Topological current algebras in two dimensions

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ABSTRACT

Two-dimensional topological field theories possessing a non-abelian current symmetry are constructed. The topological conformal algebra of these models is analysed. It differs from the one obtained by twisting the $N = 2$ superconformal models and contains generators of dimensions 1, 2 and 3 that close a linear algebra. Our construction can be carried out with one and two bosonic currents and the resulting theories can be interpreted as topological sigma models for group manifolds.
The study of two-dimensional topological conformal field theories [1,2] has attracted much interest due to its connections with non-critical string theories [3, 4]. In particular from these studies one would like to get some insight that might help to go beyond the $d = 1$ barrier in string theory [5]. A possible way to attack this problem is by considering topological theories in which additional symmetries are present. The simplest types of such theories are the ones that enjoy a current algebra symmetry. The obvious question raised in this respect is whether or not a current algebra symmetry can coexist with a topological conformal algebra. Let us recall a similar situation in ordinary conformal field theories. In this case it is well known that the Sugawara construction allows us to associate to any current algebra an energy-momentum tensor such that the currents are dimension-one operators with respect to it.

Our goal in this paper will be to find a topological symmetry in which all generators can be represented in terms of currents in a Sugawara-like form. We shall find that this symmetry is generated by operators of dimensions 1, 2 and 3, that are grouped in topological doublets. In this point our algebra is different from the one obtained by twisting [6] the $N = 2$ superconformal models [7], in which only operators of dimensions 1 and 2 appear. The presence of dimension-three operators is an unavoidable consequence of having a non-abelian current algebra in our theory. We have found this type of topological symmetry in our study of the $gl(N, N)$ affine Lie superalgebras [8]. Here we study this problem from a general point of view.

Let us consider a Lie algebra $g$ generated by the hermitian matrices $T^a$ ($a = 1, \cdots, \dim g$) that satisfy the commutation relations

$$[T^a, T^b] = i f^{abc} T^c. \quad (1)$$

The Lie algebra $g$ is assumed to be semisimple and the generators $T^a$ are chosen in such a way that $Tr(T^a T^b) = \delta_{ab}$. An holomorphic current taking values on $g$ is
an operator \( J_a(z) \) whose Laurent modes are defined by
\[
J_a(z) = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}.
\]
(2)

The \( g \)-current algebra is obtained by requiring the modes \( J^a_n \) to satisfy the following commutation relations
\[
[J^a_n, J^b_m] = i f^{abc} J^c_{n+m} + kn \delta_{ab} \delta_{n+m,0}.
\]
(3)

In eq. (3) \( k \) is a \( c \)-number constant (the level of the algebra). An alternative definition of the current algebra is obtained by giving the operator product expansion (OPE) of two currents
\[
J_a(z_1) J_b(z_2) = \frac{k \delta_{ab}}{(z_1 - z_2)^2} + i f^{abc} \frac{J_c(z_2)}{z_1 - z_2}.
\]
(4)

The Sugawara energy-momentum tensor is given by a bilinear expression in the currents. In order to unambiguously define operators in which two or more fields evaluated at the same point are multiplied, we need to adopt a normal ordering prescription. Suppose that \( A(z) \) and \( B(z) \) are two local fields whose Laurent modes are \( A^n \) and \( B^n \),
\[
A(z) = \sum_{n \in \mathbb{Z}} A^n z^{-n-\Delta_A} \quad B(z) = \sum_{n \in \mathbb{Z}} B^n z^{-n-\Delta_B},
\]
(5)

where \( \Delta_A \) and \( \Delta_B \) are the conformal weights of \( A(z) \) and \( B(z) \) respectively. All the fields we shall encounter inside normal-ordered products will have integer conformal weights and, therefore, we shall assume that this condition is satisfied in the equations that follow. The normal-ordered product of two arbitrary modes \( :A^n B^m: \) is defined as
\[
:A^n B^m: \equiv \begin{cases} 
A^n B^m, & \text{if } m \geq 1 - \Delta_B \\
(-1)^{g(A)g(B)} B^m A^n, & \text{if } m < 1 - \Delta_B,
\end{cases}
\]
(6)

where \( g(A) \) is equal to zero (one) if the operator \( A \) is bosonic (fermionic). The modes \( (: AB :)^n \) of the normal-ordered product of \( A \) and \( B \) are defined by the equation
: A(z)B(z) : \equiv \sum_{n \in \mathbb{Z}} ( : AB :)^n z^{-n - \Delta_A - \Delta_B}.

