Circularly symmetric solutions in three-dimensional Teleparallel, $f(T)$ and Maxwell-$f(T)$ gravity

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ABSTRACT: We present teleparallel 3D gravity and we extract circularly symmetric solutions, showing that they coincide with the BTZ and Deser-de-Sitter solutions of standard 3D gravity. However, extending into $f(T)$ 3D gravity, that is considering arbitrary functions of the torsion scalar in the action, we obtain BTZ-like and Deser-de-Sitter-like solutions, corresponding to an effective cosmological constant, without any requirement of the sign of the initial cosmological constant. Finally, extending our analysis incorporating the electromagnetic sector, we show that Maxwell-$f(T)$ gravity accepts deformed charged BTZ-like solutions. Interestingly enough, the deformation in this case brings qualitatively novel terms, contrary to the pure gravitational solutions where the deformation is expressed only through changes in the coefficients. We investigate the singularities and the horizons of the new solutions, and amongst others we show that the cosmic censorship can be violated. Such novel behaviors reveal the new features that the $f(T)$ structure brings in 3D gravity.

KEYWORDS: Modified Gravity, f(T) gravity, 3D Gravity, teleparallel gravity, Black Holes, BTZ solutions
1 Introduction

Although standard four-dimensional (4D) General Relativity (GR) is believed to be the correct description of gravity at the classical level, its quantization faces many well-known problems. Therefore, three-dimensional (3D) gravity has gained much interest, since classically it is much simpler and thus one can investigate more efficiently its quantization. Amongst others, in 3D gravity one obtains the Banados-Teitelboim-Zanelli (BTZ) black hole [1], which is a solution to the Einstein equations with a negative cosmological constant. This black-hole solution presents interesting properties at both classical and quantum levels, and it shares several features of the Kerr black hole of 4D GR [2, 3]. Actually it is the existence of BTZ black holes that makes 3D gravity a striking toy model.
Furthermore, remarkable attention was addressed recently to topologically massive gravity, which is a generalization of 3D GR that amounts to augment the Einstein-Hilbert action by adding a Chern-Simons gravitational term, \[4, 5\] and thus the propagating degree of freedom is a massive graviton, which amongst others also admits BTZ black-hole exact solutions. The renewed interest on topologically massive gravity relies on the possibility of constructing a chiral theory of gravity at a special point of the parameter-space, as it was suggested in \[6\]. This idea has been extensively analyzed in the last three years \[7–18\], leading to a fruitful discussion that ultimately led to a significantly better understanding of the model \[19\]. Finally, it has been shown that 3D massive gravity (where the action is given by the Einstein-Hilbert action with a square-curvature term which gives rise to field equations with a second order trace) admits exacts Lifshitz metrics and black-hole solutions which are asymptotically Lifshitz \[20\].

Despite the above efforts on 3D gravitational investigations, the formulation of a quantum theory of gravity is clearly still an open problem. Therefore, it is very interesting to study further 3D scenarios, trying to examine their features, as an interim stage to the exploration of 4D gravity. In the present work we are interested in investigating 3D gravity based on torsion. In particular, the so-called “teleparallel” equivalent of General Relativity (TEGR) \[21, 22\] is an equivalent formulation of gravity, but instead of using the curvature defined via the Levi-Civita connection it uses the Weitzenböck connection that has no curvature but only torsion. The dynamical objects in such a framework are the four linearly independent vierbeins (these are parallel vector fields which is what is implied by the appellations “teleparallel”), and the advantage of this framework is that the torsion tensor is formed solely from products of first derivatives of the tetrad. Finally, as described in \[22\], the Lagrangian density, \( T \), can then be constructed from this torsion tensor under the assumptions of invariance under general coordinate transformations, global Lorentz transformations, and the parity operation, along with requiring the Lagrangian density to be second order in the torsion tensor.

In this manuscript we will present teleparallel gravity in three dimensions in the modern language, based on the pioneering works of Kawai \[23–25\], and we will examine its solutions and in particular the BTZ black hole. Note that after Kawai’s works, the research on 3D gravity with torsion was performed under the light of the unification with electromagnetism \[26–32\] or on the chiral structure \[33\], not focusing on the pure effects of torsion which is the first goal of the present work. After this teleparallel construction, and inspired by the fact that in four dimensions one can generalize the theory considering functions \( f(T) \) of the torsion scalar \[34–71\], we extend our analysis in 3D \( f(T) \)-gravity, too. This approach has been partially followed in \[72\], and such an investigation may be helpful in a twofold way, that is it can be enlightening both for 3D gravity, since new features are induced by the \( f(T) \) structure, as well as for \( f(T) \) structure itself, since the 3D framework will bring light to the usual ambiguities concerning Lorentz invariance of 4D \( f(T) \) gravity. Finally, the main subject of the present work is to extend the analysis taking into account the electromagnetic sector, in order to extract the charged circularly symmetric solutions. As we will see, these solutions are qualitatively different than the charged BTZ solutions of 3D General Relativity.
The plan of the work is as follows: In section 2, we present a brief review of Teleparallel Equivalent to General Relativity (TEGR) in four dimensions. In section 3, we present the teleparallel 3D gravity and we extract BTZ solutions, while in section 4 we formulate the 3D $f(T)$ gravity, examining also circularly symmetric exact solutions. In section 5 we extend our analysis to 3D Maxwell-$f(T)$ gravity and we extract charged static black-hole solutions. Finally, in section 6 we discuss the physical implications of the results.

2 Teleparallel Equivalent to General Relativity (TEGR)

In this section we briefly review TEGR in four dimensions. Thus, our notation is as follows: Greek indices $\mu, \nu, \ldots$ run over all coordinate space-time 0, 1, 2, 3, lower case Latin indices $i, j, \ldots$ run over spatial coordinates 1, 2, 3, capital Latin indices $A, B, \ldots$ run over the tangent space-time 0, 1, 2, 3, and lower case Latin indices (from the beginning of the alphabet) $a, b, \ldots$ will run over the tangent space spatial coordinates 1, 2, 3.

As we stated in the Introduction, the dynamical variable of the “teleparallel” gravity is the vierbein field $e_A(x^\mu)$. This forms an orthonormal basis for the tangent space at each point $x^\mu$ of the manifold, that is $e^A \cdot e^B = \eta^{AB}$, where $\eta^{AB} = \text{diag}(1, -1, -1, -1)$. Furthermore, the vector $e_A$ can be analyzed with the use of its components $e^A_{\mu}$ in a coordinate basis, that is $e^A = e^A_{\mu} \partial^\mu$.

In such a construction, the metric tensor is obtained from the dual vierbein as

$$g_{\mu\nu}(x) = \eta_{AB} e^A_{\mu}(x) e^B_{\nu}(x) . \tag{2.1}$$

Contrary to General Relativity, which uses the torsionless Levi-Civita connection, in the present formalism one uses the curvatureless Weitzenböck connection \[73\], whose torsion tensor reads

$$T^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = e^A_{\lambda} (\partial_\mu e^A_{\nu} - \partial_\nu e^A_{\mu}) . \tag{2.2}$$

Finally, the contorsion tensor, which equals the difference between Weitzenböck and Levi-Civita connections, is defined as $K^\mu_{\rho\nu} = -\frac{1}{2} (T^\mu_{\rho\nu} - T^\nu_{\rho\mu} - T^\mu_{\nu\rho})$, and it proves useful to define $S^\rho_{\mu\nu} = \frac{1}{2} (K^\mu_{\rho\nu} + \delta^\mu_\rho T^\nu_{\alpha\alpha} - \delta^\nu_\rho T^\mu_{\alpha\alpha})$.

In summary, in the present formalism all the information concerning the gravitational field is included in the torsion tensor $T^\lambda_{\mu\nu}$. Using the above quantities one can define the simplest form of the “teleparallel Lagrangian”, which is nothing else than the torsion scalar, as \[74, 75\]

$$\mathcal{L} = T \equiv S^\rho_{\mu\nu} T^\mu_{\rho\nu} = \frac{1}{4} T^\rho_{\mu\nu} T^\rho_{\mu\nu} + \frac{1}{2} T^\rho_{\mu\nu} T^\nu_{\rho\mu} - T^\rho_{\rho\nu} T^\nu_{\mu\mu} . \tag{2.3}$$

Thus, the simplest action of teleparallel gravity reads:

$$S = \frac{1}{2\kappa} \int d^4x e (T + \mathcal{L}_m) , \tag{2.4}$$

where $\kappa = 8\pi G$, $e = \det(e^A_{\mu}) = \sqrt{-g}$ and $\mathcal{L}_m$ stands for the matter Lagrangian. We mention here that the Ricci scalar $R$ and the torsion scalar $T$ differ only by a total derivative \[76\].
Variation of the action (2.4) with respect to the vierbein gives the equations of motion
\[ e^{-1} \partial_\mu (e S_A^{\mu \nu}) - e^A_\lambda T^\rho_{\mu \lambda} S^\nu_\rho - \frac{1}{4} e^b_\lambda T = 4 \pi G e^\text{em}_A T^\rho_\nu , \] (2.5)
where the mixed indices are used as in \( S_A^{\mu \nu} = e^\rho_A S^{\mu \rho \nu} \). Note that the tensor \( T^\rho_\nu \) on the right-hand side is the usual energy-momentum tensor. These equations are exactly the same as those of GR for every geometry choice, and that is why the theory is called “Teleparallel Equivalent to General Relativity”.

3 3D Teleparallel Gravity

3.1 The Model

In this subsection we review teleparallel 3D gravity and we explore its properties. Although the first investigations on the subject were performed by Kawai almost twenty years ago [23–25], in the following we provide the corresponding formulation using the language of the modern literature on the subject.

