ON THE SIZE DISTRIBUTION OF THE FIXED-LENGTH LEVENSHTEIN BALLS WITH RADIUS ONE

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Abstract. The fixed-length Levenshtein (FLL) distance between two words $x, y \in \mathbb{Z}_n^m$ is the smallest integer $t$ such that $x$ can be transformed to $y$ by $t$ insertions and $t$ deletions. The size of a ball in the FLL metric is a fundamental yet challenging problem. Very recently, Bar-Lev, Etzion, and Yaakobi explicitly determined the minimum, maximum and average sizes of the FLL balls with radius one, respectively. In this paper, based on these results, we further prove that the size of the FLL balls with radius one is highly concentrated around its mean by Azuma’s inequality.

1. Introduction

The Levenshtein distance (also known as edit distance) between two words is the smallest number of deletions and insertions needed to transform one word to the other. This is a metric used for codes correcting synchronization errors. The theory of coding with respect to the Levenshtein distance dates back to the 1960s [6], but there has been much less progress in comparison with the classical theory of coding with respect to the Hamming distance. As commented by Mitzenmacher [7], “Channels with synchronization errors, including both insertions and deletions as well as more general timing errors, are simply not adequately understood by current theory. Given the near-complete knowledge we have for channels with erasures and errors . . . , our lack of understanding about channels with synchronization errors is truly remarkable.” Indeed, even the fundamental problem of counting the number of words formed by deleting and inserting symbols into a given word remains elusive.

2010 Mathematics Subject Classification. Primary 94B50; Secondary 05D40.

Key words and phrases. Levenshtein ball, fixed-length Levenshtein distance, Azuma’s inequality.

The paper was presented (in part) at The Twelfth International Workshop on Coding and Cryptography (WCC), Rostock, Germany, March 7–11, 2022.
In 1966, Levenshtein [6] gave the earliest bound on the number of words formed by deleting a constant number of symbols from a word. This bound was later improved by Calabi and Hartnett [3]; also by Hirschberg and Regnier [5]. On the other hand, the number of words formed by inserting $r$ symbols into $x \in \mathbb{Z}_m^n$ does not depend on $x$ itself, and is given by $\sum_{i=0}^r \binom{n+r}{i} (m-1)^i$ [6]. Motivated by estimating the rate of synchronization error-correction codes, Sala and Dolecek [8] studied the number of words formed by deleting and inserting a constant number of symbols into a given word. The fixed-length Levenshtein (FLL) distance (originally under the name “ancestor distance”) between two words $x, y \in \mathbb{Z}_m^n$ is defined as half of the traditional Levenshtein distance, i.e., the smallest $t$ such that $x$ can be transformed to $y$ by $t$ insertions and $t$ deletions. An explicit expression was given on the FLL ball size with radius one (see Lemma 2.5), and the size of the FLL balls with radius larger than one was upper bounded in [8]. Very recently, Bar-Lev, Etzion, and Yaakobi [2] found the explicit expressions for the minimum, the maximum and the average sizes of the FLL balls with radius one, respectively. A natural follow-up question is how the size of the FLL balls with radius one is distributed. In this paper, we prove that the size of the FLL balls with radius one is highly concentrated around its mean by Azuma’s inequality (see Theorem 3.3 and Theorem 3.5).

The rest of this paper is organized as follows. In Section 2, we provide some notations, definitions, and auxiliary results. In Section 3, we analyze in detail the size distribution of the FLL balls with radius one, and state the main results in Theorem 3.3 and Theorem 3.5. Finally, we conclude this paper in Section 4.

2. Preliminaries

Let $\mathbb{Z}_m = \{0, 1, \ldots, m - 1\}$ for an integer $m \geq 2$. We use $[n]$ to denote the set $\{1, \ldots, n\}$. In this paper, vectors in $\mathbb{Z}_m^n$ are called $m$-ary sequences (words) with length $n$, and are written as strings for convenience. For a word $x = x_1, \ldots, x_n \in \mathbb{Z}_m^n$ and $1 \leq i \leq j \leq n$, the subsequence $x_i, x_{i+1}, \ldots, x_j$ is denoted by $x_{[i,j]}$. A run of $x$ is a maximal subsequence $x_{[i,j]}$ with all symbols from $x_i$ to $x_j$ identical, i.e., $x_i = x_{i+1} = \cdots = x_j$, $x_{i-1} \neq x_i$ for $i \geq 2$ and $x_{j+1} \neq x_j$ for $j \leq n-1$. An alternating segment of $x$ is a maximal subsequence $x_{[i,j]}$ with the form $abab\ldots ab$ or $abab\ldots ba$, where $a, b \in \mathbb{Z}_m$ and $a \neq b$. Note that by ‘maximal’ we mean that both $x_{[i-1,j]}$ (when $i > 1$) and $x_{[i,j+1]}$ (when $j < n$) are no longer alternating. The number of runs in $x$ is denoted by $\rho(x)$, and the number of alternating segments in $x$
is denoted by \( a(x) \). For each \( x \in \mathbb{Z}_n^2 \), we have \( \rho(x) + a(x) = n + 1 \).

