New approach for normalization and photon-number distributions of photon-added (-subtracted) squeezed thermal states

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Using the thermal field dynamics theory to convert the thermal state to a “pure” state in doubled Fock space, it is found that the average value of $e^{\lambda a^\dagger a}$ under squeezed thermal state (STS) is just the generating function of Legendre polynomials, a remarkable result. Based on this point, the normalization and photon-number distributions of m-photon added (or subtracted) STS are conviently obtained as the Legendre polynomials. This new concise method can be expanded to the entangled case.

I. INTRODUCTION

Nonclassicality of optical fields is helpful in understanding fundamentals of quantum optics and have many applications in quantum information processing \cite{1}. To generate and manipulate various nonclassical optical fields, subtracting or adding photons from/to traditional quantum states or Gaussian states is proposed \cite{2–9}. For example, the photon addition and subtraction have been successfully demonstrated experimentally for probing quantum commutation rules by Parigi et al. \cite{7}. Recently, photon-added (-subtracted) Gaussian states have received more attention from both experimentalists and theoreticians \cite{10–19}, since these states exhibit an abundant of nonclassical properties and may give access to a complete engineering of quantum states and to fundamental quantum phenomena.

Theoretically, the normalization factors of such quantum states are essential for studying their nonclassical properties. Very recent, Fan and Jiang \cite{20} present a new concise approach for normalizing m-photon-added (-subtracted) squeezed thermal states (STSs) and for deriving their normalization and photon-number distributions (PNDs) which have been a remarkable result. Based on this point, in sections 4 and 5, the normalization factors and PNDs of m-photon added (or subtracted) STS are obtained as the Legendre polynomials, respectively. The last section is devoted to drawing a conclusion.

II. REPRESENTATION OF THERMAL STATE IN DOUBLED FOCK SPACE

We begin with briefly reviewing the properties of thermal state. For a single field mode with frequency $\omega$ in a thermal equilibrium state corresponding to absolute temperature $T$, the density operator is

$$\rho_{th} = \sum_{n=0}^{\infty} \frac{n_{c}^{n}}{(n_{c} + 1)^{n+1}} |n\rangle \langle n|,$$  \hspace{1cm} (1)

where $n_{c} = \exp(h\omega/(kT)) - 1\}^{-1}$ being the average photon number of the thermal state $\rho_{th}$ and $k$ being Boltzmann's constant. Note $|n\rangle = a^{\dagger n}/\sqrt{n!} |0\rangle$ and the normally ordering form of vacuum projector $|0\rangle \langle 0| = \exp(-a^\dagger a)$ (the symbol : : denotes the normal ordering), one can put Eq.(1) into the following form

$$\rho_{th} =: \frac{1}{n_{c} + 1} e^{-n_{c} a^\dagger a} = \frac{1}{n_{c} + 1} e^{a^\dagger a \ln n_{c}/n_{c} + 1},$$  \hspace{1cm} (2)

where in the last step, the operator identity $\exp(\lambda a^\dagger a) =: \exp\left(\left(e^{\lambda - 1}\right) a^\dagger a\right)$ is used.

Recalling the thermal field dynamics (TFD) introduced by Takahashi and Umezawa \cite{21,22}, its elemental spirit is to convert the calculations of ensemble averages...
for a mixed state ρ, (A) = tr (Aρ)/tr (ρ), to equivalent expectation values with a pure state |0(β)⟩, i.e.,

\[ (A) = \langle 0(β) | A | 0(β) \rangle, \]

where β = 1/kT, k is the Boltzmann constant. Thus, for the density operator ρth, by using the partial trace method [24], i.e., ρth = tr [0(β)] ⟨0(β)| where tr denotes the trace operation over the environment freedom (denoted as operator ̃a†), one can obtain the explicit expression of |0(β)⟩ in doubled Fock space,

\[ |0(β)⟩ = \text{sech}\theta \exp \left[ a†β \tan \theta \right] |00⟩ = S(θ) |00⟩, \]

where |00⟩ is annihilated by ̃a and ̃a†, [ ̃a, ̃a† ] = 1, and S(θ) is the thermal operator, S(θ) = exp \[ θ (a†βaβ − aβa†β) \] with a similar form to the a two-mode squeezing operator except for the tilde mode, and θ is a parameter related to the temperature by tanh θ = −hυ/(2kT). |0(β)⟩ is named thermal vacuum state.

