Littlewood’s fourth principle

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Abstract

In Real Analysis, Littlewood’s three principles are known as heuristics that help teach the essentials of measure theory and reveal the analogies between the concepts of topological space and continuous function on one side and those of measurable space and measurable function on the other one. They are based on important and rigorous statements, such as Lusin’s and Egoroff-Severini’s theorems, and have ingenious and elegant proofs. We shall comment on those theorems and show how their proofs can possibly be made simpler by introducing a fourth principle. These alternative proofs make even more manifest those analogies and show that Egoroff-Severini’s theorem can be considered the natural generalization of the classical Dini’s monotone convergence theorem.

1 Introduction.

John Edenson Littlewood (9 June 1885 - 6 September 1977) was a British mathematician. In 1944, he wrote an influential textbook, Lectures on the Theory of Functions ([7]), in which he proposed three principles as guides for working in real analysis; these are heuristics to help teach the essentials of measure theory, as Littlewood himself wrote in [7]:

The extent of knowledge [of real analysis] required is nothing like so great as is sometimes supposed. There are three principles, roughly expressible in the following terms: every (measurable) set is nearly a finite sum of intervals; every function (of class $L^1$) is nearly continuous; every convergent sequence is nearly uniformly convergent. Most of the results of the present section are fairly intuitive applications of these ideas, and the student armed with them should be equal to most occasions when real variable theory is called for. If one of the principles would be the obvious means to settle a problem if it were “quite” true, it is natural to ask if the “nearly” is near enough, and for a problem that is actually soluble it generally is.

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To benefit our further discussion, we shall express Littlewood’s principles and their rigorous statements in forms that are slightly different from those originally stated.

The first principle descends directly from the very definition of (Lebesgue) measurability of a set.

**First Principle.** *Every measurable set is nearly closed.*

The second principle relates the measurability of a function to the more familiar property of continuity.

**Second Principle.** *Every measurable function is nearly continuous.*

The third principle connects the pointwise convergence of a sequence of functions to the standard concept of uniform convergence.

**Third Principle.** *Every sequence of measurable functions that converges pointwise almost everywhere is nearly uniformly convergent.*

These principles are based on important theorems that give a rigorous meaning to the term “nearly”. We shall recall these in the next section along with their ingenious proofs that give a taste of the standard arguments used in Real Analysis.

In Section 3, we will discuss a **fourth principle** that associates the concept of finiteness of a function to that of its boundedness.

**Fourth Principle.** *Every measurable function that is finite almost everywhere is nearly bounded.*

In the mathematical literature (see [1], [2], [7], [9], [10], [11], [13]), the proof of the second principle is based on the third; it can be easily seen that the fourth principle can be derived from the second.

However, we shall see that the fourth principle can also be proved independently; this fact makes possible a proof of the second principle without appealing for the third, that itself can be derived from the second, by a totally new proof based on Dini’s monotone convergence theorem.

As in [7], to make our discussion as simple as possible, we shall consider the Lebesgue measure $m$ for the real line $\mathbb{R}$; then in Section 4 we shall hint at how the four principles and their rigorous counterparts can be extended to more general contexts.

## 2 The three principles

We recall the definitions of inner and outer measure of a set $E \subseteq \mathbb{R}$: they are respectively $^3$

$$m_i(E) = \sup \{ |K| : K \text{ is compact and } K \subseteq E \},$$
$$m_e(E) = \inf \{ |A| : A \text{ is open and } A \supseteq E \},$$

$^3$The number $|K|$ is the infimum of the total lengths of all the finite unions of open intervals that contain $K$. Accordingly, $|A|$ is the supremum of the total lengths of all the finite unions of closed intervals contained in $A$. 


It always holds that \( m_i(E) \leq m_e(E) \). The set \( E \) is (Lebesgue) measurable if and only if \( m_i(E) = m_e(E) \); when this is the case, the measure of \( E \) is \( m(E) = m_i(E) = m_e(E) \); thus \( m(E) \in [0, \infty] \) and it can be proved that \( m \) is a measure on the \( \sigma \)-algebra of Lebesgue measurable subsets of \( \mathbb{R} \), as specified in Section \( \text{III} \).

