Convergence to consensus for a Hegselmann-Krause-type model with distributed time delay

Alessandro Paolucci*

Abstract

In this paper we study a Hegselmann-Krause opinion formation model with distributed time delay and positive influence functions. Through a Lyapunov functional approach, we provide a consensus result under a smallness assumption on the initial delay. Furthermore, we analyze a transport equation, obtained as mean-field limit of the particle one. We prove global existence and uniqueness of the measure-valued solution for the delayed transport equation and its convergence to consensus under a smallness assumption on the delay, using a priori estimates which are uniform with respect to the number of agents.

Keywords and Phrases: Hegselmann-Krause model, opinion formation, delay, consensus

1 Introduction

In recent years, many researchers have focused their attention to multi-agent systems. One aspect of these models is the natural self-aggregation, which has been studied in different fields such as biology [1], robotics [12], sociology, economics [19], computer science, control theory [21, 22, 23], social sciences [26, 27] and many other areas. In these last decades a large number of mathematical models has been proposed to study the consensus behavior. First order models, such as the Hegselmann-Krause model [16], have been proposed to study opinion formation. We mention also [17], in which bounded confidence yields the so-called clustering phenomenon. Second order models, in particular Cucker-Smale model [11], have been studied by many authors [13, 14, 24], in order to describe, for example, flocking of birds, swarming of bacteria, or schooling of fishes.

In addition, it is reasonable to introduce a delay in the model as a reaction time or simply as a time to receive the information from outside, in order to let the dynamics more realistic. For first order models, we refer to [5, 8, 10], while for delayed Cucker-Smale-type models we mention [6, 7, 15, 25]. In particular, in very recent papers (see [9, 18, 23]), the authors analyzed modified Cucker-Smale models with distributed time delay, thanks to which agents are influenced by the other ones on a time interval $[t - \tau(t), t]$.

Furthermore, delayed and non-delayed kinetic and transport equations associated to the particle multi-agent systems have been studied in [2, 3, 4, 6, 8, 9].

In this paper, we are interested in the evolution of opinions among $N$ agents, with $N \in \mathbb{N}$. Let $x_i \in \mathbb{R}^d$ be the opinion of the $i$-th agent, for any $i = 1, \ldots, N$. Then, the dynamics is given

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*Dipartimento di Ingegneria e Scienze dell’Informazione e Matematica, Università di L’Aquila, Via Vetoio, Loc. Coppito, 67010 L’Aquila Italy (alessandro.paolucci2@graduate.univaq.it).
by the following Hegselmann-Krause-type model:

\[
\frac{dx_i(t)}{dt} = \frac{1}{Nh(t)} \sum_{j \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ij}(t; s)(x_j(s) - x_i(t)) ds, \quad t > 0
\]

\[
x_i(s) = x_{i,0}(s), \quad s \in [-\tau(0), 0],
\]

where \(\tau : [0, +\infty) \rightarrow (0, +\infty)\) is the time delay. It is a function in \(W^{1,\infty}([0, T])\), for any \(T > 0\) and we assume that \(\tau(t) \geq \tau_*\) for some \(\tau_* > 0\), and

\[
\tau'(t) \leq 0, \quad \forall \ t \geq 0.
\]

This implies that \(\tau(t) \leq \tau(0)\), for any \(t \geq 0\). We stress the fact that constant delays \(\tau(t) \equiv \bar{\tau} > 0\) are allowed.

Motivated by [11, 17, 20], we take the communication rates \(a_{ij}(t; s)\) either of the form

\[
a_{ij}(t; s) = \psi(|x_j(s) - x_i(t)|), \quad (1.3)
\]

for any \(i, j \in \{1, \ldots, N\}\), where \(\psi : [0, +\infty) \rightarrow (0, +\infty)\) is a non-increasing function, or

\[
a_{ij}(t; s) = \frac{N\psi(|x_j(s) - x_i(t)|)}{\sum_{k=1}^{N} \psi(|x_k(s) - x_i(t)|)}, \quad \forall \ t \geq 0.
\]

Without loss of generality, we can assume that \(\psi(0) = 1\). We notice that in both cases we have that

\[
\frac{1}{N} \sum_{j=1}^{N} a_{ij}(t; s) \leq 1, \quad \forall \ t \geq 0.
\]

Moreover, \(\alpha : [0, \tau(0)] \rightarrow [0, +\infty)\) is a weight function which satisfies

\[
A := \int_{0}^{\tau_*} \alpha(s) ds > 0.
\]

Furthermore, we define for any \(t \geq 0\)

\[
h(t) := \int_{0}^{\tau(t)} \alpha(s) ds.
\]

**Remark 1.1.** We notice that if \(\alpha(s) = \delta_{\tau(t)}(s)\), then system \((1.1)\) can be rewritten as

\[
\frac{dx_i(t)}{dt} = \frac{1}{N} \sum_{j \neq i} a_{ij}(t; t - \tau(t))(x_j(t - \tau(t)) - x_i(t)),
\]

\[
x_i(s) = x_{i,0}(s), \quad s \in [-\tau(0), 0],
\]

which is already analyzed in [8].

