LOCAL EXISTENCE OF STRONG SOLUTIONS TO THE $k - \varepsilon$ MODEL EQUATIONS FOR TURBULENT FLOWS

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Abstract

In this paper, we are concerned with the local existence of strong solutions to the $k - \varepsilon$ model equations for turbulent flows in a bounded domain $\Omega \subset \mathbb{R}^3$. We prove the existence of unique local strong solutions under the assumption that turbulent kinetic energy and the initial density both have lower bounds away from zero.

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1 Introduction

Turbulence is a natural phenomenon which occurs inevitably when the Reynolds number of flows becomes high enough ($10^6$ or more). In this paper, we consider the $k - \varepsilon$ model equations \[\text{[1, 16]}\] for turbulent flows in a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary,

\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
(\rho u)_t + \nabla \cdot (\rho u \otimes u) - \Delta u - \nabla (\nabla \cdot u) + \nabla p &= \frac{2}{3} \nabla (\rho k), \\
(\rho h)_t + \nabla \cdot (\rho u h) - \Delta h &= p_t + u \cdot \nabla p + S_k, \\
(\rho k)_t + \nabla \cdot (\rho u k) - \Delta k &= G - \rho \varepsilon, \\
(\rho \varepsilon)_t + \nabla \cdot (\rho u \varepsilon) - \Delta \varepsilon &= \frac{C_1 G \varepsilon}{k} - \frac{C_2 \rho \varepsilon^2}{k}, \\
(u \cdot -\vec{n}, h, \frac{\partial k}{\partial \vec{n}}, \frac{\partial \varepsilon}{\partial \vec{n}})|_{\partial \Omega} &= (0, 0, 0, 0),
\end{align*}

with

\[S_k = \left[ \mu \left( \frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \frac{\partial u^k}{\partial x_k} \right] \frac{\partial u^i}{\partial x_j} + \frac{\mu}{\rho} \frac{\partial p}{\partial x_j} \frac{\partial p}{\partial x_j}.\]
$$G = \frac{\partial u^i}{\partial x_j} \left[ \mu_e \left( \frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right) - \frac{2}{3} \delta_{ij} \left( \rho k + \mu_e \frac{\partial u^k}{\partial x_k} \right) \right], \quad (1.9)$$

$$p = \rho^\gamma, \quad (1.10)$$

where $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ij} = 1$ if $i = j$, and $\mu$, $\mu_t$, $\mu_e$, $C_1$ and $C_2$ are five positive constants satisfying $\mu + \mu_t = \mu_e$, and $\vec{n}$ is the unit outward normal to $\partial \Omega$.

The equations (1.1)-(1.10) are derived from combining the effect of turbulence on time-averaged Navier-Stokes equations with the $k - \varepsilon$ model equations. The unknown functions $\rho$, $u$, $h$, $k$ and $\varepsilon$ denote the density, velocity, total enthalpy, turbulent kinetic energy and the rate of viscous dissipation of turbulent flows, respectively. The expression of the pressure $p$ has been simplified here, which indeed has no bad effect on our study.

In partial differential equations, $k - \varepsilon$ equations belong to the compressible ones. In this regard, we will refer to the classical compressible Navier-Stokes equations and compressible MHD equations, which are also research mainstreams, to carry out our study.

For compressible isentropic Navier-Stokes equations, the first question provoking our interest is the existence of the weak solutions. P. L. Lions [14, 15] proved the global existence of weak solutions under the condition that $\gamma > \frac{3}{2}$, where $\gamma$ is the same as in (1.10) and $n$ is the dimension of space. Later, Feireisl [7, 8] improved his result to $\gamma > \frac{2}{3}$. The condition satisfied by $\gamma$ is to prove the existence of renormalized solutions, which was introduced by DiPerna and Lions [6]. When the initial data are general small perturbations of non-vacuum resting state, Hoff [10] proved the global existence of weak solutions provided $\gamma > 1$. The existence of strong solutions is another problem provoking our interest in the research of Navier-Stokes equations. It has been proved that the density will be away from vacuum at least in a small time provided the initial density is positive. If the initial data have better regularity, the compressible isentropic Navier-Stokes equations will admit unique local strong solutions under various boundary conditions [2, 3, 4, 10]. However, when initial vacuum is allowed, it was shown recently in [2] that the isentropic ones will have local strong solutions in the case that some compatibility conditions are satisfied initially. H. J. Choe and H. S. Kim [5] obtained the unique local strong solutions for full compressible polytropic Navier-Stokes equations under the similar condition as in [2]. In [5], the technic the authors used is mainly the standard iteration argument and the key point of their success is the estimate for the $L^2$ norm of the gradient of pressure. In the process of studying the condition of local solutions becoming global ones, Z. P. Xin [20] proved that the smooth solutions will blow up in finite time when initial vacuum is allowed.

As for compressible MHD equations, the research directions, which mainly contain first the existence of weak and strong solutions and second the condition of weak solutions becoming strong or even classical ones and the local becoming global ones, are similar to that of Navier-Stokes equations. For example, Hu and Wang [11, 12, 13] obtained the local existence of weak solutions to the compressible isentropic MHD equations. Rozanova [17] proved the local existence of classical solutions to the compressible barotropic MHD equations provided both the mass and energy are finite. J. S. Fan and W. H. Yu in [9] proved the existence and uniqueness of strong solutions to the full compressible MHD equations. The method used by J. S. Fan and W. H. Yu [9] is similar to that in [5], for example, they are both dependent on the standard iteration argument and the estimate for the $L^2$ norm of the gradient of pressure.

Under the hypothesis of the existence of local-in-time smooth solution, the authors of [11] prove the existence of small data smooth solution in $\mathbb{R}^3$. In this paper, we consider the local-in-time existence of strong solutions to the $k - \varepsilon$ model equations (1.1)-(1.10) in a bounded domain $\Omega \subset \mathbb{R}^3$. Our method is similar to that in [11] and [5]. However, in the process of applying the method to $k - \varepsilon$ model equations, we find that the regularity of the solutions should be higher, which is induced by higher nonlinearity than compressible Navier-Stokes equations and compressible MHD equations, than that in [9] and [5]. In fact, when we make the difference of the $n - th$ and the $(n + 1) - th$ of equation (2.4) and integrating the result, it inevitably comes
out the term $\int \partial_j \rho^{n+1} \partial_j \rho^{n+1} \cdot h^{n+1}$. Therefore, we have to use integration by parts, which leads to two terms as $\int \rho^{n+1} \partial_j \rho^{n+1} \cdot h^{n+1}$ and $\int \rho^{n+1} \partial_j \rho^{n+1} \cdot \partial_j h^{n+1}$. Then, by Hölder and Young’s inequalities, it turns out that $\|\nabla^2 \rho^{n+1}\|_{L^3}$ and $\|\nabla \rho^{n+1}\|_{L^\infty}$ should be bounded. Thus, we need $\|\rho\|_{H^3}$ be bounded for a priori estimates. Therefore, from the mass equation enough regularity of the velocity field should be imposed. Moreover, due to the strong-coupling property of $k - \varepsilon$ equations, we need corresponding high regularity of unknown functions $k$ and $\varepsilon$.

In a word, the high nonlinearity of $k - \varepsilon$ equations leads to the necessity of high regularity of some unknown functions and thus leads to much difficulties for the a priori estimates. Besides, physically, when the turbulent kinetic energy $k$ vanishes, the turbulence will disappear and the $k - \varepsilon$ model equations will degenerate into the Navier-Stokes equations, therefore, without loss of generality, we assume throughout this paper that the turbulent kinetic energy $k$ has a positive lower bound away from zero, namely, $0 < m < k$ with $m$ a constant.

To conclude this introduction, we give the outline of the rest of this paper: In section 2, we consider a linearized problem of the $k - \varepsilon$ equations and derive some local-in-time estimates for strong solutions of the original nonlinear problem. In section 3, we prove the existence theorem of the local strong solutions of the original nonlinear problem.

## 2 A priori estimates for a linearized problem

Using density equation (1.1), we could change (1.1)-(1.10) into the following equivalent form:

$$
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\rho u_t + \rho u \cdot \nabla u - \Delta u - \nabla \text{div} u + \nabla p &= -\frac{2}{3} \nabla (\rho k), \\
\rho h_t + \rho u \cdot \nabla h - \Delta h &= p_t + u \cdot \nabla p + S_k, \\
\rho k_t + \rho u \cdot \nabla k - \Delta k &= G - \rho \varepsilon, \\
\rho \varepsilon_t + \rho u \cdot \nabla \varepsilon - \Delta \varepsilon &= \frac{C_i G' \theta}{\pi} - \frac{C_2 \rho \theta^2}{\rho}, \\
(p, u, h, \varepsilon)(x,0) &= (p_0(x), u_0(x), h_0(x), k_0(x), \varepsilon_0(x)), \\
(u \cdot \nabla h, \frac{\partial k}{\partial \pi}, \frac{\partial \varepsilon}{\partial \pi})|_{\partial \Omega} &= (0,0,0).
\end{align*}
$$

(2.1)

Then, we consider the following linearized problem of (2.1):

$$
\begin{align*}
\rho_t + \nabla \cdot (\rho u) &= 0, \\
\rho u_t + \rho v \cdot \nabla u - \Delta u - \nabla \text{div} u + \nabla p &= -\frac{2}{3} \nabla (\rho \pi), \\
\rho h_t + \rho v \cdot \nabla h - \Delta h &= p_t + u \cdot \nabla p + S'_k, \\
\rho k_t + \rho v \cdot \nabla k - \Delta k &= G' - \rho \theta, \\
\rho \varepsilon_t + \rho v \cdot \nabla \varepsilon - \Delta \varepsilon &= \frac{C_1 G' \theta}{\pi} - \frac{C_2 \rho \theta^2}{\rho}, \\
(p, v, h, \pi, \theta)(x,0) &= (p_0(x), u_0(x), h_0(x), k_0(x), \varepsilon_0(x)), \\
(v \cdot \nabla h, \frac{\partial \pi}{\partial n}, \frac{\partial \theta}{\partial n})|_{\partial \Omega} &= (0,0,0).
\end{align*}
$$

(2.2)

with

$$
\begin{align*}
S'_k &= \mu \left( \frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} \right) - \frac{2}{3} \beta_{ij} \frac{\partial v^k}{\partial x_i} \frac{\partial v^j}{\partial x_j} + \mu_t \frac{\partial p}{\rho^2} \frac{\partial \rho}{\partial x_j} \frac{\partial \rho}{\partial x_j}, \\
G' &= \frac{\partial v^i}{\partial x_j} \left[ \mu_e \left( \frac{\partial v^i}{\partial x_j} + \frac{\partial v^j}{\partial x_i} \right) - \frac{2}{3} \beta_{ij} \left( \rho \pi + \mu_e \frac{\partial v^k}{\partial x_k} \right) \right].
\end{align*}
$$
where \( v, \pi \) and \( \theta \) are known quantities on \((0,T_1) \times \Omega\) with \( T_1 > 0 \).

Here we also impose the following regularity conditions on the initial data:

\[
\begin{aligned}
&0 < m < \rho_0, \quad \rho_0 \in H^3(\Omega), \\
&u_0 \in H^3(\Omega), \\
&(h_0, k_0, \varepsilon_0) \in H^2(\Omega), \\
&(u_0 \cdot n, h_0, \frac{\partial k_0}{\partial n}, \frac{\partial \varepsilon_0}{\partial n}) \big|_{\partial \Omega} = (0, 0, 0, 0), \\
&0 < m < k_0.
\end{aligned}
\]

(2.9)

For the known quantities \( v, \pi, \theta \), we assume that \( v(0) = u_0, \pi(0) = k_0, \theta(0) = \varepsilon_0 \) and

\[
\begin{aligned}
&\sup_{0 \leq t \leq T_2} (\|v\|_{H^1} + \|\pi\|_{H^1} + \|\theta\|_{H^1}) \\
&+ \int_0^{T_2} (\|\pi\|_{H^3}^2 + \|v_t\|_{H^3}^2 + \|\pi_t\|_{H^3}^2 + \|\theta_t\|_{H^3}^2) \, dt \leq c_1, \\
&\sup_{0 \leq t \leq T_2} \|v\|_{H^2} \leq c_2, \\
&\sup_{0 \leq t \leq T_2} \|\pi\|_{H^2} \leq c_3, \\
&\int_0^{T_2} \|v_t\|_{H^4}^2 \, dt \leq c_4, \\
&\sup_{0 \leq t \leq T_2} \|\theta\|_{H^2} \leq c_5.
\end{aligned}
\]

(2.10)

for some fixed constants \( c_i \) satisfying \( 1 < c_0 < c_i (i = 1, 2, \ldots, 6) \) and some time \( T_2 > 0 \). Where

\[
c_0 = 2 + \|\rho_0, u_0\|_{H^3} + \|(h_0, k_0, \varepsilon_0)\|_{H^2}.
\]

And for simplicity, we set another small time \( T = \min \{c_0^{-67}, c_1^{-10}, c_2^{-8}, c_3^{-4}, c_4^{-2}, c_5^{-2}, c_6^{-4}, T_1, T_2 \} \) and all of the \( T \) in section 2 are defined as this.

