Finite Vacuum Energy from Covariant Canonical Quantization

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ABSTRACT: In this article we present the formulation of Covariant Canonical Quantization (CCQ), where one does not incorporate any classical dynamics in the expansion in terms of creation and annihilation operators. We show that by postponing the spacetime split one gets rid of the divergences leading to the Cosmological Constant Problem. It reduces to conventional canonical quantization when constraining to the mass-shell a priori and is otherwise fully consistent with it. The physical vacuum energies are still recovered, but only at a point that is not related to cosmology any more.

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1 Introduction

Nearly twenty years after the observational evidence of cosmic acceleration, the cause of this phenomenon, labeled as “dark energy”, remains an open issue, which challenges the foundations of theoretical physics: The cosmological constant problem - why there is the apparent large disagreement between the vacuum expectation value of the energy in the universe which comes from quantum field theory on one side, and the observed value of the dark energy density on the other side [1–6] -, is in the focus of modern physics. Here we show by means of the covariant quantization formulation, rather than the canonical quantization procedure, that such divergences are only due to a hasty spacetime split regarding the calculation of the energy density.

In some textbooks on canonical quantization it is sometimes claimed that removing redundant degrees of freedom in the operators, e. g. by gauging away the Maxwell field component \(A_0\), would not change the description, but it does! The classical field dynamics must intrinsically be derived from the quantum system, and not the vice versa, as it is attempted in many articles such as [7] with a “covariant canonical quantization” prescription based on a very different mindset. Thereby, important physical properties can be uncovered which are otherwise hidden. Most prominently, the zero-point or vacuum
energy does turn out not to be a fixed infinite number, but rather a contribution subject to intrinsic uncertainty. It cancels out throughout the history of the universe in a fully covariant inspection, which seems appropriate towards gravity, and becomes infinite when a spacetime split is performed. In the following, the procedure of Covariant Canonical Quantization (CCQ) is outlined along the lines of the standard canonical quantization. The full consistency is shown and the additional outcomes are presented. A discussion of the vacuum energy is highlighted at the end.

2 Canonical vs. Covariant phase space

In the simplest version of canonical quantization [8–11], a real scalar field \( \phi \) is promoted to an operator \( \hat{\phi} \) and Fourier expanded as

\[
\hat{\phi}(x) = \int \frac{1}{(2\pi)^2} \hat{\phi}(k) e^{-ik\mu x^\mu} d^4k.
\]  

(Natural units \( \hbar = c = 1 \) are employed and the metric convention is \((+,-,-,-)\).) Usually, it would now be inserted into the Klein-Gordon equation and, therefore, one would receive a split of the coefficients in terms of “creation” and “annihilation” operators. This procedure does, however, immediately yield the flaw that classical information is here incorporated into the quantum field, right from the beginning. This makes e. g. non-Abelian field theories difficult and so standard textbooks move on to the path- or, more accurately, functional integral formulation at this point.

Very much in this spirit, the field dynamics shall be left aside here as long as possible. Albeit both approaches are covariant and all possible particles are considered, only certain distinguished ones will lead to physical results. The difference to the functional integral formalism is that the mathematical complexity of integration over all fields is omitted. Instead, the information is fully contained in the commutators of the operators. They describe all particles, including the “virtual” ones, but in the end only the physical ones will contribute to the observables.

The functional integral approach to quantum field theory uses the Lagrangian formalism and is thereby manifestly covariant. However, one might also want to formulate quantum field theory in phase space. There is a recurring dispute about which phase space in field theory is the right analog to point mechanics. On the one hand, there is the canonical phase space, spanned by the field variable \( \phi \) and their time-conjugated momentum \( \pi^0 \). This is the usual phase space applied in canonical quantization because the number of fields and momenta is equal, thus inducing a symplectic structure. The price for this is that the instantaneous phase space, which shall preserve the information of the system, is infinite-dimensional [13].

The alternative to this procedure is the covariant phase space with four conjugated momenta \( \pi^\mu \) per field variable \( \phi \). Geometrically this leads to a multisymplectic structure and it is finite-dimensional [14]. The De Donder-Weyl Hamiltonian

\[
\mathcal{H}_{DW} := \pi^\mu \frac{\partial \phi}{\partial x^\mu} - \mathcal{L}
\]  

(2.2)
carries as much information about the system as the Lagrangian $L$ does, provided a regular Legendre transform exists. The corresponding equations of motion are determined by

$$\frac{\partial \phi}{\partial x^\mu} = \frac{\partial H_{DW}}{\partial \pi^\mu} \quad \text{and} \quad \frac{\partial \pi^\mu}{\partial x^\mu} = -\frac{\partial H_{DW}}{\partial \phi}.$$  \hspace{1cm} (2.3)

The dimensionality and the information content suggest that this is the correct phase space to work with. It will be seen below that the covariant Hamiltonian in the case considered here is just equal to the energy density.

3 Covariant Canonical Quantization (CCQ)

3.1 Quantization postulates

Because the concrete Lagrangian of the system is ignored at the beginning, $\phi$ shall simply be called a “real scalar field” rather than a “Klein-Gordon field”. It can easily be extended to complex and spinorial fields and such work is, in fact, in progress. However, here we outline the principles of the process only; this suffices for the present work.

