EULERIAN POLYNOMIALS, STIRLING PERMUTATIONS AND INCREASING TREES

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Abstract. We study two generalizations of the $\gamma$-expansion of Eulerian polynomials from the viewpoint of the decompositions of statistics. We first present an expansion formula of the trivariate Eulerian polynomials, which are the enumerators for the joint distribution of descents, big ascents and successions of permutations. And then, inspired by the work of Chen and Fu on the trivariate second-order Eulerian polynomials, we show the $e$-positivity of the multivariate $k$-th order Eulerian polynomials, which are the enumerators for the joint distribution of ascents, descents and $j$-plateaux of $k$-Stirling permutations. We provide combinatorial interpretations for the coefficients of these two expansions in terms of increasing trees.

Keywords: Eulerian polynomials; Successions; Stirling permutations; Increasing trees

1. Introduction

Let $f(x) = \sum_{i=0}^{n} f_i x^i$ be a symmetric polynomial of degree $n$, i.e., $f_i = f_{n-i}$ for any $0 \leq i \leq n$. Then $f(x)$ can be expanded uniquely as

$$f(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \gamma_k x^k (1 + x)^{n-2k}.$$ 

We say that $f(x)$ is $\gamma$-positive if $\gamma_k \geq 0$ for $0 \leq k \leq \lfloor n/2 \rfloor$ (see [21, 26] for instance). Notably, $\gamma$-positivity of a polynomial implies that its coefficients are symmetric and unimodal, and the coefficients of $\gamma$-positive polynomials often have nice combinatorial interpretations. We refer the reader to Athanasiadis’s survey article [1] for details.

This paper is devoted to generalize the $\gamma$-expansion of Eulerian polynomials. There are two objects of this paper. We first study the enumerators for the joint distribution of descents, big ascents and successions of permutations, and then we study the enumerators for the joint distribution of ascents, descents and $j$-plateaux of $k$-Stirling permutations.

Let $S_n$ denote the symmetric group of all permutations of $[n]$, where $[n] = \{1, 2, \ldots, n\}$. As usual, we write $\pi = \pi(1)\pi(2)\cdots\pi(n) \in S_n$. A descent (resp. ascent, excedance) of $\pi$ is an index $i \in [n-1]$ such that $\pi(i) > \pi(i+1)$ (resp. $\pi(i) < \pi(i+1)$, $\pi(i) > i$). Let $\text{des}(\pi)$ (resp. $\text{asc}(\pi)$, $\text{exc}(\pi)$) denote the number of descents (resp. ascents, excedances) of $\pi$. It is well known that descents, ascents and excedances are equidistributed over the symmetric groups, and their common enumerative polynomials are the Eulerian polynomials $A_n(x)$, i.e.,

$$A_n(x) = \sum_{\pi \in S_n} x^{\text{des}(\pi)} = \sum_{\pi \in S_n} x^{\text{asc}(\pi)} = \sum_{\pi \in S_n} x^{\text{exc}(\pi)}.$$ 

Thus any other statistic that is equidistributed with des or exc is called an Eulerian statistic.

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There has been much work on the combinatorial expansions of Eulerian polynomials, see e.g. [10, 30, 37]. Let \( \left\{ \binom{n}{k} \right\} \) be the Stirling number of the second kind, which counts the number of partitions of \( |n| \) into \( k \) nonempty blocks. A famous combinatorial expansion of the Eulerian polynomial is the Frobenius formula:

\[
x A_n(x) = \sum_{k=1}^{n} k! \left\{ \binom{n}{k} \right\} x^k (1-x)^{n-k} \quad \text{for any } n \geq 1.
\]

We say that an index \( i \) is a double descent of \( \pi \in S_n \) if \( \pi(i-1) > \pi(i) > \pi(i+1) \), where \( \pi(0) = \pi(n+1) = 0 \). Another famous combinatorial expansion was first established by Foata and Schützenberger [18], which says that

\[
A_n(x) = \gamma_{n,i} x^i (1+x)^{n-1-2i},
\]

where \( \gamma_{n,i} \) is the number of permutations \( \pi \in S_n \) which have no double descents and \( \text{des}(\pi) = i \). The \( \gamma \)-expansion (2), along several multivariate refinements and \( q \)-analogues, were frequently discovered in the past decades, see [11, 27, 37] and references therein. For example, by using the theory of enriched \( P \)-partitions, Stembridge [36, Remark 4.8] showed that

\[
A_n(x) = \frac{1}{2^{n-1}} \sum_{i=0}^{[(n-1)/2]} 4^i P(n,i) x^i (1+x)^{n-1-2i},
\]

where \( P(n,i) \) is the the number of permutations in \( S_n \) with \( i \) interior peaks, i.e., the indices \( i \in \{2, \ldots, n-1\} \) such that \( \pi(i-1) < \pi(i) > \pi(i+1) \). By using modified Foata-Strehl action [3], one can see that the expansion (3) is equivalent to (2).

In the past decades, the bijections between permutations and increasing trees are repeatedly discovered (see [35, Section 1.5] for instance). In this paper, by using the theory of context-free grammars, we shall present two generalizations of (2), and the combinatorial interpretations for the coefficients of the generalized expansions are given in terms of increasing trees. The organization of this paper is as follows. In the next section, we give a short survey about the theory of context-free grammars. In Section 3, we study the joint distribution of descents, big ascents and successions of permutations. In Section 4, we study the joint distribution of descents, ascents and \( j \)-plateaux of \( k \)-Stirling permutations.

2. The theory of context-free grammars

For an alphabet \( A \), let \( \mathbb{Q}[[A]] \) be the rational commutative ring of formal power series in monomials formed from letters in \( A \). Following Chen [6], a context-free grammar (also known as Chen's grammar) over \( A \) is a function \( G : A \rightarrow \mathbb{Q}[[A]] \) that replaces each letter in \( A \) by a formal function over \( A \). The formal derivative \( D_G \) with respect to \( G \) satisfies the derivation rule:

\[
D_G(u + v) = D_G(u) + D_G(v), \quad D_G(uv) = D_G(u)v + uD_G(v).
\]

Context-free grammar is a powerful tool for studying exponential structures in combinatorics. We refer the reader to [7, [15, 28, 27] for further information. For example, for a large number of classical combinatorial polynomials, Dumont [15] gave a natural explanation of the mysterious
EULERIAN POLYNOMIALS, STIRLING PERMUTATIONS AND INCREASING TREES

coincidences between calculus and enumeration. Let us now recall a grammatical interpretation of Eulerian polynomials.

**Proposition 1** ([15] Section 2.1). Let \( G_1 = \{ x \rightarrow xy, y \rightarrow xy \} \). Then
\[
D_{G_1}^n(x) = x \sum_{k=0}^{n-1} \binom{n}{k} x^k y^{n-k} \quad \text{for} \quad n \geq 1.
\]
Setting \( y = 1 \), one has \( D_{G_1}^n(x) \big|_{y=1} = x A_n(x) \).

The following two definitions will be used repeatedly in our discussion.

**Definition 2** ([7]). A grammatical labeling is an assignment of the underlying elements of a combinatorial structure with variables, which is consistent with the substitution rules of a grammar.

