DEFORMATION CLASSES OF REAL CAYLEY M-OCTADS

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We study 8-point configurations in the real projective space forming an intersection
locus of three quadrics and containing no coplanar quadruples. We found that there
exists precisely 8 mirror-pairs of deformation classes of such configurations. We
describe also the mutual position of these 8 pairs and find the real monodromy
groups acting on the 8-point configurations, for each deformation class.

1. Introduction

1.1. Cayley octads. A Cayley octad $X \subset \mathbb{P}^3$ is an 8-point configuration obtained
as the intersection locus of three quadric surfaces $X = Q_0 \cap Q_1 \cap Q_2$. We allow
multiple points of $X$, which appear if the intersection is not transverse, in which
case 8 is the sum of multiplicities. A simple analysis shows that for any Cayley
octad $X$ the net of quadrics
$$
\mathcal{N} = \{Q_t = t_0 Q_0 + t_1 Q_1 + t_2 Q_2\}, \ t = [t_0 : t_1 : t_2] \in \mathbb{P}^2,
$$
is the complete linear system of quadrics passing through $X$, and so, $\mathcal{N}$ and $X$
determine each other.

Singular quadrics $Q_t \in \mathcal{N}$ are parameterized by a quartic curve $\mathcal{H} \subset \mathcal{N} \cong \mathbb{P}^2$
called the Hessian curve (Hessian quartic): it is defined by the determinant of the
symmetric $4 \times 4$-matrix defining $Q_t$. The following conditions are known to be
equivalent (cf., [Dv, Sect. 6.3.2], or [GH, Lemma 6.4]):

1. The Hessian curve $\mathcal{H}$ associated to a Cayley octad $X$ is non-singular.
2. The Cayley octad $X$ contains neither multiple points, nor coplanar subsets
   of more than 3 points.
3. Each quadric $Q_t$ from the net $\mathcal{N}$ associated to $X$ is irreducible, and the
   points of $X$ are non-singular on $Q_t$ for all $t \in \mathbb{P}^2$.

A Cayley octad $X$ is called regular if these equivalent conditions are satisfied, and
singular otherwise.

The reality condition (invariance under the complex conjugation) for Cayley
octad $X$ and for the net of quadrics $\mathcal{N}$ are obviously equivalent and imply reality
of the Hessian curve $\mathcal{H}$. We say that a real Cayley octad $X$ is maximal or $M$-octad
if its eight points are all real. If $X$ contains $k > 0$ pairs of conjugate complex points
and $8 - 2k$ real ones, we call it $(M - k)$-octad.

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1.2. Principal Results. Our principal aim is to enumerate the deformation classes that is path-connected components in the space of regular M-octads and to describe the mutual position of these classes.

Our first principal result says that there exist precisely 16 deformation classes of regular M-octads, which can be grouped into 8 pairs of mirror partner classes (see Theorems 3.4.1 and 4.1.2). Such pairs of classes, \([X]\) and \([\bar{X}]\), are represented by a regular M-octad \(X\) and its image \(\bar{X}\) under an orientation reversing projective transformation of \(\mathbb{R}P^3\). The union \([X]\cup[\bar{X}]\) is called the coarse deformation class of an M-octad \(X\).

The 8 pairs of deformation classes are named \((O^+_{\alpha\beta}, O^-_{\alpha\beta})\), where \((\alpha, \beta)\in\{0, 2, 4\}\times\{0, 3, 4\}\setminus\{(4, 3)\}\), and the course deformation classes are \(O_{\alpha\beta} = O^+_{\alpha\beta} \cup O^-_{\alpha\beta}\). The meaning of indices \(\alpha\) and \(\beta\) can be explained in terms of decorated graph \(\Gamma_X\) related to some degenerations of \(X\). This graph has the vertex set \(X\) and its edges are line segments joining the pairs of vertices, which can be merged by a real variation of \(X\) formed by regular M-octad. More precisely, among the two line segments in \(\mathbb{R}P^3\) joining a pair of points of \(X\) we select the one homotopic to the trace of these points under the merging variation. It follows that an edge with endpoints \(x, y\in X\) cannot cross any of the 20 planes passing though the triples of points of \(X\setminus\{x, y\}\).

The combinatorial types of such graphs are presented on Fig. 1, Table 1.

**Fig. 1. Graphs \(\Gamma_X\) and monodromy groups**

| Table 1. Combinatorial types of \(\Gamma_X\) for \(X\in O_{\alpha\beta}\) | Table 2. Monodromy groups |
|---|---|
| ![Graphs](image) | \(\mathbb{D}_4\) | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) | \(\mathbb{D}_4\) |
| | \(S_3\) | \(\mathbb{Z}_2\) |
| | \(S_4\) | \(\mathbb{Z}_2 \oplus \mathbb{Z}_2\) | \(S_4\) |
| | ![Graphs](image) | ![Graphs](image) |

The edges of graph \(\Gamma_X\) split into two types: oval-type labeled by “\(\alpha\)” on Table 1, and bridge-type labeled by “\(\beta\)”. These types indicate the degeneration of the Hessian curve (collapsing of an oval or a bridge) as the endpoints of an edge are merging. The numbers of edges of the corresponding types are denoted by \(\alpha(X)\) and \(\beta(X)\) and called respectively the oval-index and the bridge-index of \(X\).

Our second principal result (see Theorems 3.2.1 and 3.4.1) shows that the pair \(\alpha(X), \beta(X)\) is a complete invariant of coarse deformation equivalence, and any
combination \((\alpha, \beta) \in \{0, 2, 4\} \times \{0, 3, 4\}\) except \((4, 3)\) is realizable by some Cayley M-octad \(X\). Our next result is description of the mutual position (adjacency) of the corresponding coarse deformation components \(O_{\alpha \beta}\): in Fig. 1, Table 1 adjacent components stand next to each other (in the same row or column). After proving chirality of all Cayley octads (see Theorem 4.1.2), we deduce adjacency of the pure deformation components \(O_{\pm \alpha \beta}\).

Finally, in Theorem 5.2.1 we described the real monodromy groups \(\text{Aut}(X) \subset S_8\) of M-octads \(X\): they are indicated in Table 2 (the 8 cells of this table correspond to the cells of Table 1). Here group \(D_4\) is dihedral of order 8. These groups act on the graphs \(\Gamma_X\) preserving the decoration of edges. Table 3 gives the number of orbits of \(\text{Aut}(X)\) acting on \(X\), whose meaning is clarified in Sect. 6.1.

2. Preliminaries

In this Section, we outline some essentials on the Cayley octads and their Hessian curves with spectral theta-characteristics (for more details see [Do] and [GH]) and recall a few well-known facts on real plane quartics.

2.1. Degeneration of Cayley octads and nets of quadrics. The variety of Cayley octads form a Zariski-open subset of the Grassmannian of 2-planes in the projective space of quadrics, \(\mathcal{N} \subset P(\text{Sym}^2(C^4))\), namely, the nets of quadrics \(\mathcal{N}\) that have purely zero-dimensional basepoint locus, \(X = X(\mathcal{N})\). The Hessian curve \(\mathcal{H}\) of a Cayley octad \(X\) is interpreted as the intersection \(\mathcal{H} = \mathcal{N} \cap \Delta\), where \(\mathcal{N}\) is the associated net of quadrics and \(\Delta \subset P(\text{Sym}^2(C^4))\) the discriminant hypersurface.

