CARLEMAN ESTIMATE FOR SOLUTIONS TO A DEGENERATE CONVECTION-DIFFUSION EQUATION

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Abstract. This paper concerns a control system governed by a convection-diffusion equation, which is weakly degenerate at the boundary. In the governing equation, the convection is independent of the degeneracy of the equation and cannot be controlled by the diffusion. The Carleman estimate is established by means of a suitable transformation, by which the diffusion and the convection are transformed into a complex union, and complicated and detailed computations. Then the observability inequality is proved and the control system is shown to be null controllable.

1. Introduction. Controllability theory has been widely investigated for nondegenerate parabolic equations over the last forty years and there have been a great number of results (see for instance [2, 13, 14, 18] and the references therein for a detailed account). However, the study on the controllability of degenerate parabolic equations just began several years ago and a few results have been known ([1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26, 27]). Among these, the null controllability of the following problem of a degenerate diffusion equation was investigated

\[ u_t - (x^\alpha u_x)_x + c(x, t)u = h(x, t)\chi_\omega, \quad (x, t) \in (0, 1) \times (0, T), \quad (1) \]

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where $\alpha > 0$, $h$ is the control function, $\chi_\omega$ is the characteristic function of $\omega \subset (0, 1)$. As a typical equation with boundary degeneracy, (1) is degenerate at $\{0\} \times (0, T)$ and the degeneracy is weak if $0 < \alpha < 1$ and strong if $\alpha \geq 1$. Equations with such degeneracy can be used to describe some physical models, such as the simplified Crocco-type equation, the Black-Scholes option pricing model and the Budyko-Sellers climate model ([27]). It is shown that the problem (1)–(3) is null controllable if $0 < \alpha < 2$ ([1, 10, 11, 23]), while not if $\alpha \geq 2$ ([9]). Besides, [24, 25, 26] and [5, 8, 9] proved that the problem is approximately controllable in $L^2(0, 1)$ and regional null controllable for each $\alpha > 0$, respectively. In [15], the authors studied the null controllability of the following degenerate convection-diffusion equation

$$u_t - (x^\alpha u_x)_x + x^{\alpha/2}b(x, t)u_x + c(x, t)u = h(x, t)\chi_\omega, \quad (x, t) \in (0, 1) \times (0, T)$$

(4)

with $b \in L^\infty((0, 1) \times (0, T))$ and proved that the problem (4), (2), (3) is null controllable for $0 < \alpha < 1/2$. Clearly, the convection term can be controlled by the diffusion term in (4). A progress on the null controllability of degenerate convection-diffusion equations is [27], where the authors considered

$$u_t - (x^\alpha u_x)_x + b(x, t)u_x + c(x, t)u = h(x, t)\chi_\omega, \quad (x, t) \in (0, 1) \times (0, T)$$

(5)

and proved that the problem (5), (2), (3) is null controllable for $0 < \alpha < 1/2$. But the other case $1/2 \leq \alpha < 2$ is unknown. Since (5) is degenerate, the convection term can cause essential differences. For example, the problem (5), (2), (3) is well-posed in the weakly degenerate case $0 < \alpha < 1$, while may be ill-posed in the strongly degenerate case $\alpha \geq 1$ when a different boundary condition at $\{0\} \times (0, T)$ should be prescribed ([28]).

In this paper, we study the null controllability of the problem (5), (2), (3) in the weakly degenerate case $0 < \alpha < 1$. In order to treat the convection term, we have to assume that $b \in W^{2,1}_\infty((0, 1) \times (0, T))$, i.e. $b, b_x, b_{xx}, b_t \in L^\infty((0, 1) \times (0, T))$. Since $b_x \in L^\infty((0, 1) \times (0, T))$, the convection term in (5) can be rewritten into a nondivergence form. Precisely, we investigate the null controllability of the following problem

$$u_t - (x^\alpha u_x)_x + b(x, t)u_x + c(x, t)u = h(x, t)\chi_\omega, \quad (x, t) \in (0, 1) \times (0, T),$$

(6)

$$u(0, t) = u(1, t) = 0, \quad t \in (0, T),$$

(7)

$$u(x, 0) = u_0(x), \quad x \in (0, 1),$$

(8)

where $0 < \alpha < 1$, $b \in W^{2,1}_\infty((0, 1) \times (0, T))$, $c \in L^\infty((0, 1) \times (0, T))$, $h$ is the control function, $\chi_\omega$ is the characteristic function of $\omega = (x_0, x_1)$ with $0 < x_0 < x_1 < 1$, $u_0 \in L^2(0, 1)$.

As many studies on null controllability, the key to prove the null controllability of the problem (6)–(8) is the Carleman estimate for its conjugate problem

$$v_t + (x^\alpha v_x)_x + b(x, t)v_x - c(x, t)v = 0, \quad (x, t) \in (0, 1) \times (0, T),$$

(9)

$$v(0, t) = v(1, t) = 0, \quad t \in (0, T),$$

(10)

$$v(x, T) = v_T(x), \quad x \in (0, 1),$$

(11)

where $v_T \in L^2(0, 1)$. In [15], the authors established the Carleman estimate in the same way as the case without convection term since the convection term can
be controlled by the diffusion term in the governing equation. As to the equation with a general convection term, [27] treated the case where $0 < \alpha < 1/2$. In [27], in order to get the Carleman estimate, both the reaction term and the convection term were regarded as a source term in the governing equation and the similar auxiliary functions to the case without convection term were used. The key for the Carleman estimate is to estimate the effect of the convection term. Furthermore, to show that the restriction $0 < \alpha < 1/2$ is optimal when one establishes the Carleman estimate in such a way, the authors choose auxiliary functions for the Carleman estimate by the method of undetermined coefficients. Therefore, to establish the Carleman estimate for the problem (9)–(11) with $0 < \alpha < 1$, one must treat the convection in a different way but not regard it as a source term as done in [27].

For the problem (9)–(11), the classical solution may not exist and weak solution should be considered since (9) is degenerate. Therefore, it is more convenient to establish the uniform Carleman estimate for the regularized problem

\begin{align*}
  v^\eta_t + ((x + \eta)^\alpha v^\eta_x) + (b(x,t)v^\eta)_x - c(x,t)v^\eta &= 0, \quad (x,t) \in (0,1) \times (0,T), \\
  v^\eta(0,t) &= v^\eta(1,t) = 0, \quad t \in (0,T), \\
  v^\eta(x,T) &= v_T(x), \quad x \in (0,1)
\end{align*}

with $0 < \eta < 1$. As mentioned above, one must treat the convection in a different way but not regard it as a source term as done in [27]. To get the uniform Carleman estimate, we introduce an auxiliary function by which the diffusion and the convection are transformed into a union. The transformation depends on the coefficient of the convection and we have to assume that $b \in W^{2,1}_c((0,1) \times (0,T))$ for the desired uniform Carleman estimate. Furthermore, the union transformed from the diffusion and the convection is so complex that the computations for the desired uniform Carleman estimate are more complicated and detailed than the ones in [15, 27]. After the Carleman estimate, one can get the observability inequality and further prove the null controllability of the problem (6)–(8) in a standard way. It is noted that the strongly degenerate case ($1 \leq \alpha < 2$) still remains open. Indeed, the key auxiliary function in the paper, by which the diffusion and the convection are transformed into a union, is bounded in the weakly degenerate case while unbounded in the strongly degenerate case. Therefore, the method in the paper is not suitable for the strongly degenerate case.