The point of departure from “standard” quantization is to take the commutators as the fundamental postulates. In order to recover the usual results, one nonetheless has to impose the usual “mass-shell condition” for the commutator of the Fourier modes:

$$\left[\hat{\phi}(k), \hat{\phi}(k')\right] = (2\pi)^{\frac{5}{2}} \delta^{(4)}(k + k') \delta(k_\mu k^\mu - m^2) \text{sign}(k^0).$$  \hspace{1cm} (3.1)

The prefactor is a normalization convention. The term sign$(k^0)$ distinguishes the time direction, thus making the commutator well-defined, as it must be anti-symmetric:

$$\left[\hat{\phi}(k), \hat{\phi}(-k)\right] = -\left[\hat{\phi}(-k), \hat{\phi}(k)\right].$$

The presence of the 0-component of the 4-momentum in sign$(k^0)$ restricts the invariance of (3.1) under Lorentz transformations to orthochronous ones.

Fourier-reverse transformed, (3.1) is nothing else than the general covariant field commutator, compare eq. (A.7) in the appendix. It also hints towards the Klein-Gordon equation as off-shell components commute with each other and so can be seen as commutating “c-numbers”. When these are set to 0, it reduces to conventional canonical quantization. It is not meant to provide alternative physics, it is a technical subtlety and offers a more fundamental explanation, from the quantum level up, rather than the other way round. It is already known from quantum mechanics that for relativistic systems one has to embed the motion into an extended phase space to get a well-defined global formulation of the problem and then constrain to the mass-shell and this leads to the correct Klein-Gordon instead of the Schrödinger equation [16, 17].

In addition, another postulate is important: The covariant postulate (3.1) on the fields must be supplemented by a canonical postulate for the conjugated variables, thus naturally exhausting all degrees of freedom of the system: We demand the equal-time commutation relations for the field $\hat{\phi}$ and the momenta $\hat{\pi}^\mu$, which are actually at the heart of the quantization prescription (let boldface type letters denote only the spatial coordinates):

$$\left[\hat{\phi}(x, t), \hat{\pi}^\mu(y, t)\right] = i\delta^\mu_0 \delta^{(3)}(x - y).$$  \hspace{1cm} (3.2)
3.2 Covariant particle creation and annihilation

For a meaningful particle interpretation it is useful to split the Fourier expansion (2.1) as follows:

\[
\hat{\phi}(x) = \int \frac{1}{(2\pi)^2} \hat{\phi}(k)e^{-ik_{\mu}x^{\mu}}d^4k
\]

\[
= \int \frac{1}{(2\pi)^2} \left( \frac{1}{2} \hat{\phi}(k)e^{-ik_{\mu}x^{\mu}} + \frac{1}{2} \hat{\phi}(-k)e^{ik_{\mu}x^{\mu}} \right) d^4k
\]

\[
= \int \frac{1}{(2\pi)^2} \left( \hat{a}(k)e^{-ik_{\mu}x^{\mu}} + \hat{a}^\dagger(k)e^{ik_{\mu}x^{\mu}} \right) d^4k,
\]

where we defined the \textbf{covariant creation operator}

\[
\hat{a}^\dagger(k) := \frac{1}{2} \hat{\phi}(-k),
\]

and the \textbf{covariant annihilation operator}

\[
\hat{a}(k) := \frac{1}{2} \hat{\phi}(k).
\]

It can readily be seen that they are not independent of each other. Their commutator

\[
[\hat{a}(k), \hat{a}^\dagger(k')] = \left( \frac{2\pi}{2} \right)^{\frac{5}{2}} \frac{\delta^{(4)}(k - k')}{\delta(k_{\mu}k^{\mu} - m^2)} \text{ sign}(k^0)
\]

has the expected sign in the energy-momentum-\( \delta \), justifying the shape of the commutator (3.1).

Furthermore, define the \textbf{covariant number operator}

\[
\hat{N}(k) := \frac{4}{(2\pi)^2} \hat{a}^\dagger(k)\hat{a}(k).
\]

With \([\hat{N}(k), \hat{N}(k')] = 0\) (see appendix A.1), this operator generates a Fock space as the product of eigenstates \(|n_k\rangle\) of \(\hat{N}(k)\). We set

\[
\hat{N}(k)|n_k\rangle = \langle 0 |(\delta^{(4)}(0) - \delta^{(4)}(2k))\delta(k_{\mu}k^{\mu} - m^2) \text{ sign}(k^0)n_k\rangle|n_k\rangle.
\]

Of course, it now needs to be shown that the operators \(\hat{a}^\dagger\) and \(\hat{a}\) do really act as “creation” and “annihilation” operators. It is already clear from the definition that they are a mix of the respective on-shell operators and so \(\hat{N}(k)\) has contributions of two “creation” resp. two “annihilation” operators depending on the 4-momentum, but this does not yet mean much for its spectrum. With the identities (A.1) and (A.2) it is straightforward to calculate

\[
\hat{N}(k)\hat{a}^\dagger(k)|n_k\rangle = \hat{a}^\dagger(k)\hat{N}(k)|n_k\rangle + [\hat{N}(k), \hat{a}^\dagger(k)]|n_k\rangle
\]

\[
= (\delta^{(4)}(0) - \delta^{(4)}(2k))\delta(k_{\mu}k^{\mu} - m^2) \text{ sign}(k^0)(n_k + 1)\hat{a}^\dagger(k)|n_k\rangle
\]

