Log-affine geodesics in the manifold of vector states on a von Neumann algebra

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Abstract
This paper introduces the notion of a log-affine geodesic connecting two vector states on a von Neumann algebra. The definition is linked to the standard notion of Boltzmann-Gibbs states in Statistical Physics and the related notion of quantum statistical manifolds. In the abelian case it is linked to the notion of exponential tangent spaces.

1 Introduction
The existing approach to quantum information geometry focuses on non-degenerate density matrices. See for instance [1]. In [2] the present author proposes a reformulation which makes use of operators in the commutant of the Gelfand-Naimark-Segal (GNS) representation of the algebra of n-by-n matrices, induced by a faithful quantum state. The generalization of this approach to the context of infinite-dimensional Hilbert spaces is still open. As a first step in this direction the present paper discusses a possible definition of log-affine geodesics in a manifold of vector states, while leaving open all other aspects of a full theory of statistical manifolds in a non-commutative setting.

If the geodesic is log-affine then one expects that there exists a generator $H$ reproducing the geodesic. Consider a von Neumann algebra $\mathcal{A}$ of operators on a Hilbert space $\mathcal{H}$, $\Omega$ a normalized element of $\mathcal{H}$, and $T_s$ a semi-group of normal operators. Define states $\omega_s$ on $\mathcal{A}$ by

$$\omega_s(A) = (A\Omega_s, \Omega_s), \quad A \in \mathcal{A},$$

with

$$\Omega_s = \frac{T_s^{1/2}\Omega}{\|T_s^{1/2}\Omega\|}, \quad s \geq 0.$$

Then $s \mapsto \omega_s$ is an example of what is meant here by a geodesic connecting $\omega_1$ to $\omega_0$. By Stone’s theorem there exists a normal operator $H$ such that $T_s = \exp(sH)$. For this reason the geodesic is said to be log-affine.

In a statistical context it is natural to assume that the generator $H$ is a self-adjoint operator. A strongly continuous one-parameter group of unitary operators $(U_t)_t$ is defined by $U_t = \exp(itH)$. They are linked to the operator $T$ by analytic continuation: $T^{1/2} = U_{-1/2}$.

The setting as described above is too limited for the purposes of the present paper. A more general setting is borrowed from the theory of the modular automorphism group [3], which is associated with the Kubo-Martin-Schwinger (KMS) condition [4], a notion coming from Statistical Physics. The proposal is to give up the requirement that the unitary operators $(U_t)$, form a group. It is replaced by the cocycle condition [5]

$$U_{r+t} = U_r \tau_r(U_t), \quad r, t \in \mathbb{R}, \quad (1)$$

$$Ω_s = Ω_s,$$
where $(\tau_t)_t$ is a one-parameter group of automorphisms of the von Neumann algebra $\mathcal{A}$. Note that in Physics the notion of a cocycle is known as a set of unitary operators implementing the time evolution of the interaction picture of Quantum Mechanics.

The present work has been influenced by the approach of Pistone and Sempi [6] to statistical manifolds of probability distributions, and, in particular, by the recent works of Newton [7] and of Montrucchio and Pistone [8, 9] on the linear growth case. Part of the results of [8] can be translated to the non-commutative context in a rather straightforward manner [10]. Translation of other parts is hindered by difficulties due to non-commutativity. The connection with the present work is subject of future investigation.

In the next two sections the correspondence between cocycles and vector states of the von Neumann algebra is worked out. A definition of a log-affine geodesic is given in Section 4. Section 5 shows that any two faithful quantum states defined by $n$-by-$n$ density matrices are always connected by a log-affine geodesic. For the abelian case, the correspondence with the notion of exponential tangent spaces is worked out in Section 6. The final Section contains a short summary.

2 States labeled with cocycles

Throughout the text $\mathcal{A}$ denotes a von Neumann algebra which acts on a Hilbert space $\mathcal{H}$. The symbol $\mathbb{M}$ denotes a manifold of faithful vector states on $\mathcal{A}$. A single state $\omega_0$ is fixed in $\mathbb{M}$. It is generated by a cyclic and separating vector $\Omega_0 \in \mathcal{H}$. The one-parameter group of modular automorphisms leaving $\omega_0$ invariant is denoted $(\tau_t)_t$. The modular conjugation is denoted $S$, the modular operator equals $S^*S$ and leaves the vector $\Omega_0$ invariant.

Recall the following result.

**Lemma 2.1 (Lemma 2.1 of [5])** Let be given a strongly continuous one-parameter family of unitary operators $U_t$ belonging to $\mathcal{A}$. Assume they satisfy the cocycle condition w.r.t. a group of automorphisms $(\tau_t)_t$, which are of the form $\tau_t(A) = \Delta^{-it}A\Delta^{it}$. Then there exist a positive operator $T$ such that

$$U_t = T^{-it}\Delta^{it}, \quad t \in \mathbb{R}.$$  

**Proof**

By the cocycle condition the unitary operators $V_t$, defined by

$$V_t = U_t\Delta^{-it},$$

satisfy

$$V_{t+r} = U_{t+r}\Delta^{-i(t+r)} = U_t\tau_t[U_r]\Delta^{-i(t+r)} = U_t\Delta^{-it}U_r\Delta^{it}\Delta^{-i(t+r)} = U_t\Delta^{-it}U_r\Delta^{-ir}.$$  

Hence $t \mapsto V_t$ is a strongly continuous one-parameter group of unitaries. By Stone’s theorem these unitaries can be written as powers of a positive operator $T$. By construction one has

$$T^{-it} = V_t = U_t\Delta^{-it}$$

and hence

$$U_t = T^{-it}\Delta^{it}.$$  

**Proposition 2.2** Let be given a cocycle $(U_t)_t$ of $(\tau_t)_t$. Let $T$ be the positive operator defined by the previous Lemma. Are equivalent:
1) The map \( t \mapsto U_t \Omega_0 \) has a continuous extension \( z \mapsto \Xi_z \in \mathcal{H} \) on the strip \( 0 \leq \text{Im} \, z \leq i/2 \) and this extension is analytic in the interior \( 0 < \text{Im} \, z < i/2 \).