(7)

Substituting the mode expansions of \( A(z) \) and \( B(z) \) in the left-hand side of (7) and using (6) we get

\[
( : AB :)^n = \sum_{p=1-\Delta_B}^{\infty} A^{n-p} B^p + (-1)^{g(A)g(B)} \sum_{p=\Delta_B}^{\infty} B^{-p} A^{n+p}.
\]

(8)

The Sugawara energy-momentum tensor for the currents \( J_a \) is given by [9,10,11]

\[
T(z) = \frac{1}{2k + C_A} : J_a(z)J_a(z) :
\]

(9)

where \( C_A \) is the quadratic Casimir of the adjoint representation of \( g \):

\[
f^{acd}f^{bcd} = C_A \delta_{ab}.
\]

(10)

For simply-laced algebras \( C_A \) is given by

\[
C_A = \theta^2 \left( \frac{\dim g}{r} - 1 \right),
\]

(11)

where \( r \) is the rank of \( g \) and \( \theta^2 \) is the square of the length of the highest root of \( g \). For an \( su(N) \) algebra, our conventions for the normalization of the \( T^a \) matrices correspond to \( \theta^2 = 2 \) and, therefore, eq. (11) gives \( C_A = 2N \). It can be checked using standard methods that

\[
T(z_1)J_a(z_2) = \frac{J_a(z_2)}{(z_1 - z_2)^2} + \frac{\partial J_a(z_2)}{z_1 - z_2},
\]

\[
T(z_1)T(z_2) = \frac{c}{2(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{z_1 - z_2},
\]

(12)

where the central charge \( c \) in eq. (12) is given by [11]

\[
c = \frac{2k \dim g}{2k + C_A}.
\]

(13)

When dealing with normal-ordered products as those appearing in eq. (9), one has to bear in mind that in general they are non-commutative and non-associative.
For example suppose that two fields \(A\) and \(B\), whose Laurent models are defined as in eq. (5), satisfy

\[
[A^n, B^m] = D^{n+m} + nk_D \delta_{n+m, 0},
\]

where \(k_D\) is a constant and the brackets denote a commutator if some of the fields is bosonic and an anticommutator if both are fermionic. Then using eq. (8) one can easily prove that

\[
(\{ AB \}_n^n = (-1)^{g(A)g(B)} (\{ BA \})^n_n - (\partial D)^n_n \frac{\Delta_D}{2}(\Delta_A - \Delta_B)k_D \delta_{n, 0},
\]

where \((\partial D)^n_n = -(n + \Delta_D)D^n\). Unless otherwise specified, when more than two fields are multiplied the normal order is defined inductively according to the rule

\[
: A_n \cdots A_1 : \equiv (: (:\cdots (: A_n A_{n-1} : \cdots : A_2 : A_1) : :)
\]

i.e., the product \(A_n \cdots A_1 : \) is considered as the product of \(A_n \cdots A_2 : \) with \(A_1\) and so on. A reordering formula like (15) can also be obtained for normal-ordered products of more than two fields. Proceeding as we did to get eq. (15) and using the prescription (16) one obtains

\[
(\{ ABC \}_n^n = (-1)^{g(A)g(B)} (\{ BAC \})^n_n - (\partial DC)^n_n \frac{\Delta_D}{2}(\Delta_A - \Delta_B)k_D C^n,
\]

\[
(\{ CAB \}_n^n = (-1)^{g(A)g(B)} (\{ CBA \})^n_n - (C \partial D)^n_n \frac{\Delta_D}{2}(\Delta_A - \Delta_B)C^n k_D,
\]

where we have supposed that the bracket (14) still holds.