**Remark 2.1.** Here it should be emphasized that throughout this paper, \( C \) denotes a generic positive constant which is only dependent on \( m, \gamma \) and \( |\Omega| \), but independent of \( c_i \ (i = 0, 1, 2, \ldots, 6) \).

**Remark 2.2.** From the physical viewpoint, we assume that the turbulent kinetic energy \( k \) has a positive lower bound away from zero, namely, \( 0 < m < k \) with \( m \) a constant. We do not know whether \( 0 < m < k \) holds afterwards if its initial value \( k_0 > m \).

Next, we would like to prove the following local existence theorem of the linearized system (2.2)-(2.9).

**Theorem 2.1.** There exists a unique strong solution \( (\rho, u, h, k, \varepsilon) \) to the linearized problem (2.2)-(2.9) in \([0,T]\) satisfying the estimates (2.9), and (2.10) as well as the regularity

\[
\begin{aligned}
\rho &\in C(0,T; H^1), \quad \rho_t \in C(0,T; H^3), \quad u \in C(0,T; H^3) \cap L^2(0,T; H^4), \\
u_t &\in L^2(0,T; H^1), \quad k \in C(0,T; H^2) \cap L^2(0,T; H^4), \quad k_t \in L^2(0,T; H^1), \\
\varepsilon &\in C(0,T; H^2), \quad \varepsilon_t \in L^2(0,T; H^1), \quad h \in C(0,T; H^2), \quad h_t \in L^2(0,T; H^1), \\
(\sqrt{\rho} u_t, \sqrt{\rho} k_t, \sqrt{\rho} \varepsilon_t, \sqrt{\rho} h_t) &\in L^\infty(0,T; L^2).
\end{aligned}
\]

In the following part, we decompose the proof of Theorem 2.1 into some lemmas.

**Lemma 2.1.** There exists a unique strong solution \( \rho \) to the linear transport problem (2.2) and (2.9) such that

\[
\rho \geq \frac{m}{e}, \quad \|\rho\|_{H^3(\Omega)} \leq Cc_0, \quad \|\rho_t\|_{H^4(\Omega)} \leq Cc_0c_2
\]

for \( 0 \leq t \leq T \).
Proof. First, applying the particle trajectory method to equation (2.3), we easily deduce
\[
\rho \geq \rho_0 \exp \left( -\int_0^T \|\nabla v\|_{L^\infty} \, dt \right) \geq \rho_0 \exp(-c_3T) \geq \frac{\rho_0}{e} \geq \frac{m}{e}
\]
and thus
\[
\frac{1}{\rho} \leq \frac{e}{m} \leq C.
\]
Second, by simple calculation, we have
\[
\frac{d}{dt}\|\rho\|_{H^3} \leq C\|v\|_{H^3}\|\rho\|_{H^3} + C\|\nabla^4 v\|_{L^2},
\]
applying Gronwall and Hölder’s inequalities, one gets
\[
\|\rho\|_{H^3} \leq \left[ \exp \left( C \int_0^t \|v\|_{H^3} \, dt \right) \right] \left( \|\rho_0\|_{H^3} + C \int_0^t \|v\|_{H^3} \, dt \right) \leq C_0
\]
for \(0 \leq t \leq T\).
Next, from equation (2.2), one obtains
\[
\|\rho_t\|_{H^1} = \|\nabla \cdot (\rho v)\|_{H^1} \leq C \|\rho\|_{H^3}\|v\|_{H^2} \leq C_0c_2
\]
for \(0 \leq t \leq T\).
Thus, we complete the proof of Lemma 2.1.

Next, we estimate the velocity field \(u\).

Lemma 2.2. There exists a unique strong solution \(u\) to the initial boundary value problem (2.3) and (2.7) such that
\[
\|\sqrt{\rho}u_t\|_{L^2}^2 + \|u\|_{H^1}^2 + \int_0^t \|\nabla u\|_{L^2}^2 \, ds \leq Cc_0^{5+2\gamma}c_1^2, \quad \|u\|_{H^2} \leq Cc_0^{5+3\gamma}c_1^2
\]
for \(0 \leq t \leq T\).

Proof. We only need to prove the estimates. Differentiating equation (2.3) with respect to \(t\), then multiplying both sides of the result by \(u_t\) and integrating over \(\Omega\), we derive that
\[
\frac{1}{2} \frac{d}{dt} \int \rho u_t^2 \, dx + \|\nabla u_t\|_{L^2}^2 + \|\text{div} u_t\|_{L^2}^2
\]
\[
= -\int \rho v \cdot \nabla u \cdot u_t - \int \rho v_t \cdot \nabla u \cdot u_t - 2 \int \rho v \cdot \nabla u_t \cdot u_t - 2 \int \nabla \rho_t \cdot u_t - \frac{2}{3} \int [\nabla (\rho \pi)]_t \cdot u_t
\]
\[
= I_1 + I_2 + I_3 + I_4 + I_5,
\]
where we have used equation (2.2) and integration by parts. We will estimate \(I_i\) \((i = 1, 2, \cdots, 5)\) item by item.
First, because \(\rho\) has lower bound away from zero, we easily deduce \(\|u_t\|_{L^2} \leq C\|\sqrt{\rho}u_t\|_{L^2}\). Therefore, using Hölder, Sobolev and Young’s inequalities and (2.10), we have
\[
I_1 \leq C\|v\|_{L^\infty}\|\rho_t\|_{L^3}\|\nabla u\|_{L^2}\|u_t\|_{L^6} \leq C\|v\|_{L^\infty}\|\rho_t\|_{L^3}\|\nabla u\|_{L^2} (\|\sqrt{\rho}u_t\|_{L^2} + \|\nabla u_t\|_{L^2})
\]
\[
\leq Cc_0^5c_2^4\|\nabla u\|_{L^2}^2 + C\|\sqrt{\rho}u_t\|_{L^2}^2 + \frac{1}{8} \|\nabla u_t\|_{L^2}^2, \quad (2.15)
\]
\[ I_3 \leq C\|\rho\|_{L^\infty}^2 \|v\|_{L^\infty} \|\nabla u_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \leq C_0 c_2^2 \|\sqrt{\rho} u_t\|_{L^2}^2 + \frac{1}{8} \|\nabla u_t\|_{L^2}^2, \quad (2.16) \]

\[ I_2 \leq C\|\rho\|_{L^\infty}^2 \|v_t\|_{L^2} \|\nabla u\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \leq C_1 \eta^{-1} c_0 \|\nabla u\|_{L^2}^2 + \|v_t\|_{H^\gamma} \|\sqrt{\rho} u_t\|_{L^2}, \quad (2.17) \]

where \( \eta > 0 \) is a small number to be determined later.

Next, to evaluate \( \|\nabla u\|_{H^1}^2 \) in \( (2.17) \), we can first use Sobolev’s interpolation inequality to get

\[
\|\nabla u\|_{H^1}^2 \leq C\|\nabla u\|_{L^2} \|\nabla u\|_{L^6} \leq C\|\nabla u\|_{L^2} \|\nabla u\|_{H^1}, \quad (2.18)
\]

Then, applying the standard elliptic regularity result to equation \( (2.3) \) and using \( (2.18) \), we have

\[
\|\nabla u\|_{H^1} \leq C_0 c_1 (\|\sqrt{\rho} u_t\|_{L^2} + \|v\|_{L^6} \|\nabla u\|_{L^2}^\frac{1}{2} \|\nabla u\|_{H^1}^\frac{1}{2} + \|\nabla \rho\|_{L^2} + \|\nabla \rho\|_{L^4} \|\pi\|_{L^4} + \|\nabla \pi\|_{L^2}),
\]

thus Young’s inequality and \( (2.10) \) yield

\[
\|\nabla u\|_{H^1} \leq C_0^2 \gamma (\|\sqrt{\rho} u_t\|_{L^2} + c_1^2 \|\nabla u\|_{L^2} + c_0 c_1). \quad (2.19)
\]

Combining \( (2.17), (2.18) \) and \( (2.19) \) and using Young’s inequality, we get

\[
I_2 \leq C_0 \eta^{-1} c_0 \gamma^2 c_0 (\|\sqrt{\rho} u_t\|_{L^2} + \|v\|_{L^6} \|\nabla u\|_{L^2}^\frac{1}{2} \|\nabla u\|_{H^1}^\frac{1}{2} + \|\nabla \rho\|_{L^2} + \|\nabla \rho\|_{L^4} \|\pi\|_{L^4} + \|\nabla \pi\|_{L^2}),
\]

By integration by parts, we have

\[
I_4 = \int \rho_t \text{div} u_t \leq C_0 \gamma c_0 \|\nabla u_t\|_{L^2} \leq C_0 \gamma^2 c_2 + \frac{1}{8} \|\nabla u_t\|_{L^2}^2, \quad (2.21)
\]

\[
I_5 = \frac{2}{3} \int \rho_t \pi \nabla \cdot u_t - \frac{2}{3} \int \pi_t \nabla \r - \frac{2}{3} \int \rho \nabla \pi_t \cdot u_t \\
\leq C \|\rho\|_{L^3} \|\pi\|_{L^6} \|\nabla u_t\|_{L^2} + C_0 \|\nabla \rho\|_{L^2} \|\pi\|_{L^6} \|\sqrt{\rho} u_t\|_{L^2} + C_0 \|\nabla \pi\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2} \\
\leq C_0 c_1^2 c_2 + C_0 \eta^{-1} c_0 \gamma^3 + C \eta \|\pi_t\|_{L^2} \|\sqrt{\rho} u_t\|_{L^2}^2 + \frac{1}{8} \|\nabla u_t\|_{L^2}^2. \quad (2.22)
\]

On the other hand, we easily have

\[
\frac{d}{dt} \int |\nabla u|^2 = 2 \int \nabla u \cdot \nabla u_t \leq \frac{1}{8} \|\nabla u_t\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2, \quad (2.23)
\]

and

\[
\frac{d}{dt} \int |u|^2 \leq C_0 \|\sqrt{\rho} u_t\|_{L^2} \|u\|_{L^2} \leq C_0 \|\sqrt{\rho} u_t\|_{L^2}^2 + C \|u\|_{L^2}^2. \quad (2.24)
\]

Combining \( (2.14), (2.16) \) and \( (2.20), (2.22), \) we get

\[
\frac{d}{dt} \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|u\|_{H^\gamma}^2 \right) + \|\nabla u_t\|_{L^2}^2 \leq C \left( c_0^2 c_1^2 + \eta^{-1} c_0 \gamma^2 c_0 c_2^2 + \eta \|\pi_t\|_{H^\gamma}^2 + \eta \|v_t\|_{H^\gamma}^2 \right) \left( \|\sqrt{\rho} u_t\|_{L^2}^2 + \|u\|_{H^\gamma}^2 \right) \\
+ C \left( c_0^5 c_1^4 c_2^2 + \eta^{-1} c_0 \gamma^3 c_1^2 \right),
\]

setting \( \eta = \frac{1}{c_1} \) and using Gronwall’s inequality, we derive

\[
\|\sqrt{\rho} u_t\|_{L^2}^2 + \|u\|_{H^\gamma}^2 + \int_0^t \|\nabla u_t\|_{L^2}^2 ds \leq C_0 c_0 \gamma^2 \quad (2.26)
\]
for $0 \leq t \leq T$, where we have used the fact that $\lim_{t \to 0} (\| \sqrt{\mu} u(t) \|_{L^2}^2 + \| u(t) \|_{H^1}^2) \leq C c_0^{5+2\gamma}$. 

Next, by (2.19) and (2.26), we deduce

$$\| \nabla u \|_{H^1} \leq C c_0^{\frac{5}{2}+3\gamma} c_1^2,$$  \hspace{1cm} (2.27)

which implies (2.12) by (2.26).

