MOUNTAIN PASS ENERGIES BETWEEN HOMOTOPY CLASSES OF MAPS

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Abstract. For non-homotopic maps \( u, v \in C^\infty(M, N) \) between closed Riemannian manifolds, we consider the smallest energy level \( \gamma_p(u, v) \) for which there exist paths \( u_t \in W^{1,p}(M, N) \) connecting \( u_0 = u \) to \( u_1 = v \) with \( \|du_t\|_{L^p}^p \leq \gamma_p(u, v) \). When \( u \) and \( v \) are \((k-2)\)-homotopic, work of Hang and Lin shows that \( \gamma_p(u, v) < \infty \) for \( p \in [1, k) \), and using their construction, one can obtain an estimate of the form \( \gamma_p(u, v) \leq C(u,v)^{k-p} \).

When \( M \) and \( N \) are oriented, and \( u \) and \( v \) induce different maps on real cohomology in degree \( k-1 \), we show that the growth \( \gamma_p(u, v) \sim \frac{1}{k-p} \) as \( p \to k \) is sharp, and obtain a lower bound for the coefficient \( \lim \inf_{p \to k} (k-p)^{\gamma_p(u, v)} \) in terms of the min-max masses of certain non-contractible loops in the space of codimension-\( k \) integral cycles in \( M \).

In the process, we establish lower bounds for a related smaller quantity \( \gamma^*_p(u, v) \leq \gamma_p(u, v) \), for which there exist critical points \( u_p \in W^{1,p}(M, N) \) of the \( p \)-energy functional satisfying \( \gamma_p^*(u, v) \leq \|du_p\|_{L^p}^p \leq \gamma_p(u, v) \).

1. Introduction

Let \( M^n \) and \( N \) be closed Riemannian manifolds, with \( N \subset \mathbb{R}^L \) isometrically embedded in some higher-dimensional Euclidean space. For \( p \geq 1 \), the space \( W^{1,p}(M, N) \) of Sobolev maps from \( M \) to \( N \) is defined by

\[
W^{1,p}(M, N) := \{ u \in W^{1,p}(M, \mathbb{R}^L) \mid u(x) \in N \text{ for almost every } x \in M \}.
\]

In \[19\], Hang and Lin characterized the path components of \( W^{1,p}(M, N) \) in the low regularity setting \( p \in [1, n] \), showing that two maps \( u \) and \( v \) lie in the same component of \( W^{1,p}(M, N) \) if and only if they are \([p-1]\)-homotopic—i.e., if their restrictions to the \([p-1]\)-skeleton of a generic triangulation of \( M \) are homotopic in the usual sense. The results of \[19\] build on earlier work of White, who showed in \[39\] that maps in \( W^{1,p}(M, N) \) have well-defined \([p-1]\)-homotopy classes that are closed under bounded weak convergence, and deduced that each \([p-1]\)-homotopy class contains a representative minimizing the \( p \)-energy

\[
E_p(u) := \int_M |du|^p.
\]

As a corollary of the results in \[19\], Hang and Lin confirm a conjecture of Brezis and Li (\[19\], Conjecture 2), which posits that, as \( p \) varies in \([1, n]\), the spaces \( W^{1,p}(M, N) \) undergo a “change of topology”—i.e., a change in the number of path components—only when \( p \) passes through integer thresholds.
The results of the present paper are motivated by a desire to obtain a more quantitative understanding of this “change of topology” phenomenon, with the aim of gaining new insight into how the topology of \( M \) and \( N \) influences the variational landscape of the \( p \)-energy functionals.

Fix two maps \( u,v \in C^\infty(M,N) \), and let \( 2 \leq k \leq n = \dim M \) be the largest integer such that \( u \) and \( v \) are homotopic on the \((k-2)\)-skeleton of some—hence, any (see [19], Section 2.2)—triangulation of \( M \). By the results of [19] and [39], we then see that \( u \) and \( v \) lie in a common path component of \( W^{1,p}(M,N) \) if and only if \( p < k \). For \( p \) close to \( k \), we would like to characterize those energy levels \( c > 0 \) for which the maps \( u \) and \( v \) lie in a common path component of the energy sublevel set

\[
E^c_p := \{ w \in W^{1,p}(M,N) \mid \|dw\|_{L^p} \leq c \}.
\]

That is, we are interested in estimating the mountain-pass energies

\[
\gamma_p(u,v) := \inf\{ c > 0 \mid \exists \text{ a path } u_t \in E^c_p \text{ connecting } u \text{ to } v \};
\]

in the limit \( p \to k \) where the change of topology occurs.

First, we observe that a careful examination of the path constructed by Hang and Lin in the proof of ([19], Theorem 1.1) yields the following upper bound. (Since these estimates are not explicitly addressed in [19], we provide a proof in Section 6.2 of the appendix.)

**Theorem 1.1.** There exists a constant \( C = C(u,v) < \infty \) such that

\[
(1.2) \quad \gamma_p(u,v) \leq \frac{C}{k - p}
\]

for every \( p \in [1,k) \).

Our main result shows that, when \( M \) and \( N \) are oriented, and \( u \) and \( v \) induce different maps on the real cohomology \( u^*, v^* : H^{k-1}(N;\mathbb{R}) \to H^{k-1}(M;\mathbb{R}) \), the growth \( \gamma_p(u,v) \sim \frac{1}{k-p} \) is in fact optimal, with an explicit lower bound on the coefficient

\[
\liminf_{p \to k} (k-p)\gamma_p(u,v).
\]

In fact, we establish a lower bound for a possibly smaller quantity \( \gamma_p^*(u,v) \leq \gamma_p(u,v) \), defined roughly as the smallest energy level \( c > 0 \) for which \( u \) and \( v \) can be connected by sequences \( u = u_0, u_1, \ldots, u_{r-1}, u_r = v \) in the sub-level set \( E^c_p \) for which adjacent maps \( u_i, u_{i+1} \) are arbitrarily close in \( L^p \) norm. (See Section 4.3 below for a careful definition.)

To state the lower bound precisely, we briefly introduce some relevant notation (to be defined in greater detail in Sections 2-4). First, we denote by \( A^{k-1}(N) \) the space of closed \((k-1)\)-forms on \( N \) with the property that

\[
\langle \alpha, \Sigma \rangle \in \mathbb{Z}
\]
for every integral \((k-1)\)-cycle \(\Sigma\) in \(N\), and for \(\alpha \in \mathcal{A}^{k-1}(N)\), we use \(S_\alpha(v) - S_\alpha(u)\) to denote the dual \((n+1-k)\)-current associated to \(v^*(\alpha) - u^*(\alpha)\) by

\[
\langle S_\alpha(v) - S_\alpha(u), \zeta \rangle := \int_M (v^*(\alpha) - u^*(\alpha)) \wedge \zeta \quad \text{for} \quad \zeta \in \Omega^{n+1-k}(M).
\]

Next, we recall from Almgren’s dissertation [5] that there exists an isomorphism

\[
\Phi : \pi_1(\mathcal{Z}_m(M; \mathbb{Z}), \{0\}) \to H_{m+1}(M; \mathbb{Z})
\]

relating loops in the space of integral \(m\)-cycles (with the flat topology) to integral \((m+1)\)-homology classes in \(M\). In Section 4.1 below, we define for each \(\xi \in H_{m+1}(M; \mathbb{Z})\) a min-max width \(L_m(\xi) > 0\), which corresponds roughly to the min-max mass

\[
\inf \{ \sup_{t \in S^1} M(\gamma(t)) \mid \gamma : S^1 \to \mathcal{Z}_m(M; \mathbb{Z}), \ \Phi(\gamma) = \xi \}
\]

associated to the class \(\Phi^{-1}(\xi) \in \pi_1(\mathcal{Z}_m(M; \mathbb{Z}), \{0\})\). For any real homology class \(\overline{\xi} \in H_{m+1}(M; \mathbb{R})\) that can be represented by integral cycles, we then define

\[
L_{m,R}(\overline{\xi}) := \min \{L_m(\xi) \mid \xi \in H_{m+1}(M; \mathbb{Z}), \ \xi \equiv \overline{\xi} \text{ in } H_{m+1}(M; \mathbb{R}) \}.
\]

Our main theorem then reads as follows.

**Theorem 1.2.** For any \(\alpha \in \mathcal{A}^{k-1}(N)\) and maps \(u, v \in C^\infty(M, N)\) such that

\[
[u^*(\alpha) - v^*(\alpha)] \neq 0 \in H_{dR}^{k-1}(M),
\]

there is a constant \(\lambda(\alpha) < \infty\) such that

\[
\lambda(\alpha) \frac{k}{k-1} \lim_{p \to k} \inf (k-p)^\gamma_p(u, v) \geq \sigma_{k-1} L_{m-k,R}(|[S_\alpha(v) - S_\alpha(u)]|).
\]

**Remark 1.3.** Here, \(\sigma_{k-1}\) denotes the \((k-1)\)-volume of the standard unit \((k-1)\)-sphere. The definition of \(\lambda(\alpha)\) is given in Section 3.1 for now, but we only remark that \(\lambda(\alpha)\) is easy to estimate for specific choices of target \(N\) and \(\alpha \in \mathcal{A}^{k-1}(N)\). When \(N = S^{k-1}\) is the standard \((k-1)\)-sphere and \(\alpha = \frac{1}{\sigma_{k-1}}\), for example, one has \(\lambda(\alpha) = \frac{1}{(k-1)^{\frac{1}{k-1}}} \).

Though the details of the proof are somewhat delicate, the intuition underlying Theorem 1.2 is relatively straightforward. For any map \(w \in W^{1,p}(M, N), \ p \in (k-1, k)\), the pullback \(w^*(\alpha)\) is well-defined as a \((k-1)\)-form with coefficients in \(L^1\); as in (14), Section 5.4.2, we can then define an \((n-k)\)-current \(T_\alpha(w)\) corresponding to the distributional exterior derivative of \(w^*(\alpha)\). In Section 3 we develop a compactness theory for the so-called homological singularities \(T_\alpha(w_p)\) for families of maps with \((k-p)E_p(w_p) \leq \Lambda\) as \(p \to k\) (based largely on ideas from the \(\Gamma\)-convergence results of [2] and [25] for functionals of Ginzburg-Landau type), showing that the currents \(T_\alpha(w_p)\) converge subsequentially in \((C^1)^k\) to an integral \((n-k)\)-cycle, whose mass we can bound explicitly in terms of the limiting energy \(\liminf_{p \to k} (k-p)E_p(w_p)\).
(In fact, a much more careful description of the convergence of $T_\alpha(w_p)$ is necessary for our applications.)

To an $L^p$-fine sequence $u = u_0, u_1, \ldots, u_r = v$ of maps with $p$ close to $k$ and energy bounded above by $\gamma_p^*(u, v) + \epsilon$, we can then associate a family of integral $(n-k)$-cycles $0 = T_0, T_1, \ldots, T_r = 0$, with mass bounded in terms of $(k-p)\gamma_p^*(u, v)$. Using results of Sections 2.2 and 3 and some additional technical lemmas, we then show that the difference of adjacent cycles $T_i - T_{i-1} = \partial S_i$ for integral $(n+1-k)$-currents $S_i$ of small mass, such that

$$[\Sigma_{i=1}^r S_i] = S_\alpha(v) - S_\alpha(u) \text{ in } H_{n+1-k}(M; \mathbb{R}).$$

The conclusion of Theorem 1.2 follows from these observations.

We suspect that the lower bound

$$(1.4) \quad \liminf_{p \to k}(k-p)\gamma_p^*(u, v) > 0$$

holds for non-$(k-1)$-homotopic maps $u, v \in C^\infty(M, N)$ in much greater generality. If $N$ is simply connected and $k > 2$, for example, one might try to show this by associating to each path $u_t \in W^{1,p}(M, N)$ from $u$ to $v$ a loop of “topological singularities” $T(u_t)$ given by flat $(n-k)$-cycles with coefficients in $\pi_{k-1}(N)$, with limiting behavior as $p \to k$ similar to that described in Section 3 for the homological singularities considered here. The non-contractibility of the resulting loops in $Z_{n-k}(M; \pi_{k-1}(N))$ may then be related to the nontriviality of classical cohomological obstructions to extending a given $(k-2)$-homotopy to a $(k-1)$-homotopy (see, e.g., [22], Sections VI.4 and VI.8). If the target $N$ is $(k-2)$-connected and $\pi_1(N)$ is abelian, if $k = 2$—the analysis of topological singularities pursued by Pakzad-Rivière [30] and Canévar-Olndari [10] may provide a useful starting point for investigations in this direction. For general $N$ and $k$, however, addressing these questions will require the introduction of some new machinery.

**Remark 1.4.** In the case $N = S^1$, we observe that the general statement (1.4) follows immediately from Theorem 1.2 since the homotopy classes $[u] \in [M : S^1]$ are determined by the pullback $[u^*(\alpha)] \in H^1_{dR}(M)$ of the generator $\alpha = \frac{1}{2\pi}d\theta \in A^1(S^1)$ of $H^1_{dR}(S^1)$. In this case, Theorem 1.2 is closely related to questions raised by the author in [33] concerning the mountain-pass energies for complex Ginzburg-Landau functionals $E_\epsilon : W^{1,2}(M, \mathbb{C}) \to \mathbb{R}$ over paths in $W^{1,2}(M, \mathbb{C})$ connecting distinct classes in $[M : S^1]$. On a two-dimensional annulus $\Omega$, these questions had previously been studied by Almeida [3, 4], who obtained precise estimates for the $E_\epsilon$-mountain pass energies separating the components of $W^{1,2}(\Omega, S^1)$ in $W^{1,2}(\Omega, \mathbb{C})$. As discussed in [4], these results have some physical relevance, since the presence of high energy walls separating local minimizers of Ginzburg-Landau energies on $\Omega$ is related to the appearance of permanent currents in superconducting cylinders.

Finally, in Section 5 we apply standard mountain pass arguments to the generalized Ginzburg-Landau functionals $E_{p, \epsilon}$ studied by Wang in [37].
to demonstrate the existence of critical points for the $p$-energy functional with energy lying between $\gamma_p^*(u,v)$ and $\gamma_p(u,v)$. In particular, we obtain the following result.

**Theorem 1.5.** If $u, v \in C^\infty(M,N)$ are $(k-2)$-homotopic, $p \in (k-1,k)$, and

$$\gamma_p^*(u,v) > \max\{E_p(u), E_p(v)\},$$

then there exists a stationary $p$-harmonic map $w \in W^{1,p}(M,N)$ with

$$\gamma_p^*(u,v) \leq E_p(w) \leq \gamma_p(u,v).$$

As a consequence, whenever (1.4) holds—as it does if $u$ and $v$ induce different maps on real cohomology, by Theorem 1.2—we obtain for some $q = q(u,v) \in (k-1,k)$ a family of stationary $p$-harmonic maps

$$(q,k) \ni p \mapsto u_p \in W^{1,p}(M,N)$$

whose energy grows like $\frac{1}{k-p}$ as $p \to k$. In the case $N = S^1$, the asymptotic behavior as $p \to 2$ of stationary $p$-harmonic maps with $E_p(u_p) = O(\frac{1}{k-p})$ was studied by the author in [36], with the conclusion that the singular sets $\text{Sing}(u_p)$ converge subsequentially—in the Hausdorff sense—to the support of a stationary, rectifiable $(n-2)$-varifold in $M$. Though we expect the situation for general $N$ and $k$ to be considerably more complicated, it would likewise be interesting to understand the asymptotic behavior of stationary $p$-harmonic maps $u_p \in W^{1,p}(M,N)$ with $E_p(u_p) = O(\frac{1}{k-p})$ as $p \to k$.

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2. Preliminaries

2.1. Cubeulations, Slicing, and Retraction to Skeleta.

Throughout the paper, we will make repeated use of the fact that any compact Riemannian manifold $M$ admits a cubeulation (see, for instance, [38]). That is, we can find an $n$-dimensional cubical complex $K$ whose faces are isometric to $[-1,1]^n$, and a bi-Lipschitz map $h : |K| \to M$ from the underlying space $|K|$ onto $M$. A key tool in the study of Sobolev maps (see, e.g., [38], [39], [7], [19]) is the ability to “slice” a given Sobolev function $f$ by the skeleta of a well-chosen cubeulation $h : |K| \to M$, in such a way that the composition $f \circ h$ is well-behaved on every lower-dimensional skeleton $|K^j|$ of $K$.

To make this idea precise, we review some notation from [19]. For an $n$-dimensional cubical complex $K$ and a Lipschitz map $f \in Lip(|K|, \mathbb{R})$, define the $W^{1,p}(K)$ norm of $f$ by

$$\|f\|^p_{W^{1,p}(K)} := \sum_{\sigma \in K} \int_{\sigma} (|f|^p + |df|^p) d\mathcal{H}^{\dim(\sigma)},$$

and denote by $\mathcal{W}^{1,p}(K,\mathbb{R})$ the completion of $Lip(|K|,\mathbb{R})$ with respect to this norm. Note that the functions in $\mathcal{W}^{1,p}(K,\mathbb{R})$ then lie in the usual Sobolev space $W^{1,p}(\Delta,\mathbb{R})$ for every cell $\Delta \in K$ of any dimension. Following arguments from Section 3 of [19] (see also the proof of Lemma 2.2 in [18]), we obtain the following slicing lemma for Sobolev functions.

**Lemma 2.1.** Given a closed Riemannian manifold $M$, there exists a constant $C(M) < \infty$ and for each $\delta \in (0,1]$, there exists a cubical complex $K_\delta$ whose $n$-cells are isometric to $[-\delta,\delta]^n$, such that for any $f \in W^{1,p}(M,\mathbb{R})$, we can find a cubulation $h : |K_\delta| \to M$ for which $f \circ h \in W^{1,p}(K_\delta,\mathbb{R})$, satisfying

$$\text{Lip}(h) + \text{Lip}(h^{-1}) \leq C, \quad (2.2)$$

$$\int_{|K_\delta^j|} |f \circ h|^p \, d\mathcal{H}^j \leq C \delta^{j-n} \int_M |f|^p, \quad (2.3)$$

and

$$\int_{|K_\delta^j|} |d(f \circ h)|^p \, d\mathcal{H}^j \leq C \delta^{j-n} \int_M |df|^p, \quad (2.4)$$

for every $0 \leq j \leq n$.

**Remark 2.2.** In Section 6.1 of the appendix, we indicate how the methods of Sections 3 and 4 of [19] can be used to find such cubulations. It is useful to note that the complexes $K_\delta$ are obtained (up to minor rescalings) by subdividing the faces of an initial unit-size complex $K = K_1$: in particular, it follows that the maximum number $\nu_j(K_\delta)$ of $j$-cells containing a given $(j-1)$-cell as a face is fixed independent of $\delta$. As a consequence, whenever we have an estimate of the form $\int_{\sigma} f_1 \leq \int_{\partial \sigma} f_2$ for every $j$-cell $\sigma \in K_\delta$, we can sum over all $j$-cells to obtain an estimate of the form $\int_{|K_\delta^j|} f_1 \leq C \int_{|K_\delta^{j-1}|} f_2$, where the constant $C$ is independent of $\delta$.