The paper is organized as follows. In §2 we recall the well-posedness of the problem (6)–(8) and some a priori estimates. Some uniform estimates for smooth solutions to a regularized equation are established in §3. By means of these estimates, we first prove the Carleman estimate for the problem (12)–(14) in §4, and subsequently, the observability inequality for the problem (12)–(14) and the null controllability of the problem (6)–(8) are shown.

2. Well-posedness and some a priori estimates. Let us recall the well-posedness of the problem (6)–(8) and some a priori estimates. Some uniform estimates for smooth solutions to a regularized equation are established in §3. By means of these estimates, we first prove the Carleman estimate for the problem (12)–(14) in §4, and subsequently, the observability inequality for the problem (12)–(14) and the null controllability of the problem (6)–(8) are shown.

**Definition 2.1.** A function $u$ is called to be a solution to the problem (6)–(8), if $u \in \mathcal{H}_a((0,1) \times (0,T))$, and for any $\phi \in \mathcal{H}_a((0,1) \times (0,T))$ with $\frac{\partial \phi}{\partial t} \in L^2((0,1) \times (0,T))$ and $\phi(\cdot, T)\big|_{(0,1)} = 0$, the following integral equality holds

$$
\int_0^T \int_0^1 (u \phi_t + x^\alpha u_x \phi_x - u(b\phi)_x + cu\phi) dx dt = \int_0^T \int_0^1 h x \phi dx dt + \int_0^1 u_0(x) \phi(x, 0) dx,
$$
and (7) holds in the trace sense. Here,

$$\mathcal{H}_\alpha((0,1) \times (0,T)) = \{ u \in L^2((0,1) \times (0,T)); x^{\alpha/2}u_x \in L^2((0,1) \times (0,T)) \}.$$ 

Lemma 2.2. ([27]) Assume that $0 < \alpha < 1$ and $b, b_x, c \in L^\infty((0,1) \times (0,T))$. Then for any $h \in L^2((0,1) \times (0,T))$ and $u_0 \in L^2(0,1)$, there exists uniquely a solution $u \in \mathcal{H}_\alpha((0,1) \times (0,T))$ to the problem (6)–(8). Furthermore, $u \in L^\infty(0,T; L^2(0,1)) \cap C_w([0,T]; L^2(0,1))$. Here, $u \in C_w([0,T]; L^2(0,1))$ means that $\int_0^T u(x,t) \gamma(x)dx \in C([0,T])$ for each $\gamma \in L^2(0,1)$.

Consider the nondegenerate linear problem

$$u_t - ((x + \eta)^\alpha u_x)_x + b(x,t)u_x + c(x,t)u = f(x,t), \quad (x,t) \in (0,1) \times (0,T), \quad (15)$$

$$u^\alpha(0,t) = u^\alpha(1,t) = 0, \quad t \in (0,T), \quad (16)$$

$$u^\alpha(x,0) = u_0(x), \quad x \in (0,1), \quad (17)$$

where $0 < \eta < 1$, $b, b_x, c \in L^\infty((0,1) \times (0,T))$, $f \in L^2((0,1) \times (0,T))$ and $u_0 \in L^2(0,1)$. The problem (15)–(17) admits a unique solution $u^\alpha \in L^\infty(0,T; L^2(0,1)) \cap L^2(0,T; H^1(0,1)) \cap C_w([0,T]; L^2(0,1))$. Moreover, $u^\alpha$ satisfies the following energy estimates.

Lemma 2.3. ([27]) Assume that $\|b\|_{L^\infty((0,1) \times (0,T))} \leq K$, $\|b_x\|_{L^\infty((0,1) \times (0,T))} \leq K$ and $\|c\|_{L^\infty((0,1) \times (0,T))} \leq K$. Then the solution $u^\alpha$ to the problem (15)–(17) satisfies

$$\|u^\alpha\|_{L^\infty(0,T; L^2(0,1))} + \|(x + \eta)^{\alpha/2}u_x\|_{L^2((0,1) \times (0,T))} \leq N(\|f\|_{L^2((0,1) \times (0,T))} + \|u_0\|_{L^2(0,1)}),$$

$$\left| \int_0^1 (u^\alpha(x,t_2) - u^\alpha(x,t_1)) \gamma(x)dx \right| \leq N(t_2 - t_1)^{1/2},$$

$$\left( \|f\|_{L^2((0,1) \times (0,T))} + \|u_0\|_{L^2(0,1)} \right) \|\gamma\|_{H^1(0,1)}, 0 \leq t_1 < t_2 \leq T, \quad \gamma \in H^1(0,1)$$

and

$$\int_0^{T-\delta} \left( \int_0^1 (u^\alpha(x,\tau + \delta) - u^\alpha(x,\tau))^2 dx d\tau \right) \leq N\delta^{1/2} \left( \|f\|_{L^2((0,1) \times (0,T))}^2 + \|u_0\|_{L^2(0,1)}^2 \right),$$

$$0 < \delta < T,$$

where $N > 0$ depends only on $K$, $T$ and $\alpha$.

3. Uniform estimates for smooth solutions. In this section, we always assume that $w, w_x \in H^1((0,1) \times (0,T))$ and $w$ solves

$$w_t + ((x + \eta)^\alpha w_x)_x + (b(x,t)w)_x - c(x,t)w = \rho, \quad (x,t) \in (0,1) \times (0,T), \quad (18)$$

$$w(0,t) = w(1,t) = 0, \quad t \in (0,T), \quad (19)$$

where $0 < \eta < 1$ and $\rho \in L^2((0,1) \times (0,T))$. Let us do some uniform estimates for $w$. For convenience, we use $C_i \ (1 \leq i \leq 9)$ to denote a generic positive constant depending only on $\|b\|_{W^{2,1}_\alpha(Q_T)}$, $\|c\|_{L^\infty((0,1) \times (0,T))}$, $T$ and $\alpha$.

