VIRTUAL HOMOLOGICAL TORSION OF CLOSED HYPERBOLIC 3-MANIFOLDS

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Abstract

In this paper, we will use Kahn and Markovic’s immersed almost totally geodesic surfaces ([KM1]) to construct certain immersed \( \pi_1 \)-injective 2-complexes in closed hyperbolic 3-manifolds. Such 2-complexes are locally almost totally geodesic except along a 1-dimensional subcomplex. By using Agol’s result that the fundamental groups of closed hyperbolic 3-manifolds are virtually compact special ([Ag], [Wi]) and other works on geometric group theory, we will show that any closed hyperbolic 3-manifold virtually contains any prescribed subgroup in the homological torsion. More precisely, our main result is, for any finite abelian group \( A \), and any closed hyperbolic 3-manifold \( M \), \( M \) admits a finite cover \( N \), such that \( A \) is a direct summand of \( \text{Tor}(H_1(N; \mathbb{Z})) \).

1. Introduction

1.1. Background. In [Lü1], Lück showed that the \( L^2 \)-betti numbers of a CW-complex with residually finite fundamental group can be approximated by the betti numbers of a cofinal tower of finite regular cover. For the definitions of various \( L^2 \)-invariants, see [Lü2].

**Theorem 1.1 ([Lü1]).** Let \( X \) be a finite, connected CW-complex with residually finite fundamental group \( \Gamma \). Let \( \Gamma \supset \Gamma_1 \supset \cdots \supset \Gamma_n \supset \cdots \) be a nested sequence of finite index normal subgroups of \( \Gamma \) with \( \cap \Gamma_n = \{1\} \), and let \( X_n \) be the finite cover of \( X \) associated with \( \Gamma_n \subset \Gamma \), then

\[
\lim_{n \to \infty} \frac{b_p(X_n)}{[\Gamma : \Gamma_n]} = b_p^{(2)}(X).
\]

Since finite volume hyperbolic 3-manifolds have vanishing \( L^2 \)-betti numbers ([LL]), by applying the above result to hyperbolic 3-manifolds, we have the following immediate corollary.

**Corollary 1.2.** For any hyperbolic 3-manifold \( M \) with finite volume, and any tower of finite regular covers \( \cdots \rightarrow M_n \rightarrow \cdots \rightarrow M_1 \rightarrow M \) with

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\[ \cap \pi_1(M_i) = \{1\}, \]

\[ \lim_{n \to \infty} \frac{b_1(M_n)}{\pi_1(M) : \pi_1(M_n)} = 0. \]

Along with Agol’s virtually infinite first betti number theorem ([Ag]), these results imply that the first betti numbers of finite covers of a fixed hyperbolic 3-manifold can go to infinity, but this trend does not grow very fast, which is a very interesting phenomenon.

On the other hand, a natural question is, whether the above approximation of the \(L^2\)-betti number can be generalized to some approximation of the \(L^2\)-torsion.

In particular, in [LS], Lück and Schick showed that, for a finite volume hyperbolic 3-manifold \(M\), its \(L^2\)-torsion is related with its hyperbolic volume by the following equality:

\[ \rho^{(2)}(\tilde{M}) = -\frac{Vol(M)}{6\pi}. \]

So there arises the following natural question (see [Lü2] Question 13.73, [Lü3] Question 1.12 and [BV]).

**Question 1.3.** Let \(M\) be a hyperbolic 3-manifold with finite volume, does there exist a cofinal tower of finite regular covers \(\cdots \rightarrow M_n \rightarrow \cdots \rightarrow M_1 \rightarrow M\), such that

\[ \lim_{n \to \infty} \frac{\ln |\text{Tor}(H_1(M_n;\mathbb{Z}))|}{\pi_1(M) : \pi_1(M_n)} = \frac{Vol(M)}{6\pi}? \]

In [Le], Le claim that

\[ \lim_{n \to \infty} \frac{\ln |\text{Tor}(H_1(M_n;\mathbb{Z}))|}{\pi_1(M) : \pi_1(M_n)} \leq \frac{Vol(M)}{6\pi} \]

holds for any cofinal tower of finite regular covers.

If the answer of Question 1.3 is yes, any hyperbolic 3-manifold admits a certain cofinal tower of finite regular covers, with exponential growth on their homological torsion. However, a much weaker question, whether any hyperbolic 3-manifold virtually has nontrivial homological torsion, was still unknown. In the survey paper [AFW], Aschenbrenner, Friedl and Wilton asked the following question.

**Question 1.4.** Let \(M\) be a hyperbolic 3-manifold with finite volume, does \(M\) admit a finite cover \(N\) such that \(\text{Tor}(H_1(N;\mathbb{Z})) \neq 0\)?

This paper is devoted to answer Question 1.4 for closed hyperbolic 3-manifolds. Actually, we will prove that any closed hyperbolic 3-manifold virtually contains any prescribed subgroup in its homological torsion.

**Theorem 1.5.** For any finite abelian group \(A\), and any closed hyperbolic 3-manifold \(M\), \(M\) admits a finite cover \(N\), such that \(A\) is a direct summand of \(\text{Tor}(H_1(N;\mathbb{Z}))\).
Since Agol showed that hyperbolic 3-manifolds have virtually infinite first betti number ([\text{Ag}]) and the first betti number does not decrease under taking finite cover, we have the following immediate corollary.

**Corollary 1.6.** For any finitely generated abelian group \( A \), and any closed hyperbolic 3-manifold \( M \), \( M \) admits a finite cover \( N \), such that \( A \) is a direct summand of \( H_1(N; \mathbb{Z}) \).

**Remark 1.7.** In a previous draft of this paper, the author used the result that \( \pi_1(M) \) is LERF ([\text{Ag}],[\text{Wi}]), and only showed that \( A \) embeds into \( H_1(N; \mathbb{Z}) \). Then Agol and Friedl informed the author about the virtual retract property of quasi-convex subgroups in \( \pi_1(M) \) ([\text{HW}]), then we could promote the result to make \( A \) to be a direct summand.

In the proof of Theorem 1.5, we will use Kahn and Markovic’s construction of immersed almost totally geodesic surfaces in closed hyperbolic 3-manifolds ([\text{KM1}]). Since Kahn and Markovic’s construction requires the manifold has a positive injectivity radius, it does not work for hyperbolic 3-manifolds with cusps. So we can not show the same result for cusped 3-manifolds, and we have the following natural question.

**Question 1.8.** Whether Theorem 1.5 holds for finite volume hyperbolic 3-manifolds with cusps?

1.2. **Sketch of the Proof.** In this paper, we will always use the symbols \( l, d \) to denote the complex length and complex distance. The definitions of \( l, d, \text{hl}_{\Pi} \) and \( s \) are given in Section 2, which are standard notations in [\text{KM1}].

We will use Kahn and Markovic’s construction of immersed almost totally geodesic surfaces in closed hyperbolic 3-manifolds ([\text{KM1}]) to do the following construction. For any closed hyperbolic 3-manifold \( M \) and any positive integer \( p \geq 2 \), we will construct an immersed \( \pi_1 \)-injective 2-complex \( f : X_p \hookrightarrow M \), which provides us the virtual homological torsion.

More precisely, suppose \( f : S \hookrightarrow M \) is a Kahn-Markovic surface, and \( S \) is equipped with a pants decomposition \( C \). Then Kahn and Markovic’s theorem implies that, there exist some small number \( \epsilon > 0 \) and some large number \( R > 0 \), such that for any simple closed curve \( C \in C, |\text{hl}(C) - \frac{R}{2}| < \epsilon \) and \( |s(C) - 1| < \frac{\epsilon}{R} \) holds. The exponential mixing property of the frame flow ([\text{Mo}],[\text{Po}]) implies that there exists a closed geodesic \( \gamma \) in \( M \), such that \(|l(\gamma) - \frac{R+2\pi i}{p}| < \frac{\epsilon}{p}\). Moreover, we can choose \( S \) such that \( f(C) \) goes along \( \gamma \) for \( p \) times for some \( C \in C \).

By passing to a two-fold cover of \( S \) if necessary, we can cut \( S \) along \( C \) to get a connected surface \( S' \) with two oriented boundary components \( C_1 \) and \( C_2 \), with \([C_1] - [C_2] = 0 \) in \( H_1(S'; \mathbb{Z}) \). Then \( X_p \) is defined to be the quotient space of \( S' \) under the \( \frac{2\pi i}{p} \)-rotations on \( C_1 \) and \( C_2 \) respectively. Let \( c_1 \) and \( c_2 \) denote the image of \( C_1 \) and \( C_2 \) in \( X_p \) respectively (with
induced orientations), and we still use \( f \) to denote the map \( f : X_p \leftrightarrow M \)
induced by the immersion \( S \leftrightarrow M \).

Geometrically, in the closed hyperbolic 3-manifold \( M \), away from
points in \( c_1 \cup c_2 \), \( f(X_p) \) locally looks like an almost totally geodesic
surface. On a neighborhood of \( f(c_1) \), \( f(X_p) \) is almost a \((p\text{-prong}) \times I \)
with the top and bottom identified by the \( \frac{2\pi}{p} \)-rotation. Here the \( p \)-prong
satisfies that any two adjacent edges have angle \( \frac{2\pi}{p} \).

By doing cut-and-paste surgeries on \( X_p \) with other Kahn-Markovic
surfaces, we can assume that any essential arc in \( \partial X_p \) with end points
in \( c_1 \cup c_2 \) is very long. In this case we will show that \( f : X_p \leftrightarrow M \)
is \( \pi_1 \text{-injective} \).

Now we give two strategies to construct virtual homological torsions for closed hyperbolic 3-manifolds. One strategy uses LERF and
the other one uses the virtual retract property. The strategy using
LERF can only give an embedding of the finite abelian group \( A \) into
\( Tor(H_1(N, \mathbb{Z})) \) for some finite cover \( N \); while the second strategy can show that \( A \) is actually a virtual direct summand, which is stronger.
However, the first strategy gives us an interesting codimension-0 submanifold in some finite cover \( N \), which might be useful in some further
research, so we give both strategies here.

**Strategy I:** Let \( \tilde{M} \) be the infinite cover of \( M \) associate to \( f_*(\pi_1(X_p)) \)
\( \subset \pi_1(M) \), then \( \tilde{M} \) is a geometric finite hyperbolic 3-manifold. Let \( \tilde{M} \)
be a compact core of \( \tilde{M} \), then the boundary of \( \tilde{M} \) is incompressible, and
there exists an order-\( p \) element \( \alpha = [c_1] - [c_2] \in H_1(\tilde{M}; \mathbb{Z}) \). It is also
easy to show that, in the order-\( p \) subgroup of \( H_1(\tilde{M}; \mathbb{Z}) \) generated by \( \alpha \),
only 0 and \( \frac{p}{2} \alpha \) can be carried by \( H_1(\partial \tilde{M}; \mathbb{Z}) \) when \( p \) is even, and only 0
is carried by \( H_1(\partial \tilde{M}; \mathbb{Z}) \) when \( p \) is odd.

Since fundamental groups of hyperbolic 3-manifolds are LERF ([Ag],
[Wi]), by Scott’s criterion of LERF ([Sc]), there exists an intermediate
finite cover \( N \rightarrow M \) of \( \tilde{M} \rightarrow M \) such that \( \tilde{M} \) embeds into \( N \). By an
M-V sequence argument, \( \tilde{M} \) gives a \( \mathbb{Z}_{\sigma(p)} \) subgroup in \( Tor(H_1(N; \mathbb{Z})) \).
Here \( \sigma(p) = p \) when \( p \) is an odd number, and \( \sigma(p) = p/2 \) if \( p \) is even.

For two such geometrically finite subgroups \( G_1 = (f_1)_*(\pi_1(X_{p_1})) \) and
\( G_2 = (f_2)_*(\pi_1(X_{p_2})) \), we can find \( g \in \pi_1(M) \) such that both of the
limit points of \( g \) in \( S_\infty^2 \) do not lie in the limit sets \( \Lambda(G_1) \) and \( \Lambda(G_2) \).
Then for a large enough positive integer \( n \), the same argument as above shows that the geometric finite subgroup \( G_1 \ast g^n G_2 \ast g^{n-1} \subset \pi_1(M) \) gives
a \( \mathbb{Z}_{\sigma(p_1)} \oplus \mathbb{Z}_{\sigma(p_2)} \) subgroup in the homology of some finite cover \( N \). The result for a general finite abelian group \( A \) can be shown by induction as the \( \mathbb{Z}_{\sigma(p_1)} \oplus \mathbb{Z}_{\sigma(p_2)} \) case.

**Strategy II:** Agol and Wise showed that \( \pi_1(M) \) is virtually special
([Ag],[Wi]), so we can suppose that \( \pi_1(M) \) is already the group of a
special cube complex. Since quasi-convex subgroups of special groups
are virtual retract ([HW]), there exists a finite cover \( N \) of \( M \), such that the following conditions hold.

1) \( \pi_1(X_p) \subset \pi_1(N) \).

2) For the inclusion map \( i : \pi_1(X_p) \to \pi_1(N) \), there exists a retract homomorphism \( r : \pi_1(N) \to \pi_1(X_p) \) such that \( r \circ i = id_{\pi_1(X_p)} \).

The maps on fundamental groups induce maps on the first homology:

\[
H_1(X_p;\mathbb{Z}) \xrightarrow{i_*} H_1(N;\mathbb{Z}) \xrightarrow{r_*} H_1(X_p;\mathbb{Z}).
\]

Since \( r_* \circ i_* = id \), we know that \( H_1(X_p;\mathbb{Z}) \) is a direct summand of \( H_1(N;\mathbb{Z}) \).

It is easy to compute that \( H_1(X_p;\mathbb{Z}) \cong \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_p \), so \( \mathbb{Z}_p \) is a direct summand of \( H_1(N;\mathbb{Z}) \).

For a general finite abelian group \( A \), we can do the induction as in Strategy I and use the virtual retract property to construct our desired finite cover \( N \).

This paper is organized as the following. In Section 2, we will give a quick review of Kahn and Markovic’s result on constructing immersed almost totally geodesic surfaces in closed hyperbolic 3-manifolds ([KM1]), and prove some related lemmas. In Section 3, we will carry out the above discussion more concretely and rigorously, modulo the \( \pi_1 \)-injectivity result (Theorem 3.4). The \( \pi_1 \)-injectivity property of \( f : X_p \hookrightarrow M \) is a technical result and the proof will be deferred to Section 4.

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2. A Review of Kahn and Markovic’s Works and Further Results

In this section, we give a quick review of Kahn and Markovic’s works on constructing immersed almost totally geodesic surfaces in closed hyperbolic 3-manifolds (see [KM1]). After introducing their works, we will develop a few related lemmas.

In [KM1], Kahn and Markovic proved the following Surface Subgroup Theorem, which is the first step to prove Thurston’s Virtual Haken and Virtual Fibered Conjectures. (The conjectures were raised in [Th2], and settled in [Ag]).

**Theorem 2.1 ([KM1]).** For any closed hyperbolic 3-manifold \( M \), there exists an immersed closed hyperbolic surface \( f : S \hookrightarrow M \), such that \( f_* : \pi_1(S) \to \pi_1(M) \) is an injective map.
Actually, the surfaces constructed in Theorem 2.1 are almost totally geodesic surfaces, which are constructed by pasting oriented good pants together along oriented good curves in an almost totally geodesic way. In the following, we will describe Kahn and Markovic’s construction with more details.

At first, we need to give some geometric definitions.

Let \( \alpha \) be an oriented geodesic arc in a closed hyperbolic 3-manifold with initial point \( p \) and terminal point \( q \). For two unit normal vectors \( \vec{v} \) and \( \vec{w} \) of \( \alpha \) at \( p \) and \( q \) respectively, we define \( d_\alpha(\vec{v}, \vec{w}) \) by the following way. Let \( \vec{v}' \) be the parallel transportation of \( \vec{v} \) to \( q \) along \( \alpha \), \( \theta \in \mathbb{R}/2\pi\mathbb{Z} \) be the oriented angle between \( \vec{v}' \) and \( \vec{w} \) (with respect to the orientation of \( \alpha \)), and the length of \( \alpha \) be \( l > 0 \). Then the complex distance between \( \vec{v} \) and \( \vec{w} \) along \( \alpha \) is defined to be \( d_\alpha(\vec{v}, \vec{w}) = l + \theta i \).

For an oriented closed geodesic \( \gamma \) in a hyperbolic 3-manifold, we define its complex length in a similar way. Choose an arbitrary point \( p \) on \( \gamma \) and a unit normal vector \( \vec{v} \) of \( \gamma \) at \( p \), then we can consider \( \gamma \) as an oriented geodesic arc from \( p \) to \( p \). Then the complex length of \( \gamma \) is defined to be \( l(\gamma) = d_\gamma(\vec{v}, \vec{v}) \). This complex length not only measures the length of \( \gamma \) in the usual sense, but also measures the rotation angle of the corresponding hyperbolic isometry. Note that the complex length of a closed geodesic does not depend on the orientation and the choices we made.

In the following, we will use \( \Pi^0 \) to denote the oriented pair of pants.