Let Tr denote the trace operation over both the system freedom (expressed by tr) and the environment freedom by tr, i.e., Tr = tr tr, then we have

\[ \text{tr} (Aρth) = \text{Tr} [A |0(β)⟩ ⟨0(β)|] \]

\[ = \text{tr} [A ̃tr |0(β)⟩ ⟨0(β)|], \] (5)

and the average photon number of the thermal state ρth is

\[ n_c = \text{Tr} [a†a |0(β)⟩ ⟨0(β)|] = \sinh^2 θ. \] (6)

Here we should emphasize that ̃tr |0(β)⟩ ⟨0(β)| ≠ ⟨0(β)| 0(β)⟩, since |0(β)⟩ involves both real mode a and fictitious mode ̃a. From Eqs. (3) and (4) one can see that the worthwhile convenience in Eq. (4) is at the expense of introducing a fictitious field (or called a tilde-conjugate field) in the extended Hilbert space, i.e., the original optical field state |ν⟩ in the Hilbert space $\mathcal{H}$ is accompanied by a tilde state | ̃ν⟩ in ̃$\mathcal{H}$. A similar rule holds for operators: every Bose annihilation operator a acting on $\mathcal{H}$ has an image ̃a acting on ̃$\mathcal{H}$. These operators in $\mathcal{H}$ are commutative with those in ̃$\mathcal{H}$.

### III. SQUEEZED THERMAL VACUUM STATE

To realize our purpose, we first introduce the squeezed thermal vacuum state, defined as $S_1 (r) |0(β)⟩$, where $S_1 (r) = \exp [r (a^2 − a^2)];/2$ is the single-mode squeezing operator for the real mode with r being squeezing parameter. Note that Eq. (4) and the Baker-Hausdorff lemma

\[ S_1 (r) a^† S_1^† (r) = a^† \cosh r + a \sinh r, \] (7)

then we get

\[ S_1 (r) |0(β)⟩ = S_1 (r) \text{sech} \exp [a^† β \tan \theta] |00⟩ \]

= \text{sech}θ \exp [a^† β \tan \theta] \exp \left[ \left( a^† \cosh r + a \sinh r \right) β \tan \theta \right] \exp \left[ \frac{− β^2}{2} \tan r \right] |00⟩, \] (8)

where we have used $S_1 (λ) |0⟩ = \text{sech}^1/2 λ \exp [− a^2/2 tanh λ] |0⟩$. Further, note $e^{arβ} a^† e^{−rβ} = a^† + rβ$, and for operators $A, B$ satisfying the conditions $[A, [A, B]] = [B, [A, B]] = 0$, we have $e^{A+B} = e^A e^B e^{−[A,B]/2}$, then Eq. (8) can be put into the following form

\[ S_1 (r) |0(β)⟩ = \text{sech}θ \exp [\frac{− β^2}{2} \tan r] \exp \left[ \frac{− β^2}{2} \tan r \right] \exp \left[ \frac{− β^2}{2} \tan r \right] |00⟩, \] (9)

Next, we shall use Eq. (9) to derive the average of operator $e^{fa^†a}$ under the squeezed thermal vacuum state $S_1 (r) |0(β)⟩$, which is a bridge for our whole calculations. Notice $e^{fa^†a} a^† e^{−f/2a^†a} = a^† e^{f/2}$ and $e^{−f/2a^†a} a^† e^{f/2a^†a} = a^† e^{f/2}$, so we have

\[ e^{fa^†a} S_1 (r) |0(β)⟩ = \text{sech}θ \exp [\frac{− β^2}{2} \tan r] \exp \left[ \frac{− β^2}{2} \tan r \right] \exp \left[ \frac{− β^2}{2} \tan r \right] |00⟩, \] (10)

which leads to

\[ \langle e^{fa^†a} \rangle = \langle 0(β) | S_1^† (r) e^{fa^†a} S_1 (r) |0(β)⟩ \]

\[ = \text{sech}^2 θ \text{sech} \langle 00⟩ \exp \left[ \frac{− β^2}{2} \tan r \right] \exp \left[ \frac{− β^2}{2} \tan r \right] \exp \left[ \frac{− β^2}{2} \tan r \right] |00⟩ \]

\[ = \left[ C e^{2f} − 2B e^f + A \right]^{-1/2}, \] (11)

where we have set

\[ A = \cosh^4 θ + \cosh 2θ \sinh^2 r \]

\[ = n_c^2 + (2n_c + 1) \cosh^2 r, \]

\[ B = \sinh^2 θ \cosh θ = n_c (n_c + 1), \]

\[ C = \cosh^4 θ − \cosh 2θ \sinh^2 r \]

\[ = n_c^2 − (2n_c + 1) \sinh^2 r, \] (12)

and we have used the completeness relation of coherent state $\int d^2 z d^2 \tilde{z} \langle z | \tilde{z} \rangle / π^2 = 1$, here |z⟩ and |z⟩ is the coherent state in real and fictitious modes, respectively, and the formula [24]

