Dynamic localization and transport in complex crystals

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The behavior of a Bloch particle in a complex crystal with \(\mathcal{PT}\) symmetry subjected to a sinusoidal ac force is theoretically investigated. For unbroken \(\mathcal{PT}\) symmetry and in the single-band approximation, it is shown that time reversal symmetry of the ac force preserves the reality of the quasienergy spectrum. Like in ordinary crystals, exact band collapse, corresponding to dynamic localization, is attained for a sinusoidal band shape. The wave packet dynamics turns out to be deeply modified at the \(\mathcal{PT}\) symmetry breaking point, where band merging occurs and Bragg scattering in the crystal becomes highly non-reciprocal.

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I. INTRODUCTION

The dynamics of matter or classical waves in periodic potentials subjected to external dc or ac forces is strongly influenced by Bragg scattering, which is responsible for such important effects as Bloch oscillations (BOs) and dynamic localization (DL). Dynamic localization was originally proposed by Dunlap and Kenkre [1] as a suppression of the broadening of a charged-particle wave packet as it moves along a tight-binding lattice driven by a sinusoidal ac electric field. DL was then explained in terms of quasienergy band collapse [2], and the general conditions of DL beyond the nearest-neighbor tight-binding (NNTB) approximation used by Dunlap and Kenkre were subsequently investigated in Ref. [3]. The interest on DL has been recently renewed since the first experimental observations of DL have been reported for matter waves trapped in dynamical optical lattices [4–6], and for light waves in curved waveguide arrays [7, 11]. Matter or light waves may also interact with complex potentials. Complex crystals for matter waves emerge, for instance, in the near resonant interaction of light with open two-level systems [12], whereas in optics complex crystals are realized by waveguide arrays that include gain and/or loss regions [13]. As compared to ordinary crystals, complex crystals exhibit some unique properties, such as violation of the Friedel’s law of Bragg scattering and nonreciprocal diffraction [12, 14]. A special class of complex crystals is provided by complex potentials possessing parity-time (\(\mathcal{PT}\)) symmetry [13, 15, 16]. An important property of \(\mathcal{PT}\) crystals is to admit of an entirely real-valued energy spectrum below a phase transition (symmetry-breaking) point, in spite of the non-Hermiticity of the underlying Hamiltonian [13]. A recent study on BOs in \(\mathcal{PT}\) crystals [17] has shown that the common wisdom of coherent quantum transport in a crystal is greatly modified when dealing with a complex crystal.

It is the aim of this work to investigate the coherent motion of wave packets in a complex \(\mathcal{PT}\) crystal driven by an ac-like force. In particular, it is shown that in the unbroken \(\mathcal{PT}\) symmetry region time reversal symmetry of the ac-like force preserves the reality of the quasienergy spectrum, and that a full band collapse, corresponding to dynamic localization, occurs within the single-band and NNTB approximations like in ordinary crystals. However, the transport properties of the lattice are deeply modified at the \(\mathcal{PT}\) symmetry breaking, where Bragg scattering in the crystal becomes highly non-reciprocal.

II. DYNAMIC LOCALIZATION IN COMPLEX CRYSTALS WITH UNBROKEN \(\mathcal{PT}\) SYMMETRY

In optics, the coherent motion of charged quantum particles in periodic potentials driven by an ac electric field can be mimicked by the propagation of light waves in a periodically-curved photonic lattice [7, 18]. In the scalar and paraxial approximations, light propagation at wavelength \(\lambda\) in the lattice is described by the Schrödinger-type wave equation

\[
i\hbar \partial_x \psi = -\frac{\mathbf{A}^2}{2n_s} \partial_x^2 \psi + V(x)\psi - Fx\psi = (\mathcal{H}_0 - Fx)\psi. \tag{1}\]

where \(\mathbf{A} = \lambda/(2\pi)\) is the reduced wavelength, \(n_s\) is the substrate refractive index, \(V(x) = n_s - n(x)\) is the potential, \(n(x) = n(x + a)\) is the refractive index profile of the lattice (spatial period \(a\)), and \(F = F(z)\) is a fictitious refractive index gradient proportional to the local waveguide axis curvature which mimics the action of a driving force \(\mathbf{A}\). In particular, a sinusoidal ac-like force is simply mimicked by a sinusoidal bending profile of the waveguides [2]. In a complex lattice, the refractive index is complex, and the \(\mathcal{PT}\) symmetry requirement \(V(-x) = V^*(x)\) corresponds to suitable combinations of optical gain and loss regions in the lattice as discussed in [13]. The real and imaginary parts of the potential are denoted by \(V_R(x)\) and \(\alpha V_I(x)\), respectively, where \(\alpha \geq 0\) is a dimensionless parameter that measures
the anti-Hermitian strength of $H_0$. The spectrum of $H_0$ turns out to be real-valued for $\alpha < \alpha_c$, where $\alpha_c \geq 0$ corresponds to the transition from unbroken to broken $\mathcal{PT}$ symmetry. Numerical studies generally show that for $\alpha < \alpha_c$ the spectrum is composed by bands separated by gaps like in an ordinary crystal, whereas for $\alpha \geq \alpha_c$ band merging is observed with the appearance of pairs of complex-conjugate eigenvalues \cite{13,14}. For instance, for the potential defined by

