On a partition identity of Lehmer

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A B S T R A C T

Euler’s identity equates the number of partitions of any non-negative integer $n$ into odd parts and the number of partitions of $n$ into distinct parts. Beck conjectured and Andrews proved the following companion to Euler’s identity: the excess of the number of parts in all partitions of $n$ into odd parts over the number of parts in all partitions of $n$ into distinct parts equals the number of partitions of $n$ with exactly one even part (possibly repeated). Beck’s original conjecture was followed by generalizations and so-called “Beck-type” companions to other identities.

In this paper, we establish a collection of Beck-type companion identities to the following result mentioned by Lehmer at the 1974 International Congress of Mathematicians: the excess of the number of partitions of $n$ with an even number of even parts over the number of partitions of $n$ with an odd number of even parts equals the number of partitions of $n$ into distinct, odd parts. We also establish various generalizations of Lehmer’s identity, and prove related Beck-type companion identities. We use both analytic and combinatorial methods in our proofs.

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1. Introduction and statement of results

Many results in the theory of partitions concern identities asserting that the set $P_X(n)$ of partitions of $n$ satisfying condition $X$ and the set $P_Y(n)$ of partitions of $n$ satisfying condition $Y$ are equinumerous. Likely the oldest such result is Euler’s identity that the number of partitions of $n$ into odd parts is equal to the number of partitions of $n$ into distinct parts. In 2017, Beck made the following conjecture ([10], [2, Conjecture]):

Conjecture 1 (Beck). The excess of the number of parts in all partitions of $n$ into odd parts over the number of parts in all partitions of $n$ into distinct parts equals the number of partitions of $n$ with exactly one even part (possibly repeated).

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Beck’s conjecture was quickly proved analytically by Andrews [2], who additionally showed that this excess also equals the number of partitions of $n$ with exactly one part repeated (and all other parts distinct). The conjecture was also proved combinatorially by Yang [11] and Ballantine–Bielak [4] independently. This work was followed by generalizations and Beck-type companions to other well known identities (e.g., [3], [5], [9], [11]). In general, a Beck-type companion identity to $|\mathcal{P}_X(n)| = |\mathcal{P}_Y(n)|$ is an identity that equates the excess of the number of parts in all partitions in $\mathcal{P}_X(n)$ over the number of parts in all partitions in $\mathcal{P}_Y(n)$ to the number of partitions of $n$ satisfying a condition closely related to $X$ (or $Y$).

In this article, we establish a number of Beck-type identities related to a result of Lehmer, which he informally mentioned at the 1974 International Congress of Mathematicians [8]: for every non-negative integer $n$, we have that

$$2p_e(n, 2) = p(n) + q_o(n),$$

where

$$p_e(n, 2) := p(n \mid \text{ the number of even parts is even})$$

and

$$q_o(n) := p(n \mid \text{ distinct, odd parts}).$$

Here and throughout we use the standard notations $p(n)$ and $p(n \mid X)$ to denote the number of partitions of $n$, and the number of partitions of $n$ satisfying condition $X$, respectively. If we also denote by

$$p_o(n, 2) := p(n \mid \text{ the number of even parts is odd}),$$

identity (1) is equivalent to the following statement which we refer to as Lehmer’s identity.

**Theorem 1.1.** For any $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, we have

$$p_e(n, 2) = p_o(n, 2) + q_o(n).$$

An analytic proof of Theorem 1.1 is immediate: The generating series for $p_e(n, 2)$ and $q_o(n)$ are given by $(q; q^2)_{\infty}^{-1}(-q^2; q^2)_{\infty}^{-1}$ and $(-q; q^2)_{\infty}$, respectively. Then Theorem 1.1 follows from the fact that

$$(-q; q^2)_{\infty} = \frac{(-q; q)_{\infty}}{(-q^2; q^2)_{\infty}}$$

and Euler’s identity

$$(-q; q)_{\infty} = \frac{1}{(q; q^2)_{\infty}}.$$

Here and throughout, the $q$-Pochhammer symbol is given by

$$(a; q)_n := \begin{cases} 1, & \text{for } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{for } n > 0; \end{cases}$$

$$(a; q)_{\infty} := \lim_{n \to \infty} (a; q)_n.$$  

In [8], Gupta provided a beautiful combinatorial proof of Theorem 1.1. We also note that (2) is equivalent to the following identity due to Glaisher ([6, p.129] [7, p.256])

$$p_e(n) - p_o(n) = (-1)^n q_o(n),$$

where

$$p_{e/o}(n) := p(n \mid \text{even/odd number of parts}).$$

Our first main result, Theorem 1.2 below, is a Beck-type companion identity to Lehmer’s identity (2). To state it, we first set some additional notation. We begin by formally defining a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_j)$ of size $n \in \mathbb{N}_0$ to be a non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j$ called parts that add up to $n$. For convenience, we abuse notation and use $\lambda$ to denote either the multiset of its parts or the non-increasing sequence of parts. We write $a \in \lambda$ to mean the positive integer $a$ is a part of $\lambda$. The empty partition is the only partition of size 0. Thus, $p(0) = 1$. We write $|\lambda|$ for the size of $\lambda$, and $|\lambda| \vdash n$ to mean that $\lambda$ is a partition of size $n$. For a pair of partitions $(\lambda, \mu)$ we also write $(\lambda, \mu) \vdash n$ to mean $|\lambda| + |\mu| = n$. We use the convention that $\lambda_k = 0$ for all $k$ greater than the number of parts. When convenient we will also use the exponential notation for parts in a partition: the exponent of a part is the multiplicity of the part in the partition, e.g., we write $(a^b)$ for the partition consisting of $b$ parts equal to $a$. Further, we denote by calligraphy style capital letters the set of partitions enumerated by the function denoted by the same letter. For example, $\mathcal{P}(n)$ denotes the set of partitions of $n$ into distinct odd parts. We also define $\mathcal{Q} := \bigcup_{n \geq 0} \mathcal{Q}_n$.  

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Theorem 1.2. Let \( n \in \mathbb{N}_0 \). The excess of the number of parts in all partitions in \( \mathcal{P}_e(n, 2) \) over the number of parts in all partitions in \( \mathcal{P}_o(n, 2) \cup \mathcal{Q}_o(n) \) equals the number of partitions of \( n \) with exactly one even part, possibly repeated, and all other parts odd and distinct.

Remark 1. As proved in [3], the excess in Theorem 1.2 is almost always equal to the number of parts in all self-conjugate partitions of \( n \). Hence, the excess in the number of parts in all partitions in \( \mathcal{P}_e(n, 2) \) over the number of parts in all partitions in \( \mathcal{P}_o(n, 2) \) is almost always equal to the total number of parts in all self-conjugate partitions of \( n \) and in all partitions of \( n \) into distinct odd parts. More precisely, if \( N(x) \) is the number of times the above statement is true for \( n \leq x \), then \( \lim_{x \to \infty} N(x)/x = 1 \).

We also establish a restricted Beck-type identity accompanying (2) in which we only count the number of even parts in partitions in \( \mathcal{P}_e(n, 2) \) and \( \mathcal{P}_o(n, 2) \); this result is given in Theorem 1.3 below. To ease notation in the statement of this result and other Beck-type identities that follow, we introduce the following definition. Let \( n, r, a, b \) be non-negative integers such that \( 1 \leq ab \leq n \). We define

\[
B_1(n, a, b) := \left\{ \lambda \vdash n - rab \mid \begin{array}{l}
\lambda \neq (ra, r(a - 2)), \\
r(a + b + 1) \neq \lambda, \\
\lambda_1 - \lambda_2 \leq 2r(a + b + 1) \text{ or } \lambda_1 = 3r(a + b + 1) 
\end{array} \right\}.
\]

We write \( B(n, a, b) \) for \( B_1(n, a, b) \).

Theorem 1.3. Let \( n \in \mathbb{N}_0 \). The excess of the number of parts in all partitions in \( \mathcal{Q}_o(n) \) plus the number of even parts in all partitions in \( \mathcal{P}_o(n, 2) \) over the number of even parts in all partitions in \( \mathcal{P}_e(n, 2) \) equals the number of pairs of partitions \( (\lambda, (a^b)) \) satisfying the following conditions:

1. \( a, b \) are both odd,
2. \( \lambda \in \mathcal{Q}_o \cap B(n, a, b) \), i.e., \( \lambda \) has distinct odd parts, is not equal to \( (a, a - 2) \), does not have \( a + b + 1 \) as a part, and satisfies \( \lambda_1 - \lambda_2 \leq 2(a + b + 1) \) or \( \lambda_1 = 3(a + b + 1) \).

Remark 2. If \( n \) is even, the condition \( \lambda \neq (a, a - 2) \) in ii. is vacuously true.