In the variety of Cayley octads regular ones form a Zariski-open subset and singular ones form a certain discriminant locus, whose principal stratum (of codimension 1) is characterized in terms of nets \(\mathcal{N}\) and their Hessian curves \(\mathcal{H} = \mathcal{N} \cap \Delta\) as follows (cf. [GH], Lemma 6.4).

2.1.1. Lemma. A singular Cayley octad \(X\) represents a non-singular points of the discriminant locus of the variety of octads if and only if the \(\mathcal{H}\) has one node and no other singular points. Such a node may appear in two ways.

1. At the point of (simple) tangency of \(\mathcal{N}\) with \(\Delta\). Then the net \(\mathcal{N}\) contains one and only one cone with the vertex at some point of \(X\), and the Cayley octad \(X\) has one double point, whereas the other 6 points are ordinary and no 4 of them are coplanar.

2. At the point of (generic) intersection of \(\mathcal{N}\) with the singular locus \(\text{Sing}(\Delta)\) of \(\Delta\). Then \(\mathcal{N}\) contains a reducible quadric (a pair of distinct planes), and the 8 points of \(X\) are distinct and admit one and only one splitting into two coplanar quadruples.

Proof. The claim on the non-singular points is straightforward, and the description how a node on \(\mathcal{H} = \mathcal{N} \cap \Delta\) may appear is also trivial. The locus \(\text{Sing}(\Delta)\) parameterizes reducible quadrics, which proves the interpretation formulated in the case (2). In the remaining case then the description (1) follows. □

In the case (1), we say that Cayley octad \(X\) experiences a 2-collision, and in the case (2) (two coplanar quadruple of points), a 4-collision.

The space of real octads is the real locus of the space of complex octads, that is the corresponding Zariski-open subset in the real Grassmannian of 2-planes in
\( P(\text{Sym}^2(\mathbb{R}^4)) \). We will be interested in the part of this space represented by \( \text{M}-\)octads. Namely, we denote by \( \mathcal{O}^* \), \( \Delta_3^0 \) and \( \Delta_4^0 \) the strata formed by regular \( \text{M}-\)octads, by singular ones experiencing 2-collision and 4-collision respectively. Together they give space \( \mathcal{O} = \mathcal{O}^* \cup \Delta_3^0 \cup \Delta_4^0 \).

2.1.2. Lemma. Space \( \mathcal{O}^* \) is a manifold with boundary \( \Delta_3^0 \). The stratum \( \Delta_4^0 \) lies in the interior of \( \mathcal{O} \) and separates the connected components of \( \mathcal{O}^* \).

Proof. As it follows from Lemma 2.1.1, \( \Delta_3^0 \) as well as \( \Delta_4^0 \) are strata of codimension 1 in the variety of real Cayley octads, so, it is enough to check what lies from the two sides of them. The double point of an \( \text{M}-\)octad \( X \in \Delta_3^0 \) can be perturbed into a pair of real points, or a pair of conjugate imaginary ones by a small real variation of \( X \), which implies that \( \Delta_3^0 \) is bounded by \( \mathcal{O} \) from one side. On the contrary, the number 8 of real points of \( X \in \Delta_4^0 \) is preserved after any real variation, so \( X \) is adjacent to \( \Delta_4^0 \) from both sides. \( \square \)

2.2. The spectral correspondence. For a non-singular curve \( C \), let \( \Theta(C) \) be the set of theta-characteristics on \( C \). Consider function

\[
    h: \Theta(C) \to \mathbb{Z}/2, \quad \theta \mapsto \dim H^0(C; \theta) \mod 2,
\]

and let \( \Theta_i(C) = h^{-1}(i), i = 0, 1, \) which gives a partition \( \Theta(C) = \Theta_0(C) \cup \Theta_1(C) \) of theta-characteristics into even and odd, respectively. Given a regular Cayley octad \( X \) with the Hessian curve \( \mathcal{H} = \mathcal{N} \cap \Delta \), there is a natural line bundle \( \mathcal{L}_H \to \mathcal{H} \), whose fiber over \( Q_i \in \mathcal{H} \subset \mathcal{N} \) is the null-spaces of the degenerate quadratic form representing \( Q_i \). This bundle represents linear system \( |K_\mathcal{H} + \theta_X| \), where \( K_\mathcal{H} \) is the canonical class of \( \mathcal{H} \) and \( \theta_X \) is an even theta-characteristic called spectral. This linear system embeds \( \mathcal{H} \) to \( P^3 \) as a sextic, and the image, \( \mathcal{H} \), is called the Steinerian curve. More precisely, the map \( \mathcal{H} \to \mathcal{H} \) associates to a singular quadric \( Q_i \) (a cone) from the net \( \mathcal{N} \) the vertex of \( Q_i \), and so, \( \mathcal{H} \) lies in the original space \( P^3 \) containing \( X \) (see [Di] for details). This correspondence \( X \mapsto (\mathcal{H}, \theta_X) \) is bijective up to projective equivalence.

2.2.1. Theorem. (Hesse-Dixon) For any non-singular plane quartic \( C \) and even theta-characteristic \( \theta \) on \( C \) there exists a unique up to projective transformation of \( P^3 \) regular Cayley octad \( X \), such that \( C \) is projectively equivalent to its Hessian curve \( C \) and \( \theta \) is identified by this equivalence with the spectral theta-characteristic of \( X \). \( \square \)

We outline the proof in Sect. 2.3: it goes back to O.Hesse, who explained how \( X \) is reconstructed from \( (C, \theta) \), then A.Dixon elaborated and generalized it by proving a similar correspondence for nets of quadrics in any dimension. These arguments are applicable also in the real setting (cf. [DIK], Th. 3.3.4). Recall that for a real curve \( C \) reality of \( \theta \in \Theta(C) \) means that it is preserved invariant under the involution on \( \Theta(C) \) induced by the complex conjugation.

2.2.2. Theorem. For any real quartic \( C \) and an even real theta-characteristic \( \theta \), there exists a unique, up to real projective transformation of \( P^3_\mathbb{R} \), real net of quadrics in \( P^3_\mathbb{R} \) such that \( C \) is its Hessian curve and \( \theta \) its spectral theta-characteristic. \( \square \)

Pairs \( (C, \theta) \), with a non-singular quartic \( C \subset P^2 \) and an even theta-characteristic \( \theta \) will be called even spin quartics. The spectral correspondence is not only a bijective map between the projective classes of regular Cayley octads and projective
classes of non-singular even spin quartics, but is an isomorphism of the corresponding moduli spaces, see [Do], Remark 6.3.1 and [GH] Sect. 6. Restricting it to the real loci we can conclude in particular that their connected components are in one-to-one correspondence.

2.2.3. Corollary. The spectral correspondence induces a bijective correspondence between the coarse deformation classes of real regular Cayley octads and the deformation classes of real even spin quartics. □

2.3. Points of a Cayley octad as Aronhold sets for its Hessian quartics. The line $x_ix_j$ connecting a pair of points of a regular Cayley octad $X = \{x_0, \ldots, x_7\}$ determines a line $B_{ij}$ in the plane $N$ that is a pencil of quadrics whose base-locus is the union of line $x_ix_j$ with a twisted cubic passing through the other six points of $X \setminus \{x_i, x_j\}$. Line $B_{ij}$ is bitangent to the Hessian quartic $H$ and the divisor formed by the two tangency points defines an odd theta-characteristic denoted $\theta_{ij}$ (see [Do], Theorem 6.3.2). This gives one-to-one correspondence between the set $\Theta_1(H)$ of 28 odd theta-characteristics on $H$ and the 28 pairs $\{i, j\} \subset \{0, \ldots, 7\}$.