2) \( \Omega_0 \) belongs to the domain of \( T^{1/2} \).

Proof

1) \( \Rightarrow \) 2) Use \( \Delta \Omega_0 = 0 \) to obtain
\[
U_t \Omega_0 = T^{-it} \Omega_0.
\]

Let
\[
T = \int_0^\infty \lambda \, dE_\lambda
\]
denote the spectral representation of the operator \( T \). By assumption, the analytic continuation of
\[
F(t) = (T^{-it} \Omega_0, \psi) = \int_0^\infty \lambda^{-it} \, d(E_\lambda \Omega_0, \psi)
\]
exists for any \( \psi \in \mathcal{H} \) up to \( t = i/2 \). This implies
\[
F(i/2) = (\Xi_{i/2}, \psi) = \int_0^\infty \lambda^{1/2} \, d(E_\lambda \Omega_0, \psi).
\]

For \( \psi \) in the domain of \( T^{1/2} \) one has
\[
\int_0^\infty \lambda^{1/2} \, d(E_\lambda \psi, \Omega_0) = (T^{1/2} \psi, \Omega_0)
\]
and hence
\[
(T^{1/2} \psi, \Omega_0) = (\psi, \Xi_{i/2}).
\]

Because \( T^{1/2} \) is self-adjoint this implies that \( \Omega_0 \) is in its domain and \( T^{1/2} \Omega_0 = \Xi_{i/2} \).

2) \( \Rightarrow \) 1) For any \( z \) in the strip \( 0 \leq \text{Im} \, z \leq 1/2 \) one has using the concavity of the logarithmic function
\[
||T^{-iz} \Omega_0||^2 = \int_0^\infty \lambda^{2 \text{Im} \, z} \, d(E_\lambda \Omega_0, \Omega_0)
\]
\[
\leq \int_0^\infty [2 \text{Im} \, z + (1 - 2 \text{Im} \, z) \lambda] \, d(E_\lambda \Omega_0, \Omega_0)
\]
\[
= 2 \text{Im} \, z + (1 - 2 \text{Im} \, z)||T^{1/2} \Omega_0||^2.
\]

This upper bound suffices to prove 1) in a straightforward manner.

\( \square \)

Proposition 2.3 Let be given a cocycle \( (U_t) \) of \( (\tau_t)_t \). Assume that the equivalent conditions of Proposition 2.2 are satisfied. Then a vector state \( \omega_U \) is defined by
\[
\omega_U(A) = (A \Omega_U, \Omega_U), \quad A \in \mathcal{A},
\]
with \( \Omega_U = e^{-1/z(U)} \Xi_{i/2} = e^{-1/z(U)} T^{1/2} \Omega_0 \) and \( z(U) = \log ||\Xi_{i/2}||^2 = \log ||T^{1/2} \Omega_0||^2 \).

Proof

Let us show by proof \textit{ex absurdo} that \( \Omega_U \neq 0 \). From \( \Omega_U = 0 \) and \( \Xi_{i/2+t} = T^{1/2+it} \Omega_0 \) it follows that \( \Xi_z \) vanishes along the line \( \text{Im} \, z = 1/2 \). This implies that it vanishes everywhere on the strip \( 0 \leq \text{Im} \, z \leq 1/2 \). This implies that \( \Xi_0 = \Omega_0 = 0 \), in contradiction with the proper normalization of \( \Omega_0 \).

The remainder of the proof is straightforward.

\( \square \)
3 Some operators and their properties

Proposition 3.1 Let be given a cocycle \((U_t)_t\) of \((\tau_t)_t\). Assume that the equivalent conditions of Proposition 2.2 are satisfied. Then a linear operator, denoted \(U_{i/2}\), is defined by

\[ U_{i/2}Y\Omega_0 = Y\Xi_{i/2} = YT^{1/2}\Omega_0, \quad Y \in \mathcal{A}' . \]

It has the following properties.

1) For any \(Y\) in the commutant \(\mathcal{A}'\) the vector \(U_{i/2}Y\Omega_0\) is the analytic extension of the map \(t \mapsto U_tY\Omega_0\) to the point \(t = i/2\);

2) \(\Xi_{i/2} = U_{i/2}\Omega_0 = T^{1/2}\Omega_0\);

3) \(U_{i/2}\) commutes with any operator in the commutant \(\mathcal{A}'\).

Proof

Because \(\Omega_0\) is cyclic for the commutant the linear operator \(U_{i/2}\) is a densely defined operator.

