On Malcev algebras that contain the three dimensional simple central Lie algebra

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Abstract

In this paper, we describe the structure of Malcev algebras $\mathcal{M}$ that contain the 3-dimensional simple central Lie algebra $L := \mathfrak{sl}_2(F)$ such that $mL \neq 0$ for any $m \neq 0$ from $\mathcal{M}$. Also we describe the structure of Malcev superalgebras $\mathcal{M} = M_0 \oplus M_1$ such that $M_0$ contains $L$ with $mL \neq 0$ for any homogeneous elements $0 \neq m \in M_0 \cup M_1$.

Keywords: Malcev algebra, non-Lie Malcev module, current Lie algebra, current Lie superalgebra, Kronecker factorization theorem.

1 Introduction

A Malcev algebra is a vector space $\mathcal{M}$ with a bilinear binary operation $(x, y) \mapsto xy$ satisfying the following identities:

$$x^2 = 0, \quad J(x, y, xz) = J(x, y, z)x,$$

where $J(x, y, z) = (xy)z + (yz)x + (zx)y$ is the Jacobian of the elements $x, y, z \in \mathcal{M}$. 

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Lie algebras fall into the variety of Malcev algebras because the Jacobian of any three elements vanish. The tangent space $T(L)$ of an analytic Moufang loop $L$ is another example of Malcev algebra. Let $\mathcal{A}$ be an alternative algebra, if we introduce a new product by means of a commutator $[x, y] = xy - yx$ into $\mathcal{A}$, we obtain a new algebra that will be denoted by $\mathcal{A}^{(-)}$. It is easy to verify that the algebra $\mathcal{A}^{(-)}$ is a Malcev algebra. All Malcev algebras obtained in this form are called \textit{special}. A classic example of a non-Lie Malcev algebra $\mathbb{M}$ of the traceless elements of the Cayley-Dickson algebra with the commutator of dimension 7 is one of most cornerstone examples.

The description of the structure of algebras and superalgebras that contain certain finite-dimensional algebras and superalgebras has a rich history, which has important applications in representation theory and category theory (for example, see \cite{6, 8, 10, 15}). The classical Wedderburn Theorem says that if a unital associative algebra $\mathcal{A}$ contains a central simple subalgebra of finite dimension $\mathcal{B}$ with the same identity element, then $\mathcal{A}$ is isomorphic to a Kronecker product $S \otimes_F \mathcal{B}$, where $S$ is the subalgebra of the elements that commute with each $b \in \mathcal{B}$. In particular, if $\mathcal{A}$ contains $M_n(F)$ as subalgebra with the same identity element, we have $\mathcal{A} \cong M_n(S)$ “coordinated” by $S$. Kaplansky in Theorem 2 of \cite{7} generalized the Wedderburn result to the alternative algebras $\mathcal{A}$ and the split Cayley algebra $\mathcal{B}$. Jacobson in Theorem 1 of \cite{5} gave a new proof of the result of Kaplansky using his classification of completely reducible alternative bimodules over a field of characteristic different of 2 and finally the first author in \cite{11} proved that this result is valid for any characteristic. Using this result, Jacobson \cite{5} proved a Kronecker Factorization Theorem for Jordan algebras that contain the Albert algebra with the same identity element. The statements of this type are usually called \textit{Kronecker factorization theorems}. In the case of right alternative algebras, S. Pchelintsev, O. Shashkov and I. Shestakov \cite{16} proved that every unital right alternative bimodule over a Cayley algebra (over an algebraically closed field of characteristic not 2) is alternative and they used that result to prove a coordinatization theorem for unital right alternative algebras containing a Cayley subalgebra with the same unit. Also in the paper \cite{10} the author proved an analogue of the Kronecker factorization theorem for Malcev algebras that contain $\mathcal{M}$ which was used to prove certain category equivalences when the characteristic of the base field is not equal to 2 and 3.

In the case of superalgebras, M. López-Díaz and I. Shestakov \cite{9, 8} studied the representations of simple alternative and exceptional Jordan superalgebras in characteristic 3 and through these representations, they obtained some analogues of the Kronecker Factorization Theorem for these superalgebras. Also, the first author \cite{11} obtained analogues of the Kronecker Factorization Theorem for some central simple alternative superalgebras, where in particular the Kronecker Factorization Theorem for $M_{(11)}(F)$ answers the analogue for superalgebras of the Jacobson’s problem \cite{5}, which was recently solved by the first author and I. Shestakov \cite{12, 13} in the split case. Similarly, C. Martinez and E. Zelmanov \cite{14} obtained a Kronecker Factorization Theorem for the exceptional ten dimensional Kac superalgebra $K_{10}$. Also, Y. Popov \cite{17} studied the representations of simple finite-dimensional noncommutative Jordan superalgebras and proved some analogues of the Kronecker factorization.
Theorem for such superalgebras.

The paper is organized as follows: In Section 2, we provided some definitions about Malcev algebras and superalgebras and their representations. In Section 3, we describe the structure of the Malcev algebras $\mathcal{M}$ that contain the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(F)$. In Section 4, we describe the structure of the Malcev superalgebras $\mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1$ such that $\mathcal{M}_0$ contains $\mathfrak{sl}_2(F)$.

2 Preliminaries

In this section we provide background material that is used along the way and some preliminary results.

Throughout this paper $F$ will be a field of characteristic different of 2 and 3. So by (1), a Malcev algebra $M$ is an anticommutative algebra that satisfies the identity

$$(xz)(yt) = ((xy)z)t + ((yz)t)x + ((zt)x)y + ((tx)y)z.$$ (2)

Also we will consider the following functions that play an important role in the theory of Malcev algebras:

$$J(x, y, z) = (xy)z + (yz)x + (zx)y,$$ the Jacobian of $x, y, z$,

$$[x, y, z] = (xy)z + x(yz),$$ the antiassociator,

$$\{x, y, z\} = (xy)z - (xz)y + 2x(yz) = J(x, y, z) + 3x(yz) = [x, y, z] - [x, z, y],$$

$$h(y, z, t, x, u) = \{yz, t, x\}u + \{yz, t, u\}x + \{yx, z, u\}t + \{yu, z, x\}t.$$ 2.1 Malcev Modules

We first recall the Lie algebra $\mathfrak{sl}_2(F)$ that has a basis $\{e, f, h\}$ such that

$$eh = e, \quad fh = -f, \quad ef = \frac{1}{2}h.$$ This algebra is simple central and it is one of the main objects of our paper.