**Definition 3** ([27]). A change of grammars is a substitution method in which the original grammar is replaced with functions of the other grammar.

In [27], the change of grammars method was introduced and the power of it is to obtain recurrences and combinatorial interpretations of the \( \gamma \)-coefficients and partial \( \gamma \)-coefficients of various polynomials. Recently, by using the change of grammars method, Lin et al. [26] proved the partial-\( \gamma \)-positivity of trivariate enumerative polynomials of Stirling multipermutations.

Let us first give some examples of the change of grammars. Consider the change of variables:
\[
\begin{align*}
  u &= xy, \\
  v &= x + y,
\end{align*}
\]
(4)
The grammar \( G_1 = \{ x \rightarrow xy, y \rightarrow xy \} \) is transformed into a new grammar
\[
G_2 = \{ u \rightarrow uv, v \rightarrow 2u \}.
\]
(5)
Since \( D_{G_1}^n(x) = D_{G_2}^{n-1}(u) \) for \( n \geq 1 \), it immediately follows that \( A_n(x) \) is \( \gamma \)-positive (see [27] for details). In the following, we provide another illustration of the change of grammars method.

**A grammatical proof of the Frobenius formula** ([11]):

**Proof.** Let \( G_1 \) be the grammar given in Proposition [11]. Set \( u = y - x \). Observe that
\[
D_{G_1}(x) = x(u + x), \quad D_{G_1}(u) = 0.
\]
Then we get a new grammar \( G_3 = \{ x \rightarrow x(u + x), \ u \rightarrow 0 \} \). Note that
\[
D_{G_3}(x) = x(u + x), \quad D_{G_3}^2(x) = (u + x)(ux + 2x^2), \quad D_{G_3}^3(x) = (u + x)(ux^2 + 6x^2u + 6x^3).
\]
For \( n \geq 1 \), assume that \( D_{G_3}^n(x) = (u + x) \sum_{k=1}^{n} E_{n,k} x^k u^{n-k} \). Then
\[
D_{G_3}^{n+1}(x) = D_{G_3}(D_{G_3}^n(x))
\]
\[
= D_{G_3} \left( (u + x) \sum_{k=1}^{n} E_{n,k} x^k u^{n-k} \right)
\]
\[
= (u + x) \sum_{k} E_{n,k} \left( kx^k u^{n-k+1} + (1 + k)x^{k+1} u^{n-k} \right),
\]
which yields the recurrence relation $E_{n+1,k} = k(E_{n,k} + E_{n,k-1})$. From [34, A019538], we see that $E_{n,k}$ satisfy the same recurrence relation and initial conditions as $k!\binom{n}{k}$, so they agree. Setting $y = 1$, we obtain $u = 1 - x$. Therefore, we find that

$$D^n_{G_3}(x) |_{y=1} = \sum_{k=1}^{n} k! \binom{n}{k} x^k (1-x)^{n-k}.$$

Comparing with Proposition 1 yields the desired formula (1), and so the proof is complete. □

Consider the following change of variables:

$$\begin{cases}
u = 2xy, \\
u = x + y,
\end{cases} \quad (6)$$

the grammar $G_1 = \{x \to xy, y \to xy\}$ is transformed into a new grammar

$$G_4 = \{u \to uv, v \to u\}. \quad (7)$$

The grammar $G_4$ was introduced by Dumont [15] when he studied André polynomials.

A rooted tree of order $n$ with the vertices labelled $1, 2, \ldots, n$, is an increasing tree if the node labelled $1$ is distinguished as the root, and for each $2 \leq i \leq n$, the labels of the nodes in the unique path from the root to the node labelled $i$ form an increasing sequence. An increasing tree on $\{0, 1, 2, \ldots, n\}$ is a rooted tree with vertex set $\{0, 1, 2, \ldots, n\}$ in which the labels of the vertices are increasing along any path from the root 0 to a leaf. The degree of a vertex is referred to the number of its children. A 0-1-2 increasing tree is an increasing tree in which the degree of any vertex is at most two. Let $\ell(T)$ denote the number of leaves of a tree and let $r(T)$ denote the number of vertices with degree 1. The André polynomials are defined by

$$E_n(u, v) = \sum_T u^{\ell(T)} v^{r(T)}, \quad (8)$$

where the sum ranges over 0-1-2 increasing trees on $\{0, 1, 2, \ldots, n\}$. Dumont showed that

$$D^n_{G_4}(u) |_{y=1} = E_n(u, v). \quad (9)$$

Following Chen and Fu [7], the following grammatical labeling leads to the relation (9): a leaf is labeled by $u$, a vertex of degree 1 is labeled by $v$ and a vertex of degree 2 is labeled by 1.

A plane tree is a rooted tree in which the children of each vertex are linearly ordered (from left to right, say). Increasing plane trees are also called plane recursive trees, see [24, 25]. A 0-1-2 increasing plane tree on $[n]$ is an increasing plane tree for which each vertex has degree at most two. From the above discussion, we see that for $n \geq 1$,

$$xA_n(x) = D^n_{G_1}(x) |_{y=1} = D^{n-1}_{G_2}(u) |_{y=1} = \sum_{i=1}^{[(n+1)/2]} \gamma_{n,i-1} x^i (1+x)^{n+1-2i}. \quad (10)$$

By using grammatical labeling, Chen and Fu [8] found the following result.

**Proposition 4** ([8, Theorem 3.1]). Let $\gamma_{n,i-1}$ be the coefficient defined by (10), where $1 \leq i \leq [(n+1)/2]$. Then $\gamma_{n,i-1}$ equals the number of 0-1-2 increasing plane trees on $[n]$ with $i$ leaves.
Let \( X_n = \{x_1, x_2, \ldots, x_n\} \) be a set of commuting variables. Define
\[
S_n(x) = \prod_{i=1}^{n} (x - x_i) = \sum_{k=0}^{n} (-1)^k e_k x^{n-k}.
\]
Then the \( k \)-th elementary symmetric function associated with \( X_n \) is defined by
\[
e_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} x_{i_1} x_{i_2} \cdots x_{i_k}.
\]
In particular, \( e_0 = 1 \), \( e_1 = \sum_{i=1}^{n} x_i \) and \( e_n = x_1 x_2 \cdots x_n \). A function \( f(x_1, x_2, \ldots) \in \mathbb{R}[x_1, x_2, \ldots] \) is said to be \textit{symmetric} if it is invariant under any permutation of its indeterminates. We say that a symmetric function is \textit{\( e \)-positive} if it can be written as a nonnegative linear combination of elementary symmetric functions. In [5], Chen and Fu introduced a new change of variables:
\[
\begin{align*}
u &= x + y + z, \\
v &= xy + yz + zx, \\
w &= xyz.
\end{align*}
\]
By using (11), they discovered the new grammar (31), and then they proved the \( e \)-positivity of trivariate second-order Eulerian polynomials (see [8, Section 4]). As a unified extension of (4), (6) and (11), we now introduce a definition. A special case is discussed in Section 4.

