Constructions of Non-Abelian Zeta Functions for Curves

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In this paper, we initiate a geometrically oriented study of local and global non-abelian zeta functions for curves. This consists of two parts: construction and justification.

For the construction, we first use moduli spaces of semi-stable bundles to introduce a new type of zeta functions for curves defined over finite fields. Then, we prove that these new zeta functions are indeed rational and satisfy the functional equation, based on vanishing theorem, duality, Riemann-Roch theorem for cohomology of semi-stable vector bundles. With all this, we next introduce certain global non-abelian zeta functions for curves defined over number fields, via the Euler product formalism. Finally we establish the convergence of our Euler products, from the Clifford Lemma, an ugly yet explicit formula for local non-abelian zeta functions, a result of (Harder-Narasimhan) Siegel about quadratic forms, and Weil’s theorem on Riemann Hypothesis of Artin zeta functions.

As for justification, surely, we check that when only line bundles are involved, (so moduli spaces of semi-stable bundles are nothing but the standard Picard groups), our (new) zeta functions coincide with the classical Artin zeta functions for curves over finite fields and Hasse-Weil zeta functions for curves over number fields respectively. Moreover we compute the rank two zeta functions for genus two curves by studying the so-called non-abelian Brill-Noether loci and their infinitesimal structures. This is indeed a quite interesting, and in general, should be a very important aspect of the theory: We not only need to precisely describe all of the Brill-Noether loci but the so-called associated infinitesimal structures attached to all Seshadri equivalence classes, in which Weierstrass points appear naturally.

This work is motivated by our studies on new non-abelian zeta functions for number fields and Tamagawa measures associated to the Weil-Petersson metrics on moduli spaces of stable vector bundles. So related conjectures, or better, working hypothesis, are proposed. We hope that our non-abelian zeta functions, which do not really follow the present style of the theory of zeta functions, are acceptable and hence play a certain role in exploring the non-abelian aspect of arithmetic of curves.

Chapter 1. Local Non-Abelian Zeta Functions for Curves

In this chapter, we introduce our non-abelian zeta functions for curves defined over number fields. Basic properties such as meromorphic extensions, rationality and functional equations are established.

1.1. Moduli Spaces of Semi-Stable Bundles

1.1.1. Semi-Stable Bundles. Let $C$ be a regular, reduced and irreducible projective curve defined over an algebraically closed field $\bar{k}$. Then according to Mumford [M], a vector bundle $V$ on $C$ is called semi-stable (resp. stable) if for any proper subbundle $V'$ of $V$,

$$\mu(V') := \frac{d(V')}{r(V')} \leq (\text{resp.} \leq) \frac{d(V)}{r(V)} =: \mu(V).$$

Here $d$ denotes the degree and $r$ denotes the rank.

Proposition. Let $V$ be a vector bundle over $C$. Then

(a) ([HN]) there exists a unique filtration of subbundles of $V$, the Harder-Narasimhan filtration of $V$,

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{s-1} \subset V_s = V$$

such that for $1 \leq i \leq s - 1$, $V_i/V_{i-1}$ is semi-stable and $\mu(V_i/V_{i-1}) > \mu(V_{i+1}/V_i)$;
(b) (see e.g. [Se]) if moreover $V$ is semi-stable, there exists a filtration of subbundles of $V$, a Jordan-Hölder filtration of $V$,
\[ \{0\} = V^{t+1} \subset V^t \subset \ldots \subset V^1 \subset V^0 = V \]
such that for all $0 \leq i \leq t$, $V^i/V^{i+1}$ is stable and $\mu(V^i/V^{i+1}) = \mu(V)$. Moreover, the associated graded bundle $\text{Gr}(V) := \bigoplus_{i=0}^t V^i/V^{i+1}$, the (Jordan-Hölder) graded bundle of $V$, is determined uniquely by $V$.

1.1.2. Moduli Space of Stable Bundles. Following Seshadri, two semi-stable vector bundles $V$ and $W$ are called $S$-equivalent, if their associated Jordan-Hölder graded bundles are isomorphic, i.e., $\text{Gr}(V) \simeq \text{Gr}(W)$. Applying Mumford’s general result on geometric invariant theory, Seshadri proves the following

Theorem. ([Se]) Let $C$ be a regular, reduced, irreducible projective curve of genus $g \geq 2$ defined over an algebraically closed field. Then over the set $\mathcal{M}_{C,r}(d)$ (resp. $\mathcal{M}_{C,r}(L)$) of $S$-equivalence classes of rank $r$ and degree $d$ (resp. rank $r$ and determinant $L$) semi-stable vector bundles over $C$, there is a natural normal, projective $(r^2(g-1)+1)$-dimensional (resp. $(r^2-1)(g-1)$)-dimensional algebraic variety structure.

Remark. In this paper, we always assume that the genus of $g$ is at least 2. For elliptic curves, whose associated moduli spaces are very special, please see [We2].

1.1.3. Rational Points. Now assume that $C$ is defined over a finite field $k$. Naturally we may talk about $k$-rational bundles over $C$, i.e., bundles which are defined over $k$. Moreover, from geometric invariant theory, projective varieties $\mathcal{M}_{C,r}(d)$ are defined over a certain finite extension of $k$; and if $L$ itself is defined over $k$, the same holds for $\mathcal{M}_{C,r}(L)$. Thus it makes sense to talk about $k$-rational points of these moduli spaces too. The relation between these two types of rationality is given by Harder and Narasimhan based on a discussion about Brauer groups:

Proposition. ([HN]) Let $C$ be a regular, reduced, irreducible projective curve of genus $g \geq 2$ defined over a finite field $k$. Then there exists a finite field $\mathbb{F}_q$ such that for all $d$ (resp. all $k$-rational line bundles $L$), the subset of $\mathbb{F}_q$-rational points of $\mathcal{M}_{C,r}(d)$ (resp. $\mathcal{M}_{C,r}(L)$) consists exactly of all $S$-equivalence classes of $\mathbb{F}_q$-rational bundles in $\mathcal{M}_{C,r}(d)$ (resp. $\mathcal{M}_{C,r}(L)$).

From now on, without loss of generality, we always assume that finite fields $\mathbb{F}_q$ (with $q$ elements) satisfy the property in the Proposition. Also for simplicity, we write $\mathcal{M}_{C,r}(d)$ (resp. $\mathcal{M}_{C,r}(L)$) for $\mathcal{M}_{C,r}(d)(\mathbb{F}_q)$ (resp. $\mathcal{M}_{C,r}(L)(\mathbb{F}_q)$), the subset of $\mathbb{F}_q$-rational points, and call them moduli spaces by an abuse of notations.

1.2. Local Non-Abelian Zeta Functions

1.2.1. Definition. Let $C$ be a regular, reduced, irreducible projective curve of genus $g \geq 2$ defined over the finite field $\mathbb{F}_q$ with $q$ elements. Define the rank $r$ non-abelian zeta function $\zeta_{C,r,\mathbb{F}_q}(s)$ by setting
\[
\zeta_{C,r,\mathbb{F}_q}(s) := \sum_{V \in [V] \in \mathcal{M}_{C,r}(d), d \geq 0} \frac{q^{h^0(V) - 1}}{\# \text{Aut}(V)} \cdot (q^{-s})^{d(V)}, \quad \text{Re}(s) > 1.
\]

Proposition. With the same notation as above, $\zeta_{C,1,\mathbb{F}_q}(s)$ is nothing but the classical Artin zeta function for curve $C$. That is to say,
\[
\zeta_{C,1,\mathbb{F}_q}(s) = \sum_{D \geq 0} \frac{1}{N(D)^s} \quad \text{Re}(s) > 1.
\]

Here $D$ runs over all effective divisors of $C$.

Proof. By definition, the classical Artin zeta function ([A]) for $C$ is given by
\[
\zeta_{C}(s) := \sum_{D \geq 0} \frac{1}{N(D)^s}.
\]

Here $N(D) = q^{d(D)}$ with $d(D) = \sum_{P} n_P d(P)$. Thus by first grouping effective divisors according to their rational equivalence classes $\mathcal{D}$, then taking the sum on effective divisors in the same class, we obtain
\[
\zeta_{C}(s) = \sum_{D \in \mathcal{D}} \sum_{D \geq 0} \frac{1}{N(D)^s}.
\]
Proof. This comes from the following two facts: (1) there is a degree one due to the fact that Aut(V), and (2) Aut(V) \cong F_q.

