1/2}{r} - \frac{1}{2}\partial_{r} \right) J_{j+1/2 - 1/2}(\lambda r) \times \left( K_{1/2 - im\alpha}(i\lambda|\eta)|K_{1/2 - im\alpha}(-i\lambda|\eta) \right) + c.c. 
$$

$$
\times \left( K_{1/2 - im\alpha}(i\lambda|\eta)|K_{1/2 - im\alpha}(-i\lambda|\eta) \right) + c.c. 
$$

$$
= \frac{r}{\eta^2} \sum_{j=1/2}^{\infty} (2j+1)^2 \int_0^\infty dy \ e^{y(1-r^2/\eta^2)} Re \left[ K_{1/2 - im\alpha}(y) \right] \sum_{j=1/2}^{\infty} \left( I_{j+1/2 - 1/2}(yr^2/\eta^2) - I_{j+1/2 + 1/2}(yr^2/\eta^2) \right) 
$$

$$
\times \left( K_{1/2 - im\alpha}(i\lambda|\eta)|K_{1/2 - im\alpha}(-i\lambda|\eta) \right) + c.c. 
$$

Substituting now Eq. (47), into Eq. (46), we obtain after some algebra

$$
\langle 0|T_\phi^0|0 \rangle = \frac{\eta}{4\pi^2\beta^3 \alpha^4} \int_0^\infty dy \ e^{y(1-r^2/\eta^2)} Re \left[ K_{1/2 - im\alpha}(y) \right] \sum_{j=1/2}^{\infty} \left( I_{j+1/2 - 1/2}(yr^2/\eta^2) - I_{j+1/2 + 1/2}(yr^2/\eta^2) \right) 
$$

$$
\times \left( K_{1/2 - im\alpha}(i\lambda|\eta)|K_{1/2 - im\alpha}(-i\lambda|\eta) \right) + c.c. 
$$

(48)

Note that the angular symmetry of the our background Eq. (2) trivially guarantees that $\langle 0|T_\phi^0|0 \rangle = \langle 0|T_\phi^0|0 \rangle$. Finally, using Eq. (3) and Eq. (64) into Eq. (28), it can be easily seen that $\langle 0|T_\mu^\mu|0 \rangle = 0$, for all $\mu \neq \nu$, leaving us only with the diagonal components for the vacuum expectation value.

5 Pure global monopole contribution to $\langle 0|T_\mu^\mu|0 \rangle$

Now as of Section 3, we wish to extract below the pure global monopole contribution from the expressions Eq. (38), Eq. (45), Eq. (48). We write in analogy of Eq. (13)

$$
\langle 0|T_\mu^\mu|0 \rangle_{GM} = \langle 0|T_\mu^\mu|0 \rangle - \langle 0|T_\mu^\mu|0 \rangle_{dS}, \quad (49)
$$

The derivation of the pure de Sitter contribution, $\langle 0|T_\mu^\mu|0 \rangle_{dS}$ (corresponding to $\beta = 1$ in Eq. (38), Eq. (45), Eq. (48)) is very briefly sketched in Appendix C, once again for a check of consistency of our mode functions. Note that $\langle 0|T_\mu^\mu|0 \rangle_{dS}$ has been computed in numerous places earlier, e.g. in [53] in the context of de Sitter cosmic strings. Owing to the maximal symmetry of the de Sitter spacetime, each component of this vacuum expectation value is the same constant, Eq. (105).

Subtracting now the first equation of Eq. (102) respectively from Eq. (38), Eq. (45), Eq. (48), we have
the corresponding pure global monopole contributions,

\[
\langle 0 | T^0_0 | 0 \rangle_{\text{gm}} = \frac{\eta}{2\pi^2 \omega^4 r} \int_0^\infty dy \, y \, e^{y(1-r^2/\eta^2)} \text{Re}[K_{1/2-i \alpha \omega}(y)] \sum_{j=1/2}^{\infty} \frac{(j+1/2)}{\beta^2} \frac{1}{\beta^2} \left( I_{j+1/2-1/2}(yr^2/\eta^2) - I_{j+1/2-1/2}(yr^2/\eta^2) \right)
\]

\[
\langle 0 | T^r_r | 0 \rangle_{\text{gm}} = \langle 0 | T^0_0 | 0 \rangle_{\text{gm}},
\]

\[
\langle 0 | T^\theta_\theta | 0 \rangle_{\text{gm}} = \frac{\eta}{2\pi^2 \omega^4 r} \int_0^\infty dy \, y \, e^{y(1-r^2/\eta^2)} \text{Re}[K_{1/2-i \alpha \omega}(y)] \sum_{j=1/2}^{\infty} \frac{(j+1/2)^2}{\beta^2} \left( I_{j+1/2-1/2}(yr^2/\eta^2) - I_{j+1/2-1/2}(yr^2/\eta^2) \right)
\]

Applying now the Abel-Plana summation formula Eq. (15) into the above equations, we get

\[
\langle 0 | T^0_0 | 0 \rangle_{\text{gm}} = (0 | T^r_r | 0 \rangle_{\text{gm}} = \langle 0 | T^\theta_\theta | 0 \rangle_{\text{gm}} = \frac{\eta}{2\pi^2 \omega^4 r} \int_0^\infty dy \, y \, e^{y(1-r^2/\eta^2)} \text{Re}[K_{1/2-i \alpha \omega}(y)] \text{Im}[K_{1/2-i \alpha \omega}(y)]
\]

\[
\langle 0 | T^r_r | 0 \rangle_{\text{gm}} = \langle 0 | T^\theta_\theta | 0 \rangle_{\text{gm}} = \frac{\eta}{2\pi^2 \omega^4 r} \int_0^\infty dy \, y \, e^{y(1-r^2/\eta^2)} \text{Re}[K_{1/2-i \alpha \omega}(y)] \text{Re}[K_{1/2-i \alpha \omega}(y)]
\]

where the function \( g(\beta, u) \) is defined in Eq. (18). Also, in deriving \( \langle 0 | T^\theta_\theta | 0 \rangle_{\text{gm}} \) (or \( \langle 0 | T^r_r | 0 \rangle_{\text{gm}} \)), we have used,

\[
\sum_{j=1/2}^{\infty} \frac{1}{\beta^2} \left( I_{j+1/2-1/2} - I_{j+1/2-1/2} \right) = \frac{4}{\pi} \int_0^\infty du \, u^2 \text{Re}[K_{1/2-i \alpha \omega}(u)]
\]