Let $V$ be a vector space over $F$. The space $V$ is said to be module for the Malcev algebra $\mathcal{M}$ if there is an even $F$-linear map $\rho : \mathcal{M} \rightarrow \text{End}_F(V)(x \mapsto \rho_x)$, such that the split null extension $E = \mathcal{M} \oplus V$, with multiplication determined by

$$(x + v)(y + w) = xy + (\rho_y(v) - \rho_x(w))$$ is a Malcev algebra. The map $\rho$ is called a representation of $\mathcal{M}$.

The module $V$ is irreducible if $\rho \neq 0$ and does not contain any proper submodule. Also, $V$ is said to be almost faithful if $\ker \rho$ does not contain any nonzero ideal of $\mathcal{M}$.

A regular module, $\text{Reg} \mathcal{M}$, for algebra $\mathcal{M}$, is defined on the vector space $\mathcal{M}$ with the action of $\mathcal{M}$ coinciding with the multiplication in $\mathcal{M}$.
Example 2.1. The non Lie 2-dimensional module $M_2$ for the Lie algebra $\mathfrak{sl}_2(F)$ has a basis \{u, v\} such that

$$uh = u, \; vh = -v, \; ue = v, \; uf = 0, \; ve = 0, \; vf = -u.$$  

In characteristic 3, this is a Lie module.

2.2 Malcev superalgebras and their representations

Let now $M = M_0 \oplus M_1$ be a superalgebra (this is, $\mathbb{Z}_2$-graded algebra) and let $G = G_0 \oplus G_1$ be the Grassmann algebra generated by the elements $1, e_1, \ldots, e_n, \ldots$ over a field $F$. The Grassmann envelope of $M$ is defined to be $G(M) := G_0 \otimes M_0 + G_1 \otimes M_1$. Then $M$ is said to be Malcev superalgebra if $G(M)$ is a Malcev algebra. From this definition, $M$ satisfies the following superidentities

$$xy = -(-1)^{|x||y|}yx, \quad (3)$$

$$(-1)^{|y||z|}(zx)(yt) = ((xy)z)t + (-1)^{|x||y|+|z||t|}((yz)t)x$$

$$\quad + (-1)^{|z||t|(|y||z|+|t|)}((zt)x)y$$

$$\quad + (-1)^{|z||t|(|x||z|+|t|)}((tx)y)z. \quad (4)$$

where $x, y, z \in M_0 \cup M_1$ and $|x|$ denotes the parity index of a homogeneous element $x$ of $M$: $|x| = i$ if $x \in M_i$.

In any Malcev superalgebra $M = M_0 \oplus M_1$, the identities for Malcev algebras are easily superized to obtain the analogous graded identities. Thus, for homogeneous elements $x, y, z, t, u \in M$ we define

$$\tilde{J}(x, y, z) = (xy)z - x(yz) - (-1)^{|y||z|}(zx)y, \quad (super \; Jacobian),$$

$$\{x, y, z\} = (xy)z - (-1)^{|y||z|}(zx)y + 2x(yz), \quad (5)$$

$$\tilde{h}(y, z, t, x, u) = \{yz, t, u\}x + (-1)^{|x||u|}\{yz, t, x\}u$$

$$\quad + (-1)^{|x||(|z||t|+|u|)+|u||t|}}\{yx, z, u\}t + (-1)^{|u||(|z||t|+|x||t|)}\{yu, z, x\}t.$$  

Similarly, we define $\tilde{p}(x, y, z, t)$.

If $V = V_0 \oplus V_1$ is a $\mathbb{Z}_2$-graded $F$-space, then $\text{End}_F(V)$ becomes an associative superalgebra with

$$\text{End}_F(V)_0 = \{\varphi \in \text{End}_F(V) : \varphi(V_i) \subseteq V_i, \; i = 0, 1\},$$

$$\text{End}_F(V)_1 = \{\varphi \in \text{End}_F(V) : \varphi(V_i) \subseteq V_{-i}, \; i = 0, 1\}.$$  

The space $V$ is said to be module for the Malcev superalgebra $M$ if there is an even $F$-linear map $\rho : M \to \text{End}_F(V)(x \mapsto \rho_x)$, such that the split null extension $E = M \oplus V$, with
$\mathbb{Z}_2$-grading given by $E_0 = \mathcal{M}_0 \oplus V_0$, $E_1 = \mathcal{M}_1 \oplus V_1$, and with multiplication determined for homogeneous elements by

$$(x + v)(y + w) = xy + (\rho_y(v) - (-1)^{|x||w|}\rho_x(w))$$

is a Malcev superalgebra. The map $\rho$ is called a (super)-representation of $\mathcal{M}$.

When dealing with superalgebras, the ideals or submodules will always be graded. The module $V$ is irreducible if $\rho \neq 0$ and does not contain any proper submodule. Also, $V$ is said to be almost faithful if $\ker \rho$ does not contain any nonzero ideal of $\mathcal{M}$.

Let $V$ be a module over the Malcev superalgebra $\mathcal{M}$ and let $E = \mathcal{M} \oplus V$ be the corresponding split null extension. Let us consider

$$\tilde{\Gamma}_i = \{\alpha \in \operatorname{End}_F(E)_i : (xy)\alpha = x(y\alpha) = (-1)^{|y||x|}(x\alpha)y \; \forall \; x, y \in E_0 \cup E_1\}, \; i = 0, 1,$$

$$\tilde{\Gamma} = \tilde{\Gamma}(E) = \tilde{\Gamma}_0 \oplus \tilde{\Gamma}_1,$$ the supercentroid of $E$,

$$Z = Z(E) = \{\alpha \in \tilde{\Gamma} : \forall \; V \alpha \subseteq V \text{ and } \mathcal{M}\alpha \subseteq \mathcal{M}\},$$

$$K_i = K_i(V) = \{\varphi \in \operatorname{End}_F(V)_i : \varphi \rho_x = (-1)^{|x||\varphi|}\rho_x \varphi \; \forall \; x \in \mathcal{M}_0 \cup \mathcal{M}_1\}, \; i = 0, 1,$$

$$K = K(V) = K_0 \oplus K_1,$$ the supercentralizer of $V$.

The following proposition provides some basic properties of the subsuperalgebras $Z$ and $K$ when we consider irreducible almost faithful modules.

**Proposition 2.2** ([2], Proposition 4). Assume that $V$ is an irreducible almost faithful module for $\mathcal{M}$. Then

(i) $Z_1 = 0$ and $Z = Z_0$ is an integral domain which acts without torsion on $\mathcal{M}$.

