A weak dichotomy below $E_1 \times E_3$

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Abstract

We prove that if $E$ is an equivalence relation Borel reducible to $E_1 \times E_3$ then either $E$ is Borel reducible to the equality of countable sets of reals or $E_1$ is Borel reducible to $E$. The “either” case admits further strengthening.

Let $\mathbb{R} = 2^\mathbb{N}$. Recall that $E_1$ and $E_3$ are the equivalence relations defined on the set $\mathbb{R}^\mathbb{N}$ as follows:

- $x E_1 y$ iff $\exists k_0 \forall k \geq k_0 \ (x(k) = y(k))$
- $x E_3 y$ iff $\forall k \ (x(k) E_0 y(k))$

where $E_0$ is an equivalence relation defined on $\mathbb{R}$ so that

- $a E_0 b$ iff $\exists n_0 \forall n \geq n_0 \ (a(n) = b(n))$

The equivalence $E_3$ is often denoted as $(E_0)^\omega$.

Kechris and Louveau in [9] and Kechris and Hjorth in [3, 4] proved that any Borel equivalence relation $E$ satisfying $E <_B E_1$, resp., $E <_B E_3$, also satisfies the non-strict $E \leq_B E_0$. Here $<_B$ and $\leq_B$ are resp. strict and non-strict relations of Borel reducibility. Thus if $E$ is an equivalence relation on a Borel set $X$ and $F$ is an equivalence relation on a Borel set $Y$ then $E \leq_B F$ means that there exists a Borel map $\vartheta : X \to Y$ such that

$x E x' \iff \vartheta(x) F \vartheta(x')$

holds for all $x, x' \in X$. Such a map $\vartheta$ is called a (Borel) reduction of $E$ to $F$. If both $E \leq_B F$ and $F \leq_B E$ then they write $E \sim_B F$ (Borel bi-reducibility), while $E <_B F$ (strict reducibility) means that $E \leq_B F$ but not $F \leq_B E$. See the cited papers [3, 4] or e.g. [2, 8] on various aspects of Borel reducibility in set theory and mathematics in general.

The abovementioned results give a complete description of the $\leq_B$-structure of Borel equivalence relations below $E_1$ and below $E_3$. It is then a natural step
to investigate the \( \leq_B \)-structure below \( E_{13} \), where \( E_{13} = E_1 \times E_3 \) is the product of \( E_1 \) and \( E_3 \), that is, an equivalence on \( \mathbb{R}^N \times \mathbb{R}^N \) defined so that for any points \( \langle x, \xi \rangle \) and \( \langle y, \eta \rangle \) in \( \mathbb{R}^N \times \mathbb{R}^N \), \( \langle x, \xi \rangle E_{13} \langle y, \eta \rangle \) if and only if \( x E_1 y \) and \( \xi E_3 \eta \).

The intended result would be that the \( \leq_B \)-cone below \( E_{13} \) includes the cones determined separately by \( E_1 \) and \( E_3 \), together with the disjoint union of \( E_1 \) and \( E_3 \) (i.e., the union of \( E_1 \) and \( E_3 \) defined on two disjoint copies of \( \mathbb{R}^N \)), \( E_{13} \) itself, and nothing else. This is however a long shot. The following theorem, the main result of this note, can be considered as a small step in this direction.

**Theorem 1.** Suppose that \( E \) is a Borel equivalence relation and \( E \leq_B E_{13} \). Then either \( E \) is Borel reducible to \( T_2 \) or \( E_1 \leq_B E \).

Recall that the equivalence relation \( T_2 \), known as “the equality of countable sets of reals”, is defined on \( \mathbb{R}^N \) so that \( x T_2 y \) if \( \{ x(n) : n \in \mathbb{N} \} = \{ y(n) : n \in \mathbb{N} \} \). It is known that \( E_3 <_B T_2 \) strictly, and there exist many Borel equivalence relations \( E \) satisfying \( E <_B T_2 \) but incomparable with \( E_3 \) : for instance non-hyperfinite Borel countable ones like \( E_\infty \). The two cases are incompatible because \( E_1 \) is known not to be Borel reducible to orbit equivalence relations of Polish actions (to which class \( T_2 \) belongs).

A rather elementary argument reduces Theorem 1 to the following:

**Theorem 2.** Suppose that \( P_0 \subseteq \mathbb{R}^N \times \mathbb{R}^N \) is a Borel set. Then either the equivalence \( E_{13} \upharpoonright P_0 \) is Borel reducible to \( T_2 \) or \( E_1 \leq_B E_{13} \upharpoonright P_0 \).

Indeed suppose that \( Z \) (a Borel set) is the domain of \( E \), and \( \vartheta : Z \to \mathbb{R}^N \times \mathbb{R}^N \) is a Borel reduction of \( E \) to \( E_{13} \). Let \( f : Z \to 2^\mathbb{N} = \mathbb{R} \) be an arbitrary Borel injection. Define another reduction \( \vartheta' : Z \to \mathbb{R}^N \times \mathbb{R}^N \) as follows. Suppose that \( z \in Z \) and \( \vartheta(z) = \langle x, \xi \rangle \in \mathbb{R}^N \times \mathbb{R}^N \). Put \( \vartheta'(z) = \langle x', \xi \rangle \), where \( x' \), still a point in \( \mathbb{R}^N \), is related to \( x \) so that \( x'(n) = x(n) \) for all \( n \geq 1 \) but \( x'(0) = f(z) \). Then obviously \( \vartheta(z) \) and \( \vartheta'(z) \) are \( E_{13} \)-equivalent for all \( z \in Z \), and hence \( \vartheta' \) is still a Borel reduction of \( E \) to \( E_{13} \). On the other hand, \( \vartheta' \) is an injection (because so is \( f \)). It follows that its full image \( P_0 = \text{ran} \vartheta' = \{ \vartheta'(z) : z \in Z \} \) is a Borel set in \( \mathbb{R}^N \times \mathbb{R}^N \), and \( E \sim_B E_{13} \upharpoonright P_0 \).

The remainder of the paper contains the proof of Theorem 2. The partition in two cases is described in Section 2. Naturally assuming that \( P_0 \) is a lightface \( \Delta^1_1 \) set, Case 1 is essentially the case when for every element \( \langle x, \xi \rangle \in P_0 \) (note that \( x, \xi \) are points in \( \mathbb{R}^N \)) and every \( n \) we have \( x(n) = F(x|_{>n}, \xi|_{<k}, \xi|_{>k}) \) for some \( k \), where \( F \) is a \( \Delta^1_1 \) function \( E_3 \)-invariant w.r.t. the 3rd argument. It easily follows that then the first projection of the equivalence class \( \langle (x, \xi) \rangle_{E_{13} \cap P_0} \) of every point \( \langle x, \xi \rangle \in P_0 \) is at most countable, leading to the either option of Theorem 2 in Section 4.

The results of theorems 1 and 2 in their either parts can hardly be viewed as satisfactory because one would expect it in the form: \( E \) is Borel reducible to \( E_3 \). Thus it is a challenging problem to replace \( T_2 \) by \( E_3 \) in the theorems. Attempts
to improve the either option, so far rather unsuccessful, lead us to the following theorem established in sections 5 and 6:

**Theorem 3.** In the either case of Theorem 2 there exist a hyperfinite equivalence relation $G$ on a Borel set $P_0'' \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that $E_{13} \upharpoonright P_0$ is Borel reducible to the conjunction of $G$ and the equivalence relation $E_3$ acting on the 2nd factor of $\mathbb{R}^n \times \mathbb{R}^n$. ²

The equivalence $G$ as in the theorem will be induced by a countable group $G$ of homeomorphisms of $\mathbb{R}^n \times \mathbb{R}^n$ preserving the second component. (That is, if $g \in G$ and $g(x, \xi) = (y, \eta)$ then $\eta = \xi$, but $y$ generally speaking depends on both $x$ and $\xi$.) And $G$ happens to be even a *hyperfinite* group in the sense that it is equal to the union of an increasing chain of its finite sub groups. Recall that $E_3$ is induced by the product group $H = (\mathcal{P}_{\mathbb{R}_1}(N); \Delta^N_1)$ naturally acting in this case on the second factor in the product $\mathbb{R}^N \times \mathbb{R}^N$. And there are further details here that will be presented in sections 5 and 6.

Case 2 is treated in Sections 7 through 12. The embedding of $E_1$ in $E_{13} \upharpoonright P_0$ is obtained by approximately the same splitting construction as the one introduced in [9] (in the version closer to [7]).

1 Preliminaries: extension of “invariant” functions

If $E$ is an equivalence relation on a set $X$ then, as usual, $[x]_E = \{y \in X : y E x\}$ is the $E$-class of an element $x \in X$, and $[Y]_E = \bigcup_{x \in Y} [x]_E$ is the $E$-saturation of a set $Y \subseteq X$. A set $Y \subseteq X$ is $E$-invariant if $Y = [Y]_E$.

The following “invariant” Separation theorem will be used below.

**Proposition 4** (5.1 in [1]). Assume that $E$ is a $\Delta^1_1$ equivalence relation on a $\Delta^1_1$ set $X \subseteq \mathbb{N}^N$. If $A, C \subseteq X$ are $\Sigma^1_1$ sets and $[A]_E \cap [C]_E = \emptyset$ then there exists an $E$-invariant $\Delta^1_1$ set $B \subseteq X$ such that $[A]_E \subseteq B$ and $[C]_E \cap B = \emptyset$. \hfill $\square$

Suppose that $f$ is a map defined on a set $Y \subseteq X$. Say that $f$ is $E$-invariant if $f(x) = f(y)$ for all $x, y \in Y$ satisfying $x E y$.

**Corollary 5.** Assume that $E$ is a $\Delta^1_1$ equivalence relation on a $\Delta^1_1$ set $A \subseteq \mathbb{N}^N$, and $f : B \to \mathbb{N}^N$ is an $E$-invariant $\Sigma^1_1$ function defined on a $\Sigma^1_1$ set $B \subseteq A$. Then there exist an $E$-invariant $\Delta^1_1$ function $g : A \to \mathbb{N}^N$ such that $f \subseteq g$.

**Proof.** It obviously suffices to define such a function on an $E$-invariant $\Delta^1_1$ set $Z$ such that $Y \subseteq Z \subseteq A$. (Indeed then define $g$ to be just a constant on $A \setminus Z$.)

The set

$$P = \{ \langle a, x \rangle \in A \times \mathbb{N}^N : \forall b ( (b \in B \wedge a E b) \implies x = f(b) ) \}$$

² The conjunction as indicated is equal to the least equivalence relation $F$ on $P_0''$ which includes $G$ and satisfies $\xi E_3 \eta \implies \langle x, \xi \rangle F \langle y, \eta \rangle$ for all $\langle x, \xi \rangle$ and $\langle y, \eta \rangle$ in $P_0''$.
is $\Pi^1_1$ and $f \subseteq P$. Moreover $P$ is F-invariant, where F is defined on $A \times \mathbb{N}^\mathbb{N}$ so that $\langle a,x \rangle F \langle a',y \rangle$ iff $a \equiv a'$ and $x = y$. Obviously $[f]_F \subseteq P$. Hence by Proposition 4 there exists an F-invariant $\Delta^1_1$ set $Q$ such that $f \subseteq Q \subseteq P$. The set

$$R = \{ \langle a,x \rangle \in Q : \forall y (y \neq x \implies \langle a,y \rangle \notin Q) \}$$

is an F-invariant $\Pi^1_1$ set, and in fact a function, satisfying $f \subseteq R$. Applying Proposition 4 once again we end the proof.

\[\square\]

2 An important population of $\Sigma^1_1$ functions

Working with elements and subsets of $\mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}$ as the domain of the equivalence relation $E_{13}$, we’ll typically use letters $x, y, z$ to denote points of the first copy of $\mathbb{R}^\mathbb{N}$ (where $E_1$ lives) and letters $\xi, \eta, \zeta$ to denote points of the second copy of $\mathbb{R}^\mathbb{N}$ (where $E_3$ lives). Recall that, for $P \subseteq \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}$,

$$\text{dom } P = \{ x : \exists \xi (\langle x, \xi \rangle \in P) \} \quad \text{and} \quad \text{ran } P = \{ \xi : \exists x (\langle x, \xi \rangle \in P) \}. $$

Points of $\mathbb{R} = 2^\mathbb{N}$ will be denoted by $a, b, c$.

Assume that $x \in \mathbb{R}^\mathbb{N}$. Let $x|_{>n}$, resp., $x|_{\geq n}$ denote the restriction of $x$ (as a map $\mathbb{N} \to \mathbb{R}$) to the domain $(n, \infty)$, resp., $[n, \infty)$. Thus $x|_{>n} \in \mathbb{R}^{>n}$, where $>n$ means the interval $(n, \infty)$, and $x|_{\geq n} \in \mathbb{R}^{\geq n}$, where $\geq n$ means $[n, \infty)$. If $X \subseteq \mathbb{R}^\mathbb{N}$ then put $X|_{>n} = \{ x|_{>n} : x \in X \}$ and $X|_{\geq n} = \{ x|_{\geq n} : x \in X \}$.

