H-FUNCTIONAL AND MATSUSHIMA TYPE DECOMPOSITION THEOREM

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Abstract. The H-functional characterizes Kähler-Ricci solitons as its critical points, and also plays an important role of the existence problem for Kähler-Einstein metrics. In this paper we prove the Hessian formula for the H-functional at its critical points, and use this to obtain a new proof of Matsushima type decomposition theorem for holomorphic vector fields on Fano manifolds admitting Kähler-Ricci solitons.

1. Introduction

As a generalization of Kähler-Einstein metrics for Fano manifolds, Kähler-Ricci solitons were discussed by many experts. Indeed they are known to have deep relations with limit behavior of solutions of geometric PDE for finding Kähler-Einstein metrics. For instance the Kähler-Ricci flow equation and the complex Monge-Ampère equation.

In his paper [12], W. He introduced a functional on the space of Kähler metrics, called the H-functional, which characterizes Kähler-Ricci solitons as its critical points. Let $X$ be an $n$-dimensional Fano manifold and let $\omega \in 2\pi c_1(X)$ be a reference Kähler metric. We denote the volume $\int_X \omega^n$ of $X$ by $V$. Let

$$M(\omega) := \left\{ \phi \in C^\infty(X, \mathbb{R}) \mid \omega_{\phi} := \omega + \sqrt{-1} \partial \bar{\partial} \phi > 0 \right\}$$

be the space of Kähler metrics in $[\omega]$. Its tangent space $T_{\phi} M(\omega)$ is $C^\infty(X, \mathbb{R})$. The Ricci potential $f_{\phi}$ for $\phi \in M(\omega)$ is a smooth function on $X$ such that

$$\text{Ric}(\omega_{\phi}) - \omega_{\phi} = \sqrt{-1} \partial \bar{\partial} f_{\phi} \quad \text{and} \quad \int_X e^{f_{\phi}} \omega^n_{\phi} = V.$$ (1)

Recall a Kähler metric $\omega_{\phi} = \sqrt{-1} g_{ij} dz^i \wedge d\bar{z}^j$ is a Kähler-Ricci soliton if the gradient vector field $g^{ij} \frac{\partial f_{\phi}}{\partial \bar{z}^j} \frac{\partial}{\partial z^i}$ of the Ricci potential is holomorphic. The H-functional is then defined as

$$H(\phi) = \int_X f_{\phi} e^{f_{\phi}} \omega^n_{\phi}.$$ 

Following [12], the H-functional is the H-entropy in probability theory applied to the probability measure $V^{-1} e^{f_{\phi}} \omega^n_{\phi}$ with respect to the probability measure $V^{-1} \omega^n_{\phi}$. Thus it is shown that $H(\phi) \geq 0$ for any $\phi \in M(\omega)$, with equality only if $\phi$ defines a Kähler-Einstein metric, that is, $f_{\phi} = 0$. See [12] for more details. More generally, critical points of the

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H-functional define Kähler-Ricci solitons (See also section 3). W. He further showed that there exists a numerical invariant \(N\) of \((X, [\omega])\) such that \(H(\phi) \geq N\) for any \(\phi \in \mathcal{M}(\omega)\), with equality only if \(\phi\) defines a Kähler-Ricci soliton. See also \([3]\) for an algebro-geometric interpretation of the infimum of the H-functional. Thus the H-functional can be regarded as an analogue of the Calabi functional \([1]\) which characterizes Extremal Kähler metrics for compact Kähler manifolds as its critical points.

The purpose of this paper is to obtain the Hessian formula for the H-functional at its critical points, and use that formula to obtain Matsushima type decomposition theorem for holomorphic vector fields on Fano manifolds admitting Kähler-Ricci solitons. In \([18]\), L. Wang explained how to get the Hessian formula for the Calabi functional \([1]\) and extends that formula to a functional of the norm squared of a moment map in the finite dimensional setting of the framework of moment maps. Note that, following the formal picture of geometric invariant theory for polarized compact Kähler manifolds of Fujiki \([7]\) and Donaldson\([5]\), the Calabi functional is the norm squared of a moment map on the corresponding moduli space. In \([19]\), X. Wang explained how to get the Matsushima type decomposition theorem \([2]\) for holomorphic vector fields on compact Kähler manifolds admitting Extremal Kähler metrics in the finite dimensional setting of the framework of moment maps. Our proofs of results are based on the above formal arguments. It is interesting that, in spite of the fact that the H-functional is not the norm squared functional of a moment map, the formal arguments as in \([18, 19]\) give rigorous proofs of results.