Next, we will estimate $\int_0^T \| u \|_{H^2}^2 dt$. By the standard elliptic regularity result of equation (2.28), we have

$$\| \nabla^4 u \|_{L^2} \leq \| \rho u_t \|_{H^2} + \| \rho v \cdot \nabla u \|_{H^2} + \| \nabla^2 \|_{H^2} + \frac{2}{3} \| \nabla (\rho \pi) \|_{H^2}.$$  \hspace{1cm} (2.28)

By simple calculation, the first term of the right hand side of (2.28) can be controlled as

$$\| \rho u_t \|_{H^2} \leq C (\| \rho u_t \|_{L^2} + \| \rho \|_{H^2} \| u_t \|_{H^2}) \leq C c_0 \| u_t \|_{H^2}.$$  \hspace{1cm} (2.29)

In order to estimate $\| \nabla^2 u_t \|_{L^2}$, differentiating equation (2.28) with respect to $t$ yields

$$\Delta u_t + \nabla \text{div} u_t = \rho_t u_t + \rho u_{tt} + \rho v \cdot \nabla u + \rho v_t \cdot \nabla u + \rho v \cdot \nabla u_t + \nabla p_t + \frac{2}{3} (\nabla (\rho_t \pi + \rho_t \nabla \pi + \rho \nabla \pi_t) + \rho \nabla \pi_t),$$

applying the standard elliptic regularity result to (2.30) and using (2.29), one obtains

$$\| \nabla^2 u_t \|_{L^2} \leq C (\| \rho_t \|_{L^1} \| u_t \|_{L^4} + \| \rho u_{tt} \|_{L^2} + \| \rho_t \|_{L^4} \| v \|_{L^6} \| \nabla u \|_{L^4} + \| \rho \|_{L^6} \| v_t \|_{L^4} \| \nabla u \|_{L^4} + \| \rho \|_{L^6} \| \nabla u_t \|_{L^4} + \| \rho \|_{L^6} \| \nabla \pi \|_{L^4} + \| \rho \|_{L^6} \| \nabla \pi_t \|_{L^4} + \| \rho \|_{L^6} \| \nabla \pi_t \|_{L^2}) \leq C (\| \rho u_{tt} \|_{L^2} + c_0^{\frac{5}{2}+3\gamma} c_1^2 c_2 c_3 + c_0^{\frac{5}{2}+3\gamma} c_1^2 \| v_t \|_{H^1} + c_0 c_2 \| u_t \|_{H^1} + c_0 \| \pi_t \|_{H^1}),$$

therefore, the key point is to estimate $\| \rho u_{tt} \|_{L^2}$. Because we have the fact $\| \rho u_{tt} \|_{L^2} \leq C \| \nabla u_{tt} \|_{L^2}$, we could first estimate $\| \nabla u_{tt} \|_{L^2}$ as follows.

Multiplying both sides of (2.20) by $u_{tt}$ and integrating the result over $\Omega$ yield

$$\int \rho u_{tt}^2 dx \left[\frac{d}{dt} - \frac{1}{2} \| \nabla u_t \|_{L^2}^2 + \frac{1}{2} \| \text{div} u_t \|_{L^2}^2 \right]
= -\int \rho_t u_t \cdot u_{tt} - \int \rho v_t \cdot \nabla u \cdot u_{tt} - \int \rho v \cdot \nabla u_t \cdot u_{tt} - \int \nabla p_t \cdot u_{tt} - \frac{2}{3} \int (\nabla (\rho_t \pi + \rho_t \nabla \pi + \rho \nabla \pi_t) \cdot u_{tt} = J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$  \hspace{1cm} (2.32)

Using Hölder, Sobolev and Young’s inequalities and (2.11) and (2.20), we get

$$J_1 \leq C c_0^{\frac{5}{2}} \| \rho_t \|_{L^1} \| u_t \|_{L^6} \| \nabla u_t \|_{L^2} \leq C c_0^{\frac{5}{2}} \| \rho_t \|_{L^3} (\| \nabla u_{tt} \|_{L^2} + \| u_{tt} \|_{L^2}) \| \nabla u_{tt} \|_{L^2} \leq C c_0^{\frac{5}{2}} c_2 \| u_t \|_{L^2}^2 + C c_0^{\frac{5}{2}+3\gamma} c_1^2 + \frac{1}{18} \| \nabla u_{tt} \|_{L^2}^2,$$  \hspace{1cm} (2.33)

$$J_2 \leq C c_0^{\frac{5}{2}} \| \nabla u_{tt} \|_{L^2} \| \rho_t \|_{L^1} \| v \|_{L^6} \| \nabla u \|_{L^6} \| \nabla u_{tt} \|_{L^2} \leq C c_0^{\frac{5}{2}+6\gamma} c_1^4 c_2 + \frac{1}{18} \| \nabla u_{tt} \|_{L^2}^2,$$  \hspace{1cm} (2.34)

$$J_3 \leq C c_0^{\frac{5}{2}} \| \nabla u_{tt} \|_{L^2} \| u_t \|_{L^6} \| \nabla u \|_{L^6} \leq C c_0^{\frac{5}{2}+6\gamma} c_1^4 \| u_t \|_{H^1} + \frac{1}{18} \| \nabla u_{tt} \|_{L^2}^2,$$  \hspace{1cm} (2.35)
we could estimate the right hand side of (2.42) item by item.

\[ J_5 \leq Cc_2^2 \|
\sqrt{\rho u_t} \|_{L^2} \| \nabla u_t \|_{L^2} \leq Cc_0^2 \|
\nabla u_t \|_{L^2}^2 + \frac{1}{18} \| \sqrt{\rho u_t} \|_{L^2}^2, \tag{2.36} \]

\[ J_6 \leq Cc_2^2 \| \sqrt{\rho u_t} \|_{L^2} \| \nabla \rho_t \|_{L^2} \leq Cc_0^{2\gamma+1} c_2^2 + \frac{1}{18} \| \sqrt{\rho u_t} \|_{L^2}^2, \tag{2.37} \]

inserting (2.33)-(2.38) to (2.32), then integrating the result over (0, t), we derive

\[ \int_0^t \int_{\Omega} \rho u_t^2 \, dx \, dt + \| \nabla u_t \|_{L^2}^2 \leq Cc_0^{\gamma+6\gamma} c_1^2 c_2^2, \tag{2.39} \]

where we have used equation (2.33) to get \( \lim_{t \to 0} \| \nabla u_t(t) \|_{L^2}^2 \leq Cc_0^{2\gamma+4} \).

So, combining (2.29), (2.31) and (2.39), we obtain

\[ \int_0^t \| \rho u_t \|_{H^2} \leq Cc_0^{\gamma+6\gamma} c_1^2 c_2^2. \tag{2.40} \]

In the following, we shall estimate the rest terms of the inequality (2.28).

For the second term of the inequality (2.28), direct calculation yields

\[ \| \rho v \cdot \nabla u \|_{H^2} \leq C \| \rho \|_{H^2} \| v \|_{H^3} \| u \|_{H^3} \leq Cc_0 c_2 \| u \|_{H^3}. \tag{2.41} \]

therefore, we have to evaluate \( \| u \|_{H^3} \). In fact, applying the standard elliptic regularity result to equation (2.28), we obtain

\[ \| \nabla^3 u \|_{L^2} \leq C(\| \rho u_t \|_{H^3} + \| \rho v \cdot \nabla u \|_{H^3} + \| \nabla p \|_{H^1} + \| \nabla (\rho \pi) \|_{H^1}), \tag{2.42} \]

we could estimate the right hand side of (2.42) item by item.

First, from (2.26), we have \( \| u_t \|_{L^2} \leq Cc_0^{\frac{3}{2}+\gamma} \), thus

\[ \| \rho u_t \|_{H^1} \leq Cc_0 \| u_t \|_{L^2} + \| \nabla \rho \|_{L^\infty} \| u_t \|_{L^2} + \| \nabla u \|_{L^2} \| u_t \|_{L^2} \leq Cc_0^{\frac{3}{2}+\gamma} + Cc_0 \| \nabla u_t \|_{L^2}. \tag{2.43} \]

Second, using Sobolev’s interpolation inequality and Young’s inequality, we get

\[ \| \rho v \cdot \nabla u \|_{H^1} \leq C(\| \rho v \cdot \nabla u \|_{L^2} + \| \nabla (\rho v \cdot \nabla u) \|_{L^2}) \]

\[ \leq C(c_0 \| u \|_{L^\infty} \| u_t \|_{L^2} + \| \nabla p \|_{L^\infty} \| u_t \|_{L^2} + \| \nabla u \|_{L^2} \| u_t \|_{L^2} + \| \nabla v \|_{L^2} \| u_t \|_{L^2} + \| \nabla u \|_{L^2} \| \nabla^3 u \|_{L^2}) \]

\[ + c_0 \| u \|_{L^\infty} \| \nabla^2 u \|_{L^2} \leq Cc_0^{\frac{3}{2}+3\gamma} c_1^4 c_2 + \frac{3}{4} \| u \|_{H^3}. \tag{2.44} \]

Third, due to (2.11), we easily derive

\[ \| \nabla p \|_{H^1} \leq Cc_0^2. \tag{2.45} \]

Last, by simple calculation, one gets

\[ \| \nabla (\rho \pi) \|_{H^2} \leq C \| \rho \|_{H^3} \| \pi \|_{H^2} \leq Cc_0 c_5. \tag{2.46} \]
Combining (2.39) and (2.42)-(2.46), we deduce
\[ \|u\|_{H^3} \leq Cc_{13}^{13/3 + 3\gamma} c_1^2 c_5. \] (2.47)

Next, by simple calculation, the third and fourth terms on the right hand side of (2.28) can be estimated as
\[ \|\nabla p\|_{H^2} \leq Cc_{13}^3, \quad \|\nabla (\rho \pi)\|_{H^2} \leq Cc_{13} \|\pi\|_{H^3}. \] (2.48)

Combining (2.20), (2.28), (2.40), (2.41) and (2.47)-(2.48), one deduce
\[ \int_0^t \|u\|_{H^3}^2 dt \leq Cc_{13}^{10 + 6\gamma} c_1^5 c_2^2, \] (2.49)

for \(0 \leq t \leq T\).

Thus, we complete the proof of Lemma 2.2.

In the following part, we estimate the turbulent kinetic energy \(k\).

**Lemma 2.3.** There exists a unique strong solution \(k\) to the initial boundary value problem (2.5) and (2.9) such that
\[ \|\sqrt{\rho} k_t\|_{L^2}^2 + \|k\|_{H^1}^2 + \int_0^t \|\nabla k_t\|_{L^2}^2 ds \leq Cc_{13}^5, \] (2.50)
\[ \|k\|_{H^2} \leq Cc_{13}^2 c_1 c_2^2, \quad \int_0^t \|k\|_{H^3}^2 ds \leq Cc_{13}^7 \] (2.51)

for \(0 \leq t \leq T\).

**Proof.** We only need to prove the estimates. Differentiating equation (2.5) with respect to \(t\), then multiplying both sides of the result equation by \(k_t\) and integrating over \(\Omega\), we get
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} k_t\|_{L^2}^2 + \|\nabla k_t\|_{L^2}^2 = - \int \rho v \cdot \nabla k \cdot k_t - \int \rho v_t \cdot \nabla k \cdot k_t - 2 \int \rho v \cdot \nabla k_t \cdot k_t + \int G_t \cdot k_t
\]
\[
- \int \rho \theta \cdot k_t - \int \rho \theta_t \cdot k_t = \sum_{i=1}^6 K_i, \] (2.52)

we could evaluate \(K_i\) \((i = 1, \cdots, 6)\) as follows.

First, using similar method of deriving (2.15), (2.20), (2.40), respectively, one has
\[ K_1 \leq Cc_{13}^2 c_2^4 \|\nabla k\|_{L^2}^2 + C\|\sqrt{\rho} k_t\|_{L^2}^2 + \frac{1}{10} \|\nabla k_t\|_{L^2}^2, \] (2.53)
\[ K_2 \leq C\eta^{-1} c_{13}^{2\gamma + 1} (\|\sqrt{\rho} k_t\|_{L^2}^2 + c_1^2 \|\nabla k\|_{L^2}^2 + c_0^2 c_1^2 c_2^4) + \eta \|u_t\|_{H^1}^2 \|\sqrt{\rho} k_t\|_{L^2}^2, \] (2.54)
\[ K_3 \leq Cc_{13}^2 \|\sqrt{\rho} k_t\|_{L^2}^2 + \frac{1}{10} \|\nabla k_t\|_{L^2}^2. \] (2.55)
Next, differentiating \( G' \) with respect to \( t \) and inserting the result thus obtained to \( K_4 \) yield

\[
K_4 \leq C \int |\nabla v_i||\nabla v||k_i| + C \int |\rho||\pi||\nabla v||k_i| + C \int |\rho_t||\pi||\nabla v||k_i| + C \int |\rho||\pi_t||\nabla v||k_i|
\]

\[
\leq C c_0^2 \|\sqrt{\rho k_i}\|_{L^2} \|\nabla v_i\|_{L^2} + C c_0^2 \|\nabla v\|_{L^2} + C c_0^2 \|\sqrt{\rho k_i}\|_{L^2} \|\nabla k_i\|_{L^2}
\]

\[
+ C \|\pi\|_{L^\infty} \|\rho\|_{L^1} \|\nabla v\|_{L^2} \|k_i\|_{L^6} + C c_0^2 \|\sqrt{\rho k_i}\|_{L^2} \|\pi_t\|_{L^6} \|\nabla v\|_{L^3}
\]

\[
\leq C \eta^{-1} c_0^2 c_2^2 + C c_0^2 c_1^2 c_2^2 \|\sqrt{\rho k_i}\|_{L^2} + C \|v_i\|_{H^1} + C \|\pi_t\|_{H^1} \|\sqrt{\rho k_i}\|_{L^2} + \frac{1}{10} \|\nabla k_i\|_{L^2}^2.
\]

(2.56)

