The Booth Lemniscate Starlikeness Radius for Janowski Starlike Functions

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Abstract
The function $G_{\alpha}(z) = 1 + z/(1 - \alpha z^2)$, $0 \leq \alpha < 1$, maps the open unit disk $\mathbb{D}$ onto the interior of a domain known as the Booth lemniscate. Associated with this function $G_{\alpha}$ is the recently introduced class $BS(\alpha)$ consisting of normalized analytic functions $f$ on $\mathbb{D}$ satisfying the subordination $zf'(z)/f(z) \prec G_{\alpha}(z)$. Of interest is its connection with known classes $\mathcal{M}$ of functions in the sense $g(z) = (1/r)f(rz)$ belongs to $BS(\alpha)$ for some $r$ in $(0, 1)$ and all $f \in \mathcal{M}$. We find the largest radius $r$ for different classes $\mathcal{M}$, particularly when $\mathcal{M}$ is the class of starlike functions of order $\beta$, or the Janowski class of starlike functions. As a primary tool for this purpose, we find the radius of the largest disk contained in $G_{\alpha}(\mathbb{D})$ and centered at a certain point $a \in \mathbb{R}$.

Keywords Starlike functions · Janowski starlike functions · Booth lemniscate · Subordination · Radius of starlikeness

Dedicated to the memory of our dear friend, Prof. M. Ataharul Islam

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1 Introduction

Let $\mathcal{A}$ be the class of functions analytic on the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Further, let $\mathcal{S}$ be its subclass consisting of univalent functions. An analytic function $f$ is subordinate to an analytic function $g$, written $f \prec g$, if $f(z) = g(w(z))$ for some analytic self-map $w : \mathbb{D} \to \mathbb{D}$ with $w(0) = 0$. When the superordinate function $g$ is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. Several important subclasses of $\mathcal{A}$ are defined by $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$, respectively, being subordinate to a function of positive real part. For an analytic function $\phi : \mathbb{D} \to \mathbb{C}$, Ma and Minda [7] gave a unified treatment on growth, distortion, covering and coefficient problems for the two subclasses

$$S^*(\phi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} < \phi(z) \right\}$$

and

$$\mathcal{K}(\phi) := \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} < \phi(z) \right\}.$$

Here, $\phi$ is assumed to be univalent with positive real part, $\phi(\mathbb{D})$ is starlike with respect to $\phi(0) = 1$, symmetric about the real axis and $\phi'(0) > 0$. If $\phi$ has positive real part, then functions in $S^*(\phi)$ and $\mathcal{K}(\phi)$ are starlike and convex, respectively, and thus are univalent. Convolution theorems for some general classes were earlier investigated by Shanmugam [11] under the stronger assumption of convexity imposed on $\phi$. Radius problems have also been investigated but only for special cases of $\phi$.

For $0 \leq \alpha < 1$, let $G_\alpha : \mathbb{D} \to \mathbb{C}$ be the function defined by $G_\alpha(z) = 1 + z/(1 - \alpha z^2)$, and $\mathcal{BS}(\alpha) := S^*(G_\alpha)$. This class was introduced by Kargar et al. [4]. It is worth noting that $\mathcal{BS}(\alpha)$ contains non-univalent functions because $G_\alpha$ is not of positive real part. Functions belonging to the class $\mathcal{BS}(\alpha)$ are called Booth lemniscate starlike functions of order $\alpha$. Other properties of $\mathcal{BS}(\alpha)$ have been studied in [5, 6], while some closely related classes were also studied in [3, 8]. Recently, Cho et al. [1] obtained some subordination and radius results for $\mathcal{BS}(\alpha)$.

When $\phi_{A,B} : \mathbb{D} \to \mathbb{C}$ is $\phi_{A,B}(z) = (1 + Az)/(1 + Bz), -1 \leq B < A \leq 1$, then the class $S^*(\phi_{A,B}) =: S^*[A, B]$ is the well-known class of Janowski starlike functions [2]. In particular, if $0 \leq \beta < 1$, the class $S^*(\beta) := S^*[1 - 2\beta, -1]$ is the class of starlike functions of order $\beta$. The classes $S^* = S^*(0)$ and $\mathcal{K} = \{ f \in \mathcal{A} : zf'(z) \in S^* \}$ are the classical classes of starlike and convex functions. Let $\mathcal{M}$ be a given class of analytic functions in $\mathcal{A}$. To each $f \in \mathcal{M}$, let

$$R_f = \sup \left\{ r : \frac{zf'(z)}{f(z)} \in G_\alpha(\mathbb{D}), \quad |z| \leq r < 1 \right\},$$
and

\[ R_{BS(\alpha)}(\mathcal{M}) = \inf\{R_f : f \in \mathcal{M}\}. \]

The number \( R_{BS(\alpha)}(\mathcal{M}) \) is known as the \( BS(\alpha) \)-radius or the Booth lemniscate starlikeness radius of order \( \alpha \) for the class \( \mathcal{M} \). We shall use these two terms interchangeably. Thus the function \( g(z) = (1/r)f(rz) \) belongs to \( BS(\alpha) \) for every \( r \leq R_{BS(\alpha)}(\mathcal{M}) \).

In this paper, we seek to determine the \( BS(\alpha) \)-radius \( R_{BS(\alpha)}(\mathcal{M}) \) when \( \mathcal{M} \) is the class of starlike functions of order \( \beta \), or \( \mathcal{M} \) is the class of Janowski starlike functions. As a primary tool, we first obtain in Sect. 2, the largest disk contained in \( G_\alpha(D) \) and centered at a given point \( a \), as well as the smallest disk containing \( G_\alpha(D) \) and centered at \( a \). In Sect. 3, this result is applied to determine the Booth lemniscate starlikeness radius of order \( \alpha \) for the class of starlike functions of order \( \beta \). In this section too, the \( BS(\alpha) \)-radius is also determined for the class of convex functions and the class consisting of functions \( f \in \mathcal{A} \) with \( zf'(z)/f(z) \) lying in the half-plane \( \{w : \text{Re } w < \beta\}, 1 < \beta < 4/3 \). In Sect. 4, conditions on \( \mathcal{A} \) and \( \mathcal{B} \) are determined that will ensure the Janowski functions \( f \in S^*[A, B] \) also belong to the class \( BS(\alpha) \). When these conditions are not met, we find the Booth lemniscate starlikeness radius for \( S^*[A, B] \). The Booth lemniscate starlikeness radius is also deduced for other related classes.