**Definition 2.2.** For a closed hyperbolic 3-manifold \( M \), a map \( f : \Pi^0 \to M \) is called a skew pair of pants if \( f_* : \pi_1(\Pi^0) \to \pi_1(M) \) is injective, and \( f(\partial \Pi^0) \) is a union of three closed geodesics.

We will always think about homotopic skew pair of pants as the same object, and we will use \( \Pi \) to denote a skew pair of pants \( f : \Pi^0 \to M \) when it does not cause any confusion.

Let \( C_1, C_2 \) and \( C_3 \) be the three oriented boundary components of \( \Pi^0 \), then let \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) be the three oriented closed geodesics \( f(C_1) \), \( f(C_2) \) and \( f(C_3) \) respectively. Let \( a_i \) be the simple arc on \( \Pi^0 \) which connects \( C_{i-1} \) and \( C_{i+1} \), such that \( a_1, a_2, \) and \( a_3 \) are disjoint with each other (they are called seams of \( \Pi_0 \)). Then we can assume that \( f(a_i) \) is a geodesic arc perpendicular with both \( \gamma_{i-1} \) and \( \gamma_{i+1} \) for \( i = 1, 2, 3 \), and denote \( f(a_i) \) by \( \eta_i \).

Now we fix one \( \gamma_i \), and give orientations for \( \eta_{i-1} \) and \( \eta_{i+1} \) such that they are both pointing away from \( \gamma_i \). Then \( \eta_{i-1} \) and \( \eta_{i+1} \) divide \( \gamma_i \) to two oriented geodesic arcs \( \gamma_i^1 \) and \( \gamma_i^2 \), such that the orientation on \( \gamma_i^1 \) goes from \( \eta_{i-1} \cap \gamma_i \) to \( \eta_{i+1} \cap \gamma_i \). Let \( \vec{v}_{i-1} \) and \( \vec{v}_{i+1} \) be the unit tangent vectors of \( \eta_{i-1} \) and \( \eta_{i+1} \) at \( \eta_{i-1} \cap \gamma_i \) and \( \eta_{i+1} \cap \gamma_i \) respectively, then we have a pair of vectors \( (\vec{v}_{i-1}, \vec{v}_{i+1}) \) on the unit normal bundle \( N^1(\gamma_i) \), which
is called the pair of feet of \( \Pi \) on \( \gamma_i \). The hyperbolic geometry of right-angled hexagons in \( \mathbb{H}^3 \) implies that \( d_{\gamma_i}((\vec{v}_{i-1}, \vec{v}_{i+1}) = d_{\gamma_i}(\vec{v}_{i+1}, \vec{v}_{i-1}) \). So we can define the half length of \( \gamma_i \) with respect to \( \Pi \) by

\[
\text{hl}_\Pi(C_i) = d_{\gamma_i}((\vec{v}_{i-1}, \vec{v}_{i+1}) = d_{\gamma_i}(\vec{v}_{i+1}, \vec{v}_{i-1}).
\]

Now we are ready to define \textit{good curves} and \textit{good pants}.

**Definition 2.3.** Fix a small number \( \epsilon > 0 \) and a large number \( R > 0 \). For a closed oriented geodesic \( \gamma \) in \( M \), we say \( \gamma \) is an \((R, \epsilon)\)-\textit{good curve} if \( |I(\gamma) - R| < 2\epsilon \). The set of \((R, \epsilon)\)-\textit{good curves} is denoted by \( \Gamma_{R, \epsilon} \).

For a skew pair of pants \( f : \Pi^0 \rightarrow M \), we say it is an \((R, \epsilon)\)-\textit{good pants} if \( |\text{hl}_\Pi(C) - \frac{R}{2}| < \epsilon \) holds for all the three cuffs (boundary components) of \( \Pi^0 \). The set of \((R, \epsilon)\)-\textit{good pants} is denoted by \( \Pi_{R, \epsilon} \).

In the following, we will work with a very small number \( \epsilon > 0 \) and a very large number \( R > 0 \), and the precise value of \( \epsilon \) and \( R \) will be determined later. When \( R \) and \( \epsilon \) have been fixed, we will only talk about \textit{good curves} and \textit{good pants}, instead of \((R, \epsilon)\)-\textit{good curves} and \((R, \epsilon)\)-\textit{good pants}, when it does not cause any confusion. Note that oriented boundary components of \((R, \epsilon)\)-\textit{good pants} are \((R, \epsilon)\)-\textit{good curves}.

For a good curve \( \gamma \in \Gamma_{R, \epsilon} \), the normal bundle \( N^1(\gamma) \) of \( \gamma \) is naturally identified with \( \mathbb{C}/I(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z} \). If we have a skew pair of pants \( \Pi \) which has \( \gamma \) as one of its oriented boundary component, we can define the half normal bundle of \( \gamma \) by \( N^1(\sqrt{\gamma}) = \mathbb{C}/\text{hl}_\Pi(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z} \). Then the pair of feet of \( \Pi \) on \( \gamma \) are identified to one point in \( N^1(\sqrt{\gamma}) \), which is called the \textit{foot} of \( \Pi \) on \( \gamma \), and denoted by \( \text{foot}_\gamma(\Pi) \).

Now we are ready to talk about maps from surfaces to closed hyperbolic 3-manifolds. In the following, we will fix a closed hyperbolic 3-manifold and work on it.

Suppose \( S \) is a compact oriented closed surface with negative Euler characteristic, equipped with a pants decomposition \( C \). Then the closure of each component of \( S \setminus C \) is an oriented pair of pants, and we call such a component a \textit{pants in} \( S \).

**Definition 2.4.** A map \( f : S \rightarrow M \) is called \textit{viable} if the following conditions hold.

- For each pants \( \Pi \) in \( S \), \( f|_\Pi : \Pi \rightarrow M \) is a skew pair of pants.
- For any two pants \( \Pi \) and \( \Pi' \) in \( S \) sharing a curve \( C \in C \), \( \text{hl}_\Pi(C) = \text{hl}_{\Pi'}(C) \) holds.

So for a viable map \( f : S \rightarrow M \), we will use \( \text{hl}(C) \) to denote \( \text{hl}_\Pi(C) \) for each \( C \in C \). For two pants in \( S \) sharing a curve \( C \in C \), we give \( C \) an arbitrary orientation. Let \( \Pi \) be the pants lies to the left of \( C \) on \( S \), and \( \Pi' \) lies to the right. Let \( \gamma = f(C) \), and \( \bar{\gamma} \) be the same closed geodesic with the opposite orientation, then we can compare the feet of \( \Pi \) and \( \Pi' \) on \( N^1(\sqrt{\gamma}) \) by the following shearing parameter (here \( N^1(\sqrt{\gamma}) \) and
$N^1(\sqrt{\gamma})$ are naturally identified with each other):

\[
s(C) = \text{foot}_{\gamma}(f|_{\Pi}) - \text{foot}_{\gamma}(f|_{\Pi'}) - \pi i \in N^1(\sqrt{\gamma}) = \mathbb{C}/\text{hl}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z}.
\]

Now we can precisely describe the immersed almost totally geodesic surfaces constructed in [KM1].

**Theorem 2.5** ([KM1]). For any closed hyperbolic 3-manifold $M$, there exists constants $q > 0$ and $K > 0$, such that for every small enough $\epsilon > 0$ and every large enough $R > 0$, the following statement holds. There exists a closed surface $S$ equipped a pants decomposition $C$, and a viable map $f : S \to M$ such that for any $C \in C$, we have

\[
\begin{align*}
|\text{hl}(C) - \frac{R}{\pi}| &< \epsilon, \\
|s(C) - 1| &< KRe^{-qR} < \frac{\epsilon}{R}.
\end{align*}
\]

Moreover, $f_* : \pi_1(S) \to \pi_1(M)$ is injective.

We will call a viable map $f : S \to M$ an $(R, \epsilon)$-almost totally geodesic surface, if the inequality (1) holds for each $C \in C$.

The existence of such $(R, \epsilon)$-almost totally geodesic closed surfaces is proved by the following strategy in [KM1]. For any good curve $\gamma$, one can consider all the good pants in $M$ with $\gamma$ as one of its oriented boundary component, then consider all the feet $\text{foot}_{\gamma}(\Pi)$ on $N^1(\sqrt{\gamma})$. Kahn and Markovic showed that these feet on $N^1(\sqrt{\gamma})$ are very equidistributed, so they can paste all the good pants together in a proper way such that $|s(C) - 1| < \frac{\epsilon}{R}$ holds.

More precisely, Kahn and Markovic constructed an integer valued measure $\mu_0$ on $\Pi_{R,\epsilon}$, with the following nice property.

**Proposition 2.6** ([KM1]). There exists an integer valued measure $\mu_0$ on $\Pi_{R,\epsilon}$ with the following properties. Let $\hat{\partial} \mu_0$ be the counting measure on

\[
N^1(\sqrt{\Gamma_{R,\epsilon}}) = \bigcup_{\gamma \in \Gamma_{R,\epsilon}} N^1(\sqrt{\gamma})
\]

given by the feet of pants in $\Pi_{R,\epsilon}$ and weighted by $\mu_0$. Then for any $\gamma \in \Gamma_{R,\epsilon}$, there exists a constant $K_\gamma \geq 0$, such that $\hat{\partial} \mu_0|_{N^1(\sqrt{\gamma})}$ is $KRe^{-qR}$-equivalent to $K_\gamma \lambda$ for some universal constant $K > 0$. Here $\lambda$ is the standard Lebesgue measure on $N^1(\sqrt{\gamma}) \cong \mathbb{C}/\text{hl}(\gamma)\mathbb{Z} + 2\pi i\mathbb{Z}$.

For two Borel measures $\mu$ and $\nu$ on a compact metric space $X$, we say $\mu$ and $\nu$ are $\delta$-equivalent for some $\delta > 0$ if the following conditions hold.

- $\mu(X) = \nu(X)$.
- For any Borel measurable subset $A \subset X$, $\mu(A) < \nu(N_\delta(A))$ holds.

Here $N_\delta(A)$ is the $\delta$-neighborhood of $A$ in $X$.

In the proof of the existence of good pants (curves) and the equidistribution result, the following exponential mixing property of the frame flow played a crucial role.
Theorem 2.7 ([Mo],[Po]). Let $M$ be a closed hyperbolic 3-manifold, $\mathcal{F}(M)$ be the frame bundle of $M$, $\Lambda$ be the Liouville measure on $\mathcal{F}(M)$ which is invariant under the frame flow $g_t : \mathcal{F}(M) \to \mathcal{F}(M)$.

Then there exists a constant $q > 0$ that depends only on $M$, such that the following statement holds. Let $\psi, \phi : \mathcal{F}(M) \to \mathbb{R}$ be two $C^1$ functions, then for any $r \in \mathbb{R}$,

$$\left| \Lambda(\mathcal{F}(M)) \int_{\mathcal{F}(M)} (g^r_t \psi) \cdot \phi \ d\Lambda - \int_{\mathcal{F}(M)} \psi \ d\Lambda \cdot \int_{\mathcal{F}(M)} \phi \ d\Lambda \right| \leq C e^{-q|r|}.$$ 

Here $C > 0$ only depends on the $C^1$-norms of $\psi$ and $\phi$.

For technical reasons, we need a slightly stronger condition than (1) in this paper, so we need the following proposition. In [Sa], Šarić has shown that we can strengthen $|hl(C) - \frac{R}{2}| < \epsilon$ to $|hl(C) - \frac{R}{2}| < \frac{\epsilon}{P}R$, and it is the essential part of the proof, so we only give a very brief proof here.

**Proposition 2.8.** For any closed hyperbolic 3-manifold $M$, there exists a constant $q > 0$ and a polynomial $P(\cdot)$, such that for every small enough $\epsilon > 0$ and large enough $R > 0$, the following statement holds. There exists a closed surface $S$ equipped with a pants decomposition $C$, and a viable map $f : S \to M$, such that for any $C \in C$, we have

$$\begin{align*}
|hl(C) - \frac{R}{2}| < \frac{\epsilon}{P}R, \\
|s(C) - 1| < P(R)e^{-qR} < \frac{\epsilon}{P}R.
\end{align*}$$

Moreover, $f_* : \pi_1(S) \to \pi_1(M)$ is injective.

**Proof.** In the introduction of [Sa], Šarić pointed out that $|hl(C) - \frac{R}{2}| < \epsilon$ can be replaced by $|hl(C) - \frac{R}{2}| < \frac{\epsilon}{P}R$. The main reason that such a refinement is applicable is, the exponential mixing property of frame flow ([Mo], [Po]) gives the exponential rate, which beats any polynomial rate.

More precisely, in Kahn and Markovic's construction in [KM1], the following function is a crucial ingredient. For an arbitrary point $F_0$ in the frame bundle $\mathcal{F}(\mathbb{H}^3)$, we can choose a $C^1$ bump function $f^F_\epsilon : \mathcal{F}(\mathbb{H}^3) \to \mathbb{R}_{\geq 0}$ supporting on the $\epsilon$-neighborhood of $F_0$ in $\mathcal{F}(\mathbb{H}^3)$ (here $\epsilon > 0$ is smaller than the injectivity radius of $M$), such that

$$\int_{\mathcal{F}(\mathbb{H}^3)} f^F_\epsilon(x)d\Lambda(x) = 1.$$ 

By pulling back $f^F_\epsilon$ by $Isom_+(\mathbb{H}^3)$ and projecting to $\mathcal{F}(M)$, we get a function $f^F_\epsilon : \mathcal{F}(M) \to \mathbb{R}_{\geq 0}$ centered at $F$ for each $F \in \mathcal{F}(M)$. Then Kahn and Markovic’s constructions of $(R,\epsilon)$-good pants and immersed almost totally geodesic surfaces start from the function $f_\epsilon$.

In [Sa], Šarić gave the following observation. For the time $t$ frame flow, we consider an alternative bump function $f_\epsilon^t$. By taking $f_\epsilon^t(x)$ to
be \( t^\theta \cdot f_\epsilon(xt) \) up to a constant close to 1, we can suppose

\[
\int_{\mathcal{F}(\mathbb{H}^3)} f_\epsilon(t) d\Lambda(x) = 1.
\]

Since the frame flow has exponential mixing rate in term of \( t \), while the constant \( C \) in Theorem 2.7 can be estimated by the \( H^2 \)-Sobolev norm of \( f_\epsilon \), which grows in a polynomially rate, so we have

\[
\text{when } t \text{ goes to } \infty. \text{ Then all the works in [KM1] are still available under the inequality (3), and } |\mathbf{h}(C) - \frac{\mathbf{R}}{\mathbf{F}}| < \frac{\epsilon}{\mathbf{F}} \text{ holds.}
\]

For the second inequality in (1), under the new bump function \( f_\mathbf{F} \) and inequality (3), the same argument as in [KM1] gives an integer valued measure \( \mu_0 \) on \( \Pi_{\mathbf{R}, \mathbf{F}} \), such that \( \hat{\mu}_0|_{\mathcal{X}^1(\mathbf{g})} \) is \( P'(\mathbf{R})e^{-q\mathbf{R}} \)-equivalent to \( K_{\mathbf{v}} \lambda \) for each \( \mathbf{v} \in \Gamma_{\mathbf{R}, \mathbf{F}} \), with \( P' \) being a polynomial and \( K_{\mathbf{v}} \geq 0 \).

So we can get a gluing of good pants with \(|s(C) - 1| < P'(\mathbf{R})e^{-q\mathbf{R}} < \frac{\mathbf{F}}{\mathbf{P}} \), and the proof of this proposition is done. q.e.d.

We also need the following two lemmas which are basic for our construction, and their proofs are closely related with Kahn and Markovic’s works in [KM1].

**Lemma 2.9.** For any positive integers \( m \geq 2 \) and \( q \geq 1 \), then for small enough \( \epsilon > 0 \) and large enough \( \mathbf{R} > 0 \), there exists a closed geodesic \( \mathbf{g}' \) in \( \mathbf{M} \), such that \(|L(\mathbf{g'}) - \frac{\mathbf{R} + 2\pi}{\mathbf{p}}| < \frac{\epsilon}{\mathbf{qR}} |\).

**Proof.** Take an arbitrary frame \( F = (p, \overline{v}, \overline{n}) \) in \( \mathcal{F}(\mathbf{M}) \), with \( p \in \mathbf{M} \), \( \overline{v}, \overline{n} \in T^1_p(\mathbf{M}) \) with \( \overline{v} \perp \overline{n} \). Let \( \overline{n}' \) be the \( \frac{2\pi}{\mathbf{p}} \) rotation of \( \overline{n} \) along \( \overline{v} \), and let \( F' = (p, \overline{v}, \overline{n}') \).

Now we consider functions \( f_{\epsilon\mathbf{R}}^{\mathbf{F}} \) and \( f_{\epsilon\mathbf{R}}^{\mathbf{F}'} \) as in the proof of Proposition 2.8. By applying equation (3), with \( t \) replaced by \( \frac{\mathbf{R}}{\mathbf{p}} \) and \( \epsilon \) replaced by \( \frac{\epsilon}{\mathbf{4p}^q \mathbf{}} \), there exists an oriented geodesic arc \( \mathbf{a} \) in \( \mathbf{M} \), with two frames \( \hat{F} \) and \( \hat{F}' \) at its initial and terminal points respectively, such that the following conditions hold.