and

\[
\hat{N}(k)\hat{a}(k)|n_k\rangle = \hat{a}(k)\hat{N}(k)|n_k\rangle + [\hat{N}(k), \hat{a}(k)]|n_k\rangle
\]

\[
= (\delta^{(4)}(0) - \delta^{(4)}(2k))\delta(k_{\mu}k^{\mu} - m^2) \text{ sign}(k^0)(n_k - 1)\hat{a}(k)|n_k\rangle.
\]
In other words, \( n_k \) allows in fact for an interpretation as “particle number” that counts particles with 4-momentum \( k \) on mass-shell. Creation operators effectively become annihilation operators and vice versa for negative energy states. The Dirac \( \delta \)'s in front are only a formality; in discretized spacetime they are 1 at 0 and 0 otherwise. But for massless particles at rest \((k = 0)\) they tell us that nothing happens: the particle number loses its meaning, which is intuitively clear as there is nothing but a non-existing particle.  

### 3.3 Relation to conventional objects

A special feature of these operators is their complete symmetry. Consider an annihilation operator applied to a (yet unspecified) vacuum state \( |0\rangle \). The result is, by construction, a 1-particle state with the respective 4-momentum. There are two equal representations:

\[
|1 \cdot k\rangle = \hat{a}^\dagger(k)|0\rangle = \hat{a}(-k)|0\rangle = | -1 \cdot (-k)\rangle.
\]

In total, these operators resemble the Feynman-Stückelberg interpretation [18, 19].

So the (physical) vacuum \( |0\rangle \) has to be defined as the state where the application of the annihilation operator with positive energy leads to a 0, which is equivalent to the action of the creation operator with negative energy:

\[
\Theta(k^0)\hat{a}(k)|0\rangle := 0 \iff \Theta(-k^0)\hat{a}^\dagger(k)|0\rangle := 0 \quad \forall k.
\]

Thereby, the standard vacuum will be recovered. It can also be written in the more compact form

\[
\int c(k) \left( \Theta(k^0)a(k) + \Theta(-k^0)a^\dagger(k) \right) |0\rangle d^4 k = 0,
\]

where \( c(k) \) has to be introduced as an arbitrary function to ensure independence of the single conditions.

The same reasoning can be applied directly to the particle operators. The results from (3.9) and (3.10) show that real quantities should be defined as constrained on the upper mass-shell. Define \( E_k := \sqrt{k \cdot k + m^2} \). The real creation operator thus is

\[
\int \hat{a}^\dagger(k)\delta(k_\mu k^\mu - m^2)\Theta(k^0) dk^0 = \int \frac{\hat{a}^\dagger(k)}{2E_k} \delta(k^0 - E_k) dk^0 = \frac{\hat{a}^\dagger(k, E_k)}{2E_k}.
\]

and the real annihilation operator is

\[
\int \hat{a}(k)\delta(k_\mu k^\mu - m^2)\Theta(k^0) dk^0 = \int \frac{\hat{a}(k)}{2E_k} \delta(k^0 - E_k) dk^0 = \frac{\hat{a}(k, E_k)}{2E_k}.
\]

These are the operators usually encountered in canonical quantization. It can be checked that they yield the expected action on the above defined vacuum (3.13):

\[
a(k, E_k)|0\rangle = 2E_k \int \delta(k_\mu k^\mu - m^2)\Theta(k^0)a(k)|0\rangle dk^0 = 0,
\]

\[\text{From the mathematical point of view the “densitized” operators should be built as the spectrum of the total number operator}
\]

\[
\hat{N}_{\text{tot}} := \int \hat{N}(k) d^4 k
\]

as the preceding \( \delta^{(4)}(0) \) cancels out there. Nonetheless, the above number operators shall be used in the following. The same phenomenon already occurs in conventional canonical quantization.
resp.

\[ a^\dagger(k, E_k)|0\rangle = 2E_k \int \delta(k\mu k^\mu - m^2)\Theta(k^0) a^\dagger(k)|0\rangle dk^0 \]
\[ = 2E_k \int \frac{\delta(k^0 - E_k)}{2E_k} |1 \cdot k\rangle dk^0 \]
\[ = |1 \cdot k, E_k\rangle. \]

(3.17)

So far time did essentially not play a preferred role. Actually, this is first needed when making reason of scattering processes. And when introducing a time ordering, all the well-known scattering results like the LSZ reduction formula can be deduced, which are derived from the conventional creation and annihilation operators. An explicit calculation of the propagator is provided in appendix A.2.

