Learning Deep Models: Critical Points and Local Openness

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Abstract

With the increasing popularity of non-convex deep models, developing a unifying theory for studying the optimization problems that arise from training these models becomes very significant. Toward this end, we present in this paper a unifying landscape analysis framework that can be used when the training objective function is the composite of simple functions.

Using the local openness property of the underlying training models, we provide simple sufficient conditions under which any local optimum of the resulting optimization problem is globally optimal. We first completely characterize the local openness of the symmetric and non-symmetric matrix multiplication mapping. Then we use our characterization to: 1) provide a simple proof for the classical result of Burer-Monteiro and extend it to non-continuous loss functions. 2) Show that every local optimum of two layer linear networks is globally optimal. Unlike many existing results in the literature, our result requires no assumption on the target data matrix $Y$, and input data matrix $X$. 3) Develop a complete characterization of the local/global optima equivalence of multi-layer linear neural networks. We provide various counterexamples to show the necessity of each of our assumptions. 4) Show global/local optima equivalence of over-parameterized non-linear deep models having a certain pyramidal structure. In contrast to existing works, our result requires no assumption on the differentiability of the activation functions and can go beyond “full-rank” cases.

Keywords — Deep Learning, Neural Network, Local Openness, Non-convex, Global optima

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1 Introduction

Deep learning is an inference tool that has recently led to significant practical success in various fields ranging from computer vision to natural language processing. Despite its wide empirical use, the theoretical understanding of the landscapes of the optimization problems corresponding to the underlying neural network architecture models is still very limited. While some recent works have tried to explain these successes through the lens of expressivity by showing the power of these models in learning large class of mappings, other works find the root of the success in the generalizability of these models from learning perspective.

From an optimization perspective, training deep models requires solving non-convex optimization problems, where non-convexity arises from the “deep” structure of the model. In fact, it has been shown by [11] that training neural networks to global optimality is NP-complete in the worst case (even for a simple three-node network). Despite this worst case barrier, the practical success of deep learning may suggest a special structure in the landscapes of the underlying optimization problems. In particular, [14] uses spin glass theory and empirical experiments to show that local optima of deep neural network optimization problems are close to the global optima.

In an effort to better understand the landscape of training deep neural networks, [22, 27, 40, 20, 23, 18] studied deep linear neural networks and provided sufficient conditions under which critical points (or local optimal points) of the training optimization problems are globally optimal. Particularly, [18] and [23] show that every local optimum of a deep linear neural network is globally optimal when the widths of intermediate layers are wider than those of the input or output layer. The local/global equivalence for deep linear neural networks was also established by [27] under the assumption that the input matrix $X$ and label matrix $Y$ are both full row rank. Under similar assumptions, [40] show that every critical point of a deep linear network is a global optimum.

By providing multiple examples of non-linear network structures, [11, 15] show that this local/global equivalence cannot be generally extended to deep non-linear networks. Despite the existence of spurious local minima in non-linear networks, multiple works have shown that with over-parameterization and proper random initialization, local optima of the resulting optimization problems can be computed using local search procedures. These results are algorithm-dependent and require certain assumptions on the distribution of the input data. Moreover, such results either assume specific activation functions [38, 35, 16, 30] or apply to semi-unrealistic wide networks [2, 26, 1, 17, 46, 3, 24, 31, 4, 15]. For instance, [24, 31] showed that no set-wise strict local minima exist when the last layer has more neurons than the number of samples. With modest over-parameterization, [32] show that (stochastic) gradient descent with random initialization converges to a nearly global solution for neural networks with smooth activation functions.

Despite the abundance in research studying the landscape of deep optimization problems, many of the existing results are problem-specific or algorithmic-dependent. In this paper, we develop a unifying theoretical framework for studying the landscape of non-convex optimization problems that are composition of multiple simple mappings. The theoretical framework harnesses the concept of local openness from differential geometry to provide sufficient conditions under which local optima of the objective function are globally optimal. Specifically, consider the general optimization problem

$$\min_{w \in W} \ell(F(w)), \tag{1}$$

where $F: W \mapsto Z$ is a mapping and $\ell: Z \mapsto \mathbb{R}$ is the loss function. We define the auxiliary optimization problem

$$\min_{z \in Z} \ell(z), \tag{2}$$

where $Z$ is the range of the mapping $F$. In this paper we analyze the local openness of several popular mapping $F$ to establish a connection between the local optima of the optimization problems $[1]$ and $[2]$. This connection is then used to study the local/global equivalence of $[1]$. Our contributions are summarized below:

- We completely characterize the local openness of the matrix multiplication mappings $M(W_1, W_2) \equiv W_1 W_2$ and $M_+(\mathcal{W}) \equiv \mathcal{W} \mathcal{W}^\top$; extending the results shown in [9]. These mappings naturally appear in several practical problems such as non-convex matrix factorization, Burer-Monteiro approach for semi-definite programming, training deep neural networks, and matrix completion. Our openness results were utilized to establish the local/global equivalence in several non-convex models including low-rank matrix recovery, matrix completion, multi-linear neural networks, and hierarchical non-linear deep neural networks.
• We provide a set of necessary and sufficient conditions on the architecture of multi-linear neural networks for local/global optima equivalence. When those conditions are met, all local optimal points are globally optimal; otherwise there exists local optima that are not global. Unlike many existing results, our framework requires no assumptions on the adopted optimization algorithm nor on the probability distribution of the input data. More specifically, our method analyzes the underlying landscapes by studying the structure of the network regardless of what algorithm or input data are adopted for training. Moreover, all conditions required for local/global equivalence in existing literature are satisfied by our conditions.

• We use our framework to study the local/global equivalence of non-linear neural networks with pyramidal structure; networks with deeper layers having a lower number of neurons. We show that for continuous and strictly monotone activation functions, every local minimum $W$ with all weight matrices $W_i$'s being full row rank is a global minimum. Unlike the results in [29], our results hold for non-differentiable activation and loss functions.

The rest of the paper is organized as follows. In Section 2 we detail our proposed framework that utilizes local openness property to provide simple sufficient conditions under which deep learning optimization problems satisfy local/global optima equivalence. In Section 3 we completely characterize the local openness of matrix multiplication mappings that naturally appear in deep learning models. In Section 4 we show local/global optima equivalence for non-linear deep models having a certain pyramidal structure. Moreover, in Section 5 we study the equivalence of local and global optima in two layer linear networks. We extend our study to multi-layer linear neural networks in Section 6. Finally, Section 7 concludes the paper with a brief discussion. Before proceeding with our results, we define the following notation.

### 1.1 Notation

First, we use $A_{li}$ and $A_{jl}$ to denote the $l^{th}$ row and $l^{th}$ column of the matrix $A$ respectively. We denote by $I$ the identity matrix and by $I_d \in \mathbb{R}^{d \times d}$ the $d \times d$-dimensional identity matrix. Let $\|A\|$, $\mathcal{N}(A)$, $\mathcal{C}(A)$, and $\text{rank}(A)$ be respectively the Frobenius norm, null-space, column-space, and rank of the matrix $A$. Given subspaces $U$ and $V$, we say $U \perp V$ if $U$ is orthogonal to $V$, and $U = V^\perp$ if $U$ is the orthogonal complement of $V$. We say matrix $A \in \mathbb{R}^{d_1 \times d_0}$ is rank deficient if $\text{rank}(A) < \min\{d_1, d_0\}$, and full rank if $\text{rank}(A) = \min\{d_1, d_0\}$. We call a point $W = (W_h, \ldots, W_1)$, with $W_i \in \mathbb{R}^{d_i \times d_{i-1}}$, non-degenerate if $\text{rank}(\prod_{i=1}^h W_i) = \min_{0 \leq i \leq h} d_i$, and degenerate if $\text{rank}(\prod_{i=1}^h W_i) < \min_{0 \leq i \leq h} d_i$. We also say a point $W$ is a second order saddle point (strict saddle point) of an unconstrained optimization problem if the gradient of the objective function is zero at $W$ and the Hessian of the objective function at $W$ has a negative eigenvalue.

### 2 Proposed Framework

Consider the general optimization problem defined in (1) and its corresponding auxiliary problem (2). Since problem (2) minimizes the function $\ell(\cdot)$ over the range of the mapping $F$, the global optimal objective values of problems (1) and (2) are the same. Moreover, there is a clear relation between the global optimal points of the two optimization problems through the mapping $F$. However, the connection between the local optima of the two optimization problems is not clear. This connection, in particular, is important when the local optima of (2) are either globally optimal or close to optimal. In what follows, we establish the connection between the local optima of the optimization problems (1) and (2) under simple sufficient conditions. This connection is then used to study the relation between local and global optima of (1) and (2) for various non-convex learning models. In summary, our strategy is to map the original optimization problem (1) to a generally simpler auxiliary problem (2) for which the underlying landscape has a special structure (example all local optima are global). We then show that this special structure holds for (1) by establishing the connection between the local optima of the two problems. We start by defining the following important concepts:

**Definition 1. Relative openness:** Consider a set $S \in \mathbb{R}^n$. A subset $U \subseteq S$ is said to be an open set relative to $S$ if for any $u \in U$, there exists $\delta > 0$ such that $B_\delta(u) \cap S \subseteq U$. 

**Definition 2. Open mapping:** A mapping $F : W \to Z$ is said to be open, if for every open set $U \in W$, $F(U)$ is relatively open in $Z$. 

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Definition 3. **Locally open mapping:** A mapping $\mathcal{F}(\cdot)$ is said to be locally open at $w$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbb{B}_\delta(\mathcal{F}(w)) \cap R_F \subseteq \mathcal{F}(\mathbb{B}_\epsilon(w))$.

Here $\mathbb{B}_\epsilon(w) \subseteq \mathcal{W}$ is an open ball with radius $\epsilon$ centered at $w$, $\mathbb{B}_\delta(\mathcal{F}(w)) \subseteq \mathcal{Z}$ is a ball of radius $\delta$ centered at $\mathcal{F}(w)$, and $R_F$ is the range (imageset) of the mapping $\mathcal{F}$. Intuitively, we say a mapping is locally open at $w$ if for any small perturbation $\tilde{z} \in R_F$ of $\bar{z} = \mathcal{F}(w)$, there exists $\bar{w}$ which is a small perturbation of $w$, such that $\bar{z} = \mathcal{F}(\bar{w})$. This directly implies that all invertible functions are locally open at every point. By definition, openness of a mapping is stronger than local openness. Furthermore, it directly follows that a mapping is locally open everywhere if and only if it is open. A useful property of (locally) open mappings is stated below.

**Property 1.** The composition of two (locally) open maps is (locally) open at a given point.

The following simple intuitive observation which establishes the connection between the local optima of (1) and (2), is a major building block of our analyses.

**Observation 1.** Suppose that $\mathcal{F}(\cdot)$ is locally open at $\bar{w}$. If $\bar{w}$ is a local minimum of problem (1), then $\bar{z} = \mathcal{F}(\bar{w})$ is a local minimum of problem (2). Furthermore, if all local minima of the auxiliary problem (2) are global, then every local minimum of (1) is globally optimal.

**Proof.** Let $\bar{w}$ be a local minimum of problem (1). Then there exists an $\epsilon > 0$ such that $\ell(\mathcal{F}(\bar{w})) \leq \ell(\mathcal{F}(w))$, $\forall w \in \mathbb{B}_\epsilon(\bar{w})$. By the definition of local openness, $\exists \delta > 0$ such that $\mathbb{B}_\delta(\bar{z}) \cap R_F \subseteq \mathcal{F}(\mathbb{B}_\epsilon(\bar{w}))$ with $\bar{z} = \mathcal{F}(\bar{w})$. Therefore, $\ell(\bar{z}) \leq \ell(\bar{z})$, $\forall z \in \mathbb{B}_\delta(\bar{z}) \cap R_F$, which implies $\bar{z}$ is a local minimum of problem (2). Furthermore, assume $\bar{w}$ is a local minimum of (1) and let $\ell_{\text{min}}$ be the optimal objective value of problems (1) and (2), then

$$\ell(\mathcal{F}(\bar{w})) = \ell(\bar{z}) = \ell_{\text{min}},$$

where the second equality holds by assuming that every local minimum of (2) is global. We conclude that $\bar{w}$ is a global minimum of (1).

The above observation can be used to map multiple local optima of the original problem (1) to one local optimum of the auxiliary problem (2), and potentially make the problem easier to analyze. This mapping is particularly interesting in neural networks since permuting the neurons and the corresponding weights in each layer does not change the objective function. Hence, by nature, the underlying landscapes of these optimization problems have multiple local/global optima. However, collapsing these multiple local optima to one could potentially simplify the problem. In other words, instead of analyzing the original landscape with multiple disconnected local optima, we analyze the landscape of the auxiliary problem. Let us clarify this point through the following simple examples:

![Figure 1: Two local minima $w = -1$ and $w = +1$ in (a) are mapped to a single local minimum $z = 1$ in (b).](image-url)
Example 4. Consider the optimization problem

$$\min_{w \in \mathbb{R}} (w^2 - 1)^2,$$  \hspace{1cm} (3)

and its corresponding auxiliary problem

$$\min_{z \geq 0} (z - 1)^2.$$ \hspace{1cm} (4)

The plots of these two problems can be found in Figure 1a and Figure 1b. Since $F(w) = w^2$ is an open mapping in its range, it follows from Observation 1 that every local minimum in problem (3) is mapped to a local minimum of problem (4). Thus the two local minima $w = -1$ and $w = +1$ in (3) are mapped to a single local minimum $z = 1$ of problem (4). Moreover, since the optimization problem (4) is convex, the local minimum is global; and hence the original local optima $w = -1$ and $w = +1$ should be both global despite non-convexity of (3).

Example 5. Another example is related to the widely used matrix multiplication mapping $W_1 W_2$. Let $(\bar{W}_1, \bar{W}_2)$ be a local minimum of the optimization problem

$$\min_{W_1, W_2} \ell(W_1 W_2).$$

Then, any point in the set $S \triangleq \{ (\bar{W}_1 Q_1, Q_2 \bar{W}_2) \mid Q_1 Q_2 = I \}$ is also a local minimum. If the matrix product $W_1 W_2$ is locally open at the point $(\bar{W}_1, \bar{W}_2)$, then all points in $S$ are mapped to a single local minimum $\bar{Z} = \bar{W}_1 \bar{W}_2$ in the corresponding auxiliary problem. A simple one-dimensional example is plotted in Figures 2a and 2b.

![Figure 2: All the points in the set \{(w_1, w_2) \mid w_1 w_2 = 1\} are local minima in (a) and are mapped to a single local minimum $z = 1$ in (b).](image)

Example 6. To demonstrate the sufficiency of the local openness property in relating the local optima of the original and constructed auxiliary problem, we provide the following simple one-dimensional example. Consider the optimization problem

$$\min_{w \in \mathbb{R}} ((w^3 - 2w^2 + w - 1) - 1)^2,$$  \hspace{1cm} (5)

and its corresponding auxiliary problem

$$\min_{z \in \mathbb{R}} (z - 1)^2.$$ \hspace{1cm} (6)

Plots of these two problems can be found in Figure 3a and Figure 3b. Since $F(w) = w^3 - 2w^2 + w - 1$ is not locally open at the local optimum $w = 1/3$, the point $F(1/3) = -23/27$ is not a local optimum of the auxiliary problem.

Observation 1 motivates us to study the local openness of mappings that appear in widely used optimization problems. A mapping that appears naturally in machine learning optimization problems is matrix multiplication (multiplication of matrices). In the next section, we will focus on characterizing the local openness of this mapping.
Figure 3: Point $w = 1/3$ is a local minima in (a) and is mapped to the point $F(1/3) = -23/27$ which is not a local minimum in (b).

3 Local Openness of Matrix Multiplication Mappings

Motivated by Observation 1, we study the local openness/openness of matrix multiplication mappings. One example that is used in the famous Burer-Monteiro approach for semi-definite programming [13], is the symmetric matrix multiplication mapping $M_+ : \mathbb{R}^{n \times k} \mapsto \mathcal{R}_{M_+}$ defined as

$$M_+(W) \triangleq WW^T,$$

where $\mathcal{R}_{M_+} \triangleq \{ Z \in \mathbb{R}^{n \times n} \mid Z \succeq 0, \text{rank}(Z) \leq \min\{n, k\}\}$ is the range of $M_+$.

Another mapping that is widely used in many optimization problems, such as deep neural networks and matrix completion, is the non-symmetric matrix multiplication mapping $M : \mathbb{R}^{m \times k} \times \mathbb{R}^{k \times n} \mapsto \mathcal{R}_M$ defined as

$$M(W_1, W_2) \triangleq W_1W_2,$$  \hspace{1cm} (7)

where $\mathcal{R}_M \triangleq \{ Z \in \mathbb{R}^{m \times n} \mid \text{rank}(Z) \leq \min(m, n, k)\}$ is the range of the mapping $M$. The matrix multiplication mapping $M(W_1, W_2)$ naturally appears in deep models and is widely used as a non-convex factorization for rank constrained problems, see [39, 11, 19, 36, 37]. To our knowledge, the complete characterization of the local openness of this mapping has not been studied in the optimization literature before. Similarly, the symmetric matrix multiplication mapping $M_+(W)$ is widely used as a non-convex factorization in semi-definite programming (SDP), see [13, 43, 33, 12], and the characterization of the openness of this mapping remains unsolved.

While the classical open mapping theorem in [34] states that surjective continuous linear operators are open, this is not true for general bilinear mappings such as matrix product. In fact, by providing a simple counterexample of a bilinear mapping that is not open, [21] shows that the linear case cannot be generally extended to multilinear maps. Several papers, see [5, 6, 8], investigate this bilinear mapping and provide a characterization of the points where this mapping is open. The more general matrix multiplication mapping $M$ was studied in [9]. The former paper provides necessary and sufficient conditions under which the mapping is locally open in $\mathbb{R}^{m \times n}$. However, in our framework the (relative) local openness should be studied with respect to the range of the mapping $\mathcal{R}_M$ which can be different from $\mathbb{R}^{m \times n}$ when $k < \min\{m, n\}$.

