STABLE MODULI SPACES
OF HIGH DIMENSIONAL MANIFOLDS

SØREN GALATIUS AND OSCAR RANDEL-WILLIAMS

Dedicated to Ib Madsen on the occasion of his 70th birthday

Abstract. We prove an analogue of the Madsen–Weiss theorem for highdimensional manifolds. For example, we explicitly describe the ring of characteristic classes of smooth fibre bundles whose fibres are connected sums of \( g \times S^n \times S^n \), in the limit \( g \to \infty \). Rationally it is a polynomial ring in certain explicit generators, giving a high-dimensional analogue of Mumford’s conjecture.

More generally, we study a moduli space \( \mathcal{N}(P) \) of those nullbordisms of a fixed \((2n-1)\)-dimensional manifold \( P \) which are \((n-1)\)-connected relative to \( P \). We determine the homology of \( \mathcal{N}(P) \) after stabilisation using certain self-bordisms of \( P \). The stable homology is identified with that of a certain infinite loop space.

1. Introduction and statement of results

For any smooth compact manifold \( W \), the diffeomorphism group \( \text{Diff}(W) \) has a classifying space \( B\text{Diff}(W) \). This classifies smooth fibre bundles with fibre \( W \), in the sense that for a smooth manifold \( X \), there is a natural bijection between the set of isomorphism classes of smooth fibre bundles \( E \to X \) with fibre \( W \) and the set \( [X, B\text{Diff}(W)] \) of homotopy classes of maps. The cohomology groups \( H^k(B\text{Diff}(W)) \) therefore give characteristic classes of such bundles, and it is desirable to understand as much as possible about these cohomology groups. The difficulty of this question depends highly on \( W \): it is essentially completely understood when the dimension of \( W \) is 0 or 1, and much effort has been devoted to understanding the case where the dimension of \( W \) is 2. Mumford (\cite{Mum83}) formulated a conjecture about the case where \( W = \Sigma_g \) is an oriented surface of genus \( g \), in the limit \( g \to \infty \). If we let \( \text{Diff}(\Sigma_g, D^2) \) denote the diffeomorphism group which fixes some chosen disc \( D^2 \subset \Sigma_g \), Mumford’s conjecture predicted an isomorphism

\[
\lim_{\leftarrow} H^*(B\text{Diff}(\Sigma_g, D^2); \mathbb{Q}) \cong \mathbb{Q}[\kappa_1, \kappa_2, \kappa_3, \ldots]
\]

for certain classes \( \kappa_i \in H^{2i}(B\text{Diff}(\Sigma_g, D^2)) \). Mumford’s conjecture was finally proved by Madsen and Weiss (\cite{MW07}) in a strengthened form.

The goal of the present paper is to prove analogues of the Madsen–Weiss theorem and Mumford’s conjecture for manifolds of higher dimension. We have results for manifolds of any even dimension greater than 4. As an interesting special case of our results, we completely determine the stable rational cohomology ring

\[
\lim_{\leftarrow} H^*(B\text{Diff}(W_g, D^{2n}); \mathbb{Q}),
\]

Date: December 21, 2013.

2010 Mathematics Subject Classification. 57S05, 57R15, 57R50, 57R65, 57R90, 57R20, 55P47.

S. Galatius was partially supported by NSF grants DMS-0805843 and DMS-1105058 and the Clay Mathematics Institute. Both authors were supported by ERC Advanced Grant No. 228082, and the Danish National Research Foundation through the Centre for Symmetry and Deformation.
where $W_g = \#^g S^n \times S^n$ denotes the connected sum of $g$ copies of $S^n \times S^n$. To state our result, we recall that for each characteristic class of oriented $2n$-dimensional vector bundles $c \in H^{2n+k}(BSO(2n))$, we can define the associated generalised Mumford–Morita–Miller class of a smooth fibre bundle $\pi : E \to B$ with oriented $2n$-dimensional fibres as

$$\kappa_c(E) = \pi_!(c(T^*_E)) \in H^k(B),$$

where $T^*_E$ is the fibrewise tangent bundle of $\pi$. When the fibre is taken to be $W_g$, there is a corresponding universal class $\kappa_c \in H^k(B Diff(W_g, D^{2n}))$ which for $k > 0$ is compatible with increasing $g$.

**Theorem 1.1.** Let $2n > 4$ and let $B \subset H^*(BSO(2n); \mathbb{Q})$ be the set of monomials in the classes $e, p_{n-1}, p_{n-2}, \ldots, p_{\lfloor \frac{n}{2} \rfloor}$ of total degree greater than $2n$. Then the natural map

$$\mathbb{Q}[\kappa_c \mid c \in B] \to \lim_{g \to \infty} H^*(B Diff(W_g, D^{2n}); \mathbb{Q})$$

is an isomorphism.

The strengthened form of Mumford’s conjecture proved by Madsen and Weiss states that a certain map

$$\hocolim_{g \to \infty} B Diff(S_g, D^2) \to \Omega^\infty MTSO(2)$$

induces an isomorphism in integral homology. We will prove a similar homotopy theoretic strengthening of Theorem 1.1, which also applies to more general manifolds.

### 1.1. Definitions and recollections

To state the main results in their general form, we recall the following definitions.

**1.1.1. Classifying spaces.** We shall use the following model for the classifying space $B Diff(W, \partial W)$ of the topological group of diffeomorphisms of a compact manifold $W$, restricting to the identity on a neighbourhood of $\partial W$. We first pick an embedding $\partial W \hookrightarrow \{0\} \times \mathbb{R}^\infty$ and let $Emb^\partial(W, (-\infty, 0] \times \mathbb{R}^\infty)$ denote the space of all extensions to an embedding of $W$ (required to be standard on a collar neighbourhood of $\partial W$). We then let $B Diff(W, \partial W) = Emb^\partial(W, (-\infty, 0] \times \mathbb{R}^\infty)/Diff(W, \partial W)$ be the orbit space. If $W$ is closed and $A \subset W$ is a compact codimension 0 submanifold, we write $B Diff(W, A) = B Diff(W - \text{int}(A), \partial A)$. The construction of $B Diff(W, \partial W)$ has the following naturality property: any inclusion $W \subset W'$ of a codimension 0 submanifold induces a continuous map $B Diff(W, \partial W) \to B Diff(W', \partial W')$, well defined up to homotopy. (On the point-set level it depends on a choice of embedding of the cobordism $W' - \text{int}(W)$ into $[0, 1] \times \mathbb{R}^\infty$.) For example, a choice of inclusion $W_g - \text{int}(D^{2n}) \to W_{g+1}$ induces a map $B Diff(W_g, D^{2n}) \to B Diff(W_{g+1}, D^{2n})$; these define the inverse system in Theorem 1.1

**1.1.2. Thom spectra.** For any space $B$ and any map $\theta : B \to BO(d)$, where $BO(d) = Gr_d(\mathbb{R}^\infty)$, there is a Thom spectrum $MT\theta = B \theta$ constructed in the following way. First, we let $B(\mathbb{R}^n) = \theta^{-1}(Gr_d(\mathbb{R}^n))$. The Grassmannian $Gr_d(\mathbb{R}^n)$ carries a $(n-d)$-dimensional vector bundle $\gamma^\perp_n$, the orthogonal complement of the tautological bundle. Then the $n$th space of the spectrum $MT\theta$ is the Thom space $B(\mathbb{R}^n)^\theta \gamma^\perp_n$. The associated infinite loop space is the direct limit

$$\Omega^\infty MT\theta = \colim_{n \to \infty} \Omega^n\left(B(\mathbb{R}^n)^\theta \gamma^\perp_n\right),$$

and we shall write $\Omega^\infty MT\theta$ for the basepoint component. The rational cohomology of this space is easy to describe; in the case where the bundle classified by $\theta$ is oriented, it is as follows: for each $c \in H^{d+k}(B)$, there is a corresponding “generalised
Mumford–Morita–Miller class \( \kappa_c \in H^k(\Omega^n \wedge MT\theta) \), and \( H^*(\Omega^n \wedge MT\theta; \mathbb{Q}) \) is the free graded-commutative algebra on the classes \( \kappa_c \), where \( c \) runs through a basis for the vector space \( H^{>d}(B; \mathbb{Q}) \). We describe the general case in Section 2.5.

1.3. Moore–Postnikov towers. For any map \( A \to X \) of spaces and any \( n \geq 0 \), there is a factorisation \( A \to B \to X \) with the property that \( \pi_i(A) \to \pi_i(B) \) is surjective for \( i = n \) and bijective for \( i < n \), and \( \pi_i(B) \to \pi_i(X) \) is injective for \( i = n \) and bijective for \( i > n \) (the requirements are imposed for all basepoints).

This is the \( n \)-th stage of the Moore–Postnikov tower for the map \( A \to X \), and can be constructed for example by attaching cells of dimension greater than \( n \) to \( A \). It is well known that a factorisation \( A \to B \to X \) with these properties is unique up to weak homotopy equivalence. In the case where \( A \) is a point, \( B = X(n) \to X \) is the \( n \)-connective cover of the based space \( X \), characterised by the property that \( \pi_i(X(n)) = 0 \) for \( 0 \leq i \leq n \), and that \( \pi_i(X(n)) \to \pi_i(X) \) is an isomorphism for \( i > n \). (Some authors write \( X(n+1) \) for what we denote \( X(n) \).)

1.2. Connected sums of copies of \( S^n \times S^n \). We can now state our homotopy theoretic version of Theorem 1.1 generalising Madsen–Weiss’ theorem to dimension \( 2n \) (recall that we assume \( 2 \choose n \) throughout). As before, we write \( W_g = #^g S^n \times S^n \) for the connected sum of \( g \) copies of \( S^n \times S^n \).

If we pick a disc \( D^{2n} \subset W_g \), there is a classifying space \( \text{BDiff}(W_g, D^{2n}) \) and there are maps \( \text{BDiff}(W_g, D^{2n}) \to \text{BDiff}(W_{g+1}, D^{2n}) \) induced by taking connected sum with one more copy of \( S^n \times S^n \). Let \( g^n : BO(2n)(n) \to BO(2n) \) be the \( n \)-connective cover, and \( MT\theta^n \) the associated Thom spectrum. Let us also say that a continuous map is a \emph{homology equivalence} if it induces an isomorphism in integral homology (and hence in any homology or cohomology theory).

**Theorem 1.2.** Let \( 2n > 4 \). There is a homology equivalence

\[
\text{hocolim}_{g \to \infty} \text{BDiff}(W_g, D^{2n}) \to \Omega^n \wedge MT\theta^n.
\]

More generally, if \( W \) is any \( (n-1) \)-connected closed \( 2n \)-manifold which is parallelisable in the complement of a point, there is a homology equivalence

\[
\text{hocolim}_{g \to \infty} \text{BDiff}(W \# W_g, D^{2n}) \to \Omega^n \wedge MT\theta^n.
\]

It is easy to deduce Theorem 1.1 from Theorem 1.2. In [GRW12], we prove that the maps \( \text{BDiff}(W_g, D^{2n}) \to \text{BDiff}(W_{g+1}, D^{2n}) \) induce isomorphisms in integral homology up to degree \( \lfloor (g - 4)/2 \rfloor \) (cf. also [BM12]). Thus, Theorem 1.2 also determines the homology and cohomology of \( \text{BDiff}(W_g, D^{2n}) \) in this range.

1.3. The moduli space of highly connected null-bordisms. The determination, in Theorems 1.1 and 1.2, of the stable homology and cohomology of the space \( \text{BDiff}(W \# g S^n \times S^n, D^{2n}) \) is a special case of Theorem 1.3 below, in which we determine the stable homology of \( \text{BDiff}(W) \) for more general manifolds \( W \). We also consider manifolds equipped with an additional \emph{tangential structure}, defined as follows.

**Definition 1.3.** Let \( \theta : B \to BO(2n) \) be a map. A \( \theta \)-structure on a \( 2n \)-dimensional manifold \( W \) is a bundle map \( \ell : T\theta \to \theta^*\gamma \), i.e. a fibrewise linear isomorphism. Such a pair \( (W, \ell) \) will be called a \( \theta \)-manifold. A \( \theta \)-structure on a \( (2n-1) \)-dimensional manifold \( M \) is a bundle map \( \varepsilon^1 \otimes TM \to \theta^*\gamma \). If \( \ell \) is a \( \theta \)-structure on \( W \), the induced structure on \( \partial W \) is obtained by composing with a certain isomorphism \( \varepsilon^1 \otimes T\partial W \to T\theta \). In fact, there are two such isomorphisms: One comes from a collar \([0, 1] \times \partial W \to W \) of \( \partial W \). Differentiating this gives an isomorphism \( \varepsilon^1 \otimes T\partial W \to T\theta \), and the resulting \( \theta \)-structure on \( \partial W \) will be called the \emph{incoming} restriction. Another comes from a collar \((-1, 0] \times \partial W \to W \); this is the
outgoing restriction. When $W$ is a cobordism, we will generally use the incoming restriction to induce a $\theta$-structure on the source of $W$ and the outgoing restriction on the target.

Let $\ell_0 : TW|_{\partial W} \to \theta^*\gamma$ be a $\theta$-structure on $\partial W$, and $\text{Bun}^\theta(TW, \theta^*\gamma; \ell_0)$ denote the space of all bundle maps $\ell : TW \to \theta^*\gamma$ which restrict to $\ell_0$ over $\partial W$, equipped with the compact-open topology. The group $\text{Diff}(W, \partial W)$ of diffeomorphisms of $W$ which restrict to the identity near $\partial W$ acts on $\text{Bun}^\theta(TW, \theta^*\gamma; \ell_0)$ by precomposing with a bundle map with the differential of a diffeomorphism.

The most general case of our theorem concerns the moduli space of highly connected null-bordisms, defined as follows.

**Definition 1.4.** Let $P \subset \mathbb{R}^\infty$ be a closed $(2n-1)$-dimensional manifold with $\theta$-structure $\ell_P : \varepsilon^1 \oplus TP \to \theta^*\gamma$. A null-bordism is a pair $(W, \ell_W)$, where $W \subset (-\infty, 0] \times \mathbb{R}^\infty$ is a compact manifold with $\partial W = \{0\} \times P$ and $(-\varepsilon, 0] \times P \subset W$ for some $\varepsilon > 0$, and $\ell_W : TW \to \theta^*\gamma$ is a $\theta$-structure satisfying $\ell_W|_{\partial W} = \ell_P$. A null-bordism $(W, \ell_W)$ is highly connected if $(W, P)$ is $(n-1)$-connected, and the moduli space of highly connected null-bordisms is the set $\mathcal{N}^\theta(P, \ell_P)$ of all highly connected null-bordisms of $(P, \ell_P)$. It is topologised as the disjoint union

$$\prod_W (\text{Emb}^\theta(W, (-\infty, 0] \times \mathbb{R}^\infty) \times \text{Bun}^\theta(TW, \theta^*\gamma; \ell_P))/\text{Diff}(W, \partial W)$$

where the disjoint union is over compact manifolds $W$ with $\partial W = P$ for which $(W, P)$ is $(n-1)$-connected, one of each diffeomorphism class.

If $K \subset [0, 1] \times \mathbb{R}^\infty$ is a cobordism with collared boundary $\partial K = \{(0) \times P_0\} \cup \{(1) \times P_1\}$ we say that $K$ is highly connected if each pair $(K, \{0\} \times P_i)$ is $(n-1)$-connected. If $K$ is equipped with a $\theta$-structure $\ell_K$ restricting to $\ell_0$ and $\ell_1$ on the boundaries, then there is an induced map $\mathcal{N}^\theta(P_0, \ell_0) \to \mathcal{N}^\theta(P_1, \ell_1)$ defined by taking union with $K$ and subtracting $1$ from the first coordinate.

This moduli space classifies smooth families of null-bordisms of $P$, in the sense that if $X$ is a smooth manifold without boundary, there is a natural bijection between the set of homotopy classes $[X, \mathcal{N}^\theta(P, \ell_P)]$, and the set of equivalence classes of triples $(\pi, \varphi, \ell)$, where $\pi : E \to X$ is a proper submersion (i.e. smooth fibre bundle), $\varphi$ is a diffeomorphism $\partial E \cong X \times P$ over $X$, such that $(E, \partial E)$ is $(n-1)$-connected, and $\ell$ is a $\theta$-structure on the fibrewise tangent bundle $T_{\pi}E = \text{Ker}(D\pi)$.

Let us also introduce notation for each of the disjoint summands in (1.1).

**Definition 1.5.** Let $W$ be a compact $2n$-dimensional manifold, and $\ell_0 : TW|_{\partial W} \to \theta^*\gamma$ be a $\theta$-structure on $\partial W$. We shall write

$$\text{BDiff}^\theta(W; \ell_0) = (E\text{Diff}(W, \partial W) \times \text{Bun}^\theta(TW, \theta^*\gamma; \ell_0))/\text{Diff}(W, \partial W)$$

for the homotopy orbit space of the action of $\text{Diff}(W, \partial W)$ on $\text{Bun}^\theta(TW, \theta^*\gamma; \ell_0)$. If $\ell : TW \to \theta^*\gamma$ is a particular extension, we shall write $\text{BDiff}^\theta(W; \ell_0)_{\ell} \subset \text{BDiff}^\theta(W; \ell_0)$ for the path component containing $\ell$.

Using the model $E\text{Diff}(W, \partial W) = \text{Emb}^\theta(W, (-\infty, 0] \times \mathbb{R}^\infty)$, we have the homeomorphism

$$\mathcal{N}^\theta(P, \ell_P) = \prod_W \text{BDiff}^\theta(W; \ell_P).$$

**Definition 1.6.** A tangential structure $\theta : B \to BO(2n)$ is called spherical if any $\theta$-structure on the lower hemisphere $\partial_-D^{2n+1} \subset \partial D^{2n+1}$ extends to some $\theta$-structure on the whole sphere. (If $B$ is path connected, this is equivalent to the sphere $S^{2n}$ admitting a $\theta$-structure.)
Most of the usual structures, for example \( SO, \) Spin, Spin\(^c\), etc. are spherical, but some are not, e.g. framings. Theorem 1.8 below determines the homology of \( N^\theta(P, \ell_P) \) after stabilising with cobordisms in the \((P, \ell_P)\)-variable. The following definition makes the stabilisation procedure precise.

**Definition 1.7.** Let \( \theta : B \to BO(2n) \) be spherical, and \( K \subset [0, \infty) \times \mathbb{R}^\infty \) be a submanifold with \( \theta \)-structure \( \ell_K \), such that \( x_1 : K \to [0, \infty) \) has the natural numbers as regular values. For \( A \subset [0, \infty) \), we let \( (K|_A, \ell_K|_A) \) denote the \( \theta \)-manifold \( K \cap x_1^{-1}(A) \).

1. Let \( W \subset [0, 1] \times \mathbb{R}^\infty \) be a cobordism with \( \theta \)-structure \( \ell_W \), and suppose that \((W_0, \ell_W|_0) = (K|_0, \ell_K|_0) \). We say that \((K, \ell_K)\) absorbs \((W, \ell_W)\) if there exists an embedding \( j : W \to K \) which is the identity on \( W_0 = K|_0 \). Let \( \ell_K \circ Dj : TW \to \theta^\gamma \) be homotopic to \( \ell_W \) relative to \( W_0 \). That \( K|_{[i, \infty)} \) absorbs a \( \theta \)-bordism \( W \subset [i, i+1] \times \mathbb{R}^\infty \) is defined similarly.

2. We say that \((K, \ell_K)\) is a universal \( \theta \)-end if for each integer \( i \geq 0 \), \( K|_{[i, i+1]} \) is a highly connected cobordism and \( \theta \)-structure up to homotopy, relative to \( W \) such that \((W_i, \ell_W|_i) = (K|_i, \ell_K|_i) \).

For example, in dimension 2 with \( \theta = \text{id} : BO(2) \to BO(2) \), we can construct a universal \( \theta \)-end by letting each \( K_{[i, i+1]} \) be diffeomorphic to \( \mathbb{R}P^2 \) with two discs removed. For \( \theta = \theta^n : BO(2n) \to BO(2n) \), a universal \( \theta \)-end can be constructed by letting each \( K_{[i, i+1]} \) be diffeomorphic to \( S^n \times S^n \) with two discs removed. In many other cases, a universal \( \theta \)-end \( K \) can be constructed as the infinite iteration of a single self-bordism \( K|_{[0, 1]} \). In particular, this will be the case in the examples in Section 1.8 below.

As we shall see, universal \( \theta \)-ends are unique up to isomorphism in the following sense. If \((K, \ell_K)\) and \((K', \ell_K')\) are two universal \( \theta \)-ends with \( K|_0 = K'|_0 \), then there exists a diffeomorphism \( K \to K' \) preserving \( \theta \)-structure up to homotopy, relative to \( K|_0 \). More generally, given a highly connected cobordism \((W, \ell_W)\) from \( K|_0 \) to \( K'|_0 \), there exists a similar diffeomorphism from \( K \) to \( W \circ K' \).

**Theorem 1.8.** Let \( 2n > 4 \) and let \( \theta : B \to BO(2n) \) be spherical. Let \((K, \ell_K)\) be a universal \( \theta \)-end with \( N^\theta(K|_0, \ell_K|_0) \neq \emptyset \). Then there is a homology equivalence

\[
\text{hocolim}_{i \to \infty} N^\theta(K|_i, \ell_K|_i) \to \Omega^\infty MT\theta',
\]

where \( \theta' : B' \to B \xrightarrow{\theta} BO(2n) \) is the \( n \)th stage of the Moore–Postnikov tower for \( \ell_K : K \to B \).

The property of being a universal \( \theta \)-end can often be checked in practice, using the following addendum, as it is essentially a homotopical property.

**Addendum 1.9.** Let \( \theta : B \to BO(2n) \) be spherical, let \( K \subset [0, \infty) \times \mathbb{R}^\infty \) be a submanifold such that \( K|_{[i, i+1]} \) is a highly connected cobordism for each integer \( i \), and let \( \ell_K \) be a \( \theta \)-structure on \( K \). Then \((K, \ell_K)\) is a universal \( \theta \)-end if and only if the following conditions hold.

1. For each integer \( i \), the map \( \pi_n(K|_{[i, \infty)}) \to \pi_n(B) \) is surjective, for all basepoints in \( K \).
2. For each integer \( i \), the map \( \pi_{n-1}(K|_{[i, \infty)}) \to \pi_{n-1}(B) \) is injective, for all basepoints in \( K \).
3. For each integer \( i \), each path component of \( K|_{[i, \infty)} \) contains a submanifold diffeomorphic to \( S^n \times S^n \), which in addition has null-homotopic structure map to \( B \).
Remark 1.10. It is often useful to consider the homology equivalence in Theorem 1.8 one path component at a time, so we spell out the resulting statement using the notation of Definition 1.5. Any path component of the infinite loop space $\Omega^\infty MT\theta'$ is homotopy equivalent to the basepoint component $\Omega^\infty MT\theta'$. On the left hand side of the homology equivalence, the path component of an element $(W; \ell_W) \in \mathcal{N}^\theta(K_0, \ell_K|_{0})$ is the homotopy colimit of the spaces

$$BDiff^\theta(W \cup K|_{[0,1]}; \ell_\ell_W \cup \ell_K|_{[0,1]}).$$

Conversely, given a triple $(W, K, \ell)$ where $K \subset [0, \infty) \times \mathbb{R}^\infty$ is a non-compact manifold such that $K_{[i,i+1]}$ is a highly connected cobordism for each integer $i \geq 0$, $W \subset (-\infty, 0] \times \mathbb{R}^\infty$ is a compact manifold with collared boundary $\partial W = K_{[0]}$ such that $(W, \partial W)$ is $(n-1)$-connected, and $\ell : T(W \cup K) \to \theta^* \gamma$ is a bundle map, the Pontryagin–Thom construction described below provides a map

$$(1.2) \quad \text{hocolim} BDiff^\theta(W \cup K|_{[0,1]}; \ell_\ell_W) \to \Omega^\infty MT\theta',$$

where $\theta' : B' \to B \to BO(2n)$ is obtained from the $n$th Moore–Postnikov stage of the underlying map $W \cup K \to B$. By Theorem 1.8, the map $\mathcal{L}$ is a homology isomorphism, provided that $K$ is a universal $\theta'$-end. In particular, Theorem 1.2 can be deduced this way: If we let each $K_{[i,i+1]}$ be diffeomorphic to $S^n \times S^n - \text{int}(B^{2n})$ and let $\theta = \text{id} : BO(2n) \to BO(2n)$, then $\theta' = \theta^n : BO(2n) \to BO(2n)$, and $K$ is a universal $\theta'$-end. Similarly, all examples in Section 1.5 below arise in this way.

Let us also remark that the homotopy colimit $\mathcal{L}$ may be replaced by the strict colimit $BDiff^\theta(W \cup K; \ell)$, defined by

$$BDiff^\theta(S \times W \cup K; \ell) = \left(EDiff_c(S \times W \cup K, \theta^* \gamma; \ell) \right) / Diff_c(W \cup K),$$

where $Diff_c(W \cup K)$ is the topological group of compactly supported diffeomorphisms of the non-compact manifold $W \cup K$, and $Bun_c(T(W \cup K), \theta^* \gamma; \ell)$ is the space of bundle maps which agree outside of a compact subset of $W \cup K$ with $\ell$.

Remark 1.11. The maps in all the theorems above are induced by the Pontryagin–Thom construction. We shall briefly explain this in the setting of Theorem 1.8 after replacing $\mathcal{N}^\theta(P, \ell_P)$ by a weakly equivalent space, and refer the reader to [MT01 §2.3] for further details. First we say that a submanifold $W \subset (-\infty, 0] \times \mathbb{R}^r$ with collared boundary is fatly embedded if the canonical map from the normal bundle $\nu W$ to $\mathbb{R}^r$ restricts to an embedding of the disc bundle into $(-\infty, 0] \times \mathbb{R}^{r-1}$. In that case the Pontryagin–Thom collapse construction gives a continuous map from $(-\infty, 0] \times S^{r-1}$ to the Thom space of $\nu W$. Secondly we replace $\theta' : B' \to BO(2n)$ by a fibration, and redefine $\mathcal{N}^\theta(P, \ell_P)$ as a space of pairs $(W, \ell_W)$ where $W \subset (-\infty, 0] \times \mathbb{R}^r$ is a fatly embedded submanifold, collared near $\partial W = \{0\} \times P$, and $\ell_W : W \to B'$ is a continuous map such that $\theta' \circ \ell_W : W \to BO(2n) = Gr_{2n}(\mathbb{R}^{2n})$ is equal to the Gauss map and whose restriction to $\partial W$ is equal to a specified map $\ell_P : P \to B'$. There is a forgetful map from the space of such pairs to the space in Definition 1.4, and standard homotopy theoretic methods imply that it is a weak equivalence. If $P \subset S^{q-1} \subset \mathbb{R}^\infty$, the Pontryagin–Thom construction (composed with $\ell_P$) gives a point

$$\alpha(P, \ell_P) \in \Omega^{q-1}(B'(\mathbb{R}^j)^{(\theta')^* \gamma}) \subset \Omega^{q-1}MT\theta',$$

and if $(W, \ell_W) \in \mathcal{N}^\theta(P, \ell_P)$ has $W \subset (-\infty, 0] \times \mathbb{R}^{q-1}$, it gives a path

$$\alpha(W, \ell_W) : [-\infty, 0] \to \Omega^{q-1}(B'(\mathbb{R}^j)^{(\theta')^* \gamma}) \subset \Omega^{q-1}MT\theta',$$

starting at the basepoint and ending at $\alpha(P, \ell_P)$. The space of such paths is homotopy equivalent to the based loop space, which is $\Omega^{\infty}MT\theta'$. Finally, the non-compact manifold $K \subset [0, \infty) \times \mathbb{R}^\infty$ admits a homotopically unique $\theta'$-structure
lifting its $\theta$-structure and extending the canonical $\theta'$-structure on $P = K|_0$. The Pontryagin–Thom construction applied to each cobordism $K|_{i,i+1}$ then gives a path $\alpha(K|_{i,i+1}, \ell_K) : [i, i+1] \to \Omega^{\infty-1}MT\theta'$ and the entire process now commutes (strictly) with the stabilisation maps.

1.4. Algebraic localisation. There is one final algebraic version of our main theorem. Fix $P$, a closed $(2n-1)$-manifold with $\theta$-structure $\ell_P : \varepsilon^1 \oplus TP \to \theta^* \gamma$. As explained in Definition 1.4, a cobordism $(K, \ell_K)$ from $(P, \ell_P)$ to itself with $K \subset [0,1] \times \mathbb{R}^\infty$, which is $(n-1)$-connected with respect to both boundaries, gives a self-map of $N^\theta(P, \ell_P)$ defined by $W \mapsto W \cup_P K - e_1$. We shall write $K_0$ for the set of isomorphism classes of such $(K, \ell_K)$, where we identify $(K, \ell_K)$ with $(K', \ell_{K'})$ if there is a diffeomorphism $\varphi : K \to K'$ which is the identity near $\partial K$ such that $\varphi^* \ell_K$ is homotopic to $\ell_K$ relative to $\partial K$. It is clear that the homotopy class of the self-map of $N^\theta(P, \ell_P)$ induced by $(K, \ell_K)$ depends only on the isomorphism class of $(K, \ell_K)$, and we get an action of the non-commutative monoid $K_0$ on $H_*(N^\theta(P, \ell_P))$. Our theorem determines the algebraic localisation

$$H_*(N^\theta(P, \ell_P))[K^{-1}]$$

at a certain commutative submonoid $K \subset K_0$ which we now describe.

We say that a $\theta$-cobordism $K : P \leadsto P$ has support in a closed subset $A \subset P$ if it contains $[0,1] \times (P - A) : (P - A) \leadsto (P - A) \subset A$ as a sub-cobordism with the product $\theta$-structure. We let $K \subset K_0$ consist of those elements which admit a representative with support in a regular neighbourhood of a simplicial complex of dimension at most $n - 1$ inside $P$, and prove the following lemma.

**Lemma 1.12.** The subset $K \subset K_0$ is a commutative submonoid.

We may localise the $\mathbb{Z}[K]$-module $H_*(N^\theta(P, \ell_P))$ at any submonoid $L \subset K$. The content of Theorem 1.13 below is an isomorphism

$$H_*(N^\theta(P, \ell_P))[L^{-1}] \cong H_*(\Omega^{\infty}MT\theta')$$

under certain conditions, where $\theta' : B' \to B \leadsto BO(2n)$ is the $(n-1)$st stage of the Moore–Postnikov tower for $\ell_P : P \to B$. To describe the isomorphism explicitly, recall that in Remark 1.13 we described a map

$$N^\theta(P, \ell_P) \to \Omega^{\infty}MT\theta',$$

compatible with gluing highly connected cobordisms of $(P, \ell_P)$ equipped with $\theta'$-structures, and hence the induced map

$$H_*(N^\theta(P, \ell_P)) \to H_*(\Omega^{\infty}MT\theta')$$

is a map of $\mathbb{Z}[K']$-modules, where the monoid $K'$ is defined like $K$ but using $\theta'$ instead of $\theta$. An obstruction theoretic argument, which we explain in more detail in Section 7.6, shows that the natural map $K' \to K$ is a bijection, so (1.3) is naturally a homomorphism of $\mathbb{Z}[K]$-modules.

**Theorem 1.13.** Let $2n > 4$ and let $\theta : B \to BO(2n)$ be spherical. Let $P$ be a closed $(2n-1)$-manifold with $\theta$-structure $\ell_P : \varepsilon^1 \oplus TP \to \theta^* \gamma$, such that $N^\theta(P, \ell_P)$ is non-empty. Then the morphism (1.3) induces an isomorphism

$$H_*(N^\theta(P, \ell_P))[K^{-1}] \to H_*(\Omega^{\infty}MT\theta').$$

Furthermore, localisation at a submonoid $L \subset K$ agrees with localisation at $K$, provided $L$ satisfies the following conditions.

(i) The group $\pi_n(B)$ is generated by the subgroups $\Im(\pi_n(K) \to \pi_n(B))$, $K \in L$.

(ii) The subgroup of $\pi_n-1(P)$ generated by $\Ker(\pi_{n-1}(P) \to \pi_{n-1}(K))$, $K \in L$, contains $\Ker(\pi_{n-1}(P) \to \pi_{n-1}(B))$. 


(iii) There is an element of $\mathcal{L}$ containing a submanifold diffeomorphic to $S^n \times S^n - \text{int}(D^{2n})$.

(There is a bijection $\pi_0(P) = \pi_0(K)$, and if $P$ is not connected, conditions $[2]$, $[3]$ and $[6]$ are required to hold for each path component of $P$.)

Applying the functor $\text{Hom}_{\mathbb{Z}[K]}(-, \mathbb{Q})$ to both sides of the isomorphism in the theorem identifies the subring of $H^*(\mathcal{A}^0(P, \ell_P); \mathbb{Q})$ consisting of $K$-invariants with $H^*(\Omega^\infty MT\theta'; \mathbb{Q})$. Observing that these classes are also invariant under the larger monoid $K_0$, we deduce the isomorphism

$$H^*(\mathcal{A}^0(P, \ell_P); \mathbb{Q})_K \cong H^*(\Omega^\infty MT\theta'; \mathbb{Q}).$$

The left hand side can be interpreted as characteristic classes of certain bundles, invariant under fibre-wise gluing of trivial bundles.

1.5. Examples and applications. Recall that the connective cover $BO(d)(k)$ is $BSO(d)$ if $k = 1$, $BSpin(d)$ if $k = 2$ or $3$, and is often called $BString(d)$ if $k = 4, 5, 6$ or $7$. We write $MTSO(d)$, $MTSpin(d)$ and $MTString(d)$ for the corresponding Thom spectra. As special cases of Theorem $[1.3]$ we have the following maps, which become homology equivalences in the limit $g \to \infty$. All are deduced from Theorem $[1.3]$ as in Remark $[1.10]$ with $\theta = \text{id}: BO(2n) \to BO(2n)$.

$$\text{BDiff}(gS^3 \times S^3, D^6) \to \Omega^\infty MTSpin(6)$$

$$\text{BDiff}(g(HP^2 \# TP^2, D^8) \to \Omega^\infty MTSpin(8)$$

$$\text{BDiff}(gS^4 \times S^4, D^8) \to \Omega^\infty MTSpin(8)$$

$$\text{BDiff}(gS^5 \times S^5, D^{10}) \to \Omega^\infty MTSpin(10)$$

$$\text{BDiff}(gS^6 \times S^6, D^{12}) \to \Omega^\infty MTSpin(12)$$

$$\text{BDiff}(gS^7 \times S^7, D^{14}) \to \Omega^\infty MTSpin(14)$$

$$\text{BDiff}(g(\bigcup P^2 \# \bigcup P^2, D^{16}) \to \Omega^\infty MTSpin(16)$$

A slightly different type of example is given by $\text{BDiff}(CP^3 \# gS^3 \times S^3, U)$, where $U \subset CP^3$ is a tubular neighbourhood of $CP^1$. In this case the stable homology is that of $\Omega^\infty MTSpin^*(6)$, where $BSpin^*(6)$ is the homotopy fibre of the map $\beta_{g^2} : BSO(6) \to K(\mathbb{Z}, 3)$.

An example where we need a more complicated stabilisation (not induced by connected sum) comes from $\mathbb{R}P^6$. The map $\mathbb{R}P^6 \to BO(6)$ lifts canonically to a 3-connected map $\mathbb{R}P^6 \to BPin^-(6)$, where $\theta : BPin^-(6) \to BO(6)$ is the homotopy fibre of $w_1 + w_2^2 : BO(6) \to K(\mathbb{Z}/2, 2)$. The standard self-indexing Morse function $f : \mathbb{R}P^6 \to [0, 6]$ given by

$$f(x_0; \cdots ; x_6) = \sum_{i=0}^6 i \cdot x_i^2$$

has one critical point of each index, and we let $W = f^{-1}([0, 2.5]) \simeq \mathbb{R}P^2 \times D^4$.

Cutting out a parallel copy of $W$ gives a $\theta$-bordism $\tilde{K} \simeq f^{-1}([2.5, 3.5])$ from $\partial W = \mathbb{R}P^2 \times S^3$ to $-\partial W$ (i.e. $\mathbb{R}P^2 \times S^3$ equipped with the opposite $\theta$-structure). Hence $K_0 = \tilde{K} \circ (-\tilde{K})$ is a cobordism from $\partial W$ to itself, and we let $K$ be the infinite iteration. In this situation we get a stable homology equivalence

$$\text{BDiff}(\mathbb{R}P^2 \times D^4 \cup_0 gK_0, \partial) \to \Omega^\infty MTSpin^-(6).$$

Another interesting special case concerning the manifolds $W_g = \#^g S^n \times S^n$ is the following. Let $(Y, y)$ be a pointed space, and consider the homotopy orbit space

$$S^n(Y, y) = (E\text{Diff}(W_g, D^{2n}) \times \text{Map}(W_g, D^{2n}), (Y, y))/\text{Diff}(W_g, D^{2n}).$$
We can determine the stable homology of these spaces using a Pontryagin–Thom map
\[
\prod_{g \geq 0} S^n_g(Y, y) \longrightarrow \Omega^\infty(Y(n - 1)_+ \wedge MT^n),
\]
defined as in Remark 1.11. Any map \(f : (S^n, D^n) \to (Y, y)\) may be composed with the projection \(S^n \times S^n \to S^n\) to give a map \((W_1, D^n) \to (Y, y)\), which induces a map \(S^n_g(Y, y) \to S^n_{g+1}(Y, y)\). Thus each such \(f\) gives a self-map of the left hand side of (1.4) and a compatible self-map of the right hand side which is a weak equivalence. Up to homotopy, the self-maps depend only on \([f] \in \pi_n(Y, y)\) and different elements of \(\pi_n(Y, y)\) give homotopy commuting self-maps. Therefore, (1.4) induces a map from the stabilised homology
\[
\left(\bigoplus_{g \geq 0} H_*(S^n_g(Y, y))\right)[\pi_n(Y, y)^{-1}] \longrightarrow H_*(\Omega^\infty(Y(n - 1)_+ \wedge MT^n)).
\]

Applying Theorem 1.13 to the projection \(\theta : BO(2n) \times Y \to BO(2n)\) implies that (1.5) becomes an isomorphism, after restricting to appropriate path components. This result is a generalisation of the result of Cohen and Madsen [GM09], who proved the special case where \(2n = 2\) and \(Y\) is simply connected. (The case \(2n = 2\) was generalised to non-simply connected \(Y\) in [GRW10].)

As a final application, in [GRW12a] we deduce a generalisation of the detection result of Ebert ([Ebe11]). We will prove that for any abelian group \(k\) and any non-zero cohomology class \(c \in H^*(\Omega^n_{\ast}MTSO(2n); k)\), there exists a bundle \(p : E \to B\) of smooth oriented manifolds, such that the characteristic class associated to \(c\) is non-vanishing in \(H^*(B; k)\). (The case \(k = \mathbb{Q}\) was proved by Ebert.)

1.6. Cobordism categories and outline of proof. Finally, let us say a few words about our method of proof, which follows the strategy in [GRW10] and [GMTW09]. A central object is the cobordism category \(\mathcal{C}_0(\mathbb{R}^N)\), whose objects are closed \((d - 1)\)-dimensional manifolds \(M \subset \mathbb{R}^N\) and whose morphisms are \(d\)-dimensional cobordisms \(W \subset [0, \ell] \times \mathbb{R}^N\), both equipped with \(\theta\)-structures.

Remark 1.14. The applications described above use only the case where \(d\) is even. Our results about cobordism categories are valid for odd \(d\) as well, but we do not know an interpretation in terms of stable homology in that case. In fact, Ebert ([Ebe09]) has shown that there are non-trivial classes in \(H^*(\Omega^n_{\ast}MTSO(2n + 1); \mathbb{Q})\) which are trivial when restricted to any \(BDiff^+(M^{2n+1})\). Thus there can be no analogue of e.g. Theorem 1.13 expressing \(H_*(\Omega^n_{\ast}MTSO(2n + 1))\) as a direct limit of \(H_*(BDiff(W \cup K_{[0,n]}; K))\)’s. It is an interesting question to find an odd-dimensional analogue of our results.

In the limit \(N \to \infty\), the main result of [GMTW09] gives a weak equivalence
\[
\Omega\mathcal{B}_\theta \simeq \Omega^\infty MT^\theta.
\]
As in [GRW10], our strategy will be to find subcategories \(\mathcal{C} \subset \mathcal{C}_\theta\), as small as possible, such that the inclusion induces a weak equivalence \(\Omega\mathcal{B} \to \Omega\mathcal{B}_\theta\). The proof of Theorem 1.15 will consist of applying a version of the “group completion” theorem to a very small subcategory of \(\mathcal{C}_\theta\).

Let \(P\) be a \((2n - 1)\)-dimensional manifold with \(\theta\)-structure \(\ell_P : c^1 \oplus TP \to \theta^*\gamma\), and suppose the underlying map \(P \to B\) is \((n - 1)\)-connected. We pick a self-indexing Morse function \(f : P \to [0, 2n - 1]\) and set \(L = f^{-1}([0, n - \frac{1}{2}])\). The restriction \(L \to B\) is then still \((n - 1)\)-connected. Then we pick a (collared) embedding \(L \to (-\infty, 0] \times \mathbb{R}^\infty\), and consider the subcategory \(\mathcal{C}_{\theta, L} \subset \mathcal{C}_\theta\) where objects \(M \subset \mathbb{R} \times \mathbb{R}^\infty\) satisfy \(M \cap ((-\infty, 0] \times \mathbb{R}^\infty) = L\) and morphisms \(W \subset \mathbb{R}\).
$[0, t] \times \mathbb{R} \times \mathbb{R}^\infty$ satisfy $W \cap ([0, t] \times (-\infty, 0] \times \mathbb{R}^\infty) = [0, t] \times L$. For both objects and morphisms, these identities are required to hold as $\theta$-manifolds. (For later purposes, we note that the category only depends on $\partial L$: If $\partial L_1 = \partial L_2$, then there is an isomorphism of categories which cuts out $\text{int}(L_1)$ and replaces it with $L_2$. In fact, it is convenient to mentally cut out $\text{int}(L)$ and think of objects as manifolds with boundary, and morphisms as manifolds with corners.) In Section 2 we prove that the inclusion map induces a weak equivalence

$$BC_{\theta,L} \longrightarrow BC_{\theta}.$$  

Secondly, we filter $C_{\theta,L}$ by connectivity of morphisms: for $\kappa \geq -1$, the subcategory $C_{\theta,L}^\kappa$ has the same objects, but a morphism $W$ from $M_0$ to $M_1$ is required to satisfy that the inclusion $M_1 \to W$ is $\kappa$-connected, i.e., that any map $(D^n, \partial D^n) \to (W, M_1)$ is homotopic to one with image in $M_1$, for $i \leq \kappa$. In Section 3 we prove that the inclusion map induces a weak equivalence

$$BC_{\theta,L}^\kappa \longrightarrow BC_{\theta,L},$$

as long as $\kappa \leq (d-2)/2$. (In the case where $\kappa = 0$, this is the “positive boundary subcategory”, and this case was proved in GMTW09.)

