Möbius solutions of the curved $n$–body problem for positive curvature

Ernesto Pérez Chavela
Departamento de Matemáticas
UAM-Iztapalapa
México, D.F. MEXICO
epc@xanum.uam.mx

J. Guadalupe Reyes-Victoria
Departamento de Matemáticas
UAM-Iztapalapa
México, D.F. MEXICO
revg@xanum.uam.mx

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Abstract

We denote by $\mathbb{M}_2^2$ a two dimensional space of constant positive Gaussian curvature. With methods of Möbius geometry and using the classification of the Möbius group of automorphisms $\text{Mob}_2(\hat{\mathbb{C}})$ of the Riemman sphere $\hat{\mathbb{C}} = \mathbb{M}_2^2 \cup \{\infty\}$, we give algebraic conditions for the existence of Möbius solutions on $\hat{\mathbb{C}}$, getting a complete classification of them. We show several families of this kind of solutions.

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1 Introduction

The formalization of Euclidian Geometry from the Elements raised many important questions some of them about the definition of basic geometric
objects. In this way one can analyze the concept of a point whose primary
definition was “an object such that did not have any parts”, or as the line,
which was defined as “the object with length but without width”. In the
Nineteenth century several mathematicians found dark these kind of defin-
tions and show objects in suitable spaces where they are meaningless, as the
Peano spaces-filling curves.

After that, the definitions in modern geometry are written as mathe-
matical objects which do not need a primary key, because such primary
definition require of another definition of an even more basic object and so
on.

The axiomatization of modern geometry is then more suitable than hav-
ing to define fundamental objects. In the nineteenth century David Hilbert
proposed the axiomatization of the Geometry by 21 axioms that are funda-
mentals for the creation of modern non-Euclidean geometries.

In 1871, Felix Klein argued that the classes of Euclidean and Non-
euclidian objects can be studied in a suitable projective space. In this sense
the Projective Geometry establishes a self-reliance among the respective
theories. In his work, F. Klein proved that in order that the Euclidean ge-
ometry be consistent (without contradictions in its postulates), is necessary
and sufficient that the non-Euclidean geometries also be consistent.

In his famous Erlangen program, Klein offers a simple definition of what
is defining a Geometry in one space, in which are not considered important
concepts of point, line, surface, etc. In his paper proposes the idea of giving
an algebraic character of definition by using the concept of a primary group
transformations of this space (bijective applications onto itself). That is, the
geometry of the space is defined by properties invariant under such transfor-
mations. These are objects that define the geometry, and the relationships
between these objects build his theory.

As one example, in order to characterize the planar Euclidean geometry,
we must define a set of rotations, reflections and translations in the plane
(isometries). The main invariants under these applications are the points
and the lines, which is expected by the everyday experience.

With this definition, Klein did not make a distinction between the Eu-
clidian methods and those of the analytical algebraic geometry, which were
stated before the appearance of his program.

Among all these geometries defined in such way, is the Möbius geome-
try, which can be understand as the study of “Euclidean space with a point
added at infinity” endowed with a conformal metric. That is, the setting is
the compactification of the standard plane space, and the Möbius geometry
is concerned with the group of transformations preserving the conformal
form of the metric. It is well known that this group of conformal transformations of such space is infinite dimensional and it is the whole set of all its holomorphic maps (followed by a possible conjugation operation). However, the group of conformal fractional linear Möbius transformations (automorphisms) of the extended complex plane is a complex 3–dimensional, and its action under one-dimensional parametric subgroups generates the conics and loxodromic curves of such space. For more details on the development of the modern geometries and in particular, on the theory of Möbius geometry see [12].

We use here such group of conformal fractional Möbius transformations in \( \mathbb{C} \) for analyzing some distinguished motions of the \( n \)–body problem in the positively curved and compactified complex planar space.

We consider \( n \) bodies of masses \( m_1, \ldots, m_n \) moving on a 2–dimensional space of constant Gaussian curvature \( K \). It is well known that this surface is locally characterized by the sign of the curvature [7].

1. If \( K > 0 \), the surface is the two dimensional sphere \( S^2_R \) of radius \( R = 1/\sqrt{K} \) imbedded in the euclidian space \( \mathbb{R}^3 \), or the curved plane \( \mathbb{M}^2_R \), that is, the ordinary plane \( \mathbb{C} \) with the canonical complex variables \( (z, \bar{z}) \) endowed with the conformal metric (see [10] for more details)

\[
ds^2 = \frac{4R^4 \, dzd\bar{z}}{(R^2 + |z|^2)^2}. \tag{1}
\]

2. If \( K = 0 \), we recover the Euclidean space \( \mathbb{R}^2 \).

3. If \( K < 0 \), the surface is the upper part of the hyperboloid \( x^2 + y^2 - z^2 = K^{-1} \) imbedded in the three dimensional Minkowski space \( \mathbb{R}^3 \), or the hyperbolic Poincaré disk \( \mathbb{D}^2_R \) with the canonical complex variables \( (z, \bar{z}) \) and endowed with the conformal metric (see [5] for more details)

\[
-\, ds^2 = \frac{4R^4 \, dzd\bar{z}}{(R^2 - |z|^2)^2}. \tag{2}
\]

In [3], Florin Diacu and Ernesto Pérez-Chavela give an analytical definition of the so called Eulerian and Lagrangian homographic solutions for the 3–body problem both in the unitary two dimensional sphere \( S^2 \) imbedded in the euclidian space \( \mathbb{R}^3 \) and in the pseudo-sphere \( \mathbb{L}^2_R \) imbedded in the Minkowski space \( \mathbb{R}^3_1 \). Eulerian and Lagrangian motions are a generalization of the well known Euler and Lagrange homographic orbits of classical celestial mechanics. In [3], the authors classify all types of homographic
solutions, defined as the motions where the configuration of the particles is always on the same plane and it is similar with itself for all time. In this paper we generalize the above definition in terms of the action of Möbius transformations on the configuration space for the positive curvature case, that is on the spherical plane $\mathbb{M}^2_R$.

The paper is organized as follows: in section 2 we state the equation of motions of the problem by using complex intrinsic coordinates as in [10] and [3]. In section 3 we state the algebraic and geometric classification of all Möbius transformations in: Möbius-elliptic, Möbius-hyperbolic, Möbius-parabolic and Möbius-loxodromic, whose action in $\mathbb{M}^2_R$ generate the corresponding conic curves of the Möbius geometry.

Starting in section 4 we use the algebraic classification of the Möbius curves in $\text{Mob}_2(\hat{\mathbb{C}})$ to obtain a classification of motions in the $n$–body problem in the positive curvature case. First we define the Möbius-elliptic solutions, thorough the action of a one-dimensional parametric subgroup of isometries (and the corresponding Möbius subgroup of transformations) showing that they correspond to relative equilibria. We state the algebraic conditions that the solution must hold in order to be one of such orbits.

In section 5 we define the so called Möbius-hyperbolic solutions, via the action of the one-dimensional parametric subgroup of Möbius hyperbolic transformations showing that they correspond to some particular type of the homothetic orbits found in [3]. We also give the algebraic conditions on the positions of the particles for having one of such solutions and show examples of this kind of solutions in the curved 2–body problem.

In section 6 we define the Möbius-parabolic solutions. As in the previous cases we give the necessary and sufficient conditions which such orbits must hold. We show examples of this kind of orbits for the curved 2 and 3–body problem. Until we know, this is the first time that this kind of orbits are described explicitly, remarking the advantages of using the tools of the Möbius geometry in the analysis of the solutions of the curved $n$–body problem.

Finally in section 7 by using a set of one-parametric conformal transformations, we give a geometric definition of loxodromic or homographic solutions, which generalize those given in [3]. This is the largest family of motions, among all them, we define and analyze three distinguished classes. The first class is defined by combining the hyperbolic types obtained in section 5 with the Möbius-elliptic ones, arising the class of asymptotic Möbius loxodromic motions. The second class is a particular type of homothetic solutions, lying on the meridians and parametrized as a geodesic curve; the third class is obtained by the combined type of totally geodesic solutions.
with the Möbius-elliptic solutions, arising the class of homographic Möbius-loxodromic motions.

2 Equations of motion

We state in this section the equations of motion for the two-dimensional positively curved $n$–body problem.

For the pair of points $z_k$ and $z_j$ in $M^2_R$ we denote the geodesic distance between them by $d(z_k, z_j) = d_{kj}$ and define (see [10]) the cotangent relation

$$\cot_R \left( \frac{d_{kj}}{R} \right) = \frac{2(z_k \bar{z}_j + z_j \bar{z}_k)R^2 + (|z_k|^2 - R^2)(|z_j|^2 - R^2)}{[\Theta_1,(k,j)(z, \bar{z})]^{1/2}},$$

where

$$\Theta_1,(k,j)(z, \bar{z}) = 4R^2(z_j - z_k)(\bar{z}_j - \bar{z}_k)|R^2 + \bar{z}_j z_k| |R^2 + z_j z_k|.$$  

The singular set in $M^2_R$ for the $n$–body problem is the zero set of equation (4):

$$\Theta_1,(k,j)(z, \bar{z}) = 0,$$

from here we obtain the following sets:

1. The singular collision set given by $\Delta^+ = \cup_{kj} \Delta^+_{kj}$, where,

$$\Delta^+_{kj} = \{ z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid z_k = z_j \}.$$  

2. The singular antipodal set given by $\Delta^- = \cup_{kj} \Delta^-_{kj}$, where,

$$\Delta^-_{kj} = \left\{ z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n \mid z_k = \frac{-R^2}{|z_j|^2} z_j \right\}.$$  

