GLOBAL EXISTENCE AND ENERGY DECAY ESTIMATE OF SOLUTIONS FOR A CLASS OF NONLINEAR HIGHER-ORDER WAVE EQUATION WITH GENERAL NONLINEAR DISSIPATION AND SOURCE TERM

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Abstract. This paper deals with a higher-order wave equation with general nonlinear dissipation and source term

\[ u'' + (-\Delta)^m u + g(u') = b|u|^{p-2}u, \]

which was studied extensively when \( m = 1, 2 \) and the nonlinear dissipative term \( g(u') \) is a polynomial, i.e., \( g(u') = a|u'|^{q-2}u' \). We obtain the global existence of solutions and show the energy decay estimate when \( m \geq 1 \) is a positive integer and the nonlinear dissipative term \( g \) does not necessarily have a polynomial growth near the origin.

1. Introduction. In this paper we consider the following higher-order wave equation

\[
\begin{aligned}
& u'' + Au + g(u') = b|u|^{p-2}u, & x \in \Omega, t > 0, \\
& u(x, 0) = u_0(x), & u'(x, 0) = u_1(x), & x \in \Omega, \\
& u(x, t) = \frac{\partial^i}{\partial x^i} u(x, t) = 0, & i = 1, \cdots, m - 1, & x \in \partial \Omega, t > 0,
\end{aligned}
\]

where \( A = (-\Delta)^m \), \( m \geq 1 \) is a positive integer and \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) \((n \geq 1)\) with a smooth boundary \( \partial \Omega \), \( \nu \) is a unit outward normal vector on \( \partial \Omega \), and \( \frac{\partial^i}{\partial x^i} u \) denotes the \( i \)-order normal derivation of \( u \), \( b > 0 \) and \( p > 2 \) are constants. Furthermore, \( p \) and \( g \) are assumed to satisfy

\( (H1) \): \( 2 < p < \infty \) when \( n \leq 2m \) or \( 2 < p \leq \frac{2(n-m)}{n-2m} \) when \( n > 2m \);

\( (H2) \): \( g \) is an odd increasing \( C^1 \) function and \( c_1|s| \leq |g(s)| \leq c_2|s|^r \) for \( |s| > 1 \)

where \( c_1 \) and \( c_2 \) are two positive constants, \( 1 \leq r < \infty \) when \( n \leq 2m \) or \( 1 \leq r \leq \frac{n+2m}{n-2m} \) when \( n > 2m \).

Models of this type are interest in applications in various areas in mathematical physics, as well as in geophysics and ocean acoustics [12, 17].

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For $m = 1$ and $g(s) = a(s)$ $(a > 0)$, problem (1) was first studied by Levine [8, 7]. He showed that solutions with negative initial energy blow up in finite time. Ikehata and Suzuki [4] showed that for sufficiently small initial data $(u_0, u_1)$, the solution trajectory $(u(t), u'(t))$ tends to $(0, 0)$ in $H^m_0(\Omega) \times L^2(\Omega)$ as $t \to \infty$. When $m = 1$ and $g(s) = a|s|^{-2s}$ $(r \geq 2)$, Georgiev and Todorova [3] showed that if the damping term dominates over the source, then a global solution exists for any initial data. By using the stable set method due to Sattinger [13], Ikehata [5] proved that an energy decay rate is $E(t) \leq (1 + t)^{-2/(r-2)}$ for $t \geq 0$, which he used the general method on energy decay introduced by Nakao [10]. Assila [1] proved that an energy decay rate is exponentially if $p > r$ and showed that the solution decays algebraically by the method introduced by Nakao [10]. Moreover, the blow-up properties of the local solution blows up in finite time if $p > r$ and the initial energy is nonnegative.

For general $m \geq 2$ and $g(s) = a|s|^{-2s}$ $(r \geq 2)$, problem (1) was studied in [16, 18]. Ye [16] showed the solution exists global if the initial energy is sufficiently small. Zhou et al. [18] proved the global existence result without the relation between $p$ and $r$. Moreover, they showed that the solution decays exponentially if $r = 2$ whereas the decay is of a polynomial order if $r > 2$. They also proved that the local solution blows up in finite time if $p > r$ and the initial energy is nonnegative.

