Resolving compacta by free $p$-adic actions

by

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Abstract. We study compacta that can be obtained as the orbit spaces of a free action of the $p$-adic integers on a compactum of a lower dimension and focus on compacta with $\dim \mathbb{Z} [1/p] = 1$.

1. Introduction. We always assume that $p$ is a prime number; $A_p$ stands for the group of $p$-adic integers and we denote $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ (the $p$-cyclic group), $\mathbb{Z}_p\infty = \mathbb{Z}[1/p]/\mathbb{Z}$ (the $p$-adic circle).

The Hilbert–Smith conjecture asserts that a compact group effectively (and continuously) acting on a manifold must be a Lie group. This assertion is equivalent to the following one: there is no effective action of $A_p$ on a manifold. The Hilbert–Smith conjecture is proved for manifolds of dimension $\leq 3$ (see [14,15]) and open even for free actions of $A_p$ in dimension $> 3$. Yang [18] showed that if $A_p$ effectively acts on an $n$-manifold $M$ then either $\dim M/A_p = \infty$ or $\dim M/A_p = n + 2$. This naturally suggests examining if the latter dimensional relations may occur in a more general setting, namely when $M$ is just a finite-dimensional compactum (= compact metric space).

One of these relations was confirmed by Raymond and Williams [17] who constructed an effective action of $A_p$ on an $n$-dimensional compactum $X$, $n \geq 2$, with $\dim X/A_p = n + 2$. However, it remains open for more than 50 years now whether there exists a free action of $A_p$ on a finite-dimensional compactum $X$ such that $\dim X/A_p = \dim X + 2$ or $\dim X/A_p = \infty$.

An important collection of dimensional properties of the orbit spaces under actions of $A_p$ is provided by cohomological dimension. Recall that the cohomological dimension $\dim_G X$ of a compactum $X$ with respect to an abelian group $G$ is the least integer $n$ (or $\infty$ if no such $n$ exists) such that the Čech cohomology $H^{n+1}(X,F;G)$ vanishes for every closed $F \subset X$.

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Clearly $\dim_G X \leq \dim X$ for every group $G$ and, by the Alexandrov theorem, $\dim X = \dim_Z X$ if $X$ is finite-dimensional. The first example of an infinite-dimensional compactum $X$ with $\dim_Z X < \infty$ was constructed by A. Dranishnikov. His survey [3] is an excellent introduction to cohomological dimension theory.

It is known that a free action of $A_p$ on a compactum $X$ imposes the following dimensional relations between $X$ and $Y = X/A_p$: $\dim_{\mathbb{Z}[1/p]} Y = \dim_{\mathbb{Z}[1/p]} X$, $\dim_{\mathbb{Z}[p]} X \leq \dim_{\mathbb{Z}[p]} Y \leq \dim_{\mathbb{Z}[p]} X + 1$, $\dim_{\mathbb{Z}[p]} X \leq \dim_{\mathbb{Z}[p]} Y \leq \dim_{\mathbb{Z}[p]} X + 1$ and $\dim_{\mathbb{Z}} X \leq \dim_{\mathbb{Z}} Y \leq \dim_{\mathbb{Z}} X + 2$. Moreover, if $\dim_{\mathbb{Z}} Y = \dim_{\mathbb{Z}} X + 2$ then $\dim_{\mathbb{Z}[p]} Y = \dim_{\mathbb{Z}[p]} Y = \dim_{\mathbb{Z}} X + 1$. We will refer to these relations as the Yang relations [18] and, for the reader’s convenience, provide a short proof of these in Remark 3.3. Also note that $\dim X \leq \dim Y$ since light maps (= maps with 0-dimensional fibers) of compacta cannot lower the covering dimension.

There is a nice characterization of cohomological dimension in terms of extensions of maps. A CW-complex $K$ is said to be an absolute extensor for a space $X$, written $\text{ext-dim } X \leq K$, if every map from a closed subset of $X$ to $K$ continuously extends over $X$. The following well-known property establishes a link between cohomological dimension and absolute extensors: $\dim_G X \leq n$ if and only if the Eilenberg–MacLane complex $K(G, n)$ is an absolute extensor for $X$. Using this characterization and representing $K(\mathbb{Z}[1/p], 1)$ as the infinite telescope of the $p$-fold covering $S^1 \to S^1$ one can easily show

**Proposition 1.1.** Let $X$ be a compactum. Then $\dim_{\mathbb{Z}[1/p]} X \leq 1$ if and only if for every map $f : F \to S^1$ from a closed subset $F$ of $X$ to the circle $S^1$ there is a natural number $k$ such that $f$ followed by the $p^k$-fold covering map $S^1 \to S^1$ of $S^1$ extends over $X$.

Let us say that a compactum $Y$ can be resolved by a free $p$-adic action on a compactum $X$ if there is a free action of the $p$-adic integers $A_p$ on $X$ such that $Y = X/A_p$. We will then briefly say that $Y$ is $p$-resolvable by $X$ (keeping in mind that we consider only free actions of $A_p$).

In this paper we study compacta $Y$ that are $p$-resolvable by compacta of a lower dimension and mainly focus on compacta with $\dim_{\mathbb{Z}[1/p]} Y = 1$, and in particular, on compacta which are $p$-resolvable by 1-dimensional compacta (and hence, by Yang’s relations, have $\dim_{\mathbb{Z}[1/p]}$ equal to 1), a case that turns out to be highly non-trivial. More motivation for considering orbit spaces $Y$ with $\dim_{\mathbb{Z}[1/p]} Y = 1$ comes from

**Theorem 1.2.** If $A_p$ acts on a finite-dimensional compactum $X$ so that $Y = X/A_p$ is infinite-dimensional then there exists an invariant compactum $X' \subset X$ on which the action of $A_p$ is free and whose orbit space $Y' = X'/A_p$ is infinite-dimensional with $\dim_{\mathbb{Z}[1/p]} Y' = 1$. 
The most obvious obstruction to the $p$-resolvability of a compactum $Y$ by a compactum of dimension $< \dim Y$ is $H^1(Y; \mathbb{Z}_p) = 0$. Indeed, $A_p$ is the inverse limit of $\mathbb{Z}_{p^n}$ and hence every compactum that $p$-resolves $Y$ can be represented as the inverse limit of $\mathbb{Z}_{p^n}$-bundles over $Y$ with bonding maps being $\mathbb{Z}_p$-bundle projections. All the bundles in this paper are assumed to be principal. Recall that $B\mathbb{Z}_m = K(\mathbb{Z}_m, 1)$, and hence, $H^1(Y; \mathbb{Z}_m)$ represents the $\mathbb{Z}_m$-bundles over $Y$. We will also refer to connected $\mathbb{Z}_m$-bundles as $\mathbb{Z}_m$-coverings, keeping in mind that they are indeed normal covering spaces whose deck transformation group is $\mathbb{Z}_m$. Note that $H^1(Y; \mathbb{Z}_p) = 0$ implies $H^1(Y; \mathbb{Z}_{p^n}) = 0$ for every $n$ and hence every $\mathbb{Z}_{p^n}$-bundle over $Y$ is trivial. Thus any compactum that $p$-resolves $Y$ must contain a copy of every component of $Y$, and therefore cannot be of a lower dimension.

The conditions $\dim_{\mathbb{Z}_{[1/p]}}Y = 1$ and $\dim Y > 1$ eliminate this obstruction. To show that, consider a map from $f : F \to S^1$ from a closed subset $F$ of $Y$ that does not extend over $Y$. By Proposition [1.1] there is a $\mathbb{Z}_{p^k}$-covering map $\phi : S^1 \to S^1$ such that $f$ followed by $\phi$ extends to $g : Y \to S^1$. Clearly we may assume that $Y$ is connected. Take a component $Y' \subset Y \times S^1$ of the pull-back space of $g$ and $\phi$ with the induced maps (projections) $g' : Y' \to S^1$ and $\phi' : Y' \to Y$. Notice that $\phi'$ is not 1-to-1 since otherwise $g'$ would provide an extension of $g$. Thus $\phi'$ is a $\mathbb{Z}_{p^k}$-covering of $Y$ with $0 < t \leq k$. The existence of a non-trivial $\mathbb{Z}_{p^t}$-bundle of $Y$ implies $H^1(Y; \mathbb{Z}_p) \neq 0$. A similar reasoning also shows that $H^1(Y; \mathbb{Z}_p)$ is infinite.

It turns out that for a finite-dimensional compactum $Y$ with $\dim Y > 1$ the condition $\dim_{\mathbb{Z}_{[1/p]}}Y = 1$ is already sufficient to $p$-resolve $Y$ by a compactum of dimension $< \dim Y$ but not $< \dim Y - 1$.

**Theorem 1.3.** Let $Y$ be a finite-dimensional compactum with $\dim_{\mathbb{Z}_{[1/p]}}Y = 1$. Then:

(i) $Y$ is $p$-resolvable by a compactum of dimension $\leq \dim Y - 1$ if $\dim Y \geq 2$;

(ii) $Y$ is not $p$-resolvable by a compactum of dimension $\leq \dim Y - 2$ if $\dim Y \geq 4$.

The author was initially inclined to believe that for a 3-dimensional compactum $Y$ with $\dim_{\mathbb{Z}_{[1/p]}}Y = 1$ (the case not covered by (ii) of Theorem 1.3) the additional assumption $\dim_{\mathbb{Z}_p} Y = 2$ imposed by Yang’s relations would imply that $Y$ is $p$-resolvable by a 1-dimensional compactum and was surprised to find the following:

**Theorem 1.4.** There is a 3-dimensional compactum $Y$ with $\dim_{\mathbb{Z}_{[1/p]}}Y = 1$ and $\dim_{\mathbb{Z}_p} Y = 2$ that is not $p$-resolvable by a 1-dimensional compactum.

Moreover:
Theorem 1.5. There is an infinite-dimensional compactum $Y$ such that $\dim_{\mathbb{Z}[1/p]} Y = 1$ and $\dim_{\mathbb{Z}} Y = 2$ and that $Y$ is not $p$-resolvable by a finite-dimensional compactum.

The following property is of crucial importance for proving Theorems 1.2, 1.4 and 1.5.

Theorem 1.6. Let a compactum $Y$ be $p$-resolvable by an $n$-dimensional compactum and $\dim Y \geq n + 2$. Then $H^2(Y; \mathbb{Z})$ contains a subgroup isomorphic to $\mathbb{Z}_p^\infty$. Moreover, for every closed $F \subset Y$ with $\dim F \geq n + 2$, the image of this subgroup in $H^2(F; \mathbb{Z})$ under the homomorphism induced by the inclusion is non-trivial.

In the proofs of Theorems 1.6, 1.5 and 1.4 we interpret the elements of $H^2(Y; \mathbb{Z})$ as circle bundles over $Y$ (like before we interpreted the elements of $H^1(Y; \mathbb{Z}_p)$ as $\mathbb{Z}_p$-bundles over $Y$). A few comments regarding Theorem 1.6 are given in Remark 3.2.

Conjecture 1.7. No compactum $Y$ with $\dim_{\mathbb{Z}[1/p]} Y = 1$ and $\dim Y \geq n + 2$ is $p$-resolvable by an $n$-dimensional compactum.

By Theorem 1.3(ii) the conjecture holds for $3 < \dim Y < \infty$. Thus the open cases of the conjecture are $\dim Y = 3$ and $\dim Y = \infty$.

2. Proof of Theorem 1.3

Pontryagin $p$-surfaces. We will describe 2-dimensional compacta to which we will refer as Pontryagin $p$-surfaces.

Let $p$ be a prime number and $k \geq 0$ an integer. Consider a 2-simplex $\Delta$, denote by $\Omega(p^k)$ the mapping cylinder of a $p^k$-fold covering map $\partial \Delta \to S^1$ and refer to the domain $\partial \Delta$ and the range $S^1$ of this map as the bottom and the top of $\Omega(p^k)$ respectively.

Let $k_0 = 0, k_1, k_2, \ldots$ be an increasing sequence of natural numbers. We will construct a Pontryagin $p$-surface $Y$ determined by the sequence $k_n$ as the inverse limit of 2-dimensional finite simplicial complexes $\Omega_n$. Set $\Omega_0$ to be a 2-simplex $\Delta$. Assume that $\Omega_n$ is constructed. Take a sufficiently fine triangulation of $\Omega_n$ and in every 2-simplex $\Delta$ of $\Omega_n$ remove the interior of $\Delta$ and attach to $\partial \Delta$ the mapping cylinder $\Omega(p^{k_n+1})$ by identifying the bottom of $\Omega(p^{k_n+1})$ with $\partial \Delta$. Define $\Omega_{n+1}$ to be the space obtained this way from $\Omega_n$ and define the bonding map $\omega_{n+1} : \Omega_{n+1} \to \Omega_n$ to be a map that sends each mapping cylinder $\Omega(p^{k_n+1})$ to the corresponding simplex $\Delta$ such that $\omega_{n+1}$ does not move the points of $\partial \Delta$ and sends the top of $\Omega(p^{k_n+1})$ to the barycenter of $\Delta$. Denote $\Omega = \varprojlim (\Omega_n, \omega_n)$ and call $\Omega$ the Pontryagin $p$-surface determined by the prime $p$ and the sequence $k_n$. We additionally
The standard Baire category argument to the function space of maps from $X$ to $\partial T$, where $X$ is covered by finitely many disjoint compact sets of diameter $< 1/2^{n-i}$.

