RECOVERY BASED FINITE ELEMENT METHOD FOR BIHARMONIC EQUATION IN TWO DIMENSIONAL

YUNQING HUANG†, HUAYI WEI†, WEI YANG†, AND NIANYU YI†,*

ABSTRACT. We design and numerically validate a recovery based linear finite element method for solving the biharmonic equation. The main idea is to replace the gradient operator $\nabla$ on linear finite element space by $G(\nabla)$ in the weak formulation of the biharmonic equation, where $G$ is the recovery operator which recovers the piecewise constant function into the linear finite element space. By operator $G$, Laplace operator $\Delta$ is replaced by $\nabla \cdot G(\nabla)$. Furthermore the boundary condition on normal derivative $\nabla u \cdot n$ is treated by the boundary penalty method. The explicit matrix expression of the proposed method is also introduced. Numerical examples on uniform and adaptive meshes are presented to illustrate the correctness and effectiveness of the proposed method.

1. Introduction

The biharmonic equation is a fourth order equation which arises in areas of continuum mechanics, including linear elasticity theory and the solution of Stokes flow. In this work, we consider a $C^0$ linear finite element method for the biharmonic equation in two-dimensional space.

\begin{equation}
\Delta^2 u(x, y) = f(x, y), \quad \forall (x, y) \in \Omega,
\end{equation}

with boundary conditions

\begin{align*}
(1.2) \quad & u(x, y) = g_1(x, y), \quad (x, y) \in \partial \Omega, \\
(1.3) \quad & u_n(x, y) = g_2(x, y), \quad (x, y) \in \partial \Omega.
\end{align*}

Here $\Omega$ is a bounded domain in the two-dimensional space $\mathbb{R}^2$ with a Lipschitz boundary $\partial \Omega$, $u_n = \nabla u \cdot n$ is the normal derivative of $u$ on $\partial \Omega$, and $n$ is the unit normal vector pointing outward. The biharmonic operator $\Delta^2$ is defined through

$$
\Delta^2 = \nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.
$$

The basic idea of our method is applying the gradient recovery technique as pre-processing tool to solve the high-order partial differential equations.

2010 Mathematics Subject Classification. 65N30.

Key words and phrases. Biharmonic equation, finite element method, recovery, adaptive.

* Corresponding author.
The mixed form is rewrite the biharmonic equation (1.1)-(1.3) into a coupled system of Poisson equations as

\[
\begin{cases}
\Delta v(x, y) = f(x, y), & (x, y) \in \Omega, \\
\Delta u(x, y) = v(x, y), & (x, y) \in \Omega, \\
u(x, y) = g_1(x, y), & (x, y) \in \partial\Omega, \\
u_n(x, y) = g_2(x, y), & (x, y) \in \partial\Omega.
\end{cases}
\]

(1.4)

One can easily see that under this formulation, there are two boundary conditions for the solutions \(u\) but no boundary condition for the new variable \(v\). Thus, it is much more difficult to solve the biharmonic equation with the boundary conditions (1.2) and (1.3). These computations are dependent on accurate evaluation of the missing boundary values for \(v\), and the computational procedures are often unsatisfactory. The treatment of the boundary condition for the splitting method is a challenging problem since poor boundary approximations may reduce the accuracy of the numerical solution. An alternative technique is the so-called coupled equation approach,

\[
\begin{cases}
\Delta v(x, y) = f(x, y), & (x, y) \in \Omega, \\
v(x, y) = \Delta u(x, y) - c(u_n - g_2(x, y)), & (x, y) \in \partial\Omega, \\
\Delta u(x, y) = v(x, y), & (x, y) \in \Omega, \\
u(x, y) = g_1(x, y), & (x, y) \in \partial\Omega,
\end{cases}
\]

where \(c\) is a constant, see [12, 22]. For a given initial guess \(v_0(x, y)\), an iteration solution \((u_k(x, y), v_k(x, y))\) can be computed until its convergence.

There are various finite element methods to discretize the biharmonic equation in the literature. As the most classical approach, the \(C^1\) conforming finite element methods require the basis functions and their derivatives are continuous on \(\bar{\Omega}\), which are rarely used in practice for their too many degrees of freedom and implementation complexity. For example, the Argyris finite element method [10] has 21 degrees of freedom for triangles. The nonconforming finite element methods such as the Adini element or Morley element [3, 10, 25] are popular methods for numerical solution of the high-order partial differential equations. The key idea in nonconforming methods is to use the penalty term to ensure the convergence into the natural energy space of the variational problem. Mixed finite element method is another choice which is based on the equivalent form (1.4) and only require the Lagrangian finite element spaces, which are widely used in practice, but they require very careful treatment on the essential and natural boundary conditions. The literature on the mixed finite element methods is vast, and we refer to [11, 17, 18, 24] and the references therein for the detail of these methods. The discontinuous Galerkin method is also a choice which is based on standard continuous Lagrangian finite element spaces [6, 13] or completely discontinuous finite element spaces [13, 23]. Other methods which have been developed for fourth order problems include finite difference methods [2, 9, 16], and finite volume method [14].