We define the total singular set of the problem as

$$\Delta = \Delta^+ \cup \Delta^-.$$  

Let $z = (z_1, z_2, \cdots, z_n) \in \mathbb{C}^n$ be the (complex) position of $n$ point particles with masses $m_1, m_2, \cdots, m_n > 0$ in the space $M^2_R$. We assume that the particles are moving under the action of the Lagrangian

$$L_R(z, \bar{z}, \dot{z}, \dot{\bar{z}}) = K_R(z, \bar{z}, \dot{z}, \dot{\bar{z}}) + U_R(z, \bar{z}),$$

where

$$K_R(z, \bar{z}, \dot{z}, \dot{\bar{z}}) = \sum_{k} \sum_{j \neq k} \Theta_1,(k,j)(z, \bar{z}) \left( \frac{d_{kj}}{R} \right)^2,$$

and

$$U_R(z, \bar{z}) = \sum_{k} \sum_{j \neq k} \Theta_1,(k,j)(z, \bar{z}) \left( \frac{d_{kj}}{R} \right)^2.$$  

5
where

\[ K_R = K_R(z, \bar{z}, \dot{z}, \bar{\dot{z}}) = \frac{1}{2} \sum_{k=1}^{n} m_k \lambda(z_k, \bar{z}_k) |\dot{z}_k|^2 \]  

(9)

is the kinetic energy,

\[ U_R = U_R(z, \bar{z}) = \frac{1}{R} \sum_{1 \leq k < j \leq n} m_k m_j \cot_R \left( \frac{d_{kj}}{R} \right) \]  

\[ = \frac{1}{R} \sum_{1 \leq k < j \leq n} m_k m_j \frac{2(z_k \bar{z}_j + z_j \bar{z}_k)R^2 + (|z_k|^2 - R^2)(|z_j|^2 - R^2)}{2R |z_j - z_k| |R^2 + \bar{z}_j z_k|}, \]  

(10)

is the cotangent force function (i.e. the negative of the potential) defined in the set \((M^2_R)^n \setminus \Delta\), and

\[ \lambda(z_k, \bar{z}_k) = \frac{4R^4}{(R^2 + |z_k|^2)^2} \]  

(11)

is the conformal function for the Riemannian metric.

The solution of the corresponding Euler-Lagrange equations associated to the Lagrangian \((8)\) satisfies the following system of second order ordinary differential equations

\[ m_k \ddot{z}_k - \frac{2m_k \bar{z}_k \ddot{z}_k^2}{R^2 + |z_k|^2} = \frac{2}{\lambda(z_k, \bar{z}_k)} \frac{\partial U_R}{\partial \bar{z}_k}(z, \bar{z}), \]  

(12)

where

\[ \frac{\partial U_R}{\partial \bar{z}_k} = \sum_{j=1, j \neq k}^{n} m_k m_j \frac{(R^2 + |z_k|^2)(|z_j|^2 + R^2)^2 (R^2 + \bar{z}_j z_k)(z_j - z_k)}{4R^2 |z_j - z_k|^3 |R^2 + \bar{z}_j z_k|^3} \]  

\[ = \sum_{j=1, j \neq k}^{n} \frac{m_k m_j (R^2 + |z_k|^2)(|z_j|^2 + R^2)^2}{4R^2 |z_j - z_k| |R^2 + \bar{z}_j z_k| (z_j - \bar{z}_k)(R^2 + \bar{z}_j z_j)}, \]  

(13)

for \(k = 1, 2, \cdots, n\).

**Remark 2.1.** We observe that in equation \((12)\), the left hand side correspond to the equations of the geodesics. This means that if the potential is constant, then the particles move along geodesics.
3 The Möbius group \( \text{Mob}_2 (\hat{\mathbb{C}}) \)

In this section we give a complete algebraic classification of all Möbius transformations defined on the extended complex plane denoted by \( \hat{\mathbb{C}} = \mathbb{C}^2_R \cup \{ \infty \} \), which correspond to the rounded Riemann sphere of radius \( R \) endowed with the metric \( \Pi \). We start remembering the well known definition.

**Definition 3.1.** A Möbius transformation is a fractional linear transformation \( f_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \), defined by the rule

\[
 f_A(z) = \frac{az + b}{cz + d},
\]

where \( a, b, c, d \in \mathbb{C} \) and \( ad - bc = 1 \).

It is easy to verify that the set of these automorphisms with the composition form a group denoted by \( \text{Mob}_2 (\hat{\mathbb{C}}) \).

Recalling that the complex special linear group of \( 2 \times 2 \)-matrix is defined by

\[
 \text{SL}(2, \mathbb{C}) = \{ A \in \text{GL}(2, \mathbb{C}) \mid \det A = 1 \},
\]

where \( \text{GL}(2, \mathbb{C}) \) is the set of all non-singular matrices with complex entries, we obtain that any Möbius transformation \( f_A \) is associated to some non-singular matrix \( A \in \text{SL}(2, \mathbb{C}) \),

\[
 A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

Reciprocally, for any given matrix \( A \in \text{SL}(2, \mathbb{C}) \), we have a Möbius transformation \( f_A(z) = \frac{az + b}{cz + d} \), such that \( f_A(z) = f_{-A}(z) \), which shows the isomorphism between the groups

\[
 \text{Mob}_2 (\hat{\mathbb{C}}) \cong \text{SL}(2, \mathbb{C}) / \{ \pm I \}.
\]

**Definition 3.2.** We say that a point \( p \in \hat{\mathbb{C}} \) is a fixed point of the Möbius transformation \( f_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) if \( f_A(p) = p \).

It is clear that the equation for a fix point is

\[
 cz^2 + (d - a)z - b = 0
\]
and together with the relation \( ad - bc = 1 \) we obtain the discriminant condition
\[
(a + d)^2 - 4 = \text{tr}^2(A) - 4,
\]
where \( \text{tr}(A) = a + d \) denotes the trace of the matrix \( A \). This means that any Möbius transformation can have one or at most two fixed points. This property is the key point to classify all elements of \( \text{Mob}_2(\hat{\mathbb{C}}) \). From here we obtain.

**Definition 3.3.** The Möbius transformation \( f_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) is called:

1. **Elliptic**, if \( 0 \leq \text{tr}^2(A) < 4 \).
2. **Hyperbolic**, if \( \text{tr}^2(A) > 4 \).
3. **Parabolic**, if \( \text{tr}^2(A) = 4 \).
4. **Loxodromic**, if \( \text{tr}^2(A) < 0 \) or \( \text{tr}^2(A) \) is not a real number.

Since the determinant and the trace of one matrix are invariant under the operation of conjugation of matrices \( (D^{-1}AD) \), and since all the related matrices belong to \( \text{SL}(2, \mathbb{C}) \), then, in order to get a classification in \( \text{Mob}_2(\hat{\mathbb{C}}) \) it is necessary and sufficient to analyze the normal matrices of type
\[
A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix},
\]
where \( \lambda \) is a simple eigenvalue of \( A \), or the normal matrices of type
\[
A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},
\]
for one repeated eigenvalue.

Now we are in conditions to state the following important result from the Möbius geometry, which will play a main role along this paper (you can find its proof in [11, 9]).

**Theorem 3.4.** Let \( f_A : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) be a Möbius transformation, then,

1. It is elliptic, if the eigenvalues satisfy \( |\lambda| = 1, \lambda \neq \pm 1 \). In this case the corresponding normal matrix is
\[
\begin{pmatrix} e^{\frac{i\theta}{2}} & 0 \\ 0 & e^{-\frac{i\theta}{2}} \end{pmatrix},
\]
and the associated Möbius transformation is the rotation
\[ f(z) = e^{i\theta}z, \]
around a suitable angle \( \theta \).

2. It is hyperbolic, if the eigenvalues satisfy \( \lambda = e^{\pm \theta}, \lambda \neq \pm 1 \). In this case the corresponding normal matrix is
\[
\begin{pmatrix}
  e^\theta & 0 \\
  0 & e^{-\theta}
\end{pmatrix},
\]
and the associated Möbius transformation is the homothetic map
\[ f(z) = e^\theta z. \]

3. It is parabolic, if the eigenvalues are repeated: \( \lambda = 1 \) or \( \lambda = -1 \). In this case the corresponding normal matrix is
\[
\begin{pmatrix}
  1 & b \\
  0 & 1
\end{pmatrix},
\]
and the associated Möbius transformation is the translation
\[ f(z) = z + b. \]

4. It is loxodromic, if the eigenvalues satisfy \( |\lambda^2| \neq 1 \). In this case the corresponding normal matrix is
\[
A = \begin{pmatrix}
  \lambda & 0 \\
  0 & 1/\lambda
\end{pmatrix},
\]
and the associated Möbius transformation is the “helicoidal map”
\[ f(z) = \lambda^2 z. \]

Moreover, the parabolic transformation leaves fixed the infinity point whereas the elliptic, hyperbolic and the loxodromic transformations leave fixed both the origin of coordinates and the infinity point (The south pole and the north pole respectively in our case).

Remark 3.5. We observe that any hyperbolic transformation is also a loxodromic one, actually the form of the matrices in 2 and 4 of Theorem 3.4 can be written in the same way, however we choose the above notation to emphasize that in general in 4 the coefficient \( \lambda \) could be a complex number.

As we shall show in the following sections, the action of each one-dimensional subgroup of the respective class of Möbius transformations given in Theorem 3.4 define the conic curves in the Möbius geometry.
4 Möbius-elliptic solutions

The central part of this paper consists in to show how the algebraic classification of the Möbius curves in \( \text{Mob}_2(\mathbb{C}) \) allow us to obtain a nice classification of motions in the positive curvature case. We start our analysis of the Möbius solutions with the simplest ones, the so called Möbius-elliptic solutions which in fact have been widely studied in [1, 3, 10].