In order to define a topological theory we need to define a BRST symmetry. We want this symmetry to be an odd analogue of the current algebra. The best way to proceed is by imitating the gauge-fixing procedure of Yang-Mills theories. So, let us introduce a zero-dimensional anticommuting field \(\gamma_a\) and the following transformation:

\[
\delta J_a = i f^{abc} \gamma_b J_c - k \partial \gamma_a.
\]

It is easy to check that, if we want eq. (18) to be compatible with the commutation relations (3), the constant \(k\) must be the same in both equations. We also have to
specify what is the BRST transformation of the ghost field $\gamma_a$. Requiring nilpotency for the $\delta$-transformation, one easily arrives at

$$\delta \gamma_a = \frac{i}{2} f^{abc} \gamma_b \gamma_c. \quad (19)$$

Indeed, one can check using the Jacobi identity for the structure constants of $g$,

$$f^{abn} f^{ncm} + f^{bcn} f^{nam} + f^{can} f^{nbm} = 0, \quad (20)$$

that

$$\delta^2 \gamma_a = \delta^2 J_a = 0. \quad (21)$$

We shall assign to $J_a$ and $\gamma_a$ ghost numbers 0 and +1 respectively. From the transformation laws (eqs. (18) and (19)) we see that the $\delta$-variation increases in one unit the ghost number of the fields.

The inclusion of the $\gamma_a$ field and the BRST transformations (eqs. (18) and (19)) are not enough to define a topological field theory. In such a theory all the fields appearing in the topological algebra must have a BRST partner. Therefore as we want to have a current algebra compatible with the topological symmetry, it is clear that we must introduce a new field such that its $\delta$-variation is the total $g$-current of the theory. Let us denote it by $\rho_a$. As it is the partner of the total $g$-current, it must be a dimension-one fermionic operator with ghost number $-1$. With this ghost number and dimension, $\rho$ can be naturally coupled to the $\gamma$ field in the action by means of a term

$$\int \rho_a \bar{\delta} \gamma_a \quad (22)$$

which implies that $\rho$ and $\gamma$ obey the following OPE

$$\rho_a(z_1) \gamma_b(z_2) = -\frac{\delta_{ab}}{z_1 - z_2} \quad (23)$$
In terms of modes, eq. (23) is equivalent to the following anticommutators

$$\{\rho^n, \gamma^m_b\} = -\delta_{ab}\delta_{n+m,0}. \tag{24}$$

Notice that the anticommuting fields $\rho$ and $\gamma$ have color indices. In fact they give a contribution to the total $g$-current, which can be written as

$$J_a = J_a + if^{abc} :\gamma_b\rho_c :. \tag{25}$$

It is straightforward to verify that one has

$$J_a(z_1)\gamma_b(z_2) = if^{abc} \frac{\gamma_c(z_2)}{z_1 - z_2}$$

$$J_a(z_1)\rho_b(z_2) = if^{abc} \frac{\rho_c(z_2)}{z_1 - z_2} \tag{26}$$

which implies that, acting on the fields $\gamma$ and $\rho$, $J_a$ correctly generates non-abelian rotations. As was said above, we want $\rho$ to be the BRST partner of the total current $J$. Therefore we write

$$\delta \rho_a = J_a = J_a + if^{abc} :\gamma_b\rho_c :. \tag{27}$$

Of course $\delta^2 \rho_a$ must vanish. By inspecting the right-hand side of (27) we see that this requirement gives rise to a self-consistency condition. In fact, performing a $\delta$-variation of the right-hand side of (27) and using eqs. (18) and (19), we get

$$\delta^2 \rho_a = -k\partial \gamma_a - \frac{1}{2}f^{abc}f^{brs} :\gamma_r\gamma_s\rho_c : + f^{abc}f^{crs} :\gamma_b(\gamma_r\rho_s) :. \tag{28}$$

where the parentheses in the last two terms indicate the normal-ordering that results when the variation is performed. As was mentioned above, the normal
order is not associative. Taking into account that

\[ \left[ \gamma^n_b, \gamma^m_r \right] = \delta_{sb} \gamma^{n+m}_r, \]  

(29)

we can reorder the operators appearing in the last term of (28) as

\[ : \gamma_b (\gamma_r \gamma_s) := (\gamma_r \gamma_s) \gamma_b : - \delta_{sb} \partial \gamma_r = - (\gamma_r \gamma_b) \rho_s : - \delta_{sb} \partial \gamma_r, \]  