## 2 Preliminaries

Let \( D(a; r) := \{z \in \mathbb{C} : |z - a| < r\} \) be the open disk of radius \( r \) centered at \( z = a \). If \( f \in \mathcal{M} \), then \( zf'(z)/f(z) \in D(a_f(r); c_f(r)) \) for \( r \) sufficiently small. Thus the \( BS(\alpha) \)-radius for the class \( \mathcal{M} \) is found by determining the largest disk so that \( D(a_f(r); c_f(r)) \subseteq G_\alpha(D) \). For this purpose, a key objective in this section is to find the radius \( r_a \) of the largest disk \( D(a; r_a) \) contained in \( G_\alpha(D) \) and centered at a given point \( a \). We also find the radius \( R_a \) of the smallest disk \( D(a; R_a) \) containing \( G_\alpha(D) \) centered at \( a \). Since the range of \( zf'(z)/f(z) \) contains the point 1 for any \( f \in \mathcal{A} \), we may assume that the center \( a \) of the disk satisfy the inequality

\[
\frac{1 - 2\alpha}{2 - 2\alpha} < a < \frac{3 - 2\alpha}{2 - 2\alpha}.
\]

This will ensure that the disk \( D(a; r_a) \) contains the point \( w = 1 \).

In [1, Lemma 3.4], Cho et al. found the largest disk centered at \( w = 1 \) contained in \( G_\alpha(D) \) and the smallest disk centered at \( w = 1 \) containing \( G_\alpha(D) \). Specifically, they showed that

\[
D(1; 1/(1 + \alpha)) \subseteq G_\alpha(D) \subseteq D(1; 1/(1 - \alpha)).
\]

This readily follows since

\[
\frac{1}{1 + \alpha} \leq |G_\alpha(e^{it}) - 1| = \frac{1}{|1 - \alpha e^{2it}|} \leq \frac{1}{1 - \alpha}.
\]
Here, we compute the radii of these two disks when the centers are located at an arbitrary point \( a \in \mathbb{R} \cap G_\alpha(\mathbb{D}) \).

**Lemma 2.1** Let \( 0 \leq \alpha < 1 \) and \((1 - 2\alpha)/(2 - 2\alpha) < a < (3 - 2\alpha)/(2 - 2\alpha)\). Then the following inclusions hold:

\[
\mathbb{D}(a; r_a) \subseteq G_\alpha(\mathbb{D}) \subseteq \mathbb{D}(a; R_a),
\]

where \( r_a \) and \( R_a \) are given by

\[
r_a = \begin{cases} 
  a - 1 + \frac{1}{1 - \alpha}, & \frac{1 - 2\alpha}{2 - 2\alpha} < a \leq 1 - \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}, \\
  \sqrt{s(\alpha, a)}, & 1 - \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)} < a < 1 + \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}, \\
  1 - a + \frac{1}{1 - \alpha}, & 1 + \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)} \leq a < \frac{3 - 2\alpha}{2 - 2\alpha},
\end{cases}
\]

and

\[
R_a = \begin{cases} 
  1 - a + \frac{1}{1 - \alpha}, & \frac{1 - 2\alpha}{2 - 2\alpha} < a \leq 1, \\
  a - 1 + \frac{1}{1 - \alpha}, & 1 \leq a < \frac{3 - 2\alpha}{2 - 2\alpha},
\end{cases}
\]

with

\[
s(\alpha, a) = \sqrt{\alpha \left[ \alpha - (1 - a)^2(1 - \alpha^2)^2 \right] + \alpha \left( 1 + 2(1 + \alpha)^2(1 - \alpha)^2 \right) \left( 2\alpha(1 + \alpha)^2 \right)^{-1}, \quad \alpha \neq 0.}
\]

**Proof** As noted earlier, the result for \( a = 1 \) was proved in [1, Lemma 3.4]. If \( \alpha = 0 \), then \( G_0(z) = 1 + z \). In this case, readily \( r_a = a \) and \( R_a = 2 - a \) for \( 1/2 < a \leq 1 \).

Also, for \( 1 \leq a < 3/2 \), it is readily seen that \( r_a = 2 - a \) and \( R_a = a \). Thus, assume next that \( \alpha \neq 0 \) and \( a \neq 1 \). The boundary \( \partial G_\alpha(\mathbb{D}) \) of the image of the unit disk \( \mathbb{D} \) in parametric form is given by

\[
G_\alpha(e^{it}) = 1 + \frac{e^{it}}{1 - \alpha e^{2it}} = 1 + \frac{(1 - \alpha) \cos t + i(1 + \alpha) \sin t}{1 + \alpha^2 - 2\alpha \cos(2t)}.
\]

The result is proved by showing the minimum and maximum distance from the point \((a, 0)\) to the point on the boundary \( \partial G_\alpha(\mathbb{D}) \) are, respectively, \( r_a \) and \( R_a \).

