SECOND ADJOINTNESS FOR TEMPERED
ADMISSIBLE REPRESENTATIONS OF A REAL GROUP

BY

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ABSTRACT

We study second adjointness in the context of tempered admissible representations of a real reductive group. Compared to a recent result of Crisp and Higson, this generalizes from $SL_2$ to a general group, but specializes to only considering admissible representations. We also discuss Casselman’s canonical pairing in this context, and the relation to Bernstein morphisms. Additionally, we take the opportunity to discuss some relevant functors and some of their relations.

0. Introduction

0.1. SECOND ADJOINTNESS. Let $G$ be a connected reductive group over a local field $F$. Let $P, P^- \subset G$ be opposite parabolics defined over $F$, with Levi $L = P \cap P^-$. One has the functors of parabolic restriction and induction w.r.t. $P$, which form an adjunction

$$\text{pres} : \mathcal{M}(G(F)) \rightleftarrows \mathcal{M}(L(F)) : \text{pind}$$

(meaning that $\text{pres}$ is the left adjoint of $\text{pind}$). Here $\mathcal{M}(\cdot)$ is the category of smooth representations (over $\mathbb{C}$) in the case when $F$ is non-archimedean, and is the category of smooth Frechet representations of moderate growth (over $\mathbb{C}$) in the case when $F$ is archimedean. The functor $\text{pres}$ is usually also known as

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the Jacquet functor. Let us denote similarly by

$$\text{pres}^- : \mathcal{M}(G(F)) \rightarrow \mathcal{M}(L(F))$$

the parabolic restriction where we use the parabolic $P^-$ instead of $P$.

The functor $\text{pind}$ is exact. In the non-archimedean case, the functor $\text{pres}$ is exact as well (this is a basic result of Jacquet) and one has the fundamental second adjointness theorem of Joseph Bernstein:

**Theorem (J. Bernstein):** Suppose that $F$ is non-archimedean. Then there is a canonical adjunction

$$\text{pind} : \mathcal{M}(L(F)) \rightleftarrows \mathcal{M}(G(F)) : \text{pres}^-.$$  

In the archimedean case, things become more complicated—the functor $\text{pres}$ is not exact and second adjointness does not hold in its above formulation.

Let us from now on assume that $F = \mathbb{R}$.

Let us consider the subcategories $\mathcal{M}(\cdot)_{\text{temp}} \subset \mathcal{M}(\cdot)$ of tempered representations (those are, morally, representations whose matrix coefficients are close to being square integrable, and thus they have a chance of contributing to the Plancherel decomposition of $L^2(G(\mathbb{R}))$). The functor $\text{pind}$ preserves these, but $\text{pres}$ does not. Nevertheless, one still has an adjunction

$$\text{temppres} : \mathcal{M}(G(\mathbb{R}))_{\text{temp}} \rightleftarrows \mathcal{M}(L(\mathbb{R}))_{\text{temp}} : \text{pind},$$

where $\text{temppres}(V)$ is the biggest tempered quotient of $\text{pres}(V)$. Of course, we also denote by $\text{temppres}^-$ the analogous functor where we use $P^-$ instead of $P$.

It was relatively recently shown by T. Crisp and N. Higson:

**Theorem ([CrHi]):** Suppose that $G = SL_2$. Then there is a canonical adjunction

$$\text{pind} : \mathcal{M}(L(\mathbb{R}))_{\text{temp}} \rightleftarrows \mathcal{M}(G(\mathbb{R}))_{\text{temp}} : \text{temppres}^-.$$  

Let us consider the subcategories $\mathcal{M}^a(\cdot) \subset \mathcal{M}(\cdot)$ of admissible representations (we use terminology where those are the representations whose underlying $(\mathfrak{g}, K)$-module is of finite length). The main observation of this paper is that

$$\text{temppres} : \mathcal{M}^a(G(\mathbb{R}))_{\text{temp}} \rightarrow \mathcal{M}^a(L(\mathbb{R}))_{\text{temp}}$$

is exact (Proposition 3.16), and the following theorem holds:

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1 Actually, modulo the center, as usual.
THEOREM (Theorem 3.18): There is a canonical adjunction
\[ \text{pind} : M^a(L(\mathbb{R}))_{\text{temp}} \rightleftharpoons M^a(G(\mathbb{R}))_{\text{temp}} : \text{temppres}^- \].

Remark: Thus, relative to the result of [CrHi], we generalize from $SL_2$ to a general group, but specialize to only considering admissible representations. In this paper, we don’t deal with non-admissible representations.

0.2. Canonical pairing. In the admissible case, second adjointness is easily shown to be equivalent to the existence and non-degeneracy of Casselman’s canonical pairing between Jacquet modules. In our setting, this is the following. Denote by
\[ (\cdot)^\vee : M^a(G(\mathbb{R})) \xrightarrow{\cong} M^a(G(\mathbb{R}))^{\text{op}} \]
the functor of passing to the contragradient representation. Then Theorem 3.18 above is equivalent to:

THEOREM (Theorem 3.17): Let $V \in M^a(G(\mathbb{R}))_{\text{temp}}$. Then there is a canonical non-degenerate pairing
\[ \text{temppres}^-(V) \otimes \text{temppres}(V^\vee) \rightarrow \mathbb{C}. \]

The point of restricting attention to tempered representations in the archimedean case, from a technical perspective, is as follows. In the archimedean case, when one considers not necessarily tempered representations, Casselman’s canonical pairing exists between Casselman–Jacquet modules rather than Jacquet modules (in contrast with the non-archimedean case). The non-exactness of the Jacquet functor is responsible for this pairing not passing to a pairing between Jacquet modules. However, when one restricts attention to tempered representations, the possible exponents have a conical constraint, which causes the reading of temppres from the Casselman–Jacquet module to be exact, and things are again orderly.

0.3. Relation to Bernstein morphisms. In [DeKnKrSc], the authors construct Bernstein morphisms for real spherical varieties, as [SaVe] did for non-archimedean spherical varieties, both following ideas of J. Bernstein. In a special case of the general setting, relevant for the current paper, this is an isometric embedding
\[ \text{Ber}_I : L^2((G(\mathbb{R}) \times G(\mathbb{R}))/\Delta L(\mathbb{R}) \cdot (N^-(\mathbb{R}) \times N(\mathbb{R})))) \rightarrow L^2(G(\mathbb{R})) \]
(where $N, N^-$ are the unipotent radicals of $P, P^-$).
In §4 we will indicate how the canonical pairing for tempered admissible representations of this paper should be related to the construction of Ber_\mathcal{I}. The verification should be a straight-forward translation between the languages of [DeKnKrSc] and the current paper, but we don’t try to present details here.

0.4. **Non-tempered admissible representations.** The purpose of the second part of this paper is twofold. First, in §5 we would like to record some of the ideas from our Ph.D. thesis [Yo1] in a bit more organized and complete way. Second, in §6 we will use this to present the proof of Theorem 3.18 in a different way, which gives another point of view, putting an emphasis on what is the right adjoint of pind when one considers not necessarily tempered representations, and why it differs from pres^−.

Namely, it is explained that the right adjoint of

\[ \text{pind} : \mathcal{M}^a(L(\mathbb{R})) \to \mathcal{M}^a(G(\mathbb{R})) \]

is

\[ V \mapsto \mathbb{C}_{\rho_P} \otimes \mathcal{J}_P(V)^n, \]

while the functor pres^− is given by

\[ V \mapsto \mathbb{C}_{\rho_P} \otimes \mathcal{J}_P(V)/n^-\mathcal{J}_P(V), \]

and the former functor has an obvious map into the latter. Here \( \mathcal{J}_P(V) \) is the Casselman–Jacquet module, \( n, n^- \) are the Lie algebras of \( N, N^- \), and \( \mathbb{C}_{\rho_P} \otimes - \) are some standard \( \rho \)-twists.

We plan to further study this situation for non-tempered representations in the future.

0.5. **Dissatisfaction.** Throughout the paper, we use some analytical inputs, the main one being Casselman’s canonical pairing. It is our hope that in the future we will be able to treat all of these inputs algebraically.

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1. Setting and notations

1.1. The group. We fix the following. Let $G$ be a connected reductive algebraic group over $\mathbb{C}$, together with a real form $\sigma$ (so $G(\mathbb{R}) = G^\sigma$). Let $\theta$ be a Cartan involution of $(G, \sigma)$. Let $K := G^\theta$ be the resulting complexification of the maximal compact subgroup $K(\mathbb{R}) = G(\mathbb{R})^\theta$. We denote by $\mathfrak{g}, \mathfrak{g}_\mathbb{R}$ the Lie algebras of $G, G(\mathbb{R})$. We choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{g}^\theta$. We denote by $R \subset \mathfrak{a}^*$ the subset of roots. We choose a system of positive roots $R^+ \subset R$, with simple roots $\Sigma \subset R^+$. For $I \subset \Sigma$, we have the corresponding standard parabolic $G_I \cdot N(I) \subset G$ (where $G_I$ is the Levi subgroup and $N(I)$ is the unipotent radical), and also its opposite $G_I \cdot N(I)^- \subset G$. We set

$$K_I := K \cap G_I = K \cap (G_I \cdot N(I)).$$

For example, $G_{\Sigma} = G$. We use the standard Gothic notations for corresponding Lie algebras.