We define, now, the following quantity:

\[
d_X(t) := \max_{1 \leq i, j \leq N} |x_i(t) - x_j(t)|.
\]

**Definition 1.2.** We say that a solution \(\{x_i(t)\}_{i=1, \ldots, N}\) to \((1.1)\) converges to consensus if

\[
\lim_{t \rightarrow +\infty} d_X(t) = 0.
\]
We will prove the following consensus result.

**Theorem 1.3.** Let $\{x_i(t)\}_{i=1}^N$ be the solution to (1.1). Suppose that
\[
\left( e^{\tau(0)} - 1 \right) h(0) \lesssim \frac{A\psi(2R)^3}{2 + \psi(2R)^2}.
\] (1.7)
Then, there exist two positive constants $C, K$ such that
\[
d_X(t) \leq Ce^{-Kt}, \quad \forall \ t \geq 0.
\] (1.8)

**Remark 1.4.** Here, we stress the fact that the quantity
\[
\left( e^{\tau(0)} - 1 \right) \int_0^{\tau(0)} \alpha(s)ds
\] is increasing with respect to $\tau(0)$. Then, (1.7) represents a smallness assumption on $\tau(0)$. Moreover, the right-hand side of (1.7) is increasing with respect to $\psi(2R)$. Therefore, we observe that if $R$ is small enough and/or the decay of $\psi$ is not too fast, then the quantity
\[
\frac{\psi(2R)^3}{2 + \psi(2R)^2}
\] becomes large and consensus occurs for more values of $\tau(0)$.

The transport equation associated to (1.1) can be obtained as mean-field limit of the particle system (1.1) when $N \to +\infty$. Let $\mathcal{M}(\mathbb{R}^d)$ be the set of probability measures on the space $\mathbb{R}^d$. Then, the transport equation associated to (1.1) reads as
\[
\partial_t \mu_t + \text{div} \left( \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) F[\mu_s] ds \mu_t \right) = 0, \quad x \in \mathbb{R}^d, \quad t \geq 0 \tag{1.9}
\]
where $F$ is given by either
\[
F[\mu_s](x) = \int_{\mathbb{R}^d} \psi(|x-y|)(y-x)d\mu_s(y), \tag{1.10}
\]
or
\[
F[\mu_s](x) = \frac{\int_{\mathbb{R}^d} \psi(|x-y|)(y-x)d\mu_s(y)}{\int_{\mathbb{R}^d} \psi(|x-y|)d\mu_s(y)}, \tag{1.11}
\]
according to the choice of (1.3) and (1.4). Furthermore, we take $g_s \in C([-\tau(0),0];\mathcal{M}(\mathbb{R}^d))$.

**Definition 1.5.** Let $T > 0$. We say that $\mu_t \in C([0,T];\mathcal{M}(\mathbb{R}^d))$ is a weak solution to (1.9) on the time interval $[0,T]$ if for all $\varphi \in C_c^\infty(\mathbb{R}^d \times [0,T))$ we have the following result:
\[
\int_0^T \int_{\mathbb{R}^d} \left( \partial_t \varphi + \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) F[\mu_s](x)ds \cdot \nabla_x \varphi \right) d\mu_t(x) dt + \int_{\mathbb{R}^d} \varphi(x,0)dg_0(x) = 0, \tag{1.12}
\]
where $F[\mu_s]$ is defined as in (1.10) or (1.11).

We will prove the following theorem.
Theorem 1.6. Let $\mu_t \in C([0,T];P_1(\mathbb{R}^d))$ be a weak solution to (1.9), with compactly supported initial measure $g_s \in C([\tau(0),0];P_1(\mathbb{R}^d))$ and let $F$ as in (1.10) or (1.11). Suppose that
\[
(e^{\tau(0)} - 1)h(0) \leq \frac{4\psi(2R)^3}{2 + \psi(2R)^2}.
\]
Then, there exists a constant $C > 0$ independent of $t$ such that
\[
d_X(\mu_t) \leq \left( \max_{s \in [\tau(0),0]} d_X(g_s) \right) e^{-Ct},
\]
for all $t \geq 0$, where
\[
d_X(\mu_t) := \text{diam sup} \mu_t.
\]

The paper is organized as follows. In Section 2 we study the consensus behavior of solution to (1.1), after assuming an upper-bound on the initial delay $\tau(0)$, namely we will prove Theorem 1.3. In Section 3 we focus our attention on system (1.9) and we study the existence and uniqueness of the solution and its convergence to consensus.

2 Consensus results

We notice that $d_X$ may be not differentiable at some $t \geq 0$. Then, we will use a suitable generalized derivative. We define the upper Dini derivative of a continuous function $F$ as follows:
\[
D^+ F(t) := \limsup_{h \to 0^+} \frac{F(t + h) - F(t)}{h}.
\]
Before studying the convergence to consensus of the solution to (1.1), we state the following lemma.

Lemma 2.1. Let $\{x_i(t)\}_{i=1}^N$ be a solution to (1.1). Suppose that the initial functions $x_{i,0}(s)$ are continuous on the time interval $[-\tau(0),0]$ for all $i = 1, \ldots, N$. Set
\[
R := \max_{s \in [-\tau(0),0]} \max_{1 \leq i \leq N} |x_i(s)|.
\]
Then,
\[
\max_{1 \leq i \leq N} |x_i(t)| \leq R
\]
for all $t \geq 0$.

