Embedding loop quantum cosmology without piecewise linearity

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Abstract
An important goal is to understand better the relation between full loop quantum gravity (LQG) and the simplified, reduced theory known as loop quantum cosmology (LQC), directly at the quantum level. Such a firmer understanding would increase confidence in the reduced theory as a tool for formulating predictions of the full theory, as well as permitting lessons from the reduced theory to guide further development in the full theory. This paper constructs an embedding of the usual state space of LQC into that of standard LQG, that is, LQG based on piecewise analytic paths. The embedding is well defined even prior to solving the diffeomorphism constraint, at no point is a graph fixed and at no point is the piecewise linear category used. This motivates for the first time a definition of operators in LQC corresponding to holonomies along non-piecewise linear paths, without changing the usual kinematics of LQC in any way. The new embedding intertwines all operators corresponding to such holonomies, and all elements in its image satisfy an operator equation which classically implies homogeneity and isotropy. The construction is made possible by a recent result proven by Fleischhack.

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1. Introduction

Loop quantum gravity (LQG) is a well-defined, background-independent framework for quantum gravity which admits well-defined quantizations of dynamics. Loop quantum cosmology (LQC) is a simplified model which aims to describe the cosmological consequences of LQG by attempting to model the homogeneous and isotropic sectors of the theory. Concretely, this model is obtained by applying loop quantization methods to the classical symmetry reduced sector of gravity. The simplicity of the symmetry reduced sector enables one to complete the quantization procedure, and begin the process of making predictions [1–9]. In the course of developing LQC, lessons were learned, in particular, the need to change the quantization of the Hamiltonian constraint in order to recover the correct classical limit [3],...
and it is hoped that these lessons will guide further development in full LQG. Thus the value of LQC is twofold. It provides a way to derive predictions from LQG for cosmology, as well as providing a guide for further development in the full theory. However, each of these goals can be achieved with full confidence and precision only insofar as LQC can be derived from LQG as the homogeneous–isotropic sector in some sense. The most obvious way of making precise the idea of LQC describing such a sector is to construct an embedding of LQC states into such a sector. The meaning of the homogeneous–isotropic sector in a quantum field theory, and especially quantum gravity, was investigated in the work [10]. In the works [10–12], a program, to a large extent successful, has been developed to construct an embedding into such a sector of LQC. In doing this, at least until this paper, it has been necessary—though only as an intermediate step [12]—to first embed LQC into a version of LQG in which only piecewise linear paths are allowed [11, 12]. The formal reason for this necessity lies in a difference of choice made in the quantizations of LQG and LQC: the choice of configuration algebra, the algebra of functions on configuration space which are directly promoted to operators, and which, when interpreted as states in the connection representation, additionally play the role of ’test states’ in the theory. In standard LQG [13–15], this algebra includes matrix elements of all parallel transports along all piecewise analytic paths, whereas in LQC, this algebra includes only matrix elements of parallel transports along paths which are adapted to the homogeneous symmetry group, in the sense that (piecewise) they are integrals of the corresponding Killing vector fields—that is, in the case of isotropic $k = 0$ LQC, one restricts to paths which are ‘piecewise straight’.

In a recent paper [16], Fleischhack has begun to lay the foundations for an alternative construction of LQC in which one uses the configuration algebra generated by parallel transports along all piecewise analytic paths, as in the standard full theory. This enables the embedding into standard LQG to be achieved without using the piecewise linear category. However, as straightforward as this sounds, it remains to see how far this alternative framework can be systematically developed. As of yet, no canonical inner product1 or quantization of constraints and elementary observables of interest have been proposed. The architects of the currently more well-established LQC kinematics [18, 3] (which, for brevity, we refer to as ’standard LQC’) were in fact aware of the possibility of using a larger configuration algebra, but chose to restrict consideration to the algebra generated by linear paths in order to simplify the quantum theory and make it tractable with the intuition and hope that this should be sufficient [19]. For, one of the principal motivations for looking at LQC is precisely to improve the tractability of the tasks of completing, and computing consequences from, the theory.

This paper shows how, in fact, standard LQC can be naturally embedded into standard, piecewise analytic LQG, without using piecewise linearity at any stage: a new formulation of LQC is not necessary for this. This embedding has been made possible by a technical result proven for the first time by Fleischhack in [16], a technical result which has also been addressed in the work [17]. Specifically, the configuration algebra (and hence test states) proposed by Fleischhack [16] separates cleanly into a direct vector space sum of the configuration algebra of standard LQC and the space of functions vanishing at infinity and zero.

Of course, the embedding which we will introduce here is at the kinematical level. In the end, as argued in [10–12], it will be necessary to have an embedding at the diffeomorphism-invariant level, either by making the idea sketched in [10–12] precise or otherwise.

An added benefit of the new embedding of standard LQC is that it motivates a definition of operators corresponding to holonomies along curved paths in standard LQC, such that the

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1 The work [17] mentions a possible inner product which, however, depends on a choice of (Euclidean-group-equivalence class of ) three arbitrary fixed edges in space, among other choices.

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new embedding intertwines these new operators with the corresponding operators in the full theory. This was heretofore not possible. Furthermore, closely related to this is that states in the image of the embedding satisfy homogeneity and isotropy in a precise sense that does not use piecewise linearity in any way. Thus, the embedding presented here has all the strengths of that introduced in [11, 12], and more.