Define

$$z(x,t) = e^{s\varphi(x,t)}w(x,t), \quad (x,t) \in (0,1) \times (0,T), \quad s > 0,$$

where

$$\varphi(x,t) = \theta(t)(p(x) - 2\|p\|_{L^\infty(0,1)}), \quad (x,t) \in (0,1) \times (0,T)$$

with

$$\theta(t) = \frac{1}{(t(T-t))^t}, \quad t \in (0,T) \quad \text{and} \quad p(x) = \int_0^x (y + \eta)^{1-\alpha}dy, \quad x \in (0,1),$$

and (17) holds in the trace sense. Here,
and

\[ \zeta(x, t) = \int_0^T b(y, t)(y + \eta)^{-\alpha} dy, \quad (x, t) \in (0, 1) \times (0, T). \]

Then \( z, x \in \mathcal{H}^1((0, 1) \times (0, T)) \) and \( z \) solves

\begin{align*}
(e^{-s\varphi}z)_t + e^s((x + \eta)^\alpha e^{-s\varphi}z)_x = \varrho, & \quad (x, t) \in (0, 1) \times (0, T), \\
z(0, t) = z(1, t) = 0, & \quad t \in (0, T), \\
z(x, 0) = z(x, T) = 0, & \quad x \in (0, 1),
\end{align*}

where

\[ \varrho(x, t) = \rho(x, t)e^{s(x, t)} + (\zeta(x, t) + c(x, t))e^{-s\varphi(x, t)}z(x, t), \quad (x, t) \in (0, 1) \times (0, T). \]

Rewrite (20) into

\[ L_s z = L_s^+ z + L_s^- z = e^{s\varphi} \varrho, \quad (x, t) \in (0, 1) \times (0, T), \]

where

\begin{align*}
L_s^+ z &= \mathcal{A} z - s \varphi t z + s^2 (x + \eta)^\alpha \varphi_x^2 z, & \quad (x, t) \in (0, 1) \times (0, T), \\
L_s^- z &= z_t - s \varphi z - 2 s (x + \eta)^\alpha \varphi_x z_x, & \quad (x, t) \in (0, 1) \times (0, T)
\end{align*}

with

\[ \mathcal{A} u = e^s((x + \eta)^\alpha e^{-s\varphi}u)_x, \quad (x, t) \in (0, 1) \times (0, T), \quad u, u_x \in \mathcal{H}^1((0, 1) \times (0, T)). \]

According to the definitions of \( \theta \) and \( \zeta \), one has

\[ C_1 \leq e^{s(x, t)} \leq C_2, \quad (x, t) \in (0, 1) \times (0, T), \]

\[ \theta(t) \geq C_1, \quad |\theta'(t)| \leq C_2 \theta^{1/4}(t), \quad |\theta''(t)| \leq C_2 \theta^{3/4}(t), \quad t \in (0, T). \]

Lemma 3.1. It holds that

\[ \int_0^T \int_0^1 e^{-s} L_s^+ z L_s^- z dx dt \]

\[ = s \int_0^T \int_0^1 (2(x + \eta)^2 \varphi_{xx} + \alpha(x + \eta)^{2m-1} \varphi_x)e^{-s} \varphi_x^2 dx dt \]

\[ + s \int_0^T \int_0^1 (2(x + \eta)^2 \varphi_{xx} + \alpha(x + \eta)^{2m-1} \varphi_x)e^{-s} \varphi_x^2 dx dt \]

\[ + \frac{s}{2} \int_0^T \int_0^1 (e^{-s} \varphi_t e^{-s} \varphi_t e^{-s} \varphi_t) dx dt \]

\[ - \frac{s}{2} \int_0^T \int_0^1 ((x + \eta)^2 \varphi_{xx} + \alpha(x + \eta)^{2m-1} \varphi_x)e^{-s} \varphi_x^2 dx dt \]

\[ - \frac{1}{2} \int_0^T \int_0^1 (x + \eta)^2 e^{-s} \eta_x e^{-s} \varphi_x^2 dx dt + \frac{s^2}{2} \int_0^T \int_0^1 (x + \eta)^2 e^{-s} \eta_x e^{-s} \varphi_x^2 dx dt \]

\[ - 2s^2 \int_0^T \int_0^1 (x + \eta)^2 e^{-s} \varphi_x \varphi_x z^2 dx dt - s \int_0^T \int_0^1 (x + \eta)^2 e^{-s} \varphi_x \varphi_x z^2 dx dt \bigg|_{z=0}. \]
Proof. Integrating by parts with (21) and (22), one gets that

\[
\begin{align*}
&\int_0^1 \int_0^1 e^{-\xi} \mathcal{A} z L_s z dx dt \\
= &\int_0^T \int_0^1 e^{-\xi} \mathcal{A} z z dx dt - s \int_0^T \int_0^1 e^{-\xi} \mathcal{A} z \mathcal{A} \varphi z dx dt \\
&- 2s \int_0^T \int_0^1 e^{-\xi}(x + \eta)^{\alpha} \mathcal{A} z \varphi x z dx dt \\
= &- \int_0^0 \int_0^1 (x + \eta)\alpha e^{-\xi} z_x z dx dt + s \int_0^T \int_0^1 (x + \eta)^{\alpha} e^{-\xi} z_x (\mathcal{A} \varphi)_z dx dt \\
&- s \int_0^T \int_0^1 ((x + \eta)^{\alpha} e^{-\xi})_x (x + \eta)^{\alpha} \varphi_x z_x^2 dx dt \\
&+ s \int_0^1 \int_0^1 (x + \eta)^{\alpha} e^{-\xi} ((x + \eta)^{\alpha} \varphi_x)_z z_x^2 dx dt \\
&+ s \int_0^T \int_0^1 (x + \eta)^{\alpha} e^{-\xi} (x + \eta)^{\alpha} \varphi_x^2 z_x^2 dx dt - s \int_0^T (x + \eta)^{2\alpha} e^{-\xi} \varphi_x^2 z_x^2 |_{x=0}^1 \\
= &\frac{1}{2} \int_0^T \int_0^1 (x + \eta)^{\alpha} e^{-\xi} \mathcal{A} \varphi z_x^2 dx dt \\
&+ s \int_0^1 \int_0^1 (2(x + \eta)^{2\alpha} e^{-\xi} \varphi_{xx} + \alpha(x + \eta)^{2\alpha-1} e^{-\xi} \varphi_x)_z^2 dx dt \\
&- \frac{1}{2} s \int_0^T \int_0^1 ((x + \eta)^{\alpha} e^{-\xi} (\mathcal{A} \varphi)_z)_z z_x^2 dx dt - s \int_0^T (x + \eta)^{2\alpha} e^{-\xi} \varphi_x^2 z_x^2 |_{x=0}^1 \\
\end{align*}
\]