$$V_R(x) = V_0 \cos(2\pi x/a), \quad V_I(x) = V_0 \sin(2\pi x/a), \quad \text{(2)}$$

one has $\alpha_c = 1$ \cite{13}.

In this section we consider the unbroken symmetry phase, i.e. the case $\alpha < \alpha_c$. As recently shown in Ref. \cite{17}, for $\alpha < \alpha_c$ the motion of a Bloch particle in presence of an external dc force $F$ can be described following the same lines as in ordinary crystals by expanding the field $\psi(x, z)$ as a superposition of Bloch-Floquet eigenfunctions $\phi_n(x, \kappa) = u_n(x, \kappa) \exp(i x) \exp(i 2 \pi n z)$ of $H_0$, where the wave number $\kappa$ varies in the first Brillouin zone, i.e. $-k_B/2 \leq \kappa < k_B/2$, $k_B = 2\pi/a$ is the Bragg wave number, $n$ is the band index, and $u_n(x, \kappa)$ is the periodic part of the Bloch-Floquet eigenfunction. After setting $\psi(x, z) = \sum_n \int d x c_n(\kappa, z) \phi_n(\kappa, z)$ and assuming normalized eigenfunctions such that $\int d x \phi_n^*(x, -\kappa) \phi_n(x, \kappa) = \delta_{\kappa, \kappa'}$, with $D_n = \pm 1$, the evolution equations for the spectral coefficients $c_n(\kappa, z)$ read \cite{17}

$$i \hbar \left( \frac{\partial}{\partial z} + \frac{F}{\omega} \frac{\partial}{\partial \kappa} \right) c_n = E_n(\kappa) c_n - F D_n \sum_{l} X_{n,l}(\kappa) c_l \quad \text{(3)}$$

where $E_n(\kappa)$ is the energy of $\phi_n(x, \kappa)$ [$\text{with } E_n(-\kappa) = E_n(\kappa)$] and $X_{n,l}(\kappa) \equiv (2\pi i / a) \int_0^\Lambda d \xi F(\xi)$. The off-diagonal elements $X_{n,l} \equiv \delta_{n,l}$ in Eq. (3) are responsible for interband transitions, i.e. Zener tunneling (ZT). If bands $n$ and $l$ are separated by a large gap and the ac force $F(z)$ is small enough such that $|F x X_{n,l}(\kappa)| \ll |E_n(\kappa) - E_l(\kappa)|$ in the entire Brillouin zone, ZT is negligible as in ordinary lattices and one can make the single-band approximation by setting $X_{n,l} \equiv 0$ for $n \neq l$ in Eq. (3). In the single-band approximation one thus obtains

$$i \hbar \left( \frac{\partial}{\partial z} + \frac{F}{\omega} \frac{\partial}{\partial \kappa} \right) c(z, \kappa) = [E(\kappa) - i F(z) \Phi(\kappa)] c(z, \kappa) \quad \text{(4)}$$

where we omitted, for the sake of simplicity, the band index $n$ and set $i \Phi(\kappa) \equiv D_n X_{n,n}(\kappa)$. Because of the symmetry of $V(x)$, $\text{Re}(u_n(k, x))$ and $\text{Im}(u_n(k, x))$ have well defined and opposite parity under the inversion $x \to -x$; this implies that $\Phi(\kappa)$ is a real-valued function of $\kappa$, vanishing for a real potential (i.e. for $\alpha = 0$). As previously shown in \cite{17}, when a dc force $F$ is applied to the crystal, from Eq. (4) it follows that the energy spectrum is described by a complex-valued Wannier-Stark ladder. The non-reality of the energy spectrum comes from the extra-term $\Phi(\kappa)$ in Eq. (4) and is physically due to the fact that the external dc force $F$ breaks the $\mathcal{PT}$ symmetry of the full Hamiltonian $H = H_0 - F x$. For an ac-like force with period $\Lambda = 2\pi/\omega$, because of the $z$-periodicity of the Hamiltonian its energy spectrum is replaced by a quasi-energy spectrum. Moreover, in the single-band approximation DL corresponds to a complete collapse of the quasi-energy band like in an ordinary crystal \cite{14}. According to Floquet’s theorem of periodic systems, the quasi-energy $\mathcal{E}(\kappa)$ for the $n$-th lattice band can be readily calculated by looking for a solution to Eq. (4) of the form $c(z, \kappa) = a(z, \kappa) \exp[-i \mathcal{E}(\kappa) z / \Lambda]$ with $a(z + \Lambda, \kappa) = a(z, \kappa)$. One obtains