The lengths of the first and the last alternating segments in a word \( x \) are of particular interest in this work and denoted by \( h(x) \) and \( t(x) \), respectively.

**Example 2.1.** Let \( x = 001100101 \). Then the runs of \( x \) are 00, 11, 00, 1, 0, 1 and \( \rho(x) = 6 \); The alternating segments of \( x \) are 0, 01, 10, 0101 and \( a(x) = 4 \), \( h(x) = 1 \), \( t(x) = 4 \).

**Lemma 2.2.** Let \( n > 0 \), \( m > 1 \) both be integers. If \( x \) is picked uniformly at random, then we have

\[
\mathbb{E}_{x \in \mathbb{Z}_n^m} [h(x)] = \mathbb{E}_{x \in \mathbb{Z}_n^m} [h(x)|x_1] = 2 - \frac{1}{m^{n-1}},
\]

\[
\mathbb{E}_{x \in \mathbb{Z}_n^m} [t(x)] = \mathbb{E}_{x \in \mathbb{Z}_n^m} [t(x)|x_n] = 2 - \frac{1}{m^{n-1}}.
\]

**Proof.** We prove Eq. \((1)\), and Eq. \((2)\) then follows by symmetry. Let \( x \in \mathbb{Z}_n^m \) be an arbitrary sequence. Define the indicator variable \( I_i \) for \( i \in [n] \) as follows.

\[
I_i = \begin{cases} 
1, & \text{if } x_i \text{ belongs to the first alternating segment,} \\
0, & \text{otherwise.}
\end{cases}
\]

Then we have \( h(x) = \sum_{i=1}^n I_i \). Also note that \( \Pr(I_1 = 1) = 1 \), and \( \Pr(I_i = 1) = (m - 1)/m^{i-1} \) for \( i > 1 \). Therefore, we have

\[
\mathbb{E} [h(x)] = \sum_{i=1}^n \mathbb{E} [I_i] = 1 + (m - 1) \left( \frac{1}{m} + \cdots + \frac{1}{m^{n-1}} \right) = 2 - \frac{1}{m^{n-1}}.
\]

Intuitively, exposing the first symbol of a random \( x \in \mathbb{Z}_n^m \) does not provide any information about \( h(x) \) since \( h(x) \geq 1 \) for all words. Thus, we have \( \mathbb{E} [h(x)] = \mathbb{E} [h(x)|x_1] \). More precisely, we define an equivalence relation \( \sim \) on \( \mathbb{Z}_n^m \) as follows. For \( x, y \in \mathbb{Z}_n^m \), we say that \( x \sim y \) if and only if \( x_i \equiv y_i + k \mod m \) for each \( i \in [n] \) and some integer \( k \). Clearly, \( h(x) = h(y) \) if \( x \sim y \), and \( \{a\} \times \mathbb{Z}_m^{n-1} \) contains exactly one element from each equivalence class. It then follows that \( \mathbb{E} [h(x)|x_1 = a] = \mathbb{E} [h(x)|x_1 = b] \). Thus, we have

\[
\mathbb{E} [h(x)] = \sum_{a \in \mathbb{Z}_m} \frac{1}{m} \mathbb{E} [h(x)|x_1 = a] = \mathbb{E} [h(x)|x_1 = a],
\]

and the proof is then completed. \( \square \)
A sequence \( y \in \mathbb{Z}^{n-t}_m \) with \( t \in [n - 1] \) is called a \( t \)-subsequence of \( x \in \mathbb{Z}^n_m \) if \( y \) is formed by deleting \( t \) symbols from \( x \). In other words, \( y = (x_{i_1}, x_{i_2}, \ldots, x_{i_{n-t}}) \), where \( 1 \leq i_1 < i_2 < \cdots < i_{n-t} \leq n \). Likewise, \( x \) is called a \( t \)-supersequence of \( y \). The set of all \( t \)-subsequences of \( x \) is called the deletion \( t \)-sphere centered at \( x \), and denoted by \( D_t(x) \). The set of all \( t \)-supersequences of \( x \) is called the insertion \( t \)-sphere centered at \( x \), and denoted by \( I_t(x) \). The fixed-length Levenshtein distance is formally defined as follows.

**Definition 2.3** (Fixed-length Levenshtein (FLL) distance). The fixed-length Levenshtein (FLL) distance between two words \( x, y \in \mathbb{Z}^n_m \) is the smallest \( t \) such that \( D_t(x) \cap D_t(y) \neq \emptyset \), and is denoted by \( d_t(x, y) \).

It is easy to see that \( d_t(x, y) = t \) if and only if \( t \) is the smallest integer such that \( x \) can be transformed to \( y \) by \( t \) deletions and \( t \) insertions.

**Definition 2.4** (FLL ball). For each word \( x \in \mathbb{Z}^n_m \), the FLL \( t \)-ball centered at \( x \) is defined by

\[
L_t(x) \triangleq \{ y \in \mathbb{Z}^n_m \mid d_t(x, y) \leq t \},
\]

and \( t \) is called the radius.

The following results on the size of the FLL ball were given in \([8,2]\), and will be useful later.

