EXPANDABLE LOCAL AND PARALLEL TWO-GRID FINITE ELEMENT SCHEME FOR THE STOKES EQUATIONS

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Abstract. In this paper, we present a novel local and parallel two-grid finite element scheme for solving the Stokes equations, and rigorously establish its a priori error estimates. The scheme admits simultaneously small scales of subproblems and distances between subdomains and its expansions, and hence can be expandable. Based on the a priori error estimates, we provide a corresponding iterative scheme with suitable iteration number. The resulting iterative scheme can reach the optimal convergence orders within specific two-grid iterations \(O(\ln H)^{t-1})\) in 2-D and \(O(\ln H)\) in 3-D if the coarse mesh size \(H\) and the fine mesh size \(h\) are properly chosen. Finally, some numerical tests including 2-D and 3-D cases are carried out to verify our theoretical results.

Key words. two-grid method, domain decomposition method, superposition principle, local and parallel, iterative scheme

AMS subject classifications. 65N15, 65N30, 65N55

1. Introduction. Due to the limiting of computer resources, two-grid finite element methods/nonlinear Galerkin schemes [1, 2] and domain decomposition methods [3] are popular and powerful tools for numerical simulations of linear and nonlinear PDEs nowadays. Such as two-grid/two-level post-processing schemes for incompressible flow, we refer [4, 5, 6, 7, 8, 9] and the references therein for details.

In the past decades, a local and parallel two-grid finite element method for elliptic boundary value problems was initially proposed in [10]. The scheme firstly solves the elliptic equation on a coarse mesh to get an initial lower frequency guess of the solution. Then the whole computational domain is divided into a series of disjoint subdomains \(\{D_j\}\) and the driven term of the error equation, namely the residual term, is split into several parts defined only on such small subdomains. Finally the global error equation can be transferred into a series of subproblems with local driven terms. Since the solution of higher frequency to each subproblem decays very fast apart from the support of the local driven term, by suitably expanding each \(D_j\) to the domain \(\Omega_j \supset D_j\) and imposing the homogeneous boundary condition on \(\partial \Omega_j\), each subproblem can be approximated in a localized version defined in the corresponding expanded domain \(\Omega_j\). The most attractive feature of the scheme is that not any communication is required between local fine grid subproblems, which makes the scheme a highly effective parallel scheme. Such local and parallel two-grid scheme can be found in [11] and has been extended to the Stokes equations in [12]. Error estimates derived in [10, 12] show that the approximate solutions in such schemes can reach the optimal convergence orders in both \(H^1\) and \(L^2\) norms.

However, according to [13], the error estimates are limited by the usage of the superapproximation property of finite element spaces, which makes the error constant appeared in [10, 12] to be a form of \(O(t^{-1})\), where \(t\) denotes the distance between \(\partial D_j\) and \(\partial \Omega_j\). To obtain the optimal error orders, usually \(t = O(1)\) is required, which means the distance of \(\partial \Omega_j\) and \(\partial D_j\) is almost a constant. Therefore one can not expect \(\Omega_j\) to be arbitrary small. This will prevent the corresponding scheme from utilization in large parallel computer systems. In the

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previously mentioned local and parallel schemes, computational results for each \(\Omega_j\) are usually removed outside \(D_j\), and are simply pasted together to form the final approximation, in which discontinuity may appear along the boundaries of different \(D_j\). Instead, based on the method of partition of unity \([14]\), a local and parallel two-grid scheme is proposed for second order linear elliptic equations \([15, 16]\) and is also extended to dealing with the Stokes equations and Navier-Stokes equations \([17, 18]\). Although the partition of unity method can guarantee that the global approximation is continuous, usage of superposition principle causes a crucial requirement that the distance \(t\) should be \(O(1)\). To overcome such defects, some research on linear elliptic equations has been done by the first author and his collaborator in \([19]\) by iterative method.

In this paper, based on the basic idea presented in \([10, 12]\), as an important extension of the idea in \([19]\), we construct a local and parallel two-grid iterative scheme for solving the Stokes equations, in which the scale of each subproblem is \(O(H)\), just requiring that \(\text{diam}(D_j) = O(H)\) and \(t = O(H)\) (much smaller than \(O(1)\)). Since only a small overlapping in each two adjacent subproblems while \(H\) tends to zero, the scale of each subproblem can be arbitrary small as \(O(H)\), that’s a main reason why we call the scheme an expandable local and parallel two-grid scheme. Meanwhile, in each cycle of two-grid iteration, to guarantee a better \(L^2\) error estimate, we adopt a coarse grid correction. Another main contribution in this paper is that, to yield the globally continuous velocity in \(\Omega\), we use the principle of superposition based on a partition of unity to generate a series of local and independent subproblems. In particular, for patches of given size, to obtain a similar approximate accuracy as the one from the standard Galerkin method on the fine mesh, we carry out rigorous analysis, and through the a priori error estimate of the scheme, we show that a few number of iterations of order \(O(\ln H^2)\) in 2-D and \(O(\ln H)\) in 3-D is only needed, respectively. Similar technique has been successfully applied for adaptive schemes with some a posterior error estimates in \([20, 21]\).

The remainder of this paper is organized as follows. In Section 2, we will introduce the model problem and some preliminary materials. In Section 3, we will present our expandable local and parallel scheme for the Stokes equations. The a priori error estimates of the scheme are derived and then suggested iterative scheme is presented in Section 4. Some numerical experiments including 2-D and 3-D examples are carried out in a parallel computer system to support our theoretical analysis in Section 5. Finally we give some conclusions in Section 6.

2. Preliminaries. Consider the following Stokes equations in a smooth bounded convex domain \(\Omega \subset \mathbb{R}^d\), \(d = 2, 3\).

\[
\begin{aligned}
-\nu \Delta u + \nabla p &= f, \quad \text{in} \ \Omega, \\
\nabla \cdot u &= 0, \quad \text{in} \ \Omega, \\
u u &= 0, \quad \text{on} \ \partial \Omega.
\end{aligned}
\] (2.1)

For a given bounded domain \(D \subset \mathbb{R}^d\), standard notations for Sobolev spaces \(W^{s,p}(D)\) and their associated norms will be used, see, e.g., \([22, 23]\). In particular, for \(p = 2\), we simply define \(H^s(D) = W^{s,2}(D)\), \(H^s_0(D) = \{ v \in H^1(D) : v|_{\partial D} = 0 \}\), and their norms as \(\| \cdot \|_{s,D} = \| \cdot \|_{s,2,D}\) and \(\| \cdot \|_{s,D}\). We denote by \((\cdot, \cdot)_D\) the \(L^2\)-inner product on \(D\). Therefore, we have \(\| \cdot \|_{0,D} = (\cdot, \cdot)_D^{1/2}\), and we use \(\| \cdot \|_D\) to denote \(\| \cdot \|_{0,D}\) in the rest of the paper.

Hereafter, we always use boldface characters to denote vector valued functions or spaces, for instance, \(H^s(D) = (H^s(D))^d\). For simplicity, we will also use the symbols \(\lesssim, \gtrsim\) and \(\equiv\) in the following sense: \(x_1 \lesssim y_1, x_2 \gtrsim y_2\) and \(x_3 \equiv y_3\) are equivalent to \(x_1 \leq C_1 y_1, x_2 \geq c_2 y_2\) and \(c_3 x_3 \leq y_3 \leq C_3 x_3\) for some positive constants \(C_1, c_2, c_3, C_3\) independent of mesh size. We know that \(\| \cdot \|_{1,D} \equiv \| \nabla \cdot \|_{0,D}\) in \(H^1_0(D)\). For subdomains \(D_1 \subset D_2 \subset \Omega\), we use \(D_1 \Subset D_2\) to express that \(\text{dist}(\partial D_1, \partial D_2) \overset{\Delta}{=} \text{dist}(\partial D_1 \setminus \partial \Omega, \partial D_2 \setminus \partial \Omega) > 0\).
Furthermore, let us denote
\[
X(D) = H^1(D), \quad X_0(D) = H^1_0(D), \quad M(D) = L^2(D)
\]
\[
Q(D) = L^2_0(D) = \{q \in L^2(D) : \int_D q = 0\},
\]
and we introduce the following bilinear forms
\[
a_D(u, v) = \nu(\nabla u, \nabla v)_D, \quad d_D(p, v) = -(p, \nabla \cdot v)_D \quad \forall u, v \in X(D), \ p \in M(D).
\]
By these notations, when \(D = \Omega\), we simply denote by \(X = X(\Omega), \ X_0 = X_0(\Omega), \ M = M(\Omega), \ Q = Q(\Omega),\) and \(a(\cdot, \cdot) = a_\Omega(\cdot, \cdot), \ d(\cdot, \cdot) = d_\Omega(\cdot, \cdot)\) and \((\cdot, \cdot) = (\cdot, \cdot)_\Omega\). Then the weak formulation of (2.1) is: find \([u, p] \in X_0 \times Q\) such that
\[
\begin{cases}
\ a_D(u, v) + d(p, v) = (f, v) & \forall v \in X_0, \\
\ d(q, u) = 0 & \forall q \in Q.
\end{cases}
\]
If we further introduce the bilinear form
\[
B_D([u, p], [v, q]) = a_D(u, v) + d_D(p, v) - d_D(q, u),
\]
and \(B(\cdot, \cdot) = B_\Omega(\cdot, \cdot)\), then the above weak form for the Stokes equations can be rewritten as
\[
B([u, p], [v, q]) = (f, v) \quad \forall [v, q] \in X_0 \times Q.
\]
(2.2)
Assume that \(T^H(\Omega) = \{\tau^H\}\) is a regular triangulation of \(\Omega\), with the mesh size defined as \(H = \max_{\tau^H \in T^H(\Omega)}(\text{diam}(\tau^H))\). For certain given positive integer \(r \geq 1\), let \(X^H \subset X\) and \(M^H \subset M\) be \(C^0\)-finite element spaces defined on \(\Omega\) with approximation order \(r + 1\) in \(H^{r+1}(\Omega)\) and approximation order \(r\) in \(H^r(\Omega)\), respectively. We denote \(X^H_0 = X^H \cap H^1_0(\Omega), \ Q^H = M^H \cap L^2_0(\Omega)\) and assume \(X^H \times Q^H\) is a stable finite element pair. Given \(D \subset \Omega\), which aligns with \(T^H(\Omega)\), we denote \(T^H(D)\) as the restriction of \(T^H(\Omega)\) on \(D\), \(X^H(D)\) and \(M^H(D)\) as the restriction of \(X^H\) and \(M^H\) on \(D\) and \(Q^H(D) = M^H(D) \cap L^2_0(D)\).

Based on the finite element spaces defined above and weak formulation (2.2), we make the following assumptions.