1.2.2. Convergence and Rationality. Clearly, we must show that for general r, the infinite summation in the definition of our non-abelian zeta function \( \zeta_{C,r,F_q}(s) \) converges when Re(s) > 1. For this, let us start with the following vanishing result for semi-stable bundles.

Lemma 1. Let V be a degree d and rank r semi-stable vector bundle on C. Then
(a) if \( d \geq r(2g-2) + 1 \), \( h^1(C,V) = 0 \);
(b) if \( d < 0 \), \( h^0(C,V) = 0 \).

Proof. This is a direct consequence of the fact that if V and W are semi-stable bundles with \( \mu(V) > \mu(W) \), then \( H^0(C,\text{Hom}(V,W)) = \{0\} \).

Thus, from definition,

\[
\zeta_{C,r,F_q}(s) = \sum_{V \in [V] \in M_{C,r}(d), 0 \leq d \leq r(2g-2)} \frac{q^{h^0(C,V)} - 1}{\# \text{Aut}(V)} \cdot (q^{-s})^d(V)
\]

+ \[
\sum_{V \in [V] \in M_{C,r}(d), d \geq r(2g-2)+1} \frac{q^{d(V)} - r(1-1)}{\# \text{Aut}(V)} \cdot (q^{-s})^d(V).
\]

Clearly only finitely many terms appear in the first summation, so it suffices to show that when Re(s) > 1, the second term converges. For this purpose, we introduce the so-called Harder-Narasimhan numbers

\[
\beta_{C,r,F_q}(d) := \sum_{V \in [V] \in M_{C,r}(d)} \frac{1}{\# \text{Aut}(V)}.
\]

Lemma 2. With the same notation as above, for all \( n \in \mathbb{Z} \),

\[
\beta_{C,r,F_q}(d + rn) = \beta_{C,r,F_q}(d).
\]

Proof. This comes from the following two facts: (1) there is a degree one \( F_q \)-rational line bundle A on C; and (2) Aut(V) \cong Aut(V \otimes A^{\otimes n}) and \( d(V \otimes A^{\otimes n}) = d(V) + rn \).

Therefore, the second summation becomes

\[
\sum_{i=1}^{r} \beta_{C,r,F_q}(i) \sum_{n=2g-2}^{\infty} \left( q^{nr+i-r(1-1)} - 1 \right) \cdot (q^{-s})^{nr+i}
\]

\[
= \sum_{i=1}^{r} \beta_{C,r,F_q}(i) \cdot (q^{-s})^i \cdot \left( q^{-r(1-1)} \cdot \frac{q^{(1-s)r(2g-2)}}{1-q^{(1-s)r}} - \frac{q^{(-s)r(2g-2)}}{1-q^{(-s)r}} \right),
\]

provided that \( |q^{-s}| < 1 \). Thus we have proved the following

Proposition. The non-abelian zeta function \( \zeta_{C,r,F_q}(s) \) is well-defined for Re(s) > 1, and admits a meromorphic extension to the whole complex s-plane.

Moreover, if we set \( t := q^{-s} \) and introduce the non-abelian Z-function of C by setting

\[
\zeta_{C,r,F_q}(s) = Z_{C,r,F_q}(t) := \sum_{V \in [V] \in M_{C,r}(d), d \geq 0} \frac{q^{h^0(C,V)} - 1}{\# \text{Aut}(V)} \cdot t^d(V), \quad |t| < 1.
\]
Then the above calculation implies that
\[
Z_{C,r,\mathbf{F}_q}(t) = \sum_{d=0}^{r(2g-1)} \left( \sum_{v \in [V] \in \mathcal{M}_{C,r}(d)} \frac{q^{h^0(C,V)} - 1}{\# \text{Aut}(V)} \right) \cdot t^d + \sum_{i=1}^r \beta_{C,r,\mathbf{F}_q}(i) \cdot \left( \frac{q^{r(g-1)+i} - 1}{1 - q^r} \right) \cdot t^{r(2g-2)+i}.
\]
Therefore, there exists a polynomial \( P_{C,r,\mathbf{F}_q}(s) \in \mathbb{Q}[t] \) such that
\[
Z_{C,r,\mathbf{F}_q}(t) = \frac{P_{C,r,\mathbf{F}_q}(t)}{(1 - t^r)(1 - q^r t^r)}.
\]

In this way, we have established the following

**Rationality.** Let \( C \) be a regular, reduced irreducible projective curve defined over \( \mathbf{F}_q \) with \( Z_{C,r,\mathbf{F}_q}(t) \) the rank \( r \) non-abelian \( Z \)-function. Then, there exists a polynomial \( P_{C,r,\mathbf{F}_q}(s) \in \mathbb{Q}[t] \) such that
\[
Z_{C,r,\mathbf{F}_q}(t) = \frac{P_{C,r,\mathbf{F}_q}(t)}{(1 - t^r)(1 - q^r t^r)}.
\]

**1.2.3. Functional Equation.** Besides the fact that \( P_{C,r,\mathbf{F}_q}(t) \in \mathbb{Q}[t] \), we know very little about this polynomial. To understand \( P_{C,r,\mathbf{F}_q}(s) \) better, as well as for the theoretical purpose, we next study the functional equation for rank \( r \) zeta functions. As a preparation, here we introduce the rank \( r \) non-abelian \( \xi \)-function \( \xi_{C,r,\mathbf{F}_q}(s) \) by setting
\[
\xi_{C,r,\mathbf{F}_q}(s) := \zeta_{C,r,\mathbf{F}_q}(s) \cdot (q^s)^{r(\frac{g-1}{2})},
\]
where \( \chi(C,V) \) denotes the Euler-Poincaré characteristic of \( V \).

**Functional Equation.** Let \( C \) be a regular, reduced irreducible projective curve defined over \( \mathbf{F}_q \) with \( \xi_{C,r,\mathbf{F}_q}(s) \) its associated rank \( r \) non-abelian \( \xi \)-function. Then,
\[
\xi_{C,r,\mathbf{F}_q}(s) = \xi_{C,r,\mathbf{F}_q}(1 - s).
\]

Before proving the functional equation, we give the following

**Corollary.** With the same notation as above,
(a) \( P_{C,r,\mathbf{F}_q}(t) \in \mathbb{Q}[t] \) is a degree \( 2rg \) polynomial;
(b) Denote all reciprocal roots of \( P_{C,r,\mathbf{F}_q}(t) \) by \( \omega_{C,r,\mathbf{F}_q}(i), i = 1, \ldots, 2rg \). Then after a suitable rearrangement,
\[
\omega_{C,r,\mathbf{F}_q}(i) \cdot \omega_{C,r,\mathbf{F}_q}(2rg - i) = q, \quad i = 1, \ldots, rg;
\]
(c) For each \( m \in \mathbb{Z}_{\geq 1} \), there exists a rational number \( N_{C,r,\mathbf{F}_q}(m) \) such that
\[
Z_{C,r,\mathbf{F}_q}(t) = P_{C,r,\mathbf{F}_q}(0) \cdot \exp \left( \sum_{m=1}^{\infty} N_{C,r,\mathbf{F}_q}(m) \frac{\zeta}{m} \right).
\]
Moreover,
\[
N_{C,r,\mathbf{F}_q}(m) = \begin{cases} 
  r(1 + q^m) - \sum_{i=1}^{2r_g} \omega_{C,r,\mathbf{F}_q}(i)^m, & \text{if } r \mid m; \\
  -\sum_{i=1}^{2rg} \omega_{C,r,\mathbf{F}_q}(i)^m, & \text{if } r \not\mid m;
\end{cases}
\]
(d) For any \( a \in \mathbb{Z}_{>0} \), denote by \( \zeta_a \) a primitive \( a \)-th root of unity and set \( T = t^a \). Then
\[
\prod_{i=1}^{a} Z_{C,r,\mathbf{F}_q}(t^a) = (P_{C,r,\mathbf{F}_q}(0))^a \cdot \exp \left( \sum_{m=1}^{\infty} N_{r,\mathbf{F}_q}(ma) \frac{T^m}{m} \right).
\]
Theorem. For this, we have the following

\[ \sum_{i=1}^{a} (\zeta_{i}^{m})^{a} = \begin{cases} \alpha, & \text{if } a \mid m, \\ 0, & \text{if } a \not\mid m. \end{cases} \]