The notation connected with $|_{<n}$ and $|_{\leq n}$ is understood similarly.

Let $\xi \equiv_k \eta$ mean that $\xi E_{13} \eta$ and $\xi|_{<k} = \eta|_{<k}$ (that is, $\xi(j) = \eta(j)$ for all $j < k$). This is a Borel equivalence on $\mathbb{R}^\mathbb{N}$. A set $U \subseteq \mathbb{R}^\mathbb{N}$ is $\equiv_k$-invariant if $U = [U]|_{\equiv_k}$, where $[U]|_{\equiv_k} = \bigcup_{\xi \in U} [\xi]_{\equiv_k}$.

**Definition 6.** Let $\mathcal{F}^k_n$ denote the set of all $\Sigma^1_1$ functions $\varphi : U \to \mathbb{R}$, defined on a $\Sigma^1_1$ set $U = \text{dom } \varphi \subseteq \mathbb{R}^{>n} \times \mathbb{R}^\mathbb{N}$, and $\equiv_k$-invariant in the sense that if $\langle y, \xi \rangle$ and $\langle y, \eta \rangle$ belong to $U$ and $\xi \equiv_k \eta$ then $\varphi(y, \xi) = \varphi(y, \eta)$.

Let $\mathcal{F}_n^k$ denote the set of all total functions in $\mathcal{F}^k_n$, that is, those defined on the whole set $\mathbb{R}^{>n} \times \mathbb{R}^\mathbb{N}$.

**Lemma 7.** If $\varphi \in \mathcal{F}_n^k$ then there is a $\Delta^1_1$ function $\psi \in \mathcal{F}_n^k$ with $\varphi \subseteq \psi$.

**Proof.** Apply Corollary 5. \[\square\]

**Definition 8.** Let us fix a suitable coding system $\{ W^e \}_{e \in E}$ of all $\Delta^1_1$ sets $W \subseteq \mathbb{R} \times \mathbb{R}^\mathbb{N} \times \mathbb{R}$ (in particular for partial $\Delta^1_1$ functions $\mathbb{R} \times \mathbb{R}^\mathbb{N} \to \mathbb{R}$), where $E \subseteq \mathbb{N}$ is a $\Pi^1_1$ set, such that there exist a $\Sigma^1_1$ relation $\Sigma$ and a $\Pi^1_1$ relation $\Pi$ satisfying

$$\langle b, \xi, a \rangle \in W^e \iff \Sigma(e, b, a, \xi) \iff \Pi(e, b, a, \xi)$$

(1)

\[\square\]
whenever \( e \in E \) and \( a, b \in \mathbb{R}, \xi \in \mathbb{R}^\mathbb{N} \).

Let us fix a \( \Delta^1_1 \) sequence of homeomorphisms \( H_n : \mathbb{R} \onto \mathbb{R}^\mathbb{N} \). Put

\[
\begin{align*}
W^e_n &= \{ \langle H_n(b), \xi, a \rangle : \langle b, \xi, a \rangle \in W^e \} \quad \text{for } e \in E \\
T &= \{ \langle e, k \rangle : e \in E \lor W^e \text{ is a total and } \equiv_k \text{-invariant function} \}
\end{align*}
\]

Here the totality means that \( \text{dom} W^e = \mathbb{R} \times \mathbb{R}^\mathbb{N} \) while the invariance means that \( W^e(b, \xi) = W^e(b, \eta) \) for all \( b, \xi, \eta \) satisfying \( \xi \equiv_k \eta \).

Note that if \( \langle e, k \rangle \in T \) then, for any \( n \), \( W^e_n \) is a function in \( \mathcal{T} \mathcal{F}_k^n \), and conversely, every function in \( \mathcal{T} \mathcal{F}_k^n \) has the form \( W^e_n \) for a suitable \( e \in E \).

**Proposition 9.** \( T \) is a \( \Pi^1_1 \) set.

**Proof.** Standard evaluation based on the coding of \( \Delta^1_1 \) sets.

**Corollary 10.** The sets

\[
S^k_n = \{ \langle x, \xi \rangle \in \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N} : \exists \varphi \in \mathcal{T} \mathcal{F}_k^n (x(n) = \varphi(x\upharpoonright_n, \xi)) \}
\]

belong to \( \Pi^1_1 \) uniformly on \( n, k \). Therefore the set \( S = \bigcup_m \bigcap_{n \geq m} \bigcup_k S^k_n \) also belongs to \( \Pi^1_1 \).

**Proof.** The equality of the two definitions follows from Lemma 7. The definability follows from Proposition 9 by standard evaluation.

Beginning the proof of Theorem 2, we can w.l.o.g. assume, as usual, that the Borel set \( P_0 \) in the theorem is a lightface \( \Delta^1_1 \) set.

**Case 1:** \( P_0 \subseteq S \). We’ll show that in this case \( E_{13} \upharpoonright P_0 \) is Borel reducible to \( T_2 \).

**Case 2:** \( P_0 \setminus S \neq \emptyset \). We’ll prove that then \( E_1 \leq_{B} E_{13} \upharpoonright P_0 \).

### 3 Case 1: simplification

From now on and until the end of Section 4 we work under the assumptions of Case 1. The general strategy is to prove that for any \( \langle x, \xi \rangle \in P_0 \) there exist at most countably many points \( y \in \mathbb{R}^\mathbb{N} \) such that, for some \( \eta, \langle y, \eta \rangle \in P_0 \) and \( \langle x, \xi \rangle \in E_{13} \langle y, \eta \rangle \), and that those points can be arranged in countable sequences in a certain controlled way.

Our first goal is to somewhat simplify the picture.

**Lemma 11.** There exists a \( \Delta^1_1 \) map \( \mu : P_0 \to \mathbb{N} \) such that for any \( \langle x, \xi \rangle \in P_0 \) we have \( \langle x, \xi \rangle \in \bigcap_{n \geq \mu(x, \xi)} \bigcup_k S^k_n \).
Proof. Apply Kreisel Selection to the set

\[ \{(x, \xi, m) \in P_0 \times \mathbb{N} : \forall n \geq m \exists k ((x, \xi) \in S^k_n)\}. \]

\[ \square \]

Let \( 0 = 0^\mathbb{N} \in \mathbb{R} = 2^\mathbb{N} \) be the constant 0 : \( 0(k) = 0, \forall k. \) For any \( \langle x, \xi \rangle \in P_0 \) put \( f_\mu(x, \xi) = 0^\mu(x, \xi)^\wedge (x \vDash \mu(x, \xi)) : \) that is, we replace by 0 all values \( x(n) \) with \( n < \mu(x, \xi). \) Then \( P'_0 = \{(f_\mu(x, \xi), \xi) : (x, \xi) \in P_0\} \) is a \( \Sigma_1^1 \) set.

Put \( S' = \bigcap_k S_n^k \) (a \( \Pi_1^1 \) set by Corollary 10).

Corollary 12. There is a \( \Delta_1^1 \) set \( P''_0 \) such that \( P'_0 \subseteq P''_0 \subseteq S'. \) The map \( \langle x, \xi \rangle \mapsto f_\mu(x, \xi, \xi) \) is a reduction of \( E_{13} \upharpoonright P_0 \) to \( E_{13} \upharpoonright P''_0. \)

Proof. Obviously \( P'_0 \) is a subset of the \( \Pi_1^1 \) set \( S'. \) It follows that there is a \( \Delta_1^1 \) set \( P''_0 \) such that \( P'_0 \subseteq P''_0 \subseteq S'. \) To prove the second claim note that \( f_\mu(x, \xi)E_1x \) for all \( \langle x, \xi \rangle \in P_0. \) \[ \square \]

Let us fix a \( \Delta_1^1 \) set \( P''_0 \) as indicated. By Corollary 12 to accomplish Case 1 it suffices to get a Borel reduction of \( E_{13} \upharpoonright P''_0 \) to \( T_2. \)

Lemma 13. There exist: a \( \Delta_1^1 \) sequence \( \{\kappa_n\}_{n \in \mathbb{N}} \) of natural numbers, and a \( \Delta_1^1 \) system \( \{F_n^k\}_{i,n \in \mathbb{N}} \) of functions \( F_n^k \in \mathcal{P}^{\kappa_i} \), such that for all \( \langle x, \xi \rangle \in P''_0 \) and \( n \in \mathbb{N} \) there is \( i \in \mathbb{N} \) satisfying \( x(n) = F_n^k(x|\_\rangle_{>n}, \xi). \)

Remark 14. Recall that by definition every function \( F \in \mathcal{P}^k \) is invariant in the sense that if \( \langle x, \xi \rangle \) and \( \langle x, \eta \rangle \) belong to \( R \times R, \xi \langle_k \eta = \xi \langle_k \), and \( \xi \in \mathcal{E}_2 \) then \( \varphi(x, \xi) = \varphi(x, \eta). \) This allows us to sometimes use the notation like \( F_n^k(x|\_\rangle_{>n}, \xi|\_\rangle_{<k}, \xi|\_\rangle_{\geq k}) \), where \( k = \kappa_i \), instead of \( F_n^k(x|\_\rangle_{>n}, \xi) \), with the understanding that \( F_n^k(x|\_\rangle_{>n}, \xi|\_\rangle_{<k}, \xi|\_\rangle_{\geq k}) \) is \( \mathcal{E}_3 \)-invariant in the 3rd argument.

In these terms, the final equality of the lemma can be re-written as \( x(n) = F_n^k(x|\_\rangle_{>n}, \xi|\_\rangle_{<k}, \xi|\_\rangle_{\geq k}) \), where \( k = \kappa_i. \) \[ \square \]

Proof (lemma). By definition \( P''_0 \subseteq S' \) means that for any \( \langle x, \xi \rangle \in P''_0 \) and \( n \) there exists \( k \) such that \( \langle x, \xi \rangle \in S^k_n \). The formula \( \langle x, \xi \rangle \in S^k_n \) takes the form

\[ \exists \varphi \in \mathcal{P}^k_n (x(n) = \varphi(x|\_\rangle_{>n}, \xi)), \]

and further the form \( \exists (e, k) \in T (x(n) = W^e_n(x|\_\rangle_{>n}, \xi)). \) It follows that the \( \Pi_1^1 \) set

\[ Z = \{\langle (x, \xi, n), (e, k) \rangle \in (P_0 \times \mathbb{N}) \times T : x(n) = W^e_n(x|\_\rangle_{>n}, \xi)\} \]

satisfies \( \text{dom} Z = P_0 \times \mathbb{N}. \) Therefore by Kreisel Selection there is a \( \Delta_1^1 \) map \( \varepsilon : P_0 \times \mathbb{N} \to T \) such that \( x(n) = W^e_n(x|\_\rangle_{>n}, \xi) \) holds for any \( \langle x, \xi \rangle \in P_0 \) and \( n, \) where \( \langle e, k \rangle = \varepsilon(x(\xi, n)) \) for some \( k. \)

The range \( R = \text{ran} \varepsilon \) of this function is a \( \Sigma_1^1 \) subset of the \( \Pi_1^1 \) set \( T. \) We conclude that there is a \( \Delta_1^1 \) set \( B \) such that \( R \subseteq B \subseteq T. \) And since \( T \subseteq \mathbb{N} \times \mathbb{N}, \) it follows, by some known theorems of effective descriptive set theory, that the
set \( \hat{E} = \text{dom} B = \{ e : \exists k ( \langle e, k \rangle \in B ) \} \) is \( \Delta^1_1 \), and in addition there exists a \( \Delta^1_1 \) map \( K : \hat{E} \to \mathbb{N} \) such that \( \langle e, K(e) \rangle \in B \) (and \( \in T \)) for all \( e \in \hat{E} \).

And on the other hand it follows from the construction that

\[
\forall \langle x, \xi \rangle \in P_0 \forall n \exists e \in \hat{E} (x(n) = W^*_{\eta}(x|_{>n}, \xi)).
\]  

(3)

Let us fix any \( \Delta^1_1 \) enumeration \( \{ e(i) \}_{i \in \mathbb{N}} \) of elements of \( \hat{E} \). Put \( F_n^i = W^e_{\xi}(i) \). Then the last conclusion of the lemma follows from (3). Note that the functions \( F_n^i \) are uniformly \( \Delta^1_1 \), \( F_n^i \in \mathcal{P}_n^k \) for some \( k \), in particular, for \( k = \kappa_i \), where \( \kappa_i = K(e(i)) \), and \( \{ \kappa_i \}_{i \in \mathbb{N}} \) is a \( \Delta^1_1 \) sequence as well.

**Blanket Agreement 15.** Below, we assume that the set \( P_0'' \) is chosen as above, that is, \( \Delta^1_1 \) and \( P_0'' \subseteq S' \), while a system of functions \( F_n^i \) and a sequence \( \{ \kappa_i \}_{i \in \mathbb{N}} \) of natural numbers are chosen accordingly to Lemma 13.

