The groupoid-based logic for lattice effect algebras

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Abstract—The aim of the paper is to establish a certain logic corresponding to lattice effect algebras. First, we answer a natural question whether a lattice effect algebra can be represented by means of a groupoid-like structure. We establish a one-to-one correspondence between lattice effect algebras and certain groupoids with an antitone involution. Using these groupoids, we are able to introduce a suitable logic for lattice effect algebras.

Index Terms—D-poset, effect algebra, lattice effect algebra, antitone involution, effect groupoid, groupoid-based logic.

INTRODUCTION

Two equivalent quantum structures, D-posets and effect algebras were introduced in the nineties of the twentieth century. These were considered as "unsharp" generalizations of the structures which arise in quantum mechanics, in particular, of orthomodular lattices and MV-algebras. Effect algebras aim to describe "unsharp" event structures in quantum mechanics in the language of algebra.

Effect algebras are fundamental in investigations of fuzzy probability theory too. In the fuzzy probability frame, the elements of an effect algebra represent fuzzy events which are used to construct fuzzy random variables.

Effect algebras were introduced by Foulis and Bennett (10) and D-posets by Chovanec and Kôpka (8). Although the definition of an effect algebra looks elementary, these algebras have several very surprising properties. Concerning these properties the reader is referred to the monograph [9] by Dvurečenskij and Pulmannová. In particular, every effect algebra induces a natural partial order relation and thus can be considered as a bounded poset. If this poset is a lattice, the effect algebra is called a lattice effect algebra. A representation of lattice effect algebras by means of so-called basic algebras was derived in [4].

Since effect algebras describe quantum effects and are determined by behaviour of bounded self-adjoint operators on the Hilbert space of the corresponding physical system, we hope that a logic which should be reached by means of these algebras will enable us a better understanding of the logic of quantum mechanics.

The aim of the paper is to establish a certain logic corresponding to lattice effect algebras. By a logic we mean here a set of formulas in the language of lattice effect algebras enriched by logical connectives with a finite set of derivation rules. It can be noticed that for basic algebras the same task was solved in [3] and for the so-called dynamic De Morgan algebras in [6, 7]. Since effect algebras are only partial algebras, it looks as an advantage to use another algebraic structure which has everywhere defined operations and which is in a one-to-one correspondence with the given effect algebra E. If E is lattice ordered, this is possible and the corresponding structure can be e.g. the so-called effect near semiring, see e.g. [5] for details. However,
we can derive another algebra which has only one binary operation, i.e., a groupoid enriched by unary and nullary operations. Such approach enables us to reduce the set of formulas and the set of derivation rules. This is our aim in the first part of the paper. Using this groupoid which is called effect groupoid, we are able to introduce a suitable logic for lattice effect algebras which is provided in the second part.

I. PRELIMINARIES AND BASIC FACTS

We refer the reader to [2] for standard definitions and notations for lattice structures.

We start with the definition of an effect algebra.

Definition I.1. An effect algebra is a partial algebra \( E = (E; \oplus, 0, 1) \) of type \((2, 0, 0)\) satisfying conditions (E1) – (E4) for all \( x, y, z \in E \):

(E1) If \( x \oplus y \) exists, so does \( y \oplus x \) and \( x \oplus y = y \oplus x \);
(E2) if \( x \oplus y \) and \((x \oplus y) \oplus z \) exist, so do \( y \oplus z \) and \((x \oplus (y \oplus z)) \oplus (x \oplus (y \oplus z)) \);
(E3) there exists a unique \( x' \in E \) such that \( x \oplus x' = 1 \);
(E4) If \( x \oplus 1 \) exists then \( x = 0 \).

Since \( \cdot \) is a unary operation on \( E \) it can be regarded as a further fundamental operation. Hence in the following we will write \( E = (E; \oplus, \cdot, 0, 1) \) instead of \( E = (E; \oplus, 0, 1) \).

Let \( E = (E; \oplus, \cdot, 0, 1) \) be an effect algebra and \( a, b \in E \). The following facts are well-known:

(F1): By defining \( a \leq b \) if there exists some \( c \in E \) such that \( a \oplus c \) exists and \( a \oplus c = b \), \((E, \leq, 0, 1)\) becomes a bounded poset with an antitone involution. We call \( \leq \) the induced order of \( E \). Recall that the element \( c \) is unique, if it exists. Then \( c \) is equal to \((a \oplus b)' \) and it is denoted by \( b \cdot a \). \( E \) is called a lattice effect algebra if \((E, \leq)\) is a lattice.

