Fixed-time trajectory tracking control for nonholonomic mobile robot based on visual servoing

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Abstract This paper aims to discuss fixed-time tracking control problem for a nonholonomic wheeled mobile robot based on visual servoing. At first, by making use of the pinhole camera model, the robot system model with uncalibrated camera parameters is given. Then, the tracking error system between the mobile robot and desired trajectory is proposed. Thirdly, on the basis of fixed-time control theory and Lyapunov stability analysis, fixed-time tracking control laws are proposed for the mobile robot such that the robot can track the reference trajectory in a fixed time. It is well known that the convergence time for the finite-time control systems is usually dependent on the initial state of the system. However, the settling time obtained by the fixed-time control is independent of the system initial conditions and only determined by the controller parameters, which is more in line with practical application. Simulation results are given at the end.

Keywords Fixed-time stability · Tracking control · Mobile robot · Uncalibrated camera

1 Introduction

In recent years, visual servoing technologies play an important role in the robotic systems to accomplish various practical functions, such as hovering control for unmanned aerial vehicle [1], target tracking [2,3] and target capture [4]. Visual servo is an effective method to employ the visual information to establish the relationship between the robot and the environment and generates control commands through visual information feedback. Generally, visual servoing can be classified into three categories: including position-based visual servoing [5], image-based visual servoing [6] and hybrid approach [7]. For position-based visual servoing control, the reconstructed Euclidean information is used in the feedback loop to control the robot, whereas image-based visual servoing control directly deals with visual information on the image plane. A combination of the these two schemes is named hybrid approach.

During the past few decades, the tracking control of wheeled mobile robots with nonlinear constraints has become the focus of researches owing the inherent non-
linearity in robot dynamics and the usefulness in many applications. It is well known that to calibrate the intrinsic and extrinsic parameters of the camera is an important, inevitable task in visual servoing control. However, camera calibration is a repetitive and costly work. In order to avoid camera calibration, the uncalibrated visual servoing is studied by many researchers [8–10]. So far, the visual tracking control problem for nonholonomic mobile robot has been widely discussed. For a monocular camera system mounted on an underactuated wheeled mobile robot, the visual servoing tracking controller was developed in [11]. In [12], by using an uncalibrated fixed camera, a novel adaptive tracking controller was designed for a nonholonomic mobile robot. In [13], the image-based tracking problem of wheeled mobile robots with uncertain camera parameters was studied, and adaptive trajectory tracking control laws were designed.

With the development of automatic control technology and the increasing demand for stability accuracy of nonlinear systems, the finite-time control method has attracted increasing attention of many researchers [14–16], which has the characteristics of faster convergence rate, higher accuracy, better disturbance rejection properties and robustness against uncertainties [17–20]. Numerous works have been proposed related to the finite-time tracking control problem of nonholonomic mobile robot systems. In [21], based on the relay switching technique and the terminal sliding mode control scheme, a finite-time tracking controller was proposed for the nonholonomic systems with extended chained form, which ensured that the reference trajectory was accurately tracked in a finite time. The authors of [22] studied finite-time tracking control problem of multiple nonholonomic wheeled mobile robots with unknown parameters and external disturbances, finite-time disturbance observers were designed for each robot to estimate the time-varying external disturbances, and then, composite finite-time tracking control laws were designed for each robot to estimate the time-varying external disturbances. In [23], finite-time tracking control problem was discussed for a nonholonomic mobile robot system with uncalibrated camera parameters, the camera-objective visual servoing model was given and continuous finite-time controller laws were designed for the mobile robot by using finite-time control theory. In [24], continuous nonsingular finite-time tracking control laws were given for underwater robot manipulators with lumped disturbances.

Although the finite time control algorithm can ensure the convergence of the closed-loop system in finite time, the expression of finite settling-time depends on the initial state of the system, which is not easy to obtain in practical application. So it would be useful if the settling time could be predetermined no matter whether the initial conditions are known or not. Motivated by this, fixed-time stability conception was proposed in [25]. This soon attracted a lot of scholars’ attention. Fixed-time control is more preferable than finite-time control in practical applications since the fixed-time approach can generate a control law prescribing a transition time which is independent of the operation domain [26]. Based on the fixed-time stability conception, some new results are reported. For some second-order and high-order systems, the fixed-time control problem was studied in [27–30]. The authors of paper [31] discussed fixed-time tracking control problem for nonholonomic mobile robot, and fixed-time control algorithm was designed by proposing a new integral terminal sliding mode surface. Fixed-time trajectory tracking control problem for a nonholonomic mobile robot was discussed in [32]; on the basis of the fixed-time control method and Lyapunov stability analysis, continuous fixed-time tracking controllers were developed for the mobile robot. In [33], fixed-time trajectory tracking control for a group of nonholonomic mobile robots was investigated, and distributed fixed-time tracking controllers were developed for each robot, which made all states of each robot reach the desired value in a fixed time. However, the papers [31–33] did not consider visual servoing for the mobile robot system. The authors of paper [10] studied fixed-time trajectory tracking control problem for a wheeled mobile robot with uncalibrated camera parameters and designed the fixed-time adaptive tracking controllers for the mobile robot system.