For any \(Y, Z\) in the commutant \(\mathcal{A}'\) the function \(t \mapsto (U_tY\Omega_0, Z\Omega_0)\) has an analytic extension to \(t = i/2\). Indeed, one has

\[ (U_tY\Omega_0, Z\Omega_0) = (U_t\Omega_0, Y^*Z\Omega_0). \]

By assumption the r.h.s. has an analytic extension up to \(t = i/2\) where it has the value \((\Xi_{i/2}, Y^*Z\Omega_0)\).

Therefore one has

\[ (U_tY\Omega_0, Z\Omega_0) \bigg|_{t=i} = (\Xi_{i/2}, Y^*Z\Omega_0) = (YT^{1/2}\Omega_0, Z\Omega_0) = (U_{i/2}Y\Omega_0, Z\Omega_0). \]

This shows that \(U_{i/2}Y\Omega_0\) is the analytic extension of \(U_tY\Omega_0\) to \(t = i/2\). The argument can also be used to show that \(U_{i/2}\) commutes with any element of the commutant.

\[ \square \]

Proposition 3.2 Let be given a cocycle \((U_t)_t\) of \((\tau_t)_t\). Assume that the equivalent conditions of Proposition 2.2 are satisfied. Assume in addition that \(\Delta\) is the modular operator. Let \(U_{i/2}\) be the operator defined in the previous Proposition. Let \(S = J\Delta^{1/2}\) denote the polar decomposition of the modular conjugation operator. The following two conditions are equivalent.

1) \(\Xi_{i/2}\) is in the domain of \(S\).

2) \(\Omega_0\) belongs to the domain of \(U_{i/2}^*\).

If these conditions are satisfied then the following hold.

3) The operator \(U_{i/2}\) is closable.

4) \(U_{i/2}^*\Omega_0 = S\Xi_{i/2}\).

5) \(J\Xi_{i/2} = \Xi_{i/2}\).

6) The operator \(Y = JU_{i/2}^*J\) is affiliated with the commutant \(\mathcal{A}'\); The operator \(JT^{1/2}S\) is an extension of the operator \(Y\); In particular, one has \(Y\Omega_0 = \Xi_{i/2}\).

7) The state \(\omega_U\), defined in Proposition 2.3, satisfies

\[ \omega_U(A) = e^{-\zeta(U)}(AY\Omega_0, Y\Omega_0) = e^{-\zeta(U)}(AX^{1/2}\Omega_0, X^{1/2}\Omega_0), \quad A \in \mathcal{A}, \]

with \(X = Y^*Y\).

Proof
1) implies 2) One has for all $Y$ in the commutant $\mathcal{A}'$ that

\[
(S\Xi_{i/2}, Y\Omega_0) = \langle S^* Y\Omega_0, \Xi_{i/2} \rangle \\
= \langle Y^*\Omega_0, \Xi_{i/2} \rangle \\
= \langle \Omega_0, Y\Xi_{i/2} \rangle \\
= \langle \Omega_0, U_{i/2} Y\Omega_0 \rangle.
\]

This shows that $\Omega_0$ is in the domain of $U_{i/2}^*$ and that $U_{i/2}^*\Omega_0 = S\Xi_{i/2}$.

2) implies 1) and 4) For any $Y$ in the commutant $\mathcal{A}'$ one has

\[
(\Xi_{i/2}, S^* Y\Omega_0) = \langle U_{i/2}\Omega_0, Y^*\Omega_0 \rangle \\
= \langle Y\Omega_0, U_{i/2}^*\Omega_0 \rangle.
\]

This implies that $\Xi_{i/2}$ is in the domain of $S$ and that $S\Xi_{i/2} = U_{i/2}^*\Omega_0$.

2) implies 3) Assume that $Y_n\Omega_0$ converge to 0 and $U_{i/2} Y_n\Omega_0$ converge to some $\psi$ in $\mathcal{H}$. Then one has for all $Y, Z$ in the commutant that

\[
(\psi, Z\Omega_0) = \lim_n (U_{i/2} Y_n\Omega_0, Z\Omega_0) \\
= \lim_n (U_{i/2} Z^* Y_n\Omega_0, \Omega_0) \\
= \lim_n (Z^* Y_n\Omega_0, U_{i/2}^*\Omega_0) \\
= 0.
\]

One concludes that $\psi = 0$. Hence, the operator $U_{i/2}$ is closable.

5) In the case of a finite-dimensional Hilbert space $\mathcal{H}$ the proof is simple. From $U_{i/2} = T^{1/2}\Delta^{-1/2}$ it follows that $U_{i/2}^* = \Delta^{-1/2}T^{1/2}$. One then calculates, using that $U_{i/2}^* \in \mathcal{A}$,

\[
\Xi_{i/2} = U_{i/2}\Omega_0 \\
= SU_{i/2}^*\Omega_0 \\
= S\Delta^{-1/2}T^{1/2}\Omega_0 \\
= J\Xi_{i/2}.
\]

The general proof is found in the Appendix.

6) Because $U_{i/2}^*$ is affiliated with $\mathcal{A}$ the operator $Y = JU_{i/2}^*$ is affiliated with the commutant $\mathcal{A}'$.

In the case of a finite-dimensional Hilbert space $\mathcal{H}$ the proof of the equality $Y = JT^{1/2}S$ is simple. From $U_{i/2} = T^{1/2}\Delta^{-1/2}$ it follows that

\[
Y = JU_{i/2}J = JT^{1/2}\Delta^{-1/2}J = JT^{1/2}S.
\]

The general proof is found in the Appendix.

