Certain aspects of regularity in scalar field cosmological dynamics

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Abstract

We consider dynamics of the FRW Universe with a scalar field. Using Maupertuis principle we find a curvature of geodesics flow and show that zones of positive curvature exist for all considered types of scalar field potential. Usually, phase space of systems with the positive curvature contains islands of regular motion. We find these islands numerically for shallow scalar field potentials. It is shown also that beyond the physical domain the islands of regularity exist for quadratic potentials as well.

1 Introduction

It is known from 70-th of the last century that contraction phase of a closed Universe filled with a massive scalar field can be followed by expansion for some particular initial condition set [1, 2]. On the other hand, every expansion stage of such Universe is ultimately followed by a contraction one. These two features of dynamics (which are specific for a closed Universe in contrast to open or flat worlds) result in a rather complicated behavior which in some situations can be chaotic.

A chaotic dynamics in massive scalar field cosmology was first found by D. Page in [3], and have been studied in detail in [4, 5] with the corresponding discussion on the meaning of chaos in General Relativity. The chaotic dynamics in question represents an example of a transient chaos with a structure of "chaotic repellor" – a countable set of unstable periodic and uncountable set
of unstable aperiodic trajectories escaping cosmological singularity. Both sets
have zero measure in initial condition space. An useful toy model of such kind
of dynamics (elastic scattering on three discs on a plane) have been described
in [6].

All these early results have been found for a massive scalar field – scalar field
with the potential in the form $V(\phi) = \frac{m^2 \phi^2}{2}$, where $m$ is the scalar field mass.
Studies of other forms of the potential reveal several different form of dynamics
with transitions from one form to another (for a short review see [7]).

The equations of motion can be derived from the General Relativity action
with a scalar field

$$S = \int d^4 x \sqrt{-g} \left\{ \frac{m_P^2}{16\pi} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right\}, \quad (1)$$

where $m_P$ is the constant parameter called the Planck mass, $R$ is the scalar
curvature of a space-time. For a closed Friedman model with the metric

$$ds^2 = dt^2 - a^2(t) d^2 \Omega^{(3)}, \quad (2)$$

where $a(t)$ is a cosmological scale factor, $d^2 \Omega^{(3)}$ is the metric of a unit 3-sphere
and with homogeneous scalar field $\phi$ the action (1) gives the following equations:

$$\frac{m_P^2}{16\pi} \left( \ddot{a} + \frac{\dot{a}^2}{2a} + \frac{1}{2a} \right) + \frac{3\ddot{\phi}}{a} + 2V'(\phi) = 0, \quad (3)$$

$$\ddot{\phi} + \frac{3\dot{a}}{a} \phi' = 0. \quad (4)$$

with two variables - a scale factor $a$ and a scalar field $\phi$.

This system has one first integral of motion

$$-\frac{3}{8\pi} m_P^2 a (\dot{a}^2 + 1) + \frac{a^3}{2} \left( \dot{\phi}^2 + 2V(\phi) \right) = C. \quad (5)$$

This integral plays the role of energy and is equal to zero for cosmological
solutions of the system (3)-(4).
If $V(\phi) = 0$, the system is integrable (in this case a scalar field has the equation of state of stiff matter), though the dynamics is finite in time – the Universe evolves from Big Bang singularity towards a Big Crunch singularity through a point of maximal expansion. There is also another reason why we can not develop a perturbation analysis for the system (3)–(5): it can be easily checked that multiplying the scalar field potential $V$ by a constant with simultaneous rescaling of the scale factor $a$ and time $t$ does not change the system. This means that there is no continuous limit when, for example, $m \to 0$ for the massive scalar field.

An important property of the constraint equation (5) is that $\dot{a}^2$ and $\dot{\phi}^2$ enter in its left-hand side with opposite signs. Rewriting (5) in the form

$$-\frac{3m_P^2}{8\pi} \dot{a}^2 + \frac{\dot{\phi}^2}{2} = \frac{3m_P^2}{8\pi} \frac{1}{a^2} - V(\phi)$$

(6)

it is easy to see that

- There are no forbidden regions in the configuration state.
- The configuration state is divided to two regions – the region where the right-hand side of (6) is positive (and possible extrema of the scale factor are located), and the region where it is negative (where possible extrema of the scalar field are located).