**A1.** Interpolant. Let \(I^H\) be a Lagrange finite element interpolation of \(X(D)\) onto \(X^H(D)\) and \(\tilde{I}^H = \mathcal{I} - I^H\). There holds for any \(w \in H^s(\tau^H)\), \(0 \leq m \leq s\) and \(s > d/2,\)
\[
\|I^H w\|_{m, \tau^H} \lesssim H^{s-m}|w|_{s, \tau^H}.
\]

**A2.** Inverse Inequality. For any \(w \in X^H(D)\), \(0 \leq m \leq s, 1 \leq p, q \leq \infty,\)
\[
\|w\|_{s, p, \tau^H} \lesssim H^{m-s+\frac{d}{p} - \frac{d}{q}}\|w\|_{m, q, \tau^H}.
\]

**A3.** Regularity. Assume \(D\) is smooth such that for any \(f \in L^2(D)\), any solution to
\[
B_D([u, p], [v, q]) = (f, v)_D \quad \forall [v, q] \in X_0(D) \times Q(D),
\]
\[
\|u\|_{2, D} + \|p\|_{1, D} \lesssim \|f\|_D.
\]
Now we can state the standard Galerkin approximation of (2.2) as follows: find \([u_H, p_H] \in X^H_0 \times Q^H\) such that
\[
B([u_H, p_H], [v_H, q_H]) = (f, v_H) \quad \forall [v_H, q_H] \in X^H_0 \times Q^H.
\]
(2.3)
For this problem, a classical result holds, namely assuming that \([u, p] \in (H^{r+1}(\Omega) \times H^r(\Omega)) \cap (X_0 \times Q)\), one have
\[
\|u - u_H\|_\Omega + H(||\nabla(u - u_H)||_\Omega + ||p - p_H||_\Omega) = O(H^{r+1}).
\]
(2.4)
3. Local and Parallel Two-Grid Scheme. In this section, we will introduce our local and parallel two-grid scheme for solving Stokes equations. Firstly, we define the error functions by
\[
\hat{u} = u - u_H \in X_0, \quad \hat{p} = p - p_H \in Q,
\]
where \([u, p]\) and \([u_H, p_H]\) are solutions to (2.2) and (2.3), respectively. Then \([\hat{u}, \hat{p}]\) satisfies the error equation
\[
B([\hat{u}, \hat{p}],[v,q]) = (f,v) - B([u_H, p_H],[v,q]) \quad \forall [v,q] \in X_0 \times Q. \tag{3.1}
\]

For any given partition of unity of \(\Omega\), namely \(\{\phi_j\}_{j=1}^N\) with \(N \geq 1\) an integer, \(\Omega \subset \bigcup_{j=1}^N \text{supp} \phi_j\) and \(\sum_{j=1}^N \phi_j \equiv 1\) on \(\Omega\), we denote \(D_j = \text{supp} \phi_j\) and always assume that \(D_j\) aligns with \(T^h(\Omega)\). With this partition of unity, the error equation (3.1) can be rewritten as:
\[
B([\hat{u}^j, \hat{p}^j],[v,q]) = (f, \sum_{j=1}^N \phi_j v) - B([u_H, p_H],[\sum_{j=1}^N \phi_j v,q]) \quad \forall [v,q] \in X_0 \times Q.
\tag{3.2}
\]

Namely, \([\hat{u}, \hat{p}] = \sum_{j=1}^N [\hat{u}^j, \hat{p}^j]\). The most important feature for such subproblems is that all the subproblems are mutually independent when \([u_H, p_H]\) is known. Meanwhile each subproblem is globally defined with homogeneous Dirichlet boundary condition for the velocity, however is driven by right-hand-side term with very small compact support, namely \(D_j\). As is pointed out in [21] (used also in [19]) for elliptic problems, solution \(\hat{u}^j\) to each subproblem may decay very fast away from \(D_j\). So for the present Stokes problems, we will localize each subproblem to a small extension domain of \(D_j\), namely \(\Omega_j\), and impose homogeneous Dirichlet boundary condition for the velocity and discretize the subproblem on some triangulation \(T^h(\Omega_j)\) for finite element approximation.

Remark 3.1. For each \(\Omega_j\), here we give a much natural choice. The partition of unity functions \(\{\phi_j\}_{j=1}^N\) of \(\Omega\) is selected as the piecewise linear Lagrange basis functions associated with the coarse triangulation \(T^H(\Omega)\), where \(N\) is the number of all vertices in \(T^H(\Omega)\) including the boundary ones. For each vertex \(x_j\) of \(T^H(\Omega)\), we can denote \(D_j = \text{supp} \phi_j\), and construct \(\Omega_j\) by one layer extension of \(D_j\) in \(T^H(\Omega)\), namely \(\Omega_j = \bigcup_{x_i \in D_j} D_i\). We refer [17] for more details. Clearly, we know the scale estimates
\[
diam(D_j), \ dist(\partial D_j, \partial \Omega_j) \equiv H. \tag{3.3}
\]
Base on the fine mesh triangulations $T^h(\Omega_j)$ and $T^h(\Omega)$ for each $\Omega_j$ and $\Omega$, we can introduce corresponding finite element spaces $X^h(\Omega_j), X^h(\Omega_j)$ and $X^h, X^h_0$, similarly defined as $X^H$ and $X^H_0$ previously. We also introduce $M^h(\Omega_j), Q^h(\Omega_j), M^h$ and $Q^h$ for the pressure accordingly. Noting that any function in $X^h_0(\Omega_j)$ can be extended to a function in $X^h$ through zero value assignment over $\Omega \setminus \Omega_j$, we regard $X^h_0(\Omega_j)$ as a subspace of $X^h$ in the sense of such zero extension. In addition, we always assume that

$$
\begin{align*}
\begin{cases}
X^h \subset X^h, \quad X^h_0 \subset X^h_0 = \bigcup_{1 \leq j \leq N} X^h_0(\Omega_j), \\
M^H \subset M^h, \quad M^H|_{\Omega_j} \subset M^h(\Omega_j).
\end{cases}
\end{align*}
$$

(3.4)

Besides, we define discontinuous finite element spaces associated with the pressure

$$
M^h = \{ q_h : q_h|_{\Omega_j} \in M^h(\Omega_j), \; q_h|_{\Omega \setminus \Omega_j} \in M^h(\Omega \setminus \Omega_j) \}, \quad Q^h = M^h \cap L^2_0(\Omega).
$$

Clearly, $M^h \subset \bigcup_{j=1}^N M^h_j \subset M^h = \{ q_h \in L^2(\Omega) : q_h|_{\tau_H} \in M^h(\tau_H) \}$. Then for $Q^h_{\Omega_j} = M^h_{\Omega_j} \cap L^2_0(\Omega)$, we have that $Q^h \subset Q^h_{\Omega_j}$ and the finite element pair $X^h \times Q^h$ is stable (see [24]).

Right now, we can present the localized finite element approximation to error equation (3.2) as follows: find $[\hat{u}^j_{H,H}, \hat{p}^j_{H,H}] \in X^h_0(\Omega_j) \times Q^h(\Omega_j)$ such that $\forall [v_h, q_h] \in X^h_0(\Omega_j) \times Q^h(\Omega_j)$

$$
B_{\Omega_j}([\hat{u}^j_{H,H}, \hat{p}^j_{H,H}], [v_h, q_h]) = (f, \phi_j v_h)|_{\Omega_j} - \tau_{\Omega_j}([u_{H,p_H}, \phi_j v_h, q_h]),
$$

(3.5)
in which $\hat{u}^j_{H,H}$ and $\hat{p}^j_{H,H}$ are defined on $\Omega_j$. In the rest, we always use the same symbols, that is $[\hat{u}^j_{H,H}, \hat{p}^j_{H,H}]$, to denote their zero extension over $\Omega \setminus \Omega_j$. In such sense, $[\hat{u}^j_{H,H}, \hat{p}^j_{H,H}] \in X^h_0(\Omega_j) \times Q^h(\Omega_j)$. Then we construct totally

$$
\hat{u}_{H,h} = \sum_{j=1}^N \hat{u}^j_{H,h} \in X^h_0, \quad \hat{p}_{H,h} = \sum_{j=1}^N \hat{p}^j_{H,h} \in Q^h,
$$

and the intermediate approximate solution

$$
[u_{H,h}, p_{H,h}] = [u_H + \hat{u}_{H,h}, p_H + \hat{p}_{H,h}].
$$

(3.6)

Since this approximation $[u_{H,h}, p_{H,h}]$ is derived by solving a series of subproblems on local fine mesh with homogeneous boundary conditions, its high frequency error may be suppressed apparently. To balance the lower and higher frequency errors, a smooth step associated with the intermediate approximation $[u_{H,h}, p_{H,h}]$ is resorted through coarse grid correction as follows: find $[E^H_u, E^H_p] \in X^H_0 \times Q^H$ such that $\forall [v_h, q_H] \in X^H_0 \times Q^H$

$$
B([E^H_u, E^H_p], [v_h, q_H]) = (f, v_H) - B([u_{H,h}, p_{H,h}], [v_h, q_H]).
$$

(3.7)

To this end, an expected more accurate approximation than the coarse grid approximation is obtained,

$$
[u^h_H, p^h_H] = [u_H + E^H_u, p_{H,h} + E^H_p] = [u_H + \hat{u}_{H,h}, p_H + \hat{p}_{H,h} + E^H_p].
$$

(3.8)

In summary, we propose our local and parallel two-grid scheme for solving (2.1) in the following.

**Local and parallel two-grid scheme:**

**Step 0.** Deriving $[u_{H,h}, p_{H,h}]$ by solving (2.3);

**Step 1.** Solving the equation (3.5) to get $\{[\hat{u}^j_{H,H}, \hat{p}^j_{H,H}]\}_{j=1}^N$ for each $j$, then constructing $[u_{H,h}, p_{H,h}]$ by formula (3.6);

**Step 2.** Deriving $[E^H_u, E^H_p]$ by solving (3.7) and finally constructing $[u^h_H, p^h_H]$ by (3.8).
4. Theoretical Analysis and Suggested Local and Parallel Two-Grid Iterative Scheme. As the basic step for analyzing the scheme above, we follow the idea discussed in [19], and extend the local sub-problem (3.5) to the original domain Ω. For this purpose, we denote Γ = ∂Ω, Γj = ∂Ωj \ Γ, and introduce a trace space \( H^{\frac{1}{2}}(Γ_j) = H^1(Ω)|_{Γ_j} \) on Γj, which can be defined by interpolation (for example, see [26])

\[
H^{\frac{1}{2}}(Γ_j) = \begin{cases} 
[L^2(Γ_j), H^1(Γ_j)]_{\frac{1}{2}}, & \text{when } Γ_j \text{ is a closed curve or surface}, \\
[L^2(Γ_j), H^1_0(Γ_j)]_{\frac{1}{2}}, & \text{when } Γ_j \text{ is a non-closed curve or surface}.
\end{cases}
\]

We also denote by \( H^{\frac{1}{2}}(Γ_j) = X^h_0(Ω)|_{Γ_j} \subset H^{\frac{1}{2}}(Γ_j) \), a finite dimensional trace space, equipped with the same norm as in \( H^{\frac{1}{2}}(Γ_j) \), and its dual space by \( H^{-\frac{1}{2}}(Γ_j) = (H^1(Γ_j))^\prime \), which is equipped with the norm

\[
\|μ\|_{H^{-\frac{1}{2}}(Γ_j)} = \sup_{v_h \in H^{\frac{1}{2}}_0(Γ_j)} \frac{\int_{Γ_j} v_h μ}{\|v_h\|_{H^{\frac{1}{2}}(Γ_j)}}.
\]

Then based on the fictitious domain method (see e.g., [25]), we can construct a problem with multiplier as finding ((\( \hat{u}^j_{H, h}, \hat{p}^j_{H, h} \)), \( \xi^j \)) ∈ \( X^h_0 \times Q^h_0 × H^{−\frac{1}{2}}(Γ_j) \), such that

\[
B((\hat{u}^j_{H, h}, \hat{p}^j_{H, h}), (v_h, q_h)) + < \xi^j, v_h >_{Γ_j} + < μ, \hat{u}^j_{H, h} >_{Γ_j} = (f, \phi_j v_h) - B([u_H, p_H], [\phi_j v_h, \phi_j q_h]),
\]

holds for every \((v_h, q_h, μ) ∈ X^h_0 \times Q^h_0 × H^{−\frac{1}{2}}(Γ_j)\), where

\[
< μ, v_h >_{Γ_j} = \int_{Γ_j} μ v_h ds, \quad ∀ μ ∈ H^{−\frac{1}{2}}(Γ_j), \quad v_h ∈ X^h_0.
\]

In the following, we will show the equivalence between problems (3.5) and (4.1) in two steps.

Firstly, by introducing a subspace of \( X^h_0 \)

\( X^h_{j0} = \{ v_h ∈ X^h_0 : < μ, v_h >_{Γ_j} = 0 \ ∀ μ ∈ H^{−\frac{1}{2}}(Γ_j) \} = \ker(γ). \)

the above system (4.1) reduces to finding \([\hat{u}^j_{H, h}, \hat{p}^j_{H, h}] ∈ X^h_{j0} \times Q^h_0\) such that

\[
B((\hat{u}^j_{H, h}, \hat{p}^j_{H, h}), (v_h, q_h)) = (f, \phi_j v_h) - B([u_H, p_H], [\phi_j v_h, \phi_j q_h]), \forall (v_h, q_h) ∈ X^h_{j0} \times Q^h_0,
\]

which is obviously well-posed, since it actually consists of two independent Stokes problems defined in Ωj and Ω \( \setminus \Omega_j \) respectively, with homogeneous boundary conditions for the velocity.