In general, whenever we refer to pairs of the form \( (\lambda, (a^b)) \), we require \( (a^b) \) to be nonempty (i.e. \( a, b > 0 \)), while \( \lambda \) is allowed to be the empty partition.

Remark 3. Beck's Conjecture 1 can also be formulated in the language of pairs as in Theorem 1.3:

The excess of the number of parts in all partitions of \( n \) into odd parts over the number of parts in all partitions of \( n \) into distinct parts equals the number of pairs of partitions \( (\lambda, (a^b)) \vdash n \) satisfying the following conditions:

1. \( a \) is even,
2. \( \lambda \) is a partition into odd parts.

Next, we give a collection of Beck-type companion identities to the following generalization of Lehmer’s identity (2), which we prove in Section 3. For the remainder of the paper, we let \( r \in \mathbb{N} \).

Theorem 1.4. For any \( n \in \mathbb{N}_0 \), we have

\[
p_e(n, 2r) = p_o(n, 2r) + q_o(n, r), \tag{3}
\]

where

\[
p_e/o(n, 2r) := p(n \mid \text{all parts allowed, even/odd number of parts divisible by 2r})
\]

\[
q_o(n, r) := p \left( n \mid \begin{array}{l}
\text{parts are not divisible by 2r,} \\
\text{parts divisible by } r \text{ are distinct}
\end{array} \right).
\]

Note that for \( r = 1 \), identity (3) reduces to identity (2).

Our first Beck-type companion identity to (3) is given by the next theorem which becomes Theorem 1.2 when \( r = 1 \).

Theorem 1.5. Let \( n \in \mathbb{N}_0 \). The excess in the total number of parts in all partitions in \( \mathcal{P}_e(n, 2r) \) over the total number of parts in all partitions in \( \mathcal{P}_o(n, 2r) \cup \mathcal{Q}_o(n, r) \) equals the number of pairs of partitions \( (\lambda, (a^b)) \vdash n \) such that
i. $2^r \mid a_i$

ii. $\lambda \in \mathbb{Q}_0(n - ab, r)$.

**Remark 4.** Equivalently, the excess of Theorem 1.5 equals the number of partitions of $n$ in which, among the parts divisible by $r$, there is a single even multiple of $r$ and this part is possibly repeated, while all other parts divisible by $r$ are odd multiples of $r$ and they are distinct.

Theorem 1.6 below is a restricted Beck-type companion identity to (3), in which we only count the number of parts divisible by $r$ in $\mathbb{Q}_0(n, r)$, and the number of parts divisible by $2r$ in $\mathcal{P}_e(n, 2r)$ and $\mathcal{P}_o(n, 2r)$. The theorem reduces to Theorem 1.3 when $r = 1$.

**Theorem 1.6.** Let $n \in \mathbb{N}_0$. The excess of the number of parts divisible by $r$ in all partitions in $\mathbb{Q}_0(n, r)$ plus the number of parts divisible by $2r$ in all partitions in $\mathcal{P}_o(n, 2r)$ over the number of parts divisible by $2r$ in all partitions in $\mathcal{P}_e(n, 2r)$ equals the number of pairs of partitions $(\lambda, ((ar)^b))$ satisfying the following conditions:

i. $a, b$ are both odd,

ii. $\lambda \in \mathbb{Q}_0(n - rab, r)$ such that, if we write $\lambda = \lambda^{\text{div}} \cup \lambda^{\text{odd}}$ where $\lambda^{\text{div}}$ contains all parts of $\lambda$ that are divisible by $r$, then $\lambda^{\text{odd}} \in \mathcal{B}_r(n - |\lambda^{\text{div}}|, a, b)$.

Recall that $\lambda \cup \mu$ is the partition whose parts are precisely the parts of $\lambda$ and $\mu$ (with multiplicities).

Next we give another generalization of Lehmer’s identity (2). To describe this, we let $r \in \mathbb{N}$, and let $L_r \subseteq \{2, 4, 6, \ldots, 2r\}$, with $L_r \neq \emptyset$. We use the sets $L_r$ to restrict even parts of partitions to lie within certain arithmetic progressions. More precisely, we define

$$p_{e/o}(n, L_r, 2r) := p \left( n \mid \text{all parts allowed, even parts } \equiv \ell \pmod{2r}, \ell \in L_r \right),$$

$$q(n, L_r, r) := p \left( n \mid \text{all parts distinct, even parts } \not\equiv \ell \pmod{2r}, \ell \in L_r \right).$$

**Theorem 1.7.** For any $n \in \mathbb{N}_0$, we have

$$p_e(n, L_r, 2r) = p_o(n, L_r, 2r) + q(n, L_r, r).$$

Note that in the case $L_r = \{2, 4, \ldots, 2r\}$, identity (4) is equivalent to identity (2). The next theorem is a Beck-type companion identity to (4), which becomes Theorem 1.2 when $L_r = \{2, 4, \ldots, 2r\}$.

**Theorem 1.8.** Let $n \in \mathbb{N}_0$. The excess in the total number of parts in all partitions in $\mathcal{P}_e(n, L_r, 2r)$ over the total number of parts in all partitions in $\mathcal{P}_o(n, L_r, 2r) \cup \mathcal{Q}(n, L_r, r)$ equals the number of pairs of partitions $(\lambda, (a^b))$ satisfying the following conditions:

i. $a$ is even,

ii. $\lambda \in \mathbb{Q}(n - ab, L_r, r)$.

A restricted Beck-type companion identity to (4) is given by the next theorem, where we only count the number of even parts in $\mathcal{P}_e(n, L_r, 2r)$ and $\mathcal{P}_o(n, L_r, 2r)$. The theorem becomes Theorem 1.3 when $L_r = \{2, 4, \ldots, 2r\}$.

**Theorem 1.9.** Let $n \in \mathbb{N}_0$. The excess of the number of parts in all partitions in $\mathcal{Q}(n, L_r, r)$ plus the number of even parts in all partitions in $\mathcal{P}_o(n, L_r, 2r)$ over the number of even parts in all partitions in $\mathcal{P}_e(n, L_r, 2r)$ equals the number of pairs of partitions $(\lambda, (a^b))$ satisfying the following conditions:

i. $a, b$ are both odd,

ii. $\lambda \in \mathbb{Q}(n - ab, L_r, r)$ such that, if we write $\lambda = \lambda^e \cup \lambda^o$, where $\lambda^e$ consists of all the even parts of $\lambda$ and $\lambda^o$ consists of all the odd parts of $\lambda$, then $\lambda^o \in \mathcal{B}(n - |\lambda^e|, a, b)$.

The next result is a new restricted Beck-type companion identity to Lehmer’s identity (2), different from Theorem 1.3. We only count the number of parts in certain arithmetic progressions in $\mathbb{Q}_0(n, 2)$, $\mathcal{P}_e(n, 2)$ and $\mathcal{P}_o(n, 2)$.

To describe this, for $r \in \mathbb{N}$, let

$$L_r \subseteq \{2, 4, 6, \ldots, 2r\}, \quad \text{and} \quad O_r \subseteq \{1, 3, 5, \ldots, 2r - 1\}.$$
Theorem 1.10. Let $n$ be a positive integer, and $L_r$ and $O_r$ as above such that if $n \equiv 0 \pmod{2r}$, then $2 \not\equiv L_r$. The excess of the number of parts $\equiv \ell \pmod{2r}$, $\ell \in O_r$, in all partitions in $Q_0(n)$ plus the number of parts $\equiv \ell \pmod{2r}$, $\ell \in L_r$, in all partitions in $P_0(n, 2)$ over the number of parts $\equiv \ell \pmod{2r}$, $\ell \in L_r$, in all partitions in $P_q(n, 2)$ equals the number of pairs of partitions $(\lambda, (a^q)) \sim n$ satisfying the following conditions:

i. $a \equiv \ell \pmod{2r}$ for some $\ell \in L_r \cup O_r$, and $b$ is odd. Moreover, if $a$ is odd, then $b = 1$,

ii. $\lambda \in Q_0$. Moreover, if $a$ is odd, then $a \not\equiv \lambda$; if $a$ is even, then $\lambda_1 - \lambda_2 \leq a$ and $\lambda \not\equiv \{ (q^2 + 1, q^2 - 1), (q^2 + 2, q^2 - 2) \}$.

If $n \equiv 0 \pmod{4}$ and $2 \not\equiv L_r$, the excess is one less than the number of pairs counted above. Moreover, if we additionally have that $n \not\equiv \{4, 8, 12, 16, 20\}$, then the excess is equal to the number of pairs $(\lambda, (a^q))$ satisfying i. and ii. with the additional condition $(\lambda, (a^q)) \not\equiv \{ (9, 7, 5, 1), (2^6) \}$.

