STRONG DIAMAGNETISM FOR THE BALL
IN THREE DIMENSIONS

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Abstract. In this paper we give a detailed asymptotic formula for the lowest eigenvalue of the magnetic Neumann Schrödinger operator in the ball in three dimensions with constant magnetic field, as the strength of the magnetic field tends to infinity. This asymptotic formula is used to prove that the eigenvalue is monotonically increasing for large values of the magnetic field.

1. Introduction

1.1. The operator and main results. Let \( \Omega \) be the unit ball
\[
\Omega = \{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid |x| < 1 \}
\]
and let \( B \) be a constant magnetic field of magnitude \( B > 0 \) along the \( x_3 \) axis, with a corresponding choice of magnetic vector potential \( A \),
\[
B = (0, 0, B), \quad A = \frac{B}{2}(-x_2, x_1, 0).
\]
We consider the magnetic Neumann Schrödinger operator
\[
H(B) = (-i \nabla + A)^2
\]
with domain
\[
\text{Dom}(H(B)) = \{ \Psi \in W^{2,2}(\Omega) \mid N(x) \cdot (-i \nabla + A)\Psi|_{\partial \Omega} = 0 \}.
\]
Here \( N(x) \) is the interior unit normal to \( \partial \Omega \). This operator has compact resolvent and is semi-bounded from below, so it makes sense to enumerate its eigenvalues in an increasing order. For a self-adjoint operator \( \mathcal{H} \) that is semi-bounded from below we will use the notation \( \lambda_j, \mathcal{H} \) to denote its \( j \)th eigenvalue. In particular, we will write
\[
\lambda_1, \mathcal{H}(B) = \inf \text{Spec}(\mathcal{H}(B))
\]
for the lowest eigenvalue of \( \mathcal{H}(B) \). The first main theorem of this paper concerns the asymptotics of \( \lambda_1, \mathcal{H}(B) \) as \( B \to \infty \).

Theorem 1.1. There exist constants \( \lambda_j, j = 0, \ldots, 5, \) and \( \zeta_0, \zeta_1, \zeta_2, \zeta_3, \delta_0, \) and \( C \) such that with
\[
\zeta_3(m, B) = m - \frac{B}{2} - \zeta_0 \sqrt{B} - \zeta_1 B^{1/3} - \zeta_2 B^{1/6},
\]
and
\[
\Delta_B = \inf_{m \in \mathbb{Z}} |\zeta_3(m, B) - \zeta_3|
\]
it holds that
\[
\lambda_1, \mathcal{H}(B) = B \sum_{j=0}^{5} \lambda_j B^{-j/6} + \delta_0 \Delta_B^2 + C + \mathcal{O}(B^{-1/6}), \quad \text{as } B \to \infty.
\]
Remark 1.2. In the course of the proof we will obtain explicit expressions for the constants in the theorem above, especially it holds that
\[ \lambda_0 = \Theta_0, \quad \lambda_1 = 0, \quad \text{and} \quad \lambda_2 = 2^{-2/3} \hat{\nu}_0 \delta_0^{4/3}, \] (1.6)
which agrees with the asymptotics of \( \lambda_{1,H(B)} \) that is given in [14] for more general domains \( U \subset \mathbb{R}^3 \) (see Theorem 1.4). The constants \( \Theta_0, \hat{\nu}_0 \) and \( \delta_0 \) are well-known universal constants which appear in the study of two model operators, see Appendix A.

Before stating the next main theorem it is worth noticing that the constants \( \Theta_0, \delta_0 \) in the theorem satisfy
\[ \frac{1}{2} < \Theta_0 < 1 \quad \text{and} \quad 0 < \delta_0 < 1, \]
so especially, it holds that
\[ \Theta_0 - \frac{1}{2} \delta_0 > 0. \] (1.7)

Theorem 1.3. Let \( \delta_0 \) and \( \lambda_0 = \Theta_0 \) be the constants from Theorem 1.1. The directional derivatives
\[ \lambda_{1,H(B),\pm} = \lim_{\varepsilon \to 0} \frac{\lambda_{1,H(B+\varepsilon)} - \lambda_{1,H(B)}}{\varepsilon} \]
exist and satisfy
\[ \lambda_{1,H(B),+} \leq \lambda_{1,H(B),-}, \quad \text{for all } B > 0, \] (1.8)
\[ \liminf_{B \to \infty} \lambda_{1,H(B),+} \geq \Theta_0 - \frac{1}{2} \delta_0 > 0, \quad \text{and} \] (1.9)
\[ \limsup_{B \to \infty} \lambda_{1,H(B),-} \leq \Theta_0 + \frac{1}{2} \delta_0. \] (1.10)
In particular, the function \( B \mapsto \lambda_{1,H(B)} \) is monotonically increasing for sufficiently large \( B \).

1.2. Motivation.

1.2.1. Strong diamagnetism. Let \( H_{U}(B) \) denote the magnetic Neumann operator in a bounded and smooth domain \( U \subset \mathbb{R}^n \), \( n = 2, 3 \). From the diamagnetic inequality (see [16]) it follows that
\[ \lambda_{1,H_U(0)} \leq \lambda_{1,H_U(B)} \] (1.11)
for all \( B \geq 0 \). One might ask if the stronger monotonicity
\[ 0 < B_1 < B_2 \implies \lambda_{1,H_U(B_1)} \leq \lambda_{1,H_U(B_2)} \] (1.12)
holds. In [4] counter-examples are given showing that (1.12) does not hold in general for all \( B_1 \) and \( B_2 \). The examples are given in \( \mathbb{R}^2 \), for constant magnetic field and the presence of a scalar potential, and for variable magnetic field without a scalar potential.

Lately the question whether there exist a \( B_0 \) such that \( \lambda_{1,H_U(B)} \) is monotonically increasing for all \( B > B_0 \) has been studied in detail. This is well-understood in two dimensions by now, with the final affirmative answer for regular domains in [7] [8] and for domains with corners in [3]. We discuss below the progress in three dimensions so far, which motivates our analysis for the ball.

To continue, we introduce some conditions on the domain \( U \). Let \( \Gamma \subset \partial U \) be the set of all points on the boundary where the magnetic field \( B \) is tangent to \( \partial U \), i.e.
\[ \Gamma = \{ x \in \partial U \mid B \cdot N(x) = 0 \}. \] (1.13)

Assumption 1. Let \( d \) denote the differential on \( \partial U \). Then
\[ d(B \cdot N(x)) \neq 0, \quad \text{for all } x \in \Gamma. \]
If this assumption holds then $\Gamma$ consists of a disjoint union of regular curves. We can orient them and denote by $T(x)$ an oriented unit tangent vector at $x \in \Gamma$. It is noted in [9] that this implies that the magnetic normal curvature $k_{n,B}(x) = K_z(T(x) \wedge N(x), \frac{\mathbf{B}}{|\mathbf{B}|})$ is non-zero on $\Gamma$. Here $K$ denotes the second fundamental form on $\partial U$.

**Assumption 2.** The set of points in $\Gamma$ where $\mathbf{B}$ is tangent to $\Gamma$ is isolated.

The following asymptotic formula of $\lambda_{1,\mathcal{H}_U(B)}$ was proved in [14] (the upper bound was given in [20]).

**Theorem 1.4.** Let $U \subset \mathbb{R}^3$ be a bounded and smooth domain that satisfies Assumptions 1 and 2. Let

$$
\lambda_{1,\mathcal{H}_U(B)} = \Theta_0 B + \tilde{\gamma}_0 B^{2/3} + O(B^{2/3 - \eta}), \quad \text{as } B \to \infty. 
$$

The constant $\Theta_0$ is the same as in Theorem 1.1 and the constant $\tilde{\gamma}_0$ is given by $\tilde{\gamma}_0 = \inf_{x \in \Gamma} \tilde{\gamma}_0(x)$ where

$$
\tilde{\gamma}_0(x) = 2^{-2/3} \nu_0 \delta_0^{1/3} |k_{n,B}(x)|^{2/3} \left(1 + (\delta_0 - 1) \left| T(x) \cdot \mathbf{B} \left| \mathbf{B} \right|^2 \right)^{1/3},
$$

and $\delta_0$ and $\tilde{\nu}_0$ are fundamental constants given in Appendix A. We note that

$$
\tilde{\gamma}_0 = 2^{-2/3} \tilde{\nu}_0 \delta_0^{1/3}
$$

when $U = \Omega$ is the unit ball which makes our Theorem 1.4 compatible with Theorem 1.3.

Using the expansion (1.14) of the lowest eigenvalue $\lambda_{1,\mathcal{H}_U(B)}$, the following monotonicity result was proved in [9].

**Theorem 1.5.** Let $U \subset \mathbb{R}^3$ be a bounded and smooth domain that satisfies Assumptions 1 and 2. Let $\{\Gamma_1, \ldots, \Gamma_n\}$ be the collection of disjoint smooth curves making up $\Gamma$. Assume that for all $j$ there exists $x \in \Gamma_j$ such that $\tilde{\gamma}_0(x) > \tilde{\gamma}_0$. Then the function $B \mapsto \lambda_{1,\mathcal{H}_U(B)}$ is strictly increasing for sufficiently large $B$.

This shows that (1.12) holds for large values of $B_1$ and $B_2$. Even though the Assumptions 1 and 2 are fulfilled for the ball, the assumption on $\tilde{\gamma}_0$ in Theorem 1.5 is not. Indeed, for the unit ball $\Omega$, the set $\Gamma$ consists of the equator $\{x \in \partial \Omega \mid x_3 = 0\}$ and the function $\tilde{\gamma}_0(x)$ is constant

$$
\tilde{\gamma}_0(x) \equiv 2^{-2/3} \tilde{\nu}_0 \delta_0^{1/3}, \quad x \in \Gamma.
$$

### 1.2.2. Superconductivity

We consider superconductivity in the Ginzburg-Landau model. For a superconducting material of shape $U$ subject to an external magnetic field $\kappa \sigma \beta$, with $\beta = (0, 0, 1)$, the Ginzburg-Landau energy functional is as follows,

$$
E_{\kappa, \sigma} (\Psi, \mathbf{a}) = \int_U \left[ \left| (-i \nabla + \kappa \sigma \mathbf{a}) \Psi \right|^2 - \kappa^2 |\Psi|^2 + \frac{\kappa^2}{2} |\Psi|^4 \right] \, dx \\
+ (\kappa \sigma)^2 \int_{\mathbb{R}^3} |\text{curl } \mathbf{a} - \beta|^2 \, dx. \quad (1.16)
$$

Here $\kappa > 0$ is a material dependent parameter called the Ginzburg-Landau parameter. For given $\kappa$, the parameter $\sigma$ measures the strength of the external magnetic field. The function $\Psi \in H^1(U)$ is in this context called an order parameter and $|\Psi|$ measures the local superconducting properties (density of Cooper pairs) of the material. Finally $\mathbf{a} \in H^1_0(U, \mathbb{R}^3)$ is the induced magnetic vector potential. In order to get a well-defined minimization problem, one uses gauge invariance to restrict to
vector potentials satisfying \( \text{div} \ a = 0 \), and impose the finite energy condition (see [10] for details),

\[
\int_{\mathbb{R}^3} |\text{curl} \ a - \beta|^2 \, dx < \infty.
\]

A state \((\Psi, a)\) where \(\Psi \equiv 0\) and \(\text{curl} \ a = \beta\) is called trivial.

The analysis of magnetic ground state eigenvalues described above is relevant in superconductivity for the understanding of the loss of superconductivity in the presence of strong magnetic fields. The value of the magnetic field strength at which the material loses its superconducting properties, i.e., the minimizers of the Ginzburg-Landau functional have \(\Psi \equiv 0\), is called the third critical field. The calculation of this critical field has a long history, see [11, 1, 17, 18, 15, 7, 8, 5].

In [9, 5] it was proved that, for sufficiently large, the following sets are equal

\[
N(\kappa) := \{ \sigma > 0 \mid \mathcal{E}_{\kappa, \sigma} \text{ has a non-trivial minimizer} \}
= \{ \sigma > 0 \mid \mathcal{E}_{\kappa, \sigma} \text{ has a non-trivial stationary point} \},
\]

\[
(1.17)
\]

In the case where \(U\) is the unit ball \(\Omega\), we can use the monotonicity result of Theorem 1.3 to conclude that, for \(\kappa\) sufficiently large,

\[
\{ \sigma > 0 \mid \lambda_{1, \mathcal{H}_U(\kappa, \sigma)} < \kappa^2 \} = (0, H_{C_3}(\kappa)).
\]

Here \(\sigma = H_{C_3}(\kappa)\) is the unique solution to the equation

\[
\lambda_{1, \mathcal{H}_U(\kappa, \sigma)} = \kappa^2.
\]

Hereby, we get a complete determination of the third critical field, \(H_{C_3}(\kappa)\), for large values of \(\kappa\). Upon inserting the asymptotics (1.5), one gets a six-term asymptotic expansion of \(H_{C_3}(\kappa)\) for the ball,

\[
H_{C_3}(\kappa) = \frac{\kappa}{\Theta_0} - \frac{\hat{\gamma}_0}{\Theta_0^{13/6}} \kappa^{1/3} + \frac{\lambda_3}{\Theta_0^{4/3}} + \left( \frac{2 \gamma_0}{3 \Theta_0^{7/3}} - \frac{\lambda_4}{\Theta_0^{4/3}} \right) \kappa^{-1/3} + \frac{7 \lambda_3 \gamma_0}{6 \Theta_0^{13/6}} \frac{\lambda_5}{\Theta_0^{7/6}} \kappa^{-2/3} + \frac{1}{\Theta_0^{5/3}} \left( \frac{\lambda_3^2}{2 \Theta_0^6} + \frac{\lambda_4 \gamma_0}{\Theta_0^2} - \frac{\hat{\gamma}_0^3}{3 \Theta_0^4} - \frac{\Delta^2_{\kappa, \mathcal{H}_U(\kappa)} + C}{\Theta_0} \right) \kappa^{-1} + O(\kappa^{-4/3}).
\]

Thus, \(N(\kappa)\) is an interval (for large \(\kappa\)) both in the case where \(U\) satisfies the assumptions of Theorem 1.3 and in the case where \(U\) is a ball. It remains an interesting open question to prove this result in general, i.e., to prove strong diamagnetism for general (smooth) domains in \(\mathbb{R}^3\).

1.3. Organization of the paper. The main part of this paper is devoted to the proof of Theorem 1.1. The proof is divided into several parts:

We denote by \(\mathcal{H}_m(B)\) the operator \(\mathcal{H}(B)\) restricted to angular momentum \(m \in \mathbb{Z}\). In Section 2 we show a lower bound for \(\mathcal{H}_m(B)\). First we show in Lemmas 2.7 and 2.8 that (in terms of \(m\) and \(B\)) either the first eigenvalues of \(\mathcal{H}_m(B)\) are too large to be compatible with (1.5), or we can reduce \(\mathcal{H}_m(B)\) to an effective operator \(Q_m(B)\), satisfying, as \(B \to \infty\),

\[
\lambda_{j, \mathcal{H}_m(B)} = B \lambda_{j, Q_m(B)} + O(B^{1/2}), \quad \text{for } j = 1, 2.
\]

For the values of \(m\) and \(B\) such that (1.19) holds, we prove a lower bound for \(Q_m(B)\) in Proposition 2.9.

In Section 3 we use the lower bound from Section 2 to give localization properties of eigenfunctions of \(Q_m(B)\). These are used in Section 4 to obtain a spectral gap formula for \(Q_m(B)\), which together with (1.19) implies a spectral gap formula for
\( \mathcal{H}_m(B) \), showing that for some \( \gamma > 0 \) it holds that (still under some restrictions on \( m \) and \( B \))

\[
\lambda_{2, \mathcal{H}_m(B)} \geq \Theta_0 B + (\tilde{\gamma}_0 + \gamma)B^{2/3} + O(B^{7/12}), \quad \text{as } B \to \infty. \tag{1.20}
\]

In Section 5, Theorem 5.1(iii) we use the Grušin method to, for certain \( m \) and \( B \), calculate a trial state that together with the spectral gap formula gives upper and lower bounds on \( \lambda_{1, \mathcal{H}_m(B)} \) which are compatible with \( \Theta_0 \). We also give two alternative trial states, in Theorem 5.1(i) and (ii), for values of \( m \) that are further away from the optimal choice.

In Section 6 we show that \( \lambda_{1, \mathcal{H}_m(B)} \) is larger for the values of \( m \) and \( B \) not treated in Theorem 5.1(iii). Depending on \( m \) and \( B \) we use different methods to achieve this. For \( m \) and \( B \) which are far from the optimal region we use (a refined version of) the lower bound from Section 2. For \( m \) and \( B \) that are closer to the optimal region we use the trial state from Theorems 5.1(i) and (ii) which by the spectral gap formula \( \lambda_{2, \mathcal{H}_m(B)} \) must be \( \lambda_{1, \mathcal{H}_m(B)} \), but which is strictly greater than the ones obtained in Theorem 5.1(iii). Finally, in Section 7 we minimize the eigenvalue found in Theorem 5.1(iii), which proves Theorem 1.1.

The proof of Theorem 1.3 is a consequence of Theorem 1.1, using a perturbation argument from [5]. The details are given in Section 8.

We start by introducing the new coordinates and some quadratic forms and operators.

### 1.4. New coordinates, auxiliary operators and quadratic forms.

We conclude this first section by introducing the coordinates we will work in and the different quadratic forms we will work with. First, we switch to spherical coordinates \((x_1, x_2, x_3) \mapsto (r, \varphi, \theta)\),

\[
\begin{align*}
    x_1 &= r \cos \varphi \sin \theta, \\
    x_2 &= r \sin \varphi \sin \theta, \\
    x_3 &= r \cos \theta,
\end{align*}
\]

We decompose the Hilbert space as

\[
L^2(\Omega) \cong L^2((0, 1) \times (0, \pi), r^2 \sin \theta \, dr \, d\theta) \otimes L^2(S^1, d\varphi)
\]

\[
\cong \bigoplus_{m=\infty}^{\infty} L^2((0, 1) \times (0, \pi), r^2 \sin \theta \, dr \, d\theta) \otimes \frac{e^{im\varphi}}{\sqrt{2\pi}},
\]

that is, for a function \( \Psi \in L^2(\Omega) \), we write

\[
\Psi(r, \varphi, \theta) = \sum_{m \in \mathbb{Z}} \psi_m(r, \theta) \frac{e^{-im\varphi}}{\sqrt{2\pi}},
\]

where \( \psi_m \in L^2((0, 1) \times (0, \pi), r^2 \sin \theta \, dr \, d\theta) \). Next, we write the operator \( \mathcal{H}(B) \) corresponding to this decomposition as

\[
\mathcal{H}(B) = \bigoplus_{m=\infty}^{\infty} \mathcal{H}_m(B) \otimes 1
\]

where \( \mathcal{H}_m(B) \) is the self-adjoint operator acting in \( L^2((0, 1) \times (0, \pi), r^2 \sin \theta \, dr \, d\theta) \), given by

\[
\mathcal{H}_m(B) = -\partial_r^2 - \frac{2}{r} \partial_r - \frac{1}{r^2} \partial_\theta^2 - \frac{1}{r^2 \tan \theta} \partial_\theta + \left( \frac{Br \sin \theta}{2} - \frac{m}{r \sin \theta} \right)^2,
\]

with Neumann boundary condition at \( r = 1 \). In the continuation we skip the subscript \( m \) on \( \psi_m \) and write just \( \psi \). Inspired by [14] we introduce the new scaled
coordinates \((\tau, \rho)\) as
\[
\begin{aligned}
\tau &= \sqrt{B}(1 - \tau), \\
\rho &= \sqrt{B}(\theta - \pi/2),
\end{aligned}
\tag{1.21}
\]
with corresponding new domain
\[
\Omega(B) = \left\{ (\tau, \rho) \mid 0 < \tau < \sqrt{B}, \ -\frac{\pi}{2} \sqrt{B} < \rho < \frac{\pi}{2} \sqrt{B} \right\}.
\]
In fact, for a point \(x\) in \(\Omega\), \(\tau = \sqrt{B} \text{dist}(x, \partial\Omega)\) is equal to the (scaled) distance to the boundary and for a point \(x \in \partial\Omega\) we have \(\rho = \sqrt{B} \text{dist}_{\partial\Omega}(x, \Gamma)\), the (scaled) distance along the boundary to the equator.

