Estimation and Inference by Stochastic Optimization

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Abstract

In non-linear estimations, it is common to assess sampling uncertainty by bootstrap inference. For complex models, this can be computationally intensive. This paper combines optimization with resampling: turning stochastic optimization into a fast resampling device. Two methods are introduced: a resampled Newton-Raphson (rNR) and a resampled quasi-Newton (rqN) algorithm. Both produce draws that can be used to compute consistent estimates, confidence intervals, and standard errors in a single run. The draws are generated by a gradient and Hessian (or an approximation) computed from batches of data that are resampled at each iteration. The proposed methods transition quickly from optimization to resampling when the objective is smooth and strictly convex. Simulated and empirical applications illustrate the properties of the methods on large scale and computationally intensive problems. Comparisons with frequentist and Bayesian methods highlight the features of the algorithms.

JEL Classification: C2, C3
Keywords: Stochastic gradient descent, M-estimation, m out of n and multiplier Bootstrap.

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1 Introduction

Many empirical economic analyses involve the process of estimating parameters $\theta$ from a sample of $n$ observations and assessing sampling uncertainty. The typical routine is to first produce a consistent M-estimator $\hat{\theta}_n$ of the true $\theta^*$ by minimizing the sample objective $Q_n$. Then, standard errors and confidence intervals are computed to perform inferences. Because the sandwich estimator is harder to implement for more complicated models, it is very common to use a bootstrap at that stage. For more computationally intensive nonlinear estimations, it is common to report standard errors based on few bootstrap draws, e.g. 20, 25, or 50. The main bottleneck is that each replication requires another estimation on resampled data, which is nontrivial for these complex models. There is a long-standing interest in finding shortcuts to relieve this computational burden without sacrificing too much accuracy. Examples include Davidson and MacKinnon (1999), Andrews (2002), Kline and Santos (2012), and more recently Honoré and Hu (2017). These methods compute standard errors taking a converged estimate $\hat{\theta}_n$ as given. As such, estimation always precedes inference.

This paper proposes to combine resampling with optimization to produce estimates, confidence intervals, and standard errors in a single step. This is very close in spirit to Markov-chain Monte Carlo (MCMC) methods that produce draws from a Bayesian posterior to be used for both estimation and inference. Unlike MCMC, the information matrix equality is not required for valid inference. The main loop of the Algorithm is a stochastic optimization routine which evaluates the objective on batches of resampled data with size $m \leq n$. Starting from an initial guess $\theta_0$, one updates $\theta_b$ to $\theta_{b+1}$, $b \geq 0$, using the resampled gradient and inverse Hessian (or an approximation) as a conditioning matrix, with a fixed learning rate $\gamma$. The Algorithm can also be used with reweighted samples of data. For a suitable choice of $\gamma$ and a strictly convex objective $Q_n$, it is shown that the average over $B$ draws $\theta_b$ is equal to $\hat{\theta}_n$ up to order $1/min(m, B)$. Even with $m$ and $B \ll n$, the estimates are consistent and asymptotically normal as long as $m$ and $B \gg n$. The results also show that the distribution, conditional on the sample, of $\sqrt{m}(\theta_b - \hat{\theta}_n)$ is first-order equivalent to that of $\sqrt{n}(\hat{\theta}_n - \theta^*)$, up to scale, and standard errors are consistent. Inference results are valid for a resampled Newton-Raphson (rNR) and a resampled quasi-Newton (rqN) algorithm based on a new quasi-Newton update. For other conditioning matrices, the draws can produce...
consistent estimates but they need not be valid for inference.

The main theoretical insight of the paper is that \( r_{NR} \) and \( rqN \) draws are well approximated by a simple AR(1) process when the resampling size \( m \) is sufficiently large. The process is centered around the full sample estimate \( \hat{\theta}_n \), and its innovations have variance proportional to the sandwich formula. Hence, after burning-in the first couple of iterations, the distribution of the draws can be used as a bootstrap distribution – up to a simple adjustment which accounts for the Markov-chain properties of the algorithms. The main results assume the sample and resampled objective functions are smooth and strictly convex. Additional results and illustrations are given in the Supplement for some non-strictly convex settings. In particular, the algorithms can perform well in situations where their non-resampled counterparts are typically challenged and return inconsistent estimates. This is another advantage of combining optimization with resampling.

After introducing the algorithms, a small empirical example illustrates the properties of the methods. Then one simulated and two empirical examples illustrate different ways in which they can be applied in empirical work. First, the Monte-Carlo example considers the estimation of a dynamic discrete choice model with heterogeneous agents. Estimates are significantly biased because of non-linearity, resulting in poor inferences due to size distortions. A bootstrap bias correction would be computationally intensive. \( rqN \), combined with a simple split-panel bias-correction technique, produces accurate estimates and valid inferences in reasonable time. Next, using data from Helpman et al. (2008), \( r_{NR} \) and \( rqN \) are benchmarked against MCMC and stochastic gradient-descent (sGD) on a large-scale probit. \( r_{NR} \) and \( rqN \) converge faster and have better mixing so that fewer draws are needed. Allowing for dependence to compute cluster robust standard errors is straightforward, which is not the case with MCMC. Finally, a more complex model is considered. A replication of Donaldson (2018) shows how a computationally intensive grid search used for both estimation and bootstrapping on a 100-core cluster environment can be replaced with \( r_{NR} \), running in less than six hours on a desktop computer. The time saved allows to explore other model specifications. The results also highlight identification issues which are easily diagnosed with \( r_{NR} \). Additional empirical and simulated examples are given in a companion paper (Forneron and Ng, 2021). Finally, while the results in this paper are targeted at a class of smooth and convex M-estimation problems, the challenges of non-smooth and non-convex estimation constitute a serious impediment to empirical research. Another companion paper provides algorithms and theoretical results for finite-sample generalized method of moments estimation without assuming smoothness or convexity (Forneron, 2022).
Outline of the paper. Section 2 begins with a review of classical and stochastic optimization. The proposed algorithms are presented in Section 3, related methods are discussed. A simple analytical and empirical example highlight the theoretical aspects of the paper. Section 4 provides the theoretical results. Section 5 covers the simulated and empirical examples. Appendices A-H provide proofs, sample R codes, and primitive conditions for some of the assumptions. Issues related to non-convexity are covered in Appendix I.

2 Setting and Overview of Gradient-Based Optimization

Consider minimizing a twice-differentiable and convex sample objective function $Q_n(\theta)$ with respect to parameters $\theta \in \Theta \subset \mathbb{R}^d$:

$$\hat{\theta}_n = \arg\min_{\theta \in \Theta} Q_n(\theta), \quad Q_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} q(z_i; \theta),$$

using a sample of $n$ iid observations $z_i = (y_i, x_i)$. Examples include non-linear least-squares (NLS) for which $q(z_i; \theta) = [y_i - f(x_i; \theta)]^2$ where $f$ is the regression function, known up to the parameters of interest $\theta$. For (pseudo) maximum likelihood estimation (MLE), $q(z_i; \theta) = -\ell(y_i; \theta | x_i)$ is the negative of the log-likelihood. The true value, denoted by $\theta^\dagger$, minimizes the limiting objective $Q = \text{plim}_{n \to \infty} Q_n$. Under the conditions in Theorem 2.1 of Newey and McFadden [1994], $\hat{\theta}_n$ is consistent for $\theta^\dagger$. If, in addition, the assumptions in Theorem 3.1 of Newey and McFadden [1994] hold, then $\hat{\theta}_n$ is also asymptotically normal:

$$\sqrt{n}V_n^{-1/2}(\hat{\theta}_n - \theta^\dagger) \overset{d}{\to} N(0, I_d),$$

where $V_n = [H_n(\hat{\theta}_n)]^{-1} \Sigma_n[H_n(\hat{\theta}_n)]^{-1}$ is a consistent estimator of the sandwich variance, $[H_n(\hat{\theta}_n)]^{-1} \overset{p}{\to} [H(\theta^\dagger)]^{-1}$ approximates the bread, and $\Sigma_n \overset{p}{\to} \lim_{n \to \infty} \text{var}[\sqrt{n}G_n(\theta^\dagger)]$ approximates the meat. Practitioners are often interested in conducting inference for a function of the parameters $h(\theta) \in \mathbb{R}$. Examples include inference on a single coefficient or a counterfactual based on the structural estimates $\hat{\theta}_n$. When $h$ is continuously differentiable, the delta-method gives the standard error formula $\text{se}[h(\hat{\theta}_n)] = \sqrt{\nabla h(\hat{\theta}_n)V_n \nabla h(\hat{\theta}_n)' / n}$ and the 95% level confidence interval $h(\hat{\theta}_n) \pm 1.96 \times \text{se}[h(\hat{\theta}_n)]$. It is also common to compute confidence intervals from bootstrap draws using quantiles or the standard deviation of $h(\hat{\theta}_n^{(b)})$, where $\hat{\theta}_n^{(b)}$ minimizes the resampled objectives $Q_n^{(b)}$.

For most problems, the estimator $\hat{\theta}_n$ does not have closed-form and is usually obtained by numerical optimization. The methods below rely on the sample gradient and Hessian of
$Q_n$, defined respectively by:

$$G_n(\theta) = \nabla Q_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \nabla q(z_i; \theta), \quad H_n(\theta) = \nabla^2 Q_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \nabla^2 q(z_i; \theta).$$

### 2.1 Classical Optimizers

When $Q_n$ is quadratic, as in OLS or IV regressions, the solution has closed form and $\hat{\theta}_n = \theta_0 - [H_n(\theta_0)]^{-1}G_n(\theta_0)$, for any initial guess $\theta_0$. For general non-quadratic objectives, the solution does not have closed-form. A typical strategy is to consider a sequence of approximations:

$$Q_n(\theta) \simeq Q_n(\theta_k) + G_n(\theta_k)(\theta - \theta_k) + \frac{1}{2\gamma_k}(\theta - \theta_k)'P_k^{-1}(\theta - \theta_k), \quad k = 0, 1, 2, \ldots$$

where $P_k$ is a symmetric positive definite conditioning matrix, $\gamma_k \in (0, 1]$ is called the learning rate, it penalizes for the quality of the approximation \([1]\). For $\gamma_k = 1$ and $P_k^{-1} = H_n(\theta_k)$, the approximation is exact for $Q_n$ quadratic. Otherwise, it coincides with a second-order Taylor expansion around $\theta = \theta_k$, which is not exact. From a guess $\theta_k$, minimizing \([1]\) leads to:

$$\theta_{k+1} \equiv \theta_k - \gamma_k P_k G_n(\theta_k), \quad k = 0, 1, 2, \ldots$$

Different choices of conditioning matrix $P_k$ lead to different algorithms. Newton-Raphson (NR) iterations use $P_k = [H_n(\theta_k)]^{-1}$ and $\gamma_k = 1$; a damped update uses $\gamma_k < 1$. For strongly convex problems, NR enjoys a fast quadratic convergence property, but it comes at the cost of computing both the gradient and the Hessian at each iteration $k$.\footnote{To avoid the cost of computing the $(d_q + 1)d_q/2$ second-order derivatives in $H_n$. Other updates rely on different choices of $P_k$. These include quasi-Newton (qN) and gradient-descent (GD).}

The most popular quasi-Newton update is BFGS for which $P_k$ approximates $[H_n(\theta_k)]^{-1}$ iteratively using only gradients. Although not as fast as NR, convergence for quasi-Newton with an appropriate choice of $\gamma_k$ can be superlinear.\footnote{Quasi-Newton methods bypass Hessian computations, but analytical convergence results are more difficult to obtain. The gradient-descent algorithm (GD) sets $P_k = I_d$, it only requires computing a single gradient $G_n(\theta_k)$.}
Convergence, however, requires choosing $\gamma_k$ small enough that all eigenvalues of $I_d - \gamma_k H_n(\tilde{\theta}_k)$ lie in the restricted region $(-1, 1)$, for some intermediate value $\tilde{\theta}_k$. This results in very slow convergence when the ratio of the largest to smallest eigenvalue of $H_n$ is large.

The necessary conditions for a local minimum are $\|G_n(\hat{\theta}_n)\| = 0$ and $H_n(\hat{\theta}_n)$ positive semi-definite. The sufficient conditions are $\|G_n(\hat{\theta}_n)\| = 0$ and $H_n(\hat{\theta}_n)$ positive definite. Note that for non-convex objectives, any $\theta_k$ with $G_n(\theta_k) = 0$ is a fixed-point of the update rule (2).

This includes local and global minima and maxima as well as locally suboptimal solutions, called saddle points, for which $H_n$ has both positive and negative eigenvalues.

### 2.2 Stochastic Optimizers

In their seminal paper on stochastic approximation, Robbins and Monro (1951) consider the situation where only noisy gradients are available $G_n(\theta_k) + e_k$ with $e_k$ iid, mean-zero. Using the notation above, they propose the gradient-descent update:

$$\theta_{k+1} = \theta_k - \gamma_k [G(\theta_k) + e_k]$$

and show under regularity conditions that $\theta_k \xrightarrow{a.s.} \hat{\theta}_n$ when $\gamma_k > 0$ satisfy

$$\sum_{k=1}^{\infty} \gamma_k = +\infty, \quad \sum_{k=1}^{\infty} \gamma_k^2 < +\infty. \quad (3)$$

The first condition ensures that all possible solutions will be reached with high probability regardless of the $\theta_0$ while the second ensures convergence to the true value. Kiefer and Wolfowitz (1952) extend the results to non-linear least squares with gradients computed by finite-differences. Feasible choices of learning rate schedules include $\gamma_k = \gamma_0 k^{-\delta}$, with $\delta \in (1/2, 1]$ and $\gamma_0 > 0$ as choice parameters. Depending on $\delta$, convergence as measured by $E(\|\theta_k - \hat{\theta}_n\|^2)$ can occur at a $1/k$ rate or slower. Ruppert (1988) and Polyak and Juditsky (1992) show that the averaged values $\bar{\theta}_k = \frac{1}{k} \sum_{i=1}^{k} \theta_i$, converge at the fastest $1/k$ rate for all choices of $\delta \in (1/2, 1]$. This is commonly referred to as Polyak-Ruppert averaging. Building on these results, modern stochastic gradient-descent updates are given by

$$\theta_{k+1} = \theta_k - \gamma_k G_m(\theta_k)$$

where $G_m(\theta_k) = \frac{1}{m} \sum_{i=1}^{m} \nabla q(z_i^{(k)}, \theta_k)$ is an unbiased estimate of $G_n(\theta_k)$ based on $m < n$ randomly chosen observations $z_1^{(k)}, \ldots, z_m^{(k)}$ drawn at each $k$. Setting $e_k = G_m(\theta_k) - G_n(\theta_k)$ results in the same setting as stochastic approximation above. Though $m = 1$ is computationally inexpensive and is a popular choice, the stochastic gradients $G_m(\theta_k)$ are very noisy and a small $\gamma_k$ satisfying (3) is needed to compensate. This results in slow convergence making it well suited for problems where the gradients $G_m$ are very cheap to computed and
running very many iterations is feasible. Unfortunately, this is typically not the case for more complicated economic models. A number of methods can accelerate convergence, but they tend to converge slower than NR, or BFGS. More closely related to this paper, some study stochastic BFGS, but compute the full sample $G_n$ many times which is more demanding.

Beyond reducing the computational cost, stochastic optimization can improve on classical methods in non-convex settings. For instance, suboptimal solutions aren’t fixed points for SGD (Ge et al. 2015; Jin et al. 2017). Though SGD requires $\gamma_k \to 0$, it is common to use $\gamma_k = \gamma$ fixed in practice. Dieuleveut et al. (2020) consider $Q_n$ quadratic with $\gamma$ and $m$ fixed and find that it leads to estimation bias which needs to be corrected. Few results are available for inference, notably Chen et al. (2020) consider the computation of standard errors for SGD. This paper considers $\gamma$ fixed with $Q_n$ non-quadratic but requires $m \to \infty$. For inference, $P_k$ is either a Hessian or the new quasi-Newton update. The connection with bootstrap inference and the AR(1) representation are new.

3 Algorithms and Intuition

Because the iterations are used both for estimation and bootstrap inference, they will be indexed by $b = 0, 1, 2, \ldots$, rather than $k = 0, 1, 2, \ldots$ used for estimation only. The main algorithm below relies on a resampled objective $Q_m(b)$, and its derivatives $G_m(b), H_m(b)$, computed using either $m \leq n$ resampled observations, or using all $m = n$ observations but with random reweighting. Unless otherwise stated, $Q_m(b)$ will refer to either a resampled or reweighted objective. Using these quantities, Algorithm 1 describes a way to perform both estimation and inference for a function $h$ of $\theta$. Similar to MCMC, after discarding an initial burn-in period, estimates are computed by averaging over draws. Also, standard errors and confidence intervals are computed from the standard deviation and quantiles of the draws but here with an adjustment $\sqrt{m/n\phi(\gamma)}$ which accounts for $m$ potentially smaller than $n$, and $\phi(\gamma)$ adjusts for the Markov-chain properties of $\theta_b$ as explained below. Compared to SGD, Algorithm 1 requires $m \gg 1$ but allows for $m/n \to 0$ with the restriction that $\sqrt{n/m} \to 0$. This is the cost of using a fixed learning rate $\gamma$ rather than a decaying sequence $\gamma_k$ satisfying (3), used in nearly all of the literature.

The learning rate $\gamma$ controls the optimization behaviour of the Algorithm, a larger value
Algorithm 1 Resampled Estimation and Inference

1) Inputs (a) an initial guess $\theta_0$, (b) a bootstrap sample size $B$ and a burn-in period $\text{burn}$, (c) a batch size $m \leq n$ such that $m/n \to c \in [0, 1]$, and $\sqrt{n}/m \to 0$, (d) a fixed learning rate $\gamma \in (0, 1]$, (e) a resampling or re-weighting scheme.

2) Burn-in and Resample

for $b = 0, \ldots, \text{burn} + B - 1$ do

Resample, or re-weight, the data, $\triangleright$ see (4), (5)

Update the conditioning matrix $P_b$, $\triangleright$ see Algorithms 2 and 3

Compute $\theta_{b+1} = \theta_b - \gamma P_b G_m^{(b+1)}(\theta_b)$.

end for

3) Discard the first burn draws, re-index $\theta_{\text{burn}+b}$ to $\theta_b$ for $b = 1, \ldots B$.

4) Estimation

- Estimates: $\overline{\theta}_{\text{RE}} = \frac{1}{B} \sum_{b=1}^{B} \theta_b$

5) Inference (rnr and rqn)

- Standard errors: $\text{SE}[h(\hat{\theta}_n)] = \sqrt{\frac{m}{n \phi(\gamma)}} \frac{1}{B} \sum_{b=1}^{B} [h(\theta_b) - h(\overline{\theta}_{\text{RE}})]^2$

- Confidence interval $(1 - \alpha$ level$): \left[ c_{h,b}(\alpha/2), c_{h,b}(1 - \alpha/2) \right]$

where $\phi(\gamma) = \frac{\gamma^2}{1 - (1-\gamma)^2}$, $c_{h,b}(\alpha)$ is the $\alpha$-th quantile of $h(\tilde{\theta}_b) - \overline{\theta}_B$, $\alpha \in (0, 1)$, and $\tilde{\theta}_b = \overline{\theta}_{\text{RE}} + \sqrt{\frac{m}{n \phi(\gamma)}} (\theta_b - \overline{\theta}_{\text{RE}})$.

of $\gamma$ is associated with faster convergence. For quadratic $Q_n$, any $\gamma \in (0, 1]$ is feasible, otherwise a smaller $\gamma < 1$ is needed, $\gamma \in [0.1, 0.3]$ performs well in the Examples. The parameter $\gamma$ also affects the distributional properties of $\theta_b$ which can be more concentrated around $\hat{\theta}_n$ than conventional bootstrap draws. For instance, with $m = n$ and $\gamma = 0.1$ we have $1/\phi(\gamma) = 19$: the variance of $\theta_b$ is 19 times smaller than the variance of $\hat{\theta}_n$.

A number of different implementations are possible, a resampled Gradient-Descent (rgd) would use the same $P_b = I_d$ for each iteration $b$. Valid inference, however, requires implementations for which $P_b$ approximates the inverse Hessian sufficiently well. This is the case for the resampled Newton-Raphson and resampled quasi-Newton algorithms. The first is conceptually simple and provides good intuition for the results. For most applications, the second is more computationally efficient and will be preferred.

Depending on the choice of resampling or reweighting scheme, $G_m^{(b)}$, $H_m^{(b)}$ will be computed

\[ \phi \text{ is decreasing in } \gamma: \text{ the smaller } \gamma, \text{ the more localized the Markov-chain will be. For comparison } \phi(1) = 1, \phi(0.4) = 4, \phi(0.2) = 9, \text{ and } \phi(0.01) = 199. \]
differently. First, the resampled objective, gradient, and Hessian are:

\[ Q_m(b, \theta) = \frac{1}{m} \sum_{i=1}^m q(z_i^{(b)}, \theta), \]

\[ G_m(b, \theta) = \frac{1}{m} \sum_{i=1}^m \nabla q(z_i^{(b)}, \theta), \quad H_m(b, \theta) = \frac{1}{m} \sum_{i=1}^m \nabla^2 q(z_i^{(b)}, \theta), \quad (4) \]

where \( z_1^{(b)}, \ldots, z_m^{(b)} \) are drawn independently with replacement from the empirical data \((z_1, \ldots, z_n)\) at each iteration \( b \). Unlike sGD, the resampling should match the dependence structure of the data. For instance, clustered data should be resampled at the cluster rather than individual level. Second, the re-weighted objective, gradient, and Hessian are:

\[ Q_n(b, \theta) = \frac{1}{n} \sum_{i=1}^n w_i^{(b)} q(z_i, \theta), \]

\[ G_n(b, \theta) = \frac{1}{n} \sum_{i=1}^n w_i^{(b)} \nabla q(z_i, \theta), \quad H_n(b, \theta) = \frac{1}{n} \sum_{i=1}^n w_i^{(b)} \nabla^2 q(z_i, \theta), \quad (5) \]

where \( w_i^{(b)} \) are iid with mean and variance equal to 1. Examples include Gaussian weights \( w_i^{(b)} \sim \mathcal{N}(1, 1) \), exponential \( w_i^{(b)} \sim \exp(1) \), or Poisson \( w_i^{(b)} \sim \text{Pois}(1) \). This is known as a multiplier, or exchangeably weighted, bootstrap. Similar to resampling, the weights should vary only between clusters in clustered data, and more generally match the dependence of the data. Depending on the setting, reweighting can be preferred to resampling. For instance, if \( \theta \) includes fixed effects, resampling can produce samples with no observation that is informative about one of the fixed effects.

3.1 Resampled Newton-Raphson (rNR)

The resampled Newton-Raphson update described in Algorithm 2 below can be used within Algorithm 1 to produce consistent estimates and valid inferences. The implementation is very simple, as it only requires computing a Hessian. In many settings, however, repeatedly

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7 See van der Vaart and Wellner (1996, Ch3.6.2) and Kosorok (2007, Ch10). The Poisson bootstrap approximates, with computational advantages, the standard \( n \) out of \( n \) bootstrap (Chamandy et al., 2012).

8 This is also an issue for censored models where resampling can result in datasets with no censored observations. This also affects sGD; with \( m = 1 \) a gradient is only informative about at one fixed effect at a time. This makes convergence very slow, see Appendix 1.2 for an illustration.

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computing this matrix of second-order derivatives can be very demanding so that a quasi-
Newton update is often preferred. Nevertheless, the rNR implementation gives a good first
intuition about why and how Algorithm [1] produces estimates and valid inferences. This is
illustrated with a simple analytical and a small empirical example below.

A pen and pencil example. The following illustrates the main optimization and boot-
strap properties of rNR using a tractable OLS regression of $y_i$ on $x_i$: $Q_n(\theta) = \frac{1}{2n} \sum_{i=1}^{n} (y_i - x_i'\theta)^2$. Using $m$ out of $n$ resampling, we have $G_m^{(b+1)}(\theta) = -\frac{1}{m} \sum_{i=1}^{m} x_i^{(b+1)} (y_i^{(b+1)} - x_i^{(b+1)'}\theta)$, and $H_m^{(b)}(\theta) = \frac{1}{m} \sum_{i=1}^{m} x_i^{(b+1)'} x_i^{(b+1)'}$. In more compact notation: $G_m^{(b+1)}(\theta) = -\frac{1}{m} X_m^{(b+1)'} y_m^{(b+1)} - X_m^{(b+1)'} \theta$. For $b \geq 0$, one rNR update takes the form

$$
\theta_{b+1} = \theta_b + \gamma X_m^{(b+1)'} X_m^{(b+1)} \theta_b.
$$

Notice that $\left(X_m^{(b+1)'} X_m^{(b+1)}\right)^{-1} X_m^{(b+1)'} y_m^{(b+1)} = \hat{\theta}_m^{(b+1)}$ is the bootstrap OLS estimate. Subtract $\hat{\theta}_n$ on both sides and re-arrange terms to find that:

$$
\theta_{b+1} - \hat{\theta}_n = (1 - \gamma) (\theta_0 - \hat{\theta}_n) + \gamma (\hat{\theta}_m^{(b+1)} - \hat{\theta}_n)
$$

is an AR(1) process with conditionally iid innovations based on the centered bootstrap
estimates: $\hat{\theta}_m^{(b)} - \hat{\theta}_n$. Notice that neither $\hat{\theta}_n$ nor $\hat{\theta}_m^{(b)}$ are computed explicitly in the rNR update. Using a backwards recursion, we have:

$$
\theta_{b+1} - \hat{\theta}_n = (1 - \gamma)^b (\theta_0 - \hat{\theta}_n) + \gamma \sum_{j=0}^{b} (1 - \gamma)^j (\hat{\theta}_m^{(b+1-j)} - \hat{\theta}_n)
$$

Suppose, for simplicity, that under the bootstrap expectation $E^*(\hat{\theta}_m^{(b+1)}) = \hat{\theta}_n$. Then the starting value bias due to $\theta_0 \neq \hat{\theta}_n$ is

$$
E^*(\theta_{b+1} - \hat{\theta}_n) = (1 - \gamma)^b (\theta_0 - \hat{\theta}_n),
$$

which converges exponentially fast to zero. Very quickly, the bias becomes negligible and
the average draw consistently estimates $\hat{\theta}_n$. A seamless transition occurs from optimization
to resampling. Using the formula for the variance of an AR(1) process, we have

$$
\text{var}^*(\theta_{b+1}) = \gamma^2 \frac{1 - (1 - \gamma)^{b+1}}{1 - (1 - \gamma)^2} \text{var}^*(\hat{\theta}_m^{(b)}) \simeq \phi(\gamma) \text{var}^*(\hat{\theta}_m^{(b)}),
$$

Derivations for rNR and other methods are given in Appendix C.
which also converges exponentially fast. The factor \( \phi(\gamma) = \gamma^2 / [1 - (1 - \gamma)^2] \) is used in Algorithm 1. Assuming \( \text{var}^*(\hat{\theta}_m^{(b)}) \) consistently estimates standard errors, then so does \( r_{\text{NR}} \) with the suggested adjustment. The AR(1) representation only holds if \( P_b \) approximates the inverse Hessian sufficiently well, this is not the case for \( \text{GD} \) and several quasi-Newton updates.