1.2.4. Proof of Functional Equation. To understand the structure of the functional equation clearly, we need to decompose the non-abelian \( \xi \) function for curves. For this purpose, first recall that the canonical line bundle \( K_{C} \) of \( C \) is defined over \( \mathbb{F}_{q} \). Thus, for all \( n \in \mathbb{Z} \), we obtain the following natural \( \mathbb{F}_{q} \)-rational isomorphisms:

\[ \mathcal{M}_{r}(L) \to \mathcal{M}_{r}(L \otimes K_{C}^{\otimes n}); \quad \mathcal{M}_{r}(L) \to \mathcal{M}_{r}(L^{\otimes -1} \otimes K_{C}^{\otimes n}) \]

where \( V^{\vee} \) denotes the dual of \( V \). Next, introduce the union

\[ \mathcal{M}_{C,r}^{L} := \bigcup_{n \in \mathbb{Z}} \left( \mathcal{M}_{r}(L \otimes K_{C}^{\otimes n}) \cup \mathcal{M}_{r}(L^{\otimes -1} \otimes K_{C}^{\otimes n}) \right). \]

With this, clearly, we may and indeed always assume that

\[ 0 \leq d(L) \leq r(g - 1). \]

Further, introduce the partial non-abelian zeta function \( \xi_{C,r,F_{q}}^{L}(s) \) by setting

\[ \xi_{C,r,F_{q}}^{L}(s) := \sum_{V \in [V] \in \mathcal{M}_{C,r}^{L}} \frac{q^{h^{0}(C,V)} - 1}{\# \text{Aut}(V)} \cdot (q^{-s})^{\chi(C,V)}, \quad \Re(s) > 1. \]

Clearly, then

\[ \xi_{C,r,F_{q}}^{L}(s) = \sum_{L} \xi_{C,r,F_{q}}^{L}(s) \]

where \( L \) runs over all line bundles appeared in the following (disjoint) union

\[ \bigcup_{d \in \mathbb{Z}} \mathcal{M}_{C,r}(d) = \bigcup_{L,0 \leq d(L) \leq r(g - 1)} \mathcal{M}_{C,r}^{L}. \]

Remark. Here we remind the reader that the vanishing result of Lemma 1.2.2.1 has been used.

Therefore, to prove the functional equation for \( \xi_{C,r,F_{q}}^{L}(s) \), it suffices to show

\[ \xi_{C,r,F_{q}}^{L}(s) = \xi_{C,r,F_{q}}^{L}(1 - s). \]

For this, we have the following

Theorem. For \( \Re(s) > 1 \),

\[ \xi_{C,r,F_{q}}^{L}(s) \]

\[ = \frac{1}{2} \sum_{V \in [V] \in \mathcal{M}_{C,r}^{L}, 0 \leq d(V) \leq r(2g - 2)} \frac{q^{h^{0}(C,V)} - 1}{\# \text{Aut}(V)} \cdot \left[ (q^{-s})^{\chi(C,V)} + (q^{s-1})^{\chi(C,V)} \right] \]

\[ + \left[ \sum_{d \leq r(2g - 2) - 1} \frac{q^{(1-s)(d(L)-r(g-1))}}{q^{(s-1)r(2g-2)-1}} + \frac{q^{s(d(L)-r(g-1))}}{q^{(s-1)r(2g-2)-1}} + \frac{q^{(s-1)(d(L)-r(g-1))}}{q^{(s-1)r(2g-2)-1}} + \frac{q^{(s-1)(d(L)-r(g-1))}}{q^{(s-1)r(2g-2)-1}} \right] \cdot \beta_{C,r,F_{q}}(L). \]

Here \( \beta_{C,r,F_{q}}(L) := \sum_{V \in [V] \in \mathcal{M}_{C,r}(L)} \frac{1}{\# \text{Aut}(V)} \) denotes the Harder-Narasimhan number. In particular,

(a) \( \xi_{C,r,F_{q}}^{L}(s) \) satisfies the functional equation \( \xi_{C,r,F_{q}}^{L}(s) = \xi_{C,r,F_{q}}^{L}(1 - s); \)

(b) if \( d \leq r(g - 1) - 1 \), then \( \xi_{C,r,F_{q}}^{L}(s) \) satisfies the functional equation \( \xi_{C,r,F_{q}}^{L}(s) = \xi_{C,r,F_{q}}^{L}(1 - s); \)

(c) if \( d \geq r(g - 1) \), then \( \xi_{C,r,F_{q}}^{L}(s) \) satisfies the functional equation \( \xi_{C,r,F_{q}}^{L}(s) = \xi_{C,r,F_{q}}^{L}(1 - s); \)

(d) for all \( d \), \( \xi_{C,r,F_{q}}^{L}(s) \) satisfies the functional equation \( \xi_{C,r,F_{q}}^{L}(s) = \xi_{C,r,F_{q}}^{L}(1 - s); \)
(b) the Harder-Narasimhan number $\beta_{C,r,F_q}(L)$ is given by the leading term of the singularities of $\xi^L_{C,r,F_q}(s)$ at $s = 0$ and $s = 1$.

Remark. In this way, we use non-abelian zeta functions to evaluate the so-called Narasimhan-Harder numbers.

Thus, the Betti numbers for moduli spaces of stable bundles may also be read from our non-abelian zeta functions. The global version of this will be discussed in 2.2.3 below.

Proof. It suffices to prove (*). For this, set

$$ I(s) = \sum_{V \in [V] \in \mathcal{M}_{L,r}^\bullet} \frac{q^{h^0(C,V)}}{\#\text{Aut}(V)} \cdot (q^{-s}) \chi(C,V) $$

and

$$ II(s) = \sum_{V \in [V] \in \mathcal{M}_{L,r}^\bullet} \frac{q^{h^0(C,V)}}{\#\text{Aut}(V)} \cdot (q^{-s}) \chi(C,V) - \sum_{V \in [V] \in \mathcal{M}_{L,r}^\bullet} \frac{1}{\#\text{Aut}(V)} \cdot (q^{-s}) \chi(C,V). $$

Thus,

$$ \xi^L_{C,r,F_q}(s) = I(s) + II(s). $$

So it suffices to show the following

Lemma. With the same notation as above,

(a) $I(s) = \frac{1}{2} \sum_{V \in [V] \in \mathcal{M}_{L,r}^\bullet} \frac{q^{h^0(C,V)}}{\#\text{Aut}(V)} \cdot \left( (q^{-s}) \chi(C,V) + (q^{s-1}) \chi(C,V) \right); $ and

(b) $II(s) = \left[ \frac{q^{(1-s)}(d(L)-r(g-1))}{q^{s-1}} \frac{q^{(d(L)-r(g-1))}}{q^{s-1}} \right] \beta_{C,r,F_q}(L).$

Proof. (a) comes from Riemann-Roch theorem and Serre duality. Indeed,

$$ I(s) = \frac{1}{2} \sum_{V \in [V] \in \mathcal{M}_{L,r}^\bullet} \frac{q^{h^0(C,V)}}{\#\text{Aut}(V)} \cdot (q^{-s}) \chi(C,V) $$

$$ + \sum_{V \in [V] \in \mathcal{M}_{L,r}^\bullet} \frac{q^{h^0(C,V \vee \K_C)}}{\#\text{Aut}(V \vee \K_C)} \cdot (q^{-s}) \chi(C,V \vee \K_C). $$

$$ = \frac{1}{2} \sum_{V \in [V] \in \mathcal{M}_{L,r}^\bullet} \left[ \frac{q^{h^0(C,V)}}{\#\text{Aut}(V)} \cdot (q^{-s}) \chi(C,V) + \frac{q^{h^0(C,V \vee \K_C)}}{\#\text{Aut}(V \vee \K_C)} \cdot (q^{-s}) \chi(C,V \vee \K_C) \right] $$