(F2): \( a \cdot b \) exists if and only if \( a \leq b' \).

(F3): \( a \oplus 0 \) and \( 0 \oplus a \) exist and \( a \oplus 0 = 0 \oplus a = a \).

(F4): \((a')' = a \).

We recall Proposition 1.8.6 from [9]:

Proposition I.2. Let \( E = (E; \oplus, \cdot, 0, 1) \) be a lattice effect algebra, \( \lor \) and \( \land \) denote its lattice operations and \( a, b, c \in E \). If \( a \oplus c \) and \( b \oplus c \) exist then \((a \oplus b) \oplus c = (a \oplus c) \land (b \oplus c) \).

The following concepts were introduced in [9].

Definition I.3. A lattice orthoalgebra is a lattice effect algebra \( E = (E, \oplus, \cdot, 0, 1) \) satisfying condition (E5) for all \( x \in E \):

\( E \) if \( x \oplus x \) exists then \( x = 0 \).

An MV-effect algebra is a lattice effect algebra \( E \) such that \((x \land y') \oplus y = (y \land x') \oplus x \) for all \( x, y \in E \).

Recall that Riečanová (see [12]) showed that every lattice effect algebra is the set-theoretic union of maximal subalgebras which are MV-effect algebras, so-called blocks, and therefore is itself an MV-effect algebra if and only if it consists of one block only.

Definition I.4. An effect groupoid is an algebra \( R = (R, \cdot, 0, 1) \) of type \((2, 1, 0, 0)\) satisfying conditions (NG0) – (NG8): for all \( x, y, z \in R \):

(NG0) \((R, \cdot, 1)\) is a groupoid with unit 1;
(NG1) \( x \cdot 0 = 0 \cdot x = 0 \);
(NG3) \( 0' = 1 \);
(NG4) \( x \cdot (y \cdot x') = 0 = (y \cdot x') \cdot x \);
(NG5) \( x \cdot y = y \cdot [(y' \cdot x') \cdot x'] \);
(NG6) \((x \cdot y')' = (y' \cdot x') \cdot x = (x' \cdot y') \cdot y \);
(NG7) \([(x' \cdot y')' \cdot y]' \cdot z = [((x \cdot z) \cdot (y \cdot z)') \cdot (y' \cdot z)'] \);
(NG8) \( x' \cdot y' = 0 \) and \((x' \cdot y)' \cdot z = 0 \) then \( y' \cdot z' = 0 \).

A sub-effect groupoid of \( R \) is a subset \( Q \subseteq R \) such that 0, 1 \( \in Q \) and \( a, b \in Q \) implies \( a \cdot b \in Q \) and \( a' \in Q \).

The following theorem shows that to every lattice effect algebra there can be assigned an effect groupoid in some natural way.

Theorem I.1. Let \( E = (E; \oplus, \cdot, 0, 1) \) be a lattice effect algebra with lattice operations \( \land \) and \( \lor \) and put

\[ x \cdot y := ((x' \land y) \lor y)' \]

for all \( x, y \in E \). Then \( x \cdot y \) is well-defined because of \( x' \land y \leq y \) and, moreover, \( R(E) := (E; \cdot, 0, 1) \) is an effect groupoid.

Proof. Let \( a, b \in E \). Since \((E, \leq, 0, 1)\) is a bounded poset with an antitone involution we have \( a'' = a \), \( a \leq b \) implies \( b' \leq a' \), and \( 0' = 1 \). Moreover,

\[ a \cdot 1 = ((a' \land 1) \lor 1)' = (a' \lor 0)' = (a')' = a \] and

\[ 1 \cdot a = ((1' \land a) \lor a')' = ((0 \land a) \lor a')' = ((0 \land a) \lor a')' = (0 \lor a'') = (0 \lor a')' = (0 \lor a')' = (a')' = a. \]
Hence $(E, \cdot, 1)$ is a groupoid with neutral element. Moreover,
\[ a \cdot 0 = ((a \cdot 0) \oplus 0)' = (0 \oplus 1)' = 1' = 0 \] and
\[ 0 \cdot a = ((0' \oplus a) \oplus a)' = (a \oplus a)' = 1'. \]

Also, $a \leq b$ implies $b \oplus a = (b' \oplus a)' = ((b' \wedge a') \oplus a') = b' \cdot a'$. By (E3) we have that $a' \wedge b' = b'$ implies $b \oplus (a' \wedge b') = b \oplus b' = 1$ and, conversely, if $b \oplus (a' \wedge b') = 1$ then $a' \wedge b' = b'$. Hence the following (denoted by (†)) are equivalent:
\[ a \leq b, b' \leq a', a \wedge b' = b', b \oplus (a' \wedge b') = 1, \]
\[ (a' \wedge b') \oplus b' = 1, ((a' \wedge b') \oplus b')' = 0, a \cdot b' = 0. \]

Let $a, b \in E$. Then $a \wedge b = (a' \cdot b') \cdot b$. Namely, from the definition of $\cdot$, we obtain that $(a' \cdot b') = (a \wedge b) b'$.