Thus consider the function

\[
H(\cos t) = \left( 1 - a + \frac{(1 - \alpha) \cos t}{1 + \alpha^2 - 2\alpha \cos(2t)} \right)^2 + \left( \frac{(1 + \alpha) \sin t}{1 + \alpha^2 - 2\alpha \cos(2t)} \right)^2
\]

\[
= (1 - a)^2 + \frac{1 + 2(1 - a)(1 - \alpha) \cos(t)}{1 + \alpha^2 - 2\alpha \cos(2t)}, \quad (2.1)
\]
that is,

\[ H(x) = (1 - a)^2 + \frac{1 + 2(1 - a)(1 - \alpha)x}{(1 + \alpha)^2 - 4\alpha x^2}, \quad x = \cos t \in [-1, 1]. \tag{2.2} \]

A computation using (2.2) shows that

\[ H'(x) = \frac{2(1 - a)(1 - \alpha)}{(\alpha + 1)^2 - 4\alpha x^2} + \frac{8\alpha x(2(1 - a)(1 - \alpha)x + 1)}{((\alpha + 1)^2 - 4\alpha x^2)^2} \]

\[ = \frac{8\alpha(1 - \alpha)(1 - a)(x - x_1)(x - x_2)}{((\alpha + 1)^2 - 4\alpha x^2)^2}, \tag{2.3} \]

where \( x_1 \) and \( x_2 \) are the two zeros of \( H'(x) \). These are the zeros of the polynomial

\[ 4\alpha(1 - \alpha)(1 - a)x^2 + 4\alpha x + (1 - \alpha)(1 + \alpha)^2(1 - a) = 0 \tag{2.4} \]

and are given by

\[ x_1 = -\frac{\alpha + \sqrt{\alpha(\alpha - (1 - a)^2)(1 - \alpha^2)^2}}{2\alpha(1 - a)(1 - \alpha)}, \]

and

\[ x_2 = -\frac{\alpha - \sqrt{\alpha(\alpha - (1 - a)^2)(1 - \alpha^2)^2}}{2\alpha(1 - a)(1 - \alpha)}. \]

The zeros \( x_1 \) and \( x_2 \) satisfy \( x_1 x_2 = (1 + \alpha)^2/(4\alpha) \geq 1 \). Further, \( x_1, x_2 \) are real if \( \alpha \) and \( a \) satisfy \( \alpha \geq (1 - a)^2(1 - \alpha^2)^2 \), or equivalently, whenever

\[ |a - 1| \leq \frac{\sqrt{\alpha}}{1 - \alpha^2}. \]

The following notations are introduced to give greater clarity to the proof. Let

\[ a_0 := 1 - \frac{\sqrt{\alpha}}{1 - \alpha^2}, \quad a_1 := 1 - \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}, \]

\[ \tilde{a}_0 := 1 + \frac{\sqrt{\alpha}}{1 - \alpha^2}, \quad \tilde{a}_1 := 1 + \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}. \]

A little computation shows that \( x_1 < -1 \) for \( \alpha_0 < a < 1 \), while \( x_1 > 1 \) for \( 1 < a < \tilde{a}_0 \). Similarly, \( x_2 < 0 \) for \( a < 1 \); indeed, \( x_2 < -1 \) for \( \alpha_0 \leq a < \alpha_1 \), and \(-1 \leq x_2 \leq 0 \) for \( \alpha_1 \leq a < 1 \). Also, \( x_2 > 0 \) for \( a > 1 \); indeed, \( 0 \leq x_2 \leq 1 \) for \( 1 < a \leq \tilde{a}_1 \) and \( x_2 > 1 \) for \( \tilde{a}_1 < a \leq \tilde{a}_0 \). These observations together with (2.3) will be helpful in the following cases.
Case (i). If $\alpha_0 \leq a \leq \alpha_1$, it follows that the function $H$ is increasing and therefore

$$r_a = \sqrt{H(-1)} = a - 1 + \frac{1}{1 - \alpha}, \quad \text{and} \quad R_a = \sqrt{H(1)} = 1 - a + \frac{1}{1 - \alpha}.$$  

Case (ii). If $\alpha_1 < a < 1$, then $H'(x) < 0$ for $x < x_2$, while $H'(x) > 0$ for $x > x_2$. Thus, $x_2$ is a minimum point. Since $a < 1$, the maximum of $H$ occurs at $x = 1$. Therefore,

$$r_a = \sqrt{H(x_2)} \quad \text{and} \quad R_a = \sqrt{H(1)}.$$  

Note that

$$\sqrt{H(x_2)} = \sqrt{\frac{\sqrt{\alpha(\alpha - (1 - a)^2(1 - \alpha^2)^2)} + \alpha(1 + 2(1 + \alpha)^2(1 - a)^2)}{2\alpha(1 + \alpha)}}.$$  

Case (iii). If $1 < a < \tilde{\alpha}_1$, then $H'(x) < 0$ for $x < x_2$, while $H'(x) > 0$ for $x > x_2$. Thus, $x_2$ is a minimum point. Since $a > 1$, the function $H$ attains its maximum at $x = -1$ so that

$$r_a = \sqrt{H(x_2)} \quad \text{and} \quad R_a = \sqrt{H(-1)}.$$  

Case (iv). If $\tilde{\alpha}_1 \leq a \leq \tilde{\alpha}_0$, then the function $H$ is decreasing, whence

$$r_a = \sqrt{H(1)} \quad \text{and} \quad R_a = \sqrt{H(-1)}.$$  

It remains next to consider the range

$$|a - 1| > \sqrt{\alpha}/(1 - \alpha^2).$$  

In this case, $H'$ is non-vanishing in $[-1, 1]$. Since

$$H'(0) = \frac{8\alpha(1 - \alpha)(1 - a)x_1x_2}{(\alpha + 1)^4},$$  

and (2.4) yields $x_1x_2 = (1 + \alpha)^2/(4\alpha) > 0$, it follows that $H'(0) < 0$ for $a > 1$, while $H'(0) > 0$ for $a < 1$. Since $H'$ is non-vanishing, we deduce for $x \in [-1, 1]$ that $H'(x) < 0$ whenever $a > 1$, and $H'(x) > 0$ for $a < 1$. Therefore, for $a > 1$,

$$r_a = \min \sqrt{H(x)} = \sqrt{H(1)} \quad \text{and} \quad R_a = \max \sqrt{H(x)} = \sqrt{H(-1)}.$$  