Furthermore, the new embedding comes in two versions: a ‘c’ embedding and a ‘b’, or ‘holomorphic’, embedding. As in [11], the ‘c’ embedding is simpler and introduced first, and the ‘b’ embedding is defined in terms of it. The characteristic difference between the ‘c’ and ‘b’ embeddings lie in the way symmetry is satisfied by states in their image. Physically, the role of an embedding is to identify the LQC state space with a particular ‘cosmological sector’ of the full theory. In the cosmological sector corresponding to ‘c’ embeddings, only the configuration field satisfies homogeneity and isotropy, whereas in that of the ‘b’ embeddings, both configuration and momenta satisfy homogeneity and isotropy in a precise sense. The use of coherent states, here and in [10, 11] specified by a choice of positive phase space function called a ‘complexifier’ [20], is key in defining the ‘b’ embeddings. The cosmological sector of the ‘b’ embedding presented in this paper, when cut-off to a fixed graph and with a particular choice of complexifier, is furthermore the same cosmological sector used in the spin-foam cosmology literature [21–25]. However, in contrast to the spin-foam cosmology literature, we fix no graph in this paper, and every state of the cosmological sector considered here has support on every piecewise analytic graph, with no graph preferred.

The paper is organized as follows. We begin by reviewing the structures involved in LQG and LQC. Fleischhack’s proposal and corresponding embedding are presented, and the intertwining and symmetry properties of this embedding are proven. This then provides a starting point for defining an embedding of standard LQC into the full theory. New operators in standard LQC for holonomies along curved paths are motivated, and the intertwining property of these new operators using the new embedding is proven. The corresponding ‘b’ or ‘holomorphic’ embedding is then introduced, and all of the above properties are proven also for it. Lastly, in section 6, the equivalence of the strategy of embedding pursued here and the strategy of specifying a projection to relate LQC and LQG are briefly touched upon.

2. LQG and LQC structures

Loop quantum gravity

Loop quantum gravity is based on the Ashtekar–Barbero formulation of gravity, in which the (unconstrained) gravitational phase space $\Gamma$ is parameterized by an $SU(2)$ connection $A_\mu \equiv A_i^a \tau_i$ and a densitized triad $\tilde{E}^a_i$ on space, where $\tau_i := -\frac{i}{2} \sigma_i$, with $\sigma_i$ the Pauli matrices. We use the convention that lower case latin indices are spatial indices. The densitized triad field is related to a triad $e^a_i$ and its inverse $e^a_i$ via $\tilde{E}^a_i = \det(e^a_i)e^a_i$, which are related to the physical spatial metric via $g_{ab} = e^a_i e^b_i$. In terms of the generalized ADM variables, $A^i_a := \Gamma^i_a + \beta K^i_a$, where $\Gamma^i_a$ is the spin connection determined by $\tilde{E}^a_i$, $K^i_a := K_{ab} e^{a}_{b}$ with $K_{ab}$ the extrinsic curvature, and $\beta \in \mathbb{R}^+$ is the Barbero–Immirzi parameter [26–31].

The basic variables with direct quantum analogues are holonomies $A(\ell)$ of $A$ along piecewise analytic paths $\ell$, and electric fluxes through surfaces $\Sigma$: $\Sigma(S)^{\alpha} = \int_{S} \Sigma^{\alpha}$, where $\Sigma^{\alpha}_{ab} := 2 \epsilon_{abc} \tilde{E}^{c\alpha}$. In the connection representation, states are wave functionals $\Psi(A)$ of the connection $A$. One starts with a space, denoted as Cyl, of ‘nice’ functions called cylindrical, which depend only on the holonomies of $A$ along a finite set of (piecewise analytic) paths $\ell$; when these paths are chosen to be non-intersecting except possibly at end points, they are called edges and their union is called a graph. On Cyl is defined the Ashtekar–Lewandowski
inner product $\langle \cdot, \cdot \rangle$ [32, 33]; the elementary operators $A(\ell)$ and $\bar{\Sigma}(S)$ then act on the resulting completed kinematical Hilbert space $\mathcal{H}_{\text{kin}}$. The algebraic dual $\text{Cyl}^*$ of $\text{Cyl}$ provides a notion of distributional (i.e. non-normalizable) states, elements of which we write with rounded bra notation, $(\Psi | ) [34]$. 

**Loop quantum cosmology**

To define the homogeneous–isotropic sector (for the spatially flat case, which is the case we consider), one fixes a specific action of the Euclidean group, $\mathcal{E} \cong \mathbb{R}^3 \times \text{SO}(3)$, on the $SU(2)$ principal fiber bundle of the theory. Concretely, this is done by choosing an action $\triangleright$ of $\mathcal{E}$ on the basic variables through spatial diffeomorphisms and local $SU(2)$ rotations:

$$
(x, r) \triangleright (A^i, \bar{E}^a) = (r^j \phi_{(x, r)}, A^i, (r^{-1})^j \phi_{(x, r)}, \bar{E}^a).
$$

where $r^j$ denotes the adjoint action of $r \in \text{SO}(3)$. If we let $\mathcal{A}_S$ and $\Gamma_S$ denote the set of elements of $\mathcal{A}$ and $\Gamma$, respectively, fixed by this action, then $\mathcal{A}_S$ and $\Gamma_S$ are respectively one and two dimensional. Fix a reference element $A^i_o$ in $\mathcal{A}_S$ and a triad $\hat{e}^a_i$ such that $\hat{e}^a_i A^i_o = V_o^{-1/3} \delta^i_o$ for some $V_o$ with dimensions of volume. Let $q_{ab} := \hat{e}^a_i \hat{e}^b_i$. Then $(A^i_o, \bar{E}^a_o) \in \Gamma_S$ if and only if it is of the form

$$
A^i_o = c A^i_o, \quad \bar{E}^a_o = p V_o^{-2/3} \sqrt{q} e^a_i
$$

for some real $c$ and $p$. In the phase space $\Gamma_S$, $c$ and $p$ are conjugates, and in the quantum theory one considers wavefunctions $\psi(c)$ of $c$. A function $\psi(c)$ is called almost periodic if it is a linear combination of exponentials $e^{iuc}$ [18]; the space of such functions is denoted as $\text{Cyl}_S$. In analogy with LQG, one constructs an inner product $\langle \cdot, \cdot \rangle$ on $\text{Cyl}_S$ and completes to obtain a Hilbert space $\mathcal{H}_S$. Elements of $\text{Cyl}_S^*$ represent distributional states and are again denoted using rounded bras $(\psi | )$.