An alternative to aforementioned methods is the recovery based finite element method developed in recent years [8, 19, 21]. It is a nonconforming finite element method based on the discretization of the Laplace operator defined by applying the gradient recovery operator on the gradient of the \(C^0\) linear element. The variational formulation of (1.1)
RFEM FOR BIHARMONIC EQUATION

involves the term \((\Delta u, \Delta v)\). The idea in this paper is to redefine the discrete gradient operator and furthermore the Laplace operator involved in the weak formulation, by embedding a gradient recovery operator in pre-processing, such that the linear finite element can be used for solving the biharmonic equation. The resulting finite element scheme is given as follows:

\[
(1.5) \quad \int_{\Omega} \nabla \cdot (G(\nabla u_h) \nabla (\nabla v_h)) dx + \sigma \frac{h^2}{2} \int_{\partial \Omega} G(\nabla u_h) \cdot \mathbf{n} G(\nabla v_h) \cdot \mathbf{n} ds = \int_{\Omega} f v dx + \sigma \frac{h^2}{2} \int_{\partial \Omega} g_2 G(\nabla v_h) \cdot \mathbf{n} ds,
\]

where gradient recovery operator \(G\) is embedded in a priori way such that the \(\nabla \cdot G(\nabla v_h)\) is well-defined for any function \(v_h \in V_h\). The boundary condition (1.3) is incorporating in the finite element scheme by a penalty method. Notice that the difference between our scheme (1.5) and the existing recovery based finite element scheme for biharmonic equation \([8, 19, 21]\) is the treatment of boundary condition (1.3), especially for the non-homogeneous boundary data. In \([8, 19, 21]\), the boundary condition (1.3) is treated as an essential boundary condition, that is enforcing the numerical solution satisfies the boundary condition (1.3). While in our scheme (1.5), we impose the boundary condition (1.3) using the boundary penalty method. In this paper, we develop and numerically investigate the recovery based finite element method (1.5) for biharmonic equation.

The remaining parts of this paper are organized as follows. In Section 2 we introduce the gradient recovery operator and then present a recovery based linear finite element method for the biharmonic equation. In Section 3 we discuss the implementation issue. And in the following Section 4 we present some numerical experiments to show the correctness and effectiveness of our method. Finally, we make some concluding remarks in Section 5.

2. RECOVERY BASED FINITE ELEMENT METHOD

Consider the biharmonic equation

\[\Delta^2 u(x, y) = f(x, y), \quad \forall (x, y) \in \Omega = (0, 1)^2,\]

with boundary conditions

\[u(x, y) = 0, \quad \nabla u(x, y) \cdot \mathbf{n} = g(x, y), \quad \text{on } \partial \Omega\]

In weak form, this problem reads: Find \(u \in V^g\) such that

\[a(u, v) = L(v) \quad \forall v \in V^0,\]

where

\[V^g = \{ v \in H^1(\Omega) : \nabla v \in H(div) \}, \quad V^0 = \{ v \in H^1(\Omega) : \nabla v \in H(div) \},\]

\[a(u, v) = \int_{\Omega} \nabla \cdot (\nabla u) \nabla \cdot (\nabla v) dx,\]

and

\[L(v) = \int_{\Omega} f v dx.\]
2.1. **Discrete spaces.** Let $\mathcal{T}_h$ be a triangular partition of $\Omega \in \mathbb{R}^2$ with mesh size $h$, and $h_\tau := \text{diam}(\tau)$ for each element $\tau \in \mathcal{T}_h$. We denote the set of vertices and edges of $\mathcal{T}_h$ by $\mathcal{N}_h$ and $\mathcal{E}_h$, respectively. The length of $E \in \mathcal{E}_h$ is denoted by $h_E = \text{diam}(e)$. For each $E \in \mathcal{E}_h$, denote a unit vector normal to $E$ by $n_E$, and $\omega_E$ denotes the union of all elements that share $E$. On each element $\tau \in \mathcal{T}_h$, $P_k(\tau)$ denotes the polynomials on $\tau$ of degree $\leq k$.