Our purpose in this paper is to give a global solvability in the class $H^m_0(\Omega) \cap H^{2m}(\Omega)$ and energy decay estimates of the solutions for problem (1) for a general nonlinear damping $g$. We use some new techniques introduced in [8] to derive a decay rate of the solutions. So we use the argument combining the method in [8] with the concept of stable set in $H^m_0(\Omega)$.

We conclude this section by stating our plan and giving some notations. In Section 2 we formulate some lemmas need for our arguments. Sections 3 and 4 are devoted the proof of global existence and decay estimates for the problem (1).

2. Preliminaries. Firstly, we state a local existence result of problem (1), which can be obtained in a similar way as done in [2, 9, 11].

**Theorem 2.1.** Suppose (H1) and (H2) hold. If $u_0 \in H^m(\Omega) \cap H^{2m}(\Omega)$ and $u_1 \in H^m_0(\Omega)$, then there exists $T > 0$ such that problem (1) has a unique local solution $u(t)$ in the class $u \in C([0, T); H^m_0(\Omega) \cap H^{2m}(\Omega))$, $u' \in C([0, T); L^2(\Omega))$. Moreover, at least one of the following statements holds true:

1. $\|u\| + \|A^{1/2}u\| \to \infty$ as $t \to T^-$;
Let $u(t)$ be the solution of (1) got in Theorem 2.1, we introduce some functional as in [18]

\begin{align*}
I(t) &= I(u(t)) = \left\| A^{\frac{1}{2}}u \right\|^2 - b \|u\|^p_p, \\
J(t) &= J(u(t)) = \frac{1}{2} \left\| A^{\frac{1}{2}}u \right\|^2 - \frac{b}{p} \|u\|^p_p, \\
E(t) &= E(u(t), u'(t)) = \frac{1}{2} \|u'\|^2 + J(t), \\
E(0) &= E(u_0, u_1) = \frac{1}{2} \|u_1\|^2 + \frac{1}{2} \left\| A^{\frac{1}{2}}u_0 \right\|^2 - \frac{b}{p} \|u_0\|^p_p.
\end{align*}

Next we state three well known lemmas that will be needed later.

**Lemma 2.2.** (Sobolev-Poincaré’s inequality) Let $\lambda$ be a number with $2 \leq \lambda < \infty$ when $n \leq 2m$ or $2 \leq \lambda \leq \frac{2n}{n-2m}$ when $n > 2m$. Then there is a constant $C_\lambda$ depending on $\Omega$ and $\lambda$ such that

\[
\|u\|_\lambda \leq C_\lambda \left\| A^{\frac{1}{2}}u \right\|, \quad \forall u \in H^m_0(\Omega).
\]

**Lemma 2.3.** [8, Lemma 1] Let $E: \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ a strictly increasing function of class $C^1$ such that

\[
\phi(t) \to \infty \text{ as } t \to \infty.
\]

Assume that there exist $\sigma \geq 0$ and $\omega > 0$ such that

\[
\forall S \geq 1, \quad \int_S^\infty E(t)^{1+\sigma} \phi'(t)dt \leq \frac{1}{\omega} E(S).
\]

Then there exists $C_* > 0$ depending on $E(1)$ such that

1. if $\sigma = 0$, then $E(t) \leq C_* e^{-\omega \phi(t)}$ for $t \geq 1$;
2. if $\sigma > 0$, then $E(t) \leq C_*(\phi(t))^{-1/\sigma}$ for $t \geq 1$.

**Lemma 2.4.** [S (6.23)-(6.25)] Assume $(H2)$ holds. Then there exists a strictly increasing function $\phi: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions

1. $\phi$ is concave and $\phi(t) \to \infty$ as $t \to \infty$,
2. $\phi'(t) \to 0$ as $t \to \infty$,
3. $\int_1^\infty \phi'(t) \left( g^{-1}(\phi'(t)) \right)^2 dt < \infty$.

**3. Global existence.** Since $g$ is an odd increasing function, a direct calculation shows that

\[
E'(t) = -\int_\Omega g(u'(t))u'(t) \leq 0. \tag{2}
\]

Hence the energy is non-increasing and in particular $E(t) \leq E(0)$ for all $t \geq 0$.