Proof. Let $\epsilon > 0$ and define $R_\epsilon := R + \epsilon$. Set
\[
S^\epsilon = \left\{ t > 0 : \max_{1 \leq i \leq N} |x_i(s)| < R_\epsilon, \quad \forall \quad s \in [0,t] \right\}.
\]
By continuity, $S^\epsilon \neq \emptyset$. Denote $T^\epsilon := \sup S^\epsilon$ and assume by contradiction that $T^\epsilon < +\infty$. Then,
\[
\lim_{t \to T^\epsilon} \max_{1 \leq i \leq N} |x_i(t)| = R^\epsilon.
\]
On the other hand, we have that for any $t \leq T^\epsilon$,
\[
\frac{1}{2} D^+ |x_i(t)|^2 \leq \left\langle x_i(t), \frac{dx_i(t)}{dt} \right\rangle \\
= \left\langle x_i(t), \frac{1}{Nh(t)} \sum_{j \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ij}(t; s)(x_j(s) - x_i(t)) ds \right\rangle \\
= \frac{1}{Nh(t)} \sum_{j \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ij}(t; s)(x_j(t), x_j(s) - x_i(t)) ds \\
= \frac{1}{Nh(t)} \sum_{j \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ij}(t; s) (\langle x_i(t), x_j(s) \rangle - |x_i(t)|^2) ds \\
\leq \frac{1}{Nh(t)} \sum_{j \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s) a_{ij}(t; s)|x_i(t)| (|x_j(s)| - |x_i(t)|) ds.
\]
Using (1.5) and the fact that $t \leq T^\epsilon$ yield
\[
\frac{1}{2} D^+ |x_i(t)|^2 \leq \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) ds |x_i(t)| (R^\epsilon - |x_i(t)|) = |x_i(t)| (R^\epsilon - |x_i(t)|).
\]
Hence, we have that
\[
D^+ |x_i(t)| \leq R^\epsilon - |x_i(t)|.
\]
By Gronwall inequality, we obtain
\[
|x_i(t)| \leq e^{-t} (|x_i(0)| - R^\epsilon) + R^\epsilon < R^\epsilon.
\]
Therefore,
\[
\lim_{t \to T^\epsilon} \max_{1 \leq i \leq N} |x_i(t)| < R^\epsilon,
\]
which is in contradiction with (2.16). Moreover, since $\epsilon$ is arbitrary, we obtain (2.15). ■

**Remark 2.2.** Thanks to the previous lemma, we can find a control on $a_{ij}(t; s)$ from below. Indeed, for any $i, j \in \{1, \ldots, N\}$, for any $t \geq 0$ and $s \in [t-\tau(t), t]$, we have that
\[
|x_j(s) - x_i(t)| \leq |x_j(s)| + |x_i(t)| \leq 2R.
\]
Hence, from (1.3) and (1.4), we can deduce that
\[
a_{ij}(t; s) \geq \psi(2R), \quad \forall \ t \geq 0.
\] (2.17)

**Lemma 2.3.** Let $\{x_i(t)\}_{i=1}^N$ be the solution to (1.1). Moreover, define
\[
\gamma(t) := \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) \int_{s}^{t} \max_{1 \leq k \leq N} \left| \frac{dx_k(z)}{dz} \right| dz ds, \quad \forall \ t \geq 0.
\] (2.18)