**Proposition 2.1.** A compactum $Y$ with $\dim Y \leq m + 2$, $m \geq 0$, and $\dim_{\mathbb{Z}[1/p]} Y = 1$ admits an $m$-dimensional map into a Pontryagin $p$-surface.

In the proof of Proposition 2.1 we will use the following facts.

**Proposition 2.2 (B).** Let $f : X \to Y$ be an $m$-dimensional map (= a map whose fibers are of dim $\leq m$). Then $\dim X \leq \dim Y + m$ and $\dim_G X \leq \dim_G Y + m$ for every abelian group $G$.

**Lemma 2.3.** Let $T_1$ and $T_2$ be finite trees, $X$ a compactum and $X'$ a $\sigma$-compact subset of $X$ with $\dim X' \leq 2$. Then every map $f : X \to T_1 \times T_2$ can be arbitrarily closely approximated by a map $f' : X \to T_1 \times T_2$ such that $f'$ coincides with $f$ on $A = f^{-1}(\partial(T_1 \times T_2))$ and $f'$ is 0-dimensional on $X' \setminus A$ where $\partial(T_1 \times T_2) = ((\partial T_1) \times T_2) \cup (T_1 \times \partial T_2)$ and $\partial T_1$ and $\partial T_2$ stand for the sets of the end points of $T_1$ and $T_2$ respectively.

**Proof.** Represent $f$ as $f = (f_1, f_2)$ with $f_1 : X \to T_1$ and $f_2 : X \to T_2$ being the coordinate maps. Fix metrics on $X$, $T_1$ and $T_2$, consider the induced supremum metric on $Y = T_1 \times T_2$ and let $\epsilon > 0$. Since $T_1$ and $T_2$ are finite trees, there is $\delta > 0$ such that for every closed subset $F \subset X$ that does not intersect $A$ and every map $f_i^F : F \to T_i$, $i = 1, 2$, that is $\delta$-close to $f_i$ on $F$, $f_i^F$ extends over $X$ to a map that is $\epsilon$-close to $f_i$ and coincides with $f_i$ on $A$.

Consider a compact subset $F^\epsilon \subset X'$ such that $F^\epsilon \subset \{ x \in X : d(x, A) \geq \epsilon \}$ and take a finite closed cover $\mathcal{F}$ of $F^\epsilon$ by subsets of $F^\epsilon$ of diameter $< \epsilon$ such that the images of the sets in $\mathcal{F}$ under both $f_1$ and $f_2$ are of diameter $< \delta$ and $\mathcal{F}$ splits into the union $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1 \cup \mathcal{F}_2$ of collections $\mathcal{F}_i$ of disjoint sets.

Denote by $F_i$, $i = 1, 2$, the union of the sets in $\mathcal{F}_i$. Then $f_i$ restricted to $F_i$ can be $\delta$-approximated by a map $f_i^F : F_i \to T_i$ that sends the sets in $\mathcal{F}_i$ to distinct points in $T_i$. Now extend $f_i^F$ to a map $f_i^F : X \to T_i$ that is $\epsilon$-close to $f_i$ and coincides with $f_i$ on $A$, and set $f^\epsilon = (f_1^\epsilon, f_2^\epsilon) : X \to Y = T_1 \times T_2$.

Then every fiber of $f^\epsilon$ restricted to $F^\epsilon$ intersects at most one set of $\mathcal{F}_1$ and at most one set of $\mathcal{F}_2$, and hence every fiber of $f^\epsilon$ restricted to $F^\epsilon$ can be covered by finitely many disjoint compact sets of diameter $< 4\epsilon$. Recall that $X'$ is $\sigma$-compact, $f^\epsilon$ is $\epsilon$-close to $f$ and coincides with $f$ on $A$, and apply the standard Baire category argument to the function space of maps from $X$ to $Y$ that coincide with $f$ on $A$ to get the desired result.

**Proof of Proposition 2.1** Recall that a Pontryagin $p$-surface is the inverse limit of finite simplicial complexes $\Omega_n$ with the bonding maps $\omega_{n+1} : \Omega_{n+1} \to \Omega_n$ and $\Omega_0$ being a 2-simplex. Take a $\sigma$-compact subset $Y'$ of $Y$ such that $\dim Y' \leq 2$ and $\dim Y \setminus Y' \leq m - 1$ and take any $m$-dimensional
map \( f_0 : Y \to \Omega_0 \). We will construct by induction on \( n \) spaces \( \Omega_n \) and \( m \)-dimensional maps \( f_n : Y \to \Omega_n \). Assume the construction is completed for \( n \) and proceed to \( n + 1 \) as follows. Fix a triangulation of \( \Omega_n \). Recall that \( \dim_{\mathbb{Z}[1/p]} Y \leq 1 \) and by Proposition 1.1 take a sufficiently large \( k_n \) such that for every simplex \( \Delta \) of \( \Omega_n \), \( f_n \) restricted to \( f_n^{-1}(\Delta) \) and followed by a \( p^{k_n} \)-covering map \( \partial \Delta \to S^1 \) extends over \( f_n^{-1}(\Delta) \). This determines a simplicial complex \( \Omega_{n+1} \) as described in the construction of Pontryagin \( p \)-surfaces and a natural map \( f_{n+1} : Y \to \Omega_{n+1} \). Clearly, taking a sufficiently fine triangulation of \( \Omega_n \) we may assume that \( f_n \) and \( \omega_{n+1} \circ f_{n+1} \) are as close as we wish. Note that every point of every \( \Omega_i \) has a closed neighborhood homeomorphic to a product of a finite tree with a closed interval. By Lemma 2.3, we can replace \( f_{n+1} \) by a map which is 0-dimensional on \( Y' \), and since \( \dim Y \setminus Y' \leq m - 1 \), we find that \( f_{n+1} \) is \( m \)-dimensional. Then it is easy to see that the whole construction can be carried out so that the maps \( f_n \) will determine an \( m \)-dimensional map from \( Y \) to \( \Omega = \lim \Omega_n \).

**Lemma 2.4.** Let \( f : \tilde{\Omega}(p^k) \to \Omega(p^k) \) be the \( \mathbb{Z}_p \)-covering of \( \Omega(p^k) \) induced by the \( \mathbb{Z}_p \)-covering of the top of \( \Omega(p^k) \). Then \( f \) restricted to the preimage of the bottom of \( \Omega(p^k) \) extends over \( \tilde{\Omega}(p^k) \) as a map to the bottom of \( \Omega(p^k) \).

**Proof.** Represent \( \tilde{\Omega}(p^k) \) as the union of \( p^k \) copies of \( S^1 \times [0, 1] \) being glued along \( S^1 \times \{1\} \) such that under \( f \) the set \( S^1 \times \{1\} \) goes to the top of \( \Omega(p^k) \) and the sets \( S^1 \times \{0\} \) go to the bottom of \( \Omega(p^k) \). Consider a map \( f_1 \) from \( S^1 \times \{1\} \) to the bottom of \( \Omega(p^k) \) such that \( f_1 \) followed by the natural projection of \( \Omega(p^k) \) to its top coincides with \( f \) on \( S^1 \times \{1\} \). Then for each set \( S^1 \times [0, 1] \) the maps \( f_0 = f \) restricted to \( S^1 \times \{0\} \) and \( f_1 \) are homotopic as maps to the bottom of \( \Omega(p^k) \) and hence extend over \( S^1 \times [0, 1] \) to a map to the bottom of \( \Omega(p^k) \). This defines a map required in the proposition.

**Proposition 2.5.** Every Pontryagin \( p \)-surface is \( p \)-resolvable by a \( 1 \)-dimensional compactum.

**Proof.** Let \( \Omega = \lim \Omega_n \) be a Pontryagin \( p \)-surface determined by a sequence \( k_n \). Consider the first homology \( H_1(\Omega_n; \mathbb{Z}) \) and let \( C_i, i \geq 1 \), be the collection of the tops of all the mapping cylinders added while constructing \( \Omega_i, i \leq n \). Then the circles in \( C_1 \cup \cdots \cup C_n \) considered as elements of \( H_1(\Omega_n; \mathbb{Z}) \) form a collection of free generators of \( H_1(\Omega_n; \mathbb{Z}) \).

Take any sequence \( s_n \) such that \( s_1 = k_1 \) and \( s_{n+1} - k_{n+1} \geq s_n \) for every \( n \geq 1 \) and define the subgroup \( G_n \) of \( H_1(\Omega_n; \mathbb{Z}) \) by

\[
G_n = \left\{ \sum_{i=1}^{n} \sum_{\alpha \in C_i} t_{\alpha} : \sum_{i=1}^{n} \sum_{\alpha \in C_i} p^{s_i - k_i} t_{\alpha} \text{ is divisible by } p^{s_n} \right\}.
\]

Then \( H_1(\Omega_n; \mathbb{Z})/G_n = \mathbb{Z}_{p^{s_n}} \) and for the bonding map \( \omega_{n+1} : \Omega_{n+1} \to \Omega_n \) that the induced homomorphism \( (\omega_{n+1})_* : H_1(\Omega_{n+1}; \mathbb{Z}) \to H_1(\Omega_n; \mathbb{Z}) \)
is onto and sends $G_{n+1}$ into $G_n$. Indeed $(\omega_{n+1})_*(\sum_{i=1}^{n+1} \sum_{\alpha \in C_i} t_\alpha \alpha) = \sum_{i=1}^{n} \sum_{\alpha \in C_i} t_\alpha \alpha$, and if $\sum_{i=1}^{n+1} \sum_{\alpha \in C_i} p^{s_i-k_i} t_\alpha \alpha = p^{s_{n+1}-k_{n+1}} \sum_{\alpha \in C_{n+1}} t_\alpha$, then we see that $\sum_{i=1}^{n+1} \sum_{\alpha \in C_i} p^{s_i-k_i} t_\alpha$ is divisible by $p^{s_{n+1}}$ with $s_{n+1}-k_{n+1} \geq s_n$ then we see that $\sum_{i=1}^{n+1} \sum_{\alpha \in C_i} p^{s_i} t_\alpha$ is divisible by $s_n$ and therefore $(\omega_{n+1})_*(G_{n+1}) \subset G_n$.

Let $h_n : \pi_1(\Omega_n) \to H_1(\Omega_n; \mathbb{Z})$ be the Hurewicz homomorphism. Then

$$\tilde{G}_n = \pi_1(\Omega_n)/h_n^{-1}(G_n) = H_1(\Omega_n; \mathbb{Z})/G_n = \mathbb{Z}_{p^{s_n}}.$$  

Consider the corresponding $\tilde{G}_n$-covering $f_n : \tilde{\Omega}_n \to \Omega_n$ and lift the map $\omega_{n+1} : \Omega_{n+1} \to \Omega_n$ to $\tilde{\omega}_{n+1} : \tilde{\Omega}_{n+1} \to \tilde{\Omega}_n$. The homomorphism $(\omega_{n+1})_*$ induces the natural epimorphism $\tilde{g}_{n+1} : \tilde{G}_{n+1} \to \tilde{G}_n$ so that the actions of $\tilde{G}_{n+1}$ and $\tilde{G}_n$ agree with $\tilde{g}_{n+1}$ and $\tilde{\omega}_{n+1}$. This defines the compactum $X = \lim(\tilde{\Omega}_n, \tilde{\omega}_n)$, the action of $A_p = \lim(\tilde{G}_n, \tilde{g}_n)$ on $X$ and the map $f : X \to \Omega$ determined by the maps $f_n$.

Note that if we consider $\alpha \in C_n$ as a circle in $\Omega_n$ then the preimage of $\alpha$ under the map $f_n$ splits into $p^{s_n-k_n}$ components (circles), and $f_n$ restricted to each component is a $\mathbb{Z}_{p^{k_n}}$-covering of $\alpha$.