By the properties of the supremum, it is easily seen that, for any pair of subsets \( E \) and \( F \) of \( \mathbb{R} \), \( m_e(E \cup F) \leq m_e(E) + m_e(F) \) and \( m_e(E) \leq m_e(F) \) if \( E \subseteq F \).

The first principle is a condition for the measurability of subsets of \( \mathbb{R} \).

**Theorem 1** (First Principle). Let \( E \subset \mathbb{R} \) be a set of finite outer measure.

Then, \( E \) is measurable if and only if for every \( \varepsilon > 0 \) there exist two sets \( K \) and \( F \), with \( K \) closed (compact), \( K \cup F = E \) and \( m_e(F) < \varepsilon \).

This is what is meant for nearly closed.

**Proof.** If \( E \) is measurable, for any \( \varepsilon > 0 \) we can find a compact set \( K \subseteq E \) and an open set \( A \supseteq E \) such that

\[
m(K) > m(E) - \varepsilon/2 \quad \text{and} \quad m(A) < m(E) + \varepsilon/2.
\]

The set \( A \setminus K \) is open and contains \( E \setminus K \). Thus, by setting \( F = E \setminus K \), we have \( E = K \cup F \) and

\[
m_e(F) \leq m(A) - m(K) < \varepsilon.
\]

Vice versa, for every \( \varepsilon > 0 \) we have:

\[
m_e(E) = m_e(K \cup F) \leq m_e(K) + m_e(F) < m(K) + \varepsilon \leq m_i(E) + \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, then \( m_e(E) \leq m_i(E) \). \( \square \)

The second and third principles concern measurable functions from (measurable) subsets of \( \mathbb{R} \) to the extended real line \( \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} \), that is functions are allowed to have values \( +\infty \) and \( -\infty \).

Let \( f : E \to \overline{\mathbb{R}} \) be a function defined on a measurable subset \( E \) of \( \mathbb{R} \). We say that \( f \) is measurable if the level sets defined by

\[
L(f, t) = \{ x \in E : f(x) > t \}
\]

are measurable subsets of \( \mathbb{R} \) for every \( t \in \mathbb{R} \). It is easy to verify that if we replace \( L(f, t) \) with \( L^*(f, t) = \{ x \in E : f(x) \geq t \} \) we have an equivalent definition.

Since the countable union of measurable sets is measurable, it is not hard to show that the pointwise infimum and supremum of a sequence of measurable functions \( f_n : E \to \overline{\mathbb{R}} \) are measurable functions as well as the function defined for any \( x \in E \) by

\[
\limsup_{n \to \infty} f_n(x) = \inf_{k \geq 1} \sup_{n \geq k} f_n(x).
\]

Since the countable union of sets of measure zero has measure zero and the difference between \( E \) and any set of measure zero is measurable, the same definitions and conclusions hold even if the functions \( f \) and \( f_n \) are defined almost...
that is if the subsets of $E$ in which they are not defined has measure zero.

As already mentioned, the third principle is needed to prove the second and is known as Egoroff’s theorem or Egoroff-Severini’s theorem.

**Theorem 2** (Third Principle; Egoroff-Severini). Let $E \subset \mathbb{R}$ be a measurable set with finite measure and let $f : E \to \mathbb{R}$ be measurable and finite a.e. in $E$.

The sequence of measurable functions $f_n : E \to \mathbb{R}$ converges a.e. to $f$ in $E$ for $n \to \infty$ if and only if, for every $\varepsilon > 0$, there exists a closed set $K \subseteq E$ such that $m(E \setminus K) < \varepsilon$ and $f_n$ converges uniformly to $f$ on $K$.

This is what we mean for nearly uniformly convergent.

**Proof.** If $f_n \to f$ a.e. in $E$ as $n \to \infty$, the subset of $E$ in which $f_n \to f$ pointwise has the same measure as $E$; hence, without loss of generality, we can assume that $f_n(x)$ converges to $f(x)$ for any $x \in E$.