We fix more notation to state results precisely. We denote the negative Laplacian of \(\phi \in \mathcal{M}(\omega)\) by \(\Delta_{\phi}\). We define an operator on \(C^\infty(X, \mathbb{C})\) as

\[
L_{\phi}u = -\Delta_{\phi}u - \langle \partial u, \overline{\partial f_{\phi}} \rangle - u + \frac{1}{V} \int_X ue^{f_{\phi}}\omega^n_{\phi}.
\]

We also define its complex conjugate operator by \(\overline{L_{\phi}}u := \overline{L_{\phi}\overline{u}}\), that is,

\[
\overline{L_{\phi}}u = -\Delta_{\phi}u - \langle \partial u, \partial f_{\phi} \rangle - u + \frac{1}{V} \int_X ue^{f_{\phi}}\omega^n_{\phi}.
\]

We define a weighted inner product on \(C^\infty(X, \mathbb{C})\) by

\[
\langle \langle u, v \rangle \rangle = \int_X u\overline{v} e^{f_{\phi}}\omega^n_{\phi}.
\]

The following is the Hessian formula of the H-functional.

**Theorem 1.1.** The Hessian of the H-functional at every Kähler-Ricci soliton \(\phi\) along variations \(\delta\phi_1, \delta\phi_2 \in T_{\phi}\mathcal{M}(\omega)\) is given by

\[
\text{Hess}(H)(\delta\phi_1, \delta\phi_2) = \langle \langle L_{\phi} \overline{L_{\phi}} \delta\phi_1, \delta\phi_2 \rangle \rangle = \langle \langle \overline{L_{\phi}} L_{\phi} \delta\phi_1, \delta\phi_2 \rangle \rangle.
\]

As the result, operators \(L_{\phi}\) and \(\overline{L_{\phi}}\) are commutative at every Kähler-Ricci soliton \(\phi\).
A similar result of the Hessian formula of the H-functional is given by F. T. Fong [6] by a long tensor calculation. The virtue of the above theorem is that the proof is quite short, and that the result links directly to the Matsushima type decomposition theorem.

Let $\mathfrak{h}(X)$ be the Lie algebra of holomorphic vector fields on $X$. For any $u \in C^\infty(X, \mathbb{C})$, we define the gradient vector field $\text{grad}_\phi u$ for a Kähler metric $\phi \in \mathcal{M}(\omega)$ by

$$i_{\text{grad}_\phi u} \omega_\phi = \sqrt{-1} \partial u.$$ 

As an application of Theorem 1.1, we obtain the Matsushima type decomposition theorem for $\mathfrak{h}(X)$.

**Theorem 1.2.** Let $X$ be a Fano manifold admitting a Kähler-Ricci soliton $\phi \in \mathcal{M}(\omega)$. Then the semi-direct sum decomposition

$$\mathfrak{h}(X) = \bigoplus_{\lambda \geq 0} \mathfrak{h}_\lambda(X)$$

holds, where $\mathfrak{h}_\lambda(X)$ is the $\lambda$-eigenspace of the adjoint action of $-\text{grad}_\phi f_\phi$. Furthermore $\mathfrak{h}_0(X)$ is the complexification of the Lie algebra of Killing vector fields on $(X, \omega_\phi)$. In particular $\mathfrak{h}_0(X)$ is reductive.

When $\text{grad}_\phi f_\phi = 0$, this theorem is a classical result of Matsushima [15] which is well-known as an obstruction to the existence of Kähler-Einstein metrics. Theorem 1.2 was first proved by Tian-Zhu [17] (see also [13, 14]) by a similar calculation of the twisted Laplacian as in [15]. Our new proof based on the finite dimensional setting of the framework of moment maps is a linear algebraic argument using only the commutativity of $L_\phi$ and $T_\phi$.