(30)

where eqs. (15) and (17) have been taken into account. Using eq. (30) in eq. (28), we simply get

\[ \delta^2 \rho_a = -(k + C_A) \partial \gamma_a, \]  

(31)

i.e., only for the critical level \( k = -C_A \) is our BRST symmetry nilpotent. For this value of \( k \) the total energy-momentum tensor becomes

\[ T = -\frac{1}{C_A} : J_a J_a : + : \rho_a \partial \gamma_a :, \]  

(32)

where we have included the contribution of the \((\rho, \gamma)\) system. It is easy to check that the operator \( T \) in eq. (32) has a vanishing Virasoro anomaly. Indeed, particularizing eq. (13) to the case \( k = -C_A \), we obtain a value \( 2 \dim g \) for the central charge of the bosonic \( J_a \) currents. On the other hand, for fixed color index \( a \), the \((\rho_a, \gamma_a)\) fields form a spin-one \((b, c)\) system and therefore they give a contribution \(-2 \dim g\) to the central charge, which exactly cancels the one coming from the commuting sector of the theory. Thus the model we have at hand is certainly topological. It is easy to see that only for the critical value of \( k \) determined above does the energy-momentum tensor become BRST-exact. Indeed, notice that the only field whose \( \delta \)-variation produces the \( J_a \) current is \( \rho_a \). Therefore it is clear that by varying the operator \( \rho_a J_a \) we have a chance to obtain the bosonic part of the energy-momentum tensor. In fact we can easily prove that

\[ \delta\left[ \frac{1}{2k + C_A} : \rho_a J_a : \right] = \frac{1}{2k + C_A} : J_a J_a : + \frac{k}{2k + C_A} : \rho_a \partial \gamma_a :, \]  

(33)

where we have included a factor designed to reproduce the coefficient appearing in the Sugawara expression (eq. (9)). The right-hand side of eq. (33) reproduces
the energy-momentum tensor of the \((\rho, \gamma)\) system if the coefficient of the \(\rho \partial \gamma\) term is equal to one. Again this only happens for the value \(k = -C_A\). Therefore the BRST partner of \(T\) in eq. (32) is

\[
G(z) = -\frac{1}{C_A} : \rho_a J^a :. \tag{34}
\]

There is yet another argument to determine what level \(k\) gives rise to a topological theory. Consider the OPE of the total current \(J\) with itself. A short calculation leads to

\[
J_a(z_1) J_b(z_2) = \frac{(k + C_A) \delta_{ab}}{(z_1 - z_2)^2} + i f^{abc} \frac{J_c(z_2)}{z_1 - z_2}, \tag{35}
\]

and, therefore, \(k = -C_A\) is the value for which the total Kac-Moody level vanishes. Actually, for this value of \(k\), \(J_a\) and its fermionic counterpart \(\rho_a\) close an algebra without any central extension:

\[
\begin{align*}
J_a(z_1) J_b(z_1) &= i f^{abc} \frac{J_c(z_2)}{z_1 - z_2} \\
J_a(z_1) \rho_b(z_2) &= i f^{abc} \frac{\rho_c(z_2)}{z_1 - z_2} \\
\rho_a(z_1) \rho_b(z_2) &= 0. \tag{36}
\end{align*}
\]

From now on we shall put \(k = -C_A\) in all our expressions. Notice that, for an abelian theory, \(C_A\) vanishes and therefore \(T\) and \(G\) in eqs. (32) and (34) become ill-defined. In fact our non-abelian theory is only consistent at the quantum level, since without taking quantum effects into account, we would have obtained the condition \(k = 0\), and the classical energy-momentum tensor cannot be defined for such a value of \(k\).

