A PERTURBATION RESULT FOR QUASI-LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS IN UMD BANACH SPACES.

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ABSTRACT. We consider the effect of perturbations of $A$ on the solution to the following quasi-linear parabolic stochastic differential equation set in a UMD Banach space $X$:

\[ \begin{cases} dU(t) = AU(t) \, dt + F(t, U(t)) \, dt + G(t, U(t)) \, dW_H(t), & t > 0; \\ U(0) = x_0. \end{cases} \]

Here $A$ is the generator of an analytic $C_0$-semigroup on $X$, $G : [0,T] \times X \to \mathcal{L}(H, X_{\theta_G}^A)$ and $F : [0,T] \times X \to X_{\theta_F}^A$ for some $\theta_G > -\frac{1}{2}$, $\theta_F > -\frac{3}{2} + \frac{1}{\tau}$, where $\tau$ is the type of $X$. We assume $F$ and $G$ to satisfy certain global Lipschitz and linear growth conditions.

Let $A_0$ denote the perturbed operator and $U_0$ the solution to (SDE) with $A$ substituted by $A_0$. We provide estimates for $\|U - U_0\|_{L^p(\Omega; C([0,T]; X))}$ in terms of $D_\delta(A, A_0) := \|R(\lambda : A) - R(\lambda : A_0)\|_{\mathcal{L}(X_{\delta - 1}^A, X)}$. Here $\delta \in [0,1]$ is assumed to satisfy $0 \leq \delta < \min\{\frac{3}{2} - \frac{1}{\tau} + \theta_F, \frac{1}{2} - \frac{1}{\tau} + \theta_G\}$.

The work is inspired by the desire to prove convergence of space approximations of (SDE). In this article we prove convergence rates for the case that $A$ is approximated by its Yosida approximation.

Keywords: perturbations, stochastic differential equations, stochastic convolutions, stochastic partial differential equations, Yosida approximation

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1. Introduction

In this article we consider the effect of perturbations of $A$ on the solution to the following stochastic differential equation set in a UMD Banach space $X$:

\[ \begin{cases} dU(t) = AU(t) \, dt + F(t, U(t)) \, dt + G(t, U(t)) \, dW_H(t), & t > 0; \\ U(0) = x_0. \end{cases} \]

Here $A$ is the generator of an analytic $C_0$-semigroup on $X$, $G : [0,T] \times X \to \mathcal{L}(H, X_{\theta_G}^A)$ and $F : [0,T] \times X \to X_{\theta_F}^A$ for some $\theta_G > -\frac{1}{2}$, $\theta_F > -\frac{3}{2} + \frac{1}{\tau}$, where $\tau \in [1,2)$ is the type of $X$. We use $X_{\theta_F}^A$ to denote the fractional domain or extrapolation space corresponding to $A$. We assume $F$ and $G$ to satisfy certain global Lipschitz and linear growth conditions, see Section 2.2.2 below. The framework in which we consider (SDE) is precisely the one for which existence and uniqueness of a solution has been proven in [20]. A typical example of a stochastic partial differential equation that fits into this framework is a one-dimensional parabolic stochastic partial differential equation driven by white noise.

The main motivation to study the effect of perturbations of $A$ on solutions to (SDE) is the desire to prove convergence of certain numerical schemes for approximations in the space dimension. In fact, in [5] we demonstrate how the perturbation...
result proven in this article can be used to obtain pathwise convergence of certain Galerkin and finite element methods for SDE in the case that $X$ is Hilbertian. Here we focus more on the theoretical aspects, and demonstrate how our perturbation result can be used to prove convergence of the solution processes if $A$ is replaced by its Yosida approximation.

With applications to numerical approximations in mind, we assume the perturbed equation to be set in a (possibly finite dimensional) closed subspace $X_0$ of $X$. We assume that there exists a bounded projection $P_0 : X \to X_0$ such that $P_0(X) = X_0$. Let $i_{X_0}$ be the canonical embedding of $X_0$ in $X$ and let $A_0$ be a generator of an analytic $C_0$-semigroup $S_0$ on $X_0$. In the setting of numerical approximations, $A_0$ would be a suitable restriction of $A$ to the finite dimensional space $X_0$.

The perturbed equation we consider is the following stochastic differential equation:

\[
\begin{aligned}
\begin{cases}
    dU^{(0)}(t) &= A_0 U^{(0)}(t) \, dt + P_0 F(t, U^{(0)}(t)) \, dt \\
    &+ P_0 G(t, U^{(0)}(t)) \, dW(t), \\
    U^{(0)}(0) &= P_0 x_0.
\end{cases}
\end{aligned}
\]

(SDE$_0$)

Our main result, Theorem 3.1 below, states that if we have:

\[D_{\delta}(A, A_0) := \| R(\lambda_0 : A) - i_{X_0} R(\lambda_0 : A_0) P_0 \|_{L^p(X_{t,\lambda}^\delta \cdot X)} < \infty,\]

for some $\delta \geq 0$ satisfying

\[0 \leq \delta < \min\{\frac{3}{2} - \frac{1}{p} + \theta_F, \frac{1}{2} + \theta_G\},\]

and $x_0 \in L^p(\Omega; F_0; X_{t,\lambda}^\delta)$ for $p \in (2, \infty)$ such that $\frac{1}{p} \leq \frac{1}{2} + \theta_G - \delta$, then there exists a solution to (SDE$_0$) in $L^p(\Omega; C([0, T]; X))$ and moreover:

\[\|U - i_{X_0} U^{(0)}\|_{L^p(\Omega; C([0, T]; X))} \lesssim D_{\delta}(A, A_0) (1 + \|x_0\|_{L^p(\Omega; X_{t,\lambda}^\delta)}).
\]

Note that if $\delta < 1$ then a priori it is not obvious whether $D_{\delta}(A, A_0)$ is finite.

As a corollary of Theorem 3.1 we obtain an estimate in the Hölder norm provided we compensate for the initial values (see Corollary 3.4 below), i.e., for $\lambda \in [0, \frac{1}{2}]$ satisfying

\[0 \leq \lambda < \min\{\frac{3}{2} - \frac{1}{p} - (\delta - \theta_F) \vee 0, \frac{1}{2} - \frac{1}{p} - (\delta - \theta_G) \vee 0\},\]

we have:

\[\|U - S x_0 - i_{X_0} (U^{(0)} - S_0 P_0 x_0)\|_{L^p(\Omega; C ([0, T]; X))} \lesssim D_{\delta}(A, A_0) (1 + \|x_0\|_{L^p(\Omega; X_{t,\lambda}^\delta)}).
\]

Our results imply that if $(A_n)_{n \in \mathbb{N}}$ is a family of generators of analytic semigroups such that the resolvent of $A_n$ converges to the resolvent of $A$ in $L^p(\Omega; C([0, T]; X))$ for some $\delta \in [0, 1]$ and $(A_n)_{n \in \mathbb{N}}$ is uniformly analytic, then the corresponding solution processes $U_n$ converge to the actual solution in $L^p(\Omega; C([0, T]; X))$ and the convergence rate is given by $D_{\delta}(A, A_n)$.

In particular, we may apply Theorem 3.1 to the Yosida approximation of $A$. In this case we assume $\theta_F$ and $\theta_G$ are positive. The $n^{th}$ Yosida approximation of $A$ is given by $A_n = n AR(n : A)$, and we let $U^{(n)}$ denote the solution to (SDE$_0$) where $A$ is substituted by $A_n$. By applying Theorem 5.1 we obtain that for $\eta \in [0, 1]$ and $p \in (2, \infty)$ such that

\[\eta < \min\{\frac{3}{2} - \frac{1}{p} + \theta_F, \frac{1}{2} - \frac{1}{p} + \theta_G\},\]
we have, assuming \( x_0 \in L^p(\Omega, F_0; X^A) \):

\[
\| U - U^{(n)} \|_{L^p(\Omega; C([0,T]; X))} \lesssim n^{-\eta}(1 + \| x_0 \|_{L^p(\Omega; X^A)}).
\]

See also Theorem 4.1.

It was proven in [15] that if \((A_n)_{n \in \mathbb{N}}\) is a family of generators of analytic semigroups such that the resolvent of \(A_n\) converges to the resolvent of \(A\) in the strong operator topology, then the corresponding solution processes \(U_n\) converge to the actual solution in \(L^p(\Omega; C([0,T]; X))\). However, the approach taken in that article does not provide convergence rates and requires \(\theta_F, \theta_G \geq 0\).

Another article in which approximations of solutions to SDEs are considered in the context of perturbations on \(A\) is [11]. In that article, it is assumed that \(X\) is a UMD space with martingale type 2. In Section 5 of that article the author considers approximations of \(A, F, G\) and of the noise. Translated to our setting, the author assumes the perturbed operator \(A_0\) to satisfy \(X^{A_0}_{\theta_F} = X^A_{\theta_F}\) and \(X^{A_0}_{\theta_G} = X^A_{\theta_G}\) (in particular, \(X_0\) cannot be finite-dimensional).

A natural question to ask is how the type of perturbation studied here relates to the perturbations known in the literature. In [9], [12], [22] (see also [10, Chapter III.3]) one has derived conditions for perturbations of \(A\) that lead to an estimate of the type \(\|S(t) - S_0(t)\|_{L(X)} = O(t)\). In light of Proposition 3.2 below these results are comparable to our results if we were to take \(\alpha = -1\). In particular, [10, Theorem III.3.9] gives precisely the same results as Proposition 3.2, but then for the case \(\alpha = -1\) and \(\beta = 0\).

The proof of our perturbation result (Theorem 3.1) requires regularity results for stochastic convolutions. As the convolution under consideration concerns the difference between two semigroups instead of a single semigroup, the celebrated factorization method of [7] fails. Therefore we prove a new result on the regularity of stochastic convolutions, see Lemma 2.17 below. This lemma in combination with some randomized boundedness results on \(S - S_0P_0\) form the key ingredients of the proof Theorem 3.1.

The set-up of this article is as follows: Section 2 contains the preliminaries; i.e., the necessary results on analytic semigroups, vector-valued stochastic integration theory, and \(\gamma\)-boundedness. In that section we also state the precise assumptions on \(A, F, G\) in (SDE), and prove the regularity results for (stochastic) convolutions that we need in the proof of Theorem 3.1. In Section 3 we prove Theorem 3.1 and in Section 4 we prove convergence for the Yosida approximations.

**Notation.** For an operator \(A\) on a Banach space \(X\) we denote the resolvent set of \(A\), i.e., \(\rho(A)\), \(\rho(A) \subset \mathbb{C}\) is the set of all the complex numbers \(\lambda \in \mathbb{C}\) for which \(\lambda I - A\) is (boundedly) invertible. For \(\lambda \in \rho(A)\) we denote the resolvent of \(A\) in \(\lambda\) by \(R(\lambda : A)\), i.e., \(R(\lambda : A) = (\lambda I - A)^{-1}\). The spectrum of \(A\), i.e., the complement of \(\rho(A)\) in \(\mathbb{C}\), is denoted by \(\sigma(A)\).

For \(X, Y\) Banach spaces we let \(L(X, Y)\) be the Banach space of all bounded linear operators from \(X\) to \(Y\) endowed with the operator norm. For brevity we set \(L(X) := L(X, X)\).

For \(T > 0\) and \(\theta > 0\) we take the following definition for the Hölder norm:

\[
\| f \|_{C^\theta([0,T]; Y)} := \| f(0) \|_Y + \sup_{0 \leq s < t \leq T} \frac{\| f(t) - f(s) \|_Y}{(t-s)^\theta}.
\]
We write $A \lesssim B$ to express that there exists a constant $C > 0$ such that $A \leq CB$, and we write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$. Finally, for $X$ and $Y$ Banach spaces we write $X \simeq Y$ if $X$ and $Y$ are isomorphic as Banach spaces.

2. Preliminaries

Throughout this section $X$, $Y$, and $Y_i$, $i \in \{1, 2\}$, will be used to denote Banach spaces and $H$ will denote a Hilbert space.

2.1. Analytic semigroups. For $\delta \in [0, \pi]$ we define

$$\Sigma_\delta := \{ z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \delta \}.$$ 

We recall the definition of analytic $C_0$-semigroups [21, Chapter 2.5]:

**Definition 2.1.** Let $\delta \in (0, \frac{\pi}{2})$. A $C_0$-semigroup $(S(t))_{t \geq 0}$ on $X$ is called analytic in $\Sigma_\delta$ if

1. $S$ extends to an analytic function $S : \Sigma_\delta \to \mathcal{L}(X)$;
2. $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \Sigma_\delta$;
3. $\lim_{z \to 0, z \in \Sigma_\delta} S(z)x = x$ for all $x \in X$.

Typical examples of operators generating analytic $C_0$-semigroups are second-order elliptic operators. The theorem below is obtained from [21, Theorem 2.5.2] by straightforward adaptations and gives some characterizations of analytic $C_0$-semigroups that we shall need.

**Theorem 2.2.** Let $A$ be the generator of a $C_0$-semigroup $(S(t))_{t \geq 0}$ on $X$. Let $\omega \in \mathbb{R}$ be such that $(e^{-\omega t}S(t))_{t \geq 0}$ is exponentially stable. The following statements are equivalent:

1. $S$ is an analytic $C_0$-semigroup on $\Sigma_\delta$ for some $\delta \in (0, \frac{\pi}{2})$ and for every $\delta' < \delta$ there exists a constant $C_{1, \delta'}$ such that $\|e^{-zS}(z)\| \leq C_{1, \delta'}$ for all $z \in \Sigma_{\delta'}$.
2. There exists a $\theta \in (0, \frac{\pi}{2})$ such that $\omega + \Sigma_{\frac{\pi}{2} + \theta} \subset \rho(A)$, and for every $\theta' \in (0, \theta)$ there exists a constant $C_{2, \theta'} > 0$ such that:
   $$|\lambda - \omega|\|R(\lambda : A)\| \leq C_{2, \theta'} , \text{ for all } \lambda \in \omega + \Sigma_{\frac{\pi}{2} + \theta'}.$$ 
3. $S$ is differentiable for $t > 0$ (in the uniform operator topology), $\frac{d}{dt}S = AS$, and there exists a constant $C_3$ such that:
   $$t\|AS(t)\| \leq C_3e^{\omega t}, \text{ for all } t > 0.$$ 

This theorem justifies the following definition:

**Definition 2.3.** Let $A$ be the generator of an analytic $C_0$-semigroup on $X$. We say that $A$ is of type $(\omega, \theta, K)$, where $\omega \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$ and $K > 0$, if $\omega + \Sigma_{\frac{\pi}{2} + \theta} \subset \rho(A)$, $(e^{\omega t}S(t))_{t \geq 0}$ is exponentially stable, and

$$|\lambda - \omega|\|R(\lambda : A)\|_{\mathcal{L}(X)} \leq K, \text{ for all } \lambda \in \omega + \Sigma_{\frac{\pi}{2} + \theta}.$$ 

**Remark 2.4.** It follows from the aforementioned proof in [21] that the constants $\delta, C_{1, \delta'}; \delta' \in (0, \delta)$, $C_{2, \theta'}; \theta' \in (0, \theta)$, and $C_3$ in Theorem 2.2 can be expressed explicitly in terms of $\omega$, $\theta$, and $K$; for example, we may take $C_3 = \frac{K}{\pi \cos \theta}$. 
Note that if \( A \) is the generator of an analytic \( C_0 \)-semigroup of type \((\omega, \theta, K)\) then for all \( \lambda \in \omega(1 + 2(\cos(\theta)^{-1}) + \sum_{2}^{\theta} \) one has (noting that the choice of \( \lambda \) implies \( |\lambda| > 2|\omega| \) and hence \( |\lambda - \omega| = ||\lambda| - |\omega|| \geq \frac{1}{2}|\lambda| \)):

\[
\|AR(\lambda : A)\|_{L(X)} = \|\lambda R(\lambda : A) - I\| \leq 1 + 2K.
\]

Let \( A \) be the generator of an analytic semigroup of type \((\omega, \theta, K)\) on \( X \). We define the extrapolation spaces of \( A \) conform \cite{21} Section 2.6]; i.e., for \( \delta > 0 \) and \( \lambda \in \mathbb{C} \) such that \( \Re(\lambda) > \omega \) we define \( X^\delta_\lambda \) to be the closure of \( X \) under the norm \( \|x\|_{X^\delta_\lambda} := \|\lambda I - A\|^{-\delta}x\|_X \). We also define the fractional domain spaces of \( A \), i.e., for \( \delta > 0 \) we define \( X^\delta = D((\lambda I - A)^\delta) \) and \( \|x\|_{X^\delta} := \|\lambda I - A\|^{-\delta}x\|_X \). One may check that regardless of the choice of \( \lambda \) the extrapolation spaces and the fractional domain spaces are uniquely determined up to isomorphisms: for \( \delta > 0 \) one has \((\lambda I - A)^\delta(\mu I - A)^{-\delta} \in L(X)\) and:

\[
\|\lambda I - A\|^{\delta} (\mu I - A)^{-\delta} \|_{L(X)} \leq C(\omega, \theta, K, \lambda, \mu),
\]

where \( C(\omega, \theta, K, \lambda, \mu) \) denotes a constant depending only on \( \omega, \theta, K, \lambda, \) and \( \mu \). Moreover, for \( \delta, \beta \in \mathbb{R} \) one has \((\lambda I - A)^\delta(\lambda I - A)^\beta = (\lambda I - A)^{\delta + \beta} \) on \( X^\delta \), where \( \gamma = \max\{\beta, \delta + \beta\} \) (see \cite{21} Theorem 2.6.8).

