THE WEIGHT COMPLEX FUNCTOR IS SYMMETRIC MONOIDAL

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ABSTRACT. Bondarko’s (strong) weight complex functor is a triangulated functor from Voevodsky’s triangulated category of motives to the homotopy category of chain complexes of classical Chow motives. Its construction is valid for any dg enhanced triangulated category equipped with a weight structure. In this paper we consider weight complex functors in the setting of stable symmetric monoidal ∞-categories. We prove that the weight complex functor is symmetric monoidal under a natural compatibility assumption. To prove this result, we develop additive and stable symmetric monoidal variants of the ∞-categorical Yoneda embedding, which may be of independent interest.

1. Introduction

In the paper [3], Bondarko introduced the notion of a weight structure (see Definition 3.1) on a triangulated category as a variant of t-structure and then constructed a (strong) weight complex functor when the triangulated category has a dg enhancement. One primary example of a weight structure is the motivic weight structure on $\text{DM}^{(\text{eff})}(\kappa; \mathbb{Q})$, whose existence was proven in [3, Section 6]. The weight complex functor associated to that is a functor $\text{DM}^{(\text{eff})}(\kappa; \mathbb{Q}) \to K^b(\text{Chow}^{(\text{eff})}(\kappa; \mathbb{Q}))$, which was studied in [2, Section 6]. In a recent preprint [10], Sosnilo considered weight structures using stable ∞-categories and showed that the weight complex can be constructed in that setting. In this paper we consider them with symmetric monoidal structures.

Our result mentioned in the title is Corollary 4.5, which states that the weight complex functor is symmetric monoidal under a natural compatibility condition. We note that its dg variant appeared in [1, Lemma 20], but the author was unable to fill in the details of the proof presented there.

Applying this to the motivic weight structure, we have the following:

Theorem. Let $\kappa$ be a perfect field. Then the weight complex functor $\text{DM}^{(\text{eff})}(\kappa; \mathbb{Q}) \to K^b(\text{Chow}^{(\text{eff})}(\kappa; \mathbb{Q}))$ is symmetric monoidal.

We note that this theorem is mentioned in [6, Remark 37] as a desired statement which seems to have no written proof.

Outline. We begin in Section 2 by recalling some ∞-category theory which we will need in this paper. There we give the definition of an additive symmetric monoidal ∞-category and prove a version of the Yoneda embedding for it. In Section 3 we review the theory of weight structures on stable ∞-categories and define the weight complex functor. Section 4 is the main part of this paper. There we introduce...
a notion of compatibility between a symmetric monoidal structure and a weight structure and prove the main result.

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2. Preliminaries from $\infty$-category theory

When we deal with $\infty$-categories, we generally follow terminologies and notations used in [7, 8], but we will regard every category as an $\infty$-category by taking its nerve.

2.1. Additive and stable $\infty$-categories. We refer readers to [8, Chapter 1] for the theory of stable $\infty$-categories and [4] for the theory of additive $\infty$-categories.

Definition 2.1. Let $A$ be an $\infty$-category. We call $A$ additive if it has finite products and coproducts and its homotopy category is an additive category.

Let $A$ and $A'$ be additive $\infty$-categories and $f: A \to A'$ a functor between them. We call $f$ additive if it preserves finite products (or coproducts, equivalently). We write $\text{Fun}^{\text{add}}(A, A')$ for the full subcategory of $\text{Fun}(A, A')$ spanned by additive functors.

Example 2.2. Every stable $\infty$-category is additive. More generally, a full subcategory of a stable $\infty$-category closed under finite (co)products is additive.

Let $\text{Cat}^{\text{add}}_\infty$ denote the subcategory of the large $\infty$-category of small $\infty$-categories $\text{Cat}_\infty$ whose objects are additive $\infty$-categories and morphisms are additive functors. Similarly, we let $\text{Cat}^{\text{ex}}_\infty$ denote the subcategory of $\text{Cat}_\infty$ whose objects are stable $\infty$-categories and morphisms are exact functors.

We recall a stable version of the Yoneda embedding. Here $S$ and $\text{Sp}$ denote the large $\infty$-categories of spaces and spectra, respectively.