3.4 Building the Lagrangian

The commutation relations for the momenta were defined in eq. (3.2), but their representation in terms of the field variable \( \phi \) was still left open. This is because no Lagrangian or Hamiltonian has been assumed so far. Recall from eq. (A.7) that the commutator between the fields at different spacetime points is given by \((i \times)\) the Pauli-Jordan function \( \Delta(x - y) \). From eq. (A.9) it can be seen that only the time-derivative leads to a contribution when setting \( x^0 \to y^0 \), while the spatial derivatives all vanish. For the commutator relation this means:

\[-i\delta_0^{(3)}(x - y) = -i \lim_{x^0 \to y^0} \frac{\partial}{\partial x_\mu} \Delta(x - y) \]
\[ = \lim_{x^0 \to y^0} \frac{\partial}{\partial x_\mu} [\hat{\phi}(x), \hat{\phi}(y)] \]
\[ = \lim_{x^0 \to y^0} \left[ \frac{\partial}{\partial x_\mu} \hat{\phi}(x), \frac{\partial}{\partial y_\mu} \hat{\phi}(y) \right]. \]

(3.18)

This equals the value of the equal-time commutator (3.2) between the field and its conjugate momentum. Hence we can set \(^2\)

\[ \pi^\mu(x) = \frac{\partial \phi(x)}{\partial x_\mu}. \]

(3.19)

Taking the procedure one step further, again by comparison with eq. (A.10),

\[ 0 = i \lim_{x^0 \to y^0} \frac{\partial}{\partial x_\mu} \frac{\partial}{\partial y_\nu} \Delta(x - y) \]
\[ = \lim_{x^0 \to y^0} \left[ \frac{\partial}{\partial x_\mu} \hat{\phi}(x), \frac{\partial}{\partial y_\nu} \hat{\phi}(y) \right] \]
\[ = \lim_{x^0 \to y^0} \left[ \frac{\partial}{\partial x_\mu} \hat{\phi}(x), \frac{\partial}{\partial y_\nu} \hat{\phi}(y) \right]. \]

(3.20)

\(^2\)This is surely not the only possibility, but it is the simplest choice. One can always add a term linear in the \( \phi \)-field. In gauge theories this freedom is actually necessary because it then becomes a covariant derivative

\[ \tilde{\pi}^\mu(x) = \frac{\partial \phi(x)}{\partial x_\mu} + A^\mu(x) \phi(x). \]

Further derivative-dependent terms are not allowed.
The thus defined momentum fields commute, as it should be for the canonical momenta.

With this at hand the field dynamics can be derived. Here it does not matter whether
the Lagrangian or (De Donder-Weyl) Hamiltonian picture is taken as both are covariant.
However, for working in phase space, the latter is more convenient. From the canonical
equations (2.3),
\[ \frac{\partial \phi}{\partial x^\mu} = \frac{\partial H_{\text{scalar}}}{\partial \pi^\mu}, \]
and the definition of the momenta in eq. (3.19), straightforward integration yields
\[ H_{\text{scalar}} = \frac{1}{2} \pi^\mu \pi^\mu + f(\phi), \]
where \( f(\phi) \) is an integration function. By reverse Legendre transformation, the Lagrangian
can be found as
\[ L_{\text{scalar}} = \frac{\partial \phi}{\partial x^\mu} \pi^\mu - H_{\text{scalar}} = \frac{1}{2} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x_\mu} - f(\phi). \]

The residual freedom in \( f(\phi) \) can be closed by requiring that the resulting equations of
motion are “solved” by the propagators. Observe that the Pauli-Jordan function (A.7) is a
solution of the Klein-Gordon equation:
\[ (\partial_\nu \partial_\nu + m^2) \Delta(x) = \int \frac{1}{(2\pi)^{\frac{3}{2}}} (-k_\nu k^\nu + m^2) \delta(k_\mu k^\mu - m^2) \text{sign}(k^0) e^{-ik_\mu x^\mu} d^4k = 0. \]
Correspondingly, the Feynman propagator (A.5) solves the same equation, but with a \( \delta \)-
function as a source term:
\[ (\partial_\nu \partial_\nu + m^2) \Delta_F(x) = \int \frac{1}{(2\pi)^2} (-k_\nu k^\nu + m^2) \frac{1}{k_\mu k^\mu - m^2} e^{-ik_\mu x^\mu} d^4k = -\delta^{(4)}(x). \]
This suggests \( f(\phi) = \frac{1}{2} m^2 \phi^2 \). In this way, the action can be completely built up from
quantum principles instead of quantizing some classical Lagrangian / Hamiltonian. This
is reasonable as the classical world cannot contain as much or more information than the
quantum world; the classical picture is just a limiting case.

With other requirements there would also be other possibilities, as e. g. a curvature-
dependent contribution proposed in [20]. For calculating contributions to the \( S \)-matrix
there will be the usual need for adding renormalization factors. As for the \( S \)-matrix a time-
ordering is introduced explicitly, we shall stop at this point; then it is only conventional
quantization. Although this argument may seem trivial here, gauge fields will make the
treatment really non-trivial. There is e. g. the need to introduce gauge-fixing and ghost
terms and these are quite definitely determined by the commutation postulates.

4 Vacuum energy

According to the Einstein field equations
\[ G_{\mu\nu} = 8\pi G \theta_{\mu\nu} \]
the (expectation value of the) absolute energy-momentum tensor $\theta_{\mu\nu}$ contributes to the spacetime dynamics, where $G$ is the Newtonian Constant and $G_{\mu\nu}$ is the Einstein tensor. So any zero-point energy should contribute as well. According to conventional theory, a scalar field can be seen as a collection of harmonic oscillators and would contribute a vacuum energy

$$E_{0,\text{conv}} \sim \int \frac{1}{2} E_k d^4k.$$  

As this integral diverges, one has to take a cutoff at some $k_{\text{max}}$. It is further divided by the phase space volume to give a vacuum energy density of

$$\epsilon_{0,\text{conv}} := \frac{E_{0,\text{conv}}}{V} \sim k_{\text{max}}^4.$$  

With a reasonable value of $k_{\text{max}}$, however, this contribution would be those famous $10^{120}$ orders too big to match observations [1, 2]. In the following it will be shown why such a calculation is both misleading and to some extent arbitrary.