**Illustration with real data.** The main results extend the properties described for OLS above to more general non-quadratic strictly convex objectives. The main insight is that even though the AR(1) representation above does not hold exactly, there exists a sequence \( \theta_{b}^* \), called coupling, which follows an AR(1) process and is uniformly close to the Markov-chain \( \theta_{b} \) up to an error of order \( 1/m \). A good approximation requires \( m \to \infty \) but still allows for \( m/n \to 0 \). To illustrate this feature, Figure 1 shows the draws for the coefficient on education for a probit model fitted on data from Mroz (1987). This example uses \( \gamma = 0.3 \) and resamples \( m = n/2 \) out of \( n = 753 \) observations with replacement. The top left panel shows the first 20 iterations, which corresponds to the initial convergence phase. The resampled \( r_{\text{NR}} \) (black) and the deterministic \( r_{\text{NR}} \) (blue) have very similar paths, moving quickly to the true value (dashed black). Then, the top right panel shows the next 200 iterations – the resampling phase. Even with \( m < n \), the \( r_{\text{NR}} \) draws (black) are very close to the AR(1) process \( \theta_{b}^* \) (blue) used to prove the results. The bottom panel illustrates the bootstrap properties of the draws. The left panel compares a \( N(\hat{\theta}_n, V_n/n) \) with the standard \( n \) out of \( n \) bootstrap, and \( r_{\text{NR}} \) draws adjusted for \( m \) and \( \phi(\gamma) \). The distributions are very close. The right panel
further confirms this with a more detailed q-q plot view. Replication code for this example is provided in Appendix E with a detailed comparison of rNR, rqN, for different choices of \( m \), with standard errors computed using the sandwich formula, the standard bootstrap, and methods by Davidson and MacKinnon (1999), Kline and Santos (2012). Additional comparisons with these methods can be found in Forneron and Ng (2021).

3.2 Resampled quasi-Newton (rqN)

In practice, quasi-Newton methods are often preferred to Newton-Raphson because the latter requires computing the full Hessian matrix many times, which can be costly. The main idea behind the quasi-Newton update in Algorithm 3 below is that the \( d_\theta(d_\theta + 1)/2 \) unique elements in the Hessian can be recovered using only \( d_\theta \) scalar derivatives. Consider the Hessian-vector product \( y_b = H_{m+1}^{(b)}(\theta_b)s_b \), for a direction \( s_b \neq 0 \). Notice that \( y_b = \lim_{\varepsilon \to 0} \left[ G_{m+1}^{(b)}(\theta_b + \varepsilon s_b) - G_{m+1}^{(b)}(\theta_b - \varepsilon s_b) \right]/2\varepsilon \) is a scalar derivative which can be computed without the full matrix \( H_{m+1}^{(b)}(\theta_b) \). The full Hessian matrix can be recovered from Hessian-vector products computed in \( d_\theta \) linearly independent directions \( s_b, s_{b-1}, \ldots \) by running a simple linear regression of \( y_b \) on \( s_b \). Using an overdetermined system with \( L > d_\theta \) directions ensures the linear regression is well-conditioned. In practice, the different \( y_b \) are computed at different values of \( \theta_b \) so that the Hessian recovered by least-squares need not be symmetric nor positive definite. A simple transformation is used to enforce these two features, while retaining its approximation properties. The resulting quasi-Newton update is new. The while loop ensures the regression is well-conditionned. It never ran in the examples of Section 3 but should included as a safeguard. A more detailed discussion of difference with existing updates and the choice of tuning parameters is deferred to Section 4.3.

---

\(^{10}\) In some settings, it is possible to compute the Hessian ‘for free’ once the gradient is computed; see Example 3, Section 5 and Appendix H.3.
Algorithm 3 Resampled quasi-Newton (rqN)

1) **Inputs** (a) number of secants \( L \geq d_\theta \), (b) cutoffs \( \lambda_S > 0, \Lambda > 0 \)

2) **Least-Squares Approximation**
   
   if \( b = 0 \) then
   
   \( \triangleright \) Initialization
   
   Set an initial guess \( \hat{H}_0 \), e.g. \( H_m^{(0)}(\theta_0) \)
   
   Draw \( s_j \), normalize \( s_j = s_j/\|s_j\|_2 \), \( j = 0, \ldots, -L + 1 \)
   
   Compute \( y_j = \hat{H}_0 s_j \), \( j = 0, \ldots, -L + 1 \)
   
   else
   
   Compute \( s_b = \theta_b - \theta_{b-1} \), normalize \( s_b = s_b/\|s_b\|_2 \)
   
   Compute \( y_b = H_m^{(b+1)}(\theta_b)s_b \)
   
   \( \triangleright \) Update direction
   
   Combine \( S_b = (s_b, \ldots, s_{b-L+1})' \), \( Y_b = (y_b, \ldots, y_{b-L+1})' \)
   
   while \( \lambda_{\text{min}}(S_b'S_b) < \lambda_S \) do
   
   Discard \( s_{b-L+1} \), re-index \( s_j \) to \( s_{j-1} \), \( j = b, \ldots, b - L + 2 \)
   
   Draw \( s_b \), normalize \( s_b = s_b/\|s_b\|_2 \)
   
   Compute \( y_b = H_m^{(b+1)}(\theta_b)s_b \)
   
   Combine \( S_b = (s_b, \ldots, s_{b-L+1})' \), \( Y_b = (y_b, \ldots, y_{b-L+1})' \)
   
   end while
   
   Compute \( \hat{H}_b = Y_b'S_b(S_b'S_b)^{-1} \)
   
   \( \triangleright \) Least-Squares Approximation
   
   3) **Conditioning Matrix**
   
   Compute \( P_b = (\hat{H}_b'\hat{H}_b + \tau_b I_d)^{-1/2} \), where \( \tau_b = \lambda^2 \) if \( \lambda_{\text{min}}(\hat{H}_b'\hat{H}_b) \leq \lambda^2 \), \( \tau_b = 0 \) otherwise

### 3.3 Comparison with other methods

To generate \( B \) draws, the standard bootstrap with resampling or reweighting requires solving the minimization problem \( \hat{\theta}_m^{(b)} = \arg\min_{\theta \in \mathcal{D}} Q_m^{(b)}(\theta) \) \( B \) times which can be computationally intensive. Faster alternatives have been proposed in the literature. Davidson and MacKinnon (1999, DMK) consider a \( k \)-step NR update: \( \hat{\theta}_{n,k+1}^{(b)} = \hat{\theta}_{n,k}^{(b)} - [H_n^{(b)}(\hat{\theta}_{n,k}^{(b)})]^{-1}G_n^{(b)}(\hat{\theta}_{n,k}^{(b)}) \) starting at the same full sample estimate \( \hat{\theta}_{n,0}^{(b)} = \hat{\theta}_n \) for all \( b \). Andrews (2002) shows that \( k = 1 \) is enough for asymptotically valid inference, and \( k > 1 \) provides asymptotic refinements. DMK also propose a \( k \)-step quasi-Newton update which requires \( k > 1 \) steps\(^\text{11} \). In comparison, using the same \( m = n \), a \( k = 1 \)-step implementation of DMK has the same cost as rNR, but is higher than rNR for \( k > 1 \) steps. The \( k \)-step quasi-Newton update requires \( k > 1 \) making it \( k \) times more costly than rQRN for \( m = n \). Using \( m < n \) further reduces the cost of both estimation and inference for rNR, rqN. The original idea of DMK has been extended to fast

\(^{11}\)Following earlier work by Robinson (1988), the results are based on quadratic and super-linear convergence properties of NR and BFGS, respectively, when \( \gamma = 1 \) which holds only if \( \|\hat{\theta}_n^{(b)} - \theta_n\| \) is sufficiently small and if (and only if) the quasi-Newton matrix \( P_k \) satisfies certain properties (Dennis and Moré 1977, Th3.1).
subsampling (Hong and Scaillet, 2006), two-step estimation (Armstrong et al., 2014), and \( \ell_1 \)-penalized estimation (Li, 2021).

Building on Hu and Zidek (1995) and Hu and Kalbfleisch (2000), Kline and Santos (2012, KS) propose a wild score bootstrap which can be written as a one-step NR update:

\[
\hat{\theta}_n = \theta_n - G_n^{(b)}(\hat{\theta}_n) - 1 G_n^{(b)}(\hat{\theta}_n)
\]

where \( G_n^{(b)} \) is reweighted using de-meaned weights. If both the estimation and using \( m = n \) are not too demanding, this approach is computationally attractive. Rodman et al. (2019, Sec6.2) point out some caveats with this approach. By only evaluating \( G_n^{(b)} \) at \( \hat{\theta}_n \), it does not capture sampling variation in \( H_n^{(b)}(\hat{\theta}_n) \) or non-linearities in the objective. The latter also applies to DMK with \( k = 1 \). By design, \( r_{\text{NR}} \) and \( r_{\text{QN}} \) reflect variation in \( H_n^{(b)}, G_n^{(b)} \) around \( \hat{\theta}_n \). Notice that if the optimizer returns a suboptimal solution \( \tilde{\theta}_n \), then DMK with \( k > 1 \) may not be centered around \( \hat{\theta}_n \) which is a red flag. By de-meaning the weights, KS produces draws centered around \( \tilde{\theta}_n \) so that the issue goes unnoticed. Honoré and Hu (2017) propose to approximate the sandwich \( V_n \) using \( d_{\theta}(d_{\theta} + 1)/2 \times B \) scalar minimizations of \( Q_n^{(b)} \). When \( d_{\theta} \) increases, this quickly becomes taxing. Using a preliminary estimate \( \tilde{\theta}_n \) to resample is the defining characteristic of these methods compared to \( r_{\text{NR}} \) and \( r_{\text{QN}} \) which never compute \( \hat{\theta}_n \) explicitly.

For likelihood problems, it is common to use Bayesian inference and deploy the MCMC toolkit. Unlike frequentist estimation, Bayesian analyses rely on sampling from a posterior distribution rather than optimization. Well known samplers include Gibbs and Metropolis-Hastings. Because they can be slow to converge, gradient-based samplers are increasingly popular with Metropolis-adjusted Langevin dynamics (MALA), Hamiltonian Monte-Carlo (HMC), or stochastic gradient Langevin dynamics (SGLD) algorithms. A comparison between \( r_{\text{NR}}, r_{\text{QN}}, \) and MALA is given in Section 5. Unlike Bayesian inference, this paper does not require the information matrix equality for valid inference. Convergence diagnostics used for MCMC algorithms such as trace plots or formal tests (Gelman and Rubin, 1992; Brooks and Gelman, 1998; Gelman et al., 2013, Ch11.4) can also be used to monitor convergence of \( r_{\text{NR}} \) and \( r_{\text{QN}} \). Lastly, while the detailed balance condition guarantees the posterior is a stationary solution of a Metropolis-Hastings algorithm, convergence rate results typically assume a (nearly) concave log-posterior – similar to the convexity assumption in this paper. Forneron and Ng (2016) show how optimization can be used to sample from a Bayesian model.

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12 Appendix H, Table H6, illustrates failed optimizations for a dynamic discrete choice model, in comparison \( r_{\text{QN}} \) produces accurate estimates. It would be problematic to apply DMK or KS to the inconsistent estimates.

13 In Section 5, Example 2, \( 55945 \times B \) scalar optimizations would be needed which is quite large.

14 Recall that the detailed balance condition ensures that the random walk Metropolis-Hastings algorithm is ergodic but deriving the rate of convergence is more difficult. See e.g. Mengersen and Tweedie (1996), Brooks (1998), Belloni and Chernozhukov (2009).
posterior with an intractable likelihood. Here, optimization is used for resampling.

4 Statistical Properties of rNR and rqN

This section has three parts. The first subsection presents the main assumptions on $Q_n$, $Q_m(b)$, $P_b$, and derives convergence results for classical optimizers and Algorithm 1 as well as the pivotal coupling result with the AR(1) process $\theta_b^*$. These results are non-asymptotic. The second subsection focuses on large sample estimation and inference. The theorems can be applied to any implementation of Algorithm 1 for which $P_b$ has the required properties. The third subsection specializes them to rNR and rqN described in Algorithms 2 and 3.

4.1 Convergence of Classical and Resampled Optimizers

The econometric theory for extremum estimation in e.g. Newey and McFadden (1994) takes as a given that an optimizer finds the unique solution. But the econometric assumptions are not enough to ensure a numerical optimizer will provide a globally convergent solution to the sample problem $Q_n(\theta)$. The following is standard in gradient-based optimization theory:

**Assumption 1.** There exist constants $\lambda_H, \lambda_P, C_1$ such that for all $\theta \in \Theta$:

i. $0 < \lambda_H \leq \lambda_{\text{min}}(H_n(\theta)) \leq \lambda_{\text{max}}(H_n(\theta)) \leq \lambda_H < \infty$ with probability 1,

ii. $\|H_n(\theta) - H_n(\hat{\theta}_n)\|_2 \leq C_1 \|\theta - \hat{\theta}_n\|_2$ with probability 1.

The matrix $P_k$ is symmetric, such that for some $\lambda_H, \lambda_P$ and all $k \geq 1$:

iii. $0 < \lambda_P \leq \lambda_{\text{min}}(P_k) \leq \lambda_{\text{max}}(P_k) \leq \lambda_P < \infty$.

Condition i. implies that $Q_n$ is strongly convex with probability 1 while condition ii. requires Lipschitz continuity of the Hessian. Strong convexity of the sample objective function can be less restrictive than it appears as some non-convex problems can be convexified by penalization. Assumption 1 implies the so-called Polyak-Łojasiewicz inequalities:

$$\langle \theta - \hat{\theta}_n, G_n(\theta) \rangle = (\theta - \hat{\theta}_n)'H_n(\hat{\theta}_n)(\theta - \hat{\theta}_n) \geq \lambda_H \|\theta - \hat{\theta}_n\|_2^2,$$

$$\|G_n(\theta)\|_2^2 = (\theta - \hat{\theta}_n)'H_n(\hat{\theta}_n)^2(\theta - \hat{\theta}_n) \leq \lambda_H^2 \|\theta - \hat{\theta}_n\|_2^2,$$

15 A function $Q_n$ is strongly convex on $\Theta$ if for all $\theta \in \Theta$, there exists some $\lambda > 0$ such that $\nabla^2 Q_n(\theta) \geq \lambda I_d$. For bounded $\Theta$, there also exists $\lambda' \geq 0$ such that $\nabla^2 Q_n(\theta) \leq \lambda' I_d$. Then $\lambda'/\lambda$ is an upper bound on the condition number of $\nabla^2 Q_n(\theta)$.

16 In some cases, the strict convexity assumption restricts the parameter space to be bounded in the absence of penalization. See e.g. the discussion of Assumption (H3) in Moulines and Bach (2011).
where $\hat{\theta}_n$ is an intermediate value, between $\theta$ and $\hat{\theta}_n$. The first inequality follows from the positive definiteness of $H_n(\hat{\theta}_n)$. To see the usefulness of Assumption 1, note that for any symmetric positive definite conditioning matrix $P_k$:

$$
\|\theta_{k+1} - \hat{\theta}_n\|^2_2 = \|\theta_k - \hat{\theta}_n - \gamma P_k G_n(\theta_k)\|^2_2 \\
= \|\theta_k - \hat{\theta}_n\|^2_2 - 2\gamma \langle \theta_k - \hat{\theta}_n, P_k G_n(\theta_k) \rangle + \gamma^2 \|P_k G_n(\theta_k)\|^2_2 \\
\leq \left(1 - 2\gamma \lambda_p \lambda_H + \gamma^2 \|\gamma P_H\|^2_2\right) \|\theta_k - \hat{\theta}_n\|^2_2,
$$

where the last inequality follows from Assumption 1 and $0 < \lambda_p \leq \lambda_H < \infty$ bound the eigenvalues of $P_k$ (see Assumption 1 iii). Now a contraction occurs if the choice of $\gamma$ is such that $A(\gamma) \in [0, 1)$. To see that this is feasible, note that $A(0) = 1$ and $\partial_\gamma A(0) < 0$. Thus by continuity and local monotonicity of $A(\cdot)$, there exists a $\gamma^* \in (0, 1]$ such that $A(\gamma) \in [0, 1)$ for all $\gamma \in (0, \gamma^*]$ as desired. For such choice of $\gamma$, define $\gamma(\lambda_p, \lambda_H, \lambda_p, \lambda_H) \in (0, 1]$ independent of $k$ be such that: $A(\gamma) = (1 - \overline{\gamma})^2 \in [0, 1)$. It follows that:

$$
\|\theta_{k+1} - \hat{\theta}_n\|^2_2 \leq \sqrt{A(\gamma)} \|\theta_k - \hat{\theta}_n\|^2_2 \\
\leq (1 - \overline{\gamma}) \|\theta_k - \hat{\theta}_n\|^2_2 \\
\leq (1 - \overline{\gamma})^k \|\theta_0 - \hat{\theta}_n\|^2_2 \rightarrow 0, \quad \text{as } k \rightarrow \infty. \quad \square
$$

Global convergence follows from iterating on the contraction at each $k$. The derivations above rely on the positive definiteness of both $H_n$ and $P_k$. A non-definite $P_k$ can be problematic for convergence. While Assumption 1 i-ii require a smooth and strictly convex objective function, Assumption 1 iii restricts the choice of conditioning matrix.

The finite upper bound in Assumption 1 iii is particularly important to ensure the optimizer is well behaved. It is automatically satisfied for GD since $P_b = I_d$ but requires the lower bound in Assumption 1 i for NR. For quasi-Newton, it needs to be enforced as in Algorithm 3. Note that for NR, with an adjustment similar to the $\tau_b$ in Algorithm 3, it possible to get convergence even when $H_n$ is singular away from $\hat{\theta}_n$.

**Lemma 1.** Under Assumption 1 i-iii there exists $\gamma \in (0, 1]$ such that $A(\gamma) \in [0, 1)$. Let $\gamma^*$ be such that $A(\gamma^*) = (1 - \overline{\gamma})^2$. Then $\|\theta_k - \hat{\theta}_n\|^2_2 \leq (1 - \overline{\gamma})^k \|\theta_0 - \hat{\theta}_n\|^2_2 \rightarrow 0, \quad \text{as } k \rightarrow \infty.$

The first Lemma derives the deterministic global convergence of a classical gradient-based optimizer for an appropriate choice of $\gamma$ at a dimension-free linear rate of $(1 - \overline{\gamma})$. Convergence

\[\text{This is illustrated in Appendix 1.1. Note that modifications to the resampled hessian is also present in Gonçalves and White (2005) when deriving consistency of bootstrap standard errors in OLS regressions.}\]
of Newton-Raphson with $P_k = H_k^{-1}$ is implied by the Lemma under Assumption 1 only. This first result is useful for deriving the stochastic convergence of the resampled Algorithm 1. In particular, notice that the resampled update is:

$$
\theta_{b+1} = \theta_b - \gamma P_b G_m^{(b+1)}(\theta_b) = \theta_b - \gamma P_b G_n(\theta_b) + \gamma P_b [G_n(\theta_b) - G_m^{(b+1)}(\theta_b)].
$$

which is the sum of the deterministic update studied in Lemma 1 with resampling noise. The convergence has a deterministic and a stochastic component. The deterministic part is covered by Assumption 1 and the condition on $\gamma$ in Lemma 1. The following assumption will be used to study the stochastic part of the update.

**Assumption 2.** Let $E^*$ denote the expectation under resampling or reweighting, i.e. conditional on $(z_1, \ldots, z_n)$. Suppose there exists $\Delta_H, \lambda_H, C_2, C_3$ such that for $p \geq 2$:

i. $[E^*(\sup_{\theta \in \Theta} \|G_m^{(b)}(\theta) - G_n(\theta)\|^p)]^{1/p} \leq C_2 m^{-1/2}$ and $E^*[G_m^{(b)}(\theta)] = G_n(\theta)$,

ii. $[E^*(\sup_{\theta \in \Theta} \|H_m^{(b)}(\theta) - H_n(\theta)\|^p)]^{1/p} \leq C_3 m^{-1/2}$,

iii. $0 < \lambda_H \leq \lambda_{\min}(H_m^{(b)}(\theta)) \leq \lambda_{\max}(H_m^{(b)}(\theta)) \leq \lambda_H < \infty$, with probability 1.

The conditioning matrix $P_b$ is such that for all $b \geq 1$:

iv. $0 < \lambda_P \leq \lambda_{\min}(P_b) \leq \lambda_{\max}(P_b) \leq \lambda_P < \infty$, with probability 1.

These high-level conditions will be used to prove convergence of $\theta_b$ in $L_p$-norm, $p \geq 2$. Appendix 1 provides sufficient conditions for i-ii. using either $m$ out of $n$ resampling with replacement or reweighting with Gaussian multiplier weights. Consistent estimation and valid inference using quantiles in Algorithm 1 (steps 4 and 5), will require $p = 2$ in Assumption 2. Consistency of standard errors further requires $p = 4$. Consistency of moments for the standard bootstrap is also associated with particular moment conditions, see Gonçalves and White (2005), Kato (2011), Cheng (2015). Since the estimator in Algorithm 1 is computed by averaging draws, the derivations are fundamentally moment-based.

Assumption 2 iv reiterates the importance of the bounds for $P_b$ for convergence but in the stochastic case. Under Assumptions 1 2 and the same choice of $\gamma$ used in Lemma 1:

$$
\|\theta_{b+1} - \hat{\theta}_n\|_2 \leq (1 - \gamma)\|\theta_b - \hat{\theta}_n\|_2 + \gamma \lambda_P \sup_{\theta \in \Theta} \|G_m^{(b+1)}(\theta) - G_n(\theta)\|_2,
$$

Dependence of $C_2, C_3$ on $n$ is omitted to simplify notation. The conditions under Gaussian reweighting are the same for $p = 2$ and $p > 2$ but not with resampling. The results can be extended to other weighting distributions with the same conditions as resampling by adapting the proof of Theorem 3.6.13 in van der Vaart and Wellner (1996) similar to Cheng (2015, pp18-20).
with probability 1. Iterating the recursion reveals the general behaviour of \( \theta_b \):

\[
\|\theta_{b+1} - \hat{\theta}_n\|_2 \leq (1 - \tau)^b \|\theta_0 - \hat{\theta}_n\|_2 + \gamma \lambda \sum_{j=0}^{b-1} (1 - \tau)^j \sup_{\theta \in \Theta} \|G_{m}^{(b-j)}(\theta) - G_n(\theta)\|_2,
\]

with probability 1. The leading term converges deterministically and exponentially fast to zero. It corresponds to a starting values bias. When Assumption 2 holds for some \( p \geq 2 \) and \( (1 - \tau) \in [0, 1) \), the second term has bounded \( L_p \)-norm, of order \( 1/\sqrt{m} \). This characterizes the stochastic convergence of \( \theta_b \) as show in the following Lemma.

**Lemma 2.** Suppose Assumptions 1-2 hold and \( \gamma \in (0, 1] \) is such that \( (1 - \tau)^2 = A(\gamma) \in [0, 1) \), as defined in Lemma 1. Then there exists a constant \( C_4 \) which only depends on \( \gamma, \lambda, \alpha \) such that:

\[
\left[ \mathbb{E}^* \left( \|\theta_{b+1} - \hat{\theta}_n\|_2^p \right) \right]^{1/p} \leq \left( 1 - \tau \right)^{b+1} \left[ \mathbb{E}^* \left( \|\theta_0 - \hat{\theta}_n\|_2^p \right) \right]^{1/p} + \frac{C_4}{\sqrt{m}}.
\]

Lemma 2 shows that \( \theta_b \) converges in \( L_p \)-norm to a \( 1/\sqrt{m} \)-neighborhood of \( \hat{\theta}_n \). Unlike classical optimizers or SGD with \( \gamma_k \to 0 \) where \( \theta_k \to \hat{\theta}_n \) either deterministically or in probability, here \( \theta_b \not\to \hat{\theta}_n \) as \( b \) increases – this is similar to MCMC which produces draws around \( \hat{\theta}_n \). Except for quadratic objectives, the Markov-process followed by \( \theta_b \) around \( \hat{\theta}_n \) is intractable. Nevertheless, the following will show that

\[
\theta_{b+1} - \hat{\theta}_n = \Psi_n(\theta_b^* - \hat{\theta}_n) - \gamma \lambda \|\theta^*_b - \hat{\theta}_n\|_2^2 + \gamma \lambda \|\theta^*_b - \hat{\theta}_n\|_2^2 G_{m}^{(b+1)}(\hat{\theta}_n), \quad \theta_0^* = \theta_0,
\]

(8)
gives a good approximation of \( \theta_b \) and its distribution. The autoregressive coefficient \( \Psi_n \) and \( \lambda \) depend on the choice of update \( P_b \). Table 1 makes them explicit for rGD, rNR, and rqN.

| Algorithm                  | \( P_b \)   | \( \lambda \) | \( \psi_n \)                   |
|----------------------------|--------------|----------------|-------------------------------|
| Resampled Gradient Descent (rGD) | \( I_d \)   | \( I_d \)   | \( I_d - \gamma \lambda \) |
| Resampled Newton-Raphson (rTR) | \( \left[H_{m}^{(b+1)}(\theta_b)^{-1} \right]^{-1} \) | \( \left[H_{n}(\hat{\theta}_n)^{-1} \right]^{-1} \) | \( (1 - \gamma)I_d \) |
| Resampled quasi-Newton (rqN) | \( \text{Algorithm (3)} \) | \( \left[H_{n}(\hat{\theta}_n)^{-1} \right]^{-1} \) | \( (1 - \gamma)I_d \) |

**Assumption 3.** Let \( d_{0,n} = \mathbb{E}^* \left( \|\theta_0 - \hat{\theta}_n\|_2^p \right) \), suppose that there exists a matrix \( \lambda \) symmetric positive definite and constants \( C_5 \) such that for \( \psi_n = I - \gamma \lambda \):

1. \( 0 \leq \lambda_{\max}(\psi_n) < 1 \),
2. \( \left[ \mathbb{E}^* \left( \|\theta - P_b \right|^{-1} \|_2^p \right) \right]^{1/p} \leq C_5 \left( \rho d_{0,n} + 1/\sqrt{m} \right) \).
In the literature, $\theta^*_b$ is known as a coupling of $\theta_b$. The two have different marginal distributions because one follows a linear and the other a non-linear process. Nonetheless, they live on the same probability space because they are generated from the same resampled objective $Q_m(b)$. Hence if we can show that $\|\theta_{rnr} - \theta^*_{rnr}\|$ is negligible, we can work with the distribution of $\theta^*_b$ which is more tractable.

