The seed of magnetic monopoles in the early inflationary universe
from a 5D vacuum state

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Abstract
Starting from a 5D Riemann flat metric, we have induced an effective 4D Hermitian metric which has an antisymmetric part which is purely imaginary. We have worked an example in which both, non-metricity and cotorsion are zero. We obtained that the production of monopoles should be insignificant at the end of inflation and the tensor metric should come asymptotically diagonal and describing a nearly 4D de Sitter expansion.

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I. INTRODUCTION

The possibility that our world may be embedded in a \((4 + d)\)-dimensional universe with more than four large dimensions has attracted the attention of a great number of researches. One of these higher-dimensional theories, where the cylinder condition of the Kaluza-Klein theory\(^{1}\) is replaced by the conjecture that the ordinary matter and fields are confined to a 4D subspace usually referred to as a brane is the Randall and Sundrum model\(^{2}\). The original version of the KK theory assures, as a postulate, that the fifth dimension is compact. A few years ago, a non-compactified approach to KK gravity, known as Space-Time-Matter (STM) theory was proposed by Wesson and collaborators\(^{3}\). In this theory all classical physical quantities, such as matter density and pressure, are susceptible of a geometrical interpretation. Wesson’s proposal also assumes that the fundamental 5D space in which our usual spacetime is embedded, should be a solution of the classical 5D vacuum Einstein equations: \(R_{AB} = 0\). The mathematical basis of it is the Campbell’s theorem\(^{4}\), which ensures an embedding of 4D general relativity with sources in a 5D theory whose field equations are apparently empty. That is, the Einstein equations \(G_{\alpha\beta} = -8\pi G T_{\alpha\beta}\) (we use \(c = \hbar = 1\) units), are embedded perfectly in the Ricci-flat equations \(R_{AB} = 0\).

The theory of magnetic monopoles was formulated many decades ago by Dirac in \(^{5}\). In his classical works, Dirac showed that the existence of a magnetic monopole would explain the electric charge quantization. Also, a lagrangian formulation describing the electromagnetic interaction mediated by topologically massive vector bosons-between charged, spin-1/2 fermions with an abelian magnetic monopoles in a curved spacetime with nonminimally coupling and torsion potential, was presented in\(^{6}\). In the framework of inflation, magnetic monopole solutions of the Einstein-Yang-Mill-Higgs equations with a positive cosmological constant approach asymptotically the de Sitter spacetime background and exist only for a nonzero Higgs potential\(^{7}\). More recently, Maxwell equations with massive photons and magnetic monopoles were formulated using spacetime algebra\(^{8}\). From the quantum-theoretical standpoint monopoles and massive photons were discussed widely during the last decades\(^{9}\). To discuss massive photons one has to consider the Proca equations rather than the Maxwell ones. Further, if a magnetic charge (monopole) really exists, the Maxwell

\(^{1}\) In our conventions capital Latin indices run from 0 to 4, greek indices run from 0 to 3 and latin indices run from 1 to 3.
electrodynamics must be replaced by a generalized theory, with the dual field tensor having a non vanishing divergence. Some years ago the Weyl-Dirac formalism was generalized in order to obtain a geometrically based general relativistic theory, possessing electric and magnetic currents and admitting massive photons. In this letter we are interested to study the evolution of primordial magnetic monopoles in the inflationary epoch by extending Gravitoelectromagnetic Inflation and using some ideas of Dirac and Einstein in the framework of the Induced Matter theory, where the extra dimension is space-like and noncompact. To make it we shall start from a 5D vacuum state, on which we define null all the external sources and magnetic monopoles [the absence of magnetic monopoles is a characteristic of any ($N > 4$) theory]. It means that the universe in higher dimensions will be considered as empty. If we consider the extra dimension as a complex function of other coordinates, it is possible to obtain an effective 4D Hermitian tensor metric which could be relevant to describe extended gravitation and electrodynamics, where 4D magnetic monopoles are taken into account. In this context, we shall study the production of magnetic monopoles in the early inflationary universe.