Remark 5. If $n \not\equiv 0 \pmod{4}$, then the condition $\lambda \not\equiv \{ (q^2 + 1, q^2 - 1), (q^2 + 2, q^2 - 2) \}$ is vacuously true.

Generally speaking, our proofs are both analytic and combinatorial in nature. In Sections 2 to 4, we prove Theorems 1.2 through 1.9. In Section 5, we prove Theorem 1.10 and give several important examples. In Section 6, we establish the non-negativity of the coefficients of some related $q$-series.

2. Proofs of Theorems 1.2 and 1.3

Consider the generating series

$$F(z; q) := \frac{1}{(zq; q^2)_{\infty}(-zq^2; q^2)_{\infty}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(n \mid m \text{ parts, of which } s \text{ parts are even})(-1)^s z^m q^n,$$

$$E(z; q) := \frac{1}{(q; q^2)_{\infty}(-zq^2; q^2)_{\infty}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(n \mid \text{ the number of even parts is } m)(-z)^m q^n,$$

and

$$Q_o(z; q) := (-zq; q^2)_{\infty} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(n \mid \text{ parts must be odd and distinct, } m \text{ parts}) z^m q^n.$$

To prove Theorem 1.2, note that $\frac{d}{dz} F(z; q) - Q_o(z; q))$ gives the generating series for the excess of the number of parts in all partitions in $P_q(n, 2)$ over the number of parts in all partitions in $Q_o(n) \cup P_0(n, 2)$. We have

$$\frac{d}{dz} F(z; q) = (-q; q^2)\sum_{k=0}^{\infty} \frac{q^{2k+1}}{1 - q^{2k+1}} - \sum_{k=1}^{\infty} \frac{q^{2k}}{1 + q^{2k}} - \sum_{k=0}^{\infty} \frac{q^{2k+1}}{1 + q^{2k+1}}$$

$$= (-q; q^2)\sum_{k=0}^{\infty} \frac{q^{2k+1}}{1 - q^{2k+1}} - \sum_{k=0}^{\infty} \frac{q^{2k}}{1 + q^{2k}}$$

$$= (-q; q^2)\sum_{k=0}^{\infty} \frac{q^{2k+1}}{1 - q^{2k+1}} - \sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}}$$

$$= (-q; q^2)\sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}}.$$

The last expression is the generating series for the number of partitions of $n$ with exactly one even part, possibly repeated, and all other parts odd and distinct. This proves Theorem 1.2.

To prove Theorem 1.3 we note that $\frac{d}{dz} Q_o(z; q) - E(z; q))$ is the generating series for the excess of the number of parts in all partitions in $Q_o(n)$ plus the number of even parts in all partitions in $P_0(n, 2)$ over the number of even parts in all partitions in $P_q(n, 2)$. We compute

$$\frac{d}{dz} Q_o(z; q) - E(z; q) = (-q; q^2)\sum_{k=0}^{\infty} \frac{q^{2k+1}}{1 + q^{2k+1}} + \sum_{k=1}^{\infty} \frac{q^{2k}}{1 + q^{2k}}.$$
\[
\begin{align*}
&= (-q; q^2) \sum_{k=1}^{\infty} \frac{q^k}{1 + q^k} \\
&= (-q; q^2) \left( \sum_{k=1}^{\infty} \frac{q^k}{1 - q^{2k}} - \sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}} \right).
\end{align*}
\]

Let
\[
p_{\text{oe}}(n) := p(n \mid \text{odd number of identical even parts}),
\]
i.e.,
\[
p_{\text{oe}}(n) := |\{\lambda \vdash n \mid \lambda = (a^b), a \text{ even and } b \text{ odd}\}|.
\]
Define \(p_{\text{oe}}(n), p_{\text{oe'}}(n),\) and \(p_{\text{oe'}}(n)\) similarly.

Then
\[
\sum_{k=1}^{\infty} \frac{q^k}{1 - q^{2k}} = \sum_{n=1}^{\infty} (p_{\text{oe}}(n) + p_{\text{oe'}}(n))q^n,
\]
and
\[
\sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}} = \sum_{n=1}^{\infty} (p_{\text{oe}}(n) + p_{\text{oe'}}(n))q^n.
\]
Since conjugation gives a bijection between \(P_{\text{oe}'}(n)\) and \(P_{\text{oe}'}(n)\), we further have
\[
\sum_{k=1}^{\infty} \frac{q^k}{1 - q^{2k}} - \sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}} = \sum_{n=1}^{\infty} (p_{\text{oe}}(n) - p_{\text{oe'}}(n))q^n.
\]

Therefore
\[
(-q; q^2) \sum_{k=1}^{\infty} \frac{q^k}{1 + q^k} = \left( \sum_{n=0}^{\infty} q_0(n)q^n \right) \left( \sum_{n=1}^{\infty} (p_{\text{oe}}(n) - p_{\text{oe'}}(n))q^n \right) = \sum_{n=1}^{\infty} \left( \sum_{m=0}^{n-1} q_0(m)p_{\text{oe}}(n-m) - q_0(m)p_{\text{oe'}}(n-m) \right) q^n.
\]
and the excess in question is given by
\[
\sum_{m=0}^{n-1} (q_0(m)p_{\text{oe}}(n-m) - q_0(m)p_{\text{oe'}}(n-m)).
\]

Equivalently, this is the excess of the number of elements in
\[
B(n) := \{\lambda, (a^b) \vdash n \mid \lambda \in \mathcal{Q}_o, a, b \text{ odd}\}
\]
over that in
\[
A(n) := \{\lambda, (a^b) \vdash n \mid \lambda \in \mathcal{Q}_o, a, b \text{ even}\}.
\]

To measure this excess, we construct an injection \(T\) from \(A(n)\) to \(B(n)\) as follows. We partition the set \(A(n)\) into three disjoint subsets:

- \(A_1(n) := \{\lambda, (a^b) \in A(n) \mid a + b - 1 \notin \lambda\};\)
- \(A_2(n) := \{\lambda, (a^b) \in A(n) \mid a + b - 1 \in \lambda \text{ and } \lambda \text{ has at least two parts}\};\)
- \(A_3(n) := \{\lambda, (a^b) \in A(n) \mid \lambda = (a + b - 1)\}.
\]

We define \(T\) on each \(A_i(n)\) in the following way.

1. If \((\lambda, (a^b)) \in A_1(n)\) (including the case where \(\lambda\) is empty), then

\[
T(\lambda, (a^b)) := \left( \lambda \cup \{a + b - 1\}, (a - 1)^{b-1} \right).
\]
2. If \((\lambda, (a^b)) \in A_2(n)\), then let \(m\) denote the largest part of \(\lambda\) that is not \(a + b - 1\) and define

\[
T(\lambda, (a^b)) := \left( (\lambda \setminus [m, a + b - 1]) \cup [2a + 2b - 2 + m], \left( (a - 1)^{b-1} \right) \right),
\]

where \(\lambda \setminus [m, a + b - 1]\) is the partition obtained by removing parts \(a + b - 1\) and \(m\) from \(\lambda\).

3. If \((\lambda, (a^b)) \in A_3(n)\), then \(T(\lambda, (a^b)) := ((a + 1, a - 1), (a + 1)^{b-1})\).

The image sets are thus

\[
T(A_1(n)) = \{ (\mu, (c^d)) \in B(n) \mid c + d + 1 \in \mu \};
\]

\[
T(A_2(n)) = \left\{ (\mu, (c^d)) \in B(n) \mid c + d + 1 \notin \mu, \mu_1 \neq 3(c + d + 1), \right. \]
\[
\left. \text{and } \mu_1 - \mu_2 > 2(c + d + 1) \right\};
\]

\[
T(A_3(n)) = \{ (\mu, (c^d)) \in B(n) \mid \mu = (c, c - 2) \}.
\]

Note that \(T(A_1(n))\), \(T(A_2(n))\), and \(T(A_3(n))\) are disjoint, and their union \(T(A(n))\) is a subset of \(B(n)\). Define the map \(L\) from \(T(A(n))\) to \(A(n)\) as follows:

1. If \((\mu, (c^d)) \in T(A_1(n))\), then

\[
L(\mu, (c^d)) := \left( \mu \setminus [c + d + 1], \left( (c + 1)^{d+1} \right) \right).
\]

2. If \((\mu, (c^d)) \in T(A_2(n))\), then

\[
L(\mu, (c^d)) := \left( (\mu \setminus \{\mu_1\}) \cup [c + d + 1, \mu_1 - 2(c + d + 1)], \left( (c + 1)^{d+1} \right) \right).
\]

3. If \((\mu, (c^d)) \in T(A_3(n))\), then

\[
L(\mu, (c^d)) := \left( (c + d - 1), \left( (c - 1)^{d+1} \right) \right).
\]

Then \(L\) and \(T\) are inverses of each other. Since \(T\) gives a bijection between \(A(n)\) and \(T(A(n)) \subseteq B(n)\), the excess in question is given by the number of elements in

\[
B(n) \setminus T(A(n)) = B(n) \setminus \left( T(A_1(n)) \cup T(A_2(n)) \cup T(A_3(n)) \right)
\]

\[
= \left\{ (\mu, (c^d)) \in B(n) \mid c + d + 1 \notin \mu, \mu \neq (c, c - 2), \right. \]
\[
\left. \text{and } \mu_1 - \mu_2 \leq 2(c + d + 1) \right\};
\]

\[
= \left\{ (\mu, (c^d)) \in B(n) \mid \mu \in B(n, c, d) \right\}.
\]

Theorem 1.3 now follows.