For $W_1 \in \mathbb{R}^{m \times k}$ and $W_2 \in \mathbb{R}^{k \times n}$ with $k \geq \min\{m, n\}$, the range of the mapping $M(W_1, W_2) = W_1W_2$ is the entire space $\mathbb{R}^{m \times n}$. In this case, which we refer to as the full rank case, [9 Theorem 2.5] provides a complete characterization of the pairs $(W_1, W_2)$ for which the mapping is locally open. However, when $k < \min\{m, n\}$, which we refer to as the rank-deficient case, the mapping is not locally open in $\mathbb{R}^{m \times n}$, but can still be locally open.
in $\mathcal{R}_M$. For a simple example, consider $\mathbf{W}_1 = [1 \ 2]^T$ and $\mathbf{W}_2 = [1 \ 1]$. In this example there does not exist $\mathbf{W}_1$, $\mathbf{W}_2$ perturbations of $\mathbf{W}_1$ and $\mathbf{W}_2$ respectively such that $\mathbf{W}_1\mathbf{W}_2 = \mathbf{Z}$ when $\mathbf{Z}$ is a full rank perturbation of $\mathbf{Z} = \mathbf{W}_1\mathbf{W}_2$. However, for any rank 1 perturbation $\mathbf{Z}$, we can find a perturbed pair $(\mathbf{W}_1, \mathbf{W}_2)$ such that $\mathbf{Z} = \mathbf{W}_1\mathbf{W}_2$.

In this section we provide a complete characterization of points $(\mathbf{W}_1, \mathbf{W}_2)$ for which the mapping $\mathcal{M}$ is locally open when $k < \min\{m, n\}$. Moreover, we show in Theorem $9$ that the symmetric matrix multiplication $\mathcal{M}_+$ is open in its range $\mathcal{R}_{\mathcal{M}_+}$. The proofs of these theorems can be found in Appendices $A$ and $B$ respectively. We start by restating the main result in $9$.

**Proposition 7.** [9] *Theorem 2.5 Rephrased* Let $\mathcal{M}(\mathbf{W}_1, \mathbf{W}_2) = \mathbf{W}_1\mathbf{W}_2$ denote the matrix multiplication mapping with $\mathbf{W}_1 \in \mathbb{R}^{m \times k}$ and $\mathbf{W}_2 \in \mathbb{R}^{k \times n}$. Assume $k \geq \min\{m, n\}$. Then the following statements are equivalent:

1. $\mathcal{M}(\cdot, \cdot)$ is locally open at $(\mathbf{W}_1, \mathbf{W}_2)$.
2. $\exists \tilde{\mathbf{W}}_1 \in \mathbb{R}^{m \times k}$ such that $\tilde{\mathbf{W}}_1\mathbf{W}_2 = 0$ and $\tilde{\mathbf{W}}_1$ is full row rank.
3. $\dim \left( \mathcal{N}(\tilde{\mathbf{W}}_1) \cap \mathcal{C}(\mathbf{W}_2) \right) \leq k - m$ or $n - (\text{rank}(\mathbf{W}_2) - \dim \left( \mathcal{N}(\tilde{\mathbf{W}}_1) \cap \mathcal{C}(\mathbf{W}_2) \right)) \leq k - \text{rank}(\mathbf{W}_1)$.

The above proposition provides a checkable condition which completely characterizes the local openness of the mapping $\mathcal{M}$ at different points when the range of the mapping is the entire space. Now, let us state our result that characterizes the local openness of the mapping $\mathcal{M}$ in its range, i.e., when $k < \min\{m, n\}$.

**Theorem 8.** Let $\mathcal{M}(\mathbf{W}_1, \mathbf{W}_2) = \mathbf{W}_1\mathbf{W}_2$ denote the matrix multiplication mapping with $\mathbf{W}_1 \in \mathbb{R}^{m \times k}$ and $\mathbf{W}_2 \in \mathbb{R}^{k \times n}$. Assume $k < \min\{m, n\}$. Then if rank$(\mathbf{W}_1) \neq$ rank$(\mathbf{W}_2)$, $\mathcal{M}(\cdot, \cdot)$ is not locally open at $(\mathbf{W}_1, \mathbf{W}_2)$. Else, if rank$(\mathbf{W}_1) = \text{rank}(\mathbf{W}_2)$, then the following statements are equivalent:

1. $\exists \tilde{\mathbf{W}}_1 \in \mathbb{R}^{m \times k}$ such that $\tilde{\mathbf{W}}_1\mathbf{W}_2 = 0$ and $\tilde{\mathbf{W}}_1$ is full column rank.
2. $\exists \tilde{\mathbf{W}}_2 \in \mathbb{R}^{k \times n}$ such that $\mathbf{W}_1\tilde{\mathbf{W}}_2 = 0$ and $\tilde{\mathbf{W}}_2$ is full row rank.
3. $\dim \left( \mathcal{N}(\mathbf{W}_1) \cap \mathcal{C}(\mathbf{W}_2) \right) = 0$.
4. $\dim \left( \mathcal{N}(\mathbf{W}_2^T) \cap \mathcal{C}(\mathbf{W}_1^T) \right) = 0$.
5. $\mathcal{M}(\cdot, \cdot)$ is locally open at $(\mathbf{W}_1, \mathbf{W}_2)$ in its range $\mathcal{R}_M$.

Note that the proof of Theorem $8$ which can be found in Appendix $A$ is different than the proof of Proposition $7$ as in the former we need to work with the set of low-rank matrices. Besides, the conditions in Theorem $8$ are different than the ones in Proposition $7$. For example, while conditions i) and ii) are equivalent in the rank-deficient case, they are not equivalent in the full-rank case. Moreover, unlike the full-rank case, the condition rank$(\mathbf{W}_1) = \text{rank}(\mathbf{W}_2)$ is necessary for local openness in the former result.

**How much perturbation is needed?** As previously mentioned, local openness can be described in terms of perturbation analysis. In particular, $\mathcal{M}(\cdot, \cdot)$ is said to be locally open at $(\mathbf{W}_1, \mathbf{W}_2)$ if for a given $\epsilon > 0$, there exists $\delta > 0$ such that for any $\tilde{\mathbf{Z}} = \mathbf{Z} + \mathbf{R}_\delta \in \mathcal{R}_M$ with $\|\mathbf{R}_\delta\| \leq \delta$, there exists $\tilde{\mathbf{W}}_1$, $\tilde{\mathbf{W}}_2$ such that $\|\tilde{\mathbf{W}}_1\| \leq \epsilon$, $\|\tilde{\mathbf{W}}_2\| \leq \epsilon$ such that $\tilde{\mathbf{Z}} = (\tilde{\mathbf{W}}_1 + \mathbf{W}_1)(\tilde{\mathbf{W}}_2 + \mathbf{W}_2)$. Given $\epsilon > 0$, we show that for any locally open $(\mathbf{W}_1, \mathbf{W}_2)$ we need to choose $\delta = \Theta(\epsilon)$. The details of our analysis can be found in the proof of Theorem $8$ in Appendix $A$.

Now we state our result for the mapping $\mathcal{M}_+$.

**Theorem 9.** Let $\mathcal{M}_+(\mathbf{W}) = \mathbf{W}\mathbf{W}^T$ be the symmetric matrix multiplication mapping. Then $\mathcal{M}_+(\cdot)$ is open in its range $\mathcal{R}_{\mathcal{M}_+}$.

**How much perturbation is needed?** A perturbation bound for the symmetric matrix multiplication was also derived, details in Appendix $B$. Specifically, given an $\epsilon > 0$, we show that for any $\mathbf{W}$ the chosen $\delta$ is of order $\epsilon$, i.e., $\delta = \Theta(\epsilon)$. 7
Remark 10. Since the mapping $\mathcal{M}_+$ is open in $\mathcal{R}_+=\mathcal{M}$, then by Observation 7 any local minimum of the optimization problem $\min_{W\in\mathcal{W}} \ell(WW^T)$ leads to a local minimum in the optimization problem $\min_{Z\in Z} \ell(Z)$ where $Z = \{Z \mid Z \geq 0, Z = WW^T, W \in \mathcal{W}\}$. Consequently, if every local minimum of the optimization problem on $Z$ is globally optimal, then every local minimum of the first optimization problem is global. This provides a simple and intuitive proof for the Burrr-Montiero result [13, Proposition 2.3]; moreover, it extends it by relaxing the continuity assumption on $\ell(\cdot)$.

Remark 11. It follows from Theorem 8 that when $W_1$ is full column rank and $W_2$ is full row rank, the mapping $\mathcal{M}(\cdot, \cdot)$ is locally open at $(W_1, W_2)$. This special case of our result was observed in other works; see, e.g., [37, Proposition 4.2].

In the next sections, we use our local openness result to characterize the cases where the local optima of various training optimization problems of the form (1) are globally optimal.

4 Non-linear Deep Neural Network with a Pyramidal Structure

In this section, we utilize Observation 1 to study the local/global optima equivalence in non-linear deep neural networks having a specific pyramidal structure. Towards that end, consider the non-linear deep neural network optimization problem with a pyramidal structure

$$\min_{W} \ell(F_h(W))$$

with

$$F_i(W) \triangleq \sigma_i(W_iF_{i-1}(W)), \text{ for } i \in \{2, \ldots, h\}, \text{ and } F_1(W) \triangleq \sigma_1(W_1X)$$

where $\sigma_i(\cdot) : \mathbb{R} \to \mathbb{R}$ is a continuous and strictly monotone activation function applied component-wise to the entries of each layer, i.e., $\sigma_i(A) = [\sigma_i(A_{jk})]_{j,k}$. Here $W = (W_i)_{i=1}^h$ where $W_i \in \mathbb{R}^{d_i \times d_{i-1}}$ is the weight matrix of layer $i$, and $X \in \mathbb{R}^{d_0 \times n}$ is the input training data. In this section, we consider the pyramidal network structure with $d_0 > n$ and $d_i \leq d_{i-1}$ for $1 \leq i \leq h$; see [29] for more details on these types of networks. In their paper, [29] show that every critical point $W$ of problem (8) with $W_i$'s being full row rank is a global optimum when both $\sigma(\cdot)$ and $\ell(\cdot)$ are differentiable. In this section, we relax the differentiability assumption on both the activation and loss functions and establish local/global optima equivalence for problem (8).

We first show that when $X$ is full column rank and the functions $\sigma_i$'s are all continuous and strictly monotone, the image set of the mapping $F_h$ is convex. First notice that the full rankness of $X$ and continuity and strict monotonicity of $\sigma_1$ yield a range of $\mathbb{R}^{d_1 \times n}$ for $F_1(W) = \sigma(W_1X)$. By the same reasoning, we get that the range of the mapping $F_i(W)$ is $\mathbb{R}^{d_i \times n}$. Hence, the image set of the mapping $F_h$ is the full space $\mathbb{R}^{d_n \times n}$ which is clearly convex. Furthermore, if $\ell(\cdot)$ is convex, then every local optimum of the corresponding auxiliary optimization problem (2) is global. We now show that when $W_i$'s are all full row rank and the functions $\sigma_i$'s are all strictly monotone, the mapping $F_h$ is locally open at $W$.

Lemma 12. Assume the functions $\sigma_i(\cdot) : \mathbb{R} \to \mathbb{R}$ are all continuous and strictly monotone. If $W_i$'s are all full row rank, then the mapping $F_h$ defined in (8) is locally open at the point $W = (W_1, \ldots, W_h)$.

Before proving this result, we would like to remark that many of the popular activation functions such as logit, tangent hyperbolic, and leaky ReLU are strictly monotone and satisfy the assumptions of this lemma. Our result shows that using such activation functions on pyramidal neural networks yield underlying landscapes with all local minima having full-rank weight parameters being global. Our result does not hold when using ReLU activation function as it is not an open mapping.

Proof. We prove the result by means of induction. First notice that $F_1$ can be seen as the composition of the $\sigma_1(\cdot)$ and the linear mapping $WX$. Using the strict monotonicity assumption, we get that $\sigma_1(\cdot)$ is invertible. By the definition of open mappings, it follows that $\sigma_1(\cdot)$ is an open mapping. Hence, using the local openness property of linear maps and the openness of $\sigma_1(\cdot)$, the composition property of open maps detailed in Property 2 directly implies that $F_1$ is open. Now assume $F_{k-1}((W_i)_{i=1}^{k-1})$ is locally open at $(W_i)_{i=1}^{k-1}$, then using Proposition 1 for a full rank $W_k$, the mapping $W_kF_{k-1}((W_i)_{i=1}^{k-1})$ is locally open at $(W_k, (W_i)_{i=1}^{k-1})$. Finally, the composition property of open maps and strict monotonicity of $\sigma_k(\cdot)$ imply that $F_k((W_i)_{i=1}^{k})$ is locally open at $(W_i)_{i=1}^{k}$. □
Lemma 12 in conjunction with Observation 1 implies that if $\bar{W}$ is a local optimum of problem (8) with $\bar{W}_i$’s being full row rank, then $\bar{Z} \in F_\lambda(\bar{W})$ is a local optimum of the corresponding auxiliary problem $\min_{Z \in \mathcal{Z}} \ell(Z)$ where $\mathcal{Z}$ is a convex set. Consequently, $\bar{Z}$ is a global optimum of problem (8) when the loss function $\ell(\cdot)$ is convex. Given an oracle that returns a local minima of problem (8), one can check whether $\bar{W}_i$’s are full row rank. If true, we guarantee that the point is globally optimal. Unlike the results in [29], our lemma holds for non-differentiable activation and loss functions. A popular activation function that is strictly monotone and not differentiable is the Leaky ReLU, for which our result holds.

5 Two-Layer Linear Neural Network

Consider the two layer linear neural network optimization problem

$$\min_{\bar{W}} \frac{1}{2} \| \bar{W}_2 \bar{W}_1 \bar{X} - \bar{Y} \|^2,$$

where $\bar{W}_2 \in \mathbb{R}^{d_2 \times d_1}$, and $\bar{W}_1 \in \mathbb{R}^{d_1 \times d_0}$ are weight matrices, $\bar{X} \in \mathbb{R}^{d_0 \times n}$ is the input data, and $\bar{Y} \in \mathbb{R}^{d_2 \times n}$ is the target training data. Using our transformation, the corresponding auxiliary optimization problem can be written as

$$\min_{\bar{Z}} \frac{1}{2} \| \bar{Z} \bar{X} - \bar{Y} \|^2 \quad \text{s.t.} \quad \text{rank}(\bar{Z}) \leq \min\{d_2, d_1, d_0\}. \quad (10)$$

Theorem 2.3] shows that when $\bar{X} \bar{X}^\top$ and $\bar{Y} \bar{Y}^\top$ are full rank, $d_2 \leq d_0$, and when $\bar{Y} \bar{X}^\top (\bar{X} \bar{X}^\top)^{-1} \bar{X} \bar{Y}^\top$ has $d_2$ distinct eigenvalues, every local optimum of (9) is global and all saddle points are second order saddles. While the local/global equivalence result holds for deeper networks, the property that all saddles are second order does not hold in that case. Another result by [40, Theorem 2.2] show that when $\bar{X} \bar{X}^\top$, $\bar{Y} \bar{Y}^\top$, and $\bar{Y} \bar{X}^\top (\bar{X} \bar{X}^\top)^{-1} \bar{X} \bar{Y}^\top$ are full rank, every local optimum of (9) is global. In this section, without any assumptions on both $\bar{X}$ and $\bar{Y}$, we use local openness to show that the latter result holds for 2-layer linear networks. Moreover, we show that every degenerate local optima is global even when replacing the square loss error by a general convex loss function, see Corollary 13. We start by relaxing the full rankness assumption on $\bar{X}$.

Lemma 13. Every local minimum of problem (10) is global.

Proof. Let $r_\bar{X} = \text{rank}(\bar{X})$ and $U_{\bar{X}} \Sigma_{\bar{X}} V_{\bar{X}}^\top$ with $U_{\bar{X}} \in \mathbb{R}^{d_0 \times d_0}$, $\Sigma_{\bar{X}} \in \mathbb{R}^{d_0 \times n}$, and $V_{\bar{X}} \in \mathbb{R}^{n \times n}$ be a singular value decomposition of $\bar{X}$. Then

$$\| \bar{Z} \bar{X} - \bar{Y} \|^2 = \| U_{\bar{X}} \Sigma_{\bar{X}} V_{\bar{X}}^\top - \bar{Y} \|^2$$

$$= \| (U_{\bar{X}} \Sigma_{\bar{X}} V_{\bar{X}}^\top - \bar{Y}) V_{\bar{X}} \|^2$$

$$= \| U_{\bar{X}} [(\Sigma_{\bar{X}})_{:,1:r_{\bar{X}}} 0] - \bar{Y} V_{\bar{X}} \|^2$$

$$= \| U_{\bar{X}} (\Sigma_{\bar{X}})_{:,1:r_{\bar{X}}} - (\bar{Y} V_{\bar{X}})_{:,1:r_{\bar{X}}} \|^2 + \| (\bar{Y} V_{\bar{X}})_{:,r_{\bar{X}}+1:n} \|^2.$$

where the second and third equalities hold since $V_{\bar{X}}^\top V_{\bar{X}} = I$. Since $U_{\bar{X}} (\Sigma_{\bar{X}})_{:,1:r_{\bar{X}}}$ is full column rank, then the linear mapping $U_{\bar{X}} (\Sigma_{\bar{X}})_{:,1:r_{\bar{X}}}$ is open, and

$$\text{rank}(U_{\bar{X}} (\Sigma_{\bar{X}})_{:,1:r_{\bar{X}}}) \leq \min\{\text{rank}(\bar{Z}), r_{\bar{X}}\} \leq \min\{d_2, d_1, d_0, r_{\bar{X}}\}.$$

Consequently, every local minimum of (10) corresponds to a local minimum in problem

$$\min_{\bar{Z} \in \mathbb{R}^{d_2 \times r_{\bar{X}}}} \frac{1}{2} \| \bar{Z} - \bar{Y} \|^2 \quad \text{s.t.} \quad \text{rank}(\bar{Z}) \leq \min\{d_2, d_1, d_0, r_{\bar{X}}\}, \quad (11)$$

where $\bar{Y} = (\bar{Y} V_{\bar{X}})_{:,1:r_{\bar{X}}}$. The result follows using [27, Theorem 2.2].