Motivated by the aforementioned discussion, the paper aims to consider the fixed-time trajectory tracking problem for a nonholonomic mobile robot using an uncalibrated, fixed (ceiling-mounted) camera system. Since the robot system based on visual servoing consists of nonholonomic constraint and uncalibrated camera parameters, it is more harder to design the fixed-time tracking control laws. The key contributions of this paper can be summarized as follows: (1) The camera-objective visual servoing kinematic model is given, and the tracking error system between camera-objective visual servoing model and the desired refer-
ence trajectory is introduced. (2) Fixed-time trajectory tracking controllers are proposed for the nonholonomic mobile robot based on visual servoing, which guarantees the desired trajectory can be tracked by the mobile robot in a fixed time. (3) Under the fixed-time control, the closed-loop systems with fixed-time convergence exhibit some nice features such as better disturbance rejection properties, faster convergence and robustness against uncertainties. In addition, the settling time is independent of the system initial conditions and only determined by the controller parameters.

The rest of this paper is organized as following arrangements: Sect. 2 gives some useful lemmas, and problem formulation is provided. The tracking error dynamic system is considered and fixed-time tracking controllers are designed for the nonholonomic mobile robot in Sect. 3, and the validity of the controllers for the system is proved. Section 4 presents the simulation results to illustrate the controller’s performance. Section 5 gives some conclusions.

2 Preliminaries

For convenience, we give the following definition

\[ \text{sig}^α(X) = |X|^α \text{sgn}(X), \]  

(1)

where \( \alpha > 0 \) and \( \text{sgn}(\cdot) \) represents the standard symbolic function, and it is easy to verify that

\[ \frac{d}{dy} \text{sig}^{1+α}(y) = (1 + α)|y|^α. \]  

(2)

2.1 Some useful Lemmas

**Lemma 2.1** [25] Consider a system as follows

\[ \dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n, \]  

(3)

if there exists a positive-definite continuous function \( V(x) : \mathbb{R}^n \rightarrow \mathbb{R} \) satisfying \( \dot{V}(x) \leq -αV^p(x) - βV^q(x) \) for all \( x \in U_0 \), where \( α > 0, β > 0, 0 < p < 1, q > 1 \), then the system is fixed-time stable and the fixed settling time \( T \) can be upper bounded by

\[ T \leq \frac{1}{α(1−p)} + \frac{1}{β(q−1)}. \]

**Lemma 2.2** [34] The inequality \( \sum_{i=1}^{n} |x_i|^p \leq \sum_{i=1}^{n} |x_i|^q \) holds for any \( x_i \in \mathbb{R}, i = 1, 2, \ldots, n \), where \( p \) is a real number satisfying \( 0 < p \leq 1 \).

**Lemma 2.3** [35] The inequality \( n^{1−p}(\sum_{i=1}^{n} |x_i|)^p \leq \sum_{i=1}^{n} |x_i|^p \) holds for any \( x_i \in \mathbb{R}, i = 1, 2, \ldots, n \) and \( p > 1 \).

**Lemma 2.4** [34] If \( 0 < l \leq 1 \), then \( |x|^l - |y|^l \leq 2^{1−l}|x - y|^l \), where \( l \) is a ratio of two odd integers.

**Lemma 2.5** [36] The inequality \( |x|^c |y|^d \leq \frac{c}{c+d} \gamma \) holds for any \( x, y \in \mathbb{R} \) and \( \forall c, d, γ > 0 \).

2.2 Problem formulation

2.2.1 Kinematic model with monocular camera

As we know, nonholonomic wheeled mobile robots are divided into four types: (2, 0), (2, 1), (1, 1) and (1, 2) in [37]. In this paper, we will discuss tracking control problem for (2,0)-type nonholonomic mobile robot system based on visual servoing; as shown in Fig. 1, it has two steering wheels and one castor wheel. Letter \( P \) denotes the center of mass of the robot. If we assume the centroid coincides with the geometric center, then the nonholonomic constraint model of the robot is as follows

\[ \dot{x} \sin θ - \dot{y} \cos θ = 0, \]  

(4)

where \( (x, y) \) represents the position \( P \) of the center of mass, the angle between \( X \) axis and \( X_1 \) axis is \( θ \), and the counterclockwise direction is defined as the positive direction for angle \( θ \). By this formula, the kinematic system can be written as:

\[ \dot{x} = v \cos θ, \]  

(5a)

\[ \dot{y} = u \sin θ, \]  

(5b)

\[ \dot{θ} = ω, \]  

(5c)

where \( v \) and \( ω \) are the translational velocity and angular velocity of the mobile robot, respectively.

Figure 2 shows the robot–camera system. In this paper, we assume that the robot presented in Fig. 1 moves under a pinhole camera fixed to the ceiling, where the camera plane and the robot plane are parallel. There are three coordinate systems, inertial coordinate system \( X − Y − Z \), the camera coordinate system
of the intersection of the optical axis with the image plane, and $\alpha_1$ and $\alpha_2$ are positive constants, which are determined by the pinhole camera that is fixed to the ceiling.

$$R(\theta_0) = \begin{pmatrix} \cos \theta_0 & \sin \theta_0 \\ -\sin \theta_0 & \cos \theta_0 \end{pmatrix},$$

(7)

where $\theta_0$ means the angle between $y$ axis and $X$ axis with a positive anticlockwise orientation.

For parameters $\alpha_1$, $\alpha_2$ and $\theta_0$, the following assumption is required for controller design.

**Assumption 1** $\theta_0$ is unknown, and $\alpha_1 = \alpha_2 = \alpha$ are unknown, $\alpha \leq \alpha \leq \alpha$, where $\alpha$ and $\alpha$ are positive known constants.

