MUTATION-INVARIANCE OF KHOVANOV-FLOER THEORIES

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Abstract. Khovanov-Floer theories are a class of homological link invariants which admit spectral sequences from Khovanov homology. They include Khovanov homology, Szabó’s geometric link homology, singular instanton homology, and various Floer theories applied to branched double covers. In this short note we show that certain strong Khovanov-Floer theories, including Szabó homology and singular instanton homology, are invariant under Conway mutation. Along the way we prove two other conjectures about the structure of Szabó homology.

1. Introduction

Let \( L \subset S^3 \) be a link. Suppose that there is a sphere in \( S^3 \) which intersects \( L \) transversely in four points, splitting the link into two tangles, each in a three-ball. Mutation is the operation of regluing the three-balls by some orientation-preserving involution of the sphere. Mutation can also be represented diagrammatically: if \( \mathcal{D} \) is a link diagram and \( s \) is a circle which intersects \( \mathcal{D} \) transversely in four points, then mutation is the operation of rotating the tangle inside \( s \) by 180°. If \( L' \) is the result of mutation on \( L \), then we say that \( L \) and \( L' \) are mutants.

Perhaps surprisingly, many link invariants cannot detect mutation: if \( L \) and \( L' \) are mutants, then the invariants agree on \( L \) and \( L' \). These include the signature and HOMFLY-PT polynomial (and therefore the Jones and Alexander polynomials).

Write \( \Sigma(L) \) for the double cover of \( S^3 \) branched along \( L \). Viro showed that \( \Sigma(L) \) and \( \Sigma(L') \) are diffeomorphic [11]. Therefore any invariant which is also an invariant of branched double covers cannot detect mutation. On the other hand, genus and knot Floer homology can distinguish mutants. So the ability to detect mutants is an important measure of the power of a link invariant.

In this note we prove that a certain class of link invariants called Khovanov-Floer theories do not detect mutation. The definition of a Khovanov-Floer theory was introduced by Baldwin, Hedden, and Lobb [1] to encompass the many link homology theories which admit spectral sequences from Khovanov homology. These include singular instanton homology, Szabó’s “geometric” link homology, and various Floer...
homologies applied to $\Sigma(-L)$. Viro’s theorem implies that the latter are all mutation-invariant, and Khovanov homology (with coefficients in $\mathbb{Z}/2\mathbb{Z}$) is itself mutation-invariant, see [12], [4], and [6]. So it’s natural to ask if the other Khovanov-Floer theories are mutation-invariant as well. All the theories in this paper have coefficients in $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

**Theorem 1.1.** Let $\mathcal{K}$ be a conic, strong Khovanov-Floer theory over the ring $\mathbb{F}[X]/(X^2)$. Let $\mathcal{D}$ and $\mathcal{D}'$ be mutants. Then $H(\mathcal{K}(\mathcal{D})) \cong H(\mathcal{K}(\mathcal{D}'))$.

In [7] we defined strong Khovanov-Floer theories and showed that Szabó homology and singular instanton homology count among them.

**Corollary 1.2.** Szabó homology and singular instanton homology are mutation-invariant.

The proof of the theorem uses only the structure of strong Khovanov-Floer theories and makes no reference to, say, instantons. The corollary was conjectured by Peter Lambert-Cole in [6] (and earlier by Seed, see below). He defined a particular class of Khovanov-Floer theories, so-called extended theories, characterized by the fact that their reduced versions satisfy a Künneth theorem for connected sums.

**Theorem** (Lambert-Cole, [6]). Extended Khovanov-Floer theories are mutation-invariant.

Lambert-Cole conjectures that Szabó homology and singular instanton homology are extended Khovanov-Floer theories. We prove Theorem 1.1 by proving that every
conic, strong Khovanov-Floer theory over $\mathbb{F}[X]/(X^2)$ is extended. The advantage of working with conic, strong Khovanov-Floer theories is that they “factor through” Bar-Natan’s formal Khovanov bracket; this is the main technical result of [7]. (See Proposition 2.3 in the next section for a precise statement.) So nearly any property of Khovanov homology which can be proved via the formal bracket is also a property of conic, strong Khovanov-Floer theories. Therefore it suffices to frame the Künneth formula for reduced Khovanov homology in terms of the formal bracket.