As it is known, in standard 3D gravity one is inspired by the standard 4D GR, writing:

\[ S = \frac{1}{2 \kappa} \int d^3 x (R - 2 \Lambda) , \] (3.1)

where \( \kappa \) is the three-dimensional gravitational constant, \( R \) is the Ricci scalar in 3 dimensions and \( \Lambda \) the cosmological constant. Thus, in teleparallel 3D gravity we start with the action

\[ S = \frac{1}{2 \kappa} \int d^3 x (T - 2 \Lambda) , \] (3.2)

where \( T \) is the torsion scalar given by (2.3), but in 3 dimensions, since the vierbeins and the metric are now three-dimensional (the vierbeins are now a triad field instead of a tetrad one). Therefore, in the following all the conventions that were described in the beginning of section 2 run to one dimension less.

As usual it is convenient to consider the spacetime coordinates to be \( x^\mu = t, r, \phi \). Thus, the torsion \( T^a \) will simply be \( T^a = de^a \). Let us first see what the Lagrangian of teleparallel 3D gravity could be. The more general quadratic Lagrangian in the torsion, written in differential forms for the vielbein 1-form \( e^a \), and under the assumption of zero spin-connection, is given by \([77, 78]\)

\[ S = \frac{1}{2 \kappa} \int (\rho_0 \mathcal{L}_0 + \rho_1 \mathcal{L}_1 + \rho_2 \mathcal{L}_2 + \rho_3 \mathcal{L}_3 + \rho_4 \mathcal{L}_4) , \] (3.3)

where \( \rho_i \) are parameters and

\[ \mathcal{L}_0 = \frac{1}{4} e^a \wedge \ast e_a , \quad \mathcal{L}_1 = de^a \wedge \ast de_a , \quad \mathcal{L}_2 = (de_a \wedge \ast e^a) \wedge \ast (de_b \wedge e^b) , \]
\[ \mathcal{L}_3 = (de^a \wedge e^b) \wedge \ast (de_a \wedge e_b) , \quad \mathcal{L}_4 = (de_a \wedge \ast e^b) \wedge \ast (de_b \wedge e^a) , \] (3.4)
with $\star$ denoting the Hodge dual operator and $\wedge$ the wedge product. The coupling constant $\rho_0 = -\frac{2}{3} \Lambda$ represents the cosmological constant term, and moreover since $\mathcal{L}_3$ can be written completely in terms of $\mathcal{L}_1$, in the following we set $\rho_3 = 0$ [77].

Action (3.3) can be written in a more convenient form as

$$S = \frac{1}{2\kappa} \int (T - 2\Lambda) \star 1 ,$$

where $\star 1 = e^0 \wedge e^1 \wedge e^2$, and the torsion scalar $T$ is given by

$$T = \star \left[ \rho_1 (de^a \wedge \star de_a) + \rho_2 (de_a \wedge e^a) \wedge \star (de_b \wedge e^b) + \rho_4 (de_a \wedge e^b) \wedge \star (de_b \wedge e^a) \right].$$

Expanding this expression in terms of its components it is easy to obtain the following relation

$$T = \frac{1}{2} (\rho_1 + \rho_2 + \rho_4) T^{abc} T_{abc} + \rho_2 T^{abc} T_{bca} - \rho_4 T_{a}^{ac} T_{bc} ,$$

(this is the same to the one in [23–25] but with different definitions of the corresponding constants). Therefore, we straightforwardly see that for $\rho_1 = 0$, $\rho_2 = -\frac{1}{2}$ and $\rho_4 = 1$ the above expression coincides with (2.3) in 3D, namely

$$T = \frac{1}{4} T^{abc} T_{abc} - \frac{1}{2} T^{abc} T_{bca} - T_a^{ac} T_{bc} .$$

Finally, variation of the action (3.5) with respect to the vierbein triad provides the following field equations:

$$\delta \mathcal{L} = \delta e^a \wedge \left\{ \rho_1 \left[ 2d \star de_a + i_a (de^b \wedge \star de_b) - 2i_a (de^b) \wedge \star de_b \right] 
+ \rho_2 \left\{ -2e_a \wedge d \star (de^b \wedge e_a) + 2de_a \wedge \star (de^b \wedge e_b) + i_a \left[ de^c \wedge e_c \wedge \star (de^b \wedge e_b) \right] 
- 2i_a (de^b) \wedge e_b \wedge \star (de^c \wedge e_c) \right\} 
+ \rho_4 \left\{ -2e_b \wedge d \star (e_a \wedge de^b) + 2de_b \wedge \star (e_a \wedge de^b) + i_a \left[ e_c \wedge de^b \wedge \star (de^c \wedge e_b) \right] 
- 2i_a (de^b) \wedge e_c \wedge \star (de^c \wedge e_b) \right\} \right\} 
- 2\Lambda \star e_a \right\} = 0 ,$$

where $i_a$ is the interior product and for generality we have kept the general coefficients $\rho_i$.

### 3.2 Circularly symmetric Solutions

We are interesting in circularly symmetric solutions of the above constructed 3D teleparallel gravity. Since for the moment we neglect the electromagnetic sector focusing on the gravitational features of the theory, we consider a metric ansatz of the form

$$ds^2 = N^2 dt^2 - N^{-2} dr^2 - r^2 (d\phi + N_\phi dt)^2 ,$$

where $N$ and $N_\phi$ are the lapse and shift functions respectively. We mention that for the purpose of the present work we use a different ansatz than the corresponding one in [23–25]. Such an $SO(2)$ symmetric metric arises from the following triad field up to a Lorentz transformation:

$$e^0 = N dt , \quad e^1 = N^{-1} dr , \quad e^2 = r (d\phi + N_\phi dt) .$$
We stress here that for a linear-in-$T$ 3D of 4D teleparallel gravity, the metric is related to the vierbeins in a simple way, and thus relation (3.11) is a safe result of (3.10). Note that this property holds for every coefficient choice in (3.7) [22–25], however only for the choice (3.8) teleparallel gravity coincides with the usual curvature-formulation of General Relativity.

Now, replacing the vierbein in the field equation (3.9), we obtain the following separate equations:

\[
\left( N r \frac{d^2N_\phi}{dr^2} + 3 N \frac{dN_\phi}{dr} \right) (\rho_1 + \rho_2 + \rho_4) = 0 ,
\]

\[
-T + 2 \rho_1 \left( N \frac{d^2N}{dr^2} + \frac{N \frac{dN}{dr}}{r} - \frac{N^2}{r^2} \right) - 2\Lambda = 0 ,
\]

\[
2 \left\{ \rho_1 \frac{dN_\phi}{dr} \left( -r \frac{dN}{dr} + N \right) - \rho_2 \left[ \frac{dN_\phi}{dr} \left( N + 2r \frac{dN}{dr} \right) + rN \frac{d^2N_\phi}{dr^2} \right] \right\}
+ 2 \rho_4 \frac{dN_\phi}{dr} \left( N - r \frac{dN}{dr} \right) - 2\Lambda - T = 0 ,
\]

\[
2 \left\{ \rho_1 \left[ 2 \frac{N \frac{dN}{dr}}{r} - \left( \frac{dN}{dr} \right)^2 - \left( \frac{N}{r} \right)^2 \right] + 2 (\rho_1 + \rho_2 + \rho_4) \left( r \frac{dN_\phi}{dr} \right)^2 \right\}
+ 2 \rho_4 \left[ -N \frac{d^2N}{dr^2} - \left( \frac{dN}{dr} \right)^2 + \frac{N dN}{r dr} \right] - 2\Lambda - T = 0 ,
\]

\[ T + 2\Lambda = 0 . \]

Therefore, one can extract the general solutions of these equations resulting in the lapse and shift functions of the form:

\[
N_\phi(r) = -\frac{\tilde{J}}{2r^2} ,
\]

\[
N(r) = Ar + \frac{B}{r} ,
\]

with the integration constants $A$ and $B$ given as

\[
A^2 = \frac{-\Lambda}{\rho_4 - \rho_1} , \quad B^2 = \frac{\tilde{J}^2(\rho_1 + \rho_2 + \rho_4)}{2(\rho_1 + \rho_4)} ,
\]

where $\tilde{J}$ is a constant. These solutions coincide with those obtained in [72] in the case $\rho_1 = 0$, $\rho_2 = -1/2$ and $\rho_4 = 1$. Additionally, the horizons of the aforementioned circular solution read just $r_\pm^2 = -B/A$. The above metric is similar to the extremal BTZ metric of 3D General Relativity, which reads [1]:

\[
N = \frac{r}{l} - \frac{4GMl}{r} , \quad N_\phi = -\frac{4GJ}{r^2} , \quad J = \pm Ml ,
\]

where the two constants of integration $M$ and $J$ are the usual conserved charges associated with asymptotic invariance under time displacements (mass) and rotational invariance.
(angular momentum) respectively, given by flux integrals through a large circle at spacelike infinity, and \(-1/l^2\) is the cosmological constant [1].

