Anisotropic Kappa Distributions. I. Formulation Based on Particle Correlations

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Abstract

We develop the theoretical basis for the connection of the variety of anisotropic distributions with the statistical correlations among particles’ velocity components. By examining the most common anisotropic distribution function, we derive the correlation coefficient among particle energies, show how this correlation is connected to the effective dimensionality of the velocity distribution, and derive the connection between anisotropy and adiabatic polytropic index. Having established the importance of the correlation among particles in the formulation of anisotropic kappa distributions, we generalize these distributions within the framework of nonextensive statistical mechanics and based on the types of homogeneous or heterogeneous correlations among the particles’ velocity components. The formulation of the developed generalized distributions mediates the main two types of anisotropic distributions that consider either (a) equal correlations, or (b) zero correlations, among different velocity components. Finally, the developed anisotropic kappa distributions are expressed in terms of the energy and pitch angle in arbitrary reference frames.

Unified Astronomy Thesaurus concepts: Space plasmas (1544); Polytropes (1281); Spectral energy distribution (2129)

1. Introduction

The temperature is a well-defined quantity within the theoretical framework of kappa distributions and nonextensive statistical mechanics (Livadiotis & McComas 2009, 2010; Tsallis 2009). It follows the same physical definitions as in classical statistical mechanics and thermodynamics: namely, both the kinetic (Maxwell 1866) and thermodynamic (Clausius 1862; Abe 2001; Livadiotis 2018a) definitions of temperature are equivalent, determining exactly the same physical quantity. The kinetic definition is given by the mean kinetic energy per kinetic degrees of freedom (dof), here noted by \(d\), that is, the mean kinetic energy corresponding to a single velocity component, \(\frac{k_B}{2} T\). While this is trivial for the isotropic distributions where the thermal energy is the same for any dof (equidistribution theorem), in the case of anisotropic distributions the temperature is derived by the mean kinetic energy averaged over all dof:

\[
\frac{1}{2} k_B T = \frac{\langle u^2 \rangle}{d} = \frac{1}{d} \sum_{i=1}^{d} \left( \frac{1}{2} m \langle u_i^2 \rangle \right) = \frac{1}{d} \sum_{i=1}^{d} \left( \frac{1}{2} k_B T_i \right), \quad \text{or} \quad (1a)
\]

\[
T = \frac{1}{d} \sum_{i=1}^{d} T_i \equiv \langle T_i \rangle_d, \quad (1b)
\]

where the kinetic energy \( \varepsilon = \frac{1}{2} m u^2 \), and its mean \( \langle \varepsilon \rangle = \frac{1}{2} m \langle u^2 \rangle \), is written setting zero bulk (or mean) velocity, \( u_0 \equiv \langle u \rangle = 0 \). The fractions of thermal components define an array of anisotropy components, \( \alpha_i \equiv T_i/T, \quad i = 1, \ldots, d \), but we are interested in the typical case of 3D distribution, where one velocity component (denoted as parallel to a reference direction, e.g., that of the ambient magnetic field, \( u_l \)) is different than the other velocity components (forming a perpendicular manifold, \( u_\perp \)), with a scalar anisotropy defined by \( \alpha \equiv T_\perp/T_\parallel \).

The kappa index, the parameter that labels and governs the kappa distributions, together with the temperature, constitute two intensive physical quantities, that is, they are independent of the size of the system and characterize its thermodynamics. Similar to the temperature, the kappa index has both thermodynamic and kinetic definitions; the former is connected with the partition of entropy (Livadiotis 2018a, 2018b) and the latter is connected with the correlation of particle energies (Abe 1999; Livadiotis & McComas 2011; Livadiotis 2015c, 2017, Ch. 5.4). Therefore, the mean of the particle kinetic energies determines the temperature, while their correlation determines the kappa index.

The correlation of particle energies has been determined for the isotropic kappa distributions. The calculation of correlation involves speed moments higher than the mean kinetic energy. It is known, though, that statistical moments of velocity with order higher than the second moment, which corresponds to the mean energy and the definition of temperature, diverge for sufficiently small values of the kappa index; remarkably, the calculation of correlation converges for all the values of the kappa index even though it involves moments of fourth order (see: Appendix B in Livadiotis & McComas 2011). Therefore, both the kinetic definitions of temperature (mean value of particle kinetic energy) and kappa (correlation of particle kinetic energy) are well-defined for all kappa indices.

The mean and the correlation of the particle kinetic energies are given as follows:

\[
\langle \varepsilon \rangle = \frac{1}{2} d \cdot k_B T; \quad \rho = \frac{1}{2} d \cdot \frac{1}{k}, \quad (2a)
\]

We observe that they are both proportional to half the dof, \( \frac{1}{2} d \), and to the corresponding defined thermodynamic quantity, i.e.,
the thermal energy and (inverse) kappa index, respectively, i.e.,

\[
\begin{align*}
\text{(1/2) Thermal Energy} & \quad (k_B T) = \text{Mean kinetic Energy per dof} \\
\text{(1/2) Inverse Kappa (1/κ)} & \quad (\rho) = \text{Correlation Coefficient per dof.}
\end{align*}
\]  

(2b)

The correlation between particles’ kinetic energies is expected to have an inverse relationship with the kappa index, e.g., to be proportional to the inverse of kappa. Indeed, for the Maxwell–Boltzmann case where the correlation is zero, the kappa index is infinity. On the other hand, when a physical mechanism generates a kappa distribution of particle energies/velocities, it basically acts on the connection of particles through long-range interactions that add statistical correlations among particles; for instance, the coupling between plasma constituents and the embedded magnetic field occurring on various temporal and spatial scales “binds” particles together (Livadiotis et al. 2018). This inverse proportionality between correlation coefficient and kappa index is shown in Figure 1.

The mean kinetic energy per dof determines the temperature, while the correlation of kinetic energy per dof determines the kappa index. However, while the mean kinetic energy per dof determines the temperature either for isotropic or anisotropic distributions, it is unknown whether the correlation per dof can consistently define the kappa index in the case of anisotropic distributions. The reason is that the correlation of kinetic energy may be expressed by a complex function of dimensionality, kappa, and anisotropy, instead of only the kappa index. In such a case, the correlation per dof would have led to an ill definition of the kappa index. Then, how would the kappa index have been (kinetically) defined in the case of anisotropic kappa distributions?

As we will see in this paper, the dependence on anisotropy falls into the notion of dimensionality. Let the dependence of the correlation coefficient on the dimensionality, written as \( \rho = f_d(d) \). The involvement of the anisotropy \( \alpha \) in the formulation of \( \rho = g(d, \alpha) \), can be understood as a deeper involvement of the anisotropy with the dimensionality \( d \), leading to an effective dimensionality \( d_{\text{eff}} = f_d(d, \alpha) \) that substitutes the dependence of correlation, \( \rho = f_d(d) \rightarrow \rho = f_d(d_{\text{eff}}) \), so that \( \rho = g(f_d, f_{\text{eff}})(d, \alpha) \), i.e., \( g = f_d \circ f_d \).

The impact of anisotropy on dimensionality can be shown as follows. While the “overall” dimensionality (or dof) is given by \( d = 3 \) for any anisotropy, the limiting cases, where the parallel or perpendicular directions are neglected, should characterize a degeneration to 2D or 1D velocity distributions, respectively. Then, there is an effective dimensionality, \( d_{\text{eff}} \), which can be defined accordingly to satisfy:

\[
\begin{align*}
\alpha = 0 & \quad \Leftrightarrow d_{\text{eff}} = 1, \\
\alpha = 1 & \quad \Leftrightarrow d_{\text{eff}} = 3, \\
\alpha = \infty & \quad \Leftrightarrow d_{\text{eff}} = 2.
\end{align*}
\]  

(3)

Yet, how would the correlation be formulated in the case of the anisotropic kappa distributions? How does the expression of the correlation coefficient \( \rho \) for the anisotropic distributions define an effective dimensionality \( d_{\text{eff}} \), which would match the dimensionalities of the limiting cases of anisotropy in Equation (3)?

The purpose of this analysis is to (i) determine the correlation coefficient and the involved effective dimensionality of anisotropic kappa distributions characterized with homogeneous or heterogeneous correlations among their velocity components; (ii) indicate the connection of the adiabatic polytropic index with temperature anisotropy; (iii) characterize and study the types of homogeneous/heterogeneous correlations among the particles’ velocity components; (iv) formulate the correlation relationship that characterizes the partition of a 2D joint kappa distribution into the two marginal 1D kappa distributions, as emerges from nonextensive statistical mechanics; (v) generalize the formulae of anisotropic kappa distributions, based on the various types of homogeneous/heterogeneous correlations; (vi) describe and examine the anisotropic kappa distributions in (a) the comoving reference frame with respect to the velocity components, (b) arbitrary S/C frame with respect to to the triplet of energy, pitch angle, and azimuth; and (c) the more complicated form of azimuth-independent distributions with respect to energy and pitch angle.

We point out that the paper does not examine the possible mechanisms that generate the anisotropies in the velocity distribution (in the solar wind or other space plasmas, e.g., see Ao et al. 2003), but certainly it provides the impact and consequences on the particle velocity/energy distributions and the associated statistics. Note: all symbols used in this paper are defined in Table 1.

The paper is organized as follows. Section 2 briefly presents the formulation of kappa distributions and the involved parameters. In particular, we present the formulation of 3D anisotropic kappa distributions, the concept of the invariant kappa index, and the multi-dimensional anisotropic kappa distributions, which is critical for deriving the correlation coefficient. In Section 3, we calculate the correlation coefficient, that is, the normalized covariance among any two particle energies; then, we derive the effective dimensionality, which leads to the connection between anisotropy and adiabatic polytropic index. Having established the importance of correlation among particles in the formulation of anisotropic kappa distributions, in Section 4 we reverse the concept, in order to determine the variety of anisotropic formulations based on their correlations; namely, we develop and study the formulae of anisotropic kappa distributions by classifying the types of correlations among the particles’ velocity components. These can be distinguished in homogeneous and heterogeneous correlations, where the latter type can be further separated into a variety of different correlations that may exist among different velocity components. Section 5 shows the generalization of anisotropic kappa distributions based on the heterogeneous correlations among the particle velocity components. The generalization is developed using the partition of joint probability distribution to its marginal distributions that helps to describe and generalize the possible types of correlations. The developed generalization of the distributions actually mediates the main two types of anisotropic kappa distributions, where the first type considers equal correlations.
among particles’ velocity components and the second type considers zero correlation among different velocity components. In Section 6, we derive the formalism of anisotropic kappa distributions in arbitrary reference frames; in particular, we transform the developed anisotropic kappa distributions to spherical coordinates with respect to the energy, pitch angle, and azimuth, respectively; the corresponding thermal speeds are explained below.

Introducing the temperature anisotropy by

\[
\alpha = \frac{T_{\perp}}{T_{\parallel}}
\]

the temperature is written as

\[
T = \frac{3}{5} (T_{\parallel} + 2T_{\perp}),
\]

and the temperature-like components are expressed in terms of the actual temperature and anisotropy

\[
T_{\parallel} = \frac{3\alpha}{1 + 2\alpha} \cdot T, \quad T_{\perp} = \frac{3\alpha}{1 + 2\alpha} \cdot T.
\]

In terms of thermal speeds, these are written as

\[
\theta^2 = \frac{\frac{3}{5} (\theta_{\parallel}^2 + 2\theta_{\perp}^2)}{\theta_{\parallel}^2}, \quad \Theta_{\parallel}^2 = \frac{\frac{3}{5} (\theta_{\parallel}^2 + 2\theta_{\perp}^2)}{\theta_{\parallel}^2}, \quad \theta_{\perp}^2 = \frac{\frac{3}{5} (\theta_{\parallel}^2 + 2\theta_{\perp}^2)}{\theta_{\parallel}^2},
\]

where \(\theta = \sqrt{2k_{\parallel}T/m}\) denotes the thermal speed of a particle of mass \(m\), that is, the temperature \(T\) expressed in speed dimensions; \(\theta^2/2\) provides the second statistical moment of the velocities in the comoving reference frame. Therefore, the distribution (4a) can be written in terms of the temperature and anisotropy, i.e.,

\[
P(u_{\parallel}, u_{\perp}; \theta_{\parallel}, \theta_{\perp}, \kappa) = \pi^{-\frac{1}{2}} \left( \kappa - \frac{3}{2} \right)^{-\frac{3}{2}} \Gamma(\kappa + 1) \Gamma(\kappa - \frac{1}{2})
\]

\[
\times \left( 1 + 2\alpha \right) \left( \frac{3\alpha}{\theta_{\parallel}^2} \right)^{-3} \left( 1 + \frac{\theta_{\perp}^2}{\theta_{\parallel}^2} \right)^{-\frac{3}{2}} \left( 1 + \frac{1 + 2\alpha}{3\theta_{\parallel}^2} \right)^{-\frac{3}{2}} \left( u_{\parallel}^2 + \frac{1}{\alpha} u_{\perp}^2 \right)^{\kappa - 1}.
\]