**Proposition 1.** Suppose that Assumptions 1, 2, and 3 hold for $p \geq 2$ and $\gamma \in (0, 1]$ is such that Lemma 2 is satisfied. Let $\rho = \max\{1 - \gamma, \rho, \lambda_{\text{max}}(\Psi_n)\}$, and $d_{0,n} = \left[\mathbb{E}^*(\|\theta_0 - \hat{\theta}_n\|^2)\right]^{1/p}$. There exists a $C_6$, which depends on $C_1, \ldots, C_5, \rho, \gamma, \lambda_H, \lambda_P$ such that:

$$\mathbb{E}^*(\|\theta_b - \theta^*_b\|^2) \leq C_6 \left(\frac{1}{m} + \rho [d_{0,n} + d_{0,n}^2]\right).$$

(9)

The constant $\gamma \in (0, 1]$ is the global rate of convergence in Lemma 2. Putting $p = 2$, the lemma states that the expected deviation $\mathbb{E}^*(\|\theta_b - \theta^*_b\|)$ is of order $1/m$ plus a term that vanishes exponentially fast in $b$. This in turn implies that for $p = 2$, upon averaging, $\mathbb{E}^*(\|\theta_{rnr} - \theta^*_{rnr}\|)$ is of order $1/m$ plus a $1/B$ term due to $d_{0,n}$. Using these non-asymptotic bounds, the large sample estimation and inference properties of Algorithm 1 can be derived.

### 4.2 Large Sample Estimation and Inference

Proposition 1 is pivotal for what follows. Suppose for simplicity that there is no approximation error, i.e. $\theta_b = \theta^*_b$, then:

$$\mathbb{E}^*(\theta_b) = \hat{\theta}_n + [\Psi_n]^b(\theta_0 - \hat{\theta}_n),$$

converges to $\hat{\theta}_n$ exponentially fast under Assumption 3 i. This implies that rGD, rNR, and rqN deliver consistent estimates. This should be of interest for practitionners using SGD with $\gamma_k = \gamma$ fixed and Polyak-Ruppert averaging, which is common in practice, since the theory gives restrictions on $m$ for the bias to be negligible. Furthermore, $\theta^*_b$ being an ergodic first-order vector autoregressive process:

$$m \times \text{var}^*(\theta_b) \to \gamma^2 \Psi_n \text{var}^*\left[\sqrt{m}G_m(\hat{\theta}_n)\right]\Psi_n:\'

converging exponentially fast. It consistently estimates the asymptotic variance, up to scale, when $\text{var}^*\left[\sqrt{m}G_m(\hat{\theta}_n)\right]$ consistently estimates the meat and $\Psi_n$ approximates the bread $[H_n(\hat{\theta}_n)]^{-1}$ (up to scale). This is the case for rNR and rqN, but not rGD. Suppose $\sqrt{m}G_m(\hat{\theta}_n) \sim \mathcal{N}(0, \Sigma_n)$ then, for large $b$ we approximately have $\theta_b \sim \mathcal{N}(\hat{\theta}_n, \gamma^2 \Psi_n \Sigma_n \Psi_n^*/n)$ which matches the asymptotic distribution, after re-scaling the variance, if $\Psi_n$ approximates
the bread (up to scale). The following Theorem gives a non-asymptotic bound for the approximation error \( \| \hat{\theta}_{\text{RR}} - \hat{\theta}_n \|_2 \) and the large-sample consistency result.

**Theorem 1** (Large Sample Estimation). Suppose that \( \Sigma_n = \text{var}^*[\sqrt{m}G_m^{(b)}(\hat{\theta}_n)] \) is finite and bounded. If the conditions of Proposition \( \square \) hold for \( p = 2 \), then \( \hat{\theta}_{\text{RR}} \) satisfies:

i. \( \text{E}^* \left( \| \hat{\theta}_{\text{RR}} - \hat{\theta}_n \|_2 \right) \leq C_T \left( \frac{1}{m} + \frac{d_{0,n} + d_{0,n}}{B} + \frac{1}{\sqrt{m}B} \right) \) where \( C_T \) depends on \( C_0 \) and \( \text{trace}(\Sigma_n) \).

ii. If \( \sqrt{n}/\min(B, m) \to 0 \): \( \sqrt{n}(\hat{\theta}_{\text{RR}} - \theta^i) = \sqrt{n}(\hat{\theta}_n - \theta^i) + o_p(1) \).

First-order asymptotic equivalence of \( \hat{\theta}_{\text{RR}} \) and \( \hat{\theta}_n \) requires both \( B \) and \( m \) to be large relative to \( \sqrt{n} \). Using fixed \( m \) and \( \gamma \) does not produce consistent estimates. This is relevant for SGD with \( m \ll n \) and \( \gamma_k = \gamma \). For inference, consider the standard bootstrap as a benchmark. Let \( d^r \) denote convergence in distribution conditional on \( (z_1, \ldots, z_n) \). Under regularity conditions, an \( n \) out of \( n \) bootstrap estimate \( \hat{\theta}_n^{(b)} = \arg\min_{\theta \in \Theta} G_n^{(b)}(\theta) \) is such that

\[
\sqrt{n}(\hat{\theta}_n^{(b)} - \hat{\theta}_n) = -[H_n(\hat{\theta}_n)]^{-1}\sqrt{n}G_n^{(b)}(\hat{\theta}_n) + o_p(1).
\]

If \( \sqrt{n}G_n^{(b)}(\hat{\theta}_n) \) satisfies a conditional CLT then the bootstrap distribution targets the correct asymptotic distribution. The bootstrap is often used to compute standard errors. Consistency of moments further requires uniform integrability\(^\text{[19]}\). Here, convergence in distribution of \( \hat{\theta}_b \) comes from Proposition \( \square \) and a conditional CLT for \( \hat{\theta}_b^* \), i.e. the growing sum \( \sum_{j=0}^b (1 - \gamma)^j [H_n(\hat{\theta}_n)]^{-1}\sqrt{m}G_m^{(b)}(\hat{\theta}_n) \). Assumption \( \square \) is used to ensure that distributional convergence for \( \sqrt{m}G_m^{(b)}(\hat{\theta}_n) \) extends to \( \hat{\theta}_b^* \). Primitive conditions are given in Appendix \( \square \) for \( m \) out of \( n \) resampling, and reweighting with Gaussian weights.

**Assumption 4.** Let \( V_n = [H_n(\hat{\theta}_n)]^{-1}\Sigma_n[H_n(\hat{\theta}_n)]^{-1} \), where \( \Sigma_n \) \( m \times \text{var}^*[G_m^{(b)}(\hat{\theta}_n)] \). There exists constants \( \beta \in (0, 1/2] \), \( \kappa > 0 \), and a remainder \( \| r_m(\tau) \| \leq C_\psi \| \tau \|^{\kappa} \) such that for \( i^2 = -1 \), \( \text{E}^* \left( \exp \left[ \sqrt{m}i^\tau \left( V_n^{-1/2}H_n(\hat{\theta}_n)^{-1}G_m^{(b)}(\hat{\theta}_n) \right) \right] \right) = \exp \left( -\frac{\| \tau \|^2}{2} \right) \times \left( 1 + \frac{r_m(\tau)}{\kappa} \right) \).

**Assumption 5.** Suppose the following holds:

i. \( V_n \xrightarrow{p} V = [H(\theta^i)]^{-1}\Sigma[H(\theta^i)]^{-1}, \sqrt{n}(\hat{\theta}_n - \theta^i) \xrightarrow{d} \mathcal{N}(0, V), \)

ii. \( h : \Theta \to \mathbb{R} \) is continuously differentiable, \( \nabla h \) is Lipschitz-continuous, \( \nabla h(\theta^i) \neq 0 \).

Assumption \( \square \) is standard. It requires asymptotic normality of \( \hat{\theta}_n \), consistency of the sandwich estimator, and validity of the delta method for inference on a function \( h \) of \( \theta \).

\(^{19}\)See Billingsley (2013, Th16.14). Specifically, to consistently estimate the variance \( V_n \), the condition requires \( \sup_n \text{E}^* (\sqrt{n}(\hat{\theta}_n^{(b)} - \hat{\theta}_n)^{2+\varepsilon}) \) finite for some \( \varepsilon > 0 \). See Gonçalves and White (2005), Kato (2011), Cheng (2015), for applications to moment convergence of the bootstrap in OLS regressions and M-estimation.
Theorem 2 presents the large sample inference. First, the asymptotic validity quantile-based confidence intervals. Second, consistency of standard errors computed in Algorithm 1. Both results require a choice of $P_b$ satisfying $P_m = [H_n(\hat{\theta}_n)]^{-1}$.

**Theorem 2** (Large Sample Inference). Suppose that the conditions of Proposition 1 hold with $P_m = [H_n(\hat{\theta}_n)]^{-1}$ and $p = 2$, that $\Sigma_n$ is bounded and non-singular, and Assumptions 4-5 hold, then as $m, n, b \to \infty$, with $\log(m)/b \to 0$, for any $\alpha \in (0, 1)$:

1. $\lim_{n \to \infty} P_n[c_{h,b}(\alpha/2) \leq h(\hat{\theta}_n) - h(\hat{\theta}_n) \leq c_{h,b}(1-\alpha/2)] = 1 - \alpha$,

where $c_{h,b}(\alpha)$ is the $\alpha$-th quantile of the adjusted draws $h(\hat{\theta}_b)$, $\hat{\theta}_b = \hat{\theta}_n + \sqrt{m/n\phi(\gamma)}(\theta_b - \hat{\theta}_n)$. If, in addition, the assumptions hold with $p = 4$:

2. $\frac{m}{\phi(\gamma)} \text{var}^* [h(\theta_b)] = \nabla h(\hat{\theta}_n) V_n \nabla h(\hat{\theta}_n)' + o(1)$.

### 4.3 Implications for rnr and rqN

Having the general estimation and inference results for Algorithm 1, the following verifies the implementation-specific Assumptions 2 iv, 3 to show that Theorems 1-2 apply to rnr and rqN. For rqN, it is assumed, without loss of generality, that $\lambda = \Lambda_{H}/2^{20}$.

**Corollary 1** (Estimation and Inference with rnr and rqN). Consider an implementation of Algorithm 2 using either rnr or rqN, i.e. the choice of $P_b$ described in Algorithms 2 and 3. Suppose Assumptions 4 i-ii and 2 i-iii hold with $p \geq 2$, then Assumption 2 iv holds for the same $p$. If the assumptions for Lemma 3 also hold, then Assumption 3 holds with the same $p$ and $P_m = [H_n(\hat{\theta}_n)]^{-1}$. Then, under the additional assumptions required for each set of results, Theorems 4 and 2 apply to rnr and rqN.

Consistency of standard errors implies that they can be used to compute t-statistics, confidence intervals of the form $h(\bar{\theta}_{nc}) \pm 1.96 \sqrt{m/[n\phi(\gamma)]\text{var}[h(\theta_b)]}$, where $\text{var}[h(\theta_b)]$ is the sample variance of $h(\theta_b)$, after discarding the first $\text{BURN} \gg \log(m)$ iterations. A simple rule-of-thumb to gauge the order of magnitude for $\text{BURN}$ is that $(1 - \gamma) \geq (1 - \gamma)$ and the initial value bias is of order $(1 - \gamma)^b d_{0,n}$. Suppose, we target a bias of order $d_{0,n}/100$, then we need $b > \log(1/100)/\log(1 - \gamma) \geq \log(1/100)/\log(1 - \gamma) \simeq 43$ for $\gamma = 0.1$. This implies that the burn-in should include at least 40 – 50 draws. A burn-in of 225 draws corresponds to setting $\gamma = 0.02$ in the same calculations.

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20The proof only relies on $\lambda < \lambda_{\text{min}}(H_n(\theta))$ around $\hat{\theta}_n$. 

21The proof relies on $\lambda < \lambda_{\text{min}}(H_n(\theta))$ around $\hat{\theta}_n$. 

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The quasi-Newton update proposed in Algorithm 3 differs from BFGS and other methods in the literature to ensure the following three properties are satisfied: i) $P_b$ is symmetric, and ii) positive definite with strictly positive eigenvalues, iii) Assumption 3 holds when $\|\theta_b - \hat{\theta}_n\|$ is small. Conditions i-ii) are required to ensure optimization is stable and apply Lemma 2. Condition iii) is needed for valid inference. Other methods do not satisfy i), ii), and iii) simultaneously. Amongst the most widely used methods, BFGS enforces i) and ii) plus the so-called secant equation: $P_by_k = s_k$ so that $P_bH_n(\theta_k)s_k = s_k$ always holds but is only guaranteed in this direction so that iii) need not hold for non-quadratic $Q_n$.\footnote{See Ren-Pu and Powell (1983) for counter-examples. The BFGS matrix $P_k$ can have negative eigenvalues when $Q_n$ is non-convex. This is problematic for convergence as illustrated in Section I.2.} The SR1 update, used with trust-region algorithms, satisfies i) and iii) but not ii) which makes it unstable in estimation using $\hat{H}_b$\footnote{See Fiacco and McCormick (1968), Conn et al. (1991) for derivations of iii) with SR1.}. Both methods update $P_k$ from $P_{k-1}$ using a rank-one update based on the secant equation above. Here $P_b$ is fully updated using OLS, this is related to the multi-secant update in Schnabel (1983) which can satisfy iii) but typically not i-ii), and requires $L \leq d_\theta$.

In the proof of the Corollary, the OLS estimate $\hat{H}_b$ is shown to satisfy iii) but not necessarily i)-ii). The transformation $\hat{H}_b^T\hat{H}_b$ enforces i), and the regularization $\tau_b$ enforces ii)\footnote{These two transformations can also be useful for NR and iNR as illustrated in Section I.}. Algorithm 3 requires three inputs. At least $L \geq d_\theta$ secant updates are needed to compute $\hat{H}_b$ by OLS; $L \geq \max(25, 1.5 \times d_\theta)$ works well in the examples. The cutoff $\Lambda_S$ ensures the least-squares problem is well-conditionned. With $\Lambda_S = 10^{-6}$, the while loop never runs in the applications, but it should be included as a safeguard. The requirement for $\Lambda$ is to be small enough to satisfy $0 < \Lambda < \lambda_{\min}(H_n)$ around $\hat{\theta}_n$.

5 Three Examples

The following provides one Monte-Carlo and two empirical examples which illustrate the properties of iNR and rQN for estimation and inference.

5.1 Example 1: Dynamic Discrete Choice with Unobserved Heterogeneity

The first example considers likelihood estimation and inference in a single-agent dynamic discrete choice model with unobserved heterogeneity\footnote{See e.g. Aguirregabiria and Mira (2010) and Arcidiacono and Ellickson (2011) for reviews on the estimation of dynamic discrete choice models.}. It is common in this class of models to use a parametric mixture distribution to model heterogeneity since it is parsimonious.
relative to a more computationally demanding non-linear fixed effect estimation. There are two challenges in this example. First, the likelihood is non-convex because of the unobserved mixture component. Second, the non-linear transformation used to integrate out the unobserved heterogeneity results in finite sample bias and sizable distortion for inference. Analytical bias correction can be challenging to implement, because the terms are not easily tractable. Standard bootstrap bias correction would be computationally intensive. Using a large-T linearization of the bias, it is shown that the split panel jackknife of [Dhaene and Jochmans (2015)] can reduce this bias, and straightforward to use with rqN.25 A comparison of rNR and rqN with MLE and the standard bootstrap for a simpler homogeneous agents model are given in Appendix H.1, Table H5.

The setting is similar to the seminar model of Rust (1987), each agent $i$ solves a dynamic programming problem at each period $t$ with a flow utility for the set of actions $a_{it} = 0$ or 1:

$$U_{i0}(x_{it}, a_{it-1}) = \varepsilon_{it}(0), \quad U_{i1}(x_{it}, a_{it-1}) = \beta_{0i} + \beta_{1i} x_{it} - (1 - a_{it-1}) \delta_1 + \varepsilon_{it}(1),$$

where $\delta_1$ is the entry cost, $\varepsilon_{it}(0)$ and $\varepsilon_{it}(1)$ follow an extreme-value type I distribution, $x_{it}$ is an exogenous regressor which has finite support and follows a first-order Markov process with transition probability $\Pi$. Both $a_{it}$ and $x_{it}$ are observed for $i = 1, \ldots, n$ and $t = 1, \ldots, T$. Here the heterogeneity is modeled on the intercept and slope parameters of the utility function. The intercept $\beta_{0i}$ equals $\mu_0^1$ with probability $\omega \in [0, 1]$, and $\mu_0^2$ with probability $1 - \omega$. The slope is continuously distributed $\beta_{1i} \sim \omega N(\mu_1, \sigma_1^2) + (1 - \omega) N(\mu_2, \sigma_2^2)$. The parameters of interest are $\theta = (\mu_0^1, \mu_1^1, \mu_0^2, \mu_1^2, \sigma_1, \sigma_2, \omega, \delta_1)$, the discount factor $\rho = 0.98$ is fixed. The matrix $\Pi$ is estimated by sample average and taken as input in the log-likelihood below:

$$\ell_{nT}(\theta, \Pi) = \frac{1}{nT} \sum_{i=1}^{n} \log \left( \int \exp \left[ \sum_{t=1}^{T} \ell_{it}(\theta, \Pi, \beta_i) \right] f(\beta_i|\theta) d\beta_i \right),$$

where $\ell_{it}(\theta, \Pi, \beta_i) = \log[P(a_{it}|x_{it}, a_{it-1}, \theta, \Pi, \beta_i)]$ is the conditional probability of choice $a_{it} \in \{0, 1\}$ computed by solving the dynamic programming problem by fixed point iterations, and $f(\cdot|\theta)$ is the mixture distribution for $\beta_i = (\beta_{0i}, \beta_{1i})$.

The split panel jackknife is implemented using a full sample implementation of rqN $\theta_{b,nT}$ and two half panel implementations $\theta_{b,nT/2}, \theta_{b,nT/2}$ based on the first and last $T/2$ time-observations each. All three use the same $m = n/2$ out of $n$ resampled observations at each $b$. Bias-corrected draws are given by $\hat{\theta}_{b,nT} = 2\theta_{b,nT} - [\theta_{b,nT/2}^1 + \theta_{b,nT/2}^2]/2$. Table 2 summarizes the results from 400 Monte Carlo replications. Full panel rqN estimates (denoted as rqN)

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25See Appendix H for the derivations under simplifying assumptions.
are close to $\theta^\dagger$. Some coefficients have a bias comparable in magnitude to sampling uncertainty. Inference for these coefficients is far from the nominal 5% size using either quantiles or standard errors. The split panel correction (rqN-bc) reduces bias by an order of magnitude and bias-corrected inference is much closer to nominal size. Table H6 provides additional results with $T = 50$ and a comparison with R’s bfgs optimizer. In that configuration, BFGS estimates are very inaccurate, and far less reliable than rqN.

Table 2: Dynamic Discrete Choice Model with Heterogeneity

|          | $\mu_0^\dagger$ | $\mu_1^\dagger$ | $\mu_0^2$ | $\mu_1^2$ | $100\sigma_1$ | $100\sigma_2$ | $\omega$ | $\delta_1$ |
|----------|-----------------|-----------------|-----------|-----------|----------------|----------------|---------|-----------|
| $\theta^\dagger$ | -2.000          | 0.300           | -1.000    | 0.900     | 1.000          | 1.000          | 0.300   | 1.000     |
| Average Estimates |               |                 |           |           |                |                |         |           |
| rqN      | -2.033          | 0.307           | -0.982    | 0.903     | 1.015          | 1.154          | 0.302   | 0.976     |
| rqN-bc   | -2.019          | 0.307           | -0.991    | 0.896     | 1.006          | 1.046          | 0.302   | 1.001     |
| Standard Deviation |               |                 |           |           |                |                |         |           |
| rqN      | 0.044           | 0.013           | 0.020     | 0.009     | 0.127          | 0.181          | 0.012   | 0.018     |
| rqN-bc   | 0.043           | 0.013           | 0.021     | 0.011     | 0.146          | 0.155          | 0.013   | 0.019     |
| Rejection Rate |               |                 |           |           |                |                |         |           |
| rqN      | 0.315           | 0.025           | 0.030     | 0.041     | 0.036          | 0.330          | 0.028   | 0.122     |
| rqN-bc   | 0.025           | 0.030           | 0.028     | 0.030     | 0.030          | 0.033          | 0.028   | 0.046     |
| rqN-bcse | 0.023           | 0.033           | 0.030     | 0.033     | 0.028          | 0.033          | 0.030   | 0.041     |

Legend: rqN-bc = split panel bias-corrected rqN; rqN, rqN-bc = quantile-based CIs, rqN-bcse = std-error based CIs. $\gamma = 0.1$, $n = 1000$, $T = 100$, $m = n/2$, $B = 2000$, burn = 250, nominal level = 5%.

5.2 Example 2: A Probit with Many Regressors

It is common to use MCMC methods for large-scale estimation problems. The second example compares the properties of rnr and rqN with a Metropolis-Hastings algorithm called MALA (Metropolis-adjusted Langevin Algorithm). The idea is to compare the convergence and mixing properties of the draws when there are hundreds of parameters to estimate.\footnote{A description of and implementation details for MALA are given in the Appendix.}

The model is a simple Probit using the data from Helpman et al. (2008). There are 10 regressors plus 324 exporter and importer fixed effects for a total of 334 parameters to be estimated using 248,060 country/year observations. rnr and rqN are implemented with $\gamma = 0.1$ by re-weighting with Gaussian multiplier weights. Implementation details for the different algorithms are given in the Appendix. While rqN accepts all draws by design, MALA only accepts 57% of them which makes the Markov-chain more persistent as illustrated in Figure 2 below. The top panel shows the faster convergence rnr and rqN.
After discarding the initial burn-in, the autocorrelation coefficients are 0.889 for rNR and rqN; 0.995 for MALA, which indicates better mixing\(^{27}\). The bottom panel shows that rNR and rqN draws are more localized around \(\hat{\theta}_n\) than MCMC. Indeed, \(\gamma = 0.1\) implies \(\phi(\gamma) = 19\): the variance of \(\theta^*_{\text{rNR}}\) is 19 times smaller than \(\theta^*_{\text{MCMC}}\).

Table H7 in the Appendix further reports estimates and standard errors for the 10 regressors and the intercept. MLE and its standard errors are computed using R’s glm function and vcovHC for robust standard errors. Results for rqN are close to MLE for all 11 coefficients. Bayesian posterior standard deviations tend to underestimate. Clustered re-weighting with exponential weights are also reported for comparison, it only requires modifying the line of code generating the weights. In terms of computation, the likelihood, gradient, and hessian are computed in C++ using Rcpp for performance. R’s built-in glm takes about 2min for estimation using iteratively reweighted least-squares, rqN 4min, rNR 10min, and MCMC 1h30min. As for most applications of gradient-based optimization on large datasets with many parameters, a good implementation of the code computing the gradient and Hessian is crucial for fast computation: unoptimized code runs is about 2 hours for rqN, and 1.5 days for MCMC. Results for sGD and an infeasible implementation of sNR, which relies on the full sample \(H_n(\hat{\theta}_n)\), are reported in the Appendix. Even after 5 million iterations (> 2 hours), sGD is still far from the solution \(\hat{\theta}_n\).

Figure 2: Probit - Comparison of rNR, rqN and MCMC

Legend: Solid line: rNR(black), rqN(blue), and MCMC (black) draws. Dashed line: full sample MLE estimate.

\(^{27}\)Note that \(0.995^{25} \approx 0.889\) this is why 25 times more draws are generated for MCMC than rqN.
5.3 Example 3: NLS Estimation of Transportation Costs

The third and final example replicates an estimation in Donaldson (2018) of the transportation costs of salt in India between 1861-1930. The model is a simple NLS regression using the log-price of salt of type \( o \) at destination \( d \) in year \( t \):

\[
\text{log}(\text{price}_{dt}) = \beta_0 + \delta \log(\text{LCRED}(R_t, \alpha)_{odt}) + \text{controls} + \varepsilon_{odt}.
\]

The main regressor \( \text{LCRED}(R_t, \alpha)_{odt} \), measures the lowest-cost route effective distance between origin \( o \) and destination \( d \) in year \( t \) given the transportation network \( R_t \) and the transportation costs \( \alpha = (\alpha_{\text{rail}}, \alpha_{\text{road}}, \alpha_{\text{river}}, \alpha_{\text{coast}}) \) for each mode of transportation for the salt products. The regressor \( \text{LCRED}(R_t, \alpha)_{odt} \) is not observed directly. For a given value \( \alpha \), it is constructed using Dijkstra’s shortest-path algorithm applied to each triplet \( o,d,t \) to find the most cost-effective route along the transportation network \( R_t \). The parameters of interest are transportation costs \( \alpha \), inferred from the regression and the price elasticity \( \delta \).

This particular example is interesting for two reasons. First, the model is fairly difficult and time-consuming to estimate. Donaldson (2018) uses a grid search for both estimation and bootstrap inference in a 100 core cluster environment. Second, the model is potentially unidentified for several values. This is the case if \( \delta = 0 \), or for \( \delta \neq 0 \) when a transportation mode systematically dominates the others, e.g. if \( \alpha_{\text{road}} \) is small and \( \alpha_{\text{river}}, \alpha_{\text{coast}} \) are large. The specific values for which this occurs depend on the structure of the transportation network \( R_t \). A quadratic penalty is added to the objective: \( \text{pen}(\theta) = \lambda(\|\alpha - \bar{\alpha}\|_2^2 + \delta^2) \), with \( \lambda = 0.1/n \) and \( \bar{\alpha} = (\bar{\alpha}_{\text{road}}, \bar{\alpha}_{\text{river}}, \bar{\alpha}_{\text{coast}}) = (4.5, 3.0, 2.25) \) are observed historical relative freight rate estimates reported in Donaldson (2018, p916). Notice that if e.g. \( \alpha_{\text{river}} \) is unidentified, then \( \partial_{\alpha_{\text{river}}} Q_{n}^{(b)}(\theta) = 0 \) around \( \hat{\theta}_n \) so that only the penalty determines the path of \( \alpha_{\text{river}} \), and, after convergence, the draws are degenerate \( \alpha_{b,\text{river}} = \bar{\alpha}_{\text{river}} \). With a penalty, degeneracy is indicative that the coefficient is unindentified. This is not the case for a Bayesian posterior.

Figure 3 shows the draws and the adjusted rnr distribution for \( (\alpha, \delta) \). Baseline OLS results evaluated for \( \alpha = \bar{\alpha} \) are reported with bootstrap standard errors clustered at the district level – indexed by \( d \). There are also 142 origin/destination fixed effects, a linear time trend and the intercept, totalling in 147 parameters to be estimated using 7345 obser-

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28 The normalization \( \alpha_{\text{rail}} = 1 \) is used so that all costs are relative to rail.