In order to see the similarity more transparently, let us for simplicity, and without loss of generality, set \(\rho_1 = 0\) (note that this is what is expected for the standard teleparallel Lagrangian (3.8)). In this case (3.17) can be re-written as

\[
N_\phi(r) = -\frac{\bar{J}}{2r^2}, \\
N^2(r) = -\frac{\Lambda}{\rho_4} r^2 + \frac{(\rho_2 + \rho_4)}{2\rho_4} \frac{\bar{J}^2}{r^2} - \frac{\bar{M}}{\rho_4},
\]

where \(\bar{M}\) is a constant. Additionally, the horizons of the aforementioned circular solution read:

\[
r^2_{\pm} = \frac{\bar{M} \pm \sqrt{\bar{M}^2 + 2\Lambda(\rho_2 + \rho_4)\bar{J}^2}}{-2\Lambda}.
\]

Now we can immediately compare the above solution with the standard BTZ solution of 3D General Relativity, which reads [1]:

\[
N^2 = -8GM + \frac{\bar{J}^2}{r^2} + \frac{16G^2 J^2}{r^2}, \quad N_\phi = -\frac{4GJ}{r^2}.
\]

If we want solution (3.20) to coincide with (3.22), we have to impose the identifications that \(\bar{M}\) must be proportional to \(M\), \(\bar{J}\) proportional to \(J\), and \(\Lambda\) proportional to \(-1/l^2\). However, apart from \(\rho_1 = 0\), we have to additionally fix \(\rho_2 = -2\rho_4\), which up to an overall coefficient leads exactly to the standard teleparallel Lagrangian (3.8). This was expected since, as we already mentioned in the previous section, it is just the form (3.8) that leads to a complete equivalence with General Relativity. Finally, note that in this case the torsion-scalar can be easily calculated, leading to the constant value

\[
T = -2\Lambda,
\]

that is the cosmological constant is the sole source of torsion.

At this point we have to mention that apart from the above BTZ solution, which arises for a negative cosmological constant \(\Lambda = -1/l^2\) (under the fixing \(\rho_1 = 0\), \(\rho_2 = -\frac{1}{2}\) and \(\rho_4 = 1\)), we can immediately see that in the case of positive \(\Lambda\) we obtain the 3D Deser-de-Sitter solution [79].

In summary, we saw that the 3D teleparallel gravity accepts the BTZ solution (3.20), which coincides with that of the standard (GR-like) 3D gravity (3.22), if we use the standard teleparallel Lagrangian (3.8). Additionally, for positive cosmological constant we also obtain the 3D Deser-de-Sitter solution of the standard 3D gravity. However, this coincidence with General Relativity solutions is not the case if one goes beyond the linear order in the torsion scalar, as we will see in the following.
4 3D $f(T)$ Gravity

4.1 The Model

In this section we will extend the above discussion considering arbitrary functions of the torsion scalar $f(T)$ in the 3D gravitational action. This procedure is inspired by the corresponding one in 4D teleparallel gravity, where the $f(T)$ generalization exhibits many novel features [34–66], although it seems to spoil the Lorentz invariance of the linear theory [80–82]. Thus, we consider an action of the form

$$S = \frac{1}{2\kappa} \int d^3x \left[ T + f(T) - 2\Lambda \right], \quad (4.1)$$

with the torsion scalar $T$ given by (3.7), that is we keep the general coefficients $\rho_i$. In differential forms the above action can be written as:

$$S = \frac{1}{2\kappa} \int \{[f(T) + T - 2\Lambda] \ast 1\}, \quad (4.2)$$

where now $T$ is given by (3.6). Finally, note the difference in the various conventions in 4D $f(T)$ literature, since some authors replace $T$ by $f(T)$, while the majority replace $T$ by $T + f(T)$. In this work we follow the second convention, that is the teleparallel 3D gravity discussed in the previous section is obtained by setting $f(T) = 0$.

Thus, variation with respect to the vierbein leads to the following field equations:

$$\delta L = \delta e^a \wedge \left\{ 1 + \frac{df}{dT} \right\} \left\{ \rho_1 \left[ 2d \ast de_a + i_a (de_b \wedge \ast de_b) - 2i_a (de_b \wedge \ast de_b) \right] \\
+ \rho_2 \left\{ -2e_a \wedge d \ast (de_b \wedge e_b) + 2de_a \wedge \ast (de_b \wedge e_b) + i_{de} (de_c \wedge e_b \wedge \ast (de_b \wedge e_b)) \right\} \\
+ \rho_4 \left\{ -2e_b \wedge d \ast (e_a \wedge de_b) + 2de_b \wedge \ast (e_a \wedge de_b) \\
+ i_{de} (e_c \wedge de_b \wedge \ast (de_c \wedge e_b)) - 2i_a (de_b \wedge e_c \wedge \ast (de_c \wedge e_b)) \right\} \\
+ 2 \frac{df}{dT} \frac{dT}{dt} \left[ \rho_1 \ast de_a + \rho_2 e_a \wedge \ast (de_b \wedge e_b) + \rho_4 e_b \wedge \ast (de_a \wedge e_b) \right] \\
+ \left[ f(T) - T \frac{df}{dT} \right] \ast e_a - 2\Lambda \ast e_a \right\} = 0. \quad (4.3)$$

4.2 Circularly symmetric Solutions

Similarly to the simple teleparallel case, we are interesting in circularly symmetric solutions, and thus we consider the metric (3.10). However, in the present case one must be careful relating to what vierbein choice to use. In particular, as we mentioned below relation (3.11), in the case of linear-in-$T$ 3D or 4D gravity, such a simple relation between the metric and the vierbeins is always allowed. But in the general $f(T)$ gravity in 3D or 4D this is not the case anymore, and in general one has a more complicated relation connecting the vierbein tetrad with the metric, with the former being non-diagonal even for a diagonal metric [80]. However, in the 4D cosmological investigations of $f(T)$ gravity [34–68], as well
as in its black hole solutions [69–72], the authors still use the simple relation between the vierbeins and the metric, as a first approach to reveal the structure and the feature of the theory. Therefore, in the present work, in the case of 3D $f(T)$ we do assume the simple relation between the vierbeins and the metric, as a first approach to reveal the structure and the feature of the theory. Clearly a detailed investigation of the general vierbein choice in 3D and 4D $f(T)$ gravity, and its relation to extra degrees of freedom, is a necessary step for the understanding of this novel theory [83].

Thus, following the above discussion we impose the vierbein ansatz (3.11), and for this choice the torsion scalar (3.6) in differential forms reads:

$$T = -\rho_1 \left[ \left( \frac{dN}{dr} \right)^2 + \left( \frac{N}{r} \right)^2 - \left( \frac{r d\phi}{dr} \right)^2 \right] + \rho_2 \left( r \frac{d\phi}{dr} \right)^2 + \rho_4 \left[ \frac{2N dN}{r} + \left( \frac{r d\phi}{dr} \right)^2 \right].$$

(4.4)

Inserting this expression in the field equations (4.3), we finally acquire the following separate equations for the metric functions:

$$\left( 1 + \frac{df}{dT} \right) \left( \frac{rN^2 N_\phi^2 + 3N dN_\phi}{dr^2} \right) \left( \rho_1 + \rho_2 + \rho_4 \right) + Nr \frac{d^2 f}{dT^2} \frac{dN_\phi}{dr} \left( \rho_1 + \rho_2 + \rho_4 \right) = 0,$$

(4.5)

$$- \left( 1 + \frac{df}{dT} \right) T + 2\rho_1 \left( 1 + \frac{df}{dT} \right) \left( \rho_1 \frac{dN_\phi}{dr} + N_\phi \frac{dN}{dr} - N^2 \right) + 2 \rho_2 \frac{df}{dT} \frac{dT}{dr} = 0,$$

(4.6)

$$2 \left( 1 + \frac{df}{dT} \right) \left\{ \rho_1 \frac{dN_\phi}{dr} \left( -r \frac{dN}{dr} + N \right) - \rho_2 \left[ \frac{dN_\phi}{dr} \left( N + 2r \frac{dN}{dr} \right) + rN \frac{d^2 N_\phi}{dr^2} \right] \right\} + 2 \rho_4 \left( 1 + \frac{df}{dT} \right) \frac{dN_\phi}{dr} \left( N - r \frac{dN}{dr} \right) - 2\rho_2 NrN_\phi \frac{dT}{T^2 dr^2} = 0,$$

(4.7)

$$2 \left( 1 + \frac{df}{dT} \right) \left\{ \rho_1 \frac{2N dN}{r} - \left( \frac{dN}{dr} \right)^2 - \left( \frac{N}{r} \right)^2 \right\} + 2 \left( \rho_1 + \rho_2 + \rho_4 \right) \left( r \frac{d\phi}{dr} \right)^2$$

$$+ 2 \rho_4 \left( 1 + \frac{df}{dT} \right) \left[ -N \frac{d^2 N}{dr^2} - \left( \frac{dN}{dr} \right)^2 + \frac{N dN}{r} \right]$$

$$+ f(T) - T \frac{df}{dT} - 2\Lambda - 2 \rho_4 \frac{dT}{T^2} \frac{dN_\phi}{dr} \left( \rho_4 \frac{dN}{dr} - \rho_1 \frac{N}{r} \right) - N - \left( 1 + \frac{df}{dT} \right) T = 0,$$

(4.8)

$$\frac{1 + df}{dT} T = \left[ f(T) - T \frac{df}{dT} \right] + 2\Lambda = 0.$$

(4.9)

Although the above equations seem to have a complicated form, one is able to perform an analytical elaboration. In particular, it is worth noting that if the form of $f(T)$ is specified, then one can use equation (4.9) in order to extract explicitly the value of $T$ through an algebraic equation. For instance, setting $f(T) = 0$ we obtain $T = -2\Lambda$ as
expected, since it is just the simple teleparallel result (3.23) of the previous section. For the simplest non-trivial case which has been used in 4D $f(T)$ gravity, namely the quadratic ansatz $f(T) = \alpha T^2$, which corresponds to an ultraviolet (UV) modification of the theory, we obtain

$$T = \frac{-1 \pm \sqrt{1 - 24\alpha \Lambda}}{6\alpha}, \quad (4.10)$$

and similarly one can find the solution for the general power-law case $f(T) = \alpha T^n$ or even for a fully general ansatz $f(T)$. Although solving the algebraic equation (4.9) is not possible in general, we can straightforwardly see that the corresponding solution will not depend on $r$, that is we can consider a form $T = \beta$, with $\beta$ the specific constant solution. Since $\frac{dT}{dr} = 0$, equations (4.5)-(4.9) can be simplified significantly. Let us investigate various solution subclasses. Observing the form of equation (4.5) we deduce that we have to consider two separate cases, namely $\rho_1 + \rho_2 + \rho_4 \neq 0$ and $\rho_1 + \rho_2 + \rho_4 = 0$.