Last, direct calculation leads to

\[
K_5 \leq \|\rho_t\|_{L^1} \|\theta\|_{L^2} \|k_i\|_{L^6} \leq C c_0^2 c_1^2 c_2^2 + C \|\sqrt{\rho k_i}\|_{L^2} + \frac{1}{10} \|\nabla k_i\|_{L^2}^2,
\]

(2.57)

\[
K_6 \leq C c_0^2 \|\sqrt{\rho k_i}\|_{L^2} \|\theta_t\|_{L^2} \leq C \eta^{-1} c_0 + \eta \|\theta_t\|_{L^2} \leq C \eta^{-1} c_0 + \eta \|\theta_t\|_{L^2} \|\sqrt{\rho k_i}\|_{L^2}^2.
\]

(2.58)

On the other hand, we easily get

\[
\frac{d}{dt} \|\nabla k\|_{L^2}^2 \leq \frac{1}{10} \|\nabla k\|_{L^2}^2 + C \|\nabla k\|_{L^2}^2,
\]

(2.59)

\[
\frac{d}{dt} \|k\|_{L^2}^2 \leq C c_0 \|\sqrt{\rho k_i}\|_{L^2}^2 + C \|k\|_{L^2}^2.
\]

(2.60)

Combining (2.52) and (2.60), we obtain

\[
\frac{d}{dt} \|\sqrt{\rho k_i}\|_{L^2}^2 + \|k\|_{H^1}^2 + \|\nabla k_i\|_{L^2}^2
\]

\[
\leq C (c_0^2 c_1^2 + \eta^{-1} c_0^2 \|v_i\|_{H^1} + \eta \|\pi_t\|_{H^1} + \eta \|\theta_t\|_{L^2}) \|\sqrt{\rho k_i}\|_{L^2}^2 + \|k\|_{H^1}^2 + C (c_0^2 c_1^2 c_2^2 + c_0^2 c_1^2 c_2^2).
\]

(2.61)

setting \( \eta = c_1^{-1} \) and using Gronwall’s inequality, we deduce

\[
\|\sqrt{\rho k_i}\|_{L^2}^2 + \|k\|_{H^1}^2 + \int_0^t \|\nabla k_i\|_{L^2}^2 ds \leq C c_0^5
\]

(2.62)

for \( 0 \leq t \leq T \), where we have used the fact that \( \lim_{t \to 0} \|\sqrt{\rho k_i}\|_{L^2}^2 + \|k\|_{H^1}^2 = C c_0^5. \)

Then, by the standard elliptic regularity result of equation (2.5) and using (2.62), we have

\[
\|\nabla k\|_{H^1} \leq C c_0 \|\sqrt{\rho k_i}\|_{L^2} + C c_0 \|v_i\|_{L^\infty} \|\nabla k\|_{L^2} + C \|\nabla v\|_{L^2}^2
\]

\[
+ C c_0 \|\pi_t\|_{L^1} \|\nabla v\|_{L^4} + C c_0 \|\theta\|_{L^2} \leq C c_0 \|\sqrt{\rho k_i}\|_{L^2} + c_1 c_2^2,
\]

(2.63)

and

\[
\|\nabla k\|_{H^1} \leq C (\|\rho k_i\|_{H^1} + \|\rho v \cdot \nabla k\|_{H^1} + \|G'\|_{H^1} + \|\rho \theta\|_{H^1}).
\]

(2.64)

To evaluate \( \int_0^t \|k\|_{H^2}^2 dt \), we will estimate the right hand side of (2.64) item by item. In fact, we derive by using (2.62) and (2.63) that

\[
\|\rho k_i\|_{H^1} \leq C \|\rho k_i\|_{L^2} + \|\nabla (\rho k_i)\|_{L^2} \leq C c_0 \|\sqrt{\rho k_i}\|_{L^2} + C c_0 \|\nabla k_i\|_{L^2},
\]

(2.65)

\[
\|\rho v \cdot \nabla k\|_{H^1} \leq C (\|\rho v \cdot \nabla k\|_{L^2} + \|\nabla (\rho v \cdot \nabla k)\|_{L^2})
\]

\[
\leq C (c_0 \|v\|_{L^\infty} \|\nabla k\|_{L^2} + \|\nabla \rho\|_{L^\infty} \|v\|_{L^\infty} \|\nabla k\|_{L^2}
\]

\[
+ c_0 \|\nabla v\|_{L^4} \|\nabla k\|_{L^4} + c_0 \|v\|_{L^\infty} \|\nabla^2 k\|_{L^2} \leq C c_0 \|\sqrt{\rho k_i}\|_{L^2} + c_1 c_2^2,
\]

(2.66)
There exists a unique strong solution for \( E \). We only need to prove the estimates. Differentiating equation (2.6) with respect to \( t \),

\[
\|G'\|_{H^1} \leq C(\|\nabla v\| + \|\nabla \cdot \rho \cdot \pi\|_{L^2} + \|\nabla v \cdot \nabla^2 v\|_{L^2} + \|\nabla (\nabla \cdot \rho \cdot \pi)\|_{L^2})
\]

\[
\leq C(\|\nabla v\|_{L^2} + c_0\|\pi\|_{L^\infty}\|\nabla v\|_{L^2} + \|\nabla v\|_{L^4}\|\nabla^2 v\|_{L^4} + c_0\|\pi\|_{L^\infty}\|\nabla^2 v\|_{L^2}
\]

\[
+ \|\pi\|_{L^\infty}\|\nabla \rho\|_{L^\infty}\|\nabla v\|_{L^2} + c_0\|\nabla v\|_{L^\infty}\|\nabla \pi\|_{L^2}) \leq Cc_0c_2c_3c_5,
\]

and

\[
\|\rho \theta\|_{H^1} \leq C\|\rho\|_{H^1}\|\theta\|_{H^1} \leq Cc_0c_1.
\]

Therefore, inserting (2.65)–(2.68) to (2.64) and integrating the result thus obtained over \((0, t)\), one gets

\[
\int_0^t \|k\|_{H^1}^2 \, dt \leq Cc_0^2
\]

for \( 0 \leq t \leq T \).

Combining (2.62), (2.68) and (2.69), we complete the proof of Lemma 2.3.

In the next part, we estimate the viscous dissipation rates of the turbulent flows \( \varepsilon \).

**Lemma 2.4.** There exists a unique strong solution \( \varepsilon \) to the initial boundary value problem (2.6) and (2.7) such that

\[
\|\sqrt{\rho \varepsilon_t}\|_{L^2}^2 + \|\varepsilon_t\|_{H^1}^2 + \int_0^t \|
abla \varepsilon_t\|_{L^2}^2 \, ds \leq Cc_0^5,
\]

\[
\|\varepsilon_t\|_{H^2} \leq Cc_0^2 c_2^2 c_3^2
\]

for \( 0 \leq t \leq T \).

**Proof.** We only need to prove the estimates. Differentiating equation (2.6) with respect to \( t \), then multiplying both sides of the result by \( \varepsilon_t \) and integrating over \( \Omega \), one obtains

\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho \varepsilon_t}\|_{L^2}^2 + \|\nabla \varepsilon_t\|_{L^2}^2
\]

\[
= - \int \rho_t v \cdot \nabla \varepsilon \cdot \varepsilon_t - \int \rho v_t \cdot \nabla \varepsilon \cdot \varepsilon_t - 2 \int \rho v \cdot \nabla \varepsilon_t \cdot \varepsilon_t
\]

\[
+ \int \left( \frac{C^G \theta}{\pi} \right)_t \cdot \varepsilon_t - \int \left( \frac{C^2 \rho \theta^2}{\pi} \right)_t \cdot \varepsilon_t = \sum_{i=1}^5 E_i.
\]

We could evaluate \( E_4 \) and \( E_5 \) in the first place. Because \( \pi \) has upper and lower bound away from zero, direct calculation yields

\[
E_4 \leq C \int (|G'\theta| + |G' \theta_t| + |G' \pi \varepsilon_t|) \varepsilon_t
\]

\[
\leq C \int (|\nabla v| + |\rho_t \pi \nabla v| + |\rho v_t \nabla v| + |\rho v \nabla v_t|) \varepsilon_t
\]

\[
+ C \int (\|\nabla v\|^2 + \|\rho v \nabla v\|) |\varepsilon_t| \varepsilon_t + C \int (\|\nabla v\|^2 + \|\rho v \nabla v\|) \|\varepsilon_t\| \varepsilon_t
\]

\[
\leq Cc_0^2 \|\theta\|_{L^\infty} \|\nabla v\|_{L^2} \|\nabla v_t\|_{L^2} \|\sqrt{\rho \varepsilon_t}\|_{L^2} + Cc_0^2 \|\pi\|_{L^\infty} \|\nabla v\|_{L^2} \|\sqrt{\rho \varepsilon_t}\|_{L^2} \|\theta\|_{L^6}
\]

\[
+ Cc_0 \|\sqrt{\rho \varepsilon_t}\|_{L^2} \|\pi\|_{L^6} \|\nabla v\|_{L^6} \|\theta\|_{L^6} + Cc_0 \|\pi\|_{L^\infty} \|\theta\|_{L^\infty} \|\nabla v\|_{L^2} \|\sqrt{\rho \varepsilon_t}\|_{L^2}
\]

\[
+ Cc_0^2 \|\sqrt{\rho \varepsilon_t}\|_{L^2} \|\pi\|_{L^\infty} \|\nabla v\|_{L^\infty} \|\theta\|_{L^6} + Cc_0^2 \|\pi\|_{L^\infty} \|\theta\|_{L^\infty} \|\nabla v\|_{L^6} \|\sqrt{\rho \varepsilon_t}\|_{L^2}
\]

\[
\leq Cc_0^{-1}c_2^2c_0^2c_3^2c_5^2 + Cc_0^2c_2^2c_3^2c_5^2 + Cc_0^2 \|\nabla v\|_{L^2} + Cc_0 \|\pi\|_{L^2} + \|\pi\|_{L^\infty} + \|\varepsilon_t\|_{H^1} + \|\theta_t\|_{L^2} + \|\theta\|_{L^6}
\]

(2.73)
and

\[ E_5 \leq C \int \rho \theta^2 \varepsilon_t^2 \, dt + C \int \theta \rho \varepsilon_t \, dt + C \int \rho \theta^2 \pi_t \varepsilon_t \, dt \]

\[ \leq C \| \rho \|_{L^2} \| \theta \|_{L^2} \| \varepsilon_t \|_{L^6}^2 \right) + C \| \varepsilon_t \|_{L^2} \| \theta \|_{L^2} \| \pi_t \|_{L^2} \| \theta \|_{L^2}^2. \]  

(2.74)

\[ \leq C \eta^{-1} c_0 c_4^2 + C c_0^2 \left( \| \varepsilon_t \|_{L^2}^2 + \| \varepsilon_t \|_{H^1} \| \varepsilon_t \|_{H^2} \right) \right) + \frac{1}{8} \| \nabla \varepsilon_t \|_{L^2}^2. \]  

(2.75)

\[ \leq C \eta^{-1} c_0 c_4^2 + C c_0 \| \varepsilon_t \|_{L^2}^2, \]

(2.76)

\[ \frac{1}{10} \| \nabla \varepsilon_t \|_{L^2}^2, \]  

(2.77)

\[ \frac{d}{dt} \| \varepsilon_t \|_{L^2}^2 \leq C \| \varepsilon_t \|_{L^2}^2 + C \| \varepsilon_t \|_{H^1}^2. \]  

(2.78)

and finally

\[ \frac{d}{dt} \| \nabla \varepsilon_t \|_{L^2}^2 \leq \frac{1}{8} \| \nabla \varepsilon_t \|_{L^2}^2 + C \| \nabla \varepsilon_t \|_{L^2}^2. \]

(2.79)

Combining (2.72)–(2.73), one obtains

\[ \frac{1}{2} \frac{d}{dt} \left( \| \varepsilon_t \|_{L^2}^2 + \| \varepsilon_t \|_{H^1}^2 \right) + \| \nabla \varepsilon_t \|_{L^2}^2 \]

\[ \leq C (c_0^2 c_4^2 + \eta^{-1} c_0^2 c_1^2 + \eta \| \varepsilon_t \|_{H^1}^2 + \eta \| \theta_t \|_{H^1} + \eta \| \pi_t \|_{H^1}^2) \left( \| \varepsilon_t \|_{L^2}^2 + \| \varepsilon_t \|_{H^1}^2 \right) + C \eta^{-1} c_0 c_4^2 c_1 c_2 c_3 c_5 + C c_0^2 c_4^2 c_1 c_2 c_3. \]

(2.80)

setting \( \eta = c_1^{-1} \) and using Gronwall’s inequality, one obtains

\[ \| \sqrt{\varepsilon_t} \|_{L^2}^2 + \| \varepsilon_t \|_{H^1}^2 + \int_0^t \| \nabla \varepsilon_t \|_{L^2}^2 \, ds \leq C c_0^5 \]  

(2.81)

for \( 0 \leq t \leq T \), where we have used the fact that \( \lim_{\varepsilon \to 0} (\| \sqrt{\varepsilon_t} \|_{L^2}^2 + \| \varepsilon_t \|_{H^1}^2) \leq C c_0^5 \).

