New exact wave solutions to the space-time fractional coupled Burgers equations and the space-time fractional foam drainage equation

M. Nurul Islam and M. Ali Akbar*

Abstract: The space-time fractional-coupled Burgers equations and the space-time fractional foam drainage equation are important as an electro-hydro-dynamical model to progress the local electric field and ion acoustic waves in plasma, the shallow water wave problems, and also fluid flow of liquid through foam arisen by gravity and capillarity. In this article, we determine new and further general exact wave solutions to the above-mentioned space-time fractional equations using the generalized \((G'/G)-\)expansion method with the assistance of the fractional complex transformation. It is shown that the method is further effective, convenient, and can be used to establish new solutions for other kind non-linear fractional differential equations arising in mathematical physics. Finally, we depict the 3D and 2D figures of the obtained wave solutions in order to interpret them in geometrical sense.

Subjects: Non-Linear Systems; Mathematical Physics; Theoretical Physics

Keywords: generalized \((G'/G)-\)expansion method; fractional differential equation in mathematical physics; non-linear evolution equations; coupled Burgers equations; foam drainage equation; traveling wave transformation

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PUBLIC INTEREST STATEMENT

Fractional differential equations have gained much importance and popularity to the researchers. In order for better understanding the complex phenomena, exact solutions play a vital role. It has recently proved to be a valuable tool to the modeling in various fields of mathematical physics, biology, population dynamics, engineering, fluid-dynamic traffic model, etc. The results obtained from the fractional system are of a more general nature. In the present article, we use the generalized \((G'/G)-\)expansion method to investigate closed form wave solutions of the space-time fractional-coupled Burgers equations and the space-time fractional foam drainage equation. Consequently, we obtain abundant closed form wave solutions of these two equations among them some are new solutions. We expect that the new closed form solutions will be helpful to explain the associated phenomena. Therefore, diverse group of researchers developed and extended different methods for investigating closed form solutions to NLEEs.
1. Introduction

In recent years, non-linear fractional differential equations (NFDEs) have been attracted much interest and it has gained popularity among the researchers. On the other hand, traveling wave solutions of fractional non-linear evolution equations (NLEEs) plays a fundamental and imperative role in mathematical physics, applied sciences and engineering. NFDEs appear in various scientific and engineering fields, such as fluid mechanics, plasma physic, the modeling of earthquake, electricity, quantum mechanics, wave water mechanics, chemical kinematics, optical fibers, biology. There are different types of definitions of fractional derivative such as Jumarie’s modified Riemann–Liouville (Jumarie, 2006) derivative, Caputo (Caputo & Fabrizio, 2015) derivative are in common. In order to know the inner infrastructure of complex tangible events, exact solutions of NFDEs are very much important. Therefore, at present some useful methods have been established and enhanced for obtaining exact solution to the fractional evolution equations such as, the generalized \((G'/G)\)-expansion method (Akgul, Kilicman, & Inc, 2013; Alam & Akbar, 2014a, 2014b; Bekir & Guner, 2013; Bekir & Güner, 2014; Younis & Zafar, 2014), the modified simple equation method (MSE) (Kaplan, Bekir, Akbulut, & Aksoy, 2015; Zayed, Amer, & Al-Nowehy, 2016), the exp-function method (El-Borai, El-Sayed, & Al-Masry, 2015; Zheng, 2013), the tanh-function method (Ali, 2007; Inc, Kilic, Karatas, Al-Qurashi, & Tchier, 2016), the sub-equation method (Alzaidy, 2013; Guo, Mei, Li, & Sun, 2012; Wang & Xu, 2014; Zhao, Tang, Kumar, & Hou, 2012), the Adomian decomposition method (El-Sayed, Behiry, & Raslan, 2010), the ansatz method (Guner & Bekir, 2017; Inc, Kilic, Kılıç, & Akgül, 2017), the Jacobi elliptic method (Zheng & Feng, 2014), the differential transformation method (Moman, Odibat, & Erturk, 2007; Rabat, Erturk, & Moman, 2010), the finite element method (Deng, 2009), the variational iteration method (Neamaty, Agheli, & Darzi, 2015), the Backlund transformation of fractional Riccati equation method (Lu, 2012a), the first integral method (Lu, 2012b; Younis, 2013), the reproducing kernel method (Akgul, Inc, Karatas, & Baleanu, 2015; Akgül, Khan, Akgül, Baleanu, & Al Quran, 2017; Ali, Inc, & Karatas, 2015), the MQL-GPS method (Hashemi, Inc, Karatas, & Akgül, 2017).

In recent literature, the space and time fractional-coupled Burgers equations (Kurudy, 2010) and the space and time fractional foam drainage equation (Hutzler, Weaire, Saugey, Cox, & Peron, 2014) is investigated through the modified trial equation method (Bulut, Baskonus, & Pandir, 2013), the Adomian decomposition method (Dahmani, Mesmoudi, & Bebbouchi, 2008), the tanh-function method (Helal & Mehanna, 2007), the variation iteration method (Inc, 2008), the exp-function method (Khani, Hamedi-Nezhad, Darvishi, & Ryu, 2009), the homotopy perturbation method (Fereidoon, Yaghoobi, & Davoudabadi, 2011), the He’s projected differential transform method (Elzaki, 2015), the modified Kudryashov method (Ege & Misirli, 2014) etc. To the best of our knowledge, the space-time fractional-coupled Burgers equations and the space-time fractional foam drainage equation have not been investigated by means of the generalized \((G'/G)\)-expansion method. Therefore, our object is to examine the further general and new exact wave solutions to the space-time fractional-coupled Burgers equations and the space-time fractional foam drainage equation by making use of the generalized \((G'/G)\)-expansion method and study the effect of the included emerging parameters to the obtained solutions and the attained solutions might be useful and realistic to analyze for both the evolution equations. The generalized \((G'/G)\)-expansion method is a recently developed efficient, potential and rising method to search new wave solutions to the non-linear fractional equations. Its finding results are straightforward, more general, efficient, useful, and no need to use the symbolic computation software to manipulate the algebraic equations.

