Type II superstring field theory revisited

Hiroshi Kunitomo

Center for Gravitational Physics, Yukawa Institute for Theoretical Physics
Kyoto University, Kitashirakawa Oiwakecho, Sakyo-ku, Kyoto 606-8502, Japan
kunitomo@yukawa.kyoto-u.ac.jp

Abstract

We reconstruct a complete type II superstring field theory with $L_{\infty}$ structure in a symmetric way concerning the left- and right-moving sectors. Based on the new construction, we show again that the tree-level S-matrix agrees with that obtained using the first-quantization method. Not only is this a simple and elegant reconstruction, but it also enables the action to be mapped to that in the WZW-like superstring field theory, which has not yet been constructed and fills the only gap in the WZW-like formulation.
Contents

1 Introduction .................................................. 2

2 Type II superstring field theory with $L_{\infty}$ structure .................. 3

3 Symmetric construction of string products .................................. 6
   3.1 $L_{\infty}$ triplet .............................................. 7
   3.2 Explicit construction .......................................... 9

4 Tree-level S-matrix ........................................... 18

5 Gauge-invariant action in WZW-like formulation ......................... 24
   5.1 WZW-like action for the NS-NS sector ............................. 24
   5.2 Complete WZW-like action and gauge invariance .................. 26

6 Conclusion and discussion ........................................ 29

A Projected commutators .......................................... 30

B Expansion of $B_3^{(*)}(s, \bar{s})$, $\lambda_3^{(*)}(s, \bar{s})$ and $\bar{\lambda}_3^{(*)}(s, \bar{s})$ .... 31
1 Introduction

Recent significant developments in the superstring field theory [1–18] have ended long-term stagnation and made it possible to construct classical gauge-invariant actions for open [5,12,18], heterotic [14,16,17], and type II [15–17] superstring theories. It is based on three complementary construction methods: the formulation with the homotopy algebra structure [9–12,14,15], the Wess-Zumino-Witten (WZW) -like formulation [1–7,13–15], and the formulation accompanied with an extra free field [16–18]. The only exception is the complete WZW-like action for the type II superstring field theory, which has not yet been constructed. One of the purposes of this paper is to give it and fills a gap in the WZW-like formulation.

In the previous papers [14,15], we have attempted to construct complete WZW-like actions, including all sectors representing the space-time bosons and fermions, for the heterotic and the type II superstring field theories by mapping from those of the formulation with the $L_\infty$ algebra method. This construction worked well for the heterotic string theory but not for the type II superstring theory. The reason is that we have taken the left-right asymmetric method for constructing the type II string field theory with an $L_\infty$ structure. In this case, we could only construct an action that is a hybrid of the WZW-like action and the action with an $L_\infty$ structure, which we called the half-WZW-like action. Therefore, we first revisit the formulation based on $L_\infty$ structure and provide an alternative to the asymmetric method, the symmetric way to construct the action for the type II superstring field theory. It is another purpose of this paper.

The paper is organized as follows. In section 2, we summarize how to construct a gauge-invariant action for the type II superstring field theory based on an $L_\infty$ structure. The string field is suitably constrained and the gauge-invariant action is written by using the string products satisfying the $L_\infty$ algebra relations. We propose in section 3 a new construction method of the string products with cyclic $L_\infty$ structure symmetric with respect to the left- and right-moving sectors. After introducing the coalgebra representation, we suppose $L_\infty$ algebra using the string products with fixed cyclic Ramond numbers. The significant point, which is different from the other cases, is that their RR output part is including the picture-changing operator (PCO) explicitly. This breaks the cyclicity of the algebra but instead suitably decomposes it into three commutative $L_\infty$ algebras (rather than four), two constraint $L_\infty$ algebras and a dynamical $L_\infty$ algebra. A similarity transformation allows us to transform dynamical $L_\infty$ algebra into the one used for writing down the action and the other two into the constraints that impose that the first algebra closes in the small Hilbert space. In order to explicitly construct such cyclic string products, we give differential equations for their generating functional. These differential equations recursively determine the string products with respect to the number of input string fields, with bosonic products as the initial condition. We show that the new
action correctly derives the same tree-level physical S-matrix as that calculated using the first-quantization method in section 4. The proof is based on the homological perturbation theory (HPT), which provides the explicit form of the tree-level S-matrix in closed form and makes it possible to demonstrate the agreement. In section 5, we write down a complete WZW-like action for the type II superstring field theory, which we could not construct previously. After summarizing how the NS-NS action with \( L_\infty \) structure was rewritten as a WZW-like action through the map between string fields in two formulations, we extend it to all four sectors. It fills the gap in the WZW-like formulation and should be significant to deepen the understanding of the superstring field theory. Section 6 is devoted to the conclusion and discussion. Finally, it contains two appendices. In Appendix A, we define a projected commutator that plays a significant role in constructing the cyclic products with the \( L_\infty \) structure. Appendix B contains the \( s \) and \( \bar{s} \) expansions of the generating functionals of cyclic three-string products and corresponding gauge products to help for understanding the flow how they are recursively determined.

## 2 Type II superstring field theory with \( L_\infty \) structure

Let us recall how the type II superstring field theory with an \( L_\infty \) structure has been constructed [10,15]. The first-quantized Hilbert space, \( \mathcal{H} \), of type II superstring is composed of four sectors corresponding to the combination of the NS and Ramond boundary conditions for the left- and right-moving fermionic coordinates:

\[
\mathcal{H} = \mathcal{H}_{NS-NS} + \mathcal{H}_{R-NS} + \mathcal{H}_{NS-R} + \mathcal{H}_{R-R}.
\]  

(2.1)

Correspondingly, the type II superstring field \( \Phi \) has four components:

\[
\Phi = \Phi_{NS-NS} + \Phi_{R-NS} + \Phi_{NS-R} + \Phi_{R-R} \in \mathcal{H},
\]  

(2.2)

which is Grassmann even\(^1\) and has ghost number 2. We take the picture number of each component as \((-1,-1)\), \((-1/2,-1)\), \((-1,-1/2)\) and \((-1/2,-1/2)\), respectively. The NS-NS and R-R components, \( \Phi_{NS-NS} \) and \( \Phi_{R-R} \), represent space-time bosons, and the R-NS and NS-R components, \( \Phi_{R-NS} \) and \( \Phi_{NS-R} \), represent space-time fermions. The string field \( \Phi \) is restricted by the closed string constraints

\[
b_0^+ \Phi = L_0^- \Phi = 0,
\]  

(2.3)

with \( b_0^+ = b_0 \pm \delta_0 \), \( L_0^\pm = L_0 \pm \bar{L}_0 \), and \( c_0^\pm = (c_0 \pm \bar{c}_0)/2 \). We call the Hilbert space restricted by constraints (2.3) the closed string Hilbert space. In addition, it is necessary to impose an

\(^1\)Here, for consistent Grassmann property, we should generally assume that \( \Phi \) is GSO projected.
extra constraint by introducing a projection operator $\mathcal{P}_G = \mathcal{G}^{-1}$ with
\[ \mathcal{G} = \pi^{(0,0)} + X\pi^{(1,0)} + \bar{X}\pi^{(0,1)} + X\bar{X}\pi^{(1,1)}, \]
\[ \mathcal{G}^{-1} = \pi^{(0,0)} + Y\pi^{(1,0)} + \bar{Y}\pi^{(0,1)} + Y\bar{Y}\pi^{(1,1)}. \]  
(2.4)
Here, $\pi^{(a,b)} (a, b = 0, 1)$ is a projection operator onto a component
\[ \pi^{(0,0)}\Phi = \Phi_{NS-NS}, \pi^{(1,0)}\Phi = \Phi_{R-NS}, \pi^{(0,1)}\Phi = \Phi_{NS-R}, \pi^{(1,1)}\Phi = \Phi_{R-R}, \]  
(2.5)
and, we take the PCO’s as
\[ X = -\delta(\beta_0)G + \frac{1}{2}(\gamma_0\delta(\beta_0) + \delta(\beta_0)\gamma_0)b_0^+, \quad Y = -\frac{2G}{L_0}\delta(\gamma_0), \]  
(2.6)
\[ \bar{X} = -\delta(\bar{\beta}_0)\bar{G} + \frac{1}{2}(\bar{\gamma}_0\delta(\bar{\beta}_0) + \delta(\bar{\beta}_0)\bar{\gamma}_0)b_0^+, \quad \bar{Y} = -\frac{2\bar{G}}{L_0}\delta(\bar{\gamma}_0). \]  
(2.7)
Note that $\mathcal{G}$ satisfies
\[ \mathcal{G}\mathcal{G}^{-1}\mathcal{G} = \mathcal{G}, \quad \mathcal{G}^{-1}\mathcal{G}\mathcal{G}^{-1} = \mathcal{G}^{-1}, \quad [Q, \mathcal{G}] = 0, \]  
(2.8)
and is almost exact in the large Hilbert space $\mathcal{H}_l$: 
\[ \mathcal{G} = \pi^{(0,0)} + [Q, \mathcal{G}], \quad \mathcal{G} = \Xi\pi^{(1,0)} + \bar{\Xi}\pi^{(0,1)} + \frac{1}{2}(\Xi\bar{X} + X\bar{\Xi})\pi^{(1,1)}, \]  
(2.9)
with$^2$
\[ \Xi = \xi + (\Theta(\beta_0)\eta\xi - \xi)P_{-3/2} + (\xi\eta\Theta(\beta_0) - \xi)P_{-1/2}, \]  
(2.10)
\[ \bar{\Xi} = \bar{\xi} + (\bar{\Theta}(\bar{\beta}_0)\bar{\eta}\bar{\xi} - \bar{\xi})P_{-3/2} + (\bar{\xi}\bar{\eta}\bar{\Theta}(\bar{\beta}_0) - \bar{\xi})P_{-1/2}. \]  
(2.11)
The operator $P_{-3/2}$ and $P_{-1/2}$ are the projector onto the states with ghost number $-3/2$ and $-1/2$, respectively. The BRST operator consistently acts on the string field restricted by (2.12) thanks to the property: $\mathcal{P}_G Q\mathcal{P}_G = Q\mathcal{P}_G$. Then, the last constraint
\[ \mathcal{P}_G \Phi = \Phi \]  
(2.12)
restricts the dependence of the ghost zero modes of each component of the string field as
\[ \Phi_{NS-NS} = \phi_{NS-NS} - c_0^+\psi_{NS-NS}, \]  
(2.13a)
\[ \Phi_{R-NS} = \phi_{R-NS} - \frac{1}{2}(\gamma_0 + 2c_0^+G)\psi_{R-NS}, \]  
(2.13b)
\[ \Phi_{NS-R} = \phi_{NS-R} - \frac{1}{2}(\gamma_0 + 2c_0^+G)\psi_{NS-R}, \]  
(2.13c)
\[ \Phi_{R-R} = \phi_{R-R} - \frac{1}{2}(\gamma_0\bar{G} - \bar{\gamma}_0G + 2c_0^+G\bar{G})\psi_{R-R}. \]  
(2.13d)

$^2$For notational simplicity, we denote the zero modes of fermionic ghosts ($\xi(z), \eta(z)$) and ($\bar{\xi}(\bar{z}), \bar{\eta}(\bar{z})$) as ($\xi, \eta$) and ($\bar{\xi}, \bar{\eta}$), respectively.
We call the Hilbert space further restricted by (2.12) as the restricted Hilbert space, $\mathcal{H}^{\text{res}}$. The ghost-zero-mode independent components $\phi_i$ ($\psi_i$) ($i = \text{NS-NS}, \text{R-NS}, \text{NS-R}, \text{R-R}$) are Grassmann even (odd) and correspond to the fields (anti-fields) in the gauge-fixed basis when they are quantized by the Batalin-Vilkovisky (BV) formalism. The simplest and practical gauge is obtained by setting $\psi = 0$, which we call the Siegel-Ramond (SR) gauge and denote its Hilbert space as $\mathcal{H}_{SR}$.

A natural symplectic form in the closed string Hilbert space is defined by using the BPZ-inner product as

$$\omega(\Phi_1, \Phi_2) = (-1)^{|\Phi_1|} \langle \Phi_1 | c_0^- | \Phi_2 \rangle,$$

where $\langle \Phi \rangle$ is the BPZ conjugate of $|\Phi\rangle$. The symbol $|\Phi\rangle$ denotes the Grassmann property of string field $\Phi$: $|\Phi\rangle = 0$ or 1 if $\Phi$ is Grassmann even or odd, respectively. A natural symplectic form in the restricted Hilbert space $\Omega$ is defined by using $\omega$ as

$$\Omega(\Phi_1, \Phi_2) = \omega(\Phi_1, G^{-1}\Phi_2).$$

Due to the asymmetry of the inner product among sectors, it is nontrivial to make the $L_\infty$ algebra being cyclic across all the sectors. For later use, it is also convenient to introduce the symplectic form $\omega_l$ in the large Hilbert space. It is related to $\omega$ as

$$\omega_l(\xi \bar{\xi} \Phi_1, \Phi_2) = \omega(\Phi_1, \Phi_2)$$

if we embed $\Phi_1, \Phi_2 \in \mathcal{H}$ into $\mathcal{H}_l$ as the fields satisfying the constraint $\eta \Phi_i = \bar{\eta} \Phi_i = 0$ ($i = 1, 2$).