**Lemma 2.5.** \([8, \text{ Theorem 1}]\) For all \( x \in \mathbb{Z}^n_m \),

\[
|L_1(x)| = \rho(x)(mn - n - 1) + 2 - \frac{1}{2} \sum_{i=1}^{a(x)} s_i^2 + \frac{3}{2} \sum_{i=1}^{a(x)} s_i - a(x),
\]

where \( s_i \), for \( 1 \leq i \leq a(x) \), is the length of the \( i \)-th alternating segment of \( x \).

Lemma 2.5 was proved by a careful argument based on the principle of inclusion and exclusion. Moreover, it can be shown that \( \rho(x)(mn - n - 1) + 2 \) is an upper bound of \( |L_1(x)| \), and Eq. (3) was obtained by subtracting the number of overcounted sequences.
Lemma 2.6. \cite{2} Lemma 11, Corollary 8, Lemma 12, and Lemma 16] \[n \geq 2\]

Let the notations be the same as above. For integers \(m, n > 1\), we have

\[
\begin{align*}
\mathbb{E}_{x \in \mathbb{Z}_m^n} \left[ \sum_{i=1}^{a(x)} s_i \right] &= n + (n - 2) \cdot \frac{(m - 1)(m - 2)}{m^2}, \\
\mathbb{E}_{x \in \mathbb{Z}_m^n} [a(x)] &= 1 + \frac{(n - 2)(m - 1)(m - 2)}{m^2} + \frac{n - 1}{m}, \\
\mathbb{E}_{x \in \mathbb{Z}_m^n} [\rho(x)] &= n - \frac{n - 1}{m}, \\
(4) &
\end{align*}
\]

Remark 2.7. Eq. (4) was proved in \cite{2} Lemma 16]. However, the last term \(\frac{2}{m^n}\) was missing by mistake during the calculation\footnote{In \cite{2}, on Page 2334, left column, from the fourth equation to the fifth equation, the last term of the fifth equation should not be \(\frac{2}{m^n}\) but \(\frac{2}{m^{n-1}}\). This error spreads out in the following calculations therein.}. Theorem 2.8] has also been corrected accordingly.

Theorem 2.8. \cite{2} Theorem 13] For integers \(m, n > 1\), we have

\[
\begin{align*}
\mathbb{E}_{x \in \mathbb{Z}_m^n} \left[ |L_1(x)| \right] &= n^2 \left( m + \frac{1}{m} - 2 \right) + 2 - \frac{n}{m} + \frac{m^{n-1} - 1}{m^{n-1}(m - 1)}.
\end{align*}
\]

Note that the proof follows from Lemma 2.5, Lemma 2.6 and the linearity of expectation.

A martingale is a sequence of real random variables \(Z_0, \ldots, Z_n\) with finite expectation such that for each \(0 \leq i < n\),

\[\mathbb{E} [Z_{i+1}|Z_i, Z_{i-1}, \ldots, Z_0] = Z_i.\]

A classical martingale named Doob martingale (see for example \cite{1}) will be used in this paper. Let \(X_1, \ldots, X_n\) be the underlying random variables (not necessarily independent) and \(f\) be a function over \(X_1, \ldots, X_n\). The Doob martingale \(Z_0, \ldots, Z_n\) is defined by

\[
\begin{align*}
Z_0 &= \mathbb{E} [f(X_1, \ldots, X_n)]; \\
Z_i &= \mathbb{E} [f(X_1, \ldots, X_n)|X_1, \ldots, X_i] \text{ for } i \in [n].
\end{align*}
\]

In other words, \(Z_i\) is defined by the expected value of \(f\) after exposing \(X_1, \ldots, X_i\). The following classical result \cite{1} plays a key role in the proof of our main results.
Theorem 2.9 (Azuma’s inequality). Let $Z_0, Z_1, \ldots, Z_n$ be a martingale such that for each $1 \leq i \leq n,$

$$|Z_i - Z_{i-1}| \leq c_i.$$ 

Then for every $\lambda > 0,$ we have

$$\Pr(Z_n - Z_0 \geq \lambda) \leq \exp \left( \frac{-\lambda^2}{2(c_1^2 + \cdots + c_n^2)} \right),$$

and

$$\Pr(Z_n - Z_0 \leq -\lambda) \leq \exp \left( \frac{-\lambda^2}{2(c_1^2 + \cdots + c_n^2)} \right).$$

3. The size distribution of the FLL balls with radius one

In this section, we discuss the size distribution of the FLL balls with radius one. We start with the binary case and then deal with the general $m$-ary case.