Let us now take the Minkowski limit of the above expressions. This should correspond to \( t/\alpha \to 0 \) in the metric, where \( t \) is the cosmological time. This means the conformal time \( \eta \approx -\alpha \) in this limit. Thus in Eq. (51), defining a variable \( z = yr^2/\alpha^2 \), we have for a fixed \( z \)-value and large \( \alpha \) [84],

\[
\text{Re}[K_{1/2-i \alpha \omega}(z \alpha^2)] \approx \sqrt{\frac{\pi}{2z \alpha^2}} e^{-\frac{\pi^2}{8z \alpha^2}}
\]

Substituting the above into Eq. (51), we have

\[
\langle 0 | T^0_0 | 0 \rangle_{\text{gm}}^{(M)} = \langle 0 | T^r_r | 0 \rangle_{\text{gm}}^{(M)} = \langle 0 | T^\theta_\theta | 0 \rangle_{\text{gm}}^{(M)} = \frac{2^{1/2}}{\pi^{5/2} \omega^4} \int_0^\infty du \, u \, g(\beta, u) \int_0^\infty dz \, z^{1/2} e^{-\frac{\pi^2}{8z \alpha^2}} \text{Im}[K_{1/2-i \alpha \omega}(z)]
\]

\[
\langle 0 | T^\mu_\mu | 0 \rangle_{\text{gm}}^{(M)} = \langle 0 | T^\nu_\nu | 0 \rangle_{\text{gm}}^{(M)} = \frac{2^{1/2}}{\pi^{5/2} \omega^4} \int_0^\infty du \, u \, g(\beta, u) \int_0^\infty dz \, z^{1/2} e^{-\frac{\pi^2}{8z \alpha^2}} \text{Re}[K_{1/2-i \alpha \omega}(z)]
\]

Setting \( m^2 = 0 \) above, let us compute the trace of \( \langle 0 | T^\mu_\mu | 0 \rangle_{\text{gm}}^{(M)} \). We have

\[
\langle 0 | T^\mu_\mu | 0 \rangle_{\text{gm}}^{(M)} |_{m^2=0} = \frac{2^{1/2}}{\pi^{5/2} \omega^4} \int_0^\infty du \, u \, g(\beta, u) \int_0^\infty dz \, z^{1/2} e^{-\frac{\pi^2}{8z \alpha^2}} \left( \text{Im}[K_{1/2-i \alpha \omega}(z)] + u \text{Re}[K_{1/2-i \alpha \omega}(z)] \right)
\]
We now separate the right hand side of Eq. (21) into real and imaginary parts and substitute into the above integral. It is easy to see that both the $z$-integrals vanish. Thus the trace anomaly induced by the global monopole in the flat spacetime for fermionic field is vanishing.

Likewise, we can show that the trace anomaly induced by the global monopole in the de Sitter spacetime is vanishing, too. Setting $m = 0$ in Eq. (51), we have

$$
\langle 0| T^{\mu}_{\nu} | 0 \rangle_{gm} \bigg|_{m=0} = \frac{2^{3/2}}{\pi^{5/2} (\alpha \eta)^{\frac{1}{2}} (\eta^{2} \pi^{2})^{\frac{1}{2}}} \int_{0}^{\infty} du \, g(\beta, u) \int_{0}^{\infty} dx \, x^{1/2} \, e^{-x} \left( \text{Im} \left[ K_{1/2 - iu} (x) \right] + u \text{Re} \left[ K_{1/2 - iu} (x) \right] \right)
$$

(55)

where we have used $K_{1/2} = \sqrt{\pi} e^{-y/2y}$ [53], and the variable transformation $x = y r^{2} / \eta^{2}$ to arrive at the above expression. The above integral is formally similar to Eq. (54) and hence is vanishing.

The vanishing of the above trace anomalies can also be seen from the vanishing of the condensate $(0 | \Psi \Psi | 0)_{gm}$ for $m = 0$, as discussed in Section 3. From Eq. (26) and the Dirac equation, the trace anomaly for the fermionic field is given by

$$
\langle 0 | T^{\mu}_{\nu} | 0 \rangle_{gm} \bigg|_{m=0} = \lim_{m \to 0} m (0 | \Psi \Psi | 0)
$$

(56)

Thus it is clear that the trace anomaly induced by the global monopole in the de Sitter spacetime (and hence in the Minkowski spacetime) should be vanishing. This also shows that the decompositions of Eq. (13) and Eq. (49) are consistent with each other.

It can also be easily checked that the global monopole contribution to the energy momentum tensor Eq. (51), obeys the conservation equation, $\nabla_{\mu} \langle T^{\mu}_{\nu} \rangle_{gm} = 0$. (The pure de Sitter part Eq. (105), satisfies the same trivially). In the background Eq. (2), we have two independent components of $\nabla_{\mu} \langle T^{\mu}_{\nu} \rangle_{gm}$,

$$
\partial_{\eta} \langle T^{\eta}_{0} \rangle_{gm} + \frac{1}{r} (\langle T^{r}_{r} \rangle_{gm} + 2 \langle T^{\phi} \rangle_{gm} - 3 \langle T^{0}_{0} \rangle_{gm}), \quad \text{and} \quad \partial_{r} \langle T^{r}_{0} \rangle_{gm} + \frac{2}{r} (\langle T^{r}_{r} \rangle_{gm} - \langle T^{0}_{0} \rangle_{gm})
$$

(57)

Substituting the components from Eq. (51), it is easy to check that both the above expressions vanish.