29 The estimation relied on a coarse grid \( (\alpha_{\text{road}}, \alpha_{\text{river}}, \alpha_{\text{coast}}) \in [1, 10]^3 \) with step size 0.125 for a total of 373248 grid points, and bootstrap inference was conducted using 200 replications on a coarser grid with step size 0.5 totalling in 5832 grid points.
Figure 3: Transportations Costs: Estimation and Inference using rNR

Legend: Top panel: solid line rNR draws for transportation costs $\alpha$ and price elasticity $\delta$. Dashed line: average historical costs $\bar{\alpha}_j$ and OLS estimate $\hat{\delta}_{OLS}(\bar{\alpha})$. Bottom panel: distribution after discarding the first $b = 75$ iterations, and adjusting for $\phi(\gamma)$.

RNR converges quickly in about 75 iterations. 1000 draws are produced in 5h40m on a desktop computer with 7000 calls to Dijkstra’s algorithm, the main computational bottleneck. The bottom panel shows the distribution of the draws after discarding the first 75 iterations, and adjusting for $\phi(\gamma)$. The distributions for coast and road are degenerate, suggesting they are not identified. The results are not sensitive to the choice of $\theta_0$ and $\lambda$. Table 3 reports estimates and standard errors with and without fixed effects. The price elasticity is very similar to the one estimated by OLS when fixing $\bar{\alpha} = \bar{\alpha}$. The estimated coefficient $\alpha_{river}$ is less than half the historical average $\bar{\alpha}_{river}$. The null that river transportation is more cost effective than rail, $\alpha_{river} < 1$, cannot be rejected at the 5% significance level.

$^{30}$ Donaldson (2018) includes origin/time fixed effects instead of a linear time trend but the design matrix is singular with these controls in the baseline OLS regression. Hence, they are replaced with the linear trend.

$^{31}$ For reference, a grid search using a small set of 19 equally spaced points for each $\alpha_j$ requires $19^3 = 6859$ calls to the algorithm.
Table 3: Transportations Costs: Estimates and Standard Errors

|                  | OLS  | rNR  |                  | OLS  | rNR  |
|------------------|------|------|------------------|------|------|
|                  | δ    | α_river | α_coast | α_road | δ    | α_river | α_coast | α_road | δ    |
| Estimates        | 0.22 | 1.29  | 2.25  | 4.50  | 0.24 | 1.86  | 2.25  | 4.50  | 0.22 |
| Standard Errors  | 0.02 | 0.34  | 0.00  | 0.00  | 0.03 | 0.04  | 0.61  | 0.00  | 0.04 |
| Controls?        | Yes  |       |       |       | No   |       |       |       |      |

Legend: OLS: estimated using $\alpha = \bar{\alpha}$, the observed historical relative freight rate estimates. rNR: $\gamma = 0.1$. Both: standard errors computed with Gaussian multiplier reweighting, clustered at the district level. Controls: origin/destination fixed effects.

6 Conclusion

This paper proposes Algorithms that produce estimates, bootstrap confidence intervals and standard errors in a single run. While the theory relies on convexity of the sample and resample objectives, Appendix I discusses implementation and convergence in some non-convex settings. In Examples 1 and 3, rNR and rqN perform well without convexity. Extending the Algorithms to handle general non-convex problems would be a useful avenue for future research. Forneron (2022) shows how to modify a local gradient-based algorithm to design a fast globally convergent one without using multiple starts. This could also be very useful for sampling, as considered here.
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Appendix A  Preliminary Results

Lemma A1 (Taylor Approximation in $L_p$-norm). Suppose Assumptions 2 hold for $p \geq 2$ then for any $\theta \in \Theta$:

\[
\left[ \mathbb{E}^* \left( \|G_m^{(b)}(\theta) - G_m^{(b)}(\hat{\theta}_n) - H_n(\hat{\theta}_n)(\theta - \hat{\theta}_n) \|_p \right) \right]^{1/p} \leq \frac{C_3}{\sqrt{m}} \|\theta - \hat{\theta}_n\|_2 + C_1 \|\theta - \hat{\theta}_n\|_2.
\]

Lemma A2 (Asymptotic Normality). Suppose the Assumptions for Theorem 2 i. hold then as $m, n \to \infty$ with log$(m)/b \to 0$: $\sqrt{\frac{m}{\hat{\sigma}^2(\gamma)}} \mathbb{V}^{-1/2} (\hat{\theta}_b - \hat{\theta}_n) \xrightarrow{d} \mathcal{N}(0, I_d)$.

Appendix B  Proofs for the Main Results

Proof of Lemma 2: For any $\theta_b$, we have $\theta_{b+1} - \hat{\theta}_n = \theta_b - \hat{\theta}_n - \gamma P_b G_n(\theta_b) + \gamma P_b [G_n(\theta_b) - G_m^{(b)}(\theta_b)]$. The first term is the classical update covered in Lemma 1, the last term is uniformly bounded in $L_p$-norm using Assumption 2 i. and 2 iv. Taking the $\| \cdot \|_2$ norm on both sides and applying the triangular inequality that:

\[
\|\theta_{b+1} - \hat{\theta}_n\|_2 \leq \|\theta_b - \hat{\theta}_n - \gamma P_b G_n(\theta_b)\|_2 + \gamma \lambda_P \left( \sup_{\theta \in \Theta} \|G_n(\theta_b) - G_m^{(b)}(\theta_b)\|_2 \right)
\]

\[
\leq (1 - \gamma)\|\theta_b - \hat{\theta}_n\|_2 + \gamma \lambda_P \left( \sup_{\theta \in \Theta} \|G_n(\theta_b) - G_m^{(b)}(\theta_b)\|_2 \right),
\]

where the last inequality comes from Lemma 1. Taking expectations on both sides:

\[
\left[ \mathbb{E}^* \left( \|\theta_{b+1} - \hat{\theta}_n\|_2^p \right) \right]^{1/p} \leq (1 - \gamma) \left[ \mathbb{E}^* \left( \|\theta_b - \hat{\theta}_n\|_2^p \right) \right]^{1/p} + \gamma \lambda_P C_3 \frac{\lambda_P C_3}{\sqrt{m}}
\]

\[
\leq (1 - \gamma)^b \left[ \mathbb{E}^* \left( \|\theta_0 - \hat{\theta}_n\|_2^p \right) \right]^{1/p} + \gamma \lambda_P C_3 \frac{\lambda_P C_3}{\gamma \sqrt{m}},
\]

where the last inequality follows from iterating on the contraction and the identity $\sum_{j=0}^{+\infty} (1 - \gamma)^j = 1/\gamma$. The desired result is then obtained using $C_4 = \gamma \lambda_P C_3$.

Proof of Proposition 1: From the definition of $\theta_b$ and $\theta_b^*$, their difference is:

\[
\theta_{b+1} - \theta_{b+1}^* = (\theta_b - \gamma P_b G_m^{(b+1)}(\theta_b)) - (\hat{\theta}_n + \Psi_n(\theta_b^* - \hat{\theta}_n) - \gamma P_m G_m^{(b+1)}(\hat{\theta}_n))
\]

\[
= \Psi_n(\theta_b - \theta_b^*) + (I_d - \Psi_n)(\theta_b - \hat{\theta}_n) - \gamma \left( P_b G_m^{(b+1)}(\theta_b) - P_m G_m^{(b+1)}(\hat{\theta}_n) \right)
\]

\[
= \Psi_n(\theta_b - \theta_b^*) + \gamma P_m H_n(\hat{\theta}_n)[\theta_b - \hat{\theta}_n] - \gamma \left( P_b G_m^{(b+1)}(\theta_b) - P_m G_m^{(b+1)}(\hat{\theta}_n) \right)
\]

32
where the third equality follows from the fact that

$$ I_d - \Psi_n = \gamma P_m H_n(\hat{\theta}_n). $$

Now note that by construction,

$$ \gamma \left( P_b G_m^{(b+1)}(\theta_b) - P_m G_m^{(b+1)}(\hat{\theta}_n) \right) = \gamma P_m H_n(\hat{\theta}_n)(\theta_b - \hat{\theta}_n) $$

$$ + \gamma (P_b - P_m) \left( G_m^{(b+1)}(\theta_b) - G_m^{(b+1)}(\hat{\theta}_n) \right). $$

(B.1)

Assumption 3 i. implies that $P_m = \gamma^{-1}(I_d - \Psi_n)\gamma_n(\hat{\theta}_n)^{-1}$ has bounded eigenvalues, without loss of generality assume $\lambda_{\min}(\gamma P_m) \leq \lambda_{\max}(\gamma P_m) \leq \lambda_{\max}(P_m)$; Together with Lemma A1 this implies that conditional on $P_b, \theta_b$:

$$ \left[ \mathbb{E}^*\left( \left\| B.1 \right\|^2 \right) \right]^{2/p} \leq \gamma \lambda_P \left[ C_3 \left\| \theta_b - \hat{\theta}_n \right\|^2 + C_1 \left\| \theta_b - \hat{\theta}_n \right\|^2 \right], $$

the conditioning is possible because resampling and reweighting are done independently between iterations $b$ and $b + 1$. Now using the law of iterated expectations and Lemma 2:

$$ \left[ \mathbb{E}^*\left( \left\| B.1 \right\|^2 \right) \right]^{2/p} \leq \gamma \lambda_P \left[ C_3 \left\| \theta_b - \hat{\theta}_n \right\|^2 + C_1 \left\| \theta_b - \hat{\theta}_n \right\|^2 \right] $$

$$ \leq \gamma \lambda_P \left[ C_3 \left( 1 - \gamma \right)^{d_{0,n}} + \frac{C_4}{\gamma \sqrt{m}} \right] + C_1 \left( 1 - \gamma \right)^{b_{0,n}} + \frac{C_5}{\gamma \sqrt{m}}. $$

By Assumptions 3 iii., 2 iii., Lemma 2, mean-value theorem, and Cauchy-Schwarz inequality,

$$ \left[ \mathbb{E}^*\left( \left\| B.2 \right\|^2 \right) \right]^{2/p} \leq \gamma \left[ \mathbb{E}^*\left( \left\| P_b - P_m \right\|^2 \right)^{1/p} \right] \left[ \mathbb{E}^*\left( \left\| H_m^{(b+1)}(\tilde{\theta}_b)(\theta_b - \hat{\theta}_n) \right\|^2 \right)^{1/p} \right] $$

$$ \leq \gamma \lambda_H C_5 \left( \left\| P_b \right\| d_{0,n} + \frac{1}{\sqrt{m}} \right) \left( 1 - \gamma \right)^{b_{0,n}} + \frac{C_5}{\gamma \sqrt{m}}, $$

where $\tilde{\theta}_b$ is some intermediate value between $\theta_b$ and $\hat{\theta}_n$, and an upper bound defined in terms of $\tilde{\theta}_b = \max[1 - \gamma, \rho, \lambda_{\max}(\Psi_n)] < 1$ to simplify notation.

The two bounds lead to the following recursion on the coupling distance:

$$ \left[ \mathbb{E}^*\left( \left\| \theta_{b+1} - \theta_{b+1}^* \right\|^2 \right) \right]^{2/p} \leq \tilde{\rho} \left[ \mathbb{E}^*\left( \left\| \theta_b - \theta_b^* \right\|^2 \right)^{2/p} \right] + \left[ \mathbb{E}^*\left( \left\| B.1 \right\|^2 \right) \right]^{2/p} + \left[ \mathbb{E}^*\left( \left\| B.2 \right\|^2 \right) \right]^{2/p} $$

$$ \leq \tilde{\rho} \left[ \mathbb{E}^*\left( \left\| \theta_b - \theta_b^* \right\|^2 \right)^{2/p} \right] + C_{B_{1.2}} \left( \left\| \theta_{b} \right\|^2 + \left\| \theta_{b}^* \right\|^2 \right) + \frac{1}{m} $$

$$ \leq C_{\tilde{\rho}} \left( \left\| \theta_{b} \right\|^2 + \left\| \theta_{b}^* \right\|^2 + \frac{1}{m} \right), $$

where

32 This is satisfied for rNR and rQN with $(\Delta_P, \lambda_P) = (\Delta_H, \lambda_H)$. 33
where $C_{B_1,2}$ is a constant which depends on the terms used to bound (B.1) and (B.2). Recall that $\theta_0 = \theta_n^*$ so that the coupling distance is zero for $b = 0$. Using $C_6 = C_{B_1,2} / (1 - \rho)$ yields the desired result.

**Proof of Theorem 1**: To bound the distance $\mathbb{E}^* \left( \| \hat{\theta}_{\text{re}}^* - \hat{\theta}_n \|_2 \right)$, we use the recursive representation (8) given below:

$$
\theta_{b+1}^* - \hat{\theta}_n = \Psi_n(\theta_b^* - \hat{\theta}_n) - \gamma \mathcal{P}_m G_m^{(b+1)}(\hat{\theta}_n), \quad \theta_0^* = \theta_0.
$$

It can be re-written as:

$$
\theta_{b+1}^* - \hat{\theta}_n = \Psi_n^b(\theta_0 - \hat{\theta}_n) - \gamma \sum_{j=0}^b \Psi_n^j \mathcal{P}_m G_m^{(b-j)}(\hat{\theta}_n),
$$

and take the average $\bar{\theta}_{\text{re}}^* = 1 / B \sum_{b=1}^B \theta_b^*:

$$
\bar{\theta}_{\text{re}}^* - \hat{\theta}_n = \frac{1}{B} \sum_{b=1}^B \Psi_n^{b-1}(\theta_0 - \hat{\theta}_n) - \gamma \frac{1}{B} \sum_{b=1}^B \sum_{j=0}^{b-1} \Psi_n^j \mathcal{P}_m G_m^{(b-j)}(\hat{\theta}_n).
$$

Assumption 3 i. implies that $\| \Psi(\hat{\theta}_n)^b(\theta_0 - \hat{\theta}_n) \|_2 \leq \rho^b \| \theta_0 - \hat{\theta}_n \|_2$, so the first term is less than $\frac{d_{0,n}}{(1 - \rho)B}$ in expectation. Consider now the second term. Using the relationship between $\| \cdot \|_2$ and the Frobenius $\| \cdot \|_F$ norms, (Bhatia 2013, p7), the zero-mean condition (Assumption 2 i.), and the conditional iid assumption, we have:

$$
\mathbb{E}^* \left( \left\| \frac{1}{B} \sum_{b=1}^B \sum_{j=0}^{b-1} \Psi_n^j \mathcal{P}_m G_m^{(b-j)}(\hat{\theta}_n) \right\|_2^2 \right) \leq \mathbb{E}^* \left( \left\| \frac{1}{B} \sum_{b=1}^B \sum_{j=0}^{b-1} \Psi_n^j \mathcal{P}_m G_m^{(b-j)}(\hat{\theta}_n) \right\|^2_F \right)
$$

$$
= \text{trace} \left( \text{var}^* \left[ \frac{1}{B} \sum_{b=1}^B \sum_{j=0}^{b-1} \Psi_n^j \mathcal{P}_m G_m^{(b-j)}(\hat{\theta}_n) \right] \right)
$$

$$
= \frac{1}{B^2} \sum_{b=1}^B \text{trace} \left( \text{var}^* \left[ \sum_{j=0}^{B-b} \Psi_n^j \mathcal{P}_m G_m^{(b)} \right] \right)
$$

$$
= \frac{1}{B^2} \sum_{b=1}^B \text{trace} \left( \left[ \sum_{j=0}^{B-b} \Psi_n^j \mathcal{P}_m \Sigma_n \mathcal{P}_m^t \left[ \sum_{j=0}^{B-b} \Psi_n^j \right] \right] \right)
$$

$$
\leq \frac{\lambda_p^2 \text{trace}(\Sigma_n)}{(1 - \rho)^2} \frac{1}{mB^*}
$$

where $\Sigma_n = m \times \text{var}^*[G_m^{(b)}(\hat{\theta}_n)]$ is bounded by assumption. Putting the two inequalities together, we have the first result:

$$
\mathbb{E}^* \left( \| \hat{\theta}_{\text{re}} - \hat{\theta}_n \|_2 \right) \leq \frac{d_{0,n}}{(1 - \rho)B} + \frac{\lambda_p}{1 - \rho} \sqrt{\frac{\text{trace}(\Sigma_n)}{mB}}.
$$
The second result is then derived using this together with Proposition 1:

\[ \mathbb{E}^* \left( \| \tilde{\theta}_{RE} - \hat{\theta}_n \|_2 \right) \leq \mathbb{E}^* \left( \| \tilde{\theta}^*_{RE} - \hat{\theta}_n \|_2 \right) + \mathbb{E}^* \left( \| \tilde{\theta}^*_{RE} - \tilde{\theta}^*_{RE} \|_2 \right) \]

\[ \leq \mathbb{E}^* \left( \| \tilde{\theta}^*_{RE} - \hat{\theta}_n \|_2 \right) + \frac{1}{B} \sum_{b=1}^{B} \mathbb{E}^* \left( \| \theta_b - \theta_b^* \|_2 \right) \]

\[ \leq \frac{\lambda_p \sqrt{\text{trace} \left( \Sigma_n \right)}}{1 - \bar{p}} \frac{1}{\sqrt{mB}} + C_0 \left( \frac{1}{m} + \frac{d_{0,n} + d_{0,n}^2}{(1 - \bar{p})B} \right), \]

which is a \( o \left( \frac{1}{\sqrt{n}} \right) \) when \( \frac{\sqrt{n}}{\min(m,B)} \to 0 \) and \( d_{0,n} = O(1) \). This implies the second result:

\[ \sqrt{n}(\tilde{\theta}_{RE} - \theta^*) = \sqrt{n}(\hat{\theta}_n - \theta^*) + o^*_p(1). \]

\[ \square \]

**Proof of Theorem 2**: First, note that Lemma A2 implies that \( \sqrt{mV^{-1/2}(\tilde{\theta}_b - \hat{\theta}_n)} \xrightarrow{d^*} \mathcal{N}(0, I_d) \). Under Assumption 5 and Lemma 2 we have:

\[ \frac{h(\tilde{\theta}_b) - h(\hat{\theta}_n)}{\sqrt{\nabla h(\hat{\theta}_n)V_n \nabla h(\hat{\theta}_n)' / n}} = \frac{\nabla h(\hat{\theta}_n)(\tilde{\theta}_b - \hat{\theta}_n)}{\sqrt{\nabla h(\hat{\theta}_n)V_n \nabla h(\hat{\theta}_n)' / n}} + o^*_p(1) \xrightarrow{d^*} \mathcal{N}(0, 1). \]

Convergence in distribution implies convergence of the CDF (Billingsley, 2013, p323), which in turn implies convergence of quantiles at every point of continuity of the limiting CDF (van der Vaart, 1998, Lem21.2). By continuity everywhere of the normal CDF this implies that for any \( \alpha \in (0, 1) \):

\[ \frac{c_{h,b}(\alpha)}{\sqrt{\nabla h(\hat{\theta}_n)V_n \nabla h(\hat{\theta}_n)' / n}} - q(\alpha) \xrightarrow{p} 0, \]

where \( q(\alpha), c_{h,b}(\alpha) \) are the \( \alpha \)-th quantile of a standard Gaussian and \( \tilde{\theta}_b - \hat{\theta}_n \), respectively. Apply this to \( \alpha / 2 \) and \( 1 - \alpha / 2 \) to get the first result:

\[ \mathbb{P}_n \left( c_{h,b}(\alpha / 2) \leq h(\theta^*) - h(\hat{\theta}_n) \leq c_{h,b}(1 - \alpha / 2) \right) \]

\[ = \mathbb{P}_n \left( q(\alpha / 2) + o_p(1) \leq \frac{\sqrt{n}[h(\theta^*) - h(\hat{\theta}_n)]}{\sqrt{\nabla h(\hat{\theta}_n)V_n \nabla h(\hat{\theta}_n)'}} \leq q(1 - \alpha / 2) + o_p(1) \right) \to 1 - \alpha, \]

using Assumption 5 i. to get the desired limit on the right-hand-side.
For the second result, note that we have

\[ m \times \| \text{var}^*[h(\theta_b)] - \text{var}^*[h(\theta_b^*)] \| \]

\[ = m \times \| \mathbb{E}^* \left( [h(\theta_b) - h(\theta_b^*)][h(\theta_b) - h(\hat{\theta}_n) + h(\theta_b^*) - h(\hat{\theta}_n)] \right) \]

\[ - \mathbb{E}^*(h(\theta_b) - h(\theta_b^*)) \mathbb{E}^*(h(\theta_b) - h(\hat{\theta}_n) + h(\theta_b^*) - h(\hat{\theta}_n)) \| \]

\[ \leq 2m \times \left[ \mathbb{E}^* \left( \| h(\theta_b) - h(\theta_b^*) \|_2^2 \right)^{1/2} \left[ \mathbb{E}^* \left( \| h(\theta_b) - h(\hat{\theta}_n) \|_2^2 \right) \right]^{1/2} + \left[ \mathbb{E}^* \left( \| h(\theta_b^*) - h(\hat{\theta}_n) \|_2^2 \right) \right]^{1/2} \right]. \]

Then using Assumption 5 ii. with Proposition 1 for \( p = 4 \):

\[ \mathbb{E}^* \left( \| h(\theta_b) - h(\theta_b^*) \|_2^2 \right)^{1/2} \leq \| \nabla h \|_\infty C_6 \left( \frac{1}{m} + \rho^b d_{0,n} + d^2_{0,n} \right), \]

where \( \| \nabla h \|_\infty = \sup_{\theta \in \Theta} \max_{j=1,\ldots,d_\theta} |\partial_{\theta_j} h(\theta)| \) is the largest entry-wise element, which is finite by continuity and compactness. Also, using Assumption 5 ii., Lemma 2 we have:

\[ \mathbb{E}^* \left( \| h(\theta_b) - h(\hat{\theta}_n) \|_2^2 \right)^{1/2} \leq \| \nabla h \|_\infty \left( 1 - \gamma \right)^b d_{0,n} + \frac{C_4}{\gamma \sqrt{m}}. \]

Using Assumption 5 ii., equation 8 with \( \Psi_n = (1 - \gamma) I_d \), and Assumption 2 i., we have:

\[ \mathbb{E}^* \left( \| h(\theta_b^*) - h(\hat{\theta}_n) \|_2^2 \right)^{1/2} \leq \| \nabla h \|_\infty \left( 1 - \gamma \right)^b d_{0,n} + \frac{C_2}{\sqrt{m}}. \]

Putting the three inequalities back into (B.3), we get:

\[ \mathbb{E}^* \left( \| h(\theta_b^*) - h(\hat{\theta}_n) \|_2^2 \right)^{1/2} \leq C_{\text{B.3}} \left( m^{-1/2} + m \rho^b d_{0,n} + d^2_{0,n} \right), \]

where \( C_{\text{B.3}} \) involves \( C_2, C_4, C_6, \gamma, \) and \( \| \nabla h \|_\infty \). Since \( m^{-1/2} \) and \( m \rho^b \rightarrow 0 \), the right-hand-side is negligible. Hence, we can focus on the quantity \( \text{var}^*[h(\theta_b^*)] \).

Now repeat the steps above with:

\[ m \times \| \text{var}^*[h(\theta_b^*)] - \text{var}^*[\nabla h(\hat{\theta}_n)(\theta_b^* - \hat{\theta}_n)] \|. \]

Using the mean-value theorem, the Cauchy-Schwarz inequality, and Assumption 5 ii.:

\[ \mathbb{E}^* \left( \| h(\theta_b^*) - h(\hat{\theta}_n) - \nabla h(\hat{\theta}_n)(\theta_b^* - \hat{\theta}_n) \|_2^2 \right)^{1/2} \leq C_h \left[ \mathbb{E}^* \left( \| \theta_b^* - \hat{\theta}_n \|_2^4 \right) \right]^{1/2} \]

\[ \leq C_h \left( 1 - \gamma \right)^b d_{0,n} + \frac{C_2}{\sqrt{m}}. \]

The identity comes from \( \text{var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 \) for a random variable \( X \), and the identity \( a^2 - b^2 = (a - b)(a + b) \) for any numbers \( a, b \). The inequality follows from Cauchy-Schwarz and the triangular inequality.
where $C_h$ is the Lipschitz constant of the gradient $\nabla h$. The bound for $[E^*([h(\theta^*_b) - h(\hat{\theta}_n)])]^1/2$ is given above and the same upper-bound applies to $[E^*([\nabla h(\hat{\theta}_n)[\theta^*_b - \hat{\theta}_n]])]^1/2$. Putting everything together:

$$
(B.4) \leq C_{(B.4)} \left( m^{-1/2} + m(1 - \gamma)^b d_{0,n} \right),
$$

which is also negligible; $C_{(B.4)}$ depends on $C_h, \|\nabla h\|_\infty$, and $C_2$.

From equation $(8)$ and $\text{var}^* (\sqrt{m}[H_n(\hat{\theta}_n)]^{-1}C_{m}^{(h)}(\hat{\theta}_n)) = V_n$, it is immediate that:

$$
m \times \text{var}^* \left( \nabla h(\hat{\theta}_n)(\theta^*_b - \hat{\theta}_n - (1 - \gamma)^b (\theta_0 - \hat{\theta}_n)) \right) = \gamma^2 \frac{1 - (1 - \gamma)^{b+1}}{1 - (1 - \gamma)^2} \nabla h(\hat{\theta}_n)V_n \nabla h(\hat{\theta}_n)' = [\phi(\gamma) + o(1)] \nabla h(\hat{\theta}_n)V_n \nabla h(\hat{\theta}_n)'.
$$

Now note that:

$$
|m \times \text{var}^* \left( \nabla h(\hat{\theta}_n)(\theta^*_b - \hat{\theta}_n - (1 - \gamma)^b (\theta_0 - \hat{\theta}_n)) \right) - m \times \text{var}^* \left( \nabla h(\hat{\theta}_n)(\theta^*_b - \hat{\theta}_n) \right)| \\
\leq 2m(1 - \gamma)^{2b} d_{0,n}^2 \times \text{var}^* \left( \nabla h(\hat{\theta}_n)(\theta^*_b - \hat{\theta}_n) \right) + (1 - \gamma)^{2b} d_{0,n}^2 \rightarrow 0,
$$

since $\text{var}^*(\nabla h(\hat{\theta}_n)(\theta^*_b - \hat{\theta}_n)) = O(1)$. Putting everything together, we get the desired result:

$$
\frac{m}{\phi(\gamma)} \text{var}^* (h(\theta_b)) = \nabla h(\hat{\theta}_n)V_n \nabla h(\hat{\theta}_n)' + o(1).
$$

\[\square\]

**Proof of Corollary 1 for rNR:** Assumption 2 ii. implies Assumption 2 holds with $\lambda P = \lambda_H^{-1}$ and $\lambda_P = \lambda_H^{-1}$.