Thirdly, we filter $C_{\theta,L}^\kappa$ by connectivity of objects: for $l \geq -1$, the subcategory $C_{\theta,L}^{\kappa,l} \subset C_{\theta,L}^\kappa$ is the full subcategory on those objects where the structure map $M \to B$ induces an injection $\pi_i(M) \to \pi_i(B)$ for all $i \leq l$ and all basepoints, or equivalently the inclusion $L \to M$ is $l$-connected. In Section 4 we prove that the inclusion map induces a weak equivalence

$$BC_{\theta,L}^{\kappa,l} \longrightarrow BC_{\theta,L}^\kappa,$$

provided $l \leq (d-3)/2$ and $l \leq \kappa$. (In the case where $l = 0$ and $B$ is connected, this is the full subcategory on objects which are path connected, and this case was proved in GRW10.)

Fourthly, we focus on the case where $d = 2n > 4$, where we have now reduced to $C_{\theta,L}^{n-1,n-2}$, the full subcategory on those objects for which the inclusion $L \to M$ is $(n-2)$-connected. In the final step we let $C$ denote the full subcategory on those objects $M$ which can be obtained from $L$ by attaching handles of index at least $n$. (This is equivalent to the condition that $M - \text{int}(L)$ is diffeomorphic to a handlebody with handles of index at most $(n-1)$, which if $n > 3$ is in turn equivalent to the inclusion $L \to M$ being $(n-1)$-connected.) In Section 5 we prove that the inclusion map induces a weak equivalence

$$\Omega BC \longrightarrow \Omega BC_{\theta,L}^{n-1,n-2},$$

provided that $\theta$ is spherical.

In the setup and notation of Theorem 1.8 let us suppose for simplicity that the map $\ell_K : K \to B$ is $n$-connected (so that $B' = B$), and apply the above discussion with $P = K_{[0]}$. This gives a $\theta$-manifold $L$, and we will show how to construct a canonical “double” $\theta$-manifold $D(L)$ having the following special property: for any object $P \in C$ there is a homotopy equivalence

$$C(D(L), P) \cong N^\theta(P, \ell_P).$$

The weak equivalences (1.6), (1.7), (1.8), (1.9) and (1.10) establish the homotopy equivalence

$$\Omega BC \cong \Omega^\infty \text{MT}\theta,$$

and the proof of Theorem 1.8 in this case will be completed by applying a suitable version of the “group completion” theorem to the canonical map $C(D(L), P) \to \Omega BC$. 

The weak equivalences \([\ref{L8}], \ref{L9}\) and \([\ref{L10}]\) are established using a parametrised surgery procedure, and the proof depends on the contractibility of certain spaces of surgery data. Contractibility is proved in a similar way in all three cases, and we defer this to Section \(\ref{L3}\). Finally, in Section \(\ref{L4}\) we explain how to use a version of the group completion theorem to prove Theorem \(\ref{L5}\) and tie things together.

Sections \(\ref{L5}\)–\(\ref{L6}\) contain the main technical steps, but on a first reading it is possible to skip to Section \(\ref{L7}\) after reading Section \(\ref{L2}\) to see the overall structure of the argument. The reader mainly interested in Theorems \(\ref{L1}\) and \(\ref{L2}\) can take to Section 7.1 after reading Section 2, to see the overall structure of the paper. Considering only this special case would not significantly simplify the main technical steps in Sections \(\ref{L5}\)–\(\ref{L6}\) but the group completion arguments in Section \(\ref{L7}\) do simplify, and we incorporate a separate discussion of this case in Section \(\ref{L7.1}\).

2. Definitions and recollections

2.1. Tangential structures. Throughout this paper, an important role will be played by the notion of a tangential structure on manifolds. This will be important even for the proof of theorems which do not explicitly mention tangential structures on manifolds. However, for the proofs of Theorems \(\ref{L1}\) and \(\ref{L2}\) the structure \(\theta = \theta^n : BO(2n)(n) \to BO(2n)\) suffices.

**Definition 2.1.** A tangential structure is a map \(\theta : B \to BO(d)\). A \(\theta\)-structure on a \(d\)-manifold \(W\) is a bundle map (i.e. fibrewise linear isomorphism) \(\ell : TW \to \theta^*\gamma\). A \(\theta\)-manifold is a pair \((W, \ell)\). More generally, a \(\theta\)-structure on a \(k\)-manifold \(M\) (with \(k \leq d\)) is a bundle map \(\ell : \mathbb{R}^d - k \oplus TM \to \theta^*\gamma\).

Given vector bundles \(U\) and \(V\) of the same dimension, but not necessarily over the same space, we write \(\text{Bun}(U, V)\) for the subspace of map\((U, V)\) (with the compact-open topology) consisting of the bundle maps. Thus, \(\text{Bun}(TW, \theta^*\gamma)\) is the space of \(\theta\)-structures on \(W\).

2.2. Spaces of manifolds. We recall the definition and main properties of spaces of submanifolds, from \([GRW10]\). Fix a tangential structure \(\theta : B \to BO(d)\).

**Definition 2.2.** For an open subset \(U \subset \mathbb{R}^n\), we denote by \(\Psi_{\theta}(U)\) the set of pairs \((M^d, \ell)\) where \(M^d \subset U\) is a smooth \(d\)-dimensional submanifold that is closed as a topological subspace, and \(\ell\) is a \(\theta\)-structure on \(M\).

We denote by \(\Psi_{\theta_{d_{\geq 0}}}(U)\) the set of pairs \((M, \ell)\) where \(M \subset U\) is a smooth \((d - m)\)-dimensional submanifold that is closed as a topological subspace, and \(\ell\) is a \(\theta\)-structure on \(M\), i.e. a bundle map \(\mathbb{R}^m - k \oplus TM \to \theta^*\gamma\).

In \([GRW10]\) \(\S\)2 we have defined a topology on these sets so that \(U \mapsto \Psi_{\theta_{d_{\geq 0}}}(U)\) defines a continuous sheaf of topological spaces on the site of open subsets of \(\mathbb{R}^n\). We will not give full details of the topology again here, but remind the reader that the topology is “compact-open” in flavour: disregarding tangential structures, points nearby to \(M\) are those which near some large compact subset \(K \subset U\) look like small normal deformations of \(M\). In particular, a typical neighbourhood of the empty manifold \(\emptyset \in \Psi_{\theta}(U)\) consists of all those manifolds in \(U\) disjoint from some compact \(K\).

**Definition 2.3.** We define \(\psi_{\theta}(n, k) \subset \Psi_{\theta}(\mathbb{R}^n)\) to be the subspace consisting of those \(\theta\)-manifolds \((M, \ell)\) such that \(M \subset \mathbb{R}^k \times (-1, 1)^{n-k}\). We make the analogous definition of \(\psi_{\theta_{d_{\geq 0}}}(n, k)\).
2.3. Semi-simplicial spaces and non-unital categories. Let $\Delta$ denote the category of finite non-empty totally ordered sets and monotone maps, the simplicial indexing category. Let $\Delta_{\text{inj}} \subset \Delta$ denote the subcategory with the same objects but only injective monotone maps as morphisms. For a category $\mathcal{C}$, a simplicial object in $\mathcal{C}$ is a contravariant functor $X : \Delta \to \mathcal{C}$, and a semi-simplicial object in $\mathcal{C}$ is a contravariant functor $X : \Delta_{\text{inj}} \to \mathcal{C}$. A map of (semi-)simplicial objects is a natural transformation of functors.

We call a semi-simplicial object in the category of topological spaces a semi-simplicial space. More concretely, it consists of a space $X_n = X(0 < 1 < \cdots < n)$ for each $n \geq 0$ and face maps $d_i : X_n \to X_{n-1}$ defined for $i = 0, \ldots, n$ satisfying the simplicial identities $d_i d_j = d_{j-1} d_i$ for $i < j$. We often denote a semi-simplicial space by $X_\bullet$, where we treat $\bullet$ as a place-holder for the simplicial degree.

The geometric realisation of a semi-simplicial space $X_\bullet$ is defined to be

$$|X_\bullet| = \coprod_{n \geq 0} X_n \times \Delta^n / \sim$$

where $\Delta^n$ denotes the standard topological $n$-simplex and the equivalence relation is generated by $(d_i(x), y) \sim (x, d_i'(y))$ where $d_i' : \Delta^n \to \Delta^{n+1}$ the inclusion of the $i$th face. This space is given the quotient topology.

The $k$-skeleton of $|X_\bullet|$ is

$$|X_\bullet|^{(k)} = \coprod_{n=0}^k X_n \times \Delta^n / \sim$$

with the quotient topology, and one easily checks that $|X_\bullet| = \bigcup_{k \geq 0} |X_\bullet|^{(k)}$ with the direct limit topology. A useful consequence of this is the following: a map from a compact space to $|X_\bullet|$ lands in a finite skeleton. We recall the following result.

**Lemma 2.4.** If $X_\bullet \to Y_\bullet$ is a map of semi-simplicial spaces such that each $X_n \to Y_n$ is a weak homotopy equivalence, then $|X_\bullet| \to |Y_\bullet|$ is too.

**Remark 2.5.** The term semi-simplicial object we have defined above is not quite standard (though is gaining popularity) and deserves some justification. Our justification is that it agrees with Eilenberg and Zilber’s original usage of “semi-simplicial complex” [EZ50]. Another is that the alternative used in the literature is $\Delta$-space, but as $\Delta$ is the indexing category for full simplicial objects this seems counterintuitive.

A non-unital topological category $\mathcal{C}$ consists of a pair of spaces $(\mathcal{O}, \mathcal{M})$ of objects and morphisms, equipped with source and target maps $s, t : \mathcal{M} \to \mathcal{O}$. We let $\mathcal{M} \times_{t \mathcal{O}s} \mathcal{M}$ denote the fibre product made with the maps $t$ and $s$, and require in addition a composition map $\mu : \mathcal{M} \times_{t \mathcal{O}s} \mathcal{M} \to \mathcal{M}$ which satisfies the evident associativity requirement.

A non-unital topological category $\mathcal{C}$ has a semi-simplicial nerve, generalising the simplicial nerve of a topological category [Seg68]. Define $N_\bullet \mathcal{C}$ by $N_0 \mathcal{C} = \mathcal{O}$ and

$$N_k \mathcal{C} = \mathcal{M} \times_{t \mathcal{O}s} \mathcal{M} \times_{t \mathcal{O}s} \cdots \times_{t \mathcal{O}s} \mathcal{M} \quad k > 0$$

being the space of $k$-tuples of composable morphisms, and let the face maps be given by composing and forgetting morphisms, as in the simplicial nerve of a topological category. We define the classifying space of a non-unital topological category by

$$B \mathcal{C} = |N_\bullet \mathcal{C}|.$$
2.4. Definition of the cobordism categories. For convenience in the rest of the paper, we introduce the following notation. All of our constructions will take place inside $\mathbb{R} \times \mathbb{R}^N$, and we write $x_1 : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ for the projection to the first coordinate. Given a manifold $W \subset \mathbb{R} \times \mathbb{R}^N$ and a set $A \subset \mathbb{R}$, we write

$$W|_A = W \cap x_1^{-1}(A),$$

and we also write $\ell|_A$ for the restriction of a $\theta$-structure $\ell$ on $W$ to this manifold.

Our definition of the cobordism category of $\theta$-manifolds is similar to that of [GMTW09] (the only difference is that here we will only define a non-unital category); it follows that of [GMTW09] in spirit, but is different in some technical points. We use the spaces of manifolds of the last section in order to describe the point-set topology of these categories.

**Definition 2.6.** For each $\varepsilon > 0$ we let the non-unital topological category $C_\theta(\mathbb{R}^N)_\varepsilon$ have space of objects $\psi_{\theta_{\mathbb{R}^N}}(N, 0)$. The space of morphisms from $(M_0, \ell_0)$ to $(M_1, \ell_1)$ is the subspace of those $(W, \ell)$ in $\mathbb{R} \times \psi(\mathbb{R}^N + 1, 1)$ such that $t > 0$ and

$$W|_{(-\infty, \varepsilon)} = (\mathbb{R} \times M_0)|_{(-\infty, \varepsilon)} \in \Psi_\theta((-\infty, \varepsilon) \times \mathbb{R}^N)$$

and

$$W|_{(\varepsilon, \infty)} = (\mathbb{R} \times M_1)|_{(\varepsilon, \infty)} \in \Psi_\theta((\varepsilon, \infty) \times \mathbb{R}^N).$$

Here $\mathbb{R} \times M_i$ denotes the $\theta$-manifold with underlying manifold $\mathbb{R} \times M_i \subset \mathbb{R} \times \mathbb{R}^N$ and $\theta$-structure

$$T(\mathbb{R} \times M_i) \to \varepsilon^1 \oplus TM_i \xrightarrow{\ell_i} \theta^*\gamma.$$ 

Composition in this category is defined by

$$(t, W) \circ (t', W') = (t + t', W|_{(-\infty, t]} \cup (W' + t \cdot e_1)|_{[t, \infty)})$$

where $W' + t \cdot e_1$ denotes the manifold $W'$ translated by $t$ in the first coordinate.

We topologise the total space of morphisms as a subspace of $(0, \infty) \times \psi(\mathbb{R}^N + 1, 1)$.

If $\varepsilon < \varepsilon'$ there is an inclusion $C_\theta(\mathbb{R}^N)_{\varepsilon'} \subset C_\theta(\mathbb{R}^N)_{\varepsilon}$, and we define $C_\theta(\mathbb{R}^N)$ to be the colimit over all $\varepsilon > 0$.

Note that a morphism $(t, (W, \ell))$ in this category is uniquely determined by the restriction $(t, (W|_{[0,t]}, \ell|_{[0,t]}))$. We often think of morphisms in this category as being given by such restricted manifolds, but the topology on the space of morphisms is best described as we did above.

As explained in the introduction, we will also require a version of this category where the objects and morphisms contain a fixed codimension zero submanifold. In order to define this, we let

$$L \subset (-1/2, 0] \times (-1, 1)^{N-1}$$

be a compact $(d - 1)$-manifold which near $\{0\} \times \mathbb{R}^{N-1}$ agrees with $(-1, 0] \times \partial L$. Furthermore, we let $\ell|_L : \varepsilon^1 \oplus TL \to \theta^*\gamma$ be a $\theta$-structure on $L$. Near $\partial L$ we require that the structure is a product (i.e. that translation in the collar direction preserves the structure). Such an $\ell$ makes $\mathbb{R} \times L$ into a $\theta$-manifold with boundary, and we make the following definition.

**Definition 2.7.** The topological subcategory $C_{\theta,L}(\mathbb{R}^N) \subset C_\theta(\mathbb{R}^N)$ has space of objects those $(M, \ell)$ such that

$$M \cap ((-\infty, 0] \times \mathbb{R}^{N-1}) = L$$

as $\theta$-manifolds. It has space of morphisms from $(M_0, \ell_0)$ to $(M_1, \ell_1)$ given by those $(t, (W, \ell))$ such that

$$W \cap (\mathbb{R} \times ((-\infty, 0] \times \mathbb{R}^{N-1}) = \mathbb{R} \times L$$

as $\theta$-manifolds.
Remark 2.8. The category $C_{\theta,L}(\mathbb{R}^N)$ does not really depend on $L$, but only on $\partial L$. It is sometimes convenient to think of the interior of $L$ as being cut out, so that objects in the category are manifolds with boundary $\partial L$ and morphisms are cobordisms between manifolds with boundary which are trivial along the boundary.

If we take $L = D^{d-1}$ then the category $C_{\theta,L}(\mathbb{R}^N)$ is equivalent to the category of “manifolds with basepoint” defined in [GRW10, Definition 4.2]. That case is sufficient for the proofs of Theorems 1.1 and 1.2.

The subject of our main technical theorem, from which we shall show how to obtain results on diffeomorphism groups in Section 7, is certain subcategories of $\text{manifolds with basepoint}$ defined in [GRW10, Definition 4.2]. That case is certain subcategories of “manifolds with basepoint” defined in [GRW10, Definition 4.2]. That case is sufficient for the proofs of Theorems 1.1 and 1.2.

Definition 2.9. The topological subcategory $C_{\theta,L}^\infty(\mathbb{R}^N) \subset C_{\theta,L}(\mathbb{R}^N)$ has the same space of objects. It has space of morphisms from $(M_0,\ell_0)$ to $(M_1,\ell_1)$ given by those $(t,(W,\ell))$ such that the pair $(W|_{[0,1]},W|_{t})$ is $\kappa$-connected, i.e. such that $\pi_i(W|_{[0,1]},W|_{t}) = 0$ for all basepoints and all $i \leq \kappa$. Thus this is the subcategory on those morphisms which are $\kappa$-connected relative to their outgoing boundary.

The category $C^0_{\theta}$ is the “positive boundary category” as in [GMTW09], where each path component of a cobordism is required to have non-empty outgoing boundary.

Definition 2.10. The topological subcategory $C_{\theta,L}^{\infty,l}(\mathbb{R}^N) \subset C_{\theta,L}(\mathbb{R}^N)$ is the full subcategory on those objects $(M,\ell)$ such that the map $\ell_* : \pi_i(M) \to \pi_i(B)$ is injective for all $i \leq l$ and all basepoints. (In our main application in Section 7 the map $L \to B$ will be $(l+1)$-connected. In that case the requirement is equivalent to $(M,L)$ being $l$-connected.)

For our final definition we specialise to even dimensions.

Definition 2.11. Let $d = 2n$ and let

$$A \subset \pi_0\left(\text{Ob}(C_{\theta,L}^{n-1,n-2}(\mathbb{R}^N))\right)$$

be a collection of path components of the space of objects. The topological subcategory $C_{\theta,L}^{n-1,A}(\mathbb{R}^N) \subset C_{\theta,L}(\mathbb{R}^N)$ is the full subcategory on the subspace of those objects in $A$.

For $N = \infty$, we shall often denote $C_{\theta}(\mathbb{R}^\infty) = \text{colim}_N C_{\theta}(\mathbb{R}^N)$ by $C_{\theta}$, and similarly with any decorations.

2.5. The homotopy type of the cobordism category. The main theorem of [GMTW09] identifies the homotopy type $\Omega BC_{\theta}$ in terms of the infinite loop space of a certain Thom spectrum $MT\theta$.

Recall from the introduction that given a map $\theta : B \to BO(d) = \text{Gr}_d(\mathbb{R}^\infty)$ we let $B(\mathbb{R}^n) = \theta^{-1}(\text{Gr}_d(\mathbb{R}^n))$ and define $\gamma^\perp_n \to \text{Gr}_d(\mathbb{R}^n)$ to be the orthogonal complement of the tautological bundle. The canonical map $B(\mathbb{R}^n) \to B(\mathbb{R}^{n+1})$ pulls back $\theta^*\gamma^\perp_n$ to $\theta^*\gamma^\perp_n \oplus \varepsilon^1$ and hence we obtain pointed maps

$$\left(B(\mathbb{R}^n)\theta^*\gamma^\perp_n\right) \wedge S^1 \to B(\mathbb{R}^{n+1})\theta^*\gamma^\perp_{n+1}$$

of Thom spaces, which form a spectrum $MT\theta$. Its associated infinite loop space is

$$\Omega^\infty MT\theta = \text{colim}_{n \to \infty} \Omega^n \left(B(\mathbb{R}^n)\theta^*\gamma^\perp_n\right).$$
Theorem 2.12 (Galatius–Madsen–Tillmann–Weiss [GMTW09]). There is a canonical map

$$\Omega BC \theta \longrightarrow \Omega^\infty MT \theta$$

which is a weak homotopy equivalence.

We write $\Omega^\infty MT \theta$ for the basepoint component of $\Omega^\infty MT \theta$, and now describe the rational cohomology of this space. The map $B \overset{\theta}{\longrightarrow} BO(d) \overset{det}{\longrightarrow} BO(1)$ on fundamental groups defines a character $u_1 : \pi_1(B) \rightarrow \mathbb{Z}^\times$, and we write $H^*(B; \mathbb{Q}u_1)$ for the rational cohomology of $B$ with local coefficients given by this character. For each $n$ there are evaluation maps

$$ev : \Sigma^n \Omega^n \left( B(\mathbb{R}^n)^{\theta^* \gamma_n} \right) \longrightarrow B(\mathbb{R}^n)^{\theta^* \gamma_n}$$

and so we can define the dotted map in the diagram

$$H^{*+d}(B(\mathbb{R}^n); \mathbb{Q}u_1) \overset{Thom \ iso.}{\longrightarrow} H^*(\Omega^n(B(\mathbb{R}^n)^{\theta^* \gamma_n}); \mathbb{Q})$$

by commutativity. Taking limits and restricting to the basepoint component, we obtain a map

$$\sigma : H^{*+d}(B; \mathbb{Q}u_1) \longrightarrow H^*(\Omega^\infty MT \theta; \mathbb{Q})$$

and the right-hand side is a graded-commutative algebra, so $\sigma$ extends to the free graded-commutative algebra on the part of $H^{*+d}(B; \mathbb{Q}u_1)$ of degree $> 0$,

$$\Lambda(H^{*+d>0}(B; \mathbb{Q}u_1)) \longrightarrow H^*(\Omega^\infty MT \theta; \mathbb{Q}).$$

This is an isomorphism of graded-commutative algebras.

2.6. Poset models. A key step in the proofs of [GMTW09] and [GRW10] identifying the infinite loop space $BC \theta$ is to first identify this classifying space with the classifying space of a certain topological poset. The result holds for all variations of the cobordism category mentioned above; we prove the general result in Proposition 2.14 below.

Definition 2.13. Let $\mathcal{C} \subset \mathcal{C}_0(\mathbb{R}^N)$ be a subcategory. Let

$$D^\mathcal{C}_\theta \subset \mathbb{R} \times \mathbb{R}_{>0} \times \psi_\theta(N + 1, 1)$$

denote the subspace of triples $(t, \varepsilon, (W, \ell))$ such that $(t - \varepsilon, t + \varepsilon)$ consists of regular values for $x_1 : W \rightarrow \mathbb{R}$, and $W|_t \in \text{Ob}(\mathcal{C})$. Define a partial order on $D^\mathcal{C}_\theta$ by

$$(t, \varepsilon, (W, \ell)) < (t', \varepsilon', (W', \ell'))$$

if and only if $(W, \ell) = (W', \ell')$, $t + \varepsilon < t' - \varepsilon$ and $W|_{[t, t']} \in \text{Mor}(\mathcal{C})$.

Proposition 2.14. Let $\mathcal{C} \subset \mathcal{C}_{0,L}(\mathbb{R}^N) \subset \mathcal{C}_0(\mathbb{R}^N)$ be a subcategory which consists of entire path components of the object and morphism spaces of $\mathcal{C}_{0,L}(\mathbb{R}^N)$. Then there is a weak homotopy equivalence

$$BC \simeq BD^\mathcal{C}_\theta.$$

Proof. We introduce an auxiliary topological poset $D^\mathcal{C}_\theta$ which maps to both $D^\mathcal{C}_\theta$ and $\mathcal{C}$. It is the subposet of $D^\mathcal{C}_\theta$ consisting of $(t, \varepsilon, (W, \ell))$ such that $(W, \ell)$ is a product over $(t - \varepsilon, t + \varepsilon)$. This conditions means that if we write $W|_t = \{t\} \times M$ and give $M$ the inherited $\theta$-structure, then

$$W|_{(t - \varepsilon, t + \varepsilon)} = (t - \varepsilon, t + \varepsilon) \times M$$
as $\theta$-manifolds. Then there is a zig-zag of functors
\[
D_0^\infty \leftarrow D_0^{C_{\ell_{\infty}}} \longrightarrow C,
\]
where the first arrow is the inclusion of the subposet and the second is the functor that sends a morphism $(a < b, W, \ell)$ to the manifold $(W|_{[a,b]} - a, c_1)$ extended cylindrically in $(-\infty, 0) \times \mathbb{R}^n$ and $(b - a, \infty) \times \mathbb{R}^n$. This induces a zig-zag diagram
\[
N_k D_0^\infty \leftarrow N_k D_0^{C_{\ell_{\infty}}} \longrightarrow N_k C,
\]
and we prove that both maps are weak equivalence for all $k$ in the same way as in [GRW10] Theorem 3.9.

Applying the above construction to the categories $C_{\theta,L}^{n,f}(\mathbb{R}^N)$ we obtain topological posets $D_{n,f}^{\infty}(\mathbb{R}^N)$ and weak homotopy equivalences
\[
(2.1) \quad BC_{\theta,L}^{n,f}(\mathbb{R}^N) \simeq BD_{n,f}^{\infty}(\mathbb{R}^N).
\]

Similarly, when we specialise to the case $d = 2n$ and let $A \subset \pi_0(\text{Ob}(C_{\theta,L}^{n-1,n-2}(\mathbb{R}^N)))$ be a collection of path components of objects, we obtain weak homotopy equivalences
\[
(2.2) \quad BC_{\theta,L}^{n-1,A}(\mathbb{R}^N) \simeq BD_{n,L}^{n-1,A}(\mathbb{R}^N).
\]

2.7. The homotopy type of $C_{\theta,L}(\mathbb{R}^N)$. In [GRW10] Theorems 3.9 and 3.10 we proved that there is a weak homotopy equivalence $BD_{\theta}(\mathbb{R}^N) \simeq \psi\theta(N + 1, 1)$, which combined with Proposition 2.14 gives
\[
(2.3) \quad BC_{\theta}(\mathbb{R}^N) \simeq BD_{\theta}(\mathbb{R}^N) \simeq \psi\theta(N + 1, 1).
\]

(Strictly speaking, in that paper we worked with a version of $D_{\theta}(\mathbb{R}^N)$ where $\varepsilon = 0$, but the obvious map induces a levelwise weak equivalence of nerves.) For the purposes of this paper we require a slightly stronger version of this result, taking into account the submanifold $L$.

**Proposition 2.15.** There are weak homotopy equivalences
\[
BC_{\theta,L}(\mathbb{R}^N) \simeq BD_{\theta,L}(\mathbb{R}^N) \simeq \psi\theta,L(N + 1, 1)
\]
where $\psi\theta,L(N + 1, 1) \subset \psi\theta(N + 1, 1)$ is the subspace consisting of those $(W, \ell)$ such that $W \cap (\mathbb{R} \times (-\infty, 0] \times \mathbb{R}^{N-1}) = \mathbb{R} \times L$ as $\theta$-manifolds.

**Proof.** The proof of [GRW10] Theorem 3.10] applies verbatim. \qed

**Proposition 2.16.** The inclusion
\[
i : \psi\theta,L(N + 1, 1) \longrightarrow \psi\theta(N + 1, 1)
\]
is a weak homotopy equivalence.

**Proof.** This is similar to [GRW10] Lemma 4.6], which is essentially the case $L = D^{d-1}$. It requires careful analysis of $\theta$-structures, so let us, for this proof only, denote the $\theta_{d-1}$-structure on $L$ by $\ell_L : \varepsilon_1 \oplus TL \rightarrow \theta^\star \gamma$. We first want to construct the *double* $D(L)$ of $L$ as a $\theta_{d-1}$-manifold, and a canonical $\theta$-null bordism of it. Recall that $L$ is a submanifold of $(-1/2, 0] \times (-1, 1)^{N-1}$ which we identify with $\{0\} \times (-1/2, 0] \times (-1, 1)^{N-1} \subset (-1, 0] \times (-1/2, 0] \times (-1, 1)^{N-1}$. Let $V \subset (-1, 0] \times (-1/2, 1/2] \times (-1, 1)^{N-1}$ denote the subset swept out by rotating $L$ around $(0, 0)$ in the half-plane $(-1, 0] \times (-1, 1)$. As $L$ was collared, this subset is a $d$-dimensional submanifold with boundary, and $L$ lies in its boundary. We define $D(L) = \partial V$, and
$\overline{L} = D(L) - \text{int}(L)$. The inclusion $L \hookrightarrow V$ is a homotopy equivalence, so there is a unique extension up to homotopy

$$\varepsilon^1 \oplus TL \xrightarrow{\ell_L} \theta^* \gamma \xrightarrow{\downarrow} TV,$$

where the vertical map sends $\varepsilon^1$ to the outwards pointing vector. This restricts to a $\theta$-structure on $D(L)$, and hence on $\overline{L}$, and $V$ gives a $\theta$-cobordism $V : \emptyset \sim D(L)$.

Similarly, we can rotate $L$ in the half-plane $[0, 1) \times (-1, 1)$ around the point $(0, -1/2)$ to obtain a submanifold of $[0, 1] \times [-1, 0] \times (-1, 1)^{N-1}$, extending to a $\theta$-cobordism $U \subset [0, 1] \times [-1, 0] \times (-1, 1)^{N-1}$, ending at $\{1\} \times [-1, 0] \times \partial L$ and starting at $\{0\} \times (L \cup (T - e_1))$, where $T - e_1 \subset [-1, \frac{1}{2}) \times (-1, 1)^{N-1}$ denotes the parallel translate of $T$.

The $\theta$-manifolds $U$ and $V$ give us the tools we need. $D(L)$ is a submanifold of $(-1/2, 1/2) \times (-1, 1)^{N-1}$, so we have a $\theta$-manifold $\mathbb{R} \times D(L) \subset \mathbb{R} \times (-1/2, 1/2) \times (-1, 1)^{N-1}$. We define a map

$$r : \psi_\theta(N + 1, 1) \longrightarrow \psi_\theta(L, N + 1, 1)$$

which given $(W, \ell) \subset \mathbb{R} \times (-1, 1) \times (-1, 1)^{N-1}$ applies the affine diffeomorphism $(-1, 1) \cong (1/2, 1)$ to its second coordinate, and then takes the (disjoint) union with $\mathbb{R} \times D(L)$.

The composition $i \circ r$ is homotopic to the identity as the $\theta$-null-bordism $V$ of $D(L)$ may be used to push the cylinder $\mathbb{R} \times D(L)$ off to the right. A similar argument, pushing $U$ to the left, proves that the composition $r \circ i$ is homotopic to the identity. Figure 1 shows how.

Combining this proposition with Proposition 2.15 and the homotopy equivalence (2.3) gives the following corollary.

**Corollary 2.17.** For any pair $(L, \ell_L)$ as in Definition 2.7, the inclusion

$$BC_{\theta, L} (\mathbb{R}^N) \longrightarrow BC_{\theta} (\mathbb{R}^N)$$

is a weak homotopy equivalence.
2.8. A more flexible model. From the poset models of Section 2.6 we construct the semi-simplicial spaces
\[ D_{\theta,L}^{\kappa,l}(\mathbb{R}^N) \cong N_* D_{\theta,L}^{\kappa,l}(\mathbb{R}^N). \]

The remarks of Section 2.6 and Proposition 2.15 show that the geometric realisations of these semi-simplicial spaces are models for the classifying spaces of the categories \( C_{\theta,L}^\kappa(\mathbb{R}^N) \) in which we are interested. The benefit of working with these semi-simplicial spaces instead of the cobordism categories is that we can often make constructions which are not functorial, yet give well-defined maps between geometric realisations of the semi-simplicial spaces involved.

To make this technique easier to apply, we will define an auxiliary semi-simplicial space \( X_{\theta,L}^{\kappa,l} \). We will prove that its geometric realisation is weakly equivalent to \( B C_{\theta,L}^\kappa(\mathbb{R}^N) \) and that \( X_{\theta,L}^{\kappa,l} \) depends on \( N \), but we omit that from the notation.

**Definition 2.18.** Let \( \theta : B \to BO(d) \), \( N \) and \( L \) be as before. Let \(-1 \leq \kappa \leq \frac{d-1}{2} \), \(-1 \leq l \leq \kappa \) and \(-1 \leq l \leq d - \kappa - 2 \). Define \( X_{\theta,L}^{\kappa,l} \) to be the semi-simplicial space with \( p \)-simplices consisting of certain tuples \((a, \varepsilon, (W, \ell))\) such that \( a = (a_0, \ldots, a_p) \in \mathbb{R}^{p+1}, \varepsilon = (\varepsilon_0, \ldots, \varepsilon_p) \in (\mathbb{R}_{>0})^{p+1}, \) and \( (W, \ell) \in \Psi_{\theta}((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N) \), satisfying
   
   (i) \( W \subset (a_0 - \varepsilon_0, a_p + \varepsilon_p) \times (-1, 1)^N \),
   
   (ii) \( W \) and \((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times L \) agree as \( \theta \)-manifolds on the subspace \( x_\theta^{-1}(\{ -\infty, 0 \}) \),
   
   (iii) \( a_{i-1} + \varepsilon_{i-1} < a_i - \varepsilon_i \) for all \( i = 1, \ldots, p \),
   
   (iv) for each pair of regular values \( t_0 < t_1 \in \bigcup_i (a_i - \varepsilon_i, a_i + \varepsilon_i) \), the cobordism \( W|_{[t_0, t_1]} \) is \( \kappa \)-connected relative to its outgoing boundary,
   
   (v) for each regular value \( t \in (a_i - \varepsilon_i, a_i + \varepsilon_i) \), the map \( \pi_j(W|_t) \to \pi_j(B) \), induced by \( \ell|_t \), is injective for all basepoints and all \( j \leq l \).

We topologise this set as a subspace of \( \mathbb{R}^{p+1} \times (\mathbb{R}_{>0})^{p+1} \times \Psi_{\theta}((-1, 1) \times \mathbb{R}^N) \), where we use the standard affine diffeomorphism \((-1, 1) \cong (a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N \) to identify \( \Psi_{\theta}((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N) \) with \( \Psi_{\theta}((-1, 1) \times \mathbb{R}^N) \). The \( j \)th face map is given by forgetting \( a_j \) and \( \varepsilon_j \), and if \( j = 0 \), composing with the restriction map \( \Psi_{\theta}((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N) \to \Psi_{\theta}((a_1 - \varepsilon_1, a_p + \varepsilon_p) \times \mathbb{R}^N) \), and similarly if \( j = p \).

There are semi-simplicial maps \( D_{\theta,L}^{\kappa,l}(\mathbb{R}^N) \to X_{\theta,L}^{\kappa,l} \), which on \( p \)-simplices are given by sending \((a, \varepsilon, (W, \ell)) \) with \((W, \ell) \in \Psi_{\theta}(\mathbb{R}^N) \) to the same thing restricted down to \( \Psi_{\theta}((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N) \).

The semi-simplicial space \( X_{\theta,L}^{\kappa,l} \) is easier to map into (by a semi-simplicial map) than \( D_{\theta,L}^{\kappa,l}(\mathbb{R}^N) \), for two reasons. Firstly, we do not require that the intervals \((a_i - \varepsilon_i, a_i + \varepsilon_i) \) consist entirely of regular values; instead we allow critical values, and conditions (15) ensure that the critical values do not affect the essential properties of the space. Secondly, we discard those parts of the manifold outside of \((a_0 - \varepsilon_0, a_p + \varepsilon_p) \), and so do not need to worry about controlling parts of the manifold outside of the region.

**Definition 2.19.** In the case \( d = 2n \), with \( A \subset \pi_0(\text{Ob}(C_{\theta,L}^{n-1,n-2}(\mathbb{R}^N))) \) a collection of path components of objects, we make the entirely analogous definition of \( X_{\theta,L}^{-n+A} \). Precisely, in Definition 2.18, we replace condition (17) by (15) for each regular value \( t \in (a_i - \varepsilon_i, a_i + \varepsilon_i) \), the \( \theta_{d-1} \)-manifold \((W|_t, \ell|_t) \) lies in \( A \).
The following is our main result concerning these models, and together with \([\text{2.10}]\) and \([\text{2.20}]\) provides weak homotopy equivalences \(BC^{\kappa,\ell}_{\theta,L}(\mathbb{R}^N) \simeq |X^{\kappa,\ell}_*|\) and, in the case \(d = 2n\), \(BC^{\kappa,\ell}_{\theta,L-1}(\mathbb{R}^N) \simeq |X^{\kappa,\ell-1}_*|\).

**Proposition 2.20.** Let \(\kappa\) and \(\ell\) satisfy the inequalities in Definition \([\text{2.18}]\). The semi-simplicial map \(D^{\kappa,\ell}_{\theta,L}(\mathbb{R}^N)_* \to X^{\kappa,\ell}_*\), and in the case \(d = 2n\) also the map \(D^{\kappa,\ell}_{\theta,L-1}(\mathbb{R}^N)_* \to X^{\kappa,\ell-1}_*\), induce weak homotopy equivalences after geometric realisation.

**Proof.** For the proof we introduce an auxiliary semi-simplicial space \(\hat{X}^{\kappa,\ell}_*\). Its \(p\)-simplices are those tuples

\[
(a, \varepsilon, (W, \ell)) \in \mathbb{R}^{p+1} \times (\mathbb{R}_{>0})^{p+1} \times \psi_0(N + 1, 1)
\]
satisfying the conditions of Definition \([\text{2.18}]\) except that the interval \((a_0 - \varepsilon_0, a_p + \varepsilon_p)\) is replaced with \(\mathbb{R}\) in \([\text{1}]\) and \([\text{4}]\). We can regard \(D^{\kappa,\ell}_{\theta,L}(\mathbb{R}^N)_*\) as a subspace of \(\hat{X}^{\kappa,\ell}_*\), and we have a factorisation

\[
D^{\kappa,\ell}_{\theta,L}(\mathbb{R}^N)_* \to \hat{X}^{\kappa,\ell}_* \to X^{\kappa,\ell}_*.
\]

The map \(\hat{X}^{\kappa,\ell}_* \to X^{\kappa,\ell}_*\) is a weak homotopy equivalence in each simplicial degree, by methods similar to \([\text{GRW10}]\) Theorem 3.9. Briefly, in simplicial degree \(p\) choose—continuously in the data \((a_0, a_p, \varepsilon_0, \varepsilon_p)\)—diffeomorphisms \((a_0 - \varepsilon_0, a_p + \varepsilon_p) \cong \mathbb{R}\) which are the identity on \([0, a_0]\). Using this family of diffeomorphisms to stretch gives a map \(X^{\kappa,\ell}_p \to X^{\kappa,\ell}_p\), which is homotopy inverse to the restriction map \(X^{\kappa,\ell}_p \to X^{\kappa,\ell}_p\).

To show that the first map induces a weak homotopy equivalence on geometric realisation, we use a technique which we shall use many times in this paper. That is, we consider a map

\[
f : (D^n, \partial D^n) \to (|\hat{X}^{\kappa,\ell}_*|, |D^{\kappa,\ell}_{\theta,L}(\mathbb{R}^N)_*|)
\]

representing an element of the \(n\)th relative homotopy group, and show that it may be homotoped through maps of pairs to a map with image in \(|D^{\kappa,\ell}_{\theta,L}(\mathbb{R}^N)_*|\).

For each \(x \in D^n\) the point \(f(x)\) is a tuple \((t, a, \varepsilon, (W(x), \ell))\), and we may choose a pair \((a^x, \varepsilon^x)\) such that \([a^x - \varepsilon^x, a^x + \varepsilon^x] \subseteq \bigcup_i (a_i - \varepsilon_i, a_i + \varepsilon_i) - \{a_i\}\) and that \([a^x - \varepsilon^x, a^x + \varepsilon^x]\) consists of regular values of \(x_1 : W(x) \to \mathbb{R}\). By properness of \(x_1 : W(a) \to \mathbb{R}\), there is a neighbourhood \(U_x \ni x\) for which \([a^x - \varepsilon^x, a^x + \varepsilon^x]\) still consists of regular values. The \(U_x\)'s cover \(D^n\) and we let \(\{U_j\}_{j \in J}\) be a finite subcover. We may suppose that \(a_j \neq a_k\), as otherwise we may change the cover by letting \(U_j' = U_j \cup U_k\) with \((a_j')^i = a_j^i = a_k^i\) and \((\varepsilon^i)' = \min(\varepsilon^i, \varepsilon^k)\). Once the \(a^i\) are distinct, we may shrink the \(\varepsilon^i\) so that the intervals \([a^i + \varepsilon^i, a^i - \varepsilon^i]\) are pairwise disjoint, and so that no \(a_i\) lies in such an interval.

As the intervals \([a^i + \varepsilon^i, a^i - \varepsilon^i]\) are chosen to consist of regular values, the data \(\{(a_j, a^i, \varepsilon^i)\}_{j \in J}\), together with a choice of partition of unity subordinate to the cover by the \(U_j\)'s, determine a map \(\hat{f} : D^n \to |D^{\kappa,\ell}_{\theta,L}(\mathbb{R}^N)_*|\) with the same underlying family of \(\theta\)-manifolds. As \([a^j - \varepsilon^j, a^j + \varepsilon^j] \subseteq \bigcup_i (a_i - \varepsilon_i, a_i + \varepsilon_i)\), this new family satisfies conditions \([\text{18}]\) and \([\text{19}]\) of Definition \([\text{2.18}]\) (as the old family did) so \(\hat{f}\) actually has image in the subspace \(|D^{\kappa,\ell}_{\theta,L}(\mathbb{R}^N)_*|\). There is a homotopy \(H\) of \(p \circ \hat{f}\) to \(f\) as follows: on underlying \(\theta\)-manifolds it is constant, but on the interval data we first use the straight-line homotopy from the data \(\{(a^j, \varepsilon^j)\}\) to the data \(\{(a_i, \varepsilon)\}\) where we choose \(\varepsilon \leq \min(\varepsilon_i)\) small enough so that \([a_i - \varepsilon, a_i + \varepsilon]\) is disjoint from the \([a^j - \varepsilon^j, a^j + \varepsilon^j]\). This straight-line homotopy is in the barycentric coordinates: as the intervals are all disjoint, the join of the simplices they describe also lies in \(|D^{\kappa,\ell}_{\theta,L}(\mathbb{R}^N)_*|\), and so there is a canonical straight line between them. Then we use
the obvious homotopy from the data \( \{ (a_i, \varepsilon) \} \) to the data \( \{ (a_i, \varepsilon_i) \} \) that stretches the \( \varepsilon \)'s. The restriction of \( H \) to \( \partial D^n \) remains in the subspace \( |D_{\theta, L}^{n, 1}(\mathbb{R}^N)_*| \), and so \( H \) gives a relative null-homotopy of \( f \).

The case when \( d = 2n \) and \( A \) is chosen is identical. \( \square \)

### 3. Surgery on Morphisms

In this section we wish to study the filtration
\[
C_{\theta, L}^n(\mathbb{R}^N) \subset \cdots \subset C_{\theta, L}^1(\mathbb{R}^N) \subset C_{\theta, L}^0(\mathbb{R}^N) \subset C_{\theta, L}^{-1}(\mathbb{R}^N) = C_{\theta, L}(\mathbb{R}^N)
\]
and in particular establish the following theorem. The reader mainly interested in Theorems 1.1 and 1.2 can take \( d = 2n, \theta = \theta^0 : BO(2n)(n) \to BO(2n), L \cong D^{2n-1} \), and \( N = \infty \) (but the proof does not simplify much in this special case).