The rest of the article is patterned as follows: In Section 2, we explain the Jumarie’s modified Riemann–Liouville derivative. In Section 3, we describe the outline of the generalized \((G'/G)\)-expansion method. In Section 4, we search new and further general solutions to the time fractional evolution equations mentioned above. In Section 5, we discuss the results and discussion. In Section 6, we describe the physical explanation and show graphically representation and in Section 7, we draw our conclusions.
2. Modified Riemann–Liouville derivative

The Jumarie’s modified Riemann–Liouville derivative of order \( \alpha \) is defined as follows (Jumarie, 2006):

\[
D_\alpha^x f(x) = \begin{cases} 
\frac{1}{(1-\alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{1-\alpha} \left(f(\xi) - f(0)\right) d\xi, & \alpha < 0, \\
\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x - \xi)^{-\alpha} \left(f(\xi) - f(0)\right) d\xi, & 0 < \alpha < 1, \\
\left[f^{n-\alpha}(x)\right]^{(n)}, & n \leq \alpha < n + 1, n \geq 1.
\end{cases}
\]

(2.1)

Some properties for the proposed modified Riemann–Liouville derivative are listed below:

\[
D_\alpha^x x^\gamma = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 - \alpha)} x^{\gamma - \alpha}, \quad \gamma > 0,
\]

(2.2)

\[
D_\alpha^x (f(x)g(x)) = g(x)D_\alpha^x f(x) + f(x)D_\alpha^x g(x),
\]

(2.3)

\[
D_\alpha^x \left[g(x)\right] = f_\alpha^\prime [g(x)] D_\alpha^x g(x),
\]

(2.4)

\[
D_\alpha^x \left[g(x)\right] = D_\alpha^x [g(x)] \left(g'(x)\right)^{\alpha},
\]

(2.5)

**Remark 2.1** In the above formulas, (2.2)–(2.5), and (2.2) is non-differentiable function and the function \( f(x) \) is non-differentiable function in (2.3) and (2.4) and differentiable in (2.5), also \( g(x) \) is non-differentiable in (2.4), and \( f(x) \) is differentiable in (2.4) but non-differentiable in (2.5). In this article, based on the generalized \((G/G)\)-expansion method, we use formulas (2.3) and (2.5) to obtain the solutions of mentioned fractional differential equations. We will derive effective way for solving fractional partial differential equations. It will be shown that the use of the generalized \((G/G)\)-expansion method permit us to obtain new exact close form solutions from the known seed solutions. In (He, 2012; He, Elagan, & Li, 2012), He et al. introduced the fractional complex transform to convert an FDE into its differential partner easily. Hence, above formulae play an important role in fractional calculus and also fractional differential equations.

3. Outline of the method

Let us consider a general non-linear fractional differential equation in the form:

\[
H\left(u, D_\alpha^x u, D_\alpha^{2x} u, D_\alpha^{3x} u, \ldots \right) = 0,
\]

(3.1)

where \( u = u(x, t) \) is wave function, \( H \) is a polynomial in \( u = u(x, t) \) and its fractional derivatives, containing the highest order derivatives and non-linear terms of the highest order, and subscripts denote fractional derivatives. To obtain the solution of Equation (3.1) using the generalized \((G/G)\)-expansion method, we have to execute the subsequent steps:

**Step 1:** Let us consider, \( u(x, t) = \phi(\xi)e^{\eta} \) and the traveling wave variable,

\[
\xi = k \left(x - \frac{ct^a}{\Gamma(1 + a)} \right) \quad \text{and} \quad \eta = mx - \frac{wt^a}{\Gamma(1 + a)},
\]

(3.2)

permits us to transform the Equation (3.1) into the following ordinary differential equation (ODE):

\[
R\left(u, u', u'', \ldots \right) = 0,
\]

(3.3)

where \( R \) is a polynomial in \( u(\xi) \) and its derivatives, wherein \( u'(\xi) = \frac{du}{d\xi} \).
Step 2: According to possibility, Equation (3.3) can be integrated term-by-term one or more times, yields constant(s) of integration. The integral constant can be considered as zero since we look for solitary wave solutions.

Step 3: Assume that the wave solution of Equation (3.3) can be written in the form:

\[ u(\xi) = \sum_{i=0}^{N} a_i (d + M)^i + \sum_{i=1}^{N} b_i (d + M)^{-i}, \]  

(3.4)

where either \(a_i\) or \(b_i\) maybe zero, but both \(a_i\) and \(b_i\) cannot be zero at a time, \(a_i, (i = 0, 1, 2, \ldots N)\) and \(b_i, (i = 1, 2, \ldots N)\) and \(d\) are arbitrary constants to be evaluated and \(M(\xi)\) is given by:

\[ M(\xi) = (G'/G) \]  

(3.5)

where \(G = G(\xi)\) satisfies the following auxiliary non-linear ordinary differential equation:

\[ \text{AGG}'' - \text{BGG} - \text{EG}^2 - C(G')^2 = 0 \]  

(3.6)

where the prime stands for derivative with respect to \(\xi\); \(A, B, C\) and \(E\) are real parameters.

Step 4: The positive integer \(N\) arises in Equation (3.3) can be determined by balancing the highest order non-linear terms and the derivatives of highest order appearing in Equation (3.3).

Step 5: Inserting Equations (3.4)–(3.6) into Equation (3.3) with the value of \(N\) obtained in Step 4, we attain polynomial in \((d + M)^N (N = 0, 1, 2, \ldots)\) and \((d + M)^{M}(N = 1, 2, 3, \ldots)\). Then, we collect each coefficient of the resulted polynomials to zero gives a set of algebraic equations for \(a_i, (i = 0, 1, 2, \ldots)\), \(b_i, (i = 1, 2, 3 \ldots)\), \(d\) and \(k\).

Step 6: Suppose that the value of the constants \(a_i, (i = 0, 1, 2, \ldots)\), \(b_i, (i = 1, 2, 3 \ldots)\), \(d\) and \(k\) can be found by solving the algebraic equations obtained in Step 5. Since the general solution of Equation (3.6) is well known to us, putting the values of \(a_i, (i = 0, 1, 2, \ldots)\), \(b_i, (i = 1, 2, 3 \ldots)\), \(d\) and \(k\) into Equation (3.4), we attain more general type and new exact traveling wave solutions of the non-linear fractional differential Equation (3.1).