\textbf{Symmetric transformation of grammar.} Let \( G \) be the grammar defined by
\[
G = \{ x_1 \to f_1(x_1, x_2, \ldots, x_n), x_2 \to f_2(x_1, x_2, \ldots, x_n), \ldots, x_n \to f_n(x_1, x_2, \ldots, x_n) \}.
\]
Suppose that \( D^n_{G}(F(x_1, x_2, \ldots, x_n)) \) are symmetric functions. The symmetric transformation of \( G \) is defined by \( u_j = g_j(e_1, e_2, \ldots, e_n) \), where \( F \) and \( g_j \) are functions, \( e_\ell \) are the \( \ell \)-th elementary symmetric functions associated with \( \{x_1, x_2, \ldots, x_n\} \), and \( 1 \leq j, \ell \leq n \).

\section{Trivariate Eulerian Polynomials}

\subsection{Preliminary.}

The enumeration of permutations according to the number of successions was initiated by Riordan [31]. A \textit{succession} of \( \pi \in \mathfrak{S}_n \) is an index \( k \in [n-1] \) such that \( \pi(k+1) = \pi(k) + 1 \). Let \( P(n, r, s) = \#\{ \pi \in \mathfrak{S}_n : \text{asc}(\pi) = r, \text{suc}(\pi) = s \} \), where \( \text{suc}(\pi) \) is the number of successions of \( \pi \). Roselle [32, Eq. (2.1)] proved that
\[
P(n, r, s) = \binom{n-1}{s} P(n-s, r-s, 0).
\]
Denote by \( P^+(n, r) \) the number of permutations of \( \mathfrak{S}_n \) with \( r \) ascents, no successions and \( \pi(1) > 1 \). Let \( P_n^+(x) = \sum_{k=0}^{n-1} P^+(n, r)x^r \). According to [32, Eq. (4.3)]}, one has
\[
\sum_{n=0}^{\infty} P_n^+(x) \frac{z^n}{n!} = \frac{1-x}{e^{xz} - xe^{z}}.
\]  
(12)

The reader is referred to [5, 12, 13, 28] for recent progresses in studies of succession statistics. In particular, Diaconis, Evans and Graham [13] found the following remarkable result, which was generalized soon by Brenti and Marietti [5].
Theorem 6. Let \( \mathfrak{S}_n \) be the set of fixed points of \( A \). Note that \( \{k \in [n-1] : \pi(k+1) = \pi(k) + 1\} = I \) \( \Rightarrow \) \( \{k \in [n-1] : \pi(k) = k\} = I \).

A fixed point of \( \pi \in \mathfrak{S}_n \) is an index \( k \in [n] \) such that \( \pi(k) = k \). Let \( \text{fix}(\pi) \) be the number of fixed points of \( \pi \). There is a wealth of literature on the joint distributions of Eulerian and fixed point statistics, see e.g. [4, 12, 17, 33]. Motivated by Proposition 5, we shall study the joint distribution of Eulerian and succession statistics from the viewpoint of the decomposition of the ascent statistic, i.e., write asc as a sum of the numbers of successions and big ascents.

3.2. Main results.

The number of big ascents of \( \pi \in \mathfrak{S}_n \) is defined by

\[
\text{bas}(\pi) = \#\{i \in [n-1] : \pi(i+1) \geq \pi(i) + 2\}.
\]

Note that \( \text{asc}(\pi) = \text{suc}(\pi) + \text{bas}(\pi) \). Consider the trivariate Eulerian polynomials

\[
A_n(x, y, s) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{bas}(\pi)} y^{\text{des}(\pi)} s^{\text{suc}(\pi)}.
\] \quad (13)

Below are the polynomials \( A_n(x, y, s) \) for \( 0 \leq n \leq 5 \):

\[
\begin{align*}
A_0(x, y, s) &= A_1(x, y, s) = 1, \quad A_2(x, y, s) = s + y, \\
A_3(x, y, s) &= (s + y)^2 + 2xy, \quad A_4(x, y, s) = (s + y)^3 + 6xy(s + y) + 2xy(x + y), \\
A_5(x, y, s) &= (s + y)^4 + 12xy(s + y)^2 + 8xy(s + y)(x + y) + 2xy(x + y)^2 + 16x^2y^2.
\end{align*}
\]

Note that \( A_n(x) = A_n(x, 1, x) = A_n(1, x, 1) \). The main result of this section is the following.

Theorem 6. Let \( A_n(x, y, s) \) be the trivariate Eulerian polynomials defined by \( \text{[13]} \).

(i) We have

\[
A(x, y, s; z) = \sum_{n=0}^{\infty} A_{n+1}(x, y, s) \frac{z^n}{n!} = e^{(y+s)} \left( \frac{y - x}{ye^{xz} - xe^{yz}} \right)^2;
\] \quad (14)

(ii) For any \( n \geq 0 \), we have

\[
A_{n+1}(x, y, s) = \sum_{i=0}^{n} (s + y)^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \gamma_{n,i,j} (2xy)^j (x+y)^{n-i-2j},
\] \quad (15)

where the numbers \( \gamma_{n,i,j} \) satisfy the recurrence relation

\[
\gamma_{n+1,i,j} = \gamma_{n,i-1,j} + (1+i)\gamma_{n,i+1,j-1} + j\gamma_{n,i,j} + (n-i-2j+2)\gamma_{n,i,j-1},
\] \quad (16)

with the initial conditions \( \gamma_{0,0,0} = 1 \) and \( \gamma_{0,i,j} = 0 \) for \( i, j \neq (0, 0) \);

(iii) The number \( \gamma_{n,i,j} \) equals the number of \( 0\text{-}1\text{-}2 \) increasing rooted forests on \( \{0,1,\ldots,n\} \) with \( i+j \) leaves, among which \( i \) leaves are children of the root, where a \( 0\text{-}1\text{-}2 \) increasing rooted forest on \( \{0,1,2,\ldots,n\} \) is a tree on \( \{0,1,\ldots,n\} \) with the restriction that the root 0 has only children with degree 0 or 1, and the degree of any other vertex is at most two.
The reader is referred to Fig. 1 for an example of a 0-1-2 increasing rooted forest, where the root 0 has three 0-1-2 subtrees. Setting \( y = 1 \) in (15), we see that

\[
\sum_{\pi \in \mathcal{S}_{n+1}} x^{\text{basc} (\pi)} s^{\text{suc} (\pi)} = \sum_{i=0}^{n} (1 + s)^i \sum_{j=0}^{\left\lceil \frac{(n-i)}{2} \right\rceil} \gamma_{n,i,j} (2x)^j (1 + x)^{n-i-2j},
\]

and it reduces to (2) when \( s = x \).

