Note on the representation of the gap formation probability for real and quaternion Wishart matrices

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Wishart random matrices are often used to model multivariate systems in physics, finance, biology and wireless communication. Extreme value statistics, such as those of the smallest eigenvalue, can be used to test the accuracy of the model. In this article we study the gap formation probability (cumulative distribution function of the smallest eigenvalue) for real and quaternion $N \times (N + \nu)$ Wishart random matrices in the large $N$ limit. We derive compact expressions in terms of determinants of known functions. As a consequence of these representations, the gap formation probabilities solve the Toda lattice equation, in the index $\nu$ for $\nu$ even and for $\nu$ odd separately.

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Introduction.— Wishart random matrices with real, complex and quaternion entries are used to model the statistics of data and systems in a wide variety of disciplines. They are used to model time series in financial data\textsuperscript{3–8}, human EEG data\textsuperscript{15}, the bipartite entanglement for a generic quantum system\textsuperscript{5,21} or the Hamiltonian of topological insulators. In QCD they are referred to as Chiral Ensembles and they describe the Dirac spectrum while in multivariate analysis they are used for principal components analysis of large data sets\textsuperscript{2}. In multivariate analysis the Wishart ensemble with correlation matrix equal to the identity is called the null case and knowledge about the null case allows one to perform tests of the null hypothesis on data.

For most of these applications the matrices are real which makes the real Wishart model perhaps the most interesting and there has been a continued effort\textsuperscript{3,6–8} to characterize it. One of the key quantities that has been studied is the distribution of the smallest eigenvalue. In general, the smallest eigenvalue serves as an estimator for particular quantities of interest under consideration. For example, in quantum mechanics, when considering the question of how entangled a generic bipartite system is when in a random pure state, the eigenvalues of the reduced density matrix of one of the systems determines the degree of entanglement\textsuperscript{5,21}. If the smallest eigenvalue is zero the state space of the density matrix loses effectively a dimension and is therefore the system is less entangled. If the smallest eigenvalue acquires its maximum value, all eigenvalues must be equal and therefore it is fully entangled. The smallest eigenvalue therefore gives information on the degree of entanglement.

Our focus will be on the compact representations of the distribution of the smallest eigenvalue, $P_{\nu}^{(3)}(s)$ and the gap formation probability, $Q_{\nu}^{(3)}(s)$, in the large $N$ limit, for real ($\beta = 1$) and quaternion ($\beta = 4$) matrices. Since $P_{\nu}^{(3)}(s)$ can be computed from $Q_{\nu}^{(3)}(s)$ by differentiation, we will only discuss the representation of the gap formation probability. We will show that the gap formation probability $Q_{\nu}^{(3)}(s)$ can be represented in compact form as a determinant, for any integer $\nu$. The appearance of a determinant representation reflects strikingly different integrable properties then those known. As a result of this representation it will be clear that the nearest neighboring (in the index $\nu$) gap formation probabilities, $Q_{\nu}^{(1)}(s)$ and $Q_{\nu+2}^{(1)}(s)$, are linked together via the Toda lattice equation.

Known results.— Real Wishart random matrices have been extensively studied with regards to the statistics of the smallest eigenvalue and the gap formation probability and there are a wide variety of results. For $\nu$ odd, $P_{\nu}^{(1)}(s)$ and $Q_{\nu}^{(1)}(s)$ have a representation in terms of Hypergeometric function of matrix arguments\textsuperscript{12,23} and a representation in terms of a Pfaffian was derived in\textsuperscript{22,24}, where the dimension of the matrix in the Pfaffian is $(\nu + 1) \times (\nu + 1)$. It is not until recently that a theorem where $\nu$ is even was finally tackled\textsuperscript{16,20} and further Pfaffian forms were uncovered, where the dimension of the matrix in the Pfaffian is $\frac{n}{2} \times \frac{n}{2} \times \left( \frac{n}{2} + 1 \right) \times \left( \frac{n}{2} + 1 \right)$ when $\frac{n}{2}$ is even(odd). Much is also known of the behavior of the gap probability, $Q_{\nu}^{(3)}(s)$, in terms of solutions to the Painlevé V equation\textsuperscript{11,13}, i.e. it was shown that, for arbitrary real $\nu$, $Q_{\nu}^{(3)}(s)$ solves a differential equation with some given boundary conditions. Our analysis will be almost completely based on these results. In a nutshell we will put forward an ansatz for $Q_{\nu}^{(1)}(s)$ and $Q_{\nu}^{(3)}(s)$ and show they solve these differential equations with the same boundary conditions.

