POLYNOMIAL ALGEBRAS ON COADJOINT ORBITS OF SEMISIMPLE LIE GROUPS

Mark J. Gotay*  
Department of Mathematics  
University of Hawai‘i  
2565 The Mall  
Honolulu, HI 96822 USA

Janusz Grabowski†  
Institute of Mathematics  
University of Warsaw  
ul. Banacha 2  
02-097 Warsaw, Poland

Bryon Kaneshige‡  
Department of Mathematics  
University of Hawai‘i  
2565 The Mall  
Honolulu, HI 96822 USA

August 19, 2000

Abstract

We study the algebraic structure of the Poisson algebra \( P(O) \) of polynomials on a coadjoint orbit \( O \) of a semisimple Lie algebra. We prove that \( P(O) \) splits into a direct sum of its center and its derived ideal. We also show that \( P(O) \) is simple as a Poisson algebra iff \( O \) is semisimple.

1 Structure Theorems

Let \( g \) be a real (finite-dimensional) semisimple Lie algebra with corresponding 1-connected Lie group \( G \). It is well known that the dual space \( g^* \) carries the structure of a linear Poisson manifold under the Lie-Poisson bracket. The symplectic leaves of this Poisson structure are the orbits of the coadjoint representation of \( G \) on \( g^* \).

As the elements of \( g \) may be regarded as linear functions on \( g^* \), the symmetric algebra \( S(g) \) may be identified with the algebra of polynomial functions on \( g^* \). Consequently, \( S(g) \) can be realized as a Poisson subalgebra of \( C^\infty(g^*) \). (Equivalently, the Poisson bracket \( \{ , \} \) on \( S(g) \) can be obtained by setting \( \{ \xi, \eta \} = [\xi, \eta] \) for \( \xi, \eta \in g \) and extending to all of \( S(g) \) via the Leibniz rule.)

*Supported in part by NSF grants 96-23083 and 00-72434. E-mail: gotay@math.hawaii.edu
†Supported by KBN, grant No. 2 P03A 031 17. E-mail: jagrab@mimuw.edu.pl
‡E-mail: bryon@math.hawaii.edu
Let \( S(\mathfrak{g})' = \{S(\mathfrak{g}), S(\mathfrak{g})\} \) be the derived ideal, and let \( C(\mathfrak{g}) \) denote the Lie center of the Poisson algebra \( S(\mathfrak{g}) \).

**Proposition 1** \( S(\mathfrak{g}) = C(\mathfrak{g}) \oplus S(\mathfrak{g})' \).

**Proof.** We have the decomposition

\[
S(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} S_n(\mathfrak{g})
\]

of the symmetric algebra into the finite-dimensional subspaces \( S_n(\mathfrak{g}) \) of homogeneous elements of degree \( n \). Each \( S_n(\mathfrak{g}) \) is invariant with respect to the adjoint action of \( \mathfrak{g} \) on \( S(\mathfrak{g}) \). Since every finite-dimensional representation of a semisimple Lie algebra is completely reducible, it follows that the adjoint action of \( \mathfrak{g} \) on \( S(\mathfrak{g}) \) is itself completely reducible. According to [Di, §1.2.10] we can then split

\[
S(\mathfrak{g}) = C(\mathfrak{g}) \oplus \{\mathfrak{g}, S(\mathfrak{g})\}.
\]

So we need only show that \( \{\mathfrak{g}, S(\mathfrak{g})\} = S(\mathfrak{g})' \).

Now, applying the identity

\[
\{fg, h\} = \{f, gh\} + \{g, fh\}
\]

to \( f, g \in \mathfrak{g} \) and \( h \in S_n(\mathfrak{g}) \), we see that \( \{S_2(\mathfrak{g}), S_n(\mathfrak{g})\} \subset \{\mathfrak{g}, S_{n+1}(\mathfrak{g})\} \). Arguing recursively, we obtain

\[
\{S_m(\mathfrak{g}), S_n(\mathfrak{g})\} \subset \{\mathfrak{g}, S_{n+m-1}(\mathfrak{g})\},
\]

from which the desired result follows. \( \Box \)

Let \( \mathcal{O} \) be an orbit in \( \mathfrak{g}^* \). We can restrict polynomials on \( \mathfrak{g}^* \) to functions on \( \mathcal{O} \) thereby obtaining the (orbit) polynomial algebra \( P(\mathcal{O}) \) (which, however, may not be freely generated as an associative algebra). We may identify \( P(\mathcal{O}) \) with the quotient algebra \( S(\mathfrak{g})/I(\mathcal{O}) \), where \( I(\mathcal{O}) \) is the ideal of elements vanishing on \( \mathcal{O} \), with the canonical projection

\[
\rho_{\mathcal{O}}: S(\mathfrak{g}) \to S(\mathfrak{g})/I(\mathcal{O}) \cong P(\mathcal{O}).
\]