\[ \int \frac{d^2 z}{π} \exp \left( \frac{− β^2}{2} \tan r \right) \exp \left( \frac{− β^2}{2} \tan r \right) \exp \left( \frac{− β^2}{2} \tan r \right) |00⟩ \]

\[ = \frac{1}{\sqrt{β^2 − 4fg}} \exp \left[ \frac{− β^2}{2} \tan r \right] \exp \left[ \frac{− β^2}{2} \tan r \right] \exp \left[ \frac{− β^2}{2} \tan r \right] |00⟩, \] (13)
whose convergent condition is \( \text{Re}(\zeta \pm f \pm g) < 0, \text{Re}(\zeta^2 - 4fg)/|\zeta| < 0 \). Eq. (11) is very important for later calculation of photon-number distribution (PND) and normalization of photon-added (subtracted) squeezed thermal states (PASTS, PSSTS).

It is interesting to notice that the standard generating function of Legendre polynomials \([20]\) is given by

\[
\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{m=0}^{\infty} P_m (x) t^m, \tag{14}
\]

thus comparing Eq. (11) with Eq. (14) we find

\[
\langle e^{fa^\dagger a} \rangle = A^{-1/2} \left[ C e^{2f} - 2 B A^{f} + 1 \right]^{-1/2} = A^{-1/2} \sum_{m=0}^{\infty} P_m \left( B / \sqrt{AC} \right) \left( \sqrt{C/Ae^f} \right)^m, \tag{15}
\]

which indicates that the average value of \( e^{fa^\dagger a} \) under squeezed thermal state (STS) is just the generating function of Legendre polynomials, a remarkable result. Next, we shall examine the normalizations and PNDs of PASTS and PSSTS by using Eqs. (11) and (15).

### IV. Normalization and PND of PASTS

The \( m \)-photon-added scheme, denoted by the mapping \( \rho \rightarrow a^{m} \rho a^{m} \), was first proposed by Agarwal and Tara [2]. Here, we introduce the PASTS. Theoretically, the PASTS can be obtained by repeatedly operating the photon creation operator \( a^{\dagger} \) on a STS, so its density operator is given by

\[
\rho_{ad} = C_{a,m}^{-1} a^{m} S_1 \rho_{th} S_1^{\dagger} a^{m}, \tag{16}
\]

where \( m \) is the added photon number (a non-negative integer), \( C_{a,m}^{-1} \) is the normalization constant to be determined.

#### A. Normalization of PASTS

To fully describe a quantum state, its normalization is usually necessary. Next, we shall employ the fact [23] and (14), (15) to realize our aim. According to the normalization condition \( \text{tr} \rho_{ad} = 1 \) and the TFD, we have

\[
C_{a,m} = \text{Tr} \left[ a^{m} S_1 \rho_{th} S_1^{\dagger} a^{m} \right]
= \text{Tr} \left[ a^{m} S_1 |0(\beta)\rangle \langle 0(\beta)| S_1^{\dagger} a^{m} \right]
= |\langle 0(\beta)| S_1^{\dagger} a^{m} a^{m} S_1 |0(\beta)\rangle|,
\]

which implies that the calculation of normalization factor \( C_{a,m} \) is converted to a matrix element after introducing the thermal vacuum state \( |0(\beta)\rangle \).

Note the operator identity \([27] e^{ra^\dagger a} = e^{-r}; \exp[(1 - e^r)a^\dagger a] \rangle \), we see

\[
\sum_{m=0}^{\infty} \frac{r^m}{m!} a^{m} a^{m} = \langle e^{ra^\dagger a} \rangle = \left( \frac{1}{1 - r} \right)^{a^\dagger a + 1}, \tag{18}
\]

where the symbol \( \langle \rangle \) denotes antinormally ordering. Thus using Eqs. (11), (17) and (18), \( (e^f \rightarrow 1/\tau) \) we have

\[
\sum_{m=0}^{\infty} \frac{r^m}{m!} C_{a,m} = \frac{1}{1 - \tau} \langle 0(\beta)| S_1^{\dagger} e^{a^\dagger a} \ln \frac{1}{r} S_1 |0(\beta)\rangle
= |A\tau^2 - 2D\tau + 1|^{-1/2}, \tag{19}
\]

where we have set

\[
D = \cosh^2 \theta \cosh 2r - \sinh^2 r = n_n \cosh 2r + \cosh^2 r. \tag{20}
\]