Then,
\[
D^+ dx_X(t) \leq \frac{2}{\psi(2R)} \gamma(t) - \psi(2R) dx_X(t), \quad \forall \ t \geq 0.
\] (2.19)
Proof. Due to continuity of \( x_i(t) \), for any \( i \in \{1, \ldots, N\} \), there exists a sequence of times \( \{t_k\}_{k \in \mathbb{N}} \) such that 
\[
\bigcup_{k \in \mathbb{N}} [t_k, t_{k+1}) = [0, +\infty),
\]
and for each \( k \in \mathbb{N} \) and for any \( t \in (t_k, t_{k+1}) \) there exist \( i, j \in \{1, \ldots, N\} \) such that
\[
d_X(t) = |x_i(t) - x_j(t)|.
\]
Hence, we have that
\[
\frac{1}{2} D^+ d_X^2(t) \leq \left\langle x_i(t) - x_j(t), \frac{dx_i(t)}{dt} - \frac{dx_j(t)}{dt} \right\rangle = \frac{1}{Nh(t)} \left\langle x_i(t) - x_j(t), \sum_{k \neq i} \int_{t-\tau(t)}^t \alpha(t-s) a_{ik}(t; s)(x_k(s) - x_i(t))ds \right\rangle 
\]
(2.20)
\[
= \frac{1}{Nh(t)} \left\langle x_i(t) - x_j(t), \sum_{k \neq i} \int_{t-\tau(t)}^t \alpha(t-s) a_{ik}(t; s)(x_k(s) - x_j(t))ds \right\rangle
\]
(2.21)
\[=: I_1 + I_2.\]
Now, \( I_1 \) and \( I_2 \) can be rewritten in the following way:
\[
I_1 = \frac{1}{Nh(t)} \sum_{k \neq i} \int_{t-\tau(t)}^t \alpha(t-s) a_{ik}(t; s) \langle x_i(t) - x_j(t), x_k(s) - x_k(t) \rangle ds
\]
(2.22)
and
\[
I_2 = -\frac{1}{Nh(t)} \sum_{k \neq j} \int_{t-\tau(t)}^t \alpha(t-s) a_{jk}(t; s) \langle x_i(t) - x_j(t), x_k(s) - x_k(t) \rangle ds.
\]
We observe (as in [3]) that for any \( t \geq 0 \),
\[
\langle x_i(t) - x_j(t), x_k(t) - x_i(t) \rangle \leq 0, \quad \forall k \in \{1, \ldots, N\}.
\]
Moreover, we notice that for any \( i, j \in \{1, \ldots, N\} \)
\[
a_{ij}(t; s) \leq \frac{1}{\psi(2R)}
\]
in both cases [1.3] and [1.4]. Indeed, if \( a_{ij} \) are as in [1.4], for any \( i, j = 1, \ldots, N \), then we obtain (2.22), using (2.17) and the fact that \( \psi \) is a non-increasing function with \( \psi(0) = 1 \). Moreover, if we take \( a_{ij} \) as in [1.3], then (2.22) immediately follows, using the fact that \( a_{ij}(t; s) \leq 1 \), for any \( i, j = 1, \ldots, N \), and \( \psi(2R) \leq 1 \). Therefore, using (2.17) and (2.22) in (2.21) yield
\[
I_1 \leq \frac{1}{Nh(t)} \frac{d_X(t)}{\psi(2R)} \sum_{k \neq i} \int_{t-\tau(t)}^t \alpha(t-s)|x_k(s) - x_k(t)|ds
\]
(2.23)
\[+ \frac{\psi(2R)}{N} \sum_{k=1}^N \langle x_i(t) - x_j(t), x_k(t) - x_i(t) \rangle.
\]
As before, we observe that for any \( t \geq 0 \)
\[-\langle x_i(t) - x_j(t), x_k(t) - x_j(t) \rangle \leq 0, \quad \forall k \in \{1, \ldots, N\}.\]

Hence, using again (2.17) and (2.22), we can obtain a similar estimate for \( I_2 \), namely
\[ I_2 \leq \frac{1}{Nh(t)} \frac{dX(t)}{\psi(2R)} \sum_{k \neq j} \int_{t-\tau(t)}^{t} \alpha(t-s)|x_k(s) - x_k(t)|ds \]
\[ + \frac{\psi(2R)}{N} \sum_{k=1}^{N} \langle x_i(t) - x_j(t), x_j(t) - x_k(t) \rangle. \quad (2.24) \]

Using (2.23) and (2.24) in (2.20), we have that
\[ \frac{1}{2} D^+ dX(t)^2 \leq \frac{2}{Nh(t)} \frac{dX(t)}{\psi(2R)} \sum_{k=1}^{N} \int_{t-\tau(t)}^{t} \alpha(t-s)|x_k(s) - x_k(t)|ds - \psi(2R)dX(t)^2. \quad (2.25) \]

Moreover, we notice that, for \( s < t \),
\[ \sum_{k=1}^{N} |x_k(s) - x_k(t)| \leq \sum_{k=1}^{N} \int_{s}^{t} \left| \frac{dx_k(z)}{dz} \right| dz \leq N \int_{s}^{t} \max_{1 \leq k \leq N} \left| \frac{dx_k(z)}{dz} \right| dz. \]

Substituting this estimate in (2.25), we obtain
\[ \frac{1}{2} D^+ dX(t)^2 \leq \frac{2dX(t)}{\psi(2R)} \gamma(t) - \psi(2R)dX(t)^2, \]
which yields (2.19). \( \blacksquare \)

**Lemma 2.4.** Let \( \{x_i(t)\}_{i=1}^{N} \) be the solution to (1.1). Then, for any \( t \geq 0 \)
\[ \max_{1 \leq i \leq N} \left| \frac{dx_i(t)}{dt} \right| \leq \frac{1}{\psi(2R)} \gamma(t) + \frac{1}{\psi(2R)} dX(t). \quad (2.26) \]

**Proof.** We have that for any \( i \in \{1, \ldots, N\} \),
\[ \left| \frac{dx_i(t)}{dt} \right| \leq \frac{1}{Nh(t)} \sum_{k \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s)a_{ik}(t; s)|x_k(s) - x_k(t)|ds \]
\[ + \frac{1}{Nh(t)} \sum_{k \neq i} \int_{t-\tau(t)}^{t} \alpha(t-s)a_{ik}(t; s)|x_k(t) - x_i(t)|ds. \]

Using (2.22) yields
\[ \left| \frac{dx_i(t)}{dt} \right| \leq \frac{1}{\psi(2R)} \gamma(t) + \frac{1}{\psi(2R)} dX(t). \]

Taking the maximum, we obtain (2.26). \( \blacksquare \)