We say that \( \pi \) is a derangement if it has no fixed points. Let \( \mathcal{D}_n \) be the set of derangements in \( \mathfrak{S}_n \). The derangement polynomials are defined by \( d_n(x) = \sum_{\pi \in \mathcal{D}_n} x^{\text{exc} (\pi)} \). The generating function of \( d_n(x) \) is given as follows (see [11 Proposition 6]):

\[
d(x, z) = \sum_{n=0}^{\infty} d_n(x) \frac{z^n}{n!} = \frac{1 - x}{e^{xz} - xe^z}.
\]

Comparing (12) with (17), we see that \( P_n^*(x) = d_n(x) \). A anti-excedance of \( \pi \in \mathfrak{S}_n \) is an index \( i \in [n-1] \) such that \( \pi(i) < i \). Let \( \text{aexc} (\pi) \) denote the number of anti-excedances of \( \pi \). Clearly, \( \text{exc} (\pi) + \text{aexc} (\pi) + \text{fix} (\pi) = n \). Define

\[
C_n(x, y, s) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{exc} (\pi)} y^{\text{aexc} (\pi)} s^{\text{fix} (\pi)}.
\]

In particular, \( d_n(x) = C_n(x, 1, 0) \) and \( A_n(x) = C_n(x, 1, 1) \). It is well known (see [18, 33]) that

\[
C(x, y, s; z) = \sum_{n=0}^{\infty} C_n(x, y, s) \frac{z^n}{n!} = \frac{(y - x)e^{sz}}{ye^{xz} - xe^{yz}}.
\]

Let \( A_n(x, y) \) be the bivariate Eulerian polynomials defined by

\[
\sum_{n=0}^{\infty} A_n(x, y) \frac{z^n}{n!} = \frac{(y - x)e^{yz}}{ye^{xz} - xe^{yz}},
\]

which can also be defined by

\[
A_n(x, y) = \sum_{\pi \in \mathfrak{S}_n} x^{\text{asc} (\pi)} y^{\text{des} (\pi) + 1}, \quad A_0(x, y) = 1.
\]

Combining (14), (18) and (19), we get the following result.

**Corollary 7.** For \( n \geq 0 \), we have

\[
A_{n+1}(x, y, s) = \sum_{i=0}^{n} \binom{n}{i} A_i(x, y) C_{n-i}(x, y, s),
\]

In particular,

\[
A_{n+1}(x, 1, 0) = \sum_{i=0}^{n} \binom{n}{i} A_i(x) d_{n-i}(x), \quad A_{n+1}(x, 1, 1) = \sum_{i=0}^{n} \binom{n}{i} A_i(x) A_{n-i}(x).
\]

Let \( \gamma_{n,i,j} \) be the numbers defined by (16). Now define

\[
\gamma_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{\left\lceil \frac{(n-i)}{2} \right\rceil} \gamma_{n,i,j} x^i y^j, \quad \gamma(x, y; z) = \sum_{n=0}^{\infty} \gamma_n(x, y) \frac{z^n}{n!}.
\]

We end this subsection by giving the following result.
Proposition 8. We have
\[
\gamma(x, y; z) = e^{z(x-1)} \left( \frac{\sqrt{2y-1} \sec \left( \frac{\pi}{2} \sqrt{2y-1} \right)}{\sqrt{2y-1} - \tan \left( \frac{\pi}{2} \sqrt{2y-1} \right)} \right)^2. \tag{20}
\]

Proof. Multiplying both sides of (16) by \(x^i y^j\) and summing over all \(i, j\), we obtain
\[
\gamma_{n+1}(x, y) = (x + ny) \gamma_n(x, y) + y(1-x) \frac{\partial}{\partial x} \gamma_n(x, y) + y(1-2y) \frac{\partial}{\partial y} \gamma_n(x, y),
\]
which can be rewritten as
\[
(1 - yz) \frac{\partial}{\partial x} \gamma(x, y; z) = x \gamma(x, y; z) + y(1-x) \gamma(x, y; z) + y(1-2y) \gamma(x, y; z). \tag{21}
\]
It is routine to check that the generating function
\[
\hat{\gamma}(x, y; z) = e^{z(x-1)} \left( \frac{\sqrt{2y-1} \sec \left( \frac{\pi}{2} \sqrt{2y-1} \right)}{\sqrt{2y-1} - \tan \left( \frac{\pi}{2} \sqrt{2y-1} \right)} \right)^2
\]
satisfies (21). Note that this generating function gives \(\hat{\gamma}(x, y; 0) = 1\) and \(\hat{\gamma}(x, 0; 0) = e^{xz}\). Hence \(\gamma(x, y; z) = \hat{\gamma}(x, y; z)\).

When \(x = 1\), the explicit formula (20) reduces to the exponential generating function of the descent polynomials of simsun permutations, which was obtained by Chow and Shiu [11, Theorem 1]. Let \(E_n(u, v)\) be the André polynomials defined by (5). It should be noted that
\[
\gamma(1, y; z) = 1 + \sum_{n=1}^{\infty} \frac{1}{y} E_n(y, 1) \frac{z^n}{n!},
\]
and an equivalent explicit formula of \(\gamma(1, y; z)\) has been obtained by Foata and Schützenberger [19] as well as Foata and Han [16, Section 7].

3.3. A key Lemma.

Lemma 9. If
\[
G_5 = \{L \to Ly, M \to Ms, s \to xy, x \to xy, y \to xy\},
\]
then we have
\[
D_{G_5}^n(LM) = LM A_{n+1}(x, y, s). \tag{23}
\]

Proof. Here we introduce a grammatical labeling of \(\pi = \pi(1)\pi(2) \cdots \pi(n) \in \mathcal{S}_n\) as follows:

(i) Put a superscript label \(L\) at the front of \(\pi\);

(ii) Put a superscript label \(M\) right after the maximum entry \(n\);

(iii) If \(i\) is a big ascent, then put a superscript label \(x\) right after \(\pi(i)\);

(iv) If \(i\) is a descent and \(\pi(i) \neq n\), then put a superscript label \(y\) right after \(\pi(i)\);

(v) If \(\pi(n) \neq n\), then put a superscript label \(y\) at the end of \(\pi\);

(vi) If \(i\) is a succession, then put a superscript label \(s\) right after \(\pi(i)\).

The weight of \(\pi\) is defined to be the product of its labels. Thus the weight of \(\pi\) is given by
\[
w(\pi) = LM x^{\text{base}(\pi)} y^{\text{des}(\pi)} s^{\text{suc}(\pi)}.
\]
Note that \(\mathcal{S}_1 = \{L^1 M\}\) and \(\mathcal{S}_2 = \{L^1 s^2 M, L^2 M^1 y\}\). Note that \(D_{G_5}(LM) = LM(s + y)\). Hence the weight of the element in \(\mathcal{S}_1\) is \(LM\) and the sum of weights of the elements in \(\mathcal{S}_2\) is given...
by $D_{G_5}(LM)$. Suppose we get all labeled permutations in $\pi \in S_{n-1}$, where $n \geq 2$. Let $\pi$ be obtained from $\pi \in S_{n-1}$ by inserting the entry $n$. There are six cases to label $n$ and relabel some elements of $\pi$. The changes of labeling are illustrated as follows:

$$
\begin{align*}
L\pi(1) \cdots (n-1)^M \rightarrow & \ L \pi(1) \cdots (n-1)^y \cdots; \\
L\pi(1) \cdots (n-1)^M \rightarrow & \ L \pi(1) \cdots (n-1)^s n^M \cdots; \\
\cdots \pi(i)^x \cdots (n-1)^M \rightarrow & \ \cdots \pi(i)^x n^M \cdots (n-1)^y \cdots; \\
\cdots \pi(i)^y \pi(i+1) \cdots (n-1)^M \rightarrow & \ \cdots \pi(i)^x n^M \pi(i+1) \cdots (n-1)^y \cdots; \\
\cdots (n-1)^M \pi(n-1)^y \rightarrow & \ \cdots (n-1)^y \pi(n-1)^x n^M; \\
\cdots \pi(i)^s \pi(i+1) \cdots (n-1)^M \rightarrow & \ \cdots \pi(i)^x n^M \pi(i+1) \cdots (n-1)^y \cdots.
\end{align*}
$$