From differential geometry we know that the group of proper isometries of \( \text{Mob}_2^2 \) is the quotient

\[
\text{SU}(2) / \{\pm I\}
\]

of the special unitary subgroup

\[
\text{SU}(2) = \{ A \in \text{SL}(2, \mathbb{C}) \mid A^T A = I \} \subset \text{SL}(2, \mathbb{C})
\]

of the special linear group \( \text{SL}(2, \mathbb{C}) \). Each matrix \( A \in \text{SU}(2) \) has the form

\[
A = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix},
\]

with \( a, b \in \mathbb{C} \) satisfying \( |a|^2 + |b|^2 = 1 \). See [8] for more details.

We denote by \( \{G_e(t)\} \) the one-parametric subgroup of the Lie group \( \text{SU}(2)/\{\pm I\} \), which acts coordinate-wise in \( \mathbb{M}_2^2 \setminus \Delta \) and in \( \Delta \) leaving them invariant,

\[
G_e(t) = \begin{pmatrix} e^{it/2} & 0 \\ 0 & e^{-it/2} \end{pmatrix},
\]

we associate it to the \textit{Killing vector field}

\[
\begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}
\]

in the corresponding Lie algebra \( \text{su}(2) \). Such vector field defines the one-parametric family of acting elliptic Möbius transformations

\[
f_{G_e(t)}(z) = e^{it} z,
\]

which are solutions of the complex differential equation

\[
\dot{z} = iz,
\]

which implies that all orbits are circular periodic orbits.

Reciprocally the above differential equation generates the aforementioned \textit{Killing vector field} associated to the flow \( f_t(z) = e^{it} z \), and to the one parametric subgroup of Möbius transformations \( f_{G_e(t)} \). Figure 1 shows the Möbius-elliptic orbits of the action of the one-parametric subgroup \( \{G_e(t)\} \) in \( \mathbb{M}_2^2 \) and the corresponding circular orbits in the two dimensional sphere.
Figure 1: The Möbius-elliptic orbits on $\mathbb{M}^2_R$ and on the sphere

**Definition 4.1.** A solution $z(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ of equation (12) is called Möbius-elliptic if its invariant under the one parametric subgroup of isometries (14).

All the above prove the following result.

**Theorem 4.2.** The Möbius-elliptic solutions of the positive curved $n$-body problem founded with algebraic techniques correspond with the relative equilibria, defined in [10] as those solutions $z(t)$ of (12) which are invariant relative to the subgroup $\{G_e(t)\}$.

Then, in order to obtain the corresponding Möbius-elliptic solutions in $\mathbb{M}^2_R$, we must analyze just the class of Möbius transformations given by

$$w_k(t) = e^{i t} z_k(t),$$

(16)

where $z(t) = (z_1(t), \cdots, z_n(t))$ is a solution of equation (12).

Introducing (16) in (12) we can obtain the algebraic equations which characterize all relative equilibria which, for the above Theorem they correspond to the Möbius-elliptic solutions. This result has been proved in [10]. In order to have a self-contained paper, we reproduce it here without proof.

**Theorem 4.3.** Consider $n$ point particles with masses $m_1, m_2, \cdots, m_n > 0$ moving in $\mathbb{M}^2_R$. A necessary and sufficient condition for the solution $z(t) = (z_1(t), z_2(t), \cdots, z_n(t))$ of (12) to be a Möbius-elliptic solution (relative equilibrium) is that the coordinates satisfy the following system given by the
rational functions:

\[ \frac{2 R^6 (r_k^2 - R^2) z_k}{(R^2 + r_k^2)^4} = \sum_{j=1, j \neq k}^{n} \frac{m_j (r_j^2 + R^2)^2 (R^2 + \bar{z}_j z_k)(z_j - z_k)}{|z_j - z_k|^3 |R^2 + \bar{z}_j z_k|^3} \]  

(17)

where \( |z_l(t)| = r_l \in [0, \pi R) \), and the velocity in each particle is given by the relation

\[ 2i \dot{z}_k = z_k, \]  

(18)

for \( k = 1, 2, \cdots, n \). Moreover, all obtained solutions \( z_k = z_k(t) \) are circular periodic orbits.

### 4.1 Examples of Möbius-elliptic solutions

As we have seen in Theorem 4.2, the relative equilibria in the positively curved \( n \)-body problem correspond to the Möbius-elliptic solutions, so in [1, 3] the reader can find several families of relative equilibria defined on the sphere, in [10] you can find most of the relative equilibria in the two and three body problems defined on \( \mathbb{M}_2^2 \). All these examples correspond in our context to Möbius-elliptic solutions.

### 5 Möbius-hyperbolic solutions

We start this section by defining the concept of homothetic solution. We recall that in a Riemannian two dimensional surface, a pair of points \( p_1 \) and \( p_2 \) are conjugated if there exists a pair of different geodesics passing through them. We have that in \( \mathbb{M}_2^2 \) any pair of antipodal points \( z_k = \frac{-R^2}{|z_j|^2} z_j \) are conjugated and the whole space is foliated by all the geodesic curves passing through such pair of points. The set of all such curves is called the geodesic conjugated class foliation.

**Definition 5.1.** A solution \( z(t) = (z_1(t), z_2(t), \cdots, z_n(t)) \) of equation (12) is called homothetic if all the particles move on curves whose path belong to the same geodesic conjugated class foliation of one pair of conjugated (antipodal) points.

From the Principal Axis Theorem which states that any rotation in \( \mathbb{R}^3 \) is around a fix axis, we can assume that one point is the origin of coordinates \( z = 0 \) with conjugated point \( z = \infty \), and the geodesic foliation of \( \mathbb{M}_2^2 \) in this case, is the singular set of straight lines passing through such point, called by short meridians.
Remark 5.2. In general the path of one particle moving along one homothetic solution is parametrized by $z_k(t) = \phi(t)z_{k,0}$ a suitable real function $\phi = \phi(t)$ which holds the equations of motion (12). We observe that with this parametrization the length of the velocity along the curve can vary, however the geodesics are always parameterized such that their tangent vectors have constant speed (see [7] for more details). In other words the path of one particle moving along a homothetic solution not necessarily does it with constant speed, that is in a geodesic way.

Among the whole set of homothetic solutions of (12) there is a subclass generated by the action of a particular one-parametric subgroup of hyperbolic M"obius transformations. Let us denote by $\{G_h(t)\}$ the one-parametric subgroup of $\text{Mob}_2(\hat{\mathbb{C}})$ which acts coordinatewise in $\mathbb{M}_R^2 \setminus \Delta$ and in $\Delta$ leaving them invariant, and defined by
\[ G_h(t) = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \]
associated to the hyperbolic M"obius group of transformations
\[ f_{G_h(t)}(z) = e^t z, \quad (19) \]
also called hyperbolic group.

Figure 2: The M"obius-hyperbolic orbits on $\mathbb{M}_R^2$ and on the sphere.

Figure 2 shows the M"obius-hyperbolic orbits of the action of the one parametric subgroup $\{G_h(t)\}$ in $\mathbb{M}_R^2$ and the corresponding hyperbolic orbits.
in the two dimensional sphere, which become the geodesics (great circles) of such space.

**Definition 5.3.** A homothetic solution \( z(t) = (z_1(t), z_2(t), \cdots, z_n(t)) \) of equation (12) is called Möbius-hyperbolic if it is invariant under the one parametric subgroup of Möbius transformations \( \{19\} \).

We state the main result of this section.

**Theorem 5.4.** Consider \( n \) point particles with masses \( m_1, m_2, \cdots, m_n > 0 \) moving in \( M^2_R \). A necessary and sufficient condition for the solution \( z(t) = (z_1(t), z_2(t), \cdots, z_n(t)) \) of (12) to be a Möbius-hyperbolic solution is that the coordinates satisfy the following system given by the rational functions:

\[
2 R^6 \frac{(R^2 - |z_k|^2)z_k}{(R^2 + |z_k|^2)^4} = \sum_{j=1, j \neq k}^n \frac{m_j (|z_j|^2 + R^2)^2 (R^2 + \bar{z}_j z_k)(z_j - z_k)}{|z_j - z_k|^3 |R^2 + \bar{z}_j z_k|^3}
\]

and the velocity in each particle is given by the relation

\[
2 \ddot{z}_k = -z_k,
\]

for \( k = 1, 2, \cdots, n \). Moreover, all the solutions \( \dot{z}_k = z_k(t) \) are forward asymptotic to the origin of coordinates, which implies that there are not collisions between the particles in finite time.

**Proof.** If we suppose that \( w_k = e^t z_k \) is a solution of equation (12), then by differentiating we obtain

\[
\dot{w}_k = (\dot{z}_k + z_k) e^t,
\]

\[
\ddot{w}_k = (\ddot{z}_k + 2\dot{z}_k + z_k) e^t,
\]

which when we substitute into the equation (12) gives us the relation

\[
m_k (\ddot{z}_k + 2\dot{z}_k + z_k) e^t - \frac{2m_k \ddot{z}_k (\dot{z}_k + z_k)^2 e^{3t}}{R^2 + |z_k|^2 e^{2t}} = \frac{2}{\lambda(z_k e^t, \bar{z}_k e^t)} \frac{\partial U_R(z, \bar{z})}{\partial z_k} e^{-t},
\]

where \( \lambda(z_k, \bar{z}_k) \) is the conformal function defined in (11).