Let

\[
H = \{ u \in H_0^m(\Omega) : I(u) > 0 \} \cup \{0\}. \tag{3}
\]

The main result of this section is the following theorem:
Theorem 3.1. Assume the assumptions in Theorem 2.1 hold. Let $u(t)$ be the solution of problem (1) with initial data satisfying

$$I(u_0) = \|A^{\frac{1}{2}}u_0\|^2 - b\|u_0\|^p > 0. \quad (4)$$

If $E(0)$ satisfies

$$\eta := 1 - bC_p \left(\frac{2p}{p-2}E(0)\right)^{\frac{p-2}{2}} > 0, \quad (5)$$

where $C_p$ is the positive constant defined in Lemma 2.2, then $u(t)$ exists globally.

Proof. Let $u \in C([0,T);H_0^m(\Omega) \cap H^2_m(\Omega))$ with $u' \in C([0,T);L^2(\Omega))$ be the solution got in Theorem 2.1. Since $I(u_0) > 0$, it follows from the continuity of $u(t)$ that

$$I(u(t)) \geq 0, \quad (7)$$

for some interval near $t = 0$. Let $t_{\text{max}}$ be the maximal time when such that (7) holds.

Next we will prove $t_{\text{max}} = T$ by contradiction. Assume $t_{\text{max}} < T$. Then (7) holds for $t \in [0,t_{\text{max}}]$ and

$$I(u(t_{\text{max}})) = 0. \quad (8)$$

So,

$$J(t) = \frac{1}{2} \left\|A^{\frac{1}{2}}u\right\|^2 - \frac{b}{p}\|u\|^p$$

$$= \frac{p-2}{2p} \left\|A^{\frac{1}{2}}u\right\|^2 + \frac{1}{p}I(t)$$

$$\geq \frac{p-2}{2p} \left\|A^{\frac{1}{2}}u\right\|^2$$

for $t \in [0,t_{\text{max}}]$. Hence, by (2), we get

$$\left\|A^{\frac{1}{2}}u\right\| \leq \left(\frac{2p}{p-2}J(t)\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{2p}{p-2}E(t)\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{2p}{p-2}E(0)\right)^{\frac{1}{2}}$$

for $t \in [0,t_{\text{max}}]$. Using (H1), Lemma 2.2, (5), (6), we deduce that

$$b\|u\|^p \leq bC_p \left\|A^{\frac{1}{2}}u\right\|^p$$

$$= bC_p \left\|A^{\frac{1}{2}}u\right\|^{p-2} \left\|A^{\frac{1}{2}}u\right\|^2$$
\[ \leq bC_p^p \left( \frac{2p}{p-2} E(0) \right)^{\frac{p^2}{2}} \| A^\frac{1}{2} u \|^2 \]

\[ \left\| A^\frac{1}{2} u \right\|^2 \]

for \( t \in [0,t_{\text{max}}] \). In particular, we get

\[ I(u(t_{\text{max}})) > 0, \]

which contradicts to (8).

The above discussions imply \( u(t) \in H \) for \( t \in [0,T] \). Next, we prove \( T = \infty \) and (6) to complete the proof. By the fact that the energy \( E(t) \) is non-increasing, we have

\[ E(0) \geq E(t) \]

\[ = \frac{1}{2} \| u' \|^2 + \frac{1}{2} \left\| A^\frac{1}{2} u \right\|^2 - \frac{b}{p} \| u \|^p_p \]

\[ = \frac{1}{2} \| u' \|^2 + \frac{p-2}{2p} \left\| A^\frac{1}{2} u \right\|^2 + \frac{1}{p} I(t) \]

\[ \geq \frac{1}{2} \| u' \|^2 + \frac{p-2}{2p} \left\| A^\frac{1}{2} u \right\|^2 \]

for \( t \in [0,T] \) since \( I(t) \geq 0 \), and hence

\[ \| u' \| + \left\| A^\frac{1}{2} u \right\| \leq \sqrt{2E(0)} + \sqrt{\frac{2p}{p-2} E(0)}. \]

The above estimate implies \( T = \infty \). Furthermore, it follows from (2) that

\[ \int_0^t \int_\Omega g(u'(s))u'(s)ds = -\int_0^t E'(s)ds = E(0) - E(t) \leq 2E(0). \]

By choosing

\[ M = \max \left\{ \sqrt{2E(0)} + \sqrt{\frac{2p}{p-2} E(0)}, 2E(0) \right\}, \]

we complete the proof. \( \square \)