Now, we are ready to prove Theorem 1.3.
Proof of Theorem 1.3. Define the following Lyapunov functional:

\[
\mathcal{L}(t) := d_X(t) + \beta \int_0^{\tau(t)} \int_s^t e^{-(t-s)} \int_\sigma^t \max_{1 \leq k \leq N} \left| \frac{dx_k(\rho)}{d\rho} \right| d\rho d\sigma dt,
\]

with \(\beta > 0\). Then,

\[
D^+ \mathcal{L}(t) = D^+ d_X(t) + \beta \tau'(t) \alpha(t) \left( \int_{t-\tau(t)}^t e^{-(t-s)} \int_\sigma^t \max_{1 \leq k \leq N} \left| \frac{dx_k(\rho)}{d\rho} \right| d\rho d\sigma dt \right)
\]

\[
- \beta \int_0^{\tau(t)} \alpha(s) e^{-s} \int_{t-s}^t \max_{1 \leq k \leq N} \left| \frac{dx_k(\rho)}{d\rho} \right| d\rho ds
\]

\[
- \beta \int_0^{\tau(t)} \alpha(s) \int_{t-s}^t e^{-(t-s)} \int_\sigma^t \max_{1 \leq k \leq N} \left| \frac{dx_k(\rho)}{d\rho} \right| d\rho d\sigma ds
\]

\[
+ \beta \max_{1 \leq k \leq N} \left| \frac{dx_k(t)}{dt} \right| \int_0^{\tau(t)} \alpha(s) \int_{t-s}^t e^{-(t-s)} d\sigma ds.
\]

Using \(A \leq h(t) \leq h(0)\) and \(\tau'(t) \leq 0\), we deduce

\[
D^+ \mathcal{L}(t) \leq D^+ d_X(t) - \beta e^{-\tau(0)} A \gamma(t) - \beta \int_0^{\tau(t)} \alpha(s) \int_{t-s}^t e^{-(t-s)} \int_\sigma^t \max_{1 \leq k \leq N} \left| \frac{dx_k(\rho)}{d\rho} \right| d\rho d\sigma ds
\]

\[
+ \beta h(0)(1 - e^{-\tau(0)}) \max_{1 \leq k \leq N} \left| \frac{dx_k(t)}{dt} \right|.
\]

Now, since \((2.19)\) and \((2.26)\) hold, we have that

\[
D^+ \mathcal{L}(t) \leq \left( \frac{2}{\psi(2R)} - \beta e^{-\tau(0)} A + \beta h(0)(1 - e^{-\tau(0)}) \frac{1}{\psi(2R)} \right) \gamma(t)
\]

\[
+ \left( -\psi(2R) + \beta h(0) \frac{1 - e^{-\tau(0)}}{\psi(2R)} \right) d_X(t) - \beta \int_0^{\tau(t)} \alpha(s) \int_{t-s}^t e^{-(t-s)} \int_\sigma^t \max_{1 \leq k \leq N} \left| \frac{dx_k(\rho)}{d\rho} \right| d\rho d\sigma ds.
\]

We want to show that for \(\tau(0)\) sufficiently small we obtain the existence of \(K > 0\) such that

\[
D^+ \mathcal{L}(t) \leq -K \mathcal{L}(t), \quad \forall \ t \geq 0.
\]

This is true if the following two conditions hold:

\[
\frac{2}{\psi(2R)} - \beta e^{-\tau(0)} A + \beta h(0)(1 - e^{-\tau(0)}) \frac{1}{\psi(2R)} \leq 0,
\]

\[
-\psi(2R) + \beta h(0) \frac{1 - e^{-\tau(0)}}{\psi(2R)} < 0.
\]

The inequality \((2.29)\) is satisfied for

\[
\beta < \frac{\psi(2R)^2}{h(0)(1 - e^{-\tau(0)})}.
\]

Now, in order to have \((2.28)\), we need

\[
h(0) \left( e^{\tau(0)} - 1 \right) < A \psi(2R).
\]
Hence, (2.28) is satisfied if
\[ \beta \geq \frac{2}{e^{-\tau(0)}A\psi(2R) - h(0)(1 - e^{-\tau(0)})}. \] (2.31)

Then, in order to have the existence of the parameter \( \beta > 0 \) such that (2.30) and (2.31) hold, we need
\[ \frac{2}{e^{-\tau(0)}A\psi(2R) - h(0)(1 - e^{-\tau(0)})} < \frac{\psi(2R)^2}{h(0)(1 - e^{-\tau(0)})}, \]
which is true for any \( \tau(0) \) satisfying (1.7). Choosing
\[ K = \min \left\{ \beta, \psi(2R) - \beta h(0) \frac{1 - e^{-\tau(0)}}{\psi(2R)} \right\}, \]
we obtain (2.27). We notice that since \( \beta \) satisfies (2.30), then \( K > 0 \). This implies immediately (1.8). Hence, the theorem is proved. \( \blacksquare \)

3 Consensus of solution to (1.9)

In this section we want to analyse the transport equation (1.9) associated to (1.1), obtained as mean-field limit of the particle system when \( N \to +\infty \). To do so, we consider \( \psi \) Lipschitz continuous and we denote by \( L \) its Lipschitz constant.