Let us show that $\dim X \leq 1$. Consider the triangulation of $\Omega_n$ used for constructing $\Omega_{n+1}$ and a simplex $\Delta$ of this triangulation. Let $\Omega(p^{k_{n+1}}) = \omega_{n+1}^{-1}(\Delta)$ and let $\tilde{\Omega}_{n+1}(p^{k_{n+1}})$ be a component of $f_{n+1}^{-1}(\Omega(p^{k_{n+1}}))$. Note that $f_{n+1}$ restricted to $\tilde{\Omega}(p^{k_{n+1}})$ is a $\mathbb{Z}_{p^{k_{n+1}}}$-covering of $\Omega(p^{k_{n+1}})$. Apply Lemma 2.4 to extend the map $f_{n+1}$ restricted to the preimage in $\tilde{\Omega}(p^{k_{n+1}})$ of the bottom $\partial \Delta$ of $\Omega(p^{k_{n+1}})$ to a map $\phi : \tilde{\Omega}(p^{k_{n+1}}) \to \partial \Delta$. Now consider the triangulation of $\tilde{\Omega}_n$ induced by the triangulation of $\Omega_n$. The map $\phi$ lifts to $\phi : \tilde{\Omega}_{n+1}(p^{k_{n+1}}) \to \tilde{\Omega}_n$ so that $\phi(\tilde{\Omega}_{n+1}(p^{k_{n+1}})) \subset \tilde{\omega}_{n+1}(\tilde{\Omega}_{n+1}(p^{k_{n+1}}))$. Doing that for every simplex of $\Omega_n$ we get a map from $\tilde{\Omega}_{n+1}$ to the 1-skeleton of $\tilde{\Omega}_n$ and this shows that $\dim X \leq 1$. ■

**Proof of Theorem 1.3.** (i) Let $\dim Y = m + 2$. By Proposition 2.1 there is an $m$-dimensional map $f : Y \to \Omega$ to a Pontryagin $p$-surface $\Omega$. By Proposition 2.5 there is a free action of $A_p$ on a 1-dimensional compactum $\tilde{\Omega}$ such that $\tilde{\Omega}/A_p = \Omega$. Let $X$ be the pull-back of $f$ via the projection of $\tilde{\Omega}$ to $\Omega$. Then the projection of $X$ to $\tilde{\Omega}$ is an $m$-dimensional map and hence, by Proposition 2.2, $\dim X \leq m + 1$ and for the pull-back action of $A_p$ on $X$ we have $X/A_p = Y$.

(ii) Assume that $Y$ is $p$-resolvable by a finite-dimensional compactum $X$. Then, by Yang’s relations, $\dim_{\mathbb{Z}[1/p]} X = 1$ and $\dim_{\mathbb{Z}_{p^\infty}} X \geq \dim_{\mathbb{Z}_{p^\infty}} Y - 1$. The Bockstein inequalities (see [3] for the basics of Bockstein theory) and $\dim Y \geq 4$ imply that $\dim_{\mathbb{Q}} X = \dim_{\mathbb{Q}} Y = 1$ and $\dim_{\mathbb{Z}_{p^\infty}} Y = \dim Y - 1 \geq 3$, and hence $\dim_{\mathbb{Z}_{p^\infty}} X \geq 2$ and $\dim X = \dim_{\mathbb{Z}_{p^\infty}} X + 1 \geq \dim Y - 1$. ■

3. **Proof of Theorem 1.6.** A bundle will always mean a locally trivial principal bundle. The circle $S^1$ is considered as the group $S^1 = U(1) = \mathbb{R}/\mathbb{Z}$.  

Recall that the classifying space for circle bundles over compacta is $BS^1 = CP^\infty = K(\mathbb{Z}, 2)$ and therefore every circle bundle $Z \to Y$ over a compactum $Y$ is represented by an element $\alpha$ of the second Čech cohomology $H^2(Y; \mathbb{Z})$ that being considered as a map $\alpha : Y \to K(\mathbb{Z}, 2)$ allows one to obtain the bundle $Z \to Y$ as the pull-back of the universal circle bundle $E \to K(\mathbb{Z}, 2)$ via the map $\alpha$.

Let $Y$ be a compactum and $f : Z \to Y$ a circle bundle over $Y$. Consider the subgroup $\mathbb{Z}_m$ of $S^1$ and in each fiber of $f$ collapse the orbits under the action of $\mathbb{Z}_m$ in $S^1$ to singletons. This way we obtain the circle bundle $f' : Z' \to Y$ determined by the action of $S^1/\mathbb{Z}_m$ on $Z'$. We will refer to $f' : Z' \to Y$ as the circle bundle induced by $f : Z \to Y$ and $\mathbb{Z}_m$.

**Lemma 3.1.** Let $f : Z \to Y$ be a circle bundle over a compactum $Y$ represented by $\alpha \in H^2(Y; \mathbb{Z})$ and let $f' : Z' \to Y$ be the circle bundle induced by $f$ and $\mathbb{Z}_m$. Then $f'$ is represented by $\alpha' = m\alpha \in H^2(Y; \mathbb{Z})$.

**Proof.** Consider the short exact sequence

$$0 \to \mathbb{Z}_m \to S^1 \to S^1 \to 0.$$ 

It defines a fiber sequence

$$B\mathbb{Z}_m \to BS^1 \to BS^1$$

with the long exact sequence of the fibration

$$\cdots \to \pi_2(B\mathbb{Z}_m) \to \pi_2(BS^1) \to \pi_2(BS^1) \to \pi_1(B\mathbb{Z}_m) \to \pi_1(BS^1) \to \cdots.$$ 

Recall that $B\mathbb{Z}_m = K(\mathbb{Z}_m, 1)$ and $BS^1 = K(\mathbb{Z}, 2)$ and get

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_m \to 0.$$ 

Thus the homomorphism $S^1 \to S^1$ induces the map $h : BS^1 \to BS^1$ which acts on the second homotopy group of $BS^1$ as the multiplication by $m$. Represent $\alpha$ as a map $\alpha : Y \to BS^1 = K(\mathbb{Z}, 2)$. Then $\alpha'$ is represented by $h \circ \alpha$ that translates in $H^2(Y; \mathbb{Z})$ to $\alpha' = m\alpha$.

Let $A_p$ act freely on a finite-dimensional compactum $X$. Consider $A_p$ as a subgroup of the $p$-adic solenoid $\Sigma_p$ and the induced action of $A_p$ on $X \times \Sigma_p$. Then $\Sigma_p$ naturally acts on $X \times A_p \Sigma_p = (X \times \Sigma_p)/A_p$ with $(X \times A_p \Sigma_p)/\Sigma_p = X/A_p$ and there is a natural projection of $X \times A_p \Sigma_p$ to $S^1 = \Sigma_p/A_p$ induced by the projections $X \times \Sigma_p \to \Sigma_p \to \Sigma_p/A_p$. Note that the fibers of the projection $X \times A_p \Sigma_p \to S^1 = \Sigma_p/A_p$ are homeomorphic to $X$ and hence, by Proposition 2.2, $\dim X \times A_p \Sigma_p \leq \dim X + 1$.

Consider a decreasing sequence of subgroups $A_p^k$ of $A_p$ such that $A_p^k$ is isomorphic to $A_p$, $A_p^0 = A_p$, $A_p^{k}/A_p^{k+1} = \mathbb{Z}_p$ and the intersection of all $A_p^k$ contains only 0. Denote $Y = X/A_p$ and $Z_k = (X \times A_p \Sigma_p)/A_p^k$. Note that the circle $S^1 = \Sigma_p/A_p^k$ acts on $Z_k$ with $Y = Z_k/S^1$. This turns each $Z_k$ into a circle bundle $f_k : Z_k \to Y$ over $Y$ and the inclusion of $A_p^{k+1}$ into $A_p^k$ defines
the natural bundle map $g_{k+1} : Z_{k+1} \to Z_k$ that witnesses that the bundle $f_k : Z_k \to Y$ is induced by the bundle $f_{k+1} : Z_{k+1} \to Y$ and $\mathbb{Z}_p$. Moreover, the bundle $f_0 : Z_0 \to Y$ is trivial since $(X \times A_p, \Sigma_p)/A_p = (X/A_p) \times (\Sigma_p/A_p) = Y \times S^1$. Also note that $Z = X \times A_p \Sigma_p = \limleft(\Omega_k, g_k\right)$ and the projection $f : Z = X \times A_p \Sigma_p \to Y = (X \times A_p \Sigma_p)/\Sigma_p$ coincides with the projection of $Z$ to $Z_k$ followed by $f_k$.

**Proof of Theorem 1.6.** Let $A_p$ act freely on an $n$-dimensional compactum $X$ with $Y = X/A_p$ of dimension $\geq n + 2$. Consider the circle bundles $f_k : Z_k \to Y$ described above and let $\alpha_k \in H^2(Y; \mathbb{Z})$ represent $f_k$. By Lemma 3.1, $\alpha_k = p\alpha_{k+1}$. Recall that $f_0 : Z_0 \to Y$ is a trivial bundle and hence $\alpha_0 = 0$. Denote by $H$ the subgroup of $H^2(Y; \mathbb{Z})$ generated by $\alpha_k$, $k = 0, 1, \ldots$. We are going to show that $H$ is isomorphic to $\mathbb{Z}_{p^\infty}$ and satisfies the conclusions of the theorem.

Let $F \subset Y$ be a compactum of dimension $\geq n + 2$. Aiming at a contradiction assume that the image of $H$ under the inclusion of $F$ into $Y$ is trivial in $H^2(F; \mathbb{Z})$. Then every bundle $f_k : Z_k \to Y$ is trivial over $F$. Let $\epsilon > 0$ be such that $F$ does not admit an open cover of order $\leq n + 2$ by sets of diameter $\leq \epsilon$. Recall that $\dim Z \leq n + 1$. Then, since $Z = \limleft(\Omega_k, g_k\right)$, there is a sufficiently large $k$ such that $Z_k$ admits an open cover $\mathcal{U}$ of order $\leq n + 2$ such that the images of the sets in $\mathcal{U}$ under $f_k$ are of diameter $\leq \epsilon$. Take a section $\phi : F \to Z_k$ over $F$. Then $\mathcal{U}$ restricted to $\phi(F)$ and mapped by $f_k$ to $F$ provides an open cover of $F$ of order $\leq n + 2$ by sets of diameter $\leq \epsilon$, a contradiction.

Now assuming that $F = Y$ we deduce that $H$ is non-trivial, and since $\alpha_k = p\alpha_{k+1}$ and $\alpha_0 = 0$, we conclude that $H$ is isomorphic to $\mathbb{Z}_{p^\infty}$. ■

**Remark 3.2.** In general the assumption $\dim Y \geq n + 2$ in Theorem 1.6 cannot be weakened to $\dim Y \geq n + 1$. Indeed, it is easy to see that a Pontryagin $p$-surface $\Omega$ has $H^2(\Omega; \mathbb{Z}) = 0$ and, as was shown in the previous section, $\Omega$ is $p$-resolvable by a 1-dimensional compactum. In this connection we point out an interesting phenomenon that occurs in the proof of Theorem 1.6 for $Y = \Omega$: all the compacta $Z_k$ are homeomorphic to $Y \times S^1$ and the bonding maps $g_{k+1} : Z_{k+1} \to Z_k$ look alike, and still $Z = \limleft(Z_k\right)$ is 2-dimensional even though each $Z_k$ is 3-dimensional. This happens because we cannot fix a trivialization $Z_k = Y \times S^1$ for each $Z_k$ so that with respect to these trivializations each $g_{k+1}$ will have the form $g_{k+1}(y, s) = (y, ps), (y, s) \in Y \times S^1$.

Let us note that for a “nice” compactum $X$, namely if $X$ is close to being simply connected or more precisely if $X$ is connected and $H^1(X; \mathbb{Z}) = 0$, the group $\mathbb{Z}_{p^\infty}$ will always be present in $H^2(X/A_p; \mathbb{Z})$ for a free $p$-adic action on $X$ without any dimensional restrictions on the orbit space. Indeed, consider a coarse 2-dimensional compact classifying space $BA_p$ of $A_p$ obtained as the orbit space $BA_p = E/A_p$ of a free action of $A_p$ on a 2-dimensional
cell-like compactum $E$ constructed by Bestvina and Edwards (unpublished). Then the projection from $X \times_{A_p} E$ to $Y = X/A_p$ is a cell-like map and the projection from $X \times_{A_p} E$ to $BA_p$ is acyclic in dimension $\leq 1$ if $X$ is connected and $H^1(X; \mathbb{Z}) = 0$. The Bégle–Vietoris theorem implies that the homomorphism $H^2(Y; \mathbb{Z}) \to H^2(X \times_{A_p} E; \mathbb{Z})$ induced by the projection $X \times_{A_p} E \to Y$ is an isomorphism and the homomorphism $H^2(BA_p; \mathbb{Z}) \to H^2(X \times_{A_p} E; \mathbb{Z})$ induced by the projection $X \times_{A_p} E \to BA_p$ is injective, and hence $H^2(BA_p; \mathbb{Z}) = \mathbb{Z}_{p^\infty}$ is a subgroup of both $H^2(X \times_{A_p} E; \mathbb{Z})$ and $H^2(Y; \mathbb{Z})$. In this connection let us mention that free $p$-adic actions on “nice” spaces, namely Menger compacta, were constructed by Dranishnikov [2] and by Mayer and Stark [12].

Let us also note that our use of Theorem 1.6 in this paper is exactly in those cases when $X$ is far from being “nice". Indeed, assume that $\dim \mathbb{Z}[1/p] Y = 1$ and $\dim Y > 1$. Consider a closed subset $F$ of $Y$ such that $F$ admits a map to $S^1$ that does not extend over $Y$. Then $H^1(F; \mathbb{Z}) \neq 0$, and since $H^1(F; \mathbb{Z})$ is torsion free, we have $H^1(F; \mathbb{Z}[1/p]) = H^1(F; \mathbb{Z}) \otimes \mathbb{Z}[1/p] \neq 0$. The condition $\dim \mathbb{Z}[1/p] Y = 1$ implies $H^2(Y, F; \mathbb{Z}[1/p]) = 0$ and hence the homomorphism $H^1(Y; \mathbb{Z}[1/p]) \to H^1(F; \mathbb{Z}[1/p])$ induced by the inclusion of $F$ into $Y$ is onto. Thus $H^1(Y; \mathbb{Z}[1/p]) = 0$. Note that the homomorphism $\pi^*: H^1(Y; \mathbb{Z}[1/p]) \to H^1(X; \mathbb{Z}[1/p])$ induced by the projection $\pi: X \to Y = X/A_p$ is injective; see the second paragraph of Remark 3.3. Then $H^1(X; \mathbb{Z}[1/p]) = H^1(X; \mathbb{Z}) \otimes \mathbb{Z}[1/p] \neq 0$ and therefore $H^1(X; \mathbb{Z}) \neq 0$.