The boundary between these two regions is the curve

$$a^2 = \frac{3}{8\pi} \frac{m_P^2}{V(\phi)}$$

(7)

Extrema of the scale factor can exist only in the region where

$$a^2 \leq \frac{3}{8\pi} \frac{m_P^2}{V(\phi)}.$$

Using (3) it can be shown that the possible points of maximal expansion ($\dot{a} = 0$, $\ddot{a} < 0$) are localized inside the region

$$a^2 \leq \frac{1}{4\pi} \frac{m_P^2}{V(\phi)}.$$
while the possible points of transition from contraction to expansion (often called bounce) lie outside this region being at the same time inside the region of scalar field extrema.

Zero-velocity points ($\dot{a} = \phi = 0$) of a trajectory, if they exist, should lie on the curve (7). Numerical studies show that trajectories with these points play an important role in the described chaotic structure. In particular, all primary (i.e., having one bounce per period) periodical trajectories have zero-velocity points as the points of bounce (see numerical examples in [4]).

Numerical integrations show also that there are regions on the curve (7) which can not contain points of bounce. If a trajectory, starting from the curve (7) is directed inside the zone of possible extrema of the scale factor, it rapidly goes through a point of maximal expansion and evolves further towards a singularity. The condition for a trajectory to be directed into the opposite zone (the zone of possible extrema of the scalar field) can be written as

$$\frac{\ddot{\phi}}{\dot{a}} > d\phi(a)/da$$  \hspace{1cm} (8)

where the function $\phi(a)$ in the RHS is the equation of the curve (7).

The case of equality in (8) corresponds to a trajectory, tangent to the curve (7). This situation was first described in [3], and we call such point as a Page point. For the system (1)-(3) the equation for the Page point is [8]

$$V(\phi_{\text{page}}) = \sqrt{\frac{3m_P^2}{16\pi}V'(\phi_{\text{page}})}$$  \hspace{1cm} (9)

For power-law scalar field potentials the condition (8) is satisfied if $\phi > \phi_{\text{page}}$, and the corresponding part of the curve (7) contains zero-velocity bounce points of periodical trajectories. For exponential potentials the condition (8) can be violated for all points on the curve (7), and the whole chaotic structure disappears [8]. The dynamics in this case can be called regular in the sense that the structure of chaotic repellor is absent, and there are no trajectories escaping...
a cosmological singularity. Global structure of trajectories is the same as for
$V = 0$ case – any trajectory starts and ends in a singularity. This does not
mean that there are no local instabilities of trajectories (see the next section),
however, as the dynamics is finite in time these instabilities are not important
for the global picture.

In the next section we study local instabilities of trajectories, in Sec.3 we
describe a new regular regime for the system (3)–(5), which appears when the
scalar field potential is sufficiently shallow.

2 Local instability

The system (3)-(4) can be considered as a Hamiltonian system with

$$
\mathcal{H} = -\frac{2\pi}{3m^2_P} p_a^2 + \frac{1}{2a^3} p_\phi^2 - \frac{3m^2_P a^3}{8\pi} + V(\phi)a^3, \quad (10)
$$

where the canonical variables are $q^i = (a, \phi)$.

We rewrite (10) in the form

$$
\mathcal{H} = a^{ij} p_i p_j + V \quad (11)
$$

where

$$
a^{ij} = \text{diag}( -\frac{2\pi}{3am^2_P}, \frac{1}{2a^3}),
$$

$$
V = V(\phi)a^3 - 3m^2_P a/(8\pi).
$$

According to Maupertuis principle we can introduce an auxiliary metric
space with geodesics coinciding with trajectories of initial system. This method
of geodesics have been used actively since the middle of the last century [9] in
various physical problems from Yang-Mills fields [10] to cosmology [11]. Calculating
the Riemann curvature $\mathcal{R}$ of this space we get the criterion of local
instability in the form $\mathcal{R} < 0$. For the system with a Hamiltonian in the form
(11), the resulting curvature is [12, 13]
\[ R = \frac{(n - 1)}{2(C - V)^3} \sum \left[ (C - V) \frac{\partial^2 V}{\partial q^i \partial q^j} a^{ij} - (n - 6) \frac{\partial V}{\partial q^i} \frac{\partial V}{\partial q^j} a^{ij} \right], \]

where \( n \) is the number of degrees of freedom.

Taking \( C = 0 \) we get the curvature for cosmological solutions of our system:

\[ R = \frac{-1}{2\left(\frac{3}{8} \frac{m_p^2}{\pi} a - a^3 V \right)^3} \left[ aV'' \left( \frac{3 m_p^2}{8 \pi} - a^2 V \right) + a^3 \left( V'^2 - 4 \frac{m_p^2}{m_p^2} V^2 \right) - \frac{3}{16} \frac{m_p^2}{\pi} a \right], \]

where \( ' \) denotes partial differentiating with respect to a scalar field \( \phi \).