Secondly, for any given \( g ∈ H^{\frac{1}{2}}(Γ_j) \), we introduce two auxiliary elliptic problems

\[
a_{Ω_j}(u^j_h, v_h) = 0, \quad u^j_h|_{Γ_j} = g, \quad u^j_h|_{∂Ω_j \setminus Γ_j} = 0, \quad ∀ v_h ∈ X^h_0(Ω_j),
\]

and

\[
a_{Ω_\Omega_j}(u^j_h, v_h) = 0, \quad u^j_h|_{Γ_j} = g, \quad u^j_h|_{∂Ω(Ω \setminus Ω_j)} = 0, \quad ∀ v_h ∈ X^h_0(Ω \setminus Ω_j).
\]

Clearly, these two problems establish two mappings \( γ_1^{-1} \) and \( γ_2^{-1} \) from \( H^{\frac{1}{2}}(Γ_j) \) into \( X^h_0(Ω_j) \) and \( X^h_0(Ω \setminus Ω_j) \) respectively, with

\[
X^h_0(Ω_j) = \{ v_h ∈ X^h_0(Ω_j) : v_h|_{∂Ω_j \cap Ω} = 0 \},
\]

\[
X^h_0(Ω \setminus Ω_j) = \{ v_h ∈ X^h_0(Ω \setminus Ω_j) : v_h|_{∂Ω \setminus ∂Ω_j} = 0 \}.
\]
We can simply write as
\[ u_h^1 = \gamma_1^{-1} g, \quad u_h^2 = \gamma_2^{-1} g, \]
which have the following estimates
\[ \| \gamma_1^{-1} g \|_{H^1(\Omega_j)}, \quad \| \gamma_2^{-1} g \|_{H^1(\Omega_j)} \lesssim \| g \|_{H^\frac{1}{2}(\Gamma_j)}. \]

We can also define a lifting operator \( \gamma^{-1} \) from \( H_h^\frac{1}{2}(\Gamma_j) \) into \( X_h^0 \): for any given \( g \in H_h^\frac{1}{2}(\Gamma_j) \)
\[ \gamma^{-1} g = \begin{cases} \gamma_1^{-1} g, & \text{in } \Omega_j, \\ \gamma_2^{-1} g, & \text{in } \Omega \setminus \Omega_j, \end{cases} \]
which is the right inverse operator of the trace operator \( \gamma \) from \( X_h^0 \) onto \( H_h^\frac{1}{2}(\Gamma_j) \), with the property of
\[ \| \gamma^{-1} g \|_{H^1(\Omega)} \lesssim \| g \|_{H_h^\frac{1}{2}(\Gamma_j)} \quad \forall g \in H_h^\frac{1}{2}(\Gamma_j). \]

Then we have for any \( \mu \in H_h^{-\frac{1}{2}}(\Gamma_j) \)
\[ \| \mu \|_{H_h^{-\frac{1}{2}}(\Gamma_j)} = \sup_{g \in H_h^\frac{1}{2}(\Gamma_j)} \frac{\langle \mu, \gamma^{-1} g \rangle_{\Gamma_j}}{\| g \|_{H^\frac{1}{2}(\Gamma_j)}} \lesssim \sup_{g \in H_h^\frac{1}{2}(\Gamma_j)} \frac{\| \mu \|_{H_h^{-\frac{1}{2}}(\Gamma_j)}}{\| \gamma^{-1} g \|_{H^1(\Omega)}} \sup_{v_h \in X_h^0(\Omega)} \frac{\| v_h \|_{H^1(\Omega)}}{\| g \|_{H^1(\Omega)}}. \tag{4.2} \]

which verifies that the bilinear form \( \langle \cdot, \cdot \rangle_{\Gamma_j} \) on \( H_h^{-\frac{1}{2}}(\Gamma_j) \times X_h^0 \) satisfies the inf-sup condition.
Therefore, for any solution \( [\hat{u}_H, \hat{p}_H] \in X_h^0 \times Q_h \) to (3.5), there exists a unique \( \xi^j \in H_h^{-\frac{1}{2}}(\Gamma_j) \) such that \( ([\hat{u}^j_H, \hat{p}^j_H], \xi^j) \) satisfies the problem (4.4). In such sense, the problems (3.5) and (4.4) are equivalent.

Now let us turn back to the global and local problems (3.1) and (3.2) related to the residuals of approximate solution. For the fine triangulation \( \mathcal{T}^h(\Omega) \) and the associated finite element space \( X_h^0 \), their corresponding Galerkin approximations are finding \( [\hat{u}_H, \hat{p}_H] \in X_h^0 \times Q_h \) and \( [\hat{u}^j_H, \hat{p}^j_H] \in X_h^0 \times Q_h, \ j = 1, 2, \cdots, N \), such that \( \forall [v_h, q_h] \in X_h^0 \times Q_h \)
\[ B([\hat{u}_H, \hat{p}_H], [v_h, q_h]) = (f, v_h) - B([u_H, p_H], [v_h, q_h]), \tag{4.3} \]
and
\[ B([\hat{u}^j_H, \hat{p}^j_H], [v_h, q_h]) = (f, v_h) - B([u_H, p_H], [\phi_j v_h, \phi_j q_h]). \tag{4.4} \]

We know that \( [\hat{u}_H, \hat{p}_H] = \sum_{j=1}^N [\hat{u}_H^j, \hat{p}_H^j] \) is the Galerkin approximation of \( [\hat{u}, \hat{p}] \) in \( X_h^0 \times Q_h \), and meanwhile \( [u_h, p_h] = [u_H, p_H] + [u_H, \hat{p}_H] \in X_h^0 \times Q_h \) is the finite element approximation of \( [u, p] \) on fine mesh.

Then if denoting by
\[ e_{u_h}^j = \hat{u}_H^j - \hat{u}_{H,H}, \quad e_{u_h} = \sum_{j=1}^N e_{u_h}^j = u_h - u_{H,H}, \]
\[ e_{p_h}^j = \hat{p}_H^j - \hat{p}_{H,H}, \quad e_{p_h} = \sum_{j=1}^N e_{p_h}^j = p_h - p_{H,H}, \]
we have
\[ e_{u_h} = \sum_{j=1}^N e_{u_h}^j = u_h - u_{H,H}, \]
\[ e_{p_h} = \sum_{j=1}^N e_{p_h}^j = p_h - p_{H,H}. \]
the local and the global error of \([u_{H,h}, p_{H,h}],\) respectively, subtracting \((3.2)\) from \((4.1)\) gives
\[
a(e_{u,h}, v_h) + d(e_{p,h}, v_h) + \xi^j, v_h > \Gamma_j = 0, \quad \forall v_h \in X_0^h, \tag{4.5}
\]
and further summing over all \(j\)'s yields
\[
a(e_{u,h}, v_h) + d(e_{p,h}, v_h) + \sum_{j=1}^N \xi^j, v_h > \Gamma_j = 0, \quad \forall v_h \in X_0^h. \tag{4.6}
\]

In the following, we will carry out estimates on the local quantities \(e_{u,h}^j, e_{p,h}^j\) and \(\xi^j\), which are very important in the error analysis of our scheme. Firstly, for \(\xi^j\), by the previously defined operator \(\gamma^{-1}\) and the fact that for every point \(x \in \Omega\), there exists a positive integer \(\kappa\), which is independent of \(N\) and \(x\), such that each \(x\) belongs to at most \(\kappa\) different \(\Gamma_j\), we can easily derive the estimates as follows.

**Lemma 4.1.** The multiplier \(\xi^j\) in \((4.1)\), also in \((4.5)\), satisfies
\[
\|\xi^j\|_{H^{-\frac{1}{2}}(\Gamma_j)} \lesssim \|\nabla e_{u,h}^j\|_{\Omega} + \|e_{p,h}^j\|_{\Omega},
\]
and
\[
\sum_{j=1}^N \xi^j, v_h > \Gamma_j \lesssim \kappa^2 \left( \sum_{j=1}^N \|\xi^j\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \right)^{\frac{1}{2}} \|\nabla v_h\|_{\Omega}, \quad \forall v_h \in X_0^h.
\]

**Proof.** The first estimate is a direct result of property \((4.2)\). For the second estimate, by the definition of \(\|\cdot\|_{H^{-\frac{1}{2}}(\Gamma_j)}\), we have \(\forall v_h \in X_0^h\)
\[
\sum_{j=1}^N \xi^j, v_h > \Gamma_j \leq \sum_{j=1}^N \|\xi^j\|_{H^{-\frac{1}{2}}(\Gamma_j)} \|v_h\|_{H^{\frac{1}{2}}(\Gamma_j)} \lesssim \sum_{j=1}^N \|\xi^j\|_{H^{-\frac{1}{2}}(\Gamma_j)} \|v_h\|_{H^{\frac{1}{2}}(\partial \Omega_j)}
\]
\[
\lesssim \sum_{j=1}^N \|\xi^j\|_{H^{-\frac{1}{2}}(\Gamma_j)} \|v_h\|_{H^{1}(\Omega_j)} \leq \left( \sum_{j=1}^N \|\xi^j\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \right)^{\frac{1}{2}} \left( \sum_{j=1}^N \|v_h\|_{H^{1}(\Omega_j)}^2 \right)^{\frac{1}{2}}
\]
\[
\leq \kappa^2 \left( \sum_{j=1}^N \|\xi^j\|_{H^{-\frac{1}{2}}(\Gamma_j)}^2 \right)^{\frac{1}{2}} \|\nabla v_h\|_{\Omega}.
\]
This is the second estimate of the lemma. \(\square\)

As to the quantity of \(\|\nabla e_{u,h}^j\|_{\Omega} + \|e_{p,h}^j\|_{\Omega}\), combining the identities
\[
\|\nabla e_{u,h}^j\|_{\Omega}^2 = \|\nabla (\hat{u}_H^j - \tilde{u}_H^j)\|_{\Omega}^2 = \|\nabla \hat{u}_H^j\|_{\Omega(\Omega_j)}^2 + \|\nabla (\hat{u}_H^j - \tilde{u}_H^j)\|_{\Omega_j}^2,
\]
\[
\|e_{p,h}^j\|_{\Omega}^2 = \|\hat{p}_H^j - \tilde{p}_H^j\|_{\Omega}^2 = \|\hat{p}_H^j\|_{\Omega(\Omega_j)}^2 + \|\hat{p}_H^j - \tilde{p}_H^j\|_{\Omega_j}^2,
\]
and the equations satisfied by \([e_{u,h}^j, e_{p,h}^j]\)
\[
\begin{cases}
a_{\Omega_j}(e_{u,h}^j, v_h) + d_{\Omega_j}(e_{p,h}^j, v_h) = 0, \quad \forall v_h \in X_0^h(\Omega_j), \\
d_{\Omega_j}(q_h, e_{u,h}^j) = 0, \quad \forall q_h \in Q^h(\Omega_j), \\
e_{u,h}^j|_{\partial \Omega_j} = \hat{u}_H|_{\partial \Omega_j},
\end{cases}
\]
we have by virtue of \(\hat{u}_H^j \in X_0^h\)
\[
\|\nabla e_{u,h}^j\|_{\Omega_j} + \|e_{p,h}^j\|_{\Omega_j} \lesssim \|\hat{u}_H^j\|_{H^{\frac{1}{2}}(\partial \Omega_j)} \lesssim \|\nabla \hat{u}_H^j\|_{\Omega(\Omega_j)} + \|\hat{u}_H^j\|_{\Omega(\Omega_j)}.\]
which is arriving at

\[ \| \nabla e_{u,h} \|_{\Omega_j} + \| e_{u,h}^j \|_{\Omega_j} \lesssim \| \nabla \hat{u}_H \|_{\Omega_j}. \tag{4.7} \]

Then we present the following two results related with the Sobolev space \( H^1_0(\Omega) \) to make further analysis concerning on (4.7).