**Theorem 3.1.** Suppose that the following conditions are satisfied

(i) \( 2k \leq d - 2 \),
(ii) \( \kappa + 1 + d < N \),
(iii) \( L \) admits a handle decomposition only using handles of index \( d - \kappa - 1 \).

Then the map
\[
BC_{\theta, L}^n(\mathbb{R}^N) \longrightarrow BC_{\theta, L}^{n-1}(\mathbb{R}^N)
\]
is a weak homotopy equivalence.

The proof of Theorem 3.1 consists of performing surgery on morphisms, in order to make them more highly connected relative to their outgoing boundary. Making this idea into a proof has two main ingredients. Firstly, we construct for each morphism in \( C_{\theta, L}^{n-1} \) a contractible space of surgery data. The space is defined in Definition 3.2, and the precise statement is Theorem 3.4. Secondly, we implement the surgery described by the surgery data, using a standard one-parameter family of manifolds defined in Section 3.2.

In order to motivate some of the more technical constructions, let us first give an informal account of this technique. For simplicity, we suppose that \( N = \infty \), that we have no tangential structure, that \( L = \emptyset \), and that \( \kappa = 0 \). We first apply the equivalence 2.1 to reduce the problem to studying the map
\[
BD^0 \longrightarrow BD^{-1}
\]
of classifying spaces of posets. Let
\[
\sigma = (t_0, t_1: a_0, a_1; \varepsilon_0, \varepsilon_1; W) \in BD^{-1}
\]
be a point on a 1-simplex (for example), where \( (t_0, t_1) \in \Delta^1 \) are the barycentric coordinates. We will describe a way of producing a path from its image in \( |X_{\bullet -1}| \) into the subspace \( |X_{\bullet 0}| \). The proof of Theorem 3.1 will be a systematic, parametrised version of this construction.

If the cobordism \( W|_{[a_0, a_1]} \) is already 0-connected relative to its outgoing boundary, then the image of \( \sigma \) in \( |X_{\bullet -1}| \) already lies in the subspace \( |X_{\bullet 0}| \), and we are done. If not, we may choose distinct points
\[
\{ f_\alpha : * \to W|_{[a_0, a_1]} \}_{\alpha \in \Lambda}
\]
such that the pair \( (W|_{[a_0, a_1]}, W|_{a_1}) \cup \bigcup_\alpha f_\alpha(\ast) \) is 0-connected. We then choose tubular neighbourhoods of these points to obtain codimension 0 embeddings \( f_\alpha : D^d \to W|_{[a_0, a_1]} \), which we can extend to an embedding
\[
e_\alpha : (S^0, \{ +1 \}) \times D^d \longrightarrow ([a_0, \infty) \times \mathbb{R}^\infty, [a_1 + \varepsilon_1, \infty) \times \mathbb{R}^\infty).
\]
As the original points \( f_\alpha(\ast) \) were distinct, we may suppose the embeddings \( e_\alpha \) are disjoint. Now on each \( e_\alpha(S^0 \times D^d) \) we do the surgery move shown in Figure 2.
a move similar in spirit, though much simpler, than that described in [GMTW09 §6.2].

More precisely, Figure 2 describes a continuous 1-parameter family of $d$-manifolds $P_t$, $t \in [0,1]$, depicted (for $d = 2$) by its values at times $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$. The family comes equipped with functions to $\mathbb{R}$, depicted in the figure as the height function. The family starts at the manifold $P_0 = S^0 \times D^d$, and we may cut out each $e_\alpha(S^0 \times D^d)$ from $W$ and glue in $P_t$, to obtain a 1-parameter family of manifolds $W_t$, each equipped with a height function $W_t \to \mathbb{R}$, with $W_0 = W$. The values $\{a_0, a_1\}$ do not remain regular throughout this move, so this does not describe a path in the space $BD^{-1}$. However, the intervals $(a_i - \varepsilon_i, a_i + \varepsilon_i)$ do only contain isolated critical values, so it does describe a path in the space $|X^-|$. Furthermore, at the end of the move we obtain a manifold $W_1 = W$ such that $(W|_{[a_0,a_1]}, W|_{a_1})$ is 0-connected, and hence a point in $|X_0^-|$. By Proposition 2.20 this proves that $\pi_0(BD^0) \to \pi_0(BD^{-1})$ is surjective, as required.

This surgery move generalises easily to the case when $N$ is finite (but large enough), $L \neq \emptyset$, and $\kappa > 0$ (the analogue of the surgery move will start with $S^\kappa \times D^{d-\kappa}$). However, it does not generalise well to the case of arbitrary tangential structures (to understand how it can fail, we suggest that the reader attempt to impose a family of framings to the family of 2-manifolds in Figure 2). One way to fix this would be to use the surgery move described in [GMTW09 §6.2], but that does not seem to generalise to $\kappa > 0$. Instead we modify the surgery move in Figure 2 as shown in Figure 3. As we shall see (in the proof of Proposition 3.6 where we also explain the analogous process for $\kappa > 0$) there is a canonical way of extending any tangential structure on $\{-1\} \times D^d$ to the resulting 1-parameter family of manifolds.

3.1. Surgery data. In order to implement the ideas discussed above, we will fatten the semi-simplicial space $D^\kappa_{\partial,L} (\mathbb{R} N)$ up to a bi-semi-simplicial space $D^\kappa_{\partial,L} (\mathbb{R} N)_{\bullet,\bullet}$, which includes suitable surgery data. The space $D^\kappa_{\partial,L} (\mathbb{R} N)_{\bullet,\bullet}$ is described in Definition 3.3 below, using the following notation. Let $V \subset \overline{V} \subset \mathbb{R}^{\kappa+1} \times \mathbb{R}^{d-\kappa}$ be the subspaces

$$V = (-2,0) \times \mathbb{R}^d \quad \overline{V} = [-2,0] \times \mathbb{R}^d$$

and let $h : \overline{V} \to [-2,0] \subset \mathbb{R}$ denote projection to the first coordinate, which we call the height function. Let $\partial_- D^{\kappa+1} \subset \partial D^{\kappa+1}$ denote the lower hemisphere (i.e. $\partial_- D^{\kappa+1} = \partial D^{\kappa+1} \cap (-1,0] \times \mathbb{R}^\kappa$). We shall also use the notation $[p]^\kappa = \Delta([p], [1])$.
when \([p] \in \Delta_m\). The elements of \([p]^{\vee}\) are in bijection with \([0, \ldots, p+1]\), using the convention that \(\varphi : [p] \to [1]\) corresponds to the number \(i\) with \(\varphi^{-1}(1) = \{i, i+1, \ldots, p\}\). Finally, we fix once and for all an uncountable set \(\Omega\).

**Definition 3.2.** Let \(x = (a, e, (W, \ell_W)) \in D^{\kappa-1}_0(\mathbb{R}^N)_p\) and define \(Z_q(x)\) to be the set of triples \((\Lambda, \delta, e)\), where \(\Lambda \subset \Omega\) is a finite set, \(\delta : \Lambda \to [p]^{\vee} \times [q]\) is a function, and 
\[
e : \Lambda \times \nabla \to \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}
\]
is an embedding, satisfying the conditions below. We shall write \(\Lambda_{i,j}\) for \(\delta^{-1}(i, j)\), 
\(e_{i,j} = e|_{\Lambda_{i,j} \times \nabla}\) and \(D_{i,j} = e_{i,j}(\Lambda_{i,j} \times \partial_- D^{\kappa+1} \times \{0\})\) for \(0 \leq i \leq p+1\) and \(0 \leq j \leq q\).

(i) On each subset \((x_1 \circ e|_{\{\lambda\} \times \nabla})^{-1}(a_k - \varepsilon_k, a_k + \varepsilon_k) \subset \{\lambda\} \times \nabla\), the height function \(x_1 \circ e\) coincides with the height function \(h\) up to an affine transformation.

(ii) \(e\) sends \(\Lambda \times h^{-1}(0)\) into \(x_1^{-1}(a_p + \varepsilon_p, \infty)\).

(iii) For \(i > 0\), \(e\) sends \(\Lambda_{i,j} \times h^{-1}(-3/2)\) into \(x_1^{-1}(a_{i-1} + \varepsilon_{i-1}, \infty)\).

(iv) \(e\) sends \(\Lambda \times h^{-1}(-2)\) into \(x_1^{-1}(-\infty, a_0 - \varepsilon_0)\).

(v) \(e^{-1}(W) = \Lambda \times \partial_- D^{\kappa+1} \times \mathbb{R}^{d-\kappa}\).

(vi) For each \(j\) and each \(i \in \{1, \ldots, p]\), the pair 
\[(W|_{[a_{i-1}, a_i]}), W|_{a_i} \cup D_{i,j}|_{[a_{i-1}, a_i]}\]
is \(\kappa\)-connected.

For each \(x\), \(Z_q(x)\) is a semi-simplicial set: Given an injective map \(k : [q] \to [q']\), we replace \(\Lambda\) by the subset \(\delta^{-1}([p]^{\vee} \times \text{Im}(k))\), compose \(\delta\) with \([p]^{\vee} \times k^{-1}\), and restrict \(e\). Explicitly, the face map \(d_j\) forgets the embeddings \(e_{i,j}\).

Note that the set \(Z_q(x)\) consists of those \([q+1]\)-tuples of elements of \(Z_0(x)\) which are disjoint.

**Definition 3.3.** We define a bi-semi-simplicial space \(D^{\kappa-1}_0(\mathbb{R}^N)_{\bullet, \bullet}\) as a set by
\[
D^{\kappa-1}_0(\mathbb{R}^N)_p = \{(x, y) \mid x \in D^{\kappa-1}_0(\mathbb{R}^N)_p, y \in Z_q(x)\}
\]
topologised as a subspace of
\[
D^{\kappa-1}_0(\mathbb{R}^N)_p \times \left(\prod_{\Lambda \subset \Omega} C^\infty(\Lambda \times \nabla, \mathbb{R}^{N+1})\right)^{(p+2)(q+1)}.
\]
The space $D^s_{\partial,L}(\mathbb{R}^N)_{p,q}$ is functorial in $[p] \in \Delta_n$ by composing $\delta : \Lambda \rightarrow [p]^\vee \times [q]$ with the induced map $[p']^\vee \rightarrow [p]^\vee$ and functorial in $[q] \in \Delta_m$ in the same way as in Definition 3.2. Explicitly, the face map $d_i$ in the $q$ direction forgets the embeddings $e_{i,*}$ and in the $p$ direction takes the union of $e_{i,*}$ and $e_{i+1,*}$. We shall write $D^s_{\partial,L}(\mathbb{R}^N)_{p,-1} = D^s_{\partial,L}(\mathbb{R}^N)_{p}$, and there is an augmentation map $D^s_{\partial,L}(\mathbb{R}^N)_{p,q} \rightarrow D^s_{\partial,L}(\mathbb{R}^N)_{p-1}$ which forgets all surgery data.

The main result concerning this bi-semi-simplicial space is the following, whose proof we defer until Section 6.

**Theorem 3.4.** Under the assumptions of Theorem 3.1, the augmentation map

$$D^s_{\partial,L}(\mathbb{R}^N)_{*,*} \rightarrow D^s_{\partial,L}(\mathbb{R}^N)_{*,*}$$

induces a weak homotopy equivalence after geometric realisation.

In fact, we shall prove this theorem with condition (i) of Theorem 3.1 replaced by the weaker condition $2\kappa \leq d - 1$. The stronger assumption $2\kappa \leq d - 2$ will be used in Lemma 3.7.

### 3.2. The standard family.

We will now construct a one-parameter family of submanifolds of $V = (-2,0) \times \mathbb{R}^d$ which formalises the family of manifolds depicted in Figures 2 and 3. Let us write coordinates in $\mathbb{R}^{\kappa+1} \times \mathbb{R}^{d-\kappa}$ as $(u,v)$. First define an element $\tilde{P}_0 \in \Psi_d(\mathbb{R} \times \mathbb{R}^{\kappa} \times \mathbb{R}^{d-\kappa})$ as

$$\tilde{P}_0 = \partial D^{\kappa+1} \times \mathbb{R}^{d-\kappa}.$$  

Choose a function $\varphi : [0,\infty) \rightarrow [0,\infty)$ that is the identity function on a neighbourhood of $[1/2,\infty)$, takes value 1/4 near 0, and has $\varphi'' \geq 0$. We then define an embedding by

$$g' : \mathbb{R}^{\kappa+1} \times \partial D^{d-\kappa} \rightarrow D^{\kappa+1} \times \mathbb{R}^{d-\kappa}$$

$$(u,v) \mapsto (u/\varphi(|u|), \varphi(|u|) \cdot v).$$

and another embedding $g : \mathbb{R}^{\kappa+1} \times \partial D^{d-\kappa} \rightarrow [-2,1] \times \mathbb{R}^{\kappa} \times \mathbb{R}^{d-\kappa}$ by

$$g(u,v) = g'(u,v) + \tau(u) \left( \frac{v_1 - 1}{2}, 0, 0 \right)$$

where $\tau : \mathbb{R}^{\kappa+1} \rightarrow [0,1]$ is a bump function supported in a small neighbourhood of the point $u_0 = (-1/2,0) \in \mathbb{R} \times \mathbb{R}^\kappa$, having $\tau(u_0) = 1$, $\tau(u) < 1$ otherwise, and no critical points in $\tau^{-1}((0,1))$. We can arrange that the support of $\tau$ be small enough that it is contained in the region where $\varphi(|u|) = |u|$. We let $\tilde{P}_1 \subset \mathbb{R}^{d+1}$ denote the image of $g$ and $\tilde{P}_1'$ the image of $g'$. We then define

$$\mathcal{P}_0, \mathcal{P}_1 \in \Psi_d(V)$$

by intersecting the manifolds $\tilde{P}_0, \tilde{P}_1$ with the open set $V = (-2,0) \times \mathbb{R}^d$.

To construct $\mathcal{P}_t \in \Psi_d(V)$ for intermediate values of $t \in [0,1]$, we first observe that $\tilde{P}_0$ and $\tilde{P}_1'$ agree on the subset $|v| \geq 1/2$ and that $\tilde{P}_1$ agrees with them on the smaller subset $|v| \geq 1$ (when the support of the bump function $\tau$ is sufficiently small). Starting with the two submanifolds $\tilde{P}_0$ and $\tilde{P}_1 \subset \mathbb{R} \times \mathbb{R}^\kappa \times \mathbb{R}^{d-\kappa}$, we then pull the region $\{(u,v) \ | \ |v| < 1\}$ downwards by decreasing the first coordinate in $\mathbb{R} \times \mathbb{R}^d$, until the region where the submanifolds may disagree is moved completely outside of $V$. This gives two one-parameter families of submanifolds which, upon restricting to $V$, give two paths in $\Psi_d(V)$ starting at $\mathcal{P}_0$ and $\mathcal{P}_1$ and ending at the same point in $\Psi_d(V)$. Concatenating one path with the reverse of the other, we get the desired path from $\mathcal{P}_0$ to $\mathcal{P}_1$. 

STABLE MODULI SPACES 23
Spelling this process out in a little more detail, we first choose a function \( \rho : [0, \infty) \to [0, 1) \) taking the value 1 near \([0, 1]\), the value 0 near \([2, \infty)\), and which is strictly decreasing on \(\rho^{-1}(0, 1)\). We then define embeddings
\[
H_t : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{d-\kappa} \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{d-\kappa}
\]
\[
(s, x, y) \mapsto (s - t \cdot \rho(|y|), x, y)
\]
which for all \(t\) restrict to the identity for \(|y| \geq 2\). Define one-parameter families of manifolds by
\[
P_t^0 = V \cap H_t(\tilde{P}_0) = (H_{-t}|_V)^{-1}(\tilde{P}_0)
\]
\[
P_t^1 = V \cap H_t(\tilde{P}_1) = (H_{-t}|_V)^{-1}(\tilde{P}_1).
\]
The second description shows that these are closed subsets of \(V\) and describe continuous functions \(\mathbb{R} \to \Psi_d(V)\). It is easy to see that we have \(P_t^0 = P_t^1 \in \Psi_d(V)\) for \(t \geq 3\), and we then define the path \(P_t\) as the concatenation
\[
P_0 = P_0^0 \smallfrown P_3^0 = P_3^1 \smallfrown P_1^0 = P_1
\]
in \(\Psi_d(V)\), reparametrised so that the path has length 1. We collect the most important properties of this family in Proposition 3.6 below. The following remark partially explains how it relates to an ordinary \(\kappa\)-surgery.

**Remark 3.5.** Let \(Q(u, v) = -|u|^2 + |v|^2\), where as usual \((u, v) \in \mathbb{R}^{\kappa+1} \times \mathbb{R}^{d-\kappa}\). For \(t \in [0, 3]\) the function \((u, v) \mapsto H_t(u/|u|, v)\) defines a diffeomorphism to \(P_t^0\) from an open subset of \(Q^{-1}(t - 3)\) (namely the inverse image of \(V\) by that function) and similarly the function \((u, v) \mapsto H_t \circ g(u, v/|v|)\) defines a diffeomorphism to \(P_t^1\) from an open subset of \(Q^{-1}(3 - t)\). The inverses of these diffeomorphisms give smooth embeddings \(P_t^0 \to Q^{-1}(t - 3)\) and \(P_t^1 \to Q^{-1}(3 - t)\) and it is easy to verify that for \(t = 3\) the two resulting embeddings \(P_0^0 = P_3^1 \to Q^{-1}(0)\) agree, and so glue to a continuous family of embeddings \(P_t \to Q^{-1}(6t - 3)\).

The continuous map \(t \mapsto P_t\) has graph given by \(P = \{(t, x) \in [0, 1] \times V \mid x \in P_t\}\).

The above remarks give an embedding \(P \to Q^{-1}([-3, 3])\) and it is easy to verify that the image is disjoint from the straight lines from 0 to \(p_0 = (-1/2, 0, -1/2, 0) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^{d-\kappa-1}\) and from \(p_0\) to \(p_1 = (-1/2, 0, -\sqrt{13}/2, 0)\). Thus we get a diffeomorphism from \(P\) to an open subset of the contractible set
\[
Q = Q^{-1}([-3, 3]) - ([0, p_0] \cup [p_0, p_1]).
\]

**Proposition 3.6.** For \(2\kappa \leq d - 1\), the 1-parameter family \(P_t \in \Psi_d(V)\), defined for \(t \in [0, 1]\), has the following properties.

(i) The height function, i.e. the restriction of \(h : V \to (-2, 0)\) to \(P_t \subset V\), has isolated critical values.

(ii) \(P_0 = \text{int}(\partial_- D^{\kappa+1}) \times \mathbb{R}^{d-\kappa}\), where \(\partial_- D^{\kappa+1} = \partial D^{\kappa+1} \cap ([0, 1] \times \mathbb{R}^n)\).

(iii) Independently of \(t \in [0, 1]\) we have
\[
P_t - (\mathbb{R}^{\kappa+1} \times B_3^{d-\kappa}(0)) = \text{int}(\partial_- D^{\kappa+1}) \times (\mathbb{R}^{d-\kappa} - B_3^{d-\kappa}(0)).
\]

For ease of notation we write \(P_t^0\) for this closed subset of \(P_t\).

(iv) For all \(t \) and each pair of regular values \(-2 < a < b < 0\) of the height function, the pair
\[
(P_t|_{[a, b]} , P_t|_{[b, a]} )
\]
is \(\kappa\)-connected.

(v) For each pair of regular values \(-2 < a < b < 0\) of the height function, the pair
\[
(P_t|_{[a, b]} )
\]
is \(\kappa\)-connected.
Furthermore, if \( \mathcal{P}_0 \) is equipped with a \( \theta \)-structure \( \ell \) we can upgrade this, continuously in \( \ell \), to a 1-parameter family \( \mathcal{P}_t(\ell) \in \Psi_\theta(V) \) starting from \( \mathcal{P}_0, \ell \) such that

\[ (\text{iii}) \text{ The path } \mathcal{P}_t(\ell) \text{ is constant as } \theta \text{-manifolds near } \mathcal{P}_t. \]

**Proof.** We have seen properties [3]–[3] during the construction (the statement in [3] would still be true with 3 replaced by 2, but we wish to emphasise the smaller set). For property [4] we consider two cases depending on the value of \( a \).

In the case \( a > -1 \), the pair \([3.1]\) is homotopy equivalent to the pair

\[ (\mathcal{P}_t|_{[a,b]}, \mathcal{P}_t|_{b}), \]

using e.g. the gradient flow trajectories of \( h \) to deform \( \mathcal{P}_t|_{[a,b]} \) back to \( \mathcal{P}_t|_{b} \). In the case \( a < -1 \) we consider the modified height function, defined using the coordinates \((u, v) \in \mathbb{R}^{k+1} \times \mathbb{R}^{d-\kappa} \) as \( h(u, v) = h(u, v) + \lambda(|v|) \), where \( \lambda : [0, \infty) \to [0, \infty) \) is a smooth function which is 0 on \([0, 4]\) and restricts to a diffeomorphism \((4, \infty) \to (0, \infty) \).

We claim that the inclusion of pairs

\[ (\mathcal{P}_t \cap \mathcal{H}^{-1}([a, b]), \mathcal{P}_t \cap \mathcal{H}^{-1}(b)) \hookrightarrow (\mathcal{P}_t|_{[a,b]}, \mathcal{P}_t|_{b} \cup \mathcal{P}_0|_{[a,b]}) \]

is a homotopy equivalence. To define a homotopy inverse, we first consider the continuous, piecewise smooth function \( \rho_t : [0, \infty) \to (0, \infty) \) defined for \( t \leq b \) by

\[ \rho_t(s) = 1 \quad \text{for } s \in [0, 2], \]

\[ \rho_t(s) = \frac{\lambda^{-1}(b - t)}{s} \quad \text{for } s \in [3, \infty), \]

and by linear interpolation for \( s \in [2, 3] \). Then the function \((u, v) \mapsto (u, v \cdot \rho_u(|v|))\) restricts to a homotopy inverse of \([3.2]\), where both homotopies are given by straight lines in \( \mathbb{R}^{d+1} \).

In either case, the connectivity question is reduced to studying the inverse image of an interval relative to its outgoing boundary and can be studied as in ordinary Morse theory one critical level at a time. The proof of [4] will be finished once we establish that for each critical value of \( \mathcal{H} : \mathcal{P} \to \mathbb{R} \) in the interval \((a, b)\), the function can be perturbed in a neighbourhood of the critical set contained in \( \mathcal{H}^{-1}(a, b) \) to a Morse function with at most a critical point of index \( \leq d - \kappa - 1 \). (In the case \( a > -1 \) we have \( h = \mathcal{H} \) near any critical point of \( h \), so it suffices to consider \( \mathcal{H} \).) It is easy to verify that \( \mathcal{H} : \mathcal{P}_t^0 \to \mathbb{R} \) has at most two critical values \((-2, 0)\). One critical value moves with \( t \) and is homotopically Morse of index 0 for \( 0 \leq t < 1 \) and index \( \kappa \) for \( 1 < t < 3 \) (meaning that the function can be perturbed to a Morse function with one critical point of that index). The other is at \(-1\) and can be cancelled (meaning that the function can be perturbed to a non-singular function there). Since \( 2\kappa \leq d - 1 \) and hence \( \kappa < d - \kappa - 1 \), the index is at most \( d - \kappa - 1 \) as claimed. Similarly, one verifies that \( \mathcal{H} : \mathcal{P}_t^1 \to \mathbb{R} \) has at most two critical values in \((-2, 0)\), one of which is \(-1\) and can be cancelled, the other of which moves with \( t \) and is homotopically Morse of index \( d - \kappa - 1 \).

Property [5] can be proved in a similar way. In the case \( a < -1 \) \( b \) the pair is a relative \((d - 1)\)-cell, so it is \((d - 2)\)-connected and hence \( \kappa \)-connected (since \( d \geq 2 \) and \( 2\kappa \leq d - 1 \)). In all other cases the inclusion \( \mathcal{P}_t|_{b} \to \mathcal{P}_t|_{[a,b]} \) is a homotopy equivalence.

To establish the extra properties which can be obtained given a \( \theta \)-structure \( \ell \) on \( \mathcal{P}_0 = \text{Int}((\partial D^{d+1}) \times \mathbb{R}^{d-\kappa}) \), we again use the graph \( \mathcal{P} = \{ (t, x) \in [0, 1] \times \mathbb{R}^{d+1} | x \in \mathcal{P}_t \} \) and its identification with an open subset of the manifold \( Q \) from Remark [3.3]. The tangent bundles \( T \mathcal{P}_t \) assemble to a \( d \)-dimensional vector bundle \( T_\ast \mathcal{P} \to \mathcal{P} \) which then becomes identified with the restriction of the vector bundle \( T_\ast \mathcal{Q} = \text{Ker}(DQ : T \mathcal{Q} \to T[-3, 3]) \) and since both \( \mathcal{P}_0 \) and \( \mathcal{Q} \) are contractible, there is no obstruction to picking a vector bundle map \( r : T_\ast \mathcal{Q} \to T \mathcal{P}_0 \) which is the
identity (with respect to the identifications) over $\mathcal{P}_0$ and each $\mathcal{P}_t^0 = \mathcal{P}_0^0 \subset \mathcal{P}_0$. We can then restrict $r$ to $r_t : TP_t \to TP_0$ and let $\mathcal{P}_t(\ell)$ have the $\theta$-structure $\ell \circ r_t$. □

Let $(a, \varepsilon, (W, \ell_W), e) \in D_0^T(\mathbb{R}^N)_{p, 0}$, with $e = \{e_i, 0\}_{i=0}^{\mathbb{N}}$ (where we omit $\Lambda$ and $\delta : \Lambda \to [p]^\mathbb{N}$ from the notation). We construct a 1-parameter family of $\theta$-manifolds

$$K^t_\varepsilon(W, \ell_W) \in \Psi_0((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N),$$

$t \in [0, 1]$, by letting it be equal to $W|_{(a_0 - \varepsilon_0, a_p + \varepsilon_p)}$ outside of the images of the $e_i, 0|_{\Lambda_i \times V}$, and on each $e_i, 0(\{\lambda\} \times V)$ we let it be given by $e_i, 0(\{\lambda\} \times \mathcal{P}_t(\ell_W \circ De_i, 0))$. This gives a $\theta$-manifold, as the properties established above, $\mathcal{P}_t(\ell_W \circ De_i, 0)$ and $\mathcal{P}_0(\ell_W \circ De_i, 0)$ agree as $\theta$-manifolds near the set $(-2, 0) \times \mathbb{R}^\kappa \times (\mathbb{R}^{d-\kappa} - B^d_{d-\kappa}(0))$.

**Lemma 3.7.** Let $2\kappa \leq d - 2$. The tuple $(a, \varepsilon, K^t_\varepsilon(W, \ell_W))$ is an element of $X_p^{\kappa-1}$. If either $t = 1$ or $(W, \ell_W) \in D_0^T(\mathbb{R}^N)^p$, then $(a, \varepsilon, K^t_\varepsilon(W, \ell_W))$ lies in the subspace $X_p^\kappa \subset X_p^{\kappa-1}$.

**Proof.** We must verify conditions (i)-(v) of Definition 2.18. Condition (i) is true by definition, and certainly (ii) is satisfied as the embeddings $e_i, 0$ are disjoint from $\mathbb{R} \times L$. For (iii) and (v) there is nothing to say.

For each $\varepsilon$, consider regular values $a < b \in \cup_i (a_i - \varepsilon_i, a_i + \varepsilon_i)$ of the height function $x_t : W_t = K^t_\varepsilon(W, \ell_W) \to \mathbb{R}$. The cobordism $W_t|_{[a, b]}$ is obtained from $W_t|_{[a, b]}$ by cutting out embedded images of cobordisms $\mathcal{P}_0|_{[a_0, b_0]}$ indexed by $\lambda \in \Lambda = \Pi_\Lambda i, 0$ and gluing in $\mathcal{P}_t|_{[a_0, b_0]}$, where $a_0 < b_0$ are regular values of the height function on $\mathcal{P}_0$ and $\mathcal{P}_t$. If we denote by $X$ the complement of the embedded $e_i, 0(\int(\partial_\ast D^{\kappa+1}) \times B^d_{d-\kappa}(0))$ in the manifold $W_t|_{[a, b]}$, there are homotopy push-out squares

$$
\begin{array}{ccc}
X|_b & \longrightarrow & W_t|_b \\
\downarrow & & \downarrow \\
X & \longrightarrow & W_t|_b \cup X \\
\end{array}
$$

and

$$
\begin{array}{ccc}
\prod_{\lambda \in \Lambda} \mathcal{P}_t|_{[a_\lambda, b_\lambda]} \cup (\mathcal{P}_0^0|_{[a_0, b_0]}) & \longrightarrow & W_t|_b \cup X \\
\downarrow & & \downarrow \\
\prod_{\lambda \in \Lambda} \mathcal{P}_t|_{[a_\lambda, b_\lambda]} & \longrightarrow & W_t|_{[a, b]} \\
\end{array}
$$

The left hand map of the second square is a disjoint union of the maps discussed in property (xi) of Proposition 3.6 so it is $\kappa$-connected. As this square is a homotopy push-out, the right hand map is also $\kappa$-connected.

The pair $(X, X|_b)$ is obtained from the manifold pair $(W_t|_{[a, b]}, W_t|_b)$ by cutting out embedded copies of $(D^\kappa, \partial D^\kappa)$. By transversality we see that this does not change relative homotopy groups in dimensions $s \leq d - \kappa - 2$, which includes $s \leq \kappa$ by our assumption that $2\kappa \leq d - 2$. In particular, suppose the pair $(W_t|_{[a, b]}, W_t|_b)$ is $k$-connected, with $k \leq \kappa$, then the pair $(X, X|_b)$ is $k$-connected too. As the first square above is a homotopy push-out square, the inclusion $W_t|_b \to W_t|_b \cup X$ also has this connectivity.

Hence the composition $W_t|_b \to W_t|_b \cup X$ has the same connectivity as $W_t|_b \to W_t|_{[a, b]}$, up to a maximum of $\kappa$. This establishes that the tuple $(a, \varepsilon, K^t_\varepsilon(W, \ell_W))$ is an element of $X_p^{\kappa-1}$, and also that it lies in $X_p^\kappa$ if $(W, \ell_W)$ lies in $D_0^T(\mathbb{R}^N)$. When $t = 1$, there is a little more to say.

**Step 1.** Suppose $a < b \in (a_\varepsilon, a_i + \varepsilon_i)$. Then $(W_t|_{[a, b]}, W_t|_b)$ is $\infty$-connected and so $(W_t|_{[a, b]}, W_t|_b)$ is $\kappa$-connected, by the discussion above.
Step 2. Suppose \( a \in (a_{i-1} - \epsilon_{i-1}, a_{i-1} + \epsilon_{i-1}) \) and \( b \in (a_i - \epsilon_i, a_i + \epsilon_i) \). We now do the surgeries for \( \Lambda_0 \subset \Lambda \) first, giving a family of manifolds \( \tilde{W}_2 \). We claim that the pair \((\tilde{W}_1|_{[a,b]}, \tilde{W}_1|_{[a,b]})\) is \( \kappa \)-connected. Once this is established, doing the remaining surgeries to obtain \( W_1 \) does not change this property, as we have seen above.

Recall from Definition 3.2[7] that the pair \((W_0|_{[a,b]}, (D_{i,0}|_{[a,b]}))\) is \( \kappa \)-connected, where \( D_{i,0} = \epsilon_{i,0}(A_{i,0} \times \partial_-' D^{k+1} \times \{0\}) \subset W = W_0 \). If we write \( \tilde{D}_{i,0} = \epsilon_{i,0}(A_{i,0} \times \partial_-' D^{k+1} \times \{v\}) \subset W = W_0 \) for some \( v \in \mathbb{R}^{\kappa} - \mathbb{B}_i^{k-\kappa}(0) \), then the pair \((W_0|_{[a,b]}, (\tilde{W}_0|_{[a,b]} \sqcup (\tilde{D}_{i,0}|_{[a,b]}))\) is also \( \kappa \)-connected. Now the subset \( \tilde{D}_{i,0} \subset W \) is contained in \( \epsilon_{i,0}(A_{i,0} \times \mathbb{P}_{0}^{d}) \), so we can regard \( \tilde{D}_{i,0} \) as a subset of \( \tilde{W}_1 \) for all \( t \in [0,1] \).

The same transversality argument as before now shows that \((X, X|_b \cup \tilde{D}_{i,0}|_{[a,b]}))\) is also \( \kappa \)-connected, and the same gluing argument shows that \((\tilde{W}_1|_{[a,b]}, \tilde{W}_1|_{[a,b]} \cup \tilde{D}_{i,0}|_{[a,b]}))\) is \( \kappa \)-connected for all \( t \in [0,1] \). When \( t = 1 \), Proposition 3.6[v] shows that the inclusion \( \tilde{D}_{i,0}|_{[a,b]} \to \tilde{W}_1|_{[a,b]} \) is homotopic relative to \( \tilde{D}_{i,0}|_{[a,b]} \) to a map into \( \tilde{W}_1|_{[a,b]} \), and hence \((\tilde{W}_1|_{[a,b]}, \tilde{W}_1|_{[a,b]}))\) is \( \kappa \)-connected.

Step 3. For general \( a < b \in \cup_i(a_i - \epsilon_i, a_i + \epsilon_i) \), we may choose regular values in each intermediate interval \((a_j - \epsilon_j, a_j + \epsilon_j)\). By the previous case, this expresses \( W_1|_{[a,b]} \) as a composition of cobordisms which are each \( \kappa \)-connected relative to their outgoing boundaries, and the hence the composition also has that property. □

3.3. Proof of Theorem 3.3. We begin with the composition

\[
|D_{\theta,L}^k(\mathbb{R}^N)|_{\bullet, \bullet} \to |D_{\theta,L}^{k-1}(\mathbb{R}^N)|_{\bullet, \bullet} \to |X^\kappa_{\epsilon, \bullet}|
\]

where the first map (induced by the augmentation) is a homotopy equivalence by Theorem 3.3 and the second is a homotopy equivalence by Proposition 2.20. We will define a homotopy

\[
\mathcal{F} : [0,1] \times |D_{\theta,L}^k(\mathbb{R}^N)|_{\bullet, \bullet} \to |X^\kappa_{\epsilon, \bullet}|
\]

starting from this map so that \( \mathcal{F}(1,-) \) factors through the continuous injection \( |X^\epsilon_{\bullet, \bullet}| \to |X^\kappa_{\epsilon, \bullet}| \). Furthermore, there is an inclusion

\[
|D_{\theta,L}^k(\mathbb{R}^N)|_{\bullet, \bullet} \subseteq |D_{\theta,L}^k(\mathbb{R}^N)|_{\bullet, 0} \subseteq |D_{\theta,L}^k(\mathbb{R}^N)|_{\bullet, \bullet}
\]

as manifolds equipped with no surgery data, and \( \mathcal{F} \) will be constant on this subspace. The existence of a homotopy with these properties establishes Theorem 3.3 as follows: there is a diagram

\[
\begin{array}{ccc}
|D_{\theta,L}^k(\mathbb{R}^N)|_{\bullet, \bullet} & \to & |X^\kappa_{\epsilon, \bullet}|
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \\
|D_{\theta,L}^k(\mathbb{R}^N)|_{\bullet, \bullet} & \to & |X^\kappa_{\epsilon, \bullet}|
\end{array}
\]

where the square commutes, the horizontal maps are weak homotopy equivalences, the top triangle commutes exactly and the bottom triangle commutes up to the homotopy \( \mathcal{F} \). Taking homotopy groups we see that the vertical maps are also weak equivalences. Under the equivalence \( BC_{\theta,L}(\mathbb{R}^N) \simeq |X^\kappa_{\epsilon, \bullet}| \), and similarly for \((\kappa - 1)\), we obtain Theorem 3.3.

To define the surgery map \( \mathcal{F} \) we will give a collection of maps

\[
\mathcal{F}_{p,q} : [0,1] \times D_{\theta,L}^k(\mathbb{R}^N)|_{p,q} \times \Delta^d \to X^\kappa_{p-1}
\]

compatible on their faces. The construction of the last section gives a 1-parameter family

\[
\mathcal{K}^\epsilon : D_{\theta,L}^k(\mathbb{R}^N)|_{p,0} \to X^\kappa_{p-1}
\]

\[
(a, \epsilon, W, e) \mapsto (a, \epsilon, \mathcal{K}^\epsilon(W)),
\]
for \( r \in [0, 1] \), such that \( K^1 \) lands in \( X_p^r \). When \( q = 0 \), we set
\[
\mathcal{F}_{p,0}(r, (a, \varepsilon, W, e)) = (a, \varepsilon, K_p^r(W)) \in X_p^0,
\]
More generally, for \( q \geq 0 \) we have \( e = \{e_{i,j}\} \), and for each \( j \) we get an element
\[
(a, \varepsilon, W, e_{i,j}) \in D_{\theta,L}^p(\mathbb{R}^N)_{p,0}.
\]
We then set
\[
\mathcal{F}_{p,q}(r, (a, \varepsilon, W, e), s) = (a, \varepsilon, K_{\varepsilon,s,r}^q \circ \cdots \circ K_{\varepsilon,s,a}^q(W)),
\]
where \( \tilde{s}_j = s_j / \max(s_k) \). Note that some \( \tilde{s}_j \) is always equal to 1, so when \( r = 1 \), some \( K_{\varepsilon,s,a}^q \) is applied to \( W \) making each morphism \( \kappa \)-connected relative to its outgoing boundary. The remaining \( K_{\varepsilon,s,a}^q \) do not change this property, by Lemma 3.7, and so the map \( \mathcal{F}_{p,q}(1, \cdot, \cdot) \) factors through the subspace \( X_p^\kappa \).

The resulting map from \( \Pi_q[0, 1] \times D_{\theta,L}^p(\mathbb{R}^N)_{p,q} \times \Delta^q \) factors through a map
\[
\mathcal{F}_p : [0, 1] \times D_{\theta,L}^p(\mathbb{R}^N)_{p,\cdot} \to X_p^\kappa,
\]
which together form a map of semi-simplicial spaces with geometric realisation \( \mathcal{F} : [0, 1] \times D_{\theta,L}^p(\mathbb{R}^N)_{\cdot,\cdot} \to [X_p^\kappa]^1 \). On the subspace \( D_{\theta,L}^p(\mathbb{R}^N)_{\cdot,\cdot} \to D_{\theta,L}^p(\mathbb{R}^N)_{\cdot,0} \), the homotopy is constant as there is no surgery data. At \( r = 1 \) it factors through \( |X_p^\kappa| \).

4. Surgery on objects below the middle dimension

In this section we wish to study the filtration
\[
\mathcal{C}_{\theta,L}^{\kappa,i}(\mathbb{R}^N) \subset \cdots \subset \mathcal{C}_{\theta,L}^{\kappa,1}(\mathbb{R}^N) \subset \mathcal{C}_{\theta,L}^{\kappa,0}(\mathbb{R}^N) \subset \mathcal{C}_{\theta,L}^{\kappa,-1}(\mathbb{R}^N) = \mathcal{C}_{\theta,L}^{\kappa}(\mathbb{R}^N)
\]
and in particular establish the following theorem. The reader mainly interested in Theorems [11] and [12] can take \( d = 2n \), \( \kappa = n - 1 \), \( \theta = \theta^0 : BO(2n)(n) \to BO(2n) \), \( L \cong D^{2n-1} \), and \( N = \infty \) (but the proof does not simplify much in this special case).

**Theorem 4.1.** Suppose that the following conditions are satisfied
\[(i) \quad 2(l + 1) < d,
(ii) \quad l \leq \kappa,
(iii) \quad l \leq d - \kappa - 2,
(iv) \quad l + 2 + d < N,
(v) \quad \text{L admits a handle decomposition only using handles of index } < d - l - 1,
(vi) \quad \text{the map } \ell_L : L \to B \text{ is } (l + 1)-\text{connected}.
\]
Then the map
\[
\mathcal{B}_{\theta,L}^{\kappa,l}(\mathbb{R}^N) \to \mathcal{B}_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)
\]
is a weak homotopy equivalence.

The proof will be similar in spirit to that of the last section, in so far as we will define a contractible space of surgery data and describe a surgery move which compresses \( \mathcal{B}_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N) \) into the subspace \( \mathcal{B}_{\theta,L}^{\kappa,l}(\mathbb{R}^N) \). In the same way that the surgery move of the last section was a refinement of that of [GMTW09], the surgery move we use in this and the next section is a refinement of that of [GRW10]. Let us first give an informal account of this move, and for simplicity suppose that \( N = \infty \), that we have no tangential structure (i.e. we consider \( \theta = \text{id} : BO(d) \to BO(d) \)), that \( L = \varnothing \), and that \( d > 2 \), \( l = 0 \) and \( \kappa = 0 \). We first apply the equivalence (2.1) to reduce the problem to studying the map
\[
BD^{0,0} \to BD^{0,-1}
\]
of classifying spaces of posets. Let
\[
\sigma = (\ell_0, \ell_1; a_0, a_1; \varepsilon_0, \varepsilon_1; W) \in BD^{0,-1}
\]
be a point on a 1-simplex (for example), and let us suppose that $W|_{a_1}$ is already connected (so $\pi_0(W|_{a_1})$ injects into $\pi_0(BO(d))$). We will describe a way of producing a path from the image of this point in $|X_0^{0, -1}|$ into the subspace $|X_0^{0, 0}|$.

If $W|_{a_1}$ is already connected, then the point $\sigma$ already lies in $|X_0^{0, 0}|$ and there is nothing to prove. Otherwise, let us choose disjoint embeddings

$$\{ f_\alpha : S^0 \hookrightarrow W|_{a_0} \}_{\alpha \in \Lambda}$$

such that if we perform 0-surgery along all of these embeddings, the resulting $(d-1)$-manifold is connected. As $\kappa = 0$, the cobordism $W|_{[a_0, a_1]}$ is path connected relative to its top, and so we can extend the $f_\alpha$ to smooth maps

$$\hat{f}_\alpha : (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times S^0 \longrightarrow W$$

such that the standard height function (i.e. the projection to $(a_0 - \varepsilon_0, a_1 + \varepsilon_1)$) and $x_1 \circ \hat{f}_\alpha$ agree inside $(x_1 \circ \hat{f}_\alpha)^{-1}(\cup (a_i - \varepsilon_i, a_i + \varepsilon_i))$. As we have supposed that $d > 2$, we may assume that these $\hat{f}_\alpha$ are mutually disjoint embeddings. By taking a tubular neighbourhood, we extend the $\hat{f}_\alpha$ to embeddings

$$\hat{e}_\alpha : (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times \mathbb{R}^{d-1} \times S^0 \hookrightarrow W$$

which are still mutually disjoint, and extend this further to disjoint embeddings

$$e_\alpha : (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times \mathbb{R}^{d-1} \times D^1 \hookrightarrow \mathbb{R} \times \mathbb{R}^\infty$$

such that $e_\alpha^{-1}(W) = (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times \mathbb{R}^{d-1} \times S^0$. It is clear that we can arrange the same relationship between the standard height function on $(a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times \mathbb{R}^{d-1} \times D^1$ and the function $x_1 \circ e_\alpha$ as we have over $(a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times \mathbb{R}^{d-1} \times S^0$.