Definition 2.3. Let $C$ be a stable $\infty$-category. We write $\mathcal{P}^{\text{ex}}(C)$ for the $\infty$-category $\text{Fun}^{\text{ex}}(C^{\text{op}}, \text{Sp})$, which is equivalent to the full subcategory of $\mathcal{P}(C) = \text{Fun}(C^{\text{op}}, S)$ spanned by left exact functors. In this case the Yoneda embedding $C \to \mathcal{P}(C)$ factors through $\mathcal{P}^{\text{ex}}(C)$. We call the functor $C \to \mathcal{P}^{\text{ex}}(C)$ the stable Yoneda embedding.

There is also an additive version of the Yoneda embedding. Let $\text{Sp}_{\geq 0}$ denote the large $\infty$-category of connective spectra. The $\infty$-category $\text{Sp}_{\geq 0}$ is a full subcategory of the stable $\infty$-category $\text{Sp}$ closed under coproducts, hence additive.

Definition 2.4. Let $A$ be an additive $\infty$-category. We write $\mathcal{P}^{\text{add}}(A)$ for the $\infty$-category $\text{Fun}^{\text{add}}(A^{\text{op}}, \text{Sp}_{\geq 0})$, which is equivalent to the full subcategory of $\mathcal{P}(A)$ spanned by functors which preserve finite products (see for example [4, Corollary 2.10 (iii) and Example 5.3 (ii)]). In this case the Yoneda embedding $A \to \mathcal{P}(A)$ factors through $\mathcal{P}^{\text{add}}(A)$. We call the functor $A \to \mathcal{P}^{\text{add}}(A)$ the additive Yoneda embedding.
Applying \cite[Lemma 5.5.4.18–19]{7}, we immediately see that \( \mathcal{P}^{\text{add}}(\mathcal{A}) \) can be regarded as a strongly reflective subcategory of \( \mathcal{P}(\mathcal{A}) = \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{S}) \), hence is presentable. Similarly, \( \mathcal{P}^{\text{ex}}(\mathcal{C}) \) can be regarded as a strongly reflective subcategory of \( \mathcal{P}(\mathcal{C}) \) for a stable \( \infty \)-category \( \mathcal{C} \).

**Remark 2.5.** In fact, these constructions are special cases of \cite[Definition 5.3.6.5]{7}. Let \( \mathcal{K}, \mathcal{K}^{\text{ex}} \) and \( \mathcal{K}^{\text{add}} \) denote the classes of all simplicial sets which are small, finite and finite discrete, respectively. Comparing the constructions given here with that in the proof of \cite[Proposition 5.3.6.2]{7}, we can see that \( \mathcal{P}^{\text{ex}} \) and \( \mathcal{P}^{\text{add}} \) are the restrictions of \( \mathcal{P}^{\mathcal{K}^{\text{ex}}}_{\mathcal{K}^{\text{ex}}} \) and \( \mathcal{P}^{\mathcal{K}^{\text{add}}}_{\mathcal{K}^{\text{add}}} \), respectively. As a consequence, we can view them as functors:

\[
\begin{align*}
\mathcal{P}^{\text{ex}} : & \text{Cat}_{\infty}^{\text{ex}} \to \mathcal{P}_{\mathcal{K}^{\text{ex}}}^{\text{L-ex}}, \\
\mathcal{P}^{\text{add}} : & \text{Cat}_{\infty}^{\text{add}} \to \mathcal{P}_{\mathcal{K}^{\text{add}}}^{\text{L-add}}.
\end{align*}
\]

Here \( \mathcal{P}_{\mathcal{K}^{\text{ex}}}^{\text{L-ex}} \) and \( \mathcal{P}_{\mathcal{K}^{\text{add}}}^{\text{L-add}} \) denote full subcategories of the very large \( \infty \)-category \( \mathcal{P}_{\mathcal{K}}^{\text{L}} \) (see \cite[Definition 5.5.3.1]{7} for the definition) spanned by exact and additive presentable \( \infty \)-categories, respectively.

**Lemma 2.6.** Let \( \mathcal{A} \) be an additive \( \infty \)-category. Then the exact functor \( \text{Fun}^{\text{add}}(\mathcal{A}^{\text{op}}, \mathcal{S}) \to \mathcal{S} \) induced by the truncation functor \( \text{Fun}^{\text{add}}(\mathcal{A}^{\text{op}}, \mathcal{S}) \to \text{Fun}^{\text{add}}(\mathcal{A}^{\text{op}}, \mathcal{S}_{\geq 0}) = \mathcal{P}^{\text{add}}(\mathcal{A}) \) is an equivalence of stable \( \infty \)-categories.

