QUASICONFORMAL MAPPINGS AND HÖLDER CONTINUITY

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Abstract. We establish that every $K$-quasiconformal mapping $w$ of the unit ball $\mathbb{B}$ onto a $C^2$-Jordan domain $\Omega$ is Hölder continuous with constant $\alpha = 2 - \frac{n}{p}$, provided that its weak Laplacean $\Delta w$ is in $L^p(\mathbb{B})$ for some $n/2 < p < n$. In particular it is Hölder continuous for every $0 < \alpha < 1$ provided that $\Delta w \in L^n(\mathbb{B})$.

1. Introduction

In this paper $\mathbb{B}$ denotes the unit ball in $\mathbb{R}^n$, $n \geq 2$ and $S^{n-1}$ denotes the unit sphere. Also we will assume that $n > 2$ (the case $n = 2$ has been already treated in [17]). We will consider the vector norm $|x| = (\sum_{i=1}^n x_i^2)^{1/2}$ and the matrix norms $|A| = \text{sup} \{ |Ax| : |x| = 1 \}$.

A homeomorphism $u : \Omega \to \Omega'$ between two open subsets $\Omega$ and $\Omega'$ of Euclidian space $\mathbb{R}^n$ will be called a $K (K \geq 1)$ quasi-conformal or shortly a q.c mapping if

(i) $u$ is absolutely continuous function in almost every segment parallel to some of the coordinate axes and there exist the partial derivatives which are locally $L^n$ integrable functions on $\Omega$. We will write $u \in ACL^n$ and

(ii) $u$ satisfies the condition

$$|\nabla u(x)|^n / K \leq J_u(x) \leq Kl(\nabla u(x))^n,$$

at almost everywhere $x$ in $\Omega$ where

$$l(\nabla u(x)) := \text{inf} \{ |\nabla u(x)\zeta| : |\zeta| = 1 \}$$

and $J_u(x)$ is the Jacobian determinant of $u$ (see [21]).

Notice that, for a continuous mapping $u$ the condition (i) is equivalent to the condition that $u$ belongs to the Sobolev space $W^{1,1}_{\text{loc}}(\Omega)$.

Let $P$ be Poisson kernel i.e. the function

$$P(x, \eta) = \frac{1 - |x|^2}{|x - \eta|^n},$$

and let $G$ be the Green function i.e. the function

$$G(x, y) = c_n \left\{ \begin{array}{ll}
\frac{1}{|x-y|^{n-2}} & \text{if } n \geq 3; \\
\log \frac{1}{|x-y|} & \text{if } n = 2 \text{ and } x, y \in \mathbb{C} \cong \mathbb{R}^2.
\end{array} \right.$$
where $c_n = \frac{1}{(n-2)\omega_{n-1}}$, and $\Omega_{n-1}$ is the measure of $S^{n-1}$. Both $P$ and $G$ are harmonic for $|x| < 1$, $x \neq y$.

Let $f: S^{n-1} \to \mathbb{R}^n$ be a $L^p$, $p > 1$ integrable function on the unit sphere $S^{n-1}$ and let $g: \mathbb{B}^n \mapsto \mathbb{R}^n$ be continuous. The weak solution of the equation (in the sense of distributions) $\Delta u = g$ in the unit ball satisfying the boundary condition $u|_{S^{n-1}} = f \in L^1(S^{n-1})$ is given by

$$u(x) = P[f](x) - G[g](x) := \int_{S^{n-1}} P(x, \eta)f(\eta)d\sigma(\eta) - \int_{B^n} G(x, y)g(y)dy,$$

for $|x| < 1$. Here $d\sigma$ is Lebesgue $n - 1$ dimensional measure of Euclid sphere satisfying the condition: $P[1](x) \equiv 1$. It is well known that if $f$ and $g$ are continuous in $S^{n-1}$ and in $\overline{B^n}$ respectively, then the mapping $u = P[f] - G[g]$ has a continuous extension $\bar{u}$ to the boundary and $\bar{u} = f$ on $S^{n-1}$. If $g \in L^\infty$ then $G[g] \in C^{1,\alpha}(\overline{B^n})$. See [6, Theorem 8.33] for this argument.

We will consider those solutions of the PDE $\Delta u = g$ that are quasiconformal as well and investigate their Lipschitz character.

A mapping $f$ of a set $A$ in Euclidean $n$-space $\mathbb{R}^n$ into $\mathbb{R}^n$, $n \geq 2$, is said to belong to the Hölder class $\text{Lip}_\alpha(A)$, $\alpha > 0$, if there exists a constant $M > 0$ such that

$$|f(x) - f(y)| \leq M|x - y|^\alpha$$

for all $x$ and $y$ in $A$. If $D$ is a bounded domain in $\mathbb{R}^n$ and if $f$ is quasiconformal in $D$ with $f(D) \subset \mathbb{R}^n$, then $f$ is in $\text{Lip}_\alpha(A)$ for each compact $A \subset D$, where $\alpha = K(f)^{1/(1-n)}$ and $K(f)$ is the inner dilatation of $f$. Simple examples show that $f$ need not be in $\text{Lip}_\alpha(D)$ even when $f$ is continuous in $D$.

