MAT-free reflection arrangements

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Abstract

We introduce the class of MAT-free hyperplane arrangements which is based on the Multiple Addition Theorem by Abe, Barakat, Cuntz, Hoge, and Terao. We also investigate the closely related class of MAT2-free arrangements based on a recent generalization of the Multiple Addition Theorem by Abe and Terao. We give classifications of the irreducible complex reflection arrangements which are MAT-free respectively MAT2-free. Furthermore, we ask some questions concerning relations to other classes of free arrangements.

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1 Introduction

A hyperplane arrangement $A$ is a finite set of hyperplanes in a finite dimensional vector space $V \cong \mathbb{K}^t$ where $\mathbb{K}$ is some field. The intersection lattice $L(A)$ of $A$ encodes its combinatorial properties. It is a main theme in the study of hyperplane arrangements to link algebraic properties of $A$ with the combinatorics of $L(A)$.

The algebraic property of freeness of a hyperplane arrangement $A$ was first studied by Saito [Sai80] and Terao [Ter80a]. In fact, it turns out that freeness of $A$ imposes strong combinatorial constraints on $L(A)$: by Terao’s Factorization Theorem [OT92, Thm. 4.137] its characteristic polynomial factors over the integers. Conversely, sufficiently strong conditions on $L(A)$ imply the freeness of $A$. One of the main tools to derive such conditions
is Terao’s Addition-Deletion Theorem 8. It motivates the class of \textit{inductively free} arrangements (see Definition 9). In this class the freeness of $\mathcal{A}$ is combinatorial, i.e. it is completely determined by $L(\mathcal{A})$ (cf. Definition 5). Recently, a remarkable generalization of the Addition-Deletion theorem was obtained by Abe. His Division Theorem \cite[Thm. 1.1]{Abe16} motivates the class of \textit{divisionally free} arrangements. In this class freeness is a combinatorial property too.

Despite having these useful tools at hand, it is still a major open problem, known as Terao’s Conjecture, whether in general the freeness of $\mathcal{A}$ actually depends only on $L(\mathcal{A})$, provided the field $\mathbb{K}$ is fixed (see \cite{Zie90} for a counterexample when one fixes $L(\mathcal{A})$ but changes the field). We should also mention at this point the very recent results by Abe further examining Addition-Deletion constructions together with divisional freeness \cite{Abe18b}, \cite{Abe18a}.

A variation of the addition part of the Addition-Deletion theorem 8 was obtained by Abe, Barakat, Cuntz, Hoge, and Terao in \cite{ABC+16}: the Multiple Addition Theorem 12 (MAT for short). Using this theorem, the authors gave a new uniform proof of the Kostant-Macdonald-Shapiro-Steinberg formula for the exponents of a Weyl group. In the same way the Addition-Theorem defines the class of inductively free arrangements, it is now natural to consider the class $\mathfrak{M}_3^\mathfrak{F}$ of those free arrangements, called \textit{MAT-free}, which can be built inductively using the MAT (Definition 13). It is not hard to see (Lemma 18) that MAT-freeness only depends on $L(\mathcal{A})$. In this paper, we investigate classes of MAT-free arrangements beyond the classes considered in \cite{ABC+16}.

Complex reflection groups (classified by Shephard and Todd \cite{ST54}) play an important role in the study of hyperplane arrangements: many interesting examples and counterexamples are related or derived from the reflection arrangement $\mathcal{A}(W)$ of a complex reflection group $W$. It was proven by Terao \cite{Ter80b} that reflection arrangements are always free. There has been a series of investigations dealing with reflection arrangements and their connection to the aforementioned combinatorial classes of free arrangements (e.g. \cite{BC12}, \cite{HR15}, \cite{Abe16}). Therefore, it is natural to study reflection arrangements in conjunction with the new class of MAT-free arrangements.

Our main result is the following.

\textbf{Theorem 1.} Except for the arrangement $\mathcal{A}(G_{32})$, an irreducible reflection arrangement is \textit{MAT-free} if and only if it is \textit{inductively free}. The arrangement $\mathcal{A}(G_{32})$ is \textit{inductively free} but not \textit{MAT-free}. Thus every reflection arrangement is \textit{MAT-free} except the reflection arrangements of the imprimitive reflection groups $G(e,e,\ell)$, $e > 2$, $\ell > 2$ and of the reflection groups $G_{24}, G_{27}, G_{29}, G_{31}, G_{32}, G_{33}, G_{34}$.

A further generalization of the MAT 12 was very recently obtained by Abe and Terao \cite{AT19}: the Multiple Addition Theorem 2 14 (MAT2 for short). Again, one might consider the inductively defined class of arrangements which can be build from the empty arrangement using this more general tool, i.e. the class $\mathfrak{M}_3^\mathfrak{F}$ of \textit{MAT2-free} arrangements (Definition 15). By definition, this class contains the class of \textit{MAT-free} arrangements. Regarding reflection arrangements we have the following:
**Theorem 2.** Let $A = A(W)$ be an irreducible reflection arrangement. Then $A$ is MAT2-free if and only if it is MAT-free.

In contrast to (irreducible) reflection arrangements, in general the class of MAT-free arrangements is properly contained in the class of MAT2-free arrangements (see Proposition 28).

Based on our classification of MAT-free (MAT2-free) reflection arrangements and other known examples ([ABC+16], [CRS19]) we arrive at the following question:

**Question 3.** Is every MAT-free (MAT2-free) arrangement inductively free?

In [CRS19] the authors proved that all ideal subarrangements of a Weyl arrangement are inductively free by extensive computer calculations. A positive answer to Question 3 would directly imply their result and yield a uniform proof (cf. [CRS19, Rem. 1.5(d)]).

Looking at the class of divisionally free arrangements which properly contains the class of inductively free arrangements [Abe16, Thm. 4.4] a further natural question is:

**Question 4.** Is every MAT-free (MAT2-free) arrangement divisionally free?

This article is organized as follows: in Section 2 we briefly recall some notions and results about hyperplane arrangements and free arrangements used throughout our exposition. In Section 3 we give an alternative characterization of MAT-freeness and two easy necessary conditions for MAT/MAT2-freeness. Furthermore, we comment on the relation of the two classes $\mathcal{MF}$ and $\mathcal{MF}'$ and on the product construction. Section 4 and Section 5 contain the proofs of Theorem 1 and Theorem 2. In the last Section 6 we comment on Question 3 and further problems connected with MAT-freeness.

**2 Hyperplane arrangements and free arrangements**

Let $A$ be a hyperplane arrangement in $V \cong K^\ell$ where $K$ is some field. If $A$ is empty, then it is denoted by $\Phi_\ell$.

The intersection lattice $L(A)$ of $A$ consists of all intersections of elements of $A$ including $V$ as the empty intersection. Indeed, with the partial order by reverse inclusion $L(A)$ is a geometric lattice [OT92, Lem. 2.3]. The rank $\text{rk}(A)$ of $A$ is defined as the codimension of the intersection of all hyperplanes in $A$.

If $x_1, \ldots, x_\ell$ is a basis of $V^*$, to explicitly give a hyperplane we use the notation $(a_1, \ldots, a_\ell) \perp := \ker(a_1 x_1 + \cdots + a_\ell x_\ell)$.

**Definition 5.** Let $\mathcal{C}$ be a class of arrangements and let $A \in \mathcal{C}$. If for all arrangements $B$ with $L(B) \cong L(A)$, (where $A$ and $B$ do not have to be defined over the same field), we have $B \in \mathcal{C}$, then the class $\mathcal{C}$ is called combinatorial.