Although we analyze the case where the Wishart matrix average correlation function is the identity it worthy to note that numerical evidence has shown\textsuperscript{17,20} that the distribution of the smallest eigenvalue is universal, meaning it remains unchanged even after introduction of non trivial correlations.

Representation of $Q_{\nu}^{(1)}(s)$ as an integral over the symplectic group for $\nu$ odd.— In the Wishart Ensembles the $N \times (N + \nu)$ random matrix $W$ has real, complex or quaternion ($\beta = 1, 2$ or $4$) entries which are Gaussian distributed.

$$P(WW^\dagger) \sim e^{-\frac{1}{4} \text{Tr}[WW^\dagger C^{-1}]}$$

where $C$ is the average correlation matrix and taken to
be $\Delta_N$ here. The joint probability distribution function (j.p.d.f.) of the eigenvalues of $WW'$ is known to be

$$P(WW') = \frac{1}{C_{N,\nu}} |\Delta_N(w_k)|^\beta \prod_{j=1}^{N} e^{-\frac{1}{2} w_j} w_j^{(\nu+1)-1}. \quad (2)$$

with the Vandermonde determinant given by

$$\Delta_N(w_k) = \prod_{1 \leq j < t \leq N} (w_j - w_t) \quad (3)$$

and with $\beta = 1, 4$ for the real and quaternion ensemble and $C_{N,\nu}$ the normalization constant. This j.p.d.f. has also been studied for arbitrary real values of $\nu$ and it is generally referred to as the Laguerre Ensemble. We will specify when $\nu$ is integer, odd or even, but in general view it as an arbitrary real number. We are interested in the gap formation probability, i.e. the probability that there are no eigenvalues in the interval $[0, s]$, denoted by $q_{N,\nu}^{(1)}(s)$,

$$q_{N,\nu}^{(1)}(s) = \frac{1}{C_{N,\nu}} \prod_{j=1}^{N} \int_s^\infty dw_j |\Delta_N(w_k)| e^{-\frac{1}{2} w_j} w_j^{(\nu+1)-1} \quad (4)$$

and in particular the large $N$ limit of it

$$Q_{\nu}^{(1)}(s) = \lim_{N \to \infty} q_{N,\nu}^{(1)} \left( \frac{s}{4N} \right). \quad (5)$$

The distribution of the smallest eigenvalue, $P_{N,\nu}^{(1)}(s)$, is given through the derivative of the gap formation probability

$$P_{N,\nu}^{(1)}(s) = -\frac{\partial}{\partial s} Q_{N,\nu}^{(1)}(s) \quad (6)$$

and thus completely determined by $Q_{N,\nu}^{(1)}(s)$. For brevity and clarity we will use the following function of the gap formation probability

$$Q_{\nu}^{(1)}(s) = Q_{\nu}^{(2)}(s^2) \quad (7)$$

since most equations acquire a simpler form when written for $Q_{\nu}^{(2)}(s^2)$. We note in passing that $Q_{\nu}^{(2)}(s)$ is the gap formation probability for the smallest eigenvalue of the Dirac spectrum.

An often overlooked result, proven in (Eq. 5.44), is that for $\nu = 2m + 1$ odd the gap formation probability is equal to an integral over the symplectic group.

$$Q_{2m+1}^{(1)}(s) = e^{-\frac{a_2}{8}} \int_{Sp(m)} dU e^{\check{T} \check{U}} \quad (8)$$

where $Sp(m)$ is the symplectic group and the integration measure on the group is the Haar measure (normalized). The proof consisted of showing that the same Painlevé V equation was satisfied by both the left hand side and the right hand side. In addition it was shown, both sides have the same boundary conditions, thereby proving Eq. 8. Denoting by $e^{i\theta_j}$, the eigenvalues of $U$, in Eq. 8, and setting $\lambda_j = \cos \theta_j$, we can write this integral as follows

$$Q_{2m+1}^{(1)}(s) = e^{-\frac{a_2}{8}} \frac{C^m}{\prod_{j=1}^{m} \int_{-1}^{1} d\lambda_j |\Delta_m(\lambda_j)|^2 e^{s\lambda_j} (1 - \lambda_j^2)\frac{1}{2}} \quad (9)$$

with $C_m$ the normalization constant. $C_m$ will be used in general to denote the normalization constants but will not always be the same. From this expression, and using the Andréief-de Bruijn integration theorem, we clearly have a representation in terms of a determinant.

$$Q_{2m+1}^{(1)}(s) = e^{-\frac{a_2 m^2}{8}} \frac{\det}{C^m} \left[ \left( \frac{\partial}{\partial s} \right)^{j+k} \left( \frac{\pi I_1(s)}{s} \right) \right] \quad (10)$$

with

$$\int_{-1}^{1} d\lambda (1 - \lambda^2)\frac{1}{2} e^{s\lambda} = \frac{\pi I_1(s)}{s} \quad (11)$$

We stress that this representation is valid for $\nu = 2m + 1$ odd. The determinant in Eq. 10 is a Hankel determinant, which is known to solve the Toda lattice equation.