(ii) $K_0$ is a skew field and any nonzero homogeneous element in $K$ acts bijectively on $V$.

(iii) The restriction homomorphism $\phi : Z \longrightarrow K_0(\alpha \longmapsto \alpha|_V)$ is one-to-one.

The Proposition 2.2 can be easily statement for algebras and its proof is analogous.

### 3 Algebras satisfying the identity $h = 0$

Let $\mathcal{M}$ be a Malcev algebra and let $H(\mathcal{M})$ be the subspace spanned by the elements $h(y, z, t, u, x)$; $H(\mathcal{M})$ is an ideal of $\mathcal{M}$ (see [3]). The variety $\mathcal{H}$ is defined as the class of Malcev algebras over $F$ that satisfy the identity $h(y, z, t, u, x) = 0$, that is, $H(\mathcal{M}) = 0$.

Throughout this section, let us consider Malcev algebras in the category $\mathcal{H}$.

The most important finite-dimensional elements of the variety $\mathcal{H}$ are the 3-dimensional simple Lie algebra $\mathfrak{sl}_2(F)$ and the 7-dimensional simple non-Lie Malcev algebra $\mathcal{M}$ over its centroid $\Gamma$, which is a field (see [1]).
Let $M$ be a Malcev algebra and consider the following function (see [4])
\[ p(x, y, z, t) = \{-zt, x, y\} - \{yt, z, x\} + \{xt, y, z\}. \]

The following lemma will be needed to establish some important elements in $\Gamma(M)$ (the centroid of $M$).

\textbf{Lemma 3.1 (18, Lemma 2).} Let $M$ be a Malcev algebra. Then for any $x, y, z, t, u \in M$:
\[ p(x, y, z, t)u = p(xu, y, z, t), \] (8)
\[ p(x, y, z, ut) = p(x, u, t, yz). \]

The following theorem is a consequence of (1, Theorem 5) for algebras.

\textbf{Theorem 3.2.} Let $M$ be a simple algebra and let $V$ be a module for $M$. Then $V$ is completely reducible.

Let us define the operator $\alpha(y, z, t) \in \text{End}_F(M)$ by
\[ x\alpha(y, z, t) = p(x, y, z, t). \]

It follows from (8) that $\alpha(y, z, t) \in \Gamma(M)$ for any $y, z, t \in M$.

Now let $V$ be a Malcev module over $L := \mathfrak{sl}_2(F)$. The following Lemma was proved in [18] under the assumption that $V = VL$, but in the Theorem 13 of [1] was proved without that condition. The result says that $V$ is decomposable as the direct sum of a Lie module and a non-Lie module over $L$.

\textbf{Lemma 3.3.} Let $V$ be a Malcev module over $L$ such that $vL \neq 0$ for any $v \neq 0$ from $V$. Then $V = V_l \oplus V_m$, where $V_l$ is a $L$-module, and the $L$-submodule $V_m$ satisfies the condition that for any $a, b \in L$ and $v \in V_m$
\[ vab - vba + 2v \cdot ab = 0. \]

Also $V_l = \{v \in V : J(v, L, L) = 0\}$ and $V_m = \{v \in V : \{v, L, L\} = 0\}$.

Let $M$ be a Malcev algebra that contains $L$ such that $mL \neq 0$ for any $m \neq 0$ from $M$. So we can consider $M$ as a Malcev module over $L$, then by Lemma 3.3 and by the proof of Theorem 3.2 we have
\[ M = M_l \oplus M_m, \]
where $\mathcal{M}_l = \sum_{0 \neq \alpha_i \in \alpha(M, L, L)} L\alpha_i$ and $\mathcal{M}_m = \sum_i M_{2i}$, $M_{2i}$ is a non Lie 2-dimensional module for the Lie algebra $L$ that has a basis $\{u_i, v_i\}$ (see Example 2.1) satisfying
\[ u_i h = u_i, \quad v_i h = -v_i, \quad u_i e = v_i, \quad u_i f = 0, \quad v_i e = 0, \quad v_i f = -u_i. \] (9)

The following result provides that $\mathcal{M}_l$ is a subalgebra of $\mathcal{M}$.

**Lemma 3.4.** $\mathcal{M}_l$ is a subalgebra of $\mathcal{M}$ and $\mathcal{M}_l \mathcal{M}_m \subseteq \mathcal{M}_m$.

**Proof:** By Lemma 3.3 we have $\mathcal{M}_l = \{m \in \mathcal{M} : J(m, L, L) = 0\}$ and $\mathcal{M}_m = \{m \in \mathcal{M} : \{m, L, L\} = 0\}$. Then for any $\alpha, \beta$ in $\Gamma(\mathcal{M})$
\[ J((L\alpha)(L\beta), L, L) \subseteq J(L^2 \alpha \beta, L, L) \subseteq J(L, L) \alpha \beta = 0, \]
so $(L\alpha)(L\beta) \subseteq \mathcal{M}_l$ and $\mathcal{M}_l \mathcal{M}_l \subseteq \mathcal{M}_l$. This show that $\mathcal{M}_l$ is a subalgebra of $\mathcal{M}$.

Similarly
\[ \{(L\alpha)\mathcal{M}_m, L, L\} \subseteq \{(L\mathcal{M}_m)\alpha, L, L\} \subseteq \{\mathcal{M}_m, L, L\} \alpha = 0, \]
then $(L\alpha)\mathcal{M}_m \subseteq \mathcal{M}_m$ and $\mathcal{M}_l \mathcal{M}_m \subseteq \mathcal{M}_m$.

Q.E.D.

The following proposition provides an analogous of the Kronecker factorization theorem for $A := \mathcal{M}_l$.

**Proposition 3.5.** Let $A$ be the subalgebra of $\mathcal{M}$. Then $A$ is a current Lie algebra, that is, $A = L \otimes_F U$, where $U$ is a certain commutative associative algebra.

**Proof:** We have $A = \sum_{0 \neq \alpha_i \in \alpha(M, L, L)} L\alpha_i$. Let $U = \sum_i F\alpha_i$ denote the span of all elements $\alpha_i$. Then $A = LU$. So let us prove that $U$ is a subalgebra of $\Gamma(\mathcal{M})$.