Any quadruple $\{i, j, k, l\} \subset \{0, \ldots, 7\}$ defines an even theta-characteristic $\theta_{ijkl} = \theta_{ij} + \theta_{ik} + \theta_{il} - K_H$. The choice of a distinguished index $i$ here is not essential, and moreover, the complementary quadruple $\{0, \ldots, 7\} \setminus \{i, j, k, l\}$ gives the same theta-characteristic as $\theta_{ijkl}$. This describes $35 = \frac{1}{2} \binom{8}{4}$ even theta-characteristics of the set $\Theta_0(H) \setminus \{\theta_X\}$. The remaining even theta-characteristic is

$$\theta_X = \theta_{01} + \theta_{02} + \cdots + \theta_{07} - 3K_H.$$

The set of 7 bitangents $B_{0i}$ corresponding to the 7 summands $\theta_{0i}$ in $\theta_X$ form an Aronhold set (see [Do], Sect.6.3.3 for details). There exist precisely $288 = 8 \times 36$ Aronhold sets representing 36 even theta-characteristics, so that every chosen $\theta \in \Theta_0(H)$ is represented precisely by 8 Aronhold sets, and each Aronhold set match precisely to one vertex of the Cayley octad $X$ defined by $(H, \theta), \theta \in \Theta_0(H)$.

Namely, given such $(H, \theta)$, we embed $H$ to $P^3$ by the linear system $K_H + \theta$. Each binangent $B_i$ of a fixed Aronhold set $B_1, \ldots, B_7$ representing $\theta$, gives a pair of points on $H$. The seven lines passing through such pairs of points must be concurrent and intersect at the point of $X$ corresponding to the fixed Aronhold set. This is how $X$ is reconstructed from a given even spin quartic.

2.4. Theta-characteristics as quadratic functions. With $\theta$ we associate

$$q_\theta : H_1(C; \mathbb{Z}/2) \to \mathbb{Z}/2, \quad x \mapsto h(\theta) + h(\theta + x^*) \mod 2,$$

where $x^* \in H^1(C; \mathbb{Z}/2)$ is Poincare dual to $x \in H_1(C; \mathbb{Z}/2)$ (see [X] and [M]). It is a quadratic function in the sense that

$$(1) \quad q_\theta(x + y) = q_\theta(x) + q_\theta(y) + x \cdot y \mod 2,$$

This gives the well-known identification of the set $\Theta(C)$ with the set of quadratic functions on $H_1(C; \mathbb{Z}/2)$, compatible with the action of $H^1(C; \mathbb{Z}/2)$ on the both
elements are identified with the classes in $H^2.5.1. Proposition.

The function $h$ is identified then with the Arf invariant of quadratic functions, $\text{Arf}(q_\theta) \in \mathbb{Z}/2$. Note that by definition of $q_\theta$

(2) $\text{Arf}(q_{\theta+x^*}) = h(\theta + x^*) = h(\theta) + q_\theta(x) = \text{Arf}(q_\theta) + q_\theta(x) \mod 2$

In the case of Cayley octad $X$ and its Hessian quartic $\mathcal{H}$ and $x \in H_1(\mathcal{H}; \mathbb{Z}/2)$ we obtain

(3) $q_{\theta X}(x) = \text{Arf}(q_{\theta + x^*}) = h(\theta_X + x^*) \mod 2$

2.5. Bipartitions of Cayley octads. For a a regular Cayley octad $X = \{x_0, \ldots, x_7\}$ we consider its even bipartitions $X = A \cup B$, $A \cap B = \emptyset$, into subsets of even cardinalities $|A|$ and $|B|$, and denote by $\Lambda(X)$ the set formed by 64 such unordered pairs $\{A, B\}$. Then the assignments

$$\{x_i, x_j\} \mapsto \theta_{ij}, \quad \{x_i, x_j, x_k, x_l\} \mapsto \theta_{ijkl}, \quad X \mapsto \theta_X$$

induces a natural one-to-one correspondence $\Phi^\theta : \Lambda(X) \to \Theta(\mathcal{H})$.

Note that $\Lambda(X)$ is a $\mathbb{Z}/2$-vector space as a subquotient of the power set $\mathcal{P}(X)$, with the sum operation induced by the symmetric difference $\Delta$:

$$\{A, B\} + \{C, D\} = \{A \Delta C, A \Delta D\} = \{B \Delta D, B \Delta C\}$$

and is endowed with the non-degenerate $\mathbb{Z}/2$-valued inner product

$$\{A, B\} \cdot \{C, D\} = |A \cap B| \mod 2.$$

The difference $\theta - \theta'$ for $\theta, \theta' \in \Theta(\mathcal{H})$ is a 2-torsion element of Pic$^0(\mathcal{H})$ and such elements are identified with the classes in $H^1(\mathcal{H}; \mathbb{Z}/2)$, so, we obtain the associated one-to-one correspondence $\Phi^H : \Lambda(X) \to H^1(\mathcal{H}; \mathbb{Z}/2)$ which is the composition of $\Phi^\theta$ and the map $\theta \mapsto \theta - \theta_X$. Composing $\Phi^H$ with the Poincare duality we obtain also a map $\Phi_H : \Lambda(X) \to H_1(\mathcal{H}; \mathbb{Z}/2)$.

2.5.1. Proposition.

(1) The maps $\Phi_H^H$ and $\Phi_H$ are linear isomorphisms of $\mathbb{Z}/2$-vector spaces.

(2) Isomorphism $\Phi^\theta$ identifies $h : \Theta(\mathcal{H}) \to \mathbb{Z}/2$, with the map $\Lambda(X) \to \mathbb{Z}/2$, $\{A, B\} \mapsto \frac{1}{2}|A| \mod 2$.

(3) If $x = \Phi_H((A, B)) \in H_1(\mathcal{H}; \mathbb{Z}/2)$, then $q_{\theta X}(x) = h(x) = \frac{1}{2}|A| \mod 2$.

(4) $\Phi_H$ preserves the inner product, namely, for all $x, y \in H_1(\mathcal{H}; \mathbb{Z}/2)$

$$\Phi_H^{-1}(x) \cdot \Phi_H^{-1}(y) = x \cdot y, \quad \text{where } x \cdot y \text{ is the homology intersection index}.$$

Proof. (1) Linearity of $\Phi^H$ (and thus, $\Phi_H$) follows from the relation $\theta_{ij} + \theta_{jk} + \theta_{ik} = \theta_X + K_M$, or equivalently, $\theta_{ik} = \theta_{ij} + \theta_{jk} - \theta_X$ (cf. [GH Sect. 6] or [Do] Theorem 6.3.3)). For instance, we can deduce from it

$$\Phi^H(\{x_i, x_j\}, X \setminus \{x_i, x_j\}) + \Phi^H(\{x_j, x_k\}, X \setminus \{x_j, x_k\}) =$$

$$(\theta_{ij} - \theta_X) + (\theta_{jk} - \theta_X) = \theta_{ik} - \theta_X = \Phi^H(\{x_i, x_k\}, X \setminus \{x_i, x_k\}),$$

and similarly, linearity in the other cases. Bijectivity of $\Phi^\theta$ implies that the linear maps $\Phi^H$ and $\Phi_H$ are isomorphisms.

Part (2) holds by definition of $\Phi^\theta$, since $\theta_{ij}$ are odd and $\theta_{ijkl}$ are even.

Part (3) follows from relation (3) of Sect. 2.4, which taking into account also relation (1) applied to $q_{\theta X}$ implies part (4). □
2.6. Picard-Lefschetz monodromy transformation in $\Theta(\mathcal{H})$.