**Proof.** Since the full subcategory \( \mathcal{P}^{\text{add}}(\mathcal{A}) \subset \text{Fun}(\mathcal{A}^{\text{op}}, \mathcal{S}_{\geq 0}) \) is closed under limits, under the equivalence \( \text{Fun}(\mathcal{S}_{\infty}^{\text{fin}}, \text{Fun}(\mathcal{A}, \mathcal{S}_{\geq 0})) \simeq \text{Fun}(\mathcal{S}_{\infty}^{\text{fin}} \times \mathcal{A}^{\text{op}}, \mathcal{S}_{\geq 0}) \) we can regard the \( \infty \)-category \( \mathcal{S}^{\text{add}}(\mathcal{A}) \) as the full subcategory of the \( \infty \)-category \( \text{Fun}(\mathcal{S}_{\infty}^{\text{fin}} \times \mathcal{A}^{\text{op}}, \mathcal{S}_{\geq 0}) \) consisting of functors \( f : \mathcal{S}_{\infty}^{\text{fin}} \times \mathcal{A}^{\text{op}} \to \mathcal{S}_{\geq 0} \) which are reduced and excisive in the first variable and additive in the second variable. Moreover, since the truncation functor \( \tau_{\geq 0} : \mathcal{S} \to \mathcal{S}_{\geq 0} \) induces a equivalence \( \mathcal{S} \simeq \mathcal{S}_{\geq 0} \), this \( \infty \)-category is equivalent to the \( \infty \)-category \( \mathcal{S}^{\text{add}}(\mathcal{A}^{\text{op}}, \mathcal{S}) \) under the equivalence \( \text{Fun}(\mathcal{S}_{\infty}^{\text{fin}} \times \mathcal{A}^{\text{op}}, \mathcal{S}_{\geq 0}) \simeq \text{Fun}(\mathcal{A}^{\text{op}}, \text{Fun}(\mathcal{S}_{\infty}^{\text{fin}}, \mathcal{S}_{\geq 0})) \). \( \square \)

2.2. **Symmetric monoidal structure.** For the basic theory of symmetric monoidal \( \infty \)-categories, we refer readers to \cite{8}.

**Definition 2.7.** We call a symmetric monoidal \( \infty \)-category \( \mathcal{A}^{\otimes} \) **additive symmetric monoidal** if the underlying \( \infty \)-category \( \mathcal{A} \) is additive and tensor product operations are additive in each variable.

We call a symmetric monoidal \( \infty \)-category \( \mathcal{C}^{\otimes} \) **stable symmetric monoidal** if the underlying \( \infty \)-category \( \mathcal{C} \) is stable and tensor product operations are exact in each variable.

We present additive and stable symmetric monoidal versions of the (\( \infty \)-categorical) Yoneda embedding, which might be of independent interest. We note that these “additive and stable monoidal Yoneda embeddings” are different from what Nikolaus called by the same name in \cite[Section 6]{9}.

**Proposition 2.8.** Let \( \mathcal{A}^{\otimes} \) and \( \mathcal{C}^{\otimes} \) be additive and stable symmetric monoidal \( \infty \)-categories respectively. Then the following hold:

1. The additive \( \infty \)-category \( \mathcal{P}^{\text{add}}(\mathcal{A}) \) admits an additive symmetric monoidal structure whose tensor product operations preserve colimits in each variable. Moreover, there exists a symmetric monoidal functor \( \mathcal{A}^{\otimes} \to \mathcal{P}^{\text{add}}(\mathcal{A})^{\otimes} \) whose underlying functor is the additive Yoneda embedding.
(2) The stable ∞-category \( \mathcal{P}^{ex}(C) \) admits a stable symmetric monoidal structure whose tensor product operations preserve colimits in each variable. Moreover, there exists a symmetric monoidal functor \( C^\otimes \to \mathcal{P}^{ex}(C)^\otimes \) whose underlying functor is the stable Yoneda embedding.

Proof. By Remark 2.5 and [8, Remark 4.8.1.9], this is a corollary of [8, Proposition 4.8.1.10].