and

\[
\begin{align*}
&\int_0^T \int_0^1 e^{-\xi}(-s \varphi_t z + s^2(x + \eta)^{\alpha} \varphi_x^2 z) L_s z dx dt \\
= &\int_0^T \int_0^1 (-s \varphi_t e^{-\xi} + s^2(x + \eta)^{\alpha} e^{-\xi} \varphi_x^2 z) z dx dt + s^2 \int_0^T \int_0^1 e^{-\xi} \varphi_t \mathcal{A} \varphi z^2 dx dt \\
&- s^3 \int_0^T \int_0^1 (x + \eta)^{\alpha} e^{-\xi} \mathcal{A} \varphi_x^2 z^2 dx dt + 2s^2 \int_0^T \int_0^1 (x + \eta)^{\alpha} e^{-\xi} \varphi \varphi_t z z_x dx dt \\
&- 2s^3 \int_0^T \int_0^1 (x + \eta)^{2\alpha} e^{-\xi} \varphi_x^2 z z_x dx dt \\
= &\frac{1}{2} \int_0^T \int_0^1 (s \varphi_t e^{-\xi} - s^2(x + \eta)^{\alpha} e^{-\xi} \varphi_x^2 z)^2 dx dt - s^2 \int_0^T \int_0^1 (x + \eta)^{\alpha} e^{-\xi} \varphi_x \varphi_t z z_x^2 dx dt \\
&+ s^3 \int_0^T \int_0^1 (x + \eta)^{\alpha} e^{-\xi} \varphi_x^2 (2(x + \eta)^{\alpha} \varphi_{xx} + \alpha(x + \eta)^{\alpha-1} \varphi_x)_z^2 dx dt \\
= &\frac{1}{2} s \int_0^T \int_0^1 (\varphi_t e^{-\xi} - \varphi e^{-\xi} \omega_t)_z^2 dx dt + \frac{1}{2} s^2 \int_0^T \int_0^1 (x + \eta)^{\alpha} e^{-\xi} \varphi_x^2 z^2 dx dt 
\end{align*}
\]
Then, (25) follows from (27) and (28).

Recall the following Hardy inequality proved in [10].

**Lemma 3.2. (Hardy inequality)** Assume that $u \in H^1(0,1)$ with $u(0) = 0$. Then

$$\int_0^1 (x + \eta)^{\alpha-2} u^2 dx \leq \frac{4}{(1-\alpha)^2} \int_0^1 (x + \eta)^\alpha u^2 x dx.$$  

**Proposition 1.** There exist two positive constants $s_1$ and $M_1$ depending only on $\|b\|_{W^{1,1}_+ (Q_T)}$, $T$ and $\alpha$, but independent of $\eta$, such that for each $s \geq s_1$,

$$\int_0^T \int_0^1 e^{-\xi} L_+^s z L_-^s z dx dt \geq M_1 s \int_0^T \int_0^1 (x + \eta)^\alpha z^2 z dx dt$$

$$+ M_1 s^3 \int_0^T \int_0^1 (x + \eta)^{2\alpha-1} \varphi_x^2 e^{-\xi} z^2 dx dt - s \int_0^T e^{-\xi(1,t)} \theta(t) z^2 (1,t) dt \tag{29}$$

**Proof.** Let us estimate the terms on the right side of (25). First, it follows from the definition of $\varphi$ and (23) that

$$s \int_0^T \int_0^1 (2(x + \eta)^{2\alpha} \varphi_{xx} + \alpha(x + \eta)^{2\alpha-1} \varphi_x) e^{-\xi} z^2 dx dt$$

$$+ s^3 \int_0^T \int_0^1 (2(x + \eta)^{2\alpha} \varphi_{xx} + \alpha(x + \eta)^{2\alpha-1} \varphi_x) \varphi_x^2 e^{-\xi} z^2 dx dt$$

$$= (2 - \alpha) s \int_0^T \int_0^1 (x + \eta)^\alpha e^{-\xi} \theta z^2 dx dt + (2 - \alpha) s^3 \int_0^T \int_0^1 (x + \eta)^{2\alpha-1} e^{-\xi} \theta z^2 dx dt$$

$$\geq \frac{2 - \alpha}{C_2} s \int_0^T \int_0^1 (x + \eta)^\alpha \theta z^2 dx dt + \frac{2 - \alpha}{C_2} s^3 \int_0^T \int_0^1 (x + \eta)^{2\alpha-1} \theta z^2 dx dt. \tag{30}$$

Second, the definitions of $\varphi$ and $\zeta$, (24), the Hölder inequality and Lemma 3.2 lead to

$$\left| \frac{s}{2} \int_0^T \int_0^1 (e^{-\xi} \varphi_{ut} - e^{-\xi} \zeta (\varphi_1)) z^2 dx dt \right|$$

$$= \left| \frac{s}{2} \int_0^T \int_0^1 e^{-\xi} (\theta'' - \theta') \int_0^x b_t(y,t) (y + \eta)^{-\alpha} dy (p(x) - 2 \|p\|_{L^\infty(0,1)} z^2 dx dt \right|$$

$$\leq C_2 \|p\|_{L^\infty(0,1)} s \int_0^T \int_0^1 (\theta^{3/2} + \frac{2^{1-\alpha} \|b_t\|_{L^\infty(0,1) \times (0,T)} \theta^{3/4}}{1-\alpha}) z^2 dx dt$$

$$\leq C_3 s \int_0^T \int_0^1 \theta^2 z^2 dx dt$$

$$\leq C_3 s \left( \int_0^T \int_0^1 (x + \eta)^{2\alpha-1} \theta z^2 dx dt \right)^{1/2} \left( \int_0^T \int_0^1 (x + \eta)^{2\alpha-1} z^2 dx dt \right)^{1/2}$$

$$\leq C_3 s \left( \frac{4}{(1 - \alpha)^2} \int_0^T \int_0^1 (x + \eta)^\alpha \theta z^2 dx dt \right)^{1/2} \left( \int_0^T \int_0^1 (x + \eta)^{2\alpha-1} \theta z^2 dx dt \right)^{1/2}.$$


\[
\begin{align*}
&\leq \frac{2-\alpha}{4C_2}s \int_0^T \int_0^1 (x+\eta)^\alpha \theta z^2 dxdt + \frac{4C_2 C_3^2}{(2-\alpha)(1-\alpha)^2}s \int_0^T \int_0^1 (x+\eta)^{2-\alpha} \theta^3 z^2 dxdt.
\end{align*}
\]

(31)