If $\mathcal{C}$ is a combinatorial class of arrangements such that every arrangement in $\mathcal{C}$ is free than $A \in \mathcal{C}$ is called combinatorially free.
For $X \in L(A)$ the localization $A_X$ of $A$ at $X$ is defined by:

$$A_X := \{ H \in A \mid X \subseteq H \},$$

and the restriction $A^X$ of $A$ to $X$ is defined by:

$$A^X := \{ X \cap H \mid H \in A \setminus A_X \}.$$

Let $A_1$ and $A_2$ be two arrangements in $V_1$ respectively $V_2$. Then their product $A_1 \times A_2$ is defined as the arrangement in $V = V_1 \oplus V_2$ consisting of the following hyperplanes:

$$A_1 \times A_2 := \{ H_1 \oplus V_2 \mid H_1 \in A_1 \} \cup \{ V_1 \oplus H_2 \mid H_2 \in A_2 \}.$$

We note the following facts about products (cf. [OT92, Ch. 2]):

- $|A_1 \times A_2| = |A_1| + |A_2|.$
- $L(A_1 \times A_2) = \{ X_1 \oplus X_2 \mid X_1 \in L(A_1) \text{ and } X_2 \in L(A_2) \}.$
- $(A_1 \times A_2)^x = A_1^{x_1} \times A_2^{x_2}$ if $X = X_1 \oplus X_2$ with $X_i \in L(A_i)$.

Let $S = S(V^*)$ be the symmetric algebra of the dual space. We fix a basis $x_1, \ldots, x_\ell$ for $V^*$ and identify $S$ with the polynomial ring $\mathbb{K}[x_1, \ldots, x_\ell]$. The algebra $S$ is equipped with the grading by polynomial degree: $S = \bigoplus_{p \in \mathbb{Z}} S_p$, where $S_p$ is the set of homogeneous polynomials of degree $p$ ($S_p = \{0\}$ for $p < 0$).

A $\mathbb{K}$-linear map $\theta : S \to S$ which satisfies $\theta(fg) = \theta(f)g + f\theta(g)$ is called a $\mathbb{K}$-derivation. Let $\text{Der}(S)$ be the $S$-module of $\mathbb{K}$-derivations of $S$. It is a free $S$-module with basis $D_1, \ldots, D_\ell$ where $D_i$ is the partial derivation $\partial/\partial x_i$. We say that $\theta \in \text{Der}(S)$ is homogeneous of polynomial degree $p$ provided $\theta = \sum_{i=1}^\ell f_i D_i$ with $f_i \in S_p$ for each $1 \leq i \leq \ell$. In this case we write $p\text{deg} \theta = p$. We obtain a $\mathbb{Z}$-grading for the $S$-module $\text{Der}(S)$: $\text{Der}(S) = \bigoplus_{p \in \mathbb{Z}} \text{Der}(S)_p$.

**Definition 6.** For $H \in A$ we fix $\alpha_H \in V^*$ with $H = \ker(\alpha_H)$. The module of $A$-derivations is defined by

$$D(A) := \{ \theta \in \text{Der}(S) \mid \theta(\alpha_H) \in \alpha_H S \text{ for all } H \in A \}.$$

We say that $A$ is free if the module of $A$-derivations is a free $S$-module.

If $A$ is a free arrangement we may choose a homogeneous basis $\{\theta_1, \ldots, \theta_\ell\}$ for $D(A)$. Then the polynomial degrees of the $\theta_i$ are called the *exponents* of $A$ and they are uniquely determined by $A$, [OT92, Def. 4.25]. We write $\exp(A) := (p\text{deg} \theta_1, \ldots, p\text{deg} \theta_\ell)$. Note that the empty arrangement $\Phi_\ell$ is free with $\exp(\Phi_\ell) = (0, \ldots, 0) \in \mathbb{Z}^\ell$. If $d_1, \ldots, d_\ell \in \mathbb{Z}$ with $d_1 \leq d_2 \leq \cdots \leq d_\ell$ we write $(d_1, \ldots, d_\ell)_{\leq}$.

The notion of freeness is compatible with products of arrangements:
Proposition 7 ([OT92, Prop. 4.28]). Let $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ be a product of two arrangements. Then $\mathcal{A}$ is free if and only if both $\mathcal{A}_1$ and $\mathcal{A}_2$ are free. In this case if $\exp(\mathcal{A}_i) = (d_{i1}^1, \ldots, d_{i\ell_i}^i)$ for $i = 1, 2$ then
\[
\exp(\mathcal{A}) = (d_{11}^1, \ldots, d_{1\ell_1}^1, d_{21}^2, \ldots, d_{2\ell_2}^2).
\]

The following theorem provides a useful tool to prove the freeness of arrangements.

Theorem 8 (Addition-Deletion [OT92, Thm. 4.51]). Let $\mathcal{A}$ be a hyperplane arrangement and $H_0 \in \mathcal{A}$. We call $(\mathcal{A}, \mathcal{A}' = \mathcal{A}\setminus\{H_0\}, \mathcal{A}'' = \mathcal{A}^{H_0})$ a triple of arrangements. Any two of the following statements imply the third:

1. $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (b_1, \ldots, b_{\ell - 1}, b_{\ell})$,
2. $\mathcal{A}'$ is free with $\exp(\mathcal{A}') = (b_1, \ldots, b_{\ell - 1}, b_{\ell} - 1)$,
3. $\mathcal{A}''$ is free with $\exp(\mathcal{A}'') = (b_1, \ldots, b_{\ell - 1})$.

The preceding theorem motivates the following definition.

Definition 9 ([OT92, Def. 4.53]). The class $\mathcal{IF}$ of inductively free arrangements is the smallest class of arrangements which satisfies:

1. the empty arrangement $\Phi_\ell$ of rank $\ell$ is in $\mathcal{IF}$ for $\ell \geq 0$,
2. if there exists a hyperplane $H_0 \in \mathcal{A}$ such that $\mathcal{A}'' \in \mathcal{IF}$, $\mathcal{A}' \in \mathcal{IF}$, and $\exp(\mathcal{A}'') \subset \exp(\mathcal{A}')$, then $\mathcal{A}$ also belongs to $\mathcal{IF}$.

Here $(\mathcal{A}, \mathcal{A}', \mathcal{A}'') = (\mathcal{A}, \mathcal{A}\setminus\{H_0\}, \mathcal{A}^{H_0})$ is a triple as in Theorem 8.

The class $\mathcal{IF}$ is easily seen to be combinatorial [CH15, Lem. 2.5].

The following result was a major step in the investigation of freeness properties for reflection arrangements.

Theorem 10 ([HR15, Thm. 1.1], [BC12, Thm. 5.14]). For $W$ a finite complex reflection group, the reflection arrangement $\mathcal{A}(W)$ is inductively free if and only if $W$ does not admit an irreducible factor isomorphic to a monomial group $G(r, r, \ell)$ for $r, \ell \geq 3$, $G_{21}$, $G_{27}$, $G_{29}$, $G_{31}$, $G_{33}$, or $G_{34}$.

Definition 11 (cf. [AT16]). Let $\mathcal{A}$ be an arrangement with $|\mathcal{A}| = n$. We say that $\mathcal{A}$ has a free filtration if there are subarrangements
\[
\emptyset = \mathcal{A}_0 \subsetneq \mathcal{A}_1 \subsetneq \cdots \subsetneq \mathcal{A}_{n-1} \subsetneq \mathcal{A}_n = \mathcal{A}
\]
such that $|\mathcal{A}_i| = i$ and $\mathcal{A}_i$ is free for all $1 \leq i \leq n$.