**4 Case 1: countability of projections of equivalence classes**

We prove here that in the assumption of Case 1 the equivalence \( E_{13} \upharpoonright P_0'' \) is Borel reducible to \( T_2 \), the equality of countable sets of reals. The main ingredient of this result will be the countability of the sets

\[
C^*_x = \text{dom}(\{ \langle x, \xi \rangle | E_{13} \cap P_0'' \}) = \{ y \in \mathbb{R}^\mathbb{N} : y E_1 x \land \exists \eta (\xi E_3 \eta \land \langle y, \eta \rangle \in P_0'' ) \},
\]

where \( \langle x, \xi \rangle \in P_0'' \) — projections of \( E_{13} \)-classes of elements of the set \( P_0'' \).

**Lemma 16.** If \( \langle x, \xi \rangle \in P_0'' \) then \( C^*_x \subseteq [x]_{E_1} \) and \( C^*_x \) is at most countable.

**Proof.** That \( C^*_x \subseteq [x]_{E_1} \) is obvious. The proof of countability begins with several definitions. In fact we are going to organize elements of any set of the form \( C^*_x \) in a countable sequence.

Recall that \( \mathbb{R} = 2^{\mathbb{N}} \). If \( u \subseteq \mathbb{N} \) and \( b \in \mathbb{R} \) then define \( u \cdot a \in \mathbb{R} \) so that \( (u \cdot a)(j) = a(j) \) whenever \( j \notin u \), and \( (u \cdot a)(j) = 1 - a(j) \) otherwise.

If \( f \subseteq \mathbb{N} \times \mathbb{N} \) and \( a \in \mathbb{R}^k \) then define \( f \cdot a \in \mathbb{R}^k \) so that \( (f \cdot a)(j) = (f^j \cdot a)(j) \) for all \( j < k \), where \( f^j = \{ m : \langle j, m \rangle \in f \} \). Note that \( f \cdot a \) depends in this case only on the restricted set \( f | k = \{ \langle j, m \rangle \in f : j < k \} \).

Put \( \Phi = \mathcal{P}_{\text{fin}}(\mathbb{N} \times \mathbb{N}) \) and \( D = \bigcup_n D_n \), where for every \( n \):

\[
D_n = \{ (a, \varphi) : a \in \mathbb{N}^n \land \varphi \in \Phi^n \land \forall j < n (\varphi(j) \subseteq \kappa_{a(j)} \times \mathbb{N}) \}. \tag{4}
\]

(The inclusion \( \varphi(j) \subseteq \kappa_{a(j)} \times \mathbb{N} \) here means that the set \( \varphi(j) \subseteq \mathbb{N} \times \mathbb{N} \) satisfies \( \varphi(j) = \varphi(j) | \kappa_{a(j)} \), that is, every pair \( \langle k, l \rangle \in \varphi(j) \) satisfies \( k < \kappa_{a(j)} \).)

If \( \langle a, \varphi \rangle \in D_n \) and \( \langle x, \xi \rangle \in \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N} \) then we define \( y = \tau^*_x(a, \varphi) \in \mathbb{R}^\mathbb{N} \) as follows: \( y = \langle b_0, b_1, \ldots, b_{n-1} \rangle^{\langle x \rangle}_{\geq n} \), where the reals \( b_m \in \mathbb{R} \) (\( m < n \)) are defined by inverse so that

\[
b_m = F_m^{a(m)}(\langle b_{m+1}, b_{m+2}, \ldots, b_{n-1} \rangle^{\langle x \rangle}_{\geq n}, \varphi(m) \cdot (\xi |_{<\kappa_{a(m)}}), \xi |_{\geq \kappa_{a(m)}}).
\]  

(4)
Proof. The “if” direction is rather easy. If \( \xi \) holds if and only if \( \eta \) holds, then \( \xi = \xi \). Thus \( \xi \xi \), the trace of \( \xi \xi \), is a countable sequence, that is, a function defined on \( D = \bigcup_n D_n \), a countable set, and the set \( \text{ran} \xi \xi \) of all terms of this sequence is at most countable and satisfies \( x = \xi \xi (\Lambda, \Lambda) \in \text{ran} \xi \xi \subseteq |x|_{E_1} \).

**Claim 17.** Suppose that \( \langle x, \xi \rangle \in P''_0 \). Then \( C^E_0 \subseteq \text{ran} \xi \xi \) and hence \( C^E_0 \) is at most countable. More exactly if \( y \in C^E_0 \) and \( y|_{\geq n} = x|_{\geq n} \) then there is a pair \( \langle a, \varphi \rangle \in D_n \) such that \( y = \xi \xi (a, \varphi) \).

We prove the second, more exact part of the claim. By definition there is \( \eta \in \mathbb{R}^N \) such that \( \langle y, \eta \rangle \in P''_0 \) and \( \xi \in E_3 \eta \). Put \( b_m = y(m), \forall m \). Note that for every \( m < n \) there is a number \( a(m) \) such that

\[
b_m = F_m^a(m)(\langle b_{m+1}, \ldots, b_{n-1} \rangle \wedge (y|_{\geq n}), \eta) = F_m^a(m)(\langle b_{m+1}, \ldots, b_{n-1} \rangle \wedge (y|_{\geq n}), \eta|_{\prec \kappa_a(m)}, \eta|_{\prec \kappa_a(m)})
\]

for all \( m < n \) (see Blanket Agreement 15), and hence

\[
b_m = F_m^a(m)(\langle b_{m+1}, \ldots, b_{n-1} \rangle \wedge (x|_{\geq n}), \eta|_{\prec \kappa_a(m)}, \xi|_{\geq \kappa_a(m)})
\]

by the invariance of functions \( F_m^i \) and because \( x|_{\geq n} = y|_{\geq n} \). On the other hand, it follows from the assumption \( \xi \in E_3 \eta \) that for every \( m < n \) there is a finite set \( \varphi(m) \subseteq \kappa_a(m) \times \mathbb{N} \) such that \( \eta|_{\prec \kappa_a(m)} = \varphi(m) \cdot (\xi|_{\prec \kappa_a(m)}) \). Then

\[
b_m = F_m^a(m)(\langle b_{m+1}, \ldots, b_{n-1} \rangle \wedge (x|_{\geq n}), \varphi(m) \cdot (\xi|_{\prec \kappa_a(m)}), \xi|_{\geq \kappa_a(m)})
\]

for every \( m < n \), that is, \( y = \xi \xi (a, \varphi) \), as required. \( \square \) (Claim and Lemma 16)

The next result reduces the equivalence relation \( E_{13} \parallel P''_0 \) to the equality of sets of the form \( \text{ran} \xi \xi \), that is essentially to the equivalence relation \( T_2 \) of "equality of countable sets of reals".

**Corollary 18.** Suppose that \( \langle x, \xi \rangle \) and \( \langle y, \eta \rangle \) belong to \( P''_0 \). Then \( \langle x, \xi \rangle E_{13} \langle y, \eta \rangle \) holds if and only if \( \xi \in E_3 \eta \) and \( \text{ran} \xi \xi = \text{ran} \eta \).

**Proof.** The "if" direction is rather easy. If \( \xi \in E_3 \eta \) and \( \text{ran} \xi \xi = \text{ran} \eta \), then \( x \in E_1 y \) because \( \text{ran} \xi \xi \subseteq [y]_{E_1} \) and \( \text{ran} \xi \xi \subseteq [x]_{E_1} \) by Lemma 16.

To prove the converse suppose that \( \langle x, \xi \rangle E_{13} \langle y, \eta \rangle \). Then \( \xi \in E_3 \eta \), of course. Furthermore, \( x \in E_1 y \), therefore \( x|_{\geq n} = y|_{\geq n} \) for an appropriate \( n \). Let us prove
that $\text{ran } \tau^\xi_y = \text{ran } \tau^\xi_{z}$. First of all, by definition we have $y \in C^\xi_z$, and hence (see the proof of Claim 17) there exists a pair $\langle a, \varphi \rangle \in D_n$ such that $y = \tau^\xi_z(a, \varphi)$.

Now, let us establish $\text{ran } \tau^\xi_z = \text{ran } \tau^\xi_y$ (with one and the same $\xi$). Suppose that $z \in \text{ran } \tau^\xi_z$, that is, $z = \tau^\xi_z(b, \psi)$ for a pair $\langle b, \psi \rangle \in D_m$ for some $m$. If $m \geq n$ then obviously $z = \tau^\xi_z(b, \psi) = \tau^\xi_y(b, \psi)$, and hence (as $\langle x \rangle \geq \langle y \rangle \geq \langle m \rangle$) $z \in \text{ran } \tau^\xi_y$. If $m < n$ then $z = \tau^\xi_z(b, \psi) = \tau^\xi_y(a', \varphi')$, where $a' = b \wedge \langle a \rangle \geq \langle m \rangle$ and $\varphi' = \psi \wedge \langle \varphi \rangle \geq \langle m \rangle$, and once again $z \in \text{ran } \tau^\xi_y$. Thus $\text{ran } \tau^\xi_z \subseteq \text{ran } \tau^\xi_y$. The proof of the inverse inclusion $\text{ran } \tau^\xi_y \subseteq \text{ran } \tau^\xi_z$ is similar.

Thus $\text{ran } \tau^\xi_y = \text{ran } \tau^\xi_z$. It remains to prove $\text{ran } \tau^\xi_y = \text{ran } \tau^\xi_y$ for all $y, \xi, \eta$ such that $\xi E_3 \eta$. Here we need another block of definitions.

Let $\mathcal{H}$ be the set of all sets $\delta \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta''j = \{m : \langle j, m \rangle \in \delta\}$ is finite for all $j \in \mathbb{N}$. For instance if $\xi, \eta \in \mathbb{R}^\mathbb{N}$ satisfy $\xi E_3 \eta$ then the set

$$\delta_{\xi \eta} = \{\langle j, m \rangle : \xi(j)(m) \neq \eta(j)(m)\}$$

belongs to $\mathcal{H}$. The operation of symmetric difference $\Delta$ converts $\mathcal{H}$ into a Polish group equal to the product group $\langle \mathcal{B}_{21n}(\mathbb{N}) ; \Delta \rangle^\mathbb{N}$.

If $n \in \mathbb{N}$, $\langle a, \varphi \rangle \in D_n$, and $\delta \in \mathcal{H}$ then we define a sequence $\varphi' = H^\delta_{\varphi}(\varphi) \in \Phi^n$ so that $\varphi'(m) = (\delta \upharpoonright \kappa_m) \Delta \varphi(m)$ for every $m < n$. Then the pair $\langle a, H^\delta_{\varphi}(\varphi) \rangle$ obviously still belongs to $D_n$ and $H^\delta_{\varphi}(H^\delta_{\varphi}(\varphi)) = \varphi$.

Coming back to a triple of $y, \xi, \eta \in \mathbb{R}^\mathbb{N}$ such that $\xi E_3 \eta$, let $\delta = \delta_{\xi \eta}$.

A routine verification shows that $\tau^\xi_y(a, \varphi) = \tau^\xi_z(a, H^\delta_{\varphi}(\varphi))$ for all $\langle a, \varphi \rangle \in D$. It follows that $\text{ran } \tau^\xi_y = \text{ran } \tau^\xi_z$, as required. \hfill $\Box$

**Corollary 19.** The restricted relation $E_{13} \mid P'_0$ is Borel reducible to $T_2$.

**Proof.** Since all $\tau^\xi_z$ are countable sequences of reals, the equality $\text{ran } \tau^\xi_y = \text{ran } \tau^\xi_z$ of Corollary 18 is Borel reducible to $T_2$. Thus $E_{13} \mid P'_0$ is Borel reducible to $E_3 \times T_2$ by Corollary 18. However it is known that $E_3$ is Borel reducible to $T_2$, and so does $T_2 \times T_2$. \hfill $\Box$

$\Box$ (Case 1 of Theorem 2)

## 5 Case 1: a more elementary (?) transformation group

Here we begin the proof of Theorem 3. Our plan is to define a countable group $G$ of homeomorphisms of $\mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}$ such that the induced equivalence relation $G$ satisfies Theorem 3. We continue to argue under the assumptions of Case 1.