Let $I \subset \Sigma$. We denote

$$R^+_I := R^+ \cap \left( \sum_{\alpha \in I} \mathbb{Z}_{\geq 0} \cdot \alpha \right), \quad R^+_I := R^+ \setminus R^+_I.$$

We denote

$$\rho_I = \frac{1}{2} \sum_{\alpha \in R^+_I} \alpha \in \mathfrak{a}^*, \quad \rho(I) = \frac{1}{2} \sum_{\alpha \in R^+_I} \alpha \in \mathfrak{a}^*.$$

We denote

$$\mathfrak{a}_{\text{cent}, I} := \mathfrak{z}_{\mathfrak{a}}(\mathfrak{g}_I) = \{ H \in \mathfrak{a} \mid \alpha(H) = 0 \ \forall \alpha \in I \} \subset \mathfrak{a}.$$

Also, we denote

$$\mathfrak{a}^{+, I} := \{ H \in \mathfrak{a} \mid \alpha(H) \geq 0 \ \forall \alpha \in I \} \subset \mathfrak{a}.$$

Finally, we denote by $\leq_I$ the partial order on $\mathfrak{a}^*$ given by $\lambda \leq_I \mu$ if $(\mu - \lambda)(H) \geq 0$ for all $H \in \mathfrak{a}^{+, I}$.

1.2. Modules. Let $\mathfrak{h}$ be a reductive Lie algebra. We denote by $\mathcal{M}(\mathfrak{h})$ the abelian category of $\mathfrak{h}$-modules. By an admissible $\mathfrak{h}$-module, we understand an $\mathfrak{h}$-module $V$ which is finitely generated over $U(\mathfrak{h})$ and is $Z(\mathfrak{h})$-finite. We denote by

$$\mathcal{M}^a(\mathfrak{h}) \subset \mathcal{M}(\mathfrak{h})$$

the full subcategory of admissible modules.
For an Harish-Chandra pair \((\mathfrak{h}, L)\), we denote by \(M(\mathfrak{h}, L)\) the abelian category of \((\mathfrak{h}, L)\)-modules. We say that an \((\mathfrak{h}, L)\)-module is admissible if it is admissible as an \(\mathfrak{h}\)-module, and denote by
\[ \mathcal{M}^a(\mathfrak{h}, L) \subset M(\mathfrak{h}, L) \]
the full subcategory of admissible modules.

For a complex reductive group \(L\), we denote by \(\hat{L}\) the set of isomorphism classes of irreducible algebraic representations of \(L\). Given an algebraic representation \(V\) of \(L\), and \(\alpha \in \hat{L}\), we denote by \(V^{[\alpha]} \subset V\) the \(\alpha\)-isotypic subspace.

Given a commutative real Lie algebra \(\mathfrak{b}\) and a locally-finite complex \(\mathfrak{b}\)-module \(V\), we denote by \(\text{wt}_\mathfrak{b}(V) \subset \mathfrak{b}_C^*\) the set of generalized eigenweights of \(\mathfrak{b}\) on \(V\), and for \(\lambda \in \mathfrak{b}_C^*\) we denote by \(V_{\mathfrak{b},\lambda}\) the subspace of \(V\) consisting of vectors with generalized eigenweight \(\lambda\) with respect to \(\mathfrak{b}\). The following are very useful claims about admissibility.

**Lemma 1.1:** For \(V \in \mathcal{M}(\mathfrak{g}, K)\), the following are equivalent:

1. \(V\) is admissible.
2. \(V\) is finitely generated over \(U(\mathfrak{g})\) and \(V^{[\alpha]}\) are finite-dimensional for all \(\alpha \in \hat{K}\).
3. \(V\) is \(Z(\mathfrak{g})\)-finite and \(V^{[\alpha]}\) are finite-dimensional for all \(\alpha \in \hat{K}\).
4. \(V\) has finite length.

**Proof.** Given in Appendix A.

**Lemma 1.2:** For \(V \in \mathcal{M}(\mathfrak{g}, K_I N(I))\), the following are equivalent:

1. \(V\) is admissible.
2. \(V\) is \(Z(\mathfrak{g})\)-finite and \(V^{n(I)}\) is an admissible \((\mathfrak{g}_I, K_I)\)-module.
3. \(V\) is \(Z(\mathfrak{g})\)-finite and \(V^{n(I)}\) are admissible \((\mathfrak{g}_I, K_I)\)-modules for every \(k \in \mathbb{Z}_{\geq 1}\).
4. \(V\) is \(Z(\mathfrak{g})\)-finite, \(a_{\text{cent}, I}\)-locally finite, and every generalized eigenspace \(V_{a_{\text{cent}, I}, \lambda}\) is an admissible \((\mathfrak{g}_I, K_I)\)-module.
5. \(V\) has finite length.

**Proof.** Given in Appendix A.
Recall also that the forgetful functor $\mathcal{M}(\mathfrak{g}, K_I N(I)) \to \mathcal{M}(\mathfrak{g}, K_I)$ is fully faithful, and the essential image consists of $(\mathfrak{g}, K_I)$-modules which are locally $n_{(I)}$-torsion. We will therefore think of $\mathcal{M}(\mathfrak{g}, K_I N(I))$ as a full subcategory of $\mathcal{M}(\mathfrak{g}, K_I)$ in what follows.

**Lemma 1.3:** For $V \in \mathcal{M}(\mathfrak{g}, K_I)$, the following are equivalent:

1. $V$ is a $(\mathfrak{g}, K_I N(I))$-module, and admissible as such.
2. $V$ is $Z(\mathfrak{g})$-finite, $a_{\text{cent}, I}$-locally finite, and every generalized eigenspace $V_{\lambda}$ is an admissible $(\mathfrak{g}_I, K_I)$-module. In addition, there exists a finite set $S \subset (a_{\text{cent}, I})^*$ such that $\text{wt}_{a_{\text{cent}, I}}(V) \subset S - \sum_{\alpha \in \text{wt}_{a_{\text{cent}, I}}(n_{(I)})} Z_{\geq 0} \cdot \alpha$.

**Proof.** Given in Appendix A.

1.3. DUALITIES. Recall the contragradient duality

$$(\cdot)^\vee : \mathcal{M}^a(\mathfrak{g}, K_I N(I)) \xrightarrow{\approx} \mathcal{M}^a(\mathfrak{g}, K_I N(I))^{\text{op}}$$

given by

$$V^\vee := (V^*)^{K_I \text{-finite}, n_{(I)}^- \text{-torsion}}.$$  

In particular, for $I = \Sigma$, we obtain the contragradient duality

$$(\cdot)^\vee : \mathcal{M}^a(\mathfrak{g}, K) \xrightarrow{\approx} \mathcal{M}^a(\mathfrak{g}, K)^{\text{op}}.$$  

**Lemma 1.4:** The formula given for $(\cdot)^\vee$ indeed defines a duality as stated. One also has the description:

$$(V^*)^{K_I \text{-finite}, n_{(I)}^- \text{-torsion}} = (V^*)^{K_I \text{-finite}, a_{\text{cent}, I} \text{-finite}}.$$  

**Proof.** This is well-known, and not hard to establish based on all the admissibility Lemmas of this paper, so left as an exercise.

1.4. REPRESENTATIONS. In this paper we prefer to work with $(\mathfrak{g}, K)$-modules rather than with representations. Let us briefly recall the relation.

We denote by $\mathcal{M}(G(\mathbb{R}))$ the category of smooth Frechet representations of $G(\mathbb{R})$ which are of moderate growth. We have the functor

$$(\cdot)^{[K]} : \mathcal{M}(G(\mathbb{R})) \to \mathcal{M}(\mathfrak{g}, K)$$
of passing to $K(\mathbb{R})$-finite vectors, and we say that a representation $\mathcal{V} \in \mathcal{M}(G(\mathbb{R}))$ is **admissible** if $\mathcal{V}^{[K]}$ is an admissible $(\mathfrak{g}, K)$-module. We denote by

$$\mathcal{M}^a(G(\mathbb{R})) \subset \mathcal{M}(G(\mathbb{R}))$$

the full subcategory of admissible representations.

The following is the basic theorem:

**Theorem 1.5** (Casselman–Wallach, [Ca], [Wa1], [Wa3 §11]): The functor

$$(\cdot)^{[K]} : \mathcal{M}^a(G(\mathbb{R})) \to \mathcal{M}^a(\mathfrak{g}, K)$$

is an equivalence of categories.

We will denote by $(\cdot)^\infty$ the equivalence of categories inverse to that in Theorem 1.5.

### 2. Casselman’s canonical pairing

In this section we recall Casselman’s canonical pairing, which plays a key role in second adjointness.

#### 2.1. Definition of the Casselman–Jacquet functor.

Recall the Casselman–Jacquet functor $J_I : \mathcal{M}^a(\mathfrak{g}, K) \to \mathcal{M}^a(\mathfrak{g}, K_I N_I)$

given by

$$J_I(V) := (\lim_{k \in \mathbb{Z} \geq 1} V/(n_{(I)}^{-} k V))^{K_I \text{-finite, } n_{(I)} \text{-torsion}.}$$

**Lemma 2.1**: The formula given for $J_I$ indeed defines a functor as stated. One also has the description:

$$(\lim_{k \in \mathbb{Z} \geq 1} V/(n_{(I)}^{-} k V))^{K_I \text{-finite, } n_{(I)} \text{-torsion}} = (\lim_{k \in \mathbb{Z} \geq 1} V/(n_{(I)}^{-} k V))^{a_{\text{cent, } I} \text{-finite}}.$$  

**Proof.** This is well-known, and not hard to establish based on all the admissibility Lemmas of this paper, so left as an exercise.