1) The parallel transportation of \( \hat{F} \) along \( \mathbf{a} \) to its terminal point equals \( \hat{F}' \).

2) The first vector component of \( \hat{F} \) is tangent to \( \mathbf{a} \).

3) The distance between \( F \) and \( \hat{F} \), and the distance between \( F' \) and \( \hat{F}' \) in \( \mathcal{F}(\mathbf{M}) \) are both smaller than \( \frac{\epsilon}{\mathbf{qR}} \).

Then by connecting the initial and terminal points of \( \mathbf{a} \) by an \( \frac{\epsilon}{\mathbf{2qR}} \)-short geodesic in \( \mathbf{M} \), we get a closed path which is homotopic to a closed geodesic \( \mathbf{g}' \). Then this \( \mathbf{g}' \) satisfies \(|L(\mathbf{g'}) - \frac{\mathbf{R} + 2\pi}{\mathbf{p}}| < \frac{\epsilon}{\mathbf{qR}} |\), by elementary estimations in the hyperbolic geometry. q.e.d.
Lemma 2.10. There exists a universal constant $D > 0$, such that for any small enough $\epsilon > 0$ and large enough $R > 0$, the following statement holds. For any closed geodesic $\gamma \in \Gamma_{R, \frac{\pi}{R^3}}$, there exists a closed surface $S$ with a pants decomposition $C$, such that there exists an $(R, \frac{\pi}{R})$-almost totally geodesic immersion $f : S \looparrowright M$, and $f(C) = \gamma$ for some $C \in C$.

Proof. In [KM1], the integer valued measure $\mu_0$ in Proposition 2.6 is given by a real valued measure $\mu$ on $\Pi_{R, \epsilon}$ ($\Pi_{R, \frac{\pi}{R}}$ in our case), by first perturbing $\mu$ to a rational valued measure, then take an integer multiple.

So we need only to show that $\hat{\partial}\mu(N^1(\sqrt{\gamma})) > 0$. In this case, we can take a small enough perturbation of $\mu$ so that $\hat{\partial}\mu_0(N^1(\sqrt{\gamma})) > 0$ holds. Then in the construction of almost totally geodesic surfaces instructed by $\mu_0$, we must use some good pants with one of its cuff being $\gamma$. Then we take the component of the Kahn-Markovic surface which contains this good pants.

By the proof in Section 4.8 of [KM1], $\hat{\partial}\mu(N^1(\sqrt{\gamma})) > 0$ if and only if there exist two frames $F_1 = (p_1, \vec{v}_1, \vec{n}_1)$ and $F_2 = (p_2, \vec{v}_2, \vec{n}_2)$, and two geodesic arcs $\alpha_1$ and $\alpha_2$ in $M$, such that the following conditions hold.

1) $\alpha_1$ has initial point $p_1$ and terminal point $p_2$, while $\alpha_2$ has initial point $p_2$ and terminal point $p_1$.  
2) $\alpha_1\alpha_2$ is homotopic to $\gamma$.  
3) Let $\omega(F_1) = (p_1, \omega(\vec{v}_1), \vec{n}_1)$ be the $\frac{2\pi}{3}$-rotation of $F_1$ with respect to $\vec{n}_1$, and $\bar{\omega}(F_2) = (p_2, \bar{\omega}(\vec{v}_2), \vec{n}_2)$ be the $\frac{4\pi}{3}$-rotation of $F_2$ with respect to $\vec{n}_2$, then both $a_{\alpha_1}(F_1, F_2)$ and $a_{\alpha_2}(\omega(F_1), \bar{\omega}(F_2))$ are positive.

Under the modified Kahn-Markovic condition (2), for two frames $F_1$ and $F_2$ in $M$ with a geodesic arc $\alpha$ connecting their base points, $a_{\alpha}(F_1, F_2)$ is defined by the following way. Take two frames $\hat{F}_1$ and $\hat{F}_2$ in $\mathcal{F}(\mathbb{H}^3)$ projecting to $F_1$ and $F_2$ respectively, such that the geodesic arc in $\mathbb{H}^3$ connecting the base points of $\hat{F}_1$ and $\hat{F}_2$ projects to $\alpha$. Let $r = \frac{R}{2} + \ln \frac{4}{3}$, and let $g_\frac{r}{\sqrt{3}}(\hat{F}_1) = (p'_1, \vec{v}'_1, \vec{n}'_1)$, $g_\frac{r}{\sqrt{3}}(\hat{F}_2) = (p'_2, \vec{v}'_2, \vec{n}'_2)$, then $a_{\alpha}(F_1, F_2)$ is defined by:

$$a_{\alpha}(F_1, F_2) = \int_{\mathcal{F}(\mathbb{H}^3)} (g_\frac{r}{\sqrt{3}} f(p'_1, \vec{v}'_1, \vec{n}'_1)(x)) f(p'_2, -\vec{v}'_2, \vec{n}'_2)(x) d\Lambda(x).$$

Here $D > 0$ is some universal constant.

Then it is easy to check that if $\gamma \in \Gamma_{R, \frac{\pi}{R^3}}$, frames $F_1$ and $F_2$ satisfying the above conditions do exist.

q.e.d.

3. Geometric Constructions

In this section, for each closed hyperbolic 3-manifold $M$, we will construct an immersed $\pi_1$-injective 2-complex $X_p \looparrowright M$ which has good
pants as its building blocks. Here \( X_p \) is a local model of the homological \( \mathbb{Z}_p \)-torsion. Then we will show that the immersion \( X_p \hookrightarrow M \) provides homological torsion in some finite cover of \( M \).

### 3.1. Construction of a 2-complex

We first give a brief sketch of our construction \( X_p \hookrightarrow M \).

At first, there exists a Kahn-Markovic surface \( f : S \hookrightarrow M \), such that for some \( C \in \mathcal{C} \), \( f(C) \) goes along some closed geodesic \( \gamma' \) for \( p \) times with \( |I(\gamma') - \frac{R + 2\pi i}{p}| < \frac{\epsilon}{pD_\mathbb{R}} \) (here \( R > 0 \) is the constant in Lemma 2.10). Let \( \gamma \) be the closed geodesic which travels around \( \gamma' \) for \( p \) times, then \( \gamma \) is a nonprimitive closed geodesic with \( |I(\gamma) - R| < \frac{\epsilon}{pD_\mathbb{R}} \), so \( \gamma \in \Gamma_{R, \frac{\epsilon}{pD_\mathbb{R}}} \).

**Step I.** By Lemma 2.9, for any positive integer \( p \geq 2 \), there exists a closed geodesic \( \gamma' \) in \( M \) with \( |I(\gamma') - \frac{R + 2\pi i}{p}| < \frac{\epsilon}{pD_\mathbb{R}} \) (here \( D > 0 \) is the constant in Lemma 2.10). Let \( \gamma \) be the closed geodesic which travels around \( \gamma' \) for \( p \) times, then \( \gamma \) is a nonprimitive closed geodesic with \( |I(\gamma) - R| < \frac{\epsilon}{pD_\mathbb{R}} \), so \( \gamma \in \Gamma_{R, \frac{\epsilon}{pD_\mathbb{R}}} \).

**Step II.** By Lemma 2.10, there exists an immersed \( (R, \frac{\epsilon}{D_\mathbb{R}}) \)-almost totally geodesic closed surface \( f : S \hookrightarrow M^3 \), such that for the corresponding pants decomposition \( C \) of \( S \), there exists \( C \in \mathcal{C} \) such that \( f(C) = \gamma \). By taking a two-fold cover of \( S \) if necessary, we can suppose that \( C \) is a non-separating curve on \( S \) and the two pants adjacent to \( C \) are distinct. Let \( S' \) be the surface obtained from \( S \) by cutting along \( C \), then \( S' \) has an induced pants decomposition \( (S', C') \) (here \( C' \) does not contain the boundary of \( S' \)). Let \( C_1 \) and \( C_2 \) be the two boundary components of \( S' \), and they are given orientations such that \( [C_1] - [C_2] = 0 \in H_1(S'; \mathbb{Z}) \).

**Step III.** Let \( \rho_i : C_i \rightarrow C_i, i = 1, 2 \) be the \( \frac{2\pi}{p} \)-rotation on the circle. Then we define \( X_p \) to be the 2-complex obtained from \( S' \) quotient by the \( \rho_i \)-action for \( i = 1, 2 \), and let \( c_i \) be the oriented embedded circle in \( X_p \) which is the image of \( C_i \). Since \( C_1 \) and \( C_2 \) are both mapped to \( \gamma' \) for \( p \) times, \( f : S \hookrightarrow M \) induces a map \( f : X_p \hookrightarrow M \) with \( c_i \) mapped to \( \gamma' \) for \( i = 1, 2 \). Then the pants decomposition \( C' \) on \( S' \) and two curves \( C_1, C_2 \) induce a ”pants decomposition” on \( X_p \), which is denoted by \( (X_p, C', \{C_1, C_2\}) \). Note that \( C_1 \) and \( C_2 \) are not embedded curves in \( X_p \).

**Step IV.** Now we define a graph \( G(X_p) \) from \( (X_p, C', \{C_1, C_2\}) \). Vertices of \( G(X_p) \) are pants in \( X_p \), two vertices are connected by an edge if the corresponding two pants share some \( C \in C' \). \( G(X_p) \) is a trivalent graph except at two vertices \( v_1, v_2 \). These two vertices correspond with the two pants in \( X_p \) containing \( c_1 \) and \( c_2 \) respectively, and both of them are degree-2 vertices.
By a path in a graph $G$, we mean a sequence of oriented edges in $G$, such that for two adjacent edges $e$ and $e'$ in the path, the terminal vertex of $e$ equals the initial vertex of $e'$. For a path in $G$, its (combinatorial) length is defined to be the number of oriented edges it contains, counted with multiplicity. We say a path in $G$ is inessential, if its initial and terminal vertices are the same vertex $v$, and the corresponding map between topological spaces $(I, \partial I) \to (G, \{v\})$ is homotopic to the constant map. We say a path is essential if it is not inessential.

Let $l(G(X_p))$ be the length of the shortest essential path in $G(X_p)$ with end points in $\{v_1, v_2\}$, and $n(G(X_p))$ be the number of such paths. We define the complexity of $G(X_p)$ to be

$$c(G(X_p)) = (l(G(X_p)), -n(G(X_p)))$$

and we will do inductive constructions to make $G(X_p)$ more and more complicated until $l(G(X_p)) > \text{Re} \frac{R}{\pi}$. 

For any shortest essential path $\alpha$ with length $l \geq 1$, let $k = \lceil\frac{l+1}{2}\rceil$ and $e$ be the $k$-th edge on $\alpha$. Let $C_0 \in C'$ be the curve in the pants decomposition of $X_p$ corresponding with $e$, and let $C_0'$ denote the corresponding curve in $S$. Take a copy of $S$, and pass to a two-fold cover if necessary, such that $C_0'$ is a non-separating curve in $S$. Then we cut $X_p$ and $S$ along $C_0$ and $C_0'$ respectively, and re-paste them together to get a connected 2-complex $X_p'$ with an induced pants decomposition $C''$ (such kind of surgeries have appeared in [KM2]). Since the pants in $S$ and $X_p$ have the same feet on $N^1(\sqrt{f(C_0)})$, we still have

$$\begin{align*}
\left\{ \begin{array}{l}
|\text{hl}(C) - \frac{R}{2}| < \frac{R}{2 \pi}, \\
|s(C) - 1| < \frac{R}{2 \pi^2},
\end{array} \right.
\end{align*}$$

for any $C \in C''$.

After this surgery, the shortest essential paths in $G(X_p)$ going through the edge $e$ have been broken, and the corresponding length increases at least by 1 in $G(X_p')$. So the complexity $c(G(X_p')) = (l(G(X_p')), -n(G(X_p'))) > c(G(X_p)) = (l(G(X_p)), -n(G(X_p)))$, i.e. either $l(G)$ increases, or $l(G)$ does not change and $n(G)$ increases. After finitely many steps of such constructions, we can assure $l(G) > \text{Re} \frac{R}{\pi}$. For simplicity, we still denote the 2-complex by $X_p$, and denote the pants decomposition by $(X_p, C', \{C_1, C_2\}).$

**Definition 3.1.** A representation $\rho : \pi_1(X_p) \to PSL_2(\mathbb{C})$ is called a viable representation if the following conditions hold.

1) For each $C \in C \cup \{C_1, C_2\}$, let $g_C$ be a generator of $\pi_1(C)$, then $\rho(g_C)$ is a hyperbolic element in $PSL_2(\mathbb{C})$.

2) For each pants $\Pi$ in $X_p$, $\rho|_{\pi_1(\Pi)}$ is an injective map, and $\rho(\pi_1(\Pi))$ is a discrete subgroup of $PSL_2(\mathbb{C})$. 

VAG.
3) For any two pants \( \Pi, \Pi' \) sharing some \( C \in \mathcal{C} \cup \{C_1, C_2\} \), \( \text{hl}_{\Pi}(C) = \text{hl}_{\Pi'}(C) \) holds.

Note that although \( \rho(\pi_1(X_p)) \) may not be a discrete subgroup of \( \text{PSL}_2(\mathbb{C}) \), \( \text{hl}_{\Pi}(C) \) can still be defined.

A map \( f: X_p \to M \) is called a viable map if \( f_*: \pi_1(X_p) \to \pi_1(M) \subset \text{PSL}_2(\mathbb{C}) \) is a viable representation.

As a summary of the above construction, we have the following proposition which guarantees the existence of an immersed 2-complex \( X_p \hookrightarrow M \).

**Proposition 3.2.** For any closed hyperbolic 3-manifold \( M \), and any positive integer \( p \geq 2 \), there exists a constant \( \hat{\epsilon} > 0 \), such that for any \( 0 < \epsilon < \hat{\epsilon} \) and any \( R \) sufficiently large, the following statement holds. There exists a 2-complex \( X_p \) as above with a pants decomposition \((X_p, C', \{C_1, C_2\})\), and a viable map \( f: X_p \hookrightarrow M \) such that the following conditions hold.

1) The induced graph \( G(X_p) \) satisfies \( l(G(X_p)) > \frac{R}{4} \).
2) For any \( C \in C' \cup \{C_1, C_2\} \), \( \text{hl}(C) - \frac{R}{2} < \frac{\epsilon}{R} \).
3) For any \( C \in C' \), \( |s(C) - 1| < \frac{\epsilon}{R^2} \).
4) Let \( c_k \) be the image of \( C_k \) in the 2-complex \( X_p \), then \( |l(c_k) - \frac{R+2\pi i}{p}| < \frac{\epsilon}{pR} \) for \( k = 1, 2 \).

**Remark 3.3.** The construction of \( X_p \hookrightarrow M \) is very similar to the construction in [KM2].

In [KM2], Kahn and Markovic constructed immersed quasi-fuchsian surfaces \( S \hookrightarrow M \) by pasting (generalized) good pants, the immersion satisfies inequality (1) for curves \( C \in \mathcal{C} \), except bending along a sparse collection of curves in \( \mathcal{C} \) that are far away from each other. Our construction is almost following the same idea with theirs, but we construct immersed 2-complexes, instead of surfaces.

Moreover, the cut-and-paste technique in Step IV of our construction also appeared in [KM2]. In [KM2], Kahn and Markovic amalgamated two immersed almost totally geodesic surfaces to one immersed quasi-fuchsian surface, and they used the cut-and-paste technique to make sure that the bending curves are far away from each other.

### 3.2. Finite Cyclic Subgroups in Virtual Homology

The following theorem is the most technical theorem in this paper, which is an analogue of Theorem 2.2 in [KM1].

**Theorem 3.4.** There are universal constants \( \hat{\epsilon} > 0 \) and \( \hat{R} > 0 \) depend only on \( p \) and \( M \), such that for any \( 0 < \epsilon < \hat{\epsilon} \) and any \( R > \hat{R} > 0 \), the following statement holds. If \( X_p \) is a 2-complex with a pants decomposition \((X_p, C', \{C_1, C_2\})\) constructed as last section, and \( \rho : \pi_1(X_p) \to \text{PSL}_2(\mathbb{C}) \) is a viable representation such that the following conditions hold.
1) The induced graph $G(X_p)$ satisfies $l(G(X_p)) > \Re \rho$.  
2) For any $C \in \mathcal{C}' \cup \{C_1, C_2\}$, $|\mathbf{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R}$.
3) For any $C \in \mathcal{C}'$, $|s(C) - 1| < \frac{\epsilon}{R^2}$.
4) $|l(c_k) - \frac{R+2\pi i}{p}| < \frac{\epsilon}{pR}$ for $k = 1, 2$.

Then $\rho : \pi_1(X_p) \to \text{PSL}_2(\mathbb{C})$ is an injective map and $\rho(\pi_1(X_p))$ is a convex-cocompact subgroup of $\text{Isom}_+ (\mathbb{H}^3)$.