Similarly, for $a < 1$,

$$r_a = \sqrt{H(-1)} \quad \text{and} \quad R_a = \sqrt{H(1)}.$$  

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The results now follow because

\[ \sqrt{H(-1)} = a - 1 + \frac{1}{1 - \alpha}, \quad \text{and} \quad \sqrt{H(1)} = 1 - a + \frac{1}{1 - \alpha}. \]

\[
\Box
\]

3 Starlike Functions of Order \( \beta \)

A function \( f \in A \) is starlike if \( tf(z) \in f(D) \) whenever \( 0 \leq t \leq 1 \). Analytically, this is equivalent to the condition \( \Re(zf'(z)/f(z)) > 0 \) for all \( z \in D \). A generalization is a function \( f \in A \) satisfying \( \Re(zf'(z)/f(z)) > \beta \) for \( z \in D \), where \( 0 \leq \beta < 1 \). This function is known as starlike of order \( \beta \), and the class consisting of such functions is denoted by \( S^\ast(\beta) \). In terms of subordination, \( f \in S^\ast(\beta) \) is subordinate to the function \( (1 + (1 - 2\beta)z)/(1 - z) \). It is readily seen that the function \( k_\beta(z) = \frac{z}{(1 - z)^2 - 2\beta} \) satisfies the equation \( zf'(z)/f(z) = (1 + (1 - 2\beta)z)/(1 - z) \). This function \( k_\beta \) is called the generalized Koebe function. It serves as the extremal function for the radius problem considered in the next theorem.

**Theorem 3.1** Let \( 0 < \alpha < 1 \) and \( 0 \leq \beta < 1 \). If \( f \in S^\ast(\beta) \), then \( f \) is Booth lemniscate starlike of order \( \alpha \) in the disk of radius

\[
R_{BS(\alpha)}(S^\ast(\beta)) = \begin{cases} 
\frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{1 + 16\alpha(1 - \beta)^2}}, & 0 \leq \beta < \max \left\{ 0; \frac{9\alpha - 1}{8\alpha} \right\}, \\
\frac{1}{1 + 2(1 - \alpha)(1 - \beta)}, & \max \left\{ 0; \frac{9\alpha - 1}{8\alpha} \right\} \leq \beta < 1.
\end{cases}
\]

**Proof** It is known (see [10]) that functions \( f \in S^\ast(\beta) \) satisfy

\[
\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\beta)r^2}{1 - r^2} \right| \leq \frac{2(1 - \beta)r}{1 - r^2}, \quad |z| \leq r < 1.
\]

Thus \( zf'(z)/f(z) \in D(a_f(r); c_f(r)) \) where

\[
a_f(r) := \frac{1 + (1 - 2\beta)r^2}{1 - r^2} \quad \text{and} \quad c_f(r) := \frac{2(1 - \beta)r}{1 - r^2}.
\]

We wish to find \( \rho \) so that \( D(a_f(\rho); c_f(\rho)) \subseteq G_\alpha(D) \) for every \( f \in S^\ast(\beta) \). Since \( a_f'(r) > 0 \), we note that \( a_f(r) \geq 1 \). Now Lemma 2.1 shows that the disk
\[\mathbb{D}(a_f(r); c_f(r)) \subset G_\alpha(\mathbb{D}) \text{ provided} \]
\[
c_f(r) = \begin{cases} 
\sqrt{s(\alpha, a_f(r))}, & a_f(r) < 1 + \frac{4\alpha}{(1-\alpha)(1+6\alpha+\alpha^2)}, \\
1 - a_f(r) + \frac{1}{1 - \alpha}, & 1 + \frac{4\alpha}{(1-\alpha)(1+6\alpha+\alpha^2)} \leq a_f(r), 
\end{cases} \tag{3.4}
\]

where
\[
s(\alpha, a_f(r)) = \frac{\sqrt{\alpha[\alpha - (1 - a_f(r))^2(1 - \alpha^2)^2] + \alpha(1 + 2(1 + \alpha)^2(1 - a_f(r))^2)}}{2\alpha(1 + \alpha)^2}. \tag{3.5}
\]

Let us write
\[
\rho_0 := \frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{(1 + 16\alpha(1 - \beta))^2}} \quad \text{and} \quad \tilde{\rho}_0 := \frac{1}{1 + 2(1 - \alpha)(1 - \beta)}.
\]

The number \(\rho_0 < 1\) is the positive root of the equation \(c_f(r) = \sqrt{s(\alpha, a_f(r))}\) given by (3.5). Indeed, this equation has the form
\[
\left(2\alpha(1 + \alpha)^2(c_f(r)^2 - (1 - a_f(r))^2) - \alpha\right)^2 = \alpha^2 - \alpha(1 - \alpha^2)^2(1 - a_f(r))^2,
\]
which upon solving and replacing \(a_f\) and \(c_f\) by the expressions given by (3.3), yields the solution \(\rho_0\).

Also, the number \(\tilde{\rho}_0 < 1\) is the root of the equation
\[
c_f(r) = 1 - a_f(r) + \frac{1}{1 - \alpha}.
\]

Further,
\[
\rho_1 := \frac{\sqrt{2\alpha}}{\sqrt{2\alpha + (1 - \alpha)(1 - \beta)(1 + 6\alpha + \alpha^2)}} < 1
\]
is the positive root of the equation
\[
\frac{(1 - \beta)r^2}{1 - r^2} = \frac{2\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)},
\]

obtained from rewriting the equation
\[
a_f(r) = 1 + \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}.
\]
Let us also write

\[ \beta_0 := 1 - \frac{1 - \alpha}{8\alpha} = \frac{9\alpha - 1}{8\alpha}, \quad \tilde{\beta}_0 := 1 - \frac{1 - \alpha}{2(1 + \alpha)^2} = \frac{1 + 5\alpha + 2\alpha^2}{2(1 + \alpha)^2}. \]

Then, \( \rho_0 = \tilde{\rho}_0 \) if \( \beta = \beta_0 \). Since \( 0 < \alpha < 1 \), a calculation shows that \( \beta_0 < \tilde{\beta}_0 \).