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### Table 1

| Symbol | Description |
|--------|-------------|
| \(n\) | Number of correlated particles |
| \(\Gamma\) | Gamma function |
| \(\kappa\) | Kappa index; kappa index for different velocity components |
| \(\rho\) | Correlation coefficient (Pearson) |
| \(\theta_{\parallel}, \theta_{\perp}\) | Thermal speed; thermal speed component; parallel and perpendicular components |
| \(A, \lambda\) | Auxiliary arguments used in the azimuth-independent distributions |
| \(\beta\) | Boltzmann constant |
| \(\xi\) | Polytropic index |
| \(\alpha\) | Anisotropy |
| \(\kappa, \kappa_0, \kappa_{\infty}\) | Kappa index; kappa index for d-D distribution; invariant kappa index (d = 0) |
| \(\kappa_{\infty}\) | Kappa index, corresponding to the correlation among different velocity components |
| \(d_{\text{eff}}\) | Effective dimensionality |
| \(d\) | Dimensionality or degrees of freedom (dof) |
| \(d_i\) | Dimensionality of the velocity vector of the \(i\)th particle |
| \(N\) | Number of correlated particles (described by a kappa distribution) |
| \(f = N \cdot d\) | Total number of correlated kinetic dof |
| \(\Gamma\) | Gamma function |
| \(\gamma\) | Polytropic index |
| \(\kappa\) | Kappa index; kappa index for d-D distribution; invariant kappa index (d = 0) |
| \(\sigma_{\gamma, \gamma}^2, \sigma_{\gamma, u}^2\) | Covariance of the energies and velocity squares of any two particles |
| \(\sigma_{\gamma, \gamma}^2, \sigma_{u, u}^2\) | Variance of the energies and velocity squares of a particle |
| \(S\) | Entropy |
| \(B, B\) | Magnetic field vector, magnitude |

### 2. Formulation of Kappa Distributions—A Brief Review

#### 2.1. Three-dimensional Anisotropic Kappa Distributions

There is a variety of 3D anisotropic kappa distributions used for describing particle populations in space plasmas (e.g., see the primary paper of Summers & Thorne 1991, as well as the reviews of Pierrard & Lazar 2010; Livadiotis 2015a; see also
When the anisotropy is $\alpha = 1$, then Equation (10) recovers the standard isotropic distribution,

$$P(u; \theta, \alpha = 1, \kappa) = \pi^{-\frac{3}{2}} \left( \kappa - \frac{3}{2} \right)^{-\frac{3}{2}} \Gamma(\kappa + 1) \cdot \Theta^{-3} \cdot \left( 1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{u^2}{\Theta^2} \right)^{-\kappa-1}.$$

(11)

On the other hand, the extreme anisotropy values of $\alpha = 0$ and $\alpha \to \infty$ lead to the absolute zero of the respective temperature-like component, where the distribution in Equation (10) is characterized by the known freezing behavior (delta function; Livadiotis & McComas 2010), becoming narrow as a cigar (restricted along the parallel direction) or flat as a pie (restricted on the perpendicular plane), and thus, reducing the effective dimensionality (or effective degrees of freedom) to $d_{\text{eff}} = 1$ and $d_{\text{eff}} = 2$, respectively (see Equation (3)).

Figure 2 plots the anisotropic kappa distribution, as shown in Equation (10), for $\theta = 1$, and for various values of anisotropy $\alpha$ and kappa index $\kappa$. We observe that the distribution is concentrated near $u_\perp = 0$ as $\alpha$ decreases tending to $\alpha = 0$ (or $T_\perp = 0$), and near $u_\parallel = 0$ as $\alpha$ increases tending to $\alpha \to \infty$ (or $T_\parallel = 0$). Also, similar for the isotropic case, the distribution is denser around the lines $u_\perp = 0$ and $u_\parallel = 0$ as the kappa index decreases reaching the limit of $\kappa \to \frac{3}{2}$ (or $\kappa_0 \to 0$).

2.2. Invariant Kappa Index

The kappa index together with the temperature constitute two intensive parameters characterizing the thermodynamics of the system (Abe 2001; Livadiotis 2018a). The kappa index depends on the dimensionality, that is, the number of the correlated dof $d$, i.e., $\kappa = \kappa(d)$, also noted as $\kappa_d$. This expression is quite simple and arose from the theoretical
observation that the difference $\kappa_d - \frac{1}{2}d$ is an invariant quantity independent of $d$, hence, $\kappa_d = \text{const.} + \frac{1}{2}d$. The involved constant, noted by $\kappa_0$, indicates an invariant expression of the kappa index, so that the kappa index remains invariant under variations of the dimensionality or the number of the correlated dof. Using the invariant kappa index, $\kappa_0$, the kappa distributions may be written in expressions for any dimensionality (or dof) and number of particles (Livadiotis & McComas 2011; Livadiotis 2015b).

Hereafter, we may use either the invariant kappa index $\kappa_0$, or the standard kappa index for the 3D case $\kappa_3$, accordingly, recalling that $\kappa_3 = \kappa_0 + \frac{3}{2}$. Therefore, the $d$-D kappa distribution is parameterized using either $\kappa_0$ or $\kappa_3$, by substituting $\kappa_d = \kappa_0 + \frac{1}{2}d$ or $\kappa_d = \kappa_3 = \frac{1}{2}(d - 3)$, respectively. For example, the kappa index of 1D distributions is $\kappa_1$ and it can be replaced either by $\kappa_0$ or $\kappa_3$, according to $\kappa_0 = \kappa_3 - \frac{1}{2}$ or $\kappa_3 = \kappa_0 + 1$. The paper mostly focuses on the typical 3D distributions, thus we ignore the 3D subscript, writing simply $\kappa$ instead of $\kappa_d$.

The concept of an invariant kappa index is necessary when dealing with (one-particle or many-particle) multidimensional kappa distributions.

2.3. Multi-dimensional Anisotropic Kappa Distributions

The widely known 3D kappa distribution describes the one-particle velocities,

$$P(u; \theta, \kappa_0) = (\pi \kappa_0)^{-\frac{3}{2}} \cdot \frac{\Gamma(\kappa_0 + \frac{5}{2})}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-3} \times \left[1 + \frac{1}{\kappa_0} \cdot \frac{u^2}{\theta^2}\right]^{-\kappa_0^{-\frac{1}{2}d}}.$$  \hspace{1cm} (12)

while the $d$-D kappa distribution is written as

$$P(u; \theta, \kappa_0; d) = (\pi \kappa_0)^{-\frac{d}{2}} \cdot \frac{\Gamma(\kappa_0 + 1 + \frac{1}{2}d)}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-d} \times \left[1 + \frac{1}{\kappa_0} \cdot \frac{u^2}{\theta^2}\right]^{-\kappa_0^{-1-\frac{1}{2}d}}.$$  \hspace{1cm} (13)

In order to expose the multiparticle distribution, we have to use the dimensionality-invariance version of kappa index, $\kappa_0$. For instance the two-particle distribution is (e.g., see: Swaczyna et al. 2019),

$$P(u_1, u_2; \theta, \kappa_0; d) = (\pi \kappa_0)^{-d} \cdot \frac{\Gamma(\kappa_0 + 1 + d)}{\Gamma(\kappa_0 + 1)} \cdot \theta^{-2d} \times \left[1 + \frac{1}{\kappa_0} \cdot \frac{u_1^2 + u_2^2}{\theta^2}\right]^{-\kappa_0^{-1-d}}.$$  \hspace{1cm} (14)

Let the velocity vector of the $i$th particle be $u_i^2 = u_{i1}^2 + u_{i2}^2 + u_{i3}^2$. In a $d$-dimensional velocity space, this is $u_i^2 = u_{i1}^2 + u_{i2}^2 + \cdots + u_{id}^2$; therefore, the general case of an $N$-particle kappa distribution is

$$P(u_1, u_2, \cdots, u_N; \theta, \kappa_0; d, N) = (\pi \kappa_0)^{-\frac{d}{2}N} \times \frac{\Gamma(\kappa_0 + 1 + \frac{1}{2}Nd)}{\Gamma(\kappa_0 + 1)} \times \theta^{-Nd} \cdot \left[1 + \frac{1}{\kappa_0} \cdot \frac{u_1^2 + u_2^2 + \cdots + u_N^2}{\theta^2}\right]^{-\kappa_0^{-1-\frac{1}{2}Nd}}.$$  \hspace{1cm} (15)

Furthermore, we follow the convention of undistinguished dof, namely, all systems’ correlated dof are assembled together independently of the associated particles; then, we write $\{u_i\}_{i=1}^f = \{u_{i_1}, \cdots, u_{i_f}\}, f = Nd$, meaning the notation:

$$u_1 = u_{11}, \cdots, u_d = u_{1d},$$

$$u_{d+1} = u_{21}, \cdots, u_{2d} = u_{2d},$$

$$u_{2d+1} = u_{31}, \cdots, u_{3d} = u_{3d},$$

$$\cdots$$

$$u_{(N-1)d+1} = u_{N1}, \cdots, u_{Nd} = u_{Nd}.$$  \hspace{1cm} (16)

Hence, the $N$-particle, $d$-D per particle, kappa distribution (15) can be simply expressed as an $f$-dimensional distribution with $f = Nd$, i.e.,

$$P(u_1, u_2, \cdots, u_f; \theta, \kappa_0; f) = (\pi \kappa_0)^{-\frac{d}{2}f} \cdot \frac{\Gamma(\kappa_0 + 1 + \frac{1}{2}f)}{\Gamma(\kappa_0 + 1)} \times \theta^{-f} \cdot \left[1 + \frac{1}{\kappa_0} \cdot \frac{u_1^2 + u_2^2 + \cdots + u_f^2}{\theta^2}\right]^{-\kappa_0^{-1-\frac{1}{2}f}}.$$  \hspace{1cm} (17)

The concept of multidimensional kappa distributions is necessary for determining the covariance and correlation of kinetic energies among different velocity components (see next section). Using these distributions is essential for understanding several important properties of kappa distributions, which is impossible to be done using only the one-particle distributions (e.g., Gravanis et al. 2020).

3. Derivation of Correlation Coefficient

The correlation coefficient of particle energy is given by the covariance of any two particle energies normalized to their energy variance; covariance and variance are, respectively,

$$\sigma_{e_1 e_2}^{2} = \langle e_1 e_2 \rangle - \langle e \rangle^2 = \frac{1}{2}m(\langle u_1^2 u_2^2 \rangle - \langle u^2 \rangle^2) = \frac{1}{2}m \sigma_{u_1 u_2}^{2},$$  \hspace{1cm} (18a)

and

$$\sigma_{e_2}^{2} = \langle e^2 \rangle - \langle e \rangle^2 = \frac{1}{2}m(\langle u^2 \rangle - \langle u^2 \rangle^2) = \frac{1}{2}m \sigma_{u_1 u_2}^{2},$$  \hspace{1cm} (18b)

leading to the Pearson’s correlation coefficient (Abe 1999; Livadiotis & McComas 2011; Livadiotis 2015c, 2017, Ch. 5.4):

$$\rho = \frac{\sigma_{e_1 e_2}^{2}}{\sigma_{e_2}^{2}} = \frac{\sigma_{u_1 u_2}^{2}}{\sigma_{u_1 u_2}^{2}},$$  \hspace{1cm} (19)

where all the particles have the same kinetic energy, $\langle e_1 \rangle = \langle e_2 \rangle = \langle e \rangle$.