Let $K_\delta$ be an $n$-dimensional cubical complex as in the conclusion of Lemma 2.1. For every $j \leq n$ and $s \in [0,\delta]$, we define the map $\phi_{j,s} : |K^j| \to |K^j|$ by identifying each $j$-cell $\sigma \in K^j$ with $[-\delta,\delta]^j$, and setting

$$\phi_{j,s}(x) := \frac{\delta \cdot x}{\max\{s,|x|_{\infty}\}}. \quad (2.5)$$

(Here we use the notation $|x|_{\infty} := \max_{1 \leq i \leq n} |x_i|$.) The family $\phi_{j,s}$ then interpolates between the identity at $s = \delta$ and a retraction to the $(j-1)$-skeleton at $s = 0$. For $1 < p < j$, the pullbacks $\phi_{j,s}^* f = f \circ \phi_{j,s}$ then define endomorphisms of $W^{1,p}(K^j_\delta,\mathbb{R})$, whose properties we record in the following lemma. (Compare [19], Sections 4 and 5.)

**Lemma 2.3.** For $1 < p < j \leq n$ and $f \in L^\infty \cap W^{1,p}(K^j_\delta,\mathbb{R})$, the family $[0,\delta] \ni s \mapsto \phi_{j,s}^* f$ of pullbacks is a continuous path in $W^{1,p}(K^j_\delta,\mathbb{R})$, for
which

\[ \int_{|K^j_{\delta}|} |d(\phi^s_{j,s}f)|^p \leq C(M) \left( \frac{\delta}{j-p} \int_{|K^j_{\delta-1}|} |df|^p + \int_{|K^j_{\delta}|} |df|^p \right) \]

and

\[ \int_{|K^j_{\delta}|} |f - \phi^s_{j,s}f|^p \leq C(M) \delta^p \left( \frac{\delta}{j-p} \int_{|K^j_{\delta-1}|} |df|^p + \int_{|K^j_{\delta}|} |df|^p \right). \]

**Proof.** Since the restriction of \( \phi_{j,s} \) to the \((j-1)\)-skeleton \(|K^j_{\delta-1}|\) is given by the identity map for all \(s\), it is enough to show that \( \phi^s_{j,s}f \) defines a continuous path in \( W^{1,p}((\sigma,\mathbb{R}) \) on each \( j \)-cell \( \sigma \sim [\delta,\delta] \), and to establish \((2.5)\) and \((2.6)\), it is enough to show (per Remark 2.2) that the estimates

\[ \int_{\sigma} |d(\phi^s_{j,s}f)|^p \leq C(M) \delta^p \left( \frac{\delta}{j-p} \int_{\partial\sigma} |df|^p + \int_{\sigma} |df|^p \right) \]

and

\[ \int_{\sigma} |f - \phi^s_{j,s}f|^p \leq C(M) \delta^p \left( \frac{\delta}{j-p} \int_{\partial\sigma} |df|^p + \int_{\sigma} |df|^p \right) \]

hold on every \( j \)-cell \( \sigma \).

To see this, note that, by the scale-invariance of \((2.5)\) and \((2.6)\), it is enough to establish continuity and the desired estimates in the case \( \delta = 1 \). Moreover, by virtue of the usual bi-Lipschitz correspondence

\[ [-1,1]^j \to B^j_1, \quad x \mapsto \frac{|x|}{|x|} x \]

between the unit cube and the unit ball, we can identify the maps \( \phi^s_{j,s}f \) with the maps \( f_s \in W^{1,p}(B^j_1, \mathbb{R}) \) given by

\[ f_s(x) := f(x/\max\{s,|x|\}), \quad s \in [0,1]. \]

It is then straightforward to check that

\[ \int_{B_1} |df_s|^p = \int_1^{|x| \rightarrow \frac{|x|}{|x|}} \left( \int_0^{\delta-1-p} |df|^{p} + \int_{B_1} |df|^p \right), \]

from which \((2.7)\) follows immediately. And since \( f_s = f \) on \( \partial B_1 \), the \( L^p \) estimate \((2.8)\) follows from \((2.7)\) and the \( L^p \) Poincaré inequality (see, e.g., [11], Section 4.5.2).

To see the continuity of \( s \mapsto f_s \) in \( W^{1,p}(B_1) \), we appeal first to the fact that \( f \in L^\infty \) and the bounded convergence theorem to conclude that \( s \mapsto f_s \) is a continuous path in \( L^p \). And since we see from \((2.9)\) that the \( p \)-energy \( s \mapsto E_p(f_s) \) is a continuous function of \( s \), \( f_s \) must give a continuous path in \( W^{1,p}(B_1) \) as well.

\[ \square \]
Next, we define the retraction maps $\Phi_j : |K_\delta| \rightarrow |K_\delta^j|$, sending almost every point in the cubical complex into the $(j - 1)$-skeleton, by
\begin{equation}
\Phi_j := \phi_{j,0} \circ \cdots \circ \phi_{n,0}.
\end{equation}
By definition of the radial retractions $\phi_{j,0}$, we observe that $\Phi_j$ is locally Lipschitz away from an $(n - j)$-dimensional set, called the dual $(n - j)$-skeleton $L^{n-j}$ to $K$.

Note that the dual skeleton $L^{n-j}$ can be expressed as the union (disjoint except for a set of dimension $(n - j - 1)$) over all $j$-cells $\sigma \in K$ of the sets
\begin{equation}
P(\sigma) := \Phi_{j+1}^{-1}(a_\sigma),
\end{equation}
where $a_\sigma$ is the center of $\sigma$; and each $P(\sigma)$, in turn, can be decomposed into the union of the intersections
\[P(\sigma) \cap \Delta\]
over all $n$-cells $\Delta$ containing $\sigma$ as a $j$-face. Identifying a given $n$-cell $\Delta \in K_\delta$ with $[-\delta, \delta]^n$ and the $j$-cell $\sigma$ with $[-\delta, \delta]^j \times \{(\delta, \ldots, \delta)\}$, we see that $P(\sigma) \cap \Delta$ is given by the box $\{0\} \times [0, \delta]^{n-j}$. Since, per Remark 2.2, the number of $n$-cells intersecting a given $j$-cell is bounded independent of $\delta$ for the cubeulations $K_\delta$ of Lemma 2.1, it follows in this case that
\begin{equation}
H^{n-j}(P(\sigma)) \leq C(M)\delta^{n-j}
\end{equation}
for every $j$-cell $\sigma \in K_\delta$.

Remark 2.4. Observe also that, if $\Delta \in K_\delta$ is an $n$-face of $K_\delta$, $\sigma$ is a $j$-cell in $\partial \Delta$, and $f : S^{j-1} \rightarrow \Delta \setminus L^{n-j}$ is an embedding of the $(j - 1)$-sphere $S^{j-1}$ that links with $P(\sigma)$ (e.g., sending $S^{j-1}$ to the sphere $S^{j-1}$ of radius $\delta/2$), then $f$ can be homotoped through $\Delta \setminus L^{n-j}$ to a homeomorphism with $\partial \sigma$.

As in [19], we will make repeated use of the retraction maps $\Phi_j$ to deform given maps in $W^{1,p}(M,N)$ to ones which are continuous away from $L^{n-j}$. For the arguments of Section 3 in particular, it will be important to have some precise estimates for the maps produced in this way, which we record in the following lemma.

Lemma 2.5. For $f \in W^{1,p}(M,\mathbb{R})$ and $\delta \in (0,1]$, choose a cubulation $h : |K_\delta| \rightarrow M$ as in the conclusion of Lemma 2.1. Then for $k \in \{2,\ldots, n\}$ and $1 < p < k$, the function $\tilde{f} := f|_{|K_\delta^{k-1}|} \circ \Phi_k \circ h^{-1}$ satisfies
\begin{equation}
E_p(\tilde{f}) \leq \frac{C(M)}{k-p} E_p(f)
\end{equation}
and
\begin{equation}
\int_M |f - \tilde{f}|^p \leq \delta^p \frac{C(M)}{k-p} E_p(f).
\end{equation}
Proof. Since $K_\delta$ is chosen according to Lemma 2.1, we know that $f \in W^{1,p}(K_\delta, \mathbb{R})$, and
\[
\int_{|K_\delta^j|} |df|^p \leq C(M)\delta^{j-n}E_p(f)
\]
for every $0 \leq j \leq n$. Since $\tilde{f} = f \circ \phi_{k,0}$ on the $k$-skeleton $|K^k|$, we can then apply the estimates of Lemma 2.3 to conclude that
\[
\int_{|K^k_\delta|} |d\tilde{f}|^p \leq C(M) \left( \frac{\delta}{k-p} \cdot \delta^{k-1-n}E_p(f) + \delta^{k-n}E_p(f) \right)
\]
\[
\leq C'(M) \frac{\delta^{k-n}}{k-p}E_p(f)
\]
and
\[
(2.15) \quad \int_{|K^k_\delta|} |f - \tilde{f}|^p \leq C'(M)\delta^p \cdot \frac{\delta^{k-n}}{k-p}E_p(f).
\]

Next, we can apply the preceding estimates together with the conclusion of Lemma 2.3 on the $(k+1)$-skeleton, to see that $\tilde{f}|_{|K^{k+1}|} = \phi_{k+1,0}^* \phi_{k,0}^* f$ satisfies
\[
\int_{|K^{k+1}_\delta|} |d\tilde{f}|^p \leq C(M) \left( \frac{\delta}{k+1-p} \cdot \frac{\delta^{k-n}}{k-p}E_p(f) + \delta^{k+1-n}E_p(f) \right)
\]
\[
\leq C'(M) \frac{\delta^{k+1-n}}{k-p}E_p(f).
\]
To estimate $|f - \tilde{f}|^p$ on $|K^{k+1}|$, we again apply the scaled $L^p$ Poincaré inequality
\[
\int_{\sigma} |f - \tilde{f}|^p \leq C \left( \delta^p \int_{\sigma} |d(f - \tilde{f})|^p + \delta \int_{\partial\sigma} |f - \tilde{f}|^p \right)
\]
to every $(k+1)$-cell $\sigma = [-\delta, \delta]^{k+1}$ in $K_\delta$, and sum over $\sigma$ (again appealing to Remark 2.2), to obtain
\[
\int_{|K^{k+1}_\delta|} |f - \tilde{f}|^p \leq C(M) \left( \delta^p \int_{|K^{k+1}|} |df|^p + |d\tilde{f}|^p + \delta \int_{|K^k_\delta|} |f - \tilde{f}|^p \right)
\]
\[
\leq C'(M) \left( \delta^p |\delta^{k+1-n}E_p(f) + \frac{\delta^{k+1-n}}{k-p}E_p(f)| + \delta \cdot \delta^p \frac{\delta^{k-n}}{k-p}E_p(f) \right).
\]
In particular, we conclude that
\[
\int_{|K^{k+1}_\delta|} |d\tilde{f}|^p \leq C(M)\frac{\delta^{k+1-n}}{k-p}E_p(f)
\]
and
\[
\int_{|K^{k+1}_\delta|} |f - \tilde{f}|^p \leq C(M)\delta^p \frac{\delta^{k+1-n}}{k-p}E_p(f).
\]
Carrying on by induction on $j$, for each $j$-skeleton $|K_j^j|$ with $j \geq k$, we find that
\[ \int_{|K_j^j|} |d\tilde{f}|^p \leq \frac{C}{k-p} \delta^{j-n} E_p(f) \]
and
\[ \int_{|K_j^j|} |f - \tilde{f}|^p \leq \frac{C}{k-p} \delta^p \delta^{j-n} E_p(f) \]
for every $k \leq j \leq n$; in particular, taking $j = n$, we obtain the desired estimates for $\tilde{f}$.

\[ \Box \]

2.2. Homological Singularities.

In this section, we define the homological singularities of Sobolev maps associated to real cohomology classes in the target manifold $N$. All of the results of this section are contained, with slightly different terminology, in Section 5.4.2 of [14], but we opt for a self-contained treatment more directly suited to the purposes of this paper.

First, we fix some notation from the theory of currents (see [12], [13], or [33] for an introduction). Denote by $\Omega^m(M)$ the space of smooth $m$-forms on a compact manifold $M$, and by $\Omega^m_c(U)$ the space of compactly supported $m$-forms in an open set $U$. Following [13], we use $L^q_m(M)$ to denote the closure of $\Omega^m(M)$ with respect to the $L^q$ norm. The space of general $m$-currents will be denoted by $\mathcal{D}_m(M)$, and for each $T \in \mathcal{D}_m(M)$, following [33], we define the mass $M(T)$ by
\[ M(T) := \sup\{\langle T, \zeta \rangle \mid \zeta \in \Omega^m(M), \|\zeta\|_{L^\infty} \leq 1\}. \]
We use $\mathcal{I}_m(M; \mathbb{Z})$ to denote the space of integer rectifiable $m$-currents, and $\mathcal{Z}_m(M; \mathbb{Z})$ for the subspace of integral $m$-cycles.

Now, let $N$ be a closed, oriented Riemannian manifold. For every integer $1 \leq m \leq \dim N$, denote by $\mathcal{A}^m(N)$ the collection of closed $m$-forms on $N$ satisfying
\begin{equation}
\langle \Sigma, \alpha \rangle \in \mathbb{Z} \text{ for every } \Sigma \in \mathcal{Z}_m(N; \mathbb{Z}).
\end{equation}
Observe that the image of $\mathcal{A}^m(N)$ in de Rham cohomology defines a lattice of full rank in $H^m_{dR}(N)$. Indeed, given integral $m$-cycles $\Sigma_1, \ldots, \Sigma_q$ in $N$ generating $H_m(N; \mathbb{R})$, we can find corresponding cohomology classes $[\alpha_1], \ldots, [\alpha_m] \in H^m_{dR}(N)$ for which $\langle \Sigma_i, \alpha_j \rangle = \delta_{ij}$ (see, e.g., [13], Section 5.4.1). These $\alpha_i$ evidently lie in $\mathcal{A}^m(N)$, and give a basis for $H^m_{dR}(N)$.

We now fix some $k \in \{2, \ldots, \dim N + 1\}$, and $\alpha \in \mathcal{A}^{k-1}(N)$. Appealing to Nash’s embedding theorem, we also fix an isometric embedding $N \subset \mathbb{R}^L$ of $N$ into some Euclidean space. We can then easily extend our $(k-1)$-form $\alpha$ to a compactly supported form
\[ \bar{\alpha} = \Sigma_{|I|=k-1} \tilde{\alpha}_I(x) dx^I \in \Omega^{k-1}_c(\mathbb{R}^L), \]
for instance by taking the pullback \( \pi_N \star \alpha \) of \( \alpha \) to a tubular neighborhood of \( N \) by the nearest-point projection \( \pi_N \), then multiplying by a suitable cut-off function.

Now, let \( M^n \) be a compact, oriented manifold, possibly with boundary. We record next some important estimates for the pullback \( u^*(\alpha) \) of \( \alpha \) by smooth maps \( u \in C^\infty(M, \mathbb{R}^L) \).

**Lemma 2.6.** For \( u, v \in C^\infty(M, \mathbb{R}^L) \), there exist a \((k-1)\)-form \( \beta(u,v) \in \Omega^{k-1}(M) \) and a \((k-2)\)-form \( \eta(u,v) \in \Omega^{k-2}(M) \) such that

\[
(2.17) \quad v^*(\alpha) - u^*(\alpha) = \beta + d\eta,
\]

and the following pointwise estimates hold:

\[
(2.18) \quad |v^*(\alpha) - u^*(\alpha)| \leq C(\alpha)|u - v|(|du|^{k-1} + |dv|^{k-1})
+ C(\alpha)|d\alpha - dv|(|du|^{k-2} + |dv|^{k-2}),
\]

\[
(2.19) \quad |\beta(u,v)| \leq C(\alpha)|u - v|(|du|^{k-1} + |dv|^{k-1}),
\]

and

\[
(2.20) \quad |\eta(u,v)| \leq C(\alpha)|u - v|(|du|^{k-2} + |dv|^{k-2}).
\]

**Remark 2.7.** Here, \( C(\alpha) \) denotes a constant depending on \( \|\alpha\|_{C^1} \).

**Proof.** Write

\[
v^*(\alpha) - u^*(\alpha) := \Sigma_I(\alpha_I(v) dv^I - \alpha_I(u) du^I).
\]

Fixing a multi-index \( I = (i_1, \ldots, i_{k-1}) \), we begin by rearranging

\[
\alpha_I(v) dv^I - \alpha_I(u) du^I = (\alpha_I(v) - \alpha_I(u)) du^I + \alpha_I(u) (dv^I - du^I),
\]

and noting that

\[
|\alpha_I(v) - \alpha_I(u)||dv^I| \leq \|\nabla\alpha\|_{L^\infty}|u - v||dv|^{k-1}.
\]

The first estimate (2.18) follows immediately, and absorbing the terms

\[
(\alpha_I(v) - \alpha_I(u)) dv^I
\]

into \( \beta(u,v) \), we see that, to complete the proof of (2.17), it suffices to exhibit a decomposition of the form (2.17) for the remaining terms \( \alpha_I(u)(dv^I - du^I) \).

To this end, writing \( I_2 \) for the multi-index \( (i_2, \ldots, i_{k-1}) \), we observe that

\[
\alpha_I(u)(dv^I - du^I) = \alpha_I(u)(d(v^{i_1} - u^{i_1}) \wedge dv^{i_2} + du^{i_1} \wedge (dv^{i_2} - du^{i_2}))
= d[\alpha_I(u)(v^{i_1} - u^{i_1}) \wedge dv^{i_2}] - (v^{i_1} - u^{i_1})d(\alpha_I(u)) \wedge dv^{i_2}
+ du^{i_1} \wedge \alpha_I(u)(dv^{i_2} - du^{i_2}).
\]

Now, the \((k-2)\)-form \( \alpha_I(u)(v^{i_1} - u^{i_1}) \wedge dv^{i_2} \) evidently satisfies an estimate of the form (2.20), and so can be absorbed into \( \eta(u,v) \), while the \((k-1)\)-form \( (v^{i_1} - u^{i_1})d(\alpha_I(u)) \wedge dv^{i_2} \) can likewise be absorbed into \( \beta(u,v) \). To deal with the leftover term

\[
du^{i_1} \wedge \alpha_I(u)(dv^{i_2} - du^{i_2}),
\]
we apply the same argument to the \((k-2)\)-form \(\alpha_I(u)(dv^I - du^I)\) that we did to the \((k-1)\)-form \(\alpha_I(u)(dv^I - du^I)\), and carrying on in this way, we eventually arrive at the desired decomposition. \(\square\)

Integrating the estimates \((2.18)-(2.20)\) of Lemma 2.6 and making liberal use of Hölder’s inequality, we obtain for any \(p > k - 1\) the bounds
\[
\|v^*(\bar{\alpha}) - u^*(\bar{\alpha})\|_{L^1} \leq C(\alpha)(\|du\|_{L^p}^{k-1} + \|dv\|_{L^p}^{k-1})\|u - v\|_{L^\infty}^{k-p}\|u - v\|_{L^p}^{p+1-k} + (\|du\|_{L^k-1}^{k-2} + \|dv\|_{L^k-1}^{k-2})\|du - dv\|_{L^k-1},
\]
(2.21) \(\|\beta(u,v)\|_{L^1} \leq C(\alpha)(\|du\|_{L^p}^{k-1} + \|dv\|_{L^p}^{k-1})\|u - v\|_{L^\infty}^{k-p}\|u - v\|_{L^p}^{p+1-k},\)
and
(2.22) \(\|\eta(u,v)\|_{L^1} \leq C(\alpha)(\|du\|_{L^k-1}^{k-2} + \|dv\|_{L^k-1}^{k-2})\|u - v\|_{L^k-1}.\)

For the remainder of this section, we will have \(p \in (k-1,k)\). It then follows from the estimates above that the pullback assignment
\(u \mapsto u^*(\alpha)\)
gives a well-defined, continuous map from \(W^{1,p}(M^n, N)\) to the space \(L^k_{-1}(M)\) of \((k-1)\)-forms with coefficients in \(L^1\). For any map \(u \in W^{1,p}(M, N)\), we can, in particular, define the \((n+1-k)\)-current \(S_\alpha(u) \in D_{n+1-k}(M)\) dual to \(u^*(\alpha)\) by
\[
\langle S_\alpha(u), \zeta \rangle := \int_M u^*(\alpha) \wedge \zeta.
\]
(2.23) By virtue of (2.21) and (2.22), we then have the following decomposition lemma for the difference \(S_\alpha(v) - S_\alpha(u)\).