Next, applying the standard elliptic regularity result to equation (2.6) and using (2.81), we have

\[ \| \nabla \varepsilon \|_{H^1} \leq C (c_0^2 \| \sqrt{\varepsilon_t} \|_{L^2} + \eta \| \varepsilon_t \|_{L^6} + \| \nabla v \|_{L^6} \| \theta \|_{L^6} + \| \nabla v \|_{L^6} \| \theta \|_{L^6} \| \pi \|_{L^6} + c_0 \| \theta \|_{L^6}^2) \]

\[ \leq C (c_0^2 c_4^2 + c_0 c_1 \| \nabla \varepsilon \|_{L^2} + \| \varepsilon_t \|_{L^2}^2). \]

(2.82)

therefore, by Young’s inequality and (2.81), one deduce

\[ \| \varepsilon_t \|_{H^2} \leq C c_0^2 c_4^2 c_1 c_2. \]

Thus, we complete the proof of Lemma 2.4. □
Finally, we estimate the total enthalpy \( h \).

**Lemma 2.5.** There exists a unique strong solution \( h \) to the initial boundary value problem (2.4) and (2.9) such that

\[
\| \sqrt{r} h_t \|_{L^2}^2 + \| h \|_{Y_1}^2 + \int_t^T \| \nabla h_t \|_{L^2}^2 \, ds \leq Cc_0^2, \tag{2.83}
\]

\[
\| h \|_{H^2} \leq Cc_0^{2+\gamma} c_1^2 c_2^2 \tag{2.84}
\]

for \( 0 \leq t \leq T \).

**Proof.** We only need to prove the estimates. Differentiating equation (2.4) with respect to \( t \), multiplying both sides of the result equation by \( h_t \) and integrating over \( \Omega \), one obtains

\[
\frac{d}{dt}(\| \sqrt{r} h_t \|_{L^2}^2 + \| h \|_{Y_1}^2) + \| \nabla h_t \|_{L^2}^2 = - \int \rho_t v \cdot \nabla h \cdot h_t - \int \rho u \cdot \nabla h \cdot h_t - 2 \int \rho v \cdot \nabla h \cdot h_t + \int p_t \cdot h_t
\]

\[
+ \int u_t \cdot \nabla p \cdot h_t + \int u \cdot \nabla p \cdot h_t + \int S_{kt} \cdot h_t = \sum_{i=1}^7 H_i. \tag{2.85}
\]

Firstly, using similar method of deriving the estimates (2.15), (2.20) and (2.16), respectively, one has

\[
H_1 \leq Cc_0^2 c_2^2 \| \nabla h \|_{L^2}^2 + C \| \sqrt{r} h_t \|_{L^2}^2 + \frac{1}{20} \| \nabla h_t \|_{L^2}, \tag{2.86}
\]

\[
H_2 \leq C \eta^{-1} c_0^{2\gamma+1} (c_0^2 c_2^4 + \| \sqrt{r} h_t \|_{L^2}^2 + c_1^2 \| \nabla h \|_{L^2}^2) + \eta \| v_t \|_{Y_1}^2 \| \sqrt{r} h_t \|_{L^2}^2, \tag{2.87}
\]

\[
H_3 \leq Cc_0 c_2^2 \| \sqrt{r} h_t \|_{L^2}^2 + \frac{1}{20} \| \nabla h_t \|_{L^2}. \tag{2.88}
\]

Secondly, differentiating equation (2.22) with respect to \( t \) yields

\[
\rho_{tt} = -\rho_t \nabla \cdot v + \rho \nabla \cdot v_t + v_t \cdot \nabla \rho + v \cdot \nabla \rho_t. \tag{2.89}
\]

Therefore, by direct calculation and using (2.89), we derive

\[
H_4 = \int [\gamma(\gamma - 1) \rho^\gamma - \gamma \rho^{\gamma-1}(\rho_t \nabla \cdot v + \rho \nabla \cdot v_t + v_t \cdot \nabla \rho + v \cdot \nabla \rho_t)] \cdot h_t
\]

\[
\leq Cc_0^{\gamma-2} \rho \| \nabla h_t \|_{L^2}^2 + Cc_0^{\gamma-2} \| \rho_t \|_{L^2} \| \nabla h \|_{L^\infty} \| \sqrt{r} h_t \|_{L^2} + Cc_0^{\gamma-2} \| \sqrt{r} h_t \|_{L^2} \| \nabla h_t \|_{L^2}
\]

\[
+ Cc_0^{\gamma-2} \| \sqrt{r} h_t \|_{L^2} \| v_t \|_{L^\infty} \| \nabla \rho \|_{L^3} + Cc_0^{\gamma-2} \| \nabla \rho_t \|_{L^2} \| v \|_{L^\infty} \| \sqrt{r} h_t \|_{L^2}
\]

\[
\leq C(c_0^{\gamma+1} c_2^4 + \eta^{-1} c_0^{\gamma+1} + \| \sqrt{r} h_t \|_{L^2}^2 + \eta \| v_t \|_{Y_1}^2 \| \sqrt{r} h_t \|_{L^2}^2) + \frac{1}{20} \| \nabla h_t \|_{L^2}^2. \tag{2.90}
\]

Thirdly, simple calculation and (2.26) lead to

\[
H_5 \leq Cc_0^{\gamma-2} \| \sqrt{r} h_t \|_{L^2} \| \nabla \rho \|_{L^2} \| u_t \|_{L^\infty} \leq Cc_0^{\gamma-2} \| \sqrt{r} h_t \|_{L^2} \| \nabla \rho \|_{L^3} (\| u_t \|_{L^2} + \| \nabla u_t \|_{L^2})
\]

\[
\leq Cc_0^{\gamma+1} \| \sqrt{r} h_t \|_{L^2}^2 + Cc_0^{\gamma+5} + C \| \nabla u_t \|_{L^2}^2. \tag{2.91}
\]
Next, by direct calculation, we know that $\nabla p_t = \gamma(\gamma - 1)\rho^{\gamma - 2}\rho_t \nabla \rho + \gamma \rho^{\gamma - 1}\nabla \rho_t$. Therefore,

$$H_6 \leq C\gamma^{-2} \int |p_t||u||\nabla \rho|h_t| + C\gamma^{-1} \int |u||\nabla \rho_t|h_t|$$

$$\leq C\gamma^{-2} \|\nabla p\|_L^\infty \|p_t\|_L^3 \|u\|_L^2 (\|\sqrt{p_t}\|_L^2 + \|\nabla h_t\|_L^2)$$

$$+ C\gamma^{-2} \|u\|_L^3 \|\nabla p\|_L^2 (\|\sqrt{p_t}\|_L^2 + \|\nabla h_t\|_L^2)$$

$$\leq C\gamma^{7+2\gamma} c_2^2 + C \|\nabla h_t\|_L^2 + \frac{1}{20} \|\nabla h_t\|_L^2. \quad \text{(2.92)}$$

Consequently, combining (2.85)-(2.95), one deduces

$$H_7 \leq C \int |\nabla u||\nabla v||h_t| + C\gamma^{-1} \|p_t\|^2 \|h_t| + C\gamma^{-1} \int |\nabla \rho||\nabla \rho|\|h_t|$$

$$\leq C\gamma^{-2} \|\nabla \rho\|_L^\infty \|\nabla u\|_L^2 \|\sqrt{h_t}\|_L^2 + C\gamma^{-2} \|p_t\|_L^3 \|\nabla \rho\|_L^2 \|\sqrt{h_t}\|_L^2$$

$$+ C\gamma^{-2} \|\nabla \rho\|_L^2 (\|\nabla p\|_L^2 + \|\nabla h_t\|_L^2)$$

$$\leq C(\eta^{-1} c_0^2 + c_0^{5+2\gamma} c_2^2 + \eta \|\nabla v\|_L^2 \|\sqrt{h_t}\|_L^2 + \|\nabla h_t\|_L^2) \quad \text{(2.93)}$$

Furthermore, we easily have

$$\frac{d}{dt} \|h_t\|_L^2 \leq C \gamma c_0 \|\sqrt{h_t}\|_L^2 + C \|h_t\|_L^2, \quad \text{(2.94)}$$

and

$$\frac{d}{dt} \|\nabla h_t\|_L^2 \leq C \|\nabla h_t\|_L^2 + \frac{1}{10} \|\nabla h_t\|_L^2. \quad \text{(2.95)}$$

Consequently, combining (2.85)-(2.95), one deduces

$$\frac{d}{dt} (\|\sqrt{h_t}\|_L^2 + \|h_t\|_{H^1}^2) + \|\nabla h_t\|_L^2$$

$$\leq C(c_0^{2\gamma+1} c_2^2 + \eta^{-1} c_0^{2\gamma+1} c_1^2 + \eta \|\nabla v\|_{H^1}^2 (\|\sqrt{h_t}\|_L^2 + \|h_t\|_{H^1}^2)$$

$$+ C(c_0^{7+2\gamma} c_2^2 + \eta^{-1} c_0^{5+2\gamma} c_2^2 c_3^2), \quad \text{(2.96)}$$

setting $\eta = c_1^{-1}$ and using Gronwall’s inequality, we get

$$\|\sqrt{h_t}\|_L^2 + \|h_t\|_{H^1}^2 + \int_0^t \|\nabla h_t\|_{L^2}^2 ds \leq C c_0^5 \quad \text{(2.97)}$$

for $0 \leq t \leq T$, where we have used the fact that $\lim_{t \to 0} (\|\sqrt{h_t}\|_L^2 + \|h_t\|_{H^1}^2) \leq C c_0^5$.

Next, using (2.97) and the standard elliptic regularity result of equation (2.4), one obtains

$$\|\nabla h\|_{H^1} \leq C(c_0^{\frac{7}{2}} \|\sqrt{h_t}\|_L^2 + c_0 \|v\|_{L^4} \|\nabla h\|_{L^3} + c_0 \gamma^{-1} \|p_t\|_L^2 + c_0^{-1} \|u\|_L^4 \|\nabla p\|_L^3$$

$$+ \|\nabla v\|_{L^4} + c_0^{-1} \|\nabla \rho\|_{L^2}^2) \leq C(c_0^{\frac{7}{2}+\gamma} c_2^2 + C_0 c_1 \|\nabla h\|_L^2 \|\nabla h\|_{H^1}^\frac{7}{2}, \quad \text{(2.98)}$$

then, Young’s inequality and (2.97) yields

$$\|h_t\|_{H^2} \leq C c_0^{\frac{7}{2}+\gamma} c_1 c_2^2. \quad \text{□}$$

Thus, we have finished the proof of Lemma 2.5.
Next, let us define $c_i$ ($i = 1, \cdots, 6$) as follows:

\[
c_1 = Cc_0^{7+2\gamma}, \quad c_2 = Cc_0^{2+3\gamma}c_1^2, \quad c_3 = Cc_0^{2}c_1^2c_2, \quad c_4 = Cc_0^{1+6\gamma}c_1^2c_2^2,
\]

then we conclude from Lemma 2.1 to Lemma 2.5 that

\[
\begin{aligned}
\sup_{0 \leq t \leq T} (\|u\|_{H^1} + \|k\|_{H^1} + \|\varepsilon\|_{H^1}) \\
+ \int_0^T (\|k\|_{H^2}^2 + \|u_t\|_{H^1}^2 + \|k_t\|_{H^1}^2 + \|\varepsilon_t\|_{H^1}^2) dt \
\leq c_1,
\end{aligned}
\]

\begin{equation}
(2.99)
\end{equation}

and

\[
\begin{aligned}
\|\rho\|_{H^1(\Omega)} & \leq Cc_0, \quad \|\rho_t\|_{H^1(\Omega)} \leq Cc_0c_2 \\
\|\sqrt{\rho}h_t\|_{L^2}^2 + \|h\|_{H^2}^2 + \int_0^T \|
abla h_t\|_{L^2}^2 ds & \leq Cc_0^2,
\end{aligned}
\]

\begin{equation}
(2.100)
\end{equation}

for $0 \leq t \leq T$.

Using standard proof as that in [5], we can complete the proof of Theorem 2.1.

\[\square\]

3 Existence of strong solutions to the $k - \varepsilon$ equations

**Theorem 3.1.** There exists a small time $T^* > 0$ and a unique strong solution $(\rho, u, h, k, \varepsilon)$ to the initial boundary value problem (1.1)-(1.10) such that

\[
\begin{align*}
\rho & \in C(0, T^*; H^3), \rho_t \in C(0, T^*; H^1), u \in C(0, T^*; H^3) \cap L^2(0, T^*; H^4), \\
u_t & \in L^2(0, T^*; H^1), k \in C(0, T^*; H^3) \cap L^2(0, T^*; H^3), k_t \in L^2(0, T^*; H^1), \\
\varepsilon & \in C(0, T^*; H^2), \varepsilon_t \in L^2(0, T^*; H^1), h \in C(0, T^*; H^3), h_t \in L^2(0, T^*; H^1), \\
(\sqrt{\rho} u_t, \sqrt{\rho} k_t, \sqrt{\rho} \varepsilon_t, \sqrt{\rho} h_t) & \in L^\infty(0, T^*; L^2).
\end{align*}
\]

(3.1)

**Proof.** Our proof will be based on the iteration argument and on the results in the last section (especially Theorem 2.1).