For \( \alpha < 1/9 \), or equivalently for \( \beta_0 < 0 \), we shall show that \( R_{BS(\alpha)} = \tilde{\rho}_0 \) for all \( 0 \leq \beta < 1 \). When \( \alpha \geq 1/9 \), or equivalently \( \beta_0 \geq 0 \), we shall show that \( R_{BS(\alpha)} = \rho_0 \) for \( 0 \leq \beta < \beta_0 \), while \( R_{BS(\alpha)} = \tilde{\rho}_0 \) for \( \beta_0 \leq \beta < 1 \). Thus there are two cases to consider: \( 0 \leq \beta < \beta_0 \), and \( \beta_0 \leq \beta < 1 \).

Case (i). Let \( 0 \leq \beta < \beta_0 \). A little calculation shows that the inequality \( \rho_0 < \rho_1 \) is equivalent to

\[ (2(1 + \alpha)^2 \beta - (1 + 5\alpha + 2\alpha^2))(1 - 9\alpha + 8\alpha\beta) > 0. \] (3.6)

Inequality (3.6) holds if and only if

\[ \beta < \min \{ \beta_0, \tilde{\beta}_0 \} = \beta_0, \quad \text{or} \quad \beta > \max \{ \beta_0, \tilde{\beta}_0 \} = \tilde{\beta}_0. \]

In this case, \( \rho_0 \leq \rho_1 \) and

\[ a_f(\rho_0) \leq a_f(\rho_1) = 1 + \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}. \]

From Lemma 2.1, it follows that \( \mathbb{D}(a_f(\rho_0); c_f(\rho_0)) \subset G_\alpha(\mathbb{D}) \), whence the \( BS(\alpha) \)-radius for \( S^*(\beta) \) is at least \( \rho_0 \).

To validate the sharpness of \( \rho_0 \), we consider the generalized Koebe function \( k_\beta \) given by (3.1). We shall show the existence of a point on \( |z| = \rho_0 \) that is mapped to a point on \( \partial G_\alpha(\mathbb{D}) \). In other words, we prove that there is some \( t \) such that the image of the point \( z = \rho_0 e^{it} \) under the map \( zk_\beta(z)/k_\beta(z) \) belongs to \( G_\alpha(e^{it}) \).

For this purpose, let us write \( G_\alpha(e^{it}) = u(t) + i v(t) \) so that

\[ u(t) = 1 + \frac{(1 - \alpha) \cos t}{1 + \alpha^2 - 2\alpha \cos(2t)}, \quad \text{and} \quad v(t) = \frac{(1 + \alpha) \sin t}{1 + \alpha^2 - 2\alpha \cos(2t)}. \]

From this, it can be shown that \( u \) and \( v \) satisfy the equation

\[ ((u - 1)^2 + v^2)^2 = \left( \frac{u - 1}{1 - \alpha} \right)^2 + \left( \frac{v}{1 + \alpha} \right)^2. \] (3.7)
The representation of the function \( z k'_\beta(z)/k_\beta(z) \) at \( z = \rho_0 e^{it} \) in Cartesian coordinates is

\[
\frac{z k'_\beta(z)}{k_\beta(z)} = \frac{1 + (1 - 2\beta)z}{1 - z} = \frac{1 - z + (1 - 2\beta)z - (1 - 2\beta)|z|^2}{|1 - z|^2} = \frac{1 - 2\beta \rho_0 \cos t - (1 - 2\beta)(\rho_0)^2}{1 + (\rho_0)^2 - 2\rho_0 \cos t} + i \frac{2(1 - \beta)\rho_0 \sin t}{1 + (\rho_0)^2 - 2\rho_0 \cos t}.
\]

By taking

\[
u(t) = \frac{1 - 2\beta \rho_0 \cos t - (1 - 2\beta)(\rho_0)^2}{1 + (\rho_0)^2 - 2\rho_0 \cos t}\]

and \( v(t) = \frac{2(1 - \beta)\rho_0 \sin t}{1 + (\rho_0)^2 - 2\rho_0 \cos t} \),

it is readily seen that

\[(u - 1)^2 + v^2 = \frac{16(\rho_0)^4 (1 - \beta)^4}{(1 + (\rho_0)^2 - 2\rho_0 \cos t)^2},
\]

and

\[
\left( \frac{u - 1}{1 - \alpha} \right)^2 + \left( \frac{v}{1 + \alpha} \right)^2 = \frac{4(\rho_0)^2 (1 - \beta)^2 [(\rho_0 - \cos t)^2 (1 + \alpha)^2 + (\sin t)^2 (1 - \alpha)^2]}{(1 + (\rho_0)^2 - 2\rho_0 \cos t)^2 (1 - \alpha)^2 (1 + \alpha)^2}.
\]

From (3.7), we seek to find a \( t \) satisfying the equation

\[
4(\rho_0)^2 (1 - \beta)^2 = \frac{(\rho_0 - \cos t)^2}{(1 - \alpha)^2} + \frac{(\sin t)^2}{(1 + \alpha)^2}.
\]

Replacing the value \( \rho_0 \) and writing \( x = \cos t \) yield

\[
\frac{16\alpha (1 - \beta)^2}{(1 + \alpha)^2 (1 + 16\alpha (1 - \beta)^2)} - \frac{1 - x^2}{(1 + \alpha)^2} - \frac{1}{(1 - \alpha)^2} \left( x - \frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{1 + 16\alpha (1 - \beta)^2}} \right)^2 = 0,
\]

or equivalently, the equation

\[
4\alpha \left( 1 + 16\alpha (1 - \beta)^2 \right) x^2 - 4(1 + \alpha)\sqrt{\alpha (1 + 16\alpha (1 - \beta)^2)} x + (1 + \alpha)^2 = 0. \tag{3.8}
\]
Clearly, the number
\[ x_0 = \frac{1 + \alpha}{2\sqrt{\alpha(1 + 16\alpha(1 - \beta)^2)}} \]
is the positive double real root of (3.8). Since \(0 < \beta \leq \beta_0\), a computation shows that \(x_0 < 1\). With \(t = \arccos x_0\) and \(z = \rho_0 e^{i\theta}\), the point \(zk_\beta'(z)/k_\beta(z)\) lies on \(\partial G_\alpha(\mathbb{D})\). This proves the sharpness for \(\rho_0\).