The quadratic form corresponding to \(\frac{1}{B} \hat{H}_m\) (the prefactor \(1/B\) is just for convenience) transforms into the quadratic form
\[
\hat{q}_m[\psi] = \int_{\Omega(B)} \left[ |\partial_\tau \psi|^2 + \frac{B^{-1/3}}{(1 - B^{-1/2}\tau)^2} |\partial_\rho \psi|^2 \right. \\
+ \left. \frac{1}{B} \left( \frac{B(1 - B^{-1/2}\tau) \cos(B^{-1/3}\rho)}{2} - \frac{m}{(1 - B^{-1/2}\tau) \cos(B^{-1/3}\rho)} \right)^2 |\psi|^2 \right] \times \\
\times (1 - B^{-1/2}\tau)^2 \cos(B^{-1/3}\rho) B^{-5/6} \, d\tau \, d\rho
\tag{1.22}
\]
in the Hilbert space
\[
L^2 \left( \Omega(B), (1 - B^{-1/2}\tau)^2 \cos(B^{-1/3}\rho) B^{-5/6} \, d\tau \, d\rho \right).
\]
We apply the unitary transform \(U\psi = B^{-5/12}\psi\) to get rid of the factor \(B^{-5/6}\) in the measure and work in the Hilbert space
\[
L^2 \left( \Omega(B), (1 - B^{-1/2}\tau)^2 \cos(B^{-1/3}\rho) \, d\tau \, d\rho \right)
\]
instead. We will, by abuse of notation, continue to write \(\psi\) instead of \(U\psi\). We denote by \(\hat{Q}_m(B)\) the operator corresponding to the quadratic form \(\hat{q}_m\).

Next we want to define a quadratic form \(\hat{q}_m\) by the same integral expression as for \(\hat{q}_m\) but in the Hilbert space \(L^2(\mathbb{R}_+^2, \, d\tau \, d\rho)\), where
\[
\mathbb{R}_+^2 = \{ (\tau, \rho) \mid 0 < \tau < \infty, \ -\infty < \rho < \infty \}.
\]
However, since neither \(1 - B^{-1/2}\tau\) nor \(\cos(B^{-1/3}\rho)\) are strictly positive in \(\mathbb{R}_+^2\), we make a technical modification to be able to talk about the corresponding operator. We define smooth functions \(\ell : \mathbb{R}_+ \to \mathbb{R}_+\) and \(\cos : \mathbb{R} \to \mathbb{R}\) that satisfy
\[
\ell(x) = \begin{cases} 
 x, & x \leq \frac{1}{2}, \\
 \frac{1}{2}, & x \geq \frac{1}{2},
\end{cases}
\text{ and } \cos(x) = \begin{cases} 
 \cos(x), & |x| \leq \frac{\pi}{4}, \\
 \frac{\pi}{4}, & |x| \geq \frac{\pi}{4},
\end{cases}
\tag{1.23}
\]
and such that \(\ell\) is monotonically increasing in the interval \((\frac{1}{4}, \frac{1}{2})\) and \(\cos\) is even and monotonically decreasing in the interval \((\frac{\pi}{4}, \frac{\pi}{2})\). The quadratic form \(\hat{q}_m\) is defined by
\[
\hat{q}_m[\psi] = \int_{\mathbb{R}_+^2} \left[ |\partial_\tau \psi|^2 + \frac{B^{-1/3}}{(1 - \ell(B^{-1/2}\tau))^2} |\partial_\rho \psi|^2 \right. \\
+ \left. \frac{1}{B} \left( \frac{B(1 - \ell(B^{-1/2}\tau)) \cos(B^{-1/3}\rho)}{2} - \frac{m}{(1 - \ell(B^{-1/2}\tau)) \cos(B^{-1/3}\rho)} \right)^2 |\psi|^2 \right] \times \\
\times (1 - \ell(B^{-1/2}\tau))^2 \cos(B^{-1/3}\rho) \, d\tau \, d\rho.
\tag{1.24}
\]
We denote by \( \hat{Q}_m(B) \) the self-adjoint operator that corresponds to \( \hat{q}_m \). It is an operator in \( L^2(\mathbb{R}^3_+, 1 - \ell(B^{-1/2}\tau))^2 \cos(B^{-1/3}\rho) \) with Neumann condition at \( \tau = 0 \). An integration by parts, show that it acts as

\[
\hat{Q}_m(B) = -\partial_\tau^2 + \frac{2\ell'(B^{-1/2}\tau)B^{-1/2}}{(1 - \ell(B^{-1/2}\tau))} \partial_\tau - \frac{B^{-1/3}}{(1 - \ell(B^{-1/2}\tau))^2} (\partial_\rho^2 + \frac{\cos'(B^{-1/3}\rho)B^{-1/3}}{\cos(B^{-1/3}\rho)} \partial_\rho) \\
+ \frac{1}{B} \left( \frac{B(1 - \ell(B^{-1/2}\tau))}{2} \cos(B^{-1/3}\rho) - \frac{m}{(1 - \ell(B^{-1/2}\tau))^2} \cos(B^{-1/3}\rho) \right)^2
\]

(1.25)

Finally, we also define the quadratic form \( q_m \) in \( L^2(\mathbb{R}^3_+, d\tau d\rho) \) by

\[
q_m[\psi] = \int_{\mathbb{R}^3_+} |\partial_\tau \psi|^2 + \left( \tau + \frac{1}{\sqrt{B}} (m - B/2) + B^{-1/6} \rho^2 \right)^2 |\psi|^2 + B^{-1/3} |\partial_\rho \psi|^2 \ d\tau \ d\rho
\]

(1.26)

with corresponding self-adjoint operator \( Q_m(B) \) with Neumann boundary condition for \( \tau = 0 \).

2. A ROUGH LOWER BOUND

In this section we recall the localization formulas of the lowest eigenfunction of \( \mathcal{H}(B) \) obtained in [14] and written out in detail in [8], use them to reduce the study of \( \mathcal{H}(B) \) to the study of \( Q_m(B) \), and give a rough lower bound of its quadratic form \( q_m \) in Proposition 2.9.

**Theorem 2.1.** Suppose that \( U \subset \mathbb{R}^3 \) satisfies the Assumptions [4] and [5] and that \( \Psi \) satisfies \( \mathcal{H}(B)\Psi = \lambda \Psi \) with \( \lambda(B) \leq \Theta_0 B + \omega B^{2/3} \). Then there exist positive constants \( a_1, a_2, d_0, C, B_0 \) such that

\[
\int_U e^{2a_1 B^{1/2} \text{dist}(x, \partial U)} (|\Psi|^2 + B^{-1} (-i \nabla + A)\Psi|^2) \ d\mathbf{x} \leq C \|\Psi\|^2
\]

and

\[
\int_{\{\text{dist}(x, \partial U) \leq d_0\}} e^{2a_2 B^{1/2} \text{dist}_{\Omega}(x, \Gamma)^{3/2}} (|\Psi|^2 + B^{-1} (-i \nabla + A)\Psi|^2) \ d\mathbf{x} \leq C e^{CB^{1/3}} \|\Psi\|^2
\]

for all \( B \geq B_0 \).

From now on we assume that \( U = \Omega \) is the unit ball. Then the set \( \Gamma \), introduced in [14], consists of the equator and we can extend the distance function \( \text{dist}_{\partial U}(x, \Gamma) \) to all of \( \Omega \) (except the origin) as \( \text{dist}_{\partial U}(x, \Gamma) = \text{dist}_{\partial \Omega}(\hat{x}, \Gamma) \) where \( \hat{x} = \frac{x}{|x|} \in \partial \Omega \). By the exponential decay away from the boundary, the second inequality in Theorem 2.1 is then valid with the integral on the left-hand side being over all of \( \Omega \), with possible changes of the constants. We will use the following corollary.

**Corollary 2.2.** Suppose that \( \Psi \) satisfies \( \mathcal{H}(B)\Psi = \lambda \Psi \) with \( \lambda(B) \leq \Theta_0 B + \omega B^{2/3} \). Then for all \( n \in \mathbb{N} \) there exist positive constants \( C_n \) and \( B_n \) such that

\[
\int_{\Omega} \text{dist}(x, \partial \Omega)^n \ (|\Psi|^2 + B^{-1} (-i \nabla + A)\Psi|^2) \ d\mathbf{x} \leq C_n B^{-n/2} \|\Psi\|^2
\]

(2.1)

and

\[
\int_{\Omega} \text{dist}_{\partial \Omega}(x, \Gamma)^n \ (|\Psi|^2 + B^{-1} (-i \nabla + A)\Psi|^2) \ d\mathbf{x} \leq C_n B^{-n/4} \|\Psi\|^2
\]

(2.2)
for all $B \geq B_n$.

**Remark 2.3.** The order in (2.2) is not optimal. The calculations below indicate that the same estimate is true with $B^{-n/3}$ instead of $B^{-n/4}$ in the right-hand-side.

Let $0 < \varepsilon < 1/12$ be given. We introduce a smooth cut-off function $0 \leq \chi_B \leq 1$ such that

$$
\chi_B = \begin{cases}
1, & \text{if } \text{dist}(x, \partial \Omega) \leq B^{-1/2+\varepsilon} \text{ and dist}_{\partial \Omega}(x, \Gamma) \leq B^{-1/4+\varepsilon}, \\
0, & \text{if } \text{dist}(x, \partial \Omega) \geq 2B^{-1/2+\varepsilon} \text{ or dist}_{\partial \Omega}(x, \Gamma) \geq 2B^{-1/4+\varepsilon},
\end{cases}
$$

and such that $|\nabla \chi_B| \leq CB^{1/2-\varepsilon}$ for some $C > 0$.

**Lemma 2.4.** Suppose that $\Psi$ satisfies $\mathcal{H}(B)\Psi = \lambda \Psi$ with $\lambda(B) \leq \Theta_0 B + \omega B^{2/3}$. For any $N > 0$ it holds that

$$
\int_{\Omega} |(-i\nabla + A)\Psi|^2 \, dx = \int_{\Omega} |(-i\nabla + A)(\chi_B \Psi)|^2 \, dx + \mathcal{O}(B^{-N}) \|\Psi\|^2, \quad \text{as } B \to \infty.
$$

**Proof.** This follows by commuting $(-i\nabla - A)$ and $\chi_B$ and using Corollary 2.2. \hfill \Box

We remind the reader of the quadratic forms $\tilde{q}_m$ and $q_m$, introduced in (1.22) and (1.24), with corresponding self-adjoint operators $\tilde{Q}_m(B)$ and $Q_m(B)$. First we note that

$$
\inf \text{Spec}(\mathcal{H}(B)) = \inf_{m \in \mathbb{Z}} \inf \text{Spec}(\mathcal{H}_m(B)) = B \inf_{m \in \mathbb{Z}} \inf \text{Spec}(\tilde{Q}_m(B)).
$$

We use Lemma 2.4 to reduce the study of $\mathcal{H}(B)$ to the study of the quadratic form $\tilde{q}_m$ in the half-space $\mathbb{R}^2_+$. We will denote by $\psi_B$ the function

$$
\psi_B = \chi_B \psi,
$$

where $\chi_B$ is the cut-off function from (2.3). Notice that $\psi$, and by consequence $\psi_B$, depends on $m$, but we do not include this in the notation.

**Lemma 2.5.** Assume that $\psi_1$ and $\psi_2$ satisfies $\tilde{Q}_m(B)\psi_j = \lambda_j(B)\psi_j$ with $\lambda_j(B) \leq \Theta_0 B + \omega B^{2/3}$. For any number $N > 0$ there exist constants $B_N$ and $C_N$ (independent of $m$) such that if $B > B_N$ then

$$
|\tilde{q}_m(\psi_1, \psi_2) - q_m(\chi_B \psi_1, \chi_B \psi_2)| \leq C_N B^{-N} \|\psi_1\| \cdot \|\psi_2\|.
$$

**Proof.** This follows from Lemma 2.4 transforming to the coordinates $(\tau, \rho)$. \hfill \Box

We recall that $0 < \varepsilon < 1/12$ and note that

$$
\text{supp } \psi_B \subset \{ (\tau, \rho) \mid 0 < \tau < 2B^\varepsilon, \ -2B^{1/12+\varepsilon} < \rho < 2B^{1/12+\varepsilon} \},
$$

which implies that $B^{-1/2} \tau \leq 2B^{\varepsilon-1/2}$ and $|B^{-1/3} \rho| \leq 2B^{\varepsilon-1/4}$ on the support of $\psi_B$.

**Lemma 2.6.** Assume that $\psi$ satisfies $\tilde{Q}_m(B)\psi = \lambda(B)\psi$ with $\lambda(B) \leq \Theta_0 B + \omega B^{2/3}$. For any number $n > 0$ there exist constants $B_n$ and $C_n$ (independent of $m$) such that if $B > B_n$ then

$$
\int_{\mathbb{R}_+^2} \tau^n (|\psi_B|^2 + |\partial_\tau \psi_B|^2 + B^{-1/3} |\partial_\rho \psi_B|^2) \, d\tau \, d\rho \leq C_n \|\psi_B\|^2
$$

and

$$
\int_{\mathbb{R}_+^2} \rho^n (|\psi_B|^2 + |\partial_\tau \psi_B|^2 + B^{-1/3} |\partial_\rho \psi_B|^2) \, d\tau \, d\rho \leq C_n B^{n/12} \|\psi_B\|^2.
$$

**Proof.** This follows directly from Corollary 2.2 by transforming into the new coordinates $(\tau, \rho)$. \hfill \Box
We will use the estimates in Lemma 2.6 to reduce the values of the angular momentum \( m \) that we must consider, which in the end will enable us to study the effective quadratic form \( q_m \) instead of \( \tilde{q}_m \). Let \( \tilde{\zeta}_0 = \frac{1}{\sqrt{B}}(m - B/2) \).

**Lemma 2.7.** Assume that \( \psi \) satisfies \( \tilde{Q}_m(B)\psi = \lambda(B)\psi \) with \( \lambda(B) \leq \Theta_0 B + \omega B^{2/3} \). There exist positive constants \( \tilde{D} \) and \( B_0 \) such that if \( |\tilde{\zeta}_0| > \tilde{D} \) and \( B > B_0 \) then

\[
q_m[\psi_B] \geq \frac{1}{2} \tilde{\zeta}_0^2 \|\psi_B\|^2. \tag{2.8}
\]

**Proof.** We expand the potential in \( \tilde{q}_m \) and collect the terms in front of the different degrees of \( \tilde{\zeta}_0 \). By (1.23) and (2.5) we see that if \( B > \max\{(8/\pi)^{4/(1-4\epsilon)}, 6^{2/(1-2\epsilon)}\} \) then it holds that \( \ell(B^{-1/2}) = B^{-1/2} \) and \( \cos(B^{-1/3} \rho) = \cos(B^{-1/3} \rho) \) on the support of \( \psi_B \), and so we can write the potential in \( \tilde{q}_m \) as

\[
\left[ \tilde{\zeta}_0 - \left( \frac{\sqrt{B}(1 - B^{-1/2})^2 \cos^2(B^{-1/3} \rho) - \sqrt{B}}{2} \right) \right]^2 \frac{1}{\cos(B^{-1/3} \rho)} \]

Using that \( 0 < \cos(B^{-1/3} \rho) < 1 \) on the support of \( \psi_B \) and the general inequality \( (x - y)^2 \geq \frac{1}{4}x^2 - \frac{1}{4}y^2 \), valid for real \( x \) and \( y \), we find that \( \tilde{q}_m[\psi_B] \) is bounded from below by

\[
\int_{\mathbb{R}^d_+} \left( \frac{3 \tilde{\zeta}_0}{4} - \frac{1}{3} \left( \frac{\sqrt{B}(1 - B^{-1/2})^2 \cos^2(B^{-1/3} \rho) - \sqrt{B}}{2} \right) \right)^2 \frac{1}{\cos(B^{-1/3} \rho)} |\psi_B|^2 \, d\tau \, d\rho.
\]

Using (2.3) and (2.7) we get the existence of a constant \( C \) such that

\[
\int_{\mathbb{R}^d_+} \left( \frac{\sqrt{B}(1 - B^{-1/2})^2 \cos^2(B^{-1/3} \rho) - \sqrt{B}}{2} \right)^2 \frac{1}{\cos(B^{-1/3} \rho)} |\psi_B|^2 \, d\tau \, d\rho \leq C \|\psi_B\|^2.
\]

This clearly implies (2.8).

By Lemma 2.7 above we need only to consider bounded \( \tilde{\zeta}_0 \). This enables to study the quadratic form \( q_m \) instead of \( \tilde{q}_m \). We assume that \( |\tilde{\zeta}_0| < \tilde{D}_0 \) for some \( \tilde{D}_0 > 0 \) and also let \( D_0 = \tilde{D}_0 + |\tilde{\zeta}_0| \) so that the inequality \( |\tilde{\zeta}_0 - \tilde{\zeta}_0| < D_0 \) holds. Here \( \tilde{\zeta}_0 = \hat{\zeta}_0 \) is the constant from Lemma A.2.

**Lemma 2.8.** Assume that \( \psi_1 \) and \( \psi_2 \) satisfies \( \tilde{Q}_m(B)\psi_j = \lambda_j(B)\psi_j \) with \( \lambda_j(B) \leq \Theta_0 B + \omega B^{2/3} \). If \( |\tilde{\zeta}_0| < \tilde{D}_0 \) for some constant \( \tilde{D}_0 > 0 \) then there exist constants \( C > 0 \) and \( B_0 > 0 \) (independent of \( m \) and \( B \)) such that

\[
|q_m(\chi_B \psi_1, \chi_B \psi_2) - q_m(\chi_B \psi_1, \chi_B \psi_2)| \leq C B^{-1/2} \|\psi_1\| \cdot \|\psi_2\|
\]

for \( B > B_0 \).

**Proof.** This follows by expanding the terms in \( \tilde{q}_m \) and estimating using (2.6) and (2.7).

Next, we introduce

\[
\tilde{\zeta}_1 = (\tilde{\zeta}_0 - \tilde{\zeta}_0)B^{1/6} = \left( \frac{1}{\sqrt{B}}(m - B/2) - \tilde{\zeta}_0 \right)B^{1/6}, \tag{2.9}
\]

and note that if \( |\tilde{\zeta}_0| < \tilde{D}_0 \) then

\[
|\tilde{\zeta}_1| \leq D_0 B^{1/6}. \tag{2.10}
\]

With this notation, the form \( q_m \) reads

\[
q_m[\psi] = \int_{\mathbb{R}^d_+} |\partial_\tau \psi|^2 + \left( \tau + \tilde{\zeta}_0 + B^{-1/6} \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right) \right)^2 |\psi|^2 + B^{-1/3} |\partial_\rho \psi|^2 \, d\tau \, d\rho.
\]
Given constants $C_1$, $C_2$ and $M$ we define the function $W_B$ as
\begin{equation}
W_B(\rho) = \begin{cases} 
\Theta_0 + C_1 B^{-1/3}, & |\rho| < M, \\
\Theta_0 + C_2 M^4 B^{-1/3}, & |\rho| \geq M.
\end{cases}
\end{equation}

**Proposition 2.9.** Assume that $|\zeta_1| \leq D_0 B^{1/6}$. There exist positive constants $C_1$, $C_2$, $M$ and $B_0$ such that for all $m \in \mathbb{Z}$
\[ q_m[\psi] \geq \int_{\mathbb{R}^2} W_B(\rho)|\psi|^2 \, d\tau \, d\rho \]
for all $B > B_0$ and $\psi \in \text{Dom}(q_m)$ if $M > M$. 

Before proving this proposition we emphasize that these estimates do not only hold for ground states, but for all functions $\psi$ in the domain of $q_m$.

**Proof.** We will prove a slightly stronger statement than the one in Proposition 2.9. Let $D > 0$ be a large number, to be specified below. In the following lemmas we consider the three cases:

1. $\tilde{\zeta}_1 > D$ as $B > B_1$ for some $B_1 > 0$ (Lemma 2.10),
2. $\tilde{\zeta}_1 < -D$ as $B > B_2$ for some $B_2 > 0$ (Lemma 2.11),
3. $|\zeta_1| \leq D$ for all $B$ (Lemma 2.12).

In fact, for the cases (1) and (2) we will prove stronger estimates. \(\Box\)

**Lemma 2.10.** Assume that $\tilde{\zeta}_1 > D$ and that $B > D^6$ for some $D > 0$. Then there exists a positive constant $C$, independent of $D$, such that
\[ q_m[\psi] \geq (\Theta_0 + CD^2 B^{-1/3})\|\psi\|^2 \]
for all $\psi \in \text{Dom}(q_m)$.