### 3. Proofs of Theorems 1.4, 1.5, and 1.6

For \(r \in \mathbb{N}\), we define

\[
F_r(z; q) := \frac{1}{(2z; q^2)_{\infty} (2z^2; q^2)_{\infty} \cdots (2z^{q-1}; q^2)_{\infty} \cdot (-zq^2; q^2)_{\infty}} \]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{m} p \left( \begin{array}{c} n \\ m \end{array} \right) \left( \begin{array}{c} m \\ s \end{array} \right) (-1)^s z^m q^n, \]

\[
R_r(z; q) := \frac{(-zq^r; q^2)_{\infty}}{(zq^r; q^2)_{\infty} (zq; q^2)_{\infty} \cdots (z^{q-1}; q^2)_{\infty}} \]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p \left( \begin{array}{c} n \\ m \end{array} \right) \left( \begin{array}{c} m \\ \text{parts are not divisible by } 2r \end{array} \right) (-1)^s z^m q^n, \]

Hence, the generating series for \(p_e(n, 2r) - p_o(n, 2r)\) and \(q_o(n, r)\) are \(F_r(1; q)\) and \(R_r(1; q)\), respectively. We have
\[ F_r(1; q) = \frac{(q^{2r}; q^{2r})_\infty}{(q; q)_\infty} \cdot \frac{1}{(q^{2r}; q^{2r})_\infty} = \frac{(q^{2r}; q^{2r})_\infty}{(q; q)_\infty} \cdot (q^{2r}; q^{2r})_\infty = \frac{(q^r; q^r)_\infty}{(q; q)_\infty} \cdot (q^{2r}; q^{2r})_\infty = R_r(1; q). \]

Here we used the facts
\[ (q^{2r}; q^{2r})_\infty (q^{2r}; q^{2r})_\infty = (q^r; q^r)_\infty \quad \text{and} \quad (q^{2r}; q^{2r})_\infty (q^{2r}; q^{2r})_\infty = (q^r; q^r)_\infty \]
in the second and fourth equality respectively, and used Euler’s identity
\[ (q; q)_\infty = \frac{1}{(q; q^2)_\infty} \]
(by replacing \( q \) by \( q^2 \)) in the third equality. Theorem 1.4 now follows.

To prove Theorem 1.5, we have that \( \frac{d}{dz} \bigg|_{z=1} (F_r(z; q) - R_r(z; q)) \) is the generating series for the excess of the total number of parts in all partitions in \( \mathcal{P}_r(n, 2r) \) over the total number of parts in all partitions in \( \mathcal{P}_r(n, 2r) \cup \mathcal{Q}_r(n, r) \). We have
\[
\frac{d}{dz} \bigg|_{z=1} (F_r(z; q) - R_r(z; q)) = R_r(1; q) \left( \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 - q^{k+2kr}} + \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 + q^{k+2kr}} - \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 - q^{k+2kr}} + \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 + q^{k+2kr}} \right)
\]
\[
= R_r(1; q) \left( \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 - q^{k+2kr}} + \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 + q^{k+2kr}} - \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 - q^{k+2kr}} + \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 + q^{k+2kr}} \right)
\]
\[
= R_r(1; q) \left( \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 - q^{k+2kr}} + \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 + q^{k+2kr}} - \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 - q^{k+2kr}} + \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 + q^{k+2kr}} \right)
\]
\[
= R_r(1; q) \sum_{k=0}^{\infty} \frac{q^{k+2kr}}{1 - q^{k+2kr}}. \quad (5)
\]

This is the generating series for the number of pairs of partitions \((\lambda, (ab^d)) \vdash n\) so that
\[
\begin{align*}
& \text{i. } 2r \mid a, \\
& \text{ii. } \lambda \in \mathcal{Q}_r(n - ab, r).
\end{align*}
\]

Equivalently, (5) is the generating series for the number of partitions of \( n \) in which among the parts divisible by \( r \) there is exactly one even multiple of \( r \), possibly repeated, and all other parts divisible by \( r \) are odd multiples of \( r \) and are distinct. This proves Theorem 1.5.

To prove Theorem 1.6, we define
\[
E_r(z; q) := \frac{1}{(q; q^{2r})_\infty (q^2; q^{2r})_\infty \cdots (q^{2r-1}; q^{2r})_\infty (-zq^{2r}; q^{2r})_\infty}
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{p(n \mid \text{all parts allowed}, m \text{ parts divisible by } 2r)(-z)^m q^n},
\]
\[
Q_r(z; q) := \frac{(-zq^r; q^{2r})_\infty}{(q; q^r)_\infty (q^2; q^r)_\infty \cdots (q^{r-1}; q^r)_\infty}
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{p(n \mid \text{parts not divisible by } 2r, \text{ parts divisible by } r \text{ are distinct}, m \text{ parts divisible by } r)}{z^n q^n}.
\]

As in the proof of Theorem 1.5, we compute
\[ \frac{\partial}{\partial z} \bigg|_{z=1} (Q_r(z; q) - E_r(z; q)) = \frac{(-q^r; q^r)_\infty}{(q; q)_\infty(q^2; q^2)_\infty \cdots (q^{r-1}; q^r)_\infty} \sum_{k=1}^{\infty} \frac{q^{kr}}{1 + q^{kr}}. \]  

In the proof of Theorem 1.3, we have shown that
\[ (-q^r; q^r)_\infty \sum_{k=1}^{\infty} \frac{q^{kr}}{1 + q^{kr}} \]
is the generating series for the number of pairs of partitions \((\lambda, (a^b)) \vdash n\) satisfying the following conditions:

i. \(a, b\) are both odd,
ii. \(\lambda \in Q_0(\mathcal{B}(n, a, b))\).

For each \(r \in \mathbb{N}\), replacing \(q\) by \(q^r\) in (7) implies that
\[ (-q^r; q^r)_\infty \sum_{k=1}^{\infty} \frac{q^{kr}}{1 + q^{kr}} \]
is the generating series for the number of pairs \((\lambda^{div}, ((a^b)^r)) \vdash n\) satisfying the following conditions:

i. \(a, b\) are both odd,
ii. \(\lambda^{div} \in Q_0((n - rab, r) \cap \mathcal{B}(n, a, b))\) and every part of \(\lambda^{div}\) is divisible by \(r\).

Theorem 1.6 follows from equation (6).

4. Proofs of Theorems 1.7, 1.8, and 1.9

For \(r \in \mathbb{N}\), \(L_r \subseteq \{2, 4, \ldots, 2r\}\) as in Section 1, we define
\[
E_{r, L_r}(z; q) := \frac{1}{(q; q^2)_\infty} \prod_{\ell \in L_r} (-zq^\ell; q^2)_\infty \\
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p \left( n \left| \begin{array}{c} \text{all parts allowed,} \\ \text{even parts} \equiv \ell \pmod{2r} \end{array} \right., \ell \in L_r \right) \left( -z \right)^m q^n,
\]
\[
F_{r, L_r}(z; q) := \frac{1}{(2q; q^2)_\infty} \prod_{\ell \in L_r} (-qz^\ell; q^2)_\infty \\
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} p \left( n \left| \begin{array}{c} \text{all odd parts allowed,} \\ \text{even parts} \equiv \ell \pmod{2r} \end{array} \right., \ell \in L_r \right) (-1)^s z^m q^n,
\]
\[
Q_{r, L_r}(z; q) := \prod_{\ell \not\in L_r} (-zq^\ell; q^2)_\infty \\
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{s=0}^{\infty} p \left( n \left| \begin{array}{c} \text{all odd parts allowed,} \\ \text{even parts} \not\equiv \ell \pmod{2r} \end{array} \right., \ell \in L_r \right) z^m q^n.
\]

Theorem 1.7 now follows from the fact that \(E_{r, L_r}(1; q) = Q_{r, L_r}(1; q)\), which is not difficult to obtain after a short calculation using Euler’s identity.