7) The polar decomposition of the operator $Y$ can be written as $Y = KX^{1/2}$. Because the isometry $K$ belongs to $\mathcal{A}'$ one has for all $A$ in $\mathcal{A}$ that

\[
\omega_U(A) = e^{-\zeta(U)}(A\Xi_{i/2}, \Xi_{i/2}) \\
= e^{-\zeta(U)}(AY\Omega_0, Y\Omega_0) \\
= e^{-\zeta(U)}(AKX^{1/2}\Omega_0, KX^{1/2}\Omega_0) \\
= e^{-\zeta(U)}(AX^{1/2}\Omega_0, X^{1/2}\Omega_0).
\]

□
4 Log-affine geodesics

Let $A$, $M$, $\omega_0$ and $\Omega_0$ be as before. The state $\omega_0$ is invariant for the one-parameter group $(\tau_t)_t$ of automorphisms of $A$. These automorphisms are of the form $\tau_t(A) = e^{-itA}\Delta e^{itA}$, where $\Delta$ is a positive operator satisfying $\Delta\Omega_0 = 0$.

Let us now consider a geodesic $s \mapsto \omega_s$ which connects two states $\omega_0$ and $\omega_1$ in $M$. It is assumed that each state $\omega_s$ is labeled by a cocycle $U(s)$ which satisfies the equivalent conditions of Proposition 3.2. The state $\omega_s$ defined in Proposition 2.3 coincides with the state $\omega_s$. The normalization function is denoted $\zeta'(s)$ instead of $\zeta(U(s))$. The operator $T$ constructed in Lemma 2.1 is denoted $T_s$.

4.1 Logarithmic derivatives

Logarithmic derivatives play an essential role in Quantum Information Theory. Consider an operator-valued function $s \mapsto T_s \equiv \exp(H_s)$. In general, the derivative $dH_s/ds$ does not commute with the operator $T_s$. As a consequence, the derivative $dT_s/ds$ can differ from $T_s dH_s/ds$ or $dH_s T_s ds$.

It would be fair to call $dH_s/ds$ the logarithmic derivative of $s \mapsto T_s$. However, this does not help us in the calculation of $dT_s/ds$. Following [11] let us call the left logarithmic derivative of $s \mapsto T_s$ any solution of the problem

$$
\frac{dT_s}{ds} = L_s^r T_s.
$$

Similarly, the right logarithmic derivative $L_s^r$ is any solution of the problem

$$
\frac{dT_s}{ds} = T_s L_s^r.
$$

On the other hand, the symmetric logarithmic derivative is the solution of

$$
\frac{dT_s}{ds} = \frac{1}{2} (L_s T_s + T_s L_s).
$$

See for instance [12] and the references quoted there. If $L_s^l = L_s^r$ then they coincide with the symmetric logarithmic derivative $L_s$.

Proposition 4.1 Assume the following.

- The equivalent conditions of Proposition 3.2 are satisfied.
- There exists an operator $H_s^L$ which solves the equation

$$
\frac{d}{ds} T_s^{1/2} = \frac{1}{2} H_s^L T_s^{1/2}
$$

on a domain which includes $\Omega_0$.
- The normalization function $\zeta(s)$ is differentiable.

Then one has

1) $\frac{d}{ds} \Omega_s = \left[ H_s^L - \frac{1}{2} \zeta'(s) \right] \Omega_s$.

2) $\frac{d}{ds} Y_s = (JH_s^L J) Y_s$ holds on a domain which includes $\Omega_0$.

3) An element $f_s$ of the dual of $A$ is defined by

$$
f_s(A) = \left( \frac{1}{2} [JH_s^L J A + AJH_s^L J] \Omega_s, \Omega_s \right) - \omega_s(A) \frac{d\zeta}{ds}, \quad A \in A;
$$

It satisfies $f_s(\mathbb{1}) = 0$. 

6
\[
\frac{d}{ds} \omega_s(A) = f_s(A) \quad \text{for all } A \in \mathcal{A}.
\] (4)

**Proof**

1) One calculates

\[
\frac{d}{ds} \Omega_s = \frac{d}{ds} e^{-\frac{1}{2}s^2T^1/2_{s/2} \Omega_s} = -\frac{1}{2} \zeta'(s) \Omega_s + e^{-\frac{1}{2}s^2H \Omega_s} T^1/2_{s/2} \Omega_s = \left[ H_s - \frac{1}{2} \zeta'(s) \right] \Omega_s.
\]

2) This follows immediately from \(Y_s \subset JT^{1/2} S\), proved in Proposition 3.2.

3) and 4) The proof is straightforward. \(\square\)

The functional \(f_s\) is a tangent vector, tangent to the geodesic \(s \mapsto \omega_s\) at the point \(s\). The assumptions made in the above Proposition are technical requirements which guarantee the existence of tangent vectors. They do not suffice to say that the geodesic is log-affine. A proposal to define the latter follows below.

### 4.2 Definition of a log-affine geodesic

The operator \(T_s\) is uniquely determined by the cocycle \((U_t^{(s)})_t\). This justifies the introduction of the following definition.