As of Section 3, let us now look into the two special cases of Eq. (51), i.e. small and large proper distances from the monopole. Using for the real part of the modified Bessel function [84] for small argument,

$$
\text{Re} \left[ K_{1/2 - ima} (y) \right] \approx \sqrt{\frac{\pi}{2y}} e^{-y} \left( 1 - \frac{m^{2} \alpha^{2}}{2y} \right)
$$

Substituting the above into the first of Eq. (51), we have

$$
\langle 0 | T^{0}_{0} | 0 \rangle_{gm} \bigg|_{r/|\eta| \to 0} = \frac{2^{1/2} \eta}{\pi^{5/2} \alpha^{2} r^{4}} \int_{0}^{\infty} du \, u (\beta, u) \int_{0}^{\infty} dy \, y^{1/2} \, e^{-y r^{2} / \eta^{2}} \left( 1 - \frac{m^{2} \alpha^{2}}{2y} \right) \text{Im} \left[ K_{1/2 - iu} (y r^{2} / \eta^{2}) \right]
$$

$$
= \langle 0 | T^{0}_{0} | 0 \rangle_{gm} \bigg|_{r/|\eta| \to 0}
$$

(58)

which can be evaluated exactly using Eq. (21), Eq. (23)

$$
\langle 0 | T^{0}_{0} | 0 \rangle_{gm} \bigg|_{r/|\eta| \to 0} = -\frac{1}{72 \pi} \left( \frac{7}{40 (\alpha \eta)^{3}} \left( \frac{1}{\beta^{2}} - 1 \right) + \frac{3}{4 \pi^{3} (\alpha \eta)^{3}} \sum_{n=0}^{\infty} (-1)^{n} (\zeta (4, 1 + (n + 1) \beta) - \zeta (4, 1 + (n + 1))) \right)
$$

$$
- \frac{\pi m^{2}}{3 (\alpha \eta)^{2}} \left( \frac{1}{\beta^{2}} - 1 \right) + \frac{9 m^{2}}{2 \pi (\alpha \eta)^{2}} \sum_{n=0}^{\infty} (-1)^{n} \left( \zeta (2, 1 + (n + 1) \beta) - \zeta (2, 1 + (n + 1)) \right)
$$

$$
= \langle 0 | T^{0}_{0} | 0 \rangle_{gm} \bigg|_{r/|\eta| \to 0}
$$

(59)
Eq. (51)) and the rapid fall-off with oscillation (Eq. (61)) as a function of the dimensionless mass parameter $m \alpha$ (left) and the dimensionless radial distance (right). The $m \to 0$ limit does not diverge, but touch the $y$-axis at some high negative value. On the other hand, the $r/|\eta| \to 0$ limit is divergent, owing to the curvature singularity located there due to the global monopole. See main text for detail.

where the multiple $\zeta$ function is defined below Eq. (23). Likewise we have

$$
\langle 0 | T^0_{\theta} | 0 \rangle_{gm} \big|_{r/|\eta| \to 0} = \frac{1}{72\pi} \left( \frac{7}{40 (r\alpha/|\eta|)^4} \left( \frac{1}{\beta^4} - 1 \right) + \frac{3}{4\pi^3 (r\alpha/|\eta|)^4} \sum_{n=0}^{\infty} \left( -1 \right)^n (\zeta(4, 1 + (n + 1)\beta) - \zeta(4, 1 + (n + 1)\beta)) - \frac{9m^2\zeta(3)}{\pi^2 (r\alpha/|\eta|)^2} \left( \frac{1}{\beta^3} - 1 \right) + \frac{18m^2}{\pi^2 (r\alpha/|\eta|)^2} \sum_{n=0}^{\infty} \left( -1 \right)^n (\zeta(3, 1 + (n + 1)\beta) - \zeta(3, 1 + (n + 1)\beta)) \right)
$$

Note that all the components diverge at the location of the monopole ($r = 0$), due the curvature singularity present there.

As earlier, the next special case is to take small value of $y$ in Eq. (51), corresponding to large proper radial distances from the global monopole. Following procedure similar to that of described at the end of Section 3, we find

$$
\langle 0 | T^0_{\theta} | 0 \rangle_{gm} \big|_{r/|\eta| \to \infty} = \frac{f(\beta, m\alpha)}{\sqrt{2}\pi^{5/2} (r\alpha/|\eta|)^4} \cos \left( 2m\alpha \ln (2r/|\eta|) - \phi_0 \right) = \langle 0 | T^0_{\theta} | 0 \rangle_{gm} \big|_{r/|\eta| \to \infty}
$$

$$
\langle 0 | T^0_{\phi} | 0 \rangle_{gm} \big|_{r/|\eta| \to \infty} = \langle 0 | T^0_{\phi} | 0 \rangle_{gm} \big|_{r/|\eta| \to \infty} = \frac{f(\beta, m\alpha)}{\sqrt{2}\pi^{5/2} (r\alpha/|\eta|)^4} \left( 1 + m^2\alpha^2 \right)^{1/2} \sin \left( 2m\alpha \ln (2r/|\eta|) - \phi_0 + \phi_1 \right) \tag{61}
$$

where $\phi_1 = \cot^{-1} m\alpha$, and the function $f(\beta, m\alpha)$ and the phase factor $\phi_0$ are defined in Eq. (25). Note that both the short distance divergence (Eq. (59), Eq. (60)) and the rapid fall-off with oscillation (Eq. (61)) are qualitatively similar to that of the vacuum condensate discussed in Section 3. This is also similar to the case of the de Sitter cosmic string [53]. However, this is not exactly the case for a scalar field located in the de Sitter global monopole with a finite core background [80]. For this case we have off-diagonal components
of the energy-momentum tensor as well as a milder divergence at short distance.

Finally, we have plotted the variations of Eq. (51) in Fig. 2 and Fig. 3, with respect to the dimensionless mass parameter \( m/\eta \), as well as the dimensionless radial distance \( r/|\eta| \), with different values of the defect parameters. Note that for heavily massive cases, \( m/\eta \gg 1 \), the expectation values are highly suppressed.