2.6.1. Lemma. The Picard-Lefschetz monodromy transformation associated with the vanishing class $v \in H_1(\mathcal{H}; \mathbb{Z}/2)$ induces map $T^\Theta_v : \Theta(\mathcal{H}) \to \Theta(\mathcal{H})$

\[ \theta \mapsto \theta + (q_\theta(v) + 1)v^*. \]

In particular, $\theta$ is preserved if and only if $q_\theta(v) = 1$.

Proof. As is well-known, the monodromy action in $H_1(\mathcal{H}; \mathbb{Z}/2)$ associated with $v$ is the Picard-Lefschetz transformation $x \mapsto x + (x \cdot v)v$, so, the corresponding monodromy transforms in $\Theta(\mathcal{H})$, $q_\theta \mapsto q_\theta'$ should satisfy modulo 2 relation $q_\theta'(x) = q_\theta(x + (x \cdot v)v) = q_\theta(x) + (x \cdot v)q_\theta(v) + (x \cdot v)^2 = q_\theta(x) + (q_\theta(v) + 1)v^*(x)$. \qed

2.6.2. Proposition. The Picard-Lefschetz transformation $T^\Theta_v$ preserves the pairity of theta-characteristics $\theta$ ($\mathrm{Arf}$-invariant of $q_\theta$). The action induced by $T^\Theta_v$ in $\Lambda(X)$ via $\Phi^\Theta$ depends on the bipartition $\Phi^\Theta_H(v) = \{A_v, B_v\} \in \Lambda(X)$ as follows.

1. If one of the sets $A_v$ or $B_v$ is a 2-elements set $\{x_i, x_j\} \subset X$ then the action of $T^\Theta_v$ in $\Lambda(X)$ is induced by the transposition of $x_i$ and $x_j$.

2. If $A_v$ and $B_v$ are 4-element sets, then $T^\Theta_v$ replaces 2-element subsets $\{x_i, x_j\} \subset A_v$ (or $\{x_i, x_j\} \subset B_v$) by their complements $A_v \setminus \{x_i, x_j\}$ (respectively $B_v \setminus \{x_i, x_j\}$) and keep $\{x_i, x_j\}$ unchanged if it has one common point with $A_v$ and $B_v$.

Proof. Applying formula (3) and using relation (1), (2) we obtain

$$\mathrm{Arf}(q_\theta') = \mathrm{Arf}(q_\theta) + q_\theta((q_\theta(v) + 1)v) = \mathrm{Arf}(q_\theta) \mod 2,$$

since $q_\theta((q_\theta(v) + 1)v) = (q_\theta(v) + 1)q_\theta(v)$, which means preserving of pairity of $\theta$.

The action on $\{A, B\} \in \Lambda(X)$ induced by $T^\Theta_v$ due to Lemma 2.6.1 in terms of $\{A_v, B_v\}$ is as follows:

$$\{A, B\} \mapsto \{A, B\} + \varepsilon\{A_v, B_v\}, \quad \text{where} \quad \varepsilon = \frac{|A_v|}{2} + |A \cap A_v| + 1 \mod 2,$$

which gives $\varepsilon = |A \cap A_v| \mod 2$ if $A_v = \{x_i, x_j\}$, and so, in the case (1) $T^\Theta_v$ is induced by transposition of $x_i$ and $x_j$. In the case (2) $\varepsilon = |A \cap A_v| + 1 \mod 2$ and thus, $T^\Theta_v$ acts also as is described. \qed

Transformations part (2) of Proposition 2.6.2 were named *bifid substitutions* by Cayley, who found them as the monodromy of 4-collisions in the corollary below.

2.6.3. Corollary. Let $\mathcal{H}$ be the Hessian quartic of a regular Cayley octad $X$ with the spectral theta-characteristic $\theta \in \Theta(\mathcal{H})$ and associated quadratic function $q_\theta$. Assume that $v \in H_1(\mathcal{H}; \mathbb{Z}/2)$ is a vanishing class associated with some nodal degeneration of $\mathcal{H}$, $v = \Phi_H(\{A_v, B_v\})$. Then $q_\theta(v) = \frac{|A|}{2} \mod 2$ and the degeneration of $X$ corresponding to the nodal degeneration of $\mathcal{H}$ is

1. a 2-collision of $x_i, x_j \in X$ that form 2-element set, $A_v$ or $B_v$, in the case of $q_\theta(v) = 1$;
2. a 4-collision involving $A_v$ and $B_v$, in the case of $q_\theta(v) = 0$.

Proof. It follows from the description of the Picard-Lefschetz transformation corresponding to the vanishing class $v$ in terms of $X$ given in Proposition 2.6.2 \qed
2.7. Real non-singular quartics. The real deformation classification of non-singular real plane quartics $C$ goes back to F. Klein: there exist 6 real deformation classes characterized by the number $0 \leq k \leq 4$ of components of $C_R$ called ovals and in the case $k = 2$ by their mutual position: in one class two ovals bound disjoint pair of discs in $P^2_R$, and in the other two ovals are nested: one oval bounds a disc containing another oval. By $M$-quartics we mean real non-singular quartics with the maximal number 4 of ovals. The ovals of such quartics can be arbitrarily permuted by the monodromy.

2.7.1. Proposition. (Klein) $M$-quartics form one real deformation class. The ovals of an $M$-quartic can be arbitrarily permuted by the real deformation monodromy: the corresponding monodromy group is the symmetric group $S_4$. □

2.7.2. Proposition. A real regular Cayley octad $X$ has $8 - 2d$ real points, $0 \leq d \leq 3$, if and only if its Hessian quartic $H$ has $4 - d$ real ovals, which (for $d = 2$) are not nested.

Proof. The real structure from $H$ can be lifted to the del Pezzo surface $Z$ obtained by double covering of $P^2$ branched along $H$, so that the real locus $Z_R$ is projected to the non-orientable region of $P^2_R$ bounded by $H_R$. Then $\chi(Z_R) = 2(\chi(P^2_R) - k) = 2 - 2k$, where $k$ is the number of components of $H_R$. On the other hand, if $X$ has $8 - 2d > 0$ real points, then in the blowup model of $Z$, its real locus $Z_R$ is obtained by blowing up $P^2_R$ at $8 - 2d - 1$ points, thus, $\chi(Z_R) = 1 - (8 - 2d - 1) = 2 - 2(4 - d)$. Thus, $k = 4 - d$. □

3. Real Hessian curve with real $\theta$-characteristics

3.1. The ovals and bridges of $M$-quartics. Given an $M$-quartic $C \subset P^2$ we consider real vanishing cycles, which are conj-invariant simple closed curves on $C$ that can be contracted to the nodal point by some real degeneration of $C$. Such cycles include the the four ovals, $a_0, \ldots, a_3 \subset C_R$: contraction of an oval yields a node of solitary type.

Another kind of vanishing cycles are bridges $b_{ij}$, $0 \leq i < j \leq 3$, connecting ovals $a_i$ and $a_j$. Namely, the quartic splitting into four real lines in general position in $P^2$ can be perturbed into $M$-quartic, and the six intersection points of the lines yield the six bridges $b_{ij}$, which are real vanishing cycles having precisely two real points at the intersection with ovals $a_i$ and $a_j$. Connectedness of the space of $M$-quartics yields such bridges for any given $M$-quartic $C$ and Proposition 3.1.1 below shows that the classes $[b_{ij}] \in H_1(C; Z/2)$ are well-defined (unique).