Consider the functions defined by

$$g_n(x) = \sup_{k \geq n} |f_k(x) - f(x)|, \quad x \in E$$

and the sets

$$E_{n,m} = \left\{ x \in E : g_n(x) < \frac{1}{m} \right\} \quad \text{for } n, m \in \mathbb{N}. \quad (2.2)$$

Observe that, if $x \in E$, then $g_n(x) \to 0$ as $n \to \infty$ and hence for any $m \in \mathbb{N}$

$$E = \bigcup_{n=1}^{\infty} E_{n,m}.$$

As $E_{n,m}$ is increasing with $n$, the monotone convergence theorem implies that $m(E_{n,m})$ converges to $m(E)$ for $n \to \infty$ and for any $m \in \mathbb{N}$. Thus, for every $\varepsilon > 0$ and $m \in \mathbb{N}$, there exists an index $\nu = \nu(\varepsilon, m)$ such that $m(E \setminus E_{\nu,m}) < \varepsilon/2^{m+1}$.

The measure of the set $F = \bigcup_{m=1}^{\infty} (E \setminus E_{\nu,m})$ is arbitrary small, in fact

$$m(F) \leq \sum_{m=1}^{\infty} m(E \setminus E_{\nu,m}) < \varepsilon/2.$$

Also, since $E \setminus F$ is measurable, by Theorem there exists a compact set $K \subseteq E \setminus F$ such that $m(E \setminus F) - m(K) < \varepsilon/2$, and hence

$$m(E \setminus K) = m(E \setminus F) + m(F) - m(K) < \varepsilon.$$
Since $K \subseteq E \setminus F = \bigcap_{m=1}^{\infty} E_{\nu(m),m}$ we have that

$$|f_n(x) - f(x)| < \frac{1}{m} \quad \text{for any} \quad x \in K \quad \text{and} \quad n \geq \nu(m),$$

by the definitions of $E_{\nu,m}$ and $g_n$; this means that $f_n$ converges uniformly to $f$ on $K$ as $n \to \infty$.

Viceversa, if for every $\varepsilon > 0$ there is a closed set $K \subseteq E$ with $m(E \setminus K) < \varepsilon$ and $f_n \to f$ uniformly on $K$, then by choosing $\varepsilon = 1/m$ we can say that there is a closed set $K_m \subseteq E$ such that $f_n \to f$ uniformly on $K_m$ and $m(E \setminus K_m) < 1/m$.

Therefore, $f_n(x) \to f(x)$ for any $x$ in the set $F = \bigcup_{m=1}^{\infty} K_m$ and

$$m(E \setminus F) = m\left(\bigcap_{m=1}^{\infty} (E \setminus K_m)\right) \leq m(E \setminus K_m) < \frac{1}{m} \quad \text{for any} \quad m \in \mathbb{N},$$

which implies that $m(E \setminus F) = 0$. Thus, $f_n \to f$ a.e. in $E$ as $n \to \infty$.

The second principle corresponds to Lusin’s theorem (see [8],[5]) that we state here in a form similar to Theorems 1 and 2.

**Theorem 3** (Second Principle; Lusin). Let $E \subseteq \mathbb{R}$ be a measurable set with finite measure and let $f : E \to \mathbb{R}$ be finite a.e. in $E$.

Then, $f$ is measurable in $E$ if and only if, for every $\varepsilon > 0$, there exists a closed set $K \subseteq E$ such that $m(E \setminus K) < \varepsilon$ and the restriction of $f$ to $K$ is continuous.

This is what we mean for nearly continuous.

The proof of Lusin’s theorem is done by approximation by simple functions. A simple function is a measurable function that has a finite number of real values. If $c_1, \ldots, c_n$ are the distinct values of a simple function $s$, then $s$ can be conveniently represented as

$$s = \sum_{j=1}^{n} c_j \chi_{E_j},$$

where $\chi_{E_j}$ is the characteristic function of the set $E_j = \{x \in E : s(x) = c_j\}$. Notice that the $E_j$’s form a covering of $E$ of pairwise disjoint measurable sets.

Simple functions play a crucial role in Real Analysis; this is mainly due to the following result of which we shall omit the proof.