Let $\overline{P}_m = [H_n(\hat{\theta}_n)]^{-1}$, $\Psi_n = I_d - \gamma \overline{P}_m H_n(\hat{\theta}_n) = (1 - \gamma)I_d$ so Assumption 3 i. holds. Since for rNR we use $P_b = [H_m^{(b+1)}(\theta_b)]^{-1}$, Assumption 2 implies:

$$
[E^* \left( \|I_d - P_b[P_m]^{-1}\|^{p/2} \right)]^{2/p} \leq \overline{\lambda}_P \left[ E^* \left( \|H_m^{(b+1)}(\theta_b) - H_n(\hat{\theta}_n)\|^{p/2} \right) \right]^{2/p} \\
\leq \frac{\overline{\lambda}_P C_3}{\sqrt{m}} + \overline{\lambda}_P C_1 \left[ E^* \left( \|\theta_b - \hat{\theta}_n\|^{p/2} \right) \right]^{2/p},
$$

using the triangular inequality, Assumptions 2 ii., and 1 ii. Applying Lemma 2 into the last inequality to find:

$$
[E^* \left( \|I_d - P_b[P_m]^{-1}\|^{p/2} \right)]^{2/p} \leq \frac{\overline{\lambda}_P C_3}{\sqrt{m}} + \overline{\lambda}_P C_1 \left( (1 - \gamma)^b d_{0,n} + \frac{C_4}{\gamma \sqrt{m}} \right).
$$

Set $\rho = (1 - \gamma)$ and group terms to verify Assumption 3 ii. Now all the assumptions required for Theorems 1 and 2 are satisfied which proves Corollary 1 i-ii.  

\[\square\]
Proof of Corollary 1 for rQN:  The proof follows the same outline as the proof of Corollary 1 for rnr above. The main challenge is to show that Assumption 2 holds under Assumptions 1, 2. Then under the conditions of Lemma 2 establish that Assumption 3 holds with $\overline{P}_m = [H_n(\theta_n)]^{-1}$. Then, all the desired results will follow.

1) Verifying Assumption 2. First, recall the quantities involved in the quasi-Newton least-squares update:

$$P_b = (\hat{H}_b^H\hat{H}_b + \tau_b^2 I_d)^{-1/2}, \quad \hat{H}_b = Y_b^rS_b(\hat{S}'_bS_b)^{-1},$$

with $S_b = (s_b, \ldots, s_{b-L+1})'$, $s_{b-j} = (\theta_{b-j} - \theta_{b-j-1})/\|\theta_{b-j} - \theta_{b-j-1}\|_2$, and $Y_b = (y_b, \ldots, y_{b-L+1})'$, $y_{b-j} = H_m^{(b+1-j)}(\theta_{b-j})s_{b-j}$. The modification term is $\tau_b = \max[\Delta_H/2 - \sigma_{\min}(\hat{H}_b), 0]$, where $\sigma_{\min}(\hat{H}_b)$ is the smallest singular value of $\hat{H}_b$.

By positive semi-definite transform, $\lambda_{\min}(\hat{H}_b^H\hat{H}_b) \geq 0$ and $\lambda_{\min}((\hat{H}_b^H\hat{H}_b + \tau_b^2 I_d)^{1/2}) \geq \Delta_H/2$. This implies the first inequality: $\lambda_{\max}(P_b) \leq 2\lambda_H^{-1} < \infty$. For the lower bound, note that $\hat{H}_b^H\hat{H}_b = (S'_bS_b)^{-1}S'_bY_b^rS_b(\hat{S}'_bS_b)^{-1}$ from which we can derive:

$$\lambda_{\max}(\hat{H}_b^H\hat{H}_b) \leq \frac{\lambda_{\max}(S'_bS_b/L)}{[\lambda_{\min}(S'_bS_b/L)]^2} \lambda_{\max}(Y_b^rY'_b/L),$$

where $\lambda_{\min}(S'_bS_b/L) \geq \lambda_S > 0$ is enforced in the Algorithm. By normalization of the $s_{b-j}$, $\lambda_{\max}(S'_bS_b/L) \leq \text{trace}(S'_bS_b/L) = 1$. Similarly:

$$\lambda_{\max}(Y_b^rY'_b/L) \leq \text{trace}(Y_b^rY'_b/L) = \frac{1}{L} \sum_{j=0}^{L-1} s'_{b-j}H_m^{(b-j)}(\theta_{b-j})H_m^{(b-j)'}(\theta_{b-j})s_{b-j} \leq \lambda_H^2,$$

using the upper bound in Assumption 2 ii. Hence $\lambda_{\max}(\hat{H}_b^H\hat{H}_b) \leq [\lambda_H/\lambda_S]^2$ and $\lambda_{\min}(P_b) \geq ([\lambda_H/\lambda_S]^2 + \lambda_H^2/4)^{-1/2}$. Altogether, Assumption 2 holds with:

$$0 < \Delta_P = ([\lambda_H/\lambda_S]^2 + \lambda_H^2/4)^{-1/2} \leq \lambda_{\min}(P_b) \leq \lambda_{\max}(P_b) \leq 2\lambda_H^{-1} = \bar{\lambda}_P < \infty.$$

2) Verifying Assumption 3. Given that Assumptions 1, 2 and 3 hold, Lemma 2 is satisfied for an appropriate choice of $\gamma \in (0, 1]$. Let $\overline{P}_m = [H_n(\theta_n)]^{-1}$, the goal is to prove that:

$$\left[\mathbb{E}^* \left( \|I_d - P_bH_n(\theta_n)\|_2^2 \right) \right]^{1/p} \leq C_5 \left( (1 - \gamma)^b d_0 + m^{-1/2} \right).$$

Using Assumption 2, the identity $H_n(\theta_n) = [H_n(\hat{\theta}_n)'H_n(\hat{\theta}_n)]^{1/2}$, and the Ando-Hemmen
inequality\footnote{\cite{van1980}} we have under the Frobenius norm $\| \cdot \|_F$:

$$
\| I_d - P_b H_n(\hat{\theta}_n) \|_F \leq P \left[ \| \hat{H}_b \hat{H}_b - \hat{H}_n(\hat{\theta}_n)^{\dagger} \|_F + \| P \|_F \right]
$$

Next, recall that $\| I_d - P_b H_n(\hat{\theta}_n) \|_2 \leq \| I_d - P_b H_n(\hat{\theta}_n) \|_F$, the desired result follows from bounding $\| \hat{H}_b - H_n(\hat{\theta}_n) \|_F^2$ and $\tau_b$ in $L_p$-norm.

First, consider $\tau_b = \max[\Delta_H/2 - \sigma_{\min}(\hat{H}_b), 0]$. By Lipschitz-continuity of the soft-max operator $(x)^+ = \max(x, 0)$, and the identity $\lbrack \Delta_H/2 - \lambda_{\min}(H_n(\hat{\theta}_n)) \rbrack^+ = 0$, we have:

$$
\lbrack \Delta_H/2 - \sigma_{\min}(\hat{H}_b) \rbrack^+ = \| \hat{H}_b - H_n(\hat{\theta}_n) \|_F^2
$$

The second equality follows from \cite{Hoffman1953}, see also \cite{Bhatia2013} p153. Using $\tau_b^2 \leq \Delta_H/2$, the bound will follow from the derivations for $\| \hat{H}_b - H_n(\hat{\theta}_n) \|_F$ given below.

Next, recall that $\hat{H}_b = Y_b' S_b (S_b' S_b)^{-1}$, where $y_{b-j} = H_n^{(b+1-j)}(\theta_{b-j}) s_{b-j} = H_n(\hat{\theta}_n) s_{b-j} + r_{b-j}$, where $r_{b-j} = [H_n^{(b+1-j)}(\theta_{b-j}) - H_n(\hat{\theta}_n)] s_{b-j}$. Let $R_b = (r_{b}, \ldots, r_{b-L+1})'$, using standard OLS calculations we have:

$$
\hat{H}_b - H_n(\hat{\theta}_n) = (R_b S_b / L) (S_b' S_b / L)^{-1}.
$$

Now recall that $\lambda_{\min}(S_b' S_b / L) \geq \Delta_S > 0$ and $\| s_{b-j} \|_2 = 1$ are enforced algorithmically, also

\footnote{\cite{van1980} and \cite{Higham2008}, for any two positive definite matrices $A, B$, $\| A^{1/2} - B^{1/2} \| \leq \| \sqrt{\lambda_{\min}(A)} + \sqrt{\lambda_{\min}(B)} \|^{-1} \| A - B \|$ where $\| \cdot \|$ is a unitarily invariant norm. Recall that $\| \cdot \|_F$ is unitarily invariant.
\[ \| \cdot \|_F \leq \sqrt{d_\theta} \cdot \| \cdot \|_2 \text{ for matrices. Thus, we have:} \]

\[ \| \hat{H}_b - H_n(\hat{\theta}_n) \|_F \leq \lambda_{\theta}^{-1} \frac{1}{L} \sum_{j=0}^{L-1} |r_{b-j}^t s_{b-j}| \]

\[ \leq \lambda_{\theta}^{-2} \frac{1}{L} \sum_{j=0}^{L-1} \| H_m^{b-j}(\theta_{b-j}) - H_n(\hat{\theta}_n) \|_F \]

\[ \leq \lambda_{\theta}^{-2} \frac{\sqrt{d_\theta}}{L} \left( \sum_{j=0}^{L-1} \sup_{\theta \in \Theta} \| H_m^{(b+1-j)}(\theta) - H_n(\theta) \|_2 + C_1 \| \theta_{b-j} - \hat{\theta}_n \|_2 \right). \]

Taking the $L_p$-norm on both sides, we have:

\[ \left[ E^\star \left( \| \hat{H}_b - H_n(\hat{\theta}_n) \|_F^p \right) \right]^{1/p} \leq \lambda_{\theta}^{-2} C_2 \frac{\sqrt{d_\theta}}{m} + \lambda_{\theta}^{-2} \sqrt{d_\theta} C_1 \left( \frac{d_{0,n} (1 - \gamma)^{b-L+1}}{\gamma} + \frac{C_4}{\sqrt{m}} \right). \]

Finally, putting everything together we get:

\[ \left[ E^\star \left( \| I_d - P_b H_n(\hat{\theta}_n) \|_2^p \right) \right]^{1/p} \]

\[ \leq \frac{\chi_p}{\lambda_p^{-1} + \Delta_H} (\sigma_{\max}(\hat{H}_b) + \overline{\lambda}_H) \left( 1 + \frac{\sqrt{d_\theta} \Delta_H / 2}{\sigma_{\min}(\hat{H}_b) + \Delta_H} \right) \left[ E^\star \left( \| \hat{H}_b - H_n(\hat{\theta}_n) \|_F^p \right) \right]^{1/p}, \]

which yields the desired result and concludes the proof. \qed
Appendix C  Derivations for the Least-Squares Example

In this example, $y_n = X_n \hat{\theta} + \hat{\epsilon}_n$, $Q_n(\theta) = \frac{1}{2m} (y_n - X_n \theta)' (y_n - X_n \theta)$, $H_n = X_n' X_n / n$, $G_n = -X_n' \hat{\epsilon}_n / n$, and $Q_m^{(b)}(\theta) = \frac{1}{2m} (y_m^{(b)} - X_m^{(b)} \theta)' (y_m^{(b)} - X_m^{(b)} \theta)$, $H_b = H_m^{(b+1)}(\theta_b) = X_m^{(b+1)}' X_m^{(b+1)} / m$. $G_b(\theta) = -X_m^{(b+1)}' [y_m^{(b+1)} - X_m^{(b+1)} \theta] / m$. Let $\hat{\theta}_m^{(b+1)} = (X_m^{(b+1)}' X_m^{(b+1)})^{-1} X_m^{(b+1)}' y_m^{(b+1)}$ be the $m$ out of $n$ bootstrap estimate. Orthogonality of least squares residuals will be used repeatedly.

**Gradient Descent**  $\theta_{k+1} = \theta_k - \gamma [-X_n' (y_n - X_n \theta_k) / n]$. Subtract $\hat{\theta}_n$ on both sides and note that $y_n = X_n \hat{\theta}_n + \hat{\epsilon}_n$ (full sample estimates), then:

$$\theta_{k+1} - \hat{\theta}_n = \theta_k - \hat{\theta}_n - \gamma \left[ -X_n' (X_n \hat{\theta}_n + \hat{\epsilon}_n - X_n \theta_k) / n \right]$$

$$= \theta_k - \hat{\theta}_n - (\gamma H_n)(\theta_k - \hat{\theta}_n) + \gamma X_n' \hat{\epsilon}_n / n = (I - \gamma H_n)(\theta_b - \hat{\theta}_n) \text{ since } X_n' \hat{\epsilon}_n = 0.$$

**Newton-Raphson**  $\theta_{k+1} = \theta_k - \gamma [H_n]^{-1} [-X_n' (y_n - X_n \theta_k) / n]$. Subtract $\hat{\theta}_n$ on both sides:

$$\theta_{k+1} - \hat{\theta}_n = \theta_k - \hat{\theta}_n - \gamma H_n^{-1} \left[ -[X_n X_n / n][\hat{\theta}_n - \theta_k] + X_n' \hat{\epsilon}_n / n \right] = (1 - \gamma)(\theta_k - \hat{\theta}_n) \text{ since } X_n' \hat{\epsilon}_n = 0.$$

**Stochastic Gradient Descent**  $\theta_{b+1} = \theta_b - \gamma_b \left[ -X_m^{(b)}' (y_m^{(b)} - X_m^{(b)} \theta_b) / m \right]$. Thus

$$\theta_{b+1} - \hat{\theta}_n = \theta_b - \hat{\theta}_n - \gamma_b \left[ -X_m^{(b+1)}' (y_m^{(b+1)} - X_m^{(b+1)} \hat{\theta}_n - X_m^{(b+1)}[\theta_b - \hat{\theta}_n]) / m \right]$$

$$= (I - \gamma_b H_b)(\theta_b - \hat{\theta}_n) + \gamma_b X_m^{(b+1)}' (y_m^{(b+1)} - X_m^{(b+1)} \hat{\theta}_n) / m$$

$$= (I - \gamma_b H_b)(\theta_b - \hat{\theta}_n) - \gamma_b G_b(\hat{\theta}_n) \text{ since } X_m^{(b+1)}' \hat{\epsilon}_m^{(b+1)} = 0.$$

**Resampled Gradient Descent**  $\theta_{b+1} = \theta_b - \gamma \left[ -X_m^{(b+1)}' (y_m^{(b+1)} - X_m^{(b+1)} \theta_b) / m \right]$. Subtract $\hat{\theta}_n$ on both sides and note that $y_m^{(b+1)} = X_m^{(b+1)} \hat{\theta}_m^{(b+1)} + \epsilon_m^{(b+1)}$ (bootstrap estimates). Then

$$\theta_{b+1} - \hat{\theta}_n = \theta_b - \hat{\theta}_n - \gamma \left[ -X_m^{(b+1)}' (X_m^{(b+1)} \hat{\theta}_m^{(b+1)} - \hat{\theta}_n) + \epsilon_m^{(b+1)} - X_m^{(b+1)}[\theta_b - \hat{\theta}_n] \right] / m$$

$$= \theta_b - \hat{\theta}_n - (\gamma H_b)(\theta_b - \hat{\theta}_n) + \gamma H_b(\hat{\theta}_m^{(b+1)} - \hat{\theta}_n)$$

$$= (I - \gamma H_b)(\theta_b - \hat{\theta}_n) + \gamma H_b(\hat{\theta}_m^{(b+1)} - \hat{\theta}_n) \text{ since } X_m^{(b+1)}' \hat{\epsilon}_m^{(b+1)} = 0.$$

**Resampled Newton-Raphson**  $\theta_{b+1} = \theta_b - \gamma [H_b]^{-1} \left[ -X_m^{(b+1)}' (y_m^{(b+1)} - X_m^{(b+1)} \theta_b) / m \right]$. Then

$$\theta_{b+1} - \hat{\theta}_n = \theta_b - \hat{\theta}_n - \gamma [H_b]^{-1} \left[ -X_m^{(b+1)}' (X_m^{(b+1)} \hat{\theta}_m^{(b+1)} - \hat{\theta}_n) + \epsilon_m^{(b+1)} - X_m^{(b+1)}[\theta_b - \hat{\theta}_n] \right] / m$$

$$= (1 - \gamma)(\theta_b - \hat{\theta}_n) + \gamma (\theta_{b+1} - \hat{\theta}_n) \text{ since } X_m^{(b+1)}' \hat{\epsilon}_m^{(b+1)} = 0.$$
Supplement to
"Estimation and Inference by Stochastic Optimization"

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This Supplemental Material consists of Appendices D, E, F, G, H, and I to the main text.

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Appendix D  Proofs for the Preliminary Results

Proof of Lemma [A1]: Add and subtract $G_n(\theta) - G_n(\hat{\theta}_n)$ to the desired quantity, apply a mean-value expansion to get $G_n(\theta) - G_n(\hat{\theta}_n) = H_n(\hat{\theta}_n)(\theta - \hat{\theta}_n)$ and note that $\|H_n(\hat{\theta}_n) - H_n(\hat{\theta}_n)\| \leq C_1\|\theta - \hat{\theta}_n\|$ by Lipchitz continuity of the Hessian and intermediate value. Now using the triangular inequality:

$$\left[ \mathbb{E}^* \left( \|G_m^{(b)}(\theta) - G_m^{(b)}(\hat{\theta}_n) - H_n(\hat{\theta}_n)(\theta - \hat{\theta}_n)\|_2^p \right) \right]^{1/p} \leq \left[ \mathbb{E}^* \left( \|G_m^{(b)}(\theta) - G_m^{(b)}(\hat{\theta}_n)\|_2 + C_1\|\theta - \hat{\theta}_n\|_2 \right)^p \right]^{1/p} + C_1\|\theta - \hat{\theta}_n\|^2,$$

using the intermediate value Theorem again but to the difference $G_m^{(b)} - G_n$ this time, we have: $[G_m^{(b)}(\theta) - G_m^{(b)}(\hat{\theta}_n)] - [G_n(\theta) - G_n(\hat{\theta}_n)] = [H_m^{(b)}(\hat{\theta}_n) - H_n(\hat{\theta}_n)](\theta - \hat{\theta}_n)$. Plugging this back into the inequality above, we have:

$$\left[ \mathbb{E}^* \left( \|G_m^{(b)}(\theta) - G_m^{(b)}(\hat{\theta}_n) - H_n(\hat{\theta}_n)(\theta - \hat{\theta}_n)\|_2^p \right) \right]^{1/p} \leq \left[ \mathbb{E}^* \left( \sup_{\theta \in \Theta} \|H_m^{(b)}(\theta) - H_n(\theta)\|_2^p \right) \right]^{1/p} \|\theta - \hat{\theta}_n\|_2 + C_1\|\theta - \hat{\theta}_n\|^2 \leq \frac{C_3}{\sqrt{m}}\|\theta - \hat{\theta}_n\|_2 + C_1\|\theta - \hat{\theta}_n\|^2.$$

\[ \square \]

Proof of Lemma [A2]: The property that $\overline{\mathcal{P}}_m = [H_n(\hat{\theta}_n)]^{-1}$ is crucial for what follows. To prove the Lemma, first substitute $\theta_b$ for the linear process $\theta_b^\ast$ using Proposition [1]

$$\frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2}(\theta_b - \hat{\theta}_n) = \frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2}(\theta_b^\ast - \hat{\theta}_n) + \frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2}(\theta_b - \theta_b^\ast)$$

$$= \frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2}(\theta_b^\ast - \hat{\theta}_n) + o_p^\ast(1),$$

when $\log(m)/b \to 0$ since it implies $\sqrt{m}p^b = \exp(b[\log(m)/(2b) + \log(p)]) \to 0$, using $\log(m)/(2b) + \log(p) \to \log(p) < 0$. For $\overline{\mathcal{P}}_m = [H_n(\hat{\theta}_n)]^{-1}$, $\Psi_n = (1 - \gamma)I_d$. Using the recursion [8], we have:

$$\frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2}(\theta_b^\ast - \hat{\theta}_n) = \frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2}(1 - \gamma)^b(\theta_0 - \hat{\theta}_n)$$

$$- \gamma \sum_{j=0}^{b-1} (1 - \gamma)^j \frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2}[H_n(\hat{\theta}_n)]^{-1}G_m^{(b-j)}(\hat{\theta}_n).$$

1
The first term is a $o_p^*(1)$ as long as $\log(m)/b \to 0$. For the second term, since the $[H_n(\hat{\theta}_n)]^{-1}G^{(b-j)}_m(\hat{\theta}_n)$ are conditionally independent and identically distributed, we have by a convolution argument:

$$
\mathbb{E}^* \left( \exp(i\tau' \sqrt{m} \gamma \sum_{j=0}^{b-1} (1 - \gamma)^j \frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2} [H_n(\hat{\theta}_n)]^{-1}G^{(b-j)}_m(\hat{\theta}_n) \right)
$$

$$
= \prod_{j=0}^{b-1} \mathbb{E}^* \left( \exp(i\tau' \sqrt{m} \gamma (1 - \gamma)^j \frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2} [H_n(\hat{\theta}_n)]^{-1}G^{(b-j)}_m(\hat{\theta}_n) \right)
$$

$$
= \prod_{j=0}^{b-1} \left[ \exp \left( - \frac{\|\tau\|_2^2 \gamma^2 (1 - \gamma)^{2j}}{2} \frac{\sqrt{m}}{\phi(\gamma)} \right) \left( 1 + \frac{r_m(\gamma(1 - \gamma)^j \tau/\phi(\gamma))}{m^\beta} \right) \right]
$$

$$
= \exp \left( - \frac{\|\tau\|_2^2}{2} (1 + o(1)) \right)
$$

To show that the last product is convergent under the stated assumptions, take logs and use the inequality $\frac{\log(1 + x)}{1 + x} \leq \log(1 + x) \leq x$ for $x > -1$. Then

$$
\log \left( \|\text{error}\| \right) = \sum_{j=0}^{b-1} \log \left( 1 + \frac{|r_m(\gamma(1 - \gamma)^j \tau/\phi(\gamma))|}{m^\beta} \right) \leq \sum_{j=0}^{b-1} \frac{|r_m(\gamma(1 - \gamma)^j \tau/\phi(\gamma))|}{m^\beta}
$$

$$
\leq \sum_{j=0}^{b-1} \frac{\|\gamma \tau/\phi(\gamma)\|^\kappa (1 - \gamma)^{\kappa j}}{m^\beta} \leq \frac{\|\gamma \tau/\phi(\gamma)\|^\kappa}{[1 - (1 - \gamma)^\kappa]m^\beta} \to 0.
$$

Note that $\frac{\gamma}{\phi(\gamma)} = 2 - \gamma \geq 1$ for $\gamma \in (0, 1]$. Putting everything together we have:

$$
\mathbb{E}^* \left( \exp \left( i\tau' \frac{\sqrt{m}}{\sqrt{\phi(\gamma)}} V_n^{-1/2} (\theta_b - \hat{\theta}_n) \right) \right) = \exp \left( - \frac{\|\tau\|_2^2}{2} \right) \left( 1 + O \left( \frac{\|\tau\|_2^\kappa (2 - \gamma)^\kappa}{m^\beta [1 - (1 - \gamma)^\kappa]} \right) \right).
$$

Now since convergence of the characteristic function implies weak convergence (Billingsley 2013, Th26.3, p349), which implies the desired convergence in distribution.