**Lemma 4.2.** Let \( D \subset \Omega \) with diam \( (D) \approx H \) be any convex subdomain of \( \Omega \). Then we have

\[ \| \mathbf{w} \|^2_{L^2(\partial D)} \lesssim H^{d-1} \beta_d(H) \| \nabla \mathbf{w} \|^2_{L^2(\Omega \setminus D)} \quad \forall \mathbf{w} \in H^1_0(\Omega), \]

where \( \beta_d(H) = H^{-1} \) when \( d = 3 \) and \( \beta_d(H) = | \ln H | \) when \( d = 2 \).

**Proof.** Since \( D \) and \( \Omega \) are convex domains, there exists a point \( P \in D \) such that dist \( (P, \partial D) \approx H \) and \( \overline{PP'} \subset D \) for any \( P' \in \partial D \) and \( \overline{PP'} \subset \Omega \) for any \( P' \in \partial \Omega \). If we denote \( \Omega^c = \Omega \setminus D \), we can establish a local polar or spherical coordinate \((\rho, \omega)\) with origin at this point \( P \) corresponding to 2-D to 3-D case, see Fig. 4.1 for the sketch of such setting in the 2-D case. Here \( \rho \)

![Fig. 4.1. Local polar coordinate in 2-dimensional domain.](image)

is the radius distant between any point \( P' \) in \( \Omega \) and the origin \( P \), and \( \omega = (\omega_1, \omega_2) \) in 3-D case. The Jacobi determinant of the transformation between the Cartesian coordinate \((x_1, \ldots, x_d)\) and \((\rho, \omega)\) is

\[ J = \frac{D(x_1, \ldots, x_d)}{D(\rho, \omega_1 \cdot, \omega_{d-1})} = \rho^{d-1} \delta(\omega), \quad \text{with} \quad \delta(\omega) = \sin^{d-2} \omega_1. \]

Moreover, we use \( \rho_\Omega(\omega) \) and \( \rho_D(\omega) \) to characterize the boundary points of \( \Omega \) and \( D \), respectively.

For any \( \mathbf{w} \in H^1_0(\Omega) \), since \( \rho_D(\omega) \equiv H \) and \( H \lesssim \rho_\Omega(\omega) \lesssim 1 \), and \( \mathbf{w}(\rho_\Omega(\omega), \omega) = 0 \), we have

\[
| \mathbf{w}(\rho_D(\omega), \omega) | = \int_{\rho_D(\omega)}^{\rho_\Omega(\omega)} \frac{1}{\rho^{d-1}} ( \int_{\rho_D(\omega)}^{\rho_\Omega(\omega)} \rho^{d-1} | \frac{\partial \mathbf{w}}{\partial \rho} |^2 d\rho )^{\frac{1}{2}} d\rho \\
\lesssim \beta_d^2(H) ( \int_{\rho_D(\omega)}^{\rho_\Omega(\omega)} \rho^{d-1} | \frac{\partial \mathbf{w}}{\partial \rho} |^2 d\rho )^{\frac{1}{2}}.
\]
Thus, if we denote by \( \omega_d \) the unit ball in \( \mathbb{R}^d \), we have

\[
\|w\|^2_{L^2(\partial \Omega)} = \int_{\partial \omega_d} (\rho \partial w(\omega))^{d-1}\rho(\omega)^2 d\omega
\]

\[
\lesssim \int_{\partial \omega_d} (\rho \partial w(\omega))^{d-1}\beta_d(H)(\rho d\rho)\rho^{d-1}\rho(\omega)^2 d\omega
\]

\[
\lesssim H^{d-1}\beta_d(H)\|\nabla w\|^2_{\Omega_1\setminus \Omega_2}.
\]

The proof is complete. \( \Box \)

**Lemma 4.3.** If \( \Omega' \subset \Omega \subset \Omega \) are any two convex subdomains of \( \Omega \) with \( \text{diam}(\Omega) \approx H \), \( \text{dist}(\partial \Omega', \partial \Omega) \approx h \) and \( h < H \). Then we have

\[\|w\|^2_{\Omega \setminus \Omega'} \leq hH^{d-1}\beta_d(H)\|\nabla w\|^2_{\Omega_1\setminus \Omega_2}, \quad \forall w \in H^1(\Omega).\]

**Proof.** Like what has been done in the proof of Lemma 4.2 for \( \Omega' \subset \Omega \), we can choose some \( P \in \Omega' \) such that \( \text{dist}(P, \partial \Omega') \approx H \), then we can establish a local polar or spherical coordinate \( (\rho, \omega) \) with origin at this point \( P \) corresponding to 2-D to 3-D case. And from the proof of Lemma 4.2 we know for any point \( (\rho, \omega) \in \Omega_1 \setminus \Omega_2 \):

\[
|w(\rho, \omega)| = \left| \int_{\rho}^{\rho_{\partial D}(\omega)} \rho d\rho \rho(\omega)^2 \right| \leq \left( \int_{\rho}^{\rho_{\partial D}(\omega)} \frac{1}{\rho d\rho} \right)^{\frac{1}{2}}\left( \int_{\rho}^{\rho_{\partial D}(\omega)} \rho^{d-1} d\rho \right)^{\frac{1}{2}}
\]

\[
\lesssim \beta_d(H)\left( \int_{\rho_{\partial D}(\omega)}^{\rho_{\partial D}'(\omega)} \rho^{d-1} d\rho \right)^{\frac{1}{2}}
\]

Then we have

\[
\|w\|^2_{\Omega_1\setminus \Omega_2} = \int_{\partial \omega_d} \int_{\rho_{\partial D}(\omega)}^{\rho_{\partial D}'(\omega)} w^2 \rho d\rho d\omega
\]

\[
\lesssim \int_{\partial \omega_d} \int_{\rho_{\partial D}(\omega)}^{\rho_{\partial D}'(\omega)} \beta_d(H)\left( \int_{\rho_{\partial D}(\omega)}^{\rho_{\partial D}'(\omega)} \rho^{d-1} d\rho \right)^{\frac{1}{2}}\left( \int_{\rho_{\partial D}(\omega)}^{\rho_{\partial D}'(\omega)} \rho^{d-1} \rho(\omega)^2 d\rho \right)^{\frac{1}{2}} d\omega
\]

\[
\lesssim H^{d-1}\beta_d(H)\int_{\partial \omega_d} \int_{\rho_{\partial D}(\omega)}^{\rho_{\partial D}'(\omega)} \beta_d(H)\left( \int_{\rho_{\partial D}(\omega)}^{\rho_{\partial D}'(\omega)} \rho^{d-1} \rho(\omega)^2 d\rho \right)^{\frac{1}{2}} d\omega
\]

\[
\lesssim \beta_d(H)\left( \int_{\rho_{\partial D}(\omega)}^{\rho_{\partial D}'(\omega)} \rho^{d-1} \rho(\omega)^2 d\rho \right)^{\frac{1}{2}} d\omega = hH^{d-1}\beta_d(H)\|\nabla w\|^2_{\Omega_1\setminus \Omega_2},
\]

which completes the proof. \( \Box \)

Right now, we can give a more rigorous estimate for \( \|\nabla e^j_{u,h}\|^2_{\Omega} + \|e^j_{p,h}\|^2_{\Omega} \) based on \( 4.7 \), which will play a crucial role in the analysis of this section.

**Lemma 4.4.** Defining

\[
\alpha_d = \left\{ \frac{c}{\ln H^d}, \quad d = 2, \quad \frac{1}{\ln H^d}, \quad d = 3, \right\}
\]

where \( c > 0 \) is a positive constant independent of \( H, h \) and any subdomain \( \Omega \). Then we have

\[
\|\nabla e^j_{u,h}\|_{\Omega} + \|e^j_{p,h}\|_{\Omega} \lesssim H^{\alpha_d}\|\nabla \hat{\omega}_H\|_{\Omega} + \|\hat{p}_H\|_{\Omega},
\]

(4.8)
Proof. The proof of this lemma is divided into four steps.

Step 1. Firstly, we use the same argument to split the region \( \Omega_j \setminus D_j \) for each \( j \) as in \([19]\). We recall the sketch of a domain partition in Fig. 4.2.

![Fig. 4.2. Division of the region \( \Omega_j \setminus D_j \)](image)

Right now, we can define a series of disjoint annular zones as \( \hat{\Omega}_j^0, \hat{\Omega}_j^1, \ldots, \hat{\Omega}_j^M \), and construct a sequence of subdomains as

\[
\hat{\Omega}_j^k = \bigcup_{i=0}^{k} \Omega_j^i, \quad k = 0, 1, 2, \ldots, M,
\]

with the property

\[
\hat{\Omega}_j^0 \subset \hat{\Omega}_j^1 \subset \cdots \subset \hat{\Omega}_j^M.
\]

Then we define

\[
\gamma_j^k = \partial \hat{\Omega}_j^k \setminus \partial \Omega, \quad \Gamma_j^k = \partial \hat{\Omega}_j^k \setminus \gamma_j^k, \quad \text{and} \quad \partial \hat{\Omega}_j^k = \gamma_j^k \cup \Gamma_j^k, \quad k = 0, 1, 2, \ldots, M.
\]

Step 2. Note that, since \( \text{supp} \phi_j = D_j \), the right hand sides in (4.4) will be equal to 0 for the testing functions in \( X^0_h(\hat{\Omega}_j^k) \times M^h(\hat{\Omega}_j^k) \) defined on the domain \( \hat{\Omega}_j^k \subset \subset \Omega \setminus D_j \). By denoting \( M^h(\hat{\Omega}_j^k) = M^h(\hat{\Omega}_j^k) \cap H^1_0(\hat{\Omega}_j^k) \), we clearly have

\[
B_{\hat{\Omega}_j^k}([\hat{u}_H^j, \hat{p}_H^j], [v_h, q_h]) = 0 \quad \forall [v_h, q_h] \in X^0_h(\hat{\Omega}_j^k) \times M_h^h(\hat{\Omega}_j^k).
\]  

(4.9)

For any \( \theta \in R \), since \( v_h \in X^h_0(\hat{\Omega}_j^k) \), we have

\[
d_{\hat{\Omega}_j^k}(\theta, v_h) = \int_{\hat{\Omega}_j^k} \theta \nabla \cdot v_h \, dx = \theta \int_{\hat{\Omega}_j^k} \nabla \cdot v_h \, dx = \theta \int_{\partial \hat{\Omega}_j^k} v_h \cdot n \, ds = 0.
\]
By the definition of $B$, we know

$$0 = B_{\tilde{\Omega}_j^k}([\tilde{u}_j^{i}, \tilde{p}_j^{i}]^T, [v_h, q_h]) = B_{\tilde{\Omega}_j^k}([\tilde{u}_j^{i}, \tilde{p}_j^{i}], [v_h, q_h]) - d_{\tilde{\Omega}_j^k}(\theta, v_h) = B_{\tilde{\Omega}_j^k}([\tilde{u}_j^{i}, \tilde{p}_j^{i} - \theta], [v_h, q_h]).$$

Then, we can split it into two parts since $\tilde{\Omega}_j^k = \tilde{\Omega}_j^{k-1} \cup \Omega_j^k$

$$0 = B_{\tilde{\Omega}_j^k}([\tilde{u}_j^{i}, \tilde{p}_j^{i} - \theta], [v_h, q_h]) = B_{\tilde{\Omega}_j^{k-1}}([\tilde{u}_j^{i}, \tilde{p}_j^{i} - \theta], [v_h, q_h]) + B_{\Omega_j^k}([\tilde{u}_j^{i}, \tilde{p}_j^{i} - \theta], [v_h, q_h]),$$

therefore, by simply using

$$\tilde{v}_h^k = v_h|_{\tilde{\Omega}_j^k}, \tilde{v}_h^{k-1} = v_h|_{\tilde{\Omega}_j^{k-1}}, \quad v_h = v_h|_{\Omega_j^k},$$

we have for any $\theta \in R$

$$B_{\tilde{\Omega}_j^{k-1}}([\tilde{u}_j^{i}, \tilde{p}_j^{i} - \theta], [\tilde{v}_h^{k-1}, \tilde{q}_h^{k-1}]) = -B_{\Omega_j^k}([\tilde{u}_j^{i}, \tilde{p}_j^{i} - \theta], [v_h^k, q_h^k]).$$

(4.10)