The following is a basic fact proved by Casselman:

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3 By $\lim_{k \in \mathbb{Z} \geq 1}$ we understand the inverse limit, where the transition maps $V/(n_{(I)}^{-} k V) \to V/(n_{(I)}^{-} k+1 V)$ are the standard projections.
Proposition 2.2 (Casselman): The functor
\[ J_I : \mathcal{M}^a(\mathfrak{g}, K) \to \mathcal{M}^a(\mathfrak{g}, K_I N(I)) \]
is exact.

Proof. See, for example, [Wa2, §4.1.5] (the case \( I = \emptyset \) is considered there, but the general case is completely analogous).

Analogously one has the functor
\[ J_I^- : \mathcal{M}^a(\mathfrak{g}, K) \to \mathcal{M}^a(\mathfrak{g}, K_I N_I^{-}(I)) \]
(where one swaps the opposite parabolics).

2.2. The canonical pairing.

Theorem 2.3 (Casselman’s canonical pairing): Let \( V \in \mathcal{M}^a(\mathfrak{g}, K) \). There exists a canonical \((\mathfrak{g}, K_I)\)-invariant pairing
\[
J_I(V) \otimes J_I^-(V^\vee) \to \mathbb{C}.
\]
Moreover, this pairing is non-degenerate; this means that the induced map
\[ J_I^-(V^\vee) \to J_I(V)^\vee \]
is an isomorphism.

Remark 2.4: The construction of the pairing of Theorem 2.3 is analytical. We hope to have an algebraic treatment in the future. In the works [ChGaYo], [GaYo] some conjectural algebraic (or algebro-geometric) reformulations are given.

Proof of Theorem 2.3 Let us recall the construction of the pairing, due to Casselman, and provide a reference for the proof of non-degeneracy.

For \( v \in V \) and \( \alpha \in V^\vee \) one has a corresponding matrix coefficient
\[ m_{v,\alpha} \in C^\infty(G(\mathbb{R})). \]
It has a convergent expansion
\[
m_{v,\alpha}(e^{H}) = \sum_{\lambda} e^{\lambda(H)} \cdot p_\lambda(H) \quad (H \in \mathfrak{a}_{\text{cent}, I}^*)
\]
where \( \lambda \) runs over a subset of \( \mathfrak{a}_C^* \) of the form
\[
\text{finite subset } - \sum_{\alpha \in \Sigma} \mathbb{Z}_{\geq 0} \cdot \alpha,
\]
and $p_{\lambda}(H)$ are polynomials. Let us denote by $\text{Exp}(a_{\text{cent},I})$ the space of formal expressions as the sum in (2.2). We have the subspace

$$\text{Exp}^\text{fin}(a_{\text{cent},I}) \subset \text{Exp}(a_{\text{cent},I})$$

consisting of finite sums. Then the asymptotic expansion of matrix coefficients is a map

$$V \otimes V^\vee \to \text{Exp}(a_{\text{cent},I}).$$

Completely formally (by “continuity”) this extends to a map

$$\lim_{k \in \mathbb{Z} \geq 1} (V/(n(I)_k V) \otimes \lim_{k \in \mathbb{Z} \geq 1} (V^\vee/(n(I)_k V^\vee)) \to \text{Exp}(a_{\text{cent},I})$$

and then restricts to a map

$$\mathcal{J}(V) \otimes \mathcal{J}^-(V^\vee) \to \text{Exp}^\text{fin}(a_{\text{cent},I}).$$

Composing with the map

$$\text{Exp}^\text{fin}(a_{\text{cent},I}) \to \mathbb{C}$$

given by evaluation at $0 \in H$, one obtains the desired pairing (2.1).

That this pairing is non-degenerate is non-trivial, first proven by Milicic ([Mi]) for $I = \emptyset$, and then by Hecht and Schmid ([HeSc]) in general. ■

3. Tempered admissible modules

In this section we describe the functor $\text{temppres}_I$ of tempered parabolic re-
striction, Casselman’s canonical pairing for tempered admissible modules, and second 
adjointness for tempered admissible modules.

3.1. The functors $\text{pres}_I$ and $\text{pind}_I$.

**Definition 3.1:**

1. We define the **parabolic restriction** functor

$$\text{pres}_I : \mathcal{M}(\mathfrak{g},K) \to \mathcal{M}(\mathfrak{g}_I,K_I)$$

by

$$\text{pres}_I(V) := \mathbb{C}_{-\rho(I)} \otimes V/n(I)V.$$

2. We define the **parabolic induction** functor

$$\mathcal{M}(\mathfrak{g},K) \leftarrow \mathcal{M}(\mathfrak{g}_I,K_I) : \text{pind}_I$$

as the right adjoint of the functor $\text{pres}_I$.
Remark 3.2: It is easy to see that the right adjoint $\text{pind}_I$ exists abstractly, by an adjoint functor theorem. Alternatively, one might interpret the relation $\text{pind}_I \cong B_I \circ \Delta_I$ which we will prove later (Proposition 5.9) as a concrete description of the functor $\text{pind}_I$, which in particular shows its existence.

Remark 3.3: We similarly denote by

$$\text{pres}_{-I} : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(\mathfrak{g}_I, K_I)$$

the functor analogous to $\text{pres}_I$, where we use $n_{-I}$ instead of $n_I$; thus

$$\text{pres}_{-I}(V) := C_{\rho_{(I)}} \otimes V / n_{-I} V.$$

**Lemma 3.4:** The functors $\text{pind}_I, \text{pres}_I$ preserve the subcategories of admissible modules.

**Proof.** Given in Appendix A. 

3.2. Definition of tempered admissible modules.

**Definition 3.5:** A module $V \in \mathcal{M}^a(\mathfrak{g}, K)$ is called **tempered**, if all $\lambda \in \text{wt}_a(\text{pres}_\emptyset(V))$ satisfy\(^4\) $\Re(\lambda) \geq \Sigma 0$. We denote by

$$\mathcal{M}^a(\mathfrak{g}, K)_{\text{temp}} \subset \mathcal{M}^a(\mathfrak{g}, K)$$

the full subcategory consisting of tempered modules.

Remark 3.6: By considering the symmetry given by $w_0 \in K$, one can reformulate the above definition as: $V$ is tempered if all $\lambda \in \text{wt}_a(\text{pres}_{\emptyset}(V))$ satisfy $\Re(\lambda) \leq \Sigma 0$.

Remark 3.7: It is known that under the Casselman–Wallach equivalence (Theorem 1.3), tempered modules as defined above match with tempered representations in the usual sense (in particular, as used in [CrHi]). See, for example, [Ba] and in particular [Ba] Lemma 4.4, or [Yo2] and in particular [Yo2] Lemma 5.4.2], where the relation of the above definition of temperedness with decay of matrix coefficients is discussed.

Remark 3.8: Let $W \in \mathcal{M}^a(\mathfrak{g}_I, K_I)$ be tempered. Then, in particular, all $\omega \in \text{wt}_{\text{cent},I}(W)$ satisfy

$$\Re(\omega) = 0.$$

\(^4\) By $\Re(\cdot)$ we denote the real part of a complex-valued functional on a real vector space.
3.3. PARABOLIC INDUCTION AND RESTRICTION IN THE TEMPERED CASE.

Remark 3.9: The parabolic induction of a tempered module is tempered (see, for example, [Yo2, Corollary 5.5.2]), and the contragradient of a tempered module is tempered (see, for example, [Yo2, Corollary 6.1.6]). However, the parabolic restriction of a tempered module is not necessarily tempered.

In view of the last Remark, let us define:

Definition 3.10: We define

\[ \text{temppres}_I : M^a(g, K)_{\text{temp}} \to M^a(g, K_I)_{\text{temp}} \]

to be the left adjoint of

\[ M^a(g, K)_{\text{temp}} \leftarrow M^a(g, K_I)_{\text{temp}} : \text{pind}_I. \]

Remark 3.11: Of course, we similarly define

\[ \text{temppres}^-_I : M^a(g, K)_{\text{temp}} \to M^a(g, K_I)_{\text{temp}}, \]

using the opposite parabolic.

Let us now describe \( \text{temppres}_I \) more concretely (and thus, in particular, deduce its existence). We denote by

\[ M^a(g, K_I)_{\Sigma-\text{temp}} \subset M^a(g, K_I) \]

the full subcategory consisting of modules \( W \) for which one has \( \Re(\lambda) \leq_\Sigma 0 \) for all \( \lambda \in \text{wt}_a(\text{pres}_\emptyset(W)) \) (notice, in contrast, that the condition for the \((g, K_I)\)-module \( W \) to be tempered is \( \Re(\lambda) \leq_I 0 \) for all \( \lambda \in \text{wt}_a(\text{pres}_\emptyset(W)) \)). We then have

\[ M^a(g, K_I)_{\text{temp}} \subset M^a(g, K_I)_{\Sigma-\text{temp}} \]

(because \( \leq_I \) is a finer partial order than \( \leq_\Sigma \)) and also

\[ \text{pres}_I(M^a(g, K)_{\text{temp}}) \subset M^a(g, K_I)_{\Sigma-\text{temp}} \]

(by the transitivity of parabolic restriction).

