Consideration of Socium as Mechanical System with Stream Control

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Abstract. Mathematical modeling of socium as mechanical system with the material components determined by parts of system with qualitatively excellent complexes of properties concerning production, storage and consumption of production is considered. Existence of objectively operating feedback in the representing system is established. The mathematical model formed in work appears the nonlinear system of the ordinary differential equations and is far-reaching expansion of classical models Lotka and Volterra. The mathematical analysis shows existence of a fixed point in phase space of solutions of system and gives for her analytical expressions. Results of private numerical modeling of the offered model are presented. Results of modeling show manifestations of difficult oscillatory dynamics of phase trajectories of the solution of system and serve as an essential argument of an explanation of real oscillatory processes in the history of social systems.

1. Introduction
Mathematical modeling of mechanical systems in the generalized case begins with allocation of material components of system between which the interactions represented by quantity which have numerical values are seen or established. Though traditional methods of studying of societies consist in the verbal humanitarian description, it is represented useful and perspective to apply to these systems and the specified mechanics mathematical approach. The empirical models of societies based on a mathematization of the description of these systems as the connections of three dynamic components relying on J. Goldstone [1, 2] works and realized by S.A. Nefyodov [3, 4] and Turchin P. V. [5] are so far known. The mathematical model intended for the description of biogeocenoses and population dynamics is offered in works [6, 7]. Modification of this model can be used also for the description of the societies considered as natural-science systems. As the initial stage of the specified approach the understanding of socium as the dynamic system consisting of four components is used. ”Native habitat”, people, elite and state are these natural components. Communication between the specified components develops the idea of trophic communication for the first time offered in works Lotka and Volterra [8, 9], but applied not to two, and four components, and the closed contour of the sequence of communications provides feedback in all to system and describes difficult dynamic process of the regulating management. The model considers and describes constantly functioning real communication between environment opportunities for granting appliances of existence and the current number of socium. Need of allocation for actually human part of system more than one component is caused by their relation to the environment and to each other. It is in what generalized the people component only consumes subsistence, in the simplest and generalized
contents "food", in generalized objects of consumption. The elite component except actually consumption receives from "people" and accumulates the specified subsistence. Thereby this dynamic a component in relation to trophic streams in society holds a specific place and performs special function in their dynamics.

2. Formulation of the Problem
The substantial party of the considered model consists that between two consecutive levels - system components constantly and steadily the unidirectional interactions work, and from the last level it is also possible to state a special stream of impact on the lower level providing steady reproduction of what consumes the first level. The regulating component role "state" well is looked through in functioning of the first states which for development and maintenance demanded large-scale land development works. In modern conditions the role of the state is determined by need of performance of infrastructure and scientific and technical developments with long or uncertain by the next economic efficiency results when the private sector and individual efforts can’t provide them. Large-scale activity of the state can increase real restriction for the initial productivity of the first component an ecological niche. Such restriction is inherent in natural biogeocenoses, and is reflected in logistic models of her dynamics [10]. This increase is reached in reality in the way, both melioration, and development of production technologies of artificial fertilizers, their mass introduction and the state activities for fight against gully desertification arid areas by creation of protective forest belts. These reasons and generalizations demand to change the mathematical description of influence of the operating stream of the last component to the first. Such influence was described for the natural systems of biogeocenoses that the numerical value of the first component was multiplied by additional coefficient \(1 - \frac{x}{R}\) where \(x\) - value of the first component "native habitat", and \(R\) - restriction for this quantity. Thereby formation of positive increments of the first component \(x\) after achievement of restriction \(R\) wasn’t allowed. When using human technologies of impact on efficiency of the environment this value isn’t limited anymore and as a first approximation directly depends on influence of the last component. We will designate numerical values of the specified "native habitat", "people", "elite" and "state" components as \(x, y, z, u\). We will enter influence coefficient \(M\) from the operating impact on efficiency of the environment.