(a) First, we derive the variance of the particles’ kinetic energies.
The statistical moment of the kinetic energy of order \( a \) is given by (e.g., Livadiotis 2017, Ch. 5.2; Livadiotis 2019a):

\[
\left\langle \left( \frac{1}{kT} \right)^a \right\rangle = \kappa_0^a \cdot \frac{\Gamma(a + \frac{5}{2})}{\Gamma(\frac{5}{2})} \cdot \frac{\Gamma(\kappa_0 + 1 - a)}{\Gamma(\kappa_0 + 1)},
\]

(20a)

or, equivalently, the moment of order \( 2a \) of the \( d \)-dimensional velocity with magnitude \( u = |\mathbf{u}| \) is

\[
\left\langle |\mathbf{u}|^{2a} \right\rangle = \theta^a \cdot \kappa_0^a \cdot \frac{\Gamma(a + \frac{4}{k})}{\Gamma(\frac{4}{k})} \cdot \frac{\Gamma(\kappa_0 + 1 - a)}{\Gamma(\kappa_0 + 1)}.
\]

(20b)

Then, for the parallel component (1D), we have

\[
\left\langle |\mathbf{u}_1|^2 \right\rangle = \frac{1}{2} \theta^2 \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}} \cdot \frac{1}{2} \left( \frac{1}{2} d_i \cdot \frac{1}{2} d_j \right) \quad \text{if } i \neq j,
\]

\[
\left\langle \mathbf{u}_1 \cdot \mathbf{u}_j^2 \right\rangle = \theta^2 \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}} \cdot \frac{1}{2} \left( \frac{1}{2} d_i \cdot \frac{1}{2} d_i + 1 \right) \quad \text{if } i = j,
\]

(21a)

where \( d_i \) is the dimensionality of the velocity vector of the \( i \)th particle. Then, for the case of the 1D \( u_1 \) and 2D \( u_2 \) velocity components, we calculate

\[
\left\langle |\mathbf{u}_1|^2 |\mathbf{u}_2|^2 \right\rangle = \frac{1}{2} \theta^2 \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}}.
\]

(22)

Then, we derive the energy variance \( \sigma^2_{u^4} = \right\langle u^4 \rangle - \left\langle u^2 \right\rangle^2 \), as follows:

\[
\left\langle u^4 \right\rangle = \left\langle (u_1^2 + u_2^2)^2 \right\rangle = \left\langle u_1^4 \right\rangle + \left\langle u_2^4 \right\rangle + 2 \left\langle u_1^2 u_2^2 \right\rangle,
\]

(23)

and substituting Equations (21), (23), we find

\[
\left\langle u^4 \right\rangle = \frac{3}{4} \theta^4 + 2 \theta^4 + \theta^2 \theta^2 \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}}.
\]

(24)

Also, we have

\[
\left\langle u^2 \right\rangle^2 = \frac{9}{4} \theta^4 \cdot \frac{9}{4} \left( \frac{1}{2} \theta^2 \right)^2 \left( \frac{1}{4} \theta^4 + \theta^4 + \theta^2 \theta^2 \right) \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}}.
\]

(25)

Hence, from Equations (25), (26), we find the variance

\[
\sigma^2_{u^4} = \left\langle u^4 \right\rangle - \left\langle u^2 \right\rangle^2 = \frac{3}{4} \theta^4 + 2 \theta^4 + \theta^2 \theta^2 \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}} - \left( \frac{1}{4} \theta^4 + \theta^4 + \theta^2 \theta^2 \right) \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}}.
\]

(26)

Next, we derive the covariance between two particle energies. We calculate:

\[
\left\langle u_i^2 u_j^2 \right\rangle = \left\langle (u_i^2 + u_j^2) u_i^2 + u_j^2 \right\rangle = \frac{1}{4} \theta^4 + \frac{1}{4} \theta^4 + \frac{1}{4} \theta^4 + \frac{1}{4} \theta^4 + \theta^2 \theta^2 \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}}.
\]

(27)

Therefore, Equation (28) gives the covariance:

\[
\sigma^2_{u_i u_j} = \left\langle u_i^2 u_j^2 \right\rangle - \left\langle u_i^2 \right\rangle \left\langle u_j^2 \right\rangle = \frac{1}{4} \theta^4 + \frac{1}{4} \theta^4 + \frac{1}{4} \theta^4 + \frac{1}{4} \theta^4 + \theta^2 \theta^2 \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}} - \left( \frac{1}{4} \theta^4 + \theta^4 + \theta^2 \theta^2 \right) \cdot \frac{\kappa - \frac{3}{2}}{\kappa - \frac{5}{2}}.
\]

(28)

(c) Finally, we derive the correlation. Substituting Equations (27), (31) in Equation (19), we obtain
Given the expressions in Equation (9), we end up with

\[
\rho = \frac{1}{2} \frac{(2\alpha + 1)^2}{2n^2 + 1} \left( \kappa - \frac{3}{2} \right) + \frac{1}{2} \frac{(2\alpha + 1)^2}{2n^2 + 1} \left( \alpha + \frac{1}{2} \right)^2
\]

\[
= \frac{(\alpha + 1)^2}{2n^2 + 1} \left( \kappa - \frac{3}{2} \right) + \frac{(\alpha + 1)^2}{2n^2 + 1} \left( \alpha + \frac{1}{2} \right)^2
\]

\[
= \frac{(\alpha + 1)^2}{2n^2 + 1} \left( \kappa - \frac{3}{2} \right) + \frac{(\alpha + 1)^2}{2n^2 + 1} \left( \alpha + \frac{1}{2} \right)^2
\]

(33)

As stated in Equation (2), the mean kinetic energy and correlation provide, respectively, the kinetic definitions of thermal energy, \( k_B T \), and the inverse kappa index, \( 1/\kappa \). The latter is rewritten as an expression of the dimensionality and the invariant kappa index \( \kappa_0 = \kappa - \frac{3}{2} \), i.e.,

\[
\rho = \frac{1}{\kappa_0 + \frac{1}{2}d} \quad \text{or} \quad \rho = \frac{1}{\left( \kappa_0 - \frac{3}{2} \right) + \frac{1}{2}d}.
\]

(34)

In the case of anisotropic distributions, the effective dimensionality interwoven with the correlations among particles should be different from the dimensionality of the embedded 3D velocity space; instead, it should follow the limiting cases stated in Equation (3).

The effective dimensionality is derived from comparing the correlation of particle kinetic energies for a given anisotropy \( \alpha \),

\[
\rho = \frac{1}{2} \frac{\sigma_1^2 + \sigma_2^2 + \alpha^2 \sigma_3^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \frac{1}{2} \frac{(2\alpha + 1)^2}{2n^2 + 1} \left( \kappa - \frac{3}{2} \right) + \frac{(2\alpha + 1)^2}{2n^2 + 1} \left( \alpha + \frac{1}{2} \right)^2,
\]

which is written in terms of the dimensionality as

\[
\rho = \frac{1}{\kappa_0 + \frac{1}{2}d} \quad \text{or} \quad \rho = \frac{1}{\left( \kappa_0 - \frac{3}{2} \right) + \frac{1}{2}d},
\]

(35)

where the effective dimensionality (or effective dof) is determined by

\[
\frac{1}{2} \frac{\sigma_1^2 + \sigma_2^2 + \alpha^2 \sigma_3^2}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2} = \frac{1}{2} \frac{(2\alpha + 1)^2}{2n^2 + 1}, \quad \text{or}
\]

\[
d_{\text{eff}}(\alpha) = \frac{(2\alpha + 1)^2}{2\alpha^2 + 1}.
\]

(36)

Next, we connect the expression in Equation (38) with the polytropic index that describes adiabatic thermodynamic processes. The adiabatic polytropic index is given in terms of the effective kinetic degrees of freedom (Sckopke et al. 1981; Newbury et al. 1997; Kartalev et al. 2006; Nicolaou et al. 2014a, 2014b, 2015; Livadiotis 2015c, 2016; Pang et al. 2015a, 2015b; Dialynas et al. 2018; Nicolaou & Livadiotis 2019) as

\[
\gamma = 1 + 2d_{\text{eff}}^{-1},
\]

(39a)

Figure 3 plots the effective dimensionality \( d_{\text{eff}} \) (red) and the respective adiabatic polytropic index \( \gamma \) (blue) as a function of the anisotropy \( \alpha \), emphasizing the cases of \( \alpha = 0, 1, \infty \), corresponding to \( d_{\text{eff}} = 1, 3, 2 \), and \( \gamma = 3, 5/3, 2 \), where the velocity distribution takes the form of a cigar, sphere, and pie.

that is,

\[
\gamma = 1 + \frac{\alpha^2 + \frac{1}{2}}{(\alpha + \frac{1}{2})^2}.
\]

(39b)

4. Anisotropic Kappa Distributions with Heterogeneously Correlated Variables

4.1. General

Having established the importance of correlations among particles in the formulation of anisotropic kappa distributions, we now reverse the concept in order to determine the variety of anisotropic expressions, which can be formulated based on the variety of particle correlations. In particular, here we develop and examine the various types of anisotropic kappa distributions, similar to the standard formula of Equation 4(a), which they can be derived from considering heterogeneous correlations among
particles and their kinetic dof; first, we recognize the various types of homogeneous correlations (Section 4.2), and then we proceed to the types of heterogeneous correlations (Section 4.3).

4.2. Formulation of Anisotropic Kappa Distributions Based on Homogeneous Correlations

The anisotropic kappa distribution shown in Equation 4(a) is a one-particle distribution that refers to systems with homogeneous correlation among the particle velocity components. The respective N-particle kappa distribution is given by

\[ P(\{u_i\}_i=1^N; \theta_{\parallel}, \theta_{\perp}, \kappa_0; N) = (\pi \kappa_0)^{-2N} \times \frac{\Gamma(\kappa_0 + 1 + \frac{3}{2}N)}{\Gamma(\kappa_0 + 1)} \cdot \theta_{\parallel}^{-N} \theta_{\perp}^{-2N} \times \left[ 1 + \frac{1}{\kappa_0} \left( \frac{1}{\theta_{\parallel}^2} \sum_{i=1}^{N} u_{i\parallel}^2 + \frac{1}{\theta_{\perp}^2} \sum_{i=1}^{N} u_{i\perp}^2 \right) \right]^{-\kappa_0-1-\frac{1}{2}N} \]  

with normalization similar to that in Equation 40(b).

The base of the kappa distributions includes those velocity components that are characterized by the same correlation. Therefore, the correlation among the particles’ parallel velocity components requires the inclusion of these components into the “base” of the kappa distribution:

\[ \{u_{i\parallel}\}_i=1^N; \theta_{\parallel}; \kappa_0; N \]  

\[ P(\{u_{i\parallel}\}_i=1^N; \theta_{\parallel}; \kappa_0; N) \sim \left[ 1 + \frac{1}{\kappa_0} \left( \frac{1}{\theta_{\parallel}^2} \sum_{i=1}^{N} u_{i\parallel}^2 \right) \right]^{-\kappa_0-1-\frac{1}{2}N} \]  

where the existence of a correlation is noted with “\(\leftrightarrow\)”.

by the symbol of the kappa index, \(\kappa_0\) (or the 3D kappa index, here noted by \(\kappa\)).

Similarly, the existence of correlation among the particles’ perpendicular velocity components requires the inclusion of the perpendicular components into the base of the kappa distribution:

\[ \{u_{i\perp}\}_i=1^N; \theta_{\perp}; \kappa_0; N \]  

\[ P(\{u_{i\perp}\}_i=1^N; \theta_{\perp}; \kappa_0; N) \sim \left[ 1 + \frac{1}{\kappa_0} \left( \frac{1}{\theta_{\perp}^2} \sum_{i=1}^{N} u_{i\perp}^2 \right) \right]^{-\kappa_0-1-\frac{1}{2}N} \]  

Furthermore, the correlation among the particles’ parallel and perpendicular velocity components requires the inclusion of both the perpendicular and the parallel components into the base of the kappa distribution, i.e.,

\[ \{u_{i\parallel}\}_i=1^N; \theta_{\parallel}; \theta_{\perp}; \kappa_0; N \]  

\[ P(\{u_{i\parallel}\}_i=1^N; \theta_{\parallel}; \theta_{\perp}; \kappa_0; N) \sim \left[ 1 + \frac{1}{\kappa_0} \left( \frac{1}{\theta_{\parallel}^2} \sum_{i=1}^{N} u_{i\parallel}^2 + \frac{1}{\theta_{\perp}^2} \sum_{i=1}^{N} u_{i\perp}^2 \right) \right]^{-\kappa_0-1-\frac{1}{2}N} \]  

Since the correlation among the same particles’ velocity components equals the correlation between different velocity components, the particle system is characterized by homogeneous correlations among particles’ velocity components; symbolically, all correlations are characterized by the same kappa index, \(\kappa_0\). We note that the one-particle kappa distribution, which is convenient and frequently used due to simplicity, should not be interpreted as characterizing a case where the particles are not correlated to each other. On the contrary, the one-particle kappa distribution is simply a convenient mathematical tool, but physically an artifact; the correlations among the same velocity components of all the
correlated particles can be ignored when using the convenient one-particle kappa distribution, but still is implied through the concept of the kappa index (see Gravanis et al. 2020).

Next, we examine the case of particle systems with heterogeneous correlations among their velocity components.