We fix arbitrary elements $\alpha, \beta \in U$ and $a, b \in L$. So $a\alpha \beta \in A$ because $J(a\alpha \beta, L, L) \subseteq J(a, L, L) \alpha \beta = 0$, then $a\alpha \beta = \sum_i a_i \alpha_i$ for some $a_i \in L$ and $\alpha_i \in U$. We denote $w = a\alpha \beta - \sum_i a_i \alpha_i = 0$,
\[ w = (\delta e + \lambda f + \gamma h)\alpha \beta - \sum_i (\delta_i e + \lambda_i f + \gamma_i h)\alpha_i = 0, \] (10)
where $a = \delta e + \lambda f + \gamma h$, $a_i = \delta_i e + \lambda_i f + \gamma_i h$ and $0 \neq \delta, \lambda, \gamma, \delta_i, \lambda_i, \gamma_i$ are elements of $F$. As $\alpha \beta \in \Gamma(\mathcal{M})$ and using the multiplication table of $L$, from (10)
\[ 0 = ew = (\frac{1}{2}\lambda h + \gamma e)\alpha \beta - \sum_i (\frac{1}{2}\lambda_i h + \gamma_i e)\alpha_i, \]
\[ 0 = e(ew) = \left( \frac{1}{2} \lambda e \right) \alpha \beta - \sum_i \left( \frac{1}{2} \lambda_i e \right) \alpha_i, \]

which implies, \( e\alpha\beta = e\tilde{\alpha}, \) where \( \tilde{\alpha} = \sum_i \lambda_i \lambda^{-1} \alpha_i \in U. \) Then \( e(\alpha\beta - \tilde{\alpha}) = 0. \) Also, it is easy to see \( f(\alpha\beta - \tilde{\alpha}) = 0 \) and \( h(\alpha\beta - \tilde{\alpha}) = 0; \) so

\[ L(\alpha\beta - \tilde{\alpha}) = 0 \]

and \( \alpha\beta - \tilde{\alpha} = 0. \) Therefore \( \alpha\beta = \tilde{\alpha} \in U; \) \( UU \subseteq U. \)

Now consider \( m \in \mathcal{M} \) and \( a \in L, \) then

\[ ((m\alpha)\beta) a = (m\alpha)(a\beta) = ((m\alpha)a)\beta = (m\beta)(a\alpha) = ((m\beta)a)\alpha. \]

If \( [\alpha, \beta] = \alpha\beta - \beta\alpha, \) we have

\[ ([\mathcal{M}\alpha, \beta])L = 0; \]

so \( \mathcal{M}[\alpha, \beta] = 0 \) and \( [\alpha, \beta]|_{\mathcal{M}} = 0 \) for any irreducible component \( V_i \) of \( \mathcal{M}, \) then by a version for algebras of the Proposition 2.2 (iii) we get \( [\alpha, \beta] = 0 \) because \( \phi : Z \longrightarrow K(\alpha \mapsto \alpha|_{V_i}) \) is one-to-one. Therefore \( [U, U] = 0; \) hence \( U \) is a commutative and associative algebra.

Also

\[ (a\alpha)(b\beta) = (a(b\beta))\alpha = ((ab)\beta)\alpha = (ab)(\beta\alpha) = (ab)(\alpha\beta). \]

Let \( v = ea_1 + fa_2 + ha_3 = 0 \) be, where \( \alpha_i \in U. \) Then \( 0 = ev = \frac{1}{2}ha_2 + ea_3 \) and \( 0 = e(ev) = \frac{1}{2}e\alpha_2 \) which implies \( e\alpha_2 = 0. \) Similarly \( f\alpha_2 = 0 \) and \( h\alpha_2 = 0; \) so

\[ L\alpha_2 = 0 \]

and \( \alpha_2 = 0. \) Hence \( \alpha_i = 0 \) for \( i = 1, 3. \) Therefore \( A \cong L \otimes_F U. \)

Q.E.D.

The following lemma provides that the multiplication in \( \mathcal{M}_m \) is zero.

**Lemma 3.6.** \( \mathcal{M}_m^2 = 0. \)

**Proof:** We have \( \mathcal{M}_m = \sum_i \mathcal{M}_{2i}, i \in I. \) Let \( M_{2i} \) and \( M_{2j} \) be the non Lie 2-dimensional Malcev modules over \( L \) with the standard bases \( \{u_i, v_i\} \) and \( \{u_j, v_j\} \) respectively. Then from (9) for all \( i, j \)

\[-(u_i v_j)f = (u_i v_j)(fh) \overset{\otimes}{=} ((u_i f)v_j)h + ((f v_j)h)u_i + ((v_j h) u_i)f + ((h u_i)f)v_j = (u_j h)u_i - (v_j u_i)f - (u_i f)v_j = u_j u_i - (v_j u_i)f \]

\[ 8 \]
we get
\[(u_i v_j)f = -\frac{1}{2} u_i u_j.\]  
(11)

Also
\[-(v_i v_j)f = (v_i v_j)(f h) = ((v_i f)v_j)h + ((f v_j)h)v_i + ((v_j h)v_i)f + ((h v_i)f)v_j\]
\[= -(u_i v_j)h + (u_j h)v_i - (v_j v_i)f + (v_i f)v_j = -(u_i v_j)h + u_j v_i + (v_i v_j)f - u_i v_j\]
and
\[(v_i v_j)f = \frac{1}{2} \{(u_i v_j)h - u_j v_i + u_i v_j\}.\]  
(12)

Similarly
\[\frac{1}{2}(u_i v_j)h = (u_i v_j)(e f) = ((u_i e)v_j)f + ((e v_j)f)u_i + ((v_j f)u_i)e + ((f u_i)e)v_j\]
\[= (v_i v_j)f - (u_j u_i)e = \frac{1}{2} \{(u_i v_j)h - u_j v_i + u_i v_j\} - (u_j u_i)e.\]
Hence
\[(u_j u_i)e = \frac{1}{2} \{u_i v_j + v_i u_j\}.\]  
(13)

Moreover
\[-(u_i u_j)f = (u_i u_j)(f h) = ((u_i f)u_j)h + ((f u_j)h)u_i + ((u_j h)u_i)f + ((h u_i)f)u_j\]
\[= (u_j u_i)h - (u_i f)u_j = (u_j u_i)h\]
and
\[(u_i u_j)f = (u_i u_j)h.\]  
(14)