It follows that $a \wedge b = (a' \cdot b') \oplus b' = (a' \cdot b')'$. 

Altogether, the conditions (NG0) – (NG3) of Definition [4] are valid. Now it remains to prove the conditions (NG4) – (NG8).

(NG4): We have $a \leq a \oplus (b' \wedge a') = (b' \wedge a') \oplus a = (b' \cdot a')'$ and hence $a \cdot (b' \cdot a') = 0$ according to (†).

Moreover, $(b' \cdot a') \cdot a = ((b' \cdot a') \wedge a \oplus a')' = (a' \oplus a')' = 1' = 0$.

(NG5): We know that $b' \leq a \vee b' = (a' \cdot b')$. In this case $a \cdot b = (a' \wedge b) \oplus b' = (b' \oplus (a' \wedge b)')' = (b' \wedge (a' \wedge b') \oplus (a' \wedge b'))' = b' \cdot [(a' \wedge b')] = b' \cdot [(b' \wedge a')''] = b' \cdot [(b' \cdot a')']$.

(NG6): Since $a \wedge b = (a' \cdot b') \cdot b$ we have from the commutativity of $\wedge$ that $(b' \cdot a') \cdot a = (a' \cdot b') \cdot b$. From the fact that $b' \cdot a' = ((b' \wedge a') \oplus a') \cdot a = 1$ we know that $a$ and $(b' \cdot a')$ are both in some block of the lattice effect algebra $E$ (see [2]) and hence $[(b' \cdot a') \cdot a'] = ((b' \cdot a') \cdot a) = a \cdot (b' \cdot a') = [a \cdot (b' \cdot a')]$.

(NG7): Using Proposition [2] we have $[(b' \cdot a') \cdot b'] = c \cdot [(a' \wedge b') \oplus c'] = (b' \wedge (a' \wedge b') \oplus c') = (c \cdot (a' \wedge b') \oplus c') = ((b' \wedge c) \oplus c') = [(a' \cdot c) \oplus (b' \cdot c)]$.

(NG8): Assume $a \cdot b = 0$ and $(a' \cdot c) = 0$. Then there exists $a' \oplus b', a \cdot b = (a' \oplus b')$ and there exists $(a' \wedge b') \cdot c'$. Hence there exist $b' \cdot c'$ and $a' \oplus (b' \cdot c')$ and $(a' \wedge b') \cdot c' = a' \oplus (b' \cdot c')$. This shows $b' \leq c$, $b' \cdot c' = 0$, $(b' \cdot c')' = b' \cdot c$ and $(a' \cdot (b' \cdot c')) = 0$.

Therefore
\[ (a' \cdot b') \cdot c' = ((a' \wedge b') \cdot c')' = (a' \oplus (b' \cdot c'))' = a' \cdot (b' \cdot c'). \]

Now we show that to every effect groupoid we can assign a lattice effect algebra in some natural way.

**Theorem 1.2.** Let $R = (R; \cdot', 0, 1)$ be an effect groupoid and for $x, y \in R$ put
\[ x \oplus y := (a' \cdot y)' = (a' \cdot y)'. \] Then $E(R) := (R; \cdot', 0, 1)$ is a lattice effect algebra.

**Proof.** Let $a, b, c \in R$.

(E1): Assume $a \oplus b$ exists. Then $a \cdot b = 0$ according to the definition of $\oplus$ and hence $b' \cdot a' = a' \cdot [a \cdot b] = a' \cdot [0 \cdot b'] = a' \cdot b' \cdot c' = a' \cdot b'$ according to (NG5), (NG3) and (NG0). Moreover, $b = 1 \cdot (a' \cdot b') = (b' \cdot a')' \cdot a'$ by (NG6). Hence $b \cdot a = ((b' \cdot a')' \cdot a') \cdot a = 0$ by (NG4). It follows that $b \oplus a$ exists and $a \oplus b = b \oplus a$.