### 4.3. Anisotropic Distributions with Heterogeneously Correlated Velocity Components

We consider the simplest case of unequal correlations among different velocity components. This is the case where the three components, the parallel and the two perpendicular ones, are independent of each other, while the correlation among the same components of different particles is kept constant ($\kappa$), i.e.,

$$\{u_1 \leftrightarrow u_{1,2}\} \rightarrow \{u_{1,2} \leftrightarrow u_{1,2}\} \rightarrow \{u_{1,3} \leftrightarrow u_{1,3}\},$$  \hspace{1cm} (47a)

where the notation among different dof means $\kappa \rightarrow \infty$, that is, uncorrelated or independent velocity components; for instance, Equation 47(a) reads that the $x$- and $y$-perpendicular components, as well as the parallel ($z$-) component, are all independent of each other. This simple type of heterogeneously correlated velocity components defines the distribution:

$$P(u_1; u_2; \theta_1, \kappa) = \left[ \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \right]^{\frac{2}{\kappa}} \cdot \theta_1^{-1} \theta_2^{-2} \cdot \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{u_1^2}{\theta_1^2} \right)^{-\kappa} \cdot \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{u_2^2}{\theta_2^2} \right)^{-\kappa},$$

with normalization

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(u_1; u_2; \theta_1, \kappa) du_1 du_2 du_3 = 1.$$  \hspace{1cm} (47b)

A more general type of anisotropic distribution, with heterogeneous correlations, is to have different correlations (or kappa indices) characterizing each dof, i.e.,

$$\{u_2 \leftrightarrow u_{3,4}\} \rightarrow \{u_{3,4} \leftrightarrow u_{3,4}\} \rightarrow \{u_2 \leftrightarrow u_{2,4}\},$$  \hspace{1cm} (48a)

that defines the distribution

$$P(u; \theta, \kappa) = \left[ \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \right]^{\frac{2}{\kappa}} \cdot \theta_1^{-1} \theta_2^{-1} \cdot \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{u_1^2}{\theta_1^2} \right)^{-\kappa_1} \cdot \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{u_2^2}{\theta_2^2} \right)^{-\kappa_2},$$

with normalization

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(u; \theta, \kappa) du_1 du_2 du_3 = 1.$$  \hspace{1cm} (48b)

A more complicated case is when the heterogeneous correlations vary among different pairs of components, e.g., the two perpendicular components $x$ and $y$ are correlated with a common kappa index, while both of them are independent with the parallel component ($\kappa \rightarrow \infty$), that is,

$$\{u_1 \leftrightarrow u_{1,2}\} \rightarrow \{u_{1,2} \leftrightarrow u_{1,2}\} \rightarrow \{u_{1,3} \leftrightarrow u_{1,3}\}.$$  \hspace{1cm} (49a)

defines the distribution

$$P(u_1; u_2; \theta_1, \kappa) = \left[ \frac{\Gamma(\kappa)}{\Gamma(\kappa - \frac{1}{2})} \right]^{\frac{2}{\kappa}} \cdot \theta_1^{-1} \theta_2^{-2} \cdot \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{u_1^2}{\theta_1^2} \right)^{-\kappa_1} \cdot \left(1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{u_2^2 + u_3^2}{\theta_1^2 + \theta_3^2} \right)^{-\kappa_2},$$  \hspace{1cm} (49b)

with normalization

$$\int_{0}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(u_1; u_2; \theta_1, \kappa) du_1 du_2 du_3 = 2\pi \theta_1 \theta_3 = 1.$$  \hspace{1cm} (49c)

(e.g., for applications, see Summers & Thorne 1991; Dos Santos et al. 2015).

The above anisotropic distributions constitute trivial types of heterogeneous correlations among the particles’ velocity components. This is because the heterogeneous correlation, i.e., the correlation connecting different dof, is either zero or equal to the correlation connecting the same dof; namely, the corresponding kappa value is either infinity or equal to the kappa index of the distribution. Next, we finalize this section, by deriving the corresponding correlation coefficient. In Section 5, we will present the general case that describes the nontrivial heterogeneous correlations, by expressing the correlations in terms of probability distributions; for this, we focus on the expression of the multi-dimensional kappa probability distribution in terms of its marginal probability distributions.

### 4.4. Correlation Coefficients for Anisotropic Distributions Subject to Heterogeneous Correlations

Let the anisotropic kappa distribution in Equation 47(b) with heterogeneous correlation equal to zero, $\kappa^\text{int} \rightarrow \infty$, i.e., the three velocity components are independent of each other; hence, the ensemble of all particles’ components constitutes a system with one correlated component, that is, with effective dimensionality $d_{\text{eff}} = 1$; thus, the correlation coefficient is

$$\rho = \frac{1}{\kappa_0 + \frac{1}{2}} = \frac{1}{\kappa_3 - 1} = \kappa_1,$$

which is written in terms of the invariant kappa index $\kappa_0$, the typical 3D kappa index $\kappa_3$, or the 1D kappa index $\kappa_1$ (compare to Equation (34)).

In the case of distributions with unequal heterogeneous correlations, the derivations are more complicated and will be presented elsewhere. Here we just mention the case of distributions of $d$-D velocity, decomposed to 1D parallel component and a $(d-1)$-D perpendicular component. The correlation coefficient is

$$\rho = \frac{1}{\kappa_0 + 1 + (\kappa_0 + d - 1)(d - 1)\alpha^2},$$

where we set the triads (not vectors) $\theta \equiv (\theta_1, \theta_2, \theta_3)$ and $\kappa \equiv (\kappa_1, \kappa_2, \kappa_3)$. 

A more complicated case is when the heterogeneous correlations vary among different pairs of components, e.g., the two perpendicular components $x$ and $y$ are correlated with a common kappa index, while both of them are independent with the parallel component ($\kappa \rightarrow \infty$), that is,
which can be written again in its standard formulation, as in Equation (36),

$$\rho = \frac{1}{\kappa_0 + \frac{1}{d_{\text{eff}}}} \text{ or } \rho = \frac{1}{\kappa_0 + \frac{d_{\text{eff}}}{d - 1}}.$$  \hspace{1cm} (52)

where the effective dimensionality and adiabatic polytropic index are respectively given by

$$d_{\text{eff}}(\alpha, d) = \frac{1 + (d - 1)\alpha^2}{1 + (d - 1)\alpha^2},$$  \hspace{1cm} (53)

and

$$\gamma(\alpha, d) = 1 + 2d_{\text{eff}}(\alpha, d)^{-1} = 1 + 2\frac{1 + (d - 1)\alpha^2}{1 + (d - 1)^2\alpha^2}.$$  \hspace{1cm} (54)

5. Generalized Anisotropic Kappa Distributions Based on Heterogeneity

5.1. General

Previously, we have shown the types of homogeneous and heterogeneous correlations among the particle velocity components. We examined the typical cases, where the velocity components of each particle are either (a) homogeneously correlated, that is, the correlations from all components are characterized by the same kappa index, or (b) uncorrelated, that is, the correlations between different components are characterized by a kappa index equal to infinity. We now develop the general case, where the velocity components can be heterogeneously correlated with an arbitrary heterogeneity, namely, the correlations between different components can be characterized by any kappa index.

The generalization of anisotropic kappa distributions takes place within the framework of Tsallis nonextensive statistical mechanics (e.g., 2009; Tsallis et al. 1998). This is based on a natural generalization of entropy, the so-called q-entropy (or Tsallis entropy, e.g., see Tsallis 1988; Livadiotis 2018a, 2018b), while the statistical origin of kappa distributions is connected with nonextensive statistical mechanics, where the associated entropy is maximized under the constraints of canonical ensemble. Certainly, Tsallis nonextensive statistical mechanics is well known in the community of statistical (Tsallis 2009), space (Livadiotis 2017), and plasma (Yoon 2019) physics. On the other hand, other similar promising theories exist, but they need to be further developed, e.g., superstatistics (Beck & Cohen 2003), Kaniadakis entropy and its maximization (Kaniadakis 2001; Macedo-Filho et al. 2013).

As shown in Livadiotis & McComas (2009, 2013), the kappa distribution coincides exactly with the q-exponential distribution, the one that emerges from the maximization of q-entropy, under the trivial transformation of their indices $\kappa = 1/(q - 1)$. The equivalence between the formulations of kappa and q-exponential distributions becomes obvious in the case of particle systems with continuous energy density. Surely, there other cases, such as systems with quantum energy states, where the distribution resulting from the maximization of entropy is not given by a q-exponential or kappa distribution. Moreover, it has to be noted that kappa distributions characterize systems residing in stationary states, and they are consistent with the concept of thermal equilibrium (Livadiotis 2018a). While there are various mechanisms responsible for generating kappa distributions (Livadiotis et al. 2018; Livadiotis 2019b), it is their consistency with thermal equilibrium that allows the existence of these distributions.

The concept of correlation is interwoven with the theory of kappa distributions and nonextensive statistical mechanics. Let two sets $A$ and $B$ of discrete probability distributions be $\left( p_i^A \right)_{i=1}^{W} = p_1^A, p_2^A, \ldots, p_W^A$ and $\left( p_j^B \right)_{j=1}^{W} = p_1^B, p_2^B, \ldots, p_W^B$ as well as their 2D joint probability distribution, $\left( p_{ij}^{AB} \right)_{i=1}^{W} = p_{1j}^{AB}, \ldots, p_{Wj}^{AB}$. The classical Boltzmann–Gibbs statistical mechanics requires that there is no correlation between the particles (Gibbs 1902), i.e., $p_{ij}^{AB} = p_i^A \cdot p_j^B$, which leads to the additivity of Boltzmann’s entropy, i.e., $S^{A+B} = S^A + S^B$ (Tsallis 2005). On the other hand, nonextensive statistical mechanics implies a certain type of correlation, that is $p_{ij}^{AB} = g(p_i^A; p_j^B; a)$, where $g(x, y; a) \equiv [x^{-a} + y^{-a} - 1]^{-1/a}$. Then we can easily show that the Boltzmann’s entropy is nonadditive, where the additivity rule is also characterized by the nonlinear coupling term $S^{A+B} = S^A + S^B - a \cdot S^A \cdot S^B$, but, the Tsallis entropy is still additive under this type of correlation (Tsallis 2005; Livadiotis 2018a, 2018b). The exponent can be easily related to the kappa index $\kappa = 1/a$, while the above type of correlations will be used to construct the formulation of anisotropic kappa distributions with nontrivial heterogeneity among the particle velocity components.

5.2. Joint Probability Distribution and Its Partition to Marginal Probability Distributions

Let the $f$-dimensional kappa distribution describing $f = N \cdot d$ velocity components, $\{ u_i \}_{i=1}^{f} = \{ u_1, \cdots, u_f \}$, according to the notation in Section 2.3; that is,

$$P(u_1, u_2, \cdots, u_f; \kappa_0, f) = \left( \frac{\Gamma(\kappa_0 + 1 + \frac{1}{2f})}{\Gamma(\kappa_0)} \right) \cdot \theta^f \cdot \left[ 1 + \frac{1}{\kappa_0} \cdot \frac{u_1^2 + u_2^2 + \cdots + u_f^2}{\theta^2} \right]^{-\kappa_0 - 1 - \frac{1}{2f}},$$  \hspace{1cm} (55)

where we recall again the requirement of using the invariant kappa index $\kappa_0$ in the formulation of multidimensional distributions.

Furthermore, we use the variables of the total kinetic energy and the kinetic energy of the $i$th component, which are, respectively,

$$\varepsilon_{\text{tot}} = \frac{1}{2} \sum_{i=1}^{f} u_i^2 = \frac{1}{2} \sum_{i=1}^{f} \varepsilon_i \text{ and } \varepsilon_i = \frac{1}{2} m u_i^2.$$  \hspace{1cm} (56)

Hence, we rewrite the distribution (55) as

$$P(\varepsilon_{\text{tot}}) = P(0) \cdot \left( 1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon_{\text{tot}}}{k_B T} \right)^{-\kappa_0 - 1 - \frac{1}{2f}},$$  \hspace{1cm} (57)

while the one-particle distribution of any $i$th component is similarly written as

$$P(\varepsilon_i) = P(0) \cdot \left( 1 + \frac{1}{\kappa_0} \cdot \frac{\varepsilon_i}{k_B T} \right)^{-\kappa_0 - 1 - \frac{1}{2f}}, \text{ i: } 1, \ldots, f.$$  \hspace{1cm} (58)
Then, we obtain
\[
\left[ \frac{P(\varepsilon_i)}{P(0)} \right]^{-\frac{1}{\kappa_0 k B T}} - 1 = \frac{1}{\kappa_0 k B T} \cdot \varepsilon_i,
\]
and
\[
\left[ \frac{P(\varepsilon_{i\sigma})}{P(0)} \right]^{-\frac{1}{\kappa_0 k B T}} - 1 = \frac{1}{\kappa_0 k B T} \cdot \varepsilon_{i\sigma} = \frac{1}{\kappa_0 k B T} \cdot \sum_{i=1}^{f} \varepsilon_i = \sum_{i=1}^{f} \left[ \left[ \frac{P(\varepsilon_i)}{P(0)} \right]^{-\frac{1}{\kappa_0 k B T}} - 1 \right].
\]
(60)

Therefore, the partition of the joint probability distribution, \(P(\varepsilon_{i\sigma})\) or \(P(u_1, u_2, \ldots, u_f)\), with its marginal probability distributions, \(P(\varepsilon_i)\) or \(P(u_i)\), is given by the relation
\[
\left[ \frac{P(\varepsilon_{i\sigma})}{P(0)} \right]^{-\frac{1}{\kappa_0 k B T}} - 1 = \sum_{i=1}^{f} \left[ \left[ \frac{P(\varepsilon_i)}{P(0)} \right]^{-\frac{1}{\kappa_0 k B T}} - 1 \right].
\]
(61a)

which generalizes the respective independence relation of Maxwell–Boltzmann distributions, corresponding to \(\kappa_0 \to \infty\),
\[
\frac{P(\varepsilon_{i\sigma})}{P(0)} = \lim_{\kappa_0 \to \infty} \left[ \frac{P(\varepsilon_i)}{P(0)} \right]^{-\frac{1}{\kappa_0 k B T}} - 1 = \sum_{i=1}^{f} \left[ \left[ \frac{P(\varepsilon_i)}{P(0)} \right]^{-\frac{1}{\kappa_0 k B T}} - 1 \right].
\]
(61b)

The finite value of the kappa index involved in the partition relation 61(a) is the key-parameter setting the correlations of the joint probability distribution, \(P(\varepsilon_{i\sigma})\) or \(P(u_1, u_2, \ldots, u_f)\). For an arbitrary kappa index, a unique correlation type is determined among the velocity components. These statistical correlations can be caused by existing, or even pre-existing, interactions among particles, which reserve the correlations, especially in the case of collisionless particle systems; the phenomenological kappa index is noted by \(\kappa^{int}_0\) (i.e., due to “interactions”), and Equation 61(a) leads to the partition relation
\[
\left[ \frac{P(\varepsilon_{i\sigma})}{P(0)} \right]^{-\frac{1}{\kappa^{int}_0 k B T}} - 1 = \sum_{i=1}^{f} \left[ \left[ \frac{P(\varepsilon_i)}{P(0)} \right]^{-\frac{1}{\kappa^{int}_0 k B T}} - 1 \right].
\]
(62a)

leading to the distribution
\[
P(\varepsilon_{i\sigma}) = P(0) \times \left\{ 1 + \sum_{i=1}^{f} \left[ \left[ \frac{P(\varepsilon_i)}{P(0)} \right]^{-\frac{1}{\kappa^{int}_0 k B T}} - 1 \right] \right\}^{-\kappa^{int}_0 - 1 - \frac{1}{\kappa^{int}_0 k B T}}.
\]
(62b)

Next, we see how the partition relation 62(a) is used for developing the formulation of more complicated types of anisotropic distributions, covering all combinations of heterogeneous correlations among the particles’ velocity components.