There are two other extant proofs that Khovanov homology is mutation-invariant. Wehrli’s was the first [12], and it applies only to component-preserving mutations. It uses some subtle structure of the formal bracket; in fact, it uses more structure than Proposition 2.3 provides. Unfortunately, we are not yet able to show that strong Khovanov-Floer theories carry all this structure. If they do, then one could upgrade Theorem 1.1 to a statement about chain homotopy types rather than homology groups. See the remark following the proof of Corollary 3.7. Bloom offered another proof for odd Khovanov homology using techniques which have been since unexplored, as best we can tell. It is interesting to consider their relevance their relevance to other Khovanov-Floer theories.

1.1. The structure of Szabó homology. Szabó homology is a link homology theory which interpolates between the combinatorial and analytic Khovanov-Floer theories. To a link diagram $\mathcal{D}$ it assigns a filtered chain complex $\text{CSz}(\mathcal{D})$ whose underlying vector space is identical to the Khovanov chain group. Its homology, $\text{Sz}(\mathcal{D})$, is a link invariant. The differential is equal to the Khovanov differential plus maps along the diagonals of the cube of the resolutions of $\mathcal{D}$. There is a spectral sequence from $\text{Kh}(\mathcal{D})$ to $\text{Sz}(\mathcal{D})$ which is formally similar to Ozsváth-Szabó’s spectral sequence from $\text{Kh}(\mathcal{D})$ to $\widehat{\text{HF}}(\Sigma(-\mathcal{D}))$. Conjecturally, this similarity is more than formal.

**Conjecture (Szabó [10], Seed [8]).** $\text{Sz}(\mathcal{D}) \cong \widehat{\text{HF}}(\Sigma(-\mathcal{D}))$, and the Ozsváth-Szabó spectral sequence agrees with the Leray spectral sequence from $\text{Kh}(\mathcal{D})$ to $\text{Sz}(\mathcal{D})$.

Seed used a computer program to provide numerical evidence for the conjecture. Drawing on these computations and the spirit of the conjecture above, he made several other conjectures about $\text{Sz}$.

**Conjecture (Seed [8]).**

1. Write $\text{CSz}(\mathcal{D}, p)$ for the Szabó homology of $\mathcal{D}$ reduced at $p$. Then $\text{CSz}(\mathcal{D}, p) \cong \text{CSz}(\mathcal{D}, p')$ for any two points $p, p'$ which are not double-points of $\mathcal{D}$.

2. (“Twin arrows”) Let $E^k$ be the Leray spectral sequence from the homological filtration on $\text{CSz}(\mathcal{D})$. Then

$$E^k(\mathcal{D}) \cong \tilde{E}^k(L)[-1] \oplus \tilde{E}^k(L)[1]$$
where $\tilde{E}^k$ denotes the spectral sequence on $\tilde{\text{CSz}}(\mathcal{D})$ and $\{\pm 1\}$ denotes a shift in the quantum grading.

3. Let $K$ be a knot. Then $E^k(K)$ is invariant under mutation for $k \geq 2$.

4. Szabó homology is isomorphic to mirror Szabó homology.

We prove the first and second of Seed’s conjectures in the course of proving Theorem 1.1 and of course the third is the main subject of this paper. We do not have anything original to say about the fourth conjecture and include it only for completeness.

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2. Strong Khovanov-Floer theories

The following definition is from [7], following [1].

Definition 2.1. A strong Khovanov-Floer theory is a rule which assigns to a link diagram $\mathcal{D}$ and some auxiliary data $A$ a filtered chain complex $\mathcal{K}(\mathcal{D}, A)$ so that

1. For any two collections of auxiliary data $A_\alpha$, $A_\beta$ there is a filtered chain homotopy equivalence

$$a^\beta_\alpha: \mathcal{K}(\mathcal{D}, A_\alpha) \to \mathcal{K}(\mathcal{D}, A_\beta)$$

so that the collection of all such complexes over all choices of auxiliary data and all the maps $a^\beta_\alpha$ is a transitive system. Write $\mathcal{K}(\mathcal{D})$ for the canonical representative (i.e. the inverse limit of the system).