Notation 3.12: In what follows it will be convenient, given \( W \) which lies in \( M^a(g, K_I) \) or in \( M^a(g, K_IN(I)) \) and given \( \lambda \in \mathfrak{a}^* \), to denote

\[ W_{(\lambda)} := \bigoplus_{\omega \in (\mathfrak{a}_{\text{cent}}, I)^*} W_{\mathfrak{a}_{\text{cent}}, I, \omega} \quad \text{s.t. } \Re(\omega) = \lambda|_{\mathfrak{a}_{\text{cent}}, I} \]

(this is a \((g, K_I)\)-module).
Lemma 3.13: A module $W \in \mathcal{M}^a(g_I, K_I)_{\Sigma-temp}$ lies in $\mathcal{M}^a(g_I, K_I)_{temp}$ if and only if $W = W_{(0)}$.

Proof. Notice that, in view of Casselman’s submodule theorem, one has
\[
\text{wt}_{a_{cent,I}}(W) = \text{wt}_a(\text{pres}_0(W))|_{a_{cent,I}}.
\]
Therefore, we need to check that for $\lambda \in \text{wt}_a(\text{pres}_0(W))$, one has $R(\lambda|_{a_{cent,I}}) = 0$ if and only if $\lambda \leq I$. So, we are reduced to checking that for $\lambda \in a^*$ satisfying $\lambda \leq \Sigma$, one has $\lambda \leq I$ if and only if $\lambda|_{a_{cent,I}} = 0$. This is clear in view of the equality $a^+ + I = a^+ + \Sigma + a_{cent,I}$.

From the last Lemma we see that the functor
\[
(\cdot)_{(0)} : \mathcal{M}^a(g_I, K_I)_{\Sigma-temp} \rightarrow \mathcal{M}^a(g_I, K_I)_{temp}
\]
is both the right and the left adjoint of the inclusion
\[
\mathcal{M}^a(g_I, K_I)_{temp} \subset \mathcal{M}^a(g_I, K_I)_{\Sigma-temp}.
\]
We thus conclude:

Claim 3.14: One has
\[
\text{temppres}_I = (\cdot)_{(0)} \circ \text{pres}_I : \mathcal{M}^a(g, K)_{temp} \rightarrow \mathcal{M}^a(g_I, K_I)_{temp}.
\]

3.4. Exactness. The following lemma has a simple proof, but it is key.

Lemma 3.15: Let $V \in \mathcal{M}^a(g, K)_{temp}$. The projection map
\[
J_I^-(V) \rightarrow C_{\rho(I)} \otimes \text{pres}_I(V)
\]
induces an isomorphism
\[
J_I^-(V)_{(\rho(I))} \rightarrow \text{pres}_I(V)_{(0)} = \text{temppres}_I(V).
\]

Proof. One needs to see that
\[
J_I^-(V)_{(\rho(I))} \rightarrow \text{pres}_I(V)_{(0)}
\]
is injective. This will follow if we see that
\[
(n_I J_I^-(V))_{(\rho(I))} = 0.
\]
To that end, notice that all $\omega \in \text{wt}_{a_{cent,I}}(n_I J_I^-(V))$ are contained in
\[
\text{wt}_{a_{cent,I}}(V/n_I V) + \left(\sum_{\alpha \in R_{(I)}^+} \mathbb{Z}_{\geq 0} \cdot \alpha \right) \backslash \{0\}|_{a_{cent,I}}.
\]
Thus, since $V$ is tempered, the real part of every $\omega \in \text{wt}_{a_{\text{cent} \cdot I}}(n(I) \mathcal{J}_I^{-}(V))$ is the restriction to $a_{\text{cent} \cdot I}$ of some weight of the form

$$\rho(I) + \lambda + \left( \sum_{\alpha \in R^+_{(I)}} \mathbb{Z}_{\geq 0} \cdot \alpha \right) \setminus \{0\}$$

where $\lambda \in a^*$ satisfies $\lambda \geq \Sigma 0$. In particular, this real part clearly can not be $\rho(I)|_{a_{\text{cent} \cdot I}}$.  

The following can be thought of as the main difference between the tempered and non-tempered cases, explaining why the archimedean tempered case regains similarity to the non-archimedean case.

**Proposition 3.16:** The functor

$$\text{temppres}_I : M^a(g, K)_{\text{temp}} \to M^a(g_I, K_I)_{\text{temp}}$$

is exact.

**Proof.** This follows immediately from Lemma 3.15, as $\mathcal{J}_I^{-}$ is exact.  

3.5. **Casselman’s canonical pairing for tempered admissible modules.** Let $V \in M^a(g, K)$. Recall (Theorem 2.3) Casselman’s canonical pairing

$$\mathcal{J}_I(V) \otimes \mathcal{J}_I^{-}(V) \to \mathbb{C}.$$ 

It induces a pairing

$$\mathcal{J}_I(V)_{\langle \rho(I) \rangle} \otimes \mathcal{J}_I^{-}(V)_{\langle -\rho(I) \rangle} \to \mathbb{C}.$$ 

Since the former pairing is non-degenerate, so is the latter. Now, assume that $V$ is tempered. By Lemma 3.15 the latter pairing can be rewritten as

$$\text{temppres}_I(V) \otimes \text{temppres}_I(V^\vee) \to \mathbb{C}.$$ 

Let us thus summarize:

**Theorem 3.17:** Let $V \in M^a(g, K)_{\text{temp}}$. There exists a canonical non-degenerate pairing

$$\text{temppres}_I(V) \otimes \text{temppres}_I(V^\vee) \to \mathbb{C}.$$ 

In other words, one has a canonical isomorphism

$$\text{temppres}_I(V)^\vee \cong \text{temppres}_I(V^\vee).$$
3.6. SECOND ADJOINTNESS FOR TEMPERED ADMISSIBLE MODULES. One can quite formally rewrite Theorem 3.17 as follows:

**THEOREM 3.18:** There is a natural adjunction

$$\text{pind}_I : \mathcal{M}^a(g_I, K_I)_{\text{temp}} \rightleftarrows \mathcal{M}^a(g, K)_{\text{temp}} : \text{temppres}_I^-.$$ 

**Proof.** Let $V \in \mathcal{M}^a(g, K)_{\text{temp}}$ and $W \in \mathcal{M}^a(g_I, K_I)_{\text{temp}}$. One has:

$$\text{Hom}(\text{pind}_I(W), V) \cong \text{Hom}(V^\vee, \text{pind}_I(W)^\vee) \cong \text{Hom}(V^\vee, \text{pind}_I(W^\vee))$$

$$\cong \text{Hom}(\text{temppres}_I(V^\vee), W^\vee) \cong \text{Hom}(\text{temppres}_I^-(V^\vee), W^\vee)$$

$$\cong \text{Hom}(W, \text{temppres}_I^-(V)).$$

Here, we used the well-known isomorphism $\text{pind}_I(W)^\vee \cong \text{pind}_I(W^\vee)$.

4. Relation to Bernstein morphisms

In this section we briefly record how the canonical pairing for tempered admissible modules should be related to the construction of Bernstein morphisms.

4.1. BOUNDARY DEGENERATIONS AND BERNSTEIN MORPHISMS. Let us denote

$$Y_I := (G(\mathbb{R}) \times G(\mathbb{R}))/\Delta G_I(\mathbb{R}) \cdot (N_I(\mathbb{R}) \times N_I(\mathbb{R})).$$

One has

$$Y_\Sigma \cong G(\mathbb{R}),$$

and the $Y_I$’s are “boundary degenerations” of $Y_\Sigma$. According to ideas of J. Bernstein, one should have $(G(\mathbb{R}) \times G(\mathbb{R})$-equivariant) **Bernstein morphisms**

$$\text{Ber}_I : L^2(Y_I) \to L^2(Y_\Sigma),$$

which are (not necessarily surjective) isometries, and which should provide a conceptual derivation of the Plancherel formula for $L^2(Y_\Sigma)$ (modulo knowledge of twisted discrete spectrum).

And indeed, such Bernstein morphisms (in a much greater generality, of some spherical varieties) were constructed in [DeKnKrSc] (see also [SaVe] for the non-archimedean case).
4.2. Relation of the canonical pairing to boundary degenerations.

The following is a (presumably) well-known “automatic continuity” result:

**Lemma 4.1:** Let $U \in \mathcal{M}^a(g \oplus g, K \times K)$. Then the map

$$\text{Hom}_{g \oplus g, K \times K}(U, C^\infty(Y_I)) \to U^* \Delta g_I + (n_I) \oplus n(I)$$

given by evaluation at 1 is a bijection.