The surgery move is then given by gluing the trace of a 0-surgery on $\mathbb{R}^{d-1} \times S^0$ inside of $(a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times \mathbb{R}^{d-1} \times D^1$, into $\mathbb{R} \times \mathbb{R}^\infty$ using each of the embeddings $e_\alpha$, as shown in Figure 4. This does not define a path in $BD^{0, -1}$, as $(a_1 - \varepsilon_1, a_1 + \varepsilon_1)$ will contain a critical value at some points during the path. However, it does define a path in $|X_0^{0, -1}|$. Furthermore, if we let $\overline{W}$ be the manifold obtained at the end of the path, then $\overline{W}|_{a_0}$ is obtained from $W|_{a_0}$ by doing 0-surgery along the data $\{ \hat{e}_\alpha|_{(a_0) \times \mathbb{R}^{d-1} \times S^0} \}_{\alpha \in \Lambda}$ and so is connected. Also, $\overline{W}|_{a_1}$ is obtained from $W|_{a_1}$ by doing 0-surgery along the data $\{ \hat{e}_\alpha|_{(a_1) \times \mathbb{R}^{d-1} \times S^0} \}_{\alpha \in \Lambda}$, and as it was connected to start with (and $d > 2$), it remains connected. Hence $(t_0, t_1; a_0, a_1; \varepsilon_0, \varepsilon_1; \overline{W}) \in |X_0^{0, 0}|$, as required.

This surgery move generalises well to $l > 0$, to finite (but large enough) $N$, and to non-empty $L$, but to make it work with general tangential structures $\theta$ we must equip the surgery data $\{ e_\alpha \}_{\alpha \in \Lambda}$ with extra data describing how to induce a

![Figure 4. The surgery move for surgery on objects below the middle dimension.](image-url)
\(\theta\)-structure on the surgered manifold. We will first give a definition of \(\theta\)-surgery, then describe the standard family, and finally go on to describe the semi-simplicial space of surgery data analogous to that of Section 4.1.

4.1. \(\theta\)-surgery. Consider a \(\theta\)-manifold \((M, \ell_M)\) and an embedding \(e : \mathbb{R}^{d-l-1} \times S^l \hookrightarrow M\), and let \(C\) be the \(d\)-dimensional cobordism obtained as the trace of the surgery along \(e\). Thus \(\partial_{in}C = M\) and \(\partial_{out}C = \overline{M}\) is the result of the surgery.

The data of a \(\theta\)-surgery on \(M\) is an embedding \(e\) as above along with a \(\theta\)-structure \(\ell\) on \(C\) which agrees with \(\ell_M\) on \(M\). This induces a \(\theta\)-structure on \(\overline{M}\).

We will typically give the data of a \(\theta\)-surgery extending an embedding \(e\) by giving an extension of the \(\theta\)-structure \(\ell_M \circ De\) on \(\mathbb{R}^{d-l-1} \times S^l\) to \(\mathbb{R}^{d-l-1} \times D^{l+1}\). Up to homotopy, this is the same as specifying a null-homotopy of the map \(S^l \to B\) underlying the \(\theta\)-structure on \(\mathbb{R}^{d-l-1} \times S^l\).

4.2. The standard family. Let us construct the one-parameter family of manifolds depicted in Figure 2. Choose a function \(\rho : \mathbb{R} \to \mathbb{R}\) which is the identity on \((-\infty, 1/2)\), has nowhere negative derivative, and has \(\rho(t) = 1\) for all \(t \geq 1\). We define

\[K = \{(x, y) \in \mathbb{R}^{d-l} \times \mathbb{R}^{l+1} | |y|^2 = \rho(|x|^2 - 1)\},\]

a smooth \(d\)-dimensional submanifold, contained in \(\mathbb{R}^{d-l} \times D^{l+1}\), which outside of the set \(B^{d-l}(0) \times D^{l+1}\) is identically equal to \(\mathbb{R}^{d-l} \times S^l\). Let

\[h = x_1 : K \to \mathbb{R}\]
denote the first of the \(x\) coordinates, which is the height function we will consider on \(K\). This function is Morse with precisely two critical points: \((-1, 0, \ldots, 0)\) of index \(l+1\) and \((1, 0, \ldots, 0)\) of index \(d-l-1\).

We now define a 1-parameter family of \(d\)-dimensional submanifolds \(\mathcal{P}_t\) inside \((-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1}\) in the following way. Pick a smooth one-parameter family of embeddings \(\lambda_s : (-6, -2) \to (-6, 0)\), such that \(\lambda_0 = \text{Id}\), that \(\lambda_s|_{(-6, -4)} = \text{id}\) for all \(s\), and that \(\lambda_1(-3) = -1\). Then we get an embedding \(\lambda_t \times \text{Id}_{\mathbb{R}^d} : (-6, -2) \times \mathbb{R}^d \to (-6, 0) \times \mathbb{R}^d\) and define

\[\mathcal{P}_t = (\lambda_t \times \text{Id}_{\mathbb{R}^d})^{-1}(K) \in \Psi_d((-6, -2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1}).\]

It is easy to verify that \(\mathcal{P}_t\) agrees with \((-6, -2) \times \mathbb{R}^{d-l-1} \times S^l\) outside \((-4, -2) \times B^{d-l}(0) \times D^{l+1}\), independently of \(t\).

We shall also need a tangentially structured version of this construction, given a structure \(\ell : TK|_{(-6, 0)} \to \theta^*\gamma\). For this purpose, let \(\omega : [0, \infty) \to [0, 1]\) be a smooth function such that \(\omega(r) = 0\) for \(r \geq 2\) and \(\omega(r) = 1\) for \(r \leq \sqrt{2}\). We define a 1-parameter family of embeddings by

\[\psi_t : (-6, -2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1} \to (-6, 0) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1}\]

\[(s, x, y) \mapsto (\lambda_{t \omega(|x|)}(s), x, y),\]

It is easy to see that we also have \(\psi_t^{-1}(K) = (\lambda_t \times \text{Id}_{\mathbb{R}^d})^{-1}(K) = \mathcal{P}_t\), and we define a \(\theta\)-structure on \(\mathcal{P}_t\) by pullback along \(\psi_t\). This gives a family

\[\mathcal{P}_t(t) \in \Psi_d((-6, -2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1}),\]

and we record some important properties in the following proposition. We will omit \(\ell\) from the notation when it is unimportant.

**Proposition 4.2.** The elements \(\mathcal{P}_t(t) \in \Psi_d((-6, -2) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1})\) are \(\theta\)-submanifolds of \((-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1}\) satisfying

(i) \(\mathcal{P}_0(t) = K|_{(-6, -2)} = (-6, -2) \times \mathbb{R}^{d-l-1} \times S^l\) as \(\theta\)-manifolds.

(ii) For all \(t\), \(\mathcal{P}_t(t)\) agrees with \(K|_{(-6, -2)}\) as \(\theta\)-manifolds outside of \((-4, -2) \times B^{d-l}(0) \times D^{l+1}\).
For the embedding obtained by restricting \( e \) is an embedding, and \( \ell \) sending the embedding.

We will describe how to construct this map. In the following section we will describe the semi-simplicial space of surgery data up to a homotopy of bundle maps which is constant outside \((-4, -2) \times B_2^{d-1-1}(0) \times D^{l+1}\).

Proof. (iii) and (iv) follow easily from the properties of \( \lambda_t \) and \( \ell \), and the fact that \( K \) agrees with \( \mathbb{R}^{d-l} \times S^l \) outside \( B_2^{d-l} \times \mathbb{R}^{l+1} \). It follows from the properties of \( \omega \) that the \( \theta \)-structures agree outside \( B_2^{d-l} \times \mathbb{R}^{l+1} \). For \( \Omega \), the Morse function \( \mathcal{P} \) \( \rightarrow \) \((-6, -2) \) has at most one critical point, and that has index \( l + 1 \). If the critical value is in \((a, b)\), then the pair is \((d-l-2)\)-connected, otherwise \( \mathcal{P}_{\mid a,b} \) deformation retracts to \( \mathcal{P} \mid b \). The fact that the Morse function has at most one critical point, of index \( l + 1 \), also implies (iv) by definition of surgery (and \( \theta \)-surgery, cf. Section 4.1). Finally, the property that \( \lambda_1(-3) = -1 \) and \( \lambda_1(-4) = -4 \) implies that \( h : \mathcal{P} \rightarrow \) \((-6, -2) \) does have a critical point of index \( l + 1 \), with critical value \(-3\), which proves (iv).

4.3. Surgery data. We can now describe the semi-simplicial space of surgery data out of which we will construct a “perform surgery” map. In the following section we will describe how to construct this map.

Before doing so, we choose once and for all, smoothly in the data \((a_t, \varepsilon_t, a_p, \varepsilon_p)\), increasing diffeomorphisms

\[
\varphi = \varphi(a_t, \varepsilon_t, a_p, \varepsilon_p) : (-6, -2) \cong (a_t - \varepsilon_t, a_p + \varepsilon_p)
\]

sending \(-3\) to \(a_t - \frac{1}{3}\varepsilon_t\) and \(-4\) to \(a_t - \frac{4}{3}\varepsilon_t\).

Definition 4.3. Let \( x = (a, \varepsilon, (W, \ell_W)) \in D_{\theta,\delta,\ell}^e(\mathbb{R}^N)_p \), and write \( M_t = W \mid a_t \). Define the set \( Y_p(x) \) to consist of tuples \((\Lambda, \delta, e, \ell)\), where \( \Lambda \subset \Omega \) is a finite set, \( \delta : \Lambda \rightarrow |p| \times |q| \) is a function,

\[
e : \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}
\]

is an embedding, and \( \ell : T(\Lambda \times K \mid (-6, 0)) \rightarrow \theta^*\gamma \) is a bundle map, satisfying the conditions below. We shall write \( \Lambda_{i,j} \cong \delta^{-1}(i, j) \) and

\[
e_{i,j} : \Lambda_{i,j} \times (a_i - \varepsilon_i, a_p + \varepsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}
\]

for the embedding obtained by restricting \( e \) and reparametrising using (4.1).

(i) \( e^{-1}(W) = \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times S^l \). We let

\[
\partial e : \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times S^l \hookrightarrow W
\]

denote the embedding restricted to the boundary.

(ii) For any \( t = 0, \ldots, p \) and \( t \in (a_t - \varepsilon_t, a_t + \varepsilon_t) \), we have \((x_1 \circ e_{i,j})^{-1}(t) = \Lambda_{i,j} \times \{t\} \times \mathbb{R}^{d-l-1} \times D^{l+1} \).
(iii) The composition \( \ell_W \circ D \partial e : T(\Lambda \times K|_{(-6,-2)}) \to \theta^* \gamma \) agrees with the restriction of \( \ell \).

If we let \( \ell_{i,j} \) denote the restriction of \( \ell \) to \( T(A_{i,j} \times K|_{(-6,0)}) \), the data \((e_{i,j}, \ell_{i,j})\) is enough to perform \( \theta \)-surgery on \( M_i \) (as \( K|_{(-6,0)} \) is the trace of an \( l \)-surgery), and we further insist that

(iv) For each \( j = 0, \ldots, q \) and \( i = 0, \ldots, p \), the resulting \( \theta_{q+1} \)-manifold \( \overline{M}_i \) has the property that \( \pi_k(\overline{M}_i) \to \pi_k(B) \) is injective for \( k \leq l \).

For each \( x \), \( Y_\bullet(x) \) is a semi-simplicial set in the same way as in Definition 4.2.

Note that the set \( Y_q(x) \) consists of those \((q+1)\)-tuples of elements of \( Y_0(x) \) which are disjoint.

**Definition 4.4.** We define a bi-semi-simplicial space \( D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{\bullet,\bullet} \) (augmented in the second semi-simplicial direction) as a set by

\[
D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{p,q} = \{(x,y) \mid x \in D_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_p, y \in Y_q(x)\},
\]

and topologise it as a subspace of

\[
D_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_p \times \left( \prod_{A \subset \Omega} C^\infty(A \times V, \mathbb{R}^{N+1}) \times \text{Bun}(T(\Lambda \times K|_{(-6,0)}), \theta^* \gamma) \right)^{(p+1)(q+1)}
\]

where \( V \) denotes the manifold \((-6,-2) \times \mathbb{R}^{d-l-1} \times D^{l+1}) \). Explicitly, the face map \( d_q \) in the \( q \) direction forgets the surgery data \((e_{i,j}, \ell_{i,j})\) with \( j = k \), and the face map \( d_p \) in the \( p \) direction forgets both the surgery data \((e_{i,j}, \ell_{i,j})\) with \( i = k \) and the \( q \)th regular value.

The main result about this bi-semi-simplicial space of manifolds equipped with surgery data is the following, whose proof we defer until Section 6.

**Theorem 4.5.** Under the assumptions of Theorem 4.1, the maps

\[
|D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{\bullet,0}| \rightarrow |D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{\bullet,1}| \rightarrow |D_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_{\bullet,0}|
\]

are weak homotopy equivalences, where first map is the inclusion of \( 0 \)-simplices and the second is the augmentation, in the second simplicial direction.

In fact, we shall prove this theorem assuming the conditions of Theorem 3.1 (except (iii)). That condition will be used in the proof of Lemma 4.6.

4.4. **Proof of Theorem 4.1** We now go on to prove Theorem 4.1 so suppose that the inequalities in the statement of that theorem are satisfied: \( 2(l+1) < d, l \leq \kappa, l \leq d - \kappa - 2, l + 2 + d < N, L \) admits a handle decomposition using only handles of index \( < d - l - 1 \), and and the map \( \ell_L : L \to B \) is \((l+1)\)-connected.

Let \((a, \epsilon, (W, \ell_W), e, \ell) \in D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{p,0}\). For each \( i = 0, \ldots, p \), we have an embedding \( e_i = e_{i,0} \) and a bundle map \( \ell_i = \ell_{i,0} \), from which we shall construct a 1-parameter family of elements \( K_{e_i,\ell_i}(W, \ell_W) \in \Psi_{\theta}((a_0 - \epsilon_0, a_p + \epsilon_p) \times \mathbb{R}^N), t \in [0, 1] \) as follows. Changing the first coordinate of the manifolds \( \mathcal{P}_t(\ell_i) \) by composing with the reparametrisation functions of \( \ell_i \), we get a family of manifolds

\[
\overline{\mathcal{P}}_t(\ell_i) \in \Psi_{\theta}((a_i - \epsilon_i, a_p + \epsilon_p) \times \mathbb{R}^{d-l-1} \times \mathbb{R}^{l+1})
\]

having all the properties of Proposition 4.2, where property (ii) now holds for all regular values in \((a_i - \frac{1}{2} \epsilon_i, a_p + \epsilon_p)\). Then for \( t \in [0, 1] \), let

\[
K_{e_i,\ell_i}(W, \ell_W) \in \Psi_{\theta}((a_0 - \epsilon_0, a_p + \epsilon_p) \times \mathbb{R}^N)
\]

be equal to \( W|_{(a_0 - \epsilon_0, a_p + \epsilon_p)} \) outside the image of \( e_{i,0} \), and on \( e_i(A_i \times (a_i - \epsilon_i, a_p + \epsilon_p) \times \mathbb{R}^{d-l-1} \times D^{l+1}) \) be given by \( e_i(A_i \times \overline{\mathcal{P}}_t(\ell_i)) \). This gives a \( \theta \)-manifold, because
\[ \Lambda_i \times \overline{\cal P}_i(\ell_i) \text{ and } \Lambda_i \times \overline{\cal P}_0(\ell_i) \text{ agree as } \theta\text{-manifolds outside of } (a_i - \frac{3}{2}\varepsilon_i, a_p + \varepsilon_p) \times B_2^{d-l-1}(0) \times D^1. \]

As the embeddings \( e_i \) are all disjoint, this procedure can be iterated, and for a tuple \( t = (t_0, \ldots, t_p) \in [0, 1]^{p+1} \) we let
\[
K^t_{e,t}(W, \ell_W) = K^{t_0}_{e_0,t_0}(W, \ell_W) \circ \cdots \circ K^{t_p}_{e_p,t_p}(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N).
\]

**Lemma 4.6.** Firstly, the tuple \( (a, \frac{1}{2}\varepsilon, K^t_{e,t}(W, \ell_W)) \) is an element of \( X^\kappa_{d-l-1} \). Secondly, if \( t_i = 1 \) — so the surgery for the regular value \( a_i \) is fully done — then for any regular value \( b \) of \( x_1 : K^t_{e,t}(W, \ell_W) \to \mathbb{R} \) in the interval \( (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i) \) we have that
\[
\pi_j(K^t_{e,t}(W, \ell_W)|_b) \to \pi_j(B)
\]
is injective for \( j \leq l \).

**Proof.** For the first part we must verify the conditions of Definition 2.18. Conditions (i)–(iii) are immediate from the properties of \( \Lambda_i \). As the embeddings \( e_i \) are all disjoint, this procedure can be iterated, and for a tuple \( t = (t_0, \ldots, t_p) \in [0, 1]^{p+1} \) we let
\[
K^t_{e,t}(W, \ell_W) = K^{t_0}_{e_0,t_0}(W, \ell_W) \circ \cdots \circ K^{t_p}_{e_p,t_p}(W, \ell_W) \in \Psi_\theta((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N).
\]

For condition (iv), let \( b \in (a_i - \frac{1}{2}\varepsilon_i, a_i + \frac{1}{2}\varepsilon_i) \) be a regular value of the height function on \( K^t_{e,t}(W, \ell_W) \), and define the \( \theta_{d-1} \)-manifolds
\[
\overline{M} = K^t_{e,t}(W, \ell_W)|_b
\]
\[
M = W|_b.
\]

By Proposition 4.12 (iv), the \( \theta_{d-1} \)-manifold \( \overline{M} \) is obtained from \( M \) by performing \( \theta \)-surgery. Let \( C : M \sim \overline{M} \) be the \( \theta \)-cobordism given by the trace of these surgeries. We have the commutative diagram
\[
\pi_j(M) \xrightarrow{i} \pi_j(C) \xleftarrow{\pi_j} \pi_j(\overline{M})
\]
and \( C \) is obtained by attaching \( (l+1) \)-cells to \( M \) or by attaching \( (d-l-1) \)-cells to \( \overline{M} \). Hence \( i \) is surjective for \( j \leq l \) and \( \pi \) is bijective for \( j \leq d-l-3 \). Since our assumption \( 2l+1 < d \) implies \( l \leq d-l-3 \), as long as \( j \leq l \), the right hand diagonal map is injective whenever the left hand one is, and in particular for all \( j \leq l-1 \).

We now prove the second part, so suppose \( t_i = 1 \). We construct the manifold \( K^t_{e,t}(W, \ell_W) \) by first taking \( K^1_{e_1,t_1}(W, \ell_W) \) and then performing the remaining surgeries to it. Let \( \overline{M} = K^1_{e_1,t_1}(W, \ell_W)|_b \), so that \( \overline{M} \) is obtained from \( M \) by \( l \)-surgery.

We first show that \( \pi_j(M) \to \pi_j(B) \) is injective for \( j \leq l \). By property (viii) of the complex of surgery data, \( (e_i, \ell_i) \) is enough surgery data on \( M = W|_b \) to make the map on \( \pi_0 \) be injective after performing it. By property (vi) of the standard family, as \( b > a_i - \frac{1}{2}\varepsilon_i \) the manifold \( \overline{M} \) has all of this surgery done, and so \( \pi_j(M) \to \pi_j(B) \) is injective for \( j \leq l \).
By the previous argument, with $M$ replaced with $\tilde{M}$ in (1.2), the remaining surgeries do not change this injectivity property. \hfill \square

In the composition
\[ |D_{\theta,L}^{r,i}(\mathbb{R}^N)\cdot| \longrightarrow |D_{\theta,L}^{r,i-1}(\mathbb{R}^N)\cdot| \longrightarrow |X_\kappa^{r,i-1}| \]
both maps are homotopy equivalences by Theorem 4.5 and Proposition 2.20 respectively. There is also an inclusion
\[ |D_{\theta,L}^{r,i}(\mathbb{R}^N)\cdot| \longrightarrow |D_{\theta,L}(\mathbb{R}^N)\cdot,0| \longrightarrow |D_{\theta,L}(\mathbb{R}^N)\cdot\cdot| \]
as the subspace of manifolds equipped with no surgery data, and the second map is a weak homotopy equivalence by Theorem 4.6.

We define a map
\[ \mathcal{F}_p : [0,1]^{p+1} \times D_{\theta,L}^{r,i}(\mathbb{R}^N)_{p,0} \longrightarrow X_\kappa^{r,i-1} \]
\[ (t,(a,\varepsilon,(W,\ell_W),e,\ell)) \longmapsto (a,\frac{1}{2}\varepsilon,\kappa,e,\ell(W,\ell_W)), \]
which has the desired range by the first part of Lemma 4.6 and furthermore sends $(1,\ldots,1) \times D_{\theta,L}^{r,i}(\mathbb{R}^N)_{p,0}$ into $X_\kappa^{r,i-1}$. On the boundary of the cube this map has further distinguished properties: one is given by the second part of Lemma 4.6.

The second is that, by Proposition 4.2 (i), we have an equality $K_{\varepsilon,\ell}(W') = W'$ of $\theta$-submanifolds of $(a_0 - \varepsilon_0,a_p + \varepsilon_p) \times \mathbb{R}^N$. Thus we obtain the formula
\[ (4.3) \]
\[ d_i\mathcal{F}_p(d^i t,x) = \mathcal{F}_{p-1}(t,d_i x) \]
where $d^i : [0,1]^p \rightarrow [0,1]^{p+1}$ adds a zero in the $i$th position, and the $d_i$ are the face maps of the semi-simplicial spaces $D_{\theta,L}^{r,i}(\mathbb{R}^N)\cdot\cdot$ and $X_\kappa^{r,i-1}$.

We wish to assemble the maps $\mathcal{F}_p$ to a homotopy $\mathcal{F} : [0,1] \times |D_{\theta,L}^{r,i}(\mathbb{R}^N)\cdot,0| \rightarrow |X_\kappa^{r,i-1}|$. Hence we define $\lambda,\psi : \Delta^p \rightarrow [0,1]^{p+1}$ by the formulae
\[ \lambda_i(t) = \min (1,2\bar{t}_i) \]
\[ \psi_i(t) = \max (0,2\bar{t}_i - 1), \]
where again $\bar{t}_i = t_i / \max(t_j)$, and a map $H : [0,1] \times \Delta^p \rightarrow [0,1]^{p+1} \times \Delta^p$ by
\[ H(s,t) = \left( s \cdot \lambda(t) \cdot \sum_j \psi_j(t) \right). \]
These may be used to form the composition
\[ F_p : [0,1] \times D_{\theta,L}^{r,i}(\mathbb{R}^N)_{p,0} \times \Delta^p \overset{H}{\longrightarrow} D_{\theta,L}^{r,i}(\mathbb{R}^N)_{p,0} \times [0,1]^{p+1} \times \Delta^p \overset{\mathcal{F}_p \times \Delta}{\longrightarrow} X_\kappa^{r,i-1} \times \Delta^p. \]

**Lemma 4.7.** These maps glue to a homotopy $\mathcal{F} : [0,1] \times |D_{\theta,L}^{r,i}(\mathbb{R}^N)\cdot,0| \rightarrow |X_\kappa^{r,i-1}|$.

*Proof.* The points $F_p(s,x,d^i t)$ and $F_{p-1}(s,d_i x,t)$ are identified under the usual face maps among the $X_\kappa^{r,i-1} \times \Delta^p$. This follows immediately from the formula (4.3) and the observation that $\lambda(d^i t) = d^i(\lambda(t))$, $\psi(d^i t) = d^i(\psi(t))$ and $\sum_j \psi_j(d^i t) = \sum_j \psi_j(t)$. \hfill \square

By construction, the map $\mathcal{F}(1,-) : |D_{\theta,L}^{r,i}(\mathbb{R}^N)\cdot,0| \rightarrow |X_\kappa^{r,i-1}|$ factors through the continuous injection $|X_\kappa^{r,i}| \rightarrow |X_\kappa^{r,i-1}|$. This may be seen at the level of the maps $F_p$, since the domain of $F_p$ is covered by the $2^{p+1}$ closed sets obtained by requiring for each $i$ either $\lambda_i(t) = 1$ or $\psi_i(t) = 0$, on each of which the map $F_p(i,-)$ composed with $X_\kappa^{r,i-1} \times \Delta^p \rightarrow |X_\kappa^{r,i-1}|$ factors through $|X_\kappa^{r,i}|$: If $\lambda_i(t) = 1$, the surgery near the regular value $a_i$ is completely done (and so by the second part of Lemma 1.6 it does not matter what the remaining surgeries do near the regular
value \(a_1\); if not, then \(\psi_1(t) = 0\) and by the face identifications, we can forget the regular value \(a_1\).

The homotopy \(\mathcal{F}\) is constant on the subspace \(|D_{\theta,L}^e,\ell|(\mathbb{R}^N)|_•| → |D_{\theta,L}^e,\ell|(\mathbb{R}^N)|_•,0|\), and by the argument in Section 3.3 we deduce the weak equivalence in Theorem 4.1.

**Remark 4.8.** It is possible to weaken condition (iii) of Theorem 4.1 to the map \(\ell_L : L → B\) being \(l\)-connected, in which case the method of Section 5 below can be used to prove that the inclusion gives a weak equivalence from \(BC_{\theta,L}^{n-1}\) to a union of path components of \(BC_{\theta,L}^{n-1,1-1}(\mathbb{R}^N)\).

### 5. Surgery on objects in the middle dimension

We now restrict our attention to even dimensions, and write \(d = 2n\). Given a collection of path components \(\mathcal{A} \subset \pi_0(\text{Ob}(C_{n-1,n-2}(\mathbb{R}^N)))\), in Definition 2.11 we defined

\[ C_{\theta,L}^{n-1,0}(\mathbb{R}^N) \subset C_{\theta,L}^{n-1,n-2}(\mathbb{R}^N) \]

to be the full subcategory on this collection of objects. To state our main theorem concerning these subcategories, we first need a definition.

**Definition 5.1.** We say a tangential structure \(\theta\) is **reversible** if whenever there is a morphism \(M : M → N\) in \(C_{\theta,L}\), there also exists a morphism \(N → M\) in this category, whose underlying manifold is the reflection of \(C\).

In Proposition 5.6 we prove that this property is equivalent to \(\theta\) being **spherical**, as defined in Section 1 (i.e. the \(2n\)-sphere admits a \(\theta\)-structure extending any given structure on a disc). We can now state our main theorem concerning these subcategories, analogous to Theorem 1.1 but in the middle dimension. The reader mainly interested in Theorems 1.1 and 1.2 can take \(\theta = \theta^n : BO(2n)/(n) → BO(2n)\), \(L \cong D^{2n-1}\), \(N = \infty\), and \(\mathcal{A}\) the class of objects which are either diffeomorphic to \(S^{2n-1}\) with its standard smooth structure and \(\theta\)-structure or are not \(\theta\)-bordant to \(S^{2n-1}\). (Again, the proof does not simplify much in this special case).

**Theorem 5.2.** Suppose that

(i) \(2n \geq 6\),
(ii) \(3n + 1 < N\),
(iii) \(\theta\) is reversible,
(iv) \(L\) admits a handle decomposition only using handles of index less than \(n\),
(v) the map \(\ell_L : L → B\) is \((n-1)\)-connected,
(vi) the natural map \(\mathcal{A} → \pi_0(BC_{\theta,L}^{n-1,n-2}(\mathbb{R}^N))\) is surjective.

Then

\[ BC_{\theta,L}^{n-1,0}(\mathbb{R}^N) \rightarrow BC_{\theta,L}^{n-1,n-2}(\mathbb{R}^N) \]

is a weak homotopy equivalence.

The surgery move that we will employ is similar to that of the last section, but has a crucial difference. In the last section, when we performed the surgery move to make \(a_1\) a good regular value, we glued a family of manifolds having the effect of performing \(l\)-surgery on the level sets \(W|_{a_1}\), but at the same time performing \(l\)-surgery on all higher level sets. In Section 3.1 \(l = (d-2)/2 = n-1\), and therefore performing \(l\)-surgery on a \((2n-1)\)-manifold which is \(l\)-connected preserves its \(l\)-connectedness. In this section, we will need to change level sets by doing \((n-1)\)-surgery on \((2n-1)\)-manifolds, and this is much more delicate. For example, any 1-manifold can be made connected by performing 0-surgeries, but performing further 0-surgeries will disconnect it again.

Instead we use a modified surgery move, which will let us perform \((n-1)\)-surgery on a level set \(W|_{a}\) and leave all other level sets \(W|_{b}\) unchanged, except when \(b\) is
very close to $a$. For $n = 1$, this was done in [GRW10], and the construction there generalises to higher $n$. Let us briefly recall and depict the case $n = 1$. We start with the same surgery data as in Section 4, a collection of embeddings
\[
\{e_\alpha : (a_0 - \varepsilon_0, a_1 + \varepsilon_1) \times D^1 \times D^1 \hookrightarrow \mathbb{R} \times \mathbb{R}^\infty\}_{\alpha \in \Lambda},
\]
but glue in to the image of each $e_\alpha$ the path of manifolds shown in Figure 5. This defines a path in the space $|X^{0,-1}|$, and if the handle in Figure 5 which we have moved into the manifold is “thin” enough (with respect to the height function) then the manifold $W$ obtained at the end of the path has $W|_{a_0}$ and $W|_{a_1}$ both connected, and so lies in $|X^{0,0}|$.

In order to make sense of this surgery move in the presence of $\theta$-structures, we must equip the 1-parameter family of manifolds shown in Figure 5 with $\theta$-structures which start at a given structure, are constant near the vertical boundaries, and at the end of the path the level sets above and below the handle should be isomorphic as $\theta$-manifolds to the level sets before the handle was added. This last property does not hold in general: for example, if we equip the original manifold in Figure 5 with a framing, one may easily see (using the Poincaré–Hopf theorem) that there is no framing on the final manifold consistent with these requirements. As we will see, this problem goes away when $\theta$ is assumed to be reversible. Let us first discuss the reversibility condition in more detail.

5.1. **Reversibility.** Recall that a tangential structure $\theta : B \to BO(d)$ is called spherical if any structure on a disc $D \subset S^d$ extends to one on $S^d$. (When $B$ is path connected, this is equivalent to the $d$-sphere admitting any structure at all.) Let us first discuss some related conditions on tangential structures $\theta : B \to BO(d)$.

**Definition 5.3.** A tangential structure $\theta : B \to BO(d)$ is once-stable if there exists a map $\tilde{\theta} : \tilde{B} \to BO(d + 1)$ and a commutative diagram
\[
\begin{array}{ccc}
B & \xrightarrow{\theta} & \tilde{B} \\
\downarrow & & \downarrow \\
BO(d) & \longrightarrow & BO(d + 1)
\end{array}
\]
which is homotopy pullback.

A tangential structure $\theta$ is weakly once-stable if there exists such a diagram for which $\pi_i(BO(d), B) \to \pi_i(BO(d + 1), B)$ is surjective for $i = d + 1$ and bijective for $i \leq d$, for all basepoints.
From the commutative diagram in the definition, there is a bundle map \( \varepsilon^1 \oplus \theta^* \gamma \rightarrow \theta^* \gamma \). Hence a \( \theta \)-structure \( TW \rightarrow \theta^* \gamma \) on a \( d \)-manifold \( W \) induces a bundle map \( \varepsilon^1 \oplus TW \rightarrow \theta^* \gamma \). If \( \theta \) is weakly once-stable we may deduce the converse, that a bundle map \( \varepsilon^1 \oplus TW \rightarrow \theta^* \gamma \) is homotopic to one that arises from a \( \theta \)-structure. More precisely, we have the following useful lemma.

**Lemma 5.4.** Let \( \theta : B \rightarrow BO(d) \) be weakly once-stable. Let \( W \) be a \( d \)-manifold and let \( \ell : TW|_A \rightarrow \theta^* \gamma \) be a \( \theta \)-structure defined on a closed submanifold \( A \subset W \). Then \( \ell \) extends to a \( \theta \)-structure \( TW \rightarrow \theta^* \gamma \) if and only if the stabilised bundle map \( \varepsilon^1 + TW|_A \rightarrow \varepsilon^1 + \theta^* \gamma \) extends to a bundle map over all of \( W \).

**Proof.** Without loss of generality, we may assume that \( \theta \) and \( \tilde{\theta} \) are Serre fibrations. Let us write \( s : BO(d) \rightarrow BO(d + 1) \) for the stabilisation map, and let us pick a classifying map \( t : W \rightarrow BO(d) \) for the tangent bundle. Tangential structures on \( TW \) then correspond to lifts of \( t \) along some fibration, and tangential structures on \( \varepsilon^1 + TW \) correspond to lifts of \( s \circ t \) along some fibration.

We write \( \tilde{\theta} : \tilde{B} \rightarrow BO(d) \) for the pullback of \( \theta \), so the commutative diagram in Definition 5.3 gives a map \( i : B \rightarrow \tilde{B} \) over \( BO(d) \). A \( \theta \)-structure on \( \varepsilon^1 + TW \) is then nothing but a \( \tilde{\theta} \)-structure on \( TW \).

The long exact sequence on homotopy for the various fibrations combine to give

\[
\cdots \rightarrow \pi_i(\tilde{B}, B) \rightarrow \pi_i(BO(d), B) \rightarrow \pi_i(BO(d + 1), B) \rightarrow \pi_{i-1}(\tilde{B}, B) \rightarrow \cdots
\]

from which we deduce that \( (\tilde{B}, B) \) is \( d \)-connected. Now, the situation described in the statement is a lifting problem

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
W & \rightarrow & \tilde{B},
\end{array}
\]

which has a solution as \( (W, A) \) has cells of dimension at most \( d \), and \( (\tilde{B}, B) \) is \( d \)-connected. \( \square \)

**Lemma 5.5.** The tangential structure \( \theta : B \rightarrow BO(d) \) is weakly once-stable if and only if it is spherical.

**Proof.** Given any bundle map \( \ell : TS^d|_D \rightarrow \theta^* \gamma \) we can of course extend the stabilised map to \( \varepsilon^1 \oplus TS^d \rightarrow \varepsilon^1 + \theta^* \gamma \), and if \( \theta \) is weakly once stable, the above lemma implies that the \( \theta \) structure extends to all of \( S^d \).

Conversely, given a spherical structure \( \theta : B \rightarrow BO(d) \) we may pick \( \theta \)-structures \( \ell_i : TS^d \rightarrow \theta^* \gamma \), one for each path component of \( B \), and form \( \tilde{B} \) by attaching an \((n+1)\)-cell to \( B \) along each map. The compositions \( S^d \rightarrow B \rightarrow BO(d) \rightarrow BO(d+1) \) are null-homotopic, so we obtain an extension \( \theta : \tilde{B} \rightarrow BO(d+1) \).

It follows that \( H_i(\tilde{B}, B) \rightarrow H_i(BO(d+1), BO(d)) \) is surjective for \( i = d + 1 \) and an isomorphism for \( i \leq d \), even with local coefficients. By the Hurewicz theorem, \( \pi_i(BO(d), B) \rightarrow \pi_i(BO(d+1), BO(d)) \) is surjective for \( i = d + 1 \) and bijective for \( i \leq d \), for all basepoints. It follows that \( \pi_i(BO(d), B) \rightarrow \pi_i(BO(d+1), B) \) is surjective for \( i = d + 1 \) and bijective for \( i \leq d \). \( \square \)

We now show that these conditions on \( \theta \) are also equivalent to reversibility.

**Proposition 5.6.** The tangential structure \( \theta \) is reversible if and only if it is spherical.

**Proof.** If \( \theta \) is reversible and a structure on \( D^d \) is given, we think of \( D^d \) as a morphism from the empty set to \( S^{d-1} \). By assumption, a compatible structure exists on the disc, thought of as a morphism from \( S^{d-1} \) to the empty set.
For the reverse direction we use Lemma 5.4. Suppose given a cobordism $C : M \leadsto N$ with $\theta$-structure $\ell : TC \to \theta^* \gamma$. Let $\overline{C} : N \leadsto M$ be the cobordism whose underlying manifold is $C$, but regarded as a morphism in the other direction. Since $TC_{|\partial C} = \varepsilon^1 \oplus T(\partial C)$, we may reflect in the $\varepsilon^1$-direction to get a reversed $\theta$-structure near $\partial \overline{C} \cong N \amalg M$, and our task is to extend the reversed structure to $C$. By the lemma, it suffices to extend the stabilised bundle map, but that is easy: Pick a non-zero section of the vector bundle $\varepsilon^1 \oplus TC$ which over $\partial C$ is the inwards pointing normal to $T(\partial C) \subset TC_{|\partial C}$, and reflect the stabilised bundle map in that field. $\Box$

One key property of reversible tangential structures is that they allow us to connect-sum $\theta$-manifolds, which of course is not possible in general: the connect-sum of framed manifolds is not typically framable. In fact, more is true. We can perform arbitrary surgeries on a $\theta$-manifold and find a $\theta$-structure on the new manifold.

**Proposition 5.7.** Let $(M, \ell_M)$ be a $d$-dimensional $\theta$-manifold, and suppose that $e : S^{n-1} \times D^{d-n+1} \hookrightarrow M$ is a piece of surgery data such that the map $S^{n-1} \to B$ induced by $e \circ \ell_M$ is null-homotopic. Then if $\theta$ is reversible, the surgered manifold

$\overline{M} = (M - \text{int}(S^{n-1} \times D^{d-n+1})) \cup_{S^{n-1} \times S^{d-n}} (D^n \times S^{d-n})$

admits a $\theta$-structure which agrees with $\ell_M$ on $(M - \text{int}(S^{n-1} \times D^{d-n+1}))$. 

**Proof.** If we let $V$ denote the trace of the surgery, then the $\theta$-structure on $M$ and a choice of null-homotopy of $e \circ \ell_M$ induces a bundle map $TV \to \varepsilon^1 \oplus \theta^* \gamma$, and by restriction a bundle map $\varepsilon^1 \oplus TV \to \varepsilon^1 \oplus \theta^* \gamma$, which we can assume agrees with the stabilisation of $\ell_M$ on $(M - \text{int}(S^{n-1} \times D^{d-n+1})) \subset \overline{M}$. But when $\theta$ is weakly once-stable, Lemma 5.4 says that this bundle map can be replaced with one induced from a $\theta$-structure. $\Box$

For tangential structures that are once-stable (not just weakly), we can say that for a $d$-manifold $W$ with a fixed $\theta$-structure $\ell_0 : TW_{|_{\partial W}} \to \theta^* \gamma$, the stabilisation map

$$\text{Bun}^0(TW, \theta^* \gamma; \ell_0) \longrightarrow \text{Bun}^0(\varepsilon^1 \oplus TW, \theta^* \gamma; \varepsilon^1 \oplus \ell_0)$$

is a weak homotopy equivalence. (Weakly once-stable only implies that this map is 0-connected.) We shall not make explicit use of the stronger condition in this paper, but point out that most of the naturally occurring tangential structures are once-stable. In particular, the following construction will be our main source of once-stable tangential structures. Let $W$ be a connected $d$-dimensional manifold with basepoint, and $\tau : W \to BO(d)$ be its Gauss map, which we may assume to be pointed. For each $k$ there are Moore–Postnikov factorisations of $\tau$

$$W \xrightarrow{\gamma_k} B_W(k) \xrightarrow{\text{proj}} BO(d)$$

where $\gamma_k$ is an isomorphism for $* < k$ and an epimorphism for $* = k$, and $\text{proj} : p_k$ is an isomorphism for $* > k$ and a monomorphism for $* = k$. These connectivity properties characterise $B_W(k)$, by obstruction theory. Then $\theta_W(k) = p_k$ is a tangential structure.

**Lemma 5.8.** The tangential structure $\theta_W(k) : B_W(k) \to BO(d)$ is once-stable for any $k \leq d$.

**Proof.** We let $B_W(k)$ denote the same Moore–Postnikov construction applied to the composition $W \to BO(d) \to BO(d + 1)$. The claim then follows as $BO(d) \to BO(d + 1)$ is $d$-connected. $\Box$
Remark 5.9. There do exist tangential structures which are reversible but not once-stable, which justifies our emphasis on reversibility. An interesting example is $BU(3) \to BO(6)$, which is reversible as $S^n$ admits an almost complex structure, but is not once-stable: if it were pulled back from a fibration $f : B \to BO(7)$, one can easily use the Serre spectral sequence to check that the kernel of the map $f^*$ on $\mathbb{F}_2$-cohomology would be the ideal $I = \langle w_1, w_3, w_5 \rangle \subset H^*(BO(7); \mathbb{F}_2)$, but this is not closed under the action of the Steenrod algebra as $Sq^5(w_5) = w_1 \cdot w_5 + w_1 \cdot w_6 + w_2 \cdot w_7 \notin I$.

5.2. The standard family. We will prove Theorem 5.2 by performing $(n - 1)$-surgery on objects until we reach an object in $A$, just as in Section 4 we performed $l$-surgery on objects to make them $l$-connected (relative to $L$). As in that section, the surgery shall be performed by gluing in a suitable family of manifolds along certain families of embeddings, whose existence we shall prove in Section 6. The standard family to be glued in is very similar to that in Section 4 where we started with a certain submanifold $K \subset \mathbb{R}^{d-1} \times \mathbb{R}^{l+1}$. In this section, $l = n - 1$, so we have a submanifold $K \subset \mathbb{R}^{n+1} \times \mathbb{R}^n$ defined as follows. We first chose a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ which is the identity on $(-\infty, \frac{1}{2})$, has nowhere negative derivative, and has $\rho(t) = 1$ for all $t \geq 1$, and we let

$$K = \{(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \mid |y|^2 = \rho(|x|^2 - 1)\}.$$  

The first coordinate restricts to a Morse function $h = x_1 : K \to \mathbb{R}$ with exactly two critical points: $(-1, 0, \ldots, 0; 0)$ and $(+1, 0, \ldots, 0; 0)$ both of index $n$.

In Section 4, we constructed from $K$ a one-parameter family of manifolds $P_t \subset (\mathbb{R}^{d-1} \times \mathbb{R}^{l+1})$, obtained from $K$ by moving the lowest critical point down as $t \in [0, 1]$ increases, as in Figure 4. In this section we shall need a two-parameter family $P_{t, w} \subset \mathbb{R} \times (\mathbb{R}^{d-1} \times \mathbb{R}^{l+1})$ which is constructed from $K$ by moving both critical points down as $t \in [0, 1]$ increases, as in Figure 5. As $w \in [0, 1]$ decreases, we shrink the width of the handle so that the distance between the two critical values is $2w$. In order for the manifold to stay embedded in the limit $w = 0$, we need an extra ambient dimension.