We also use the bilinear map representation of symplectic forms defined by

$$\langle \omega_l \rangle : \mathcal{H}_l \otimes \mathcal{H}_l \longrightarrow \mathbb{C}$$

$$\Phi_1 \otimes \Phi_2 \longmapsto \omega_l(\Phi_1, \Phi_2),$$

and

$$\langle \omega_s \rangle = \langle \omega_l | \xi \bar{\xi} \otimes \mathbb{I} \rangle, \quad \langle \Omega \rangle = \langle \omega_l | \xi \bar{\xi} \otimes G^{-1} \rangle.$$

The action of the type II superstring field theory is defined using the string products $L_n$ mapping $n$ string fields to a string field as

$$L_n : (\mathcal{H}^{\text{res}})^n \longrightarrow \mathcal{H}^{\text{res}}, \quad (n \geq 1),$$

$$\Phi_1 \wedge \cdots \wedge \Phi_n \longmapsto L_n(\Phi_1, \cdots, \Phi_n).$$

We identify the one-string product as the BRST operator $L_1 = Q$, and each $(\mathcal{H}^{\text{res}})^n$ ($n \geq 2$) is the symmetrized tensor product of $\mathcal{H}^{\text{res}}$ whose element, $\Phi_1 \wedge \cdots \wedge \Phi_n \in (\mathcal{H}^{\text{res}})^n$, is defined
by
\[ \Phi_1 \land \cdots \land \Phi_n = \sum_{\sigma} (-1)^{\epsilon(\sigma)} \Phi_{\sigma(1)} \otimes \cdots \otimes \Phi_{\sigma(n)}, \quad \Phi_i \in \mathcal{H}^{\text{res}}, \] (2.20)
where \( \sigma \) and \( \epsilon(\sigma) \) denote all the permutations of \( \{1, \cdots, n\} \) and the sign factor coming from the exchange \( \{\Phi_1, \cdots, \Phi_n\} \) to \( \{\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(n)}\} \). Note that the string field \( L_n(\Phi_1, \cdots, \Phi_n) \) must satisfy the constraints (2.3) and (2.12):
\[ b^{-} L_n(\Phi_1, \cdots, \Phi_n) = L_0^{-} L_n(\Phi_1, \cdots, \Phi_n) = 0, \]
\[ P_g L_n(\Phi_1, \cdots, \Phi_n) = L_n(\Phi_1, \cdots, \Phi_n). \] (2.21)

If the string products are equipped with the cyclic \( L_\infty \) structure, that is, satisfy the \( L_\infty \) relations
\[ \sum_{m=1}^{n} \sum_{\sigma} (-1)^{\epsilon(\sigma)} \frac{1}{m!(n-m)!} L_{n-m+1}(L_m(\Phi_{\sigma(1)}, \cdots, \Phi_{\sigma(m)}), \Phi_{\sigma(m+1)}, \cdots, \Phi_{\sigma(n)}) = 0, \] (2.22)
and the cyclicity condition,
\[ \Omega(\Phi_1, L_n(\Phi_2, \cdots, \Phi_{n+1})) = -(-1)^{|\Phi_1|} \Omega(L_n(\Phi_1, \cdots, \Phi_n), \Phi_{n+1}), \] (2.23)
the action of the type II superstring field theory is given by
\[ I = \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \Omega(\Phi, L_{n+1}(\Phi_1, \cdots, \Phi_n)), \] (2.24)
which is invariant under the gauge transformation
\[ \delta \Phi = \sum_{n=0}^{\infty} \frac{1}{n!} L_{n+1}(\Phi_1, \cdots, \Phi_n, \Lambda). \] (2.25)

The gauge parameter \( \Lambda \) has also four components: \( \Lambda = \Lambda_{\text{NS}-\text{NS}} + \Lambda_{\text{R}-\text{NS}} + \Lambda_{\text{NS}-\text{R}} + \Lambda_{\text{R}-\text{R}} \in \mathcal{H} \).

A set of string products satisfying (2.22) and (2.23) defines a cyclic \( L_\infty \) algebra \( (\mathcal{H}^{\text{res}}, \Omega, \{L_n\}) \).

The nontrivial task is to provide a prescription that gives these string products.

3 Symmetric construction of string products

In the previous paper [15], we gave a prescription to construct a set of string products with an \( L_\infty \) structure required by complete action of the type II superstring field theory by extending the asymmetric construction proposed in [10]. It is sufficient for defining an action and gauge transformation but is slightly complicated and could not be mapped to those in the WZW-like formulation. In this paper, we propose another prescription by extending the symmetric construction in [10].
3.1 $L_\infty$ triplet

We use the coalgebra representation. The coderivation $L = \sum_{n=0}^{\infty} L_n$ is defined as an operator acting on symmetrized tensor algebra

$$SH = \mathcal{H}^{\wedge 0} \oplus \mathcal{H} \oplus \mathcal{H}^{\wedge 2} \oplus \cdots ,$$

(3.1)

by

$$L_n \Phi_1 \wedge \cdots \wedge \Phi_m = 0 ,$$

(3.2)

for $m < n$,

$$L_n \Phi_1 \wedge \cdots \wedge \Phi_m = L_n(\Phi_1, \cdots, \Phi_n) ,$$

(3.3)

for $m = n$,

$$L_n \Phi_1 \wedge \cdots \wedge \Phi_m = (L_n \wedge \mathbb{I}_{m-n}) \Phi_1 \wedge \cdots \wedge \Phi_m ,$$

(3.4)

for $m < n$.

We denote the projection onto $\mathcal{H}^{\wedge n}$ as $\pi_n$: $\pi_n SH = \mathcal{H}^{\wedge n}$. The string products satisfying $L_\infty$ relation (2.22) is given by a nilpotent odd coderivation, $L = \sum_{n=1}^{\infty} L_n$ satisfying $[L, L] = 0$.

The basic building block of our construction is the odd coderivation

$$B = \sum_{p,r=0}^{\infty} \sum_{\bar{p}, \bar{r}=0}^{\infty} B^{(p, \bar{p})}_{p+r+1, \bar{p}+\bar{r}+1} |^{(2r, 2\bar{r})} ,$$

(3.5)

acting on the symmetrized tensor algebra, $SH_t$, generated by the large Hilbert space. The superscripts $p$ ($\bar{p}$) denotes the left- (right-) moving cyclic Ramond number defined by

Cyclic Ramond No. $= \text{No. of Ramond inputs} + \text{No. of Ramond outputs}$.

We used a matrix notation describing the string products as a diagonal matrix [10]:

$$B^{(p, \bar{p})}_{p+r+1, \bar{p}+\bar{r}+1} |^{(2r, 2\bar{r})} = \delta_{p+r, \bar{p}+\bar{r}} B^{(p, \bar{p})}_{p+r+1} |^{(2r, 2\bar{r})} .$$

(3.6)

The coderivation $B$ can be decomposed into four components according to the sector of outputs as:

$$\pi_1^{(0,0)} B = \sum_{p, r=0}^{\infty} \sum_{\bar{p}, \bar{r}=0}^{\infty} \pi_1 B^{(p, \bar{p})}_{p+r+1, \bar{p}+\bar{r}+1} |^{(2r, 2\bar{r})} ,$$

(3.7a)

$$\pi_1^{(1,0)} B = \sum_{p, r=0}^{\infty} \sum_{\bar{p}, \bar{r}=0}^{\infty} \pi_1 B^{(p, \bar{p})}_{p+r+1, \bar{p}+\bar{r}+1} |^{(2r, 2\bar{r})} ,$$

(3.7b)

$$\pi_1^{(0,1)} B = \sum_{p, r=0}^{\infty} \sum_{\bar{p}, \bar{r}=0}^{\infty} \pi_1 B^{(p, \bar{p})}_{p+r+1, \bar{p}+\bar{r}+1} |^{(2r, 2\bar{r})} ,$$

(3.7c)

$$\pi_1^{(1,1)} B = \sum_{p, r=0}^{\infty} \sum_{\bar{p}, \bar{r}=0}^{\infty} \pi_1 B^{(p, \bar{p})}_{p+r+1, \bar{p}+\bar{r}+1} |^{(2r, 2\bar{r})} ,$$

(3.7d)

$^3$The identity operator $\mathbb{I}_n$ acting on $\mathcal{H}^{\wedge n}$ is given by $\mathbb{I}_n = \frac{1}{n!} \mathbb{I} \wedge \cdots \wedge \mathbb{I} = \mathbb{I} \otimes \cdots \otimes \mathbb{I}$.

$^4$In this paper the square bracket $[,]$ denotes the graded commutator.
where $\pi_{1}^{(a,b)} = \pi^{(a,b)} \pi_{1}$. The subscripts $2r$ ($2\bar{r}$) denotes the left- (right-) moving Ramond number defined by

$$\text{Ramond No.} = \text{No. of Ramond inputs} - \text{No. of Ramond outputs}.$$ 

Although it is not difficult to construct the products $B$ to be cyclic (with respect to $\omega_l$), they cannot be used as $L$ in the action (2.24) since they have wrong picture numbers except for the first component (3.7a). The picture number deficit of the components (3.7b), (3.7c), and (3.7d) from those of $L$ are $[1,0]$, $[0,1]$ and $[1,1]$, respectively: the picture number of their outputs are $(-3/2,-1)$, $(-1,-2/3)$, and $(-3/2,-3/2)$, respectively.

Using the coderivation $B$, we can construct three independent $L_\infty$ algebras, $(\mathcal{H}_l, D)$, $(\mathcal{H}_l, C)$ and $(\mathcal{H}_l, \bar{C})$, which are called an $L_\infty$ triplet in [13], with

$$\pi_{1}D = \pi_{1}Q + \pi_{1}^{(0,0)} B, \quad (3.8a)$$

$$\pi_{1}C = \pi_{1}\eta - \pi_{1}^{(1,0)} B - \frac{1}{2} X \pi_{1}^{(1,1)} B, \quad (3.8b)$$

$$\pi_{1}\bar{C} = \pi_{1}\bar{\eta} - \pi_{1}^{(0,1)} B - \frac{1}{2} X \pi_{1}^{(1,1)} B. \quad (3.8c)$$

We can merge them into an $L_\infty$ algebra, $(\mathcal{H}_l, D - C - \bar{C})$, as

$$\pi_{1}(D - C - \bar{C}) = \pi_{1}Q - \pi_{1}\eta - \pi_{1}\bar{\eta} + (\pi_{1} - \pi_{1}^{(1,1)}) B + \frac{1}{2}(X + \bar{X}) \pi_{1}^{(1,1)} B, \quad (3.9)$$

because the $L_\infty$ relation

$$[D - C - \bar{C}, D - C - \bar{C}] = 0, \quad (3.10)$$

holds for each picture number deficit and can decompose as

$$[D, D] = [C, C] = [\bar{C}, C] = 0,$$

$$[D, C] = [D, \bar{C}] = [C, \bar{C}] = 0. \quad (3.11)$$

Note that it is difficult to make this merged $L_\infty$ algebra $(\mathcal{H}_l, D - C - \bar{C})$ cyclic, unlike the heterotic string field theory case, due to the last (R-R) component. Once this $L_\infty$ triplet is constructed, we can transform it into the triplet $(\eta, \bar{\eta}; L)$ closed in $\mathcal{H}^{\text{res}}$ as

$$\pi_{1}\hat{F}^{-1}\bar{C}\hat{F} = \pi_{1}\eta, \quad \pi_{1}\hat{F}^{-1}C\hat{F} = \pi_{1}\bar{\eta}, \quad (3.12a)$$

$$\pi_{1}L \equiv \pi_{1}\hat{F}^{-1}D\hat{F} = \pi_{1}Q + G\pi_{1}b, \quad \pi_{1}b = \pi_{1}B\hat{F}. \quad (3.12b)$$

by using the cohomomorphism

$$\pi_{1}\hat{F}^{-1} = \pi_{1}\mathbb{I} - \mathcal{G}\pi_{1}B, \quad \mathcal{G} = \Xi\pi^{(1,0)} + \bar{\Xi}\pi^{(0,1)} + \frac{1}{2}(\Xi X + \bar{X}\bar{\Xi})\pi^{(1,1)}. \quad (3.13)$$

As was shown in [14], if $B$ is cyclic with respect to $\omega_l$, $L$ is cyclic with respect to $\Omega$, and thus gives a cyclic $L_\infty$ algebra $(\mathcal{H}^{\text{res}}, \Omega, L)$ used for the action.
3.2 Explicit construction

Now, the task is to construct odd coderivation $B$ satisfying the $L_\infty$ relations (3.11) written for $B$ as

$$[Q, B] + \frac{1}{2}[B, B]^{11} = 0,$$  
(3.14a)

$$[\eta, B] - \frac{1}{2}[B, B]^{21} - \frac{1}{4}[B, B]^{22}_\bar{X} = 0,$$  
(3.14b)

$$[\bar{\eta}, B] - \frac{1}{2}[B, B]^{12} - \frac{1}{4}[B, B]^{22} = 0.$$  
(3.14c)

Here, the square bracket $[\cdot, \cdot]_{ab}$ with $a, b = 1$ or 2 denotes the projected commutator defined by projecting onto specific cyclic Ramond numbers, an extension of that introduced in [14] for the heterotic string field theory. In addition, similar bracket with subscript $\bar{X} = X$ or $\bar{X}$ is defined by inserting $X$ at the intermediate state. We give an explicit definition of these brackets in Appendix A, in which we also summarize the Jacobi identities they satisfy.

Assume that the cyclic $L_\infty$ algebra $(\mathcal{H}_l, \omega_l, L^{(0,0)} = \sum_{n=0}^{\infty} L^{(0,0)}_{n+1})$ without picture number, which is straightforwardly constructed similar to that of the bosonic string field theory [19–21], is known. We define a generating functional

$$L^{(0,0)}(s, \bar{s}) = Q + \sum_{m, \bar{m}, r, \bar{r}} s^m \bar{s}^{\bar{m}} L^{(0,0)}_{m+r+1, \bar{m}+\bar{r}+1} \mid^{(2r, 2\bar{r})} \equiv Q + L^B(s, \bar{s}),$$  
(3.15)

with

$$L^{(0,0)}_{m+r+1, \bar{m}+\bar{r}+1} \mid^{(2r, 2\bar{r})} = \delta_{m+r, \bar{m}+\bar{r}} L^{(0,0)}_{m+r+1} \mid^{(2r, 2\bar{r})},$$  
(3.16)

which reduces to $L^{(0,0)}$ at $(s, \bar{s}) = (1, 1)$. The parameter $s$ or $\bar{s}$ is counting the left- or right-moving picture number deficit from $B$, respectively. We can show that $L^B(s, \bar{s})$ satisfies

$$[Q, L^B(s, \bar{s})] + \frac{1}{2}[L^B(s, \bar{s}), L^B(s, \bar{s})]^{11} + \frac{s}{2}[L^B(s, \bar{s}), L^B(s, \bar{s})]^{21} + \frac{\bar{s}}{2}[L^B(s, \bar{s}), L^B(s, \bar{s})]^{12} + \frac{s\bar{s}}{2}[L^B(s, \bar{s}), L^B(s, \bar{s})]^{22} = 0,$$  
(3.17a)

derived from the $L_\infty$ relation $[L^{(0,0)}, L^{(0,0)}] = 0$. It is also closed in the small Hilbert space,

$$[\eta, L^B(s, \bar{s})] = 0, \quad [\bar{\eta}, L^B(s, \bar{s})] = 0,$$  
(3.17b)

since it can be constructed without using $\xi$ or $\bar{\xi}$. Then we extend $B$ to include those with

5This bracket cannot always be well-defined since $X$ and $\bar{X}$ are the PCO’s acting on the states with picture number $-3/2$.