Consider the canonical energy-momentum tensor corresponding to the Klein-Gordon Lagrangian $L_{\text{KG}}$:

$$\theta_{\mu\nu} := \frac{\partial L_{\text{KG}}}{\partial (\partial_\mu \phi)} \frac{\partial \phi}{\partial x^\nu} - \eta_{\mu\nu} L_{\text{KG}} = \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x^\nu} - \eta_{\mu\nu} \frac{1}{2} \left( \frac{\partial \phi}{\partial x^\xi} \frac{\partial \phi}{\partial x^\eta} - m^2 \phi^2 \right). \quad (4.1)$$

As one cannot find a direct expression for its elements in terms of a number operator, one is forced to integrate out the $x$-dependence and look at its representation in momentum space. The instantaneous energy of the field is defined as the space-integral of the 00-component of this tensor:

$$E(x^0) := \int \theta_{00}(x) d^3x. \quad (4.2)$$

In this covariant approach, the total energy, i.e. its integral also over time, is the fundamental object. There seems to be no reason why time should play a preferred role already at this stage, particularly as the instantaneous energy is not the ultimate quantity in question. The corresponding quantum expression can be expanded in terms of the covariant creation and annihilation operators:

$$\hat{E}_{\text{tot}} := \int \hat{E}(x^0) dx^0$$  

$$= \int \frac{1}{2} \left( \frac{\partial \hat{\phi}(x)}{\partial x^0} \frac{\partial \hat{\phi}(x)}{\partial x^0} - \frac{\partial \hat{\phi}(x)}{\partial x^i} \frac{\partial \hat{\phi}(x)}{\partial x^i} + m^2 \hat{\phi}(x)^2 \right) d^4x$$  

$$= \frac{1}{2(2\pi)^4} \int \int (-k_0 k'_0 + k^i k'_i + m^2) \hat{\phi}(k) \hat{\phi}(k') e^{-i(k_0 k'_0 + k_i k'_i)} d^4k d^4k'$$  

$$= \frac{1}{2(2\pi)^2} \int (k_0 k_0 - k^i k'_i + m^2) \hat{\phi}(k) \hat{\phi}(-k) d^4k$$  

$$= \frac{2}{(2\pi)^2} \int (k_0^2 + E_k^2) \hat{a}^\dagger(k) \hat{a}(k) d^4k$$  

$$= \frac{\sqrt{\pi}}{2} \int (k_0^2 + E_k^2) \hat{N}(k) d^4k.$$  

- 8 -
The quadratic dependence on the energy is a bit fallacious as the mass-shell, implicitly contained in $\hat{N}$ (compare eq. (3.8)), actually reduces its order by 1. To see this explicitly take the trace. By means of the identity (A.6) and the interpretation (3.11) this gives

\[
E_{\text{tot}} := \int \langle n_k | \hat{E}_{\text{tot}} | n_k \rangle d^4k = \frac{\sqrt{\pi}}{2} \int (k_0^2 + E_k^2)(\delta^{(4)}(0) - \delta^{(4)}(2k))\delta(k_\mu k^\mu - m^2) \text{sign}(k_0)n(k)d^4k \\
= \frac{\sqrt{\pi}}{2} \delta^{(4)}(0) \int E_k (n(k, E_k) - n(-k, -E_k)) d^3k \\
= \sqrt{\pi} \delta^{(4)}(0) \int E_k n(k)d^3k.
\]

(4.4)

In this scheme there is no remaining zero-point contribution! When the total energy is finite, the energy density cannot be a fixed infinite term which is integrated over. Of course, one can wonder why to take this representation, but for the purpose of gravity a covariant procedure should be appropriate. It is not stated that there would be no vacuum energy, the density is actually undetermined and cannot logically be deduced by dividing by a simple phase space volume.

In fact, if one had not used the identity $\hat{a}^\dagger(-k) = \hat{a}(k)$ and segregated a commutator $[\hat{a}(k), \hat{a}^\dagger(k)]$, its contribution would have canceled anyway, after (anti-symmetric) integration over $k_0$. This eliminates the (artificial) need for a “normal ordering”. If on the other hand a “Weyl ordering” $\mathbf{m}(\cdot)$ is employed explicitly, i.e. the particle operator contributions are shared equally between terms of positive and negative 4-momentum, such a representation for its density can be recovered:

\[
\mathbf{m} \left( \hat{E}_{\text{tot}}(k) \right) = \mathbf{m} \left( \frac{4}{(2\pi)^2} (k_0^2 + E_k^2) \hat{a}^\dagger(k)\hat{a}(k) \right) \\
= \frac{2}{(2\pi)^2} (k_0^2 + E_k^2)(\hat{a}^\dagger(k)\hat{a}(k) + \hat{a}(k)\hat{a}^\dagger(k)) \\
= \frac{2}{(2\pi)^2} (k_0^2 + E_k^2)(2\hat{a}^\dagger(k)\hat{a}(k) + [\hat{a}(k), \hat{a}^\dagger(k)]) \\
= \sqrt{\pi} \delta^{(4)}(0) \left( \hat{N}(k) + \frac{1}{2} \delta^{(4)}(0)\delta(k_\mu k^\mu - m^2) \text{sign}(k_0) \right).
\]