Before proving the existence and uniqueness of solutions to (1.9), we first recall some tools on probability spaces and measures.

**Definition 3.1.** Let \( \mu, \nu \in \mathcal{M}(\mathbb{R}^d) \) be two probability measures on \( \mathbb{R}^d \). We define the 1-Wasserstein distance between \( \mu \) and \( \nu \) as
\[ d_1(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y| d\pi(x, y), \]
where \( \Pi(\mu, \nu) \) is the space of all couplings for \( \mu \) and \( \nu \), namely all those probability measures on \( \mathbb{R}^{2d} \) having as marginals \( \mu \) and \( \nu \):
\[ \int_{\mathbb{R}^d \times \mathbb{R}^d} \varphi(x) d\pi(x, y) = \int_{\mathbb{R}^d} \varphi(x) d\mu(x), \quad \int_{\mathbb{R}^d} \varphi(y) d\pi(x, y) = \int_{\mathbb{R}^d} \varphi(y) d\nu(y), \]
for all \( \varphi \in C_b(\mathbb{R}^d) \).

It’s well-known that \((\mathcal{P}_1(\mathbb{R}^d), d_1)\) (where \( \mathcal{P}_1 \) is the space of all probability measures with finite first-order moment) is a complete metric space. Moreover, in order to prove the existence of solution to (1.9), we need the following definition.

**Definition 3.2.** Let \( \mu \) be a Borel measure on \( \mathbb{R}^d \) and let \( T : \mathbb{R}^d \to \mathbb{R}^d \) be a measurable map. We define the push-forward of \( \mu \) via \( T \) as the measure
\[ T\#\mu(A) := \mu(T^{-1}(A)), \]
for all Borel sets \( A \subset \mathbb{R}^d \).
Then, we have the following theorem.

**Theorem 3.3.** Consider the system (1.9) with \( g_s \in C([-\tau(0), 0]; \mathcal{P}_1(\mathbb{R}^d)) \). Suppose that there exists a constant \( R > 0 \) such that

\[
\text{supp } g_s \in B^d(0, R),
\]

for all \( t \in [-\tau(0), 0] \), where \( B^d(0, R) \) denotes the ball of radius \( R \) in \( \mathbb{R}^d \) centered at the origin. Then, for any \( T > 0 \) there exists a unique weak solution \( \mu_t \in C([0, T); \mathcal{P}_1(\mathbb{R}^d)) \) of (1.9) in the sense of (1.12). Moreover, \( \mu_t \) is uniformly compactly supported and

\[
\mu_t = X(t; \cdot)\#\mu_0, \tag{3.32}
\]

where \( X(t; \cdot) \) is the solution of the characteristic system associated to (1.9) for any \( t \in [0, T) \).

**Proof.** First of all we claim that for any \( t \in [0, T] \), there exist two positive constants \( C, K > 0 \) such that

\[
\left| \frac{1}{h(t)} \int_{t-\tau(t)}^t \alpha(t-s)F[\mu_s](x)ds - \frac{1}{h(t)} \int_{t-\tau(t)}^t \alpha(t-s)F[\mu_s](\tilde{x})ds \right| \leq C|x - \tilde{x}|,
\]

for any \( x, \tilde{x} \in B^d(0, R) \), and

\[
\left| \frac{1}{h(t)} \int_{t-\tau(t)}^t \alpha(t-s)F[\mu_s](x)ds \right| \leq K,
\]

for all \( x \in B^d(0, R) \), with \( F \) as in (1.10) or in (1.11). The proof of this claim is very similar to [8, Lemma 3.4]. Then, from [2, Theorem 3.10], we deduce that there exists a unique weak solution \( \mu_t \) in the sense of (1.12) and it exists as long as \( \mu_t \) is compactly supported. Hence, we need to estimate the growth of support. To do so, we set

\[
R_X[\mu_t] := \max_{x \in \text{supp } \mu_t} |x|,
\]

for \( t \in [0, T] \) and we define

\[
R_X(t) := \max_{-\tau(0) \leq s \leq t} R_X[\mu_s].
\]

Now, we proceed by steps. We consider \( t \in [0, \tau_*] \) and we construct the system of characteristics \( X(t; x) : [0, \tau_*] \times \mathbb{R}^d \to \mathbb{R}^d \) associated to (1.9):

\[
\frac{dX(t; x)}{dt} = \frac{1}{h(t)} \int_{t-\tau(t)}^t \alpha(t-s)F[\mu_s](X(s; x))ds, \quad X(0; x) = x, \quad x \in \mathbb{R}^d. \tag{3.33}
\]

We notice that the system (3.33) is well-defined, since the velocity field

\[
\frac{1}{h(t)} \int_{t-\tau(t)}^t \alpha(t-s)F[\mu_s]ds
\]

is locally Lipschitz and locally bounded. Then, arguing as in Lemma 2.1, we have that

\[
\frac{d|X(t; x)|}{dt} \leq R_X(t) - |X(t; x)|,
\]

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which yields
\[ R_X(t) < R_X(0), \]
for any \( t \in [0, \tau_s] \). Thus, we obtain a unique solution \( \mu_t \) to (1.9) on the time interval \([0, \tau_s]\). We can iterate this process on all the intervals of the type \([k\tau_s, (k+1)\tau_s]\), with \( k = 1, 2, \ldots \), until we reach the final time \( T \). Moreover, following \[2\], it’s possible to find a measure \( \mu_t \) which satisfies (3.32) and this is equivalent to the definition of weak solution (1.12).