We next state the main results for problem (9).

Theorem 14. Every local minimum of problem (9) is global. Moreover, every degenerate saddle point of problem (9) is a second order saddle.
Consider the training problem of multi-layer deep linear neural networks:

**Multi-Layer Linear Neural Network**

Relaxing the assumptions on both \( X \) and \( Y \) separately considered the cases of degenerate and non-degenerate critical points. For the former case, we construct a descent direction for critical points that are not global. Such directions are constructed using the null-space of the rank deficient matrices, and can be of independent interest when developing algorithms for training deep neural networks. For the non-degenerate case, it follows by Theorem 9 that the matrix product is locally open at a given non-degenerate local minimum. Then by Observation 1, the local minimum can be mapped to a local minimum of problem (10) which is globally optimal by Lemma 13.

**Corollary 15.** Let the square loss error in (10) be replaced by a general convex loss function \( \ell(\cdot) \). Then every degenerate critical point is either a global minimum or a second order saddle.

**Proof.** The proof of the corollary is relegated to Appendix C.

[27] and [36] show the same result when both \( X \) and \( Y \) are full row rank. Theorem 14 generalizes their results by relaxing the assumptions on both \( X \) and \( Y \).

### 6 Multi-Layer Linear Neural Network

Consider the training problem of multi-layer deep linear neural networks:

\[
\min_{W} \frac{1}{2} \| W_h \cdots W_1 X - Y \|^2. \tag{12}
\]

Here \( W = (W_i)_{i=1}^d \), \( W_i \in \mathbb{R}^{d_i \times d_{i-1}} \) are the weight matrices, \( X \in \mathbb{R}^{d_0 \times n} \) is the input training data, and \( Y \in \mathbb{R}^{d_n \times n} \) is the target training data. Based on our general framework, the corresponding auxiliary optimization problem is given by

\[
\min_{Z \in \mathbb{R}^{d_h \times n}} \frac{1}{2} \| ZX - Y \|^2 \quad \text{s.t.} \quad \text{rank}(Z) \leq d_p \triangleq \min_{0 \leq i \leq d} d_i. \tag{13}
\]

[27] showed that when \( X \) and \( Y \) are full row rank, every local minimum of (12) is global. We show that the full rankness assumption on \( Y \) cannot be simply relaxed by providing the following counterexample:

\[
X = I, \quad \bar{W}_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{W}_2 = [0], \quad \bar{W}_1 = [1 \ 0], \quad Y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]

In this example, the point \( \bar{W} = (\bar{W}_1, \bar{W}_2, \bar{W}_3) \) is a local optimum of a 3-layer deep linear model that is not global. Despite this counterexample, we will provide a set of conditions on the network architecture for which all local minima of (12) are global even if \( Y \) is not full rank. Before proceeding to the proof we define the mapping

\[ M_{i,j}(W_i, \ldots, W_j) : \{W_i, \ldots, W_j\} \to R_{M_{i,j}} \quad \text{for} \ i > j, \]

where \( R_{M_{i,j}} \triangleq \{Z = W_i \cdots W_j \in \mathbb{R}^{d_i \times d_{j-1}} | \text{rank}(Z) \leq \min_{j-1 \leq l \leq d} d_l\} \). We start by re-stating Theorem 3.1 of [27] using our notation.

**Lemma 16.** If \( W \) is non-degenerate, then \( M_{h,1}(W) = W_h \cdots W_1 \) is locally open at \( W \).

**Proof.** We construct a proof by induction on \( h \) to show the desired result. When \( h = 2 \), we either have \( d_1 < \min\{d_2, d_0\} \) or \( d_1 \geq \min\{d_2, d_0\} \). In the first case,

\[
d_1 = \text{rank}(W_2 W_1) \leq \text{rank}(W_1) \leq d_1 \Rightarrow \text{rank}(W_1) = d_1,
\]

and

\[
d_1 = \text{rank}(W_2 W_1) \leq \text{rank}(W_2) \leq d_1 \Rightarrow \text{rank}(W_2) = d_1.
\]

Since \( W_1 \) is full row rank and \( W_2 \) is full column rank, then by Theorem 3 choosing \( \bar{W}_1 = \bar{W}_2 = 0 \) yields \( M_{2,1}(\cdot) \) is locally open at \((W_2, W_1)\). In the second case, either

\[
d_2 = \text{rank}(W_2 W_1) \leq \text{rank}(W_2) \leq d_2 \Rightarrow \text{rank}(W_2) = d_2,
\]

or

\[
d_0 = \text{rank}(W_2 W_1) \leq \text{rank}(W_1) \leq d_0 \Rightarrow \text{rank}(W_1) = d_0.
\]
Thus, either $W_2$ is full row rank or $W_1$ is full column rank, then by Proposition 7, $M_{2,1}(\cdot)$ is locally open at $(W_2, W_1)$. Now assume the result holds for the product of $h$ matrices $M_{h,1}(W)$, we show it is true for $M_{h+1,1}(W)$. Since

$$d_p = \text{rank}(W_h \cdots W_1) \leq \text{rank}(W_{p+1}W_p) \leq d_p \Rightarrow \text{rank}(W_{p+1}W_p) = d_p,$$

then using Proposition 7, we get $M_{p+1,p}(\cdot)$ is locally open at $(W_{p+1}, W_p)$. So we can replace $W_{p+1}W_p$ by a new matrix $Z_p$ with rank $d_p$. Then by induction hypothesis, the product mapping $M_{h+1,1}(W) = W_{h+1} \cdots W_{p+2}Z_pW_{p-1} \cdots W_1$ is locally open at $W$. Since the composition of locally open maps is locally open, the result follows.

We next show that under a set of necessary conditions, every local minimum of problem (12) is global.

**Lemma 17.** Every non-degenerate local minimum of (12) is global minimum.

**Proof.** Suppose $W = (W_h, \ldots, W_1)$ is a non-degenerate local minimum. Then it follows by Lemma 16 that $M_{h,1}$ is locally open at $W$. Then by Observation Z $Z = M_h(W_h, \ldots, W_1)$ is a local optimum of problem (13) which is in fact global by Lemma 13.

As previously mentioned, due to a simple counterexample, we cannot in general relax the full rankness assumption on $Y$. We now determine problem structures for which every degenerate local minimum is global, i.e., (due to Lemma 4) problem structures for which every local minimum is global.

**Theorem 18.** If there does not exist $p_1$ and $p_2$, $1 \leq p_1 < p_2 \leq h - 1$ with $d_h > d_{p_2}$ and $d_0 > d_{p_1}$, then every local minimum of problem (12) is a global minimum.

**Proof.** The proof of the theorem is relegated to Appendix D.

**Remark 19.** Following the same steps of the proof of Theorem 18, we get the same result when replacing the square loss error by a general convex and differentiable function $\ell(\cdot)$. Moreover, if the range of the mapping $M_h$ is the entire space, i.e., $\min_{0 \leq i \leq h} d_i = \min\{d_h, d_0\}$, the auxiliary problem (13) is unconstrained and convex. Then, as we show in Corollary 21, every non-degenerate critical point is global, and every degenerate critical point is either a saddle point or a global minimum; which generalizes [40, Theorem 2.1].

**Remark 20.** Practically, Theorem 18 provides a simple test that uses the network structure to determine whether every local minimum of the underlying landscape is global.

**Corollary 21.** Consider problem (12) with general convex and differentiable loss function $\ell(\cdot)$. When $\min_i d_i = \min(d_h, d_0)$, every non-degenerate critical point is global, and every degenerate critical point is either a saddle point or a global minimum.

**Proof.** Suppose $\bar{W}$ is a degenerate critical point, then by replacing the square loss error by a general convex and differentiable function $\ell(\cdot)$ in Theorem 18 we get that $\bar{W}$ is either a saddle or a global minimum. Suppose $\bar{W} = (\bar{W}_h, \ldots, \bar{W}_1)$ is a non-degenerate critical point and $k = \min_i d_i = \min(d_h, d_0)$, we follow the same steps of the proof of [40, Theorem 2.1] to show the desired result. First note that

$$\frac{\partial \ell(\bar{W}_h \cdots \bar{W}_1X)}{\partial \bar{W}_1} \bigg|_{\bar{W} = \bar{W}} = W_2^T \cdots W_k^T \nabla \ell(\bar{W}_h \cdots \bar{W}_1X)X^T,$$

and $\frac{\partial \ell(\bar{W}_h \cdots \bar{W}_1X)}{\partial \bar{W}_h} \bigg|_{\bar{W} = \bar{W}} = \nabla \ell(\bar{W}_h \cdots \bar{W}_1X)X^T \bar{W}_1^T \cdots \bar{W}_{k-1}^T$, where $\nabla \ell$ is the gradient mapping of the function $\ell(\cdot)$. If $k = d_h$, let $S = W_2^T \cdots W_k^T \in \mathbb{R}^{d_h \times k}$ and $T = \nabla \ell(\bar{W}_h \cdots \bar{W}_1X)X^T$. It follows that

$$k = \text{rank} (\bar{W}_h \cdots \bar{W}_1) \leq \text{rank}(S^T) \leq k \Rightarrow \text{rank}(S) = k.$$

Since $\bar{W}$ is a critical point and $S^T$ is full row rank, we get

$$0 = \left\| \frac{\partial \ell(\bar{W}_h \cdots \bar{W}_1X)}{\partial \bar{W}_1} \bigg|_{\bar{W} = \bar{W}} \right\|^2 = \text{tr}(T^T S^T ST) \geq \sigma_{\min}(S)||T||^2.$$

Thus, $T = \nabla \ell(\bar{W}_h \cdots \bar{W}_1X)X^T = 0$, which by convexity $\ell(\cdot)$ implies that $\bar{W}$ is a global minimum. Similarly, we can show that the case of $k = d_0$ results in the global optimality of $\bar{W}$ as well.
7 Conclusion

In this paper, we develop a unifying landscape analysis framework for studying the local/global equivalence for several non-convex objective functions that arise in statistical machine learning settings. In particular, our proposed framework utilizes the concept of local openness from differential geometry to provide sufficient conditions under which local optima of the objective function are global. While we narrow down our focus to a certain class of non-convex problems, the studied class is general enough to cover many practical applications such as matrix completion, low-rank matrix recovery, and deep learning. In our work, we completely characterize the local openness of matrix multiplication mapping in its range. More specifically, we provide necessary and sufficient conditions under which the matrix multiplication mapping is locally open. Based on this theoretical result, we develop a complete characterization of the local/global optima equivalence of multi-layer linear neural networks and provide sufficient conditions for which no spurious local optima exist under hierarchical non-linear deep neural networks. Unlike many existing results that focus on a particular algorithm (example gradient descent) and specific input data distribution, our result depends on the network structure and does not rely on the probability distribution of the input data.

Leveraging on our results, [44] show that every second-order stationary point of the low-rank matrix factorization problem is global. Our framework was also used to show similar results for shallow linear neural networks [43], low-rank matrix recovery [25], and meta learning objectives on several reinforcement learning tasks [28]. Such favorable geometry directly implies that local search methods that compute second-order stationary solutions will converge to global minima of the objective functions.

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References

[1] Zeyuan Allen-Zhu, Yuanzhi Li, and Yingyu Liang. Learning and generalization in overparameterized neural networks, going beyond two layers. *arXiv preprint arXiv:1811.04918*, 2018.

[2] Zeyuan Allen-Zhu, Yuanzhi Li, and Zhao Song. A convergence theory for deep learning via overparameterization. *arXiv preprint arXiv:1811.03962*, 2018.

[3] Sanjeev Arora, Nadav Cohen, Noah Golowich, and Wei Hu. A convergence analysis of gradient descent for deep linear neural networks. *arXiv preprint arXiv:1810.02281*, 2018.

[4] Sanjeev Arora, Simon S Du, Wei Hu, Zhiyuan Li, Ruslan Salakhutdinov, and Rusosong Wang. On exact computation with an infinitely wide neural net. *arXiv preprint arXiv:1904.11955*, 2019.

[5] M. Balcerzak, A. Majchrzycki, and A. Wachowicz. Openness of multiplication in some function spaces. *Taiwanese J. Math*, 17:1115–1126, 2013.

[6] M. Balcerzak, A. Wachowicz, and W. Wilczyński. Multiplying balls in the space of continuous functions on [0,1]. *Studia Mathematica*, 170:203–209, 2005.

[7] P. Baldi and K. Hornik. Neural networks and principal component analysis: Learning from examples without local minima. *Neural networks*, 2(1):53–58, 1989.

[8] E. Behrends. Products of $n$ open subsets in the space of continuous functions on $[0,1]$. *Studia Mathematica*, 204:73–95, 2011.

[9] E. Behrends. Where is matrix multiplication locally open? *Linear Algebra and its Applications*, 517:167–176, 2017.

[10] S. Bhajanapalli, B. Neyshabur, and N. Srebro. Global optimality of local search for low rank matrix recovery. In *Advances in Neural Information Processing Systems*, pages 3873–3881, 2016.

[11] A. Blum and R. L. Rivest. Training a 3-node neural network is np-complete. In *Advances in neural information processing systems*, pages 494–501, 1989.
[12] N. Boumal, V. Voroninski, and A. Bandeira. The non-convex burer-monteiro approach works on smooth semidefinite programs. In *Advances in Neural Information Processing Systems*, pages 2757–2765, 2016.

[13] S. Burer and R. D. C. Monteiro. Local minima and convergence in low-rank semidefinite programming. *Mathematical Programming*, 103(3):427–444, 2005.

[14] A. Choromanska, M. Henaff, M. Mathieu, G. B. Arous, and Y. LeCun. The loss surfaces of multilayer networks. In *Artificial Intelligence and Statistics*, pages 192–204, 2015.

[15] Tian Ding, Dawei Li, and Ruoyu Sun. Sub-optimal local minima exist for almost all over-parameterized neural networks. *arXiv preprint arXiv:1911.01413*, 2019.

[16] Simon S Du and Jason D Lee. On the power of over-parametrization in neural networks with quadratic activation. *arXiv preprint arXiv:1803.01206*, 2018.

[17] Simon S Du, Xiyu Zhai, Barnabas Poczos, and Aarti Singh. Gradient descent provably optimizes over-parameterized neural networks. *arXiv preprint arXiv:1810.02054*, 2018.

[18] C. Daniel Freeman and Joan Bruna. Topology and geometry of half-rectified network optimization. *arXiv preprint arXiv:1611.01540*, 2016.

[19] R. Ge, J. D. Lee, and T. Ma. Matrix completion has no spurious local minimum. In *Advances in Neural Information Processing Systems*, pages 2973–2981, 2016.

[20] Moritz Hardt and Tengyu Ma. Identity matters in deep learning. *arXiv preprint arXiv:1611.04231*, 2016.

[21] C. Horowitz. An elementary counterexample to the open mapping principle for bilinear maps. *Proceedings of the American Mathematical Society*, 53(2):293–294, 1975.

[22] K. Kawaguchi. Deep learning without poor local minima. In *Advances in Neural Information Processing Systems*, pages 586–594, 2016.

[23] Thomas Laurent and James Brecht. Deep linear networks with arbitrary loss: All local minima are global. In *International conference on machine learning*, pages 2902–2907. PMLR, 2018.

[24] Dawei Li, Tian Ding, and Ruoyu Sun. Over-parameterized deep neural networks have no strict local minima for any continuous activations. *arXiv preprint arXiv:1812.11039*, 2018.

[25] Shuang Li, Qiuwei Li, Zhihui Zhu, Gongguo Tang, and Michael B Wakin. The global geometry of centralized and distributed low-rank matrix recovery without regularization. *arXiv preprint arXiv:2003.10981*, 2020.

[26] Yuanzhi Li and Yingyu Liang. Learning overparameterized neural networks via stochastic gradient descent on structured data. In *Advances in Neural Information Processing Systems*, pages 8157–8166, 2018.

[27] H. Lu and K. Kawaguchi. Depth creates no bad local minima. *arXiv preprint arXiv:1702.08580*, 2017.

[28] Igor Molybog and Javad Lavaei. Global convergence of maml for lqr. *arXiv preprint arXiv:2006.00453*, 2020.

[29] Q. Nguyen and M. Hein. The loss surface of deep and wide neural networks. *arXiv preprint arXiv:1704.08045*, 2017.

[30] Quynh Nguyen. On connected sublevel sets in deep learning. In *International Conference on Machine Learning*, pages 4790–4799. PMLR, 2019.

[31] Quynh Nguyen, Mahesh Chandra Mukkamala, and Matthias Hein. On the loss landscape of a class of deep neural networks with no bad local valleys. *arXiv preprint arXiv:1809.10749*, 2018.

[32] Samet Oymak and Mahdi Soltanolkotabi. Towards moderate overparameterization: global convergence guarantees for training shallow neural networks. *arXiv preprint arXiv:1902.04674*, 2019.

[33] D. Park, A. Kyrillidis, C. Caramanis, and S. Sanghavi. Non-square matrix sensing without spurious local minima via the burer-monteiro approach. *arXiv preprint arXiv:1609.03240*, 2016.

[34] W. Rudin. Functional analysis, mcgraw-hill series in higher mathematics. 1973.

[35] Mahdi Soltanolkotabi, Adel Javanmard, and Jason D Lee. Theoretical insights into the optimization landscape of over-parameterized shallow neural networks. *IEEE Transactions on Information Theory*, 65(2):742–769, 2019.
[36] N. Srebro and T. Jaakkola. Weighted low-rank approximations. In Proceedings of the 20th International Conference on Machine Learning (ICML-03), pages 720–727, 2003.

[37] R. Sun. Matrix completion via nonconvex factorization: Algorithms and theory. PhD thesis, University of Minnesota, 2015.