**Theorem 4** (Approximation by Simple Functions). Let $E \subseteq \mathbb{R}$ be a measurable set and let $f : E \to [0, +\infty]$ be a measurable function.

Then, there exists an increasing sequence of non-negative simple functions $s_n$ that converges pointwise to $f$ in $E$ for $n \to \infty$.

Moreover, if $f$ is bounded, then $s_n$ converges to $f$ uniformly in $E$.

We can now give the proof of Lusin’s theorem.

\[N. N. Lusin or Luzin was a student of Egoroff. For biographical notes on Egoroff and Lusin see [6].\]
Proof. Any measurable function $f$ can be decomposed as $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$ are measurable and non-negative functions. Thus, we can always suppose that $f$ is non-negative and hence, by Theorem 4 it can be approximated pointwise by a sequence of simple functions.

We first prove that a simple function $s$ is nearly continuous. Since the sets $E_j$ defining $s$ are measurable, if we fix $\varepsilon > 0$ we can find closed subsets $K_j$ of $E_j$ such that $m(E_j \setminus K_j) < \varepsilon/n$ for $j = 1, \ldots, n$. The union $K$ of the sets $K_j$ is also a closed set and, since the $E_j$’s cover $E$, we have that $m(E \setminus K) < \varepsilon$. Since the closed sets $K_j$ are pairwise disjoint (as the $E_j$’s are pairwise disjoint) and $s$ is constant on $K_j$ for all $j = 1, \ldots, n$, we conclude that $s$ is continuous in $K$.

Now, if $f$ is measurable and non-negative, let $s_n$ be a sequence of simple functions that converges pointwise to $f$ and fix an $\varepsilon > 0$.

As the $s_n$’s are nearly continuous, for any natural number $n$, there exists a closed set $K_n \subseteq E$ such that $m(E \setminus K_n) < \varepsilon/2^{n+1}$ and $s_n$ is continuous in $K_n$. By Theorem 2 there exists a closed set $K_0 \subseteq E$ such that $m(E \setminus K_0) < \varepsilon/2$ and $s_n$ converges uniformly to $f$ in $K_0$ as $n \to \infty$. Thus, in the set

$$K = \bigcap_{n=0}^{\infty} K_n$$

the functions $s_n$ are all continuous and converge uniformly to $f$. Therefore $f$ is continuous in $K$ and

$$m(E \setminus K) = m\left(\bigcup_{n=0}^{\infty} (E \setminus K_n)\right) \leq \sum_{n=0}^{\infty} m(E \setminus K_n) < \varepsilon.$$

Viceversa, if $f$ is nearly continuous, fix an $\varepsilon > 0$ and let $K$ be a closed subset of $E$ such that $m(E \setminus K) < \varepsilon$ and $f$ is continuous in $K$. For any $t \in \mathbb{R}$, we have:

$$L^*(f, t) = \{x \in K : f(x) \geq t\} \cup \{x \in E \setminus K : f(x) \geq t\}.$$

The former set in this decomposition is closed, as the restriction of $f$ to $K$ is continuous, while the latter is clearly a subset of $E \setminus K$ and hence its outer measure must be less than $\varepsilon$. By Theorem 1 $L^*(f, t)$ is measurable (for any $t \in \mathbb{R}$), which means that $f$ is measurable. 

3 The fourth principle

We shall now present alternative proofs of Theorems 2 and 3. They are based on a fourth principle, that corresponds to the following theorem.

Theorem 5 (Fourth Principle). Let $E \subset \mathbb{R}$ be a measurable set with finite measure and let $f : E \to \mathbb{R}$ be a measurable function.

Then, $f$ is finite a.e. in $E$ if and only if, for every $\varepsilon > 0$, there exists a closed set $K \subseteq E$ such that $m(E \setminus K) < \varepsilon$ and $f$ is bounded on $K$.

This is what we mean for nearly bounded.