Appendix E  Implementing rNR in R

To illustrate how the rNR is implemented in a real data setting, the following provides detailed commented R code for a probit model using the Mroz (1987) data.
Illustration using Labor Force Participation  The table below presents the estimates and standard errors for all methods and coefficients in the Mroz (1987) application.

| MLE | rNRn | rNR100 | rQNn | rQN100 | rQN200 |
|-----|------|--------|------|--------|--------|
| nwifeinc | -0.012 | -0.013 | -0.014 | -0.012 | -0.011 | -0.012 |
| educ | 0.131 | 0.138 | 0.143 | 0.131 | 0.129 | 0.129 |
| exper | 0.123 | 0.124 | 0.123 | 0.123 | 0.124 | 0.125 |
| exper2 | -0.002 | -0.002 | -0.002 | -0.002 | -0.002 | -0.002 |
| age | -0.053 | -0.053 | -0.055 | -0.052 | -0.052 | -0.052 |
| kidslt6 | -0.868 | -0.892 | -0.902 | -0.864 | -0.855 | -0.844 |
| kidsge6 | 0.036 | 0.038 | 0.041 | 0.036 | 0.035 | 0.032 |
| const. | 0.270 | 0.216 | 0.234 | 0.248 | 0.256 | 0.249 |

| ASE | BOOT | DMK | KS | rNRn | rNR100 | rQNn | rQN100 | rQN200 |
|-----|------|-----|----|------|--------|------|--------|--------|
| nwifeinc | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 |
| educ | 0.025 | 0.026 | 0.026 | 0.025 | 0.027 | 0.028 | 0.027 | 0.025 |
| exper | 0.019 | 0.020 | 0.019 | 0.019 | 0.020 | 0.021 | 0.019 | 0.018 |
| exper2 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
| age | 0.008 | 0.009 | 0.008 | 0.009 | 0.009 | 0.009 | 0.009 | 0.008 |
| kidslt6 | 0.119 | 0.120 | 0.118 | 0.119 | 0.129 | 0.117 | 0.113 | 0.117 |
| kidsge6 | 0.043 | 0.046 | 0.045 | 0.045 | 0.048 | 0.047 | 0.044 | 0.042 |
| const. | 0.509 | 0.512 | 0.507 | 0.505 | 0.494 | 0.535 | 0.544 | 0.494 |

Sample R code to implement the Mroz Example.

```r
set.seed(123) # set the seed
library(numDeriv) # compute numerical derivatives using finite differences, alternative: library(pracma) is usually faster
library(foreign) # to load the data set in Stata dta format

data = read.dta('mroz.dta') # read the mroz data
y = data$inlf # outcome variable
X = cbind(data$nwifeinc, data$educ, data$exper, data$exper^2, data$age, data$kidslt6, data$kidsge6, 1)
colnames(X) = c('nwifeinc', 'educ', 'exper', 'exper2', 'age', 'kidslt6', 'kidsge6', 'constant')
n = 753 # sample size
index0 = 1:n # indices for the sample data

loglik <- function(coef, index=index0) {
  ...
}````
```r
# compute the log-likelihood for the Probit model on the
# observations indexed by index (default 1:n, the original sample)
# at theta = coef

score = X[index,]%*%coef # compute the z-scores
ll = y[index]*log(pnorm(score)) +
    (1-y[index])*log(1-pnorm(score))
return(sum(ll))
}
d_loglik <- function(coef,index=index0) {
  # compute the gradient of the log-likelihood for the Probit model on
  # the observations indexed by index (default 1:n, the original
  # sample) at theta = coef
  # In this example, the gradient is analytically tractable, it could
  # be evaluated by finite differences by using the following:
  # d_loglik <- function(coef,index=index0) { return(jacobian(loglik,
  # coef,index=index)) }

  yy = y[index] # keep observations indexed by index
  XX = X[index,] # keep observations indexed by index
  score = XX%*%coef # compute the z-score
  dll = 0 # initialize the gradient
  for (i in 1:length(index)) {
    dll = dll +
    (yy[i]*XX[i,]*dnorm(score[i])/pnorm(score[i]) -
     (1-yy[i])*XX[i,]*dnorm(score[i])/(1-pnorm(score[i])))
  }
  return(dll)
}

rNR <- function(coef0, learn = 0.1, iter = 500, m = n) {
  # generate 'B = iter' rNR draws with learning rate 'gamma = learn'
  # with m out of n resampling

  coefs = matrix(NA,iter,length(coef0)) # matrix where draws will
  be stored
  coefs[1,] = coef0 # initialize the first-draw

  for (i in 2:iter) {
    index = sample(1:n,m,replace=TRUE) # sample m out of n
    observations with replacement

    G = d_loglik(coefs[i-1,],index=index) # compute the resampled
    gradient G using analytical derivatives. Alternative using
    finite differences:
    # G = jacobian(loglik,coefs[i-1,],index=index)
    H = hessian(loglik,coefs[i-1,],index=index) # compute the
    resampled hessian H using finite differences; we could also
```
compute the jacobian of the gradient \( d_{\loglik} \)

\[
\text{coefs}[i,] = \text{coefs}[i-1,] - \text{learn}*\text{solve}(H,G) \quad \# \text{update}
\]

\[
\text{colnames(coefs)} = \text{colnames}(X) \quad \# \text{label the coefficients}
\]

\[
\text{return( list(coefs = coefs) )} \quad \# \text{return draws}
\]

# estimates and standard errors (source: Introductory Econometrics, A Modern Approach 2nd Edition, Wooldridge)

\[
\text{coef} = c(-0.012,0.131,0.123,-0.0019,-0.053,-0.868,0.036,0.270)
\]

\[
\text{ses} = c(0.005,0.025,0.019, 0.0006, 0.008, 0.119, 0.043, 0.509)
\]

\[
\text{iter\_rNR} = 2e3 \quad \# \text{number of rNR draws}
\]

\[
\text{learn} = 0.3 \quad \# \text{learning rate}
\]

\[
\text{coef0} = \text{coef} \times 3.25 \quad \# \text{starting value}
\]

\[
\text{m1} = 753 \quad \# m = n
\]

\[
\text{m2} = 200 \quad \# m = 200
\]

\[
\text{m3} = 100 \quad \# m = 100
\]

# adjustments to get valid standard errors

\[
\text{adj\_rnr1} = \sqrt{\frac{\text{m1}}{\text{n}}} \times \sqrt{\frac{(1-(1-\text{learn})^2)}{\text{learn}^2}}
\]

\[
\text{adj\_rnr2} = \sqrt{\frac{\text{m2}}{\text{n}}} \times \sqrt{\frac{(1-(1-\text{learn})^2)}{\text{learn}^2}}
\]

\[
\text{adj\_rnr3} = \sqrt{\frac{\text{m3}}{\text{n}}} \times \sqrt{\frac{(1-(1-\text{learn})^2)}{\text{learn}^2}}
\]

\[
\text{b1} = 1 + \text{round}(\log(0.01)/\log(1-\text{learn})) \quad \# \text{burn-in sample size}
\]

# generate rNR draws

\[
\text{out\_rNR1} = \text{rNR(coef0,learn,b1 + iter\_rNR, m1)}
\]

\[
\text{out\_rNR2} = \text{rNR(coef0,learn,b1 + iter\_rNR, m2)}
\]

\[
\text{out\_rNR3} = \text{rNR(coef0,learn,b1 + iter\_rNR, m3)}
\]

# format output

\[
\text{estimates} = \text{rbind}\begin{pmatrix}
\text{coef}, \\
\text{apply(out\_rNR1$coef[b1:(iter\_rNR+b1),],2,mean)}, \\
\text{apply(out\_rNR2$coef[b1:(iter\_rNR+b1),],2,mean)}, \\
\text{apply(out\_rNR3$coef[b1:(iter\_rNR+b1),],2,mean)}
\end{pmatrix}
\]

\[
\text{std\_errs} = \text{rbind}\begin{pmatrix}
\text{ses}, \\
\text{apply(out\_rNR1$coef[b1:(iter\_rNR+b1),],2,sd)}*\text{adj\_rnr1}, \\
\text{apply(out\_rNR2$coef[b1:(iter\_rNR+b1),],2,sd)}*\text{adj\_rnr2}, \\
\text{apply(out\_rNR3$coef[b1:(iter\_rNR+b1),],2,sd)}*\text{adj\_rnr3}
\end{pmatrix}
\]

\[
\text{estimates} = \text{as.data.frame}(\text{estimates})
\]

\[
\text{colnames(\text{estimates})} = \text{colnames}(X)
\]

\[
\text{rownames(\text{estimates})} = c(\text{’MLE’}, \text{’rNRn’}, \text{’rNR200’}, \text{’rNR100’})
\]

\[
\text{std\_errs} = \text{as.data.frame(\text{std\_errs})}
\]
```r
colnames(std_errs) = colnames(X)
rownames(std_errs) = c('ase','rNRn','rNR200','rNR100')