**Proof.** By the assumption on $\tilde{\zeta}_1$ we have
\[ B^{-1/6} (\tilde{\zeta}_1 + \frac{\rho^2}{2}) > DB^{-1/6} \]
for all $\rho \in \mathbb{R}$. Using the de Gennes model operator $\mathcal{G}$ from Appendix A.2 we get a positive constant $C$ such that
\[ \lambda_{1,\mathcal{G}} (\hat{\zeta}_0 + B^{-1/6} (\tilde{\zeta}_1 + \frac{\rho^2}{2})) \geq \lambda_{1,\mathcal{G}} (\hat{\zeta}_0 + DB^{-1/6}) \]
\[ \geq \Theta_0 + CD^2 B^{-1/3}, \quad B > D^6, \]
from which (2.12) follows. \(\Box\)

**Lemma 2.11.** Assume that $|\zeta_1| \leq D_0 B^{1/6}$. Then there exist positive constants $C$ and $\tilde{D}$ such that if $B > D_0^{18}$, $D > \tilde{D}$ and $\tilde{\zeta}_1 < -D$ then
\[ q_m[\psi] \geq (\Theta_0 + CD^{1/2} B^{-1/3})\|\psi\|^2 \]
for all $\psi \in \text{Dom}(q_m)$.

**Proof.** We assume that $\tilde{\zeta}_1 < -D$, for some constant $D > 0$. Along the proof we will get some constraints on $D$ that finally will determine $\tilde{D}$.

We first assume that $D > 1$. Let $0 \leq \chi_{1,B}(\rho) \leq 1$ be a smooth cut-off function that satisfies:

(A) $\chi_{1,B}(\rho) = 1$ if $(1 - \frac{1}{4}|\tilde{\zeta}_1|^{-1/4}) \sqrt{2|\tilde{\zeta}_1|} \leq |\rho| \leq (1 + \frac{1}{4}|\tilde{\zeta}_1|^{-1/4}) \sqrt{2|\tilde{\zeta}_1|}$,
(B) $\chi_{1,B}(\rho) = 0$ if $|\rho| \leq (1 - |\tilde{\zeta}_1|^{-1/4}) \sqrt{2|\tilde{\zeta}_1|}$ or $|\rho| \geq (1 + |\tilde{\zeta}_1|^{-1/4}) \sqrt{2|\tilde{\zeta}_1|}$,
(C) $|\chi'_{1,B}(\rho)| \leq h_1 |\tilde{\zeta}_1|^{-1/4}$ for some constant $h_1 \geq 0$. 

By the assumption $|\tilde{\zeta}_1| \leq D_0 B^{1/6}$, we get $\chi_{1,B}(\rho) \equiv 1$ and in this case
\[ q_m[\psi] \geq (\Theta_0 + CD^{1/2} B^{-1/3})\|\psi\|^2 \]
for all $\psi \in \text{Dom}(q_m)$. \(\Box\)
(D) The function $\chi_{2,B}(\rho) = \sqrt{1 - \chi_{1,B}^2(\rho)}$ satisfies $|\chi_{2,B}'(\rho)| \leq l_2|\tilde{\zeta}_1|^{-1/4}$ for some constant $l_2 \geq 0$.

The IMS formula gives

$$q_m[\psi] = q_m[\chi_{1,B}\psi] + q_m[\chi_{2,B}\psi] - B^{-1/3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} |\chi_{j,B}(\rho)|^2 \cdot |\psi|^2 \, d\tau \, d\rho.$$  

The error term above is bounded as

$$B^{-1/3} \sum_{j=1}^{2} \int_{\mathbb{R}^3} |\chi_{j,B}(\rho)|^2 \cdot |\psi|^2 \, d\tau \, d\rho \leq (l_1^2 + l_2^2)\tilde{\zeta}_1^{-1/2} B^{-1/3} \|\psi\|^2 \leq (l_1^2 + l_2^2)D^{-1/3} B^{-1/3} \|\psi\|^2.$$  

From (B) we see that the support of $\chi_{1,B}$ is included in the set

$$\left\{ \rho \mid \tilde{\zeta}_1 + \frac{\rho^2}{2} > (-2\tilde{\zeta}_1)^{3/4} + |\tilde{\zeta}_1|^{1/2} \right\} \text{ or } \tilde{\zeta}_1 + \frac{\rho^2}{2} < (2\tilde{\zeta}_1)^{3/4} + |\tilde{\zeta}_1|^{1/2},$$

so especially if $|\tilde{\zeta}_1| > 16$ it holds that

$$|\tilde{\zeta}_1 + \frac{\rho^2}{2}| < \frac{5}{2}|\tilde{\zeta}_1|^{3/4}$$

on the support of $\chi_{1,B}$. By the estimate $|\tilde{\zeta}_1| \leq D_6 B^{1/6}$ we have

$$B^{-1/6} \tilde{\zeta}_1 + \frac{\rho^2}{2} \leq \frac{5}{2} D_6^{3/4} B^{-1/24}$$

on the support of $\chi_{1,B}$. If $B$ is sufficiently large ($B > D_6^{18}$) we can Taylor expand $\lambda_{1,G}$ to find a positive constant $C$ such that

$$\lambda_{1,G} \left( \tilde{\zeta}_0 + B^{-1/6} \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right) \right) \geq \Theta_0 + C \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right)^2 B^{-1/3}$$

for all $\rho$ on the support of $\chi_{1,B}$. We insert this into $q_m$, to get

$$q_m[\chi_{1,B}\psi] \geq \int_{\mathbb{R}^3} \left[ \Theta_0 |\chi_{1,B}\psi|^2 + B^{-1/3} \left( |\partial_\rho(\chi_{1,B}\psi)|^2 + C \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right)^2 |\chi_{1,B}\psi|^2 \right) \right] \, d\tau \, d\rho \geq \left( \Theta_0 + C \tilde{\zeta}_1^{-1/2} B^{-1/3} \right) \|\chi_{1,B}\psi\|^2,$$

where we have used the notation of the Montgomery model in Appendix A.3. By Lemma A.7 it follows that if $|\tilde{\zeta}_1|$ is sufficiently large then

$$\lambda_{1,M} \left( C^{1/3} B^{1/3} \tilde{\zeta}_1 \right) \geq 2^{1/6} D^{1/6} \tilde{\zeta}_1^{-1/2},$$

and so

$$q_m[\chi_{1,B}\psi] \geq \left( \Theta_0 + 2^{1/2} C^{1/2} \tilde{\zeta}_1^{1/2} B^{-1/3} \right) \|\chi_{1,B}\psi\|^2 \geq \left( \Theta_0 + 2^{1/2} C^{1/2} D^{1/2} B^{-1/3} \right) \|\chi_{1,B}\psi\|^2.$$  

In the same way we show that on the support of $\chi_{2,B}$ it holds that

$$|\tilde{\zeta}_1 + \frac{\rho^2}{2}| > \frac{1}{2}|\tilde{\zeta}_1|^{3/4}$$

if $|\tilde{\zeta}_1| > 1/16$. This implies that there exists a constant $C > 0$ such that

$$q_m[\chi_{2,B}\psi] \geq \lambda_{1,G} \left( \tilde{\zeta}_0 \pm \frac{1}{2}|\tilde{\zeta}_1|^{3/4} B^{-1/6} \right) \|\chi_{2,B}\psi\|^2 \geq \left( \Theta_0 + C \tilde{\zeta}_1^{3/2} B^{-1/3} \right) \|\chi_{2,B}\psi\|^2 \geq \left( \Theta_0 + CD^{3/2} B^{-1/3} \right) \|\chi_{2,B}\psi\|^2.$$  

(2.14)

(2.15)
The localization error is bounded by

By the IMS formula, it holds that

On the support of \( \psi \)

Since \( \psi \)

the notation

sufficiently large

The proof is completed by combining the equations (2.13), (2.14) and (2.15) for a sufficiently large \( D \).

The case left to study is when there exists a constant \( D \) such that \( |\tilde{\zeta}_1| < D \).

**Lemma 2.12.** Assume that \( |\tilde{\zeta}_1| \leq D \). There exist positive constants \( C_1, C_2, \tilde{M} \) and \( B_0 \) such that with \( W_B \) from (2.11)

\[
q_m[\psi] \geq \int_{\mathbb{R}^2_+} W_B(\rho)|\psi|^2 \, d\tau \, d\rho
\]

for all \( B > B_0 \) and \( \psi \in \text{Dom}(q_m) \) if \( M > \tilde{M} \).

**Proof.** Fix \( M > 0 \), to be specified below. Let us introduce a smooth cut-off function \( \chi_{1,M} \) that satisfies the following properties

(i) \( 0 \leq \chi_{1,M}(\rho) \leq 1 \)

(ii) \( \chi_{1,M}(\rho) = 1 \) if \( |\rho| < M \)

(iii) \( \chi_{1,M}(\rho) = 0 \) if \( |\rho| \geq 2M \)

(iv) There exists a constant \( l_1 > 0 \) such that \( |\chi_1'(\rho)| \leq l_1/M \) for all \( \rho \).

(v) The function \( \chi_{2,M}(\rho) = \sqrt{1 - \chi_{1,M}^2(\rho)} \) satisfies \( |\chi_{2,M}(\rho)| \leq l_2/M \) for some constant \( l_2 > 0 \).

By the IMS formula, it holds that

\[
q_m[\psi] = q_m[\chi_{1,M}\psi] + q_m[\chi_{2,M}\psi] - B^{-1/3} \sum_{j=1}^{2} \int_{\mathbb{R}^2_+} |\chi_{j,M}'(\rho)|^2 \cdot |\psi|^2 \, d\tau \, d\rho.
\]

The localization error is bounded by

\[
B^{-1/3} \sum_{j=1}^{2} \int_{\mathbb{R}^2_+} |\chi_{j,M}'(\rho)|^2 \cdot |\psi|^2 \, d\tau \, d\rho \leq (l_1^2 + l_2^2) \frac{1}{M^2} B^{-1/3} ||\psi||^2
\]

(2.16)

so by choosing \( M \) large, we can make this error small. We locally introduce the notation \( \psi_j = \chi_{j,M}\psi \), for \( j = 1, 2 \). On the support of \( \psi_1 \) we can use the Taylor expansion of \( \lambda_{1,\tilde{\omega}} \) to get the existence of a constant \( C > 0 \) such that

\[
\lambda_{1,\tilde{\omega}} \left( \tilde{\sigma}_0 + B^{-1/6} \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right) \right) \geq \Theta_0 + CB^{-1/3} \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right)^2,
\]

for \( B \) large enough. This gives

\[
q_m[\psi_1] \geq \int_{\mathbb{R}^2_+} \Theta_0 |\psi_1|^2 + B^{-1/3} \left( |\partial_\rho \psi_1|^2 + C \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right)^2 |\psi_1|^2 \right) \, d\tau \, d\rho.
\]

Next, we use the modified Montgomery operator in Appendix A.3 to get

\[
q_m[\psi_1] \geq \int_{\mathbb{R}^2_+} \Theta_0 |\psi_1|^2 + B^{-1/3} C^{1/3} 2^{-2/3} \lambda_{1,M} \left( C^{1/3} 2^{1/3} \tilde{\zeta}_1 \right) |\psi_1|^2 \, d\tau \, d\rho.
\]

Since \( \lambda_{1,M} \) has minimum \( \hat{\nu}_0 \) we get a constant \( C_1 = \frac{1}{2} C^{1/3} 2^{-2/3} \hat{\nu}_0 \) such that

\[
q_m[\psi_1] \geq \left( \Theta_0 + 2C_1 B^{-1/3} \right) ||\psi_1||^2
\]

On the support of \( \psi_2 \) we have

\[
|\tilde{\zeta}_1 + \frac{\rho^2}{2}| \geq \frac{M^2}{8} - D \geq \frac{M^2}{10}
\]
where $\bar{M}$ is chosen large enough, so that the last inequality holds for $M > \bar{M}$. There exists a constant $C_2 > 0$, independent of $M$, such that

$$\lambda_{1, \mathcal{G}} \left( \frac{\zeta_0 + B^{-1/6} (\zeta_1 + \frac{\rho^2}{2})}{\rho} \right) \geq \lambda_{1, \mathcal{G}} \left( \frac{\zeta_0 \pm B^{-1/6} M^2 / 10}{\rho} \right) \geq \Theta_0 + 2 C_2 M^4 B^{-1/3}$$

and so we get

$$q_m [\psi_2] \geq \left( \Theta_0 + 2 C_2 M^4 B^{-1/3} \right) \| \psi_2 \|^2.$$

This finishes the proof if we choose $\bar{M}$ so large that the localization error (2.15) is dominated by $C_1 B^{-1/3} \| \psi_1 \|^2 + C_2 M^4 B^{-1/3} \| \psi_2 \|^2$.

3. An improved localization formula

**Proposition 3.1.** Let $\omega > 0$ and $a > 0$. Then there exist positive constants $B_0$ and $C_0$ such that if $B > B_0$ and $\psi$ satisfy $Q_m(B) \psi = \lambda \psi$ with $\lambda \leq \Theta_0 + \omega B^{-1/3}$ then

$$\int_{\mathbb{R}^3} e^{2a |\rho| \left( |\psi|^2 + |\partial_\tau \psi|^2 + B^{-1/3} |\partial_\rho \psi|^2 \right)} \, d\tau \, d\rho \leq C_0 \| \psi \|^2. \quad (3.1)$$

**Proof.** Let $\chi_{1,M}$ and $\chi_{2,M}$ be the cut-off functions from the proof of Lemma 2.12, where $M$ is going to be specified below. Also, for $\varepsilon > 0$, let

$$v_\varepsilon (\rho) = \frac{|\rho|}{1 + \varepsilon |\rho|}. \quad (3.2)$$

This function $v_\varepsilon$ is bounded and continuous on $\mathbb{R}$ and differentiable everywhere except at 0. Moreover $|v'_\varepsilon (\rho)| \leq 1$ for all $\rho \neq 0$. In particular the function $\chi_{2,M} \psi e^{av_\varepsilon (\rho)}$ belongs to the domain of $q_m$. We use integration by parts (the IMS formula) to get

$$\lambda \| \chi_{2,M} \psi e^{av_\varepsilon (\rho)} \|^2 = q_m [\chi_{2,M} \psi e^{av_\varepsilon (\rho)}] - \int_{\mathbb{R}^3} |\partial_\rho (\chi_{2,M} \psi e^{av_\varepsilon (\rho)}) \psi |^2 \, d\tau \, d\rho. \quad (3.3)$$

Next, we choose $M$ such that both $(C_2 M^4 - \omega - 2a^2) \geq 1$ and $M > \bar{M}$ hold, where $\bar{M}$ is the constant in Proposition 2.10 and we choose $B$ greater than the constant $B_0$ in Proposition 2.10. Using the assumption on $\lambda$ and the lower bound on $q_m$ from Lemma 2.12 we get

$$\int_{\mathbb{R}^3} \left( C_2 M^4 - \omega - 2a^2 v'_\varepsilon (\rho)^2 \right) \| \chi_{2,M} \psi e^{av_\varepsilon (\rho)} \|^2 \, d\tau \, d\rho \leq 2 \int_{\mathbb{R}^3} |\chi_{2,M} (\rho)|^2 e^{2av_\varepsilon (\rho)} |\psi|^2 \, d\tau \, d\rho. \quad (3.4)$$

The function $\chi_{2,M}' (\rho)$ is supported in the set $\{ \rho \in \mathbb{R} \mid M < \rho < 2M \}$, where also the inequality $e^{av_\varepsilon (\rho)} \leq e^{4aM}$ holds. Inserting this and the choice of $M$ in (3.4) we get

$$\int_{\mathbb{R}^3} |\chi_{2,M} \psi e^{av_\varepsilon (\rho)} |^2 \, d\tau \, d\rho \leq 2 e^{4aM} \int_{\mathbb{R}^3} |\psi|^2 \, d\tau \, d\rho. \quad (3.5)$$

Since the right-hand side is independent of $\varepsilon$ we can let $\varepsilon$ tend to zero and use monotone convergence to get

$$\int_{\mathbb{R}^3} \chi_{2,M}^2 e^{2a \rho |\psi|^2} \, d\tau \, d\rho \leq 2 e^{4aM} \int_{\mathbb{R}^3} |\psi|^2 \, d\tau \, d\rho. \quad (3.6)$$

Since $\chi_{1,M}$ is supported in $\{ \rho \in \mathbb{R} \mid |\rho| < 2M \}$, we have

$$\int_{\mathbb{R}^3} \chi_{1,M}^2 e^{2a \rho |\psi|^2} \, d\tau \, d\rho \leq e^{4aM} \int_{\mathbb{R}^3} |\psi|^2 \, d\tau \, d\rho. \quad (3.7)$$
Combining (3.6) and (3.7) gives the $L^2$-bound in (3.1). Next we turn to the terms involving derivatives. Using the triangle inequality, we have for the $\rho$-derivative
\[
\int_{\mathbb{R}^+} e^{2av(x)}(\chi_{2,M})^2 |\partial_\rho \psi|^2 \, d\tau \, d\rho \\
\leq 2 \int_{\mathbb{R}^+} |\partial_\rho (e^{av(x)}(\chi_{2,M})\psi)|^2 \, d\tau \, d\rho + 2 \int_{\mathbb{R}^+} |\partial_\rho (e^{av(x)}(\chi_{2,M})\psi)|^2 \, d\tau \, d\rho \\
\leq 2B^{1/3}q_m [e^{av(x)}(\chi_{2,M})\psi] + 2 \int_{\mathbb{R}^+} |\partial_\rho (e^{av(x)}(\chi_{2,M})\psi)|^2 \, d\tau \, d\rho.
\]
The corresponding inequality for the $\tau$-derivative is, since $v_\epsilon$ and $\chi_{2,M}$ are independent of $\tau$,
\[
\int_{\mathbb{R}^+} e^{2av(x)}(\chi_{2,M})^2 |\partial_\tau \psi|^2 \, d\tau \, d\rho \leq q_m [e^{av(x)}(\chi_{2,M})\psi].
\]
Combining these two inequalities with (3.3) gives
\[
\int_{\mathbb{R}^+} e^{2av(x)}(\chi_{2,M})^2 \left(|\partial_\tau \psi|^2 + B^{-1/3}|\partial_\rho \psi|^2\right) \, d\tau \, d\rho \\
\leq 2B^{-1/3} \int_{\mathbb{R}^+} |\partial_\rho (e^{av(x)}(\chi_{2,M})\psi)|^2 \, d\tau \, d\rho + 3q_m [e^{av(x)}(\chi_{2,M})\psi] \\
\leq (2B^{-1/3} + 3) \int_{\mathbb{R}^+} |\partial_\rho (e^{av(x)}(\chi_{2,M})\psi)|^2 \, d\tau \, d\rho + 3\lambda \|\chi_{2,M} \psi e^{av(x)}(\rho)\|^2.
\]
Moreover,
\[
|\partial_\rho (e^{av(x)}(\chi_{2,M})\psi)| = |av(\rho) + \chi_{2,M}(\rho)e^{av(x)}| \leq (a + l_2/M)e^{av(x)},
\]
so for $B > 1$ we can use (3.3) to get
\[
\int_{\mathbb{R}^+} e^{2av(x)}(\chi_{2,M})^2 \left(|\partial_\tau \psi|^2 + B^{-1/3}|\partial_\rho \psi|^2\right) \, d\tau \, d\rho \\
\leq (5(a + l_2/M) + 3(\Theta_0 + \omega)) \|\chi_{2,M} \psi e^{av(x)}\|^2 \\
\leq 2[5(a + l_2/M) + 3(\Theta_0 + \omega)]e^{4aM} \int_{\mathbb{R}^+} |\psi|^2 \, d\tau \, d\rho.
\]
By monotone convergence we have
\[
\int_{\mathbb{R}^+} e^{2av(x)}(\chi_{2,M})^2 \left(|\partial_\tau \psi|^2 + B^{-1/3}|\partial_\rho \psi|^2\right) \, d\tau \, d\rho \\
\leq 2[5(a + l_2/M) + 3(\Theta_0 + \omega)]e^{4aM} \int_{\mathbb{R}^+} |\psi|^2 \, d\tau \, d\rho. \quad (3.8)
\]
The same estimate with $\chi_{1,M}$ in place of $\chi_{2,M}$ is easier since we do not have to use $v_\epsilon$ and the functions involved have compact support. The result is
\[
\int_{\mathbb{R}^+} e^{2av(x)}(\chi_{1,M})^2 \left(|\partial_\tau \psi|^2 + B^{-1/3}|\partial_\rho \psi|^2\right) \, d\tau \, d\rho \\
\leq 2[5(a + l_1/M) + 3(\Theta_0 + \omega)]e^{4aM} \int_{\mathbb{R}^+} |\psi|^2 \, d\tau \, d\rho. \quad (3.9)
\]
Finally, a combination of the equations (3.6), (3.7), (3.8) and (3.9) implies (3.1). 
\[\square\]
Corollary 3.2. For all $\omega > 0$ and $n \in \mathbb{N}$ there exist positive constants $B_n$ and $C_n$ such that if $B > B_n$ and $\psi$ satisfies $Q_m \psi = \lambda \psi$ with $\lambda \leq \Theta_0 + \omega B^{-1/3}$ then
\[
\int_{B^2} |\rho|^n (|\psi|^2 + |\partial_r \psi|^2 + B^{-1/3} |\partial_\rho \psi|^2) \, d\tau \, d\rho \leq C_n \|\psi\|^2.
\]

Proof. This follows from Proposition 5.1 by noting that all the terms in the Taylor expansion of $e^{2a|\rho|}$ are positive, and thus for all non-negative integers $n$ it holds that $e^{2a|\rho|} \geq (2a|\rho|)^n / n!$.