Consider $\pm 1$-eigengroups $H^+_1(C) = \{x \in H_1(C) | c(x) = \pm x\}$ along with their images $H^+_1(C; Z/2) \subset H_1(C; Z/2)$ under the modulo 2 reduction homomorphism. It is a well-known fact that the action of $c$ in $H_1(C; Z/2)$ is trivial for $M$-curves, which leads to the direct sum decompositions

$$H_1(C) = H^+_1(C) \oplus H^-_1(C), \quad H_1(C; Z/2) = H^+_1(C; Z/2) \oplus H^-_1(C; Z/2).$$

3.1.1. Proposition. (1) Classes $[a_i]$, $0 \leq i \leq 3$ span $Z/2$ vector space $H^+_1(C; Z/2)$ with the only relation $[a_0] + \cdots + [a_3] = 0$. In particular, any three of these classes form a basis.

(2) The homology classes $[b_{ij}] \in H_1(C; Z/2)$, $0 \leq i < j \leq 3$ span $H^-_1(C; Z/2)$ and satisfy relations $[b_{ij}] + [b_{jk}] + [b_{ik}] = 0$ for any triple of indices $0 \leq i < j < k \leq 3$. In particular, classes $[b_{0i}]$, $1 \leq i \leq 3$ form a basis of $H^-_1(C; Z/2)$. 

3.2. Lemma. The 64 theta-characteristics of $C$ classified by the three Arf-invariant.

Proof. It follows trivially from Proposition 3.1.1 and from the definition of the Arf-invariant. □

3.2.1. Theorem. Space $C$ has precisely 11 connected components: 8 components in $C^{ev}$ and 3 components in $C^{odd}$. The components of $C^{ev}$ are $C^{ev}_{\alpha,\beta}$ where $a \in \{0, 2, 4\}$, $b \in \{0, 3, 4\}$, with exception of the pair $(\alpha, \beta) = (2, 4)$, and the components of $C^{odd}$ are $C^{odd}_{\alpha,\beta}$ where $(\alpha, \beta)$ is $(2, 4)$, $(2, 3)$, or $(4, 3)$.

We start proving it with two lemmas.

3.2.2. Lemma. The 64 theta-characteristics $\theta \in \Theta(C)$ on an M-quartic $C$ are classified by the three $\mathbb{Z}/2$-values $q_0([a_i])$ and three $\mathbb{Z}/2$-values $q_0([b_{ij}])$, $i = 1, 2, 3$. The Arf-invariant of $q_0$ is equal to

$$Arf(q_0) = \sum_{i=1}^{3} q_0([a_i])q_0([b_{0i}]).$$

Proof. It follows trivially from Proposition 3.1.1 and from the definition of the Arf-invariant. □

So, we can encode the theta-characteristics $\theta \in \Theta(C)$ by binary $2 \times 3$-matrices

$$\begin{bmatrix}
q_0([a_1]) & q_0([a_2]) & q_0([a_3]) \\
q_0([b_{01}]) & q_0([b_{02}]) & q_0([b_{03}])
\end{bmatrix}.$$ Symmetric group $S_4$ permuting the ovals (or in the other words, changing ordering $a_0, \ldots, a_3$ acts on the set $\Theta(C)$. Namely, $\sigma \in S_4$ induces an action on the bridges classes sending $[b_{ij}]$ to $[b_{\sigma(i)\sigma(j)}]$, which determines the induced action on the set of 64 binary $2 \times 3$-matrices. We will list the orbits of this actions.
3.2.3. Lemma. There exist precisely 11 orbits of the $S_4$-action on $\Theta(C)$: eight in $\Theta_0(C)$ and three in $\Theta_1(C)$. In terms of binary matrices they are represented by

\begin{equation}
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}.
\end{equation}

Proof. Enumeration of orbits is a straightforward exercise. The parity of $\theta \in \Theta(C)$ represented by the matrices listed are determined by Lemma 3.2.2. □

Proof of Theorem 3.2.1. Connectedness of the space of M-quartics implies immediately that the set of connected components of the space $C$ is in one-to-one correspondence with the set of orbits of the real deformation monodromy action on the set $\Theta(C)$ for any chosen M-quartic $C$. Since the real monodromy group acting on the ovals of $C$ is $S_4$ (see 2.7.1), Lemma 3.2.3 implies the first part of the Theorem. For the second part we need just to determine the values of $\alpha$ and $\beta$ (numbers of non-zero values of $q_{\theta}$ on the 4 ovals and 6 bridges) for the 11 matrices in Lemma 3.2.3. It is straightforward because both subgroups $H^\pm(C; \mathbb{Z}/2)$ are isotropic with respect to the intersection form, and thus, the restriction of $q_{\theta}$ to each of them is linear. Namely, in the list (1) values of $\alpha$ are respectively 0, 2 and 4 in the first, the second and the third columns, while the values of $\beta$ are respectively 4, 3 and 0 in the first, second and third rows.

In part (2) the three matrices represent pairs $(\alpha, \beta)$ which are, respectively, (2, 4), (2, 3), and (4, 3). □

3.3. Theta-diagrams of M-quartics. We give below a simple graphical interpretation of the $2 \times 3$-matrices listed in Lemma 3.2.2, which reveals the geometry behind the enumeration of orbits in Lemma 3.2.3 and simplifies it.

The four ovals $a_i$ of an M-quartic $C$ are represented by four vertices depicted as small circles on the plane and the six bridges are represented as edges depicted as line segments connecting these circles pairwise: $b_{ij}$ connects circle $a_i$ with $a_j$ in $\mathbb{RP}^2$, so that two of the six bridges pass “through infinity”, as it is shown on Fig. 2. One can view this diagram as a complete graph $K_4$ embedded in $\mathbb{RP}^2$.

A theta-characteristic $\theta \in \Theta(C)$ can be described as a coloring of such diagram: black color for an oval $a_i$ or bridge $b_{ij}$ means that $q_{\theta}$ takes value 0 on the corresponding class $[a_i]$ or $[b_{ij}]$, and white means values 1. Black colored ovals are depicted as filled circles, and black bridges are usual edges, while white ovals and bridges look like empty circles and dotted edges. A diagram decorated in this way will be called the theta-diagram of $\theta \in \Theta(C)$. Such diagram with even $\theta$ will be denoted $D_{\alpha \beta}$, where $\alpha$ and $\beta$ are respectively the numbers of white vertices (ovals) and white edges (bridges) in it.

3.4. Coarse Deformation Components.
3.4.1. **Theorem.** There exist eight coarse deformation classes of regular Cayley M-octads which are in spectral correspondence with the eight components $C_{\alpha\beta}^{ev}$ of $C^{ev}$ listed in Theorem 3.2.1.

*Proof.* Theorem 2.2.2 with Corollary 2.2.3 and Proposition 2.7.2 (in the case of $d = 0$) give a bijective correspondence between the set of coarse deformation classes of real regular M-octads and the set of deformation classes of non-singular real even spin M-quartics. So, it is left to apply Theorem 3.2.1 that enumerates them. □

Let us denote by $O_{\alpha\beta}$ the coarse deformation class corresponding to $C_{\alpha\beta}^{ev}$ by Theorem 3.4.1.

3.5. **Adjacency of the coarse deformation components.** Two real deformation classes of M-octads (connected components of $O^*$) lying on the opposite sides from a wall of $\Delta_4^*$ (formed by M-octad with a 4-collision) are said to be adjacent. A pair of coarse deformation classes $O_{\alpha\beta}$ containing such an adjacent pair of components will be called adjacent too.

