Expected Number of Vertices of a Hypercube Slice

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Abstract. Given a random $k$-dimensional cross section of a hypercube, what is its expected number of vertices? We show that, for a suitable distribution of random slices, the answer is $2^k$, independent of the dimension of the hypercube.

To understand a high-dimensional object, a good starting point is to take a look at lower dimension cross sections which can be more easily visualized. For this to be effective, though, we have to have some point of comparison—i.e., an idea of what such cross sections look like for “typical” high dimensional objects. This note considers what cross sections “look like” for the case of a hypercube. In particular, we ask: What is the expected number of vertices for a random $k$-dimensional slice of an $n$-dimensional hypercube? We will show that, for a suitable distribution of slices, the answer is $2^k$, regardless of the dimension $n$ of the hypercube.

Throughout our discussion, we fix our hypercube to be $C := [-1, 1]^n$. A $k$-dimensional slice (i.e., cross section) of the cube is the intersection of $C$ with a $k$-dimensional affine subspace of $\mathbb{R}^n$ (henceforth referred to as a $k$-flat), and is characterized by the following two properties.

- An orientation, specified by $k$ unit vectors $n_1, \ldots, n_k \in S^{n-1}$.
- A translation, specified by some $\tau \in \mathbb{R}^n$, which may be taken WLOG to be perpendicular to the unit vectors $n_i$.

The slice is then given explicitly by $C \cap (\tau + \text{span}\{n_1, \ldots, n_k\})$.

There are various different ways to choose a random slice (i.e., distributions), but one of the most natural is the so-called distribution of isotropic random $k$-flats [4]. This comes from choosing the unit vectors $n_i$ from $S^{n-1}$ uniformly and independently and then choosing the translation $\tau$ uniformly from all those translations for which the resulting $k$-flat still intersects $C$. The result of this note holds for this distribution, and in fact holds much more generally. We will show that, for any fixed orientation, by choosing the translation $\tau$ uniformly from all translations for which the resulting flat still intersects the hypercube, the expected number of vertices of the resulting slice is $2^k$. As a corollary, for any distribution of orientations, so long as we then choose the translation uniformly, the expected number of vertices is $2^k$.

To put this in perspective, we remark that a particular 2-slice of a $n$-dimensional hypercube (for $n > 2$) can have anywhere from 3 to $2n$ vertices. An interpretation of this result for the case $k = 2$ is thus that there is a very small chance of finding a slice of a hypercube with very many vertices.

To compute the expected number of vertices, first fix the orientation of a $k$-slice, as given by (linearly independent) unit vectors $n_1, \ldots, n_k$. The translation $\tau$ may then be taken to be a vector in the $(n-k)$-dimensional subspace normal to $n_1, \ldots, n_k$, which we denote by $N$. The range of values $\tau$ may take so that the slice still intersects the hypercube $C$ is given by the projection $P_N(C)$ of $C$ onto $N$, where $P_N(\cdot)$ is the projection operator onto $N$.

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Figure 1. A typical 2-dimensional slice $S$ of a 3-cube. The vertices of the slice result from intersections with 1-dimensional faces (edges) of the cube, such as the one labeled $F$. $N$ is the 1-dimensional subspace normal to $S$.

A vertex results from the intersection of a $k$-flat with an $(n - k)$-dimensional face $F$ of $C$. For a given translation $\tau$, the resulting slice $(\tau + \text{span}\{n_1, \ldots, n_k\}) \cap C$ intersects a particular $(n - k)$-dimensional face $F$ of $C$ if and only if $\tau \in P_N(F)$. Since $\tau$ is assumed to be chosen uniformly within $P_N(C)$, the probability that the slice intersects $F$ is just the ratio of the $(n - k)$-volume of $P_N(F)$ to that of $P_N(C)$. We will need a formula for the volume of these regions, and to accomplish this we first introduce the concept of a zonotope.

Given arbitrary vectors $v_1, \ldots, v_m \in \mathbb{R}^n$, the zonotope $Z(v_1, \ldots, v_m)$ is defined as

$$Z(v_1, \ldots, v_m) := \{\lambda_1 v_1 + \cdots + \lambda_m v_m | \lambda_1, \ldots, \lambda_m \in [0, 1]\}.$$