Firstly, using the regularity effect of classical heat equation, we can construct functions $(u^0 = u^0(x, t), k^0 = k^0(x, t), \varepsilon^0 = \varepsilon^0(x, t))$ satisfying $(u^0(x, 0), k^0(x, 0), \varepsilon^0(x, 0)) = (u_0(x), k_0(x), \varepsilon_0(x))$ and

\[
\begin{aligned}
\sup_{0 \leq t \leq T} (\|u^0\|_{H^1} + \|k^0\|_{H^1} + \|\varepsilon^0\|_{H^1}) \\
+ \int_0^T (\|k^0\|_{H^2}^2 + \|u_t^0\|_{H^1}^2 + \|k_t^0\|_{H^1}^2 + \|\varepsilon_t^0\|_{H^1}^2) dt \
\leq c_1,
\end{aligned}
\]

\[
\begin{aligned}
\sup_{0 \leq t \leq T} (\|u^0\|_{H^2} \leq c_2, \quad \sup_{0 \leq t \leq T} (\|u^0\|_{H^3} + \|k^0\|_{H^3} + \|\varepsilon^0\|_{H^3}) dt \leq c_4, \\
\sup_{0 \leq t \leq T} (\|k^0\|_{H^2} \leq c_5, \quad \sup_{0 \leq t \leq T} (\|\varepsilon^0\|_{H^2} \leq c_6).
\end{aligned}
\]

Therefore it follows from Theorem 2.1 that there exists a unique strong solution $(\rho^1, u^1, h^1, k^1, \varepsilon^1)$ to the linearized problem (2.2)-(2.6) with $v, \pi, \theta$ replaced by $u^0, k^0, \varepsilon^0$, respectively, which satisfies the regularity estimates (2.99) and (2.100). Similarly, we construct approximate solutions $(\rho^n, u^n, h^n, k^n, \varepsilon^n)$, inductively, as follows: assuming that $u^{n-1}, k^{n-1}, \varepsilon^{n-1}$ have been defined for $n \geq 1$, let $(\rho^n, u^n, h^n, k^n, \varepsilon^n)$ be the unique solution to the linearized problem (2.2)-(2.6) with $v, \pi, \theta$ replaced by $u^{n-1}, k^{n-1}, \varepsilon^{n-1}$, respectively. Then it follows from Theorem 2.1 that there
exists a constant $\tilde{C} > 1$ such that

$$\sup_{0 \leq t \leq T} (\|\rho^n\|_{H^2} + \|\rho^n\|_{H^3}) + \sup_{0 \leq t \leq T} (\|u^n\|_{H^2} + \|k^n\|_{H^2} + \|\varepsilon^n\|_{H^2} + \|h^n\|_{H^2})$$

$$\sup_{0 \leq t \leq T} (\|\sqrt{\rho^n} u^n_{tt}\|_{L^2} + \|\sqrt{\rho^n} h^n_{tt}\|_{L^2} + \|\sqrt{\rho^n} k^n_{tt}\|_{L^2} + \|\sqrt{\rho^n} \varepsilon^n\|_{L^2})$$

$$+ \int_0^T (\|u^n\|^2_{H^1} + \|h^n\|^2_{H^1} + \|k^n\|^2_{H^1} + \|\varepsilon^n\|^2_{H^1} + \|u^n\|^2_{H^1} + \|k^n\|^2_{H^3}) \leq \tilde{C}$$  (3.2)

for all $n \geq 1$. Throughout the proof, we denote by $\tilde{C}$ a generic constant depending only on $m$, $\gamma$, $|\Omega|$ and $c_0$, but independent of $n$. Next, we will show that the full sequence $(\rho^n, u^n, h^n, k^n, \varepsilon^n)$ converges to a solution to the original nonlinear problem (1.1)-(1.10) in the strong sense.

Define $\overline{\rho}^{n+1} = \rho^{n+1} - \rho^n$, $\overline{u}^{n+1} = u^{n+1} - u^n$, $\overline{h}^{n+1} = h^{n+1} - h^n$, $\overline{k}^{n+1} = k^{n+1} - k^n$, $\overline{\varepsilon}^{n+1} = \varepsilon^{n+1} - \varepsilon^n$, $\overline{\rho}^{n+1} = \rho^{n+1} - \rho^n = (\rho^{n+1} - \gamma) - (\rho^n)^\gamma$.

Then, by equations (2.2)-(2.4), we deduce that $(\overline{\rho}^{n+1}, \overline{u}^{n+1}, \overline{h}^{n+1}, \overline{k}^{n+1}, \overline{\varepsilon}^{n+1}, \overline{\rho}^{n+1})$ satisfy the following equations:

$$\overline{\rho}^{n+1} + \nabla \cdot (\overline{\rho}^{n+1} u^{n+1} + \rho^n \overline{u}^{n+1}) = 0$$  (3.3)

$$\rho^{n+1} \overline{u}^{n+1} + \rho^{n+1} u^{n+1} \cdot \nabla u^{n+1} + \rho^n \overline{u}^{n+1} \cdot \nabla u^n + \rho^n \overline{\rho}^{n+1} \nabla u^n$$

$$= -\Delta \overline{\rho}^{n+1} - (\nabla \cdot \overline{\rho}^{n+1} + \rho^n \overline{u}^{n+1}) \nabla \overline{\rho}^{n+1} + \frac{2}{3} \Delta (\overline{\rho}^{n+1} k^n + \rho^n \overline{k}^{n+1})$$  (3.4)

$$\rho^{n+1} \overline{h}^{n+1} + \rho^{n+1} u^{n+1} \cdot \nabla h^{n+1} + \rho^n \overline{u}^{n+1} \cdot \nabla h^n + \rho^n \overline{\rho}^{n+1} \nabla h^n$$

$$= -\Delta \overline{h}^{n+1} = \overline{p}^{n+1} + \overline{\rho}^{n+1} + \rho^n \overline{\rho}^{n+1} + \rho^n \overline{u}^{n+1} \cdot \nabla u^n + \overline{h}^{n+1} + \overline{\rho}^{n+1} + S'_{n, k, n+1} - S'_{n, k, n+1}$$  (3.5)

$$\rho^{n+1} \overline{k}^{n+1} + \rho^{n+1} u^{n+1} \cdot \nabla k^{n+1} + \rho^n \overline{u}^{n+1} \cdot \nabla k^n + \rho^n \overline{\rho}^{n+1} \nabla k^n$$

$$= -\Delta \overline{k}^{n+1} = G'_{n+1} - G'_{n} - (\rho^{n+1} \varepsilon^n - \rho^n \varepsilon^{n-1})$$  (3.6)

$$\rho^{n+1} \overline{\varepsilon}^{n+1} + \rho^{n+1} u^{n+1} \cdot \nabla \varepsilon^{n+1} + \rho^n \overline{u}^{n+1} \cdot \nabla \varepsilon^n + \rho^n \overline{\rho}^{n+1} \nabla \varepsilon^n$$

$$= -\Delta \overline{\varepsilon}^{n+1} = C_1 \left( \frac{G_{n+1} \varepsilon^n}{k^n} - \frac{G_n \varepsilon^{n-1}}{k^{n-1}} \right) - C_2 \left( \frac{(\rho^{n+1} \varepsilon^n)^2}{k^n} - \frac{(\rho^n \varepsilon^{n-1})^2}{k^{n-1}} \right)$$  (3.7)

where

$$S'_{n, k, n+1} = [\mu (\partial_j u^{n+1}_j + \partial_i u^{n+1}_i) - \frac{2}{3} \delta_{ij} \mu \partial_i u^{n+1}_k \partial_j u^{n+1}_i - \frac{\mu_4}{(\rho^{n+1})^2} \partial_j \rho^{n+1} \partial_j \rho^{n+1}]$$

$$G'_{n+1} = \partial_j u^{n+1}_j [\mu_4 (\partial_j u^{n+1}_j + \partial_i u^{n+1}_i) - \frac{2}{3} \delta_{ij} (\rho^{n+1} k^n + \mu_4 \partial_i u^{n+1}_j)]$$  (3.9)

To evaluate $\|\overline{\rho}^{n+1}\|_{L^2}$, multiplying both sides of equation (3.3) by $\overline{\rho}^{n+1}$ and integrating the result over $\Omega$, we get

$$\frac{1}{2} \frac{d}{dt} \|\overline{\rho}^{n+1}\|^2_{L^2} = - \int \nabla \cdot (\overline{\rho}^{n+1} u^{n+1} + \rho^n \overline{u}^{n+1}) \cdot \overline{\rho}^{n+1}$$

$$= - \int (\overline{\rho}^{n+1})^2 \nabla \cdot u^n + \overline{\rho}^{n+1} u^n \cdot \nabla \overline{\rho}^{n+1} + \rho^n \overline{\rho}^{n+1} \nabla \cdot u^n + \overline{\rho}^{n+1} \nabla \cdot \overline{\rho}^{n+1} \nabla \rho^n$$  (3.10)

Applying integration by parts to the second term of the second equality of (3.10) and using Hölder, Sobolev, and Young’s inequalities yield

$$\frac{d}{dt} \|\overline{\rho}^{n+1}\|^2_{L^2} \leq C (\|\nabla u^n\|_{L^\infty} + \|\overline{\rho}^{n+1}\|_{L^2} + \|\nabla \overline{\rho}^{n+1}\|_{L^2} + \|\overline{\rho}^{n+1}\|_{L^2} + \|\nabla \rho^n\|_{L^2} + \|\overline{\rho}^{n+1}\|_{L^2})$$

$$\leq C (1 + \eta^{-1}) \|\overline{\rho}^{n+1}\|^2_{L^2} + \tilde{C} \eta \|\nabla \overline{\rho}^{n+1}\|^2_{H^1},$$  (3.11)
where (3.2) has been used and $0 < \eta < 1$ is a small constant to be determined later.

Next, multiplying both sides of (3.14) by $\mathbf{u}^{i+1}$ and integrating the result thus derived over $\Omega$, one obtains

\[
\frac{1}{2} \frac{d}{dt} \left( \sqrt{\rho \mathbf{u}^{i+1} \cdot \mathbf{u}^{i+1}} \right)^2 + \left\| \nabla \mathbf{u}^{i+1} \right\|^2_{L^2} + \left\| \mathbf{v} \cdot \mathbf{u}^{i+1} \right\|^2_{L^2} = - \int \mathbf{p}^{i+1} \mathbf{u}^i \cdot \mathbf{u}^{i+1} - \int \mathbf{p}^{i+1} \mathbf{u}^n \cdot \nabla \mathbf{u}^n \cdot \mathbf{u}^{i+1} - \int \rho^n \mathbf{u}^n \cdot \nabla \mathbf{u}^n \cdot \mathbf{u}^{i+1} - \int \nabla \mathbf{p}^{i+1} \cdot \mathbf{u}^{i+1} + \int \frac{2}{3} \nabla (\mathbf{p}^{i+1} \mathbf{k}^n + \rho^n \mathbf{K}^{i+1}) \cdot \mathbf{u}^{i+1} = \sum_{i=1}^{5} L_i. \tag{3.12}
\]

Using Hölder, Sobolev and Young’s inequalities and (3.2), we estimate $L_1$, $L_2$ and $L_3$, respectively, as follows:

\[
L_1 \leq C \left\| \mathbf{p}^{i+1} \right\|^2_{L^2} + \left\| \mathbf{u}^n \right\|^2_{L^2} \left\| \mathbf{u}^{i+1} \right\|^2_{L^6} \leq C \left\| \mathbf{p}^{i+1} \right\|^2_{L^2} \left\| \mathbf{u}^n \right\|^2_{L^2} \left( \left\| \sqrt{\rho \mathbf{u}^{i+1} \cdot \mathbf{u}^{i+1}} \right\|_{L^2} + \left\| \nabla \mathbf{u}^{i+1} \right\|_{L^2} \right) \leq \bar{C} \left\| \mathbf{u}^n \right\|^2_{L^2} \left\| \mathbf{p}^{i+1} \right\|^2_{L^2} + \tilde{C} \left\| \mathbf{u}^n \right\|^2_{L^2} \left\| \mathbf{p}^{i+1} \right\|^2_{L^2} + \frac{1}{8} \left\| \nabla \mathbf{u}^{i+1} \right\|^2_{L^2}, \tag{3.13}
\]

\[
L_2 \leq C \left\| \mathbf{p}^{i+1} \right\|^2_{L^2} + \left\| \mathbf{u}^n \right\|^2_{L^2} \left\| \nabla \mathbf{u}^n \right\|^2_{L^6} \left\| \mathbf{u}^{i+1} \right\|^2_{L^6} \leq \tilde{C} \left\| \mathbf{u}^n \right\|^2_{L^2} + \tilde{C} \left\| \mathbf{u}^n \right\|^2_{L^2} \left\| \mathbf{p}^{i+1} \right\|^2_{L^2} + \frac{1}{8} \left\| \nabla \mathbf{u}^{i+1} \right\|^2_{L^2}, \tag{3.14}
\]

\[
L_3 \leq C \left\| \mathbf{u}^n \right\|^2_{L^6} \left\| \nabla \mathbf{u}^n \right\|^2_{L^3} \left\| \sqrt{\rho \mathbf{u}^{i+1} \cdot \mathbf{u}^{i+1}} \right\|^2_{L^2} \leq \bar{C} \eta^{-1} \left\| \sqrt{\rho \mathbf{u}^{i+1} \cdot \mathbf{u}^{i+1}} \right\|^2_{L^2} + \eta \left\| \mathbf{u}^n \right\|^2_{H^1}. \tag{3.15}
\]