Adjacency of classes $O_{\alpha\beta}$ are analyzed below through adjacency of the corresponding components $C_{\alpha\beta}^{ev}$. Namely, let us fix an M-quartic $C$ with the ovals $a_i$, $i = 0, \ldots, 3$ and bridges $b_{ij}$, $0 \leq i < j \leq 3$. We say that theta-characteristics $\theta$ and $\theta'$ in $\Theta_0(C)$ are adjacent theta-characteristics if the pairs $(C, \theta)$ and $(C, \theta')$ represent adjacent components $C_{\alpha\beta}^{ev}$ and $C_{\alpha'\beta'}^{ev}$ (corresponding to adjacent classes $O_{\alpha\beta}$ and $O_{\alpha'\beta'}$).

3.5.1. **Proposition.**

1. Group $H_1(C; \mathbb{Z}/2)$ contains precisely 10 real vanishing classes: four oval-classes $[a_i]$ and six bridge-classes $[b_{ij}]$.

2. A pair of theta-characteristics $\theta, \theta' \in \Theta_0(C)$ are adjacent if and only if $
\theta = \theta + v^*$, where $v^*$ is Poincare dual to some real vanishing class $v \in H_1(C; \mathbb{Z}/2)$ satisfying the condition $q_\theta(v) = 0$.

*Proof.* Any nodal degeneration of $C$ is either a contraction of some oval $a_i$ with the vanishing class $[a_i]$, or merging of two ovals, $a_i$ and $a_j$ with the vanishing class $[b_{ij}]$. 
(see Proposition 3.1.1(3)). This proves part (1).

By Lemma 2.1.2 and Corollary 2.6.3, the condition \( q_\theta(v) = 0 \) in part (2) means that the corresponding to this nodal degeneration wall is internal (deformation components of spin M-quartics lie on the both sides of it).

For proving the wall-crossing formula \( \theta' = \theta + v^* \), recall that the forgetful map \((C, \theta) \mapsto C\) from the variety of spin quartics to the space of quartics is unramified near nodal quartic if for the vanishing class \( v \) we have \( q_\theta(v) = 1 \) and has ramification of index 2 along the stratum of nodal quartics with \( q_\theta(v) = 0 \). It follows for instance from Lemma 2.6.1. The Picard-Lefschetz transformation in \( \Theta(C) \), as it was observed in Lemma 2.6.1, is \( q_\theta \mapsto q_\theta + (q_\theta(v) + 1)v^* \), which in the case of \( q_\theta(v) = 0 \) is just just adding \( v^* \), so, it implies \( q_\theta'(v) = q_\theta(v) + v^*(v) = q_\theta(v) \).

Finally, it is left to notice that a loop around a wall in the space of spin quartics after lifting to the covering space with the ramification of index 2 becomes a path into the adjacent deformation component and the Picard-Lefschetz formula becomes the wall-crossing formula for the markings that define a given branched covering. □

In terms of M-quartic theta-diagrams \( D_{\alpha\beta} \) and \( D_{\alpha'\beta'} \) of adjacent deformation classes this proposition means that \( D_{\alpha'\beta'} \) is obtained from \( D_{\alpha\beta} \) after one of the following two modifications (see Fig.3):

**Black edge and vertex modifications**

1. **Black edge modification:** each endpoint, \( a_i \) and \( a_j \), of some black edge \( \beta_{ij} \) changes its color, while the other 2 vertices and all 6 edges preserve.
2. **Black vertex modification:** each edge \( b_{ij} \) incident to some black vertex \( a_i \) changes its color, while the other 3 edges and all 4 vertices preserve.

The following theorem shows that adjacency of coarse deformation classes \( O_{\alpha\beta} \) corresponds to vertical and horizontal adjacency of theta-diagrams \( D_{\alpha\beta} \) on Fig.2.

**3.5.2. Theorem.** Two different coarse deformation classes of M-octads \( O_{\alpha\beta} \) and \( O_{\alpha'\beta'} \) are adjacent if and only if one of the following holds

1. \( \alpha = \alpha' \) and numbers in the pair \( \{\beta, \beta'\} \subset \{0, 3, 4\} \) have odd difference,
2. \( \beta = \beta' \) and \( |\alpha - \alpha'| = 2 \).

Besides, a coarse deformation component \( O_{\alpha\beta} \) is adjacent to itself if and only if \((\alpha, \beta)\) is either \((2, 3)\) or \((2, 0)\).

**Proof.** If the vanishing cycle \( v \) representing the wall between adjacent deformation classes is a bridge-class \( b_{ij} \), then according to Lemma 3.5.1 the value of \( q_\theta \) changes on the two oval-classes \( a_i \) and \( a_j \) and is preserved on the other oval classes and on all bridge-classes. Then \( \beta = \beta' \) and \( |\alpha - \alpha'| \) is either 2 or 0. Moreover, 0 may appear only if some black edge has endpoints of different colour, which is possible only for \( O_{03} \) and \( O_{20} \).

If \( v \) is an oval-class \( a_i \), then for the same reason \( q_\theta \) changes on the three bridge-classes \( [b_{ij}], 0 \leq j \leq 3, j \neq i \) and so, \( \alpha = \alpha' \) and \( |\beta - \beta'| \) is either 1 or 3. □
3.5.3. Corollary. Coarse deformation classes of regular M-octads are related through a finite number of wall-crossings (passing to an adjacent coarse deformation class of M-octads). □

3.6. Graphs \( \Gamma_X \). Assume that \( X \) is a regular M-octad and \((H, \theta)\) is the associated even spin Hessian quartic. Consider a pair of real vanishing cycles \( v_1, v_2 \in H_1(H; \mathbb{Z}/2)\) representing 2-collisions of \( X \), and let \( e_1, e_2 \) be the corresponding edges of \( \Gamma_X \).

3.6.1. Lemma. (1) Edges \( e_1 \) and \( e_2 \) are different for different \( v_1 \) and \( v_2 \).

(2) The intersection index \( v_1 \cdot v_2 \) is 1 if and only if \( e_1 \) and \( e_2 \) have a common vertex; in particular, edges \( e_1 \) and \( e_2 \) are disjoint if \( v_1, v_2 \) are both oval-cycles, or both bridge-cycles.

Proof of Lemma 3.6.1. It follows immediately from that \( \Phi_H \) preserves \( \mathbb{Z}/2 \)-inner product by Proposition 2.5.1. □

3.6.2. Proposition. The graphs \( \Gamma_X \) for \( X \in \mathbb{O}_{\alpha \beta} \) have combinatorial types as presented in Table 1 of Fig. 1.

Proof. By Proposition 2.5.1, white ovals and bridges of \( H \) represent edges of graph \( \Gamma_X \), so that an incident pair of oval end bridge represents a pair of adjacent edges. Note that white ovals and edges shown on each of the theta-diagrams of Fig. 2 split into several chains formed by consecutively incident ovals and bridges which alternate in the chain. The corresponding edges in \( \Gamma_X \) then split into chains of consecutively adjacent edges. Applying it to each of the eight theta-diagrams, we obtain the graphs \( \Gamma_X \) presented in Table 1 of Fig. 1.