6This is actually a cyclic $L_\infty$ algebra $(\mathcal{H}, \omega, L^{(0,0)})$ closed in the small Hilbert space.
non-zero picture number deficit from $B$ and define a generating functional

$$B(s, \bar{s}, t) = \sum_{p,m,r=0}^{\infty} \sum_{\bar{p},\bar{r},\bar{m}=0}^{\infty} t^{p+\bar{p}} s^m \bar{s}^\bar{m} B^{(p,\bar{p})}_{p+m+r+1,\bar{p}+\bar{m}+r+1} \bigg|^{(2r,2\bar{r})}$$

(3.18)

by introducing another parameter $t$ counting the (total) picture number. The $L_\infty$ relations (3.14) are extended for $B(s, \bar{s}, t)$ to

$$I(s, \bar{s}, t) \equiv [Q, B(s, \bar{s}, t)] + \frac{1}{2}[B(s, \bar{s}, t), B(s, \bar{s}, t)]^{11}$$

$$+ \frac{s}{2} \left( [B(s, \bar{s}, t), B(s, \bar{s}, t)]^{21} + t[B(s, \bar{s}, t), B(s, \bar{s}, t)]^{22}_X \right)$$

$$+ \frac{\bar{s}}{2} \left( [B(s, \bar{s}, t), B(s, \bar{s}, t)]^{12} + t[B(s, \bar{s}, t), B(s, \bar{s}, t)]^{22}_X \right)$$

$$+ \frac{s \bar{s}}{2} [B(s, \bar{s}, t), B(s, \bar{s}, t)]^{22} = 0,$$  \hspace{1cm} (3.19a)

$$J(s, \bar{s}, t) \equiv [\eta, B(s, \bar{s}, t)]$$

$$- \frac{t}{2} \left( [B(s, \bar{s}, t), B(s, \bar{s}, t)]^{21} + \frac{t}{2}[B(s, \bar{s}, t), B(s, \bar{s}, t)]^{22}_X \right) = 0,$$  \hspace{1cm} (3.19b)

$$\bar{J}(s, \bar{s}, t) \equiv [\bar{\eta}, B(s, \bar{s}, t)]$$

$$- \frac{t}{2} \left( [B(s, \bar{s}, t), B(s, \bar{s}, t)]^{12} + \frac{t}{2}[B(s, \bar{s}, t), B(s, \bar{s}, t)]^{22}_X \right) = 0. \hspace{1cm} (3.19c)$$

The string product $B$ is obtained as $B = B(0,0,1)$, and the relations (3.19) reduce to (3.14) for $B$. We can show that if $B(s, \bar{s}, t)$ satisfies the differential equations,

$$\partial_s B(s, \bar{s}, t) = [Q, (\lambda + \bar{\lambda})(s, \bar{s}, t)] + [B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{11}$$

$$+ s \left( [B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{21} + t[B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{22}_X \right)$$

$$+ \bar{s} \left( [B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{12} + t[B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{22}_X \right)$$

$$+ s \bar{s} [B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{22},$$  \hspace{1cm} (3.20a)

$$\partial_s B(s, \bar{s}, t) = [\eta, \lambda(s, \bar{s}, t)]$$

$$- t \left( [B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{21} + \frac{t}{2}[B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{22}_X \right),$$  \hspace{1cm} (3.20b)

$$\partial_s B(s, \bar{s}, t) = [\bar{\eta}, \bar{\lambda}(s, \bar{s}, t)]$$

$$- t \left( [B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{12} + \frac{t}{2}[B(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{22}_X \right),$$  \hspace{1cm} (3.20c)

10
the $L_\infty$ relations (3.19) hold. Here, we introduced two degree-even coderivations $\lambda(s, \bar{s}, t)$ and $\bar{\lambda}(s, \bar{s}, t)$ satisfying
\[
[\eta, \bar{\lambda}(s, \bar{s}, t)] = 0, \quad [\bar{\eta}, \lambda(s, \bar{s}, t)] = 0,
\]
and used an abbreviated notation
\[
\lambda(s, \bar{s}, t) + \bar{\lambda}(s, \bar{s}, t) = (\lambda + \bar{\lambda})(s, \bar{s}, t).
\]
These degree-even coderivations, $\lambda(s, \bar{s}, t)$ and $\bar{\lambda}(s, \bar{s}, t)$, are called (generating functionals of) gauge products and can be expanded in the parameters as
\[
\lambda(s, \bar{s}, t) = \sum_{p, m, r, \bar{p}, \bar{m}} \sum_{\bar{\bar{p}}, \bar{\bar{m}}} t^{p+\bar{p}} s^{m} \bar{s}^{\bar{m}} \lambda_{p+p+2, \bar{\bar{p}}+\bar{m}+\bar{\bar{r}}+1} \equiv \sum_{p, \bar{p}=0}^{\infty} t^{p+\bar{p}} \lambda_{p+1, \bar{p}}(s, \bar{s}), \quad (3.23)
\]
\[
\bar{\lambda}(s, \bar{s}, t) = \sum_{p, m, r, \bar{p}, \bar{m}} \sum_{\bar{\bar{p}}, \bar{\bar{m}}} t^{p+\bar{p}} s^{m} \bar{s}^{\bar{m}} \bar{\lambda}_{p+p+1, \bar{\bar{p}}+\bar{m}+\bar{\bar{r}}+2} \equiv \sum_{p, \bar{p}=0}^{\infty} t^{p+\bar{p}} \bar{\lambda}_{p+1, \bar{p}}(s, \bar{s}). \quad (3.24)
\]
The bracket with subscript, $[\cdot, \cdot]_{\mathfrak{X}}$ with $\mathfrak{X} = \Xi$ or $\bar{\Xi}$, is the (graded) commutator with $\mathfrak{X}$ inserted at the intermediate state, whose explicit definition is given in Appendix A. By differentiating (3.19a) by $t$ and using the differential equation (3.20a), we find that
\[
\partial_t I(s, \bar{s}, t) = [I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{11} + t[I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{21}
\]
\[
+ \bar{s}[I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{12} + t[I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{22}
\]
\[
+ s[\bar{I}(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{22} + s\bar{s}[I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{22}
\]
\[
+ s[I(s, \bar{s}, t), B(s, \bar{s}, t)]^{22} + \bar{s}[I(s, \bar{s}, t), B(s, \bar{s}, t)]^{22}, \quad (3.25)
\]
which implies if $I(s, \bar{s}, 0) = 0$, then $I(s, \bar{s}, t) = 0$ for $\forall t$. Similarly, by differentiating (3.19b)
(3.19c) by \( t \) and using the differential equations (3.20) we obtain

\[
\partial_t J(s, \bar{s}, t) = - \partial_s I(s, \bar{s}, t) + [J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{11}
\]

\[
+ s \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{21} + t \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{22}
\]

\[
+ s \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{12} + t \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{22}
\]

\[
+ ss\bar{J}(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{22}
\]

\[
- t \left[ I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{12} + t \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{22}
\]

\[
+ s J(s, \bar{s}, t), \bar{B}(s, \bar{s}, t) \right]^{22} + \bar{s}[J(s, \bar{s}, t), \bar{B}(s, \bar{s}, t)]^{22},
\]  

(3.26a)

\[
\partial_t \bar{J}(s, \bar{s}, t) = - \partial_s I(s, \bar{s}, t) + [\bar{J}(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t)]^{11}
\]

\[
+ s \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{21} + t \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{22}
\]

\[
+ s \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{12} + t \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{22}
\]

\[
+ ss\bar{J}(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{22}
\]

\[
- t \left[ I(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{12} + t \left[ J(s, \bar{s}, t), (\lambda + \bar{\lambda})(s, \bar{s}, t) \right]^{22}
\]

\[
+ s J(s, \bar{s}, t), \bar{B}(s, \bar{s}, t) \right]^{22} + \bar{s}[J(s, \bar{s}, t), \bar{B}(s, \bar{s}, t)]^{22},
\]  

(3.26b)

which imply if \( I(s, \bar{s}, t) = 0 \) and \( J(s, \bar{s}, 0) = \bar{J}(s, \bar{s}, 0) = 0 \), then \( J(s, \bar{s}, t) = \bar{J}(s, \bar{s}, t) = 0 \) for \( t \). On the other hand, (3.19) reduce to

\[
I(s, \bar{s}, 0) = [Q, B(s, \bar{s}, 0)] + \frac{1}{2} \left[ B(s, \bar{s}, 0), B(s, \bar{s}, 0) \right]^{11} + \frac{s}{2} \left[ B(s, \bar{s}, 0), B(s, \bar{s}, 0) \right]^{21}
\]

\[
+ \frac{s^2}{2} \left[ B(s, \bar{s}, 0), B(s, \bar{s}, 0) \right]^{12} + \frac{ss^2}{2} \left[ B(s, \bar{s}, 0), B(s, \bar{s}, 0) \right]^{22} = 0, \]  

(3.27a)

\[
J(s, \bar{s}, 0) = [\eta, B(s, \bar{s}, 0)] = 0, \quad \bar{J}(s, \bar{s}, 0) = [\bar{\eta}, B(s, \bar{s}, 0)] = 0, \]  

(3.27b)

at \( t = 0 \), which hold if we set

\[
B(s, \bar{s}, 0) = B^{(0,0)}(s, \bar{s}) = L^B(s, \bar{s})
\]  

(3.28)

as was seen in (3.17). Thus, if \( B(s, \bar{s}, t) \) satisfies the differential equations (3.20) with the
initial condition (3.28), the relations (3.19) hold. We obtain the string product \( B \) satisfying
the relations (3.2) as \( B = B(0, 0, 1) \).

Under the initial condition (3.28), we can explicitly solve the differential equations (3.20)
and find \( B(s, \bar{s}, t) = \sum_{n=2} B_n(s, \bar{s}, t) \) in ascending order of \( n \). First, the 2-string products
\( B_2(s, \bar{s}, t) \), \( \lambda_2(s, \bar{s}, t) \), and \( \bar{\lambda}_2(s, \bar{s}, t) \) are expanded in \( t \) as

\[
B_2(s, \bar{s}, t) = B_2^{(0,0)}(s, \bar{s}) + t \left( B_2^{(1,0)}(s, \bar{s}) + B_2^{(0,1)}(s, \bar{s}) \right) + t^2 B_2^{(1,1)}(s, \bar{s})
\]  

(3.29a)

\[
\lambda_2(s, \bar{s}, t) = \lambda_2^{(0,0)}(s, \bar{s}) + t \lambda_2^{(1,0)}(s, \bar{s})
\]  

(3.29b)

\[
\bar{\lambda}_2(s, \bar{s}, t) = \bar{\lambda}_2^{(0,1)}(s, \bar{s}) + t \bar{\lambda}_2^{(1,1)}(s, \bar{s})
\]  

(3.29c)
We can decompose the equations (3.20) to

\[ \begin{align*}
B_2^{(1,0)}(s, \bar{s}) &= \left[ Q, \lambda_2^{(1,0)}(s, \bar{s}) \right], \\
\partial_s B_2^{(0,0)}(s, \bar{s}) &= \left[ \eta, \lambda_2^{(1,0)}(s, \bar{s}) \right], \\
B_2^{(0,1)}(s, \bar{s}) &= \left[ Q, \bar{\lambda}_2^{(1,0)}(s, \bar{s}) \right], \\
\partial_s B_2^{(0,0)}(s, \bar{s}) &= \left[ \bar{\eta}, \bar{\lambda}_2^{(1,0)}(s, \bar{s}) \right], \\
2B_2^{(1,1)}(s, \bar{s}) &= \left[ Q, \lambda_2^{(1,1)}(s, \bar{s}) + \bar{\lambda}_2^{(1,1)}(s, \bar{s}) \right], \\
\partial_s B_2^{(0,1)}(s, \bar{s}) &= \left[ \eta, \lambda_2^{(1,1)}(s, \bar{s}) \right], \\
\partial_s B_2^{(0,0)}(s, \bar{s}) &= \left[ \bar{\eta}, \bar{\lambda}_2^{(1,1)}(s, \bar{s}) \right],
\end{align*} \tag{3.30a-d} \]

for each order of \( t \) and each picture number. Note that there is no nonlinear term in these equations (3.30) for the 2-string products. First, eqs. (3.30b) can be solved for \( \lambda_2^{(1,0)}(s, \bar{s}) \) and \( \bar{\lambda}_2^{(1,0)}(s, \bar{s}) \) as

\[ \begin{align*}
\pi_1 \lambda_2^{(1,0)}(s, \bar{s}) &= \frac{1}{3} \left( \xi \partial_s L_2^B(s, \bar{s}) - \partial_s L_2^B(s, \bar{s}) (\xi \pi_1 \wedge \mathbb{I}_2) \right) \equiv \xi \circ \pi_1 \partial_s L_2^B(s, \bar{s}), \\
\pi_1 \bar{\lambda}_2^{(1,0)}(s, \bar{s}) &= \frac{1}{3} \left( \bar{\xi} \partial_s L_2^B(s, \bar{s}) - \partial_s L_2^B(s, \bar{s}) (\bar{\xi} \pi_1 \wedge \mathbb{I}_2) \right) \equiv \bar{\xi} \circ \pi_1 \partial_s L_2^B(s, \bar{s}),
\end{align*} \tag{3.31a-b} \]

under the initial condition \( B_2^{(0,0)}(s, \bar{s}) = L_2^B(s, \bar{s}) \). Then, eqs. (3.30a) determine \( B_2^{(1,0)}(s, \bar{s}) \) and \( B_2^{(0,1)}(s, \bar{s}) \) as

\[ \begin{align*}
\pi_1 B_2^{(1,0)}(s, \bar{s}) &= \frac{1}{3} \left( X_0 \partial_s L_2^B(s, \bar{s}) + \partial_s L_2^B(s, \bar{s}) (X_0 \pi_1 \wedge \mathbb{I}_2) \right) \equiv X \circ \pi_1 \partial_s L_2^B(s, \bar{s}), \\
\pi_1 B_2^{(0,1)}(s, \bar{s}) &= \frac{1}{3} \left( \bar{X}_0 \partial_s L_2^B(s, \bar{s}) + \partial_s L_2^B(s, \bar{s}) (\bar{X}_0 \pi_1 \wedge \mathbb{I}_2) \right) \equiv \bar{X} \circ \pi_1 \partial_s L_2^B(s, \bar{s}).
\end{align*} \tag{3.32a-b} \]