# print results
print(round(cbind(t(estimates), t(std_errs)), digits = 3))
```

|        | MLE | rNRn | rNR200 | rNR100 | ase | rNRn | rNR200 | rNR100 |
|--------|-----|------|--------|--------|-----|------|--------|--------|
| nwifeinc | -0.012 | -0.012 | -0.013 | -0.014 | 0.005 | 0.005 | 0.005 | 0.005  |
| educ    | 0.131  | 0.132 | 0.136  | 0.140  | 0.025 | 0.026 | 0.026 | 0.028  |
| exper   | 0.123  | 0.123 | 0.123  | 0.125  | 0.019 | 0.020 | 0.021 |        |
| exper2  | -0.002 | -0.002 | -0.002 | -0.002 | 0.001 | 0.001 | 0.001 | 0.001  |
| age     | -0.053 | -0.053 | -0.054 | -0.055 | 0.008 | 0.008 | 0.009 | 0.009  |
| kidslt6 | -0.868 | -0.872 | -0.895 | -0.917 | 0.119 | 0.121 | 0.121 | 0.126  |
| kidsge6 | 0.036  | 0.038 | 0.040  | 0.043  | 0.045 | 0.047 | 0.049 |        |
| constant| 0.270  | 0.272 | 0.282  | 0.276  | 0.509 | 0.506 | 0.505 | 0.535  |
Appendix F  Primitive Conditions for Assumption 2

The following provides primitive conditions for Assumption 2. In the main text, although the dimension of \( \theta \) is fixed in the main results, the derivation make explicit the dependence of \( C_2, C_3 \) on the dimension \( d_\theta \).

Setup and Notation. Let \( z_i = (y_i, x_i) \) be iid, the parameter space \( \Theta \) is a compact, convex subset of \( \mathbb{R}^{d_\theta} \), the sample objective, gradient and hessian are given by:

\[
Q_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} q(z_i, \theta), \quad G_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \nabla q(z_i, \theta), \quad H_n(\theta) = \frac{1}{n} \sum_{i=1}^{n} \partial^2_{\theta \theta} q(z_i, \theta).
\]

The re-sampled objective relies on \( m \) draws \( z^{(b)}_i \) taken with equal probability and with replacement from \( (z_1, \ldots, z_n) \), the re-sampled objective, gradient and hessian are given by:

\[
Q_m^{(b)}(\theta) = \frac{1}{m} \sum_{i=1}^{m} q(z^{(b)}_i, \theta), \quad G_m^{(b)}(\theta) = \frac{1}{m} \sum_{i=1}^{m} \nabla q(z^{(b)}_i, \theta), \quad H_m^{(b)}(\theta) = \frac{1}{m} \sum_{i=1}^{m} \nabla^2 q(z^{(b)}_i, \theta).
\]

Alternatively, the re-weighted objective relies on \( m = n \) iid random weights \( w^{(b)}_i \) with mean and variance equal to one, the re-sampled objective, gradient and hessian are given by:

\[
Q_m^{(b)}(\theta) = \frac{1}{m} \sum_{i=1}^{m} w^{(b)}_i q(z_i, \theta), \quad G_m^{(b)}(\theta) = \frac{1}{m} \sum_{i=1}^{m} w^{(b)}_i \nabla q(z_i, \theta), \quad H_m^{(b)}(\theta) = \frac{1}{m} \sum_{i=1}^{m} w^{(b)}_i \nabla^2 q(z_i, \theta),
\]

for Gaussian multiplier weights we have \( w^{(b)}_i \sim \mathcal{N}(1, 1) \).

Assumption F6 (Lipschitz Derivatives). There exists a measurable function \( C_q \) such that for any two \( \theta_1, \theta_2 \in \Theta \) and any \( z_i \):

\[
\| \nabla q(z_i, \theta_1) - \nabla q(z_i, \theta_2) \| \leq C_q(z_i) \| \theta_1 - \theta_2 \|, \quad (F.5)
\]

\[
\| \nabla^2 q(z_i, \theta_1) - \nabla^2 q(z_i, \theta_2) \| \leq C_q(z_i) \| \theta_1 - \theta_2 \|, \quad (F.6)
\]

Let \( p \geq 2 \), \( C_q \) and the derivatives have finite \( p \)-th moment:

\[
\mathbb{E}(\|C_q(z_i)\|^p) < +\infty, \quad \mathbb{E}(\|\nabla q(z_i, \theta^1)\|^p) < +\infty, \quad \mathbb{E}(\|\nabla^2 q(z_i, \theta^1)\|^p) < +\infty \quad (F.7)
\]

The following subsections will prove the Lemma below under resampling and then under re-weighting with Gaussian multiplier weights.

Lemma F3. Suppose Assumption F6 holds, then we have:

i. \( \mathbb{E}^* [G_m^{(b)}(\theta)] = G_n(\theta) \),

ii. \( (\mathbb{E}^* \|G_m^{(b)}(\theta) - G_n(\theta)\|^p_2)^{1/p} \leq C_{2n}^{-1/2} \), where \( C_{2n} \xrightarrow{p} C_2 \) finite,

iii. \( (\mathbb{E}^* \|H_m^{(b)}(\theta) - H_n(\theta)\|^p_2)^{1/p} \leq C_{3n}^{-1/2} \), where \( C_{3n} \xrightarrow{p} C_3 \) finite.
F.1 Re-sampled objective

Proof of Lemma F3. Condition i. is immediate from iid resampling and the additivity over $i$ of the sample and re-sampled gradients. Lemma F3 ii. and iii. are proved the same way so the below will only focus on ii. for brevity.

Let $F_q(z_i, \theta) = C_q(z_i)\|\theta - \theta^i\| + \|\partial q(z_i, \theta^i)\|$ be the envelope function which satisfies: $\|\partial q(z_i, \theta)\| \leq F_q(z_i, \theta)$ for all $\theta \in \Theta$. We have $F_q(z_i, \theta) \leq F_q(z_i) = C_q(z_i)\text{diam}(\Theta) + \|\partial q(z_i, \theta^i)\|$, which does not depend on $\theta$. Here $\text{diam}(\Theta) = \sup_{\theta_1, \theta_2 \in \Theta} ||\theta_1 - \theta_2||$, by compactness it is finite. The main idea is to apply Theorem 2.14.5 in van der Vaart and Wellner (1996), under the distribution $\mathbb{P}^*$, for a $p \geq 2$ such that Assumption F6 holds:

$$\mathbb{E}^* \left( \sup_{\theta \in \Theta} \sqrt{m} \|G_m^{(1)}(\theta) - G_n(\theta)\|^p \right)^{1/p} \lesssim \mathbb{E}^* \left( \sup_{\theta \in \Theta} \sqrt{m} \|G_m^{(1)}(\theta) - G_n(\theta)\| \right) + m^{1/p-1/2} \left[ \frac{1}{n} \sum_{i=1}^n |F_q(z_i)|^p \right]^{1/p},$$

where $\lesssim$ stands for less or equal than, up to universal constants. By the strong law of large numbers and the finite moment Assumption F6, we have, as $n \to \infty$, $\frac{1}{n} \sum_{i=1}^n |F_q(z_i)|^p \to [\mathbb{E}(|F_q(z_i)|^p)]^{1/p}$. Also for $p \geq 2$ and $m \geq 1$, $0 \leq m^{1/p-1/2} \leq 1$. Now using Theorem 2.14.2 in van der Vaart and Wellner (1996), again under the distribution $\mathbb{P}^*$, we have:

$$\mathbb{E}^* \left( \sup_{\theta \in \Theta} \sqrt{m} \|G_m^{(1)}(\theta) - G_n(\theta)\| \right) \lesssim J_{[1]}(1, \mathcal{F}, L_2(\mathbb{P}_n)) \left[ \frac{1}{n} \sum_{i=1}^n |F_q(z_i)|^2 \right]^{1/2},$$

where $J_{[1]}(1, \mathcal{F}, L_2(\mathbb{P}_n)) = \int_0^1 \sqrt{1 + \log N_{[1]}(\varepsilon \|F_q\|, \mathcal{F}, L_2(\mathbb{P}_n))} d\varepsilon$ is the bracketing integral of the functions class $\mathcal{F} = \{ \theta \to \partial q(z_i, \theta) \}$ and $N_{[1]}(\varepsilon \|F_q\|, \mathcal{F}, L_2(\mathbb{P}_n))$ its bracketing number. By Theorem 9.23 in Kosorok (2007), we have $N_{[1]}(\varepsilon \|F_q\|, \mathcal{F}, L_2(\mathbb{P}_n)) \leq N(\varepsilon/2, \Theta, \|\cdot\|) \leq (6/\varepsilon)^d \text{vol}(\Theta)/\text{vol}(B)$ where $\text{vol}(\Theta), \text{vol}(B)$ are the volumes of $\Theta$ and the unit sphere $B$ in $\mathbb{R}^d$, respectively. As a result, we have:

$$\mathbb{E}^* \left( \sup_{\theta \in \Theta} \sqrt{m} \|G_m^{(1)}(\theta) - G_n(\theta)\| \right) \lesssim d_\theta \left[ \frac{1}{n} \sum_{i=1}^n |F_q(z_i)|^2 \right]^{1/2},$$

and $\left[ \frac{1}{n} \sum_{i=1}^n |F_q(z_i)|^2 \right]^{1/2} \to [\mathbb{E}(|F_q(z_i)|^2)]^{1/2}$ finite. Using the inequality: $\left[ \frac{1}{n} \sum_{i=1}^n |F_q(z_i)|^2 \right]^{1/2} \leq \left[ \frac{1}{n} \sum_{i=1}^n |F_q(z_i)|^p \right]^{1/p}$, we can conclude that:

$$\mathbb{E}^* \left( \sup_{\theta \in \Theta} \sqrt{m} \|G_m^{(1)}(\theta) - G_n(\theta)\|^p \right)^{1/p} \lesssim (1 + \sqrt{d_\theta}) \left[ \frac{1}{n} \sum_{i=1}^n |F_q(z_i)|^p \right]^{1/p},$$

where the last term converges in probability to $[\mathbb{E}(|F_q(z_i)|^p)]^{1/p}$, finite. \(\square\)
F.2 Re-weighted objective with Gaussian multiplier weights.

The following specializes to the case where \( w_i^{(b)} \sim \mathcal{N}(1, 1) \), iid. Notice that:

\[
G_m^{(b)}(\theta) = G_n(\theta) + \frac{1}{m} \sum_{i=1}^{m} (w_i^{(b)} - 1)\nabla q(z_i, \theta) \Rightarrow G_m^{(b)}(\theta) \sim \mathcal{N}
\left(G_n(\theta), \frac{1}{m} V_n(\theta)\right),
\]

where \( V_n(\theta) = 1/n \sum_{i=1}^{n} \nabla q(z_i, \theta)\nabla q(z_i, \theta)' \) and \( w_i^{(b)} - 1 \sim \mathcal{N}(0, 1) \). Note that the proof below only requires Assumption \( F6 \) to hold with \( p = 2 \), even if when the desired result is stated for \( p > 2 \).

**Proof of Lemma \( F3 \)** Condition i. is immediate from the above. For any \( \theta \in \Theta \),

\[
\sqrt{m} \left(G_m^{(b)}(\theta) - G_n(\theta)\right) \sim \mathcal{N}(0, V_n(\theta)),
\]

is a vector-valued Gaussian process, conditional on the sample of data \( z_1, \ldots, z_n \). For each

\[
i = 1, \ldots, n \quad \text{we have:}
\]

\[
||w_i^{(b)} - 1||\nabla q(z_i, \theta)\| \leq |w_i^{(b)} - 1||[G_q(z_i)\text{diam}(\Theta) + ||\nabla q(z_i, \theta')||]| = |w_i^{(b)} - 1|F_q(z_i),
\]

which defines the envelope function for the re-weighted objective. For any \( a \in S_{d_\theta} = \{\ a \in \mathbb{R}^{d_\theta}, \|a\| = 1\} \), the surface of the unit sphere, define \( \sqrt{ma'}(G_m^{(b)}(\theta) - G_n(\theta)) \), a scalar-value Gaussian process defined on \( S_{d_\theta} \times \Theta \) a compact subset of \( \mathbb{R}^{d_\theta} \times \mathbb{R}^{d_\theta} \). Let \( \sigma_n^2 = \sup_{(a,\theta) \in S_{d_\theta} \times \Theta} \mathbb{E}^* \left(m|a' (G_m^{(b)}(\theta) - G_n(\theta))|^2 \right) = \sup_{(a,\theta) \in S_{d_\theta} \times \Theta} a' V_n(\theta)a \) which is finite and converges in probability to \( \sup_{\theta \in \Theta} [\lambda_{\max}(\mathbb{E}[\nabla q(z_i, \theta)\nabla q(z_i, \theta)'])] \), also finite. We can now apply results for scalar valued Gaussian processes. In particular, using Theorem 5.8 in \cite{Boucheron2013}, see also Proposition 3.19 in \cite{Massart2007}, we have for any \( u \geq 0 \):

\[
\mathbb{P}^* \left( \sup_{(a,\theta) \in S_{d_\theta} \times \Theta} \sqrt{m}|a' (G_m^{(b)}(\theta) - G_n(\theta))| \geq M_n + u \right) \leq \exp \left(-\frac{u^2}{2\sigma_n^2} \right),
\]

where \( M_n = \mathbb{E}^* \left[ \sup_{(a,\theta) \in S_{d_\theta} \times \Theta} \sqrt{m}|a' (G_m^{(b)}(\theta) - G_n(\theta))| \right] \).

Let \( Z_n^{(b)} = \sup_{(a,\theta) \in S_{d_\theta} \times \Theta} \sqrt{m}|a' (G_m^{(b)}(\theta) - G_n(\theta))| \). The main idea is to notice that

\[
\sup_{(a,\theta) \in S_{d_\theta} \sqrt{m}|a' (G_m^{(b)}(\theta) - G_n(\theta))| \text{ is atained at } a = [G_m^{(b)}(\theta) - G_n(\theta)]/||G_m^{(b)}(\theta) - G_n(\theta)||
\]

so that \( |Z_n^{(b)}|^p = \sup_{\theta \in \Theta} ||G_m^{(b)}(\theta) - G_n(\theta)||^p \) which is the quantity we want to bound in
expectations. Given that \( Z_m^{(b)} \geq 0 \) we can write its \( p \)-th moment as:

\[
\mathbb{E}^*(|Z_m^{(b)}|^p) = p \int_0^\infty z^{p-1} \mathbb{P}^*(Z_m^{(b)} > z) dz \\
= p \int_0^{M_n} z^{p-1} \mathbb{P}^*(Z_m^{(b)} > z) dz + p \int_0^\infty (M_n + z)^{p-1} \mathbb{P}^*(Z_m^{(b)} > M_n + z) dz \\
\leq M_n^p + p \int_0^\infty (M_n + z)^{p-1} \exp\left(-\frac{z^2}{2\sigma_n^2}\right) dz \\
= M_n^p + p\sigma_n \int_0^\infty (M_n + \sigma_n z)^{p-1} \exp\left(-\frac{z^2}{2}\right) dz.
\]

We already known that \( \sigma_n \) is bounded and converges in probability to a finite limit. It remains to bound the moment \( M_n \) before we can conclude.

Let \( X_m^{(b)}(\theta) = \sqrt{m}(G_m^{(b)}(\theta) - G_n(\theta)) \), by Gaussian re-weighting it is a Gaussian process. We will use an inequality for separable sub-Gaussian processes, specifically Corollary 2.2.8 in \textbf{van der Vaart and Wellner (1996)}, to bound \( M_n \). The following verifies the assumptions required to apply the result. For any \( \theta_1, \theta_2 \in \Theta \), we have:

\[
X_m^{(b)}(\theta_1) - X_m^{(b)}(\theta_2) = \frac{1}{\sqrt{m}} \sum_{i=1}^m (w_i^b - 1)[\partial_q(z_i, \theta_1) - \partial_q(z_i, \theta_2)] \sim \mathcal{N}(0, V_m(\theta_1, \theta_2)),
\]

where \( V_m(\theta_1, \theta_2) = \frac{1}{m} \sum_{i=1}^m [\partial_q(z_i, \theta_1) - \partial_q(z_i, \theta_2)][\partial_q(z_i, \theta_1) - \partial_q(z_i, \theta_2)]' \). For any \( \theta \), let \( X_{m,j}^{(b)}(\theta) \) be the \( j \)-th row of \( X_m^{(b)}(\theta) \). For any \( j \in \{1, \ldots, d_b\} \), we have \( X_{m,j}^{(b)}(\theta_1) - X_{m,j}^{(b)}(\theta_2) \sim \mathcal{N}(0, V_{m,jj}(\theta_1, \theta_2)) \) where \( V_{m,jj}(\theta_1, \theta_2) \) is the \( j \)-th diagonal element of \( V_m(\theta_1, \theta_2) \) which satisfies \( V_{m,jj}(\theta_1, \theta_2) \leq \text{trace}[V_m(\theta_1, \theta_2)] = \frac{1}{m} \sum_{i=1}^m ||\partial_q(z_i, \theta_1) - \partial_q(z_i, \theta_2)||^2 \leq \frac{1}{m} \sum_{i=1}^m F_q(z_i)^2 ||\theta_1 - \theta_2||^2 \), by Lipschitz continuity. This implies the tail inequality:

\[
\mathbb{P}^*(|X_{m,j}^{(b)}(\theta_1) - X_{m,j}^{(b)}(\theta_2)| > x) \leq 2 \exp\left(-\frac{-x^2}{2\frac{1}{m} \sum_{i=1}^m F_q(z_i)^2 ||\theta_1 - \theta_2||^2}\right),
\]

which implies that each \( X_{m,j}^{(b)} \) is a separable sub-Gaussian process under the semi-metric \( d(\theta_1, \theta_2) = [\frac{1}{m} \sum_{i=1}^m F_q(z_i)^2]^{1/2} ||\theta_1 - \theta_2|| \). Using Corollary 2.2.8 in \textbf{van der Vaart and Wellner (1996)}, we have for some universal constant \( K \):

\[
\mathbb{E}^*(\sup_{\theta \in \Theta} |X_{m,j}^{(b)}(\theta)|) \leq \mathbb{E}^*(|X_{m,j}^{(b)}(\theta^1)|) + K \int_0^\infty \sqrt{\log D(\varepsilon, d)} d\varepsilon,
\]

where by Gaussianity of \( X_{m,j}^{(b)} \), we have \( \mathbb{E}^*(|X_{m,j}^{(b)}(\theta^1)|) = 2/\pi V_{n,jj}^{1/2} \), \( V_{n,jj} \) is the \( j \)-th diagonal
element of $V_n$ defined earlier and $D(\varepsilon, d) \leq 3\left[\frac{1}{n} \sum_{i=1}^{n} F_q(z_i)^2\right]^{1/2} \text{diam}(\Theta)/\varepsilon$. Now notice that:

$$M_n = \mathbb{E}^* \left( \sup_{\theta \in \Theta} \|X_m^{(b)}(\theta)\| \right) \leq \sum_{j=1}^{d_\theta} \mathbb{E}^* \left( \sup_{\theta \in \Theta} |X_m^{(b)}(\theta)| \right) \leq d_\theta \left[ \frac{2}{\pi} (\max_j V_{1/2}^{1/2}) + K \int_0^\infty \sqrt{\log D(\varepsilon, d)} d\varepsilon \right].$$

Without loss of generality, assume that $M_n \geq 1$ then we have:

$$\mathbb{E}^*(|Z_m^{(b)}|^p) \leq M_n^p \left[ 1 + p\sigma_n \int_0^\infty (1 + \sigma_n z)^{p-1} \exp \left( -\frac{z^2}{2} \right) dz \right].$$

Putting everything together, we have the following inequality for any $p \geq 2$:

$$\left[ \mathbb{E}^* \left( \sup_{\theta} \|\sqrt{m}(G_m^{(b)}(\theta) - G_n(\theta))\|^p \right) \right]^{1/p} \leq d_\theta \left[ \frac{2}{\pi} (\max_j V_{1/2}^{1/2}) + K \int_0^\infty \sqrt{\log D(\varepsilon, d)} d\varepsilon \right] \left[ 1 + p\sigma_n \int_0^\infty (1 + \sigma_n z)^{p-1} \exp \left( -\frac{z^2}{2} \right) dz \right]^{1/p}.$$

where $\int \sqrt{\log D(\varepsilon, d)} d\varepsilon$ and $\sigma_n$ converge in probability to a finite limit if Assumption $\text{F6}$ holds for $p = 2$ since the quantities involved only depend on second moments.
Appendix G  Primitive Conditions for Assumption \ref{assumption:4}

The derivations below use the same setup and notation as Appendix \[\text{F}\] above.

**Re-sampled objective.** Suppose the gradient has finite third moment, i.e. \( \mathbb{E}(\|\nabla q(z_i, \theta)\|_3) < \infty \). The calculations below are based on the derivations in \cite[Ch6.2, pp147-148]{Lahiri2013}. Note that \( V_n^{-1/2}[H_n(\hat{\theta}_n)]^{-1}G_m^{(b)}(\hat{\theta}_n) \overset{d}{=} \Sigma_n^{-1/2}G_m^{(b)}(\hat{\theta}_n) \) so that we can focus on the latter. Also, note that \( \mathbb{E}^*[\Sigma_n^{-1/2}G_m^{(b)}(\hat{\theta}_n)] = 0 \), \( \text{var}^*[\Sigma_n^{-1/2}G_m^{(b)}(\hat{\theta}_n)] = I_d \). Now compute the characteristic function under \( \mathbb{E}^* \):

\[
\varphi_m^*(\tau) = \mathbb{E}^* \left( \exp[i\tau V_n^{-1/2}\sqrt{m}G_m^{(b)}(\hat{\theta}_n)] \right) = \left[ \mathbb{E}^* \left( \exp \left[ i\tau V_n^{-1/2} \frac{\nabla q(z_i^{(b)}, \hat{\theta}_n)}{\sqrt{m}} \right] \right) \right]^m.
\]

Using a Taylor expansion with remainder, we have for each \( \tau \):

\[
\mathbb{E}^* \left( \exp \left[ i\tau V_n^{-1/2} \frac{\nabla q(z_i^{(b)}, \hat{\theta}_n)}{\sqrt{m}} \right] \right) = 1 - \frac{\|\tau\|_2^2}{2m} + R_m(\tau),
\]

where \( |R_m(\tau)| \leq \mathbb{E}^*(\|\nabla q(z_i^{(b)}, \hat{\theta}_n)\|_3^3)\|\tau\|_2^3 m^{-3/2} \) because all derivatives of the exponential term have modulus less than one. Apply a Taylor expansion with integral remainder to the logarithm:

\[
\log[\varphi_m^*(\tau)] = -\frac{\|\tau\|_2^2}{2} + mR_m(\tau) - m \int_0^{\|\tau\|_2^2/R_m(\tau)} \frac{(-\|\tau\|_2^2/[2m] + R_m(\tau) - t)}{(1 + t)^2} dt,
\]

where \( mR_m(\tau) \leq \mathbb{E}^*(\|\nabla q(z_i^{(b)}, \hat{\theta}_n)\|_3^3)\|\tau\|_2^3 m^{-1/2}; \mathbb{E}^*(\|\nabla q(z_i^{(b)}, \hat{\theta}_n)\|_3^3) \overset{p}{\rightarrow} \mathbb{E}(\|\nabla q(z_i, \theta^*)\|_3^3) < \infty \) using a uniform law of large numbers. For \( m \) sufficiently large, \( | -\frac{\|\tau\|_2^2}{2m} - R_m(\tau) | < 1/2 \) with probability approaching 1, so we can use the following bound:

\[
m \left| \int_0^{\|\tau\|_2^2/R_m(\tau)} \frac{(-\|\tau\|_2^2/[2m] + R_m(\tau) - t)}{(1 + t)^2} dt \right| \leq 6m \left[ \frac{\|\tau\|_2^4}{4m^2} + |R_m(\tau)|^2 \right],
\]

which satisfies the conditions required for Assumption \ref{assumption:4}.

**Re-weighted objective with Gaussian multiplier weights.** Assumption \ref{assumption:4} automatically holds because as noted in Appendix \[\text{F}\] conditionally on the sample: \( G_m^{(b)}(\hat{\theta}_n) \sim \mathcal{N}(0, \Sigma_n) \) which implies \( V_n^{-1/2}[H_n(\hat{\theta}_n)]^{-1}G_m^{(b)}(\hat{\theta}_n) \sim \mathcal{N}(0, I_d) \) for which the characteristic function is the same as in Assumption \ref{assumption:4} with \( r_m(\tau) = 0 \).
Appendix H  Additional Results for Section 5

H.1  Example 1: Dynamic Discrete Choice with Unobserved Heterogeneity

Baseline results with homogeneity. Table H5 provides baseline results for homogeneous dynamic discrete choice model where estimation is much faster so that a comparison with rNR, and the standard bootstrap is feasible. The specification is taken from Abbring and Klein (2020). The first three columns correspond to MLE estimates and rejection rates using the standard $m$ out of $n$ bootstrap. The other 6 columns correspond to rNR and rqN.

Table H5: Baseline estimation and inference results with homogeneous agents

|     | MLE/bootstrap | rNR | rqN |
|-----|--------------|-----|-----|
|     | $\beta_0$  | $\beta_1$ | $\delta_1$ | $\beta_0$  | $\beta_1$ | $\delta_1$ | $\beta_0$  | $\beta_1$ | $\delta_1$ |
|     | Average Estimate |                      |               |               |               |               |               |               |               |
| 1000| -0.500 | 0.200 | 1.000 | -0.500 | 0.200 | 1.000 | -0.500 | 0.200 | 1.000 |
| 500 | -    | -    | -    | -    | -    | -    | -    | -    | -    |
| 100 | -    | -    | -    | -    | -    | -    | -    | -    | -    |
|     | Standard Deviation |               |               |               |               |               |               |               |               |
| 1000| 0.014 | 0.004 | 0.013 | 0.014 | 0.004 | 0.013 | 0.014 | 0.004 | 0.013 |
| 500 | -    | -    | -    | 0.014 | 0.004 | 0.014 | 0.014 | 0.004 | 0.014 |
| 100 | -    | -    | -    | 0.014 | 0.004 | 0.013 | 0.014 | 0.004 | 0.013 |
|     | Rejection Rates |               |               |               |               |               |               |               |               |
| 1000| 0.053 | 0.051 | 0.055 | 0.054 | 0.050 | 0.051 | 0.051 | 0.049 | 0.052 |
| 500 | 0.052 | 0.053 | 0.062 | 0.053 | 0.052 | 0.066 | 0.051 | 0.051 | 0.061 |
| 100 | 0.064 | 0.060 | 0.061 | 0.057 | 0.052 | 0.059 | 0.061 | 0.058 | 0.055 |
|     | Legend: $n = 1000$, $T = 100$, $\gamma = 0.1$, nominal size = 5%. |

Results with $T = 50$, comparison with BFGS
Table H6: Dynamic Discrete Choice Model with Heterogeneity ($T = 50$)

| $\theta^T$ | $\mu_0^1$ | $\mu_0^2$ | $\mu_1^0$ | $\mu_1^1$ | $100\sigma_1$ | $100\sigma_2$ | $\omega$ | $\delta_1$ |
|------------|------------|------------|------------|------------|----------------|----------------|---------|----------|
| -2.000     | 0.300      | -1.000     | 0.900      | 0.010      | 0.010          | 0.300          | 1.000   |

Average Estimates

| rqN       | -2.032     | 0.302      | -0.983     | 0.906      | 1.132          | 0.907          | 0.294   | 0.954    |
| rqN-bc    | -2.014     | 0.303      | -1.000     | 0.894      | 1.140          | 0.793          | 0.294   | 1.000    |
| BFGS      | -1.823     | 0.222      | -0.950     | 0.945      | 4.029          | 3.620          | 0.329   | 0.860    |
| BFGS-bc   | -1.996     | 0.269      | -1.014     | 0.923      | 1.167          | 1.938          | 0.320   | 0.955    |

Standard Deviation

| rqN       | 0.044      | 0.012      | 0.025      | 0.015      | 0.165          | 0.365          | 0.013   | 0.023    |
| rqN-bc    | 0.047      | 0.013      | 0.025      | 0.017      | 0.242          | 0.429          | 0.013   | 0.026    |
| BFGS      | 0.049      | 0.018      | 0.031      | 0.015      | 0.306          | 0.180          | 0.011   | 0.028    |
| BFGS-bc   | 0.056      | 0.021      | 0.032      | 0.016      | 0.368          | 0.258          | 0.013   | 0.031    |

Rejection Rate

| rqN       | 0.035      | 0.035      | 0.068      | 0.035      | 0.033          | 0.393          | 0.018   | 0.255    |
| rqN-bc    | 0.022      | 0.033      | 0.013      | 0.015      | 0.030          | 0.370          | 0.018   | 0.043    |
| rqN-bc$_{se}$ | 0.022    | 0.030      | 0.015      | 0.015      | 0.033          | 0.367          | 0.018   | 0.045    |

Legend: rqN-bc = split panel bias-corrected rqN; rqN, rqN-bc = quantile-based CIs, rqN-bc$_{se}$ = std-error based CIs. $\gamma = 0.1, n = 1000, T = 50, m = n/2, B = 2000, \text{burn} = 250$, nominal level = 5%. BFGS: R’s optim with BFGS optimizer and tight convergence criteria, BFGS-bc: split panel biased corrected.

Implementation details. A grid of 60 sobol points is used to integrate out the random coefficients and solve the fixed point problem. During the burn-in of 250 draws, a smaller grid of 30 points is used. The split panel $\theta_{b,nT/2}^1,\theta_{b,nT/2}^2$ are only computed after $\frac{2}{3}$burn draws and initialized at $\theta_{b,nT}$ to save time. The quasi-Newton matrix $P_b$ is pooled between split panel draws but computed separately for the full panel ones. The regularization schedule is $\lambda = 20$ for the first burn/2 draws and then an exponentially decaying schedule is used: $\lambda_{b+1} = 0.9\lambda_b$.

Derivations for the second-order bias. The following derives the second-order asymptotic bias of the random coefficient dynamic discrete choice model in Section 5.1 under large-$T$ asymptotics and simplifying assumptions. In particular, the transition probability $\Pi$ is assumed to be known and a quadratic assumption is used to shorten integral derivations. Let $Q_{nT}(\theta)$ be the negative sample log-likelihood:

$$Q_{nT}(\theta, \Pi) = -\frac{1}{nT} \sum_{i=1}^{n} \log [\int \exp(T\ell_{iT}(\theta, \Pi, \beta))f(\beta|\theta)d\beta],$$
where $\ell_{it}(\theta, \Pi, \beta) = \sum_{t=1}^{T} \ell_{it}(\theta, \Pi, \beta)/T$ is the log-likelihood for individual $i$ using observations $t = 1, \ldots, T$ for a given set of parameters $(\theta, \Pi, \beta)$, $f(\beta|\theta)$ is the mixture distribution. The function $G_{nT}$ of the objective function $Q_{nT}$ is given by:

$$
G_{nT}(\theta, \Pi) = -\frac{1}{nT} \sum_{i=1}^{n} \frac{\int(T \partial_{\theta} \ell_{iT}(\theta, \Pi, \beta) + \partial_{\beta} \log[f(\beta|\theta)]) \exp(T \ell_{iT}(\theta, \Pi, \beta) + \log[f(\beta|\theta)]) d\beta}{\int \exp(T \ell_{iT}(\theta, \Pi, \beta) + \log[f(\beta|\theta)]) d\beta}.
$$

Under regularity conditions, the estimator $\hat{\theta}_{nT}$ is $\sqrt{n}$-consistent and asymptotically normal, with $T$ fixed with second order representation:

$$
\sqrt{n} \left( \hat{\theta}_{nT} - \theta^\dagger \right) = -[H_{nT}(\theta^\dagger, \Pi^\dagger)]^{-1} G_{nT}(\theta^\dagger, \Pi^\dagger) + \frac{1}{\sqrt{n}} B_{nT}(\theta^\dagger, \Pi^\dagger) + o_p(n^{-1/2}),
$$

and $B_{nT}(\theta^\dagger, \Pi^\dagger)$ converges in probability to some limit which depends on $T$. The main idea for the following is to capture some of the terms in $B_{nT}(\theta^\dagger, \Pi^\dagger)$ using a large-$T$ asymptotic framework where $T$ and $n$ grow at the same rate. The main idea here is to approximate $\sqrt{n}G_{nT}(\theta^\dagger, \Pi^\dagger)$ as a asymptotically normal term which does not involve the log/integral/exponential transformation plus a $1/\sqrt{T} \approx 1/\sqrt{n}$ term which accounts for the transformation. The formula is very similar to the incidental parameter bias but is of second rather than first order. To further simplify the derivations below, the following assumption will be used.

**Assumption H7.** The functions $\ell_{iT}(\theta, \Pi, \beta)$, $\partial_{\theta} \ell_{iT}(\theta, \Pi, \beta)$, $\log[f(\beta|\theta)]$ are quadratic in $\beta$ for each $\theta, \Pi$.

Let $\beta_i$ be the true value of $\beta$ for each individual $i$, using the change of variable $\beta = \beta_i + hT^{-1/2}$, re-write:

$$
G_{nT}(\theta, \Pi) = \frac{1}{n} \sum_{i=1}^{n} \partial_{\theta} \ell_{iT}(\theta, \Pi, \beta_i) \quad (H.8)
$$

$$
- \frac{1}{n\sqrt{T}} \sum_{i=1}^{n} \partial_{\beta} \ell_{iT}(\theta, \Pi, \beta_i) \int \frac{h \exp(T \ell_{iT}(\theta, \Pi, \beta_i + hT^{-1/2}) + \log[f(\beta_i + hT^{-1/2})]) dh}{\int \exp(T \ell_{iT}(\theta, \Pi, \beta_i + hT^{-1/2}) + \log[f(\beta_i + hT^{-1/2})]) dh} \quad (H.9)
$$

$$
- \frac{1}{2nT} \sum_{i=1}^{n} \sum_{j=1}^{d_{\theta}} \partial_{\beta} \ell_{iT}(\theta, \Pi, \beta_i) \int \frac{h \partial_{\beta} \ell_{iT}(\theta, \Pi, \beta_i + hT^{-1/2}) + \log[f(\beta_i + hT^{-1/2})]) dh}{\int \exp(T \ell_{iT}(\theta, \Pi, \beta_i + hT^{-1/2}) + \log[f(\beta_i + hT^{-1/2})]) dh} \quad (H.10)
$$

---

*The estimator is $\sqrt{n}$ not $\sqrt{nT}$ asymptotically normal because of the dependence over $t$ for each $i$ due to $\beta_i$. See [Rilstone et al. (1996)] for general results on second-order bias in nonlinear models.*
The first term, (H.8), is $\sqrt{n}$-asymptotically normal; it corresponds to a likelihood where all the heterogeneity is fully observed. The second term, (H.9), is the linear expansion term of the transformation and is also $\sqrt{n}$-asymptotically normal; i.e. it contributes to the asymptotic variance by taking into account the estimation of the unobserved heterogeneous distribution. The third term, (H.9), is the quadratic expansion term of the transformation which accounts for the non-linear effect of the transformation. It will contribute to the second-order asymptotic bias. Notice that, with the simplifying assumptions, the terms inside the exponential function are quadratic so that the corresponding distribution is Gaussian. To derive the mean and variance-covariance matrix of the Gaussian let $Q_{IT}(\theta, \Pi, \beta_i + hT^{-1/2}) = -\ell_{IT}(\theta, \Pi, \beta_i + hT^{-1/2}) - \frac{1}{2} \log[f(\beta_i + hT^{-1/2})]$ and $G_{IT}(\theta, \Pi, \beta_i), H_{IT}(\theta, \Pi, \beta_i)$ be its gradient and hessian. Given the quadratic assumption, we have:

$$\exp(-TQ_{IT}(\theta, \Pi, \beta_i + hT^{-1/2})) =$$

$$\exp\left(-TQ_{IT}(\theta, \Pi, \beta_i) + \frac{T}{2} G_{IT}(\theta, \Pi, \beta_i) H_{IT}(\theta, \Pi, \beta_i) G_{IT}(\theta, \Pi, \beta_i) \right)$$

$$\times \exp\left(- \frac{1}{2} \left[h + \sqrt{T}H_{IT}(\theta, \Pi, \beta_i)^{-1}G_{IT}(\theta, \Pi, \beta_i)\right]^T H_{IT}(\theta, \Pi, \beta_i) \left[h + \sqrt{T}H_{IT}(\theta, \Pi, \beta_i)^{-1}G_{IT}(\theta, \Pi, \beta_i)\right]\right),$$

which is, up to normalizing constants, equal to the density of a multivariate Gaussian with mean $-\sqrt{T}H_{IT}(\theta, \Pi, \beta_i)^{-1}G_{IT}(\theta, \Pi, \beta_i)$ and variance $H_{IT}(\theta, \Pi, \beta_i)^{-1}$. Notice that this is the asymptotic distribution of $\sqrt{T}(\hat{\beta}_i - \beta_i)$, the individual fixed effect estimator of $\beta$ when the information matrix equality holds. This implies that (H.9) simplifies to:

$$(H.9) = \frac{1}{n} \sum_{i=1}^{n} \partial_{\theta, \beta} \ell_{IT}(\theta, \Pi, \beta_i) H_{IT}(\theta, \Pi, \beta_i)^{-1} G_{IT}(\theta, \Pi, \beta_i).$$

Likewise, (H.10) simplifies to:

$$(H.10) = -\frac{1}{2nT} \sum_{i=1}^{n} \sum_{j=1}^{d_0} \partial_{\theta, \beta} \ell_{IT}(\theta, \Pi, \beta_i) [H_{IT}(\theta, \Pi, \beta_i)^{-1}]_{j},$$

where $[H_{IT}(\theta, \Pi, \beta_i)^{-1}]_{j}$ is the $j$-th column of $H_{IT}(\theta, \Pi, \beta_i)^{-1}$.

**Assumption H8.** Suppose that as both $n, T \to \infty$:

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left[ \partial_{\theta} \ell_{IT}(\theta^t, \Pi^t) + \partial_{\theta} \ell_{IT}(\theta^t, \Pi^t, \beta_i) H_{IT}(\theta^t, \Pi^t, \beta_i)^{-1} \sqrt{T} H_{IT}(\theta^t, \Pi^t, \beta_i) \right] \overset{d}{\to} N(0, \Sigma),$$