5.3. Anisotropic Distributions with Nontrivial Heterogeneity

Let the kappa index \(\kappa^{int}\) characterize the heterogeneous correlations between any two dof, either perpendicular or parallel velocity components, namely,
\[
\{u_{ix} \leftrightarrow u_{iy}\} \leftrightarrow \{u_{iy} \leftrightarrow u_{iy}\} \leftrightarrow \{u_{iz} \leftrightarrow u_{iz}\}
\]
(63a)

This defines the distribution
\[
P(u; \theta, \kappa) \sim \left\{ -1 + \left[ -1 + \left( 1 + \frac{1}{\kappa_{\perp} - \frac{u_x^2}{\theta^2}} \right)^{\frac{\kappa_{\parallel}}{\theta^2}} + \frac{1}{\kappa_{\perp} - \frac{u_x^2}{\theta^2}} \right]^{\frac{\kappa_{\parallel}}{\theta^2}} \right\}^{-\kappa_{\perp} - 1}.
\]
(63b)

A more general case concerns heterogeneous correlations with different kappa value between different velocity components. As a typical example, we consider the case where two different kappa values, \(\kappa^{int}_1\) and \(\kappa^{int}_2\), describe the heterogeneous correlations among different dof; in particular, \(\kappa^{int}_1\) describes the correlations between components \(x\) and \(y\), while \(\kappa^{int}_2\) describes correlations between \(x\) or \(y\) components with the \(z\) component, i.e.,
\[
\{u_{ix} \leftrightarrow u_{iy}\} \leftrightarrow \{u_{iy} \leftrightarrow u_{iy}\} \leftrightarrow \{u_{iz} \leftrightarrow u_{iz}\},
\]
(64a)

which defines the distribution
\[
P(u; \theta, \kappa) \sim \left\{ -1 + \left[ -1 + \left( 1 + \frac{1}{\kappa_{\perp} - \frac{u_x^2}{\theta^2}} \right)^{\frac{\kappa_{\parallel}}{\theta^2}} + \frac{1}{\kappa_{\perp} - \frac{u_x^2}{\theta^2}} \right]^{\frac{\kappa_{\parallel}}{\theta^2}} \right\}^{-\kappa_{\perp} - 1}.
\]
(64b)

This can be focused to the perpendicular/parallel notation, as follows:
\[
\{u_{ix} \leftrightarrow u_{iy}\} \leftrightarrow \{u_{iy} \leftrightarrow u_{iy}\} \leftrightarrow \{u_{iz} \leftrightarrow u_{iz}\},
\]
(65a)

and
\[
P(u; \theta, \kappa) \sim \left\{ -1 + \left[ -1 + \left( 1 + \frac{1}{\kappa_{\perp} - \frac{u_x^2}{\theta^2}} \right)^{\frac{\kappa_{\parallel}}{\theta^2}} + \frac{1}{\kappa_{\perp} - \frac{u_x^2}{\theta^2}} \right]^{\frac{\kappa_{\parallel}}{\theta^2}} \right\}^{-\kappa_{\perp} - 1}.
\]
(65b)

A more interesting case is when \(\kappa_{\perp} = \kappa_{\perp} = \kappa_{\perp} = \kappa^{int}_{\perp}\), i.e., the correlations become
\[
\{u_{ix} \leftrightarrow u_{iy}\} \leftrightarrow \{u_{iy} \leftrightarrow u_{iy}\} \leftrightarrow \{u_{iz} \leftrightarrow u_{iz}\},
\]
(66a)
with distribution

\[
P(u_{||}, u_{\perp}; \theta_{||}, \theta_{\perp}, \kappa_{||}, \kappa_{\perp}, \kappa_{\text{int}}) \\
\sim \left\{ -1 + \left( 1 + \frac{1}{\kappa_{||}} \cdot \frac{u_{\perp}^2}{\theta_{\perp}} \right)^{-\frac{\kappa_{||} + \frac{1}{2}}{\kappa_{||} - \frac{1}{2}}} \right\}^{\kappa_{\text{int}} - 1} \\
+ \left( 1 + \frac{1}{\kappa_{\perp}} \cdot \frac{u_{\parallel}^2}{\theta_{\parallel}} \right)^{-\frac{\kappa_{\text{int}} - 1}{2}} .
\]

(66b)

This is even more simplified when \( \kappa_{||} = \kappa_{\perp} = \kappa \), i.e., the correlations are written as

\[
\{ u_{||} \rightarrow u_{i}, u_{i} \} \approx \{ u_{\parallel} \rightarrow u_{i}, u_{i} \} \rightarrow \{ \kappa_{||} \rightarrow \kappa_{\perp} \} \rightarrow \{ u_{\parallel} \rightarrow u_{i} \} ,
\]

(67a)

with distribution

\[
P(u_{||}, u_{\perp}; \theta_{||}, \theta_{\perp}, \kappa, \kappa_{\text{int}}) \\
\sim \left\{ -1 + \left( 1 + \frac{1}{\kappa} \cdot \frac{u_{\perp}^2}{\theta_{\perp}} \right)^{-\frac{\kappa + \frac{1}{2}}{\kappa - \frac{1}{2}}} \right\}^{\kappa_{\text{int}} - 1} \\
+ \left( 1 + \frac{1}{\kappa} \cdot \frac{u_{\parallel}^2}{\theta_{\parallel}} \right)^{-\frac{\kappa_{\text{int}} - 1}{2}} .
\]

(67b)

The normalization constant of these distributions does not exist in a closed-form expression. It is given by

\[
C(\kappa_{\text{int}}, \kappa) \cdot \alpha^{-1} \left[ \frac{1}{3} (1 + 2 \alpha) \right] \theta^{-3} ,
\]

where

\[
C(\kappa_{\text{int}}, \kappa)^{-1} = \int_{0}^{\infty} \int_{-\infty}^{\infty} \left[ -1 + \left( 1 + \frac{\kappa_x^2}{\kappa - \frac{1}{2}} \right)^{-\frac{\kappa + \frac{1}{2}}{\kappa - \frac{1}{2}}} \right]^{-\kappa_{\text{int}} - 1} \\
+ \left( 1 + \frac{\kappa_y^2}{\kappa - \frac{1}{2}} \right)^{-\frac{\kappa_{\text{int}} - 1}{2}} d\kappa_x \cdot 2\pi \kappa_x d\kappa_y .
\]

(68a)

In the Appendix, we show that

\[
C(\kappa_{\text{int}}, \kappa) = \pi^{-1} \cdot \frac{\kappa_{\text{int}} \kappa}{\kappa_{\text{int}} + \frac{1}{2}} \cdot \frac{\kappa}{\kappa + \frac{1}{2}} \cdot \frac{\Gamma\left(\frac{\kappa_{\text{int}} + 1}{2}\right)}{\Gamma\left(\frac{\kappa + \frac{1}{2}}{2}\right)} \\
\times \int_{0}^{\infty} \int_{-\infty}^{\infty} \left( \frac{(2m)!}{4^m(2\pi)^m} \right)^{1/2} \left( \kappa_{\text{int}} + 1, 1, 1; \kappa_{\text{int}} + 2 \right) \left( \kappa + \frac{1}{2}, 1; \kappa_{\text{int}} + 2 + \frac{\kappa_{\text{int}} + \frac{1}{2}}{2 + \frac{\kappa_{\text{int}} + \frac{1}{2}}{m - \frac{1}{2}}}; 1 \right) \\
\times \left( \frac{2}{\kappa + 1} \right)^{\frac{1}{2}} \left( \frac{u_\parallel^2}{\theta_\parallel^2} + \frac{u_\perp^2}{\theta_\perp^2} \right) \right)^{-\kappa_{\text{int}} - 1} .
\]

(68b)

5.4. Summary of Anisotropic Distributions Formulæ

Table 2 summarizes the main formulæ of anisotropic kappa distributions, as determined from the type of homogeneous/heterogeneous correlations.

The main three and more useful anisotropic distributions are the following:

(i) Heterogeneous correlation equal to the correlation of each component, \( \kappa_{\text{int}} = \kappa \),

\[
P(u; \theta, \alpha, \kappa; \kappa_{\text{int}} = \kappa) = \pi^{-\frac{3}{2}} \cdot \left( \kappa - \frac{3}{2} \right)^{-\frac{1}{2}} \Gamma(\alpha + 1) \frac{1}{\Gamma(\frac{k}{2})} \\
\times \alpha^{-1} \left[ \frac{1}{3} (1 + 2 \alpha) \right] \theta^{-3} \\
\times \left[ 1 + \frac{1}{\kappa} \cdot \frac{1 + 2 \alpha}{3 \alpha \theta^2} \cdot (\kappa u_\parallel^2 + u_\perp^2) \right]^{-\kappa_{\text{int}} - 1} .
\]

(70a)
Table 2
Main Formulae of Anisotropic Kappa Distributions in Association with the Correlation Type

| Correlation Type\(^a\) | Distribution \(P(\text{uv})\) |
|------------------------|-------------------------|
| 1                      | \(\pi^{-2} \left( \kappa - \frac{3}{2} \right)^{\frac{3}{2}} \left( \Gamma\left(\kappa + 1\right) \Gamma\left(\frac{3}{2}\right) \right)^{-1} \theta_1^{-2} \left( 1 + \frac{\theta_1^2}{\kappa - \frac{3}{2}} \right)^{-\kappa}\) |
| 2a                     | \(\pi^{-2} \left( \kappa - \frac{3}{2} \right)^{\frac{3}{2}} \left( \Gamma\left(\kappa + 1\right) \Gamma\left(\frac{3}{2}\right) \right)^{-1} \theta_1^{-2} \left( 1 + \frac{\theta_1^2}{\kappa - \frac{3}{2}} \right)^{-\kappa}\) |
| 2b                     | \(\pi^{-2} \left( \kappa - \frac{3}{2} \right)^{\frac{3}{2}} \left( \Gamma\left(\kappa + 1\right) \Gamma\left(\frac{3}{2}\right) \right)^{-1} \theta_1^{-2} \left( 1 + \frac{\theta_1^2}{\kappa - \frac{3}{2}} \right)^{-\kappa}\) |
| 3                      | \(\pi^{-2} \left( \kappa - \frac{3}{2} \right)^{\frac{3}{2}} \left( \Gamma\left(\kappa + 1\right) \Gamma\left(\frac{3}{2}\right) \right)^{-1} \theta_1^{-2} \left( 1 + \frac{\theta_1^2}{\kappa - \frac{3}{2}} \right)^{-\kappa}\) |
| 4a                     | \(\pi^{-2} \left( \kappa - \frac{3}{2} \right)^{\frac{3}{2}} \left( \Gamma\left(\kappa + 1\right) \Gamma\left(\frac{3}{2}\right) \right)^{-1} \theta_1^{-2} \left( 1 + \frac{\theta_1^2}{\kappa - \frac{3}{2}} \right)^{-\kappa}\) |
| 4b                     | \(\pi^{-2} \left( \kappa - \frac{3}{2} \right)^{\frac{3}{2}} \left( \Gamma\left(\kappa + 1\right) \Gamma\left(\frac{3}{2}\right) \right)^{-1} \theta_1^{-2} \left( 1 + \frac{\theta_1^2}{\kappa - \frac{3}{2}} \right)^{-\kappa}\) |

Notes.