**3. The Fleischhack state space**

We here review the configuration algebra, and hence ‘test states’, proposed by Fleischhack [16], Brunnemann and Koslowski [17] for cosmology, as well as the embedding of these states into full LQG mentioned in [16]. We also prove that this embedding satisfies all of the properties satisfied by the embedding in [11], except with the piecewise linear category replaced by the piecewise analytic category. The proofs of these properties constitute new results, though the arguments are strongly modeled on those already present in [11].

**Definition and embedding**

The reference connection $\hat{A}_o$ provides a map $r : \mathbb{R} \rightarrow \mathcal{A}_S \subset \mathcal{A}$ via

$$
r : c \mapsto c \hat{A}_o.
$$

Let $\text{Cyl}_F := r^*[\text{Cyl}]$. This is the configuration algebra, and hence the space of ‘test’ states, which Fleischhack and others [16, 17] have advocated as an alternative foundation for quantum cosmology. In such an alternative framework, $\text{Cyl}_F^*$ would play the role of ‘distributional states.’ The advantage of such a framework lies in the existence of an embedding into full theory states, $\iota_F : \text{Cyl}_F^* \hookrightarrow \text{Cyl}^*$, defined by $(\iota_F \alpha)(\Phi) := (\alpha | r^* \Phi)$. As we will see, $\iota_F$ is injective, thus justifying the term ‘embedding.’ In fact, it is formally identical to the embedding defined in [11, 12], with piecewise linearity replaced by piecewise analyticity. As was the case in [11, 12], $\iota_F$ intertwines operators central to the quantizations, and its image satisfies an operator equation implying homogeneity and isotropy, as we shall also prove.
Lemma 1. $\iota_F$ is injective and hence an embedding.

Proof. It is sufficient to show that $\iota_F$ has a trivial kernel. Suppose $\iota_F \alpha = 0$. Then for all $\Phi \in \text{Cyl}$,

$$0 = (\iota_F \alpha | \Phi) = (\alpha | r^* \Phi).$$

Because $r^*[\text{Cyl}] = \text{Cyl}_F$, the above implies $\alpha = 0$. □

Intertwining of operators

Let $F(A)$ denote any cylindrical function, considered as a full theory phase space function depending only on $A$. The restriction of $F(A)$ to the homogeneous–isotropic sector is $F(r(c))$. Thus, in the quantum theory, the full theory operator $\hat{F}(A)$ corresponds to the reduced theory operator $\tilde{F}(r(c))$. We therefore adopt the notation $\hat{F}(A) : = \tilde{F}(r(c))$. Using logic identical to that in proposition 2 of [11] (which we do not repeat), one proves the following.

Theorem 1. $\iota_F$ intertwines $\hat{F}(A)$ and $\tilde{F}(A)$ in the sense $\hat{F}(A)^* \circ \iota_F = \iota_F \circ \tilde{F}(A)^*$.

Homogeneity and isotropy

States in the image of $\iota_F$ are furthermore in a precise sense homogeneous and isotropic. To discuss homogeneity and isotropy at the quantum level, one must formulate it in terms of holonomies and fluxes. The condition of homogeneity and isotropy on the connection takes the form [10, 11]

$$(g \triangleright A)(\ell)^A_{\; B} = A(\ell)^A_{\; B},$$

for all piecewise analytic $\ell$, all $g \in \mathcal{E}$ and all $A, B = 0, 1$.

Theorem 2. Every element $\Psi$ in the image of $\iota_F$ satisfies, for all piecewise analytic $\ell$ and $g \in \mathcal{E}$,

$$(g \triangleright A)(\ell)^A_{\; B}^* \Psi = A(\ell)^A_{\; B}^* \Psi. \quad (3)$$

Proof. Let $g \in \mathcal{E}$, $\ell$, and $A, B \in [0, 1]$ be given, and let $F(A) := A(\ell)^A_{\; B}$ and $F'(A) := (g^* F)(A) = (g \triangleright A)(\ell)^A_{\; B}$. As $r(c)$ is invariant under $\mathcal{E}$, $F'(r(c)) := F(g \triangleright r(c)) = F(r(c))$. Applying theorem 1 to $F(A)$ and $F'(A)$ and equating yields

$$\tilde{F}(A)^* \iota_F \psi = \tilde{F}(A)^* \iota_F \psi$$

for all $\psi \in \text{Cyl}_F^*$. □

Note this holds for all piecewise analytic $\ell$, in contrast to the piecewise linear result in [11].

The fact that states in the image of $\iota_F$ satisfy a condition of homogeneity and isotropy only on the connection variable is due to the fact that it is a ‘c’-embedding (see [10, 11]). Better in this respect are the ‘b’ or holomorphic embeddings, in which symmetry is imposed on both configuration and momenta by using coherent states [10, 11]. See section 5.
4. Embedding of standard LQC into standard LQG

We now come to the embedding of standard LQC states into standard, piecewise analytic LQG—the central subject of this paper—and prove its properties. This embedding is constructed using the embedding discussed above together with a recent decomposition of \( \text{Cyl}_F \) proven by Fleischhack [16]. We begin by reviewing this decomposition.

Decomposition of the Fleischhack state space

Let \( \mathcal{V} \) denote the set of all functions on \( \mathbb{R} \) vanishing at \( \pm \infty \) and zero. One then has the following result proven in [16].

**Theorem 3.** As vector spaces, 
\[
\text{Cyl}_F = \text{Cyl}_S \oplus \mathcal{V}.
\] (4)

This furthermore implies that \( \text{Cyl}_F^* \), the algebraic dual of \( \text{Cyl}_F \), physically representing ‘distributional states’ in the framework proposed by Fleischhack, is naturally isomorphic to \( \text{Cyl}_S^* \oplus \mathcal{V}^* \). Let \( \text{PS} : \text{Cyl}_F \to \text{Cyl}_S \) and \( \text{PV} : \text{Cyl}_F \to \mathcal{V} \) denote canonical projection onto the two components in equation (4).