**Proof.** Given in Appendix A. ■

Let $V \in \mathcal{M}^a(g, K)_{\text{temp}}$. Corresponding to the tautological pairing

$$V \otimes V^\vee \to \mathbb{C},$$

under the identification of Lemma 4.1 is the matrix coefficients map

$$\alpha : V \otimes V^\vee \to C^\infty(Y_\Sigma).$$

Additionally, corresponding to the pairing of Theorem 3.17, again using Lemma 4.1 one obtains a map

$$\alpha' : V \otimes V^\vee \to C^\infty(Y_I).$$

4.3. Bernstein morphism via canonical pairing. Let us fix a Plancherel decomposition for $L^2(Y_\Sigma)$ (see [Be] for more details): A measure space $(\Omega, \mu)$, and for each $\omega \in \Omega$ a tempered irreducible module $V_\omega \in \mathcal{M}^a(g, K)_{\text{temp}}$. The matrix coefficient map

$$\alpha_\omega : V_\omega \otimes V_\omega^\vee \to C^\infty(Y_\Sigma)$$

gives rise to the “adjoint” map

$$\beta_\omega : C_c^\infty(Y_\Sigma) \to (V_\omega \otimes V_\omega^\vee)^{(2)}$$

(here $(\cdot)^{(2)}$ denotes the completion w.r.t. the inner product—$V_\omega \otimes V_\omega^\vee$ has a canonical one)—again, see [Be] for details. The data is required to give rise to an isomorphism of Hilbert spaces

$$pl : L^2(Y_\Sigma) \xrightarrow{\sim} \int_{\omega \in \Omega} (V_\omega \otimes V_\omega^\vee)^{(2)} d\mu : \phi \mapsto [\beta_\omega(\phi)]_{\omega \in \Omega}.$$

---

5 By $W^{*,\mathfrak{h}}$ we denote the space of functionals on $W$ which are invariant under $\mathfrak{h}$, i.e., annihilating $\mathfrak{h}W$. 
Now, by §4.2 we also have maps
\[ \alpha'_\omega : V_\omega \otimes V_\omega^\vee \rightarrow C^\infty(\Sigma_I), \]
and to them correspond the “adjoint” maps
\[ \beta'_\omega : C^\infty_c(\Sigma_I) \rightarrow (V_\omega \otimes V_\omega^\vee)^{(2)}. \]

**Expectation 4.2:** The Bernstein morphism
\[ \text{Ber}_I : L^2(\Sigma_I) \rightarrow L^2(\Sigma) \]
is given by
\[ \phi \mapsto \text{pl}^{-1}([\beta'_\omega(\phi)]_{\omega \in \Omega}) \]
for \( \phi \in C^\infty_c(\Sigma_I). \)

**Remark 4.3:** As far as we understand, establishing the above expectation should be simply a matter of comparing the languages of [DeKnKrSc] and the current paper.

5. Functors

In this section, we describe the functors \( B_I \) and \( C_I \) which we studied in [Yo1], and their relation with \( \text{pind}_I \) and \( \text{pres}_I \).

One can summarize the functors in the following diagram:
Here, all functors preserve the admissible subcategories. We have three adjunctions

\[(\mathcal{B}_I, \mathcal{C}_I); \quad (\Delta_I, \text{cofib}_I); \quad (\text{pres}_I, \text{pind}_I),\]

the relation

\[
\text{pind}_I \cong \mathcal{B}_I \circ \Delta_I,
\]
a morphism

\[
\text{cofib}_I \rightarrow \text{fib}^{-}_I,
\]
and on the admissible subcategories an isomorphism

\[
\text{fib}^{-}_I \circ \mathcal{C}_I \cong \text{pres}^{-}_I
\]
(where \(\text{pres}^{-}_I\) is analogous to \(\text{pres}_I\), but using the opposite parabolic).

5.1. The functors \(\mathcal{B}_I\) and \(\mathcal{C}_I\).

**Definition 5.1:**

1. We define the functor\(^6\)

\[
\mathcal{B}_I : \mathcal{M}(\mathfrak{g}, K_I N(I)) \rightarrow \mathcal{M}(\mathfrak{g}, K)
\]

by

\[
\mathcal{B}_I(V) := (\mathcal{O}(K) \otimes V)^{K_I}.
\]

Here the notations are as follows. The \(K\)-action on \(\mathcal{O}(K) \otimes V\) is the left regular one on \(\mathcal{O}(K)\). The \(\mathfrak{g}\)-action on \(\mathcal{O}(K) \otimes V\) is \(\xi(f)(k) = k^{-1} \xi \cdot f(k)\), where we think about \(f \in \text{Fun}(K, V) \cong \mathcal{O}(K) \otimes V\). The \(K_I\)-action w.r.t. which we take invariants is \(m(f \otimes v) = R_m f \otimes mv\) (here \(R_m\) denotes the right regular action of \(m\)). The \(\mathfrak{t}\)-action w.r.t. which we take coinvariants is the difference between the \(\mathfrak{t}\)-action obtained by differentiating the \(K\)-action, and the \(\mathfrak{t}\)-action obtained by restricting the \(\mathfrak{g}\)-action. The actions of \(\mathfrak{g}\) and \(K\) are well-defined after passing to the invariants and coinvariants, and we obtain a \((\mathfrak{g}, K)\)-module in this way.

2. We define the functor\(^7\)

\[
\mathcal{M}(\mathfrak{g}, K_I N(I)) \leftarrow \mathcal{M}(\mathfrak{g}, K) : \mathcal{C}_I
\]

as the right adjoint of \(\mathcal{B}_I\).

---

\(^6\) This can be called Bernstein’s functor, as it is similar to a functor Bernstein has studied, which in turn is a version of Zuckerman’s functor.

\(^7\) This can be called the Casselman–Jacquet functor, in view of Theorem 5.10.
Remark 5.2: In more geometric terms, say using $D$-algebras, the functor $\mathcal{B}_I$ is given by forgetting the $N(I)$-equivariancy, followed by performing $\ast$-averaging from $K_I$-equivariancy to $K$-equivariancy. See [Yo1] for this as well as a more detailed (although, at some points, yet premature) discussion of the functors $\mathcal{B}_I$ and $\mathcal{C}_I$.

Remark 5.3: Let us describe the functor $\mathcal{C}_I$ more concretely (again, see [Yo1] for details). It is given by

$$\mathcal{C}_I(V) = \left( \prod_{\alpha} V^{[\alpha]} \right)^{K_I\text{-finite, } n(I)\text{-torsion}}.$$  

Lemma 5.4: The functors $\mathcal{B}_I, \mathcal{C}_I$ preserve the subcategories of admissible modules.

Proof. Given in Appendix A.

5.2. THE FUNCTORS $\Delta_I$, cofib$^I$, AND fib$^{-I}$.

Definition 5.5:

1. We define the functor

$$\mathcal{M}(g_I, K_I) \leftarrow \mathcal{M}(g, K_I N(I)) : \text{cofib}_I$$

by

$$\text{cofib}_I(V) := C_{\rho(I)} \otimes V^{n(I)}.$$  

2. We define the functor

$$\Delta_I : \mathcal{M}(g_I, K_I) \to \mathcal{M}(g, K_I N(I))$$

as the left adjoint of cofib$_I$.

3. We define the functor

$$\text{fib}^{-I} : \mathcal{M}(g, K_I N(I)) \to \mathcal{M}(g_I, K_I)$$

by

$$\text{fib}^{-I}(V) := C_{\rho(I)} \otimes V^{n^{-I}(I)}.$$  

Remark 5.6: Let us describe the functor $\Delta_I$ more concretely. It is given by

$$\Delta_I(V) := U(g) \otimes_{U(g_I + n(I))} (C_{-\rho(I)} \otimes V),$$

where $C_{-\rho(I)} \otimes V$ is considered as a $U(g_I + n(I))$-module by making $n(I)$ act by zero.
Remark 5.7: Notice that we have a morphism
$$\text{cofib}_I \to \text{fib}_I^{-},$$
given by $V^{n(I)} \hookrightarrow V \rightarrow V/n_{-}^{(I)}V$.

Lemma 5.8: The functors $\text{cofib}_I, \Delta_I, \text{fib}_I^{-}$ preserve the subcategories of admissible modules.

Proof. Given in Appendix A.

Proposition 5.9: One has
$$\mathcal{B}_I \circ \Delta_I \cong \text{pind}_I.$$

Proof. One first checks that the map
$$(\mathbb{O}(K) \otimes V)^{K_I} \to \left(\mathbb{O}(K) \otimes \left(U(g) \otimes_{U(g_I + n(I))} V\right)^{K_I}\right)_I,$$
given by inserting 1 at the $U(g)$-component, is an isomorphism of $K$-representations (this is the analog of the “compact picture” for parabolic induction).