**Lemma 2.8.** For \(u, v \in W^{1,p}(M, N)\), the difference \(S_\alpha(v) - S_\alpha(u)\) admits a decomposition of the form
\[
S_\alpha(v) - S_\alpha(u) := S_\alpha(u,v) + \partial R_\alpha(u,v),
\]
(2.24) for some \(R_\alpha(u,v) \in D_{n-k+2}(M)\) and \(S_\alpha(u,v) \in D_{n+1-k}(M)\) satisfying the mass bounds
\[
M(S_\alpha(u,v)) \leq C(\alpha)[E_p(u)^\frac{k-1}{\nu} + E_p(v)^\frac{k-1}{\nu}]\|u - v\|_{L^p}^{1+p-k}.
\]
(2.25) and
\[
M(R_\alpha(u,v)) \leq C(\alpha)[\|du\|_{L^{k-1}}^{k-2} + \|dv\|_{L^{k-1}}^{k-2}]\|u - v\|_{L^{k-1}}.
\]
(2.26)

For \(u \in W^{1,p}(M, N)\), we now define the homological singularity \(T_\alpha(u) \in D_{n-k}(M)\) associated to \(\alpha\) to be the \((n-k)\)-boundary
\[
\langle T_\alpha(u), \zeta \rangle := \langle \partial S_\alpha(u), \zeta \rangle = \int_M u^*(\alpha) \wedge d\zeta.
\]
(2.27) Homological singularities of this sort have been been considered by various authors—as, for instance, [14], or [3], which studies their role as obstructions to the approximation of Sobolev maps by smooth maps. In the special case
\[ N = S^{k-1}, \quad \alpha = \frac{dvol_{\partial K}}{\sigma_{k-1}}, \]
the current \( T_\alpha \) coincides with the distributional Jacobian, whose geometric properties have been well studied in recent decades (see, for instance, [1], [26], and references therein).

When \( u \) is smooth on the support of an \((n-k)\)-form \( \zeta \in \Omega^{n-k}_c(\bar{M}) \) supported in the interior \( \bar{M} \) of \( M \), Stokes’s theorem and the naturality of the exterior derivative give
\[
\langle T_\alpha(u), \zeta \rangle = \int u^*(\alpha) \wedge d\zeta = (-1)^{k-1} \int d(u^*(\alpha)) \wedge \zeta = 0.
\]

In fact, if \( u \) is continuous on an open set containing \( spt(\zeta) \), then one again has
\[
\langle T_\alpha(u), \zeta \rangle = 0,
\]
since we can find a sequence of smooth maps \( u_j \in C^\infty(M, N) \) approaching \( u \) in \( W^{1,p} \) on a neighborhood of \( spt(\zeta) \) (see, e.g., [7], [19]). In particular, if \( u \) is continuous away from a closed set \( \text{Sing}(u) \subset M \), it follows that
\[
spt(T_\alpha(u)) \subset \text{Sing}(u) \cup \partial M.
\]

Next, define
\[
\mathcal{E}^p(M, N) \subset W^{1,p}(M, N)
\]
to be the collection of maps \( u \in W^{1,p}(M, N) \) of the form
\[
u = f \circ \Phi_k \circ h^{-1}
\]
for some cubeulation \( h : |K| \to M \) of \( M \) and some Lipschitz map \( f \in \text{Lip}(|K^{k-1}|, N) \) from the \((k-1)\)-skeleton. For such maps, the set \( \text{Sing}(u) \) of discontinuities is evidently contained in the dual \((n-k)\)-skeleton \( L^{n-k} \) to \( K \), and the homological singularity \( T_\alpha(u) \) is given by an integral \((n-k)\)-cycle that we can describe explicitly.

**Proposition 2.9.** (cf. [GMS2] Section 5.4.2, Theorem 1) If \( u \in \mathcal{E}^p(M, N) \) is given by \( u = f \circ \Phi_k \circ h^{-1} \) for some \( f \in \text{Lip}(|K^{k-1}|, N) \) and a cubeulation \( h : |K| \to M \), then for any \((n-k)\)-form \( \zeta \in \Omega^{n-k}_c(\bar{M}) \) supported in the interior of \( M \), the pairing with \( T_\alpha(u) \) is given by
\[
\langle T_\alpha(u), \zeta \rangle = \sum_{\sigma \in K^k \setminus K^{k-1}} \theta(\sigma) \cdot \int_{P(\sigma)} \zeta,
\]
where
\[
|\theta(\sigma)| = |\int_{\partial \sigma} f^*(\alpha)|,
\]
and \( P(\sigma) \) is defined as in (2.11) to be the component of \( L^{n-k} \) intersecting the \( k \)-cell \( \sigma \in K \).
Remark 2.10. The integrality $\theta(\sigma) \in \mathbb{Z}$ follows from the fact that $\alpha \in \mathcal{A}^{k-1}(N)$, since the pushforward $f_*(\partial \sigma)$ of the $(k-1)$-cycle $\partial \sigma$ by the Lipschitz map $f$ is an integral $(k-1)$-cycle in $N$.

The proposition follows from results in Section 5.4.2 of [14], but in the interest of keeping the discussion self-contained (and because our terminology differs somewhat from that of [14]) we provide a proof below. In fact, the conclusion of Proposition 2.9 applies to a much larger collection of maps than $\mathcal{E}^p(M,N)$—namely, any $W^{1,p}$ map which is continuous away from the dual $(n-k)$-skeleton of some cubeulation (see [14]).

Proof. To begin, we claim that it is enough to establish (2.29) for forms $\zeta \in \Omega^{n-k}_c(\hat{M})$ supported in the interior of a single $n$-face $\Delta$ of the cubeulation. To see this, denote by $\Xi$ the $(n-k-1)$-dimensional intersection $\Xi := L^{n-k} \cap |K^{n-1}|$, and for $\epsilon \in (0,1/2)$, define the cutoff functions
\[ \chi_{\epsilon}(x) := \psi(\epsilon^{-1} \text{dist}(x, L^{n-k})) \]
and
\[ \varphi_{\epsilon}(x) := \psi(\epsilon^{-1} \text{dist}(x, \Xi)), \]
where $\psi \in C^\infty(\mathbb{R})$ satisfies
\[ \psi(t) = 0 \text{ for } t \geq 1 \text{ and } \psi(t) = 1 \text{ for } t \leq \frac{1}{2}. \]

Then $\chi_{\epsilon}$ is supported on the $\epsilon$-neighborhood of $L^{n-k}$ and $\chi_{\epsilon} \equiv 1$ near $L^{n-k}$, while $\varphi_{\epsilon}$ is supported on the $\epsilon$-neighborhood of $\Xi$, with $\varphi_{\epsilon} \equiv 1$ near $\Xi$.

For any $\zeta \in \Omega^{n-k}_c(M)$, it follows from (2.28) that
\[ \langle T_\alpha(u), \zeta \rangle = \langle T_\alpha(u), \chi_{\epsilon}\zeta \rangle. \]

For $\epsilon > 0$ sufficiently small, we observe that the form
\[ \chi_{\epsilon}\zeta - \varphi_{\epsilon}\zeta \]
is supported away from the $(n-1)$-skeleton $|K^{n-1}|$, and can therefore be written as a sum of forms supported in the interiors of the $n$-faces $\Delta$ of $K$. In particular, to justify our claim that it suffices to establish (2.29) for forms $\zeta$ supported in an $n$-face $\Delta$, it is enough to show that
\[ \lim_{\epsilon \to 0} \langle T_\alpha(u), \varphi_{\epsilon}\zeta \rangle = 0. \]

To establish (2.32), we first observe that
\[
|\langle T_\alpha(u), \varphi_{\epsilon}\zeta \rangle| = \left| \int u^*(\alpha) \wedge d\varphi_{\epsilon} \wedge \zeta + \int u^*(\alpha) \wedge \varphi_{\epsilon} d\zeta \right| \\
\leq C \left( \frac{1}{\epsilon} \|\zeta\|_{L^\infty} + \|d\zeta\|_{L^\infty} \right) \int_{\{\text{dist}_{\Xi} \leq \epsilon\}} |du|^{k-1}.
\]
To prove the orthogonality condition (2.34), write \( \langle y \rangle \) for the coordinates of (2.34) \( \langle \rangle \). It follows from standard constancy theorems (e.g., Theorem 2 in Section T.5.3.1 of [13]) that (2.33)\( \langle \rangle \) for any \( \zeta_n \) with \( \zeta_n \). Since \( \zeta \) and we can check by direct computation on each \( n \)-cell \( \Delta \) that
\[
\int_{\{ \text{dist} \leq \epsilon \}} |du|^{k-1} \leq C L^{k-1} \int_{\{ \text{dist} \leq \epsilon \}} (\text{dist}_{L^{n-1}}(x))^{1-k} dH^n(x)
\]
\[
\leq C L^{k-1} \cdot \epsilon^2.
\]
Returning to our estimate for \( \langle T_\alpha(u), \varphi_\epsilon \zeta \rangle \), we then see that
\[
\lim_{\epsilon \to 0} |\langle T_\alpha(u), \varphi_\epsilon \zeta \rangle| \leq \lim_{\epsilon \to 0} C \left( \frac{1}{\epsilon} \| \zeta \|_{L^\infty} + \| d\zeta \|_{L^\infty} \right) \cdot L^{k-1} \epsilon^2
\]
\[
\leq C(K, \zeta, f) \lim_{\epsilon \to 0} \epsilon,
\]
so (2.32) holds, and we can restrict our attention to forms \( \zeta \) supported in the interior of a single \( n \)-cell \( \Delta \).

In fact, if we modify the definition of \( \Xi \) above by adding the \( (n-k-1) \)-dimensional set given by the union of all intersections \( P(\sigma_1) \cap P(\sigma_2) \) for distinct \( k \)-cells \( \sigma_1, \sigma_2 \in K \), then the same argument shows that it is enough to establish (2.29) for \( \zeta \) supported in the interior of \( \Delta \cap \Phi_{k+1}^{-1}(\sigma) \) with \( n \)-face \( \Delta \in K \) and \( k \)-cell \( \sigma \in K \).

Thus, identifying \( \Delta \) homothetically with \( I^n := [-1,1]^n \), \( \sigma \) with \{1, \ldots, 1\} \( \times \) \( I^k \), and (consequently) \( \Delta \cap \Phi_{k+1}^{-1}(\sigma) \) with \( [0,1]^{n-k} \times I^k \), it remains to show that for a \( W^{1, p} \) map
\[
u : E = (0,1)^{n-k} \times I^k \to N
\]
with
\[
\text{Sing}(u) \subset [0,1]^{n-k} \times \{0\},
\]
and any \( \zeta \in \Omega_c^{n-k}((0,1)^{n-k} \times (-1,1)^k) \), we have
\[
(2.33) \quad \langle T_\alpha(u), \zeta \rangle = \theta \int_{[0,1]^{n-k} \times \{0\}} \zeta, \text{ where } |\theta| = |\int_{\partial \alpha} u^*(\alpha)|.
\]

Since \( \partial T_\alpha(u) = 0 \) and the support \( spt(T_\alpha(u)) \) satisfies (by (2.28))
\[
\text{spt}(T_\alpha(u)) \cap E \subset [0,1]^{n-k} \times \{0\},
\]
it follows from standard constancy theorems (e.g., Theorem 2 in Section 5.3.1 of [13]) that \( T_\alpha(u) \) has the form (2.33) for some \( \theta \in \mathbb{R} \), provided that
\[
(2.34) \quad \langle T_\alpha(u), \zeta \rangle = 0 \text{ for every } \zeta \in \Omega_c^{n-k}(E) \text{ with } \langle \zeta, dy \wedge \ldots \wedge dy^{n-k} \rangle = 0.
\]

To prove the orthogonality condition (2.34), write \( (y^1, \ldots, y^{n-k}, z^1, \ldots, z^k) \) for the coordinates of \( E \), and consider \( \zeta \in \Omega_c^{n-k}(E) \) of the form
\[
(2.35) \quad \zeta = dz^j \wedge \omega \text{ for } \omega \in \Omega_c^{n-k-1}(E),
\]
and let \( \chi \in C^\infty_c((-\delta, \delta)) \) be a bump function with \( \chi(t) = 1 \) for \( t \in [-\frac{\delta}{2}, \frac{\delta}{2}] \). Since \( spt(T_\alpha(u)) \subset \{(y, z) \mid z = 0\} \), we then have

\[
|\langle T_\alpha(u), \zeta \rangle| = |\langle T_\alpha(u), \chi(z^j)dz^j \wedge \omega \rangle| = |\int u^*(\alpha) \wedge \chi(z^j)dz^j \wedge d\omega| \leq C\|d\omega\|_{L^\infty} E_p(u) \frac{k+1}{p} \cdot Vol(\{ |z| < \delta \})^{\frac{1}{k+1}}.
\]

Since \( \delta > 0 \) was arbitrary, we can then take \( \delta \to 0 \), to see that \( \langle T_\alpha(u), \zeta \rangle = 0 \) for any \( \zeta \) of the form (2.35). In particular, it follows that (2.34) holds, so that \( T_\alpha(u) \) indeed has the form (2.33) for some \( \theta \in \mathbb{R} \).

To determine the constant \( \theta \) in (2.33), we test \( T_\alpha(u) \) against a form

\[
\zeta(x) = \zeta(y, z) = \varphi(y)\psi(|z|)dy^1 \wedge \cdots \wedge dy^{n-k},
\]

where \( \varphi \in C^\infty_c((0, 1)^{n-k}) \) and \( \psi \) is given by (2.31). By direct computation, we see that

\[
\langle T_\alpha(u), \zeta \rangle = \int u^*(\alpha) \wedge d\zeta = \int \varphi(y)u^*(\alpha) \wedge \psi'(|z|)dz|dy^{1} \wedge \cdots \wedge dy^{n-k} = (-1)^{k(n-k)} \int_{y \in (0, 1)^{n-k}} \varphi(y) \left( \int_{y \times (-1, 1)^k} \psi'(|z|)u^*(\alpha) \wedge dz \right) dy = (-1)^{k(n-k)}(-1)^{k-1} \int_{y \in (0, 1)^{n-k}} \varphi(y) \int_{0}^{1} \psi'(r) \left( \int_{y \times S^{k-1}(0)} u^*(\alpha) \right) dr dy.
\]

Now, since \( u \) is locally Lipschitz away from \([0, 1]^{n-k} \times \{0\}\), it follows from the observations in Remark 2.4 that

\[
\int_{y \times S^{k-1}(0)} u^*(\alpha) = \int_{\partial \sigma} u^*(\alpha)
\]

for every sphere \( y \times S^{k-1}(0) \) linking with \([0, 1]^{n-k} \times \{0\}\). Using this in the preceding computation, we see that

\[
\langle T_\alpha(u), \zeta \rangle = (-1)^{k(n-k+1)-1} \langle \alpha, u_\ast(\partial \sigma) \rangle \int_{y \in (0, 1)^{n-k}} \varphi(y)dy \int_{0}^{1} \psi'(r)dr = (-1)^{k(n-k+1)} \langle \alpha, u_\ast(\partial \sigma) \rangle \int_{[0, 1]^{n-k} \times \{0\}} \zeta.
\]

Thus, the constant \( \theta \) in (2.33) must be given by

\[
\theta = (-1)^{k(n-k+1)} \int_{\partial \sigma} u^*(\alpha),
\]

as desired. \( \square \)
3. Limits of Homological Singularities as $p \to k$

3.1. Degree-type Estimates in $k$-Dimensional Domains.

In this section, we are concerned with estimating the topological quantity

$$\int_{\partial U} u^*(\alpha)$$

for maps $u \in W^{1,p}(U, N) \cap W^{1,p}(\partial U, N)$ on a $k$-dimensional domain $U \subset \mathbb{R}^k$ in terms of the $p$-energy $\int_U |du|^p$. Our arguments are modeled very closely on those used by Jerrard [24] to estimate the degrees of $\mathbb{R}^k$-valued maps in terms of Ginzburg-Landau energies (see also [32], [25]). In the case $N = S^{k-1}$, $\alpha = \frac{dvol}{\sigma_{k-1}}$, estimates similar to the ones we consider here can also be found in [21] (see also [20]), where they are used to study the asymptotic behavior of $p$-energy minimizing maps from $U$ to $S^{k-1}$ as $p \to k$.

Fix a closed $(k-1)$-form $\alpha \in \mathcal{A}^{k-1}(N)$ as before, and define the constant

$$\lambda(\alpha) := \sigma_{k-1} \sup \left\{ \frac{\int_{S^{k-1}} u^*(\alpha)}{\int_{S^{k-1}} |du|^{k-1}} \mid u \in C^\infty(S^{k-1}, N) \right\}.$$

That $\lambda(\alpha) < \infty$ is clear from the estimates of Section 2.2, and when working with specific examples, it is not difficult to obtain explicit bounds for $\lambda(\alpha)$. When $N = S^{k-1}$ and $\alpha$ is the normalized volume form $\frac{dvol}{\sigma_{k-1}}$, for example, one can check (see [21], Section 1) that

$$\lambda(\alpha) = (k-1)\frac{k^{k-1}}{k^{k-1}}.$$

Next, for $p \in (k-1, k)$, we define the constants

$$c(N, \alpha, p) := \frac{\sigma_{k-1}}{\lambda(\alpha)^{\frac{1}{k-p}}},$$

and set

$$F_p(s) := \frac{c(N, \alpha, p)}{k-p}s^{k-p}.$$

The functions $F_p$ will take on the role in our setting played by the functions $\Lambda^\epsilon(s)$ in [24], [25], [32]. Since $0 < k-p < 1$, we easily check that

$$\frac{d}{ds} \left( \frac{F_p(s)}{s} \right) < 0 \text{ for } s > 0,$$

and, by the concavity of $s \mapsto s^{k-p}$, we have the subadditivity

$$F_p(s_1 + s_2) \leq F_p(s_1) + F_p(s_2)$$

for all $s_1, s_2 > 0$. Our estimates begin with the following simple lemma (compare [24], Proposition 3.2).

**Lemma 3.1.** Let $u \in C^\infty(B_{r_2}^k(0) \setminus B_{r_1}^k(0), N)$ be a smooth map from the annulus $B_{r_2}^k \setminus B_{r_1}^k$ to $N$, and set

$$d := \left| \int_{\partial B_{r_2}^k} u^*(\alpha) \right|$$
for some (hence every) \( r \in [r_1, r_2] \). The \( p \)-energy of \( u \) on \( B_{r_2} \setminus B_{r_1} \) then satisfies the lower bound

\[
E_p(u, B_{r_2} \setminus B_{r_1}) \geq d[F_p(r_2/d) - F_p(r_1/d)].
\]

**Proof.** For any \( r \in (r_1, r_2) \), by definition of \( \lambda(\alpha) \), we know that

\[
\sigma_{k-1} d \leq \lambda(\alpha) \int_{\partial B_r} |u|^k - 1.
\]

Raising both sides to the power \( \frac{p}{k-1} \) and applying Hölder’s inequality to the integral on the right-hand side, we then see that

\[
(\sigma_{k-1} d)^{\frac{p}{k-1}} \leq \lambda(\alpha)^{\frac{p}{k-1}} |\partial B_r|^{\frac{p}{k-1} - 1} \int_{\partial B_r} |u|^p,
\]

which we can rearrange to read

\[
\sigma_{k-1} d^\frac{p}{k-1} \leq \lambda(\alpha)^{\frac{p}{k-1}} \int_{\partial B_r} |u|^p.
\]

Integrating the latter relation over \( r \in [r_1, r_2] \), we arrive at the estimate

\[
\sigma_{k-1} d^\frac{p}{k-1} r^k_p - r^{k-1}_p \leq \lambda(\alpha)^{\frac{p}{k-1}} E_p(u, B_{r_2} \setminus B_{r_1}).
\]

The desired estimate (3.4) now follows from the trivial observation that

\[
d^\frac{p}{k-1} \geq d \geq d^{1+p-k},
\]

for \( p \in (k-1, k) \), so that

\[
d[F_p(r_2/d) - F_p(r_1/d)] = d^{1+p-k} r^k_p - r^{k-1}_p \leq d^\frac{p}{k-1} r^k_p - r^{k-1}_p.
\]

\( \square \)

We next record an analog of [25], Proposition 6.4 (see also [24], Proposition 4.1 or [32], Proposition 3.1), from which the main estimates of this section will follow.