The proof of Theorem 3.4 will be deferred to Section 4. In this section, we will focus on proving Theorem 1.5 by assuming Theorem 3.4.

To precisely describe a viable representation $\rho : \pi_1(X_p) \to \text{PSL}_2(\mathbb{C})$ as in Theorem 3.4, we first give parameters for such a representation. For each curve $C \in \mathcal{C}'$, it is associate with two complex numbers $\xi_C$ and $\eta_C$ such that $|\xi_C|, |\eta_C| < \epsilon$; for each $C \in \{C_1, C_2\}$, it is associate with a complex number $\xi_i$ with $|\xi_i| < \epsilon$.

We can choose parameters such that the representation $\rho : \pi_1(X_p) \to \text{PSL}_2(\mathbb{C})$ satisfies the following conditions.

1) For any $C \in \mathcal{C}' \cup \{C_1, C_2\}$, $\mathbf{hl}(C) = \frac{R}{2} + \frac{\xi_C}{R}$.
2) For any $C \in \mathcal{C}'$, $s(C) = 1 + \frac{\eta_C}{R^2}$.
3) $l(c_k) = \frac{R+2\pi i}{p} + \frac{\xi_k}{pR}$ for $k = 1, 2$.

Let $\mathbb{D}(0, 1)$ be the disc in the complex plane centered at 0 with radius 1. Then for each $\tau \in \mathbb{D}(0, 1)$, there is a small deformation of $\rho$, denote by $\rho_\tau : \pi_1(X_p) \to \text{PSL}_2(\mathbb{C})$, which is defined by the following conditions.

1) For any $C \in \mathcal{C}' \cup \{C_1, C_2\}$, $\mathbf{hl}(C) = \frac{R}{2} + \frac{\tau C}{R}$.
2) For any $C \in \mathcal{C}'$, $s(C) = 1 + \frac{\tau C}{R^2}$.
3) $l(c_k) = \frac{R+2\pi i}{p} + \frac{\tau C}{pR}$ for $k = 1, 2$.

Then $\{\rho_\tau\}_{\tau \in \mathbb{D}(0, 1)}$ is a continuous family of representations from $\pi_1(X_p)$ to $\text{PSL}_2(\mathbb{C})$, such that $\rho_1 = \rho$, and $\rho_0$ provides us a standard model of studying the representation $\rho : \pi_1(X_p) \to \text{PSL}_2(\mathbb{C})$.

Let $\theta : \tilde{X}_p \to X_p$ be the universal cover of $X_p$. There is a natural map $\tilde{f}_0 : \tilde{X}_p \to \mathbb{H}^3$ to realize the representation $\rho_0$. $\tilde{f}_0$ maps each component of $\tilde{X}_p \setminus \theta^{-1}(c_1 \cup c_2)$ to a totally geodesic subsurface in $\mathbb{H}^3$, and two such totally geodesic subsurfaces sharing a geodesic has angle equal to $\frac{2\pi k}{p}$ for some integer $k \neq 0$. For each pants $\Pi \subset X_p$, the induced map $\Pi \to \mathbb{H}^3/\rho_0(\pi_1(\Pi))$ maps $\Pi$ to a totally geodesic pants with $\mathbf{hl}_\Pi(C) = \frac{R}{2}$. $\tilde{f}_0$ induces a path metric on $\tilde{X}_p$: for any $x, y \in \tilde{X}_p$, define $d(x, y) = \inf \{l(\tilde{f}_0(\gamma)) \mid \gamma \text{ is a path in } \tilde{X}_p \text{ with end points } x \text{ and } y\}$. By using elementary hyperbolic geometry, we have the following lemma.

**Lemma 3.5.** For $R$ large enough, if the induced graph $G(X_p)$ satisfies $l(G(X_p)) > \Re \frac{R}{2}$, then $\tilde{f}_0 : (\tilde{X}_p, d) \to (\mathbb{H}^3, d_{\mathbb{H}^3})$ is injective and is a
quasi-isometric embedding. In particular, \( \rho : \pi_1(X_p) \to PSL_2(\mathbb{C}) \) is an injective map.

**Proof.** For any two points \( x, y \in \tilde{X}_p \), the shortest path \( \alpha \) connecting \( x \) and \( y \) is a piecewise geodesic. Let \( L_1, L_2, \ldots, L_m \) be consecutive geodesics in \( q^{-1}(c_1 \cup c_2) \) intersecting with \( \alpha \), and let \( \alpha_i \) be the segment of \( \alpha \) between \( L_i \) and \( L_{i+1} \). Give an arbitrary orientation for each \( L_i \), then the angle between \( \alpha_{i-1}, L_i \) and the angle between \( \alpha_i, L_i \) sum to \( \pi \). Since \( l(c_k) = \frac{R + 2\pi i}{p} \), the angle between \( \tilde{f}_0(\alpha_{i-1}) \) and \( \tilde{f}_0(\alpha_i) \) in \( \mathbb{H}^3 \) is greater or equal to \( \frac{2\pi}{p} \).

For any \( i \in \{1, \ldots, m-1\} \), \( \alpha_i \) lies in a component of \( \tilde{X}_p \setminus q^{-1}(c_1 \cup c_2) \) and connects two components of \( q^{-1}(c_1 \cup c_2) \). So \( q(\alpha_i) \) is a homotopic nontrivial path in \( X_p \) with end points in \( c_1 \cup c_2 \), and it induces a path \( \beta_i \) in the graph \( G(X_p) \) with end points in \( \{v_1, v_2\} \).

If \( \beta_i \) is an essential path in \( G(X_p) \), since \( l(G(X_p)) > Re^\frac{R}{4} \), the combinatorial length of \( \beta_i \) is greater than \( Re^\frac{R}{4} \). Since the distance between two different cuffs in the pair of pants with cuff length \( R \) is roughly \( 2e^{-\frac{R}{4}} \), the length of \( \alpha_i \) is greater than \( R \). If \( \beta_i \) is an inessential path in \( G(X_p) \), then it contains a segment \( ee \) for some oriented edge \( e \) of \( G(X_p) \), or \( \beta_i \) is just a point. Since \( \alpha_i \) is a geodesic in \( \tilde{X}_p \), it contains a segment which is an essential path in the pair of pants with end points lying on the same cuff. Since the pants have cuff length \( R \), such a path has length greater than \( R/2 \). So in this case, \( \alpha_i \) has length greater than \( R/2 \).

As a summary, for any \( i \in \{1, \ldots, m-1\} \), \( \alpha_i \) has length greater than \( R/2 \).

Now the proof reduces to an elementary exercise in hyperbolic geometry. Let \( \alpha \) be a piecewise geodesic in \( \mathbb{H}^3 \) consists of geodesic segments \( \alpha_0, \ldots, \alpha_m \), and let the length of \( \alpha_i \) be \( l_i \). If \( l_i \geq R/2 \) for each \( i \in \{1, \ldots, m-1\} \), and if the angle between \( \alpha_{i-1} \) and \( \alpha_i \) is greater or equal to \( \frac{2\pi}{p} \) for \( i \in \{1, \ldots, m\} \). Then for large enough \( R \) (depending on \( p \)), the distance between the end points of \( \alpha \) in \( \mathbb{H}^3 \) is greater than \( \frac{1}{2} \sum_{i=0}^{m} l_i - \frac{R}{4} \).

Since \( d(x, y) = \sum_{i=0}^{m} l(\alpha_i) \), we have

\[
    d_{\mathbb{H}^3}(\tilde{f}_0(x), \tilde{f}_0(y)) \leq \sum_{i=0}^{m} d_{\mathbb{H}^3}(\tilde{f}_0(x_i), \tilde{f}_0(x_{i+1})) = \sum_{i=0}^{m} l(\alpha_i) = d(x, y)
\]

and

\[
    d_{\mathbb{H}^3}(\tilde{f}_0(x), \tilde{f}_0(y)) \geq \frac{1}{2} \sum_{i=0}^{m} d_{\mathbb{H}^3}(\tilde{f}_0(x_i), \tilde{f}_0(x_{i+1})) - \frac{R}{4} = \frac{1}{2} d(x, y) - \frac{R}{4}.
\]
So $\tilde{f}_0$ is a quasi-isometry, and $\rho$ is injective. The above inequality also implies that $\tilde{f}_0(x) \neq \tilde{f}_0(y)$ if $\alpha$ intersects with at least two components of $q^{-1}(c_1 \cup c_2)$, and the injectivity property obviously holds for the remaining cases.

We will first prove Theorem 1.5 for finite cyclic abelian groups. As we mentioned in the introduction, we will give two proofs here. The first proof will only prove a weaker statement: the finite cyclic group embeds into the virtual homology of any closed hyperbolic 3-manifold, but this proof is more geometric flavor. The second proof proves the original statement about virtual direct summand.

**Proposition 3.6.** For any finite cyclic abelian group $\mathbb{Z}_n$, and any closed hyperbolic 3-manifold $M$, $M$ admits a finite cover $N$ such that $\mathbb{Z}_n$ embeds into $\text{Tor}(H_1(N;\mathbb{Z}))$.

**Proof.** Let $p = 2n$, then Proposition 3.2 gives an immersed 2-complex $f : X_p \hookrightarrow M$ with a pants decomposition $(X_p, C', \{C_1, C_2\})$ such that $f_* : \pi_1(X_p) \to \pi_1(M)$ satisfies the conditions in Theorem 3.4.

By Lemma 3.5, $\rho_0(\pi_1(X_p))$ is a convex cocompact Kleinian group, so $\rho_0$ lies in $\text{int}(AH(\pi_1(X_p)))$. Here $AH(\pi_1(X_p))$ is the set of equivalent classes in

$$\{\rho : \pi_1(X_p) \to PSL_2(\mathbb{C}) | \rho \text{ is a discrete, faithful representation}\}/\sim,$$

and the relation is given by conjugations.

Since the map $f : X_p \hookrightarrow M$ induces a viable representation $f_* : \pi_1(X_p) \to \pi_1(M) \subset PSL_2(\mathbb{C})$ satisfying the assumption of Theorem 3.4, $f_*$ is $\pi_1$-injective.

We have pointed out that $f_*$ lies in a continuous family of viable representations $\rho_\tau : \pi_1(X_p) \to PSL_2(\mathbb{C})$ for $\tau \in \mathbb{D}(0, 1)$, with $\rho_1 = f_*$. By Theorem 3.4, $\{\rho_\tau(\pi_1(X_p))\}_{\tau \in \mathbb{D}(0, 1)}$ is a continuous family of convex cocompact Kleinian groups. So $f_*$ and $\rho_0$ lie in the same component of $\text{int}(AH(\pi_1(X_p)))$, and $\mathbb{H}^3/f_*(\pi_1(X_p))$ is homeomorphic to $\mathbb{H}^3/\rho_0(\pi_1(X_p))$.

Let $f_0 : X_p \to \mathbb{H}^3/\rho_0(\pi_1(X_p))$ be the map induced by $\tilde{f}_0$. Then Lemma 3.5 implies that $f_0$ is an embedding, and a neighborhood $\hat{M} = N(f_0(X_p))$ is a compact core of $\mathbb{H}^3/\rho_0(\pi_1(X_p))$. It is easy to figure out the topological type of $\hat{M}$.

Recall that $S'$ is an orientable surface with two oriented boundary components $C_1, C_2$ with $[C_1] - [C_2] = 0 \in H_1(S';\mathbb{Z})$, and $X_p$ is the quotient of $S'$. Let $V$ be the oriented solid torus and $\alpha \subset \partial V$ be the oriented $(p, 1)$-curve, then $\alpha$ has a neighborhood $\alpha \times [-1, 1] \subset \partial V$. Take two copies of $(V, \alpha \times [-1, 1])$, and denote them by $(V_1, \alpha_1 \times [-1, 1])$ and $(V_2, \alpha_2 \times [-1, 1])$ respectively. Let $\psi_1 : C_1 \to \alpha_1$ and $\psi_2 : C_2 \to \alpha_2$ be two orientation preserving homeomorphisms. Let $\phi_1 : C_1 \times [-1, 1] \to \alpha_1 \times [-1, 1]$ equals $\psi_1 \times id$, and $\phi_2 : C_2 \times [-1, 1] \to \alpha_2 \times [-1, 1]$ equals
ψ_2 \times (-id). Then \( \hat{M} = N(f_0(X_p)) \) is homeomorphic to \( V \cup_{\phi_1} S' \times [-1,1] \cup_{\phi_2} V_2 \) and the boundary of \( \hat{M} \) is incompressible. Such an \( \hat{M} \) is called a "book of I-bundles" by Culler and Shalen in [CS].

Since \( \mathbb{H}^3/f_*(\pi_1(X_p)) \) is homeomorphic to \( \mathbb{H}^3/\rho_0(\pi_1(X_p)) \), \( \mathbb{H}^3/f_*(\pi_1(X_p)) \) has a submanifold \( \hat{M}' \) homeomorphic to \( \hat{M} \). It is known that fundamental groups of hyperbolic 3-manifolds are LERF ([Ag], [Wi]), and by Scott’s criterion of LERF ([Sc]), \( M \) admits an intermediate cover \( N \rightarrow M \) of \( \mathbb{H}^3/f_*(\pi_1(X_p)) \rightarrow M \) such that \( \hat{M}' \) projects to \( N \) by homeomorphism.

By the description of the topological type of \( \hat{M}' \), \( \hat{M}' \) has only one boundary component, and \( H_1(\hat{M}';\mathbb{Z}) = \mathbb{Z}^{2g} \oplus \mathbb{Z} \oplus \mathbb{Z}_p \) for \( g = g(S) \). The \( \mathbb{Z} \)-component is generated by \( [c_1] \) and the \( \mathbb{Z}_p \)-component is generated by \( [c_1] - [c_2] \). Since the image of \( i_* : H_1(\partial \hat{M}' ; \mathbb{Z}) \rightarrow H_1(\hat{M}';\mathbb{Z}) \) is \( \mathbb{Z}^{2g} + \mathbb{Z}[pc_1] + \mathbb{Z}[[c_1] + [c_2]] \), it is easy to show that \( \mathbb{Z}_p \cap i_*(H_1(\partial \hat{M}' ; \mathbb{Z})) = \{0, \frac{p}{2}([c_1] - [c_2])\} \).

Then an M-V sequence argument shows that \( \mathbb{Z}_{p/2} = \mathbb{Z}_n \) embeds into \( H_1(N;\mathbb{Z}) \).

q.e.d.

Now let’s prove the statement for virtual direct summand.

**Proposition 3.7.** For any finite cyclic abelian group \( \mathbb{Z}_n \), and any closed hyperbolic 3-manifold \( M \), \( M \) admits a finite cover \( N \), such that \( \mathbb{Z}_n \) is a direct summand of Tor(\( H_1(N;\mathbb{Z}) \)).

**Proof.** Since \( M \) is a hyperbolic 3-manifold, \( \pi_1(M) \) is virtually compact special by [Ag] and [Wi]. So we can suppose \( \pi_1(M) \) is already the fundamental group of a compact special cube complex.

Let \( f : X_n \hookrightarrow M \) be the \( \pi_1 \)-injective immersion we have constructed in Proposition 3.6. Since \( f_*(\pi_1(X_n)) \) is a convex-cocompact subgroup of \( PSL_2(\mathbb{C}) \), it is a quasi-convex subgroup of the hyperbolic group \( \pi_1(M) \).

Since \( f_*(\pi_1(X_n)) \) is a quasi-convex subgroup of the special group \( \pi_1(M) \), \( f_*(\pi_1(X_n)) \) is a virtual retract of \( \pi_1(M) \) ([HW]), i.e. \( M \) admits a finite cover \( N \), such that the following conditions hold.

1) \( f_*(\pi_1(X_n)) \subset \pi_1(N) \);
2) There exists a retraction homomorphism \( r : \pi_1(N) \rightarrow \pi_1(X_n) \) such that \( r \circ f_*: \pi_1(X_n) \rightarrow \pi_1(X_n) \) is identity.

So we have the induced maps on homology:

\[ H_1(X_n;\mathbb{Z}) \xrightarrow{f_*} H_1(N;\mathbb{Z}) \xrightarrow{r_*} H_1(X_n;\mathbb{Z}). \]

Since the composition is the identity map and \( H_1(X_n;\mathbb{Z}) = \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_n \), we know that \( \mathbb{Z}^{2g+1} \oplus \mathbb{Z}_n \) is a direct summand of \( H_1(N;\mathbb{Z}) \). In particular, \( \mathbb{Z}_n \) is a direct summand of Tor(\( H_1(N;\mathbb{Z}) \)).

q.e.d.

**Remark 3.8.** If \( n \) is an odd number, the proof of Proposition 3.6 also shows that \( \mathbb{Z}_n \) is virtually a direct summand of the homology, by taking \( p = n \).
3.3. Finite Abelian Subgroups in Virtual Homology. We will finish the proof of Theorem 1.5 in this subsection. As in the finite cyclic group case, we will also give two proofs. One for virtual embedding, and the other one for virtual direct summand.