Using these results, the equations (3.30d) are solved for \( \pi_1 \lambda_2^{(1,1)}(s, \bar{s}) \) and \( \pi_1 \bar{\lambda}_2^{(1,1)}(s, \bar{s}) \) as

\[ \begin{align*}
\pi_1 \lambda_2^{(1,1)}(s, \bar{s}) &= \xi \circ \pi_1 \partial_s B_2^{(0,1)}(s, \bar{s}) = \xi \circ \bar{X} \circ \pi_1 \partial_s L_2^B(s, \bar{s}), \\
\pi_1 \bar{\lambda}_2^{(1,1)}(s, \bar{s}) &= \bar{\xi} \circ \pi_1 \partial_s B_2^{(1,0)}(s, \bar{s}) = \bar{\xi} \circ X \circ \pi_1 \partial_s L_2^B(s, \bar{s}),
\end{align*} \tag{3.33a-b} \]

which determine \( B_2^{(1,1)}(s, \bar{s}) \) as

\[ \pi_1 B_2^{(1,1)}(s, \bar{s}) = X \circ \bar{X} \circ \pi_1 \partial_s \partial_s L_2^B(s, \bar{s}), \quad \tag{3.34} \]

from (3.30c). We illustrate in Fig. 1 the flow of how the 2-string (gauge) products are determined. The string and gauge products with specific cyclic Ramond numbers are obtained as coefficients by expanding in \( s \) and \( \bar{s} \):

\[ \begin{align*}
B_2^{(0,0)}(s, \bar{s}) &= B_2^{(0,0)} \mid (2, 2) + s B_2^{(0,0)} \mid (0, 2) + \bar{s} B_2^{(0,0)} \mid (2, 0) + s \bar{s} B_2^{(0,0)} \mid (0, 0), \\
B_2^{(1,0)}(s, \bar{s}) &= B_2^{(1,0)} \mid (0, 2) + s B_2^{(1,0)} \mid (0, 0), \\
B_2^{(0,1)}(s, \bar{s}) &= B_2^{(0,1)} \mid (2, 0) + s B_2^{(0,1)} \mid (0, 0), \\
B_2^{(1,1)}(s, \bar{s}) &= B_2^{(1,1)} \mid (0, 0), \\
\lambda_2^{(1,0)}(s, \bar{s}) &= \lambda_2^{(1,0)} \mid (0, 2) + \bar{s} \lambda_2^{(1,0)} \mid (0, 0), \\
\bar{\lambda}_2^{(1,0)}(s, \bar{s}) &= \bar{\lambda}_2^{(1,0)} \mid (2, 0) + s \bar{\lambda}_2^{(1,0)} \mid (0, 0), \\
\lambda_2^{(1,1)}(s, \bar{s}) &= \lambda_2^{(1,1)} \mid (0, 0), \\
\bar{\lambda}_2^{(1,1)}(s, \bar{s}) &= \bar{\lambda}_2^{(1,1)} \mid (0, 0),
\end{align*} \tag{3.35-40} \]
The explicit form of each product is found by expanding the solutions (3.31)-(3.34) in $s$ and $\bar{s}$. In particular, the 2-string product satisfying (3.14) is given by

$$
\pi_1 B_2 = \pi_1 B_2^{(0,0)} |^{(2,2)} + \pi_1 B_2^{(1,0)} |^{(0,2)} + \pi_1 B_2^{(0,1)} |^{(2,0)} + \pi_1 B_2^{(1,1)} |^{(0,0)} \\
= \pi_1 L_2^B |^{(2,2)} + X \circ \pi_1 L_2^B |^{(0,2)} + \bar{X} \circ \pi_1 L_2^B |^{(2,0)} + X \circ \bar{X} \circ \pi_1 L_2^B |^{(0,0)}. \quad (3.41)
$$

The flow of determining each component is given in Fig. 1.

Similarly, expanding the 3-string products as

$$
B_3(s, \bar{s}, t) = B_3^{(0,0)}(s, \bar{s}) + t \left( B_3^{(1,0)}(s, \bar{s}) + B_3^{(0,1)}(s, \bar{s}) \right) \\
+ t^2 \left( B_3^{(2,0)}(s, \bar{s}) + B_3^{(1,1)}(s, \bar{s}) + B_3^{(0,2)}(s, \bar{s}) \right) \\
+ t^3 \left( B_3^{(2,1)}(s, \bar{s}) + B_3^{(1,2)}(s, \bar{s}) \right) + t^4 B_3^{(2,2)}(s, \bar{s}), \quad (3.42)
$$

$$
\lambda_3(s, \bar{s}, t) = \lambda_3^{(1,0)}(s, \bar{s}) + t \left( \lambda_3^{(2,0)}(s, \bar{s}) + \lambda_3^{(1,1)}(s, \bar{s}) \right) \\
+ t^2 \left( \lambda_3^{(2,1)}(s, \bar{s}) + \lambda_3^{(1,2)}(s, \bar{s}) \right) + t^3 \lambda_3^{(2,2)}(s, \bar{s}), \quad (3.43)
$$

$$
\bar{\lambda}_3(s, \bar{s}, t) = \bar{\lambda}_3^{(0,1)}(s, \bar{s}) + t \left( \bar{\lambda}_3^{(1,1)}(s, \bar{s}) + \bar{\lambda}_3^{(0,2)}(s, \bar{s}) \right) \\
+ t^2 \left( \bar{\lambda}_3^{(2,1)}(s, \bar{s}) + \bar{\lambda}_3^{(1,2)}(s, \bar{s}) \right) + t^3 \bar{\lambda}_3^{(2,2)}(s, \bar{s}), \quad (3.44)
$$

Figure 1: The flow of how the 2-string (gauge) products are determined.
Then, all the terms on the right-hand sides of eqs. (3.45a) and (3.45b) are given, and thus, they determine the differential equations (3.20) for 3-string products at $O(t^0)$ become

$$B_3^{(1,0)}(s, \bar{s}) = \left[ Q, \lambda_3^{(1,0)}(s, \bar{s}) \right] + \left[ B_2^{(0,0)}(s, \bar{s}), \lambda_2^{(1,0)}(s, \bar{s}) \right]^{11} + s[ B_2^{(0,0)}(s, \bar{s}), \lambda_2^{(1,0)}(s, \bar{s}) ]^{21} + \bar{s}[ B_2^{(0,0)}(s, \bar{s}), \lambda_2^{(1,0)}(s, \bar{s}) ]^{12} + s\bar{s}[ B_2^{(0,0)}(s, \bar{s}), \lambda_2^{(1,0)}(s, \bar{s}) ]^{22} + \frac{s}{2} \left[ B_2^{(0,0)}(s, \bar{s}), B_2^{(0,0)}(s, \bar{s}) \right]^{22}, \quad (3.45a)$$

$$B_3^{(0,1)}(s, \bar{s}) = \left[ Q, \lambda_3^{(0,1)}(s, \bar{s}) \right] + \left[ B_2^{(0,0)}(s, \bar{s}), \lambda_2^{(0,1)}(s, \bar{s}) \right]^{11} + s[ B_2^{(0,0)}(s, \bar{s}), \lambda_2^{(0,1)}(s, \bar{s}) ]^{21} + \bar{s}[ B_2^{(0,0)}(s, \bar{s}), \lambda_2^{(0,1)}(s, \bar{s}) ]^{12} + s\bar{s}[ B_2^{(0,0)}(s, \bar{s}), \lambda_2^{(0,1)}(s, \bar{s}) ]^{22} + \frac{s}{2} \left[ B_2^{(0,0)}(s, \bar{s}), B_2^{(0,0)}(s, \bar{s}) \right]^{22}, \quad (3.45b)$$

$$\partial_5 B_3^{(0,0)}(s, \bar{s}) = [ \eta, \lambda_3^{(1,0)}(s, \bar{s}) ], \quad \partial_5 B_3^{(0,0)}(s, \bar{s}) = [ \eta, \lambda_3^{(0,1)}(s, \bar{s}) ]. \quad (3.45c)$$

Under the initial condition $B_3^{(0,0)}(s, \bar{s}) = L_3^B(s, \bar{s})$, eqs. (3.45c) are solved for $\lambda_3^{(1,0)}(s, \bar{s})$ and $\lambda_3^{(0,1)}(s, \bar{s})$ as

$$\pi_1 \lambda_3^{(1,0)}(s, \bar{s}) = \xi \circ \partial_5 L_3^B(s, \bar{s}) = \frac{1}{4} \left( \xi \partial_5 L_3^B(s, \bar{s}) - \partial_5 L_3^B(s, \bar{s})(\xi \pi_1 \wedge \eta_2) \right), \quad (3.46)$$

$$\pi_1 \lambda_3^{(0,1)}(s, \bar{s}) = \tilde{\xi} \circ \partial_5 L_3^B(s, \bar{s}) = \frac{1}{4} \left( \tilde{\xi} \partial_5 L_3^B(s, \bar{s}) - \partial_5 L_3^B(s, \bar{s})(\tilde{\xi} \pi_1 \wedge \eta_2) \right). \quad (3.47)$$

Then, all the terms on the right-hand sides of eqs. (3.45a) and (3.45b) are given, and thus, they determine $B_3^{(1,0)}(s, \bar{s})$ and $B_3^{(0,1)}(s, \bar{s})$. We can repeat similar procedures at each order of $t$ and
determine all the products $B_3^{(\bullet \bullet)}(s, \bar{s})$, $\lambda_3^{(\bullet \bullet)}(s, \bar{s})$, and $\bar{\lambda}_3^{(\bullet \bullet)}(s, \bar{s})$. In general, the equations (3.20) at each order of $t$ have the form

$$B_3^{(\bullet \bullet)}(s, \bar{s}) = [Q, \lambda_3^{(\bullet \bullet)}(s, \bar{s}) + \bar{\lambda}_3^{(\bullet \bullet)}(s, \bar{s})] + \cdots, \quad (3.48a)$$

$$[\eta, \lambda_3^{(\bullet \bullet)}(s, \bar{s})] = \cdots, \quad [\eta, \bar{\lambda}_3^{(\bullet \bullet)}(s, \bar{s})] = \cdots, \quad (3.48b)$$

where $\cdots$ denotes the terms including only the products already given in the lower order. We can solve eqs. (3.48b) for $\lambda_3^{(\bullet \bullet)}(s, \bar{s})$ or $\bar{\lambda}_3^{(\bullet \bullet)}(s, \bar{s})$, and then eq. (3.48a) determine $B_3^{(\bullet \bullet)}(s, \bar{s})$.

We give the flow of how the 3-string (gauge) products are determined in Fig. 3. Their expansion in $s$ and $\bar{s}$ and the flow of how each component, (gauge) products with specific cyclic Ramond number, are determined are given in Appendix B. In principle, we can repeat a similar procedure

---

Figure 3: The flow of how 3-string (gauge) products are determined.

---

7Depending on the picture number, $\lambda_3^{(\bullet \bullet)}(s, \bar{s})$ or $\bar{\lambda}_3^{(\bullet \bullet)}(s, \bar{s})$ is missing in eq. (3.48a).
for higher-string products $B_n^{(s)}(s, \bar{s})$, $\lambda_n^{(s)}(s, \bar{s})$, and $\bar{\lambda}_n^{(s)}(s, \bar{s})$ in ascending order of $n$ and determine all the string and gauge products $B(s, \bar{s}, t)$, $\lambda(s, \bar{s}, t)$ and $\bar{\lambda}(s, \bar{s}, t)$.

Before closing this section, we rewrite the (extended) $L_\infty$-relations (3.19) and the differential equations (3.20) as

\begin{align}
[Q, B_{n+2}(s, \bar{s}, t)] + \sum_{m=0}^{n-1} B_{m+2}(s, \bar{s}, t) \left( \pi(s, \bar{s}, t) \pi_1 B_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right) &= 0, & \text{(3.49a)} \\
[\eta, B_{n+2}(s, \bar{s}, t)] + \sum_{m=0}^{n-1} B_{m+2}(s, \bar{s}, t) \left( \pi(t) \pi_1 B_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right) &= 0, & \text{(3.49b)} \\
[\bar{\eta}, B_{n+2}(s, \bar{s}, t)] + \sum_{m=0}^{n-1} B_{m+2}(s, \bar{s}, t) \left( \bar{\pi}(t) \pi_1 B_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right) &= 0, & \text{(3.49c)}
\end{align}

and

\begin{align}
\partial_s B_{n+2}(s, \bar{s}, t) &= [Q, (\lambda + \bar{\lambda})_{n+2}(s, \bar{s}, t)] \\
&+ \sum_{m=0}^{n-1} \left( B_{m+2}(s, \bar{s}, t) \left( \pi(s, \bar{s}, t) \pi_1 (\lambda + \bar{\lambda})_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right) \\
&- (\lambda + \bar{\lambda})_{m+2}(s, \bar{s}, t) \left( \pi(s, \bar{s}, t) \pi_1 B_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right) \right) \\
&+ \sum_{m=0}^{n-1} B_{m+2}(s, \bar{s}, t) \left( (s\bar{\Xi} + \bar{s}\Xi) \pi_1 B_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right), & \text{(3.50a)}
\end{align}

\begin{align}
\partial_s B_{n+2}(s, \bar{s}, t) &= [\eta, \lambda_{n+2}(s, \bar{s}, t)] \\
&+ \sum_{m=0}^{n-1} \left( B_{m+2}(s, \bar{s}, t) \left( \pi(t) \pi_1 (\lambda + \bar{\lambda})_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right) \\
&- (\lambda + \bar{\lambda})_{m+2}(s, \bar{s}, t) \left( \pi(t) \pi_1 B_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right) \right), & \text{(3.50b)}
\end{align}

\begin{align}
\partial_s B_{n+2}(s, \bar{s}, t) &= [\bar{\eta}, \bar{\lambda}_{n+2}(s, \bar{s}, t)] \\
&+ \sum_{m=0}^{n-1} \left( B_{m+2}(s, \bar{s}, t) \left( \bar{\pi}(t) \pi_1 (\lambda + \bar{\lambda})_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right) \\
&- (\lambda + \bar{\lambda})_{m+2}(s, \bar{s}, t) \left( \bar{\pi}(t) \pi_1 B_{n-m+1}(s, \bar{s}, t) \wedge I_{m+1} \right) \right), & \text{(3.50c)}
\end{align}

where

\begin{align}
\pi(s, \bar{s}, t) &= \pi^{(0,0)} + s\pi^{(1,0)} + \bar{s}\pi^{(0,1)} + t(s\bar{X} + \bar{s}X) + s\bar{s}\pi^{(1,1)}, \\
\pi(t) &= -t \left( \pi^{(1,0)} + \frac{t}{2} \bar{X}\pi^{(1,1)} \right), \\
\bar{\pi}(t) &= -t \left( \pi^{(0,1)} + \frac{t}{2} X\pi^{(1,1)} \right). & \text{(3.51)}
\end{align}

These alternative forms are convenient for use in the next section [24].
4 Tree-level S-matrix

Using HPT, we can show that the tree-level S-matrix derived from (super)string field theory agrees with that calculated using the first-quantization method [22–24]. We show in this section that the new prescription proposed in the previous section simplifies this proof for the type II superstring field theory.