**Remark 3.3.** As we promised in the introduction we provide here a proof of Yang’s relations. Let $A_p$ act freely on a compactum $X$ and let $\pi: X \to Y = X/A_p$ be the projection.

Let $F_Y \subset Y$ be closed and $F_X = \pi^{-1}(F_Y)$. The homomorphism $\pi^*: H^n(Y, F_Y; \mathbb{Z}[1/p]) \to H^n(X, F_X; \mathbb{Z}[1/p])$ induced by $\pi$ is injective for every $n$. Indeed, representing $X$ as the inverse limit of $\mathbb{Z}_p$-bundles we may assume that $\pi$ is a $\mathbb{Z}_p$-bundle as well, and approximating $X$ and $Y$ by finite simplicial complexes we may also assume that $X$ and $Y$ are finite simplicial complexes, $\pi$ is simplicial, $F_Y$ is a subcomplex of $Y$ and the action of $\mathbb{Z}_p$ on $X$ is simplicial. All the cochain complexes that we consider here are with coefficients in $\mathbb{Z}[1/p]$, and $\pi^*$ also denotes the induced homomorphism between the cochain complexes of $(Y, F_Y)$ and $(X, F_X)$. Let $g$ be a generator of $\mathbb{Z}_p$, consider $g$ acting on the cochain complex of $(X, F_X)$ and take a cocycle $\alpha_Y$ in $(Y, F_Y)$ such that $\alpha_X = \pi^*(\alpha_Y)$ is the coboundary of a cochain $\beta_X$ in $(X, F_X)$. Consider the cochain $\gamma_X = \frac{1}{p}(\beta_X + g\beta_X + \cdots + g^{n-1}\beta_X)$ in $(X, F_X)$. Note that $\alpha_X = g\alpha_X$ and $\gamma_X = g\gamma_X$. Then $\alpha_X$ is the coboundary of $\gamma_X$ and there is a cochain $\gamma_Y$ in $(Y, F_Y)$ such that $\pi^*(\gamma_Y) = \gamma_X$. It is easy to see that $\alpha_Y$ is the coboundary of $\gamma_Y$ and hence $\pi^*$ is injective at the level of cohomology groups.
The injectivity of $\pi^*$ implies that $\dim_{\mathbb{Z}[1/p]} Y \leq \dim_{\mathbb{Z}[1/p]} X$. By Dranishnikov–Uspenskij’s theorem on light maps \cite{7}, $\dim_G X \leq \dim_G Y$ for every group $G$. Split $S^1$ into the union of two intervals $I_1$ and $I_2$ and denote by $M_1$ and $M_2$ the preimages of $I_1$ and $I_2$ respectively under the projection $X \times_{A_p} \Sigma_p \to S^1$. Note that $M_i$ is homeomorphic to $X \times I_i$ and hence $\dim_{G} X \times_{A_p} \Sigma_p = \dim_{G} X + 1$ for every group $G$. The $p$-adic solenoid $\Sigma_p$ is both $\mathbb{Z}_p$- and $\mathbb{Z}_{p\infty}$-acyclic, and applying the Begle–Vietoris theorem to the projection $X \times_{A_p} \Sigma_p \to Y$ we get $\dim_{G} Y \leq \dim_{G} X \times_{A_p} \Sigma_p = \dim_{G} X + 1$ for $G = \mathbb{Z}_p$ and $G = \mathbb{Z}_{p\infty}$. The rest of the Yang relations follow from the Bockstein inequalities \cite{3}.

4. Proof of Theorem 1.2. Let $P$ denote the set of all primes. The Bockstein basis is the collection of groups $\sigma = \{ \mathbb{Q}, \mathbb{Z}_p, \mathbb{Z}_{p\infty}, \mathbb{Z}(p) : p \in P \}$ where $\mathbb{Z}(p) = \{ m/n : n \text{ is not divisible by } p \} \subset \mathbb{Q}$ is the $p$-localization of integers \cite{3}.

By a Moore space $M(G, n)$ for a group in the Bockstein basis $\sigma$ we always mean the standard Moore space; a description of the standard models of Moore spaces can be found in \cite{10}. All CW-complexes are assumed to be countable and all the spaces are assumed to be separable metrizable. An infinite-dimensional compactum $X$ is said to be hereditarily infinite-dimensional if every closed subset of $X$ is either 0-dimensional or infinite-dimensional. The notation ext-dim $X \leq K$ means that a CW-complex $K$ is an absolute extensor for a space $X$. We denote by $H_n(K)$ the reduced $n$-dimensional homology of $K$ with coefficients in $\mathbb{Z}$. We need the following results:

1. (Dranishnikov’s first extension criterion \cite{4}) Let $X$ be a compactum and $K$ a CW-complex such that ext-dim $X \leq K$. Then $\dim_{H_n(K)} X \leq n$ for every $n$.
2. (Dranishnikov’s second extension criterion \cite{4}) Let $K$ be a simply connected CW-complex and $X$ a finite-dimensional compactum with $\dim_{H_n(K)} X \leq n$ for every $n$. Then ext-dim $X \leq K$.
3. (Dranishnikov’s splitting theorem \cite{5, 6}) Let $X$ be a space and $K_1$ and $K_2$ CW-complexes such that ext-dim $X \leq K_1 \ast K_2$. Then $X$ splits into $X = X_1 \cup X_2$ with ext-dim $X_1 \leq K_1$, ext-dim $X_2 \leq K_2$ and $X_1$ being $F_\sigma$ in $X$.
4. \cite{9} Let $X$ be a compactum with dim$_\mathbb{Q} X \leq n$. Then ext-dim $X \leq M(\mathbb{Q}, n)$.
5. (A factorization theorem that was actually proved in \cite{10}) Let $\sigma'$ be a subcollection of the Bockstein basis $\sigma$ and $X$ a compactum (not necessarily finite-dimensional) such that ext-dim $X \leq M(G, n_G)$ for every $G \in \sigma'$. Then every map $f : X \to K$ to a finite CW-complex $K$
can be arbitrarily closely approximated by a map \( f' : X \to K \) such that \( f' \) factors through a compactum \( Z \) with \( \dim Z \leq \dim K \) and \( \text{ext-dim } Z \leq M(G, n_G) \) for every \( G \in \sigma' \).

(6) It follows from the results of Ancel [1] and Pol [16] (see also [8]) on \( C \)-spaces that every infinite-dimensional compactum of finite integral cohomological dimension contains a hereditarily infinite-dimensional compactum.

(7) \[\text{Let } A_p \text{ act on an } n\text{-dimensional compactum } X. \text{ Then } \dim_{\mathbb{Z}} X/A_p \leq n + 3.\]

**Proposition 4.1.** A compactum \( X \) is finite-dimensional if and only if there is a natural number \( n \) such that \( \text{ext-dim } X \leq M(\mathbb{Q}, n) \) and \( \text{ext-dim } X \leq M(\mathbb{Z}_p, n) \) for every prime \( p \).

\[\text{Proof. If } \dim X = k > 0 \text{ then } \dim_G \leq k \text{ for every group } G \text{ and, by } (2) \text{ ext-dim } X \leq M(G, n) \text{ for } n = k + 1 \text{ and every } G \in \sigma.\]

Now assume that \( \text{ext-dim } X \leq M(\mathbb{Q}, n) \) and \( \text{ext-dim } X \leq M(\mathbb{Z}_p, n) \) for every prime \( p \). Fix \( \epsilon > 0 \) and take an \( \epsilon \)-map (a map with fibers of diameter \( < \epsilon \)) \( f : X \to K \) to a finite-dimensional cube \( K \). By \((5)\) \( f \) factors through a \((2\epsilon)\)-map \( g : X \to Z \) with \( Z \) being finite-dimensional with \( \text{ext-dim } Z \leq M(\mathbb{Z}_p, n) \) for every prime \( p \) and \( \text{ext-dim } Z \leq M(\mathbb{Q}, n) \). By \((1)\) \( \dim_{\mathbb{Z}_p} Z \leq n \) for every prime \( p \) and \( \dim_{\mathbb{Q}} Z \leq n \) and hence, by the Bockstein inequalities, \( \dim_\mathbb{Z} Z \leq n + 1 \) and since \( Z \) is finite-dimensional, \( \dim Z = \dim_{\mathbb{Z}} Z \leq n + 1 \). Thus for every \( \epsilon > 0 \), \( X \) admits a \((2\epsilon)\)-map to a compactum of dimension \( \leq n + 1 \) and hence \( \dim X \leq n + 1 \). \( \blacksquare \)

**Proof of Theorem 1.2.** By \((7)\) and \((6)\) we may replace \( X \) and \( Y \) by closed subsets and assume that \( Y \) is hereditarily infinite-dimensional. Let \( \dim X = n \) and let \( Y_* \subset Y \) be the subset of \( Y \) of all finite orbits. Then \( \dim Y_* \leq n \). Replace \( Y_* \) by a larger a \( G_\delta \)-subset of \( Y \) with \( \dim Y_* \leq n \). Since \( Y \) is hereditarily infinite-dimensional and \( Y \setminus Y_* \) is \( \sigma \)-compact, there is an infinite-dimensional compactum in \( Y \setminus Y_* \). Thus replacing \( Y \) and \( X \) by this compactum and its preimage respectively we may assume that the action of \( A_p \) is free.

Let us show that \( Y \) contains an infinite-dimensional compactum with \( \dim_{\mathbb{Z}[1/p]} \) equial to 1. Aiming at a contradiction assume that every compactum \( Y' \) in \( Y \) with \( \dim Y' > 0 \) has \( \dim_{\mathbb{Z}[1/p]} Y' > 1 \). Then \( \dim_{\mathbb{Z}[1/q]} Y' > 1 \) for every prime \( q \neq p \) because otherwise \( H^2(Y'; \mathbb{Z}[1/q]) = 0 \) and hence, by the universal coefficients theorem, \( H^2(Y'; \mathbb{Z}) \) is \( q \)-torsion and cannot contain a copy of \( \mathbb{Z}_{p^\infty} \), which violates Theorem 1.6.

Note that for every prime \( q \) the join \( M(\mathbb{Z}[1/q], 1) \ast M(\mathbb{Z}_q, n + 3) \) is contractible and hence \( \text{ext-dim } Y \leq M(\mathbb{Z}[1/q], 1) \ast M(\mathbb{Z}_q, n + 3) \). Then, by \((3)\) \( Y = Y_q^0 \cup Y_q^1 \) with \( \text{ext-dim } Y_q^0 \leq M(\mathbb{Z}[1/q], 1) \), \( \text{ext-dim } Y_q^1 \leq M(\mathbb{Z}_q, n + 3) \)
and $Y_q^0$ being $\sigma$-compact. By (1) $\dim_{\mathbb{Z}[1/q]} Y_q^0 \leq 1$ and $\dim_{\mathbb{Z}} Y_q^1 \leq n + 3$. Since any compactum $Y'$ in $Y$ of positive dimension has $\dim_{\mathbb{Z}[1/q]} Y' > 1$, we deduce that $\dim Y_q^0 \leq 0$.

Denote by $Y^0$ the union of $Y_q^0$ for all prime $q$ and let $Y^1 = Y \setminus Y^0$. Then $\dim Y^0 \leq 0$ and $\text{ext-dim} Y^1 \leq M(\mathbb{Z}_q, n + 3)$ for every prime $q$. Replacing $Y^0$ by a larger 0-dimensional $G_\delta$-subset of $Y$, we find that $Y^1$ is $\sigma$-compact and infinite-dimensional and hence contains an infinite-dimensional compactum $Y'$. By (7) $\dim_{\mathbb{Z}} Y \leq n + 3$ and hence $\dim_{\mathbb{Q}} Y' \leq n + 3$. Then, by (4) $\text{ext-dim} Y' \leq M(\mathbb{Q}, n + 3)$. Recall that $\text{ext-dim} Y' \leq M(\mathbb{Z}_q, n + 3)$ for every prime $q$. By 4.1 $Y'$ is finite-dimensional, and we arrive at a contradiction.

Thus $Y$ contains an infinite-dimensional compactum $Y'$ with $\dim_{\mathbb{Z}[1/p]} Y' = 1$. Define $X'$ to be the preimage of $Y'$ in $X$ and we are done. ■

5. Auxiliary constructions. In this section we provide auxiliary tools used in constructing the examples in Theorems 1.4 and 1.5. We say that a map between simplicial or CW-complexes is combinatorial if the preimage of every subcomplex of the range is a subcomplex of the domain.