Using the general solution of Equation (3.6), we attain the following solutions of Equation (3.5):

Family 1: When \(B \neq 0, \psi = A - C\) and \(\Omega = B^2 + 4E(A - C) > 0,\)

\[ M(\xi) = (G'/G) = \frac{B}{2\psi} + \frac{\sqrt{\Omega}}{2\psi} \times \frac{r_1 \sinh(\frac{\sqrt{\Omega}}{2A} \xi)}{r_1 \cosh(\frac{\sqrt{\Omega}}{2A} \xi)} \]  

(3.7)

Family 2: When \(B \neq 0, \psi = A - C\) and \(\Omega = B^2 + 4E(A - C) < 0,\)

\[ M(\xi) = (G'/G) = \frac{B}{2\psi} + \frac{\sqrt{-\Omega}}{2\psi} \times \frac{-r_1 \sin(\frac{\sqrt{-\Omega}}{2A} \xi)}{r_1 \cos(\frac{\sqrt{-\Omega}}{2A} \xi)} \]  

(3.8)

Family 3: When \(B \neq 0, \psi = A - C\) and \(\Omega = B^2 + 4E(A - C) = 0,\)

\[ M(\xi) = (G'/G) = \frac{B}{2\psi} \]
4. Formulation of the solutions

In this section, we will determine the new and further general useful solutions to the space-time fractional-coupled Burgers equations and the space-time fractional foam drainage equation.

4.1. The coupled burgers equations

In this sub-section, we determine some new close form traveling wave solutions to the space-time fractional-coupled Burgers equations by making use of the generalized \((G'/G)\)-expansion method. Let us consider the space-time fractional-coupled Burgers equations of the form:

\[
D_{\alpha}^{\alpha}u - D_{\alpha}^{\alpha}u + 2uD_{\alpha}^{\alpha}u + pD_{\alpha}^{\alpha}(uv) = 0
\]

\[
D_{\alpha}^{\alpha}v - D_{\alpha}^{\alpha}v + 2vD_{\alpha}^{\alpha}v + qD_{\alpha}^{\alpha}(uv) = 0; \quad 0 < 0 \leq 1,
\]

where \(\alpha, p,\) and \(q\) are emerging parameters. The fractional-coupled Burgers equations are mathematical equations that describe the variation overtime of a physical structure on the fractional fluid mechanics system, ion acoustic waves through a gas-filled pipe, model of turbulent motion and certain steady-state viscous fluid and waves in bubbly liquids. It is also an important equation in financial mathematics. By means of the traveling transformation (3.2), Equation (4.1.1) is converted into the following non-linear ODE:

\[
k^{\prime}u - m^{2}u'' + 2mu' + pmv' + pmuv' = 0
\]

\[
k^{\prime}v - m^{2}v'' + 2mv' + pmv' + pmuv' = 0
\]

Now, balancing the linear term of the highest order and the non-linear term of the highest order occurring in (4.1.2), yields \(N_{1} = N_{2} = 1\). Then the solution of Equation (4.1.2) is the form:

\[
u = e_{1}(d + M)^{-1} + c_{0} + c_{1}(d + M)
\]

\[
u = e_{1}(d + M)^{-1} + c_{0} + c_{1}(d + M)
\]

where \(a_{0}, a_{1}, b_{1}, c_{0}, c_{1}\) and \(e_{1}\) are arbitrary constants to be determined, such that either \(a_{1}\) or \(b_{1}\) maybe zero, but both \(a_{0}\) and \(b_{1}\) cannot be zero at a time and also either \(a_{1}\) or \(b_{1}\) maybe zero, but both \(c_{1}\) and \(e_{1}\) cannot be zero at a time.
Substituting Equation (4.1.3) together with Equations (3.5) and (3.6) into Equation (4.1.2), the left-hand side is converted into polynomials in \((d + M)^N (N = 0, 1, 2, \ldots)\) and \((d + M)^{-N} (N = 1, 2, 3, \ldots)\). We collect each coefficient of these resulted polynomials and setting zero polynomials and setting zero yields a set of simultaneous algebraic equations (for simplicity, the equations are not present here) for \(a_0, a_1, b_0, c_0, c_1, d, k\) and \(v\). Solving these algebraic equations with the help of symbolic computation software, such as Maple, we obtain the following 3 (three) sets of solutions:

**Set-1**

\[
\begin{align*}
    a_0 &= a_0, a_1 = \frac{m\psi(p - 1)}{A(1 - pq)}, b_0 = \frac{(q - 1)a_0}{p - 1}, c_0 = \frac{m\psi(q - 1)}{A(1 - pq)}, d = -\frac{A}{2\psi}, \\
    b_1 &= \frac{m(p - 1)(4E\psi + B^2)}{4A(1 - pq)\psi}, e_1 = \frac{m(q - 1)(4E\psi + B^2)}{4A(1 - pq)\psi}, k = \frac{2a_0m(1 - pq)}{(p - 1)},
\end{align*}
\]

(4.1.4)

**Set-2**

\[
\begin{align*}
    a_0 &= a_0, a_1 = 0, b_0 = \frac{m(p - 1)(d^2\psi + Bd - E)}{A(pq - 1)}, c_0 = \frac{(q - 1)a_0}{p - 1}, c_1 = 0, d = d,
    e_1 &= \frac{m(q - 1)(d^2\psi + Bd - E)}{A(pq - 1)}, k = -\frac{m(p - 1)(m(2d\psi + B)) - 2Ac_0(1 - pq)}{(p - 1)A},
\end{align*}
\]

(4.1.5)

**Set-3**

\[
\begin{align*}
    a_0 &= a_0, a_1 = \frac{m\psi(p - 1)}{A(1 - pq)}; b_1 = 0, c_0 = \frac{(q - 1)a_0}{p - 1}, c_1 = \frac{m\psi(q - 1)}{A(1 - pq)}, \\
    d &= d, e_1 = 0, k = \frac{m(p - 1)(2d\psi + B) + 2A^2c_0(1 - pq)}{(p - 1)A},
\end{align*}
\]

(4.1.6)

where \(\psi = A - C, a_0, A, B, C, E\) and \(d\) are free parameters.

For simplicity, we have discussed only on the solutions Set-1 of the mentioned equations arranged in Equation (4.1.4) as follows and other sets of solutions are omitted here.