4. IMPROVED LOWER BOUNDS AND A SPECTRAL GAP

Proposition 4.1. Let $\tilde{\zeta}_1$ be as in (2.10), i.e., $\tilde{\zeta}_1 = (\frac{1}{\sqrt{B}} (m - B/2) - \zeta_0) B^{1/6}$ and $\lambda_{1,\tilde{\zeta}_1}$ be the lowest eigenvalue of the Montgomery operator, see Appendix A.3.

(A) For all $C_1 > 0$ there exist constants $B_0$ and $C_0$ (independent of $m$) such that if $|\tilde{\zeta}_1| < C_1$ and $B > B_0$ then
\[
q_m[\psi] \geq (\Theta_0 + \lambda_{1,\tilde{\zeta}_1}(B^{1/6} - C_0 B^{-3/8}) \|\psi\|^2
\]
for all $\psi \in \text{Dom}(Q_m(B)).$

(B) For all $C_1 > 0$ there exist positive constants $\gamma$, $B_0$ and $C_0$ (independent of $m$) such that if $|\tilde{\zeta}_1| < C_1$ it holds that
\[
\lambda_{2,\tilde{\zeta}_1}(B) \geq \Theta_0 B + (\zeta_0 + \gamma) B^{2/3} - C_0 B^{7/12}
\]
if $B > B_0$. In particular, if $\tilde{\zeta}_1$ is bounded there exists a positive constant $B_1$ (independent of $m$), for $B > B_1$ then the set
\[
\text{Spec}(\mathcal{H}_m(B)) \cap (-\infty, \Theta_0 B + (\zeta_0 + \gamma/2) B^{2/3})
\]
is either empty or consists of the lowest eigenvalue of $\mathcal{H}_m(B)$.

Proof. We recall that
\[
q_m[\psi] = \int_{\mathbb{R}^n_+} |\partial_r \psi|^2 + \left( \tau \tilde{\zeta}_1 + \hat{\zeta}_0 + B^{-1/6} \left( \hat{\zeta}_1 + \frac{\rho^2}{2} \right) \right)^2 = B^{-1/3} |\partial_\rho \psi|^2 \, d\tau \, d\rho.
\]
We start with the proof of (A). Fix $0 < \zeta < 1/12$. Let us introduce a smooth cut-off function $0 \leq \chi_{1,B}(\rho) \leq 1$ that satisfies the following properties
\[
(i) \quad \chi_{1,B}(\rho) = 1 \text{ if } |\rho| < B^\zeta
\]
\[
(ii) \quad \chi_{1,B}(\rho) = 0 \text{ if } |\rho| \geq 2B^\zeta
\]
\[
(iii) \quad \text{There exists a constant } l_1 > 0 \text{ such that } |\chi_{1,B}'(\rho)| \leq l_1 B^{-\zeta} \text{ for all } \rho.
\]
\[
(iv) \quad \text{The function } \chi_{2,B}(\rho) = \sqrt{1 - \chi_{1,B}^2(\rho)} \text{ satisfies } |\chi_{2,B}'(\rho)| \leq l_2 B^{-\zeta} \text{ for some constant } l_2 > 0.
\]

We denote by $\psi_j = \chi_{j,B} \psi$. Then clearly both $\psi_1$ and $\psi_2$ belong to the domain of $q_m$ and by the IMS formula
\[
q_m[\psi] = q_m[\psi_1] + q_m[\psi_2] - (B^{-1/3} \|\psi_1\|^2 + \|\psi_2\|^2).
\]

The IMS error is easily seen to be bounded from below by some negative constant times $B^{-1/3 - \zeta} \|\psi\|^2$. By Proposition 4.1
\[
q_m[\psi_2] \geq (\Theta_0 + C_0 B^{-1/3}) \|\psi_2\|^2,
\]
where we can make the constant $C_0$ as large as we want, by choosing $B$ large (using the properties of the support of $\chi_{2,B}$).
We turn to \( q_m[\psi_1] \), and note that, with \( \lambda_{1,G} \) from Appendix A.2
\[
q_m[\psi_1] \geq \int_{\mathbb{R}_+^2} \lambda_{1,G} \left( \tilde{\zeta}_0 + B^{-1/6} \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right) \right) |\psi_1|^2 + B^{-1/3} |\partial_\rho \psi_1|^2 \, d\tau \, d\rho.
\]
Since \( \zeta < 1/12 \) we can Taylor expand the first eigenvalue \( \lambda_{1,G} \) to get a constant \( C > 0 \) such that, on the support of \( \psi_1 \),
\[
\lambda_{1,G} \left( \tilde{\zeta}_0 + B^{-1/6} \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right) \right) \
\geq \lambda_{1,G}(\tilde{\zeta}_0) + \frac{1}{2} \lambda'_{1,G}(\tilde{\zeta}_0) \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right)^2 B^{-1/3} - C \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right)^3 B^{-1/2} \
\geq \Theta_0 + \delta_0 \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right)^2 B^{-1/3} - CB^{-1/2+6c}
\]
We insert this into \( q_m \) to get
\[
q_m[\psi_1] \geq \Theta_0 ||\psi_1||^2 + B^{-1/3} \int_{\mathbb{R}_+^2} |\partial_\rho \psi_1|^2 + \delta_0 \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right)^2 |\psi_1|^2 \, d\tau \, d\rho - CB^{-1/2+6c} ||\psi_1||^2.
\]
Next, we use the modified Montgomery model from Appendix A.3 to estimate the integral above,
\[
\int_{\mathbb{R}_+^2} |\partial_\rho \psi_1|^2 + \delta_0 \left( \tilde{\zeta}_1 + \frac{\rho^2}{2} \right)^2 |\psi_1|^2 \, d\tau \, d\rho \geq \lambda_{1,G}(\tilde{\zeta}_1) ||\psi_1||^2.
\]
We choose \( \zeta = 1/48 \) and put the pieces together to obtain (4.1).

We continue with the proof of (B). It is enough to prove (4.2) for \( Q_m(B) \), i.e.,
\[
\lambda_{2,Q_m(B)} \geq \Theta_0 B + (\bar{\zeta}_0 + \gamma) B^{2/3} + O(B^{7/12}), \quad \text{as } B \to \infty,
\]
The inequality for \( H_m(B) \) is then a direct consequence of Lemmas 2.5 and 2.8.

Let \( \psi^{(1)} \) and \( \psi^{(2)} \) denote the first two normalized eigenfunctions of \( Q_m(B) \) and assume that \( m \) is such that \( \lambda_{2,Q_m(B)} \leq \Theta_0 + \omega B^{-1/3} \) for some \( \omega > 0 \), otherwise there is nothing to prove.

We use the same cut-off function \( 0 \leq \chi_{1,B}(\rho) \leq 1 \) as in the proof of (A), but with \( \zeta = 1/72 \). We also introduce the quadratic form \( q_m^D \) with the same action as \( q_m \), but with an additional Dirichlet condition at \( |\rho| = 2B^2 \). For simplicity, we extend functions in the domain of \( q_m^D \) by zero for \( |\rho| > 2B^2 \). We also denote by \( Q_m^D(B) \) the corresponding self-adjoint operator.

We start by showing that
\[
\lambda_{2,Q_m(B)} \geq \lambda_{2,Q_m^D(B)} + O(B^{-\infty}), \quad \text{as } B \to \infty. \quad (4.3)
\]
Let us write \( \psi_j^{(k)} = \chi_{j,B} \psi_j^{(k)} \), \( j, k = 1, 2 \). By the IMS formula, it holds that
\[
q_m[\psi_j^{(k)}] = q_m[\psi_j^{(1)}] + q_m[\psi_j^{(2)}] - B^{-1/3} \sum_{j=1}^2 \int_{\mathbb{R}_+^2} \left| \chi_{j,B}(\rho) \right|^2 \cdot |\psi_j^{(k)}|^2 \, d\tau \, d\rho, \quad k = 1, 2.
\]
By Proposition 3.1, we have for \( k = 1, 2 \), as \( B \to \infty \),
\[
\|\psi_j^{(k)}\| = O(B^{-\infty})\|\psi_j^{(k)}\|, \quad q_m[\psi_j^{(k)}] = O(B^{-\infty})\|\psi_j^{(k)}\|^2, \quad \text{and} \quad B^{-1/3} \sum_{j=1}^2 \int_{\mathbb{R}_+^2} \left| \chi_{j,B}(\rho) \right|^2 \cdot |\psi_j^{(k)}|^2 \, d\tau \, d\rho = O(B^{-\infty})\|\psi_j^{(k)}\|^2.
\]
By the min-max principle we have
\[
\lambda_{2,Q_m(B)} = \max_{\psi \in \text{span}(\psi^{(1)}, \psi^{(2)})} \frac{q_m[\psi]}{\|\psi\|^2} = \max_{|\alpha|, |\beta|} q_m[\alpha\psi^{(1)} + \beta\psi^{(2)}]. \quad (4.5)
\]
Using Proposition 3.1 and (4.4) we see that
\[ q_m[\alpha\psi_1^{(1)} + \beta\psi_1^{(2)}] = q_m[\alpha\psi_1^{(1)} + \beta\psi_1^{(2)}] + O(B^{-\infty}), \quad \text{as } B \to \infty. \]
In the right-hand side we can write \( q_m^D \) instead of \( q_m \). It follows from (4.4) that
\[ \|\alpha\psi_1^{(1)} + \beta\psi_1^{(2)}\|^2 = 1 + O(B^{-\infty}), \quad \text{as } B \to \infty. \]
Using the min-max principle for \( q_m^D \) we have
\[
\lambda_2, q_m^D(B) = \min_{\dim V = 2} \max_{\psi \in V} \frac{q_m^D[\psi]}{\|\psi\|^2} \leq \max_{\alpha, \beta} \frac{q_m[\alpha\psi_1^{(1)} + \beta\psi_1^{(2)}]}{\|\alpha\psi_1^{(1)} + \beta\psi_1^{(2)}\|^2} = \max_{\alpha, \beta} \frac{q_m[\psi]}{\|\psi\|^2} = \lambda_2, B, q_m^D. \]
Combining this with (4.3) we get (4.5).

Next, we show the existence of a positive constant \( \gamma \) such that the inequality
\[ \lambda_2, q_m^D(B) \geq \Theta_0 + (\overline{\gamma} + \gamma)B^{-1/3} + O(B^{-5/12}), \quad \text{as } B \to \infty, \quad (4.6) \]
holds for all \( m \in \mathbb{Z} \) for which \( |\overline{\zeta}| \) is bounded. Let \( \epsilon = B^{-1/3} \) and \( \psi \in \text{Dom}(q_m^D) \).
We write \( q_m^D[\psi] \) as
\[
q_m^D[\psi] = \epsilon \int_{\mathbb{R}^d} |\partial_\tau \psi|^2 + (\tau + \overline{\zeta})^2 |\psi|^2 \, d\tau \, d\rho
+ (1 - \epsilon) \int_{\mathbb{R}^d} |\partial_\rho \psi|^2 + \left( |\partial_\rho \psi| \right) \left( \frac{B^{-1/3}}{1 - \epsilon} \left( \overline{\zeta} + \frac{\rho^2}{2} \right) \right)^2 |\psi|^2 + \left( \frac{B^{-1/3}}{1 - \epsilon} \right) |\partial_\rho \psi|^2 \, d\tau \, d\rho
- \frac{\epsilon}{1 - \epsilon} \int_{\mathbb{R}^d} \left( \frac{B^{-1/6}}{1 - \epsilon} \right)^2 |\psi|^2 \, d\tau \, d\rho.
\]
For sufficiently large \( B \) we use the support of \( \psi \) and the assumption \( |\overline{\zeta}| \leq C_1 \) to bound the last integral, uniformly in \( m \),
\[
\left| \frac{\epsilon}{1 - \epsilon} \int_{\mathbb{R}^d} \left( \frac{B^{-1/6}}{1 - \epsilon} \left( \overline{\zeta} + \frac{\rho^2}{2} \right) \right)^2 |\psi|^2 \, d\tau \, d\rho \right| = O(B^{-2/3 + \epsilon}) \|\psi\|^2, \quad \text{as } B \to \infty.
\]
We get that \( q_m^D[\psi] \) satisfies
\[
q_m^D[\psi] \geq \epsilon \int_{\mathbb{R}^d} |\partial_\tau \psi|^2 + (\tau + \overline{\zeta})^2 |\psi|^2 \, d\tau \, d\rho
+ (1 - \epsilon) \int_{\mathbb{R}^d} \lambda_{1, \varphi} \left( \overline{\zeta} + \frac{B^{-1/6}}{1 - \epsilon} \left( \overline{\zeta} + \frac{\rho^2}{2} \right) \right) |\psi|^2 + \left( \frac{B^{-1/3}}{1 - \epsilon} \right) |\partial_\rho \psi|^2 \, d\tau \, d\rho
+ O(B^{-2/3 + \epsilon}) \|\psi\|^2, \quad \text{as } B \to \infty,
\]
where \( \lambda_{1, \varphi} \) is the lowest eigenvalue of the de Gennes model, see Appendix A.2. We use that \( \psi \) has bounded support and estimate, using the Taylor expansion of \( \lambda_{1, \varphi} \), as \( B \to \infty, \)
\[
\int_{\mathbb{R}^d} \lambda_{1, \varphi} \left( \overline{\zeta} + \frac{B^{-1/6}}{1 - \epsilon} \left( \overline{\zeta} + \frac{\rho^2}{2} \right) \right) |\psi|^2 \, d\tau \, d\rho
\geq \int_{\mathbb{R}^d} \left( \Theta_0 + \frac{\delta_0}{(1 - \epsilon)^2} \right) \left( \frac{B^{-1/3}}{1 - \epsilon} \right)^2 |\psi|^2 \, d\tau \, d\rho + O(B^{-1/3 + \epsilon}) \|\psi\|^2
\geq \int_{\mathbb{R}^d} \left( \Theta_0 + \frac{\delta_0}{(1 - \epsilon)^2} \right) \left( \frac{B^{-1/3}}{1 - \epsilon} \right)^2 |\psi|^2 \, d\tau \, d\rho + O(B^{-1/3 + \epsilon}) \|\psi\|^2.
\]
In the last inequality we also used that \((1 - \varepsilon)^{-2} - (1 - \varepsilon)^{-1} = \mathcal{O}(B^{-1/3})\) together with Corollary 3.2. Inserting in \(q^D_m\) we have, with the choice \(\varsigma = 1/72\), as \(B \to \infty\),
\[
q^D_m[\psi] \geq \varepsilon \int_{\mathbb{R}^*_+} |\partial_\tau \psi|^2 + (\tau + \tilde{\gamma}_0)^2 |\psi|^2 \, d\tau + (1 - \varepsilon) \Theta_0 \int_{\mathbb{R}^*_+} |\psi|^2 \, d\rho + (1 - \varepsilon) \Theta_0 \int_{\mathbb{R}^*_+} |\psi|^2 \, d\rho + \mathcal{O}(B^{-5/12})\|\psi\|^2 \\
+ \frac{B^{-1/3}}{1 - \varepsilon} \int_{\mathbb{R}^*_+} |\partial_\rho \psi|^2 + \delta_0 (\tilde{\gamma}_1 + \frac{\rho^2}{2}) |\psi|^2 \, d\tau \, d\rho + \mathcal{O}(B^{-5/12})\|\psi\|^2 \\
= \varepsilon (\mathcal{G}(\tilde{\gamma}_0) \otimes 1)[\psi] + (1 - \varepsilon) \Theta_0 \|\psi\|^2 + \varepsilon \int_{\mathbb{R}^*_+} |\psi|^2 \, d\rho + \mathcal{O}(B^{-5/12}).
\]

Here the operators \(\mathcal{G}\) and \(\tilde{\mathcal{M}}\) were introduced in Appendix A. We note that the variables \(\tau\) and \(\rho\) are separated in the last expression, so if we denote by \(T\) the operator corresponding to the form on the right-hand side above then we have
\[
\lambda_{1,T} = \varepsilon A_{1,\mathcal{G}}(\tilde{\gamma}_0) + (1 - \varepsilon) \Theta_0 + \lambda_{1,\tilde{\mathcal{M}}}(\tilde{\gamma}_1) B^{-1/3} + \mathcal{O}(B^{-5/12}) \\
= \Theta_0 + \lambda_{1,\tilde{\mathcal{M}}}(\tilde{\gamma}_1) B^{-1/3} + \mathcal{O}(B^{-5/12}), \quad \text{as } B \to \infty.
\]

Denote by
\[
\gamma_{\mathcal{G}} = \lambda_{2,\mathcal{G}}(\tilde{\gamma}_0) - \lambda_{1,\mathcal{G}}(\tilde{\gamma}_0) \quad \text{and} \quad \gamma_{\tilde{\mathcal{M}}} = \inf_{\tilde{\gamma}_1 \leq \tilde{\gamma}_0} \left( \lambda_{2,\tilde{\mathcal{M}}}(\tilde{\gamma}_1) - \lambda_{1,\tilde{\mathcal{M}}}(\tilde{\gamma}_1) \right)
\]
the spectral gaps for the de Gennes model and Montgomery model respectively. Since both \(A_{1,\mathcal{G}}(\tilde{\gamma}_0)\) and \(\lambda_{1,\tilde{\mathcal{M}}}(\tilde{\gamma}_1)\) are simple eigenvalues, and \(\tilde{\gamma}_1\) is varying in a compact set, it follows that both \(\gamma_{\mathcal{G}}\) and \(\gamma_{\tilde{\mathcal{M}}}\) are strictly positive.

For the second eigenvalue of \(T\) we get
\[
\lambda_{2,T} = \Theta_0 + (\lambda_{1,\tilde{\mathcal{M}}}(\tilde{\gamma}_1) + \min(\gamma_{\mathcal{G}}, \gamma_{\tilde{\mathcal{M}}})) B^{-1/3} + \mathcal{O}(B^{-5/12}), \quad \text{as } B \to \infty.
\]

If we choose \(\gamma = \frac{1}{4} \min(\gamma_{\mathcal{G}}, \gamma_{\tilde{\mathcal{M}}} )\), we see that
\[
\lambda_{2,T} \geq \Theta_0 + (\lambda_{1,\tilde{\mathcal{M}}}(\tilde{\gamma}_1) + \gamma) B^{-1/3} + \mathcal{O}(B^{-5/12}) \\
\geq \Theta_0 + (\tilde{\gamma}_0 + \gamma) B^{-1/3} + \mathcal{O}(B^{-5/12}), \quad \text{as } B \to \infty, \quad (4.8)
\]
where in the last inequality we use that \(\lambda_{1,\tilde{\mathcal{M}}}(\tilde{\gamma}_1) \geq \lambda_{1,\tilde{\mathcal{M}}}(\tilde{\gamma}) = \tilde{\gamma}_0\). By (4.7) it follows that \(\lambda_j Q_m(B) \geq \lambda_{j,T}\) for all \(j\), so by (4.8) we get (4.9).