Since \( \mathcal{O} \) is a symplectic leaf of the Poisson structure on \( \mathfrak{g}^* \), \( I(\mathcal{O}) \) is a Lie ideal as well. Thus \( I(\mathcal{O}) \) is a Poisson ideal (i.e., an associative ideal which is also a Lie ideal) and hence \( P(\mathcal{O}) \) is a Poisson algebra of polynomial functions on the symplectic leaf \( \mathcal{O} \). Note that since \( \mathcal{O} \) is symplectic, the Lie center \( Z(P(\mathcal{O})) = \mathbb{R} \).

We now show that the decomposition in Proposition 1 projects to a similar decomposition of \( P(\mathcal{O}) \).

**Theorem 2** \( P(\mathcal{O}) = \mathbb{R} \oplus P(\mathcal{O})' \).
Decomposing \( f = \{ I \} \)

**Proof.** It is clear that \( C(\mathfrak{g}) \) projects onto constants on \( \mathcal{O} \) and \( \rho_\mathcal{O}(S(\mathfrak{g})') = P(\mathcal{O})' \), so that \( \mathbb{R} + P(\mathcal{O})' = P(\mathcal{O}) \) by Proposition \( \mathbb{[1]} \). It remains to show that \( P(\mathcal{O})' \cap \mathbb{R} = \{0\} \).

Now the restriction of the adjoint action of \( \mathfrak{g} \) on \( S(\mathfrak{g})' \) to the invariant subspace \( I(\mathcal{O}) \) is also completely reducible, so we can again use \( \mathbb{[1]} \) §1.2.10] to split \( I(\mathcal{O}) = I(\mathcal{O})_1 \oplus I(\mathcal{O})_2 \), where \( I(\mathcal{O})_1 = I(\mathcal{O}) \cap C(\mathfrak{g}) \) and \( I(\mathcal{O})_2 = \{\mathfrak{g}, I(\mathcal{O})\} \subset S(\mathfrak{g})' \).

If \( P(\mathcal{O})' \cap \mathbb{R} \neq \{0\} \), then there is an \( f \in I(\mathcal{O}) \) such that \( 1 + f \in S(\mathfrak{g})' \). Decomposing \( f = f_1 + f_2 \) with \( f_1 \in I(\mathcal{O})_1 \) and \( f_2 \in I(\mathcal{O})_2 \), we get \( 1 + f_1 + f_2 \in S(\mathfrak{g})' \), so \( 1 + f_1 \in C(\mathfrak{g}) \cap S(\mathfrak{g})' = \{0\} \). Hence \( f_1 = -1 \), and this contradicts the fact that \( f_1 \in I(\mathcal{O})_1 \subset I(\mathcal{O}) \). \( \square \)

Theorem \( \mathbb{2} \) was already known in the case when \( \mathcal{O} \) is compact \( \mathbb{GGG} \). In the \( C^\infty \) context, one knows that if \( M \) is a compact symplectic manifold, then its Poisson algebra

\[
C^\infty(M) = \mathbb{R} \oplus C^\infty(M)' \]

\( \mathbb{[Ax]} \), while if \( M \) is noncompact

\[
C^\infty(M) = C^\infty(M)' \]

\( \mathbb{[A]} \). Since in the smooth case \( f \in C^\infty(M)' \) if and only if \( f\eta \) is an exact form, where \( \eta \) is the Liouville volume form, Theorem \( \mathbb{2} \) thus suggests that the polynomial Poisson (resp. de Rham) cohomology of a noncompact coadjoint orbit \( \mathcal{O} \) may differ from its smooth Poisson (resp. de Rham) cohomology. For example, take \( \mathcal{O} \subset sl(2, \mathbb{R})^* \) to be the one-sheeted hyperboloid \( x^2 + y^2 - z^2 = 1 \). The Poisson tensor

\[
\Lambda = x \partial_y \wedge \partial_z + y \partial_x \wedge \partial_z - z \partial_x \wedge \partial_y
\]

on \( sl(2, \mathbb{R})^* \) is polynomial and the induced symplectic form

\[
\omega = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy
\]

on \( \mathcal{O} \) is also polynomial. As \( \omega \) is a volume form on the non-compact manifold \( \mathcal{O} \), it is exact in the smooth category. It is, however, not exact in the polynomial category. Indeed, if \( \omega = d\alpha \) for some polynomial 1-form \( \alpha \) on \( \mathcal{O} \), then, according to the well-known isomorphism between Poisson and de Rham cohomology on a symplectic manifold, we would have \([\Lambda, i_\alpha \Lambda] = \Lambda\), where \([\ , \] \) is the Schouten bracket and \([\Lambda, \cdot] \) is the Poisson cohomology differential \( \mathbb{[V]} \). Writing \( \alpha = f \, dx + g \, dy + h \, dz \), where \( f, g, h \) are polynomials, this gives

\[
\Lambda = H_x \wedge H_f + H_y \wedge H_g + H_z \wedge H_h,
\]

where \( H_a \) is the Hamiltonian vector field of \( a \). Contracting \( \Lambda \) with \( \omega \) then yields

\[
1 = \{x, f\} + \{y, g\} + \{z, h\} -- \text{a contradiction with Theorem } \mathbb{2}.
\]