**Lemma 2.** Define \( \Phi : \text{Cyl}_S^* \oplus \mathcal{V}^* \to \text{Cyl}_F^* \) by
\[
(\Phi \psi_S, \psi_V) = (\psi_S | \text{PS} \phi) + (\psi_V | \text{PV} \phi).
\]

\( \Phi \) is one-to-one and onto, yielding a natural isomorphism \( \text{Cyl}_F^* \cong \text{Cyl}_S^* \oplus \mathcal{V}^* \).

**Proof.** One-to-one.
Suppose \( (\psi_S, \psi_V) = 0 \). Then for all \( \phi \in \text{Cyl}_S \subset \text{Cyl}_F \), \( 0 = (\psi_S | \phi) = (\psi_S | \text{PS} \phi) \), so that \( \psi_S = 0 \), and for all \( \phi \in \mathcal{V} \subset \text{Cyl}_F \), \( 0 = (\psi_V | \phi) = (\psi_V | \text{PV} \phi) \), so that \( \psi_V = 0 \).

Onto.
Let \( \psi \in \text{Cyl}_F^* \) be given. Define \( \psi_S \in \text{Cyl}_S^* \) and \( \psi_V \in \mathcal{V}^* \) as the restriction of \( \psi \) to \( \text{Cyl}_S \subset \text{Cyl}_F \) and \( \mathcal{V} \subset \text{Cyl}_F \), respectively. Then \( \Phi \psi = (\psi | \Phi) \).

Definition of the embedding

Define \( \iota : \text{Cyl}_S^* \hookrightarrow \text{Cyl}_F^* \equiv \text{Cyl}_S^* \oplus \mathcal{V}^* \) as the inclusion map via the isomorphism proved in lemma 2. One then defines \( \iota : \text{Cyl}_S^* \hookrightarrow \text{Cyl}_S^* \) by \( \iota := \text{PS} \circ \iota_F \). Explicitly, for all \( \psi \in \text{Cyl}_S^* \) and \( \Phi \in \text{Cyl}_S^* \),
\[
(\iota \psi | \Phi) = (\psi | \text{PS} \Phi).
\] (5)

This is the embedding of central interest to this paper. Because \( \text{Im} \iota \subset \text{Im} \iota_F \), elements in the image of \( \iota \) are also homogeneous and isotropic in the sense of theorem 2 above. Furthermore, as we will see below, \( \iota \) satisfies direct analogues of all other above properties of \( \iota_F \) as well.

New operators: curved holonomies in standard LQC

The fact that \( \text{Cyl}_S \) can be identified as a subspace of \( \text{Cyl}_F \) offers a method to define operators in standard LQC corresponding to holonomies along curved paths, operators which heretofore were simply not defined.

The program initiated by Fleischhack [16] does not yet include the specification of an inner product on \( \text{Cyl}_F \). However, let us suppose an inner product is chosen. Let us furthermore
assume that the restriction of the inner product on \( \text{Cyl}_F \) to \( \text{Cyl}_S \) is the same as the usual Bohr inner product on \( \text{Cyl}_S \), and that \( \mathcal{V} \) and \( \text{Cyl}_S \) are mutually orthogonal in this inner product\(^2\). Given an operator \( \hat{O}_F \) on \( \mathcal{H}_F \), one can then define a corresponding operator \( \hat{O}_S \) on \( \mathcal{H}_S \) simply by matrix elements

\[
\langle \psi_S | \hat{O}_S | \phi_S \rangle = \langle \psi_S | \hat{O}_F | \phi_S \rangle.
\]

From this one deduces

\[
\hat{O}_S = P_S \circ \hat{O}_F.
\]  

(6)

(7)

Let \( F(A) \) be any cylindrical function, considered as a full theory phase space function depending only on \( A \), and let \( \hat{F}(A) \) denote the corresponding quantum operator. As noted earlier, the corresponding operator on \( \mathcal{H}_F \) is \( \hat{F}(A)_F := F(r(c)) \), so that, from equation (7), the corresponding operator on the standard LQC Hilbert space \( \mathcal{H}_S \) is given by \( \hat{F}(A)_S = \hat{F}(r(c))_S := P_S \circ \hat{F}(r(c)) \).

Let us apply this operator definition to the matrix elements of the \( SU(2) \) holonomy along an arbitrary piecewise analytic path \( \ell \), \( F(A)_B := A(\ell)^B \). The corresponding operator in standard LQC is then

\[
\left( \hat{\mathcal{A}}(\ell)_S \right)_B := P_S \circ (r(c))(\ell)^B.
\]

(8)

To write this operator more explicitly and prove that it continues to behave as an \( SU(2) \) holonomy, we use the following key lemma which will again be important later.

**Lemma 3.** \( P_S : \text{Cyl}_F \to \text{Cyl}_S \) is a multiplicative homomorphism.

**Proof.** Let \( f, g \in \text{Cyl}_F \) be given. Then,

\[
f \cdot g = (P_S f + P_V f)(P_S g + P_V g) \\
= (P_S f)(P_S g) + (P_V f)(P_V g) + (P_S f)(P_V g) + (P_V f)(P_S g).
\]

As the first term is almost periodic and the last three terms vanish at infinity, it follows

\[
P_S(f \cdot g) = (P_S f)(P_S g).
\]

\(\square\)

**Remark.** The key in the proof above is that any element of \( \mathcal{V} \) times any element of \( \text{Cyl}_F \) is in \( \mathcal{V} \), that is, \( \mathcal{V} \) is an ideal.