By evaluating at \( t = 0 \) we obtain the condition for the infinitesimal generator of the vector field associated to the solution \( z_k = z_k(t) \) in one arbitrary point. This gives us the relation

\[
m_k (\ddot{z}_k + 2\dot{z}_k + z_k) - \frac{2m_k \ddot{z}_k (\dot{z}_k + z_k)^2}{R^2 + |z_k|^2} = \frac{2}{\lambda(z_k, \bar{z}_k)} \frac{\partial U_R(z, \bar{z})}{\partial z_k},
\]
and using that $z_k = z_k(t)$ holds equation (12), then this last equation becomes into

$$
2 \ddot{z}_k + z_k = \frac{2 z_k (2 \dot{z}_k \dot{z}_k + z_k^2)}{R^2 + |z_k|^2} = \frac{2|z_k|^2 (2 \dot{z}_k + z_k)}{R^2 + |z_k|^2}.
$$

(25)

The equation (25) holds for the infinitesimal conditions

$$
|z_k| = R,
$$

(26)

which corresponds to the geodesic equator, or for

$$
\dot{z}_k = -\frac{z_k}{2},
$$

(27)

which indicate that the solution through the point $z(0)$ is in the direction of the meridian passing by the origin of coordinates and the given point.

If we derive the infinitesimal condition (27), we obtain $\ddot{z}_k = \frac{z_k}{4}$, which when is substituted in equation of motion (12) allows to the system (20), this ends the proof of the Theorem.

5.1 Examples of Möbius-hyperbolic solutions

For the two body problem we have the following result

**Theorem 5.5.** Suppose that two particles with masses $m_1$ and $m_2$ and positions $z_1(t)$ and $z_2(t)$ in $M_2^2_R$ move as a Möbius-hyperbolic solution. Then the masses are equal if and only if $z_1(t) = -z_2(t)$.

**Proof.** For two bodies in $M_2^2_R$ with masses $m_1$ and $m_2$ moving as a Möbius-hyperbolic solution, the system (20) becomes into,

$$
\frac{2 R^6 (R^2 - |z_1|^2) z_1}{(R^2 + |z_1|^2)^4} = \frac{m_2 (|z_2|^2 + R^2)^2 (R^2 + \ddot{z}_2 z_1) (z_2 - z_1)}{|z_2 - z_1|^5 |R^2 + \ddot{z}_2 z_1|^3},
$$

$$
\frac{2 R^6 (R^2 - |z_2|^2) z_2}{(R^2 + |z_2|^2)^4} = \frac{m_1 (|z_1|^2 + R^2)^2 (R^2 + \ddot{z}_1 z_2) (z_1 - z_2)}{|z_1 - z_2|^5 |R^2 + \ddot{z}_1 z_2|^3},
$$

(28)

and if we divide term to term both sides of this system avoiding singularities and conjugated points, after a straightforward algebraic manipulation, we obtain the relation

$$
0 = m_1 z_1 (R^2 - |z_1|^2) (|z_2|^2 + R^2)^2 (R^2 + \ddot{z}_1 z_2)
+ m_2 z_2 (R^2 - |z_2|^2) (|z_1|^2 + R^2)^2 (R^2 + \ddot{z}_2 z_1) = 0.
$$

(29)
If we put $z_1(t) = -z_2(t)$ into equation (29), then necessarily the masses must be equal.

On the other hand, if $m_1 = m_2$, without lose of generality (up a rotation) we take $z_1 = r$ as one real number, then equation (29) implies that necessarily $z_2$ is also a real number, say $z_2 = \alpha$. Therefore such equation becomes into the algebraic equation in the unknown $\alpha$,

$$0 = [r(R^2 - r^2)(\alpha^2 + R^2)^2 + \alpha(R^2 - \alpha^2)(r^2 + R^2)^2](R^2 + r\alpha). \quad (30)$$

Since we avoid conjugated points, then $R^2 + r\alpha \neq 0$, and therefore we obtain the real equation

$$0 = r(R^2 - r^2)(\alpha^2 + R^2)^2 + \alpha(R^2 - \alpha^2)(r^2 + R^2)^2, \quad (31)$$

which has the only real roots $\alpha = \pm r$. This ends the proof.

\[\square\]

**Remark 5.6.** We observe that if we put two equal masses on initial positions $z_1(t_0) = -z_2(t_0)$, then we can easily generate a Möbius-hyperbolic solution.

In [3], the authors show that a necessary and sufficient condition for having a homothetic solution in the curved 3–body problem is that the configuration be always an equilateral triangle and that the masses be equal (they called it Lagrangian homothetic solution). So in order to study this kind of motion it is enough to analyze the case of equal masses. Proceeding as in the above reference, we get the same type of Möbius hyperbolic solution, which also call them Lagrangian, we omit here the details.

In [10], the authors prove that a necessary and sufficient condition in order to have a collinear Möbius-hyperbolic solution in $\mathbb{M}_R^2$, with one particle at the origin $z_3(t) = 0$, is that the masses of the other two particles with positions $z_1(t)$ and $z_2(t)$ be equal, and that $z_1(t) = -z_2(t)$. This is another example of a Möbius-hyperbolic solution.

### 6 Möbius-parabolic solutions

In this section we do the analysis of the Möbius-parabolic solutions corresponding to the solutions associated to the one parametric subgroup

$$G_p(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix},$$

which defines the one-parametric subgroup of acting Möbius transformations

$$f_{G_p(t)}(z) = z + t, \quad (32)$$
in $\mathbb{M}^2_R$. Figure 3 shows the straight lines or parabolic orbits of the action of the one parametric subgroup $\{G_p(t)\}$ in $\mathbb{M}^2_R$ and the corresponding looped M"obius-parabolic orbits on the two dimensional sphere.

Let $w = (w_1, \ldots, w_n)$, with $w_k(t) = z_k(t) + t$, be the action orbit for $z = (z_1, \ldots, z_n)$, which is a solution of the equations of motion (12). Then

$$\dot{w}_k = \dot{z}_k + 1 \quad \text{and} \quad \ddot{w}_k = \ddot{z}_k,$$

$k = 1, \ldots, n,$

therefore $w$ is a solution of system (12), if and only if

$$m_k \ddot{z}_k = \frac{2m_k(\dot{z}_k + t)(\dot{z}_k + 1)^2}{R^2 + |z_k + t|^2} + \frac{(R^2 + |z_k + t|^2)^2}{2R^4} \frac{\partial U_R}{\partial \bar{z}_k}, \quad k = 1, \ldots, n,$$

to get the above expression we have used the fact that $\frac{d\bar{w}_k}{d\bar{z}_k} = 1$.

Once again, if we take $t = 0$ for finding the condition for the infinitesimal generator of the vector field associated to such motions we have the system of algebraic equations

$$m_k \ddot{z}_k = \frac{2m_k \ddot{z}_k (\dot{z}_k + 1)^2}{R^2 + |z_k|^2} + \frac{(R^2 + |z_k|^2)^2}{2R^4} \frac{\partial U_R}{\partial z_k}, \quad k = 1, \ldots, n.$$

Now, since $z$ is a solution of system (12), we obtain the condition for the infinitesimal generator of the vector field

$$2\dot{z}_k = -1, \quad k = 1, \ldots, n,$$

(33)
which holds if and only if
\[ z_k(t) = -\frac{t}{2} + z_k(0), \quad k = 1, \ldots, n, \] (34)
where \( z_k(0), \quad k = 1, \ldots, n, \) are initial conditions.

Consequently, a necessary condition for the particles \( m_1, \ldots, m_n \) to form a Möbius-parabolic solution is that they move along horizontal straight lines in \( \mathbb{M}^2_R \), with constant negative velocity, passing through the initial conditions \( z_k(0), \quad k = 1, \ldots, n. \)

We can now state the following result, whose proof follows by straightforward computations.

**Theorem 6.1.** Consider \( n \geq 2 \) point particles of masses \( m_1, \ldots, m_n > 0 \) moving in \( \mathbb{M}^2_R \). Then a necessary and sufficient condition for the function \( z = (z_1, \ldots, z_n) \) to be a Möbius-parabolic solution of system (12) is that the coordinate functions satisfy the equations
\[
-\frac{4R^6\ddot{z}_k}{(R^2 + |z|^2)^4} = \sum_{j=1}^{n} \frac{m_j (R^2 + |z_j|^2)^2 (R^2 + z_k \bar{z}_j)(z_j - z_k)}{|z_j - z_k|^3 |R^2 + \bar{z}_j z_k|^3},
\] (35)
for \( k = 1, \ldots, n. \)

Notice that equations (33) give the conditions for the velocities of the particles in case they form a Möbius-parabolic solution.

In the next subsection, using the tools of the Möbius geometry, we show how the Möbius-parabolic solutions appear in a natural way when the one-parametric group \( \{G_p(t)\} \) acts in \( \mathbb{M}^2_R \). These kind of parabolic motions obtained with the Möbius geometric tools could hardly be obtained using the traditional approach.

### 6.1 Examples of Möbius-parabolic solutions

We start this subsection by giving a concrete example of a Möbius-parabolic solution in the following lemma.

**Lemma 6.2.** For small \( m > 0 \) there exist a pair of functions \( z_1(t) = -z_2(t) \), such that conform a Möbius-parabolic solution of the problem.

**Proof.** If \( z_1(t) = -\frac{t}{2} + \alpha Ri \) and \( z_1(t) = \frac{t}{2} - \alpha Ri \) are substituted in the equations of motion (12), then the condition \( z_1(t) = -z_2(t) \) with equal
masses $m_1 = m_2 = m$ in the particles carries such system into the single equation

$$\frac{\ddot{z}_1(t)}{(R^2 + r(t)^2)^6} = -\frac{m z_1(t)}{16R^6r(t)^3(R^2 - r(t)^2)^2}. \quad (36)$$

Now taking $t = 0$ in the above equation we have $z_1 = \alpha R i$ and $r(0) = \alpha R$, from here we get the condition for $\alpha$ given by the equation

$$16\alpha^3(1 - \alpha^2)^2 = \frac{m}{R}(1 + \alpha^2)^6. \quad (37)$$

If we do $m = 0$ (or $R = \infty$, the planar case) in equation (37), the roots for the above equation are $\alpha = 0, \pm 1$, and the particles are located at the origin of coordinates, or on the geodesic circle. Therefore, by using an argument of continuity and the generic property of transversality for the smooth real functions on the left and right hand sides of (37), for fixed $R > 0$ and small enough $m > 0$ (or for a fixed $m > 0$ and big enough $R > 0$), it follows that there exists a solution of such equation depending of $m$ and $R$, say $\alpha = \alpha(R, m)$ such that $1 < \alpha(R, m) < \pi$. We define the functions $z_1(t) = -\frac{t}{2} + \alpha(R, m) R i$ and $z_1(t) = \frac{t}{2} - \alpha(R, m) R i$, which by construction conform a Möbius-parabolic solution.