The curvature diverges at the curve (7), and this is a consequence of the fact that the supermetrics \( a^{ij} \) is not positively definite. This property is typical for dynamical systems of General Relativity and discussed in [13].

Three different cases are plotted in Fig.1. The first plot (Fig.1(a)) represents the situation of a massive scalar field with the potential \( V = m^2 \phi^2 / 2 \). We know that a chaotic repellor exists in the corresponding dynamical system. The Fig.1(b) is created for the potential \( V = \cosh(\phi) - 1 \), for which it is known that the structure of chaotic repellor is absent.

We can see that in both cases there are zones with \( R > 0 \) as well as zones with \( R < 0 \). It means that studying only local properties of our system, it is impossible to understand its global features. The chaotic system does not belong to the class of Anosov’s U-systems (this would require \( R < 0 \) in all points of the configuration space), which posses maximally strong statistical properties [14].

However, some qualitative difference in configuration of \( R < 0 \) and \( R > 0 \) zones exists between the chaotic case in Fig.1(a) and the regular case in Fig. 1(b). It can be easily checked that the curve \( R = 0 \) intersects the boundary curve (7) at the Page point (9), so the picture for the system with Page points (as for power-law potentials) and without Page points (for steep potentials) should be different. Moreover, we can see from the plots that the area of possible bounces
(which is located between two hyperbolae) belongs to a stable zone in a regular case, and some part of this area is located in unstable zone for the chaotic case. This can be checked further for the potential \( V = (\exp(\phi/\phi_0)^2 + \exp(-\phi/\phi_0)^2) - 2 \) with \( \phi_0 < 0.96m_P \), where chaotic repellor exists for intermediate values of \( \phi \) (Fig.1(c)) [8]. All this lead to a suggestion that negative curvature in zone of bounces is required for the described type of chaotic behavior, though more effort is needed for better understanding this connection.

3 Islands of regularity

The main goal of this section is to claim that another type (different from described in Introduction) of regular dynamics exists for the system (3)–(4). This type of dynamics is known for a variety of dynamical systems [15–18]. We have found it in the cosmological dynamics \( (C = 0) \) with shallow potentials. In a general situation this behavior exists even for quadratic potential if the constant \( C \) from the constraint equation (5) is sufficiently large and negative.

In all our numerical studies we use the units \( m_P/\sqrt{16\pi} = 1 \).

It has been already reported that in these two cases the picture of chaotic dynamics changes significantly in comparison with the cosmological case of power-law potentials. In Fig.2 we reproduce the plot from [7], where zones in initial condition space leading to at least one bounce is shown for the potential \( V = m^2\phi^2/2 \) and negative \( C \). A trajectory starts at the point of maximal expansion, so \( \dot{a} = 0 \), and using the constraint equation (5) for eliminating \( \dot{\phi} \) we get 2-dimensional space of initial conditions \( (a, \phi) \). In Fig.4 the \( \phi = 0 \) section of this space is shown depending on \( C \). When some bands, being separate for small \( |C| \) merge with \( |C| \) increasing, structure of trajectories changes. Trajectories starting from the wide zone formed by merged bands show complicated

\[ \text{Footnote 1:} \text{In nonstandard cosmology this type of dynamics exists also in braneworlds [19] and in theories with quadratic curvature corrections to Einstein gravity [20].} \]
behavior, which can not be described in a simple way, like in the $C = 0$ case. That is why this type of behavior have been called as "strong" or "less ordered" chaos. However, the dynamical behavior in this case appeared to be even more complicated in the sense that islands of regular dynamics can be distinguished in a chaotic "sea".

To see this we have studied an appropriate Poincare section, taking for this purpose a point of maximal expansion, i.e. the point with $\dot{a} = 0, \ddot{a} < 0$. In Fig. 3 the points of this Poincare section are plotted for quadratic scalar field potential with $m = 1.0$ and $C = -300$. Initial values of the scale factor in the point of maximal expansion vary from 7 to 12. Island of regular dynamics is easily distinguishable. It should be pointed out that this regular regime is of completely different nature than regular dynamics for a steep scalar field potential. Now we can see a quasi-periodic behavior, and trajectories from this part of the phase space never fall into a singularity. A particular strongly stable periodical trajectory exists for initial $a \sim 8$ (in general, this value is a function of $C$).