As a preparation, let us define the on-shell subspace,

$$\mathcal{H}^p = \{ \Phi \in \mathcal{H}^{res} \mid L_0^+ \Phi_{NS-NS} = G_0 \Phi_{R-NS} = \bar{G}_0 \Phi_{NS-R} = G_0 \Phi_{R-R} = \bar{G}_0 \Phi_{R-R} = 0 \} ,$$  

(4.1)

and the BRST invariant projection operator,

$$P_0 : \mathcal{H}^{res} \to \mathcal{H}^p, \quad P_0^2 = P_0, \quad [Q, P_0] = 0 .$$  

(4.2)

The homotopy operator $Q^+$ of $Q$ satisfying

$$Q^+ Q + QQ^+ + P_0 = 1 ,$$  

(4.3)

$$Q^+ P_0 = P_0 Q^+ = Q^+ Q^+ = 0 ,$$  

(4.4)

is defined by

$$Q^+ = \frac{1}{L_0} b_0^+(1 - P_0) ,$$  

(4.5)

and provides a Hodge-Kodaira decomposition of $\mathcal{H}^{res}$,

$$\mathcal{H}^{res} = \mathcal{H}^p + \mathcal{H}^t + \mathcal{H}^u ,$$  

(4.6)

with

$$\mathcal{H}^p = P_0 \mathcal{H}^{res} , \quad \mathcal{H}^t = QQ^+ \mathcal{H}^{res} , \quad \mathcal{H}^u = Q^+ Q \mathcal{H}^{res} .$$  

(4.7)

It is compatible with $\Omega$:

$$\Omega(\mathcal{H}^p, \mathcal{H}^u) = \Omega(\mathcal{H}^u, \mathcal{H}^u) = 0 .$$  

(4.8)

Consider two chain complexes, $(\mathcal{H}^{res}, Q)$ and $(\mathcal{H}^p, QP_0)$, with chain maps

$$p = P_0 : \mathcal{H}^{res} \to \mathcal{H}^p, \quad i = P_0 : \mathcal{H}^p \hookrightarrow \mathcal{H}^{res} ,$$  

(4.9)

satisfying the relations

$$pi = P_0 , \quad ip = 1 - Q^+ Q - QQ^+ .$$  

(4.10)

Under the SR gauge condition $\psi = 0$ , $\mathcal{H}^t = \emptyset$ , and the Hilbert space of the quantum string field $\phi$ is decomposed to the on-shell and off-shell components:

$$\phi = P_0 \phi + (1 - P_0) \phi \in \mathcal{H}_0 + \overline{\mathcal{H}}_0 ,$$  

(4.11)
where
\[ H_0 = \mathcal{H}^p \cap \mathcal{H}_{SR} = P_0 \mathcal{H}_{SR}, \quad \mathcal{P}_0 = \mathcal{H}^u \cap \mathcal{H}_{SR} = (1 - P_0) \mathcal{H}_{SR}. \] (4.12)

The chain complex \((\mathcal{H}^p, Q P_0)\), with the SR gauge condition, defines the relative BRST cohomology. Lift these (equivalence) data to the chain complexes \((\mathcal{S H}^{res}, Q)\) and \((\mathcal{S H}^p, Q P)\) with the chain maps
\[ \hat{p} = \hat{P} : \mathcal{S H}^{res} \to \mathcal{S H}^p, \quad \hat{i} = \hat{P} : \mathcal{S H}^p \hookrightarrow \mathcal{S H}^{res}, \] (4.13)
satisfying
\[ \hat{p} \hat{i} = \hat{P}, \quad \hat{i} \hat{p} = \hat{I} + HQ + QH, \] (4.14)

where
\[ \hat{P} = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \frac{1}{n!} P_0 \wedge \cdots \wedge P_0, \] (4.15)
\[ \hat{I} = \sum_{n=0}^{\infty} I_1 \wedge \cdots \wedge I_1, \] (4.16)
\[ H = \sum_{r,s=0}^{\infty} \frac{1}{(r+s+1)!} (-Q^+) \wedge I_1 \wedge \cdots \wedge I_1 \wedge P_0 \wedge \cdots \wedge P_0. \] (4.17)

The projection, identity, and homotopy operators, \(\hat{P}, \hat{I},\) and \(H\), satisfy the relations
\[ \hat{P} = \hat{I} + HQ + QH, \] (4.18)
\[ \hat{P}^2 = \hat{P}, \quad [Q, \hat{P}] = 0, \] (4.19)
\[ H \hat{P} = \hat{P} H = HH = 0. \] (4.20)

If we perturb \(Q\) by \(L_{int} = \sum_{n=0}^{\infty} L_{n+2}\) so that \((Q + L_{int})^2 = 0\), the homological perturbation lemma tells us that chain complexes and chain maps are deformed as
\[ (\mathcal{S H}^{res}, L = Q + L_{int}) \xrightarrow{\nu'} (\mathcal{S H}^p, S = Q P + S_{int}), \] (4.21)
with
\[ \nu' = \hat{P} (\hat{I} - L_{int} H)^{-1}, \] (4.22)
\[ \hat{\nu}' = (\hat{I} - HL_{int})^{-1} \hat{P}, \] (4.23)
\[ S_{int} = \hat{P} L_{int} (\hat{I} - HL_{int})^{-1} \hat{P}. \] (4.24)

\(^8\)To be precise, we must remove the ghost number constraint for the quantum string field, but components with non-zero space-time ghost number do not contribute the tree-level S-matrix.
Here, we defined $\left( \hat{I} - \mathcal{O} \right)^{-1}$ by the formal series:

$$\left( \hat{I} - \mathcal{O} \right)^{-1} = \sum_{n=0}^{\infty} \mathcal{O}^n.$$  \hfill (4.25)

For convenience, introduce $\Sigma$ with

$$\pi_1 S_{int} = G P_0 \pi_1 \Sigma, \quad \Sigma = B \hat{F} \hat{i}.$$  \hfill (4.26)

The multi-linear representation,

$$\langle S \rangle = \langle \Omega | P_0 \otimes \pi_1 S_{int} = \langle \omega_l | \xi \bar{\xi} P_0 \otimes P_0 \pi_1 \Sigma,$$  \hfill (4.27)

defines a map from $\mathcal{H}^p \otimes S \mathcal{H}^p$ to $\mathbb{C}$. The total $S$-matrix (at the tree level) is given by $\langle S \rangle$ by restricting the external states onto the gauge-fixed states, that is, $\mathcal{H}^p \otimes S \mathcal{H}^p$ to $\mathcal{H}_0 \otimes S \mathcal{H}_0$. If we expand $\Sigma$ in the number of inputs, $\Sigma = \sum_{n=0}^{\infty} \Sigma_{n+2}$, it induces the expansion of the $S$-matrix in the number of external states:

$$\langle S \rangle = \sum_{n=0}^{\infty} \langle S_{n+3} \rangle, \quad \langle S_{n+3} \rangle = \langle \omega_l | \xi \bar{\xi} P_0 \otimes P_0 \pi_1 \Sigma_{n+2}.$$  \hfill (4.28)

Each term with specific number of external states is further classified by the number of external Ramond states:

$$\langle S_{n+3} \rangle = \sum_{0 \leq r, \bar{r} \leq (n+3)/2} \langle S_{n+3} \rangle^{(2r, 2\bar{r})} = \sum_{0 \leq r, \bar{r} \leq (n+3)/2} \langle \omega_l | \xi \bar{\xi} P_0 \otimes P_0 \pi_1 \Sigma_{n+2} \rangle^{(2r, 2\bar{r})}.$$  \hfill (4.29)

For example, $\langle S_3 \rangle$ contains four terms:

$$\langle S_3 \rangle = \langle S_3 \rangle^{(0, 0)} + \langle S_3 \rangle^{(0, 2)} + \langle S_3 \rangle^{(2, 0)} + \langle S_3 \rangle^{(2, 2)}.$$  \hfill (4.30)

The first, second, and third terms give $(\text{NS-NS})^3$, $(\text{NS-NS})-(\text{R-NS})^2$, and $(\text{NS-NS})-(\text{NS-R})^2$ amplitudes, respectively. The fourth term, on the other hand, includes two amplitudes $(\text{NS-NS})-(\text{R-R})^2$ and $(\text{R-NS})-(\text{NS-R})-(\text{R-R})$, which cannot be distinguished by the Ramond number of the external states\(^9\).

Since the Hilbert space $\mathcal{H}_0$ still contains unphysical states in general, the physical $S$-matrix is defined by projecting it onto the physical subspace, that is, by taking physical states defined by the relative BRST cohomology, $\mathcal{H}_Q = \text{Ker} Q/\text{Im} Q \subset \mathcal{H}_0$, as external states:

$$\langle S^{\text{phys}} \rangle = \langle S \rangle | P \rangle,$$  \hfill (4.31)

\(^9\)To be precise, the second, third, or fourth term contains two different representations for an amplitude due to the asymmetry that distinguishes one external state (output) from the others (inputs).
with
\[
\hat{P} = \sum_{n=0}^{\infty} \frac{1}{n!} P \wedge \cdots \wedge P, \quad \mathcal{P} : \mathcal{H}_0 \rightarrow \mathcal{H}_Q.
\] (4.32)

The unitarity of the physical S-matrix is guaranteed by the BRST invariance, \([Q, S_{int}] = 0\).

By using the relations of the cohomomorphisms
\[
\pi_1 \hat{F} = \pi_1 \mathbb{1} + \mathcal{G} \pi_1 \hat{B} \hat{F}, \quad \pi_1 \hat{\iota}' = P_0 \pi_1 - Q^+ \mathcal{G} \pi_1 \hat{B} \hat{\iota}'
\] (4.33)
following from the definitions (3.13) of \(\hat{F}^{-1}\) and (4.23) of \(\hat{\iota}'\), respectively, we can show that the Dyson-Schwinger (DS) equation for \(\Sigma\),
\[
\pi_1 \Sigma = \sum_{n=0}^{\infty} \pi_1 B_{n+2} \left( \frac{1}{(n+2)!} \left( P_0 \pi_1 - (Q^+ \mathcal{G} - \mathcal{G}) \pi_1 \Sigma \right)^{\wedge (n+2)} \right),
\] (4.34)
holds. By expanding \(\Sigma\) in the number of inputs, it becomes the recurrence relation
\[
\pi_1 \Sigma_{n+2} = \sum_{m=0}^{n} \pi_1 B_{m+2} \left( \frac{1}{(m+2)!} \left( P_0 \pi_1 - (Q^+ \mathcal{G} - \mathcal{G}) \pi_1 \sum_{l=0}^{n-m-1} \Sigma_{l+2}^{(s, \bar{s}, t)} \right)^{\wedge (m+2)} \right) \pi_{n+2},
\] (4.35)
which determine \(\Sigma_{n+2}\) recursively. We extend it with three parameters \(s, \bar{s},\) and \(t\) to the extended DS equation,
\[
\pi_1 \Sigma_{n+2}^{(s, \bar{s}, t)} = \sum_{m=0}^{n} \pi_1 B_{m+2}^{(s, \bar{s}, t)} \left( \frac{1}{(m+2)!} \left( P_0 \pi_1 - \Delta(s, \bar{s}, t) \pi_1 \sum_{l=0}^{n-m-1} \Sigma_{l+2}^{(s, \bar{s}, t)} \right)^{\wedge (m+2)} \right) \pi_{n+2},
\] (4.36)
which reduces to the DS equation (4.35) at \((s, \bar{s}, t) = (0, 0, 1)\). Here, \(\Delta(s, \bar{s}, t)\) defined by
\[
\Delta(s, \bar{s}, t) = Q^+ \mathcal{G}(s, \bar{s}, t) - \mathcal{G}(t),
\] (4.37)
with
\[
\mathcal{G}(s, \bar{s}, t) = \pi^{(0,0)} + (tX + s) \pi^{(1,0)} + (tX + \bar{s}) \pi^{(0,1)} + (t^2 X \bar{X} + t(sX + \bar{s}X) + s\bar{s}) \pi^{(1,1)},
\] (4.38)
\[
\mathcal{G}(t) = t \Xi \pi^{(1,0)} + t \Xi \pi^{(0,1)} + \frac{t^2}{2} (\Xi X + X \Xi) \pi^{(1,1)},
\] (4.39)
was determined to satisfy the relations
\[
[Q, \Delta(s, \bar{s}, t)] = \pi(s, \bar{s}, t) - P_0 \mathcal{G}(s, \bar{s}, t),
\] (4.40a)
\[
[\eta, \Delta(s, \bar{s}, t)] = \pi(t), \quad [\bar{\eta}, \Delta(s, \bar{s}, t)] = \bar{\pi}(t),
\] (4.40b)
Combining the relations (4.42), we can derive the key relation

\[
\partial_t \Delta(s, \bar{s}, t) = - \left[ Q, Q^+ \left( \partial_t \mathcal{S}(t) + (s \Xi + \bar{s} \Xi \pi^{(1,1)}) \right) \right] + (s \Xi + \bar{s} \Xi \pi^{(1,1)}) + P_0 \left( \partial_t \mathcal{S}(t) + (s \Xi + \bar{s} \Xi \pi^{(1,1)}) \right),
\]

which can be shown, similar to the case in [24], using the alternative form of the (extended) \( L_{\infty} \)-relations (3.49) and the fact that generic intermediate states are off-shell. Next, differentiating eq. (4.36) by parameters, we obtain the relations

\[
\pi_1 \partial_t \Sigma(s, \bar{s}, t) = \pi_1 \left[ Q, \rho(s, \bar{s}, t) \right],
\]

using the (alternative form of) the differential equations (3.50), where \( \rho(s, \bar{s}, t) \) is the degree even map determined by the recurrence relation

\[
\pi_1 \rho_{n+2}(s, \bar{s}, t) = \sum_{m=0}^{n} (\lambda + \bar{\lambda})_{m+2}(s, \bar{s}, t) \left( D_{m+2}(s, \bar{s}, t) \right) P_{n+2} \pi_{n+2} - \sum_{m=0}^{n} \pi_1 B_{m+2}(s, \bar{s}, t) \left( D_{m+1}(s, \bar{s}, t) \wedge \pi_1 E(s, \bar{s}, t) \right) P_{n+2} \pi_{n+2},
\]

with

\[
D_{M}(s, \bar{s}, t) = \frac{1}{M!} \left( P_0 \pi_1 - \Delta(s, \bar{s}, t) \pi_1 \Sigma(s, \bar{s}, t) \right)^{\wedge M},
\]

\[
\pi_1 E(s, \bar{s}, t) = \Delta(s, \bar{s}, t) \pi_1 \rho + Q^+ \left( \partial_t \mathcal{S} + (s \Xi + \bar{s} \Xi \pi^{(1,1)}) \right) \pi_1 \Sigma(s, \bar{s}, t).
\]

Combining the relations (4.42), we can derive the key relation

\[
\pi_1 \partial_t \Sigma(s, \bar{s}, t) - X_0 \circ \pi_1 \partial_s \Sigma(s, \bar{s}, t) - \bar{X}_0 \circ \pi_1 \partial_{\bar{s}} \Sigma(s, \bar{s}, t) = \left[ Q, [\eta, [\eta, T(s, \bar{s}, t)]] \right],
\]

with \( \pi_1 T(s, \bar{s}, t) = \xi \circ \xi \circ \pi_1 \rho(s, \bar{s}, t) \).