5.1. Extending partial maps. Let $M$ be a finite simplicial complex, $K$ a connected CW-complex and $f : F \to K$ a cellular map from a subcomplex $F$ of $M$. We will present a certain construction (that will be referred to as Construction 5.1, and analogously for those in the forthcoming subsections) of a CW-complex $M'$ and a map $\mu : M' \to M$ such that $\mu$ restricted to $F' = \mu^{-1}(F)$ and followed by $f$ extends to a map $f' : M' \to K$.

Let $F_i = F \cup M^{(i)}$ be the union of $F$ with the $i$-skeleton $M^{(i)}$ of $M$. Extend $f$ over $F_1$ to a cellular map $f_1 : F_1 \to K$, set $M_1 = F_1$ and let $\mu_1 : M_1 \to F_1$ be the identity map. We will construct by induction a CW-complex $M_i$, a combinatorial map $\mu_i : M_i \to F_i$ and a cellular map $f_i : M_i \to K$ so that $M_i \subset M_{i+1}$ and $f_{i+1}$ and $\mu_{i+1}$ extend $f_i$ and $\mu_i$ respectively.

The construction proceeds from $i$ to $i+1$ as follows. If $F_i$ already contains $M^{(i+1)}$, set $M_{i+1} = M_i$, $f_{i+1} = f_i$ and $\mu_{i+1} = \mu_i$. Otherwise for every $(i+1)$-simplex $\Delta$ of $M$ not contained in $F_i$ attach to $M_i^{\Delta} = \mu_i^{-1}(\partial \Delta)$ the mapping cylinder of $f_i|_{M_i^{\Delta}}$. The map $f_i$ naturally extends over each mapping cylinder defining $f_{i+1} : M_{i+1} \to K$. Extend the map $\mu_i$ to $\mu_{i+1}$ by sending each mapping cylinder to the corresponding simplex $\Delta$ so that the top ($K$-level) of the mapping cylinder goes to the barycenter of $\Delta$ and $\mu_{i+1}$ is linear on the intervals of the mapping cylinder.

Finally for $i = \dim M$ denote $M' = M_i$, $\mu = \mu_i$ and $f' = f_i$. Note that $M'$ admits a triangulation for which $\mu$ is combinatorial provided the map $f$ is simplicial.
Proposition 5.1. Assume that in the above construction $K = S^1$, the subcomplex $F$ contains the 1-skeleton of $M$ and the map $f : F \to S^1$ is of degree $p^k$ (that is, $f$ lifts to a $\mathbb{Z}_{p^k}$-covering of $S^1$). Let $N$ be a subcomplex of $M$ with $\dim N \leq m$ and let $N' = \mu^{-1}(N)$. Consider the homomorphisms

\begin{align*}
(*) & \quad H_m(N'; \mathbb{Z}_{p^t}) \to H_m(N; \mathbb{Z}_{p^t}), \\
(**) & \quad H_m(N; \mathbb{Z}_p) \to H_m(N; \mathbb{Z}_{p^t}) \to H_m(M; \mathbb{Z}_{p^t}), \\
(***) & \quad H_m(N'; \mathbb{Z}_p) \to H_m(N'; \mathbb{Z}_{p^t}) \to H_m(M'; \mathbb{Z}_{p^t})
\end{align*}

induced by the map $\mu$, the monomorphism $\mathbb{Z}_p \to \mathbb{Z}_{p^t}$, and the inclusions of $N$ into $M$ and $N'$ into $M'$ respectively. Then for $t \leq k$:

1. the homomorphism in (*) is an isomorphism, and
2. if the composition of the homomorphisms in (**) is trivial then the composition of the homomorphisms in (***) is trivial as well.

Proof. Note that $\mu$ is 1-to-1 over the 1-skeleton of $M$ and for every 2-simplex $\Delta$ of $M$, either $\mu$ is 1-to-1 over $\Delta$ or $\mu^{-1}(\Delta)$ is the mapping cylinder of $f$ restricted to $\partial\Delta$ and attached to $\partial\Delta$. Then, since $f$ is of degree $p^k$, one can easily see that (1) holds for $m \leq 2$.

Now let $m > 2$. Note that if $\dim N < m$ then $\dim N' < m$ as well and the proposition trivially holds. Thus assume $\dim N = m$ and let $L$ be the union of all the simplexes of dimension $\leq m - 3$ of the first barycentric subdivision of $M$. Note that $\dim \mu^{-1}(L) \leq m - 2$. Collapse the fibers of $\mu$ over $L \cap N$ to singletons and denote by $N^*$ the CW-complex obtained this way from $N'$ and let $\mu' : N' \to N^*$ and $\mu^* : N^* \to N$ be the induced natural maps.

Let $N_\ast = N^{(m-1)}_\ast$ and $N'_\ast = \mu^{-1}(N_\ast)$. Note that $\mu^*$ is 1-to-1 over $N_\ast$ and hence we can identify $(\mu^*)^{-1}(N_\ast)$ with $N_\ast$. Then, since $\dim \mu^{-1}(L) \leq m - 2$, the map $\mu'$ induces isomorphisms $H_m(N'; \mathbb{Z}_{p^t}) \to H_m(N^*; \mathbb{Z}_{p^t})$ and $H_m(N', N'_\ast; \mathbb{Z}_p) \to H_m(N^*, N'_\ast; \mathbb{Z}_p)$ and $H_m(N', N'_\ast; \mathbb{Z}_{p^t}) \to H_m(N^*, N'_\ast; \mathbb{Z}_{p^t})$. Moreover, it follows from Construction 5.1 that, since $t \leq k$, for every $m$-simplex $\Delta$ of $N$ (with respect to the original triangulation of $M$), we have $H_m((\mu^*)^{-1}(\Delta), (\mu^*)^{-1}(\partial\Delta); \mathbb{Z}_{p^t}) = \mathbb{Z}_{p^t}$ and $\mu^*$ induces an isomorphism

$$H_m((\mu^*)^{-1}(\Delta), (\mu^*)^{-1}(\partial\Delta); \mathbb{Z}_{p^t}) \to H_m(\Delta, \partial\Delta; \mathbb{Z}_{p^t})$$

(everything here can be easily visualized for $m = 3$). Then $\mu^*$ also induces an isomorphism $H_m(N^*, N_\ast; \mathbb{Z}_{p^t}) \to H_m(N, N_\ast; \mathbb{Z}_{p^t})$. Now apply the long exact sequences for the pairs $(N, N_\ast)$ and $(N^*, N'_\ast)$ and the 5-lemma to show, by induction on $m$, that (1) holds. Note that we have also showed that

$$(\dagger) \quad \mu \text{ induces an isomorphism } H_m(N', N'_\ast; \mathbb{Z}_{p^t}) \to H_m(N, N_\ast; \mathbb{Z}_{p^t}).$$

Let us turn to (2). Clearly we can replace $M$ by its $(m+1)$-skeleton and assume that $\dim M \leq m + 1$. Let $M_\ast$ be the $m$-skeleton of $M$ and $M'_\ast = \mu^{-1}(M_\ast)$. By (1) and (\dagger) we see that $\mu$ induces isomorphisms $H_m(M'_\ast; \mathbb{Z}_{p^t}) \to$
Let $M$ be a finite simplicial complex and $g : L \to M$ a circle bundle. We will say that $g$ is $p$-flexible if for every $k > 0$ there is a section $f : M^{(1)} \to L$ over the 1-skeleton of $M$ such that for every
2-simplex $\Delta$ of $M$ and a trivialization $g^{-1}(\Delta) = \Delta \times S^1$ the map $f$ restricted to $\partial \Delta$ and followed by the projection $\Delta \times S^1 \to S^1$ lifts to the $\mathbb{Z}_{p^k}$-covering of $S^1$. We will refer to $p^k$ as the degree of $f$. Clearly this definition does not depend on the trivializations over 2-simplexes of $M$.

We will describe two versions of killing non-trivial $p$-flexible bundles that will be used in different contexts.

**5.2.1. Killing non-trivial flexible bundles: a version for Theorem 1.4.** Let $g : L \to M$ be a $p$-flexible circle bundle over a finite simplicial complex $M$. Let $f : M^{(1)} \to L$ be a section of degree $p^k$ over the 1-skeleton of $M$. We will construct a CW-complex $M'$ and a map $\mu : M' \to M$ so that the pull-back bundle $g' : L' \to M'$ of $f$ via $\mu$ admits a section $f' : M' \to L'$.

Set $M_1 = M^{(1)}$, $\mu_1 : M_1 \to M$ to be the inclusion and $f_1 = f : M_1 \to L$. We will construct by induction finite CW-complexes $M_i$ and maps $\mu_i : M_i \to M^{(i)}$ and $f_i : M_i \to L$ so that $g \circ f_i = \mu_i$.

The construction proceeds from $i$ to $i + 1$ as follows. Fix an $(i + 1)$-simplex $\Delta$ of $M$ and consider a trivialization $g^{-1}(\Delta) = \Delta \times S^1$ and let $f_\Delta : \mu_i^{-1}(\partial \Delta) \to S^1$ be the map $f_i$ restricted to $\mu_i^{-1}(\partial \Delta)$ and followed by the projection $g^{-1}(\Delta) = \Delta \times S^1 \to S^1$.

Consider the mapping cylinder $M_{\Delta}$ of $f_\Delta$ and let $\mu_\Delta : M_{\Delta} \to \Delta$ be the map sending the top of $M_{\Delta}$ to the barycenter of $\Delta$ and linearly extending $\mu_i$ restricted to $\mu_i^{-1}(\partial \Delta)$. Then the map $f_i$ restricted to $\mu_i^{-1}(\partial \Delta)$ naturally extends over $M_{\Delta}$ to a map $f_\Delta : M_{\Delta} \to g^{-1}(\Delta)$ such that $\mu_\Delta = g \circ f_\Delta$.

Now attach $M_{\Delta}$ to a map $f_\Delta : M_{\Delta} \to g^{-1}(\Delta)$ such that $\mu_\Delta = g \circ f_\Delta$. Note that $M_{\Delta}$ admits a triangulation for which $\mu$ is combinatorial.

**Proposition 5.3.** The conclusions of Proposition 5.1 hold for the construction above.

The proof of Proposition 5.1 applies to prove Proposition 5.3 as well.

**5.2.2. Killing non-trivial flexible bundles: a version for Theorem 1.5.** Let $g : L \to M$ be a $p$-flexible circle bundle over a finite simplicial complex $M$. Let $m = \dim M$ and $k = tm$, and let $f : M^{(1)} \to L$ be a section of degree $p^{km}$ over the 1-skeleton of $M$. We will construct a CW-complex $M'$ and a map $\mu : M' \to M$ so that the pull-back bundle $g' : L' \to M'$ of $f$ via $\mu$ admits a section $f' : M' \to L'$.

Set $M_1 = M^{(1)}$, $\mu_1 : M_1 \to M$ to be the inclusion and $f_1 = f : M_1 \to L$. We will construct by induction finite CW-complexes $M_i$ and maps $\mu_i :
Let $M_i \rightarrow M^{(i)}$ and $f_i : M_i \rightarrow L$ so that $g \circ f_i = \mu_i$ and $f_i$ is a map of degree $p^{(m-i)k}$. By this we mean that for every $(i+1)$-simplex $\Delta$ of $M$ and a trivialization $g^{-1}(\Delta) = \Delta \times S^1$ of $g$ over $\Delta$ the map $f_i$ restricted to $\mu_i^{-1}(\partial \Delta)$ and followed by the projection $\Delta \times S^1 \rightarrow S^1$ lifts to the $\mathbb{Z}_p^{(m-i)k}$-covering of $S^1$.

The construction proceeds from $i$ to $i+1$ as follows. Fix an $(i+1)$-simplex $\Delta$ of $M$, a trivialization $g^{-1}(\Delta) = \Delta \times S^1$ and a map $f_\Delta : \mu_i^{-1}(\partial \Delta) \rightarrow S^1$ witnessing that $f_i$ restricted to $\mu_i^{-1}(\partial \Delta)$ is a map of degree $p^{(m-i)k}$. Consider the finite telescope $T$ of maps $S^1 \rightarrow S^1$ each of them being the $\mathbb{Z}_p$-covering of $S^1$, and denote by $M_\Delta$ the mapping cylinder of $f_\Delta$ followed by the embedding of $S^1$ in $T$ as the first circle of $T$. Let $\mu_\Delta : M_\Delta \rightarrow \Delta$ be the map sending the top ($T$-level) of $M_\Delta$ to the barycenter of $\Delta$ and linearly extending $\mu_i$ restricted to $\mu_i^{-1}(\partial \Delta)$. Then the map $f_\Delta$ restricted to $\mu_i^{-1}(\partial \Delta)$ naturally extends over $M_\Delta$ to a map $f_\Delta : M_\Delta \rightarrow g^{-1}(\Delta)$ of degree $p^{(m-i-1)k}$ such that $\mu_\Delta = g \circ f_\Delta$. Then attach $M_\Delta$ to $\mu_i^{-1}(\partial \Delta)$; doing that for every $(i+1)$-simplex $\Delta$ of $M$ we obtain the CW-complex $M_{i+1}$ and the maps $\mu_{i+1} : M_{i+1} \rightarrow M^{(i+1)}$ and $f_{i+1} : M_{i+1} \rightarrow L$ induced by the maps $\mu_\Delta$ and $f_\Delta$ respectively. Clearly $\mu_{i+1} = g \circ f_{i+1}$.