Very recently, Abe [Abe18a] introduced the class $\mathcal{AF}$ of additionally free arrangements. Arrangements in $\mathcal{AF}$ are by definition exactly the arrangements admitting a free filtration. Furthermore, it is a direct consequence of [Abe18a, Thm. 1.4] that the class $\mathcal{AF}$ is combinatorial.
3 Multiple Addition Theorem

The following theorem presented in [ABC+16] is a variant of the addition part ((2) and (3) imply (1)) of Theorem 8.

**Theorem 12 (Multiple Addition Theorem (MAT)).** Let \( A' \) be a free arrangement with \( \exp(A') = (d_1, \ldots, d_\ell) \leq 1 \leq p \leq \ell \) the multiplicity of the highest exponent, i.e.,

\[
d_{\ell-p} < d_{\ell-p+1} = \cdots = d_\ell =: d.
\]

Let \( H_1, \ldots, H_q \) be hyperplanes with \( H_i \not\in A' \) for \( i = 1, \ldots, q \). Define

\[ A''_j := (A' \cup \{H_j\})^{H_j} = \{H \cap H_j \mid H \in A'\}, \quad j = 1, \ldots, q. \]

Assume that the following three conditions are satisfied:

1. \( X := H_1 \cap \cdots \cap H_q \) is \( q \)-codimensional.
2. \( X \not\subseteq \bigcup_{H \in A'} H \).
3. \( |A'| - |A''_j| = d \) for \( 1 \leq j \leq q \).

Then \( q \leq p \) and \( A := A' \cup \{H_1, \ldots, H_q\} \) is free with \( \exp(A) = (d_1, \ldots, d_{\ell-q}, d + 1, \ldots, d + 1) \leq 1 \).

Note that in contrast to Theorem 8 no freeness condition on the restriction is needed to conclude the freeness of \( A \) in Theorem 12. The MAT motivates the following definition.

**Definition 13.** The class \( \mathcal{M}_\mathcal{F} \) of MAT-free arrangements is the smallest class of arrangements subject to

(i) \( \Phi_\ell \) belongs to \( \mathcal{M}_\mathcal{F} \), for every \( \ell \geq 0 \);

(ii) if \( A' \in \mathcal{M}_\mathcal{F} \) with \( \exp(A') = (d_1, \ldots, d_\ell) \leq 1 \leq p \leq \ell \) the multiplicity of the highest exponent \( d = d_\ell \), and if \( H_1, \ldots, H_q, q \leq p \) are hyperplanes with \( H_i \not\in A' \) for \( i = 1, \ldots, q \) such that:

1. \( X := H_1 \cap \cdots \cap H_q \) is \( q \)-codimensional,
2. \( X \not\subseteq \bigcup_{H \in A'} H \),
3. \( |A'| - |(A' \cup \{H_j\})^{H_j}| = d \), for \( 1 \leq j \leq q \),

then \( A := A' \cup \{H_1, \ldots, H_q\} \) also belongs to \( \mathcal{M}_\mathcal{F} \) and has exponents \( \exp(A) = (d_1, \ldots, d_{\ell-q}, d + 1, \ldots, d + 1) \leq 1 \).

Abe and Terao [AT19] proved the following generalization of Theorem 12:
Theorem 14 (Multiple Addition Theorem 2 (MAT2), [AT19, Thm. 1.4]). Assume that $A'$ is a free arrangement with $\exp(A') = (d_1, d_2, \ldots, d_\ell)_\leq$. Let

$$t := \begin{cases} \min\{i \mid d_i \neq 0\} & \text{if } A' \neq \Phi_\ell, \\ 0 & \text{if } A' = \Phi_\ell. \end{cases}$$

For $H_s, \ldots, H_\ell \not\in A$ with $s > t$, define $A'' := (A' \cup \{H_j\})^{H_j}$, $A := A' \cup \{H_s, \ldots, H_\ell\}$ and assume the following conditions:

1. $X := \bigcap_{i=s}^\ell H_i$ is $(\ell - s + 1)$-codimensional,
2. $X \not\subseteq \bigcup_{K \in A'} K$, and
3. $|A'| - |A''| = d_j$ for $j = s, \ldots, \ell$.

Then $A$ is free with exponents $(d_1, d_2, \ldots, d_{s-1}, d_s + 1, \ldots, d_\ell + 1)_\leq$. Moreover, there is a basis $\theta_1, \theta_2, \ldots, \theta_{s-1}, \eta_s, \ldots, \eta_\ell$ for $D(A')$ such that $\deg \theta_i = d_i$, $\deg \eta_j = d_j$, $\theta_i \in D(A)$ and $\eta_j \in D(A \setminus \{H_j\})$ for all $i$ and $j$.

This in turn motivates:

Definition 15. The class $\mathcal{M}_A'$ of MAT2-free arrangements is the smallest class of arrangements subject to

1. $\Phi_\ell$ belongs to $\mathcal{M}_A'$, for every $\ell \geq 0$;
2. if $A' \in \mathcal{M}_A'$ with $\exp(A') = (d_1, d_2, \ldots, d_\ell)_\leq$ and if $H_s, \ldots, H_\ell$ are hyperplanes with $H_i \not\in A'$ for $i = s, \ldots, \ell$, where

$$s > \begin{cases} \min\{i \mid d_i \neq 0\} & \text{if } A' \neq \Phi_\ell, \\ 0 & \text{if } A' = \Phi_\ell. \end{cases}$$

and with

1. $X := H_s \cap \cdots \cap H_\ell$ is $(\ell - s + 1)$-codimensional,
2. $X \not\subseteq \bigcup_{H \in A'} H$,
3. $|A'| - |(A' \cup \{H_j\})^{H_j}| = d_j$ for $s \leq j \leq \ell$,

then $A := A' \cup \{H_s, \ldots, H_\ell\}$ also belongs to $\mathcal{M}_A'$ and has exponents $\exp(A) = (d_1, \ldots, d_{s-1}, d_s + 1, \ldots, d_\ell + 1)_\leq$.

We note the following:

Remark 16. 1. We have $\mathcal{M}_A \subseteq \mathcal{M}_A'$.
2. If $A$ is a free arrangement with $\exp(A) = (0, \ldots, 0, 1, \ldots, 1, d, \ldots, d)_\leq$, i.e. $A$ has only two distinct exponents $\neq 0$, then it is clear from the definitions that $A$ is MAT2-free if and only if $A$ is MAT-free.
Example 17. 1. If rk($\mathcal{A}$) = 2 then $\mathcal{A}$ is MAT-free and therefore MAT2-free too.

2. Every ideal subarrangement of a Weyl arrangement is MAT-free and therefore also MAT2-free, [ABC+16].

Lemma 18. The classes $\mathcal{M}_3$ and $\mathcal{M}_3'$ are combinatorial.

Proof. The class of all empty arrangements is combinatorial and contained in $\mathcal{M}_3$. Let $\mathcal{A} \in \mathcal{M}_3$ ($\mathcal{A} \in \mathcal{M}_3'$). Since conditions (1)–(3) in Definition 13 (respectively Definition 15) only depend on $L(\mathcal{A})$ the claim follows. See also [AT19, Thm. 5.1]. □

If an arrangement $\mathcal{A}$ is MAT-free, the MAT-steps yield a partition of $\mathcal{A}$ whose dual partition gives the exponents of $\mathcal{A}$. Vice versa, the existence of such a partition suffices for the MAT-freeness of the arrangement:

Lemma 19. Let $\mathcal{A}$ be an $\ell$-arrangement. Then $\mathcal{A}$ is MAT-free if and only if there exists a partition $\pi = (\pi_1|\cdots|\pi_n)$ of $\mathcal{A}$ where for all $0 \leq k \leq n - 1$,

1. $\text{rk}(\pi_{k+1}) = |\pi_{k+1}|$, 
2. $\cap_{H \in \pi_{k+1}} H = X_{k+1} \not\subset \bigcup_{H' \in A_k} H'$ where $A_k = \bigcup_{i=1}^k \pi_i$, 
3. $|A_k| - |(A_k \cup \{H\})^H| = k$ for all $H \in \pi_{k+1}$.