Consider the $C^0$ linear finite element space $S_h$ associated with $\mathcal{T}_h$ and defined by

$$S_h = \{v \in H^1(\Omega) : v \in P_1(\tau), \forall \tau \in \mathcal{T}_h\} = \text{span}\{\phi_z : z \in \mathcal{N}_h\}.$$  

The node basis functions of $S_h$ are the standard Lagrangian basis functions. The element patch is defined by $\omega_z = \text{supp}\phi_z$. Furthermore, the piecewise constant function space is denoted as

$$W_h := \{w_h \in (L^\infty(\Omega))^2 : w_h|_\tau \in (P_0(\tau))^2, \forall \tau \in \mathcal{T}_h\}.$$  

2.2. **Recovery operator.** In this subsection, we introduce the recovery operator which can recover a piecewise constant function into the continuous piecewise linear finite element space. For simplicity, we take the weighted averaging recovery operator $G : W_h \rightarrow S_h \times S^h$, which is defined as follows: for $w_h \in W_h$,

\begin{equation}
G(w_h) := \sum_{z \in \mathcal{N}_h} G(w_h)(z) \phi_z, \quad G(w_h)(z) := \sum_{\tau \in \omega_z} w_\tau w_h|_\tau,
\end{equation}

where the weights can be chosen as following \[20\]

\begin{equation}
\text{Simple averaging :} \quad w_\tau = \frac{1}{|\omega_z|},
\end{equation}

\begin{equation}
\text{Harmonic averaging :} \quad w_\tau = \frac{1/|\tau|}{\sum_{\tau \in \omega_z} 1/|\tau|}.
\end{equation}

Given $u_h \in S_h$, its gradient $\nabla u_h$ is piecewise constant and may discontinuous across each element, thus $\Delta u_h$ is not well-defined. To fix this problem, we use the recovery operator $G$ to 'lift' the gradient $\nabla u_h$ into a vector finite element space in which $\nabla \cdot G(\nabla u_h)$ is well-defined. In other words, we define the discrete Laplace operator by $\Delta u_h := \nabla \cdot G(\nabla u_h)$ for piecewise linear function $u_h \in S_h$, where

$$G(\nabla u_h) = (G(\partial_x u_h), G(\partial_y u_h))^T.$$  

2.3. **Recovery based linear finite element scheme.** After defining the finite element spaces and the gradient recovery operators, we now introduce the recovery based finite element method with a penalty for the biharmonic equation. Let

$$S^0_h = S_h \cap H^1_0(\Omega) = \{v_h \in S_h : \forall v_h|_{\partial \Omega} = 0\}.$$  

The recovery based finite element scheme is to find $u_h \in S^0_h$ such that

\begin{equation}
a_h(u_h, v_h) = \int_{\Omega} f v_h dx + \frac{\sigma}{h^2} \int_{\Gamma} g_2 G(\nabla v_h) \cdot \mathbf{n} ds, \quad \forall v_h \in S^0_h,
\end{equation}

where

$$a_h(u_h, v_h) := \int_{\Omega} \nabla \cdot G(\nabla u_h) \nabla \cdot G(\nabla v_h) dx + \frac{\sigma}{h^2} \int_{\Gamma} G(\nabla u_h) \cdot \mathbf{n} G(\nabla v_h) \cdot \mathbf{n} ds,$$  

with $G(\nabla u_h) = (G(\partial_x u_h), G(\partial_y u_h))^T.$
and the recovery operator $G$ is defined in (2.1). Notice that the boundary conditions (1.2) and (1.3) are treated in different ways. The boundary condition (1.2) is treated as an essential boundary condition, while the boundary condition (1.3) is imposed weakly with a boundary penalty term in the discrete scheme.

**Theorem 2.1.** For the recovery based linear finite element scheme (2.4), there exists a unique solution $u_h \in S_0^h$.

**Proof.** Based on the scheme (2.4), we define the following functional:

$$J(u_h) := \frac{1}{2} \int_\Omega (\nabla \cdot G(\nabla u_h))^2 \, dx + \frac{\sigma}{2h^2} \int_\Gamma (G(\nabla u_h) \cdot n)^2 \, ds$$

$$- \int_\Omega f u_h \, dx - \frac{\sigma}{h^2} \int_\Gamma g_2 G(\nabla u_h) \cdot n ds.$$

(2.5)

Notice that the first and second terms of $J(u_h)$ are convex, and the third and fourth terms of $J(u_h)$ are linear with respect to $u_h$, then the functional $J(u_h)$ is a convex functional.

Take the derivative of the functional $J(u_h)$, and for any $v_h \in S_0^h$ we have

$$\left( \frac{\delta J(u_h)}{\delta u_h}, v_h \right) = a_h(u_h, v_h) - \int_\Omega f v_h \, dx - \frac{\sigma}{h^2} \int_\Gamma g_2 G(\nabla v_h) \cdot n ds = 0.$$

Then the uniqueness of the solution of scheme (2.4) is approved. \qed

### 3. Implementation

In this section, we discuss the implementation of the term $(\nabla \cdot G(\nabla u_h), \nabla \cdot G(\nabla v_h))$ in details, the calculation of the other terms in recovery based finite element scheme (2.4) are similar.