**Lemma 3.2.** Let \( U \subset \mathbb{R}^k \) be a bounded Lipschitz domain, and let \( u : U \rightarrow N \) be smooth away from a finite set \( \Sigma = \{a_1, \ldots, a_m\} \subset U \) of singularities with

\[
d_j := \lim_{r \rightarrow 0} \int_{\partial B_r(a_j)} u^*(\alpha).
\]

Then for every \( \sigma > 0 \), there exists a family \( \mathcal{B}(\sigma) = \{B^\sigma_j\}_{j=1}^m(\sigma) \) of \( m(\sigma) \leq m \) disjoint closed balls of radius \( r^\sigma_j \) such that, defining

\[
d^\sigma_j := |\Sigma_{a_\ell \in B^\sigma_j} \cap \Sigma\text{d}_\ell|,
\]
we have

\[ \Sigma \subset \bigcup_{j=1}^{m(\sigma)} B_j^\sigma \quad \text{and} \quad \Sigma \cap B_j^\sigma \neq \emptyset \text{ for each } j, \]

(3.6) \[ \int_{U \cap B_j^\sigma} |du|^p \geq \frac{r_j^\sigma}{|\sigma|} F_p(\sigma) \text{ if } d_j^\sigma > 0, \]

and

(3.7) \[ r_j^\sigma \geq |\sigma| d_j^\sigma \text{ if } B_j^\sigma \subset U. \]

Proof. Denote by \( S \) the collection of \( \sigma > 0 \) for which such a family \( \mathcal{B}(\sigma) \) exists. To see that \( S \) is nonempty, for each \( a_j \in \Sigma \), set

\[ d_j := \lim_{r \to 0} \int_{\partial B_r(a_j)} u^*(\alpha) \]

and

\[ D := 1 + \max_{1 \leq j \leq m} |d_j|, \]

and choose \( \sigma_0 > 0 \) such that the balls \( B_{D\sigma_0}(a_1), \ldots, B_{D\sigma_0}(a_m) \) are disjoint. Taking

\[ r_j^{\sigma_0} := \sigma_0 |d_j| \text{ if } d_j \neq 0, \]

and

\[ r_j^{\sigma_0} := \sigma_0 \text{ if } d_j = 0, \]

it’s then clear that the collection

\[ \mathcal{B}(\sigma_0) := \{ B_{r_j^{\sigma_0}(a_j)} \}_{j=1}^{m} \]

satisfies (3.5) and (3.7), as well as (3.6), by Lemma 3.1. In particular, \( \sigma_0 \in S \), so \( S \neq \emptyset \).

Since the functions \( F_p(s) \) satisfy the growth conditions (3.2) and (3.3), we can apply Steps 2 and 3 in the proof of Proposition 6.4 of \cite{25} directly (with \( F_p \) in place of \( \Lambda^c \)) to see that the set \( S \) is open, and closed away from 0. In particular, we deduce that \( S = (0, \infty) \), as desired.

\[ \square \]

With Lemma 3.2 in hand, we arrive at the following proposition. (Compare \cite{24}, Theorem 1.2.)

**Proposition 3.3.** Let \( U \subset \mathbb{R}^k \) and \( u \in W^{1,p}(U, N) \) satisfy the hypotheses of Lemma 3.2, and suppose that the singular set \( \Sigma = \{ a_1, \ldots, a_m \} \) satisfies

\[ \inf_{1 \leq j \leq m} \text{dist}(a_j, \partial U) \geq r > 0. \]

Setting

\[ d := |\int_{\partial U} u^*(\alpha)| = |\Sigma_j=1 d_j|, \]
we then have the lower bound

\[(3.9) \quad \int_U |du|^p \geq d \cdot F_p(r/2d) = \frac{c(N, \alpha, p)}{k - p} (r/2d)^{k-p} d.\]

Proof. Again, we can argue just as in [24], [25]. Suppose that (3.9) doesn’t hold, to obtain a contradiction. Setting \( \sigma = \frac{r}{d} \), we then have

\[\int_U |du|^p < d \cdot F_p(r/2d) = \frac{r F_p(\sigma)}{\sigma}.\]

Choosing a collection of balls \( \mathcal{B}(\sigma) = \{B^\sigma_j\} \) according to Lemma 3.2, it follows from (3.6) that

\[(3.10) \quad r^\sigma_j \leq \frac{\sigma}{F_p(\sigma)} \int_{U \cap B^\sigma_j} |du|^p < \frac{r}{2}\]

whenever \( d^\sigma_j > 0 \). In particular, if \( d^\sigma_j > 0 \), we then deduce from (3.5) and (3.8) that

\[B^\sigma_j \subset U,\]

and therefore, by (3.7), we have

\[r^\sigma_j \geq \sigma d^\sigma_j.\]

Finally, summing (3.6) over \( 1 \leq j \leq m(\sigma) \), we see that

\[\int_U |du|^p \geq \sum_{j=1}^{m(\sigma)} \int_{U \cap B^\sigma_j} |du|^p \geq \sum_{j=1}^{m(\sigma)} \frac{r^\sigma_j}{\sigma} F_p(\sigma) \geq F_p(\sigma) \sum_{j=1}^{m(\sigma)} d^\sigma_j \geq d \cdot F_p(\sigma),\]

a contradiction. Thus, (3.9) holds. \( \square \)

Remark 3.4. By the density results of Bethuel (namely, [7], Theorem 2), we can remove the requirement that \( u \) have finite singular set from the hypotheses of Proposition 3.3: the conclusion applies to any map \( u \in W^{1,p}(U, N) \) for which \( u \) is continuous on the \( r \)-neighborhood of \( \partial U \) in \( U \).

The final estimate of this section is a simple consequence of Proposition 3.3, modeled on ([2], Lemma 3.10). Arguing much as in [2], we will employ this estimate repeatedly in the following sections to obtain the needed compactness results as \( p \to k \) for the homological singularities \( T_\alpha(u_p) \) in higher-dimensional manifolds. In what follows we denote by \( I^k_\delta \) the \( k \)-cube

\[I^k_\delta := [-\delta, \delta]^k.\]

Proposition 3.5. Let \( u \in W^{1,p}(I^k_\delta, N) \) such that \( u|_{\partial I^k_\delta} \in W^{1,p}(\partial I^k_\delta, N) \), and set

\[d := |\int_{\partial I^k_\delta} u^*(\alpha)|.\]
Then for any $r > 0$, we have the estimate
\[ \sigma_{k-1}d^{1+p-k} \leq \lambda(\alpha)^{\frac{p}{k-p}}(r/2)^{p-k}(k-p)[E_p(u, I^k_\delta) + C(k)rE_p(u, \partial I^k_\delta)]. \]

**Proof.** We argue as in [2]. Extend $u$ to a map $\tilde{u}$ on $I^k_{\delta + r}$ by setting
\[ \tilde{u}(x) := u(\delta \cdot x/|x|_{\infty}) \text{ when } \delta \leq |x|_{\infty} \leq r, \]
so that
\[ E_p(\tilde{u}, I^k_{\delta + r}) \leq E_p(u, I^k_\delta) + C(k)rE_p(u, \partial I^k_\delta). \]

In view of Remark 3.3, we can then apply Proposition 3.3 to the map $\tilde{u}$ on $I^k_{\delta + r}$ to see that
\[ c(N, \alpha, p) := \sigma_{k-1}d^{1+p-k} \lambda(\alpha)^{\frac{p}{k-p}}, \]
the desired estimate follows immediately. \qed

### 3.2. Compactness Results for $T_\alpha(u_p)$ as $p \to k$.

Henceforth, let $M^n$ be a closed, oriented Riemannian manifold of dimension $n \geq k$. In this section and the next, we analyze the limiting behavior as $p \to k$ of the homological singularities $T_\alpha(u_p)$ for maps $u_p \in W^{1,p}(M, N)$ with energy growth of the form $E_p(u_p) = O(\frac{1}{k-p})$. Our results are inspired in large part by those of [2] and [25], concerning the limiting behavior of Jacobian currents for maps of controlled energy growth with respect to functionals of Ginzburg-Landau type.

The starting point for our compactness results is the following proposition—inspired by arguments in [2]—in which we construct good approximations $\tilde{u} \in \mathcal{E}^p(M, N)$ to given maps $u \in W^{1,p}(M, N)$, such that the mass $T_\alpha(\tilde{u})$ is controlled uniformly.

**Proposition 3.6.** For any $u \in W^{1,p}(M, N)$ with $k - \frac{1}{2} < p < k$ and
\[ E_p(u) \leq \frac{\Lambda}{k-p}, \]
there exists a map $\tilde{u} \in \mathcal{E}^p(M, N)$ for which
\[ \|u - \tilde{u}\|_{L^p(M)} \leq C(M)(k-p)^{3p-2}\Lambda, \]
\[ E_p(\tilde{u}) \leq \frac{C(M)\Lambda}{(k-p)^2}, \]
and
\[ M(T_\alpha(\tilde{u})) \leq C(M, \alpha, \Lambda) \]
where

Now, recalling (2.12), we have the volume bound

\[ k \]

for every \( k \). The bounds (3.12) and (3.13) follow immediately.

(3.17)

\[ T(3.17) \]

On the other hand, Proposition 3.5 (with \( r = \delta \)) furnishes us with an estimate of the form

\[ |\theta(u, \sigma)| \leq C(M, \alpha) \left( \delta^{p-k}(k-p)|E_p(u, \sigma) + \delta E_p(u, \partial \sigma)| \right)^{\frac{1}{1+p-k}} \]

on every \( k \)-cell \( \sigma \). Now, since the cubeulation \( |K_{\delta}| \to M \) was chosen according to Lemma 2.1, we know that

(3.19)

\[ E_p(u, |K^k|) + \delta E_p(u, |K^{k-1}|) \leq C(M)\delta^{k-n} \frac{A}{k-p}, \]

By Lemma 2.1, we know that

\[ \delta |\sigma| \leq \delta \alpha \]

on every \( |\sigma| \). Therefore, we have the estimates

(3.16)

\[ E_p(\bar{u}_0) \leq \frac{C(M)}{k-p} E_p(u) \leq \frac{C(M)\delta}{(k-p)^2}. \]

Since \( p > k-1 \), we can then find \( f \in Lip(|K_{\delta}^{k-1}|, N) \) homotopic to \( u|_{K_{\delta}^{k-1}} \) on \( |K_{\delta}^{k-1}| \) and arbitrarily close in \( W^{1,p}(|K_{\delta}^{k-1}|) \). In particular, we can choose \( f \) homotopic to \( u|_{K_{\delta}^{k-1}} \) such that

\[ \tilde{u} := f \circ \Phi_k \circ h^{-1} \in \mathcal{E}^p(M, N) \]

satisfies (3.15) and (3.16)—modifying the constant \( C(M) \) if necessary. Defining \( \tilde{u} \) in this way, and taking

\[ \delta = \delta_p := (k-p)^3, \]

the bounds (3.12) and (3.13) follow immediately.

To estimate the mass of \( T_\alpha(\tilde{u}) \), we first appeal to Lemma 2.9 to see that

(3.17)

\[ T_\alpha(\tilde{u}) = \sum_{\sigma \in K_{\delta}^{k-1}} \theta(u, \sigma) \cdot [P(\sigma)], \]

where

\[ |\theta(u, \sigma)| = |\int_{\partial \sigma} \tilde{u}^*(\alpha)| = |\int_{\partial \sigma} u^*(\alpha)|. \]

Now, recalling (2.12), we have the volume bound

\[ \mathcal{H}^{n-k}(P(\sigma)) \leq C(M)\delta^{n-k} \]

for every \( k \)-cell \( \sigma \in K_{\delta} \), so by (3.17), the mass of \( T_\alpha(\tilde{u}) \) is bounded by

(3.18)

\[ M(T_\alpha(\tilde{u})) \leq \sum_{\sigma \in K_{\delta}^{k-1}} |\theta(u, \sigma)|\delta^{n-k}. \]

Proof. For \( \delta \in (0, 1) \), choose a cubeulation \( h : |K_{\delta}| \to M \) satisfying the conclusions of Lemma 2.1 and let \( \bar{u}_0 = u|_{K_{\delta}^{k-1}} \circ \Phi_k \circ h^{-1} \). By Lemma 2.3, we then have the estimates

(3.15)

\[ \|u - \bar{u}_0\|_{L^p(M)} \leq \delta^p \frac{C(M)}{k-p} \leq E_p(u) \leq \delta^p \frac{C(M)\Lambda}{(k-p)^2} \]

and

(3.16)

\[ E_p(\bar{u}_0) \leq \frac{C(M)}{k-p} E_p(u) \leq \frac{C(M)\Lambda}{(k-p)^2}. \]
from which we immediately obtain the simple-minded estimate
\[(k - p)[E_p(u, \sigma) + \delta E_p(u, \partial \sigma)] \leq C(M)\Lambda \delta^{k-n}\]
for every \(k\)-cell \(\sigma \in K_\delta\). Using this to bound the last factor on the right-hand side of the preceding estimate for \(|\theta(u, \sigma)|\), we find that
\[|\theta(u, \sigma)| \leq \frac{C(M, \alpha)}{(k - p)^{\frac{k-p}{p-k}}} \cdot \left[\Lambda \delta^{k-1-n} + \Lambda \delta^{k-n}\right]^{\frac{k-p}{p-k}}.
\]
On the other hand, summing \(E_p(u, \sigma) + \delta E_p(u, \partial \sigma)\) over all \(k\)-cells \(\sigma \in K_\delta\), we also have the bound
\[\Sigma_\sigma [E_p(u, \sigma) + \delta E_p(u, \partial \sigma)] \leq C(M)\Lambda \delta^{k-n}\]
In particular, summing \((3.20)\) over all \(k\)-cells and employing the estimate above, we find that
\[\Sigma_\sigma |\theta(u, \sigma)| \leq \frac{C(M, \alpha)}{(k - p)^{\frac{k-p}{p-k}}} \cdot \left[\Lambda \delta^{k-1-n} + \Lambda \delta^{k-n}\right]^{\frac{k-p}{p-k}}.
\]
Given a family of maps \((k - 1, k) \ni p \mapsto u_p \in W^{1,p}(M, N)\) with \(E_p(u_p) = O(k-p)^{\frac{k-p}{p-k}}\), Proposition \((3.18)\) gives us an associated family of integral \((n-k)\)-cycles \(T_\alpha(\tilde{u}_p)\) with uniform mass bounds. By showing that \(T_\alpha(u_p) - T_\alpha(\tilde{u}_p) \rightarrow 0\) in \((C^1)^*\) as \(p \rightarrow k\), and applying the Federer-Fleming compactness theorem to the cycles \(T_\alpha(\tilde{u}_p)\), we arrive at the following preliminary compactness result.

**Corollary 3.7.** Let \(p_j \in (k - 1, k)\) be a sequence with \(p_j \rightarrow k\), and let \(u_j \in W^{1,p_j}(M, N)\) be a sequence of maps satisfying
\[(3.21) \limsup_{j \rightarrow \infty} (k - p_j)E_{p_j}(u_j) \leq \Lambda < \infty.
\]
Then there exists a subsequence (unrelabelled) \(p_j \rightarrow k\) such that \(T_\alpha(u_j)\) converges in \((C^1)^*\) to an integer rectifiable cycle \(T \in Z_{n-k}(M; \mathbb{Z})\) of finite mass.
Proof. To each map \( u_j \), by Proposition 3.6 we can associate a map \( \tilde{u}_j \in E_p(M,N) \) for which
\[
\| u_j - \tilde{u}_j \|_{L^p}^p \leq C \Lambda (k - p_j)^{3p_j - 2},
\]
\[
E_p(\tilde{u}_j) \leq \frac{C}{(k - p_j)^2} \Lambda,
\]
and
\[
\mathcal{M}(T_\alpha(\tilde{u}_j)) \leq C(M,\alpha,\Lambda).
\]
On the other hand, by Lemma 2.8 we know that
\[
T_\alpha(u_j) - T_\alpha(\tilde{u}_j) = \partial S_\alpha(u_j, \tilde{u}_j),
\]
where
\[
\mathcal{M}(S_\alpha(u_j, \tilde{u}_j)) \leq C(\alpha)[E_{p_j}(u_j) \frac{k-1}{p_j} + E_{p_j}(\tilde{u}_j) \frac{k-1}{p_j}] \| u_j - \tilde{u}_j \|_{L^p_j}^{1+p_j-k}
\]
\[
\leq C[\Lambda/(k - p_j)^2] \frac{k-1}{p_j} \cdot (\Lambda(k - p_j)^{3p_j - 2}) \frac{k-1}{p_j}^{1+p_j-k}
\]
\[
\leq C(M,\alpha,\Lambda)(k - p_j)^{3(p_j+1-k)-2},
\]
so in particular,
\[
\lim_{j \to \infty} \mathcal{M}(S_\alpha(u_j, \tilde{u}_j)) = 0.
\]
Since the currents \( T_\alpha(\tilde{u}_j) \) are integral cycles with uniformly bounded mass, it follows from the Federer-Fleming compactness theorem (see [12], Theorem 4.2.17) that–after passing to a subsequence–there exists an integral cycle \( T \in \mathcal{Z}_{n-k}(M;\mathbb{Z}) \) and a sequence of integer-rectifiable \((n + 1 - k)\)-currents \( \Gamma_j \in \mathcal{I}_{n+1-k}(M;\mathbb{Z}) \) such that
\[
\lim_{j \to \infty} \mathcal{M}(\Gamma_j) = 0
\]
and
\[
\partial \Gamma_j = T_\alpha(\tilde{u}_j) - T.
\]
Putting all this together, we see that
\[
T_\alpha(u_j) - T = \partial(S_\alpha(u_j, \tilde{u}_j) + \Gamma_j)
\]
and
\[
\mathcal{M}(S_\alpha(u_j, \tilde{u}_j) + \Gamma_j) \to 0,
\]
from which it clearly follows that \( T_\alpha(u_j) - T \to 0 \) in \((C^1)^*\). \( \square \)

Remark 3.8. For a simple consequence of Corollary 3.7 consider a map \( u \in W^{1,p}(M,N) \) for which \(|du| \in L^{k,\infty}(M)\)–that is, for which
\[
(3.22) \quad \|du\|_{L^{k,\infty}}^k := \sup\{t^k Vol(\{|du| > t\}) \mid t \in (0,\infty)\} < \infty
\]
–and note that for $p < k$, we have the straightforward $L^p$ estimate

$$
\int_M |du|^p = \int_0^\infty pt^{p-1}Vol(\{|du| > t\})dt \\
\leq \int_0^1 pt^{p-1}Vol(M)dt + \int_1^\infty pt^{p-1}Vol(\{|du| > t\})dt \\
= Vol(M) + \|du\|_{L^k,\infty}^k \frac{p}{k-p}.
$$

In particular, the hypotheses of Corollary 3.7 hold with $u_j = u$, so we see that $T_\alpha(u)$ must be an integral cycle.