[38] Luca Venturi, Afonso S Bandeira, and Joan Bruna. Spurious valleys in two-layer neural network optimization landscapes. arXiv preprint arXiv:1802.06384, 2018.

[39] L. Wang, X. Zhang, and Q. Gu. A unified computational and statistical framework for nonconvex low-rank matrix estimation. arXiv preprint arXiv:1610.05275, 2016.

[40] C. Yun, S. Sra, and A. Jadbabaie. Global optimality conditions for deep neural networks. arXiv preprint arXiv:1707.02444, 2017.

[41] Chulhee Yun, Suvrit Sra, and Ali Jadbabaie. Small nonlinearities in activation functions create bad local minima in neural networks. arXiv preprint arXiv:1802.03487, 2018.

[42] Li Zhang. Depth creates no more spurious local minima. arXiv preprint arXiv:1901.09827, 2019.

[43] Q. Zheng and J. Lafferty. Convergence analysis for rectangular matrix completion using burer-monteiro factorization and gradient descent. arXiv preprint arXiv:1605.07051, 2016.

[44] Zhihui Zhu, Qiuwei Li, Xinshuo Yang, Gongguo Tang, and Michael B Wakin. Distributed low-rank matrix factorization with exact consensus. In Advances in Neural Information Processing Systems, pages 8422–8432, 2019.

[45] Zhihui Zhu, Daniel Soudry, Yonina C Eldar, and Michael B Wakin. The global optimization geometry of shallow linear neural networks. Journal of Mathematical Imaging and Vision, pages 1–14, 2019.

[46] Difan Zou, Yuan Cao, Dongruo Zhou, and Quanquan Gu. Stochastic gradient descent optimizes over-parameterized deep relu networks. arXiv preprint arXiv:1811.08888, 2018.
Appendices

A Proof of Theorem 8

Before proceeding to the proof of Theorem 8, we need to state and prove few lemmas.

Lemma 22. Let $V \in \mathbb{R}^{m \times n}$ be a matrix with rank$(V) = r < m$. Then there exists an index set $B = \{1, \ldots, r\} \subseteq \{1, \ldots, m\}$ and a matrix $A \in \mathbb{R}^{(m-r) \times r}$ such that

$$||A||_\infty = \max_{i,j} |A_{ij}| \leq 2^{m-r-1}$$

and $V_{B^c} = AV_B$, where $V_B \in \mathbb{R}^{r \times n}$ is a matrix with rows $\{V_{i,:}\}_{i \in B}$ and $V_{B^c} \in \mathbb{R}^{(m-r) \times n}$ is a matrix with rows $\{V_{i,:}\}_{i \in B^c}$.

Notice that in the above lemma, the bound on the norm of matrix $A$ is independent of the dimension $n$ and the choice of matrix $V$.

Proof. To ease the notation, we denote the $i^{th}$ row of $V$ by $v_i$. We use induction on $m$ to show that there exists a basis $B = \{i_1, \ldots, i_r\}$ and a vector $a_j \in \mathbb{R}^r$ such that $v_j = \sum_{i \in B} a_j v_i$ with $|a_j| \leq 2^{m-r-1}$ $\forall i \in B$.

• Induction Base Case $m = r + 1$: Without loss of generality, assume $B = \{1, \ldots, r\}$. Since the case of $v_{r+1} = 0$ trivially holds, we consider $v_{r+1} \neq 0$. By the property of basis, there exists a non-zero vector $a_{r+1} \in \mathbb{R}^r$ such that $v_{r+1} = \sum_{i=1}^r a_{r+1,i} v_i$.

Let $i^* = \arg \max_{i \in B} |a_{r+1,i}|$. If $|a_{r+1,i^*}| \leq 1$, then the induction hypothesis is true. Otherwise, when $|a_{r+1,i^*}| > 1$, we have

$$v_{i^*} = \frac{1}{a_{r+1,i^*}} v_{r+1} - \sum_{i=1; i \neq i^*}^r \frac{a_{r+1,i}}{a_{r+1,i^*}} v_i = \sum_{i \in B^*} \tilde{a}_{r+1,i} v_i,$$

where $B^* = (B \cup \{r + 1\}) \backslash \{i^*\}$, i.e., we remove the item $i^*$ from $B$ and include the item $r + 1$ instead. Since $|\tilde{a}_{r+1,i}| \leq 1$, the induction base case holds.

• Inductive Step: Assume the induction hypothesis is true for $m > r$, we show it is also true for $m + 1$. Without loss of generality we can assume that $B = \{1, \ldots, r\}$. By induction hypothesis, $v_j = \sum_{i=1}^r a_{j,i} v_i$ with $|a_{j,i}| \leq 2^{m-r-1}$ $\forall j \in \{r + 1, \ldots, m\}$. Since the case of $v_{m+1} = 0$ trivially holds, we consider $v_{m+1} \neq 0$. Since $B$ is a basis, there exists $a_{m+1} \neq 0$ such that $v_{m+1} = \sum_{i=1}^r a_{m+1,i} v_i$. Let $i^* = \arg\max_{i \in B} |a_{m+1,i}|$. If $|a_{m+1,i^*}| \leq 2^{m-r}$, the induction step is done. Otherwise, for the case of $|a_{m+1,i^*}| > 2^{m-r}$, we have

$$v_{i^*} = \frac{1}{a_{m+1,i^*}} v_{m+1} - \sum_{i=1; i \neq i^*}^r \frac{a_{m+1,i}}{a_{m+1,i^*}} v_i = \sum_{i \in B^*} \tilde{a}_{m+1,i} v_i,$$

where $B^* = (B \cup \{m+1\}) \backslash \{i^*\}$ and clearly $|\tilde{a}_{m+1,i}| \leq 1$, $\forall i \in B^*$ according to the definition of $i^*$. For all $j \in \{r + 1, \ldots, m\}$

$$v_j = \sum_{i=1; i \neq i^*}^r a_{j,i} v_i + a_{j,i^*} v_{i^*} = \sum_{i \neq i^*} a_{j,i} v_i + \frac{a_{j,i^*}}{a_{m+1,i^*}} v_{m+1} - \sum_{i \neq i^*} \frac{a_{m+1,i}}{a_{m+1,i^*}} a_{j,i} v_i$$

$$= \sum_{i=1; i \neq i^*} \left( a_{j,i} - \frac{a_{j,i^*}}{a_{m+1,i^*}} \frac{a_{m+1,i}}{a_{j,i}} \right) v_i + \frac{a_{j,i^*}}{a_{m+1,i^*}} v_{m+1} = \sum_{i \in B^*} \tilde{a}_{j,i} v_i.$$

It remains to show that $|\tilde{a}_{j,i}| \leq 2^{m-r}$ for all $i \in B^*$, $j \in \{r + 1, \ldots, m\}$. Let us first consider $i \in B^\prime \{m + 1\}$ and $j \in \{r + 1, \ldots, m\}$:

$$|\tilde{a}_{j,i}| \leq |a_{j,i}| + |a_{j,i} \frac{a_{m+1,i}}{a_{m+1,i^*}}| \leq 2^{m-r-1} + 2^{m-r-1} \frac{a_{m+1,i}}{a_{m+1,i^*}} \leq 2^{m-r},$$
where the first inequality holds by triangular inequality, the second inequality holds by the induction hypothesis, and the last inequality holds by the definition of $i^*$. For $i = m + 1$, $|\bar{a}_{j,m+1}| = |a_{j,m+1,i^*}| |\bar{a}_{m+1,i^*}| \leq \frac{2^{m-r-1}}{a_{m+1,i^*}} \leq 2^{m-r}$.

This concludes the inductive step and completes our proof.

The following results show that by using local openness of linear mappings and standard SVD, without loss of generality, we can assume that the product $\bar{W}_1 \bar{W}_2$ is a diagonal matrix.

**Lemma 23.** Let $W_1 \in \mathbb{R}^{m \times k}$ and $W_2 \in \mathbb{R}^{k \times n}$. Assume further that $W_1 W_2 = U \Sigma V^T$ is a singular value decomposition of the matrix product $W_1 W_2$ with $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$, and $\Sigma \in \mathbb{R}^{m \times n}$. Then $\mathcal{M}(\cdot, \cdot)$ is locally open at $(W_1, W_2)$ if and only if $\mathcal{M}(\cdot, \cdot)$ is locally open at $(U^T W_1, W_2 V)$.

The proof of this Lemma is a direct consequence of the definition of local openness. Lemma 23 implies that for proving Theorem 8 without loss of generality, we can assume that the product $\bar{W}_1 \bar{W}_2$ is a diagonal matrix.

**Lemma 24.** Let $W_1 \in \mathbb{R}^{m \times k}$ and $W_2 \in \mathbb{R}^{k \times n}$. Assume further that $W_1 W_2 = U \Sigma V^T$ is a singular value decomposition of the matrix product $W_1 W_2$ with $U \in \mathbb{R}^{m \times m}$, $V \in \mathbb{R}^{n \times n}$, and $\Sigma \in \mathbb{R}^{m \times n}$. Define $W_1 \doteq U^T W_1$ and $W_2 \doteq W_2 V$. Then the condition $(A)$ below holds true if and only if the condition $(B)$ is true. Similarly, condition $(C)$ is true if and only if condition $(D)$ is true.

\begin{enumerate}
  \item[(A)] $\exists \tilde{W}_1 \in \mathbb{R}^{m \times k}$ such that $\tilde{W}_1 W_2 = 0$ and $W_1 + \tilde{W}_1$ is full column rank.
  \item[(B)] $\exists \tilde{W}_1 \in \mathbb{R}^{m \times k}$ such that $\tilde{W}_1 \bar{W}_2 = 0$ and $\bar{W}_1 + \tilde{W}_1$ is full column rank.
  \item[(C)] $\exists \tilde{W}_2 \in \mathbb{R}^{k \times n}$ such that $W_1 \tilde{W}_2 = 0$ and $W_2 + \tilde{W}_2$ is full row rank.
  \item[(D)] $\exists \tilde{W}_2 \in \mathbb{R}^{k \times n}$ such that $W_1 \bar{W}_2 = 0$ and $\bar{W}_2 + \tilde{W}_2$ is full row rank.
\end{enumerate}

**Proof.** Setting $\tilde{W}_1 = U^T \bar{W}_1$ and $\tilde{W}_2 = \bar{W}_2 V$ leads to the desired result.

We next show in Lemma 25 that if $k < \min\{m, n\}$ and rank$(W_1) = \text{rank}(W_2)$, then statements $i, ii, iii,$ and $iv$ in Theorem 8 are all equivalent.

**Lemma 25.** Let $W_1 \in \mathbb{R}^{m \times k}$, $W_2 \in \mathbb{R}^{k \times n}$ with rank$(W_1) = \text{rank}(W_2)$. Assume further that $k < \min\{m, n\}$. Then, the following conditions are equivalent

\begin{enumerate}
  \item[i)] $\exists \tilde{W}_1 \in \mathbb{R}^{m \times k}$ such that $\tilde{W}_1 W_2 = 0$ and $W_1 + \tilde{W}_1$ is full column rank.
  \item[ii)] $\exists \tilde{W}_2 \in \mathbb{R}^{k \times n}$ such that $W_1 \tilde{W}_2 = 0$ and $W_2 + \tilde{W}_2$ is full row rank.
  \item[iii)] $\dim(\mathcal{N}(W_1) \cap \mathcal{C}(W_2)) = 0$.
  \item[iv)] $\dim(\mathcal{N}(W_1^T) \cap \mathcal{C}(W_2^T)) = 0$.
\end{enumerate}

**Proof.** To prove the desired result we show the equivalences $ii \iff iii$, and $i \iff iv$. Then we complete the proof by showing $iii \iff iv$.

We first show the direction “$ii \Rightarrow iii$”. Consider $W_1 \in \mathbb{R}^{m \times k}, W_2 \in \mathbb{R}^{k \times n}$ with both being rank $r$ matrices. Suppose $ii$ holds, then

$$C(\tilde{W}_2) \subseteq \mathcal{N}(W_1)$$

which implies $\text{rank}(\tilde{W}_2) \leq \dim(\mathcal{N}(W_1)) = k - r$. (14)

Also, $k = \text{rank}(W_2 + \tilde{W}_2) \leq \text{rank}(W_2) + \text{rank}(\tilde{W}_2) = r + \text{rank}(\tilde{W}_2)$. This inequality combined with (14) implies that $\text{rank}(\tilde{W}_2) = k - r$. Note that $\dim(\mathcal{C}(\tilde{W}_2)) = \dim(\mathcal{N}(W_1))$ and $\mathcal{C}(\tilde{W}_2) \subseteq \mathcal{N}(W_1)$, which implies that $\mathcal{C}(\tilde{W}_2) = \mathcal{N}(W_1)$. Then, since $\text{rank}(W_2 + \tilde{W}_2) = \text{rank}(W_2) + \text{rank}(\tilde{W}_2)$, we get

$$\emptyset = \mathcal{C}(\tilde{W}_2) \cap \mathcal{C}(W_2) = \mathcal{N}(W_1) \cap \mathcal{C}(W_2) \Rightarrow \dim(\mathcal{N}(W_1) \cap \mathcal{C}(W_2)) = 0.$$
we get \( \text{rank}(W_2 + \tilde{W}_2) = k \) for generic choice of \( \epsilon \). This completes the proof.

Note that by setting \( W_1 = W_2^* \) and \( W_2 = W_1^\top \), the same proof can be used to show \( i \Leftrightarrow iv \). Next, we will prove the equivalence \( iii \Leftrightarrow iv \). Notice that

\[
\dim \left( \text{span}(N(W_1) \cup C(W_2)) \right) = \dim (N(W_1)) + \dim (C(W_2)) - \dim (N(W_1) \cap C(W_2))
\]

\[
= k - r + r - \dim (N(W_1) \cap C(W_2))
\]

\[
= k - \dim (N(W_1) \cap C(W_2)).
\]

Thus,

\[
\dim (N(W_1) \cap C(W_2)) \neq 0 \Leftrightarrow \dim \left( \text{span}(N(W_1) \cup C(W_2)) \right) < k
\]

\[
\Leftrightarrow \exists a \neq 0 \text{ such that } a \perp C(W_2), \text{ and } a \perp N(W_1)
\]

\[
\Leftrightarrow \exists a \neq 0 \text{ such that } a \in N(W_2^\top), \text{ and } a \in C(W_1^\top)
\]

\[
\Leftrightarrow \dim (N(W_2^\top) \cap C(W_1^\top)) \neq 0,
\]

which completes the proof. \( \Box \)

**Lemma 26.** Let \( W_1 \in \mathbb{R}^{m \times k} \), \( W_2 \in \mathbb{R}^{k \times n} \) with \( k < \min\{m, n\} \) and let \( r \triangleq \text{rank}(W_1W_2) \). Assume further that \( W_1W_2 = U\Sigma V^\top \) is an SVD of \( W_1W_2 \) with \( U \in \mathbb{R}^{m \times m} \), and \( V \in \mathbb{R}^{n \times n} \), and \( \Sigma \in \mathbb{R}^{m \times n} \). If

\[
\begin{align*}
(i) & \exists \tilde{W}_1 \in \mathbb{R}^{m \times k} \text{ such that } \tilde{W}_1W_2 = 0 \text{ and } W_1 + \tilde{W}_1 \text{ is full column rank.} \\
(ii) & \exists \tilde{W}_2 \in \mathbb{R}^{k \times n} \text{ such that } W_1\tilde{W}_2 = 0 \text{ and } W_2 + \tilde{W}_2 \text{ is full row rank.}
\end{align*}
\]

then

\[
\text{rank}(W_1) = \text{rank}(W_2), \quad (W_2V)_{:,r+1:n} = 0, \quad \text{and} \quad (U^\top W_1)_{r+1:n, :} = 0.
\]

**Proof.** Suppose that \( ii \) holds, then

\[
\mathcal{C}(\tilde{W}_2) \subseteq \mathcal{N}(W_1) \Rightarrow \text{rank}(\tilde{W}_2) \leq \dim (\mathcal{N}(W_1)) = k - \text{rank}(W_1). \tag{15}
\]

Also, \( k = \text{rank}(W_2 + \tilde{W}_2) \leq \text{rank}(W_2) + \text{rank}(\tilde{W}_2) \). This inequality combined with \( 15 \) implies

\[
k - \text{rank}(W_2) \leq \text{rank}(\tilde{W}_2) \leq k - \text{rank}(W_1) \Rightarrow \text{rank}(W_2) \geq \text{rank}(W_1). \tag{16}
\]

Similarly, condition \( i \) implies \( \text{rank}(W_1) \geq \text{rank}(W_2) \). Combined with \( 16 \), we obtain \( \text{rank}(W_1) = \text{rank}(W_2) \).