The proof of (4.2) for \(Q_m(B)\) now follows by combining (4.3) and (4.6). \(\square\)

5. Calculating good trial states

5.1. Statement. We provide three different estimates of (the lowest point in) the spectrum of \(\mathcal{H}_m(B)\). Each is superior to the others in a specific parameter regime. When combined with Proposition 4.1, Theorem 5.1 will give two-sided bounds on the ground state energy \(\lambda_{1,\mathcal{H}_m(B)}\).

Theorem 5.1. Let \(\delta_0 > 0\) be the constant from Lemma 4.3. There exist constants \(\tilde{\gamma}_0, \tilde{\gamma}_1, \tilde{\gamma}_2, \lambda_j, j = 0, \ldots, 5,\) and polynomials \(\lambda_4(\delta) = \delta_0 \delta^2 + \lambda_4, \lambda_5(\delta)\) with constant term \(\lambda_5\), and \(\lambda_6(\varsigma)\) quadratic with quadratic term \(\delta_0 \varsigma^2\), such that with
\[
\delta = \left( m - \frac{B}{2} - \tilde{\gamma}_0 \sqrt{B} - \tilde{\gamma}_1 B^{1/3} \right) B^{-1/6} - \tilde{\gamma}_2, \quad \text{and}
\]
\[
\varsigma = m - \frac{B}{2} - \tilde{\gamma}_0 \sqrt{B} - \tilde{\gamma}_1 B^{1/3} - \tilde{\gamma}_2 B^{1/6},
\]
the following holds:
(i) For any $\bar{K}_{2/6} > 0$ there exist $\bar{K}_{2/6}$ and $\bar{B}_{2/6}$ such that if $|\delta| < \bar{K}_{2/6}B^{1/6}$ and $B > \bar{B}_{2/6}$ then
\[ \text{dist} \left[ \sum_{j=0}^{3} \lambda_j B^{-j/6} + \lambda_4(\delta) B^{-4/6}, \text{Spec} \left( \frac{1}{B} \mathcal{H}_m(B) \right) \right] \leq \bar{K}_{2/6}(1 + |\delta|^2)B^{-5/6}. \] (5.1)

(ii) For any $\bar{K}_{1/6} > 0$ there exist $\bar{K}_{1/6}$ and $\bar{B}_{1/6}$ such that if $|\delta| < \bar{K}_{1/6}B^{1/6}$ and $B > \bar{B}_{1/6}$ then
\[ \text{dist} \left[ \sum_{j=0}^{3} \lambda_j B^{-j/6} + \lambda_4(\delta) B^{-4/6} + \lambda_5(\delta) B^{-5/6}, \text{Spec} \left( \frac{1}{B} \mathcal{H}_m(B) \right) \right] \leq \bar{K}_{1/6}B^{-6/6}. \] (5.2)

(iii) For any $\bar{K}_0 > 0$ there exist $\bar{K}_0$ and $\bar{B}_0$ such that if $|\zeta| \leq \bar{K}_0$ and $B > \bar{B}_0$ then
\[ \text{dist} \left[ \sum_{j=0}^{5} \lambda_j B^{-j/6} + \lambda_6(\zeta) B^{-6/6}, \text{Spec} \left( \frac{1}{B} \mathcal{H}_m(B) \right) \right] \leq \bar{K}_0B^{-7/6}. \] (5.3)

We emphasize that the constants $\lambda_j$, $j = 0, \ldots, 5$ and $\zeta_0$, $\zeta_1$ and $\zeta_2$ agree with the constants in Theorem 1.1. In particular $\lambda_0$, $\lambda_1$ and $\lambda_2$ are given in (1.6).

We will spend the rest of this section to prove this theorem. We start by proving part (iii), which is most detailed, and then go back and implement the necessary modifications for (i) and (ii).

5.2. Proof of Theorem 5.1 (iii).

5.2.1. Outline. We will use a trial function $\tilde{\psi}$ on the support of which the operators $\tilde{Q}_m(B)$ and $\mathcal{H}_m(B)$ agree (after a change of coordinates), see Section 5.2.1.3 below. This implies that it is enough to prove the theorem for $\tilde{Q}_m(B)$ instead of $\mathcal{H}_m(B)$.

We start by expanding the operator $\tilde{Q}_m(B)$ in powers of $B^{-1/6}$. Then we use the Grušin method [12, 23] to produce a trial state that agrees with (5.3).

5.2.2. Expansion of $\tilde{Q}_m(B)$. We will write
\[ \zeta_3 = m - \frac{B}{2} - \zeta_0 \sqrt{B} - \zeta_1 B^{1/3} - \zeta_2 B^{1/6}, \]
and once the optimal values of $\zeta_0$, $\zeta_1$ and $\zeta_2$ are determined we will write those as $\bar{\zeta}_0$, $\bar{\zeta}_1$ and $\bar{\zeta}_2$ respectively and insert them in the definition of $\zeta_3$.

We expand the operator $\mathfrak{h} := \tilde{Q}_m(B)$ in the form
\[ \mathfrak{h} \sim \sum_{j=0}^{\infty} \mathfrak{h}_j B^{-j/6}. \] (5.4)

The expansion (5.4) is to be understood as follows: For any function $f$ in $S(\mathbb{R}^3_+)$ and for all $N$ it holds that
\[ \mathfrak{h} f = \sum_{j=0}^{N} B^{-j/6} \mathfrak{h}_j f + O(B^{-(N+1)/6}) \]
in the sense of $L^2(\mathbb{R}^3_+)$. We recall the formula (1.25) for $\tilde{Q}_m(B)$ and expand each term in a Taylor series, valid for small values of $B^{-1/3} \rho$ and $B^{-1/2} \tau$. We will need
Remark 5.2. The first seven operators, 

\[ h_0 = -\partial_\rho^2 + (\tau + \zeta_0)^2, \]
\[ h_1 = 2\left(\zeta_1 + \frac{\rho^2}{2}\right)(\tau + \zeta_0), \]
\[ h_2 = -\partial_\rho^2 + \left(\zeta_1 + \frac{\rho^2}{2}\right)^2 + 2\zeta_2(\tau + \zeta_0), \]
\[ h_3 = 2\zeta_3(\tau + \zeta_0) + 2\zeta_2\left(\zeta_1 + \frac{\rho^2}{2}\right) + \tilde{h}_3, \]
\[ h_4 = 2\zeta_3\left(\zeta_1 + \frac{\rho^2}{2}\right) + \zeta_3^2 + \tilde{h}_4, \]
\[ h_5 = 2\zeta_2\zeta_3 + \tau(3\tau + 4\zeta_0)\zeta_2 + \tilde{h}_5, \]
\[ h_6 = \zeta_3^2 + \tau(3\tau + 4\zeta_0)\zeta_3 + 2((\zeta_0 + \tau)\rho^2 + 2\tau\zeta_1)\zeta_3 + \tilde{h}_6. \]

with 

\[ \tilde{h}_3 = 2\partial_\rho + \tau(\tau + 2\zeta_0)(\tau + \zeta_0) \]
\[ \tilde{h}_4 = \zeta_0^2\rho^2 + \frac{1}{2}(6\zeta_1 + \rho^2)\tau^2 + 4\zeta_0\left(\zeta_1 + \frac{\rho^2}{2}\right)\tau \]
\[ \tilde{h}_5 = -2\tau\partial_\rho^2 + \frac{\rho^4}{6}(4\zeta_0 + \tau) + 2\zeta_1\rho^2(\tau + \zeta_0) + 2\zeta_1^2\tau \]
\[ \tilde{h}_6 = \rho\partial_\rho + 2\tau\partial_\rho + \frac{1}{12}\rho^6 + \frac{2\zeta_1}{3}\rho^4 + \frac{5}{4}\tau^4 + 4\zeta_0\tau^3 + 3\zeta_0^2\tau^2. \]

Remark 5.2. The operators \( h_j, 3 \leq j \leq 6 \) are chosen to be independent of \( \zeta_2 \) and \( \zeta_3 \). For future reference, we also note that the linear terms in \( \zeta_2 \) in the operators \( h_3, h_4 \) and \( h_5 \) re-appear as linear terms in \( \zeta_3 \) in the operators \( h_4, h_5 \) and \( h_6 \) respectively.

We study asymptotic expansions on the form

\[ \lambda \sim \sum_{j=0}^{\infty} \lambda_j B^{-j/6} \quad \text{and} \quad \psi(\tau, \rho) \sim \sum_{j=0}^{\infty} \psi_j(\tau, \rho) B^{-j/6}. \]

We want to find \( \lambda_j \) (as small as possible!) and \( \psi_j \) such that

\[ (h - \lambda)\psi \sim 0. \] (5.6)

The expansion (5.6) is to be understood term-wise, i.e.,

\[ \sum_{j=0}^{k} (h_j - \lambda_j)\psi_{k-j} = 0. \] (5.7)

To start the Grušin approach, we need a function to project on. A study of (5.7) for \( k = 0 \) provides us with that.

5.2.3. Order \( B^{0/6} \), a starting point. At this order equation (5.7) reads

\[ (h_0 - \lambda_0)\psi_0 = 0. \]

Notice that \( h_0 \) does not act in the \( \rho \) variable. We let \( \psi_0(\tau, \rho) = u_0(\tau)\varphi_0(\rho) \). We do not want \( \varphi_0 \) to be identically equal to zero, so we are led to solve

\[ -u_0'' + (\tau + \zeta_0)^2u_0 = \lambda_0u_0, \quad u_0'(0) = 0. \]

This is the eigenvalue equation for the well-known de Gennes operator \( \mathcal{G}(\zeta_0) \) from Appendix A.2. The smallest eigenvalue \( \lambda_0 \) is simple and given by

\[ \lambda_0 = \Theta_0 \]

which is obtained for

\[ \zeta_0 = \zeta_0 = \xi_0, \] (5.8)
see Lemma A.2. The eigenfunction \( u_0 \) is positive and belongs to \( S(\mathbb{R}_+) \) (see Lemma A.3). We may also assume that \( u_0 \) is normalized, \( \int_0^\infty u_0^2 \, d\tau = 1 \).

5.2.4. Higher orders, the Grušin approach. In this subsection we will implement the Grušin method [12, 23] which provides us with a systematic way of calculating a trial state for \( h := Q_m(B) \). We start by introducing some notation. First, we let

\[
\delta h = h - (h_0 - \lambda_0) \sim \sum_{j=1}^\infty h_j B^{-j/6} + \lambda_0,
\]

and notice that

\[
\delta h - \lambda \sim \sum_{j=1}^\infty (h_j - \lambda_j) B^{-j/6}.
\]

Next, we introduce operators \( R^+: L^2(\mathbb{R}) \to L^2(\mathbb{R}_+^2) \), \( R^-: L^2(\mathbb{R}_+^2) \to L^2(\mathbb{R}) \) and \( E_0: L^2(\mathbb{R}_+^2) \to L^2(\mathbb{R}_+^2) \) (and in the corresponding Schwartz spaces) as

\[
(R^+)\varphi(\tau, \rho) = \varphi(\rho) a_0(\tau),
\]

\[
(R^-)f(\rho) = \int_0^{\infty} f(\tau, \rho) u_0(\tau) \, d\tau,
\]

and

\[
E_0 = I \otimes R_{\text{reg}}.
\]

Here \( R_{\text{reg}} \) is the regularized resolvent introduced in (A.2). We introduce the matrix operators \( \mathcal{H} \) and \( \mathcal{E}_0 \), both acting in the Hilbert space \( L^2(\mathbb{R}_+^2) \oplus L^2(\mathbb{R}) \), as

\[
\mathcal{H} = \begin{pmatrix} h - \lambda & R^+ \\ R^- & 0 \end{pmatrix}, \quad \text{and} \quad \mathcal{E}_0 = \begin{pmatrix} E_0 & R^+ \\ R^- & 0 \end{pmatrix}.
\]

By noting that

\[
\begin{pmatrix} h_0 - \lambda_0 & R^+ \\ R^- & 0 \end{pmatrix} \mathcal{E}_0 = I,
\]

we see that

\[
\mathcal{K} := \mathcal{H} \mathcal{E}_0 - I = \begin{pmatrix} (\delta h - \lambda) E_0 & (\delta h - \lambda) R^+ \\ 0 & 0 \end{pmatrix}.
\]

Notice that \( \mathcal{K} = \mathcal{O}(B^{-1/6}) \) as an operator on the Schwartz space.

Let \( N \in \mathbb{N} \). Then by (5.9),

\[
\mathcal{H} \mathcal{E}_0 \sum_{j=0}^{N} (-1)^j \mathcal{K}^j = I + (-1)^N \mathcal{K}^{N+1} = I + \mathcal{O}(B^{-(N+1)/6}), \quad \text{as} \; B \to \infty. \tag{5.10}
\]

By (5.9) we see that \( \mathcal{K}^j \) is given by

\[
\mathcal{K}^j = \begin{pmatrix} (\delta h - \lambda) E_0 & (\delta h - \lambda) E_0 \mathcal{K}^{j-1} \\ 0 & 0 \end{pmatrix}.
\]

Let us define \( E^N, E^{N,+}, E^{N,-} \) and \( E^{N,\pm} \) via

\[
\begin{pmatrix} E^N \\ E^{N,-} \\ E^{N,\pm} \end{pmatrix} := \mathcal{E}_0 \sum_{j=0}^{N} (-1)^j \mathcal{K}^j.
\]

Then, by (5.10),

\[
(h - \lambda) E^N + R^+ E^{N,-} = 1 + \mathcal{O}(B^{-(N+1)/6}),
\]

\[
(h - \lambda) E^{N,+} + R^+ E^{N,\pm} = \mathcal{O}(B^{-(N+1)/6}),
\]

\[
R^- E^N = \mathcal{O}(B^{-(N+1)/6}),
\]

\[
R^- E^{N,+} = 1 + \mathcal{O}(B^{-(N+1)/6}), \tag{5.11}
\]

\[
R^- E^{N,-} = \mathcal{O}(B^{-(N+1)/6}).
\]
as $B \to \infty$. Assume that

$$
\varphi(\rho) = \sum_{j=0}^{N} \varphi_j(\rho)B^{-j/6}, \quad \text{with} \quad E^{N,\pm}\varphi = \mathcal{O}(B^{-(N+1)/6}), \quad \text{as} \quad B \to \infty, \quad (5.12)
$$

with $\varphi_j \in \mathcal{S}(\mathbb{R})$ for all $j$. Then by inserting the vector $(0, \varphi)$ in $(5.10)$ and using the second formula of $(5.11)$ we find that

$$(\mathfrak{h} - \lambda)E^{N,+}\varphi = \mathcal{O}(B^{-(N+1)/6}), \quad \text{as} \quad B \to \infty. \quad (5.13)$$

We expand

$$
E^{N,+} = \sum_{j=0}^{N} E_j^+ B^{-j/6} + \mathcal{O}(B^{-(N+1)/6}), \quad \text{and}
$$

$$
E^{N,\pm} = \sum_{j=0}^{N} E_j^\pm B^{-j/6} + \mathcal{O}(B^{-(N+1)/6}), \quad \text{as} \quad B \to \infty, \quad (5.14)
$$

where the operators

$$(E_j^+)\varphi = \sum_{l_1, \ldots, l_j \in \{1, \ldots, j\}} (-1)^j \left( \prod_{m=1}^{j} E_0(\mathfrak{h}_{l_m} - \lambda_{l_m}) \right) \varphi(\rho)$$

and

$$(E_j^\pm)\varphi = \int_{0}^{\infty} u_0(\tau) \left[ \sum_{i=0}^{j-1} (-1)^i \left( \prod_{l_1, \ldots, l_{i+1} \in \{1, \ldots, j\}} E_0(\mathfrak{h}_{l_i} - \lambda_{l_i}) \right) \varphi(\rho) \right] \varphi(\rho) d\tau$$

are independent of $N$. Here we use the convention that the summand is just $(\mathfrak{h}_j - \lambda_j)$ for $i = 0$. We write

$$E^{N,\pm}\varphi = \sum_{0 \leq j, k \leq N} E_j^\pm \varphi_k B^{-(j+k)/6} + \mathcal{O}(B^{-(N+1)/6}), \quad \text{as} \quad B \to \infty, \quad (5.17)$$

and define for $j \in \{0, \ldots, N\}$ the function $\psi_j$ to be the coefficient in front of $B^{-j/6}$ in the sum on the right-hand side in $(5.17)$. We use $(5.15)$ to get a formula for $\psi_j$,

$$\psi_j(\tau, \rho) = \sum_{l_1, \ldots, l_j \in \{1, \ldots, j\}} (-1)^j \left( \prod_{m=1}^{j} E_0(\mathfrak{h}_{l_m} - \lambda_{l_m}) \right) u_0(\rho) \varphi_k.$$  

We define our trial state by

$$\psi(\tau, \rho) = \sum_{j=0}^{N} \psi_j(\tau, \rho) B^{-j/6}. \quad (5.19)$$

Since the operators involved are continuous (uniformly in $B > 1$), considered on Schwartz functions (the functions $\psi_j$ will be Schwartz functions, see Section $5.2.12$), it follows from $(5.13)$ that

$$(\mathfrak{h} - \lambda)\psi = \mathcal{O}(B^{-(N+1)/6}), \quad \text{as} \quad B \to \infty.$$ 

In particular $(5.7)$ holds for all $0 \leq k \leq N$. 

Before we start with the calculations, let us note that the condition \((5.12)\) on the right reads that

\[
\sum_{j=1}^{k} E_j^{\pm} \varphi_{k-j} = 0, \quad \text{for all } 0 \leq k \leq N. \tag{5.20}
\]

We also introduce the notation

\[
\tilde{E}_j^{\pm} = E_j^{\pm} - \lambda_j.
\]

5.2.5. Order \(B^{-1/6}\), calculation of \(\lambda_1\). To calculate \(\lambda_1\) we need the operator \(E_1^{\pm}\). By \((A.3)\) we see that, for any function \(f \in L^2(\mathbb{R})\), it holds that

\[
E_1^{\pm} f = -R^{-}(\vartheta_1 - \lambda_1)R^{+} f = -R^{-} \left( 2 \left( \zeta_1 + \frac{\rho^2}{2} \right)(\tau + \zeta_0) - \lambda_1 \right) R^{+} f = \lambda_1 f,
\]

and so \(E_1^{\pm}\) is just a multiplication operator. The equation \((5.20)\) reads for \(k = 1\)

\[
0 = E_1^{\pm} \varphi_0 = \lambda_1 \varphi_0.
\]

Since we do not want \(\varphi_0\) to be identically zero, we find that

\[
\lambda_1 = 0 \quad \text{and} \quad E_1^{\pm} = 0
\]
as an operator.

5.2.6. Order \(B^{-1/3}\), calculation of \(\lambda_2\). Using that \(E_1^{\pm} = 0\), the equation \((5.20)\) for \(k = 2\) reads

\[
0 = -E_2^{\pm} \varphi_0 = -R^{-} \left( \vartheta_1 E_0 \vartheta_1 - (\vartheta_2 - \lambda_2) \right) R^{+} \varphi_0
\]

\[
= -\int_0^{\infty} u_0 \left[ 2 \left( \zeta_1 + \frac{\rho^2}{2} \right)(\tau + \zeta_0) R_{reg} 2 \left( \zeta_1 + \frac{\rho^2}{2} \right)(\tau + \zeta_0)
\]

\[
- \left( -\frac{\partial^2}{\partial \rho^2} + (\zeta_1 + \frac{\rho^2}{2} - \lambda_2) \right) \varphi_0 u_0 \, d\tau
\]

\[
= \left( -\frac{\partial^2}{\partial \rho^2} + (1 - 4k_1) \left( \zeta_1 + \frac{\rho^2}{2} \right)^2 - \lambda_2 \right) \varphi_0. \tag{5.21}
\]

The number \(k_1\) is defined in \((A.4)\) and satisfies \((1 - 4k_1) = \delta_0\) by \((A.5)\). We choose \(\lambda_2 = \lambda_2(\zeta_1)\) to be the smallest eigenvalue of the operator \(-\frac{\partial^2}{\partial \rho^2} + \delta_0(\zeta_1 + \frac{\rho^2}{2})^2\), and optimize in \(\zeta_1\), using the analysis in Appendix \(A.3\) to get

\[
\lambda_2 = 2^{-2/3} \rho_0(\rho_0 (\rho_0(\zeta_1 + \frac{\rho^2}{2})) = \zeta_1, \quad \zeta_1 = \hat{\zeta}_1 = \bar{\zeta} = (2\delta_0)^{-1/3} \bar{\zeta}. \tag{5.22}
\]

From now on \(\varphi_0\) denotes the corresponding eigenfunction, normalized in \(L^2(\mathbb{R})\). We note that this also fixes the function \(\psi_0(\tau, \rho) = u_0(\tau) \varphi_0(\rho)\).