$$\mathcal{E}(\kappa) = \frac{1}{\Lambda} \int_0^\Lambda d z \left[ E(\kappa') - i F(z) \Phi(\kappa') \right] \quad \text{(5)}$$

where we have set $\kappa' \equiv \kappa - k(\Lambda) + k(z)$ and $k(z) = (1/\Lambda) \int_0^\Lambda d\xi F(\xi)$. Let us assume that the ac forcing $F(z)$ is an odd function with respect to some point $z_0$, i.e. that $F(z - z_0) = -F(z_0 - z)$ for some $z_0$ in the oscillation cycle. This condition is satisfied, for instance, for the important case of a harmonic (e.g. sinusoidal or cosinusoidal) ac driving force, originally considered by Dunlap and Kenkre \cite{1} and that will be assumed in the following. Owing to this additional temporal symmetry on the driving force, which is absent for the BO problem \cite{17}, a real-valued quasi-energy spectrum for the non-Hermitian time-periodic Hamiltonian $H_0 - F(z)x$ is obtained. In fact, in this case the imaginary term on the right hand side of Eq. (5) vanishes after integration because $F(z)$ and $\Phi(\kappa')$ have opposite parity for the inversion $(z - z_0) \to -(z - z_0)$. The quasi-energy spectrum is thus real-valued and its expression takes the usual form as in a conventional crystal. DL corresponds to a collapse of the quasiesenergy band $\mathcal{E}(\kappa)$, i.e. to $d \mathcal{E}(\kappa)/d z = 0$. For most driving fields like a sinusoidal field, DL can be attained exactly solely in the NNTB approximation, i.e. when the band shape $E(\kappa)$ is sinusoidal \cite{13}, $\mathcal{E}(\kappa) = E_0 - \Delta \cos(\kappa a)$. In this case, assuming for the sake of definiteness a sinusoidal ac-like force $F(z) = F_0 \cos(\omega z)$, the explicit form of the quasi-energy reads

$$\mathcal{E}(\kappa) = E_0 - \Delta J_0 \left( \frac{F_0 a}{\pi \omega} \right) \cos(\kappa a). \quad \text{(6)}$$

Band collapse, leading to DL, is thus attained when $J_0(F_0 a / \pi \omega) = 0$ \cite{1,2,17}.

To check the correctness of the analysis, we investigated DL for the complex crystal $V(x) = V_0 \cos(2\pi x/a) + \alpha \sin(2\pi x/a)$ in the unbroken $\mathcal{PT}$ symmetry phase ($\alpha < 1$) by a direct numerical analysis of Eq. (1) using a pseudo-spectral split-step method with absorbing boundary conditions. Figure 1 shows a typical example of band diagram below the phase transition point ($\alpha = 0.3$), numerically computed by a spectral analysis of the Hamiltonian $H_0$. Note that the lowest band of the array is with excellent accuracy approximated by a sinusoidal curve and turns out to be separated by a large gap from the second
FIG. 1: (color online) Band diagram of the complex potential defined by Eq.(2) for parameter values $\lambda = 633$ nm, $a = 8 \mu m$, $n_s = 1.42$, $V_0 = 0.002$, and $\alpha = 0.3$. The dotted curve shows, for comparison, the parabolic dispersion relation $E(k) = (\lambda_0)^2/(2n_s)$, folded inside the first Brillouin zone, corresponding to the critical case $\alpha = \alpha_c = 1$. The picture on the right hand side is an enlargement of the lowest band (solid curve), fitted by a sinusoidal curve (dotted curve), almost overlapped with the solid one.

FIG. 2: (color online) Wave packet broadening (discrete diffraction) and dynamic localization in the lattice of Fig.1 for broad beam excitation at normal incidence (left panels) and for single site excitation at normal incidence (right panels). In (a) and (b), the external force is absent ($F_0 = 0$). In (c), (d) a sinusoidal force with period $\Lambda = 1$ cm and amplitude $F_0 = 13.32 \mu m^{-1}$, corresponding to $\Gamma = 1.684$, is applied. In (e) and (f), the forcing amplitude is increased to $F_0 = 19.03 \mu m^{-1}$, corresponding to the DL condition $\Gamma = 2.405$.