Let us first construct a 1-parameter family of submanifolds $K_w \subset \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^n$ such that $K_1 = \{0\} \times K$. Let $\mu : \mathbb{R} \to [0, 1]$ be a smooth function which is zero on $(2, \infty)$ and identically 1 on $(-\infty, \sqrt{2})$, and define a 1-parameter family of embeddings

$$\varphi_w : \mathbb{R}^{n+1} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^n$$

$$(x, y) \mapsto (x_1(1 - w)\mu(|x|), x_1(1 - (1 - w)\mu(|x|)), x_2, \ldots, x_{n+1}, y).$$

We now let

$$K_w = \varphi_w(K) \subset \mathbb{R} \times \mathbb{R}^{n+1} \times \mathbb{R}^n$$

for $w \in [0, 1]$. A calculation shows that (for $w > 0$) the critical points of the height function $h : K_w \to \mathbb{R}$, which is now projection to the second coordinate, are $\varphi_w(\pm 1, 0, \ldots, 0)$ and so lie at heights $\pm w$. They remain Morse of index $n$.

We now define a 2-parameter family of $l$-dimensional submanifolds $P_{t, w} \subset \mathbb{R} \times (\mathbb{R}^{d-1} \times \mathbb{R}^n \times \mathbb{R}^n)$ in much the same way as $P_t$ was constructed from $K$ in Section 4. Apart from the extra width parameter, the main difference is that in this section we will use a larger part of $K$, including both critical points. Pick a smooth one-parameter family of embeddings $\lambda_s : (\mathbb{R}^{d-1} \times \mathbb{R}^n \times \mathbb{R}^n)$ such that $\lambda_0 = \text{Id}$, that $\lambda_1|_{(-\infty, -5]} = \text{Id}$ for all $s$, and that $\lambda_1(-4) = -1$ and $\lambda_1(-3) = 1$. Then we get embeddings $\text{Id}_\mathbb{R} \times \lambda_t \times \text{Id}_\mathbb{R}^{2n} : \mathbb{R} \times (\mathbb{R}^{d-1} \times \mathbb{R}^{2n}) \to \mathbb{R} \times (\mathbb{R}^{d-1} \times \mathbb{R}^{2n})$ and define

$$P_{t, w} = (\text{Id}_\mathbb{R} \times \lambda_t \times \text{Id}_\mathbb{R}^{2n})^{-1}(K_w) \in \Psi_\mathbb{R}(\mathbb{R} \times (\mathbb{R}^{d-1} \times \mathbb{R}^{2n} \times \mathbb{R}^n)).$$
It is easy to verify that $\mathcal{P}_{t,w}$ agrees with $\{0\} \times (-6,-2) \times \mathbb{R}^n \times S^{n-1}$ outside $(-2,2) \times (-5,-2) \times B^n_\ell(0) \times D^n$, independently of $t$ and $w$.

We shall also need a tangentially structured version of this construction, given a structure $\ell : TK(-6,2) \to \theta^*\gamma$. For this purpose, let $\omega = \mu : \mathbb{R} \to [0,1]$ be the function defined above and define a 1-parameter family of embeddings by

$$\psi_t : \mathbb{R} \times (-6,-2) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \times (-6,2) \times \mathbb{R}^n \times \mathbb{R}^n$$

$$(s;x_1,\ldots,x_{n+1};y) \mapsto (s;\omega_t(x_1),x_2,\ldots,x_{n+1};y),$$

It is easy to see that we also have $\psi_t^{-1}(K_w) = (\text{Id}_\mathbb{R} \times \lambda_t \times \text{Id}_\mathbb{R}^{2n})^{-1}(K_w) = \mathcal{P}_{t,w}$, and we define a $\theta$-structure on $\mathcal{P}_{t,w}$ by pullback along $\psi_t$. This gives a two-parameter family

$$\mathcal{P}_{t,w}(\ell) \in \Psi_\theta(\mathbb{R} \times (-6,-2) \times \mathbb{R}^n \times \mathbb{R}^n).$$

Let $P : [0,1] \to [0,1]^2$ be the piecewise linear path with $P(0) = (0,0)$, $P(\frac{1}{2}) = (1,0)$ and $P(1) = (1,1)$, and define a one-parameter family

$$\mathcal{P}_t(\ell) = \mathcal{P}_{P(t)}(\ell) \in \Psi_\theta(\mathbb{R} \times (-6,-2) \times \mathbb{R}^n \times \mathbb{R}^n).$$

We will omit $t$ from the notation when it is unimportant. We record some important properties of this family in Proposition 5.11 below, using the following definition.

**Definition 5.10.** Let $\ell : TK \to \theta^*\gamma$ be a $\theta$-structure on $K$. Recall that outside of $\mathbb{R} \times B^n_\ell(0) \times D^n$ the manifold $K$ agrees with $\mathbb{R} \times \mathbb{R}^n \times S^{n-1}$. We say that $\ell$ is extendible if the $\theta$-structure $\ell|_{\mathbb{R} \times (\mathbb{R} \times B^n_\ell(0)) \times S^{n-1}}$ extends to a $\theta$-structure on the whole of $\mathbb{R} \times \mathbb{R}^n \times S^{n-1}$.

**Proposition 5.11.** Suppose $\ell$ is extendible. The elements $\mathcal{P}_t(\ell) \in \Psi_\theta(\mathbb{R} \times (-6,-2) \times \mathbb{R}^n \times \mathbb{R}^n)$ are $\theta$-submanifolds of $\mathbb{R} \times (-6,-2) \times \mathbb{R}^n \times D^n$ satisfying

(i) $\mathcal{P}_0(\ell) = K|_{(-6,-2)} = \{0\} \times (-6,-2) \times \mathbb{R}^n \times S^{n-1}$ as $\theta$-manifolds.

(ii) For all $t$, $\mathcal{P}_t(\ell)$ agrees with $K_1|_{(-6,-2)}$ as a $\theta$-manifold, outside of $(-2,2) \times (-5,-2) \times B^n_\ell(0) \times D^n$.

(iii) For all $t$ and each pair of regular values $-6 < a < b < -2$ of the height function $h : \mathcal{P}_t \to \mathbb{R}$, the pair

$$\mathcal{P}_{t|_{[a,b]}},\mathcal{P}_{t|a}$$

is $(n-1)$-connected.

(iv) Let $a$ be a regular value of $h : \mathcal{P}_t(\ell) \to (-6,-2)$. If $a$ is outside of $(-4,-3)$ then the manifold $\mathcal{P}_t(\ell)|_a$ is isomorphic to $\mathcal{P}_0(\ell)|_a = \{a\} \times \{0\} \times \mathbb{R}^n \times S^{n-1}$ as a $\theta$-manifold. If $a$ is inside of $(-4,-3)$ then the manifold $\mathcal{P}_t(\ell)|_a$ is either isomorphic to $\mathcal{P}_0(\ell)|_a$ as a $\theta$-manifold, or is obtained from it by $(n-1)$-surgery along the standard embedding.

(v) The critical values of $h : \mathcal{P}_t(\ell) \to (-6,-2)$ are $-4$ and $-3$. For $a \in (-4,-3)$, $\mathcal{P}_t(\ell)|_a$ is obtained by $(n-1)$-surgery from $\mathcal{P}_0(\ell)|_a = \{0\} \times \mathbb{R}^n \times S^{n-1}$ along the standard embedding.

In (iii) and (iv), the $\theta$-structure on the surgered manifold is determined (up to homotopy) by the $\theta$-structure $\ell$ on $K|_{(-6,-2)}$.

The precise meaning of the word isomorphic in (iv) above is the following: By (iv) we know that the manifolds are equal outside $(-2,2) \times (-5,-2) \times B^n_\ell(0) \times D^n$. Being isomorphic means that the identity extends to a diffeomorphism which preserves $\theta$-structures up to a homotopy of bundle maps which is constant outside $(-2,2) \times (-5,-2) \times B^n_\ell(0) \times D^n$.

**Proof.** (i) and (ii) follow easily from the properties of $\lambda_t$ and $\psi_t$, and the fact that $K$ agrees with $\mathbb{R}^{n+1} \times S^{n-1}$ outside $B^n_\ell(0) \times \mathbb{R}^n$. For (iii), the Morse function $\mathcal{P}_{t,w} \to (-6,-2)$ has at most two critical point, both of index $n$. If a critical value is
in \((a,b)\), then the pair is \((n-1)\)-connected, otherwise it is \(\infty\)-connected as \(\mathcal{P}_{t,w}|_{[a,b]}\) deformation retracts to \(\mathcal{P}_{t,w}|_{b}\).

To see (i) and (ii), first suppose that \(t \in [0,\frac{1}{2}]\). Then \(P(t) = (\ast,0)\) so \(\mathcal{P}_t(\ell) = \mathcal{P}_{t,0}(\ell)\). The function \(K_0 \to \mathbb{R}\) has exactly one critical value, 0, so \(K_0|_a\) is diffeomorphic to \((a) \times \mathbb{R}^n = S^{n-1}\) for all regular values \(a\), and extendibility implies that they are also isomorphic as \(\theta\)-manifolds. If instead \(t \in [\frac{1}{2},1]\) then \(P(t) = (1,\ast)\), and so \(\mathcal{P}_t(\ell) = \mathcal{P}_{1,\ast}(\ell)\). The fact that the function \(K_w \to \mathbb{R}\) has exactly two critical points with value \(\pm w\) implies that \(K_w|_a\) is diffeomorphic to \((a) \times \mathbb{R}^n = S^{n-1}\) for regular values \(a \in \mathbb{R} - (-1,1) \subset \mathbb{R} - (-w, w)\) and extendibility implies that they are also isomorphic as \(\theta\)-manifolds. When \(w = 1\), for regular values \(a \in (-1,1)\) we have that \(K_1|_a\) is obtained from \((a) \times \mathbb{R}^n = S^{n-1}\) by \((n-1)\)-surgery along the standard embedding.

\[\square\]

5.3. Surgery data. We can now describe the semi-simplicial space of surgery data in the middle dimension. It is similar to the space of surgery data below the middle dimension, but taking into account the slightly different range of definition of the standard family in this case.

Before doing so, we choose once and for all, smoothly in the data \((a_i, \epsilon_i, a_p, \epsilon_p)\) increasing diffeomorphisms

\[
\psi = \psi(a_i, \epsilon_i, a_p, \epsilon_p) : (-6, -2) \cong (a_i - \epsilon_i, a_p + \epsilon_p) \quad \text{sending} \quad [-4, -3] \quad \text{linearly onto} \quad [a_i - \frac{1}{6} \epsilon_i, a_i + \frac{1}{6} \epsilon_i].
\]

**Definition 5.12.** Let \(x = (a, \epsilon, (W, \ell_W)) \in D^{n-1,n-2}(\mathbb{R}^N)_p\), and write \(M_t = W|_a\). Define the set \(Y_q(x)\) to consist of tuples \((\Lambda, \delta, e, \ell)\), where \(\Lambda\) and \(\delta\) are as in Definition 4.3

\[
e : \Lambda \times \mathbb{R} \times (-6, -2) \times \mathbb{R}^n \times D^n \hookrightarrow \mathbb{R} \times (0,1) \times (-1,1)^{N-1}
\]

is an embedding, and \(\ell\) is a bundle map \(T(\Lambda \times K) \to \theta^* \gamma\). (In Definition 4.3, it was only defined on \(T(\Lambda \times K)|_{(-6,0)}\).) Define \(\Lambda_{i,j}, e_{i,j}\) and \(\ell_{i,j}\) in the same way as in Definition 4.3. This data is required to satisfy the following conditions.

(i) \(e^{-1}(W) = \Lambda \times \{0\} \times (-6, -2) \times \mathbb{R}^n \times \partial D^n\). We let \(\partial e : \Lambda \times \{0\} \times (-6, -2) \times \mathbb{R}^n \times \partial D^n \hookrightarrow W\)

denote the embedding restricted to the boundary.

(ii) For \(t \in \bigcup (a_i - \epsilon_i, a_i + \epsilon_i)\), we have \((x_t \circ e_{i,j})^{-1}(t) = \Lambda_{i,j} \times \mathbb{R} \times \{t\} \times \mathbb{R}^n \times D^n\).

(iii) The composition \(\ell_W \circ D(\partial e) : T(\Lambda \times K)_{(-6,-2)} \to \theta^* \gamma\) agrees with the restriction of \(\ell\).

(iv) For each \(\lambda \in \Lambda\), the restriction of \(\ell\) to \(T(\{\lambda\}) \times K\) is extendible.

For each \(j\), the data \((e_{i,j}, \ell_{i,j})\) is enough to perform \(\theta\)-surgery on \(M_t\) (as \(K|_{(-6,0)}\) is the trace of an \((n-1)\)-surgery), and we further insist that

(v) The resulting \(\theta\)-manifold \(\overline{M}_t\) lies in \(\mathcal{A}\).

For each \(x\), \(Y_q(x)\) is a semi-simplicial set.

Define a bi-semi-simplicial space \(D^{n-1,A}(\mathbb{R}^N)_{\bullet,\bullet}\) (augmented in the second semi-simplicial direction) from this, as in Definition 4.3. The main result about this bi-semi-simplicial space of manifolds equipped with surgery data is the following, whose proof we defer until Section 6.

**Theorem 5.13.** Under the assumptions of Theorem 5.2, the maps

\[
|D^{n-1,A}_{0,\ast}(\mathbb{R}^N)_{\bullet,0}| \longrightarrow |D^{n-1,A}_{0,\ast}(\mathbb{R}^N)_{\bullet,1}| \longrightarrow |D^{n-1,n-2}_{0,\ast}(\mathbb{R}^N)_{\ast,\ast}|
\]

are weak homotopy equivalences, where the first map is the inclusion of 0-simplices and the second is the augmentation in the simplicial direction.
5.4. **Proof of Theorem 5.2** The proof of this theorem will be almost identical with that of Theorem 4.1. Thus, suppose that the conditions in the statement of Theorem 5.2 are satisfied, and let \((a, \varepsilon, (W, \ell_W), e, \ell) \in D_{p,L}^{n-1, A}(\mathbb{R}^N)_{p,0}\). For each \(i = 0, \ldots, p\), we have an embedding \(e_i = e_{i,0}\) and a bundle map \(\ell_i = \ell_{i,0}\), and precisely as in Section 4.4 we may construct a one-parameter family of elements \(K_{e_i, \ell_i}(W, \ell_W) \in \Psi_b((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N)\) for \(t \in [0, 1]\). From this, for each tuple \(t = (t_0, \ldots, t_p) \in [0, 1]^{p+1}\) we may form the element
\[
K_{e, \ell}^t(W, \ell_W) = K_{e_{t_p}, \ell_{t_p}} \circ \cdots \circ K_{e_{t_0}, \ell_{t_0}}(W, \ell_W) \in \Psi_b((a_0 - \varepsilon_0, a_p + \varepsilon_p) \times \mathbb{R}^N).
\]
To apply the same proof as that of Theorem 4.1 we need an analogue of Lemma 4.6 to tell us how the manifold improves when we apply the various surgery operations.

**Lemma 5.14.** Firstly, the tuple \((a, \frac{1}{2} \varepsilon, K_{e, \ell}^1(W, \ell_W))\) is an element of \(X_{p}^{n-1,n-2}\). Secondly, if \(t_i = 1\) then the surgery for the regular value \(a_i\) is fully done—then for each regular value \(b \in (a_i - \frac{1}{2} \varepsilon_i, a_i + \frac{1}{2} \varepsilon_i)\) of \(x_1 : K_{e, \ell}^1(W, \ell_W) \to \mathbb{R}\), the \(\theta\)-manifold \(K_{e, \ell}^t(W, \ell_W) \mid_b\) lies in \(\mathcal{A}\).

**Proof.** For the first part we must verify the conditions of Definition 2.18. This part of the argument of Lemma 4.6 applies equally well when \(\kappa = n - 1, l = n - 1\).

For the second part, we suppose \(t_i = 1\). Let \(b \in (a_i - \frac{1}{2} \varepsilon_i, a_i + \frac{1}{2} \varepsilon_i)\) be a regular value of the height function on \(K_{e, \ell}^1(W, \ell_W)\) and define \(\theta\)-manifolds
\[
\overline{M} = (K_{e, \ell}^1(W, \ell_W))_b,
\]
\[
\overline{M} = (K_{e, \ell}^t(W, \ell_W))_b
\]
\[
M = W_b.
\]
By Definition 5.12 (viii), performing surgery on \(M\) using the data \((\varepsilon, \ell_i)\) gives a \(\theta\)-manifold in \(\mathcal{A}\). By Proposition 5.11 (viii), \(K_{e, \ell}^t(W, \ell_W)\) has this surgery done, so \(\overline{M}\) lies in \(\mathcal{A}\). Now \(\overline{M}\) is obtained from \(\overline{M}\) by applying the remaining operations \(K_{e, \ell}^j\) for \(j \neq i\), but by Proposition 5.11 (viii), applying each of these only changes \(\overline{M}\) up to isomorphism (because \(b \in (a_i - \frac{1}{2} \varepsilon_i, a_i + \frac{1}{2} \varepsilon_i)\), so it is not in \((a_j - \frac{1}{2} \varepsilon_j, a_j + \frac{1}{2} \varepsilon_j)\), so \(\overline{M}\) lies in \(\mathcal{A}\).

As in Section 4.4 we define a map
\[
\mathcal{F}_p : [0, 1]^{p+1} \times D_{p,L}^{n-1, A}(\mathbb{R}^N)_{p,0} \longrightarrow X_p^{n-1,n-2}
\]
\[
(t, (a, \varepsilon, (W, \ell_W), e, \ell)) \quad \longrightarrow \quad (a, \frac{1}{2} \varepsilon, K_{e, \ell}^t(W, \ell_W)),
\]
which has the desired range by the first part of Lemma 5.14. The argument of Section 4.4 gives maps
\[
F_p : [0, 1] \times D_{p,L}^{n-1, A}(\mathbb{R}^N)_{p,0} \times \Delta^p \longrightarrow X_p^{n-1,n-2} \times \Delta^p
\]
which glue to a homotopy \(\mathcal{F} : [0, 1] \times |D_{p,L}^{n-1, A}(\mathbb{R}^N)_{p,0}| \to |X_p^{n-1,n-2}|\) which is constant on the subspace \(|D_{p,L}^{n-1, A}(\mathbb{R}^N)_{\bullet,0}| \to |D_{p,L}^{n-1, A}(\mathbb{R}^N)_{\bullet,0}|\) of manifolds equipped with no surgery data. It also provides a factorisation of the map \(\mathcal{F}(1, -)\) through the continuous injection \(|X^{n-1, A}_p| \to |X^{n-1,n-2}_p|\). The argument in Section 4.3 then gives the weak equivalence in Theorem 5.2.

6. **Contractibility of spaces of surgery data**

In order to finish the proofs of the results of the last three sections, we must supply proofs of Theorems 5.3, 5.5, and 5.13 concerning the bi-semi-simplicial spaces of manifolds equipped with surgery data. For convenience we assume that the domain \(B\) of the map \(\theta\) defining the tangential structure is path connected. In the
category $\mathcal{C}^n_{\kappa,l}$, this implies that objects are path connected as long as $l > -1$, and morphisms are path connected if in addition $\kappa > -1$.

6.1. The first part of Theorems 4.5 and 5.13. Theorems 4.5 and 5.13 both assert that two maps are weak equivalences. In either theorem, the proof for the first part of Theorems 4.5 and 5.13 respectively, but it clearly factors as $\theta,L \rightarrow R,N$. We define, for this proof only, a bi-semi-simplicial space $D_{\bullet,\bullet}$ in the same way as $D_{\bullet,\bullet}$ except that the usual inequalities $a_i + \epsilon_i < a_{i+1} - \epsilon_{i+1}$ and $\epsilon_i > 0$ are replaced by $a_i \leq a_{i+1}$ and $\epsilon_i \geq 0$ (so the intervals $[a_i, a_{i+1}]$ are allowed to overlap).

The inclusion $D_{\bullet,\bullet} \hookrightarrow D'_{\bullet,\bullet}$ is easily seen to be a levelwise weak homotopy equivalence, by spreading the $a_i$ out and making the $\epsilon_i$ positive but small, so it is enough to work with $D'_{\bullet,\bullet}$ throughout and show that $[D_{\bullet,\bullet}] \to [D'_{\bullet,\bullet}]$ is a weak homotopy equivalence.

To do so, we describe a retraction $r : [D_{\bullet,\bullet}] \to [D'_{\bullet,\bullet}]$ which will be a weak homotopy inverse to the inclusion. The map $r$ does not change the underlying manifold $W \in \psi_0(N + 1, 1)$, but only modifies the $a_i$ and barycentric coordinates. There is a map

$$D_{p,q} \rightarrow D'_{(p+1)(q+1)-1,0}$$

given by considering $(p+1)$ regular values, each equipped with $(q+1)$ pieces of surgery data, as $(p+1)(q+1)$ not-necessarily distinct regular values, each with a single piece of surgery data. There is also a map $\Delta^p \times \Delta^q \to \Delta^{(p+1)(q+1)-1} \subset \mathbb{R}^{(p+1)(q+1)}$ with $(j + (q+1)i)$th coordinate given by $(t,s) \mapsto s_it_j$. Taking the product of these maps gives

$$r_{p,q} : D_{p,q} \times \Delta^p \times \Delta^q \to D'_{(p+1)(q+1)-1,0} \times \Delta^{(p+1)(q+1)-1}$$

which glue together to give the map $r : [D_{\bullet,\bullet}] \to [D'_{\bullet,\bullet}]$. It is clear that $r$ is a retraction (i.e. left inverse to the inclusion), so the induced map on homotopy groups is surjective. To see that it is injective, we use the map $[D_{\bullet,\bullet}] \to [D']$ induced by the augmentation in the second bi-semi-simplicial direction (by forgetting all surgery data). This is a weak equivalence by the second part of Theorem 4.5 or 5.13 respectively, but it clearly factors as

$$[D_{\bullet,\bullet}] \xrightarrow{\sim} [D'_{\bullet,\bullet}] \to [D']$$

where the second map is again induced by the augmentation in the second bi-semi-simplicial direction. Therefore $r$ is also injective on homotopy groups, and hence a weak homotopy equivalence.

6.2. A simplicial technique. In order to give the proofs of Theorems 3.4, 4.5 and 5.13 we need a technique for showing that for certain augmented semi-simplicial spaces $X_{\bullet} \to X_{-1}$, the map $[X_{\bullet}] \to X_{-1}$ is a weak homotopy equivalence. The semi-simplicial spaces occurring in those theorems are all of the following special type.

Definition 6.1. Let $X_{\bullet} \to X_{-1}$ be an augmented semi-simplicial space. We say it is an augmented topological flag complex if

(i) The map $X_{\bullet} \to X_0 \times X_{-1} : X_0 \times X_{-1} : \cdots \times X_{-1} X_0$ to the $(n+1)$-fold product—which takes an $n$-simplex to its $(n+1)$ vertices—is a homeomorphism onto its image, which is an open subset.
A tuple \((v_0, \ldots, v_\ell) \in X_0 \times X_{-1}, X_0 \times X_{-1} \cdots \times X_{-1}, X_0 \) lies in \(X_n\) if and only if \((v_i, v_j) \in X_1\) for all \(i < j\).

If elements \(v, w \in X_0\) lie in the same fibre over \(X_{-1}\) and \((v, w) \in X_1\), we say \(w\) is orthogonal to \(v\). (We do not require the relation to be symmetric, in which case it will be.) If \(X_{-1} = *\) we omit the adjective augmented.

The semi-simplicial space \(Z_\bullet(a, \varepsilon, (W, \varepsilon W)) \to \ast\) from Definition 3.2 and the semi-simplicial spaces \(Y_\bullet(a, \varepsilon, (W, \varepsilon W)) \to \ast\) from Definitions 4.3 and 5.12 are topological flag complexes. Furthermore, \(D^0_{\theta, L}(\mathbb{R}^N)_{p, \bullet} \to D^0_{\theta, L}^{-1}(\mathbb{R}^N)_{p, \bullet}\) from Definition 6.3 and (ii) \(D^0_{\theta, L}^{n, l}(\mathbb{R}^N)_{p, \bullet} \to D^0_{\theta, L}^{n, l-1}(\mathbb{R}^N)_{p, \bullet}\) from Definition 4.3 and \(D^0_{\theta, L}^{n-1, -1}(\mathbb{R}^N)_{p, \bullet} \to D^0_{\theta, L}^{n-1, n-2}(\mathbb{R}^N)_{p, \bullet}\) from Section 5.3 are all augmented topological flag complexes. In all cases this is immediate from the definition: firstly, a \(p\)-simplex of these semi-simplicial spaces consists of \((p + 1)\)-tuples of surgery data, which are each 0-simplices; secondly, the pieces of surgery data are subject to the requirement that they are all disjoint, but disjointness is a property that can be verified pairwise.

**Theorem 6.2.** Let \(X_\bullet \to X_{-1}\) be an augmented topological flag complex. Suppose that

(i) The map \(\varepsilon : X_0 \to X_{-1}\) has local sections (in the strong sense that given any \(x \in X_0\), there is a neighbourhood \(U \subset X_{-1}\) of \(\varepsilon(x)\) and a section \(s : U \to X_0\) with \(s(\varepsilon(x)) = x\)).

(ii) \(\varepsilon : X_0 \to X_{-1}\) is surjective.

(iii) For any \(p \in X_{-1}\) and any (non-empty) finite set \(\{v_1, \ldots, v_\ell\} \subset \varepsilon^{-1}(p)\) there exists a \(v \in \varepsilon^{-1}(p)\) with \((v_i, v) \in X_1\) for all \(i\).

Then \(|X_\bullet| \to X_{-1}\) is a weak homotopy equivalence.

Condition (iii) can be viewed as the special case \(n = 0\) of condition (iii), but we prefer to keep the cases \(n = 0\) and \(n > 0\) separate.

**Remark 6.3.** To motivate the proof of this theorem, let us first consider the case where \(X_{-1} = *\) and that each \(X_i\) is discrete, so \(|X_\bullet|\) has the structure of a \(\Delta\)-complex. Then any map \(f : S^n \to |X_\bullet|\) may be homotoped to be simplicial, for some triangulation of \(S^n\), and so hits finitely many vertices \(v_1, \ldots, v_k\). By (iii) there exists a \(v \in X_0\) such that \((v_i, v)\) is a 1-simplex for all \(i\). But then the map \(f\) extends to the join

\[f * \{v\} : S^n * \{v\} \to |X_\bullet|\]

and so \(f\) is null-homotopic.

The proof we give below follows this in spirit, although is necessarily more complicated when the \(X_i\) carry a topology. To deal with the topology, we require the following technical result.

**Proposition 6.4.** Let \(Y_\bullet\) be a semi-simplicial set, and \(X\) be a Hausdorff space. Let \(Z_\bullet \subset Y_\bullet \times X\) be a sub-semi-simplicial set which in each degree is an open subset. For \(x \in X\), let \(Z_\bullet(x) \subset Y_\bullet\) be the sub-semi-simplicial set defined by \(Z_\bullet \cap (Y_\bullet \times \{x\}) = Z_\bullet(x) \times \{x\}\) and suppose that \(|Z_\bullet(x)|\) is contractible for all \(x \in X\). Then the map \(\pi : |Z_\bullet| \to X\) is a Serre fibration with contractible fibres.

**Proof.** This follows from [GRW12, Proposition 2.7] and [Wei05, Lemma 2.2].

**Corollary 6.5.** Let \(\Omega\) be a set and \(X\) a Hausdorff space and let

\[P \subset \mathbb{N} \times \Omega \times X\]

be a subset which is open (when \(\mathbb{N}\) and \(\Omega\) are given the discrete topology) and such that the projection \(P \to \mathbb{N} \times X\) is surjective. We give \(\mathbb{N} \times \Omega \times X\) the partial order defined by

\[(n, \alpha, x) < (m, \beta, y)\text{ if and only if }n < m \text{ and } x = y\]
Remark 6.3. \(\square\)

We begin with an element of the relative homotopy group \(\pi\) of the pair of spaces \((X, V)\) defined on a neighbourhood \(D\) of \(x\). The properness of \(\pi\) restricts to a continuous map \(f: D \mapsto X\), and we will show that after changing \(D\) by a fibrewise homotopy there is a diagonal map \(D^k \mapsto |X_\bullet|\) making the lower triangle commute, ignoring the upper triangle for a moment. To do this we first pick an infinite set \(\Omega\) (topologised discretely) and note that it suffices to find open sets \(P_\bullet \subset \Omega \times D^k\) together with maps \(g_n : P_\bullet \mapsto X\) with the properties that the projection \(\pi_n : P_\bullet \mapsto D^k\) is surjective, that \(\varepsilon \circ g_n = f \circ \pi_n\), and that for all \(x \in D^k\) and \(n < m\), any \(p \in \pi_n^{-1}(x)\) and \(q \in \pi_m^{-1}(x)\) have \((g_n(p), g_m(q)) \in X_1\). Namely, given such \((P_\bullet, g_n)\) we can let \(P = \bigcup \{n\} \times P_\bullet \subset \Omega \times D^k\) and assemble the \(g_n\) to a simplicial map \(g : N_\bullet P \mapsto X_\bullet\). By Corollary 6.5 the map \(\pi : |N_\bullet P| \mapsto D^k\) is a Serre fibration with contractible fibres, so we may pick a section \(s : D^k \mapsto |N_\bullet P|\). Then the composition \(|g| \circ s : D^k \mapsto |X_\bullet|\) gives a diagonal map in the diagram, making the lower triangle commute.

The \((P_\bullet, g_n)\) will be constructed by an inductive procedure, for which it is useful to construct a slightly stricter structure. If we write \(\overline{P}_\bullet \subset \Omega \times D^k\) for the closure, we will demand an extension \(\overline{g}_\bullet : \overline{P}_\bullet \mapsto X\) satisfying

(i) the projection \(\pi_\bullet : \overline{P}_\bullet \mapsto D^k\) is proper, and the restriction \(\pi_\bullet : P_\bullet \mapsto D^k\) is surjective,

(ii) \(\varepsilon \circ \overline{g}_\bullet = f \circ \pi_\bullet\),

(iii) for all \(x \in D^k\) and \(n < m\), any \(p \in \pi_n^{-1}(x)\) and \(q \in \pi_m^{-1}(x)\) have \((\overline{g}_n(p), \overline{g}_m(q)) \in X_1\).

The properness of \(\pi_\bullet\) is equivalent to the compactness of \(\overline{P}_\bullet\), which in turn is equivalent to the compactness of \(\overline{P}_\bullet\) in \(\Omega\) being finite. For the construction, we first pick for each \(x \in D^k\) an element \(g_x(x) \in \varepsilon^{-1}(f(x))\) which is orthogonal to each element of the finite set \(\bigcup_{i<n} \overline{g}_i((\varepsilon^{-1}(x)))\), as is possible by assumption. Then, since \(\varepsilon\) is proper, we can extend to a map \(g_x : V_x \mapsto X_0\) which is a lift of \(D^k \mapsto X_{-1}\), defined on a neighbourhood \(V_x\) of \(x\). The maps

\[ \overline{g}_i \times g_x : \overline{P}_i \times D^k \times V_x \mapsto X_0 \times \times_x X_0 \]

for \(i < n\) send \(\overline{P}_i \times D^k \times \{x\}\) into the open subset \(X_1\), so by properness of \(\pi_\bullet\) we can ensure that all these maps have image in \(X_1\), after perhaps shrinking the open set \(V_x\). If we let \(U_x \subset V_x\) be a smaller neighbourhood of \(x\) with \(\overline{U}_x \subset V_x\), then \(g_x\) restricts to a continuous map \(U_x \mapsto X_0\). The sets \(U_x\) give an open cover of \(D^k\), and we let \(U_{x_1}, \ldots, U_{x_m}\) be a finite subcover. Finally, we pick distinct \(\omega_1, \ldots, \omega_m \in \Omega\), disjoint from the image of \(\bigcup_{i<n} P_i \mapsto \Omega\) and let

\[ P_n = \bigcup_{i=1}^m \{\omega_i\} \times U_{x_i} \subset \Omega \times D^k \]
and define the map \( \overline{f}_n : \overline{P} \to X_0 \) by \( \overline{g}_n(w, y) = g_n(y) \). The sequence of \((P_n, \overline{g}_n)\) thus constructed will satisfy the properties (i), (ii) and (iii) above, and hence gives a lift \( D^k \to |X| \).

The construction of the lift \( D^k \to |X| \) so far has not used the given \( \hat{f} \) in any way, so we should not expect the upper triangle to commute. We shall add an extra step preceding the above inductive construction of \((P_n, \overline{g}_n)\), in order to fix this. Namely, we shall construct a compact subset \( \overline{P}_1 \subset \Omega \times \partial D^k \) and a continuous \( \overline{f}_1 : \overline{P}_1 \to X_0 \) such that after changing \( \hat{f} \) by a fibrewise homotopy, all vertices of \( \hat{f}(x) \) are contained in \( \overline{g}_1^{-1}(\overline{f}_1^{-1}(x)) \), where \( \overline{g}_1 : \overline{P}_1 \to \partial D^k \) again denotes the projection. In the inductive construction of \((P_n, \overline{g}_n)\), we can then ensure that condition (iii) above is satisfied also for \( n = -1 \) when \( x \in \partial D^k \). Then all vertices of \( \hat{f}(x) \) will be orthogonal to all vertices of \( \overline{f} \circ s(x) \), so these two points are connected by the straight line inside the join of the simplices that contain them. These straight lines then assemble to a fibrewise homotopy between \( \hat{f} \) and \( \overline{f} \circ s \).

To construct \( P_1 \) and \( \overline{g}_1 \) we shall, for the rest of this proof, replace the usual coordinates \((t_0, \ldots, t_p) \in \Delta^k \) (which are non-negative numbers with \( \sum t_i = 1 \)) by the coordinates \( s_i = t_i / \max(t_i) \) (which are non-negative numbers with \( \max(s_i) = 1 \)). Points in \(|X|\) are then written as \((y, s_0, \ldots, s_q)\), where \( y \in X_q \), \( s_i \geq 0 \) and \( \max s_i = 1 \). For \( t \in (0, 1) \), we shall write \( U_t \subset |X| \) for the subset where no \( s_i \) is equal to \( t \). There is a function \( U_t \mapsto \Pi_p X_{p} \times \Delta^p \) which to \((y, s_0, \ldots, s_q) \in |X| \) associates \((\theta^* y, s_{q}(0), \ldots, s_{q}(p))\), where \( \theta : [p] \to [q] \) is the order-preserving monomorphism defined as the composition of the unique order-preserving bijection \([p] \cong \{ i \in [q] \mid s_i > t \}\) and the inclusion to \([q]\), and \( \theta^* : X_q \to X_p \) is the corresponding face map. It is easy to verify that \( U_t \subset |X| \) is open (in the usual quotient topology from \( \Pi_p X_p \times \Delta^p \to |X| \)) and that the function \( U_t \mapsto \Pi_p X_p \times \Delta^p \) is continuous (when \( U_t \subset |X| \) is given the subspace topology). Furthermore it is clear that any infinite collection of numbers \( t \in (0, 1) \) will give a cover of \(|X|\) by the corresponding \( U_t \)'s.

Proceeding with the construction of \( P_1 \) and \( \overline{g}_1 \), we may cover \( \partial D^k \) by the open sets \( \hat{f}(x) \). (By compactness we can find a finite cover of \( \partial D^k \) by open sets \( U_i \subset \partial D^k \) such that each closure \( \overline{U}_i \) is contained in some \( \hat{f}(x) \), and hence we get a continuous mapping \( U_i \mapsto \Pi_p X_p \). Writing \( \overline{U}_i \) for the subspace mapping into \( X_p \subset X_{p+1} \), we get a continuous adjoint \( \overline{g}_1 : \overline{P}_1 \to X_0 \) such that \( \overline{g}_1(p \times \{ x \}) \subset X_0 \) consists of the vertices of \( \hat{f}(x) \) with simplicial coordinate greater than \( t_i \). In particular, this set contains all vertices of \( \hat{f}(x) \) with simplicial coordinate \( \geq \frac{1}{2} \). We can then pick an injection \( \Pi_p [p] \to \Omega \) and let \( \overline{P}_1 \) be the image of the resulting embedding

\[
\bigcup_{i,p} [p] \times \overline{U}_i \longrightarrow \Omega \times \partial D^k,
\]

and assemble the \( \overline{g}_1 \) to a map \( \overline{g}_1 : \overline{P}_1 \to X_0 \). After changing \( \hat{f} \) by composing with the self-map of \(|X|\) which replaces all simplicial coordinates \( s_i \) by \( \max(0, 2s_i - 1) \) (which is obviously continuous and fibrewise homotopic to the identity), the finite set \( \overline{g}_1^{-1}(\overline{f}_1^{-1}(x)) \subset X_0 \) contains all vertices of \( \hat{f}(x) \), as required.

After replacing the simplicial coordinates of \( \hat{f} \) as described, the restriction to \( U_{i,p} \subset \partial D^k \) factors through \( X_p \times \Delta^p \to |X| \). This implies that the linear homotopy from \( \hat{f} \) to \( \overline{f} \circ s \) is continuous on each \( U_{i,p} \). \( \square \)

6.3. \textbf{Proof of Theorem 3.4} Recall that this theorem states that the augmentation

\[
D_{\theta,L}^k(\mathbb{R}^N)_{\bullet, \bullet} \longrightarrow D_{\theta,L}^{k-1}(\mathbb{R}^N)_{\bullet, \bullet}
\]
induces a weak homotopy equivalence after geometric realisation, as long as the conditions of Theorem 3.1 are satisfied. In fact, we only require the following weaker set of conditions:

(i) $2\kappa \leq d - 1$,
(ii) $\kappa + 1 + d < N$,
(iii) $L$ admits a handle decomposition only using handles of index $< d - \kappa - 1$.

We will use Theorem 6.2 to prove that for each $p$ the augmentation map induces a weak equivalence 

$$\left| D_{\theta,L}^\kappa(\mathbb{R}^N),\bullet \right| \rightarrow D_{\theta,L}^{\kappa - 1}(\mathbb{R}^N).$$

Theorem 6.2 does not apply directly to the augmentation $D_{\theta,L}^\kappa(\mathbb{R}^N),\bullet \rightarrow D_{\theta,L}^{\kappa - 1}(\mathbb{R}^N)$, but we will show that it does apply after replacing with weakly equivalent spaces.

Recall that an element of $D_{\theta,L}^\kappa(\mathbb{R}^N),p,q$ consists of an element $(a,\varepsilon,(W,\ell_W)) \in D_{\theta,L}^{\kappa - 1}(\mathbb{R}^N),p$, together with an element $(\Lambda,\delta,e) \in Z_0(a,\varepsilon,(W,\ell_W))$, where $\Lambda \subset \Omega$ is a finite set equipped with a map $\delta : \Lambda \rightarrow [p] \times [q] = \{0,\ldots,p+1\} \times \{0,\ldots,q\}$ and $e$ is an embedding $e : \Lambda \times V \hookrightarrow \mathbb{R} \times (0,1) \times (-1,1)^{N-1}$.

**Definition 6.6.** The core of $\nabla$ is the submanifold $C = [-2,0] \times D^\kappa \times \{0\} \subset \nabla = [-2,0] \times \mathbb{R} \times \mathbb{R}^{d-\kappa}$. Let $\tilde{Z}_a(a,\varepsilon,(W,\ell_W))$ be the semi-simplicial space defined as in Definition 3.2 except that instead of demanding that $e : \Lambda \times \nabla \rightarrow \mathbb{R} \times (0,1) \times (-1,1)^{N-1}$ be an embedding, we demand only that it be a smooth map which restricts to an embedding of a neighbourhood of $\Lambda \times C$. We still require that $e$ satisfy the numbered conditions listed in Definition 3.2. Let $\tilde{D}_{\theta,L}^{\kappa}(\mathbb{R}^N),\bullet \hookrightarrow \tilde{D}_{\theta,L}^{\kappa - 1}(\mathbb{R}^N),\bullet$ be the augmented bi-semi-simplicial space defined as in Definition 3.3 but using $\tilde{Z}_a(x)$ instead of $Z_a(x)$.

**Proposition 6.7.** The inclusion $D_{\theta,L}^{\kappa}(\mathbb{R}^N),\bullet \hookrightarrow D_{\theta,L}^{\kappa - 1}(\mathbb{R}^N),\bullet$ induces a weak homotopy equivalence in each bidegree, and so on geometric realisation.

**Proof.** It is easy to see that there is an isotopy of embeddings $j_t : \nabla \rightarrow \nabla$, $t \in [1,\infty)$, such that $j_1 = \text{Id}$, $j_t|\nabla = \text{Id}$ for all $t$ and $j_t(\nabla)$ is contained in the $(1/t)$-neighbourhood of $\nabla$ for large $t$, and also such that every $j_t$ preserves the submanifold $\text{int}(\partial_1 D^{\kappa+1}) \times \mathbb{R}^{d-\kappa}$ and preserves the height function $h : \nabla \rightarrow [-2,0]$.

Precomposing the embedding $e : \Lambda \times \nabla \rightarrow \mathbb{R} \times (0,1) \times (-1,1)^{N-1}$ with the maps $\text{Id}_\Lambda \times j_t$ induces a deformation $[1,\infty) \times \tilde{Z}_a(a,\varepsilon,(W,\ell_W)) \rightarrow \tilde{Z}_a(a,\varepsilon,(W,\ell_W))$ and in turn $[1,\infty) \times \tilde{D}_{\theta,L}^{\kappa}(\mathbb{R}^N),p,q \rightarrow \tilde{D}_{\theta,L}^{\kappa - 1}(\mathbb{R}^N),p,q$. Elements of $\tilde{Z}_a(a,\varepsilon,(W,\ell_W))$ have disjoint cores, so in a compact family $K \rightarrow \tilde{D}_{\theta,L}^{\kappa - 1}(\mathbb{R}^N),p,q$, there exists an $\varepsilon > 0$ such that the $\varepsilon$-neighbourhoods of all cores are also disjoint. Composing with the deformation of $\tilde{D}_{\theta,L}^{\kappa - 1}(\mathbb{R}^N),p,q$, the map from $K$ will eventually deform into $D_{\theta,L}^{\kappa - 1}(\mathbb{R}^N),p,q$. It follows easily from this that the relative homotopy groups vanish.

In order to prove Theorem 6.4 we will show that for each $p$ the map

$$\tilde{D}_{\theta,L}^{\kappa}(\mathbb{R}^N),p,\bullet \rightarrow \tilde{D}_{\theta,L}^{\kappa - 1}(\mathbb{R}^N)$$

is a weak homotopy equivalence after geometric realisation, by applying Theorem 6.2. Hence we must verify the conditions of that theorem. First we establish condition (i).