The only thing that we need to check is that $f_{i+1}$ is of degree $p^{(m-i-1)k}$ on $\mu_{i+1}^{-1}(\partial \Delta)$ for every $(i+2)$-simplex $\Delta$ of $M$. Consider a trivialization $g^{-1}(\Delta) = \Delta \times S^1$ of $g^{-1}(\Delta)$ and denote by $\Delta_0, \ldots, \Delta_{i+1}$ the $(i+1)$-simplexes contained in $\Delta$. Note that for $\mu_{i+1}^{-1}((\Delta_0 \cup \cdots \cup \Delta_s) \cap \Delta_{s+1})$ is connected for every $0 \leq s \leq i$. Then the cycles of $H_1(\mu_{i+1}^{-1}(\Delta_s))$, $0 \leq s \leq i + 1$, generate $H_1(\mu_{i+1}^{-1}(\partial \Delta))$. Since $f_{i+1}$ restricted to each $\Delta_s$ and followed by the projection $\alpha : \Delta \times S^1 \rightarrow S^1$ is of degree $p^{(m-i-1)k}$, the homomorphism induced by $\alpha \circ f_{i+1}$ sends $H_1(\mu_{i+1}^{-1}(\partial \Delta))$ into the elements of $H_1(S^1) = \mathbb{Z}$ divisible by $p^{(m-i-1)k}$. This means that $f_{i+1}$ restricted to $\mu_{i+1}^{-1}(\partial \Delta)$ is of degree $p^{(m-i-1)k}$.

Finally set $m = \dim M$, $M' = M_m$, $\mu' = \mu_m$, $L' = \text{the pull-back of } L$ under $\mu$, $g$ and $g' : L' \rightarrow M'$ the pull-back of $g$. Then $f_m$ induces a section $f' : M' \rightarrow L'$. Note that $M'$ admits a triangulation for which $\mu$ is combinatorial.

Let $T^j$, $1 \leq j \leq m$, be the subtelescope of $T$ consisting of the first $j$ maps $S^1 \rightarrow S^1$. Thus $T^m = T$ and in the construction above we can consider the subcomplexes $M^j \subset M' = M^m$ obtained from $M'$ by leaving only the subtelescope $T_j$ from $T$ in all the mapping cylinders $M_\Delta$.

**Proposition 5.4.** Let $N$ be a connected CW-complex whose homotopy groups are finite $p$-groups. Consider Construction 5.2.2 with $t$ being such that $\alpha^p = 1$ for every $\alpha \in \pi_n(N)$ and $1 \leq n \leq m$ and assume that $f_N : F_N \rightarrow N$ is a map from a subcomplex $F_N$ of $M$ such that $f_N$ does not extend over $M$. Then $\mu$ restricted to $\mu^{-1}(F_N)$ and followed by $f_N$ does not extend over $M'$. 


Let $M_*$ be the space obtained from $M'$ by collapsing the fibers of $\mu$ over $F_N$ to singletons, $\mu_* : M_* \to M$ the map induced by $\mu$ and $M_*^j = \mu_*(M^j)$, $1 \leq j \leq m$. Recall that $M^m = M'$ and hence $M_*^m = M_*$. Thus we can consider $F_N$ as a subcomplex of $M_*$ and aiming for a contradiction assume that $f_N$ extends to $f_* : M_* \to N$. Recall that in Construction 5.2.2 we denote by $T$ the telescope of $m$ copies of $\mathbb{Z}_{p'}$-coverings $S^1 \to S^1$ and by $T^j$, $1 \leq j \leq m$, the subtelescope of the first $j$ maps of $T$ ($T^m = T$).

Consider the first barycentric subdivision $\beta M$ of the triangulation of $M$. Note that for every 0-dimensional simplex (vertex) $\Delta$ of $\beta M$, $(\mu_*^{-1}(\Delta))$ is either a singleton or homeomorphic to $T = T^m$. Then $(\mu_*^{-1}(\Delta)) \cap M_*^{m-1}$ is either a singleton or homeomorphic to the subtelescope $T^{m-1}$ of $T$. Note that the inclusion of $T^{m-1}$ into $T = T^m$ represents up to homotopy a map of degree $p'$ of the circle $S^1 \to S^1$ and hence $f_*$ is null-homotopic on $(\mu_*^{-1}(\Delta)) \cap M_*^{m-1}$. Thus we can replace $M_*^{m-1}$ by the space $M_0^{m-1}$ obtained from $M_*^{m-1}$ by collapsing $(\mu_*^{-1}(\Delta)) \cap M_*^{m-1}$ to singletons for every 0-simplex $\Delta$ of $\beta M$, and $\mu_*$ and $f_*$ by the induced maps $\mu_0^{m-1}$ and $f_0^{m-1}$ to $M$ and $N$ respectively, and assume that $\mu_0^{m-1}$ is 1-to-1 over the 0-simplices of $\beta M$.

Now consider a 1-simplex $\Delta$ of $\beta M$. Then $(\mu_0^{m-1})^{-1}(\Delta)$ is either contractible or homotopy equivalent to $\Sigma T^{m-1}$. Denote by $M_0^{m-2}$ the image of $M_0^{m-2}$ in $M_0^{m-1}$ under the natural projection from $M' = M^m$ to $M_0^{m-1}$. Then $(\mu_0^{m-1})^{-1}(\Delta) \cap M_0^{m-2}$ is either contractible or homeomorphic to the subtelescope $\Sigma T^{m-2}$ of $\Sigma T^{m-1}$ and hence $f_0^{m-1}$ is null-homotopic on $(\mu_0^{m-1})^{-1}(\Delta) \cap M_0^{m-2}$ since the inclusion of $\Sigma T^{m-2}$ into $\Sigma T^{m-1}$ represents up to homotopy a map of degree $p'$ of the 2-sphere $S^2 \to S^2$. Thus $f_0^{m-1}$ restricted to $M_0^{m-2}$ factors up to homotopy through the space obtained from $M_0^{m-2}$ by collapsing the fibers of $\mu_0^{m-1}|M_0^{m-2}$ over the simplex $\Delta$. Doing that consecutively for all the 1-simplices of $\beta M$ we obtain the space $M_1^{m-2}$ and the maps $\mu_1^{m-2} : M_1^{m-1} \to M$ and $f_1^{m-2} : M_1^{m-2} \to N$ induced by $\mu_0^{m-1}$ and $f_0^{m-1}$ respectively such that $\mu_1^{m-2}$ is 1-to-1 over the 1-simplices of $\beta M$.

Proceed by induction and construct for every $i \leq m-1$ the space $M_i^{m-i-1}$ and the maps $\mu_i^{m-i-1} : M_i^{m-i-1} \to M$ and $f_i^{m-i-1} : M_i^{m-i-1} \to N$ and finally deduce for $i = m-1 = \dim M - 1$ that $M_0^{m-1} = M$ and $f_0^{m-1}$ extends $f_N$, which contradicts the assumptions of the proposition.

### 5.3. Constructing inverse limits

We will describe here how to construct an inverse limit $Y = \lim_\leftarrow (M_i, \mu_i)$ of finite simplicial complexes $M_i$ performing countably many procedures on certain objects determined by each $M_i$. We mainly deal with the objects described in Section 5 like partial maps and circle bundles and the procedures like extending partial partial maps and killing some non-trivial circle bundles. We assume that any object on $M_j$ can be transferred to any $M_i$ with $i \geq j$ via the map $\mu_i^j =$
\( \mu_i \circ \cdots \circ \mu_{j+1} : M_i \to M_j \) (\( \mu^i \) is the identity map of \( M_i \)). By transferring a partial map on \( M_j \) to \( M_i \) we just mean that a partial map \( f : F \to K \) from a closed subset \( F \) of \( M_j \) to a CW-complex \( K \) goes to the partial map \( f \circ \mu^i_j : (\mu^i_j)^{-1}(F) \to K \) on \( M_i \). And by transferring a circle bundle over \( M_j \) to \( M_i \) we mean the pull-back of the bundle to \( M_i \) via the map \( \mu^i_j \). We also assume that for each finite simplicial complex we can fix countably many objects on which we want to perform appropriate procedures.

The inverse system \((Y_i, \mu_i)\) is constructed as follows. Consider a bijection \( \beta : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \) such that \( \beta(j, n) \geq j \). Assume that \( Y_j \) and \( \mu_j \) are already constructed for \( j \leq i \). Moreover, for each \( j \leq i \) we have also fixed countably many objects \( Q_j \) for \( Y_j \) that we need to take care of and the objects in \( Q_j \) are indexed by natural numbers. Proceed to \( i + 1 \) as follows. Let \( (j, n) = i \).

Transfer the object indexed by \( n \) in \( Q_j \) to \( M_i \), and construct \( M_{i+1} \) and \( \mu_{i+1} \) in order to perform the procedure appropriate for this object. Finally, pick out countably many objects \( Q_{i+1} \) for \( Y_{i+1} \) needed to be taken care of and index the objects in \( Q_{i+1} \) by the natural numbers.

Thus we ensure that in the inverse limit determining \( Y \) we have taken care of all the objects picked out for each \( M_i \).

6. Proof of Theorem 1.4 Theorem 1.4 is a consequence of Theorem 1.6 and the case \( n = 1 \) of the following proposition.

**Proposition 6.1.** For every \( n \geq 1 \) there is a compactum \( Y \) with \( \dim_{\mathbb{Z}[1/p]} Y = 1 \), \( \dim_{\mathbb{Z}_p} Y = n + 1 \) and \( \dim Y = n + 2 \) such that \( H^2(Y; \mathbb{Z}) \) does not contain a subgroup isomorphic to \( \mathbb{Z}_p^{\infty} \).

**Proof.** We will construct \( Y \) as the inverse limit of \((n + 2)\)-dimensional finite simplicial complexes \( M_i \) and combinatorial bonding maps \( \mu_{i+1} : M_{i+1} \to M_i \). In order to show that \( Y \) has the required properties we consider for each \( i \) a subcomplex \( F_i \) of \( M_i \) such that \( F_{i+1} = \mu_{i+1}^{-1}(F_i) \) and two natural numbers \( k_i \) and \( t_i \) such that \( k_{i+1} \geq k_i \), \( t_{i+1} \geq t_i \) and \( k_i \geq t_i \). Set \( M_0 \) to be an \((n + 2)\)-ball, \( F_0 = S^{n+1} \) the boundary of this ball, and \( k_0 = t_0 = 1 \). We require that:

1. \( \dim F_i = n + 1 \), \( H_{n+1}(F_i; \mathbb{Z}_p) = \mathbb{Z}_p \) and the homomorphism \( H_{n+1}(F_{i+1}; \mathbb{Z}_p) \to H_{n+1}(F_i; \mathbb{Z}_p) \) induced by \( \mu_{i+1} \) is an isomorphism.
2. Let \( H_{n+1}(F_i; \mathbb{Z}_p) \to H_{n+1}(F_i; \mathbb{Z}_p^{t_i}) \) and \( H_{n+1}(F_i; \mathbb{Z}_p^{t_i}) \to H_{n+1}(M_i; \mathbb{Z}_p^{t_i}) \) be the homomorphisms induced by the monomorphism \( \mathbb{Z}_p \to \mathbb{Z}_p^{t_i} \) and the inclusion of \( F_i \) into \( M_i \) respectively. Then the following composition is trivial:

\[
H_{n+1}(F_i; \mathbb{Z}_p) \to H_{n+1}(F_i; \mathbb{Z}_p^{t_i}) \to H_{n+1}(M_i; \mathbb{Z}_p^{t_i}).
\]
Clearly the relevant parts of (1) and (2) hold for $i = 0$. Assuming that the
construction is completed for $i$, we proceed to $i + 1$ by performing one of the
following procedures.

**PROCEDURE I** (taking care of non-flexible bundles). Let $g : L \to M_i$ be
a circle bundle which is not $p$-flexible. Then take any natural number $k$
such that there is no section over $M_i^{(1)}$ of degree $p^k$. Set $M_{i+1} = M_i$, $\mu_{i+1} = \mu$
the identity map, $k_{i+1} = \max\{k, k_i\}$ and $t_{i+1} = t_i$.

**PROCEDURE II** (taking care of flexible bundles). Let $g : L \to M_i$ be a
flexible bundle. Apply 5.2.1 with $M = M_i$ and $k = k_i$, construct $M_{i+1} = M'$
and $\mu_{i+1} = \mu$. Set $k_{i+1} = k_i$ and $t_{i+1} = t_i$.

**PROCEDURE III** (extending partial maps to a circle). Let $F$ be a subcomplex
of $M_i$ and $f : F \to S^1$ a map. Extend $f$ over the 1-skeleton $M^{(1)}$
of $M$ and replace $f$ by a cellular map homotopic to $f$ followed by a map
$S^1 \to S^1$ of degree $k_i$. Thus we assume that $F$ contains the 1-skeleton of $M_i$
and $f : F \to S^1$ is a cellular map of degree $p^{k_i}$ (recall that “of degree $p^{k_i}$”
means that $f$ lifts to a $\mathbb{Z}_p^{k_i}$-covering of $S^1$). Apply 5.1 with $M = M_i$
and $K = S^1$ to construct $M_{i+1} = M'$ and $\mu_{i+1} = \mu$. Set $k_{i+1} = k_i + 1$
and $t_{i+1} = t_i + 1$.

