ACTORS IN CATEGORIES OF INTEREST

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Abstract. For an object $A$ of a category of interest $\mathcal{C}$ we construct the group with operations $\mathfrak{B}(A)$ and the semidirect product $\mathfrak{B}(A) \rtimes A$ and prove that there exists an actor of $A$ in $\mathcal{C}$ if and only if $\mathfrak{B}(A) \rtimes A \in \mathcal{C}$. The cases of groups, Lie, Leibniz, associative, commutative associative, alternative algebras, crossed and precrossed modules are considered. In particular, we give examples of categories of interest, where always exist actors. The paper contains some results for the case $\Omega_2 = \{+,*,*^c\}$.

1. Introduction

The paper is dedicated to the question of the existence and construction of actors for the objects in categories of interest (see below Section 2 for the definitions and examples). This kind of categories were introduced by G. Orzech \cite{Orz72}. Actions in algebraic categories were studied by G. Hochschild \cite{Hoc47}, S. Mac Lane \cite{Mac58}, A. S.-T. Lue \cite{Lue68}, K. Norrie \cite{Nor90}, J.-L. Loday \cite{Lod93}, R. Lavendhomme and Th. Lucas \cite{LL96} and others. The authors were looking for the analogs of automorphisms of groups in associative algebras, rings, Lie algebras, crossed modules and Leibniz algebras. We see different approaches to this question. Lue and Norrie (on the base of the results of Lue \cite{Lue79} and Whitehead \cite{Whi48}), to any object associate a certain type of object, the construction in the corresponding category, called

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actor of this object [Nor90], with special properties, analogous to group automorphisms, under which is meant that the actor fits into a certain commutative diagram (see Section 2, diagram (2.7)). In [LL96] Lavendhomme and Lucas introduce the notion of a \( \Gamma \)-algebra of derivations for an algebra \( A \), which is the terminal object in the category of crossed modules under \( A \). Recently F. Borceux, G. Janelidze and G. M. Kelly [BJK05] proposed a categorical approach to this question. They study internal object actions and introduce the notion of a representable action, which in the case of a category of interest is equivalent to the definition of an actor given in this paper (see Section 3).

Let \( \mathcal{C} \) be a category of interest with a set of operations \( \Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2 \) and a set of identities \( E \). We define a general category of groups with operations \( \mathcal{C}_G \) with the same set of operations and a set of identities \( E_G \) for which \( \mathcal{C} \hookrightarrow \mathcal{C}_G \). We introduce the notions of an actor and of a general actor object for the objects of \( \mathcal{C} \). For any object \( A \in \mathcal{C} \) we give a construction of the universal algebra \( \mathcal{B}(A) \) with the operations from \( \Omega \). We show that in general \( \mathcal{B}(A) \) is an object of \( \mathcal{C}_G \).

For any \( A \in \mathcal{C} \) we define an action of \( \mathcal{B}(A) \) on \( A \), which is a \( \mathcal{B}(A) \)-structure on \( A \) in \( \mathcal{C}_G \) (i.e. the derived action appropriate to \( \mathcal{C}_G \), see Section 2 below for the definitions). In a well-known way we define the universal algebra \( \mathcal{B}(A) \hookrightarrow A \) which is an object of \( \mathcal{C}_G \). We define the homomorphism \( A \rightarrow \mathcal{B}(A) \) in \( \mathcal{C}_G \), which turned out to be a crossed module in \( \mathcal{C}_G \). We show that the general actor object always exists and \( \mathcal{B}(A) = \text{GActor}(A) \) (Theorem 3.7). The main theorem states that an object \( A \) from \( \mathcal{C} \) has an actor in \( \mathcal{C} \) if and only if \( \mathcal{B}(A) \hookrightarrow A \) is an object in \( \mathcal{C} \) and in this case \( A \rightarrow \mathcal{B}(A) \) is an actor of \( A \) in \( \mathcal{C} \) (Theorem 3.6).

From the results of [BJK05, Theorem 6.3] and from Theorem 3.6 of this paper we conclude that a category of interest \( \mathcal{C} \) has representable object actions in the sense of [BJK05] if and only if \( \mathcal{B}(A) \hookrightarrow A \in \mathcal{C} \) for any \( A \in \mathcal{C} \), and if it is the case the corresponding representing objects are \( \mathcal{B}(A) \), \( A \in \mathcal{C} \).

We consider separately the case \( \Omega_2 = \{+, \ast, \ast^o\} \). In the case of groups (\( \Omega_2 = \{+\} \)) we obtain that \( \mathcal{B}(A) \approx \text{Aut}(A) \), \( A \in \text{Gr} \). In the case of Lie algebras (\( \Omega_2 = \{+, [, ,]\} \)) for \( A \in \text{Lie} \) we obtain \( \mathcal{B}(A) \approx \text{Der}(A) \). In the case of Leibniz algebras we have \( \mathcal{B}(A) \in \text{Leib} \), for any \( A \in \text{Leib} \); \( \mathcal{B}(A) \) has a derived set of actions on \( A \) if and only if for any \( B, C \in \text{Leib} \) which have a derived action on \( A \) we have \([c, [a, b]] = -[c, [b, a]],\) for any \( a \in A, b \in B, c \in C, \) (which we call Condition 1 and it is equivalent to the existence of an \( \text{Actor}(A) \)). In this case \( \mathcal{B}(A) = \text{Actor}(A) \) (Theorem 4.5). We give examples of such Leibniz algebras. In particular Leibniz algebras \( A \) with \( \text{Ann}(A) = (0) \), where \( \text{Ann}(A) \) denotes the annihilator of \( A \), and perfect Leibniz algebras (i.e. \( A = [A, A] \)) satisfy Condition...
1. We have an analogous picture for associative algebras. In this case $\mathfrak{B}(A)$ is always an associative algebra, but the action of $\mathfrak{B}(A)$ on $A$ defined by us is not a derived action on $A$. Here we introduce Condition 2: for any $B$ and $C \in \text{Ass}$, which have a derived action on $A$ we have $c \ast (a \ast b) = (c \ast a) \ast b$, for any $a \in A$, $b \in B$, $c \in C$, where $\ast$ denotes the action. The action of $\mathfrak{B}(A)$ on $A$ is a derived action if and only if $A$ satisfies Condition 2 and it is equivalent to the existence of an Actor($A$). In this case $\mathfrak{B}(A) = \text{Actor}(A)$ (Proposition 4.6). Associative algebras with conditions Ann($A$) = (0) or with $A^2 = A$ satisfy Condition 2. These kind of associative algebras are considered in [LL96, Mac58]. The cases of modules over some ring, commutative associative algebras, alternative algebras, crossed modules and precrossed modules in the category of groups are discussed. Note, that the construction and the results given in our work enabled us to prove the existence of an actor in the category of precrossed modules. We consider the case where $\Omega_2 = \{+, \ast, \ast'\}$. The necessary and sufficient conditions for the existence of an actor is determined in the case where $\mathcal{E}$ contains only all identities from $\mathcal{E}_G$ and Axiom 1 and Axiom 2, and Axiom 2 has not consequence identities (Theorem 4.11). We consider separately the algebras of the same type as in Theorem 4.11 with additional commutativity or anticommutativity condition. We obtain the necessary and sufficient conditions for the existence of an actor in the corresponding category (Theorem 4.12) and give examples of such categories. The paper contains a comment to the formulation of Proposition 1.1 of [Dat95], which we apply in this paper. The results obtained here enabled us to find the construction of an actor of a precrossed module, which will be the subject of the forthcoming paper. The results of the paper are included in the Doctoral dissertation of the second author [Dat06] Chapter V.

In Section 2 we present the main definitions and results which are used in what follows. In Section 3 we give the main construction of the object $\mathfrak{B}(A)$ and the corresponding results. In Section 4 we consider the case of groups, Lie algebras, associative algebras and Leibniz algebras. For the special types of objects in $\text{Ass}$ and $\text{Leib}$ it is proved that $\mathfrak{B}(A) \approx \text{Bim}(A)$, $\mathfrak{B}(A) \approx \text{Bider}(A)$ respectively, (Propositions 4.7, 4.8), where $\text{Bim}(A)$ denotes the associative algebra of bimultipliers defined by G. Hochschild and by S. Mac Lane for rings (called bimultiplications in [Mac58] and multiplications in [Hoc47] from where the notion comes) and $\text{Bider}(A)$ denotes the Leibniz algebra of biderivations of $A$ defined in Section 2, which is isomorphic for these special types of Leibniz algebras to the biderivation algebra defined by J.-L. Loday in [Lod93]. At the end of the section we summarize for the special type of categories
of interest the results of Sections 3 and 4 and obtain the necessary and sufficient conditions for the existence of an actor.

2. Preliminary definitions and results

This section contains well-known definitions and results which will be used in what follows.

Let $C$ be a category of groups with a set of operations $\Omega$ and with a set of identities $E$, such that $E$ includes the group laws and the following conditions hold. If $\Omega_i$ is the set of $i$-ary operations in $\Omega$, then:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$;

(b) The group operations (written additively : 0, $-$, $+$) are elements of $\Omega_0$, $\Omega_1$ and $\Omega_2$ respectively. Let $\Omega'_2 = \Omega_2 \setminus \{+\}$, $\Omega'_1 = \Omega_1 \setminus \{-\}$ and assume that if $* \in \Omega_2$, then $\Omega'_2$ contains $*^0$ defined by $x *^0 y = y * x$.

Assume further that $\Omega_0 = \{0\}$;

(c) for each $* \in \Omega'_2$, $E$ includes the identity $x * (y + z) = x * y + x * z$;

(d) for each $\omega \in \Omega'_1$ and $* \in \Omega'_2$, $E$ includes the identities $\omega(x + y) = \omega(x) + \omega(y)$ and $\omega(x) * y = \omega(x * y)$.

We formulate more axioms on $C$ (Axiom (7) and Axiom (8) of [Orz72]).

If $C$ is an object of $C$ and $x_1, x_2, x_3 \in C$:

**Axiom 1.** $x_1 + (x_2 \ast x_3) = (x_2 \ast x_3) + x_1$, for each $* \in \Omega'_2$.

**Axiom 2.** For each ordered pair $(\ast, \overline{\ast}) \in \Omega'_2 \times \Omega'_2$ there is a word $W$ such that

$$(x_1 \ast x_2)\overline{\ast}x_3 = W(x_1(x_2x_3), x_1(x_3x_2), (x_2x_3)x_1,$$

$$(x_3x_2)x_1, x_2(x_1x_3), x_2(x_3x_1), (x_1x_3)x_2, (x_3x_1)x_2),$$

where each juxtaposition represents an operation in $\Omega'_2$.

We will write the right side of Axiom 2 as $W(x_1, x_2; x_3; \ast, \overline{\ast})$. A category of groups with operations satisfying Axiom 1 and Axiom 2 is called a category of interest by Orzech [Orz72] (see also [Por87]).

Note that from the equalities $(x + y) * (z + t) = x * z + x * t + y * z + y * t$ follows that $x * t + y * z = y * z + x * t$, for $* \in \Omega'_2$, $x, y, z, t \in C$, $c \in C$.