**Remark 1** [39] $\alpha_1 = \alpha_2 = \alpha$ shows that the scalar factor along $u_1$ axis is the same as that one along $v_1$ axis. Some charge coupled device cameras (CCD cameras) are made like this. Commonly, the upper and lower bounds of the scalar factor, the depth and the focal length can be estimated in advance.

The tracking control problem is considered in this paper. Firstly, the kinematic model of the mobile robot in the image coordinate system $u_1 - o_1 - v_1$ is assumed as follows:

$$\dot{x}_m = v_\alpha_1 \cos(\theta - \theta_0), \quad \dot{y}_m = v_\alpha_2 \sin(\theta - \theta_0), \quad \dot{\theta}_m = \omega_m,$$

(8)

where $(x_m, y_m)$ is the path of the mass center $(x, y)$ in the image coordinate system, and $\theta_m$ is the angular in the image coordinate system. $v_m$ and $\omega_m$ are the translational velocity and the angular velocity of the robot, respectively.

According to equation (5), calculating the time derivative of system (7), the camera-objective visual servoing model yields as follows

$$\dot{x}_m = v_\alpha \cos(\theta - \theta_0), \quad \dot{y}_m = v_\alpha \sin(\theta - \theta_0), \quad \dot{\theta}_m = \omega.$$

(9)

Combining systems (8) with (9), we have

$$v_\alpha \cos(\theta - \theta_0) = v_m \cos \theta_m,$$

$$v_\alpha \sin(\theta - \theta_0) = v_m \sin \theta_m.$$

(10)
From (10), it has

\[ v \alpha [\sin(\theta - \theta_0) \cos \theta_m - \cos(\theta - \theta_0) \sin \theta_m] = v \alpha \sin(\theta - \theta_0 - \theta_m) \]

\[ = 0. \tag{11} \]

Since Assumption 1 holds and \( v \) is arbitrary, we get

\[ \sin(\theta - \theta_0 - \theta_m) = 0, \tag{12} \]

so we can get

\[ \theta - \theta_0 - \theta_m = N \pi, \tag{13} \]

where \( N \) is a positive integer. According to (10), we obtain

\[ v \alpha \cos(\theta - \theta_0) \cos \theta_m = v_m \cos^2 \theta_m, \]

\[ v \alpha \sin(\theta - \theta_0) \sin \theta_m = v_m \sin^2 \theta_m, \]

hence

\[ v_m = v \alpha \cos(\theta - \theta_0 - \theta_m). \tag{15} \]

From (13), we can also obtain \( \dot{\theta} - \dot{\theta}_m = 0 \), and it means that \( \dot{\theta} = \dot{\theta}_m \), thus,

\[ \omega_m = \omega. \tag{16} \]

On the basis of (15) and (16), we can rewrite system (8) as follows

\[ \dot{x}_m = v \alpha \cos(\theta - \theta_0 - \theta_m) \cos \theta_m, \tag{17a} \]

\[ \dot{y}_m = v \alpha \cos(\theta - \theta_0 - \theta_m) \sin \theta_m, \tag{17b} \]

\[ \dot{\theta}_m = \omega. \tag{17c} \]

From (13), we have \( \cos(\theta - \theta_0 - \theta_m) = \pm 1 \). Because the inertial coordinate system and camera coordinate system are fixed, it is easy to see that \( \theta - \theta_m \) and \( \theta_0 \) are constants, and it means that \( \cos(\theta - \theta_0 - \theta_m) \) is a fixed value. In this paper, we let \( \cos(\theta - \theta_0 - \theta_m) = 1 \), and then, system (17) can be written as

\[ \dot{x}_m = v \alpha \cos \theta_m, \tag{18a} \]

\[ \dot{y}_m = v \alpha \sin \theta_m, \tag{18b} \]

\[ \dot{\theta}_m = \omega. \tag{18c} \]

### 2.2.2 Control problem description and control objective

The fixed-time tracking problem will be discussed in this paper, and fixed-time control laws \( v \) and \( \omega \) for the nonholonomic robot system (18) will be designed. The model of reference robot is presented as follows

\[ \dot{x}_r = v_r \alpha_1 \cos \theta_r, \tag{19a} \]

\[ \dot{y}_r = v_r \alpha_2 \sin \theta_r, \tag{19b} \]

\[ \dot{\theta}_r = \omega_r, \tag{19c} \]

where \( (x_r, y_r) \) is the desired path of the mass center \((x, y)\) in the coordinate system, and \( \theta_r \) is the desired direction. \( v_r \) and \( \omega_r \) are the desired translational velocity and the angular velocity of the reference mobile robot, respectively. They satisfy the following assumption:

**Assumption 2** Suppose the velocities \( v_r, \omega_r \) and the derivative \( \dot{\omega}_r \) are bounded, \(|v_r(t)| \leq v_r^{\text{max}}, 0 < \omega_r^{\text{min}} \leq |\omega_r(t)| \leq \omega_r^{\text{max}} \) and \(|\dot{\omega}_r(t)| < \omega_r^{\text{max}}\) for any \( 0 \leq t \leq t_0 \) where \( v_r^{\text{max}}, \omega_r^{\text{min}}, \omega_r^{\text{max}}, \) and \( \omega_r^{\text{max}} \) are known positive constants.