3.1. The binary case. Let $n, n'$ be positive integers. In order to estimate $|L_1(x)|$, we define the following notation: for each $y \in \mathbb{Z}_2^{n'},$ define

$$f_n(y) = \rho(y)n - \frac{1}{2} \sum_{i=1}^{a(y)} s_i^2,$$

where $s_i$ is the length of the $i$-th alternating segment of $y$ for $1 \leq i \leq a(y)$. Note that $\sum_{i=1}^{a(y)} s_i = n'$ for all $y \in \mathbb{Z}_2^{n'},$ and $\rho(y) + a(y) = n' + 1.$ Then by Eq. (3), we have $|L_1(x)| = f_n(x) + \frac{n}{2} + 1$ for each $x \in \mathbb{Z}_2^{n}.\ \text{Therefore, it suffices to find the distribution of } f_n(x) \text{ for } x \in \mathbb{Z}_2^{n}. To this end, we express } f_n(y) \text{ by two partial values } f_n(y_{[1,i]}) \text{ and } f_n(y_{[i+1,n']}) \text{ in the following lemma.}

Lemma 3.1. Let $n, n'$ be positive integers. For each $i \in [n'-1]$ and $y \in \mathbb{Z}_2^{n'},$ we have

$$f_n(y) = \begin{cases} f_n(y_{[1,i]}) + f_n(y_{[i+1,n']}) - n, & \text{if } y_i = y_{i+1}, \\ f_n(y_{[1,i]}) + f_n(y_{[i+1,n']}) - t(y_{[1,i]})h(y_{[i+1,n']}), & \text{if } y_i \neq y_{i+1}. \end{cases}$$

Proof. It is important to use the additive behavior of $f_n(y)$ in Eq. (6). The first case follows from the equations of $\rho(y) = \rho(y_{[1,i]}) + \rho(y_{[i+1,n']}) - 1$ and $a(y) = a(y_{[1,i]}) + a(y_{[i+1,n']}).$
When \( y_i \neq y_{i+1} \), we have \( \rho(y) = \rho(y_{[1,i]}) + \rho(y_{[i+1,n']}) \). Then the difference of \( f_n(y) \) and \( f_n(y_{[1,i]}) + f_n(y_{[i+1,n']}) \) is given by

\[
\frac{1}{2} \left[ (t(y_{[1,i]}) + h(y_{[i+1,n']}))^2 - t(y_{[1,i]})^2 - h(y_{[i+1,n']}))^2 \right] = t(y_{[1,i]})h(y_{[i+1,n']}).
\]

The proof is then completed.

\( \square \)

**Corollary 3.2.** Let \( n, n' \) be positive integers. For all \( y \in \mathbb{Z}_2^n \), we have

\[
f_n(y) - f_n(y_{[1,n'-1]}) = \begin{cases} 
-\frac{1}{2}, & \text{if } y_{n'-1} = y_n', \\
\frac{1}{2} - t(y_{[1,n'-1]}), & \text{if } y_{n'-1} \neq y_n'.
\end{cases}
\]

**Proof.** Note that by the definition of \( f_n(y) \) in Eq. (6), we have \( f_n(y_{n'}) = n - \frac{1}{2} \). By setting \( i = n'-1 \), the result then follows from Lemma 3.1 \( \square \)

Now we are ready to discuss the distribution of \(|L_1(x)|\) for uniformly distributed \( x \in \mathbb{Z}_2^n \). Since the case that \( n \leq 3 \) is trivial, we now focus on the cases that \( n \) is large.

**Theorem 3.3.** Let \( n > 3 \) be an integer and \( x_1, \ldots, x_n \) be independent random variables such that \( \Pr(x_i = 0) = \Pr(x_i = 1) = \frac{1}{2} \) for \( i \in [n] \). Then for the word \( x = x_1 \cdots x_n \), we have

\[
\Pr \left( |L_1(x)| - \mathbb{E}_{x \in \mathbb{Z}_2^n} [|L_1(x)|] \geq cn\sqrt{n - 1} \right) \leq e^{-2c^2},
\]

and

\[
\Pr \left( |L_1(x)| - \mathbb{E}_{x \in \mathbb{Z}_2^n} [|L_1(x)|] \leq -cn\sqrt{n - 1} \right) \leq e^{-2c^2},
\]

where \( \mathbb{E}_{x \in \mathbb{Z}_2^n} [|L_1(x)|] = \frac{n^2}{2} - \frac{n}{2} - \frac{1}{2n-1} + 3 \), and \( c \) is a positive constant.

**Proof.** We define the Doob martingale \( Z_0 = \mathbb{E}[f_n(x)] \), and \( Z_i = \mathbb{E}[f_n(x)|x_{[1,i]}] \) by exposing one \( x_i \) at a time for \( i \in [n] \). Clearly, by Eq. (6) and Lemma 2.6, we have \( Z_n = f_n(x) \), and

\[
Z_0 = \mathbb{E} \left[ \rho(x)n - \frac{1}{2} \sum_{i=1}^{a(x)} s_i^2 \right] = n \left( n - \frac{n}{2} - 1 \right) - \frac{1}{2} \left[ \frac{n(4 \cdot 2^2 - 3 \cdot 2 + 2)}{2^2} + \frac{6 \cdot 2 - 4}{2^2} - 4 \left( 1 - \frac{1}{2^n} \right) + \frac{2}{2^n} \right]
= \frac{n^2}{2} - n + 2 - \frac{1}{2n-1}.
\]
For every $1 \leq i < n$, we have

\[
Z_i = \mathbb{E}[f_n(x)|x_{[1,i]}] = \mathbb{E}[f_n(x)|x_{[1,i]}, x_i = x_{i+1}] \Pr(x_i = x_{i+1}) + \mathbb{E}[f_n(x)|x_{[1,i]}, x_i \neq x_{i+1}] \Pr(x_i \neq x_{i+1}).
\]