Combining this proof with [8, Proposition 4.8.1.5] and [4, Corollary 5.5 (ii)], we get the following counterpart for the construction of Lemma 2.6.

Corollary 2.9. Let \( A^\otimes \) be an additive symmetric monoidal ∞-category. Then the stable ∞-category \( \text{Sp}(\mathcal{P}^{\text{add}}(A)) \) admits a stable symmetric monoidal structure whose tensor product operations preserve colimits in each variable and there exists a symmetric monoidal refinement of the composition \( A \to \mathcal{P}^{\text{add}}(A) \to \text{Sp}(\mathcal{P}^{\text{add}}(A)) \).

By construction, these three can be seen as functors between appropriate ∞-categories, but we only need their 1-categorical functoriality in this paper.

Remark 2.10. Let \( A^\otimes \) be an additive symmetric monoidal ∞-category. By mimicking the proof of [9] Proposition 4.9, we obtain a symmetric monoidal structure on \( \mathcal{P}^{\text{add}}(A) \) concretely as a localization of the symmetric monoidal structure on \( \mathcal{P}(A) \) of [5, Section 3]. Since the symmetric monoidal structure on \( \mathcal{P}^{\text{add}}(A) \) of Proposition 2.8 can be characterized by the property that the additive Yoneda embedding is symmetric monoidal and the tensor product operations preserve colimits in each variable, these two constructions coincide.

Similarly, the symmetric monoidal structure on \( \text{Sp}(\mathcal{P}^{\text{add}}(A)) \) can be obtained by localizing that on \( \text{Fun}(A^{\text{op}}, \text{Sp}) \). In particular, for a stable symmetric monoidal ∞-category \( C \), the functor \( \text{Sp}(\mathcal{P}^{\text{add}}(C)) \cong \text{Fun}^{\text{add}}(C^{\text{op}}, \text{Sp}) \to \text{Fun}^{\text{ex}}(C^{\text{op}}, \text{Sp}) = \mathcal{P}^{\text{ex}}(C) \) has a symmetric monoidal refinement.

3. Weight structures

3.1. Basic definitions and properties. We present the basic theory of weight structures here. We refer readers to [3] for a detailed study of weight structures.

Definition 3.1. Let \( D \) be a triangulated category. A weight structure on \( D \) is a pair of full subcategories \( (D_{w\geq 0}, D_{w\leq 0}) \) satisfying the following conditions:

1. \( D_{w\geq 0} \) and \( D_{w\leq 0} \) are closed under retracts in \( D \). In particular, they are closed under isomorphism.
2. We have inclusions \( D_{w\geq 0}[1] \subset D_{w\geq 0} \) and \( D_{w\leq 0}[-1] \subset D_{w\leq 0} \).
3. For \( X \in D_{w\leq 0} \) and \( Y \in D_{w\geq 0}[1] \), we have \( \text{Hom}_D(X,Y) = 0 \).
4. For any \( Z \in D \), there exists a distinguished triangle \( X \to Z \to Y \) where \( X \in D_{w\leq 0} \) and \( Y \in D_{w\geq 0}[1] \).

If \( D \) is equipped with a weight structure, we will write \( D_{w\geq n} \) and \( D_{w\leq n} \) for \( D_{w\geq 0}[n] \) and \( D_{w\leq 0}[n] \), respectively.

Definition 3.2. Let \( C \) be a stable ∞-category. A weight structure on \( C \) is a weight structure on the homotopy category \( hC \). When \( C \) is equipped with a weight structure, we will write \( C_{w\geq n} \) and \( C_{w\leq n} \) for the full subcategories of \( C \) determined by \( hC_{w\geq n} \) and \( hC_{w\leq n} \), respectively.

Definition 3.3. Let \( C \) be a stable ∞-category equipped with a weight structure.
(1) The heart $C^w_w$ of the weight structure is the full subcategory $C_{w \geq 0} \cap C_{w \leq 0} \subset C$.

(2) We denote the full subcategory of $C$ consisting of objects $X$ satisfying $X \in C_{w \geq m} \cap C_{w \leq n}$ for some $m, n$ by $C^b$. The weight structure on $C$ is called **bounded** if the equality $C^b = C$ holds.

**Remark 3.4.** Unlike the case of $t$-structures, the heart of a weight structure is generally not equivalent to an ordinary category, i.e., the mapping spaces may not be (homotopy) discrete.