The proof of Theorem 1.8 is similar to the proofs of Theorems 1.2 and 1.5, and can be seen from
\[
\frac{\partial}{\partial z} \bigg|_{z=1} (Q_{r, L_r}(z; q) - F_{r, L_r}(z; q)) = Q_{r, L_r}(1; q) \sum_{k=1}^{\infty} \frac{q^{2k}}{1 - q^{2k}}.
\]
To prove Theorem 1.9, we compute
\[
\left. \frac{\partial}{\partial z} \right|_{z=1} (Q_{r,L_r}(z; q) - E_{r,L_r}(z; q)) = \prod_{j=1}^{2r} (-q^j; q^{2r})_\infty \left( \sum_{k=1}^{\infty} \frac{q^k}{1 + q^k} \right) \\
= \left( \prod_{j=1}^{2r} (-q^j; q^{2r})_\infty \right) (-q; q^{2r})_\infty \sum_{k=1}^{\infty} \frac{q^k}{1 + q^k}.
\]
(8)

Using the combinatorial interpretation of (7) in the proof of Theorem 1.3, Theorem 1.9 follows from (8).

5. Proof of Theorem 1.10

Let \( r \in \mathbb{N} \), \( L_r \subseteq \{2, 4, \ldots, 2r\} \) and \( O_r \subseteq \{1, 3, \ldots, 2r-1\} \) as in Section 1. Also let \( L_r^{c} = \{2, 4, \ldots, 2r\} \setminus L_r \) and \( O_r^{c} = \{1, 3, \ldots, 2r-1\} \setminus O_r \). Define
\[
\tilde{E}_{r,L_r}(z; q) := \frac{1}{(q; q^2)_\infty \prod_{j \in L_r^c} (-q^j; q^{2r})_\infty \prod_{\ell \in L_r} (-q^\ell; q^{2r})_\infty}
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (p_e(n, m; L_r) - p_0(n, m; L_r)) z^m q^n,
\]
\[
Q_{r,O_r}(z; q) := \prod_{j \in 0_r^c} (-q^j; q^{2r})_\infty \prod_{\ell \in O_r} (-q^\ell; q^{2r})_\infty
\]
\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_0(n, m; O_r) z^m q^n,
\]
where
\[
p_{e/o}(n, m; L_r) := p \left( n \left| \begin{array}{l}
\text{all parts allowed,}
\text{even/odd no. of even parts,}
\text{m (even) parts } \equiv \ell \text{ (mod 2r), } \ell \in L_r
\end{array} \right. \right).
\]
\[
q_0(n, m; O_r) := p \left( n \left| \begin{array}{l}
\text{all parts odd and distinct,}
\text{m (odd) parts } \equiv \ell \text{ (mod 2r), } \ell \in O_r
\end{array} \right. \right).
\]

When \( z = 1 \), \( \tilde{E}_{r,L_r}(1; q) = Q_{r,O_r}(1; q) \) recovers Lehmer’s identity (2) in Theorem 1.1.

We compute that
\[
\left. \frac{\partial}{\partial z} \right|_{z=1} (Q_{r,O_r}(z; q) - \tilde{E}_{r,L_r}(z; q)) = (-q; q^2)_\infty \sum_{\ell \in L_r \cup O_r} \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1 + q^{2kr+\ell}}.
\]
(9)

To prove Theorem 1.10, it suffices to prove the case where \( L_r \cup O_r = \{\ell\} \) for each positive integer \( \ell \leq 2r \). In Section 5.1, we state and prove Proposition 5.1, which establishes the non-negativity of the q-series coefficients of the series in (9) (noting that special case \( \ell = 2 \) is more delicate). We then prove Theorem 1.10, making use of Proposition 5.1 and its proof.

5.1. Non-negativity of q-series coefficients

We use the notation \( F(q) \geq 0 \) to mean that the coefficients of \( F(q) \) when expanded as a q-series are all non-negative.

Proposition 5.1. Let \( r \in \mathbb{N} \), and \( \ell \) a positive integer such that \( \ell \leq 2r \).
If \( \ell \neq 2 \), then
\[
(-q; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1 + q^{2kr+\ell}} \geq 0.
\]
If \( \ell = 2 \), then the only possible negative coefficients of
\[
(-q; q^2) \sum_{k=0}^{\infty} \frac{q^{2kr+2}}{1 + q^{2kr+2}}
\]
are
\[
(-q; q^2) \sum_{k=0}^{\infty} \frac{q^{2kr}}{1 + q^{2kr+2}}
\]
(when expanded as a \( q \)-series) are the coefficients of \( q^4 \), \( q^8 \), \( q^{12} \), \( q^{16} \), and \( q^{20} \), and any such negative coefficient is equal to \(-1\). Precisely, the set of all \( n \) such that the coefficient of \( q^n \) in (10) is negative (and thus equal to \(-1\)) is given as a function of \( r \) in the following table:

| \( r \)  | \( \{n\} \)            |
|---------|-------------------------|
| 1       | \{1\}                   |
| 2, 4, 7 | \{4, 8, 12\}            |
| 3       | \{4\}                   |
| 5       | \{4, 8\}                |
| 6, 9    | \{4, 8, 12, 16\}        |
| 8, or \( \geq 10 \) | \{4, 8, 12, 16, 20\} |

**Proof of Proposition 5.1.** We divide our proof into three cases: \( \ell \) odd, \( \ell \) even but \( \ell \neq 2 \), and \( \ell = 2 \).

For \( 0 < \ell \leq 2r \) odd, we have that
\[
(-q; q^2) \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1 + q^{2kr+\ell}} = \sum_{k=0}^{\infty} \prod_{m=0}^{\infty} q^{2kr+\ell}(1 + q^{2m+1}) \geq 0.
\]

For \( 0 < \ell \leq 2r \) even, we note that
\[
(-q; q^2) \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1 + q^{2kr+\ell}} = \sum_{k=0}^{\infty} (-q; q^2) \sum_{m=0}^{\infty} q^{2kr+\ell}(1 + q^{2m+1}) \frac{q^{2kr+\ell}}{1 - q^{2(2kr+\ell)}}.
\]

We first assume that \( \ell \neq 2 \). Using (12), it suffices to show that \(-q; q^2) \sum_{m=0}^{\infty} (1 - q^{2a}) \geq 0 \) for any integer \( a \geq 2 \). We apply the well-known identity (see, e.g., [1, (2.2.6) with \( q \mapsto q^4 \), \( t \mapsto q \))
\[
(-q; q^2) = \sum_{n=0}^{\infty} \frac{q^n}{(q^2; q^2)_n}
\]
to re-write
\[
(-q; q^2) (1 - q^{2a}) = (1 - q^{2a}) \sum_{n=0}^{\infty} \frac{q^n}{(q^2; q^2)_n}
\]
\[
= (1 - q^{2a}) \left( 1 - \frac{1}{q^2} \right) \left( \sum_{n=1}^{\infty} \frac{q^n}{(q^4; q^2)_{n-1}} \right)
\]
\[
= 1 - q^{2a} - \left( \sum_{t=0}^{a-1} q^{2t} \right) \left( \sum_{n=1}^{\infty} \frac{q^n}{(q^4; q^2)_{n-1}} \right).
\]

Thus, it suffices to show that the coefficient of \( q^{2a} \) in the \( q \)-series expansion of
\[
\left( \sum_{t=0}^{a-1} q^{2t} \right) \left( \sum_{n=1}^{\infty} \frac{q^n}{(q^4; q^2)_{n-1}} \right)
\]
is strictly positive. We re-write \( 2a = u^2 + v \), where \( u^2 \) is the largest even perfect square at most equal to \( 2a \) (with \( u \) a non-negative even integer), and \( v \) is a non-negative integer. Note that \( u^2 \geq 4 \) (so \( u \geq 2 \)), since \( 2a \geq 4 \). Since \( u \) is even, \( v \) is even, and since \( u^2 \geq 4 \), we have that \( 0 \leq v \leq 2(a - 2) \). That is, \( v = 2t \) for some \( 0 \leq t \leq a - 2 \). For this \( t \), we consider
\[
q^{2t} \sum_{n=1}^{\infty} \frac{q^{2n}}{(q^4; q^2)_{n-1}},
\]
which appears in (14). We extract the \( n = u^2 \) term \( q^{u^2}/(q^4; q^2)_{u-1} \) from the sum in (15), noting that \( u \geq 2 \). Expanding this as a \( q \)-series, we obtain...
Theorem 5.2. Combinatorial interpretation of Proposition 5.1

In this section, we give a combinatorial interpretation of the coefficient of $q^m$ in the $q$-series of Proposition 5.1 in terms of the number of pairs of partitions $(\lambda, (\alpha^\ell)) \vdash n$ satisfying certain conditions. This will complete the proof of Theorem 1.10.