Then by Lemma 3.1, we have

\[
Z_i = \frac{1}{2} \mathbb{E}\left[f_n(x_{[1,i]}) + f_n(x_{[i+1,n]}) - n|x_{[1,i]}, x_i = x_{i+1}\right] + \frac{1}{2} \mathbb{E}\left[f_n(x_{[1,i]}) + f_n(x_{[i+1,n]}) - t(x_{[i+1,n]})h(x_{[i+1,n]})|x_{[1,i]}, x_i \neq x_{i+1}\right] = f_n(x_{[1,i]}) + \mathbb{E}\left[f_n(x_{[i+1,n]})\right] - \frac{n}{2} - \frac{1}{2}t(x_{[1,i]})\mathbb{E}\left[h(x_{[i+1,n]})|x_i \neq x_{i+1}\right].
\]

Note that by Eq. (6) and Lemma 2.6, we have

\[
\mathbb{E}\left[f_n(x_{[i+1,n]})\right] = \mathbb{E}\left[\rho(x_{[i+1,n]})n - \frac{1}{2}\sum_{i=1}^{o(x_{i+1,n})} s_i^2\right] = n \cdot \frac{n - i + 1}{2} - \frac{1}{2}\left[3(n - i) - 4 + \frac{1}{2^{n-i-2}}\right] = \frac{n^2}{2} - \frac{in}{2} + 2 - \frac{3}{2}(n - i) + \frac{n}{2} - \frac{1}{2^{n-i-1}},
\]

and by Lemma 2.2, we have

\[
\mathbb{E}\left[h(x_{[i+1,n]})|x_i \neq x_{i+1}\right] = 2 - \frac{1}{2^{n-i-1}}.
\]

It then follows that

\[
(9) \quad Z_i = f_n(x_{[1,i]}) + \frac{n^2}{2} - \frac{in}{2} + 2 - \frac{3}{2}(n - i) - \frac{1}{2^{n-i-1}} - t(x_{[1,i]})\left(1 - \frac{1}{2^{n-i}}\right).
\]

Notice that $Z_n = f_n(x)$ also satisfies Eq. (9). Thus, for $1 < i \leq n$, we have

\[
Z_i - Z_{i-1} = f_n(x_{[1,i]}) - f_n(x_{[1,i-1]}) - \frac{n}{2} + \frac{3}{2} - \frac{1}{2^{n-i}} - t(x_{[1,i]})\left(1 - \frac{1}{2^{n-i}}\right) + t(x_{[1,i-1]})\left(1 - \frac{1}{2^{n-i+1}}\right).
\]

The difference between $f_n(x_{[1,i]})$ and $f_n(x_{[1,i-1]})$ is given by Corollary 3.2 also note that $t(x_{[1,i]}) = 1$ if $x_{i-1} = x_i$ and $t(x_{[1,i]}) = t(x_{[1,i-1]}) +$
1 if \(x_{i-1} \neq x_i\). Therefore, we have

\[
Z_i - Z_{i-1} = \begin{cases} 
\frac{n}{2} - t(x_{[1, i-1]})\left(1 - \frac{1}{2^{n-1}}\right), & \text{if } x_{i-1} \neq x_i, \\
-\frac{n}{2} + t(x_{[1, i-1]})\left(1 - \frac{1}{2^{n-1}}\right), & \text{if } x_{i-1} = x_i.
\end{cases}
\]

It is straightforward to check that \(|Z_i - Z_{i-1}| \leq \frac{n}{2}\) for \(1 < i \leq n\) and \(Z_1 - Z_0 = 0\). Then by Theorem \ref{thm:2.9} we have

\[
\Pr(Z_n - Z_0 \geq \lambda) \leq \exp\left(\frac{-\lambda^2}{2(0^2 + (n-1)(\frac{n}{2})^2)}\right) = \exp\left(\frac{-2\lambda^2}{n^2(n-1)}\right).
\]

Take \(\lambda = cn\sqrt{n-1}\), where \(c\) is a positive constant, then we have

\[
(10) \quad \Pr(Z_n - Z_0 \geq cn\sqrt{n-1}) \leq e^{-2c^2},
\]

and

\[
(11) \quad \Pr(Z_n - Z_0 \leq -cn\sqrt{n-1}) \leq e^{-2c^2}.
\]

Note that \(|L_1(x)| = Z_n + \frac{n}{2} + 1\), and \(\mathbb{E}_{x \in \mathbb{Z}_n^m} [\|L_1(x)\|] = Z_0 + \frac{n}{2} + 1\).