However O. Martio and R. Nääkö in [20] showed that if $f$ induces a boundary mapping which belongs to $\text{Lip}_\alpha(\partial D)$, then $f$ is in $\text{Lip}_\beta(D)$, where

$$\beta = \min(\alpha, K(f)^{1/(1-n)});$$

the exponent $\beta$ is sharp.

In a recent paper of the second author and Saksman [10] it is proved the following result, if $f$ is quasiconformal mapping of the unit disk onto a Jordan domain with $C^2$ boundary such that its weak Laplacean $\Delta f \in L^p(\mathbb{B}^2)$, for $p > 2$, then $f$ is Lipschitz continuous. The condition $p > 2$ is necessary also. Further in the same paper they proved that if $p = 1$, then $f$ is absolutely continuous on the boundary of $\partial \mathbb{B}^2$. The results from [10] optimise in certain sense the results of the first author, Mateljević, Pavlović, Partyka, Sakan, Manojlović, Astala ([13, 14, 15, 16, 23, 24, 25, 26, 11, 12, 3]), since it does not assume that the mapping is harmonic, neither its weak Laplacean is bounded.
We are interested in the condition under which the quasiconformal mapping is in $\text{Lip}_\alpha(B^n)$, for every $\alpha < 1$. It follows from our results that the condition that $u$ is quasiconformal and $|\Delta u| \in L^p$, such that $p > n/2$ guarantees that the selfmapping of the unit ball is in $\text{Lip}_\alpha(B^n)$, where $\alpha = 2 - \frac{2}{n}$. In particular if $p = n$, then $f \in \text{Lip}_\alpha(B^n)$ for $\alpha < 1$.

Our result in several-dimensional case is the following:

**Theorem 1.** Let $n \geq 2$ and let $p > n/2$ and assume that $g \in L^p(B^n)$. Assume that $w$ is a $K$-quasiconformal solution of $\Delta w = g$, that maps the unit ball onto a bounded Jordan domain $\Omega \subset \mathbb{R}^n$ with $C^2$-boundary.

- If $p < n$, then $w$ is Hölder continuous with the Hölder constant $\alpha = 2 - \frac{2}{p}$.
- If $p = n$, then $w$ is Hölder continuous for every $\alpha \in (0, 1)$.
- If $n > p$ then $w$ is Lipschitz continuous.

The proof is given in the next section.

### 2. Proofs of the results

In what follows, we say that a bounded Jordan domain $\Omega \subset \mathbb{R}^n$ has $C^2$-boundary if it is the image of the unit disc $B^n$ under a $C^2$-diffeomorphism of the whole complex plane onto itself. For planar Jordan domains this is well-known to be equivalent to the more standard definition, that requires the boundary to be locally isometric to the graph of a $C^2$-function on $\mathbb{R}^{n-1}$.

In what follows, $\Delta$ refers to the distributional Laplacian. We shall make use of the following well-known facts.

**Proposition 2.1** (Morrey’s inequality). Assume that $n < p \leq \infty$ and assume that $U$ is a domain in $\mathbb{R}^n$ with $C^1$ boundary. Then there exists a constant $C$ depending only on $n$, $p$ and $U$ so that

$$
||u||_{C^{\alpha}(U)} \leq C||u||_{W^{1,p}(U)}
$$

for every $u \in C^1(U) \cap L^p(U)$, where

$$
\alpha = 1 - \frac{n}{p}.
$$

**Lemma 1.** See e.g. [3]. Suppose that $w \in W^{2,1}_{\text{loc}}(B^n) \cap C(\overline{B^n})$, that $h \in L^p(B^n)$ for some $1 < p < \infty$ and that

$$
\Delta w = h \text{ in } B^n, \text{ with } w|_{\partial B^n} = 0,
$$

a) If $1 < p < n$, then

$$
||\nabla w||_{L^q(B^n)} \leq c(p, n)||h||_{L^p(B^n)}, \quad q = \frac{pn}{n-p}.
$$
b) If \( p = n \) and \( 1 < q < \infty \) then
\[
\| \nabla w \|_{L^q(\mathbb{R}^n)} \leq c(q, n)\| h \|_{L^p(\mathbb{R}^n)}.
\]

c) if \( p > n \), then
\[
\| \nabla w \|_{L^q(\mathbb{R}^n)} \leq c(p, n)\| h \|_{L^p(\mathbb{R}^n)}.
\]

Now we prove

**Lemma 2.** If \( \Delta u = g \in L^p \) and \( r < 1 \), then \( Du \in L^q(\mathbb{R}^n) \) for \( q \leq \frac{np}{n-p} \).