Now
\[u_i u_j = (u_i h)(u_j h) = ((u_i u_j)h)h + ((u_j h)h)u_i + ((h u_i)u_j)h + ((h u_i)u_j)h\]
\[= ((u_i u_j)h)h + (u_j h)u_i - (u_i u_j)h = ((u_i u_j)f)h + u_j u_i - (u_i u_j)h,\]
then
\[2u_i u_j = ((u_i u_j)f)h - (u_i u_j)h\]
\[= (u_i f)(u_j h) - ((u_j f)h)u_i - ((f h)u_i)u_j - ((h u_i)u_j)f - (u_i u_j)h\]
\[= (f u_i)u_j + (u_i u_j)f - (u_i u_j)h = 0;\]
therefore
\[u_i u_j = 0.\]  
(15)
Now, replacing (15) in (11) and (13) we have \((u_iv_j)f = 0\) and
\[
u_iv_j = -v_iu_j.
\] (16)

We next find that
\[
(v_iu_j)h = ((u_ih)u_j)h = (u_iu_j)(eh) - ((eu_j)h)u_i - ((u_jh)u_i)e - ((hu_i)e)u_j \quad \text{(15)}
\]
\[
\equiv (v_jh)u_i - (u_ju_i)e + (u_i)e u_j \equiv -v_ju_i + v_iu_j = 0,
\] then
\[
(v_iu_j)h = 0. \quad \text{(17)}
\]

Also
\[
(u_iu_j)h = ((u_ih)v_j)h = (u_iu_j)(he) - ((hv_j)e)u_i - ((v_jh)e)u_i - ((ue_i)v_j)h
\]
\[
= -(v_iu_j)e - (v_jh)u_i + (v_i)e u_j
\]
\[
= -(u_iu_j)e - v_iu_j
\]
\[
\text{and}
\]
\[
(u_iu_j)e = -\frac{1}{2}v_iu_j. \quad \text{(18)}
\]

Also
\[
v_iu_j = (u_iu_j)(ue) \equiv (u_iu_j)e + ((u_jh)e)u_i + ((ee)i)u_j + ((ue_i)u_j)e
\]
\[
= (v_jh)u_i - (v_iu_j)e \equiv -\frac{1}{2}v_jv_i,
\]

which implies that
\[
v_iu_j = 0. \quad \text{(19)}
\]

Finally
\[
u_iu_j = (u_ih)(u_je) \equiv ((u_iu_j)h)e + ((u_jh)e)u_i + ((he)i)u_j + ((eu_i)u_j)h
\]
\[
= (u_iu_j)h - (eu_i)u_j - (v_iu_j)h \equiv v_ju_i + v_iu_j \equiv -2u_iu_j,
\]

and \(3u_iu_j = 0\). So
\[
u_iu_j = 0. \quad \text{(20)}
\]

Therefore from (15), (19) and (20) we get \(M_2, M_2j = 0\) and \(M_m^2 = 0\).

**Q.E.D.**

**Example 3.7.** We have the non Lie 5-dimensional Malcev algebra \(M_5 = sl_2(F) \oplus M_2\) that contains \(sl_2(F)\) with \(M_2^2 = 0\). So \(M_5\) is the split null extension of \(M_2\) by \(sl_2(F)\).
Consider \( \mathcal{N} := \mathcal{M}_m \). Then from (9) for any \( \alpha \in \mathcal{A} \) is easy to proof that \( \mathcal{N}\alpha \) is a non-Lie Malcev module over \( L \) and \( \{ \mathcal{N}\alpha, L, L \} = 0 \), then \( \mathcal{N}\alpha \subseteq \mathcal{N} \). So \( (LU)\mathcal{N} \subseteq (LN)U \subseteq \mathcal{N}U \subseteq \mathcal{N} \). Hence \( 0 \neq \mathcal{N} \triangleleft \mathcal{M} \) and from Lemma 3.6 we have \( \mathcal{N}^2 = 0 \).

Therefore, by Lemmas 3.4 and 3.6 we have that \( \mathcal{M} = \mathcal{A} \oplus \mathcal{N} \) with \( \mathcal{N}^2 = 0 \) is the split null extension of \( \mathcal{N} \) by Lie algebra \( \mathcal{A} \).

With this, we reach our main results in the case of Malcev algebras.

**Theorem 3.8.** Let \( \mathcal{M} \) be a Malcev algebra that contains \( L \) such that \( mL \neq 0 \) for any \( 0 \neq m \) from \( \mathcal{M} \). Then

\[
\mathcal{M} = L \otimes_F U + \mathcal{N},
\]

where \( \mathcal{A} \) is a certain commutative associative algebra, with \( 0 \neq \mathcal{N} \triangleleft \mathcal{M} \) and \( \mathcal{N}^2 = 0 \), that is, \( \mathcal{N} \) is a nilpotent ideal with index of nilpotency 2. Also \( \frac{\mathcal{M}}{\mathcal{N}} \) is the Lie algebra \( \mathfrak{sl}_2(U) \) over \( U \).

**Proof:** It follows directly from Proposition 3.5 and Lemma 3.6.

Q.E.D.

In Theorem 3.8 we can say that the Lie algebra \( \frac{\mathcal{M}}{\mathcal{N}} \cong \mathfrak{sl}_2(U) \) is “coordinated” by \( U \).

**Corollary 3.9.** There are no semiprime Malcev algebras \( \mathcal{M} \) that contain \( L \) such that \( mL \neq 0 \) for any \( 0 \neq m \) from \( \mathcal{M} \).

**Proof:** Let \( \mathcal{M} \) be a semiprime Malcev algebra with the indicated conditions. Then by Theorem 3.8 there is an ideal \( 0 \neq \mathcal{N} \) of \( \mathcal{M} \) such that \( \mathcal{N}^2 = 0 \), a contradiction.

Q.E.D.

## 4 Superalgebras satisfying the identity \( \tilde{h} = 0 \)

The variety \( \mathcal{H} \) of superalgebras is the class of Malcev superalgebras in which \( \tilde{h} = 0 \) (\( \tilde{h} \) is defined in (7)). Throughout this section we will assume that all Malcev superalgebra belong to \( \mathcal{H} \). Let \( \mathcal{M} \) a Malcev superalgebra, then by Lemma 3.1

\[
\tilde{p}(x, y, z, t)u = (-1)^{|u||y|+|z|+|t|}\tilde{p}(xu, y, z, t),
\]

\[
\tilde{p}(x, y, z, ut) = (-1)^{(|u||y|+|z|)(|y|+|z|)}\tilde{p}(x, u, t, yz)
\]

for any homogeneous elements \( x, y, z, t, u \in \mathcal{M} \).