**Proposition 3.9.** For any finite abelian group \( A \), and any closed hyperbolic 3-manifold \( M \), \( M \) admits a finite cover \( N \), such that \( A \) embeds into \( \text{Tor}(H_1(N; \mathbb{Z})) \).

*Proof.* We will prove the statement by induction on the number of generators of the finite abelian group \( A \), and the proof of one-generator case has been done in Proposition 3.6.

For \( A = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \), take \( p_k = 2n_k \) for \( k = 1, 2 \). Then Proposition 3.2 and Theorem 3.4 provide us two immersed \( \pi_1 \)-injective 2-complexes \( f_1 : X_{p_1} \rightarrow M \) and \( f_2 : X_{p_2} \leftarrow M \).

Let \( G_1 = (f_1)_*(\pi_1(X_{p_1})) \) and \( G_2 = (f_2)_*(\pi_1(X_{p_2})) \) be the two convex cocompact subgroups of \( \pi_1(M) \) given by Theorem 3.4. Let \( M_k \) be the compact 3-manifold whose interior is homeomorphic to \( \mathbb{H}^3/G_k \) for \( k = 1, 2 \).

Take \( g \in \pi_1(M^3) \) such that both of the two limit points of \( g \) on \( S^2_\infty \) do not lie in the limit sets \( \Lambda(G_1) \) and \( \Lambda(G_2) \). Then for a large enough positive integer \( n \), by the Kleinian combination Theorem, \( j : G_1 * (g^nG_2g^{-n}) \rightarrow \pi_1(M^3) \) is an embedding, and \( j(G_1 * (g^nG_2g^{-n})) \) is a convex cocompact subgroup of \( \pi_1(M^3) \). So \( \mathbb{H}^3/j(G_1 * (g^nG_2g^{-n})) \) is homeomorphic to the interior of the boundary connected sum of \( M_1 \) and \( M_2 \).

Since hyperbolic 3-manifold groups are LERF, by running the argument in the proof of Proposition 3.6, we can find a finite cover \( N \) of \( M \), such that \( \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \) embeds into \( H_1(N; \mathbb{Z}) \).

For finite abelian groups with more generators, the result can be shown by induction as the two-generator case. q.e.d.

**Remark 3.10.** As in Remark 3.8, if the finite abelian group \( A \) has odd order, the proof of Proposition 3.9 also shows that \( A \) is a virtual direct summand.
4. The $\pi_1$-injectivity Property of Immersed 2-Complexes

This section is devoted to prove Theorem 3.4. Actually, we will show a $\pi_1$-injectivity result for more general immersed 2-complexes in closed hyperbolic 3-manifolds, with good pants as building blocks.

For the topological pair of pants $\Pi_0$, let $\partial_k \Pi_0, k = 1, 2, 3$ denote the three boundary components of $\Pi_0$. Suppose we have finitely many pair of pants $\mathcal{P} = \{\Pi_i\}_{i=1}^m$ and finitely many circles $\mathcal{C} = \{C_j\}_{j=1}^n$. By using these building blocks and some additional data, we will construct a 2-complex with a ”pants decomposition”.

**Definition 4.1.** For each pair $(i, k), i \in \{1, \cdots, m\}, k \in \{1, 2, 3\}$, suppose it is associated with a unique $j \in \{1, \cdots, n\}$ and a homeomorphism $f_{ik} : \partial_k \Pi_i \to C_j$. For each $C \in \mathcal{C}$, suppose it is associated with a positive integer $d_C > 0$.

Let $(\cup_{i=1}^m \Pi_i) \cup (\cup_{j=1}^n C_j) \to X'$ be the quotient space quotient by the relation given by $\{\phi_{ik}\}$. Let $X$ be a further quotient space of $X'$ quotient by the $\frac{2\pi}{d_C}$-rotation on each $C \in \mathcal{C}$, and let $q : (\cup_{i=1}^m \Pi_i) \cup (\cup_{j=1}^n C_j) \to X$ be the quotient map giving $X$.

For each circle $C \in \mathcal{C}$, let $D_C = d_C \cdot \# \{ (k, i) | \partial_k \Pi_i \text{ is mapped to } C \}$. For any point $x \in p(C) \subset X$, a neighborhood of $x$ in $X$ is homeomorphic to the quotient space of the union of $D_C$ half-discs, by identifying their diameters together. So $D_C$ measures the local singularity near $p(C)$.

Let $C_1 = \{ C \in \mathcal{C} | D_C = 2, d_C = 1 \}$ and $C_2 = \{ C \in \mathcal{C} | D_C > 2 \text{ or } d_C > 1 \}$. If $D_C > 1$ for each $C \in \mathcal{C}$ (X does not have ”boundary”), we say $X$ is a 2-complex with a pants decomposition $(X, C_1, C_2)$.

We will call the curves in $C_1$ regular curves, since each of these curves has a neighborhood in $X$ which is homeomorphic to the annulus. Curves in $C_2$ are called singular curves. We can also define a graph $G(X)$ from $X$ as in step IV of our construction of $X_\rho$ in Section 3.1. Here vertices are given by pants in $X$, and edges are given by regular curves. In $G(X)$, all the vertices are trivalent except those vertices corresponding with pants adjacent to singular curves. $l(G(X))$ can also be defined similarly, by considering the shortest essential path in $G(X)$ with end points corresponding with pants adjacent to singular curves. Let $S(X) = \max \{ D_C \}$, which measures the maximal singularity of $X$.

The following definition of viable representation for more general 2-complexes is almost the same with Definition 3.1.

**Definition 4.2.** A representation $\rho : \pi_1(X) \to \text{PSL}_2(\mathbb{C})$ is called viable if the following conditions hold.

1) For each $C \in C_1 \cup C_1$, let $g_C$ be a generator of $\pi_1(C)$, then $\rho(g_C)$ is a hyperbolic element in $\text{PSL}_2(\mathbb{C})$.

2) For each pants $\Pi$ in $X$, $\rho|_{\pi_1(\Pi)}$ is an injective map, and $\rho(\pi_1(\Pi))$ is a discrete subgroup of $\text{PSL}_2(\mathbb{C})$.  


3) For any two pants $\Pi, \Pi'$ sharing some $C \in C_1 \cup C_2$, $\text{hl}_{\Pi}(C) = \text{hl}_{\Pi'}(C)$ holds.

For a singular curve $C \in C_2$, let $N^1(\sqrt{C})$ be the half unit normal bundle of $C$ (under a viable representation $\rho$), then $N^1(\sqrt{C}) \cong \mathbb{C}/\text{hl}(C)\mathbb{Z} + 2\pi i\mathbb{Z}$. There is a flow $\Psi_t$ on $N^1(\sqrt{C})$ along the direction of $\text{hl}(C)$, and all the orbits of $\Psi_t$ are closed orbits. Suppose $\Pi_1, \cdots, \Pi_k$ are the pants adjacent to $C$, then $D_C = d_C \cdot k$. Let $\text{foot}_{\Pi_j}(C) \in N^1(\sqrt{C})$ be the foot of $\Pi_j$ on $C$.

Let $F' = \{\text{foot}_C(\Pi_j) + \frac{2\pi l}{d_C} | j = 1, \cdots, k; \ l = 1, \cdots, d_C\} \subset N^1(\sqrt{C})$. If a point appears more than once in the definition of $F'$, we will count it with multiplicity. Let $\Psi(F')$ be the union of closed orbits passing through $F'$ under the flow $\Psi_t$. Let $F = \Psi(F') \cap \{ti | t \in \mathbb{R}/2\pi\mathbb{Z}\} \subset N^1(\sqrt{C})$, and we will consider $F$ as a subset of $S^1 = \mathbb{R}/2\pi\mathbb{Z}$ (with multiplicity).

**Definition 4.3.** We say the pants $\Pi_1, \cdots, \Pi_n$ are $p$-separated along $C$ if the distance between any two distinct points in $F$ is greater or equal to $\frac{2\pi}{p}$ on $S^1$.

Then we can state our main technical theorem in this section.

**Theorem 4.4.** Fix an positive integer $p \geq 2$, there are universal constants $\hat{\epsilon} > 0$ and $\hat{R} > 0$ depending only on $p$, such that for any $0 < \epsilon < \hat{\epsilon}$ and $R > \hat{R} > 0$, the following statement holds. If $X$ is a connected 2-complex with a pants decomposition $(X, C_1, C_2)$, and $\rho : \pi_1(X) \to PSL_2(\mathbb{C})$ is a viable representation such that:

1) $S(X) \leq p$ and the induced graph $G(X)$ satisfies $l(G(X)) > R e^{\frac{R}{2}}$,

2) For any $C \in C_1 \cup C_2$, it satisfies the first modified Kahn-Markovic condition:

$$|\text{hl}(C) - \frac{R}{2}| < \frac{\epsilon}{R},$$

3) For any $C \in C_1$, it satisfies the second modified Kahn-Markovic condition:

$$|s(C) - 1| < \frac{\epsilon}{R^2},$$

4) For any $C \in C_2$, $|l(q(C)) - \frac{R + 2k\pi l}{d_C} | < \frac{\epsilon}{d_C R}$ for some $k$ coprime with $d_C$, and all the pants adjacent to $C$ are $p$-separated along $C$.

Then $\rho : \pi_1(X) \to PSL_2(\mathbb{C})$ is injective, and $\rho(\pi_1(X))$ is a convex-cocompact subgroup of $PSL_2(\mathbb{C})$.

It is obviously that Theorem 3.4 is a special case of Theorem 4.4, so we need only to prove Theorem 4.4 in the remaining part of this section.

We will prove Theorem 4.4 by showing that a partially defined map $i : \tilde{X} \to \mathbb{H}^3$ satisfying $i \circ x = \rho(x) \circ i$ for any $x \in \pi_1(X)$ is a quasi-isometric embedding. Actually, $i$ is defined on a 1-subcomplex of $\tilde{X}$, and $i$ maps each 1-cell in the 1-subcomplex to a geodesic arc in $\mathbb{H}^3$. 


For each component \( W \) of \( \tilde{X} \setminus p^{-1}(C_2) \), we will use the work in [KM1] and some classical results in hyperbolic geometry to show that \( i|_W : W \to \mathbb{H}^3 \) is a quasi-isometric embedding, by comparing \( i|_W \) to some quasi-isometric embedding. Then we estimate the “angle” between two adjacent components of \( \tilde{X} \setminus p^{-1}(C_2) \), and give it a definite lower bound. Given these estimations, we can show the globally quasi-isometric property, as the proof of Lemma 3.5.

### 4.1. Piecewise Linear Map is a Quasi-isometry.

To prove Theorem 4.4, we need to study more detail about Kahn-Markovic surfaces under the modified condition. The following Theorem in [KM1] helps us to understand the boundary behavior of the universal cover of Kahn-Markovic surfaces in \( \mathbb{H}^3 \).

**Theorem 4.5.** There are universal constants \( \hat{\epsilon}, \hat{R}, K_0 > 0 \), such that the following statement holds. For any \( \epsilon, R \) such that \( 0 < \epsilon < \hat{\epsilon} \) and \( R > \hat{R} \), suppose \( (S, C) \) is a closed surface with a pants decomposition \( C \), and \( f : S \dashrightarrow M \) is a viable map such that

\[
\begin{align*}
|\mathrm{hl}(C) - \frac{R}{2}| &< \epsilon, \\
|s(C) - 1| &< \frac{\epsilon}{R}
\end{align*}
\]

holds for any \( C \in C \). Then \( \rho = f_* : \pi_1(S) \to \PSL_2(\mathbb{C}) \) is injective and \( \rho(\pi_1(S)) \) is a quasi-fuchsian group.

Moreover, suppose \( S \) is endowed with the hyperbolic metric with \( \mathrm{hl}(C) = \frac{R}{2} \) and \( s(C) = 1 \), then \( \partial \tilde{f} : \partial S = \partial \mathbb{H}^2 \to \partial \tilde{M}^3 = \partial \mathbb{H}^3 \) extends to a \( \pi_1(S) \)-equivariant \( (1 + K_0\epsilon) \)-quasiconformal map \( \partial \tilde{g} : \partial \mathbb{H}^3 \to \partial \mathbb{H}^3 \) (by considering \( \pi_1(S) \subset \Isom_+((\mathbb{H}^2) \subset \Isom_+((\mathbb{H}^3)) \)).

Since we can suppose that \( f \) satisfies the modified Kahn-Markovic condition:

\[
\begin{align*}
|\mathrm{hl}(C) - \frac{R}{2}| &< \frac{\epsilon}{R}, \\
|s(C) - 1| &< \frac{\epsilon}{R^2},
\end{align*}
\]

the induced map \( \partial \tilde{f} : \partial S = \partial \mathbb{H}^2 \to \partial \tilde{M}^3 = \partial \mathbb{H}^3 \) extends to a \( \pi_1(S) \)-equivariant \( (1 + \frac{K_0\epsilon}{R}) \)-quasiconformal conjugacy between \( \rho_0(\pi_1(S)) \) and \( \rho(\pi_1(S)) \).

Let \( \rho_0 : \pi_1(S) \to \PSL_2(\mathbb{C}) \) be the representation near \( \rho \) satisfying \( \mathrm{hl}(C) = \frac{R}{2} \), \( s(C) = 1 \) for any \( C \in C \), then the map \( \partial \tilde{g} : S_\infty^2 \to S_\infty^2 \) is a \( (1 + \frac{K_0\epsilon}{R}) \)-quasiconformal conjugacy between \( \rho_0(\pi_1(S)) \) and \( \rho(\pi_1(S)) \).

Such a quasiconformal conjugacy between two Kleinian groups can be extended to a quasi-isometry from \( \mathbb{H}^3 / \rho_0(\pi_1(S)) \) to \( \mathbb{H}^3 / \rho(\pi_1(S)) \), and a quantitative version of this result is proved in [Th1] Chapter 11 and [McM] Corollary B.23.

**Theorem 4.6.** Let \( M_i = \mathbb{H}^3 / \Gamma_i \), \( i = 1, 2 \) be two hyperbolic 3-manifolds with isomorphic fundamental groups, and let \( \phi : S_\infty^2 \to S_\infty^2 \) be a \( K \)-quasiconformal conjugation between \( \Gamma_1 \) and \( \Gamma_2 \). Then \( \phi \) extends to
an equivariant $K^{\frac{3}{2}}$-quasi-isometry $\Phi: \mathbb{H}^3 \to \mathbb{H}^3$. In particular, $M_1$ and $M_2$ are $K^{\frac{3}{2}}$-quasi-isometric with each other.

The proof of Theorem 4.6 is quite differential geometry style, and the $L$-quasi-isometry showed up in the statement means $(L, 0)$-quasi-isometry in the coarse geometry sense. In the following part of this paper, we will use the language in the coarse geometry.

Given the previous a few theorems and the modified Kahn-Markovic condition, we have the following quick corollary.

**Corollary 4.7.** For any closed hyperbolic 3-manifold $M$, there exist constants $K_0, \hat{\epsilon} > 0, \hat{R} > 0$ such that the following statement hold. Suppose $S$ is a hyperbolic surface with a pants decomposition $C$, and $f: S \to M$ is a viable map satisfying the modified Kahn-Markovic condition

$$\left\{ \begin{array}{l} |\text{hl}(C) - \frac{R}{2}| < \frac{\hat{\epsilon}}{R}, \\ |s(C) - 1| < \frac{\hat{\epsilon}}{R^2}, \end{array} \right.$$ 

for each $C \in C$, with $0 < \epsilon < \hat{\epsilon}$ and $R > \hat{R}$.

Let $\rho = f_* : \pi_1(S) \to \text{PSL}_2(\mathbb{C})$ be the induced map on the fundamental group, and $\rho_0 : \pi_1(S) \to \text{PSL}_2(\mathbb{C})$ be the representation near $\rho$ and satisfying $\text{hl}(C) = \frac{R}{2}, s(C) = 1$.

Then there exists a $(1 + 2K_0\epsilon, 0)$-quasi-isometry $\tilde{g}: \mathbb{H}^3 \to \mathbb{H}^3$ such that $\tilde{g} \circ \rho_0(x) = \rho(x) \circ \tilde{g}$ for any $x \in \pi_1(S)$. In particular, there exists a $(1 + 2K_0\epsilon, 0)$-quasi-isometry $g: M_1 \to M_2$.

Although we know the existence of the equivariant quasi-isometry $\tilde{g}: \mathbb{H}^3 \to \overline{\mathbb{H}}^3$, we need to know more detail about it. So we will use a more concrete partially defined map to approximate $\tilde{g}$.

Let $(S, C)$ be the pants decomposition in Corollary 4.7. Endow $S$ with the hyperbolic metric with $\text{hl}(C) = \frac{R}{2}$ and $s(C) = 1$ for any $C \in C$, then $C$ is a union of simple closed geodesics on $S$.