6 Conclusions

In this work we have computed the vacuum expectation values of \( \bar{\Psi}\Psi \) and the energy-momentum tensor of the Dirac fermions in the de Sitter spacetime endowed with a pointlike global monopole defect. We have extracted the pure global monopole contributions for various quantities, Eq. (19), Eq. (51), by subtracting the pure de Sitter part from the full contribution. These results contain radial dependences as a consequence of the breaking of the translational symmetry of the de Sitter spacetime due to the presence of the global monopole located at \( r = 0 \). The general results Eq. (19), Eq. (51), could not further be simplified analytically and hence was plotted in Fig. 1, Fig. 2, Fig. 3. However, in the special scenarios when we are sufficiently close to, or far away from the monopole are also analysed analytically in Section 3, Section 5.

Note that all the results we have found diverge at the the location of the monopole due to the curvature singularity introduced by it and fall of sufficiently rapidly for large proper radial distances. We also note that the vanishing of the fermionic condensate, Section 3, for the global monopole indicates vanishing of the trace anomaly due to itself as well, which has been confirmed in Section 5. This also shows that the decompositions of Eq. (13) and Eq. (49) are consistent with each other. However, we must also note here an essential caveat regarding what we mean by the ‘global monopole induced’ part in all the above results. Precisely, the expressions for anomaly are universal, given by curvature invariants, e.g. [85]. The curvature of Eq. (1) should also get contribution from the defect \( \beta < 1 \), even for \( \alpha \to \infty \). Indeed, by this method we
directly have the anomaly

\[ \langle T_{\mu}^{\mu} \rangle = \frac{4b(1 - \beta^2)^2}{3\beta^4(\alpha r/\eta)^4} + \frac{8b'(r^4(9 - 8(1 - \beta^2)) - (\eta^2 - r^2)(1 - \beta^2))}{3\eta^2\beta^2(\alpha r/\eta)^4} + \frac{4b''(1 - \beta^2)(\eta^2 + r^2)}{\eta^2\beta^2(\alpha r/\eta)^4} \]

where \( b = 1/320\pi^2 \) and \( b' = -11/5760\pi^2 \). The coefficient \( b'' \) is not fixed, and can be changed by adding a term proportional to \( R^2 \) in the gravitational action. Assuming \( \beta \) to be close to unity, we may identify the terms dependent on \((1 - \beta^2)\) or its powers in the above expression to be the part induced by the global monopole, it is clear that no choice of \( b'' \) can make the anomaly vanishing, leading to a contradiction to what we have found above using the Dirac modes, Eq. (3), even though Fig. 2, Fig. 3 show that an individual massless \( \langle T_{\mu}^{\mu} \rangle \) is indeed dependent on the defect, \( \beta \). It seems similar mismatch is present for a de Sitter cosmic string as well [53]. Keeping in mind that the anomaly is essentially an ultraviolet phenomenon, this ambiguity could probably be related to the particular regularisation scheme we have adopted here, i.e. subtracting the de Sitter contribution from the full expression, Eq. (13), Eq. (49). The standard regularisation techniques in curved spacetime on the other hand, apart from the dimensional regularisation, either use the so called adiabatic subtraction, or subtract the flat spacetime contribution from the Hadamard expansion of the Green function, obtained via the Riemann normal coordinates [85]. Thus possibly we are loosing some essential curvature contribution, e.g. via equations like Eq. (17), Eq. (18). It is well known that the results of quantum field theory in curved spacetimes can be regularisation dependent. Perhaps one should compute the fermion Green function using the modes of Eq. (3) and use some other regularisation technique to see if it reproduces the above mentioned non-vanishing expression of anomaly due to the global monopole. We reserve this for a future work.

There can be further some directions towards which our present study can be extended. For example, the orthonormal Dirac modes found here may be used to compute cosmological correlation functions. In particular as we stated above, we may use them to find out the Green or the Wightman functions, which can be used in perturbative calculations in the presence of some interactions. It will further be interesting to investigate the vacuum expectation value of the conserved current and in particular the chiral anomaly in this background. We hope to return to these issues in our future works.

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A  Detail for Section 2

In this Appendix we shall provide the computational details for Section 2. The Dirac equation in curved spacetime reads,

\[ (i\gamma^\mu \nabla_\mu - m) \Psi(x) = 0, \]  (62)
where $\nabla_\mu \equiv \partial_\mu + \Gamma_\mu$ is the spin covariant derivative, and $\Gamma_\mu$’s are the spin connection matrices,

$$\Gamma_\mu = \frac{1}{4} \gamma^{(a)} \gamma^{(b)} e^{\alpha}_{(a)} e^{\beta}_{(b)}$$

(63)

where the Latin indices within parenthesis stand for the local Lorentz frame and $\gamma^\mu = e^{\mu}_{(a)} \gamma^{(a)}$, where $e^{\mu}_{(a)}$’s are the tetrads.

Since the metric Eq. (2) is just conformally related to the flat metric with a global monopole, it is clear that the angular part of our modes will be the same as that of the latter, can be seen in e.g. [64].