Denote by $E_G$ the subset of identities of $E$ which includes the group laws and the identities (c) and (d). We denote by $C_G$ the corresponding category of groups with operations. Thus we have $E_G \hookrightarrow E$, $C = (\Omega, E)$, $C_G = (\Omega, E_G)$ and there is a full inclusion functor $C \hookrightarrow C_G$.

In the case of associative algebras with multiplication represented by $\ast$, we have $\Omega'_2 = \{\ast, \ast^0\}$. For Lie algebras take $\Omega'_2 = \{[\ , ], [\ , ]^0\}$ (where $[a, b]^0 = [b, a] = -[a, b]$). For Leibniz algebras (see the definition
below), take $\Omega_2' = ([ , ], [ , ]^\circ)$, (here $[a, b]^\circ = [b, a]$). It is easy to see that all these algebras are categories of interest. In the example of groups $\Omega_2 = \emptyset$. As it is mentioned in [Orz72] Jordan algebras do not satisfy Axiom 2.

**Definition 2.1.** [Orz72] Let $C \in \mathcal{C}$. A subobject of $C$ is called an ideal if it is the kernel of some morphism.

**Theorem 2.2.** [Orz72] Let $A$ be a subobject of $B$ in $\mathcal{C}$. Then $A$ is an ideal of $B$ if and only if the following conditions hold:

(i) $A$ is a normal subgroup of $B$;

(ii) For any $a \in A$, $b \in B$ and $* \in \Omega_2'$, we have $a * b \in A$.

**Definition 2.3.** [Orz72] Let $A, B \in \mathcal{C}$. An extension of $B$ by $A$ is a sequence

\begin{equation}
0 \longrightarrow A \overset{i}{\longrightarrow} E \overset{p}{\longrightarrow} B \longrightarrow 0
\end{equation}

in which $p$ is surjective and $i$ is the kernel of $p$. We say that an extension is split if there is a morphism $s : B \longrightarrow E$ such that $ps = 1_B$.

**Definition 2.4.** [Orz72] A split extension of $B$ by $A$ is called a $B$-structure on $A$.

As in [Orz72], for $A, B \in \mathcal{C}$ we will say we have “a set of actions of $B$ on $A$”, whenever there is a set of maps $f_* : B \times A \longrightarrow A$, for each $* \in \Omega_2$.

A $B$-structure induces a set of actions of $B$ on $A$ corresponding to the operations in $\mathcal{C}$. If (2.1) is a split extension, then for $b \in B$, $a \in A$ and $* \in \Omega_2'$ we have

\begin{align}
(2.2) & \quad b \cdot a = s(b) + a - s(b), \\
(2.3) & \quad b * a = s(b) * a.
\end{align}

(2.2) and (2.3) are called derived actions of $B$ on $A$ in [Orz72] and split derived actions in [Dat95], since we considered there actions derived from non split extensions too when $A$ is a singular object.

Given a set of actions of $B$ on $A$ (one for each operation in $\Omega_2$), let $B \ltimes A$ be a universal algebra whose underlying set is $B \times A$ and whose operations are

\begin{align}
& (b', a') + (b, a) = (b' + b, a' + b' \cdot a), \\
& (b', a') * (b, a) = (b' * b, a' * a + a' * b + b' * a).
\end{align}

**Theorem 2.5.** [Orz72] A set of actions of $B$ on $A$ is a set of derived actions if and only if $B \ltimes A$ is an object of $\mathcal{C}$. 
Together with the description of the set of derived actions given in the Theorem above, we will need the identities which satisfies a set of derived actions in case $A, B \in C_G$ and which guarantee that the set of actions is a set of derived actions in $C_G$.

**Proposition 2.6.** [Dat95] A set of actions in $C_G$ is a set of derived actions if and only if it satisfies the following conditions:

1. $0 \cdot a = a$,
2. $b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2$,
3. $(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$,
4. $b * (a_1 + a_2) = b * a_1 + b * a_2$,
5. $(b_1 + b_2) * a = b_1 * a + b_2 * a$,
6. $(b_1 \cdot b_2) \cdot (a_1 \cdot a_2) = a_1 \cdot a_2$,
7. $(b_1 \cdot b_2) \cdot (a * b) = a * b$,
8. $a_1 * (b * a_2) = a_1 * a_2$,
9. $b * (b_1 \cdot a) = b * a$,
10. $\omega(b \cdot a) = \omega(b) \cdot \omega(a)$,
11. $\omega(a * b) = \omega(a) * b = a * \omega(b)$,
12. $x \cdot y + z * t = z * t + x \cdot y$,

for each $\omega \in \Omega'_1$, $* \in \Omega'_2$, $b, b_1, b_2 \in B$, $a, a_1, a_2 \in A$ and for $x, y, z, t \in A \cup B$ whenever each side of 12 has a sense.

Note that in the formulation of Proposition 1.1 in [Dat95] we mean that the set of identities of the category of groups with operations contains only identities from $E_G$, but it is not mentioned there. The same concerns to some other statements of [Dat95]. In the case where we have a category $C$ with the set of identities $\mathbb{E}$, the conditions 1-12 of the above proposition are necessary conditions. Of course it is possible according to other identities included in $\mathbb{E}$ to write down the corresponding conditions for derived actions which will be necessary and sufficient for the set of actions to be a set of derived actions (i.e. for $B \ltimes A \in C$). Denote all these identities by $\tilde{E}_G$ and $\tilde{E}$ respectively.

If the addition is commutative in $C$, then $\tilde{E}$ (resp. $\tilde{E}_G$) consists of the same kind of identities that we have in $\mathbb{E}$ (resp. in $E_G$), written down for the elements from the set $A \cup B$, whenever each identity has a sense.

We will denote by Axiom 2 the identities for the action in $C$, which correspond to Axiom 2 (see [Dat95]). In the category of groups, Lie, associative, Leibniz algebras derived actions are called simply actions. We will use this terminology in these special cases; we will also say “an action in $C$”, if it is a derived action, and we will say a set of actions is not an action in $C$ if this set is not a set of derived actions. Recall that a left action of a group $B$ on $A$ is a map $\varepsilon : B \times A \rightarrow A$, which
we denote by \( \varepsilon(b, a) = b \cdot a \), with conditions
\[
(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a),
\]
\[
0 \cdot a = a,
\]
\[
b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2.
\]

The right action is defined in an analogous way.

All algebras below are considered over a commutative ring \( k \) with unit.

In the case of associative algebras an action of \( B \) on \( A \) is a pair of bilinear maps
\[
(2.4) \quad B \times A \longrightarrow A, \quad A \times B \longrightarrow A
\]
which we denote respectively as \((b, a) \mapsto b \ast a\), \((a, b) \mapsto a \ast b\), with conditions
\[
(b_1 \ast b_2) \ast a = b_1 \ast (b_2 \ast a),
\]
\[
a \ast (b_1 \ast b_2) = (a \ast b_1) \ast b_2,
\]
\[
(b_1 \ast a) \ast b_2 = b_1 \ast (a \ast b_2),
\]
\[
b \ast (a_1 \ast a_2) = (b \ast a_1) \ast a_2,
\]
\[
(a_1 \ast a_2) \ast b = a_1 \ast (a_2 \ast b),
\]
\[
a_1 \ast (b \ast a_2) = (a_1 \ast b) \ast a_2.
\]

Here the associative algebra operation is denoted by \( \ast \) (resp. \( a_1 \ast a_2 \))
and the corresponding action by the same sign \( \ast \) (resp. \( b \ast a \)).

For Lie algebras an action of \( B \) on \( A \) is a bilinear map \( B \times A \longrightarrow A \),
which we denote by \((b, a) \mapsto [b, a]\), with conditions
\[
[[b_1, b_2], a] = [b_1, [b_2, a]] - [b_2, [b_1, a]],
\]
\[
[b, [a_1, a_2]] = [a_1, [b, a_2]] + [[b, a_1], a_2].
\]

Note that we actually have above again two bilinear maps (2.4): \((b, a) \mapsto [b, a]\), \((a, b) \mapsto [a, b]\) with conditions
\[
[b, a] = -[a, b],
\]
\[
[a, [b_1, b_2]] + [b_1, [b_2, a]] + [b_2, [a, b_1]] = 0,
\]
\[
[a_1, [b, a_2]] + [b, [a_2, a_1]] + [a_2, [a_1, b]] = 0.
\]

Recall from \([\text{Lod}93]\) that a Leibniz algebra \( L \) over a commutative
ring \( k \) with unit is a \( k \)-module equipped with a bilinear map \([-, -]\) :
L × L → L which satisfies the following identity, called the Leibniz identity:

\[[x, [y, z]] = [[x, y], z] - [[x, z], y]\]

for all x, y, z ∈ L.

Obviously, when \[[x, x] = 0 \text{ for all } x ∈ L\], the Leibniz bracket is skew-symmetric, therefore the Leibniz identity comes down to the Jacobi identity, and a Leibniz algebra is then just a Lie algebra.

For Leibniz algebras an action of \(B\) on \(A\) is a pair of bilinear maps

\[(2.4) \quad (b, a) \mapsto [b, a], (a, b) \mapsto [a, b]\]

with conditions:

\[[a_1, [a_2, b]] = [[a_1, a_2], b] - [[a_1, b], a_2],\]
\[[a_1, [b, a_2]] = [[a_1, b], a_2] - [[a_1, a_2], b],\]
\[[b, [a_1, a_2]] = [[b, a_1], a_2] - [[b, a_2], a_1],\]
\[[a, [b_1, b_2]] = [[a, b_1], b_2] - [[a, b_2], b_1],\]
\[[b_1, [a, b_2]] = [[b_1, a], b_2] - [[b_1, b_2], a],\]
\[[b_1, [b_2, a]] = [[b_1, b_2], a] - [[b_1, a], b_2].\]

Recall \[Ser92\] that a derivation for a Lie algebra \(A\) over a ring \(k\) is a \(k\)-linear map \(D : A \rightarrow A\) with

\[D[a_1, a_2] = [D(a_1), a_2] + [a_1, D(a_2)].\]

The set of all derivations \(\text{Der}(A)\) of \(A\), with the operation defined by

\[[D, D'] = DD' - D'D\]

is a Lie algebra.