2. If $\mathcal{D}$ is a crossingless diagram of the unknot, then $H(\mathcal{K}(\mathcal{D})) \cong \text{Kh}(\mathcal{D})$.

3. Let $\mathcal{D} \cup \mathcal{D}'$ be a disjoint union of diagram. Then

$$\mathcal{K}(\mathcal{D} \cup \mathcal{D}') \cong \mathcal{K}(\mathcal{D}) \otimes \mathcal{K}(\mathcal{D}')$$

4. Suppose that $\mathcal{D}'$ is obtained from $\mathcal{D}$ by a diagrammatic handle attachment. There is a function $\phi$ from the auxiliary data for $\mathcal{D}$ to the auxiliary data for $\mathcal{D}'$ and a map

$$\mathfrak{h}_{A_\alpha, \phi(A_\alpha), B}: \mathcal{K}(\mathcal{D}, A_\alpha) \to \mathcal{K}(\mathcal{D}', \phi(A_\alpha'))$$

where $B$ is some additional auxiliary data. For fixed $B$, these maps form a map of transitive systems and therefore form a map $\mathfrak{h}_\beta: \mathcal{K}(\mathcal{D}) \to \mathcal{K}(\mathcal{D}')$. For any two sets of additional data $B$ and $B'$, $\mathfrak{h}_\beta \simeq \mathfrak{h}_{\beta'}$.

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1The first conjectured property is part of Lambert-Cole’s definition of an extended Khovanov-Floer theory and therefore plays a central rule in proving the third conjecture. But Seed’s conjecture predates Lambert-Cole’s argument by six years – he likely did not know that they were related in this way!
Let $U$ be a crossingless diagram of an unknot. Then $\mathcal{K}(U)$ is a Frobenius algebra with operations given by the handle attachment maps. We say that $\mathcal{K}$ is a strong Khovanov-Floer theory over this Frobenius algebra.

If $D'$ is obtained from $D$ by a planar isotopy, then $\mathcal{K}(D)$ is filtered chain homotopy equivalent to $\mathcal{K}(D')$.

Let $D$ be the disjoint union of two diagrams $D_0$ and $D_1$. Let $\Sigma$ a diagrammatic cobordism from $D$ to $D'$. Suppose that $D'$ is the disjoint union of $D'_0$ and $D'_1$ and that $\Sigma$ is the disjoint union of $\Sigma_0$ and $\Sigma_1$, where $\Sigma_i$ is a cobordism from $\Sigma_i$ to $\Sigma'_i$. Then $\mathcal{K}(\Sigma) \simeq \mathcal{K}(\Sigma_0) \otimes \mathcal{K}(\Sigma_1)$.

Handle attachment maps with disjoint supports commute up to filtered chain homotopy. The map attached to a pair of canceling diagrammatic handle attachments is chain homotopic to the identity.

Khovanov homology, Bar-Natan homology, Heegaard Floer homology of branched double covers, singular instanton homology, and Szabó homology are strong Khovanov-Floer theories. Every strong Khovanov-Floer theory we are aware of satisfies an additional condition.

**Definition 2.2.** Let $D$ be a link diagram with crossings. Pick a crossing and write $D_0$ and $D_1$ for the zero- and one-resolution of that crossing, respectively. Let $\mathcal{K}$ be a strong Khovanov-Floer theory. $\mathcal{K}$ is conic if

$$\mathcal{K}(D) \simeq \text{cone}(h_c: \mathcal{K}(D_0) \to \mathcal{K}(D_1))$$

where $h_c$ is the diagrammatic one-handle attachment along the arc shown in Figure 2.

![Figure 2](image)

**Figure 2.** A diagrammatic one-handle attachment along the dotted green arc.

Being conic essentially means that $\mathcal{K}(D)$ can be computed from a cube of resolutions. The central technical result of [7] connects these theories to Bar-Natan’s formal Khovanov bracket $[-]$, see [2].