Proof. If $f$ is finite a.e., we have that

$$m(\{x \in E : |f(x)| = \infty\}) = 0.$$
As $f$ is measurable, $|f|$ is also measurable and so are the sets

$$L(|f|, n) = \{x \in E : |f(x)| > n\}, \quad n \in \mathbb{N}.$$  

Observe that the sequence of sets $L(|f|, n)$ is decreasing and

$$\bigcap_{n=1}^{\infty} L(|f|, n) = \{x \in E : |f(x)| = \infty\}.$$  

As $m(L(|f|, 1)) \leq m(E) < \infty$, we can apply the (downward) monotone convergence theorem and infer that

$$\lim_{n \to \infty} m(L(|f|, n)) = m(\{x \in E : |f(x)| = \infty\}) = 0.$$  

Thus, if we fix $\varepsilon > 0$, there is an $n_\varepsilon \in \mathbb{N}$ such that $m(L(|f|, n_\varepsilon)) < \frac{\varepsilon}{2}$. Also, we can find a closed subset $K$ of the measurable set $E \setminus L(|f|, n_\varepsilon)$ such that $m(E \setminus L(|f|, n_\varepsilon)) - m(K) < \frac{\varepsilon}{2}$. Finally, since $K \subseteq E \setminus L(|f|, n_\varepsilon)$, $|f|$ is obviously bounded by $n_\varepsilon$ on $K$ and

$$m(E \setminus K) = m(E \setminus L(|f|, n_\varepsilon)) + m(L(|f|, n_\varepsilon) \setminus K) < \varepsilon.$$  

Vice versa, if $f$ is nearly bounded, then for any $n \in \mathbb{N}$ there exists a closed set $K_n \subseteq E$ such that $m(E \setminus K_n) < 1/n$ and $f$ is bounded (and hence finite) in $K_n$. Thus, $\{x \in E : |f(x)| = \infty\} \subseteq E \setminus K_n$ for any $n \in \mathbb{N}$, and hence

$$m(\{x \in E : |f(x)| = \infty\}) \leq \lim_{n \to \infty} m(E \setminus K_n) = 0,$$

that is $f$ is finite a.e.. \qed

**Remark 6.** Notice that this theorem can also be derived from Theorem 3. In fact, without loss of generality, the closed set $K$ provided by Theorem 5 can be taken to be compact and hence, $f$ is surely bounded on $K$, being continuous on a compact set.

More importantly for our aims, Theorem 5 enables us to prove Theorem 3 without using Theorem 2.

**Alternative proof of Lusin’s theorem.** The proof runs similarly to that presented in Section 2. If $f$ is measurable, without loss of generality, we can assume that $f$ is non-negative and hence $f$ can be approximated pointwise by a sequence of simple functions $s_n$, which we know are nearly continuous. Thus, for any $\varepsilon > 0$, we can still construct the sequence of closed subsets $K_n$ of $E$ such that $m(E \setminus K_n) < \varepsilon/2^{n+1}$ and $s_n$ is continuous in $K_n$.

Now, as $f$ is finite a.e., Theorem 5 implies that it is nearly bounded, that is we can find a closed subset $K_0$ of $E$ in which $f$ is bounded and $m(E \setminus K_0) < \varepsilon/2$. We apply the second part of the Theorem 4 and infer that $s_n$ converges uniformly to $f$ in $K_0$. As seen before, we conclude that $f$ is continuous in the intersection $K$ of all the $K_n$’s, because in $K$ it is the uniform limit of the sequence of continuous functions $s_n$. As before $m(E \setminus K) < \varepsilon$.

The reverse implication remains unchanged. \qed

In order to give our alternative proof of Theorem 2 we need to recall a classical result for sequences of continuous functions.
**Theorem 7** (Dini). Let $K$ be a compact subset of $\mathbb{R}$ and let be given a sequence of continuous functions $f_n : K \to \mathbb{R}$ that converges pointwise and monotonically in $K$ to a function $f : K \to \mathbb{R}$.

If $f$ is also continuous, then $f_n$ converges uniformly to $f$.

*Proof*. We shall prove the theorem when $f_n$ is monotonically increasing.

For each $n \in \mathbb{N}$, set $h_n = f - f_n$; as $n \to \infty$ the continuos functions $h_n$ decrease pointwise to 0 on $K$.