For simplicity, we only take the simple averaging (the weights are chosen as (2.2)) for illustration. For a mesh node $z_i \in \mathcal{N}_h$, let $\phi_i$ denotes the basis function at node $z_i$, $\omega_i$ denotes the element patch of $z_i$, and $\mathcal{N}(i)$ denotes the mesh nodes in $\omega_i$. Then

$$V_h = \text{span}\{\phi_i\}_{i=1}^N, \quad N = \#\mathcal{N}_h,$$

and

$$u_h = \sum_{i=1}^N = [\phi_1, \ldots, \phi_N] U, \quad U = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}.$$

From (2.1), we have

$$G(\nabla u_h) = \begin{bmatrix} G(\partial_x u_h) \\ G(\partial_y u_h) \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^N G(\partial_x u_h)(z_j) \phi_j \\ \sum_{j=1}^N G(\partial_y u_h)(z_j) \phi_j \end{bmatrix} = \begin{bmatrix} [\phi_1, \ldots, \phi_N] AU \\ [\phi_1, \ldots, \phi_N] BU \end{bmatrix},$$

$$G(\nabla \phi_i) = \begin{bmatrix} G(\partial_x \phi_i) \\ G(\partial_y \phi_i) \end{bmatrix} = \begin{bmatrix} [C_{i,1}, \ldots, C_{i,N}] [\phi_1, \ldots, \phi_N]^T \\ [D_{i,1}, \ldots, D_{i,N}] [\phi_1, \ldots, \phi_N]^T \end{bmatrix}.$$
where

\[
A_{i,j} = \begin{cases} 
\sum_{\tau \in \omega_i} \frac{1}{\nu_i} \partial_x \phi_j|_{x}, & \text{if } j = i \in \mathcal{N}(i), \\
\sum_{\tau \in \omega_i \cap \omega_j} \frac{1}{\nu_i} \partial_x \phi_j|_{x}, & \text{if } j \neq i \in \mathcal{N}(i), \\
0, & \text{if } j \notin \mathcal{N}(i), 
\end{cases}
\]

\[
B_{i,j} = \begin{cases} 
\sum_{\tau \in \omega_i} \frac{1}{\nu_i} \partial_y \phi_j|_{y}, & \text{if } j = i \in \mathcal{N}(i), \\
\sum_{\tau \in \omega_i \cap \omega_j} \frac{1}{\nu_i} \partial_y \phi_j|_{y}, & \text{if } j \neq i \in \mathcal{N}(i), \\
0, & \text{if } j \notin \mathcal{N}(i), 
\end{cases}
\]

\[
C_{i,j} = \begin{cases} 
\sum_{\tau \in \omega_j} \frac{1}{\nu_j} \partial_x \phi_i|_{x}, & \text{if } j = i \in \mathcal{N}(i), \\
\sum_{\tau \in \omega_j \cap \omega_i} \frac{1}{\nu_j} \partial_x \phi_i|_{x}, & \text{if } j \neq i \in \mathcal{N}(i), \\
0, & \text{if } j \notin \mathcal{N}(i), 
\end{cases}
\]

\[
D_{i,j} = \begin{cases} 
\sum_{\tau \in \omega_j} \frac{1}{\nu_j} \partial_y \phi_i|_{y}, & \text{if } j = i \in \mathcal{N}(i), \\
\sum_{\tau \in \omega_j \cap \omega_i} \frac{1}{\nu_j} \partial_y \phi_i|_{y}, & \text{if } j \neq i \in \mathcal{N}(i), \\
0, & \text{if } j \notin \mathcal{N}(i). 
\end{cases}
\]

By taking \(v_h = \phi_i, i = 1, \cdots, N\), in matrix form, we obtain

\[
(\nabla \cdot G(\nabla u_h), \nabla \cdot G(\nabla v_h)) = (\partial_x G(\partial_x u_h) + \partial_y G(\partial_y u_h), \partial_x G(\partial_x v_h) + \partial_y G(\partial_y v_h))
\]

\[
= ([\partial_x \phi_1, \cdots, \partial_x \phi_N]AU + [\partial_y \phi_1, \cdots, \partial_y \phi_N]BU,
C[\partial_x \phi_1, \cdots, \partial_x \phi_N]^T + D[\partial_y \phi_1, \cdots, \partial_y \phi_N]^T)
\]

\[
= (CPA + CQB + DSA + DTB)U.
\]

where the matrices are calculated as following

\[
P = \int_{\Omega} \begin{bmatrix} 
\partial_x \phi_1 \\
\partial_x \phi_2 \\
\vdots \\
\partial_x \phi_N 
\end{bmatrix} \begin{bmatrix} 
[\partial_x \phi_1, \partial_x \phi_2, \cdots, \partial_x \phi_N] 
\end{bmatrix} dxdy,
\]