**Proposition 6.8.** The map $\tilde{D}_{\theta,L}^{\kappa}(\mathbb{R}^N),p,0 \rightarrow \tilde{D}_{\theta,L}^{\kappa - 1}(\mathbb{R}^N),p$ has local sections.

**Proof.** Let’s consider a point $x \in \tilde{D}_{\theta,L}^{\kappa}(\mathbb{R}^N),p,0$, given by elements $(a,\varepsilon,(W,\ell_W)) \in D_{\theta,L}^{\kappa - 1}(\mathbb{R}^N),p$ and $(\Lambda,\delta,e) \in Z_0(a,\varepsilon,(W,\ell_W))$. Choose $t_0 < a_0 - \varepsilon_0$ and $t_1 > a_p + \varepsilon_p$ which are regular values for $x_1 : W \rightarrow \mathbb{R}$, and such that $(x_1 \circ e)(\Lambda \times \nabla) \subset (t_0,t_1)$. 


Proof.\(\phi\) of regular values). If \(x\) has fibres contained in level sets of \(\pi\) and has \(g\) near to a point \(W \in \psi_g(N + 1, 1) \subset \psi_g(R \times R^N)\) look like a section of the normal bundle of \(W\) inside a compact set, e.g. inside \([0, t_1] \times [-1, 1]^N\).

We now require two results on spaces of embeddings. Firstly, the map

\[\Emb(\partial M, [t_0, t_1] \times (-1, 1)^N) \to \Diff(M)\]

is well-known to be a principal \(\Diff(M)\)-bundle, and has local sections (see e.g. \[\text{BFS}\]). Thus, after perhaps passing to a smaller open neighbourhood, which we will still call \(U\), \(F\) has a lift \(\tilde{F} : U \to \Emb(\partial M, [t_0, t_1] \times (-1, 1)^N)\), and we call \(f = \tilde{F}(a, e, (W, \ell W))\).

Secondly, we need the following generalisation of a technical theorem of Cerf [Cer61] 2.2.1 Théorème 5 (the “first isotopy and extension theorem”), an especially elementary proof of which was given by Lima [Lim69]. We follow Lima’s proof.

**Lemma 6.9.** Let \(C \subset [t_0, t_1]\) be a closed subset and let \(S \subset \Emb(\partial M, [t_0, t_1] \times (-1, 1)^N)\) be the open subset of those embeddings \(e\) for which \(\pi_1 \circ e : M \to [t_0, t_1]\) has no critical values inside \(C\).

Given an \(f \in S\), there is a neighbourhood \(U\) of \(f\) in \(S\) and a continuous map \(\varphi : U \to \Diff([t_0, t_1] \times (-1, 1)^N)\) such that \(\varphi(g) \circ f\) and \(g\) have the same image, and \(\varphi(g)\) is height-preserving over \(C\).

**Proof.** Consider \(M\) to be a submanifold of \([t_0, t_1] \times (-1, 1)^N\) via \(f\). We choose a tubular neighbourhood \(\pi : T \to M\) of radius \(\varepsilon\) which over the boundary and \(x_1^{-1}(C)\) has fibres contained in level sets of \(x_1\) (this is possible as \(C\) is closed and consists of regular values). If \(g \in S\) is sufficiently close to \(f\), it will have image in \(T\) and we may define an element \(\varphi(g) \in C^{\infty}(M, M)\) by

\[\varphi(g)(x) = \pi(g(x)).\]

This is a diffeomorphism for \(g = f\), and so there is a neighbourhood \(U'\) of \(f\) in \(S\) where this remains true. We get a function \(\tilde{\varphi} : U' \to \Diff(M)\) and for each \(g \in U'\) we define a new embedding \(G = G(g) : M \to [t_0, t_1] \times (-1, 1)^N\) by \(G = g \circ (\varphi(g)^{-1})\). It has the same image as \(g\) and has \(\pi(G(x)) = x\). Therefore \(x\) and \(G(x)\) have the same height when \(x \in x_1^{-1}(C)\).

Let \(\lambda\) be a bump function which is 1 on \([0, \varepsilon/4]\) and 0 on \([\varepsilon/2, \infty)\). Now let

\[\varphi(g)(x) = x + \lambda(|x - \pi(x)|) \cdot (G(\pi(x)) - \pi(x))\]

define a compactly-supported smooth map \(\varphi(g) \in C^{\infty}([t_0, t_1] \times (-1, 1)^N)\). For \(g = f\) it is a diffeomorphism, and so there is a smaller neighbourhood \(U\) of \(f\) in \(S\) where this remains true. We get a function \(\varphi : U \to \Diff([t_0, t_1] \times (-1, 1)^N)\).

By construction \(\varphi(g) \circ f(x) = \varphi(g)(x) = x + (G(x) - x) = G(x)\), so \(\varphi(g) \circ f\) has the same image as \(g\). Also, if \(x \in x_1^{-1}(C)\) then the vector \(G(\pi(x)) - \pi(x)\) has no component in the \(x_1\) direction, so \(x_1(\varphi(g)(x)) = x_1(x)\) and \(\varphi(g)\) is height function preserving over \(C\).
It now follows that $\pi$ also has this local structure: after possibly shrinking $U$, there is a map

$$\varphi : U \rightarrow \text{Diff}(]t_0, t_1[ \times (-1, 1)^N)$$

with the properties described in the lemma, such that $\Gamma[[t_0, t_1[ \subset U \times ]t_0, t_1[ \times (-1, 1)^N$ is obtained from $W[[t_0, t_1[\times (-1, 1)^N$ by applying the family of diffeomorphisms $\varphi$.

The element $(\Lambda, \delta, \epsilon) \in \tilde{Z}_0(a, \epsilon, (W, \ell_W))$ has surgery data

$$e : \Lambda \times V \rightarrow ]t_0, t_1[ \times (0, 1) \times (-1, 1)^{N-1},$$

so we attempt to define a section $U \rightarrow \tilde{D}_{\delta, \ell}^0(\mathbb{R}^N)_{p,0}$ by sending $u$ to the point $(\Lambda, \delta, \varphi(u) \circ \epsilon)$. We must verify that this is indeed an element of $\tilde{Z}_0(u)$ by checking the conditions of Definition 5.1. Conditions (i)–(iv) hold as required element $\tilde{U}$ necessarily forming a $\varphi$-function $\delta$ extending this perturbation to a map $\Lambda \times \varphi : \Lambda \rightarrow [p]^X$ so we attempt to define a section $\tilde{v}_k \in \tilde{Z}_0(a, \epsilon, (W, \ell_W))$ be a non-empty collection of pieces of surgery data (not necessarily forming a $(k-1)$-simplex). Then, if $2k < d$ and $\kappa + 1 + d < N$, there exists a piece of surgery data $v \in \tilde{Z}_0(a, \epsilon, (W, \ell_W))$ such that each $(v, v)$ is a 1-simplex.

**Proof.** Each $v_j$ is given by a set $\Lambda^j$ (which is a subset of the uncountable set $\Omega$), a function $\delta^j : \Lambda^j \rightarrow [p]^Y$ and a map $e^j : \Lambda^j \times V \rightarrow \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}$, satisfying certain properties. We first pick a set $\Lambda$ which is disjoint from all $\Lambda^j$ and a bijection $\varphi : \Lambda \rightarrow \Lambda^1$, let $\delta = \delta^1 \circ \varphi : \Lambda \rightarrow [p]^X$, and then set

$$\tilde{c} = e^1 \circ (\varphi \times \text{Id}_V) : \Lambda \times V \rightarrow \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}.$$ 

This gives a new element of $\tilde{Z}_0(a, \epsilon, (W, \ell_W))$, but it is of course not orthogonal to $v_1$ (and not necessarily orthogonal to the other $v_j$). We then perturb $\tilde{c}$ inside the class of functions satisfying the requirements of Definition 5.2 to a new function $e : \Lambda \times V \rightarrow \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}$ whose core is in general position with respect to the cores of the $v_j$. More explicitly, $\tilde{c}$ restricts to a map

$$\Lambda \times \partial_- D_{\kappa+1} \times \mathbb{R}^{d-k} \rightarrow W,$$

and we first perturb this so that $\Lambda \times \partial_- D_{\kappa+1} \times \{0\}$ is transverse in $W$ to the corresponding part of the other embeddings, and remains disjoint from $L$, then we extend this perturbation to a map $e : \Lambda \times V \rightarrow \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1}$ whose restriction to the interior of $C$ is transverse to the corresponding part of the other embeddings. In the first step we make $\kappa$-dimensional manifolds transverse in a $d$-dimensional manifold, and in the second we make $(\kappa + 1)$-dimensional manifolds disjoint in an $(N + 1)$-dimensional manifold. As $2k < d$ and $2(\kappa + 1) \leq \kappa + d + 2 < N + 1$, the new core will actually be disjoint from all other cores, producing the required element $v \in \tilde{Z}_0(a, \epsilon, (W, \ell_W))$. 

Finally, we establish condition (ii) of Theorem 5.2.

**Proposition 6.11.** $\tilde{Z}_0(a, \epsilon, (W, \ell_W))$ is non-empty as long as $2k < d$, $\kappa + 1 + d < N$, and $L$ admits a handle decomposition only using handles of index $\kappa < d - k - 1$. 


Proof. For each \( i = 1, \ldots, p \) we consider the pair \((W_{\|a_{i-1},a_i\|}, W_{|a_i|})\). Since it is \((\kappa - 1)\)-connected, it is homotopy equivalent to a finite relative CW complex \((X, W_{|a_i|})\) with cells of dimension \(\geq \kappa\) only. Since \(2\kappa < d\), the homotopy equivalence \((X, W_{|a_i|}) \to (W_{\|a_{i-1},a_i\|}, W_{|a_i|})\) may be assumed to restrict to a smooth embedding of the relative \(\kappa\)-cells. If we pick a subset \(\Lambda_{i,0} \subset \Omega\) with one element for each relative \(\kappa\)-cell (choosing disjoint sets for each \(i\)), we may therefore pick an embedding
\[
\tilde{e}_{i,0} : \Lambda_{i,0} \times (D^\kappa, \partial D^\kappa) \to (W_{\|a_{i-1}+\varepsilon_{i+1},a_i+\varepsilon_i\|}, W_{|a_i|}),
\]
which we may assume collared on \([a_i - \varepsilon_i, a_i + \varepsilon_i]\), such that the pair
\[
(W_{\|a_{i-1},a_i\|}, W_{|a_i|} \cup \text{Im}(\tilde{e}_{i,0})_{\|a_{i-1},a_i\|})
\]
is \(\kappa\)-connected. Furthermore, \(\mathbb{R} \times \mathbb{L} \subset W\) has a core of dimension \(d - \kappa\), by our assumption on the indices of handles of \(\mathbb{L}\), and so we may suppose that the embedding \(\tilde{e}_{i,0}\) is disjoint from \(\mathbb{R} \times \mathbb{L}\). As \(2\kappa < d\) we may also suppose that the images of the \(\tilde{e}_{i,0}\) are mutually disjoint.

The embedding
\[
\tilde{e}_{i,0}|_{\Lambda_{i,0} \times \partial D^\kappa} : \Lambda_{i,0} \times \partial D^\kappa \times \{0\} \to W_{|a_i+\varepsilon_i|} \subset W_{\|a_i+\varepsilon_i,a_{i+1}+\varepsilon_{i+1}\|}
\]
extends to an embedding of \(\Lambda_{i,0} \times \partial D^\kappa \times [0,1]\), where \(\Lambda_{i,0} \times \partial D^\kappa \times \{1\}\) is sent into \(W_{\|a_{i+1}+\varepsilon_{i+1}\|}\) and is collared on the \(\varepsilon_i\)-neighbourhoods of both boundaries. This may be seen as follows: to extend \(\tilde{e}_{i,0}|_{\Lambda_{i,0} \times \partial D^\kappa}\) to a continuous map having this property is possible as \(\pi_{\kappa-1}(W_{\|a_{i-1},a_i\|}, W_{|a_i|}) = 0\), but this may then be perturbed to be an embedding as \(2\kappa < d\). As above, this may be made disjoint from \(\mathbb{R} \times \mathbb{L}\), and they can be made mutually disjoint.

We may glue the two embeddings together. Using a suitable diffeomorphism \(D^\kappa \approx \partial D^\kappa \cup \partial D^\kappa \times [0,1]\), this gives a new embedding of \(\Lambda_{i,0} \times D^\kappa\). Continuing in this way, we obtain an extension of \(\tilde{e}_{i,0}\) to an embedding
\[
\tilde{e}_{i,0} : \Lambda_{i,0} \times (D^\kappa, \partial D^\kappa) \to (W_{\|a_{i-1}+\varepsilon_{i+1},a_i+\varepsilon_i\|}, W_{|a_i|}),
\]
which is disjoint from \(\mathbb{R} \times \mathbb{L}\), and which are mutually disjoint. Identifying \(D^\kappa\) with the disc \(\partial_1 D^{\kappa+1} \subset [-1,0] \times \mathbb{R}^{\kappa+1}\) gives a height function \(D^\kappa \to [-1,0]\) and if we pick the diffeomorphisms \(D^\kappa \approx \partial D^\kappa \cup \partial D^\kappa \times [0,1]\) carefully, we can arrange that on each \(\tilde{e}_{i,0}|(W_{\|a_{i-1}+\varepsilon_{i+1},a_i+\varepsilon_i\|})\), the embedding \(\tilde{e}_{i,0}\) is height function preserving up to an affine transformation.

We now want to extend the \(\tilde{e}_{i,0}\) from \(\Lambda_{i,0} \times \partial D^\kappa \times \{0\} \subset \lambda_{i,0} \times \mathbb{L}\) to the whole of \(\lambda_{i,0} \times \mathbb{L}\) so that it satisfies the conditions of Definition 3.2. As \(\kappa + 1 < N\), there is no trouble with extending the maps \(e_{i,0}\) to disjoint maps \(e_{i,0}\) from \(\Lambda_{i,0} \times \mathbb{L}\) to \(\mathbb{R} \times (0,1) \times \mathbb{R}^{N-1}\) satisfying conditions b–d of Definition 3.2, we first extend each \(e_{i,0}\) to an embedding of \([-2,0] \times \mathbb{R}^\kappa \times \{0\}\) (which is possible as \(2(\kappa + 1) < d + \kappa + 1 < N\)), then make this intersect \(W\) only in \(\partial_\lambda D^{\kappa+1}\) (which is possible as \(\kappa + 1 < N\)), and finally thicken it up by \(\mathbb{R} \times \mathbb{L}\). Property (2) is ensured by the way we chose \(e_{i,0}\).

Finally, we let \(\Lambda = \prod_{i=1}^p \Lambda_{i,0}\), \(\delta : \Lambda \to [p]^{\psi}\) be given by \(\delta(\Lambda_{i,0}) = i \in [p]^{\psi}\), and \(\epsilon = \prod_{i=1}^p \varepsilon_{i,0}\). The data (\(\Lambda, \delta, \epsilon\)) thus lies in \(\tilde{Z}_0(\mathcal{A}, (\mathbb{W}, \ell(W)))\).

6.4. Proof of Theorem 4.5. We have already proved the first part of this theorem in Section 6.1. Recall that the second part states that the augmentation map
\[
D^d_{\theta, L}(\mathbb{R}^N)_{\psi} \to D^d_{\theta, L}(\mathbb{R}^N)_{\psi},
\]
induces a weak homotopy equivalence after geometric realisation, as long as the conditions of Theorem 4.1 are satisfied. In fact, we only require the following weaker set of conditions:
(i) \(2(l + 1) < d\),
(ii) \( l \leq \kappa \),

(iii) \( l + 2 + d < N \),

(iv) \( L \) admits a handle decomposition only using handles of index \( d - l - 1 \),

(v) the map \( \ell_L : L \to B \) is \((l + 1)\)-connected.

We will proceed much as in the previous section. Recall that each point of \( D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{p,0} \) lying over \( (a, \varepsilon, (W, \ell_W)) \in D_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_p \) is a tuple \((\Lambda, \delta, e, \ell)\) where \( \Lambda \subset \Omega \) is a subset, \( \delta : \Lambda \to [p] \times [0] \) is a function,

\[ e : \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1} \to \mathbb{R} \times (0, 1) \times (-1, 1)^{N-1} \]

is an embedding, and \( \ell : T(\Lambda \times K_{(-6,0)}) \to \theta^* \gamma \) is a bundle map (where \( K \) is defined in Section 4.2). Let us define

\[ C = (-6, -2) \times \{ 0 \} \times D^{l+1} \subset (-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1} \]

and call it the core. Shrinking in the \( \mathbb{R}^{d-l-1}\)-direction gives an isotopy from the identity map of \((-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1}\) into any neighbourhood of its core.

**Definition 6.12.** Let \( \widetilde{\Psi}_*(a, \varepsilon, (W, \ell_W)) \) be the semi-simplicial space defined as in Definition 4.3 expect we only ask for \( e \) to be a smooth map which restricts to an embedding on a neighbourhood of \( \Lambda \times C \subset \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1} \). Note that condition 4.1 still makes sense: although the surgery data is no longer disjoint, it is still disjoint when restricted to a small enough neighbourhood of each core.

Let \( \widetilde{D}_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{*} \rightarrow \tilde{D}_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_* \) be the augmented bi-semi-simplicial space defined as in Definition 4.3 but using \( \widetilde{\Psi}_*(a, \varepsilon, (W, \ell_W)) \) instead of \( \Psi_*(a, \varepsilon, (W, \ell_W)) \).

We have the following analogue of Proposition 6.7 although the proof is slightly more complicated in this case, due to the tangential structures on the surgery data.

**Proposition 6.13.** The inclusion \( D_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{*} \rightarrow \tilde{D}_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_* \) induces a weak homotopy equivalence in each bidegree, and so on geometric realisation.

**Proof.** This is very similar to Proposition 6.7. We pick an isotopy of maps \( \psi_t : \mathbb{R}^{d-l-1} \to \mathbb{R}^{d-l-1}, t \in [0, \infty) \) which starts at the identity, has \( \psi_t(0) = 0 \) for all \( t \), and has image in the ball of radius \( 1/t \) for all \( t \). Applying \( \psi_t \) in the \( \mathbb{R}^{d-l-1} \) direction gives an isotopy of self-embeddings of \( \Lambda \times (-6, -2) \times \mathbb{R}^{d-l-1} \times D^{l+1} \). Similarly, we can get an isotopy of self-embeddings of the manifold \( K_{(-6,0)} \) from Section 4.2 which applies \( \psi_t \) in the \( \mathbb{R}^{d-l-1} \) direction on \( h^{-1}((-6, -2)) \), is the identity on \( h^{-1}((-\sqrt{2}, 0)) \), and interpolates inbetween. Precomposing with these isotopies gives a homotopy of self-maps of \( \tilde{D}_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_* \), which eventually deforms any compact space into \( \tilde{D}_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_* \).

Therefore it is enough to show that for each \( p \), the augmentation map

\[ \tilde{D}_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_p \to \tilde{D}_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_p \]

which forgets all surgery data induces a weak homotopy equivalence after geometric realisation, which we do by establishing the conditions of Theorem 6.2. The proofs that conditions 4 and 4.1 hold are very similar to the analogous case in Section 6.3 so we consider those first.

**Proposition 6.14.** The map \( \tilde{D}_{\theta,L}^{\kappa,l}(\mathbb{R}^N)_{p,0} \to \tilde{D}_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_p \) has local sections.

**Proof.** Exactly as in the proof of Proposition 6.8.

**Proposition 6.15.** Fix a point \( (a, \varepsilon, (W, \ell_W)) \in \tilde{D}_{\theta,L}^{\kappa,l-1}(\mathbb{R}^N)_p \), and let \( v_1, \ldots, v_k \in \tilde{Y}_0(a, \varepsilon, (W, \ell_W)) \) be a non-empty collection of pieces of surgery data. Then, if \( 2|l + 1| < d \) and \( l + 2 + d < N \), there exists a piece of surgery data \( v \in \tilde{Y}_0(a, \varepsilon, (W, \ell_W)) \) such that \( \langle v, v \rangle \) is a 1-simplex.
Proof. This is essentially the same as Proposition 6.10 first we let \( v = v_1 \), then we perturb it to have its cores transverse to the cores of all the \( v_j \). We first do the perturbation on the part of the cores inside \( W \). On the boundary the cores are \((l+1)\)-dimensional, so disjoint when they are transverse as \( 2(l+1) < d \). We now make sure the cores intersect \( W \) only on their boundary, which is possible as \( l+2+d < N \). We finally make sure that the cores are also disjoint on their interiors, which is possible as \((l+2) + (l+2) \leq (l+2) + d < N \).

Finally, we establish condition (ii).

Proposition 6.16. \( \tilde{Y}_0(\alpha, \varepsilon, (W, \ell_W)) \) is non-empty as long as \( 2(l+1) < d \), \( l \leq \kappa \), \( l+2+d < N \), \( L \) admits a handle decomposition only using handles of index \( < d-l-1 \), and the map \( \ell_L : L \to B \) is \((l+1)\)-connected.

Proof. For each \( i \), we consider the map \( \pi_\ast(W|_{\alpha_i}) \to \pi_\ast(B) \), induced by the tangential structure. By assumption, this map is injective for \( * \leq l-1 \). Since \( \{\alpha_i\} \times L \subset W|_{\alpha_i} \), and \( L \to B \) is assumed \((l+1)\)-connected, we deduce that the map \( L \to W|_{\alpha_i} \) is \((l-1)\)-connected, \( W|_{\alpha_i} \to B \) is \( l \)-connected, that

\[
(6.1) \quad \pi_i(W|_{\alpha_i}) \to \pi_i(B) \approx \pi_i(L)
\]

is split surjective, and that \( \pi_i(L) \to \pi_i(W|_{\alpha_i}) \) is split injective. We first claim that the kernel of \((6.1)\) is finitely generated as a module over \( \pi_1(L) \) (interpreted appropriately when \( l = 0 \) and \( l = 1 \); we shall leave the necessary modifications of the following argument in those two cases to the reader). Since the kernel is isomorphic to the cokernel of the splitting, we deduce the exact sequence

\[
(6.2) \quad \pi_i(W|_{\alpha_i}, L) \to \pi_i(W|_{\alpha_i}) \to \pi_i(B) \to 0.
\]

As \((W|_{\alpha_i}, L)\) is \((l-1)\)-connected, we can find a relative CW-complex \((K, L)\), where \( K \) is built from \( L \) by attaching only cells of dimension \( \geq l \), and a weak homotopy equivalence \( p : (K, L) \to (W|_{\alpha_i}, L) \). Since \((W|_{\alpha_i}, L)\) has the homotopy type of a CW pair, this map has a homotopy inverse \( q : (W|_{\alpha_i}, L) \to (K, L) \), and since \( W|_{\alpha_i} \) is compact, its image in \( K \) is contained in a finite subcomplex \( K' \subset K \). Then \( \pi_i(K') \to \pi_i(W|_{\alpha_i}) \) is surjective. Since \( \pi_i(W|_{\alpha_i}) \to \pi_i(W|_{\alpha_i}, L) \) is also surjective (as \( \pi_{i-1}(L) \to \pi_{i-1}(W|_{\alpha_i}) \) is split injective), we conclude that \( \pi_i(K', L) \to \pi_i(W|_{\alpha_i}, L) \) is surjective, and hence that \( \pi_i(W|_{\alpha_i}, L) \) is a finitely generated module over \( \pi_1(L) \), as claimed.

Let \( \{f_\alpha : S^l \to W|_{\alpha_i}\}_{\alpha \in \Lambda_i} \) be a finite collection of elements which generate the kernel of \( \pi_\ast(W|_{\alpha_i}) \to \pi_\ast(B) \), where \( \Lambda_i \subset \Omega \) are disjoint subsets. As the vector bundle \( \varepsilon^1 \oplus TW|_{\alpha_i} \) is pulled back from \( B \), it becomes trivial when pulled back via \( f_\alpha \) so we can pick an isomorphism \( \varepsilon^1 \oplus f_\alpha^\ast(TW|_{\alpha_i}) \cong e^d \). As \( l+1 < d \) this isomorphism can be destabilised to an isomorphism \( f_\alpha^\ast(TW|_{\alpha_i}) \cong e^{d-1} \cong e^{d-l-1} \oplus TS^l \) and by Smale–Hirsch theory \( f_\alpha \) is then homotopic to an immersion with trivial normal bundle. We can make this immersion self-transverse, and as \( 2l < d-1 \) it is then an embedding with trivial normal bundle. Thus each \( f_\alpha \) gives rise to an embedding \( f|_{\alpha_i} : R^{d-l-1} \times S^l \to W|_{\alpha_i} \) representing the same homotopy class. As \( 2l < d-1 \) we may also assume that the \( f_\alpha \) are disjoint, so we obtain an embedding

\[
f|_{\alpha_i} : \Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times S^l \hookrightarrow W|_{\alpha_i},
\]

and as \( L \) only has handles of index \( < d-l-1 \), we may suppose this embedding is disjoint from \( L \). As \( l+1+d < N \), this extends to an embedding

\[
e|_{\alpha_i} : \Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow \{a_i\} \times (0, 1) \times (-1, 1)^{N-1}
\]
which intersects $W|_{a_i}$ precisely on the boundary. Furthermore, as each $S^l \xrightarrow{f_i} W|_{a_i} \to B$ is null-homotopic, the $\theta$-structure $\ell|_{a_i} \circ Df_i$ extends to a $\theta$-structure on $\mathbb{R}^{d-l-1} \times S^l$ and so gives $f_i|_{a_i}$ the data of a $\theta$-surgery, cf. Section 4.3.

We can extend the map $e|_{a_i}$ to an embedding $\Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \{0, 1\} \times (-1, 1)^{N-1}$ using just the cylindrical structure of $W$ over $(a_i - \varepsilon_i, a_i + \varepsilon_i)$, but we wish to extend it to an embedding of $\Lambda_i \times (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \mathbb{R}^{d-l-1} \times D^{l+1}$, which is cylindrical over each $(a_j - \varepsilon_j, a_j + \varepsilon_j)$ and intersects $W$ precisely on the boundary. We will do this by extending it step-by-step over each interval $[a_j, a_{j+1}]$: if it is defined up to $a_j$ we have an embedding

$$e|_{[a_j, a_{j+1}]} : \Lambda_i \times \{a_j\} \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow \{a_j\} \times \{0, 1\} \times (-1, 1)^{N-1},$$

and as the pair $(W|_{[a_j, a_{j+1}]}, W|_{a_{j+1}})$ is $\kappa$-connected, and $l \leq \kappa$, on the boundary this extends to a continuous map

$$f_i|_{[a_j, a_{j+1}]} : \Lambda_i \times [a_j, a_{j+1}] \times \mathbb{R}^{d-l-1} \times S^l \longrightarrow W|_{[a_j, a_{j+1}]}.$$ 

By the Smale–Hirsch argument above, we may perturb this to be a self-transverse immersion of the core, and hence an embedding of the core as $2(l+1) < d$, while keeping it as it was near $a_j$. Shrinking in the $\mathbb{R}^{d-l-1}$-direction, we can ensure that it is an embedding of the whole manifold, and then make the embedding cylindrical over the necessary $\varepsilon$-neighbourhood of the ends and disjoint from $[a_j, a_{j+1}] \times L$.

Finally, as $l + 2 + d < N$ we may extend this to an embedding

$$e|_{[a_j, a_{j+1}]} : \Lambda_i \times \{a_j\} \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow \{a_j\} \times \{0, 1\} \times (-1, 1)^{N-1},$$

which is cylindrical over each $(a_j - \varepsilon_j, a_j + \varepsilon_j)$ and intersects $W$ precisely on the boundary. In total we obtain an embedding

$$e_i : \Lambda_i \times (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \mathbb{R}^{d-l-1} \times D^{l+1} \hookrightarrow (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \{0, 1\} \times (-1, 1)^{N-1}$$

which is cylindrical over each $(a_j - \varepsilon_j, a_j + \varepsilon_j)$ and intersects $W$ precisely on the boundary. Furthermore, by doing the above in increasing order of $i$, we can ensure that the different $e_i$ have disjoint cores: while constructing $e_i$ make sure that its core stays disjoint from those of the $e_j$ for all $j < i$, which is possible as $2(l+1) < d$ and $2(l + 2) < N$.

We let $\Lambda = \prod_{i=0}^N \Lambda_i$, $\delta : \Lambda \to \{0, 1\}$ be given by $\delta(\Lambda_i) = (i, 0)$, and $e$ be given by $\prod_{i=0}^N e_i$, reparametrised using the $\varphi(a_i, \varepsilon_i, a_p, \varepsilon_p)$. Then the data $(\Lambda, \delta, e)$ gives the embedding part of the data of an element of $\hat{Y}_d(a, \varepsilon, (W, \ell W))$, and we must now provide the bundle part. Under the chosen diffeomorphism

$$K|_{(-6,-2)} = (-6, -2) \times \mathbb{R}^{d-l-1} \times S^l \simeq \varphi(a_i, \varepsilon_i, a_p, \varepsilon_p) \times \mathbb{R}^{d-l-1} \times S^l,$$

the embedding $f_i : \Lambda_i \times (a_i - \varepsilon_i, a_i + \varepsilon_i) \times \mathbb{R}^{d-l-1} \times S^l \hookrightarrow W$ gives a $\theta$-structure $\ell_i|_{(-6,-2)} = \ell W \circ Df_i$ on $\Lambda_i \times K|_{(-6,-2)}$. For $\ell_i$ we may take any extension of this $\theta$-structure to $\Lambda_i \times K$, and so only need to know that such an extension exists. This is a purely homotopical problem, and homotopically $K|_{(-6,0)}$ is obtained from $K|_{(-6,-2)}$ by attaching a $D^{l+1}$, so the extension problem can be solved if and only if $\ell_W \circ Df|_{a_i} : T(\Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times S^l) \to \theta \gamma$ extends over $\Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times D^{l+1}$, but we have seen above that it does, because $\Lambda_i \times \{a_i\} \times \mathbb{R}^{d-l-1} \times S^l \to W|_{a_i} \to B$ is null-homotopic.

6.5. **Proof of Theorem 5.13.** Recall that the statement of the theorem is as follows. We work in dimension $2n$, and fix a tangential structure $\theta$ which is reversible (cf. Definition 5.1), a $(2n-1)$-manifold with boundary $L$ equipped with $\theta$-structure, and a collection $A \subset \pi_0(\text{Ob}(\mathcal{C}^{n-1,n-2}_{\theta,L}((\mathbb{R}^N))))$ of objects. This allows us to define the augmented bi-semi-simplicial space

$$D_{\theta,L}^{n-1,A}(\mathbb{R}^N), \bullet \longrightarrow D_{\theta,L}^{n-1,n-2}(\mathbb{R}^N), \bullet.$$
of surgery data, and the second part of Theorem 5.13 states that if the conditions of Theorem 5.2 are satisfied, then the induced map on geometric realisation is a weak homotopy equivalence. (We have already proved the first part of Theorem 5.13 in Section 6.1.) We recall that these conditions are:

(i) \( 2n \geq 6 \),
(ii) \( 3n + 1 < N \),
(iii) \( \theta \) is reversible,
(iv) \( L \) admits a handle decomposition only using handles of index \( < n \),
(v) \( \ell_L : L \to B \) is \((n-1)\)-connected,
(vi) the natural map \( A \to \pi_0(\mathcal{B}G_{\theta,L}^{-1.1-n^2}(\mathbb{R}^N)) \) is surjective.

Note that the penultimate condition implies that for any object, the map \( M \to B \) induced by the tangential structure induces a surjection on \( \pi_* \) for \( * \prec n \).

In many respects the proof of this theorem is very similar to what we did in Section 6.1, but in that section we often used the inequality \( 2(l+1) < d \) so that pairs of transverse \((l+1)\)-dimensional submanifolds of a \( d \)-manifold are automatically disjoint. In Theorem 5.13 \( d = 2n \) and the analogue of \( l \) is \((n-1)\) so this observation fails. Instead, we will use a version of the Whitney trick to separate \( n \)-dimensional submanifolds of our \( 2n \)-manifolds; this accounts for the restriction \( 2n \geq 6 \) in the statement of the theorem.

We proceed precisely as in Definition 6.12 by for \((a,\varepsilon,(W,\ell_W)) \in D_{\theta,L}^{n-1.1-n^2}(\mathbb{R}^N)_p \) letting \( \tilde{Y}_{\bullet}(a,\varepsilon,(W,\ell_W)) \) be the analogue of \( Y_{\bullet}(a,\varepsilon,(W,\ell_W)) \) from Definition 6.12 where instead of asking that \( e \) be an embedding, we only ask for it to be a smooth map which restricts to an embedding on a neighbourhood of \( \Lambda \times C \). We use this to define the bi-semi-simplicial space \( \tilde{D}_{\theta,L}^{n-1.1}(\mathbb{R}^N)_{\bullet,\bullet} \), and by the same argument as Proposition 6.13 the inclusion

\[
D_{\theta,L}^{n-1.1}(\mathbb{R}^N)_{\bullet,\bullet} \hookrightarrow \tilde{D}_{\theta,L}^{n-1.1}(\mathbb{R}^N)_{\bullet,\bullet}
\]

is a weak homotopy equivalence in each bidegree. We are now left to verify the conditions of Theorem 6.2 for the augmented semi-simplicial spaces

\[
\tilde{D}_{\theta,L}^{n-1.1}(\mathbb{R}^N)_{\bullet,\bullet} \to \tilde{D}_{\theta,L}^{n-1.1-n^2}(\mathbb{R}^N)_p.
\]

That the map on 0-simplices has local sections is proved as in the previous two sections.

**Proposition 6.17.** Fix a point \((a,\varepsilon,(W,\ell_W)) \in D_{\theta,L}^{n-1.1-n^2}(\mathbb{R}^N)_p \), and let \( v_1, \ldots, v_k \in \tilde{Y}_{\circ}(a,\varepsilon,(W,\ell_W)) \) be a non-empty collection of pieces of surgery data. Then if \( 2n \geq 6 \) and \( 3n + 1 < N \) there exists a piece of surgery data \( v \in \tilde{Y}_{\circ}(a,\varepsilon,(W,\ell_W)) \) such that each \((v_i,v)\) is a 1-simplex.

**Proof.** Let us write \( v_j = (A_j,\delta^j,\epsilon^j,\ell^j) \). First we let \( v = v_1 \), then we perturb it to have its cores transverse to the cores of all the \( v_j \). We first do the perturbation on the part of the cores inside \( W \). On the boundary the cores are \( n \)-dimensional, so when they are transverse, they intersect in a finite set of points. We now make sure the cores intersect \( W \) only on their boundary, which is possible as \((n+1) + 2n < N \). We finally make sure that the cores are also disjoint on their interiors, which is possible as \( 2(n+1) < N \).

We are left with surgery data \( v \) whose core is disjoint from the cores of \( v_j \) away from \( W \), and on \( W \) intersects the other cores transversely. It has a finite number of transverse intersections with all the other cores in \( W \), so it is enough to give a procedure which reduces the number of intersections by 1. Let \( x \) be such an intersection point, between \( v \) and some \( v_j \). More precisely, suppose it is a point of intersection of the cylinders

\[
e_i(A_i \times (-6, -2) \times \{0\} \times S^{n-1}) \quad e_k^j(A_k^j \times (-6, -2) \times \{0\} \times S^{n-1}).
\]
Claim 6.18. Let $T \subset \mathbb{R}^2$ denote the triangle $\{(x, y) \mid y \leq 0, y + 1 \geq |x|\}$ and $U$ a small open neighbourhood of it, e.g. defined by $y < \varepsilon$, $y + 1 < |x| + \varepsilon$. There is a Whitney disc $w : U \to W$ such that

(i) $w$ is disjoint from $\mathbb{R} \times L$.

(ii) $w|_{[-1,1] \times \{0\}}$ is a path in $W|_{a_p}$ which on its interior is disjoint from all the cores.

(iii) The inverse image of the first cylinder is the line on $\partial T$ from $(0, -1)$ to $(-1, 0)$.

(iv) The height functions $x_1 \circ w$ and $y : T \to \mathbb{R}$ agree up to an affine transformation inside each $(x_1 \circ w)^{-1}(a_j - \varepsilon_j, a_j + \varepsilon_j)$.

Given such a disc, we can extend it to a standard neighbourhood $w(U) \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \subset W$ as in the proof of [Mil65 Theorem 6.6]. Note the argument is easier in this case as we are canceling intersection points against the boundary instead of against each other, and so no framing problems arise. We can further extend this to a neighbourhood

$$w(U) \times \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \times \mathbb{R}^{N+1-2n} \subset \mathbb{R} \times (0, 1) \times \mathbb{R}^{N-1}.$$ 

There is a compactly supported vector field on $U$ which is $\partial/\partial x$ on $D^2$, and we extend it using bump functions in the euclidean directions to this open subset of $\mathbb{R} \times (0, 1) \times \mathbb{R}^{N-1}$. The flow associated to this vector field gives a 1-parameter family of diffeomorphisms $\varphi_t$, and flowing $e_i$ along using $\varphi_t$ will eventually lead to a new $e_i$ whose core has one fewer intersection point with other cores (at least inside $W|_{[a_0-c_0,a_1]}$), but we can then use the cylindrical structure of $W|_{[a_0-c_0,a_p+c_p]}$ to remove any interactions above $a_p$). It will still satisfy condition (iv) of Definition 5.12 by property (iv) above, and the other conditions are clear.

It remains to prove the claim. If it were not for property (iv), the argument is clear: choose an embedded path from $x$ in each cylinder up to $W|_{a_p}$. Together these give an element of $\pi_1(W|_{[a_i,a_j]}), W|_{a_p})$ which is 0 as $\kappa = n - 1 \geq 2$, and so this extends to a continuous map $w|_T : T \to W$ which gives these two paths along the lower part of its boundary and lies in $W|_{a_p}$ in the top part of its boundary. As $2 \cdot 2 < 2n$, this map may be perturbed to be an embedding into $W$, still enjoying these two properties. Finally, as $2 + n < 2n$, $w|_{D^2}$ can be made disjoint from the other cores on its interior. This may now be extended to a map on $U$, enjoying properties (iii) and (iv).

To obtain property (iv) as well, we instead build up the embedding $w|_T$ in pieces inside each $W|_{[a_i,a_j+1]}$, which is possible as each $\pi_1(W|_{[a_i,a_j+1]}), W|_{a_j+1})$ is 0.

In the proof of Proposition 6.10 it was easy to see that for an object $M \in C^{n,l-1}_{\theta,L}(\mathbb{R}^N)$ there exists a piece of $\theta$-surgery data $e : \Lambda \times \mathbb{R}^{d-l-1} \times S^i \hookrightarrow M$ such that the resulting manifold $\overline{M}$ has $\pi_1(\overline{M}) \to \pi_1(B)$ injective, so satisfies condition [v] of Definition 5.13. In the present situation we have $M \in C^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$ and require surgery data so that $\overline{M} \in \mathcal{A}$, to satisfy condition [v] of Definition 5.12.

This is rather more difficult, and we first describe how to accomplish this step.

Lemma 6.19. Let $M \in C^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$ be an object, and suppose that $\theta$ is reversible, $2n \geq 6$, $L$ has a handle structure with only handles of index $< n$, $\ell_L : L \to B$ is $(n-1)$-connected, and $\mathcal{A}$ contains an object in the same path component of $BC^{n-1,n-2}_{\theta,L}(\mathbb{R}^N)$ as $M$. Then there is a piece of $\theta$-surgery data, given by an embedding $e : \Lambda \times \mathbb{R}^{d-l-1} \times S^i \hookrightarrow M$ disjoint from $L$ and a compatible bundle map $T(\Lambda \times \mathbb{R}^{d-l-1} \times D^{i+1}) \to \theta^*g$, such that the resulting surgered manifold $\overline{M}$ lies in $\mathcal{A}$.

Proof. Part of this proof is very similar to [Kri99 pp. 722–724].
We first claim that if there is a morphism $W : M_0 \twoheadrightarrow M_1 \in C_{n-1,n-2}^0(\mathbb{R}^N)$, then there is another, $W'$ say, with the property that $(W', M_0)$ is also $(n-1)$-connected. (By definition, $(W', M_1)$ is $(n-1)$-connected.) In fact, we claim that it is possible to do surgery along a finite set of embeddings of $S^{n-1} \times D^{n+1}$ into the interior of $W$ (and disjoint from $L$), such that the resulting cobordism $W'$ is $(n-1)$-connected with respect to both boundaries. Let us first point out that doing any such $(n-1)$-surgery does not change the property that $\pi_k(W, M_1) = 0$ for $k \leq (n-1)$: up to homotopy it amounts to cutting out a manifold of codimension $(n+1)$ and then attaching a cells of dimension $n$ and $2n$. We have assumed that $L \to B$ is $(n-1)$-connected so $\pi_k(M_0) \to \pi_k(B)$ is surjective for $k \leq n - 1$. Since $M_0 \in C_{n-1,n-2}^0$, it is an isomorphism for $k \leq n - 2$ and similarly for $M_1$. Since $\pi_k(M_1) \to \pi_k(W)$ is an isomorphism for $k \leq n - 2$, we conclude that $\pi_k(M_0) \to \pi_k(W)$ is an isomorphism for $k \leq n - 2$, but it need not be surjective for $k = n - 1$. In fact, the long exact sequence in homotopy groups identifies the cokernel with $\pi_{n-1}(W, M_0) \cong H_{n-1}(W, M_0)$. The fact that $\pi_{n-1}(M_0) \to \pi_{n-1}(B)$ is surjective implies that the composition

$$\text{Ker}(\pi_{n-1}(W) \to \pi_{n-1}(B)) \to \pi_{n-1}(W) \to \pi_{n-1}(W, M_0)$$

is still surjective. By the Hurewicz theorem, $\pi_{n-1}(W, M_0) \cong H_{n-1}(W, \tilde{M}_0)$ is finitely generated as a module over $\pi_1$, and we have proved that there exist finitely many elements $\alpha_i \in \text{Ker}(\pi_{n-1}(W) \to \pi_{n-1}(B))$ which generate the cokernel of $\pi_{n-1}(M_0) \to \pi_{n-1}(W)$. These elements may be represented by disjoint embedded framed spheres in the interior of $W$, and as $L$ has a handle structure with only handles of index $< n$ they can be made disjoint from $L$, and we let $W'$ denote the result of performing surgery. Both pairs $(W', M_0)$ and $(W', M_1)$ are now $(n-1)$-connected, and by Proposition 5.7, $W'$ again admits a $\theta$-structure.

We now return to the proof of the lemma. There is a zig-zag of morphisms in the category $C_{n-1,n-2}^0(\mathbb{R}^N)$ from $M$ to an object of $A$, as $A$ was assumed to hit the path component of $M$. By the above discussion we can suppose that it is a zig-zag of $\theta$-cobordisms which are $(n-1)$-connected relative to both ends. Then, by reversibility, we can reverse the backwards-pointing arrows and obtain a single morphism

$$(C, \ell_C) : (M, \ell_M) \twoheadrightarrow (A, \ell_A) \in C_{n-1,n-2}^0(\mathbb{R}^N),$$

which is $(n-1)$-connected relative to both ends, so $\pi_*(C, A) = \pi_*(C, M) = 0$ for $* \leq n - 1$.