Let $\ell$ be a positive integer such that $\ell \leq 2r$.

If $\ell$ is odd, then the coefficient of $q^m$ in (11) is the number of pairs of partitions $(\lambda, (\alpha)) \vdash n$ satisfying $a \equiv \ell \pmod{2r}$, $\lambda \in \mathcal{Q}_\ell$, and $a \not\equiv \lambda$. 

When $\ell = 2$, we begin with the identity in (13), which also holds for $a = 1$. In this case, the $q$-series expansion for the expression in (14) is $q + q^4 + O(q^6)$ (and has non-negative coefficients); that is, the coefficient of $q^2$ is 0. Thus, the $q$-expansion of (13) in this case is $1 + q - q^2 + q^4 + O(q^6)$, and has non-negative coefficients for all powers of $q$ greater than 4. Referring to (12), as above, we have that

$$
\sum_{k=1}^{\infty} (-q; q^2)^k (1 - q^{2k+\ell}) \frac{q^{2k+\ell}}{1-q^{2(2k+\ell)}} \geq 0
$$

for any even $\ell \in \mathcal{L}$, including $\ell = 2$. We also have from the above that the $k = 0$ term from (12) satisfies

$$( -q; q^2)_\infty (1-q^\ell) \frac{q^\ell}{1-q^\ell} \geq 0
$$

for even $\ell \geq 4$. For $\ell = 2$, we have shown that the only negative term appearing in the $q$-expansion

$$( -q; q^2)_\infty (1-q^2) = 1 + q - q^2 + q^4 + q^8 + q^9 + q^{12} + q^{13} + q^{15} + 2q^{16} + O(q^{17})
$$

is $-q^2$. We multiply (19) by

$$
\frac{q^2}{1-q^2} = q^2 + q^6 + q^{10} + q^{14} + q^{18} + q^{22} + O(q^{26})
$$

(whcih clearly has non-negative $q$-series coefficients) to obtain

$$
q^2 - q^3 - q^4 + 2q^6 + q^7 - q^8 + 3q^{10} + 2q^{11} - q^{12} + 4q^{14} + 3q^{15} - q^{16} + q^{17} + 6q^{18} + 4q^{19} - q^{20} + \Sigma(q)
$$

where $\Sigma(q) = O(q^{21})$. We now argue that $\Sigma(q) \geq 0$. It is not diffeicult to see that the only powers of $q$ in the expansion for $\Sigma(q)$ which may possibly have negative coefficients are those $q^m$ such that $m \equiv 0 \pmod{4}$, $m \geq 24$ (where we have also used that $\Sigma(q) = O(q^{21})$). Now, any $m \equiv 0 \pmod{4}$ such that $m \geq 24$ can also be written as $m = (3+1)^2 + 6 + (2+4c)$ for some integer $c \geq 0$. Thus, we also obtain the term $+q^m$ after multiplying (19) and (20) as follows. We use the expression in (13) (with $a = 1$ and $t = 0$) for (19), and take the numerator $q^{(3+1)^2}$ of the $n = 3$ term and also $q^6$ from the expansion of the denominator $(q^4; q^2)_3$ of that same term. This yields a term $q^{(3+1)^2 + 6}$ after multiplying. We now multiply by the term $q^2 + 4q^4$ from the expansion of $q^2/(1-q^4)$ in (20). Overall, this yields after multiplication the term $q^{(3+1)^2 + 6 + (2+4c)} = q^m$, which cancels with the earlier $-q^m$. This shows that $\Sigma(q) \geq 0$.

Thus, the only negative coefficients of the series in (21) are $q^4, q^8, q^{12}, q^{16}$, and $q^{20}$, and these coefficients are all equal to $-1$. When added to the rest of the sum in (17) (which has non-negative coefficients), this argument shows that the only powers of $q$ in the expansion of (10) (equivalently, (12) with $\ell = 2$) with potentially negative coefficients are $q^4, q^8, q^{12}, q^{16}, q^{20}$, and that any such negative coefficient must be $-1$, as claimed. Moreover, for $r \geq 10$, and any $k \geq 1$, we have that

$$
\frac{q^{2k+2}}{1-q^{2(2k+2)}} = O(q^{22}),
$$

which, when combined with the above argument, proves that the coefficients of $q^4, q^8, q^{12}, q^{16}$, and $q^{20}$ are all equal to $-1$. The remaining negative coefficients as given in the table in Proposition 5.1 for $1 \leq r \leq 9$ are easily calculated directly. 



If \( \ell \) is even, \( \ell \neq 2 \), we substitute (13) in (12) to obtain
\[
(-q; q^2)^\infty \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1 + q^{2r+\ell}} =
\sum_{a=\ell \pmod{2r}}^{} \sum_{a>0} \left( 1 + \left( \sum_{l=0}^{d-1} a^{2l} \right) \left( \sum_{n=0}^{\infty} q^{(n+1)/2} (q^4; q^2)_n \right) \right) \frac{q^a}{1 - q^{2a}}
\]
(22)

\[
- \sum_{a=\ell \pmod{2r}}^{} q^a \frac{q^a}{1 - q^{2a}}
\]
(23)

The q-series
\[
\sum_{n=0}^{\infty} \frac{q^{(n+1)/2}}{(q^4; q^2)_n}
\]
is the generating series for the number of self-conjugate partitions of \( n \geq 1 \) which are either (1) or whose smallest part is at least 2. Thus, this is also the generating series of the number of partitions of \( n \geq 1 \) into distinct odd parts which are either (1) or partitions whose first two parts differ by exactly 2. Moreover,
\[
\sum_{l=0}^{d-1} q^{2l} \left( \sum_{n=0}^{\infty} q^{(n+1)/2} \left( (q^4; q^2)_n \right) \right)
\]
is the generating series for partitions of \( n \) into a single even part no larger than \( a-2 \).

By adding a non-negative even integer no larger than \( a-2 \) to (1) or the first part of a partition into distinct odd parts whose first two parts differ by exactly 2, we see that
\[
\left( \sum_{l=0}^{d-1} q^{2l} \right) \left( \sum_{n=0}^{\infty} q^{(n+1)/2} \left( (q^4; q^2)_n \right) \right)
\]
is the generating series for the number of partitions \( \lambda \) of \( n \geq 1 \) into distinct odd parts with \( \lambda_1 - \lambda_2 \leq a \). Note that \( \lambda_2 \) can be 0.

Thus, (22) is the generating series for the number of pairs of partitions \( (\lambda, (a^b)) \vdash n \) with \( b \) odd, \( a \equiv \ell \pmod{2r} \), and \( \lambda \in \mathcal{Q}_a \) satisfying \( \lambda_1 - \lambda_2 \leq a \). Note that \( \lambda \) may be empty.

To interpret (23), for \( a \geq 4 \) and even, define \( \mu(a) \in \mathcal{Q}_a(a) \) to be the partition
\[
\mu(a) := \begin{cases} \left( \frac{a}{2} + 1, \frac{a}{2} - 1 \right) & \text{if } \frac{a}{2} \text{ is even} \\ \left( \frac{a}{2} + 2, \frac{a}{2} - 2 \right) & \text{if } \frac{a}{2} \text{ is odd} \end{cases}
\]
(24)

Thus, (23) is the generating series for the number of pairs of partitions \( (\mu(a), (a^b)) \vdash n \) with \( b \) odd, \( a \equiv \ell \pmod{2r} \).

Therefore, the coefficient of \( q^n \) in
\[
(-q; q^2)^\infty \sum_{k=0}^{\infty} \frac{q^{2kr+\ell}}{1 + q^{2r+\ell}}
\]
is equal to the number of the pairs of partitions \( (\lambda, (a^b)) \vdash n \) with \( b \) odd, \( a \equiv \ell \pmod{2r} \), and \( \lambda \in \mathcal{Q}_a \) satisfying \( \lambda_1 - \lambda_2 \leq a \) and \( \lambda \neq \mu(a) \).

If \( \ell = 2 \), the argument above fails only in the interpretation of \( q^\ell \frac{q^d}{1-q^d} \) when \( a = 2 \). However, this q-series is just \( \sum_{k=0}^{\infty} q^{4k} \). Thus, if \( \ell = 2 \) and \( n \equiv 0 \pmod{4} \), the coefficient of \( q^n \) in (12) is one less than the number of pairs of partitions described above. If \( n \geq 24 \), \( n \equiv 0 \pmod{4} \), then \((9, 7, 5, 1), (2^{(n-22)/2})/2)\) is a pair counted by the sequence whose generating series is (22). Thus, if \( n \not\in \{4, 8, 12, 16, 20\} \), the coefficient of \( q^n \) in (12) is non-negative and equal to the number of pairs of partitions \( (\lambda, (a^b)) \vdash n \) with \( b \) odd, \( a \equiv \ell \pmod{2r} \), and \( \lambda \in \mathcal{Q}_a \) satisfying \( \lambda_1 - \lambda_2 \leq a \), \( \lambda \neq \mu(a) \) and, if \( n = 0 \pmod{4} \), also \((\lambda, (a^d)) \not\in \{(9, 7, 5, 1), (2^{(n-22)/2})/2)\)\).