\[
Q = \int_{\Omega} \begin{bmatrix} 
\partial_y \phi_1 \\
\partial_y \phi_2 \\
\vdots \\
\partial_y \phi_N 
\end{bmatrix} \begin{bmatrix} 
[\partial_y \phi_1, \partial_y \phi_2, \cdots, \partial_y \phi_N] 
\end{bmatrix} dxdy,
\]
\[ S = \int_{\Omega} \left[ \begin{array}{c} \partial_y \phi_1 \\ \partial_y \phi_2 \\ \vdots \\ \partial_y \phi_N \end{array} \right] \left[ \begin{array}{c} \partial_x \phi_1 \\ \partial_x \phi_2 \\ \vdots \\ \partial_x \phi_N \end{array} \right] dxdy, \]

\[ T = \int_{\Omega} \left[ \begin{array}{c} \partial_y \phi_1 \\ \partial_y \phi_2 \\ \vdots \\ \partial_y \phi_N \end{array} \right] \left[ \begin{array}{c} \partial_y \phi_1 \\ \partial_y \phi_2 \\ \vdots \\ \partial_y \phi_N \end{array} \right] dxdy. \]

4. Numerical Examples

In this section, we present some numerical examples to demonstrate the performance of the recovery based linear finite element for the biharmonic equation presented in (2.4). We investigate the proposed recovery based finite element method on the uniform regular mesh and the Centroidal Voronoi-Delaunay Triangulation (CVDT) mesh. Also, we are interesting the performance of the recovery based finite element method on adaptive meshes when the solution of biharmonic equation appears singularity.

Example 4.1. We first consider the biharmonic equation with homogeneous boundary conditions

\[ \begin{align*}
\Delta^2 u(x, y) &= f(x, y), \quad \forall (x, y) \in \Omega = (0, 1)^2, \\
u &= 0, \quad \nabla u \cdot \mathbf{n} = 0, \quad \text{on } \partial \Omega.
\end{align*} \]

The exact solution is chosen the following function:

\[ u = \sin^2(\pi x) \sin^2(\pi y). \]

Hence we choose \( f = \Delta^2 u \) as the function defined by

\[ f(x, y) = 8\pi^4(\sin^2(\pi x) - \cos^2(\pi x))\sin^2(\pi y) + 8\pi^4\sin^2(\pi x)(\sin^2(\pi y) - \cos^2(\pi y)) + 8\pi^4(\sin^2(\pi x) - \cos^2(\pi x))(\sin^2(\pi y) - \cos^2(\pi y)). \]

The errors \( \|u - u_h\|, \|\nabla u - \nabla u_h\|, \|\nabla u - \nabla G(\nabla u_h)\|, \|\Delta u - \nabla \cdot G(\nabla u_h)\| \) and corresponding rates of convergence are reported in Table 1 and Table 2. Table 1 shows the numerical results on the uniform mesh in regular pattern and Table 2 shows the numerical results on the CVDT mesh. We see clearly that: i) The \( L^2 \) errors \( \|u - u_h\| \) and the gradient errors \( \|\nabla u - \nabla u_h\| \) converge at the rate of second order and first order, respectively, which are optimal for the linear approximation; ii) The recovered gradient \( G(\nabla u_h) \) converges to the exact gradient \( \nabla u \) under the second order rate, and one order higher than the gradient of the finite element approximation. This shows that the recovered gradient is superclose to the exact one; iii) The convergence rate of the error \( \|\Delta u - \nabla \cdot G(\nabla u_h)\| \) is first order.

Example 4.2. For the second example, we consider the biharmonic equation with non-homogeneous boundary condition

\[ \begin{align*}
\Delta^2 u(x, y) &= f(x, y), \quad \forall (x, y) \in \Omega = (0, 1)^2, \\
u &= g, \quad \nabla u \cdot \mathbf{n} = h, \quad \text{on } \partial \Omega.
\end{align*} \]
Table 1. Example 4.1, regular mesh, errors and convergence rates

| Dof   | 441   | 1681  | 6561  | 25921 |
|-------|-------|-------|-------|-------|
| $\|u - u_h\|$ | 0.01394 | 0.00344 | 0.00086 | 0.00021 |
| Order | –     | 2.02  | 2.00  | 2.00  |
| $\|\nabla u - \nabla u_h\|$ | 0.39714 | 0.19032 | 0.09431 | 0.0471 |
| Order | –     | 1.06  | 1.01  | 1.00  |
| $\|\nabla u - G(\nabla u_h)\|$ | 0.0418 | 0.01051 | 0.00264 | 0.00066 |
| Order | –     | 1.99  | 1.99  | 2.00  |
| $\|\Delta u - \nabla \cdot G(\nabla u_h)\|$ | 1.64494 | 0.80249 | 0.3989 | 0.19915 |
| Order | –     | 1.04  | 1.01  | 1.00  |