**Definition 4.2** Let be given a map \(s \in [0,1] \mapsto \omega_s \in \mathcal{M}\). Assume that each state \(\omega_s\) is labeled with a strongly continuous map \(t \mapsto U_t^{(s)}\), which forms a cocycle for the automorphism group \((\tau_t)_t\), in such a way that

\[
\omega_s(A) = e^{-\zeta(s)}(AU_t^{(s)} \Omega_0, U_t^{(s)} \Omega_0), \quad A \in \mathcal{A}.
\]

Let \(T_s\) denote the corresponding positive operator, as defined in Lemma 2.1. The map \(s \mapsto \omega_s\) is said to be a log-affine geodesic connecting \(\omega_1\) to \(\omega_0\) if \(U_t^{(0)} = I\) for all \(t\) and there exist a self-adjoint operator \(H\) and a function \(\Phi(s)\) such that for all \(s, r \in [0, 1]\) one has

\[
[\log T_s + \Phi(s)] - [\log T_r + \Phi(r)] = (s - r)H.
\] (5)

From \(U_t^{(0)} = I\) for all \(t\) it follows that \(T_0 = \Delta\). From 5 it follows that the derivative exists and is given by

\[
\frac{d}{ds} [\log T_s + \Phi(s)] \psi = H \psi, \quad \psi \in \text{dom} \, H.
\] (6)

The cocycles of a log-affine geodesic can be written as

\[
U_t^{(s)} = e^{it(\Phi(s) - \Phi(0))} e^{-it[\log \Delta - sH]} \Delta^i t.
\]

In the example of a semigroup, discussed in the Introduction, the operators \(U_t^{(s)}\) and \(T_s\) are of the form \(U_t^{(s)} = e^{istH}\), respectively \(T_s = e^{sH}\). This corresponds with \(\Delta = I\) and \(\Phi(s)\) a constant function. Hence, one has \(U_t^{(s)} = T_{t/2}^{s/2}\) with \(T_s = \exp[-sH]\). The state \(\omega_0\) is then a tracial
Hence, (2) is satisfied with \( \zeta \) is a straightforward generalization from a tracial state together with a semigroup to a faithful vector state together with a family of cocycles.

A further justification for the above definition comes from the compatibility with the existing notions of quantum exponential families and of exponential tangent spaces. See Sections 5 respectively.

It is straightforward to verify that if \( s \mapsto \omega_s \) is a log-affine geodesic connecting \( \omega_1 \) to \( \omega_0 \) then for any \( \lambda \in [0, 1] \) the map \( s \mapsto \omega_{\lambda s} \) is a log-affine geodesic connecting \( \omega_\lambda \) to \( \omega_0 \).

5 The matrix algebra

Let \( H_1, H_2, \ldots, H_k \) be self-adjoint \( n \)-by-\( n \) matrices. They define a density matrix \( \rho_0 \) by

\[
\rho_0 = \frac{1}{Z(\theta)} \exp(-\sum_{j=1}^{k} \theta^j H_j).
\]

Here, \( \theta \) belongs to some open convex domain \( D \subset \mathbb{R}^k \). The partition sum \( Z(\theta) \) is given by

\[
Z(\theta) = \text{Tr} \exp \left( -\sum_{j=1}^{k} \theta^j H_j \right).
\]

The expression (7) is the quantum analogue of a Boltzmann-Gibbs distribution. The operators \( H_1, H_2, \ldots, H_k \) have the meaning of parts of a Hamiltonian.

A state \( \omega_\rho \) on the algebra \( \mathcal{A} \) of all \( n \)-by-\( n \) matrices is defined by

\[
\omega_\rho(A) = \text{Tr} \rho_0 A, \quad A \in \mathcal{A}.
\]

By the Gelfand-Naimark-Segal (GNS) construction the elements \( \rho \) of \( \mathcal{A} \) act on a Hilbert space \( \mathcal{H} \) in which there exists a cyclic and separating vector \( \Omega_0 \) such that

\[
\text{Tr} \rho_0 A = (\Lambda \Omega_0, \Omega_0) \quad \text{for all} \quad A \in \mathcal{A}.
\]

Fix yet another set of parameters \( \eta \) in \( D \) and let \( H = -\sum_{j=1}^{k} (\eta^j - \theta^j) H_j \). Use the abbreviation \( \omega_\eta = \omega_{(1-s)\theta + s\eta} \). The corresponding density matrix \( \rho_{(1-s)\theta + s\eta} \) is denoted \( \rho_s \).

Let \( \Delta \) denote the modular operator corresponding to the state \( \omega_\rho \), and select \( \Omega_0 \) such that \( \Delta \Omega_0 = \Omega_0 \). The modular automorphisms \( \tau_i \) satisfy

\[
\tau_i[A] = \Delta^{-it} AA^\dagger = \rho_0^{-it} A \rho_0^{it}, \quad A \in \mathcal{A}.
\]

Let \( U^{(s)}_t = \rho_0^{-it} \rho_s^t \). It is the product of two unitary operators and hence it is itself a unitary operator as well. It belongs to \( \mathcal{A} \). One verifies that the cocycle condition is satisfied:

\[
U^{(s)}_{t+t} = \rho_0^{-it} \rho_s^{(r+t)} = \rho_0^{-ir} \rho_0^{-it} \rho_s^{(r+t)} = U^{(s)}_r \tau_i[U^{(s)}_t] = U^{(s)}_r \tau_t[U^{(s)}_r].
\]

Let \( \Xi^{(s)}_t = \rho_0^{-iz} \rho_s^{iz} \Omega_0 \). Then one has \( \Xi^{(s)}_{t/2} = \rho_s^{1/2} \rho_0^{-1/2} \Omega_0 \) so that for all \( A \in \mathcal{A} \)

\[
\left( A \Xi^{(s)}_{t/2}, \Xi^{(s)}_{t/2} \right) = \left( A \rho_0^{1/2} \rho_0^{-1/2} \Omega_0, \rho_s^{1/2} \rho_0^{-1/2} \Omega_0 \right) = \text{Tr} \rho_0 \left( \rho_0^{-1/2} \rho_s^{1/2} \rho_0^{-1/2} \Omega_0 \right) = \text{Tr} \rho_0 A = \omega_\rho(A).
\]

Hence, (2) is satisfied with \( \zeta(s) = 1 \) for all \( s \).