Under Assumption 1, by using the results of paper [40], the tracking error dynamic equation can be described as follows

\[ \begin{pmatrix} x_e \\ y_e \\ \theta_e \end{pmatrix} = \begin{pmatrix} \cos \theta_m & \sin \theta_m & 0 \\ -\sin \theta_m & \cos \theta_m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_r - x_m \\ y_r - y_m \\ \theta_r - \theta_m \end{pmatrix}, \tag{20} \]

i.e.,

\[ x_e = (x_r - x_m) \cos \theta_m + (y_r - y_m) \sin \theta_m, \]

\[ y_e = -(x_r - x_m) \sin \theta_m + (y_r - y_m) \cos \theta_m, \]

\[ \theta_e = \theta_r - \theta_m. \tag{21} \]

Taking the time derivative of \( x_e, y_e, \theta_e \) along system (18) and (19), the error dynamics equations can be obtained as

\[ \dot{x}_e = (\dot{x}_r - \dot{x}_m) \cos \theta_m - (x_r - x_m) \dot{\theta}_m \sin \theta_m + (\dot{y}_r - \dot{y}_m) \sin \theta_m + (y_r - y_m) \dot{\theta}_m \cos \theta_m = (v_r \alpha \cos \theta_r - v \alpha \cos \theta_m) \cos \theta_m - \omega (x_r - x_m) \sin \theta_m + (v_r \alpha \sin \theta_r - v \alpha \sin \theta_m) \sin \theta_m \]
Firstly, subsystem (22c) will be discussed and the angular velocity controller will be designed, which can make error state $\theta_e$ achieve zero in fixed time.

From the transformation equation (20), if $x_e = y_e = \theta_e = 0$, it will mean that $x_m = x_r, y_m = y_r$ and $\theta_m = \theta_r$. In the following section, the object of this paper is to design control laws $v$ and $\omega$

$$v = v(t, x_e, y_e, \theta_e), \quad \omega = \omega(t, x_e, y_e, \theta_e),$$

(23a) (23b)

to make the tracking error system (22) fixed-time stable.

3 Control design

The main task in this part is to design fixed-time controllers $v$ and $\omega$ such that system (18) can track the desired trajectory (19) in fixed time.

Obviously, the tracking error dynamic equation (22) consists of two subsystems: a first-order subsystem (22c) and a second-order subsystem (22a) and (22b); therefore, in the following part, two subsections will be given to study these two subsystems, respectively. Firstly, subsystem (22c) will be discussed and the angular velocity $\omega$ will be designed. Secondly, subsystem (22a) and (22b) will be investigated and the translational velocity $v$ will be proposed.

3.1 Control law design of $\omega$

Under Assumption 1, the subsystem (22c) will be discussed, angular velocity controller $\omega$ will be proposed, which can make error state $\theta_e$ achieve zero in fixed time.

**Theorem 3.1** Consider system (18), if Assumption 1 holds and the control law $\omega$ is designed as follows

$$\omega = \omega_r + k_1 s g^\beta_1 \theta_e + k_2 s g^\beta_2 \theta_e,$$

(24)

where $k_1, k_2 > 0, 0 < \beta_1 < 1, \beta_2 > 1$, then the control law (24) can make the error state $\theta_e$ of subsystems (22c) converge to zero in fixed time; in other words, the desired trajectory $\theta_r$ can be tracked by $\theta_m$ in fixed time.

Proof. The Lyapunov function is chosen as

$$V(\theta) = \frac{1}{2} \theta_e^2.$$  

(25)

According to system (24), computing the time derivative of $V(\theta)$, we obtain

$$\dot{V}(\theta) = \theta_e \dot{\theta}_e = -k_1 |\theta_e|^{1+\beta_1} - k_2 |\theta_e|^{1+\beta_2}$$

$$= -k_1 2^{1+\beta_1} \left( \frac{1}{2} \theta_e^2 \right)^{1+\beta_1} - k_2 2^{1+\beta_2} \left( \frac{1}{2} \theta_e^2 \right)^{1+\beta_2}$$

$$\leq -k_1 2^{1+\beta_1} (V(\theta))^{1+\beta_1} - k_2 2^{1+\beta_2} (V(\theta))^{1+\beta_2}.$$  

(26)

Noticing that $0 < \beta_1 < 1, \beta_2 > 1$, it can be calculated that $0 < \frac{1+\beta_1}{2} < 1, \frac{1+\beta_2}{2} > 1$. By virtue of Lemma 2.1, it can be seen that $\dot{V}(\theta)$ converges to zero in a fixed time

$$T_\theta \leq \frac{1}{T_1} + \frac{1}{T_2},$$  

(27)

where $T_1 = k_1 2^{1+\beta_1} \frac{1-\beta_1}{2}, T_2 = k_2 2^{1+\beta_2} \frac{1-\beta_2}{2}$. On the other hand, if $V(\theta) = 0$ in fixed time, then it means that $\theta_e = 0$ in fixed time, i.e., the state $\theta_m$ can converge to the desired value $\theta_r$ in fixed time.

3.2 Translational velocity design

Under Assumptions 1–2, the subsystem (22a) and (22b) will be considered and the fixed-time control law $v$ will
be given, and we also prove that the controller \( v \) can make the error states \( x_e \) and \( y_e \) converge to zero in a fixed time.