Let $\tilde{C} \subset \mathbb{H}^2$ be the preimage of $C$. For each pair of pants $\Pi$ in $S$ with three cuffs $c_i$, $i = 1, 2, 3$, there are three seams $a_i$, $i = 1, 2, 3$ in $\Pi$ with $a_i$ connects $c_{i-1}$ and $c_{i+1}$. We suppose that $a_i$ is perpendicular with both $c_{i-1}$ and $c_{i+1}$ in the hyperbolic surface $S$. Let $A$ be the union of all the seams in $S$, and $\tilde{A} \subset \mathbb{H}^2$ be the preimage of $A$. Let $Y = \tilde{C} \cup \tilde{A} \subset \mathbb{H}^2 \subset \mathbb{H}^3$, then $Y$ is a $\rho_0(\pi_1(S))$-equivariant 1-subcomplex in $\mathbb{H}^3$, and $Y \subset \mathbb{H}^2$ gives a tessellation of $\mathbb{H}^2$ by isometric right-angled hexagons.

In Corollary 4.7, we can suppose $\tilde{g}: \mathbb{H}^3 \to \overline{\mathbb{H}}^3$ has been extended to a self-map on $\mathbb{H}^3 \cup S^2_\infty$ and we still denote it by $\tilde{g}$.

Let $Y'$ be the $\rho(\pi_1(S))$-equivariant 1-complex in $\mathbb{H}^3$ defined as the following. For any geodesic $\gamma \subset \tilde{C}$ with end points $x, y \in S^2_\infty$, we use $\partial \tilde{g}(\gamma)$ to denote the geodesic with end points $\tilde{g}(x)$ and $\tilde{g}(y)$. For any geodesic arc $\alpha \subset \tilde{A}$ orthogonal with geodesics $\gamma_1, \gamma_2 \subset \tilde{C}$, we use $\partial \tilde{g}(\alpha)$
to denote the common perpendicular geodesic of $\partial \tilde{g}(\gamma_1)$ and $\partial \tilde{g}(\gamma_2)$. Then we define $Y' = \partial \tilde{g}(\tilde{C}) \cup \partial \tilde{g}(\tilde{A})$.

Now we can define a piecewise linear $\pi_1(S)$-equivariant homeomorphism $h : Y \to Y'$. $h$ maps each $\alpha \subset \tilde{A}$ to $\partial \tilde{g}(\alpha)$ linearly, and $h$ maps each $\gamma \subset \tilde{C}$ to $\partial \tilde{g}(\gamma)$ piecewise linearly such that the restriction of $h$ on each component of $\gamma \setminus \tilde{A}$ is linear.

Since the modified Kahn-Markovic condition is satisfied by $\rho$, it is easy to check that $h$ is a $(1 + \frac{2\epsilon}{\eta}, 0)$-quasi-isometry on each geodesic segment of $Y = \tilde{C} \cup \tilde{A}$. However, we do not know whether $h : Y \to Y'$ is a globally quasi-isometry yet. By using the information given by $\tilde{g}$, we will show that $h$ is actually a quasi-isometry.

**Theorem 4.8.** Under the induced metric from $\mathbb{H}^3$, $h : Y \to Y'$ is a $(1 + \frac{2\epsilon}{\eta}, K(\epsilon + \frac{1}{\eta})\frac{1}{\delta})$-quasi-isometry for some universal constant $K > 0$.

We need to show the following elementary lemma first, such kind of statements are well-known and we give a quantitative version here.

**Lemma 4.9.** For small enough $\delta > 0$, suppose $\gamma : [0, n] \to \mathbb{H}^3$ is a $(1 + \delta, \delta)$-quasigeodesic in $\mathbb{H}^3$, and let $\tilde{\gamma}$ be the geodesic with end points $\gamma(0)$ and $\gamma(n)$, then $\gamma \subset N_\eta(\tilde{\gamma})$ for $\eta = 5\delta$.

**Proof.** Suppose $\gamma(t_0)$ is the point on $\gamma([0, n])$ which is the farthest one from $\tilde{\gamma}$, and let $D = d(\gamma(t_0), \tilde{\gamma})$. We can suppose $D > 2\delta$, or the lemma holds trivially. In this case, $t_0, n - t_0 > \frac{2\delta}{1 + \delta} > \delta^\frac{1}{2}$.

Let $t_1 = t_0 - \delta^\frac{1}{2}$, $t_2 = t_0 + \delta^\frac{1}{2}$, let $l_i = d(\gamma(t_0), \gamma(t_i))$ for $i = 1, 2$, and $d_i = d(\gamma(t_i), \gamma(t_0)) \leq D$. We use $x_j$ to denote the orthogonal projection of $\gamma(t_j)$ on $\tilde{\gamma}$ for $j = 0, 1, 2$.

For two points $x, y \in \mathbb{H}^3$, we will use $\overline{xy}$ to denote the geodesic segment with end points $x$ and $y$.

Let $\phi_i$ be the angle difference between $\overline{\gamma(t_0)x_0}$ and $\overline{\gamma(t_i)x_i}$ along geodesic $\tilde{\gamma}$ for $i = 1, 2$ (by using the parallel transportation from $x_0$ to $x_i$). Let $\theta_i$ be the angle between $\overline{\gamma(t_i)\gamma(t_0)}$ and $\overline{\gamma(t_0)x_0}$, and let $\theta$ be the angle between $\overline{\gamma(t_1)\gamma(t_0)}$ and $\overline{\gamma(t_2)\gamma(t_0)}$. Then $\theta \leq \theta_1 + \theta_2$.

A computation in hyperbolic geometry gives:

$$\cos \theta_i = \frac{\sinh D \cosh l_i - \sinh d_i \cos \phi_i}{\cosh D \sinh l_i} \geq \frac{\sinh D \cosh l_i - \sinh D}{\cosh D \sinh l_i} = \tanh D \tanh \frac{l_i}{2}.$$

$$\cos \theta = \frac{\sinh D \cosh l_1 - \sinh D}{\cosh D \sinh l_1} \geq \frac{2\delta^\frac{1}{2}}{1 + \delta} - \delta.$$
By the hyperbolic cosine law and (4):
\[
\cosh d(\gamma(t_1), \gamma(t_2)) = \cosh l_1 \cosh l_2 - \sinh l_1 \sinh l_2 \cos \theta \\
\leq \cosh l_1 \cosh l_2 - \sinh l_1 \sinh l_2 \cos (\theta_1 + \theta_2)
\]
\[
\leq \cosh l_1 \cosh l_2 + \sinh l_1 \sinh l_2 \left(1 - \tanh^2 D \tanh \frac{l_1}{2} \tanh \frac{l_2}{2}\right).
\]
So we have
\[
\tanh^2 D \leq \frac{\cosh(l_1 + l_2) - \cosh d(\gamma(t_1), \gamma(t_2))}{\sinh l_1 \sinh l_2 \tanh \frac{l_1}{2} \tanh \frac{l_2}{2}}.
\]
Since \(\frac{\delta}{1+\delta} - \delta \leq l_1, l_2 \leq (1 + \delta)\delta^{1/3} + \delta\) and \(d(\gamma(t_1), \gamma(t_2)) \geq \frac{2\delta}{1+\delta} - \delta\), we have:
\[
\tanh^2 D \leq \frac{\cosh (2(1 + \delta)\delta^{1/3} + 2\delta) - \cosh \left(\frac{2\delta}{1+\delta} - \delta\right)}{\sinh^2 \left(\frac{\delta}{1+\delta} - \delta\right) \tanh^2 \left(\frac{\delta}{2(1+\delta)} - \frac{\delta}{2}\right)} = 24\delta^2 (1 + O(\delta^{2/3})).
\]
So \(D = \sqrt{24\delta^2 (1 + O(\delta^{2/3}))} < 5\delta^{1/3}\). \hspace{1cm} \text{q.e.d.}

Since \(\tilde{g} : \mathbb{H}^3 \to \mathbb{H}^3\) is a \((1 + \frac{2K_0\epsilon}{\rho}, 0)\)-quasi-isometry, by Lemma 4.9, we know that \(\tilde{g}(\tilde{C}) \subset N_{\eta}(\partial \tilde{g}(\tilde{C}))\) for \(\eta = 5(\frac{2K_0\epsilon}{\rho})^{1/3}\). Let \(p : \tilde{g}(\tilde{C}) \to \partial \tilde{g}(\tilde{C})\) be the nearest point projection from \(\tilde{g}(\gamma)\) to \(\partial \tilde{g}(\gamma)\) for each \(\gamma \subset \tilde{C}\). Since \(p\) moves every point in \(\tilde{g}(\tilde{C})\) by at most \(\eta\). Let \(\tilde{g}' = p \circ \tilde{g}|_{\tilde{C}} : \tilde{C} \to \partial \tilde{g}(\tilde{C})\), then \(\tilde{g}'\) is a \(\pi_1(S)\)-equivariant \((1 + \frac{2K_0\epsilon}{\rho}, 2\eta)\)-quasi-isometry. We will compare \(\tilde{g}'\) and \(h\), which shows that \(h\) is a quasi-isometry.

Now we are ready to prove Theorem 4.8.

**Proof.** For an oriented geodesic \(\gamma \in \tilde{C}\) corresponding with a curve \(C \subset \mathcal{C}\), we will also use \(\gamma\) to denote the hyperbolic isometry corresponding with the oriented curve \(C \subset S\).

For any \(\alpha \in \mathcal{A}\) which is orthogonal to \(\gamma_1, \gamma_2 \in \tilde{C}\), we give orientations for \(\alpha, \gamma_1, \gamma_2\) as in Figure 1. Let \(x = \alpha \cap \gamma_2, y_1 = \gamma_1(x)\) and \(y_2 = \gamma_1(y_1)\).

![Figure 1](image)

We will compare \(\tilde{g}'(x)\) and \(h(x)\) on \(\partial \tilde{g}(\gamma_2)\), and show that they are close to each other.
Let $d_1 = d(\gamma_1, \gamma_2)$. By the hyperbolic cosine law of right-angled hexagons, and all the curves $C \subset C$ satisfy $\text{hl}(C) = \frac{R}{2}$, we have $\cosh d_1 = \frac{\cosh \frac{R}{2}}{\cosh \frac{R}{2} - 1}$. Let $d_2 = d(y_1, \gamma_1)$. Since $d(x, y) = R$, we have $\sinh d_2 = \cosh d_1 \cosh R$, and

\[
\sinh^2 d_2 = \cosh^2 R \frac{2 \cosh \frac{R}{2} - 1}{(\cosh \frac{R}{2} - 1)^2}.
\]

By computations in hyperbolic geometry of $\mathbb{H}^2$, we have

\[
cosh d(y_1, y_2) = \cosh^2 d_2 \cosh R - \sinh^2 d_2 = \cosh R + \cosh^2 R (\cosh R - 1) \frac{2 \cosh \frac{R}{2} - 1}{(\cosh \frac{R}{2} - 1)^2} = \frac{1}{2} e^\frac{R}{2} (1 + O(e^{-\frac{R}{2}})).
\]

So $d(y_1, y_2) = \frac{5}{2} R + O(e^{-\frac{R}{2}})$.

Now we think about the position of $\tilde{g}'(x)$ on $\partial \tilde{g}(\gamma_2)$. Let $l$ be the oriented distance between $h(x) = h(\gamma_2) \cap h(\alpha)$ and $\tilde{g}'(x)$ on $h(\gamma_2)$. We will prove that $l$ is very small.

Let $d'_1 = d_\alpha(\gamma_2, \gamma_1)$, by the hyperbolic cosine law of right-angled hexagons,

\[
d'_1 = 2e^{-\frac{R}{2} + \frac{1}{2\pi}(\epsilon_3 - \epsilon_2 - \epsilon_1)} + O(e^{-\frac{3R}{4}})
\]

for complex numbers $\epsilon_i$ with $|\epsilon_i| < \epsilon$ for $i = 1, 2, 3$. Let $d'_2 = d(\tilde{g}'(y_1), \partial \tilde{g}(\gamma_1))$, and let the real length of $\partial \tilde{g}(\gamma_2)$ be $R + \frac{2\epsilon_4}{R}$ for some real number $\epsilon_4$ with $|\epsilon_4| < \epsilon$, then

\[
\sinh^2 d'_2 = \cosh^2 \left( l + R + \frac{2\epsilon_4}{R} \right) \sinh^2 (\Re(d'_1)) + \sinh^2 \left( l + R + \frac{2\epsilon_4}{R} \right) \sin^2 (\Im(d'_1)) \geq \cosh^2 \left( l + R - \frac{2\epsilon}{R} \right) \sinh^2 \left( (2 + O(e^{-\frac{R}{2}}))e^{-\frac{R}{2} - \frac{3\pi}{2}} \right).
\]

Here $\Re(z)$ and $\Im(z)$ are the real and imaginary part of a complex number $z$ respectively.
Let \( R_1 = 1(h(\gamma_1)) \) be the complex translation length of \( h(\gamma_1) \), then |\( R_1 - R \)| < \( \frac{2\epsilon}{R} \). So by (10) and (11), we have

\[
(12) \quad \cosh (d(g'(y_1), g'(y_2))) = \cosh^2 d_2' \cosh \Re (R_1) - \sinh^2 d_2' \cos \Im (R_1)
\]

\[
\geq \cosh^2 d_2' \cosh \left( R - \frac{2\epsilon}{R} \right) - \sinh^2 d_2'
\]

\[
= \cosh \left( R - \frac{2\epsilon}{R} \right) + \left( \cosh \left( R - \frac{2\epsilon}{R} \right) - 1 \right) \sinh^2 d_2'
\]

\[
\geq \left( \cosh \left( R - \frac{2\epsilon}{R} \right) - 1 \right) \cosh^2 \left( l + R - \frac{2\epsilon}{R} \right) \sinh^2 \left( (2 + O(e^{-\frac{R}{2}})) e^{-\frac{R}{4} - \frac{\eta}{2}} \right)
\]

\[
+ \cosh \left( R - \frac{2\epsilon}{R} \right)
\]

\[
= \frac{1}{2} e^{2l + \frac{5\eta}{2} - \frac{\eta}{R}} (1 + O(e^{-\frac{R}{2}})).
\]

So \( d(g'(y_1), g'(y_2)) \geq |2l + \frac{5\eta}{2} - \frac{\eta}{R} + O(e^{-\frac{R}{2}})| \). Since \( g' \) is a \((1 + 2\frac{K_0\epsilon}{R}, 2\eta)\)-quasi-isometry and \( d(y_1, y_2) = \frac{5}{2} R + O(e^{-\frac{R}{2}}) \), the following inequality holds:

\[
(13) \quad \left( 1 + \frac{2K_0\epsilon}{R} \right) \left( \frac{5}{2} R + O(e^{-\frac{R}{2}}) \right) + 2\eta \geq |2l + \frac{5\eta}{2} - \frac{\eta}{R} + O(e^{-\frac{R}{2}})|.
\]

By considering \( y_1' = \gamma_2^{-1}(x) \) and \( y_2' = \gamma_1^{-1}(y_1) \) and run the same argument, we have

\[
(14) \quad \left( 1 + \frac{2K_0\epsilon}{R} \right) \left( \frac{5}{2} R + O(e^{-\frac{R}{2}}) \right) + 2\eta \geq \left| -2l + \frac{5\eta}{2} - \frac{\eta}{R} + O(e^{-\frac{R}{2}}) \right|.
\]

By (13) and (14) and \( \eta = 5(\frac{2K_0\epsilon}{R})^{\frac{1}{2}} \), we get

\[
(15) \quad |l| \leq \frac{5}{2} K_0 \epsilon + \frac{9\epsilon}{2R} + \eta + O(e^{-\frac{R}{2}}) \leq K_1 \left( \epsilon + \frac{1}{R} + \left( \frac{\epsilon}{R} \right)^{\frac{1}{2}} \right).
\]

Since \( l \) is the oriented distance between \( g'(x) \) and \( h(x) \), \( d(g'(x), h(x)) \leq K_1(\epsilon + \frac{1}{R} + \left( \frac{\epsilon}{R} \right)^{\frac{1}{2}}) \) holds for any \( x \in \tilde{C} \cap \tilde{A} \).

Since \( g' : \tilde{C} \to \partial \tilde{\gamma}(\tilde{C}) \) is a \((1 + 2\frac{K_0\epsilon}{R}, 2\eta)\)-quasi-isometry and the restriction of \( h : \tilde{C} \to \partial \tilde{\gamma}(\tilde{C}) \) on each single geodesic \( C \subset \tilde{C} \) is a \((1 + 2\frac{K_2\epsilon}{R}, 0)\)-quasi-isometry. So \( d(g'(y), h(y)) \leq 2\epsilon + 2K_0\epsilon + 2\eta + K_1(\epsilon + \frac{1}{R} + \left( \frac{\epsilon}{R} \right)^{\frac{1}{2}}) \) for each \( y \in \tilde{C} \).