For Eq. (2), we choose the tetrads to be

$$e^{\mu}_{(a)} = \eta^{\alpha r} \left( \begin{array}{cccc}
1 & 0 & 0 & 0 \\
0 & \sin \theta \cos \phi & 0 & -\sin \phi \\
0 & \sin \theta \sin \phi & 0 & \cos \phi \\
0 & 0 & 1 & 0
\end{array} \right)$$

(64)

where the rows of the matrix are specified by the local Lorentz index $a$ and the columns by the spacetime index $\mu$. From Eq. (63), Eq. (64), we find the non-vanishing spin connection matrices

$$\Gamma_1 = -\frac{g_{rr}}{2 \eta} \gamma^1 \gamma^0, \quad \Gamma_2 = \frac{g_{\theta\theta}}{2 \eta} \gamma^2 \gamma^0 + \frac{g_{\theta\phi}}{2 r} \left( 1 - \frac{1}{\beta} \right) \gamma^2 \gamma^1, \quad \Gamma_3 = -\frac{g_{\phi\phi}}{2 \eta} \gamma^3 \gamma^0 + \frac{g_{\phi\phi}}{2 r} \left( 1 - \frac{1}{\beta} \right) \gamma^3 \gamma^1$$

(65)

which yields,

$$\gamma^\mu \Gamma_\mu = -\frac{3}{2 \alpha} \gamma^{(0)} + \frac{\beta - 1}{\beta r} \gamma^{(1)}$$

(66)

We take the representation of the $\gamma$-matrices,

$$\gamma^{(0)} = \left( \begin{array}{cc}
I & 0 \\
0 & -I
\end{array} \right), \quad \gamma^{(i)} = \left( \begin{array}{cc}
0 & \sigma^i \\
-\sigma^i & 0
\end{array} \right)$$

(67)

where $\sigma^i$’s are the Pauli matrices. Putting now things in together, the Dirac equation Eq. (62) takes the form,

$$\left[ \left( \gamma^{(0)} \partial_\eta - \frac{3}{2 \eta} \gamma^{(0)} + \frac{ima}{\eta} \right) + \gamma^i \hat{r} \left( \partial_r + \frac{\beta - 1}{\beta r} \right) + \frac{1}{\beta r} \left( \gamma^i \hat{\theta} \partial_\theta + \gamma^i \hat{\phi} \right) \right] \Psi = 0,$$

(68)

where $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are the unit vectors in spherical polar coordinates. We now take

$$\Psi = \left( \begin{array}{c}
\Psi_1 \\
\Psi_2
\end{array} \right)$$

(69)

where $\Psi_1$ and $\Psi_2$ are both two component spinors. Using now Eq. (67), we have from Eq. (68),

$$\left( \partial_\eta - \frac{1}{\eta} \left( \frac{3}{2} - \frac{ima}{\eta} \right) \right) \Psi_1 + \left( \hat{\sigma} \cdot \hat{r} \left( \partial_r + \frac{\beta - 1}{\beta r} \right) + \frac{1}{\beta r} \left( \hat{\sigma} \cdot \hat{\theta} \partial_\theta + \hat{\sigma} \cdot \hat{\phi} \right) \right) \Psi_2 = 0,$$

$$\left( \partial_\eta - \frac{1}{\eta} \left( \frac{3}{2} + \frac{ima}{\eta} \right) \right) \Psi_2 + \left( \hat{\sigma} \cdot \hat{r} \left( \partial_r + \frac{\beta - 1}{\beta r} \right) + \frac{1}{\beta r} \left( \hat{\sigma} \cdot \hat{\theta} \partial_\theta + \hat{\sigma} \cdot \hat{\phi} \right) \right) \Psi_1 = 0$$

(70)
Using now (e.g. \[87\])

\[
\left(\bar{\sigma} \cdot \bar{\partial} \theta + \frac{\bar{\sigma} \cdot \hat{\omega}}{\sin \theta} \partial_{\theta}\right) \equiv -\bar{\sigma} \cdot \hat{r} \left(\hat{K} - 1\right), \quad \text{with} \quad \hat{K} = \bar{\sigma} \cdot \bar{L} + 1,
\]  

(71)

where \(\bar{L}\) is the standard orbital angular momentum operator. Eq. (70) can be rewritten as,

\[
\left(\partial_\eta - \frac{1}{\eta} \left(\frac{3}{2} - \frac{im\alpha}{\eta}\right)\right) \Psi_1 + \bar{\sigma} \cdot \hat{r} \left(\partial_r + \frac{1}{r} - \frac{\hat{K}}{\beta r}\right) \Psi_2 = 0,
\]

\[
\left(\partial_\eta - \frac{1}{\eta} \left(\frac{3}{2} + \frac{im\alpha}{\eta}\right)\right) \Psi_2 + \bar{\sigma} \cdot \hat{r} \left(\partial_r + \frac{1}{r} - \frac{\hat{K}}{\beta r}\right) \Psi_1 = 0
\]

(72)

We now choose the ansatz for the variable separation

\[
\Psi_1 = f^\sigma (r, \eta) \Omega_{j^\sigma, m}(\theta, \phi), \quad \Psi_2 = (-1)^\sigma g^\sigma (r, \eta) \Omega_{j^\sigma, m}(\theta, \phi) \quad (\sigma = 0, 1, \text{ no sum on } \sigma)
\]

where

\[
l^\sigma = j - \frac{(-1)^\sigma}{2}, \quad l_-^\sigma = j + \frac{(-1)^\sigma}{2},
\]

(74)

and \(\Omega_{j^\sigma, m}\) are the standard spin-1/2 spherical harmonics

\[
\Omega_{l-1/2,l,m} (\theta, \phi) = \left(\begin{array}{c} C^+_{lm} Y_{l,m-1/2}(\theta, \phi) \\ C^-_{lm} Y_{l,m+1/2}(\theta, \phi) \end{array}\right), \quad \Omega_{l+1/2,l,m} (\theta, \phi) = \left(\begin{array}{c} -C^+_{lm} Y_{l,m-1/2}(\theta, \phi) \\ C^-_{lm} Y_{l,m+1/2}(\theta, \phi) \end{array}\right)
\]

where

\[
C^\pm_{lm} = \sqrt{\frac{l \pm m + 1/2}{2l + 1}}
\]

(73)

\(j(j + 1)\) is the eigenvalue of the total angular momentum \((J^2)\), \(j = 1/2, 3/2, \ldots\) and \(m = -j, \ldots, j\) are its projections. \(\Omega_{j^\sigma, m}\)'s are the simultaneous eigenfunctions of the operators \(L^2, S^2, J^2, J_z\) and \(\hat{K}\). In particular (e.g. \[87, 88, 64\] and references therein),

\[
\hat{K} \Omega_{j^\sigma, m} = -k^\sigma \Omega_{j^\sigma, m}, \quad k^\sigma = -(-1)^\sigma (j + 1/2).
\]