We recall the construction of the \(k\)-algebra \(\text{Bim}(A)\) of bimultipliers of an associative \(k\)-algebra \(A\) \[Hoc47, Mac58\]. An element of \(\text{Bim}(A)\) is a pair \(f = (f*, *f)\) of \(k\)-linear maps from \(A\) to \(A\) with

\[f * (a * a') = (f * a) * a',\]
\[(a * a') * f = a * (a' * f),\]
\[a * (f * a') = (a * f) * a'.\]

We prefer to use the notation \(*f\) instead of \(f^o\). We denote by \(f * a\) (resp. \(a * f\)) the value \(f * (a)\) (resp. \(*f(a)\)). \(\text{Bim}(A)\) is a \(k\)-module in an obvious way. The operation in \(\text{Bim}(A)\) is defined by

\[f * f' = (f * f', *f * f')\]

and \(\text{Bim}(A)\) becomes a \(k\)-algebra. Note that here we use different notations than in \[Mac58\] and \[LL96\]. Here as above * denotes an operation
in associative algebra, and \( f \ast f' \ast, \ast f \ast f' \) denote the compositions of maps. Thus
\[
(f \ast f')(a) = f \ast (f' \ast a),
\]
\[
(*f \ast f')(a) = (a \ast f) \ast f'.
\]

For the addition we have
\[
f + f' = ((f \ast) + f' \ast, f + (*f')),
\]
where
\[
((f \ast) + f' \ast)(a) = f \ast a + f' \ast a,
\]
\[
(*f + (*f'))(a) = a \ast f + a \ast f'.
\]

For a Leibniz \( k \)-algebra \( A \) we define the \( k \)-algebra \( \text{Bider}(A) \) of biderivations in the following way. An element of \( \text{Bider}(A) \) is a pair \( \varphi = ([\cdot, \varphi], [\varphi, \cdot]) \) of \( k \)-linear maps \( A \rightarrow A \) with
\[
[[a_1, a_2], \varphi] = [a_1, [a_2, \varphi]] + [[a_1, \varphi], a_2],
\]
\[
[\varphi, [a_1, a_2]] = [[\varphi, a_1], a_2] - [[a_2, \varphi], a_1],
\]
\[
[a_1, [a_2, \varphi]] = -[a_1, [\varphi, a_2]].
\]

We used above the notation: \([\varphi, \cdot](a) = [\varphi, a], [\cdot, \varphi](a) = [a, \varphi]\). Biderivations where defined by Loday in [Lod93], where another notation is used; biderivation is a pair \((d, D)\), where according to our definition \([\varphi, \cdot] = D, [\cdot, \varphi] = -d\) and instead of the third condition we have in [Lod93] \([a_1, d(a_2)] = [a_1, D(a_2)]\).

The operation in \( \text{Bider}(A) \) is defined by:
\[
[\varphi, \varphi'] = ([\cdot, [\varphi, \varphi']], [[\varphi, \varphi'], \cdot]),
\]
where
\[
(2.5.1) \quad [a, [\varphi, \varphi']] = [[a, \varphi], \varphi'] - [[a, \varphi'], \varphi],
\]
\[
(2.5.2) \quad [[\varphi, \varphi'], a] = [\varphi, [\varphi', a]] + [[\varphi, a], \varphi'].
\]

Note that we could define \([[\varphi, \varphi'], \cdot]] by
\[
(2.5.2') \quad [[\varphi, \varphi'], a] = -[[\varphi, a, \varphi']] + [[\varphi, a], \varphi'].
\]

To avoid confusions we forget about \( \ast' \) in special cases, e.g. for the \([\cdot, \cdot] \) operation. The above given both operations define a Leibniz algebra structure on \( \text{Bider}(A) \). It is easy to see that the second definition \((2.5.1), (2.5.2')\) gives the algebra which is isomorphic to the biderivation algebra defined in [Lod93]; according to this definition \([(d, D), (d', D')] = (dd' - d'd, Dd' - d'D)\).
We have a set of actions of $\text{Der}(A)$, $\text{Bim}(A)$ and $\text{Bider}(A)$ on $A$. These actions are defined by

\[
[D, a] = D(a), \\
f * a = f * (a), \\
a * f = *f(a), \\
[\varphi, a] = [\varphi, (a), \ [a, \varphi] = [\ , \varphi](a),
\]

where $a \in A$, $D \in \text{Der}(A)$, $f = (f\ast, *f) \in \text{Bim}(A)$, $\varphi = (([\ , \varphi], [\varphi, \ ])) \in \text{Bider}(A)$ and $A$ is a Lie algebra, an associative algebra and a Leibniz algebra respectively.

In the case of Lie algebras the action of $\text{Der}(A)$ on $A$ is a set of derived actions, thus this action satisfies the corresponding conditions of an action in $\text{Lie}$, but for the case of associative and Leibniz algebras these actions do not satisfy all the conditions given above respectively for the action in $\text{Ass}$ and $\text{Leib}$. Note that for the case of Leibniz algebras if $[\varphi, [\varphi', a]] = -[[\varphi, [a, \varphi']]]$ for any $a \in A$ and $\varphi, \varphi' \in \text{Bider}(A)$, then above two ways of defining operations in $\text{Bider}(A)$ are equal and also the action of $\text{Bider}(A)$ becomes a derived action (see below Proposition 4.8).

We have an analogous situation for associative algebras. The action of $\text{Bim}(A)$ on $A$ is not a derived action because the condition

\[(f * a) * f' = f * (a * f')\]

fails. So if we would have the condition for associative algebra $A$ that for any two bimultipliers is fulfilled (2.6), then the action of $\text{Bim}(A)$ on $A$ defined above is a set of derived actions on $A$ (see below Proposition 4.7).

An alternative algebra $A$ over a field $F$ is an algebra which satisfies the identities

\[x^2y = x(xy)\]

and

\[yx^2 = (yx)x\]

for all $x, y \in A$. These identities are known respectively as the left and right alternative laws. We denote the corresponding category of alternative algebras by $\text{Alt}$. Clearly any associative algebra is alternative. The class of 8-dimensional Cayley algebras is an important class of alternative algebras which are not associative $\text{Sch66}$.

The axioms above for alternative algebras are equivalent to the following:

\[x(yz) = (xy)z + (yx)z - y(xz)\]
and

\[(xy)z = x(yz) - (xz)y + x(zy)\]

We consider these conditions as Axiom 2 and consequently alternative algebras can be interpreted as categories of interest.

For alternative algebras over a field \(F\) an action of \(B\) on \(A\) is a pair of bilinear maps (2.4), which we denote again by \((b, a) \mapsto ba, (a, b) \mapsto ab\) with conditions:

\[
b(a_1a_2) = (ba_1)a_2 + (a_1b)a_2 - a_1(ba_2),
\]

\[
(a_1a_2)b = a_1(a_2b) - (a_1b)a_2 + a_1(ba_2),
\]

\[
(ba_1)a_2 = b(a_1a_2) - (ba_2)a_1 + b(a_2a_1),
\]

\[
a_1(a_2b) = (a_1a_2)b + (a_2a_1)b - a_2(a_1b),
\]

\[
(b_1b_2)a = b_1(b_2a) - (b_1a)b_2 + b_1(ab_2),
\]

\[
a(b_1b_2) = (ab_1)b_2 + (b_1a)b_2 - b_1(ab_2),
\]

\[
(ab_1)b_2 = a(b_1b_2) - (ab_2)b_1 + a(b_2b_1),
\]

\[
b_1(b_2a) = b_1(b_2a) + (b_2b_1)a - b_2(b_1a).
\]

A crossed module in \(\mathbb{C}\) is a triple \((C_0, C_1, \partial)\), where \(C_0, C_1 \in \mathbb{C}\), \(C_0\) acts on \(C_1\) (i.e. we have a derived action in \(\mathbb{C}\)) and \(\partial : C_1 \rightarrow C_0\) is a morphism in \(\mathbb{C}\) with conditions:

(i) \(\partial(r \cdot c) = r + \partial(c) - r\);

(ii) \(\partial(c) \cdot c' = c + c' - c\);

(iii) \(\partial(c) \cdot c' = c \cdot c'\);

(iv) \(\partial(r \cdot c) = r \cdot \partial(c), \partial(c \cdot r) = \partial(c) \cdot r\)

for any \(r \in C_0, c, c' \in C_1, \) and \(* \in \Omega_2'\).

A morphism between two crossed modules \((C_0, C_1, \partial) \rightarrow (C_0', C_1', \partial')\) is a pair of morphisms \((T_0, T_1)\) in \(\mathbb{C}\), \(T_0 : C_0 \rightarrow C_0', T_1 : C_1 \rightarrow C_1'\), such that

\[
T_0\partial(c) = \partial'T_1(c),
\]

\[
T_1(r \cdot c) = T_0(r) \cdot T_1(c),
\]

\[
T_1(r \cdot c) = T_0(r) \cdot T_1(c)
\]

for any \(r \in C_0, c \in C_1\) and \(* \in \Omega_2'\).

**Definition 2.7.** For any object \(A\) in \(\mathbb{C}\) an actor of \(A\) is a crossed module \(\partial : A \rightarrow \text{Actor}(A)\), such that for any object \(C\) of \(\mathbb{C}\) and an action of \(C\) on \(A\) there is a unique morphism \(\varphi : C \rightarrow \text{Actor}(A)\) with \(c \cdot a = \varphi(c) \cdot a, c \ast a = \varphi(c) \ast a\) for any \(* \in \Omega_2', a \in A\) and \(c \in C\).

See the equivalent Definition 3.9. in Section 3.
From this definition it follows that an actor object \( \text{Actor}(A) \), for the object \( A \in C \), with this properties is a unique object up to an isomorphism in \( C \).

Note that according to the universal property of an actor object, for any two elements \( x, y \in \text{Actor}(A) \) from \( x \cdot a = y \cdot a \), (here we mean equalities for the dot action and the action \( * \), for any \( * \in \Omega_2' \) and any \( a \in A \)) and \( (w_1 \cdots w_n x) \cdot a = (w_1 \cdots w_n y) \cdot a, w_1 \cdots w_n \in \Omega'_1 \), it follows that \( x = y \).

It is well-known that for the case of groups \( \text{Actor}(G) = \text{Aut}(G) \); the corresponding crossed module is \( \partial : G \to \text{Aut}(G) \), where \( \partial \) sends any \( g \in G \) to the inner automorphism of \( G \) defined by \( \partial(g) = g + g' - g, g' \in G \). For the case of Lie algebras \( \text{Actor}(A) = \text{Der}(A), A \in \text{Lie} \), and the operator homomorphism \( \partial : A \to \text{Der}(A) \) is defined by \( \partial(a) = [a, \cdot] \), so \( \partial(a)(a') = [a, a'] \).

As we have seen above, in general, in \( \text{Ass} \) and \( \text{Leib} \) the objects \( \text{Bim}(A) \) and \( \text{Bider}(A) \) do not have derived actions on \( A \) in the corresponding categories. So the obvious homomorphisms \( A \to \text{Bim}(A), A \to \text{Bider}(A) \) do not define crossed modules in \( \text{Ass} \) and \( \text{Leib} \) for any \( A \) from \( \text{Ass} \) and \( \text{Leib} \) respectively.

It is well-known [Nor90] that for the case of groups if \( N \) is a normal subgroup of \( G \) and \( \tau : N \to \text{Inn}(N) \) is the homomorphism sending any element \( n \) to the corresponding inner automorphism \( (\tau(n)(n') = n + n' - n) \), since \( G \) acts on \( N \) by conjugation, we have a unique homomorphism \( \theta : G \to \text{Actor}(N) \), with \( \theta(g) \cdot n = g \cdot n \). \( \text{Inn}(N) \) is a normal subgroup of \( \text{Actor}(N) \), \( \theta \) extends \( \tau \) and we have a commutative diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & G/N & \longrightarrow & 0 \\
& \tau \downarrow & & \theta \downarrow & & \downarrow & & 0 \\
0 & \longrightarrow & \text{Inn}(N) & \longrightarrow & \text{Actor}(N) & \longrightarrow & \text{Out}(N) & \longrightarrow & 0.
\end{array}
\]

According to the work of R. Lavendhomme and Th. Lucas [LL96] in the categories \( \text{Gr}, \text{Lie} \) the actor crossed modules \( A \to \text{Actor}(A) \) are terminal objects in the categories of crossed modules under \( A \). If \( \text{Ann}(A) = (0) \) or \( A^2 = A \) then \( \text{Bim}(A) \) acts on \( A \) and the corresponding crossed module \( A \to \text{Bim}(A) \) is a terminal object in the category of crossed modules under \( A \). It is easy to see that in this case

\[
\text{Bim}(A) = \text{Actor}(A).
\]

**Definition 2.8.** A general actor object \( G\text{Actor}(A) \) for \( A, A \in C \), is an object from \( C_G \), which has a set of actions on \( A \), which is a set of derived actions in \( C_G \), i.e. satisfies conditions of Proposition 2.6, there
is a morphism \( d : A \rightarrow \text{GActor}(A) \) in \( \mathcal{C}_G \) which defines a crossed module in \( \mathcal{C}_G \) and for any object \( C \in \mathcal{C} \) and a derived action of \( C \) on \( A \), there exists in \( \mathcal{C}_G \) a unique morphism \( \varphi : C \rightarrow \text{GActor}(A) \) such that \( c \ast a = \varphi(c) \ast a \), for any \( c \in C \), \( a \in A \), \( \ast \in \Omega'_2 \).