With this lemma, the explicit action of the operator (8) on an element \( \psi \in \text{Cyl}_S \) is

\[
\left( \hat{\mathcal{A}}(\ell)_S \right)_B \psi(c) := P_S((r(c))(\ell)^B \psi(c)) = (P_S(r(c))(\ell)^B)\psi(c),
\]

(9)

where \( P_S \psi = \psi \) has been used. Using again lemma 3 and the fact that \( P_S f = P_S \hat{f} \), it is straightforward to prove that

\[
\left( \hat{\mathcal{A}}(\ell)_S \right)_B^A = \delta^A_C \mathbb{1},
\]

\[
e^A_B e^{CDE}(\hat{\mathcal{A}}(\ell)_S)^C = 2 \mathbb{1},
\]

and

\[
\left( \hat{\mathcal{A}}(\ell \circ e^C) \right)^A_\ell = \delta^A_B \hat{\mathcal{A}}(\ell)_S^B.
\]

\(^2\) The usual way [35, 32, 18] of constructing an inner product would be to use the Gel’fand transform [36] to identify \( \text{Cyl}_F \) with continuous functions on its Gel’fand spectrum [16] \( \mathbb{R}_{\text{Bohr}} \cup \mathbb{R} \setminus \{0\} \), and then use an \( L^2 \) inner product determined by a choice of measure on this spectrum. If one chooses the measure on the Gel’fand spectrum by combining the Haar measure on \( \mathbb{R}_{\text{Bohr}} \) and any other measure on \( \mathbb{R} \setminus \{0\} \), then the resulting inner product will satisfy the above assumptions. An interesting question is whether the condition of invariance under residual diffeomorphisms used in [37] would lead to such a measure in this case.
for all piecewise analytic ℓ and ℓ′, and where, in the first equation, † denotes Hermitian
conjugation both as an operator and as a 2 × 2 matrix, that is,

\[(\hat{A}(\ell)_S)_{\beta} := [(\hat{A}(\ell)_S)^{\dagger}]^{\dagger}B_{\alpha}.\]

It follows that \((\hat{A}(\ell)_S)_{\beta}\) indeed has eigenvalues only in \(SU(2)\), and obeys the composition
law for parallel transports, as one would hope.

For completeness, we also give the explicit expression for \((\hat{A}(\ell)_S)_{\beta}\) by writing out
explicitly the multiplicative factor in (9). Let a piecewise analytic path \(\ell(t)\), where \(t\) is an arc
length parameter with respect to the background metric \(\hat{q}\), be given. From equations (2)–(7)
in [16] and proposition 5.13 in [16], one deduces that the multiplicative factor is given by

\[P_S(r(c))(\ell) = A_+ e^{i\frac{\pi}{2}} - A_- e^{-i\frac{\pi}{2}},\]

where \(\hat{V}^{1/3}\mu\) is the geometric length of \(\ell\) with respect to the background metric \(\hat{q}_{ab}\) and \(A_\pm\)
are two 2 × 2 matrices, independent of \(c\), given by

\[A_\pm = \frac{m(\hat{V}^{1/3}\mu)}{2} e^{\pm iR} \begin{pmatrix} m(0) \frac{1}{2} (n(0) \pm 1) & m(0)^{\frac{1}{2}} (n(0) \pm 1) \\ -m(0)^{\frac{1}{2}} & m(0) \frac{1}{2} (n(0) \pm 1) \end{pmatrix},\]

where

\[\hat{V}^{1/3}\hat{\lambda}_a e^{\hat{q}} := -\frac{i}{2} \begin{pmatrix} n & m \\ \bar{m} & -n \end{pmatrix},\]

\[R := n(\hat{V}^{1/3}\mu) - n(0) - \int_0^{\hat{V}^{1/3}\mu} \frac{\bar{m}}{m} n dt,\]

and the square root \(m(t)^{1/2}\) is chosen such that it is continuous in \(t\). Note in particular that the
multiplicative factor (10) is not only almost periodic, but sinusoidal with period \(4\pi/\mu\), just as
in the piecewise straight case.

**Intertwining**

With the above definition of \(\hat{F}(A)_S\), we show that the embedding \(\iota : \text{Cyl}_S^* \hookrightarrow \text{Cyl}^*\) in (5)
**intertwines** the operators \(\hat{F}(A)_S^*\) and \(\hat{F}(A)_S\) in full LQG and standard LQC, giving yet further
support for both the definition (6) of the operators on \(\mathcal{H}_S\) and the embedding \(\iota\).

**Theorem 4.** \(\iota\) intertwines \(\hat{F}(A)_S^*\) and \(\hat{F}(A)_S^*\).

**Proof.** For all \(\alpha \in \text{Cyl}_S^*\) and \(\Phi \in \text{Cyl}_S\),

\[\left(\hat{F}(A)^*\right) (\alpha | \Phi) := (\alpha | \hat{F}(A)\Phi) := (\alpha | P_S r^* (F(A)\Phi)) = (\alpha | P_S (F(r(c))\Phi(r(c)))) = (\alpha | (P_S (F(r(c))) r^* (\Phi(r(c))))) = (\alpha | \hat{F}(A) \circ P_S r^* (\Phi)) = (\hat{F}(A)^* \alpha | P_S r^* \Phi) = (\alpha | \hat{F}(A) \circ r \alpha | \Phi),\]

where lemma 3 was used in the second line.

This extends the intertwining result of [11] fully to the piecewise analytic category without
modifying the standard LQC Hilbert space \(\mathcal{H}_S\) in anyway.