In this case $\mathcal{A}$ has exponents $\exp(\mathcal{A}) = (d_1,\ldots,d_\ell)_{\leq}$ with $d_i = \{|k| \mid \pi_k \geq \ell - i + 1\}|$.

Proof. This is immediate from the definition. □

Definition 20. If $\pi$ is a partition as in Lemma 19 then $\pi$ is called an MAT-partition for $\mathcal{A}$.

If we have chosen a linear ordering $\mathcal{A} = \{H_1,\ldots,H_m\}$ of the hyperplanes in $\mathcal{A}$, to specify the partition $\pi$, we give the corresponding ordered set partition of $[m] = \{1,\ldots,m\}$.

Example 21. Supersolvable arrangements, a proper subclass of inductively free arrangements [OT92, Thm. 4.58], are not necessarily MAT2-free: an easy calculation shows that the arrangement denoted $\mathcal{A}(10,1)$ in [Grü09] is supersolvable but not MAT2-free. In particular $\mathcal{A}(10,1)$ is neither MAT-free.

Restrictions of MAT2-free (MAT-free) arrangements are not necessarily MAT2-free (MAT-free):

Example 22. Let $\mathcal{A} = \mathcal{A}(E_6)$ be the Weyl arrangement of the Weyl group of type $E_6$. Then $\mathcal{A}$ is MAT-free by Example 17(2). Let $H \in \mathcal{A}$. A simple calculation (with the computer) shows that $\mathcal{A}^H$ is not MAT2-free.

We have two simple necessary conditions for MAT-freeness respectively MAT2-freeness. The first one is:
Lemma 23. Let $\mathcal{A}$ be a non-empty MAT2-free arrangement with exponents $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell) \in \mathbb{N}$. Then there is an $H \in \mathcal{A}$ such that $|\mathcal{A}| - |\mathcal{A}^H| = d_\ell$. In particular, the same holds, if $\mathcal{A}$ is MAT-free.

Proof. By definition there are $H_q, \ldots, H_\ell \in \mathcal{A}$, $2 \leq q$ such that $\mathcal{A}':= \mathcal{A} \setminus \{H_q, \ldots, H_\ell\}$ is MAT2-free. Furthermore by condition (1) the hyperplanes $H_q, \ldots, H_\ell$ are linearly independent. Let $H := H_\ell$. By condition (2), we have $X = \cap_{i=q}^{\ell} H_i \nsubseteq \cup_{H' \in \mathcal{A}\setminus H} H'$ and thus $|\mathcal{A}^H| = |(\mathcal{A}' \cup \{H\})^H| + \ell - q$. Now

$$|\mathcal{A}'| - |(\mathcal{A}' \cup \{H\})^H| = d_\ell - 1$$

by condition (3) and hence

$$|\mathcal{A}| - |\mathcal{A}^H| = |\mathcal{A}'| + \ell - q + 1 - |(\mathcal{A}' \cup \{H\})^H| - \ell + q = d_\ell.$$

The second one is:

Lemma 24. Let $\mathcal{A}$ be an MAT2-free arrangement. Then $\mathcal{A}$ has a free filtration, i.e. $\mathcal{A}$ is additionally free. In particular, the same is true, if $\mathcal{A}$ is MAT-free.

Proof. Let $\mathcal{A}$ be MAT2-free. Then by definition there are $H_q, \ldots, H_\ell \in \mathcal{A}$ such that $\mathcal{A}' := \mathcal{A} \setminus \{H_q, \ldots, H_\ell\}$ is MAT2-free and conditions (1)–(3) are satisfied. Set $B := \{H_q, \ldots, H_\ell\}$. By [AT19, Cor. 3.2] for all $C \subseteq B$ the arrangement $\mathcal{A}' \cup C$ is free. Hence by induction $\mathcal{A}$ has a free filtration. 

An MAT2-free but not MAT-free arrangement

We now provide an example of an arrangement which is MAT2-free but not MAT-free.

Example 25. Let $\mathcal{A}$ be the arrangement defined by

$$\mathcal{A} := \{H_1, \ldots, H_{10}\} = \{(1,0,0)^\perp, (0,1,0)^\perp, (0,0,1)^\perp, (1,1,0)^\perp, (1,2,0)^\perp, (0,1,1)^\perp, (1,3,0)^\perp, (1,1,1)^\perp, (2,3,0)^\perp, (1,3,1)^\perp\}.$$

It is not hard to see that $\mathcal{A}$ is inductively free (actually supersolvable) with $\exp(\mathcal{A}) = (1,4,5)$.

Proposition 26. The arrangement $\mathcal{A}$ from Example 25 is MAT2-free.

Proof. Let $\mathcal{B}_1 = \{H_1, H_2, H_3\}$, $\mathcal{B}_2 = \{H_4\}$, $\mathcal{B}_3 = \{H_5, H_6\}$, $\mathcal{B}_4 = \{H_7, H_8\}$, $\mathcal{B}_5 = \{H_9, H_{10}\}$, and $\mathcal{A}_k = \cup_{i=1}^{k} \mathcal{B}_i$ for $1 \leq k \leq 5$. It is clear that $\mathcal{A}_1$ is MAT2-free. A simple linear algebra computation shows that the addition of $\mathcal{B}_{i+1}$ to $\mathcal{A}_i$ for $1 \leq i \leq 4$ satisfies Condition (1)–(3) of Definition 15. Hence $\mathcal{A} = \mathcal{A}_5$ is MAT2-free.

Proposition 27. The arrangement $\mathcal{A}$ from Example 25 is not MAT-free.
Proof. Suppose $A$ is MAT-free and $\pi = (\pi_1, \ldots, \pi_5)$ is an MAT-partition. Since $\exp(A) = (1, 4, 5)$ the last block $\pi_5$ has to be a singleton, i.e. $\pi_5 = \{H\}$. By Condition (3) of Lemma 19 we have $|A^H| = 5$ and the only hyperplane with this property is $H_0 = (2, 3, 0)^\perp$. Similarly $\pi_4$ can only contain one of $H_3, H_6, H_8, H_{10}$. But looking at their intersections we see that all of the latter are contained in another hyperplane of $A$, e.g. $H_3 \cap H_8 \subseteq H_4$. This contradicts Condition (2). Hence $A$ is not MAT-free. \hfill \Box

As a direct consequence we get:

**Proposition 28.** We have

$$\mathfrak{M}_S \subseteq \mathfrak{M}'_S.$$ 

**Products of MAT-free and MAT2-free arrangements**

As for freeness in general (Proposition 7), the product construction is compatible with the notion of MAT-freeness:

**Theorem 29.** Let $A = A_1 \times A_2$ be a product of two arrangements. Then $A \in \mathfrak{M}_S$ if and only if $A_1 \in \mathfrak{M}_S$ and $A_2 \in \mathfrak{M}_S$.