Let us define the operator \( \tilde{\alpha}(y, z, t) \in \text{End}_F(\mathcal{M}) \) by

\[
x\tilde{\alpha}(y, z, t) = \tilde{p}(x, y, z, t).
\]

It follows from (21) that \( \tilde{\alpha}(y, z, t) \in \tilde{\Gamma}(\mathcal{M}) \) for any \( y, z, t \in \mathcal{M} \).

Now let \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) be a Malcev superalgebra such that the even part \( \mathcal{M}_0 \) contains \( L \) with \( mL \neq 0 \) for all homogeneous elements \( 0 \neq m \in \mathcal{M}_0 \cup \mathcal{M}_1 \). Then we consider \( \mathcal{M} \) as a Malcev module over \( L \), so by Lemma 3.3 and by the proof of Theorem 3.2 we have \( \mathcal{M}_0 = \mathcal{M}_0 \oplus \mathcal{M}_m_0, \mathcal{M}_1 = \mathcal{M}_1 \oplus \mathcal{M}_m_1 \) and

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\[ \mathcal{M} = \mathcal{M}_{t_0} \oplus \mathcal{M}_{m_0} \oplus \mathcal{M}_{l_1} \oplus \mathcal{M}_{m_1}, \]

where

\[ \mathcal{M}_{t_0} = \sum_{0 \neq \alpha \in \mathcal{M}_{t_0}(L, L)} L \alpha_i, \quad \mathcal{M}_{l_1} = \sum_{0 \neq \beta \in \mathcal{M}_{l_1}(L, L)} L \beta_i, \]

\[ \mathcal{M}_{m_0} = \sum_i M_{2i} \quad \text{and} \quad \mathcal{M}_{m_1} = \sum_i \tilde{M}_{2i}, \]

here \( M_{2i} \) and \( \tilde{M}_{2i} \) are non Lie 2-dimensional modules for the Lie algebra \( L \) with bases \( \{ u_i, v_i \} \) and \( \{ \tilde{u}_i, \tilde{v}_i \} \) respectively and both satisfy (9) (see Example 2.1).

From (3), using the same arguments of the proof of Lemma 3.4 and by the \( \mathbb{Z}_2 \)-grading of \( \mathcal{M} \) is easy to proof that \( M = \mathcal{M}_{t_0} + \mathcal{M}_{l_1} \) is a subsuperalgebra of \( \mathcal{M} \).

The following proposition provides an analogous of the Kronecker factorization theorem for \( \mathcal{M} \).

**Proposition 4.1.** Let \( M \) be the subsuperalgebra of \( \mathcal{M} \). Then \( M \) is a current Lie superalgebra, that is, \( M = L \otimes_F \mathcal{U} \), where \( \mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1 \) is a certain supercommutative associative superalgebra.

**Proof:** We have \( \mathcal{M}_{t_0} = \sum_{0 \neq \alpha \in \mathcal{M}_{t_0}(L, L)} L \alpha_i \) and \( \mathcal{M}_{l_1} = \sum_{0 \neq \beta \in \mathcal{M}_{l_1}(L, L)} L \beta_j \). Let \( \mathcal{U}_0 = \sum_i F \alpha_i \) and \( \mathcal{U}_1 = \sum_j F \beta_j \) denote the span of all elements \( \alpha_i \) and \( \beta_j \) respectively. Then \( \mathcal{M}_{t_0} = L \mathcal{U}_0 \), \( \mathcal{M}_{l_1} = L \mathcal{U}_1 \) and \( M = L \mathcal{U}_0 + L \mathcal{U}_1 \).

So let us go to prove that \( \mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1 \) is a subsuperalgebra of \( \tilde{\Gamma}(\mathcal{M}) \). For all \( a, b \in L \) we have

\[ (a \alpha)(b \beta) = (-1)^{|a|(|b| + |\beta|)} (a(b \beta)) \alpha = (-1)^{|\alpha|(|b| + |\beta|)} ((ab) \beta) \alpha = (-1)^{|\alpha|(|b| + |\beta|)} (ab)(\beta \alpha). \] (22)

So by (22) and using the relationships

\[ M_{t_1}M_{t_j} \subseteq M_{t_{(i+j) \bmod 2}}, \quad i, j = 0, 1 \]

of the \( \mathbb{Z}_2 \)-grading of \( \mathcal{M} \) and the proof of the Proposition 3.5 we get

\[ \mathcal{U}_i \mathcal{U}_j \subseteq \mathcal{U}_{(i+j) \bmod 2}, \quad i, j = 0, 1. \]

Hence \( \mathcal{U} = \mathcal{U}_0 \oplus \mathcal{U}_1 \) is a subsuperalgebra.

We fix arbitrary homogeneous elements \( \alpha, \beta \in \mathcal{U}, \ m \in \mathcal{M} \) and \( \alpha \in L \), then

\[ ((m \alpha) \beta) a = (-1)^{|\alpha||\beta|}(m \alpha)(a \beta) = (-1)^{|\alpha|(|\beta| + |\alpha| + |\alpha||\beta|)}(m \alpha a) \beta = (-1)^{|\alpha|(|\alpha| + |\beta|)}((m \beta)a) \alpha = (-1)^{|\alpha||\beta|}(m \beta a) \alpha. \]

If \( [\alpha, \beta] = \alpha \beta - (-1)^{|\alpha||\beta|} \beta \alpha \), we have
(M[\alpha, \beta])L = 0,

so M[\alpha, \beta] and [\alpha, \beta]|_M = 0. In particular, [\alpha, \beta]|_{V_i} = 0 for any irreducible components V_i of M, then by Proposition 2.2 iii) [\alpha, \beta] = 0 because \phi : Z \rightarrow U(\alpha \mapsto \alpha|_{V_i}) is one-to-one. Therefore [U, U] = 0; hence U is a supercommutative and associative superalgebra.

Also, by (22) and [U, U] = 0

\[(a\alpha)(b\beta) = (-1)^{|\alpha||\beta|}|ab|(ab)(\alpha\beta)\]

\[= (-1)^{|\alpha||\beta|}(ab)(\alpha\beta)\]

for all a, b \in L.

As in the Proposition 3.5, using the multiplication table of L, it is easy to see that the superalgebra U is free over L. Therefore M \cong L \otimes_F U.

Q.E.D.

As in Lemma 3.6 the following lemma provides that the multiplication in M_{m0} and M_{m1} is zero.

Lemma 4.2. M^2_{m0} = M^2_{m1} = 0.