When \(B \neq 0, \psi = A - C\) and \(\Omega = B^2 + 4E(A - C) > 0\), substituting the values of the constants arranged in Equation (4.1.4) into Equation (4.1.3) and simplifying, we attained the traveling wave solutions

\[
\begin{align*}
    u(\xi) &= a_0 + \frac{m\psi(p - 1)}{A(1 - pq)} \left( \frac{\sqrt{\Omega} r_1 \sinh\left(\frac{\sqrt{\Omega} \xi}{2A}\right) + r_2 \cosh\left(\frac{\sqrt{\Omega} \xi}{2A}\right)}{2\psi} \right) \left( \frac{\sqrt{\Omega} r_1 \sinh\left(\frac{\sqrt{\Omega} \xi}{2A}\right) + r_2 \cosh\left(\frac{\sqrt{\Omega} \xi}{2A}\right)}{2\psi} \right)^{-1} \\
    &+ \frac{m(p - 1)(4E\psi + B^2)}{4A(1 - pq)\psi} \left( \frac{\sqrt{\Omega} r_1 \sinh\left(\frac{\sqrt{\Omega} \xi}{2A}\right) + r_2 \cosh\left(\frac{\sqrt{\Omega} \xi}{2A}\right)}{2\psi} \right)^{-1},
\end{align*}
\]

(4.1.7)

and
\[ v(\xi) = \frac{(q-1)a_0}{p-1} + \frac{m\psi(q-1)}{A(1-pq)} \left( \frac{\sqrt{\Omega}r_1 \sinh\left(\frac{\sqrt{\Omega} \xi}{2A}\right) + r_2 \cosh\left(\frac{\sqrt{\Omega} \xi}{2A}\right)}{2\psi} \right)^{-1} \]

\[ + \frac{m(q-1)(4\psi + B^2)}{4A(1-pq)} \left( \frac{\sqrt{\Omega}r_1 \sinh\left(\frac{\sqrt{\Omega} \xi}{2A}\right) + r_2 \cosh\left(\frac{\sqrt{\Omega} \xi}{2A}\right)}{2\psi} \right)^{-1} \]

where \( \xi = x - \frac{ct^*}{\Gamma(1+\alpha)} \)

Since \( r_1 \) and \( r_2 \) are integral constants, so we might choose arbitrarily their values. If we choose \( r_1 = 0 \) but \( r_2 \neq 0 \), then the solutions (4.1.7) and (4.1.8) is simplified as:

\[ u_1(x,t) = a_0 + \frac{m\psi(p-1)}{A(1-pq)} \left( \frac{\sqrt{\Omega}r_1 \cosh\left(\frac{\sqrt{\Omega} k}{2A}\right)\left(x - \frac{ct^*}{\Gamma(1+\alpha)}\right)}{2\psi} \right)^{-1} \]

\[ + \frac{m(p-1)(4\psi + B^2)}{4A(1-pq)} \left( \frac{\sqrt{\Omega}r_1 \cosh\left(\frac{\sqrt{\Omega} k}{2A}\right)\left(x - \frac{ct^*}{\Gamma(1+\alpha)}\right)}{2\psi} \right)^{-1} \]

Again if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then the solutions (4.1.7) and (4.1.8) is simplified as:

\[ u_2(x,t) = a_0 + \frac{m\psi(q-1)}{A(1-pq)} \left( \frac{\sqrt{\Omega}r_1 \tanh\left(\frac{\sqrt{\Omega} k}{2A}\right)\left(x - \frac{ct^*}{\Gamma(1+\alpha)}\right)}{2\psi} \right)^{-1} \]

\[ + \frac{m(q-1)(4\psi + B^2)}{4A(1-pq)} \left( \frac{\sqrt{\Omega}r_1 \tanh\left(\frac{\sqrt{\Omega} k}{2A}\right)\left(x - \frac{ct^*}{\Gamma(1+\alpha)}\right)}{2\psi} \right)^{-1} \]

When \( B \neq 0, \psi = A - C and \Omega = B^2 + 4E(A - C) < 0 \), substituting the values of the constants arranged in Equation (4.1.4) into Equation (4.1.3) and if we choose \( r_1 = 0 \) but \( r_2 \neq 0 \) and simplifying, we attained the traveling wave solutions as:

\[ u_3(x,t) = a_0 + \frac{m\psi(p-1)}{A(1-pq)} \left( \frac{\sqrt{\Omega}r_2 \cot\left(\frac{\sqrt{\Omega} k}{2A}\right)\left(x - \frac{ct^*}{\Gamma(1+\alpha)}\right)}{2\psi} \right)^{-1} \]

\[ + \frac{m(p-1)(4\psi + B^2)}{4A(1-pq)} \left( \frac{\sqrt{\Omega}r_2 \cot\left(\frac{\sqrt{\Omega} k}{2A}\right)\left(x - \frac{ct^*}{\Gamma(1+\alpha)}\right)}{2\psi} \right)^{-1} \]
\[ u_4(x,t) = a_0 + \frac{m\psi(p - 1)}{A(1 - pq)} \left( \frac{1}{k} \left( x - \frac{ct^u}{\Gamma(1 + \alpha)} \right) \right) + \frac{m(p - 1)(4\psi + B^2)}{4A(1 - pq)\psi} \left( \frac{1}{k} \left( x - \frac{ct^u}{\Gamma(1 + \alpha)} \right) \right)^{-1} \]

Again if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then we attained the traveling wave solutions as:

\[ u_5(x,t) = a_0 + \frac{m\psi(q - 1)}{A(1 - pq)} \left( \frac{1}{k} \left( x - \frac{ct^v}{\Gamma(1 + \alpha)} \right) \right) + \frac{m(q - 1)(4\psi + B^2)}{4A(1 - pq)\psi} \left( \frac{1}{k} \left( x - \frac{ct^v}{\Gamma(1 + \alpha)} \right) \right)^{-1} \]

When \( B \neq 0, \psi = A - C \) and \( \Omega = B^2 + 4E(A - C) = 0 \), substituting the values of the constants arranged in Equation (4.1.4) into Equation (4.1.3) and if we choose \( r_1 = 0 \) but \( r_2 \neq 0 \) and simplifying, we attained the traveling wave solutions as:

\[ u_2(x,t) = a_0 + \frac{m\psi(p - 1)}{A(1 - pq)} \left( \frac{1}{k} \left( x - \frac{ct^u}{\Gamma(1 + \alpha)} \right) \right)^{-1} \] + \[ \frac{m(p - 1)(4\psi + B^2)}{4A(1 - pq)\psi} \left( \frac{1}{k} \left( x - \frac{ct^u}{\Gamma(1 + \alpha)} \right) \right)^{-1} \]

\[ v_2(x,t) = \frac{(q - 1)a_0}{p - 1} + \frac{m\psi(q - 1)}{A(1 - pq)} \left( \frac{1}{k} \left( x - \frac{ct^v}{\Gamma(1 + \alpha)} \right) \right)^{-1} \] + \[ \frac{m(q - 1)(4\psi + B^2)}{4A(1 - pq)\psi} \left( \frac{1}{k} \left( x - \frac{ct^v}{\Gamma(1 + \alpha)} \right) \right)^{-1} \]

But if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then in this case we get trivial solution, which is not recorded here.