**Theorem 3.2** For the system (18), if Assumptions 1–2 hold and control law \( v \) is given as follows

\[
v = v_r - \frac{1}{\omega_r}(\eta_3 \sigma^{1+m_1}(y_e) + \sigma_3(x_e, y_e) + 1) \xi r_1^{m_1-1} - \eta_1 \sigma_{e1}(x_e, y_e) + \sigma_1(y_e) + 1) \xi r_1^{2r_1-1},
\]

where

\[
\eta_1 = \frac{r_1 \tau_1^{2r_1-1}}{(1 + r_1)^{1 + \frac{1}{r_1}}},
\]

\[
\eta_2 = \frac{r_1^{2r_1+3r_1-1}}{(1 + r_1)^{1 + r_1}}
\]

\[
\eta_3 = \frac{m_1^{2r_1-1} r_1^{1+m_1}}{(1 + m_1)^{1+m_1}}
\]

\[
\sigma(y_e) = 2^{\frac{2}{r_1}}(2 - r_1) \left( 1 + y_e^{m_1-1} \right)^{\frac{1}{r_1}}
\]

\[
\sigma_1(x_e, y_e) = \frac{2 - r_1}{1 + r_1} \left( \frac{\eta_1}{\omega_r} \right)^{1 + r_1} \left| \sigma \right|^{1 + r_1}
\]

\[
\sigma_2(x_e, y_e) = \frac{4r_1}{1 + r_1} \left( \frac{\eta_2}{\omega_r} \right)^{1 + r_1} \left| \sigma \right|^{1 + r_1}
\]

\[
\sigma_3(x_e, y_e) = \frac{2^{m_1 + 1}}{1 + m_1} \left( \frac{\eta_3}{\omega_r} \right)^{1 + m_1} \left| \sigma \right|^{1 + m_1}
\]

\[
\xi = (-\omega_r x_e)^{\frac{1}{r_1}} - (-2y_e^{m_1} - 2e_m^{m_1})^{\frac{1}{r_1}}
\]

\[
\eta_1, \eta_2, \eta_3, m_1, r_1 \text{ are defined as above and}
\]

\[
\sigma(e_1) = 2^{\frac{1}{r_1}}(2 - r_1) \left( 1 + e_m^{m_1-1} \right)^{\frac{1}{r_1}}
\]

\[
\sigma_1(e_1) = \frac{2 - r_1}{1 + r_1} \left( \frac{\eta_1}{\omega_r} \right)^{1 + r_1} \left| \sigma \right|^{1 + r_1}
\]

\[
\sigma_2(e_1, e_2) = \frac{4r_1}{1 + r_1} \left( \frac{\eta_2}{\omega_r} \right)^{1 + r_1} \left| \sigma \right|^{1 + r_1}
\]

\[
\sigma_3(e_1, e_2) = \frac{2^{m_1 + 1}}{1 + m_1} \left( \frac{\eta_3}{\omega_r} \right)^{1 + m_1} \left| \sigma \right|^{1 + m_1}
\]

**Proof** We let

\[
e_1 = y_e, \quad e_2 = -\omega_r x_e, \quad \theta_e = \theta_e.
\]

Combing system (22) with control laws (24) and (28), the time derivative of (29) is given

\[
\dot{e}_1 = \frac{\omega_r e_2}{\omega_r} + v_r \alpha \sin \theta_e,
\]

\[
\dot{e}_2 = \frac{\omega_r e_2}{\omega_r} - \omega_r \omega e_1 + \omega_r v \alpha - \omega_r v_r \alpha
\]

\[
- \frac{\alpha}{2} (\eta_3 \sigma^{1+m_1} (e_1) + \sigma_3 (e_1, e_2) + 1) \xi r_1^{m_1-1} - \frac{\alpha}{2} (\eta_2 \sigma^{1+r_1} (e_1) + \eta_1 \sigma_1 (e_1) + 1) \xi r_1^{2r_1-1},
\]

where \( r_1 = 1 + \tau_1, \) \( m_1 = 1 + \tau_2, \) \( -\frac{1}{2} < \tau_1 = -\frac{q_1}{p_1} < 0, \) \( \tau_2 = \frac{q_2}{p_2} > 0, \) and \( q_1(i = 1, 2) \) are positive even integer, \( p_i(i = 1, 2) \) are positive odd integer, then the control law (28) can make the error states \( x_e \) and \( y_e \) of subsystems (22a)–(22b) converge to zero in fixed time.
\[
\frac{\dot{V}_1}{\alpha r} e_2 - \omega_2^r e_1 = -\frac{\alpha}{\omega_r} \left( \eta_3 \sigma^{1+m_1}(e_1) + \sigma_3(e_1) + 1 \right) \xi^{r+1+m_1-1} \\
- \frac{\alpha}{\omega_2^r} \left( 2^{1-r_1}\sigma(e_1) + \eta_2 \sigma^{1+r_1}(e_1) + \eta_1 \right) + \sigma_4(e_1, e_2) + \sigma_1(e_1) + 1 \right) \xi^{2r_1-1}.
\]

(31b)

Two steps will be given to prove this theorem, and fixed-time control theory and adding a power integrator technique will be employed.

