An inverse problem for a wave equation with sources and observations on disjoint sets

Matti Lassas and Lauri Oksanen

Department of Mathematics and Statistics, University of Helsinki, PO Box 68 FI-00014, Finland
E-mail: matti.lassas@helsinki.fi and lauri.oksanen@helsinki.fi

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Abstract

We consider an inverse problem for a hyperbolic partial differential equation on a compact Riemannian manifold. Assuming that $\Gamma_1$ and $\Gamma_2$ are two disjoint open subsets of the boundary of the manifold we define the restricted Dirichlet-to-Neumann operator $\Lambda_{\Gamma_1, \Gamma_2}$. This operator corresponds the boundary measurements when we have smooth sources supported on $\Gamma_1$ and the fields produced by these sources are observed on $\Gamma_2$. We show that when $\Gamma_1$ and $\Gamma_2$ are disjoint but their closures intersect at least at one point, the restricted Dirichlet-to-Neumann operator $\Lambda_{\Gamma_1, \Gamma_2}$ determines the Riemannian manifold and the metric on it up to an isometry. In the Euclidian space, the result yields that an anisotropic wave speed inside a compact body is determined, up to natural coordinate transformations, by measurements on the boundary of the body even when wave sources are kept away from receivers. Moreover, we show that if we have three arbitrary nonempty open subsets $\Gamma_1, \Gamma_2$ and $\Gamma_3$ of the boundary, then the restricted Dirichlet-to-Neumann operators $\Lambda_{\Gamma_j, \Gamma_k}$ for $1 \leq j < k \leq 3$ determine the Riemannian manifold up to an isometry. A similar result is proven also for the finite-time boundary measurements when the hyperbolic equation satisfies an exact controllability condition.

1. Introduction and main results

Let $M$ be a compact and connected $C^\infty$-smooth manifold of dimension $n$ and let $g$ be a $C^\infty$-smooth Riemannian metric on $M$. Let $q$ be a real-valued $C^\infty$-smooth function on $M$, and denote by $\Delta_g$ the Laplace–Beltrami operator on $M$. We consider a hyperbolic inverse problem corresponding to the second-order elliptic operator

$$a(x, D) := -\Delta_g + q(x).$$
In local coordinates $g$ is a positive-definite $C^\infty$-smooth matrix $(g^{jk}(x))_{j,k=1}^n$ with the inverse $(g^{jk}(x))_{j,k=1}^n$ and
\[
a(x, D)u = -|g|^{-1/2} \sum_{j,k=1}^n \frac{\partial}{\partial x^j} \left( g^{jk}|g|^{1/2} \frac{\partial}{\partial x^k} u \right) + qu, \tag{1}
\]
where $|g| := \det(g_{jk})$. Hence our results cover the setting, where $M \subset \mathbb{R}^n$ is an open domain with a smooth boundary and $a(x, D)$ is an elliptic operator of the form (1).

Let $H^s(M)$ be the Sobolev space of $s \in \mathbb{N}$ times weakly differentiable functions on $M$, and let $H^s_0(M)$ be the $H^1(M)$ closure of $C^\infty_0(M)$, the space of smooth compactly supported functions. The operator
\[
Au(x) := a(x, D)u, \quad D(A) := H^2(M) \cap H^1_0(M)
\]
is self-adjoint in $L^2(M) = L^2(M, dV_g)$, where $dV_g$ is the Riemannian volume measure. In local coordinates $dV_g = |g|^{1/2} \, dx$.

Denote by $v^f(x, t) = v(x, t)$ the solution of the initial boundary value problem
\[
\begin{align*}
\partial_t^2 v + a(x, D)v &= 0 \quad \text{in} \quad M \times (0, \infty), \\
v|_{\partial M \times (0, \infty)} &= f, \\
v|_{t=0} &= \hat{v}, \quad \partial_t v|_{t=0} = 0,
\end{align*}
\tag{3}
\]
for $f \in C^\infty_0(\partial M \times (0, \infty))$, and define the hyperbolic Dirichlet-to-Neumann operator
\[
\Lambda : C^\infty_0(\partial M \times (0, \infty)) \to C^\infty(\partial M \times (0, \infty)), \quad A \hat{v} := \partial_{\nu} v^f|_{\partial M \times \mathbb{R}_+},
\]
where $\partial_\nu$ is the normal derivative on $\partial M$. In local coordinates the exterior conormal $\nu$ is the covector $(\nu_1, \ldots, \nu_n)$ with
\[
\sum_{j,k=1}^n \nu_j g^{jk}(x) \nu_k(x) = 1, \quad x \in \partial M,
\]
and
\[
\partial_\nu = \sum_{j,k=1}^n \nu_j g^{jk} \frac{\partial}{\partial x^k}.
\]

Denote by $\Lambda_{\Gamma_1, \Gamma_2}^T$ the restriction of the Dirichlet-to-Neumann operator
\[
\Lambda_{\Gamma_1, \Gamma_2}^T : C^\infty_0(\Gamma_1 \times (0, T)) \to C^\infty(\Gamma_2 \times (0, T)),
\]
where $\Gamma_1, \Gamma_2 \subset \partial M$ are open. Furthermore, denote $\Lambda_{\Gamma_1, \Gamma_2} := \Lambda_{\Gamma_1, \Gamma_2}^{\infty}$.

It is well known that the map $\Lambda$ determines the manifold $(M, g)$ up to an isometry [5]. This is also true for the restriction $\Lambda_{\Gamma}^T$ when $\Gamma$ is nonempty and $T$ is sufficiently large [24].

In many applications observations of physical fields cannot be done on the same locations where the sources of the fields are. For instance, in imaging in earth sciences, elastic or acoustic fields are often implemented using explosions [38, 40]. In such a case observation devices need to be far away from the sources.

Similarly, in electromagnetic imaging, it is technically difficult to use electrodes at the same time as sources and for making observations. These are typical examples of cases where the observation devices and the sources of the fields are supported on disjoint sets.

In this paper we show that for certain collections of pairs $(\Gamma_1, \Gamma_2)$ of open and disjoint subsets of $\partial M$, the operators $\Lambda_{\Gamma_1, \Gamma_2}^T$ determine the manifold $(M, g)$ up to an isometry. We point out that the question whether the operator $\Lambda_{\Gamma_1, \Gamma_2}^T$ determines the manifold $(M, g)$ up to an isometry for arbitrary nonempty open sets $\Gamma_1, \Gamma_2 \subset \partial M$ remains open.
**Theorem 1.** Let $\Gamma_1$, $\Gamma_2$, $\Sigma \subset \partial M$ be open sets such that $\Gamma_1 \cap \Gamma_2 \subset \Sigma$ and $\Gamma_1 \cap \Gamma_2 \neq \emptyset$. Then $\Sigma$, given as a smooth manifold, and the operator $\Lambda_{\Gamma_1, \Gamma_2}$ determine the manifold $(M, g)$ up to an isometry.

Let us point out that the auxiliary set $\Sigma$ appears in theorem 1 because we need to use smooth coordinates near a point $x \in \overline{\Gamma_1} \cap \overline{\Gamma_2}$. These coordinates are not provided if we know only $\Gamma_1$ and $\Gamma_2$ as smooth manifolds.

**Theorem 2.** Let $\Gamma_1$, $\Gamma_2$, $\Gamma_3 \subset \partial M$ be open and nonempty. Then the smooth manifolds $\Gamma_p$, $p = 1, 2, 3$, and the operators

$$
\Lambda_{\Gamma_1, \Gamma_2}, \quad \Lambda_{\Gamma_1, \Gamma_3}, \quad \Lambda_{\Gamma_2, \Gamma_3},
$$

determine the manifold $(M, g)$ up to an isometry.

Note that in theorems 1 and 2 we do not assume that the sets $\Gamma_p$ for $p = 1, 2$ and $p = 1, 2, 3$, respectively, are disjoint, but the disjoint case is the focus of this paper. In the non-disjoint case, the manifold is determined by $\Lambda_{\Gamma_1, \Gamma_2}$, where $\Gamma = \Gamma_p \cap \Gamma_q$ is nonempty and $p \neq q$ [24].

Let us comment on theorem 2 in terms of applications when the sets $\Gamma_p$, $p = 1, 2, 3$, are far apart. Even though the measurements $\Lambda_{\Gamma_1, \Gamma_2}$ require the use of sources on $\Gamma_2$ and the measurements $\Lambda_{\Gamma_2, \Gamma_3}$ require the use of observation devices on $\Gamma_2$, any single measurement $\Lambda_{\Gamma_p, \Gamma_q} f$, where $1 \leq p < q \leq 3$ and $f \in C_0^\infty(\Gamma_p \times (0, \infty))$, can be performed with observation devices far away from the source.

For measurements on a finite time interval, we prove a theorem similar to theorem 2 under an additional controllability assumption.