**Example 3.5.** Let $B$ be an additive category. Then the stable $\infty$-category $K^b(B) = N_{dg}(Ch^b(B))$ has a canonical weight structure, where $N_{dg}$ denotes the dg nerve construction given in [8, Construction 1.3.1.6]. In this case, the weight structure is bounded by definition. Its heart is the essential image under the canonical embedding $B \to K^b(B)$.

We present some basic facts concerning weight structures here.

**Lemma 3.6.** Let $C$ be a stable $\infty$-category equipped with a weight structure. Then the following hold:

(1) The heart $C^w_w$ is an additive subcategory of $C$.

(2) The full subcategory $C^b$ is closed under finite limits and colimits. In particular, $C^b$ is a stable $\infty$-category. Moreover, the pair $(C^b \cap C_{w \geq 0}, C^b \cap C_{w \leq 0})$ defines a weight structure on $C^b$, whose heart coincides with that of $C$.

(3) If the weight structure is bounded, the $\infty$-category $C$ is generated by the heart $C^w_w$ under finite limits and colimits.

**Proof.** Part (1) is immediate from the definition. Parts (2) and (3) are $\infty$-categorical reformulations of [3, Proposition 1.3.6] and [3, Corollary 1.5.7], respectively. □

**3.2. Weight complex.** Before stating Sosnilo’s result, which we use to define the weight complex functor, we give a definition of a morphism between stable $\infty$-categories equipped with weight structures.

**Definition 3.7.** Let $C$ and $C'$ be stable $\infty$-categories equipped with weight structures. A functor $f: C \to C'$ is called **weight exact** if it is exact and carries $C_{w \geq 0}$ and $C_{w \leq 0}$ into $C'_{w \geq 0}$ and $C'_{w \leq 0}$, respectively.

The following result is due to Sosnilo:

**Proposition 3.8** (Sosnilo). Let $C$ be a stable $\infty$-category equipped with a bounded weight structure. Then the following hold:

(1) The composition

$$C \to \mathcal{P}^{ex}(C) = \text{Fun}^{ex}(C^{\text{op}}, \text{Sp}) \to \text{Fun}^{\text{add}}((C^w)^{\text{op}}, \text{Sp})$$

is exact and fully faithful. Here the second arrow is given by restriction.

(2) Let $C'$ be a stable $\infty$-category equipped with a weight structure. Then the restriction functor $\text{Fun}^{w-\text{ex}}(C, C') \to \text{Fun}^{\text{add}}((C^w)^{\text{op}}, C'^{\text{op}})$ is an equivalence of $\infty$-categories.

**Proof.** See [10, Proposition 3.3]. □
Let $\mathcal{C}$ be a stable $\infty$-category equipped with a bounded weight structure. Combining part (1) of Proposition 3.8 and part (3) of Lemma 3.6, we can identify $\mathcal{C}$ with the full subcategory $\text{Sp}(P^{\text{add}}(\mathcal{C}^w))$ generated by the essential image of the embedding $\mathcal{C}^w \to \text{Sp}(P^{\text{add}}(\mathcal{C}^w))$ under finite limits and colimits. We denote this subcategory by $\text{Sp}(P^{\text{add}}(\mathcal{C}^w))_{\text{fin}}$; beware that this $\infty$-category is not determined by $\text{Sp}(P^{\text{add}}(\mathcal{C}^w))$ itself but by the additive $\infty$-category $\mathcal{C}^w$. Under the hypothesis of part (2) of Proposition 3.8, we can describe the weight exact functor $g: \mathcal{C} \to \mathcal{C}'$ which corresponds to an additive functor $f: \mathcal{C}^w \to \mathcal{C}'^w$ by the equivalence under this identification: The functor $g$ can be regarded as the restriction of the functor $\text{Sp}(P^{\text{add}}(\mathcal{C}^w)) \to \text{Sp}(P^{\text{add}}(\mathcal{C}'^w))$ determined by $f$ to the full subcategory $\text{Sp}(P^{\text{add}}(\mathcal{C}^w))_{\text{fin}}$.