Step 3. We firstly recall the following classical result (see for instance (2.11) in [24]):

$$\|q\|_{0, \Omega} = \inf_{c \in R} \|q + c\|_{0, \Omega} \quad \forall q \in L_0^2(\Omega).$$

Then, for convenience of expression, we denote by $X_{h}^b(\Omega_j^k) = X_0^b(\Omega \setminus \Omega_j^k)|_{\Omega_j^k}$, $M_h^b(\Omega_j^k) = M_0^b(\Omega \setminus \Omega_j^k)|_{\Omega_j^k}$, and introduce

$$\tilde{\theta}_{j, q_h}^k = \frac{1}{|\Omega_j^k|} \int_{\tilde{\Omega}_j^k} q_h, \quad \tilde{\theta}_{j, q_h}^k = \frac{1}{|\Omega_j^k|} \int_{\Omega_j^k} q_h.$$

Since $X_{h}^b(\Omega_j^k) \subset X_{k}^b(\Omega_j^k)$, $M_h^b(\Omega_j^k) \subset M_{k}^b(\tilde{\Omega}_j^k)$ and $X_0^b(\tilde{\Omega}_j^k) \times M_0^b(\tilde{\Omega}_j^k) / R$ is a stable finite element pair, we know $\Sigma_j^k = X_{h}^b(\Omega_j^k) \times M_h^b(\Omega_j^k) / R$ is also a stable finite element pair, that is

$$\sup_{\tilde{\varphi}_h^k \in X_{h}^b(\Omega_j^k)} \frac{\|\nabla \tilde{\varphi}_h^k\|_{\tilde{\Omega}_j^k}}{\|\tilde{\varphi}_h^k\|_{\tilde{\Omega}_j^k}} \succeq \inf_{\theta \in R} \|\tilde{\varphi}_h^k - \theta\|_{\tilde{\Omega}_j^k} = \|\tilde{\varphi}_h^k - \tilde{\theta}_{j, q_h}^k\|_{\tilde{\Omega}_j^k}.$$

We define a smooth function $\phi \in C^\infty(\tilde{\Omega}_j^k)$ such that $\phi|_{\tilde{\Omega}_j^{k-1}} = 0$ and

$$\text{supp} \phi = \tilde{\Omega}_j^k, \quad \phi(x) = 1 \forall x \in \tilde{\Omega}_j^{k-1}, \quad 0 \leq \phi \leq 1, \text{and } \phi(x) \lesssim C.$$

In the following, we still use $I_k$ to denote the scalar valued Lagrange finite element interpolation, which share the same property in A1. Firstly, we can arrive at

$$\|\nabla \tilde{u}_j^i\|_{\tilde{\Omega}_j^k} + \|\tilde{p}_j^i - \tilde{\theta}_{j, \tilde{p}_j^i}\|_{\tilde{\Omega}_j^k} = \|\nabla \tilde{u}_j^i\|_{\tilde{\Omega}_j^k} + \|\nabla \tilde{u}_j^i\|_{\tilde{\Omega}_j^k} + \inf_{\phi \in R} \|\tilde{p}_j^i - \theta\|_{\tilde{\Omega}_j^k}$$

$$\lesssim \sup_{[\tilde{v}_j^i, \tilde{q}_j^i] \in \Sigma_j^k} \frac{B_{\tilde{\Omega}_j^k}([\tilde{u}_j^i, \tilde{p}_j^i - \theta_{j, \tilde{p}_j^i}], [\tilde{v}_j^i, \tilde{q}_j^i])}{\|\nabla \tilde{v}_j^i\|_{\tilde{\Omega}_j^k} + \|\tilde{q}_j^i\|_{\tilde{\Omega}_j^k}}$$

$$\lesssim \sup_{[\tilde{v}_j^i, \tilde{q}_j^i] \in \Sigma_j^k} \frac{B_{\tilde{\Omega}_j^{k-1}}([\tilde{u}_j^i, \tilde{p}_j^i - \theta_{j, \tilde{p}_j^i}], [\tilde{v}_j^{k-1}, \tilde{q}_j^{k-1}]) + B_{\Omega_j^k}([\tilde{u}_j^i, \tilde{p}_j^i - \theta_{j, \tilde{p}_j^i}], [v_h^k, q_h^k])}{\|\nabla \tilde{v}_j^i\|_{\tilde{\Omega}_j^k} + \|\tilde{q}_j^i\|_{\tilde{\Omega}_j^k}}$$

$$= \sup_{[\tilde{v}_j^i, \tilde{q}_j^i] \in \Sigma_j^k} \frac{-B_{\Omega_j^k}([\tilde{u}_j^i, \tilde{p}_j^i - \theta_{j, \tilde{p}_j^i}], [I_h(\phi v_h^k), I_h(\phi q_h^k)]) + B_{\Omega_j^k}([\tilde{u}_j^i, \tilde{p}_j^i - \theta_{j, \tilde{p}_j^i}], [v_h^k, q_h^k])}{\|\nabla \tilde{v}_j^i\|_{\tilde{\Omega}_j^k} + \|\tilde{q}_j^i\|_{\tilde{\Omega}_j^k}}$$

$$\lesssim \sup_{[\tilde{v}_j^i, \tilde{q}_j^i] \in \Sigma_j^k} \frac{-B_{\Omega_j^k}([\tilde{u}_j^i, \tilde{p}_j^i - \theta_{j, \tilde{p}_j^i}], [I_h(\phi v_h^k), I_h(\phi q_h^k)])}{\|\nabla \tilde{v}_j^i\|_{\tilde{\Omega}_j^k} + \|\tilde{q}_j^i\|_{\tilde{\Omega}_j^k}} + \sup_{[\tilde{v}_j^i, \tilde{q}_j^i] \in \Sigma_j^k} \frac{B_{\Omega_j^k}([\tilde{u}_j^i, \tilde{p}_j^i - \theta_{j, \tilde{p}_j^i}], [v_h^k, q_h^k])}{\|\nabla \tilde{v}_j^i\|_{\tilde{\Omega}_j^k} + \|\tilde{q}_j^i\|_{\tilde{\Omega}_j^k}}$$

$$:= I_1 + I_2.$$
In the following, we will carry out further estimates for the two items.

\[ I_1 \lesssim \sup_{q_k^j \in \mathcal{M}_h^k(\hat{\Omega}_j^k)} \frac{[d_{h,j}^k(I_h(\phi v^k_h))]}{\|\nabla \hat{v}_h^k\|_{\hat{\Omega}_j^k}} + \sup_{q_k^j \in \mathcal{M}_h^k(\hat{\Omega}_j^k)} \frac{[d_{h,j}^k(I_h(\phi v^k_h), \tilde{u}_H^j)]}{\|\tilde{q}_h^k\|_{\hat{\Omega}_j^k}} \]
\[ \lesssim \sup_{q_k^j \in \mathcal{M}_h^k(\hat{\Omega}_j^k)} \frac{\|\nabla I_h(\phi v^k_h)\|_{\Omega_j^k}}{\|\nabla \hat{v}_h^k\|_{\hat{\Omega}_j^k}} + \sup_{q_k^j \in \mathcal{M}_h^k(\hat{\Omega}_j^k)} \frac{\|\nabla \hat{u}_H^j\|_{\Omega_j^k}}{\|\tilde{q}_h^k\|_{\hat{\Omega}_j^k}} \]
\[ \lesssim \sup_{q_k^j \in \mathcal{M}_h^k(\hat{\Omega}_j^k)} \frac{\|\nabla \hat{u}_H^j\|_{\Omega_j^k}}{\|\tilde{q}_h^k\|_{\hat{\Omega}_j^k}} + \sup_{q_k^j \in \mathcal{M}_h^k(\hat{\Omega}_j^k)} \frac{\|\tilde{q}_h^k\|_{\hat{\Omega}_j^k}}{\|\tilde{q}_h^k\|_{\hat{\Omega}_j^k}}. \]

and

\[ I_2 \lesssim \sup_{q_k^j \in \mathcal{M}_h^k(\Omega_j^k \setminus \hat{\Omega}_j^k)} \frac{[a_{\Omega_j^k}(u_H^j, v^k_H) + b_{\Omega_j^k}(\tilde{p}_H^j - \theta^k, \nabla \hat{v}_h^k)]}{\|\nabla \hat{v}_h^k\|_{\hat{\Omega}_j^k}} + \sup_{q_k^j \in \mathcal{M}_h^k(\hat{\Omega}_j^k)} \frac{[d_{h,j}^k(q_k^j, \tilde{u}_H^j)]}{\|\tilde{q}_h^k\|_{\hat{\Omega}_j^k}} \]
\[ \lesssim \frac{h^{-1}\|u^k_H\|_{\Omega_j^k}}{\|\nabla \hat{u}_H^j\|_{\Omega_j^k}} + \frac{h^{-1}\|\tilde{p}_H^j - \theta^k\|_{\Omega_j^k}}{\|\nabla \hat{v}_h^k\|_{\hat{\Omega}_j^k}} \]
\[ \lesssim \frac{h^{-1}\|\tilde{p}_H^j - \theta^k\|_{\Omega_j^k}}{\|\nabla \hat{v}_h^k\|_{\hat{\Omega}_j^k}}. \]

We can choose \( D = \Omega_j^k \setminus \hat{\Omega}_j^k \), \( D' = \Omega_j^k \setminus \hat{\Omega}_j^k \) in Lemma 4.3. Noting that \( \text{diam}(D) \approx H \), \( \text{dist}(\partial D, \partial D') \approx h \) by the definitions of \( \hat{\Omega}_j^k \) and \( \Omega_j^k \), then we can derive by virtue of (A2) and Lemma 4.3

\[ h^{-1}\|u^k_H\|_{\Omega_j^k} \lesssim h^{-\frac{1}{2}} H^\frac{d+1}{4} \beta_d^\frac{1}{2}(H) \|\nabla \hat{u}_H^j\|_{\Omega_j^k} \|\nabla \hat{v}_h^k\|_{\hat{\Omega}_j^k}, \]
\[ h^{-1}\|\tilde{p}_H^j - \theta^k\|_{\Omega_j^k} \lesssim h^{-\frac{1}{2}} H^\frac{d+1}{4} \beta_d^\frac{1}{2}(H) \|\tilde{q}_h^k\|_{\hat{\Omega}_j^k} \|\nabla \hat{v}_h^k\|_{\hat{\Omega}_j^k}. \]

From \( q_k^j \in \mathcal{M}_h^k(\hat{\Omega}_j^k) \), Lemma 4.3 and the inverse inequality, we also have

\[ \|q_k^j\|_{\hat{\Omega}_j^k} \|\nabla \hat{u}_H^j\|_{\Omega_j^k} \lesssim h^{\frac{1}{2}} H^\frac{d+1}{4} \beta_d^\frac{1}{2}(H) \|\nabla \hat{u}_H^j\|_{\Omega_j^k} \]
\[ \lesssim \frac{h^{-1}\|\tilde{p}_H^j - \theta^k\|_{\Omega_j^k} \|\nabla \hat{v}_h^k\|_{\hat{\Omega}_j^k}}{\|\tilde{q}_h^k\|_{\hat{\Omega}_j^k}} \|\nabla \hat{v}_h^k\|_{\hat{\Omega}_j^k}. \]

Combining the estimates above leads to

\[ \|\nabla \hat{u}_H^j\|_{\Omega_j^k} + \inf_{\theta \in R} \|\tilde{p}_H^j - \theta\|_{\Omega_j^k} = \|\nabla \hat{u}_H^j\|_{\Omega_j^k} + \|\tilde{p}_H^j - \theta^k\|_{\Omega_j^k} \]
\[ \lesssim h^{-\frac{1}{2}} H^\frac{d+1}{4} \beta_d^\frac{1}{2}(H) (\|\nabla \hat{u}_H^j\|_{\Omega_j^k} + \|\tilde{p}_H^j - \theta^k\|_{\Omega_j^k}). \]