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3 If we identify \(\hat{q}_{ab}\) in this paper with the background Euclidean metric in [16, 38], then \(A_\pm\) in [16, 38] becomes
identified with \(2^{\ell(0)/2}\lambda_a\) here, leading to the extra factor of 2 in the exponentials (10) and the extra factor of \(\hat{V}^{1/3}\)
in the interpretation of \(\mu\) as compared with [16].
Explicit expression in a common case

In the case where the argument of the embedding $\iota$ is the dual $\theta^\ast|/Phi_1$ of an element $\theta \in \text{Cyl}_S$, a more direct expression is possible. For all $f \in \text{Cyl}_F$, define the mean

$$M(f) := \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(c) \, dc.$$ 

When $f$ is almost periodic, this is the mean used by Bohr in [39]. The fact that it is well defined also for $f \in \text{Cyl}_F$ follows from the decomposition

$$f = P_S f + P_V f \quad (11)$$

ensured by theorem 3: because $P_V f$ vanishes at infinity, $M(f) = M(P_S f) + M(P_V f) = M(P_S f)$. For $\theta, \psi \in \text{Cyl}_S$, $M$ is related to the inner product in standard LQC via

$$\langle \theta, \psi \rangle = M(\theta \psi).$$

Expression (5) then takes the form

$$\langle \iota \theta^\ast|/Phi_1 \rangle = \langle \theta^\ast|/Phi_1 \rangle = M(\theta^\ast|/Phi_1) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \theta(c) \Phi_1(r(c)) \, dc.$$ 

In this expression, note that it is theorem 3 which, by ensuring the decomposition (11), ensures the convergence of the limit.

5. b-embeddings

The embedding discussed thus far is of the type named ‘c’ in the work [10, 11], because the states in its image satisfy homogeneity and isotropy only of the configuration field. To overcome this, and provide a better capacity to adapt to dynamics, the ‘b’ embeddings were introduced [10, 11], where ‘b’ refers to the “balanced” way in which homogeneity and isotropy are imposed on both configuration and momenta.

The basic idea of the ‘b’-embeddings is to use coherent states to define an embedding of the reduced theory into the full theory. In the work [11], complexifier coherent states [20] were used for this purpose. These are directly related to a choice of complex coordinates on phase space. The resulting embeddings, in contrast to the ‘c’ embedding, intertwine an algebra of operators whose classical analogues separate points on phase space, and the states in the image of each ‘b’ embedding satisfy an operator equation whose classical analogue implies homogeneity and isotropy of both configuration and momentum fields. However, the ‘b’ embeddings defined in [11] were limited to the piecewise linear category. In the present section, we apply ideas in [11] to the foregoing work of this paper to obtain a ‘b’ embedding of LQC into piecewise analytic LQG. Because most of the derivations are formally the same as elsewhere, we skip almost all details and primarily state definitions and results.

Definition

The complex coordinates used in complexifier coherent states are generated by a choice of positive function on phase space, called a complexifier [20]. Let $C : \Gamma \to \mathbb{R}^+$ and $C_S : \Gamma_S \to \mathbb{R}^+$ denote complexifiers on the full and reduced phase spaces of general relativity, and let $\hat{C}$ and $\hat{C}_S$ denote their quantizations on $\mathcal{H}$ and $\mathcal{H}_S$, respectively. The corresponding classical complex coordinates $\mathfrak{z}$ and $\mathfrak{z}$ are then

$$\mathfrak{z}_r(x) := (\varphi_{C}(t)^\ast A_r(x))_{t \to \iota}, \quad \mathfrak{z} := (\varphi_{C_S}(t)^\ast c)_{t \to \iota}, \quad (12)$$
where $\psi_C(t)$ and $\psi_C(t)$ respectively denote the one-parameter Hamiltonian flows on $\Gamma$ and $\Gamma_S$ generated by the phase space functions $C$ and $C_S$, and $t \to i$ denotes complex analytic continuation. We make the same assumptions about $C$ and $C_S$ as were made in [11], namely (1) they are pure momentum, $C = C(\hat{E}_r)$, $C_S = C_S[p]$, (2) $\frac{\partial}{\partial t_i}$ and $\frac{\partial}{\partial p}$ vanish only at $\hat{E}_r = 0$ and $p = 0$, and (3) if $s : \mathbb{C} \to \mathbb{A}_C$ denotes the inclusion map $\Gamma_S \hookrightarrow \Gamma$ in the coordinates $z$ and $\tilde{z}$, then $s$ is holomorphic. The necessary and sufficient conditions on $C$ and $C_S$ for $s$ to be holomorphic were derived in lemma 1 of the work [11]. The above three conditions furthermore imply that $s$ is the analytic continuation of the map $r$ introduced in equation (2).

We begin by stating the definition of the ‘b’ embedding in terms of the ‘c’ embedding. Motivated by equation (68) of [11], define $t_b : \text{Cyl}_p^{\epsilon} \to \text{Cyl}_p$ by

$$ t_b := e^{-C} \circ i \circ e^\hat{C}. \quad (13) $$

The injectivity of $t_b$ follows from the injectivity of $i$. $t_b$ furthermore maps complexifier coherent states to complexifier coherent states—see the appendix for an exposition of this fact. The other properties of $t_b$ are proven below, in turn.

**Intertwining of a set of operators whose classical analogues separate points**

Consider a function $F(\tilde{z})$ depending holomorphically on a finite number of parallel transports of $\tilde{z}$ along piecewise analytic paths—i.e. a holomorphic, piecewise analytic cylindrical function. In contrast to the cylindrical functions of $A$, the holomorphic cylindrical functions of $\tilde{Z}$ separate points on $\Gamma$. From the fact that $F(\tilde{z})$ is holomorphic, together with (12), one has

$$ F(\tilde{z}) = (\psi_C(t)\psi(A))_{t \to i} $$

from which follows the quantization [20]

$$ F(\tilde{z}) = e^\hat{C} F(A) e^{-\hat{C}}. $$

Because $s$ is holomorphic, $F(s(z))$ is likewise holomorphic, so that $F(s(z))$ can be quantized as an operator on $\mathcal{H}_S$ in a similar way, yielding

$$ F(s(z))_S := e^{\hat{s}_S} F(r(c)) e^{-\hat{s}_S} := e^{\hat{s}_S} \circ P_S \circ F(r(c)) \circ e^{-\hat{s}_S}. \quad (14) $$