**Proof of Lemma** By writing \( u = v + w \) from (2), and differentiating it we have

\[
Du(x) = Dv + Dw = \int_{S^{n-1}} \nabla P(x, \eta) f(\eta) d\sigma(\eta) - \int_{\mathbb{R}^n} \nabla_x G(x, y) g(y) dy.
\]

Then
\[
\int_{r\mathbb{B}} |Du(x)|^q dx = \int_{r\mathbb{B}} \left| \int_{S^{n-1}} \nabla_x P(x, \eta) f(\eta) d\sigma(\eta) - \int_{\mathbb{R}^n} \nabla_x G(x, y) g(y) dy \right|^q dx.
\]

Thus
\[
\| Du \|_{L^q(r\mathbb{B})} = \| Dv \|_{L^q(r\mathbb{B})} + \| Dw \|_{L^q(r\mathbb{B})} \\
\leq \left( \int_{r\mathbb{B}} \left| \int_{S^{n-1}} \nabla_x P(x, \eta) f(\eta) d\sigma(\eta) \right|^q dx \right)^{1/q} \\
+ \left( \int_{r\mathbb{B}} \left| \int_{\mathbb{R}^n} \nabla_x G(x, y) g(y) dy \right|^q dx \right)^{1/q}.
\]

There is a constant \( C \) so that

\[
|\nabla_x P(x, \eta)| \leq \frac{C}{(1 - |x|)^{n+1}}.
\]

From Lemma 1 and (6) we have \( \| Du \|_{L^q(r\mathbb{B})} < \infty \). \( \square \)

Now we formulate the following fundamental result of Gehring

**Proposition 2.2.** \( \square \) Let \( f \) be a quasiconformal mapping of the unit ball \( \mathbb{B}^n \) onto a Jordan domain \( \Omega \) with \( C^2 \) boundary. Then there is a constant \( p = p(K, n) > n \) so that

\[
\int_{\mathbb{B}^n} |Df|^p < C(n, K, f(0), \Omega).
\]

Then we prove

**Lemma 3.** If \( H : \mathbb{R}^n \rightarrow \mathbb{R} \) and \( w = (w_1, \ldots, w_n) : A \rightarrow B \) (where \( A, B \) are open subsets in \( \mathbb{R}^n \)) are functions from \( C^2 \) class, then:

\[
\Delta(H \circ w) = \sum_{i=1}^{n} \frac{\partial^2 H}{\partial w_i^2} |\nabla w_i|^2 + 2 \sum_{1 \leq i < j \leq n} \frac{\partial^2 H}{\partial w_i \partial w_j} \langle \nabla w_i, \nabla w_j \rangle + \sum_{i=1}^{n} \frac{\partial H}{\partial w_i} \Delta w_i
\]
Thus

\[
\mathcal{A}(H \circ w) = \sum_{k=1}^{n} \frac{\partial^{2}(H \circ w)(x_1, \ldots, x_n)}{\partial x_k^2}
\]

\[
= \sum_{k=1}^{n} \left[ \sum_{i=1}^{n} \frac{\partial^2 H}{\partial w_i \partial w_j} \left( \frac{\partial w_i}{\partial x_k} \frac{\partial w_j}{\partial x_k} \right) + \sum_{i=1}^{n} \frac{\partial H}{\partial w_i} \frac{\partial^2 w_i}{\partial x_k^2} \right]
\]

\[
= \sum_{k=1}^{n} \frac{\partial^2 H}{\partial w_i \partial w_j} \left( \frac{\partial w_i}{\partial x_k} \frac{\partial w_j}{\partial x_k} \right) + \sum_{i=1}^{n} \frac{\partial H}{\partial w_i} \sum_{k=1}^{n} \frac{\partial^2 w_i}{\partial x_k^2}
\]