Therefore we have

\[ \|\nabla \hat{u}_H^j\|_{\Omega_j^k}^2 + \|\tilde{p}_H^j - \theta^k\|_{\Omega_j^k}^2 \lesssim h^{-1} H^{d-1} \beta_d(H) (\|\nabla \hat{u}_H^j\|_{\Omega_j^k}^2 + \|\tilde{p}_H^j - \theta^k\|_{\Omega_j^k}^2). \]
It is obvious that
\[
\|\nabla \hat{u}_H^j\|^2_{\Omega_j^{2M}} + \|\hat{P}_H^j - \hat{\theta}^{k-1}_{j,\hat{P}_H}\|^2_{\Omega_j^{2M-1}} \lesssim h^{-1}H^{d-1} \beta_d(H) (\|\nabla \hat{u}_H^j\|^2_{\Omega_j^{2k}} + \|\hat{P}_H^j - \hat{\theta}^k_{j,\hat{P}_H}\|^2_{\Omega_j^{2k-1}}),
\]
or equivalently, there exists a mesh independent constant \(c > 0\) so that
\[
\|\nabla \hat{u}_H^j\|^2_{\Omega_j^{2M}} + \|\hat{P}_H^j - \theta^k_{j,\hat{P}_H}\|^2_{\Omega_j^{2M}} \geq c h H^{1-d} \beta^{-1}_d(H) (\|\nabla \hat{u}_H^j\|^2_{\Omega_j^{2M-1}} + \|\hat{P}_H^j - \theta^{k-1}_{j,\hat{P}_H}\|^2_{\Omega_j^{2M-1}}). \tag{4.11}
\]

**Step 4.** By using the above inequality successively, we can deduce that
\[
\|\nabla \hat{u}_H^j\|^2_{\Omega_j^{2M}} + \|\hat{P}_H^j\|^2_{\Omega_j^{2M}} \geq (1 + c h H^{1-d} \beta^{-1}_d(H)) (\|\nabla \hat{u}_H^j\|^2_{\Omega_j^{2M-1}} + \|\hat{P}_H^j - \theta^{k-1}_{j,\hat{P}_H}\|^2_{\Omega_j^{2M-1}}) \geq \cdots \geq (1 + c h H^{1-d} \beta^{-1}_d(H)) M (\|\nabla \hat{u}_H^j\|^2_{\Omega_j} + \|\hat{P}_H^j\|^2_{\Omega_j}) = (1 + c h H^{1-d} \beta^{-1}_d(H)) M (\|\nabla \hat{u}_H^j\|^2_{\Omega_j} + \|\hat{P}_H^j\|^2_{\Omega_j}).
\]

Noting that
\[
\|\nabla \hat{u}_H^j\|^2_{\Omega_j} + \|\hat{P}_H^j\|^2_{\Omega_j} = \|\nabla \hat{u}_H^j\|^2_{\Omega_j} + \|\hat{P}_H\|^2_{\Omega_j} \leq \|\nabla \hat{u}_H^j\|^2_{\Omega_j} + \|\hat{P}_H\|^2_{\Omega_j},
\]
we can get
\[
\|\nabla \hat{u}_H^j\|^2_{\Omega_j} + \|\hat{P}_H\|^2_{\Omega_j} \leq (1 + c h H^{1-d} \beta^{-1}_d(H))^{-M} (\|\nabla \hat{u}_H^j\|^2_{\Omega_j} + \|\hat{P}_H\|^2_{\Omega_j}).
\]

Finally, due to (4.7), we can arrive at
\[
\|\nabla e_{\Omega,H}^j\|^2_{\Omega_j} + \|e_{p,H}^j\|^2_{\Omega_j} \lesssim (1 + c h H^{1-d} \beta^{-1}_d(H))^{-M} (\|\nabla \hat{u}_H^j\|^2_{\Omega_j} + \|\hat{P}_H\|^2_{\Omega_j}).
\]

Since \(M \approx \frac{h}{\lambda}\), for small \(H\), simple calculation leads to the result of this lemma. \(\Box\)

**Remark 4.1.** For small \(H\), due to a basic inequality \(\lambda/2 \leq \ln(1 + \lambda) \leq \lambda\) for any \(0 \leq \lambda \leq 1\), we have
\[
(1 + c h H^{1-d} \beta^{-1}_d(H))^{-M} \approx \exp (-M \ln(1 + c h H^{1-d} \beta^{-1}_d(H))) \approx \exp (-M \ln(c h H^{1-d} \beta^{-1}_d(H))) \approx \exp (\alpha_d \ln H) = H^{\alpha_d},
\]
with
\[
\alpha_d = \frac{c M h H^{1-d} \beta^{-1}_d(H)}{|\ln H|} = \begin{cases} 
\frac{(M \frac{h}{\lambda})}{|\ln H|} \approx \frac{c}{|\ln H|}, & d = 2, \\
(M \frac{h}{\lambda}) \approx \frac{c}{|\ln H|}, & d = 3.
\end{cases}
\]
Since \(\beta_d(H) = H^{-1}\) when \(d = 3\) and \(\beta_d(H) = |\ln H|\) when \(d = 2\) (defined in Lemma 4.3), we can easily check that \(c h H^{1-d} \beta^{-1}_d(H) \leq 1\) holds when \(h \leq H/c\) for \(d = 3\) or \(h \leq H|\ln H|/c\) for \(d = 2\), which is a very trivial condition.

Similar with the estimate of \(\|\nabla e_{\Omega,H}^j\|_\Omega + \|e_{p,H}^j\|_\Omega\) in Lemma 4.4, we can derive an estimate for the approximate ”local” error of the coarse mesh Galerkin approximation, say \(\|\nabla \hat{u}_H^j\|_\Omega + \|\hat{P}_H\|_\Omega\) in the following.
Lemma 4.5. Under the assumptions of A1, A2 and A3, we have for \( j = 1, 2, \cdots, N \)
\[
\|\nabla \hat{u}_H^j\|_\Omega + \|\hat{p}_H^j\|_\Omega \lesssim \|\nabla (u - u_H)\|_{D_j} + \|p - p_H\|_{D_j}. \tag{4.12}
\]

Proof. First we rewrite the problem \((4.4)\) as
\[
B([\hat{u}_H^j, \hat{p}_H^j], [v_h, q_h]) = (f, \phi_j v_h) - a(u_H, \phi_j v_h) \\
- d(p_H, \phi_j v_h) + d(\phi_j q_h, u_H), \quad \forall [v_h, q_h] \in X_0^h \times Q_h,
\]
or equivalently,
\[
B([\hat{u}_H^j, \hat{p}_H^j], [v_h, q_h]) = a(u - u_H, \phi_j v_h) + d(p - p_H, \phi_j v_h) - d(\phi_j q_h, u - u_H).
\]

Since \( X_0^h \times Q_h \) is a stable pair of the finite element spaces, we have
\[
\|\nabla \hat{u}_H^j\|_\Omega + \|\hat{p}_H^j\|_\Omega \lesssim \sup_{[v_h, q_h] \in X_0^h \times Q_h} \frac{B([\hat{u}_H^j, \hat{p}_H^j], [v_h, q_h])}{\|\nabla v_h\|_\Omega + \|q_h\|_\Omega} \\
= \sup_{[v_h, q_h] \in X_0^h \times Q_h} \frac{a(u - u_H, \phi_j v_h) + d(p - p_H, \phi_j v_h) - d(\phi_j q_h, u - u_H)}{\|\nabla v_h\|_\Omega + \|q_h\|_\Omega} \\
\lesssim \sup_{v_h \in X_0^h} \frac{a(u - u_H, \phi_j v_h)}{\|\nabla v_h\|_\Omega} + \sup_{v_h \in X_0^h} \frac{d(p - p_H, \phi_j v_h)}{\|\nabla v_h\|_\Omega} + \sup_{q_h \in Q_h} \frac{d(\phi_j q_h, u - u_H)}{\|q_h\|_\Omega}.
\]

Note that \( I_H \) is the Lagrange finite element interpolation operator of \( X \) onto \( X^H \) and \( \hat{I}_H = I - I_H \), especially, \( \supp \phi_j = D_j \), and \( a(u - u_H, \phi_j v_h) + d(p - p_H, \phi_j v_h) = a(u - u_H, \hat{I}_H \phi_j v_h) + d(p - p_H, \hat{I}_H \phi_j v_h) \), it’s easy to derive
\[
a(u - u_H, \hat{I}_H \phi_j v_h) = a(u - u_H, \hat{I}_H \phi_j I_H v_h) + \hat{I}_H \phi_j \hat{I}_H v_h \]
\[
\lesssim \|\nabla (u - u_H)\|_{D_j} (\|\hat{I}_H \phi_j I_H v_h\|_{D_j} + \|\hat{I}_H \phi_j \hat{I}_H v_h\|_{D_j}),
\]
\[
d(p - p_H, \hat{I}_H \phi_j v_h) = d(p - p_H, \hat{I}_H \phi_j I_H v_h) + \hat{I}_H \phi_j \hat{I}_H v_h \]
\[
\lesssim \|p - p_H\|_{D_j} (\|\hat{I}_H \phi_j I_H v_h\|_{D_j} + \|\hat{I}_H \phi_j \hat{I}_H v_h\|_{D_j}),
\]
\[
d(\phi_j q_h, u - u_H) \lesssim \|q_h\|_{D_j} \|\nabla (u - u_H)\|_{D_j}.
\]

In view of A1, A2 and noting that \( \phi_j \) is a linear function on each mesh of \( \tau_1^H \) with \( |D \phi_j| \lesssim H^{-1} \),
\[
\|\nabla \hat{I}_H \phi_j I_H v_h\|_{D_j} = \left( \sum_{\tau_1^H \subset D_j} \|\nabla \hat{I}_H \phi_j I_H v_h\|_{\tau_1^H}^2 \right)^{\frac{1}{2}} \lesssim H \left( \sum_{\tau_1^H \subset D_j} \|D^2 (\phi_j I_H v_h)\|_{\tau_1^H}^2 \right)^{\frac{1}{2}}
\]
\[
\lesssim H \left( \sum_{\tau_1^H \subset D_j} \|D^2 (I_H v_h)\|_{\tau_1^H}^2 \right)^{\frac{1}{2}} + \|D \phi_j D (I_H v_h)\|_{\tau_1^H}^2 \lesssim H \left( \sum_{\tau_1^H \subset D_j} \|D^2 (I_H v_h)\|_{\tau_1^H}^2 \right)^{\frac{1}{2}} + \|D (I_H v_h)\|_{\tau_1^H}^2 \lesssim \|\nabla v_h\|_{D_j},
\]
\[
\|\nabla \hat{I}_H \phi_j \hat{I}_H v_h\|_{D_j} \lesssim \|D (\phi_j \hat{I}_H v_h)\|_{D_j} \lesssim \|D \phi_j \hat{I}_H v_h\|_{D_j} + \|\phi_j D (\hat{I}_H v_h)\|_{D_j} \lesssim \|\nabla v_h\|_{D_j}.
\]

Combining these estimates immediately yield
\[
\|\nabla \hat{u}_H^j\|_\Omega + \|\hat{p}_H^j\|_\Omega \lesssim \|\nabla (u - u_H)\|_{D_j} + \|p - p_H\|_{D_j}.
\]

The proof is complete. \( \square \)

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Now we can state the main theorem concerning the approximate accuracy of solutions derived by the scheme \([3.5]-[3.8]\) as follows.

**Theorem 4.6.** Assume that \(A1, A2, A3\) and \([2.4]\) hold and and \([u, p] \in H^{r+1}(\Omega) \times H^r(\Omega)\). Then we have

\[
\|\nabla(u_h - u^h)\|_\Omega + \|p_h - p^h\|_\Omega \lesssim H^{\alpha_d}(\|\nabla(u_h - u_H)\|_\Omega + \|p_h - p_H\|_\Omega),
\]

(4.13)

\[
\|u_h - u^h\|_\Omega \lesssim H(\|\nabla(u_h - u_H)\|_\Omega + \|p_h - p_H\|_\Omega),
\]

(4.14)

\[
\|\nabla(u - u^h)\|_\Omega + \|p - p^h\|_\Omega \lesssim h^r + H^{\alpha_d}(\|\nabla(u - u_H)\|_\Omega + \|p - p_H\|_\Omega),
\]

(4.15)

\[
\|u - u^h\|_\Omega \lesssim h^{r+1} + H(\|\nabla(u - u_H)\|_\Omega + \|p - p_H\|_\Omega),
\]

(4.16)

where \(\alpha_d > 0\) is defined in Lemma \([4.4]\).