Therefore, Lemma 25 implies \( \dim (\mathcal{N}(W_1) \cap \mathcal{C}(W_2)) = 0 \). It follows from the SVD of \( W_1W_2 \), that \( U^\top W_1(W_2V)_{:,r+1:n} = \Sigma_{r+1:n, :} = 0 \), or equivalently \( W_1(W_2V)_{:,r+1:n} = 0 \). On the other hand, since \( \mathcal{C}(W_2V)_{:,r+1:n} \subseteq \mathcal{C}(W_2) \) and \( \mathcal{N}(W_1) \cap \mathcal{C}(W_2) = \emptyset \), we have \( (W_2V)_{:,r+1:n} = 0 \). Similarly, we can show that \( (U^\top W_1)_{r+1:n, :} = 0 \). \( \Box \)

**Proposition 27.** Let \( M(W_1, W_2) = W_1W_2 \) be the matrix product mapping with \( W_1 \in \mathbb{R}^{m \times k} \), \( W_2 \in \mathbb{R}^{k \times n} \), and \( k < \min\{m, n\} \). Then, \( M(\cdot, \cdot) \) is locally open in its range \( \mathcal{R}_M \triangleq \{ Z \in \mathbb{R}^{m \times n} : \text{rank}(Z) \leq k \} \) at the point \( (W_1, W_2) \) if and only if the following two conditions are satisfied:

\[
\begin{align*}
(i) & \exists \tilde{W}_1 \in \mathbb{R}^{m \times k} \text{ such that } \tilde{W}_1W_2 = 0 \text{ and } W_1 + \tilde{W}_1 \text{ is full column rank.} \\
(ii) & \exists \tilde{W}_2 \in \mathbb{R}^{k \times n} \text{ such that } W_1\tilde{W}_2 = 0 \text{ and } W_2 + \tilde{W}_2 \text{ is full row rank.}
\end{align*}
\]

**Proof.** First of all, according to Lemma 23 and Lemma 24 without loss of generality we can assume that the matrix product \( W_1W_2 \) is of diagonal form. Let us start by first proving the “only if” direction. Notice that the result clearly holds when \( \text{rank}(W_1) = \text{rank}(W_2) = k \) by choosing \( \tilde{W}_1 = \tilde{W}_2 = 0 \). Moreover, the mapping \( M(\cdot, \cdot) \) cannot be locally open if only one of the matrices \( W_1 \) or \( W_2 \) is rank deficient. To see this, let us assume that \( W_1 \) is full column rank, while \( W_2 \) is rank deficient. Assume further that the mapping \( M(\cdot, \cdot) \) is locally open at \( (W_1, W_2) \), it follows from the definition
of openness that the mapping $M^I(W_1, W_2) \triangleq W_1W_2^T$ is locally open at $(W_1, W_2)$ where $W_2^T \equiv (W_2)_{1:k}$ only contains the first $k$ columns of $W_2$. Since the range of the mapping $M^I$ at $(W_1, W_2^T)$ is the entire space $\mathbb{R}^{m \times k}$, Proposition 7 implies that

$$
\begin{align*}
&\exists W_1 \text{ such that } W_1W_2^T = 0 \text{ and } W_1 + W_1^T \text{ is full row rank.} \\
&\text{or} \\
&\exists W_2 \text{ such that } W_1W_2^T = 0 \text{ and } W_2^T + W_2^T \text{ is full rank.}
\end{align*}
$$

Moreover, since $W_1 \in \mathbb{R}^{m \times k}$ and $m > k$, it is impossible for $W_1 + W_1^T$ to be full row rank. On the other hand, since $W_1$ is full column rank, $W_1W_2^T = 0$ implies that $W_2^T = 0$; and hence $W_2^T + W_2^T$ is not full column rank. Hence none of the above two conditions can hold and consequently, $M(\cdot, \cdot)$ cannot be open at the point $(W_1, W_2^T)$ in this case. Similarly, we can show that when $W_1$ is rank deficient and $W_2^T$ is full row rank, the mapping $M(\cdot, \cdot)$ cannot be locally open. Hence, if $W_1$ and $W_2^T$ are not both full rank, then they both should be rank deficient. Assume that the matrices $W_1$ and $W_2^T$ are both rank deficient and $M(\cdot, \cdot)$ is locally open at $(W_1, W_2^T)$. It follows that $M^I(W_1, W_2^T) \triangleq W_1W_2^T$ is local rank deficient and $M(\cdot, \cdot)$ is locally open at $(W_1, W_2^T)$. By Proposition 7, and since there does not exist $W_1$ such that $W_1 + W_1^T$ is full row rank, there should exist $W_2^T$ such that $W_1W_2^T = 0$ and $W_2^T + W_2^T$ is full rank. Defining $W_2^T \triangleq \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$, we satisfy the desired condition ii). Similarly, by looking at the transpose of the mapping $M$, we can show that condition i) is true when $M$ is locally open. We now prove the "if" direction. Suppose i) and ii) hold.

Let $\Sigma = \begin{bmatrix} \Sigma_{1,1,k} & 0 \end{bmatrix}$ be a rank $r$ matrix. Lemma 26 implies that rank($W_1$) = rank($W_2$), and the last $n - r$ columns of $W_2$ are all zero. We need to show that for any given $\epsilon > 0$, there exists $\delta > 0$, such that $B_{\delta}(W_1^T, W_2) \cap R_M \subseteq M(B_{\epsilon}(W_1), B_{\epsilon}(W_2))$. Consider a perturbed matrix $\tilde{\Sigma} \in B_{\delta}(\Sigma) \cap R_M$, we show that $\tilde{\Sigma} \in M(B_{\epsilon}(W_1), B_{\epsilon}(W_2))$. Without loss of generality, and by permuting the columns of $\tilde{\Sigma}$ if necessary, any perturbation of $\sigma$ with rank at most $k$ can expressed as

$$
\tilde{\Sigma} = \begin{bmatrix} \Sigma_{1,1,k} & R_1^T \\
R_1 & \Sigma_{1,1,k} + R_1^T A_1 + R_1^T A_2 \end{bmatrix}.
$$

Here $A_1 \in \mathbb{R}^{r \times (n-k)}$ and $A_2 \in \mathbb{R}^{(k-r) \times (n-k)}$ exist since rank($\tilde{\Sigma}$)$\leq k$. More specifically, these matrices exist since each of the last $n - k$ columns of $\tilde{\Sigma}$ is a linear combination of the first $k$ columns. Moreover, let the perturbation matrix $R_{\delta}$ be defined as $R_{\delta} \triangleq \|\Sigma - \Sigma\|$. Then $R_{\delta}$ can expressed as

$$
R_{\delta} \triangleq \begin{bmatrix} R_1^T & R_2^T & (\Sigma_{1,1,k} + R_1^T A_1 + R_1^T A_2) \end{bmatrix}
$$

and requires to have a norm less than or equal $\delta$, i.e., $\|R_{\delta}\| \leq \delta$. Since rank($W_2^T + \tilde{W}_2$) = $k$, there exist a unitary basis set $\{\tilde{w}_2^1, \ldots, \tilde{w}_2^{k-r}\}$ for $\tilde{W}_2$ such that span$\{\tilde{w}_2^1, \ldots, \tilde{w}_2^{k-r}\} \cap C(W_2) = \emptyset$. Define

$$
\tilde{W}_2^1 \triangleq \frac{\epsilon}{n^2 + 1} \begin{bmatrix} k \times r & k \times (k-r) \\
0 & \tilde{w}_2^1 \ldots \tilde{w}_2^{k-r} \end{bmatrix},
$$

and let us form the matrix $\tilde{W}_2^1 \in \mathbb{R}^{k \times k}$ using the first $k$ columns of $\tilde{W}_2$. Since the last $n - r$ columns of the matrix $\tilde{W}_2$ are zero, $\tilde{W}_2^1 + \tilde{W}_2^1$ is a full rank $k \times k$ matrix and $W_1W_2^T = 0$. Let us define $W_1^0 \triangleq \begin{bmatrix} R_1^T & R_2^T \end{bmatrix} (\tilde{W}_2^1 + \tilde{W}_2^1)^{-1}$,
and $\tilde{W}_2^0 \triangleq \left[ \begin{array}{c|c} \tilde{W}_2^0 & (\tilde{W}_2^0 + \tilde{W}_1^0)_{r+1:k} A_1 + (\tilde{W}_2^0 + \tilde{W}_1^0)_{r+1:n} A_2 \end{array} \right]$. Using this definition, we have

$$(\tilde{W}_1 + \tilde{W}_1^0)(\tilde{W}_2 + \tilde{W}_2^0) = \left[ \begin{array}{c|c} (\tilde{W}_1 + \tilde{W}_1^0)(\tilde{W}_2 + \tilde{W}_2^0)_{i:k} & (\tilde{W}_1 + \tilde{W}_1^0)(\tilde{W}_2 + \tilde{W}_2^0)_{k+1:n} \end{array} \right]$$

$$= \left[ \begin{array}{c|c} (\tilde{W}_1 + \tilde{W}_1^0)(\tilde{W}_2 + \tilde{W}_2^0) & (\tilde{W}_1 + \tilde{W}_1^0)(\tilde{W}_2 + \tilde{W}_2^0)_{:k+1:n} \end{array} \right]$$

$$= \left[ \begin{array}{c|c} \Sigma_{:1:k} + W_1 W_2 + [ \begin{array}{c} \Sigma_{:1:r} \end{array} ] & \Sigma_{:1:r} + R_1 \end{array} \right] (\tilde{W}_2^1 + \tilde{W}_2^0)^{-1} (\tilde{W}_1^1 + \tilde{W}_1^0)$$

$$= \tilde{W}_1 \tilde{W}_2 + [ \begin{array}{c} \Sigma_{:1:r} + R_1 \end{array} ] (\tilde{W}_2^1 + \tilde{W}_2^0)$$

To complete the proof, it remains to show that for any $\epsilon > 0$, we can choose $\delta$ small enough such that $\| \tilde{W}_1^0 \| \leq \epsilon$ and $\| \tilde{W}_2^0 \| \leq \epsilon$. In other words, we will show $\tilde{\Sigma} \in \mathcal{M}(\mathbb{B}_e(\tilde{W}_1), \mathbb{B}_e(\tilde{W}_2))$. Let $\bar{\tau}$, with $k \geq \bar{\tau} \geq r$, be the rank of $\tilde{\Sigma}$. According to Lemma 1 and by possibly permuting the columns, $\tilde{\Sigma}$ can be expressed as $\tilde{\Sigma} = [ \tilde{\Sigma}_1 \mid \tilde{\Sigma}_1 \tilde{A} ]$, where $\tilde{\Sigma}_1 \in \mathbb{R}^{m \times \bar{\tau}}$ is full column rank, and $\tilde{A}$ has a bounded norm $\| \tilde{A} \| \leq n 2^{n-\bar{\tau}-1}$. Notice that for given $\tilde{W}_1^0$ and $\tilde{W}_2^0$ satisfying (18), permuting the columns of $\tilde{\Sigma}$ corresponds to permuting the columns of $(\tilde{W}_2 + \tilde{W}_2^0)$. If we can show that the first $r$ columns are not among the permuted ones, then using the fact that $\tilde{W}_2$ has only its first $r$ columns non-zero, it follows that the permutation of the columns of $\tilde{\Sigma}$ corresponds to the same permutation of the columns of $\tilde{W}_2^0$. Moreover, if the first $r$ columns are not among the permuted ones, then without loss of generality we can express the perturbed matrix

$$\tilde{\Sigma} = \left[ \begin{array}{c|c} \Sigma_{:1:r} + R_\delta^1 & \Sigma_{:1:r} + R_\delta^2 \end{array} \right]$$

and the perturbation matrix

$$R_\delta = \left[ \begin{array}{c|c} R_\delta^1 & R_\delta^2 \end{array} \right],$$

where $[ \tilde{A}_1 \tilde{A}_2 ] = \tilde{A}$ has a bounded norm.

We now show that the first $r$ columns of $\tilde{\Sigma}$ are not among the permuted columns. Assume the contrary, then there exists at least a column $\Sigma_{:j} + (R_\delta^1)_{:j}$ with $j \leq \bar{\tau}$, that is not a column of $\tilde{\Sigma}_1$ and is thus a column of $\tilde{\Sigma}_1 \tilde{A}$. Without loss of generality let $\Sigma_{:j} + (R_\delta^1)_{:j} = \tilde{A}_1 \tilde{A}_1$. It follows that $\Sigma_{:j} + (R_\delta^1)_{:j} = (\tilde{\Sigma}_1)_{:j} \tilde{A}_1$. But since $\Sigma_{:j} + (R_\delta^1)_{:j}$ is a non-zero perturbed singular value, and since elements of $(\tilde{\Sigma}_1)_{:j}$ are all of order $\delta$, then by choosing $\delta$ sufficiently small, we get $\| \tilde{A} \| > 2^{n-\bar{\tau}-1}$, which contradicts the bound we have on $\tilde{A}$.

We now obtain an upper-bound on $\| \tilde{W}_2^0 \|$. Since the norm of $\tilde{A}$ is bounded, the norm of $\tilde{A}_2$ is also bounded by some constant $K \triangleq n 2^n > n 2^{n-\bar{\tau}-1}$. Hence,

$$\delta \geq \| R_\delta \| \geq \| (\Sigma_{:1:r} + R_\delta^1) \tilde{A}_1 + R_\delta^2 \tilde{A}_2 \| \geq \| (\Sigma_{:1:r} + R_\delta^1) \tilde{A}_1 \| - \| R_\delta^2 \tilde{A}_2 \|$$

$$\geq \| (\Sigma_{:1:r} + R_\delta^1) \tilde{A}_1 \| - K \delta \geq \frac{\sigma_{\min}}{2} \| \tilde{A}_1 \| - K \delta,$$

where $\sigma_{\min}$ is the minimum singular value of the full column rank matrix $\Sigma_{:1:r}$ which is bounded away from zero. Here, we have chosen $\delta < \sigma_{\min}/2$ so that $\| (\Sigma_{:1:r} + R_\delta^1) \tilde{A}_1 \| \geq \frac{\sigma_{\min}}{2} \| \tilde{A}_1 \|$. Rearranging the
terms, we obtain \( \| \mathbf{A}_1 \| \leq \frac{2(1 + K)\delta}{\sigma_{\text{min}}} \). Thus, for some constant \( C \equiv \| \mathbf{W}_2^1 \| \), we obtain

\[
\| \mathbf{W}_2^0 \|^2 \leq \| \mathbf{W}_2^1 \|^2 + \| \mathbf{W}_2^1 \|^2 \| \mathbf{A}_1 \|^2 + \| \mathbf{W}_2^1 \|^2 \| \mathbf{A}_2 \|^2 \\
\leq \frac{\epsilon^2}{4n^22^{2n}} + \delta^2C^2 \left( \frac{2 + 2K}{\sigma_{\text{min}}} \right)^2 + \frac{\epsilon^2K^2}{4n^22^{2n}} \\
\leq \frac{\epsilon^2}{4K^2} + \delta^2C^2 \left( \frac{2 + 2K}{\sigma_{\text{min}}} \right)^2 + \frac{\epsilon^2K^2}{4K^2} \\
\leq \frac{\epsilon^2}{2} + \delta^2C^2 \left( \frac{2 + 2K}{\sigma_{\text{min}}} \right)^2,
\]

where the first inequality holds by Cauchy-Swarz and triangular inequality. Thus, for a given \( \epsilon > 0 \), we can choose

\[
\delta \leq \min \left\{ \frac{\epsilon}{1 + \max \left\{ \| (\mathbf{W}_2^1 + \mathbf{W}_2^1)^{-1} \|, \sqrt{2C} \left( \frac{2 + 2K}{\sigma_{\text{min}}} \right) \right\}, \sigma_{\text{min}}/2 \} \right\}.
\]

This choice of \( \delta \) leads to \( \| \mathbf{W}_2^0 \| \leq \epsilon \). Moreover,

\[
\| \mathbf{W}_2^0 \| \leq \| \mathbf{R}_\delta \| \| (\mathbf{W}_2^1 + \mathbf{W}_2^1)^{-1} \| \leq \delta \| (\mathbf{W}_2^1 + \mathbf{W}_2^1)^{-1} \| \leq \frac{\epsilon \| (\mathbf{W}_2^1 + \mathbf{W}_2^1)^{-1} \|}{1 + \| (\mathbf{W}_2^1 + \mathbf{W}_2^1)^{-1} \|} \leq \epsilon,
\]

which completes the proof. \( \square \)

We now use Proposition 27, Lemma 25, and Lemma 26 to complete the proof of Theorem 8.

**Proof.** First of all, if \( \mathcal{M}(\cdot, \cdot) \) is locally open at \( (\mathbf{W}_1, \mathbf{W}_2) \), according to Proposition 27, the conditions i) and ii) must hold; and hence \( \text{rank}(\mathbf{W}_1) = \text{rank}(\mathbf{W}_2) \) due to Lemma 26. Thus, \( \mathcal{M}(\cdot, \cdot) \) cannot be locally open if \( \text{rank}(\mathbf{W}_1) \neq \text{rank}(\mathbf{W}_2) \). On the other hand, when \( \text{rank}(\mathbf{W}_1) = \text{rank}(\mathbf{W}_2) \), the conditions i), ii), iii), and iv) are equivalent due to Lemma 25. Moreover, these conditions imply local openness according to Proposition 27. \( \square \)

### B Proof of Theorem 9

Before proceeding with the proof of Theorem 9, we recall the definition of the symmetric matrix multiplication mapping \( \mathcal{M}_+ : \mathbb{R}^{n \times k} \rightarrow \mathcal{R}_{\mathcal{M}_+} \) with \( \mathcal{M}_+(\mathbf{W}) \triangleq \mathbf{W}^T \mathbf{W} \), where \( \mathcal{R}_{\mathcal{M}_+} \triangleq \{ \mathbf{Z} \in \mathbb{R}^{n \times n} \mid \mathbf{Z} \succeq 0, \text{rank} (\mathbf{Z}) \leq k \} \). In this section, we show that \( \mathcal{M}_+ \) is open in \( \mathcal{R}_{\mathcal{M}_+} \). Particularly, we show that given a matrix \( \mathbf{W} \in \mathbb{R}^{n \times k} \) and a small perturbation \( \mathbf{Z} \in \mathcal{R}_{\mathcal{M}_+} \) of \( \mathbf{Z} \triangleq \mathbf{W}^T \mathbf{W} \), there exists a small perturbation \( \mathbf{W} \) of \( \mathbf{W} \) such that \( \mathbf{Z} = \mathbf{W}^T \mathbf{W} \). Similar to the previous proof scheme, we first show that local openness of \( \mathcal{M}_+(\cdot) \) at \( \mathbf{W} \) is equivalent to local openness of \( \mathcal{M}_+(\cdot) \) at \( \mathbf{U}^T \mathbf{W} \) where \( \mathbf{U}^T \mathbf{\Sigma} \mathbf{U} \) is a symmetric singular value decomposition of the product \( \mathbf{W}^T \).