5.2.7. Order \(B^{-k/6}, k \geq 3\). On the level \(B^{-k/6}\), our unknowns are \(\lambda_k, \varphi_{k-2}\) and also, up to some level, \(\zeta_2\) and \(\zeta_3\). The following procedure will determine \(\varphi_{k-2}\) and \(\lambda_k\). For \(\varphi_{k-2}\) to exist, it must be possible to solve

\[
-E_2^{\pm} \varphi_{k-2} = \sum_{j=3}^{k} E_j^{\pm} \varphi_{k-j}.
\]

As we saw above \(-E_2^{\pm}\) is the Montgomery operator minus its lowest eigenvalue. Thus, we want the solvability of the differential equation

\[
\left( -\frac{\partial^2}{\partial \rho^2} + \delta_0(\zeta_1 + \frac{\rho^2}{2})^2 - \lambda_2 \right) \varphi_{k-2} = \sum_{j=3}^{k} E_j^{\pm} \varphi_{k-j}. \tag{5.23}
\]
But this is equivalent to the condition that the right hand side is orthogonal to the ground state \( \varphi_0 \), i.e.

\[
\int_{-\infty}^{\infty} \varphi_0 \left( \sum_{j=3}^{k} E_j^+ \varphi_{k-j} \right) \, dp = 0. \tag{5.24}
\]

From (5.24) we get a formula for the unknown \( \lambda_k \),

\[
\lambda_k = - \int_{-\infty}^{\infty} \varphi_0 \left( \sum_{j=3}^{k-1} E_j^+ \varphi_{k-j} + \bar{E_k}^+ \varphi_0 \right) \, dp.
\]

Note that the right hand side will at some levels depend on \( \zeta_2 \) and \( \zeta_3 \). As soon as it does, we will minimize \( \lambda_k \) over \( \zeta_2 \) and \( \zeta_3 \). When \( \lambda_k \) is determined we find \( \varphi_{k-2} \) by inverting the Montgomery operator in (5.23).

We calculate the first terms, the coefficients that appear are introduced in Appendix [A]

5.2.8. Order \( B^{-1/2} \), calculation of \( \lambda_3 \). For \( \lambda_3 \) we get

\[
\lambda_3 = - \int_{-\infty}^{\infty} \varphi_0 \left( \frac{\hat{E}_3^+ \varphi_0}{2} \right) \, dp
\]

\[= \langle \varphi_0 | 0, [h_3 - h_1 E_0(h_2 - \lambda_2) - (h_2 - \lambda_2)E_0 h_2 + h_1 E_0 h_1 E_0 h_2] u_0 \varphi_0 \rangle. \]

Calculations, using the formulas (5.5) for \( h_j \) and the calculations in Appendix [A] give

\[
\langle \varphi_0 u_0, h_3 u_0 \varphi_0 \rangle = -\frac{7}{6} u_0(0)^2 + \bar{\zeta}_3 + \frac{1}{2} \zeta_2,
\]

\[
\langle \varphi_0 u_0, h_1 E_0(h_2 - \lambda_2)u_0 \varphi_0 \rangle = 0,
\]

\[
\langle \varphi_0 u_0, (h_2 - \lambda_2)E_0 h_1 u_0 \varphi_0 \rangle = 0,
\]

\[
\langle \varphi_0 u_0, h_1 E_0 h_1 E_0 h_2 u_0 \varphi_0 \rangle = 8k_2 M_{0,0}^3,
\]

which implies that

\[
\lambda_3 = -\frac{7}{6} u_0(0)^2 + \bar{\zeta}_3 - \frac{1}{2} \zeta_2 + 8k_2 M_{0,0}^3. \tag{5.25}
\]

5.2.9. Order \( B^{-2/3} \), calculation of \( \lambda_4 \). For \( \lambda_4 \) we get

\[
\lambda_4 = - \int_{-\infty}^{\infty} \varphi_0 \left( \frac{\hat{E}_4^+ \varphi_1 + \bar{E}_4^+ \varphi_0}{3} \right) \, dp
\]

\[= \langle \varphi_0 | 0, [h_4 - h_1 E_0(h_3 - \lambda_3) - (h_2 - \lambda_2)E_0 h_2 - \lambda_2 + (h_2 - \lambda_2)E_0 h_2 - \lambda_2 - (h_3 - \lambda_3)E_0 h_1 + h_1 E_0 h_1 E_0 h_2 + h_1 E_0 h_1 - h_1 E_0 h_1 E_0 h_2 - h_1 E_0 h_1 E_0 h_2] u_0 \varphi_0 \rangle. \]

Instead of calculating all these integrals explicitly we only look for their dependence on \( \zeta_2 \) and \( \zeta_3 \). It turns out that they are all independent of \( \zeta_3 \) and that \( \zeta_2 \) appears in the following integrals, which we calculate with help of the relations in Appendix [A] (we let const denote any constant independent of \( \zeta_2 \)): The quadratic dependence on \( \zeta_2 \) is given by

\[
\langle \varphi_0 u_0, h_4 u_0 \varphi_0 \rangle = \zeta_2^2 + \text{const},
\]

\[-\langle \varphi_0 u_0, (h_2 - \lambda_2)E_0(h_2 - \lambda_2)u_0 \varphi_0 \rangle = -4k_1 \zeta_2^2. \]
The linear dependence on $\zeta_2$ is given by
\[
\langle \varphi_0 u_0, (h_3 - \lambda_3) u_0 \varphi_1 \rangle = 2\zeta_2 M_{0,1}^1 + \text{const},
\]
\[
-\langle \varphi_0 u_0, (h_2 - \lambda_2) u_0 \varphi_1 \rangle = -4M_{0,1}^1 k_1 \zeta_2,
\]
\[
-\langle \varphi_0 u_0, (h_2 - \lambda_2) E_0 h_1 u_0 \varphi_1 \rangle = -4M_{0,1}^2 k_1 \delta_0 \zeta_2,
\]
\[
\langle \varphi_0 u_0, (h_2 - \lambda_2) E_0 h_1 E_0 (h_2 - \lambda_2) u_0 \varphi_0 \rangle = 8M_{0,0}^2 k_2 \zeta_2,
\]
\[
\langle \varphi_0 u_0, (h_2 - \lambda_2) E_0 h_1 E_0 (h_2 - \lambda_2) u_0 \varphi_0 \rangle = 8M_{0,0}^2 k_2 \zeta_2 + \text{const},
\]
\[
\langle \varphi_0 u_0, (h_2 - \lambda_2) E_0 h_1 E_0 (h_2 - \lambda_2) u_0 \varphi_0 \rangle = 8M_{0,0}^2 k_2 \zeta_2.
\]

The result is
\[
\lambda_4 = (1 - 4k_1) \zeta_2^2 + 2(M_{0,1}^1 (1 - 4k_1) + 12M_{0,0}^2 k_2) \zeta_2 + \text{const}
\]
\[
= \delta_0 \zeta_2^2 + 2(M_{0,1}^1 \lambda_0 + 12M_{0,0}^2 k_2) \zeta_2 + \text{const}.
\]

We see that $\lambda_4$ depends on $\zeta_2$ as a parabola and is minimal if
\[
\zeta_2 = \hat{\zeta}_2 = -\frac{1}{\delta_0} (M_{0,1}^1 \lambda_0 + 12M_{0,0}^2 k_2) = -M_{0,1}^1 - M_{0,0}^2 12k_2 \delta_0^{-1}.
\]

This fixes the value of $\lambda_4$. 

5.2.10. Order $B^{-5/6}$, calculation of $\lambda_5$. For $\lambda_5$ we get

\[
\lambda_5 = -\int_{-\infty}^{\infty} \varphi_0 (E_3^+ \varphi_2 + E_3^+ \varphi_1 + \hat{E}_3^+ \varphi_0) d\rho.
\]

To calculate all the integrals corresponding to $\lambda_5$ in full detail would be too cumbersome for the presentation. The importance of this step is the dependence on $\zeta_2$. Thus we only calculate the integrals involving $\zeta_3$. First, we get some integrals involving $\zeta_3$ and $\zeta_2$.

\[
\langle \varphi_0 u_0, h_3 u_0 \varphi_0 \rangle = 2\hat{\zeta}_2 \zeta_3,
\]
\[
-\langle \varphi_0 u_0, (h_2 - \lambda_2) E_0 (h_3 - \lambda_3) u_0 \varphi_0 \rangle = -4k_1 \hat{\zeta}_2 \zeta_3 + \text{const},
\]
\[
-\langle \varphi_0 u_0, (h_3 - \lambda_3) E_0 (h_2 - \lambda_2) u_0 \varphi_0 \rangle = -4k_1 \hat{\zeta}_2 \zeta_3 + \text{const}.
\]

Here const denotes a constant that is independent of $\zeta_3$. By Remark 5.2 the linear terms in $\zeta_3$ (independent of $\hat{\zeta}_2$) are the same as the linear terms in $\zeta_2$ for $\lambda_4$, see (5.26). The result is that $\lambda_5$ is given by

\[
\lambda_5 = [2M_{0,1}^1 (1 - 4k_1) + 2(1 - 4k_1)\hat{\zeta}_2 + 24M_{0,0}^2 k_2] \zeta_3 + \text{const}.
\]

By the choice of $\hat{\zeta}_2$ in (5.28) we see that the coefficient in front of $\zeta_3$ is zero, so $\lambda_5$ is independent of $\zeta_3$. We continue with the calculation of the dependence of $\zeta_3$ in $\lambda_6$.

5.2.11. Order $B^{-1}$, calculation of $\lambda_6$. We do not calculate $\lambda_6$ in its full detail. The important part is its dependence on $\zeta_3$. There are two integrals that give rise to quadratic terms $\zeta_3^2$:

\[
\langle \varphi_0 u_0, h_0 u_0 \varphi_0 \rangle = \zeta_3^2 + \text{lower order terms in } \zeta_3, \quad \text{and}
\]
\[
-\langle \varphi_0 u_0, (h_3 - \lambda_3) E_0 (h_3 - \lambda_3) u_0 \varphi_0 \rangle = -4k_1 \zeta_3^2 + \text{lower order terms in } \zeta_3.
\]

The result is that $\lambda_6$ depends on $\zeta_3$ quadratically with coefficient $1 - 4k_1 = \delta_0 > 0$ in front of $\zeta_3^2$. We introduce the constants $\zeta_3$ and $\mathcal{C}$ via

\[
\lambda_6 = \lambda_6 (\zeta_3) = \delta_0 (\zeta_3 - \hat{\zeta}_3)^2 + \mathcal{C}.
\]
5.2.12. Order $B^{-7/6}$ and $B^{-8/6}$, regularity properties. We note in order to obtain the functions $\varphi_5$ and $\varphi_6$ we should continue the calculations to the scales $B^{-7/6}$ and $B^{-8/6}$ respectively.

We end this section by noting that the facts that $u_0$ and $\varphi_0$ are Schwartz functions (Lemmas A.3 and A.8) and that the resolvents of the de Gennes and Montgomery operators both maps the corresponding Schwartz space continuously to itself (Lemmas A.3 and A.9), imply that the functions $\psi_j$, $j = 0, \ldots, 6$, given by (5.18) all belong to $\mathcal{S}(\mathbb{R}_+^2)$.

5.2.13. End of proof of Theorem 5.1 (iii). The calculations above provide us with the differences that $
abla \varphi_5 \eta$ and $\nabla \varphi_6 \eta$ are Schwartz functions $(\varphi_5, \varphi_6)$, and functions $\psi_j \in \mathcal{S}(\mathbb{R}_+^2)$, $j = 0, \ldots, 6$. We note that among the constants $\lambda_j$, $\lambda_0$ is the only one that depends on $\zeta$, as in (5.29). Moreover, by carefully following $\zeta$ through the calculations of $\psi_j$, we find that $\psi_0, \psi_1$ and $\psi_2$ do not depend on $\zeta$, while

\[
\begin{align*}
\psi_3 &= \psi_{3,0} + \psi_{3,1}\zeta_3 \\
\psi_4 &= \psi_{4,0} + \psi_{4,1}\zeta_3 + \psi_{4,2}\zeta_3^2 \\
\psi_5 &= \psi_{5,0} + \psi_{5,1}\zeta_3 + \psi_{5,2}\zeta_3^2 + \psi_{5,3}\zeta_3^3 \\
\psi_6 &= \psi_{6,0} + \psi_{6,1}\zeta_3 + \psi_{6,2}\zeta_3^2 + \psi_{6,3}\zeta_3^3 + \psi_{6,4}\zeta_3^4,
\end{align*}
\]

where all the involved functions belong to $\mathcal{S}(\mathbb{R}_+^2)$. We let $\chi_B$ be a usual smooth cut-off function, satisfying

\[
\begin{align*}
\chi_B(\tau, \rho) &= 1 \text{ on } \{(\tau, \rho) \mid 0 < \tau < \frac{1}{6}B^{1/2}, \ |\rho| < \frac{\pi}{8}B^{1/3}\}, \text{ and} \\
\operatorname{supp}(\chi_B(\tau, \rho)) &\subset \{(\tau, \rho) \mid 0 < \tau < \frac{1}{3}B^{1/2}, \ |\rho| < \frac{\pi}{4}B^{1/3}\},
\end{align*}
\]

and with Neumann condition at $\tau = 0$, $(\partial_\tau \chi_B)(0, \rho) = 0$. With

\[
\lambda = 6 \sum_{j=0}^{6} \lambda_j B^{-j/6}, \quad \tilde{\psi}(\tau, \rho) = \chi_B \sum_{j=0}^{6} \psi_j B^{-j/6}, \quad \text{and} \quad \hat{Q}_m = \sum_{j=0}^{6} h_j B^{-j/6} + R_\tau
\]

we can write

\[
\begin{align*}
(\hat{Q}_m - \lambda) \hat{\psi} &= \sum_{k=0}^{6} \sum_{j=0}^{k} (h_j - \lambda_j) \psi_{k-j} B^{-k/6} + \sum_{k=1}^{6} \sum_{j=1}^{k} (h_j - \lambda_j) \psi_{k+j-6} B^{-(k+6)/6} + R_\tau \hat{\psi} \\
&+ \sum_{k=0}^{6} \sum_{j=0}^{k} (h_j - \lambda_j)(1 - \chi_B) \psi_{k-j} B^{-k/6} + \sum_{k=1}^{6} \sum_{j=1}^{k} (h_j - \lambda_j)(1 - \chi_B) \psi_{k+j-6} B^{-(k+6)/6}.
\end{align*}
\]

The first sum vanishes according to (5.27). Since $|\zeta_3| \leq \tilde{K}_0$ the second sum and $R_\tau \hat{\psi}$ are both bounded by a constant (independent of $\zeta_3$) times $B^{-7/6}$. The last two sums are of order $O(B^{-\infty})$, since all functions $\psi_j \in \mathcal{S}(\mathbb{R}_+^2)$. We therefore get the existence of constants $\tilde{K}_0$ and $\tilde{B}_0$ such that

\[
\| (\hat{Q}_m - \lambda) \hat{\psi} \| \leq \tilde{K}_0 B^{-7/6} \| \hat{\psi} \|
\]

for $B > \tilde{B}_0$. By the spectral theorem we conclude (5.3).

5.3. Proof of Theorem 5.1 (i). We use the same approach as in the proof of (iii). Actually, we repeat the same calculations, but with the differences that $\zeta_3 = 0$ and $\zeta_4 = \zeta_2 + \delta$. The result is that the operators $h_0$ and $h_1$ are independent of $\delta$, while $h_2$ involves $\delta$ linearly in the form $2\delta(\tau + \zeta_0)$, $h_3$ involves $\delta$ linearly and $h_4$ quadratically. By keeping track of the $\delta$ in the calculations, we find that $\lambda$ will have the form

\[
\lambda = \lambda_0 + \lambda_2 B^{-2/6} + \lambda_3 B^{-3/6} + \lambda_4 (\delta) B^{-4/6}, \quad (5.31)
\]

Thus, we conclude the proof.
where $\lambda_0 = \Theta_0$, $\lambda_2 = \tilde{\gamma}_0$, $\lambda_3$ is given in \[ \text{(5.20)} \] and $\lambda_4(\delta) = \lambda_4 + \delta_0\delta^2$ where $\lambda_4$ is the coefficient calculated in Section \[ \text{(5.2.10)} \] and $\delta_0 > 0$ is the constant from Lemma \[ \text{(A.2)} \]

The trial state $\hat{\psi}$ has the form

$$
\hat{\psi} = \chi_B \left( \psi_0 + \psi_1 B^{-1/6} + (\psi_{2,0} + \psi_{2,1}\delta)B^{-2/6} + (\psi_{3,0} + \psi_{3,1}\delta + \psi_{3,2}\delta^2)B^{-3/6} + (\psi_{4,0} + \psi_{4,1}\delta + \psi_{4,2}\delta^2 + \psi_{4,3}\delta^3)B^{-4/6} \right),
$$

(5.32)

where all involved functions belong to $\mathcal{S}(\mathbb{R}^2_+)$. We write $\hat{Q}_m(B) = \sum_{j=0}^4 \mathfrak{h}_j B^{-j/6} + R_5$ and organize the terms in $(\hat{Q}_m(B) - \lambda)\hat{\psi}$ as

$$(\hat{Q}_m(B) - \lambda)\hat{\psi} = \sum_{k=0}^4 \sum_{j=0}^4 (\mathfrak{h}_j - \lambda_j)\psi_{k-j} B^{-k/6} + \sum_{k=1}^4 \sum_{j=k}^4 (\mathfrak{h}_j - \lambda_j)\psi_{k+j-4-j} B^{-(k+4)/6} + R_5 \hat{\psi} + \sum_{k=1}^4 \sum_{j=k}^4 (\mathfrak{h}_j - \lambda_j)(1 - \chi_B)\psi_{k-j} B^{-k/6} + \sum_{k=1}^4 \sum_{j=k}^4 (\mathfrak{h}_j - \lambda_j)(1 - \chi_B)\psi_{k+j-4-j} B^{-(k+4)/6}.$$

The first double sum is zero by \[ \text{(5.7)} \]. For the second one, we use \[ \text{(5.32)} \] and \[ \text{(5.31)} \] to bound it in $L^2$-norm by (recall that $|\delta| \leq \tilde{K}_{2/6} B^{1/6}$)

$$
C' (1 + |\delta|^2) B^{-5/6} (1 + |\delta| B^{-1/6} + |\delta|^2 B^{-2/6} + |\delta|^3 B^{-3/6}) \leq C'' (1 + |\delta|^2) B^{-5/6}.
$$

Since the original operator depends quadratically on $\delta$, it follows that

$$
\|R_5 \hat{\psi}\| \leq (1 + |\delta|^2) B^{-5/6}\|\hat{\psi}\|.
$$

Again, the last two sums are $O(B^{-\infty})$, since the involved functions belong to the Schwartz space. We get, with $\lambda$ as in \[ \text{(5.31)} \], that there exist constants $\hat{K}_{2/6}$ and $\hat{B}_{2/6}$ such that

$$
\|((\hat{Q}_m - \lambda)\hat{\psi})\| \leq \hat{K}_{2/6} (1 + |\delta|^2) B^{-5/6}\|\hat{\psi}\|,
$$

for $B > \hat{B}_{2/6}$. By the spectral theorem we conclude \[ \text{(5.1)} \].