\(^a\) (1) Homogeneous correlations. (2) Heterogeneous correlations, among the same dof: (a) \(\kappa\), (b) \((\kappa_x, \kappa_y, \kappa_z)\); among different dof: \(\kappa \rightarrow \infty\). (3) Heterogeneous correlations, among the same dof: \(\kappa\); among different dof: \(\kappa \rightarrow \infty\) between perpendicular and parallel. (4) Heterogeneous correlations, among the same dof: (a) \(\kappa\), (b) \((\kappa_x, \kappa_y, \kappa_z)\); among different dof: (a) \(\kappa \rightarrow \infty\) between perpendicular and parallel, (b) \(\kappa_{\text{int}}\) between perpendicular \(x\) and \(y\), \(\kappa_{\text{int}}\) between perpendicular and parallel.

\(^b\) The normalization constant \(C\) of the last two examples does not exist in a closed-form expression; it can be given in a series expression and derived numerically (see Appendix).
(ii) Heterogeneous correlation (between perpendicular and parallel components) equal to zero, $\kappa_{\text{int}} \rightarrow \infty$, 

$$P(\mathbf{u}; \theta, \alpha, \kappa; \kappa_{\text{int}} \rightarrow \infty) = \left[ \frac{\pi}{\Gamma(\kappa - \frac{1}{2})} \right]^{\frac{3}{2}} \alpha^{-1} \left[ \frac{1}{3} (1 + 2\alpha) \right]^{\frac{3}{2}} \theta^{-3} \times \left( 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1 + 2\alpha}{3\theta^2} \cdot u^2 \right)^{-\kappa - \frac{3}{2}} \times \left( 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1 + 2\alpha}{3\theta^2} \cdot u^2 \right)^{-\kappa}.$$  

(70b)

(iii) Heterogeneous correlation between perpendicular and parallel components) has an arbitrary value, $\kappa_{\text{int}} < \infty$, 

$$P(\mathbf{u}; \theta, \alpha, \kappa; \kappa_{\text{int}}) = C(\kappa_{\text{int}}, \kappa) \cdot \alpha^{-1} \left[ \frac{1}{3} (1 + 2\alpha) \right]^{\frac{3}{2}} \theta^{-3} \times \left[ -1 + \left( 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1 + 2\alpha}{3\theta^2} \cdot u^2 \right)^{\frac{3}{2} \alpha + \frac{1}{2}} \right]^{-\kappa_{\text{int}} - \frac{3}{2}} \times \left[ 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1 + 2\alpha}{3\theta^2} \cdot u^2 \right]^{\frac{3}{2} \alpha + \frac{1}{2}}.$$  

(70c)
The developed distribution in Equation 70 mediates the main two types of anisotropic kappa distributions, where the first in Equation 70 considers equal correlations among particles' velocity components, while the second in Equation 70 considers zero correlation among different velocity components. (The Appendix helps on the normalization of those distributions.) Similarly to Figure 2 plotting the distribution 70(a), Figure 4 plots the distribution 70(b), that is, for \( \kappa^{\text{int}} = \infty \), while Figures 5 and 6 plot the distribution 70(c), for \( \kappa^{\text{int}} = 3 \) and 1.5, respectively; the plotted velocity components are normalized to the thermal speed, \( \theta \).

6. Formalism of Anisotropic Kappa Distributions in Arbitrary Reference Frames

6.1. Distributions of Energy \( E \), Pitch Angle \( \vartheta \), and Azimuth \( \varphi \)

We derive the expressions of the anisotropic kappa distributions in an arbitrary reference frame. We focus on the three main distributions shown in Section 5.4; there, the distribution had been expressed in the comoving system (where the bulk velocity is set to zero); using the transformations \( u_{||} = u \cos \vartheta \), \( u_{\perp\perp} = u \sin \vartheta \cos \varphi \), \( u_{\perp\parallel} = u \sin \vartheta \sin \varphi \), (i.e., \( \cos \vartheta = u_{||} / u \) and \( \cos \varphi = u_{\perp\parallel} / u_{\perp\perp} \)), with \( E = \frac{1}{2}m u^2 \) (kinetic energy in the S/C reference frame), and \( u_{||} = u_b \cos \vartheta_b \), \( u_{\perp\parallel} = u_b \sin \vartheta_b \cos \varphi_b \), \( u_{\perp\perp} = u_b \sin \vartheta_b \sin \varphi_b \), we express the distributions as follows:

(i) Homogeneous correlations (correlations between velocity components of each particle, are equal to the correlations between the same components of different particles, \( \kappa^{\text{int}} = \kappa \)):

\[
P(u - u_b; \vartheta, \alpha, \kappa; \kappa^{\text{int}} = \kappa) = \pi^{-2} \times \left( \kappa - \frac{3}{2} \right)^{\frac{3}{2}} \Gamma \left( \frac{3}{2} \kappa - \frac{3}{2} \right) \times \theta^{-3} \times \left\{ 1 + \frac{1}{\kappa - 1} \cdot \frac{1 + 2\alpha}{3\alpha \theta^2} \cdot [\alpha (u_{||} - u_{b||})]^2 + (u_{\perp\parallel} - u_{b\perp\parallel})^2 \right\}^{-\frac{\kappa - 1}{2}},
\]

or, in terms of the kinetic energy \( E \), pitch angle \( \vartheta \), and azimuth angle \( \varphi \):

\[
\frac{100}{100} \begin{array}{c}
0.1 \ \ \ 0.3 \ \ \ 0.5 \ \ \ 0.7 \ \ \ 0.9 \\
1.1 \ \ \ 1.3 \ \ \ 1.5 \ \ \ 1.7 \ \ \ 1.9 \\
2.1 \ \ \ 2.3 \ \ \ 2.5 \ \ \ 2.7 \ \ \ 2.9 \\
3.1 \ \ \ 3.3 \ \ \ 3.5 \ \ \ 3.7 \ \ \ 3.9 \\
4.1 \ \ \ 4.3 \ \ \ 4.5 \ \ \ 4.7 \ \ \ 4.9 \\
5.1 \ \ \ 5.3 \ \ \ 5.5 \ \ \ 5.7 \ \ \ 5.9 \\
6.1 \ \ \ 6.3 \ \ \ 6.5 \ \ \ 6.7 \ \ \ 6.9 \\
7.1 \ \ \ 7.3 \ \ \ 7.5 \ \ \ 7.7 \ \ \ 7.9 \\
8.1 \ \ \ 8.3 \ \ \ 8.5 \ \ \ 8.7 \ \ \ 8.9 \\
9.1 \ \ \ 9.3 \ \ \ 9.5 \ \ \ 9.7 \ \ \ 9.9 \\
10.1 \ \ \ 10.3 \ \ \ 10.5 \ \ \ 10.7 \ \ \ 10.9 \\
\end{array}
\]

Figure 5. Anisotropic kappa distributions with heterogeneous correlation corresponding to \( \kappa^{\text{int}} = 3 \), for various kappa indices \( \kappa \) and anisotropies \( \alpha \).
Figure 6. Anisotropic kappa distributions with heterogeneous correlation corresponding to $\kappa^{\text{int}} = 1.5$, for various kappa indices $\kappa$ and anisotropies $\alpha$.

\[
P(E, \vartheta, \varphi; E_b, \vartheta_b, \varphi_b; T, \kappa, \kappa^{\text{int}} = \kappa) \\
= \pi^{-\frac{1}{2}} \left( \kappa - \frac{3}{2} \right)^{\frac{1}{2}} \frac{\Gamma(\kappa + 1)}{\Gamma(\kappa - \frac{1}{2})} \\
\times \alpha^{-1} \left[ \frac{1}{16} (1 + 2\alpha) \right]^2 (2k_b T/m)^{\frac{1}{2}} \\
\times \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{1 + 2\alpha}{3\kappa^2} \cdot L(E, \theta, \varphi; E_b, \theta_b, \varphi_b; \alpha) \right]^{-\alpha^{-1}},
\]

where
\[
L(E, \theta, \varphi; E_b, \theta_b, \varphi_b; \alpha) = E \left[ 1 + (\alpha - 1)\cos^2 \theta \right] \\
+ E_b \left[ 1 + (\alpha - 1)\cos^2 \theta_b \right] \\
- 2\sqrt{E E_b} \left[ \alpha \cos \vartheta \cos \theta_b + \sin \vartheta \sin \theta_b \cos (\varphi - \varphi_b) \right].
\]

(ii) Heterogeneous correlations equal to zero, (no correlations between velocity components of each particle, $\kappa^{\text{int}} \to \infty$, and finite correlations between the same components of different particles, $\kappa < \infty$):

\[
P(u - u_b; \theta, \alpha, \kappa; \kappa^{\text{int}} \to \infty) = \left[ \pi \left( \kappa - \frac{3}{2} \right) \right]^{\frac{1}{2}} \\
\times \frac{\Gamma(\kappa - \frac{3}{2})}{\Gamma(\kappa - \frac{1}{2})} \cdot \alpha^{-1} \left[ \frac{1}{3} (1 + 2\alpha) \right]^2 \theta^{-3} \\
\times \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \frac{1 + 2\alpha}{3\kappa^2} \cdot (u - u_b)^2 \right]^{-\kappa^{-\frac{1}{2}}}.
\]
or, in terms of the kinetic energy $E$, pitch angle $\vartheta$, and azimuth angle $\varphi$:

$$
P(E, \vartheta, \varphi; E_b, \vartheta_b, \varphi_b; T, \alpha, \kappa; \kappa_{\text{int}} \to \infty)$$

$$= \left[ \pi \left( \kappa - \frac{3}{2} \right) \right]^{-\frac{3}{2}}$$

$$\times \frac{\Gamma\left(\frac{1}{3}\kappa\right) \Gamma\left(\frac{1}{3}\kappa - \frac{1}{2}\right)}{\Gamma\left(\kappa - \frac{1}{2}\right)} \cdot \alpha^{-1} \left[\frac{1}{3}(1 + 2\alpha)\right]^{-\frac{1}{2}} (2k_b T/m)^{-\frac{1}{2}}$$

$$\times \left\{ 1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \left[ \frac{1}{3} + \frac{2\alpha}{3k_b T} \right] (E \sin^2 \vartheta + E_b \sin^2 \vartheta_b)$$

$$- 2\sqrt{E E_b} \sin \vartheta \sin \vartheta_b \cos(\varphi - \varphi_b) \right\}^{-\frac{1}{2}}$$

$$\times \left\{ 1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \left[ \frac{1}{3} + \frac{2\alpha}{3k_b T} \right] (E \cos^2 \vartheta + E_b \cos^2 \vartheta_b)$$

$$- 2\sqrt{E E_b} \cos \vartheta \cos \vartheta_b \right\}^{-\kappa}.$$

(72b)

(iii) Arbitrary heterogeneous correlation (finite correlations between velocity components of each particle, $\kappa_{\text{int}} < \infty$, and finite correlations between the same components of different particles, $\kappa < \infty$):

$$
P(\mathbf{u} - \mathbf{u}_b; \vartheta, \alpha, \kappa; \kappa_{\text{int}}) = C(\kappa_{\text{int}}, \kappa)$$

$$\times \alpha^{-1} \left[\frac{1}{3}(1 + 2\alpha)\right]^{\frac{1}{2}} \vartheta^{-3}$$

$$\times \left\{ -1 + \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \left( \frac{1 + 2\alpha}{3k_b T} \cdot (u_{\vartheta \vartheta} - u_{\vartheta_b \vartheta_b}) \right) \right]^{\frac{1}{2}}$$

$$+ \left[ 1 + \frac{1}{\kappa - \frac{3}{2}} \cdot \left( \frac{1 + 2\alpha}{3k_b T} \cdot (u_{\alpha \alpha} - u_{\alpha_b \alpha_b}) \right) \right]^{\frac{1}{2}} \right\}^{-\frac{1}{2}} \kappa_{\text{int}}^{\frac{1}{2}} \kappa_{\text{int}}^{-1},$$

(73a)

or, in terms of the kinetic energy $E$, pitch angle $\vartheta$, and azimuth angle $\varphi$:

Figure 7. Anisotropic kappa distributions with heterogeneous correlation equal to zero, $\kappa_{\text{int}} \to \infty$, for $\vartheta_b = 60^\circ$, $E_b/(k_b T) = 0.5$, kappa index $\kappa = 3$ and various anisotropies $\alpha$, in an arbitrary reference frame.
Figure 7 plots the distribution $P(E,\theta,\varphi;E_b,\theta_b,\varphi_b; T,\alpha,\kappa;\kappa^{\text{int}})$

$$P(E,\theta,\varphi;E_b,\theta_b,\varphi_b; T,\alpha,\kappa;\kappa^{\text{int}}) = C(\kappa^{\text{int}},\kappa) \cdot \alpha^{-1} \left[ \frac{1}{3} (1 + 2\alpha) \right] \left( 2 k_B T / m \right)^{\frac{1}{2}}$$

$$\times \left\{ -1 + \left[ 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1 + 2\alpha}{3 k_B T} \cdot (E \cos^2 \varphi + E_b \sin^2 \varphi_b) \right]^{\frac{1}{\kappa - 1}} \right. \bigg[ E \sin^2 \varphi \sin \varphi_b + E_b \sin \varphi \sin \varphi_b \bigg]^{\frac{1}{\kappa - 1}} \left. \right\}^{ \frac{\kappa^{\text{int}} - 1}{\kappa - 1}} \bigg[ E \sin \varphi \sin \varphi_b \cos(\varphi - \varphi_b) \bigg]^{\frac{\kappa^{\text{int}} - 1}{\kappa - 1}} \bigg[ E \sin \varphi \sin \varphi_b \cos(\varphi - \varphi_b) \bigg]^{\frac{\kappa^{\text{int}} - 1}{\kappa - 1}} \cdot (73b)$$

Note that in all the above equations, we used $(u_L - u_{b,L})^2 = (u_{L,x} - u_{b,L,x})^2 + (u_{L,y} - u_{b,L,y})^2$.