3.3. Sharp Mass Bounds for the Limiting Current.

Our goal in this section is to establish a sharp upper bound for the mass of the limiting current in Corollary 3.7; namely, we prove the following proposition.

**Proposition 3.9.** For a sequence $p_j \in (k-1,k)$ with $\lim_{j \to \infty} p_j = k$ and a sequence of maps $u_j \in W^{1,p_j}(M,N)$ satisfying

$$
\limsup_{j \to \infty} (k-p_j)E_{p_j}(u_j) \leq \Lambda
$$

and

$$
(3.23) \quad \lim_{j \to \infty} T_\alpha(u_j) = T
$$

in $(C^1)^*$, the limit current $T$ satisfies

$$
(3.24) \quad \sigma_{k-1} |M(T)| \leq \lambda(\alpha)^{\frac{k}{k-1}} \Lambda.
$$

To prove Proposition 3.9 we continue to model our arguments on those of [2, Section 3], proving first the following lemma for maps from the Euclidean unit ball $B_1^n(0)$.

**Lemma 3.10.** Let $p_j \in (k-1,k)$ be a sequence with $\lim_{j \to \infty} p_j = k$, and let $u_j \in W^{1,p_j}(B_1^n(0),N)$ be a family of maps for which

$$
(3.25) \quad \limsup_{j \to \infty} (k-p_j)E_{p_j}(u_j,B_1^n(0)) \leq \Lambda < \infty.
$$

Then for any simple unit $(n-k)$-covector $\beta \in \bigwedge^{n-k}(\mathbb{R}^n)$ and $\varphi \in C_\infty^\infty(B_1^n)$, we have the estimate

$$
(3.26) \quad \sigma_{k-1} \limsup_{j \to \infty} |\langle T_\alpha(u_j), \varphi \cdot \beta \rangle| \leq \lambda(\alpha)^{\frac{k}{k-1}} \Lambda \|\varphi\|_{L^\infty}.
$$

**Proof.** After a rotation, it is enough to prove (3.26) in the case

$$
\beta = dx^1 \wedge \cdots \wedge dx^{n-k}.
$$

Following the notation of [2], for $a \in \mathbb{R}^n$ and $\delta > 0$, let $G(\delta, a)$ denote the grid

$$
G(\delta, a) := a + \delta \cdot \mathbb{Z}^n,
$$
and let $R_j(\delta, \alpha)$ denote the $j$-skeleton of the associated $n$-dimensional cubical complex for which $\mathcal{G}(\delta, \alpha)$ gives the vertices. Denote by $\tilde{R}_k(\delta, \alpha)$ the component
\[
\tilde{R}_k(\delta, \alpha) := a + (\delta \mathbb{Z}^{n-k} \times \mathbb{R}^k)
\]
of $R_k(\delta, \alpha)$ parallel to $\{0\} \times \mathbb{R}^k$. As in Lemma 3.11 of [2], a simple Fubini argument shows that for $u \in W^{1,p}(M, N)$ and $\eta > 0$, we can find $a(\delta, \eta) \in \mathbb{R}^n$ such that
\[
\int_{R_j(\delta, \alpha) \cap B_1} |du|^p \leq \frac{C}{\eta} \delta^{j-n} \int_{B_1^p} |du|^p
\]
for all $0 \leq j \leq n$ and
\[
\int_{\tilde{R}_k(\delta, \alpha) \cap B_1} |du|^p \leq (1 + \eta) \delta^{k-n} \int_{B_1^p} |du|^p.
\]

Now, fix some arbitrary $\varphi \in C_\infty^0(B_1^n)$ and $\eta > 0$, and consider a family of maps $u_j \in W^{1,p}(B_1, N)$ satisfying (3.25). As in the proof of Proposition 3.6 we let
\[
\delta_j := (k - p_j)^3,
\]
and let $\tilde{u}_j = u_j \circ \Phi_k$ with respect to the cubical complex associated to the grid $\mathcal{G}(\delta_j, a_j(\varphi))$—where $a_j(\varphi)$ is chosen to satisfy (3.27) and (3.28) with respect to $u_j$. Of course, $\tilde{u}_j$ is only well-defined on those $n$-cells strictly contained in $B_1^n(0)$, but since $\varphi$ is supported in the interior of $B_1$ and $\lim_{j \to \infty} \delta_j = 0$, we see that $\tilde{u}_j$ is defined on $spt(\varphi)$ for $j$ sufficiently large.

Setting $\zeta = \varphi dx^1 \wedge \cdots \wedge dx^{n-k}$, we can then argue as in the proof of Corollary 3.7 to see that, for $j$ sufficiently large,
\[
|\langle T_\alpha(u_j) - T_\alpha(\tilde{u}_j), \zeta \rangle| = |\langle S_\alpha(u_j, \tilde{u}_j), d\zeta \rangle| \leq C(n, \alpha) \eta^{-1} \Lambda(k - p_j) \|d\zeta\|_{L^\infty}.
\]
In particular, it follows that
\[
\lim_{j \to \infty} |\langle T_\alpha(u_j) - T_\alpha(\tilde{u}_j), \zeta \rangle| = 0.
\]

On the other hand, by Proposition 2.9 we know that
\[
\langle T_\alpha(\tilde{u}_j), \zeta \rangle = \sum_{\sigma \subset \tilde{R}_k(\delta_j, a_j) \cap B_1} \theta(u_j, \sigma) \int_{P(\sigma)} \varphi,
\]
where the sum is over all $k$-cells $\sigma \cong [0, \delta]^k$ contained in $\tilde{R}_k(\delta_j, a_j) \cap B_1$, and
\[
\theta(u_j, \sigma) = \pm \int_{\partial \sigma} u_j^*(\alpha).
\]
In this Euclidean setting, the component $P(\sigma)$ of the dual $(n-k)$-skeleton intersecting $\sigma$ is given by a single $(n-k)$-cell isometric to $[0, \delta]^{n-k}$, and as a consequence, we see that
\[
|\langle T_\alpha(\tilde{u}_j), \zeta \rangle| \leq \sum_{\sigma \subset \tilde{R}_k(\delta_j, a_j) \cap B_1} |\theta(u_j, \sigma)| \delta^{n-k} \|\varphi\|_{L^\infty}.
\]
To estimate the coefficients $|\theta(u_j, \sigma)|$, we first appeal to Proposition 3.5 and (3.27) to get the crude estimate
\[ |\theta(u, \sigma)|^{1+p_j - k} \leq C(k, \alpha) \delta_j^{p_j} (k - p_j) [E_{p_j}(u_j, \sigma) + \delta_j E_{p_j}(u_j, \partial\sigma)] \]
\[ \leq C(k, n, \alpha, \eta) \delta_j^{p_j - n} \Lambda. \]
In particular, setting
\[ c_j := [C(k, n, \alpha, \eta) \delta_j^{p_j - n} \Lambda]^{-\frac{k-p_j}{k+p_j-k}}, \]
we have the bound
\[ |\theta(u_j, \sigma)|^{k-p_j} \leq c_j, \]
and recalling that $\delta_j = (k - p_j)^3$, we observe that
\[ \lim_{j \to \infty} c_j = 1. \]
For a finer estimate, we appeal again to Proposition 3.5 to see that, for any $r > 0$ and any $k$-cell $\sigma$,
\[ \sigma_{k-1} |\theta(u, \sigma)|^{1+p_j - k} \leq \lambda(\alpha) \frac{p_j}{k-r} (\delta_j r)^{p_j} (k - p_j) [E_{p_j}(u_j, \sigma) + C(k) r \delta_j E_{p_j}(u_j, \partial\sigma)]. \]
Multiplying both sides above by $|\theta(u_j, \sigma)|^{k-p_j}$ and appealing to (3.31), we then arrive at the bound
\[ \sigma_{k-1} |\theta(u_j, \sigma)| \leq c_j \lambda(\alpha) \frac{p_j}{k-r} (\delta_j r)^{p_j-1} (k - p_j) [E_{p_j}(u, \sigma) + C(k) r \delta_j E_{p_j}(u, \sigma)]. \]
Summing over $k$-cells $\sigma \subset \tilde{R}_k(\delta_j, a_j)$, and appealing to (3.27) and (3.28), we find that
\[ \sigma_{k-1} \cdot \Sigma_{\sigma \subset \tilde{R}_k(\delta_j, a_j)} |\theta(u_j, \sigma)| \leq c_j \lambda(\alpha) \frac{p_j}{k-r} (\delta_j r)^{p_j-1} (k - p_j)
\]
\[ \cdot \left( \int_{\tilde{R}_k \cap B_1} |du_j|^{p_j} + C(k) r \delta \int_{\tilde{R}_{k-1} \cap B_1} |du_j|^{p_j} \right) \]
\[ \leq c_j \lambda(\alpha) \frac{p_j}{k-r} (\delta_j r)^{p_j-1} (k - p_j)
\]
\[ \cdot [(1 + \eta) + \frac{C(n, k)}{\eta}] (\delta_j r)^{k-n} E_{p_j}(u_j, B_1). \]
Choosing $r = \eta^2$ above, and returning to (3.30), we arrive at the estimate
\[ \sigma_{k-1} |\langle T_\alpha(\tilde{u}_j), \zeta \rangle| \leq c_j \lambda(\alpha) \frac{p_j}{k-r} (\delta_j r^2)^{p_j-1} [1 + C'(n, k) \eta] \Lambda \|\varphi\|_{L^\infty}. \]
Now, since $\lim_{j \to \infty} c_j = 1$, and likewise $\lim_{j \to \infty} (\delta_j r^2)^{p_j-1} = 1$, we deduce that
\[ \limsup_{j \to \infty} \sigma_{k-1} |\langle T_\alpha(\tilde{u}_j), \zeta \rangle| \leq \lambda(\alpha) \frac{k}{k-r} [1 + C \eta] \Lambda \|\varphi\|_{L^\infty}. \]
By (3.29), this is equivalent to the statement that
\[ \limsup_{j \to \infty} \sigma_{k-1} |\langle T_\alpha(u_j), \zeta \rangle| \leq \lambda(\alpha) \frac{k}{k-r} [1 + C \eta] \Lambda \|\varphi\|_{L^\infty}; \]
finally, taking η → 0, we arrive at the desired estimate. □

With Lemma 3.10 in hand, we can now prove Proposition 3.9 via a blow-up argument.

Proof. (Proof of Proposition 3.9)

Let \( u_j \in W^{1,p_j}(M,N) \) be a sequence of maps as in Proposition 3.7, for which

\[
\limsup_{j \to \infty} (k - p_j) E_{p_j}(u_j) \leq \Lambda
\]

and

\[
\lim_{j \to \infty} T_\alpha(u_j) = T \in \mathcal{N}_{n-k}(M; \mathbb{Z}).
\]

Passing to a further subsequence, we can also assume that the normalized energy measures

\[
\mu_j := (k - p_j)|du_j|^{p_j} dv_g
\]

converge weakly in \( (C^0)^\ast \) to a limiting Radon measure \( \mu \) satisfying

\[
\mu(M) \leq \Lambda.
\]

Denote by \( |T| \) the weight measure associated to the current \( T \). By standard results on derivates of Radon measures (see, e.g., [33], Section 4 or [12], Section 2.9), the quantity

\[
D_{\mu}|T|(x) := \lim_{r \to 0} \frac{|T|(B_r(x))}{\mu(B_r(x))}
\]

is well-defined for \( |T| \)-a.e. \( x \in M \), and to establish the desired mass bound for \( T \), it will suffice to show that

\[
(3.35) \quad D_{\mu}|T|(x) \leq \sigma_{k-1}^{-1} \lambda(\alpha)^{\frac{k}{k-1}} \text{ for } |T| \text{-a.e. } x \in M.
\]

Now, on a small geodesic ball \( B_r(x) \subset M \), denote by

\[
\Phi_{x,r} : B_r(x) \to B^n_1(0) \subset T_x M
\]

the dilation map

\[
\Phi_{x,r}(y) := \frac{1}{r} \exp_x^{-1}(y),
\]

and set

\[
\mu_{x,r} := (\Phi_{x,r})\# \mu, \quad T_{x,r} := (\Phi_{x,r})\# T.
\]

Since \( T \) is integer rectifiable, for \( |T| \)-almost every \( x \in M \), the currents \( T_{x,r} \) converge weakly

\[
(3.36) \quad T_{x,r} \rightharpoonup \theta(x)[P] \in \mathcal{D}_{n-k}(B^n_1),
\]

to an oriented \((n-k)\)-plane \( P \) in \( \mathbb{R}^n \) with multiplicity

\[
\theta(x) := \lim_{r \to 0} \frac{|T|(B_r(x))}{\omega_{n-k} r^{n-k}},
\]

where \( \omega_{n-k} := \mathcal{L}^{n-k}(B^n_{1}(0)) \) (see, for instance, [33], Section 32).
Now, let \( x \in M \) be a point at which \( D_\mu \left| T \right|(x) \) is defined and (3.36) holds, and observe that the density \( \Theta_{n-k}(\mu, x) \) is then well-defined (though possibly infinite), as

\[
\Theta_{n-k}(\mu, x) = \lim_{r \to 0} \frac{\mu(B_r(x))}{\omega_{n-k} r^{n-k}}
\]

\[
= \theta(x) \lim_{r \to 0} \frac{\mu(B_r(x))}{|T|(B_r(x))}
\]

\[
= \theta(x) \frac{1}{D_\mu \left| T \right|(x)}.
\]

In particular, to prove (3.35), we just need to show that (3.37)

\[
\sigma_{k-1} \theta(x) \leq \lambda(\alpha) k^{-1} \Theta_{n-k}(\mu, x).
\]

If \( \Theta_{n-k}(\mu, x) = \infty \), then (3.37) holds trivially, so assume that \( \Theta_{n-k}(\mu, x) < \infty \), and consider a sequence \( r_\ell \to 0 \) for which \( \mu(\partial B_{r_\ell}(x)) = 0 \).

For each \( r_\ell \), we then have

\[
\mu(B_{r_\ell}(x)) = \lim_{j \to \infty} (k - p_j) \int_{B_{r_\ell}(x)} |du_j|^{p_j},
\]

and consequently

\[
r_\ell^{k-n} \mu(B_{r_\ell}(x)) = \lim_{j \to \infty} r_\ell^{p_j - n} (k - p_j) \int_{B_{r_\ell}(x)} |du_j|^{p_j}.
\]

Moreover, since the convergence \( T_\alpha(u_j) \to T \) established in Proposition 3.7 is convergence in the \((C^1)^*\) norm, we also see that

\[
\lim_{j \to \infty} \|T - T_\alpha(u_j)\|_{(C^1_{\alpha}(B_{r_\ell}(x)))^*} = 0.
\]

It follows from (3.38) and (3.39) that for each \( r_\ell \), we can select \( p_\ell = p_{j_\ell} \) and \( u_\ell = u_{j_\ell} \) such that

\[
| \frac{\mu(B_{r_\ell}(x))}{r_\ell^{n-k}} - r_\ell^{p_\ell - n} (k - p_\ell) \int_{B_{r_\ell}(x)} |du_\ell|^{p_\ell} | < \frac{1}{\ell}
\]

and

\[
r_\ell^{k-n-1} \|T - T_\alpha(u_\ell)\|_{(C^1_{\alpha}(B_{r_\ell}(x)))^*} < \frac{1}{\ell}.
\]

Defining the maps \( v_\ell \in W^{1,p_\ell}(B^n_1(0), N) \) by

\[
v_\ell := u \circ \Phi_{x,r_\ell}^{-1},
\]

we then see that

\[
\lim_{\ell \to \infty} (k - p_\ell) E_{p_\ell}(v_\ell, B_1) = \omega_{n-k} \Theta_{n-k}(\mu, x)
\]
and

\[
\lim_{\ell \to \infty} \langle T_\alpha(v_\ell), \zeta \rangle = \lim_{r \to 0} \langle T_{x,r}, \zeta \rangle = \theta(x) \langle [P], \zeta \rangle
\]

for all \( \zeta \in \Omega^{n-k}_c(B^n_\epsilon(0)) \).

Now, applying Lemma 3.10 to the maps \( v_\ell \) and the simple unit \((n-k)\)-covector \( \beta \) orienting \([P]\), we deduce from (3.42) and (3.43) that

\[
(3.44) \\
\sigma_{k-1}\theta(x) \int_P \varphi = \sigma_{k-1}\theta(x) \langle [P], \varphi \cdot \beta \rangle \leq \lambda(\alpha) \frac{k}{k-1} \omega_{n-k} \Theta_{n-k}(\mu, x) \|\varphi\|_{L^\infty}
\]

for any \( \varphi \in C_\infty^c(B_1) \). Finally, applying (3.44) to a reasonable approximation \( \varphi_j \to \chi_{B_1(0)} \) with \( \|\varphi_j\|_{L^\infty} \leq 1 \), we obtain in the limit

\[
\sigma_{n-k}\theta(x) \omega_{n-k} \leq \lambda(\alpha) \frac{k}{k-1} \omega_{n-k} \Theta_{n-k}(\mu, x)
\]

Dividing through by \( \omega_{n-k} \) gives precisely (3.37), and the proposition follows. \( \Box \)

Finally, combining the results of Proposition 3.6, Corollary 3.7, and Proposition 3.9 above, together with a simple contradiction argument, we obtain the following strong version of the compactness result.

**Theorem 3.11.** For any \( \Lambda < \infty \) and \( \eta > 0 \), there exists \( q(M, \alpha, \Lambda, \eta) \in (k-1, k) \) such that if \( p \in (q, k) \), and \( u \in W^{1,p}(M, N) \) satisfies

\[
(k-p)E_p(u) \leq \Lambda,
\]

then there exists a map \( \tilde{u} \in E^p(M, N) \) satisfying

\[
(3.45) \quad E^p(\tilde{u}) \leq C(M) \frac{\Lambda}{(k-p)^2},
\]

\[
(3.46) \quad \|u - \tilde{u}\|_{L^p(M)}^p \leq (k-p)^{3p-2}C(M)\Lambda,
\]

and an integral \((n-k)\)-cycle \( T \in Z_{n-k}(M; \mathbb{Z}) \) and integral \((n+1-k)\)-current \( \Gamma \in I_{n+1-k}(M; \mathbb{Z}) \) such that

\[
(3.47) \quad T_\alpha(\tilde{u}) - T = \partial \Gamma,
\]

\[
(3.48) \quad \sigma_{k-1}M(T) \leq \lambda(\alpha)^{\frac{k}{k-1}}\Lambda,
\]

and

\[
(3.49) \quad M(\Gamma) < \eta.
\]
4. The Lower Bounds for $\gamma^*_p(u, v)$

4.1. The Almgren Isomorphism and Min-Max Widths.

Before proving Theorem 1.2, we recall some basic facts about the map $\pi_1(\mathcal{Z}_m(M; \mathbb{Z}), 0) \to H_{m+1}(M; \mathbb{Z})$ constructed by Almgren in [5], and make precise the definition of the min-max widths that we use to obtain lower bounds.

As in [5], we topologize the space $\mathcal{Z}_m(M; \mathbb{Z})$ of integral $m$-cycles via the flat norm $F(T) := \inf \{ M(T') + M(S) \mid T = T' + \partial S, T' \in \mathcal{Z}_m(M; \mathbb{Z}), S \in \mathcal{I}_{m+1}(M; \mathbb{Z}) \}$.