As for (b), clearly by the vanishing result,

$$ T_{r,L}(s) = \sum_{V \in [V] \in \mathcal{M}_{L,r}^\bullet} \sum_{d(E) \leq r(2g-2)} \frac{1}{\#\text{Aut}(E)} \cdot (q^{-s}) \chi(C,V) $$

$$ - \sum_{V \in [V] \in \mathcal{M}_{L,r}^\bullet} \sum_{d(E) \geq 0} \frac{1}{\#\text{Aut}(E)} \cdot (q^{-s}) \chi(C,E) $$

$$ = \left( \sum_{V \in [V] \in \mathcal{M}_{L,r}(L \otimes K_C^{r/2}); d(E) > r(2g-2)} \frac{1}{\#\text{Aut}(E)} \cdot (q^{-s}) \chi(C,E) \right) $$

$$ - \left( \sum_{V \in [V] \in \mathcal{M}_{L,r}(L \otimes K_C^{r/2}); d(E) > r(2g-2)} \frac{1}{\#\text{Aut}(E)} \cdot (q^{-s}) \chi(C,E) \right) $$

$$ + \left( \sum_{V \in [V] \in \mathcal{M}_{L,r}(L \otimes K_C^{r/2}); d(E) > r(2g-2)} \frac{1}{\#\text{Aut}(E)} \cdot (q^{-s}) \chi(C,E) \right) $$

$$ - \left( \sum_{V \in [V] \in \mathcal{M}_{L,r}(L \otimes K_C^{r/2}); d(E) > r(2g-2)} \frac{1}{\#\text{Aut}(E)} \cdot (q^{-s}) \chi(C,E) \right). $$
But $\chi(C, V)$ depends only on $d(V)$. Thus, accordingly,

$$II(s) = \left[ \sum_{n=1}^{\infty} q^{-s} \frac{d(L) + nr(2g-2) - r(g-1)}{(q^{-s})^{-d(L) + nr(2g-2) - r(g-1)}} \right] \
+ \left[ \sum_{n=1}^{\infty} q^{-s} \frac{d(L) + nr(2g-2) - r(g-1)}{(q^{-s})^{-d(L) + nr(2g-2) - r(g-1)}} \right] : \beta_{C,r}(L)$$

This completes the proof of the lemma, and hence the Theorem and the Functional Equation for rank $r$ zeta functions of curves.

**Chapter 2. Global Non-Abelian Zeta Functions for Curves**

In this chapter, we introduce new non-abelian zeta functions for curves defined over number fields via the Euler product formalism. Thus the global construction is based on our study for the non-abelian zeta of curves defined over finite fields in Chapter 1. Main result of this chapter is about a convergence region of such Euler products. Key ingredients of the proof are a result of (Harder-Narasimhan) Siegel, an ugly yet very precise formula for our local zeta functions, Clifford Lemma for semi-stable vector bundles, and Weil’s theorem on Riemann Hypothesis for Artin zeta functions.

**2.1. Preparations**

**2.1.1. Invariants $\alpha, \beta$ and $\gamma$.** Let $C$ be a regular, reduced, irreducible projective curve of genus $g$ defined over the finite field $\mathbb{F}_q$ with $q$ elements. As in Chapter 1, we then get (the subset of $\mathbb{F}_q$-rational points of) the associated moduli spaces $\mathcal{M}_{C,r}(L)$ and $\mathcal{M}_{C,r}(d)$. Recall that in Chapter 1, motivated by a work of Harder-Narasimhan [HN], we, following Desale-Ramanan [DR], define the Harder-Narasimhan numbers $\beta_{C,r,\mathbb{F}_q}(L), \beta_{C,r,\mathbb{F}_q}(d)$, which are very useful in the discussion of our zeta functions. Now we introduce new invariants for $C$ by setting

$$\alpha_{C,r, \mathbb{F}_q}(d) := \sum_{V \in [V] \in \mathcal{M}_{C,r}(\mathbb{F}_q)} \frac{q^{h^0(C, V)}}{\# \text{Aut}(V)}, \quad \gamma_{C,r, \mathbb{F}_q}(d) := \sum_{V \in [V] \in \mathcal{M}_{C,r}(\mathbb{F}_q)} \frac{q^{h^0(C, V)} - 1}{\# \text{Aut}(V)},$$

and similarly define $\alpha_{C,r, \mathbb{F}_q}(L)$ and $\gamma_{C,r, \mathbb{F}_q}(L)$.

**Lemma.** With the same notation as above,

(i) for $\alpha_{C,r, \mathbb{F}_q}(d)$,

$$\alpha_{C,r, \mathbb{F}_q}(d) = \begin{cases} \beta_{C,r, \mathbb{F}_q}(d); & \text{if } d < 0; \\ \alpha_{C,r, \mathbb{F}_q}(r(2g-2) - d) \cdot q^{d - r(g-1)}, & \text{if } 0 \leq d \leq r(2g-2); \\ \beta_{C,r, \mathbb{F}_q}(d) \cdot q^{d - r(g-1)}, & \text{if } d > r(2g-2); \end{cases}$$

(ii) for $\beta_{C,r, \mathbb{F}_q}(d)$,

$$\beta_{C,r, \mathbb{F}_q}(\pm d + rn) = \beta_{C,r, \mathbb{F}_q}(d) \quad n \in \mathbb{Z};$$

(iii) for $\gamma_{C,r, \mathbb{F}_q}(d)$,

$$\gamma_{C,r, \mathbb{F}_q}(d) = \alpha_{C,r, \mathbb{F}_q}(d) - \beta_{C,r, \mathbb{F}_q}(d).$$

**Proof.** (iii) is simply the definition, while (ii) is a direct consequence of Lemma 1.2.2 and the fact that $\text{Aut}(V) \simeq \text{Aut}(V^\vee)$ for a vector bundle $V$. So it suffices to prove (i).

When $d < 0$, the relation is deduced from the fact that $h^0(C, V) = 0$ if $V$ is a semi-stable vector bundle with strictly negative degree; when $0 \leq d \leq r(2g-2)$, the result comes from the Riemann-Roch and Serre duality; finally when $d > r(2g-2)$, the result is a direct consequence of the Riemann-Roch and the fact that $h^1(C, V) = 0$ if $V$ is a semi-stable vector bundle with degree strictly bigger than $r(2g-2)$. 7
We here remind the reader that this Lemma and Lemma 1.2.2.2 tell us that all $\alpha_{C,r,F_q}(d), \beta_{C,r,F_q}(d)$ and $\gamma_{C,r,F_q}(d)$’s may be calculated from $\alpha_{C,r,F_q}(i), \beta_{C,r,F_q}(j)$ with $i = 0, \ldots, r(g - 1)$ and $j = 0, \ldots, r - 1$.

2.1.2. Asymptotic Behaviors of $\alpha, \beta$ and $\gamma$

For later use, we here discuss the asymptotic behavior of $\alpha_{C,r,F_q}(d), \beta_{C,r,F_q}(d), \gamma_{C,r,F_q}(0)$ when $q \to \infty$.

**Proposition.** With the same notation as above, when $q \to \infty$,

(a) For all $d$,\[
\beta_{C,r,F_q}(d) = O\left(q^{r^2(g-1)}\right);
\]

(b)\[
\frac{q^{(r-1)(g-1)}}{\gamma_{C,r,F_q}(0)} = O(1)
\]

(c) For $0 \leq d \leq r(g - 1)$,

\[
\frac{\alpha_{C,r,F_q}(d)}{q^{d/2+g/2}(g-1)} = O(1).
\]

**Proof:** Following Harder and Narasimhan [HN], a result of Siegel on quadratic forms, which is equivalent to the fact that Tamagawa number of $SL_r$ is 1, may be understood via the following relation on automorphism groups of rank $r$ vector bundles:

\[
\sum_{V: \det(V) = L} \frac{1}{\#Aut(V)} = \frac{q^{(r^2-1)(g-1)}}{q-1} \cdot \zeta_C(2) \ldots \zeta_C(r).
\]

Here $V$ runs over all rank $r$ vector bundles with determinant $L$ and $\zeta_C(s)$ denotes the Artin zeta function of $C$. Thus,

\[
0 < \beta_{C,r,F_q}(L) \leq \frac{q^{(r^2-1)(g-1)}}{q-1} \cdot \zeta_C(2) \ldots \zeta_C(r).
\]

This implies

\[
\beta_{C,r,F_q}(d) = \prod_{i=1}^{2g} (1 - \omega_{C,1,F_q}(i)) \cdot \beta_{C,r,F_q}(L) \leq \prod_{i=1}^{2g} (1 - \omega_{C,1,F_q}(i)) \cdot \frac{q^{(r^2-1)(g-1)}}{q-1} \cdot \zeta_C(2) \ldots \zeta_C(r).
\]

Here two facts are used:

(1) The number of $F_q$-rational points of degree $d$ Jacobian $J^d(C)$ is equal to $\prod_{i=1}^{2g} (1 - \omega_{C,1,F_q}(i))$; and

(2) a result of Desale and Ramanan, which says that for any two $L, L' \in \text{Pic}^d(C)$, $\beta_{C,r,F_q}(L) = \beta_{C,r,F_q}(L')$.