- **Case $\rho_1 + \rho_2 + \rho_4 \neq 0$.**

  In this case, and assuming that $f(T) \neq -T$ (which is a trivial and unphysical case since it leads to a zero total gravitational Lagrangian), from (4.5) we obtain the simple equation

  $$\frac{d^2 N_\phi}{dr^2} = -\frac{3}{r} \frac{dN_\phi}{dr}.$$  \hspace{1cm} (4.11)

  Therefore, we acquire

  $$N_\phi(r) = -\frac{\tilde{J}}{2r^2}, \quad (4.12)$$

  where $\tilde{J}$ is the non-trivial integration constant. Going further, from (4.7) we obtain two subcases, that is $\rho_1 + 2\rho_2 + \rho_4 \neq 0$, which proves to lead to no solution, and $\rho_1 + 2\rho_2 + \rho_4 = 0$. In the later case (4.7) is an identity, however (4.4) leads to

  $$N(r) = Ar + \frac{B}{r}, \quad (4.13)$$

  with the integration constants $A$ and $B$ given as

  $$A^2 = \frac{\beta}{2(\rho_4 - \rho_1)}, \quad B^2 = \frac{\tilde{J}^2(\rho_1 + \rho_2 + \rho_4)}{2(\rho_1 + \rho_4)}, \quad (4.14)$$

  with $\rho_1 \neq \rho_4$ and $\rho_1 \neq -\rho_4$, in order for (4.8) to be satisfied ($T = \beta$ is the $r$-independent solution of (4.9)). These solutions coincide with those obtained in [72] in the case $\rho_1 = 0$, $\rho_2 = -1/2$ and $\rho_4 = 1$.

  Comparing the obtained solution (4.12) and (4.13) with the BTZ solution (3.22), we straightforwardly observe that the present solution is of a BTZ-like structure, however the effective cosmological constant proportional to $A^2$ depends on the constant $\beta$, that is on the constant solution of (4.9) (which includes the initial cosmological constant $\Lambda$ as well as the parameters of the used $f(T)$ ansatz). Therefore, even if we use the
standard teleparallel Lagrangian (3.8) (that is for \( \rho_1 = 0, \rho_2 = -\frac{1}{2} \) and \( \rho_4 = 1 \)), we
obtain

\begin{align*}
    N_\phi(r) &= -\frac{\tilde{J}}{2r^2}, \\
    N^2(r) &= \frac{\beta}{2} r^2 + \frac{\tilde{J}^2}{4r^2} - \tilde{M},
\end{align*}

(4.15)

that is a solution that is different from the BTZ solution (3.22) of standard 3D (GR-like) gravity, since the first term in the second equation has a different coefficient.

We stress that the above BTZ-like solution, corresponding to an effective cosmological constant, does not require a negative initial cosmological constant \( \Lambda \), but only a positive \( \beta \). This is a radical difference with standard 3D gravity, and indicates the novel features that the \( f(T) \) structure induces in the gravitational theory. Similarly, for a negative \( \beta \) (and the standard torsion scalar (3.8)) we can immediately see that we obtain a Deser-de-Sitter-like solution [79] corresponding to an effective cosmological constant, however again we mention that this does not require a positive initial cosmological constant.

In the specific case where \( \rho_1 = \rho_4 \) we acquire

\( \rho_1 = \rho_4, \quad \beta = 0, \quad B^2 = \frac{\tilde{J}^2(2\rho_1 + \rho_2)}{4\rho_1}, \)

(4.16)

while for \( \rho_1 = -\rho_4 \) we obtain

\( \rho_1 = -\rho_4, \quad \tilde{J} = 0, \quad A^2 = -\frac{\beta}{4\rho_1}. \)

(4.17)

Finally, if \( \frac{dN_\phi}{dr} = 0 \) in (4.11), we result to \( N_\phi = 0 \) (this integration constant is not relevant) and to (4.13), but now with

\( A^2 = \frac{\beta}{2(\rho_4 - \rho_1)}, \quad 2B^2(\rho_1 + \rho_4) = 0, \)

(4.18)

with \( \rho_2 \) unspecified.

• Case \( \rho_1 + \rho_2 + \rho_4 = 0 \).

In this case equation (4.5) is identically satisfied. Similarly to the previous solution subclass, from equation (4.7) we have two subcases, namely \( \rho_1 + 2\rho_2 + \rho_4 = 0 \) and \( \rho_1 + 2\rho_2 + \rho_4 \neq 0 \).

The first subcase leads to the simpler expressions \( \rho_2 = 0 \) and \( \rho_1 + \rho_4 = 0 \), and thus from (4.6) we result to

\( N(r) = Ar + \frac{B}{r}. \)

(4.19)

Note however that now equation (4.4) provides only the \( A \) constant:

\( A^2 = \frac{\beta}{2(\rho_4 - \rho_1)}, \)

(4.20)
while $B$ remains unspecified. Additionally, equations (4.7) and (4.8) are trivially satisfied, and therefore $N_\phi$ remains unspecified.

In the second subcase, namely $\rho_1 + 2\rho_2 + \rho_4 \neq 0$, we result to the following solution

$$N(r) = Ar, \quad A^2 = \frac{\beta}{2(\rho_4 - \rho_1)}, \quad N_\phi = -\frac{j}{2r^2}. \quad (4.21)$$

5 3D Maxwell-$f(T)$ Gravity

In this section we extend the previous discussion, incorporating additionally the electromagnetic sector. In particular, we consider an action of the form

$$S = \frac{1}{2\kappa} \int \left\{ [f(T) + T - 2\Lambda] \ast 1 \right\} + \int \mathcal{L}_F, \quad (5.1)$$

where

$$\mathcal{L}_F = -\frac{1}{2} F \wedge \ast F, \quad (5.2)$$

corresponds to the Maxwell Lagrangian and $F = dA$, with $A \equiv A_\mu dx^\mu$ is the electromagnetic potential 1-form. In this case action variation leads to the following field equations:

$$\delta \mathcal{L} = \delta e^a \wedge \left\{ 1 + \frac{df}{dT} \right\} \left\{ \rho_1 \left[ 2d \ast de_a + i_a (de_b \wedge \ast de_b) - 2i_a (de_b \wedge \ast de_b) \right] + \rho_2 \left[ -2e_a \wedge d \ast (de_b \wedge e_b) + 2de_a \wedge \ast (de_b \wedge e_b) + i_a \left[ de_c \wedge e_c \wedge \ast (de_b \wedge e_b) \right] - 2i_a (de_b \wedge e_b) \wedge \ast (de_c \wedge e_c) \right\} + \rho_4 \left[ -2e_b \wedge d \ast (e_a \wedge de_b) + 2de_b \wedge \ast (e_a \wedge de_b) + i_a \left[ e_c \wedge de_b \wedge \ast (de_c \wedge e_b) \right] - 2i_a (de_b \wedge e_c) \wedge \ast (de_e \wedge e_b) \right\} + 2 \frac{df}{dT} \left[ \rho_1 e_a + \rho_2 e_a \wedge \ast (de_b \wedge e_b) + \rho_4 e_b \wedge \ast (de_b \wedge e_a) \right] + \left[ f(T) - T \frac{df}{dT} \right] \wedge \ast e_a - 2\Lambda \ast e_a - \epsilon_{abc} s^b e^c \right\} + \delta A (d^\ast F) = 0. \quad (5.3)$$

In the above relation we have defined

$$s^a = -\left( S^a_b - \frac{1}{2} \delta^a_b S \right) e^b, \quad (5.4)$$

where

$$S^a_b = -F^{ac} F_{bc} + \frac{1}{4} s^a_b \left( F^{cd} F_{cd} \right), \quad (5.5)$$

is the energy momentum tensor for the electromagnetic field and $S = S^a_a$ its trace. Although one could investigate solution subclasses with general coupling parameters $\rho_i$, in the following for simplicity we restrict to the usual case $\rho_1 = 0$, $\rho_2 = -1/2$ and $\rho_4 = 1$ of (3.8).
In order to extract the static, circularly symmetric solutions for such a theory, we consider the diagonal ansatz

$$e^0 = N dt , \quad e^1 = K^{-1} dr , \quad e^2 = r d\phi ,$$

which yields the usual metric form [2]

$$ds^2 = N(r)^2 dt^2 - K(r)^{-2} dr^2 - r^2 d\phi^2 ,$$

having in mind the discussion of the beginning of subsection 4.2 on the alternative vierbein choices corresponding to the same metric. Concerning the electric sector of electromagnetic 2-form we assume [30]

$$F = E_r e^0 \wedge e^1 + E_\phi e^2 \wedge e^0 ,$$

where $E_r$ and $E_\phi$ are the radial and the azimuthal electric field respectively. Contracting the electromagnetic tensor with itself we obtain the electromagnetic invariant

$$F_{ab} F^{ab} = -2(E_r^2 + E_\phi^2) ,$$

and thus we extract the Maxwell energy momentum tensor

$$S^a_b = \begin{pmatrix} \frac{1}{2}(E_r^2 + E_\phi^2) & 0 & 0 \\ 0 & \frac{1}{2}(E_r^2 - E_\phi^2) & -E_r E_\phi \\ 0 & -E_r E_\phi & \frac{1}{2}(-E_r^2 + E_\phi^2) \end{pmatrix} ,$$

and its trace:

$$S = \frac{1}{2}(E_r^2 + E_\phi^2) .$$

Inserting the above ansatzes in the field equations (5.3), we finally obtain

$$T - f(T) + 2T \frac{df}{dT} + 2\Lambda + \frac{1}{2} (E_r^2 - E_\phi^2) = 0 ,$$

$$\left[ 1 + \frac{df}{dT} \right] \left( -2 K \frac{dK}{dr} + 2 K^2 \frac{dN}{dr} \right) - \frac{2}{r^2} \frac{dT}{N} \frac{dr}{dN} = E_r^2 - E_\phi^2 = 0 ,$$

$$\left[ 1 + \frac{df}{dT} \right] \left( -2 K \frac{dK}{dr} \frac{dN}{dr} - 2 K^2 \frac{d^2 N}{dr^2} + 2 K^2 \frac{dN}{r N} \frac{dN}{dr} \right) - \frac{2}{r^2} \frac{dT}{N} \frac{dr}{dN} + E_r^2 - E_\phi^2 = 0 ,$$

along with

$$E_r E_\phi = 0 ,$$

$$d^* F = 0 ,$$

where

$$T = \frac{2K(r)^2 N'(r)}{r N(r)} .$$

A first observation is that, contrary to the simple $f(T)$ case of the previous section where the torsion scalar $T$ was a constant, in the present case $T$ has in general an $r$-dependence, which disappears for a zero electric charge. Such a behavior reveals the new
features that are brought in by the richer structure of the addition of the electromagnetic sector.