**Step 1** The Lyapunov function is chosen as

\[
V_1(e_1) = \frac{1}{2} e_1^2.
\]

(32)

From (31a), the derivative of \(V_1(e_1)\) is

\[
\dot{V}_1(e_1) = e_1 e_2 = e_1 e_2^* + e_1(e_2 - e_2^*).
\]

(33)

With the help of the backstepping design idea, let us design a virtual control law as follows

\[
e_2^* = -2e_1^{r_1} - 2e_1^{m_1},
\]

(34)

which leads to

\[
\dot{V}_1(e_1) \leq -2e_1^{1+r_1} - 2e_1^{1+m_1} + e_1(e_2 - e_2^*).
\]

(35)

**Step 2** Consider the Lyapunov function \(V_2(e_1, e_2)\), which can be described as

\[
V_2(e_1, e_2) = V_1(e_1) + \int_{e_2^*}^{e_2} \left( \frac{1}{2} - e_2^* \right)^{2-r_1} ds.
\]

(36)

For brevity, denote \(\xi = \frac{1}{2} - e_2^*\). The derivative of \(V_2(e_1, e_2)\) along systems (31a) and (35) is

\[
\dot{V}_2(e_1, e_2) \leq -2e_1^{1+r_1} - 2e_1^{1+m_1} + e_1(e_2 - e_2^*) + \xi^{2-r_1} \dot{e}_2 \\
+ \left( 2 - r_1 \right) \frac{d(-e_2^*)}{dt} \int_{e_2^*}^{e_2} \left( \frac{1}{2} - e_2^* \right)^{1-r_1} ds.
\]

(37)

Using Lemmas 2.4 and 2.5, one obtains

\[
e_1(e_2 - e_2^*) \leq |e_1||e_2^*|^{r_1} - (e_2^*)^r_1 \\
\leq 2^{1-r_1}|e_1||\xi|^{r_1} \\
\leq \frac{1}{4}|e_1|^{1+r_1} + \eta_1 |\xi|^{1+r_1}.
\]

(38)

Notice that

\[
-e_2^* \frac{1}{\sqrt{\xi}} = \left( 2e_1^{r_1}(1+e_1^{m_1-r_1}) \right) \frac{1}{\sqrt{\xi}} = 2^{\frac{1}{4}}e_1(1+e_1^{m_1-r_1}) \frac{1}{\sqrt{\xi}}.
\]

(39)

which leads to

\[
(2 - r_1)^{-\frac{d(-e_2^*)}{dt}} = 2^{\frac{1}{4}}(2 - r_1)(1 + e_1^{m_1-r_1}) \frac{1}{\sqrt{\xi}} \\
+ 2^{\frac{1}{4}}(2 - r_1)e_1^{m_1-r_1} \\
\times (1 + e_1^{m_1-r_1}) \frac{1}{\sqrt{\xi}} \frac{1}{\sqrt{e_1^{m_1-r_1}}} \triangleq \sigma(e_1).
\]

(40)

In addition, based on Lemma 2.2, from (34) and the definition of \(\xi\), we have

\[
|e_2| = |\xi + e_2^*|^{\frac{1}{r_1}} \\
\leq |\xi|^{\frac{1}{r_1}} + |e_2^*| \leq |\xi|^{\frac{1}{r_1}} + 2|e_1|^{\frac{1}{r_1}} + 2|e_1|^{m_1}.
\]

(41)

By Lemma 2.4, we can also obtain that

\[
\int_{e_2}^{e_2} \left( \frac{1}{2} - e_2^* \right)^{1-r_1} ds \leq |\xi|^{1-r_1}|\xi - e_2^*| \\
= |\xi|^{1-r_1}(e_2^*|^{1-r_1} - (e_2^*)^{1-r_1}) \leq 2^{1-r_1}|\xi|.
\]

(42)

From (40), (41), (42) and Lemma 2.5, one obtains

\[
(2 - r_1)^{-\frac{d(-e_2^*)}{dt}} \int_{e_2^*}^{e_2} \left( \frac{1}{2} - e_2^* \right)^{1-r_1} ds \\
= (2 - r_1)^{-\frac{d(-e_2^*)}{dt}} \int_{e_2^*}^{e_2} \left( \frac{1}{2} - e_2^* \right)^{1-r_1} ds \\
\leq \sigma(e_1)(|\xi|^{1-r_1} + 2|e_1|^{1-r_1} + 2|e_1|^{m_1}) \frac{1}{\sqrt{\xi}} |\xi| \\
\leq \frac{1}{4}|e_1|^{1+r_1} + \frac{1}{2}|e_1|^{1+m_1} + (2^{1-r_1} \sigma(e_1) + \eta_2 \sigma^{1+r_1}(e_1))|\xi|^{1+r_1} + \eta_3 \sigma^{1+m_1}(e_1)|\xi|^{1+m_1}.
\]

(43)

Substituting (38) and (43) into (37) yields

\[
\dot{V}_2(e_1, e_2) \leq \frac{3}{4}|e_1|^{1+r_1} + \frac{3}{2}|e_1|^{1+m_1} \\
+ \eta_3 \sigma^{1+m_1}(e_1)|\xi|^{1+m_1} \\
+ (2^{1-r_1} \sigma(e_1) + \eta_2 \sigma^{1+r_1}(e_1))|\xi|^{1+r_1} \\
+ \frac{1}{4}|e_1|^{1+r_1} + \eta_1 |\xi|^{1+r_1} \\
+ \xi^{2-r_1} \dot{e}_2.
\]