(A) For any $w \in L^2(M)$ there is a boundary value $f \in L^2(\partial M \times (0, \infty))$ satisfying

$$
v_f(T/2) = w, \quad \text{supp}(f) \subset \Gamma_3 \times (0, T),
$$

where $v_f$ is the solution of equation (3) and $\Gamma_3 \subset \partial M$.

Let us comment on the controllability assumption (A) when $M$ is embedded in $\mathbb{R}^n$. Property (A) follows from the geometric control condition of Bardos, Lebeau and Rauch [2], which yields exact controllability of the wave equation. Property (A) also follows from the existence of a strictly convex function $h$ on $M$ with respect to the Riemannian metric $g$.

Suppose that $h \in C^2(M)$ is strictly convex and that $\rho > 0$ is a lower bound for the Hessian of $h$ in the Riemannian metric $g$, that is

$$
D^2 h(X, X) \geq \rho |X|_g, \quad X \in T_x M, \quad x \in M.
$$

By [31], (A) holds if

$$
T > \frac{4}{\rho} \max_{x \in M} |\nabla_g h|_g, \quad \sup_{x \in \Gamma_3} \nabla_g h(x) \cdot \overline{v}(x) \leq 0,
$$

where $\nabla_g$ and $| \cdot |_g$ are the gradient and length with respect to the Riemannian metric $g$, $\overline{v}$ is the Euclidean unit outward normal to $\partial M \subset \mathbb{R}^n$ and $\nabla_g h \cdot \overline{v}$ is the Euclidean inner product.

We refer to [31] for examples of Riemannian manifolds $(M, g)$ having a strictly convex function $h$.

**Theorem 3.** Let $\Gamma_1$, $\Gamma_2$, $\Gamma_3 \subset \partial M$ be open and nonempty. If the controllability assumption (A) holds for $\Gamma_3$ and $T > 0$, and

$$
4d(x, y) < T, \quad x \in \Gamma_p, \quad y \in M, \quad p = 1, 2, 3,
$$

then
then the Riemannian manifolds \((\Gamma_p, g|_{\Gamma_p})\), \(p = 1, 2, 3\), and the operators
\[
\Lambda^T_{\Gamma_1,\Gamma_2}, \quad \Lambda^T_{\Gamma_1,\Gamma_3}, \quad \Lambda^T_{\Gamma_2,\Gamma_3}
\]
determine the manifold \((M, g)\) up to an isometry.

The proofs of these theorems consist of showing that the data determine, up to a gauge transformation, the boundary spectral data of the operator \(A\) on a part of the boundary. The manifold is then determined up to an isometry, as can be seen using the boundary control method \([3, 5, 24, 25]\).

Note also that the operator \(\Lambda^T_{\Gamma_1,\Gamma_2}\) determines the operator \(\Lambda^T_{\Gamma_2,\Gamma_1}\) by a time reversal argument.

**Lemma 1.** Let \(\Gamma_1, \Gamma_2 \subset \partial M\) be open and nonempty. Let \(T > 0\), and define the time reversal operator \(R f(x, t) := f(x, T - t)\). If \(f \in C_0^\infty(\Gamma_1 \times (0, T))\) and \(h \in C_0^\infty(\Gamma_2 \times (0, T))\), then
\[
(f, \Lambda^T_{\Gamma_1,\Gamma_2}h)_{L^2(\partial M \times (0, T))} = (R \Lambda^T_{\Gamma_2,\Gamma_1}Rf, h)_{L^2(\partial M \times (0, T))}.
\]
Hence \((\Gamma_j, g|_{\Gamma_j})\), \(j = 1, 2\), given as Riemannian manifolds, and the operator \(\Lambda^T_{\Gamma_1,\Gamma_2}\) determine the operator \(\Lambda^T_{\Gamma_2,\Gamma_1} = (R \Lambda^T_{\Gamma_1,\Gamma_2}R)^\dagger\).

This result is relatively well known, see e.g. \([6, 12]\), but for the sake of completeness, we will give a proof in the appendix.

Let us review previous results on the topic. The inverse problem for the isotropic wave equation on a compact manifold with measurements on the whole boundary was solved by Belishev and Kurylev \([5]\), see also \([37]\). Their proof was based on the boundary control method originally developed in \([3]\) for the wave equation on a bounded domain of \(\mathbb{R}^n\). The inverse problems for more general hyperbolic equations on a compact Riemannian manifold with sources and observations on the same open subset \(\Gamma\) of the boundary have been studied by Katchalov and Kurylev \([24]\), see also \([27, 29]\). A similar problem has recently been studied for non-compact manifolds in \([23, 26]\).

Inverse problems for elliptic equations with data on a part of the boundary have been studied intensively as they are the natural generalization of the Calderón’s inverse problem for the conductivity equation \([10]\).
When measurements are given on the whole boundary, the inverse problem for the Schrödinger equation on a bounded domain of $\mathbb{R}^n$, $n \geq 3$, and hence for the isotropic conductivity equation was solved by Sylvester and Uhlmann in [39]. The corresponding two-dimensional problem for the isotropic conductivity equation was solved first by Nachman in [36] for $C^2$ conductivities, and for $L^\infty$ conductivities, for which the Calderón’s inverse problem was originally posed, by Astala and Päivärinta in [1]. Recently, also the inverse problem for the Schrödinger equation on a bounded domain of dimension 2 with measurements on the whole boundary was solved by Bukhgeim in [8]. The corresponding problem on a compact Riemannian surface was later solved in [17].

The inverse problem for the Schrödinger equation on a bounded domain of $\mathbb{R}^n$, $n \geq 3$, with observations on an open subset $\Gamma_1$ of the boundary was solved in [28]. The inverse problem for the Schrödinger equation on a bounded domain of $\mathbb{R}^n$, $n = 2$, with sources and observations on the same open subset $\Gamma_1$ of the boundary was solved by Imanuvilov, Uhlmann and Yamamoto in [21]. The corresponding problem on a compact Riemannian surface was later solved in [16]. For related results with measurements on a part of the boundary, see [9, 15, 22].

The inverse problem for the Laplace–Beltrami operator $\Delta_1$ on a compact Riemannian manifold with sources and observations on the same open subset $\Gamma_1$ of the boundary has been studied on analytic Riemannian manifolds of dimension $n \geq 3$ in [32, 34], and on Riemannian surfaces in [33], see also [4, 18, 19]. The inverse problem for the Laplace–Beltrami operator in dimensions $n \geq 3$ is open in general, even when measurements are given on the whole boundary. For positive results under certain geometrical conditions, see [13].

2. Spectral analysis of the data

Denote by $(\lambda_j)_{j \in \mathbb{N}}$ the increasing sequence of distinct eigenvalues of the operator $A$ and let $(\phi_k)_{k \in \mathbb{N}}$ be an orthonormal basis of real-valued $C^\infty$-smooth eigenfunctions. Moreover, let $(I_j)_{j \in \mathbb{N}}$ be a partition of $\mathbb{N}$ such that $(\phi_k)_{k \in I_j}$ is a basis for the space of eigenfunctions corresponding $\lambda_j$.

Let $f \in C_0^\infty(\partial M \times (0, \infty))$, and consider $\Lambda f$ also as a function in $C^\infty(\partial M \times \mathbb{R})$ by defining $\Lambda f(\cdot, t) = 0$ for $t \leq 0$. There is a constant $C > 0$ such that for $x \in \partial M$, the Fourier transform $\mathcal{F}_{t \to \tau} \Lambda f(x)$ is an analytic function of $\tau$ when $\text{Im} \tau < -C$. It is known (see e.g. [25]) that $\mathcal{F}_{t \to \tau} \Lambda f(x)$ extends to a meromorphic function of $\tau \in \mathbb{C}$, and that it may have poles only at points $\sqrt{\lambda_j}$. Moreover, the residues at these points are

$$\text{res}_{\tau = \sqrt{\lambda_j}} \mathcal{F}_{t \to \tau} \Lambda f(x) = \sum_{k \in I_j} (\hat{f}(\cdot, \sqrt{\lambda_j}), \partial_\nu \phi_k)_{L^2(\partial M, dS_x)} \partial_\nu \phi_k(x),$$

where $\hat{f}(x, \tau) = (\mathcal{F}_{t \to \tau} f)(x, \tau)$ and $dS_x$ is the Riemannian surface measure.

If $j \in \mathbb{N}$, $a_k$ are constants for $k \in I_j$ and the linear combination

$$\sum_{k \in I_j} a_k \partial_\nu \phi_k = \partial_\nu \left( \sum_{k \in I_j} a_k \phi_k \right)$$

vanishes on a nonempty open subset of $\partial M$, then $a_k = 0$ for all $k \in I_j$ by unique continuation, see e.g. [35]. Hence for an open nonempty set $\Gamma \subset \partial M$ and $j \in \mathbb{N}$, the functions $(\partial_\nu \phi_k|\Gamma)_{k \in I_j}$ are linearly independent.
Let $\Gamma_1, \Gamma_2 \subset \partial M$ be open and nonempty. By linear independence and smoothness of $\partial_n \phi_k$, there are $f \in C^\infty_0(\Gamma_1)$ and $x \in \Gamma_2$ such that

$$\sum_{k \in I_j} \langle f, \partial_n \phi_k \rangle_{L^2(\partial M, \delta S)} \partial_n \phi_k(x) \neq 0.$$ 

Moreover, for fixed $\tau \in \mathbb{C}$, the map $f \mapsto \hat{\tau}(\cdot)$ from $C^\infty_0(\Gamma_1 \times (0, \infty))$ to $C^\infty_0(\Gamma_1)$ is surjective.