Now we define the weight complex functor as follows:

**Definition 3.9.** Let $\mathcal{C}$ be a stable $\infty$-category equipped with a bounded weight structure. The weight complex functor is the weight exact functor $\mathcal{C} \to K^b(h\mathcal{C}^w)$ which is mapped to (an additive functor equivalent to) the additive functor $\mathcal{C}^w \to h\mathcal{C}^w$ under the equivalence $\text{Fun}^{w-ex}(\mathcal{C}, K^b(h\mathcal{C}^w)) \to \text{Fun}^{add}(\mathcal{C}^w, h\mathcal{C}^w)$ of part (2) of Proposition 3.8.

**Remark 3.10.** As shown in [10, Corollary 3.5], the functor of Definition 3.9 is an $\infty$-categorical enhancement of the functor constructed in [3] under the name “strong weight complex functor”.

### 4. Main theorem

To state our main result, we need a notion of compatibility between a stable symmetric monoidal structure and a weight structure.

**Definition 4.1.** Let $\mathcal{C}^\otimes$ be a stable symmetric monoidal $\infty$-category. We call a weight structure $(\mathcal{C}^w_{\geq 0}, \mathcal{C}^w_{\leq 0})$ on the underlying $\infty$-category $\mathcal{C}$ compatible with the symmetric monoidal structure if $\mathcal{C}^w_{\geq 0}$ and $\mathcal{C}^w_{\leq 0}$ are closed under tensor product operations.

By [8, Proposition 2.2.1.1], if the underlying $\infty$-category of a stable symmetric monoidal $\infty$-category $\mathcal{C}^\otimes$ has a compatible weight structure, the symmetric monoidal structure on $\mathcal{C}^\otimes$ can be restricted to its full subcategories $\mathcal{C}^w_{\geq 0}$, $\mathcal{C}^w_{\leq 0}$, and $\mathcal{C}^w$. We let $(\mathcal{C}^w)^\otimes$ denote the restriction to the heart.

**Lemma 4.2.** Let $\mathcal{C}^\otimes$ be a stable symmetric monoidal $\infty$-category whose underlying $\infty$-category is equipped with a bounded compatible weight structure. Then the fully faithful functor $\mathcal{C} \to \text{Sp}(P^{\text{add}}(\mathcal{C}^w))$ given in part (1) of Proposition 3.8 admits a canonical symmetric monoidal refinement.

**Proof.** We consider the following diagram of symmetric monoidal $\infty$-categories commutative up to homotopy:

$$
\begin{array}{ccc}
(C^w)^\otimes & \xrightarrow{i^\otimes} & C^\otimes \\
\downarrow{j^\otimes} & & \downarrow{j^\otimes'} \\
\text{Sp}(P^{\text{add}}(C^w)) & \xrightarrow{I^\otimes} & \text{Sp}(P^{\text{add}}(C))^\otimes \\
\end{array}
$$
The square is constructed from the inclusion \( i^\otimes : (\mathcal{C}_w^\otimes) \to \mathcal{C}^\otimes \) using the functoriality of the construction \( \text{Sp}(\mathcal{P}^{\text{add}}(-))^\otimes \). The functor \( j'^\otimes \) is the stable symmetric monoidal Yoneda embedding for \( \mathcal{C}^\otimes \) and the functor \( L^\otimes \) is a symmetric monoidal refinement of the reflector \( L \) (see Remark 2.10). Note that the functors \( I \) and \( L \) have right adjoints whose composition is the restriction functor \( \mathcal{P}^{\text{ex}}(\mathcal{C}) = \text{Fun}^{\text{ex}}(\mathcal{C}^{\text{op}}, \mathcal{S}p) \to \text{Fun}^{\text{add}}((\mathcal{C}_w^\otimes)^{\text{op}}, \mathcal{S}p) \simeq \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes)) \). In particular, both functors are exact.

Let \( \mathcal{C}' \) be the full subcategory of \( \mathcal{P}^{\text{ex}}(\mathcal{C}) \) generated by the essential image of the functor \( j''o i \simeq Lo I o j \) under finite limits and colimits. Since the functors \( j'' \), \( L \) and \( I \) are exact, \( \mathcal{C}' \) is equal to the essential image of \( j'' \) and also that of the restriction of \( L o I \) to the full subcategory \( \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^{\text{fin}} \). From the former equality and the symmetric monoidality of \( j''^\otimes \), we can see that \( \mathcal{C}' \) is closed under the tensor product operations in \( \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C})) \). Hence we can give a symmetric monoidal structure on \( \mathcal{C}' \) by restriction.