**Proof.** We know from \([3.7]\) that

\[
\|\nabla E_H^u\|_\Omega + \|E_H^p\|_\Omega \lesssim \|\nabla(u_h - u_{H,h})\|_\Omega + \|p_h - p_{H,h}\|_\Omega.
\]

Since

\[
[u - u^h, p - p^h] = [u - u_h, p - p_h] + [u_h - u_{H,h}, p_h - p_{H,h}] + [E_H^u, E_H^p],
\]

and

\[
\|\nabla(u - u_h)\|_\Omega + \|p - p_h\|_\Omega \lesssim h^r,
\]

the conclusions \([4.13]\) and \([4.15]\) follows immediately due to Lemma \([4.4]\) and Lemma \([4.5]\) and the triangle inequality.

To estimate the \(L^2\)-norms of the velocity, we firstly introduce a Stokes projection \(P_H = [P_H^u, P_H^p]\) defined from \(X_0 \times Q\) onto \(X_0^H \times Q^H\): for given \([w, r] \in X_0 \times Q\), find \([P_H^u w, P_H^p r] \in X_0^H \times Q^H\) such that

\[
B([v_H, q_H], [w - P_H^u w, r - P_H^p r]) = 0, \quad \forall [v_H, q_H] \in X_0^H \times Q^H.
\]

In view of the definitions of \(\hat{u}_{H,h}\) and \(\hat{p}_{H,h}\), and \(\sum_{j=1}^N \phi_j = 1\) in \(\Omega\), adding up all the equations of \([4.1]\) over \(j\) with \(\mu = 0\) gives

\[
B([\hat{u}_{H,h}, \hat{p}_{H,h}], [v_h, q_h]) = (f, v_h) - B([u_H, p_H], [v_h, q_h])
\]

(4.17)

\[
- \sum_{j=1}^N \int_{\Gamma_j} \xi_j^3 v_h ds, \quad \forall [v_h, q_h] \in X_0^h \times Q^h.
\]

Since \([u_{H,h}, p_{H,h}] = [u_H + \hat{u}_{H,h}, p_H + \hat{p}_{H,h}]\), then we have

\[
B([u_{H,h}, p_{H,h}], [v_h, q_h]) = (f, v_h) - \sum_{j=1}^N \int_{\Gamma_j} \xi_j^3 v_h ds, \quad \forall [v_h, q_h] \in X_0^h \times Q^h.
\]

On the another aspect, by the definition of the Stokes projection \(P_H\) above, we can rewrite \([3.7]\) associated with the coarse mesh correction as

\[
B([E_H^u, E_H^p], [v_h, q_h]) = (f, P_H^u v_h) - B([u_{H,h}, p_{H,h}], [P_H^u v_h, P_H^p q_h]).
\]
By using the triangle inequality again, we prove which immediately yields

\[ B([u^h, p^h], [v, q]) = (f, v) + (f, P^h_H v) - B([u^h_H, p^h_H], [P^0_H v, P^p_H q]) \]

\[ - \sum_{j=1}^{N} \int_{\Gamma_j} \xi^j v h ds, \quad \forall [v, q] \in X_0^h \times Q^h. \]

Finally, we obtain \( \forall [v, q] \in X_0^h \times Q^h \)

\[ B([u^h, p^h], [v, q]) = (f, v) - \sum_{j=1}^{N} \int_{\Gamma_j} \xi^j (I - P^n_H) v h ds, \]

and \( \forall [v, q] \in X_0^h \times Q^h \)

\[ B([u^h - u^h_H, p^h - P^n_H], [v, q]) = \sum_{j=1}^{N} \int_{\Gamma_j} \xi^j (I - P^n_H) v h ds. \] (4.18)

From this error equation about \( [u^h - u^h_H, p^h - P^n_H] \), and by virtue of Lemma 4.1, 4.4 and 4.5, we can easily get

\[ \|\nabla (u^h - u^h_H)\| + \|p^h - P^n_H\| \leq H^{nu}(\|\nabla (u - u^H)\| + \|p - P_H\|). \]

Then we can derive the first result by using the triangle inequality.

Next, we also use the Aubin-Nitsche duality argument to show \( L^2 \)-error estimate of the velocity. By A3, for \( u^h - u^h_H \in L^2(\Omega) \), there exists \( [w, r] \in H^2(\Omega) \times H^1(\Omega) \) satisfing

\[ B([v, q], [w, r]) = (u^h - u^h_H, v), \quad \forall [v, q] \in X_0(\Omega) \times Q, \]

and

\[ \|w\|_{2, \Omega} + \|r\|_{1, \Omega} \leq \|u^h - u^h_H\|_{\Omega}. \]

If taking \( [v, q] = [u^h - u^h_H, p^h - P^n_H] \) here and in view of (4.18), we have

\[ \|u^h - u^h_H\|_{\Omega}^2 = B([u^h - u^h_H, p^h - P^n_H], [w, r]) \]

\[ = B([u^h - u^h_H, p^h - P^n_H], [w - P^0_H w, r - P^p_H r]). \]

Then we can proceeds the estimate as

\[ \|u^h - u^h_H\|_{\Omega}^2 \leq (\|\nabla (u^h - u^h_H)\| + \|p^h - P^n_H\|)\|\nabla (I - P^n_H) w\|_{\Omega} \]

\[ + \|\nabla (u^h - u^h_H)\|_{\Omega}(\|I - P^n_H\| r)_{\Omega} \]

\[ \leq H(\|\nabla (u^h - u^h_H)\| + \|p^h - P^n_H\|)(\|w\|_{2, \Omega} + \|r\|_{1, \Omega}) \]

\[ \leq H(\|\nabla (u^h - u^h_H)\| + \|p^h - P^n_H\|)\|u^h - u^h_H\|_{\Omega}, \]

which immediately yields

\[ \|u^h - u^h_H\|_{\Omega} \leq H(\|\nabla (u^h - u^h_H)\| + \|p^h - P^n_H\|). \]

By using the triangle inequality again, we prove \( L^2 \)-error estimate for the velocity. □

**Remark 4.2.** By the results in Theorem 4.6 above, especially (4.13) and (4.14), we can clearly improve the convergence orders by introducing a two-grid iteration. Actually, we can verify that all the lemmas and theorems above still hold when replacing \([u^h_H, p^h_H]\) by \([u^h_H, p^h_H]\).
This suggests us to introducing the following two-grid iterative scheme with a iteration number of $K$.

$$K = [\alpha^{-1} + 0.5] = \begin{cases} \mathcal{O}(\ln H^2), & d = 2, \\ \mathcal{O}(\ln H), & d = 3. \end{cases} \quad (4.19)$$

Local and parallel two-grid iterative scheme:

**Step 0.** Setting $k = 0$; Deriving $[\mathbf{u}_H, p_H]$ by solving (2.3) and denoting by $[\mathbf{u}^{0,h}, p^{0,h}] = [\mathbf{u}_H, p_H]$;

**Step 1.** For $k \geq 0$, solving the equations (3.5) with $[\mathbf{u}_H, p_H] = [\mathbf{u}^{k,h}, p^{k,h}]$ to get $\{[\mathbf{u}^{j+1,h}, p^{j+1,h}]\}_{j=1}^N$ for each $j$, which are denoted as $\{[\mathbf{u}_H, p_H] \}_{j=1}^N$. Then we construct $[\mathbf{u}^{k+1,h}, p^{k+1,h}]$ by formula (3.6);

**Step 2.** Deriving $[\mathbf{E}^{k+1}_u, E^{k+1}_p]$ by solving (3.7) with $[\mathbf{u}_H, p_H] = [\mathbf{u}^{k+1,h}, p^{k+1,h}]$, and constructing

$$[\mathbf{u}^{k+1,h}, p^{k+1,h}] = [\mathbf{u}^{k+1,h}, p^{k+1,h}] + \mathbf{v}_H = [\mathbf{u}^{k+1,h}, p^{k+1,h}] + \mathbf{v}_H, \quad \mathbf{v}^{k+1,h} \in \mathbf{V}^h$$

**Step 3.** Checking whether $k + 1 > K$, if yes, terminating the iteration and deriving $[\mathbf{u}^h, p^h] = [\mathbf{u}^{k+1,h}, p^{k+1,h}]$; otherwise, letting $k := k + 1$ and turning to Step 1.

**Theorem 4.7.** Suppose $[\mathbf{u}, p] \in H^{r+1}(\Omega) \times H^r(\Omega)$, the final approximation $[\mathbf{u}_H, p_H]$ for the iterative scheme with $K$ defined by (4.19) has the following error bounds

$$\|\nabla (\mathbf{u} - \mathbf{u}_H)\|_\Omega + \|p - p_H\|_\Omega \lesssim H^r + H^{r+1}, \quad (4.20)$$

$$\|\mathbf{u} - \mathbf{u}_H\|_\Omega \lesssim H^{r+1} + H^{r+2}. \quad (4.21)$$

**Proof.** Note that, the Ritz-Projection of $[\mathbf{u}_H, p_H]$ onto the corresponding course grid finite element spaces is nothing but the standard Galerkin approximation, see (4.18), when selecting $[\mathbf{v}_H, q_H] = [\mathbf{v}_H, q_H]$, we have

$$B([\mathbf{u}_H - \mathbf{u}_H, p_H - p_H], [\mathbf{v}_H, q_H]) = 0,$$

then,

$$B([\mathbf{u}_H, p_H], [\mathbf{v}_H, q_H]) = B([\mathbf{u}_H, p_H], [\mathbf{v}_H, q_H]) = (f, v_H) = B([\mathbf{u}_H, p_H], [\mathbf{v}_H, q_H]).$$

Then, we can update $[\mathbf{u}^{0,h}, p^{0,h}]$ by $[\mathbf{u}^{1,h}, p^{1,h}]$, and execute the previous local and parallel two-grid iterative scheme (without Step 0) again to get $[\mathbf{u}^{2,h}, p^{2,h}]$, then follow (4.13) and (4.14) to derive the error estimates of the form, equivalently, there exists a mesh independent constant $c > 0$,

$$\|\nabla (\mathbf{u}_H - \mathbf{u}^{2,h}_H)\|_\Omega + \|p_H - p^{2,h}_H\|_\Omega \lesssim H^{2\alpha d}(\|\nabla (\mathbf{u}_H - \mathbf{u}^{1,h}_H)\|_\Omega + \|p_H - p^{1,h}_H\|_\Omega)$$

$$\lesssim H^{2\alpha d}(\|\nabla (\mathbf{u}_H - \mathbf{u}^{0,h}_H)\|_\Omega + \|p_H - p^{0,h}_H\|_\Omega),$$

$$\|\mathbf{u}_H - \mathbf{u}^{2,h}_H\|_\Omega \lesssim H(\|\nabla (\mathbf{u}_H - \mathbf{u}^{2,h}_H)\|_\Omega + \|p_H - p^{2,h}_H\|_\Omega).$$

Therefore, cyclically executing our two-grid iterative scheme with updating data by a iteration of $K$ times, we can finally derive

$$\|\nabla (\mathbf{u}_H - \mathbf{u}^{K,h}_H)\|_\Omega + \|p_H - p^{K,h}_H\|_\Omega \lesssim H^{K\alpha d}(\|\nabla (\mathbf{u}_H - \mathbf{u}^{0,h}_H)\|_\Omega + \|p_H - p^{0,h}_H\|_\Omega)$$

$$\lesssim H^{K\alpha d}(\|\nabla (\mathbf{u}_H - \mathbf{u}_H)\|_\Omega + \|p_H - p_H\|_\Omega),$$

$$\|\mathbf{u}_H - \mathbf{u}^{K,h}_H\|_\Omega \lesssim H(\|\nabla (\mathbf{u}_H - \mathbf{u}^{K,h}_H)\|_\Omega + \|p_H - p^{K,h}_H\|_\Omega).$$
By the choice of $K$ within (4.19) and the triangle inequality, since $[u_h^K, p_h^K] = [u_h^K, p_h^K]$ now, we finally get (4.20) and (4.21).