 III. WAVE PACKET DYNAMICS AT THE $\mathcal{PT}$ SYMMETRY-BREAKING POINT

As $\alpha$ is increased to reach and cross the $\mathcal{PT}$ symmetry-breaking point $\alpha_c$, gap narrowing till band merging, associated to the appearance of pairs of complex-conjugate eigenvalues, is observed [13]. For instance, for the potential defined by Eq.(2), at the transition point the band diagram is given by the free-particle energy dispersion curve $E = \lambda^2 k^2/(2n_s)$, periodically folded inside the first Brillouin zone [13] (see the dotted curve in Fig.1). In this case, as previously noticed for BOs [17], wave packet transport is deeply modified and can not be described by means of the canonical model (3) introduced in the previous section. From a physical viewpoint, this is related to the highly non-reciprocal behavior of Bragg scattering in the crystal and violation of Friedel’s law of Bragg diffraction for crystal inversion [12, 14]. As in Ref. 17, we limit here to consider the dynamical behavior of a broad wave packet in the complex lattice defined by Eq.(2) at $\alpha = \alpha_c = 1$, i.e. $V(x) = V_0 \exp(i k_B x)$, which enables a rather simple analytical and physical analysis. As opposed to Ref. 17, we assume here an ac-like force, namely $F(z) = F_0 \cos(\omega z)$. Figure 3 shows a few typical examples of wave packet dynamics as obtained by a numerical analysis of Eq.(1) when the lattice is excited at $z = 0$ with a broad Gaussian beam $\psi(x, 0) = \exp(-x^2/w^2)$ for a fixed value of the ac modulation period $\Lambda = 2\pi/\omega$ and for increasing values of the amplitude $F_0$. For a small amplitude $F_0$, it turns out that the wave packet propagates as if the lattice were absent [Fig.3(a)], following an oscillatory path as predicted by the semiclassical analysis of Eq.(1) with $V = 0$. Owing to the external force, the mean wave packet momentum varies periodically according to $\tilde{\mathcal{M}}(z) = \int_0^z d\xi F(\xi) = (F_0/\omega) \sin(\omega z)$. As the forcing $F_0$ reaches and crosses the critical value $F_c = k_B \lambda_0/2$, new wave packets, arising from first-order Bragg diffraction, periodically bifurcate from the primary beam at propagation distances $z$ satisfying the Bragg condition $k(z) = -k_B/2$ [see Figs.3(b) and 3(c)]. The mean momentum of the first-order diffracted wave packets differs from that of the primary wave packet by an additional term $k_B$, which explains the refraction of the first-order diffracted beams at the angle $\theta = dx/dz = k_B \lambda /n_s$ observed in Figs.3(b) and 3(c). At stronger forcing, namely for $F \geq 3F_c$, additional wave packets bifurcate from the
first-order wave packets at propagation distances $z$ such that $k(z) = -3k_B/2$, as shown in Fig.3(d). These wave packets originate from second-order Bragg diffraction in the crystal and greatly complicate the pattern scenario. The mean momentum of second-order diffracted wave packets differs from that of the primary wave packet by the additional term $2k_B$, which explains the larger (twice) refraction angle of second-order diffracted beams as compared to that of first-order diffracted beams [see Fig.3(d)]. At even higher forcing, i.e. at $F \geq 5F_c$, new wave packets bifurcate from the second-order wave packets because of third-order Bragg diffraction at propagation distances $z$ such that $k(z) = -5k_B/2$, and so on. The appearance of wave packets generated by Bragg diffraction at various orders is rather abrupt, as indicated by the behavior of the normalized beam power $P(z) = \int dx |\psi(x, z)|^2 / \int dx |\psi(x, 0)|^2$ versus propagation distance $z$ shown in the right panels of Fig.3.