Since \( L^\otimes \) and \( I^\otimes \) are symmetric monoidal, the restriction of \( L^\otimes o I^\otimes \) defines a symmetric monoidal functor \( f^\otimes : (\text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^{\text{fin}})^\otimes \to \mathcal{C}'^\otimes \), where the left hand side denotes the restriction of \( \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^\otimes \) to the full subcategory. According to part 1 of Proposition 3.8, the right adjoint of \( L o I \) induces an equivalence \( \mathcal{C}' \to \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^{\text{fin}} \), so we deduce that its underlying functor \( f \) is an equivalence, which means that \( f^\otimes \) is itself an equivalence by [8, Remark 2.1.3.8]. Therefore, the composition

\[
\mathcal{C}^\otimes \xrightarrow{j''^\otimes} \mathcal{C}'^\otimes \xleftarrow{f^\otimes} (\text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^{\text{fin}})^\otimes \subset \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^\otimes
\]

is the desired functor.

Given a stable \( \infty \)-category \( \mathcal{C} \) equipped with a bounded weight structure and a symmetric monoidal structure on its heart \( \mathcal{C}_w^\otimes \), we can construct a stable symmetric monoidal structure on \( \mathcal{C} \) by restricting the symmetric monoidal structure on \( \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes)) \) to its full subcategory \( \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^{\text{fin}} \simeq \mathcal{C} \), which is closed under the tensor product operations in \( \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes)) \). In particular, we can attach a symmetric monoidal structure to the stable \( \infty \)-category \( \mathcal{K}^\otimes(\mathcal{B}) \) of Example 3.5 for an additive symmetric monoidal category \( \mathcal{B}^\otimes \). Lemma 4.2 ensures that this construction is the only way to do this, if it is required that the symmetric monoidal structure on \( \mathcal{C} \) is compatible with its weight structure and its restriction to the heart \( \mathcal{C}_w^\otimes \) coincides with the given symmetric monoidal structure.

**Theorem 4.3.** Let \( \mathcal{C}^\otimes \) and \( \mathcal{C}_w^\otimes \) be a stable symmetric monoidal \( \infty \)-categories whose underlying \( \infty \)-categories are equipped with bounded compatible weight structures. Let \( f : \mathcal{C} \to \mathcal{C}' \) be a weight exact functor whose restriction to their hearts \( \mathcal{C}_w^\otimes \to \mathcal{C}_w^\otimes \) admits a symmetric monoidal refinement \( g^\otimes : (\mathcal{C}_w^\otimes)^\otimes \to (\mathcal{C}_w^\otimes)^\otimes \). Then \( f \) admits a canonical symmetric monoidal refinement whose restriction to their hearts coincides with \( g^\otimes \).

**Proof.** Let \( G^\otimes : \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^\otimes \to \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^\otimes \) be the symmetric monoidal functor determined by \( g^\otimes \) and the functoriality of the construction \( \text{Sp}(\mathcal{P}^{\text{add}}(-))^\otimes \). According to Lemma 4.2 and the description of the correspondence of part 2 of Proposition 3.8, the restriction \( (\text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^{\text{fin}})^\otimes \to \text{Sp}(\mathcal{P}^{\text{add}}(\mathcal{C}_w^\otimes))^{\text{fin}}) \) of \( G^\otimes \) can be regarded as a symmetric monoidal refinement of the functor \( f \).

\( \square \)
Remark 4.4. By taking the full subcategory of $\mathcal{C}^{\otimes}$ spanned by objects which can be written as $Y_1 \otimes \cdots \otimes Y_n$ for some $Y_1, \ldots, Y_n \in \mathcal{C}^b$ up to equivalence, we can see that the conclusion of Theorem 4.3 still holds when the weight structure on $\mathcal{C}$ is not necessarily bounded.

Applying this theorem to the case $\mathcal{C}^{\otimes} = K^b(h\mathcal{C}^{\otimes})^{\otimes}$, we have the following main result of this paper:

**Corollary 4.5.** For a stable symmetric monoidal $\infty$-category whose underlying stable $\infty$-category has a bounded weight structure compatible with the symmetric monoidal structure, the weight complex functor has a symmetric monoidal refinement.

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