Composing the inverse of this isomorphism with the evaluation at $1 \in K$, we obtain a map
$$\left(\mathbb{O}(K) \otimes \left(U(g) \otimes_{U(g_I + n(I))} V\right)^{K_I}\right)_I \to V.$$

One now routinely checks that for a $(g, K)$-module $W$, by composing with this map one obtains a bijection
$$\text{Hom}_{g, K} \left(W, \left(\mathbb{O}(K) \otimes \left(U(g) \otimes_{U(g_I + n(I))} V\right)^{K_I}\right)_I \right) \cong \text{Hom}_{g_I, K_I} \left(W/n_{(I)}W, C_{2\rho(I)} \otimes V\right).$$

5.3. Casselman’s canonical pairing in terms of the functors.

Casselman’s canonical pairing (Theorem 2.3) has the following reformulation:

Theorem 5.10: There exists a canonical isomorphism of functors
$$\mathcal{C}_I \cong \mathcal{J}_I : \mathcal{M}^a(g, K) \to \mathcal{M}^a(g, K_I N_{(I)}).$$

Proof. This will be clearly a reformulation of Theorem 2.3 once we establish an isomorphism
$$\mathcal{C}_I \cong (\cdot)_{\lor} \circ \mathcal{J}_{I}^{-} \circ (\cdot)_{\lor} : \mathcal{M}^a(g, K) \to \mathcal{M}^a(g, K_I N_{(I)}).$$
This is established using the concrete description of $\mathcal{C}_I$ in Remark 5.3. Indeed, clearly

$$\prod_\alpha V^{[\alpha]} \cong (V^\vee)^*,$$

and then

$$\left(\prod_\alpha V^{[\alpha]}\right)^{n(I)\text{-torsion}} \cong (V^\vee)^{*,n(I)\text{-torsion}} \cong \left(\lim_{k \in \mathbb{Z}} (V^\vee/n_k^k V^\vee)^*) \cong \ldots$$

where $(\lim V^\vee/n_k^k V^\vee)^*$ denotes the subspace of the space of functionals, consisting of those which factor through the projection onto one of the $V^\vee/n_k^k V^\vee$'s. By $a_{\text{cent},I}$-weight consideration, we can continue:

$$\cong \left(\lim_{k \in \mathbb{Z}} (V^\vee/n_k^k V^\vee)^{a_{\text{cent},I}\text{-finite}}\right)^*, a_{\text{cent},I}\text{-finite}.$$

Therefore, we obtain

$$\mathcal{C}_I(V) = \left(\prod_\alpha V^{[\alpha]}\right)^{K_I\text{-finite}, n(I)\text{-torsion}} \cong \left(\lim_{k \in \mathbb{Z}} (V^\vee/n_k^k V^\vee)^{a_{\text{cent},I}\text{-finite}}\right)^*, K_I\text{-finite} a_{\text{cent},I}\text{-finite} \cong J_I^-(V^\vee)^\vee.$$

For our current purposes, only the following corollary will be needed:

**Corollary 5.11:** One has an isomorphism of functors

$$\text{fib}_I^\circ \mathcal{C}_I \cong \text{pres}_I^- : \mathcal{M}^a(g, K) \to \mathcal{M}^a(g_I, K_I).$$

**Proof.** In view of Theorem 5.10 this follows from the easy relation

$$\text{fib}_I^\circ J_I^\circ \cong \text{pres}_I^-.$$

**6. Second adjointness—second take**

In this section we describe again second adjointness for tempered admissible modules, but with an emphasis on trying to work with all admissible modules (rather than just the tempered ones).
6.1. SECOND “preadjointness” FOR ADMISSIBLE MODULES. From §5 we see that we have an adjunction

$$pind_I : M^a(g_I, K_I) \rightleftharpoons M^a(g, K) : cofib_I \circ \mathcal{C}_I,$$

and a morphism

$$cofib_I \to fib_I^-.$$

Thus, we obtain a morphism of functors

$$cofib_I \circ \mathcal{C}_I \to fib_I^- \circ \mathcal{C}_I \cong \text{pres}_I^-,$$

where the latter isomorphism is Corollary 5.11 which, we recall, uses the non-trivial Casselman’s canonical pairing (Theorem 5.10). We see that the failure of the naive second adjointness, that is, of \((pind_I, \text{pres}_I^-)\) being an adjoint pair, is encoded by the non-isomorphicity of \(cofib_I \to fib_I^-\). Nevertheless, we have a “candidate for a unit” for an adjunction between \(pind_I\) and \(\text{pres}_I^-\), namely the composition

$$\text{Id} \to (cofib_I \circ \mathcal{C}_I) \circ pind_I \to \text{pres}_I^- \circ pind_I.$$

In other words, we have maps

$$\text{Hom}(pind_I(W), V) \to \text{Hom}(W, \text{pres}_I^-(V))$$

functorial in \(W \in M^a(g_I, K_I)\) and \(V \in M^a(g, K)\). One might call this the second “preadjointness”.

6.2. SECOND ADJOINTNESS FOR TEMPERED ADMISSIBLE MODULES.

**Claim 6.1:** Let \(V \in M^a(g, K)_{\text{temp}}\) and let \(W \in M^a(g_I, K_I)\) be such that

$$W = W_{(0)}.$$

Then the morphism (6.1) is an isomorphism.

**Proof.** It is enough to show that the map

$$\mathcal{C}_I(V)^{n(I)} \to \mathcal{C}_I(V)/n_I^- \mathcal{C}_I(V)$$

induces an isomorphism

$$\left(\mathcal{C}_I(V)^{n(I)}\right)_{(-\rho(I))} \to \left(\mathcal{C}_I(V)/n_I^- \mathcal{C}_I(V)\right)_{(-\rho(I))}$$

\([\text{Correlation} 3.12] [\text{Notation} 3.12]\)
In fact, decomposing this map as
\[ C_I(V)^n(I) \hookrightarrow C_I(V) \twoheadrightarrow C_I(V)/n(I)C_I(V), \]
we will see that these two maps separately become an isomorphism after applying \((\cdot)(-\rho(I))\).

Let us argue by contradiction, assuming that one of these two isomorphisms fails. Then it is easy to see that there exists \( \omega \in \text{wt}_{a_{\text{cent},I}}(C_I(V)/n(I)C_I(V)) \) such that
\[ \Re(\omega) \in \left( -\rho(I) + \sum_{\alpha \in R^+_I} \mathbb{Z}_{\geq 0} \cdot \alpha \right) \setminus \{-\rho(I)\} \]
(here in the right-hand side we understand restrictions to \( a_{\text{cent},I} \)). Then, by Casselman’s submodule theorem, there will exist \( \lambda \in \text{wt}_a(C_I(V)/n(0)C_I(V)) \) such that
\[ \lambda|_{a_{\text{cent},I}} = \omega. \]
In other words, there will exist \( \lambda' \in \text{wt}_a(\text{pres}_{(0)}(V)) \) such that
\[ \Re(\lambda')|_{a_{\text{cent},I}} \in \left( \sum_{\alpha \in R^+_I} \mathbb{Z}_{\geq 0} \cdot \alpha \right) \setminus \{0\}; \]
here we used
\[ \text{pres}_{(0)}(V) \cong \text{pres}_{\emptyset}(\text{pres}_{I} V) \cong \text{pres}_{\emptyset}(\text{fib}_{I}(C_I(V))) \cong \text{fib}_{\emptyset}(C_I(V)) \]
(where some of the functors were not formally defined with their current domain, but their meaning is completely clear). But clearly then \( \Re(\lambda') \leq_{\Sigma} 0 \) does not hold, contradicting \( V \) being tempered.

**Corollary 6.2:** The preadjointness morphism (6.1) is an isomorphism when \( V \) and \( W \) are tempered.

**Proof.** This follows from Claim 6.1 because, in view of Remark 3.8 if \( W \) is tempered then \( W = W_{(0)} \).

Notice, finally, that Corollary 6.2 gives one more proof of Theorem 3.18.
Appendix A. Proofs of Lemmas

A.1. Proofs of the Lemmas characterizing admissibility.

Proof of Lemma 1.1. (1) $\implies$ (2): Since $V$ is finitely generated over $U(g)$, a Theorem of Harish-Chandra ([Wa2, §3.4.1]) implies that each $V[\alpha]$ is finitely generated over $Z(g)$. Since $Z(g)$ acts finitely on $V$, we deduce that each $V[\alpha]$ is in fact finite-dimensional.

(2) $\implies$ (3): Since $V$ is finitely generated over $U(g)$, it is generated over $Z(g)$ by finitely many of the $V[\alpha]$'s, so it is enough to show that $Z(g)$ acts finitely on each $V[\alpha]$. This, in turn, is clear since $Z(g)$ preserves each $V[\alpha]$ and each $V[\alpha]$ is finite-dimensional by our assumption.

(3) $\implies$ (4): This follows from the fact that there are, up to isomorphism, only finitely many irreducible $(g, K)$-modules with a given infinitesimal character (for that fact, see [Wa2, §5.5.6]; alternatively (and algebraically), it can be easily deduced from Beilinson–Bernstein localization theory). Indeed, that fact implies, since $V$ is $Z(g)$-finite, that there exists a finite set $S$ of isomorphism classes of irreducible $(g, K)$-modules such that the isomorphism class of every irreducible subquotient of $V$ lies in $S$. Then we can pick a finite set $T \subset \hat{K}$ such that for every irreducible $(g, K)$-module $W$ of isomorphism class in $S$, one has $W[\alpha] \neq 0$ for some $\alpha \in T$. Now, the functor from the category of $(g, K)$-modules all of whose irreducible subquotients are of isomorphism class in $S$, to the category of vector spaces, given by $W \mapsto \bigoplus_{\alpha \in T} W[\alpha]$, is exact, conservative (i.e., maps a non-zero object to a non-zero object), and the image of $V$ under it is of finite length (i.e., a finite-dimensional vector space). This implies that $V$ has finite length.

(4) $\implies$ (1): One reduces immediately to the case when $V$ is irreducible. Then that $V$ is finitely generated over $U(g)$ is clear. The center $Z(g)$ acts finitely because it in fact acts by scalars, by Schur’s Lemma ([Wa2, §0.5.2, §3.3.2]).