The dynamical scenario observed in numerical simulations can be analytically captured by considering the limiting case of a plane wave exciting the crystal at $z = 0$ with initial wave number $k = 0$. In fact, the solution to Eq.(1) with the initial condition $\psi(x, 0) = 1$ is given by the superposition of diffracted plane waves at different Bragg orders according to

$$\psi(x, z) = \sum_{n=0}^{\infty} a_n(z) \exp[i(k(z) - ink_B)z - i\gamma_n(z)]. \quad (7)$$

In Eq.(7) we have set

$$k(z) = \frac{1}{\lambda} \int_0^z d\xi F(\xi) = \frac{F_0}{\lambda \omega} \sin(\omega z) \quad (8)$$

$$\gamma_n(z) = \frac{\lambda}{2nk_0} \int_0^z d\xi [nk_0 + k(\xi)]^2, \quad (9)$$

whereas the amplitudes $a_n(z)$ are calculated from the recurrence relations

$$a_n(z) = -iV_0 \int_0^z d\xi a_{n-1}(\xi) \exp[i\gamma_n(z) - i\gamma_{n-1}(z)] \quad (10)$$

with $a_0(z) = 1$. The amplitude $a_0$ corresponds, in Fig.3, to the primary wave packet, and the independence of $a_0$ from $z$ indicates that this wave packet propagates as if the lattice were absent. The amplitude $a_1$ corresponds to the first-order diffracted wave packets, $a_2$ to second-order diffracted wave packets, and so on. According to Eq.(10), a wave packet corresponding to Bragg diffraction at order $n$ bifurcates from a wave packet of order $(n-1)$, and Bragg diffraction is effective provided that the phase difference $\varphi_n(z) = \gamma_n(z) - \gamma_{n-1}(z)$ entering in the exponential of the integral on the right-hand-side of Eq.(10) has a stationary point. The condition $d\varphi_n(z)/dz = 0$ is satisfied at propagation distances $z_0$ such that

$$k(z_0) = -k_B \left( n - \frac{1}{2} \right) \quad (n = 1, 2, 3, ...). \quad (11)$$

i.e. for a wave number that reaches the edge of the Bratt

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**FIG. 3:** (color online) Propagation of a broad Gaussian wave packet (input beam size $w = 80 \mu m$) in the complex lattice $V(x) = V_0 \exp(2\pi i x/a)$ for $V_0 = 0.0002, a = 6 \mu m$, $\lambda = 633$ nm, $n_s = 1.42$, subjected to a sinusoidal ac-like force $F(z) = F_0 \cos(2\pi z/\Lambda)$ with period $\Lambda = 1$ cm and with increasing values of force amplitude $F_0$: (a) $F_0/F_c = 0.5$, (b) $F_0/F_c = 1$, (c) $F_0/F_c = 1.5$, and (d) $F_0/F_c = 3$, where the critical forcing $F_c$ is defined by Eq.(15). Left panels show snapshots of $|\psi(x, z)|$; the central panels depict the behavior of $k(z)$, normalized to the Bragg wave number $k_B$, as given by Eq.(8); the right panels show the evolution of normalized beam power $P(z)$. The horizontal dotted lines indicate the crossing points $z = z_0$ where new wave packets are generated by Bragg scattering. For the sake of clarity, in (d) only the first two crossing points are indicated. At the crossing points a rather abrupt increase of beam power $P(z)$ is observed. The crossing is linear in (b) and (d), and parabolic in (c).
The coherent motion of a Bloch wave packet in a tight-binding lattice driven by an ac electric field is shown to demonstrate a self-imaging effect that arises from quasi-energy band collapse of the time-periodic Hamiltonian. This phenomenon, referred to as dynamic localization [1, 2], has been newly observed for both matter and optical waves as a suppression of wave packet broadening in the lattice. In this work, we investigated theoretically the behavior of a Bloch particle in a complex crystal with \( \mathcal{PT} \) symmetry subjected to a sinusoidal ac-like force. As compared to ordinary crystals, complex crystals exhibit some unique properties, such as violation of the Friedel’s law of Bragg scattering and nonreciprocal diffraction. For an unbroken \( \mathcal{PT} \) symmetry and in the single-band approximation, it has been shown that the quasi-energy spectrum of the time-periodic non-Hermitian Hamiltonian remains real-valued. In this regime, like in an ordinary crystal exact band collapse is possible within the
NNTB approximation, i.e. for a sinusoidal band shape. At the $\mathcal{PT}$ symmetry breaking transition point, band merging greatly modifies the wave packet dynamics as compared to an ordinary crystal. Here we have investigated in details the dynamics of a broad wave packet in the $\mathcal{PT}$-symmetric potential $V(x) = V_0 \exp(2\pi ix/a)$ and shown that the complexity of the dynamical scenario greatly increases as the strength of forcing is increased, with the appearance of a cascading of bifurcating wave packets. The main features observed in numerical simulations have been analytically explained by considering the Bragg scattering of a plane wave in the complex lattice subjected to the sinusoidal ac driving force. These results are complementary to the recent study of Bloch oscillations in complex crystals [17], and are expected to motivate further investigations aimed to explore the exotic transport properties of complex crystals.

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