(4.5)

But it is very important to note that the expression in Fourier space, i.e. the term under the integral, is not the energy density! In contrast to the canonical approach, here one has to impose some ordering for receiving the common zero-point energy, rather than eliminating it. If one restricted the integral to the mass-shell $\delta(k_\mu k^\mu - m^2)\Theta(k_0)$ right from the beginning, like in conventional canonical quantization, this would lead to the familiar expression

\[
\int_{\text{restr}} \mathbf{m} \left( E_{\text{tot}}(k) \right) d^4k = \sqrt{\pi} \delta^{(4)}(0) \int E_k \left( n(k) + \frac{1}{2} \right) d^3k.
\]

(4.6)

Thereby, it can be seen that the infinity originates from the immediate incorporation of the classical field equation, but it does not have to hold strictly on the quantum level. With
other orderings one could create virtually any zero-point contribution. The meaning of this
is that it is simply fallacious to deduce a definite value of vacuum energy density, especially
an infinite one. It exists, as seen e. g. in the Casimir effect [21], but there is a principle
uncertainty, in line with Heisenberg’s uncertainty principle in quantum mechanics.
As the instantaneous values are uncertain, the densities are even more ambiguous. Such
an arbitrariness is also seen in renormalization, where various approaches exist, and the
resulting values are always valid only with respect to the employed renormalization scheme.
It is also in line with certain other modifications where the zero-point energy is damped
with the frequency [22] so that the usual value of the vacuum energy is physical at low
frequencies as a “semi-classical” effect, whereas at high frequencies the description becomes
fully quantum and the zero-point energy fades out.

5 Summary and outlook
This paper outlined the method of Covariant Canonical Quantization to question the infinite
vacuum energy argument, using the simple example of a real scalar field. The key features
of this approach are: not to restrict to the mass-shell in the expansion of creation and
annihilation operators and not to split space and time before looking at scattering processes.
In its mindset this procedure is similar to the functional integral formulation, but stays on
the operator level. Standard canonical quantization can be recovered easily, while the
pathologies around the zero-point energy do not emerge. The claim is not that there would
be no vacuum energy - as it is physically relevant e. g. in the Casimir effect [21] -, but
only no divergence. Its appearance in the second quantized regime is subject to intrinsic
uncertainty and the infinite value is recovered when ignoring this fact, i. e. employing a
specific choice of representation and the restriction to the mass-shell a priori.

This result directly impacts the discussion of the cosmological constant problem, which
was so far determined to be at a value $10^{120}$ too high as compared to such naive summations
[1–6]. Here, no contradiction to the observed value of the cosmological constant is produced.
It could be a leftover of this vacuum energy fluctuation - much like the prevalence of matter
over anti-matter has also arisen from a small breaking of symmetry after all particle species
have pair-annihilated and only a fraction of about $10^{-10}$ has remained [3] - or it is simply
an external field that has nothing to do with the zero-point energy.

Fields with internal structure like Dirac or Maxwell fields were not covered here. How-
ever, similar features will emerge in a straightforward manner. There are more conserved
quantities to be considered, such as the charge operator. This usually has the problem of
infinite vacuum values, too. But with the same argumentation as above, this divergence
emerges only out of an enforced specific representation, but is not a fundamental problem
of the theory.

A big advantage of the method presented here, as compared to conventional canonical
quantization, is its manifest covariance. Therefore, it may even facilitate a smart road to a
quantum theory of gravity. Since the pathologies around the energy-momentum tensor can
be explained this way, there is the hope that some of the curvature-related pathologies may
also be clarified in a covariant operator approach. At the end of their book [12], Nakanishi
and Ojima suspect that a rigorous theory based on covariant operators might even remove all divergences in quantum field theory. Work along these lines is in progress.

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A Appendix

A.1 Number operator algebra

For the calculation with the covariant number operators some identities are needed. The commutator of the number operator with the creation operator is given by

\[
[\hat{N}(k), \hat{a}(k')] = \frac{4}{(2\pi)^2} (\hat{a}(k)[\hat{a}(k), \hat{a}(k')] + [\hat{a}(k), \hat{a}(k')][\hat{a}(k)]
\]

\[
= \hat{a}(k)\delta(\frac{1}{2})\delta(k-k')\delta(k_\mu k_\mu - m^2)\text{sign}(k^0)
+ \delta(k+k')\delta(k_\mu k_\mu - m^2)\text{sign}(-k^0)\hat{a}(k')
= (\delta(k-k') - \delta(k+k'))\delta(k_\mu k_\mu - m^2)\text{sign}(k^0)\hat{a}(k')
\]

and the one with the annihilation operator by

\[
[\hat{N}(k), \hat{a}(k')] = \frac{4}{(2\pi)^2} (\hat{a}(k)[\hat{a}(k), \hat{a}(k')] + [\hat{a}(k), \hat{a}(k')][\hat{a}(k)]
\]

\[
= \hat{a}(-k)\delta(\frac{1}{2})(k-k')\delta(k_\mu k_\mu - m^2)\text{sign}(k^0)
+ \delta(k-k')\delta(k_\mu k_\mu - m^2)\text{sign}(-k^0)\hat{a}(k)
= -(\delta(k-k') - \delta(k+k'))\delta(k_\mu k_\mu - m^2)\text{sign}(k^0)\hat{a}(k').
\]