**Remark 4.3.** By the estimates in Theorem 4.7, we can clearly know that in order to get the optimal convergence orders of the velocity - pressure pair in the sense of $H^1 - L^2$-norms, or that of the velocity in $L^2$-norm, we should choose $H$ and $h$ such that

$$h \sim H^{\frac{r+1}{2}} \quad \text{or} \quad h \sim H^{\frac{r+2}{r+1}},$$

respectively.

5. **Numerical Tests.** In all the numerical experiments below, the algorithms are implemented using public domain finite element software Freefem++[27]. All simulations were performed on a Dawning parallel cluster composed of 64 nodes (each node consists of eight-core 2.0 GHz CPU, 8×2 GB DRAM, and all the nodes are connected via 20Gbps InfiniBand). The message-passing interface is supported by MPICH.

We introduce the following notations for convenience:

- SFEM means the standard finite element method.
- EPLP denotes the expandable local and parallel two-grid finite element iterative scheme.

Wall time covers the maximal CPU time among all processors used for EPLP, including the CPU time for solving the two global coarse grid problems and the parallel time for solving all the subproblems.

5.1. **Problem 1.** Firstly, to verify the theoretical results, we consider the following 2D numerical example (referred as Problem 1) with exact solution

$$u = (10x^2(x-1)^2y(y-1)(2y-1), -10x(x-1)(2x-1)y^2(y-1)^2),$$

$$p = 10(2x-1)(2y-1).$$

The domain is selected as the unit square $\Omega = [0, 1] \times [0, 1]$ with a uniform triangulation $T^H$. The Taylor-Hood finite element pairs are used in solving the Stokes equation.

According to Theorem 4.7 for EPLP, we know that the estimate has the form of

$$\|\nabla (u - u_H^h)\|_\Omega + \|p - p_H^h\|_\Omega = O(h^2 + H^3),$$

which suggests us to select $H$ and $h$ such that $h \approx H^2$ when verifying the optimal convergence orders for $H^1$-norm of velocity and $L^2$-norm of pressure.

To this end, for deriving the approximate $H^1$-error of velocity and $L^2$-error of pressure, we implement EPLP in parallel with coarse meshes of the sizes $H=1/16$, $1/25$, $1/36$, $1/49$, $1/64$. 

![Fig. 5.1. Structured mesh for Problem 1, $D_j$: local domain (red) and $\Omega_j$: expanded domain of $D_j$ with one mesh layer (both red and green).](image)
and corresponding fine meshes with the sizes \( h = 1/64, 1/125, 1/216, 1/343, 1/512 \), under the parallel environment of fixed 64 processors. Meanwhile, SFEM is executed for the same fine mesh on a single processor. The results for both methods are presented in Table 5.2, 5.1, respectively.

Table 5.1
The errors of SFEM for Problem 1.

| \( h \) | \( \| u - u_h \|_{1, \Omega} \) | \( \| p - p_h \|_{0, \Omega} \) | Order | \( \| p - p_h \|_{0, \Omega} \) | Order | CPU |
|-------|-----------------|-----------------|------|-----------------|------|------|
| 1/64  | 0.000205741     | 0.000630369     | 0    | 0.000165247     | 2    | 2.718|
| 1/125 | 5.396e-05       | 1.99927         | 2.718| 0.000165247     | 2    | 68.49|
| 1/216 | -               | -               | -    | -               | -    | -    |
| 1/343 | -               | -               | -    | -               | -    | -    |
| 1/512 | -               | -               | -    | -               | -    | -    |

Table 5.2
The errors of EPLP for Problem 1, \( h \cong H^\frac{3}{2} \), 64 processors.

| \( H \) | \( h \) | \( K \) | \( \| u - u_h^K \|_{1, 0, \Omega} \) | Order | \( \| p - p_h^K \|_{0, 0, \Omega} \) | Order | Wall time |
|-------|-------|-------|-----------------|------|-----------------|------|----------|
| \( 1/16 \) | 1/64  | 2     | 0.000299829     | 0    | 0.00077599      | 10.54| 1.00     |
| \( 1/25 \) | 1/125 | 3     | 5.44032e-05     | 2.718| 2.54961         | 230623| 91.51   |
| \( 1/36 \) | 1/216 | 3     | 1.81385e-05     | 2.718| 2.00815         | 200391| 465.08  |
| \( 1/49 \) | 1/343 | 3     | 7.19776e-06     | 2.718| 1.99862         | 199882| 2084.38 |
| \( 1/64 \) | 1/512 | 4     | 3.25815e-06     | 2.718| 1.97859         | 192844| 7892.11 |

Table 5.3
The \( L^2 \)-error of Velocity by EPLP for Problem 1, \( h \cong H^\frac{4}{3} \), 64 processors.

| \( H \) | \( h \) | \( K \) | \( \| u - u_h^K \|_{0, 0, \Omega} \) | Order | Wall time |
|-------|-------|-------|-----------------|------|----------|
| \( 1/32 \) | 1/96  | 3     | 1.0881e-07      | 0.00077599| 80.44   |
| \( 1/64 \) | 1/256 | 4     | 5.71659e-09     | 3.00382| 2032.04 |
| \( 1/96 \) | 1/384 | 4     | 1.74889e-09     | 2.92107| 10435.2 |

From Table 5.2, the optimal orders for the \( H^1 \)-error of velocity and \( L^2 \)-error of pressure by EPLP are observed, which can verify the theoretical result. Also the approximate accuracy by EPLP are comparable with those of SFEM under same fine meshes in Table 5.1. Noting that, in the present computational environment, SFEM fails for mesh with \( h = 1/216 \) or finer, meanwhile our EPLP still works well for much finer meshes including \( h = 1/512 \).

Similarly, by Theorem 4.7, we choose mesh pairs with a relation of \( h \cong H^\frac{4}{3} \) to check the order for \( L^2 \)-error of the velocity computed by our EPLP. The parameters and computational results are shown in Table 5.3, from which the optimal order of \( O(h^3) \) is verified. This also supports the result in the theoretical analysis.

Table 5.4
Wall time \( T(J) \) in seconds, speedup \( S_p \) and parallel efficiency \( E_p \) for Example 1, with \( H = 1/36 \).

| \( J \) | 2   | 4   | 8   | 16  | 32  | 64  |
|-------|-----|-----|-----|-----|-----|-----|
| \( T(J) \) | 9204.77 | 4684.47 | 2520.25 | 1392.31 | 747.51 | 465.08 |
| \( S_p = \frac{T(2)}{T(J)} \) | 1.00 | 1.96 | 3.65 | 6.61 | 12.31 | 19.79 |
| \( E_p = \frac{2\times T(2)}{J\times T(J)} \) | 1.00 | 0.98 | 0.91 | 0.83 | 0.77 | 0.62 |
The performance of a parallel algorithm in a homogeneous parallel environment is usually measured by speedup and parallel efficiency, commonly defined as
\[ S_p = \frac{T(J_1)}{T(J_2)}, \quad E_p = \frac{J_1 \times T(J_1)}{J_2 \times T(J_2)}, \]
where \( T(J_1) \) and \( T(J_2) \) \((J_1 \leq J_2)\) are wall times of the parallel program when using \( J_1 \) and \( J_2 \) processors, respectively.

Table 5.4 reports the wall time of EPLP in a parallel environment using processors of number \( J = 2, 4, 8, 16, 32, 64 \), and presents the corresponding speedup and parallel efficiency, which are computed by comparison with \( J_1 = 2 \). These results show good parallel performance of our EPLP.

![Fig. 5.2. A regular unstructured mesh for Problem 2, \( D_j \): a local domain (red) and \( \Omega_j \): an expanded domain of \( D_j \) (both red and green).](image)

### Table 5.5

The errors of EPLP for Problem 2 with Taylor-Hood elements, \( h \equiv H^2 \), with 64 processors.

| \( H \) | \( h \) | \( K \) | \( \| u - u^h \|_{1, \Omega} \) | Order | \( \| p - p^h \|_{0, \Omega} \) | Order | Wall time |
|---|---|---|---|---|---|---|---|
| 1/16 | 1/64 | 2 | 0.00237151 | | 9.76556e-05 | | 10.75 |
| 1/32 | 1/160 | 3 | 0.000415357 | 1.90130 | 1.51911e-05 | 2.03073 | 196.61 |
| 1/48 | 1/288 | 3 | 0.000137697 | 1.87837 | 4.88635e-06 | 1.92972 | 1137.06 |
| 1/64 | 1/448 | 4 | 4.91655e-05 | 2.33089 | 2.05054e-06 | 1.96532 | 6254.84 |

### Table 5.6

The errors of EPLP for Problem 2 with Mini-elements, \( h \equiv H^2 \), with 64 processors.

| \( H \) | \( h \) | \( K \) | \( \| u - u^h \|_{1, \Omega} \) | Order | \( \| p - p^h \|_{0, \Omega} \) | Order | Wall time |
|---|---|---|---|---|---|---|---|
| 1/8 | 1/64 | 2 | 0.151405 | | 0.0550405 | | 14.67 |
| 1/16 | 1/256 | 2 | 0.0361013 | 1.03415 | 0.0131228 | 1.03421 | 538.43 |
| 1/32 | 1/1024 | 3 | 0.00924041 | 0.983011 | 0.00212624 | 1.31285 | 52038.4 |

### 5.2. Problem 2

To further test our EPLP, we also consider another smooth problem (referred as Problem 2) with exact solution of the trigonometric form
\[
\begin{align*}
u &= (\sin(\pi x)^2 \sin(2\pi y) - \sin(2\pi x) \sin(\pi y)^2), \\
p &= \cos(\pi x) \cos(\pi y),
\end{align*}
\]
on the domain \( \Omega = [0, 1] \times [0, 1] \). The regular unstructured triangulations, and Taylor-Hood elements and Mini-elements are used for our EPLP. The corresponding approximate results are
listed in Table 5.5 and 5.6, respectively. From these two tables, one can observe that, for EPLP with K iterations, the $H^1$-error of velocity and $L^2$-error of pressure achieve the optimal orders, namely, $O(h^2)$ by Taylor-Hood elements, and $O(h)$ by Mini-elements, which support theoretical results in Theorem 4.7.

The errors of EPLP for Problem 3 with Taylor-Hood element pair, $h \equiv H^\frac{3}{2}$, with 64 processors.

| $H$ | $h$  | $K$ | $\|u - u^h\|_{1, \Omega}$ | Order | $\|p - p^h\|_{0, \Omega}$ | Order |
|-----|------|-----|--------------------------|-------|--------------------------|-------|
| 1/4 | 1/8  | 1   | 0.763                    | 1     | 0.0408459                | 0     |
| 1/8 | 1/24 | 1   | 0.104595                 | 1.8079| 0.00564176               | 1.80192|
| 1/16| 1/64 | 2   | 0.0154352               | 1.95084| 0.000769727              | 2.03085|

5.3. Problem 3. As the final experiment, we will test our EPLP using a three dimensional problem (referred as Problem 3) with $\nu = 0.05$ and exact solution

$$u = \left(\sin(\pi x)^2 \sin(2\pi y) \sin(2\pi z), -\sin(2\pi x) \sin(\pi y)^2 \sin(2\pi z), \sin(2\pi x) \sin(2\pi y)\right),$$

$$p = \cos(2\pi x) \cos(2\pi y) \cos(2\pi z).$$

The uniform triangulation and Taylor-Hood elements are used for $T^H$ with mesh size $H$ and the computational domain is chosen as the unit cube $\Omega = [0,1] \times [0,1] \times [0,1]$. The computational results are shown in Table 5.7, from which one can observe that, while $h$ deceases, by suitable iteration of $K$, both the $H^1$-error of velocity and $L^2$-error of pressure can reach the optimal orders of $O(h^2)$, which also support our theoretical analysis.

6. Conclusions. In this paper, we have designed an expandable local and parallel two-grid finite element iterative scheme based on superposition principle for the Stokes problem. The optimal convergence orders of the scheme are analyzed and obtained within suitable two-grid iterations while numerical tests in 2D and 3D are carried out to show the flexible and high efficiency of the scheme. The extension of the scheme to the time-dependent problems or nonlinear problems, e.g., Navier-Stokes equations, will be our further work.

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