**Lemma 28.** Consider \( \mathbf{W} \in \mathbb{R}^{n \times k} \) and assume that \( \mathbf{W}^T = \mathbf{U} \mathbf{\Sigma} \mathbf{U}^T \) is a symmetric singular value decomposition of the matrix product \( \mathbf{W}^T \mathbf{W} \) with \( \mathbf{U} \in \mathbb{R}^{n \times n} \) and \( \mathbf{\Sigma} \in \mathbb{R}^{n \times n} \). Then, \( \mathcal{M}_+(\cdot) \) is locally open at \( \mathbf{W} \) if and only if \( \mathcal{M}_+(\cdot) \) is locally open at \( \mathbf{U}^T \mathbf{W} \).

The proof of this lemma is a direct consequence of local openness definition.

According to Lemma 28, proving local openness of \( \mathcal{M}_+(\cdot) \) at \( \mathbf{W} \) is equivalent to proving local openness of \( \mathcal{M}_+(\cdot) \) at \( \mathbf{U}^T \mathbf{W} \). To ease the notation, denote \( \mathbf{U}^T \mathbf{W} \) by \( \mathbf{W} \). Notice that when \( \mathbf{W} \in \mathbb{R}^{n \times n} \) is a full rank square matrix, for any symmetric perturbation \( \mathbf{R}_\delta \) with \( \| \mathbf{R}_\delta \| \leq \delta \) sufficiently small, \( \mathbf{\Sigma} = \mathbf{W}^T \mathbf{W} + \mathbf{R}_\delta \) is a full rank symmetric positive definite matrix. Then finding a perturbation \( \mathbf{W} + \mathbf{A}_\epsilon \) of \( \mathbf{W} \) such that \( (\mathbf{W} + \mathbf{A}_\epsilon)^T(\mathbf{W} + \mathbf{A}_\epsilon) = \mathbf{\Sigma} \) is
equivalent to solving the matrix equation $A_W^T + \tilde{W}A_x^T + A_x^T = R_\delta$. Substituting $A_x = P(W^{-1})^T$ for some matrix $P \in \mathbb{R}^{n \times n}$, we obtain the following quadratic matrix equation

$$P + P^T + P\Sigma^{-1}P^T = R_\delta,$$

where $\Sigma = \tilde{W}W^T$. In the next Lemma, we show how to find a solution matrix $P$ with $\|P\| = \mathcal{O}(\delta)$ that satisfies [19]; thus proving local openness of $\mathcal{M}_+(\cdot)$ at any full rank square matrix $\tilde{W}$.

**Lemma 29.** Let $\Sigma \in \mathbb{R}^{n \times n}$ be a full rank diagonal positive definite matrix. There exists $\delta_0 > 0$ such that for any positive $\delta < \delta_0$ and any symmetric matrix $R \in \mathbb{R}^{n \times n}$ with $\|R\|_\infty \leq \delta$, there exists an upper-triangular matrix $P \in \mathbb{R}^{n \times n}$ with $\|P\|_\infty \leq 3\delta$ satisfying the equation $P + P^T + P\Sigma^{-1}P^T = R$.

Before proving this lemma, let us emphasize that the value of $\delta_0$ depends on $\Sigma$, but is independent of the choice of $R$.

**Proof.** Let us start by simplifying the equation of interest. For all $i = 1, \ldots, n$, let $s_i = \Sigma_{ii}^{-1}$, which is positive by the positive definiteness of $\Sigma$. Then,

$$P + P^T + P\Sigma^{-1}P^T = R \Leftrightarrow \begin{cases} 2P_{ii} + \sum_l l s_l P_{il}^2 = R_{ii} \quad \forall i \\ P_{ij} + P_{ji} + \sum_l l s_l P_{il} P_{jl} = R_{ij} \quad \forall i < j \\ (s_i P_{ii} + 1)^2 + \sum_{l \neq i} s_l s_i P_{il}^2 = s_i R_{ii} + 1 \quad \forall i \\ P_{ij} (s_j P_{jj} + 1) + P_{ji} (s_i P_{ii} + 1) + \sum_{l \neq i, j} s_l P_{il} P_{jl} = R_{ij} \quad \forall i < j \\ P_{ii} = \frac{1}{s_i} \left( \pm \sqrt{s_i R_{ii} + 1 - \sum_{l \neq i} s_l s_i P_{il}^2} - 1 \right) \\ P_{ij} (s_j P_{jj} + 1) + P_{ji} (s_i P_{ii} + 1) + \sum_{l \neq i, j} s_l P_{il} P_{jl} = R_{ij} \quad \forall i < j. \end{cases}$$

An upper-triangular solution $P$ can be generated using the following pseudo-code:

**Algorithm 1** Pseudo-code for generating matrix $P$

1: For all $(i, j)$ with $i > j$, set $P_{ij} = 0$.
2: for $j = n \rightarrow 1$ do
   \[ P_{jj} = \frac{1}{s_j} \left( \pm \sqrt{s_j R_{jj} + 1 - \sum_{l > j} s_l s_j P_{jl}^2} - 1 \right) \]  
3: \quad for $i = j - 1 \rightarrow 1$ do
   \[ P_{ij} = \frac{R_{ij} - \sum_{l > j} s_l P_{il} P_{jl}}{s_j P_{jj} + 1} \]
4: \quad end for
5: end for

Notice that at each iteration of the algorithm corresponding to the $(i, j)$-th index, the corresponding equation is satisfied. Moreover, once an equation is satisfied, the variables in that equation are not going to change anymore; and thus it remains satisfied. We proceed by showing that Algorithm 1 generates a matrix $P$ with $\|P\| \leq 3\delta$ for $\delta$ small enough. In particular, we show that for sufficiently small $\delta > 0$, $|P_{ij}| \leq 2\delta + \mathcal{O}(\delta^2)$ for all $i \leq j$. We prove our result by a reverse induction on $j$:

**Base step, $j = n$ (last column of $P$):** Using (20),

\[ |P_{nn}| = \frac{1}{s_n} \left| \sqrt{s_n R_{nn} + 1} - 1 \right| \leq \frac{1}{s_n} (s_n |R_{nn}| + 1 - 1) = |R_{nn}| \leq \delta. \]
Moreover, \( (21) \) implies \( |P_{in}| = |R_{in}| / |s_n P_{nn} + 1| \). For sufficiently small \( \delta \), \( |s_n P_{nn} + 1| \geq 1/2 \). It follows that \( |P_{in}| \leq 2|R_{in}| \leq 2\delta \).

Induction hypothesis: Assume \( |P_{ij}| \leq 2\delta + O(\delta^2) \) for all \( i \leq j, j = n, \ldots, k \). We show that the result holds for \( k - 1 \). First of all, \( (20) \) implies

\[
|P_{(k-1)(k-1)}| = \frac{1}{s_{k-1}} \left| \sqrt{s_{k-1} R_{(k-1)(k-1)} + 1 - \sum_{l>k-1} s_{k-1} s_l P_{(k-1)l}^2} - 1 \right|
\]

\[
\leq \frac{1}{s_{k-1}} \left| s_{k-1} R_{(k-1)(k-1)} + 1 + \sum_{l>k-1} s_{k-1} s_l P_{(k-1)l}^2 - 1 \right| \leq |R_{(k-1)(k-1)}| + O(\delta^2).
\]

Also, \( |P_{ii(k-1)}| = \frac{|R_{i(k-1)} - \sum_{l>k-1} s_l P_{il}^2 P_{(k-1)l}|}{|s_{k-1} P_{(k-1)(k-1)} + 1|} \), which implies

\[
|P_{ii(k-1)}| \leq \frac{|R_{i(k-1)}| + |\sum_{l>k-1} s_l P_{il}^2| + O(\delta^2)}{|s_{k-1} P_{(k-1)(k-1)} + 1|}.
\]

Thus, for sufficiently small \( \delta \), we have \( |s_{k-1} P_{(k-1)(k-1)} + 1| \geq 1/2 \). Consequently, \( |P_{ii(k-1)}| \leq 2|R_{i(k-1)}| + O(\delta^2) \leq 2\delta + O(\delta^2) \).

Using the above two lemmas we complete the proof of Theorem 9 i.e showing local openness of \( M_+ \) at non-square matrices \( W \).

**Proof.** Proof for the Local Openness of \( M_+ \) at non-square matrices \( \bar{W} \).

To show the openness of the mapping, it suffices to show that it is locally open everywhere. Consider an arbitrary point \( \bar{W} \in \mathbb{R}^{n \times k} \), and let \( U \Sigma U^T \) be a singular value decomposition of the symmetric matrix product \( WW^T \). To ease the notation, denote \( U^T \bar{W} \) by \( \bar{W} \). By Lemma 28, \( M_+ (\cdot) \) is locally open at \( \bar{W} \) if and only if \( M_+ (\cdot) \) is locally open at \( \bar{W} \). When \( WW^T \) is rank deficient, we can write \( \Sigma = \bar{W} \bar{W}^T = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \), where \( \Sigma_1 \in \mathbb{R}^{r \times r} \) is a positive definite diagonal matrix and \( r \) is the rank of \( WW^T \). It is easy to show that the last \( n - r \) rows of \( \bar{W} \) are all zeros, i.e., for all \( j > r \), \( \langle W_{j,:}, (W_{j,:})^T \rangle = \| W_{j,:} \|^2 = 0 \), or equivalently, \( W_{j,:} = 0 \). To show local openness of \( M_+ (\cdot) \) at \( \bar{W} \), we consider a perturbation \( \bar{\Sigma} \triangleq \Sigma + R_\epsilon \) of \( \Sigma \) in the range \( R_{M_+^r} \), and show that there exists a small perturbation \( \bar{\bar{W}} + A_\epsilon \) of \( \bar{W} \) such that \( (\bar{W} + A_\epsilon)^T (\bar{W} + A_\epsilon)^T = \bar{\Sigma} \). By possibly permuting the columns of \( \Sigma \), the perturbed matrix which we know is symmetric positive semi-definite with rank at most \( k \) can be expressed as

\[
\begin{bmatrix}
\Sigma_1 + R_1 \\
R_2^T \\
R_3^T \\
B^T \begin{bmatrix} \Sigma_1 + R_1^T \\ R_2^T \\ R_3^T \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Sigma_1 + R_1 \\
R_2^T \\
R_3^T \\
B^T \begin{bmatrix} \Sigma_1 + R_1^T \\ R_2^T \\ R_3^T \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Sigma_1 + R_1 \\
R_2^T \\
R_3^T \\
B^T \begin{bmatrix} \Sigma_1 + R_1^T \\ R_2^T \\ R_3^T \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Sigma_1 + R_1 \\
R_2^T \\
R_3^T \\
B^T \begin{bmatrix} \Sigma_1 + R_1^T \\ R_2^T \\ R_3^T \end{bmatrix}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\Sigma_1 + R_1 \\
R_2^T \\
R_3^T \\
B^T \begin{bmatrix} \Sigma_1 + R_1^T \\ R_2^T \\ R_3^T \end{bmatrix}
\end{bmatrix}
\]
Let
\[
\mathbf{R}_3 = \begin{bmatrix}
\mathbf{R}_3 & [\mathbf{R}_2^\top | \mathbf{R}_3] \mathbf{B}
\end{bmatrix},
\]
and \(\mathbf{R}_2 = [\mathbf{R}_2 | \mathbf{R}_1 | \mathbf{R}_2] \mathbf{B}\).

Here \(\mathbf{B} \in \mathbb{R}^{k \times (n-k)}\) exists since \(\text{rank}(\Sigma) \leq k\). Moreover, \(\tilde{\Sigma} \geq 0\) for small enough perturbation. Therefore, the Schur complement theorem implies \(\mathbf{R}_3 \succeq \mathbf{R}_2\) \((\Sigma_1 + \mathbf{R}_1)^{-1}\mathbf{R}_2\). Thus \(\tilde{\Sigma} \in \mathcal{R}_{M^+}\) requires \(\mathbf{R}_1\) to be a symmetric \(\mathbb{R}^{r \times r}\) matrix, \(\mathbf{R}_2\) to be an \(\mathbb{R}^{r \times n-r}\) matrix, and \(\mathbf{R}_3\) to be a symmetric \(\mathbb{R}^{(n-r) \times (n-r)}\) matrix with \(\mathbf{R}_3 \succeq \mathbf{R}_2\) \((\Sigma_1 + \mathbf{R}_1)^{-1}\mathbf{R}_2\). For every small perturbation \(\mathbf{R}_\delta = \begin{bmatrix} \mathbf{R}_1 & \mathbf{R}_2 \\ \mathbf{R}_2 & \mathbf{R}_3 \end{bmatrix}\), with \(\|\mathbf{R}_\delta\| \leq \delta\), we need to find \(\mathbf{A}_\epsilon \in \mathbb{R}^{n \times k}\) such that

\[
(\hat{\mathbf{W}} + \mathbf{A}_\epsilon)(\hat{\mathbf{W}} + \mathbf{A}_\epsilon)^\top = \tilde{\Sigma} \text{ or equivalently } \hat{\mathbf{W}} \mathbf{A}_\epsilon^\top + \mathbf{A}_\epsilon \hat{\mathbf{W}}^\top + \mathbf{A}_\epsilon \mathbf{A}_\epsilon^\top = \mathbf{R}_\delta.
\] (22)

Since the last \(n-r\) rows of \(\hat{\mathbf{W}}\) are all zeros, we obtain

\[
\hat{\mathbf{W}} \mathbf{A}_\epsilon^\top = \begin{bmatrix} \mathbf{W}_1 \\ 0 \end{bmatrix} [ (\mathbf{A}_1^{(1)})^\top (\mathbf{A}_2^{(1)})^\top ] = \begin{bmatrix} \hat{\mathbf{W}}_1 (\mathbf{A}_1^{(1)})^\top \\ 0 \end{bmatrix}, \text{ and}
\]

\[
\mathbf{A}_\epsilon \mathbf{A}_\epsilon^\top = \begin{bmatrix} \mathbf{A}_1^2 \\ \mathbf{A}_2^2 \end{bmatrix} [ (\mathbf{A}_1^{(1)})^\top (\mathbf{A}_2^{(1)})^\top ] = \begin{bmatrix} \hat{\mathbf{W}}_1 (\mathbf{A}_1^{(1)})^\top \\ 0 \end{bmatrix} (\mathbf{A}_1^{(1)})^\top
\]

where \(\hat{\mathbf{W}}_1 = (\hat{\mathbf{W}})_{1,r,:} \in \mathbb{R}^{r \times k}\) is a full row rank matrix, \(\mathbf{A}_1^{(1)} \in \mathbb{R}^{r \times k}\), and \(\mathbf{A}_2^{(1)} \in \mathbb{R}^{(n-r) \times k}\). From Equation (22), we get the following three expressions:

\[
\hat{\mathbf{W}}_1 (\mathbf{A}_1^{(1)})^\top + \mathbf{A}_1^2 \hat{\mathbf{W}}_1^\top + \mathbf{A}_1^{(1)} (\mathbf{A}_1^{(1)})^\top = \mathbf{R}_1,
\]

(23)

\[
\hat{\mathbf{W}}_1 (\mathbf{A}_2^{(1)})^\top + \mathbf{A}_2^2 (\mathbf{A}_2^{(1)})^\top = \mathbf{R}_2,
\]

(24)

\[
\mathbf{A}_2^{(1)} (\mathbf{A}_2^{(1)})^\top = \mathbf{R}_3.
\]

(25)

Setting \(\mathbf{A}_1^{(1)} \triangleq \mathbf{P}(\hat{\mathbf{W}}_1^\top)^\top\), where \((\hat{\mathbf{W}}_1)^\top \triangleq \hat{\mathbf{W}}_1^\top (\hat{\mathbf{W}}_1 \hat{\mathbf{W}}_1^\top)^{-1}\), we obtain

\[
\hat{\mathbf{W}}_1 (\mathbf{A}_1^{(1)})^\top + \mathbf{A}_1^2 \hat{\mathbf{W}}_1^\top + \mathbf{A}_1^{(1)} (\mathbf{A}_1^{(1)})^\top = \mathbf{W}_1 \hat{\mathbf{W}}_1 \mathbf{P}^{-1} \mathbf{P}^\top + \mathbf{P} \hat{\mathbf{W}}_1 \mathbf{P}^\top + \mathbf{P} \Sigma_1^{-1} \hat{\mathbf{W}}_1 \mathbf{P}^\top = \mathbf{P}^\top + \mathbf{P} + \mathbf{P} \Sigma_1^{-1} \mathbf{P}^\top.
\]

Using Lemma [29], we can choose \(\delta\) small enough so that for any perturbation matrix \(\mathbf{R}\) with \(\|\mathbf{R}\| < \delta\), there exists a solution \(\mathbf{P}\) with \(\|\mathbf{P}\| = O(\delta)\). More precisely, we can generate \(\mathbf{P} \in \mathbb{R}^{r \times r}\) that satisfies expression (23), with \(\|\mathbf{P}\|_\infty \leq 3\delta\). Also, since \((\hat{\mathbf{W}}_1^\top)^\top = \Sigma_1^{-1}\), we obtain \(\|\mathbf{(W)}_1\|_j^2 \leq \frac{1}{\sigma_{min}}\) \(\forall j \leq r\), where \(\sigma_{min}\) is the minimum singular value for \(\Sigma_1\). Then by definition of \(\mathbf{A}_1^{(1)}\), we can bound its norm:

\[
\|\mathbf{A}_1^{(1)}\| \leq \|\hat{\mathbf{W}}_1\| \|\mathbf{P}\| \leq \frac{\sqrt{r}}{\sigma_{min}} \delta \leq \frac{3r^2 \delta}{\sqrt{\sigma_{min}}}.
\]

(26)