So \( h|_{\tilde{C}} : \tilde{C} \to \partial \tilde{\gamma}(\tilde{C}) \) is a \( \pi_1(S) \)-equivariant \((1 + 2\frac{K_0\epsilon}{R}, K_2(\epsilon + \frac{1}{R} + \left( \frac{\epsilon}{R} \right)^{\frac{1}{2}}))\)-quasi-isometry. Since \( \tilde{C} \subset Y \) is \( 2e^{-\frac{R}{4}} \)-dense, \( h : Y \to Y' \) is a \((1 + 2\frac{K_0\epsilon}{R}, K_3(\epsilon + \frac{1}{R} + \left( \frac{\epsilon}{R} \right)^{\frac{1}{2}}))\)-quasi-isometry.

q.e.d.
4.2. Estimation of the Angle Change. Now we consider two points $x, y \in Y$ with $x$ lying on a geodesic $\gamma \subset \tilde{C}$ and $d(x, y) \geq \frac{R}{2}$. We give $\gamma$ an arbitrary orientation. Let $\alpha \subset \tilde{A}$ be the geodesic arc which is at the same side of $\gamma$ as $\overline{xy}$ on $\mathbb{H}^2$, such that it is the closest such geodesic arc to $x$ (it is possible there are two choices and we choose either of them). Let $\bar{e}_1 \in T^1_x(\mathbb{H}^2)$ be the unit vector in the direction of $\gamma$, $\bar{e}_2 \in T^1_x(\mathbb{H}^3)$ be the unit vector in the direction of the parallel transportation of $\alpha$ to $x$, and take the third unit vector $\bar{e}_3 \in T^1_x(\mathbb{H}^3)$ such that $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$ is an orthonormal frame of $T_x(\mathbb{H}^3)$ and gives the orientation of $\mathbb{H}^3$. Let $\bar{e}$ be the unit vector in $T^1_x(\mathbb{H}^3)$ in the direction of $\overline{xy}$.

Now we define $\Theta(\gamma, \alpha, \overline{xy}) = (\theta, \phi) \in \mathbb{R}^2$ for $\theta = \angle(\bar{e}, \bar{e}_1) \in [0, \pi]$ and $\phi = \angle(\bar{e}, \bar{e}_3) \in [0, \pi]$. $\Theta(\gamma, \alpha, \overline{xy})$ measures the direction of $\bar{e}$ under the coordinate $(\bar{e}_1, \bar{e}_2, \bar{e}_3)$. Since $\rho_0(\pi_1(S))$ is a Fuchsian group, $\phi = \frac{\pi}{2}$. $\Theta(h(\gamma), h(\alpha), h(x)h(y)) = (\theta', \phi') \in \mathbb{R}^2$ is defined similarly.

The main result of this subsection is the following statement.

**Theorem 4.10.** There exists constants $\hat{\epsilon} > 0$, $\hat{R} > 0$ depend only on $p$, such that for any $0 < \epsilon < \hat{\epsilon}$ and $R > \hat{R}$, $|\Theta(\gamma, \alpha, \overline{xy}) - \Theta(h(\gamma), h(\alpha), h(x)h(y))| \leq \frac{1}{p}$.

Since each hexagon in $\mathbb{H}^2 \setminus Y$ has diameter $\leq \frac{R}{2} + 1$, $\overline{xy} \cap Y = \{x_0, x_1, \cdots, x_n\}$ for $x = x_0, y = x_n$ and $d(x_i, x_{i+1}) \leq \frac{R}{2} + 1$. Let $h(x_i)h(x_{i+1})$. There is a natural piecewise linear map $h' : \overline{xy} \to h(\overline{xy})$ such that $h'(x_i) = h(x_i)$. Since $h : Y \to Y'$ is a $(1 + \mu_1, \mu_2)$-quasi-isometry for $\mu_1 = \frac{K\epsilon}{R}$, and $d(x_i, x_{i+1}) \leq \frac{R}{2} + 1$, it is easy to check that $h'$ is a $(1 + \mu_1, 3\mu_2 + 4K\epsilon)\text{-quasi-isometry}$. Since $\mu_2 = K(\epsilon + \frac{1}{R})^\frac{1}{2}$, $h'$ is a $(1 + \mu_1, 10\mu_2)\text{-quasi-isometry}$ in particular.

Let $x_k$ be the point in $\{x_0, \cdots, x_n\}$ which is the nearest one to $x$ and such that

$$l(p) \leq d(x, x_k) \leq R$$

holds for $l(p) = \frac{1}{10000p^2}$. If $x_k \in \alpha \subset \tilde{A}$, let $x'$ be one of the intersection points of $\alpha \cap \tilde{C}$. If $x_k$ does not lie in $\tilde{A}$, simply let $x' = x_k$. By Lemma 4.9 and the choice of $x'$, we have $d(h(x'), h(x)h(y)) \leq 5(\mu_2)^\frac{1}{2} + 2e^{-\frac{p}{4}} \leq 6(\mu_2)^\frac{1}{2}$. Since $d(h(x), h(x')) \geq \frac{l(p)}{1+\mu_1} - \mu_2 - 2e^{-\frac{p}{4}} \geq \frac{l(p)}{2}$, by the hyperbolic sine law,

$$\sin (\angle h(x')h(x)y) = \frac{\sinh (d(h(x'), h(x)h(y)))}{\sinh (d(h(x), h(x')))}$$

$$\leq \frac{\sinh 6(\mu_2)^\frac{1}{2}}{\sinh \frac{l(p)}{2}} \leq \frac{15(\mu_2)^\frac{1}{2}}{l(p)}.$$
So $\angle h(x') h(x) h(y) \leq \frac{20(\mu_2)^{\frac{1}{2}}}{l(p)}$, and the same argument shows $\angle x' x y \leq \frac{20(\mu_2)^{\frac{1}{2}}}{l(p)}$. Now it suffices to show that

$$|\Theta(\gamma, \alpha, x' x') - \Theta(h(\gamma), h(\alpha), h(x) h(x'))| \leq \frac{1}{2p}.$$ 

Even if $x'$ may not be same with $x_k$, we will abuse the notation and still use $x_0 = x, x_1, \ldots, x_k = x'$ to denote the intersection points in $\overline{xx'} \cap Y$. Moreover, we still use the notation $\Theta(\gamma, \alpha, x' x') = (\theta, \phi)$ and $\Theta(h(\gamma), h(\alpha), h(x) h(x')) = (\theta', \phi')$, with $\phi = \frac{\pi}{2}$.

**Proposition 4.11.** Let $\theta$ and $\theta'$ be the first coordinate of $\Theta(\gamma, \alpha, x' x')$ and $\Theta(h(\gamma), h(\alpha), h(x) h(x'))$ respectively, then $|\theta - \theta'| \leq \frac{1}{4p}$.

**Proof.** Let $d_1 = d(x, x')$, $d_2 = d(x', \gamma)$ and $d_1' = d(h(x), h(x'))$, $d_2' = d(h(x'), h(\gamma))$, then $d_2 \leq d_1$ and $d_2' \leq d_1'$. Since $h : Y \to Y'$ is a $(1 + \mu_1, \mu_2)$-quasi-isometry, and $l(p) \leq d_1 \leq R$, we have $|d_1 - d_1| \leq K\epsilon + \mu_2$ and $|d_2' - d_2| \leq K\epsilon + \mu_2$. In particular, $d_1' \geq \frac{1}{2} l(p)$.

Since $\sin \theta = \frac{\sinh d_2}{\sinh d_1'}$ and $\sin \theta' = \frac{\sinh d_2'}{\sinh d_1}$, we have the following estimation:

$$|\sin \theta - \sin \theta'| = \frac{\sinh d_2 \cdot |\sinh d_1 - \sinh d_1'| + \sinh d_1 \cdot |\sinh d_2 - \sinh d_2'|}{\sinh d_1 \sinh d_1'}$$

$$\leq \frac{|d_1 - d_1'| \cdot \cosh (\max(d_1, d_1')) + |d_2 - d_2'| \cdot \cosh (\max(d_2, d_2'))}{\sinh d_1'}$$

$$\leq 2(K\epsilon + \mu_2) \frac{\cosh (\max(d_1, d_1'))}{\sinh d_1'}$$

$$\leq 2(K\epsilon + \mu_2) e^{K\epsilon + \mu_2} \coth d_1'$$

$$\leq 2(K\epsilon + \mu_2) e^{K\epsilon + \mu_2} \coth \left(\frac{1}{2} l(p)\right).$$

Let $\nu = 2(K\epsilon + \mu_2) e^{K\epsilon + \mu_2} \coth \left(\frac{1}{2} l(p)\right)$. If both $\theta$ and $\theta'$ are acute angles, then

$$\nu \geq |\sin \theta - \sin \theta'| = 2 \sin \frac{\theta - \theta'}{2} \cos \frac{\theta + \theta'}{2} \geq 2 \sin^2 \frac{\theta - \theta'}{2} \geq \frac{(\theta - \theta')^2}{8}.$$

So in this case $|\theta - \theta'| \leq \sqrt{8\nu}$.

Without lose of generality, we can suppose that $\theta$ is an acute angle. If $\theta'$ is also acute, the above inequality gives $|\theta - \theta'| \leq \sqrt{8\nu} \leq \frac{1}{4p}$, so the lemma is proved.

If $\theta'$ is not acute, we will show that $\theta$ is very close to $\pi/2$. Let $\beta$ be the subray of $\gamma$ starting at $x$ and has acute angle with $xx'$. In
this case, the angle between $h(x)h(x')$ and $h(\beta)$ is an obtuse angle, so $d(h(x'), h(\beta)) = d(h(x'), h(x))$. By this geometric observation, we have:

$$d(x', \gamma) = d(x', \beta) \geq \frac{d(h(x'), h(\beta))}{1 + \mu_1} - \mu_2$$

(20)

$$= \frac{d(h(x'), h(x))}{1 + \mu_1} - \mu_2 \geq \frac{d(x', x)}{(1 + \mu_1)^2} - 2\mu_2.$$

Since $d(x', x) \leq R$ and $\mu_1 = \frac{K\epsilon}{R}$, $d(x', \gamma) \geq \frac{d(x', x)}{(1 + \mu_1)^2} - 2\mu_2 \geq d(x', x) - (3K\epsilon + 2\mu_2)$. So

$$\sin \theta = \frac{\sinh d(x', \gamma)}{\sinh d(x', x)} \geq \frac{\sinh (d(x', x) - (3K\epsilon + 2\mu_2))}{\sinh (d(x', x))}$$

(21)

$$\geq \frac{\sinh (l(p) - (3K\epsilon + 2\mu_2))}{\sinh (l(p))} \geq 1 - (3K\epsilon + 2\mu_2) \coth (l(p)).$$

So $\frac{\pi}{2} - \theta \leq 2\sqrt{(3K\epsilon + 2\mu_2) \coth (l(p))} \leq \frac{1}{8p}$, and the same argument shows that $\theta' - \frac{\pi}{2} \leq \frac{1}{8p}$. So we have $|\theta' - \theta| \leq \frac{1}{4p}$, q.e.d.

Since the second coordinate of $\Theta(\gamma, \alpha, \overline{x^'}) = (\theta, \phi)$ is $\phi = \frac{\pi}{2}$, we need only to estimate the second coordinate $\phi'$ of $\Theta(h(\gamma), h(\alpha), h(x)h(x')) = (\theta', \phi')$.

**Proposition 4.12.** For small enough $\epsilon > 0$ and large enough $R$, we have $|\phi' - \frac{\pi}{2}| \leq \frac{1}{4p}$.

We first need the following local estimation.

**Lemma 4.13.** If $\alpha \subset \partial \tilde{g}(\tilde{A})$ is the common perpendicular of geodesics $\gamma_1, \gamma_2 \subset \partial \tilde{g}(\tilde{C})$ such that $z_1 = \gamma_1 \cap \alpha$, and $y \in \gamma_1$ is a point on $\gamma_1$ such that $d = d(y, z_1) \leq R$. Let $P_1$ be the hyperbolic plane containing $\gamma_1$ and $z_2$, $P_2$ be the hyperbolic plane containing $\gamma_2$ and $y$, and $\psi$ be the angle between $P_1$ and $P_2$, then $\psi \leq \frac{10\epsilon}{R^2}$.

**Proof.** Let $\beta = \angle z_1z_2y$, and $b + i\xi = d_\alpha(\gamma_1, \gamma_2)$. Here we choose orientations for $\alpha, \gamma_1, \gamma_2$ such that $b > 0$ and $\xi$ is close to 0.

By the hyperbolic cosine law of right-angled hexagons,

$$d_\alpha(\gamma_1, \gamma_2) = b + i\xi = 2e^{-\frac{R}{4}} + \frac{\sin^{\frac{3}{2}} - \sin^{\frac{1}{2}}}{2R} + O(e^{-\frac{3R}{4}}) + O(e^{-\frac{3R}{4}}) \frac{\epsilon}{R} i,$$

(22)

here the two $O(e^{-\frac{3R}{4}})$s are two different real functions. So $b \geq e^{-\frac{R}{4}}$, and $|\xi| \leq \frac{4\epsilon}{R} e^{-\frac{R}{4}}$.

An elementary computation gives $\cos \psi = \frac{\cos \beta \cos \xi}{\sqrt{1 - \sin^2 \beta \cos^2 \xi}}$, and

$$\sin^2 \psi = \frac{\sin^2 \xi}{1 - \sin^2 \beta \cos^2 \xi}.$$

(23)
For the angle $\beta$, we have the following estimation:

$$\sin^2 \beta = \frac{\sinh^2 d}{\sinh^2 d(y, z_2)} = \frac{\cosh^2 d - 1}{\cosh^2 d \cosh^2 b - 1} \leq \frac{\cosh^2 R - 1}{\cosh^2 R \cosh^2 (e^{-\frac{R}{4}}) - 1} = 1 - e^{-\frac{R}{2}} + O(e^{-R}).$$

(24)

By (23) and (24)

$$\sin^2 \psi \leq \frac{\sin^2 \xi}{1 - (1 - e^{-\frac{R}{2}} + O(e^{-R})) \cos^2 \xi} \leq \frac{\sin^2 \left(\frac{4\epsilon e^{-\frac{R}{4}}}{R}\right)}{1 - (1 - e^{-\frac{R}{2}} + O(e^{-R})) \cos^2 \left(\frac{4\epsilon e^{-\frac{R}{4}}}{R}\right)} \leq \frac{32\epsilon^2}{R^2}.$$

(25)

So $\psi \leq \frac{10\epsilon}{R}$.

q.e.d.

Now we are ready to prove Proposition 4.12.

**Proof.** Let $x = x_0, x_1, \ldots, x_k = x'$ be the consecutive intersection points of $xx'$ with $Y = \tilde{C} \cup \tilde{A}$, and $y_i = h(x_i)$. Let $C_i$ be the geodesic (segment) in $\partial \tilde{g}(\tilde{C})$ or $\partial \tilde{g}(\tilde{A})$ containing $y_i$.

Let $P_0$ be the hyperbolic plane containing $C_0 = h(\gamma)$ and $h(\alpha)$, $P_0'$ be the hyperbolic plane containing $C_0$ and $y_1$. Let $P_i$ be the hyperbolic plane containing $C_i$ and $y_{i-1}$, and $P_i'$ be the hyperbolic plane containing $C_i$ and $y_{i+1}$ for $i = 1, \ldots, k - 1$. When considering about $\rho_0$, all the corresponding hyperbolic planes coincide with each other. So for each hyperbolic plane $P_i$ and $P_i'$, we can give it an orientation such that the corresponding orientations coincide with each other, when considering $\rho_0$.

Since $d(x, x_{k-1}) \leq l(p) \leq \frac{1}{10000p^2}$, by Lemma 2.3 of [KM1], $k \leq (2 + R)e^5l(p) \leq 500RL(p)$. The possible position of $y_{i-1}, y_i$ and geodesics $C_{i-1}, C_i$ looks like a) or b) in Figure 2 for $i = 1, \ldots, k - 1$. By the choice of $x'$, the possible position of $y_{k-1}, y_k$ and $C_{k-1}, C_k$ looks like a), b) or c) in Figure 2.

Now we study two possible positions of $y_{i-1}, y_i$ and $C_{i-1}, C_i$. 