Note that the operator \( \Delta \rho_0^{-1} \) belongs to the commutant \( \mathcal{A}' \). This follows from

\[
(\Delta \rho_0^{-1} AB \Omega_0, C \Omega_0) = (C^* \Omega_0, B^* A^* \rho_0^{-1} \Omega_0) = \text{Tr} ABC^* = \text{Tr} BC^* A = (\Delta \rho_0^{-1} B \Omega_0, A^* C \Omega_0).
\]
Hence \( T_s = \Delta \rho_0^{-1} \rho_s \) implies that

\[
\log T_s = \log \rho_s - \log \rho_0 \Delta^{-1} 
\]

and

\[
\frac{d}{ds} \left[ \log T_s + \Phi(s) \right] = H,
\]

with

\[
\Phi(s) = \log Z(s) \equiv \log Z((1 - s)\theta + s\eta).
\]

One also has

\[
\log T_s - \log T_0 = sH - \Phi(s) + \Phi(0).
\]

On the other hand, the left logarithmic derivative is given by

\[
\frac{1}{2} H_s^L = \left( \frac{d}{ds} T_s^{1/2} \right) T_s^{-1/2} = \frac{1}{2} \int_0^1 du \, T_s^{u/2} (H - \Phi'(s)) T_s^{(1-u)/2}.
\]

It is self-adjoint and therefore it coincides with the right logarithmic derivative and the symmetric logarithmic derivative.

In [2] an operator \( X_s \) is introduced which belongs to the commutant \( A' \) and is defined by

\[
X_s \Omega_0 = \rho_s \rho_0^{-1} \Omega_0, \quad \text{in the notations of the present paper. A short calculation shows that this operator satisfies}
\]

\[
X_s = \rho_0^{-1} S^* \rho_s S = S^* T_s S \quad \text{and hence that it coincides with the operator}
\]

\[
X_s = Y_s^* Y_s \quad \text{as introduced in Proposition 3.2.}
\]

### 6 The abelian case

In this section the von Neumann algebra \( \mathcal{A} \) is (isomorphic with) the space \( L_{\infty}(\mathbb{R}^n, \mathbb{C}) \) of all essentially bounded complex functions on \( \mathbb{R}^n \) with its Lebesgue measure. The Hilbert space \( \mathcal{H} \) coincides with the space of square-integrable complex functions. A function \( f(x) \), element of \( \mathcal{A} \), acts on a square-integrable function \( g(x) \) by pointwise multiplication \( (fg)(x) = f(x)g(x) \). The commutant \( \mathcal{A}' \) of \( \mathcal{A} \) coincides with \( \mathcal{A} \). See [13], Part I, Ch. 7, Thm. 2.

Consider states \( \omega_\psi \) of \( \mathcal{A} \) of the form

\[
\omega_\psi(f) = \int_{\mathbb{R}^n} f(x) |\psi(x)|^2 \, dx,
\]

where \( \psi \in \mathcal{H} \) is a square-integrable function with \( L_2 \) norm

\[
||\psi||_2 = \left( \int_{\mathbb{R}^n} |\psi(x)|^2 \, dx \right)^{1/2}
\]

equal to 1. The states \( \omega_\psi \) can be identified with probability measures on \( \mathbb{R}^n \). Of interest is the set \( \mathcal{M} \) of states \( \omega_\psi \) where \( |\psi(x)|^2 \) is non-vanishing almost everywhere. This implies that the state \( \omega_\psi \) is faithful and that the vector \( \psi \) is cyclic and separating for the von Neumann algebra \( \mathcal{A} \).

The modular operator \( \Delta \) is the identity, i.e. the constant function 1. Therefore, any cocycle \( U(t) \) is of the form \( U(t) = \exp(ith) \) with \( h \) a measurable function. The analytic continuation \( \Xi_z \) of \( U(t)\psi \) to \( t = i/2 \) should belong to \( \mathcal{H} \). This is,

\[
\Xi_{i/2}(x) = e^{-\frac{i}{2}h(x)}\psi(x)
\]

must be square integrable. If this is the case, then a state \( \omega_h \) is defined by

\[
\omega_h(f) = e^{-\zeta(h)}(f\Xi_{i/2},\Xi_{i/2}) = e^{-\zeta(h)} \int \, dx \, f(x)e^{-\text{Re}h(x)}|\psi(x)|^2.
\]
The normalization $\zeta(h)$ is given by

$$
\zeta(h) = \log \int dx \ e^{-\text{Re} h(x)} |\psi(x)|^2 = \omega_0(e^{-\text{Re} h(x)}).
$$

**Definition 6.1** (Definition 3.14 of [14]) Fix a single $\omega_0$. A measurable real function $k(x)$ is said to belong to the exponential tangent space at the point $\omega_0$ if it satisfies the condition

$$
\int \left[ e^{|tk|} - 1 \right] |\psi(x)|^2 \ dx < +\infty \quad \text{for some } t \in \mathbb{R}, t \neq 0.
$$

(8)

**Lemma 6.2** Assume that the real function $k(x)$ belongs to the exponential space tangent at $\omega_0$. Then the integrals

$$
\int e^{\pm tk(x)} |\psi(x)|^2 \ dx
$$

converge for any $t$ for which (8) holds.