(44)
Based on Assumption 1, we can obtain that $1 \leq \frac{\sigma}{2}$. Combining (31b) with (44) yields

$$V_2(e_1, e_2) \leq -\frac{3}{2} |e_1|^{1+r_1} - \frac{3}{2} |e_1|^{1+m_1} + \eta_3 \sigma |e_1|^{1+m_1} + \frac{\eta_3 \sigma^{1+m_1}}{\sigma (e_1 + \eta_2 \sigma^{1+r_1} (e_1))} + \sigma_1 (e_1, e_2) + \sigma_1 (e_1) + 1 |\xi|^{1+r_1}$$

$$\leq -\frac{3}{2} |e_1|^{1+r_1} - \frac{3}{2} |e_1|^{1+m_1} + \frac{\eta_3 \sigma (e_1) + \sigma_1 (e_1) + 1 |\xi|^{1+r_1}}{\sigma (e_1 + \eta_2 \sigma^{1+r_1} (e_1))} + \sigma_1 (e_1, e_2) + \sigma_1 (e_1) + 1 |\xi|^{1+r_1}$$

$$\leq -\frac{3}{2} |e_1|^{1+r_1} - \frac{3}{2} |e_1|^{1+m_1} + \frac{\eta_3 \sigma (e_1) + \sigma_1 (e_1) + 1 |\xi|^{1+r_1}}{\sigma (e_1 + \eta_2 \sigma^{1+r_1} (e_1))} + \sigma_4 (e_1, e_2) + \sigma_4 (e_1) + 1 |\xi|^{1+r_1}$$

(45)

By Lemma 2.5, we obtain

$$\|\xi\|^{2-r_1} |\omega_1^2| |e_1| \leq \frac{2}{\alpha \omega_1^2} e_1^{2-r_1} \|\xi\|^{2-r_1} |e_1|^{2r_1-1} \leq \sigma_1 (e_1) \|\xi\|^{1+r_1} + \frac{1}{3} |e_1|^{1+r_1}. \tag{46}$$

Note that $\xi = e_1^{\frac{1}{2}} - \frac{e_1}{r_1}$, $e_2^* = -2e_1^{\frac{1}{2}} - 2e_1^{m_1}$, using Lemmas 2.2 and 2.5, we have

$$\|\omega_1 \|^{\|\xi\|^{2-r_1} |e_2| \leq \frac{\alpha \omega_1 \|\xi\|^{2-r_1} |\xi| + e_2 |e_1|}{r_1} \leq \frac{\alpha \omega_1 \|\xi\|^{2-r_1} |\xi| + e_2 |e_1|}{r_1} \leq \frac{\alpha \omega_1 \|\xi\|^{2-r_1} |\xi| + e_2 |e_1|}{r_1}$$

(47)

According to (46) and (47), equation (45) can be written as

$$V_2(e_1, e_2) \leq -|e_1|^{1+r_1} - |e_1|^{1+m_1} - |\xi|^{1+r_1} \tag{48}$$

Meanwhile, by using Lemma 2.4, from the Lyapunov function $V_2(e_1, e_2)$ defined in (36), it has

$$V_2(e_1, e_2) \leq \frac{1}{2} e_1^2 + 2 |e_1|^{2-r_1} |e_2|^2 \leq \lambda (e_1^2 + \xi^2), \tag{49}$$

where $\lambda = \max \{\frac{1}{2}, 2^{1-r_1}\}$. By Lemmas 2.2–2.3, it can be concluded that

$$(e_1^2 + \xi^2)^{1+r_1} \leq |e_1|^{1+r_1} + |\xi|^{1+r_1}, \tag{50}$$

and

$$(e_1^2 + \xi^2)^{1+m_1} \leq \frac{e_1^{1+m_1}}{1-r_1} + |e_1|^{1+m_1} + |\xi|^{1+m_1}. \tag{51}$$

In view of inequalities (48), (49), (50) and (51), it yields

$$V_2(e_1, e_2) \leq -\lambda^{\frac{1}{2}} \frac{1}{2} V_2^{\frac{1}{2}}(e_1, e_2) \tag{52}$$

On the one hand, according to the inequality equation (52) and by using Lemma 2.1, it can easily be obtained that there exists a fixed time constant

$$T_0 = \frac{1}{1-r_1} \frac{2}{\lambda} + \frac{2}{\lambda} \frac{1}{1-r_1} \frac{2}{m_1-1} \frac{2}{m_1-1} < \infty$$

satisfying $V_2(e_1, e_2) = 0$ for any $t \geq T_0$. In addition, based on equation (36), $V_2(e_1, e_2) = 0$ indicates that $e_1 = 0$ and $e_2 = 0$. Therefore, it can be concluded that the subsystem (22a)–(22b) with the controller (28) is globally fixed-time stable.

Remark 2 The boundedness of states $e_j (j = 1, 2)$ in the time interval $[0, T_0]$ does not prove in Theorem 3.2, and it is mainly because the closed-loop system consists of complex nonlinear items and is difficult to analyze. In order to illustrate the effectiveness of the theoretical results, with the controllers (24) and (28), a large number of simulations have been done for systems (18) and (19) in the following simulation section, and we do not observe any divergence phenomenon. In many situations, we can assume the control law is a bounded
value in the time interval \( [0, T_2] \) to guarantee the boundedness of system states.

Based upon the above analysis, we can directly obtain the following theorem.

**Theorem 3.3** If Assumptions 1-2 are satisfied and the control laws \( \omega \) and \( v \) are designed as (24) and (28), then the mobile robot system (18) can globally track the desired reference trajectory (19) in a fixed time, where the control parameters used in (24) and (28) are chosen as those in Theorems 3.1 and 3.2.