The self-commutator of the number operator reads

\[
[\hat{N}(k), \hat{N}(k')] = \frac{4}{(2\pi)^2} (\hat{a}(k')[\hat{N}(k), \hat{a}(k')] + [\hat{N}(k), \hat{a}(k')]\hat{a}(k'))
\]

\[
= \frac{4}{(2\pi)^2} \left( -(\delta(k-k') - \delta(k+k'))\delta(k_\mu k_\mu - m^2)\text{sign}(k^0)\hat{a}(k')\hat{a}(k')
+ (\delta(k-k') - \delta(k+k'))\delta(k_\mu k_\mu - m^2)\text{sign}(k^0)\hat{a}(k')\hat{a}(k') \right)
= 0.
\]
A.2 Feynman propagator

For calculating the propagators in this covariant treatment, one has to respect the condition (3.12). And this is actually the only the difference to the usual calculation. Then the 2-point function as the expectation value with respect to the (physical) vacuum state can be calculated as

\[
\langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle = \frac{1}{(2\pi)^4} \int \int \langle 0|\hat{\phi}(k)\hat{\phi}(k')|0\rangle e^{-ik_\mu x^\mu - ik'_\mu y^\mu} d^4k d^4k'
\]

\[
= \frac{4}{(2\pi)^4} \int \int \langle 0|\hat{\phi}(k)\hat{\phi}(k')|0\rangle e^{-ik_\mu x^\mu + ik'_\mu y^\mu} d^4k d^4k'
\]

\[
= \frac{4}{(2\pi)^4} \int \int \Theta(k^0)\Theta(k^0') \delta(k - k') |0\rangle e^{-ik_\mu x^\mu + ik'_\mu y^\mu} d^4k d^4k'
\]

\[
= \int \Theta(k^0)\Theta(k^0') \frac{1}{(2\pi)^2 2E_k} e^{-iE_k(x^0-y^0)} e^{-ik_\mu (x^\mu - y^\mu)} d^4k.
\]

The result is the same as in standard textbooks, compare e. g. [9, section 4.5].

The actual quantity that appears in the LSZ reduction formula is the vacuum expectation value of the time-ordered product, denoted \( T(\cdot) \). This and an application of the residue theorem lead to the canonical result for the Feynman propagator

\[
\Delta_F(x-y) := -i \langle 0|T(\hat{\phi}(x)\hat{\phi}(y))|0\rangle
\]

\[
:= -i \left( \Theta(x^0 - y^0) \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle + \Theta(y^0 - x^0) \langle 0|\hat{\phi}(y)\hat{\phi}(x)|0\rangle \right)
\]

\[
= \frac{1}{(2\pi)^2} \int \frac{1}{k_\mu k^\nu - m^2 + i\epsilon} e^{-ik_\mu (x^\mu - y^\mu)} d^4k.
\]

The additional \( i\epsilon \) is added to the denominator to indicate the integration contour. The poles are then at \( k^0 = \pm(E_k - i\epsilon) \), so the integration is performed above the real axis for positive energy and below the real axis for negative energy.

A.3 Pauli-Jordan function

Another quantity, which essentially represents the commutator between the field at completely different points, is called the Pauli-Jordan function. Its properties shall be outlined here as they are needed in the above argumentation. Using the identities

\[
\delta^{(4)}(k_\mu k^\mu - m^2) = \frac{1}{2E_k} \left( \delta(k^0 + E_k) + \delta(k^0 - E_k) \right), \quad \int e^{-iyx} dx = \sqrt{2\pi} \delta(y)
\]
for $E_k := \sqrt{k \cdot k + m^2}$, and some trigonometric formulas, it can be written in several equivalent ways:

$$
\Delta(x - y) := \frac{1}{i} [\hat{\phi}(x), \hat{\phi}(y)]
= \int \int \frac{1}{(2\pi)^4} e^{-ik_y (x^\mu - y^\mu)} d^4k d^4k'
= \int \frac{1}{(2\pi)^2} \frac{1}{i} \delta(k_y k - m^2) \text{sign}(k_0) e^{-ik_y (x^\mu - y^\mu)} d^4k
= \int \frac{1}{(2\pi)^2} \frac{1}{2i E_k} \left( \delta(k_0^0 + E_k) + \delta(k_0^0 - E_k) \right) \text{sign}(k_0) e^{-ik_y (x^\mu - y^\mu)} d^4k
= \int \frac{1}{(2\pi)^2} \frac{1}{2i E_k} e^{-ik_y (x^\mu - y^\mu)} \left( -e^{iE_k (x^0 - y^0)} + e^{-iE_k (x^0 - y^0)} \right) d^3k
= -\int \frac{1}{(2\pi)^2} \frac{1}{2i E_k} e^{-ik_y (x^\mu - y^\mu)} \sin(E_k (x^0 - y^0)) d^3k.
$$

(A.7)

Instead of the energy, one could also single out a spatial momentum, but then receive a cosine dependence and an additional sign($k^0$). It can thus be seen that it is an antisymmetric function in time:

$$
\Delta(x^0, x) = -\Delta(-x^0, x).
$$

This means that at $x = 0$ and already for $x^0 = 0$ the Pauli-Jordan function vanishes. In total, the equal-time commutator

$$
[\hat{\phi}(x, t), \hat{\phi}(y, t)] = 0
$$

is recovered.