We shall call an algebra as the one displayed in eq. (36) a **topological current algebra**. These algebras, which are the main subject of this paper, are generated by two dimension-one operators, one bosonic \((J_a)\) and the other fermionic \((\rho_a)\),
related by a topological symmetry, in such a way that they form a BRST doublet.
We now want to construct a topological algebra (containing $T$ and its partner $G$ among its generators) compatible with the algebra (36). Let us see how this construction can be carried out. First of all, let us write a local expression for the generator of the topological BRST algebra:

$$Q = - : \gamma_a J_a : - \frac{i}{2} f^{abc} : \gamma_a \gamma_b \rho_c : .$$  \hspace{1cm} (37)

It can be checked that the $\delta$-variations (18), (19) and (27) are obtained by (anti)commuting with the zero-modes of $Q$. In fact the OPE’s of $Q$ with the fields of the theory are

$$Q(z_1) J_a(z_2) = \frac{1}{z_1 - z_2} (i f^{abc} \gamma_b(z_2) J_c(z_2) + C_A \partial \gamma_a(z_2)) + \frac{C_A}{(z_1 - z_2)^2} \gamma_a(z_2)$$

$$Q(z_1) \gamma_a(z_2) = \frac{i}{2} f^{abc} \gamma_b(z_2) \gamma_c(z_2)$$

$$Q(z_1) \rho_a(z_2) = \frac{1}{z_1 - z_2} (J_a(z_2) + i f^{abc} \gamma_b(z_2) \rho_c(z_2)),$$  \hspace{1cm} (38)

while $Q$ acts on the $(\rho, J)$ currents as

$$Q(z_1) \rho_a(z_2) = \frac{J_a(z_2)}{z_1 - z_2}$$

$$Q(z_1) J_a(z_2) = 0.$$  \hspace{1cm} (39)

From the explicit expressions of $Q$ and $G$ (eqs. (37) and (34)), one can work out the OPE’s of $Q$ with itself and with $G$, with the result

$$Q(z_1) Q(z_2) = 0$$

$$Q(z_1) G(z_2) = \frac{T(z_2)}{z_1 - z_2} + \frac{R(z_2)}{(z_1 - z_2)^2} + \frac{d}{(z_1 - z_2)^3},$$  \hspace{1cm} (40)

where $R$ and $d$ are, respectively, a $U(1)$ current and a c-number given by

$$R = : \rho_a \gamma_a : \quad d = \text{dim } g.$$  \hspace{1cm} (41)

The first equation in (40) confirms that there are no anomalies that could spoil the nilpotency of $Q$. The second one gives us how $Q$ acts on $G$. Notice that, acting with
the non-zero modes of $Q$, the $R$ operator and a c-number anomaly (proportional to the dimension of $g$) show up. The interpretation of $R$ is quite clear: it is nothing but the ghost number current. Indeed, one can check that all the fields transform under $R$ with the ghost number we have assigned them. The second equation in (40) is characteristic of the topological sigma models [12], where the parameter $d$ is the dimension of the target space in which the bosonic sector of the theory is embedded. For this reason $d$ is called the dimension of the topological algebra. Notice that, in our case, $d$ is equal to the dimension of the Lie group whose Lie algebra is $g$, which leads us to conclude that we are describing a topological sigma model for the group manifold. After all, this interpretation is quite natural, since what we are considering is nothing but the Wess-Zumino-Witten model supplemented with a ghost sector—the level of the bosonic currents being chosen in such a way that all local degrees of freedom can be gauged away. In order to establish the nature of the topological symmetry, we have to compute all OPE’s involving $Q$, $R$, $G$ and $T$. After some calculations we get

$$T(z_1)Q(z_2) = \frac{Q(z_2)}{(z_1 - z_2)^2} + \frac{\partial Q(z_2)}{z_1 - z_2}$$

$$T(z_1)R(z_2) = -\frac{d}{(z_1 - z_2)^3} + \frac{R(z_2)}{(z_1 - z_2)^2} + \frac{\partial R(z_2)}{z_1 - z_2}$$

$$T(z_1)G(z_2) = \frac{2G(z_2)}{(z_1 - z_2)^2} + \frac{\partial G(z_2)}{z_1 - z_2}$$

$$R(z_1)R(z_2) = -\frac{d}{(z_1 - z_2)^2}$$

$$R(z_1)Q(z_2) = \frac{Q(z_2)}{z_1 - z_2}$$

$$R(z_1)G(z_2) = -\frac{G(z_2)}{z_1 - z_2}.$$  

The OPE’s of eqs. (40) and (42) are also obtained when one performs a twist to an $N = 2$ superconformal algebra. However the $G(z_1)G(z_2)$ OPE vanishes in these algebras, while in our model we get

$$G(z_1)G(z_2) = \frac{W(z_2)}{z_1 - z_2}.$$  

(43)
where $W$ is a bosonic dimension-three operator with ghost number $-2$, whose explicit expression is

$$ W = \frac{i}{C_A^2} f^{abc} : J_a \rho_b \rho_c : - \frac{1}{C_A} : \partial \rho_a \rho_a : . \quad (44) $$