The coefficient \(d\) will represent the speed of increase of numerical value of efficiency of a component \(x\) only on the basis of external natural sources of his supply with the necessary external making elements (energy and substance). The actual contribution of this external factor is modified by the additional member of the expression corresponding to the defining logistic dependence \(d(1 - \frac{x}{R})\) that provides accounting of limitation of an external contribution and quantitative value of this limitation with value \(R\). Let \(a\) - production of unit of the environment which is taken from her by socium and arrive on use by component unit \(y\) (to the certain person from the people). Production of unit of component \(y\), arriving on use to component unit \(z\) (to the certain person from elite) we will designate \(e\). Let \(h\) - production of unit of a component \(z\) arriving on use to component unit \(u\) (public servant). At last, \(b\) - the production of unit of a component \(u\) arriving on use to component unit \(x\). Coefficients \(c_1, c_2, c_3, c_4\) presents speeds of loss of the value material components \(x, y, z, u\) because of their internal consumption for activity maintenance

3. Theory
As result of the previous analysis, we receive that total influence natural and artificial sources on efficiency of the environment can be presented by expression \(bM u + d(1 - \frac{x}{R})\). The cumulative system of the differential equations of model describing the interactions stated above can be
written down in the following look now:

$$
\begin{align*}
\dot{x} &= (-ay + bMu - c_1 + d(1-x/R))x,
\dot{y} &= (ax - ez - c_2)y,
\dot{z} &= (ey - hu - c_3)z,
\dot{u} &= (hz - bx - c_4)y.
\end{align*}
$$

(1)

We will pay attention that impact of the managing director of component on the initial environment is described through coefficient $M > 1$ that leads to the corresponding increment $bMu$ on unit of this component. Only those solutions of system of the equations (1) which it isn’t less than zero are almost interesting, it directly follows from a target task of the description of real-life objects or masses. Taking into account orientation of a research to dynamics the foremost attention should be given findings of fixed points of model that is defined by zero values of the right parts of the equations.

$$
\begin{align*}
(-ay + bMu - c_1 + d(1-x/R))x &= 0,
(ax - ez - c_2)y &= 0,
(ey - hu - c_3)z &= 0,
(hz - bx - c_4)y &= 0.
\end{align*}
$$

(2)

Here perhaps trivial decision $x = y = z = u = 0$ which doesn’t represent direct practical value, but can theoretically takes place as result of some dynamics from a nonzero starting point. In the field of nonzero values of variables we have the system of the linear equations for definition of a fixed point

$$
\begin{align*}
-ay + bMu - c_1 + d(1-x/R) &= 0,
ax - ez - c_2 &= 0,
ey - hu - c_3 &= 0,
hz - bx - c_4 &= 0.
\end{align*}
$$

To the system of the equations there corresponds the matrix

$$
A = \begin{bmatrix}
-d/R & -a & 0 & bM \\
 a & 0 & -e & 0 \\
0 & e & 0 & -h \\
-b & 0 & h & 0
\end{bmatrix}
$$

The determinant of this matrix as it is easily calculated is equal $(ah - be)(ah - bMe)$. For uniqueness of the decision receives a necessary condition

$$(ah - be)(ah - bMe) \neq 0.$$  

(3)

This condition will consider further executed. The degenerate case of equality of this determinant to zero in this work won’t be considered. We will enter for reduction of further calculations of quantity

$$
\begin{align*}
q_1 &= 1/(ah - be), 
q_2 &= 1/(ah - bMe).
\end{align*}
$$

For determination of coordinates of a fixed point in the specified condition it is possible to proceed from the matrix equation directly

$$
A \begin{bmatrix}
x_0 \\
y_0 \\
z_0 \\
u_0
\end{bmatrix}^T = \begin{bmatrix}
c_1 - d & c_2 & c_3 & c_4
\end{bmatrix}^T.
$$
It is possible and to receive directly the decision for values \( x_0, z_0 \) from the second and fourth equations of system which aren’t containing other variables and then similarly to arrive with remained two, using already found value \( x_0 \). We have

\[
\begin{bmatrix}
  a & -e \\
  -b & h
\end{bmatrix}
\begin{bmatrix}
  x_0 \\
  z_0
\end{bmatrix} =
\begin{bmatrix}
  c_2 \\
  c_4
\end{bmatrix},
\]

\[
\begin{bmatrix}
  -a & bM \\
  e & -h
\end{bmatrix}
\begin{bmatrix}
  y_0 \\
  u_0
\end{bmatrix} =
\begin{bmatrix}
  c_1 - d(1 - x_0/R) \\
  c_3
\end{bmatrix}.
\]