Hence the operator $\Lambda_{\Gamma_1, \Gamma_2}$ determines the eigenvalues $\lambda_j$ and the operators

$$L_{\Gamma_1, \Gamma_2} : C^\infty_0(\Gamma_1 \times (0, \infty)) \to C^\infty(\Gamma_2 \times (0, \infty)),$$

$$L_{\Gamma_1, \Gamma_2} f := \sum_{k \in I_j} \langle f, \partial_n \phi_k \rangle_{L^2(\partial M, \delta S)} \partial_n \phi_k|_{\Gamma_2}.$$ 

(4)

3. Inverse problem with disjoint sources and observations

In this section we prove theorem 1.

Lemma 2. Suppose that $f, h \in C^\infty(\mathbb{R})$ are such that

$$\partial^j \partial^k \hat{h}(x)f(x)f(y)|_{x=0, y=0} = \partial^j \partial^k \hat{h}(x)f(x)f(y)|_{x=0, y=0}$$

for all $j, k \in \mathbb{N}$. Then $\partial^j f(0) = 0$ for all $j \in \mathbb{N} := \{0, 1, 2, \ldots\}$ or $\partial^k h(0) = 0$ for all positive $k \in \mathbb{N}$.

Proof. Assume that the claim is not valid. Then there exist $j \in \mathbb{N}$ and $k \in \mathbb{N}\{0\}$ such that $\partial^j f(0) \neq 0$ and $\partial^k h(0) \neq 0$. Let us next consider the smallest such integers $j$ and $k$.

By Leibniz’s formula

$$0 = \partial^{j+k} \hat{h}(x)f(x)f(y)|_{x=0, y=0} - \partial^j \partial^k \hat{h}(x)f(x)f(y)|_{x=0, y=0}$$

$$= \sum_{l=0}^{j+k} \binom{j+k}{l} \hat{h}(0) \partial^{j+k-l} f(0) \partial^l f(0) - \sum_{m=0}^{j} \binom{j}{m} \partial^m h(0) \partial^{j-m} f(0) \partial^k f(0)$$

$$= S_1 + \binom{j+k}{k} \hat{h}(0) \partial^j f(0) \partial^k f(0) + S_2 - S_3,$$

where

$$S_1 := \sum_{l=1}^{k-1} \binom{j+k}{l} \partial^l h(0) \partial^{j+k-l} f(0) \partial^j f(0),$$

$$S_2 := \sum_{l=k+1}^{j+k} \binom{j+k}{l} \partial^l h(0) \partial^{j+k-l} f(0) \partial^j f(0),$$

$$S_3 := \sum_{m=1}^{j} \binom{j}{m} \partial^m h(0) \partial^{j-m} f(0) \partial^k f(0),$$

and the terms with indices $l = 0$ and $m = 0$ have canceled each other out.

As $k$ is the smallest positive integer such that $\partial^k h(0) \neq 0$, we have $\partial^k h(0) = 0$ in the sum $S_1$, and so $S_1 = 0$. As $j$ is the smallest integer such that $\partial^j f(0) \neq 0$, we have $\partial^{j+k-l} f(0) = 0$ in the sum $S_2$ and $\partial^{j-m} f(0) = 0$ in the sum $S_3$, thus $S_2 = S_3 = 0$.

Hence $\partial^k h(0) (\partial^j f(0))^2 = 0$, which is a contradiction with the assumption that $\partial^j f(0) \neq 0$ and $\partial^k h(0) \neq 0$. This proves the claim. \(\square\)
In the proof of the next lemma we use the equation
\[ \partial_j f(tv) = \sum_{|\alpha|=j} \frac{j!}{\alpha!} \partial^\alpha f(tv) v^\alpha, \]  
(5)
where \( f \in C^\infty(\mathbb{R}^n), t \in \mathbb{R}, v \in \mathbb{R}^n \) and \( j \in \mathbb{N} \).

**Lemma 3.** Suppose that \( f, h \in C^\infty(\mathbb{R}^n) \) are such that
\[ \partial^\alpha \partial^\beta (h(x) f(x) f(y))|_{x=0, y=0} = \partial^\alpha \partial^\beta (h(x) f(x) f(y))|_{x=0, y=0} \]
for all multi-indices \( \alpha, \beta \in \mathbb{N}^n \). Then \( \partial^\alpha f(0) = 0 \) for all multi-indices \( \alpha \in \mathbb{N}^n \) or \( \partial^\beta h(0) = 0 \) for all nonzero multi-indices \( \beta \in \mathbb{N}^n \).

**Proof.** Let \( j, k \in \mathbb{N} \) and \( v \in \mathbb{R}^n \). By (5)
\[ \partial^j \partial^k (h(tv) f(tv) f(sv))|_{t=0, s=0} = \partial^j \partial^k (h(tv) f(tv) f(sv))|_{t=0, s=0} \]
\[ = \sum_{|\alpha|=j} \frac{j! k!}{\alpha! \beta!} v^\alpha \partial^\alpha (h f)(0) \partial^\beta f(0) = \partial^\alpha (h f)(0) \partial^\beta f(0) \]
Hence
\[ \partial^j \partial^k (h(tv) f(tv) f(sv))|_{t=0, s=0} = \partial^j \partial^k (h(tv) f(tv) f(sv))|_{t=0, s=0} \]
for all \( j, k \in \mathbb{N} \) and \( v \in \mathbb{R}^n \).

Define the sets
\[ F := \{ v \in \mathbb{R}^n : \partial^j f(tv)|_{t=0} = 0 \text{ for all } j \in \mathbb{N} \}, \]
\[ H := \{ v \in \mathbb{R}^n : \partial^j h(tv)|_{t=0} = 0 \text{ for all positive } k \in \mathbb{N} \}. \]
The sets \( F \) and \( H \) are closed by smoothness of \( f \) and \( h \), respectively. Lemma 2 gives that \( F \cup H = \mathbb{R}^n \). If \( F \neq \mathbb{R}^n \), then \( \mathbb{R}^n \setminus F \) is open, nonempty and contained in \( H \). Thus \( F \) or \( H \) contains an open nonempty subset.

Suppose that \( U \subset F \) is open and nonempty. Let \( j \in \mathbb{N} \), and define the polynomial
\[ p(v) := \sum_{|\alpha|=j} \frac{j!}{\alpha!} \partial^\alpha f(0) v^\alpha. \]
By (5), \( p(v) = \partial^j f(tv)|_{t=0} \), and \( p \) vanish in \( U \). Using unique continuation for real analytic functions we see that \( p = 0 \) in \( \mathbb{R}^n \), and so the coefficients of \( p \) vanish. As \( j \) can be chosen freely, \( \partial^\alpha f(0) = 0 \) for all multi-indices \( \alpha \).

Similarly, if there exists an open and nonempty \( V \subset H \), then \( \partial^\beta h(0) = 0 \) for all nonzero multi-indices \( \beta \).

**Remark 1.** Let \( U \) be a \( C^\infty \)-smooth manifold of dimension \( n \), \( f \in C^\infty(U) \) and \( p \in U \). If \( \partial^\beta f(0) = 0 \) for all multi-indices \( \alpha \in \mathbb{N}^n \) in some local coordinates taking \( p \) to 0, then \( \partial^\beta f(0) = 0 \) for all multi-indices \( \alpha \in \mathbb{N}^n \) in all local coordinates taking \( p \) to 0.

**Lemma 4.** Let \( \phi \) be an eigenfunction of the operator \( A \) corresponding to an eigenvalue \( \lambda \), and let \( p_0 \in \partial M \). Then in any local coordinates of \( \partial M \) taking \( p_0 \) to 0, there is a multi-index \( \alpha \in \mathbb{N}^{n-1} \) such that \( \partial^\alpha \partial_\nu \phi(0) \neq 0 \).

**Proof.** Assume that the claim is not valid. Then \( \partial^\alpha \partial_\nu \phi(0) = 0 \) for all \( \alpha \in \mathbb{N}^{n-1} \) in some local coordinates of \( \partial M \) taking \( p_0 \) to 0.

Consider boundary normal coordinates of \( \overline{M} \) taking \( p_0 \) to 0. We may suppose that the coordinates map a small neighborhood \( V \) of \( p_0 \) onto \( B(0, \epsilon) \times [0, \epsilon) \), where \( B(0, \epsilon) \subset \mathbb{R}^{n-1} \).
is a ball of radius \( r > 0 \) centered at the origin. Then these coordinates take a boundary point \( p' \in \partial M \cap V \) to a point \((x',0) \in B(0,\epsilon) \times \{0\} \), where \( x' = (x^1, \ldots, x^{n-1}) \).