Figure 8. Anisotropic kappa distributions with heterogeneous correlation corresponding to $\kappa^{\text{int}} = 3$, for $\varphi_b = 60^\circ$, $E_b/(k_B T) = 0.5$, kappa index $\kappa = 3$ and various anisotropies $\alpha$, in an arbitrary reference frame.

6.2. Azimuth-integrated Distributions of Energy $E$ and Pitch Angle $\varphi$

Measurements taken from various S/C instruments that collect at certain energy and pitch angle channels can be used to fit the theoretical distributions, however, the latter must be azimuth-independent. For this, we use the azimuth-integrated
Figure 9. Anisotropic kappa distributions with heterogeneous correlation corresponding to $\kappa^{\text{int}} = 1.5$, for $\vartheta_b = 60^\circ$, $E_b/(k_bT) = 0.5$, kappa index $\kappa = 3$ and various anisotropies $\alpha$, in an arbitrary reference frame.

distributions, that is, functions of energy $E$ and pitch angle $\vartheta$, and not of azimuth $\varphi$. These are derived as follows:

In general, for the azimuthal function $f(\varphi)$, we find its expectation value, i.e., its average, as:

$$\langle f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(\varphi) d\varphi,$$

that is, \[ \langle f \rangle = \left( \int_{-\infty}^{\infty} f(\varphi) \right) \]  

Moreover, since we deal with an expectation value, it is reasonable to also seek for the standard deviation, or the $1\sigma$ uncertainty, given by:

$$\delta \langle f \rangle = \sqrt{\frac{1}{2\pi} \int_0^{2\pi} \left( f(\varphi) - \langle f \rangle \right)^2 d\varphi},$$

that is, \[ \delta \langle f \rangle = \sqrt{\left( \int_{-\infty}^{\infty} \left( f(\varphi) - \langle f \rangle \right)^2 \right) d\varphi} \].

Below we apply Equations 74(a), (b) to each of the $(E, \vartheta, \varphi)$-depended distributions expressed by Equations 71(b), 72(b), 73(b). (Note that we neglected writing the dependence on parameters, $E_b$, $\vartheta_b$, $T$, $\alpha$, $\kappa$, for simplicity.)

(i) Homogeneous correlations, $\kappa^{\text{int}} = \kappa$.

The azimuth-independent kappa distribution of energy $E$ and pitch angle $\vartheta$, corresponding to Equation 71(b), is given by:

$$P(E, \vartheta; \kappa^{\text{int}} \rightarrow \kappa) = \pi \left( \kappa - \frac{3}{2} \left( \frac{2k_bT}{m} \right) \right)^{\frac{3}{2} \Gamma(\kappa + 1)} \Gamma(\kappa) \left( \frac{1 + 2\alpha}{3} \right)^{\frac{1}{3} - \kappa}$$

$$\times \left[ \frac{\sqrt{E_b\sin\vartheta\sin\vartheta_b}}{(\kappa - \frac{3}{2})k_bT A(\lambda)} \right]^{\kappa - 1} \cdot 2F_1[\kappa + 1, \kappa + 1, 1, A(\lambda)^2],$$

\[ \text{Equation 75a} \]
where we have set $A = A[\lambda(E, \vartheta)]$ as:

$$A(\alpha) \equiv x - \sqrt{x^2 - 1}, \quad \text{and}$$

$$\lambda(E, \vartheta) \equiv -\alpha \tan \vartheta \tan \vartheta_b + \frac{(\varepsilon - \frac{1}{2}) k\theta \frac{\sin \vartheta}{\sin \vartheta_b} + E \left[(\alpha - 1) \cos^2 \vartheta + 1\right] + E_b[(\alpha - 1) \cos^2 \vartheta_b + 1]}{2\sqrt{E_b \sin \vartheta \sin \vartheta_b}}.$$  \hfill (75b)

Figure 10. Azimuth-independent anisotropic kappa distribution of energy $E$ and pitch angle $\vartheta$, in an arbitrary reference frame, in the case of homogeneous correlations, Equation 76(a), i.e., $\kappa^\text{int} = \kappa$, plotted for $\vartheta_b = 0^\circ$, $V_b = 5000 \text{ km s}^{-1}$ ($E_b \sim 70 \text{ eV}$), $T = 10^7 \text{ K}$ ($\sim 0.86 \text{ keV}$), and various anisotropies $\alpha$ and kappa indices $\kappa$.

Then, the theoretical uncertainty, caused by substituting the azimuth with its expectation value, is

$$\delta P(E, \vartheta; \kappa^\text{int} \to \kappa) = \left[\frac{E_b \sin \vartheta \sin \vartheta_b}{\varepsilon - \frac{1}{2} k \theta A(\lambda)}\right]^{-\kappa - 1} \times 2 F_1[2(\kappa + 1), 2(\kappa + 1), 1, A(\lambda)^2]^2 -_{\kappa + 1, \kappa + 1, 1, 1, A(\lambda)^2}^2.$$  \hfill (75d)
Figure 11. Similar to Figure 10 but for $\vartheta_b = 90^\circ$.

(ii) Heterogeneous correlations equal to zero, $\kappa^{\text{int}} \to \infty$:

The azimuth-independent kappa distribution of energy $E$ and pitch angle $\vartheta$, corresponding to Equation 72(b), is given by

$$P(E, \vartheta; \kappa^{\text{int}} \to \infty) = \left[\pi \left(\kappa - \frac{3}{2}\right) \left(\frac{2kT_m}{m}\right)^{\frac{1}{2}} \Gamma(\kappa - \frac{1}{2}) \Gamma(\kappa - \frac{1}{2}) \alpha^2 \left(1 + \frac{2a}{3}\right)^{1 - \kappa}ight]$$

$$\times \left[1 + \left(\frac{1}{\kappa - \frac{1}{2}} \right) \cdot \left(\frac{1 + \frac{2a}{3kT}}{\kappa - \frac{1}{2}}\right) \left(\sqrt{E \cos \vartheta - \sqrt{E_b \cos \vartheta_b}}\right)^2\right]^{-\kappa}$$

$$\times \left[\left(\frac{EE_b \sin \vartheta \sin \vartheta_b}{(\kappa - \frac{1}{2}) E_b A(\lambda)}\right)^{\kappa - \frac{1}{2}} \cdot {}_2F_1\left[\kappa + \frac{1}{2}, \kappa + \frac{1}{2}, 1, A(\lambda)^2\right]\right]$$

(76a)
Figure 12. Azimuth-independent anisotropic kappa distribution of energy $E$ and pitch angle $\vartheta$, in an arbitrary reference frame, in the case of heterogeneous correlations, Equation 77(a), i.e., $\kappa^{\text{int}} \to \infty$, plotted for $\vartheta_b = 0^\circ$, $V_b = 5000 \text{ km s}^{-1}$ ($E_b \sim 70 \text{ eV}$), $T = 10^7 \text{ K}$ ($\sim 0.86 \text{ keV}$), and various anisotropies $\alpha$ and kappa indices $\kappa$.

The corresponding theoretical uncertainty, caused by substituting the azimuth with its expectation value, is now given by

$$
\delta P(E, \vartheta; \kappa^{\text{int}} \to \infty) = \pi \left( \kappa - \frac{1}{2} \right) \left( \frac{2k_B T}{m} \right) \frac{1}{\Gamma \left( \frac{\kappa - 1}{2} \right)} \alpha^2 \left( \frac{1 + 2\alpha}{3\alpha} \right)^{1-\kappa} 
\times \left[ 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1 + 2\alpha}{3k_B T} \cdot (\sqrt{E} \cos \vartheta - \sqrt{E_b} \cos \vartheta_b)^2 \right]^{-\kappa} 
\times \left[ \frac{\sin \vartheta \sin \vartheta_b}{(\kappa - \frac{1}{2})k_B T A(\lambda)} \right]^{-\kappa - \frac{1}{2}} 
\times \sqrt{\frac{\pi}{2}} F_1[2\kappa + 1, 2\kappa + 1, 1, A(\lambda^2)] - 2 F_1[\kappa + \frac{1}{2}, \kappa + \frac{1}{2}, 1, A(\lambda^2)]^2,
$$

(76b)

where we have now set $A = A[\lambda(E, \vartheta)]$ as:

$$
A(\chi) \equiv x - \sqrt{x^2 - 1},
$$

(76c)

$$
\lambda(E, \vartheta) \equiv \frac{(\kappa - \frac{1}{2})k_B T \frac{1}{\alpha} + E \sin^2 \vartheta + E_b \sin^2 \vartheta_b}{2 \sqrt{E_b \sin \vartheta \sin \vartheta_b}}.
$$

(76d)
(iii) Arbitrary heterogeneous correlation, \( \kappa_{\text{int}} < \infty \):

In the general case corresponding to Equation (73b), the azimuth-independent kappa distribution of energy \( E \) and pitch angle \( \vartheta \) and the corresponding theoretical uncertainty are respectively given by:

\[
P(E, \vartheta; \kappa_{\text{int}}) = C(\kappa_{\text{int}}, \kappa) \cdot \alpha^2 \left[ \frac{3\alpha}{1 + 2\alpha} \left( \frac{2k_BT}{m} \right) \right] \frac{1}{\pi} \cdot \pi^{-1} \\
\int_{0}^{\pi} \left\{ -1 + \left[ 1 + \frac{1}{\kappa - \frac{1}{\alpha^2}} \cdot \left( \sqrt{E \cos \vartheta - \sqrt{E_b \cos \vartheta_b} \, 2} \right) \right] \right\} \frac{\sin \vartheta}{E_b} \sin \vartheta_b \cos (X) \right] dX, \quad (77a)
\]

\[
\int_{0}^{\pi} \left\{ -1 + \left[ 1 + \frac{1}{\kappa - \frac{1}{\alpha^2}} \cdot \left( \sqrt{E \cos \vartheta - \sqrt{E_b \cos \vartheta_b} \, 2} \right) \right] \right\} \frac{\sin \vartheta}{E_b} \sin \vartheta_b \cos (X) \right] dX, \quad (77a)
\]

\[
\int_{0}^{\pi} \left\{ -1 + \left[ 1 + \frac{1}{\kappa - \frac{1}{\alpha^2}} \cdot \left( \sqrt{E \cos \vartheta - \sqrt{E_b \cos \vartheta_b} \, 2} \right) \right] \right\} \frac{\sin \vartheta}{E_b} \sin \vartheta_b \cos (X) \right] dX, \quad (77a)
\]
Figure 14. Azimuth-independent anisotropic kappa distribution of energy $E$ and pitch angle $\vartheta$, in an arbitrary reference frame, in the case of heterogeneous correlations with $\kappa^{\text{int}} = 1.5$, Equation 77(a), plotted for $\vartheta_b = 0^\circ$, $u_b = 5000$ km s$^{-1}$ ($E_b \sim 70$ eV), $T = 10^7$ K ($\sim 0.86$ keV), and various anisotropies $\alpha$ and kappa indices $\kappa$.

and

$$
\delta P(E, \vartheta; \kappa^{\text{int}}) = C(\kappa^{\text{int}}, \kappa) \cdot \alpha^{\frac{3}{2}} \left[ \frac{3 \alpha}{1+2 \alpha} \left( \frac{2kT}{m} \right) \right]^{\frac{3}{2}} \cdot \pi^{-1}
$$

$$
\times \left\{ \begin{array}{l}
\int_0^\pi \left\{ -1 + \left[ 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1+2 \alpha}{3kT} \cdot (\sqrt{E \cos \vartheta - \sqrt{E_b \cos \vartheta_b}}) \right]^\frac{3}{2} \\
+ \left\{ 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1+2 \alpha}{3kT} \cdot [E \sin^2 \vartheta + E_b \sin^2 \vartheta_b - 2\sqrt{E E_b \sin \vartheta \sin \vartheta_b \cos(X)}] \right\}^{\frac{3}{2}} \right\}^{2} \cdot dX
\end{array} \right\}^{\frac{1}{2}}
$$

$$
\times \left\{ \begin{array}{l}
\int_0^\pi \left\{ -1 + \left[ 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1+2 \alpha}{3kT} \cdot (\sqrt{E \cos \vartheta - \sqrt{E_b \cos \vartheta_b}}) \right]^\frac{3}{2} \\
+ \left\{ 1 + \frac{1}{\kappa - \frac{1}{2}} \cdot \frac{1+2 \alpha}{3kT} \cdot [E \sin^2 \vartheta + E_b \sin^2 \vartheta_b - 2\sqrt{E E_b \sin \vartheta \sin \vartheta_b \cos(X)}] \right\}^{\frac{3}{2}} \right\}^{2} \cdot dX
\end{array} \right\}^{\frac{1}{2}}
$$

(77b)
Figures 10 and 11 plot the azimuth-independent kappa distribution of energy $E$ and pitch angle $\vartheta$, for both the cases of $\vartheta_b = 0^\circ$ and $\vartheta_b = 90^\circ$, respectively, for $\kappa_{\text{int}} = \kappa$, and for various values of the kappa index and anisotropy; (the case $\vartheta_b = 180^\circ$ is symmetric to case of $\vartheta_b = 0^\circ$ and is ignored). Similar are the plots in Figures 12 and 13, for $\kappa_{\text{int}} \to \infty$, and the plots in Figures 14 and 15, for for $\kappa_{\text{int}} = 1.5$. We observe that the panels in Figures 10–15 are smoother compared to those in Figures 7–9, because the former figures are plotted after the integration of $\varphi$, while the latter figures are plotted for specific values of $\varphi$.