In his dissertation [5], Almgren exhibited an isomorphism

$$\pi_\ell(\mathcal{Z}_m(M; \mathbb{Z}), 0) \cong H_{\ell+m}(M; \mathbb{Z})$$

between the homotopy groups of $\mathcal{Z}_m$ and the homology groups of $M$, which he later employed in [6] for the purpose of constructing minimal submanifolds via min-max methods. Recent years have seen a tremendous renewal of interest in the topology of spaces of cycles and related min-max constructions of minimal submanifolds (particularly in codimension one, where Pitts’s work provides a powerful regularity theory [31]): we refer the interested reader to [17], [28], [27], [23], [29], [34], and references therein for some recent developments.

In the case $\ell = 1$ of interest for our present purposes, the map

$$\Psi : \pi_1(\mathcal{Z}_m(M; \mathbb{Z}), 0) \to H_{m+1}(M; \mathbb{Z})$$

is fairly simple to describe. First, by two applications of the Federer-Fleming isoperimetric inequality on manifolds ([12], Theorem 4.4.2), there exists a constant $\epsilon(M) > 0$ such that if

$$R \in \mathcal{Z}_{m+1}(M; \mathbb{Z}) \text{ with } M(R) < \epsilon,$$

then $R = \partial \Omega$ for some $\Omega \in \mathcal{I}_{m+2}(M; \mathbb{Z})$, and there exists also $\delta > 0$ such that if

$$T \in \mathcal{Z}_m(M; \mathbb{Z}) \text{ with } F(T) < \delta,$$

then $T = \partial S$ for some $S \in \mathcal{I}_{m+1}(M; \mathbb{Z})$ such that

$$M(S) < \frac{\epsilon}{2}.$$

As a consequence, if $T_0, T_1, \ldots, T_r$ is a finite sequence in $\mathcal{Z}_m(M; \mathbb{Z})$ with $T_0 = T_r = 0$ and

$$F(T_i - T_{i-1}) < \delta \quad \text{for every } i = 1, \ldots, r \text{ (as can be obtained, for instance, by sampling points from a loop in } \pi_1(\mathcal{Z}_m(M; \mathbb{Z}), 0)),$$

then we can find $S_i \in \mathcal{I}_{m+1}$ for which

$$\partial S_i = T_i - T_{i-1} \text{ and } M(S_i) < \frac{\epsilon}{2}.$$
The sum \( S = \sum_{i=1}^{r} S_i \) then defines a cycle in \( Z_{m+1}(M;\mathbb{Z}) \), and for any other choices \( S'_i \in I_{m+1} \) satisfying (4.2), the differences \( R_i := S'_i - S_i \) are \((m+1)\)-cycles of mass \( M(R_i) < \epsilon \), so that

\[
S'_i = S_i + \partial \Omega_i
\]

for some \( \Omega_i \in I_{m+2}(M;\mathbb{Z}) \). In particular, it follows that the homology class \([S] \in H_{m+1}(M;\mathbb{Z})\) of \( S \) is independent of the choice of \( S_i \) in (4.2). In a similar way (taking \( \delta \) in (4.1) smaller, if necessary), it is shown in [5] that the homology class \([S]\) produced in this way remains constant over sequences \( \{T_i\}, \{T'_i\}\) which are close in an appropriate sense, and it is this observation which accounts for the well-definedness of the map \( \Psi : \pi_1(\mathbb{Z}_m(M;\mathbb{Z}),0) \to H_{m+1}(M;\mathbb{Z}) \).

Given a homotopy class \( \Pi \in \pi_1(\mathbb{Z}_m(M;\mathbb{Z}),0) \), for the purposes of intuition, the min-max width \( L(\Pi) \) can be identified with the quantity

\[
\inf_{F \in \Pi} \sup_{y \in S^m} M(F(y)),
\]

giving the infimum over all families \( F \in \Pi \) of the maximal mass attained by a cycle in the family \( F \). For the purposes of this paper, however, it is convenient to define the min-max widths in terms of finite sequences \( \{T_i\} \) in \( \mathbb{Z}_m \) for which adjacent cycles are close in flat norm. The interested reader can compare the definition given below with those of [6], [31] (which require fineness in stronger norms), referring to interpolation procedures like those described in [16], Section 8).

For \( \delta > 0 \), we denote by \( \mathcal{S}_{m,\delta}(M) \) the collection of all finite sequences \( \{T_i\}_{i=0}^{r} \) of integral \( m \)-cycles \( T_i \in Z_{m}(M;\mathbb{Z}) \) for which

\[
T_0 = T_r = 0 \quad \text{and} \quad F(T_{i-1} - T_i) < \delta \quad \text{for every} \quad i = 1, \ldots, r.
\]

By the discussion in the preceding paragraphs (or see again [5], [6] Chapter 13), there are constants \( \epsilon_0(M) > 0 \) and \( \delta_0(M) > 0 \) such that for \( \delta < \delta_0 \), the map

\[
\Psi : \mathcal{S}_{m,\delta}(M) \to H_{m+1}(M;\mathbb{Z})
\]

given by

\[
\Psi(\{T_i\}) = [\sum_{i=1}^{r} S_i]
\]

for some \( S_i \in I_{m+1}(M;\mathbb{Z}) \) satisfying

\[
\partial S_i = T_i - T_{i-1} \quad \text{and} \quad M(S_i) < \frac{\epsilon_0}{2}
\]

is well-defined, independent of the choice of \( \{S_i\} \) satisfying (4.4).

Given a homology class \( \xi \in H_{m+1}(M;\mathbb{Z}) \) and \( \delta < \delta_0 \), we then set

\[
L_{m,\delta}(\xi) := \inf_{0 \leq i \leq r} \max_{\{T_i\} \in \mathcal{S}_{m,\delta}(M), \ \Psi(\{T_i\}) = \xi} M(T_i)
\]

and define

\[
L_m(\xi) := \lim_{\delta \to 0} L_{m,\delta}(\xi) = \sup_{\delta > 0} L_{m,\delta}(\xi).
\]
By another simple application of the isoperimetric inequality, one finds that
\[
\inf_{\xi \neq 0} L_m(\xi) > 0;
\]
to see this, note that there exists a constant \(\eta_1(M) > 0\) such that for every
\(T \in \mathcal{I}_m(M; \mathbb{Z})\) with \(M(T) < \eta_1\), there is some \(R \in \mathcal{I}_{m+1}(M; \mathbb{Z})\) satisfying
\[
T = \partial R \text{ and } M(R) < \frac{\epsilon_0}{4}.
\]
In particular, if we have \(\{T_i\} \in \mathcal{I}_{m,\delta}(M)\) with \(\max_i M(T_i) < \eta_1\), then for
each \(i = 1, \ldots, r - 1\) we can choose \(R_i \in \mathcal{I}_{m+1}(M; \mathbb{Z})\) satisfying \((4.7)\) with
respect to \(T_i\), and setting \(R_0 = R_r = 0\), we see that the currents
\[
S_i := R_i - R_{i-1}
\]
satisfy \((4.8)\). But evidently \(\Sigma_{i=1}^r S_i = 0\), so it follows that \(\Psi(\{T_i\}) = 0\).

For the families of cycles \(\{T_i\} \in \mathcal{I}_{n-k,\delta}(M)\) arising in the proof of
Theorem 1.2 we can determine the associated homology class \(\Psi(\{T_i\})\) only
at the level of real homology. For any real homology class \(\xi \in H_{m+1}(M; \mathbb{R})\)
containing an integral representative \(S \in \mathcal{I}_{m+1}(M; \mathbb{Z})\), we therefore define
the real-homological widths
\[
(4.8) \quad L_{m,\mathbb{R}}(\xi) := \min \{L_m(\xi) \mid \xi \in H_{m+1}(M; \mathbb{Z}), \, \xi \equiv \xi \bmod H_{m+1}(M; \mathbb{R})\}.
\]
Equivalently, we can set
\[
L_{m,\mathbb{R},\delta}(\xi) := \min \{L_{m,\delta}(\xi) \mid \xi \equiv \xi \bmod H_{m+1}(M; \mathbb{R})\},
\]
and define \(L_{m,\mathbb{R}}(\xi)\) by
\[
L_{m,\mathbb{R}}(\xi) := \lim_{\delta \to 0} L_{m,\mathbb{R},\delta}(\xi) = \sup_{\delta > 0} L_{m,\mathbb{R},\delta}(\xi).
\]

The need to work with real homology in the proof of Theorem 1.2 is
due in part to the fact that the currents \(S_i \in \mathcal{D}_{n+1-k}(M)\) that we use to
connect adjacent \((n-k)\)-cycles \(T_i - T_{i-1} = \partial S_i\) are not integer-rectifiable.
However, from the results of Section 4.2.2 we will see that they have the
form \(S_i = \Gamma_i + \partial R_i\), where \(\Gamma_i \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})\) and \(R_i \in \mathcal{D}_{n+2-k}(M)\). The
following lemma then allows us to compare the masses \(M(T_i)\) to the
real-homological widths \(L_{m,\mathbb{R}}(\xi)\).

**Lemma 4.1.** Given \(\delta > 0\) and \(L_1 < \infty\), there exists \(\eta(M, L_1, \delta) > 0\) such
that if \(T_0, T_1, \ldots, T_r \in \mathcal{I}_m(M; \mathbb{Z})\) is a sequence of integral \(m\)-cycles of mass
\[
(4.10) \quad M(T_i) \leq L_1,
\]
with \(T_0 = T_r = 0\), for which there exist \((m+1)\)-currents of the form
\[
S_1, \ldots, S_r \in \mathcal{I}_{m+1}(M; \mathbb{Z}) + \partial \mathcal{D}_{m+2}(M)
\]
such that
\[
(4.11) \quad \partial S_i = T_i - T_{i-1} \text{ and } M(S_i) < \eta,
\]
then \(\{T_i\} \in \mathcal{I}_{m,\delta}(M)\), with
\[
(4.12) \quad \Psi(\{T_i\}) \equiv [\Sigma_{i=1}^r S_i] \in H_{m+1}(M; \mathbb{R}).
\]
Proof. To begin, we claim that there exists $\eta(M, L_1, \delta) > 0$ such that $F(T) < \delta$ for any integral cycle $T$ with

$$M(T) \leq 2L_1 \text{ and } \|T\|_{C^1} < \eta.$$  

Indeed, this is a simple consequence of the Federer-Fleming compactness theorem ([12], Theorem 4.2.17), since any sequence of integral cycles converging weakly to 0 with uniformly bounded mass must also converge to 0 in the flat norm. Applying this claim to the differences $T_i - T_{i-1}$ for a family of cycles $\{T_i\}$ satisfying (4.10)-(4.11), we immediately deduce that $\{T_i\} \in S_{m, \delta}(M)$ for $\eta(M, L_1, \delta) > 0$ sufficiently small.

To check (4.12), fix (as in [13], Sect. 5.4.1) a collection $\omega_1, \ldots, \omega_{b_{m+1}(M)} \in A_{m+1}(M)$ of closed $(m+1)$-forms generating the integer lattice in $H_{dR}^{m+1}(M)$, and let

$$C(M) := \max_{1 \leq i \leq b_{m+1}(M)} \|\omega^i\|_{L^\infty}.$$  

Given $\{T_i\} \in S_{m, \delta}(M)$ satisfying (4.10)-(4.11), let $S_i' \in I_{m+1}(M; \mathbb{Z})$ be a family of integer rectifiable $(m+1)$-currents satisfying

$$\partial S_i' = T_i - T_{i-1} \text{ and } M(S_i') < \frac{\epsilon_0}{2}.$$  

For each $i = 1, \ldots, r$, the difference

$$R_i := S_i - S_i'$$  

is then a cycle of the form $R_i \in Z_{m+1}(M; \mathbb{Z}) + \partial D_{m+2}(M)$, and as a consequence, we see that

(4.14) \hspace{1cm} \langle R_i, \omega \rangle \in \mathbb{Z}  

for every $\omega \in A^{m+1}(M)$. In particular, (4.14) holds for the generators $\omega_1, \ldots, \omega_{b_{m+1}(M)}$ chosen above.

On the other hand, the mass bounds in (4.10) and (4.13) imply that

$$|\langle R_i, \omega \rangle| \leq C(M)(\eta + \frac{\epsilon_0}{2})$$  

whenever $\|\omega\|_{L^\infty} \leq 1$. Thus, taking $\epsilon_0(M)$ and $\eta(M, L_1, \delta) > 0$ small enough that

$$C(M)(\eta + \epsilon_0/2) < 1,$$  

it follows from (4.14) that $\langle R_i, \omega^j \rangle = 0$ for each $i = 1, \ldots, r$ and $j = 1, \ldots, b_{m+1}(M)$. Summing over $i = 1, \ldots, r$, we therefore have

$$\langle \Sigma_{i=1}^r S_i', \omega^j \rangle = \langle \Sigma_{i=1}^r S_i, \omega^j \rangle$$  

for each $j = 1, \ldots, b_{m+1}(M)$, and (4.12) follows. \qed
4.2. A Decomposition Lemma for $S_\alpha(u, v)$.

In this section, we prove that for weakly close maps $u, v \in \mathcal{E}^p(M, N)$, the current $S_\alpha(u, v) \in D_{n+1-k}(M)$ of Lemma 2.8 admits a decomposition of the form

$$S_\alpha(u, v) := \Gamma + \partial R,$$

where $R \in D_{n+2-k}(M)$, and $\Gamma \in I_{n+1-k}(M)$ is integer rectifiable.

**Lemma 4.2.** For $p \in (k - 1, k)$ and $L_2 < \infty$, there exists $\epsilon(M, N, L_2, p) > 0$ such that if $u, v \in \mathcal{E}^p(M, N)$ satisfy

$$E_p(u) + E_p(v) \leq L_2$$

and

$$\|u - v\|_{L^p} < \epsilon,$$

then there exist $\Gamma \in I_{n+1-k}(M)$ and $R \in D_{n+2-k}(M)$ for which

$$S_\alpha(u, v) = \Gamma + \partial R.$$

The same result holds if either $u$ or $v \in C^\infty(M, N)$.

**Remark 4.3.** By Lemma 2.8 (4.17) is clearly equivalent to the statement that

$$S_\alpha(v) - S_\alpha(u) = \Gamma + \partial R'$$

for some $R' \in D_{n+2-k}(M)$.

**Proof.** By the Fubini-type arguments of [38] and [19], for maps $u, v \in \mathcal{E}^p(M, N)$ satisfying (4.15) and (4.16), we can find a cubeulation $h : |K| \to M$ such that

$$E_p(u \circ h, |K^{k-1}|) + E_p(v \circ h, |K^{k-1}|) \leq C(M)L_2,$$

(4.18)

$$\|u \circ h - v \circ h\|_{L^p(|K^{k-1}|)} \leq C(M)\epsilon,$$

(4.19)

where $K$ is a fixed cubical complex and the Lipschitz constants $\text{Lip}(h)$ and $\text{Lip}(h^{-1})$ are bounded independent of $u$ and $v$.

Moreover, since the singular sets $\text{Sing}(u)$ and $\text{Sing}(v)$ are contained in a finite union of $(n - k)$-dimensional submanifolds of $M$, we can choose this $h : |K| \to M$ such that the $(k - 1)$-skeleton $h(|K^{k-1}|)$ lies a positive distance from $\text{Sing}(u) \cup \text{Sing}(v)$. Since $u$ and $v$ are Lipschitz away from $\text{Sing}(u) \cup \text{Sing}(v)$ by definition of $\mathcal{E}^p(M, N)$, it follows in particular that the restrictions $(u \circ h)|_{K^{k-1}}$ and $(v \circ h)|_{K^{k-1}}$ to the $(k - 1)$-skeleton are again Lipschitz.

Next (as in, e.g., [39], Theorem 1.1), we note that the compactness of the embedding $W^{1,p}(|K^{k-1}|) \hookrightarrow C^0(|K^{k-1}|)$ implies the existence of a constant $\epsilon(M, N, L_2, p) > 0$ such that (4.18) and (4.19) imply

$$\|u \circ h - v \circ h\|_{C^0(|K^{k-1}|)} \leq \delta(N).$$

Here, $\delta(N)$ is chosen such that the $\delta(N)$-neighborhood $U_\delta$ of $N$ in $\mathbb{R}^L$ retracts $\pi_N : U_\delta \to N$ onto $N$.  

Choosing such an \( \epsilon \), we proceed to define a map \( w : M \times [0, 1] \rightarrow N \) satisfying \( w(x, 0) = u(x) \) and \( w(x, 1) = v(x) \) (compare [13], Lemma 2.2); throughout, we use the bi-Lipschitz map \( h \) to identify \( M \) and \( |K| \), without comment. First, we set

\[
w(x, 0) := u(x) \quad \text{and} \quad w(x, 1) = v(x) \quad \text{for} \quad x \in |K|,
\]

and on \( |K^{k-1}| \times [0, 1] \), we define

\[
w(x, t) := \pi_N(t v(x) + (1-t) u(x)).
\]

For each \( k \)-cell \( \sigma \) in \( K \), the restriction \( w | \partial(\sigma \times [0, 1]) \) of \( w \) to \( \partial(\sigma \times [0, 1]) \) is then a well-defined Sobolev map, satisfying an estimate of the form

\[
\sup_{x \in \partial(\sigma \times [0, 1])} \text{dist}(x, \Sigma_0) \cdot |dw(x)| < \infty,
\]

where \( \Sigma_0 = \text{Sing}(u) \times \{0\} \cup \text{Sing}(v) \times \{1\} \). Identifying \( \sigma \times [0, 1] \) with the \((k+1)\)-ball \( B_{\frac{1}{4}}^{k+1} \) in a bi-Lipschitz way, we can then extend \( w \) to \( \sigma \times [0, 1] \) radially, setting \( w(x) = \frac{x}{|x|} \).

We have now extended \( w \) to a \( W^{1,p} \) map on the whole \((k+1)\)-skeleton \( |\tilde{K}^{k+1}| \) of \( \tilde{K} = K \otimes [0, 1] \), satisfying

\[
\sup_{x \in |\tilde{K}^{k+1}|} \text{dist}(x, \Sigma_1) \cdot |dw(x)| < \infty,
\]

where \( \Sigma_1 \) is contained in a finite collection of Lipschitz curves in \( |\tilde{K}^{k+1}| \). On each \((k+1)\)-cell \( \sigma \) of \( K \), we can then extend \( w \) from \( \partial(\sigma \times [0, 1]) \) to \( \sigma \times [0, 1] \) as above, and carry on in this way, until finally we have the desired map

\[
w \in W^{1,p}(M \times [0, 1], N)
\]
satisfying

\[
w(x, 0) = u(x) \quad \text{and} \quad w(x, 1) = v(x) \quad \text{in the trace sense},
\]

and

\[
(4.20) \quad \sup_{x \in M} \text{dist}(x, \Sigma)|dw(x)| < \infty,
\]

where \( \Sigma \subset M \times [0, 1] \) is a \((n+1-k)\)-rectifiable set with \( \mathcal{H}^{n+1-k}(\Sigma) < \infty \).