**Case (ii).** Let \(\beta_0 \leq \beta < 1\). Here \(\tilde{\rho}_0 \geq \rho_1\) and
\[
af(\tilde{\rho}_0) \geq af(\rho_1) = 1 + \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}.
\]
From Lemma 2.1, it follows that \(\mathbb{D}(af(\tilde{\rho}_0); cf(\tilde{\rho}_0)) \subset G_\alpha(\mathbb{D})\) showing that the \(BS(\alpha)\)-radius for the class of starlike functions of order \(\beta\) is at least \(\tilde{\rho}_0\).

To show the sharpness of \(\tilde{\rho}_0\), consider again the generalized Koebe function \(k_\beta\) given by (3.1). We shall find a point on \(|z| = \tilde{\rho}_0\) such that it is mapped to a point on \(\partial G_\alpha(\mathbb{D})\). Evidently,
\[
zk_\beta'(z) = 1 + 2(1 - \beta)\frac{z}{1 - z}.
\]
Since
\[
\frac{\tilde{\rho}_0}{1 - \tilde{\rho}_0} = \frac{1}{2(1 - \alpha)(1 - \beta)},
\]
evaluating at \(z = \tilde{\rho}_0\) gives
\[
zk_\beta'(z) = 1 + \frac{1}{1 - \alpha} = G_\alpha(1) \in \partial G_\alpha(\mathbb{D}).
\]
This proves the sharpness of \(\tilde{\rho}_0\).

Condition (3.2) suggests that the \(BS(\alpha)\)-radius in the case \(\alpha = 0\) is \(1/(3 - 2\beta)\). That this is indeed the case follows easily from Lemma 2.1.

**Theorem 3.2** Let \(0 \leq \beta < 1\). The Booth lemniscate starlikeness radius (of order 0) for the class of starlike functions of order \(\beta\) is \(1/(3 - 2\beta)\).

Theorems 3.1 and 3.2 also readily yield the following results for starlike and convex functions.
Corollary 3.3  Let $0 \leq \alpha < 1$. The Booth lemniscate starlikeness radius of order $\alpha$ for the class $S^*$ of starlike functions is

$$R_{BS(\alpha)}(S^*) = \begin{cases} 
\frac{1}{3 - 2\alpha}, & 0 \leq \alpha \leq \frac{1}{3}, \\
\frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{1 + 16\alpha}}, & \frac{1}{3} \leq \alpha \leq 1.
\end{cases}$$  \hspace{1cm} (3.9)$$

Corollary 3.4  Let $0 \leq \alpha < 1$. The Booth lemniscate starlikeness radius of order $\alpha$ for the class $K$ of convex functions is

$$R_{BS(\alpha)}(K) = \begin{cases} 
\frac{1}{2 - \alpha}, & 0 \leq \alpha \leq \frac{1}{5}, \\
\frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{1 + 4\alpha}}, & \frac{1}{5} \leq \alpha \leq 1.
\end{cases}$$

Proof Every convex function is also starlike of order $1/2$. Thus the $BS(\alpha)$-radius is at least as big as that given by Lemma 2.1 with $\beta = 1/2$. However, the extremal starlike function $k_{1/2}$ given by (3.1) is also convex, whence the result. \hfill $\Box$

Next let $1 < \beta < 4/3$, and $M(\beta)$ be the class consisting of functions $f \in A$ for which $\Re(\frac{zf'(z)}{f(z)}) < \beta$. This class was introduced by Uralegaddi et al. [13] who investigated functions in the class with positive coefficients. The following result gives the $BS(\alpha)$-radius for the class $M(\beta)$.

Theorem 3.5  Let $0 < \alpha < 1$ and $1 < \beta < 4/3$. The Booth lemniscate starlikeness radius of order $\alpha$ for the class $M(\beta)$ is

$$R_{BS(\alpha)}(M(\beta)) = \begin{cases} 
\frac{1}{1 + 2(1 - \alpha)(\beta - 1)}, & 1 < \beta \leq 1 + \frac{1 - \alpha}{8\alpha}, \\
\frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{1 + 16\alpha(\beta - 1)^2}}, & 1 + \frac{1 - \alpha}{8\alpha} \leq \beta < \frac{4}{3}.
\end{cases}$$

Proof Every function $f \in M(\beta)$ satisfies the inequality

$$\left| \frac{zf'(z)}{f(z)} - \frac{1 + (1 - 2\beta)r^2}{1 - r^2} \right| \leq \frac{2(\beta - 1)r}{1 - r^2}, \quad |z| \leq r < 1.$$ 

Define $a_f$ and $c_f$ by

$$a_f(r) := \frac{1 + (1 - 2\beta)r^2}{1 - r^2} \quad \text{and} \quad c_f(r) := \frac{2(\beta - 1)r}{1 - r^2}.$$ 

As $\beta > 1$, it follows that $a_f$ is decreasing, whence $a_f(r) \leq 1$ for all $0 \leq r < 1$. Recall that this function was increasing in the case of starlike functions of order $\beta$. \hfill $\Box$
Since \( a_f(r) \leq 1 \), Lemma 2.1 shows that the disk \( \mathbb{D}(a_f(r); c_f(r)) \subset G_\alpha(\mathbb{D}) \) provided

\[
c_f(r) = \begin{cases} \sqrt{s(\alpha, a)}, & a_f(r) > 1 - \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}, \\ a_f(r) - 1 + \frac{1}{1 - \alpha}, & 1 - \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)} \geq a_f(r), \\ \end{cases}
\]

(3.10)

where \( s(\alpha, a_f(r)) \) is given by (3.5).