**PROCEDURE IV** (extending partial maps to $M(\mathbb{Z}_p, n + 1)$). Let $F$
be a subcomplex of $M_i$ and $f : F \to M(\mathbb{Z}_p, n + 1)$ a map. Extend $f$
over the $(n + 1)$-skeleton on $M_i$ and replace $f$ by a homotopic cellular map.
Thus we assume that $F$ contains the $(n + 1)$-skeleton $M^{(n+1)}$ of $M_i$
and $f : F \to M(\mathbb{Z}_p, n + 1)$ is a cellular map. Apply 5.1 with $M = M_i$
and $K = M(\mathbb{Z}_p, n + 1)$ to construct $M_{i+1} = M'$ and $\mu_{i+1} = \mu$. Set $k_{i+1} = k_i + 1$
and $t_{i+1} = t_i + 1$.

Let us check that after preforming the above procedures the conditions (1)
and (2) hold. This is obvious for Procedure I. Procedures II and III preserve (1)
and (2) by Propositions 5.1 and 5.3 because $t_{i+1} = t_i \leq k_i$. Procedure IV
preserves (1) and (2) by Proposition 5.2.

Let us show that $Y = \lim_{i \to \infty} M_i$ is $(n + 2)$-dimensional. Clearly $\dim Y$
$\leq n + 2$. Consider the map $\mu_i|F_i : F_i \to F_0 = S^{n+1}$. Since $\dim F_i \leq n + 1$,
the long exact sequence generated by

$$0 \to \mathbb{Z}_p \to \mathbb{Z}_{p^{k_i}} \to \mathbb{Z}_{p^{k_i}+1} \to 0$$

implies that $H_{n+1}(F_i; \mathbb{Z}_p) \to H_{n+1}(F_i; \mathbb{Z}_p)$ is injective for every $i$ and hence,
by (1) and (2), the map $\mu_i^0 = \mu_i \circ \cdots \circ \mu_0 : M_i \to M_0$ restricted to $F_i$ and $F_0$
induces a non-trivial homomorphism

$$\ker[H_{n+1}(F_i; \mathbb{Z}_{p^{k_i}}) \to H_{n+1}(M_i; \mathbb{Z}_{p^{k_i}})] \to H_{n+1}(F_0; \mathbb{Z}_{p^{k_i}}).$$

This implies that $\mu_i^0$ restricted to $F_i$ does not extend over $M_i$ as a map to
$F_0 = S^{n+1}$ and hence $\dim Y = n + 2$. 

Assume that $Y$ is constructed as described in 5.3. The objects that we pick out for each $M_i$ are all the (non-isomorphic) circle bundles over $M_i$ and countably many maps from subcomplexes of $M_i$ to $S^1$ and $M(Z_p, n + 1)$ representing all possible maps up to homotopy. The procedures that we perform are Procedures I–IV.

Assuming that on each $M_i$ we fix a sufficiently finite triangulation, we find that:

- Procedure III together with Proposition 1.1 implies that $\dim_{\mathbb{Z}[1/p]} Y \leq 1$;
- Procedure IV implies that $\text{ext-dim } Y \leq M(Z_p, n + 1)$ and hence, by Dranishnikov’s extension criterion, $\dim_{\mathbb{Z}_p} Y \leq n + 1$.

Then $\dim Y = n + 2$ and the Bockstein inequalities imply that $\dim_{\mathbb{Z}[1/p]} Y = 1$ and $\dim_{\mathbb{Z}_p} Y = n + 1$.

Let us show that $H^2(Y; \mathbb{Z})$ does not contain a subgroup isomorphic to $\mathbb{Z}_{p^\infty}$. Since $\dim_{\mathbb{Z}[1/p]} Y = 1$ we see that $H^2(Y; \mathbb{Z}) \otimes \mathbb{Z}[1/p] = 0$ and hence $H^2(Y; \mathbb{Z})$ is $p$-torsion. We will show that $H^2(Y; \mathbb{Z})$ does not contain a non-trivial element $\alpha$ that is divisible by $p^k$ for every $k$. Aiming at a contradiction assume that such $\alpha$ does exist. Consider the circle bundle $g_Y : Z \to Y$ corresponding to $\alpha$. Then Construction 5.3 guarantees that at a certain step $i$ of the construction we will consider a circle bundle $g : L \to M_i$ such that $g_Y$ is the pull-back of $g$ under the projection $\mu^i : Y \to M_i$.

If $g$ is flexible then we apply Procedure II and, by Construction 5.2.1 the pull-back bundle $g_{i+1} : L_{i+1} \to M_{i+1}$ of $g$ under the map $\mu_{i+1}$ admits a section and hence both $g_{i+1}$ and $g_Y$ are trivial. Thus we arrive at a contradiction with $\alpha \neq 0$.

If $g : L \to M_i$ is non-flexible then we apply Procedure I. Denote by $g_j : L_j \to M_j$, $j \geq i$, the pull-back bundle of $g$ to $M_j$ under $\mu_j^i = \mu_j \circ \cdots \circ \mu_{i+1} : M_j \to M_i$ with $\mu_i^i$ being the identity map and $g_i = g$ and $L_i = L$. Denote by $G_j^i : L_j \to L_i$ the induced map. Recall that $\alpha$ is divisible by $p^k$ for every $k$. Then for a sufficiently large $j$ there is a circle bundle $g^*_j : L_j^* \to M_j$ such that $g_j : L_j \to M_j$ is induced by $g^*_j$ and $\mathbb{Z}_{p^{k_{i+1}}}$ (see Lemma 3.1). Let $G_j^* : L_j^* \to L_j$ be a bundle map induced by the homomorphism $S^1 \to S^1/\mathbb{Z}_{p^{k_{i+1}}}$. Take a section $s^*_j : M_j^{(1)} \to L_j^*$.

Consider a 2-simplex $\Omega_i$ of $M_i$ and let $q_i^{-1}(\Omega_i) = \Omega_i \times S^1$ be a trivialization of $L_i$ over $\Omega_i$. Consider $\Omega_j = (\mu_j^i)^{-1}(\Omega_i)$. Analyzing Procedures I–IV we deduce that $\dim \Omega_j = 2$, for Procedures I and IV we have $\Omega_j^{k+1} = \Omega_j$, and for Procedures II and III we find that $\Omega_j^{k+1}$ is obtained from $\Omega_j$ by taking a triangulation of $\Omega_j$ and replacing each 2-simplex $\Delta$ of $\Omega_j$ by a mapping cylinder of a map $\partial \Delta \to S^1$ of degree $k_j$ attached to $\partial \Delta$. Note that $k_j \geq k_{i+1}$ for $j > i$. Also note that $\mu_j^i$ is 1-to-1 over the 1-skeleton $M_i^{(1)}$ of $M_i$, and hence $M_i^{(1)}$ can be considered as a subset $M_i^{(1)} \subseteq M_j^{(1)}$ of the 1-skeleton.
of \(M_j\), and then \(\partial \Omega_j\) considered as a subset of \(\Omega_j\) is homologous to 0 in \(H_1(\Omega_j; \mathbb{Z}_{p^{k_{i+1}}})\), \(j > i\).

Consider the trivialization \(g_j^{-1}(\Omega_j) = \Omega_j \times S^1\) of \(L_j\) over \(\Omega_j\) induced by the trivialization of \(g_i^{-1}(\Omega_i) = \Omega_i \times S^1\) and consider the section \(s_j : M_j^{(1)} \to L_j\) which is \(s_j^*\) followed by \(G_j^*\). Note that for every 2-simplex \(\Delta\) of \(\Omega_j\), the map \(s_j\) restricted to \(\partial \Delta\) and followed by the projection of \(g_j^{-1}(\Omega_j) = \Omega_j \times S^1\) to \(S^1\) is of degree \(p^{k_{i+1}}\). Then, since \(\partial \Omega_i\) is homologous to 0 in \(H_1(\Omega_j; \mathbb{Z}_{p^{k_{i+1}}})\), we see that \(s_j\) restricted to \(\partial \Omega_i\) and followed by the projection of \(g_j^{-1}(\Omega_j) = \Omega_j \times S^1\) to \(S^1\) is of degree \(p^{k_{i+1}}\) as well. Thus the section \(s_i = \mu_j^i \circ s_j|M_i^{(1)} : M_i^{(1)} \to L_i\) is of degree \(p^{k_{i+1}}\) and this violates our choice of \(k_{i+1}\) in Procedure 1. ■

7. Proof of Theorem 1.5. Theorem 1.5 is a consequence of Theorem 1.6 and the following proposition.

**Proposition 7.1.** There is an infinite-dimensional compactum \(Y\) with \(\dim_{\mathbb{Z}[1/p]} Y = 1\) and \(\dim_{\mathbb{Z}} Y = 2\) such that \(H^2(Y; \mathbb{Z})\) does not contain a subgroup isomorphic to \(\mathbb{Z}_{p^\infty}\).

Let \(K\) be a CW-complex and \(N\) a connected CW-complex. We denote by map\((K, N)\) the space of pointed maps from \(K\) to \(N\) with the compact-open topology. We will write map\((K, N)\) \(\cong 0\) if map\((K, N)\) is weakly homotopy equivalent to a point. Note that map\((K, N)\) \(\cong 0\) if and only if for every \(n \geq 0\), every map from \(\Sigma^n K\) to \(N\) is null-homotopic.

In the proof of Proposition 7.1 we will use the following facts.

**Theorem 7.2** (Miller’s theorem (the Sullivan conjecture), [13]). Let \(G\) be a finite group and \(N\) a connected finite CW-complex. Then map\((K(G, 1), N)\) \(\cong 0\).

**Proposition 7.3** ([11]). Let \(M\) be a countable CW-complex, \(N\) a connected CW-complex whose homotopy groups are finite, \(F_N\) a subcomplex of \(M\), and \(f : F_N \to N\) a map that cannot be continuously extended over \(M\). Then there exists a finite subcomplex \(M_N\) of \(M\) such that \(f_N|F_N \cap M_N : F_N \cap M_N \to N\) cannot be continuously extended over \(M_N\).

**Proposition 7.4.** Let \(N\) be a CW-complex whose homotopy groups are finite \(p\)-groups. Then map\((K(\mathbb{Z}[1/p], 1), N)\) \(\cong 0\).

**Proof.** Note that \(\Sigma^n K(\mathbb{Z}[1/p], 1)\) can be represented as the infinite telescope of a map \(S^{n+1} \to S^{n+1}\) of degree \(p\). Then \(\pi_n^{n+1}(\Sigma^n K(\mathbb{Z}[1/p], 1)) = \mathbb{Z}[1/p]\) and every finite subtelescope of \(\Sigma^n K(\mathbb{Z}[1/p], 1)\) is homotopy equivalent to \(S^{n+1}\). Since \(\pi_n^{n+1}(N)\) is a finite \(p\)-group every map \(f : \Sigma^n K(\mathbb{Z}[1/p], 1) \to N\) sends \(\pi_n^{n+1}(\Sigma^n K(\mathbb{Z}[1/p], 1))\) to 0 in \(\pi_n^{n+1}(N)\) and hence \(f\) is null-
homotopic one very finite subtelescope of $\Sigma^n K(\mathbb{Z}[1/p], 1)$. Then $f$ extends over $\Sigma(\Sigma^n K(\mathbb{Z}[1/p], 1))$ since otherwise it would contradict Proposition 7.3. Thus $f$ is null-homotopic.

**Proposition 7.5.** Consider Construction 5.1. Let $N$ be a connected CW-complex such that $\text{map}(K, N) \cong 0$ and $f_N : F_N \to N$ a map from a subcomplex $F_N$ of $M$ such that $f_N$ does not extend over $M$. Then $\mu$ restricted to $\mu^{-1}(F_N)$ and followed by $f_N$ does not extend over $M'$.

**Proof.** Let $M^*$ by the space obtained from $M'$ by collapsing the fibers of $\mu$ over $N$ to singletons and $\mu^* : M^* \to M$ the induced map. Thus we can consider $F_N$ as a subcomplex of $M^*$ and aiming at a contradiction assume that $f_N$ extends to $f^* : M^* \to N$.

Consider the first barycentric subdivision $\beta M$ of the triangulation of $M$. Note that for every 0-dimensional simplex (vertex) $\Delta$ of $\beta M$, $(\mu^*)^{-1}(\Delta)$ is either a singleton or homeomorphic to $K$. Then $f^*$ is null-homotopic on $(\mu^*)^{-1}(\Delta)$. Thus we can replace $M^*$ by the space $M_0^*$ obtained from $M^*$ by collapsing $(\mu^*)^{-1}(\Delta)$ to singletons for every 0-simplex $\Delta$ of $\beta M$, and $\mu^*$ and $f^*$ by the induced maps $\mu_0^*$ and $f_0^*$ to $M$ and $N$ respectively, and assume that $\mu_0^*$ is 1-to-1 over the 0-simplexes of $\beta M$.