And then, one deduces by integration by parts that

\[
L_4 = \int \mathbf{p}^{i+1} \mathbf{v} \cdot \mathbf{u}^{i+1} \leq C \int \mathbf{p}^{i+1} \mathbf{v} \cdot \mathbf{u}^{i+1} \leq \tilde{C} \left\| \mathbf{p}^{i+1} \right\|^2_{L^2} + \frac{1}{8} \left\| \nabla \mathbf{u}^{i+1} \right\|^2_{L^2}, \tag{3.16}
\]

and

\[
L_5 = \frac{2}{3} \int \mathbf{p}^{i+1} \mathbf{k}^n \mathbf{v} \cdot \mathbf{u}^{i+1} \leq \tilde{C} \left\| \mathbf{p}^{i+1} \right\|^2_{L^2} + \left\| \mathbf{u}^n \right\|^2_{L^2} \left\| \mathbf{u}^{i+1} \right\|^2_{L^2} \left\| \nabla \mathbf{u}^{i+1} \right\|^2_{L^2} + \left\| \mathbf{u}^n \right\|^2_{L^2} \left\| \mathbf{u}^n \right\|^2_{L^2} \left\| \nabla \mathbf{u}^{i+1} \right\|^2_{L^2} \leq \tilde{C} (1 + \eta^{-1}) \left\| \mathbf{p}^{i+1} \right\|^2_{L^2} + \left\| \sqrt{\rho \mathbf{u}^{i+1} \cdot \mathbf{u}^{i+1}} \right\|^2_{L^2} + \frac{1}{8} \left\| \nabla \mathbf{u}^{i+1} \right\|^2_{L^2} + \tilde{C} \eta \left\| \mathbf{u}^n \right\|^2_{H^1}. \tag{3.17}
\]

Inserting (3.13)-(3.17) to (3.12) and using inequality $\left\| \mathbf{u}^{i+1} \right\|^2_{L^2} \leq \tilde{C} \left\| \sqrt{\rho \mathbf{u}^{i+1} \cdot \mathbf{u}^{i+1}} \right\|^2_{L^2}$, one has

\[
\frac{d}{dt} \left\| \sqrt{\rho \mathbf{u}^{i+1} \cdot \mathbf{u}^{i+1}} \right\|^2_{L^2} + \left\| \mathbf{u}^{i+1} \right\|^2_{H^1}, \tag{3.18}
\]

\[
\leq \tilde{C} (1 + \eta^{-1}) \left\| \mathbf{u}^n \right\|^2_{L^2} + \left\| \sqrt{\rho \mathbf{u}^{i+1} \cdot \mathbf{u}^{i+1}} \right\|^2_{L^2} + \tilde{C} \eta \left\| \mathbf{u}^n \right\|^2_{H^1} + \tilde{C} \eta \left\| \mathbf{u}^n \right\|^2_{H^1}.
\]

Then, multiplying both sides of (3.15) by $\mathbf{h}^{i+1}$ and integrating the result thus got over $\Omega$, one obtains

\[
\frac{1}{2} \frac{d}{dt} \left\| \sqrt{\rho \mathbf{h}^{i+1} \cdot \mathbf{h}^{i+1}} \right\|^2_{L^2} + \left\| \nabla \mathbf{h}^{i+1} \right\|^2_{L^2} = - \int \mathbf{p}^{i+1} \mathbf{h}^i \cdot \mathbf{h}^{i+1} - \int \mathbf{p}^{i+1} \mathbf{u}^n \cdot \nabla \mathbf{h}^n \cdot \mathbf{h}^{i+1} - \int \rho^n \mathbf{u}^n \cdot \nabla \mathbf{h}^n \cdot \mathbf{h}^{i+1} + \int \left( S_{k,n+1} - S_{k,n} \right) \cdot \mathbf{h}^{i+1} = \sum_{i=1}^{5} M_i. \tag{3.19}
\]

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First, using similar method of deriving (3.13), (3.14) and (3.15), respectively, one easily obtains

\[ M_1 \leq \tilde{C} h_t^n \frac{1}{2} \| \mathbf{p}^{n+1} \|_{L^2}^2 + \tilde{C} \| \rho^{n+1} h^{n+1} \|_{L^2} + \frac{1}{20} \| \nabla h^{n+1} \|_{L^2}^2, \tag{3.20} \]

\[ M_2 \leq \tilde{C} \| \mathbf{p}^{n+1} \|_{L^2}^2 + \tilde{C} \| \rho^{n+1} h^{n+1} \|_{L^2} + \frac{1}{20} \| \nabla h^{n+1} \|_{L^2}^2, \tag{3.21} \]

\[ M_3 \leq \tilde{C} \eta^{-1} \| \rho^{n+1} h^{n+1} \|_{L^2}^2 + \eta \| \mathbf{v} \|_{H^1}, \tag{3.22} \]

Second, simple calculation leads to

\[ M_4 = \int [\gamma (\rho^{n+1}) \gamma^{-1} \rho^{n+1} - \gamma (\rho^n) \gamma^{-1} \rho^n] \cdot \mathbf{H}^n + \int \mathbf{p}^{n+1} \cdot \nabla \rho^{n+1} \mathbf{H}^n + \int u^n \cdot \nabla \rho^{n+1} \mathbf{H}^n. \tag{3.23} \]

By the differential mean value theorem, the first integral of (3.23) can be controlled as

\[ \int [\gamma (\rho^{n+1}) \gamma^{-1} \rho^{n+1} - \gamma (\rho^n) \gamma^{-1} \rho^n] \cdot \mathbf{H}^n + \int \mathbf{p}^{n+1} \cdot \nabla \rho^{n+1} \mathbf{H}^n \leq C \int \mathbf{p}^{n+1} \| \rho^{n+1} \|_{H^1} + \int \gamma (\rho^n) \gamma^{-1} \rho^{n+1} \cdot \mathbf{H}^n. \tag{3.24} \]

By equation (3.23), the second integral on the right hand side of (3.24) can be estimated as

\[ \int \gamma (\rho^n) \gamma^{-1} \rho^{n+1} \cdot \mathbf{H}^n = -\int \gamma (\rho^n) \gamma^{-1} \nabla \cdot (\mathbf{p}^{n+1} u^n + \rho^n \mathbf{v}^n) \cdot \mathbf{H}^n \leq C \int |\nabla \rho^n| \mathbf{H}^n \| \mathbf{p}^{n+1} \|_{H^1} + C \int \mathbf{p}^{n+1} \| u^n \|_2 \| \nabla \mathbf{H}^n \|_{H^1} + C \int (|\nabla \rho^n| \mathbf{p}^{n+1} + |\rho^n| \| \nabla \mathbf{p}^{n+1} \|_{H^1}). \tag{3.25} \]

Then, the second integral on the right hand side of (3.23) can be controlled as

\[ \int \mathbf{p}^{n+1} \cdot \nabla \rho^{n+1} \mathbf{H}^n \leq C \int \mathbf{p}^{n+1} \| \nabla \rho^{n+1} \|_{H^1}. \tag{3.26} \]

Next, applying integration by parts to the third integral on the right hand side of (3.23), we easily get

\[ \int u^n \cdot \nabla \rho^{n+1} \mathbf{H}^n \leq C \int |\nabla u^n| \| \mathbf{p}^{n+1} \|_{H^1} + C \int \| u^n \|_2 \| \nabla \mathbf{p}^{n+1} \|_{H^1}. \tag{3.27} \]

Consequently, combining (3.23)-(3.27) and using Hölder, Sobolev and Young’s inequalities and (3.2), one obtains

\[ M_4 \leq \tilde{C} (1 + \eta^{-1}) (\| \mathbf{p}^{n+1} \|_{L^2}^2 + \| \rho^{n+1} h^{n+1} \|_{L^2}^2) + \frac{1}{4} \| \mathbf{p}^{n+1} \|_{H^1}^2 + \frac{1}{20} \| \nabla \rho^{n+1} \|_{L^2}^2 + \tilde{C} \eta \| \mathbf{p}^{n+1} \|_{H^1}. \tag{3.28} \]
Finally, we evaluate $M_5$. Direct calculation yields

$$M_5 \leq C \int (|\nabla u^n| + |\nabla u^{n-1}|)|\nabla \pi^n||\tilde{\kappa}^{n+1}| + C \int |\tilde{\kappa}^{n+1}||\nabla \rho^{n+1}||\tilde{\kappa}^{n+1}|$$

$$+ \int \frac{\mu_t}{(\rho^n)^{\gamma}} \partial_j \tilde{p}^{n+1} \partial_j \rho^{n+1} \cdot \tilde{\kappa}^{n+1} + \int \frac{\mu_t}{(\rho^n)^{\gamma}} \partial_j \rho^n \partial_j \tilde{p}^{n+1} \cdot \tilde{\kappa}^{n+1}$$

$$\leq C \int (|\nabla u^n| + |\nabla u^{n-1}|)|\nabla \pi^n||\tilde{\kappa}^{n+1}| + C \int |\tilde{\kappa}^{n+1}||\nabla \rho^{n+1}||\tilde{\kappa}^{n+1}|$$

$$+ C \int |\nabla \rho^n||\nabla \rho^{n+1}||\tilde{\kappa}^{n+1}||\tilde{\kappa}^{n+1}| + C \int |\nabla \rho^n||\nabla \rho^{n+1}||\tilde{\kappa}^{n+1}|$$

$$+ C \int |\nabla^2 \rho^n||\tilde{\kappa}^{n+1}||\tilde{\kappa}^{n+1}| + C \int |\nabla \rho^n||\nabla \rho^{n+1}||\tilde{\kappa}^{n+1}|. \quad (3.29)$$

Then, applying similar method of deriving (3.28), one deduces

$$M_5 \leq \tilde{C}(1 + \eta^{-1})(||\tilde{\kappa}^{n+1}||^2_{L^2} + ||\sqrt{\rho^{n+1} \tilde{\kappa}^{n+1}}||^2_{H^1}) + \eta ||\tilde{\kappa}^n||^2_{H^1} + \frac{1}{20} ||\nabla \tilde{\kappa}^{n+1}||^2_{L^2}. \quad (3.30)$$

Consequently, inserting (3.20), (3.22), (3.28) and (3.30) to (3.19), one gets

$$\frac{d}{dt} ||\sqrt{\rho^{n+1} \tilde{\kappa}^{n+1}}||^2_{L^2} + ||\tilde{\kappa}^{n+1}||^2_{H^1}$$

$$\leq \tilde{C}(1 + \eta^{-1} + ||\tilde{\kappa}^n||^2_{L^2}) ||\tilde{\kappa}^{n+1}||^2_{L^2} + ||\sqrt{\rho^{n+1} \tilde{\kappa}^{n+1}}||^2_{L^2}$$

$$+ \frac{1}{4} ||\tilde{\kappa}^{n+1}||^2_{H^1} + \tilde{C} \eta ||\tilde{\kappa}^n||^2_{H^1}. \quad (3.31)$$

For the turbulent kinetic energy $k$, using similar method of deriving (3.19), one easily deduces from equation (3.6) that

$$\frac{1}{2} \frac{d}{dt} ||\sqrt{\rho^{n+1} \tilde{K}^{n+1}}||^2_{L^2} + ||\tilde{K}^{n+1}||^2_{L^2} = - \int \tilde{\kappa}^{n+1} k^n \cdot \tilde{K}^{n+1} - \int \tilde{\kappa}^{n+1} u^n \cdot \nabla k^n \cdot \tilde{K}^{n+1}$$

$$- \int \rho^n \nabla k^n \cdot \tilde{K}^{n+1} + \int (G'_{n+1} - G'_n) \cdot \tilde{K}^{n+1} - \int (\rho^{n+1} \varepsilon^n - \rho^n \varepsilon^{n-1}) \cdot \tilde{K}^{n+1} = \sum_{i=1}^{5} N_i. \quad (3.32)$$