(75)

Putting \(j = l + 1/2\), we have

\[
(\bar{\sigma} \cdot \hat{r}) \Omega_{l+1/2,l,m} = -\Omega_{l-1/2,l+1,m}, \quad \hat{K} \Omega_{l+1/2,l,m} = (l + 1) \Omega_{l+1/2,l,m}
\]

\[
(\bar{\sigma} \cdot \hat{r}) \Omega_{l+1/2,l+1, m} = -\Omega_{l-1/2,l,m}, \quad \hat{K} \Omega_{l+1/2,l+1, m} = -(l + 1) \Omega_{l+1/2,l+1, m}
\]

(76)

Using now

\[
\Omega_{j^\sigma, l} = \bar{t}^{l^\sigma - l_-^\sigma} (\hat{r} \cdot \bar{\sigma}) \Omega_{j^\sigma, m},
\]

(77)

Eq. (72) can be recast as

\[
\left(\frac{\partial}{\partial r} + \frac{\beta + k^\sigma}{\beta r}\right) f^\sigma (r, \eta) = -(-1)^\sigma \bar{t}^{l^\sigma - l_-^\sigma} \left(\partial_\eta - \frac{1}{\eta} \left(\frac{3}{2} + \frac{im\alpha}{\eta}\right)\right) g^\sigma (r, \eta)
\]

\[
\left(\frac{\partial}{\partial r} + \frac{\beta - k^\sigma}{\beta r}\right) g^\sigma (r, \eta) = (-1)^\sigma \bar{t}^{l^\sigma - l_-^\sigma} \left(\partial_\eta - \frac{1}{\eta} \left(\frac{3}{2} - \frac{im\alpha}{\eta}\right)\right) f^\sigma (r, \eta)
\]

(78)
Squaring the above equations leads to two second order differential equations,

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{k_\sigma (k_\sigma + \beta)}{r^2} \right) f^\sigma(r, \eta) - \left( \frac{\partial^2}{\partial \eta^2} - \frac{3}{\eta} \frac{\partial}{\partial \eta} + \frac{(15/4 - ima + m^2\alpha^2)}{\eta^2} \right) f^\sigma(r, \eta) = 0
\]

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{k_\sigma (k_\sigma - \beta)}{r^2} \right) g^\sigma(r, \eta) - \left( \frac{\partial^2}{\partial \eta^2} - \frac{3}{\eta} \frac{\partial}{\partial \eta} + \frac{(15/4 + ima + m^2\alpha^2)}{\eta^2} \right) g^\sigma(r, \eta) = 0
\]

(79)

Separating now the variables, \( f^\sigma(r, \eta) = T^\sigma_1(\eta)R^\sigma_1(r) \) (no sum on \( \sigma \)) we have

\[
\frac{d^2 R^\sigma_1(r)}{dr^2} + \frac{2}{r} \frac{d R^\sigma_1(r)}{dr} + \left( \lambda^2 - \frac{k_\sigma (k_\sigma + \beta)}{r^2} \right) R^\sigma_1(r) = 0,
\]

\[
\frac{d^2 T^\sigma_1(\eta)}{d\eta^2} - \frac{3}{\eta} \frac{dT^\sigma_1(\eta)}{d\eta} + \left( \lambda^2 + \frac{15/4 + m^2\alpha^2 - ima}{\eta^2} \right) T^\sigma_1(\eta) = 0,
\]

(80)

where \( \lambda \) is a separation constant. Likewise we take \( g^\sigma(r, \eta) = T^\sigma_2(\eta)R^\sigma_2(r) \), to obtain

\[
\frac{d^2 R^\sigma_2(r)}{dr^2} + \frac{2}{r} \frac{d R^\sigma_2(r)}{dr} + \left( \lambda^2 - \frac{k_\sigma (k_\sigma - \beta)}{r^2} \right) R^\sigma_2(r) = 0
\]

\[
\frac{d^2 T^\sigma_2(\eta)}{d\eta^2} - \frac{3}{\eta} \frac{dT^\sigma_2(\eta)}{d\eta} + \left( \lambda^2 + \frac{15/4 + m^2\alpha^2 + ima}{\eta^2} \right) T^\sigma_2(\eta) = 0
\]

(81)

The solutions to the spatial parts of Eq. (80) and Eq. (81) are given by the spherical Bessel’s functions

\[
R^\sigma_1(r) \sim \frac{1}{\sqrt{r}} J_{\nu_\sigma}(\lambda r), \quad R^\sigma_2(r) \sim \frac{1}{\sqrt{r}} J_{\nu_\sigma + (-1)^\sigma}(\lambda r)
\]

(82)

where

\[
\nu_\sigma = k_\sigma / \beta + 1/2
\]

and \( k_\sigma \) is given by Eq. (75).

We take the temporal parts of Eq. (80) and Eq. (81) as the Hankel functions of the first kind

\[
T^\sigma_1(\eta) \sim \eta^2 H^{(1)}_{1/2-ima}(\lambda |\eta|), \quad T^\sigma_2(\eta) \sim \eta^2 H^{(1)}_{-1/2-ima}(\lambda |\eta|)
\]

(83)

Thus, for the spinor \( \Psi_1 \) in Eq. (69), we may take

\[
\Psi_1 = \sqrt{\frac{\pi}{2\lambda \eta}} \eta^2 H^{(1)}_{1/2-ima}(\lambda |\eta|) J_{\nu_\sigma}(\lambda r) \Omega_{j_\lambda m}
\]

(84)

Using then Eq. (72), Eq. (77) we also have

\[
\Psi_2 = -i(\hat{r} \cdot \sigma) \sqrt{\frac{\pi}{2\lambda \eta}} \eta^2 H^{(1)}_{-1/2-ima}(\lambda |\eta|) J_{\nu_\sigma + (-1)^\sigma}(\lambda r) \Omega_{j_\lambda m}
\]