Doubling this construction, we can evidently extend \( w \) to a map

\[
w : M \times S^1 \cong M \times [-3, 3]/6\mathbb{Z} \rightarrow N
\]
satisfying

\[
(4.21) \quad w(x, t) = u(x) \quad \text{for} \quad t \in [-2, -1], \quad w(x, t) = v(x) \quad \text{for} \quad t \in [1, 2],
\]

and, by virtue of \((4.20)\),

\[
(4.22) \quad \|dw\|_{L^{k,\infty}(M)} < \infty.
\]
In particular, it follows from Remark 3.8 that the homological singularity $T_\alpha(w)$ is an integral cycle in $M \times S^1$:

$$T_\alpha(w) = \partial S_\alpha(w) \in \mathcal{I}_{n+1-k}(M \times S^1; \mathbb{Z}).$$

Now, let $\pi : M \times S^1 \rightarrow M$ be the obvious projection, and define currents $R \in \mathcal{D}_{n+2-k}(M)$ and $\Gamma \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})$ to be the pushforwards

$$R := \pi_\#(S_\alpha(w)[(M \times [-1.5, 1.5])]$$

and

$$\Gamma := \pi_\#(T_\alpha(w)[(M \times [-1.5, 1.5])]).$$

Fix a sequence $\psi_j \in C^\infty_c((-1.5, 1.5))$ satisfying $0 \leq \psi_j \leq 1$ and

$$\psi_j \equiv 1 \text{ on } [-1.5 + \frac{1}{j}, 1.5 - \frac{1}{j}];$$

since

$$S_\alpha(w) = S_\alpha(u) \times (-2, 1) \text{ on } M \times (-2, -1),$$

and

$$S_\alpha(w) = S_\alpha(v) \times (1, 2) \text{ on } M \times (1, 2),$$

it’s clear that

$$R = \lim_{j \to \infty} R_j = \lim_{j \to \infty} \pi_\#(\psi_j(t)S_\alpha(w))$$

and

$$\Gamma = \lim_{j \to \infty} \Gamma_j = \lim_{j \to \infty} \pi_\#(\psi_j(t)S_\alpha(w)).$$

For any $\zeta \in \Omega^{n+1-k}(M)$, we now compute

$$\langle \partial R_j, \zeta \rangle = \langle R_j, d\zeta \rangle$$

$$= \int_{M \times [-1.5, 1.5]} \psi_j(t)w^*(\alpha) \wedge d(\pi^*\zeta)$$

$$= \int_{M \times [-1.5, 1.5]} w^*(\alpha) \wedge d(\psi_j(t)\pi^*\zeta)$$

$$- \int_{M \times [-1.5, 1.5]} w^*(\alpha) \wedge \psi'_j(t)dt \wedge \pi^*\zeta,$$

which, by definition of $\Gamma_j$, gives

$$\langle \partial R_j - \Gamma_j, \zeta \rangle = (-1)^{n+2-k} \int_{M \times [-1.5, 1.5]} w^*(\alpha) \wedge \pi^*\zeta \wedge \psi'_j(t)dt$$

$$+ (-1)^{n+2-k} \int_{-1.5}^{1.5} \psi'_j(t)dt \left( \int_M w^*(\alpha) \wedge \zeta \right)$$

$$+ (-1)^{n+2-k} \int_{-1.5}^{1.5} \psi'_j(t)dt \left( \int_M w^*(\alpha) \wedge \zeta \right)$$

$$= (-1)^{n+2-k} \langle S_\alpha(u) - S_\alpha(v), \zeta \rangle$$

for $j$ sufficiently large.
Passing to the limit $j \to \infty$, we conclude that
\[ \partial R - \Gamma = (-1)^{n+2-k}(S_\alpha(u) - S_\alpha(v)), \]
and in particular,
\[ S_\alpha(v) - S_\alpha(u) \in I_{n+1-k}(M; Z) + \partial D_{n+2-k}(M). \]
Recalling from Lemma 2.8 that the current $S_{\alpha}(u,v)$ differs from $S_{\alpha}(v) - S_{\alpha}(u)$ by the boundary of an $(n+2-k)$-current, it follows that
\[ S_\alpha(u,v) \in I_{n+1-k}(M; Z) + \partial D_{n+2-k}(M) \]
as well, as desired. □

4.3. Proof of Theorem 1.2

Now, let $u, v \in C^\infty(M,N)$ be $(k-2)$-homotopic, and suppose there exists $\alpha \in A^{k-1}(N)$ such that
\[ [u^*(\alpha)] - [v^*(\alpha)] \neq 0 \in H_{dR}^{k-1}(M), \]
or, equivalently,
\[ [S_\alpha(v) - S_\alpha(u)] \neq 0 \in H_{n+1-k}(M; \mathbb{R}). \]
Evidently, such an $\alpha$ exists if and only if $u$ and $v$ induce different maps on the de Rham cohomology $H_{dR}^{k-1}(N) \to H_{dR}^{k-1}(M)$.

For every $\delta > 0$, we define $C^\delta_p(u,v)$ to be the collection of all finite sequences $u_0, u_1, \ldots, u_r \in W^{1,p}(M,N)$ such that $u_0 = u$, $u_r = v$, and
\[ \|u_i - u_{i-1}\|_{L^p(M)} < \delta \]
for every $i = 1, \ldots, r$. We then define
\[ \gamma^\delta_p(u,v) := \inf \{ \max_{0 \leq j \leq r} E_p(u_j) \mid \{u_j\}_{j=1}^r \in C^\delta_p(u,v) \}, \]
and set
\[ (4.25) \quad \gamma^*_p(u,v) := \lim_{\delta \to 0} \gamma^\delta_p(u,v) = \sup_{\delta > 0} \gamma^\delta_p(u,v). \]
Remark 4.4. As observed in the introduction, it’s clear from the definitions that $\gamma^*_p(u,v) \leq \gamma_p(u,v)$, since for any path $u_t \in W^{1,p}(M,N)$ from $u$ to $v$ and any $\delta > 0$, we can find a finite sequence $0 = t_0, t_1, \ldots, t_r = 1$ such that $\{u_{t_i}\} \in C^\delta_p(u,v)$. In particular, it follows from Theorem 1.1 that
\[ \sup_{p < k}(k-p)\gamma^*_p(u,v) < \infty. \]

We recall now the statement of Theorem 1.2.
Theorem 4.5. For closed, oriented Riemannian manifolds $M^n, N$, maps $u, v \in C^\infty(M, N)$, and a $(k-1)$-form $\alpha \in \mathcal{A}^{k-1}(N)$ as above, such that
\[ \bar{\xi} := [S_\alpha(v) - S_\alpha(u)] \neq 0 \in H_{n+1-k}(M; \mathbb{R}), \]
define
\[ \Lambda(u, v) := \liminf_{p \to k} (k-p)\gamma_p^*(u, v). \]
Then we have the lower bound
\[ (4.26) \quad \sigma_{k-1}L_{n-k, \mathbb{R}}(\bar{\xi}) \leq \lambda(\alpha)^{k-1}k^{-1}\Lambda(u, v). \]

Most of the work in the proof of Theorem 4.5 is contained in the following lemma, which combines the results of Section 3 and Lemma 4.2.

Lemma 4.6. For any $\eta \in (0, 1)$, there exists $q(\eta) \in (k-1, k)$ with the property that for every $p \in (q, k)$, there exists $\delta_1(p) > 0$ such that for any $\{u_i\}_{i=0}^r \in C^p(u, v)$ satisfying
\[ (4.27) \quad (k-p)\max_j E_p(u_j) \leq \Lambda(u, v) + \eta, \]
we can find cycles $T_0, T_1, \ldots, T_r \in \mathbb{Z}_{n-k}(M; \mathbb{Z})$ with $T_0 = T_r = 0$ for which
\[ \max_{0 \leq i \leq r} \sigma_{k-1}M(T_i) \leq \lambda(\alpha)^{k-1}(\Lambda(u, v) + \eta), \]
and currents
\[ S_1, \ldots, S_r \in \mathcal{I}_{n+1-k}(M; \mathbb{Z}) + \partial \mathcal{D}_{n+2-k}(M) \]
such that
\[ \partial S_i = T_i - T_{i-1}, \]
\[ (4.28) \quad M(S_i) < 3\eta, \]
and
\[ [\Sigma_{i=1}^r S_i] = [S_\alpha(v) - S_\alpha(u)] \in H_{n+1-k}(M; \mathbb{R}). \]

Proof. First, we appeal to Theorem 3.11 to guarantee the existence of $q_0(\eta) = q_0(\Lambda(u, v) + \eta, \eta) > 0$ such that for any $p \in (q_0, k)$ and any sequence of maps $u_1, \ldots, u_{r-1} \in W^{1,p}(M, N)$ satisfying (4.27), there exists a corresponding sequence $\tilde{u}_1, \ldots, \tilde{u}_{r-1} \in \mathcal{E}^p(M, N)$ satisfying
\[ (4.29) \quad E_p(\tilde{u}_i) \leq \frac{C}{(k-p)^2}, \]
\[ (4.30) \quad \|u_i - \tilde{u}_i\|_{L^p} \leq C(k-p)^{3p-2}, \]
and a sequence of integral cycles $T_i \in \mathbb{Z}_{n-k}(M; \mathbb{Z})$ and integral $(n+1-k)$-currents $\Gamma_i \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})$ such that
\[ (4.31) \quad \sigma_{k-1}M(T_i) \leq \lambda(\alpha)^{k-1}(\Lambda(u, v) + \eta) \]
\[ (4.32) \quad T_\alpha(\tilde{u}_i) - T_i = \partial \Gamma_i, \]
and
\[(4.33) \quad \mathcal{M}(\Gamma_i) < \eta.\]

Now, consider a family \(\{u_i\}_{i=0}^r \in C^p_\delta(u, v)\) for \(p \in (q_0, k)\) satisfying \((4.27)\). For \(1 \leq i \leq r-1\), choose
\[\tilde{u}_i \in \mathcal{E}^p(M, N), \quad T_i \in \mathcal{I}_{n-k}(M; \mathbb{Z}), \quad \text{and} \quad \Gamma_i \in \mathcal{I}_{n+1-k}(M; \mathbb{Z})\]
satisfying \((4.29)\)-(4.33), and extend these sequences trivially by setting
\[\tilde{u}_0 = u, \quad \tilde{u}_r = v, \quad T_0 = T_r = 0, \quad \text{and} \quad \Gamma_0 = \Gamma_r = 0.\]

Next, setting
\[S_i := S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i) + \Gamma_{i-1} - \Gamma_i,\]
for \(i = 1, \ldots, r\), we see that
\[\partial S_i = T_i - T_{i-1},\]
and, by Lemma \(4.2\) these \(S_i\) have the form
\[S_i \in \mathcal{I}_{n+1-k}(M; \mathbb{Z}) + \partial D_{n+2-k}(M).\]

Moreover, since
\[\Sigma_{\tau=1}^r(\Gamma_{i-1} - \Gamma_i) = \Gamma_0 - \Gamma_r = 0,\]
and \((by \quad \text{Lemma} \quad 2.8)\)
\[S_\alpha(\tilde{u}_i) - S_\alpha(\tilde{u}_{i-1}) - S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i) \in \partial D_{n+2-k}(M),\]
it follows that
\[(4.34) \quad [\Sigma_{\tau=1}^r S_i] = [S_\alpha(v) - S_\alpha(u)] \in H_{n+1-k}(M; \mathbb{R}).\]

To complete the proof of the lemma, it remains to establish the mass bound \((4.28)\) for these \(S_i\), for \(p > q(\eta)\) sufficiently large and \(\delta = \delta_1(p)\) sufficiently small. To estimate the mass of
\[S_i = S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i) + \Gamma_{i-1} - \Gamma_i,\]
we first apply \((4.33)\) to see that
\[\mathcal{M}(S_i) \leq \mathcal{M}(S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i)) + 2\eta.\]

Now, by Lemma \(2.8\) and the energy bound \((4.29)\), we know that
\[\mathcal{M}(S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i)) \leq C[E_p(\tilde{u}_{i-1}) \frac{\xi}{L^p} + E_p(\tilde{u}_i) \frac{\xi}{L^p}] \Vert \tilde{u}_i - \tilde{u}_{i-1} \Vert_{L^{1+p-k}} \]
\[\leq C(k - p)^{-2(k-1)/p} \Vert \tilde{u}_i - \tilde{u}_{i-1} \Vert_{L^{1+p-k}}.\]

Moreover, by \((4.30)\), we have
\[\Vert u_i - \tilde{u}_i \Vert_{L^p} \leq C(k - p)^{3p-2},\]
while by definition of \(C^p_\delta(u, v)\), we have also
\[\Vert u_i - u_{i-1} \Vert_{L^p} < \delta^p.\]

Taking \(\delta = \delta_1(p) = (k - p)^{3-2/p}\) and combining the estimates above, it follows in particular that
\[\Vert \tilde{u}_i - \tilde{u}_{i-1} \Vert_{L^p} \leq C'(k - p)^{3p-2},\]
and consequently,
\[
\mathbb{M}(S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i)) \leq C(k-p)^{-2(k-1)/p}\|\tilde{u}_i - \tilde{u}_{i-1}\|_{L^p}^{1+p-k} \\
\leq C'(k-p)^{-2(k-1)/p + (3p-2)p/2^{1+p-k}} \\
= C'(k-p)^{1-3(k-p)}.
\]

Since \(\lim_{p \to k}(k-p)^{1-3(k-p)} = 0\), we can therefore choose \(q(\eta) \in (k-1,k)\) such that
\[
\mathbb{M}(S_\alpha(\tilde{u}_{i-1}, \tilde{u}_i)) < \eta,
\]
and, consequently,
\[
\mathbb{M}(S_i) < 3 \eta
\]
for \(p \in (q,k)\). This completes the proof. \(\square\)

Combining the preceding lemma with the results of Lemma 4.1, we complete the proof of Theorem 4.5 as follows.

**Proof.** Fix some \(\delta \in (0,1)\). By Lemma 4.1, we can find \(\eta(\delta) \in (0,\delta)\) such that for any sequence \(\{T_i\}_{i=0}^r \subset \mathcal{Z}_{n-k}(M;\mathbb{Z})\) satisfying \(T_0 = T_r = 0\),
\[
\mathbb{M}(T_i) \leq C(\alpha)[\Lambda(u,v) + 1]
\]
and
\[
T_i - T_{i-1} = \partial S_i
\]
for some \(S_1, \ldots, S_r \in \mathcal{L}_{n+1-k}(M;\mathbb{Z}) + \partial D_{n-k+2}(M)\) with
\[
\mathbb{M}(S_i) < 3 \eta,
\]
we have \(\{T_i\} \in \mathcal{S}_{n-k,\delta}(M)\), with the associated homology class \(\Psi(\{T_i\})\) satisfying
\[
\Psi(\{T_i\}) \equiv [\sum_{i=1}^r S_i] \in H_{n-k+1}(M;\mathbb{R}).
\]

Now, with \(\eta(\delta) \in (0,\delta)\) as above, let \(q(\eta)\) be as in Lemma 4.6 and choose \(p \in (q,k)\) such that
\[
(k-p)\gamma_p^*(u,v) \leq \Lambda(u,v) + \eta/2.
\]
Then, choose \(\delta_1(p) > 0\) according to Lemma 4.6 and select some family \(\{u_i\}_{i=0}^r \subset C^p_{\delta_1}(u,v)\) such that
\[
(k-p) \max_i E_p(u_i) \leq (k-p)\gamma_p^{\delta_1}(u,v) + \eta/2 \\
\leq \Lambda(u,v) + \eta.
\]
By Lemma 4.6, we can associate to these \(\{u_i\}_{i=0}^r\) a family of cycles \(T_0, \ldots, T_r \in \mathcal{Z}_{n-k}(M;\mathbb{Z})\) and currents \(S_1, \ldots, S_r \in \mathcal{L}_{n+1-k}(M;\mathbb{Z}) + \partial D_{n-k+2}(M)\) satisfying (4.35) - (4.37), as well as the sharper mass bound
\[
\sigma_{k-1} \max_i \mathbb{M}(T_i) < \lambda(\alpha)\gamma^{\delta_1}_p \Lambda(u,v) + C(\alpha)\eta \\
< \lambda(\alpha)\gamma^{\delta_1}_p \Lambda(u,v) + C(\alpha)\delta,
\]
for \(p \in (q,k)\). This completes the proof.
and the homological condition
\[ \left[ \sum_{i=1}^{r} S_i \right] = [S_\alpha(v) - S_\alpha(u)] \text{ in } H_{n-k+1}(M; \mathbb{R}). \]

In particular, it follows that there exists \( \{T_i\} \in S_{n-k,\delta}(M) \) satisfying
\[ \Psi(\{T_i\}) \equiv [S_\alpha(v) - S_\alpha(u)] \in H_{n+1-k}(M; \mathbb{R}) \]
and
\[ \max_i \sigma_{k-1} M(T_i) \leq \lambda(\alpha)^{\frac{k}{k+1}} \Lambda(u,v) + C(\alpha) \delta. \]

Recalling the notation of Section 4.1, this means precisely that
\[ \sigma_{k-1} L_{n-k,\delta}(\gamma^*_{p}(u,v)) \leq \lambda(\alpha)^{\frac{k}{k+1}} \Lambda(u,v). \]

Finally, taking \( \delta \to 0 \), we arrive at the desired inequality
\[ \sigma_{k-1} L_{n-k,\delta}(\gamma^*_{p}(u,v)) \leq \lambda(\alpha)^{\frac{k}{k+1}} \Lambda(u,v). \]

\[ \square \]

5. Associated \( p \)-Harmonic Maps

In this short section, we demonstrate the existence of mountain pass critical points for the \( p \)-energy functional associated with energy lying between \( \gamma^*_{p}(u,v) \) and \( \gamma_{p}(u,v) \).

**Theorem 5.1.** For any \((k-2)\)-homotopic maps \( u, v \in C^\infty(M, N) \) and \( p \in (1, k) \setminus \mathbb{N} \) such that
\[ \max\{E_p(u), E_p(v)\} < \gamma^*_{p}(u,v), \]
there exists a stationary \( p \)-harmonic map \( w \in W^{1,p}(M, N) \) such that
\[ \gamma^*_{p}(u,v) \leq E_p(w) \leq \gamma_{p}(u,v). \]

We produce these \( p \)-harmonic maps by applying standard mountain pass methods to the generalized Ginzburg-Landau functionals studied by Wang in \[37\]. Fixing once again an isometric embedding
\[ N \subset \mathbb{R}^L \]
of our target manifold \( N \) into some higher-dimensional Euclidean space, we consider a function \( F : C^\infty(\mathbb{R}^L) \) satisfying
\[ F(x) = \text{dist}(x, N)^2 \text{ when dist}(x, N) < \delta_{N} \]
on the \( \delta_{N} \)-tubular neighborhood of \( N \),
\[ F(x) \geq \delta_{N}^2 \text{ when dist}(x, N) \geq \delta_{N}, \]
and (for technical reasons)
\[ F(x) = |x| \text{ for } |x| > R_0, \text{ some large radius}. \]

For \( p \in (1, \infty) \) and \( \epsilon > 0 \), the generalized Ginzburg-Landau functionals
\[ E_{p,\epsilon} : W^{1,p}(M, \mathbb{R}^L) \to \mathbb{R} \]
can then be defined by

\[(5.2) \quad E_{p,\epsilon}(w) := \int_M (|dw|^p + \epsilon^{-p} F(w)).\]

For the \((k - 2)\)-homotopic maps \(u, v \in C^\infty(M, N)\) and \(p \in (1, k)\), we then define the mountain pass energies \(\gamma_{GL,p,\epsilon}(u, v)\) to be the infimum

\[(5.3) \quad \gamma_{GL,p,\epsilon}(u, v) := \inf \{ \max_{t \in [0, 1]} E_{p,\epsilon}(u_t) \mid u_0 = u, \ u_1 = v \}\]

of the maximum energy \(\max_{t \in [0, 1]} E_{p,\epsilon}(u_t)\) over all continuous paths \(t \mapsto u_t\) in \(C^0([0, 1], W^{1,p}(M, \mathbb{R}^L))\) from \(u_0 = u\) to \(u_1 = v\). It follows immediately that

\[(5.4) \quad \gamma_{GL,p,\epsilon}(u, v) \leq \gamma_p(u, v)\]

for every \(\epsilon > 0\), since any continuous family \(u_t \in W^{1,p}(M, N)\) connecting \(u\) to \(v\) through \(N\)-valued maps satisfies

\[\gamma_{GL,p,\epsilon}(u, v) \leq \max_{t \in [0, 1]} E_{p,\epsilon}(u_t) = \max_{t \in [0, 1]} E_p(u_t).\]

Now, since the generalized Ginzburg-Landau energies \(E_{p,\epsilon}\) are \(C^1\) functionals on the Banach space \(W^{1,p}(M, \mathbb{R}^L)\), and satisfy a Palais-Smale condition (see, e.g., \[36\], Section 7.1), we can appeal to standard existence results for critical points of mountain pass type (see \[15\], Chapter 6) to arrive at the following lemma.