**Representation of $Q_{\nu}^{(1)}(s)$ for arbitrary integer $\nu$.**— Before discussing the $\nu$ even case we make the following observation about the $\nu$ odd case. It is easily seen, by comparing the j.p.d.f. of the eigenvalues, that the integral over the symplectic group, in Eq. 9, is equal to one over the orthogonal matrices with determinant equal to $-1$ and of even dimension equal to $2m + 2$, $O_{2m+2}^-$. It was already noted in, that the j.p.d.f of the integral over these two ensembles are the same. Concretely we have

$$\int_{Sp(m)} dO e^{\check{T} \check{U}} = e^{-\frac{a_2}{8}} \prod_{j=1}^{m} \int_{-1}^{1} d\lambda_j |\Delta_m(\lambda_j)|^2 e^{s\lambda_j} (1 - \lambda_j^2)\frac{1}{2}. \quad (12)$$

The condition that the determinant of the orthogonal matrices be equal to $-1$ fixes one eigenvalue to 1 and the other to $-1$. We have therefore from Eq. 12 and for $\nu = 2m + 1$ odd

$$Q_{2m+1}^{(1)}(s) = e^{-\frac{a_2 m^2}{8}} \int_{Sp(m)} dO e^{\check{T} \check{U}}. \quad (13)$$

From this equation we put forward the ansatz that, for $\nu = 2m$ even, the gap formation probability is equal to the integral over orthogonal matrices with odd dimension and determinant equal to $-1$, i.e. as we decrease $\nu = 2m + 1$ by 1 we also decrease the dimension of the orthogonal matrix by 1:

$$Q_{2m}^{(1)}(s) = e^{-\frac{a_2}{8}} \int_{O_{2m+1}^-} dO e^{\check{T} \check{U}}. \quad (14)$$
As was done in\textsuperscript{11} one can prove Eq. (13) by showing that the left and right hand side solve the same Painlevé V equation, with the same boundary conditions. More specifically it was shown\textsuperscript{12} that, for arbitrary real $\nu$, the following function of the gap formation probability

\begin{equation}
F(s) = s \frac{\partial}{\partial s} \log \Omega^{(1)}_{\nu}(s)
\end{equation}

is in fact related to the Painlevé V solution through

\begin{equation}
F(s) = \sigma_V(s) = -\frac{s^2}{4} + \frac{\nu - 1}{2} s - \frac{\nu(\nu - 1)}{4}
\end{equation}

with $\sigma_V(s)$ solving the following Painlevé V equation with $s \to 2t$

\begin{equation}
(t \sigma^\prime)^2 - \bigg((s-s')^2 + 2(s')^2 + \mu \sigma\bigg)^2 + 4(\mu_0 + \sigma')(\mu_1 + \sigma')(\mu_2 + \sigma') (\mu_3 + \sigma') = 0
\end{equation}

and with the following coefficients

\begin{equation}
\mu = \nu - 1, \quad \mu_0 = 0, \quad \mu_1 = \nu, \quad \mu_2 = \frac{2\nu - 1}{\nu}, \quad \mu_3 = -\frac{\nu}{2}.
\end{equation}

The boundary condition is given by

\begin{equation}
\lim_{s \to 0} F(s) = -\frac{s}{2} J_\nu(s) - \frac{s^2}{4} (J_{\nu}^2(s) - J_{\nu-1}(s)J_{\nu+1}(s))
\end{equation}

\begin{equation}
= -\left(\frac{s}{2}\right)^{\nu+1} \frac{1}{\nu!}
\end{equation}