Statement \( \text{iii} \) in Theorem 2.2 can be extended; from the proof of \cite{21} Theorem 2.6.13) we obtain that for an analytic \( C_0 \)-semigroup \( S \) of type \((\omega, \theta, K)\) generated by \( A \) one has, for \( \delta > 0 \):

\[
\|S(t)\|_{L(X, X^{\delta})} \leq 2\left(\frac{K}{\pi \cos(\theta)}\right)^{[\delta]} t^{-\delta} e^{\omega t}.
\]

The following interpolation result holds for the fractional domain spaces (see \cite{21} Theorem 2.6.10):}

\textbf{Theorem 2.5.} Let \( A \) be the generator of an analytic \( C_0 \)-semigroup on \( X \) of type \((\omega, \theta, K)\). Let \( \delta \in (0, 1) \) and \( \lambda \in \mathbb{C} \) such that \( \Re(\lambda) > \omega \). Then for every \( x \in D(A) \) we have:

\[
\|\lambda I - A\|^{\delta} x \| \leq 2(1 + K)\|x\|^{1-\delta} \|\lambda I - A\|^{\delta} x.
\]

For more properties of \( X^\delta \), \( \delta \in \mathbb{R} \), we refer to \cite{21} Section 2.6).

\textbf{2.2. Stochastic differential equations.} Throughout this section \( X \) and \( Y \) denote UMD Banach spaces, and \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) denotes a probability space.

\textbf{2.2.1. Stochastic integration in UMD Banach spaces.} Before turning to stochastic differential equations, we recall the basics concerning stochastic integration in UMD Banach spaces as presented in \cite{19}. Recall that the UMD property is a geometric Banach space property that is satisfied by all Hilbert spaces and the ‘classical’ reflexive function spaces, e.g. the \( L^p \)-spaces and Sobolev spaces \( W^{k,p} \) for \( k \in \mathbb{N} \) and \( p \in (1, \infty) \). For the precise definition of the UMD property and for a more elaborate treatment of spaces satisfying this property we refer to \cite{2}.

Fix \( T > 0 \). An \( H \)-cylindrical Brownian motion over \((\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) is a mapping \( W_H : L^2(0, T; H) \rightarrow L^2(\Omega) \) with the following properties:

(i) for all \( h \in L^2(0, T; H) \) the random variable \( W_H(h) \) is Gaussian;
(ii) for all \( h_1, h_2 \in L^2(0, T; H) \) we have \( \mathbb{E}W_H(h_1)W_H(h_2) = \langle h_1, h_2 \rangle \);
(iii) for all \( h \in H \) and all \( t \in [0, T] \) we have that \( W_H(1[0,t] \otimes h) \) is \( \mathcal{F}_t \)-measurable;
(iv) for all \( h \in H \) and all \( s, t \in [0, T], s \leq t \) we have that \( W_H(1_{[s,t]} \otimes h) \) is independent of \( \mathcal{F}_s \).
Formally, an $H$-cylindrical Brownian motion can be thought of as a standard Brownian motion taking values in the Hilbert space $H$.

For the precise definition of the stochastic integral of a process $\Phi : [0,T] \times \Omega \to \mathcal{L}(H, X)$ with respect to $W_H$ we refer to [19]. For our purposes it suffices to cite the characterization of such stochastically integrable processes in terms of the so-called $\gamma$-radonifying norm of the process.

Let $\mathcal{H}$ be a Hilbert space (we will take $\mathcal{H} = L^2(0,T; H)$ later on). The Banach space $\gamma(\mathcal{H}, X)$ is defined as the completion of $\mathcal{H} \otimes X$ with respect to the norm

$$
\left\| \sum_{n=1}^{N} h_n \otimes x_n \right\|_{\gamma(\mathcal{H}, X)}^2 := \mathbb{E} \left\| \sum_{n=1}^{N} \gamma_n \otimes x_n \right\|^2.
$$

Here we assume that $(h_n)_{n=1}^{\infty}$ is an orthonormal sequence in $\mathcal{H}$, $(x_n)_{n=1}^{\infty}$ is a sequence in $X$, and $(\gamma_n)_{n=1}^{\infty}$ is a standard Gaussian sequence on some probability space. The space $\gamma(\mathcal{H}, X)$ embeds continuously into $\mathcal{L}(\mathcal{H}, X)$ and it elements are referred to as the $\gamma$-radonifying operators from $\mathcal{H}$ to $X$. For properties of this norm and further details we refer to the survey paper [17].

Let $(R, \mathcal{R}, \mu)$ be a measure space. In that case $\gamma(R; H; X)$ is used as shorthand notation for $\gamma(L^2(R; H), X)$; in particular, $\gamma(0, T; H; X)$ is short-hand notation for $\gamma(L^2(0,T; H); X)$. We use $\gamma(0, T; X)$ to denote $\gamma(L^2(0, T), X)$. If $X$ is a Hilbert space, and $(R, \mathcal{R}, \mu)$ is a $\sigma$-finite measure space, then $\gamma(R; H; X) = L^2(R, \mathcal{L}_2(H, X))$ where $\mathcal{L}_2(H, X)$ denotes the space of Hilbert-Schmidt operators from $H$ to $X$.

A process $\Phi : [0, \infty) \times \Omega \to \mathcal{L}(H, X)$ is called $H$-strongly measurable if for every $h \in H$ the process $\Phi h$ is strongly measurable. The process is called adapted if $\Phi h$ is adapted for each $h \in H$.

We cite [19, Theorem 3.6]:

**Theorem 2.6** ($L^p$-stochastic integrability). Let $p \in (1, \infty)$ and $T > 0$ be fixed. For an $H$-strongly measurable adapted process $\Phi : (0, T) \times \Omega \to \mathcal{L}(H, X)$ the following are equivalent:

(i) $\Phi$ is $L^p$-stochastically integrable with respect to $W_H$;

(ii) we have $\Phi^* x^* \in L^p(\Omega; L_2^2(0,T; H))$ for all $x^* \in X^*$ and there exists a (necessarily unique) $R_\Phi \in L^p(\Omega; \gamma(0, T; H; X))$ such that for all $x^* \in X^*$ we have

$$
R_\Phi^* x^* = \Phi^* x^*
$$

in $L^p(\Omega; L^2(0,T; H))$.

In this situation one has

$$
\mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^t \Phi \, dW_H \right\|_Y^p \asymp_p \mathbb{E} \|R_\Phi\|_{\gamma(0,T; H; Y)}^p,
$$

the implied constants being independent of $\Phi$.

The inequalities in equation (4) are referred to as Burkholder-Davis-Gundy inequalities.

From now on, if $\Phi$ is stochastically integrable, we shall simply use $\Phi$ to denote both a process and the (unique) $R_\Phi \in L^p(\Omega; \gamma(0, t, H; X))$ that satisfies $R_\Phi^* x^* = \Phi^* x^*$.
2.2.2. The SDE. In Section we will prove a perturbation result for the following stochastic differential equation:

\[
\begin{align*}
\text{(SDE)} \quad \begin{cases} dU(t) &= AU(t) \, dt + F(t, U(t)) \, dt + G(t, U(t)) \, dW_H(t); \quad t \in [0, T], \\
U(0) &= x_0.
\end{cases}
\end{align*}
\]

where \( A, F \) and \( G \) are assumed to satisfy conditions (A), (F), and (G) below. The assumptions on \( F \) depend on the type of the Banach space \( F \). The type of a Banach space is defined based on the behavior of moments of randomized sums, and it always takes values in the interval \([1, 2]\). The greater the type, i.e., the closer to 2, the more the space behaves like a Hilbert space; every Banach space has type at least 1, and all Hilbert spaces have type 2. The \( L^p \)-spaces have type \( \min\{p, 2\} \).

We refer to [13] for a precise definition of type (and co-type) and further details. Both UMD and type are preserved under Banach space isomorphisms.

(A): A generates an analytic \( C_0 \)-semigroup on a UMD Banach space \( X \).

(F): For some \( \theta_F > -1 + \left( \frac{1}{2} - \frac{1}{2} \right) \), where \( \tau \) is the type of \( X \), the function \( F : [0, T] \times X \to X_{\theta_F} \) is measurable in the sense that for all \( x \in X \) the mapping \( F(\cdot, x) : [0, T] \to X_{\theta_F} \) is strongly measurable. Moreover, \( F \) is uniformly Lipschitz continuous and uniformly of linear growth on \( X \).

That is to say, there exist constants \( C_0 \) and \( C_1 \) such that for all \( t \in [0, T] \) and all \( x, y \in X \):

\[
\|F(t, x) - F(t, y)\|_{X_{\theta_F}} \leq C_0 \|x - y\|_X,
\]

\[
\|F(t, x)\|_{X_{\theta_F}} \leq C_1 (1 + \|x\|_X).
\]

The least constant \( C_0 \) such that the above holds is denoted by Lip\((F)\), and the least constant \( C_1 \) such that the above holds is denoted by \( M(F) \).

(G): For some \( \theta_G > -\frac{1}{2} \), the function \( G : [0, T] \times X \to \mathcal{L}(H, X_{\theta_G}) \) is measurable in the sense that for all \( h \in H \) and \( x \in X \) the mapping \( G(\cdot, x)h : [0, T] \to X_{\theta_G} \) is strongly measurable. Moreover, \( G \) is uniformly \( L^2 \)-Lipschitz continuous and uniformly of linear growth on \( X \).

That is to say, there exist constants \( C_0 \) and \( C_1 \) such that for all \( \alpha \in [0, \frac{1}{2}) \), all \( t \in [0, T] \), and all simple functions \( \phi_1, \phi_2, \phi : [0, T] \to X \) one has:

\[
\|s \mapsto (t - s)^{-\alpha} G(s, \phi_1(s)) - G(s, \phi_2(s))\|_{\gamma(0, t; H, X_{\theta_G})} \leq C_0 \|s \mapsto (t - s)^{-\alpha} [\phi_1 - \phi_2]\|_{L^2(0, t; X) \cap \gamma(0, t; X)};
\]

\[
\|s \mapsto (t - s)^{-\alpha} G(s, \phi(s))\|_{\gamma(0, t; H, X_{\theta_G})} \leq C_1 (1 + \|s \mapsto (t - s)^{-\alpha} \phi(s)\|_{L^2(0, t; X) \cap \gamma(0, t; X)}).
\]

The least constant \( C_0 \) such that the above holds is denoted by Lip\((G)\), and the least constant \( C_1 \) such that the above holds is denoted by \( M_r(G) \).

If \( Y_2 \) is a type 2 space and \( G : [0, T] \times Y_1 \to \gamma(H, Y_2) \) is Lipschitz-continuous, uniformly in \([0, T]\), then \( G \) is \( L^2 \)-Lipschitz continuous (see [20] Lemma 5.2]). More examples of \( L^2 \)-Lipschitz continuous operators can be found in [20].

2.2.3. Existence and uniqueness. We recall an existence and uniqueness result for the problem \( \text{(SDE)} \). This result is formulated in a space of continuous, ‘weighted’ stochastically integrable processes which is defined as follows:
Definition 2.7. For $\alpha \geq 0$, $1 \leq p < \infty$ and $0 \leq a \leq b < \infty$, we denote by $V^\alpha,p_c([a, b] \times \Omega; Y)$ the space of adapted, continuous processes $\Phi : [a, b] \times \Omega \to Y$ for which the following norm is finite:

$$
\| \Phi \|_{V^\alpha,p_c([a, b] \times \Omega; Y)} = \| \Phi \|_{L^p(\Omega; C([a, b]; Y))} + \sup_{a \leq t \leq b} \| s \mapsto (t - s)^{-\alpha} \Phi(s) \|_{L^p(\Omega; Y)}.
$$

One easily checks that for $a \leq c < d < b$,

$$
(5) \quad \| \Phi \|_{V^\alpha,p_c([c, d] \times \Omega; X)} = \| \Phi \|_{V^\alpha,p_c([c, d] \times \Omega; X)}.
$$

Moreover, for $0 \leq \beta < \alpha < \frac{1}{2}$ and $\Phi \in V^\alpha,p([a, b] \times \Omega; X)$ one has:

$$
(6) \quad \| \Phi \|_{V^{\beta,p}([a, b] \times \Omega; X)} \leq (b - a)^{\alpha - \beta} \| \Phi \|_{V^\alpha,p([a, b] \times \Omega; X)}.
$$

Note also that we have $V^\alpha,p([0, T] \times \Omega; X) \subset L^p(\Omega; C([0, T]; X))$. On the other hand, we have the following embedding (see [20, Lemma 3.3]):

Lemma 2.8. Let $X$ be a Banach space with type $\tau$. Then for all $T > 0$, $\varepsilon > 0$ and $\alpha \in [0, \frac{1}{2})$ there exists a constant $C$ such that for all $T_0 \subset [0, T]$ one has:

$$
(7) \quad V^\alpha,p_c([0, T_0] \times \Omega; X) \subset CL^p(\Omega; C^{\frac{1}{2} + \varepsilon}([0, T_0]; X)).
$$

If $G : [0, T] \times X \to L(H, X_{\theta G})$ satisfies (G) and $\Phi_1, \Phi_2 \in V^\alpha,p([0, T] \times \Omega; X)$ for some $p \geq 2$, then:

$$
(8) \quad \sup_{0 \leq t \leq T} \| s \mapsto (t - s)^{-\alpha}[G(s, \Phi_1(s)) - G(s, \Phi_2(s))] \|_{L^p(\Omega; Y)} \\
\leq (1 + T^{\frac{\alpha}{2} - \alpha}) \text{Lip}_\gamma(G) \| \Phi_1 - \Phi_2 \|_{V^\alpha,p([0, T] \times \Omega; X)},
$$

and, for $\Phi \in V^\alpha,p_c([0, T] \times \Omega; X)$:

$$
(9) \quad \sup_{0 \leq t \leq T} \| s \mapsto (t - s)^{-\alpha}G(s, \Phi(s)) \|_{L^p(\Omega; Y)} \\
\leq (1 + T^{\frac{\alpha}{2} - \alpha}) M_\gamma(G)(1 + \| \Phi \|_{V^\alpha,p([0, T] \times \Omega; X)}).
$$

The following existence and uniqueness result for solutions to (SDE) is presented in [20, Theorem 6.2].