(85)
Recalling now the asymptotic form of the Hankel functions \[84\] in the asymptotic past, \(\eta \to -\infty\),

\[
H_{1/2-ima}^{(1)}(\lambda|\eta|) \approx \sqrt{\frac{2}{\pi|\eta|}} e^{i(-\lambda\eta - \pi/4)} e^{-\text{m}\pi\lambda/2}
\]  \(86\)

we identify the two positive frequency solutions (corresponding to \(\sigma = 0,1\) for \(\lambda > 0\)) as

\[
\Psi_{\sigma jlm}^{(+)} = \sqrt{\frac{\pi}{2\lambda r}} N_\sigma \left( \eta^2 H_{1/2-ima}^{(1)}(\lambda|\eta|) J_{\nu_\sigma}(\lambda r) \Omega_{j l m} \right)
\]

The normalisation constant \(N_\sigma\) can be determined via the relation,

\[
\frac{\beta^2 \alpha^3}{|\eta|^3} \int_0^{\infty} r^2 \sin \theta dr d\theta d\phi \left( \Psi_{\sigma jlm}^{(+)} \right)^\dagger \Psi_{\sigma j' l m'}^{(+)} = \delta_{\sigma\sigma'} \delta_{jj'} \delta_{ll'} \delta_{mm'}
\]

Using Eq. \(87\), the left hand side of the above equation becomes

\[
\frac{|N_\sigma|^2 \beta^2 |\eta|^2 \pi}{2\lambda \alpha^{-3}} \left( |H_{1/2-ima}^{(1)}(\lambda|\eta|)|^2 + |H_{-1/2-ima}^{(1)}(\lambda|\eta|)|^2 \right) \int_0^{\infty} r dr J_{\nu_\sigma}(\lambda r) J_{\nu_\sigma}(\lambda r) \int_0^{\pi} \int_0^{2\pi} d\phi d\theta \sin \theta \Omega_{j l m}^{\dagger} \Omega_{j' l m'}
\]

Using now the integrals,

\[
\int_0^{\pi} \int_0^{2\pi} d\theta d\phi \sin \theta \Omega_{j l m}^{\dagger} \Omega_{j' l m'} = \delta_{jj'} \delta_{ll'} \delta_{mm'}
\]

and the asymptotic form of the Hankel function, Eq. \(86\), we have

\[
|N_\sigma|^2 = \frac{\lambda^3}{2\beta^2 \alpha^3} e^{-\pi m \alpha}
\]  \(91\)

Note that the normalisation is independent of \(\sigma\).

The negative frequency modes are found via the charge conjugation : \(\Psi_{\sigma jlm}^{(-)} = i \gamma^{(2)} \Psi_{\sigma jlm}^{(+)}\).

\[
\Psi_{\sigma jlm}^{(-)}(\eta, r, \theta, \phi) = \sqrt{\frac{\pi}{2\lambda r}} M_\sigma \left( -i(-1)^\sigma (r \cdot \vec{\sigma}) \eta^2 H_{-1/2+ima}^{(1)}(\lambda|\eta|) J_{\nu_\sigma} + (-1)^\sigma (\lambda r) (i\sigma^2 \Omega_{j l m}^{*}) \right)
\]

where we have used \((H_{\alpha}^{(1)}(x))^* = H_{\alpha}^{(2)}(x)\), for real \(x\). Using now the well known properties of the spin harmonics,

\[
(i\sigma^2) \Omega_{l+1/2,l,m} = (-1)^{m+1/2} \Omega_{l+1/2,l,-m}, \quad (i\sigma^2) \Omega_{l-1/2,l,m} = (-1)^{m+3/2} \Omega_{l-1/2,l,-m}, \quad \Omega_{j l m} = (-1)^{m+1/2} \Omega_{j l m}
\]

Eq. \(92\) simplifies to

\[
\Psi_{\sigma jlm}^{(-)}(\eta, r, \theta, \phi) = \sqrt{\frac{\pi}{2\lambda r}} M_\sigma \left( i(-1)^\sigma (r \cdot \vec{\sigma}) \eta^2 H_{-1/2+ima}^{(1)}(\lambda|\eta|) J_{\nu_\sigma} + (-1)^\sigma (\lambda r) \Omega_{j l m} \right)
\]

where

\[
|M_\sigma|^2 = \frac{\lambda^3}{2\beta^2 \alpha^3} e^{-\pi m \alpha}
\]

\[22\]
B  $\langle 0|\overline{\Psi}\Psi|0\rangle$ for pure de Sitter spacetime

The fermionic vacuum condensate for the pure dS spacetime is obtained by setting $\beta = 1$ in Eq. (12),

$$\langle 0|\overline{\Psi}\Psi|0\rangle_{dS} = \frac{\eta}{\pi^2\alpha^3r} \int_0^\infty dy\; e^{y^{1-r^2/\eta^2}} \text{Im}[K_{1/2-\text{im} \alpha}(y)] \sum_{j=1/2}^\infty (j + 1/2) \left( I_{(j+1/2)+1/2}(y\eta^2/\eta^2) + I_{(j+1/2)-1/2}(y\eta^2/\eta^2) \right)$$

(96)

Note that $(j + 1/2)$ appearing above takes values 1, 2, 3, .... Using then the formula [89]

$$\sum_{k=1}^\infty k I_{k+\nu}(z) = \frac{e^z}{2} \int_0^z dse^{-s} I_\nu(s),$$

(97)

we obtain after some algebra

$$\langle 0|\overline{\Psi}\Psi|0\rangle_{dS} = \frac{\eta}{2\pi^2\alpha^3r} \int_0^\infty dy\; y e^{y^{1-r^2/\eta^2}} \text{Im}[K_{1/2-\text{im} \alpha}(y)] \int_0^{y\eta^2/\eta^2} dse^{-s} \left( I_{1/2}(s) + I_{-1/2}(s) \right)$$

(98)