Then Eq.s (7) and (8) follow from Eq.s (10) and (11), respectively. The proof is completed. \(\square\)

3.2. The \(m\)-ary case. Let \(n, n'\) be positive integers. Again, to analyze \(|L_1(x)|\), for each \(y \in \mathbb{Z}_m^n\), we define

\[
f_{m,n}(y) = \rho(y)(mn - n - 1) - \frac{1}{2} \sum_{i=1}^a(y) s_i^2 + \frac{3}{2} \sum_{i=1}^a(y) s_i - a(y),
\]

where \(s_i\) is the length of the \(i\)-th alternating segment of \(y\), and then \(|L_1(x)| = f_{m,n}(x) + 2\). In parallel to Lemma \ref{lem:3.1}, we try to express \(f_{m,n}(y)\) by \(f_{m,n}(y_{[1,i]})\) and \(f_{m,n}(y_{[i+1,n']})\) as follows.

**Lemma 3.4.** Let \(n > 1\) and \(m, n' > 2\). For each \(i \in [n' - 1]\) and \(y \in \mathbb{Z}_m^n\), we have

- If \(y_i = y_{i+1}\),

\[
f_{m,n}(y) = f_{m,n}(y_{[1,i]}) + f_{m,n}(y_{[i+1,n']}) - mn + n + 1;
\]

- If \(y_i \neq y_{i+1}\),

\[
(12) \quad f_{m,n}(y) \leq f_{m,n}(y_{[1,i]}) + f_{m,n}(y_{[i+1,n']});
\]

\[
(13) \quad f_{m,n}(y) \geq f_{m,n}(y_{[1,i]}) + f_{m,n}(y_{[i+1,n']}) + 1 - t(y_{[1,i]}) h(y_{[i+1,n']}).
\]

\footnote{It can be verified by taking \(i = 1\) in Eq. (9). Alternatively, it also follows from the symmetry induced by the equivalence relation defined in the proof of Lemma 2.2.}
Therefore, Eq. (12) follows from the additive behavior of $f_{m,n}(y)$. It remains to discuss the case when $y_i \neq y_{i+1}$. Observe that in this case, we split exactly one alternating segment $y_{i-[l+1,i+r]}$ into two segments: $y_{i-[l+1,i]}$ and $y_{[i+1,i+r]}$, where $l, r > 0$. In addition, $a(y) = a(u) + a(v) - 1$. The difference between $f_{m,n}(y)$ and $f_{m,n}(u) + f_{m,n}(v)$ is given by

$$-\frac{1}{2}(l + r)^2 + \frac{1}{2}(l^2 + r^2) + 1 = 1 - lr,$$

and

$$f_{m,n}(y) = f_{m,n}(y_{1,i}) + f_{m,n}(y_{[i+1,i+r]}) + 1 - lr.$$

Note that if $l, r > 1$, then $l = t(u), r = h(v)$. Thus, Eq. (13) holds. 

**Theorem 3.5.** Let $m > 2, n > 3$ be integers, and $x_1, \ldots, x_n$ be independent random variables such that $\Pr(x_i = j) = \frac{1}{m}$ for $i \in [n], j \in \mathbb{Z}_m$. Then for the word $x = x_1, \ldots, x_n$, we have

$$\Pr\left(|L_1(x)| - \mathbb{E}_{x \in \mathbb{Z}_m} |L_1(x)| \geq c(m + \frac{1}{m})n\sqrt{n-1}\right) \leq e^{-c^2/2},$$

and

$$\Pr\left(|L_1(x)| - \mathbb{E}_{x \in \mathbb{Z}_m} |L_1(x)| \leq -c(m + \frac{1}{m})n\sqrt{n-1}\right) \leq e^{-c^2/2},$$

where $\mathbb{E}_{x \in \mathbb{Z}_m} |L_1(x)|$ is given in Eq. (5), and $c$ is a positive constant.

**Proof.** As in Section 3.1, define the Doob martingale $Z_0 = \mathbb{E}[f_{m,n}(x)]$, and $Z_i = \mathbb{E}[f_{m,n}(x)|x_{[1,i]}]$ for $1 \leq i \leq n$. Note that by symmetry defined in the proof of Lemma 2.2

$$Z_1 = Z_0 = n^2(m + \frac{1}{m}) - \frac{n}{m} - \frac{1}{m(m-1)} + \frac{1}{m-1} - \frac{1}{m^n}.$$

For $1 < i < n$,

$$Z_i = \mathbb{E}[f_{m,n}(x)|x_{[1,i]}] = \mathbb{E}[f_n(x)|x_{[1,i]}, x_i = x_{i+1}] \Pr(x_i = x_{i+1}) + \mathbb{E}[f_n(x)|x_{[1,i]}, x_i \neq x_{i+1}] \Pr(x_i \neq x_{i+1}).$$
By Lemma 3.4, for $1 < i < n$,

\begin{equation}
Z_i \leq \frac{1}{m} \cdot \mathbb{E} \left[ f_{m,n}(x_{[1,i]}) + f_{m,n}(x_{[i+1,n]}) - mn + n + 1 \big| x_i = x_{i+1}, x_{[1,i]} \right] \\
+ \frac{m-1}{m} \cdot \mathbb{E} \left[ f_{m,n}(x_{[1,i]}) + f_{m,n}(x_{[i+1,n]}) \big| x_i \neq x_{i+1}, x_{[1,i]} \right] \\
= f_{m,n}(x_{[1,i]}) + g_{m,n}(i) + \frac{1}{m}(-mn + n + 1) \\
\end{equation}