Remarkably, $W$ is $\delta$-exact. In order to check this fact, let us consider the following operator:

$$ V = \frac{i}{3C_A^2} f^{abc} : \rho_a \rho_b \rho_c : . \quad (45) $$

It is obvious that $V$ is an anticommuting dimension-three field with ghost number $-3$ and thus it is a good candidate to become the BRST partner of $W$. Let us compute the OPE of $Q$ and $V$. From eq. (39) we get

$$ Q(z_1) V(z_2) = \frac{i}{C_A^2} \frac{f^{abc}}{z_1 - z_2} : (J_a \rho_b \rho_c - \rho_a J_b \rho_c + \rho_a \rho_b J_c) : . \quad (46) $$

Let us now reorder the last two terms in (46) by using our general expressions (17). Taking the OPE among the topological currents (eq. (36)) into account, we get

$$ f^{abc} : \rho_a J_b \rho_c := - f^{abc} : J_a \rho_b \rho_c : - iC_A : \partial \rho_a \rho_a : , $$

$$ f^{abc} : \rho_a \rho_b J_c := f^{abc} : J_a \rho_b \rho_c : + 2iC_A : \partial \rho_a \rho_a : . \quad (47) $$

Substituting (47) back into (46) and using the fact that

$$ f^{abc} f^{ade} : \gamma_d \rho_b \rho_c \rho_e := 0 , \quad (48) $$

which is a consequence of the Jacobi identity (eq. (20)), we can write

$$ Q(z_1) V(z_2) = \frac{W(z_2)}{z_1 - z_2} . \quad (49) $$

which proves our statement that $W$ is $\delta$-exact. We must add these new fields to the set of generators of our topological algebra and we must compute their OPE’s
with any other generator. There is no a priori guarantee that the algebra close within a finite number of fields. However this is the case, as one can explicitly check. One has:

\[
Q(z_1)W(z_2) = 0
\]
\[
R(z_1)W(z_2) = -\frac{2}{z_1 - z_2} W(z_2)
\]
\[
T(z_1)W(z_2) = \frac{2}{(z_1 - z_2)^2} W(z_2) + \frac{\partial W(z_2)}{z_1 - z_2}
\]
\[
G(z_1)W(z_2) = \frac{2}{(z_1 - z_2)^2} V(z_2) + \frac{\partial V(z_2)}{z_1 - z_2}
\]
\[
R(z_1)V(z_2) = -\frac{3}{z_1 - z_2} V(z_2)
\]
\[
T(z_1)V(z_2) = \frac{2}{(z_1 - z_2)^2} V(z_2) + \frac{\partial V(z_2)}{z_1 - z_2}.
\]

Other OPE’s vanish, \textit{i.e.}

\[
G(z_1)V(z_2) = W(z_1)W(z_2) = V(z_1)W(z_2) = V(z_1)V(z_2) = 0
\] (51)

Notice that, although our algebra involves dimension-three fields, it is linear, contrary to what happens with the $W_3$ algebras.

We have thus found a topological algebra closed by three BRST doublets of operators \(((Q, R), (G, T)\text{ and } (V, W))\text{ of dimensions 1, 2 and 3. This algebra is not unknown. It has been previously found by Kazama [13] as a consistent non-trivial extension of the twisted } N = 2 \text{ superconformal algebra. In Kazama's analysis no explicit representation of the algebra was given, and it was related to an } N = 1 \text{ superconformal symmetry. A main outcome of our representation is the connection between the extended character of the algebra and non-abelian topological current algebras. In fact, apart from those in eq. (39), the only non-vanishing OPE’s among the topological currents } (\rho, \mathcal{J}) \text{ and the generators of the extended algebra}
are

\[ G(z_1)\mathcal{J}_a(z_2) = \frac{\rho_a(z_2)}{(z_1 - z_2)^2} + \frac{\partial \rho_a(z_2)}{z_1 - z_2} \]

\[ G(z_1)\rho_a(z_2) = 0, \]

which proves that, indeed, the topological and current algebras are compatible. Notice that eq. (52) is the odd analogue of the first equation in (12). It is worth pointing out that one can construct an \( N = 2 \) superconformal theory associated to the BRST charge \( Q \) in eq. (37) [14]. However, the energy-momentum tensor for this theory does not have the canonical Sugawara form (eq. (32)) and, therefore, the currents are not primary fields.