When \( B = 0, \psi = A - C \) and \( \Delta = \psi E > 0 \), substituting the values of the constants arranged in Equation (4.1.4) into Equation (4.1.3) and if we choose \( r_1 = 0 \) but \( r_2 \neq 0 \) and simplifying, we attained the traveling wave solutions as:

\[ u_6(x,t) = a_0 + \frac{m\psi(p - 1)}{A(1 - pq)} \left( \frac{1}{k} \left( x - \frac{ct^u}{\Gamma(1 + \alpha)} \right) \right) + \frac{m(p - 1)(4\psi + B^2)}{4A(1 - pq)\psi} \left( \frac{1}{k} \left( x - \frac{ct^u}{\Gamma(1 + \alpha)} \right) \right)^{-1} \]

\[ v_6(x,t) = \frac{(q - 1)a_0}{p - 1} + \frac{m\psi(q - 1)}{A(1 - pq)} \left( \frac{1}{k} \left( x - \frac{ct^v}{\Gamma(1 + \alpha)} \right) \right) + \frac{m(q - 1)(4\psi + B^2)}{4A(1 - pq)\psi} \left( \frac{1}{k} \left( x - \frac{ct^v}{\Gamma(1 + \alpha)} \right) \right)^{-1} \]
Again if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then we attained the traveling wave solutions as:

\[
\begin{align*}
    u_2(x,t) &= a_0 + \frac{m v(p-1)}{A(1-pq)} \left( -\frac{B}{2\psi} + \sqrt{\Delta} r_1 \tanh\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right) \\
    &\quad + \frac{m(p-1)(4\psi + B^2)}{4A(1-pq)\psi} \left( -\frac{B}{2\psi} + \sqrt{\Delta} r_1 \tanh\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right)^{-1}
\end{align*}
\]

\[
\begin{align*}
    v_2(x,t) &= \frac{(q-1)a_0}{p-1} + \frac{m v(q-1)}{A(1-pq)} \left( -\frac{B}{2\psi} + \sqrt{\Delta} r_1 \tanh\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right) \\
    &\quad + \frac{m(q-1)(4\psi + B^2)}{4A(1-pq)\psi} \left( -\frac{B}{2\psi} + \sqrt{\Delta} r_1 \tanh\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right)^{-1}
\end{align*}
\]

When \( B = 0, \psi = A - C \) and \( \Delta = \psi E < 0 \), substituting the values of the constants arranged in Equation (4.1.4) into Equation (4.1.3) and if we choose \( r_1 = 0 \) but \( r_2 \neq 0 \) and simplifying, we attained the traveling wave solutions as:

\[
\begin{align*}
    u_3(x,t) &= a_0 + \frac{m v(p-1)}{A(1-pq)} \left( -\frac{B}{2\psi} + \sqrt{\Delta} r_2 \cot\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right) \\
    &\quad + \frac{m(p-1)(4\psi + B^2)}{4A(1-pq)\psi} \left( -\frac{B}{2\psi} + \sqrt{\Delta} r_2 \cot\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right)^{-1}
\end{align*}
\]

\[
\begin{align*}
    v_3(x,t) &= \frac{(q-1)a_0}{p-1} + \frac{m v(q-1)}{A(1-pq)} \left( -\frac{B}{2\psi} + \sqrt{\Delta} r_2 \cot\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right) \\
    &\quad + \frac{m(q-1)(4\psi + B^2)}{4A(1-pq)\psi} \left( -\frac{B}{2\psi} + \sqrt{\Delta} r_2 \cot\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right)^{-1}
\end{align*}
\]

Again if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then we attained the traveling wave solutions as:

\[
\begin{align*}
    u_5(x,t) &= a_0 + \frac{m v(p-1)}{A(1-pq)} \left( -\frac{B}{2\psi} - \frac{\sqrt{\Delta}}{A} r_1 \tan\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right) \\
    &\quad + \frac{m(p-1)(4\psi + B^2)}{4A(1-pq)\psi} \left( -\frac{B}{2\psi} - \frac{\sqrt{\Delta}}{A} r_1 \tan\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right)^{-1}
\end{align*}
\]

\[
\begin{align*}
    v_5(x,t) &= \frac{(q-1)a_0}{p-1} + \frac{m v(q-1)}{A(1-pq)} \left( -\frac{B}{2\psi} - \frac{\sqrt{\Delta}}{A} r_1 \tan\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right) \\
    &\quad + \frac{m(q-1)(4\psi + B^2)}{4A(1-pq)\psi} \left( -\frac{B}{2\psi} - \frac{\sqrt{\Delta}}{A} r_1 \tan\left( \frac{\sqrt{\Delta}}{A} k \left( x - \frac{ct^*}{\Gamma(1+\alpha)} \right) \right) \right)^{-1}
\end{align*}
\]

where \( k = \frac{2a_v m(1-pq)}{(p-1)} \).
foam drainage equation are fresh and further more general and have not been established in the previous solutions. Obtaining solutions are to be convenient to search the demandable model of waves on the variation overtime of a physical structure on the fractional fluid mechanics system, Sallow water waves, model of turbulent motion, and electro-hydro-dynamical model to progress the local electric field and also ion acoustic waves in plasma.