Proof: As M_0 is a subalgebra of M. Therefore, by Lemma 3.6 we have M^2_{m0} = 0.

We have M_{m1} = \sum_i \tilde{M}_{2i}, i \in J. Let \tilde{M}_{2i} and \tilde{M}_{2j} be non Lie 2-dimensional Malcev modules over L with the standard bases \{\tilde{u}_i, \tilde{v}_i\} and \{\tilde{u}_j, \tilde{v}_j\} respectively. Then from (9) for all i, j

\[-(\tilde{u}_i\tilde{v}_j)f = (\tilde{u}_i\tilde{v}_j)(fh) \quad \text{by (1)}\]

\[= ((\tilde{u}_i f)\tilde{v}_j)h - ((f\tilde{v}_j)h)\tilde{u}_i - (h\tilde{u}_i)f + ((h\tilde{u}_i)f)\tilde{v}_j\]

\[= -(\tilde{u}_j h)\tilde{u}_i + (\tilde{v}_j\tilde{u}_i)f - (\tilde{u}_i f)\tilde{v}_j = -\tilde{u}_j\tilde{u}_i + (\tilde{u}_i\tilde{v}_j)f \quad \text{by (8)}\]

and

\[(\tilde{u}_i\tilde{v}_j)f = \frac{1}{2} \tilde{u}_i\tilde{u}_j. \quad \text{(23)}\]

Moreover

\[-(\tilde{v}_i\tilde{v}_j)f = (\tilde{v}_i\tilde{v}_j)(fh) \quad \text{by (1)}\]

\[= ((\tilde{v}_i f)\tilde{v}_j)h - ((f\tilde{v}_j)h)\tilde{v}_i - (h\tilde{v}_i)f + ((h\tilde{v}_i)f)\tilde{v}_j\]

\[= -(\tilde{u}_i\tilde{v}_j)h - (\tilde{u}_j h)\tilde{v}_i + (\tilde{v}_j\tilde{v}_i)f + (\tilde{u}_i f)\tilde{v}_j = -(\tilde{u}_i\tilde{v}_j)h - \tilde{u}_j\tilde{v}_i + (\tilde{v}_j\tilde{v}_i)f - \tilde{u}_i\tilde{v}_j\]

and

\[(\tilde{v}_i\tilde{v}_j)f = \frac{1}{2} \{(\tilde{u}_i\tilde{v}_j)h + \tilde{u}_j\tilde{v}_i + \tilde{u}_i\tilde{v}_j\}. \quad \text{(24)}\]
Similarly
\[
\frac{1}{2}(\tilde{u}_i \tilde{v}_j)h = (\tilde{u}_i \tilde{v}_j)(ef) = ((\tilde{u}_i e)\tilde{v}_j)f - ((e\tilde{v}_j)f)\tilde{u}_i - ((\tilde{v}_j f)\tilde{u}_i)e + ((f\tilde{u}_i)e)\tilde{v}_j
\]
\[
= (\tilde{v}_j \tilde{v}_j)f + (\tilde{u}_j \tilde{u}_i)e = \frac{1}{2}\{(\tilde{u}_i \tilde{v}_j)h + \tilde{u}_j \tilde{v}_i + \tilde{u}_i \tilde{v}_j\} + (\tilde{u}_j \tilde{u}_i)e.
\]
Hence
\[
(\tilde{u}_j \tilde{u}_i)e = -\frac{1}{2}\{\tilde{u}_i \tilde{v}_j + \tilde{u}_j \tilde{v}_i\}. \tag{25}
\]
Also
\[
-(\tilde{u}_i \tilde{u}_j)f = (\tilde{u}_i \tilde{v}_j)(fh) = ((\tilde{u}_i f)\tilde{v}_j)h - ((f\tilde{v}_j h)\tilde{u}_i) - ((\tilde{v}_j f)\tilde{u}_i)h + ((h\tilde{u}_i)f)u_j
\]
\[
= -(\tilde{u}_j \tilde{u}_i)h - (\tilde{u}_i f)\tilde{u}_j = -(\tilde{u}_j \tilde{u}_i)h - (\tilde{u}_i f)\tilde{u}_j
\]
and
\[
(\tilde{u}_i \tilde{u}_j)f = (\tilde{u}_i \tilde{v}_j)h. \tag{26}
\]
Now
\[
\tilde{u}_i \tilde{u}_j = (\tilde{u}_i h)(\tilde{u}_j h) = ((\tilde{u}_i \tilde{u}_j)h)h - ((\tilde{u}_j h)\tilde{u}_i) + ((h\tilde{u}_i)\tilde{u}_j) + ((h\tilde{u}_i)\tilde{u}_j)h
\]
\[
= ((\tilde{u}_i \tilde{u}_j)h)h - (\tilde{u}_j h)\tilde{u}_i - (\tilde{u}_i \tilde{u}_j)h \tag{26}
\]
then from (3)
\[
2\tilde{u}_i \tilde{u}_j = ((\tilde{u}_i \tilde{u}_j)h)h - (\tilde{u}_i \tilde{u}_j)h \tag{24}
\]
\[
= (\tilde{u}_i f)(\tilde{u}_j h) + ((\tilde{u}_j f)h)\tilde{u}_i - ((f\tilde{u}_j h)\tilde{u}_i) - ((h\tilde{u}_i)\tilde{u}_j) \tag{26}
\]
\[
= (f\tilde{u}_i)\tilde{u}_j + (\tilde{u}_i \tilde{u}_j)f - (\tilde{u}_i \tilde{u}_j)h \tag{26}
\]
so
\[
\tilde{u}_i \tilde{u}_j = 0. \tag{27}
\]
Replacing (27) in (23) and (25) we have \((\tilde{u}_i \tilde{v}_j)f = 0\) and
\[
\tilde{u}_i \tilde{v}_j = -\tilde{u}_j \tilde{v}_i. \tag{28}
\]
Next, we have
\[
(\tilde{v}_j \tilde{u}_i)h = ((\tilde{v}_j e)\tilde{u}_i)h = (\tilde{v}_j \tilde{u}_i)(eh) + ((e\tilde{u}_i)h)\tilde{v}_j + ((\tilde{u}_i h)\tilde{v}_j)e - ((h\tilde{u}_i)h)e\tilde{v}_j
\]
\[
= (\tilde{u}_i \tilde{u}_j)e - (\tilde{v}_j h)\tilde{u}_i + (\tilde{u}_j \tilde{u}_i)e + (\tilde{u}_i e)\tilde{u}_j \tag{27}
\]
\[
\tilde{v}_j \tilde{u}_i \tag{28}
\]
so
\[
\tilde{v}_j \tilde{u}_i = -\tilde{u}_j \tilde{v}_i. \tag{28}
\]
then
\[(\tilde{v}_i \tilde{u}_j) h = 0.\] (29)