First of all let us define the basic domain of transformations,

$$\Pi = \{\langle x, \xi \rangle \in \mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N} : \forall n \exists \langle a, \varphi \rangle \in D_n (x = \tau^\xi_z(a, \varphi))\}.$$ 

This is a closed subset of $\mathbb{R}^\mathbb{N} \times \mathbb{R}^\mathbb{N}$. Applying Claim 17 with $y = x$ we obtain

\footnote{Recall that $\delta \upharpoonright k = \{\langle j, i \rangle : j < k\}$.}
Corollary 20. $P''_0 \subseteq \Pi$. \hfill \Box

Suppose that pairs $\langle a, \varphi \rangle$ and $\langle b, \psi \rangle$ belong to $D_n$ for one and the same $n$, and $\langle x, \xi \rangle \in \mathbb{R}^N \times \mathbb{R}^N$. We define $G_{a,\varphi}^{b,\psi}(x, \xi) = \langle y, \xi \rangle \in \mathbb{R}^N \times \mathbb{R}^N$ so that

$$y = \begin{cases} \tau^\xi_x(b, \psi) & \text{whenever } x = \tau^\xi_x(a, \varphi) \\ \tau^\xi_x(a, \varphi) & \text{whenever } x = \tau^\xi_x(b, \psi) \\ x & \text{whenever } \tau^\xi_x(a, \varphi) \neq x \neq \tau^\xi_x(b, \psi) \end{cases}$$

Note that if $\tau^\xi_x(a, \varphi) = x = \tau^\xi_x(b, \psi)$ then still $y = x$ by either of the two first cases of the definition. And in any case $y|_{\geq n} = x|_{\geq n}$ provided $\langle a, \varphi \rangle \in D_n$.

Lemma 21. Suppose that $n \in \mathbb{N}$ and pairs $\langle a, \varphi \rangle$, $\langle b, \psi \rangle$ belong to $D_n$. Then $G_{a,\varphi}^{b,\psi}$ is a homeomorphism of $\mathbb{R}^N \times \mathbb{R}^N$ onto itself, and $G_{a,\varphi}^{b,\psi} = G_{b,\psi}^{a,\varphi}$.

In addition, $G_{a,\varphi}^{b,\psi}$ is a homeomorphism of $\Pi$ onto itself.

Proof. Suppose that $\langle x, \xi \rangle$ belongs to $\Pi$ and prove that so does $\langle y, \xi \rangle = G_{a,\varphi}^{b,\psi}(x, \xi)$. By definition $y$ coincides with one of $x, \tau^\xi_x(a, \varphi), \tau^\xi_x(b, \psi)$. So assume that $y = \tau^\xi_x(b, \psi)$. Consider any $m$, we have to show that $y = \tau^\xi_y(b, \psi)$ for some $\langle a', \varphi' \rangle \subseteq D_m$. If $m \leq n$ then the pair of $a' = b \upharpoonright m$ and $\varphi' = \psi \upharpoonright m$ obviously works. If $m > n$ then take the pair of $a' = b^\upharpoonright m \upharpoonright n$ and $\varphi' = \psi^\upharpoonright n$ where $\langle b', \psi' \rangle \subseteq D_m$ is an arbitrary pair satisfying $x = \tau^\xi_x(b', \psi')$. \hfill \Box

Lemma 22. Suppose that $\langle x, \xi \rangle \in \Pi$. Then:

(i) if $\langle a, \varphi \rangle$, $\langle b, \psi \rangle \in D_n$ and $\langle y, \xi \rangle = G_{a,\varphi}^{b,\psi}(x, \xi)$ then $\text{ran } \tau^\xi_{\langle a, \varphi \rangle} \subseteq \text{ran } \tau^\xi_{\langle b, \psi \rangle}$;

(ii) if $y \in \text{ran } \tau^\xi_{\langle b, \psi \rangle}$ then there exist $n$ and pairs $\langle a, \varphi \rangle$, $\langle b, \psi \rangle \in D_n$ such that $\langle y, \xi \rangle = G_{a,\varphi}^{b,\psi}(x, \xi)$.

Proof. (i) Consider an arbitrary $z = \tau^\xi_x(a', \varphi') \in \text{ran } \tau^\xi_x$, where $\langle a', \varphi' \rangle \subseteq D_m$. Once again $y$ coincides with one of $x, \tau^\xi_x(a, \varphi), \tau^\xi_x(b, \psi)$, so assume that $y = \tau^\xi_x(b, \psi)$. If $m \geq n$ then obviously $z = \tau^\xi_y(a', \varphi') \in \text{ran } \tau^\xi_y$. If $m < n$ then we have $z = \tau^\xi_y(b', \psi')$, where $b' = a' \upharpoonright m$ and $\psi' = \varphi' \upharpoonright n$. (ii) If $y \in \text{ran } \tau^\xi_x$ then by definition there is a pair $\langle b, \psi \rangle$ in some $D_n$ such that $y = \tau^\xi_x(b, \psi)$. Then by the way $x|_{\geq n} = y|_{\geq n}$. As $\langle x, \xi \rangle \in \Pi$, there is a pair $\langle a, \varphi \rangle \in D_n$ such that $x = \tau^\xi_x(a, \varphi)$. Then $\langle y, \xi \rangle = G_{a,\varphi}^{b,\psi}(x, \xi)$. \hfill \Box

Let $G$ denote the group of all finite superpositions of maps of the form $G_{a,\varphi}^{b,\psi}$, where $\langle a, \varphi \rangle$, $\langle b, \psi \rangle$ belong to one and the same set $D_n$ as in the lemma. Thus $G$ is a countable group of homeomorphisms of $\mathbb{R}^N \times \mathbb{R}^N$. (We’ll prove that $G$ is even an increasing union of its finite subgroups!) Note that a superposition of the form $G_{a,\varphi}^{b,\psi} \circ G_{a',\varphi'}^{b',\psi'}$ does not necessarily coincide with $G_{a,\varphi}^{b,\psi}$. 

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We are going to prove that the equivalence relation $\mathcal{G}$ induced by $\mathcal{G}$ on $\mathcal{H}$ satisfies Theorem 3. To be more exact, $\mathcal{G}$ is defined on $\mathcal{H}$ so that $\langle x, \xi \rangle \mathcal{G} \langle y, \eta \rangle$ if there exists a homeomorphism $g \in \mathcal{G}$ such that $g(x, \xi) = \langle y, \eta \rangle$. Note that then by definition $\eta = \xi$.

The hyperfiniteness $\mathcal{G}$ will be established in the next Section. Now let us study relations between $\mathcal{G}$ and $\mathcal{H}$, the other involved group introduced in the proof of Corollary 18. For any $\delta \in \mathcal{H}$ define a homeomorphism $H_\delta$ of $\mathbb{R}^N \times \mathbb{R}^N$ so that $H_\delta(x, \xi) = \langle x, \eta \rangle$, where simply $\eta = \delta \Delta \xi$ in the sense that

$$\eta(m, j) = \begin{cases} \xi(m, j) & \text{whenever } \langle m, j \rangle \notin \delta \\ 1 - \xi(m, j) & \text{whenever } \langle m, j \rangle \in \delta \end{cases}$$

(Then obviously $\delta = \delta_{\xi, \eta}$.) If $\gamma, \delta \in \mathcal{H}$ then the superposition $H_\delta \circ H_\gamma$ coincides with $H_{\gamma \Delta \delta}$, where $\Delta$ is the symmetric difference, as usual.

Transformations of the form $G_{a,\varphi}$ do not commute with those of the form $H_\delta$, yet there exists a convenient law of commutation:

**Lemma 23.** Suppose that $n \in \mathbb{N}$ and pairs $\langle a, \varphi \rangle$ and $\langle b, \psi \rangle$ belong to $D_n$, and $\delta \in \mathcal{H}$. Then the superposition $G_{a,\varphi} \circ H_\delta$ coincides with $H_\delta \circ G_{b,\psi}$, where $\varphi' = H_\delta^g(\varphi)$ and $\psi' = H_\delta^b(\psi)$.

**Proof.** A routine argument is left for the reader. □

Let us consider the group $\mathcal{S}$ of all homeomorphisms $s : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ of the form

$$s = H_\delta \circ g_{t-1} \circ g_{t-2} \cdots \circ g_1 \circ g_0,$$

where $t \in \mathbb{N}$, $\delta \in \mathcal{H}$, and each $g_t$ is a homeomorphism of $\mathbb{R}^N \times \mathbb{R}^N$ of the form $G_{a_t,\varphi_t}$, where the pairs $\langle a_t, \varphi_t \rangle$, $\langle b_t, \psi_t \rangle$ belong to one and the same set $D_n$, $n = n_t$. (It follows that $g_{t-1} \circ g_{t-2} \cdots \circ g_1 \circ g_0 \in \mathcal{G}$.)

Lemma 23 implies that $\mathcal{S}$ is really a group under the operation of superposition. For instance if $g = G_{a,\varphi}$ and $g_1$ belong to $\mathcal{G}$ (and $\langle a, \varphi \rangle$, $\langle b, \psi \rangle$ belong to one and the same $D_n$) then the superposition $H_\delta \circ g \circ H_{\delta_1} \circ g_1$ coincides with $H_\delta \circ H_{\delta_1} \circ g' \circ g_1 = H_{\Delta \delta_1} \circ (g' \circ g_1)$, where $g' = G_{a,\varphi'}$ and $\varphi' = H_{\delta_1}(\varphi)$, $\psi' = H_{\delta_1}(\psi)$ as in Lemma 23.

Thus $\mathcal{S}$ seems to be a more complicated group than the direct cartesian product of $\mathcal{G}$ and $\mathcal{H}$, but on the other hand more elementary than the free product (of all formal superpositions of elements of both groups). A natural action of $\mathcal{S}$ on $\mathbb{R}^N \times \mathbb{R}^N$ is defined as follows: if $s$ is as in (5) then $s \cdot \langle x, \xi \rangle = H_\delta(g_{t-1}(g_{t-2}(\cdots g_1(g_0(x, \xi)) \ldots )))$. Let $\mathcal{S}$ denote the induced orbit equivalence relation. One can easily check that both the group $\mathcal{S}$ and the action are Polish. On the other hand, $\mathcal{S}$ is obviously the conjunction of $\mathcal{G}$ and the equivalence relation $E_3$ acting on the 2nd factor of $\mathbb{R}^N \times \mathbb{R}^N$, in the sense of Theorem 3 and footnote 2 on page 3. Thus the next lemma, together with the result of Lemma 25 on the hyperfiniteness of $\mathcal{G}$, accomplish the proof of Theorem 3.
Lemma 24. Suppose that $\langle x, \xi \rangle, \langle y, \eta \rangle \in P'_0$. Then $\langle x, \xi \rangle E_{13} \langle y, \eta \rangle$ if and only if $\langle x, \xi \rangle S \langle y, \eta \rangle$.

Proof. Suppose that $\langle x, \xi \rangle E_{13} \langle y, \eta \rangle$. Then $y \in \text{ran} \tau^\xi_x$ by Corollary 18, and further $\langle x, \xi \rangle S \langle y, \eta \rangle$ by Lemma 22(ii). It remains to note that $\langle y, \xi \rangle S \langle y, \eta \rangle$ by obvious reasons.

Now suppose that $\langle x, \xi \rangle S \langle y, \eta \rangle$. Then $\xi E_3 \eta$, and hence by Corollary 19 it suffices to prove that $\text{ran} \tau^\xi_x = \text{ran} \tau^\eta_y$. This follows from two observations saying that transformations in $H$ and in $G$ preserve $\text{ran} \tau^\xi_x$. First, if $\langle x, \xi \rangle \in \mathbb{R}^N \times \mathbb{R}^N$, $\delta \in H$, and $\langle y, \xi \rangle = H_3(x, \xi)$ then $\tau^\delta_y$ obviously is a permutation of $\tau^\eta_y$, and hence $\text{ran} \tau^\xi_x = \text{ran} \tau^\eta_y$. Second, if $\langle x, \xi \rangle \in \mathbb{R}^N \times \mathbb{R}^N$, pairs $\langle a, \varphi \rangle, \langle b, \psi \rangle$ belong to one and the same set $D_n$, and $\langle y, \xi \rangle = G_{a, \varphi}(x, \xi)$, then $\text{ran} \tau^\xi_x = \text{ran} \tau^\xi_x$ by Lemma 22.

$\square$ (Theorem 3 modulo Lemma 25)

6 Case 1: the “hyperfiniteness” of the countable group $G$

Lemma 24 reduces further study of Case 1 of Theorem 2 to properties of the group $S$ and its Polish actions. This is an open topic, and maybe the next result, the “hyperfiniteness” of $G$, one of the two components of $S$, can lead to a more comprehensive study. One might think that $G$ is a rather complicated countable group, perhaps close to the free group on two (or countably many) generators. The reality is different:

Lemma 25. $G$ is the union of an increasing sequence of finite subgroups, therefore the induced equivalence relation $G$ is hyperfinite.

Proof. Let us show that a finite set of “generators” $G_{a, \varphi}$ produces only finitely many superpositions — this obviously implies the lemma. Suppose that $m \in \mathbb{N}$, and $\langle a_i, \varphi_i \rangle \in D_n(i)$ for all $i < m$. Put $G_{ij} = G_{a_i, \varphi_i}$ provided $n(i) = n(j)$, and let $G_{ij}$ be the identity otherwise. Thus all $G_{ij}$ are homeomorphisms of $\Pi$. We are going to prove that the set of all superpositions of the form $f_0 \circ f_1 \circ \cdots \circ f_\ell$, where $\ell$ is an arbitrary natural number and each of $f_k$ is equal to one of $G_{ij}$ ($i,j$ depend on $k$) contains only finitely many really different functions.