Proof of Lemma 1.2. Let us first assume that the $(g, K_{I N(I)})$-module $V$ is $Z(g)$-finite and deduce some preliminary observations. One has the Harish-Chandra homomorphism $Z(g) \to Z(g_I)$, which is finite, and from its definition one sees that the action of $Z(g)$ on $V^{n(I)}$ factors through this homomorphism. Therefore, we deduce that $V^{n(I)}$ is $Z(g_I)$-finite, and hence $a_{cent, I}$-finite. Considering, for $k \in \mathbb{Z}_{\geq 1}$, the exact sequence of $g_I$-modules

\[(A.1) \quad 0 \to V^{n_k(I)} \to V^{n_{k+1}(I)} \to \text{Hom}_C(n^k_I), V^{n(I)}\]
(where the last arrow is given by acting on $V^{n_{I}^{k+1}}$ by $n_{I}^{k}$), we by induction deduce that $V^{n_{I}^{k}}$ are $a_{\text{cent},I}$-finite for all $k \in \mathbb{Z}_{\geq 1}$. In particular, $V$ is $a_{\text{cent},I}$-locally finite. Moreover, the above exact sequence shows that

$$\text{wt}_{a_{\text{cent},I}}(V^{n_{I}^{k+1}}/V^{n_{I}^{k}}) \subset \text{wt}_{a_{\text{cent},I}}(V_{n_{I}}^{n_{I}^{(I)}}) - k \cdot \text{wt}_{a_{\text{cent},I}}(n_{I}).$$

Now we will proceed with the steps.

$(1) \implies (2)$: We remarked above that $V_{n_{I}}^{n_{I}^{(I)}}$ is $Z(g_{I})$-finite. Since $V$ is finitely generated over $U(g)$, there exists an $a_{\text{cent},I}$-stable finite-dimensional subspace $V_{0} \subset V$ such that $V = U(n_{I}^{-})U(g_{I})V_{0}$. It is then clear by $a_{\text{cent},I}$-weight consideration that there exists $k \in \mathbb{Z}_{\geq 1}$ such that $V^{n_{I}^{(I)}} \subset (n_{I}^{-})^{k}U(g_{I})V_{0}$. Therefore, as $(n_{I}^{-})^{k}U(g_{I})V_{0}$ is finitely generated over $U(g_{I})$, so is $V^{n_{I}^{(I)}}$.

$(2) \implies (3)$: One shows that $V^{n_{I}^{k}}$ is admissible for any $k \in \mathbb{Z}_{\geq 1}$ by induction on $k$, using the exact sequence (A.1).

$(3) \implies (4)$: Since $V = \bigcup_{k \in \mathbb{Z}_{\geq 1}} V^{n_{I}^{k}}$, it is clear that $V$ is $a_{\text{cent},I}$-locally finite. It is enough now to show that for every $\lambda \in (a_{\text{cent},I})^{*}_{C}$ there exists $k \in \mathbb{Z}_{\geq 1}$ such that $V^{n_{I}^{(I)}} \subset V^{n_{I}^{k}}$. This is clear by $a_{\text{cent},I}$-weight consideration, from the last preliminary observation.

$(4) \implies (5)$: Let $I \subset Z(g)$ be an ideal of finite codimension that acts by zero on $V$. There exists, depending only on $I$, a finite set $S \subset (a_{\text{cent},I})^{*}_{C}$ such that $\text{wt}_{a_{\text{cent},I}}(V^{n_{I}^{(I)}}) \subset S$. Consider now the functor from the category of $(g, K_{I}N_{I})$-modules on which $I$ acts by zero and which are $a_{\text{cent},I}$-locally finite, to the category of $(g_{I}, K_{I})$-modules, given by

$$W \mapsto \bigoplus_{\lambda \in S} W_{a_{\text{cent},I}, \lambda}.$$

This functor is exact, conservative, and the image of $V$ under it is of finite length. This implies that $V$ has finite length.

$(5) \implies (1)$: One reduces immediately to the case when $V$ is irreducible. Then that $V$ is finitely generated over $U(g)$ is clear. The center $Z(g)$ acts finitely because it in fact acts by scalars, by Schur’s Lemma ([Wa2, §0.5.2]).

Proof of Lemma 1.3 $(1) \implies (2)$: This is clear, in view of the implication $(1) \implies (4)$ of Lemma 1.2 as well as the final preliminary observation in the proof of Lemma 1.2.

$(2) \implies (1)$: The last condition makes it clear that $V$ is locally $n_{I}$-torsion. Then the implication follows from implication $(4) \implies (1)$ of Lemma 1.2.
A.2. Proofs of the Lemmas about Preservation of Admissibility.

Proof of Lemma 3.4. We first address $\text{pres}_I$. Using the definition and finiteness of the Harish-Chandra homomorphism $h_I : Z(\mathfrak{g}) \to Z(\mathfrak{g}_I)$, it is clear that $\text{pres}_I$ sends $Z(\mathfrak{g})$-finite modules to $Z(\mathfrak{g}_I)$-finite modules. More precisely, one sees that given $z \in Z(\mathfrak{g})$, applying the functor $\text{pres}_I$ to the morphism $V \to V$ given by multiplication by $z$, one obtains the morphism $\text{pres}_I(V) \to \text{pres}_I(V)$ given by multiplication by $h_I(z)$. Also, since $\mathfrak{g} = n(I) + \mathfrak{g}_I + \mathfrak{k}$, it is clear that $\text{pres}_I$ sends modules which are finitely generated over $U(\mathfrak{g})$ to modules which are finitely generated over $U(\mathfrak{g}_I)$.

We now address $\text{pind}_I$, solely exploiting it being the right adjoint of $\text{pres}_I$. Let $W$ be a $(\mathfrak{g}_I, K_I)$-module having finite-dimensional isotypic components. We will show that $\text{pind}_I(W)$ also has finite-dimensional isotypic components. Let $E$ be a finite-dimensional $K$-module. Denote

$$V_E := U(\mathfrak{g}) \otimes_{U(\mathfrak{k})} E.$$  

Then, for a $(\mathfrak{g}_I, K_I)$-module $W$, we have

$$\text{Hom}_{K}(E, \text{pind}_I(W)) \cong \text{Hom}_{(\mathfrak{g}, K)}(V_E, \text{pind}_I(W)) \cong \text{Hom}_{(\mathfrak{g}_I, K_I)}(\text{pres}_I(V_E), W).$$

Since $V_E$ is finitely generated over $U(\mathfrak{g})$, by what we have seen $\text{pres}_I(V_E)$ is finitely generated over $U(\mathfrak{g}_I)$. As $W$ has finite-dimensional isotypic components, it is clear that the last Hom-space is finite-dimensional, and thus so is the first, showing that $\text{pind}_I(W)$ has finite-dimensional isotypic components.

Finally, let us show that if $W$ is a $Z(\mathfrak{g}_I)$-finite $(\mathfrak{g}_I, K_I)$-module, then $\text{pind}_I(W)$ is $Z(\mathfrak{g})$-finite. More precisely, we will show that given $z \in Z(\mathfrak{g})$, the morphism $\text{pind}_I(W) \to \text{pind}_I(W)$ given by multiplication by $z$ is equal to the morphism obtained by applying $\text{pind}_I$ to the morphism $W \to W$ given by multiplication by $h_I(z)$. For this, it is enough to show that for every $(\mathfrak{g}, K)$-module $V$, two endomorphisms of

$$\text{Hom}_{\mathfrak{g}, K}(V, \text{pind}_I(W)),$$

the first obtained via the multiplication by $z$ on $\text{pind}_I(W)$ and the second obtained via the multiplication by $h_I(z)$ on $W$, coincide. We can interpret the first endomorphism as given via the multiplication by $z$ on $V$, and identifying

$$\text{Hom}_{\mathfrak{g}_I, K_I}(\text{pres}_I(V), W),$$

with the second, we conclude that

$$\text{Hom}_{\mathfrak{g}, K}(V, \text{pind}_I(W)) \cong \text{Hom}_{\mathfrak{g}_I, K_I}(\text{pres}_I(V), W).$$
we further interpret it, in view of what was said about $\text{pres}_I$ above, as given via the multiplication by $h_I(z)$ on $\text{pres}_I(V)$. On the other hand, the second endomorphism gets interpreted on the latter Hom-space still as given via the multiplication by $h_I(z)$ on $W$. These interpretations show that our two endomorphisms indeed coincide. □

Proof of Lemma 5.8 That cofib$_I$ preserves admissibility is the content of the implication $(1) \implies (2)$ of Lemma 1.2

Let $W$ be a $(\mathfrak{g}_I, K_I)$ module. Since $\Delta_I(W)$ is generated over $U(\mathfrak{g})$ by (a twist of) $W$, it is clear that $\Delta_I(W)$ is finitely generated over $U(\mathfrak{g})$ if $W$ is finitely generated over $U(\mathfrak{g}_I)$, and, using the Harish-Chandra homomorphism $Z(\mathfrak{g}) \to Z(\mathfrak{g}_I)$, that $\Delta_I(W)$ is $Z(\mathfrak{g}_I)$-finite if $W$ is $Z(\mathfrak{g}_I)$-finite.