**Proof**

Split the function $k(x)$ into its positive and its negative part: $k = k_+ - k_-$. Then (8) can be written as

$$
\int e^{tk_+} |\psi(x)|^2 \ dx < +\infty \quad \text{and} \quad \int e^{tk_-} |\psi(x)|^2 \ dx < +\infty.
$$

Since the exponential function is bounded for negative values of the argument and $\psi(x)$ is square-integrable one concludes that (9) holds.

\[ \Box \]

**Lemma 6.3** Assume that $h(x)$ is locally square-integrable. Then the operator $H$, defined by $(H\phi)(x) = h(x)\phi(x)$, is a densely-defined self-adjoint operator on $H$ affiliated with $A$.

**Proof**

The continuous functions $\phi$ vanishing outside a compact subset $C_\phi$ of $\mathbb{R}^n$ form a dense subspace of $H$ and belong to the domain of $H$ because

$$
\int |H\phi(x)|^2 \ dx \leq ||\phi||^2 \int_{C_\phi} |h(x)|^2 \ dx < +\infty.
$$

Hence the operator $H$ is densely defined. It is self-adjoint on the domain

$$
\left\{ \phi : \int h^2(x) |\phi(x)|^2 \ dx < +\infty \right\}.
$$

\[ \Box \]

**Proposition 6.4** Fix a normalized square-integrable function $\psi(x)$ which is non-vanishing almost everywhere. Let be given a real function $k(x)$ which belongs to the exponential space tangent at $\omega_0$. Assume in addition that $k(x)$ is locally square-integrable. Select a $t$ for which (8) holds and let $h(x) = tk(x)$. Let $H$ be the self-adjoint operator discussed in Lemma 6.3. Then one has

1) $\psi$ belongs to the domain of the operators $\exp(\pm H/2)$;

2) Let $\Omega_1$ be given by

$$
\Omega_1 = [e^{H - \zeta(1)}]^{1/2} \psi, \quad \text{with} \quad \zeta(1) = \log ||e^{H/2} \psi||^2.
$$

Then the state $\omega_1$ defined by the vector $\Omega_1$ is connected to the state $\omega_0$ by a log-affine geodesic.

**Proof**
1) This follows immediately from Lemma 6.2.

2) Let

\[ H = \int \lambda \, dE_\lambda \]

be the spectral decomposition of \( H \). Let \( 0 \leq s \leq 1 \) and \( \beta \in \mathbb{R} \). Convexity of the exponential function implies that

\[ \int \exp(s \lambda - \beta) \, d(E_\lambda \psi, \psi) \leq e^{-\beta} \left[ 1 - s + s \int \exp(\lambda) \, d(E_\lambda \psi, \psi) \right]. \]

The latter integral is finite because \( \psi \) belongs to the domain of \( \exp H/2 \). One concludes from the inequality that \( \psi \) belongs to the domain of \( \exp(s \lambda - \beta)^{1/2} \) for all real \( \beta \).

Let

\[ \omega_s(A) = (A \Omega_s, \Omega_s) \quad \text{for all } A \in \mathcal{A}, \]

with

\[ \Omega_s = [\exp(sH - \zeta(s))]^{1/2} \psi \quad \text{and} \quad \zeta(s) = \log ||\exp(sH/2)\psi||. \]

Note that \( \Omega_0 = \psi \) and \( \omega_0 = \omega_\psi \).

Let us now verify that \( s \mapsto \omega_s \) is a log-affine connection of \( \omega_1 \) with \( \omega_0 \). In the present commutative context the one-parameter group of automorphisms \( \{\tau_t\}_t \) is the trivial one. The modular operator \( \Delta \) is the identity operator. The cocycle \( (U_t^{(s)})_t \) is the one-parameter group of unitaries defined by

\[ U_t^{(s)} = e^{istH}, \quad t \in \mathbb{R}. \]

The analytic continuation of \( t \mapsto U_t^{(s)} \Omega_0 = e^{istH} \psi \) is

\[ \Xi_{z}^{(s)} = e^{iszH} \psi. \]

The vector \( \Xi_{z}^{(s)} \) belongs to the Hilbert space \( \mathcal{H} \) because \( |s \, \text{Im } z| < 1/2 \) implies that

\[ \int dx \, |\Xi_{z}^{(s)}(x)|^2 = \int dx \, e^{-2s(\text{Im } z)h(x)} |\psi(x)|^2 \leq \int dx \, e^{-h(x)} |\psi(x)|^2 \]

\[ < +\infty. \]

The latter inequality holds because of item 1) of the Proposition.

The vector \( \Omega_s \) is given by

\[ \Omega_s = e^{-\frac{1}{2} \zeta(s)} \Xi_{\zeta(s)/2}^{(s)} = e^{-\frac{1}{2} \zeta(s)} e^{-\frac{1}{2} sH} \psi, \]

with

\[ \zeta(s) = \log \int dx \, e^{-sh(x)} |\psi(x)|^2. \]

The operator \( T_s \) is given by

\[ T_s = U_t^{(s)} = e^{-\frac{1}{2} sH}. \]

It satisfies \( \log T_s = -\frac{1}{2} sH \). Therefore, the requirement for the geodesic to be log-affine is fulfilled with a vanishing function \( \Phi(s) \).