Fix $\varepsilon > 0$. The sets $A_n = \{x \in K : h_n(x) < \varepsilon\}$ are open, since the $h_n$’s are continuous; also, $A_n \subseteq A_{n+1}$ for every $n \in \mathbb{N}$, since the $h_n$’s decrease; finally, the $A_n$’s cover $K$, since the $h_n$ converge pointwise to 0.

By the compactness, $K$ is then covered by a finite number $m$ of the $A_n$’s, which means that $A_m = K$ for some $m \in \mathbb{N}$. This implies that $|f(x) - f_n(x)| < \varepsilon$ for all $n \geq m$ and $x \in K$, as desired.

**Remark 8.** The conclusion of Theorem 7 still holds true if we assume that the sequence of $f_n$’s is increasing (respectively decreasing) and $f$ and all the $f_n$’s are lower (respectively upper) semicontinuous.

Now, Theorem 2 can be proved by appealing for Theorems 3 and 7.

*Alternative proof of Egoroff’s theorem*. As in the classical proof of this theorem, we can always assume that $f_n(x) \to f(x)$ for every $x \in E$.

Consider the functions and sets defined in (2.1) and (2.2), respectively. We shall first show that there exists an $\nu \in \mathbb{N}$ such that $g_n$ is nearly bounded for every $n \geq \nu$. In fact, as already observed, since $g_n \to 0$ pointwise in $E$ as $n \to \infty$, we have that

$$E = \bigcup_{n=1}^{\infty} E_{n,1},$$

and the $E_{n,1}$’s increase with $n$. Hence, if we fix $\varepsilon > 0$, there is a $\nu \in \mathbb{N}$ such that $m(E \setminus E_{\nu}) < \varepsilon / 2$. Since $E_{\nu}$ is measurable, by Theorem 1 we can find a closed subset $K$ of $E_{\nu}$ such that $m(E_{\nu} \setminus K) < \varepsilon / 2$.

Therefore, $m(E \setminus K) < \varepsilon$ and for every $n \geq \nu$

$$0 \leq g_n(x) \leq g_\nu(x) < 1, \quad \text{for any } x \in K.$$

Now, being $g_n$ nearly bounded in $E$ for every $n \geq \nu$, the alternative proof of Theorem 3 implies that $g_n$ is nearly continuous in $E$, that is for every $n \geq \nu$ there exists a closed subset $K_n$ of $E$ such that $m(E \setminus K_n) < \varepsilon / 2^{n-\nu+1}$ and $g_n$ is continuous on $K_n$. The set

$$K = \bigcap_{n=\nu}^{\infty} K_n$$

is closed, $m(E \setminus K) < \varepsilon$ and on $K$ the functions $g_n$ are continuos for any $n \geq \nu$ and monotonically decrease to 0 as $n \to \infty$.

By Theorem 7, the $g_n$’s converge to 0 uniformly on $K$. This means that the $f_n$’s converge to $f$ uniformly on $K$ as $n \to \infty$.

The reverse implication remains unchanged.  

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*We say that $f$ is lower (respectively upper) semicontinuous if the level sets $\{x \in E : f(x) > t\}$ (respectively $\{x \in E : f(x) < t\}$) are open for every $t \in \mathbb{R}$. 

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Remark 9. Egoroff’s theorem can be considered, in a sense, as the natural substitute of Dini’s theorem, in case the monotonicity assumption is removed. In fact, notice that the sequence of the $g_n$’s defined in (2.1) is decreasing; however, the $g_n$’s are in general no longer upper semicontinuous (they are only lower semicontinuous) and Dini’s theorem (even in the form described in Remark 8) cannot be applied. In spite of that, the $g_n$’s remain measurable if the $f_n$’s are so.

4 Extensions.

Of course, all the proofs presented in Sections 2 and 3 work if we replace the real line $\mathbb{R}$ by an Euclidean space of any dimension.

Theorems 2, 3 and 5 can also be generalized replacing $\mathbb{R}$ by a general measure space not necessarily endowed with a topology.