It is easy to see that \( \text{Bim}(A) \) and \( \text{Bider}(A) \) are general actor objects for \( A \in \text{Ass} \), \( A \in \text{Leib} \) respectively. These constructions satisfy the existence of the commutative diagram like (2.7).

3. The main construction

In this section \( \mathcal{C} \) will denote a category of interest with a set of operations \( \Omega \) and with a set of identities \( \mathcal{E} \). Let \( \mathcal{C}_G \) be the corresponding general category of groups with operations. According to the definition given in Section 2, \( \mathcal{C}_G \) is a \( \{ \Omega, \mathcal{E}_G \} \)-algebra, where \( \Omega \) is the same set of operations as we have in \( \mathcal{C} \), and \( \mathcal{E}_G \) includes group laws and identities \( (c) \) and \( (d) \) from the definition of Section 2. Thus we have \( \mathcal{E}_G \hookrightarrow \mathcal{E} \) and \( \mathcal{C} \) is the full subcategory of \( \mathcal{C}_G \), \( \mathcal{C} \hookrightarrow \mathcal{C}_G \).

Let \( A \in \mathcal{C} \); consider all split extensions of \( A \) in \( \mathcal{C} \):

\[
\begin{array}{ccccccc}
E_j : 0 & \longrightarrow & A & \overset{i_j}{\longrightarrow} & C_j & \overset{p_j}{\longrightarrow} & B_j & \longrightarrow & 0, & j \in J.
\end{array}
\]

Note that it may happen that \( B_j = B_k = B \), for \( j \neq k \), then these extensions will correspond to different actions of \( B \) on \( A \). Let \( \{ b_j \cdot, b_j \ast | b_j \in B_j, \ast \in \Omega'_2 \} \) be the corresponding set of derived actions for \( j \in J \). For any element \( b_j \in B_j \) denote \( b_j = \{ b_j \cdot, b_j \ast, \ast \in \Omega'_2 \} \). Let \( \mathcal{B} = \{ b_j | b_j \in B_j, j \in J \} \).

Thus each element \( b_j \in \mathcal{B}, j \in J \) is a special type of function \( b_j : \Omega_2 \longrightarrow \text{Maps}(A \rightarrow A), b_j(\ast) = b_j \ast - : A \longrightarrow A \).

According to Axiom 2 from the definition of a category of interest, we define \( \ast \) operation, \( b_i \ast b_k, \ast \in \Omega'_2 \), for the elements of \( \mathcal{B} \) by the equalities

\[
\begin{align*}
(b_i \ast b_k)\bar{\ast}(a) & = W(b_i, b_k; a; \ast, \bar{\ast}), \\
(b_i \ast b_k) \cdot (a) & = a.
\end{align*}
\]

We define the operation of addition by

\[
\begin{align*}
(b_i + b_k) \cdot (a) & = b_i \cdot (b_k \cdot a), \\
(b_i + b_k) \ast (a) & = b_i \ast a + b_k \ast a.
\end{align*}
\]

For a unary operation \( \omega \in \Omega'_1 \) we define

\[
\begin{align*}
\omega(b_k) \cdot (a) & = \omega(b_k) \cdot (a), \\
\omega(b_k) \ast (a) & = \omega(b_k) \ast (a),
\end{align*}
\]
\[ \omega(b * b') = \omega(b) * b' \] and we will have \( \omega(b) * b' = b * \omega(b') \),
\[ \omega(b_1 + \cdots + b_n) = \omega(b_1) + \cdots + \omega(b_n), \]
\[ (-b_k) \cdot (a) = (-b_k) \cdot a, \]
\[ (-b) \cdot (a) = a \]
\[ (-b_k) * (a) = -(b_k * a), \]
\[ (-b) * (a) = -(b * (a)), \]
\[ -(b_1 + \cdots + b_n) = -b_n - \cdots - b_1, \]

where \( b, b', b_1, \ldots, b_n \) are certain combinations of star operations on the elements of \( B \), i.e. the elements of the type \( b_{i_1} * \cdots * b_{i_n}, n > 1 \).

We do not know if the new functions defined by us are again in \( B \). Denote by \( \mathfrak{B}(A) \) the set of functions \( (\Omega_2 \rightarrow \text{Maps}(A \rightarrow A)) \) obtained by performing all kind of above defined operations on elements of \( B \) and new obtained elements as the results of operations. Note that \( b = b' \) in \( \mathfrak{B}(A) \) means that \( b * a = b' * a, w_1 \ldots w_n b \cdot a = w_1 \ldots w_n b' \cdot a \) for any \( a \in A, * \in \Omega_2', w_1 \ldots w_n \in \Omega_1' \) and for any \( n \). It is an equivalence relation and under \( \mathfrak{B}(A) \) we mean the corresponding quotient object.

**Proposition 3.1.** \( \mathfrak{B}(A) \) is an object of \( C_G \).

**Proof.** Direct easy checking of the identities. \( \square \)

As above, we will write for simplicity \( b \cdot (a) \) and \( b * (a) \) instead of \( (b(+))(a) \) and \( (b(*))(a) \) for \( b \in \mathfrak{B}(A) \) and \( a \in A \). Define the set of actions of \( \mathfrak{B}(A) \) on \( A \) in a natural way. For \( b \in \mathfrak{B}(A) \) we define
\[ b \cdot a = b \cdot (a), b * a = b * (a), * \in \Omega_2'. \]
Thus if \( b = b_{i_1} * \cdots * b_{i_n} \), where we mean certain brackets, we have
\[ b \mathfrak{\bar{x}} a = (b_{i_1} * \cdots * b_{i_n}) \mathfrak{\bar{x}} (a), \]
\[ b \cdot a = a. \]

The right side of the equality is defined inductively according to Axiom 2. For \( b_k \in B_k, k \in \mathbb{J} \), we have
\[ b_k * a = b_k * (a) = b_k * a, \]
\[ b_k \cdot a = b_k \cdot (a) = b_k \cdot a. \]

Also
\[ (b_1 + b_2 + \cdots + b_n) * a = b_1 * (a) + \cdots + b_n * (a), \]
for \( b_i \in \mathfrak{B}(A), \ i = 1, \ldots, n \)
\[ (b_1 + b_2 + \cdots + b_n) \cdot a = b_1 \cdot (b_2 \cdots (b_n \cdot (a)) \cdots), \]
\[ b_i \in \mathcal{B}(A), \quad i = 1, \ldots, n \]
\[ \omega(b) \cdot a = a \quad \text{if} \quad b = b_1 \cdots b_n, \quad b_i \in \mathcal{B}(A), \quad i = 1, \ldots, n \]
\[ \omega(b_k) \cdot a = \omega(b_k) \cdot a, \quad k \in \mathbb{J}, \quad b_k \in B_k. \]

**Proposition 3.2.** The set of actions of \( \mathcal{B}(A) \) on \( A \) is a set of derived actions in \( \mathcal{C}_G \).

**Proof.** The checking shows that the set of actions of \( \mathcal{B}(A) \) on \( A \) satisfies conditions of Proposition 2.6, which proves that it is a set of derived actions in \( \mathcal{C}_G \). \( \Box \)

Define the map \( d : A \rightarrow \mathcal{B}(A) \) by \( d(a) = a \), where \( a = \{a, a*, * \in \Omega'_2\} \). Thus we have by definition
\[ d(a) \cdot a' = a + a' - a, \]
\[ d(a) * a' = a * a', \quad \forall a, a' \in A, \quad * \in \Omega'_2. \]

**Lemma 3.3.** \( d \) is a homomorphism in \( \mathcal{C}_G \).

**Proof.** We have to show that \( d(\omega a) = \omega d(a) \) for any \( \omega \in \Omega'_1 \). For this we need to show that
\[ d(\omega a) \cdot (a') = (\omega d(a)) \cdot (a') \]
\[ \omega'(d(\omega a)) \cdot a' = \omega'(\omega d(a)) \cdot a', \quad \text{for any} \ \omega' \in \Omega'_1 \]
\[ d(\omega a) * (a') = (\omega d(a)) * (a'), \quad \text{for any} \ * \in \Omega'_2. \]

We have
\[ d(\omega a) \cdot a' = \omega a + a' - \omega a, \]
\[ \omega d(a) \cdot a' = \omega(a) \cdot a' = \omega a + a' - \omega a, \]

The second equality follows form the first one. For the third equality we have
\[ d(\omega a) * a' = (\omega a) * a', \]
\[ (\omega d(a)) * a' = \omega(a) * a' = \omega(a) * a' \]
for \( \omega = - \) we have to show \( d(-a) \cdot (a') = (-da) \cdot a' \) and \( d(-a) * a' = (-d(a)) * a' \). The checking of these equalities is an easy exercise.

Now we will show that \( d(a_1 + a_2) = d(a_1) + d(a_2) \). The direct computation of both sides for each \( a \in A \) gives
\[ d(a_1 + a_2) \cdot (a) = a_1 + a_2 + a - a_2 - a_1, \]
\[ (d(a_1) + d(a_2)) \cdot (a) = d(a_1) \cdot (d(a_2) \cdot a), \]
which shows that the desired equality holds for the dot action. The proof of \( \omega(d(a_1 + a_2)) \cdot a = \omega(d(a_1) + d(a_2)) \cdot a \) is based on the first
equality, the property of unary operations to respect the addition and the fact that \( d \) commutes with unary operations.

For any \( \ast \in \Omega'_2 \) we shall show that
\[
d(a_1 + a_2) \ast (a) = (d(a_1) + d(a_2)) \ast (a).
\]
We have
\[
d(a_1 + a_2) \ast (a) = (a_1 + a_2) \ast a = a_1 \ast a + a_2 \ast a,
\]
and
\[
(d(a_1) + d(a_2)) \ast (a) = d(a_1) \ast a + d(a_2) \ast a = a_1 \ast a + a_2 \ast a
\]
which proves the equality.