Therefore to prove Eq. (13) it suffices to show that the right-hand side also solves this equation and with the same boundary condition. The integrals over $O_{2m+1}^{(\nu)}$ of Eq. (13) was studied in\textsuperscript{11}. Using the theory of $\tau$-functions it was shown that the right-hand side also solves this equation and with the following change of variables $s \to 2t$

\begin{equation}
\lim_{s \to 0} F(s) = \frac{1}{2} \left(\tau_V^+(s) + \tau_V^-(s)\right)
\end{equation}

such that $F^\pm(s)$, given by

\begin{equation}
F^\pm(s) = s \partial_s \log \tau_V^\pm(s)
\end{equation}

is related to a solution of the Painlevé V equation, Eq. (10), with coefficients (17), through the formula (15), with the following change $\nu \to 2\nu$ in the coefficients of Eqs. (17). The two solutions differ in their boundary conditions

\begin{equation}
\lim_{s \to 0} F^\pm(s) = \pm \left(\frac{s}{2}\right)^{2\nu+1} \frac{1}{(2\nu)!}.
\end{equation}

For $2(\nu + 1) - 1 = 2m$ a compact representation of $Q^{(4)}_{\nu}(s)$ in terms of a determinant was derived\textsuperscript{12}. However the condition on $\nu$ implies $\nu$ is a half-integer and therefore this case does not include the quaternion Wishart matrices (we recall $\nu$ is the difference between the amount of columns and rows and thus an integer). For $\nu = \frac{2m+1}{2}$ a half integer it was shown\textsuperscript{11} that

\begin{equation}
Q^{(4)}_{\nu=\frac{2m+1}{2}} \left(\frac{s}{2}\right) = e^{-\frac{s^2}{4}} \int_{O_{2m+2}} dO e^{\frac{s}{2} Tr[O]}.
\end{equation}

By decomposing the integral in two integrals, one over orthogonal matrices with positive determinant and one over orthogonal matrices with negative determinant,

\begin{equation}
Q^{(4)}_{\nu=\frac{2m+1}{2}} \left(\frac{s}{2}\right) = e^{-\frac{s^2}{4}} \left(\int_{O_{2m+2}^+} dO e^{\frac{s}{2} Tr[O]} + \int_{O_{2m+2}^-} dO e^{\frac{s}{2} Tr[O]}\right)
\end{equation}

it was shown that these corresponded to the $\tau$-functions $\tau_V^\pm$ solving the Painlevé equation with the appropriate
boundary conditions. Following suit, we put forward the ansatz
\[ Q_{\nu,m}^{(4)} \left( \frac{s}{2} \right) = e^{-s^2/2} \int_{O_{2m+1}} dO e^{s Tr[O]}, \]
and split this integral into the integrals over two ensembles
\[ Q_{\nu,m}^{(4)} \left( \frac{s}{2} \right) = e^{-s^2/2} \left( \int_{O_{2m+1}^{(21)}} dO e^{s Tr[O]} + \int_{O_{2m+1}^{(01)}} dO e^{s Tr[O]} \right). \]
As done previously, we can use the results to show both of these integrals satisfy the Painlevé V equation with the appropriate boundary conditions given by Eq. (23). The sign of the orthogonal matrix ensemble over which is integrated corresponds then to the upper index of the function defined in Eq. (21) and determines the boundary conditions. We note that the second integral in Eq. (27) corresponds to our previous ansatz Eq. (13). Finally this leads to
\[ Q_{\nu,m}^{(4)} \left( \frac{s}{2} \right) = e^{-s^2/2} \left( Q_{2m}^{(1)}(s) + Q_{2m}^{(1)}(-s) \right). \]
We have compared our results with various known particular cases (see and references therein) and have found them to agree.

Conclusions.— We have derived, utilizing known results form the literature, compact closed representations as a determinant of known functions, for the gap formation probability in the large N limit, for real Wishart matrices of size \( N \times N + \nu \). We have also shown that the gap formation probability solves a Toda lattice equation in the index \( \nu \) for \( \nu \) even and for \( \nu \) odd separately. In the quaternion case, for which no previous results exist, we have shown that it can be written as the sum of two determinants each of which satisfies the Toda Lattice with different initial conditions. Although the recently derived representations of the gap formation probability are quite compact, they do not show Toda Lattice relationship between different indexes.

It is interesting to note that for \( \nu \) even the gap formation probability can be written as the average of half integer powers of a characteristic polynomial. Although half integer powers of characteristic polynomials appear in many physics applications not much is known about their integrability properties, if they even have any. Thus our results provide an example of such properties, albeit a simple one.

The determinant representation is also interesting from the perspective of representations of Hypergeometric Functions of scalar Matrix Argument. To our knowledge there is only a Pfaffian representation available for this type of Hypergeometric Function of Matrix Argument.

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