Third, for each \(0 < \kappa \leq 1\), it follows from the definition of \(\mathcal{A}\), (23), Lemma 3.2 and (24) that

\[
\begin{align*}
&\left| -\frac{s}{2} \int_0^T \int_0^1 ((x+\eta)^\alpha e^{-\varsigma (\mathcal{A}\varphi)_{x,z}}z^2 dxdt \right| \\
&= \left| \frac{s}{2} \int_0^T \int_0^1 e^{-\varsigma} (b_{xx}(x+\eta) + (2-\alpha) b_x - bb_x(x+\eta)^{1-\alpha} - (1-\alpha) \theta^2 (x+\eta)^{-\alpha}) \theta z^2 dxdt \right| \\
&\leq C_4s \int_0^T \int_0^{\max(0,\kappa-\eta)} (x+\eta)^{-\alpha} \theta z^2 dxdt + C_4s \int_0^T \int_0^{\max(0,\kappa-\eta)} (x+\eta)^{-\alpha} \theta z^2 dxdt \\
&\leq C_4 \kappa^{2-2\alpha} s \int_0^T \int_0^1 (x+\eta)^{2-\alpha} \theta z^2 dxdt + C_4 \kappa^{-2} s \int_0^T \int_0^1 (x+\eta)^{2-\alpha} \theta z^2 dxdt \\
&\leq \frac{4C_4 \kappa^{2-2\alpha}}{(1-\alpha)^2} \int_0^T \int_0^1 (x+\eta)^{\alpha} \theta z^2 dxdt + C_5 \kappa^{-2} \int_0^T \int_0^1 (x+\eta)^{2-\alpha} \theta^3 z^2 dxdt,
\end{align*}
\]

which yields

\[
\begin{align*}
&\left| -\frac{s}{2} \int_0^T \int_0^1 ((x+\eta)^\alpha e^{-\varsigma (\mathcal{A}\varphi)_{x,z}}z^2 dxdt \right| \\
&\leq \frac{2-\alpha}{4C_2}s \int_0^T \int_0^1 (x+\eta)^\alpha \theta z^2 dxdt + C_5 \kappa^{-2} \int_0^T \int_0^1 (x+\eta)^{2-\alpha} \theta^3 z^2 dxdt
\end{align*}
\]

(32)

by choosing

\[
\kappa = \min \left\{ 1, \left( \frac{(2-\alpha)(1-\alpha)^2}{16C_2 C_4} \right)^{1/(2-2\alpha)} \right\}.
\]

Forth, the definitions of \(\varphi\) and \(\varsigma\), (23) and (24) show

\[
\begin{align*}
&\left| -\frac{1}{2} \int_0^T \int_0^1 (x+\eta)^\alpha e^{-\varsigma \varsigma_1 z^2 dxdt} + \frac{s^2}{2} \int_0^T \int_0^1 (x+\eta)^\alpha e^{-\varsigma \varsigma_1^2 \varphi^2 z^2 dxdt} \\
&- \int_0^T \int_0^1 (x+\eta)^\alpha e^{-\varsigma \varphi z^2 dxdt} \right| \\
&\leq \frac{1}{2} \left| \int_0^T \int_0^1 (x+\eta)^\alpha e^{-\varsigma} \left( \int_0^x b_t(y,t)(y+\eta)^{-\alpha} dy \right)(-z^2 + s^2 \varphi^2 z^2) dxdt \right| \\
&+ \left| 2s^2 \int_0^T \int_0^1 (x+\eta)^{2-\alpha} e^{-\varsigma} \theta^3 z^2 dxdt \right| \\
&\leq \frac{\|b_t\|_{L^\infty((0,1)\times(0,T))}}{2(1-\alpha)C_1} \left( \int_0^T \int_0^1 (x+\eta)^2 dxdt + s^2 \int_0^T \int_0^1 (x+\eta)^{3-2\alpha} \theta^2 z^2 dxdt \right) \\
&+ \frac{2C_2 C_3^2 s^2}{C_4} \int_0^T \int_0^1 (x+\eta)^{2-\alpha} \theta^3 z^2 dxdt \\
&\leq C_6 \int_0^T \int_0^1 (x+\eta)^\alpha \theta z^2 dxdt + C_8 s^2 \int_0^T \int_0^1 (x+\eta)^{2-\alpha} \theta^3 z^2 dxdt.
\end{align*}
\]

(33)
Fifth, the definition of \( \varphi \) gives
\[
-s \int_0^T (x + \eta)^2 \varphi_x e^{-\varsigma} z_x^2 \, dx \, dt \bigg|_{x=0}^{x=1} \geq -s \int_0^T e^{-\varsigma(1,t)} \theta z_x^2 (1, t) \, dt.
\] (34)

Then, (29) follows from Lemma 3.1 and (30)–(34).

**Theorem 3.3.** There exist two positive constants \( s_2 \) and \( M_2 \) depending only on \( \|b\|_{W^{1,1}_0((0,1) \times (0,T))} \), \( \|c\|_{L^\infty(0,1) \times (0,T)} \), \( T \) and \( \alpha \), but independent of \( \eta \), such that for each \( s \geq s_2 \),
\[
s \int_0^T \int_0^1 (x + \eta)^2 \theta^2 \tau^2 \, dx \, dt + s^3 \int_0^T \int_0^1 (x + \eta)^{2-\alpha} e^{2 \tau \theta} \theta^3 \, w^2 \, dx \, dt
\leq M_2 \left( \int_0^T \int_0^1 \rho^2 e^{2 \tau \theta} \theta^2 \, dx \, dt + s \int_0^T \theta(t) e^{2 \tau \theta} \theta^2 (1, t) \, dt \right).
\] (35)

**Proof.** For each \( s \geq s_1 \), it follows from Proposition 1 and the definitions of \( L^\pm \) that
\[
s \int_0^T \int_0^1 (x + \eta)^2 \theta^2 \tau^2 \, dx \, dt + s^3 \int_0^T \int_0^1 (x + \eta)^{2-\alpha} \theta^3 z^2 \, dx \, dt
\leq \frac{1}{M_1} \int_0^T \int_0^1 L^+ \tau L^- \theta z e^{-\varsigma} \, dx \, dt + \frac{s}{M_1} \int_0^T e^{-\varsigma(1,t)} \theta(t) \theta^2 (1, t) \, dt
\leq \frac{1}{M_2 C_1} \int_0^T \int_0^1 \rho^2 e^{2 \tau \theta} \theta^2 \, dx \, dt + \frac{1}{M_1 C_1} \int_0^T \theta(t) \theta^2 (1, t) \, dt
\leq C_7 \left( \int_0^T \int_0^1 \rho^2 e^{2 \tau \theta} \theta^2 \, dx \, dt + \int_0^T \int_0^1 z^2 \, dx \, dt + s \int_0^T \theta(t) \theta^2 (1, t) \, dt \right).
\] (36)