The next equality we have to prove is \( d(a_1 \ast a_2) = d(a_1) \ast d(a_2) \). For this we need to show that \( d(a_1 \ast a_2) \cdot (a) = (d(a_1) \ast d(a_2)) \cdot (a), \omega(d(a_1 \ast a_2)) \cdot a = \omega(d(a_1) \ast d(a_2)) \cdot a \) and \( d(a_1 \ast a_2)\bar{\omega}(a) = (d(a_1) \ast d(a_2))\bar{\omega}(a) \), for any \( \bar{\omega} \in \Omega'_2 \).

We have \( d(a_1 \ast a_2) \cdot a = a_1 \ast a_2 + a - a_1 \ast a_2 = a \), since \( A \in \mathbb{C} \) and therefore it satisfies Axiom 1.

\( (d(a_1) \ast d(a_2)) \cdot a = a \), by the definition of the action of \( \mathfrak{B}(A) \) on \( A \).

The next equality is proved in a similar way applying that \( d \) commutes with \( \omega \) and \( \omega(a_1 \ast a_2) = \omega(a_1) \ast a_2 \).

For the next above given identity we have the following computations:

\[
d(a_1 \ast a_2)\bar{\omega}(a) = (a_1 \ast a_2)\bar{\omega}a = W(a_1, a_2; a; \ast, \bar{\omega}),
\]
\[
(d(a_1) \ast d(a_2))\bar{\omega}(a) = W(d(a_1), d(a_2); a; \ast, \bar{\omega}).
\]

These two expressions on the right sides of above equalities are equal, by the type of the word \( W \) in Axiom 2 and the definition of \( d \).

\textbf{Proposition 3.4.} \( d: A \longrightarrow \mathfrak{B}(A) \) is crossed module in \( C_G \).

\textit{Proof.} We have to check conditions (i)-(iv) from the definition of a crossed module given in Section 2.

(i) condition states that \( d(b \ast a) = b + d(a) - b \) for \( a \in A, b \in \mathfrak{B}(A) \); so we have to show that \( d(b \ast a) \ast a' = (b + da - b) \ast a' \) and \( \omega_1 \ldots \omega_n(d(b \ast a)) \cdot a' = \omega_1 \ldots \omega_n(b + da - b) \cdot a' \). Below we compute each side for the dot action of the first equality:

\[
d(b \ast a) \cdot a' = b \ast a + a' - b \ast a,
\]
\[
(b + d(a) - b) \cdot a' = b \cdot (d(a) \cdot (-b \ast a'))
\]
\[
= b \cdot (a - b \ast a' - a) = b \cdot a + a' - b \ast a.
\]

The second equality is proved in a similar way. Now we compute each side of the first equality for the * action. \( d(b \ast a) \ast a' = (b \ast a) \ast a' = a \ast a' \) by Proposition 2.6; \( (b + da - b) \ast a' = b \ast a' + d(a) \ast a' - b \ast a' = b \ast a' + a \ast a' \).
Suppose $b$ is the sum of the elements of the type $\mathcal{A}$, here we applied Axiom 1, that $\mathcal{A} + a \cdot b = a \cdot b + \mathcal{A}$, for any element $\mathcal{A}$ of $A$.

We have to show: (ii) $d(a_1) \cdot a_2 = a_1 + a_2 - a_1$, (iii) $d(a_1) \cdot a_2 = a_1 \cdot a_2$; both (ii) and (iii) are true by definition of $d$. Note that $a_1 \cdot (d(a_2)) = d(a_2) \cdot a_1 = a_2 \cdot a_1$. The first condition of (iv) states

$$d(b \cdot a) = b \cdot d(a), \text{ for any } b \in \mathfrak{B}(A), \ a \in A, \ * \in \Omega'_2.$$ 

Thus we have to show

$$d(b \cdot a) \cdot a' = b \cdot a + a' - b \cdot a.$$ 

If $b = b_i$, then $b \cdot a = b_i \cdot a$ and since $B_i \in \mathfrak{C}$, and $B_i$ acts on $A$ (action is in $\mathfrak{C}$), by Axiom 1 for the action of $B_j$ on $A$ we shall have $b \cdot a + a' = a' + b \cdot a$ and so $d(b \cdot a) \cdot a' = a'$.

If $b = b_{i_1} \cdot \cdots \cdot \cdot n-1 b_{i_n}$ then, by the definition of $\ast$ operation in $\mathfrak{B}(A)$, $b \cdot a$ is the sum of the elements of the type $b_{i_t} \ast a_t$ for certain $i_t$ and the element $a_t \in A$; this kind of element again commutes with any element of $A$. So that $d(b \cdot a) \cdot a' = a'$. We will have the same result if $b$ is the sum of the elements of the type $b_{i_1} \ast \cdots \cdot s_{n-1} b_{i_n}$.

Now we shall show (3.1) for the $\ast$ operation. By the definition of $d$ we have

$$d(b \cdot a) \ast a' = (b \cdot a) \ast a'.$$

In the case $b = b_i$, $i \in \mathfrak{I}$, $b \cdot a = b_i \cdot a = b_i \cdot a$, so we obtain

$$d(b \cdot a) \ast a' = (b_i \cdot a) \ast a = W(b_i, a; a'; \ast, \ast).$$

We have the last equality according to the properties of an action in $\mathfrak{C}$, which correspond to Axiom 2. For the right side of (3.1) in case $b = b_i$ we have

$$(b \cdot d(a)) \ast a' = (b_i \cdot a) \ast a' = W(b_i, a; a'; \ast, \ast).$$

Suppose $b = b_{i_1} \ast \cdots \cdot s_{n-1} b_{i_n}$, then in the same way as it was in the previous proof, $b \cdot a$ is the sum of the elements of the type $b_{i_t} \ast a_t$ and $(b \cdot a) \ast a'$ is the sum of the elements of the type $(b_{i_t} \ast a_t) \ast a'$. The element from the right side of (3.1) will be the same type of the sum of the elements $(b_{i_t} \ast a_t) \ast a'$. Applying Axiom 2 to the element $(b_{i_t} \ast a_t) \ast a'$, by the definition of the operation for the elements of $\mathfrak{B}(A)$ (for the element $(b_{i_t} \ast a_t) \ast a'$) and from the facts that $b_{i_t} \ast a = b_{i_t} \ast a$,
$\overline{a} * a = \overline{a} * a$, we will have the desired equality (3.1). In the analogous way we will prove (3.1) for $*$ operation in case $b$ is a sum of the elements of the form $b_i*1\ldots*n-1b_{i,n}$. The second condition of (iv) can be proved in a similar way. □

**Proposition 3.5.** If $A$ has an actor in $\mathbb{C}$, then $\mathfrak{B}(A) = \text{Actor}(A)$.

**Proof.** From the existence of $\text{Actor}(A)$ it follows that $\text{Actor}(A)$ is one of the objects $B_i$, which acts on $A$. We have a natural homomorphism $e : \text{Actor}(A) \rightarrow \mathfrak{B}(A)$ in $\mathbb{C}_G$ sending $b_i$ to $b_i$. According to the note made in Section 2, if $b_i \neq b'_i$ in $\text{Actor}(A)$, then $b_i \neq b'_i$; thus $e$ is an injective homomorphism. Let $\varphi_j : B_j \rightarrow \text{Actor}(A)$ be a unique morphism with $\varphi_j(b_j) * a = b_j * a$, $b_j \in B_j$, $j \in J$, $a \in A$. $e$ is a surjective homomorphism, since for any element $b_{i} * 1 \ldots * n-1 b_{i,n}$ of $\mathfrak{B}(A)$ there exists the element $\varphi_i(b_i) * 1 \ldots * n-1 \varphi_{i,n}(b_{i,n})$ in $\text{Actor}(A)$ with $e(\varphi_i(b_i) * 1 \ldots * n-1 \varphi_{i,n}(b_{i,n})) = b_{i} * 1 \ldots * n-1 b_{i,n}$ which ends the proof. □

**Theorem 3.6.** Let $\mathbb{C}$ be a category of interest and $A \in \mathbb{C}$. A has an actor if and only if $\mathfrak{B}(A) \ltimes A \in \mathbb{C}$. If it is the case, then $\text{Actor}(A) = \mathfrak{B}(A)$.

**Proof.** From the Proposition 3.5 it follows that if $A$ has an actor then $\mathfrak{B}(A) \in \mathbb{C}$ and $\mathfrak{B}(A)$ has a derived action on $A$. By the theorem of Orzech [Orz72] (see Section 2, Theorem 2.5) we will have $\mathfrak{B}(A) \ltimes A \in \mathbb{C}$. The converse is also easy to prove. Since $\mathfrak{B}(A) \ltimes A \in \mathbb{C}$, from the split exact sequence $0 \rightarrow A \xrightarrow{i} \mathfrak{B}(A) \ltimes A \xrightarrow{\pi} \mathfrak{B}(A) \rightarrow 0$ in $\mathbb{C}_G$, $\mathfrak{B}(A) = \text{Coker } i$ and thus it is an object of $\mathbb{C}$; again by Theorem 2.5 $\mathfrak{B}(A)$ has a derived action on $A$ in $\mathbb{C}$ (it is the action we have defined). By Proposition 3.4, $d : A \rightarrow \mathfrak{B}(A)$ is a crossed module in $\mathbb{C}_G$; since $\mathfrak{B}(A) \in \mathbb{C}$, and the action of $\mathfrak{B}(A)$ on $A$ is a derived action in $\mathbb{C}$, it follows that $d : A \rightarrow \mathfrak{B}(A)$ is a crossed module in $\mathbb{C}$. Now we have to show the universal property of this crossed module. For any action of $B_k$ on $A$, $k \in J$, we define $\varphi_k : B_k \rightarrow \mathfrak{B}(A)$ by $\varphi_k(b_k) = b_k$, for any $b_k \in B_k$, where $b_k \in \mathbb{B}$. By definition of $\mathbb{B}$, $b_k * a = b_k * a, * \in \Omega_2'$, and we obtain $\varphi_k(b_k) * a = b_k * a; \varphi_k$ is a homomorphism in $\mathbb{C}$. For another homomorphism $\varphi'_k$ we would have $\varphi'_k(b_k) * a = b_k * a = \varphi_k(b_k) * a$, $\omega(\varphi'_k(b_k)) * a = \varphi'_k(\omega b_k) * a = (\omega b_k) * a = \omega(\varphi(b_k)) * a$, for any $b_k \in B_k$, $a \in A$, $\omega \in \Omega_1'$, and $* \in \Omega_2'$, which means that $\varphi_k(b_k) = \varphi'_k(b_k)$, for any $b_k \in B_k$, this gives the equality $\varphi_k = \varphi'_k$, which proves the theorem. □

**Theorem 3.7.** Let $\mathbb{C}$ be a category of interest. For any $A \in \mathbb{C}$, $\mathfrak{B}(A) = G\text{Actor}(A)$. 
Proof. By Propositions 3.2 and 3.4 and Lemma 3.3 we have the crossed module $d : A \to \mathfrak{B}(A)$ in $\mathbb{C}_G$. For any object $C \in \mathbb{C}$ which has a derived action on $A$ we construct the homomorphism $\varphi : C \to \mathfrak{B}(A)$ in $\mathbb{C}_G$ with the property $c * a = \varphi(c) * a$ and show that $\varphi$ is unique with this property in the similar way as we have done for $\varphi_k$ in the proof of Theorem 3.6. □

Below we give a categorical presentation of the necessary and sufficient conditions for the existence of an actor in the category of interest $\mathbb{C}$, i.e. for any object $A \in \mathbb{C}$. Actually we have constructed the functor $T : \mathfrak{B}(-) \ltimes (-) : \mathbb{C} \to \mathbb{C}_G$. This functor is defined in a natural way $T(A) = \mathfrak{B}(A) \ltimes A, A \in \mathbb{C}$. For the definition of $T(\alpha), \alpha : A \to A'$ in $\mathbb{C}$, we apply the universality property of the general actor object in the following way. The pushout diagram in $\mathbb{C}_G$

$$
\begin{array}{c}
0 \\ \downarrow \alpha \\
A' \\
\downarrow \alpha \\
C \\
\downarrow \alpha \\
\mathfrak{B}(A) \to 0
\end{array}
$$

where the first sequence is split, implies that the second one is also split. Thus by Theorem 2.5, $\mathfrak{B}(A)$ has a derived set of actions on $A$ in $\mathbb{C}_G$. The object $\mathfrak{B}(A')$ is a general actor object for $A'$ in $\mathbb{C}_G$. Thus there exists a unique arrow $\varphi : \mathfrak{B}(A) \to \mathfrak{B}(A')$ in $\mathbb{C}_G$ such that $\varphi(b) * a = b * a$ and $\varphi(b) \cdot a = b \cdot a$. We define $T(\alpha) : \mathfrak{B}(A) \ltimes A \to \mathfrak{B}(A') \ltimes A'$ by $T(\alpha)(b, a) = (\varphi(b), \alpha(a))$. It is easy to check that $T(\alpha)$ is a homomorphism in $\mathbb{C}_G$.