Let

\[
\rho_0 := \frac{1}{1 + 2(1 - \alpha)(\beta - 1)} \quad \text{and} \quad \tilde{\rho}_0 := \frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{(1 + 16\alpha(\beta - 1)^2)}},
\]

Then, \( \rho_0 \) satisfies the equation

\[
c_f(r) = a_f(r) - 1 + \frac{1}{1 - \alpha},
\]

while \( \tilde{\rho}_0 \) is the solution of the equation

\[
c_f(r)^2 = s(\alpha, a_f(r)).
\]

Also,

\[
\rho_1 = \frac{\sqrt{4\alpha}}{\sqrt{4\alpha} + (2\beta - 2)(1 - \alpha)(1 + 6\alpha + \alpha^2)}
\]

is the positive root of the equation

\[
a_f(r) = 1 - \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}.
\]

Evidently, \( \rho_1 \leq \rho_0 \) holds if and only if

\[
\beta \leq 1 + \frac{1 - \alpha}{8\alpha}.
\]

Case (i). \( 1 < \beta \leq 1 + ((1 - \alpha)/8\alpha) \). Here \( \rho_1 \leq \rho_0 \), and because the center \( a_f(r) \) is decreasing, then

\[
a_f(\rho_0) \leq a_f(\rho_1) = 1 - \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}.
\]

Thus, it follows from (3.10) that \( \mathbb{D}(a_f(\rho_0); c_f(\rho_0)) \subset G_\alpha(\mathbb{D}) \) for every \( f \in M(\beta) \), or the \( \mathcal{BS}(\alpha) \)-radius for \( M(\beta) \) is at least \( \rho_0 \).
Case (ii). $1 + ((1 - \alpha)/8\alpha) \leq \beta < 4/3$. In this case, $\rho_1 \geq \rho_0$, and because the center $a_f(r)$ is decreasing, then

\[ a_f(\rho_0) \geq a_f(\rho_1) = 1 - \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}. \]

Thus, $\mathbb{D}(a_f(\rho_0); c_f(\rho_0)) \subset G_\alpha(\mathbb{D})$ from (3.10).

To complete the proof, we observe that the function $k_\beta$ given by $k_\beta(z) = z/(1 - z)^{2-2\beta}$ shows that the radius in each case above is best possible. \qed

4 Janowski Starlike Functions

Let $-1 < B < A \leq 1$. The class $S^*[A, B]$ of Janowski starlike functions [2] consists of $f \in A$ satisfying the subordination $zf'(z)/f(z) < (1+Az)/(1+Bz)$. For judicious choices of $A$ and $B$, $S^*[A, B]$ reduces to several widely studied subclasses of $A$. For instance, the choice $\beta = (1 - A)/2$ yields $S^*[A, -1] = S^*(\beta)$, the class which was studied in the previous section.

When $B \neq -1$, the image of $zf'(z)/f(z)$ lies in a disk. Further, if $A$ and $B$ are close to 0, then this disk is small, and whence the class $S^*[A, B]$ must be contained in the class $BS(\alpha)$. This is the inclusion result given below.

**Theorem 4.1** Let $-1 < B < A \leq 1$. The inclusion $S^*[A, B] \subset BS(\alpha)$ holds if either

(i) $(1-\alpha)(1+6\alpha+\alpha^2)|B|(A-B) \leq 4\alpha(1-B^2)$ and $(1+\alpha)^2(4\alpha(A-B)^2 + B^2) \leq 4\alpha$, or

(ii) $(1-\alpha)(1+6\alpha+\alpha^2)|B|(A-B) \geq 4\alpha(1-B^2)$ and $(1-\alpha)(A-B) + |B| \leq 1.$

**Proof** Every function $f \in S^*[A, B]$ satisfies (see [10])

\[ \left| \frac{zf'(z)}{f(z)} - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leq \frac{(A-B)r}{1 - B^2r^2}, \quad |z| \leq r < 1. \]  

(4.1)

This shows that $zf'(z)/f(z) \in \mathbb{D}(a_f; c_f)$ where

\[ a_f = \frac{1 - AB}{1 - B^2} \quad \text{and} \quad c_f = \frac{A - B}{1 - B^2}. \]

We first prove the result for $B < 0$. Here note that $a_f > 1$.

**Case (i)** Assume that $(1-\alpha)(1+6\alpha+\alpha^2)|B|(A-B) \leq 4\alpha(1-B^2)$ and $(1+\alpha)^2(4\alpha(A-B)^2 + B^2) \leq 4\alpha$. The first inequality reduces to

\[ a_f \leq 1 + \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}, \]

and so the result will follow if

\[ c_f^2 \leq \sqrt{\frac{\alpha[a - (1 - a_f)^2(1 - a_f^2)] + \alpha(1 + 2(1 + \alpha)^2(1 - a_f)^2)}{2(1 + \alpha)^2}}. \]

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The latter inequality is the statement of the second inequality \((1 + \alpha)^2(4\alpha(A - B)^2 + B^2) \leq 4\alpha\).

**Case (ii).** Assume that \((1 - \alpha)(1 + 6\alpha + \alpha^2)|B|(A - B) \geq 4\alpha(1 - B^2)\) and \((1 - \alpha)(A - B) - B \leq 1\). Then

\[
a_f \geq 1 + \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)},
\]

whence the result will follow if

\[
c_f \leq 1 - a_f + \frac{1}{1 - \alpha},
\]

or equivalently, when \((1 - \alpha)(A - B) - B \leq 1\).

When \(B \geq 0\), the center \(a_f \leq 1\), and the proof proceeds similarly as before, and is thus omitted. \(\square\)

We next turn our attention when the conditions in Theorem 4.1 fail to hold. In this case, we seek the Booth lemniscate starlikeness radius for the class \(S^*[A, B]\). The following result is also an extension of Theorem 3.1.