(E2): Assume $a \oplus b$ and $(a \oplus b) \oplus c$ exist. Then $a \cdot b = 0$ and $(a' \cdot b')' \cdot c = 0$ according to the definition of $\oplus$. Hence $b \cdot c = 0$, $a \cdot (b' \cdot c')' = 0$ and $(a' \cdot b')' \cdot c = a' \cdot (b' \cdot c')$ according to (NG8). This finally implies that there exist $b \wedge c$ and $a \oplus (b \wedge c)$ and
\[ (a \oplus b) \oplus c = ((a' \cdot b')' \cdot c') = (a' \cdot (b' \cdot c')) = a \oplus (b \wedge c). \]

(E3): If $a \oplus b$ exists and $a \oplus b = 1$ then $a \cdot b = 0$, i.e., $b' \cdot a = 0$ according to (E1), and $(a' \cdot b')' = 1$ according to the definition of $\oplus$ and hence $a \cdot b = 0$. It follows $a = 1 \cdot a = (a' \cdot b')' = b' \cdot 1 = b' = b'$ according to (NG0) and (NG6). On the other hand, $a' \cdot 0 = 0$ and $a' \cdot c = 0$ by (NG4). Hence $a \oplus a' \exists$ and therefore $a \oplus a' = (a' \cdot a') = 0' = 1'$ according to (NG3). On that matter $a \oplus b = 1 \text{ if and only if } b = a'$.

(E4): If $a \oplus 1$ exists then, using (NG0), $a = a' = 1$ and hence $a = 0$.

Hence $E(R)$ is an effect algebra. Let $\leq$ denote its induced order. Then $a \leq b'$ if and only if $a \oplus b$ exists and if only if $a \cdot b = 0$. It is enough to check that the operation $\wedge$ defined by $a \wedge b = (a' \cdot b') \cdot b$ is a meet with respect to $\leq$. From (NG4) we obtain that $(a' \cdot b') \cdot b = 0$. Hence $a \wedge b \leq b$. Since also $a \wedge b = (b' \cdot a') \cdot a$ by (NG6) we obtain $a \wedge b \leq a$. Let $x, y \in R$, $x \leq a$ and $x \leq b$. Then $a' \cdot x = b' \cdot x$. It follows by (NG7) that $(a \wedge b)' \cdot x = (a' \cdot b')' \cdot b' \cdot c' = (a' \cdot c) \cdot (b' \cdot c')' \cdot (b' \cdot c') = (0 \cdot a) \cdot 0' = 0' = 1' = 0$. Therefore $x \leq a \wedge b$. 

Next we show that the described correspondence between lattice effect algebras and effect groupoids is one-to-one.
Theorem 1.3. Let $E = (E; \otimes', 0, 1)$ be a lattice effect algebra. Then $E(\mathbb{R}(E)) = E$.

Proof. Let $\mathbb{R}(E) = (E; \cdot, 0, 1)$, $E(\mathbb{R}(E)) = (E; \otimes_1, 0, 1)$ and $a, b \in E$. Then the following are equivalent: $a \otimes_1 b$ exists, $a \cdot b = 0$, $a \oplus b$ exists. If this is the case then $a \otimes_1 b = (a' \cdot b')' = (a \land b) \oplus b = a \oplus b$.

Corollary I.5. Theorem I.5. by means of effect groupoids as follows:

Proof. Let $E$ be an effect algebra. Then $R(E)$ is commutative.

Corollary I.6. Every effect groupoid is a set-theoretic union of associative and commutative sub-effect groupoids.

II. THE GROUPOID-BASED LOGIC FOR LATTICE EFFECT ALGEBRAS

We know that the logic associated to MV-algebras is already desired as many-valued Lukasiewicz logic and its axioms and reference rules are well-known, the same can be said on the logic induced by orthomodular lattices (see e.g. [1]). The previous Corollaries I.5 and I.6 motivate us to set up an appropriate logic also for lattice effect algebras. Of course, we will formulate the axioms and rules in the language of effect groupoids as derived in the previous part.

In what follows, similarly as in [13], we denote propositional variables by $p, q, r; \ldots$, the logical binary connective by $\cdot$, the logical unary connective negation by $\neg$, and two logical constants $\bot$ and $\top$ where $\bot$ stands for the contradiction and $\top$ stands for the tautology. So formulae are inductively defined by the following BNF:

\[
\phi ::= p \mid \phi \cdot \phi \mid \neg \phi \mid \bot \mid \top.
\]