In each case, the insertion of $n$ corresponds to one substitution rule in $G_5$. By induction, it is routine to check that the action of the formal derivative $D_{G_5}$ on the set of weights of permutations in $S_{n-1}$ gives the set of weights of permutations in $S_n$. This completes the proof of (23). \qed

3.4. Proof of Theorem 5.

(A) Let $G_5$ be the grammar given by (22). From (23) we see that there exist nonnegative integers $a_{n,i,j}$ such that $D^n_{G_5}(LM) = LM \sum_{i,j=0}^n a_{n,i,j} x^i y^j s^{n-i-j}$. Then we have

$$D_{G_5}(D^n_{G_5}(LM)) = LM \sum_{i,j=0}^n a_{n,i,j} (x^i y^{j+1} s^{n-i-j} + x^i y^j s^{n+1-i-j}) + LM \sum_{i,j=0}^n a_{n,i,j} (i x^i y^j s^{n-i-j} + j x^{i+1} y^j s^{n-i-j} + (n-i-j) x^{i+1} y^{j+1} s^{n-1-i-j}).$$

Comparing the coefficients of $LM x^i y^j s^{n+1-i-j}$ in both sides of the above expression, we get

$$a_{n+1,i,j} = a_{n,i,j} + (1+i)a_{n,i,j-1} + ja_{n,i-1,j} + (n-i-j+2)a_{n,i-1,j-1}.$$  \hspace{1cm}(24)

Multiplying both sides of (24) by $x^i y^j s^{n+1-i-j}$ and summing over all $i,j$, we obtain

$$A_{n+2}(x, y, s) = (s+y)A_{n+1}(x, y, s) + xy \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial s} \right) A_{n+1}(x, y, s).$$ \hspace{1cm}(25)

By rewriting (25) in terms of generating function $A := A(x, y, s)$, we have

$$\frac{\partial}{\partial z} A = (s+y)A + xy \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial s} \right) A.$$ \hspace{1cm}(26)

It is routine to check that the generating function

$$\tilde{A} = e^{2(y+s)} \left( \frac{y-x}{y e^{2z} - x e^{y+z}} \right)^2$$

satisfies (26). Moreover, $\tilde{A}(0,0,0; z) = 1$, $\tilde{A}(x,0,s;z) = e^{sz}$ and $\tilde{A}(0,y,s;z) = e^{2(y+s)}$. The proof of $A = \tilde{A}$ follows.

(B) Setting $u = 2xy, v = x+y, t = s+y$ and $I = LM$, we get $D_{G_5}(u) = uv, D_{G_5}(v) = u, D_{G_5}(t) = u$ and $D_{G_5}(I) = It$. Thus we get a new grammar

$$G_6 = \{ I \rightarrow It, t \rightarrow u, u \rightarrow uv, v \rightarrow u \}.$$ \hspace{1cm}(27)
Figure 1. The labeling of a 0-1-2 increasing rooted forest on \{0, 1, 2, \ldots, 8\}.

Note that \( D_{G_6}(I) = It, \ D_{G_6}^2(I) = I(t^2 + u) \) and \( D_{G_6}^3(I) = I(t^3 + 3tu + uv) \). Then by induction, it is easy to verify that there exist nonnegative integers \( \gamma_{n,i,j} \) such that

\[
D_{G_6}^n(I) = I \sum_{i=0}^{n} t^i \sum_{j=0}^{\lfloor (n-i)/2 \rfloor} \gamma_{n,i,j} u^j v^{n-i-2j}.
\]  

(28)

Then upon taking \( u = 2xy, v = x + y, t = s + y \) and \( I = LM \), we get (15). In particular, \( \gamma_{0,0,0} = 1 \) and \( \gamma_{0,i,j} = 0 \) if \((i,j) \neq (0,0) \). Since \( D_{G_6}^{n+1}(I) = D_{G_6}(D_{G_6}^n(I)) \), we obtain

\[
D_{G_6}(D_{G_6}^n(I)) = I \sum_{i,j} \gamma_{n,i,j} \left( t^{i+1}u^j v^{n-i-2j} + it^{i-1}u^{j+1}v^{n-i-2j} \right) +
I \sum_{i,j} \gamma_{n,i,j} \left( jt^i u^j v^{n+1-i-2j} + (n-i-2j)t^i u^j v^{n-1-i-2j} \right).
\]

Comparing the coefficients of \( t^i u^j v^{n+1-i-2j} \) in both sides of the above expansion, we get (16).

(C) The combinatorial interpretation of \( \gamma_{n,i,j} \) can be found by using the following grammatical labeling. Given a 0-1-2 increasing rooted forest \( T \), the root 0 is labeled by \( I \). For the children of the root, each child with degree 0 (a leaf of the root) is labeled by \( t \) and each child with degree 1 is labeled by 1. For the other vertices (not the children of the root), each leaf is labeled by \( u \), each vertex with degree 1 is labeled by \( v \) and each vertex of degree 2 is labeled by 1. See Fig. for an example, where the grammatical labels are in parentheses. Let \( T \) be the 0-1-2 increasing rooted forest given in Fig. Then there are four cases to be considered:

(i) If we add 9 as a child of the root 0, then the vertex 9 becomes a leaf of the root with label \( t \). This corresponds to the substitution rule \( I \to It \);

(ii) If we add 9 as a child of the vertex 4, the label \( t \) of 4 becomes 1, and the vertex 9 gets the label \( u \). This corresponds to the substitution rule \( t \to u \);

(iii) If we add 9 as a child of the vertex 5 (resp. 7, 8), the label \( u \) of 5 (resp. 7, 8) becomes \( v \), and the vertex 9 gets the label \( u \). This corresponds to the substitution rule \( u \to uv \);

(iv) If we add 9 as a child of the vertex 6, the label \( v \) of 6 becomes 1, and the vertex 9 gets the label \( u \). This corresponds to the substitution rule \( v \to u \).
The aforementioned four cases exhaust all the cases to construct a 0-1-2 increasing rooted forest \( T' \) on \( \{0, 1, 2, \ldots, n, n+1\} \) from a 0-1-2 increasing rooted forest \( T \) on \( \{0, 1, 2, \ldots, n\} \) by adding \( n+1 \) as a leaf. Since \( D^2_{m,n}(I) \) equals the sum of the weights of 0-1-2 increasing rooted forests on \( \{0, 1, 2, \ldots, n\} \), the coefficient \( \gamma_{n,i,j} \) equals the number of 0-1-2 increasing rooted forest \( T \) on \( \{0, 1, 2, \ldots, n\} \) with \( i+j \) leaves, among which \( i \) leaves are the children of the root 0. This completes the proof.