5.4. Proof of Theorem \[ \text{5.1} \] (ii). We repeat the Grušin calculation from the previous step once more, but this time we take one more term in the expansion. The result is a function $\tilde{\psi}$ of the form

$$
\tilde{\psi} = \chi_B \left( \psi_0 + \psi_1 B^{-1/6} + (\psi_{2,0} + \psi_{2,1}\delta)B^{-2/6} + (\psi_{3,0} + \psi_{3,1}\delta + \psi_{3,2}\delta^2)B^{-3/6} + (\psi_{4,0} + \psi_{4,1}\delta + \psi_{4,2}\delta^2 + \psi_{4,3}\delta^3)B^{-4/6} + (\psi_{5,0} + \psi_{5,1}\delta + \psi_{5,2}\delta^2 + \psi_{5,3}\delta^3 + \psi_{5,4}\delta^4)B^{-5/6} \right)
$$

where all involved functions belongs to $\mathcal{S}(\mathbb{R}^2_+)$. We also get

$$
\lambda = \lambda_0 + \lambda_2 B^{-2/6} + \lambda_3 B^{-3/6} + \lambda_4(\delta) B^{-4/6} + \lambda_5(\delta) B^{-5/6},
$$

(5.33)

with the same $\lambda_0$, $\lambda_2$, $\lambda_3$ and $\lambda_4(\delta)$ as in \[ \text{(5.31)} \]. Moreover, $\lambda_5(\delta)$ depends on $\delta$ as a polynomial with constant coefficient $\lambda_5$, calculated in Section \[ \text{5.2.10} \]

$$
\lambda_5(\delta) = \lambda_5 + a_1\delta + a_2\delta^2 + a_3\delta^3
$$

for some $a_1$, $a_2$ and $a_3$ in $\mathbb{R}$. 

We write \( \hat{Q}_m(B) = \sum_{j=0}^5 b_j B^{-j/6} + R_6 \) and organize the terms in \((\hat{Q}_m - \lambda)\hat{\psi}\) as
\[
(\hat{Q}_m(B) - \lambda)\hat{\psi} = \sum_{k=0}^5 \sum_{j=0}^5 (b_j - \lambda_j)\psi_{k-j}B^{-k/6} + \sum_{k=1}^5 \sum_{j=k}^5 (b_j - \lambda_j)\psi_{k+j-5}B^{-(k+5)/6}
+ R_0\hat{\psi} + \sum_{k=0}^5 \sum_{j=0}^5 (b_j - \lambda_j)(1 - \chi_B)\psi_{k-j}B^{-k/6}
+ \sum_{k=1}^5 \sum_{j=k}^5 (b_j - \lambda_j)(1 - \chi_B)\psi_{k+j-5}B^{-(k+5)/6}.
\]

Again, the first double sum is zero. The second two terms are of order \(B^{-6/6}\), uniformly for bounded \(\delta\), and the last two sums are of order \(O(B^{-\infty})\). We get, with \(\lambda\) as in \((5.3)\), the existence of constants \(\tilde{K}_{1/6}\) and \(\tilde{B}_{1/6}\) such that
\[
\| (\hat{Q}_m - \lambda)\hat{\psi} \| \leq \tilde{K}_{1/6}B^{-6/6}\|\hat{\psi}\|
\]
for \(B > \tilde{B}_{1/6}\). We use the spectral theorem to conclude inequality \((5.2)\). \(\square\)

6. Refined lower bounds

6.1. Statement. We combine the lower bounds from Section 2 with part (i) and (ii) from Theorem 5.1 to have the improved lower bound.

**Theorem 6.1.** Let \(K > 0\). With the constants \(\tilde{C}_0, \tilde{C}_1, \tilde{C}_2, \lambda_j, j = 0, \ldots, 5\) and \(\delta_0\), and with \(\zeta_3\) from Theorem 5.7 there exist constants \(B_0\) and \(K_0\) such that if \(|\zeta_3| \geq K_0\) and \(B > B_0\) then
\[
\lambda_1, H_m(B) \geq \lambda_0 B + \lambda_2 B^{4/6} + \lambda_3 B^{3/6} + \lambda_4 B^{2/6} + \lambda_5 B^{3/6} + K.
\]

6.2. Proof. We divide the proof into several parts, depending on the size of \(\zeta_3\). Actually, for most values of \(\zeta_3\) we prove stronger results. We remind the reader that \(\lambda_0 = \Theta_0\) and \(\lambda_2 = \tilde{\gamma}_0\).

6.2.1. Proof for \(K_{1/2}B^{1/2} \leq |\zeta_3|\). It follows from Lemmas 2.16 and 2.27 that there exist constants \(K_{1/2}\) and \(B_{1/2}\) such that for all \(m \in \mathbb{Z}\), \(|\zeta_3| \geq K_{1/2}B^{1/2}\), \(B > B_{1/2}\) and for all \(\psi\) that satisfies \(\hat{Q}_m(B)\psi = \lambda(B)\psi\) with \(\lambda(B) \leq \Theta_0 B + \omega B^{2/3}\) it holds that
\[
\tilde{q}_m[\psi] \geq (\Theta_0 + 1)\|\psi\|^2,
\]
which clearly implies \((6.1)\).

6.2.2. Proof for \(K_{1/3}B^{1/3} \leq |\zeta_3| \leq K_{1/2}B^{1/2}\). Assume that \(|\zeta_3| \leq K_{1/2}B^{1/2}\) and that \(B > B_{1/2}\). By Lemmas 2.19 and 2.11 it follows that there exist constants \(K_{1/3}\) and \(B_{1/3}\) such that for all \(m \in \mathbb{Z}\), \(|\zeta_3| \geq K_{1/3}B^{1/3}\) and \(B > B_{1/3}\) it holds that
\[
q_m[\psi] \geq (\Theta_0 + (\tilde{\gamma}_0 + 1)B^{-1/3})\|\psi\|^2, \quad \forall \psi \in \text{Dom}(q_m).
\]
By possibly changing \(K_{1/3}\) and \(B_{1/3}\) slightly, it follows from Lemmas 2.5 and 2.8 that for all \(m \in \mathbb{Z}\) such that \(|\zeta_3| \geq K_{1/2}B^{1/3}\), \(B > B_{1/3}\) and for all \(\psi\) that satisfies \(\hat{Q}_m(B)\psi = \lambda(B)\psi\) with \(\lambda(B) \leq \Theta_0 B + \omega B^{2/3}\) it holds that
\[
\tilde{q}_m[\psi] \geq (\Theta_0 + (\tilde{\gamma}_0 + 1)B^{-1/3})\|\psi\|^2,
\]
from which \((6.1)\) follows.
6.2.3. **Proof for** \(K_{5/16}B^{5/16} \leq |\zeta_3| \leq K_{1/3}B^{1/3}\). Assume that \(|\zeta_3| \leq K_{1/3}B^{1/3}\) and that \(B > B_{1/3}\) with the constants \(K_{1/3}\) and \(B_{1/3}\) from the previous step. We define \(\delta\) by the equation

\[
m = \frac{B}{2} + \tilde{\zeta}_0 B^{1/2} + (\tilde{\zeta}_1 + \delta) B^{1/3},
\]

where \(\tilde{\zeta}_0\) and \(\tilde{\zeta}_1\) are the constants from (5.8) and (5.22), and note that the condition \(|\zeta_3| < K_{1/3}B^{1/3}\) implies that \(|\delta| \leq C\) for some constant \(C\). From Lemma A.7 it follows that for all \(|\delta| \leq C\) there exist a positive constant \(C_{\text{pos}}\) such that

\[
\lambda_1,\tilde{\lambda}(\tilde{\zeta}_1 + \delta) \geq \lambda_1,\tilde{\lambda}(\tilde{\zeta}_1) + C_{\text{pos}}|\delta|^2.
\]

(6.2)

For \(|\delta| \geq C_1 B^{-1/48}\) we combine (6.2) with Proposition 4.1 (A) to find that

\[
q_m[\psi] \geq (\Theta_0 + \tilde{\gamma}_0 B^{-1/3} + (C_{\text{pos}} C_1^2 - C) B^{-3/8}) \|\psi\|^2.
\]

for sufficiently large \(B\). If we choose \(C_1\) sufficiently large, we get the existence of a positive constant \(C_{\text{pos}}\) such that for \(C_1 B^{-1/48} \leq |\delta| \leq C\) and for all \(B\) large enough it holds that

\[
q_m[\psi] \geq (\Theta_0 + \tilde{\gamma}_0 B^{-1/3} + C_{\text{pos}} B^{5/8} B^{-3/8}) \|\psi\|^2.
\]

Finally, we invoke Lemmas 2.5 and 2.8 decrease the constant \(C_{\text{pos}}\) slightly to \(C_{\text{pos}}'\) if necessary, to get existence of positive constants \(K_{5/16}\) and \(B_{5/16}\) such that if \(K_{5/16}B^{5/16} \leq |\zeta_3| \leq K_{1/3}B^{1/3}\) and \(B > B_{5/16}\), then for all \(\psi\) that satisfies

\[
\tilde{Q}_m(B) = \lambda(B)\psi \text{ with } \lambda(B) \leq \Theta_0 B + \omega B^{2/3}
\]

it holds that

\[
\tilde{q}_m[\psi] \geq (\Theta_0 B + \tilde{\gamma}_0 B^{2/3} + C_{\text{pos}}' B^{5/8} B^{-3/8}) \|\psi\|^2.
\]

This inequality is also stronger than (6.1).

6.2.4. **Proof for** \(K_{1/6}B^{1/6} \leq |\zeta_3| < K_{5/16}B^{5/16}\). Assume that \(|\zeta_3| < K_{5/16}B^{5/16}\) and \(B > B_{5/16}\) with the constants \(K_{5/16}\) and \(B_{5/16}\) from the previous step. We introduce \(\delta\) as

\[
m = \frac{B}{2} + \tilde{\zeta}_0 B^{1/2} + \tilde{\zeta}_1 B^{1/3} + (\tilde{\zeta}_2 + \delta) B^{1/6},
\]

where \(\tilde{\zeta}_0\), \(\tilde{\zeta}_1\) and \(\tilde{\zeta}_2\) are the constants from (5.8), (5.22) and (5.28) respectively, and note that \(|\delta| \leq C B^{-7/48}\) for some constant \(C\).

Since \(7/48 < 1/6\) we may apply Theorem 5.1(i). It follows by Proposition 4.1 that

\[
\frac{1}{B} \lambda_1,\mathcal{M}_m(B) = \lambda_0 + \lambda_2 B^{-2/6} + \lambda_3 B^{-3/6} + \lambda_4(\delta) B^{-4/6} + O((1 + |\delta|^2) B^{-5/6}),
\]

as \(B \to \infty\).

For large \(|\delta|\) and \(B\), we have

\[
\lambda_4(\delta) B^{-4/6} + O((1 + |\delta|^2) B^{-5/6}) \geq \lambda_4 + 2
\]

where \(\lambda_4\) is the constant in (5.27). We let \(K_{1/6}\) and \(B_{1/6}\) correspond to the constants for which \(K_{1/6}B^{1/6} \leq |\zeta_3| \leq K_{5/16}B^{5/16}\) and \(B > B_{1/6}\) implies that

\[
\lambda_1,\mathcal{M}_m(B) \geq \lambda_0 B + \lambda_2 B^{1/6} + \lambda_3 B^{3/6} + (\lambda_4 + 1) B^{2/6}.
\]

This clearly implies (5.21).
6.2.5. Proof for \( K_0 \leq |\zeta_3| < K_{1/6}B^{1/6} \), final step. Assume that \( |\zeta_3| < K_{1/6}B^{1/6} \) and that \( B > B_1/6 \), with the constants \( K_{1/6} \) and \( B_1/6 \) from the previous step. Again, we let \( \delta \) be given by

\[
m = \frac{B}{2} + \hat{\zeta}_0 B^{1/2} + \hat{\zeta}_1 B^{1/3} + (\hat{\zeta}_2 + \delta) B^{1/6}.
\]

This time \( |\delta| \leq C \), for some constant \( C \), so we can apply Theorem 5.1(ii) combined with Proposition 4.1. Recall that \( \lambda_4(\delta) = \delta_0 \delta^2 + \lambda_4 \) and \( \lambda_5(\delta) = \lambda_5 + a_1 \delta + a_2 \delta^2 + a_3 \delta^3 \). We rewrite the last two terms in the eigenvalue expansion from Theorem 5.1(ii) as

\[
\lambda_4(\delta) B^{-4/6} + \lambda_5(\delta) B^{-5/6}
= \lambda_4 B^{-4/6} + \lambda_5 B^{-5/6} + \delta_0 \delta^2 B^{-4/6} + (a_1 \delta + a_2 \delta^2 + a_3 \delta^3) B^{-5/6}.
\]

If we choose \( \hat{C} \) sufficiently large and \( C \geq \hat{C} B^{-1/6} \) then the term \( \delta_0 \delta^2 B^{-4/6} \) will dominate both \( (a_1 \delta + a_2 \delta^2 + a_3 \delta^3) B^{-5/6} \) and the error which is bounded by some constant times \( B^{-6/6} \), indeed, we can get that all these three terms are bounded from below by \( \frac{1}{2} \delta_0 C^2 B^{-6/6} \).

Therefore we find that there exist constants \( B_0 \) and \( K_0 \) such that if it holds that

\[
\lambda_1, \lambda_0(B) \geq \lambda_0 B + \lambda_2 B^{4/6} + \lambda_3 B^{3/6} + \lambda_4 B^{2/6} + \lambda_5 B^{1/6} + K.
\]

This finishes the proof of Theorem 5.1 \( \square \)

7. Proof of Theorem 1.3

For bounded \( \zeta_3 \) we combine Theorem 5.1(iii) with Proposition 4.1 to get the asymptotic formula

\[
\lambda_1, \lambda_0(B) = \lambda_0 B + \lambda_2 B^{4/6} + \lambda_3 B^{3/6} + \lambda_4 B^{2/6} + \lambda_5 B^{1/6} + \lambda_6(\zeta_3) + O(B^{-1/6}) \tag{7.1}
\]

as \( B \to \infty \). Comparing the lower bound from (6.1) with (7.1), we find that the lowest eigenvalue is smallest for bounded \( \zeta_3 \), and that its asymptotic expansion then is given by (7.1). For bounded \( \zeta_3 \) we see from (5.29) that the smallest value of \( \lambda_6(\zeta_3) \) is given for \( \zeta_3 = \hat{\zeta}_1 \). However, since \( m \) must be an integer, we are not free to choose \( \zeta_3 = \zeta_3(m, B) \) arbitrarily. With

\[
\Delta_B = \inf_{m \in \mathbb{Z}} |\zeta_3(m, B) - \hat{\zeta}_1|
\]

as in (1.4) we find that the smallest possible \( \lambda_6(\zeta_3) \) is given by

\[
\lambda_6 = \delta_0 \Delta_B^2 + C. \tag{7.2}
\]

This finishes the proof of Theorem 5.1 \( \square \)

8. Monotonicity of \( \lambda_1, \lambda_0(B) \), Proof of Theorem 1.3

We first note that by perturbation theory it holds that

\[
\lambda_1', \lambda_0(B), + \leq \lambda_1', \lambda_0(B), - \tag{8.1}
\]

for all \( B > 0 \).

From Theorem 1.1 we know that the lowest eigenvalue \( \lambda_1, \lambda_0(B) \) of \( \mathcal{H}(B) \) satisfies

\[
\lambda_1, \lambda_0(B) = \Theta_0 B + \lambda_2 B^{2/3} + \lambda_3 B^{1/2} + \lambda_4 B^{1/3} + \lambda_5 B^{1/6} + \delta_0 \Delta_B^2 + C + O(B^{-1/6}) \tag{8.2}
\]

where

\[
\Delta_B = \inf_{m \in \mathbb{Z}} |\zeta_3(m, B) - \hat{\zeta}_1|
\]

and \( \zeta_3(m, B) = m - \frac{B}{2} - \hat{\zeta}_0 \sqrt{B} - \hat{\zeta}_1 B^{1/3} - \hat{\zeta}_2 B^{1/6} \).
It is proved in [5] that the derivatives \( \lambda'_1, \mathcal{H}(B), \pm \) satisfies

\[
\liminf_{B \to \infty} \frac{\lambda'_1, \mathcal{H}(B), +}{\varepsilon} \geq \limsup_{B \to \infty} \frac{1}{\varepsilon} \liminf_{\varepsilon \to 0^+} \left( \lambda_1, \mathcal{H}(B + \varepsilon) - \lambda_1, \mathcal{H}(B) \right),
\]

\[
\limsup_{B \to \infty} \frac{\lambda'_1, \mathcal{H}(B), -}{\varepsilon} \leq \liminf_{B \to \infty} \frac{1}{\varepsilon} \limsup_{\varepsilon \to 0^+} \left( \lambda_1, \mathcal{H}(B) - \lambda_1, \mathcal{H}(B - \varepsilon) \right).
\]

We start with the right derivative \( \lambda'_1, \mathcal{H}(B), + \), and use (8.2) to write

\[
\frac{\lambda_1, \mathcal{H}(B + \varepsilon) - \lambda_1, \mathcal{H}(B)}{\varepsilon} = \Theta_0 + \frac{g(B + \varepsilon) - g(B)}{\varepsilon} + \delta_0 \frac{\Delta^2_{B + \varepsilon} - \Delta^2_B}{\varepsilon} + \frac{f(B + \varepsilon) - f(B)}{\varepsilon}
\]

where

\[
g(B) = \lambda_2 B^{2/3} + \lambda_3 B^{1/2} + \lambda_4 B^{1/3} + \lambda_5 B^{1/6} + C
\]

and \( f(B) \) is a function satisfying \( \lim_{B \to \infty} f(B) = 0 \). For any fixed \( \varepsilon > 0 \) we clearly have

\[
\lim_{B \to \infty} g(B + \varepsilon) - g(B) = 0.
\]

Consider the term involving \( \frac{\Delta^2_{B + \varepsilon} - \Delta^2_B}{\varepsilon} \). We note that there exist integers \( m_B \) and \( m_{B + \varepsilon} \) such that

\[
\Delta_B = \left| m_B - B - \frac{\Delta 0 \sqrt{B} - \hat{\lambda}_1 B^{1/3} - \hat{\lambda}_2 B^{1/6} - \hat{\lambda}_3 \right|, \quad \text{and}
\]

\[
\Delta_{B + \varepsilon} = \left| m_{B + \varepsilon} - B + \varepsilon - \frac{\Delta 0 \sqrt{B + \varepsilon} - \hat{\lambda}_1 (B + \varepsilon)^{1/3} - \hat{\lambda}_2 (B + \varepsilon)^{1/6} - \hat{\lambda}_3 \right|.
\]

We note that

\[
\Delta_B = \inf_{m \in \mathbb{Z}} \left| m - B - \frac{\Delta 0 \sqrt{B} - \hat{\lambda}_1 B^{1/3} - \hat{\lambda}_2 B^{1/6} - \hat{\lambda}_3 \right|
\]

\[
\leq \left| m_{B + \varepsilon} - B + \frac{\Delta 0 \sqrt{B + \varepsilon} - \hat{\lambda}_1 (B + \varepsilon)^{1/3} - \hat{\lambda}_2 (B + \varepsilon)^{1/6} - \hat{\lambda}_3 \right|
\]

Using this, and the fact that \( 0 \leq \Delta_B \leq 1/2 \) for all \( B \), we get by the triangle inequality

\[
\frac{\Delta^2_{B + \varepsilon} - \Delta^2_B}{\varepsilon} = \left( \frac{\Delta_{B + \varepsilon} - \Delta_{B}}{\varepsilon} \right) (\Delta_{B + \varepsilon} + \Delta_B)
\]

\[
\geq -\frac{1}{2} \left( 1 + \frac{\Delta 0 \sqrt{B + \varepsilon} - \sqrt{B}}{\varepsilon} + \frac{\hat{\lambda}_1 (B + \varepsilon)^{1/3} - B^{1/3}}{\varepsilon} + \frac{\hat{\lambda}_2 (B + \varepsilon)^{1/6} - B^{1/6}}{\varepsilon} \right)
\]

The right-hand side tends to \( -1/2 \) as \( B \to \infty \), so for any fixed \( \varepsilon > 0 \) we get

\[
\frac{1}{\varepsilon} \liminf_{B \to \infty} (\Delta^2_{B + \varepsilon} - \Delta^2_B) \geq -\frac{1}{2}
\]

Inserting these calculations in (8.3) it follows that

\[
\liminf_{B \to \infty} \lambda'_1, \mathcal{H}(B), + \geq \Theta_0 - \frac{1}{2} \delta_0
\]

According to (1.7) the right-hand side \( \Theta_0 - \frac{1}{2} \delta_0 > 0 \). This finishes the proof of (1.9). The same calculations give (1.10) for the left-derivative \( \lambda'_1, \mathcal{H}(B), - \). We conclude the proof of Theorem 1.3 by noting that the equations (1.9) and (8.1) imply that \( \lambda_1, \mathcal{H}(B) \) is increasing for large \( B \).

\[
\square
\]

Appendix A. Model operators

In this appendix, we consider two self-adjoint model operators. The first one is an operator in \( L^2(\mathbb{R}_+^+) \) that was introduced by Saint-James and de Gennes [22]. The second one is an operator in \( L^2(\mathbb{R}) \) first studied by Montgomery in [19].

\[
\]
A.1. A general Lemma. We start by giving a general lemma that will enable us to give moment formulas for the two operators under study.