Furthermore, form (5.15) we deduce that either $E_\phi = 0$ or $E_r = 0$, that is we cannot have simultaneously non-zero radial and azimuthal electric field. This is an interesting result, since it shows that the known no-go theorem of 3D GR-like gravity [29, 84], that configurations with two non-vanishing components of the Maxwell field are dynamically not allowed, holds in 3D $f(T)$ gravity too. At this point one could ask whether this result is accidental, holding only for the diagonal vierbein choice (5.6). However, as we show in appendix A, a general vierbein choice, although it will change the solution structure, it will still lead to the above no-go theorem, which is thus a general result. So let us investigate these two electric-field cases separately.

5.1 Absence of azimuthal electric field

In the case $E_\phi = 0$, that is in the absence of azimuthal electric field, equation (5.16) leads to

$$E_r = \frac{Q}{r},$$

(5.18)

where $Q$ is an integration constant, that as usual coincides with the electric charge of the circular object (black hole). In order to proceed, we will consider Ultraviolet (UV) and Infrared (IR) corrections to $f(T)$ gravity respectively.

5.1.1 UV modified 3D gravity

In order to examine the modifications on the circular solutions caused by UV modifications of 3D gravity we consider a representative ansatz of the form $f(T) = \alpha T^2$. Thus, for $\alpha \neq 0$ equation (5.12) gives:

$$T = \frac{-1 \pm \sqrt{1 - 12\alpha \left(2\Lambda + \frac{Q^2}{2r^2}\right)}}{6\alpha},$$

(5.19)

with the upper and lower signs corresponding to the positive and negative branch solutions respectively. Choosing for simplicity and without loss of generality that $\Lambda = 0$, we obtain the solution

$$T(r) = \frac{-1 \pm \sqrt{1 - 6\alpha Q^2}}{6\alpha},$$

(5.20)

corresponding to

$$N(r)^2 = \frac{1}{108} \left\{-\frac{1}{\alpha} \left\{r^2 \mp P(r)[12\alpha Q^2 + r^2] + 36\alpha Q^2 \ln r \pm 18\alpha Q^2 \ln\{r[1 + P(r)]\}\right\} + const\right\},$$

$$K(r)^2 = N(r)^2 \left[\frac{2}{3} \pm \frac{1}{3}P(r)\right]^{-2},$$

(5.21)

We mention here that if $\alpha = 0$ then (5.12) becomes linear and it has only one solution, which is given by the $\alpha \to 0$ limit of the positive branch of (5.19), namely $T(r) = -Q^2/(2r)$. In this case teleparallel gravity is restored and the corresponding solutions coincide with the BTZ ones of General Relativity.
where

\[ P(r) = \sqrt{1 - \frac{6\alpha Q^2}{r^2}}. \quad (5.22) \]

As we observe, in the zero-electric-charge limit, the above solutions coincide with those of (4.10), (4.15), with \( \tilde{J} = 0, \Lambda = 0 \) and \( f(T) = \alpha T^2 \), as expected.

Let us now investigate the singularities and the horizons of the above solutions. The first step is to find at which \( r \) do the functions \( N(r) \) and \( K(r) \) become zero or infinity. However, since these singularities may correspond to just coordinate singularities, the usual procedure is to investigate various invariants, since if these invariants diverge at one point they will do that independently of the specific coordinate basis. In standard black-hole literature of the curvature-formulated gravity (either General Relativity or its modifications) one usually studies the Ricci scalar, the Kretschmann scalar, or other invariants constructed by the Riemann tensor and its contractions. In teleparallel description of gravity one needs to examine curvature invariants too, but using the standard Levi-Civita connection, since the use of the Weitzenböck connection leads to zero curvature invariants by construction. The use of curvature invariants instead of torsion ones is also indicated by the fact that in a realistic theory matter is coupled to the gravitational sector through the metric and not the vierbeins (with the interesting exception of fermionic matter), and particles follow geodesics defined by the Levi-Civita connection. Whether one can formulate everything in terms of the Weitzenböck connection and torsion invariants, as he can do with the Levi-Civita connection and curvature invariants, is an open subject and needs further investigation, in particular relating to the quantization procedure (where the fundamental field, the metric or the vierbeins, should be determined).

Thus, in order to proceed to the singularities and horizons investigation along the above lines, we have to first solve the equation \( K(r)^2 = 0 \). However, from the form of \( K(r)^2 \) in (5.21) we can clearly see that in general this is a transcendental equation, whose roots cannot be obtained analytically apart from the root at \( r = 0 \). Thus, in the following we will examine numerically a specific case, choosing without loss of generality \( \alpha = -1 \) (note that for \( \alpha < 0 \), the torsion scalar (5.20) is always real) \( Q = 1 \) and \( \text{const} = -1 \), and in Fig. 1 we depict \( K(r)^2 \) as a function of \( r \) for both the positive and negative branch.

For the positive branch we observe that apart from the root at \( r = 0 \), there is another value \( r = r_H \) where \( K(r_H)^2 = N(r_H)^2 = 0 \). Additionally, in Fig. 2 and Fig. 3 we respectively depict the Ricci scalar \( R(r) \) and the Kretschmann scalar \( R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}(r) \), calculated using the Levi-Civita connections (which are easily calculated since the metric is known) as described above. As we can see, both these invariants exhibit a physical singularity at \( r = 0 \), however they are regular at \( r = r_H \).

In order to ensure whether \( r = r_H \) is a physical singularity or a horizon one, we consider the Painlevé-Gullstrand coordinates [85–88] through the transformation

\[ dt = d\tau + f(r) \, dr, \quad (5.23) \]

with \( f(r) \) a function of the radial coordinate \( r \). Thus, the metric (5.7) writes as

\[ ds^2 = N(r)^2 \, d\tau^2 + 2f(r) \, N(r)^2 \, drd\tau - \left[ \frac{1}{K(r)^2} - N(r)^2 f(r)^2 \right] dr^2 - r^2 d\phi^2, \quad (5.24) \]
and choosing \( f(r)^2 = \frac{1}{N(r)^2} \left[ \frac{1}{K(r)^2} - 1 \right] \) and setting \( h(r)^2 = N(r)^2 / K(r)^2 = [2 + P(r)]/3 \) we can write it in a flat Euclidean form

\[
\begin{align*}
    ds^2 &= N(r)^2 d\tau^2 + 2h(r) \sqrt{1 - K(r)^2} dr d\tau - dr^2 - r^2 d\phi^2, \\
    &\text{(5.25)}
\end{align*}
\]

which is regular at \( r = r_H \). Therefore, \( r = r_H \) is a coordinate singularity.

Furthermore, in order to examine whether \( r = r_H \) is a Killing horizon we observe that the timelike Killing vector of the metric is \( \epsilon^\mu \partial_\mu = \partial_t \), with norm \( \epsilon^\mu \epsilon^\nu = g_{tt} = N(r)^2 \) which vanishes at \( r = r_H \). Outside the horizon the Killing vector field is spacelike, while inside it is timelike and thus it corresponds to a null hypersurface, that is a cosmological Killing horizon.

For the negative branch \( K(r)^2 = 0 \) at \( r_s = 0 \) and at \( r = r_H \), however between these two values, and contrary to the positive branch, we have a third singularity where \(|K(r_{inf})|^2 \rightarrow \infty\), and according to (5.21) this happens when \( P(r_{inf}) = 2 \), that is at \( r_{inf} = \sqrt{-2\alpha Q^2} \) (in our specific numerical example \( r_{inf} = \sqrt{2} \)). Now, from Fig. 2 and Fig. 3 we observe that the curvature invariants diverge at \( r_s = 0 \) and at \( r_{inf} \), while they remain regular at \( r = r_H \). Therefore, we conclude that \( r_{inf} \) is a physical singularity while \( r = r_H \) is a Killing horizon, corresponding to an event horizon since the Killing vector field is timelike outside the horizon and it is spacelike inside. Similarly, to the previous case, in the Painleve-Gullstrand coordinates the metric is regular at the event horizon, and therefore we have a coordinate singularity. Finally, in the specific numerical example of Figures 1, 2 and 3 we observe that \( r_{inf} < r_H \), and thus the singularity will be shielded by the horizon. However, in general we can have \( r_{inf} > r_H \), that is a naked singularity, and

\[\text{Since none of the metric coefficients depends on time, the manifold has a timelike Killing vector } \partial_t, \text{ and similarly since none of the metric coefficients depends on } \phi \text{ then there exist a spacelike Killing vector field } \partial_\phi \text{ (the metric is circularly symmetric).}\]
Figure 2. The Ricci scalar $R(r)$ as a function of $r$, for the positive (thick curve) and negative (thin curve) branch of the UV modified 3D Maxwell-$f(T)$ gravity, for $\alpha = -1$, $Q = 1$ and $\const = -1$.