II. FORMALISM IN A 5D VACUUM

We consider the 5D Riemann flat metric

\[ dS^2 = \psi^2 dN^2 - \psi^2 e^{2N} \left( dr^2 + r^2 d\Omega^2 \right) - d\psi^2, \tag{1} \]

with the action

\[ I = \int d^4xd\psi \sqrt{\left( \frac{g}{g_0} \right)} \left[ \frac{1}{16\pi G} R^{(5)} + \mathcal{L}(A_B, A_{C;B}) \right], \tag{2} \]

where $\left( \frac{g}{g_0} \right) = \psi^8 e^{6N} r^4 \sin^2(\theta)$ is the determinant of the covariant metric tensor $g_{AB}$, on a Riemman spacetime. The metric (1) is Riemann-flat ($R_{ABCD} = 0$) and is a particular case of the so called canonical metric: $dS^2 = \psi^2 g_{\alpha\beta}(x^\mu, \psi) dx^\alpha dx^\beta - d\psi^2$, where $\frac{\partial g_{\alpha\beta}}{\partial \psi} = 0$, so that $8\pi G G_{\alpha\beta} = -3g_{\alpha\beta}/\psi_0^2$, $\Lambda = 3/(8\pi G \psi_0^2)$ being the cosmological constant. Physically, this metric removes the potentials of electromagnetic type and flattens the potential of scalar type, so that the fields $A^B$ in the action (2) must be considered as test ones. Here, $\left( \frac{g}{g_0} \right) \equiv \left( \frac{g}{g_0} \right)_{|N=0, \psi=\psi_0, \theta=\pi/2}$ is a constant. As in previous papers, we shall consider a
Lagrangian density given by

\[ \mathcal{L}(A_B, A_C; B) = -\frac{1}{4} Q_{BC} Q^{BC}, \]

(3)

where \( Q^{AB} \) is an operator given by

\[ Q^{AB} = F^{AB} + \gamma g^{AB} A_D;D, \]

(4)

such that \( A_B = (A_\mu, \varphi), A^B = (A^\mu, -\varphi) \), \( \varphi \) being the inflaton field.

Since we are considering a 5D vacuum, one obtains absence of both, 5D gravitoelectric \( J^B \) and gravitomagnetic \( K^B \) currents

\[ \bigg( Q^{AB} \bigg)_{;A} = 0, \]

(5)

\[ \bigg( Q^{\dagger AB} \bigg)_{;A} = 0, \]

(6)

where the dual tensor of \( Q^{AB} \) is

\[ \bigg( Q^{\dagger AB} \bigg) = \frac{1}{2} \frac{\epsilon^{ABCD}}{\sqrt{|(5)g|}} Q_{CD}. \]

(7)

III. INDUCED GRAVITOELECTROMAGNETIC EQUATIONS FROM A 5D VACUUM

Now we consider the 4D embedding \( \psi \equiv \psi(r, N) \) on the 5D metric (1), such that if we consider physical coordinates \( d\psi = \frac{\partial \psi}{\partial \mu} dr + \frac{\partial \psi}{\partial N} dN \). The induced operators \( Q^{\alpha\beta} \) will be

\[ Q^{\alpha\beta} = F^{\alpha\beta} + \gamma g^{\alpha\beta} \left( A^\delta;\delta \right), \]

(8)

\[ \left( Q^{\alpha\beta} \right)_{;\alpha} = -4\pi J^{\beta}, \]

(9)

\[ \left( Q^{\dagger \alpha\beta} \right)_{;\alpha} = -4\pi K^{\beta}, \]

(10)

where \( \left( Q^{\dagger} \right)^{\alpha\beta} = \frac{1}{2} \frac{\epsilon^{\alpha\beta\mu\nu}}{\sqrt{|(4)g|}} Q_{\mu\nu} \). The induced gravitoelectric current is

\[ J^\alpha = -\frac{1}{4\pi} \left\{ \left( F^{\alpha}(\psi) \right)_{;\alpha} + \gamma \left[ g^{\alpha\beta} A^{\beta;\alpha} + g^{\alpha}(\psi) A^{D;\alpha} + g^{\alpha}(\psi) \left( A^{D;\alpha} + A^{\alpha;\alpha} \right) \right] \right\}. \]

(11)

The induced gravitomagnetic current is given by

\[ K^{\beta} = \left( Q^{\dagger \alpha\beta} \right)_{;\alpha} = -\frac{1}{8\pi} \epsilon^{\alpha\beta\mu\nu} \left\{ \frac{A_{\sigma} \left( \Gamma_{\nu\mu}^{\epsilon} - \Gamma_{\mu\nu}^{\epsilon} \right) + \gamma g_{\mu\nu} \Gamma_{\rho\eta}^{\epsilon} A^\rho}{\sqrt{|(4)g|}} \right\} ;\alpha, \]

(12)

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2 We denote with comma ordinary derivatives and with (;) covariant derivatives.
where $\varepsilon^{\alpha\beta\mu\nu}$ is the Ricci tensor density. Notice that when torsion $T^\sigma_{\nu\mu} = \frac{\Gamma^\sigma_{\mu
u} - \Gamma^\sigma_{\nu\mu}}{2}$ is zero and the metric tensor is diagonal, hence gravitomagnetic current vanishes.