The special property of the boundary normal coordinates is that a point \( p \in \overline{M} \cap V \) has coordinates \((x',x^n) \in B(0,\epsilon) \times \{0,\epsilon\} \), where \( x^n = d(p,\partial M) \) and \( x' \) are the coordinates of the unique boundary point \( p' \in \partial M \) such that \( d(p, p') = d(p, \partial M) \).

Moreover, in the coordinates \((x', x^n)\) the equation

\[
(-\Delta_g + q)\phi = \lambda \phi
\]

has the form

\[
-\partial_{x'}^2 \phi - \sum_{j,k=1}^{n-1} g^{jk} \partial_j \partial_k \phi + \sum_{j=1}^{n} a^j \partial_j \phi + a^0 \phi = \lambda \phi,
\]

for some \( a^j \in C^\infty(B(0,\epsilon) \times \{0,\epsilon\}), \ j = 0, \ldots, n, \) see e.g. [11].

Let us show that \( \phi \equiv 0 \). Let \( b \in \mathbb{N} \) and \( \alpha \in \mathbb{N}^{n-1} \). By applying the operator \( \partial_{x'}^\alpha \partial_v^b \) on both the sides of (6), we get

\[
\partial_{x'}^\alpha \partial_v^b \phi(0) = \partial_{x'}^\alpha \partial_v^b \left( \left. -\sum_{j,k=1}^{n-1} g^{jk} \partial_j \partial_k \phi + \sum_{j=1}^{n} a^j \partial_j \phi + a^0 \phi - \lambda \phi \right|_{x=0} \right).
\]

The right-hand side of this equation is a linear combination of functions

\[
\partial_{x'}^\alpha \partial_v^b \phi, \quad \alpha' \in \mathbb{N}, b' \leq b + 1,
\]

at the point \( x = 0 \).

The equations \( \partial_{x'}^\alpha \phi(0) = 0 \) hold for all \( \alpha \in \mathbb{N}^{n-1} \) by the boundary condition \( \phi|_{\partial M} = 0 \). Furthermore, we have by remark 1 that \( \partial_{x'}^\alpha \partial_v^b \phi(0) = 0 \) for all \( \alpha \in \mathbb{N}^{n-1} \) and \( b \in \mathbb{N} \).

Define the odd and even reflection operators

\[
R_o f(x', x^n) := (\text{sign } x^n) f(x', |x^n|), \quad R_e f(x', x^n) := f(x', |x^n|),
\]

where \( f \in C(B(0,\epsilon) \times \{0,\epsilon\}), x' \in B(0,\epsilon) \) and \( x^n \in (-\epsilon, \epsilon) \).

Also, define \( \tilde{\phi} := R_o \phi, \tilde{g}^{jk} := R_o g^{jk}, \tilde{a}^j := R_o a^j \) for \( j = 0, \ldots, n-1 \). Denote \( U := B(0,\epsilon) \times (-\epsilon, \epsilon) \).

As \( \phi|_{x=0} = 0 \) we see that \( \phi \in H^2(U) \) and

\[
\partial_{x'}^\alpha \partial_v^b \tilde{\phi}(x', x^n) = (\text{sign } x^n)^{b+1} \partial_{x'}^\alpha \partial_v^b \phi(x', |x^n|), \quad |\alpha| + b \leq 2.
\]

Moreover, \( \tilde{g}^{jk} \) is Lipschitz continuous in \( U, \tilde{a}^j \in L^\infty(U) \) for \( j = 0, \ldots, n \), and

\[
-\tilde{\partial}_{x'}^2 \tilde{\phi} - \sum_{j,k=1}^{n-1} \tilde{g}^{jk} \partial_j \partial_k \tilde{\phi} + \sum_{j=1}^{n} \tilde{a}^j \partial_j \tilde{\phi} + \tilde{a}^0 \tilde{\phi} = \lambda \tilde{\phi},
\]

where both the sides are considered as functions in \( L^2(U) \). Hence for some constant \( C > 0 \)

\[
|\tilde{\partial}_{x'}^2 \tilde{\phi} + \sum_{j,k=1}^{n-1} \tilde{g}^{jk} \partial_j \partial_k \tilde{\phi}| \leq C \sum_{|\alpha|+b \leq 1} |\partial_{x'}^\alpha \partial_v^b \tilde{\phi}|, \quad \text{in } U.
\]

Since \( \phi \in C^\infty(B(0,\epsilon) \times \{0,\epsilon\}) \) vanishes up to arbitrary degree in origin, Taylor’s formula gives for any \( m \in \mathbb{N} \) a constant \( C_m > 0 \) such that

\[
\int_{B(0,r)} \int_0^r |\phi(x)|^2 dx' dx' \leq \int_{B(0,\epsilon)} \int_0^\epsilon C_m r^m dx' dx', \quad \text{as } r \to 0.
\]
Hence for any \( m \in \mathbb{N} \), there is a constant \( C'_m > 0 \) such that
\[
\int_{B(0,r)} \int_{\partial B(0,r)} |\tilde{\phi}(x)|^2 \, dx \, ds(x) \leq C'_m r^m, \quad \text{as} \quad r \to 0.
\]

By Hörmander’s strong unique continuation result [20] this yields that \( \tilde{\phi} = 0 \) in \( U \). In particular, \( \phi = 0 \) around some point \( q \in M \). As \( M \) is connected, unique continuation gives that \( \phi = 0 \) in \( M \). This is a contradiction with the assumption that \( \phi \) is an eigenfunction, and the claim is proved. \( \square \)

**Remark 2.** Let \( X \) and \( Y \) be Hilbert spaces, and let \( u_1, \ldots, u_N \in X \) and \( v_1, \ldots, v_N \in Y \) be linearly independent. Suppose that \( D \subset X \) is a dense subspace, and define
\[
L : D \to Y, \quad Lf := \sum_{k=1}^N (f, u_k) X v_k.
\]
Then \( L \) determines the unique bounded extension \( \tilde{L} : X \to Y \), and its adjoint \( \tilde{L}^* : Y \to X \). Hence, \( L \) determines the spaces
\[
\text{span}(u_1, \ldots, u_N) = \tilde{L}(X), \quad \text{span}(v_1, \ldots, v_N) = \tilde{L}^*(Y).
\]

**Theorem 4.** Let \( \Gamma_1, \Gamma_2, \Sigma \subset \partial M \) be open, \( \Gamma_1, \Gamma_2 \subset \Sigma \) and \( \Gamma_1 \cap \Gamma_2 \neq \emptyset \). Then the smooth manifold \( \Sigma \) and the collection
\[
\left\{ (\lambda_j, L_{\Gamma_1, \Gamma_2,j}) \mid j \in \mathbb{N} \right\}
\]
determine boundary spectral data up to a constant gauge transformation on \( \Gamma_2 \). That is, one can determine the set
\[
\left\{ (\lambda_j, (\partial_\nu \phi|_{\Gamma_1})_{k \in \mathbb{N}}) \mid j \in \mathbb{N} \right\},
\]
where for an unknown constant \( C > 0 \) not depending on \( j \) or \( k \), \( (C \psi_k)_{k \in \mathbb{N}} \), is an orthonormal basis of eigenfunctions in \( L^2(M) \) corresponding the eigenvalue \( \lambda_j \).

**Proof.** Choose a smooth positive measure \( d\mu \) on \( \Sigma \). Then there is a positive function \( \eta \in C^\infty(\Sigma) \) such that \( \eta d\mu = dS_x \mid_\Sigma \). We denote by \( \phi_k, k \in I_j, j \in \mathbb{N} \), the eigenfunctions of \( \Lambda \) as in section 2. As \( \eta > 0 \), the functions \( (\eta \partial_\nu \phi_k|_{\Gamma_1})_{k \in \mathbb{N}} \), are linearly independent for all \( j \in \mathbb{N} \).

For all \( j \in \mathbb{N} \), denote \( L_j := L_{\Gamma_1, \Gamma_2,j} \), and define
\[
L_j(x, y) := \sum_{k \in I_j} \eta(x) \partial_\nu \phi_k(x) \partial_\nu \phi_k(y), \quad x, y \in \partial M,
\]
\[
\tilde{E}_j^1 := \text{span}(\eta \partial_\nu \phi_k|_{\Gamma_1})_{k \in I_j}, \quad E_j^2 := \text{span}(\partial_\nu \phi_k|_{\Gamma_2})_{k \in I_j}.
\]

Note that for the smallest eigenvalue \( \lambda_0 \), the space of eigenfunctions is one dimensional (see e.g. [14, theorem 6.5.2]), and so
\[
l_0(x, y) = \eta(x) \partial_\nu \phi_0(x) \partial_\nu \phi_0(y).
\]

Consider a positive function \( \eta \in C^\infty(\Sigma) \) and real-valued functions \( e_k \in C^\infty(\Sigma), k \in \mathbb{N} \), such that the following three conditions hold:

(A1) If \( x_0 \in \Gamma_1 \cap \Gamma_2 \), then in local coordinates of \( \Sigma \) taking \( x_0 \) to 0
\[
\frac{\partial_\alpha \partial_\beta (l_0(x, y))}{\eta(x)} \bigg|_{x=0, y=0} = \frac{\partial_\alpha \partial_\beta (l_0(x, y))}{\eta(x)} \bigg|_{x=0, y=0}
\]
for all multi-indices \( \alpha, \beta \in \mathbb{N}^{n-1} \).
(A2) \( \text{span}(\tilde{\eta}e_k|_{\Gamma_1})_{k \in I} = \tilde{E}_j^1 \) and \( \text{span}(e_k|_{\Gamma_2})_{k \in I} = E_j^2 \) for all \( j \in \mathbb{N} \).