Also
\[
(\tilde{u}_i \tilde{v}_j) e = ((\tilde{u}_i h) \tilde{v}_j) e = (\tilde{u}_i \tilde{v}_j)(he) + ((h \tilde{v}_j) e) \tilde{u}_i + ((\tilde{v}_j e) \tilde{u}_i) h - ((e \tilde{u}_i) h) \tilde{v}_j
\]
\[= - (\tilde{u}_i \tilde{v}_j) e + (\tilde{v}_j e) \tilde{u}_i + (\tilde{v}_i h) \tilde{v}_j
\]
and
\[(\tilde{u}_i \tilde{v}_j) e = - \frac{1}{2} \tilde{v}_i \tilde{v}_j.\] (30)

Furthermore,
\[
\tilde{v}_i \tilde{v}_j = (\tilde{u}_i e) (\tilde{u}_j e) = ((\tilde{u}_i \tilde{u}_j) e) e - ((\tilde{u}_j e) e) \tilde{u}_i + ((e e) \tilde{u}_i) \tilde{u}_j + ((e \tilde{u}_i) \tilde{u}_j) e
\]
\[= - (\tilde{v}_j e) \tilde{u}_i - (\tilde{v}_i \tilde{u}_j) e = \frac{1}{2} \tilde{v}_j \tilde{v}_i,
\]
which implies that
\[
\tilde{v}_i \tilde{v}_j = 0.\] (31)

Similarly
\[
\tilde{u}_i \tilde{v}_j = (\tilde{u}_i h) (\tilde{u}_j e) = ((\tilde{u}_i \tilde{u}_j) h) e - ((\tilde{u}_j h) e) \tilde{u}_i + ((h e) \tilde{u}_i) \tilde{u}_j + ((e \tilde{u}_i) \tilde{u}_j) h
\]
\[= - (\tilde{v}_j e) \tilde{u}_i - (e \tilde{u}_i) \tilde{u}_j - (\tilde{v}_i \tilde{u}_j) h \quad \text{(29)}
\]
\[- \tilde{v}_j \tilde{u}_i + \tilde{v}_i \tilde{u}_j \quad \text{(30)}
\]
then
\[2 \tilde{u}_i \tilde{v}_j = \tilde{v}_i \tilde{u}_j = - \tilde{u}_i \tilde{v}_j, \quad 3 \tilde{u}_i \tilde{v}_j = 0 \quad \text{and}
\]
\[\tilde{u}_i \tilde{v}_j = 0.\] (32)

Therefore from (27), (31) and of the last equality we get \(\tilde{M}_2, \tilde{M}_2j = 0\) and \(M^2_{m1} = 0\).

Q.E.D.

The proof of the following result is straightforward and analogous of the proof of Lemma 3.6. We do not include the process because basically use the identities (3) and (4) with respect to the parities of the homogeneous elements as in Lemma 4.2.

**Lemma 4.3.** \(M_{m0}, M_{m1} = 0\).

Denote \(\tilde{N} := M_{m0} \oplus M_{m1}\). Then for any homogeneous element \(0 \neq \alpha \in U_0 \cup U_1\) and from (29) it is easy to prove that \(\tilde{N} \alpha\) is a non-Lie Malcev module over \(L\) and \{\(\tilde{N} \alpha, L, L\}\ \subset \{\tilde{N}, L, L\}\ \alpha = 0, \) then \(\tilde{N} \alpha \subset \tilde{N}\). So (\(LU)\tilde{N} \subset (L\tilde{N}) U \subset \tilde{N} U \subset \tilde{N}\). Hence \(0 \neq \tilde{N} \subset \mathcal{M}\) and by Lemmas 4.2 and 4.3 we have \(\tilde{N}^2 = 0\).

With this, we reach our main results in the case of Malcev superalgebras.
**Theorem 4.4.** Let \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) be a Malcev superalgebra such that \( \mathcal{M}_0 \) contains \( L \) with \( mL \neq 0 \) for all homogeneous elements \( 0 \neq m \in \mathcal{M}_0 \cup \mathcal{M}_1 \). Then

\[
\mathcal{M} = L \otimes_F U + \tilde{\mathcal{N}},
\]

where \( U \) is a certain supercommutative associative superalgebra, with \( 0 \neq \tilde{\mathcal{N}} \triangleleft \mathcal{M} \) and \( \tilde{\mathcal{N}}^2 = 0 \), that is, \( \tilde{\mathcal{N}} \) is a nilpotent ideal with index of nilpotency 2. Also \( \frac{\mathcal{M} \setminus \tilde{\mathcal{N}}}{\tilde{\mathcal{N}}} \) is the Lie superalgebra \( \mathfrak{sl}_2(U) \) over \( U \).

**Proof:** The Theorem follows from Proposition 4.1 and Lemmas 4.2 and 4.3.

Q.E.D.

Also in Theorem 4.4 we can say that the Lie superalgebra \( \frac{\mathcal{M} \setminus \tilde{\mathcal{N}}}{\tilde{\mathcal{N}}} \cong \mathfrak{sl}_2(U) \) is “coordinated” by \( U \). In addition, \( \mathcal{M} = M \oplus \tilde{\mathcal{N}} \) with \( \tilde{\mathcal{N}}^2 = 0 \) is the split null extension of \( \tilde{\mathcal{N}} \) by Lie superalgebra \( M \).

**Corollary 4.5.** There are no semiprime Malcev superalgebras \( \mathcal{M} = \mathcal{M}_0 \oplus \mathcal{M}_1 \) such that \( \mathcal{M}_0 \) contains \( L \) with \( mL \neq 0 \) for all homogeneous elements \( 0 \neq m \in \mathcal{M}_0 \cup \mathcal{M}_1 \).

**Proof:** Let \( \mathcal{M} \) be a semiprime Malcev superalgebra with the indicated conditions. Then by Theorem 4.4 there is an ideal \( 0 \neq \tilde{\mathcal{N}} \) of \( \mathcal{M} \) such that \( \tilde{\mathcal{N}}^2 = 0 \), a contradiction.

Q.E.D.

## 5 Declaration of competing interest

There is no competing interest.

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