We can also extend the S-matrix using \( \Sigma(s, \bar{s}, t) = \sum_{n=0}^\infty \Sigma_{n+2}(s, \bar{s}, t) \) as

\[
\langle S(s, \bar{s}, t) | = \sum_{n=0}^\infty \langle S_{n+3}(s, \bar{s}, t) | = \sum_{n=0}^\infty \langle \omega_l | \xi P_0 \otimes P_0 \pi_1 \Sigma_{n+2}(s, \bar{s}, t),
\]

where

\[
\omega_l = \sum_{l=-\infty}^\infty \omega_{l,0} (s l) \Xi + (s \Xi + \bar{s} \Xi \pi^{(1,1)}) + P_0 \left( s \Xi + \bar{s} \Xi \pi^{(1,1)} \right) \pi_1 \Sigma(s, \bar{s}, t).
\]
which reduces the S-matrix (4.28) at \((s, \bar{s}, t) = (0, 0, 1)\). The extended S-matrix element \(\langle S_{n+3}(s, \bar{s}, t) \rangle\) is expanded in the parameters as

\[
\langle S_{n+3}(s, \bar{s}, t) \rangle = \sum_{m=0}^{s} S^{m \bar{m}} \langle S_{n+3}^{[m, \bar{m}]} \rangle ,
\]

(4.48a)

\[
\langle S_{n+3}^{[m, \bar{m}]} \rangle (t) = \sum_{p=0}^{n-m+1} \sum_{\bar{p}=0}^{n-m+1} t^{p+\bar{p}} \langle S_{n+3}^{(p, \bar{p})} \rangle (2(n-m-p+1), 2(n-m-\bar{p}+1)) .
\]

(4.48b)

The key relation (4.46) induces the equation for the extended S-matrix,

\[
\partial_t \langle S(s, \bar{s}, t) \rangle - \partial_s \langle S(s, \bar{s}, t) \rangle (X_0)_\text{cyc} \partial_t \langle S(s, \bar{s}, t) \rangle (\bar{X}_0)_\text{cyc} = \langle \omega l [\xi \xi P_0 \otimes P_0 \pi_1 | Q, [\eta, \bar{\eta}, T](s, \bar{s}, t) \rangle ,
\]

(4.49)

where

\[
\langle S(s, \bar{s}, t) \rangle (X_0)_\text{cyc} = \sum_{n=0}^{\infty} \langle S_{n+3}(s, \bar{s}, t) \rangle (X_0 \otimes I_{n+2} + I \otimes I_{n+1} \wedge X_0 \pi_1) ,
\]

(4.50)

for \(X_0 = X_0\) or \(\bar{X}_0\). Note that the right hand side of (4.49) does not contribute to the physical S-matrix. For each amplitude, we have

\[
(p + \bar{p}) \langle S_{n+3}^{(p, \bar{p})} \rangle (2r, 2\bar{r}) = (n - p + 1 - r) \langle S_{n+3}^{(p-1, \bar{p})} \rangle (2r, 2\bar{r}) (X_0)_\text{cyc} + (n - \bar{p} + 1 - \bar{r}) \langle S_{n+3}^{(p-1, \bar{r})} \rangle (2r, 2\bar{r}) (\bar{X}_0)_\text{cyc} + \cdots ,
\]

(4.51)

where dots on the right-hand side represent the terms vanishing on the physical subspace \(H_Q\). Let us apply this relation on \((p + \bar{p}) \langle S_{n+3}^{(p, \bar{p})} \rangle (2r, 2\bar{r})\), repeatedly. By using the relation once, we can reduce \(p\) or \(\bar{p}\) by 1 with the factor \((n - r - p + 2)\) or \((n - \bar{r} - \bar{p} + 2)\), respectively. By repeatedly using the relation \((p + \bar{p})\) times, we eventually reach \(\langle S_{n+3}^{(0, 0)} \rangle (2r, 2\bar{r})\) through \(p + \bar{p} C_p\) paths. The number of paths can be obtained by counting the places to reduce \(p\) in the \((p + \bar{p})\) steps. The coefficients coming from the relation are the same for all the paths, and equal to

\[
\frac{(n - r + 1)!}{(n - r - p + 1)!} \frac{(n - \bar{r} + 1)!}{(n - \bar{r} - \bar{p} + 1)!} .
\]

(4.52)

Thus, we eventually find that

\[
\langle S_{n+3}^{(p, \bar{p})} \rangle (2r, 2\bar{r}) = \binom{n - r + 1}{p} \binom{n - \bar{r} + 1}{\bar{p}} \langle S_{n+3}^{(0, 0)} \rangle (2r, 2\bar{r}) (X_0)_\text{cyc} (\bar{X}_0)_{\text{cyc}}^\bar{p} ,
\]

(4.53)

except for the terms vanishing on \(H_Q\). Then, the physical amplitudes obtained by projecting \(\langle S_{n+3}^{(p, \bar{p})} \rangle (2r, 2\bar{r})\) with \((p, \bar{p}) = (n - r + 1, n - \bar{r} + 1)\) onto \(H_Q\) is written as

\[
\langle S_{n+3}^{\text{phys}} \rangle (2r, 2\bar{r}) = \langle (S_B)^{\text{phys}} \rangle (2r, 2\bar{r}) (X_0)^{n-r+1} (\bar{X}_0)^{n-\bar{r}+1} ,
\]

(4.54)

\[10\text{We consider that } \langle S_{n+3}^{(p, \bar{p})} \rangle (2r, 2\bar{r})\text{ with the parameters outside the range } (0, 0) \leq (p, \bar{p}) \leq (n + 1 - r, n + 1 - \bar{r}) , \quad 0 \leq 2r, 2\bar{r} \leq n + 3 \text{ is equal to zero.}\]
where \( \langle (S_B)^{phys}_{n+3}|^{(2r,2r)} \) in the right-hand side is the \textit{bosonic} physical amplitude defined by

\[
\langle (S_B)^{phys}_{n+3}|^{(2r,2r)} = \langle S_{n+3}^{(0,0)}|^{(2r,2r)} \hat{P} \rangle.
\]

(4.55)

Since the physical amplitude is independent of how we insert PCO’s, we can appropriately deform and move them to agree with that obtained using the first-quantization method.

5 Gauge-invariant action in WZW-like formulation

Using an alternative method, symmetric construction, we have constructed the string products (interactions) with \( L_\infty \) structure for the type II superstring. The new gauge-invariant action looks different from the previous one but is the same and nothing essentially new. However, a difference appears when we try to map it to the WZW-like action through a field redefinition. Previously, we could only find a map to the half-WZW-like action [15], but the new construction enables us to construct a map to the complete WZW-like action.

5.1 WZW-like action for the NS-NS sector

We first summarize the results obtained in [13] on the construction of the WZW-like action for the NS-NS sector.

For the NS-NS sector, generating functional of string products with the \( L_\infty \) structure, \( L_{NS-NS}(s,\bar{s},t) \), has been constructed by imposing the differential equations [10],

\[
\begin{align*}
\partial_t L_{NS-NS}(s,\bar{s},t) &= [L_{NS-NS}(s,\bar{s},t), (\lambda_{NS-NS}^{NS-NS} + \bar{\lambda}_{NS-NS}^{NS-NS})(s,\bar{s},t)], \\
\partial_s L_{NS-NS}(s,\bar{s},t) &= [\eta, \lambda_{NS-NS}^{NS-NS}(s,\bar{s},t)], \quad [\eta, \lambda_{NS-NS}^{NS-NS}(s,\bar{s},t)] = 0, \\
\partial_{\bar{s}} L_{NS-NS}(s,\bar{s},t) &= [\bar{\eta}, \bar{\lambda}_{NS-NS}^{NS-NS}(s,\bar{s},t)], \quad [\bar{\eta}, \lambda_{NS-NS}^{NS-NS}(s,\bar{s},t)] = 0,
\end{align*}
\]

(5.1a)

(5.1b)

(5.1c)

introducing the (generating functional of) gauge products \( \lambda_{NS-NS}^{NS-NS}(s,\bar{s},t) \) and \( \bar{\lambda}_{NS-NS}^{NS-NS}(s,\bar{s},t) \).

The type II superstring field theory with the \( L_\infty \) structure is characterized by the \( L_\infty \) triplet \( (\eta, \bar{\eta}; L_{NS-NS}) \) through the equations

\[
\begin{align*}
\eta \Phi_{NS-NS} &= 0, \quad \bar{\eta} \bar{\Phi}_{NS-NS} = 0, \\
\pi_1 L_{NS-NS}(e^{\Phi_{NS-NS}}) &= 0.
\end{align*}
\]

(5.2a)

(5.2b)

(5.2c)

The first two (5.2a) and (5.2b) are the constraints imposing \( \Phi_{NS-NS} \) is in the small Hilbert space, and the last one (5.2c) is the equation of motion. Using the cohomomorphism

\[
\hat{g} = \hat{P} \exp \left( \int_0^1 dt(\lambda_{NS-NS}^{NS-NS} + \bar{\lambda}_{NS-NS}^{NS-NS})(0,0,t) \right),
\]

(5.3)
where \( \tilde{\mathcal{D}} \) denotes the path-ordered product from left to right, we can transform the original triplet \((\eta, \bar{\eta}; L_{NS-NS}^\dagger)\) and equations (5.2) to the dual \( L_\infty \) triplet \((L^\eta = \hat{g}\eta \hat{g}^{-1}, L^{\bar{\eta}} = \hat{g}\bar{\eta} \hat{g}^{-1}; Q)\) and the equations

\[
\begin{align*}
\pi_1 L^\eta (e^{\pi_1 \hat{g}(e^{\Lambda \Phi_{NS-NS}})}) &= 0, \quad (5.4a) \\
\pi_1 L^{\bar{\eta}} (e^{\pi_1 \hat{g}(e^{\Lambda \Phi_{NS-NS}})}) &= 0, \quad (5.4b) \\
Q \pi_1 \hat{g}(e^{\Lambda \Phi_{NS-NS}}) &= 0, \quad (5.4c)
\end{align*}
\]

respectively, which characterize the WZW-like formulation. The constraint equations,

\[
\begin{align*}
\pi_1 L^\eta (e^{G_{\eta\bar{\eta}}(V)}) &= 0, \quad (5.5a) \\
\pi_1 L^{\bar{\eta}} (e^{G_{\eta\bar{\eta}}(V)}) &= 0, \quad (5.5b)
\end{align*}
\]

are identically satisfied by the pure-gauge (functional) string field\(^{11}\)

\[
G_{\eta\bar{\eta}}(V) = \eta\bar{\eta}V + \frac{1}{2} \left( L_2^\eta(\eta\bar{\eta}V, \bar{\eta}V) + \eta L_2^{\bar{\eta}}(\eta\bar{\eta}V, V) \right) + \cdots, \quad (5.6)
\]

with the string field \( V \) of the WZW-like formulation having the ghost number 0 and the picture number \((0, 0)\). Therefore, if we identify

\[
G_{\eta\bar{\eta}}(V) = \pi_1 \hat{g}(e^{\Lambda \Phi_{NS-NS}}), \quad (5.7)
\]

it gives a map between the string fields in two formulations. By this identification, the last equation (5.2c) becomes the equation of motion of the WZW-like formulation:

\[
Q G_{\eta\bar{\eta}}(V) = 0. \quad (5.8)
\]

In order to write down the WZW-like action, it is also necessary to introduce another important functional field called the associated string field,

\[
B_d(V(t)) = -dV(t) + \frac{1}{2} (L_2^\eta(V(t), \bar{\eta}dV(t)) + L_2^{\bar{\eta}}(\eta V(t), dV(t))) + \cdots, \quad (d = \partial_t, \delta, Q). \quad (5.9)
\]

Here, \( V(t) \) is an extension of \( V \) by a parameter \( t \in [0, 1] \) satisfying \( V(0) = 0 \) and \( V(1) = V \). The identification (5.7) induces the map

\[
B_d(V(t)) = \pi_1 \hat{g} D_{\xi\xi}(e^{\Lambda \Phi_{NS-NS}}), \quad (5.10)
\]

where \( D_{\xi\xi} \) is the coderivation derived from \( \pi_1 D_{\xi\xi} = \xi\bar{\xi} \pi_1 D \) with \( D = d \) for \( d = \partial_t, \delta \) and \( D = \pi_1 L_{NS-NS}^\dagger \) for \( d = Q \). These maps (5.7) and (5.10) are consistent with the identities characterizing the associated field,

\[
\begin{align*}
d G_{\eta\bar{\eta}}(V(t)) &= D_{\bar{\eta}}(t) D_{\eta}(t) B_d(V(t)), \quad (5.11a) \\
D_{\bar{\eta}}(t) D_{\eta}(t) \left( \partial_t B_d(V(t)) - \delta B_{\partial_t}(V(t)) \right) &= 0, \quad (5.11b)
\end{align*}
\]

\(^{11}\)The method how we find the explicit form of \( G_{\eta\bar{\eta}}(V) \) and \( B_d(V) \) is given in [13].
where we introduced the nilpotent linear operators $D_\eta(t)$ and $D_{\bar{\eta}}(t)$ as
\[ D_\eta(t) \varphi = \pi_1 L^\eta(e^{\wedge G_{\eta\bar{\eta}}(V(t))} \wedge \varphi), \quad D_{\bar{\eta}}(t) \varphi = \pi_1 L^\bar{\eta}(e^{\wedge G_{\eta\bar{\eta}}(V(t))} \wedge \varphi), \] (5.12)
for a general string field $\varphi \in \mathcal{H}_{\text{NS-NS}}$. Then, we can map the action to the WZW-like action
\[ I_{\text{WZW}}^{\text{NS-NS}} = \int_0^1 dt \omega_l(B_{\partial t}(V(t)), QG_{\eta\bar{\eta}}(V(t))). \] (5.13)
Using the identities (5.11), we can calculate an arbitrary variation of the action as
\[ \delta I_{\text{WZW}}^{\text{NS-NS}} = \omega_l \left( B_\delta(V), QG_{\eta\bar{\eta}}(V) \right), \] (5.14)
which derives the equation of motion (5.8) and the gauge invariance under the transformation
\[ B_\delta(V) = Q\Lambda + D_\eta\Omega + D_{\bar{\eta}}\bar{\Omega}. \] (5.15)
The gauge transformation generated by $\Lambda$ is mapped from that generated by $\Lambda_{\text{NS-NS}}$ in the formulation with the $L_\infty$ structure with the identification
\[ \Lambda = \pi_1 \hat{g}(e^{\wedge \Phi_{\text{NS-NS}}} \wedge \xi\Lambda_{\text{NS-NS}}). \] (5.16)
The extra gauge invariances come from the fact that the map (5.7) is not one-to-one due to the invariance of $G_{\eta\bar{\eta}}(V)$ under the variation $B_\delta(V) = D_\eta\Omega + D_{\bar{\eta}}\bar{\Omega}$ following from the identity (5.11a).