Theorem 2.9 (Van Neerven, Veraar and Weis, 2008). Consider (SDE) under the assumptions (A), (F), and (G). Let $x_0 \in L^p(\Omega, F_0; X_\eta)$ for $p \in (2, \infty)$ and $\eta > 0$ satisfying

$$
0 \leq \eta < \min\{\frac{3}{2} - \frac{1}{p} + \theta_F, \frac{1}{2} - \frac{1}{p} + \theta_C\}.
$$

Then for any $T > 0$ and any $\alpha \in [0, \frac{1}{2})$ there exists a unique $U \in V^\alpha,p_c([0, T] \times \Omega; X_\eta^A)$ such that $s \mapsto S(t - s)G(s, U(s))$ is stochastically integrable for all $t \in [0, T]$, and $U$ satisfies:

$$
(10) \quad U(t) = S(t)x_0 + \int_0^t S(t - s)F(s, U(s)) \, ds + \int_0^t S(t - s)G(s, U(s)) \, dW_H(s)
$$

almost surely for all $t \in [0, T]$. Moreover:

$$
(11) \quad \| U \|_{V^\alpha,p([0, T] \times \Omega; X_\eta^A)} \leq 1 + \| x_0 \|_{L^p(\Omega; X_\eta^A)}.
$$

Remark 2.10. In [20] the authors assume $\theta_F \leq 0$ and $\theta_C \leq 0$ (and promptly refer to them as $-\theta_F$ and $-\theta_C$). However, one may check that the theorem remains valid for $\theta_F, \theta_C \geq 0$, which leads to extra space regularity of the solution (i.e., greater values for $\eta$ in (11)).
Moreover, in [20] the authors assume \( \alpha > \frac{1}{p} - \theta_G \), this assumption can be dropped: existence of a solution in \( V_{c,p}^\alpha([0,t] \times \Omega; X) \) for any \( \alpha \in [0, \frac{1}{2}) \) follows by equation (6). Uniqueness of a solution for any \( \alpha \in [0, \frac{1}{2}) \) follows by observing that if \( \Phi \in V_{c,p}^\alpha([0,t] \times \Omega; X^A) \) for some \( \alpha \in [0, \frac{1}{2}) \), and \( \Phi \) satisfies (10), then \( \Phi \in V_{c,p}^{\beta,p}([0,t] \times \Omega; X^A) \) for \( 0 \leq \beta < \alpha + \frac{1}{2} + \theta_G - \eta \) (this follows by the proof of Proposition 2.12).

2.3. \( \gamma \)-Boundedness. For vector-valued stochastic integrals, the concept of \( \gamma \)-boundedness plays the same role as uniform boundedness does for ordinary integrals: the Kalton-Weis multiplier theorem (Proposition 2.13 below) allows one to estimate terms out of a stochastic integral, provided they are \( \gamma \)-bounded. Note that any \( \gamma \)-bounded set of operators is automatically uniformly bounded, and the reverse holds if \( X \) is a Hilbert space.

A family \( \mathcal{B} \subset \mathcal{L}(X,Y) \) is called \( \gamma \)-bounded if there exists a constant \( C \) such that for all \( N \geq 1 \), all \( x_1, \ldots, x_N \in X \), and all \( B_1, \ldots, B_N \in \mathcal{B} \) we have:

\[
E \left\| \sum_{n=1}^{N} \gamma_n B_n x_n \right\|_Y^2 \leq C^2 E \left\| \sum_{n=1}^{N} \gamma_n x_n \right\|_X^2.
\]

The least admissible constant \( C \) is called the \( \gamma \)-bound of \( \mathcal{B} \), notation: \( \gamma_{[X,Y]}(\mathcal{B}) \).

The following lemma is a direct consequence of the Kahane contraction principle:

Lemma 2.11. If \( \mathcal{B} \subset \mathcal{L}(X,Y) \) is \( \gamma \)-bounded and \( M > 0 \) then \( M \mathcal{B} := \{ aB : a \in [-M,M], B \in \mathcal{B} \} \) is \( \gamma \)-bounded with \( \gamma_{[X,Y]}(M \mathcal{B}) \leq M \gamma_{[X,Y]}(\mathcal{B}) \).

The following proposition, which is a variation of a result of Weis [24] Proposition 2.5, gives a sufficient condition for \( \gamma \)-boundedness.

Proposition 2.12. Let \( f : [0,T] \to \mathcal{L}(X,Y) \) be a function such that for all \( x \in X \) the function \( t \mapsto f(t)x \) is continuously differentiable. Suppose \( g \in L^1(0,T) \) is such that for all \( t \in (0,T) \):

\[
\left\| \frac{d}{dt} f(t)x \right\|_Y \leq g(t) \|x\|_X, \quad \text{for all } x \in X.
\]

Then the set \( \mathcal{B} := \{ f(t) : t \in (0,T) \} \) is \( \gamma \)-bounded in \( \mathcal{L}(X,Y) \) and

\[
\gamma_{[X,Y]}(\mathcal{B}) \leq \|f(0+)\| + \|g\|_{L^1(0,T)}.
\]

The following \( \gamma \)-multiplier result, due to Kalton and Weis [13] (see also [17]), establishes a relation between stochastic integrability and \( \gamma \)-boundedness.

Proposition 2.13 (\( \gamma \)-Multiplier theorem). Suppose \( X \) does not contain a closed subspace isomorphic to \( c_0 \). Suppose \( M : (0,T) \to \mathcal{L}(X,Y) \) is a strongly measurable function with \( \gamma \)-bounded range \( \mathcal{M} = \{ M(t) : t \in (0,T) \} \). If \( \Phi \in \gamma(0,T,H;X) \) then \( M \Phi \in \gamma(0,T,H;Y) \) and:

\[
\|M\Phi\|_{\gamma(0,T,H;Y)} \leq \gamma_{[X,Y]}(\mathcal{M}) \|\Phi\|_{\gamma(0,T,H;X)}.
\]

Remark 2.14. The assumption that \( X \) should not contain a copy of \( c_0 \) can be avoided, provided one replaces \( \gamma(0,T,H;X) \) by \( \gamma_{\alpha_0}(L^2(0,T;H),X) \), the space of all \( \gamma \)-summing operators from \( L^2(0,T;H) \) to \( X \). We refer to [17] for more details. In all applications in this paper, \( X \) is a UMD space and therefore does not contain a copy of \( c_0 \).

Finally, we recall the following \( \gamma \)-boundedness estimate for analytic semigroups (see e.g. [20] Lemma 4.1]).
Lemma 2.15. Let $X$ be a Banach space and let $A$ be the generator of an analytic $C_0$-semigroup $S$ of type $(\omega, \theta, K)$ on $X$. Then for all $0 \leq \delta < \alpha$ and $T > 0$ there exists a constant $C$ depending on $\omega$, $\theta$, and $K$, such that for all $t \in (0, T]$ the set $\mathcal{I}_{\alpha,t} = \{ s^\alpha S(s) : s \in [0, t] \}$ is $\gamma$-bounded in $\mathcal{L}(X, X_\delta^\alpha)$ and we have
\[
\gamma_{[X, X_\delta^\alpha]}(\mathcal{I}_{\alpha,t}) \leq C t^{\alpha-\delta}, \quad t \in (0, T).
\]

Note that the constant $C$ in the lemma above may depend on $T$.

2.4. Estimates for (stochastic) convolutions. In this section we provide the estimates for (stochastic) convolutions necessary to derive the perturbation result given in Theorem 3.1. In order to avoid confusion further on, we shall use $Y_1$ and $Y_2$ to denote UMD Banach spaces in this section.

The following lemma is proven in [6]. It is an adaptation of [20, Proposition 4.5].

Lemma 2.16. Let $(R, \mathcal{R}, \mu)$ be a finite measure space and $(S, \mathcal{S}, \nu)$ a $\sigma$-finite measure space. Let $\Phi_1 : [0, T] \times \Omega \rightarrow \mathcal{L}(H, Y_1)$, let $\Phi_2 \in L^1(R; \mathcal{L}(Y_1, Y_2))$, and let $f \in L^\infty(R \times [0, T]; L^2(S))$. If $\Phi_1$ is $L^p$-stochastically integrable for some $p \in (1, \infty)$, then
\[
\left\| \int_0^T \int_R f(r, u)(s)\Phi_2(r)\Phi_1(u) \, d\mu(r) \, dW_H(u) \right\|_{L^p(\Omega; \gamma(0, T; Y_2))} \lesssim \operatorname{ess sup}_{(r,u) \in R \times [0, T]} \|f(r, u)\|_{L^2(S)} \|\Phi_2\|_{L^1(R; \mathcal{L}(Y_1, Y_2))} \|\Phi_1\|_{L^p(\Omega; \gamma(0, T; H; Y_1))},
\]
with implied depending only on $p$, $Y_1$, $Y_2$, provided the right-hand side is finite.

To our knowledge, most regularity results for stochastic convolutions are based on the factorization method introduced in [3]. The result below is merely based on the regularity of the convolving functions.

Lemma 2.17. Let $T > 0$, $p \in [1, \infty)$ and $\eta > 0$. Suppose the process $\Phi \in L^p(\Omega; \gamma(0, T, H; Y_1))$ is adapted to $(\mathcal{F}_t)_{t \geq 0}$ and satisfies:
\[
\sup_{0 \leq t \leq T} \|s \mapsto (t-s)^{-\eta} \Phi(s)\|_{L^p(\Omega; \gamma(0, t, H; Y_1))} < \infty.
\]

Let $\Psi : [0, T] \rightarrow \mathcal{L}(Y_1, Y_2)$ be such that $\Psi x$ is continuously differentiable on $(0, T)$ for all $x \in Y_1$. Suppose moreover there exists a $g \in L^1(0, T)$ and $0 \leq \theta < \eta$ such that
\[
v^\theta \|\frac{d}{dv}\Psi(v)x\|_{Y_2} + \theta v^{\theta-1}\|\Psi(v)x\|_{Y_2} \leq g(v)\|x\|_{Y_1}, \quad \text{for all } x \in Y_1.
\]

Then the stochastic convolution process
\[
t \mapsto \int_0^t \Psi(t-s)\Phi(s) \, dW_H(s)
\]
is well-defined and
\[
\left\| t \mapsto \int_0^t \Psi(t-s)\Phi(s) \, dW_H(s) \right\|_{C^{\eta-\delta}([0, T]; L^p(\Omega; Y_2))} \leq 2 C_p \|g\|_{L^1(0, T)} \sup_{0 \leq t \leq T} \|s \mapsto (t-s)^{-\eta} \Phi(s)\|_{L^p(\Omega; \gamma(0, t, H; Y_1))},
\]
where $C_p$ is the constant in the Burkholder-Davis-Gundy inequality for the $p$th moment, for the space $Y_1$, see equation [4].
Before proving the Lemma, observe that the corollary below follows directly from Kolmogorov’s continuity criterion (see Theorem I.2.1 in Revuz and Yor).

**Corollary 2.18.** Let the setting be as in Lemma 2.17 and assume in addition that $\frac{1}{p} < \eta - \theta$. Let $0 < \beta < \eta - \theta - \frac{1}{p}$. There exists a modification of the stochastic convolution process $t \mapsto \int_0^t \Psi(t - s)\Phi(s)\,dW_H(s)$, which we shall denote by $\Psi \circ \Phi$, such that:

$$
\|\Psi \circ \Phi\|_{L^p(\Omega; C^\beta([0,T];Y_2))} \leq \tilde{C} \|g\|_{L^p([0,T])} \sup_{0 \leq t \leq T} \|s \mapsto (t - s)^{-\eta} \Phi(s)\|_{L^p(\Omega; \gamma(0,t,H;Y_1))},
$$

where $\tilde{C}$ depends only on $\eta$, $\beta$, and $p$ and $\tilde{C}$.

**Proof of Lemma 2.17.** By Proposition 2.12 and assumption it follows that $\{s^\theta \Psi(s) : s \in [0, T]\}$ is $\gamma$-bounded. Thus by the Kalton-Weis multiplier Theorem (see Proposition 2.13), and the fact that

$$
\sup_{0 \leq t \leq T} \|s \mapsto (t - s)^{-\eta} \Phi(s)\|_{L^p(\Omega; \gamma(0,t,H;Y_1))} < \infty,
$$

it follows that $s \mapsto \Psi(t - s)\Phi(s)1_{s \in [0,t]} \in L^p(\Omega; \gamma(0,t,H;Y_2))$ for all $t \in [0, T]$. By Theorem 2.6 this process is stochastically integrable.

In what follows we let $\frac{d}{dt}$ denote the derivative with respect to the strong operator topology. By the triangle inequality we have:

$$
\left\| \int_0^t \Psi(t - u)\Phi(u)\,dW_H(u) - \int_0^s \Psi(s - u)\Phi(u)\,dW_H(u) \right\|_{L^p(\Omega;Y_2)}
\leq \left\| \int_0^s [\Psi(t - u) - \Psi(s - u)]\Phi(u)\,dW_H(u) \right\|_{L^p(\Omega;Y_2)}
+ \left\| \int_s^t \Psi(t - u)\Phi(u)\,dW_H(u) \right\|_{L^p(\Omega;Y_2)}
\leq \left\| \int_0^s (s - u)\Phi(u)\,dW_H(u) \right\|_{L^p(\Omega;Y_2)}
+ \left\| \int_s^t \Psi(t - u)\Phi(u)\,dW_H(u) \right\|_{L^p(\Omega;Y_2)}
+ \left\| \int_s^t \Psi(t - u)\Phi(u)\,dW_H(u) \right\|_{L^p(\Omega;Y_2)}.
$$

We now wish to apply the stochastic Fubini theorem (see [3] Lemma 2.7, [18]). Consider $\Upsilon : [0, s] \times [0, t] \to \mathcal{L}(H,Y)$ defined by $\Upsilon(u, v) = 1_{\{s - u \leq v \leq t - u\}} \frac{d}{dv} \Psi(v)\Phi(u)$. As $\frac{d}{dv} \Psi$ is strongly continuous and $\Phi$ is $H$-strongly measurable, we have that $\Upsilon$ is $\mathcal{H}$-strongly measurable. Moreover, as $\Phi$ is adapted it follows that $\Upsilon_v := \Upsilon(\cdot, v)$ is adapted for almost all $v \in [0, t]$. Finally, we have that $\Upsilon \in L^1([0,t]; \gamma(0,s,H;Y_2))$ by assumption:

$$
\|\Upsilon(\cdot, v)\|_{\gamma(0,s,H;Y_2)} \leq v^{\eta - \theta} g(v) \|u \mapsto (s - u)^{-\eta} \Phi(u)\|_{\gamma(0,s,H;Y_1)},
$$

where we use that $v \geq s - u$ on supp($\Upsilon$).