Table 2. Example 4.1, CVDT mesh, errors and convergence rates

| Dof   | 499   | 1920  | 7566  | 29952 |
|-------|-------|-------|-------|-------|
| $\|u - u_h\|$ | 0.01205 | 0.00296 | 0.00069 | 0.00017 |
| Order | –     | 2.03  | 2.10  | 2.03  |
| $\|\nabla u - \nabla u_h\|$ | 0.37264 | 0.17001 | 0.07189 | 0.03457 |
| Order | –     | 1.13  | 1.24  | 1.06  |
| $\|\nabla u - G(\nabla u_h)\|$ | 0.03859 | 0.00981 | 0.00234 | 0.00059 |
| Order | –     | 1.98  | 2.07  | 1.99  |
| $\|\Delta u - \nabla \cdot G(\nabla u_h)\|$ | 1.34359 | 0.65669 | 0.34152 | 0.17329 |
| Order | –     | 1.03  | 0.94  | 0.98  |

Table 3. Example 4.2, regular mesh, errors and convergence rates

| Dof   | 441   | 1681  | 6561  | 25921 |
|-------|-------|-------|-------|-------|
| $\|u - u_h\|$ | 0.04686 | 0.01166 | 0.00292 | 0.00073 |
| Order | –     | 2.01  | 2.00  | 2.00  |
| $\|\nabla u - \nabla u_h\|$ | 1.02312 | 0.48499 | 0.23852 | 0.11861 |
| Order | –     | 1.08  | 1.02  | 1.01  |
| $\|\nabla u - G(\nabla u_h)\|$ | 0.18431 | 0.04731 | 0.01207 | 0.00307 |
| Order | –     | 1.96  | 1.97  | 1.98  |
| $\|\Delta u - \nabla \cdot G(\nabla u_h)\|$ | 4.90213 | 2.47485 | 1.27612 | 0.66004 |
| Order | –     | 0.99  | 0.96  | 0.95  |

We take

$$u = \sin(2\pi x) \sin(2\pi y)$$

and then the corresponding problem have following type of boundary conditions

$$u|_{\partial \Omega} = 0, \quad \nabla u \cdot n|_{\partial \Omega} \neq 0.$$ 

The corresponding right hand side function $f$ is then take

$$f = 64\pi^4 \sin(2\pi x) \sin(2\pi y).$$

The numerical results are reported in Table 3 and Table 4. The results indicate that both $u$ and $\nabla u$ achieve optimal convergence order, and the recovered gradient $G(\nabla u_h)$
Table 4. Example 4.2, CVDT mesh, errors and convergence rates

| Dof          | 499  | 1920 | 7566 | 29952 |
|--------------|------|------|------|-------|
| $\|u - u_h\|$| 0.03992 | 0.00983 | 0.00238 | 0.00059 |
| Order        | –    | 2.02 | 2.05 | 2.01 |
| $\|\nabla u - \nabla u_h\|$| 0.89327 | 0.3836 | 0.16833 | 0.07981 |
| Order        | –    | 1.22 | 1.19 | 1.08 |
| $\|\nabla u - G(\nabla u_h)\|$| 0.16318 | 0.04231 | 0.01067 | 0.00275 |
| Order        | –    | 1.95 | 1.99 | 1.96 |
| $\|\Delta u - \nabla \cdot G(\nabla u_h)\|$| 4.59341 | 2.38537 | 1.29143 | 0.67979 |
| Order        | –    | 0.95 | 0.89 | 0.93 |

is superclose to $\nabla u$. These numerical results show that the recovery based finite element method also converges with optimal rates for the biharmonic equation with non-homogeneous boundary conditions.

In the following, we apply the recovery based linear finite element method for the biharmonic equation with a singular solution. An adaptive algorithm is used to resolve the singularity. Note that $\nabla \cdot G(\nabla u_h)$ is a piecewise constant function, and it can be restored to the continuous piecewise linear space by recovery operator $G$. Since $G(\nabla \cdot G(\nabla u_h))$ is better approximation of $\Delta u$ than $\nabla \cdot G(\nabla u_h)$, we can use

$$
\|G(\nabla \cdot G(\nabla u_h)) - \nabla \cdot G(\nabla u_h)\|
$$

as an recovery type a posteriori error estimator to guide the mesh refinement. In the adaptive procedure, the Döfler marking strategy with bulk parameter $\theta = 0.2$ is used for marking the elements to be refined. We present three numerical examples to investigate the performance of recovery based linear finite element method on the adaptive meshes.

**Example 4.3.** We consider the model problem on a L-shaped domain $\Omega = (-1, 1)^2 \setminus ([0, 1] \times (-1, 0))$ with the following exact singular solution:

\begin{equation}
(4.3)
\quad u(r, \theta) = (r^2 \cos^2 \theta - 1)^2(r^2 \sin^2 \theta - 1)^2 r^{(1+\alpha)} g_{\alpha,\omega}(\theta)
\end{equation}

where $\alpha = 0.54483736782464$ is a noncharacteristic root of $\sin^2(\alpha \omega) = \alpha^2 \sin^2 \omega$, $\omega = \frac{3\pi}{2}$ and

\begin{equation}
(4.4)
\quad g_{\alpha,\omega}(\theta) = \left( \frac{1}{\alpha - 1} \sin((\alpha - 1)\omega) - \frac{1}{\alpha + 1} \sin((\alpha + 1)\omega) \right) \times (\cos((\alpha - 1)\theta) - \cos((\alpha + 1)\theta))
\end{equation}

\begin{equation}
- \left( \frac{1}{\alpha - 1} \sin((\alpha - 1)\theta) - \frac{1}{\alpha + 1} \sin((\alpha + 1)\theta) \right) \times (\cos((\alpha - 1)\omega) - \cos((\alpha + 1)\omega)).
\end{equation}