Figure 3. The Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}(r)$ as a function of $r$, for the positive (thick curve) and negative (thin curve) branch of the UV modified 3D Maxwell-$f(T)$ gravity, for $\alpha = -1$, $Q = 1$ and $\const = -1$.

this happens when

$$\const - 18Q^2 - 18Q^2 \ln \left(\sqrt{-2\alpha Q^2}\right) + 18Q^2 \ln 3 > 0 ,$$  \quad (5.26)

as can be seen by investigating the singularities and the root structure of $N^2(r)$ in (5.21) (examining its first and second derivatives), and calculating also $N^2(r_{inf})$.

Going beyond the above specific numerical example, we may still obtain analytical expressions for the horizon $r_H$, in specific limits. In particular, if $|6\alpha Q^2| \ll r_H^2$ for the positive branch we obtain

$$r_H \approx 2^{-\frac{1}{3}} e^{\frac{1}{2} \left(1 + \frac{\const}{3Q^2}\right)} .$$ \quad (5.27)
Similarly, for the negative branch we acquire
\[ r_H \approx 2e^\frac{1}{2}(-1+\text{const})e^{-\frac{1}{2}W_0\left(\frac{8}{9\alpha \lambda^2}e^{-\frac{1}{2}+\text{const}}\right)} , \tag{5.28} \]
where \( W_0 \) stands for the main branch of the Lambert function that is single-valued since \( W_0(x) \geq -1 \) \( [89] \) (note that this is real if the argument of the Lambert function is greater than \(-1/e\), that is if \( \frac{8}{9\alpha \lambda^2} > -e^{-\text{const}} \)).

We close this paragraph by mentioning that in the scenario at hand the physical singularities are not always shielded by the horizon. Thus, the cosmic censorship does not always hold for 3D Maxwell-\( f(T) \) gravity in the absence of azimuthal electric field.

### 5.1.2 IR modified 3D gravity

In order to examine the modifications on the circular solutions caused by IR modifications of 3D gravity we consider a representative ansatz of the form \( f(T) = \alpha T^{-1} \). In this case equation (5.12) gives:
\[ T(r) = -\left(\Lambda + \frac{Q^2}{4r^2}\right) \pm \frac{1}{2} \sqrt{12\alpha + \left(2\Lambda + \frac{Q^2}{2r^2}\right)^2} , \tag{5.29} \]

with the upper and lower signs corresponding to the positive and negative branch solution respectively. Choosing for simplicity \( \Lambda = 0 \), we result to:
\[ N(r)^2 = -\frac{Q^6}{1728\alpha r^4} \pm \left[\frac{1}{18} - \frac{Q^4}{1728\alpha r^4}\right] Y(r) - \frac{1}{3} Q^2 \ln r \pm \frac{1}{12} Q^2 \ln \left(\frac{r^2}{2Q^2 + 2Y(r)}\right) + \text{const} , \]
\[ K(r)^2 = N(r)^2 \left\{1 - \frac{16\alpha r^4}{[-Q^2 \pm Y(r)]^2}\right\}^{-2} , \tag{5.30} \]
where
\[ Y(r) = \sqrt{Q^4 + 48\alpha r^4} . \tag{5.31} \]

As we observe, in the zero-electric-charge limit, the above solutions coincide with those of (4.15), with \( \tilde{J} = 0 \), \( \Lambda = 0 \) and \( f(T) = \alpha T^{-1} \), as expected (in this case \( T = \beta \) will be the specific constant solution of the algebraic equation (4.9)).

Similarly to the previous paragraph 5.1.1, we will investigate the singularities and the horizons. Since the transcendental equation \( K(r)^2 = 0 \), apart from the root at \( r = 0 \), cannot be solved analytically, in the following we examine numerically a specific result, choosing without loss of generality \( \alpha = 1 \) (for \( \alpha > 0 \), the torsion scalar is always real), \( Q = 1 \) and \( \text{const} = -1 \). Thus, in Fig. 4 we depict \( K(r)^2 \) as a function of \( r \) for both the positive and negative branch, while in Figures 5 and 6 we respectively present the Ricci and Kretschmann scalar.

For the positive branch we have a singularity at \( r_s = 0 \) and one at \( r_H \), both of them corresponding to \( K(r)^2 = 0 \). However, for both of them the curvatures scalars remain finite, while in the Painleve-Gullstrand coordinates the metric is regular and therefore these are just coordinate singularities. The timelike Killing vector of the metric is \( e^\mu \partial_\mu = \partial_t \) and thus its norm \( e^\mu e^\mu = g_{tt} = N(r)^2 \) vanishes at \( r = r_H \), since \( N(r_H)^2 = 0 \). Outside \( r = r_H \),
Figure 4. The solution $K(r)^2$ of (5.30) as a function of $r$, for the positive (thick curve) and negative (thin curve) branch of the IR modified 3D Maxwell-$f(T)$ gravity, for $\alpha = 1$, $Q = 1$ and $const = -1$. For the negative branch, $|K(r)^2| \to \infty$ at $r = 0$, which cannot be clearly seen in the figure scale.

Figure 5. The Ricci scalar $R(r)$ as a function of $r$, for the positive (thick curve) and negative (thin curve) branch of the IR modified 3D Maxwell-$f(T)$ gravity, for $\alpha = 1$, $Q = 1$ and $const = -1$.

the Killing vector field is timelike, while inside it is spacelike, and thus it corresponds to a null hypersurface, that is a Killing horizon. Additionally, we have a third singularity when $|K(r_{inf})^2| \to \infty$, and according to (5.30) this happens when $16\alpha r_{inf}^4 = \left[-Q^2 + Y(r_{inf})\right]^2$, that is at $r_{inf} = Q/(2\alpha^{1/4})$ (in our specific numerical example $r_{inf} = 1/2$). As we observe from Figures 5 and 6, the Ricci and Kretschmann scalar diverge at $r_{inf}$ too, and thus it is a physical singularity. Although in the specific numerical example of Figures 4, 5 and 6 we observe that $r_{inf} < r_H$, and therefore the singularity will be shielded by the horizon, in
Figure 6. The Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}(r)$ as a function of $r$, for the positive (thick curve) and negative (thin curve) branch of the IR modified 3D Maxwell-$f(T)$ gravity, for $\alpha = 1$, $Q = 1$ and $\text{const} = -1$.

In general we can have a naked singularity if

$$108\text{const} + 9Q^2 - 18Q^2 \ln \left( \frac{1}{2\alpha^{1/4}} Q \right) - 9Q^2 \ln (6Q^2) > 0,$$

as can be seen from the root structure of $N^2(r)$ in (5.30).

For the negative branch we have two singularities. The first is at $r_{\text{inf}} = 0$, corresponding to $|K(r_{\text{inf}})| \to \infty$ (it cannot be clearly seen in the scale of Fig. 4, but it can be immediately verified by considering the $r \to 0$ limit of (5.30)). This singularity is accompanied by a divergence in the curvature scalars in Figures 5 and 6, and thus it is a physical and not a coordinate one. The second singularity is at $r_H$, corresponding to $K(r)^2 = 0$, where the curvature scalars remain finite. Since in the Painleve-Gullstrand coordinates one can show that the metric is regular at $r_H$, we deduce that it is just a coordinate singularity.

In the same lines as above, $r_H$ is a cosmological Killing horizon which moreover in this specific example shields the physical singularity at $r_{\text{inf}} = 0$. Note that in the present case, independently of the parameter values we cannot have a naked singularity.

As in previous subsection, we can go beyond the specific numerical example and obtain analytical expressions for the horizon $r_H$, in specific limits. In particular, if $\left| \frac{Q^2}{48\alpha} \right| \ll r_H^4$ we obtain

$$r_H \approx \left( 2\alpha \sqrt{\frac{48}{48}} \right)^{1/4} e^{-\frac{1}{3} \text{const}} - \frac{1}{3} W_0 \left( \frac{\sqrt{\frac{48}{48}}}{3Q^2} (2\alpha \sqrt{48})^{1/4} e^{-\frac{1}{3} \text{const}} \right),$$

where as usual the upper sign corresponds to the positive branch and the lower sign corresponds to the negative branch. This is real if

$$\frac{\sqrt{48\alpha}}{3Q^2} (2\alpha \sqrt{48})^{1/4} e^{-\frac{1}{3} \text{const}} > -\frac{1}{e}.$$
Finally, similarly to the UV modification, in the present IR modified case we observe that the physical singularities are not always shielded by the horizon. Thus, we also verify that the cosmic censorship does not always hold for 3D Maxwell-$f(T)$ gravity in the absence of azimuthal electric field.

5.1.3 Comparison with the BTZ-like solution in the absence of azimuthal electric field

Let us compare the above solutions (5.21) and (5.30) with the charged BTZ-like solution of 3D General Relativity in the absence of azimuthal electric field [2]:

$$N(r)^2 = K(r)^2 = -8GM + \frac{r^2}{l^2} - \frac{1}{2}Q^2 \ln\left(\frac{r}{r_0}\right).$$

As we observe, solutions (5.21) and (5.30) correspond to a “deformed”, charged BTZ-like solution, and they completely coincide with it in the limit $f(T) \rightarrow 0$ (that is when $\alpha \rightarrow 0$). Finally, as we have mentioned, in the zero electric charge limit we re-obtain the results of the previous section.