We first evaluate $N_4$. Using inserting items technic, one easily gets

$$N_4 \leq \tilde{C}(1 + \eta^{-1})(||\tilde{\kappa}^{n+1}||^2_{L^2} + ||\sqrt{\rho^{n+1} \tilde{\kappa}^{n+1}}||^2_{H^1}) + \tilde{C} \eta ||\tilde{\kappa}^n||^2_{H^1}. \quad (3.33)$$

Using Hölder, Sobolev, and Young’s inequalities and (5.2), we have

$$N_4 \leq \tilde{C}(1 + \eta^{-1})(||\tilde{\kappa}^{n+1}||^2_{L^2} + ||\sqrt{\rho^{n+1} \tilde{\kappa}^{n+1}}||^2_{H^1}) + \tilde{C} \eta ||\tilde{\kappa}^n||^2_{H^1}. \quad (3.34)$$

Second, we estimate $N_5$. Using similar method of deriving (3.33) and (5.34), we have

$$N_5 = \int (\tilde{\kappa}^{n+1} \varepsilon^n + \rho^n \varepsilon^n) \cdot \tilde{K}^{n+1} \leq C(||\tilde{\kappa}^{n+1}||^2_{L^2} ||\varepsilon^n||_{L^\infty} + ||\tilde{\kappa}^n||^2_{L^6} ||\rho^n||_{L^3} ||\sqrt{\rho^{n+1} \tilde{\kappa}^{n+1}}||_{L^2}$$

$$\leq \tilde{C}(1 + \eta^{-1})(||\sqrt{\rho^{n+1} \tilde{\kappa}^{n+1}}||^2_{L^2} + ||\tilde{\kappa}^{n+1}||^2_{L^2} + \tilde{C} \eta ||\tilde{\kappa}^n||^2_{H^1}. \quad (3.35)$$
Next, using similar method of deriving the estimates of (3.13), (3.14) and (3.15), respectively, one easily gets

\[ N_1 \leq \tilde{C}k^2\|\rho^{\eta^{-1}}\|_L^2 + \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \frac{1}{8}\|\nabla\rho^{\eta^{-1}}\|_L^2, \]  
(3.36)

\[ N_2 \leq \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \frac{1}{8}\|\nabla\rho^{\eta^{-1}}\|_L^2, \]  
(3.37)

\[ N_3 \leq \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \eta\|\nabla\|_H^2. \]  
(3.38)

Consequently, inserting (3.34)- (3.38) to (3.32), one deduces

\[ \frac{d}{dt}\|\rho^{\eta^{-1}}\|_L^2 + \frac{1}{2}\|\nabla\|_L^2 \leq \tilde{C}(1 + \eta^{-1} + \|k_n\|_L^2)(\|\nabla\rho^{\eta^{-1}}\|_L^2 + \|\rho^{\eta^{-1}}\|_L^2) + \tilde{C}\|\rho^{\eta^{-1}}\|_H^2 + \|\rho^{\eta^{-1}}\|_H^2 + \|\nabla\|_H^2. \]  
(3.39)

Next, multiplying both sides of (3.37) by \(\phi^{\eta^{-1}}\) and integrating the result over \(\Omega\), one gets

\[ \frac{1}{2}\frac{d}{dt}\|\rho^{\eta^{-1}}\|_L^2 + \|\nabla\|_L^2 = -\int \rho^{\eta^{-1}}n \cdot \nabla n + \int \rho^{\eta^{-1}}(\nabla n) \cdot \nabla n + C_1 \int \left( G_n^{\eta^{-1}} - G_n^{\eta^{-1}} \right) \cdot \nabla n + C_2 \int \left( \rho^{\eta^{-1}}(\nabla n)^2 - \rho^{\eta^{-1}}(\nabla n)^2 \right) \cdot \nabla n + 5 \sum_{i=1}^n Q_i. \]  
(3.40)

Using an argument similar to that used in deriving (3.38), (3.11) and (3.12), respectively, we obtain

\[ Q_1 \leq \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \frac{1}{8}\|\nabla\|_L^2, \]  
(3.41)

\[ Q_2 \leq \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \frac{1}{8}\|\nabla\|_L^2, \]  
(3.42)

\[ Q_3 \leq \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \tilde{C}\|\rho^{\eta^{-1}}\|_L^2 + \tilde{C}\|\rho^{\eta^{-1}}\|_H^2. \]  
(3.43)

Next, direct calculation leads to

\[ Q_4 \leq C \int \left( (\nabla n \cdot \nabla u) + |\nabla u| |\nabla u^-1|\right) \|\rho^{\eta^{-1}}\|_L^2 + C \int \left( |\rho^{\eta^{-1}}| + |\rho^{\eta^{-1}}| |\nabla u| \right)^2 \|\rho^{\eta^{-1}}\|_L^2 \]  
(3.44)

\[ \leq \tilde{C}(1 + \eta^{-1})(\|\rho^{\eta^{-1}}\|_L^2 + \|\nabla\|_L^2) + \tilde{C}\|\rho^{\eta^{-1}}\|_H^2 + \|\rho^{\eta^{-1}}\|_H^2 + \|\nabla\|_H^2. \]  
(3.44)
Finally, using similar methods in deriving the estimate of $Q_4$, one deduces

$$Q_5 \leq \tilde{C}(1 + \eta^{-1})(\sqrt{\rho^{n+1}}\|\nabla \psi\|_{L^2} + \|\psi^n\|_{L^2}) + \tilde{C}\eta\|\nabla \psi^n\|_{L^2} + \frac{1}{8}\|\nabla \psi^{n+1}\|_{L^2}^2. \quad (3.45)$$

Consequently, inserting (3.41), (3.45) to (3.46), one derives

$$\frac{d}{dt}\|\sqrt{\rho^{n+1}}\psi^{n+1}\|_{L^2}^2 + \|\psi^{n+1}\|_{H^1}^2 \leq \tilde{C}(1 + \eta^{-1} + \|u^n\|_{L^2}^2 + \|h^n\|_{L^2}^2 + \|k^n\|_{L^2}^2 + \|\varepsilon^n\|_{L^2}^2)\varphi^{n+1}(t) + \tilde{C}\eta\|\sqrt{\rho^n}\|_{H^1}^2 + \|\psi^n\|_{H^1}^2 + \|\psi^n\|_{H^1}^2. \quad (3.47)$$

where it should be noted that $\varphi^{n+1}(0) = 0$.

Since

$$\int_0^t \int_0^s I^n_{\eta}(s)ds \leq \tilde{C}t + \tilde{C}\eta^{-1}t + \tilde{C}, \quad (3.49)$$

setting $\bar{T} \leq \eta < 1$, then we have

$$\int_0^t \int_0^s I^n_{\eta}(s)ds \leq C\tilde{C}. \quad (3.50)$$

for $t \leq \bar{T}$.

By (3.48), (3.50), integrating (3.47) from $[0, t]$, one derives

$$\varphi^{n+1}(t) + \int_0^t (\|\psi^n\|_{H^1}^2 + \|\psi^{n+1}\|_{H^1}^2 + \|\nabla \psi^n\|_{H^1}^2 + \|\nabla \psi^{n+1}\|_{H^1}^2)ds \leq C\tilde{C}\eta\left(\int_0^t \|\psi^n\|_{H^1}^2 + \|\psi^{n+1}\|_{H^1}^2 + \|\nabla \psi^n\|_{H^1}^2 + \|\nabla \psi^{n+1}\|_{H^1}^2)ds\right) \left(\int_0^t \int_0^s I^n_{\eta}(s)ds \right) + 1$$

$$\leq C\eta \exp(\tilde{C}) \int_0^t (\|\psi^n\|_{H^1}^2 + \|\psi^{n+1}\|_{H^1}^2 + \|\nabla \psi^n\|_{H^1}^2 + \|\nabla \psi^{n+1}\|_{H^1}^2)ds \quad (3.51)$$

for $T^* := \min\{T, \bar{T}\}$.

Therefore, we have

$$\sum_{n=1}^{\infty} sup_{0 \leq t \leq T^*} \varphi^{n+1}(t) + \sum_{n=1}^{\infty} \int_0^t (\|\psi^n\|_{H^1}^2 + \|\psi^{n+1}\|_{H^1}^2 + \|\nabla \psi^n\|_{H^1}^2 + \|\nabla \psi^{n+1}\|_{H^1}^2)ds \leq C\eta \exp(\tilde{C}) \sum_{n=1}^{\infty} \int_0^t (\|\psi^n\|_{H^1}^2 + \|\psi^{n+1}\|_{H^1}^2 + \|\nabla \psi^n\|_{H^1}^2 + \|\nabla \psi^{n+1}\|_{H^1}^2)ds. \quad (3.52)$$
Thus, choosing $\eta$ such that $C\eta \exp(\tilde{C}) \leq \frac{1}{2}$, one deduce
\[
\sum_{n=1}^{\infty} \sup_{0 \leq t \leq T} \rho^n \varphi^{n+1}(t) + \sum_{n=1}^{\infty} \int_{0}^{t} \|\tilde{\varphi}^{n+1}\|_{H^1}^2 ds
\]
\[
+ \frac{1}{2} \sum_{n=1}^{\infty} \int_{0}^{t} (\|\tilde{\varphi}^{n+1}\|_{H^1}^2 + \|\tilde{\varphi}^{n+1}\|_{H^1}^2 + \|\tilde{\varphi}^{n+1}\|_{H^1}^2) ds
\]
\[
\leq C\tilde{C} < \infty.
\] (3.53)

Therefore, we conclude that the full sequence $(\rho^n, u^n, h^n, k^n, \varepsilon^n)$ converges to a limit $(\rho, u, h, k, \varepsilon)$ in the following strong sense: $\rho^n \to \rho$ in $L^\infty(0, T; L^2(\Omega))$; $(u^n, h^n, k^n, \varepsilon^n) \to (u, h, k, \varepsilon)$ in $L^2(0, T; H^1(\Omega))$. It is easy to prove that the limit $(\rho, u, h, k, \varepsilon)$ is a weak solution to the original nonlinear problem. Furthermore, it follows from (3.2) that $(\rho, u, h, k, \varepsilon)$ satisfies the following regularity estimates:
\[
\sup_{0 \leq t \leq T} (\|\rho\|_{H^3} + \|\rho\|_{H^1}) + \sup_{0 \leq t \leq T} (\|u\|_{H^3} + \|u\|_{H^1} + \|\varepsilon\|_{H^2} + \|\varepsilon\|_{H^1})
\]
\[
+ \sup_{0 \leq t \leq T} (\|\sqrt{\rho} u\|_{L^2} + \|\sqrt{\rho} h\|_{L^2} + \|\sqrt{\rho} k\|_{L^2} + \|\sqrt{\rho} \varepsilon\|_{L^2})
\]
\[
+ \int_{0}^{T} (\|u_t\|_{H^1}^2 + \|u_t\|_{H^1}^2 + \|k_t\|_{H^1}^2 + \|\varepsilon_t\|_{H^1}^2 + \|u_t\|_{H^3}^2 + \|k_t\|_{H^3}^2) \leq \tilde{C} < \infty.
\]

This proves the existence of strong solution. Then, we can easily prove the time continuity of the solution $(\rho, u, h, k, \varepsilon)$ by adapting the arguments in [2, 5]. Finally, we prove the uniqueness. In fact, assume that $(\rho_1, u_1, h_1, k_1, \varepsilon_1)$ and $(\rho_2, u_2, h_2, k_2, \varepsilon_2)$ are two strong solutions to the problem (1.1)–(1.10) with the regularity (3.1). Let $(\overline{\rho}, \overline{\pi}, \overline{\eta}, \overline{\varepsilon}) = (\rho_1 - \rho_2, u_1 - u_2, h_1 - h_2, k_1 - k_2, \varepsilon_1 - \varepsilon_2)$. Then using the same argument as in the derivations of (3.11), (3.13), (3.39) and (3.40), we can prove that
\[
\frac{d}{dt} (\|\overline{\rho}\|_{L^2}^2 + \|\sqrt{\rho_1} u\|_{L^2}^2 + \|\sqrt{\rho_1} h\|_{L^2}^2 + \|\sqrt{\rho_1} k\|_{L^2}^2 + \|\sqrt{\rho_1} \varepsilon\|_{L^2}^2)
\]
\[
\leq R(t)(\|\overline{\rho}\|_{L^2}^2 + \|\sqrt{\rho_1} u\|_{L^2}^2 + \|\sqrt{\rho_1} h\|_{L^2}^2 + \|\sqrt{\rho_1} k\|_{L^2}^2 + \|\sqrt{\rho_1} \varepsilon\|_{L^2}^2)
\]
for some $R(t) \in L^1(0, T^*)$. Thus, by Gronwall’s inequality, we conclude that $(\overline{\rho}, \overline{\pi}, \overline{\eta}, \overline{\varepsilon}) = (0, 0, 0, 0)$ in $(0, T^*) \times \Omega$. This completes the proof of Theorem 3.1.

Competing interests
The authors declare that they have no competing interests.

Authors contributions
The authors contributed equally in this article. They read and approved the final manuscript.

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