The reductivity of the identity component of the automorphism group preserving the holomorphic vector field $\text{grad}_\phi f_\phi$ in Theorem 1.2 has a crucial role in the recent work of Darvas and Rubinstein [11] on the implication from the existence of a Kähler-Ricci soliton to the coercivity of an energy functional.

Similar results for the Hessian formula for the norm squared functional of a moment map and the Matsushima type decomposition theorem of holomorphic vector fields were observed for various situations. See for instance, [9] (for a Calabi type functional and the perturbed extremal Kähler metric), [10] (for a Calabi type functional and the $f$-extremal Kähler metric), [11] (for the norm squared functional of the Cahen-Gutt moment map and the closed Fedosov star product) and [16] (for the Ricci-Calabi functional and the Mabuchi’s generalized Kähler Einstein metric).

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2. Foundations of operators \(L_\phi\) and \(\overline{L}_\phi\)

In this section we refer to Futaki’s book \[8, Section 2.4\] for fundamental properties of operators \(L_\phi\) and \(\overline{L}_\phi\).

**Lemma 2.1.** For any \(\phi \in \mathcal{M}(\omega)\), operators \(L_\phi\) and \(\overline{L}_\phi\) are self-adjoint and non-negative with respect to the inner product \(\langle \cdot, \cdot \rangle\).

**Proof.** It is crucial that the operator \(u \mapsto \Delta_\phi u + \langle \overline{\partial} u, \overline{\partial} f_\phi \rangle\) is the negative Laplacian with respect to the inner product \(\langle \cdot, \cdot \rangle\) on \(C^\infty(X)\). Indeed, the self-adjointness follows from the following integration by parts:

\[
\langle \Delta_\phi u - \langle \overline{\partial} u, \overline{\partial} f_\phi \rangle, v \rangle = \int_X \left( \langle \overline{\partial} u, \overline{\partial} (v e^{f_\phi}) \rangle - \langle \overline{\partial} u, \overline{\partial} (e^{f_\phi}) \rangle \right) \omega^n_\phi
\]

\[
= \int_X \langle \overline{\partial} u, \overline{\partial} v \rangle e^{f_\phi} \omega^n_\phi
\]

\[
= \int_X \left( \langle \overline{\partial} (ue^{f_\phi}), \overline{\partial} v \rangle - \langle \overline{\partial} e^{f_\phi}, \overline{\partial} v \rangle \right) \omega^n_\phi
\]

\[
= \langle u, -\Delta_\phi v - \langle \overline{\partial} v, \overline{\partial} f_\phi \rangle \rangle.
\]

On the other hand, the non-negativity follows from the following Bochner type formula \[8, Proof of Theorem 2.4.3\]:

\[
\int_X |\Delta_\phi u + \langle \overline{\partial} u, \overline{\partial} f_\phi \rangle|^2 e^{f_\phi} \omega^n_\phi = \int_X |\nabla_1 \nabla_3 u|^2 e^{f_\phi} \omega^n_\phi + \int_X |\overline{\partial} u|^2 e^{f_\phi} \omega^n_\phi.
\]

Indeed we can see the first eigenvalue of the operator \(u \mapsto -\Delta_\phi u - \langle \overline{\partial} u, \overline{\partial} f_\phi \rangle\) is greater than or equal to 1. This complete the proof. \(\square\)

In view of Theorem 1.1 we can see the Hessian of the H-functional is non-negative at Kähler-Ricci solitons. However we shall not use this fact in this paper.

The following lemma tells us that non-trivial kernels of the operator \(L_\phi\) have a geometric interpretation on Fano manifolds.

**Lemma 2.2.** For any \(\phi \in \mathcal{M}(\omega)\), the kernel \(\ker L_\phi/\mathbb{C}\) (modulo additive constants) of the operator \(L_\phi\) is identified with the space of holomorphic vector fields \(\mathfrak{h}(X)\) on \(X\) as the vector space through \(u \mapsto \nabla_\phi u\).