Translations of such sets we also refer to as zonotopes. For a set of vectors $S = \{v_1, \ldots, v_m\}$ we also use the notation $Z(S)$ to refer to the zonotope $Z(v_1, \ldots, v_m)$. When the vectors $v_i$ are linearly independent, the zonotope $Z(v_1, \ldots, v_m)$ is just a parallelotope, and conversely any parallelotope is a zonotope, but in general a zonotope is a more complicated object (e.g., when $m > n$). We note that all of $C$, $F$, $P_N(C)$, and $P_N(F)$ are zonotopes. For $C$ and $F$ this is obvious, because they are both parallelotopes. For the other two, we see from the linearity of $P_N(\cdot)$ that the projection of a zonotope is again a zonotope, i.e.,

$$P_N(Z(v_1, \ldots, v_m)) = \{\lambda_1 P_N(v_1) + \cdots + \lambda_m P_N(v_m) | \lambda_1, \ldots, \lambda_m \in [0, 1]\} = Z(P_N(v_1), \ldots, P_N(v_m))$$

As an aside, this fact that a zonotope may be realized as the projection of a parallelotope is true in general. Any zonotope is the projection of some parallelotope (which is not hard to see from the definition we gave above), and this can actually be taken as a definition of a zonotope.\footnote{An informal interpretation of this definition is that “a zonotope is the shadow cast by a box of arbitrary orientation.”}

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in the current problem, where we are concerned with projections of the parallelotopes $C$ and $F$ into the space $N$.

Zonotopes have a number of interesting combinatorial properties\(^2\). Most importantly for our purposes, a zonotope $Z(v_1, \ldots, v_m)$ which has dimension $d$ can be partitioned into a number of parallelotopes [1], with one parallelotope for each maximal linearly independent subset $S \subseteq \{v_1, \ldots, v_m\}$. These partitioning parallelotopes have the form $Z(S)$ for each such subset $S$. An example of this partitioning can be seen in Figure 2. This decomposition allows us to evaluate the $d$-volume of $Z(v_1, \ldots, v_n)$ as the sum of the $d$-volumes of the partitioning parallelotopes $Z(S)$, leading to the following formula for the volume of a zonotope \([5, 1]\):

$$V_d(Z(v_1, \ldots, v_m)) = \sum_{S \subseteq \{v_1, \ldots, v_m\}: |S|=d} V_d(Z(S)),$$

(1)

where $V_d(\cdot)$ denotes $d$-dimensional volume. (Note that in this formula we sum over $d$-subsets of $\{v_1, \ldots, v_m\}$, whereas we expressed the partitioning of $Z(v_1, \ldots, v_m)$ in terms of maximal linearly independent subsets. In fact, we could just as well sum over maximal linearly independent subsets in the volume formula, but the form we have written will be more useful to us later. The equality of the two sums follows from the fact that every maximal linearly independent subset has $d$ vectors, and any $d$-subset which is not linearly independent yields a degenerate parallelotope with zero volume.) The utility of this formula in general is that when $Z(S)$ is a parallelotope, its volume can be found via the usual determinant formula\(^3\). In the present case, it turns out that we will never actually need to evaluate a determinant, as all factors of $V_d(\cdot)$ will eventually cancel.

\[\text{Figure 2.} \quad \text{A 2-dimensional zonotope and its partitioning into parallelotopes (parallelograms, in this case). The solid black arrows are the vectors defining the zonotope, and the zonotope itself is the entire hexagonal region. The smaller parallelogram regions outlined by the blue and black arrows are those that partition the zonotope. Fun fact: this partitioning is not unique. Can you find another?}\]

The cube $C$ is, up to a translation, $Z(2e_1, \ldots, 2e_n)$, where $e_i$ are the standard basis vectors for $\mathbb{R}^n$. So $P_N(C) = Z(2P_N(e_1), \ldots, 2P_N(e_n))$. Similarly, an $(n - k)$-dimensional face $F$ has the form (again up to translation) $Z(2e_{i_1}, \ldots, 2e_{i_{n-k}})$ for some

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\(^2\)One of the chief motivations for studying zonotopes comes from their combinatorial relation to arrangements of hyperplanes [3].

\(^3\)The fact that zonotopes admit such a concise analytical expression for their volume makes them very special in the world of polytopes, where computing volumes is generally difficult [2, 1].
standard basis vectors $e_{ij}$, so $P_N(F) = Z(2P_N(e_{i_1}), \ldots, 2P_N(e_{i_{n-k}}))$. As mentioned above, the probability that face $F$ is intersected by a random flat is given by the ratio of the volumes

$$\frac{V_{n-k}(P_N(F))}{V_{n-k}(P_N(C))} = \frac{V_{n-k}(Z(2P_N(e_{i_1}), \ldots, 2P_N(e_{i_{n-k}})))}{V_{n-k}(Z(2P_N(e_1), \ldots, 2P_N(e_n)))}.$$