That fib$_I^{-1}$ sends $Z(\mathfrak{g})$-finite modules to $Z(\mathfrak{g}_I)$-finite modules is shown exactly as the corresponding claim for pres$_I$. That fib$_I^{-1}$ sends modules which are finitely generated over $U(\mathfrak{g})$ to modules which are finitely generated over $U(\mathfrak{g}_I)$ is again shown similarly to the corresponding claim for pres$_I$, where we use now

$$\mathfrak{g} = n_{(I)}^+ + \mathfrak{g}_I + n_{(I)}^-$$

Proof of Lemma 5.4 Let us first notice that $\mathcal{B}_I$ sends $Z(\mathfrak{g})$-finite modules to $Z(\mathfrak{g})$-finite modules. More precisely, one sees that given $z \in Z(\mathfrak{g})$, applying the functor $\mathcal{B}_I$ to the morphism $W \to W$ given by multiplication by $z$, one obtains the morphism $\mathcal{B}_I(W) \to \mathcal{B}_I(W)$ given by multiplication by $z$. This is clear from the defining formula for $\mathcal{B}_I$.

To show that $\mathcal{B}_I$ preserves admissibility, let us fix an ideal of finite codimension $J \subset Z(\mathfrak{g})$ and consider an admissible $(\mathfrak{g}, K_I N(I))$-module $W$ on which $J$ acts by zero. Depending only on $J$, there exists a finite set $S \subset (a_{\text{cent}}, I)_C^*$ such that $\text{wt}_{a_{\text{cent}}, I}(W^{n(I)}) \subset S$. We will prove the admissibility of $\mathcal{B}_I(W)$ by induction on the number of elements in $\text{wt}_{a_{\text{cent}}, I}(W) \cap S$ (if this number is zero, then $W = 0$ and the claim is clear). Notice that the counit map $\Delta_I(\text{cofib}_I(W)) \to W$ is an isomorphism on a generalized eigenspace $(\cdot)_{a_{\text{cent}}, I, \lambda}$ whenever $\lambda \in \text{wt}_{a_{\text{cent}}, I}(W) \cap S$ is maximal w.r.t. the partial order given by $\lambda_1 \leq \lambda_2$ if $\lambda_2 - \lambda_1 \in \sum_{\alpha \in \text{wt}_{a_{\text{cent}}, I}(n_{(I)})} \mathbb{Z}_{\geq 0} \cdot \alpha$. Therefore, considering the exact sequence

$$\Delta_I(\text{cofib}_I(W)) \to W \to C \to 0$$

(where $C$ simply denotes the cokernel of the counit map), and applying $\mathcal{B}_I$ to it, we reduce ourselves to showing that $\mathcal{B}_I(\Delta_I(\text{cofib}_I(W)))$ and $\mathcal{B}_I(C)$ are
admissible. But

$$\mathcal{B}_I(\Delta_I(\text{cofib}_I(W))) \cong \text{pind}_I(\text{cofib}_I(W))$$

(here we used Proposition 5.9, which is admissible since $W$ is (as we have already shown that $\text{pind}_I$ and $\text{cofib}_I$ preserve admissibility), and we are thus reduced to showing that $\mathcal{B}_I(C)$ is admissible. Notice that $C$ is again an admissible $(\mathfrak{g}, K_1 N_I)$-module on which $J$ acts by zero, and that $\text{wt}_{a_{\text{cent}},I}(C) \cap S$ is contained properly in $\text{wt}_{a_{\text{cent}},I}(W) \cap S$, as it lacks the maximal elements. Therefore, by the induction hypothesis, $\mathcal{B}_I(C)$ is admissible.

We now address $\mathcal{C}_I$, solely exploiting it being the right adjoint of $\mathcal{B}_I$. We first show that $\mathcal{C}_I$ sends $Z(\mathfrak{g})$-finite modules to $Z(\mathfrak{g})$-finite modules. More precisely, given a $(\mathfrak{g}, K)$-module $V$ and $z \in Z(\mathfrak{g})$, the morphism $\mathcal{C}_I(V) \rightarrow \mathcal{C}_I(V)$ given by multiplication by $z$ is equal to the morphism obtained by applying $\mathcal{C}_I$ to the morphism $V \rightarrow V$ given by multiplication by $z$. In fact, one deduces this from the corresponding fact for $\mathcal{B}_I$ noted above, in complete analogy with the parallel treatment for $\text{pind}_I$ in the last paragraph of the proof of Lemma 3.4, so we skip this.

Finally, we will show that given an admissible $(\mathfrak{g}, K)$-module $V$, the $(\mathfrak{g}, K_1 N_I)$-module $\mathcal{C}_I(V)$ is also admissible. We just mentioned that $\mathcal{C}_I(V)$ is $Z(\mathfrak{g})$-finite, therefore by Lemma 1.2 it is enough to show that $\text{cofib}_I(\mathcal{C}_I(V))$ is an admissible $(\mathfrak{g}_I, K_I)$-module. Moreover, again since $\mathcal{C}_I(V)$ is $Z(\mathfrak{g})$-finite, we already know that $\text{cofib}_I(\mathcal{C}_I(V))$ is $Z(\mathfrak{g}_I)$-finite (see the preliminary observations in the proof of 1.2), and it is therefore enough to see that $\text{cofib}_I(\mathcal{C}_I(V))$ has finite-dimensional isotypic components. Let $E$ be a finite-dimensional $K_I$-module. Denote

$$W_E := U(\mathfrak{g}_I) \otimes_{U(\mathfrak{g}_I)} E$$

(a $(\mathfrak{g}_I, K_I)$-module) and denote by $J \subset Z(\mathfrak{g}_I)$ an ideal of finite codimension which acts on $\text{cofib}_I(\mathcal{C}_I(V))$ by zero. Then $W_E/JW_E$ is an admissible $(\mathfrak{g}_I, K_I)$-module, and we have

$$\text{Hom}_{K_I}(E, \text{cofib}_I(\mathcal{C}_I(V))) \cong \text{Hom}_{(\mathfrak{g}_I, K_I)}(W_E/JW_E, \text{cofib}_I(\mathcal{C}_I(V))) \cong \text{Hom}_{(\mathfrak{g}, K)}(\mathcal{B}_I(\Delta_I(W_E/JW_E)), V) \cong \text{Hom}_{(\mathfrak{g}, K)}(\text{pind}_I(W_E/JW_E), V).$$

Since both $W_E/JW_E$ and $V$ are admissible, the last Hom-space is finite-dimensional, and therefore so is the first, and hence the desired conclusion. ■
A.3. **Proof of Lemma 4.1**  Let us fix
\[ \ell \in U^* \Delta g + (n_{I}) \oplus n_{(I)} \].

Using the Casselman–Wallach Theorem (Theorem 1.5) and standard Frobenius reciprocity, one has an identification
\[ \text{Hom}_{g \oplus g, K \times K}(U, C^\infty(Y_I)) \cong \text{Hom}_{G(\mathbb{R}) \times G(\mathbb{R})}(U^\infty, C^\infty(Y_I)) \]
\[ \cong (U^\infty)^* \Delta g + (n_{I}) \oplus n_{(I)}) \],

where \((\cdot)^*\) denotes the space of continuous functionals. Therefore, we see that we simply need to show that \(\ell\) extends to a continuous functional on \(U^\infty\).

We consider the parabolic subgroup \(G_I N_I \times G_I N_I\) in \(G \times G\) (defined over \(\mathbb{R}\)), and denote (just for this proof) by
\[ \text{pres} : M^a(g \oplus g, K \times K) \rightleftharpoons M^a(g_I \oplus g_I, K_I \times K_I) : \text{pind} \]
the corresponding unnormalized parabolic restriction and induction functors. We want to see first that a continuous dashed arrow making the following diagram commutative, exists:

\[ \begin{array}{ccc}
U^\infty & \to & \text{pres}(U)^\infty \\
\uparrow & & \uparrow \\
U & \to & \text{pres}(U)
\end{array} \]

One has the unit map \(U \to \text{pind}(\text{pres}(U))\), and corresponding to it the map of representations \(U^\infty \to \text{pind}(\text{pres}(U))^\infty\). It is well-known and not hard to establish, for \(W \in M^a(g_I \oplus g_I, K_I \times K_I)\), an isomorphism \(\text{pind}(W)^\infty \cong \text{pind}(W^\infty)\), where \(\text{pind}(\cdot)\) is the “usual” parabolic induction construction, consisting of smooth functions on \(G(\mathbb{R}) \times G(\mathbb{R})\) which satisfy a transformation rule, etc. We clearly have a map \(\text{pind}(W^\infty) \to W^\infty\) given by evaluating at 1, which gives us the composition
\[ U^\infty \to \text{pind}(\text{pres}(U))^\infty \cong \text{pind}(\text{pres}(U)^\infty) \to \text{pres}(U)^\infty, \]
which is the desired arrow.

The functional \(\ell\) factors as the projection \(U \to \text{pres}(U)\) followed by a functional \(\ell' \in \text{pres}(U)^* \Delta g\). We therefore see, using the commutative diagram above, that it is enough to show that \(\ell'\) extends to a continuous functional on \(\text{pres}(U)^\infty\). This, in its turn, is a well-known “automatic continuity” for symmetric subgroups ([BaDe Théorème 1]).
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