Note that \(\|\mathbf{A}_1^{(1)}\|\) is of order \(\delta\) which can be chosen arbitrarily small so that \(\mathbf{(W)}_1 + \mathbf{A}_1^{(1)}\) is full row rank. Define

\[
(\mathbf{A}_2^{(1)})^\top \triangleq (\mathbf{(W)}_1 + \mathbf{A}_1^{(1)})^\top \mathbf{R}_2 + \mathbf{M},
\]

where \(\mathbf{M} \subset \{\mathbf{M} \in \mathbb{R}^{k \times (n-r)} | \|\mathbf{M}\| \leq \delta, \mathcal{C}(\mathbf{M}) \subset \mathcal{N}(\mathbf{(W)}_1 + \mathbf{A}_1^{(1)})\}\), and

\[
(\mathbf{(W)}_1 + \mathbf{A}_1^{(1)})^\top \triangleq (\mathbf{(W)}_1 + \mathbf{A}_1^{(1)})^\top [\mathbf{(W)}_1 + \mathbf{A}_1^{(1)}(\mathbf{(W)}_1 + \mathbf{A}_1^{(1)})^\top]^{-1} = (\mathbf{(W)}_1 + \mathbf{A}_1^{(1)})^\top (\Sigma_1 + \mathbf{R}_1)^{-1}
\]

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with the last equality obtained using (23). Substituting $A_{\epsilon}^2$ in (24), we obtain

$$(\bar{W}_1 + A_{\epsilon}^1)(A_{\epsilon}^2)^\top = (\bar{W}_1 + A_{\epsilon}^1)(\bar{W}_1 + A_{\epsilon}^1)^\top \bar{R}_2 + (\bar{W}_1 + A_{\epsilon}^1)M = \bar{R}_2,$$

where the last equality is valid since $C(M) \subset N(\bar{W}_1 + A_{\epsilon}^1)$. Substituting $A_{\epsilon}^2$ in (25), we obtain

$$A_{\epsilon}^2(A_{\epsilon}^2)^\top = \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}(\Sigma_1 + R_1)(\Sigma_1 + R_1)^{-1}\bar{R}_2 + M^\top (\bar{W}_1 + A_{\epsilon}^1)^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2 + M^\top M$$

$$\quad + \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}(\bar{W}_1 + A_{\epsilon}^1)M + M^\top M$$

$$\quad = \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2 + M^\top M,$$

where the second inequality holds since $C(M) \subset N(\bar{W}_1 + A_{\epsilon}^1)$. Expression (25) can be satisfied if for any symmetric $R_3 \geq \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2$, there exists $M$ such that $M^\top M = R_3 - \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2$. Since $(\bar{W}_1 + A_{\epsilon}^1) \in \mathbb{R}^{r \times k}$ is a full row rank matrix, then $\dim(N(\bar{W}_1 + A_{\epsilon}^1)) = k - r$. Let $Q \in \mathbb{R}^{k \times (k-r)}$ be a basis for $N(\bar{W}_1 + A_{\epsilon}^1)$. Then for every $M \in \{M \in \mathbb{R}^{k \times (n-r)} \mid \|M\| \leq \delta, C(M) \subset N(\bar{W}_1 + A_{\epsilon}^1)\}$, there exist $N \in \mathbb{R}^{(k-r) \times (n-r)}$ with $M = QN$, which implies $M^\top M = N^\top Q^\top QN = N^\top N$. Since $R_3 - \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2$ is the schur complement of $\Sigma + R_3$, then by the Guttman rank additivity formula, we get $k \geq \text{rank}(\Sigma) = \text{rank}(\Sigma_1 + R_1) + \text{rank}(R_3 - \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2)$, which implies $\text{rank}(R_3 - \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2) \leq k - r$. Thus for any symmetric positive semi-definite matrix $R_3 \geq \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2$, there exist a matrix $N \in \mathbb{R}^{(k-r) \times (n-r)}$ such that $N^\top N = R_3 - \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2$. It follows that there exist a matrix $M \in \mathbb{R}^{k \times (n-r)}$, $M \triangleq QN$, with $M^\top M = N^\top N = \bar{R}_3 - \bar{R}_2^\top (\Sigma_1 + R_1)^{-1}\bar{R}_2$. We have defined $A_{\epsilon} = \begin{bmatrix} A_{\epsilon}^1 \\ A_{\epsilon}^2 \end{bmatrix}$ such that $(\bar{W} + A_{\epsilon})(\bar{W} + A_{\epsilon})^\top = \bar{\Sigma}$.

We now obtain an upper-bound on $\|A_{\epsilon}\|$. Since $((\bar{W}_1 + A_{\epsilon}^1)^\top(\bar{W}_1 + A_{\epsilon}^1)^\top)^\top(\bar{W}_1 + A_{\epsilon}^1)(\bar{W}_1 + A_{\epsilon}^1)^\top(\bar{W}_1 + A_{\epsilon}^1)^\top(\Sigma_1 + R_1)^{-1}(\bar{W}_1 + A_{\epsilon}^1)(\bar{W}_1 + A_{\epsilon}^1)^\top(\Sigma_1 + R_1)^{-1} = (\Sigma_1 + R_1)^{-1}$, we obtain

$$\|(\bar{W}_1 + A_{\epsilon}^1)^\top\| \leq \frac{1}{\sigma_{\min} - \delta} \quad \forall j \leq r.$$

Then by the definition of $A_{\epsilon}^2$, we can bound its norm as follows

$$\|A_{\epsilon}^2\| \leq \|(\bar{W}_1 + A_{\epsilon}^1)^\top\| \|\bar{R}_2\| + \|M\| \leq \frac{\delta \sqrt{r}}{\sqrt{\sigma_{\min} - \delta}} + \delta.$$

Using (26) and (27), we obtain

$$\|A_{\epsilon}\| \leq \frac{3r^{2.5} \delta}{\sqrt{\sigma_{\min}}} + \frac{\delta \sqrt{r}}{\sqrt{\sigma_{\min}} - \delta} + \delta \leq \frac{3r^{2.5} \delta}{\sqrt{\sigma_{\min}}} + \frac{\delta \sqrt{2r}}{\sqrt{\sigma_{\min}}} + \delta$$

$$\leq \delta \frac{3r^{2.5} + \sqrt{2r} + \sqrt{\sigma_{\min}}}{\sqrt{\sigma_{\min}}},$$

where the second inequality assumes $\delta \leq \sigma_{\min}/2$. Now, for a given $\epsilon > 0$, choose

$$\delta \leq \min \left\{ \frac{\epsilon \sqrt{\sigma_{\min}}}{3r^{2.5} + \sqrt{2r} + \sqrt{\sigma_{\min}}}, \frac{\sigma_{\min}}{2} \right\}.$$

This choice of $\delta$ leads to $\|A_{\epsilon}\| \leq \epsilon$, which completes the proof.
C Proof of the Theorem 14

Proof. The proof for the degenerate case is done by constructing a descent direction if the point is critical but not global. Let \((\bar{W}_2, \bar{W}_1)\) be a degenerate critical point, i.e., \(\text{rank}(\bar{W}_2) < \min\{d_2, d_1, d_0\}\). Then, based on the dimensions of \(d_0, d_1, \) and \(d_2\), we have one of the following cases:

- \(d_2 < d_1\) then \(\exists b \neq 0\) such that \(b \in N(\bar{W}_2)\).
- \(d_0 < d_1\) then \(\exists b \neq 0\) such that \(b \in N(\bar{W}_1)\).
- \(d_1 \leq d_2\) and \(d_1 \leq d_0\) then either \(\bar{W}_2\) is rank deficient and \(\exists b \neq 0\) s.t. \(b \in N(\bar{W}_2)\) or \(\bar{W}_1\) is rank deficient and \(\exists b \neq 0\) such that \(b \in N(\bar{W}_1)\).

So in all cases either \(N(\bar{W}_2) \neq \emptyset\) or \(N(\bar{W}_1) \neq \emptyset\). Also, let \(\Delta = \bar{W}_2 \bar{W}_1 X - Y\). If \(\Delta X^T = 0\), then by convexity of the square loss error function, the point \((\bar{W}_2, \bar{W}_1)\) is a global minimum of \((\bar{Y})\). Else, there exists \((i, j)\) such that \(\langle X_{i:,j:}, \Delta_{i:j}\rangle \neq 0\). We now use first and second order optimality conditions to construct a descent direction when the current critical point is not global.

First order optimality condition: By considering perturbations in the directions \(A \in \mathbb{R}^{d_2 \times d_1}\) and \(B \in \mathbb{R}^{d_1 \times d_0}\) for the optimization problem

\[
\min_i \frac{1}{2} \| (\bar{W}_2 + tA)(\bar{W}_1 + tB)X - Y \|^2, \tag{28}
\]

we obtain the first order optimality condition:

\[
\langle AW_1 X + WBX, \Delta \rangle = 0, \quad \forall A \in \mathbb{R}^{d_2 \times d_1}, B \in \mathbb{R}^{d_1 \times d_0}.
\]

and second order optimality condition:

\[
2 \langle ABX, \Delta \rangle + \| AW_1 X + WBX \|^2 \geq 0 \quad \forall A \in \mathbb{R}^{d_2 \times d_1}, B \in \mathbb{R}^{d_1 \times d_0}.
\]

Suppose \((\bar{W}_2, \bar{W}_1)\) is a critical point and there exists \(b \neq 0, b \in N(\bar{W}_2)\). Define

\[
B_{\alpha, l} = \begin{cases} 
\alpha b & \text{if } l = i, \\
0 & \text{otherwise}
\end{cases} 
\]

\[
A_l: \triangleq \begin{cases} 
\alpha b^T & \text{if } l = j, \\
0 & \text{otherwise}
\end{cases}
\]

where \(\alpha\) is a scalar constant. Then, using the second order optimality condition, for \(c = \| AW_1 X \|^2\), we get

\[
2\alpha \sum_{\neq 0} \| X_{:,l:}, \Delta_{:,l:} \|^2 + c \geq 0.
\]

Since this is true for every value of \(\alpha\), \(b\) should be zero which contradicts the assumption on the choice of \(b\). Hence \(N(\bar{W}_2) = \emptyset\). Similarly, when \((\bar{W}_2, \bar{W}_1)\) is a critical point and there exists \(a^T \neq 0, a^T \in N(\bar{W}_1)\), we can show that \((\bar{W}_2, \bar{W}_1)\) is a second order saddle point of \((\bar{Y})\). Combining these results, we get that every degenerate critical point that is not a global optimum is a second-order saddle point.

We now show the result for the non-degenerate case. Let \((\bar{W}_2, \bar{W}_1)\) be a non-degenerate local minimum, i.e., \(\text{rank}(\bar{W}_2) = \min\{d_2, d_1, d_0\}\). It follows by Theorem 8, that the matrix product is locally open at \(\bar{W}_2 \bar{W}_1\). Then by Observation 1, \(Z = \bar{W}_2 \bar{W}_1\) is a local optimum of problem 10, which is in fact global by Lemma 13. \(\square\)

C.1 Proof of Corollary 15

Proof. We follow the same steps used in the proof of Theorem 14 to show the result. Similar to the proof of Theorem 14 we obtain the following first and second order optimality conditions:

\[
\langle AW_1 X + WBX, \nabla \ell(\bar{W}_2 W_1 X - Y) \rangle = 0 \quad \forall A \in \mathbb{R}^{d_2 \times d_1}, B \in \mathbb{R}^{d_1 \times d_0}
\]

\[
2 \langle ABX, \nabla \ell(\bar{W}_2 W_1 X - Y) \rangle + h(AW_1 X, WBX, W_2 W_1 X) \geq 0 \quad \forall A \in \mathbb{R}^{d_2 \times d_1}, B \in \mathbb{R}^{d_1 \times d_0},
\]

where \(h(\cdot)\) is a function that has a tensor representation. But we only need to know that it is a function of \(AW_1 X, WBX\), and \(W_2 W_1 X\). If \(\nabla \ell(\bar{W}_2 W_1 X - Y)X^T = 0\), then by convexity of \(\ell(\cdot)\), \((\bar{W}_2, W_1)\) is a global minimum. Otherwise, there exists \((i, j)\) such that \(\langle X_{i:,j:}, \nabla \ell(\bar{W}_2 W_1 X - Y)_{j,:}\rangle \neq 0\). Using the same former argument in proof of Theorem 14, we choose \(A\) and \(B\) such that \(h(AW_1 X, WBX, W_2 W_1 X)\) is some constant that does not depend on \(\alpha\), and

\[
\langle ABX, \nabla \ell(\bar{W}_2 W_1 X - Y) \rangle = \alpha \langle X_{i:,j:}, \nabla \ell(\bar{W}_2 W_1 X - Y)_{j,:}\rangle, \quad \forall \neq 0
\]

Then by proper choice of \(\alpha\) we show that the point \((\bar{W}_2, \bar{W}_1)\) is a second order saddle point. \(\square\)
### D Proof of Theorem 18

Consider the training problem of a multi-layer deep linear neural network:

$$\min_{\mathbf{W}} \frac{1}{2} \| \mathbf{W}_h \cdots \mathbf{W}_1 \mathbf{X} - \mathbf{Y} \|^2. \tag{29}$$

Here $\mathbf{W} = (\mathbf{W}_i)_{i=1}^h$, $\mathbf{W}_i \in \mathbb{R}^{d_i \times d_{i-1}}$ are the weight matrices, $\mathbf{X} \in \mathbb{R}^{d_0 \times n}$ is the input training data, and $\mathbf{Y} \in \mathbb{R}^{d_h \times n}$ is the target training data. Based on our general framework, the corresponding auxiliary optimization problem is given by

$$\min_{\mathbf{Z} \in \mathbb{R}^{n \times d_0}} \frac{1}{2} \| \mathbf{Z} \mathbf{X} - \mathbf{Y} \|^2$$

subject to $\text{rank}(\mathbf{Z}) \leq d_p \triangleq \min_{0 \leq i \leq h} d_i$. \tag{30}

**Lemma 30.** Consider a degenerate critical point $\mathbf{W} = (\mathbf{W}_h, \ldots, \mathbf{W}_1)$ with $\mathcal{N}(\mathbf{W}_i)$ and $\mathcal{N}(\mathbf{W}_i^\top)$ for $h-1 \leq i \leq 2$ all non-empty. If

$$\mathcal{N}(\mathbf{W}_h) \text{ is non-empty or } \mathcal{N}(\mathbf{W}_1^\top) \text{ is non-empty},$$

then $\mathbf{W}$ is either a global minimum or a saddle point of problem (29).

**Proof.** Suppose that $\mathcal{N}(\mathbf{W}_h)$ is non-empty. Let $\Delta = \mathbf{W}_h \cdots \mathbf{W}_1 \mathbf{X} - \mathbf{Y}$. If $\Delta \mathbf{X}^\top = 0$, by convexity of the loss function, the point $\mathbf{W} = (\mathbf{W}_h, \ldots, \mathbf{W}_1)$ is a global minimum of (29). Else, there exist $(i, j)$ such that $(\mathbf{X}_i, \Delta_{ij}) \neq 0$. We define the set $\mathcal{K} \triangleq \{ k \in \mathbb{N} | 3 \leq k \leq h \}, \mathcal{N}(\mathbf{W}_k) \perp \mathcal{N}(\mathbf{W}_{k-1} \mathbf{W}_{k-2} \cdots \mathbf{W}_2) \}$. We split the rest of the proof into two cases that correspond to $\mathcal{K}$ being empty and non-empty.

**Case a:** Assume $\mathcal{K}$ is non-empty. We define $k^* \triangleq \max k$. By definition of the set $\mathcal{K}$ and choice of $k^*$, the null space $\mathcal{N}(\mathbf{W}_{k^*})$ is orthogonal to the null-space $\mathcal{N}(\mathbf{W}_{k^* - 1} \cdots \mathbf{W}_2)$. This implies there exists a non-zero $\mathbf{b} \in \mathbb{R}^{d_{k^* - 1}}$ such that $\mathbf{b} \in \mathcal{N}(\mathbf{W}_{k^*}) \cap \mathcal{C}(\mathbf{W}_{k^* - 1} \cdots \mathbf{W}_2)$. By considering perturbation directions $\mathbf{A} = (\mathbf{A}_h, \ldots, \mathbf{A}_1)$, $\mathbf{A}_i \in \mathbb{R}^{d_i \times d_{i-1}}$ for the optimization problem

$$\min_{t} g(t) \triangleq \frac{1}{2} \| (\mathbf{W}_h + t \mathbf{A}_h) \cdots (\mathbf{W}_1 + t \mathbf{A}_1) \mathbf{X} - \mathbf{Y} \|^2, \tag{31}$$

we examine the optimality conditions for a specific direction $\mathbf{A}$.