IV. INDUCED GRAVITOELECTROMAGNETIC EQUATIONS ON AN HERMITIC METRIC

In order to work an example with nonzero gravitomagnetic current we shall study an example where the fifth coordinate $\psi \equiv \psi(r, N)$ is a complex function of $N$ and $r$.

A. An example with null cotorsion and non-metricity on an Hermitian metric

We are interested to study the case where the resultant 4D effective metric is Hermitian

$$dS^2 = \left[|\psi|^2 + \left|\frac{\partial \psi}{\partial N}\right|^2\right] dN^2 - \left[|\psi|^2 e^{2N} + \left|\frac{\partial \psi}{\partial r}\right|^2\right] dr^2 - \frac{\partial \psi}{\partial N} \frac{\partial \psi}{\partial N} (dN dr - dr dN) - |\psi|^2 e^{2N} r^2 d\Omega^2,$$

such that $|\psi|^2 = \psi \psi^*$ and $U^A = \frac{dx^A}{ds}$ are the penta-velocities such that $g_{AB} U^A U^B = 1$. Furthermore, the conditions to obtain an Hermitian metric tensor are:

$$\frac{\partial \psi}{\partial r} = \pm \frac{\partial \psi^*}{\partial r}, \quad \frac{\partial \psi}{\partial N} = \mp \frac{\partial \psi^*}{\partial N}.$$  \hspace{1cm} (14)

Hence, the metric (13) results to be represented by an Hermitian tensor with $g_{\alpha\beta} = g^*_{\beta\alpha}$, where the asterisk denotes the complex conjugate. The determinant $g = det |g_{\alpha\beta}|$ is nonzero and real. Furthermore, as in a real tensor metric, one has $g_{\alpha\beta} g^{\gamma\beta} = \delta^\gamma_\beta$, where $\delta^\gamma_\beta$ is the Kronecker tensor. Here, the order of indices is important and, for example: $g_{\alpha\beta} g^{\beta\gamma} \neq \delta^\gamma_\beta$. Recently, Hermitian metrics has been subject of interest to describe cosmology\cite{15}. The effective 4D connections are given by

$$\Gamma^\lambda_{\nu\mu} + g^{\lambda\sigma} \Gamma^\sigma_{\mu\sigma} g_{\nu\sigma} = \left|\frac{\lambda}{\mu}\right| + C^\lambda_{\mu\nu} + \frac{1}{2} g^{\lambda\sigma} (Q_{\nu\mu\sigma} + Q_{\mu\nu\sigma} - Q_{\sigma\mu\nu}),$$

where the non-metricity terms are due to $Q_{\nu\mu\sigma} \equiv -g_{\mu\sigma,\nu}$, torsion are related with cotorsion terms by $2T^\alpha_{\mu\nu} = C^\alpha_{\mu\nu} - C^\alpha_{\nu\mu}$ and

$$\left|\frac{\lambda}{\mu}\right| = \frac{1}{2} g^{\lambda\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}).$$ \hspace{1cm} (16)
Finally, the non-zero gravitomagnetic currents components on the effective 4D metric, are

\[ K^N = -\frac{e^{\alpha N \mu \nu}}{4\pi} \frac{A_\sigma T^\sigma_{\mu \nu}}{\sqrt{|(4) g|}} ;^\alpha, \]  

(17)

\[ K^r = -\frac{e^{\alpha r \mu \nu}}{4\pi} \frac{A_\sigma T^\sigma_{\mu \nu}}{\sqrt{|(4) g|}} ;^\alpha, \]  

(18)

\[ K^\theta = \frac{1}{4\pi \sqrt{|(4) g|}} \left\{ 2 \left[ (A_\sigma T^\sigma_{N r})_{:\varphi} - (A_\sigma T^\sigma_{N \varphi})_{:r} - (A_\sigma T^\sigma_{r \varphi})_{:N} \right] + \gamma \left( g_{[N r]} \Gamma^\eta_{\rho \eta} A^\rho \right)_{:\varphi} \right\}, \]  