The Hölder inequality and Lemma 3.2 lead to
\[
\int_0^T \int_0^1 z^2 \, dx \, dt
\leq \left( \int_0^T \int_0^1 (x + \eta)^{2-\alpha} \theta^2 \, dx \, dt \right)^{1/2} \left( \int_0^T \int_0^1 (x + \eta)^{2-\alpha} z^2 \, dx \, dt \right)^{1/2}
\leq \frac{4}{C_1^{1/2}(1-\alpha)^2} \int_0^T \int_0^1 (x + \eta)^2 \theta^2 \, dx \, dt \left( \frac{1}{C_1^{3/2}} \int_0^T \int_0^1 (x + \eta)^{2-\alpha} \theta^3 z^2 \, dx \, dt \right)^{1/2}
\leq \frac{2}{C_1(1-\alpha)} \int_0^T \int_0^1 (x + \eta)^2 \theta^2 \, dx \, dt + \frac{\alpha}{C_1^2} \int_0^T \int_0^1 (x + \eta)^{2-\alpha} \theta^3 z^2 \, dx \, dt.
\] (36)

For each \( s \geq s_1 + 2C_7/(C_1^2 - C_1^2 \alpha) + 1 \), it follows from (35) and (36) that
\[
s \int_0^T \int_0^1 (x + \eta)^2 \theta^2 \, dx \, dt + s^3 \int_0^T \int_0^1 (x + \eta)^{2-\alpha} \theta^3 z^2 \, dx \, dt
\leq 2C_7 \left( \int_0^T \int_0^1 \rho^2 e^{2 \tau \theta} \theta^2 \, dx \, dt + s \int_0^T \theta(t) \theta^2 (1, t) \, dt \right).
\] (37)

The definition of \( z \) yields
\[
w = e^{-s \varphi} e^{-\varsigma} z, \quad w_x = e^{-s \varphi} e^{-\varsigma} (z_x - s \varphi_x z - b(x + \eta)^{-\alpha} z), \quad (x, t) \in (0, 1) \times (0, T).
\] (38)
For each $s \geq s_1 + 2C_7/(C_2^2 - C_2^2\alpha) + 1$, (37), (38), (23) and Lemma 3.2 lead to
\[ s \int_0^T \int_0^1 (x + \eta)^\alpha \theta e^{2s\varphi} w^2_x \ dx \ dt + s^3 \int_0^T \int_0^1 (x + \eta)^{2-\alpha} e^{2s\varphi} \theta^3 w^2 \ dx \ dt \]
\[ \leq C_8 \left( s \int_0^T \int_0^1 (x + \eta)^\alpha \theta z^2_x \ dx \ dt + s^3 \int_0^T \int_0^1 (x + \eta)^{2-\alpha} \theta^3 z^2 \ dx \ dt \right) \]
\[ + s \int_0^T \int_0^1 (x + \eta)^{-\alpha} \theta^2 z^2 \ dx \ dt \]
\[ \leq C_9 \left( s \int_0^T \int_0^1 (x + \eta)^\alpha \theta z^2_x \ dx \ dt + s^3 \int_0^T \int_0^1 (x + \eta)^{2-\alpha} \theta^3 z^2 \ dx \ dt \right) \]
\[ \leq C_7 C_9 \left( \int_0^T \int_0^1 \rho^2 e^{2s\varphi} \ dx \ dt + s \int_0^T \theta z^2_x (1,t) \ dt \right) \]
\[ \leq C_7 C_9 \left( \int_0^T \int_0^1 \rho^2 e^{2s\varphi} \ dx \ dt + C_2^2 s \int_0^T \theta(t) e^{2s\varphi(1,t)} w^2_x (1,t) \ dt \right), \]
which completes the proof of the theorem. \hfill \qed

4. Carleman estimate, observability inequality and null controllability.

In this section, we establish the Carleman estimates and the observability inequalities for the problems (12)–(14) and (9)–(11), and then show the null controllability of the problem (6)–(8).

**Theorem 4.1. (Carleman estimate)** There exist $s_0 > 0$ and $M_0 > 0$ depending only on $\|b\|_{W^{1,1}_2(Q_T)}$, $\|c\|_{L^\infty((0,1) \times (0,T))}$, $x_0$, $x_1$, $T$ and $\alpha$, but independent of $\eta$, such that for each $v_T \in L^2(0,1)$ and each $s \geq s_0$, the solution $v^n$ to the problem (12)–(14) satisfies
\[ s \int_0^T \int_0^1 (x + \eta)^\alpha \theta e^{2s\varphi} (v^n_x)^2 \ dx \ dt + s^3 \int_0^T \int_0^1 (x + \eta)^{2-\alpha} e^{2s\varphi} \theta^3 (v^n)^2 \ dx \ dt \]
\[ \leq M_0 \int_0^T \int_\omega (v^n)^2 \ dx \ dt. \]

**Proof.** For convenience, $v^n$ is abbreviated by $v$ in the proof. By a standard mollification process (see, for example, [27]), we can assume that $v, v_x \in H^1((0,1) \times (0,T))$ without loss of generality. Denote $\tilde{\omega} = ((2x_0 + x_1)/3, (x_0 + 2x_1)/3)$ and define
\[ w(x, t) = \psi(x) v(x, t), \quad (x, t) \in [0, 1] \times [0, T], \] (39)
where $\psi \in C^\infty([0, 1])$ satisfies
\[
\psi \begin{cases} 
= 1, & 0 \leq x \leq \frac{2x_0 + x_1}{3}, \\
\in [0, 1], & x \in \tilde{\omega}, \\
= 0, & \frac{x_0 + 2x_1}{3} \leq x \leq 1.
\end{cases}
\]