If \( \ell = 2, \ell \geq 10, \) and \( n \in \{4, 8, 12, 16, 20\} \), then \( q^n \) appears only when \( a = 2 \) in (22). If \( b \) is odd, \( n - 2b \equiv 2 \pmod{4} \) and \( n - 2b \leq 18 \). Thus, a partition \( \lambda \vdash n - 2b \) into distinct parts would have two parts. However, since \( n - 2b \equiv 2 \pmod{4} \), it follows that \( \lambda_1 - \lambda_2 \geq 4 \). Therefore, there are no pairs of partitions \( (\lambda, (a^b)) \) counted by the sequence whose generating series is (22) and the coefficient of \( q^n \) in (12) is \(-1\). For \( r < 10 \), one can easily verify that the values of \( n \) giving negative coefficients are as in the statement of the theorem.
5.3. Examples of Theorem 1.10

In this section, we give some examples of Theorem 1.10 for specific choices of $L_r$ and $O_r$ in which the excess is non-negative for all $n$. Example 1 gives a new interpretation for the excess studied in Theorem 1.3. Examples 2 and 3, when specialized to $r = 1$, are also related to the excess studied in Theorem 1.3.

5.3.1. Example 1: $L_r \cup O_r = \{1, \ldots, 2r\}$

The excess in this case is the same as that in Theorem 1.3, but the combinatorial description given by Theorem 1.3 and Theorem 1.10 are different.

When $n \not\equiv 0 \pmod{4}$, Theorem 1.10 becomes: The excess of the number of parts in all partitions in $Q_0(n)$ plus the number of even parts in all partitions of $P_r(n, 2)$ over the number of even parts in all partitions in $P_o(n, 2)$ equals the number of pairs of partitions $(\lambda, (a^2)) \vdash n$ satisfying the following conditions:

i. $b$ is odd. Moreover, if $a$ is odd, then $b = 1$.
ii. $\lambda \in Q_0$. Moreover, if $a$ is odd, then $a \neq \lambda$; if $a$ is even, then $\lambda_1 - \lambda_2 \leq a$.

When $n \in \{4, 8, 12, 16, 20\}$, since $2 \in L_r \cup O_r$, Theorem 1.10 does not guarantee the non-negativity of the excess. However, since $L_r \supseteq \{2, 4, \ldots, 2r\}$, when $n \equiv 0 \pmod{4}$, we can exclude $(\varnothing, (n))$ instead of $((9, 7, 5, 1), (2^3))$, proving non-negativity of the excess. Specifically, the excess is now equal to the number of pairs of partitions $(\lambda, (a^2)) \vdash n$ satisfying the following conditions:

i. $b$ is odd. Moreover, if $a$ is odd, then $b = 1$.
ii. $\lambda \in Q_0$. Moreover, if $a$ is odd, then $a \neq \lambda$; if $a$ is even, then $\lambda_1 - \lambda_2 \leq a$ and $\lambda \neq \mu(a)$; if $a \equiv 0 \pmod{4}$ and $b = 1$, then $\lambda \neq \varnothing$.

With the modification explained above we have the following corollary of Theorem 1.10:

**Corollary 5.2.** The excess of the number of even parts in all partitions of $P_o(n, 2)$ over the number of even parts in all partitions of $P_r(n, 2)$ equals the number of partitions $\lambda$ of $n$ such that exactly one part is even, all other parts are odd and distinct, the even part may be repeated an odd number of times and, if we write $\lambda = (\lambda^o \cup (2^k))^o$ with $\lambda^o \in Q_0$, $k \geq 1$, then $\lambda_1^o - \lambda_2^o \leq 2k$, $\lambda^o \neq \mu(2k)$, and if $k$ is even and $b = 1$ then $\lambda^o \neq \varnothing$.

**Proof.** Corollary 5.2 follows from the fact that counting $(\lambda, (a)) \vdash n$ with $a$ odd and $a \neq \lambda$ is the same as counting parts in $Q_0(n)$. When $a = 2k$, we insert the even (possibly repeated) part into the partition into distinct odd parts to obtain a statement similar to the original Beck conjecture. □

Combining Theorem 1.2 and Corollary 5.2, we arrive at the following corollary.

**Corollary 5.3.** The excess of the number of odd parts in all partitions in $P_o(n, 2)$ over the number of odd parts in $P_o(n, 2) \cup Q_0(n)$ equals the number of partitions $\lambda$ of $n$ satisfying either

i. $\lambda$ has exactly one even part, possibly repeated, and all other parts are odd and distinct, or
ii. all parts of $\lambda$ are odd and exactly one part $b$ is repeated. Moreover, let $\lambda^o = \lambda \setminus (b^2)$ be the partition obtained by removing from $\lambda$ the largest even number of parts equal to $b$, then $\lambda_1^o - \lambda_2^o \leq 2k$, $\lambda^o \neq \mu(2k)$, and if $k$ is even and $b = 1$ then $\lambda^o \neq \varnothing$.

**Proof.** The excess in Corollary 5.3 is the sum of the excess in Theorem 1.2 and Corollary 5.2. The partitions described in ii. are disjoint from the partitions described in i. They are in one-to-one correspondence with the partitions described in Corollary 5.2. To see this correspondence, consider a partition $\mu = (\mu^o \cup (2^k))^o$ as in Corollary 5.2. Then define $\lambda = \mu^o \cup (b^2)$. Now $b$, which is odd, is a repeated part. The part $b$ may have already existed in $\mu^o$, so its multiplicity in $\lambda$ can be even or odd. □

5.3.2. Example 2: $L_r \cup O_r = \{r, 2r\}$

In this case,

$$\frac{\partial}{\partial z} \Big|_{z=1} (\tilde{Q}_r, O_r(z; q) - \tilde{E}_r, L_r(z; q)) = (-q; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{kr}}{1 + q^{kr}}.$$ 

Theorem 1.10 implies that the series has non-negative coefficients when $r \geq 3$, because $2 \not\equiv L_r \cup O_r$. When $r = 1$, both Example 1 and Proposition 5.1 show that the series has non-negative coefficients. When $r = 2$, the series also has non-negative coefficients. As in Example 1, when $n \equiv 0 \pmod{4}$, we can exclude $(\varnothing, (n))$ instead of $((9, 7, 5, 1), (2^3))$. Alternatively, one
can see that in Proposition 5.1, the coefficient of $q^n$ in $(-q; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{2k+2}}{1-q^{2k+2}}$ is negative (and equal to $-1$) only for $n = 4, 8, 12$, while it can be computed that the coefficient of $q^4, q^8, q^{12}$ in $(-q; q^2)_\infty \sum_{k=0}^{\infty} \frac{q^{4k+4}}{1+q^{4k+4}}$ are 1, 4, respectively.

5.3.3. Example 3: $L_r \cup O_r = \{1, 2r\}$

In this case,

$$\left. \frac{d}{dz} \right|_{z=0} (\tilde{Q}_{r, 0}(z; q) - \tilde{E}_{r, L_r}(z; q)) = (-q; q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2kr+1}}{1+q^{2kr+1}} + \sum_{k=1}^{\infty} \frac{q^{2kr}}{1+q^{2kr}} \right).$$

Theorem 1.10 (for $r \geq 2$) and Example 1 (for $r = 1$) imply that the series has non-negative coefficients.

Remark 6. The derivative differences given in Examples 2 and 3 also have interpretations as generating series for the number of pairs of partitions satisfying certain conditions as in Theorem 1.10.

6. Further non-negativity results

The derivative difference in Example 3 can be expressed as

$$(-q; q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2kr}}{1-q^{2kr}} - \sum_{k=0}^{\infty} \frac{q^{2(k+1)r}}{1-q^{2(k+1)r}} \right)$$

and,

$$+ \sum_{k=0}^{\infty} \frac{q^{2kr+1}}{1-q^{2kr+1}} - \sum_{k=1}^{\infty} \frac{q^{4kr}}{1-q^{4kr}}.$$}

In Theorem 6.1, we show that (25) has non-positive coefficients, and, in Theorem 6.2, we show that (26) has non-negative coefficients.