In the vicinity of this trajectory the points of Poincare sections form rather regular ovals, those from more distant orbits show some structure in their Poincare set. In the transition zone between regular and chaotic zones non-trivial structures can appear, like shown in Fig. 4. Here we can see a Poincare set consisting of 9 disconnected ovals. Structure of zone between chaos and regularity requires further investigations.

We have seen similar islands of regularity in the cosmological solutions (i. e. with $C = 0$) of the system (3)–(4) with a family of shallow Damour-Mukhanov potentials [21] studied with respect to their chaotic properties in [22]. This family has the form

$$V(\varphi) = \frac{M^4}{q} \left[ \left( 1 + \frac{\varphi^2}{\varphi^2_0} \right)^{q/2} - 1 \right].$$

(13)
with three parameters – \( M_0, q \) and \( \varphi_0 \) and interpolates between a quadratic form of the potential for small \( \phi \) and a power-law form \( V(\phi) \sim \phi^q \) for large \( \phi \).

It was found numerically that the described event of bounce band merging exists for a subfamily of (13) with \( q < 1.24 \) and \( \phi_0 < \tilde{\phi}_0 \), where \( \tilde{\phi}_0 \) is some function of \( q \) \cite{22}. In the table below we compare \( \tilde{\phi}_0(q) \) with the value of \( \phi'_0 \) for which the island of regularity appears (it exists for \( \phi_0 < \phi'_0 \)). They are close enough, and it indicates that bounce zone merging is a good indicator of regularity in our system. The difference between these two values grows significantly only in the vicinity of the critical value of \( q \sim 1.24 \) when this type of dynamics disappears for any \( \phi_0 \).

| \( q \) | \( \phi_0 \) | \( \phi'_0 \) |
|-------|-------|-------|
| 0.5   | 0.5   | 0.48  |
| 1.0   | 0.25  | 0.23  |
| 1.2   | 0.06  | 0.045 |
| 1.21  | 0.045 | 0.035 |
| 1.22  | 0.035 | 0.015 |

To our knowledge, this is the first reported case of stable periodical cosmological regime in General Relativity (other known stable examples like described in \cite{23} require modifications of Einstein gravity, some other solutions periodic with respect to scale factor evolution in time are accompanied by a monotonic growth of a scalar field \cite{24}). This regular island is surrounded by a "sea" of chaotic trajectories. In general all of them are finite and end in a cosmological singularity, though existence of chaotic infinite trajectories of non-zero measure is not currently ruled out in our numerical studies.

## 4 Conclusions

We have studied local instability of trajectories representing evolution of a closed isotropic Universe with a scalar field using the Maupertuis principle. Zones of positive and negative curvature \( \mathcal{R} \) of the configuration space are constructed. It is shown that the system in question is not a U-system, and there are zones with
positive and negative $\mathcal{R}$. It is interesting that though zones with different sign of $\mathcal{R}$ exist for both power-law and exponentially steep potential, in the latter case (when the structure of a transient chaos is absent) the configuration of zones differs qualitatively from the case of less steep potential (when the chaos is present). Our empirical data suggests that a crucial feature for chaos to exist is local instability of trajectories in the narrow zones when bounce can occur.

A typical property of dynamical system with nonnegative $\mathcal{R}$ is that their phase space contains positive-measured islands of regular dynamics [10]. We show that the dynamics of this type exists in the case of sufficiently shallow potential as well as in another similar systems with merged bands of initial conditions leading to bounces. Trajectories in these islands never reach the cosmological singularity, as well as inflationary regime, and have trapped for infinite time intervals in a certain zone of small scale factors.

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Figure 1: Zones of negative curvature (dashed) for the quadratic scalar field potential (a), for the potential $V = \cosh(\phi) - 1$ (b) and $V = \left(\exp(\phi/\phi_0)^2 + \exp(-\phi/\phi_0)^2\right) - 2$ with $\phi_0 < 0.96m_P$ (c). Possible bounces can be located between bold and dashed lines.
Figure 2: Zones of initial scale factor for trajectories which have bounce if started from their points of maximal expansion. For large enough negative $C$ separate intervals of initial scale factors merge.

Figure 3: A Poincare section for trajectories starting in the zone of bounce intervals merging. The scalar field potential $V(\phi) = m^2 \phi^2 / 2$, $m = 1.0$, the constant $C = -300$. Regular structure around initial $a \sim 8, \phi = 0$ is clearly seen.
Figure 4: A Poincare section for a particular trajectory near the boundary of the island of regularity. The potential and $C$ are the same as in Fig.3.