(A3) \( \tilde{L}_j = L_j \) for all \( j \in \mathbb{N} \), where

\[
\tilde{L}_j f(y) := \sum_{k \in I_j} (f, \tilde{\eta}e_k)_{L^2(\partial M, d\mu)} e_k(y), \quad f \in C_0^\infty(\Gamma_1), \quad y \in \Gamma_2.
\]

Such functions \( \tilde{\eta} \) and \( e_k \) exist. For example, \( \tilde{\eta} = \eta \) and \( e_k = \partial_v \phi_k|_\Sigma \) satisfy the conditions.

Next we show the following two statements.

(i) We can verify using the data (7) whether any given functions \( \tilde{\eta} \in C^\infty(\Sigma) \) and \( e_k \in C^\infty(\Sigma) \), \( k \in \mathbb{N} \), satisfy conditions (A1), (A2) and (A3).

(ii) There is an orthonormal basis of eigenfunctions \( (\psi_k)_{k \in \mathbb{N}} \) of operator \( A \) and a constant \( C > 0 \), not depending on \( k \), such that

\[
e_k|_{\Gamma_1} = C \partial_v \psi_k|_{\Gamma_2}, \quad k \in \mathbb{N}.
\]

If (i) holds, then the data (7) determine the nonempty collection

\[
\mathcal{B} := \{ (e_k|_{\Gamma_2})_{k \in \mathbb{N}} \mid (A1)–(A3) \text{ hold with a positive } \tilde{\eta} \in C^\infty(\Sigma) \},
\]

and if (ii) holds, then any element in \( \mathcal{B} \) determines a collection of type (8). So the claim of the theorem is proved after proving (i) and (ii).

We point out that our argument does not give an explicit method to choose functions \( \tilde{\eta} \) and \( (e_k)_{k \in \mathbb{N}} \). Instead, we give a method to verify whether a pair \( (\tilde{\eta}, (e_k)_{k \in \mathbb{N}}) \in C^\infty(\Sigma) \times C^\infty(\Sigma)^0 \) is in the set

\[
\mathcal{R} := \{(\tilde{\eta}, (e_k)_{k \in \mathbb{N}}) \in C^\infty(\Sigma) \times C^\infty(\Sigma)^0 \mid (A1)–(A3) \text{ hold}\}.
\]

The sets \( \mathcal{R} \) and \( \mathcal{B} = \{ (e_k|_{\Gamma_2})_{k \in \mathbb{N}} \mid \exists \tilde{\eta} \in C^\infty(\Sigma) \text{ s.t. } (\tilde{\eta}, (e_k)_{k \in \mathbb{N}}) \in \mathcal{R} \} \) are determined in this sense.

Let us show claim (i). Clearly, condition (A3) can be verified using the data (7). As

\[
L_j f(y) = \left( f, \sum_{k \in I_j} (\partial_v \phi_k(y)) \tilde{\eta} \partial_v \phi_k \right)_{L^2(\partial M, d\mu)}, \quad f \in C_0^\infty(\Gamma_1), \quad y \in \Gamma_2,
\]

by varying \( f \in C_0^\infty(\Gamma_1) \), we see that the map \( L_j \) determines the function \( l_j|_{\Gamma_1 \times \Gamma_2} \).

Let \( x_0 \in \Gamma_1 \cap \Gamma_2 \). Given \( l_0|_{\Gamma_1 \times \Gamma_2} \) and \( \tilde{\eta} \), it is possible to compute

\[
\partial_\alpha^\alpha \partial_\beta^\beta \left( \frac{l_0(x, y)}{\tilde{\eta}(x)} \right), \quad (x, y) \in \Gamma_1 \times \Gamma_2, \quad \alpha, \beta \in \mathbb{N}^{n-1},
\]

in any local coordinates \( \Sigma \) taking \( x_0 \) to 0. By smoothness of \( \eta, \tilde{\eta} \) and \( \partial_v \phi_0 \), these functions are known also at \( (x, y) = (0, 0) \). Hence condition (A1) can be verified using the data (7).

Taking \( X = L^2(\Gamma_1, d\mu), Y = L^2(\Gamma_2, d\mu) \) and \( D = C_0^\infty(\Gamma_1) \) in the formulation of remark 2, we see that the map \( L_j \) determines the spaces \( \tilde{E}_j^1 \) and \( E_j^2 \). Hence condition (A2) can be verified using the data (7), and claim (i) is proved.

Let us show claim (ii). Let \( x_0 \in \Gamma_1 \cap \Gamma_2 \). Lemma 4 gives that, in local coordinates of \( \partial M \) taking \( x_0 \) to 0, there is a multi-index \( \alpha \in \mathbb{N}^{n-1} \) such that \( \partial_\alpha^\alpha \partial_v \phi_0(0) \neq 0 \). Hence condition (A1) and Lemma 3 imply that \( \partial_\beta^\beta (\eta \tilde{\eta}^{-1})(0) = 0 \) for all nonzero multi-indices \( \beta \in \mathbb{N}^{n-1} \).

Fix \( j \in \mathbb{N} \) and, to simplify the notation, drop the subindices \( j \) from now on. By condition (A2)

\[
\tilde{\eta} e_l|_{\Gamma_1} = \sum_{k \in I_l} a_{lk} \eta \partial_v \phi_k|_{\Gamma_1}, \quad e_l|_{\Gamma_2} = \sum_{k \in I_l} b_{lk} \partial_v \phi_k|_{\Gamma_2}, \quad l \in I,
\]

for some constant matrices \( A := (a_{lk})_{l,k \in I} \) and \( B := (b_{lk})_{l,k \in I} \).
Fix $x_0 \in \Gamma_1 \cap \Gamma_2$, let $l \in I$ and define the function
\[
\phi(p) := \sum_{k \in l} \left( \frac{\eta(x_0)}{\eta(x)} - b_{lk} \right) \phi_k(p), \quad p \in M.
\]
We have seen that, in local coordinates of $\partial M$ taking $x_0$ to 0, the equation $\partial^\beta (\eta \tilde{\eta}^{-1})(0) = 0$ holds for all nonzero multi-indices $\beta \in \mathbb{N}^{n-1}$. Hence for any multi-index $\alpha \in \mathbb{N}^{n-1}$
\[
0 = \partial^\alpha e_l(0) - \tilde{\partial}^\alpha e_l(0)
\]
\[
= \partial^\alpha \left( \sum_{k \in l} a_{lk} \eta(x) \partial_\alpha \phi_k(x) \right) \bigg|_{x=0} - \tilde{\partial}^\alpha \left( \sum_{k \in l} b_{lk} \partial_\alpha \phi_k(y) \right) \bigg|_{y=0}
\]
\[
= \sum_{k \in l} \left( a_{lk} \eta(0) \eta(0)^{-1} - b_{lk} \right) \partial^\alpha \phi_k(0) = \partial^\alpha \phi(0).
\]
By lemma 4, the coefficients
\[
a_{lk} \eta(0) \eta(0)^{-1} - b_{lk}, \quad k, l \in I,
\]
vanish, and so $\eta(0)\tilde{\eta}(0)^{-1}A = B$.

Using notations $L^2(dS) := L^2(\partial M, dS_\gamma)$ and $L^2(d\mu) := L^2(\partial M, d\mu)$, we have
\[
(h, Lf)_{L^2(dS)} = \sum_{k \in l} (f, \eta \eta \partial_\alpha \phi_k)_{L^2(d\mu)} (h, \sum_{m \in l} b_{km} \partial_\alpha \phi_m)_{L^2(dS)}
\]
\[
= \sum_{l, m \in l} \left( \sum_{k \in l} a_{kl} b_{km} \right) (f, \partial_\alpha \phi_k)_{L^2(dS)} (h, \partial_\alpha \phi_m)_{L^2(dS)},
\]
for all $f \in C_0^\infty(\Gamma_1)$ and $h \in C_0^\infty(\Gamma_2)$. On the other hand, condition (A3) gives
\[
(h, Lf)_{L^2(dS)} = (h, Lf)_{L^2(dS)}
\]
\[
= \sum_{l \in l} (f, \partial_\alpha \phi_k)_{L^2(dS)} (h, \partial_\alpha \phi_k)_{L^2(dS)},
\]
for all $f \in C_0^\infty(\Gamma_1)$ and $h \in C_0^\infty(\Gamma_2)$.