**Proof.** Assume $A_i$ is an arrangement in the vector space $V_i$ of dimension $\ell_i$ for $i = 1, 2$. We argue by induction on $|A|$. If $|A| = 0$, i.e. $A_1 = \Phi_{\ell_1}$, and $A_2 = \Phi_{\ell_2}$ then the statement is clear. Assume $A_1$ is MAT-free with $\exp(A_1) = (d_1^1, \ldots, d_\ell_1) \in \Phi_{\ell_1}$ and $A_2$ is MAT-free with $\exp(A_2) = (d_1^2, \ldots, d_\ell_2) \in \Phi_{\ell_2}$. Let $q_i$ be the multiplicity of the exponent $d_i$ in $\exp(A_i)$ for $i = 1, 2$ (note that $q_1 = 0$ if $d > d_\ell_1^2$. Then without loss of generality $d := d_\ell_1^1 \geq d_\ell_2^2$. Let $d_i$ be the multiplicity of the exponent $d_i$ in $\exp(A_i)$ for $i = 1, 2$. Then since $A_i$ is MAT-free there are hyperplanes $\{H_1^i, \ldots, H_{q_i}^i\} \subseteq A_i$ such that $A_i' := A_i \setminus \{H_1^i, \ldots, H_{q_i}^i\}$ is MAT-free, i.e. they satisfy Conditions (1)–(3) from Definition 13. Now by the induction hypothesis $A' = A_1' \times A_2'$ is MAT-free and clearly $\{H_1^1 \oplus V_2, \ldots, H_{q_1}^1 \oplus V_2\} \cup \{V_1 \oplus H_1^2, \ldots, V_1 \oplus H_{q_2}^2\}$ satisfy Conditions (1)–(3). Hence $A$ is MAT-free.

Conversely assume $A$ is MAT-free with $\exp(A) = (d_1, \ldots, d_\ell) \in \Phi_{\ell}$. By Proposition 7 both factors $A_1$ and $A_2$ are free with $\exp(A_i) = (d_i^1, \ldots, d_\ell_i) \in \Phi_{\ell_i}$ and without loss of generality $d_i = d_\ell_1^i \geq d_\ell_2^i$. Assume further that $q_i$ is the multiplicity of $d_i$ in $\exp(A_i)$ and $q$ is the multiplicity of $d_\ell$ in $\exp(A)$, i.e. $q = q_1 + q_2$. There are hyperplanes $\{H_1, \ldots, H_q\} \subseteq A$ such that $A' = A \setminus \{H_1, \ldots, H_q\}$ is MAT-free with $\exp(A') = (d_1, \ldots, d_{\ell-q}, d_{\ell-q+1} - 1, \ldots, d_\ell - 1) \in \Phi_{\ell}$, and Conditions (1)–(3) are satisfied. We may further assume that $H_i = H_i^1 \oplus V_2$ for $1 \leq i \leq q_1$ and $H_j = V_1 \oplus H_{j-q_1}^2$ for $q_1 + 1 \leq j \leq q$. Let $A_i' = A_i \setminus \{H_1^i, \ldots, H_{q_i}^i\}$ for $i = 1, 2$. Note that if $d_\ell > d_\ell_2^i$ we have $q_2 = 0$ and $A_2' = A_2$. But at least we have $A_i' \subseteq A_i$. Then $A' = A_1' \times A_2'$, $|A'| < |A|$ and by the induction hypothesis $A_1'$ and $A_2'$ are MAT-free and Conditions (1) and (2) are clearly satisfied for $A_1'$ and $\{H_1^1, \ldots, H_{q_i}^1\}$. But since

$$d_\ell - 1 = |A'| - |(A' \cup \{H_i\})^H_i|$$

$$= |A_1'| + |A_2'| - |(A_1 \cup \{H_1^1\})^H_1| + |A_2'|$$

$$= |A_1'| - |(A_1 \cup \{H_1^1\})^H_1|$$

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for $1 \leq i \leq q_1$ and
\[ d_i - 1 = |A'_i| - |(A'_i \cup \{H_j\})^{H_j}| \]
\[ = |A'_1| + |A'_2| - (|A_1 \cup \{H_{j-q_1}\})^{H_{j-q_1}}| + |A'_2|) \]
\[ = |A'_1| - |(A_1 \cup \{H_{j-q_1}\})^{H_{j-q_1}}| \]
for $q_1 + 1 \leq j \leq q_2$, Condition (3) is also satisfied for $A'_1$ and $A'_2$. Hence both $A_1$ and $A_2$ are MAT-free.

Alternatively, one can prove Theorem 29 by observing that MAT-Partitions for $A_1$ and $A_2$ are directly obtained from an MAT-Partition for $A$: take the non-empty factors of each block in the same order, and vise versa: take the products of the blocks of partitions for $A_1$ and $A_2$.

**Remark 30.** Thanks to the preceding theorem, our classification of MAT-free irreducible reflection arrangements proved in the next 2 sections gives actually a classification of all MAT-free reflection arrangements: a reflection arrangement $A(W)$ is MAT-free if and only if it has no irreducible factor isomorphic to one of the non-MAT-free irreducible reflection arrangements listed in Theorem 1.

In contrast to MAT-freeness, the weaker notion of MAT2-freeness is not compatible with products as the following example shows:

**Example 31.** Let $A_1$ be the MAT2-free but not MAT-free arrangement of Example 25 with exponents $\exp(A_1) = (1, 4, 5)$. Let $\zeta = \frac{1}{2}(-1 + i\sqrt{3})$ be a primitive cube root of unity, and let $A_2$ be the arrangement defined by the following linear forms:

$$A_2 := \{H^2_1, \ldots, H^2_{10}\}$$

$$= \{(1, 0, 0)^\perp, (0, 1, 0)^\perp, (0, 0, 1)^\perp, (1, -\zeta, 0)^\perp, (1, 0, -\zeta)^\perp,$$

$$(1, -\zeta^2, 0)^\perp, (1, 0, -\zeta^2)^\perp, (1, -1, 0)^\perp, (1, 0, -1)^\perp, (0, 1, -\zeta)^\perp\}.$$

A linear algebra computation shows that $\pi = (1, 2, 3|4, 5|6, 7|8, 9|10)$ is an MAT-partition for $A_2$. In particular $A_2$ is MAT2-free with $\exp(A_2) = (1, 4, 5)$.

Now by Proposition 7 the product $A := A_1 \times A_2$ is free with $\exp(A) = (1, 1, 4, 4, 5, 5)$. Suppose $A$ is MAT2-free. Then either there are hyperplanes $H_i \in A_1$ and $H_2 \in A_2$ such that $A' = A'_1 \times A'_2$ is MAT2-free with exponents $\exp(A') = (1, 1, 4, 4, 4, 4)$ where $A'_i = A_i \setminus \{H_i\}$. Or there are hyperplanes $H^1_1, H^2_1 \in A_1, H^2_2 \in A_2$ such that $A' = A'_1 \times A'_2$ is MAT2-free with exponents $\exp(A') = (1, 1, 3, 3, 4, 4)$ where $A'_i = A_i \setminus \{H^1_1, H^2_2\}$.

In the first case $A'$ is actually MAT-free by Remark 16. But then by Theorem 29 $A'_2$ is MAT-free and $A_2$ is MAT-free too which is a contradiction.

In the second case $H^1_1 \oplus V_2, H^2_1 \oplus V_2, V_1 \oplus H^1_1, V_1 \oplus H^2_2$ satisfy Condition (1)–(3) of Definition 15. But by Condition (3) we have

$$|A'_1| - |(A'_1 \cup \{H^1_1\})^{H^1_1}| = 4$$
and

\[ |A'_i| - |(A'_i \cup \{ H_j \})_{H_j}| = 3. \]

But an easy calculation shows that there are no two hyperplanes in \( A_1 \) with this property and which also satisfy Condition (2)–(3). This is a contradiction and hence \( A = A_1 \times A_2 \) is not MAT2-free.