We denote by $Q : \mathbb{C}_G \to \mathbb{C}$ the functor which assigns to each object $C \in \mathbb{C}_G$ the greatest quotient object of $C$ which belongs to $\mathbb{C}$. The above description gives to Theorem 3.6 the following form.

**Theorem 3.6'** Let $\mathbb{C}$ be a category of interest. There exists an Actor$(A)$ for any $A \in \mathbb{C}$ if and only if the following diagram commutes

$$
\begin{array}{c}
\mathbb{C} \\
\downarrow T \\
\mathbb{C}_G \\
\downarrow Q \\
\mathbb{C}_G \\
\downarrow E \\
\mathbb{C}
\end{array}
$$

where $E$ denotes the natural inclusion functor.

Suppose $I$ is an ideal of $C$ in $\mathbb{C}$ and Actor$(I)$ exists. Thus we have the crossed module $d : I \to \text{Actor}(I)$. Denote $\text{Im} d = \text{Inn}(I)$. Thus
we have
\[ \text{Inn}(I) = \{ a \in \text{Actor}(I) | a \in I \}. \]
Recall that by definition of \( d \), \( d(a) = a \), and \( a \) is defined by
\[ a \cdot (a') = a + a' - a, \]
\[ a * (a') = a * a'. \]
It is easy to see that \( \text{Inn}(I) \) is an ideal of \( \text{Actor}(I) \). It follows from the fact that \( d : I \rightarrow \text{Actor}(I) \) is a crossed module and it can also be checked directly. Since \( I \) is an ideal of \( C \), we have an action of \( C \) on \( I \), defined by
\[ c \cdot a = c + a - c, \]
\[ c * a = c * a, \]
\[ * \in \Omega'_2. \]
It is a derived action. Thus there exists a unique homomorphism \( \theta : C \rightarrow \text{Actor}(I) \), such that
\[ \theta(c) \ast a = c \ast a, \quad a \in I, \quad c \in C, \quad * \in \Omega'_2. \]
Let \( \tau : I \rightarrow \text{Inn}(I) \) be a homomorphism defined by \( d \), then \( \theta \) induces the commutative diagram
\[ \begin{array}{ccccccccc}
0 & \rightarrow & I & \rightarrow & C & \rightarrow & C/I & \rightarrow & 0 \\
\tau \downarrow & & \downarrow \theta & & \downarrow \gamma & & \downarrow \gamma & & \downarrow \theta & & 0,
\end{array} \]
which is well-known for the case of groups \([\text{Nor90}]\) (see Section 2).

For any object \( C \in \mathcal{C} \) there is an action of \( A \) on itself defined by
\[ a \cdot a' = a + a' - a; a * a' = a * a', \]
for any \( a, a' \in A, * \in \Omega'_2 \), where \( * \) on the left side denotes the action and on the right side the operation in \( A \). We call this action the conjugation.

Let \( E_A : \rightarrow \rightarrow A \rightarrow \rightarrow A \rightarrow \rightarrow 0 \) be the split extension which corresponds to the action of \( A \) on itself by conjugation. Consider the category of all split extensions with fixed \( A \); thus the objects are
\[ 0 \rightarrow A \rightarrow \rightarrow C \rightarrow \rightarrow C' \rightarrow \rightarrow 0, \]
and the arrows are triples \((1_A, \gamma, \gamma')\) between extensions which commute with section homomorphism too.

**Proposition 3.8.** If \( E_t : \rightarrow \rightarrow A \rightarrow \rightarrow C \rightarrow \rightarrow B \rightarrow \rightarrow 0 \) is a terminal object in the category of split extensions with fixed \( A \), then the unique arrow \((1, \gamma, \beta) : E_A \rightarrow E_t \) defines a crossed module \( \beta : A \rightarrow B \), which is an actor of \( A \).

**Proof.** The prove is similar to that of Proposition 3.4. It is obvious that \( B \) has the universal property of an actor. We have to prove that
$\beta : A \to B$ is a crossed module, thus we shall show the following identities
\[
\begin{align*}
\beta(a) \cdot a' &= a + a' - a, \\
\beta(b \cdot a) &= b + \beta(a) - b, \\
\beta(a) \ast a' &= a \ast a' \\
\beta(b \ast a) &= b \ast \beta(a).
\end{align*}
\]
for any $a \in A, b \in B, \ast \in \Omega'_2$. We have the commutative diagram
\[
\begin{array}{c}
E_A : 0 \longrightarrow A \xrightarrow{A \times A} A \xrightarrow{\beta} 0 \\
E_t : 0 \longrightarrow A \xrightarrow{C} B \xrightarrow{\beta} 0
\end{array}
\]
from which we obtain $\beta(a) \cdot a' = a + a' - a$ and $\beta(a) \ast a' = a \ast a'$ for any $a, a' \in A, \ast \in \Omega'_2$, which proves the first and third equalities. Since $E_t$ is a terminal extension, it has the following property: if for $b, b' \in B$ we have $b \ast a = b' \ast a, \omega_1 \cdots \omega_n(b) \ast a = \omega_1 \cdots \omega_n(b') \ast a$ for any $a \in A$ and any unary operations $\omega_1, \cdots, \omega_n \in \Omega'_1$, $n \in \mathbb{N}$, then $b = b'$. For the second equality we have
\[
(\beta(b \cdot a)) \cdot a' = b \cdot a + a' - b \cdot a \\
(b + \beta(a) - b) \cdot a' = b \cdot (\beta(a) \cdot (-b \cdot a')) = b \cdot (a - b \cdot a' - a) = b \cdot a + a' - b \cdot a
\]
by condition 8 of Proposition 2.6.

For the forth equality we have:
\[
\beta(b \ast a) \cdot a' = (b \ast a) \cdot a' = a'
\]
it follows from the property of the derived action in categories of interest, as a result of Axiom 1 [Dat95]. The same property gives
\[
(b \ast \beta(a)) \cdot a' = a'.
\]
For a star operation we have:
\[
\beta(b \ast a) \ast a' = (b \ast a) \ast a' \\
(b \ast \beta(a)) \ast a' = (b \ast a) \ast a',
\]
here we apply Axiom 2 for the set $(A \cup B)$ and the fact $\beta(a) \ast a' = a \ast a'$.

For any unary operation $\omega \in \Omega'_1$,
\[
\omega(\beta(b \cdot a)) = \beta(\omega(b \cdot a)) = \beta(\omega(b) \cdot \omega(a)),
\]
here we apply condition 10 of Proposition 2.6,
\[ \omega(b + \beta(a) - b) = \omega(b) + \beta(\omega(a)) - \omega(b). \]

As we have proved above these elements are equal.

Below we apply condition 11 of Proposition 2.6 and obtain:
\[ \omega(\beta(b * a)) = \beta(\omega(b * a)) = \beta(\omega(b) * a), \]
\[ \omega(b * \beta(a)) = \omega(b) * \omega(a). \]

As we have shown above these elements are equal. For \( \omega_1, \cdots, \omega_n \) the corresponding equalities are obtained similarly.

By Proposition 3.8, Definition 2.7 is equivalent to the following one.

**Definition 3.9.** For any object \( A \) in \( \mathbb{C} \) an actor of \( A \) is an object \( \text{Actor}(A) \), which acts on \( A \) in \( \mathbb{C} \), and for any object \( C \) of \( \mathbb{C} \) and an action of \( C \) on \( A \) there is a unique morphism \( \varphi : C \rightarrow \text{Actor}(A) \) with \( c \cdot a = \varphi(c) \cdot a, c * a = \varphi(c) * a \) for any \( * \in \Omega_2', a \in A \) and \( c \in C \).

It is a well-known fact that the category of crossed modules in the category of groups \( \text{XMod}(\mathbb{G}r) \) is equivalent to the category \( \mathbb{G} \) with objects groups with the additional two unary operations \( \omega_0, \omega_1 : G \rightarrow G, G \in \mathbb{G}r \) which are group homomorphisms satisfying conditions

\[
\begin{align*}
(1) \quad & \omega_0 \omega_1 = \omega_1, \omega_1 \omega_0 = \omega_0 \\
(2) \quad & \omega_1(x) + y - \omega_1(x) = x + y - x, \quad x, y \in \text{Ker} \omega_0
\end{align*}
\]

This category is a category of interest. The computations and properties of actions in this category and the direct checking of identities (1), (2) show that \( \mathfrak{B}(A) \) is an actor of \( A \in \mathbb{G} \). Thus the same is true for the category of crossed modules \( \text{XMod}(\mathbb{G}r) \). From the results of Norrie [Nor90] it follows that the constructed by her the object \( A(T, G, \partial) \) for any crossed module \( (T, G, \partial) \) is an actor in the sense of Definition 2.7. Thus it follows that in the category of interest \( \mathbb{G} \) there exists an actor for any \( A \in \mathbb{G} \). By the Proposition 3.5 it follows that \( \mathfrak{B}(A) \) is an actor for any \( A \in \mathbb{G} \). It is another way of proving that \( \mathfrak{B}(A) = \text{Actor}(A) \) in \( \mathbb{G} \).

The category of precrossed modules is equivalent to the category of interest \( \mathbb{G} \), which objects are groups with additional two unary operations \( \omega_0, \omega_1 \), which are group homomorphisms satisfying identity (1). By Theorem 3.7, \( \mathfrak{B}(A) = \text{GActor}(A) \), for any \( A \in \mathbb{G} \). It is easy to check that \( \mathfrak{B}(A) \) satisfies identity (1) and thus \( \mathfrak{B}(A) \in \mathbb{G} \), therefore \( \mathfrak{B}(A) = \text{Actor}(A) \). From this we conclude that in the category of precrossed modules always exists an actor.