**Proposition 2.3.** Let $\mathcal{K}(-)$ be a conic, strong Khovanov-Floer theory. Let $[-]$ be Bar-Natan’s formal Khovanov bracket. $\mathcal{K}(-)$ factors through $[-]$ as a functor.
More precisely, recall that \([-\] is a functor from a category of diagrams and cobordisms to the category \(\mathcal{Mat}(\text{Cob})\). There is a functor \(F_K\) from a certain subcategory of \(\mathcal{Mat}(\text{Cob})\) to \(\text{Kom}\) so that \(K = F_K([-])\).

This is the point of Definition 2.1: theorems about the formal Khovanov bracket become theorems about every strong Khovanov-Floer theory. So to prove the conjectures, we will reframe the proofs of the equivalent facts for Khovanov homology in terms of the formal bracket.

3. The basepoint action and dotted cobordisms

By a *basepoint* of a link diagram \(D\) we mean a point which is not a double point.

**Definition 3.1.** Let \(K\) be a strong Khovanov-Floer theory over the Frobenius algebra \(R = \mathbb{F}[X]/(r(X))\) for some polynomial \(r\). Let \(U\) be a crossingless diagram of the unknot. Fix a chain homotopy equivalence \(K(U) \simeq R\). Multiplication by \(X\) induces a chain homotopy class of map

\[
\chi: K(\circ) \to K(\circ).
\]

Let \(D\) be a link diagram and let \(p \in D\) be a basepoint. Define

\[
\chi_p: K(D) \to K(D)
\]

as the composition

\[
K(D) \xrightarrow{z} K(D) \otimes K(\circ) \xrightarrow{\text{Id} \otimes \chi} K(D) \otimes K(\circ) \xrightarrow{\text{Id} \otimes h_\gamma} K(D)
\]

The first map is a zero-handle attachment near \(p\). The third map is a one-handle attachment along an embedded arc \(\gamma\) which connects the new component to \(p\).

**Lemma 3.2.** Let \(p \in D\) be a basepoint. The map \(X_p\) defines an action of \(R\) on \(K(D)\).

**Proof.** It suffices to show that \(r(\chi_p) \simeq 0\). Let \(p \in U\). By Condition 5 of Definition 2.1 the map \(r(\chi_p)\) induces \(r(\chi)\) on \(R\) via the chain homotopy equivalence \(K(U) \simeq R\). Therefore \(r(\chi_p)\) is null-homotopic on \(K(\circ)\).

Observe that \(\chi_p \circ \chi_p\) is chain homotopic to the following composition:

\[
K(D) \xrightarrow{z} K(D) \otimes K(U) \xrightarrow{z} K(D) \otimes K(U) \otimes K(U) \xrightarrow{\text{Id} \otimes X \otimes X} K(D) \otimes K(U) \otimes K(U) \xrightarrow{\text{Id} \otimes h_\gamma'} K(D) \otimes K(U) \xrightarrow{\text{Id} \otimes h_\gamma} K(D)
\]
Here $\gamma'$ is an embedded arc connecting the two copies of $U$. This follows from writing down the definition of $\chi_p \circ \chi_p$ and swapping the order of the two one-handles. In general, $r(\chi_p)$ is chain homotopic to a map

$$K(D) \xrightarrow{Id \otimes Z} K(D) \otimes K(\circ) \xrightarrow{Id \otimes r(\chi)} K(D) \otimes K(\circ) \xrightarrow{b_{\gamma}} K(D)$$

where $\gamma$ connects the new component to $p$. So $r(X) \simeq 0$ implies that $r(X_p) \simeq 0$. □

This is not necessarily the Khovanov basepoint action, even if $K(D)$ can be arranged to have the same rank as $CKh(D)$. It is a fun exercise to characterize the basepoint action on $CSz(D)$ in terms of Szabó’s decorations and configurations. (See the proof of Corollary 3.7 for the answer.)

3.1. Dotted cobordisms. Over $\mathbb{F}$, the map $Id \otimes X$ in Definition 3.1 does not have an obvious cobordism-theoretic interpretation. In $\mathbb{R}$, multiplication by $X$ is represented by dots on a cobordism. In this section we check carefully that the dotted cobordism regime makes sense for strong Khovanov-Floer theories.