The expected number of vertices, which we denote as $\#$, is the sum

$$\# = \sum_F \frac{V_{n-k}(P_N(F))}{V_{n-k}(P_N(C))} = \sum_F \frac{V_{n-k}(P_N(F))}{V_{n-k}(P_N(C))}$$

of this probability over all $(n-k)$-dimensional faces $F$. Note that for any set of $n-k$ standard basis vectors $e_{i_1}, \ldots, e_{i_{n-k}}$ there are $2^k$ faces $F$ which are translations of the zonotope $Z(2e_{i_1}, \ldots, 2e_{i_{n-k}})$; this is because such a face $F$ has $k$ fixed coordinates $x_i$, each of which is either $+1$ or $-1$. Since each such face has the same probability of being intersected by a random flat, we can replace the sum over faces $F$ in the above equation by a sum over $(n-k)$-subsets of $\{2e_1, \ldots, 2e_n\}$:

$$\# = \sum_{S \subseteq \{2e_1, \ldots, 2e_n\} : |S| = n-k} \frac{2^k V_{n-k}(Z(P_N(S)))}{V_{n-k}(P_N(C))},$$

The only thing that remains is to expand $V_{n-k}(P_N(C))$ using the volume formula for zonotopes:

$$\# = 2^k \left( \sum_{S \subseteq \{2e_1, \ldots, 2e_n\} : |S| = n-k} \frac{V_{n-k}(Z(P_N(S)))}{V_{n-k}(Z(P_N(2e_1), \ldots, P_N(2e_n)))} \right)$$

$$= \frac{2^k \left( \sum_{S \subseteq \{2e_1, \ldots, 2e_n\} : |S| = n-k} \frac{V_{n-k}(Z(P_N(S)))}{V_{n-k}(Z(P_N(S)))} \right)}{2^k}$$

$$= \frac{1}{2^k} \left( \sum_{S \subseteq \{2e_1, \ldots, 2e_n\} : |S| = n-k} \frac{V_{n-k}(Z(P_N(S)))}{V_{n-k}(Z(P_N(S)))} \right)$$

$$= 2^k.$$

Which is our desired result.

Note that this argument generalizes immediately to the case where the hypercube $C$ is replaced by an arbitrary parallelotope.

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**REFERENCES**

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Proofs Without Using Limits of Both Calculus Power Rules

Based on Newton’s “generated by a continual motion” view, the area proof uses the slope rule in an example application of the fundamental theorem of calculus.

**The Slope Rule.** For \( y = x^n \), integer \( n > 1 \), real \( x > 0 \), the tangent slope is \( nx^{n-1} \).

**Proof.** For fixed \( a = x > 0 \), the line \( y = na^{n-1}x - (n - 1)a^n \) passes through the point \((a, a^n)\). The curve is above that line by \( x^n - [na^{n-1}x - (n - 1)a^n] \), and it suffices to show that distance is positive when \( x \neq a \). Putting \( z = x/a \), multiplying by \((z - 1)/(z - 1)\), and simplifying, the distance is \( a^n(z - 1)[(z^n - 1)/(z - 1) - n] \). The fraction \((z^n - 1)/(z - 1)\) is less than \( n \) when \( 0 \leq z < 1 \) and greater than \( n \) when \( z > 1 \). So the distance is the product of three positive factors when \( x > a \) and one positive and two negative factors when \( x < a \).

**The Area Rule.** For integer \( n > 1 \), fixed real \( a = x \geq 0 \), the area under \( y = x^n \) from 0 to \( a \) is \( a^{n+1}/(n + 1) \).

**Proof.** Enclose the curve \( y = x^n \), \( 0 \leq x \leq a \), with the minimal rectangle; its area will be \( a(a^n) = a^{n+1} \). At \( x \), the area under the curve increases at a rate of its height times the rate of its rightward movement, or \( x^n dx \). Simultaneously, the area over the curve increases at a rate of its width times the rate of its upward movement, or \( x dy \). Applying the power rule for slope that is \( x(nx^{n-1})dx = nx^n dx \). The upward/rightward ratio of the area growth rates is \( nx^n dx/(x^n dx) = n \). So as \( x \) moves rightward, area accumulates \( n \) times more over than under. So, put the under area equal to 1, then the over area will be \( n \), and the total area is \( n + 1 \). So the under the curve area fraction of the rectangular area is \( 1/(n + 1) \). So the area under the curve is \( a^{n+1}/(n + 1) \).

—Submitted by Peter McCauley