Note that if $i,j < m$ and $n(i) < n(j)$ then the pair

$\langle a_i, b_{n(i)}, \varphi_i, b_{n(i)} \rangle$

belongs to $D_{n(j)}$. We can w.l.o.g. assume that every such a pair occurs in the list of pairs $\langle a_i, \varphi_i \rangle, i < m$.

Let us associate a pair $q(x, \xi) = \langle u_x, \xi, w_x \rangle$ of finite sets

$u_x = \{ i < m : \tau^\xi_x(a_i, \varphi_i) = x \}$, and

$w_x = \{ (i,j) : i,j < m \land \tau^\xi_x(a_i, \varphi_i) = \tau^\xi_x(a_j, \varphi_j) \}$
with every point \( \langle x, \xi \rangle \in \Pi \). Put \( Q = \mathcal{P}(m) \times \mathcal{P}(m \times m) \), a (finite) set including all possible values of \( q(\pi) \).

**Claim 26.** For every \( q = \langle u, w \rangle \in Q \) and \( i, j < m \) there exists \( \tilde{q} = \langle \tilde{u}, \tilde{w} \rangle \in Q \) such that \( q(G_{ij}(x, \xi)) = \tilde{q} \) for all \( \langle x, \xi \rangle \in \Pi \) with \( q(x, \xi) = q \).

**Proof (Claim).** We can assume that \( i \neq j \) and \( n(i) = n(j) \) since otherwise \( G_{ij}(x, \xi) = \langle x, \xi \rangle \), and hence \( \tilde{q} = q \) works. By the same reason we can w.l.o.g. assume that either \( i \in u \land j \notin u \) or \( i \notin u \land j \in u \). Let say \( i \in u \land j \notin u \), that is, \( \tau^\xi_i(a_i, \varphi_i) = x \neq \tau^\xi_j(a_j, \varphi_j) \). Then by definition the element \( \langle y, \xi \rangle = G_{ij}(x, \xi) = \alpha_{ai \varphi_j}(x, \xi) \) coincides with \( \langle \tau^\xi_i(a_i, \varphi_i), \xi \rangle \). Let us compute \( \tilde{q} = q(y, \xi) \).

Consider an arbitrary \( k < m \). To figure out whether \( k \in \bar{u} = u_y \xi \) we have to determine whether \( \tau^\xi_y(a_k, \varphi_k) = y \) holds. If \( n(k) \geq n(i) = n(j) \) then obviously \( \tau^\xi_y(a_k, \varphi_k) = \tau^\xi_y(a_k, \varphi_k) \), and hence \( \tau^\xi_y(a_k, \varphi_k) = y \) iff \( \langle j, k \rangle \in w \). Suppose that \( n(k) < n(i) = n(j) \). Then

\[
\tau^\xi_y(a_k, \varphi_k) = \tau^\xi_y(a_j, \varphi_j)(a_k, \varphi_k) = \tau^\xi_y(b, \psi),
\]

where the pair \( \langle b, \psi \rangle = \langle a_k \land (a_j \mid \geq n(k)) \rangle \) is equal to one of the pairs \( \langle a_{\nu}, \varphi_{\nu} \rangle \), \( \nu < m \) (and then \( n(\nu) = n(i) = n(j) \)). Thus \( \tau^\xi_y(a_k, \varphi_k) = y \) iff \( \tau^\xi_y(a_{\nu}, \varphi_{\nu}) = \tau^\xi_y(a_j, \varphi_j) \) iff \( \langle j, \nu \rangle \in w \).

Now consider arbitrary numbers \( k, k' < m \). To figure out whether \( \langle k, k' \rangle \in \bar{w} = w_{y \xi} \) we have to determine whether \( \tau^\xi_y(a_k, \varphi_k) = \tau^\xi_y(a_{k'}, \varphi_{k'}) \) holds. As above in the first part of the proof of the claim, there exist indices \( \nu, \nu' < m \) (that depend on \( q(\pi) = \langle u, v \rangle \) but not directly on \( \langle x, \xi \rangle \)) such that \( \tau^\xi_y(a_k, \varphi_k) = \tau^\xi_y(a_{\nu}, \varphi_{\nu}) \) and \( \tau^\xi_y(a_{k'}, \varphi_{k'}) = \tau^\xi_y(a_{\nu'}, \varphi_{\nu'}) \). And then the equality \( \tau^\xi_y(a_k, \varphi_k) = \tau^\xi_y(a_{k'}, \varphi_{k'}) \) is equivalent to \( \langle \nu, \nu' \rangle \in w \). \( \Box \) (Claim)

Come back to the proof of Lemma 25.

Consider any \( q = \langle u, w \rangle \in Q \). Then \( \Pi_q = \{ \langle x, \xi \rangle \in \Pi : q(x, \xi) = q \} \) is a Borel subset of \( \Pi \). It follows from the claim that for every superposition of the form \( f = f_0 \circ f_1 \circ \cdots \circ f_\ell \), where each of \( f_k \) is equal to one of \( G_{ij} \) \( (i, j \) depend on \( k \) \) there exists a sequence \( k_0, k_1, \ldots, k_\ell \) of numbers \( k_i < m \) such that

\[
\begin{align*}
f(x, \xi) = (g_{a_{k_0}, \varphi_{k_0}} \circ g_{a_{k_1}, \varphi_{k_1}} \circ \cdots \circ g_{a_{k_\ell}, \varphi_{k_\ell}})(x, \xi)
\end{align*}
\]

for all \( \langle x, \xi \rangle \in \Pi_q \), where \( g_{a, \varphi} \) is a map of \( \Pi \to \Pi \) defined so that \( g_{a, \varphi}(x, \xi) = \langle \tau^\xi(a, \varphi), \xi \rangle \) for all \( \langle x, \xi \rangle \in \mathbb{R}^N \times \mathbb{R}^N \). In other words \( f = f_0 \circ \cdots \circ f_\ell \) coincides with the superposition \( g_{a_{k_0}, \varphi_{k_0}} \circ \cdots \circ g_{a_{k_\ell}, \varphi_{k_\ell}} \) on \( \Pi_q \).

Note finally that if \( (a, \varphi) \in D_{n_0}, \langle b, \psi \rangle \in D_{n'}, \) and \( n' \leq n \) then \( g_{a, \varphi}(g_{b, \psi}(x, \xi)) = g_{a, \varphi}(x, \xi) \) for all \( \langle x, \xi \rangle \in \Pi \). It follows that the superposition \( g_{a_{k_0}, \varphi_{k_0}} \circ \cdots \circ g_{a_{k_\ell}, \varphi_{k_\ell}} \) will not change as a function if we remove all factors \( g_{a_{k_j}, \varphi_{k_j}} \) such that \( n(k_j) \leq n(k_j) \) for some \( j < i \). The remaining superposition obviously contains at most
$n = \max_{i \leq m} n(i)$ terms, and hence there exist only finitely many superpositions of such a reduced form.

As $Q$ itself is finite, this ends the proof of the lemma. \hfill \Box \quad \text{(Lemma 25)}

\hfill \Box \quad \text{(Theorem 3)}

7 Case 2

Then the $\Sigma^1_1$ set $R = P_0 \cap H$, where $H = 2^N \setminus S$ is the chaotic domain, is non-empty. Our goal will be to prove that $E_1 \leq H E_{13} \upharpoonright R$ in this case. The embedding $\vartheta : \mathbb{R}^N \rightarrow R$ will have the property that any two elements $\langle x, \xi \rangle$ and $\langle x', \xi' \rangle$ in the range $\text{ran} \vartheta \subseteq R$ satisfy $\xi E_3 \xi'$, so that the $\xi'$-component in the range of $\vartheta$ is trivial. And as far as the $x$-component is concerned, the embedding will resemble the embedding defined in Case 1 of the proof of the 1st dichotomy theorem in [9] (see also [6, Ch. 8]).

Recall that sets $S^k_n$ were defined in Corollary 10, and by definition

\begin{equation}
\langle x, \xi \rangle \in H \implies \forall m \exists n \geq m \forall k \left( \langle x, \xi \rangle \notin S^k_n \right)
\end{equation}

\begin{equation}
\implies \forall m \exists n \geq m \forall k \forall \varphi \in \mathcal{F}_n^k \left( x(n) \neq \varphi(x|_{>n}, \xi) \right)
\end{equation}

in Case 2. Prove a couple of related technical lemmas.

Lemma 27. Each set $S^k_n$ is invariant in the following sense: if $\langle x, \xi \rangle \in S^k_n$, $\langle y, \eta \rangle \in \mathbb{R}^N \times \mathbb{R}^N$, $x|_{>n} = y|_{>n}$, and $\xi E_3 \eta$ then $\langle y, \eta \rangle \in S^k_n$.

Proof. Otherwise there is a $\Delta^1_1$ function $\varphi \in \mathcal{F}_n^k$ such that $y(n) = \varphi(y|_{>n}, \eta)$. Then $x(n) = \varphi(x|_{>n}, \eta)$ as well because $x|_{>n} = y|_{>n}$. We put

$$u_j = \xi(j) \Delta \eta(j) = \{ m : \xi(j)(m) \neq \eta(j)(m) \}$$

for every $j < k$, these are finite subsets of $\mathbb{N}$. If $a \in 2^N$ and $u \subseteq \mathbb{N}$ then define $u \cdot a \in 2^N$ so that $(u \cdot a)(m) = a(m)$ for $m \notin u$, and $(u \cdot a)(m) = a(m)$ for $m \in u$. If $\xi \in \mathbb{R}^N$ then define $f(\xi) \in \mathbb{R}^N$ so that $f(\xi)(j) = u_j \cdot \xi(j)$ for $j < k$, and $f(\xi)(j) = \xi(j)$ for $j \geq k$.

Finally, put $\psi(z, \zeta) = \varphi(z, f(\zeta))$ for every $\langle z, \zeta \rangle \in \mathbb{R}^{\geq n} \times \mathbb{R}^N$. The map $\psi$ obviously belongs to $\mathcal{F}_n^k$ together with $\varphi$. Moreover

$$x(n) = \varphi(x|_{>n}, \eta) = \psi(x|_{>n}, f(\eta)) = \psi(x|_{>n}, \xi)$$

because $f(\eta)|_{<k} = \xi|_{<k}$, and this contradicts to the choice of $\langle x, \xi \rangle$. \hfill \Box

The next simple lemma will allow us to split $\Sigma^1_1$ sets in $\mathbb{R}^N \times \mathbb{R}^N$.

Lemma 28. If $P \subseteq \mathbb{R}^N \times \mathbb{R}^N$ is a $\Sigma^1_1$ set and $P \not\subseteq S^k_n$ then there exist points $\langle x, \xi \rangle$ and $\langle y, \eta \rangle$ in $P$ with

$$y|_{>n} = x|_{>n}, \quad \eta E_3 \xi, \quad \eta|_{<k} = \xi|_{<k}, \quad \text{but} \quad y(n) \neq x(n).$$

Proof. Otherwise $\psi = \{ \langle y|_{>n}, \eta \rangle, y(n) : \langle y, \eta \rangle \in P \}$ is a map in $\mathcal{F}_n^k$, and hence $P \subseteq S^k_n$, contradiction. \hfill \Box
8 Case 2: splitting system

We apply a splitting construction, developed in [5] for the study of “ill”-founded Sacks iterations. Below, $2^n$ will typically denote the set of all dyadic sequences of length $n$, and $2^{<\omega} = \bigcup_n 2^n$ all finite dyadic sequences.

The construction involves a map $\varphi : \mathbb{N} \to \mathbb{N}$ assuming infinitely many values and each its value infinitely many times (but $\text{ran} \varphi$ may be a proper subset of $\mathbb{N}$), another map $\pi : \mathbb{N} \to \mathbb{N}$, and, for each $u \in 2^{<\omega}$, a non-empty $\Sigma^1_1$ subset $P_u \subseteq R = H \cap \mathcal{P}_b$ — which satisfy a quite long list of properties.

First of all, if $\varphi$ is already defined at least on $[0, n)$ and $u \neq v \in 2^n$ then let $\nu_{\varphi}[u, v] = \max\{\varphi(\ell) : \ell < n \land u(\ell) \neq v(\ell)\}$. And put $\nu_{\varphi}[u, u] = -1$ for any $u$.

Now we present the list of requirements $1^a - 8^a$.

$1^a$: if $\varphi(n) \notin \{\varphi(\ell) : \ell < n\}$ then $\varphi(n) > \varphi(\ell)$ for each $\ell < n$;

$2^a$: if $u \in 2^n$ then $P_u \cap (\bigcup_{k<\omega} S^k_{\varphi(\ell)}) = \emptyset$ for each $\ell < n$;

$3^a$: every $P_u$ is a non-empty $\Sigma^1_1$ subset of $R \cap H$;

$4^a$: $P_{u \cup i} \subseteq P_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

Two further conditions are related rather to the sets $X_u = \text{dom} P_u$.