We denote formulae by $\phi, \psi, \chi, \mu, \ldots$ and let $\Phi$ and $\Lambda$ be the set of all propositional variables and the set of all formulae. Let $\Gamma, \Delta, \Sigma, \Pi$ be arbitrary (possibly empty) finite lists of formulae, $\varphi, \nu$ a list of at most one formula. A logical consequence relation $\Rightarrow$ is a binary relation on $\Lambda$. We may interpret $\phi \Rightarrow \psi$ as “if $\phi$ then $\psi$.” So we call the left-hand formulae premises and the right-hand formulae conclusions. We may sometimes call logical consequences sequents. In the following $\Gamma \Rightarrow \varphi \Delta \Rightarrow \nu$ will be short for the two rules:

\[
\varphi \Rightarrow \psi
\]

and if, conversely, $\cdot$ is commutative then

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(x \land y') \oplus y = (x' \cdot y')' = (y' \cdot x')' = (y \land x') \oplus x
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It is an easy observation that a commutative effect groupoid is associative (cf. [11], Theorem 2). Due to Riečanová’s theorem (cf. [12], Theorem 3.2) we conclude

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Corollary I.6. Every effect groupoid is a set-theoretic union of associative and commutative sub-effect groupoids.
The groupoid-based logic for lattice effect algebras \( L_{LEA} \) is sound for the class of effect groupoids. That is, for every sequent \( \phi \models \psi \) in \( L_{LEA} \), \( s_\phi \leq t_\psi \) is valid on all effect groupoids \( \mathcal{R} \), where \( s_\phi \) and \( t_\psi \) are the corresponding term functions for \( \phi \) and \( \psi \) and \( \leq \) is the order in the corresponding lattice effect algebra \( \mathbb{E}(\mathcal{R}) \).
Proof. Let \( R = (R; \cdot', 0, 1) \) be an effect groupoid. The axiom (DN-1) follows from the fact that \( a = a'' \). For the inductive steps, (\( \neg r \)) follows from the fact that \( a \leq b \Rightarrow b' \leq a' \) and \( a = a'' \), (itm1) and (itm2) follow from the antisymmetry of \( \leq \), and (cut) follows from the transitivity of \( \leq \). Since 0 is the bottom element of \( \mathbb{E}(R) \) we have as in the proof of Theorem [2] that \( a \leq b \) if and only if \( a \cdot b' \leq 0 \). This yields that both (m-\( \perp \)) and (\( \perp \)-m) are valid. Since \( 1 \cdot a \leq a \) and \( a \cdot 1 \leq a \) we get (1-l) and (1-r). Similarly, since \( 0 \cdot a \leq 0 \) and \( 0 \leq 0 \cdot a \), and \( a \cdot 0 \leq 0 \) and \( 0 \leq a \cdot 0 \) we get (0-l) and (0-r). The axioms (ol) and (or) follow immediately for \( \leq \). Since \( \bot \) yields that both (m-\( \neg \)) and (\( \neg \)-m) are valid. Since \( \exists \phi \equiv \exists \psi \iff \phi \Rightarrow \psi \) in \( L_{\text{LEA}} \) and \( \psi \Rightarrow \phi \) in \( L_{\text{LEA}} \). It is plain that \( \equiv \) is really an equivalence relation. On this quotient set \( \Lambda_{/\equiv} \), we can define

\[
0 := [\bot]_{/\equiv}, \quad [\phi]_{/\equiv} := [\neg \phi]_{/\equiv},
\]

\[
1 := [\top]_{/\equiv}, \quad [\phi]_{/\equiv} \cdot [\psi]_{/\equiv} = [\phi \cdot \psi]_{/\equiv}.
\]

First, we have to verify that the definitions of \( \cdot \) and \( \cdot' \) do not depend on representatives. Assume that \( \phi \Rightarrow \phi, \psi \Rightarrow \psi, \text{ and } \psi \Rightarrow \phi \). Using Proposition [11] we get by (\( \neg r \)) that \( \neg \phi \Rightarrow \neg \phi \) and \( \neg \phi \Rightarrow \neg \phi \). Hence \( \neg \phi \equiv \neg \phi \). Similarly, we have from (itm2) that \( \phi \cdot \psi \Rightarrow \phi \cdot \psi \) and by (itm1) that \( \phi \cdot \psi \Rightarrow \phi \cdot \psi \). Using (cut) we obtain that \( \phi \cdot \psi \Rightarrow \phi \cdot \psi \). By symmetric considerations we obtain that \( \phi \cdot \psi \Rightarrow \phi \cdot \psi \), i.e., \( \phi \cdot \psi \equiv \phi \cdot \psi \).

It is a transparent task to show that the Lindenbaum-Tarski algebra \( R_{\text{LEA}} = (\Lambda_{/\equiv}; \cdot', 0, 1) \) is an effect groupoid.

\( \square \)

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