(See e.g., [DR, Prop 1.7.1])

Thus by Weil’s theorem on Riemann Hypothesis on Artin zeta functions ([W]),

\[
|\omega_{C,1,F_q}(i)| = O(q^{1/2}), \quad i = 0, \ldots, 2g.
\]

This then completes the proof of (a).

To prove (b), first note that (b) is equivalent to that, asymptotically, the lower bound of $\gamma_{C,r,F_q}(0)$ is at least $q^{(r-1)(g-1)}$. To show this, note that

\[
\gamma_{C,r,F_q}(0) \geq \sum_{\substack{V = \mathcal{O}_C \oplus L_2 \oplus \cdots \oplus L_r, L_2, \ldots, L_r \in \text{Pic}^0(C), \#(\mathcal{O}_C, L_2, \ldots, L_r) = r}} \frac{q^h(C,V) - 1}{\#Aut(V)}
\]

\[
= \frac{1}{(q-1)^{r-1}} \sum_{\substack{V = \mathcal{O}_C \oplus L_2 \oplus \cdots \oplus L_r, L_2, \ldots, L_r \in \text{Pic}^0(C), \#(\mathcal{O}_C, L_2, \ldots, L_r) = r}} 1.
\]
Now, by the above cited result of Weil again, as $q \to \infty$, 

$$\sum_{V=\mathcal{O}_C \otimes L_2 \otimes \ldots \otimes L_r, L_2, \ldots, L_r \in \text{Pic}^d(C), \#\{\mathcal{O}_C, L_2, \ldots, L_r\} = r} 1 = O(q^{g(r-1)}) .$$

So we have (b) as well.

Just as (a), (c) is about to give an upper bound for $\alpha_{C,r,F_q}(d)$ for $0 \leq d \leq r(2g - 2)$. For this, we first recall the following Clifford Lemma.

**Clifford Lemma.** (See e.g., [B-PBGN, Theorem 2.1]) Let $V$ be a semi-stable bundle of rank $r$ and degree $d$ with $0 \leq \mu(V) \leq 2g - 2$. Then 

$$h^0(C, V) \leq r + \frac{d}{2} .$$

Thus,

$$\alpha_{C,r,F_q}(d) \leq q^{\frac{d}{2} + r} \cdot \beta_{C,r,F_q}(d).$$

With this, (c) is a direct consequence of (a).

### 2.1.3. Ugly Formula

Recall that the rationality of $\zeta_{C,r,F_q}(s)$ says that there exists a degree $2rg$ polynomial $P_{C,r,F_q}(t) \in \mathbb{Q}[t]$ such that 

$$Z_{C,r,F_q}(t) = \frac{P_{C,r,F_q}(t)}{(1 - t^r)(1 - q^r t^r)} .$$

Thus we may set 

$$P_{C,r,F_q}(t) = \sum_{i=0}^{2rg} a_{C,r,F_q}(i)t^i .$$

On the other hand, by the functional equation for $\zeta_{C,r,F_q}(t)(s)$, we have 

$$P_{C,r,F_q}(t) = P_{C,r,F_q}(\frac{1}{qt}) \cdot q^rg \cdot t^{2rg} .$$

Thus by comparing coefficients on both sides, we get the following

**Lemma.** With the same notation as above, for $i = 0, 1, \ldots, rg - 1$,

$$a_{C,r,F_q}(2rg - i) = a_{C,r,F_q}(i) \cdot q^{rg-i} .$$

Now, to determine $P_{C,r,F_q}(t)$ and hence $\zeta_{C,r,F_q}(s)$ it suffices to find $a_{C,r,F_q}(i)$ for $i = 0, 1, \ldots, rg$.

**Proposition.** (An Ugly Formula) With the same notation as above,

$$a_{C,r,F_q}(i) = \begin{cases} 
\alpha_{C,r,F_q}(d) - \beta_{C,r,F_q}(d), & \text{if } 0 \leq i \leq r - 1; \\
\alpha_{C,r,F_q}(d) - (q^r + 1)\alpha_{C,r,F_q}(d - r) + q^r \beta_{C,r,F_q}(d - r), & \text{if } r \leq i \leq 2r - 1; \\
\alpha_{C,r,F_q}(d) - (q^r + 1)\alpha_{C,r,F_q}(d - r) + q^r \alpha_{C,r,F_q}(d - 2r), & \text{if } 2r \leq i \leq r(g - 1) - 1; \\
-(q^r + 1)\alpha_{C,r,F_q}(r(g - 2)) + \alpha_{C,r,F_q}(r(g - 3)) + \alpha_{C,r,F_q}(r(g - 1)), & \text{if } i = r(g - 1); \\
\alpha_{C,r,F_q}(d) - (q^r + 1)\alpha_{C,r,F_q}(d - r) + \alpha_{C,r,F_q}(d - 2r)q^r, & \text{if } r(g - 1) + 1 \leq i \leq rg - 1; \\
2q^r\alpha_{C,r,F_q}(r(g - 2)) - (q^r + 1)\alpha_{C,r,F_q}(r(g - 1)), & \text{if } i = rg;
\end{cases}$$
Proof. By definition,
\[
Z_{C,r,F_v}(t) = \sum_{d=0}^{r(2g-2)} \sum_{d=r(2g-2)+1}^{\infty} \sum_{V \in |V| \in M_{C,r}(d), d \geq 0} \frac{q^{h(C,V)} - 1}{# \text{Aut}(V)} t^d
\]
\[
= \sum_{d=0}^{r(2g-2)} \sum_{V \in |V| \in M_{C,r}(d)} \frac{q^{h(C,V)} - 1}{# \text{Aut}(V)} t^d
\]
\[
+ \sum_{i=1}^{\infty} \sum_{n=2g-2}^{\infty} \sum_{d=rn+i}^{\infty} \sum_{V \in |V| \in M_{C,r}(d)} \frac{q^{rn+i-r(g-1)} - 1}{# \text{Aut}(V)} t^{rn+i}
\]
\[
= \sum_{d=0}^{r(2g-2)} \sum_{V \in |V| \in M_{C,r}(d)} \frac{q^{h(C,V)} - 1}{# \text{Aut}(V)} t^d
\]
\[
+ \frac{q^{r(1-g)}}{1 - (qt)^r} \sum_{i=1}^{r} \beta_{C,r,F_v}(i)(qt)^i - \frac{1}{1 - t} t^{r} \sum_{i=1}^{r} \beta_{C,r,F_v}(i)t^i,
\]
and by a similar calculation as in the proof of Lemma 1.2.4(b). Now
\[
\sum_{d=0}^{r(2g-2)} = \sum_{d=0, r(2g-2)} + \sum_{d=1, r(2g-2)-1} + \ldots + \sum_{d=r(g-1)-1, r(g-1)+1} + \sum_{d=r(g-1)}.
\]
Thus, by Riemann-Roch, Serre duality and Lemma 2.1.1, we conclude that
\[
\sum_{d=0}^{r(2g-2)} \sum_{V \in |V| \in M_{C,r}(d)} \frac{q^{h(C,V)} - 1}{# \text{Aut}(V)} t^d
\]
\[
= \sum_{d=0}^{r(g-1)-1} \left[ \alpha_{C,r,F_v}(d)(t^d + q^{r(g-1)-d}t^{r(2g-2)-d}) - \beta_{C,r,F_v}(d)(t^d + t^{r(2g-2)-d}) \right]
\]
\[
+ \left( \alpha_{C,r,F_v}(r(g-1)) - \beta_{C,r,F_v}(r(g-1)) \right) \cdot r^{r(g-1)}.
\]
With all this, together with Lemmas 1.2.2.2 and 2.1.1, by a couple of pages routine calculation, we are lead to the ugly yet very precise formula in the proposition.

2.2. Global Non-Abelian Zeta Functions for Curves

2.2.1. Definition. Let \( C \) be a regular, reduced, irreducible projective curve of genus \( g \) defined over a number field \( F \). Let \( S_{\text{bad}} \) be the collection of all infinite places and these finite places of \( F \) at which \( C \) do not have good reductions. As usual, a place \( v \) of \( F \) is called good if \( v \notin S_{\text{bad}} \).