**Proof.** The kernel \(\ker L_\phi/\mathbb{C}\) is identified with \\(\{ u \in C^\infty(X, \mathbb{C}) \mid \Delta_\phi u + \langle \overline{\partial} u, \overline{\partial} f_\phi \rangle = -u \} \) through \(u \mapsto u - \frac{1}{\nu} \int_X ue^{f_\phi} \omega^n_\phi\). By \[8, Theorem 2.4.3\], this is identified with \(\mathfrak{h}(X)\) through \(u \mapsto \nabla_\phi u\). \(\square\)

3. The Hessian formula for the H-functional

We first give the first variation of the H-functional.
Lemma 3.1. (c.f. [12, Proposition 2.2]) For any $\phi \in \mathcal{M}(\omega)$ and any variation $\delta \phi \in T_\phi \mathcal{M}(\omega)$, we have

$$\delta H(\delta \phi) = \langle L_\phi(f_\phi), \delta \phi \rangle = \langle L_\phi(f_\phi), \delta \phi \rangle.$$ 

Proof. Note that the first variation of the Ricci potential is given by

$$\delta f_\phi = -\Delta_\phi \delta \phi - \delta \phi + \frac{1}{V} \int_X \delta \phi e^{f_\phi} \omega^n_\phi.$$ 

Indeed, the variation of the first equation in (1) shows

$$\delta f_\phi = -\Delta_\phi \delta \phi - \delta \phi + C$$

for some constant $C$, and this constant is determined by the variation $\int_X (\delta f_\phi + \Delta \delta \phi) e^{f_\phi} \omega^n_\phi = 0$ of the second equation in (1).

By noting the integration by parts

$$\int_X (\Delta_\phi \delta \phi) e^{f_\phi} \omega^n_\phi = \int_X \delta \phi (\Delta_\phi f_\phi + \langle \overline{\partial f_\phi}, \overline{\partial f_\phi} \rangle) f_\phi \omega^n_\phi,$$

we then have

$$\delta H(\delta \phi) = \int_X \delta f_\phi e^{f_\phi} \omega^n_\phi + f_\phi \delta f_\phi e^{f_\phi} \omega^n_\phi + f_\phi e^{f_\phi} (\Delta_\phi \delta \phi) \omega^n_\phi$$

$$= \int_X - (\Delta_\phi \delta \phi) e^{f_\phi} \omega^n_\phi + f_\phi (\delta \phi + \frac{1}{V} \int_X \delta \phi e^{f_\phi} \omega^n_\phi) e^{f_\phi} \omega^n_\phi$$

$$= \int_X \delta \phi (-\Delta_\phi f_\phi - \langle \overline{\partial f_\phi}, \overline{\partial f_\phi} \rangle - f_\phi + \frac{1}{V} \int_X f_\phi e^{f_\phi} \omega^n_\phi) e^{f_\phi} \omega^n_\phi$$

$$= \langle L_\phi(f_\phi), \delta \phi \rangle.$$ 

Similarly we have $\delta H(\delta \phi) = \langle L_\phi(f_\phi), \delta \phi \rangle.$ \hfill $\Box$

We can see that a critical point of the H-functional defines a Kähler Ricci soliton by the above lemma and Lemma 2.2.

We now prove Theorem 1.1.

Proof of Theorem 1.1. We must compute the variation $(\delta L_\phi)(f_\phi)$ at a Kähler-Ricci soliton $\phi \in \mathcal{M}(\omega)$ since

$$\text{Hess}(H)(\delta \phi_1, \delta \phi_2) = \langle \delta_1(L_\phi(f_\phi)), \delta \phi_2 \rangle = \langle (\delta_1 L_\phi)(f_\phi) + L(\delta_1 f_\phi), \delta \phi_2 \rangle,$$

where $\delta_1$ stands the variation along $\delta \phi_1 \in T_\phi \mathcal{M}(\omega)$. Now the gradient vector field $Z := \text{grad}_\phi f_\phi$ is holomorphic. Since we can see that