### 5.2 Complete WZW-like action and gauge invariance

It is straightforward to extend the results on the NS-NS sector to all four sectors. We first note that the similarity transformation by $\hat{F}$ is trivial except for the NS-NS sector, and thus, the string product $L$ we constructed reduces to
\[ L^{(0,0)} = Q + \sum_{p=0}^{\infty} B^{(p,p)} \big|^{(0,0)}(0,0) \equiv Q + B^{(0,0)}, \] (5.17)
in the NS-NS sector. Then, the differential equations (3.20) implies that the generating functional $L^{(0,0)}(s, \bar{s}, t) = Q + B^{(0,0)}(s, \bar{s}, t)$ of string product in the NS-NS sector satisfies
\begin{align*}
\partial_s L^{(0,0)}(s, \bar{s}, t) = & \left[ L^{(0,0)}(s, \bar{s}, t), (\lambda + \bar{\lambda}) \big|^{(0,0)}(s, \bar{s}, t) \right], \quad \text{(5.18a)} \\
\partial_{\bar{s}} L^{(0,0)}(s, \bar{s}, t) = & \left[ \eta, \lambda \big|^{(0,0)}(s, \bar{s}, t) \right], \quad \left[ \eta, \lambda \big|^{(0,0)}(s, \bar{s}, t) \right] = 0, \quad \text{(5.18b)} \\
\partial_t L^{(0,0)}(s, \bar{s}, t) = & \left[ \eta, \lambda \big|^{(0,0)}(s, \bar{s}, t) \right], \quad \left[ \eta, \lambda \big|^{(0,0)}(s, \bar{s}, t) \right] = 0 \quad \text{(5.18c)}
\end{align*}
where

\[
B^{(0,0)}(s, \bar{s}, t) = \sum_{p, \bar{p}, m, \bar{m}} t^{p+\bar{p}} s^m \bar{s}^\bar{m} B_{p+m+1, \bar{p}+\bar{m}+1}^{(0,0)}, 
\]
(5.19a)

\[
\lambda^{(0,0)}(s, \bar{s}, t) = \sum_{p, \bar{p}, m, \bar{m}} t^{p+\bar{p}} s^m \bar{s}^\bar{m} \lambda_{p+m+2, \bar{p}+\bar{m}+1}^{(0,0)}, 
\]
(5.19b)

\[
\bar{\lambda}^{(0,0)}(s, \bar{s}, t) = \sum_{p, \bar{p}, m, \bar{m}} t^{p+\bar{p}} s^m \bar{s}^\bar{m} \bar{\lambda}_{p+m+1, \bar{p}+\bar{m}+2}^{(0,0)}. 
\]
(5.19c)

Since the equations (5.18) agrees with the equations (5.1), we can identify the new string product in the NS-NS sector, \(L^{(0,0)} = L^{(0,0)}(0, 0, 1)\), to that constructed in [10]. The cohomomorphism (5.3) is then given by

\[
\hat{g} = \bar{\mathcal{P}} \exp \left( \int_0^1 dt (\lambda + \bar{\lambda}) \right). 
\]
(5.20)

With this identification, the \(L_\infty\) triplet \((\eta, \bar{\eta}; L)\) can transform to the dual triplet \((L^\eta, \bar{L}^\eta; \tilde{L})\), where

\[
\pi_1 \tilde{L} = \pi_1 \hat{g} L \hat{g}^{-1} = \pi_1 Q + \mathcal{G} \pi_1 \tilde{b}, 
\]
(5.21)

\[
\pi_1 \tilde{b} = \pi_1 \hat{g} (b - B^{(0,0)}) \hat{g}^{-1}. 
\]
(5.22)

Since \(\hat{g}\) is identity except in the NS-NS sector, we find that

\[
\pi_1 \hat{g} (e^{\Psi}) = \pi_1 \hat{g} (e^{\Phi_{NS-NS}}) + \Phi_{R-NS} + \Phi_{NS-R} + \Phi_{R-R}, 
\]
(5.23)

and can identify \(\Phi_{R-NS}, \Phi_{NS-R},\) and \(\Phi_{R-R}\) with the corresponding string fields \(\Psi, \bar{\Psi},\) and \(\Sigma,\) in the WZW-like formulation, respectively:

\[
\Phi_{R-NS} = \Psi, \quad \Phi_{R-NS} = \bar{\Psi}, \quad \Phi_{R-R} = \Sigma. 
\]
(5.24)

The identification (5.7) is now extended as

\[
\pi_1 \hat{g} (e^{\Phi}) = G_{\eta \bar{\eta}}(V) + \Psi + \bar{\Psi} + \Sigma \equiv G(V), 
\]
(5.25)

where we denoted the string fields in the WZW-formulation as \(V\) collectively. The dual constraint equations (5.4a) and (5.4b) map to

\[
0 = \pi_1 L^\eta (e^{\Lambda V}) = \pi_1 L^\eta (e^{\Lambda G_{\eta \bar{\eta}}(V)}) + \eta \Psi + \bar{\eta} \bar{\Psi} + \eta \Sigma, 
\]
(5.26)

\[
0 = \pi_1 L^{\bar{\eta}} (e^{\Lambda V}) = \pi_1 L^{\bar{\eta}} (e^{\Lambda G_{\eta \bar{\eta}}(V)}) + \bar{\eta} \Psi + \bar{\eta} \bar{\Psi} + \bar{\eta} \Sigma, 
\]
(5.27)
which decompose to the Maurer-Cartan equations (5.5) and the constraints that Ψ, ¯Ψ, and Σ are the fields in the small Hilbert space. The map for the associated string field (5.10) can also be extended to that for all four sectors as

\[
\pi_1 \hat{g} D_{\xi \xi}(e^{\lambda \Phi}) = B_d(V(t)) + \xi \xi d\Psi(t) + \xi \xi d\bar{\Psi}(t) + \xi \xi d\Sigma(t) \equiv B_d(V(t)). \tag{5.28}
\]

We can find that the identities (5.11) are then extended to

\[
dG(V(t)) = D_\eta(t)D_\eta(t)B_d(V(t)), \tag{5.29}
\]

\[
D_\eta(t)D_\eta(t)\left(\partial_t B_\delta(V(t)) - \delta B_\delta(V(t))\right) = 0. \tag{5.30}
\]

Here, for general string field \(\varphi \in \mathcal{H}\),

\[
D_\eta(t)\varphi = \pi_1^{(0,0)} L^\eta(e^{\lambda \Phi(0(V))} \wedge \varphi_{NS-NS}) + \eta \varphi_{NS} + \eta \varphi_{R-NS} + \eta \varphi_{R-R}, \tag{5.31}
\]

\[
D_\eta(t)\varphi = \pi_1^{(1,0)} L^\eta(e^{\lambda \Phi(0(V))} \wedge \varphi_{NS-NS}) + \bar{\eta} \varphi_{NS} + \bar{\eta} \varphi_{NS-R} + \bar{\eta} \varphi_{R-R}, \tag{5.32}
\]

with

\[
\varphi_{NS-NS} = \pi^{(0,0)} \varphi, \quad \varphi_{R-NS} = \pi^{(1,0)} \varphi, \quad \varphi_{NS-R} = \pi^{(0,1)} \varphi, \quad \varphi_{R-R} = \pi^{(1,1)} \varphi. \tag{5.33}
\]

It is easy to show that an arbitrary variation of the action

\[
I_{WZW} = \int_0^1 dt \omega_l \left(B_{\delta l}(V(t)) , G^{-1} \pi_1 \hat{L}(e^{\lambda \Phi(V(t)))}\right) \tag{5.34}
\]

gives

\[
\delta I_{WZW} = \omega_l \left(B_{\delta l}(V) , G^{-1} \pi_1 \hat{L}(e^{\lambda \Phi(V)})\right). \tag{5.35}
\]

It derives the equations of motion

\[
\hat{L}(e^{\lambda \Phi(V)}) = 0 \tag{5.36}
\]

and yields the invariance of the action under the gauge transformation

\[
B_{\delta \Lambda}(V) = \pi_1^{(0,0)} \hat{L}(e^{\lambda \Phi} \wedge \hat{\Lambda}) + \xi \bar{\xi}(\pi_1^{(1,0)} + \pi_1^{(0,1)} + \pi_1^{(1,1)}) \hat{L}(e^{\lambda \Phi} \wedge \bar{\eta} \hat{\Lambda}), \tag{5.37}
\]

where

\[
\pi_1^{(0,0)} \hat{\Lambda} = \Lambda, \quad \pi_1^{(1,0)} \hat{\Lambda} = \xi \bar{\xi} \Lambda, \quad \pi_1^{(0,0)} \hat{\Lambda} = \xi \bar{\xi} \Lambda, \quad \pi_1^{(0,0)} \hat{\Lambda} = \xi \bar{\xi} \rho, \tag{5.38}
\]

with \(\eta \lambda = \bar{\eta} \lambda = \eta \bar{\lambda} = \bar{\eta} \bar{\lambda} = \eta \rho = \bar{\eta} \rho = 0\). The identification with the parameters of the formulation with \(L_\infty\) structure is given by

\[
\hat{\Lambda} = \pi_1^{(0,0)} g(e^{\Phi_{NS-NS}} \wedge \xi \bar{\xi} \Lambda_{NS-NS}), \quad \lambda = \Lambda_{R-NS}, \quad \bar{\lambda} = \Lambda_{NS-R}, \quad \rho = \Lambda_{R-R}. \tag{5.39}
\]
6 Conclusion and discussion

In this paper, we have revisited the type II superstring field theory with $L_\infty$ structure and proposed an alternative method to construct string products, which is symmetric with respect to the left- and right-moving sectors. The symmetric method makes transparent not only the construction of string products but also the proof that the tree-level S-matrix agrees with that calculated using the first-quantization method. Another advantage of the symmetric construction is that it enables us to write down a WZW-like action through a map between the string fields of the two formulations, which was not possible with the previous (asymmetric) construction method. The complete WZW-like action of type II superstring field theory, which was the only missing piece, has now been constructed. We have completed the superstring field theory for all three complementary formulations, which allows us to use a convenient formulation depending on what we are studying.

There is another interesting superstring field theory to consider: the (oriented) open-closed superstring field theory. This is the one that should be derived as type II superstring field theory on a non-trivial D-brane background but is worth constructing independently as it helps to calculate non-perturbative effects. In fact, the one based on the formulation with extra free field has already been constructed [25] and used for studying some non-perturbative effect [26–30]. It is interesting to construct it based on the open-closed homotopy algebra (OCHA) [31] and study the relation to the one in the WZW-like formulation.

Acknowledgments

This work is supported in part by JSPS Grant-in-Aid for Scientific Research (C) Grant Number JP18K03645.
A Projected commutators

The definition of the projected commutator introduced in (3.14) is given by

\[
[\mathcal{D}, \mathcal{D}']^{11} = \sum_{n,r,f,m,s,\bar{s}} [\mathcal{D}_n|^{(2r,2f)}_n, \mathcal{D}'_m|^{(2s,2\bar{s})}_m] |^{(2(r+s),2(f+\bar{s}))}, \quad (A.1a)
\]

\[
[\mathcal{D}, \mathcal{D}']^{21} = \sum_{n,r,f,m,s,\bar{s}} [\mathcal{D}_n|^{(2r,2f)}_n, \mathcal{D}'_m|^{(2s,2\bar{s})}_m] |^{(2(r+s-1),2(f+\bar{s}))}, \quad (A.1b)
\]

\[
[\mathcal{D}, \mathcal{D}']^{12} = \sum_{n,r,f,m,s,\bar{s}} [\mathcal{D}_n|^{(2r,2f)}_n, \mathcal{D}'_m|^{(2s,2\bar{s})}_m] |^{(2(r+s),2(f+\bar{s}-1))}, \quad (A.1c)
\]

\[
[\mathcal{D}, \mathcal{D}']^{22} = \sum_{n,r,f,m,s,\bar{s}} [\mathcal{D}_n|^{(2r,2f)}_n, \mathcal{D}'_m|^{(2s,2\bar{s})}_m] |^{(2(r+s-1),2(f+\bar{s}-1))}, \quad (A.1d)
\]

for coderivations \( \mathcal{D} = \sum_{n,r,f} \mathcal{D}_n|^{(2r,2f)}_n \) and \( \mathcal{D}' = \sum_{m,s,\bar{s}} \mathcal{D}'_m|^{(2s,2\bar{s})}_m \). An alternative expressions obtained by projecting the intermediate state \([24]\),

\[
[\mathcal{D}, \mathcal{D}']^{11} = \sum_n \left( \mathcal{D}_n(\pi^{(0,0)}_1 \mathcal{D}' \land \mathbb{I}_{n-1}) - (-)^{|D||D'|} \mathcal{D}'_n(\pi^{(0,0)}_1 \mathcal{D} \land \mathbb{I}_{n-1}) \right), \quad (A.2a)
\]

\[
[\mathcal{D}, \mathcal{D}']^{21} = \sum_n \left( \mathcal{D}_n(\pi^{(1,0)}_1 \mathcal{D}' \land \mathbb{I}_{n-1}) - (-)^{|D||D'|} \mathcal{D}'_n(\pi^{(1,0)}_1 \mathcal{D} \land \mathbb{I}_{n-1}) \right), \quad (A.2b)
\]

\[
[\mathcal{D}, \mathcal{D}']^{12} = \sum_n \left( \mathcal{D}_n(\pi^{(0,1)}_1 \mathcal{D}' \land \mathbb{I}_{n-1}) - (-)^{|D||D'|} \mathcal{D}'_n(\pi^{(0,1)}_1 \mathcal{D} \land \mathbb{I}_{n-1}) \right), \quad (A.2c)
\]

\[
[\mathcal{D}, \mathcal{D}']^{22} = \sum_n \left( \mathcal{D}_n(\pi^{(1,1)}_1 \mathcal{D}' \land \mathbb{I}_{n-1}) - (-)^{|D||D'|} \mathcal{D}'_n(\pi^{(1,1)}_1 \mathcal{D} \land \mathbb{I}_{n-1}) \right), \quad (A.2d)
\]

are also useful. Here, \(|\mathcal{D}| (|\mathcal{D}'|)\) is equal to 0 or 1 when the degree of coderivation \( \mathcal{D} (\mathcal{D}') \) is even or odd, respectively. The square bracket with subscript \( \mathcal{X} = X \) or \( \overline{X} \) is defined by inserting \( \mathcal{X} \) at the intermediate state\(^{12}\):

\[
[\mathcal{D}, \mathcal{D}']^{22}_{\mathcal{X}} = \sum_n \left( \mathcal{D}_n(\pi^{(1,1)}_1 \mathcal{X} \mathcal{D}' \land \mathbb{I}_{n-1}) - (-)^{|D||D'|} \mathcal{D}'_n(\pi^{(1,1)}_1 \mathcal{X} \mathcal{D} \land \mathbb{I}_{n-1}) \right). \quad (A.3)
\]