Let

$$(\mathbf{A}_h)_{l, j} \triangleq \begin{cases} \alpha_h \mathbf{p}_h^\top & \text{if } l = j, \\ 0 & \text{otherwise} \end{cases} \quad (\mathbf{A}_1)_{l, j} \triangleq \begin{cases} \alpha_1 \mathbf{b}_1 & \text{if } l = i, \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbf{A}_k \triangleq \begin{cases} \mathbf{b}_k \mathbf{p}_k^\top & \text{if } k^* + 1 \leq k \leq h - 1, \\ \mathbf{b}_k \mathbf{b}_k^\top & \text{if } k = k^*, \\ 0 & \text{if } 2 \leq k \leq k^* - 1, \end{cases}$$

where $\alpha_h$ and $\alpha_1$ are scalar constants, $\mathbf{b}_1 \in \mathbb{R}^{d_1}$ such that $\mathbf{W}_{k^* - 1} \cdots \mathbf{W}_2 \mathbf{b}_1 = \mathbf{b}$, and

$$\mathbf{p}_k \in \mathcal{N}(\mathbf{W}_{k-1} \cdots \mathbf{W}_2)^\top, \mathbf{b}_{k-1} \in \mathcal{N}(\mathbf{W}_k), \text{ and } \langle \mathbf{p}_k, \mathbf{b}_{k-1} \rangle \neq 0 \forall k^* + 1 \leq k \leq h. \tag{32}$$

Notice that such $\mathbf{p}_k$ and $\mathbf{b}_{k-1}$ exist from the definition of $\mathcal{K}$ and choice of $k^*$. For this particular choice of $\mathbf{A} = (\mathbf{A}_h, \ldots, \mathbf{A}_1)$, we obtain

$$\mathbf{W}_{k+1} \mathbf{A}_k = 0 \text{ for } k^* \leq k < h - 1; \text{ and } \mathbf{A}_k \mathbf{W}_{k-1} \cdots \mathbf{W}_2 = 0 \text{ for } k^* + 1 \leq k \leq h. \tag{33}$$

We now show that $(\mathbf{A}_h, \ldots, \mathbf{A}_1)$ is in fact a descent direction. Before proceeding, let us define some notation to ease the expressions of the optimality conditions. Let $\mathcal{V}$ be an index set that is a subset of $\{1, \ldots, h\}$. We define the function $f(\mathbf{A}^\top, \mathbf{W}^\top)$ which is the matrix product attained from $\mathbf{W}_h \cdots \mathbf{W}_1 \mathbf{X}$ by replacing matrices $\mathbf{W}_v$ by matrices $\mathbf{A}_v$ for every $v \in \mathcal{V}$. For instance, if $h = 5$ and $\mathcal{V} = \{2, 3, 5\}$, then $f(\mathbf{A}^\top, \mathbf{W}^\top) = \mathbf{A}_5 \mathbf{W}_4 \mathbf{A}_3 \mathbf{A}_2 \mathbf{W}_1 \mathbf{X}$. We now determine index sets $\mathcal{V}$, with $|\mathcal{V}| \geq 1$, that correspond to non-zero $f(\mathbf{A}^\top, \mathbf{W}^\top)$. First note by definition of $\mathbf{A}$, if $\mathcal{V} \cap \{k^* - 1, \ldots, 2\} \neq \emptyset$, then $f(\mathbf{A}^\top, \mathbf{W}^\top) = 0$. Also by (33), for any $k^* \leq v \leq h - 1$, if $v \in \mathcal{V}$ then either $\{k^*, \ldots, h\} \in \mathcal{V}$ or $f(\mathbf{A}^\top, \mathbf{W}^\top) = 0$. This implies that $\mathbf{A}_h \cdots \mathbf{A}_k \mathbf{W}_{k-1} \cdots \mathbf{W}_1 \mathbf{X}$ and
\( \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_2 \bar{A}_1 X \) are the only terms that can take non-zero values. Using the definition equation \([31]\) we obtain

\[
g(t) = \frac{1}{2} \| t^{h-k^*+1} \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_1 X + t^{h-k^*+2} \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_2 \bar{A}_1 X + \Delta \|^2.
\]

It follows that

\[
\left. \frac{\partial^r g(t)}{\partial t^r} \right|_{t=0} = 0 \quad \text{for all } r \leq h-k^*
\]

and

\[
\left. \frac{\partial^{h-k^*+1} g(t)}{\partial t^{h-k^*+1}} \right|_{t=0} = c_1 \langle \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_1 X, \Delta \rangle,
\]

where \( c_1 > 0 \) is a scalar. If \( \langle \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_1 X, \Delta \rangle \neq 0 \), then by properly choosing the sign of \( \alpha_h \) such that \( \langle \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_1 X, \Delta \rangle < 0 \), we get a descent direction. Otherwise,

\[
\left. \frac{\partial^{h-k^*+2} g(t)}{\partial t^{h-k^*+2}} \right|_{t=0} = c_1 \langle \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_2 \bar{A}_1 X, \Delta \rangle + h(\bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_1 X).
\]

where \( c_1 > 0 \) is a scalar, and \( h(\cdot) \) is a function of \( \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_1 X \).

We now evaluate the term \( \langle \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_2 \bar{A}_1 X, \Delta \rangle \). Since \( (\bar{A}_h)_{i,j} = 0 \) for all \( l \neq j \) and \( (\bar{A}_1)_{i,l} = 0 \) for all \( l \neq i \), we only need to compute the \((j,i)\) index \( \langle \bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_2 \bar{A}_1 \rangle_{(j,i)} \) as all other indices are zero.

For some constant \( c = p^h_1 b_{h-1} p^T_{h-1} b_{h-2} \cdots p^1_{k+1} b_k b^T b \), we obtain

\[
c_1(\bar{A}_h \cdots \bar{A}_k \cdot W_{k-1} \cdots W_2 \bar{A}_1)_{(j,i)}
\]

\[
= c_1 \alpha_h \alpha_1 c \langle X_i, \Delta_j \rangle + c_0 < 0, \text{ we get a descent direction. This completes the first case.}
\]

**Case b:** Assume \( K \) is empty. We consider

\[
(\bar{A}_h)_{i,l} \begin{cases} 
\alpha_h p^T_i & \text{if } l = j, \\
0 & \text{otherwise}
\end{cases}
\begin{align*}
(\bar{A}_1)_{i,l} & \begin{cases} 
\alpha_1 b_1 & \text{if } l = i, \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

\[
\bar{A}_k \begin{cases} 
b_k p^T_k & \text{if } 3 \leq k \leq h - 1, \\
b_k b^T_k & \text{if } k = 2,
\end{cases}
\]

where \( \alpha_h \) and \( \alpha_1 \) are scalar constants, \( b_1 \in N(\bar{W}_2) \), and

\[
p_k \in N((\bar{W}_{k-1} \cdots \bar{W}_1)^T), b_{k-1} \in N(\bar{W}_k), \text{ and } (p_k, b_{k-1}) \neq 0, \forall 3 \leq k \leq h.
\]

For this particular choice of \( \bar{A} = (\bar{A}_h, \ldots, \bar{A}_1) \), we obtain \( \bar{W}_{k+1} \bar{A}_k = 0 \) for \( 2 \leq k \leq h - 1 \); and \( \bar{A}_k \bar{W}_{k-1} \cdots \bar{W}_2 = 0 \) for \( 3 \leq k \leq h \). We now determine index sets \( V \), with \( |V| \geq 1 \), that correspond to non-zero \( f(\bar{A}^V, \bar{W}^V) \). By \([34]\), for any \( 2 \leq v \leq h - 1 \), if \( v \in V \) then either \( \{2, \ldots, h\} \in V \) or \( f(\bar{A}^V, \bar{W}^V) = 0 \). This directly imply that \( \bar{A}_h \cdots \bar{A}_2 \bar{W}_1 X \) and \( \bar{A}_h \cdots \bar{A}_1 X \) are the only terms that can take non-zero values. Using the definition of equation \([31]\) we obtain

\[
g(t) = \frac{1}{2} \| t^{h-1} \bar{A}_h \cdots \bar{A}_2 \bar{W}_1 X + t^h \bar{A}_h \cdots \bar{A}_1 X + \Delta \|^2.
\]

It follows that

\[
\left. \frac{\partial^r g(t)}{\partial t^r} \right|_{t=0} = 0 \quad \text{for all } r \leq h - 2,
\]

and

\[
\left. \frac{\partial^{h-1} g(t)}{\partial t^{h-1}} \right|_{t=0} = c_1 \langle \bar{A}_h \cdots \bar{A}_2 \bar{W}_1 X, \Delta \rangle, \text{ where } c_1 > 0 \text{ is a scalar. If } \langle \bar{A}_h \cdots \bar{A}_2 \bar{W}_1 X, \Delta \rangle \neq 0, \text{ then by properly choosing the sign of } \alpha_h \text{ such that } \langle \bar{A}_h \cdots \bar{A}_2 \bar{W}_1 X, \Delta \rangle < 0, \text{ we get a descent direction. Otherwise,}
\]

\[
\left. \frac{\partial^h g(t)}{\partial t^h} \right|_{t=0} = c_1 \langle \bar{A}_h \cdots \bar{A}_1 X, \Delta \rangle + h(\bar{A}_h \cdots \bar{A}_2 \bar{W}_1 X),
\]

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where \( c_1 > 0 \) is a scalar, and \( h(\cdot) \) is a function of \( \bar{A}_h \cdots \bar{A}_2 \bar{W}_1 \mathbf{X} \). We now evaluate the term \( \langle \bar{A}_h \cdots \bar{A}_1 \mathbf{X}, \Delta \rangle \). Since \( (\bar{A}_h)_{l,j} = 0 \) for all \( l \neq j \) and \( (\bar{A}_1)_{l,j} = 0 \) for all \( l \neq i \), we only need to compute the \((j, i)\) index \( (\bar{A}_h \cdots \bar{A}_1)_{(j, i)} \) as all other indices are zero. For some constant

\[
c = p_h \bar{b}_{h-1} p_{h-1}^T \bar{b}_{h-2} \cdots p_2^T \bar{b}_2 \bar{b}_1^T \mathbf{b}_1,
\]

we obtain

\[
c_1 (\bar{A}_h \cdots \bar{A}_1)_{(j, i)} = c_1 \alpha_h \alpha_1 p_h \bar{b}_{h-1} p_{h-1}^T \bar{b}_{h-2} \cdots p_2^T \bar{b}_2 \bar{b}_1^T \mathbf{b}_1 = \alpha_h \alpha_1 c,
\]

where \( c \) is non-zero by our choice of \( \mathbf{b}, p_h \) and \( b_{k-1} \) for \( 3 \leq k \leq h \) as defined in (34). For a fixed \( \alpha_h \neq 0 \), \( h(\bar{A}_h \cdots \bar{A}_2 \bar{W}_1 \mathbf{X}) \) is a constant scalar we denote by \( c_{\alpha} \). Then by properly choosing \( \alpha_1 \) such that

\[
\frac{\alpha_h}{c_{\alpha}} < 0,
\]

we get a descent direction. This completes the second case.

Now if \( \mathcal{N}(\bar{W}_1^T) \) is non-empty, we define the set

\[
\mathcal{K} \triangleq \{k \mid 1 \leq k \leq h - 2, \mathcal{N}(\bar{W}_{h-1} \cdots \bar{W}_{k+1}) \perp \mathcal{N}(\bar{W}_k^T)\},
\]

and use a similar proof scheme to show the result. More specifically, we split the proof into two cases that correspond to \( \mathcal{K} \) being empty and non-empty.

**Case a:** Assume \( \mathcal{K} \) is non-empty. We define \( k^* \triangleq \min \mathcal{K} \). By definition of the set \( \mathcal{K} \) and choice of \( k^* \), the null space \( \mathcal{N}(\bar{W}_k^T) \) is orthogonal to the null-space \( \mathcal{N}(\bar{W}_{h-1} \cdots \bar{W}_{k^*+1}) \). This implies there exists a non-zero \( \mathbf{p} \in \mathbb{R}^{d_{h-1}} \) such that \( \mathbf{p} \in \mathcal{N}(\bar{W}_k^T) \cap \mathcal{C}(\bar{W}_{h-1} \cdots \bar{W}_{k^*+1}) \). By considering perturbation in directions \( \mathbf{A} = (\bar{A}_h, \ldots, \bar{A}_1) \), \( \bar{A}_i \in \mathbb{R}^{d_i \times d_{i-1}} \) for the optimization problem

\[
\min_{\mathbf{t}} g(t) \triangleq \frac{1}{2} \| (\bar{W}_h + t \bar{A}_h) \cdots (\bar{W}_1 + t \bar{A}_1) \mathbf{X} - \mathbf{Y} \|^2,
\]

we examine the optimality conditions for a specific direction \( \bar{A} \).

Let

\[
(\bar{A}_h)_{l,j} \triangleq \begin{cases} \alpha_h \mathbf{p}_h^T & \text{if } l = j, \\ 0 & \text{otherwise} \end{cases}, \quad (\bar{A}_1)_{l,j} \triangleq \begin{cases} \alpha_1 \mathbf{b}_1 & \text{if } l = i, \\ 0 & \text{otherwise} \end{cases},
\]

\[
\bar{A}_k \triangleq \begin{cases} \mathbf{b}_k \mathbf{p}_k^T & \text{if } 2 \leq k \leq k^* - 1, \\ \mathbf{p} \mathbf{p}_k^T & \text{if } k = k^* \\ 0 & \text{if } k^* + 1 \leq k \leq h - 1, \end{cases}
\]

where \( \alpha_h \) and \( \alpha_1 \) are constants and \( \mathbf{p}_h \in \mathbb{R}^{d_{h-1}} \) with

\[
\mathbf{p}_h^T \bar{W}_{h-1} \cdots \bar{W}_{k^*+1} = \mathbf{p}^T, \quad \mathbf{p}_k \in \mathcal{N}(\bar{W}_k^T), \quad \mathbf{b}_k \in \mathcal{N}(\bar{W}_{h-1} \cdots \bar{W}_k), \quad \text{and } (\mathbf{p}_k, \mathbf{b}_{k-1}) \neq 0
\]

for all \( 2 \leq k \leq k^* \). Notice that such \( \mathbf{p}_k \) and \( \mathbf{b}_{k-1} \) exist from the definition of \( \mathcal{K} \) and choice of \( k^* \). For this particular choice of \( \mathbf{A} = (\bar{A}_h, \ldots, \bar{A}_1) \), we obtain

\[
\bar{A}_k \bar{W}_{k-1} = 0 \quad \text{for } 2 \leq k \leq k^*; \quad \text{and } \bar{W}_{h-1} \cdots \bar{W}_{k+1} \bar{A}_k = 0 \quad \text{for } 1 \leq k \leq k^* - 1.
\]

(36)

The same argument used above can be used to show that \( (\bar{A}_h, \ldots, \bar{A}_1) \) is actually a descent direction. This completes the proof of the first case.

**Case b:** Assume \( \mathcal{K} \) is empty. We consider

\[
(\bar{A}_h)_{l,j} \triangleq \begin{cases} \alpha_h \mathbf{p}_h^T & \text{if } l = j, \\ 0 & \text{otherwise} \end{cases}, \quad (\bar{A}_1)_{l,j} \triangleq \begin{cases} \alpha_1 \mathbf{b}_1 & \text{if } l = i, \\ 0 & \text{otherwise} \end{cases},
\]

\[
\bar{A}_k \triangleq \begin{cases} \mathbf{b}_k \mathbf{p}_k^T & \text{if } 2 \leq k \leq h - 2, \\ \mathbf{p}_h \mathbf{p}_k^T & \text{if } k = h - 1, \end{cases}
\]

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where \( \alpha_h \) and \( \alpha_1 \) are scalar constants, \( p_h \in \mathcal{N}(\bar{W}_{h-1}^T) \), and \( p_k \in \mathcal{N}(\bar{W}_{k-1}^T) \), \( b_{k-1} \in \mathcal{N}(\bar{W}_{h-1} \cdots \bar{W}_k) \), and \( \langle p_k, b_{k-1} \rangle \neq 0 \) \( \forall 2 \leq k \leq h-1 \). For this particular choice of \( \bar{A} \), we obtain \( \bar{A}_k \bar{W}_{k-1} = 0 \) for \( 2 \leq k \leq h-1 \) and \( \bar{W}_{h-1} \cdots \bar{W}_{k+1} \bar{A}_k = 0 \) for \( 1 \leq k \leq h-2 \). The same argument used above can be used to show that \((\bar{A}_h, \ldots, \bar{A}_1)\) is actually a descent direction. This completes the second case and thus completes the proof. 

Following the same steps of the proof in Lemma 30, we get the same result when replacing the square loss error by a general convex and differentiable function \( \ell(\cdot) \). We are now ready to prove the main result restated below.

**Theorem 18** If there does not exist \( p_1 \) and \( p_2 \), \( 1 \leq p_1 < p_2 \leq h-1 \) with \( d_h > d_{p_2} \) and \( d_0 > d_{p_1} \), then every local minimum of problem (16) is a global minimum.

**Proof.** We now show that if such a pair \( \{p_2, p_1\} \) does not exist, then if \( \bar{W} \) is not a global minimum, we can construct a descent direction.

First notice that if for some \( 1 \leq i \leq h-1 \), \( \bar{W}_i \) is full column rank, then using Proposition 1, \( \mathcal{M}_{i+1, i}(\cdot) \) is locally open at \((\bar{W}_{i+1}, \bar{W}_i)\) and \( \bar{W}_{i+1} \bar{W}_i \in \mathbb{R}^{d_{i+1} \times d_i} \). Using Observation 1, we conclude that any local minimum of problem (33) relates to a local minimum of the problem obtained by replacing \( \bar{W}_{i+1} \bar{W}_i \) by \( \bar{Z}_{i+1, i} \in \mathbb{R}^{d_{i+1} \times d_i} \). By a similar argument, we conclude that if \( \bar{W}_i \) is a full row rank for some \( 2 \leq i \leq h \), any local minimum of problem (33) relates to local minimum of the problem obtained by replacing \( \bar{W}_i \bar{W}_{i-1} \) by \( \bar{Z}_{i, i-1} \in \mathbb{R}^{d_i \times d_{i-1}} \). Thus, if \( \bar{W} = (\bar{W}_h, \ldots, \bar{W}_1) \) is a local minimum of problem (33), the new point \( \bar{Z} = (\bar{Z}'_h, \ldots, \bar{Z}'_1) \), where \( \bar{Z}_i \in \mathbb{R}^{d'_i \times d'_{i-1}} \) and \( h' \leq h \), is a local minimum of the problem attained by applying the replacements discussed above. If \( h' = 1 \), we get the desired result from Lemma 7. Else, if \( h' > 2 \), the auxiliary problem becomes a two layer linear network for which Theorem 8 provides the desired result. When \( h' > 2 \), examine \( d'_{h'}, d'_{h'-1}, d'_1 \) and \( d'_0 \). If \( d'_{h'} > d'_{h'-1} \) and \( d'_0 > d'_1 \), then there exist \( 1 \leq p_1 < p_2 \leq h-1 \) with \( d_h > d_{p_2} \) and \( d_0 > d_{p_1} \), which contradicts our assumption. It follows by construction of \( \bar{Z}_i \), that either \( d'_{h'} \leq d'_{h'-1} \) and \( \bar{Z}'_h \) is not full column rank or \( d'_0 \leq d'_1 \) and \( \bar{Z}'_1 \) is not full column rank; thus at least one of the null spaces \( \mathcal{N}(\bar{Z}'_{h'}) \), \( \mathcal{N}(\bar{Z}'_1) \) is non-empty. Moreover, \( \bar{Z}_i \) has non-empty right and left null spaces for \( 2 \leq i \leq h-1 \). The result follows using Lemma 30. \( \square \)