(19)

\[ K^\varphi = -\frac{1}{4\pi \sqrt{|(4) g|}} \left\{ 2 \left[ (A_\sigma T^\sigma_{N r})_{:\theta} - (A_\sigma T^\sigma_{N \theta})_{:r} - (A_\sigma T^\sigma_{r \theta})_{:N} \right] + \gamma \left( g_{[N r]} \Gamma^\eta_{\rho \eta} A^\rho \right)_{:\theta} \right\}. \]  

(20)

Now we consider the equation (15) with both, zero cotorsion and non-metricity. In this case the last two terms in the right side are zero, and we obtain

\[ \Gamma^\lambda_{\mu \nu} + g^{\lambda \sigma} \Gamma^\kappa_{\mu \sigma} g_{[\nu \kappa]} = \left| \lambda \right|_{[\mu \nu]}, \]  

(21)

and the gravitomagnetic currents for this particular case results to be

\[ K^N = K^r = 0, \]  

(22)

\[ K^\theta = -\frac{\gamma g_{[r \rho]} \left\{ \left( \frac{g_{[N r]}}{g_{[r \rho]}} \right) + \left\{ \frac{\varphi}{\rho \varphi} \right\} \right\} A^\rho}{4\pi \sqrt{|(4) g|}} ;^\varphi, \]  

(23)

\[ K^\varphi = -\frac{\gamma g_{[N r]} \left\{ \left( \frac{g_{[N r]}}{g_{[r \rho]}} \right) + \left\{ \frac{\varphi}{\rho \varphi} \right\} \right\} A^\rho}{4\pi \sqrt{|(4) g|}} ;^\theta, \]  

(24)

where \( g_{[\alpha \beta]} \) denotes the antisymmetric components of the tensor metric and \( \left\{ \frac{\alpha}{\beta \gamma} \right\} \) are the second kind Christoffel symbols. Notice that the existence of non-zero gravitomagnetic current components depends of the antisymmetry of the tensor metric. An interesting example, which is relevant for cosmology is the case in which the complex function \( \psi(r, N) \) is given by

\[ \psi(N, r) = \psi_0 \left[ e^{-r} + i e^{-N} \right], \]

such that, at the end of inflation, the non-diagonal part of the metric tensor becomes negligible with respect to the diagonal one. If we take \( \psi_0 = 1/H \) (when \( H \) is the Hubble parameter, which in this case is a constant) and \( N = Ht \), at the end of inflation the asymptotic tensor metric is diagonal.
\[ g_{\mu\nu}|_{t \gg 1/H} \simeq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -H^{-2}e^{2Ht} & 0 & 0 \\ 0 & 0 & -H^{-2}e^{2Ht}r^2 & 0 \\ 0 & 0 & 0 & -H^{-2}e^{2Ht}r^2\sin^2(\theta) \end{pmatrix}, \] 

which describes a de Sitter expansion [16]. Notice that, at the end of inflation \( U^\psi|_{t \gg 1/H} \to 0 \), and \( U^t|_{t \gg 1/H} \to 1 \) for \( U^\theta = U^\phi = 0 \), so that observers are at this time in a nearly comoving frame: \( U^r|_{t \gg 1/H} \to 0 \).

V. FINAL REMARKS

In this letter we have extended Gravitoelectromagnetic Inflation using some ideas of Dirac [11] and Einstein [12], in the framework of the Induced Matter theory, where the extra dimension is space-like, noncompact and complex. Starting from a 5D Riemann flat metric (1), we have induced an effective 4D Hermitian metric which has an antisymmetric part \( g_{[\mu\nu]} \) which is purely imaginary. We have worked an example where the only non-diagonal components of the tensor metric are \( g_{Nr} = i \frac{\partial \psi}{\partial N} \frac{\partial \bar{\psi}}{\partial r} \) and \( g_{rN} = -i \frac{\partial \psi}{\partial r} \frac{\partial \bar{\psi}}{\partial N} \) in which both, non-metricity and cotorsion are zero. In this case the non-zero components of the gravitomagnetic current during inflation are \( K^\theta \) and \( K^\phi \). However, at the end of inflation the production of monopoles should be insignificant and the tensor metric should describe a nearly 4D de Sitter expansion. The evolution of the universe from a de Sitter expansion to other Friedmann-Robertson-Walker cosmology was studied using different approaches [17].

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