An interesting feature of the realization we have obtained is that it can be deformed in the following way:

\[ T \rightarrow T + \sum_a \alpha_a \partial \mathcal{J}_a \]

\[ G \rightarrow G + \sum_a \alpha_a \partial \rho_a \]

\[ R \rightarrow R + \sum_a \alpha_a \mathcal{J}_a, \]

(53)

where \( \alpha_a \) are constant c-numbers. The operators \( Q, V \) and \( W \) and the topological dimension \( d \) are left unaffected by the deformation. It can be easily verified that the transformed operators also satisfy the extended topological algebra. Actually eq.(53) can be regarded as implementing a quantum Hamiltonian reduction [15,16] in our theory. Of course when the transformation (53) is performed, the currents are no longer primary dimension-one operators and, therefore, the current algebra symmetry is lost.

Our construction closely resembles the one performed in (super)string theory, where the matter content of the theory is adjusted in such a way that its total Virasoro anomaly exactly cancels the one coming from the (super)conformal ghosts. This matching condition determines the critical dimension of the theory. In our case the condition required to have a topological current algebra is that the total
level \( k \) of the “matter” currents must equal \(-C_A\), which is the contribution coming from the ghost sector of the theory. Up to now, we have only considered the situation in which only one bosonic current is present in the “matter” part. It turns out, however, that a similar construction can be performed with two bosonic currents. Suppose we have two such independent currents \( J_1^a \) and \( J_2^a \) that close an affine algebra, as in eq. (4), with levels \( k_1 \) and \( k_2 \) respectively. The total current is now

\[
J_a = J_1^a + J_2^a + i f^{abc} : \gamma_b \rho_c : .
\]  

(54)

If we assume that \( J_1^a \) and \( J_2^a \) transform under the BRST symmetry as in eq. (18) and we impose \( \delta \rho_a = J_a \) again, it is clear that we shall end up with the following condition for the two levels \( k_1 \) and \( k_2 \):

\[
k_1 + k_2 = -C_A
\]  

(55)

Observe that eq. (55) only constrains the value of the sum of the current algebra levels. Denoting \( k_1 \) simply by \( k \) and using eq. (55), we get for the energy-momentum tensor of the theory:

\[
T = \frac{1}{2k + C_A} : J_1^a J_1^a : - \frac{1}{2k + C_A} : J_2^a J_2^a : + : \rho_a \partial \gamma_a : .
\]  

(56)

It is very easy to check that the operator \( T \) in (56) has vanishing Virasoro anomaly. The currents \( J_1^a \) and \( J_2^a \) give contributions \( \frac{2k}{2k + C_A} \dim g \) and \( \frac{2k + 2C_A}{2k + C_A} \dim g \) respectively, whose sum exactly cancels the central charge of the ghost system. Moreover it can be verified that \( T \) is also \( \delta \)-exact and, in fact, one has the same topological algebra as in the one-current case. The generators are now given by

\[
Q = - : \gamma_a (J_1^a + J_2^a) : - \frac{i}{2} f^{abc} : \gamma_b \gamma_c \rho_a : .
\]

\[
G = \frac{1}{2k + C_A} : \rho_a (J_1^a - J_2^a) : .
\]

\[
W = \frac{i}{(2k + C_A)^2} f^{abc} : (J_1^a + J_2^a) \rho_b \rho_c : - \frac{C_A}{(2k + C_A)^2} : \partial \rho_a \rho_a : .
\]

\[
V = \frac{i}{3(2k + C_A)^2} f^{abc} : \rho_a \rho_b \rho_c : .
\]  

(57)
$R$ and $d$ are the same as in (41). Moreover the algebra is also compatible with the current symmetry and eqs. (39) and (52) remain valid; they are the only non-vanishing OPE’s among the topological currents and the generators of the extended algebra. This means that eq. (53) also provides a deformation of the theory that preserves its extended topological symmetry. Notice that in this two-current case, our construction can be performed for an abelian current algebra. However from eq.(57) we see that $W$ and $V$ vanish identically in that case and we are left with the standard $N = 2$ twisted algebra (for $d = 0$).