Denote $(\cdot, \cdot) := (\cdot, \cdot)_{L^2(dS)}$. By density of $C_0^\infty(\Gamma_\rho)$ in $L^2(\Gamma_\rho, dS_\gamma)$, $p = 1, 2,$
\[
\sum_{l, m \in l} \left( \sum_{k \in l} a_{kl} b_{km} \right) (f, \partial_\alpha \phi_k)_{L^2(dS)} (h, \partial_\alpha \phi_m) = \sum_{k \in l} (f, \partial_\alpha \phi_k)_{L^2(dS)},
\]
for all $f \in L^2(\Gamma_1, dS_\gamma)$ and $h \in L^2(\Gamma_2, dS_\gamma)$.

Let $(f_l)_{l \in l}$ be biorthogonal with $(\partial_\alpha \phi_l|_{\Gamma_\rho})_{l \in l}$ in $L^2(\Gamma_\rho, dS_\gamma)$, and let $(h_m)_{m \in l}$ be biorthogonal with $(\partial_\alpha \phi_m|_{\Gamma_\rho})_{m \in l}$ in $L^2(\Gamma_2, dS_\gamma)$, that is,
\[
(f_l, \partial_\alpha \phi_l) = \delta_{l,l}, \quad (h_m, \partial_\alpha \phi_m) = \delta_{m,m'}, \quad l', l, m, m' \in I.
\]
Then
\[
\sum_{k \in l} a_{kl} b_{km'} = \sum_{l, m \in l} \left( \sum_{k \in l} a_{kl} b_{km} \right) (f_l, \partial_\alpha \phi_l)(h_m, \partial_\alpha \phi_m)
\]
\[
= \sum_{k \in l} (f_l, \partial_\alpha \phi_l)(h_m, \partial_\alpha \phi_l) = \delta_{l,m'}, \quad l', m' \in I.
\]
Denote \( c := \eta(0)^{-1}\tilde{\eta}(0) > 0 \). We have shown that \( I = AT B = cB^T B \). Hence the matrix \( \sqrt{c}B \) is orthogonal. To conclude, we observe that

\[
ed_l|_{\Gamma_2} = \frac{1}{\sqrt{c}} \partial_\nu \sum_{k \in I} \sqrt{c} b_{lk} \phi_k|_{\Gamma_2}, \quad l \in I,
\]

where \( \{ \sum_{k \in I} \sqrt{c} b_{lk} \phi_k \}_{l \in I} \) is an orthonormal basis of eigenfunctions corresponding the eigenvalue \( \lambda_j \).

As discussed in the previous section, the operator \( \Lambda_{\Gamma_1,\Gamma_2} \) determines the collection (7). So by the previous theorem, if \( \Gamma_1 \cap \Gamma_2 \neq \emptyset \), then the operator \( \Lambda_{\Gamma_1,\Gamma_2} \) determines the collection (8). The collection (8) determines the manifold up to an isometry by [25, chapter 4.4]. This proves theorem 1.

4. Inverse problem with observations far away from sources

In this section we prove theorems 2 and 3.

**Proof of theorem 2.** Denote \( L_j^{p \rightarrow q} = L_{\Gamma_p,\Gamma_q;j} \). It is enough to show that the collection

\[
\{ (\lambda_j, L_j^{1 \rightarrow 2}, L_j^{1 \rightarrow 3}, L_j^{2 \rightarrow 3}) \mid j \in \mathbb{N} \}
\]

(9)
determines a collection of type (8).

Denote \( \Sigma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \), and choose a smooth positive measure \( d\mu \) on \( \Sigma \). There is a positive function \( \eta \in \mathcal{C}_\infty(\Sigma) \) such that \( \eta d\mu = dS_\Sigma \). Define for all \( j \in \mathbb{N} \)

\[
\tilde{E}_j^p := \text{span}(\eta \partial_\nu \phi_k|_{\Gamma_p})_{k \in I_j}, \quad p = 1, 2,
\]

\[
E_j^q := \text{span}(\partial_\nu \phi_k|_{\Gamma_q})_{k \in I_j}, \quad q = 2, 3.
\]

Choose a positive function \( \tilde{\eta} \in \mathcal{C}_\infty(\Sigma) \) and real-valued functions \( e_k \in \mathcal{C}_\infty(\Sigma) \), \( k \in \mathbb{N} \), such that the following two conditions hold:

(B1) for all \( j \in \mathbb{N} \)

\[
\text{span}(\tilde{\eta} e_k|_{\Gamma_p})_{k \in I_j} = \tilde{E}_j^p, \quad p = 1, 2,
\]

\[
\text{span}(e_k|_{\Gamma_q})_{k \in I_j} = E_j^q, \quad q = 2, 3.
\]

(B2) \( \tilde{L}_j^{p \rightarrow q} = L_j^{p \rightarrow q} \) for all \( j \in \mathbb{N} \) and \( (p, q) = (1, 2), (1, 3), (2, 3) \), where

\[
\tilde{L}_j^{p \rightarrow q} f(y) := \sum_{k \in I_j} (f, \tilde{\eta} e_k)_L^2 \partial M d\mu \partial_\nu e_k(y) \quad f \in \mathcal{C}_\infty(\Gamma_p), y \in \Gamma_q.
\]

Again, such functions \( \tilde{\eta} \) and \( e_k \) exist, as \( \tilde{\eta} = \eta \) and \( e_k = \partial_\nu \phi_k |_{\Sigma} \) satisfy conditions (B1) and (B2). It is enough to show the following two statements.

(i) We can verify using the data (9) whether any given functions \( \tilde{\eta} \in \mathcal{C}_\infty(\Sigma) \) and \( e_k \in \mathcal{C}_\infty(\Sigma) \), \( k \in \mathbb{N} \), satisfy conditions (B1) and (B2).

(ii) There is an orthonormal basis of eigenfunctions \( (\psi_k)_{k \in \mathbb{N}} \) of operator \( A \) and a constant \( C > 0 \), not depending on \( k \), such that

\[
e_k|_{\Gamma_2} = C \partial_\nu \psi_k|_{\Gamma_2}, \quad k \in \mathbb{N}.
\]
Analogously with the proof of theorem 4, the maps $L^1 \rightarrow L^2_j$, $L^2 \rightarrow L^3_j$ and $L^2 \rightarrow L^3_j$ determine the spaces $E^1_j$, $E^2_j$ and $E^3_j$. Hence claim (i) is proved.

Let us show claim (ii). Fix $j \in \mathbb{N}$ and, to simplify the notation, drop the subindices $j$ from now on. Condition (B1) gives that for all $l \in I$

$$\tilde{\eta}e_l|_{\Gamma_1} = \sum_{k \in I} a_{lk} \eta \partial_\nu \phi_k|_{\Gamma_1}, \quad \tilde{\eta}e_l|_{\Gamma_2} = \sum_{k \in I} b_{lk} \eta \partial_\nu \phi_k|_{\Gamma_2},$$

$$e_l|_{\Gamma_1} = \sum_{k \in I} b_{lk} \partial_\nu \phi_k|_{\Gamma_1}, \quad e_l|_{\Gamma_2} = \sum_{k \in I} c_{lk} \partial_\nu \phi_k|_{\Gamma_2},$$

for some constant matrices

$$A := (a_{lk})_{l,k \in I}, \quad \tilde{B} := (\tilde{b}_{lk})_{l,k \in I}, \quad B := (b_{lk})_{l,k \in I}, \quad C := (c_{lk})_{l,k \in I}.$$

For the smallest eigenvalue $\lambda_0$, the space of eigenfunctions is one dimensional, and so

$$0 = e_0(y) - e_0(y) = \left( \frac{\eta(y)\tilde{\eta}(y)}{\eta(y)\tilde{\eta}(y)} - b_{00} \right) \partial_\nu \phi_0(y)$$

for all $y \in \Gamma_2$. By lemma 4, the set

$$N := \{ y \in \Gamma_2 : \partial_\nu \phi(y) = 0 \}$$

does not contain a nonempty open set of $\partial M$. Hence $\Gamma_2 \setminus N = \Gamma_2$. Moreover, $\eta\tilde{\eta}^{-1}b_{00} - b_{00} = 0$ in $\Gamma_2 \setminus N$ and by continuity also in the whole set $\Gamma_2$.

Denote by $c$ the constant $\eta^{-1}\tilde{\eta}|_{\Gamma_2} > 0$. As $(\partial_\nu \phi_k|_{\Gamma_2})_{k \in I}$ are linearly independent, $\tilde{B} = cB$ for all $j \in \mathbb{N}$. Moreover, we may use condition (B2) as we used the corresponding condition in the proof of theorem 4, and get

$$A^T B = I, \quad A^T C = I, \quad \tilde{B}^T C = I.$$

Hence $B = C$ and $cB^T B = I$.