4 MAT-free imprimitive reflection groups

Definition 32 ([OT92, §6.4]). Let \( x_1, \ldots, x_\ell \) be a basis of \( V^* \). Let \( \zeta = \exp(\frac{2\pi i}{r}) \) (\( r \in \mathbb{N} \)) be a primitive \( r \)-th root of unity. Define the linear forms \( \alpha_{ij}(\zeta^k) \in V^* \) by

\[ \alpha_{ij}(\zeta^k) = x_i - \zeta^k x_j \]

and the hyperplanes

\[ H_{ij}(\zeta^k) = \ker(\alpha_{ij}(\zeta^k)). \]

for \( 1 \leq i, j \leq \ell \) and \( 1 \leq k \leq r \). Then the reflection arrangement of the imprimitive complex reflection group \( G(r, 1, \ell) \) can be defined by:

\[ A(G(r, 1, \ell)) = \{ \ker(x_i) \mid 1 \leq i \leq \ell \} \cup \{ H_{ij}(\zeta^k) \mid 1 \leq i < j \leq \ell, 1 \leq k \leq r \}. \]

Proposition 33. Let \( A = A(G(r, 1, \ell)) \). Let

\[ \pi_{11} := \{ \ker(x_i) \mid 1 \leq i \leq \ell \}, \]

and

\[ \pi_{ij} := \{ H_{(i-1)k}(\zeta^j) \mid i \leq k \leq \ell \}, \]

for \( 2 \leq i \leq \ell, 1 \leq j \leq r \). Then

\[ \pi = (\pi_{ij})_{1 \leq i, j \leq \ell}, m_i = \begin{cases} 1 & \text{for } i = 1 \\ r & \text{for } 2 \leq i \leq \ell \end{cases} \]

\[ = (\pi_{11}|\pi_{21}| \cdots |\pi_{2r}| \cdots |\pi_{\ell r}) \]

is an MAT-partition of \( A \). In particular \( A \in \mathcal{M}_{\mathbb{R}} \) with exponents

\[ \exp(A) = (1, r + 1, 2r + 1, \ldots, (l - 1)r + 1). \]

Proof. We verify Conditions (1)–(3) from Lemma 19 in turn.

Let

\[ A_{ij} := \bigcup_{1 \leq a \leq i - 1, 1 \leq b \leq j} \pi_{ab} \cup \bigcup_{1 \leq b \leq j} \pi_{ib} \]

and

\[ A'_ij := \bigcup_{1 \leq a \leq i - 1, 1 \leq b \leq j-1} \pi_{ab} \cup \bigcup_{1 \leq b \leq j} \pi_{ib}. \]
For $\pi_{11}$ we clearly have $|\pi_{11}| = \text{rk}(\pi_{11}) = \ell$. Similarly for $2 \leq i \leq \ell$, $1 \leq j \leq r$ we have $|\pi_{ij}| = \text{rk}(\pi_{ij}) = \ell - i + 1$ since all the defining linear forms $\alpha_{(i-1)k}(\zeta^j)$ ($i \leq k \leq \ell$) for the hyperplanes in $\pi_{ij}$ are linearly independent. Thus Condition (1) holds.

Furthermore, the forms $\{\alpha_{ac}(\zeta^b)\} \cup \{\alpha_{(i-1)k}(\zeta^j) \mid i \leq k \leq \ell\}$ are linearly independent for all $1 \leq a \leq i - 1$, $1 \leq b \leq j - 1$, and $a + 1 \leq c \leq \ell$, i.e. $\cap_{H \in \pi_{ij}} H =: X_{ij} \notin H$ for all $H \in \mathcal{A}_j$. Hence Condition (2) is also satisfied.

To verify Condition (3) let $H = H_{(i-1)k}(\zeta^j) \in \pi_{ij}$ for a fixed $1 \leq k \leq r$. We show

$$|\mathcal{A}_{ij}^r| - (j + (i - 2)r) = |(\mathcal{A}_{ij}^r)^H|.$$

Let $H'_a := H_{a(i-1)}(\zeta^a) \in \mathcal{A}_{ij}^r$, $1 \leq a \leq j - 1$. Then

$$\mathcal{B} := (\mathcal{A}_{ij}^r)_{H \cap H'_a} = \{\ker(x_{i-1}), \ker(x_k)\} \cup \{H'_b \mid 1 \leq b \leq j - 1\},$$

and $\text{rk}(\mathcal{B}) = 2$. So all $H' \in \mathcal{B}$ give the same intersection with $H$ and $|\mathcal{B}| = j + 1$. For $H' = H_{a(i-1)}(\zeta^a) \in \mathcal{A}_j^r$ with $a \leq i - 2$, and $1 \leq b \leq r$ we have

$$\mathcal{C} := (\mathcal{A}_{ij}^r)_{H' \cap H'} = \{H', H_{ak}(\zeta^i+j+b)\},$$

$|\mathcal{C}| = 2$ and there are exactly $(i - 2)r$ such $H'$. All other $H'' \in \mathcal{A}_{ij}'$ intersect $H$ simply. Hence

$$|(\mathcal{A}_{ij}^r)^H| = |\mathcal{A}_{ij}^r| - (|\mathcal{B}| - 1) - (i - 2)r(|\mathcal{C}| - 1)$$

$$= |\mathcal{A}_{ij}^r| - j - (i - 2)r,$$

or $|\mathcal{A}_{ij}^r| - |(\mathcal{A}_{ij}^r)^H| = \sum_{a=1}^{i-1} m_i + (j - 1)$. This finishes the proof.

Proposition 34. Let $\mathcal{A} = \mathcal{A}(G(r, r, \ell))$ ($r, \ell \geq 3$). Then $\mathcal{A}$ is not MAT2-free. In particular $\mathcal{A}$ is not MAT-free.

Proof. By [OT92, Prop. 6.85] the arrangement $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell) = (1, r+1, 2r+1, \ldots, (\ell-2)r+1, (\ell-1)(r-1))$. In particular we have $(\ell-1)(r-1) = d_\ell$ and $|\mathcal{A}| = \frac{\ell(\ell-1)}{2} r$. But for all $H \in \mathcal{A}$ by [OT92, Prop. 6.82, 6.85] we have $|\mathcal{A}^H| = \frac{\ell(\ell-1)(\ell-2)}{2} r + 1$. Hence $|\mathcal{A}| - |\mathcal{A}^H| = (\ell - 1)r - 1 \neq d_\ell$ and by Lemma 23 the arrangement $\mathcal{A}$ is not MAT2-free.

Theorem 35. Let $\mathcal{A} = \mathcal{A}(W)$ be the reflection arrangement of the imprimitive complex reflection group $W = G(r, e, \ell)$ ($r, \ell \geq 3$). Then $\mathcal{A}$ is MAT-free if and only if it is MAT2-free if and only if $e \neq r$.

Proof. Since $\mathcal{A} = \mathcal{A}(G(r, 1, \ell))$ if and only if $r \neq e$, this is Proposition 33 and Proposition 34.
5 MAT-free exceptional complex reflection groups

To prove the MAT-freeness of one of the following reflection arrangements, we explicitly give a realization by linear forms.

First note that if $W$ is an exceptional Weyl group, or a group of rank $\leq 2$, then by Example 17 $\mathcal{A}(W)$ is MAT-free.

**Proposition 36.** Let $\mathcal{A}$ be the reflection arrangement of the reflection group $H_3$ ($G_{23}$). Then $\mathcal{A}$ is MAT-free. In particular $\mathcal{A}$ is $\text{MAT}^2$-free.