4. The multivariate \( k \)-th order Eulerian polynomials

4.1. Preliminary.

The second-order Eulerian polynomials are defined by

\[
C_n(x) = (1 - x)^{2n+1} \sum_{k=0}^{\infty} \binom{n+k}{k} x^k.
\]

In order to find a combinatorial interpretation of the coefficients of \( C_n(x) \) in terms of descents of permutations, Gessel and Stanley \[22\] introduced Stirling permutations. A Stirling permutation of order \( n \) is a permutation of \( \{1, 1, 2, 2, \ldots, n, n\} \) such that for each \( i, 1 \leq i \leq n \), all entries between the two occurrences of \( i \) are larger than \( i \). Denote by \( \mathcal{Q}_n \) the set of Stirling permutations of order \( n \). Let \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_{2n} \in \mathcal{Q}_n \). In this paper, we always set \( \sigma_0 = \sigma_{2n+1} = 0 \). Following \[2, 22\], for \( 0 \leq i \leq 2n \), we say that an index \( i \) is a descent (resp. ascent, plateau) of \( \sigma \) if \( \sigma_i > \sigma_{i+1} \) (resp. \( \sigma_i < \sigma_{i+1}, \sigma_i = \sigma_{i+1} \)). Let \( \des(\sigma), \asc(\sigma) \) and \( \plat(\sigma) \) be the number of descents, ascents and plateaux of \( \sigma \), respectively. According to \[22\] Theorem 2.1, one has

\[
C_n(x) = \sum_{\sigma \in \mathcal{Q}_n} x^{\des(\sigma)} = \sum_{j=1}^{n} C_{n,j} x^j.
\]

where \( C_{n,j} \) is called the second-order Eulerian number. Below are \( C_n(x) \) for \( n \leq 5 \):

\[
C_1(x) = x, \quad C_2(x) = x + 2x^2, \quad C_3(x) = x + 8x^2 + 6x^3, \\
C_4(x) = x + 22x^2 + 58x^3 + 24x^4, \quad C_5(x) = x + 52x^2 + 328x^3 + 444x^4 + 120x^5.
\]

The trivariate second-order Eulerian polynomials are defined as follows:

\[
C_n(x, y, z) = \sum_{\sigma \in \mathcal{Q}_n} x^{\asc(\sigma)} y^{\des(\sigma)} z^{\plat(\sigma)}.
\]

It is now well known that

\[
C_{n+1}(x, y, z) = xyz \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \right) C_n(x, y, z), \quad C_0(x, y, z) = 1. \quad (29)
\]

As pointed out by Chen and Fu \[8\], the recursion \[29\] first appeared in the work of Dumont \[14\] p. 317, which implies that \( C_n(x, y, z) \) is symmetric in the variables \( x, y \) and \( z \). The symmetry of \( C_n(x, y, z) \) was rediscovered by Janson \[24\] Theorem 2.1] by constructing an urn model. In \[23\], Haglund and Visontai introduced a refinement of the polynomial \( C_n(x, y, z) \) by indexing each ascent, descent and plateau by the value where they appear.

Let \( G_7 \) be the following grammar

\[
G_7 = \{ x \rightarrow xyz, y \rightarrow xyz, z \rightarrow xyz \}. \quad (30)
\]
It has been shown by Dumont [14], Chen et al. [9] and Ma et al. [27] that
\[ D_{G_7}^n(x) = C_n(x, y, z). \]

The following definition will be used in the following discussion.

**Definition 10.** A 0-1-2-$\cdots$-$k$ increasing plane tree on $[n]$ is an increasing plane tree with each vertex with at most $k$ children.

Using the change of variables [11], we see that $D_{G_7}(u) = 3w$, $D_{G_7}(v) = 2uw$, $D_{G_7}(w) = vw$, which yields a new grammar
\[ G_8 = \{ u \to 3w, v \to 2uw, w \to vw \}. \]

Chen and Fu [8] gave an interpretation of the grammar $G_8$ and obtained the following result.

**Theorem 11 [8].** For $n \geq 1$, one has
\[ C_n(x, y, z) = \sum_{k \geq 1} (xyz)^k \sum_{j \geq 0} \gamma_{n,k,j}(xy + yz + zx)^j (x + y + z)^{2n+1-2j-3k}, \]
where the coefficient $\gamma_{n,k,j}$ equals the number of 0-1-2-3 increasing plane trees on $[n]$ with $k$ leaves, $j$ degree one vertices and $i$ degree two vertices.

**Corollary 12.** For $n \geq 1$, one has
\[ C_n(x) = \sum_{k \geq 1} x^k \sum_{j \geq 0} \gamma_{n,k,j}(1 + 2x)^j (2 + x)^{2n+1-2j-3k}. \]

In the next section, we shall consider a decomposition of the statistic plat of $k$-Stirling permutations, i.e., write plat as a sum of the numbers of $j$-plateaux.

### 4.2. A key Lemma.

Let $k$ be a given positive integer, and let $j^k$ denote $k$ times of the letter $j$. A $k$-Stirling permutation of order $n$ is a multiset permutation of $\{1^k, 2^k, \ldots, n^k\}$ with the property that all elements between two occurrences of $i$ are at least $i$, where $i \in [n]$. Let $Q_n(k)$ be the set of $k$-Stirling permutations of order $n$. Clearly, $Q_n(1) = S_n$ and $Q_n(2) = Q_n$. Following [35, p. 657], an $k$-ary tree $T$ is either empty, or else one specially designated vertex is called the root of $T$ and the remaining vertices (excluding the root) are put into a (weak) ordered partition $(T_1, \ldots, T_k)$ of exactly $k$ disjoint (possibly empty) sets $T_1, \ldots, T_k$, each of which is an $k$-ary tree. A bijection between $Q_n(k)$ and the set of $(k + 1)$-ary increasing trees was independently established by Gessel [29] and Janson-Kuba [25, Theorem 1].

Let $\sigma \in Q_n(k)$. The ascents, descents and plateaux of $\sigma$ of are defined as before, where we always set $\sigma_0 = \sigma_{kn+1} = 0$. More precisely, an index $i$ is called an ascent (resp. descent, plateau) of $\sigma$ if $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i > \sigma_{i+1}$, $\sigma_i = \sigma_{i+1}$). It is clear that $\text{asc}(\sigma) + \text{des}(\sigma) + \text{plat}(\sigma) = kn + 1$.

As a natural refinement of ascents, descents and plateaux, Janson and Kuba [25] introduced the following definition, and related the distribution of $j$-ascents, $j$-descents and $j$-plateaux in $k$-Stirling permutations with certain parameters in $(k + 1)$-ary increasing trees.

**Definition 13 [25].** An index $i$ is called a $j$-plateau (resp. $j$-descent, $j$-ascent) if $i$ is a plateau (resp. descent, ascent) and there are exactly $j - 1$ indices $\ell < i$ such that $a_\ell = a_i$. 
Let $\text{plat}_j(\sigma)$ be the number of $j$-plateaux of $\sigma$. For $\sigma \in Q_n(k)$, it is clear that $\text{plat}_j(\sigma) \leq k - 1$.

**Example 14.** Consider the 4-Stirling permutation $\sigma = 1122333221$. The set of 1-plateaux is given by $\{1, 4, 6\}$, the set of 2-plateaux is given by $\{2, 7\}$, and the set of 3-plateaux is given by $\{8, 10\}$. Thus $\text{plat}_1(\sigma) = 3$ and $\text{plat}_2(\sigma) = \text{plat}_3(\sigma) = 2$.