Here we would like to stress that the deformation of the solutions (5.21) and (5.30), comparing to the standard charged BTZ solution (5.35), is not of a trivial type, since we obtain qualitatively different novel terms, corresponding to different behavior, as it was analyzed in detail in the previous paragraphs. This was not the case in the pure gravitational solutions of the previous section, where the deformation was expressed only through changes in the coefficients. Such a novel behavior of the Maxwell-$f(T)$ theory reveals the new features that the $f(T)$ structure brings in 3D gravity.

Despite the radical difference with the charged BTZ-like solution, away from the circular object we obtain similarities. In particular, for the UV modified case of paragraph 5.1.1, the asymptotic behavior of the solutions (5.21), that is at $r \rightarrow \infty$, is

$$N(r)^2 \rightarrow -\frac{Q^2}{2} \ln r, \quad K(r)^2 \rightarrow N(r)^2,$$

for the positive branch, while for the negative branch we obtain

$$N(r)^2 \rightarrow \frac{r^2}{54\alpha} - \frac{Q^2}{6} \ln r, \quad K(r)^2 \rightarrow 9N(r)^2.$$  \hspace{1cm} (5.36)\hspace{1cm} (5.37)

As we observe the asymptotic behavior of the negative branch coincides with that of the usual BTZ black hole (5.35), while the asymptotic behavior of the positive branch presents novel behavior.

Similarly, for the case of IR modification of paragraph 5.1.2, the asymptotic behavior of the solutions is

$$N(r)^2 \rightarrow \pm \frac{2}{9} \sqrt{3r^2} - \frac{1}{3}Q^2 \ln r, \quad K(r)^2 \rightarrow \frac{9}{4}N(r)^2,$$

with the upper and lower sign corresponding to the positive and negative branch respectively. In this case we observe that the asymptotic behavior of the positive branch coincides with that of the usual BTZ black hole (5.35), while the asymptotic behavior of the negative branch exhibits novel behavior.
5.2 Absence of radial electric field

In the case $E_r = 0$, that is in the absence of radial electric field, equation (5.16) leads to

$$E_\phi = \frac{Q}{r^2},$$  \hspace{1cm} (5.39)

where $Q$ is an integration constant, that as usual coincides with the electric charge. This case is simpler than case $E_\phi = 0$ of the previous subsection, and in particular it allows for the extraction of $N(r)$ and $K(r)$ for a general $f(T)$, namely:

$$N(r) = \gamma r,$$

$$K(r) = \sqrt{\frac{T(r)T^2}{2}},$$  \hspace{1cm} (5.40)

where $\gamma$ is an integration constant. Interestingly enough, since in the present case the metric in terms of the torsion scalar has the very simple form (5.40), one can calculate and express the Levi-Civita connections and then the Riemann tensor and the curvature scalars, in terms of the torsion scalar too. In particular, for both branches we obtain:

$$R(r) = 3T(r) + rT'(r),$$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}(r) = 3T(r)^2 + 2rT(r)T'(r) + \frac{1}{2}T''(r)^2,$$  \hspace{1cm} (5.42)

where we mention that the scalar curvatures are defined through the Levi-Civita connections, while the torsion scalar is defined using the Weitzenböck one.

For completeness, we explicitly present the $T(r)$ solution in the case of UV and IR modifications of 3D gravity of the previous subsection. In the case of a UV modification of the form $f(T) = \alpha T^2$ we obtain

$$T(r) = \frac{-1 \pm \sqrt{1 - 24\alpha\Lambda + \frac{6\alpha Q^2}{r^4}}}{6\alpha},$$  \hspace{1cm} (5.43)

with the upper and lower sign corresponding to the positive and negative branch respectively, and we can see that for both branches the torsion scalar is always real if $\alpha < (24\Lambda)^{-1}$.

For the positive branch we have a singularity at $r_s = 0$ where $N^2(r_s) = 0$, and since $T(0)$ diverges, from (5.42) we deduce that the Ricci and Kretschmann scalar diverge too, thus $r_s = 0$ is a physical singularity. However, for $\Lambda > 0$ we obtain a second singularity where $K^2(r_H) = 0$, namely at $r_H = \left(\frac{Q^2}{4\Lambda}\right)^{1/4} > 0$, which due to (5.40) corresponds to $T(r_H) = 0$. We mention here that the fact that $K(r)$ is given by a function of the covariant quantity $T(r)$ in the given coordinate system, it does not mean that is will do so in another coordinate choice, since $K(r)$ is not itself a covariant quantity. Thus, $K^2(r_H)$ can be non-zero although $T(r_H) = 0$ still holds. In particular, in the vicinity of $r_H$ we observe that $K^2(r)$ behaves like $-4r_H\Lambda(r - r_H)$, neglecting higher terms in $(r - r_H)$, and thus defining $H = r - r_H$, the metric, for $H > 0$, can be approximately written as

$$ds^2 = \gamma^2 (r_H\Lambda L^2 + r_H)^2 dt^2 + dL^2 - (r_H\Lambda L^2 + r_H)^2 d\phi^2,$$  \hspace{1cm} (5.44)

We thank an anonymous referee for his clarification on this subject.
with \( L^2 = H/(r_H \Lambda) \), while for \( H < 0 \) it becomes

\[
ds^2 = \gamma^2 (-r_H \Lambda L^2 + r_H)^2 dt^2 - dL^2 - (-r_H \Lambda L^2 + r_H)^2 d\phi^2 ,
\]

(5.45)

with \( L^2 = -H/(r_H \Lambda) \). As we can clearly see all metric components are smooth functions of the coordinates around \( L = 0 \), and therefore \( r_H \) is not a curvature singularity. However, note that the metric changes signature at \( r_H \) so it is definitely singular at \( r_H \). Additionally, by examining the Killing vector, the norm of the timelike Killing vector \( \epsilon^\mu \partial_\mu = \partial_t \) is

\[
\epsilon^\mu \epsilon^\nu = N^2(r) = \gamma^2 r^2
\]

(note that \( \gamma \) can be absorbed in \( dt \)), and thus at \( r = r_H \) we have \( \epsilon^\mu \epsilon^\mu = \gamma^2 r_H^2 \). Therefore, the norm is different from zero and we do not have a Killing horizon.

For the negative branch we have a physical singularity at \( r_s = 0 \), in which the Ricci and Kretschmann scalar diverge, however in this case there is not any other singularity at \( r > 0 \) (unless \( \alpha = 0 \), but in this case the negative branch is meaningless similarly to the case of footnote 1). Therefore, in this case \( r_s = 0 \) is a naked singularity, and the cosmic censorship does not hold.

In the case of an IR modification of the form \( f(T) = \alpha T^{-1} \) we acquire

\[
T(r) = -\Lambda + \frac{Q^2}{4r^4} \pm \sqrt{3\alpha + \left(-\Lambda + \frac{Q^2}{4r^4}\right)^2} ,
\]

(5.46)

and the torsion scalar is always real if \( \alpha > -\Lambda^2/3 \). Both branches have a physical singularity at \( r_s = 0 \), at which the torsion scalar and thus the curvature scalars in (5.42) diverge, however both branches do not have a horizon at \( r > 0 \) (unless \( \alpha = 0 \)). Therefore, in this case the singularity at \( r_s = 0 \) is a naked one and the cosmic censorship does not hold.

Let us compare the above solutions (5.40),(5.43) and (5.40),(5.46) with the charged BTZ-like solution in the absence of radial electric field [2]:

\[
N(r) = \gamma r ,
\]

\[
K(r) = \sqrt{-\Lambda r^2 + \frac{Q^2}{4r^2}} .
\]

(5.47)

As we observe, the obtained solutions correspond to a “deformed”, charged BTZ-like solution, and they completely coincide with it in the limit \( f(T) \to 0 \) (that is when \( \alpha \to 0 \)). Once again we stress that the above deformation is not of a trivial type, since we obtain qualitatively different novel terms, which was not the case in the pure gravitational solutions of the previous section.

The most important qualitative difference is that although in the case of charged BTZ-like solution (5.47) there is always a horizon at \( r_H = \left(\frac{Q^2}{4\Lambda}\right)^{1/4} \) that shields the physical singularity at \( r_s = 0 \), in our case, apart from the positive branch of UV modification, we obtained naked physical singularities at \( r_s = 0 \). Furthermore, as expected, in the zero-electric-charge limit we re-obtain the results of the previous section.

Finally, in the asymptotic region \( (r \to \infty) \), the above solutions exhibit the behavior

\[
K(r) \to \sqrt{\frac{-1 \pm \sqrt{1 - 24\alpha \Lambda}}{12\alpha}} r ,
\]

(5.48)
for the UV modification, and

$$K(r) \rightarrow \sqrt{-\frac{\Lambda}{2} \pm \frac{\sqrt{3\alpha + \Lambda^2}}{2} r},$$

(5.49)

for the IR modification, which coincides with the asymptotic behavior of the charged BTZ-like solution (5.47) with an effective cosmological constant.

For completeness we close this section by mentioning an interesting feature of the 3D $f(T)$-Maxwell theory at hand, namely that it accepts AdS $pp$-wave solutions [90-92]. The relevant calculations are shown in the Appendix B.

6 Final Remarks

In this work we presented teleparallel gravity in three dimensions and we examined its circularly symmetric solutions. Furthermore, we extended our analysis considering functions $f(T)$ of the torsion scalar, that is formulating 3D $f(T)$ gravity, and we examined the circularly symmetric solutions too. Finally, we extended our analysis taking into account the electromagnetic sector, in order to extract the charged circularly symmetric solutions.

In the simple case of teleparallel 3D gravity, we showed that for a negative cosmological constant one can obtain the BTZ solution of standard 3D (GR-like) gravity, while for a positive cosmological constant one acquires the standard Deser-de-Sitter solution. Such a complete coincidence between teleparallel 3D gravity and standard 3D gravity was expected, since the theory is linear in the torsion scalar $T$ and in this case the equivalence of the above gravitational formulations is complete in all dimensionalities.