Note that stochastic Fubini theorem in [3] Lemma 2.7, [18] requires $\Upsilon_v$ to be progressive. However, it suffices to assume that $\Upsilon_v$ is adapted, see [23]. Thus the conditions necessary to apply the stochastic Fubini theorem are satisfied, and we
have:
\[
\begin{align*}
\left\| \int_0^t \int_{t-u}^{t} \frac{d}{dv} [\Psi(v) \Phi(u)] dv \, dW_H(u) \right\|_{L^p(\Omega; Y_2)} = \left\| \int_0^t \int_{(s-v)\land 0}^{(t-v)\land s} \frac{d}{dv} [\Psi(v) \Phi(u)] dv \, dW_H(u) \right\|_{L^p(\Omega; Y_2)} \\
\leq \int_0^t v^{-\theta} g(v) \left\| \int_{(s-v)\land 0}^{(t-v)\land s} \Phi(u) \, dW_H(u) \right\|_{L^p(\Omega; Y_1)} dv \\
\leq \bar{C}_p \int_0^t v^{-\theta} g(v) \left\| 1_{[(s-v)\land 0, (t-v)\land s]} \right\|_{L^p(\Omega; \gamma(0,t; Y_1))} dv \\
\leq \bar{C}_p \int_0^t v^{-\theta} g(v) \sup_{t \in [0,T]} \left\| u \mapsto (t - u)^{-\eta} \Phi(u) \right\|_{L^p(\Omega; \gamma(0,t; Y_1))} dv \\
\leq \bar{C}_p (t - s)^{\eta - \theta} \int_0^t g(v) dv \sup_{t \in [0,T]} \left\| u \mapsto (t - u)^{-\eta} \Phi(u) \right\|_{L^p(\Omega; \gamma(0,t; Y_1))}.
\end{align*}
\]

For the final term in (12) one may also check that the conditions of the stochastic Fubini hold and thus:
\[
\begin{align*}
\left\| \int_0^t (t - u)^{-\theta} \int_u^t \frac{d}{dv} [v^{\beta} \Psi(v) \Phi(u)] dv \, dW_H(u) \right\|_{L^p(\Omega; Y_2)} \\
\leq \int_0^t \tilde{C}_p \int_0^{t-s} g(v) \left\| u \mapsto 1_{[s,v\land s]}(u)(t-u)^{-\theta} \Phi(u) \right\|_{L^p(\Omega; \gamma(0,t; H; F_1))} dv \\
\leq \bar{C}_p (t - s)^{\eta - \theta} \|g\|_{L^1(0,T)} \sup_{t \in [0,T]} \left\| u \mapsto (t - u)^{-\eta} \Phi(u) \right\|_{L^p(\Omega; \gamma(0,t; Y_1))},
\end{align*}
\]

By inserting the two estimates above in (12) we obtain that
\[
\begin{align*}
\left\| \int_0^t \Psi(t-u) \Phi(u) \, dW_H(u) - \int_0^s \Psi(s-u) \Phi(u) \, dW_H(u) \right\|_{L^p(\Omega; Y_2)} \\
\leq 2\bar{C}_p (t - s)^{\eta - \theta} \|g\|_{L^1(0,T)} \sup_{t \in [0,T]} \left\| u \mapsto (t - u)^{-\eta} \Phi(u) \right\|_{L^p(\Omega; \gamma(0,t; Y_1)},
\end{align*}
\]
which completes the proof as \(0 \leq s < t \leq T\) where chosen arbitrarily. \(\square\)

Remark 2.19. In the setting of the lemma above one may also take \(g \in L^q(0,T)\) and \(\Phi\) such that
\[
\int_0^T \left\| s \mapsto (t-s)^{-\eta} \Phi(s) \right\|^q_{L^p(\Omega; \gamma(0,t; Y_1))} dt < \infty,
\]
where \(q \in [1, \infty); \frac{1}{q} + \frac{1}{q'} = 1\). In that case one obtains:
\[
\begin{align*}
\left\| t \mapsto \int_0^t \Psi(t-s) \Phi(s) \, dW_H(s) \right\|_{C^{\eta - \theta}(0,T; L^p(\Omega; Y_2))} \\
\leq (3 + 2^\eta) \bar{C}_p \|g\|_{L^{q'}(0,T)} \left( \int_0^T \left\| s \mapsto (t-s)^{-\eta} \Phi(s) \right\|^q_{L^p(\Omega; \gamma(0,t; H; F_1))} dt \right)^{\frac{1}{q'}}.
\end{align*}
\]

We omit the proof because it requires significantly more space, and we do not need this result in what follows (for the extended proof, see [4, Lemma A.9]).
Based on the above two lemmas, we obtain the following result for stochastic convolutions in the $V_c^{\alpha, p}$-norm:

**Proposition 2.20.** Let the setting be as in Lemma 2.17 and assume in addition that $\frac{1}{p} < \frac{1}{p} - \frac{1}{p}$. Let $\alpha \in [0, \frac{1}{2})$. Then $\Psi \circ \Phi \in V_c^{\alpha, p}([0, T] \times \Omega; Y_2)$. Moreover, there exists a constant $C$ such that for all $T_0 \in [0, T]$ we have:

$$
\| \Psi \circ \Phi \|_{V_c^{\alpha, p}([0, T_0] \times \Omega; Y_2)} \leq C \| g \|_{L^1(\Omega)} \sup_{0 \leq t \leq T_0} \| s \mapsto (t-s)^{-\eta} \Phi(s) \|_{L^p(\Omega; \gamma(0, t; Y_1))}.
$$

**Proof.** For the norm estimate in $L^p(\Omega; C([0, T_0]; Y_2))$ we apply Corollary 2.18 with $\beta > 0$ such that $0 \leq \beta + \theta < \frac{1}{p} - \frac{1}{p}$. For the estimate in the weighted $\gamma$-norm fix $t \in [0, T_0]$. We apply Lemma 2.11 with $\Phi_1(u) = (t-u)^{-\eta} \Phi(u) 1_{0 \leq u < t}$, $\Phi_2(r) = \frac{\gamma}{\pi} e^{\beta} \Psi(r)$, $R = [0, t]$ and $f(r, u)(s) = (t-s)^{-\alpha}(s-u)^{-\theta}(t-u)^{\eta} 1_{0 \leq r < s-u} 1_{0 \leq u < t}$. From Lemma 2.10 it follows that:

$$
\| s \mapsto (t-s)^{-\alpha} \int_0^s (s-u) \Phi(u) dW_H(u) \|_{L^p(\Omega; \gamma(0, t; Y_2))} \lesssim t^{\frac{1}{2} + \eta - \alpha - \theta} \| g \|_{L^p(\Omega; \gamma(0, t; Y_2))} \| s \mapsto (t-s)^{-\eta} \Phi(s) \|_{L^p(\Omega; \gamma(0, t; Y_1))}.
$$

Taking the supremum over $t \in [0, T_0]$ and using that $\theta - \eta > 0$ (whence $T_0^{\frac{1}{2} + \eta - \alpha - \theta} \leq T^{\frac{1}{2} + \eta - \alpha - \theta}$) we arrive at the desired result.

For deterministic convolutions we have the following:

**Proposition 2.21.** Suppose $\Phi \in L^p(\Omega; L^\infty(0, T; Y_1))$ for some $p \in [1, \infty)$. Let $\Psi : [0, T] \rightarrow \mathcal{L}(Y_1, Y_2)$ be such that $\Psi x$ is continuously differentiable on $(0, T)$ for all $x \in Y_1$. Suppose moreover there exists a $g \in L^1(0, T)$ and a $\theta \in [0, 1]$ such that for all $v \in (0, T)$ we have:

$$
v^\theta \| \frac{\partial}{\partial \theta} \Psi(v) x \|_{Y_2} + \theta v^\theta - 1 \| \Psi(v) x \|_{Y_2} \leq g(v) \| x \|_{Y_1}, \quad \text{for all } x \in Y_1.
$$

Then there exists a constant $C$ such that for all $T_0 \in [0, T]$ we have, almost surely:

$$
\| \Psi \circ \Phi \|_{C^{1, \theta}([0, T_0]; Y_2)} \leq C \| g \|_{L^1(0, T_0)} \| \Phi \|_{L^\infty(0, T_0; Y_1)}.
$$

By Lemma 2.8 with $\varepsilon = \frac{1}{2} - \frac{1}{p} T - \theta$, we obtain the following corollary:

**Corollary 2.22.** Let the setting be as in Proposition 2.21. Assume in addition that $Y_2$ has type $\tau$, and let $0 \leq \theta < \frac{1}{2} - \frac{1}{p}$. Then for $\alpha \in [0, \frac{1}{2})$ and $p \in [1, \infty)$ there exists a constant $C$ such that for $T_0 \in [0, T]$ one has:

$$
\| \Psi \circ \Phi \|_{V_c^{\alpha, p}([0, T_0] \times \Omega; Y_2)} \leq C \| g \|_{L^1(\Omega)} \| \Phi \|_{L^p(\Omega; L^\infty(0, T_0; Y_1))}.
$$

**Proof of Proposition 2.21.** Observe that we have, for $0 \leq s < t \leq T_0$:

$$
\left\| \int_0^t (s-u) \Phi(u, \omega) du - \int_0^s \Phi(s-u) \Phi(u, \omega) du \right\|_{Y_2}
\leq \left\| \int_0^s \int_{s-u}^{t-u} \frac{d}{dv} |\Psi(v) \Phi(u, \omega)| dv du \right\|_{Y_2}
+ \left\| \int_s^t (t-u)^{-\theta} \int_0^{t-u} \frac{d}{dv} |\Psi(v) \Phi(u, \omega)| dv du \right\|_{Y_2}.
$$

(14)
Now
\[ \left\| \int_0^s \int_{s-u}^{(t-v)\wedge s} \frac{d}{dv}[\Psi(v)\Phi(u, \omega)] \, dv \, du \right\|_{Y_2} \]
\[ \leq \int_0^t \int_{(s-v)\wedge s}^{(t-v)\wedge s} du (s - v)^{-\theta} g(v) \, dv \|\Phi(\omega)\|_{L^\infty(0,t;Y_1)} \]
\[ \leq (t - s)^{1-\theta} \int_0^t g(v) \, dv \|\Phi(\omega)\|_{L^\infty(0,t;Y_1)}, \]
where we used that \( \int_{(s-v)\wedge s}^{(t-v)\wedge s} du \leq (t - s) \wedge v. \) Furthermore, we have
\[ \left\| \int_s^t (t - u)^{-\theta} \int_0^{t-u} \frac{d}{dv}[v^\theta \Psi(v)] \Phi(u, \omega) \, dv \, du \right\|_{Y_2} \]
\[ \leq (1 - \theta)^{-1} (t - s)^{1-\theta} \left\| \int_0^t g(v) \, dv \|\Phi(\omega)\|_{L^\infty(0,t;Y_1)}. \]
Inserting these two estimates in (14) completes the proof. \( \Box \)

3. A Perturbation Result

In this section we shall prove the perturbation theorem announced in the introduction. Consider (\textbf{SDE}) with \( A, F \) and \( G \) satisfying (A), (F), and (G). Keeping in mind possible applications in approximations of solutions to stochastic partial differential equations, we let \( X_0 \) be a (possibly finite-dimensional) closed subspace of \( X \). We assume there exists a bounded projection \( P_0 : X \to X_0 \) such that \( P_0(X) = X_0 \). Let \( i_{X_0} \) represent the canonical embedding of \( X_0 \) into \( X \) (note however that we shall omit \( i_{X_0} \) when it is clear from the context).

Let \( A_0 \) be the generator of an analytic \( C_0 \)-semigroup \( S_0 \) on \( X_0 \). For \( t \geq 0 \) define \( \tilde{S}_0 \in \mathcal{L}(X) \) by \( \tilde{S}_0(t) := i_{X_0} S_0(t) P_0 \), this defines a degenerate \( C_0 \)-semigroup, i.e., \( \tilde{S}_0 \) satisfies the semigroup property but \( \tilde{S}_0(0) = i_{X_0} P_0 \) (which is clearly not the identity unless \( X_0 = X \)).

Let \( x_0 \in L^p(\Omega, F_0; X) \) (where \( p > 2 \) satisfies \( \frac{1}{p} \leq \frac{1}{2} + \theta \) and let \( U \) be the solution to (\textbf{SDE}) as provided by Theorem 2.9.

**Theorem 3.1.** Let \( \omega \geq 0, \theta \in (0, \frac{1}{2}) \) and \( K > 0 \) be such that \( A \) and \( A_0 \) are both of type \((\omega, \theta, K)\). Suppose there exist \( \delta \in [0, 1] \) and \( p \in (2, \infty) \) satisfying
\[ 0 \leq \delta < \min\{ \frac{\omega}{2} - \frac{1}{p} + \theta, \frac{\omega}{2} - \frac{1}{p} + \theta \} \]
such that for some \( \lambda_0 \in \rho(A) \) we have:
\[ D_\delta(A, A_0) := \| R(\lambda_0 : A) - i_{X_0} R(\lambda_0 : A_0) P_0 \|_{\mathcal{L}(X_{\delta^{-1}}, X)} < \infty. \]
Suppose \( x_0 \in L^p(\Omega, F_0; X^A) \) and \( y_0 \in L^p(\Omega, F_0; X) \).

For any \( \alpha \in [0, \frac{1}{2}) \) there exists a unique process \( \textbf{U}(0) \in V^\alpha_{c,p}([0, T_0] \times \Omega; X_0) \) such that \( s \mapsto 1_{[0, \alpha]} S_0(t - s) P_0 G(s, U(0)(s)) \) is stochastically integrable for all \( t \in [0, T_0] \) and for all \( t \in [0, T] \) we have:
\[ U^{(0)}(t) = S_0(t - s) P_0 y_0 + \int_0^T S_0(t - s) P_0 F(s, U^{(0)}(s)) \, ds \]
\[ + \int_0^t S_0(t - s) P_0 G(s, U^{(0)}(s)) \, dW_H(s), \quad a.s. \]
Moreover:
\[
\|U - iX_0U^{(0)}\|_{V_{\alpha}^\eta([0,T] \times \Omega;X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega;X)} + D_3(A, A_0)(1 + \|x_0\|_{L^p(\Omega;X^\beta)}).
\]

The implied constant depends on \(X_0\) only in terms of \(\|P_0\|_{L(X,X_0)}\), on \(A\) and \(A_0\) only in terms of \(1 + D_3(A, A_0)\), \(\omega\), \(\theta\) and \(K\), and on \(F\) and \(G\) only in terms of their Lipschitz and linear growth constants \(\text{Lip}(F), \text{Lip}_x(G), M(F)\) and \(M_x(G)\).

To prove Theorem 3.1, we need a proposition concerning the \(\gamma\)-boundedness of \(S - \tilde{S}_0\). The proof of this proposition is postponed to the end of this section.

**Proposition 3.2.** Let \(A, A_0\) be as introduced above, i.e., \(A\) generates an analytic semigroup on \(X\) and \(A_0\) generates an analytic semigroup on \(X_0\). Let \(\omega \geq 0, \theta \in (0, \frac{\pi}{2})\) and \(K > 0\) be such that \(A\) and \(A_0\) are of type \((\omega, \theta, K)\). Suppose there exists a \(\lambda_0 \in \mathbb{C}, \Re(\lambda_0) > \omega\), and \(\delta \in \mathbb{R}\) such that \(D_3(A, A_0) < \infty\), where \(D_3(A, A_0)\) is as defined in (13). Set

\[
\omega' = \omega + |\lambda_0 - \omega|/(\cos \theta)^{-1}.
\]

Then for all \(\beta \in \mathbb{R}\) such that \(\beta \in [\delta - 1, \delta]\) one has:

\[
(18) \quad \sup_{t \in [0,\infty)} t^{\delta - \beta} e^{-\omega' t}\|S(t) - \tilde{S}_0(t)\|_{L(X^\beta, X)} \lesssim D_3(A, A_0),
\]

and

\[
(19) \quad \sup_{t \in [0,\infty)} t^{\delta - \beta + 1} e^{-\omega' t}\|\frac{d}{dt}S(t) - \frac{d}{dt}\tilde{S}_0(t)\|_{L(X^\beta, X)} \lesssim D_3(A, A_0),
\]

with implied constants depending only on \(\|P_0\|_{L(X,X_0)}, \omega, \theta, K, \delta - \beta\).

Moreover, for all \(\alpha > \delta - \beta\) we have, for \(t \in [0,T]\):

\[
\gamma_{[X^\beta, X]} \left( \left\{ s^\alpha |S(s) - \tilde{S}_0(s)|; 0 \leq s \leq t \right\} \right) \lesssim t^{\alpha + \delta} D_3(A, A_0),
\]

with implied constant depending only on \(\|P_0\|_{L(X,X_0)}\), \(\omega\), \(\theta\), \(K\), \(\delta - \beta\), and \(T\).