Figure 1 shows the adaptive meshes at refinement level 60 and level 70. The error estimator captures the singularities of the solution throughout the mesh refinement process. The numerical solution, and corresponding exact errors and error estimators are presented in Figure 2. We have observed that $\|G(\nabla \cdot G(\nabla u_h)) - \nabla \cdot G(\nabla u_h)\| \approx 1.1 \times \|\Delta u - \nabla \cdot G(\nabla u_h)\|$, which means the error estimator is reliable and efficiency.
Example 4.4. In this example, we take \( \Omega := (-1, 1)^2 \setminus \text{conv}\{(0,0), (1,-1), (1,0)\} \). We consider the model problem (1.1) on \( \Omega \) with the exact solution given by (4.3) with \( \alpha = 0.505009698896589 \), \( \omega = \frac{7\pi}{4} \) and \( g_{\alpha,\omega}(\theta) \) is of the form (4.4).

Figure 3 and Figure 4 display the adaptive meshes, numerical solution and the convergence history of the error estimators and the exact errors. As in the previous example, the error estimator yields a good approximation of the true Laplace error, and the singularities of the solution are well predicted by the error estimator throughout the mesh refinement process. On the adaptive meshes, we see clearly that the adaptive mesh-refinement mainly concentrates on the V-corner. We also observe some additional refinement near the boundary where the gradient is relatively large.

Example 4.5. In this example, we consider the problem (4.1) with \( f = 1 \) on the non-convex domain \( \Omega \) with the corners \((0,0), (1,0), (1,1), (2,0), (3,0), (1,2), (3,4), (2,4), (1,3), (1,4) \) and \((0,4)\).
Figure 3. Adaptive meshes for Example 4.4. Left: level 60; Right: level 70.

Figure 4. Numerical solution and errors history of Example 4.4.

Figure 5. Adaptive meshes for Example 4.5. Left: level 60; Right: level 70.
In this case, there appear corner singularities at the L-corner and the two V-corners. Figure 5 and Figure 6 plot the adaptive meshes, numerical solution and the convergence history of the error estimators. We see clearly that the method finds and clearly distinguishes all the corner singularities and refines locally near the L-corner and the two V-corners.

5. Concluding remarks

In this paper, we have developed a recovery based linear finite element method for solving the biharmonic equation. In the discrete weak formulation, the gradient operator $\nabla$ on the linear finite element space is replaced by $G(\nabla)$ with $G$ denotes a suitable gradient recovery operator. Thus, the Laplace operator $\Delta$ is replaced by $\nabla \cdot G(\nabla u)$. Furthermore, we impose the boundary condition $\nabla u \cdot n|_{\partial \Omega} = g_2$ by the boundary penalty method. Numerical examples for the biharmonic equation with the homogeneous or non-homogeneous boundary conditions are presented for illustrating the correctness and effectiveness of our method. They show that the recovery based linear finite element method converges with optimal rates, and the recovered gradient is superclose to the exact one. We also numerical investigate the effectiveness of the recovery based finite element method on adaptive meshes. The results show that the error estimator captures the singularities of the solution throughout the mesh refinement process.

For the recovery based finite element method for high order partial differential equations, we will continuou our works in the following issues: i) design efficient implementation of the recovery based finite element method, which incorporate with other gradient recovery operators besides the weighted averaging method; ii) derive the error estimation for recovery based finite element method; iii) design the preconditioner for the linear algebra system which is resulting from the recovery based finite element method; iv) extend the recovery based finite element method for other high order partial differential equation, such as the fourth order parabolic equation and the Cahn-Hilliard type equation arising from the phase field models. We will report these results and applications in our future works.
Acknowledgments

Huang’s research was partially supported by NSFC Project (91430213). Wei’s research was partially supported by Hunan Provincial Civil-Military Integration Industrial Development Project. Yang’s research was partially supported by NSFC Project (11771371) and Hunan Education Department Project (15B236). Yi’s research was partially supported by NSFC Project (11671341), Hunan Provincial NSF Project (2015JJ2145) and Hunan Education Department Project (16A206).