We remark that Theorem \( \mathbb{2} \) need not hold if \( \mathfrak{g} \) is not semisimple. For instance, \( \mathbb{R}^{2n} \) with its standard symplectic structure is a coadjoint orbit of the Heisenberg group \( H(2n) \), but in this case \( P(\mathbb{R}^{2n}) = P(\mathbb{R}^{2n})' \).
2 A Characterization of $P(O)$

We call a Lie algebra $L$ essentially simple if every Lie ideal of $L$ is either contained in the center $Z(L)$ of $L$ or contains the derived ideal $L' = [L, L]$. We say that a Poisson algebra $P$ is simple if the only Poisson ideals of $P$ are $P$ and $\{0\}$.

**Proposition 3** Let $P$ be a unital Poisson algebra which has no nilpotent elements with respect to the associative structure. If $P$ is simple, then it is essentially simple.

**Proof.** In view of [Gr, Thm.1.10], if $L$ is a Lie ideal of a unital Poisson algebra $P$ then

$$\{P, ad^{-1}(L)\} \subset r(J(L)),$$

where

$$ad^{-1}(L) = \{f \mid \{f, P\} \subset L\},$$

$J(L)$ is the largest associative ideal of $P$ contained in $ad^{-1}(L)$, and $r(J(L))$ is its radical,

$$r(J(L)) = \{f \mid f^n \in J(L) \text{ for some } n = 1, 2, \ldots \}.$$  

We recall from [Gr, Thm. 1.6] that $J(L)$ is in fact a Poisson ideal of $P$.

Suppose that $P' \not\subset L$. Then $ad^{-1}(L) \neq P$, so $J(L) \neq P$, and thus $J(L) = \{0\}$ as $P$ is simple. Then $r(J(L)) = \{0\}$ since by assumption $P$ has no associative nilpotents. Then (1) gives

$$\{P, L\} \subset \{P, ad^{-1}(L)\} = \{0\},$$

i.e., $L \subset Z(P)$.

In particular, the hypotheses of Proposition 3 are satisfied by the polynomial algebra $P(O)$. We now use this Proposition to prove our main result.

**Theorem 4** The Lie algebra $P(O)$ is essentially simple iff the orbit $O$ is semisimple.

**Proof.** $(\Rightarrow)$ Assume $O$ is semisimple and let $O_C$ be the complexification of $O$, i.e., the orbit in $g_C^*$ with respect to the complexified Lie group $G_C$ which contains $O$ in its real part. It is well known that $O_C$ is semisimple and that $O_C$ is an algebraic set in $g_C^*$ [Kd, §3.8]. If $P(O)$ were not essentially simple, then by Proposition 3 we would have a proper Poisson ideal $I$ in $P(O)$, and so, after complexification, a proper Poisson ideal $I_C$ in $P(O)_C := P_C(O_C)$.

Let $V(I_C)$ be the set of zeros of $I_C$ in $O_C$. Since $O_C$ is algebraic, $V(I_C) \neq \emptyset$, and since $I_C$ is a Lie ideal, $V(I_C)$ is $G_C$-invariant and hence consists of orbits. This forces $V(I_C) = O_C$ and so $I_C = \{0\}$. Hence we have a contradiction, since $I_C$ is proper.

$(\Leftarrow)$ Assume that $O$ is not semisimple. Complexifying as before, we get the complexified orbit $O_C$ which is not semisimple. Now there exists a semisimple
orbit $S$ in the Zariski closure of $\mathcal{O}_C$ [K4, §3.8]. Consider the Poisson ideal $K$ of elements of $P(O)$ which vanish on $S$. We claim that this ideal is proper. Indeed, $K = \{0\}$ implies that $I(S) = I(\mathcal{O}_C)$, whence $S = \text{cl}(\mathcal{O}_C)$ as $S$ is an algebraic set. But this is impossible as $S$ and $\mathcal{O}_C$ are distinct orbits. As well, $K = P(O)_C$ is improper as $S \neq \emptyset$.