**Proof of Theorem 3.1.** We split the proof into several parts.

**Part 1.** In order to prove existence and uniqueness of \(U^{(0)} \in V_{\alpha}^\eta([0,T_0] \times \Omega;X_0)\) satisfying (10) it suffices, by Theorem 2.9, to prove that there exist \(\eta_F > -1 - \frac{1}{p}\) and \(\eta_G > -1 - \frac{1}{p}\) such that \(P_0F: [0,T] \times X \to X_{0,\eta_F}\) is Lipschitz continuous and of linear growth and \(P_0G: [0,T] \times X \to \gamma(H, X_{0,\eta_G})\) is \(L_2^\gamma\)-Lipschitz continuous and of linear growth. If \(\theta_F \geq 0\) then clearly we may take \(\eta_F = 0\), and we have \(\text{Lip}(P_0F) \leq \|P_0\|_{L(X,X_0)}\text{Lip}(F), \|P_0G\|_{L(X,X_0)}\text{Lip}(G)\). The same goes for \(\theta_G \geq 0\).

Now suppose \(\theta_F < 0\). Recall the following representation of negative fractional powers of an operator \(A\) generating an analytic semigroup \(S\) of type \((\omega, \theta, K)\) (see [21 Chapter 2.6]):

\[
(M - A)\eta = \frac{1}{\Gamma(-\eta)} \int_0^\infty t^{-\eta - 1} e^{-Mt} dt, \quad \eta < 0, \quad \Re(e(\lambda)) > \omega.
\]

Let \(\bar{\omega} > \omega'\), where \(\omega'\) is as in Proposition 3.2. From the representation above and Proposition 3.2 it follows that for \(\beta \in [\delta - 1, \delta]\), \(\eta < \beta - \delta\) and \(x \in X\) we have:

\[
\|P_0x\|_{X_{0,\eta}} \approx \|((\bar{\omega}I - A_0)\eta P_0x\|_{X} = \left\| \frac{1}{\Gamma(-\eta)} \int_0^\infty t^{-\eta - 1} e^{-\bar{\omega}t} \tilde{S}_0(t) x dt \right\|_{X}.
\]
with implied constants depending on $X_0$ only in terms of $\|P_0\|_{L(X,X_0)}$ and on $A$ and $A_0$ only in terms of $\omega, \theta,$ and $K$. Thus for $\beta \in [\delta - 1, \delta]$, $\eta < \beta - \delta$ we have:

$$\|P_0x\|_{X^\beta_{0,\eta}} \approx \|(\tilde{\omega}I - A_0)^\beta P_0x\|_X$$

(20)

with implied constants depending on $X_0$ only in terms of $\|P_0\|_{L(X,X_0)}$ and on $A$ and $A_0$ only in terms of $\omega, \theta,$ and $K$.

Note that by assumption we have $\theta_F > -\frac{1}{2} + \frac{1}{2} + \delta \geq 1$. Hence one can pick $\eta_F$ such that $-\frac{1}{2} + \frac{1}{2} + \eta_F < \theta_F - \delta$. By (20) it follows that $P_0F : [0,T] \times X \to X^\beta_{0,\eta_F}$ is Lipschitz continuous and

$$\text{Lip}(P_0F) \lesssim (1 + D(A, A_0))\text{Lip}(F); \quad M(P_0F) \lesssim (1 + D(A, A_0))M(F),$$

with implied constant depending on $X_0$ only in terms of $\|P_0\|_{L(X,X_0)}$ and on $A$ and $A_0$ only in terms of $\omega, \theta,$ and $K$.

Similarly, if $\theta_G < 0$ there exists a $\eta_G$ such that $-\frac{1}{2} + \frac{1}{2} + \eta_G < \theta_G - \delta$ such that $P_0G : [0,T] \times X \to \gamma(H, X^\beta_{0,\eta_G})$ is $L^2_\gamma$-Lipschitz continuous and

$$\text{Lip}_\gamma(P_0G) \lesssim (1 + D(A, A_0))\text{Lip}_\gamma(G); \quad M_\gamma(P_0G) \lesssim (1 + D(A, A_0))M_\gamma(G),$$

with implied constant depending on $X_0$ only in terms of $\|P_0\|_{L(X,X_0)}$ and on $A$ and $A_0$ only in terms of $\omega, \theta,$ and $K$.

Part 2. Define $\tilde{U}^{(0)} = iX_0U^{(0)}$ and observe that if $U^{(0)}$ satisfies (16), then $\tilde{U}^{(0)}$ satisfies:

$$\tilde{U}^{(0)}(t) = \tilde{S}_0(t-s)y_0 + \int_0^T \tilde{S}_0(t-s)F(s, \tilde{U}^{(0)}(s))\,ds$$

$$+ \int_0^t \tilde{S}_0(t-s)G(s, \tilde{U}^{(0)}(s))\,dW_H(s), \quad \text{a.s.}$$

Let $T_0 \in [0,T]$ be fixed. By the above we have:

$$\|U - \tilde{U}^{(0)}\|_{V^{\alpha,r}_{0,\eta_{0,T_0} \times \Omega;X}}$$

$$\leq \|(S - \tilde{S}_0)x_0\|_{V^{\alpha,r}_{0,\eta_{0,T_0} \times \Omega;X}} + \|\tilde{S}_0(x_0 - y_0)\|_{V^{\alpha,r}_{0,\eta_{0,T_0} \times \Omega;X}}$$

$$+ \left\|t \mapsto \int_0^t \tilde{S}_0(t-s)[F(s, U(s)) - F(s, \tilde{U}^{(0)}(s))]\,ds\right\|_{V^{\alpha,r}_{0,\eta_{0,T_0} \times \Omega;X}}$$

$$+ \left\|t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]F(s, U(s))\,ds\right\|_{V^{\alpha,r}_{0,\eta_{0,T_0} \times \Omega;X}}$$

$$+ \left\|t \mapsto \int_0^t \tilde{S}_0(t-s)[G(s, U(s)) - G(s, \tilde{U}^{(0)}(s))]\,dW_H(s)\right\|_{V^{\alpha,r}_{0,\eta_{0,T_0} \times \Omega;X}}$$

$$+ \left\|t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]G(s, U(s))\,dW_H(s)\right\|_{V^{\alpha,r}_{0,\eta_{0,T_0} \times \Omega;X}}.$$
Let $\eta_F$ and $\eta_G$ be as defined in part 1. Let $\varepsilon > 0$ be such that
$$\varepsilon \leq 1 - 2\alpha;$$
$$\varepsilon < \min\{\frac{3}{2} - \frac{1}{p} + \eta_F, \frac{1}{2} - \frac{1}{p} + \eta_G\}.$$ It follows that $\varepsilon + \delta < \min\{\frac{3}{2} - \frac{1}{p} + \theta_F, \frac{1}{2} - \frac{1}{p} + \theta_G\}$. By equation (3) we may assume, without loss of generality, that $\alpha = \frac{1}{2} - \varepsilon/2$.

We will estimate each of the six terms on the right-hand side of (23) in parts 2a-2f below. In part 2c and 2e we keep track of the dependence on $T_0$, for the other parts this is not necessary.

**Part 2a.** By Proposition 3.2 with $\beta = \delta$ there exists an $M > 0$ depending on $X_0$ only in terms of $\|P_0\|_{\mathcal{L}(X, X_0)}$, and on $A$ and $A_0$ only in terms on $\omega, \theta$ and $K$, such that
$$\sup_{t \in [0, T_0]} \|S(t) - \tilde{S}_0(t)\|_{\mathcal{L}(X^A_{\delta}, X)} \leq MD_\delta(A, A_0);$$
$$\gamma_{[X^A_{\delta}, X]}\{t^{\varepsilon/2}(S(t) - \tilde{S}_0(t)) : t \in [0, T_0]\} \leq MD_\delta(A, A_0).$$

Thus by Proposition 2.13 we have
$$\|S - \tilde{S}_0\|_{V_2, p([0, T_0]; \Omega; X)} \leq MD_\delta(A, A_0) \left[ \sup_{t \in [0, T_0]} \|s \mapsto (t - s)^{-\alpha}s^{-\varepsilon/2}x_0\|_{L^p(\Omega; \gamma(0, t; X^A_{\delta}))} + \|x_0\|_{L^p(\Omega; X^A_{\delta})} \right].$$

For $f \in L^2(0, t)$ and $x \in L^p(\Omega; X^A_{\delta})$ we have
$$\|f \otimes x\|_{L^p(\Omega; \gamma(0, t; X^A_{\delta}))} = \|f\|_{L^2(0, t)}\|x\|_{L^p(\Omega; X^A_{\delta})}.$$ Thus, recalling that $\alpha = \frac{1}{2} - \varepsilon/2$, we have:
$$\sup_{t \in [0, T_0]} \|s \mapsto (t - s)^{-\alpha}s^{-\varepsilon/2}x_0\|_{L^p(\Omega; \gamma(0, t; X^A_{\delta}))} \leq \|s \mapsto (1 - s)^{-\alpha}s^{-\varepsilon/2}\|_{L^2(0, 1)}\|x_0\|_{L^p(\Omega; X^A_{\delta})} \leq C_\varepsilon\|x_0\|_{L^p(\Omega; X^A_{\delta})},$$
where $C_\varepsilon$ is a constant depending only on $\varepsilon$, and we used that $\alpha = \frac{1}{2} - \varepsilon/2$. Hence
$$\|S - \tilde{S}_0\|_{V_2, p([0, T_0]; \Omega; X)} \leq MD_\delta(A, A_0)(1 + C_\varepsilon)\|x_0\|_{L^p(\Omega; X^A_{\delta})}.$$ **Part 2b.** By assumption (see Remark 2.3) there exists an $M$ depending only on $\|P_0\|_{\mathcal{L}(X, X_0)}$, $\omega, \theta$, and $K$ and $T$ such that we have that $\sup_{t \in [0, T]} \|\tilde{S}_0(t)\|_{\mathcal{L}(X, X_0)} \leq M$. Moreover, by Lemma 2.13 we may pick $M$ such that in addition we have that $\gamma_{[X, X]}\{t^{\varepsilon/2}\tilde{S}_0(t) : t \in [0, T]\} \leq M$. Thus by the same argument as in part 2a we have:
$$\|\tilde{S}_0(x_0 - y_0)\|_{V_2, p([0, T_0]; \Omega; X)} \leq M(1 + C_\varepsilon)\|x_0 - y_0\|_{L^p(\Omega; X)}.$$ **Part 2c.** Recall that $\eta_F \leq 0$. By equation (3) there exists an $M$ depending only on $\omega$, $\theta$, $K$ and $T$ such that for all $t \in [0, T]$ we have:
$$t^{-\eta_F + \varepsilon}\|\frac{d}{dt}S_0(t)x\|_{\mathcal{L}(X_{\delta, \eta_F, X})} + (\varepsilon - \eta_F)t^{-\eta_F + \varepsilon - 1}\|S_0(t)x\|_{\mathcal{L}(X_{\delta, X_0})} \leq M t^{-\varepsilon}.$$
By Corollary \[\text{2.22}\] with \(Y_1 = X_{0,\eta_F}\), \(Y_2 = X\),
\[
\Phi(s) = P_0[F(s, U(s)) - F(s, \bar{U}(0)(s))],
\]
\(\Psi(s) = S_0(s), \theta = -\eta_F + \varepsilon\) and \(g(v) = Mv^{-1+\varepsilon}\), it follows that:
\[
\|t \mapsto \int_0^t \tilde{S}_0(t-s)[F(s, U(s)) - F(s, \bar{U}(0))]| ds \|_{W^{s, p}([0, T_0] \times \Omega; \mathbb{X})}
\lesssim T_0^s \|P_0[F(\cdot, U) - F(\cdot, \bar{U}(0))]\|_{L^p(\Omega; L^\infty(0, T_0; X_{0, \eta_F}^0))}
\lesssim T_0^s \text{Lip}(P_0 F) \|U - \bar{U}(0)\|_{L^p(\Omega; L^\infty(0, T_0; X))}
\lesssim T_0^s (1 + D_A(A, A_0)) \text{Lip}(F) \|U - \bar{U}(0)\|_{L^p(\Omega; L^\infty(0, T_0; X))},
\]
(26)
where the second-last estimate follows by Lipschitz-continuity of \(P_0\) and the final estimate follows by \[\text{21}\]. Note that the implied constants are independent of \(T_0\), and depend on \(x_0\) only in terms of \(\|P_0\|_{L(X, X_0)}\) and on \(A_0\) only in terms of \(\omega, \theta\) and \(K\).

Part 2d. By Proposition \[\text{3.2}\] with \(\beta = \theta_F \wedge \delta \in [\delta - 1, \delta]\) we have that there exists a constant \(\mathcal{M}\) depending only on \(\|P_0\|_{L(X, X_0)}\), \(\omega, \theta, K\), \((\delta - \theta_F) \lor 0\) and \(T\) such that for all \(t \in [0, T]\) we have:
\[
t^{(\delta - \theta_F) + \varepsilon} \|d[S(t) - \tilde{S}_0(t)]\|_{L(X_{\theta_F, \delta}^A, X)}
+ ((\delta - \theta_F) + \varepsilon) t^{((\delta - \theta_F) + \varepsilon - 1)} \|S(t) - \tilde{S}_0(t)\|_{L(X_{\theta_F, \delta}^A, X)} \leq \mathcal{M} D_\delta(A, A_0) t^{-1 + \varepsilon}.
\]
Thus by Corollary \[\text{2.22}\] with \(Y_1 = X_{\theta_F, \delta}^A\), \(Y_2 = X\), \(\Phi(s) = F(s, U(s))\), \(\Psi(s) = S(s) - \tilde{S}_0(s), \theta = (\delta - \theta_F) + \varepsilon\), \(g(v) = \mathcal{M} D_\delta(A, A_0) v^{-1 + \varepsilon}\), we obtain:
\[
\|t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]F(s, U(s))| ds \|_{W^{s, p}([0, T_0] \times \Omega; \mathbb{X})}
\lesssim D_\delta(A, A_0) \|F(\cdot, U)\|_{L^p(\Omega; L^\infty(0, T_0; X_{\theta_F, \delta}^A))}
\lesssim D_\delta(A, A_0) M(F) \|U\|_{L^p(\Omega; L^\infty(0, T_0; X))}
\lesssim D_\delta(A, A_0) M(F) (1 + \|x_0\|_{L^p(\Omega; X)}),
\]
(27)
where the penultimate estimate follows by the linear growth condition on \(F\) and the final estimate by \[\text{11}\]. Note that the implied constants are independent of \(T_0\), and depend on \(x_0\) only in terms of \(\|P_0\|_{L(X, X_0)}\), and on \(A\) and \(A_0\) only in terms of \(\omega, \theta\) and \(K\).

Part 2e. Recall that \(\eta_G \leq 0\). By equation \[\text{3}\] there exists an \(\mathcal{M}\) depending only on \(\omega, \theta, K\) such that for all \(t \in [0, T]\) we have:
\[
t^{-\eta_G + \varepsilon/2} \|d[S(t)]\|_{L(X_{\alpha, -\eta_G}^A, X_0)} + ((\varepsilon/2 - \eta_G) t^{-\eta_G + \varepsilon/2 - 1}) \|S_0(t)\|_{L(X_{\alpha, -\eta_G}^A, X_0)} \leq \mathcal{M} t^{-1 + \varepsilon/2}.
\]
By applying Proposition \[\text{2.20}\] with \(Y_1 = X_{\alpha, -\eta_G}^A\), \(Y_2 = X\), \(\Psi(s) = S_0(s), \eta = \alpha, \alpha = \alpha, \theta = -\eta_G + \varepsilon/2\) and \(g(v) = \mathcal{M} v^{-1+\varepsilon/2}\), we obtain:
\[
\Phi(s) = P_0[G(s, U(s)) - G(s, \bar{U}(0)(s))]
\]
such that for all $T$ and $x$ where the penultimate line follows by estimate (9). Note that the implied constants and $A$ Setting $T$ we obtain:

$$\|t \mapsto \int_0^t \tilde{S}_0(t-s)[G(s, U(s)) - G(s, U(0)(s))] dW_H(s)\|_{V^{\alpha,p}_c([0,T_0] \times \Omega; X)}$$

$$\lesssim T_0^{1+\varepsilon} \sup_{0 \leq t \leq T_0} \|s \mapsto (t-s)^{-\alpha} P_0[G(s, U(s)) - G(s, U(0)(s))]\|_{L^p(\Omega; \gamma(0, t; X^{A_0}_{s, t}))}$$

$$\lesssim T_0^{1+\varepsilon} (1 + D_\delta(A, A_0)) \text{Lip}_\gamma(G)\|U - \tilde{U}(0)\|_{V^{\alpha,p}_c([0,T_0] \times \Omega; X)},$$

where the final estimate follows from estimates (5) and (22). Note that the implied constants are independent of $T_0$, and depend on $X_0$ only in terms of $\|P_0\|_{L(\Omega, X_0)}$, and on $A$ and $A_0$ only in terms of $\omega$, $\theta$ and $K$.