Finally, note that in this abelian case the operators \( X_s \) and \( T_s \) coincide.
7 Summary

Section 4.2 proposes a definition for the notion of a log-affine geodesic $s \in [0, 1] \mapsto \omega_s$ connecting two states of a von Neumann algebra $\mathcal{A}$. With each state $\omega_s$ corresponds a cocycle $U^{(s)}$ of the modular automorphism group of the starting point $\omega_0$. The geodesic is said to be log-affine when the positive operator $T_s$ associated with the cocycle $U^{(s)}$ is a log-affine function of $s$, up to a scalar normalization factor. The definition is linked to that of existing notions in the case of a finite-dimensional Hilbert space and in the abelian case. The definition generalizes the notion of a trajectory which starts at a tracial state and which is described by the action of a semigroup.

The definition of log-affine geodesics as given here is only a first step in developing a general theory of the geometry of manifolds consisting of vector states on a von Neumann algebra and belonging to an exponential family.

A Appendix

Here, items 5) and 6) of Proposition 3.2 are proved. The essence of the proof is that $U_t \Delta^{-it} = T^{-it}$, where $T$ is a positive operator. In a finite-dimensional Hilbert space one has

$$U_{i/2} \Delta^{1/2} = T^{1/2} = \left(T^{1/2}\right)^* = \Delta^{1/2} \left(U_{i/2}\right)^*.$$ 

This implies

$$J \Xi_{i/2} = JU_{i/2} \Omega = JU_{i/2} \Delta^{1/2} \Omega = J \Delta^{1/2} \left(U_{i/2}\right)^* \Omega = S \left(U_{i/2}\right)^* \Omega = U_{i/2} \Omega = \Xi_{i/2}$$

and

$$Y = JU_{i/2} J = J \Delta^{1/2} \left(U_{i/2}\right)^* \Delta^{-1/2} J = J T^{1/2} S.$$

In the general case one has to take care of domain problems.

**Proposition A.1** For all $A$ in $\mathcal{A}$ is $A \Omega_0$ in the domain of $T^{1/2}$ and of $U_{i/2} \Delta^{1/2}$ and one has

$$T^{1/2} A \Omega_0 = U_{i/2} \Delta^{1/2} A \Omega_0 = JA^* J \Xi_{i/2}.$$

**Proof**

The vector $JA^* J \Omega_0 = JA^* \Omega_0 = \Delta^{1/2} A \Omega_0$ belongs to the domain of $\Delta^{-1/2}$ and also to the domain of $U_{i/2}$, with $U_{i/2} JA^* J \Omega_0 = JA^* JU_{i/2} \Omega_0 = JA^* J \Xi_{i/2}$. Therefore $A \Omega_0$ belongs to the domain of $U_{i/2} \Delta^{1/2}$ and one has $U_{i/2} \Delta^{1/2} A \Omega_0 = JA^* J \Xi_{i/2}^2$.

Next take any $\psi$ in the domain of $T^{1/2}$. Then $(T^{1/2} \psi, A \Omega_0)$ is the analytic continuation of $(T^{-it} \psi, A \Omega_0)$ to the point $t = i/2$. One has

$$(T^{-it} \psi, A \Omega_0) = (U_t \Delta^{it} \psi, A \Omega_0) = (\psi, U_{-i} \Delta^t A \Omega_0).$$

Take $t = i/2$ to obtain

$$(T^{1/2} \psi, A \Omega_0) = (\psi, U_{i/2} \Delta^{1/2} A \Omega_0).$$

This shows that $A \Omega_0$ belongs to the domain of $T^{1/2}$ and that $T^{1/2} A \Omega_0 = U_{i/2} \Delta^{1/2} A \Omega_0$.

**Proof of item 5)** The above Proposition is used in the following calculation. For all $Z$ in the commutant $\mathcal{A}'$ one has

$$(Z \Omega_0, \Xi_{i/2}) = (\Omega_0, Z^* \Xi_{i/2}) = (\Omega_0, T^{1/2} J Z \Omega_0) = (T^{1/2} \Omega_0, J Z \Omega_0) = (Z \Omega_0, JT^{1/2} \Omega_0).$$

This shows that $\Xi_{i/2} = JT^{1/2} \Omega_0 = J \Xi_{i/2}$. 

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Proof of item 6) Note that $\Omega_0$ is in the domain of $Y = JU_{i/2}^*J$ and that $Y\Omega_0 = JU_{i/2}\Omega_0 = J\Xi_{i/2} = \Xi_{i/2}$. This implies that $\Omega_0$ is also in the domain of $X^{1/2}$ and that $X^{1/2}\Omega_0 = K^*T^{1/2}\Omega_0 = K^*\Xi_{i/2}$, with $Y = KX^{1/2}$ the polar decomposition of $Y$. From Proposition A.1 it follows that for all $A \in \mathcal{A}$

$$T^{1/2}SA\Omega_0 = JA\Xi_{i/2} = JAY\Omega_0 = JYA\Omega_0.$$  

Because $\mathcal{A}\Omega_0$ is a core of $Y$ this implies that $JT^{1/2}S$ is an extension of $Y$.

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