Once we have show an example of a Möbius-parabolic solution, we can state a general result about these kind of motions.

**Theorem 6.3.** Suppose that two particles with masses $m_1$ and $m_2$ in $M^2_R$ located in positions $z_1(t)$ and $z_2(t)$, with $|z_1(t)| = |z_2(t)| = r(t)$ move as Möbius-parabolic solutions (34). Then, for $r(t) > R$, the masses are equal, if and only if, $z_1(t) = -z_2(t)$.

**Proof.** For such two bodies with masses moving as a Möbius-parabolic solution, the system (35) becomes into,

$$-\frac{4R^6\dot{z}_1}{(R^2 + r(t)^2)^6} = \frac{m_2 (R^2 + z_1 \bar{z}_2)(z_2 - z_1)}{|z_2 - z_1|^3 |R^2 + \bar{z}_2 z_1|^3},$$

$$-\frac{4R^6\dot{z}_2}{(R^2 + r(t)^2)^6} = \frac{m_1 (R^2 + z_2 \bar{z}_1)(z_1 - z_2)}{|z_1 - z_2|^3 |R^2 + \bar{z}_1 z_2|^3}. \quad (38)$$

Since we avoid collisions, by dividing the corresponding hand sides of first equation with those of the second one, we obtain,

$$\frac{\ddot{z}_1}{\ddot{z}_2} = -\frac{m_2 (R^2 + z_1 \bar{z}_2)}{m_1 (R^2 + z_2 \bar{z}_1)}. \quad (39)$$
If we assume that \( z_1(t) = -z_2(t) \) then \( R^2 + z_1 \bar{z}_2 = R^2 + z_2 \bar{z}_1 = R^2 - r(t)^2 \), and therefore equation (39) implies that the masses must be equal.

Conversely, we suppose that the masses are equal and the functions

\[ z_1(t) = -\frac{t}{2} + \alpha Ri \]
\[ z_2(t) = \pm \frac{t}{2} + (C + \beta Ri) \]

conform a Möbius parabolic solution for suitable \( 1 < \alpha < \pi \). By taking \( t = 0 \) in equations (34) for the corresponding solutions, we consider the first particle sited on the imaginary axis \( z_1 = \alpha Ri \) and the other one, say, in the position \( z_2(t) = C + \beta Ri \) for some real numbers \( C \) and \( \beta \). The condition \( \alpha^2 R^2 = C^2 + \beta^2 R^2 \) on the given initial positions, together with the condition (39) implies that necessarily \( C = 0 \) and \( \beta = \pm \alpha \). This is, both particles must initially be located on the imaginary axis. Using again the condition (39), it follows that necessarily \( z_1(t) = -z_2(t) \) for all time \( t \in (-\pi R, \pi R) \).

\( \square \)

Remark 6.4. The above solutions in Corollary 6.2 correspond on the two dimensional sphere to motions of particles moving symmetrically on parabolic curves, leaving the north pole (as \( t \to -\pi R \)) and going forward to the same point (as \( t \to \pi R \)) and the collision between the particles does in finite time.

Now we shall show the existence of Möbius-parabolic motions in the curved three body problem. Such solutions correspond on the two dimensional sphere to motions of particles moving symmetrically on parabolic curves, leaving the north pole (as \( t \to -\pi R \)) and converging to the same point (as \( t \to \pi R \)). The third arbitrary mass is located always in the conjugated point, that is, at the south pole.

Lemma 6.5. If \( m_1 = m_2 = m \), then, for small \( m > 4M \) there exist a pair of functions \( z_1(t) = -z_2(t) \), such that conform a Möbius-parabolic solution of the problem, and the antipodal particles collide in the north pole in finite time.

Proof. If the functions \( z_1(t) = -\frac{t}{2} + \alpha Ri \) and \( z_1(t) = \frac{t}{2} - \alpha Ri \) are substituted into the equations of motion (12), then the condition \( z_1(t) = -z_2(t) \) with equal masses \( m_1 = m_2 = m \) in the particles carries such system in the single equation

\[
\frac{4R^6 \dot{z}_1(t)}{(R^2 + r(t)^2)^3} = -\frac{m(R^2 + r(t)^2)^2 z_1(t)}{4R^2 r(t)^3(R^2 - r(t)^2)^2} + \frac{M z_1(t)}{r(t)^3}.
\] (40)

If we take \( t = 0 \), then \( z_1 = \alpha Ri \) and \( r(0) = \alpha R \), getting the condition for \( \alpha \) given by the equation

\[
16\alpha^3(1 - \alpha^2)^2 = \frac{(1 + \alpha^2)^4}{R}[m (1 + \alpha^2)^2 - 4M (1 - \alpha^2)^2].
\] (41)
If we do \( m = 0 \) and \( M = 0 \) in equation (41), the roots for the corresponding equation are \( \alpha = 0, \pm 1 \), and the particles are located at the origin of coordinates, or on the geodesic circle. By using again an argument of continuity and the generic property of transversality for the smooth functions on the corresponding left and right hand sides of (41), for fixed \( R > 0 \) and small \( m > 4M \), it follows that there exists a solution of the such equation depending of \( m \), \( M \) and \( R \), say \( \alpha = \alpha(R, m, M) \) such that \( 1 < \alpha(R, m, M) < \pi \). Once again, we define the functions 
\[
\begin{align*}
  z_1(t) &= -\frac{t}{2} + \alpha(R, m, M) Ri \\
  z_2(t) &= t - \alpha(R, m, M) Ri,
\end{align*}
\]
which by construction conform a Möbius-parabolic solution where the antipodal particles collide in the north pole in finite time.

**Theorem 6.6.** Suppose that two particles with masses \( m_1 \) and \( m_2 \) in \( \mathbb{M}_R^2 \) located in positions \( z_1(t) \) and \( z_2(t) \), with \( |z_1(t)| = |z_2(t)| = r(t) \), and the third mass \( M \) located at the origin of coordinates, move as Möbius-parabolic solutions \( (34) \). Then, for \( r(t) > R \), the masses \( m_1 \) and \( m_2 \) are equal, if and only if, \( z_1(t) = -z_2(t) \).

**Proof.** For such two bodies with masses moving as a Möbius-parabolic solution, system (35) becomes into,

\[
\begin{align*}
  -\frac{4R^6 \dot{z}_1}{(R^2 + r(t)^2)^4} &= \frac{m_2 (R^2 + r(t)^2)^2 (R^2 + z_1 \bar{z}_2)(z_2 - z_1) - m_3 z_1}{|z_2 - z_1|^3 |R^2 + \bar{z}_2 z_1|^3} - \frac{m_3 z_1}{|z_1|^3}, \\
  -\frac{4R^6 \dot{z}_2}{(R^2 + r(t)^2)^4} &= \frac{m_1 (R^2 + r(t)^2)^2 (R^2 + z_2 \bar{z}_1)(z_1 - z_2) - m_3 z_2}{|z_1 - z_2|^3 |R^2 + \bar{z}_1 z_2|^3} - \frac{m_3 z_2}{|z_2|^3}, \\
  0 &= \frac{m_1 (R^2 + r(t)^2)^2 z_1}{|z_1|^3 R^4} + \frac{m_2 (R^2 + r(t)^2)^2 z_2}{|z_2|^3 R^4},
\end{align*}
\]

(42)

From third equation of (42), we obtain the condition for the value and position of the first mass, getting

\[
0 = m_1 z_1 + m_2 z_2,
\]

(43)

which, together with the relation \( |z_1(t)| = |z_2(t)| = r(t) \) implies the assertion.

### 7 Möbius-loxodromic solutions

We study here two particular classes of Möbius loxodromic motions of the positively curved \( n \)-body problem. We recall that a loxodromic transforma-
tion has associated the matrix, which in its normal form is given by
\[
A = \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix},
\]
its corresponding Möbius transformation is the helicoidal map
\[
f(z) = \lambda^2 z.
\]

We consider the one dimensional parameterized set of loxodromic Möbius transformations acting in \( \mathbb{M}_2^2 \) given by,
\[
f_{G_l(t)}(z) = \lambda^2(t) z = \phi(t) e^{it} z(t), \quad (44)
\]
where \( \phi(t) \) is a real nonnegative function defined in a suitable interval. Figure 4 shows the Möbius-loxodromic (spiral) orbits of the action of the one-parametric set \( \{G_l(t)\} \) in \( \mathbb{M}_2^2 \) and the corresponding helicoidal (loxodromic) orbits in the two dimensional sphere when \( \phi(t) < 1 \).

![Figure 4: The Möbius-loxodromic orbits on \( \mathbb{M}_2^2 \) and on the sphere.](image)

**Definition 7.1.** A solution \( z(t) = (z_1(t), z_2(t), \ldots, z_n(t)) \) of equation (12) is called Möbius-loxodromic solution if it is invariant under the one-parametric set of Möbius transformations given in (44).