**Proof** First of all, Theorems 3.1 and 3.2 show that control laws (24) and (28) can make the states \( \theta_e \) and \( e_j (j = 1, 2) \) in system (30a) converge to zero in a fixed time. In addition, the state transformation equations (20) and (29) show that \( \theta_e = 0 \) and \( e_j (j = 1, 2) = 0 \) imply that \( x_r = x_m \), \( y_r = y_m \), \( \theta_r = \theta_m \). Therefore, system (18) can globally track the desired trajectory (19) in a fixed time.

**Remark 3** For control laws (24) and (28), the control parameters \( k_1 > 0, k_2 > 0, 0 < \beta_1 < 1, \beta_2 > 1, \tau_1 = 1 + \tau_1, m_1 = 1 + \tau_2, -0.5 < \tau_1 < 0, \tau_2 > 0 \) directly influence the closed-loop performance. Usually, when the parameters \( \beta_1, \beta_2 \) are chosen close to 1 and the parameters \( \tau_1, \tau_2 \) are chosen close to 0, the convergence rate of states will be faster.

**Remark 4** The authors of paper [23] have discussed finite-time tracking control problem for systems (18) and (19), and finite-time tracking controllers have been given as follows

\[
\omega = \omega_r + \mathcal{K}_1 \sigma g^\beta \theta_e, \quad (53a)
\]

\[
v = v_r - \frac{1}{\omega_r \alpha} \big( \mathcal{K}_2 + \rho_1(y_e) \
+ \rho_2(x_e, y_e) \big) (\mathcal{K}_2^p y_e - \omega_r^p x_e^p)^{\frac{1}{p} - 1}, \quad (53b)
\]

where \( 0 < \beta < 1, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3 \) denote positive constant control gains, \( 1 < p < 2 \) is a ratio of two odd integers, \( \rho_1(y_e) = \frac{2^{p-1}}{1+p} \omega_r^{\frac{2(2p+1)}{2p-1}} y_e^\frac{2(p+1)}{2p-1} \), \( \rho_2(x_e, y_e) = \frac{2^{p-1}}{1+p} \omega_r^{\frac{2p-1}{2p-1}} y_e^{\frac{2p-1}{2p-1}} \). Comparing with the finite-time controllers (53), the main advantage of the proposed fixed-time control result lies in the convergence time that can be predetermined but independent with any initial conditions, which will be illustrated in the next section.

4 Simulation results

For system (18), simulation examples will be presented in this section, and two cases are considered. In the first case, simulation results will be given to show the effectiveness of the proposed fixed-time control laws (24) and (28). In the second case, under different initial conditions, we will compare the convergent performance between fixed-time control laws (24) and (28) and finite-time controllers (53).

**Case 1** In this case, suppose the desired value of system (19) as \( v_r = 1.5 - \frac{15t}{1+t^{10}} \text{m/s}, \omega_r = 1 + \frac{2t}{1+t^{10}} \text{rad/s}. \) Let the initial value \( \{x_r(0), y_r(0), \theta_r(0)\} = (-4, 1, 0), \{x(0), y(0), \theta(0)\} = (2, 1.5, 0.5). \) Fixed-time parameters are taken as \( k_1 = 1, k_2 = 2, \tau_1 = -\frac{1}{2}, \tau_2 = \frac{1}{3^5}, \beta_1 = 0.65, \beta_2 = 1.5, \alpha = 10 \) and \( \alpha = 8. \) Simulation results are shown in Figs. 3, 4, 5, 6.

Figure 3 shows state curves for \( x_r, y_r, \theta_r \) and \( x, y, \theta \).

Figure 4 presents the tracking errors \( x_e, y_e \) and \( \theta_e \). Curves of desired and tracking trajectories are presented in Fig. 5. Figure 6 gives the control outputs \( v \) and \( \omega \) acted on the robot, respectively. Figures 3, 4, 5 demonstrate that control laws (24) and (28) can make the mobile robot achieve the desired value in a fixed time.

**Case 2** In this case, based on remark 4, under different initial conditions for fixed-time control laws (24) and (28) and finite-time control law (53), the convergent time performance of these two kinds of control laws is compared. Figure 7 shows that the convergent time is independent of initial state for fixed-time con-
Fixed-time trajectory tracking control for nonholonomic mobile robot

Fig. 4 Response tracking errors curves for $x_e$, $y_e$ and $\theta_e$

Fig. 5 Response desired trajectory and tracking curves

Fig. 6 Control outputs $v$ and $\omega$ acted on the robot

Fig. 7 The convergence time for the different initial conditions

ccontrol laws, where the 2 norm of initial value is defined as $\delta(0) = \sqrt{(x_r(0) - x(0))^2 + (y_r(0) - y(0))^2}$.

5 Conclusion

For the nonholonomic wheeled mobile robot with uncalibrated camera parameters, fixed-time tracking control problem has been considered. Firstly, according to the pinhole camera model, the camera-objective visual model has been established, and tracking error system between the robot and desired trajectory has been given. Then, by using the fixed-time control theory and Lyapunov stability analysis, fixed-time control laws for the robot have been proposed, which can make the wheeled mobile robot track the desired trajectory in a fixed time. Finally, the simulation results have been given to verify the validity of the conclusions. In our future research, we will try our best to solve fixed-time control problem with uncertainties and disturbances for the nonholonomic mobile robot system, which are more complicated and general.

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Data availability The datasets generated during and/or analyzed during the current study are available from the corresponding author on reasonable request.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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