3.1 Consensus behavior

In this subsection we will prove the consensus behavior of the solution to (1.9), with \( F \) as in (1.10) or (1.11). To do so, we firstly need the following stability result.

**Lemma 3.4.** Let \( \mu^1_t, \mu^2_t \in C([0,T]; \mathcal{P}_1(\mathbb{R}^d)) \) be two weak solutions to (1.9), with compactly supported initial data \( g^1_s, g^2_s \in C([-\tau(0),0]; \mathcal{P}_1(\mathbb{R}^d)) \) respectively. Then, there exists a constant \( C > 0 \) depending only on \( T \) such that
\[
d_1(\mu^1_t, \mu^2_t) \leq C \max_{s \in [-\tau(0),0]} d_1(g^1_s, g^2_s),
\]
for any \( t \in [0,T] \).

**Proof.** For \( i = 1, 2 \) let \( X^i(t; x) : [0,T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) be the characteristics associated to (1.9), which obey to
\[
\frac{dX^i(t; x)}{dt} = \frac{1}{h(t)} \int_{t-\tau(t)}^t \alpha(t-s)F[\mu^i_s](X^i(s; x))ds,
\]
\[ X^i(0; x) = x, \]
for any \( x \in \mathbb{R}^d \). We remember that the characteristics \( X^i \) are well-defined in \([0,T]\) since, by Theorem 3.3, \( \mu^i_t \) have uniformly compact support on such interval. Then, we have that
\[ \mu^i_t = X^i(t; \cdot)\#\mu^i_s, \quad \forall t, s \in [0,T]. \]
Moreover, as before, we define
\[ R^T_{i,X} := \max_{s \in [-\tau(0),T]} R_X[\mu^i_s]. \]
Then, we choose an optimal transport map between \( \mu^i_0 \) and \( \mu^2_0 \) with respect to \( d_1 \) (call it \( S_0(x) \)) such that \( \mu^2_0 = S_0\#\mu^1_0 \) and
\[
d_1(\mu^1_0, \mu^2_0) = \int_{\mathbb{R}^d} |x - S_0(x)|d\mu^1_0(x).
\]
Moreover, we define the map \( T^t \) for any \( t \in [0,T] \) as
\[ T^t := X^2(t; \cdot) \circ S_0 \circ X^1(t; \cdot)^{-1}. \]
Therefore, we can write
\[ T^t \#\mu^1_t = \mu^2_t, \quad \forall t \in [0,T] \]
and
\[
d_1(\mu^1_t, \mu^2_t) \leq \int_{\mathbb{R}^d} |x - T^t(x)|d\mu^1_t(x) := u(t).
\]
Moreover, we extend the definition of $T^t$ on the interval $[-\tau(0), 0]$ and define $u(t)$ for $t \in [-\tau(0), 0]$ as

$$u(t) := \int_{\mathbb{R}^d} |x - T^t(x)| \, d\mu^1_s(x).$$

Now, differentiating $u(t)$ and using (3.35), we obtain

$$\frac{d}{dt} u(t) \leq \frac{1}{h(t)} \int_{\mathbb{R}^d} \alpha(t-s) \left| F[\mu^1_s](x) - F[\mu^2_s](T^t(x)) \right| \, ds \, d\mu^1_s(x) =: J.$$

We consider, now, the case of $F$ as in (1.10). Then,

$$\left| F[\mu^1_s](x) - F[\mu^2_s](T^t(x)) \right|$$

$$\leq \int_{\mathbb{R}^d} \left| \psi(|x-y|)(y-x) \right| \, d\mu^1_s(y)$$

$$\leq \int_{\mathbb{R}^d} \left| \psi(|x-y|) \right| \, d\mu^1_s(y)$$

$$+ \int_{\mathbb{R}^d} \psi(|T^t(x) - T^s(y)|) \, |y-x| \, d\mu^1_s(y)$$

$$= (1) + (2).$$

Now,

$$(1) \leq L \int_{\mathbb{R}^d} |x - T^t(x) + T^s(y)| \cdot |y-x| \, d\mu^1_s(y)$$

$$\leq L(|x| + R^T_{1,\infty}) \left( |x - T^t(x)| + \int_{\mathbb{R}^d} |y - T^s(y)| \, d\mu^1_s(y) \right),$$

and

$$(2) \leq |x - T^t(x)| + \int_{\mathbb{R}^d} |y - T^s(y)| \, d\mu^1_s(y).$$

Therefore, there exists a constant $C > 0$ depending only on $T$ such that

$$J \leq C \left( u(t) + \frac{1}{h(t)} \int_{t-\tau(t)}^{t} \alpha(t-s) u(s) \, ds \right).$$