We recall that a measure space is a triple $(X, \mathcal{M}, \mu)$. Here, $X$ is any set; $\mathcal{M}$ is a $\sigma$-algebra in $X$, that is $\mathcal{M}$ is a collection of subsets of $X$ that contains $X$ itself, the complement in $X$ of any set $E \in \mathcal{M}$, and any countable union of sets $E_n \in \mathcal{M}$ (the elements of $\mathcal{M}$ are called measurable sets); $\mu$ is a function from $\mathcal{M}$ to $[0, \infty]$ which is countably additive, that is such that

$$\mu \left( \bigcup_{n=1}^{\infty} E_n \right) = \sum_{n=1}^{\infty} \mu(E_n),$$

for any sequence of pairwise disjoint sets $E_n \in \mathcal{M}$.

It descends from the definition that a measure $\mu$ is monotone, that is $\mu(E) \leq \mu(F)$ if $E, F \in \mathcal{M}$ and $E \subseteq F$. Another crucial property of a measure is the monotone convergence theorem: the sequence $\mu(E_n)$ converges to

$$\mu \left( \bigcup_{n=1}^{\infty} E_n \right) \quad \text{if } E_n \subseteq E_{n+1} \text{ for any } n \in \mathbb{N}, \text{ or to}$$

$$\mu \left( \bigcap_{n=1}^{\infty} E_n \right) \quad \text{if } E_n \supseteq E_{n+1} \text{ for any } n \in \mathbb{N} \text{ and } \mu(E_1) < \infty.$$

In this general environment, Theorems 2, 3 and 5 can be extended simply by replacing closed sets by measurable sets; the proofs run similarly.

Theorem 10. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$.

(i) (Egoroff-Severini) A sequence of measurable functions $f_n : X \to \mathbb{R}$ converges a.e. in $X$ to a measurable and finite a.e. function $f : X \to \mathbb{R}$ if and only if, for every $\varepsilon > 0$, there exists a measurable subset $E$ of $X$ such that $\mu(X \setminus E) < \varepsilon$ and $f_n$ converges uniformly to $f$ on $E$.

(ii) (Lusin) A finite a.e. function $f : X \to \mathbb{R}$ is measurable in $X$ if and only if, for every $\varepsilon > 0$, there exists a measurable subset $E$ of $X$ such that $\mu(X \setminus E) < \varepsilon$ and the restriction of $f$ to $E$ is continuous.

(iii) (Fourth Principle) A measurable function $f : X \to \mathbb{R}$ is finite a.e. if and only if, for every $\varepsilon > 0$, there exists a measurable subset $E$ of $X$ such that $\mu(X \setminus E) < \varepsilon$ and $f$ is bounded on $E$.
Stated in this forms, the second, third and fourth principles elude the necessity of a First Principle that, of course, needs the presence of a topological space \((X, \tau)\) and the definition of a suitable outer measure on \(X\).

We recall that on any set \(X\) an outer measure \(\mu_e\) can be defined as a function on the power set \(\mathcal{P}(X)\) with values in \([0, +\infty]\), which is monotone, countably subadditive and such that \(\mu(\emptyset) = 0\). Carathéodory’s extension theorem (see [13]) then states that one can always find a \(\sigma\)-algebra \(\mathcal{M}\) in \(X\) (the \(\sigma\)-algebra of the so-called \(\mu_e\)-measurable sets) on which \(\mu_e\) is actually a measure (that is \(\mu_e\) is countably additive). Also, Carathéodory’s criterion (see [5]) states that, if \(\mu_e\) is a Carathéodory measure\(^7\) on a metric space \((X, d)\), then the \(\sigma\)-algebra of \(\mu_e\)-measurable sets contains that of the Borel sets\(^8\) and hence all the compact sets.

Whenever a First Principle is valid for a metric space \((X, d)\), the statements (classical and alternative) and proofs of Theorems 2, 3 and 5 simply hold by replacing \(\mathbb{R}\) by \(X\) and \(m\) by \(\mu_e\).

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\(^7\)That is, \(\mu_e(E \cup F) = \mu_e(E) + \mu_e(F)\) for any choice of sets \(E\) and \(F\) such that \(d(E, F) > 0\).

\(^8\)That is the smallest \(\sigma\)-algebra that contains the topology in \((X, d)\).
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