4. Energy decay estimate.

**Theorem 4.1.** Assume the assumptions in Theorem 3.1 hold. Then the energy \( E(t) \) of the solution \( u(t) \) of problem (1) satisfies the decay estimates

\[ E(t) \leq C_1 g^{-1} \left( \frac{1}{t} \right), \quad t \geq \frac{1}{G(1/s_1)}, \]

where \( G(t) := t g(t) \), \( C_1 \) is a positive constant only depending on \( g'(0) = 0 \), \( E(0) \) in a continuous way, \( s_1 \geq \max \{ 1, \frac{1}{G(1)} \} \) be such that \( G(1/s_1) \leq 1 \). If, in addition, \( t \mapsto g(t)/t \) is non-decreasing in \([0,1]\) and \( g'(0) = 0 \), then we have

\[ E(t) \leq C_2 g^{-1} \left( \frac{1}{t} \right), \quad t \geq \frac{1}{g(1/s_2)}, \]

where \( C_2 \) is a positive constant only depending on \( g'(0) = 0 \), \( E(0) \) in a continuous way, \( s_2 \geq \max \{ 1, \frac{1}{g(1)} \} \) be such that \( g(1/s_2) \leq 1 \).
Proof. Let \( u \in C \left( [0, \infty); H^m_0(\Omega) \cap H^{2m}(\Omega) \right) \) with \( u' \in C \left( [0, \infty); L^2(\Omega) \right) \) be the global solution of (1). Similar to the proof of (10), we get

\[
b \| u \|_p^p \leq (1 - \eta) \left\| A^{\frac{1}{2}} u \right\|^2,
\]

where \( \eta \) is given in (5). So,

\[
b \left( 1 - \frac{2}{p} \right) \| u \|_p^p \leq (1 - \eta) \left( 1 - \frac{2}{p} \right) \left\| A^{\frac{1}{2}} u \right\|^2 \leq (1 - \eta) \left( 1 - \frac{2}{p} \right) \frac{2p}{p - 2} E(t) = 2(1 - \eta)E(t).
\]

We multiply the first equation of (1) by \( E\phi' u \), where \( \phi \) is a function satisfying all hypotheses of Lemma 2.4, and then integrate the result over \( \Omega \times (S, T), \forall 0 \leq S < T < \infty \). We obtain

\[
0 = \int_S^T \int_{\Omega} E\phi' \left( u'' + Au + g(u') - b|u|^{p-2} u \right) \, dx \, dt
\]

\[
= \left[ E\phi' \int_{\Omega} u' \, dx \right]_S^T - \int_S^T (E' \phi' + E\phi'') \int_{\Omega} u' \, dx \, dt - 2 \int_S^T E\phi' \int_{\Omega} u^2 \, dx \, dt
\]

\[
+ \int_S^T E\phi' \int_{\Omega} \left( u^2 + \left| A^{\frac{1}{2}} u \right|^2 - \frac{2b}{p} |u|^p \right) \, dx \, dt
\]

\[
+ \int_S^T E\phi' \int_{\Omega} u g(u') \, dx \, dt + \int_S^T E\phi' \int_{\Omega} b \left( \frac{2}{p} - 1 \right) |u|^p \, dx \, dt
\]

\[
\geq \left[ E\phi' \int_{\Omega} u' \, dx \right]_S^T - \int_S^T (E' \phi' + E\phi'') \int_{\Omega} u' \, dx \, dt
\]

\[
- 2 \int_S^T E\phi' \int_{\Omega} u^2 \, dx \, dt + 2 \int_S^T E^2 \phi' \, dt
\]

\[
+ \int_S^T E\phi' \int_{\Omega} u g(u') \, dx \, dt + 2(\eta - 1) \int_S^T E^2 \phi' \, dt.
\]

The above inequality with Young’s inequality imply

\[
2\eta \int_S^T E^2 \phi' \, dt
\]

\[
\leq - \left[ E\phi' \int_{\Omega} u' \, dx \right]_S^T + \int_S^T (E' \phi' + E\phi'') \int_{\Omega} u' \, dx \, dt
\]

\[
+ 2 \int_S^T E\phi' \int_{\Omega} u^2 \, dx \, dt + \int_S^T E\phi' \int_{\Omega} |u g(u')| \, dx \, dt
\]

\[
\leq - \left[ E\phi' \int_{\Omega} u' \, dx \right]_S^T + \int_S^T (E' \phi' + E\phi'') \int_{\Omega} u' \, dx \, dt
\]