**Proof.** Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio and $\tau' = 1/\tau$ its reciprocal. The arrangement $\mathcal{A}$ can be defined by the following linear forms:

$$\mathcal{A} = \{H_1, \ldots, H_{15}\} = \{(1,0,0)\perp, (0,1,0)\perp, (0,0,1)\perp, (1,\tau,\tau')\perp, (\tau',1,\tau)\perp, (\tau,\tau',1)\perp, (1,-\tau,\tau')\perp, (-\tau',1,-\tau)\perp, (1,\tau,-\tau')\perp, (-\tau',1,\tau)\perp, (\tau,-\tau',1)\perp, (-\tau',1,-\tau)\perp, (1,\tau,-\tau')\perp, (-\tau',1,\tau)\perp, (\tau,-\tau',1)\perp\}.$$  

With this linear ordering of the hyperplanes the partition

$$\pi = (13,14,15|10,12|5,6|4,11|8,9|7|3|2|1)$$

satisfies Conditions (1)–(3) of Lemma 19 as one can easily verify by a linear algebra computation. Hence $\pi$ is an MAT-partition and $\mathcal{A}$ is MAT-free. $\square$

**Proposition 37.** Let $\mathcal{A}$ be the reflection arrangement of the complex reflection group $G_{24}$. Then $\mathcal{A}$ is not $\text{MAT}^2$-free. In particular $\mathcal{A}$ is not MAT-free.

**Proof.** The arrangement $\mathcal{A}$ is free with $\exp(\mathcal{A}) = (1,9,11)$ and $|\mathcal{A}| - |\mathcal{A}^H| = 13$ for all $H \in \mathcal{A}$ by [OT92, Tab. C.5]. Hence by Lemma 23 $\mathcal{A}$ is not MAT2-free. $\square$

**Proposition 38.** Let $\mathcal{A}$ be the reflection arrangement of the complex reflection group $G_{25}$. Then $\mathcal{A}$ is MAT-free. In particular $\mathcal{A}$ is $\text{MAT}^2$-free.

**Proof.** Let $\zeta = \frac{1}{2}(-1 + i\sqrt{3})$ be a primitive cube root of unity. The reflecting hyperplanes of $\mathcal{A}$ can be defined by the following linear forms (cf. [LT09, Ch. 8, 5.3]):

$$\mathcal{A} = \{H_1, \ldots, H_{12}\} = \{(1,0,0)\perp, (0,1,0)\perp, (0,0,1)\perp, (1,1,1)\perp, (1,1,\zeta)\perp, (1,1,\zeta^2)\perp, (1,\zeta,1)\perp, (1,\zeta,\zeta)\perp, (1,\zeta,\zeta^2)\perp, (1,\zeta^2,1)\perp, (1,\zeta^2,\zeta)\perp, (1,\zeta^2,\zeta^2)\perp\}.$$  

With this linear ordering of the hyperplanes the partition

$$\pi = (7,4,3|8,5|9,6|2,1|10|11|12)$$

satisfies the three conditions of Lemma 19 as one can easily verify by a linear algebra computation. Hence $\pi$ is an MAT-partition and $\mathcal{A}$ is MAT-free. $\square$
Proposition 39. Let $\mathcal{A}$ be the reflection arrangement of the complex reflection group $G_{26}$. Then $\mathcal{A}$ is MAT-free. In particular $\mathcal{A}$ is MAT2-free.

Proof. Let $\zeta = \frac{1}{2}(-1+i\sqrt{3})$ be a primitive cube root of unity. The reflection arrangement $\mathcal{A}$ is the union of the reflecting hyperplanes of $\mathcal{A}(G_{26})$ and $\mathcal{A}(G(3, 3, 3))$ (cf. [LT09, Ch. 8, 5.5]). In particular the hyperplanes contained in $\mathcal{A}$ can be defined by the following linear forms:

$$\mathcal{A} = \{H_1, \ldots, H_{26}\}$$

$$= \{(1, 0, 0)^\perp, (0, 1, 0)^\perp, (0, 0, 1)^\perp, (1, 1, 1)^\perp, (1, 1, \zeta)^\perp, (1, 1, \zeta^2)^\perp,$$

$$(1, \zeta, 1)^\perp, (1, \zeta, \zeta)^\perp, (1, \zeta, \zeta^2)^\perp, (1, \zeta^2, 1)^\perp, (1, \zeta^2, \zeta)^\perp, (1, \zeta^2, \zeta^2)^\perp,$$

$$(1, -\zeta, 0)^\perp, (1, -\zeta^2, 0)^\perp, (1, -1, 0)^\perp, (1, 0, -\zeta)^\perp, (1, 0, -\zeta^2)^\perp,$$

$$(1, 0, -1)^\perp, (0, 1, -\zeta)^\perp, (0, 1, -\zeta^2)^\perp, (0, 1, -1)^\perp\}.$$

With this linear ordering of the hyperplanes the partition

$$\pi = (12, 19, 20|16, 18|13, 15|17, 21|10, 14|6, 11|8, 9|7|5|4|3|2|1)$$

satisfies the three conditions of Lemma 19 as one can verify by a standard linear algebra computation. Hence $\pi$ is an MAT-partition and $\mathcal{A}$ is MAT-free. \hfill \Box

Proposition 40. Let $\mathcal{A}$ be the reflection arrangement of the complex reflection group $G_{27}$. Then $\mathcal{A}$ is not MAT2-free. In particular $\mathcal{A}$ is not MAT-free.

Proof. The arrangement $\mathcal{A}$ is free with $\text{exp}(\mathcal{A}) = (1, 19, 25)$ and $|\mathcal{A}| - |\mathcal{A}^H| = 29$ for all $H \in \mathcal{A}$ by [OT92, Tab. C.8]. Hence by Lemma 23 $\mathcal{A}$ is not MAT2-free. \hfill \Box

Proposition 41. Let $\mathcal{A}$ be the reflection arrangement of the reflection group $H_4$ (G_{30}). Then $\mathcal{A}$ is MAT-free. In particular $\mathcal{A}$ is MAT2-free.

Proof. Let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio and $\tau' = 1/\tau$ its reciprocal. The arrangement $\mathcal{A}$ can be defined by the following linear forms:

$$\mathcal{A} = \{H_1, \ldots, H_{60}\}$$

$$= \{(1, 0, 0, 0)^\perp, (0, 1, 0, 0)^\perp, (0, 0, 1, 0)^\perp, (0, 0, 0, 1)^\perp, (1, \tau, \tau', 0)^\perp,$$

$$(1, 0, \tau, \tau')^\perp, (1, \tau', 0, \tau)^\perp, (\tau, 1, 0, \tau')^\perp, (\tau', 1, \tau, 0)^\perp, (0, 1, \tau', \tau)^\perp,$$

$$(\tau, \tau', 0, 1)^\perp, (0, \tau, 1, \tau')^\perp, (\tau', 0, 1, \tau)^\perp, (\tau, \tau', 0, 1)^\perp,$$

$$(0, \tau', \tau, 1)^\perp, (-1, \tau, \tau', 0)^\perp, (1, -\tau, \tau', 0)^\perp, (1, \tau, -\tau', 0)^\perp, (-1, 0, \tau, \tau')^\perp,$$

$$(1, 0, -\tau, \tau')^\perp, (1, 0, \tau, -\tau')^\perp, (-1, 0, \tau', 0)^\perp, (1, -\tau', 0, \tau)^\perp, (1, \tau', 0, -\tau)^\perp,$$

$$(1, -\tau, 0, \tau')^\perp, (1, 0, \tau, -\tau)^\perp, (-1, \tau', 0, \tau)^\perp, (1, -\tau', 0, \tau)^\perp, (\tau', -1, \tau, 0)^\perp,$$