Comparing Eq. (21) with Eq. (12), and taking \( \tau' \rightarrow \sqrt{A} \tau \), we obtain

\[
\sum_{m=0}^{\infty} \frac{r^m}{m!} A^{m/2} P_m \left( D/\sqrt{A} \right)^{r^m} = \left[ \tau'^2 - 2D/\sqrt{A}\tau' + 1 \right]^{-1/2}, \tag{21}
\]

thus the normalization constant of PASTS is given by

\[
C_{a,m} = m! A^{m/2} P_m \left( D/\sqrt{A} \right), \tag{22}
\]

which is identical with the result in Ref. [28]. It is noted that, for the case of no-photon-addition with \( m = 0 \), \( C_{a,0} = 1 \) as expected. Under the case of \( m \)-photon-addition thermal state (no squeezing) with \( D = n_n + 1 \), \( A = (n_n + 1)^2 \), and \( P_m (1) = 1 \), then \( C_{a,m} = m! (n_n + 1)^m \). The same result as Eq. (32) found in Ref. [29]. In addition, when \( r = 0 \) corresponding to photon-added thermal state, Eq. (22) just reduces to \( C_{a,m} = m! \cosh 2m \theta \) [24].

#### B. PND of PASTS

The photon-number distribution (PND) is a key characteristic of every optical field. The PND, i.e., the probability of finding \( n \) photons in a quantum state described by the density operator \( \rho \), is \( \mathcal{P}(n) = \text{tr} [a^n |n\rangle \langle n| \rho] \). In a similar spirit of deriving Eq. (22), noting \( a^{m} |n\rangle = \sqrt{n!/(n-m)!} |n-m\rangle \) and \( |n\rangle = a^{m} |0\rangle \), the PND
of the PASTS can be calculated as

\[
\mathcal{P}_a(n) = C_{a,m}^{-1} tr \left[ n \langle a^{\dagger m} S_1 | 0(\beta) \rangle S_{1}^{\dagger} a^m \right] \\
= C_{a,m}^{-1} tr \left[ n \langle a^{\dagger m} S_1 | 0(\beta) \rangle (0(\beta)) S_{1}^{\dagger} a^m \right] \\
= \frac{n! C_{a,m}^{-1}}{\beta} \langle 0(\beta) | S_{1}^{\dagger} a^m | 0(\beta) \rangle \\
= \frac{n! C_{a,m}^{-1}}{\beta} \langle 0(\beta) | S_{1}^{\dagger} a^m | 0(\beta) \rangle \\
= \frac{n! C_{a,m}^{-1}}{(\beta)^l} \langle 0(\beta) | S_{1}^{\dagger} a^m | 0(\beta) \rangle \\
\] (23)

which leads to

\[
\sum_{l=0}^{\infty} \frac{\tau^l}{n!} C_{a,m} \mathcal{P}_a(n) \\
= \sum_{l=0}^{\infty} \frac{\tau^l}{l} \langle 0(\beta) | S_{1}^{\dagger} (a^l a^\dagger a^l S_{1}^{\dagger} a^m | 0(\beta) \rangle \\
= \langle 0(\beta) | S_{1}^{\dagger} e^{a^\dagger a l \tau} S_{1} | 0(\beta) \rangle, \\
\] (24)

where \(l = n - m\) and the vacuum projector operator \(|0| 0\rangle = |e^{-1} a^\dagger a\rangle\); and the operator identity \(e^{a^\dagger a l} =: \exp(e^l - 1) a^\dagger a\): are used.

Using Eq. (11) again (\(e^{\beta} \rightarrow \tau\)) and comparing Eq. (24) with Eq. (14) we see

\[
\sum_{l=0}^{\infty} \frac{\tau^l}{n!} C_{a,m} \mathcal{P}_a(n) \\
= A^{-1/2} \left[ \frac{C}{A} \tau^2 - 2 \frac{B}{A} \tau + 1 \right]^{-1/2} \\
= A^{-1/2} \sum_{l=0}^{\infty} P_l \left( B/\sqrt{AC} \right) \left( \sqrt{C/A} \tau \right)^l, \\
\] (25)

which leads to the PND of PASTS

\[
\mathcal{P}_a(n) = \frac{n! C_{a,m}^{-1}(C/A)^{\frac{n-m}{2}}}{(n-m)! \sqrt{A}} P_{n-m} \left( B/\sqrt{AC} \right), \\
\] (26)

a Legendre polynomial with a condition \(n \geq m\) which implies that the photon-number \(n\) involved in PASTS is always no-less than the photon-number \(m\) operated on the STS, and there is no photon distribution when \(n < m\). It is obvious that when \(m = 0\) corresponding to the STS, then the PND of STS is also a Legendre distribution [30].