Using the results of [7], it suffices to define dotted cobordisms embedded in $\mathbb{R}^2 \times I$. A dotted zero-handle attachment is a zero-handle attachment with a dot at the critical point. To this we assign the map

$$K(D) \xrightarrow{z} K(D) \otimes K(\circ) \xrightarrow{Id \otimes X} K(D) \otimes K(\circ) \xrightarrow{b_{\gamma}} K(D \cup \circ)$$

A dotted two-handle attachment is two-handle attachment with a dot at the critical point. To it we assign the dual of the zero-handle map. Let $W: D \to D'$ be a cobordism without critical points. Put a dot at the point $p$ on $W$. This is shorthand for a dotted zero-handle attachment near $p$ and a one-handle connecting the new component and $p$. We call this a generic dot.

Lemma 3.3. Let $W: L \to L'$ be a link cobordism in $\mathbb{R}^2 \times I$ in Morse position. Let $p \in W$ be an interior point which is not the critical point of a one-handle attachment. Let $q$ be another such point on the same component of $W$. Write $W_p$ and $W_q$ for the cobordisms with dots at $p$ and $q$. Then $K(W_p) \simeq K(W_q)$.

Proof. Connect the dots: as long as $p$ and $q$ lie on the same component of $W$, there is a path between them through generic points except possibly at $p$ and $q$. It therefore suffices to show that $K(W_p) \simeq K(W_q)$ in the cases that

- $p$ and $q$ are generic and there is a path between them which does not pass a critical level.
- $p$ and $q$ are generic and the path between them passes through a single critical level.

$^2$Over a ring without 2-torsion, multiplication by $X$ can be represented as attaching a tube.
• \( p \) is the critical point of a zero- or two-handle attachment, \( q \) is generic, and there is a path between them which does not pass a critical level.

The first point is clear. Suppose that \( p \) and \( q \) are separated by an index one critical point so that \( p \) is above it. Replace the generic dot at \( p \) with a dotted zero-handle and a canceling one-handle. Slide the one-handle past the index one critical point.

Now slide the dotted two-handle past the index one critical point. These moves do not affect the chain homotopy type of \( \mathcal{K}(W_p) \) because of conditions 7 and 8 of Definition 2.1. So by the first point, \( \mathcal{K}(W_p) \simeq \mathcal{K}(W_q) \).

Suppose that \( p \) lies at the critical point of a zero-handle attachment and that \( q \) is a generic dot which lies just next to it. Then \( \mathcal{K}(W_p) \simeq \mathcal{K}(W_q) \) using Condition 8 of Definition 2.1. A similar argument applies to two-handle attachments.

The arguments of [7] imply the following Proposition.

**Proposition 3.4.** Let \( \text{Cob}_\bullet \) be the category whose objects are planar diagrams and whose morphisms are dotted and undotted cobordisms in \( \mathbb{R}^2 \times I \). There is a dotted Khovanov bracket \([\cdot]_\cdot\) defined in [2].

Every conic strong Khovanov-Floer theory factors through \([\cdot]_\cdot\).

The only thing to check is that dotted handles slide past each other. This is an implication of condition 7 of 2.1. From Lemma 3.3 and the 4Tu relation we get the dotted neck-cutting relation of Figure 3 and [2].

![Figure 3. The dotted neck-cutting relation.](image)

### 3.2. Reduced theories.

**Definition 3.5.** Let \( \mathcal{K} \) be a strong Khovanov-Floer theory over \( \mathbb{F}[X]/(X^2) \). The complex \( \text{Im}(X_p) \) is called \( \mathcal{K} \) reduced at \( p \) or the reduced version of \( \mathcal{K} \). We write often use the notation \( \tilde{\mathcal{K}}_p \) or just \( \tilde{\mathcal{K}} \).