References

[1] E.M. Behrens and J. Guzman, A mixed method for the biharmonic problem based on a system of first-order equations. *SIAM J. Numer. Anal.*, 49:789-817, 2011.
[2] M. Ben-Artzi, I. Chorev, J.-P. Croisille and D. Fishelov, A compact difference scheme for the biharmonic equation in planar irregular domains. *SIAM J. Numer. Anal.*, 47:3087-3108, 2009.
[3] C. Bi and L. Li, Mortar finite volume method with Adini element for biharmonic problem. *J. Comput. Math.*, 22:475-488, 2004.
[4] H. Blum and R. Rannacher, On the boundary value problem of the biharmonic operator on domains with angular corners. *Math. Mech. Appl. Sci.*, 2:556-581, 1980.
[5] J. H. Bramble, J. E. Pasciak and C. Bacuta, Shift theorems for the biharmonic Dirichlet problem. *In Recent Progress in Computational and Applied PDEs*, 1-26, New York, Kluwer Academic/Plenum Publishers, 2002.
[6] S. Brenner and L. Sung, $C^0$ interior penalty methods for fourth order elliptic boundary value problems on polygonal domains. *J. Sci. Comput.*, 22:83-118, 2005.
[7] F. Brezzi and M. Fortin, Mixed and hybrid finite element methods. *Spring-Verlag, New York*, 1991.
[8] H. Chen, H. Guo, Z. Zhang and Q. Zou, A $C^0$ finite element method for two fourth-order eigenvalue problems. *IMA J. Numer. Anal.*, DOI: https://doi.org/10.1093/imanum/drw051.
[9] G. Chen, Z. Li and P. Lin, A fast finite difference method for biharmonic equations on irregular domains and its application to an incompressible Stokes flow. *Adv. Comput. Math.*, 29:113-133, 2008.
[10] P. Ciarlet, The finite element method. In P.G. Ciarlet and J.-L. Lions, editors, Part I, Handbook of Numerical Analysis, III. *North-Holland, Amsterdam*, 1991.
[11] W. Dörfler, A convergent adaptive algorithm for Poisson’s equation. *SIAM J. Numer. Anal.*, 33:1106-1124, 1996.
[12] L.W. Ehrlich, Solving the biharmonic equation as coupled finite difference equations. *SIAM J. Numer. Anal.*, 8:278-287, 1971.
[13] G. Engel, K. Gariikipati, T. J. R. Hughes, M. G. Larson, L. Mazzei and R. L. Taylor, Continuous/discontinuous finite element approximations of fourth-order elliptic problems in structural and continuum mechanics with applications to thin beams and plates, and strain gradient elasticity. *Comput. Methods Appl. Mech. Engrg.*, 191:3669-3750, 2002.
[14] R. Eymard, T. Gallouet, R. Herbin and A. Linke, Finite volume schemes for the biharmonic problem on general meshes. *Math. Comput.*, 80:2019-2048, 2012.
[15] E.H. Georgoulis and P. Houston, Discontinuous Galerkin methods for the biharmonic problem. *IMA J. Numer. Anal.*, 29:573-594, 2009.
[16] R. Glowinski and O. Pironneau, Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem. *SIAM Rev.*, 21:167-212, 1979.
[17] P. Grisvard, Singularities in Boundary Value Problems, in: Recherches en Mathematiques Appliquees (Research in Applied Mathematics). Vol 22, Masson, Paris, 1992.
[18] T. Gudi, N. Nataraj and A.K. Pani, Mixed discontinuous Galerkin finite element method for the biharmonic equation. *J. Sci. Comput.*, 37:139-161, 2008.
[19] H. Guo, Z. Zhang and Q. Zou, A $C^0$ finite element method for biharmonic problems. *J. Sci. Comput.*, DOI 10.1007/s10915-017-0501-0.
[20] Y. Huang, K. Jiang and N. Yi, Some weighted averaging methods for gradient recovery. *Adv. Appl. Math. Mech.*, 4:131-155, 2012.
[21] B. Lamichhane, A finite element method for a biharmonic equation based on gradient recovery operators. *BIT Numer. Math.*, 235:5188-5197, 2015.
[22] J.W. McLaurin, A general coupled equation approach for solving the biharmonic boundary value problem. *SIAM J. Numer. Anal.*, 11:14-33, 1974.
[23] E. Suli and I. Mozolevski, hp-version interior penalty DGFEMs for the biharmonic equation. *Comput. Methods Appl. Mech. Engrg.*, 196:1851-1863, 2007.
[24] T. Wang, A mixed finite volume element method based on retangular mesh for biharmonic equations. *J. Comput. Appl. Math.*, 172:117-130, 2004.
[25] M. Wang and J. Xu, The Morley element for fourth order elliptic equations in any dimensions. *Numer. Math.*, 103:155-169, 2006.

† Hunan Key Laboratory for Computation and Simulation in Science and Engineering; School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, P.R.China

E-mail address: huangyq@xtu.edu.cn; weihuayi@xtu.edu.cn; yangwei@xtu.edu.cn; yinianyu@xtu.edu.cn