4.2. The foam drainage equation

In this section, we determine some appropriate close form traveling wave solutions to the space-time fractional foam drainage equation by making use of the generalized \((G'/G)\)-expansion method. Let us consider the space-time fractional foam drainage equation of the form:

\[
D_\alpha^w = \frac{1}{2} wD_\alpha^w D_\alpha^w w - 2w^2 D_\alpha^w w + (D_\alpha^w w)^2; \quad 0 < \alpha \leq 1, \tag{4.2.1}
\]

The fractional foam drainage equation is a model of the flow of liquid through the channels and nodes between the bubbles, driven by gravity and capillarity. This equation is also described as the demandable model of waves on shallow water surface, ion acoustic waves in plasma, and the waves on foam maybe explain in terms of a non-linear PDE for the foam density as a function of time and vertical position. It is also an important equation in financial mathematics. By means of the traveling transformation (3.2), Equation (4.2.1) is converted into the following non-linear ODE:

\[
2kw' - m^2 w'' + 4mw^2 w' - 2m^2 (w')^2 = 0, \tag{4.2.2}
\]

Again we assume a new transformation \(w = v^2\), then Equation (4.2.2) becomes into the following non-linear ODE:

\[
2kv^4 v' + 4m^2 (v')^2 - m^2 vv'' + 4mv' = 0, \tag{4.2.3}
\]

Now, balancing the two highest order non-linear terms occurring in (4.2.3), yields \(N = 1\). Then the solution of Equation (4.2.3) is the form:

\[
v = b_1 (d + M)^{-1} + a_0 + a_1(d + M) \tag{4.2.4}
\]

where \(a_0, a_1, b_1, d\) are arbitrary constants to be determined, such that either \(a_1\) or \(b_1\) maybe zero, but both \(a_0\) and \(b_1\) cannot be zero at a time.

Substituting Equation (4.2.4) together with Equations (3.5) and (3.6) into Equation (4.2.3), the left-hand side is converted into polynomial in \((d + M)^N\) \((N = 0, 1, 2, \ldots)\) and \((d + M)^{-N}\) \((N = 1, 2, 3, \ldots)\). We collect each coefficient of these resulted polynomials and setting them zero yields a set of simultaneous algebraic equations (for simplicity, the equations are not present here) for \(a_0, a_1, b_1, d\) and \(k\). Solving these algebraic equations with the help of symbolic computation software, such as, Maple, we obtain the following three sets of solutions:

Set-1

\[
a_0 = 0, a_1 = -\frac{A\psi}{m(4\psi^2 + B^2)} b_1 = -\frac{A}{4m\psi}, d = -\frac{B}{2\psi}, \text{ and } k = -\frac{m^3(4\psi^2 + B^2)}{4A^2}, \tag{4.2.5}
\]

Set-2

\[
a_0 = a_0, a_1 = 0, b_1 = -\frac{(4A^2 - a_0^2 m^2(4\psi^2 + B^2))}{4Am\psi}, d = -\frac{B}{2\psi}, \text{ and } k = -\frac{m^3(4\psi^2 + B^2)}{4A^2}, \tag{4.2.6}
\]
Set-3

\[
\alpha_0 = \frac{2A(2d\psi + B)}{m(4E\psi + B^2)} , \quad \alpha_1 = \frac{4A\psi}{m(4E\psi + B^2)}, \quad b_1 = 0, \quad d = d, \quad \text{and} \quad k = - \frac{m^3(4E\psi + B^2)}{4A^2} \tag{4.2.7}
\]

where \( \psi = A - C, A, B, C, E \) and \( d \) are free parameters.

For simplicity, we have discussed only on the solutions Set-1 of the mentioned equation arranged in Equation (4.2.5) as follows and other sets of solutions are omitted here.

When \( B \neq 0, \psi = A - C \) and \( \Omega = B^2 + 4E(A - C) > 0 \), substituting the values of the constants arranged in Equation (4.2.5) into Equation (4.2.4) and simplifying, we attained the traveling wave solutions as:

\[
v(\xi) = -\frac{A}{4\mu} \left( \frac{\sqrt{\Omega} r_1 \sinh \left(\frac{\sqrt{\Omega} x}{2A}\right) + r_2 \cosh \left(\frac{\sqrt{\Omega} x}{2A}\right)}{2\psi r_1 \cosh \left(\frac{\sqrt{\Omega} x}{2A}\right) + r_2 \sinh \left(\frac{\sqrt{\Omega} x}{2A}\right)} \right)^{-1} \tag{4.2.8}
\]

But when using the result \( w = v^3 \), the solution (4.2.8) attain

\[
w(\xi) = \frac{\left( \frac{\sqrt{\Omega} r_1 \sinh \left(\frac{\sqrt{\Omega} x}{2A}\right) + r_2 \cosh \left(\frac{\sqrt{\Omega} x}{2A}\right)}{2\psi r_1 \cosh \left(\frac{\sqrt{\Omega} x}{2A}\right) + r_2 \sinh \left(\frac{\sqrt{\Omega} x}{2A}\right)} \right)^2 - \frac{A}{4\mu} \left( \frac{\sqrt{\Omega} r_1 \sinh \left(\frac{\sqrt{\Omega} x}{2A}\right) + r_2 \cosh \left(\frac{\sqrt{\Omega} x}{2A}\right)}{2\psi r_1 \cosh \left(\frac{\sqrt{\Omega} x}{2A}\right) + r_2 \sinh \left(\frac{\sqrt{\Omega} x}{2A}\right)} \right)}{4\mu} \tag{4.2.9}
\]

where \( \xi = \left( x - \frac{ct^*}{\sqrt{1 + \alpha}} \right) \).

Since \( r_1 \) and \( r_2 \) are integral constants, so we might choose arbitrarily their values. If we choose \( r_1 = 0 \) but \( r_2 \neq 0 \), then the solutions (4.2.9) is simplified as:

\[
w_1(x, t) = \frac{\left( \frac{\sqrt{\Omega} r_2 \coth \left(\frac{\sqrt{\Omega} k(x - \frac{ct^*}{\sqrt{1 + \alpha}})}{2A}\right)}{2\psi} \right)}{4\mu} \tag{4.2.10}
\]

Again if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then we attained the traveling wave solutions as:

\[
w_2(x, t) = \frac{\left( \frac{\sqrt{\Omega} r_1 \tanh \left(\frac{\sqrt{\Omega} k(x - \frac{ct^*}{\sqrt{1 + \alpha}})}{2A}\right)}{2\psi} \right)}{4\mu} \tag{4.2.11}
\]

When \( B \neq 0, \psi = A - C \) and \( \Omega = B^2 + 4E(A - C) < 0 \), substituting the values of the constants arranged in Equation (4.2.5) into Equation (4.2.4) and using the result \( w = v^3 \) and also if we choose \( r_1 = 0 \) but \( r_2 \neq 0 \) and simplifying, we attained the traveling wave solutions as:
Again if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then we attained the traveling wave solutions as:

\[
W_3(x,t) = \frac{\left( \frac{\sqrt{\Delta}}{2\psi} r_2 \cot \left( \frac{\sqrt{\Delta}}{2\psi} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)}{\frac{A\psi}{m(4E^2+B^2)}} \left( \frac{\sqrt{\Delta}}{2\psi} r_2 \cot \left( \frac{\sqrt{\Delta}}{2\psi} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)^2 - \frac{A}{4m\psi}
\]