Thus, in particular, for any good place \( v \) of \( F \), the \( v \)-reduction of \( C \), denoted as \( C_v \), gives a regular, reduced, irreducible projective curve defined over the residue field \( F(v) \) of \( F \) at \( v \). Denote the cardinal number of \( F(v) \) by \( q_v \). Then, by the construction of Chapter 1, we obtain the associated rank \( r \) non-abelian zeta function \( \zeta_{C_v,r,F_{q_v}}(s) \). Moreover, from the rationality of \( \zeta_{C_v,r,F_{q_v}}(s) \), there exists a degree \( 2rg \) polynomial \( P_{C_v,r,F_{q_v}}(t) \in \mathbb{Q}[t] \) such that
\[
Z_{C_v,r,F_{q_v}}(t) = \frac{P_{C_v,r,F_{q_v}}(t)}{(1 - t^r)(1 - q^{r}t^r)}.
\]
Clearly,
\[
P_{C_v,r,F_{q_v}}(0) = \gamma_{C_v,r,F_{q_v}}(0) \neq 0.
\]
Thus it makes sense to introduce the polynomial $\tilde{P}_{C_{v,r,F,\nu}}(t)$ with constant term 1 by setting

$$\tilde{P}_{C_{v,r,F,\nu}}(t) := \frac{P_{\nu,F(t)}}{P_{C_{v,r,F,\nu}}(0)}.$$ 

Now by definition, the rank $r$ non-abelian zeta function $\zeta_{C,r,F}(s)$ of $C$ over $F$ is the following Euler product

$$\zeta_{C,r,F}(s) := \prod_{v: \text{good}} \frac{1}{P_{C_{v,r,F,\nu}}(q_v^{-s})}, \quad \text{Re}(s) >> 0.$$ 

Clearly, when $r = 1$, $\zeta_{C,r,F}(s)$ coincides with the classical Hasse-Weil zeta function for $C$ over $F$ ([H]).

2.2.2. Convergence. At this earlier stage of the study of our non-abelian zeta functions, the central problem is to justify the above definition. That is to say, to show indeed the Euler product in 2.2.1 converges. In this direction, we have the following

**Conjecture.** Let $C$ be a regular, reduced, irreducible projective curve of genus $g$ defined over a number field $F$. Then its associated rank $r$ global non-abelian zeta function $\zeta_{C,r,F}(s)$ admits a meromorphic continuation to the whole complex $s$-plane.

Recall that even when $r = 1$, i.e., for the classical Hasse-Weil zeta functions, this conjecture has not been confirmed. However, for general $r$, we have the following

**Theorem.** Let $C$ be a regular, reduced, irreducible projective curve defined over a number field $F$. Then its associated rank $r$ global non-abelian zeta function $\zeta_{C,r,F}(s)$ converges when $\text{Re}(s) \geq 1 + g + (r^2 - r)(g - 1)$.

**Proof.** Clearly, it suffices to show that for the reciprocal roots $\omega_{C,F,i}(i)$, $i = 1, \ldots, 2rg$ of $P_{C,F,i}(t)$ associated to curves $C$ over finite fields $F_q$, 

$$|\omega_{C,F,i}(i)| = O(q^{g+(r^2-1)(g-1)}).$$

Thus we are lead to estimate coefficients of $P_{C,F,i}(t)$. Since we have the ugly yet very precise formula for these coefficients, i.e., Lemma and Proposition 2.1.3, it suffices to give upper bounds for $\alpha_{C,F,i}(i), \beta_{C,F,i}(j)$ and a lower bound for $\gamma_{C,F,i}(0)$, the constant term of $P_{C,F,i}(t)$. This is achieved in Proposition 2.1.2.

2.2.3. Working Hypothesis. Like in the theory for abelian zeta functions, we want to use our non-abelian zeta functions to study non-abelian aspect of arithmetic of curves. Motivated by the classical analytic class number formula for Dedekind zeta functions and its counterpart BSD conjecture for Hasse-Weil zeta functions of elliptic curves, we expect that our non-abelian zeta function could be used to understand the Weil-Petersson volumes of moduli space of stable bundles as well as the associated Tamagawa measures.

For doing so, we then also need to introduce the local factors for ‘bad’ places. This may be done as follows: for $\Gamma$-factors, we take these coming from the functional equation for $\zeta_F(rs) \cdot \zeta_F(r(s - 1))$, where $\zeta_F(s)$ denotes the standard Dedekind zeta function for $F$; while for finite bad places, we do as follows: first, use the semi-stable reduction for curves to find a semi-stable model for $C$, then use Seshadri’s moduli spaces of parabolic bundles to construct polynomials for singular fibers, which usually have degree lower than $2rg$. With all this being done, we then can introduce the so-called completed rank $r$ non-abelian zeta function for $C$ over $F$, or better, the completed rank $r$ non-abelian zeta function $\xi_{C,r,\mathcal{O}_F}(s)$ for a semi-stable model $X \to \text{Spec}(\mathcal{O}_F)$ of $C$. Here $\mathcal{O}_F$ denotes the ring of integers of $F$. (If necessary, we take a finite extension of $F$.)

**Conjecture.** $\xi_{X,r,\mathcal{O}_F}(s)$ is holomorphic and satisfies the functional equation

$$\xi_{X,r,\mathcal{O}_F}(s) = \pm \xi_{X,r,\mathcal{O}_F}(1 + \frac{1}{r} - s).$$

Moreover, we expect that for certain classes of curves, the inverse Mellin transform of our non-abelian zeta functions are naturally associated to certain modular forms of weight $1 + \frac{1}{r}$. 

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Remark. From our study for non-abelian zeta functions of elliptic curves [We2], we obtain the following so-called ‘absolute Euler product’ for rank 2 zeta functions of elliptic curves

\[ \zeta_2(s) = \prod_{p \text{ prime}} \frac{1}{1 + (p-1)p^{-s} + (2p-4)p^{-2s} + (p^2-p)p^{-3s} + p^2p^{-4s}} \]

\[ = \prod_{p \text{ prime}} \frac{1}{A_p(s) + B_p(s)p^{-2s}}, \quad \text{Re}(s) > 2 \]

with

\[ A_p(s) = 1 + (p-1)p^{-s} + (2p-4)p^{-2s}, \quad B_p(s) = (p-2) + (p^2-p)p^{-s} + p^2p^{-2s}. \]

Set \( t := q^{-s} \) and \( a_p(t) := A_p(s), b_p(t) := B_p(s) \). Then in \( \mathbb{Z}[t] \), we have the factorization

\[ a_p(t) = (1 + (p-2)t)(1+t), \quad b_p(t) = ((p-2) + pt)(1 + pt) \]

and

\[ a_p\left(\frac{1}{pt}\right) = \frac{1}{p^2t^2} \cdot b_p(t). \]

We hope a similar discussion on Hecke operators as in the classical theory of \( L \)-functions, or better modular forms, works here as well. In particular we ask the following

**Questions.**

(a) What can we say about \( \zeta_2(s) \)?

(b) What is the importance of the factorization of \( a_p(t) \) and \( b_p(t) \) in \( \mathbb{Z}[t] \)?

Chapter 3. Non-Abelian Zeta Functions and Infinitesimal Structures of Brill-Noether Loci

In this chapter, we study the infinitesimal structures of the so-called non-abelian Brill-Noether loci for rank two semi-stable vector bundles over genus two curves. As an application, we calculate the corresponding rank two non-abelian zeta functions for genus two curves. During this process, we see clearly how Weierstrass points, intrinsic arithmetic invariants of curves [We3], contribute to our zeta functions among others.

We in this chapter assume that the characteristic of the base field is strictly bigger than 2 for simplicity.

3.1. Infinitesimal Structures of Non-Abelian Brill-Noether Loci

3.1.1. **Invariants** \( \beta_{C,2,F_4} (d) \). Let \( C \) be a regular reduced irreducible projective curve defined over \( \mathbb{F}_q \). Here we want to calculate Harder-Narasimhan numbers \( \beta_{C,2,F_4} (d) \) for all \( d \). Note that from Lemma 2.1.1,

\[ \beta_{C,2,F_4} (d) = \beta_{C,2,F_4} (d + 2n). \]

So it suffices to calculate \( \beta_{C,2,F_4} (d) \) when \( d = 0, 1 \). For this, we cite the following result of Desale and Ramanan:

**Proposition.** ([DR]) With the same notation as above, for \( L \in \text{Pic}^d(C) \), \( d = 0, 1 \),

\[ \beta_{C,2,F_4}(L) = \frac{q^3}{q-1} \cdot \zeta_C(2) - q \cdot \prod_{i=1}^4 (1 - \omega_i) \cdot \sum_{d_1 + d_2 = d, d_1 > d_2} \frac{\beta_{C,1,F_4}(d_1) \beta_{C,1,F_4}(d_2)}{q^{d_1 - d_2}}. \]

Here \( \zeta_C(s) \) denotes the Artin zeta function for \( C \) and \( \omega_1, \ldots, \omega_4 \) are the roots of the associated \( Z \)-function \( Z_C(s) \), i.e., \( \omega_{C,1,F_4}(i), i = 0, \ldots, 4 = 2 \times 2 \) in our notation.