V. NORMALIZATION AND PND OF PSST)

Next, we turn our attention to discussing the photon-subtracted squeezed thermal state (PSSTS), defined as

\[
\rho_{bh} = C_{s,m}^{-1} a^m S_{1} \rho_{th} S_{1}^{\dagger} a^{\dagger m}, \\
\] (27)

where \(m\) is the subtracted photon number (a non-negative integer), and \(C_{s,m}\) is a normalized constant.

In a similar way to deriving Eq. (22), we have

\[
C_{s,m} = \langle 0(\beta) | S_{1}^{\dagger} a^{\dagger m} a^m S_{1} | 0(\beta) \rangle, \\
\] (28)

so employing \(e^{\lambda a^\dagger a} =: \exp((e^\lambda - 1)a^\dagger a)\): and Eq. (11) \((e^{\beta} \rightarrow 1 + \tau)\) we see

\[
\sum_{m=0}^{\infty} \frac{\tau^m}{m!} C_{s,m} = \langle 0(\beta) | S_{1}^{\dagger} e^{a^\dagger a (1+\tau)} S_{1} | 0(\beta) \rangle \\
= \left[ C \tau^2 - 2E\tau + 1 \right]^{-1/2}, \\
\] (29)

where

\[
E = \cosh 2\tau \cosh^2 \theta - \cosh^2 r \\
= \frac{1}{2} \left( (2n_c + 1) \cosh 2\tau - 1 \right). \\
\] (30)

Comparing Eq. (29) with Eq. (14) yields

\[
C_{s,m} = m! C^{m/2} P_m \left( E/\sqrt{C} \right), \\
\] (31)

which is the normalization factor of PSSTS. When \(r = 0\) corresponding to photon-subtracted thermal state, Eq. (31) just reduces to \(C_{s,m} = m! \sinh^{2m} \theta\) [29].

Using the same procession as obtaining Eq. (26), the PND of PSSTS is given by

\[
\mathcal{P}_a(n) = C_{s,m}^{-1} \langle 0(\beta) | S_{1}^{\dagger} a^{\dagger m} | n \rangle \langle n | a^m S_{1} | 0(\beta) \rangle \\
= \frac{1}{n!} C_{s,m}^{-1} \langle 0(\beta) | S_{1}^{\dagger} e^{a^\dagger a n \tau} S_{1} | 0(\beta) \rangle \\
= \frac{1}{n!} C_{s,m}^{-1} \langle 0(\beta) | S_{1}^{\dagger} e^{a^\dagger a n \tau} S_{1} | 0(\beta) \rangle \\
= \frac{1}{n!} C_{s,m}^{-1} \langle 0(\beta) | S_{1}^{\dagger} e^{a^\dagger a n \tau} S_{1} | 0(\beta) \rangle \\
= \frac{1}{n!} C_{s,m}^{-1} \langle 0(\beta) | S_{1}^{\dagger} e^{a^\dagger a n \tau} S_{1} | 0(\beta) \rangle \\
\] (32)

so \((k = m + n)\)

\[
\sum_{k=0}^{\infty} \frac{\tau^k}{k!} n! C_{s,m} \mathcal{P}_a(n) = \langle 0(\beta) | S_{1}^{\dagger} e^{a^\dagger a n \tau} S_{1} | 0(\beta) \rangle \\
= R.H.S of \ (22), \\
\] (33)

which leads to the PND of PSSTS

\[
\mathcal{P}_a(n) = \frac{(m + n)!}{n! C_{s,m} \sqrt{A}} (C/A)^{m+n/2} P_{m+n} \left( B/\sqrt{AC} \right), \\
\] (34)

a Legendre polynomial, which is same as the result of Ref. [30].

VI. CONCLUSION

In this paper, we present a new concise approach for normalizing \(m\)-photon-added (-subtracted) STS and for deriving the PNDs, which improve the method used in Refs. [28] [30]. That is to say, using the thermal field dynamics theory, we convert the thermal state to a pure state in doubled Fock space in which the calculations of ensemble averages under a mixed state \(\rho\), \((A) = tr(\rho A)/tr(\rho)\) is replaced by an equivalent expectation
values with a pure state $|0(\beta)\rangle$, i.e., $\langle A \rangle = \langle 0(\beta) | A | 0(\beta) \rangle$. It is shown that the average value of $e^{f_a a^\dagger}$ under STS is just the generating function of Legendre polynomials, a remarkable result. Based on this point, the normalization and PNDs of $m$-photon added (or subtracted) STS are easily obtained as the Legendre polynomials. The generating function of the Legendre polynomials and the average value of $e^{f_a a^\dagger}$ under STS are used in the whole calculation.

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