![Figure 2](image-url)
i) For three points \(a, b, c \in \mathbb{H}^3\) not lying on a geodesic, let \(P_{abc}\) be the hyperplane in \(\mathbb{H}^3\) containing these three points. Then in Figure 2 a), since \(d(y_i, a), d(y_i, b) < R\), Lemma 4.13 implies:

\[
\angle(P_{ay_i-1y_i}, P_{aby_i}) \leq \angle(P_{ay_i-1y_i}, P_{aby_i}) + \angle(P_{aby_i}, P_{aby_{i-1}})
\]

\[
\leq \frac{10\epsilon}{R} + \frac{20\epsilon}{R} = \frac{30\epsilon}{R},
\]

and

\[
\angle(P'_{i-1}, P_i) = \angle(P_{ay_i-1y_i}, P_{by_{i-1}y_i})
\]

\[
\leq \angle(P_{ay_i-1y_i}, P_{aby_{i-1}}) + \angle(P_{aby_{i-1}}, P_{aby_{i-1}}) \leq \frac{30\epsilon}{R}.
\]

ii) In Figure 2 c), let \(\beta\) be the geodesic arc in \(\partial \tilde{g}(\tilde{A})\) containing \(y_{k-1}\) and we use \(\beta'\) to denote the bi-infinite geodesic containing \(\beta\). Let \(P_{\beta', c}\) be the hyperbolic plane containing \(\beta'\) and \(c\), and \(l = d(d, P_{\beta', c})\), then

\[
sinh l \leq \sinh(\frac{\epsilon}{R}) \sinh(3e^{-\frac{R}{4}}) \leq \frac{4\epsilon}{R} e^{-\frac{R}{4}}.
\]

So \(l \leq \frac{4\epsilon}{R} e^{-\frac{R}{4}}\). Since \(d(d, \beta') \geq \frac{R}{2} - 1\),

\[
\sin \angle(P_{\beta', c}, P_{\beta', d}) \leq \frac{\sinh(\frac{4\epsilon}{R} e^{-\frac{R}{4}})}{\sinh(\frac{R}{2} - 1)} \leq \frac{50\epsilon}{R} e^{-\frac{3R}{4}}.
\]

So \(\angle(P_{\beta', c}, P_{\beta', d}) \leq \frac{100\epsilon}{R} e^{-\frac{3R}{4}}\). The same argument shows that

\[
\angle(P_{\beta', c}, P_{\beta', f}) \leq \frac{100\epsilon}{R} e^{-\frac{3R}{4}}.
\]

Since \(\angle(P_{\beta', c}, P_{\beta', e}) \leq \frac{4\epsilon}{R} e^{-\frac{R}{4}}\) and by monotonicity, we have \(\angle(P_{\beta', c}, P_{\beta', y_k}) \leq \frac{\epsilon}{R}\).

Then by a routine case-by-case argument, and using results in i) and ii), we get that:

\[
\angle(P'_{i-1}, P_i) \leq \frac{100\epsilon}{R}
\]

for \(i = 1, \cdots, k - 1\), and

\[
\angle(P_i, P'_i) \leq \frac{100\epsilon}{R}
\]

for \(i = 0, \cdots, k - 1\).

Now let \(l_i = d(y_i, y_{i+1})\). Then by (16),

\[
\sum_{i=0}^{k-2} l_i \leq 2l(p)
\]

and

\[
\frac{l(p)}{2} \leq \sum_{i=0}^{k-1} l_i \leq R + 1.
\]
Let $\vec{v}_i \in T^1_{y_i}(\mathbb{H}^3)$ be the unit normal vector of $P_i$ at $y_i$ and $\vec{v}'_i \in T^1_{y_i}(\mathbb{H}^3)$ be the unit normal vector of $P'_i$ at $y_i$. Let $z_i$ be the orthogonal projection of $y_i$ to $P_0$ and $\vec{n}_i$ be the unit normal vector of $P_0$ at $z_i$. For $x, y \in \mathbb{H}^3$ and $\vec{\sigma} \in T_x(\mathbb{H}^3)$, we will use $\vec{\sigma}@y$ to denote the parallel transportation of $\vec{\sigma}$ to $y$ along the geodesic arc $xy$, as in [KM1].

Claim 1: For $i = 0, \cdots , k - 1$, the following inequalities hold:

$$\angle(\vec{v}_i @ y_0, n_0) \leq \frac{200i\epsilon}{R} + \sum_{j=0}^{i-1} l_j,$$

$$\angle(\vec{v}'_i @ y_0, n_0) \leq \frac{(200i + 100)\epsilon}{R} + \sum_{j=0}^{i-1} l_j.$$

We will prove Claim 1 by induction. The statement holds for $i = 0$ since $\angle(P_0, P'_0) \leq \frac{100\epsilon}{R}$. Suppose the statement holds for $i = m$, then for $i = m + 1$,

$$\angle(\vec{v}_{m+1} @ y_0, n_0) \leq \angle(\vec{v}_{m+1}, \vec{v}'_{m+1} @ y_{m+1}) + \angle(\vec{v}_m @ y_{m+1} @ y_0, \vec{n}_0) \leq \angle(P_{m+1}, P'_m) + \angle(\vec{v}'_m @ y_0, \vec{n}_0) + \angle(\vec{v}_m @ y_{m+1} @ y_0, \vec{v}'_m @ y_0).$$

The first term is less than $\frac{100\epsilon}{R}$ by (30), the second term is less than $\frac{(200m+100)\epsilon}{R} + \sum_{j=0}^{m-1} l_j$ by induction hypothesis, and the third term is less than $l_m$ by Proposition 4.1 of [KM1]. So

$$\angle(\vec{v}_{m+1} @ y_0, \vec{n}_0) \leq \frac{200(m+1)\epsilon}{R} + \sum_{j=0}^{m} l_j.$$

Since $\angle(\vec{v}_{m+1}, \vec{v}'_{m+1}) \leq \frac{100\epsilon}{R}$, the second inequality holds for $i = m + 1$ and the proof of Claim 1 is done.

Now we estimate $\angle(\vec{v}_m @ z_m, \vec{n}_m)$ for $m = 0, \cdots , k - 1$. Since $k \leq 500Rl(p)$, we have:

$$\angle(\vec{v}_m @ z_m, \vec{n}_m) \leq \angle(\vec{v}'_m @ z_m, \vec{v}'_m @ z_0 @ z_m) + \angle(\vec{v}'_m @ z_0 @ z_m, \vec{n}_m) \leq d(y_0, y_m) + \angle(\vec{v}'_m @ y_0, \vec{n}_0) \leq 2l(p) + \frac{(200m + 100)\epsilon}{R} + \sum_{j=0}^{m-1} l_j \leq 4l(p) + 2 \cdot 10^5 \epsilon l(p) \leq 10l(p).$$

Claim 2: For $d_i = d(y_i, z_i) = d(y_i, P_0)$, $i = 0, \cdots , k - 1$, we have:

$$d(y_i, z_i) \leq \frac{1}{100p} \sum_{j=0}^{i-1} l_j.$$

We will also prove Claim 2 by induction. The inequality holds trivially for $i = 0$. Suppose the inequality holds for $i = m$. For $i = m + 1$,
Since \( \vec{v}_m \perp \overline{y_my_{m+1}} \) and \( \angle(\vec{v}_m \perp z_m, \vec{n}_m) \leq 10l(p) \), an elementary computation gives:

\[
\sinh d(y_{m+1}, z_{m+1}) = \sinh d(y_m, z_m) \cosh l_m + \cosh d(y_m, z_m) \sinh l_m \sin \theta_m
\]

(40)

for some \( \theta_m \) with \( |\theta_m| \leq \angle(\vec{v}_m \perp z_m, \vec{n}_m) \leq 10l(p) \) and \( l_m \leq 2l(p) \).

Since \( d(y_m, z_m) \), \( l_m \leq 2l(p) \), we have:

\[
\sinh d(y_{m+1}, z_{m+1}) \leq \sinh d(y_m, z_m)(1 + l_m^2) + (1 + 4l(p^2)) \cdot 2l_m \cdot 10(p)
\]

(41)

\[
\leq \sinh d(y_m, z_m) + 2d(y_m, z_m)l_m^2 + (1 + 4l(p^2)) \cdot 2l_m \cdot 10l(p)
\]

\[
= \sinh d(y_m, z_m) + l_m (2d(y_m, z_m)l_m + 20l(p)(1 + 4l(p^2)) \leq \sinh d(y_m, z_m) + 100l(p) \cdot l_m.
\]

So \( d(y_{m+1}, z_{m+1}) \leq d(y_m, z_m) + 100l(p) \cdot l_m \leq d(y_m, z_m) + \frac{1}{100p} l_m \), and the proof of Claim 2 is done.

The final computation is to estimate \( \angle z_0y_0y_k \).

Since \( h \) is a \( (1 + \frac{K_0}{R}, K(\epsilon + \frac{1}{R})^2) \)-quasi-isometry, and \( \frac{l(p)}{2} \leq \sum_{i=0}^{k-1} l_i \leq R \), \( d(y_0, y_k) \geq \max(\frac{1}{2} \sum_{i=0}^{k-1} l_i, l_{k-1} - 1) \). Moreover, \( d(y_k, z_k) \) is given by:

\[
\sinh d(y_k, z_k) = \sinh d(y_{k-1}, z_{k-1}) \cosh l_{k-1} + \cosh d(y_{k-1}, z_{k-1}) \sinh l_{k-1} \sin \theta_{k-1}
\]

for some \( \theta_{k-1} \) with \( |\theta_{k-1}| \leq 10l(p) \).

If \( l_{k-1} \leq 2, (41) \) still works with \( m \) replaced by \( k - 1 \), so \( d(y_k, z_k) \leq \frac{1}{100p} \sum_{i=0}^{k-1} l_i \). Then

\[
\sin \angle z_0y_0y_k = \frac{\sinh d(y_k, z_k)}{\sinh d(y_k, y_0)} \leq \frac{\sinh \frac{\sum_{i=0}^{k-1} l_i}{100p}}{\sinh \frac{\sum_{i=0}^{k-1} l_i}{2}} \leq \frac{1}{10p}.
\]

If \( l_{k-1} \geq 2 \), we have

\[
\sin d(y_k, z_k) \leq 4l(p) \cosh l_{k-1} + 20l(p) \sinh l_{k-1} \leq 20l(p) \cosh l_{k-1}.
\]

Then

\[
\sin \angle z_0y_0y_k = \frac{\sinh d(y_k, z_k)}{\sinh d(y_k, y_0)} \leq \frac{20l(p)e^{l_{k-1}}}{\sinh l_{k-1} - 1} \leq \frac{20l(p)e^2}{\sinh 1} \leq \frac{1}{50p}.
\]

So in both of these cases, \( |y' - \frac{\pi}{2}| = \angle z_0y_0y_k \leq \frac{1}{10p} \). q.e.d.
4.3. Proof of Theorem 4.4. Given Theorem 4.10, we are ready to prove Theorem 4.4.

Proof. Given the estimations in Theorem 4.10, the proof here is similar to the proof of Lemma 3.5, so we will only point out the necessary modifications.

In Theorem 4.4, if conditions 2), 3) and 4) are replaced by $h\ell(C) = \frac{R}{2}$, $s(C) = 1$ and $l(q(C)) = \frac{R+2k+1}{dC}$ respectively, then we denote the corresponding representation by $\rho_0$. Since $l(G(X)) > Re\frac{R}{2}$, the same argument as in Lemma 3.5 shows that Theorem 4.4 is true for $\rho_0$.

Let $q : \tilde{X} \to X$ be the universal cover. When considering $\rho_0$, since $l(G(X)) > Re\frac{R}{2}$, there is a $\pi_1(X)$-equivariant embedding $\tilde{X} \to \mathbb{H}^3$ with respect to $\rho_0$, and the image is a locally finite union of subsets of hyperbolic planes. There are two metrics on $\tilde{X}$, one is the induced metric $d_0$ from $\mathbb{H}^3$ and the other one is the path metric $d$ induced by $d_0$. The proof of Lemma 3.5 implies that these two metrics are quasi-isometric, and we will always endow $\tilde{X}$ with the metric $d$. $d$ is a geodesic metric on $X$ which is locally the hyperbolic metric away from singular curves.

Let $\tilde{C} \subset \tilde{X} \subset \mathbb{H}^3$ be the preimage of the pants decomposition $C_1 \cup C_2$ of $X$, which are union of geodesics. For each pants $\Pi$ in $X$, there are three seams which are the common perpendiculars of the three pair of cuffs of $\Pi$. Let $A \subset X$ be the union of all such geodesic arcs and let $\tilde{A} \subset \tilde{X} \subset \mathbb{H}^3$ be the preimage of $A$ in $\mathbb{H}^3$. We define $Z = \tilde{C} \cup \tilde{A}$, then the embedding of $Z$ into $\mathbb{H}^3$ is $\pi_1(X)$-equivariant with respect to $\rho_0$.

Now we turn to study a general representation $\rho : \pi_1(X) \to PSL_2(\mathbb{C})$, which is a deformation of $\rho_0$. For each geodesic $C \subset \tilde{C}_2$, let $C'$ be the corresponding geodesic in $\mathbb{H}^3$ with respect to $\rho$, and let $\tilde{C}'$ be the union of such $C'$ for all $C \subset \tilde{C}$. Let $\tilde{A}'$ be the union of common perpendiculars of geodesics in $\tilde{C}'$, which correspond with geodesic arcs in $\tilde{A}$. Let $Z' = \tilde{C}' \cup \tilde{A}'$, then $Z'$ is $\rho(\pi_1(X))$-equivariant with respect to $\rho$.

There is a natural piecewise linear map $h : Z \to Z'$ defined as in Section 4.1, such that the restriction of $h$ on each geodesic arc in $\tilde{A}$ is linear, and the restriction of $h$ on each component of $\tilde{C} \setminus \tilde{A}$ is linear. To show the convex cocompact property of $\rho$, we need only to show that $h : (Z, d) \to (Z', d_{\mathbb{H}^3}|_{Z'})$ is a quasi-isometry.

For any $x, y \in Z$ with $d(x, y) \geq R$, let $\alpha$ be the shortest path in $\tilde{X}$ connecting $x$ and $y$. If $\alpha \cup q^{-1}(C_2) = \emptyset$, the Theorem 4.8 implies the quasi-isometric property. So we assume $\alpha \cap q^{-1}(C_2)$ is not empty, and the intersection points are $x_1, x_2, \ldots, x_k$. Since $l(G(X)) > Re\frac{R}{2}$, $d(x_i, x_{i+1}) \geq \frac{R}{2}$ for $i = 1, \ldots, k-1$. Let $x_0 = x$ if $d(x, x_1) \geq \frac{R}{2}$, or let $x_0 = x_1$. Similarly, define $x_{k+1} = y$ if $d(x_k, y) \geq \frac{R}{2}$, or let $x_{k+1} = x_k$. Now we consider about angles $\angle h(x_{i-1})h(x_i)h(x_{i+1})$. 

\[
\angle h(x_{i-1})h(x_i)h(x_{i+1}) \leq \frac{\pi}{2}.
\]
Let $\gamma_i$ be the geodesic in $Z$ containing $x_i$, with a preferred orientation, and let $\theta$ be the angle between $\gamma_i$ and $\overline{x_{i-1}x_i}$. Since $\alpha$ is a shortest path in $\tilde{X}$, the angle between $\gamma_i$ and $\overline{x_ix_{i+1}}$ equals $\pi - \theta$. Without lose of generality, we can suppose $\theta \leq \frac{\pi}{2}$.

Let $\theta_1$ be the angle between $h(\gamma_i)$ and $\overline{h(x_{i-1})h(x_i)}$, and $\theta_2$ be the angle between $h(\gamma_i)$ and $\overline{h(x_i)h(x_{i+1})}$. If $\theta \leq \frac{\pi}{2} - \frac{3}{2p}$, then by Theorem 4.10, we have $\theta_1 \leq \frac{\pi}{2} - \frac{1}{2p}$ and $\theta_2 \geq \frac{\pi}{2} + \frac{1}{2p}$, so $\angle h(x_{i-1})h(x_i)h(x_{i+1}) \geq \frac{1}{p}$.

If $\frac{\pi}{2} - \frac{3}{2p} \leq \theta \leq \frac{\pi}{2}$, then $\frac{\pi}{2} - \frac{5}{2p} \leq \theta_1 \leq \frac{\pi}{2} + \frac{1}{p}$ and $\frac{\pi}{2} - \frac{1}{p} \leq \theta_2 \leq \frac{\pi}{2} + \frac{5}{2p}$.

Let $\alpha_i$ be the component of $A$ which is on the same component of $\tilde{X} \setminus q^{-1}(C_2)$ as $x_{i-1}$, intersecting with $\gamma_i$, and is the closest such arc from $x_i$. We also choose $\alpha'_i$ by the same way with $x_{i-1}$ replaced by $x_{i+1}$. Let $P_i$ be the hyperbolic plane containing $h(\gamma_i)$ and $h(\alpha_i)$, and $P'_i$ be the hyperbolic plane containing $h(\gamma_i)$ and $h(\alpha'_i)$. Since pants adjacent to the same singular curve are $p$-separated, $\angle(P_i, P'_i) \geq \frac{2\pi}{p}$.

Let $\vec{n}_i$ and $\vec{n}'_i$ be the normal vectors of $P_i$ and $P'_i$ at $h(x_i)$ respectively, then $\angle(\vec{n}_i, \vec{n}'_i) \geq \frac{2\pi}{p}$. Theorem 4.10 implies that, $|\angle(h(x_{i-1})h(x_i), \vec{n}_i) - \frac{\pi}{2}| \leq \frac{1}{p}$ and $|\angle(h(x_i)h(x_{i+1}), \vec{n}'_i) - \frac{\pi}{2}| \leq \frac{1}{p}$. An elementary computation implies that

$$\angle h(x_{i-1})h(x_i)h(x_{i+1}) \geq \cos \frac{1}{p} \sin \frac{2\pi}{p} - \frac{2}{p} \geq \left( \cos \frac{1}{p} - 1 \right) \frac{2}{p} > 0.$$}

The remaining proof is same with the proof of Lemma 3.5. q.e.d.

**Remark 4.14.** Given the cited theorems, our proof of Theorem 4.4 is very elementary and quite geometric flavor, but a little bit tedious. It is possible to give alternative proofs by using geometric group theory or using the method in [Sa].

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