In the simple case of teleparallel 3D gravity, after formulating it for a general torsion scalar, we showed that one can obtain a BTZ-like solution corresponding to an effective cosmological constant, even in the case of the standard torsion scalar definition. In particular, one obtains an effective cosmological constant which depends on the initial cosmological constant as well as on the parameters of the used $f(T)$ ansatz. Moreover, we saw that a negative cosmological constant is not required for such a BTZ-like solution. This is a difference with standard 3D gravity, and indicates the novel features that the $f(T)$ structure induces in the gravitational theory. Additionally, and in the same lines, a positive cosmological constant is not required for the Deser-de-Sitter-like solution. Finally, note that the circularly symmetric solutions of 3D $f(T)$ gravity are also different from the corresponding solutions of $f(R)$ gravity in three dimensions [93], which was also expected since it is well known that $f(T)$ and $f(R)$ modified gravitational theories are quite different.

In the case of Maxwell-$f(T)$ 3D gravity, interestingly enough we found that the known no-go theorem of standard (GR-like) 3D gravity [29, 84], which dynamically excludes configurations with two non-vanishing components of the Maxwell field, is valid too, even going beyond the simple diagonal relation between the metric and the vierbeins. Thus, examining separately the case of radial or azimuthal electric field, and considering UV and IR $f(T)$ modifications of 3D gravity, we showed that the theory accepts “deformed” charged BTZ-like solutions, which coincide with the exact standard 3D result in the limit $f(T) \rightarrow 0$. Moreover, contrary to the simple $f(T)$ case where the torsion scalar $T$ was a constant, in
the Maxwell-$f(T)$ case $T$ has in general an $r$-dependence, a behavior that reveals the new features brought in by the richer structure of the addition of the electromagnetic sector.

However, the most interesting feature of the 3D $f(T)$-Maxwell theory is that the deformation of the standard charged BTZ solution is not of a trivial type, since we obtain qualitatively different novel terms and radically different behavior, contrary to the pure gravitational solutions where the deformation is expressed only through changes in the coefficients. In particular, we analyzed the singularities and the horizons of specific (but quite general) numerical examples. Although one can find Killing horizons that shield the physical singularities, in the majority of the examined cases there are always parameter choices that lead to the appearance of naked singularities. This violation of cosmic censorship, that disappears only in the limit $f(T) \to 0$, may serve as another disadvantage of the $f(T)$ extension of teleparallel gravity, although charged BTZ black holes could also exhibit naked singularities under special conditions $[94, 95]$. Moreover, we examined the asymptotic behavior of the solutions far away from the circular object, comparing it with the corresponding behavior of the usual charged BTZ-like solution (5.47). In summary, the novel obtained behavior of the $f(T)$-Maxwell theory reveals the new features that the $f(T)$ structure brings in 3D gravity. Finally, for completeness we showed that this theory supports AdS $pp$-wave solutions.

In conclusion, the analysis of the present work can be enlightening both for 3D gravity, since the new features that are brought in by the $f(T)$ structure may contribute to its quantization efforts, as well as for $f(T)$ structure itself, since it may bring light to the Lorentz invariance issues that appear in 4D.

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**A No-go theorem in 3D Maxwell-$f(T)$ gravity for non-diagonal vierbein choice**

Let us go beyond the simple diagonal vierbein choice (5.6) and consider a non-diagonal ansatz corresponding to the same metric (5.7). As we have said, this new vierbein choice will arise from a Lorentz transformation of the diagonal one, namely:

$$e^{a'} = \Lambda^a_{a'} e^a ,$$

(A.1)
denoting the new indices using primes. Without loss of generality we consider a boost transformation of the form

\[
\Lambda_{a'}^a(x) = \begin{pmatrix}
\cosh \theta & \sinh \theta & 0 \\
\sinh \theta & \cosh \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\] (A.2)

where \( \theta \equiv \theta(t,r,\phi) \), and we mention that \( \Lambda_{a'}^a \) is the inverse of \( \Lambda_a^{a'} \). Therefore, the new vierbein reads

\[
e^{0'} = N \cosh \theta dt + \frac{1}{K} \sinh \theta dr ,
\]
\[
e^{1'} = N \sinh \theta dt + \frac{1}{K} \cosh \theta dr ,
\]
\[
e^2' = e^2 .
\] (A.3)

Inserting these into the torsion scalar (3.6), with \( \rho_1 = 0, \rho_2 = -\frac{1}{2}, \rho_4 = 1 \), we find that the new torsion scalar will be

\[
T' = T - 2 \frac{K}{N r} \frac{\partial \theta}{\partial t} .
\] (A.4)

Similarly, the electric sector of the electromagnetic 2-form (5.8) will be

\[
F' = E'_r e^0' e^1' + E'_\phi e^2' e^0' ,
\] (A.5)

and correspondingly one can find the energy momentum tensor for the electromagnetic field \( S^a_b \) using (5.5). Therefore, it is easy to see that the variation of the electric part of the Lagrangian (5.2) will now give

\[
\frac{\delta L_F}{\delta e^a'} = -\epsilon_{a'b'c'} s^b'_c e^{c'} ,
\] (A.6)

that is one replaces the old quantities by the prime-ones.

So in summary, we can see that the new field equations will have the form of (5.3) but with all quantities replaced by the prime-ones. Obviously, we observe that the equations not involving the electric fields will be more complicated that those of (5.12)-(5.14), and thus the solution structure will be different. However, the contribution of the electric fields is given by

\[
- \epsilon_{b'c'} s^b' e^{c'} = \frac{1}{2} \left( E^2_r - E^2_\phi \right) e^{0'} e^{1'} + E'_r E'_\phi e^{0'} e^{1'}
\]
\[
= \frac{1}{2} \left( E^2_r - E^2_\phi \right) \left( \cosh \theta e^0 + \sinh \theta e^1 \right) e^2 - E'_r E'_\phi e^0 e^1 ,
\] (A.7)

where we take \( a' = 1' \). Then, by substituting in the fields equations we find that

\[
E'_r E'_\phi = 0,
\] (A.8)

that is the no-go theorem (5.15) is still valid in the general vierbein choice.

\[\text{4}^{\text{According to [82] D-dimensional } f(T) \text{ gravity has } D-1 \text{ new degrees of freedom, which is an indication that they will correspond to boosts instead of rotations. This can be verified by a detailed investigation of the general vierbein choice in 3D and 4D } f(T) \text{ gravity, and its relation to extra degrees of freedom [83].}}\]
pp-wave solutions in 3D $f(T)$-Maxwell theory

In this appendix we show that the 3D $f(T)$-Maxwell theory accepts the interesting class of solutions known as AdS pp-waves [90–92]. The corresponding metric reads:

$$ds^2 = h(y)^2 \left[-2H(u,y)du^2 - 2dudv + dy^2\right].$$

(B.1)

We consider the triad as

$$e^0 = h(y)\left(\frac{H + \frac{1}{2} du + dv}{h}\right), \quad e^1 = h(y)dy, \quad e^2 = h(y)\left(\frac{H - \frac{1}{2} du + dv}{h}\right),$$

(B.2)

and the electromagnetic potential as

$$A = a(u,y)du.$$  

(B.3)

Then

$$F = dA = -\frac{1}{h^2} \frac{\partial a}{\partial y} e^0 \wedge e^1 - \frac{1}{h^2} \frac{\partial a}{\partial y} e^1 \wedge e^2,$$

(B.4)

and the field equations are given by

$$\left[1 + \frac{df}{dT}\right] \left[\frac{1}{h} \frac{\partial}{\partial y} \left(\frac{1}{h} \frac{\partial H}{\partial y}\right) - 2h' \frac{\partial H}{h^2} \frac{\partial y}{\partial y}\right] + \frac{1}{h^2} \frac{\partial^2 f}{\partial T^2} \frac{\partial H}{\partial y} \frac{\partial y}{\partial y} - \left(\frac{1}{h^2} \frac{\partial a}{\partial y}\right)^2 = 0,$$

(B.5)

$$\left[1 + \frac{df}{dT}\right] \frac{1}{h} \frac{\partial}{\partial y} \left(h' \frac{h}{h^2}\right) + \frac{\partial^2 f}{\partial T^2} \frac{h'}{h^3} = 0,$$

(B.6)

$$\left[1 + 2\frac{df}{dT}\right] T - f(T) + 2\Lambda = 0,$$

(B.7)

with $h' = dh(y)/dy$. Using the vierbein choice (B.2) and the definition of the torsion scalar (3.8) we can calculate

$$T = 2\left(\frac{h'}{h^2}\right)^2.$$  

(B.8)

Now, using the Maxwell equations we get $\frac{\partial}{\partial y} \left(\frac{1}{h} \frac{\partial a}{\partial y}\right) = 0$ and in summary we result to the pp-wave solutions

$$h(y) = \frac{1}{y} \sqrt{\frac{2}{T}},$$

$$a(u,y) = \sqrt{\frac{2}{T}} k(u) \ln y + j(u),$$

$$H(u,y) = k^2(u) \left[\frac{y^2}{8 \left[1 + \frac{df}{dT}\right]} + g(u)\right],$$

(B.9)

where $k(u)$ and $j(u)$ are arbitrary function and the scalar torsion is constant. Finally, note that in the special case where $f(T) = -T + 2\Lambda + \sqrt{T}$, equation (B.7) is satisfied identically and thus the torsion scalar is not restricted to be a constant. Equation (B.6) is satisfied too, and therefore from (B.5) we obtain

$$H(u,y) = h^2(y)k(u) + g(u),$$

$$a(u,y) = j(u),$$

(B.10)

with $h(y)$, $k(u)$, $g(u)$ and $j(u)$ arbitrary functions.
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