Determinants of these matrixes are equal \( s = (ah - be) \) and \( p = (ah - bMe) \), respectively and, they can be considered as nonzero as it is already determined by a condition (3) above. Therefore through the return matrixes we have

\[
\begin{bmatrix}
  x_0 \\
  z_0
\end{bmatrix} =
\begin{bmatrix}
  q_1 h & q_1 e \\
  q_1 b & q_1 a
\end{bmatrix}
\begin{bmatrix}
  c_2 \\
  c_4
\end{bmatrix},
\]

\[
\begin{bmatrix}
  y_0 \\
  u_0
\end{bmatrix} =
\begin{bmatrix}
  -q_2 h & -q_2 bM \\
  -q_2 e & -q_2 a
\end{bmatrix}
\begin{bmatrix}
  c_1 - d(1 - x_0/R) \\
  c_3
\end{bmatrix}.
\]

As the decision we receive

\[
x_0 = q_1 (hc_2 + ec_4),
\]

\[
z_0 = q_1 (bc_2 + ac_4),
\]

\[
y_0 = q_2 (h(d(1 - x_0/R) - c_1) - bMc_4),
\]

\[
u_0 = q_2 (e(d(1 - x_0/R) - c_1) - ac_3).
\]

From here for the problem of definition of a fixed point of system stated above in the field of positive values of coordinates \( x, y, z, u \) of space, necessary conditions follow

\[
(ah - be) > 0 \quad \text{and} \quad (ah - bMe) > 0 \quad \text{or} \quad (ah - be) < 0 \quad \text{and} \quad (ah - bMe) < 0,
\]

\[
h(d(1 - q_1 (hc_2 + ec_4)/R) - c_1) > bc_3, \quad e(d(1 - q_1 (hc_2 + ec_4)/R) - c_1) > ac_3.
\]

The first of these conditions after transformation, taking into account \( M > 0 \) can be in abbreviated form written down as \( (ah - bMe) > 0 \quad (ah - be) < 0 \). There are not as obvious following conclusions as the next mathematical and applied consequences of the described model in her absence. The fixed point appears somewhat average value. More exact value of average value is possible to calculate only for concrete dynamics at the set and measured values of parameters \( a, b, e, h, M, d, c_1, c_2, c_3, c_4 \). It turns out that the value \( z_0 \) gives the estimated quantity of number of elite in socium, and according to the second expression from (4) it is proportional to value \( q_1 \). The same way, values \( y_0 \) and \( u_0 \) also give estimated values of number of the main part of the population (people) and number of people in government which according to the third and fourth expression from (4) it is proportional to quantity \( q_1 \). Also follows from analytical expressions of quantities \( q_1 \) and \( q_2 \) that for increase in number of elite the value of quantity has to decrease \( s = (ah - be) \). Taking into account the requirement of positivity of quantity stated above \( p \) and \( s \), it turns out that bigger the value of elite is reached at values of coefficient \( M \), close to unit, for efficiency of investments to the external environment. More exact conclusions describing dynamics can be received only from the solution of the considered systems of the equations. These equations are nonlinear and therefore their analytical decision is essentially complicated.
Therefore numerical methods of modeling of the specified decisions were forcedly demanded. The considered mathematical model can be investigated for determination of character of trajectories in the neighborhood of a fixed point and also for establishment of dependence of type of this fixed point on system parameters. Such research naturally leads to a characteristic polynomial of the fourth order therefore there are known difficulties of obvious expression of roots of it of decisions through initial parameters. As real values of "human elements" in components $x, y, z, u$, makes millions of units, numerical modeling with such multiple-valued quantity technically inconveniently and them it is desirable to scale at the level of the most mathematical model. For this purpose we will enter scaling coefficient $k$, having define $x = kx', y = ky', z = kz', u = ku'$. Substituting these expressions instead of initial variables in system (1), after transformations by reductions of the general multiplier, we receive

$$\begin{align*}
\dot{x}' &= (-ak'y + bMk'u - c_1 + d(1 - kx'/R))x' \\
\dot{y}' &= (akx' - ekz' - c_2)y' \\
\dot{z}' &= (eky' - hku' - c_3)z' \\
\dot{u}' &= (hkz' - bkkx' - c_4)y'
\end{align*}$$

Scaling of values of components $k$ by means of coefficient generates the equivalent system of the equations if to carry out the coordinated scaling of parameters $a, b, e, h$ by transformations $a' = ka, b' = kb, e' = ke, h' = kh$. It gives the chance when modeling real systems with values of number of the elements expressed in millions to transform the empirical or experimentally available small coefficients $a, b, e, h$, by multiplication by coefficient $10^6$ and reducing thereby records of digital observed variables of value at numerical calculations.