As we have mentioned in the introduction internal object actions were studied recently by F. Borceux, G. Janelidze and G. M. Kelly [BJK05], where the authors introduce a new notion of representable...
action. From Theorem 6.3 of [BJK05], applying Proposition 3.8, it follows that in the case of category of interest $\mathcal{C}$ the existence of representable object actions is equivalent to the existence of an $\operatorname{Actor}(A)$ for any $A \in \mathcal{C}$ in the sense of Definition 2.7. Thus by Theorem 3.6, $\mathcal{C}$ has representable object actions if and only if $\mathfrak{B}(A) \ltimes A \in \mathcal{C}$, for any $A \in \mathcal{C}$, and if it is the case, the corresponding representing objects are $\mathfrak{B}(A), A \in \mathcal{C}$.

4. The case $\Omega_2 = \{+, \*, *^\circ\}$

It is interesting to know in which kind of categories of interest $\mathcal{C}$ there exists $\operatorname{Actor}(A)$ for any object $A \in \mathcal{C}$; or what the sufficient conditions for the existence of $\operatorname{Actor}(A)$ for a certain $A \in \mathcal{C}$ are. In the case of groups ($\Omega_2 = \{+\}$), the direct checking shows that $\mathfrak{B}(A) \in \mathfrak{Gr}$, and the action of $\mathfrak{B}(A)$ on $A$ is a derived action. It follows also from Propositions 3.1 and 3.2; thus $\mathfrak{B}(A)$ is an actor of $A$ by the Theorem 3.6. This fact is also a consequence of Proposition 3.5 since it is well-known that $\operatorname{Aut}(A)$ is an actor of $A$ in $\mathfrak{Gr}$, thus $\mathfrak{B}(A) \approx \operatorname{Aut}(A)$. In the case of Lie algebras ($\Omega_2 = \{+, [], \} \}$), the object $\mathfrak{B}(A) \in \mathfrak{Lie}$ and the action of $\mathfrak{B}(A)$ on $A$ is a derived action, so $\mathfrak{B}(A)$ is an actor again in $\mathfrak{Lie}$ and therefore $\mathfrak{B}(A) \approx \operatorname{Der}(A)$.

Consider the case of Leibniz algebras. In this case we can define the bracket operation for the elements of $\mathfrak{B}$ in two ways (see Section 2 for the definition of the set $\mathfrak{B}$).

**Definition 4.1.**

$$[a, [b_i, b_j]] = [[a, b_i], b_j] - [[a, b_j], b_i],$$
$$[[b_i, b_j], a] = [b_i, [b_j, a]] + [[b_i, a], b_j].$$

**Definition 4.2.**

$$[a, [b_i, b_j]] = [[a, b_i], b_j] - [[a, b_j], b_i],$$
$$[[b_i, b_j], a] = -[b_i, [a, b_j]] + [[b_i, a], b_j].$$

The bracket operation $[b, b']$ for any $b, b'$ which are the results of bracket operations itself is defined according to above formulas.

The addition is defined by:

$$[b_i + b_j, a] = [b_i, a] + [b_j, a],$$
$$[a, b_i + b_j] = [a, b_i] + [a, b_j].$$

For any $b, b' \in \mathfrak{B}(A)$, $b + b'$ is defined by the same formulas.

The action of $\mathfrak{B}(A)$ on $A$ is defined according to Definition 4.1 or 4.2 respectively. So we have two different ways of definition of an action. It is easy to check that non of them is the derived action in $\mathfrak{Leib}$. 
The algebras $\mathfrak{B}(A)$ defined by Definitions 4.1 and 4.2 are not isomorphic.

**Condition 1.** For $A \in \text{Leib}$, and any two objects $B, C \in \text{Leib}$, which act on $A$, we have

$$[c, [a, b]] = -[c, [b, a]],$$

$a \in A$, $b \in B$, $c \in C$.

Note that in this condition under action we mean the derived action.

**Example 1.** If $\text{Ann}(A) = (0)$ or $[A, A] = A$, then $A$ satisfies Condition 1.

**Proposition 4.3.** For any object $A \in \text{Leib}$, the Definitions 4.1 and 4.2 give the same algebras if $A$ satisfies Condition 1.

The proof follows directly from the definitions of operations in $\mathfrak{B}(A)$ and Condition 1.

Below we mean that $\mathfrak{B}(A)$ is defined in one of the ways.

**Proposition 4.4.** For any $A \in \text{Leib}$, $\mathfrak{B}(A)$ is a Leibniz algebra. The set of actions of $\mathfrak{B}(A)$ on $A$ is a set of derived actions if and only if $A$ satisfies Condition 1.

**Proof.** The computation shows that if Condition 1 holds then the same kind of condition is fulfilled for $b, c \in \mathfrak{B}(A)$, from which follows the result. □

**Theorem 4.5.** For a Leibniz algebra $A$ there exists an actor if and only if $A$ satisfies Condition 1. If it is the case, then $\mathfrak{B}(A) = \text{Actor}(A)$.

**Proof.** By Proposition 4.3, $\mathfrak{B}(A)$ is always a Leibniz algebra and by Theorem 3.7, $\mathfrak{B}(A) = G\text{Actor}(A)$. If $A$ satisfies Condition 1, by Proposition 4.4, $\mathfrak{B}(A)$ has a derived action on $A$ and thus $\mathfrak{B}(A) = \text{Actor}(A)$. Conversely, if $A$ has an actor then $\mathfrak{B}(A) = \text{Actor}(A)$ by Proposition 3.5, and so the action of $\mathfrak{B}(A)$ on $A$ is a derived action, thus we have for any $a \in A$, $b_i \in B_i$, $b_j \in B_j$, $i, j \in \mathbb{J}$ the following equalities

$$[b_i, [a, b_j]] = [[b_i, a], b_j] - [[b_i, b_j], a],$$

$$[b_i, [b_j, a]] = [[b_i, b_j], a] - [[b_i, a], b_j],$$

from which follows Condition 1, which proves the theorem. □

We have an analogous picture for associative algebras. The operations for the elements of $\mathfrak{B}$ (see Section 2 for the notation) in this
category are given by
\[(b_i * b_j) * (a) = b_i * (b_j * a), \]
\[*(b_i * b_j)(a) = (a * b_i) * b_j, \]
\[(b_i + b_j) * (a) = b_i * a + b_j * a, \]
\[*(b_i + b_j)(a) = a * b_i + a * b_j. \]

The set of actions of $\mathfrak{B}(A)$ on $A$ is defined according to (4.1).

**Condition 2.** For $A \in \text{Ass}$ and any two objects $B$ and $C$ from $\text{Ass}$ which have derived actions on $A$ we have
\[c * (a * b) = (c * a) * b, \]
for any $a \in A$, $b \in B$, $c \in C$.

**Example 2.** If $\text{Ann}(A) = (0)$ or $A^2 = A$ then $A$ satisfies Condition 2. For this kind of associative algebras it is proved in [LL96] that $A \rightarrow \text{Bim}(A)$ is a terminal object in the category of crossed modules under $A$.

**Proposition 4.6.** For $A \in \text{Ass}$, the algebra $\mathfrak{B}(A)$ is an associative algebra and the set of actions of $\mathfrak{B}(A)$ on $A$ defined according to (4.1) is the set of derived actions in $\text{Ass}$ if and only if $A$ satisfies Condition 2. If it is the case $\mathfrak{B}(A) = \text{Actor}(A)$.

The proof contains the analogous arguments as for the case of Leibniz algebras and is left to the reader.

It is easy to see that in $\text{Ass}$ and $\text{Leib}$ generally we have injections
\[\mathfrak{B}(A) \rightarrow \text{Bim}(A) \quad \text{and} \quad \mathfrak{B}(A) \rightarrow \text{Bider}(A)\]
which are homomorphisms in $\text{Ass}$ and $\text{Leib}$ respectively.

**Proposition 4.7.** Let $A$ be an associative algebra with the condition $\text{Ann}(A) = 0$ or $A^2 = A$. Then $\mathfrak{B}(A) \approx \text{Bim}(A) = \text{Actor}(A)$.

**Proof.** It is well-known that $\text{Bim}(A)$ is an associative algebra [Mac58]. The action of $\text{Bim}(A)$ on $A$ (see Section 2) is not a derived action in general, and the condition which fails is
\[(4.2) \quad f * (a * f') = (f * a) * f' \]
for any $f = (f*, *f)$ and $f' = (f'**, *f'')$ from $\text{Bim}(A)$. The direct checking shows that in case $\text{Ann}(A) = (0)$ or $A^2 = A$, identity (4.2) holds for the action $\mathfrak{L}L96$. For any action of the object $B$ on $A$, $B \in \text{Ass}$, we define $\varphi : B \rightarrow \text{Bim}(A)$ by $\varphi(b) = (b*, *b)$, which is a unique homomorphism with the property that $\varphi(b) \ast a = b \ast a$, $\ast \in \Omega_2$, since in $\text{Bim}(A)$ for any two elements $f, f' \in \text{Bim}(A)$ from $f = f'$ follows that $f* = f'*$, $*f = *f'$. Thus $\text{Bim}(A)$ is an actor of $A$ in $\text{Ass}$ and the isomorphism $\mathfrak{B}(A) \approx \text{Bim}(A)$ follows from Proposition 3.5. $\square$
We have the analogous result for Leibniz algebras.

**Proposition 4.8.** Let $A \in \Leib$ and $\Ann(A) = (0)$ or $[A, A] = A$. Then $\mathfrak{B}(A) \approx \Bider(A) = \operatorname{Actor}(A)$.

**Proof.** We will follow the first definition of the bracket operation in $\Bider(A)$ (see Section 2, (2.5.1), (2.5.2)). The direct checking shows that $\Bider(A)$ is a Leibniz algebra (see Remark below and cf. [Lod93]). The action of $\Bider(A)$ on $A$ is not a derived action, fails the following condition

\begin{equation}
[\varphi, [a, \varphi']] = [[\varphi, a], \varphi'] - [[\varphi, \varphi'], a],
\end{equation}

where $\varphi = [[\cdot, \varphi], [\varphi, \cdot]]$ and $\varphi' = [[\cdot, \varphi'], [\varphi', \cdot]] \in \Bider(A)$.

From (2.5.2) we have

\begin{equation}
[\varphi, [\varphi', a]] = [[\varphi, \varphi'], a] - [[\varphi, a], \varphi'].
\end{equation}

We shall show that if $\Ann(A) = (0)$ then $[\varphi, [\varphi', a]] = -[\varphi, [a, \varphi']]$, and from (4.4) will follow (4.3).