\begin{equation}
Z_i \geq \frac{1}{m} \cdot \mathbb{E} \left[ f_{m,n}(x_{[1,i]}) + f_{m,n}(x_{[i+1,n]}) - mn + n + 1 \big| x_i = x_{i+1}, x_{[1,i]} \right] \\
+ \frac{m-1}{m} \cdot \mathbb{E} \left[ f_{m,n}(x_{[1,i]}) + f_{m,n}(x_{[i+1,n]}) + 1 \right. \\
\left. - t(x_{[1,i]})h(x_{[i+1,n]}) \big| x_i \neq x_{i+1}, x_{[1,i]} \right] \\
= f_{m,n}(x_{[1,i]}) + g_{m,n}(i) + \frac{1}{m}(-mn + n + 1) \\
+ \frac{m-1}{m} \left( 1 - t(x_{[1,i]}) \mathbb{E} \left[ h(x_{[i+1,n]}) \big| x_i \neq x_{i+1} \right] \right) \\
= f_{m,n}(x_{[1,i]}) + g_{m,n}(i) + 1 - n + \frac{n}{m} - \frac{m-1}{m}t(x_{[1,i]}) \left( 2 - \frac{1}{m^{m-i-1}} \right),
\end{equation}

where $g_{m,n}(i) := \mathbb{E} \left[ f_{m,n}(x_{[i+1,n]}) \big| x_{[1,i]} \right]$, and by Lemma 2.6 we have

\begin{align*}
g_{m,n}(i) &= \mathbb{E} \left[ f_{m,n}(x_{[i+1,n]}) \big| x_{[1,i]} \right] \\
&= \mathbb{E} \left[ f_{m,n}(x) \big| x_{[1,i]} \right] \\
&= (mn - n - 1)\mathbb{E} \left[ \rho(x) \right] - \frac{1}{2} \mathbb{E} \left[ \sum_{i=1}^{n} s_i^2 \right] + \frac{3}{2} \mathbb{E} \left[ \sum_{i=1}^{n} s_i \right] - \mathbb{E} \left[ a(x) \right] \\
&= n(n-i)(m+\frac{1}{m}-2) - \frac{n}{m} + \frac{1}{m-1} - \frac{1}{(m-1)m^{n-1}} + i - \frac{1}{m^{n-i}}.
\end{align*}

Now we claim that $|Z_i - Z_{i-1}|$ can be bounded as follows. The proof details are left in Appendix A

\begin{equation}
|Z_i - Z_{i-1}| \leq \begin{cases} 
0, & i = 1, \\
\frac{n}{m} + \frac{1}{m} - 2, & 2 \leq i \leq n.
\end{cases}
\end{equation}

The result then follows by Theorem 2.9
Remark 3.6. The key step of proving Theorem 3.3 and Theorem 3.5 is to calculate $Z_i$ to bound $|Z_i - Z_{i-1}|$. One may wonder why $m = 2$ is not a special case of the general $m$. This is because we do not have explicit expressions of $Z_i$ for $m > 2$, but it is possible to derive it. Indeed, the proof of Lemma 3.4 in fact expresses $f_{m,n}(x)$ by $f_{m,n}(x_{[1,i]}), f_{m,n}(x_{[i+1,n]})$ and $l_i, r_i$, where $x_{[i-l_i+1, i+r_i]}$ is the maximal alternating segment that lies on both $x_{[1,i]}$ and $x_{[i+1,n]}$. Furthermore, one can calculate the conditional expectation of $l_i$ and $r_i$ given $x_{[1,i]}, x_{i+1}, x_{i+2}$, which yields an explicit expression on $Z_i$ and $Z_i - Z_{i-1}$. However, this calculation is much more complex and neither significantly improves Theorem 3.5 (still, $|Z_i - Z_{i-1}| = O(n)$) nor gives more insights on this problem.

4. Conclusion

In this paper, we analyze the distribution of $|L_1(x)|$ for $x \in \mathbb{Z}_m^n$ by Azuma’s inequality. The numerical result suggests that $|L_1(x)|$ are more concentrated than we expected. Specifically, the gap between the simulation results and the bounds in Theorem 3.3 and Theorem 3.5 is still large (see Appendix B), leaving the derivation of better bounds as an open problem. Intuitively, the distribution of $|L_t(x)|$ should be more and more concentrated as $t$ grows. For example, $|L_n(x)| = m^n$ for all $x \in \mathbb{Z}_m^n$. However, finding the distribution of $|L_t(x)|$ is in general difficult and left open. Very recently, He and Ye [4] considered the case of radius two, and further gave the concentration bound for $|L_2(x)|$.

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Appendix A. Proof of Eq. (18)

We follow the notations in the proof of Theorem 3.5.