The form of $T$, $Q$ and $G$ in eqs. (56) and (57) is exactly the one that appears in the $G/G$ coset models [17,18,19,20]. These models have a rich cohomology and, when they are deformed as in eq.(53), they present striking similarities with minimal matter systems coupled to two-dimensional gravity.

One might wonder if our construction can be carried out with an arbitrary number of currents. If we had $M$ bosonic currents $J_i^a$ with levels $k_i$ ($i = 1, \cdots, M$), the nilpotency requirement imposes that $\sum_i k_i = -C_A$. It is easy to verify that when $M \geq 3$ this condition for the levels is not enough to ensure the vanishing of the Virasoro central charge. Thus, in this case, by requiring the $\delta$-exactness of the total current we do not eliminate all local degrees of freedom of the theory. We could, of course, have $c = 0$ by adjusting the $k_i$ levels without spoiling the nilpotency of the BRST symmetry. However it is easy to see that $T$ cannot be put as the $\delta$-variation of an operator local in the currents unless $M$ is 1 or 2. Let us prove it. The Sugawara energy-momentum tensor of the theory is now

$$T = \sum_{i=1}^{M} \frac{1}{2k_i + C_A} : J_i^a J_i^a : + : \rho_a \partial \gamma_a : \ ,$$  

(58)

The BRST-exactness of the total current implies that

$$\delta \rho_a = \sum_{i=1}^{M} J_i^i + i f^{abc} : \gamma_b \rho_c : .$$  

(59)

Let us study the possible form of $G$, the BRST partner of $T$. If we require that $G$
be given by a local expression in the currents, it can only have the form

\[ G = \rho_a \sum_{i=1}^{M} \lambda_i J^i_a + C f^{abc} : \rho_a \rho_b \gamma_c :, \]  

(60)

where \( \lambda_i \) and \( C \) are constants. The purely bosonic term in the variation of \( G \) can be easily computed

\[ \delta G|_{\text{bosonic}} = \sum_{i,j=i}^{M} \lambda_j : J^i_a J^j_a :. \]  

(61)

Comparing this result with eq. (58) it is clear that we must require that all crossed terms in the double sum in (61) vanish, i.e.

\[ \lambda_i + \lambda_j = 0 \quad \text{for} \quad i \neq j. \]  

(62)

Eq. (62) has non-trivial solutions only if \( M = 1, 2 \), which are precisely the ones we have written in eqs. (34) and (57).

To summarize, we have been able to formulate two-dimensional topological conformal field theories possessing a current algebra symmetry. The currents of these theories form a doublet under the topological symmetry and close what we have called a topological current algebra, in which no central extension appears. The price we have to pay for preserving the current symmetry is a topological algebra involving dimension-three operators. This makes a fundamental difference with the ordinary Virasoro algebra, in which the presence of a current symmetry does not require the modification of the conformal algebra.

There are many directions in which our results can be generalized. We could, for example, ask ourselves what happens if we want to make a supercurrent algebra and a topological symmetry compatible. Presumably this is of interest in the study of non-critical superstrings. Another topic that, in our opinion, deserves further study is whether or not there is, in these theories, an analogue of the coset construction, which would allow us to generate many classes of topological
matter. Finally, it might be possible that some of the ideas developed here in a two-dimensional context could be relevant in a three- or four-dimensional situation since, after all, gauge symmetries exist in any number of space-time dimensions.

Acknowledgements: The authors would like to thank J.M.F. Labastida, J. Mas, G. Sierra, L. Alvarez-Gaumé and J. Sánchez Guillén for discussions. One of us (A.V.R.) is grateful to the CERN Theory Division, where the last part of this work was carried out, for hospitality. This work was supported in part by DGICYT under grant PB90-0772, and by CICYT under grants AEN90-0035 and AEN93-0729.
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