\[
+ 2 \int_S^T E\phi' \int_{\Omega} u^2 \, dx \, dt + \epsilon \int_S^T E\phi' \int_{|u| \leq 1} u^2 \, dx \, dt
\]

\[
+ \frac{1}{4\epsilon} \int_S^T E\phi' \int_{|u| > 1} g(u')^2 \, dx \, dt + \int_S^T E\phi' \int_{|u| > 1} |u g(u')| \, dx \, dt.
\]
By Lemma 2.2 and the definition of $E$, we get
\[ \int_s^T E\phi' \int_{|u'|\leq 1} u^2 \, dx \, dt \leq \frac{2pC_2^2}{p-2} \int_s^T E^2 \phi' \, dt. \tag{13} \]

By $(H2)$, we know that
\[ \tilde{C} := \sup_{|s| \leq 1} \left| \frac{g(s)}{s} \right|^2 < \infty. \]

Then
\[ \int_s^T E\phi' \int_{|u'|\leq 1} g(u')^2 \, dx \, dt \leq \tilde{C} \int_s^T E\phi' \int_{\Omega} u'^2 \, dx \, dt. \tag{14} \]

Choosing $\varepsilon$ small enough such that $2pC_2^2 \varepsilon/(p-2) = \eta$, we get from (13)-(14) that
\[ \eta \int_s^T E^2 \phi' \, dt \leq - \left[ E\phi' \int_{\Omega} uu' \, dx \right]_s^T + \int_s^T (E'\phi' + E\phi'') \int_{\Omega} uu' \, dx \, dt \]
\[ + \tilde{C} \int_s^T E\phi' \int_{\Omega} u'^2 \, dx \, dt + \int_s^T E\phi' \int_{|u'|>1} |ug(u')| \, dx \, dt, \tag{15} \]

where $\tilde{C} = 2 + \tilde{C}/(4\varepsilon)$.

By (2), (9), (12), (H2), Hölder’s inequality and Young’s inequality, we get
\[ \int_s^T E\phi' \int_{|u'|>1} |ug(u')| \, dx \, dt \]
\[ \leq \int_s^T E\phi' \left( \int_{\Omega} |u|^{r+1} \right)^{1/r+1} \left( \int_{|u'|>1} |g(u')|^{r+1} \, dx \right)^{1/r} \, dt \]
\[ \leq C_{r+1} C_2^{r+1} \int_s^T E\phi' \left\| A^{\frac{1}{2}} u \right\| \left( \int_{|u'|>1} |g(u')u'| \, dx \right)^{1/r} \, dt \]
\[ \leq C_{r+1} C_2^{r+1} \left( \frac{2p}{p-2} \right)^{1/2} \int_s^T E^{1+\frac{1}{r+1}} \phi' \left( \int_{\Omega} g(u')u' \, dx \right)^{1/r} \, dt \]
\[ = \tilde{C} \int_s^T \phi' E^{1+\frac{1}{r+1}} (-E')^{\frac{1}{r+1}} \, dt = \tilde{C} \int_s^T \phi' E^{1+\frac{1}{r+1}} (-E')^{\frac{1}{r+1}} \, dt \]
\[ \leq \epsilon \int_s^T \phi' E^{1+\frac{1}{r+1}} \, dt + C(\epsilon) \int_s^T \phi' (-E') \, dt \]
\[ \leq \epsilon \int_s^T \phi' E^{1+\frac{1}{r+1}} \, dt + C(\epsilon) \int_s^T \phi' (-E') \, dt \]
\[ \leq \epsilon \int_s^T \phi' E^{1+\frac{1}{r+1}} \, dt + C(\epsilon) \int_s^T \phi' (-E') \, dt \]

for any $\epsilon > 0$, where
\[ \tilde{C} = C_{r+1} C_2^{r+1} \left( \frac{2p}{p-2} \right)^{1/2(r+1)}, \quad C(\epsilon) = \frac{r}{r+1} \left( \frac{r+1}{\epsilon} \right)^{-\frac{1}{2}}. \]

The decreasing property of $E(t)$ and the fact that $r \geq 2q + 1$ implies
\[ \int_s^T \phi' E^{1+\frac{1}{r+1}} \, dt = \int_s^T \phi' E^2 \frac{1}{r+1} \, dt \]
\[ \leq E(0) \frac{1}{r+1} \int_s^T E^2 \phi' \, dt. \tag{17} \]
Since $\phi$ is increasing and concave, we have
\[
\int_S^T \phi'(t)E'dt = \left[ -\frac{1}{2}\phi'E'(t)^2 \right]_S^T + \frac{1}{2} \int_S^T E''E'dt \leq \frac{1}{2} \phi'(0)E(S)^2.
\] (18)