**Part 2f.** By Proposition 3.2 with $\beta = \theta_G$ and $\delta \in [\delta - 1, \delta]$ we have that there exists a constant $M$ depending only on $\|P_0\|_{L(\Omega, X_0)}$, $\omega, \theta, K$, $(\delta - \theta_G) \vee 0$ and $T$ such that for all $t \in [0, T]$ we have:

$$t^{(\delta - \theta_G)^+ + \varepsilon/2} \|\frac{d}{dt}[S(t) - \tilde{S}_0(t)]\|_{L(X^{A}_\delta, X)}$$

$$+ ((\delta - \theta_G)^+ + \varepsilon/2)t^{(\delta - \theta_G)^+ + \varepsilon/2 - 1}\|S(t) - \tilde{S}_0(t)\|_{L(X^{A}_\delta, X)}$$

$$\leq MD_\delta(A, A_0)t^{-1+\varepsilon/2}.$$ 

Thus by Proposition 2.20 with $Y_1 = X^{A}_\delta$, $Y_2 = X$, $\Phi(s) = G(s, U(s))$, $\Psi(s) = S(s) - \tilde{S}_0(s)$, $\eta = \alpha$, $\alpha = \alpha$, $\theta = (\delta - \theta_G)^+ + \varepsilon/2$ and $g(v) = MD_\delta(A, A_0)v^{-1+\varepsilon/2}$ we obtain:

$$\|t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]G(s, U(s)) dW_H(s)\|_{V^{\alpha,p}_c([0,T_0] \times \Omega; X)}$$

$$\lesssim D_\delta(A, A_0) \sup_{0 \leq t \leq T_0} \|s \mapsto (t-s)^{-\alpha}G(s, U(s))\|_{L^p(\Omega; \gamma(0, t; X^{A_0}_{s, t}))}$$

$$\leq D_\delta(A, A_0) M_\gamma(G)\|U\|_{V^{\alpha,p}_c([0,T_0] \times \Omega; X)}$$

$$\lesssim D_\delta(A, A_0) M_\gamma(G)(1 + \|x_0\|_{L^p(\Omega; X)}),$$

where the penultimate line follows by estimate (9). Note that the implied constants are independent of $T_0$, and depend on $X_0$ only in terms of $\|P_0\|_{L(\Omega, X_0)}$ and on $A$ and $A_0$ only in terms of $\omega$, $\theta$ and $K$.

**Part 2g.** Inserting (24) and (29) in (24) we obtain that there exists a constant $C > 0$ independent of $x_0$ and $y_0$, depending on $X_0$ only in terms of $\|P_0\|_{L(\Omega, X_0)}$, on $A$ and $A_0$ only in terms of $1 + D_\delta(A, A_0)$, $\omega$, $\theta$ and $K$, and on $F$ and $G$ only in terms of their Lipschitz and linear growth constants $\text{Lip}(F)$, $\text{Lip}_\gamma(G)$, $M(F)$ and $M_\gamma(G)$, such that for all $T_0 \in [0, T]$ one has:

$$\|U - \tilde{U}(0)\|_{V^{\alpha,p}_c([0,T_0] \times \Omega; X)}$$

$$\leq CT_0^{-\varepsilon/2}\|U - U(0)\|_{V^{\alpha,p}_c([0,T_0] \times \Omega; X)}$$

$$+ C\left(\|x_0 - y_0\|_{L^p(\Omega; X)} + D_\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega; X)}^{a_0})\right).$$

Setting $T_0 = [2C]^{-2/\varepsilon}$ we obtain:

$$\|U - \tilde{U}(0)\|_{V^{\alpha,p}_c([0,T_0] \times \Omega; X)}$$

$$\leq 2C\left(\|x_0 - y_0\|_{L^p(\Omega; X)} + D_\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega; X)}^{a_0})\right).$$
Part 3. Let \( t_0 \geq 0, z \in L^p(\Omega, \mathcal{F}_{t_0}; X), T > 0 \) and \( \alpha \in [0, \frac{1}{2}) \). By \( U(z, t_0, \cdot) \), we denote the (unique) process in \( V^{\alpha,p}_c([t_0, t_0 + T] \times \Omega; X) \) satisfying, for \( s \in [t_0, t_0 + T] \):
\[
U(z, t_0, s) = S(t - t_0)z + \int_{t_0}^s S(t - t_0 - s)F(U(z, t_0, s)) \, ds \\
+ \int_{t_0}^s S(t - t_0 - s)G(U(z, t_0, s)) \, dW_H(s) \quad \text{a.s.}
\]
The process \( U^{(0)}(z, t_0, \cdot) \) is defined analogously.

From the proof of \([30]\) it follows that for any \( x \in L^p(\Omega, \mathcal{F}_{t_0}; X^A) \) and \( y \in L^p(\Omega, \mathcal{F}_{t_0}; X) \) we have:
\[
\|U(x, t_0, \cdot) - U^{(0)}(y, t_0, \cdot)\|_{V^{\alpha,p}_c([t_0, t_0 + T] \times \Omega; X)}
\leq 2C \left( \|x - y\|_{L^p(\Omega; X)} + D_\delta(A, A_0)[1 + \|x\|_{L^p(\Omega; X^A)}] \right),
\]
with \( C \) as in \([30]\).

Part 4. Throughout this section one may check that the implied constants always depend \( X_0 \) only in terms of \( \|P_0\|_{\mathcal{L}(X, X_0)} \), on \( A \) and \( A_0 \) only in terms of \( 1 + D_\delta(A, A_0) \), \( \omega \), \( \theta \) and \( K \), and on \( F \) and \( G \) only in terms of their Lipschitz and linear growth constants \( \text{Lip}(F), \text{Lip}_i(G), M(F) \) and \( M_i(G) \), even when this is not mentioned explicitly.

By uniqueness of the solution to \([\text{SDE}]\) it follows that for any \( 0 \leq s_0 \leq t_0 \leq t \) and any \( x, y \in L^p(\Omega, \mathcal{F}_{s_0}; X) \) one has:
\[
U(x, s_0, t) = U(U(x, s_0, t_0), t_0, t) \quad \text{and} \quad U^{(0)}(y, s_0, t) = U^{(0)}(U^{(0)}(y, s_0, t_0), t_0, t).
\]

Let \( j \in \mathbb{N} \). By the embedding \( V^{\alpha,p}_c([0, T_0] \times \Omega; X) \hookrightarrow L^\infty(0, T_0; L^p(\Omega; X)) \) and estimate \([31]\) with \( x = U(x_0, 0, (j - 1)T_0) \) and \( y = U^{(0)}(y_0, 0, (j - 1)T_0) \), we obtain:
\[
\|U(x_0, 0, jT_0) - U^{(0)}(y_0, 0, jT_0)\|_{L^p(\Omega; X)}
\leq \|U(U(x_0, 0, (j - 1)T_0), (j - 1)T_0, T_0)
- U^{(0)}(U^{(0)}(y_0, 0, (j - 1)T_0), (j - 1)T_0, T_0)\|_{L^p(\Omega; X)}
\lesssim \|U(x_0, 0, (j - 1)T_0) - U^{(0)}(y_0, 0, (j - 1)T_0)\|_{L^p(\Omega; X)}
+ D_\delta(A, A_0)[1 + \|U(x_0, 0, (j - 1)T_0)\|_{L^p(\Omega; X^A)}],
\]
with implied constant independent of \( j \) and \( n \) and the ‘initial values’ \( U(x_0, 0, (j - 1)T_0) \) and \( U^{(0)}(y_0, 0, (j - 1)T_0) \).

By equation \([11]\) it follows that:
\[
\sup_{1 \leq j \leq \lceil T/T_0 \rceil} \|U(x_0, 0, jT_0)\|_{L^p(\Omega; X^A)} \leq \sup_{s \in [0, T]} \|U(x_0, 0, s)\|_{L^p(\Omega; X^A)}
\lesssim 1 + \|x_0\|_{L^p(\Omega; X^A)}.
\]

Thus for \( j = 1, \ldots, \lceil T/T_0 \rceil \) we obtain the following relation from \([32]\):
\[
\|U(x_0, 0, jT_0) - U^{(0)}(y_0, 0, jT_0)\|_{L^p(\Omega; X)}
\lesssim \|U(x_0, 0, (j - 1)T_0) - U^{(0)}(y_0, 0, (j - 1)T_0)\|_{L^p(\Omega; X)}
+ D_\delta(A, A_0)[1 + \|x_0\|_{L^p(\Omega; X^A)}].
\]
Note that $T_0$ depends on $X_0$ only in terms of $\|P_0\|_{L(X,X_0)}$, on $A$ and $A_0$ only in terms of $1 + D_0(A,A_0)$, $\omega$, $\theta$ and $K$, and on $F$ and $G$ only in terms of $\text{Lip}(F)$, $\text{Lip}_\gamma(G)$, $M(F)$, and $M_\gamma(G)$. By induction we obtain, for $j = 1, \ldots, [T/T_0]$:

\begin{equation}
\|U(x_0, jT_0) - U(0)(y_0, jT_0)\|_{L^p(\Omega; X)} \\
\lesssim \|x_0 - y_0\|_{L^p(\Omega; X)} + D_\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega; X^d)}).
\end{equation}

(34)

Fix $j \in \mathbb{N}$, $j < [T/T_0]$. Set

\[ x = U(x_0, 0, (j - 1)T_0) \quad \text{and} \quad y = U(0)(y_0, 0, (j - 1)T_0) \]

in (31) to obtain, using (33) and (34):

\[ \|U(U(x_0, 0, (j - 1)T_0), (j - 1)T_0, \cdot) - U(0)(y_0, (j - 1)T_0)\|_{V^{\alpha,p}([0,T]; \Omega; X)} \]

\[ \lesssim \|U(x_0, 0, (j - 1)T_0) - U(0)(y_0, 0, (j - 1)T_0)\|_{L^p(\Omega; X)} \\
+ D_\delta(A, A_0)(1 + \|x_0, 0, (j - 1)T_0\|_{L^p(\Omega; X^d)}) \]

\[ \lesssim \|x_0 - y_0\|_{L^p(\Omega; X)} + D_\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega; X^d)}), \]

with implied constants independent of $j$.

Due to inequality (31) we thus obtain:

\[ \|U(x_0, 0, \cdot) - U(0)(y_0, 0, \cdot)\|_{V^{\alpha,p}([0,T]; \Omega; X)} \]

\[ \leq \sum_{j=1}^{[T/T_0]} \left( \left\| U(U(x_0, 0, (j - 1)T_0), (j - 1)T_0, \cdot) - U(0)(y_0, (j - 1)T_0) \right\|_{V^{\alpha,p}([0,T]; \Omega; X)} \right) \]

\[ \leq \sum_{j=1}^{[T/T_0]} \left( \left\| U(U(x_0, 0, (j - 1)T_0), (j - 1)T_0, \cdot) - U(0)(y_0, (j - 1)T_0) \right\|_{V^{\alpha,p}([0,T]; \Omega; X)} \right) \]

\[ \lesssim \sum_{j=1}^{[T/T_0]} \|x_0 - y_0\|_{L^p(\Omega; X)} + D_\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega; X^d)}), \]

\[ \lesssim \|x_0 - y_0\|_{L^p(\Omega; X)} + D_\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega; X^d)}). \]

\[ \square \]

It remains to provide a proof for Proposition 3.2. For that purpose, we first prove the following lemma. Given the lemma, the proof of Proposition 3.2 basically follows the lines of known proofs concerning comparison of semigroups, see Chapter III.3.b. For notational simplicity we define the pseudo-resolvent

\begin{equation}
R(\lambda : \hat{A}_0) := i_{X_0} R(\lambda : A_0) P_0, \quad \lambda \in \omega + \Sigma_{\omega + \theta}^d
\end{equation}

(35)

we leave it to the reader to verify the resolvent identity.)
Lemma 3.3. Let the setting be as in Proposition 3.2. Then for all \( \lambda \in \omega' + \Sigma_{\frac{1}{2} + \theta} \) we have:
\[
\|R(\lambda : A) - R(\lambda : \tilde{A}_0)\|_{\mathcal{L}(X^A_{\frac{1}{2}}, X)} \\
\leq C_{\omega, \theta, K, P_0} |\lambda - \omega|^\delta - 1 \|R(\lambda_0 : A) - R(\lambda_0 : \tilde{A}_0)\|_{\mathcal{L}(X^A_{\frac{1}{2}-1}, X)},
\]
where \( C_{\omega, \theta, K, P_0} \) is a constant depending only on \( \omega, \theta, K \) and \( \|P_0\|_{\mathcal{L}(X,X_0)} \).