$5^a$: if $u, v \in 2^n$ then $X_u \restriction_{>\nu_{\varphi}[u, v]} = X_v \restriction_{>\nu_{\varphi}[u, v]}$;

$6^a$: if $u, v \in 2^n$ then $X_u \restriction_{\gtrsim\nu_{\varphi}[u, v]} \cap X_v \restriction_{\gtrsim\nu_{\varphi}[u, v]} = \emptyset$.

The content of the next condition is some sort of genericity in the sense of the Gandy – Harrington forcing in the space $\mathbb{R}^N \times \mathbb{R}^N$, that is, the forcing notion

$$\mathbb{P} = \text{all non-empty } \Sigma^1_1 \text{ subsets of } \mathbb{R}^N \times \mathbb{R}^N.$$

Let us fix a countable transitive model $\mathcal{M}$ of a sufficiently large fragment of ZFC. \footnote{For instance remove the Power Set axiom but add the axiom saying that for any set $X$ there exists the set of all countable subsets of $X$.} For technical reasons, we assume that $\mathcal{M}$ is an elementary submodel of the universe w.r.t. all analytic formulas. Then simple relations between sets in $\mathbb{P}$ in the universe, like $P = Q$ or $P \subseteq Q$, are adequately reflected as the same relations between their intersections $P \cap \mathcal{M}$, $Q \cap \mathcal{M}$ with the model $\mathcal{M}$. In this sense $\mathbb{P}$ is a forcing notion in $\mathcal{M}$.

A set $D \subseteq \mathbb{P}$ is open dense iff, first, for any $P \in \mathbb{P}$ there is $Q \in D$, $Q \subseteq P$, and given sets $P \subseteq Q \in \mathbb{P}$, if $Q$ belongs to $D$ then so does $P$. A set $D \subseteq \mathbb{P}$ is coded in $\mathcal{M}$, iff the set $\{P \cap \mathcal{M} : P \in D\}$ belongs to $\mathcal{M}$. There exists at most countably many such sets because $\mathcal{M}$ is countable. Let us fix an enumeration (not in $\mathcal{M}$) $\{D_n : n \in \mathbb{N}\}$ of all open dense sets $D \subseteq \mathbb{P}$ coded in $\mathcal{M}$.

The next condition essentially asserts the $\mathbb{P}$-genericity of each branch in the splitting construction over $\mathcal{M}$.
7°: for every \( n \), if \( u \in 2^{n+1} \) then \( P_u \in D_n \).

**Remark 29.** It follows from 7° that for any \( a \in 2^N \) the sequence \( \{P_{a|n}\}_{n \in \mathbb{N}} \) is generic enough for the intersection \( \bigcap_n P_{a|n} \neq \emptyset \) to consist of a single point, say \( \langle g(a), \gamma(a) \rangle \), and for the maps \( g, \gamma : 2^N \rightarrow \mathbb{R}^N \times \mathbb{R}^N \) to be continuous.

Note that \( g \) is 1 − 1. Indeed if \( a \neq b \) belong to \( 2^N \) then \( a(n) \neq b(n) \) for some \( n \), and hence \( \nu_\alpha[a \upharpoonright m, b \upharpoonright m] \geq \varphi(n) \) for all \( m \geq n \). It follows by 6° that \( X_{a|n} \cap X_{b|n} = \emptyset \) for \( m > n \), therefore \( g(a) \neq g(b) \).

Our final requirement involves the \( \xi \)-parts of sets \( P_u \). We’ll need the following definition. Suppose that \( \langle x, \xi \rangle \) and \( \langle y, \eta \rangle \) belong to \( \mathbb{R}^N \times \mathbb{R}^N \), \( p \in \mathbb{N} \), and \( s \in \mathbb{N}^{<\omega} \), \( \mathbf{1} n s = m \) (the length of \( s \)). Define \( \langle x, \xi \rangle \cong_p^s \langle y, \eta \rangle \) iff

\[
\xi \in \mathbb{E}_3 \eta, \quad x \upharpoonright p = y \upharpoonright p, \quad \text{and} \quad (\xi(k) \Delta \eta(k)) \subseteq s(k) \quad \text{for all} \quad k < m = \mathbf{1} n s,
\]

where \( \alpha \Delta \beta = \{ j : \alpha(j) \neq \beta(j) \} \) for \( \alpha, \beta \in 2^N \). If \( P, Q \subseteq \mathbb{R}^N \times \mathbb{R}^N \) are arbitrary sets then under the same circumstances \( P \cong_p^s Q \) will mean that

\[
\forall \langle x, \xi \rangle \in P \exists \langle y, \eta \rangle \in Q \left( \langle x, \xi \rangle \cong_p^s \langle y, \eta \rangle \right) \quad \text{and vice versa}.
\]

Obviously \( \cong_p^s \) is an equivalence relation.

The following is the last condition:

8°: there exists a map \( \pi : \mathbb{N} \rightarrow \mathbb{N} \), such that \( P_u \cong_{\nu_\pi[u,v]} P_v \) holds for every \( n \) and all \( u, v \in 2^n \) (and then \( X_u \upharpoonright_{\nu_\pi[u,v]} = X_v \upharpoonright_{\nu_\pi[u,v]} \) as in 5°).

9 Case 2: splitting system implies the reducibility

Here we prove that any system of sets \( P_u \) and \( X_u = \text{dom} P_u \) and maps \( \varphi, \pi \) satisfying 1° − 8° implies Borel reducibility of \( E_1 \) to \( E_{13} \upharpoonright R \). This completes Case 2. The construction of such a splitting system will follow in the remainder.

Let the maps \( g \) and \( \gamma \) be defined as in Remark 29. Put

\[
W = \{ \langle g(a), \gamma(a) \rangle : a \in 2^N \}.
\]

**Lemma 30.** \( W \) is a closed set in \( \mathbb{R}^N \times \mathbb{R}^N \) and a function. Moreover if \( \langle x, \xi \rangle \) and \( \langle y, \eta \rangle \) belong to \( W \) then \( \xi \in \mathbb{E}_3 \eta \).

**Proof.** \( W \) is closed as a continuous image of \( 2^N \). That \( W \) is a function follows from the bijectivity of \( g \), see Remark 29. Finally any two \( \xi, \eta \) as inquired satisfy \( \xi(k) \Delta \eta(k) \subseteq \pi(k) \) for all \( k \) by 8°.

Put \( X = \text{dom} W \). Thus \( W \) is a continuous map \( X \rightarrow \mathbb{R}^N \) by the lemma.

**Corollary 31.** There exists a Borel reduction of \( E_1 \upharpoonright X \) to \( E_{13} \upharpoonright W \).
Proof. As $W$ is a function, we can use the notation $W(x)$ for $x \in X = \text{dom} W$. Put $f(x) = (x, W(x))$. This is a Borel, even a continuous map $X \to W$. It remains to establish the equivalence

$$x \in E_1 y \iff f(x) \in f(y)$$

for all $x, y \in X$. (7)

If $x \in E_1 y$ then $W(x) \in W(y)$ by Lemma 30, and hence easily $f(x) \in f(y)$. If $x \in E_1 y$ fails then obviously $f(x) \in f(y)$ fails, too. \[\square\]

Thus to complete Case 2 it now suffices to define a Borel reduction of $E_1$ to $E_1 \restriction X$. To get such a reduction consider the set $\Phi = \text{ran} \varphi$, and let $\Phi = \{p_m : m \in \mathbb{N}\}$ in the increasing order; that the set $\Phi \subseteq \mathbb{N}$ is infinite follows from 1°.

Suppose that $n \in \mathbb{N}$. Then $\varphi(n) = p_m$ for some (unique) $m : \text{we put}$ $\psi(n) = m$. Thus $\psi : \mathbb{N} \rightarrow \mathbb{N}$ and the preimage $\psi^{-1}(m) = \varphi^{-1}(p_m)$ is an infinite subset of $\mathbb{N}$ for any $m$. Define a parallel system of sets $Y_u \subseteq \mathbb{N}$, $u \in 2^\omega$, as follows. Put $Y_A = \mathbb{N}$. Suppose that $Y_u$ has been defined, $u \in 2^n$. Put $p = \varphi(n) = p_{\psi(n)}$. Let $K$ be the number of all indices $\ell < n$ still satisfying $\varphi(\ell) = p$, perhaps $K = 0$. Put $Y_{u \cup i} = \{x \in Y_u : x(p)(K) = i\}$ for $i = 0, 1$.

Each of $Y_u$ is clearly a basic clopen set in $\mathbb{N}$, and one easily verifies that conditions 4°, 5°, 6° are satisfied for the sets $Y_u$ and the map $\psi$ (instead of $\varphi$ in 5°, 6°), in particular

6*: if $u, v \in 2^n$ then $Y_u \restriction u \psi[u, v] = Y_v \restriction u \psi[u, v]$

7*: if $u, v \in 2^n$ then $Y_u \restriction \psi[u, v] \cap Y_v \restriction \psi[u, v] = \varnothing$

where $\psi[u, v] = \max\{\psi(\ell) : \ell < n \land u(\ell) \neq v(\ell)\}$ (compare with $\nu_\varphi$ above).

It is clear that for any $a \in 2^n$ the intersection $\bigcap_n Y_a \restriction n = \{f(a)\}$ is a singleton, and the map $f$ is continuous and $1 - 1$. (We can, of course, define $f$ explicitly: $f(a)(p)(K) = a(n)$, where $n \in \mathbb{N}$ is chosen so that $\psi(n) = p$ and there is exactly $K$ numbers $\ell < n$ with $\psi(\ell) = p$.) Note finally that $\{f(a) : a \in 2^n\} = \mathbb{N}$ since by definition $Y_a \cap Y_a = Y_u$ for all $u$.

We conclude that the map $\vartheta(x) = g(f^{-1}(x))$ is a continuous map (in fact a homeomorphism in this case by compactness) $\mathbb{N} \rightarrow X = \text{dom} W$.

Lemma 32. The map $\vartheta$ is a reduction of $E_1$ to $E_1 \restriction X$, and hence $\vartheta$ witnesses $E_1 \leq_\mathbb{N} E_1 \restriction X$ and $E_1 \leq_\mathbb{N} E_{13} \restriction W$ by Corollary 31.

Proof. It suffices to check that the map $\vartheta$ satisfies the following requirement: for each $y, y' \in \mathbb{N}$ and $m$, $y \restriction m = y' \restriction m$ iff $\vartheta(y) \restriction m = \vartheta(y') \restriction m$. \[\text{(8)}\]

To prove (8) suppose that $y = f(a)$ and $x = g(a) = \vartheta(y)$, and similarly $y' = f(a')$ and $x' = g(a') = \vartheta(y')$, where $a, a' \in 2^n$. Suppose that $y \restriction m = y' \restriction m$. \[\square\]
We then have \( m > \nu_\psi | a \upharpoonright n, a' \upharpoonright n \) for any \( n \) by \( 7^* \). It follows, by the definition of \( \psi \), that \( p_m > \nu_\psi | a \upharpoonright n, a' \upharpoonright n \) for any \( n \), hence \( X_{a|n} \upharpoonright p_m = X_{a'|n} \upharpoonright p_m \) for any \( n \) by \( 5^* \). Therefore \( x \upharpoonright p_m = x' \upharpoonright p_m \) by \( 7^* \), that is, the right-hand side of (8). The inverse implication in (8) is proved similarly. \( \square \) (Lemma)

It follows that we can now focus on the construction of a system satisfying \( 1^\circ - 8^\circ \). The construction follows in Section 12, after several preliminary lemmas in Sections 10 and 11.

10 Case 2: how to shrink a splitting system

Let us prove some results related to preservation of condition \( 8^\circ \) under certain transformations of shrinking type. They will be applied in the construction of a splitting system satisfying conditions \( 1^\circ - 8^\circ \) of Section 8.

Lemma 33. Suppose that \( n \in \mathbb{N}, s \in \mathbb{N}^{<\omega} \), and a system of \( \Sigma_1^1 \) sets \( \varnothing \neq P_u \subseteq \mathbb{N} \times \mathbb{N} \), \( u \in 2^n \), satisfies \( P_u \cong_{\nu_\psi[u,v]} P_v \) for all \( u, v \in 2^n \). Assume also that \( w_0 \in 2^n \), and \( \varnothing \neq Q \subseteq P_{w_0} \) is a \( \Sigma_1^1 \) set. Then the system of \( \Sigma_1^1 \) sets

\[
P'_u = \{ (x, y) \in P_u : \exists (z, \zeta) \in Q (\langle x, \xi \rangle \cong_{\nu_\psi[u,v]} \langle z, \zeta \rangle) \}, \quad u \in 2^n,
\]

still satisfies \( P'_u \cong_{\nu_\psi[u,v]} P'_v \) for all \( u, v \in 2^n \), and \( P'_{w_0} = Q \).