If such a cobordism $C$ admits a Morse function with only critical points of index $n$, then the descending manifolds of the critical points, and $\ell_C$ restricted to them, gives the required $\theta$-surgery data. It remains to produce such a Morse function.

If $\pi_1(L) = 0$ then all of the manifolds appearing above are also simply-connected, and we deduce by Poincaré duality and the Universal coefficient theorem that $H_*(C, M)$ is concentrated in degree $n$ and is free abelian. We can choose a self-indexing Morse function on $C$ and as in the proof of the $h$-cobordism theorem we can first modify it to have no critical points of index 0 or 1 [Mil65, Theorem 8.1], do the same to the negative of the Morse function to remove critical points of index $2n$ and $(2n-1)$, and finally by the Basis Theorem [Mil65, Theorem 7.6] we can diagonalise the differentials in the Morse homology complex, and so modify the Morse function to only have critical points of index $n$.

When $\pi_1(L) \neq 0$ we must go to a little more trouble, and use techniques from the proof of the $s$-cobordism theorem. As these are less well known, we go into more detail, but recommend [Lüc02] and [Ker65] for details of that argument. As above, pick a self-indexing Morse function on $C$ and let us write

$$\pi = \pi_1(L) = \pi_1(M) = \pi_1(C) = \pi_1(A)$$
for the common fundamental group, and $\mathbb{Z}[\pi]$ for its integral group ring.

When $M \hookrightarrow C$ is 1-connected, [Mil65, Theorem 8.1] is still true: we may modify the Morse function to have no critical points of index 0 or 1, and as above do the same on the opposite Morse function to eliminate critical points of index $2n$ and $(2n - 1)$. The cores of the handles given by this Morse function on the universal cover give a cell complex with cellular chain complex $C_*(\widetilde{C}, \widetilde{M})$, and $C_*(\widetilde{C}, \widetilde{A})$ for the opposite Morse function. These are chain complexes of based free $\mathbb{Z}[\pi]$-modules, and geometric Poincaré duality gives an isomorphism

$$C_*(\widetilde{C}, \widetilde{M}) \cong \text{Hom}_{\mathbb{Z}[\pi]}(C_{2n-\ell}(\widetilde{C}, \widetilde{A}), \mathbb{Z}[\pi])$$

of chain complexes, by sending basis elements to their “dual” basis elements (we use the convention of [Wal70, Ch. 2] to interchange right and left $\mathbb{Z}[\pi]$-module structures).

The chain complex $C_{2n-\ell}(\widetilde{C}, \widetilde{A})$ is one of free $\mathbb{Z}[\pi]$-modules and $0 = \pi_*(C, A) = \pi_*(\widetilde{C}, \widetilde{A}) = H_*(\widetilde{C}, \widetilde{A}; \mathbb{Z})$ for $* \leq n - 1$, so it is acyclic in degrees $2n - * \leq n - 1$. By the Universal coefficient spectral sequence, the same is true for its $\mathbb{Z}[\pi]$-dual and so $C_*(\widetilde{C}, \widetilde{M})$ is acyclic for $* \geq n + 1$. Furthermore $0 = \pi_*(C, M) = \pi_*(\widetilde{C}, \widetilde{M}) = H_* (\widetilde{C}, \widetilde{M}; \mathbb{Z})$ for $* \leq n - 1$, so the homology of $C_*(\widetilde{C}, \widetilde{M})$ is concentrated in degree $n$.

By the usual modification technique, we can use handle exchanges to modify the Morse function to only have critical points of index $n$ and $(n - 1)$. We are left with a short exact sequence of $\mathbb{Z}[\pi]$-modules

$$0 \rightarrow H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z}) \rightarrow C_n(\widetilde{C}, \widetilde{M}) \xrightarrow{\partial_n} C_{n-1}(\widetilde{C}, \widetilde{M}) \rightarrow 0.$$

The rightmost term is a free $\mathbb{Z}[\pi]$-module and so this sequence is split: in particular, $H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z})$ is stably free as a $\mathbb{Z}[\pi]$-module. If $H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z})$ is not actually free as a $\mathbb{Z}[\pi]$-module, there cannot exist a Morse function on $C$ with only critical points of index $n$. In this case we replace $C$ by $C \# S^n \times S^n$ for $g$ sufficiently large (and this manifold admits a $\theta$-structure by Proposition 5.47). This has the effect of adding on a large free $\mathbb{Z}[\pi]$-module to $H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z})$, so we may assume that this homology group is now free, and pick a basis of it.

Choosing a splitting of the short exact sequence above, we obtain an isomorphism

$$(6.3)
C_*(\widetilde{C}, \widetilde{M}) \cong H_*(\widetilde{C}, \widetilde{M}; \mathbb{Z}) \oplus C_{n-1}(\widetilde{C}, \widetilde{M})$$

of based free $\mathbb{Z}[\pi]$-modules, and so an element of $K_1(\mathbb{Z}[\pi])$. However, the basis we chose for $H_n(\widetilde{C}, \widetilde{M}; \mathbb{Z})$ was not geometrically meaningful and we are free to change it. After possibly stabilising $C$ further, it is possible to choose a basis for which (6.3) represents the zero class in $K_1(\mathbb{Z}[\pi])$, and hence in the Whitehead group $\text{Wh}(\pi)$ too. We may then use the modification lemma to rearrange the index $n$ critical points of the Morse function so that $\partial_n : C_n(\widetilde{C}, \widetilde{M}) \rightarrow C_{n-1}(\widetilde{C}, \widetilde{M})$ is simply projection onto the first few basis elements: this allows us to cancel all the critical points of index $(n - 1)$.

\begin{proposition}
$\tilde{Y}_0(a, \varepsilon, (W, \ell_W))$ is non-empty as long as $3n + 1 < N, 2n \geq 6$, $\theta$ is reversible, $L$ admits a handle structure with only handles of index $< n$, $\ell_L : L \rightarrow B$ is $(n - 1)$-connected, and the natural map $A \rightarrow \pi_0(B_{\theta_L}(\ell_L^{-1}n-2([\mathbb{R}^N]))$ is surjective.
\end{proposition}

\begin{proof}
We follow the proof of Proposition 6.10 with a few changes. Let $d = 2n$ and $l = n - 1$. The first step of Proposition 6.10 is to produce for each $W|_a$, the $\theta$-surgery data $f|_{a_i}$. The method described in that proposition no longer works, and we use Lemma 6.10 to produce the necessary data instead. From this point up to constructing the maps $e_i|_{a_i, \varepsilon, a_i + \varepsilon}$ there is no difference, and the argument given in Proposition 6.10 goes through.
\end{proof}
It remains to explain how given an embedding \( e_i|\partial j \) we can extend it to \( e_i|\{a_j, a_{j+1}\} \). We proceed in the same way: we have the embedding

\[
f_i|\{a_j, a_{j+1}\} : \Lambda_i \times \{a_j, a_{j+1}\} \times \mathbb{R}^n \times S^{n-1} \hookrightarrow W|\{a_j, a_{j+1}\}
\]
disjoint from \( L \), which extends to a continuous map

\[
f_i|\{a_j, a_{j+1}\} : \Lambda_i \times \{a_j, a_{j+1}\} \times \mathbb{R}^n \times S^{n-1} \longrightarrow W|\{a_j, a_{j+1}\}
\]
as \((W|\{a_j, a_{j+1}\}, W|\{a_{j+1}\})\) is \((n - 1)\)-connected by assumption. We can again make this be a self-transverse immersion of the core, but this no longer implies that the core is embedded: it will have isolated points of self-intersection. As \(2n \geq 6\) we can remove these using the Whitney trick, as in the proof of Proposition 6.17. The core may still intersect the core of \([a_j, a_{j+1}] \times L\), as they are both of dimension \(n\) inside a \(2n\)-manifold, but we can again use the Whitney trick to separate them. Given \(f_i|\{a_j, a_{j+1}\}\) which is an embedding of the core and whose core is disjoint from that of \([a_j, a_{j+1}] \times L\), we can shrink in the \(\mathbb{R}^n\) direction and isotope it to get an embedding disjoint from \([a_j, a_{j+1}] \times L\), and then extend this to \(e_i|\{a_j, a_{j+1}\}\) as in Proposition 6.16.

This gives the required embeddings \(e_i\), which are then combined as in Proposition 6.16 to get \((\Lambda, \delta, \varepsilon, e)\), the embedding part of the data of an element of \(\tilde{Y}_0(a, \varepsilon, (W, \ell W))\). The remaining bundle part of the data consists of an extendible (cf. Definition 5.10) \(\theta\)-structure \(\ell\) on \(\Lambda \times K\) which agrees with \(\ell_W \circ D(\partial e)\) on \(\Lambda \times K|\{(-6, -2)\}\), and such that the effect of the \(\theta\)-surgery described by this data (i.e. the restriction of \(\ell\) to \(K|\{(-6, 0)\}\)) lies in \(A\). We will describe a construction which for each \(\lambda \in \Lambda\) produces a \(\theta\)-structure \(\ell_\lambda\) on \(K \subset \mathbb{R}^{n+1} \times \mathbb{R}^n\); these are then combined in the obvious way.

Firstly, there is a unique \(\theta\)-structure on the subspace

\[
K|\{(-6, -2)\} = (\{-6, -2\} \times \mathbb{R}^n) \times S^{n-1},
\]
such that the embedding \(\partial e\) preserves \(\theta\)-structures (i.e. satisfies requirement (iii) of Definition 5.12). Secondly, the manifold

\[
K|\{(-6, 0)\} \subset \mathbb{R}^{n+1} \times \mathbb{R}^n
\]
is obtained from (6.3) by attaching an \(n\)-handle. To extend the \(\theta\)-structure requires a null-homotopy of the structure map \(S^{n-1} \rightarrow B\) from the \(\theta\)-structure on (6.3), and this is provided as part of the \(\theta\)-surgery data in Lemma 6.19. Finally, we need to prove that this structure extends to a \(\theta\)-structure over all of \(K\), which is furthermore extendible. To see this, we observe that the restriction of this structure to the subspace

\[
K|\{(-6, 0)\} - (B_2^{n+1}(0) \times \mathbb{R}^n) = ((-6, 0] \times \mathbb{R}^n - B_2^{n+1}(0)) \times S^{n-1}
\]
admits a (homotopically unique) extension to the manifold \((\mathbb{R} \times \mathbb{R}^n) \times S^{n-1}\), since this deformation retracts to (6.3). Restrict this extension to the manifold \((\mathbb{R} \times \mathbb{R}^n - B_2^{n+1}(0)) \times S^{n-1} = K - (B_2^{n+1}(0) \times \mathbb{R}^n)\) and glue it with the structure on (6.5), to get a \(\theta\)-structure on

\[
K|\{(-\infty, 0)\} \cup (K - B_2^{n+1}(0) \times \mathbb{R}^n).
\]

It remains to prove that this structure can be extended to all of \(K\). It is easy to see that the stabilised structure extends to a bundle map \(e^1 \oplus TK \rightarrow \theta^*\gamma\), but then Lemma 5.7 implies that the unstabilised bundle map also extends. □
In this section, we use the results of Sections 3–6 to prove the theorems stated in Section 1. As explained in Remark 1.14, Theorem 1.2 follows from Theorem 1.8 which we prove in full detail in Sections 7.2, 7.3, 7.4, and 7.5 below. Nevertheless, we shall first outline in some detail how to deduce Theorem 1.2 directly from the results in Sections 3–6 in the hope of putting the general case in a useful perspective.

In the following we shall work entirely in even dimension $d = 2n > 4$ and always set $N = \infty$. (Suitably interpreted, all results hold for sufficiently large finite $N$, but we shall not pursue this here.)

### 7.1. Outline of proof of Theorem 1.2

To apply the theorems in Sections 3–6, we must specify a structure $\theta : B \to BO(2n)$ and a $(2n - 1)$-dimensional manifold $L$ with $\theta$-structure $\ell_L : \varepsilon^1 \oplus TL \to \theta^* \gamma$. For the purpose of deducing Theorem 1.2 we let $\theta = \theta^n : BO(2n)/(n) \to BO(2n)$ be the $n$-connective cover, and let $L \subset (-1, 0) \times \mathbb{R}^N$ be a $(2n - 1)$-manifold with collared boundary, diffeomorphic to $D^{2n-1}$. Now, the inclusion functors induce weak equivalences

$$BC^{n-1,n-2}_\theta \simeq BC^{\alpha}_\theta \simeq BC^{\alpha}_\theta (\infty, 1) \simeq \psi^\alpha (\infty, 1) \simeq \Omega \infty^{-1} MT \theta^n,$$

obtained by applying Theorem 3.1 $(n - 1)$ times, Theorem 3.1 $n$ times, Proposition 2.10 and Proposition 2.16 respectively, composed with the weak equivalence $\psi^\alpha (\infty, 1) \simeq \Omega \infty^{-1} MT \theta^n$ from [GRW10] Theorem 3.12.

To apply the result of Section 5 we must specify a subset $\mathcal{A} \subset \pi_0(\text{Ob}(C^{n-1,n-2}_{\theta^n})).$ There is a unique path component of $\text{Ob}(C^{n-1,n-2}_{\theta^n})$ consisting of manifolds diffeomorphic to $S^{2n-1}$ (with its standard smooth structure). Letting $\mathcal{A}$ consist of this path component, $C^{n-1,A}_{\theta^n}$ is the full subcategory of $C^{n-1,n-2}_{\theta^n}$ on the objects in $\mathcal{A}$. It is clear that $\theta^n$ is spherical and hence reversible (cf. Proposition 5.6), that $L \simeq D^{2n-1}$ admits a handle decomposition using only handles of index less than $n$ (since a single 0-handle suffices), and that the map $\ell_L : D^{2n-1} \to BO(2n)/(n)$ is $(n - 1)$-connected (it is even $n$-connected). Theorem 5.2 would give the weak equivalence $BC^{n-1,A}_{\theta^n} \simeq BC^{n-1,n-2}_{\theta^n}$, except that the theorem requires $\mathcal{A}$ to contain at least one object from each path component of $BC^{n-1,n-2}_{\theta^n}$, which may not hold here. Therefore we let $\mathcal{\overline{A}} \subset \pi_0(\text{Ob}(C^{n-1,n-2}_{\theta^n}))$ be the union of $\mathcal{A}$ and the set of path components of objects which map to a path component of $BC^{n-1,n-2}_{\theta^n}$ disjoint from that of $\mathcal{A}$. Theorem 5.2 does apply to $\mathcal{\overline{A}}$ and gives the weak equivalence

$$BC^{n-1,\overline{A}}_{\theta^n} \simeq BC^{n-1,n-2}_{\theta^n} \simeq \Omega \infty^{-1} MT \theta^n.$$

By definition, the inclusion $BC^{n-1,\mathcal{A}}_{\theta^n} \subset BC^{n-1,\overline{A}}_{\theta^n}$ is just the inclusion of a path component, and hence becomes a homeomorphism after taking based loop space, so we get the weak equivalence

$$\Omega BM \simeq \Omega \infty MT \theta^n.$$

The category $C^{n-1,A}_{\theta^n}$ is not quite a monoid, since it contains multiple objects (namely all those manifolds diffeomorphic to $S^{2n-1}$), but the space of objects is path connected, and we let $M$ be the endomorphism monoid of some chosen object. Then the nerve $N_p M$ is the fibre of the fibration $N_p C^{n-1,A}_{\theta^n} \to (N_0 C^{n-1,A}_{\theta^n})_{p+1},$ and the Bousfield–Friedlander theorem ([BF78] Theorem B.4) or the earlier special case [May72] Theorem 12.7] implies that the inclusion $BM \to BC^{n-1,A}_{\theta^n}$ is a weak equivalence (this can also be seen more geometrically as in [GRW10] Proposition 4.26]). Altogether, we obtain a weak equivalence $\Omega BM \simeq \Omega \infty MT \theta^n.$
The monoid $\mathcal{M}$ is described up to homotopy as

$$\mathcal{M} \simeq \prod_{W} BD\text{iff}(W, D),$$

where $W$ ranges over $(n-1)$-connected closed $2n$-manifolds admitting a $\theta^n$-structure, and $D \subset W$ is a submanifold equipped with a diffeomorphism $D \cong D^{2n}$. (Admitting a $\theta^n$-structure is equivalent to being parallelisable over the $n$-skeleton. Since the pair $(W, D)$ is $(n-1)$-connected, the space of $\theta^n$-structures is contractible when it is non-empty.) In this description, the monoid structure corresponds to connected sum and therefore $\mathcal{M}$ is homotopy commutative. The classical “group completion” theorem (cf. [MS76]) then gives an isomorphism in homology

$$H_*(\mathcal{M})[\pi_0\mathcal{M}^{-1}] \xrightarrow{\cong} H_*(\Omega^\infty MT^\theta),$$

where the left hand side denotes the ring $H_*(\mathcal{M})$ localised by inverting the multiplicative subset $\pi_0\mathcal{M}$. Finally, we claim that the localisation on the left hand side may be calculated by inverting only the element of $\pi_0\mathcal{M}$ corresponding to $T = S^n \times S^n$. To see this, we use that if $W$ is an element of $\mathcal{M}$, then there is another element $\overline{W}$ with the same underlying manifold, but where the identification $D^{2n} \cong D \subset W$ is changed by an orientation-reversing diffeomorphism. Then the connected sum $W \# \overline{W}$ may be identified with $\partial((W - \text{int}(D)) \times [0, 1])$, and in degree 0 and $\mathbb{Z}^b$ in degree $n$ and is parallelisable; cancelling critical points in a Morse function as in [Mi65] proves that $(W - \text{int}(D)) \times [0, 1]$ is diffeomorphic to the boundary connected sum of $b$ copies of $S^n \times D^{n+1}$. Therefore the element $[W] \in \pi_0\mathcal{M}$ is invertible in the ring $H_*(\mathcal{M})[T^{-1}]$, with inverse $[T]^{-b}[\overline{W}]$. The localisation by inverting the element $[T]$ may be calculated as a direct limit, and hence we have the homology equivalence

$$\text{hocollim}(\mathcal{M} \xrightarrow{T} \mathcal{M} \xrightarrow{T} \cdots) \rightarrow \Omega^\infty MT^\theta^n,$$

which upon restricting to the appropriate path component gives Theorem 1.8 \hfill \Box

We now embark on the detailed proof of Theorem 1.8 which will occupy Sections 7.2, 7.3, and 7.4, as follows. Suppose given a spherical tangential structure $\theta : B \rightarrow BO(2n)$, a $(2n-1)$-manifold $L$ which admits a handle structure with handles of index less than $n$, and a $\theta$-structure $\ell_L : \varepsilon^1 \oplus TL \rightarrow \theta^\gamma$ such that the underlying map $L \rightarrow B$ is $(n-1)$-connected. In this situation the results of Section 3.3 apply, and will be summarised in Section 7.2 below as a weak equivalence between $\Omega^\infty MT^\theta$ and the loop space of the classifying space of a category $\mathcal{C}$. Then in Section 7.4 we apply a version of the “group completion” theorem to relate the homology of $\Omega^\infty BC$ to the homology of morphism spaces of $\mathcal{C}$, suitably localised using the theory of universal $\theta$-ends developed in Section 7.3. In Section 7.5 we explain how to apply these results to prove Theorem 1.8. Finally, in Section 7.6 we explain how to deduce the results about algebraic localisation from Theorem 1.8.

7.2. The category $\mathcal{C}$. Suppose that $2n > 4$, let $\theta : B \rightarrow BO(2n)$ be a spherical tangential structure, and $L$ be a $(2n-1)$-dimensional manifold with boundary which admits a handle structure using handles of index at most $(n-1)$. Let $\ell_L$ be a $\theta$-structure on $L$, and suppose that the underlying map $L \rightarrow B$ is $(n-1)$-connected.

Picking a collared embedding $L \hookrightarrow (-1/2, 0) \times (-1, 1)^{n-1}$, we have defined a category $\mathcal{C}_{\theta,L}$. Finally, let $A \subset \pi_0(\text{Ob}(\mathcal{C}_{\theta,L}))$ be the set of objects $(M, \ell)$ for which $M - \text{int}(L)$ is diffeomorphic to a handlebody with handles of index at
most \((n-1)\). In Definition 2.11 we have defined
\[
\mathcal{C}_{\theta,L}^{n-1,A} \subset \mathcal{C}_{\theta,L}^{n-1,n-2}
\]
as the full subcategory on those objects contained in \(\mathcal{A}\).

**Definition 7.1.** A morphism in \(\mathcal{C}_{\theta,L}^{n-1,A}\) is a manifold \(W \subset [0,t] \times (-1,1)^{\infty}\) with \(W \cap x_2^{-1}((-\infty,0]) = [0,t] \times L\) (equality as \(\theta\)-manifolds). Write
\[
W^\circ = W - ([0,t] \times \text{int}(L))
\]
for morphisms and similarly
\[
M^\circ = M - \text{int}(L)
\]
for objects. Morphisms or objects \(X \in \mathcal{C}_{\theta,L}^{n-1,A}\) are completely determined by \(X^\circ\) and we denote by \(\mathcal{C}\) the category with
\[
\text{Ob}(\mathcal{C}) = \{M^\circ \mid M \in \text{Ob}(\mathcal{C}_{\theta,L}^{n-1,A})\}
\]
\[
\text{Mor}(\mathcal{C}) = \{W^\circ \mid W \in \text{Mor}(\mathcal{C}_{\theta,L}^{n-1,A})\}
\]
made into a topological category by insisting that the functor \(\mathcal{C}_{\theta,L}^{n-1,A} \to \mathcal{C}\) given by \(X \mapsto X^\circ\) is an isomorphism of topological categories.

Our work in Sections 3, 4, 5, and 6 determines the homotopy type of the space \(\Omega BC\), as follows. (We emphasise that in this section \(L \to B\) is assumed \((n-1)\)-connected and \(\theta: B \to BO(2n)\) is assumed spherical.)

**Theorem 7.2.** There is a weak equivalence
\[
\Omega BC \simeq \Omega^{\infty} MT\theta,
\]
where loops are based at any object \(P^\circ \in \text{Ob}(\mathcal{C})\), and \(MT\theta\) is the Thom spectrum associated to \(\theta: B \to BO(2n)\).

**Proof.** This is identical with the argument given in Section 7A. Briefly, we define the set \(\overline{\mathcal{A}}\) to be the union of \(\mathcal{A}\) and all objects not in a path component of \(\mathcal{BC}_{\theta,L}^{n-1,n-2}\) containing an element of \(\mathcal{A}\), and use the string of weak equivalences
\[
\mathcal{BC}_{\theta,L}^{n-1,1} \simeq \mathcal{BC}_{\theta,L}^{n-1,n-2} \simeq \mathcal{BC}_{\theta,L}^{n-1} \simeq \mathcal{BC}_{\theta,L} \simeq \psi_{\theta}(\infty,1) \simeq \psi_{\theta}(\infty,1) \simeq \Omega^{\infty} MT\theta
\]
as well as the homeomorphism \(\mathcal{BC} \cong \mathcal{BC}_{\theta,L}^{n-1,1}\), and the fact that the inclusion \(\mathcal{BC}_{\theta,L}^{n-1,1} \to \mathcal{BC}_{\theta,L}^{n-1,1}\) is a homeomorphism onto the path components it hits. \(\square\)

From now on we will work with the category \(\mathcal{C}\), and we need a lemma to translate what the connectivity conditions in \(\mathcal{C}_{\theta,L}^{n-1,1}\) mean after cutting \(\text{int}(L)\) out.

**Lemma 7.3.**

(i) Let \(N\) be an object in \(\mathcal{C}_{\theta,L}^{n-1,1}\) and \(W: M \to N\) be a morphism in the larger category \(\mathcal{C}_{\theta,L}^{n-1,n-2}\). Then the pair \((W,M)\) is \((n-1)\)-connected.

(ii) Let \(W^\circ: M^\circ \to N^\circ\) be a morphism in \(\mathcal{C}\). Then the pairs \((W^\circ, M^\circ)\) and \((W, N)\) are \((n-1)\)-connected.

**Proof.** By definition, \(\mathcal{A}\) consists of manifolds \(M\) such that \(M - \text{int}(L)\) admits a handle structure with handles of index at most \((n-1)\), and reversing such a handle structure we see that this is equivalent to \(M - \text{int}(L)\) being obtained from \(\partial L\) by attaching handles of index at least \(n\).

The pairs \((N, \{0\} \times L)\) and \((W, N)\) are both \((n-1)\)-connected, from which we deduce that \((W, \{0\} \times L)\), and so \((W, \{0\} \times L)\), is also \((n-1)\)-connected. As \((M, \{0\} \times L)\) is \((n-2)\)-connected, we deduce that \((W, M)\) is \((n-1)\)-connected, which establishes the first part.
For the second part, observe that $W$ deformation retracts to $W^\circ \cup N$. Therefore all pairs $(N, L)$, $(M, L)$, $(W, N)$ and $(W, W^\circ)$ are homotopy equivalent to relative CW complexes with relative cells of dimension at least $n$. As $n \geq 3$, it follows that all the inclusions between $L$, $\partial L$, $N$, $N^\circ$, $M$, $M^\circ$, $W$, $W^\circ$, and $\partial W^\circ$ induce isomorphisms on fundamental groups, and we write $\pi$ for the common fundamental group. There are isomorphisms

$$H_*(W, N; \mathbb{Z}[\pi]) \cong H_*(W^\circ \cup N, N; \mathbb{Z}[\pi]) \cong H_*(W^\circ, N^\circ; \mathbb{Z}[\pi])$$

given by the homotopy equivalence $W^\circ \cup N \simeq W$ and excision of $\text{int}(L)$ respectively, and so $H_*(W^\circ, N^\circ; \mathbb{Z}[\pi]) = 0$ for $s \leq n - 1$. Hence $(W^\circ, N^\circ)$ is $(n - 1)$-connected, and the same argument applies to the pair $(W^\circ, M^\circ)$.

**Definition 7.4.** Recall from the proof of Proposition 2.16 that we constructed a $\theta$-manifold $D(L)$ which is diffeomorphic to the double of $L$. This contains $L \subset D(L)$ with its standard $\theta$-structure, and we write $\overline{\mathcal{T}}$ for the $\theta$-manifold $D(L) - \text{int}(L)$. As $L$ has a handle structure with handles of index at most $(n - 1)$, $D(L)$ can be obtained from $L$ by attaching handles of index at least $n$. We extend the embedding of $L$ to an embedding $D(L) \to (-1, 1)^\infty$ to get objects $D(L) \in ^{\theta-1,A}_{\theta,L}$, and $D(L)^\circ = \overline{\mathcal{T}} \in \mathcal{C}$.

The relevance of the category $\mathcal{C}$ to Theorem 11.8 is evident from the following proposition.

**Proposition 7.5.** For any object $P \in ^{\theta-1,A}_{\theta,L}$, there is a weak equivalence $\varphi_P : \mathcal{C}(\overline{\mathcal{T}}, P) \to N^\theta(P, \ell_P)$ such that if $K : P \rightsquigarrow P'$ is a morphism in $^{\theta-1,A}_{\theta,L}$, then the diagram

$$
\begin{array}{c}
\mathcal{C}(\overline{\mathcal{T}}, P) \xrightarrow{\circ K} \mathcal{C}(\overline{\mathcal{T}}, (P')^\circ) \\
\varphi_P \downarrow \quad \varphi_{P'} \\
N^\theta(P, \ell_P) \xrightarrow{\circ K} N^\theta(P', \ell_{P'})
\end{array}
$$

commutes, i.e. $\varphi_P$ is a natural transformation of functors $^{\theta-1,A}_{\theta,L} \to \text{Top}$.

**Proof.** $\varphi_P$ is defined as the composition

$$\varphi_P : \mathcal{C}(\overline{\mathcal{T}}, P) \cong ^{\theta-1,A}_{\theta,L}(D(L), P) \xrightarrow{\nu} ^{\theta-1}_{\theta,L}(\emptyset, P) \cong N^\theta(P, \ell_P),$$

where $V : \emptyset \rightsquigarrow D(L)$ is the $\theta$-cobordism constructed in the proof of Proposition 2.10 and the last map is $(t, W) \mapsto W - t \cdot e_1$. It is clear that the square commutes, so it remains to show that $\varphi_P$ is a homotopy equivalence. To do this, consider first the trivial cobordism $P \times [0, 1]$. This contains $L \times [1/4, 3/4]$, which is diffeomorphic to $V$ and has a homotopic $\theta$-structure. Cutting this out gives a $\theta$-cobordism $P \sqcup D(L) \rightsquigarrow P$ containing $L \times I$. Composition along the incoming $P$ of this $\theta$-cobordism defines a continuous map

$$N^\theta(P, \ell_P) \xrightarrow{\nu} ^{\theta-1}_{\theta,L}(\emptyset, P) \to ^{\theta-1}_{\theta,L}(D(L), P) = ^{\theta-1,A}_{\theta,L}(D(L), P) \cong \mathcal{C}(\overline{\mathcal{T}}, P^\circ)$$

which is homotopy inverse to $\varphi_P$. \qed

### 7.3. Universal $\theta$-ends and the proof of Addendum 11.9

**Let $\theta : B \to BO(2n)$ be spherical.** Recall from Definition 1.4 that a universal $\theta$-end is a submanifold $K \subset [0, \infty) \times \mathbb{R}^\infty$ with $\theta$-structure $\ell_K$ such that $x : K \to [0, \infty)$ has the natural numbers as regular values. We insist that

(i) Each $K|_{[i, i+1]}$ is a highly connected cobordism, i.e. is $(n - 1)$-connected relative to either end,

(ii) For each highly connected $\theta$-cobordism $W : K|_i \rightsquigarrow P$, there is an embedding $j : W \hookrightarrow K|_{[i, \infty)}$, and a homotopy $\ell_K \circ Dj \simeq \ell_W$, both relative to $K|_i$. 

We wish to have the notion of universal θ-end available to us in the cobordism category $C$. Let $K[0, K][1, \ldots]$ be a sequence of objects in $C$, and $K[|i-1, a|] : K[i-1] \to K[i]$ be a sequence of morphisms in $C$. For integers $0 \leq a < b$, let us write

$$K[|a, b|] = K[|a, a+1|] \circ K[|a+1, a+2|] \circ \cdots \circ K[|b-1, b|]$$

for the composition of the morphisms from $K[|a|]$ to $K[|b|]$. There are natural inclusions $K[|0, a|] \subset K[|0, a+1|] \subset \cdots$ and we let $K$ denote the union: a non-compact smooth manifold with θ-structure. The symbol $K[|a, b|]$ is not ambiguous, and we can also make sense of $K[|a, \infty|] = \cup_{b>a}K[|a, b|]$.

**Definition 7.6.** Say that $K$ is a universal θ-end in $C$ if, in the notation just introduced, properties (i) and (ii) above hold, while in (iii) we require $W$ to be a morphism in $C$.

Proposition 7.8 below proves Addendum 1.9 together with a version for universal θ-ends in $C$. Before giving the proof, we make some preparations.

**Lemma 7.7.** Let $W : N \to M$ be a highly connected cobordism. There exist cobordisms $F : M \to M$ and $G : N \to N$ such that $W \circ F$ and $G \circ W$ both admit handle structures using only handles of index $n$. Similarly, if $W$ is a morphism in the category $C$, then $F$ and $G$ can be taken to be morphisms in this category, with the same conclusion (in this case, attaching handles along embeddings $S^{n-1} \times D^n \to \text{int}(N)$).

**Proof.** The pairs $(W, M)$ and $(W, N)$ are both $(n-1)$-connected, so if we let $F$ and $G$ be sufficiently large multiples of $([0,1] \times M)\#(S^n \times S^n)$ and $([0,1] \times N)\#(S^n \times S^n)$ respectively, then, by the method used in the proof of Lemma 6.19, both $W \circ F$ and $G \circ W$ admit the required handle decompositions. □

**Proposition 7.8.** Let $K[|i, i+1|]$ be a sequence of composable morphisms in $C$ and let $K = \cup K[|0, a|]$ be the infinite composition. Then $(K, \ell_K)$ is a universal θ-end in $C$ if and only if the following conditions hold.

(i) For each integer $i$, the map $\pi_n(K[|i, \infty|]) \to \pi_n(B)$ is surjective, for all basepoints in $K$.

(ii) For each integer $i$, the map $\pi_{n-1}(K[|i, \infty|]) \to \pi_{n-1}(B)$ is injective, for all basepoints in $K$.

(iii) For each integer $i$, each path component of $K[|i, \infty|]$ contains a submanifold diffeomorphic to $S^n \times S^n - \text{int}(D^{2n})$, which in addition has null-homotopic structure map to $B$.

Similarly, if $K[|i, i+1|]$ is a sequence of composable, highly connected cobordisms in $C$, then $K$ is a universal θ-end if and only if conditions (i), (ii) and (iii) hold. (I.e., Addendum 1.9 is true).

**Proof.** We shall only give the proof in $C$, the other case being completely analogous. To prove the “if” direction, we must show that for each integer $i$ and each highly connected cobordism $W : K[|i|] \to P$ with θ-structure $\ell_W$, there is an embedding $j : W \hookrightarrow K[|i, \infty|]$ and a homotopy $\ell_K \circ Dj \simeq \ell_W$, all relative to $K[|i|]$.

By Lemma 7.7, for any such $W$ there is a cobordism $F : P \to P$ so that $W \circ F$ admits a handle structure with handles of index $n$ only, so it suffices to consider the case where $W$ consists of a single $n$-handle relative to $K[|i|]$, attached along an embedding $S^{n-1} \times D^n \to K[|i|]$. We need to find an extension of this embedding into $K[|i, \infty|]$ (with the correct homotopy class of θ-structure). The map $S^{n-1} \times D^n \to K[|i|] \to K[|i, \infty|]$ is null-homotopic by assumption (iii); it is certainly null-homotopic when composed with $K[|i, \infty|] \to B$, because that composition is equal to the composition $S^{n-1} \times D^n \to K_i \to W \to B$. Thus there is a continuous map
To construct such a manifold, we may represent \( K \) is homotopic relative to \( W \) that this admits an embedding into \( S^n \) manifold be the boundary connected sum of the cylinder \( K \) too.

To prove the "only if" direction, we must prove that any universal embedding \( f : W \to K|_{i, \infty} \) by assumption (i), we can change \( f \) by adding on elements of \( \pi_n(K|_{i, \infty}) \) so that

\[
W \xrightarrow{f} K|_{i, \infty} \xrightarrow{\ell_W} B
\]

is homotopic relative to \( K|_i \) to \( \ell_W \). The \( \theta \)-structures on \( W \) and \( K \) now give bundle isomorphisms

\[
TW \cong \ell_W^* \theta^* \gamma \cong f^* \ell_K^* \theta^* \gamma \quad \text{and} \quad TK|_{i, \infty} \cong \ell_K^* \theta^* \gamma,
\]

and hence an isomorphism \( TW \cong f^* TK|_{i, \infty} \) relative to \( K|_i \), i.e. \( f : W \to K \) is covered by a bundle map \( TW \to TK \), which near \( K|_i \) is the derivative of the embedding. By Smale–Hirsch theory, we may therefore homotope \( f : W \to K|_{i, \infty} \) to an immersion, without changing it near \( K|_i \).

Finally, we explain how to replace the immersion \( f : W \to K|_{i, \infty} \) by an embedding. It suffices to make \( f \) an embedding near a core \( (D^n, \partial D^n) \subset (W, K|_i) \) of the \( n \)-handle, and we shall write \( f : D^n \to K|_{i, \infty} \) for the restriction of \( f \). After changing \( f \) by a small isotopy, we may assume that all self-intersections of \( f \) are transverse. We shall explain how to remove one self-intersection point of \( f \), changing the homotopy class of \( f \) in the process. Around a self-intersection point, choose a coordinate \( \mathbb{R}^n \times \mathbb{R}^n \to K|_{i, \infty} \) so that \( \mathbb{R}^n \times \{0\} \) and \( \{0\} \times \mathbb{R}^n \) give local coordinates around the two preimages of the double point. By assumption (ii) we can find an embedded \( S^n \times S^n - \text{int}(D^{2n}) \subset K|_{i, \infty} \) with null-homotopic map to \( B \). We can also assume it is disjoint from the image of \( f \), since \( W \) is compact. Then we choose an embedded path from this \( S^n \times S^n - \text{int}(D^{2n}) \) to the patch \( \mathbb{R}^n \times \mathbb{R}^n \), and thicken it up: inside this we have a subset diffeomorphic to the boundary connect sum

\[
(D^n \times D^n) \natural (S^n \times S^n - \text{int}(D^{2n})),
\]

which the image of \( f \) intersects in \( D^n \times \{0\} \cup \{0\} \times D^n \). Inside this subset there are embedded disjoint discs which give the same embedding on the boundary, and we can modify \( f \) by redefining it to have these discs as image instead. This reduces by 1 the number of geometric self-intersections of \( f \), and up to homotopy we have added an element of \( \pi_n(S^n \times S^n - \text{int}(D^{2n})) \) to the homotopy class of \( f \). As \( S^n \times S^n - \text{int}(D^{2n}) \to K|_{i, \infty} \to B \) was null-homotopic, we have not changed the homotopy class of \( f \) in \( B \).

After finitely many steps, we have changed \( f \) to an embedding. The corresponding embedding \( f : W \to K|_{i, \infty} \) (obtained by thickening \( f \) up again) is homotopic to the original one after composing with \( \ell_K : K|_{i, \infty} \to B \), so \( \ell_K \circ f \simeq \ell_W \) relative to \( K|_i \). Hence the induced \( \theta \)-structure on \( W \) is homotopic to the given one relative to \( K|_i \).

To prove the “only if” direction, we must prove that any universal \( \theta \)-end \( (K, \ell_K) \) satisfies the three conditions. It is clear that (iii) is necessary: For any \( i \) we can let \( W \) be the boundary connected sum of the cylinder \( K|_i \times [i, i + 1] \) and the (parallelisable) manifold \( S^n \times S^n - \text{int}(D^{2n}) \) equipped with a trivial \( \theta \)-structure. Universality implies that this admits an embedding into \( K|_{i, \infty} \), and hence \( S^n \times S^n - \text{int}(D^{2n}) \) does too.

For property (i), it suffices to prove that for any \( i \) and any \( \alpha \in \pi_n(B) \), there exists a morphism \( W_\alpha \in \mathcal{C}(K|_i, P) \) for some \( P \), with \( \alpha \in \text{Im}(\pi_n(W_\alpha) \to \pi_n(B)) \). To construct such a manifold, we may represent \( \alpha \) by a map \( S^n \to B \) and lift the composition \( \theta \circ \alpha : S^n \to B \to BO(2n) \) to a map \( f : S^n \to BO(n) \). If we let \( D \to S^n \) be the disc bundle of the vector bundle classified by \( f \), the tangent bundle of \( D \) is classified by \( \theta \circ \alpha \), and therefore admits a \( \theta \)-structure whose underlying map \( S^n \simeq D \to B \), which represents \( \alpha \). We can then let \( W_\alpha \) be the boundary connected sum of \( K|_i \times [i, i + 1] \) and \( D \).
Finally, for property (ii), we use that each $K_{[i,j+1]}$ is a highly connected cobordism to see that $\pi_{n-1}(K_{[i]}) \to \pi_{n-1}(K_{[i,\infty]})$ is surjective. It therefore suffices to prove that for any $\alpha \in \text{Ker}(\pi_{n-1}(K_{[i]})) \to \pi_{n-1}(B)$, there exists a morphism $W_\alpha \in C(K_{[i]}, P)$ for some $P$, with $\alpha \in \text{Ker}(\pi_{n-1}(K_{[i]})) \to \pi_{n-1}(W_\alpha)$). We may represent $\alpha$ by an embedding $S^{n-1} \to K_{[i]}$. Since the composition $S^{n-1} \to K_{[i]} \to B \to BO(2n)$ is trivial, the normal bundle of the embedding is stably trivial and hence trivial, so we may extend to an embedding $j: S^{n-1} \times D^n \to K_{[i]}$. The underlying manifold of the morphism $W_\alpha$ is then defined as the trace of surgery along $j$, and a $\theta$-structure is constructed from a choice of null-homotopy of $S^{n-1} \to K_{[i]} \to B$. 

The following three propositions establish further useful properties of universal $\theta$-ends. The first proposition gives a refinement of property (ii), which lets us exert more control on the behaviour of the embedding $j$ which is provided by (ii). Propositions 7.10 and 7.11 give strong existence and uniqueness properties for universal $\theta$-ends (and universal $\theta$-ends in $C$), which essentially say that a universal $\theta$-end $(K, \ell_K)$ is determined up to diffeomorphism (respecting $\theta$-structures) by $(K_{[0]}, \ell_{K_{[0]}})$.

**Proposition 7.9.** If $(K, \ell_K)$ is a universal $\theta$-end (or a universal $\theta$-end in $C$) then it also satisfies

(iii) For each highly connected $\theta$-cobordism $W: K_{[i]} \to P$, there is a $k \gg i$, an embedding $j: W \to K_{[i,k]}$, and a homotopy $\ell_K \circ D_j \simeq \ell_{W_j}$, both relative to $K_{[i]}$, such that the complement of $j(W)$ is a cobordism $Z: P \to K_{[k]}$ which is highly connected.

**Proof.** Let us treat the case of a universal $\theta$-end; working in $C$ can be done in the same way. As $W$ is $(n-1)$-connected relative to either end, Lemma 7.7 applies, and for sufficiently large $g$, the manifold $W' = W \#(gS^n \times S^n) = W \circ (([0,1] \times P) \#(gS^n \times S^n))$ admits a handle structure relative to $K_{[i]}$ using handles of index $n$ only.

By universality, there is an embedding of $\theta$-manifolds $j': W' \to K_{[i,k]}$ relative to $K_{[i]}$. We wish to modify this embedding, and increase $k'$, so that

$$\{e_\alpha: (D^n \times D^n, D^n \times S^{n-1}) \to (W', K_{[i]})\}_{\alpha \in I}$$

denotes the collection of relative $n$-handles of $W'$, there exist embedded spheres $\{\alpha: S^n \to K_{[i,k]}\}_{\beta \in I}$ so that

$$e_\alpha(\{0\} \times D^n) \cap f_\beta(S^n) = \begin{cases} \emptyset & \alpha \neq \beta \\ \{\ast\} & \alpha = \beta. \end{cases}$$

We can ensure this property as follows: by property (iii) of Proposition 7.8, we may find an embedded $S^n \times S^n - \text{int}(D^{2n})$ in $K_{[k',\infty]}$ with null-homotopic structure map to $B$. We may form the connect-sum of $S^n \times D^n$ with the handle $e_\alpha$ away from the other handles, and let $f_\alpha$ be the embedding of $\{\ast\} \times S^n$. Repeating this for each handle, we can ensure the required property, and because the $S^n \times S^n - \text{int}(D^{2n})$ we used had trivial structure map to $B$, the new embedding of $W'$ we obtain still has the correct homotopy class of $\theta$-structure.