**Theorem 5.** $t_b$ intertwines the dual action of any holomorphic, piecewise analytic cylindrical function $F(\tilde{z})$ of $\tilde{Z}$, and the dual action of the corresponding reduced theory operator $F(s(z))_S$:

$$ t_b \circ F(s(z))_S^* = F(\tilde{z})^* \circ t_b. $$

**Proof.** The proof follows from theorem 4, in the same way proposition 3 follows from proposition 2 in [11].

**Homogeneity and isotropy.** Because $\tilde{z}$ is a good coordinate on $\Gamma$, there exists a unique action $\triangleright$ of the Euclidean group $\mathcal{E}$ on complex connections such that

$$ g \triangleright (\tilde{z}[p]) = \tilde{z}[g \triangleright p] \quad (15) $$

for all $g \in \mathcal{E}$ and $p \in \Gamma$. If the chosen complexifier $C$ is invariant under $E$ (which will be the case if $C$ is diffeomorphism and $SU(2)$ gauge invariant), then $\tilde{z}[p]$ will transform covariantly under $E$, and the above action $\triangleright$ will be the same as $\rhd$—i.e. for each $g \in \mathcal{E}$, $g \triangleright$ will act with the same combination of diffeomorphisms and local $SU(2)$ rotations as in equation (1).
In terms of this action, the condition of homogeneity and isotropy on the complex connection $Z$ takes the form [10, 11]

$$(g \triangleright Z)(\ell)_A^B = Z(\ell)_A^B,$$

for all piecewise analytic $\ell$, all $g \in \mathcal{E}$, and all $A, B = 0, 1$. This condition is equivalent to the homogeneity of both the real connection $A^a_i$ and its conjugate momentum $\tilde{E}^a_i$. All states in the image of $\iota_b$ satisfy the operator version of this condition.

**Theorem 6.** Every element $\Psi$ in the image of $\iota_b$ satisfies, for all piecewise analytic $\ell$ and $g \in \mathcal{E}$,

$$(g \triangleright Z)(\ell)_A^B \Psi = \tilde{Z}(\ell)_A^B \Psi.$$

**Proof.** For all $z$, $s(z)$ is by construction invariant under $\mathcal{E}$. From this and equation (14), it follows that $\tilde{F}'(s(z)_S) = \tilde{F}(s(z)_S)$. The proof then follows from theorem 5 in the same way that theorem 2 followed from theorem 1 in section 4. \qed

6. Discussion

We have shown that it is not necessary at any point to restrict to the piecewise linear category when embedding standard LQC states into a homogeneous–isotropic sector of full LQG. We have shown this by exhibiting such an embedding $\iota$, not only of normalizable, but also of all distributional states of standard LQC, $\text{Cyl}_S^*$, into standard LQG—that is, LQG based on piecewise analytic graphs. This has been done without fixing a graph. The sense in which the image of the embedding consists in homogeneous isotropic states is defined via operator equations. Furthermore, this embedding has motivated a new definition of operators in LQC for parallel transports along curved paths, heretofore undefined in LQC, which may be of use in applications. These operators, together with the corresponding operators in the full theory, are intertwined by the embedding that has been introduced. All of these results have been proven for both a ‘c’ version and ‘b’—or ‘holomorphic’—version of the embedding, in the terminology of [11].

The properties of the embeddings introduced in this paper contrast with those of the embedding suggested in [17], whose image consists in cylindrical functions based on a fixed graph with few edges and which are far from homogeneous and isotropic. Additionally, the embedding in [17] has no property similar to the intertwining properties proven in theorems 4 and 6 of this paper. However, states in the image of the embedding in [17] do consist in normalizable states, giving it the advantage of being usable in more contexts.

We have derived the embedding $\iota$ by starting from the embedding $\iota_F$ of an extended space of states $\text{Cyl}_F^*$ into standard LQG. This extended space of states has been proposed by Fleischhack [16] precisely because, by construction, it admits such an embedding. Using a result of Fleischhack’s [16], we have shown that $\text{Cyl}_F^*$ is naturally isomorphic to a subspace of $\text{Cyl}_L^*$, so that by simply restricting $\iota_F$ to $\text{Cyl}_L^*$, one obtains an embedding of $\text{Cyl}_L^*$ into $\text{Cyl}_L^*$, which is precisely the $\iota$ we have introduced. The injectivity of $\iota$ and the fact that states in its image are homogeneous and isotropic descend trivially from the corresponding properties of $\iota_F$, which we have also proven here. The fact that $\iota$ intertwines curved holonomies, on the other hand, is quite non-trivial and was by no means guaranteed.

One might take the viewpoint that $\text{Cyl}_F$ is the more ‘fundamental’ choice for the configuration algebra. If one takes such a viewpoint, there are still heuristic arguments for
why the restriction to $Cyl_S \subset Cyl_F$ is appropriate and consistent. The difference between $Cyl_S$ and $Cyl_F$ lies only in whether or not one includes holonomies along curved paths. However, as was shown in [12], in the full theory, the exclusion of curved paths, once one solves the diffeomorphism constraint, in fact does not alter the final theory\(^4\). This suggests that, also in quantum cosmology, a restriction to piecewise straight paths should be sufficient to capture all of the physics. Such a restriction is furthermore consistent with the dynamics if quantized using the same strategy as that in the well-established ‘improved dynamics’ quantization of [3]. It is easy to see that the resulting Hamiltonian constraint operator $\hat{H}$ will preserve $Cyl_S$ and hence its dual $\hat{H}^*$ will preserve $Cyl_S^* \subset Cyl_F^*$, so that one can consistently restrict to $Cyl_S^*$.