Proof. \( P'_{w_0} = Q \) holds because \( \nu_\psi[w_0, w_0] = -1 \). Let us verify \( 8^\circ \). Suppose that \( u, v \in 2^n \). Each one of the three numbers \( \nu_\psi[u, w], \nu_\psi[v, w], \nu_\psi[u, v] \) is obviously not bigger than the largest of the two other numbers. This observation leads us to the following three cases.

Case a: \( \nu_\psi[u, w] = \nu_\psi[v, w] \geq \nu_\psi[u, v] \). Consider any \( (x, \xi) \in P'_u \). Then by definition there exists \( (z, \zeta) \in Q \) with \( \langle x, \xi \rangle \cong_{\nu_\psi[u,v]} \langle z, \zeta \rangle \). Then, as \( P_{w_0} \cong_{\nu_\psi[u,v]} P_v \) is assumed by the lemma, there is \( (y, \eta) \in P_v \) such that \( \langle y, \eta \rangle \cong_{\nu_\psi[u,v]} \langle z, \zeta \rangle \). Note that \( \langle z, \zeta \rangle \) witnesses \( (y, \eta) \in P'_v \). On the other hand, \( \langle x, \xi \rangle \cong_{\nu_\psi[u,v]} \langle y, \eta \rangle \) because \( \nu_\psi[u, w] = \nu_\psi[u, v] \geq \nu_\psi[v, w] \). Conversely, suppose that \( \langle y, \eta \rangle \in P'_v \). Then there is \( (z, \zeta) \in Q \) such that \( \langle y, \eta \rangle \cong_{\nu_\psi[u,v]} \langle z, \zeta \rangle \). Yet \( P_{w_0} \cong_{\nu_\psi[u,v]} P_u \), and hence there exists \( (x, \xi) \in P'_u \) with \( \langle x, \xi \rangle \cong_{\nu_\psi[u,v]} \langle y, \eta \rangle \).

Case b: \( \nu_\psi[v, w] = \nu_\psi[u, v] \geq \nu_\psi[u, w] \). Absolutely similar to Case a.

Case c: \( \nu_\psi[u, w] = \nu_\psi[v, w] \geq \nu_\psi[u, v] \). This is a symmetric case, thus it is enough to carry out only the direction \( P'_u \rightarrow P'_v \). Consider any \( (x, \xi) \in P'_u \). As above there is \( (z, \zeta) \in Q \) such that \( \langle x, \xi \rangle \cong_{\nu_\psi[u,v]} \langle z, \zeta \rangle \). On the other hand, as \( P_u \cong_{\nu_\psi[u,v]} P_v \), there exists a point \( (y, \eta) \in P_v \) such that \( \langle y, \eta \rangle \cong_{\nu_\psi[u,v]} \langle x, \xi \rangle \). Note that \( \langle z, \zeta \rangle \) witnesses \( (y, \eta) \in P'_v \) : indeed by definition we have \( \langle y, \eta \rangle \cong_{\nu_\psi[u,v]} \langle z, \zeta \rangle \). \( \square \)
Corollary 34. Assume that \( n \in \mathbb{N}, \ s \in \mathbb{N}^{<\omega}, \) and a system of \( \Sigma_1^1 \) sets \( \varnothing \neq P_u \subseteq \mathbb{R}^N \times \mathbb{R}^N, \ u \in 2^n, \) satisfies \( P_u \cong_{\nu_u[u,v]} P_v \) for all \( u, v \in 2^n. \) Assume also that \( \varnothing \neq W \subseteq 2^n, \) and a \( \Sigma_1^1 \) set \( \varnothing \neq Q_w \subseteq P_w \) is defined for every \( w \in W \) so that still \( Q_w \cong_{\nu_w[w,w']} Q_w' \) for all \( w, w' \in W. \) Then the system of \( \Sigma_1^1 \) sets

\[
P'_u = \{ (x, \xi) \in P_u : \forall w \in W \exists \langle y, \eta \rangle \in Q_w \ (\langle x, \xi \rangle \cong_{\nu_w[u,w]} \langle y, \eta \rangle) \}
\]

still satisfies \( P'_u \cong_{\nu_u[u,v]} P'_v \) for all \( u, v \in 2^n, \) and \( P'_u = Q_w \) for all \( w \in W. \)

Proof. Apply the transformation of Lemma 33 consecutively for all \( w_0 \in W \) and the corresponding sets \( Q_{w_0}. \) Note that these transformations do not change the sets \( Q_w \) with \( w \in W \) because \( Q_w \cong_{\nu_w[w,w']} Q_w' \) for all \( w, w' \in W. \) \( \square \)

Remark 35. The sets \( P'_u \) in Corollary 34 can as well be defined by

\[
P'_u = \{ (x, \xi) \in P_u : \exists \langle y, \eta \rangle \in Q_{w_u} \ (\langle x, \xi \rangle \cong_{\nu_w[u,w_u]} \langle y, \eta \rangle) \}
\]

where, for each \( u \in 2^n, \ w_u \) is an element of \( W \) such that the number \( \nu_w[u, w_u] \) is the least of all numbers of the form \( \nu_w[u, w], \ w \in W. \) (If there exist several \( w \in W \) with the minimal \( \nu_w[u, w] \) then take the least of them.) \( \square \)

11 Case 2: how to split a splitting system

Here we consider a different question related to the construction of systems satisfying conditions 1° - 8° of Section 8. Given a system of \( \Sigma_1^1 \) sets satisfying a \( 8° \)-like condition, how to shrink the sets so that \( 8° \) is preserved and in addition \( 6° \) holds. Let us begin with a basic technical question: given a pair of \( \Sigma_1^1 \) sets \( P, Q \) satisfying \( P \cong_{\nu_p} Q \) for some \( p, s, \) how to define a pair of smaller \( \Sigma_1^1 \) sets \( P' \subseteq P, \ Q' \subseteq Q, \) still satisfying the same condition, but as disjoint as it is compatible with this condition.

Recall that \( \text{dom} P = \{ x : \exists \xi (\langle x, \xi \rangle \in P) \} \) for \( P \subseteq \mathbb{R}^N \times \mathbb{R}^N. \)

Lemma 36. If \( P, Q \subseteq \mathbb{R}^N \times \mathbb{R}^N \) are non-empty \( \Sigma_1^1 \) sets, \( p, s \in \mathbb{N}, \ s \in \mathbb{N}^{<\omega}, \) \( P \cong_{\nu_p} Q, \) and \( (P \cup Q) \cap \Sigma_p^k = \varnothing, \) where \( k = \| p \| s, \) then there exist non-empty \( \Sigma_1^1 \) sets \( P' \subseteq P, \ Q' \subseteq Q \) such that still \( P' \cong_{\nu_p} Q' \) but in addition \( (\text{dom} P') \upharpoonright \geq p \cap (\text{dom} Q') \upharpoonright \geq p = \varnothing. \)

Note that \( P \cong_{\nu_p} Q \) implies \( (\text{dom} P) \upharpoonright \geq p = (\text{dom} Q) \upharpoonright \geq p. \)

Proof. It follows from Lemma 28 that there exist points \( \langle x_0, \xi_0 \rangle \) and \( \langle x_1, \xi_1 \rangle \) in \( P \) such that \( \langle x_0, \xi_0 \rangle \cong_{\nu_p} \langle x_1, \xi_1 \rangle \) but \( x_1(p) \neq x_0(p). \) Then there exists a number \( j \) such that, say, \( x_1(p)(j) = 1 \neq 0 = x_0(p)(j). \) On the other hand, there exists \( \langle y_0, \eta_0 \rangle \in Q \) such that \( \langle x_i, \xi_i \rangle \cong_{\nu_p} \langle y_0, \eta_0 \rangle \) for \( i = 0, 1. \) Then \( y_0(p)(j) \neq x_i(p)(j) \) for one of \( i \in \{0, 1\} \). Let say \( y_0(p)(j) = 0 \neq 1 = x_0(p)(j). \) Then the \( \Sigma_1^1 \) sets

\[
P' = \{ (x, \xi) \in P : \exists \langle y, \eta \rangle \in Q \ (x(p)(j) = 1 \land y(p)(j) = 0 \land (x, \xi) \cong_{\nu_p} \langle y, \eta \rangle) \};
\]

\[
Q' = \{ (y, \eta) \in Q : \exists (x, \xi) \in P (x(p)(j) = 1 \land y(p)(j) = 0 \land (x, \xi) \cong_{\nu_p} \langle y, \eta \rangle) \}
\]
are $\Sigma^1_1$ and non-empty (contain resp. $\langle x_0, \xi_0 \rangle$ and $\langle y_0, \eta_0 \rangle$), and they satisfy $P' \equiv^s P$, but $(\dom P')\upharpoonright_{\geq 2}(\dom Q')\upharpoonright_{\geq 2} = \emptyset$ because $y(p)(j) = 0 \neq 1 = x(p)(j)$ whenever $\langle x, \xi \rangle \in P'$ and $\langle y, \eta \rangle \in Q'$. □

**Corollary 37.** Assume that $n \in \mathbb{N}$, $s \in \mathbb{N}^{<\omega}$, and a system of $\Sigma^1_1$ sets $\emptyset \neq P_u \subseteq \mathbb{R}_n \times \mathbb{R}_n$, $u \in 2^n$, satisfies $P_u \equiv^s_{\nu_\mathcal{F}[u,v]} P_v$ for all $u, v \in 2^n$. Then there exists a system of $\Sigma^1_1$ sets $\emptyset \neq P'_u \subseteq P_u$, $u \in 2^n$, such that still $P'_u \equiv^s_{\nu_\mathcal{F}[u,v]} P_v$, and in addition $(\dom P'_u)\upharpoonright_{\geq 2\nu_\mathcal{F}[u,v]} \cap (\dom Q'_v)\upharpoonright_{\geq 2\nu_\mathcal{F}[u,v]} = \emptyset$, for all $u \neq v \in 2^n$.

**Proof.** Consider any pair of $u_0 \neq v_0$ in $2^n$. Apply Lemma 36 for the sets $P = P_{u_0}$ and $Q = P_{v_0}$ and $p = \nu_\mathcal{F}[u_0, v_0]$. Let $P'$ and $Q'$ be the $\Sigma^1_1$ sets obtained, in particular $P' \equiv^s_{\nu_\mathcal{F}[u_0, v_0]} Q'$ and $(\dom P')\upharpoonright_{\geq 2\nu_\mathcal{F}[u_0, v_0]} \cap (\dom Q')\upharpoonright_{\geq 2\nu_\mathcal{F}[u_0, v_0]} = \emptyset$. Then by Corollary 34 there is a system of $\Sigma^1_1$ sets $\emptyset \neq P'_u \subseteq P_u$ such that still $P'_u \equiv^s_{\nu_\mathcal{F}[u,v]} P'_v$ for all $u, v \in 2^n$, and $P_{u_0} = P'$, $P_{v_0} = Q'$ — and hence

$$(\dom P'_{u_0})\upharpoonright_{\geq 2\nu_\mathcal{F}[u_0, v_0]} \cap (\dom P'_{v_0})\upharpoonright_{\geq 2\nu_\mathcal{F}[u_0, v_0]} = \emptyset.$$ 

Take any other pair of $u_1 \neq v_1$ in $2^n$ and transform the system of sets $P'_u$ the same way. Iterate this construction sufficient (finite) number of steps. □

**12 Case 2: the construction of a splitting system**

We continue the proof of Theorem 2 – Case 2. Recall that $R = P_0 \cap \mathcal{H}$ is a $\Sigma^1_1$ set. By Lemma 32, it suffices to define functions $\varphi$ and $\pi$ and a system of $\Sigma^1_1$ sets $P_u \subseteq R$ together satisfying conditions $1^0 - 8^2$. The construction of such a system will go on by induction on $n$. That is, at any step $n$ the sets $P_u$ with $u \in 2^n$, as well as the values of $\varphi(k)$ and $\pi(k)$ with $k < n$, will be defined.

For $n = 0$, we put $P_\Lambda = R$. ($\Lambda \in 2^0$ is the only sequence of length 0.)

Suppose that sets $P_u \subseteq R$ with $u \in 2^n$, and also all values $\varphi(\ell)$, $\ell < n$, and $\pi(k)$, $k < n$, have been defined and satisfy the applicable part of $1^0 - 8^2$. The content of the inductive step $n \mapsto n + 1$ will consist in definition of $\varphi(n)$, $\pi(n)$, and sets $P_{u \wedge i}$ with $u \wedge i \in 2^{n+1}$, that is, $u \in 2^n$ (a dyadic sequence of length $n$) and $i = 0, 1$. This goes on in four steps A,B,C,D.

**12.1 Step A: definition of $\varphi(n)$**

Suppose that, in the order of increase,

$$\{\varphi(\ell) : \ell < n\} = \{p_0 < \cdots < p_m\}.$$ 

For $j \leq m$, let $K_j$ be the number of all $\ell < n$ with $\varphi(\ell) = p_j$.