Now, if we take $F$ as in (1.11), we have that

$$\left| F[\mu^1_s](x) - F[\mu^2_s](T^t(x)) \right|$$

$$= \left| \int_{\mathbb{R}^d} \psi(|x-y|)(y-x) \, d\mu^1_s(y) \right|$$

$$\leq \frac{1}{\psi(R^T_{1,\infty})} \int_{\mathbb{R}^d} \psi(|x-y|)(y-x) \, d\mu^1_s(y)$$

$$+ \frac{1}{\psi(R^T_{2,\infty}) \psi(R^T_{2,\infty})} \left| \int_{\mathbb{R}^d} \psi(|T^t(x) - y|)(y - T^t(x)) \, d\mu^2_s(y) \right|$$

$$\times \left| \int_{\mathbb{R}^d} \psi(|x-y|) \, d\mu^1_s(y) - \int_{\mathbb{R}^d} \psi(|T^t(x) - y|) \, d\mu^2_s(y) \right|. $$
As before we have that
\[
\left| \int_{\mathbb{R}^d} \psi(|x-y|)(y-x)d\mu_s^1(y) - \int_{\mathbb{R}^d} \psi(|T_t^s(x) - y|)(y-T_t^s(x))d\mu_s^2(y) \right| \leq \left( |x| + R_{1,X}^T \right) L + 1 \left( |x - T_t^s(x)| + u(s) \right).
\]
Furthermore,
\[
\left| \int_{\mathbb{R}^d} \psi(|T_t^s(x) - y|)(y-T_t^s(x))d\mu_s^2(y) \right| \leq R_{2,X}^T + |T_t^s(x)|,
\]
and
\[
\left| \int_{\mathbb{R}^d} \psi(|x-y|)d\mu_s^1(y) - \int_{\mathbb{R}^d} \psi(|T_t^s(x) - y|)d\mu_s^2(y) \right| \leq L(|x - T_t^s(x)| + u(s)).
\]
Hence, we obtain again the existence of a constant \(C > 0\) depending only on \(L\) and \(T\) such that
\[
\frac{du(t)}{dt} \leq C \left( u(t) + \frac{1}{h(t)} \int_{t-\tau(t)}^t \alpha(t-s)u(s)ds \right).
\]
Denote
\[
\pi = \max_{s \in [-\tau(0),0]} u(s) = \max_{s \in [-\tau(0),0]} d_1(g_s^1,g_s^2),
\]
and define \(w(t) := e^{-Ct}u(t)\). Then, we have that
\[
\frac{dw(t)}{dt} \leq C \frac{1}{h(t)} \int_{t-\tau(0)}^t \alpha(t-s)w(s)ds.
\]
Thus, we can rewrite (3.36) as
\[
\frac{dw(t)}{dt} \leq K\tau(0)\pi + K \int_0^t w(s)ds,
\]
for some \(K > 0\). This gives us the following estimate:
\[
w(t) \leq \tilde{K}\pi, \quad \forall t \in [0,T],
\]
for some \(\tilde{K} > 0\). Then, by definition of \(w\) we have
\[
d_1(\mu_1^t,\mu_2^t) \leq u(t) \leq \tilde{K} e^{CT\pi}, \quad \forall t \in [0,T],
\]
which gives us the thesis of this lemma.

We are finally ready to prove Theorem 1.6.

**Proof of Theorem 1.6** Fixed \(g_s \in C([-\tau(0),0];P_1(\mathbb{R}^d))\), we construct the family of \(N\)-particle approximations of \(g_s\), which is a family \(\{g_s^N\}_{N \in \mathbb{N}}\) such that
\[
g_s^N = \sum_{i=1}^N \delta(x - x_i^N(s)),
\]
where \(x_i^N \in C([-\tau(0),0];\mathbb{R}^d)\) satisfy
\[
\max_{s \in [-\tau(0),0]} d_1(g_s^N,g_s) \to 0, \quad as \ N \to +\infty.
\]
Moreover, let \( \{x_i^N\} \) be the solution to (1.1), with initial data \( x_i(s) = x_i^0(s) \) for any \( s \in [-\tau(0), 0] \) and we denote
\[
\mu_i^N := \sum_{i=1}^{N} \delta(x - x_i^N(t)),
\]
for any \( t \in [0, T] \), which is a weak solution to (1.9). Now, since (1.13) holds, then we know that there exists a constant \( C > 0 \) such that
\[
d_X(t) \leq d_X(0)e^{-Ct} \leq \left( \max_{s \in [-\tau(0), 0]} d_X(s) \right) e^{-Ct},
\]
for any \( t \geq 0 \). Fixing \( T \geq 0 \), by Lemma 3.4 we have that there exists a constant \( K > 0 \) independent of \( N \) such that
\[
d_1(\mu_t, \mu_i^N) \leq K \max_{s \in [-\tau(0), 0]} d_1(g_s, g_s^N),
\]
for any \( t \in [0, T] \), where \( \mu_t \) is the weak solution to (1.3) with initial measure \( g_s \). Sending \( N \to +\infty \) we have that \( d_X(t) \to d_X(\mu_t) \) and for any \( s \in [-\tau(0), 0] \), \( d_X(g_s) = d_X(s) \). This gives (1.14) for any \( t \in [0, T] \). Since \( T \) can be chosen arbitrarily, then the theorem is proved.

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