The multivariate $k$-th order Eulerian polynomials $C_n(x_1, \ldots, x_{k+1})$ are defined by

$$C_n(x_1, x_2, \ldots, x_{k+1}) = \sum_{\sigma \in Q_n(k)} x_1^{\text{plat}_1(\sigma)} x_2^{\text{plat}_2(\sigma)} \cdots x_{k-1}^{\text{plat}_{k-1}(\sigma)} x_k^{\text{des}(\sigma)} x_{k+1}^{\text{asc}(\sigma)}.$$

In particular, when $x_1 = z$, $x_2 = \cdots = x_{k-1} = 0$, $x_k = y$ and $x_{k+1} = x$, the polynomials $C_n(x_1, x_2, \ldots, x_{k+1})$ reduce to $C_n(x, y, z)$; when $x_1 = x_2 = \cdots = x_{k-1} = 0$, $x_k = 1$ and $x_{k+1} = x$, the polynomials $C_n(x_1, x_2, \ldots, x_{k+1})$ reduce to $A_n(x)$.

In the following discussion, we always let $X_{k+1} = \{x_1, x_2, \ldots, x_{k+1}\}$ and let $e_i$ be the $i$-th elementary symmetric function associated with $X_{k+1}$. In particular,

$$e_0 = 1, \quad e_1 = x_1 + x_2 + \cdots + x_{k+1}, \quad e_k = \sum_{i=1}^{k} \frac{e_{k+1}}{x_i}, \quad e_{k+1} = x_1 x_2 \cdots x_{k+1}.$$

The following lemma is fundamental.

**Lemma 15.** Let $G_9 = \{x_1 \rightarrow e_{k+1}, x_2 \rightarrow e_{k+1}, \ldots, x_{k+1} \rightarrow e_{k+1}\}$, where $e_{k+1} = x_1 x_2 \cdots x_{k+1}$. For $n \geq 1$, one has

$$D_{G_9}^n(x_1) = C_n(x_1, x_2, \ldots, x_{k+1}).$$

**Proof.** We shall show that the grammar $G_9$ can be used to generate $k$-Stirling permutations. We first introduce a grammatical labeling of $\sigma \in Q_n(k)$ as follows:

- $(L_1)$ If $i$ is an ascent, then put a superscript label $x_{k+1}$ right after $\sigma_i$;
- $(L_2)$ If $i$ is a descent, then put a superscript label $x_k$ right after $\sigma_i$;
- $(L_3)$ If $i$ is a $j$-plateau, then put a superscript label $x_j$ right after $\sigma_i$.

The weight of $\sigma$ is defined as the product of the labels, that is

$$w(\sigma) = x_1^{\text{plat}_1(\sigma)} x_2^{\text{plat}_2(\sigma)} \cdots x_{k-1}^{\text{plat}_{k-1}(\sigma)} x_k^{\text{des}(\sigma)} x_{k+1}^{\text{asc}(\sigma)}.$$

Recall that we always set $\sigma_0 = \sigma_{kn+1} = 0$. Thus the index 0 is always an ascent and the index $kn$ is always a descent. Thus $Q_1(k) = \{x_{k+1}^1 x_1^2 x_2^2 \cdots x_k^2\}$. The are $k + 1$ elements in $Q_2(k)$ and they can be labeled as follows, respectively:

$$x_{k+1}^1 x_1^1 x_2^1 x_3^1 \cdots x_k^1 x_{k+1}^1 2x_1^2 2x_2^2 \cdots 2x_{k-1}^2 2x_k^2,$$

$$x_{k+1}^1 x_1^1 x_2^1 x_3^1 \cdots x_{k-2}^1 x_{k+1}^1 2x_1^2 x_2^2 \cdots 2x_{k-1}^2 2x_k^2 1x_k,$$

$$\ldots$$

$$x_{k+1}^1 2x_1^2 x_2^2 \cdots 2x_{k-1}^2 2x_k^2 1x_1^1 x_2^1 x_3^1 \cdots 1x_{k-1}^1 x_k.$$

Note that $D_{G_9}(x_1) = e_{k+1}$ and $D_{G_9}^2(x_1) = e_k e_{k+1}$. Then the weight of the element in $Q_1(k)$ is given by $D_{G_9}(x_1)$, and the sum of weights of the elements in $Q_2(k)$ is given by $D_{G_9}^2(x)$. Hence the result holds for $n = 1, 2$. We proceed by induction on $n$. Suppose we get all labeled
permutations in $Q_{n-1}(k)$, where $n \geq 3$. Let $\sigma'$ be obtained from $\sigma \in Q_{n-1}(k)$ by inserting the string $nn \cdots n$ with length $k$. Then the changes of labeling are illustrated as follows:

$$\sigma^x = \sigma_{i+1}^x \sigma_{i+1} \cdots \sigma_{i}^x \sigma_{i+1} \cdots ;$$

In each case, the insertion of the string $nn \cdots n$ corresponds to one substitution rule in $G_9$. Then the action of $D_{G_9}$ on the set of weights of all elements in $Q_{n-1}(k)$ gives the set of weights of all elements in $Q_n(k)$. Therefore, we get a grammatical interpretation of $C_n(x_1, x_2, \ldots, x_{k+1})$, and this completes the proof.

From the symmetry of the grammar $G_9$ and $D_{G_9}(x_1) = e_{k+1}$, we get the following result.

**Corollary 16.** The multivariate polynomials $C_n(x_1, x_2, \ldots, x_{k+1})$ are symmetric, i.e., the variables are exchangeable.

By combining an urn model for the exterior leaves of $(k + 1)$-ary increasing trees and a bijection between $(k + 1)$-ary increasing trees and $k$-Stirling permutations, Janson, Kuba and Panholzer [25, Theorem 2, Theorem 8] found that the variables in $C_n(x_1, x_2, \ldots, x_{k+1})$ are exchangeable. Therefore, in an equivalent form, Corollary 16 was first obtained in [25]. It should be noted that in [25], there is no explicit connection to the $k$-th order Eulerian polynomials is brought up.

### 4.3. Main results.

**Theorem 17.** For $n \geq 2$ and $k \geq n - 2$, we have

$$C_n(x_1, x_2, \ldots, x_{k+1}) = \sum \gamma(n; i_1, i_2, \ldots, i_n)e_{k-n}^{i_n}e_{k-n+2}^{i_{n-1}} \cdots e_k^{i_2}e_{k+1}^{i_1},$$  \hspace{1cm} (32)

where the summation is over all sequences $(i_1, i_2, \ldots, i_n)$ of nonnegative integers such that $i_1 + i_2 + \cdots + i_n = n$, $1 \leq i_1 \leq n - 1$, $i_n = 0$ or $i_n = 1$. When $i_n = 1$, one has $i_1 = n - 1$. The coefficients $\gamma(n; i_1, i_2, \ldots, i_n)$ equals the number of 0-1-2-$\cdots$-$k$-$(k+1)$ increasing plane trees on $[n]$ with $i_j$ degree $j - 1$ vertices for all $1 \leq j \leq n$.