**Case A:** $K_j \geq m$ for all $j \leq m$. Then consider any $u_0 \in 2^n$ and an arbitrary point $\langle x_0, \xi_0 \rangle \in P_{u_0}$. Note that by (6) of Section 7 there is a number $p > \max_{\ell < n} \varphi(\ell)$ such that $\langle x_0, \xi_0 \rangle \notin \bigcup_p S_p^k$. Put $\varphi(n) = p$. 

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Lemma 38.

We claim that the sets $P'_u = P_u \setminus \bigcup_k S^k_{\varphi(n)}$ still satisfy condition $S^8$ (and then $S^5$ for $X'_u = \text{dom } P'_u$). Indeed suppose that $u, v \in 2^n$ and $(x, \xi) \in P'_u$. Then $(x, \xi) \in S^k_{\varphi(n)}$ and hence there is a point $(\eta, \eta') \in P_v$ such that $(x, \xi) \equiv_{\nu_{v[u, v]}} (y, \eta)$. It remains to show that $(y, \eta) \notin \bigcup_k S^k_{\varphi(n)}$. Suppose towards the contrary that $(y, \eta) \in S^k_{\varphi(n)}$ for some $k$. By definition $\varphi(n) > \nu_{\varphi(u, v)}$, therefore $x|_{\varphi(n)} = y|_{\varphi(n)}$. It follows that $(x, \xi) \in S^k_{\varphi(n)}$ by Lemma 27, contradiction.

Case B. If some numbers $K_j$ are $< m$ then choose $\varphi(n)$ among $p_j$ with the least $K_j$, and among them take the least one. Thus $\varphi(n) = \varphi(\ell)$ for some $\ell < n$. It follows that in this case $P_u \cap (\bigcup_k S^k_{\varphi(n)}) = \emptyset$ for all $u \in 2^n$ by the inductive assumption of $S^2$. Put $P'_u = P_u$.

Note that this manner of choice of $\varphi(n)$ implies $1^*, 2^*$ and also implies that $\varphi$ takes infinitely many values and takes each value infinitely many times. In addition, the construction given above proves:

**Lemma 38.** There exists a system of $\Sigma^1_1$ sets $\emptyset \neq P'_u \subset P_u$ satisfying $S^8$ and $P'_u \cap (\bigcup_k S^k_{\varphi(n)}) = \emptyset$ for all $u \in 2^n$.

### 12.2 Step B: definition of $\pi(n)$

We work with the sets $P'_u$ such as in Lemma 38. The next goal is to prove the following result:

**Lemma 39.** There exist a number $r \in \mathbb{N}$ and a system of $\Sigma^1_1$ sets $\emptyset \neq P''_u \subset P'_u$ satisfying $P''_u \cong_{\nu_{\pi[u,v]}} P''_v$ for all $u, v \in 2^n$.

**Proof.** Let $2^n = \{u_j : j < K\}$ be an arbitrary enumeration of all dyadic sequences of length $n$; $K = 2^n$, of course. The method of proof will be to define, for any $k \leq K$, a number $r_k \in \mathbb{N}$ and a system of $\Sigma^1_1$ sets $\emptyset \neq Q^k_{u_i} \subset P'_{u_i}$, $j < k$, by induction on $k$ so that

\[(*)\quad Q^k_{u_i} \cong_{\nu_{\pi[u_i,u_j]}} Q^k_{u_j} \quad \text{for all } i < j < k.\] (Where $\langle \pi \upharpoonright n \rangle^r$ is the extension of the finite sequence $\pi \upharpoonright n$ by $r$ as the new rightmost term.)

After this is done, $r = r_K$ and the sets $P''_u = Q^K_{u_0}$ prove the lemma.

We begin with $k = 2$. Then $P'_{u_0} \cong_{\nu_{\pi[u_0,u_1]}} P'_{u_1}$ by $S^8$, and hence there exist points $(x_0, \xi_0) \in P'_{u_0}$, $(x_1, \xi_1) \in P'_{u_1}$ such that $(x_0, \xi_0) \equiv_{\nu_{\pi[u_0,u_1]}} (x_1, \xi_1)$. Then $\xi_0 \Delta \xi_1$, so that there is a number $r_2 \in \mathbb{N}$ with $\xi_0(n) \Delta \xi_1(n) \subseteq r_2$. Note that for any $p \in \mathbb{N}$ and any points $(x, \xi), (y, \eta) \in \mathbb{R}^n \times \mathbb{R}^n$, $(x, \xi) \equiv_{\nu_{\pi[u_0,u_1]}} (y, \eta)$ is equivalent to the conjunction

\[\langle x, \xi \rangle \equiv_{\nu_{\pi[u_0,u_1]}} \langle y, \eta \rangle \wedge \xi(n) \Delta \eta(n) \subseteq r.\] 21
It follows that the sets
\[ S_0 = \{ \langle x, \xi \rangle \in P_{u_0}': \exists (y, \eta) \in P_{u_1}' (\langle x, \xi \rangle \equiv_{\nu_\phi[u_0,u_1]} \langle y, \eta \rangle) \}, \] and
\[ S_1 = \{ (y, \eta) \in P_{u_1}' : \exists (x, \xi) \in P_{u_0}' (\langle x, \xi \rangle \equiv_{\nu_\phi[u_0,u_1]} \langle y, \eta \rangle) \} \]
are $\Sigma^1_1$ and non-empty (contain resp. $\langle x_0, \xi_0 \rangle$ and $\langle x_1, \xi_1 \rangle$), and they obviously satisfy $S_0 \equiv_{\nu_\phi[u_0,u_1]} S_1$. Therefore by Corollary 34 there exists a system of $\Sigma^1_1$ sets $\varnothing \neq Q_r^2 \subseteq P_{u'}$, $u \in 2^\omega$, such that $Q_{u_0}^2 = S_0$, $Q_{u_1}^2 = S_1$, $\delta^0$ still holds, and in addition $Q_{u_0}^2 \equiv_{\nu_\phi[u_0,u_1]} Q_{u_1}^2$. Put $r_2 = r$. 

Now let us carry out the step $k \mapsto k + 1$. Suppose that $r_k$ and sets $Q_{u_j}^k$, $j < k$, satisfy $(\ast)$. Of all numbers $\nu_\phi[u_j, u_k]$, $j < k$, consider the least one. Let this be, say, $\nu_\phi[u_\ell, u_k]$, so that $\ell < k$ and $\nu_\phi[u_\ell, u_k] \leq \nu_\phi[u_j, u_k]$ for all $j < k$. As above there exists a number $r$ and a pair of non-empty $\Sigma^1_1$ sets $S_\ell \subseteq Q_{u_\ell}^k$ and $S_k \subseteq Q_{u_k}^k$ such that $S_\ell \equiv_{\nu_\phi[u_\ell, u_k]} S_k$. We can assume that $r \geq r_k$. Put
\[ Q'_{u_j} = \{ (y, \eta) \in S_{u_j} : \exists (x, \xi) \in S_\ell (\langle x, \xi \rangle \equiv_{\nu_\phi[u_\ell, u_j]} \langle y, \eta \rangle) \} \]
for all $j < k$. The proof of Lemma 33 shows that $Q'_{u_j}$ are non-empty $\Sigma^1_1$ sets still satisfying $(\ast)$ in the form of $Q'_{u_j} \equiv_{\nu_\phi[u_\ell, u_j]} Q_{u_j}$ for $i < j < k$ — since $r \geq r_k$, and obviously $Q'_{u_\ell} = S_\ell$. In addition, put $Q'_{u_k} = S_k$. Then still $Q'_{u_i} \equiv_{\nu_\phi[u_\ell, u_k]} Q'_{u_k}$ by the choice of $S_\ell$ and $S_k$. We claim that also
\[ Q'_{u_j} \equiv_{\nu_\phi[u_\ell, u_k]} Q'_{u_k} \quad \text{for all} \ j < k. \] (9)

Indeed we have $Q'_{u_j} \equiv_{\nu_\phi[u_\ell, u_j]} Q'_{u_\ell}$ and $Q'_{u_\ell} \equiv_{\nu_\phi[u_\ell, u_k]} Q'_{u_k}$ by the above. It follows that $Q'_{u_j} \equiv_{\nu_\phi[u_\ell, u_j]} Q'_{u_k}$, where $p = \max\{\nu_\phi[u_j, u_\ell], \nu_\phi[u_\ell, u_k]\}$. Thus it remains to show that $p \leq \nu_\phi[u_j, u_k]$. That $\nu_\phi[u_\ell, u_k] \leq \nu_\phi[u_j, u_k]$ holds by the choice of $\ell$. Prove that $\nu_\phi[u_j, u_\ell] \leq \nu_\phi[u_j, u_k]$. Indeed in any case
\[ \nu_\phi[u_j, u_\ell] \leq \max\{\nu_\phi[u_j, u_k], \nu_\phi[u_\ell, u_k]\}. \]
But once again $\nu_\phi[u_\ell, u_k] \leq \nu_\phi[u_j, u_k]$, so $\nu_\phi[u_j, u_\ell] \leq \nu_\phi[u_j, u_k]$ as required.

Thus (9) is established. It follows that $Q'_{u_i} \equiv_{\nu_\phi[u_i, u_j]} Q'_{u_j}$ for all $i < j \leq k$. We end the inductive step of the lemma by putting $r_{k+1} = r$. \qed (Lemma)

### 12.3 Step C: splitting to the next level

We work with the number $r$ and sets $P''_u$ such as in Lemma 39. Put $\pi(n) = r$. (Recall that $\varphi(n)$ was defined at Step A.) The next step is to split each one of the sets $P''_u$ in order to define sets $P_{u \wedge i}$, $u \wedge i \in 2^{n+1}$, of the next splitting level.
To begin with, put \( Q_u^\iota_j = P_u^j \) for all \( u \in 2^n \) and \( i = 0, 1 \). It is easy to verify that the system of sets \( Q_u^\iota_j \), \( u^\iota_j \in 2^{n+1} \), satisfies conditions \( 1^\circ \) – \( 8^\circ \) for the level \( n + 1 \), except for \( 7^\circ \) and \( 6^\circ \). In particular, \( 2^\circ \) was fixed at Step A, and \( 8^\circ \) in the form that \( Q_u^\iota_j \cong \pi^{(n+1)}_{\nu_\iota_j} Q_{v^\iota_j} \) for all \( u^\iota_j \) and \( v^\iota_j \) in \( 2^{n+1} \) (and then \( 5^\circ \) as well) at Step B. — because \( (\pi \restriction n)^\iota r = \pi \restriction (n + 1) \).

Recall that by definition all sets involved have no common point with \( 2^\circ \). Therefore Corollary 37 is applicable. We conclude that there exists a system of non-empty \( \Sigma^1_1 \) sets \( W_u^\iota_j \subseteq Q_u^\iota_j \), \( u^\iota_j \in 2^{n+1} \), still satisfying \( 8^\circ \), and also satisfying \( 6^\circ \).

### 12.4 Step D: genericity

We have to further shrink the sets \( W_u^\iota_j \), \( u^\iota_j \in 2^{n+1} \), obtained at Step C, in order to satisfy \( 7^\circ \), the last condition not yet fulfilled in the course of the construction. The goal is to define a new system of \( \Sigma^1_1 \) sets \( \emptyset \neq P_u^\iota_j \subseteq W_u^\iota_j \), \( u^\iota_j \in 2^{n+1} \), such that still \( 8^\circ \) holds, and in addition \( P_u^\iota_j \in D_n \) for all \( u^\iota_j \in 2^{n+1} \), where \( D_n \) is the \( n \)-th open dense subset of \( \mathbb{P} \) coded in \( \mathcal{M} \).

Take any \( u_0^\iota_0 \in 2^{n+1} \). As \( D_n \) is a dense subset of \( \mathbb{P} \), there exists a set \( W_0 \in D_n \), therefore, a non-empty \( \Sigma^1_1 \) set, such that \( W_0 \subseteq W_{u_0^\iota_0} \). It follows from Lemma 33 that there exists a system of non-empty \( \Sigma^1_1 \) sets \( W'_u^\iota_j \subseteq W_u^\iota_j \), \( u^\iota_j \in 2^{n+1} \), still satisfying \( 8^\circ \), and such that \( W'_{u_0^\iota_0} = Q_0 \).

Now take any other \( u_1^\iota_1 \neq u_0^\iota_0 \) in \( 2^{n+1} \). The same construction yields a system of non-empty \( \Sigma^1_1 \) sets \( W''_u^\iota_j \subseteq W'_u^\iota_j \), \( u^\iota_j \in 2^{n+1} \), still satisfying \( 8^\circ \), and such that \( W''_{u_1^\iota_1} = W_1 \subseteq W'_{u_1^\iota_1} \) is a set in \( D_n \).

Iterating this construction \( 2^{n+1} \) times, we obtain a system of sets \( P_u^\iota_j \) satisfying \( 7^\circ \) as well as all other conditions in the list \( 1^\circ \) – \( 8^\circ \), as required.

\[ \square \text{(Construction and Case 2 of Theorem 2)} \]

\[ \square \text{(Theorems 2 and 1)} \]

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