**Theorem 4.2** Let \(0 < \alpha < 1\), \(-1 < B \leq 0\) and \(B < A \leq 1\). If neither condition (i) nor (ii) of Theorem 4.1 holds, then the Booth lemniscate starlikeness radius of order \(\alpha\) for the class \(S^*[A, B]\) is

\[
R_{BS(\alpha)}(S^*[A, B]) = \begin{cases} 
\min \left\{1, \frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{4\alpha(A - B)^2 + B^2}} \right\}, & 4A\alpha \geq (5\alpha - 1)B, \\
\min \left\{1, \frac{1}{(1 - \alpha)(A - B) - B} \right\}, & 4A\alpha \leq (5\alpha - 1)B.
\end{cases}
\]

**Proof** The inequality (4.1) gives \(zf'(z)/f(z) \in D(a_f(r); c_f(r))\), where

\[
a_f(r) := \frac{1 - ABr^2}{1 - B^2r^2} \quad \text{and} \quad c_f(r) := \frac{(A - B)r}{1 - B^2r^2}.
\]

The result follows easily for \(B = 0\), and so assume that \(B < 0\). Since

\[
a_f'(r) = \frac{2B(B - A)r}{(1 - B^2r^2)^2},
\]

and \(-1 < B < 0\), it follows that \(a_f\) is increasing with \(a_f(r) \geq 1\) for \(0 \leq r < 1\). Only a brief outline of the proof will be given here because the proof is similar to Theorem 3.1.

The numbers

\[
\tilde{\rho}_0 = \frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{4\alpha(A - B)^2 + B^2}} \quad \text{and} \quad \rho_0 = \frac{1}{(1 - \alpha)(A - B) - B}
\]
satisfy, respectively, the equations

\[ c_f(r)^2 = s(\alpha, a_f(r)) \]

with \( s(\alpha, a_f(r)) \) given by (3.5), and

\[ c_f(r) = 1 - a_f(r) + \frac{1}{1 - \alpha}. \]

Also, the number

\[ \rho_1 = \sqrt{\frac{4\alpha}{\sqrt{4\alpha B^2 + (1 - \alpha)(1 + 6\alpha + 6\alpha^2)(B^2 - AB)}}} \]

is the solution to the equation

\[ a_f(r) = 1 + \frac{4\alpha}{(1 - \alpha)(1 + 6\alpha + \alpha^2)}. \]

Here, the condition \( \rho_1 \leq \rho_0 \) holds if and only if

\[ A \leq \left(1 - \frac{1 - \alpha}{4\alpha}\right) B. \quad (4.2) \]

Thus \( a_f(\rho_1) \leq a_f(\rho_0) \) if and only if (4.2) holds. The result follows by an application of Lemma 2.1 and is sharp for the function \( f \in S^*(A, B) \) given by \( f(z) = z/(1 + Bz)^{(B-A)/B} \) for \( B \neq 0 \), while \( f(z) = ze^{Az} \) for \( B = 0 \).\( \square \)

The result in the case \( B > 0 \) is similar, which we state without proof.

**Theorem 4.3** Let \( 0 < \alpha < 1 \), and \( 0 < B < A \leq 1 \). If neither condition (i) nor (ii) of Theorem 4.1 holds, then the Booth lemniscate starlikeness radius of order \( \alpha \) for the class \( S^*[A, B] \) is

\[
\begin{align*}
R_{BS(\alpha)}(S^*[A, B]) &= \min \left\{ 1, \frac{2\sqrt{\alpha}}{(1 + \alpha)\sqrt{4\alpha(A - B)^2 + B^2}} \right\}, \quad 4A\alpha \geq (3\alpha + 1)B, \\
&\min \left\{ 1, \frac{1}{(1 - \alpha)(A - B) + B} \right\}, \quad 4A\alpha \leq (3\alpha + 1)B.
\end{align*}
\]

For \( 0 \leq \beta < 1 \), the class \( S^*[\beta, -\beta] =: S^*_\beta \) consists of functions \( f \in A \) satisfying the inequality

\[ \left| \frac{zf'(z)}{f(z)} - 1 \right| < \beta \left| \frac{zf'(z)}{f(z)} + 1 \right|. \]

Padmanabhan [9] introduced this class. The function \( f(z) = z/(1 - \beta z)^2 \) belongs to the class \( S^*_\beta \). The \( BS(\alpha) \)-radius for this class follows readily from Theorem 4.2.
Corollary 4.4 For $0 \leq \alpha < 1$, $0 < \beta < 1$, the Booth lemniscate starlikeness radius of order $\alpha$ for the class $S^*_\beta$ is

$$R_{BS(\alpha)}(S^*_\beta) = \begin{cases} 
\min \left\{ \frac{1}{\beta(3-2\alpha)}, \frac{2}{\beta(1+\alpha) \sqrt{1+16\alpha}} \right\}, & 0 \leq \alpha \leq \frac{1}{9}, \\
\min \left\{ 1, \beta(1+\alpha) \sqrt{1+16\alpha} \right\}, & \frac{1}{9} \leq \alpha < 1.
\end{cases}$$

It is worthy to note that for $\beta = 1$, Corollary 4.4 reduces to the one given by (3.9).

For $0 \leq \beta < 1$, the class $S^*[1-\beta,0] := S^*[\beta]$ consists of functions $f \in A$ satisfying the inequality

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \beta.$$

Clearly, $S^*[\beta] \subset S^*(\beta)$ and the function $f(z) = ze^{(1-\beta)z}$ belongs to the class $S^*[\beta]$. This class was introduced and studied by Singh [12], and we state its Booth lemniscate starlikeness radius.

Corollary 4.5 For $0 \leq \alpha < 1$, $0 \leq \beta < 1$, the Booth lemniscate starlikeness radius of order $\alpha$ for the class $S^*[\beta]$ is

$$R_{BS(\alpha)}(S^*[\beta]) = \min \left\{ \frac{1}{1+\alpha(1-\beta)} \right\}.$$

In particular, $S^*[\alpha/(1+\alpha)] \subset BS(\alpha)$.

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