Choosing $\epsilon$ small enough such that $\epsilon E(0)^{\frac{1}{2}} < \eta$, we get from (15)-(18) that there exist positive constants $C_1$, $C_2$, $C_3$ such that
\[
\int_S^T E^2 \phi'dt \leq -C_1 \left[ E\phi' \int_{\Omega} u'dx \right]_S^T + C_1 \int_S^T (E'\phi' + E\phi'') \int_{\Omega} u'dxdt + C_2 \int_S^T E\phi' \int_{\Omega} u'^2 dx + C_3 E(S)^2.
\] (19)

Next we estimate the first three terms on the right-hand side of above inequality. Firstly, we consider $-[E\phi' \int_{\Omega} u'dx]^T$. By Lemma 2.2 [9] and the properties of $E$ and $\phi$, we get
\[
\leq E(S)\phi'(0) \left( \int_{\Omega} |u(T)| u'(T)|dx + \int_{\Omega} |u(S)| u'(S)|dx \right)
\leq \frac{1}{2} E(S)\phi'(0) \left( ||u(T)||^2 + ||u(S)||^2 + ||u'(T)||^2 + ||u'(S)||^2 \right)
\leq \frac{1}{2} E(S)\phi'(0) \left( C_2^2 \left( \|A^\frac{1}{2} u(T)\|^2 + \|A^\frac{1}{2} u(S)\|^2 \right) + 4E(S) \right)
\leq E(S)\phi'(0) \left( C_2^2 \left( \frac{2p}{p-2} \right) E(S) + 2E(S) \right)
= C_2^2 \left( \frac{2p}{p-2} \right) \phi'(0)E(S)^2 + 2\phi'(0)E(S)^2.
\] (20)

Secondly, we study the term $\int_S^T (E'\phi' + E\phi'') \int_{\Omega} u'^2 dx$. Similar to above inequality, we have
\[
\int_S^T (E'\phi' + E\phi'') \int_{\Omega} u'dx dt
\leq - \frac{1}{2} \int_S^T (E'\phi'(0) + E\phi'') \left( C_2^2 \left( \frac{2p}{p-2} \right) E + 2E \right) dt.
\]