Similarly, the square bracket with subscript \( \mathcal{O} = \Xi \) or \( \overline{\Xi} \) is defined by

\[
[\mathcal{D}, \mathcal{D}']^{22}_{\mathcal{O}} = \sum_n \left( \mathcal{D}_n(\pi^{(1,1)}_1 \mathcal{O} \mathcal{D}' \land \mathbb{I}^{(n-1)}) + (-)^{|D||D'|+1} \mathcal{D}'_n(\pi^{(1,1)}_1 \mathcal{O} \mathcal{D} \land \mathbb{I}^{(n-1)}) \right). \quad (A.4)
\]

The projected commutators (A.1) satisfy the Jacobi identities

\[
[\mathcal{D}, [\mathcal{D}', \mathcal{D}']^{cd}]^{ab} + [\mathcal{D}, [\mathcal{D}', \mathcal{D}']^{cd}]^{ab} + (\text{cyclic perm.}) = 0, \quad (A.5)
\]

\(^{12}\)These brackets can only be applicable for the case considered since \( X \) and \( \overline{X} \) are the PCO’s acting on the states with picture number \(-3/2\).
which reduce to the conventional ones if two projected commutators are the same type. Similar identities also hold even if they include projected commutator(s) with subscript \( \bar{x} = X \) or \( \bar{X} \):

\[
[D, [D', D'']_{X^{22}}]^{ab} + [D, [D', D'']_{X^{22}}]^{ab} (\text{cyclic perm.}) = 0, \quad (A.6)
\]

\[
[D, [D', D'']_{X^{22}} |_{X^{22}} + [D, [D', D'']_{X^{22}} |_{X^{22}}] (\text{cyclic perm.}) = 0. \quad (A.7)
\]

The additional sign factor is necessary for the Jacobi identities including the one with subscript \( \mathcal{O} = \Xi \) or \( \bar{\Xi} \) since \( \Xi \) and \( \bar{\Xi} \) are the Grassmann odd operators. For example,

\[
[D, [D', D'']_{\mathcal{O}}]^{ab} + (-1)^{|D'|}[D, [D', D'']_{\mathcal{O}}]^{ab} + (\text{cyclic perm.}) = 0, \quad (A.8)
\]

\[
[D, [D', D'']_{\mathcal{O}} |_{X^{22}} + (-1)^{|D'|}[D, [D', D'']_{\mathcal{O}} |_{X^{22}}] (\text{cyclic perm.}) = 0. \quad (A.9)
\]

We should note that \( \mathcal{O} \) is not commutative to \( Q \) and either \( \eta \) or \( \bar{\eta} \). Thus, for example,

\[
[Q, [D, D']_{\Xi^{22}}] = [[Q, D'], D']_{\Xi^{22}} + (-1)^{|D|+1}[D, [Q, D']_{\Xi^{22}}]_{\Xi^{22}} + (-1)^{|D|}[D, D']_{\Xi^{22}}; \quad (A.10)
\]

\[
[\eta, [D, D']_{\Xi^{22}}] = [[\eta, D'], D']_{\Xi^{22}} + (-1)^{|D|+1}[D, [\eta, D']_{\Xi^{22}}]_{\Xi^{22}} + (-1)^{|D|}[D, D']_{\Xi^{22}}. \quad (A.11)
\]

**B Expansion of \( B_3^{(\bullet \bullet)}(s, \bar{s}) \), \( \lambda_3^{(\bullet \bullet)}(s, \bar{s}) \) and \( \bar{\lambda}_3^{(\bullet \bullet)}(s, \bar{s}) \)**

![Diagram](image.png)

Figure 4: The flow of how each component of 3-string products are determined.

In section 3, we show that how the generating functionals of 3-string and gauge products with specific picture number are determined. In this appendix, we illustrate the flow to be determined the 3-string (gauge) products with specific cyclic Ramond numbers, which are
Figure 5: The flow of how each component of 3-string products are determined 2.

given as coefficients further expanded by \( s \) and \( \bar{s} \):

\[
    B_3^{(0,0)}(s, \bar{s}) = B_3^{(0,0)}(0,4) + sB_3^{(0,0)}(2,4) + \bar{s}B_3^{(0,0)}(4,2) + s^2B_3^{(0,0)}(0,4) + s\bar{s}B_3^{(0,0)}(0,2) + B_3^{(0,0)}(2,2) + \bar{s}^2B_3^{(0,0)}(0,0),
\]

(B.1)

\[
    B_3^{(1,0)}(s, \bar{s}) = B_3^{(1,0)}(1,4) + sB_3^{(1,0)}(1,2) + \bar{s}B_3^{(1,0)}(1,0) + s\bar{s}B_3^{(1,0)}(0,2) + s^2B_3^{(1,0)}(0,0),
\]

(B.2)

\[
    B_3^{(0,1)}(s, \bar{s}) = B_3^{(0,1)}(0,4) + sB_3^{(0,1)}(2,2) + \bar{s}B_3^{(0,1)}(4,0) + s^2B_3^{(0,1)}(0,2) + \bar{s}^2B_3^{(0,1)}(0,0),
\]

(B.3)

\[
    B_3^{(2,0)}(s, \bar{s}) = B_3^{(2,0)}(0,4) + sB_3^{(2,0)}(0,2) + \bar{s}B_3^{(2,0)}(2,0) + s\bar{s}B_3^{(2,0)}(0,0),
\]

(B.4)

\[
    B_3^{(1,1)}(s, \bar{s}) = B_3^{(1,1)}(1,2) + sB_3^{(1,1)}(1,0) + \bar{s}B_3^{(1,1)}(1,0) + s\bar{s}B_3^{(1,1)}(0,0),
\]

(B.5)

\[
    B_3^{(0,2)}(s, \bar{s}) = B_3^{(0,2)}(0,4) + sB_3^{(0,2)}(2,0) + \bar{s}B_3^{(0,2)}(0,0) + s^2B_3^{(0,2)}(0,0),
\]

(B.6)

\[
    B_3^{(2,1)}(s, \bar{s}) = B_3^{(2,1)}(0,2) + sB_3^{(2,1)}(0,0),
\]

(B.7)

\[
    B_3^{(1,2)}(s, \bar{s}) = B_3^{(1,2)}(2,0) + sB_3^{(1,2)}(0,0),
\]

(B.8)

\[
    B_3^{(2,2)}(s, \bar{s}) = B_3^{(2,2)}(0,0),
\]

(B.9)
The flow of each product is shown in Figs. 4 and 5. The 3-products at the end of flows are used to make the $L_\infty$ structure for writing down the action, which are those with no picture number deficit.

References

[1] N. Berkovits, “SuperPoincare invariant superstring field theory,” Nucl. Phys. B **450**, 90 (1995) Erratum: [Nucl. Phys. B **459**, 439 (1996)] doi:10.1016/0550-3213(95)00620-6, 10.1016/0550-3213(95)00259-U [hep-th/9503099].

[2] N. Berkovits, Y. Okawa and B. Zwiebach, “WZW-like action for heterotic string field theory,” JHEP **0411**, 038 (2004) doi:10.1088/1126-6708/2004/11/038 [hep-th/0409018].

[3] H. Matsunaga, “Nonlinear gauge invariance and WZW-like action for NS-NS superstring field theory,” JHEP **1509**, 011 (2015) doi:10.1007/JHEP09(2015)011 [arXiv:1407.8485 [hep-th]].

[4] T. Erler, Y. Okawa and T. Takezaki, “$A_\infty$ structure from the Berkovits formulation of open superstring field theory,” arXiv:1505.01659 [hep-th].

[5] H. Kunitomo and Y. Okawa, “Complete action for open superstring field theory,” PTEP **2016**, no. 2, 023B01 (2016) doi:10.1093/ptep/ptv189 [arXiv:1508.00366 [hep-th]].

[6] H. Matsunaga, “Comments on complete actions for open superstring field theory,” JHEP **1611**, 115 (2016) doi:10.1007/JHEP11(2016)115 [arXiv:1510.06023 [hep-th]].
[7] T. Erler, “Superstring Field Theory and the Wess-Zumino-Witten Action,” JHEP 1710, 057 (2017) doi:10.1007/JHEP10(2017)057 [arXiv:1706.02629 [hep-th]].

[8] B. Jurco and K. Muenster, “Type II Superstring Field Theory: Geometric Approach and Operadic Description,” JHEP 1304, 126 (2013) doi:10.1007/JHEP04(2013)126 [arXiv:1303.2323 [hep-th]].

[9] T. Erler, S. Konopka and I. Sachs, “Resolving Witten’s superstring field theory,” JHEP 1404, 150 (2014) doi:10.1007/JHEP04(2014)150 [arXiv:1312.2948 [hep-th]].

[10] T. Erler, S. Konopka and I. Sachs, “NS-NS Sector of Closed Superstring Field Theory,” JHEP 1408, 158 (2014) doi:10.1007/JHEP08(2014)158 [arXiv:1403.0940 [hep-th]].

[11] T. Erler, S. Konopka and I. Sachs, “Ramond Equations of Motion in Superstring Field Theory,” JHEP 1511, 199 (2015) doi:10.1007/JHEP11(2015)199 [arXiv:1506.05774 [hep-th]].

[12] T. Erler, Y. Okawa and T. Takezaki, “Complete Action for Open Superstring Field Theory with Cyclic $A_{\infty}$ Structure,” JHEP 1608, 012 (2016) doi:10.1007/JHEP08(2016)012 [arXiv:1602.02582 [hep-th]].

[13] H. Matsunaga, “Notes on the Wess-Zumino-Witten-like structure: $L_{\infty}$ triplet and NS-NS superstring field theory,” JHEP 05, 095 (2017) doi:10.1007/JHEP05(2017)095 [arXiv:1612.08827 [hep-th]].

[14] H. Kunitomo and T. Sugimoto, “Heterotic string field theory with cyclic L-infinity structure,” PTEP 2019, no. 6, 063B02 (2019), doi:10.1093/ptep/ptz051 [arXiv:1902.02991 [hep-th]], Errata PTEP 2020, no. 1, 019201 (2020), doi:10.1093/ptep/ptz148.

[15] H. Kunitomo and T. Sugimoto, “Type II superstring field theory with cyclic $L_{\infty}$ structure,” PTEP 2020, no.3, 033B06 (2020) doi:10.1093/ptep/ptaa013 [arXiv:1911.04103 [hep-th]].

[16] A. Sen, “Gauge Invariant 1PI Effective Superstring Field Theory: Inclusion of the Ramond Sector,” JHEP 1508, 025 (2015) doi:10.1007/JHEP08(2015)025 [arXiv:1501.00988 [hep-th]].

[17] A. Sen, “BV Master Action for Heterotic and Type II String Field Theories,” JHEP 1602, 087 (2016) doi:10.1007/JHEP02(2016)087 [arXiv:1508.05387 [hep-th]].

[18] S. Konopka and I. Sachs, “Open Superstring Field Theory on the Restricted Hilbert Space,” JHEP 1604, 164 (2016) doi:10.1007/JHEP04(2016)164 [arXiv:1602.02583 [hep-th]].

[19] M. Saadi and B. Zwiebach, “Closed String Field Theory from Polyhedra,” Annals Phys. 192, 213 (1989). doi:10.1016/0003-4916(89)90126-7

[20] T. Kugo, H. Kunitomo and K. Suehiro, “Nonpolynomial Closed String Field Theory,” Phys. Lett. B 226, 48 (1989). doi:10.1016/0370-2693(89)90287-6

[21] B. Zwiebach, “Closed string field theory: Quantum action and the B-V master equation,” Nucl. Phys. B 390, 33 (1993) doi:10.1016/0550-3213(93)90388-6 [hep-th/9206084].

[22] H. Kajiura, “Noncommutative homotopy algebras associated with open strings,” Rev. Math. Phys. 19, 1 (2007) doi:10.1142/S0129055X07002912 [math/0306332 [math-qa]].
[23] S. Konopka, “The S-Matrix of superstring field theory,” JHEP 11, 187 (2015) doi:10.1007/JHEP11(2015)187 [arXiv:1507.08250 [hep-th]].

[24] H. Kunitomo, “Tree-level S-matrix of superstring field theory with homotopy algebra structure,” JHEP 03, 193 (2021) doi:10.1007/JHEP03(2021)193 [arXiv:2011.11975 [hep-th]].

[25] S. Faroogh Moosavian, A. Sen and M. Verma, “Superstring Field Theory with Open and Closed Strings,” JHEP 01, 183 (2020) doi:10.1007/JHEP01(2020)183 [arXiv:1907.10632 [hep-th]].

[26] A. Sen, “Fixing an Ambiguity in Two Dimensional String Theory Using String Field Theory,” JHEP 03, 005 (2020) doi:10.1007/JHEP03(2020)005 [arXiv:1908.02782 [hep-th]].

[27] A. Sen, “D-instanton Perturbation Theory,” JHEP 08, 075 (2020) doi:10.1007/JHEP08(2020)075 [arXiv:2002.04043 [hep-th]].

[28] A. Sen, “Divergent $\Rightarrow$ complex amplitudes in two dimensional string theory,” JHEP 02, 086 (2021) doi:10.1007/JHEP02(2021)086 [arXiv:2003.12076 [hep-th]].

[29] A. Sen, “Cutkosky Rules and Unitarity (Violation) in D-instanton Amplitudes,” [arXiv:2012.00041 [hep-th]].

[30] A. Sen, “Normalization of D-instanton Amplitudes,” [arXiv:2101.08566 [hep-th]].

[31] H. Kajiura and J. Stasheff, “Homotopy algebras inspired by classical open-closed string field theory,” Commun. Math. Phys. 263, 553-581 (2006) doi:10.1007/s00220-006-1539-2 [arXiv:math/0410291 [math.QA]].