$$(\tau', 1, -\tau, 0)^\perp, (0, -1, \tau', \tau)^\perp, (0, 1, -\tau', \tau)^\perp, (0, 1, \tau', -\tau)^\perp, (\tau', -1, \tau, 0)^\perp,$$

$$(\tau, -\tau', 1, 0)^\perp, (\tau, \tau', -1, 0)^\perp, (0, -\tau, 1, \tau')^\perp, (0, \tau, -1, \tau')^\perp, (0, \tau, 1, -\tau)^\perp, (0, \tau, 1, -\tau)^\perp,$$

$$(\tau, -\tau', 1, 0)^\perp, (\tau, \tau', -1, 0)^\perp, (0, -\tau, 1, \tau')^\perp, (0, \tau, -1, \tau')^\perp, (0, \tau, 1, -\tau)^\perp, (0, \tau, 1, -\tau)^\perp.$$
With this linear ordering of the hyperplanes the partition
\[\pi = \{(-\tau', 0, 1, \tau)^1, (\tau', 0, -1, \tau)^1, (\tau', 0, 1, -\tau)^1, (-\tau, 0, \tau', 1)^1, (\tau, 0, -\tau', 1)^1, \\
(\tau, 0, \tau', -1)^1, (-\tau', \tau, 0, 1)^1, (\tau', -\tau, 0, 1)^1, (\tau', \tau, 0, -1)^1, (0, \tau', \tau, 1)^1, \\
(0, \tau', -\tau, 1)^1, (0, \tau', \tau, -1)^1, (1, 1, 1, 1)^1, (-1, -1, 1, 1)^1, (-1, 1, 1, -1)^1, \\
(-1, -1, 1, -1)^1\}.\]

With this linear ordering of the hyperplanes the partition
\[
\pi = (31, 43, 48, 54|29, 38, 51|23, 34, 58|18, 20, 25|17, 59, 60 \\
|21, 47, 52|39, 41, 44|26, 32, 49|30, 35, 40|2, 3, 42|33, 46, 50 \\
|4, 37|27, 57|19, 24|55, 56|10, 22|12, 45|16, 28|15, 36 \\
|53|14|13|11|9|8|7|6|5|1)
\]
satisfies Conditions (1)–(3) of Lemma 19 as one can verify with a linear algebra computation. Hence \(\pi\) is an MAT-partition and \(A\) is MAT-free. In particular \(A\) is MAT2-free. \(\square\)

We recall the following result about free filtration subarrangements of \(A(G_{31})\):

**Proposition 42** ([Müc17, Pro. 3.8]). Let \(A := A(G_{31})\) be the reflection arrangement of the finite complex reflection group \(G_{31}\). Let \(A\) be a minimal (w.r.t. the number of hyperplanes) free filtration subarrangement. Then \(A \cong A(G_{29})\).

**Corollary 43.** Let \(A\) be the reflection arrangement of one of the complex reflection groups \(G_{29}\) or \(G_{31}\). Then \(A\) has no free filtration.

**Proposition 44.** Let \(A\) be the reflection arrangement of one of the complex reflection groups \(G_{29}\) or \(G_{31}\). Then \(A\) is not MAT2-free. In particular \(A\) is not MAT-free.

**Proof.** By Corollary 43 both arrangements have no free filtration and hence are not MAT2-free by Lemma 24. \(\square\)

**Proposition 45.** Let \(A\) be the reflection arrangement of the complex reflection group \(G_{32}\). Then \(A\) is not MAT2-free and also not MAT2-free.

**Proof.** Up to symmetry of the intersection lattice there are exactly 9 different choices of a basis, where a basis is a subarrangement \(B \subseteq A\) with |\(B\)| = \(r(B) = r(A) = 4\). Suppose that \(A\) is MAT-free. Then the first block in an MAT-partition for \(A\) has to be one of these bases. But a computer calculation shows that none of these bases may be extended to an MAT-partition for \(A\). Hence \(A\) is not MAT-free. A similar but more cumbersome calculation shows that \(A\) is also not MAT2-free. \(\square\)

**Proposition 46.** Let \(A\) be the reflection arrangement of one of the complex reflection group \(G_{33}\) or \(G_{34}\). Then \(A\) is not MAT2-free. In particular \(A\) is not MAT-free.

**Proof.** First, let \(A = A(G_{33})\). Then \(\exp(A) = (1, 7, 9, 13, 15)\) by [OT92, Tab. C.14]. But |\(A\)| = |\(A^H\)| = 17 for all \(H \in A\) also by [OT92, Tab. C.14]. So \(A\) is not MAT2-free by Lemma 23.

Similarly \(A = A(G_{33})\) is free with \(\exp(A) = (1, 13, 19, 25, 31, 37)\) by [OT92, Tab. C.17] and |\(A\)| = |\(A^H\)| = 41 for all \(H \in A\). Hence \(A\) is not MAT2-free by Lemma 23. \(\square\)

Comparing with Theorem 10 finishes the proofs of Theorem 1 and Theorem 2.
6 Further remarks on MAT-freeness

In their very recent note [HR19] Hoge and Röhrle confirmed a conjecture by Abe [Abe18a] by providing two examples $\mathcal{B}, \mathcal{D}$ of arrangements, related to the exceptional reflection arrangement $\mathcal{A}(E_7)$, which are additionally free but not divisionally free and in particular also not inductively free. The arrangements have exponents $\exp(\mathcal{B}) = (1,5,5,5,5,5,5)$ and $\exp(\mathcal{D}) = (1,5,5,5,5)$. Since both arrangements have only 2 different exponents by Remark 16 they are MAT-free if and only if they are MAT2-free. Now a computer calculation shows that both arrangements are not MAT-free and hence also not MAT2-free. In particular they provide no negative answer to Question 3 and Question 4.

Several computer experiments suggest that similar to the poset obtained from the positive roots of a Weyl group giving rise to an MAT-partition (cf. Example 17) MAT-free arrangements might in general satisfy a certain poset structure:

**Problem 47.** Can MAT-freeness be characterized by the existence of a partial order on the hyperplanes, generalizing the classical partial order on the positive roots of a Weyl group?

Recall that by Example 22 the restriction $\mathcal{A}^H$ is in general not MAT-free (MAT2-free) if the arrangement $\mathcal{A}$ is MAT-free (MAT2-free). But regarding localizations there is the following:

**Problem 48.** Is $\mathcal{A}_X$ MAT-free (MAT2-free) for all $X \in L(\mathcal{A})$ provided $\mathcal{A}$ is MAT-free (MAT2-free)?

Last but not least, related to the previous problem, our investigated examples suggest the following:

**Problem 49.** Suppose $\mathcal{A}'$ and $\mathcal{A} = \mathcal{A}' \cup \{H\}$ are free arrangements such that $\exp(\mathcal{A}') = (d_1, \ldots, d_\ell) \leq$ and $\exp(\mathcal{A}) = (d_1, \ldots, d_{\ell-1}, d_\ell+1) \leq$. Let $X \in L(\mathcal{A})$ with $X \subseteq H$. By [OT92, Thm. 4.37] both localizations $\mathcal{A}'_X$ and $\mathcal{A}_X$ are free. If $\exp(\mathcal{A}'_X) = (c_1, \ldots, c_r) \leq$ is it true that $\exp(\mathcal{A}) = (c_1, \ldots, c_{r-1}, c_r + 1) \leq$, i.e. if we only increase the highest exponent is the same true for all localizations?

Note that the answer is yes if we only look at localizations of rank $\leq 2$. Our proceeding investigation of Problem 47 suggests that this should be true at least for MAT-free arrangements. Furthermore, a positive answer to Problem 49 would imply (with a bit more work) a positive answer to Problem 48.

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