Then $w$ solves
\[ w_t + ((x + \eta)^\alpha w_x)_x + (b(x, t)w)_x - c(x, t)w = \rho, \quad (x, t) \in (0, 1) \times (0, T), \]
where
\[ \rho(x, t) = ((x + \eta)^\alpha \psi'(x) v(x, t))_x + \psi'(x)(x + \eta)^\alpha v_x(x, t) + b(x, t)\psi'(x) v(x, t), \]
\[ (x, t) \in (0, 1) \times (0, T). \]
Theorem 3.3 shows that
\[
s\int_0^T \int_0^1 (x+\eta)^\alpha \theta e^{2s\varphi} w^2_x dx dt + s^3 \int_0^T \int_0^1 (x+\eta)^{2-\alpha} \theta^3 w^2 dx dt \leq M_2 \left( \int_0^T \int_0^1 \rho^2 e^{2s\varphi} \theta^3 w^2 dx dt + s \int_0^T \theta(t) e^{2s\varphi(1,t)} w_x^2(1,t) dt \right) = M_2 \int_0^T \int_0^1 \rho^2 e^{2s\varphi} \theta^3 w^2 dx dt, \quad s \geq s_2,
\]
where \(s_2\) and \(M_2\), which depend only on \(\|b\|_{W^{2,1}_x((0,1) \times (0,T))}, \|c\|_{L^\infty((0,1) \times (0,T))}, T\) and \(\alpha\), are given in Theorem 3.3. According to the definition of \(\rho\), one gets that
\[
s\int_0^T \int_0^1 (x+\eta)^\alpha \theta e^{2s\varphi} w^2_x dx dt + s^3 \int_0^T \int_0^1 (x+\eta)^{2-\alpha} \theta^3 w^2 dx dt \leq M_3 \int_0^T \int_0^1 e^{2s\varphi}(v^2 + v_x^2) dx dt, \quad s \geq s_2, \tag{40}
\]
where \(M_3 > 0\) depends only on \(\|b\|_{W^{2,1}_x((0,1) \times (0,T))}, \|c\|_{L^\infty((0,1) \times (0,T))}, x_0, x_1, T\) and \(\alpha\). Let \(\xi \in C_0^\infty(0,1)\) satisfy
\[
\xi \begin{cases} 
= 1, & x \in \tilde{\omega}, \\
\in [0,1], & x \in \omega \setminus \tilde{\omega}, \\
= 0, & x \in (0, x_0] \cup [x_1, 1).
\end{cases}
\]
From (12) and (13), one gets that
\[
0 = \int_0^T \frac{d}{dt} \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt = 2s \int_0^T \int_0^1 \xi^2 \varphi_t e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} \theta^3 v d\theta dx dt = 2s \int_0^T \int_0^1 \xi^2 \varphi_t e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} \theta^3 v d\theta dx dt \leq 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2(0,v_x^2) dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt = 2s \int_0^T \int_0^1 \xi^2 \varphi_t e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt + 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt.
\]
Therefore
\[
\int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt = -2 \int_0^T \int_0^1 \xi^2 \varphi_t e^{2s\varphi} v^2 dx dt - 2 \int_0^T \int_0^1 \xi^2 e^{2s\varphi} v^2 dx dt.
\]
Theorem 4.2. (Observability inequality) There exists $M > 0$ depending only on $\|b\|_{W^{2,1}_x((0,1) \times (0,T))}$, $\|c\|_{L^\infty((0,1) \times (0,T))}$, $x_0$, $x_1$, $T$ and $\alpha$, but independent of $\eta$, such that for each $v_T \in L^2(0,1)$, the solution $v^0$ to the problem (12)–(14) satisfies
\[
\int_0^1 (v^0)^2(x,\tau)dx \leq M \int_0^T \int_\omega (v^0)^2 dx dt, \quad 0 \leq \tau \leq \frac{T}{2}.
\]

Proof. For convenience, $v^0$ is abbreviated by $v$ in the proof. By a standard mollification process (see, for example, [27]), we can assume that $v, v_x \in H^1((0,1) \times (0,T))$ without loss of generality.
For any $s > 0$, it holds that
\[
\begin{align*}
  s \int_0^T \int_0^1 \theta(x + \eta)^\alpha v_x^2 e^{2s\varphi} dxdt + s^3 \int_0^T \int_0^1 \theta^3(x + \eta)^{2-\alpha} v^2 e^{2s\varphi} dxdt \\
  \geq m(s) \int_{T/2}^{3T/4} \int_0^1 (v^2 + v_x^2) dxdt,
\end{align*}
\] (45)
where
\[
m(s) = \min_{0 < s < 1, T/2 \leq t \leq 3T/4} \{ s\theta(t)c^{2s\varphi(x,t)}, s^3\theta^3(t)c^{2s\varphi(x,t)} \}.
\]
Then, it follows from Theorem 4.1, (45) and Lemma 3.2 that
\[
\int_{T/2}^{3T/4} \int_0^1 v^2 dxdt \leq N_1 \int_0^T \int_\Omega v^2 dxdt,
\] (46)
where $N_1 > 0$ depends only on $\|b\|_{W^{2,1}_0(\Omega_T)}$, $\|c\|_{L^\infty((0,1) \times (0,T))}$, $x_0$, $x_1$, $T$ and $\alpha$. For $0 \leq \tau \leq T/2 \leq \tilde{\tau} \leq T$, multiplying (12) by $v$ and then integrating over $(0,1) \times (\tau, \tilde{\tau})$ by parts, one gets that
\[
\begin{align*}
  &\frac{1}{2} \int_0^1 v^2(x, \tau) dx - \frac{1}{2} \int_0^1 v^2(x, \tilde{\tau}) dx \\
  &= \frac{1}{2} \int_{\tau}^{\tilde{\tau}} \frac{d}{dt} \left( \int_0^1 v^2 dx \right) dt \\
  &= \int_{\tau}^{\tilde{\tau}} \int_0^1 (x + \eta)^\alpha v_x^2 + bvv_x + cv^2 dx dt \\
  \geq & \int_{\tau}^{\tilde{\tau}} \int_0^1 (x + \eta)^\alpha v_x^2 dx dt - \|b\|_{L^\infty((0,1) \times (0,T))} \int_\tau^{\tilde{\tau}} \int_0^1 |v_x| dx dt \\
  &\quad - \frac{1}{2} \|c\|_{L^\infty((0,1) \times (0,T))} \int_{\tau}^{\tilde{\tau}} \int_0^1 v^2 dx dt \\
  \geq & \frac{1}{2} \int_{\tau}^{\tilde{\tau}} \int_0^1 (x + \eta)^\alpha v_x^2 dx dt - \frac{1}{2} \|b\|_{L^\infty((0,1) \times (0,T))} \int_{\tau}^{\tilde{\tau}} \int_0^1 (x + \eta)^{-\alpha} v^2 dx dt \\
  &\quad - \frac{1}{2} \|c\|_{L^\infty((0,1) \times (0,T))} \int_{\tau}^{\tilde{\tau}} \int_0^1 v^2 dx dt.
\end{align*}
\] (47)
For each $0 < \kappa \leq 1$, it follows from Lemma 3.2 that
\[
\begin{align*}
  &\int_{\tau}^{\tilde{\tau}} \int_0^1 (x + \eta)^{-\alpha} v^2 dx dt \\
  = & \int_{\tau}^{\tilde{\tau}} \int_{\max\{0, \kappa-\eta\}}^1 (x + \eta)^{-\alpha} v^2 dx dt + \int_{\tau}^{\tilde{\tau}} \int_{\max\{0, \kappa-\eta\}}^1 (x + \eta)^{-\alpha} v^2 dx dt \\
  \leq & \kappa^{-2\alpha} \int_{\tau}^{\tilde{\tau}} \int_0^1 (x + \eta)^{2-\alpha} v^2 dx dt + \kappa^{-\alpha} \int_{\tau}^{\tilde{\tau}} \int_0^1 (x + \eta)^{2-\alpha} v^2 dx dt \\
  \leq & \frac{4\kappa^{2-2\alpha}}{(1 - \alpha)^2} \int_{\tau}^{\tilde{\tau}} \int_0^1 (x + \eta)^\alpha v_x^2 dx dt + \kappa^{-\alpha} \int_{\tau}^{\tilde{\tau}} \int_0^1 (x + \eta)^{2-\alpha} v^2 dx dt.
\end{align*}
\] (48)
Choosing
\[
\kappa = \min \left\{ 1, \left( \frac{1 - \alpha}{2\|b\|_{L^\infty((0,1) \times (0,T))}} \right)^{1/(1-\alpha)} \right\},
\]
and substituting (48) into (47) yield that
\[
\int_0^1 v^2(x, \tau)dx \leq \int_0^1 v^2(x, \tilde{\tau})dx + N_2 \int_\tau^{\tilde{\tau}} \int_0^1 v^2 dx dt,
\]
where \( N_2 > 0 \) depends only on \( \|b\|_{W^{1,1}_2(Q_T)}, \|c\|_{L^\infty((0,1) \times (0,T))}, x_0, x_1, T \) and \( \alpha \).