For instance, in the (least trivial) case of \( X \in \mathbb{O}_{44} \), there is one chain forming a cycle, \( a_0, b_01, a_1, b_12, \ldots, a_7, b_07 \) (if the ovals are numerated cyclically). This cycle should represent an cycle on \( \Gamma_X \) forming an octagon shown in the top-right cell in Table 1 of Fig. 1. Analysis of the other cases is similar. □

4. Pure deformation classification

4.1. Chirality of configurations in \( \mathbb{R}P^3 \). By a simple \( n \)-configuration in \( \mathbb{R}P^3 \), \( n \geq 4 \), we mean an \( n \)-point subset \( A \subset \mathbb{R}P^3 \) not containing coplanar quadruples of points. A pair of such configurations are said to be deformation equivalent if they can be connected by a continuous family of simple \( n \)-configurations. A mirror partner of \( A \) is a configuration \( \bar{A} \) obtained from \( A \) by an orientation reversing projective transformation (for instance, reflection across a plane). If a simple \( n \)-configuration \( A \) is deformation equivalent to \( \bar{A} \), it is called achiral and otherwise chiral. The following observation belongs to O. Viro and V. Kharlamov (see [VV]).

4.1.1. Proposition. All simple configurations of 6 and 7 points in \( \mathbb{R}P^3 \) are chiral.

Proof. Following [VV], with a triple of skew lines \( \{\ell_1, \ell_2, \ell_3\} \) in \( \mathbb{R}P^3 \) we associate sign

\[ \text{lk}(\ell_1, \ell_2, \ell_3) = \text{lk}(\bar{\ell}_1, \bar{\ell}_2)\text{lk}(\bar{\ell}_1, \bar{\ell}_3)\text{lk}(\bar{\ell}_2, \bar{\ell}_3) \in \{+1, -1\} \]

where \( \bar{\ell}_i \) is the line \( \ell_i \) endowed with an arbitrary orientation and \( \text{lk}(\bar{\ell}_i, \bar{\ell}_j) = \pm 1 \) is the normalized linking number in \( \mathbb{R}P^3 \) (normalization means multiplication by 2, since the usual linking number is \( \pm \frac{1}{2} \)). This triple index is clearly independent of the choice of the order of lines \( \ell_1, \ldots, \ell_3 \) and of their orientations, but alternates if we change the orientation of \( \mathbb{R}P^3 \), or reflect a triple of lines across a plane.
Given a simple 6-configuration $A$, we consider a triple of skew lines connecting the points of $A$ pairwise. We can obtain 15 such triples of lines corresponding to 15 splitting of 6 points into 3 pairs, and the product of the corresponding 15 signs $\text{lk}(\ell_1, \ell_2, \ell_3)$ is a deformation invariant of simple 6-configuration $A$, denoted $\text{sign}_6(A)$. Then $\text{sign}_6(A') = -\text{sign}_6(A)$ if $A'$ is a mirror of $A$, and so, $A$ is chiral.

For a simple 7-configuration $B$ we obtain seven signs for its 6-subconfigurations, and the product of these signs, $\text{sign}_7(B) \in \{+1, -1\}$, is a deformation invariant, such that $\text{sign}_7(B') = -\text{sign}_7(B)$ for a mirror image $B'$ of $B$. □

Each point $x$ of a simple 8-configuration $X$ can be also equipped with a sign, $\text{sign}(X, x) = \text{sign}_7(X \setminus \{x\})$. Then, if $X'$ is the mirror partner of $X$ and $x' \in X'$ the vertex corresponding to $x$, we have $\text{sign}(X', x') = -\text{sign}(X, x)$.

4.1.2. Theorem. All points of a regular Cayley M-octad are equipped with the same sign. In particular, any regular Cayley M-octad is chiral. Thus, there exist precisely 16 pure deformation classes of them.

We complete its prove towards the end of section 4.

4.2. Plane 4-moves for 6 and 7-configurations. We say that two deformation classes of simple $n$-configurations, $n \geq 4$, are adjacent if they lie from the opposite sides of the codimension 1 stratum formed by $n$-configurations with a coplanar quadruple of points (a deformation class can be adjacent to itself if this stratum is one-sided or the same deformation class lie from the both sides).

4.2.1. Lemma. Assume that simple $n$-configurations $A_0$ and $A_1$ represent adjacent deformation classes. Then $\text{sign}_n(A_0) = -\text{sign}_n(A_1)$ if $n = 6$ or $n = 7$. In particular, for such $n$ self-adjacent deformation classes do not exist.

Proof. Since $A_0$ and $A_1$ are adjacent, they can be connected with a path $A_t = \{x_1(t), \ldots, x_n(t)\}$, $t \in [0, 1]$, in which configurations $A_t$ are simple for all $t$ except one value $t = t_0$, and for this value precisely one quadruple of points $x_i(t)$ become coplanar. These four points can be connected pairwise by two lines in three ways. The linking number between the corresponding pairs of lines (with some auxiliary orientations) for $t \neq t_0$ alternates as we cross the wall at $t = t_0$. Thus, in the case of $n = 6$, for 3 of the 15 pairwise matchings the triple linking number alternates. For the remaining 12 matchings such number is obviously preserved, so, $\text{sign}_6(A_0) = -\text{sign}_6(A_1)$.

In the case of $n = 7$, among the seven signs of 6-subconfigurations exactly three will change: if the point that we drop is not among the four points which become coplanar in the process of deformation $A_t$. Thus, $\text{sign}_7(A_0) = -\text{sign}_7(A_1)$. □

4.2.2. Lemma. For any Cayley M-octad $X$ and any point $x \in X$ the sign $\text{sign}(X, x)$ alternates after $X$ experiences a 4-collision.

Proof. It follows from Lemma 4.2.1 applied to the residual 7-configuration $X \setminus \{x\}$, because it contains precisely one of the two complementary quadruple of points of $X$ involved into a 4-collision. □

4.2.3. Lemma. If in a Cayley M-octad $X$ points $x_i, x_j \in X$ are adjacent in the graph $\Gamma_X$, then $\text{sign}(X, x_i) = \text{sign}(X, x_j)$.

Proof. This follows from that residual 7-configurations $X_i = X \setminus \{x_i\}$, $i = 1, 2$ are deformation equivalent through simple 7-configurations, in which $x_1$ moves...
towards $x_2$ along the corresponding edge of $\Gamma_X$ and the other 6 points of $X$ remain constant. \qed

Proof of Theorem 4.1.3. The signs $\text{sign}(X, x)$ are the same for all $x \in X$ if $X \in \mathcal{O}_{44}$ due to Lemma 4.2.3, because the graph $\Gamma_X$ is connected. Since by Corollary 5.5.3 we may reach any coarse deformation classes $\mathcal{O}_{\alpha \beta}$ through wall-crossing beginning from $\mathcal{O}_{44}$, Lemma 4.2.2 implies that for any regular M-octad $X$ the signs $\text{sign}(X, x)$ are the same for all eight points of $X$. \qed

By Theorem 4.1.3 each coarse deformation class $\mathcal{O}_{\alpha \beta}$ is the union of two purely deformation classes distinguished by the common sign of vertices. According to this sign, we will denote the corresponding pair of deformation classes $\mathcal{O}_{\alpha \beta}^+$ and $\mathcal{O}_{\alpha \beta}^-$.  

4.2.4. Corollary. Any pair of deformation classes $\mathcal{O}_{\alpha \beta}^\pm$ are related by several wall-crossings.