Using now $I_{1/2}(s) = \sqrt{\frac{2}{\pi s}} \cosh s$ and $I_{-1/2}(s) = \sqrt{\frac{2}{\pi s}} \sinh s$ (e.g. [89]), Eq. (98) simplifies to

$$\langle 0|\overline{\Psi}\Psi|0\rangle_{dS} = \frac{8}{(2\pi)^{3/2}\alpha^3} \int_0^\infty dy\; y^{3/2} e^{y} \text{Im}[K_{1/2-\text{im} \alpha}(y)]$$

(99)

Expectedly, the above integral is divergent. In order to regularise it, we introduce an exponential cut-off by making the replacement, $e^y \to e^{(1-\epsilon)y}$ ($\epsilon > 0$), so that

$$\langle 0|\overline{\Psi}\Psi|0\rangle^{(\epsilon)}_{dS} = \frac{1}{\pi\alpha^3 \sinh(\pi m\alpha)} \partial^2_{\epsilon^2} \text{F}_{1} \left( im\alpha, -im\alpha + 1; 1; 1 - \frac{\epsilon}{2} \right)$$

(100)

where we also have used the integral relationship given in [82]. The above can be simplified to (e.g. [84]),

$$\langle 0|\overline{\Psi}\Psi|0\rangle^{(\epsilon)}_{dS} = -\frac{m}{4\pi^2\alpha^2} \left( \frac{2}{\epsilon} + (1 + m^2\alpha^2) \ln \frac{\epsilon}{2} + 2 \left( 1 + m^2\alpha^2 \right) \left[ \text{Re} \psi(im\alpha) - \ln(m\alpha) + \gamma - \frac{3}{4} \right] + \mathcal{O}(\epsilon) \right)$$

(101)

where $\gamma$ is Euler’s constant and $\psi$ is the digamma function. Note that the above expression is independent of the spacetime, owing to the maximal symmetry of the pure de Sitter background. Also as expected, it is the same as that of [53], found in the context of the de Sitter cosmic string background using a cylindrical coordinate.

Note also that owing to the $\epsilon$-derivative in Eq. (100), Eq. (101) is unique only up to some $\epsilon$-independent additive constant. This ambiguity can be tackled by imposing the physical condition that in the heavy mass limit, $m \to \infty$, the condensate must vanish [53]. Since the vacuum expectation value of the trace of the fermionic energy-momentum tensor is given by $m\langle\overline{\Psi}\Psi\rangle$, it is clear that the divergences of Eq. (101) can be absorbed by the renormalisation of the cosmological constant. After renormalising, and choosing the aforementioned $\epsilon$-independent additive constant appropriately, we have the final expression of Eq. (14). It is easy to check by expanding the digamma function of Eq. (14) that in the large mass limit, it indeed vanishes as $\mathcal{O}(m^{-1})$. 

23
C Energy-momentum tensor in pure de Sitter background

In this Appendix we shall very briefly sketch the derivation of the vacuum expectation value of the energy-momentum tensor in the pure de Sitter spacetime ($\beta = 1$ in Eq. (2)). For similar derivation in the context of the de Sitter cosmic strings, we refer our reader to [53].

Now, setting $\beta = 1$ in Eq. (38) or Eq. (45), and using the identity Eq. (97), we obtain

$$
\langle 0 | T_0^0 | 0 \rangle_{\text{dS}} = \frac{\eta}{4\pi^2\alpha^4} \sum_{j=1/2}^{\infty} (2j + 1) \int_0^\infty dy \, y \, e^{y(1-r^2/\eta^2)} \left( I_{j+1/2+1/2}(yr^2/\eta^2) + I_{j+1/2-1/2}(yr^2/\eta^2) \right) \text{Re} [K_{1/2-ima}(y)]
$$

$$
= \frac{4}{(2\pi)^{5/2} \alpha^4} \int_0^\infty dy \, y^{3/2} e^{y} \text{Re} [K_{1/2-ima}(y)] = \langle 0 | T_0^0 | 0 \rangle_{\text{dS}}
$$

Likewise, by setting $\beta = 1$ in Eq. (48), we have

$$
\langle 0 | T_0^0 | 0 \rangle_{\text{dS}} = \langle 0 | T_0^0 | 0 \rangle_{\text{dS}} = \frac{4}{(2\pi)^{5/2} \alpha^4} \int_0^\infty dy \, y \, e^{y(1-r^2/\eta^2)} \text{Re} \left[ K_{1/2-ima}(y) \right] \left( y \partial_y - yr^2/\eta^2 + 1/2 \right) \left( e^{yr^2/\eta^2} y^{1/2} \right)
$$

$$
= \frac{4}{(2\pi)^{5/2} \alpha^4} \int_0^\infty dy \, y^{3/2} e^{y} \text{Re} \left[ K_{1/2-ima}(y) \right]
$$

As expected, the above expressions are the same as that of [53], found in the context of the de Sitter cosmic string. Note that all the components of $\langle 0 | T_\mu^\nu | 0 \rangle_{\text{dS}}$ are equal and independent of the spacetime coordinates, as expected from the maximal symmetry of the de Sitter. The components of the energy-momentum tensor can thus be put in a compact form,

$$
\langle 0 | T_\mu^\nu | 0 \rangle_{\text{dS}} = \frac{4\delta_\mu^\nu}{(2\pi)^{5/2} \alpha^4} \int_0^\infty dy \, y^{3/2} e^{y} \text{Re} \left[ K_{1/2-ima}(y) \right]
$$

The renormalised expression can be found after using a cosmological constant counterterm [53],

$$
\langle 0 | T_\mu^\nu | 0 \rangle_{\text{Ren.}, \text{dS}} = \frac{\delta_\mu^\nu}{8\pi^2\alpha^4} \left[ m^2 \alpha^2 \left( 1 + m^2 \alpha^2 \right) \left[ \ln (ima) - \text{Re} (ima) \right] + \frac{m^2 \alpha^2}{12} + \frac{11}{120} \right]
$$

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