Among the whole set of Möbius-loxodromic solutions we study two particular families obtained by combining two different kinds of homothetic solutions with those of elliptic type analyzed in section 5.1. The first combined type of homothetic solutions are the hyperbolic motions obtained in
section 5.1, which when combine with the Möbius-elliptic ones arise the family of asymptotic Möbius loxodromic motions as it can be seen in subsection 7.1 below. The second combined type of homothetic solutions are the so called totally geodesic, which are defined in subsection 7.2, and when combine with the Möbius-elliptic solutions arise the class of homographic Möbius-loxodromic motions as it can be seen in subsection 7.3.

### 7.1 Asymptotic Möbius-loxodromic solutions

We consider the one dimensional parametric subgroup of Möbius-loxodromic transformations

\[
f_A(t) = e^{t} e^{i t} z = e^{t(1+i)} z.
\]

**Definition 7.2.** A solution \( z(t) = (z_1(t), z_2(t), \cdots, z_n(t)) \) of equation (12) is called Asymptotic Möbius-loxodromic if it is invariant under the one-dimensional parametric subgroup (45).

The following result gives necessary and sufficient conditions in order to have the above kind of motions.

**Theorem 7.3.** In the positively curved \( n \)-body problem, a necessary and sufficient condition for having Asymptotic Möbius-loxodromic solutions of the equation (12) is that the positions \( z_k(t) \) of all the particles satisfy the system of algebraic equations

\[
4 R^6 i (R^2 - |z_k|^2) z_k = \sum_{j=1, j \neq k}^{n} m_j (|z_j|^2 + R^2)^2 (R^2 + \bar{z}_j z_k)(z_j - z_k) \frac{m_j (|z_j|^2 + R^2)^2 (R^2 + \bar{z}_j z_k)(z_j - z_k)}{|z_j - z_k|^3 |R^2 + \bar{z}_j z_k|^3}
\]

and the velocity in each particle is given by the relation

\[
2 \ddot{z}_k = -(1 + i) z_k,
\]

for \( k = 1, 2, \cdots, n \). Moreover, all solutions \( z_k = z_k(t) \) are forward asymptotic in an helicoidal way to the origin of coordinates, which implies that there are not collisions among the particles.

**Proof.** If we suppose that \( w_k = e^{t(1+i)} z_k \) is a solution of equation (12), then by differentiating we obtain

\[
\dot{w}_k = (\dot{z}_k + (1 + i) z_k) e^{t(1+i)}
\]

\[
\ddot{w}_k = (\ddot{z}_k + 2(1 + i) \dot{z}_k + (1 + i)^2 z_k) e^{t(1+i)},
\]

for \( k = 1, 2, \cdots, n \).
which when we substitute in the equation \((12)\) gives us the relation
\[
m_k(\ddot{z}_k + 2(1 + i)\dot{z}_k + (1 + i)^2 z_k) e^{t(1+i)} - \frac{2m_k e^{t(1-i)} \dot{z}_k e^{2t(1+i)} (\dot{z}_k + (1 + i) z_k)^2}{R^2 + |z_k|^2 e^{t(1+i)}} = \frac{2}{\lambda(z_k e^{t(1+i)}, \bar{z}_k e^{t(1+i)})} \frac{\partial U_R}{\partial \bar{z}_k}(z, \bar{z}) e^{-t(1-i)}.
\]
(49)

Again, by evaluating at \(t = 0\) we obtain the condition for the infinitesimal generator of the vector field associated to the solution \(z_k = z_k(t)\) in one arbitrary point. This gives the relation
\[
m_k(\ddot{z}_k + 2(1 + i)\dot{z}_k + (1 + i)^2 z_k) - \frac{2m_k \ddot{z}_k + (1 + i) \dot{z}_k}{R^2 + |z_k|^2} = \frac{2}{\lambda(z_k, \bar{z}_k)} \frac{\partial U_R}{\partial \bar{z}_k}(z, \bar{z}),
\]
(50)
and using that \(z_k = z_k(t)\) holds equation \((12)\), then this last equation becomes into the one
\[
2(1 + i)\dot{z}_k + (1 + i)^2 z_k = \frac{2\ddot{z}_k + [(1 + i)^2 z_k^2 + 2(1 + i)\dot{z}_k z_k]}{R^2 + |z_k|^2},
\]
(51)
or equivalently, to the equation,
\[
(1 + i) \left[ 1 - \frac{2|z_k|^2}{R^2 + |z_k|^2} \right] [(1 + i)z_k + 2\ddot{z}_k] = 0.
\]
(52)
The equation \((52)\) holds for the infinitesimal conditions \(|z_k| = R\), which corresponds to the geodesic equator, or for
\[
2\ddot{z}_k = -(1 + i) z_k
\]
(53)
which shows that the solution through the point \(z(0)\) is an helicoidal curve converging asymptotically to the origin of coordinates. Since the solutions move along different helicoidal curves, then there are not collisions among the particles along these kind of solutions.

If we derive the infinitesimal condition given in equation \((53)\), we obtain
\[
2\ddot{z}_k = i z_k
\]
(54)
which when is substituted into the equations of motion \((12)\) gives the system \((46)\). This ends the proof of the Theorem.

By straightforward computations, very similar to the ones we did in the above sections, we can prove the following result for the asymptotic Möbius-loxodromic motion of two particles in \(M^2_R\).
Proposition 7.4. Suppose that two particles with masses $m_1$ and $m_2$ in $\mathbb{M}^2_R$ and positions $z_1(t)$ and $z_2(t)$ move as an asymptotic Möbius-loxodromic solution. Then, the masses are equal iff the particles are located on opposite sides of the same circle, that is, $z_1(t) = -z_2(t)$.

For the curved 3–body problem on $\mathbb{M}^2_R$, we get the following results whose proofs follow by straightforward computations, we omit them the proof of both results here.

Proposition 7.5. Three equal masses moving on $\mathbb{M}^2_R$ form an asymptotic Möbius-loxodromic solution iff the particles form an equilateral triangle for all time.

Proposition 7.6. A necessary and sufficient condition in order to have a collinear asymptotic Möbius-loxodromic solution in $\mathbb{M}^2_R$, with one particle at the origin $z_3(t) = 0$, is that the masses of the other two particles with positions $z_1(t)$ and $z_2(t)$ be equal, and that $z_1(t) = -z_2(t)$.

7.2 Totally geodesic solutions

In order to find another type of Möbius-loxodromic solutions for equation (12), we begin by searching homothetic solutions of the form $z_k(t) = \phi(t) z_{k,0}$ lying on the meridians through the point $z_{k,0}$ and parameterized as a geodesic curve. For this, we impose that the homothetic function $\phi(t)$ in (44) must satisfy the relations

$$
\ddot{\phi} z_{k,0} - \frac{2 \dot{z}_{k,0} z_{k,0}^2 \phi \dot{\phi}^2}{R^2 + \phi^2 |z_{k,0}|^2} = 0,
$$

$$
\frac{\partial U_R}{\partial \dot{z}_k} (\phi(t) z_{k,0}, \phi(t) \dot{z}_{k,0}) = 0,
$$

for $k = 1, 2, \ldots, n$.

In this way we obtain the function which state the geodesic parametrization of such solutions.

Lemma 7.7. The first second order differential equation of system (55) can be integrated by quadratures by a smooth real function $\phi = \phi(t)$ with initial conditions $\phi(0) = 1$ and $\dot{\phi}(0) = -1$, which when is substituted into the second equation of (55) holds the equality.

Proof. For $z_{k,0} \neq 0$, the first differential equation is equivalent to,

$$
\frac{\ddot{\phi}}{\phi} = \frac{2 |z_{k,0}|^2 \phi \dot{\phi}}{R^2 + \phi^2 |z_{k,0}|^2},
$$
which when is integrated becomes into

$$\frac{\dot{\phi}}{R^2 + \phi^2 |z_{k,0}|^2} = C_2,$$  \(56\)

for a suitable real constant \(C_2\). If we integrate directly the equation \(56\) we obtain the relation

$$\phi(t) = \frac{R^2}{|z_{k,0}|^2} \tan \left( C_2 |z_{k,0}| t + C_1 \right),$$  \(57\)

for other suitable constant of integration \(C_1\). Now, using the initial conditions, relation \(57\) we obtain the particular solution

$$\phi(t) = \frac{R^2}{|z_{k,0}|^2} \tan \left( \arctan \left( \frac{|z_{k,0}|^2}{R^2} \right) - \left( \frac{R^2 |z_{k,0}|}{|z_{k,0}|^4 + R^4} \right) t \right).$$  \(58\)

Since function \(58\) makes that all particles move locally along meridians in a geodesic way, then along all such solutions the second equation in \(55\) vanishes. This remark ends the proof of the Lemma.

**Definition 7.8.** A homothetic solution \(z(t) = (z_1(t), z_2(t), \cdots, z_n(t))\) of equation \(12\) is called totally geodesic if all the particles move along geodesic curves which hold equations \(55\).

We obtain the following result for these kind of motions.

**Theorem 7.9.** In the positively curved \(n\)-body problem, a necessary and sufficient condition for having totally geodesic solutions of the equation \(12\) is that all particles move along the straight lines through the origin of coordinates (meridians) and the whole set of solutions satisfy the system of equations

$$0 = \frac{\partial U_R}{\partial \bar{z}_k} (z_k, \bar{z}_k) = \sum_{j=1, j \neq k}^n m_k m_j \left( \frac{R^2 + |z_k|^2}{|z_j - z_k|^2} \frac{R^2 + \bar{z}_j z_k}{R^2 + \bar{z}_j z_k} \right) \left( \frac{R^2 + \bar{z}_j z_k}{R^2 + \bar{z}_j z_k} \right),$$  \(59\)

for \(k = 1, 2, \cdots, n\).
Proof. By hypothesis, any particle with mass \( m_k \) must hold the relations (55), which implies that the corresponding solution moves with the geodesic parametrization given in Lemma 7.7 in the direction of the meridian thorough such point and the origin of coordinates. Since one solution of the equation (12) moves along one geodesic curve if and only if the right hand side (equation (59)) vanishes, the claim follows.