7. Conclusions

In this paper we developed the theoretical basis for the connection of correlations among particles’ velocity components with the variety of types of anisotropic kappa distributions. The correlation coefficient between two particle kinetic energies is one-to-one connected to the kappa index, the parameter that labels and governs the kappa distributions. The kappa index and the temperature constitute two intensive parameters characterizing the thermodynamics of the system. Similar to the kinetic definition of temperature, stated by the mean particle kinetic energy per half dof, the kinetic definition of kappa is stated by the correlation coefficient (that is, the normalized covariance) of the particle kinetic energy per half dof.

We derived the correlation coefficient of anisotropic kappa distributions and showed how this is connected to the effective dimensionality. In particular, we were able to determine the effective dimensionality $d_{\text{eff}}$ and express it as a function of the anisotropy $\alpha$, satisfying the expected limiting behavior of $d_{\text{eff}} = 1$ for $\alpha = 0$, $d_{\text{eff}} = 3$ for $\alpha = 1$, and $d_{\text{eff}} = 2$ for $\alpha \to \infty$. Then, it was straightforward to connect the adiabatic polytropic index $\gamma$ to the anisotropy $\alpha$.

Having established the importance of correlation among particles in the formulation of anisotropic kappa distributions, we proceed and show how the variety of anisotropic formulations can be systematically defined and determined through the concept of correlations. The generalization of the anisotropic kappa distributions was developed by considering the appropriate homogeneous/heterogeneous correlations among the particles’ velocity components. The derived distributions mediate the main two types of anisotropic kappa distributions, where the first considers equal correlations among particles’ velocity components...
components, while the second considers zero correlation among different velocity components.

The generalization of the anisotropic kappa distributions is achieved within the framework of nonextensive statistical mechanics and its connection to the statistics of kappa distributions. There is a certain functional form that characterizes the partition of 2D joint kappa distribution into its two marginal 1D kappa distributions. This relationship leads to a statistical correlation, generalizing the relationship that leads to the statistical independence of Maxwell–Boltzmann distributions. We developed and examined the anisotropic distributions in the comoving reference frame but also in an arbitrary S/C frame, expressed in terms of the energy, pitch angle, and azimuth, including also the case where the azimuth is not measured, thus the distribution is averaged over the azimuth. The distributions have a single maximum in the comoving frame, but more complicated graphs with several peaks may characterize the same distributions in the arbitrary frame expressed in terms of energy and pitch angle.

In summary, the paper:

(i) determined the correlation coefficient and the involved effective dimensionality of anisotropic distributions with homogeneous/heterogeneous correlations;
(ii) indicated the connection of the adiabatic polytropic index with the anisotropy;
(iii) characterized and studied the homogeneous/heterogeneous correlations among the particles velocity components;
(iv) formulated the correlation relationship that characterizes the partition of 2D joint kappa distribution into the two marginal 1D kappa distributions, as emerges from nonextensive statistical mechanics;
(v) generalized the formulae of anisotropic distributions, based on the types of homogeneous/heterogeneous correlations; and
(vi) described and examined the anisotropic kappa distributions in (i) the comoving reference frame with respect to the velocity components, (ii) arbitrary S/C frame with respect to the triplet of energy, pitch angle, and azimuth, and (iii) the more complicate form of azimuth-dependent distributions with respect to energy and pitch angle.

Having developed the possible anisotropic kappa distributions for any reference frame, it is straightforward to apply this well-grounded “toolbox” for future reference in data analyses of space plasma populations such as electrons with Jovian Auroral Distributions Experiment (JADE) on board the Juno Mission at Jupiter (Allegrini et al. 2017, 2020a, 2020b; McComas et al. 2017). For instance, JADE-E measurements provide the electron distribution with respect to energy and pitch angle, ignoring the azimuth values, thus the appropriate theoretical distributions to describe the collected data sets are the azimuth-independent distributions with respect to energy and pitch angle, shown in Section 6.2.

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Appendix

Here we derive the normalization constant $C$ of the distribution

$$
C \cdot \Theta_{\perp}^{-2} \Theta_{\parallel}^{-1} \cdot \int_{0}^{\infty} \int_{-\infty}^{\infty} \left\{ -1 + \left( 1 + \frac{1}{\kappa - \frac{1}{\kappa}} \cdot \frac{u_{\perp}^{\kappa - \frac{1}{2}}}{\Theta_{\perp}^{\kappa - 1}} \right) \right. \\
+ \left. \left( 1 + \frac{1}{\kappa - \frac{1}{\kappa}} \cdot \frac{u_{\parallel}^{\kappa - \frac{1}{2}}}{\Theta_{\parallel}^{\kappa - 1}} \right) \right\}^{\kappa^{-1} - 1} \cdot du_\parallel 2\pi u_\parallel du_\parallel = 1,
$$

(A1)

from the normalization

$$
\int_{0}^{\infty} \int_{-\infty}^{\infty} P(u_\parallel; u_\perp; \Theta_\parallel, \Theta_\perp, \kappa) du_\parallel 2\pi u_\parallel du_\parallel = 1.
$$

(A2)

Setting $x_{\perp}^{2} = u_{\perp}^{2}/(1 - \frac{3}{2}\Theta_{\perp}^{2})$, $x_{\parallel}^{2} = u_{\parallel}^{2}/(1 - \frac{3}{2}\Theta_{\parallel}^{2})$, we find

$$
C^{-1} = 4\pi \left( \kappa - \frac{1}{\kappa} \right)^{\frac{1}{2}} \cdot \int_{0}^{\infty} \int_{0}^{\infty} \left[ -1 + (1 + x_{\perp}^{2})^{\kappa - \frac{1}{2}} \right. \\
+ \left. (1 + x_{\parallel}^{2})^{\kappa - \frac{1}{2}} \right]^{-1} \cdot dx_\parallel dx_{\perp}.
$$

(A3)

Next, we set $y_{\perp} = (1 + x_{\perp}^{2})^{\kappa - \frac{1}{2}} - 1$ and $y_{\parallel} = (1 + x_{\parallel}^{2})^{\kappa - \frac{1}{2}} - 1$; thus, we have

$$
x_{\parallel} dx_{\parallel} = \frac{\kappa^{\kappa - \frac{1}{2}}}{2} \cdot \left( 1 + y_{\parallel} \right)^{-\frac{1}{2}} dy_{\parallel},
$$

$$
dx_{\perp} = \frac{\kappa^{\kappa - \frac{1}{2}}}{2} \cdot \left( 1 + y_{\perp} \right)^{-\frac{1}{2}} dy_{\perp}.
$$

and we find

$$
C^{-1} = \pi \left( \kappa - \frac{3}{2} \right)^{\frac{1}{2}} \cdot \frac{\kappa^{\kappa - \frac{1}{2}}}{\kappa + \frac{1}{\kappa}} \\
\times \int_{0}^{\infty} \left\{ \int_{0}^{\infty} \left[ (1 + y_{\perp} + y_{\parallel})^{\kappa^{-1} - 1}(1 + y_{\perp})^{\kappa^{-1} - 1} \right] \cdot dy_{\perp} \right\} \\
\times \left[ (1 + y_{\parallel})^{\kappa^{-1} - 1} \right]^{-\frac{1}{2}} dy_{\parallel}.
$$

(A4)

The integration over $y_{\perp}$ leads to the Gauss hypergeometric function (Gradshteyn & Ryzhik 1965, p. 286), i.e.,

$$
\int_{0}^{\infty} \left[ (1 + y_{\perp} + y_{\parallel})^{\kappa^{-1} - 1}(1 + y_{\perp})^{\kappa^{-1} - 1} \right] \cdot dy_{\perp} = \frac{1}{\kappa + 1} \cdot \left( 1 + y_{\parallel} \right)^{-\kappa^{-1} - 1} \\
\times \frac{\kappa^{\kappa + \frac{1}{2}}}{\kappa + 1} \cdot \left( 1 + y_{\parallel} \right)^{-\frac{\kappa^{\kappa + \frac{1}{2}}}{\kappa + 1}} \cdot \left( 1 + y_{\parallel} \right)^{-\frac{\kappa^{\kappa + \frac{1}{2}}}{\kappa + 1}}.
$$

(A5)
Hence,

\[
C^{-1} = \pi \left( \kappa - \frac{3}{2} \right)^{1/2} \cdot \frac{\kappa^{m} \cdot \kappa^{\nu + 1/2}}{\kappa^{m}(\kappa - 1/2) + \kappa} \times \int_{0}^{\infty} (1 + y)^{\nu} - \kappa^{-2}[(1 + y)^{\nu} - 1]^{1/2} \times \frac{y_{1}}{1 + y_{1}} dy_{1}
\]  

(A6)

or

\[
C^{-1} = \pi \left( \kappa - \frac{3}{2} \right)^{1/2} \cdot \frac{\kappa^{m} \cdot \kappa^{\nu + 1/2}}{\kappa^{m}(\kappa - 1/2) + \kappa} \cdot I(\kappa, \kappa^{\nu}).
\]  

(A7a)

with

\[
I(\kappa, \kappa^{\nu}) = \int_{0}^{\infty} (1 + x)^{-2}[(1 + x)^{\nu} - 1]^{1/2} \times \frac{y_{1}}{1 + y_{1}} dy_{1}
\]  

(A7b)

Then, setting \( u \equiv (1 + y)^{-1} \), Equation A7(b) becomes

\[
I(\kappa, \kappa^{\nu}) = \int_{1}^{\infty} u^{\nu}(1 - u^{-1})^{1/2} \times \frac{y_{1}}{1 + y_{1}} dy_{1}
\]  

(8)

Also, we expand \((1 - x)^{1/2}\) in terms of \(x = u^{\nu}/\kappa\), thus,

\[
I(\kappa, \kappa^{\nu}) = \sum_{m=0}^{\infty} \frac{(2m)!}{4^{m}(m!)^{2}} \cdot a_{m}(\kappa, \kappa^{\nu}),
\]  

with

(A9a)

\[
a_{m}(\kappa, \kappa^{\nu}) = \int_{0}^{1} u^{\nu + 1/2}(m - 1/2) \cdot \frac{y_{1}}{1 + y_{1}} dy_{1}
\]  

(A9b)

because

\[
(1 - x)^{1/2} = \sum_{m=0}^{\infty} \left( \frac{1}{2} \right) \frac{(-1)^{m} x^{m}}{m!}, \text{ for } 0 < x < 1, \text{ and,}
\]  

(A10a)

\[
\left( \frac{1}{2} \right) = \frac{(-1)^{m}(2m)!}{4^{m}(m!)^{2}}.
\]  

(A10b)

Hence, we find

\[
a_{m}(\kappa, \kappa^{\nu}) = \int_{0}^{1} (1 - u)^{\nu + 1/2}(m - 1/2) dy_{1}
\]  

\[
\times \frac{y_{1}}{1 + y_{1}} dy_{1}
\]  

\[
= \left[ \kappa^{m} + 1 + \frac{\kappa^{m}}{\kappa}(m - 1/2) \right]^{-1} \cdot \frac{y_{1}}{1 + y_{1}} dy_{1}
\]  

(11)

Finally, we arrive at

\[
I(\kappa, \kappa^{\nu}) = \sum_{m=0}^{\infty} \frac{(2m)!}{4^{m}(m!)^{2}} \cdot \frac{y_{1}}{1 + y_{1}} dy_{1}
\]  

(12)

\[
\times \frac{y_{1}}{1 + y_{1}} dy_{1}
\]  

(13a)

or

\[
C = \pi^{1} \cdot \frac{\kappa^{m}(\kappa - 1/2) + \kappa}{\kappa^{m}(\kappa - 1/2) + \kappa} \cdot \left[ \kappa^{m} + 1 + \frac{\kappa^{m}}{\kappa}(m - 1/2) \right]^{-1}
\]  

(A13a)

\[
\times \sum_{m=0}^{\infty} \frac{(2m)!}{4^{m}(m!)^{2}} \cdot \frac{y_{1}}{1 + y_{1}} dy_{1}
\]  

(13b)

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