To conclude, we observe that

$$e_l|_{\Gamma_2} = \frac{1}{\sqrt{c}} \partial_\nu \sum_{k \in I} \sqrt{c} b_{lk} \phi_k|_{\Gamma_2}, \quad l \in I,$$

where $(\sum_{k \in I} \sqrt{c} b_{lk} \phi_k)_{l \in I}$ is an orthonormal basis of eigenfunctions corresponding the eigenvalue $\lambda_j$. □

We prove theorem 3 by reduction it to theorem 2 using a time continuation argument similar to [30].

**Lemma 5.** Suppose that $\Gamma_1, \Gamma_2 \subset \partial M$ are open and nonempty. Denote

$$T^* := 2 \max\{ d(x,y) : x \in \Gamma_1, y \in M \}.$$

If $T^* < t_0 < T$, then the smooth manifolds $\Gamma_1, \Gamma_2$, the operator $A^T_{\Gamma_1,\Gamma_2}$ and the inner products

$$(u^f(t_0), u^h(t_0))_{L^2(M)}, \quad f, h \in C^\infty_0(\Gamma_1 \times (0, T))$$

(10)

determine the operator $A^T_{\Gamma_1,\Gamma_2}$ for $\delta < t_0 - T^*$.

**Proof.** Denote by $Y_t$ the time delay operator

$$Y_t f(\cdot, t) := f(\cdot, t - s), \quad t, s \in \mathbb{R}.$$

As the coefficients of the wave equation (3) are time independent, $v^{Y_t f}(x, t) = (Y_t v^f)(x, t)$ and $(\Lambda Y_t f)(x, t) = (Y_t \Lambda f)(x, t)$. 

13
Let $f \in C^\infty_0(\Gamma_1 \times (0, T+\delta))$ and choose $h \in C^\infty_0(\Gamma_1 \times (0, t_0))$ and $h' \in C^\infty_0(\Gamma_1 \times (\delta, T+\delta))$ such that $f = h + h'$.

Let $\varepsilon \in (0, \delta)$. As supp$(Y_{-\delta}h') \subset \Gamma_1 \times (0, T)$, the operator $\Lambda^T_{\Gamma_1, \Gamma_2}$ determines the function

$$\Lambda h'(. , T + \varepsilon) = (Y_{-\delta} \Lambda h')(. , T - (\delta - \varepsilon)) = (\Lambda^T_{\Gamma_1, \Gamma_2} Y_{-\delta} h')(., T - (\delta - \varepsilon)),$$

in $\Gamma_2$. Therefore, it is enough to show that the given data also determine

$$\Lambda h(x, T + \varepsilon), \quad x \in \Gamma_2.$$

Consider a sequence $(h_j)_{j=1}^\infty \subset C^\infty_0(\Gamma_1 \times (0, t_0 - \delta))$ satisfying the following two conditions.

(C1) $\lim_{j \to \infty} v^T_{\Gamma_1 h}(t_0) = v^h(t_0)$ in $H^1(M)$,

(C2) $\lim_{j \to \infty} \partial_t v^T_{\Gamma_1 h}(t_0) = \partial_t v^h(t_0)$ in $L^2(M)$.

To see that the set of sequences satisfying (C1) and (C2) is nonempty note that, since $t_0 - \delta > T^*$, the set

$$\{(v^f(t_0 - \delta), \partial_t v^f(t_0 - \delta)) : f \in C^\infty_0(\Gamma_1 \times (0, t_0 - \delta))\}$$

is dense in $H^1_0(M) \times L^2(M)$ [25, theorem 4.28].

Next we show the following two statements.

(i) We can verify using the inner products (10) whether any given sequence $(h_j)_{j=1}^\infty \subset C^\infty_0(\Gamma_1 \times (0, t_0 - \delta))$ satisfies conditions (C1) and (C2).

(ii) If $(h_j)_{j=1}^\infty \subset C^\infty_0(\Gamma_1 \times (0, t_0 - \delta))$ satisfies conditions (C1) and (C2), then

$$\Lambda h(x, T + t) = \lim_{j \to \infty} \Lambda^T_{\Gamma_1, \Gamma_2} h_j(x, T - (\delta - t))$$

in $L^2(\Gamma_2 \times (0, \delta))$.

If (i) holds, then the inner products (10) determine the nonempty set

$$\{(h_j)_{j=1}^\infty \subset C^\infty_0(\Gamma_1 \times (0, t_0 - \delta)) \mid (C1) \text{ and } (C2) \text{ hold}\},$$

and if (ii) holds, then any sequence in this set together with $\Lambda^T_{\Gamma_1, \Gamma_2}$ determines $\Lambda h$ as a function in $L^2(\Gamma_2 \times (0, T + \delta))$. Thus the claim of the lemma is proved after proving (i) and (ii). Note that we are using here a similar kind of argument as in the proof of theorem 4.

Let us begin the proof of claim (i) by showing that (C1) is equivalent to

(C1'). For all $c > 0$

$$\lim_{j \to \infty} \left\{ (-\Delta_d w_j + q w_j, w_j)_{L^2(\partial M)} + c(w_j, w_j)_{L^2(M)} \right\} = 0,$$

where $w_j := v^T_{\Gamma_1 h}(t_0) - v^h(t_0)$.

As supp$(Y_{-\delta} h_j) \subset \Gamma_1 \times (0, t_0)$ and supp$(h) \subset \Gamma_1 \times (0, t_0)$, we have that $w_j|_{\partial M} = 0$. Hence

$$-(\Delta_d w_j, w_j)_{L^2(\partial M)} = (dw_j, dw_j)_{L^2(\partial M)},$$

where $d$ is the exterior derivative on $M$. If (C1) holds, then

$$\left| (-\Delta_d w_j + q w_j, w_j)_{L^2(\partial M)} + c(w_j, w_j)_{L^2(M)} \right| \leq \|dw_j\|_{L^2(\partial M)}^2 + \|q + c\|_{L^\infty(M)} \|w_j\|_{L^2(M)}^2 \to 0, \quad \text{as } j \to \infty.$$

For large enough $c > 0$ there is a constant $c_0 > 0$ such that $q + c \geq c_0$. Hence if (C1') holds, then

$$\|dw_j\|_{L^2(\partial M)}^2 + c_0 \|w_j\|_{L^2(M)}^2 \leq \left| (-\Delta_d w_j + q w_j, w_j)_{L^2(\partial M)} + c(w_j, w_j)_{L^2(M)} \right| \to 0, \quad \text{as } j \to \infty,$$

and (C1) holds. Therefore (C1) and (C1') are equivalent.
Next we observe that
$$\partial_s^2 \left( v^{j} (Y_{hj} - h) (t_0), v^{h} (t_0) \right)_{L^2(M)} = \left( -\frac{\partial^2}{\partial s^2} w_j, w_j \right)_{L^2(M)} = ( -\Delta g + q ) w_j, w_j )_{L^2(M)}.$$
Hence condition (C1’) can be verified for the given functions \((h_j)_{j=1}^{\infty}\) using the inner products (10).

Similarly,
$$\partial_t \partial_s \left( v^{j} (Y_{hj} - h) (t_0), v^{h} (t_0) \right)_{L^2(M)} = \left( \partial_t \partial_s w_j, \partial_t \partial_s w_j \right)_{L^2(M)}$$
and condition (C2) can be verified for the given functions \((h_j)_{j=1}^{\infty}\) using the inner products (10). Thus claim (i) is proved.

Let us show claim (ii). As \(v^{j} - v^h = 0\) on \(\partial M \times [t_0, \infty)\), conditions (C1) and (C2) together with the continuous dependence of the solution of the wave equation on the initial data, see e.g. [25, theorem 2.30], give
$$\lim_{j \to \infty} \partial_t (v^{j} (t, f, h) - v^h (t, f, h)) = 0, \quad \text{in } L^2(\partial M \times (t_0, \infty)).$$
Hence
$$\Lambda h(x, T + t) = \lim_{j \to \infty} (\Lambda Y_{hj} (x, T + t) = \lim_{j \to \infty} \Lambda^T_{Y_{hj}} (x, T - (\delta - t)) \quad \text{in } L^2(\Gamma_2 \times (0, \delta)).$$
This proves claim (ii).

Next we prove the last of the three main theorems formulated in the introduction.