Now we can prove Seed’s first two conjectures.
Proposition 3.6. Let $\mathcal{K}$ be a conic, strong Khovanov-Floer theory over $\mathbb{F}[X]/(X^2)$. Let $p$ and $q$ be two basepoints on $\mathcal{D}$. Then $\tilde{\mathcal{K}}_p(\mathcal{D}) \simeq \tilde{\mathcal{K}}_q(\mathcal{D})$ and $$\mathcal{K}(\mathcal{D}) \cong \tilde{\mathcal{K}}(\mathcal{D}) \oplus \tilde{\mathcal{K}}(\mathcal{D}).$$

Proof. First we define a map $$N : \mathcal{D} \to \mathcal{D}.$$ If $\mathcal{D}$ is a crossingless unknot, then define $N$ to be the composition of a two-handle attachment with a zero-handle attachment, see Figure 4. Extend $N$ by the Liebniz rule to crossingless unlink; for a crossingless diagram with $r$ components, $N$ is the sum of $r$ cobordisms which consist of $r - 1$ cylinders and one cobordism like the one in Figure 4. Now suppose that $N$ is defined for links with up to $k$ crossings and let $\mathcal{D}$ be a link diagram with $k + 1$ crossings. Conicity implies that

$$[\mathcal{D}] = \text{cone}(h_{k+1} : [\mathcal{D}_0] \to [\mathcal{D}_1]).$$

$N$ is defined on $[\mathcal{D}_0]$ and $[\mathcal{D}_1]$ by hypothesis. It defines a chain map on $[\mathcal{D}]$ if it commutes with $h_{k+1}$. Figure 5 shows that it does. The diagram describes four different cobordisms. Label the circles 1 to 4 from left to right. For $i,j$ distinct integers between 1 and 4, write $N_{ij}$ for the cobordism given by attaching a tube between circle $i$ and circle $j$. The 4Tu relation states that

$$N_{12} + N_{23} + N_{34} + N_{41} = 0,$$

taking the cobordisms to be morphisms in the linear cobordism category $\text{Mat}(\text{Cob})$. Observe that $N_{12} + N_{34}$ is a summand of $N \circ h_{k+1}$, that $N_{41}$ is a summand of $h_{k+1} \circ N$, and that $N_{23} = 0$. The remaining summands cancel in pairs because the handle attachments belonging to $N$ have support disjoint from the support of $h_{k+1}$. Therefore $N$ and $h_{k+1}$ commute.
Figure 6 shows that
\[ \chi_p N \chi_q N \chi_p = \chi_p \]
\[ \chi_q N \chi_p N \chi_q = \chi_q. \]

Write \( \nu = \mathcal{K}(N) \) following [9] (or [3] for a very clear exposition). It follows that
\[ X_p \nu X_q \nu X_p \simeq X_p, \]
\[ X_q \nu X_p \nu X_q \simeq X_q. \]

\( X_q \nu \) restricts to a map \( \text{Im}(X_p) \to \text{Im}(X_q) \). The equations above show that this restriction has chain homotopy inverse \( X_p \nu \), and therefore \( \tilde{K}_p(D) \simeq \tilde{K}_q(D) \).

We have shown that the short exact sequence
\[ 0 \to \tilde{K}_p(D) \to \tilde{K}(D) \xrightarrow{X_p} \tilde{K}(D) \to 0 \]
splits with section \( \nu \). The second statement follows. □

**Corollary 3.7.** \( \tilde{Sz}(D, p) \cong \tilde{Sz}(D, p') \) for any two basepoints \( p, p' \) of \( D \).

**Proof.** This follows immediately from the previous proposition as long as reduced Szabó homology as defined by Szabó agrees with the reduced strong Khovanov-Floer theory given by Szabó homology. Szabó defines \( \tilde{CSz}(D, p) \) as the subcomplex of
Figure 6. The calculation that $\chi_p N\chi_q N\chi_p = \chi_p$. It would be more precise to replace the circles labeled by $p$ and $q$ with arcs as in Figure 5 – one-handle attachments between those circles may represent either merges or splits.