When \( B \neq 0, \psi = A - C \) and \( \Omega = B^2 + 4E(A - C) = 0 \), substituting the values of the constants arranged in Equation (4.2.5) into Equation (4.2.4) and using the result \( w = v^{-1} \) and also if we choose \( r_1 = 0 \) but \( r_2 \neq 0 \) and simplifying, we attained the traveling wave solutions as:

\[
W_5(\xi) = \frac{\frac{1}{k} \left( x - \frac{c^*}{\Gamma(1+\omega)} \right)^{-1}}{\frac{A\psi}{m(4E^2+B^2)}} \left( \frac{\sqrt{\Delta}}{2\psi} r_2 \cot \left( \frac{\sqrt{\Delta}}{2\psi} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)^2 - \frac{A}{4m\psi}
\]

But if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then in this case we get trivial solution, which is not recorded here.

When \( B = 0, \psi = A - C \) and \( \Delta = \psi E > 0 \), substituting the values of the constants arranged in Equation (4.2.5) into Equation (4.2.4) and using the result \( w = v^{-1} \) and also if we choose \( r_1 = 0 \) but \( r_2 \neq 0 \) and simplifying, we attained the traveling wave solutions as:

\[
W_6(x,t) = \frac{\left( \frac{B}{2\psi} - \frac{\sqrt{\Delta}}{\psi} r_2 \coth \left( \frac{\sqrt{\Delta}}{A} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)}{\frac{A\psi}{m(4E^2+B^2)}} \left( \frac{B}{2\psi} - \frac{\sqrt{\Delta}}{\psi} r_2 \coth \left( \frac{\sqrt{\Delta}}{A} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)^2 + \frac{A}{4m\psi}
\]

Again if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then we attained the traveling wave solutions as:

\[
W_7(x,t) = \frac{\left( \frac{B}{2\psi} - \frac{\sqrt{\Delta}}{\psi} r_1 \tanh \left( \frac{\sqrt{\Delta}}{A} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)}{\frac{A\psi}{m(4E^2+B^2)}} \left( \frac{B}{2\psi} - \frac{\sqrt{\Delta}}{\psi} r_1 \tanh \left( \frac{\sqrt{\Delta}}{A} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)^2 + \frac{A}{4m\psi}
\]

When \( B = 0, \psi = A - C \) and \( \Delta = \psi E < 0 \), substituting the values of the constants arranged in Equation (4.2.5) into Equation (4.2.4) and using the result \( w = v^{-1} \) and also if we choose \( r_1 = 0 \) but \( r_2 \neq 0 \) and simplifying, we attained the traveling wave solutions as:

\[
W_8(x,t) = \frac{\left( \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{\psi} r_2 \cot \left( \frac{\sqrt{\Delta}}{A} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)}{\frac{A\psi}{m(4E^2+B^2)}} \left( \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{\psi} r_2 \cot \left( \frac{\sqrt{\Delta}}{A} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)^2 + \frac{A}{4m\psi}
\]

Again if we choose \( r_2 = 0 \) but \( r_1 \neq 0 \), then we attained the traveling wave solutions as:

\[
W_9(x,t) = \frac{\left( \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{\psi} r_1 \tan \left( \frac{\sqrt{\Delta}}{A} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)}{\frac{A\psi}{m(4E^2+B^2)}} \left( \frac{B}{2\psi} + \frac{\sqrt{\Delta}}{\psi} r_1 \tan \left( \frac{\sqrt{\Delta}}{A} k(x - \frac{c^*}{\Gamma(1+\omega)}) \right) \right)^2 + \frac{A}{4m\psi}
\]
where \( k = -\frac{m^3(4E_\text{u} + B^2)}{4A^2} \).

It is remarkable to see that the traveling wave solution \( w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8, \) and \( w_9 \) of the space-time fractional-coupled Burgers equations and space-time fractional foam drainage equation are fresh and further more general and have not been established in the previous solutions. Obtaining solutions arise to be convenient to search the demandable model of waves on shallow water surface, the waves of flow of liquid through the channels and nodes between the bubbles, driven by gravity and capillarity, ion acoustic waves in plasma, and the waves on foam density as a function of time and vertical position.

5. Results and discussion

It is very important to point out that our obtained solutions play a significant role over the already published solutions. We notice that, investigated by He’s projected differential transform method (PDTM) (Hutzler et al., 2014), Elzaki obtained only two wave solutions (see Appendix A), for space-time fractional-coupled Burgers equations. But we have obtained more new wave solutions by used the generalized \((G'/G)-expansion\) method for these equations which have not been reported in the previous literature. Also we have mentioned that investigated by the modified Kudryashov method (Bulut et al., 2013), Ege and Misirli obtained only two wave solutions (see Appendix B), for space-time fractional foam drainage equation. But from our mentioned equations, we have obtained more new wave solutions by used the generalized \((G'/G)-expansion\) method which have not been reported in the previous literature.

Hence, comparing between our solutions and their solutions; we declare that our solutions are more general and huge amount of new exact traveling wave solutions.

6. Physical explanation

In this section, we describe the physical explanation and of the graphically representations of the obtained solutions for 3D and 2D figures by means of symbolic software like Mathematica of the non-linear space-time fractional-coupled Burgers equations and the space-time fractional foam drainage equation as follows:

(1) Solutions \( u_2, u_7, v_2 \) and \( v_7 \) are the kink-type solutions. Figure 1 shows the shape of the exact kink-type solution as follows. The figures of solution \( u_2, u_7, v_2 \) and \( v_7 \) are similar and therefore, the other figures of these solutions are omitted for convenience.

(2) Solutions \( u_1, u_6, v_1 \) and \( v_6 \) are the singular kink solution. Figure 2 shows the shape of the exact singular kink-type solution as follows. The figures of solutions \( u_1, u_6, v_1 \) and \( v_6 \) are similar and therefore, the other figures of these solutions are omitted for convenience.

(3) Solutions \( u_1, u_2, u_6, u_7, v_1, v_2, v_6 \) and \( v_7 \) are the singular kink solution. Figure 3 shows the shape of the exact singular kink-type solution as follows. The figures of solutions \( u_1, u_2, u_6, u_7, v_1, v_2, v_6 \) and \( v_7 \) are similar and therefore the other figures of these solutions are omitted for convenience.