**Lemma A.1.** Let \(-\infty \leq \alpha < \beta \leq \infty\), and \(p \in C^1[\alpha, \beta]\). (If \(\alpha = -\infty\) and \(\beta = +\infty\) then we assume that \(\lim_{x \to +\infty} p(x) = +\infty\).) Assume that for some \(\lambda \in \mathbb{R}\) and \(u \in L^2(\alpha, \beta)\) it holds that
\[-u'' + pu = \lambda u \quad \text{for all } x \in [\alpha, \beta].\]
Then, for any polynomial \(b\), it holds that
\[
\int_{\alpha}^{\beta} [b'' + 4(\lambda - p)b' - 2p'b] u^2 \, dx = \left[2b(u')^2 + b'uu' - 2b'uu' + 2(\lambda - p)bu^2\right]_{\alpha}^{\beta}. \tag{A.1}
\]

**Proof.** In the case \(\alpha = -\infty\) and/or \(\beta = +\infty\), the additional assumption on \(p\) implies that \(u\) decays exponentially at \(\alpha\) and/or \(\beta\) (the proof is the same as in Lemma A.3).

One could use the same reasoning as in [1]. However, a simple calculation shows that the derivative of the expressions inside the brackets on the right-hand side of (A.1) equals the integrand on the left-hand side. \(\square\)

A.2. The de Gennes operator. For \(\xi \in \mathbb{R}\) we define the operator \(\mathcal{G}(\xi)\) as the self-adjoint Neumann extension in \(L^2(\mathbb{R}^+))\), acting as
\[(\mathcal{G}(\xi)u)(x) = -u''(x) + (x + \xi)^2 u(x),\]
\[u'(0) = 0.\]

Denote by \(\lambda_1, \mathcal{G}(\xi)\) the lowest eigenvalue of \(\mathcal{G}(\xi)\) and \(\Theta_0 = \inf_{\xi \in \mathbb{R}} \lambda_1, \mathcal{G}(\xi)\). We refer to [5] for a discussion of the results summarized in the following lemma.

**Lemma A.2.** The function \(\xi \mapsto \lambda_1, \mathcal{G}(\xi)\) is smooth. Moreover,
1. \(\lim_{\xi \to +\infty} \lambda_1, \mathcal{G}(\xi) = +\infty\).
2. \(\lim_{\xi \to -\infty} \lambda_1, \mathcal{G}(\xi) = 1\).
3. The function \(\lambda_1, \mathcal{G}(\xi)\) attains its minimum value \(\Theta_0, \frac{1}{2} < \Theta_0 < 1\), at a unique point \(\xi_0 < 0\).
4. \(\lambda_1, \mathcal{G}(\xi)\) is decreasing for \(\xi < \xi_0\) and increasing for \(\xi > \xi_0\).
5. The number \(\delta_0 := \frac{1}{2} \lambda_1, \mathcal{G}(\xi_0)\) satisfies \(0 < \delta_0 < 1\).

If we denote by \(v_\xi\) the normalized eigenfunction of \(\mathcal{G}(\xi)\) corresponding to the eigenvalue \(\lambda_1, \mathcal{G}(\xi)\), then we introduce the regularized resolvent
\[R_{reg}(\xi)g = \begin{cases} (\mathcal{G}(\xi) - \lambda_1, \mathcal{G}(\xi))^{-1}g, & g \perp v_\xi, \\ 0, & g \parallel v_\xi, \end{cases} \tag{A.2}\]
and let \(R_{reg} = R_{reg}(\xi_0)\) and \(u_0 = v_{\xi_0}\).

**Lemma A.3** (5, Lemma A.5). The function \(u_0\) belongs to \(S(\mathbb{R}^+)\) and \(R_{reg}\) maps \(S(\mathbb{R}^+)\) continuously into itself.

**Lemma A.4** (1, equations (2.34)–(2.36)). The following equalities hold
\[
\int_{0}^{\infty} u_0^2 \, dx = 1, \quad \int_{0}^{\infty} (x + \xi_0)u_0^2 \, dx = 0, \quad \int_{0}^{\infty} (x + \xi_0)^2u_0^2 \, dx = \frac{\Theta_0}{2}, \quad \int_{0}^{\infty} (x + \xi_0)^3u_0^2 \, dx = \frac{\Theta_0^2(0)}{6}. \tag{A.3}\]

We introduce the integrals
\[k_j(\xi) = \int_{0}^{\infty} (x + \xi)u[R_{reg}(\xi)(x + \xi)]^j u \, dx \tag{A.4}\]
and \(k_j = k_j(\xi_0)\).
Lemma A.5 ([6], Proposition A.3). It holds that
\[ \delta_0 = \frac{1}{2} \lambda_1' \theta(\xi_0) = 1 - 4k_1. \] (A.5)

Remark A.6. Numerical calculations of the constants \( \xi_0, \Theta_0, \delta_0 \) and \( u_0(0) \) has been carried out in [2].

We give a new approach. It is readily seen that the decaying normalized solution to \( G(\xi)u = \lambda_1 G(\xi)u \) is given by \( u(x) = ce^{-\frac{1}{2}(x + \xi)^2} H_{\frac{1}{2}(\lambda_1 G(\xi) - 1)}(x + \xi) \). Here \( c \) denotes a normalization constant and \( H_{\nu} \) is the Hermite function, that solves 
\[ -y''(x) + 2xy'(x) - 2\nu y(x) = 0. \]
The boundary condition \( u(0) = 0 \) transforms into 
\( (\lambda_1 G(\xi) - 1) H_{\frac{1}{2}(\lambda_1 G(\xi) - 1)}(\xi) - \xi H_{\frac{1}{2}(\lambda_1 G(\xi) - 1)}(\xi) = 0 \) and since \( \Theta_0 = \lambda_1 G(\xi_0) = \xi_0^2 \), we find that \( \xi_0 \) should be the largest (it is negative!) number that solves 
\[ (\xi^2 - 1) H_{\frac{1}{2}(\xi^2 - 3)}(\xi) - \xi H_{\frac{1}{2}(\xi^2 - 1)}(\xi) = 0. \]

Numerical calculations in Mathematica give 
\[ \xi_0 \approx -0.76818365314, \quad \Theta_0 \approx 0.59010612495, \quad u_0(0) \approx 0.87304313851, \quad \delta_0 \approx 0.58551290029. \]

A.3. The Montgomery operator. Next, we turn to the Montgomery operator \( \mathcal{M}(\zeta) \), \( \zeta \in \mathbb{R} \), defined as the self-adjoint operator in \( L^2(\mathbb{R}) \) acting as
\[ (\mathcal{M}(\zeta)u)(\rho) = -u''(\rho) + (\zeta + \rho^2)u(\rho), \quad -\infty < \rho < \infty \]
Denote by \( \lambda_{1,\mathcal{M}}(\zeta) \) the lowest eigenvalue of \( \mathcal{M}(\zeta) \) and \( \nu_0 = \inf_{\zeta \in \mathbb{R}} \lambda_{1,\mathcal{M}}(\zeta) \).

Lemma A.7 ([19] [21] [13]). The function \( \zeta \mapsto \lambda_{1,\mathcal{M}}(\zeta) \) is smooth and satisfies 
\( \lim_{\zeta \to \pm \infty} \lambda_{1,\mathcal{M}}(\zeta) = +\infty \). Moreover, the minimal value \( \nu_0 > 0 \) of \( \lambda_{1,\mathcal{M}}(\zeta) \) is attained at a unique point \( \zeta < 0 \), and \( \lambda''_{1,\mathcal{M}}(\zeta) > 0 \).

Let us denote by \( \varphi \) the eigenfunction corresponding to \( \lambda_{1,\mathcal{M}}(\zeta) \). It is known that such an eigenfunction belongs to \( C^\infty(\mathbb{R}) \). We show that \( \varphi \) and its derivatives decay exponentially, which implies that \( \varphi \) belongs to \( S(\mathbb{R}) \).

Lemma A.8. Let \( \varphi \) be the ground state of \( \mathcal{M}(\zeta) \). For any \( 0 < a < 1/3 \) and nonnegative integer \( \ell \) there exist a constant \( C_k \) such that
\[ \int_\mathbb{R} e^{2a|\rho|^3} \left( |\varphi|^2 + |\rho^a \varphi|^2 + |(\partial_\rho)^k \varphi|^2 \right) \, d\rho \leq C_k \int_\mathbb{R} |\varphi|^2 \, d\rho. \] (A.6)

Proof. Let \( a < 1/3 \) be given. For \( \epsilon > 0 \) we define \( \nu_\epsilon(\rho) = (|\rho|/(1 + \epsilon |\rho|))^3 \). Then, for fixed \( \rho \), \( \nu(\rho) \) is monotonically increasing to \( |\rho|^3 \) as \( \epsilon \to 0 \). Moreover it holds that
\[ |\nu'(\rho)| \leq 3|\rho|^2 \] (A.7)
for all \( \epsilon > 0 \) and all \( \rho \in \mathbb{R} \). We let \( \chi_{1,\nu} \) and \( \chi_{2,\nu} \) denote the same functions as in the proof of Lemma 2.12. The IMS formula gives
\[ \int_\mathbb{R} |\partial_\rho(\chi_{2,\nu} e^{av})\varphi|^2 + \left( \zeta + \rho^2 \right)^2 - \lambda_{1,\mathcal{M}}(\zeta) \right) |\chi_{2,\nu} e^{av} \varphi|^2 \, d\rho \]
\[ = \int_\mathbb{R} |\partial_\rho(\chi_{2,\nu} e^{av})\varphi|^2 \, d\rho \] (A.8)

We use the Cauchy-Schwarz inequality on the right-hand side, to get, for any \( \xi > 0 \),
\[ \int_\mathbb{R} |\partial_\rho(\chi_{2,\nu} e^{av})\varphi|^2 \, d\rho \leq (1 + \xi) \int_\mathbb{R} |av e^{av} \chi_{2,\nu} \varphi|^2 \, d\rho + (1 + \frac{1}{\xi}) \int_\mathbb{R} |\partial_\rho(\chi_{2,\nu} e^{av})\varphi|^2 \, d\rho \] (A.9)
We recall that $|\partial_{\rho} \chi_{2,M}| \leq l_2/M$ for all $\rho$ and that $\partial_{\rho} \chi_{2,M}$ has support in the set $\{\rho \in \mathbb{R} \mid M \leq \rho \leq 2M\}$. We implement (A.7) and (A.9) in (A.5) to find that

$$\int_{\mathbb{R}} |\partial_{\rho} (\chi_{2,M} e^{a_{\nu} \varphi})|^2 + (\hat{\zeta} + \rho^2)^2 - \lambda_{1,M}(\hat{\zeta}) - (1 + \zeta)9a^2|\rho|^4) |\chi_{2,M} e^{a_{\nu} \varphi}|^2 \, d\rho \leq (1 + \frac{1}{\zeta})^{\frac{1}{2}} \int_{\mathbb{R}} |\varphi|^2 \, d\rho \quad (A.10)$$

Since $a < 1/3$ we can choose $\zeta$ so small that $(1 + \zeta)9a^2 < 1$. With this choice of $\zeta$ we can find $M$ so large that

$$(\hat{\zeta} + \rho^2)^2 - \lambda_{1,M}(\hat{\zeta}) - (1 + \zeta)9a^2|\rho|^4 \geq 1$$

for all $\rho$ on the support of $\chi_{2,M}$. This together with the trivial bound for small $\rho$ settles the result for $\varphi$. We might also use the first term in (A.10) to prove the result for $\partial_{\rho}$. The statement for $\rho^k \varphi$ follows from this by decreasing $a$ (or to be more precise, prove the result for $\varphi$ for $1/3 > a' > a$ and then decrease this $a'$ to $a$). The result for higher derivatives is now a consequence of induction, using the eigenvalue equation.

Let us define the regularized resolvent $\tilde{R}_{\text{reg}}$ as

$$\tilde{R}_{\text{reg}} u = \begin{cases} (\mathcal{M}(\hat{\zeta}) - \lambda_{1,M}(\hat{\zeta}))^{-1} u, & u \perp \varphi, \\ 0, & u \parallel \varphi. \end{cases} \quad (A.11)$$

We show that if $u$ and its derivatives decay exponentially, then the same is true for $\tilde{R}_{\text{reg}} u$.

Lemma A.9. Let $0 < a < 1/3$. Assume that $u$, $\rho^k u$ and $(\partial_{\rho})^k u$ belong to $L^2(\mathbb{R}, e^{2b|\rho|^3} \, d\rho)$ for all non-negative integers $k$. Then, for any $b < a$, $\tilde{R}_{\text{reg}} u$, $\rho^k \tilde{R}_{\text{reg}} u$ and $(\partial_{\rho})^k \tilde{R}_{\text{reg}} u$ belong to $L^2(\mathbb{R}, e^{2b|\rho|^3} \, d\rho)$ for all non-negative integers $k$.

Proof. Let $w = \tilde{R}_{\text{reg}} u$, so that

$$-\partial_{\rho}^2 w + (\hat{\zeta} + \rho^2)^2 w - \lambda_{1,M}(\hat{\zeta}) w = u, \quad (A.12)$$

with $u$ as in the assumptions. Let $\varepsilon > 0$ and let $v_{\varepsilon}$ be the same function as in the proof of the previous lemma, $v_{\varepsilon}(\rho) = (|\rho|/(1 + \varepsilon|\rho|))^3$. We also let $\chi_{1,M}$ and $\chi_{2,M}$ be the same cut-off functions as in the proof of Lemma 2.12. An integration by parts gives

$$\int_{\mathbb{R}} |\partial_{\rho} (\chi_{2,M} e^{a_{\nu} \varphi})|^2 + (\hat{\zeta} + \rho^2)^2 \chi_{2,M} e^{a_{\nu} \varphi} |^2 \, d\rho = \lambda_{1,M}(\hat{\zeta}) \int_{\mathbb{R}} |\chi_{2,M} e^{a_{\nu} \varphi}|^2 \, d\rho + \int_{\mathbb{R}} |\partial_{\rho} (\chi_{2,M} e^{a_{\nu} \varphi})|^2 \, d\rho + \int_{\mathbb{R}} |\chi_{2,M} e^{a_{\nu} \varphi}|^2 wu \, d\rho.$$

We use the Cauchy-Schwarz inequality for the last term and move terms to the left-hand side,

$$\int_{\mathbb{R}} |\partial_{\rho} (\chi_{2,M} e^{a_{\nu} \varphi})|^2 + P(\rho) |e^{a_{\nu} \varphi}|^2 \, d\rho \leq 2 \int_{\mathbb{R}} |\chi_{2,M} e^{a_{\nu} \varphi} u|^2 \, d\rho,$$

where

$$P(\rho) = \left( (\hat{\zeta} + \rho^2)^2 \chi_{2,M}^2 - \lambda_{1,M}(\hat{\zeta}) \chi_{2,M}^2 - |\partial_{\rho} (\chi_{2,M}) + a_{\nu} \chi_{2,M}|^2 \right).$$

By (A.7), the first term in $P$ is dominant for large $\rho$, so if we choose $M$ large enough we have $P(\rho) \geq 1$ for all $\rho$ on the support of $\chi_{2,M}$, and thus we get, for such $M$, that

$$\int_{|\rho| > M} |\partial_{\rho} (\chi_{2,M} e^{a_{\nu} \varphi})|^2 + |e^{a_{\nu} \varphi}|^2 \, d\rho \leq 2 \int_{\mathbb{R}} |\chi_{2,M} e^{a_{\nu} \varphi} u|^2 \, d\rho. \quad (A.13)$$
The right-hand side is clearly bounded by $2 \int_{\mathbb{R}} |u|^2 e^{2a|\rho|^3} \, d\rho$ which is bounded by assumption. We let $\varepsilon \to 0$ and use monotone convergence to conclude
\[
\int_{\{|\rho| > M\}} |u|^2 e^{2a|\rho|^3} \, d\rho \leq 2 \int_{\mathbb{R}} |u|^2 e^{2a|\rho|^3} \, d\rho.
\] The estimate for $|\rho| < M$ is trivial. This proves the statement in our lemma for $w$ (with $b = a$). The estimate for $\rho'w$ is simple if we just decrease $a$ to $b$, so that $\rho'$ is dominated by the exponential $\exp\{2(b - a)|\rho|^3\}$ for large $\rho$. For the first derivative $\partial_\rho w$ we might use (A.13) and for higher derivatives we continue by induction, using (A.12). □

We will several times encounter a variant of the Montgomery operator. For $k > 0$, we denote by $\hat{\mathcal{M}}(\zeta)$ the operator
\[
(\hat{\mathcal{M}}(\zeta)u)(\rho) = -u''(\rho) + k\left(\zeta + \frac{\rho^2}{2}\right)^2 u(\rho), \quad -\infty < \rho < \infty
\] (A.14) A change of coordinates
\[
\tilde{\rho} = k^{1/6} 2^{-1/3} \rho
\] transforms $\hat{\mathcal{M}}(\zeta)$ into
\[
\hat{\mathcal{M}}(\zeta) = k^{1/3} 2^{-2/3} \mathcal{M}(k^{1/3} 2^{1/3} \tilde{\zeta})
\] and so for the lowest eigenvalue it holds that
\[
\lambda_{1,\hat{\mathcal{M}}}(\zeta) = k^{1/3} 2^{-2/3} \lambda_{1,\mathcal{M}}(k^{1/3} 2^{1/3} \tilde{\zeta}).
\] (A.15)
In the case when $k = \delta_0$ we write
\[
\tilde{\zeta} = (2\delta_0)^{-1/3} \zeta
\] and we get that $\lambda_{1,\hat{\mathcal{M}}}(\zeta)$ is minimal for $\zeta = \tilde{\zeta}$ and that
\[
\lambda_{1,\hat{\mathcal{M}}}(\tilde{\zeta}) = 2^{-2/3} \delta_0^{1/3} \lambda_{1,\mathcal{M}}(\tilde{\zeta}) = 2^{-2/3} \delta_0^{1/3} \tilde{\nu}_0 = \tilde{\gamma}_0.
\] (A.16)
For the case $k = \delta_0$ we also introduce the moments
\[
M_{l,k} = \int_{-\infty}^{\infty} \left(\zeta + \frac{\rho^2}{2}\right)^l \varphi_j \varphi_k \, d\rho.
\] Here $\varphi_0$ is the first normalized eigenfunction of $\hat{\mathcal{M}}(\tilde{\zeta})$ for $k = \delta_0$, and $\varphi_j, j \geq 1$ are constructed via the Grušin method in Section 5.

Lemma A.10. It holds that
\[
M_{0,0}^0 = 1, \quad M_{0,0}^1 = 0, \quad M_{0,0}^2 = \frac{\tilde{\nu}_0}{3(2\delta_0)^{2/3}}, \quad \text{and} \quad M_{0,0}^3 = \frac{1}{6\delta_0} - \frac{\tilde{\gamma}_0}{3(2\delta_0)^{2/3}}.
\]

Proof. The first formula is just the normalization of $\varphi_0$ and the second one follows by a perturbation argument, just as for the de Gennes model.

We use Lemma A.11 to calculate $M_{l,0}^0$ for $l \geq 2$. Indeed, with $p(\rho) = (\zeta + \frac{\rho^2}{2})^2$ and $\lambda = \tilde{\nu}_0$, the formula (A.11) becomes
\[
\int_{-\infty}^{\infty} \left[b'' + 4\left(\tilde{\nu}_0 - (\zeta + \frac{\rho^2}{2})^2\right)b' - 2\rho(\tilde{\zeta} + \frac{\rho^2}{2})b\right] \varphi_0^2 \, d\rho = 0.
\] (A.17)
The choice $b(\rho) = \rho^{2l-1}$ gives $M_{l,0}^0$ provided that the previous moment formulas are known. Especially the choices $b(\rho) = \rho$ and $b(\rho) = \rho^3$ give the announced formulas for $M_{0,0}^1$ and $M_{0,0}^3$. □
Acknowledgements. The authors thank Ayman Kachmar and Bernard Helffer for fruitful discussions. Both authors were supported by the Lundbeck Foundation. SF is also supported by the Danish Research Council and by the European Research Council under the European Community’s Seventh Framework Program (FP7/2007–2013)/ERC grant agreement 202859.

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