- Case $i = 1$: By symmetry, we have $|Z_1 - Z_0| = 0$.
- Case $1 < i < n$:
  
  By Eq. (16), we have
  \[
  Z_i - Z_{i-1} \geq f_{m,n}(x_{[1,i]}) + g_{m,n}(i) + 1 - n + \frac{n}{m} - \frac{m-1}{m} t(x_{[1,i]}) \left(2 - \frac{1}{m^{n-i}}\right)
  \]
  
  \[
  - \left[f_{m,n}(x_{[1,i-1]}) + g_{m,n}(i-1) - n + \frac{n}{m} + \frac{1}{m}\right]
  \]
  
  \[
  = f_{m,n}(x_{[1,i]}) - f_{m,n}(x_{[1,i-1]}) + g_{m,n}(i) - g_{m,n}(i-1)
  + \frac{m-1}{m} - \frac{m-1}{m} t(x_{[1,i]}) \left(2 - \frac{1}{m^{n-i}}\right),
  \]

  By Lemma 3.4, we have\(^3\)
  \[
  0 \leq f_{m,n}(x_{[1,i]}) - f_{m,n}(x_{[1,i-1]}) \leq mn - n - 1,
  \]
  
  and
  \[
  g_{m,n}(i) - g_{m,n}(i-1) = 1 - n(m + \frac{1}{m} - 2) - \frac{1}{m^{n-i}}.
  \]

\(^3\)Note that $f_{m,n}(x_i) = mn - n - 1$, and we then express $f_{m,n}(x_{[1,i]})$ by $f_{m,n}(x_{[1,i-1]})$ and $f_{m,n}(x_i)$.
Therefore, we have

\[ Z_i - Z_{i-1} \geq 1 - n \left( m + \frac{1}{m} - 2 \right) - \frac{1}{m^{n-i}} - \frac{m-1}{m} - \frac{m-1}{m} t(x_{[1,i]}) \left( 2 - \frac{1}{m^{n-i}} \right) \]

\[ > -n \left( m + \frac{1}{m} - 2 \right) - \frac{1}{m^{n-i}} - (n-1) \cdot 2 \]

\[ > -n \left( m + \frac{1}{m} \right) + 1, \]

and also

\[ Z_i - Z_{i-1} \leq mn - n - 1 + 1 - n \left( m + \frac{1}{m} - 2 \right) - \frac{1}{m^{n-i}} - \frac{m-1}{m} \]

\[ + \frac{m-1}{m} t(x_{[1,i-1]}) \left( 2 - \frac{1}{m^{n-i}} \right) \]

\[ = n \left( 1 - \frac{1}{m} \right) - \frac{m-1}{m} + \frac{m-1}{m} t(x_{[1,i-1]}) \left( 2 - \frac{1}{m^{n-i}} \right) \]

\[ < n \left( 1 - \frac{1}{m} \right) + 2n = n \left( 3 - \frac{1}{m} \right). \]

Note that \(| - n(m + \frac{1}{m}) + 1 | \leq n(m + \frac{1}{m})\) and \(|n(3 - \frac{1}{m})| \leq n(m + \frac{1}{m})\). Hence, we have \(|Z_i - Z_{i-1}| \leq n(m + \frac{1}{m})\) for \(1 < i < n\).

- Case \(i = n\):
  By Eq. (16) and Lemma 3.4 we have

\[ 0 \leq f_{m,n}(x) - f_{m,n}(x_{[1,n-1]}) \leq mn - n - 1, \]

and

\[ Z_n - Z_{n-1} \leq f_{m,n}(x) - \left[ f_{m,n}(x_{[1,n-1]}) + \left( 1 - \frac{1}{m} \right) (mn - n - t(x_{[1,n-1]})) \right] \]

\[ \leq mn - n - 1 - \left( 1 - \frac{1}{m} \right) (mn - n - t(x_{[1,n-1]})) \]

\[ = t(x_{[1,n-1]}) \left( 1 - \frac{1}{m} \right) - \frac{n}{m} \]

\[ < n \left( 1 - \frac{1}{m} \right), \]
and also,

\[ Z_n - Z_{n-1} \geq f_{m,n}(x) - \left[ f_{m,n}(x_{[1,n-1]}) + \left( 1 - \frac{1}{m}\right)(mn - n - 1) \right] \]
\[ \geq - \left( 1 - \frac{1}{m}\right)(mn - n - 1) \]
\[ = -n \left( m + \frac{1}{m}\right) - \frac{1}{m} + 1. \]

Thus, we have \(|Z_n - Z_{n-1}| \leq n(m + \frac{1}{m})\).

**Appendix B. Simulation results**

We independently pick \(x \in \mathbb{Z}^n_m\) uniformly at random and record the value \(|L_1(x)|\). The distribution of \(|L_1(x)|\) is then reflected by the frequency that \(|L_1(x)|\) lies in different intervals. We also compare it with the expected frequency given by the bounds in Theorems 3.3 and 3.5. For instance, the expected frequency of the event \(|L_1(x)| > \tau\) by Eq. (7) is

\[ Ne^{-2\left(\frac{\tau - E[|L_1(x)|]}{n\sqrt{n-1}}\right)^2}, \]

where \(N\) is the sample size. The simulation results for \(n = 100, m = 2, 3, 4, 5\) are depicted in Fig. 1.

**Figure 1.** \(N = 5000\) for each experiment.
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