**Proof of theorem 3.** By lemma 1 the operators \(\Lambda^T_{\Gamma_1, \Gamma_2}, \Lambda^T_{\Gamma_1, \Gamma_3}, \Lambda^T_{\Gamma_2, \Gamma_3}\) determine the operators
$$\Lambda^T_{\Gamma_1, \Gamma_2}, \quad p, q = 1, 2, 3, \quad p \neq q.$$ (11)

We use the time delay operator \(Y_s\) defined in the proof of lemma 5. Define
$$B(f, h) := \int_0^T \int_{\partial M} (\partial_s v^f - v^h - \partial_t v^h) dS_t dt, \quad f, h \in C_0^\infty(\partial M \times (0, \infty)),$$
and let \(t_0 := T/2\). We recall the Blagovestenchkii identity [25, lemma 4.16], originating from [7],
$$\left( v^f (t_0), v^h (t_0) \right)_{L^2(M)} = \frac{1}{2} \int_{-t_0}^{t_0} (\text{sign } s) B(Y_{h - s}, f, Y_{h - s} - h) ds,$$
where \(f, h \in C_0^\infty(\partial M \times (0, T))\). By this identity, the operators (11) determine the inner products
$$\left( v^f (t_0), v^h (t_0) \right)_{L^2(M)} = \left( v^f (t_0), v^h (t_0) \right)_{L^2(M)} = \left( v^f (t_0), v^h (t_0) \right)_{L^2(M)} = \left( v^f (t_0), v^h (t_0) \right)_{L^2(M)} = \left( v^f (t_0), v^h (t_0) \right)_{L^2(M)} = \left( v^f (t_0), v^h (t_0) \right)_{L^2(M)}.$$ (12)
for \(p, q = 1, 2, 3, \quad p \neq q\).

Next we will show that the operators (11) determine the inner products (12) also for \(p = q, \quad p = 1, 2, 3\).

Let \(f \in C_0^\infty(\Gamma_1 \times (0, T))\) and consider a sequence \((f_j)_{j=1}^{\infty} \subset C_0^\infty(\Gamma_3 \times (0, T))\) satisfying the following two conditions.

(D1) For all \(h \in C_0^\infty(\Gamma_1 \times (0, T))\)
$$\lim_{j \to \infty} \left( v^{j} (t_0) - v^{Y_{hj}} (t_0), v^h (t_0) \right)_{L^2(M)} = 0.$$

(D2) The sequence \((f_j)_{j=1}^{\infty}\) is bounded in \(L^2(\Gamma_3 \times (0, T))\).
By assumption (A), there is $\tilde{f} \in L^2(\Gamma_3 \times (0, T))$ such that $v^j(t_0) = v^{\tilde{j}}(t_0)$. Moreover, as $C_0^\infty(\Gamma_3 \times (0, T)) \subset L^2(\Gamma_3 \times (0, T))$ is dense, there is a sequence $(f_j)_{j=1}^\infty \subset C_0^\infty(\Gamma_3 \times (0, T))$ such that
$$\lim_{j \to \infty} f_j = \tilde{f}, \quad \text{in} \quad L^2(\Gamma_3 \times (0, T)).$$

By [25, lemma 2.42], $v^{\tilde{j}}(t_0) \to v^{\tilde{j}}(t_0)$ in $L^2(M)$ as $j \to \infty$. Hence a sequence satisfying conditions (D1) and (D2) exists.

Let us next show that
$$\|v^j(t_0)\|_{L^2(M)} = \lim_{j \to \infty} (v^{\tilde{j}}(t_0), v^j(t_0))_{L^2(M)}. \quad (13)$$

As $t_0 > 2d(x, y)$ for all $x \in \Gamma_1$ and $y \in M$ [25, theorem 3.10] gives that the set
$$\{v^p(t_0) \mid h \in C_0^\infty(\Gamma_1 \times (0, T))\}$$
is dense in $L^2(M)$. Let $\epsilon > 0$ and choose $h \in C_0^\infty(\Gamma_1 \times (0, T))$ such that
$$\|v^j(t_0) - v^h(t_0)\|_{L^2(M)} < \epsilon.$$

By [25, lemma 2.42] there is $C > 0$, and by condition (D1) there is $J \in \mathbb{N}$ such that for $j \geq J$
$$\|(v^j(t_0) - v^{\tilde{j}}(t_0), v^j(t_0))_{L^2(M)}\| \leq \|(v^p(t_0) - v^{\tilde{j}}(t_0), v^h(t_0))_{L^2(M)}\|$$
$$+ \|(v^j(t_0) - v^{\tilde{j}}(t_0), v^h(t_0))_{L^2(M)}\|$$
$$\leq C\|f - f_j\|_{L^2(M \times (0, T))} \epsilon + \epsilon.$$

By condition (D2)
$$\sup_{j \in \mathbb{N}} \|f - f_j\|_{L^2(M \times (0, T))} < \infty.$$

Hence equation (13) is valid.

By [25, theorem 3.10] the functions $v^j(t_0)$, $f \in C_0^\infty(\Gamma_2 \times (0, T))$, are dense in $L^2(M)$. Hence
$$\|v^h(t_0)\|_{L^2(M)} = \sup (v^h(t_0), v^j(t_0))_{L^2(M)} \quad (14)$$
where $h \in C_0^\infty(\Gamma_p \times (0, T))$, $p = 1, 3$, and the supremum is taken over all $f \in C_0^\infty(\Gamma_2 \times (0, T))$ such that $\|v^h(t_0)\|_{L^2(M)} = 1$.

Condition (D1) can be verified for any $f$ and $(f_j)_{j=1}^\infty$ using the inner products (12) for $p = 2, 3, q = 1$. Therefore, these inner products determine for any $f \in C_0^\infty(\Gamma_2 \times (0, T))$ the nonempty set
$$\{(f_j)_{j=1}^\infty \subset C_0^\infty(\Gamma_3 \times (0, T)) \mid (D1), (D2) hold\}.$$

By equation (13) any sequence in this set together with inner products (12) for $p = 3$ and $q = 2$ determines $\|v^j(t_0)\|_{L^2(M)}$.

As $f \in C_0^\infty(\Gamma_3 \times (0, T))$ can be chosen freely, the inner products (12) for $p = 2, 3, q = 1$ and for $p = 3, q = 2$ together with polarization identity determine the inner products (12) for $p = q = 2$.

Equation (14), polarization identity and the inner products (12) for $p = 1, 2, 3, q = 2$ determine the inner products (12) for $p = q = 1, 3$.

Therefore, the operators (11) determine the inner products
$$\langle v^j(t_0), v^h(t_0) \rangle_{L^2(M)}, \quad f, h \in C_0^\infty(\Gamma_p \times (0, T)), \quad p = 1, 2, 3. \quad (15)$$
Choose \( \delta \in (0, t_0 - T^*) \), where \( T^* \) is defined as in lemma 5. By lemma 5 the operators (11) and the inner products (15) determine the operators

\[
\Lambda_{\Gamma_{p}, \Gamma_{q}}, \quad p, q = 1, 2, 3, \quad p \neq q.
\]

Repeating this construction, we see that the operators

\[
\Lambda_{\Gamma_{p}, \Gamma_{q}}, \quad p, q = 1, 2, 3, \quad p \neq q,
\]

are determined for all \( m \in \mathbb{N} \). The claim follows from theorem 2. \( \square \)

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Appendix

Next we prove lemma 1 stating that the operator \( \Lambda_{\Gamma_{1}, \Gamma_{2}} \) determines the operator \( \Lambda_{\Gamma_{2}, \Gamma_{1}} \).

Proof of lemma 1. Define \( u := Rv^{Rf} \), where \( v_{Rf} \) is the solution of equation (3) with the boundary data \( Rf \in C_{\infty}^{0}(\Gamma_{1} \times (0, T)) \). Then \( u(x, t) = v^{Rf}(x, T - t) \) satisfies the equation

\[
\partial_{t}^{2} u + a(x, D)u = 0, \quad \text{in} \quad M \times (0, T),
\]

\[
u|_{\partial M \times (0, T)} = f,
\]

\[
u|_{t=T} = \partial_{t}u|_{t=T} = 0.
\]

Integration by parts gives

\[
( f, \Lambda_{\Gamma_{2}, \Gamma_{1}} h )_{L^{2}(\partial M \times (0, T))} - (R \Lambda_{\Gamma_{1}, \Gamma_{2}} Rf, h )_{L^{2}(\partial M \times (0, T))} = \int_{0}^{T} \int_{\partial M} (u(x, t)\partial_{t}v^{h}(x, t) - (\partial_{t}v^{Rf})(x, T - t)v^{h}(x, t))dS_{g}(x)dt
\]

\[
= \int_{0}^{T} \int_{M} (u(x, t)\Delta_{g}v^{h}(x, t) - (\Delta_{g}u)(x, t)v^{h}(x, t))dV_{g}(x)dt
\]

\[
= \int_{0}^{T} \int_{M} (u(x, t)\partial_{x}^{2}v^{h}(x, t) - (\partial_{x}^{2}u)(x, t)v^{h}(x, t))dV_{g}(x)dt
\]

\[
= \left[ \int_{M} (u(x, t)\partial_{x}v^{h}(x, t) - (\partial_{x}u)(x, t)v^{h}(x, t))dV_{g}(x) \right]_{t=0}^{t=T} = 0,
\]since \( u|_{t=T} = \partial_{t}u|_{t=T} = 0 \) and \( v^{f}|_{t=0} = \partial_{t}v^{f}|_{t=0} = 0. \) \( \square \)

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