Now consider a 1-simplex $\Delta$ of $\beta M$. Then $(\mu_0^*)^{-1}(\Delta)$ is either contractible or homotopy equivalent to $\Sigma K$ and hence $f_0^*$ is null-homotopic on $(\mu_0^*)^{-1}(\Delta)$. Thus $f_0^*$ factors up to homotopy through the space obtained from $M_0^*$ by collapsing the fibers of $\mu_0^*$ over the simplex $\Delta$. Doing that consecutively for all the 1-simplexes of $\beta M$ we obtain the space $M_1^*$ and the maps $\mu_1^* : M_1^* \to M$ and $f_1^* : M_1^* \to N$ induced by $\mu_0^*$ and $f_0^*$ respectively such that $\mu_1^*$ is 1-to-1 over the 1-simplexes of $\beta M$.

Proceed by induction and finally deduce for $m = \dim M$ that $M_m^* = M$ and $f_m^*$ extends $f_N$, which contradicts the assumptions of the proposition.

**Proof of Proposition 7.1.** We will construct $Y$ as the inverse limit of finite simplicial complexes $M_i$ and combinatorial bonding maps $\mu_{i+1} : M_{i+1} \to M_i$. In order to show that $Y$ has the required properties we consider for each $i$ a subcomplex $F_i$ of $M_i$ such that $F_{i+1} = \mu_{i+1}^{-1}(F_i)$ and a natural numbers $k_i$ such that $k_{i+1} \geq k_i$. Set $F_0$ to be a Moore space $M(\mathbb{Z}_p, 2)$, $M_0$ the cone over $F_0$ and $k_0 = 1$. We denote $\mu_j^i = \mu_j \circ \cdots \circ \mu_{i+1} : M_j \to M_i$ with $\mu_j^i$ being the identity map and require that

1. $\mu_0^i|_{F_i} : F_i \to F_0 = M(\mathbb{Z}_p, 2)$ does not extend over $M_i$ as a map to $M(\mathbb{Z}_p, 2)$.

Clearly (1) holds for $i = 0$. Assuming that the construction is completed for $i$, we proceed to $i + 1$ performing one of the following procedures.

**Procedure I** (taking care of non-flexible bundles). Let $g : L \to M_i$ be a circle bundle which is not $p$-flexible. Then take any natural number $k$...
such that there is no section over $M_i^{(1)}$ of degree $p^k$. Set $M_{i+1} = M_i$, $\mu_{i+1} = \mu_i$.

Clearly Procedure I preserves (1) for $i + 1$.

**Procedure II** (taking care of flexible bundles). Note that the homotopy groups of $M(\mathbb{Z}_p, 2)$ are finite $p$-torsion; denote by $t$ any natural number such that $t \geq k_i$ and $p^t \pi_j(M(\mathbb{Z}_p, 2)) = 0$ for every $j \leq m = \dim M_i$. Let $g : L \to M_i$ be a flexible bundle. Apply 5.2.2 with $M = M_i$, $N = M(\mathbb{Z}_p, 2)$, $F_N = F_i$, $f_N = \mu_i^0$, $F : F_i \to N = F_0 = M(\mathbb{Z}_p, 2)$ and $t$ and $m$ as above to construct $M_{i+1} = M'$ and $\mu_{i+1} = \mu'$, and set $k_{i+1} = k_i$. By Proposition 5.4 we see that (1) is preserved for $i + 1$.

**Procedure III** (extending partial maps to a circle). Let $F$ be a subcomplex of $M_i$ and $f : F \to K(\mathbb{Z}[1/p], 1)$ a map. Recall that $K(\mathbb{Z}[1/p], 1)$ can be represented as the infinite telescope of the $\mathbb{Z}_p$-covering map $S^1 \to S^1$. Extend $f$ over the 1-skeleton $M^{(1)}$ of $M$ and replace $f$ by a cellular map homotopic to $f$ such that $f(F) = S^1 \subset K(\mathbb{Z}[1/p], 1)$ and $f$ as a map to $S^1$ is of degree $k_i$. Apply 5.1 with $M = M_i$ and $K = K(\mathbb{Z}[1/p], 1)$ to construct $M_{i+1} = M'$ and $\mu_{i+1} = \mu$. By Propositions 7.3 and 5.4 we find that (1) is preserved for $i + 1$. By Proposition 7.3 we can replace in Construction 5.1 the complex $K = K(\mathbb{Z}[1/p], 1)$ by a finite subcomplex of $K(\mathbb{Z}[1/p], 1)$ containing $f(F)$ and still preserve (1). Thus we assume that $M'$ is a finite CW-complex and set $k_{i+1} = k_i$.

**Procedure IV** (extending partial maps to $K(\mathbb{Z}_{p^n}, 1)$). Let $F$ be a subcomplex of $M_i$ and $f : F \to K(\mathbb{Z}_{p^n}, 1)$ a map. Extend $f$ over the 1-skeleton of $M_i$ and assume that $M_i^{(1)} \subset F$. Represent $K(\mathbb{Z}_{p^n}, 1)$ as the infinite telescope of the maps $K(\mathbb{Z}_{p^n}, 1) \to K(\mathbb{Z}_{p^{n+1}}, 1)$ induced by the monomorphisms $\mathbb{Z}_{p^n} \to \mathbb{Z}_{p^{n+1}}$ and replace $f$ by a homotopic cellular map $f : F \to K(\mathbb{Z}_{p^n}, 1)$ such that $f(M_i^{(1)}) \subset K(\mathbb{Z}_{p^n}, 1)^{(1)}$ and $f$ restricted to $M_i^{(1)}$ and considered as a map to $S^1 = K(\mathbb{Z}_{p^n}, 1)^{(1)}$ is of degree $p^{k_i}$. Apply 5.1 with $M = M_i$ and $K = K(\mathbb{Z}_{p^n}, 1)$ to construct $M_{i+1} = M'$ and $\mu_{i+1} = \mu$. By Theorem 7.2 and Proposition 7.3 we see that (1) is preserved for $i + 1$. By Proposition 7.3 we can replace in Construction 5.1 the complex $K = K(\mathbb{Z}_{p^n}, 1)$ by a finite subcomplex of $K(\mathbb{Z}_{p^n}, 1)$ containing $f(F)$ and still preserve (1). Thus we assume that $M'$ is a finite CW-complex and set $k_{i+1} = k_i$.

Let $Y = \lim M_i$. Clearly (1) implies that $\dim Y > 2$.

Assume that $Y$ is constructed as described in 5.3. The objects that we pick out for each $M_i$ are all the (non-isomorphic) circle bundles over $M_i$ and countably many maps from subcomplexes of $M_i$ to $K(\mathbb{Z}[1/p], 1)$ and $K(\mathbb{Z}_{p^n}, 1)$ representing all possible maps up to homotopy. The procedures that we perform are Procedures I–IV.
Assuming that on each \( M_i \) we fix a sufficiently finite triangulation, we find that:

- Procedure III implies \( \text{ext-dim } Y \leq K(\mathbb{Z}[1/p], 1) \), hence \( \dim_{\mathbb{Z}[1/p]} Y \leq 1 \);
- Procedure IV implies \( \text{ext-dim } Y \leq K(\mathbb{Z}_p^{\infty}, 1) \), hence \( \dim_{\mathbb{Z}_p^{\infty}} Y \leq 1 \).

Then \( \dim Y > 2 \) and the Bockstein inequalities imply that \( \dim_{\mathbb{Z}} Y = 2 \) and hence \( Y \) is infinite-dimensional.

Let us show that \( H^2(Y; \mathbb{Z}) \) does not contain a subgroup isomorphic to \( \mathbb{Z}_p^{\infty} \). Since \( \dim_{\mathbb{Z}[1/p]} Y = 1 \), we have \( H^2(Y; \mathbb{Z}) \otimes \mathbb{Z}[1/p] = 0 \) and hence \( H^2(Y; \mathbb{Z}) \) is a \( p \)-torsion group. We will show that \( H^2(Y; \mathbb{Z}) \) does not contain a non-trivial element \( \alpha \) that is divisible by \( p^k \) for every \( k \). Aiming at a contradiction assume that such an \( \alpha \) does exist. Consider the circle bundle \( g_Y : Z \to Y \) corresponding to \( \alpha \). Then Construction 5.3 guarantees that at a certain step \( i \) of the construction we will consider a circle bundle \( g : L \to M_i \) such that \( g_Y \) is the pull-back of \( g \) under the projection \( \mu^i : Y \to M_i \).

If \( g \) is flexible then we apply Procedure II and, by Construction 5.2.1, the pull-back bundle \( g_{i+1} : L_{i+1} \to M_{i+1} \) of \( g \) under the map \( \mu_{i+1} \) admits a section and hence both \( g_{i+1} \) and \( g_Y \) are trivial. Thus we arrive at a contradiction with \( \alpha \neq 0 \).

If \( g : L \to M_i \) is non-flexible then we apply Procedure I. Denote by \( g_j : L_j \to M_j \), \( j \geq i \), the pull-back bundle of \( g \) to \( M_j \) under \( \mu^j \) and \( g_i = g \) and \( L_i = L \). Denote by \( G^i_j : L_j \to L_i \) the induced map. Recall that \( \alpha \) is divisible by \( p^k \) for every \( k \). Then for a sufficiently large \( j \) there is a circle bundle \( g^*_j : L^*_j \to M_j \) such that \( g_j : L_j \to M_j \) is induced by \( g^*_j \) and \( \mathbb{Z}_p^{k+1} \) (see Lemma 3.1). Let \( G_j^* : L^*_j \to L_j \) be a bundle map induced by the homomorphism \( S^1 \to S^1/\mathbb{Z}_p^{k+1} \). Take a section \( s^*_j : M^{(1)}_j \to L^*_j \).

Let \( \Omega \) be a 2-dimensional subcomplex of \( M_j \). Denote by \( \mu^{-\#}_{j+1}(\Omega) \) the subcomplex of \( \mu^{j+1}_j(\Omega) \) which is the closure of \( \mu^{j+1}_j(\Omega) \setminus \{ \text{the fibers of } \mu_{j+1} \text{ over the barycenters of the 2-simplexes of } \Omega \} \).

Consider a 2-simplex \( \Omega_i \) of \( M_i \) and let \( g_i^{-1}(\Omega_i) = \Omega_i \times S^1 \) be a trivialization of \( L_i \) over \( \Omega_i \). For every \( j > i \) define \( \Omega_j \) by \( \Omega_{i+1} = \mu^{-\#}_i(\Omega_i) \), \( \Omega_{i+2} = \mu^{-\#}_{i+2}(\Omega_{i+1}) \), \ldots, \( \Omega_j = \mu^{-\#}_{j}(\Omega_{j-1}) \).

Analyzing Procedures I–IV we deduce that \( \dim \Omega_j = 2 \), for Procedure I we have \( \Omega_{j+1} = \Omega_j \), and for Procedures II–IV we find that \( \Omega_{j+1} \) is obtained from \( \Omega_j \) by taking a triangulation of \( \Omega_j \) and replacing each 2-simplex \( \Delta \) of \( \Omega_j \) by the mapping cylinder of a map \( \partial \Delta \to S^1 \) of degree \( k_t \) attached to \( \partial \Delta \). Note that \( k_j \geq k_{i+1} \) for \( j > i \). Also note that \( \mu^i_j \) is 1-to-1 over the 1-skeleton \( M_{i}^{(1)} \) of \( M_i \), and hence \( M_{i}^{(1)} \) can be considered as a subset...
$M_i^{(1)} \subset M_j^{(1)}$ of the 1-skeleton of $M_j$, and then $\partial \Omega_i$ considered as a subset of $\Omega_j$ is homologous to 0 in $H_1(\Omega_j; \mathbb{Z}/p^{k_{i+1}}), j > i$.

Consider the trivialization $g_j^{-1}(\Omega_j) = \Omega_j \times S^1$ of $L_j$ over $\Omega_j$ induced by the trivialization of $g_i^{-1}(\Omega_i) = \Omega_i \times S^1$ and consider the section $s_j : M_j^{(1)} \to L_j$ which is $s_j^*$ followed by $G_j^*$. Note that for every 2-simplex $\Delta$ of $\Omega_j$, the map $s_j$ restricted to $\partial \Delta$ and followed by the projection of $g_j^{-1}(\Omega_j) = \Omega_j \times S^1$ to $S^1$ is of degree $p^{k_{i+1}}$. Then, since $\partial \Omega_i$ is homologous to 0 in $H_1(\Omega_j; \mathbb{Z}/p^{k_{i+1}})$, we see that $s_j$ restricted to $\partial \Omega_i$ and followed by the projection of $g_j^{-1}(\Omega_j) = \Omega_j \times S^1$ to $S^1$ is of degree $p^{k_{i+1}}$ as well. Thus the section $s_i = \mu_j^i \circ s_j | M_i^{(1)} : M_i^{(1)} \to L_i$ is of degree $p^{k_{i+1}}$ and this violates our choice of $k_{i+1}$ in Procedure I. ■

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