When \( n = 2 \), the following result shows the no-existence of totally geodesic orbits, an unintuitive and surprising result.

**Theorem 7.10.** For the positively curved two-body problem there are not totally geodesic orbits.

Proof. Let \( m_1 \) and \( m_2 \) be two arbitrary masses in the space \( \mathbb{M}^2_R \) moving under equations (12) and conforming an homothetic solution, that is, the particles are moving along two (possibly different) geodesics. The system (59) becomes into,

\[
0 = \frac{m_2(R^2 + z_1 \bar{z}_2)(z_2 - z_1)}{|z_2 - z_1|^3 |R^2 + \bar{z}_2 z_1|^3},
\]

\[
0 = \frac{m_1(R^2 + z_2 \bar{z}_1)(z_1 - z_2)}{|z_1 - z_2|^3 |R^2 + \bar{z}_1 z_2|^3},
\]

with the infinitesimal condition (27).

We remark that avoiding collisions \( z_i \neq z_j \) and antipodal points \( R^2 + \bar{z}_i z_j \neq 0 \), a condition necessary and sufficient for the existence of nontrivial solutions of the linear system (60) for the masses is that the determinant of such system vanishes, that is,

\[
0 = \frac{(R^2 + z_1 \bar{z}_2)(z_2 - z_1) (R^2 + z_2 \bar{z}_1)(z_2 - z_1)}{|z_2 - z_1|^3 |R^2 + \bar{z}_2 z_1|^3 |z_1 - z_2|^3 |R^2 + \bar{z}_1 z_2|^3},
\]

\[
= \frac{1}{|z_2 - z_1|^2 (z_2 - \bar{z}_1)^2 |R^2 + \bar{z}_2 z_1|^4}
\]

which never holds for \( z_1, z_2 \in \mathbb{M}^2_R \). This proves the claim.

In the spirit of [3] we obtain conditions for having totally geodesic solutions of the system (12), which have the same shape configuration for all time \( t \) and all the particles move along geodesic curves. That is, all the particles are located at any time on the same euclidian circle and they do not rotate. Moreover, the property to be totally geodesic for a solution implies that the curved gradient vanishes along all the curves conforming a solution.
Let \( m_1, m_2 \) and \( m_3 \) be three masses in the space \( \mathbb{M}_R^2 \) moving along a particular totally geodesic solution, such that the corresponding configuration satisfy \( r(t) = |z_1(t)| = |z_2(t)| \), with \( z_3(t) \) fixed at the origin. Along the solution the following system holds (these kind of orbits are called Eulerian solutions).

\[
0 = \frac{m_2(R^2 + z_1 \bar{z}_2)(z_2 - z_1)}{|z_2 - z_1|^3 |R^2 + \bar{z}_2 z_1|^3} - \frac{m_3 z_1}{R^4 |z_1|^3}, \\
0 = \frac{m_1(R^2 + z_2 \bar{z}_1)(z_1 - z_2)}{|z_2 - z_1|^3 |R^2 + \bar{z}_2 z_1|^3} - \frac{m_3 z_2}{R^4 |z_2|^3}, \\
0 = \frac{m_1 z_1}{|z_1|^3} + \frac{m_2 z_2}{|z_2|^3},
\]

(62)

**Theorem 7.11.** A necessary and sufficient condition for having an Eulerian totally geodesic solution of system (62) is that the masses \( m_1 \) and \( m_2 \) be equal and the corresponding particles are always located on the same circle but in opposite sides.

**Proof.** A necessary and sufficient condition in the linear system (62) for having nontrivial solutions for the masses is that the principal determinant vanishes, but since we avoid collisions (\( z_1 \neq z_2 \)) and antipodal points (\( R^2 + \bar{z}_j z_k \neq 0 \)), it is equivalent to the equality

\[
z_2 \bar{z}_1 - z_1 \bar{z}_2 = 0.
\]

(63)

Without loss of generality we can put \( z_1(t) = r(t) e^{i\theta_1} \), \( z_2(t) = r(t) e^{i\theta_2} \) in equation (63), for suitable constant angles \( \theta_1 \) y \( \theta_2 \). Such relation is equivalent to the trigonometric equation

\[
\sin(\theta_2 - \theta_1) = 0
\]

(64)

which, avoiding collisions holds iff \( \theta_2 = \theta_1 + \pi \).

Therefore we can do \( z_1(t) = r(t) \) and \( z_2(t) = -r(t) \). Substituting these values in the third equation of system (62) we obtain that the masses \( m_1 \) and \( m_2 \) must be equal. This ends the proof of the Theorem.

**Remark 7.12.** Assuming that the masses \( m_1 \) and \( m_2 \) are equal, and that they are symmetrically located on the same circle, in [3], the authors obtain a homothetic Eulerian solution. In this sense Theorem 7.11 is stronger, since in principle the masses could be situate on different geodesic meridians and not necessarily symmetrically located.
Corollary 7.13. There is not totally geodesic solution for the restricted eulerian positively curved problem.

Proof. The restricted eulerian positively curved problem is obtained when the third mass tends to zero, $m_3 \to 0$ in the system (62). Such system becomes in the more simple two-body system (60) plus the equation

$$\frac{m_1 z_1}{|z_1|^3} + \frac{m_2 z_2}{|z_2|^3} = 0,$$

which together with Theorem 7.11 implies that $m_1 = m_2$ and $z_1 = -z_2$. A direct substitution of these pair of relations in any equation of (62) allows us to a contradiction. This proves the claim and ends the proof.

Let $m_1$, $m_2$ and $m_3$ be three masses in the space $\mathbb{R}^2$ moving along any totally geodesic solution. Then the following system holds,

$$0 = \frac{m_2 (R^2 + z_1 \bar{z}_2)(z_2 - z_1)}{|z_2 - z_1|^3 |R^2 + \bar{z}_2 z_1|^3} + \frac{m_3 (R^2 + z_1 \bar{z}_3)(z_3 - z_1)}{|z_3 - z_1|^3 |R^2 + \bar{z}_3 z_1|^3},$$

$$0 = \frac{m_1 (R^2 + z_2 \bar{z}_1)(z_1 - z_2)}{|z_1 - z_2|^3 |R^2 + \bar{z}_1 z_2|^3} + \frac{m_3 (R^2 + z_2 \bar{z}_3)(z_3 - z_2)}{|z_3 - z_2|^3 |R^2 + \bar{z}_3 z_2|^3},$$

$$0 = \frac{m_1 (R^2 + z_3 \bar{z}_1)(z_1 - z_3)}{|z_1 - z_3|^3 |R^2 + \bar{z}_1 z_3|^3} + \frac{m_2 (R^2 + z_3 \bar{z}_2)(z_2 - z_3)}{|z_2 - z_3|^3 |R^2 + \bar{z}_2 z_3|^3}. \quad (65)$$

Suppose that a particular configuration satisfy $r(t) = |z_1(t)| = |z_2(t)| = |z_3(t)|$. We will show that if, in general, three arbitrary masses are located on the same circle, they generate a totally geodesic solution if the masses form an equilateral triangle for all time.

Theorem 7.14. The configuration of a totally geodesic solution of the positively curved 3-body problem for arbitrary masses located on the same circle for all time is always an equilateral triangle.

Proof. A necessary and sufficient condition in the linear system (65) for having nontrivial solutions for the masses is that the principal determinant vanishes, but since we avoid collisions and antipodal points, it is equivalent to the equality

$$(R^2 + z_1 \bar{z}_3)(R^2 + z_2 \bar{z}_1)(R^2 + z_3 \bar{z}_2) - (R^2 + z_1 \bar{z}_2)(R^2 + z_1 \bar{z}_3)(R^2 + z_2 \bar{z}_3) = 0, \quad (66)$$

Without loss of generality we can do $z_1(t) = r(t)$ in equation (66), $z_2(t) = r(t) e^{i\theta_2}$ and $z_3(t) = r(t) e^{i\theta_3}$, for suitable constant angles $\theta_2$ and $\theta_3$, such
relation is equivalent to the system of trigonometric equations

$$\begin{align*}
\sin \theta_2 - \sin \theta_3 + \sin(\theta_3 - \theta_2) &= 0, \\
\cos \theta_2 + \cos \theta_3 + \cos(\theta_3 - \theta_2) &= 0,
\end{align*}$$

which avoiding collisions hold iff \( \theta_2 = \frac{2\pi}{3} \) and \( \theta_3 = \frac{4\pi}{3} \).

In another word, equation (66) holds iff the configuration is an equilateral triangle.

### 7.3 Homographic Möbius-loxodromic solutions

We consider the one dimensional parametric set of Möbius–loxodromic transformations

$$f_{H(t)}(z) = \phi(t) e^{it} z,$$

where \( \phi(t) \) is the geodesic homothetic function obtained in Lemma 7.7.

**Definition 7.15.** A solution \( z(t) = (z_1(t), z_2(t), \ldots, z_n(t)) \) of equation (12) is called homographic Möbius-loxodromic if it is invariant under the one-dimensional parametric set (68).

The next step is to obtain algebraic conditions on the position of the particles for having Homographic Möbius-loxodromic solutions in the positively curved problem, these are given in the following result.