Then
\[
\int_S^T (E'\phi' + E\phi'') \int_{\Omega} u'dx dt
\leq - \frac{1}{2} C_2^2 \left( \frac{2p}{p-2} \right) \phi'(0) \int_S^T E'E'dt - \phi'(0) \int_S^T E'E'dt - \frac{1}{2} C_2^2 \left( \frac{2p}{p-2} \right)^2 \int_S^T \phi'' dt - E(S) \int_S^T \phi'' dt
\leq C_2^2 \left( \frac{2p}{p-2} \right) \phi'(0)E(S)^2 + 3\phi'(0)E(S)^2.
\] (21)
At last, we consider the term \( \int_S^T \mathcal{E} \phi' \int_\Omega u^2 dx \, dt \). By [8] page 278], for \( t \geq 1 \), we can choose
\[
\psi(t) = 1 + \int_1^t \frac{1}{g \left( \frac{t}{s} \right)} \, ds.
\]
We introduce \( h(t) = g^{-1}(\phi'(t)) \), then \( h \) is a decreasing positive function and satisfies \( h(t) \to 0 \) as \( t \to \infty \). Fix \( t \geq 1 \) we define
\[
\Omega_1 := \{ x \in \Omega : |u'| \leq h(t) \},
\]
\[
\Omega_2 := \{ x \in \Omega : h(t) \leq |u'| \leq h(1) \},
\]
\[
\Omega_3 := \{ x \in \Omega : |u'| > h(1) \}.
\]
Fix \( S \geq 1 \), we can write \( \int_S^T \mathcal{E} \phi' \int_\Omega u^2 dx \, dt \) as
\[
\int_S^T \mathcal{E} \phi' \int_\Omega u^2 dx \, dt = \int_S^T \mathcal{E} \phi' \int_{\Omega_1} u^2 dx \, dt + \int_S^T \mathcal{E} \phi' \int_{\Omega_2} u^2 dx \, dt + \int_S^T \mathcal{E} \phi' \int_{\Omega_3} u^2 dx \, dt.
\]
First we look at the part on \( \Omega_1 \)
\[
\int_S^T \mathcal{E} \phi' \int_{\Omega_1} u^2 dx \, dt \leq \int_S^T \int_{\Omega_3} h(t)^2 dx \leq |\Omega|E(S) \int_S^T \phi' (g^{-1}(\phi'))^2 \, dt. \tag{22}
\]
Next we look at the part on \( \Omega_2 \). By monotonicity, if \( x \in \Omega_2 \), we obtain
\[
\phi'(t) = g(h(t)) \leq g(|u'|) = |g(u')|.
\]
Then we get
\[
\int_S^T \mathcal{E} \phi' \int_{\Omega_2} u^2 dx \, dt \leq \int_S^T E \int_{\Omega_2} |g(u')|u^2 dx \, dt \leq h(1) \int_S^T E \int_{\Omega_2} g(u')u' dx \, dt \leq h(1)E(S)^2. \tag{23}
\]
At last we look at the part on \( \Omega_3 \). If \( h(1) \geq 1 \), we get from (H2) that
\[
|g(u')| = g(|u'|) \geq c_1 |u'|.
\]
If \( h(1) < 1 \), it follows from (H2) and the monotone of \( g \) and \( h \) that
\[
|g(u')| \geq \begin{cases} \frac{c_1 |u'|}{g(|u'|)} & \text{if } |u'| \geq 1; \\ \frac{|u'|}{|u'|} & \text{if } h(1) |u'| \geq g(h(1)) |u'| = \phi'(1) |u'|, & h(1) |u'| < 1.
\end{cases}
\]
So there exists a positive constant \( \overline{C}_1 \) such that \( |g(u')| \geq \overline{C}_1 |u'| \) if \( |u'| > h(1) \). Then we have
\[
\int_S^T \mathcal{E} \phi' \int_{\Omega_3} u^2 dx \, dt \leq \frac{1}{\overline{C}_1} \int_S^T \mathcal{E} \phi' \int_{\Omega_3} u' g(u') dx \, dt \leq \frac{1}{\overline{C}_1} \int_S^T \phi' (-E') dt
\]
By (22)-(24), we obtain
\[
\int_0^T E \phi' \int_\Omega u^2 dx dt \leq \left( h(s) + \frac{\phi'(s)}{C_1} \right) E(S)^2 + |\Omega| \int_0^T \phi' (g^{-1}(\phi'))^2 dt E(S). \tag{25}
\]

Finally, by Lemma 2.4, (19), (20), (21) and (25), we know there exist two positive constant \( \hat{C}_1 \) and \( \hat{C}_2 \) such that
\[
\int_0^T E \phi' dt \leq \hat{C}_1 E(S) + \hat{C}_2 E(S)^2 \leq \left( \hat{C}_1 + \hat{C}_2 E(0) \right) E(S).
\]

Since \( \hat{C} := \left( \hat{C}_1 + \hat{C}_2 E(0) \right) \) is independent of \( T \), we get
\[
\int_0^\infty E \phi' dt \leq \hat{C} E(S), \quad \forall S \geq 1.
\]

Thanks to Lemma 2.3, we deduce
\[
E(t) \leq \frac{C_{s_1}}{\phi(t)}, \quad \forall t \geq 1, \tag{26}
\]
where \( C_{s_1} \) is a positive constant only depending on initial energy \( E(0) \) in a continuous way.

It remains to estimate the growth of \( \phi \). This is equivalent to consider the function \( \psi := \phi^{-1} \). Let \( s_1 \geq \max\{ 1, \frac{1}{G(1)} \} \) be such that \( G(1/s_1) \leq 1 \). By monotonicity of \( g \), we have
\[
\psi(s) \leq 1 + (s - 1) \frac{1}{g(1/s)} \leq \frac{s}{g(1/s)} = \frac{1}{G(1/s)}, \quad \forall s \geq s_1,
\]
hence \( s \leq \phi(t) \) with \( t = 1/G(1/s) \). It follows \( t = 1/G(1/s) \) that \( 1/s = G^{-1}(1/t) \). Thus
\[
\frac{1}{\phi(t)} \leq G^{-1} \left( \frac{1}{t} \right), \quad t \geq \frac{1}{G(1/s_1)} \geq 1.
\]