We close with a note on the equivalence of the ‘embedding strategy’ employed in this paper with the ‘projection strategy’ for relating LQC and LQG advocated in [42, 43]. This paper has considered the problem of relating a given quantum theory, with state space $\mathcal{H}$ and operators $\hat{O}$, to some spatial symmetry reduction thereof, with state space $\mathcal{H}_S$ and corresponding operators $\hat{O}_S$. In doing this, we have followed the general strategy of specifying an embedding $\iota : \mathcal{H}_S \hookrightarrow \mathcal{H}$, such that the states in the image of the embedding satisfy operator equations expressing the relevant symmetry, and hence membership in the corresponding ‘symmetric sector’ of the full theory. We wish to note that this strategy is fully equivalent to the strategy suggested in the two papers [42, 43], in which one specifies a projection from the larger space of states $\mathcal{H}$ to the smaller $\mathcal{H}_S$, the interpretation being that of ‘integrating out the non-symmetric degrees of freedom’. This equivalence results from the fact that the adjoint of every onto projection $P : \mathcal{H} \twoheadrightarrow \mathcal{H}_S$ is injective, and hence an embedding $\iota := P^\dagger : \mathcal{H}_S \hookrightarrow \mathcal{H}$, and vice versa. Furthermore, a pair of operators $\hat{O} \text{and} \hat{O}_S$ is intertwined by $P$ if and only if their adjoints are intertwined by the corresponding embedding $\iota = P^\dagger$. Indeed, one can see from (5) that the embedding we have proposed is the adjoint of the projection $P = P_S \circ r^\ast$.

We have taken the embedding viewpoint because it permits a clear sense in which homogeneity and isotropy play a role. However, it should be emphasized that the work [42] has achieved something important. At least for the case of reducing the quantum Bianchi I model to isotropic LQC, the authors have constructed a dynamical projection, which intertwines the Hamiltonian constraints of the two models, whence the corresponding embedding also intertwines the Hamiltonian constraints. LQC thus passes a first test of its ability to represent the dynamics of a less symmetric quantum model. However, the role of homogeneity and isotropy in the definition of the dynamical projector does not have a clear generalization to the full theory. By contrast, the role of homogeneity and isotropy in the full theory embedding $\iota$ defined here is clear. A hope is that if one understands better the relation between the dynamical projector of [42] and the strategy of embedding carried out in this paper, this might lead to a way to extend the success of [42] to the full theory.

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\(^4\) As long as one uses a proposal for the extended diffeomorphism group which has already been advocated on other grounds [40, 41].
Appendix. Relation of the ‘b’ embedding to coherent states

In this appendix, we show a precise sense in which the embedding \( t_b \), defined in equation (13), maps coherent states to coherent states. The complexifiers \( \tilde{C} \) and \( \tilde{C}_b \) determine families of coherent states \( \psi^C_{\tilde{C}} \in \text{Cyl} \), \( \psi^C_{\tilde{C}_b} \in \text{Cyl}_b \) via

\[
\left( \psi^C_{\tilde{C}} | \Phi \right) := (e^{-\tilde{C}}\Phi(A'))_{A \to 3} \\
\left( \psi^C_{\tilde{C}_b} | \phi \right) := (e^{-\tilde{C}_b} \phi(c'))_{c \to z}
\]

for all \( \Phi \in \text{Cyl} \) and \( \phi \in \text{Cyl}_b \), where ‘\( \to \)’ denotes complex analytic continuation. These are quantum states in the full and reduced theory, respectively, which are ‘peaked’ at the classical phase space points labeled by the coordinates \( z \) and \( z' \) [20].

Given a graph \( \gamma \), let \( P_{\gamma} \) denote orthogonal projection, using the Ashtekar–Lewandowski inner product, from \( \text{Cyl} \) to the space \( \text{Cyl}_b \) of cylindrical functions which depend only on holonomies along edges in \( \gamma \). Define truncations of \( t_b \) and \( \psi^C_{\tilde{C}} \) to the graph \( \gamma \) by \( \gamma t_b := P_{\gamma} \circ t_b \) and \( \gamma \psi^C_{\tilde{C}} := P_{\gamma} \psi^C_{\tilde{C}} \)—this coincides with taking the ‘cut-off’ [20] or ‘shadow’ [44], on \( \gamma \), of the states in the image of \( \tilde{b} \) and \( \psi^C_{\tilde{C}} \).

**Theorem 7.** For \( \gamma \) piecewise linear,

\[
\gamma t_b \psi^C_{\tilde{C}_b} = \gamma \psi^C_{\tilde{C}_b}.
\]

**Proof.** For all \( \Phi \in \text{Cyl} \),

\[
\gamma t_b \psi^C_{\tilde{C}_b} | \Phi \rangle = (t_b \psi^C_{\tilde{C}_b} | P_{\gamma} \Phi \rangle = (\psi^C_{\tilde{C}_b} | e^{-\tilde{C}} P_{\gamma} e^{-\tilde{C}_b} P_{\gamma} \Phi \rangle.
\]

Because \( \tilde{C} \) is pure momentum and hence graph preserving, \( e^{-\tilde{C}} P_{\gamma} \Phi \in \text{Cyl}_b \) once more, so that \( r^* e^{-\tilde{C}_b} P_{\gamma} \Phi \in \text{Cyl}_b \), making the projector unnecessary, whence

\[
\gamma t_b \psi^C_{\tilde{C}_b} | \Phi \rangle = (\psi^C_{\tilde{C}_b} | e^{-\tilde{C}} P_{\gamma} e^{-\tilde{C}_b} P_{\gamma} \Phi \rangle = (\gamma r^* e^{-\tilde{C}} P_{\gamma} \Phi (c'))_{c \to z} = (e^{-\tilde{C}} P_{\gamma} \Phi (c'))_{c \to z} = (\psi^C_{\tilde{C}_b} | \Phi \rangle = (\gamma \psi^C_{\tilde{C}_b} | \Phi \rangle.
\]

\[\Box\]

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