Then, it follows from the Gronwall inequality that
\[
\int_0^1 v^2(x, \tau)dx \leq e^{N_2(\tilde{\tau} - \tau)} \int_0^1 v^2(x, \tilde{\tau})dx, \quad 0 \leq \tau \leq T/2 \leq \tilde{\tau} \leq T.
\]  
(49)

Integrating (49) over \((T/2, 3T/4)\) with respect to \(\tilde{\tau}\), one gets that
\[
\frac{1}{4}T \int_0^1 v^2(x, \tau)dx \leq e^{3/4N_2T} \int_{T/2}^{3T/4} \int_0^1 v^2 dx dt, \quad 0 \leq \tau \leq T/2,
\]
which, together with (46), completes the proof of the Theorem. \(\square\)

By a standard limit process, one can get the Carleman estimate and the observability inequality for the degenerate problem (9)–(11).

**Theorem 4.3. (Carleman estimate)** There exist \( s_0 > 0 \) and \( M_0 > 0 \) depending only on \( \|b\|_{W^{1,1}_2(Q_T)}, \|c\|_{L^\infty((0,1) \times (0,T))}, x_0, x_1, T \) and \( \alpha \), such that for each \( v_T \in L^2(0,1) \) and each \( s \geq s_0 \), the solution \( v \) to the problem (9)–(11) satisfies
\[
\int_0^1 s \int_0^T v^2 dx dt + s^3 \int_0^T \int_0^1 v^2 dx dt \
\leq \int_0^T \int_0^1 v^2 dx dt \leq M_0 \int_0^T \int_\Omega v^2 dx dt
\]
with
\[
\theta(t) = \frac{1}{(t(T-t))^4}, \quad t \in (0, T)
\]
and
\[
\varphi(x, t) = \frac{x^{2-\alpha} - 2}{(2-\alpha)(t(T-t))^4}, \quad (x, t) \in (0,1) \times (0,T).
\]

**Theorem 4.4. (Observability inequality)** There exists \( M > 0 \) depending only on \( \|b\|_{W^{1,1}_2(Q_T)}, \|c\|_{L^\infty((0,1) \times (0,T))}, x_0, x_1, T \) and \( \alpha \), such that for each \( v_T \in L^2(0,1) \), the solution \( v \) to the problem (9)–(11) satisfies
\[
\int_0^1 v^2(x, \tau)dx \leq M \int_0^T \int_\Omega v^2 dx dt, \quad 0 \leq \tau \leq T/2.
\]

Using the observability inequality, one can show the null controllability of the problem (6)–(8).

**Theorem 4.5.** The problem (6)–(8) is null controllable. More precisely, for any \( u_0 \in L^2(0,1) \), there exists \( h \in L^2((0,1) \times (0,T)) \), such that the solution \( u \in L^\infty(0,T; L^2(0,1)) \cap H^1_0((0,1) \times (0,T)) \cap C_w([0,T]; L^2(0,1)) \) to the problem (6)–(8) satisfies
\[
u(x, T) = 0, \quad x \in (0,1).
\]

Furthermore, there exists \( N > 0 \) depending only on \( \|b\|_{W^{1,1}_2(Q_T)}, \|c\|_{L^\infty((0,1) \times (0,T))}, x_0, x_1, T \) and \( \alpha \), such that
\[
\|b\|_{L^2((0,1) \times (0,T))} \leq N \|u_0\|_{L^2(0,1)}.
\]
More general, one can consider the following nonlinear equation
\[
    u_t - (x^\alpha u_x)_x + b(x,t)u_x + g(x,t,u) = h(x,t)\chi_\omega, \quad (x,t) \in (0,1) \times (0,T), \quad (50)
\]
where \(0 < \alpha < 1\), \(b \in W^{2,1}_\infty((0,1) \times (0,T))\) and
\[
    g(x,t,0) = 0, \quad |g(x,t,u) - g(x,t,v)| \leq K|u - v|,
\]
\((x,t) \in (0,T) \times (0,1), u, v \in \mathbb{R} \quad (K > 0)\).

**Definition 4.6.** A function \(u\) is called to be a solution to the problem (50), (7), (8), if \(u \in \mathcal{H}_a((0,1) \times (0,T))\), and for any \(\phi \in \mathcal{H}_a((0,1) \times (0,T))\) with \(\frac{\partial \phi}{\partial t} \in L^2((0,1) \times (0,T))\) and \(\phi(\cdot,T)|_{(0,1)} = 0\), the following integral equality holds
\[
    \int_0^T \int_0^1 (-u \phi_t + x^\alpha u_x \phi_x - u(b\phi)_x + g(x,t,u)\phi)dxdt
    = \int_0^T \int_0^1 h\chi_\omega \phi dxdt + \int_0^1 u_0(x)\phi(x,0)dx,
\]
and (7) holds in the trace sense.

**Theorem 4.7.** The problem (50), (7), (8) is null controllable. More precisely, for any \(u_0 \in L^2((0,1))\), there exists \(h \in L^2((0,1) \times (0,T))\), such that the solution \(u \in L^\infty(0,T;L^2(0,1)) \cap \mathcal{H}_a((0,1) \times (0,T)) \cap C_w([0,T];L^2(0,1))\) to the problem (50), (7), (8) satisfies
\[
    u(x,T) = 0, \quad x \in (0,1).
\]
Furthermore, there exists \(N \geq 0\) depending only on \(\|b\|_{W^{2,1}_\infty(Q_T)}, K, x_0, x_1, T \) and \(\alpha\), such that
\[
    \|h\|_{L^2((0,1) \times (0,T))} \leq N\|u_0\|_{L^2(0,1)}.
\]

Theorems 4.5 and 4.7 are based on Theorems 4.1, 4.2 and Lemma 2.3. The process of the proof is the same as the one in [27] and is omitted here.

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