Then it follows from (26) that
\[
E(t) \leq C_{s_1} G^{-1} \left( \frac{1}{t} \right), \quad t \geq \frac{1}{G(1/s_1)}.
\]

If \( H(t) := g(t)/t \) in an increasing function in \([0, 1]\) such that \( H(0) = 0 \). By [8 page 280], we can define \( h(t) = H^{-1}(\phi(t)) \) with
\[
\psi(t) = \phi^{-1}(t) = 1 + \int_1^t \frac{1}{H(1/s)} ds.
\]
On $\Omega_2$ it holds that $\varphi'(t)u'^2 \leq |H(u')|u'^2 = u'g(u')$. Then the same calculations as above yield

$$E(t) \leq C_{*2}g^{-1} \left( \frac{1}{t} \right), \quad t \geq \frac{1}{g(1/s_2)},$$

where $C_{*2}$ is a positive constant only depending on initial energy $E(0)$ in a continuous way, $s_2 \geq \max \{ 1, \frac{1}{g(1)} \}$ be such that $g(1/s_2) \leq 1$.

REFERENCES

[1] M. Aassila, Global existence of solutions to a wave equation with damping and source terms, *Diff. Inte. Equations*, 14 (2001), 1301–1314.
[2] Q. Gao, F. Li and Y. Wang, Blow up of solution for higher-order Kirchhoff-type equations with nonlinear dissipation, *Cent. Euro. J. Math.*, 9 (2011), 686–698.
[3] V. Georgiev and G. Todorova, Existence of a solution of the wave equation with nonlinear damping and source terms, *J. Differential Equations*, 109 (1994), 295–308.
[4] R. Ikehata and T. Suzuki, Stable and unstable sets for evolution equations of parabolic and hyperbolic type, *Hiroshima Math. J.*, 26 (1996), 475–491.
[5] R. Ikehata, Some remarks on the wave equations with nonlinear damping and source terms, *Nonlinear Anal.*, 27 (1996), 1165–1175.
[6] H. A. Levine, Instability and nonexistence of global solutions of nonlinear wave equation of the form $D_{tt} u = Au + f(u)$, *Trans. Am. Math. Soc.*, 192 (1974), 1–21.
[7] H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.*, 5 (1974), 138–146.
[8] P. Martinez, A new method to obtain decay rate estimates for dissipative systems, *ESAIM. Cont. Opt. Cal. Var.*, 4 (1999), 419–444.
[9] S. A. Messaoudi, Global existence and nonexistence in a system of Petrovsky, *J. Math. Anal. Appl.*, 265 (2002), 296–308.
[10] M. Nako, Asymptotic stability of the bounded or almost periodic solution of the wave equation with nonlinear dissipative term, *J. Math. Anal. Appl.*, 58 (1977), 336–343.
[11] K. Ono, On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation, *J. Math. Anal. Appl.*, 216 (1997), 321–342.
[12] M. Reed and B. Simon, *Methods of Modern Mathematical Physics*, in: *Scattering Theory*, vol. III, Academic Press, New York, London, 1979.
[13] D. H. Sattinger, On global solutions of nonlinear hyperbolic equations *Arch. Rational Mech. Anal.*, 30 (1968), 148–172.
[14] G. Todorova, Stable and unstable sets for the Cauchy problem for a nonlinear wave equation with nonlinear damping and source terms, *J. Math. Anal. Appl.*, 239 (1999), 213–226.
[15] S. T. Wu and L. Y. Tsai, On global solutions and blow-up of solutions for a nonlinearly damped Petrovsky system, *Taiwanese J. Math.*, 13 (2009), 545–558.
[16] Y. Ye, Existence and asymptotic behavior of global solutions for a class of nonlinear higher-order wave equation, *J. Ineq. Appl.*, 2010 (2010), Art. ID 394859, 14 pp.
[17] E. Zauderer, *Partial Differential Equations of Applied Mathematics*, in: *Pure and Applied Mathematics*, second edition, A Wiley-inter science Publication, John Wiely & Sons, Inc., New York, 1989.
[18] J. Zhou, X. R. Wang, X. J. Song and C. L. Mu, Global existence and blowup of solutions for a class of nonlinear higher-order wave equations, *Z. Angew. Math. Phys.*, 63 (2012), 461–473.

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