Proof. It follows from Corollary 5.5.3 and adjacency between classes $\mathcal{O}_{23}^+$ and $\mathcal{O}_{23}^-$ realized by a black-edge-move in the theta-diagram $D_{23}$, for a black edge whose endpoints are of different colors. \qed

5. Real monodromy groups

5.1. The Complex monodromy. Consider a regular Cayley Octad $X$ with its Hessian quartic $\mathcal{H}$ and spectral theta-characteristic $\theta \in \Theta_0(\mathcal{H})$. Isomorphism $\Phi_H : A(X) \cong H_1(\mathcal{H};\mathbb{Z}/2)$ due to Proposition 2.5.1 and relation (3) in Sect. 2.4 induces a homomorphism $\Phi_{\text{aut}} : S(X) \rightarrow \text{Aut}(H_1(\mathcal{H};\mathbb{Z}/2), q_\theta)$ from the permutation group $S(X) \cong S_8$ acting on $X$ to the group of automorphisms of $H_1(\mathcal{H};\mathbb{Z}/2)$ preserving the quadratic function $q_\theta$ associated to $\theta$ (and thus, preserving $\mathbb{Z}/2$-valued intersection form in $H_1(\mathcal{H};\mathbb{Z}/2)$).

5.1.1. Proposition. (cf. [GH], Proposition 2.1) $\Phi_{\text{aut}}$ is a group isomorphism.

Proof. Proposition 2.5.1 immediately implies that $\Phi_{\text{aut}}$ is monomorphic, so, it is left to verify that $|\text{Aut}(H_1(\mathcal{H};\mathbb{Z}/2), q_\theta)| = 8!$. First, we find the order of the automorphism group $\text{Aut}(H_1(\mathcal{H};\mathbb{Z}/2))$ preserving just the intersection form, by counting the number of symplectic bases $e_1, f_1, e_2, f_2, e_3, f_3$ in $H_1(\mathcal{H};\mathbb{Z}/2)$. This number is $(63 \cdot 32)(15 \cdot 8)(3 \cdot 2) = 36(8!)$, where $63 \cdot 32$ counts the number of pairs of $e_1, f_1$, etc.

Group $\text{Aut}(H_1(\mathcal{H};\mathbb{Z}/2))$ acts transitively on the set $\Theta_0(\mathcal{H})$ (identified with the set of quadratic function with even Arf invariant). Since $\text{Aut}(H_1(\mathcal{H};\mathbb{Z}/2), q_\theta)$ is the stabilizer of $\theta$, its order is $8!$, since $|\Theta_0(\mathcal{H})| = 36$. \qed

5.2. The monodromy groups of regular M-octads. By the real monodromy group, $\text{Aut}_R(X)$, of a real regular Cayley M-octad $X$ we mean the subgroup of $S(X)$ that is under the monodromy homomorphism $\pi_1(\mathcal{O}_{\alpha \beta}^\pm, X) \rightarrow S(X)$, where $\mathcal{O}_{\alpha \beta}^\pm$ is the deformation component of M-octads containing $X$.

5.2.1. Theorem. For any regular Cayley M-octad $X$ the group $\text{Aut}_R(X)$ is isomorphic to a subgroup of $S_8$ formed by the permutations of the ovals $\alpha_i$, $0 \leq i \leq 3$, of the associated Hessian quartic $\mathcal{H}$ which preserve their colors as well as the colors of bridges $b_{ij}$, $0 \leq i < j \leq 3$ under the induced permutation of them.

Proof. By Proposition 3.1.3 a permutation of ovals of M-quartic $\mathcal{H}$ determines the corresponding permutation of bridges and thus, an automorphism of $H_1(\mathcal{H};\mathbb{Z}/2)$ which is necessarily real and non-trivial for a non-trivial permutation of ovals. By
Proposition 5.1.1, an automorphism of $H_1(H; \mathbb{Z}/2)$ determines a permutation of $X$ provided it preserves the quadratic function $q_{\theta}$ associated to the spectral theta-characteristic $\theta$, which is equivalent to preserving the coloring of ovals and edges on the theta-diagram of $(H, \theta)$. The permutation of $X$ that we obtain is represented by a real monodromy, as it follows from Theorem 2.2.2. □

5.2.2. Corollary. The list of groups $\text{Aut}_R(X)$ for $X \in O_{\alpha \beta}$, $a \in \{0, 2, 4\}$ and $b \in \{0, 3, 4\}$, $(\alpha, \beta) \neq (2, 3)$ is as presented in the Table 2 of Fig. 1.

Proof. Theorem 5.2.1 reduces finding of $\text{Aut}_R(X)$ to a trivial analysis of symmetries of the eight theta-diagrams on Fig. 2. □

6. Concluding remarks

6.1. Deformation classes of marked Cayley M-octads. As an application of analysis of the real monodromy action in the last Section, we can now easily upgrade our deformation classification of Cayley M-octad to more refined classification of M-octads with a marked point. Namely, each coarse deformation class $O_{\alpha \beta}$ gives a number of coarse deformation classes or marked M-octad which is obviously equal to the number of orbits of the real monodromy action on $X \in O_{\alpha \beta}$. The numbers of orbits are indicated in the Table 3 on Figure 1 and there sum, 14, is the number of classes of marked M-octads.

Furthermore, dropping the chosen point in a marked regular M-octad gives a configuration of 7 points in $P^2_{\mathbb{R}}$. On the other hand, the central projection from this marked point send the others into a configuration of 7 points in $P^2_{\mathbb{R}}$, none of which is collinear and no 6 is conic. Following [GH] we call such 7-configurations typical. These two 7-configurations, planar and spacial, are related by the Gale duality (see [GH], Sect. 7), and knowing one of them (up to projective equivalence), one can recover the whole Cayley octad (cf., Sect. 2.3).

In particular the 14 coarse deformation classes of marked regular Cayley M-octads correspond to the 14 deformation classes of planar typical 7-configurations that were analyzed in [FZ].

6.2. Theta-characteristics for real (M-1)-quartics. In the case of regular (M-1)-octads the corresponding Hessian curves $H$ are (M-1)-quartics, by Proposition 2.7.2. The set of ovals $a_1, a_2, a_3$ of $H$ is connected pairwise by bridges: each pair $a_i, a_j$ by two bridges denoted $b_{ij}^+$: in accord with two kinds of line segments connecting a pair of points in $\mathbb{R}P^2$, see Fig. 4 for the corresponding theta-diagrams. It is not difficult to show that $[b_{ij}^+] + [b_{ij}^-] = [H_{\theta}]$, and so, $q_{\theta}$ takes opposite values on $[b_{ij}^+]$ and $[b_{ij}^-]$. A further analysis shows that there exist two deformation classes of theta-diagrams for even spin (M-1)-quartics, which we presented on Fig. 4 together with the corresponding graphs $\Gamma_X$ (which have six vertices for (M-1)-octads) and real monodromy groups $\text{Aut}_R(X)$ (the definitions and proofs are analogous to the case of M-octads).

In particular, we see that there exist precisely two coarse deformation classes of regular (M-1)-octads described by theta-diagrams and graphs $\Gamma_X$ on Fig. 4. By Proposition 4.1.1, pure deformation classification of regular (M-1)-octads gives then 4 deformation classes.

6.3. Odd spin quartics. An odd theta-characteristic, $\theta$, on a non-singular quartic $C$ defines a cubic surface $Z$ with a point $z \in Z$, so that $C$ is projectively
The three deformation classes of odd spin M-quartics \((C, \theta)\) found in Proposition 3.2.3 corresponds to the three types of real points on a real nonsingular M-cubic surface \(Z \subset \mathbb{P}^3\). Recall that \(Z\) is \(M\)-cubic if and only if all 27 lines on it are real, and the complement of these lines in \(Z\) split into polygonal regions, which can be triangular, quadrilateral, or pentagonal. These three types of regions correspond to the above three types of \((C, \theta)\), according to the total number of black ovals and bridges on an odd theta-diagram, which can be respectively 3, 4 or 5 (see the three rightmost diagrams of Fig. 3).

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