**Lemma 5.2.** For any \(p \in (1, k)\) and \(\epsilon > 0\), if

\[(5.5) \quad \gamma_{GL,p,\epsilon}(u, v) > \max\{E_{p,\epsilon}(u), E_{p,\epsilon}(v)\} = \max\{E_p(u), E_p(v)\},\]

then there exists a critical point \(w_\epsilon\) of \(E_{p,\epsilon}\) of energy

\[E_{p,\epsilon}(w_\epsilon) = \gamma_{GL,p,\epsilon}(u, v).\]

In light of the upper bound \((5.4)\) for the energies \(\gamma_{GL,p,\epsilon}(u, v)\), the critical points \(w_\epsilon\) given by Lemma 5.2 have uniformly bounded energies \(E_{p,\epsilon}(w_\epsilon)\) as \(\epsilon \to 0\). For non-integer \(p \in (1, k) \setminus \mathbb{N}\), it therefore follows from the compactness results of \[37\] (namely, \[37\], Corollary B) that some subsequence \(w_{\epsilon_j}\) converges strongly to a stationary \(p\)-harmonic map \(w \in W^{1,p}(M, N)\). In particular, we have the following existence result.

**Proposition 5.3.** For every \(p \in (1, k) \setminus \mathbb{N}\), if

\[\max\{E_p(u), E_p(v)\} < \lim_{\epsilon \to 0} \gamma_{GL,p,\epsilon}(u, v) = \sup_{\epsilon > 0} \gamma_{GL,p,\epsilon}(u, v),\]

then there exists a stationary \(p\)-harmonic map \(w \in W^{1,p}(M, N)\) of energy

\[(5.6) \quad E_p(w) = \sup_{\epsilon > 0} \gamma_{GL,p,\epsilon}(u, v).\]
From (5.4), it’s clear that the maps $w$ obtained in Proposition 5.3 satisfy
\[ E_p(w) \leq \gamma_p(u, v). \]
Thus, to complete the proof of Theorem 5.1, it remains only to establish the lower bound
\[ (5.7) \quad \gamma^*_p(u, v) \leq \sup_{\varepsilon > 0} \gamma_{GL,p,\varepsilon}(u, v). \]
This will follow fairly easily from the definition of $\gamma^*_p(u, v)$ and the following easy lemma.

**Lemma 5.4.** For every $\eta > 0$, there exists some $\varepsilon_0(p, \eta) > 0$ such that if $\varepsilon < \varepsilon_0$ and $w \in W^{1,p}(M, \mathbb{R}^L)$ satisfies
\[ E_{p,\varepsilon}(w) < \gamma_p(u, v) + 1, \]
then there exists $w' \in W^{1,p}(M, N)$ such that
\[ \|w - w'\|_{L^p} < \eta \]
and
\[ E_p(w') \leq E_{p,\varepsilon}(w) + \eta. \]

**Proof.** This is another simple proof by contradiction via compactness. If, for some $\eta > 0$, no such $\varepsilon_0(p, \eta)$ existed, then we could find a sequence $\varepsilon_j \to 0$ and $w_j \in W^{1,p}(M, \mathbb{R}^L)$ such that
\[ \limsup_{j \to \infty} E_{p,\varepsilon_j}(w_j) \leq \gamma_p(u, v) + 1 < \infty \]
and for every $j$ and $w' \in W^{1,p}(M, N)$, either
\[ (5.8) \quad \|w_j - w'\|_{L^p} > \eta \text{ or } E_p(w') > E_{p,\varepsilon}(w_j) + \eta. \]
But, passing to a further subsequence, we can find $w \in W^{1,p}(M, \mathbb{R}^L)$ for which $w_j \rightharpoonup w$ in $W^{1,p}$ and
\[ (5.9) \quad \|w_j - w\|_{L^p} \to 0. \]
Since the energies $E_{p,\varepsilon_j}(w_j)$ are uniformly bounded as $\varepsilon_j \to 0$, we see that
\[ \lim_{j \to \infty} \int_M F'(w_j) = 0, \]
and consequently $w \in W^{1,p}(M, N)$. And of course, it follows from the weak convergence that
\[ (5.10) \quad E_p(w) \leq \liminf_{j \to \infty} E_{p,\varepsilon}(w_j), \]
which, together with (5.9), contradicts (5.8). □
Now, for any $\delta > 0$, choose $\epsilon_0 = \epsilon_0(p, \delta/3)$ according to Lemma 5.4, and for $\epsilon < \epsilon_0$, consider a path $w_t \in W^{1,p}(M, \mathbb{R}^L)$ connecting $w_0 = u$ to $w_1 = v$, such that
\[
\max_{t \in [0,1]} E_{p,\epsilon}(w_t) \leq \gamma_{GL,p,\epsilon}(u,v) + \delta.
\]
Select a sequence of times $0 = t_0 < t_1 < \cdots < t_r = 1$ such that
\[
\|w_{t_i} - w_{t_{i-1}}\|_{L^p} < \frac{\delta}{3},
\]
and for each $1 \leq i \leq r - 1$, appeal to Lemma 5.4 to find a map $u_i \in W^{1,p}(M,N)$ such that
\[
\|u_i - w_{t_i}\|_{L^p} < \frac{\delta}{3}
\]
and
\[
E_{p}(u_i) \leq \gamma_{GL,p,\epsilon}(u,v) + 2\delta.
\]
It follows from (5.11) and (5.12) that the sequence
\[
u = u_0, u_1, \ldots, u_{r-1}, u_r = v
\]
belongs to $C_{\delta}^p(u,v)$, and from (5.13), we therefore see that
\[
\gamma_{\delta}(u,v) \leq \gamma_{GL,p,\epsilon}(u,v) + 2\delta.
\]
In particular, we’ve now shown that
\[
\gamma_{\delta}(u,v) \leq \sup_{\epsilon > 0} \gamma_{GL,p,\epsilon}(u,v) + 2\delta
\]
for every $\delta > 0$. Taking the limit as $\delta \to 0$, we arrive at the desired lower bound (5.7), together with (5.4) and Proposition 5.3, this completes the proof of Theorem 5.1.

6. Appendix

6.1. Remarks on the Proof of Lemma 2.1

Here, we provide a few comments on how Lemma 2.1 follows from the arguments of ([19], Section 3 and 4; see also [18], Section 2).

To begin, we fix some (piecewise) smooth cubeulation $h : |K| \to M$ of $M$ (following, for instance, the construction in [38]), where $K$ is a cubical complex all of whose faces are isometric to $[-1,1]^n$, and
\[
Lip(h) + Lip(h^{-1}) \leq C(M).
\]
We remark that it is enough to prove Lemma 2.1 for rational scales $\delta = \frac{1}{m}$, and henceforth (as in [18]) we restrict ourselves to this case. Beginning from the initial cubeulation $K = K_1$ above, we can then subdivide each $n$-cell into $m^n$ copies of $[-\delta,\delta]^n$, to obtain a new complex $K_\delta$ with the same underlying space $|K_\delta| = |K|$ as the initial one.

Now, consider an isometric embedding $M \subset \mathbb{R}^L$ into a high-dimensional Euclidean space, and fix $\epsilon(M) > 0$ such that the nearest point projection...
\( \Pi_M \) onto \( M \) is well-defined and smooth on the \( \epsilon(M) \)-neighborhood of \( M \) in \( \mathbb{R}^L \). As in [19], define the map
\[
H : |K| \times B^L_\epsilon(0) \to M
\]
by setting
\[
H(x, \xi) := \Pi_M(h(x) + \xi).
\]
By choosing \( \epsilon(M) \) sufficiently small, we can then arrange that
\[
\|H\|_{\text{Lip}} \leq C(M),
\]
and the maps
\[
h_\xi := H(\cdot, \xi) : |K| \to M
\]
are invertible, with
\[
\text{Lip}(h_\xi) + \text{Lip}(h_\xi^{-1}) \leq C(M).
\]
Moreover, we can arrange that the Jacobian determinant
\[
JH_{j, \delta}(x, \xi) := \det(DH_{j, \delta}(x, \xi) \circ [DH_{j, \delta}(x, \xi)]^*)^{1/2}
\]
of the restriction \( H_{j, \delta} := H|_{K^j_\delta} \) of \( H \) to the \( j \)-skeleton \( K^j_\delta \) has a uniform lower bound
\[
JH_{j, \delta}(x, \xi) \geq C(M)^{-1} > 0.
\]
Next, as in Section 4 of [19], fix \( y \in M \), and consider the map
\[
\psi : |K| \times \{ \xi \in \mathbb{R}^L \mid \xi \perp T_y M, \ |\xi| \leq \epsilon(M) \} \to |K| \times \mathbb{R}^L
\]
mapping the product of \( |K| \) with the normal disk \( D^\perp_\epsilon(y) \) to \( M \) at \( y \) to \( |K| \times \mathbb{R}^L \) by
\[
\psi(x, y) := (x, y + \xi - h(x)).
\]
For any subset \( A \subset |K| \), we then observe that
\[
H^{-1}(y) \cap A \subset \psi(A).
\]
In particular, for the skeleta \( |K^j_\delta| \) of \( K_\delta \), it follows that
\[
\mathcal{H}^{L-n+j}(H^{-1}(y) \cap |K^j_\delta|) \leq \mathcal{H}^{L-n+j}(\psi(|K^j_\delta| \times D^\perp_\epsilon(y)) \leq C(M)\delta^{j-n},
\]
where in the last inequality we have used the area formula for the map \( \psi \) together with the simple estimate \( \mathcal{H}^j(|K^j_\delta|) \leq C(K_1)\delta^{j-n} \) (since \( K^j_\delta \) comprises \( C(K_1)\delta^{-n} \) \( j \)-cells of size \( \delta \)).

Armed with the estimates (6.1)-(6.4), one can now employ the coarea formula and argue exactly as in Section 3 of [19] to conclude the proof of Lemma 2.1.
6.2. Upper Bounds for $\gamma_p(u,v)$ from the Hang-Lin Construction.

We recall now the construction of [19] (cf. also [9] in the case that either $u$ or $v$ is constant), and explain how it leads immediately to a proof of Theorem 1.1

**Proposition 6.1.** (cf. [19]) Let $u,v \in C^\infty(M,N)$ be $(k-2)$-homotopic for some $k \leq n = \dim(M)$. Then there is a path of maps $t \mapsto u_t$ with $u_0 = u$, $u_1 = v$, continuous in $W^{1,p}(M,N)$ for every $1 \leq p < k$, such that

\[
\sup_{t \in [0,1]} E_p(u_t) \leq \frac{C}{k-p}
\]

for some $C$ independent of $p$.

**Proof.** To begin, fix a smooth cubeulation $h : |K| \to M$, where $K$ is a cubical complex built of $n$-cells isometric to $[-1,1]^n$. In what follows, we will frequently identify $M$ and $|K|$ without comment. Since we’ve taken $u$ and $v$ to be smooth, note that the restrictions $u_{|K^j}$ and $v_{|K^j}$ of $u$ and $v$ to the lower-dimensional skeleta of $K$ define Lipschitz maps from $|K^j|$ to $N$.

Recalling the terminology of Section 2.1, we observe now that there exists a path of maps $u_t$ connecting $u$ to $v \circ \Phi_k$, such that $t \mapsto u_t$ is continuous in $W^{1,p}(M,N)$ for each $p < k$, with the desired energy bounds. Indeed, it follows directly from Lemma 2.23 that the path\[
[0,1] \ni s \mapsto u \circ \phi_{n,s}
\]
connecting $u \circ \Phi_n$ to $u$ satisfies the desired properties, as do the paths\[
[0,1] \ni s \mapsto u \circ \phi_{j,s} \circ \Phi_{j+1}
\]
connecting $u \circ \Phi_{j+1}$ to $u \circ \Phi_j$ for each $k \leq j \leq n-1$. Concatenation yields the desired path $u_t$ from $u$ to $u \circ \Phi_k$, and in the same way we can construct such a path connecting $v$ to $v \circ \Phi_k$.

It remains now to construct a path of maps $u_t$ from $M$ to $N$ connecting $u \circ \Phi_k$ to $v \circ \Phi_k$, in such a way that $t \mapsto u_t$ is continuous in $W^{1,p}(M,N)$ and

\[
\max_t E_p(u_t) \leq \frac{C}{k-p}
\]

for every $1 \leq p < k$. In fact, it is enough to construct such a path of maps\[
w_t : |K^k| \to N
\]
connecting $u \circ \phi_{k,0}$ to $v \circ \phi_{k,0}$ on the $k$-skeleton $|K^k|$, since we can then take $u_t := w_t \circ \Phi_{k+1}$ to obtain the desired path of maps on $M$. In the remainder of the proof, we construct such a path $w_t : |K^k| \to N$.

Since the maps $u$ and $v$ are $(k-2)$-homotopic, their restrictions $u_{|K^{k-2}}$ and $v_{|K^{k-2}}$ to the $(k-2)$-skeleton $|K^{k-2}|$ are homotopic, by definition. And since the pair $(|K^{k-1}|, |K^{k-2}|)$ satisfies the homotopy extension property (cf. [19], Proposition 2.1), we can therefore find a map\[
u_2 : |K^{k-1}| \to N
\]
such that \( u|_{K^{k-1}} \) is homotopic to \( u_2 \) on \( |K^{k-1}| \) and \( u_2 \) agrees with \( v \)

\[
u_2|_{K^{k-2}} = v|_{K^{k-2}}\]
on \( |K^{k-2}| \). Moreover, it’s easy to check (cf. [19], Sections 2.2-2.3) that we can take both the map \( u_2 \) and the homotopy \( f : |K^{k-1}| \times [0, 1] \to N \) from \( u|_{K^{k-1}} \) to \( u_2 \) to be Lipschitz. The precomposition \( f_t \circ \Phi_k \) of the homotopy \( f_t \) with \( \Phi_k \) then gives us a path of maps \( M \to N \) connecting \( u \circ \Phi_k \) to \( u_2 \circ \Phi_k \), which evidently satisfies the desired estimates and continuity properties in \( W^{1,p}(M, N) \) for \( 1 \leq p < k \).

In particular, to complete the proof of the proposition, we now see that it suffices to construct a path of maps \( w_2 : |K^k| \to N \), continuous in \( W^{1,p}(|K^k|, N) \) for \( 1 \leq p < k \), satisfying

\[
\max_i E_p(w_i, |K^k|) \leq \frac{C}{k-p},
\]

that connects \( v \circ \phi_{k,0} \) to \( u_2 \circ \phi_{k,0} \), where \( u_2 \in \text{Lip}(|K^{k-1}|, N) \) agrees with \( v \) on the \((k-2)\)-skeleton \( |K^{k-2}| \). To do this, we enumerate the \((k-1)\)-cells \( \sigma_1, \ldots, \sigma_m \in K^{k-1} \setminus K^{k-2} \), and define maps \( w_0, \ldots, w_m \in W^{1,p}(|K^k|, N) \) by

\[
w_i := f_i \circ \phi_{k,0},
\]

where the maps \( f_i \in \text{Lip}(|K^{k-1}|, N) \) are defined by \( f_0 = v|_{K^{k-1}} \), \( f_m = u_2 \), and

\[
f_i(x) := v(x) \text{ for } x \in |K^{k-1}| \setminus (\sigma_1 \cup \cdots \cup \sigma_i),
\]

\[
f_i(x) := u_2(x) \text{ for } x \in \sigma_1 \cup \cdots \cup \sigma_i,
\]

for \( 1 \leq i \leq m \). (That these \( f_i \) are Lipschitz follows from the fact that \( u_2 = v \) on \( |K^{k-2}| \).) We claim that each \( w_i \) can be deformed into \( w_{i+1} \) through a path of maps \( w_t \) satisfying the desired properties; once we have constructed these paths, concatenation evidently gives the desired path from \( v \circ \phi_{k,0} \) to \( u_2 \circ \phi_{k,0} \).

Now, fix \( i \in \{1, \ldots, m\} \). By construction, the maps \( f_{i-1}, f_i \in \text{Lip}(|K^{k-1}|, N) \) coincide on the complement \( |K^{k-1}| \setminus \sigma_i \) of the \((k-1)\)-cell \( \sigma_i \). Consider the star neighborhood

\[
V := \bigcup \{ \Delta \in K^k \text{ a } k\text{-cell} \mid \sigma_i \subset \partial \Delta \}
\]
of \( \sigma_i \), which we can identify in a bi-Lipschitz way with

\[
W = \bigcup_{j=1}^a W_j \subset \mathbb{R}^{k-1+a},
\]

where

\[
W_j := \{(x, 0, \ldots, 0) + te_{k-1+j} \mid x \in [-1,1]^{k-1}, \ 0 \leq t \leq 1\},
\]

and \( a \) is simply the number of distinct \( k \)-cells for which \( \sigma \cong [-1,1]^{k-1} \times \{0\} \subset \mathbb{R}^{k-1+a} \) is a face.
Next, note that the boundary
\[ \partial V \cong W \cap \partial[-1, 1]^{k-1+a} \]
lies in \(|K^{k-1}| \setminus \sigma_i\), so that the maps
\[ f_i = f_{i-1} =: g \in Lip(\partial V, N) \]
agree on \(\partial V\). For \(t \in [0, \frac{1}{2}]\), we can then define maps \(w_{i-1+t} : |K^k| \to N\) by setting
\[ w_{i-1+t} := w_{i-1} = w_i \text{ on } |K^k| \setminus V, \]
and (identifying \(V\) with \(W\))
\[ w_{i-1+t}(x) := w_{i-1} \left( \frac{x}{\max\{1-(2t), |x|_\infty\}} \right) \text{ for } x \in V. \]
We can then check by direct computation, as in the proof of Lemma 2.3, that \([0, \frac{1}{2}] \ni t \mapsto w_{i-1+t}\) satisfies the desired energy estimates and continuity properties, while connecting \(w_{i-1}\) to the map \(w_{i-0.5}\) given by
\[ w_{i-0.5} := w_i \text{ on } |K^k| \setminus V \]
and
\[ w_{i-0.5}(x) := g(x/|x|_\infty) \text{ for } x \in V. \]
Since \(w_i|_{\partial V} = g\) as well, we can employ the same construction to obtain a path
\[ \left[ \frac{1}{2}, 1 \right] \ni t \mapsto w_{i-1+t} \]
connecting \(w_{i-0.5}\) to \(w_i\) in the desired way. We have thereby constructed a path \([0, 1] \ni t \mapsto w_{i-1+t} : |K^k| \to N\) from \(w_{i-1}\) to \(w_i\) satisfying the desired estimates, completing the proof. □

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