**Theorem 6.1.** For $r \in \mathbb{N}$, we have that

$$(-q; q^2)_\infty \left( \sum_{k=0}^{\infty} \frac{q^{2(k+1)r+1}}{1-q^{2(k+1)r+1}} - \sum_{k=1}^{\infty} \frac{q^{2kr}}{1-q^{4kr}} \right) \geq 0.$$

**Proof.** It suffices to construct an injection $T$ from

$$A(n) := \{ (\lambda, (a^b)) \mid n | \lambda \in Q_0, a \equiv 0 \pmod{2r}, b \text{ odd} \}$$

to

$$B(n) := \{ (\lambda, (a^b)) \mid n | \lambda \in Q_0, a \equiv 1 \pmod{2r}, b \text{ even} \}.$$

Let $(\lambda, (a^b)) \in A(n)$. Let $0 \leq c < 2r$ be the remainder of $b - 1$ when divided by $2r$. Note that $c$ is even. We partition the set $A(n)$ into two disjoint subsets:

$$A_1(n) := \{ (\lambda, (a^b)) \in A(n) \mid \lambda \neq \emptyset \};$$

$$A_2(n) := \{ (\lambda, (a^b)) \in A(n) \mid \lambda = \emptyset \}.$$

We define $T$ on each $A_1(n)$ in the following way.

1. If $\lambda \neq \emptyset$, then

$$T(\lambda, (a^b)) = (\lambda \setminus \{\lambda_1\} \cup \{\lambda_1 + ab - (a - c)(b - c)\}, ((b - c)^{a-c})).$$

2. If $\lambda = \emptyset$, then

$$T(\emptyset, (a^b)) = (\mu(ab - (a - c)(b - c)), ((b - c)^{a-c})).$$

The image sets are thus

$$T(A_1(n)) = \{ (\mu, (x^y)) \in B(n) \mid \mu_1 - \mu_2 > (y + z)(x + z) - xy \},$$

$$T(A_2(n)) = \{ (\mu, (x^y)) \in B(n) \mid \mu = \mu((x + z)(y + z) - xy) \}.$$
where $z$ is the remainder of $-y$ when divided by $2r$.

Note that $T$ maps $(\lambda, (a^\ell)) \in A(n)$ with $b \equiv \ell (\text{mod } 2r)$ to $(\mu, (x^\ell)) \in B(n)$ with $y \equiv -\ell + 1 (\text{mod } 2r)$. When $b \equiv 1 (\text{mod } 2r)$, $T(\varnothing, (a^\ell)) = (\varnothing, (b^\ell)) \notin T(A_1(n))$. When $b \equiv \ell \neq 1 (\text{mod } 2r)$, because $(y + z)(x + z) - xy \geq 2(x + y) + 4 \geq 4$, $T(A_1(n))$ and $T(A_1(n))$ are disjoint.

Define the map $L$ from $T(A(n))$ to $A(n)$ as follows:

1. If $(\mu, (x^\ell)) \in T(A_1)$, then
   \[
   L(\mu, (x^\ell)) = (\mu \setminus \{\mu_1\} \cup \{\mu_1 - (y + z)(x + z) + xy\}, ((y + z)^{x + z})).
   \]

2. If $(\mu, (x^\ell)) \in T(A_2)$, then
   \[
   L(\mu, (x^\ell)) = (\varnothing, ((y + z)^{x + z})).
   \]

Then $L$ and $T$ are inverse to each other. Hence $T$ is an injection and Theorem 6.1 follows. \qed

**Remark 7.** When $r = 1$, the injection $T$ is the bijection $(\lambda, (a^\ell)) \mapsto (\lambda, (b^\ell))$, where the conjugation $(a^\ell) \mapsto (b^\ell)$ was used in the proof of Theorem 1.3.

**Theorem 6.2.** For $r \in \mathbb{N}$, we have that

\[
(-q; q^2) \sum_{\ell=0}^{\infty} \frac{q^{\ell r+1}}{1 - q^{2(\ell r+1)}} - \sum_{\ell=1}^{\infty} \frac{q^{\ell r}}{1 - q^{4\ell r}} \geq 0.
\]

**First Proof.** Because the derivative difference in Example 3 has non-negative coefficients, Theorem 6.2 follows from Theorem 6.1. \qed

**Second Proof.** Alternatively, we can also prove Theorem 6.2 directly. It suffices to construct an injection $T$ from

\[
A(n) := \{(\lambda, (a^\ell)) \mid \lambda \in \mathcal{Q}_\alpha, a \equiv 0 \pmod{2r}, b \text{ even}\}
\]

to

\[
B(n) := \{(\lambda, (a^\ell)) \mid \lambda \in \mathcal{Q}_\alpha, a \equiv 1 \pmod{2r}, b \text{ odd}\}.
\]

We partition the set $A(n)$ into three disjoint subsets:

- $A_1(n) := \{(\lambda, (a^\ell)) \in A(n) \mid a + (2r - 1)(b - 1) \notin \lambda\};$
- $A_2(n) := \{(\lambda, (a^\ell)) \in A(n) \mid a + (2r - 1)(b - 1) \in \lambda \text{ and } \lambda \text{ has at least two parts}\};$
- $A_3(n) := \{(\lambda, (a^\ell)) \in A(n) \mid \lambda = (a + (2r - 1)(b - 1))\}.$

We define $T$ on each $A_i(n)$ in the following way.

1. If $(\lambda, (a^\ell)) \in A_1(n)$ (including the case where $\lambda$ is empty), then
   \[
   T(\lambda, (a^\ell)) := (\lambda \cup \{a + (2r - 1)(b - 1)\}, (a + 1 - 2r)^{(b - 1)})).
   \]

2. If $(\lambda, (a^\ell)) \in A_2(n)$, let $m$ denote the largest part of $\lambda$ that is not $(a + (2r - 1)(b - 1))$. Then $T(\lambda, (a^\ell))$ equals
   \[
   ((\lambda \setminus \{a + (2r - 1)(b - 1), m\}) \cup \{2(a + (2r - 1)(b - 1)) + m\}, ((a + 1 - 2r)^{(b - 1)})).\]

3. If $(\lambda, (a^\ell)) \in A_3(n)$, then
   \[
   T(\lambda, (a^\ell)) := ((a + 1, a + (2r - 2)b - (2r - 1)), ((a + 1)^{(b - 1)})).
   \]

The image sets are thus

\[
T(A_1(n)) = \{(\mu, (c^d)) \in B(n) \mid c + (2r - 1)(d + 1) \notin \mu, \mu \neq \mu_1 - \mu_2 > 2(c + (2r - 1)(d + 1)), \mu_1 \neq 3(c + (2r - 1)(d + 1))\},
\]
\[
T(A_2(n)) = \{(\mu, (c^d)) \in B(n) \mid c + (2r - 1)(d + 1) \notin \mu, \mu_1 - \mu_2 > 2(c + (2r - 1)(d + 1)), \mu_1 \neq 3(c + (2r - 1)(d + 1))\},
\]
\[
T(A_3(n)) = \{(\mu, (c^d)) \in B(n) \mid \mu = (c, c + (2r - 2)d - 2)\}.
\]
Note that when $r = 1$, $(2r - 2)d - 2 = -2 < 0$, and $2 \leq 2(c + (2r - 1)(d + 1))$. When $r > 1$, $(2r - 2)d - 2 > 0$ and $(2r - 2)d - 2 \leq 2(c + (2r - 1)(d + 1))$. Hence $T(A_1(n))$, $T(A_2(n))$, $T(A_3(n))$ are pairwise disjoint.

Define the map $L$ from $T(A(n))$ to $A(n)$ as follows:

1. If $(\mu, (c^d)) \in T(A_1(n))$, then
   \[ L(\mu, (c^d)) := (\mu \setminus \{c + (2r - 1)(d + 1), ((c + 2r - 1)^d + 1)\}). \]
2. If $(\mu, (c^d)) \in T(A_2(n))$, then we define $L(\mu, (c^d))$ by
   \[ ((\mu \setminus \{\mu_1\}) \cup \{c + (2r - 1)(d + 1), \mu_1 - 2(c + (2r - 1)(d + 1))\}, ((c + 2r - 1)^d + 1)). \]
3. If $(\mu, (c^d)) \in T(A_3(n))$, then
   \[ L(\mu, (c^d)) := ((c - 1 + (2r - 1)d), ((c - 1)^d + 1)). \]

Then $L$ and $T$ are inverses of each other. Hence, $T$ is an injection and Theorem 6.2 follows.

\[ \square \]

\textbf{Remark 8.} When $r = 1$, the second proof of Theorem 6.2 recovers the proof of Theorem 1.3.

\section*{Declaration of competing interest}

No conflicts of interest.

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