4. Experimental results

The studied system has been simulated in a tool program system of Scilab [11]. As concrete values for such private modeling values of parameters of model have been chosen $a = 0.7; b = 0.08; e = 0.6; h = 0.6; M = 1.7; c_1 = 0.2, c_2 = 0.3, c_3 = 0.3, c_4 = 0.2 ; d = 0.65; R = 10.0$. These values meet the conditions (2) of a fixed point stated above in the field of positive values. The coordinates of a fixed point calculated on them give: $x_0 = 0.161, x_0 = 0.584, x_0 = 0.441, x_0 = 0.084$. As initial values of model the values which are the fifth part from values for a fixed point are chosen. Results of modeling in the range of values of parameter from 0 to 200 by means of the standard ode procedure for the Runge-Kuta method are given in a graphic view of fig.1a and fig. 1b. Here projections of 4-dimensional phase trajectories to spaces of smaller number of coordinates are given; these images are submitted by traditional methods of graphics.

They present to dependence of values of functions of decisions $z$ (Figure 1) and $u$ Figure 2) on values $x$ and $y$. Oscillatory dynamics and gradual approach of trajectories to value of a fixed point is well visible. For additional presentation the line piece connecting a projection of a fixed point to the plane to her graphic representation in the considered system of coordinates is in addition given in drawings. This piece is represented in red color. The only schedule taking into account complexity of model appears insufficiently. The image on specialized graphical representation of the three-dimensional image of the decision also insufficiently informatively. It is necessary to notice, seen in the submitted drawings of crossing of trajectories the valid trajectories, properly actually from the equations are only crossing of projections, aren’t crossed in the field of positive values of the considered variables.

As additional information of the qualitative plan for the considered decision it is possible to use the two-dimensional dependences of values $z$ (Figure 3) and $u$ (Figure 4) on value $x$ which are projections of the general phase portrait of the considered solution of mathematical model to the corresponding coordinate planes.

The projection of a phase portrait of the solution of system to the coordinate plane $XY$ as the schedule of the received dependence $y$ from $x$ is given in fig. 1e. The dependence of the
**Figure 1.** Phase portrait of dependence of values of the elite component on food and people values

**Figure 2.** Phase portrait of dependence of values of the state component on food and people values

**Figure 3.** Phase portrait of dependence of values of the elite component on food values

**Figure 4.** Phase portrait of dependence of values of the state component on food values

**Figure 5.** Dependence of values of the elite and state components on time

**Figure 6.** Dependence of values of the food and people components on time
solution on the parameter of time $t$ is given in Figure 5 and Figure 6.

5. DISCUSSION AND ANALYSIS
The visual analysis of the received dependences directly shows that in the offered model for the difficult four-component interacting systems distinctly are looked through characteristic of natural systems oscillatory the nature of interactions. At the same time not only cycles of the big periods of approach to equilibrium state, but also the variable shortly periodic cycles enclosed in them are visible. Thereby it is directly visible that this model, potentially possessing it is considerable bigger complexity, than classical models like Lotka-Volterra, reflects the essential party of real dynamics of societies.

6. CONCLUSION
Theoretical approach to as to the mechanical system operated by material streams has allowed to construct the mathematical model reflecting the essential parties of dynamics of this system. Streams of interactions between components of system are described by material production, and can be estimated by cost or power indexes. The analysis of model has shown existence of the only fixed point of phase space of decisions for rather simple conditions on the applied sense imposed on model parameters. When performing these conditions numerical modeling of the solution of system of the equations of model shows the meeting dynamics to a fixed point. The research type of a fixed point is possible and is calculated for concrete numerical values of parameters. Numerical experiments reveal oscillatory convergence of decisions to a fixed point. The conducted researches allow representing integrated dynamics of interactions in complex multi-component societies. The established dependences and their mathematical model allow reflecting obviously the oscillatory nature of dynamics, characteristic in time for the real difficult systems of sociums.

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