Proof. Using only the resolvent identity and the definition of \( R(\lambda : \tilde{A}_0) \) (see (35)) one may verify that the following identity holds:
\[
R(\lambda : A) - R(\lambda : \tilde{A}_0) = [I + (\lambda_0 - \lambda)R(\lambda : \tilde{A}_0)][R(\lambda_0 : A) - R(\lambda_0 : \tilde{A}_0)](\lambda_0 - A)R(\lambda : A).
\]
Moreover, one may check that:
\[
\omega' + \Sigma_{\frac{1}{2} + \theta} \subset \{ \omega + \Sigma_{\frac{1}{2} + \theta} \} \cap \{ \lambda \in \mathbb{C} : |\lambda - \omega| \geq |\lambda_0 - \omega| \}.
\]
Therefore one has, for \( \lambda \in \omega' + \Sigma_{\frac{1}{2} + \theta} \),
\[
\|I + (\lambda_0 - \lambda)R(\lambda : \tilde{A}_0)\|_{\mathcal{L}(X)} \leq 1 + \frac{|\lambda_0 - \lambda|}{|\lambda - \omega|} K \|P_0\|_{\mathcal{L}(X,X_0)} \leq 1 + 2K \|P_0\|_{\mathcal{L}(X,X_0)}.
\]
From (36) one obtains:
\[
\|R(\lambda : A) - R(\lambda : \tilde{A}_0)\|_{\mathcal{L}(X^A_{\frac{1}{2}}, X)} \leq (1 + 2K) \|P_0\|_{\mathcal{L}(X,X_0)}
\times \|R(\lambda_0 : A) - R(\lambda_0 : \tilde{A}_0)\|_{\mathcal{L}(X^A_{\frac{1}{2}-1}, X)}(\lambda_0 - A)R(\lambda : A)\|_{\mathcal{L}(X)}.
\]
Let \( \bar{\lambda} \in \mathbb{C}, \text{Re}(\bar{\lambda}) > \omega \) be such that \( |\bar{\lambda} - \lambda_0| \leq 2|\bar{\lambda} - \omega| \) (if \( \text{Re}(\lambda_0) > \omega \) one may simply pick \( \bar{\lambda} = \lambda_0 \)). For \( \eta \in \mathbb{R} \) and \( x \in X^A_\eta \) set \( \|x_0\|_{X^A_\eta} := \|\tilde{A} - A\|^\eta |x| \). Then:
\[
\|R(\lambda_0 - \lambda)R(\lambda : A)\|_{\mathcal{L}(X^A_{\frac{1}{2}}, X^A_{\frac{1}{2}-1})} = \|R(\lambda : A)\|_{\mathcal{L}(X)}(\lambda - A)^{\delta - \beta - 1} \|R(\lambda_0 : A)\|_{\mathcal{L}(X)}
\leq (1 + \frac{|\lambda - \lambda_0|}{|\lambda - \omega|} K) \|R(\lambda : A)\|_{\mathcal{L}(X)}(\lambda - A)^{\delta - \beta - 1} \|R(\lambda_0 : A)\|_{\mathcal{L}(X)}
\leq (1 + 2K)\|R(\lambda : A)\|_{\mathcal{L}(X)}(\lambda - A)^{\delta - \beta - 1} \|R(\lambda : A)\|_{\mathcal{L}(X)}.
\]
If \( \delta - \beta = 1 \) then:
\[
\|R(\lambda : A)\|^\delta - \beta \|R(\lambda : A)\|_{\mathcal{L}(X)} = \|R(\lambda : A)\|^\delta - \beta \|R(\lambda : A)\|_{\mathcal{L}(X)} \leq 1 + 2K.
\]
If \( \delta - \beta = 0 \) then:
\[
\|R(\lambda : A)\|^\delta - \beta \|R(\lambda : A)\|_{\mathcal{L}(X)} = \|R(\lambda : A)\|_{\mathcal{L}(X)} \leq K|\lambda - \omega|^{-1}.
\]
For \( \delta - \beta \in (0, 1) \) we have, by Theorem 2.5:
\[
\|R(\lambda : A)\|^\delta - \beta \|R(\lambda : A)\|_{\mathcal{L}(X)} \leq 2(1 + K)\|R(\lambda : A)\|_{\mathcal{L}(X)}^{\delta - \beta - 1} \|R(\lambda : A)\|_{\mathcal{L}(X)}^{\delta - \beta - 1}
\leq 2(1 + K)(1 + 2K)\|R(\lambda : A)\|_{\mathcal{L}(X)}^{\delta - \beta - 1} \|R(\lambda : A)\|_{\mathcal{L}(X)}^{\delta - \beta - 1}
\leq 2(1 + 2K)^2|\lambda - \omega|^{\delta - \beta - 1}.
\]
Substituting this into (37) one obtains:
\[
\|R(\lambda : A) - R(\lambda : \tilde{A}_0)\|_{\mathcal{L}(X^A_{\frac{1}{2}}, X)} \\
\leq 2(1 + 2K)^4 \|P_0\|_{\mathcal{L}(X,X_0)}|\lambda - \omega|^{\delta - \beta - 1} \|R(\lambda_0 : A) - R(\lambda_0 : \tilde{A}_0)\|_{\mathcal{L}(X^A_{\frac{1}{2}-1}, X)}.
\]
□
Proof of Proposition 3.2. Let $\omega'$ be as defined in Lemma 3.3. For brevity set $\varepsilon = \delta - \beta$. First of all observe that

$$
\lim_{s \to 0} s^\varepsilon \| [S(s) - \tilde{S}_0(s)] \|_{\mathcal{L}(X^A, X)} = 0.
$$

Fix $\theta' \in (0, \theta)$. It follows from [21, Theorem 1.7.7], that one has, for all $t > 0$:

$$
S(t) = \frac{1}{2\pi i} \int_{\omega' + \Gamma_{\theta'}} e^{\lambda t} R(\lambda : A) d\lambda;
$$

where $\Gamma_{\theta'}$ is the path composed from the two rays $re^{i(\frac{\pi}{2} + \theta')}$ and $re^{-i(\frac{\pi}{2} + \theta')}$, $0 \leq r < \infty$, and is oriented such that $\operatorname{Re}(\lambda)$ increases along $\Gamma_{\theta'}$. As $\omega' \geq \omega$, the integral is well-defined as $\mathcal{L}(X)$-valued Bochner integral, and for $t > 0$ one has:

$$
\frac{d}{dt} S(t) = \frac{1}{2\pi i} \int_{\omega' + \Gamma_{\theta'}} \lambda e^{\lambda t} R(\lambda : A) d\lambda;
$$

the integral again being well-defined as $\mathcal{L}(X)$-valued Bochner integrals (see also the proof of [21, Theorem 2.5.2]). Analogous identities hold for $\tilde{S}_0$ and $R(\lambda : \tilde{A}_0)$.

First let us assume that $\varepsilon \in (0, 1)$. Below we shall apply Lemma 3.3, observing that for $r \in [0, \infty)$ we have

$$
|\omega' + re^{i(\frac{\pi}{2} + \theta')} - \omega| \geq K_{\theta} r,
$$

where $K_{\theta}$ is a constant depending only on $\theta$. Note that we use the coordinate transform $\lambda = \omega' + re^{i(\frac{\pi}{2} + \theta')}$. For $s > 0$ we have:

$$
\|S(s) - \tilde{S}_0(s)\|_{\mathcal{L}(X^A, X)} = \left\| \frac{1}{2\pi i} \int_{\Gamma_{\theta'}} e^{\lambda s} [R(\lambda : A) - R(\lambda : \tilde{A}_0)] d\lambda \right\|_{\mathcal{L}(X^A, X)}
\leq \frac{1}{2\pi} \int_0^\infty \left| e^{-i(\frac{\pi}{2} + \theta')} + (\omega' + re^{i(\frac{\pi}{2} + \theta')} ) s \right| \left| R(\omega' + re^{i(\frac{\pi}{2} + \theta')} : A) - R(\omega' + re^{i(\frac{\pi}{2} + \theta')} : \tilde{A}_0) \right|_{\mathcal{L}(X^A, X)} dr
+ \frac{1}{2\pi} \int_0^\infty \left| e^{i(\frac{\pi}{2} + \theta')} + (\omega' + re^{i(\frac{\pi}{2} + \theta')} ) s \right| \left| R(\omega' + re^{i(\frac{\pi}{2} + \theta')} : A) - R(\omega' + re^{i(\frac{\pi}{2} + \theta')} : \tilde{A}_0) \right|_{\mathcal{L}(X^A, X)} dr
\leq \frac{1}{2\pi} C_{\omega, \theta, K, \beta} K_{\theta} D_\beta (A, A_0) e^{\omega' s} \int_0^\infty \int_0^\infty r^{s-1} e^{-rs \sin \theta'} dr d\rho
= \frac{1}{2\pi} C_{\omega, \theta, K, \beta} K_{\theta} D_\beta (A, A_0) \int_0^\infty [s \sin \theta']^{s-1} e^{\omega' s} \int_0^\infty u^{s-1} e^{-u} du d\rho
= \frac{\Gamma(\varepsilon)}{\pi} |\sin \theta'|^{-\varepsilon} C_{\omega, \theta, K, \beta} K_{\theta} D_\beta (A, A_0) s^{-\varepsilon} e^{\omega' s}.
$$

For $\varepsilon = 0$ one may avoid the singularity in $0$ in the usual way: for $s > 0$ given we integrate over

$$
\omega' + \Gamma_{\theta', s} = (\omega' + \Gamma_{\theta', s}^{(1)}) \cup (\omega' + \Gamma_{\theta', s}^{(2)}) \cup (\omega' + \Gamma_{\theta', s}^{(3)}),
$$

where $\Gamma_{\theta', s}^{(1)}$ and $\Gamma_{\theta', s}^{(2)}$ are the rays $re^{i(\omega' + \theta')}$ and $re^{-i(\omega' + \theta')}$, $s^{-1} \leq r < \infty$, and $\Gamma_{\theta', s}^{(3)} = s^{-1} \omega' + \phi \in \left[ -\frac{\pi}{2} - \theta', \frac{\pi}{2} + \theta' \right]$. This leads to the following estimate:

$$
\|S(s) - \tilde{S}_0(s)\|_{\mathcal{L}(X^A, X)} = \left\| \frac{1}{2\pi i} \int_{\omega' + \Gamma'_{\theta', s}} e^{\lambda s} [R(\lambda : A) - R(\lambda : \tilde{A}_0)] d\lambda \right\|_{\mathcal{L}(X^A, X)}
\leq \frac{1}{2\pi} C_{\omega, \theta, K, \beta} K_{\theta} D_\beta (A, A_0) s^{-\varepsilon} e^{\omega' s}.
$$
\[ \leq C_{\omega,\theta,K,P_0,K_0}D_\delta(A, A_0)e^{\omega s} \left[ \int_{s-1}^{\infty} r^{-1}e^{-r\sin \theta'} dr + e \right] \]

\[ \leq C_{\omega,\theta,K,P_0,K_0}D_\delta(A, A_0) \left[ |\pi \sin \theta'|^{-1}e^{-|\pi \sin \theta'|} + e \right] e^{\omega s} \]

\[ \leq 2|\sin \theta'|^{-1}C_{\omega,\theta,K,P_0,K_0}D_\delta(A, A_0)e^{\omega s}. \]

Recalling that \( \varepsilon = \delta - \beta \) this proves the uniform boundedness estimate of (18).

Similarly to the above, for \( \varepsilon \in [0,1] \) and \( s > 0 \) we have:

\[ \frac{d}{ds}S(s) - \frac{d}{ds}\tilde{S}_0(s) \|_{\mathcal{L}(X^d_X, X)} = \left\| \frac{1}{2\pi} \int_{\Gamma_{\omega'+\theta'}} \lambda e^{\lambda s}[R(\lambda : A) - R(\lambda : \tilde{A}_0)] d\lambda \right\|_{\mathcal{L}(X^d_X, X)} \]

\[ \leq \frac{1}{2\pi}e^{\omega s} \int_0^\infty r^{-r\sin \theta'} \| R(re^{-i(\frac{\pi}{2} + \theta')} : A) - R(re^{-i(\frac{\pi}{2} + \theta')} : \tilde{A}_0) \|_{\mathcal{L}(X^d_X, X)} dr \]

\[ + \frac{1}{2\pi}e^{\omega s} \int_0^\infty r^{-r\sin \theta'} \| R(re^{i(\frac{\pi}{2} + \theta')} : A) - R(re^{i(\frac{\pi}{2} + \theta')} : \tilde{A}_0) \|_{\mathcal{L}(X^d_X, X)} dr \]

\[ = \frac{1}{\pi}C_{\omega,\theta,K,P_0,K_0}D_\delta(A, A_0) |\sin \theta'|^{-1}e^{\omega s} \int_0^\infty u^2e^{-u} du \]

\[ = e^{o(\varepsilon)} |\sin \theta'|^{-1}e^{\omega s}C_{\omega,\theta,K,P_0,K_0}D_\delta(A, A_0)s^{-1-\varepsilon}e^{\omega s}. \]

Recalling that \( \varepsilon = \delta - \beta \) this proves the uniform boundedness estimate of (19).

Concerning the \( \gamma \)-boundedness estimates, fix \( \alpha > \varepsilon \). By Proposition 2.12 one has:

\[ \gamma_{[X^d_X, X]} \left( \{ s^\alpha |S(s) - \tilde{S}_0(s)\} : s \in [0,t] \} \right) \leq \int_0^t \| \frac{d}{ds}(s^\alpha[S(s) - \tilde{S}_0(s)]) \|_{\mathcal{L}(X^d_X, X)} ds \]

\[ \leq \int_0^t s^\alpha \| S(s) - \tilde{S}_0(s) \|_{\mathcal{L}(X^d_X, X)} ds \]

\[ + \int_0^t s^\alpha \| \frac{d}{ds}S(s) - \frac{d}{ds}\tilde{S}_0(s) \|_{\mathcal{L}(X^d_X, X)} ds. \]

Substituting (18) and (19) into the above one obtains that there exists a constant \( C \) depending only on \( \omega, \theta, K, \varepsilon = \delta - \beta \), and \( \|P_0\|_{\mathcal{L}(X,X_0)} \) such that:

\[ \gamma_{[X^d_X, X]} \left( \{ s^\alpha |S(s) - \tilde{S}_0(s)\} : s \in [0,t] \} \right) \leq CD_\delta(A, A_0) \int_0^t e^{\omega s}s^{-1-\varepsilon}e^{\omega t} ds \]

\[ \leq Ce^{(\omega T)^\gamma}e^{o(\varepsilon)} s^{\alpha-1-\varepsilon} D_\delta(A, A_0), \]

as \( \alpha > \varepsilon \).

\[ \square \]

**Corollary 3.4.** Let the setting be as in Theorem 3.3. Let \( \lambda \in [0, \frac{1}{2}] \) satisfy

\[ 0 \leq \lambda < \min\{1 - (\delta - \theta_F) \vee 0, \frac{1}{2} - \frac{1}{p} - (\delta - \theta_G) \vee 0\}. \]

Suppose \( x_0 \in L^p(\Omega, \mathcal{F}_0; X^d_X) \) and \( y_0 \in L^p(\Omega, \mathcal{F}_0; X) \), then:

\[ \| U - Sx_0 - iX_0(U^{(0)} - S_0P_0y_0) \|_{L^p(\Omega; C^\lambda([0,T]; X))} \]

\[ \leq \| x_0 - y_0 \|_{L^p(\Omega; X)} + D_\delta(A, A_0)(1 + \| x_0 \|_{L^p(\Omega; X^d_X)}), \]

with implied constant depending on \( X_0 \) only in terms of \( \|P_0\|_{\mathcal{L}(X,X_0)} \), on \( A \) and \( A_0 \) only in terms of \( 1 + D_\delta(A, A_0) \), \( \omega, \theta \) and \( K \), and on \( F \) and \( G \) only in terms of their Lipschitz and linear growth constants \( Lip(F), Lip_\gamma(G), M(F), \) and \( M_\gamma(G) \).
Proof. As before, we write:

\[ (38) \]

\[ \|U - Sx_0 - iX_0(U^{(0)} - S_0P_0y_0)\|_{L^p(\Omega; C^\lambda([0,T];X))} \]

\[ = \left\| t \mapsto \int_0^t \tilde{S}_0(t-s)[F(s, U(s)) - F(s, \tilde{U}^{(0)}(s))] \, ds \right\|_{L^p(\Omega; C^\lambda([0,T];X))} \]

\[ + \left\| t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]F(s, U(s)) \, ds \right\|_{L^p(\Omega; C^\lambda([0,T];X))} \]

\[ + \left\| t \mapsto \int_0^t \tilde{S}_0(t-s)[G(s, U(s)) - G(s, \tilde{U}^{(0)}(s))] \, dW_H(s) \right\|_{L^p(\Omega; C^\lambda([0,T];X))} \]

\[ + \left\| t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]G(s, U(s)) \, dW_H(s) \right\|_{L^p(\Omega; C^\lambda([0,T];X))}. \]

For the first and second term on the right-hand side of (38) we apply Proposition 2.21. Note that as before we may pick \( \eta_F, \eta_G \leq 0 \) such that \( \eta_F < \theta_F - \delta \) and \( \eta_G < \theta_G - \delta \) and

\[ \lambda < \min\{1 + \eta_F, \frac{1}{p} - \frac{1}{p} + \eta_G\}. \]

Our choice of \( Y_1, Y_2, \Phi, \Psi \) is the same as in part 2c, respectively 2d, of the proof of Theorem 3.1 whereas we set \( \theta = 1 - \lambda \). This leads to the following estimates:

\[ \left\| t \mapsto \int_0^t \tilde{S}_0(t-s)[F(s, U(s)) - F(s, \tilde{U}^{(0)}(s))] \, ds \right\|_{L^p(\Omega; C^\lambda([0,T];X))} \]

\[ \lesssim \|U - \tilde{U}^{(0)}(s)\|_{L^p(\Omega; L^{\infty}(0,T;X))} \leq \|U - \tilde{U}^{(0)}(s)\|_{V^{\alpha,p}(0,T\times\Omega,X)}, \]

and

\[ \left\| t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]F(s, U(s)) \, ds \right\|_{L^p(\Omega; C^\lambda([0,T];X))} \]

\[ \lesssim D\delta(A, A_0)\|U\|_{L^p(\Omega; L^{\infty}(0,T;X))} \lesssim D\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega,X)}). \]