Thus, in particular, \( \beta_{C,2,F_4}(L) \) is independent of \( L \).

**Lemma.** With the same notation as above, for \( d = 0, 1 \)

\[ \beta_{C,2,F_4}(d) = \frac{q^3}{q-1} \cdot \zeta_C(2) \cdot \prod_{i=1}^4 (1 - \omega_i) - \frac{q^{d+1}}{(q-1)^2(q^2-1)} \cdot \prod_{i=1}^4 (1 - \omega_i)^4. \]
Proof. This comes from the following two facts:

(1) for all $d$,

$$\beta_{C,1,F_q}(d) = \frac{\prod_{i=1}^{d}(1 - \omega_i)}{q-1};$$

(2) the number of $F_q$-rational points of $\text{Pic}^0(C)$ is equal to $\prod_{i=1}^{d}(1 - \omega_i)$.

3.1.2. Infinitesimal structures: a taste. Here we want to calculate $\alpha_{C,2,F_q}(0)$. By Lemma 3.1.1, it suffices to give $\gamma_{C,2,F_q}(0)$. So we are lead to study $\gamma_{C,2,F_q}(L)$ which is supported over the Brill-Noether locus

$$W^0_{C,2}(L) := \{[V] \in \mathcal{M}_{C,2}(L) : h^0(C, \text{Gr}(V)) \geq 1\}.$$

(In general, as in the beautiful paper [B-PGN], we define the Brill-Noether locus

$$W^k_{C,2}(L) := \{[V] \in \mathcal{M}_{C,2}(L) : h^0(C, \text{Gr}(V)) \geq k + 1\}.$$ )

Note that no degree zero stable bundle admits non-trivial global sections, so $W^0_{C,2}(L) := \{[O_C \oplus L]\}$ consists of only a single point.

(a) If $L = O_C$, then $W^0_{C,2}(O_C) = W^1_{C,2}(O_C)$. Moreover, infinitesimally, $V = O_C \oplus O_C$ or $V$ corresponds to all non-trivial extensions

$$0 \to O_C \to V \to O_C \to 0$$

which are parametrized by $\text{PExt}^1(O_C, O_C) \simeq \mathbf{P}^1$. Thus, by definition,

$$\gamma_{C,2,F_q}(O_C) = \frac{q^2 - 1}{(q^2 - 1)(q^2 - q)} + \frac{q - 1}{q(q - 1)} = \frac{q}{q - 1}.$$

(b) If $L \neq O_C$, then, infinitesimally, $V = O_C \oplus L$ or $V$ corresponds to the single non-trivial extension

$$0 \to O_C \to V \to L \to 0.$$

Thus, by definition,

$$\gamma_{C,2,F_q}(L) = \frac{q - 1}{(q - 1)^2} + \frac{q - 1}{q - 1} = \frac{q}{q - 1}.$$

Thus we have the following

**Lemma.** With the same notation as above, for all $L \in \text{Pic}^0(C)$,

$$\gamma_{C,2,F_q}(L) = \frac{q}{q - 1}.$$

In particular,

$$\gamma_{C,2,F_q}(0) = \frac{q}{q - 1} \cdot \prod_{i=1}^{4}(1 - \omega_i).$$

3.1.3. Invariants $\alpha_{C,2,F_q}(1)$. As before, it suffices to calculate $\gamma_{C,2,F_q}(L)$ for all $L \in \text{Pic}^1(C)$. Note that in this case, all bundles are stable, so $\text{Aut}(V) \simeq F_q^*$ and

$$W^0_{C,2}(L) \simeq \{V : \text{stable}, r(V) = 2, \det(V) = L, h^0(C, V) \geq 1\}.$$

Moreover, by [B-PGN, Prop. 3.1],

$$W^0_{C,2}(L) = \{V : \text{stable}, r(V) = 2, \det(V) = L, h^0(C, V) = 1\}$$

and any $V \in W^0_{C,2}(L)$ admits a non-trivial extension

$$0 \to O_C \to V \to L \to 0.$$

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On the other hand, any non-trivial extension
\[0 \to \mathcal{O}_C \to V \to L \to 0\]
gives rise a stable bundle. So in fact
\[W^0_{C,2}(L) \simeq \text{P} \text{Ext}^1(L, \mathcal{O}_C) \simeq \text{P}^1.\]
Thus we have the following
\[\text{Lemma. With the same notation as above, for } L \in \text{Pic}^1(C),\]
\[W^0_{C,2}(L) \simeq \text{P}^1, \quad \text{and} \quad \gamma_{C,2,F_q}(L) = q + 1.\]
\[In \text{ particular},\]
\[\gamma_{C,2,F_q}(1) = (q + 1) \cdot \prod_{i=1}^{4}(1 - \omega_i).\]

3.1.4. Non-abelian Brill-Noether loci and their infinitesimal structures. We here want to calculate
\[\gamma_{C,r,F_q}(2).\]
This is the most complicated level as \(2 = r(g - 1).\) For this purpose, we need to understand the
structures of the non-abelian Brill-Noether loci \(W^0_{C,2}(L)\) and \(W^1_{C,2}(L)\) for \(L \in \text{Pic}^1(C)\).

We begin with recalling the structure of the map \(\pi : C \times C/S_2 \to \text{Pic}^2(C).\) Here \(S_2\) denotes the
symmetric group of two symbols which acts naturally on \(C \times C\) via \((x, y) \mapsto (y, x).\) One checks that \(\pi\) is a
one point blowing-up centered at the canonical line bundle \(K_C\) of \(C.\) For later use, denote by \(\Delta\) the image
of the diagonal of \(C \times C\) in \(\text{Pic}^2(C).\)

Next, we want to understand the structure of sublocus \(W^0_{C,2}(L)^{\text{ss}}\) of \(W^0_{C,2}(L)\) consisting of non-stable
but semi-stable vector bundles.

By definition, for any \(V \in [V] \in W^0_{C,2}(L)^{\text{ss}}, \text{Gr}(V) = \mathcal{O}_C(P) \oplus L(-P)\) for a suitable \((\text{F}_q\text{-rational points of})\) point \(P \in C.\) Thus accordingly,
(a) if \(L \neq K_C,\) then \(W^0_{C,2}(L)^{\text{ss}}\) is parametrized by \((\text{F}_q\text{-rational points of})\) \(C,\) due to the fact that now \(h^0(C, L) = 1.\) Write also \(L = \mathcal{O}_C(A + B)\) with two points \(A, B\) of \(C,\) which are unique from the above
discussion on the map \(\pi,\) we then conclude that
\[W^1_{C,2}(L) = \{[\mathcal{O}_C(A) \oplus \mathcal{O}_C(B)]\}.\]
(b) if \(L = K_C,\) then for any \(P, K_C = \mathcal{O}_C(P + \iota(P))\) where \(\iota : C \to C\) denotes the canonical involution on
\(C.\) So
\[W^0_{C,2}(L)^{\text{ss}} = \{[\mathcal{O}_C(P) \oplus \mathcal{O}_C(\iota(P))]: P \in C\}.\]
Thus, indeed \(W^0_{C,2}(L)^{\text{ss}}\) is parametrized by \(\text{P}^1.\) Moreover,
\[W^1_{C,2}(K_C) = W^0_{C,2}(L)^{\text{ss}} = \{[\mathcal{O}_C(P) \oplus \mathcal{O}_C(\iota(P))]: P \in C\}.\]

On the other hand, it is easily to check that every non-trivial extension
\[0 \to \mathcal{O}_V \to W \to L \to 0\]
gives rise to a semi-stable vector bundle \(W,\) and if \(W\) is not stable, then there exists a point \(Q \in C\) such
that \(W\) may also be given by the non-trivial extension
\[0 \to \mathcal{O}_C(Q) \to W \to L(-Q) \to 0.\]
Note also that the kernel of the natural map \(H^1(C, \text{Hom}(L, \mathcal{O}_C)) \to H^1(C, \text{Hom}(L(-Q), \mathcal{O}_C))\) is one
dimensional. So among all non-trivial extensions \(0 \to \mathcal{O}_C \to V \to L \to 0,\) which are parametrized by
\(\text{P} \text{Ext}^1(L, \mathcal{O}_C) \simeq \text{P}^2,\) the non-stable (yet semi-stable) vector bundles are parametrized by
\((\text{F}_q\text{-rational points of})\) \(C\) when \(L \neq K_C\) by (a) and \(\text{P}^1\) when \(L = K_C\) by (b) above respectively. (See \[\text{NR, Lemma 3.1}\]) In this
way, we have proved the following result on non-abelian Brill-Noether loci for moduli space of $M_{C,2}(L)$ with $L$ a degree 2 line bundles on a genus two curve, which is not covered by [B-PGN]:

**Lemma.** With the same notation as above, $W_{C,2}^0(L) \cong \mathbf{PE}_{1}(L, \mathcal{O}_C) \cong \mathbf{P}^2$, in which the locus $W_{C,2}^0(L)^{ss}$ of semi-stable but not stable bundles is parametrized by $C$ and $\mathbf{P}^1$ in the cases $L \neq K_C$ and $L = K_C$ respectively. More precisely,

(a) If $L = \mathcal{O}_C(A + B) \neq K_C$ with $A, B$ two points of $C$, then $W_{C,2}^0(L)^{ss}$, as a birational image of $C$ under the complete linear system $K_C(A + B)$, is a degree 4 plane curve with a single node located at $W_{C,2}^1(L) = \{[\mathcal{O}_C(A) \oplus \mathcal{O}_C(B)]\}$;

(b) If $L = K_C$, as a degree 2 regular plane curve,

$$W_{C,2}^1(K_C) = W_{C,2}^0(L)^{ss} = \{[\mathcal{O}_C(P) \oplus \mathcal{O}_C(\iota(P))] : \: P \in C\} \cong \mathbf{P}^1.$$

Next, we study the infinitesimal structures of non-abelian Brill-Noether loci. Set

$$W_{C,2}^0(L)^* := W_{C,2}^0(L) \backslash W_{C,2}^0(L)^{ss}.$$  

Then the infinitesimal structure of $W_{C,2}^0(L)$ at points $[V] \in W_{C,2}^0(L)^*$ is simple: each $[V]$ consists a stable rank two vector bundle with $\det(V) = L, h^0(C, V) = 1$ and $\text{Aut}(V) \cong \mathbf{F}_q^*.$

Now consider $W_{C,2}^0(L)^{ss}$.

(a) $L \neq K_C$. Then there exists two points $A, B$ of $C$ such that $L = \mathcal{O}_C(A + B)$. Thus, for any $V \in [\mathcal{O}_C(P) \oplus \mathcal{O}_C(A + B - P)] \notin W_{C,2}^0(L)$, $V$ is given by an extension $0 \rightarrow \mathcal{O}_C(P) \rightarrow V \rightarrow \mathcal{O}_C(A + B - P) \rightarrow 0$ due to the fact that for the non-trivial extension $0 \rightarrow \mathcal{O}_C(A + B - P) \rightarrow W \rightarrow \mathcal{O}_C(P) \rightarrow 0$, $h^0(C, W) = 0.$ Thus, each class $[\mathcal{O}_C(P) \oplus \mathcal{O}_C(A + B - P)] \notin W_{C,2}^1(L)$ consists of exact two vector bundles, i.e., $V_1 = \mathcal{O}_C(P) \oplus \mathcal{O}_C(A + B - P)$ and $V_2$ given by the non-trivial extension $0 \rightarrow \mathcal{O}_C(P) \rightarrow V \rightarrow \mathcal{O}_C(A + B - P) \rightarrow 0.$

Clearly, $h^0(C, V_1) = h^0(C, V_2) = 1$ and $\# \text{Aut}(V_1) = (q - 1)^2, \# \text{Aut}(V_2) = q - 1.$

To study $W_{C,2}^1(L) = \{[\mathcal{O}_C(A) \oplus \mathcal{O}_C(B)]\}$, we divide it into two subcases.

(i) $A \neq B$. Then there are exactly three vector bundles in the class $[\mathcal{O}_C(A) \oplus \mathcal{O}_C(B)]$. They are $V_0 = \mathcal{O}_C(A) \oplus \mathcal{O}_C(B), V_1$ given by the non-trivial extension $0 \rightarrow \mathcal{O}_C(A) \rightarrow V_0 \rightarrow \mathcal{O}_C(B) \rightarrow 0$ and $V_2$ given by the non-trivial extension $0 \rightarrow \mathcal{O}_C(B) \rightarrow V_2 \rightarrow \mathcal{O}_C(A) \rightarrow 0$. Clearly, $h^0(C, V_0) = 2, h^0(C, V_1) = h^0(C, V_2) = 1$ and $\# \text{Aut}(V_0) = (q - 1)^2, \# \text{Aut}(V_1) = \# \text{Aut}(V_2) = q - 1.$

Thus in particular,

$$\gamma_{C,2, \mathbf{F}_q}(L) = (q^2 + q + 1 - (N_1 - 1)) \frac{q - 1}{q - 1} + (N_1 - 2) \left(\frac{q - 1}{(q - 1)^2} + \frac{q - 1}{q - 1}\right) + \left(\frac{q^2 - 1}{q^2 - 1} \frac{q - 1}{q - 1}\right) + \frac{q - 1}{q - 1}.$$  

Here $N_1 = q + 1 - (\omega_1 + \ldots + \omega_4)$ denotes the number of $\mathbf{F}_q$-rational points of $C$.

(ii) $A = B$. Then the infinitesimal structure at $[\mathcal{O}_C(A) \oplus \mathcal{O}_C(A)]$ is as follows: an independent point corresponding to $V_0 = \mathcal{O}_C(A) \oplus \mathcal{O}_C(A)$ and a projective line parametrizing all non-trivial extension $0 \rightarrow \mathcal{O}_C(A) \rightarrow V \rightarrow \mathcal{O}_C(A) \rightarrow 0$. Clearly, $h^0(C, V_0) = 2, h^0(C, V) = 1$ and $\# \text{Aut}(V_0) = (q^2 - 1)(q^2 - q), \# \text{Aut}(V) = q(q - 1);$

Thus in particular,

$$\gamma_{C,2, \mathbf{F}_q}(L) = (q^2 + q + 1 - (N_1 - 1)) \frac{q - 1}{q - 1} + (N_1 - 2) \left(\frac{q - 1}{(q - 1)^2} + \frac{q - 1}{q - 1}\right) + \left(\frac{q^2 - 1}{(q^2 - 1)(q^2 - q)} + (q - 1)\right) \frac{q - 1}{q - 1}.$$
all non-trivial extension \(0 \to \mathcal{O}_C(P) \to V \to \mathcal{O}_C(P) \to 0\). Clearly, \(h^0(C, V_0) = 2, h^0(C, V) = 1\) and 
\[#\text{Aut}(V_0) = (q^2 - 1)(q^2 - q), \#\text{Aut}(V) = q(q - 1)\].

Thus, in particular,

\[
\gamma_{C,2,F_q}(K_C) = (q^2 + q + 1 - (q + 1)) \cdot \frac{q-1}{q-1} + (q + 1 - 4) \left( \frac{q^2-1}{(q^2-1)^2} + \frac{q-1}{q-1} + \frac{q-1}{q-1} \right) + 4 \left( \frac{q^2-1}{(q^2-1)(q^2-q)} + (q + 1) \frac{q-1}{q(q-1)} \right).
\]

So we have completed the proof of the following

**Proposition.** With the same notation as above,

(a) For \(L \neq K_C\),

(i) if \(L \not\in \Delta\), \(\gamma_{C,2,F_q}(L) = \frac{q^3+2q-3+N_1}{q-1}\);

(ii) if \(L \in \Delta\), \(\gamma_{C,2,F_q}(L) = \frac{q^3-2+2N_1}{q-1}\);

(b) if \(L = K_C\), \(\gamma_{C,2,F_q}(L) = q^2 + 3q - 3\).

In particular,

\[
\gamma_{C,2,F_q}(2) = \left( \prod_{i=1}^4 (1 - \omega_i) - (q + 1) \right) \cdot \frac{q^3+2q-3+N_1}{q-1} + q \cdot \frac{q^3-2+N_1}{q-1} + q^2 + 3q - 3.
\]

In this way, by using the ugly formula in Prop 2.1.3, we can finally write down the rank two non-abelian zeta functions for genus two curves, where a degree 8 polynomial is involved. We leave this to the reader.

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