Note that under the annihilator we mean both left and right annulator. For any $a' \in A$ we have the following equalities:

\[
[a', [\varphi, [\varphi', a]]] = -[a', [[\varphi', a], \varphi]] - [[a', [\varphi', a]], \varphi] + [[a', \varphi], [\varphi', a]],
\]
\[
[a', [\varphi, [a, \varphi']]] = -[[a', [a, \varphi']], \varphi] + [[a', \varphi], [a, \varphi']] = [[a', [\varphi', a]], \varphi] - [[a', \varphi], [\varphi', a]].
\]

Thus we obtain that for $a' \in A$

\[
[a', [\varphi, [\varphi', a]]] + [\varphi, [a, \varphi']] = 0.
\]

In analogous way we show that

\[
[[\varphi, [\varphi', a]] + [\varphi, [a, \varphi']], a'] = 0.
\]

From which we conclude that

\[
[\varphi, [\varphi', a]] + [\varphi, [a, \varphi']] = 0.
\]

The case $[A, A] = A$ can be proved analogously. Thus we have a derived action of $\Bider(A)$ on $A$ and the crossed module $A \to \Bider(A)$ ($a \mapsto ([\cdot, a], [a, \cdot])$ has the universal property of the actor object. By Proposition 3.5 $\mathfrak{B}(A) \approx \Bider(A)$ which ends the proof. \qed

**Remark 1.** As we have also mentioned in Section 2, if $[\varphi, [\varphi', a]] = -[\varphi, [a, \varphi']]$ for any $\varphi = ([\cdot, \varphi], [\varphi, \cdot])$ and $\varphi' = ([\cdot, \varphi'], [\varphi', \cdot])$ from $\Bider(A)$, then two definitions of $\Bider(A)$ according to (2.5.1), (2.5.2) and (2.5.1), (2.5.2') coincide and this algebra is isomorphic to the Leibniz algebra of biderivations defined by Loday [Lod93].
In the category of $R$-modules over some ring $R$, it is obvious that $\text{Actor}(A) = 0$ for any $A$ since every action is trivial in this category. The same result gives our construction, $\mathcal{B}(A) = 0$ for any $R$-module $A$.

As it is in the case of associative algebras, in the category of commutative associative algebras the condition for the action $(b_1a)b_2 = b_1(ab_2)$ fails; also in this category we must have $ba = ab$, for $b \in \mathcal{B}(A)$, and $b_1b_2 = b_2b_1$ for $b_1, b_2 \in \mathcal{B}(A)$. All these conditions are satisfied and we have $\mathcal{B}(A) = \text{Actor}(A)$ in commutative associative algebras if and only if $A$ satisfies Condition 2. If $\text{Ann}(A) = (0)$ or $A^2 = A$ then $A$ satisfies Condition 2. For this kind of commutative algebras, $\text{Actor}(A) = \text{Bim}(A) = M(A)$ where $M(A)$ is the set of multiplications (or multipliers) of $A$ [LS67, LL96], i.e., $k$-linear maps $f : A \to A$ with $f(aa') = f(a)a'$.

In the category of alternative algebras $\text{Actor}(A)$ does not exist for any $A$. The condition of Proposition 4.9 given below is not fulfilled. If $A$ satisfies the condition $b(ac) = (ba)c$, for $b, c \in B, C$, respectively, where $B, C$ are any actions on $A$, then $\mathcal{B}(A)$ is an actor of $A$. Note that the condition given above implies that $A$ is an associative algebra. So we have actors for associative algebras with this condition (i.e. Condition 2 in alternative algebras) in the category $\text{Alt}$. We can consider the weaker condition $(xy)z = x(yz) = z(yx) - (zy)x$ for $x, y, z \in A \cup (\cup B_i)$, which does not imply in general associativity of $A$. This condition is important for the fulfilment of the action conditions for the elements from $\cup B_i$. The existence of an actor in $\text{Alt}$ under this condition can be studied in the future.

Consider now a more general case, where $(C, E, \Omega)$ is a category of interest and $\Omega_2 = \{+, *, *^0\}$. In this case Axiom 2 contains two identities

$$(y * z) * x = W_1(y, z; x; *, *),$$

$$(y * z) *^0 x = W_2(y, z; x; *, *^0).$$

So under Axiom 2 we mean the two identities above. $\tilde{\text{Axiom2}}$ will denote the corresponding identities from $E$ (see Section 2). For $C \in C$ and $x, y, z \in C$ denote

$$T = \{(y * (x * z), z * (x * y)), (y * (z * x), z * (y * x)), ((y * x) * z, (z * x) * y), ((x * y) * z, (x * z) * y)\}.$$

**Condition 3.** a) The words $W_1$ and $W_2$ in Axiom 2 contain at least one element from each pair of the set $T$ so that each one can be expressed...
by $W_1$ or $W_2$ (e.g. computing from Axiom 2 we must have $y \ast (x \ast z) = W_2(x, z; y; \ast, \ast^0)$, the analogous equalities for other elements from $T$) in a direct way, i.e. not due to identities from $E$ or their consequences.

b) $W_1(x, y; z \ast t; \ast, \ast)$ and $W_2(z, t; x \ast y; \ast, \ast^0)$ are the same words up to commutativity of “juxtapositions”.

Here under “juxtaposition” we mean e.g. $x_1(x_2x_3)$ and thus each member from the eight members in $W(\ )$.

**Condition 3.** It is analogous to Condition 3 but is stated for elements of certain $A \in C$ and the elements of its different actions, thus for $x, y, z, t \in A \cup (\cup B_i)$, whenever they have a sense. We admit that a) and b) conditions are fulfilled not necessarily in a direct way. So there can be applied identities from $\tilde{E}$ and the special properties of $A$ itself.

**Condition 4.** The final decompositions of the words $W(x \ast y, z; t; \ast, \tilde{\ast})$ and $W(x, y; z; \ast, \ast) \tilde{\ast} t, W(x \ast y, z; t; \ast^0, \tilde{\ast})$ and $W(x, y; z; \ast, \ast^0) \tilde{\ast} t$ are the same up to commutativity of “juxtapositions”, $\ast \in \{\ast, \ast^0\}$. We mean the corresponding indices for $W$ in each case.

**Condition 4.** It is analogous to Condition 4 but we have $x, y, z, t \in A \cup (\cup B_i)$ and we mean that we have the equalities between pairs of words given in Condition 4, applying identities from $E$ and the special properties of $A$.

**Proposition 4.9.** i) If $\mathfrak{B}(A)$ has the derived action on $A$ then $A$ satisfies Condition 3.

ii) If $\mathfrak{B}(A) \in C$, then $A$ satisfies Condition 4.

**Proof.** i) From the Theorem 2.5 and the definition of the algebra $B \ltimes A$ it follows that the conditions for derived actions for $\ast$ operation follow from the Axioms 1 and 2, where $x_1, x_2, x_3 \in A \cup B$, whenever it has a sense. The result now follows directly from the conditions of the Proposition.

ii) It is obvious.

**Remark 2.** Leibniz algebras, associative algebras, satisfying Conditions 1, 2, respectively, are examples of i) in Proposition 4.9. Note that identities in $\tilde{E}$ involving only once the operation $\ast$, e.g. $x \ast y = y \ast x$ or $x \ast y = -y \ast x$ play an important role. This is the case e.g. of commutative associative and Lie algebras.

These conditions are not generally sufficient since we do not know what kind of identities we have in $\tilde{E}$. These conditions usually can be
not sufficient even in the case where \( E = E_G \cup \{ \text{Axiom1, Axiom2} \} \), since it may happen that Condition 3 is not fulfilled when certain \( b_i \in B_i \) is replaced by the element of \( b_i * b_p \in \mathcal{B}(A) \) in the identities involved in Condition 3. The same note we can make concerning Condition 4.

Below we summarize for the case \( E = E_G \cup \{ \text{Axiom1, Axiom2} \} \) our results and obtain

**Theorem 4.10.** Let \( \mathcal{C} \) be a category of interest, where \( \Omega_2 = \{ +, *, *^o \} \) and \( E = E_G \cup \{ \text{Axiom1, Axiom2} \} \).

a) If \( \mathcal{C} \) satisfies Condition 3, then \( \mathcal{B}(A) \) has a derived action on \( A \) for any \( A \in \mathcal{C} \).

b) If \( \mathcal{C} \) satisfies Condition 4, then \( \mathcal{B}(A) \in \mathcal{C} \) for any \( A \in \mathcal{C} \).

c) If \( \mathcal{C} \) satisfies Conditions 3 and 4, then \( \mathcal{B}(A) = \text{Actor}(A) \) for any \( A \in \mathcal{C} \).

If Conditions 3 and 4 are satisfied for any \( A \in \mathcal{C} \), then these conditions are satisfied for all free algebras; it can involve certain identities which are consequences of Axiom2. But these identities can be not true for the elements of \( A \cup \mathcal{B}(A) \). In the case, where we do not have consequence identities of Axiom 2 from the fulfilment of Conditions 3 and 4 for any \( A \) it follows that Conditions 3 and 4 are also satisfied. Thus in this case Conditions 3 and 4 are sufficient conditions for the existence of an actor. But it is important to note that if Axiom 2 has no consequence identities, then Conditions 3 and 4 are always satisfied too. Actually we obtain simpler conditions for this special case.

**Theorem 4.11.** Let \( \Omega_2 = \{ +, *, *^o \} \), \( E = E_G \cup \{ \text{Axiom1, Axiom2} \} \), and Axiom 2 does not imply new identities. \( \mathcal{B}(A) = \text{Actor}(A) \) for any \( A \in \mathcal{C} \) if and only if \( W_1, W_2 \) contain at least one element from each pair of the set \( T \).

Below we consider the algebras with additional commutativity \( (x * y = y * x) \) or anticommutativity \( (x * y = -y * x) \) condition on the binary operation \( * \). We will write \( E = E_G \cup \{ \text{Axiom 1, Axiom 2, (A)Comm} \} \).

In the corresponding category of interest \( \mathcal{C} \), our construction \( \mathcal{B}(A) \) must satisfy also (a)commutativity condition. For this category we apply weaker forms of Conditions 3 and 4. We require that they are fulfilled in a direct way using only (a)commutativity property of the \( * \) operation. In this case (a)commutativity of \( * \) operation in \( \mathcal{B}(A) \) guarantees the identity

\[
W_2(y, z; x; *, *) = (-)W_2(z, y; x; *, *)
\] (4.5)
which must be fulfilled applying only (a)commutativity of the \(*\) operation and commutativity of “juxtapositions”. Note that e.g. for commutative associative algebras (4.5) does not hold in the way it is required above. For the corresponding equality in this case we apply not only commutativity of the multiplication but also associativity, thus Axiom 2 for this case.

**Theorem 4.12.** Let \(\mathbb{C}\) be a category of interest, \(\Omega_2 = \{+, \ast\}, \mathbb{E} = \mathbb{E}_G \cup \{\text{Axiom 1, Axiom 2, (A)Com}\}\). If Axiom 2 does not imply new identities and (4.5) holds, then \(\mathfrak{B}(A)\) is an actor of \(A\) for any \(A \in \mathbb{C}\) if and only if \(W(\ )\) in Axiom 2 contains at least one element from each pair of the set \(T\).

**Proof.** Direct checking of identities. \(\square\)

**Example 3.** If Axiom 2 has the form
\[
x \ast (y \ast z) = -y \ast (z \ast x) - z \ast (x \ast y)
\] (4.6)
then the category \(\mathbb{C}\) with \(\Omega_2 = \{+, \ast\}, \mathbb{E} = \mathbb{E}_G \cup \{\text{Axiom 1, Axiom 2, Com}\}\) satisfies the conditions of Theorem 4.12. The same is true for the category \(\mathbb{C}\) with the same Axiom 2 and anticommutativity property.

Note that (4.6) is equivalent to Jacobi identity, but the addition is not commutative in our case.

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