and that:

$$\frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{d_0} \partial_{\theta} \ell_{IT}(\theta^t, \Pi^t, \beta_i) [H_{IT}(\theta^t, \Pi^t, \beta_i)^{-1}]_{j} \overset{p}{\to} 2B(\theta^t, \Pi^t),$$
where \( B(\theta^\dagger, \Pi^\dagger) = \mathbb{E}\left( \sum_{j=1}^{d_0} \partial^3_{\theta^\dagger, \beta_j} \ell_{IT}(\theta^\dagger, \Pi^\dagger, \beta_j)[H_{IT}(\theta^\dagger, P_i^\dagger, \beta_j)^{-1}]_j \right) / 2 \).

**Proposition H2** (Asymptotic Bias). Suppose Assumptions \( H7 \) \( \text{H8} \) hold and \( n/T \to \kappa^2 \in (0, +\infty) \), then:

\[
\sqrt{n}G_{nT}(\theta^\dagger, \Pi^\dagger) \xrightarrow{d} \mathcal{N}(0, \Sigma), \tag{H.11} \]
\[
\sqrt{nT}(G_{nT}(\theta^\dagger, \Pi^\dagger) - (\text{H.8}) - (\text{H.9})) \xrightarrow{p} -\kappa B(\theta^\dagger, \Pi^\dagger). \tag{H.12} \]

Suppose \( \hat{\theta}_{nT} \) satisfies the asymptotic expansion:

\[
\sqrt{n}(\hat{\theta}_{nT} - \theta^\dagger) = -\sqrt{n}[H_\infty(\theta^\dagger, \Pi^\dagger)]^{-1}G_{nT}(\theta^\dagger, \Pi^\dagger) + \frac{1}{\sqrt{n}}B_{nT}(\theta^\dagger, \Pi^\dagger) + o_p(n^{-1/2}), \tag{H.13} \]

where \( H_{nT}(\theta^\dagger, \Pi^\dagger) \xrightarrow{p} H_\infty(\theta^\dagger, \Pi^\dagger) \) positive definite. The estimator is asymptotically normal:

\[
\sqrt{n}\left(\hat{\theta}_{nT} - \theta^\dagger\right) \xrightarrow{d} \mathcal{N}(0, V), \tag{H.14} \]

where \( V = [H_\infty(\theta^\dagger, \Pi^\dagger)]^{-1}\Sigma[H_\infty(\theta^\dagger, \Pi^\dagger)]^{-1} \). The second-order bias can be decomposed in two:

\[
\sqrt{nT}\left(\hat{\theta}_{nT} - \theta^\dagger - [H_{nT}(\theta^\dagger, \Pi^\dagger)]^{-1}(\text{H.8}) + (\text{H.9})\right) \xrightarrow{p} \kappa[H_\infty(\theta^\dagger, \Pi^\dagger)]^{-1}B(\theta^\dagger, \Pi^\dagger) + \kappa^{-1}\text{plim}_{n,T \to \infty}B_{nT}(\theta^\dagger, \Pi^\dagger) \]

The asymptotic expansion condition (H.13) can be verified using conditions in Rilstone et al. (1996), the convergence in probability of the Hessian can be derived from a law of large numbers. The positive definiteness is a local identification condition.

**Proof.** Note that \( \sqrt{n}G_{nT}(\theta^\dagger, \Pi^\dagger) = \sqrt{n}[\text{H.8}] + \sqrt{n}[\text{H.9}] + \sqrt{n}[\text{H.10}] \). Assumption \( \text{H8} \) implies that the first term converges in distribution to a Gaussian distribution with zero mean and asymptotic variance-covariance matrix \( \Sigma \). Using the same Assumption, the second term converges in probability to zero. Using the same identity, \( \sqrt{nT}[G_{nT}(\theta^\dagger, \Pi^\dagger) - (\text{H.8}) - (\text{H.9})] = \sqrt{nT}[\text{H.10}] \xrightarrow{p} -\kappa B(\theta^\dagger, \Pi^\dagger) \) which is the first result. Then (H.12) comes from Assumption \( \text{H8} \) Pre-multiplying (H.12) by \( H_{nT}(\theta^\dagger, \Pi^\dagger)^{-1} \xrightarrow{p} H_\infty(\theta^\dagger, \Pi^\dagger)^{-1} \) and using the asymptotic expansion (H.13) then implies the last result. \( \square \)

**Proposition H3** (Split Panel Bias Reduction). Let \( \hat{\theta}_{nT/2}^1, \hat{\theta}_{nT/2}^2 \) be the estimators computed using the first \([T/2]\) and the last \( T - [T/2] \) time-observations, both using all individuals \( i = 1, \ldots, n \). Suppose the associated objectives \( Q_{nT/2}^1, Q_{nT/2}^2 \) satisfy Assumptions \( \text{H7} \) \( \text{H8} \) using \([T/2]\) and \( T - [T/2] \) instead of \( T \). Let \( \hat{\theta}_{nT/2}^{1,2} \) be either estimator, suppose they both satisfy the asymptotic expansions:

\[
\sqrt{n}(\hat{\theta}_{nT/2}^{1,2} - \theta^\dagger) = -\sqrt{nT}[H_\infty^{1,2}(\theta^\dagger, \Pi^\dagger)]^{-1}G_{nT/2}^{1,2}(\theta^\dagger, \Pi^\dagger) + \frac{1}{\sqrt{n}}B_{nT/2}^{1,2}(\theta^\dagger, \Pi^\dagger) + o_p(n^{-1/2}), \tag{H.15} \]
where $H_{nT/2}^{1,2}(\theta^i, \Pi^i) \xrightarrow{p} H_\infty(\theta^i, \Pi^i)$ positive definite. Then the bias corrected estimator $\tilde{\theta}_{nT} = 2\hat{\theta}_{nT} - (\hat{\theta}_{nT/2}^1 + \hat{\theta}_{nT/2}^2)/2$ satisfies:

$$\sqrt{n} \left( \tilde{\theta}_{nT} - \theta^i \right) \xrightarrow{d} \mathcal{N}(0, V), \quad (H.16)$$

$$\sqrt{nT} \left( \tilde{\theta}_{nT} - \theta^i - \left[ H_\infty(\theta^i, \Pi^i) \right]^{-1} \left[ (H.8) + (H.9) \right] \right) \xrightarrow{p} \kappa^{-1} \text{plim}_{n,T \to \infty} B_{nT}(\theta^i, \Pi^i). \quad (H.17)$$

where $V$ is the same as the asymptotic variance of $\hat{\theta}_{nT}$.

The main takeaway of the Proposition is $(H.17)$ which indicates that the part of the second-order bias which is associated with the integral transformed can be removed using the split panel jackknife.

**Proof.** Combining asymptotic expansions $(H.13)$, $(H.15)$, we have:

$$\sqrt{n} \left( \tilde{\theta}_{nT} - \theta^i \right) = -\sqrt{n} H_\infty(\theta^i, \Pi^i)^{-1} \left( 2G_{nT}(\theta^i, \Pi^i) - [G_{nT/2}^1(\theta^i, \Pi^i) + G_{nT/2}^2(\theta^i, \Pi^i)]/2 \right)$$

$$+ \frac{1}{\sqrt{n}} \left( 2B_{nT} - [B_{nT/2}^1 + B_{nT/2}^2]/2 \right).$$

Notice that:

$$2G_{nT}(\theta^i, \Pi^i) - [G_{nT/2}^1(\theta^i, \Pi^i) + G_{nT/2}^2(\theta^i, \Pi^i)]/2 = (H.8) + (H.9) + (H.18),$$

where $(H.18)$ is defined below. The identity comes from additivity over $t$ of $(H.8), (H.9)$. The $(H.18)$ term comes from combining the $(H.10)$ terms in each one of $G_{nT}$, $G_{nT/2}^1$, and $G_{nT/2}^2$:

$$(H.18) = -2 \frac{1}{2nT} \sum_{i=1}^{d_\theta} \sum_{j=1}^{d_\beta} \tilde{\ell}_{iT}(\theta, \Pi, \beta_i) \left[ H_{iT}(\theta, \Pi, \beta_i) \right]^{-1} j$$

$$- \frac{1}{2} \frac{1}{2nT/2} \sum_{i=1}^{d_\theta} \sum_{j=1}^{d_\beta} \tilde{\ell}_{iT/2}^1(\theta, \Pi, \beta_i) \left[ H_{iT/2}^1(\theta, \Pi, \beta_i) \right]^{-1} j$$

$$- \frac{1}{2} \frac{1}{2nT/2} \sum_{i=1}^{d_\theta} \sum_{j=1}^{d_\beta} \tilde{\ell}_{iT/2}^2(\theta, \Pi, \beta_i) \left[ H_{iT/2}^2(\theta, \Pi, \beta_i) \right]^{-1} j.$$
Using the assumptions and $\sqrt{n/[T/2]} \to \sqrt{2}\kappa$, $\sqrt{n}\hat{\theta}_{[H.18]}$ has the following limit:

$$\sqrt{n}\hat{\theta}_{[H.18]} = -2\sqrt{n/[T/2]} - \frac{1}{2n} \sum_{i=1}^{n} \sum_{j=1}^{d_0} \frac{\theta}{\theta_{[H.18]}}$$

$$-\frac{1}{2} \sqrt{\frac{n}{T}} \left[ \frac{1}{2nT/2} \sum_{i=1}^{n} \sum_{j=1}^{d_0} \partial_{\theta_{[H.18]}} \ell_{IT}(\theta, \Pi, \beta_i) \right]_{\theta_{[H.18]} = 0, \beta_i}$$

Putting all the derivations together, we get the two sets of results:

$$\sqrt{n} \left( 2G_{nT}(\theta^\dagger, \Pi^\dagger) - [G_{nT/2}^1(\theta^\dagger, \Pi^\dagger) + G_{nT/2}^2(\theta^\dagger, \Pi^\dagger)]/2 \right) = \sqrt{n}[H_{\infty}(\theta^\dagger, \Pi^\dagger)]^{-1} [H[8] + H[9]] + o_p(1),$$

from which the asymptotic normality result follows. Also,

$$\sqrt{n} \left( 2G_{nT}(\theta^\dagger, \Pi^\dagger) - [G_{nT/2}^1(\theta^\dagger, \Pi^\dagger) + G_{nT/2}^2(\theta^\dagger, \Pi^\dagger)]/2 - [H_{\infty}(\theta^\dagger, \Pi^\dagger)]^{-1} [H[8] + H[9]] \right) = o_p(1) + \kappa^{-1} \lim_{n,T \to \infty} B_{nT}(\theta^\dagger, \Pi^\dagger),$$

where the $o_p(1)$ term includes the bias reduction in $[H.18]$. \hfill \square

**A split panel implementation of rqN.** The split panel bias corrected estimator is $\hat{\theta}_{nT} = 2\hat{\theta}_{nT} - [\hat{\theta}_{nT/2}^1 + \hat{\theta}_{nT/2}^2]/2$. The goal is to target $\hat{\theta}_{nT}$ using rqN. Let $\theta_{b,nT}, \theta_{b,nT/2}^1, \theta_{b,nT/2}^2$ be rqN draws computed using the full and split panels, respectively but the same resampled individuals $i$. Using Proposition 1 we have the three couplings:

$$\theta_{b+1,nT}^* = \hat{\theta}_{nT} + (1 - \gamma) (\theta_{b,nT}^* - \hat{\theta}_{nT}) - \gamma [H_{nT}(\theta^\dagger, \Pi^\dagger)]^{-1} G_{nT}(\theta^\dagger, \Pi^\dagger)$$

$$\theta_{b+1,nT/2}^* = \hat{\theta}_{nT/2}^1 + (1 - \gamma) (\theta_{b,nT/2}^1 - \hat{\theta}_{nT/2}) - \gamma [H_{nT/2}^1(\theta^\dagger, \Pi^\dagger)]^{-1} G_{nT/2}^1(\theta^\dagger, \Pi^\dagger)$$

$$\theta_{b+1,nT/2}^* = \hat{\theta}_{nT/2}^2 + (1 - \gamma) (\theta_{b,nT/2}^2 - \hat{\theta}_{nT/2}) - \gamma [H_{nT/2}^2(\theta^\dagger, \Pi^\dagger)]^{-1} G_{nT/2}^2(\theta^\dagger, \Pi^\dagger).$$

Compute $\hat{\theta}_{b+1,nT} = 2\theta_{b+1,nT} - [\theta_{b+1,nT/2}^1 + \theta_{b+1,nT/2}^2]/2$, we have the associated coupling:

$$\hat{\theta}_{b+1,nT}^* = \hat{\theta}_{nT} + (1 - \gamma) (\theta_{b,nT}^* - \hat{\theta}_{nT}) - \gamma \left( 2[H_{nT}(\theta^\dagger, \Pi^\dagger)]^{-1} G_{nT}(\theta^\dagger, \Pi^\dagger) \right.$$  

$$\left. - \left( [H_{nT/2}^1(\theta^\dagger, \Pi^\dagger)]^{-1} G_{nT/2}^1(\theta^\dagger, \Pi^\dagger) + [H_{nT/2}^2(\theta^\dagger, \Pi^\dagger)]^{-1} G_{nT/2}^2(\theta^\dagger, \Pi^\dagger) \right) / 2 \right),$$

which has innovations that match the first-order expansion for $\hat{\theta}_{nT}$. 19
H.2 Example 2: A Probit with Many Regressors

Both rqN and MCMC are initialized at the same starting value \( \theta_0 \) which is computed as follows. A linear probability model is estimated by OLS yielding coefficients \( \tilde{\theta}_{n,\text{OLS}} \). The starting value \( \theta_0 \) is then computed using the linear approximation:

\[
x_i'\tilde{\theta}_{n,\text{OLS}} \approx \Phi(x_i'\theta_0) = \Phi(0) + \phi(0)x_i'\theta_0,
\]

where \( \phi, \Phi \) are the Gaussian pdf and cdf, \( \Phi(0) = 1/2, \phi(0) = [2\pi]^{-1/2} \). For the intercept, this yields \( \theta_{0,\text{const}} = [\tilde{\theta}_{n,\text{const,OLS}} - \Phi(0)]/\phi(0) \). For all other coefficients this yields \( \theta_{0,j} = \tilde{\theta}_{n,j,\text{OLS}}/\phi(0) \).

Implementation of the MALA algorithm. For Bayesian inference, let \( Q_n \) be the sample negative log-likelihood and \( \pi \) the prior distribution. Let \( G_n \) be the gradient of the negative log-posterior distribution and \( H_n \) its Hessian. Given a draw \( \theta_b \), the baseline MALA algorithm produces the next draw \( \theta_{b+1} \) using the following steps:

\[
\begin{align*}
\text{Draw } \tilde{\theta} &= \theta_b - \gamma G_n(\theta_b) + \sqrt{2\gamma}Z_b, & Z_b &\sim \mathcal{N}(0, I), \text{ and } u \sim U_{[0,1]}, \\
\text{If } u \leq \frac{\exp[-Q_n(\tilde{\theta})] \pi(\tilde{\theta}) q(\tilde{\theta}|\theta_b)}{\exp[-Q_n(\theta_b)] \pi(\theta_b) q(\theta_b|\tilde{\theta})}, & \text{set } \theta_{b+1} = \tilde{\theta}. \text{ Otherwise, set } \theta_{b+1} = \theta_b,
\end{align*}
\]

where \( q(\tilde{\theta}|\theta_b) = \phi([\tilde{\theta} - \theta_b + \gamma G_n(\theta_b)]/\sqrt{2\gamma}) \) and \( q(\theta_b|\tilde{\theta}) = \phi([\theta_b - \tilde{\theta} + \gamma G_n(\tilde{\theta})]/\sqrt{2\gamma}) \) are the transition probabilities, \( \phi \) is the pdf of the multivariate Gaussian. Compared to the random-walk Metropolis-Hastings algorithm which uses the proposal \( \tilde{\theta} = \theta_b + \sqrt{2\gamma}Z_b \), MALA adds the gradient descent term, \( -\gamma G_n(\theta_b) \), which directs the proposal towards the maximum of the posterior distribution to get faster convergence and a higher acceptance rate. Here, the proposal is modified for more direct comparison with rNR, rqN. It requires computing \( H_n(\tilde{\theta}_n) \):

\[
\tilde{\theta} = \theta_b - \gamma [H_n(\tilde{\theta}_n)]^{-1} G_n(\theta_b) + \sqrt{2\gamma}Z_b, \quad Z_b \sim \mathcal{N}(0, [H_n(\tilde{\theta}_n)]^{-1}),
\]

the transition probability \( q(\cdot|\cdot) \) is adjusted accordingly. Following Roberts and Rosenthal (1998), \( \gamma \) is tuned to get an acceptance rate of 0.57, after the initial convergence phase. This requires \( \gamma = O(1/d_0) \) as illustrated below. The desired acceptance rate is achieved for \( \gamma \approx 0.0034 \) to be compared with \( \gamma = 0.1 \) used for rNR and rqN.

Choice of learning rate \( \gamma \) for the MALA algorithm. Recall the modified proposal:

\[
\tilde{\theta} = \theta_b - \gamma [H_n(\tilde{\theta}_n)]^{-1} G_n(\theta_b) + \sqrt{2\gamma}Z_b, \quad Z_b \sim \mathcal{N}(0, [H_n(\tilde{\theta}_n)]^{-1}).
\]

\(^2\)The negative log-likelihood is minus one times the log-likelihood. The log-posterior distribution is the log-likelihood plus the log-prior distribution.
To illustrate how the dimension of the parameters $d_\theta$ affects the choice of $\gamma$, consider the case where $\theta_b = \hat{\theta}_n$ so that $G_n(\theta_b) = 0$. This implies that conditional on $\theta_b = \hat{\theta}_n$, $\tilde{\theta} = \hat{\theta}_n + \sqrt{2\gamma} Z_b \sim \mathcal{N}(\hat{\theta}_n, 2\gamma H_n(\hat{\theta}_n)^{-1})$. The log-posterior is approximately:

$$Q_n(\tilde{\theta}) - \log(p(\tilde{\theta})) \simeq Q_n(\hat{\theta}_n) - \log(p(\hat{\theta}_n)) + \gamma Z_b' H_n(\hat{\theta}_n) Z_b,$$

where $Z_b' H_n(\hat{\theta}_n) Z_b \sim \chi_p^2$. For $d_\theta$ large, we have $Z_b' H_n(\hat{\theta}_n) Z_b/d_\theta = 1 + o_p(1)$. Plug this into the accept/reject step of the algorithm:

$$P\left(u \leq \frac{\exp[-Q_n(\hat{\theta})]p(\hat{\theta})}{\exp[-Q_n(\theta_n)]p(\theta_n)}\right) \simeq P(u \leq \exp[-\gamma d_\theta]) = \exp[-\gamma d_\theta],$$

which equals 0.57 for $\gamma = -\log(0.57)/d_\theta$. At $\theta_b = \hat{\theta}_n$, the optimal choice of $\gamma$ is inversely proportional to $d_\theta$. For $\theta_b \neq \hat{\theta}_n$, the acceptance rate is higher because the gradient term sets $\tilde{\theta}$ in directions of increasing posterior, on average. This implies that a larger value $\gamma > -\log(0.57)/d_\theta$ should be used to get the desired acceptance rate overall. In the empirical application, the learning rate was set to $\gamma = -2\log(0.57)/d_\theta$ which resulted in the desired acceptance rate.

Additional Empirical Results and Comparisons.

| Table H7: Probit - Estimates and Standard Errors |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| const distance border island landlock legal language colonial currency FTA religion | Estimates | Estimates | Estimates | Estimates | Estimates | Estimates | Estimates | Estimates | Estimates | Estimates | Estimates |
| MLE | 2.537 | -0.618 | -0.380 | 0.355 | 0.220 | 0.072 | 0.275 | 0.288 | 0.530 | 1.854 | 0.249 |
| rNR | 2.541 | -0.619 | -0.381 | 0.355 | 0.219 | 0.071 | 0.275 | 0.287 | 0.529 | 1.863 | 0.249 |
| rqN | 2.539 | -0.618 | -0.380 | 0.355 | 0.219 | 0.071 | 0.275 | 0.286 | 0.529 | 1.860 | 0.249 |
| MCMC | 2.533 | -0.617 | -0.380 | 0.355 | 0.215 | 0.072 | 0.274 | 0.289 | 0.534 | 1.854 | 0.250 |
| Standard Errors (iid) | MLE | 0.068 | 0.008 | 0.033 | 0.023 | 0.031 | 0.009 | 0.012 | 0.075 | 0.039 | 0.101 | 0.018 |
| rNR | 0.066 | 0.008 | 0.035 | 0.022 | 0.030 | 0.009 | 0.011 | 0.077 | 0.038 | 0.118 | 0.017 |
| rqN | 0.065 | 0.008 | 0.034 | 0.021 | 0.030 | 0.009 | 0.011 | 0.073 | 0.038 | 0.115 | 0.017 |
| MCMC | 0.040 | 0.004 | 0.020 | 0.015 | 0.020 | 0.006 | 0.008 | 0.060 | 0.028 | 0.069 | 0.012 |
| Standard Errors (clustered) | MLE | 0.094 | 0.015 | 0.039 | 0.031 | 0.030 | 0.012 | 0.016 | 0.073 | 0.068 | 0.103 | 0.029 |
| rNR | 0.087 | 0.015 | 0.038 | 0.032 | 0.030 | 0.011 | 0.014 | 0.073 | 0.069 | 0.102 | 0.028 |
| rqN | 0.105 | 0.015 | 0.040 | 0.034 | 0.036 | 0.011 | 0.016 | 0.105 | 0.071 | 0.134 | 0.030 |

Legend: rNR/rqN computed using $\gamma = 0.1$, Gaussian (iid) and exponential (clustered) weights.

MCMC computed using $\gamma = 0.0034$, Gaussian prior with mean zero, variance 10 for fixed effects, flat uniform prior on other parameters.
Figure H4: Stochastic Gradient Descent

Legend: x-axis = $k$ in millions, $P_b = I_d$, $m = 1$, $\gamma_k = \gamma_0 k^{-5/8}$, 5 million iterations.

Figure H5: Infeasible Stochastic Newton-Raphson

Legend: x-axis = $k$ in thousands, $P_b = [H_n(\hat{\theta}_n) + \gamma_k I_d]^{-1}$, $m = 1$, $\gamma_k = \gamma_0 k^{-5/8}$, 100 thousand iterations.
H.3  Example 3: NLS Estimation of Transportation Costs

Implementation of rNR. The objective function can be written as:

$$Q_n(\theta) = \frac{1}{n} \sum_{o, d, t} [y_{o, d, t} - f(x_{o, d, t}, \theta)]^2,$$

with $\theta = (\alpha, \delta, \beta)$, $\beta$ are the intercept and fixed effect coefficients and $x_{o, d, t}$ consists of $R_t$ and the intercept, fixed effect dummies. $n$ is the total number of observations. The gradient and hessian are:

$$G_n(\theta) = -\frac{2}{n} \sum_{o, d, t} \partial_{\theta} f(x_{o, d, t}, \theta) [y_{o, d, t} - f(x_{o, d, t}, \theta)],$$
$$H_n(\theta) = \frac{2}{n} \sum_{o, d, t} \partial_{\theta} f(x_{o, d, t}, \theta) \partial_{\theta} f(x_{o, d, t}, \theta)' - \frac{2}{n} \sum_{o, d, t} \partial^2_{\theta, \theta'} f(x_{o, d, t}, \theta) [y_{o, d, t} - f(x_{o, d, t}, \theta)].$$

If $\mathbb{E}(y_{o, d, t} - f(x_{o, d, t}, \theta)) | x_{o, d, t}) = 0$, then $\hat{H}_n(\hat{\theta}_n) = \frac{2}{n} \sum_{o, d, t} \partial_{\theta} f(x_{o, d, t}, \hat{\theta}_n) \partial_{\theta} f(x_{o, d, t}, \hat{\theta}_n)'$ is a consistent estimator of the population hessian $H(\theta^\dagger) = 2\mathbb{E} [\partial_{\theta} f(x_{o, d, t}, \theta^\dagger) \partial_{\theta} f(x_{o, d, t}, \theta^\dagger)'].$

Using this estimator, a cost-effective implementation of rNR with $m = n$ and multiplier weights clustered at the district level $d$ relies on:

$$G^{(b)}_m(\theta) = -\frac{2}{n} \sum_{o, d, t} w^{(b)}_d \partial_{\theta} f(x_{o, d, t}, \theta) [y_{o, d, t} - f(x_{o, d, t}, \theta)],$$
$$H^{(b)}_m(\theta) = \frac{2}{n} \sum_{o, d, t} w^{(b)}_d \partial_{\theta} f(x_{o, d, t}, \theta) \partial_{\theta} f(x_{o, d, t}, \theta)',$$

where $w^{(b)}_d \sim \mathcal{N}(1, 1)$ are iid over districts $d$ and iterations $b$. The appeal of this approach is that $H^{(b)}_m(\theta)$ is symmetric positive semidefinite for all $\theta$, even if there are values for which $H_n(\theta)$ is non-definite. The added penalty ensures that $H^{(b)}_m(\theta) + \lambda \partial^2_{\theta, \theta'} \text{pen}(\theta)/n$ is symmetric positive definite. The derivative $\partial_{\theta} f(x_{o, d, t}, \theta)$ is computed once for each iteration $b$, and then used to evaluate both the gradient and hessian. More specifically $\partial_{\alpha} f(x_{o, d, t}, \theta)$ is computed using finite differences and $\partial_{(\delta, \beta)} f(x_{o, d, t}, \theta)$ is simply the vector of linear regressors, i.e. $\log(\text{LCRED}(R_t, \alpha)_{o, d, t})$ and the intercept and fixed effect dummies. The gradient and hessian of the penalty term are then added to the re-weighted gradient and hessian above.
Appendix I  Convergence in Some Non-Convex Settings

I.1 Recovering from a bad start

Lemmas 1 and 2 require the user-chosen learning rate \( \gamma \) to satisfy \( A(\gamma) < 1 \) to get the desired contraction property, which leads to convergence. In practice, feasible values of \( \gamma \), depend on both the choice of \( P_b \) and the hessian \( H_n \). The following discussion will focus on NR in the context of Lemma 1 for simplicity. For values of \( \gamma \) that are too large, \( A(\gamma) < 1 \) may not hold and convergence may fail. The following illustrates how introducing a certain quadratic penalty in the first few iterations can restore the contraction property.

From the starting value \( \theta_0 \), the first NR iteration has the form: 
\[
\theta_1 = \theta_0 - \gamma H_n(\theta_0)^{-1} G_n(\theta_0),
\]
which is only well-defined if \( H_n(\theta_0) \) is non-singular. Using the mean-value theorem: 
\[
\theta_1 - \hat{\theta}_n = [I_d - \gamma H_n(\theta_0)]^{-1} H_n(\theta_0) [\theta_0 - \hat{\theta}_n],
\]
for some intermediate value \( \hat{\theta}_0 \) between \( \theta_0 \) and \( \hat{\theta}_n \). A contraction only occurs if \( \sigma_{\text{max}} \) \( [I_d - \gamma H_n(\theta_0)]^{-1} H_n(\hat{\theta}_0) \) < 1. For \( ||\theta_0 - \hat{\theta}_n|| \) small, continuity of the hessian implies \( [H_n(\theta_0)]^{-1} H_n(\hat{\theta}_0) \) is close to the identity matrix \( I_d \) so that a small \( \gamma < 1 \) always leads to a contraction closer to \( \hat{\theta}_n \).

Issues can arise for distant starting values \( \theta_0 \). Suppose \( H_n(\theta_0) \) is close to singular, then 
\[
[H_n(\theta_0)]^{-1} H_n(\hat{\theta}_0)
\]
can be very large and \( \gamma \) needs to be very small to get a contraction. In that situation, NR iterations are poorly behaved if \( \gamma \) is not sufficiently small. As a solution, add a quadratic penalty: 
\[
Q_n(\theta) + \frac{\lambda}{2n} ||\theta - \theta_0||^2.
\]
Penalizing towards \( \theta_0 \) implies the gradient is unchanged but the hessian becomes \( H_n(\theta_0) + \frac{\lambda}{n} I_d \). The first NR iteration is now: 
\[
\theta_1 = \theta_0 - \gamma [H_n(\theta_0) + \lambda/n I_d]^{-1} G_n(\theta_0),
\]
which is always well-defined as long as \( H_n(\theta_0) \) is positive semi-definite. For the same intermediate value, we now have: 
\[
\theta_1 - \hat{\theta}_n = [I_d - \gamma [H_n(\theta_0) + \lambda/n I_d]^{-1} H_n(\hat{\theta}_0)] (\theta_0 - \hat{\theta}_n),
\]
which even for \( \gamma = 1 \) is a contraction, provided \( \lambda \) is sufficiently large. Note that the update above coincides with an iteration using the trust-region algorithm. The modification is also similar to the \( \tau_b \) used in the quasi-Newton update proposed in the paper. A simple strategy to is then to pick \( \gamma \in (0, 1) \), and add a large penalty \( \lambda \) in the first couple of iterations. Valid inference requires the penalty to be asymptotically negligible, so after the burn-in one can either set \( \lambda = 0 \), use a schedule where \( \lambda_b \to 0 \) sufficiently fast as \( b \) increases, or use a smaller \( \lambda_n = o(1) \). Figure 10 below illustrates the effect of different \( \lambda s \) during the initial iterations for Example 1 in Section 5 where \( H_n(\theta_0) \) is singular. The baseline, \( \lambda = 20 \), is used in the Monte-Carlo simulations, see Appendix H.1 for implementation details.

---

3If \( H_n(\theta_0) \) has negative eigenvalues it is preferable to use the positive semi-definite \( [H_n(\theta_0)'H_n(\theta_0)]^{1/2} \) instead, as illustrated in the next subsection.

4The trust-region method is an alternative to line-search which can improve on standard NR iterations, see Nocedal and Wright (2006, Ch4.1) and Bertsekas (2016, Ch1.4.2).
I.2 Suboptimal solutions

An important concern is when a classical optimizer converges to a point that is suboptimal, i.e. is neither a local nor a global optimum. In these cases, the user has to interve and restart an optimization at another value which can be a time-consuming process. The following illustrates, using a stylized example, the effect of the resampling noise in this scenario.

Consider a point $\theta_x$ for which the Hessian is indefinite, i.e. $H_n(\theta_x)$ has both strictly positive and negative eigenvalues. For simplicity, the following assumes that the objective is quadratic: $Q_n(\theta) = \frac{1}{2}(\theta - \theta_x)'H_n(\theta - \theta_x)$, with eigendecomposition $H_n = Q\Lambda Q'$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_d)$ where $0 < \lambda_S \leq \lambda_{S-1} \leq \cdots \leq \lambda_1$ and $0 > \lambda_{S+1} \geq \cdots \geq \lambda_{d_{\theta}}$. There are $S$ directions with positive curvature, and $d_{\theta} - S$ directions with negative curvature. Let $q_1, \ldots, q_{d_{\theta}}$ be the corresponding eigenvectors. Without constraints, the global minimum is $Q_n(\theta) = -\infty$ attained at $\theta = +\infty \times q_s$, $s \geq S + 1$, the optimizer should diverge: $\|\theta_b\| \to \infty$.

Issues with classical Newton-Raphson: Consider the Newton-Raphson iteration $\theta_{b+1} = \theta_b - \gamma H_n^{-1}G_n(\theta_b)$, here $G_n(\theta_b) = H_n(\theta_b - \theta_x)$ so that: $\theta_{b+1} - \theta_x = (1 - \gamma)(\theta_b - \theta_x)$, NR converges exponentially fast to the suboptimal solution $\theta_x$. It is possible to improve the behaviour of NR by using a Hessian modification: $\theta_{b+1} = \theta_b - \gamma (H_n^2)^{-1/2}G_n(\theta_b)$ which enforces $(H_n^2)^{-1/2} = Q(\Lambda^2)^{-1/2}Q'$ positive definite. NR iterations are now such that:

$$\theta_{b+1} - \theta_x = (I + Q\Gamma Q')(\theta_b - \theta_x), \quad \Gamma = \text{diag}(-\gamma, \ldots, -\gamma, +\gamma, \ldots, +\gamma).$$

If $\theta_b - \theta_x$ equals zero or is orthogonal to $q_{S+1}, \ldots, q_{d_{\theta}}$, then NR will converge to $\theta_x$ exponentially fast. However, NR will diverge exponentially fast if $\theta_b - \theta_x = q_{S+1}$, for instance, because of the explosive root $1 + \gamma > 1$ which is outside the unit circle.

5 See Section 3.4 Nocedal and Wright (2006) for more details on this issue.
The case of resampled Newton-Raphson: The main difference between NR and rNR in this example is that rNR diverges exponentially fast with high-probability even when \( \theta_0 = \theta_x \), which is always problematic for classical optimizers such as GD, NR, or BFGS. To simplify notation, assume that \( Q = I \) and \( G^{(b)}(\theta) \sim \mathcal{N}(G_n(\theta), \Sigma/m) \), \( \Sigma \) non-singular, and \( H^{(b)}_m = H_n \). For rNR: \( \theta_{b+1} = \theta_b - \gamma([H^{(b+1)}_m]^2)^{-1/2}G^{(b+1)}_m(\theta_b) \), which implies:

\[
\theta_{b+1} - \theta_x = (I + \Gamma)(\theta_b - \theta_x) + \Gamma Z^{(b+1)}_m,
\]

where \( \Gamma \) is the same as above. Notice that \( (\theta_{b+1} - \theta_x) \) follows a VAR(1) process with transition matrix \( I + \Gamma \) which has \( d_\theta - S \) explosive roots outside the unit circle, all equal to \( 1 + \gamma \).

Suppose \( \theta_0 = \theta_x \) which is problematic for NR. Let \( \theta_{j,b+1} \) be the s-th row of \( \theta_{b+1} \) and \( Z^{(b-j)}_{m,j} \) the j-th row of \( Z^{(b-j)}_m \), denote its variance by \( \sigma^2_j > 0 \). For \( j \geq S + 1 \) we have the AR(1) representation

\[
\theta_{j,b+1} - \theta_{j,x} = \gamma(1 + \gamma)^b \sum_{j=0}^{b} (1 + \gamma)^{-j} Z^{(j)}_{m,j} \sim \mathcal{N}\left(0, \gamma^2(1 + \gamma)^{-2b - 2} \sigma^2_j \right).
\]

Using Gaussianity and the Paley-Zygmund inequality we have the probability bound:

\[
P^\star\left(\frac{\left|\theta_{j,b+1} - \theta_{j,x}\right|}{\gamma(1 + \gamma)^b \sqrt{2/\pi} \sigma_j \left[1 - (1 + \gamma)^{-2b - 2}\right]^{1/2}} \geq \varepsilon\right) \geq \frac{2(1 - \varepsilon)^2}{\pi},
\]

for any \( \varepsilon \in [0, 1] \). This implies that, with high probability, \( \theta_{b+1} \) diverges exponentially fast.

Numerical Illustration: To illustrate the above numerically, consider a two-dimensional optimization problem where \( \theta_x = (0, 0) \). We will set \( \theta_0 = q_1 + cq_2 \) and vary \( c \) to compare the performance of R’s BFGS optimizer, which relies on a line-search, with NR and rNR which use a fixed learning rate \( \gamma = 0.1 \) and the hessian modification. For \( c = 0 \), NR will converge to the saddle point and \( c \neq 0 \) implies that NR will eventually diverge. Table \[8\] reports the results for a range of \( c \). Note that BFGS converges to \( \theta_x \) even when \( c \neq 0 \). This is because, the BFGS update does not guarantee positive definiteness of the conditioning matrix so that it behaves similarly to NR without the hessian modification. NR diverges for \( c \neq 0 \) but slowly for \( c < 1 \). As predicted, rNR diverges quickly for all \( c \).
Table I8: Behaviour of BFGS, NR, and rNR around a saddle point

| \( c \) | \( \theta_1 \) | \( \theta_2 \) | \( Q_n(\theta) \) | \( \theta_1 \) | \( \theta_2 \) | \( Q_n(\theta) \) | \( \theta_1 \) | \( \theta_2 \) | \( Q_n(\theta) \) |
|-------|-----|-----|-------|-----|-----|-------|-----|-----|-------|
| 0.0   | 0.00 | 0.00 | 0.00  | 0.01 | -0.01 | 0.00  | -0.01 | -90.42 | -91.35 |
| 0.1   | 0.05 | 0.05 | -0.00 | 4.78 | 5.30  | -0.25 | 5.30  | -85.65 | -81.95 |
| 0.5   | 0.23 | 0.25 | -0.00 | 23.88| 26.53 | -6.37 | 26.53 | -66.54 | -49.47 |
| 1.0   | 0.45 | 0.50 | -0.00 | 47.76| 53.06 | -25.48| 53.06 | -42.66 | -20.33 |
| 5.0   | 2.26 | 2.51 | -0.06 | 238.80| 265.32| -637.09| 265.32| 148.37 | -245.96 |

Legend: BFGS: output of R’s optimizer (optim). NR and rNR: output after 50 iterations.
True minimum is \( Q_n = -\infty \). Saddle point is \( \theta_x = (0, 0) \).