**Proof.** Let $G_9$ be the grammar given in Lemma 15. We first consider a change of $G_9$. Note that $D_{G_9}(x_1) = e_{k+1}$, $D_{G_9}(e_i) = (k - i + 2)e_{i-1}e_{k+1}$ for $1 \leq i \leq k + 1$. Thus we get a new grammar

$$G_{10} = \{x_1 \rightarrow e_{k+1}, \; e_i \rightarrow (k - i + 2)e_{i-1}e_{k+1} \text{ for } 1 \leq i \leq k + 1\},$$  \hspace{1cm} (33)

Note that $G_{10}(x_1) = e_{k+1}$, $G_{10}^2(x_1) = e_k e_{k+1}$, $G_{10}^3(x_1) = e_k^2 e_{k+1} + 2e_{k-1}e_{k+1}^2$. By induction, we assume that

$$G_{10}^n(x_1) = \sum \gamma(n; i_1, i_2, \ldots, i_n)e_{k-n}^{i_n}e_{k-n+2}^{i_{n-1}} \cdots e_k^{i_2}e_{k+1}^{i_1},$$  \hspace{1cm} (34)
Note that
\[
G_{10}^{n+1}(x_1) = G_{10} \left( \sum \gamma(n; i_1, i_2, \ldots, i_n) e_k^{i_n} e_{k-n+2}^{i_{n-1}} \cdots e_k^{i_1} \right)
\]
\[
= \sum n i_n \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+1}^{i_n} e_{k-n+2}^{i_{n-1}} \cdots e_k^{i_1} + (n-1) i_{n-1} \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+1}^{i_n} e_{k-n+2}^{i_{n-1}} \cdots e_k^{i_1} + \cdots
\]
\[
+ 2 i_2 \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_1} + i_1 \gamma(n; i_1, i_2, \ldots, i_n) e_{k-n+2}^{i_n} e_{k-n+3}^{i_{n-1}} \cdots e_k^{i_1},
\]
which yields that the expansion \((34)\) holds for \(n+1\). Combining Lemma \((15)\) and \((34)\), we get \((32)\).

By induction, one can easily verify that \(i_1 + i_2 + \cdots + i_n = n, 1 \leq i_1 \leq n - 1, i_n = 1\) or \(i_n = 0\).

By using \((33)\), the combinatorial interpretation of the coefficients \(\gamma(n; i_1, i_2, \ldots, i_n)\) can be proved along the same lines as the proof of \([8, \text{Theorem 4.1}]\). However, we give a direct proof of it for our purpose. Let \(T\) be a \(0-1-2-\cdots-(k+1)\) increasing plane tree on \([n]\). The labeling of \(T\) is given by labeling a degree \(i\) vertex by \(e_k-i+1\) for all \(0 \leq i \leq k + 1\). In particular, label a leaf by \(e_k+1\) and label a degree \(k + 1\) vertex by 1. Let \(T'\) be a \(0-1-2-\cdots-(k+1)\) increasing plane tree on \([n+1]\) by adding \(n+1\) to \(T\) as a leaf. We can add \(n+1\) to \(T\) only as a child of a vertex \(v\) that is not of degree \(k + 1\). For \(1 \leq i \leq k + 1\), if the vertex \(v\) is a degree \(k-i+1\) vertex with label \(e_i\), there are \(k-i+2\) cases to attach \(n+1\) (from left to right, say). In either case, in \(T'\), the vertex \(v\) becomes a degree \(k - i + 2\) with label \(e_{i-1}\) and \(n + 1\) becomes a leaf with label \(e_{k-1}\). Hence the insertion of \(n + 1\) corresponds to the substitution rule \(e_i \to (k-i+2)e_{i-1}e_{k-1}\). Therefore, \(G_{10}(x_1)\) equals the sum of the weights of \(0-1-2-\cdots-(k+1)\) increasing plane trees on \([n]\), and the combinatorial interpretation of \(\gamma(n; i_1, i_2, \ldots, i_n)\) follows. This completes the proof. \(\Box\)

For convenience, we present the expansions of \(G_{10}^4(x_1)\) and \(G_{10}^5(x_1)\):
\[
G_{10}^4(x_1) = e_k^3 e_{k+1} + 8 e_{k-1} e_k^2 e_{k+1} + 6 e_{k-2}^3 e_{k+1},
\]
\[
G_{10}^5(x_1) = e_k^4 e_{k+1} + 22 e_k^2 e_{k-1}^2 e_{k+1} + 16 e_{k-1}^3 e_{k+1} + 42 e_{k-2} e_k^3 e_{k+1} + 24 e_{k-3} e_k^4 e_{k+1}.
\]

By using \(G_{10}^{n+1}(x_1) = G_{10} (G_{10}^n(x_1))\), it is routine to verify that
\[
\gamma(n+1; 1, n, 0, \ldots, 0) = \gamma(n; 1, n-1, 0, \ldots, 0) = 1,
\]
\[
\gamma(n+1; n, 0, \ldots, 0, 1) = n! \gamma(n; n-1, 0, \ldots, 0, 1) = n!.
\]
\[
\gamma(n+1; i_1, i_2, \ldots, i_n, 0) = i_1 \gamma(n; i_1, i_2-1, i_3, \ldots, i_n) + \sum_{j=2}^{n-1} j(i_j+1) \gamma(n; i_1-1, i_2, \ldots, i_{j-1}, i_j+1, i_{j+1}-1, i_{j+2}, \ldots, i_n).
\]

Note that \(\gamma(3; 2, 0, 1, 0, \ldots, 0) = 2, \gamma(4; 2, 1, 1, 0, \ldots, 0) = 8\) and
\[
\gamma(n+1; 2, n-2, 1, 0, \ldots, 0) = 2 \gamma(n; 2, n-3, 1, 0, \ldots, 0) + 2(n-1) \gamma(n; 1, n-1, 0, \ldots, 0).
\]

By induction, it is easy to verify that
\[
\gamma(n; 2, n-3, 1, 0, \ldots, 0) = 2^n - 2n \text{ for } n \geq 3. \quad (35)
\]
Recall that the second-order Eulerian numbers $C_{n,j}$ satisfy the recurrence relation

$$C_{n+1,j} = jC_{n,j} + (2n + 2 - j)C_{n,j-1},$$

with the initial conditions $C_{1,1} = 1$ and $C_{1,j} = 0$ if $j \neq 1$ (see [2, 22]). In particular,

$$C_{n,2} = 2^{n+1} - 2(n + 1).$$

Comparing this with (35), we see that $\gamma(n; 2, n - 3, 1, 0, \ldots, 0) = C_{n-1,2}$ for $n \geq 3$. Moreover, $\gamma(n + 1; n, 0, \ldots, 0, 1) = C_{n,n} = n!$. Following Janson [21], the number $C_{n,j}$ equals the number of increasing plane trees on $[n + 1]$ with $k$ leaves. So we get the following result.

**Corollary 18.** For $n \geq 2$ and $1 \leq j \leq n - 1$, we have

$$C_{n-1,j} = \sum_{i_2+i_3+\cdots+i_n = n-j} \gamma(n; j, i_2, \ldots, i_{n-1}, i_n).$$

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