$\text{CSz}(D)$ in which every circle containing $p$ is labeled with $v_-$. So we will show that $\tilde{\text{CSz}}(D, p) = \text{Im}(X_p)$. 
The action of $X_p$ on $\text{CSz}(D)$ can be written

$$X_p = X_{p,0} + X_{p,1} + \cdots + X_{p,C}$$

where $C$ is the number of crossings in $D$. Here $X_{p,k}$ is homological degree $k$ part of $X_p$; informally, it “counts” the $k$-dimensional configurations counted in $\partial$ with the addition of the little circle near $p$ and the decoration $\gamma_p$. All of these configurations must be of type E, and the central circle must be the one which contains $p$.

The filtration rule of [10] implies that if $x \in \text{Im}(X_p)$, then $x$ is a sum of canonical generators in which the circle with $p$ is labeled by $v^-$.

Therefore $\text{CSz}(D, p) \subset \text{Im}(X_p)$. □

**Remark 3.8.** Let $p$ and $q$ be basepoints on opposite sides of a crossing $c$ of $D$. In the formal Khovanov bracket it holds that $X_p \simeq X_q$. The chain homotopy is given by the “backwards map” $h_c : [D_1] \to [D_0]$, the one-handle attachment dual to $h_c$.

Proving this statement for strong Khovanov-Floer theories would allow us to adapt Wehrli’s proof of mutation-invariance.

Seed’s second conjecture follows from the fact that the Leray spectral sequence for a direct sum of complexes is isomorphic to the direct sum of the individual spectral sequences and from the fact that $\nu$ has quantum grading 2.

## 4. Connected sums and mutation

**Proposition 4.1.** Let $D$ and $D'$ be link diagrams and write $D \# D'$ for their connected sum. Let $K$ be a conic, strong Khovanov-Floer theory over $\mathbb{F}[X]/(X^2)$. Then

$$\tilde{K}(D) \otimes_{\mathbb{F}} \tilde{K}(D') \simeq \tilde{K}(D \# D').$$
Fix points $p \in \mathcal{D}$ and $q \in \mathcal{D}'$ so that there is an embedded arc $\gamma$ from $p$ to $q$. Write $\nu$ and $\nu'$ for the $\nu$ maps on $K(\mathcal{D})$ and $K(\mathcal{D}')$. One isomorphism is
$$h_\gamma \circ (\text{Id} \otimes \nu').$$

It is chain homotopic to the isomorphism
$$h_\gamma \circ (\nu \otimes \text{Id}).$$

Proof. See Figures 7, 8, and 9. \hfill \Box

![Figure 7](image)

FIGURE 7. The interesting part of a cobordism which represents $h_\gamma \circ (\text{Id} \otimes \nu')$. Of course $\nu$ is a sum of many cobordisms, but the rest are each equal to zero because they contain a surface with two dots.

The following definition and result are due to Lambert-Cole in [6].

**Definition 4.2.** An extended Khovanov-Floer theory is a pair of a Khovanov-Floer theory and a reduced Khovanov-Floer theory $(A, \tilde{A})$ so that

1. $A(L) = \tilde{A}(L \cup \circ, p)$.
2. $A$ and $\tilde{A}$ satisfy unoriented skein exact triangles.
3. $\tilde{A}(L, p) \cong \tilde{A}(L, q)$.

**Theorem ([6]).** Extended Khovanov-Floer theories are invariant under Conway mutation.
Now we can prove Theorem 1.1.

**Proof of Theorem 1.1**  In [7] we constructed a Khovanov-Floer theory for every strong Khovanov-Floer theory – use the Leray spectral sequence given by the homological filtration on $\mathcal{K}(\mathcal{D})$. For a strong Khovanov-Floer theory $\mathcal{K}$ over $\mathbb{F}[X]/(X^2)$, this Khovanov-Floer $\mathcal{A}$ theory agrees with the one defined in [1].

Every conic, strong Khovanov-Floer theory and its reduced version form an extended Khovanov-Floer theory under this construction. The first requirement of
Definition 4.2 is straightforward. The second is equivalent to conicity. We have shown that $\tilde{K}(D, p) \cong \tilde{K}(D, q)$. It follows that $\tilde{A}(D, p) \cong \tilde{A}(D, q)$.

Our proof relies on Lambert-Cole’s theorem, but in fact Lambert-Cole’s proof can be adapted to any strong Khovanov-Floer theory. In other words, the proof need not use any spectral sequence techniques.

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