(4) Solutions \( u_3, u_8, v_3 \) and \( v_8 \) are the singular periodic solution. Figure 4 shows the shape of the exact singular periodic-type solution as follows. The figures of solutions \( u_3, u_8, v_3 \) and \( v_8 \) are similar and therefore, the other figures of these solutions are omitted for convenience.

(5) Solutions \( u_4, u_9, v_4 \) and \( v_9 \) are the singular periodic solution. Figure 5 shows the shape of the exact singular periodic-type solution as follows. The figures of solutions \( u_4, u_9, v_4 \) and \( v_9 \) are similar and therefore, the other figures of these solutions are omitted for convenience.

(6) Solutions \( u_5 \) and \( v_5 \) are the singular kink solution. Figure 6 shows the shape of the exact singular kink-type solution as follows. The figures of solutions \( u_5 \) and \( v_5 \) are similar and therefore, the other figures of these solutions are omitted for convenience.
Figure 1. Modulus plot of kink wave shape of $u_2$ when $r_1 = 1.2$, $A = C = E = m = 1$, $\sqrt{\Omega} = 2, \nu = A - C, A = 0, p = 0, q = 1, r_1 = 1.2$, and $-10 \leq x, t \leq 10$.

Figure 2. Modulus plot of singular kink wave shape of $u_1$ when $r_2 = 1.2$, $A = C = E = m = 1$, $\sqrt{\Omega} = 2, \nu = A - C, A = 0, p = 0, q = 1, r_1 = 1.2$, and $-10 \leq x, t \leq 10$.

Figure 3. Modulus plot of singular kink wave shape of $u_1$ when $r_2 = 1.2$, $C = E = m = 1$, $\sqrt{\Omega} = 2, \nu = A - C, A = 0, p = 0, q = 1, r_1 = 1.2$, and $-10 \leq x, t \leq 10$.

Figure 4. Modulus plot of singular periodic wave shape of $v_9$ when $A = 3$, $r_1 = 1.2$, $C = E = m = B = 1$, $\sqrt{-\Delta} = -2, \nu = A - C, A = 0, p = 0, q = 1, r_1 = 1.2$, and $-10 \leq x, t \leq 10$.

Figure 5. Modulus plot of singular periodic wave shape of $v_9$ when $A = 3$, $r_1 = 1.2$, $C = E = m = B = 1$, $\sqrt{-\Delta} = -2, \nu = A - C, A = 0, p = 0, q = 1, r_1 = 1.2$, and $-10 \leq x, t \leq 10$. 
Figure 6. Modulus plot of singular kink wave shape of $u_2$ when $A = 3, C = E = m = B = 1, \psi = A - C, c_0 = 0, p = 1.1, q = 1.5$ and $-10 \leq x, t \leq 10$.

Figure 7. Modulus plot of kink wave shape of $w_7$ when $A = 3, r_1 = 1.2, C = E = m = B = 1, \sqrt{\Omega} \approx 2, \psi = A - C$ and $-10 \leq x, t \leq 10$.

Figure 8. Modulus plot of kink wave shape of $w_1$ when $A = 3, r_2 = 1.2, C = E = m = B = 1, \sqrt{\Delta} = -2, \psi = A - C$ and $-10 \leq x, t \leq 10$.

Figure 9. Modulus plot of singular periodic wave shape of $w_8$ when $A = 3, r_2 = 1.2, C = E = m = B = 1, \sqrt{\Delta} = -2, \psi = A - C$ and $-10 \leq x, t \leq 10$.

Figure 10. Modulus plot of singular kink wave shape of $w_5$ when $A = 3, C = E = m = B = 1, \psi = A - C$ and $-10 \leq x, t \leq 10$. 
(7) Solutions $w_{1}$ and $w_{2}$ are the kink solutions. Figure 7 shows the shape of the exact singular kink-type solution as follows. The figures of solutions $w_{2}$ and $w_{7}$ are similar and therefore, the other figures of these solutions are omitted for convenience.

(8) Solutions $w_{1}$ and $w_{3}$ are the kink solutions. Figure 8 shows the shape of the exact singular kink-type solution as follows. The figures of solutions $w_{4}$ and $w_{8}$ are similar and therefore, the other figures of these solutions are omitted for convenience.

(9) Solutions $w_{5}, w_{6}, w_{8},$ and $w_{10}$ are the singular periodic solutions. Figure 9 shows the shape of the exact singular periodic-type solution as follows. The figures of solutions $w_{5}, w_{6}, w_{8},$ and $w_{10}$ are similar and therefore, the other figures of these solutions are omitted for convenience.

(10) Solution $w_{11}$ is the singular kink-type solution. Figure 10 shows the shape of the singular kink-type solutions as follows.

7. Conclusion
In this article, we have determined new and further general solitary wave solutions to the space-time fractional-coupled Burgers equations and also the space-time fractional foam drainage equation by using the efficient and powerful technique known as the generalized $(G'/G)$-expansion method. The obtained solutions are in more general forms, and many known solutions to these equations are only special cases of them and definite values of the included parameters give diverse known soliton solutions. We also have shown that the generalized $(G'/G)$-expansion method over the He’s projected differential transform method (PDTM) and the modified Kudryashov method, offers more general and huge amount of new exact traveling wave solutions with some free parameters. Finally, we have shown graphically to the sense of interpret of the physical phenomena for both the equations. The established results also have shown that the generalized $(G'/G)$-expansion method is more powerful, effective and can be helped for many other fractional equations to get feasible solutions of the tangible incidents.
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Appendix A

Elzaki’s solutions (investigated by the He’s projected differential transform method (PDTM) (Hutzler et al., 2014)) for space-time fractional-coupled Burgers equation are as follows:

\[ u_1(x,t) = u_2(t,x) = e^{-t}\sin x, \quad \text{(A)} \]

Appendix B

Ege and Misirli’s solutions (investigated by the modified Kudryashov method (Bulut et al., 2013)) for space-time fractional foam drainage equation are as follows:

\[ u_1(x,t) = \frac{1}{2} \left( \frac{1}{1 + \frac{\alpha}{\Gamma(1+\alpha)}} \right), \quad \text{(B.1)} \]

and

\[ u_2(x,t) = \frac{1}{2} \left( \frac{1}{1 - \frac{\alpha}{\Gamma(1+\alpha)}} \right), \quad \text{(B.2)} \]