Now we will show that the existence of the proper Poisson ideal $K$ in the complex Poisson algebra $P(O)_C$ implies the existence of a proper Poisson ideal $I$ in $P(O)$. First, put

$$K_R = \{ f \in P(O) \mid f + ig \in K \text{ for some } g \in P(O) \}.$$

Since for $h \in P(O)$, $f + ig \in K$ implies $(hf) + i(hg) \in K$ and $\{h, f\} + i\{h, g\} \in K$, $K_R$ is a Poisson ideal of $P(O)$. Clearly $K \subset K_R + iK_R$ so that if $K_R = \{0\}$ then $K = \{0\}$. We can thus take $I = K_R$ as long as $K_R \neq P(O)$.

If $K_R = P(O)$, then there is $g \in P(O)$ such that $1 + ig \in K$. Let

$$K_0 = \{ f \in P(O) \mid if \in K \}.$$

Similarly as for $K_R$, we can prove that $K_0$ is a Poisson ideal. Now $K_0 \neq P(O)$, for otherwise $K = P(O)_C$. We can then take $I = K_0$ provided $K_0 \neq \{0\}$. But in fact $K_0 \neq \{0\}$: Since

$$\{P(O), 1 + ig\} = i\{P(O), g\} \subset K,$$

$$\{P(O), g\} \subset K_0,$$

and so $K_0 = \{0\}$ implies that $g \in Z(P(O)) = \mathbb{R}$. So $1 + ig \in K$ is a nonzero constant, whence again $K = P(O)_C$.

In any eventuality, we now have a proper Poisson ideal $I$ of $P(O)$. Of course, $I \not\subset Z(P(O)) = \mathbb{R}$. However, it may happen that $I \supset P(O)'$, in which case Theorem 2 forces $I = P(O)'$. In this circumstance we pass to the associative ideal $I^2$. Since

$$\{P(O), I^2\} \subset \{P(O), I\}I \subset I^2,$$

$I^2$ is also a Lie ideal. If $I^2 \neq I$, then $I^2$ is a proper Lie ideal which neither is contained in the center nor contains the derived ideal.

To see that $I^2 \neq I$ for $I$ proper, we can use the following.

**Lemma 1** If $P$ is a commutative unital ring with no zero divisors and $I$ is a proper ideal which is finitely generated, then $I^2 \neq I$.

**Proof.** Assume that $x_1, \ldots, x_n$ are generators of $I$ and $I^2 = I$. Then $x_i = \sum_{j=1}^n a_{ij} x_j$ for some $a_{ij} \in I$, so that $\sum_{j=1}^n b_{ij} x_j = 0$ where $b_{ij} = \delta_{ij} - a_{ij}$. Setting $B = (b_{ij})$, Cramer’s Rule gives $x_i \det B = 0$ whence $\det B = 0$. But $\det B \in \{1\} + I$ so det $B \neq 0$. ▼

Thus $P(O)$ is not essentially simple.

The last part of this proof provides a converse to Proposition 2 when $P = P(O)$. In particular, we conclude that $P(O)$ is simple if and only if $\mathcal{O}$ is semisimple.
One can also see explicitly that $P(O)$ is not essentially simple when $O$ is nilpotent as follows. Since a nilpotent orbit is conical [Br], $I(O)$ is a homogeneous ideal. As a consequence, the notion of homogeneous polynomial makes sense in $P(O)$. Let $P_k(O)$ denote the subspace consisting of all homogeneous polynomials of degree $k$. By virtue of the commutation relations of $g$,

$$\{P_k(O), P_l(O)\} \subset P_{k+l-1}(O),$$

whence each $P_{(k)}(O) = \oplus_{l \geq k} P_l(O)$ for $k \geq 1$ is a proper Poisson ideal of $P(O)$.

References

[Av] A. Avez, *Remarques sur les automorphismes infinitésimaux des variétés symplectiques compactes*, Rend. Sem. Mat. Univers. Politecn. Torino 33 (1974–1975), 5–12.

[Br] R. Brylinski, *Geometric quantization of nilpotent orbits*, J. Diff. Geom. Appl. 9 (1998), 5–58.

[Di] J. Dixmier, *Enveloping Algebras*, North-Holland, Amsterdam (1977).

[GGG] M. J. Gotay, J. Grabowski, and H. B. Grundling, *An obstruction to quantizing compact symplectic manifolds*, Proc. Amer. Math. Soc. 28 (2000), 237–243.

[Gr] J. Grabowski, *The Lie structure of $C^*$ and Poisson algebras*, Studia Math. 81 (1985), 259–270.

[Ko] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math. 85 (1963), 327–404.

[Li] A. Lichnerowicz, *Sur l’algèbre des automorphismes infinitésimaux d’une variété symplectique*, J. Diff. Geom. 9 (1974), 1–40.

[Va] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Math. 118, Birkhäuser, Boston (1994).