\[
= \sum_{i=1}^{n} \frac{\partial^2 H}{\partial w_i^2} |\nabla w_i|^2 + 2 \sum_{1 \leq i < j \leq n} \frac{\partial^2 H}{\partial w_i \partial w_j} \langle \nabla w_i, \nabla w_j \rangle + \sum_{i=1}^{n} \frac{\partial H}{\partial w_i} \Delta w_i
\]
The quasiconformality of \( f \) and the behavior of \( \nabla H \) near \( \partial \Omega \) imply that there is \( r_0 \in (0, 1) \) so that the weak gradients satisfy
\[
|\nabla h(x)| \approx |\nabla w(x)| \quad \text{for} \quad r_0 \leq |x| < 1.
\]
Moreover, by Lemma \[\text{(2)}\]
for \( q \in (1, \frac{np}{n-p}) \), we have
\[
||\nabla h(x)||_{L^{q}(\partial \Omega)} \leq ||\nabla w(x)||_{L^{q}(\partial \Omega)} \leq C.
\]
It follows that for any \( q \in (1, \frac{np}{n-p}) \) we have that
\[
\nabla h \in L^{q}(\mathbb{B}^{n}) \quad \text{if and only if} \quad \nabla w \in L^{q}(\mathbb{B}^{n}).
\]
A direct computation (from Lemma \[\text{(3)}\]) by using the fact that \( H \in C^{2} \) is real valued, we obtain
\[
|\Delta h| \leq |\nabla w|^{2} + |g|.
\]
The higher integrability of quasiconformal self-maps of \( \mathbb{B}^{n} \) makes sure that \( \nabla (u \circ w) \in L^{q}(\mathbb{B}^{n}) \) for some \( q > n \), which implies that \( \nabla w \in L^{q}(\mathbb{B}^{n}) \). By combining this with the fact that \( g \in L^{p}(\mathbb{B}^{n}) \) with \( p > 1 \), we deduce that \( \Delta h \in L^{r}(\mathbb{B}^{n}) \) with \( r = \min(p, q/2) > 1 \). We use bootstrapping argument based on the following observation: in our situation
\[
\text{if} \quad \nabla w \in L^{q}(\mathbb{B}^{n}) \quad \text{with} \quad n < q < 2n, \quad \text{then} \quad \nabla w \in L^{q/(2n-a)}(\mathbb{B}^{n}),
\]
where \( a = q \wedge 2p \). In order to prove \[\text{(10)}\], assume that \( \nabla w \in L^{q}(\mathbb{B}^{n}) \) for an exponent \( q \in (n, 2n) \). Then \[\text{(9)}\] and our assumption on \( g \) verify that \( \Delta h \in L^{q/(2\wedge p)}(\mathbb{B}^{n}) \). Since \( h \) vanishes continuously on the boundary \( \partial \mathbb{B}^{n} \), we may apply Lemma \[\text{(1 a)}\] to obtain that \( \nabla h \in L^{q/(2n-a)}(\mathbb{B}^{n}) \) which yields the claim according to \[\text{(8)}\].

We then claim that in our situation one has \( \nabla w \in L^{q}(\mathbb{B}^{n}) \) with some exponent \( q > 2n \). To prove that, fix an exponent \( q_{0} > n \) obtained from the higher integrability of the quasiconformal map \( w \) so that \( \nabla w \in L^{q_{0}}(\mathbb{B}^{n}) \). By diminishing \( q_{0} \) if needed, we may well assume that \( q_{0} \in (n, 2n) \) and \( q_{0} \notin \{2^{m}/(2^{m-1} - 1), m = 3, 4, \ldots \} \). Then we may iterate \[\text{(10)}\] and deduce inductively that \( \nabla w \in L^{q_{k}}(\mathbb{B}^{n}) \) for \( k = 0, 1, 2 \ldots k_{0} \), where the indexes \( q_{k} \) satisfy the recursion \( q_{k+1} = \frac{np}{2n-q_{k}} \) and \( k_{0} \) is the first index such that \( q_{k_{0}} > 2n \). Such an index exists since by induction we have the relation \( (1 - n/q_{k}) = 2^{k}(1 - n/q_{0}) \), for \( k \geq 0 \). So \( q_{k} > n \). If \( q_{k} \leq 2n \), then we have \( \limsup_{k \rightarrow \infty} (1 - n/q_{k}) = \infty \) which is impossible.

Thus we may assume that \( \nabla w \in L^{q}(\mathbb{B}^{n}) \) with \( q > 2n \). At this stage \[\text{(9)}\] shows that \( \Delta h \in L^{p\wedge(q/2)}(\mathbb{B}^{n}) \). As \( p \wedge (q/2) = p \), Lemma \[\text{(1 a)}\] verifies that \( \nabla h \in L^{p/(n-a-p)}(\mathbb{B}^{n}) \). Finally, by \[\text{(8)}\] we have the same conclusion for \( \nabla w \), and hence by Morrey’s inequality \( w \) is H"{o}lder continuous with the constant \( c = \alpha = 2 - \frac{a}{p} \) as claimed. \( \square \)

If follows from the proof of the previous theorem that
Theorem 2. Assume that $g \in L^2(\mathbb{R}^n)$. If $w$ is a $K$-quasiconformal solution of $\Delta w = g$, that maps the unit disk onto a bounded Jordan domain $\Omega \subset \mathbb{R}^n$ with $C^2$-boundary, then $ Dw \in L^p(\mathbb{R}^n)$ for every $p < \infty$.

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