We denote by $j': W' \to K_{[i,k]}$ this improved embedding, and $Z'$ the complement of the image of $j'$. The cobordism $W'$ only has relative $n$-handles, and $n \geq 3$, so $K_{[i]}, W', P, Z', K_{[i,k]}, K_{[k]}$ all have the same fundamental group, which we denote by $\pi$. To understand the connectivity of the pair $(Z', K_{[k]})$, we look at the long exact sequence for $\pi$-homology of the triple $(K_{[i,k]}, Z', K_{[k]})$ and use excision $(K_{[i,k]}, Z') \sim (W', P)$ to obtain the exact sequence

$$H_n(K_{[i,k]}, K_{[k]}; \pi) \to H_n(W', P; \pi) \to H_{n-1}(Z', K_{[k]}; \pi) \to 0.$$
By Poincaré duality $H_\alpha(W',P;\mathbb{Z}[\pi]) \cong \text{Hom}_{\mathbb{Z}[\pi]}(H_\alpha(W',K_1;\mathbb{Z}[\pi]),\mathbb{Z}[\pi])$ we see that $\varphi$ is surjective, as $\varphi(\epsilon_j):H_\alpha(W',K_1;\mathbb{Z}[\pi]) = \mathbb{Z}[\pi]\langle \epsilon_\alpha | \alpha \in I \rangle \rightarrow \mathbb{Z}[\pi]$ is the dual basis element to $\epsilon_\alpha$, so $(Z',K_{[k]})$ is $(n-1)$-connected, as required. That the pair $(Z',P)$ is $(n-1)$-connected follows by the long exact sequence for $\mathbb{Z}[\pi]$-homology of the triple $(K_1[i,k],W',K_1[i])$ and excision $(Z',P) \sim (K_1[i,k],W')$.

Finally, we note that if $Z$ denotes the complement of the image of $j = j'_W$, then we have $Z = ([0,1] \times P)\#(gS^n \times S^n) \cup Z'$ so it is also a highly connected cobordism.

\[\square\]

**Proposition 7.10.** Let $(K,\ell_K)$ and $(K',\ell_{K'})$ be universal $\theta$-ends, and suppose we are given a highly connected $\theta$-cobordism $W : K_0 \rightarrow K'_0$. Then there is a diffeomorphism $\varphi : W \cup_{K_0} K' \cong K$, and a homotopy $\ell_K \circ D\varphi \simeq \ell_{W\cup K'}$, both relative to $K_0$. Furthermore, there is a weak homotopy equivalence

$$\text{hocolim}_{i \to \infty} N^\theta(K_i,\ell_{K_i}) \simeq \text{hocolim}_{i \to \infty} N^\theta(K'_i,\ell_{K'_i}).$$

**Proof.** By replacing $K'$ with $W \cup_{K_0} K'$, we may as well assume that $K_0 = K'_0$ as $\theta$-manifolds, and that $W$ is the trivial cobordism. As $K$ is a universal $\theta$-end, we may find an embedding of $\theta$-manifolds $j_1 : K_0[0,1] \hookrightarrow K'[0,0,1]$ relative to $K_0$, and by Proposition 7.9 we may suppose its complement $Z_1 : K_1 \sim K'[1]$ is highly connected. Now, as $K$ is a universal $\theta$-end, we may find an embedding of $\theta$-manifolds $j'_1 : Z_1 \hookrightarrow K_1[1,1]$ relative to $K_1$, again with highly-connected complement $Z_2 : K'[1,1] \sim K_1[1,1]$. Together, $j^{-1}_1$ and $j'_1$ give an embedding of $\theta$-manifolds $K'_0[0,0,1] \sim K_0[0,0,1]$. Continuing in this way, we produce the required diffeomorphism $\varphi$ and homotopy.

For the second part, note that we have constructed a direct system

$$N^\theta(K_0) \xrightarrow{K_0} N^\theta(K_1) \rightarrow N^\theta(K'_0) \rightarrow N^\theta(K'_1) \rightarrow \cdots$$

which contains cofinal subsystems which are also cofinal in either of the direct systems used to form the homotopy colimits in the statement. \[\square\]

**Proposition 7.11.** Let $\pi_\alpha(B)$ be countable. Then for any object $(M,\ell_M) \in \mathcal{C}$ there exists a universal $\theta$-end $(\mathcal{K},\ell_{\mathcal{K}})$ in $\mathcal{C}$ with $(K_0,\ell_{K_0}) = (M,\ell_M)$. Furthermore, $\mathcal{K} \cup ([0,\infty) \times L) \rightarrow \text{a universal } \theta$-end.

**Proof.** In the proof of Proposition 7.8 we saw that for each $\alpha \in \text{Ker}(\pi_{n-1}(M) \rightarrow \pi_{n-1}(B))$, there exists a morphism $W_\alpha \in \mathcal{C}(M,P)$ with $\alpha \in \text{Ker}(\pi_{n-1}(M) \rightarrow \pi_{n-1}(W_\alpha))$, and for each element $\alpha \in \pi_n(B)$, there exists a morphism $W_\alpha \in \mathcal{C}(M,P)$ with $\alpha \in \text{Im}(\pi_n(W_\alpha) \rightarrow \pi_n(B))$. A priori, the target $P$ depends on $\alpha$, but as $\theta$ has been assumed to be spherical, it is reversible (by Proposition 5.0), and we may find another morphism $P \sim M$; after composing, we may assume that $M = P$ so we have endomorphisms $W_\alpha \in \mathcal{C}(M,M)$. We then construct a universal $\theta$-end in $\mathcal{C}$ by letting $K_{[i]} = M$ for each integer $i \geq 0$ and letting each $K_{[i,k]}$ be of the form $W_\alpha \#(S^n \times S^n)$, where the $\alpha_k$ form a sequence of elements of $\pi_n(B) \cup \text{Ker}(\pi_{n-1}(M) \rightarrow \pi_{n-1}(B))$ in which each element occurs infinitely often. (This is possible because $\pi_n(B)$ is assumed countable and $\pi_{n-1}(M)$ is automatically countable.) It then follows from Proposition 7.8 that $K$ is a universal $\theta$-end in $\mathcal{C}$.

It is obvious that gluing $[0,\infty) \times L$ to a universal $\theta$-end in $\mathcal{C}$ gives a universal $\theta$-end, since the homotopical properties in Proposition 7.8 are clearly preserved. \[\square\]

**Corollary 7.12.** Let $(K,\ell_K)$ be a universal $\theta$-end for which $P = (K_0,\ell_{K_0})$ is an object of $\mathcal{C}_0$. Then we may isotope the proper embedding $K \rightarrow [0,\infty) \times (-1,1)^\infty$ and homotope the bundle map $\ell_K : TK \rightarrow \theta^*\gamma$, both relative to $K_0$, after which $K$ is of the form $K^o \cup ([0,\infty) \times L)$ where $K^o$ is a universal $\theta$-end in $\mathcal{C}$. 


Proof. By Proposition 7.8, the structure map \( \ell_K : K \to B \) induces a surjection on \( \pi_n \). Since \( K \) is a manifold, \( \pi_n(K) \), and hence \( \pi_n(B) \), is countable, so there exists a universal \( \theta \)-end in \( C \), by Proposition 7.11. Denoting this by \( K^\circ \), the \( \theta \)-manifold \( K^\circ \cup ([0, \infty) \times L) \) is a universal \( \theta \)-end, and hence by Proposition 7.10 is isomorphic to the original \( K \).

7.4. Group completion. Let us return to the category \( C \) of Section 7.2. Assigning to a morphism \( \mathcal{W} \in \mathcal{C}(\mathcal{P}_0, \mathcal{P}_1) \) the corresponding 1-simplex in the nerve of \( C \) gives a continuous map \( \mathcal{C}(\mathcal{P}_0, \mathcal{P}_1) \to \Omega \mathcal{P}_0, \mathcal{P}_1 B \mathcal{C} \), analogous to the map \( \mathcal{M} \to \Omega \mathcal{B} \mathcal{M} \) in the outline in Section 7.1. As in that section, the effect in homology can be studied by a version of the “group completion” theorem. The classical group completion theorem concerns a topological monoid \( M \), and says that the map \( H_n(M) \to H_n(\Omega BM) \) is an algebraic localisation at the multiplicative subset \( \pi_0(M) \subset H_\ast(M) \). The group completion theorem holds under the assumption that this localisation admits a calculus of right fractions, cf. [MS76]. A similar result holds for topological categories, and here implies that \( H_\ast(\Omega B \mathcal{C}) \) is a suitable direct limit of \( H_\ast(\mathcal{C}(\mathcal{P}_0, \mathcal{P}_1)) \), generalising the localisation in the monoid case. As in the monoid case, some assumption is needed in order to apply the group completion theorem: Lemma 7.15 below can be seen as a multi-object version of admitting a calculus of right fractions.

Theorem 7.13. Let

\[
\begin{array}{cccccccc}
K_0 & \overset{K_{[0,1]}}{\Longrightarrow} & K_1 & \overset{K_{[1,2]}}{\Longrightarrow} & K_2 & \overset{K_{[2,3]}}{\Longrightarrow} & K_3 & \overset{K_{[3,\infty]}}{\longrightarrow} & \cdots
\end{array}
\]

be a sequence of composable morphisms in \( \mathcal{C} \) such that \( K \) is a universal \( \theta \)-end in the category \( \mathcal{C} \) and \( \mathcal{C}(\mathcal{T}, K_{[0]} \neq \emptyset \). Then there is a map

\[
\text{hocolim}_{i \to \infty} \mathcal{C}(\mathcal{T}, K_{[i]}) \longrightarrow \Omega B \mathcal{C}
\]

which is a homology equivalence.

The proof will be based on Proposition 7.14 below. Let \( F_i : \mathcal{C}^{\text{op}} \to \text{Top} \) denote the representable functor \( \mathcal{C}(\mathcal{P}_0, K_{[i]}) \) and let \( F_\infty : \mathcal{C}^{\text{op}} \to \text{Top} \) denote the (object-wise) homotopy colimit of the natural transformations \( F_i \to F_{i+1} \) given by right composition with \( K_{[i,i+1]} \).

Proposition 7.14. The functor \( F_\infty \) sends each morphism in \( \mathcal{C} \) to a homology equivalence.

Given this proposition, by [GMTW09, Theorem 7.1] and the discussion following it, the pull-back square

\[
\begin{array}{ccc}
F_\infty(\mathcal{L}) & \longrightarrow & B(\mathcal{C} \downarrow F_\infty) \simeq \text{hocolim}_i B(\mathcal{C} \downarrow F(i)) \simeq \ast \\
\downarrow & & \downarrow \\
(\mathcal{L}) & \longrightarrow & \Omega B \mathcal{C}
\end{array}
\]

is homology cartesian and so \( F_\infty(\mathcal{L}) \to \Omega B \mathcal{C} \) is a homology equivalence, which establishes Theorem 7.13.

Proof of Proposition 7.14. By Lemma 7.7, it suffices to prove that \( F_\infty \) sends any cobordism admitting a handle structure with a single \( n \)-handle to a homology isomorphism: indeed, in the notation of that lemma, for any cobordism \( \mathcal{W} \), the functor \( F_\infty \) sends both \( \mathcal{W} \circ F \) and \( G \circ \mathcal{W} \) to homology isomorphisms, but then it must send \( \mathcal{W} \) to one as well. We therefore consider a cobordism \( \mathcal{W} \in \mathcal{C}(\mathcal{N}, \mathcal{M}) \) admitting a
handle structure with a single \( n \)-handle. The cobordism \( W \) gives a map of direct systems

\[
\begin{array}{c}
\mathcal{C}(M, K|_0) & \to & \mathcal{C}(M, K|_1) & \to & \mathcal{C}(M, K|_2) & \to & \mathcal{C}(M, K|_3) & \to & \cdots \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\mathcal{C}(N, K|_0) & \to & \mathcal{C}(N, K|_1) & \to & \mathcal{C}(N, K|_2) & \to & \mathcal{C}(N, K|_3) & \to & \cdots .
\end{array}
\]

Taking homotopy colimits of the rows gives a map \( F_\infty(M) \to F_\infty(N) \), and Lemma 7.15 below implies that the induced map on homology is a bijection, finishing the proof of Proposition 7.14

**Lemma 7.15.** Let \( W : N \to M \) be a cobordism which is obtained by attaching a single \( n \)-handle to \( N \). For each \( i \) there is a \( k \geq i \) such that the commutative square (7.1)

\[
\begin{array}{ccc}
\mathcal{C}(M, K|_i) & \to & \mathcal{C}(M, K|_k) \\
W_0 & \to & W_0
\end{array}
\]

admits a dotted map making the top square commute up to homotopy, and a (possibly different) dotted map making the bottom square commute up to homotopy.

**Proof.** The objects \( M \) and \( N \) in \( \mathcal{C} \) are \((2n-1)\)-dimensional submanifolds of \([0,1] \times \mathbb{R}^\infty \) (with \( \theta \)-structure), and the morphism \( W \in \mathcal{C}(N, M) \) is a submanifold of \([0, t] \times [0, 1] \times \mathbb{R}^\infty \). Rotating \( W \) in the first two coordinate directions gives a submanifold \( \overline{W} \subset [0, t] \times (-1, 0] \times \mathbb{R}^\infty \) with incoming boundary \( \{0\} \times \overline{M} \) and outgoing boundary \( \{t\} \times N \). As in the proof of Proposition 2.18 the \( \theta \)-structure on \( M \) extends to a \( \theta \)-structure on the closed manifold \( \overline{M} \cup M \subset (-1, 1) \times \mathbb{R}^\infty \), giving an object \( \langle M, M \rangle \in \mathcal{C}_n^{n-1} \) with a canonical null-bordism \( V \in \mathcal{C}^{n-1}_\emptyset (\emptyset, \langle M, M \rangle) \). Similarly, we have objects \( \langle M, K|_i \rangle = \overline{M} \cup K|_i \) and \( \langle N, K|_i \rangle = \overline{N} \cup K|_i \), and the submanifold \( \overline{W} \cup ([0, t] \times K|_i) \subset [0, t] \times (-1, 1) \times \mathbb{R}^\infty \) inherits a \( \theta \)-structure from \( W \), giving an element of \( \mathcal{C}^{n-1}_\emptyset (\langle M, K|_i \rangle, \langle N, K|_i \rangle) \) which we shall denote \( \langle W, K|_i \rangle \). The resulting diagram

\[
\begin{array}{ccc}
\mathcal{C}(M, K|_i) & \to & \mathcal{C}(N, K|_i) \\
\mathcal{N}(\langle M, K|_i \rangle) & \to & \mathcal{N}(\langle N, K|_i \rangle)
\end{array}
\]

homotopy commutes, where the vertical equivalences are as in Proposition 7.6. The diagram of solid arrows in (7.1) may now be replaced with

\[
\begin{array}{ccc}
\mathcal{N}(\langle M, K|_i \rangle) & \to & \mathcal{N}(\langle M, K|_k \rangle) \\
\mathcal{N}(\langle N, K|_i \rangle) & \to & \mathcal{N}(\langle N, K|_k \rangle)
\end{array}
\]

where \( \langle M, K|_{i,k} \rangle = ([i, k] \times \overline{M}) \cup K|_{i,k} \subset [i, k] \times (-1, 1) \times \mathbb{R}^\infty \), and similarly for \( \langle N, K|_{i,k} \rangle \).

Let us first show that there is a dotted map making the top triangle commute up to homotopy, for some \( k \gg i \). We wish to find an embedding (of \( \theta \)-manifolds) of \( \langle W, K|_i \rangle \) into \( \langle M, K|_{i,k} \rangle \) relative to \( \langle M, K|_i \rangle \), with complement a \( \theta \)-cobordism
Z : ⟨N, K|₁⟩ \rightsquigarrow ⟨M, K|₂⟩. If we can ensure that \( Z, ⟨M, K|₂⟩ \) is \((n-1)\)-connected, then gluing on \( Z \) gives a map
\[ o Z : N^\#(⟨N, K|₁⟩) \rightarrow N^\#(⟨M, K|₂⟩) \]
making the top triangle commute (as \( \langle W, K|₁ \rangle \circ Z \cong \langle M, K|₂ \rangle \) as \( \theta \)-manifolds), as required.

By definition of the category \( C \), \( \mathcal{M} \) is obtained from its boundary, \( \partial L_1 \), by attaching handles of index \( n \) and above. Thus, by transversality, the attaching map for the \( n \)-handle of \( W \) relative to \( \mathcal{M} \) may be assumed to have image in a collar neighbourhood \([-\varepsilon, 0] \times \partial L_1 \subset \mathcal{M} \). Thus \( \langle W, K|₁ \rangle \) may be obtained from \( \langle M, K|₁ \rangle \) by attaching a single \( n \)-handle along \( f : S^{n-1} \times D^n \hookrightarrow [0, \varepsilon] \times \partial L_1 \subset K|₁ \), so up to diffeomorphism (relative to its incoming boundary) the cobordism \( \langle W, K|₁ \rangle \) is of the form \( \langle M, W' \rangle \) for some cobordism \( W' : K|₁ \hookrightarrow X \) in \( C \). As \( K|₁ \) is a universal \( \theta \)-end in the category \( C \), there exists an embedding of \( \theta \)-manifolds \( j' : W' \hookrightarrow K|₂ \) relative to \( K|₁ \), for some \( k \gg i \), and by Proposition 7.9 we may assume that its complement \( Z' \) is highly connected. Gluing \( M \) back in, we obtain an embedding \( j : \langle W, K|₁ \rangle \hookrightarrow \langle M, K|₂ \rangle \) relative to \( \langle M, K|₂ \rangle \) whose complement \( Z \cong \langle M, Z' \rangle \) is highly connected, as required.

To produce the dotted map making the bottom triangle commute up to homotopy, we must produce an embedding relative to \( ⟨N, K|₁⟩ \) of \( ⟨W, K|₁⟩ \) into \( ⟨N, K|₂⟩ \), for some suitably large \( k \), with an appropriate connectivity condition on its complement. As we shall explain, this reduces to the same embedding problem as for the upper triangle. We have collar neighbourhoods \([-\varepsilon, 0] \times \partial L \subset \mathcal{N} \) and \([0, \varepsilon] \times \partial L \subset K|₁ \), and as above, we can suppose \( W \) is obtained from \( \mathcal{N} \) by attaching a single \( n \)-handle along a map \( f : S^{n-1} \times D^n \hookrightarrow [-\varepsilon, 0] \times \partial L \subset \mathcal{N} \). We now consider \( \mathcal{N} \) to lie inside
\[ ([i - \varepsilon, i] \times (\mathcal{N} \cup K|₁)) \cup K|₂ \]
where we may extend \( f \) inside \([i - \varepsilon, i] \times [-\varepsilon, \varepsilon] \times \partial L \) to an embedding of \([0, 1] \times S^{n-1} \times D^n \) so that \( \{1\} \times S^{n-1} \times D^n \) is embedded into \( \{i\} \times [0, \varepsilon] \times \partial L \subset K|₁ \). Since \( K|₂ \) is a universal \( \theta \)-end in \( C \), we may extend the embedding of \( \{1\} \times S^{n-1} \times D^n \) to an embedding of the handle \( \{1\} \times D^n \times D^n \) into \( K|₂ \) having highly connected complement, and such that the \( \theta \)-structure is homotopic to the one given on the \( n \)-handle of \( W \). By compactness, the handle has image in \( K|₂ \) for some \( k \gg i \), and we extend \( f \) cylindrically to an embedding
\[ ([{-1, 1}] \times S^{n-1} \times \{1\} \times D^n) \times D^n \hookrightarrow ([i - \varepsilon, i] \times (\mathcal{N} \cup K|₁)) \cup (K|₂ \cup [i, k] \times \mathcal{N}) \]
which sends \( \{-1\} \times S^{n-1} \times D^n \) to \( \{k\} \times \mathcal{N} \). The source of this map is diffeomorphic to a tubular neighbourhood of the \( n \)-handle in \( W \), and the target is diffeomorphic relative to \( K|₂ \cup \mathcal{N} \) to \( K|₂ \cup ([i, k] \times \mathcal{N}) \) to \( (N, K|₂) \cup ([i, k] \times \mathcal{N}) = (N, K|₂) \).

The argument above can not be improved to show that \( F_\infty \) sends each morphism in \( \mathcal{C} \) to a weak homotopy equivalence, since the dotted maps we constructed in no sense preserve basepoints. The case \( n = 0 \) gives rise to the following example from [MS76]: we have \( F_\infty(\varnothing) \cong \mathbb{Z} \times B\Sigma_\infty \) and the morphism \( 1 : \varnothing \rightarrow \varnothing \) given by a single point induces the shift map on \( \Sigma_\infty \), that is, the map induced by the self-embedding given by \( \{1, 2, \ldots\} \cong \{2, 3, \ldots\} \hookrightarrow \{1, 2, \ldots\} \). This is not surjective, so the map is not a homotopy equivalence; it is however a homology equivalence, by the argument we have presented.

7.5. Proof of Theorem 1.8. In the situation of Theorem 1.8, we have a \( \theta \)-manifold \( (K, \ell|K) \) and a proper map \( x_1 : K \rightarrow [0, \infty) \) with the integers as regular values, satisfying the property of being a universal \( \theta \)-end (cf. Definition 1.7). Let \( \theta' : B' \rightarrow B \xrightarrow{\theta} BO(2n) \) be obtained as the \( n \)-th stage of the Moore–Postnikov tower.
of $\ell_K : K \to B$, and $\ell'_K$ be the $\theta'$-structure on $K$ given by the Moore–Postnikov factorisation. By Proposition 7.8, the map $K \to B$ induces an injection in $\pi_{n-1}$ and a surjection in $\pi_n$, so the $n$th and $(n-1)$st stages of the Moore–Postnikov tower actually agree, and in particular the homotopy fibres of $B' \to B$ are $(n-2)$-types.

The following two lemmas allow us to work with $\theta'$-manifolds instead of $\theta$-manifolds for many purposes.

**Lemma 7.16.** Let $W$ be a manifold with boundary $\partial W$, and suppose that $(W, \partial_0 W)$ is $(n-1)$-connected, for $\partial_0 W \subset \partial W$ a collection of boundary components. If $\ell'_{\partial_0 W}$ is a $\theta'$-structure with underlying $\theta$-structure $\ell_{\partial_0 W}$, then

$$\text{Bun}_0(TW, (\theta')^*; \ell'_{\partial_0 W}) \to \text{Bun}_0(TW, \theta^*; \ell_{\partial_0 W})$$

is a weak homotopy equivalence. Consequently, the natural map induces a weak equivalence

$$\mathcal{N}^{\theta'}(K_1|_i, \ell'_{K_1|_i}) \xrightarrow{\simeq} \mathcal{N}^\theta(K_1|_i, \ell_{K_1|_i}).$$

**Proof.** As the homotopy fibres of $B' \to B$ are $(n-2)$-types and $(W, \partial_0 W)$ is $(n-1)$-connected, the space of lifts

$$\partial_0 W \xrightarrow{\ell'_{\partial_0 W}} B' \xrightarrow{\ell_W} B$$

is contractible, for each $\theta$-structure $\ell_W$ on $W$ restricting to $\ell_{\partial_0 W}$ on the boundary. But this space of lifts is easily identified with the homotopy fibre of the map $\text{Bun}_0(TW, (\theta')^*; \ell'_{\partial_0 W}) \to \text{Bun}_0(TW, \theta^*; \ell_{\partial_0 W})$ over the point $\ell_W$.

The last claim follows from the case $\partial_0 W = \partial W$ by forming the homotopy orbit space by the action of $\text{Diff}(W, \partial W)$ and taking disjoint union over all $W$ with $\partial W = K_1|_i$, for which $(W, \partial W)$ is $(n-1)$-connected. \qed

**Lemma 7.17.** The $\theta'$-manifold $(K, \ell'_K)$ is a universal $\theta'$-end.

**Proof.** We verify the conditions of Definition 7.7. The cobordisms $K_1|_{i-1, i}$ are highly connected, as we have assumed that $K$ is a universal $\theta$-end. If $(W : K_1|_i \to P, \ell_W)$ is a highly connected $\theta'$-cobordism, with underlying $\theta$-structure $\ell_W$, then by assumption there is an embedding $j : W \to K_1|_{i, \infty}$ and a homotopy $\ell_K \circ D j \simeq \ell_W$, all relative to $K_1|_i$, but then by Lemma 7.13 there is also a homotopy $\ell'_K \circ D j \simeq \ell'_W$ relative to $K_1|_i$. \qed

We can now give the proof of Theorem 7.8. Recall that the theorem asserts a homotopy equivalence between the homotopy colimit of the direct system

$$(7.2) \quad \mathcal{N}^\theta(K_0, \ell_K) \xrightarrow{K_0|_{n-1}} \mathcal{N}^\theta(K_1, \ell_K) \xrightarrow{K_1|_{n-1}} \mathcal{N}^\theta(K_2, \ell_K) \xrightarrow{K_2|_{n-1}} \cdots$$

and the infinite loop space $\Omega^\infty MT\theta'$. By Lemmas 7.16 and 7.17, it suffices to prove the theorem in the case $\theta = \theta'$, i.e. when $\ell_K : K \to B$ is $n$-connected. In order to apply Theorem 7.13 we first need to define a $\theta$-manifold $L$ (in order to have the category $\mathcal{C}$ defined). To do so, we pick a self-indexing Morse function $f : K_0 \to [0, 2n-1]$ and let $L = f^{-1}([0, n-1])$. Then the inclusions $L \to K_0$ and $K_0 \to K$ are both $(n-1)$-connected, so the structure map $L \to B$ is $(n-1)$-connected and we have defined the category $\mathcal{C}$, satisfying Theorem 7.13. By Proposition 7.10 we may replace $(K, \ell_K)$ with any other universal $\theta$-end without changing the homotopy type of the homotopy colimit (7.2), as long as $K_0|_0$ is unchanged, and by Corollary 7.12 there exists a universal $\theta$-end of the form $K^\circ \cup ([0, \infty) \times L)$, where $K^\circ$ is a universal
θ-end in C. Now, by Proposition 7.15 the direct system (7.2) is homotopy equivalent to
\[
\mathcal{C}(L, K_{[0,1]}^n) \xrightarrow{K_{[0,1]}} \mathcal{C}(L, K_{[0,1]}^n) \xrightarrow{K_{[0,1]}} \mathcal{C}(L, K_{[0,1]}^n) \xrightarrow{K_{[0,1]}} \cdots.
\]
By Theorem 7.13 the homotopy colimit is homotopy equivalent to $\Omega BC$, which in turn is weakly equivalent to $\Omega^\infty MT\theta = \Omega^\infty MT\theta'$, by Theorem 7.12.

7.6. Proof of Lemma 7.12 and Theorem 7.13. Let us first show that $K \subset K_0$ is a submonoid, and that it is commutative. Recall that $K_0$ was the set of isomorphism classes of highly connected cobordisms $K \subset [0,1] \times \mathbb{R}^\infty$ with θ-structure, starting and ending at $(P, \ell_P)$ and that $K$ is the subset admitting representatives containing $[0,1] \times (P - A)$ with product θ-structure, where $A \subset P$ is a closed regular neighbourhood of a simplicial complex of dimension at most $(n - 1)$ inside $P$. Let $K_0, K_1 : P \to P$ be two such cobordisms and let $K_1$ have support in $A_1$, a regular neighbourhood of a simplicial complex $X_1$ of dimension at most $(n - 1)$. As $P$ is $(2n - 1)$-dimensional, we can perturb the $X_1$ to be disjoint and then shrink the $A_i$ so they are disjoint. But if $W_0$ and $W_1$ have support in the disjoint sets $A_0$ and $A_1$, then $W_0 \cup W_1$ has support in $A_0 \cup A_1$ which is a regular neighbourhood of $X_0 \cup X_1$ which is again a simplicial complex of dimension at most $(n - 1)$. Furthermore $K_0 \circ K_1$ is isomorphic to the θ-bordism $K_{01}$ which is supported in $A_0 \cup A_1$ and agrees with $K_i$ on $[0,1] \times A_i$, and this in turn is isomorphic to $K_1 \circ K_0$, so $K$ is commutative.

Recall that we have a monoid map $K' \to K$, where $K'$ is defined like $K$, but with $\theta'$ instead of $\theta$. We saw in Lemma 7.10 that the map $N^{\theta'}(P, \ell_P) \to N^\theta(P, \ell_P)$ is a weak equivalence, and we claim that a similar obstruction theoretic argument shows that $K' \to K$ is an isomorphism. Explicitly, $[0,1] \times P$ has a canonical lift of its θ-structure to a θ'-structure. If an element of $K$ is represented by a cobordism $K$ supported in $A \subset P$, it contains the subset $\{(0) \times P\} \cup \{(0,1) \times (P - A)\} \cup \{1\} \times P$ which has a canonical θ'-structure. Because $A$ is a regular neighbourhood of a simplicial complex of dimension at most $(n - 1)$, the manifold $K$ is obtained up to homotopy from this subset by attaching cells of dimension at least $n$, so up to homotopy there is a unique extension of the lift. This shows that $K' \to K$ is a bijection.

Before embarking on the proof of Theorem 7.15 we establish the following useful strengthening of assumption (iii) of that theorem.

Lemma 7.18. Let $[W] \in K$ be such that each path component of $W$ contains a submanifold diffeomorphic to $S^n \times S^n - \text{int}(D^{2n})$. Then each path component of $3W = W \circ W \circ W$ contains such a submanifold which in addition has null-homotopic structure map to $B$.

Proof. Let us suppose that $W$ is path connected: otherwise we repeat the argument below for each path component. Finding an embedded $S^n \times S^n - \text{int}(D^{2n})$ is equivalent to finding two embedded $n$-spheres with trivial normal bundles, which intersect at a single point. By assumption, this holds for $W$ so we have
\[S^n \times S^n - \text{int}(D^{2n}) \hookrightarrow W \xrightarrow{\ell_W} B,\]
which in $\pi_n$ induces a homomorphism $\mathbb{Z} \oplus \mathbb{Z} = \pi_n(S^n \times S^n - \text{int}(D^{2n})) \to \pi_n(B)$, sending the basis elements to $x, y \in \pi_n(B)$.

In a separate copy of $W$ we have a framed embedding
\[S^n \times \{x\} \xrightarrow{\text{reflection}} S^n \times \{x\} \hookrightarrow S^n \times S^n - \text{int}(D^{2n}) \hookrightarrow W\]
which in $\pi_n(B)$ gives the element $-x$. Thus in $2W$, the connect-sum of this embedded framed sphere and the original one gives an embedded framed sphere with
null-homotopic map to $B$. Using the third copy of $W$ we can fix the remaining sphere, without changing the property that the two spheres intersect transversely in one point. □

We shall first prove Theorem 1.13 under an additional countability hypothesis, namely we prove the following.

**Proposition 7.19.** Let $\theta$, $(P, \ell_P)$ and $\mathcal{K}$ be as in Theorem 1.13 and let $\mathcal{L} \subset \mathcal{K}$ be a submonoid satisfying conditions (ii), (iii) and (iv) of that theorem. Assume in addition that $\mathcal{L}$ is countable. Then the induced morphism

$$H_* (\mathcal{N}^0 (P, \ell_P)) (\mathcal{L}^{-1}) \longrightarrow H_* (\Omega^\infty MT \theta')$$

is an isomorphism.

**Proof.** By countability of $\mathcal{L}$, we may pick a sequence of $\theta$-manifolds $(K|_{i, i+1}, \ell_i)$ which are self-bordisms of $(P, \ell_P)$ representing elements of $\mathcal{L}$, in a way that each element of $\mathcal{L}$ is represented infinitely often. We then let $K$ be the infinite composition of the $K|_{i, i+1}$, and deduce from Addendum 1.9 that $(K, \ell_K)$ is a universal $\theta$-end. (That property (iii) of the Addendum is satisfied follows from assumption (iii) and Lemma 7.18.) Then Theorem 1.8 gives a homology equivalence

$$\text{hocolim} \mathcal{N}^0 (P, \ell_P) \longrightarrow \Omega^\infty MT \theta',$$

where the homotopy colimit is over composition with the $K|_{i, i+1}$. Taking homology turns the homotopy colimit into a colimit of the $\mathbb{Z} [\mathcal{L}]$-module $H_* (\mathcal{N}^0 (P, \ell_P))$ over multiplying with elements of $\mathcal{L}$, each element occurring infinitely many times. But that precisely calculates the localisation at $\mathcal{L}$. □

The proposition above proves Theorem 1.13 in the case where $\mathcal{K}$ is countable. To apply Proposition 7.19 with $\mathcal{L} = \mathcal{K}$, we need to check that conditions (ii), (iii) and (iv) hold. This is proved using the manifolds $W_\alpha$ from the proof of Proposition 7.8.

We will deduce the general case by a colimit argument, based on the following result.

**Corollary 7.20.** Let $\theta$, $(P, \ell_P)$ and $\mathcal{K}$ be as in Theorem 1.13 and let $\mathcal{L} \subset \mathcal{K}$ be a submonoid satisfying conditions (ii) and (iii) of that theorem, but not necessarily (iv). Assume in addition that $\mathcal{L}$ is countable. Then the induced morphism

$$(7.3) \quad H_* (\mathcal{N}^{\theta \mathcal{L}} (P, \ell_P)) (\mathcal{L}^{-1}) \longrightarrow H_* (\Omega^\infty MT \theta \mathcal{L})$$

is an isomorphism, where $\theta \mathcal{L} : B_\mathcal{L} \rightarrow BO(2n)$ is obtained as the $n$th Moore-Postnikov factorisation of a certain map $\ell : X_\mathcal{L} \rightarrow B$, defined as follows. Each self-bordism $(K, \ell_K)$ representing an element of $\mathcal{L}$ has incoming boundary $P \subset K$, and we let $X_\mathcal{L}$ be obtained by gluing every such $K$ along their common incoming boundary; the structure maps $\ell_K$ then glue to the map $\ell : X_\mathcal{L} \rightarrow B$.

**Proof.** The structure $\ell_P$ lifts canonically to a $\theta \mathcal{L}$-structure $\ell_P^\mathcal{L}$, and we claim that every representative of an element of $\mathcal{L}$ admits a homotopically unique lift to a $\theta \mathcal{L}$-manifold which is an endomorphism of $(P, \ell_P^\mathcal{L})$. Granted this claim, $\mathcal{L}$ satisfies the conditions of Proposition 7.19 with respect to $\theta \mathcal{L}$, and the result follows.

To prove the claim, let $(K, \ell_K)$ be a self cobordism of $(P, \ell_P)$ which is supported inside $A \subset P$, a regular neighbourhood of a simplicial complex of dimension $(n - 1)$, and consider the lifting problem

$$\begin{align*}
\{0\} \times P & \longrightarrow (\{0, 1\} \times P) \cup ([0, 1] \times P - A) \\
\{0\} \times K & \longrightarrow (\{0, 1\} \times K) \cup ([0, 1] \times K - A) \\
K & \longrightarrow K \\
\ell_P^\mathcal{L} & \longrightarrow B_\mathcal{L} \\
\ell_K & \longrightarrow B.
\end{align*}$$
As the fibre of $B_L \to B$ is an $(n-1)$-type, and the pair $(K, ([0,1] \times P) \cup ([0,1] \times P-A))$ is $(n-1)$-connected, there is a unique obstruction
\[ \omega_n \in H^n(K, ([0,1] \times P) \cup ([0,1] \times P-A); \pi_n(B, B_L)) \]
to finding the desired lift. As $A$ is a regular neighbourhood of a simplicial complex of dimension $(n-1)$, the group $H^{n-1}(([0,1] \times P) \cup ([0,1] \times P-A), [0] \times P; \pi_n(B, B_L))$ vanishes, so $\omega_n$ is zero if and only if it is zero when restricted to the group $H^n(K, [0] \times P; \pi_n(B, B_L))$. But $K$ has a canonical lift relative to $[0] \times P$, so this last obstruction is zero.

Uniqueness of (homotopy classes of) lifts is similar, but easier. \hfill \Box

Proof of Theorem 1.13. Let $L \subset K$ satisfy the conditions of Theorem 1.13. We may replace $\theta : B \to BO(2n)$ by $\theta' : B' \to BO(2n)$, the $(n-1)$st Moore–Postnikov stage of $\ell_P : P \to B$, in the statement of Theorem 1.13. Then for each countable submonoid $L' \subset L$, the maps $B_{L'} \to B$ from Corollary 7.20 factor canonically as $B_{L'} \to B' \to BO(2n)$, and if $L'' \subset L'$ is a submonoid we also have a factorisation $B_{L''} \to B_{L'} \to B'$, as our description of $B_L$ is strictly functorial in the monoid $L$ (using a functorial model for Moore–Postnikov factorisation). Therefore we may form the colimit of the isomorphisms $\Omega^n$ over the poset of countable submonoids $L' \subset L$.

Using the manifolds $W_\alpha$ constructed in the proof of Proposition 7.8, it is easy to see that the homotopy colimit of the $B_{L'}$ is (weakly equivalent to) $B'$, and hence the homotopy colimit of the $\Omega^nMT\theta_{L'}$ is $\Omega^nMT\theta'$. If we can prove that the homotopy colimit of the spaces $N^\theta_{L'}(P, \ell'_P)$ is $N^\theta(P, \ell_P)$, $\Omega^n(P, \ell_P)$, Theorem 1.13 will therefore follow as the direct limit of the isomorphism $\Omega^n$.

We saw in Lemma 7.10 that the map $N^\theta(P, \ell'_P) \to N^\theta(P, \ell_P)$ is a weak equivalence. A similar obstruction-theoretic argument as in that lemma shows that the map
\[ N^\theta_{L'}(P, \ell'_P) \to N^\theta(P, \ell_P) \]
induces an injection on $\pi_0$ and a weak equivalence of each path component. Viz., $N^\theta_{L'}(P, \ell'_P)$ is up to homotopy a disjoint union of path components of $N^\theta(P, \ell_P)$; the component containing $(W; \ell_W)$ is included precisely when $\ell_W$ admits a lift to a $\theta_{L'}$-structure. Up to homotopy, the system of spaces $N^\theta_{L'}(P, \ell'_P)$ therefore just consists of including more and more components of $N^\theta(P, \ell_P)$, including all of them in the colimit. \hfill \Box

References

[BF78] A. K. Bousfield and E. M. Friedlander, Homotopy theory of $\Gamma$-spaces, spectra, and bisimplicial sets, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, Lecture Notes in Math., vol. 658, Springer, Berlin, 1978, pp. 80–130.

[BF81] E. Binz and H. R. Fischer, The manifold of embeddings of a closed manifold, Differential geometric methods in mathematical physics (Proc. Internat. Conf., Tech. Univ. Clausthal, Clausthal-Zellerfeld, 1978), Lecture Notes in Phys., vol. 139, Springer, Berlin, 1981, With an appendix by P. Michor, pp. 310–329.

[BM12] Alexander Berglund and Ib Madsen, Homological stability of diffeomorphism groups, arXiv:1203.4161, 2012.

[Cer61] Jean Cerf, Topologie de certains espaces de plongements, Bull. Soc. Math. France 89 (1961), 227–380.

[CM09] Ralph Cohen and Ib Madsen, Surfaces in a background space and the homology of mapping class group, Proc. Symp. Pure Math. 80 (2009), no. 1, 43–76.

[Ebe09] Johannes Ebert, A vanishing theorem for characteristic classes of odd-dimensional manifold bundles, arXiv:0902.4719, to appear in Crelle’s Journal, 2009.

[Ebe11] Johannes Ebert, Algebraic independence of generalized MMM-classes, Algebr. Geom. Topol. 11 (2011), no. 1, 69–105.

[EZ50] Samuel Eilenberg and J. A. Zilber, Semi-simplicial complexes and singular homology, Ann. of Math. (2) 51 (1950), 499–513.
[GMTW09] Søren Galatius, Ib Madsen, Ulrike Tillmann, and Michael Weiss, *The homotopy type of the cobordism category*, Acta Math. **202** (2009), no. 2, 195–239.

[GRW10] Søren Galatius and Oscar Randal-Williams, *Monoids of moduli spaces of manifolds*, Geom. Topol. **14** (2010), no. 3, 1243–1302.

[GRW12a] Søren Galatius and Oscar Randal-Williams, *Detecting and realising characteristic classes of manifold bundles*, In preparation, 2012.

[GRW12b] Søren Galatius and Oscar Randal-Williams, *Homological stability for moduli spaces of high dimensional manifolds*, arXiv:1203.6830, 2012.

[Ker65] Michel A. Kervaire, *Le théorème de Barden–Mazur–Stallings*, Comment. Math. Helv. **40** (1965), 31–42.

[Kre99] Matthias Kreck, *Surgery and duality*, Ann. of Math. (2) **149** (1999), no. 3, 707–754.

[Lim63] Elon Lima, *On the local triviality of the restriction map for embeddings*, Commentarii Mathematici Helvetici **38** (1963), 163–164.

[Lüc02] Wolfgang Lück, *A basic introduction to surgery theory*, Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001), ICTP Lect. Notes, vol. 9, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002, pp. 1–224.

[May72] J. P. May, *The geometry of iterated loop spaces*, Springer-Verlag, Berlin, 1972, Lectures Notes in Mathematics, Vol. 271.

[Mil83] John Milnor, *Lectures on the h-cobordism theorem*, Notes by L. Siebenmann and J. Sondow, Princeton University Press, Princeton, N.J., 1965.

[MS76] Dusa McDuff and Graeme Segal, *Homology fibrations and the “group-completion” theorem*, Invent. Math. **31** (1975/76), no. 3, 279–284.

[MT01] Ib Madsen and Ulrike Tillmann, *The stable mapping class group and \(Q(\mathbb{C}P^\infty_+)\)*, Invent. Math. **145** (2001), no. 3, 509–544.

[Mum83] David Mumford, *Towards an enumerative geometry of the moduli space of curves*, Arithmetic and geometry, Vol. II, Progr. Math., vol. 36, Birkhäuser Boston, Boston, MA, 1983, pp. 271–328.

[MW07] Ib Madsen and Michael Weiss, *The stable moduli space of Riemann surfaces: Mumford’s conjecture*, Ann. of Math. (2) **165** (2007), no. 3, 843–941.

[Seg68] Graeme Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. (1968), no. 34, 105–112.

[Wal70] C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, London, 1970, London Mathematical Society Monographs, No. 1.

[Wei05] Michael Weiss, *What does the classifying space of a category classify?*, Homology Homotopy Appl. **7** (2005), no. 1, 185–195.

E-mail address: galatius@stanford.edu

Department of Mathematics, Stanford University, Stanford CA, 94305

E-mail address: o.randal-williams@math.ku.dk

Institut for Matematiske Fag, Universitetsparken 5, DK-2100 København Ø, Denmark