Of special importance here are its derivatives. The first derivatives fulfill

$$
\frac{\partial}{\partial x^0} \Delta(x - y) = -\int \frac{1}{(2\pi)^2} e^{-ik_y (x^\mu - y^\mu)} \cos(E_k (x^0 - y^0)) d^3k \rightarrow -\delta(x - y),
$$

$$
\frac{\partial}{\partial x^i} \Delta(x - y) = -i \int \frac{k_i}{(2\pi)^2} E_k e^{-ik_y (x^\mu - y^\mu)} \sin(E_k (x^0 - y^0)) d^3k \rightarrow 0,
$$

(A.9)

and the second derivatives fulfill

$$
\frac{\partial}{\partial x^0} \frac{\partial}{\partial y^0} \Delta(x - y) = -\int \frac{1}{(2\pi)^2} E_k e^{-ik_y (x^\mu - y^\mu)} \sin(E_k (x^0 - y^0)) d^3k \rightarrow 0,
$$

$$
\frac{\partial}{\partial x^i} \frac{\partial}{\partial y^0} \Delta(x - y) = i \int \frac{1}{(2\pi)^2} E_k e^{-ik_y (x^\mu - y^\mu)} \cos(E_k (x^0 - y^0)) d^3k \rightarrow 0,
$$

$$
\frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \Delta(x - y) = -\int \frac{1}{(2\pi)^2} E_k k_i k_j e^{-ik_y (x^\mu - y^\mu)} \sin(E_k (x^0 - y^0)) d^3k \rightarrow 0.
$$

(A.10)

References

[1] S. Weinberg (1989): *The Cosmological Constant Problem*, Rev. Mod. Phys. 61, 1, doi:10.1103/RevModPhys.61.1
[2] S. M. Carroll (2001): *The Cosmological Constant*, Living Rev. Relativ. 4:1, Springer International Publishing
[3] A. Riotto (1998): *Theories of Baryogenesis*, High Energy Physics and Cosmology 326
[4] D. Benisty, E. I. Guendelman and O. Lahav (2019), arXiv:1904.03153 [astro-ph.GA]
[5] L. Lombriser (2019), arXiv:1901.08588 [gr-qc]
[6] C. Pagani and M. Reuter (2019): *Background Independent Quantum Field Theory and Gravitating Vacuum Fluctuations*, arXiv:1906.02507
[7] G. M. von Hippel and M. N. R. Wohlfarth (2006): *Covariant canonical quantization*, Eur. Phys. J. C 47, 861-872
[8] W. A. Fock (1932): *Konfigurationsraum und zweite Quantelung*, Zeitschrift für Physik 75, 622–647
[9] W. Greiner and J. Reinhardt (1996): *Field Quantization*, Springer Berlin Heidelberg, ISBN 3-540-59179-6
[10] M. Srednicki (2012): *Quantum Field Theory*, 7th printing, Cambridge University Press, ISBN 978-0-521-86449-7
[11] M. E. Peskin and D. V. Schroeder (2018): *An Introduction to Quantum Field Theory*, Chapman and Hall/CRC, ISBN 978-0-429-96272-1
[12] N. Nakanishi and I. Ojima (1990): *Covariant Operator Formalism of Gauge Theories and Quantum Gravity*, World Scientific Lecture Notes in Physics Vol. 27, Singapore, ISBN 9971-50-238-0
[13] M. J. Gotay, J. Isenberg and J. E. Marsden (2004): *Momentum Maps and Classical Fields Part II: Canonical Analysis of Field Theories*, arXiv:0411032 [math-ph]
[14] M. J. Gotay, J. Isenberg, J. E. Marsden and R. Montgomery (2003): *Momentum Maps and Classical Fields Part I: Covariant Field Theory*, arXiv:physics/9801019 [math-ph]
[15] J. Struckmeier and A. Redelbach (2008): *Covariant Hamiltonian Field Theory*, Int. J. Mod. Phys. E17, 435-491
[16] J. Struckmeier (2009): *Extended Hamilton-Lagrange formalism and its application to Feynman’s path integral for relativistic quantum physics*, Int. J. of Mod. Phys. E18, 79-108
[17] L. Mangiarotti and G. Sardanashvily (1998): *Gauge Mechanics*, World Scientific, Singapore
[18] E. C. G. Stückelberg (1941): *La signification du temps propre en mécanique ondulatoire*, Helv. Phys. Acta 14, 322-323
[19] R. P. Feynman (1948): *Space-time approach to non-relativistic quantum mechanics*, Reviews of Modern Physics 20 (2), 367–387
[20] J. Struckmeier, D. Vasak, A. Redelbach, P. Liebrich and H. Stoecker (2018): *Pauli-type coupling between spinors and curved spacetime*, arXiv:1812.09669
[21] H. B. G. Casimir (1948): *On the attraction between two perfectly conducting plates*, Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen B51, 793–795
[22] B. Lehnert (2014): *Some Consequences of Zero Point Energy*, Journal of Electromagnetic Analysis and Applications 6, 319-327