For the third and fourth term on the right-hand side of (38) we apply Corollary 2.28 with \( \beta = \lambda \) and \( \alpha \in (0, \frac{1}{2}) \) such that \( \alpha > \lambda + \frac{1}{p} + \eta_G \). The choice of \( Y_1, Y_2, \Phi \) and \( \Psi \) is as in parts 2e and 2f of the proof of Theorem 3.1. This leads to:

\[ \left\| t \mapsto \int_0^t \tilde{S}_0(t-s)[G(s, U(s)) - G(s, \tilde{U}^{(0)}(s))] \, dW_H(s) \right\|_{L^p(\Omega; C^\lambda([0,T];X))} \]

\[ \lesssim \|U - \tilde{U}^{(0)}(s)\|_{V^{\alpha,p}(0,T\times\Omega,X)}, \]

and

\[ \left\| t \mapsto \int_0^t [S(t-s) - \tilde{S}_0(t-s)]G(s, U(s)) \, dW_H(s) \right\|_{L^p(\Omega; C^\lambda([0,T];X))} \]

\[ \lesssim D\delta(A, A_0)\|U\|_{V^{\alpha,p}(0,T\times\Omega,X)} \lesssim D\delta(A, A_0)(1 + \|x_0\|_{L^p(\Omega,X)}). \]

Combining these estimates with Theorem 3.1 gives the desired result. It goes without saying that all the implied constants above depend on \( x_0 \) only in terms of \( \|P_0\|_{L^2(X,X_0)} \), on \( A \) and \( A_0 \) only in terms of \( 1 + D\delta(A, A_0) \), \( \omega, \theta \) and \( K \), and on \( F \) and \( G \) only in terms of their Lipschitz and linear growth constants \( \text{Lip}(F) \), \( \text{Lip}_\gamma(G) \), \( M(F) \), and \( M_\gamma(G) \).
4. YOSIDA APPROXIMATIONS

Consider (SDE) under the assumptions (A), (F), and (G) with the additional assumption that $\theta_F, \theta_G \geq 0$. We define $A_n := nAR(n : A)$ to be the $n^{th}$ Yosida approximation of $A$. Let $U$ to denote the solution to (SDE) with operator $A$ and initial data $x_0 \in L^p(\Omega, \mathcal{F}_0; X)$ and, for $n \in \mathbb{N}$, let $U^{(n)}$ denote the solution to (SDE) with operator $A_n$ instead of $A$ and initial data $y_0 \in L^p(\Omega, \mathcal{F}_0; X)$.

**Theorem 4.1.** For any $\eta \in [0, 1]$ and $p \in (2, \infty)$ such that

$$\eta < \min\left\{\frac{2}{p} - \frac{1}{p} + \theta_F, \frac{1}{p} - \frac{1}{p} + \theta_G\right\}$$

and any $\alpha \in [0, \frac{1}{2})$ we have, assuming $y_0 \in L^p(\Omega, \mathcal{F}_0; X^A_\eta)$:

$$\|U - U^{(n)}\|_{V^{\lambda, \alpha}(\Omega; X)} \lesssim \|x_0 - y_0\|_{L^p(\Omega; X)} + n^{-\eta}(1 + \|y_0\|_{L^p(\Omega; X^A)})$$

with implied constants independent of $n$, $x_0$ and $y_0$.

The following corollary is a direct consequence of the Borel-Cantelli lemma and the above theorem (see Corollary [14] Lemma 2.1):

**Corollary 4.2.** Let $\eta > 0$ and $p \in (2, \infty)$ be such that

$$\eta + \frac{1}{p} < \min\left\{\frac{2}{p} - \frac{1}{p} + \theta_F, \frac{1}{p} - \frac{1}{p} + \theta_G, 1\right\}$$

and assume $y_0 = x_0 \in L^p(\Omega, \mathcal{F}_0; X^A_\eta)$. Then there exists a random variable $\chi \in L^0(\Omega)$ such that for all $n \in \mathbb{N}$:

$$\|U - U^{(n)}\|_{C([0,T]; \mathcal{H})} \leq \chi n^{-\eta}.$$

To prove Theorem 4.1 we shall need the following lemma:

**Lemma 4.3.** Let $\beta \in [0, 1]$. Then there exists a constant $K'$ such that for all $n \geq 2\omega$ and all $x \in X^A_\beta$ one has:

$$\|(2\omega I - A_n)^\beta x\| \leq K'\|(2\omega I - A)^\beta x\|.$$

**Proof.** Observe that

(39) $$2\omega I - A_n = [(n + 2\omega)I - 4\omega^2R(2\omega : A)](2\omega I - A)(n : A).$$

Thus for $x \in D(A)$ and $n \geq 2\omega$ we have:

$$\|(2\omega I - A_n)x\| \leq \|(n + 2\omega)I - 4\omega^2R(2\omega : A)]R(n : A)\|_{\mathcal{L}(X)}\|(2\omega I - A)x\|$$

$$\leq [K + \frac{2\omega}{n - \omega} + K^2 \frac{2\omega}{n - \omega}]\|(2\omega I - A)x\| \leq 4K(1 + K)\|(2\omega I - A)x\|.$$

This proves the lemma for $\beta = 1$. For $\beta = 0$ the lemma is trivial. For $\beta \in (0, 1)$ we need two extra observations.

First of all, for $s > \omega$ and $\beta \in (0, 1)$ we have, by definition, (see [21] Section 2.6)):

$$(sI - A)^{-\beta}x = \frac{\sin(\pi \beta)}{\pi} \int_0^\infty t^{-\beta}((t + s)I - A)^{-1}x dt,$$

and hence

$$\|(sI - A)^{-\beta}\|_{\mathcal{L}(X)} \leq K \frac{\sin(\pi \beta)}{\pi} \int_0^\infty t^{-\beta}(t + \omega)^{-1} dt$$

$$\leq K \frac{\sin(\pi \beta)}{\pi \beta(1 - \beta)}(s - \omega)^{-\beta}.$$

(40)
Secondly, let $\mu, \lambda \in \omega + \Sigma_{\varphi + \theta}$. We have:

$$\|e^{-(\lambda I - A)R(\mu I)}\|_{\mathcal{L}(X)} = e^{-t}\|e^{(n - \lambda)R(\mu I)}\|_{\mathcal{L}(X)} \leq e^{-t + \frac{\lambda}{\mu - \omega} K t}.$$ 

Now suppose $n \geq 2\omega(1 + 4K)$, $\lambda = 2\omega$, $\mu = \frac{n\lambda}{\lambda + n} = \frac{2\omega n}{2\omega + n}$. In that case one may check that $\frac{\mu - \lambda}{\mu - \omega} K \leq \frac{1}{2}$, and thus that for $\beta \in (0,1)$:

$$\|(-2\omega I - A)R\left(\frac{2\omega n}{2\omega + n} : A\right)^{-\beta}\|_{\mathcal{L}(X)} = \left\|\frac{1}{\sqrt{\beta}} \int_0^\infty t^{\frac{1}{2}} e^{-(2\omega I - A)R\left(\frac{2\omega n}{2\omega + n} : A\right)t} dt\right\|_{\mathcal{L}(X)} \leq 2^\beta.$$ 

As $(-2\omega I - A)R\left(\frac{2\omega n}{2\omega + n} : A\right)^{-\beta} \in \mathcal{L}(X)$ for any $n \geq 2\omega$, it follows that there exists a constant $M > 0$ such that for all $n \geq 2\omega$:

$$\text{(41)} \quad \|(-2\omega I - A)R\left(\frac{2\omega n}{2\omega + n} : A\right)^{-\beta}\|_{\mathcal{L}(X)} \leq M.$$ 

For $\beta \in (0,1)$ and $x \in X^A$ we have, by standard theory on functional calculus (see [11]), equation [39] and the estimates (40) and (41):

$$\|(-2\omega I - A_n)^\beta x\| = \|(n + 2\omega)^\beta \left(\frac{2\omega n}{2\omega + n} I - A\right)^\beta (nI - A)^{-\beta} x\|
\leq (n + 2\omega)^\beta \left\|[\frac{2\omega I - A}{2\omega I} R\left(\frac{2\omega n}{2\omega + n} : A\right)\right]^\beta \|_{\mathcal{L}(X)}
\times \|(-2\omega I - A)^\beta x\| \|(nI - A)^{-\beta}\|_{\mathcal{L}(X)}
\leq 4\frac{\sin(\pi\beta)}{\pi\beta(1-\beta)} K M \|(2\omega I - A)^\beta x\|.$$ 

\[\Box\]

**Proof of Theorem 4.1.** Without loss of generality we may assume $\omega \geq 0$. In order to apply Theorem 3.1 we must prove that $A_n$, $n \geq 2\omega$, are of uniform type, i.e., that there exist $\bar{\omega} \in \mathbb{R}, \theta \in (0, \frac{\pi}{2})$ and $\bar{K} > 0$ such that $A_n$ is of type $(\bar{\omega}, \theta, \bar{K})$ for all $n \geq 2\omega$. Fix $n \geq 2\omega$. One checks that:

$$\text{(42)} \quad R(\lambda : A_n) = (n + \lambda)^{-1} (nI - A) R\left(\frac{\lambda n}{n + \lambda} : A\right)$$

whenever $\frac{\lambda n}{n + \lambda} \in \omega + \Sigma_{\varphi + \theta}$. Define $f : \mathbb{C} \to \mathbb{C}; f(z) = \frac{z - \bar{\omega}}{n - z}$. From (42) it follows that $\lambda \in \rho(A)$ if and only if $f(\lambda) \in \rho(\bar{A}_n)$. By standard theory on Möbius transforms we have that

$$f\left\{\{\omega + \Sigma_{\varphi + \theta}\}\right\} = \mathbb{C} \setminus (D_1 \cap D_2),$$

where $D_1$ and $D_2$ are both closed disks with radius $\frac{n}{2(n - \omega)}(2\omega - n, \tan(\theta))$ and $\frac{n}{2(n - \omega)}(2\omega - n, -\tan(\theta))$. The boundaries of these disks intersect each other on the real axis at the points $-\omega$ and $\frac{n\omega}{n - \omega}$. The angle at intersection is $\pi - 2\theta$. As $n \geq 2\omega$ we have $\frac{n\omega}{n - \omega} \leq 2\omega$ and thus $\rho(\bar{A}_n) \subset 2\omega + \Sigma_{\varphi + \theta}$. It remains to prove the desired estimate on the resolvent.

Using (42) one may check that for $\lambda \in 2\omega + \Sigma_{\varphi + \theta}$ we have:

$$\text{(43)} \quad R(\lambda : A) - R(\lambda : A_n) = -(\lambda + n)^{-1} A^2 R\left(\frac{\lambda n}{n + \lambda} : A\right) R(\lambda : A).$$

Thus by (2) we have, for $\lambda \in \omega(1 + 2(\cos\theta)^{-1}) + \Sigma_{\varphi + \theta}$:

$$\|R(\lambda : A) - R(\lambda : A_n)\|_{\mathcal{L}(X)} \leq (1 + 2K)^2|\lambda + n|^{-1} \leq (1 + 2K)^2|\lambda - \omega|^{-1}.$$
The final estimate follows from the fact that by standard theory on Möbius transforms we have that $\frac{\|X\|}{\|\gamma\|} \leq 1$ for $\lambda \in \omega + \Sigma^\theta$. In conclusion we have, for $\lambda \in \omega(1 + 2(\cos \theta)^{-1})$ + $\Sigma^\theta$:

$$
\|R(\lambda : A_n)\|_{L^2(X)} \leq \|R(\lambda : A)\|_{L^2(X)} + \|R(\lambda : A) - R(\lambda : A_n)\|_{L^2(X)} \\
\leq [K + (1 + 2K)^2]\|1 - \omega\|^{-1}.
$$

This proves that $A_n$ is of type $(\omega(1 + 2(\cos \theta)^{-1}), \theta, K, (1 + 2K)^2)$ for all $n \geq 2\omega$.

It also follows from (3.3) that if we take, for example, $\lambda_0 = \omega(1 + 2(\cos \theta)^{-1})$, then we have, for $n \geq 2\omega$:

$$
\|R(\lambda_0 : A) - R(\lambda_0 : A_n)\|_{L^2(X)} \leq (1 + 2K)^2 n^{-1}.
$$

In other words, for all $n \in \mathbb{N}$ condition (1.5) in Theorem 3.1 is satisfied with $\delta = 1$ and $\lambda_0 = \omega(1 + 2(\cos \theta)^{-1})$. In particular we can apply Theorem 3.1 to obtain the desired result for the case $\theta > -\frac{1}{2} + \frac{1}{p}$, where $\tau$ is the type of $X$, and $\theta_G > \frac{1}{2} + \frac{1}{p}$.

Concerning the dependence on $1 + D(A, A_n)$ of the implied constant in (7), note that $1 + D(A, A_n)$ is uniformly bounded in $n$, both from above and away from 0.

In order to get the desired result for general $\theta_F, \theta_G \geq 0$ we consider the difference $R(\lambda_0 : A) - R(\theta_F : A_n)$ in the $L^2(X_{\delta_{\theta_F}}^A, X)$-norm. (Note that if $A$ is unbounded then $R(\lambda_0 : A) - R(\theta_F : A_n) \notin L^2(X_{\delta_{\theta_F}}^A, X)$ for any $\delta < 1$.) For $n \geq (1 + 2(\cos \theta)^{-1})$ we have, by (2), that $\|\theta - A_n\|_{L^2(X)} \leq 2n(1 + K)$. Thus by Theorem 2.9 we have, for $\delta \in (0, 1)$:

$$
\|\theta - A_n\|^{1-\delta} = 2(1 + 2K)\|\theta - A_n\|^{1-\delta} \\
\leq 2^{1-\delta}(1 + 2K)^{2-\delta} n^{1-\delta} \|\theta - A_n\|.
$$

It follows that for $\delta \in [0, 1)$ we have:

$$
\|R(\lambda_0 : A) - R(\lambda_0 : A_n)\|_{L^2(X_{\delta_0}^A, X)} \leq 2^{1-\delta}(1 + 2K)^{4-\delta} n^{-\delta}.
$$

We are now ready to apply Theorem 3.1. First of all observe that by Lemma 2.9 we have that $F : [0, T] \times X \to X_{\theta_F}^A$ is Lipschitz continuous and of linear growth for all $n \geq 2\omega$ with Lipschitz and growth constants independent of $n$, and $G : [0, T] \times X \to \gamma(H, X_{\theta_G}^A)$ is $L^2_\gamma$-Lipschitz continuous and of linear growth for all $n \geq 2\omega$ with Lipschitz and growth constants independent of $n$. Also, $1 + D(A, A_n)$ is uniformly bounded in $n$.

Fix $\eta \in [0, 1]$ such that $\eta < \min\{\frac{1}{2} - \frac{1}{2} + \frac{1}{p} - \frac{1}{p} + \frac{1}{p} + \theta_G\}$ and suppose $y_0 \in L^p(\Omega, F_0; X_0^A)$. It follows from Theorem 3.1 with $\delta = \eta$, but with $A_n$ playing the role of $A$ and $A$ playing the role of $A_0$, that:

$$
\|U - U^{(n)}\|_{V^{p, p}_{\gamma}(\omega, X)} \lesssim \|x_0 - y_0\|_{L^p(\gamma, X)} + n^{-\eta}(1 + \|y_0\|_{L^p(\gamma, X)}),
$$

with implied constants independent of $n$, $x_0$ and $y_0$.

\[\square\]

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