**Theorem 7.16.** Consider \( n \) point particles with masses \( m_1, m_2, \ldots, m_n > 0 \) moving in \( M^2_\mathbb{R} \). A necessary and sufficient condition for the solution \( z(t) = (z_1(t), z_2(t), \ldots, z_n(t)) \) of (12) to be a homographic Möbius-loxodromic solution is that the coordinates satisfy the following system of rational equations.

$$\begin{align*}
\frac{R^6}{2(R^2 + |z_k|^2)^4} \left( 3i - 1 \right) z_k &= -\sum_{j=1, j \neq k}^{n} \frac{m_j(R^2 + |z_j|^2)^2(\bar{z}_j z_j)(|z_j - z_k|)}{|z_j - z_k|^3 |R^2 + \bar{z}_j z_k|^3},
\end{align*}$$

for \( k = 1, 2, \ldots, n \), and for the condition in the velocities \( \dot{z}_k = \frac{3i - 1}{4} z_k \).

In particular, if all the particles are located on the same euclidean circle, \( |z_k| = |z_j| = r(t) = r \), then the system of algebraic equations

$$\begin{align*}
\frac{R^6 (3i - 1)^2(R^2 - r^2)}{2(R^2 + |z_k|^2)^6} &= -\sum_{j=1, j \neq k}^{n} \frac{m_j(R^2 + z_k \bar{z}_j)(|z_j - z_k|)}{|z_j - z_k|^3 |R^2 + \bar{z}_j z_k|^3},
\end{align*}$$

must hold.
Proof. Let us suppose that \( z_k = z_k(t) \) is a solution of (12) and let \( w_k \) be the conformal function given by

\[
w_k(t) = \phi(t) e^{it} z_k(t),
\]

then, by differentiating we obtain

\[
\dot{w}_k = (\dot{\phi} z_k + i \phi \dot{z}_k + \phi \dot{z}_k) e^{it},
\]

\[
\ddot{w}_k = (\ddot{\phi} z_k + 2i \dot{\phi} z_k + 2\dot{\phi} \dot{z}_k - \phi \dot{z}_k + 2i \phi \dot{z}_k + \phi \ddot{z}_k) e^{it}.
\]  

(71)

Using the fact that \( w_k \) is a solution of (12) and the relation \( \bar{z}_k \bar{w}_k = e^{it} \), substituting in (12) we obtain

\[
m_k (\ddot{\phi} z_k + 2i \dot{\phi} z_k + 2\dot{\phi} \dot{z}_k - \phi \dot{z}_k + 2i \phi \dot{z}_k + \phi \ddot{z}_k) e^{it}
\]

\[
- \frac{2m_k \phi e^{-it} \bar{z}_k}{R^2 + \phi^2 |z_k|^2} \left( \phi z_k + i \phi \dot{z}_k + \phi \dot{z}_k \right)^2 e^{2it}
\]

\[
= \frac{(R^2 + \phi^2 |z_k|^2)^2}{2R^4} \frac{\partial U_R}{\partial \bar{z}_k} e^{it}.
\]  

(72)

Since \( e^{it} \neq 0 \), by using the condition for the first equation of (55) in equation (72), we get that for all time \( t \) the function \( \phi \) and the solution \( z_k \) must satisfy the system

\[
m_k (2i \dot{\phi} z_k + 2\dot{\phi} \dot{z}_k - \phi \dot{z}_k + 2i \phi \dot{z}_k + \phi \ddot{z}_k)
\]

\[
- \frac{2m_k \phi \ddot{z}_k}{R^2 + \phi^2 |z_k|^2} \left( -\phi^2 z_k^2 + 2\phi \dot{z}_k \ddot{z}_k + 2i \phi \dot{z}_k \dot{z}_k + 2\phi \dot{z}_k \dot{z}_k + 2i \phi \ddot{z}_k \ddot{z}_k \right)
\]

\[
= \frac{(R^2 + \phi^2 |z_k|^2)^2}{2R^4} \frac{\partial U_R}{\partial \bar{z}_k}.
\]  

(73)

Again, we search the conditions for having the infinitesimal generators of the vector field for such solutions by doing \( t = 0 \). Introducing the initial conditions \( \phi(0) = 1 \) and \( \dot{\phi}(0) = -1 \) into equation (73), we obtain the algebraic rational system of equations

\[
m_k (-2iz_k - 2\ddot{z}_k - z_k + 2i \dot{z}_k + \ddot{z}_k)
\]

\[
- \frac{2m_k \ddot{z}_k}{R^2 + |z_k|^2} \left( -\ddot{z}_k^2 + \dot{z}_k^2 - 2i \phi \ddot{z}_k^2 - 2z_k \dot{z}_k + 2i \dot{z}_k \ddot{z}_k \right)
\]

\[
= \frac{(R^2 + |z_k|^2)^2}{2R^4} \frac{\partial U_R}{\partial \bar{z}_k}.
\]  

(74)
Now, since \( m_k \neq 0 \) and \( z_k \) is a solution for equation (12) for all time \( t \), the above system (74) give us

\[
(-2i z_k - 2 \dot{z}_k - z_k + 2i \dot{z}_k) = \frac{2 \ddot{z}_k}{R^2 + |z_k|^2} \left( -z_k^2 - 2i \dot{z}_k^2 - 2z_k \dot{z}_k + 2i z_k \dot{z}_k \right).
\]

(75)

Last equation (75) holds if it satisfy at least one of the following conditions

\[
|z_k| = R, \quad \dot{z}_k = \frac{2i + 1}{2i - 2} z_k = \frac{3i - 1}{4} z_k.
\]

(76)

We observe that the first equation in (76) corresponds to the geodesic circle of radius \( R \). The second is a first order differential equation (the infinitesimal conditions for having such motions), whose integrals are loxodromic (helicoidal) curves in \( M^2_R \), parameterized by

\[
z_k(t) = z_{k0} e^{(\frac{3i - 1}{4})t},
\]

(77)

the same second equation in (76) gives us the relation of velocities that must be hold for obtaining such solutions.

Finally, deriving the second equation we obtain

\[
\ddot{z}_k = \frac{3i - 1}{4} \dot{z}_k = \frac{(3i - 1)^2}{16} z_k.
\]

(78)

A direct substitution of the second relation of (76) and the relation (78) into equations of motion (12) gives the system of algebraic rational equations (69). This ends the proof of Theorem .

We remark that the second equation in (76) implies that any Möbius-loxodromic solution intersects all the geodesics rays (the meridians on the sphere) in a constant angle. This is the reason for call those solutions loxodromic.

Now we study some examples of Möbius-loxodromic solution, we start with the particular case of two particles in the space \( M^2_R \) where \( r = r(t) = |\phi(t)| = |z_k(t)| \) (for \( k = 1, 2 \)). Actually this example has been widely studied in [10], from where we know that two particles of masses \( m_1 \) and \( m_2 \) moving on the same circle of radius \( r \neq R \) form a relative equilibriu iff the masses are equal and they are located at opposite sides of the circle. For Möbius-loxodromic solutions we have the following result.
**Theorem 7.17.** For the positively curved two-body problem with equal masses there are not homographic Möbius-loxodromic solutions.

**Proof.** In this case, we have that $m_1 = m_2 = m$, then the system of rational equations (69) becomes into

$$
\frac{R^6(3i-1)^2(R^2 - r^2)z_1}{(R^2 + r^2)^6} = -\frac{m(R^2 + z_1 \bar{z}_2)(z_2 - z_1)}{|z_2 - z_1|^3 |R^2 + \bar{z}_2 z_1|^3}
$$

$$
\frac{R^6(3i-1)^2(R^2 - r^2)z_2}{(R^2 + r^2)^6} = -\frac{m(R^2 + z_2 \bar{z}_1)(z_1 - z_2)}{|z_1 - z_2|^3 |R^2 + \bar{z}_1 z_2|^3}
$$

(79)

If we avoid collisions and conjugated points from the system (79) we obtain the condition

$$
-(R^2 + z_1 \bar{z}_2)\ddot{z}_1 = (R^2 + z_2 \bar{z}_1)\ddot{z}_2,
$$

(80)

which, since $|z_1(t)| = r(t) = r$ and $|z_2(t)| = r(t) = r$, it is equivalent to

$$
-R^2 (\ddot{z}_1 + \ddot{z}_2) = r^2 (\ddot{z}_2 + \ddot{z}_1),
$$

(81)

which does not hold for any mass.

We remark that this last result matches with the result of Theorem 7.10 which shows that there are not totally geodesic solutions in the two body problem in $M^2_R$.

Now we use the results obtained in sections 4.1 and 5.1 for obtaining Möbius-loxodromic (homographic) solutions for the particular case of three particles in $M^2_R$ when $r = r(t) = |\phi(t)| = |z_k(t)|$ (for $k = 1, 2, 3$). We start generalizing the Eulerian orbits of celestial mechanics.

**Theorem 7.18.** A necessary and sufficient condition for a solution of the curved 3-body problem in $M^2_R$, with one particle at the origin and the other two with equal mass located on the same geodesic, to be a Möbius-loxodromic solution is that the particles are located always at opposite sides of the same circle of radius $r = r(t) \neq R$.

**Proof.** The proof of this Theorem follows by straightforward computations, using the conditions of the configuration in equations (69). The point to remark here is the conditions on the velocities (76) of the particles in the same circle. This is given by $\dot{z}_k = \frac{3i-1}{4} z_k$. 

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For the generalization of the Lagrangian solutions of celestial mechanics, we have the following result, whose proof is also by straightforward computations, we omit it here, getting exactly the same condition (78) on the velocities of the particles as in the previous Theorem.

**Theorem 7.19.** Let us consider a configuration of 3 equal masses on the same circle of radius \( r = r(t) \) in \( \mathbb{M}^2_R \), then a necessary and sufficient condition to have a Möbius-loxodromic solution is that the particles form always an equilateral triangle.

**Remark 7.20.** The Möbius-loxodromic solutions are the similar to the homographic solutions studied in [3], many results for the case \( n = 3 \) were proved first in that paper, nevertheless we think that the proofs present in this work are easier, showing the big advantage to work with intrinsic coordinates and the use of complex variable for the computations.

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