$$i_Z(\omega_\phi + t \sqrt{-1} \partial \overline{\partial} \delta \phi) = \sqrt{-1} \partial(\delta f_\phi + t Z(\delta \phi))$$

any small $t \in (-\varepsilon, \varepsilon)$, this holomorphic vector field $Z$ can be re-written as $\text{grad}_{\phi+t\delta \phi}(f_\phi + tZ(\delta \phi))$. Therefore Lemma 2.2 shows $L_{\phi+t\delta \phi}(f_\phi + tZ(\delta \phi)) = 0$. Taking derivative at $t = 0$, we have

$$(\delta L_\phi)(f_\phi) = -L_\phi(Z(\delta \phi)) = -L_\phi(\partial \delta \phi, \partial f_\phi)$$
to conclude
\[
(\delta L_\phi)(f_\phi) + L_\phi(\delta f_\phi) = L_\phi \left( -\langle \partial \delta \phi, \partial f_\phi \rangle + -\Delta_\phi \delta \phi - \delta \phi + \frac{1}{V} \int_X \delta \phi e^{f_\phi} \omega_\phi^\flat \right) \\
= L_\phi \overline{L_\phi}(\delta \phi).
\]
Similarly \( \delta(\overline{L_\phi}(f_\phi)) = \overline{L_\phi}L_\phi(\delta \phi) \). This completes the proof. \( \square \)

4. The Matsushima type decomposition theorem

As an application of theorem 1.1, we now prove the Matsushima type decomposition theorem for holomorphic vector fields on \( X \) admitting a Kähler-Ricci soliton \( \phi \in \mathcal{M}(\omega) \). We only use the commutativity of operators \( L_\phi \) and \( \overline{L_\phi} \) in the following proof.

\textbf{Proof of Theorem 1.2.} The commutativity of operators \( L_\phi \) and \( \overline{L_\phi} \) for a Kähler-Ricci soliton \( \phi \in \mathcal{M}(\omega) \) shows that the operator \( \overline{L_\phi} \) maps \( \text{Ker} L_\phi \) to itself. We thus have the eigenspace decomposition \( \text{Ker} L_\phi = \sum_{\lambda \geq 0} E_\lambda \), where \( \lambda \geq 0 \) by Lemma 2.1. Let us define \( h_\lambda(X) := \text{grad}_\phi(E_\lambda) \). Then Lemma 2.2 gives the decomposition \( h(X) = \sum_{\lambda \geq 0} h_\lambda(X) \). We can see that the subspace \( h_\lambda(X) \) is the \( \lambda \)-eigenspace of the adjoint action of \( -\text{grad}_\phi f_\phi \).

Indeed, for any \( u \in E_\lambda \), we have
\[
\lambda u = \overline{L} u \\
= (\overline{L} - L) u \\
= -\langle \partial u, \partial f_\phi \rangle + \langle \overline{\partial u}, \overline{\partial f_\phi} \rangle \\
= \{ f_\phi, u \},
\]
where \( \{\cdot, \cdot\} \) is the Poisson bracket. It then follows that
\[
\lambda \text{grad}_\phi u = [-\text{grad}_\phi f_\phi, \text{grad}_\phi u].
\]

We next see the structure of \( h_0(X) \). Note that \( E_0 = \text{Ker} L \cap \text{Ker} \overline{L} \). If \( u \in E_0 \), then both the real part and the imaginary part of \( u \) are in \( E_0 \). It then follows that
\[
E_0 = \left\{ u \in \sqrt{-1} C^\infty_\mathbb{R}(X) \mid \text{grad}_\phi u \in h(X) \right\} \otimes \mathbb{C},
\]
and consequently
\[
h_0(X) = \left\{ \text{grad}_\phi u + \overline{\text{grad}_\phi u} \mid u \in \sqrt{-1} C^\infty_\mathbb{R}(X) \text{ and } \text{grad}_\phi u \in h(X) \right\} \otimes \mathbb{C}.
\]
By [8, Lemma 2.3.8], this is the complexification of the Lie algebra of Killing vector fields on \( X \). This completes the proof. \( \square \)
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