Space-filling curves of self-similar sets (I): iterated function systems with order structures

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Abstract
This paper is the first part of a series which provides a systematic treatment of the space-filling curves of self-similar sets. In the present paper, we introduce a notion of linear graph-directed IFS (linear GIFS in short). We show that to construct a space-filling curve of a self-similar set, it amounts to exploring its linear GIFS structures. Compared to the previous methods, such as the $L$-system or recurrent set method, the linear GIFS approach is simpler, more rigorous and leads to further studies on this topic. We also propose a new algorithm for the beautiful visualization of space-filling curves.

In a series of papers Dai et al (2015 arXiv:1511.05411 [math.GN]), Rao and Zhang (2015) and Rao and Zhang (2015), we investigate for a given self-similar set how to get ‘substitution rules’ for constructing space-filling curves, which was obscure in the literature. We solve the problem for self-similar sets of finite type, which covers most of the known results on constructions of space-filling curves.

Keywords: space-filling curve, linear GIFS, self-similar set, optimal parametrization
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(Some figures may appear in colour only in the online journal)

1. Introduction

Since the monumental construction of Peano in 1890 ([22]), space-filling curves have fascinated mathematicians for over a century. A survey of early works on space-filling curves can
be found in Sagan [26]. Around the 1970s, two remarkable space-filling curves were found: the Heighway dragon curve ([8, 13]) and the Gosper curve ([13]). After that, many reptiles with space-filling curves have been found by professional mathematicians or by amateurs; nice collections of them can be found, for example, on the websites ‘www.fractalcurves.com’ ([30]) and ‘teachout1.net/village/’ ([28]).

It is natural to seek systematic methods to construct space-filling curves. There are two well-known methods, the \textit{L-system} introduced in 1968 by Lindenmayer, a biologist [17], and the \textit{recurrent set} method introduced in 1982 by Dekking [9]. Dekking’s paper [9] is also important in fractal geometry, since it leads to the emergence of the notion of the graph-directed iterated function system. Another work which is worth mentioning is Hata [15], which deals with a class of self-similar sets satisfying a certain ‘chain condition’.

All the known constructions of space-filling curves depend on certain ‘substitution rules’, which are somehow mysterious. However, for a given reptile, how to find a substitution rule leading to space-filling curves was obscure in the literature. In this paper and the subsequent papers [7, 23] and [24], we unveil the mystery of space-filling curves by providing a rigorous and systematic treatment; in particular, we find a method to construct substitution rules.

The main purpose of the present paper is to introduce a notion of a \textit{linear graph-directed iterated function system} (linear GIFS) in order to describe and handle space-filling curves. The linear GIFS can be regarded as an improvement of the \textit{L}-system, Dekking’s recurrent set method, and Hata’s method; moreover, it is the basis of our further studies.

First, let us specify what we mean by space-filling curves. We call an onto mapping from an interval \([a, b]\) to a self-similar set \(K\) an \textit{optimal parametrization}, if it is almost one-to-one, measure-preserving and \(1/s\)-Hölder continuous, where \(s = \dim_H K\) is the Hausdorff dimension of \(K\) (For the precise definition, see section 2.). It is observed that most classical space-filling curves fulfill the above requirements ([14, 21]), while some others, like the Lebesgue curve (figure 1), do not. It may be better to introduce a distinction—to call an optimal parametrization a \textit{space-filling curve} if \(K\) has a non-empty interior, and to call it a \textit{fractal-filling curve} otherwise. However, for simplicity, we shall just call an optimal parametrization a space-filling curve. Under this convention, we roughly describe our results within this section.

1.1. Linear GIFS

GIFS is an important notion in fractal geometry. We equip a GIFS with an order structure (and call it an ordered GIFS), which induces a lexicographical order of the associated symbolic space. An ordered GIFS is called a linear GIFS if every two consecutive cylinders have non-empty intersections (see section 3 for precise definitions of all the above terminologies). We prove the following result.

\textbf{Theorem 1.1.} \textit{Let} \(\{E_j\}_{j=1}^N\) \textit{be the invariant sets of a linear graph-directed IFS satisfying the open set condition and} \(0 < \mathcal{H}^b(E_j) < \infty\) \textit{for} \(j = 1, \ldots, N\), \textit{where} \(b\) \textit{is the similarity dimension, then} \(E_j\) \textit{admits optimal parameterizations for every} \(j = 1, \ldots, N\).

The proof of theorem 1.1 is constructive; hence, to construct space-filling curves amounts to seeking a linear GIFS structure of the given set.

\textbf{Remark 1.1.} In [1, 2], Akiyama and Loridant used an idea similar to our linear GIFS to show that the boundaries of certain planar self-affine tiles are simple Jordan curves, and hence the tiles are topological disks.

\textbf{Example 1.2 (Heighway dragon curve ([8])).} The Heighway dragon is a reptile generated by the IFS \(\{S_1, S_2\}\) where

\[ \begin{align*}
S_1(x, y) & = \left( \frac{1}{2}x + \frac{1}{2}y, \frac{1}{2}y + \frac{1}{2} \right) \\
S_2(x, y) & = \left( x + \frac{1}{2}, y + \frac{1}{2} \right) 
\end{align*} \]
We use the dragon curve to compare the three methods: the L-system, the recurrent set method, and our linear GIFS method.

(i) **L-system** ([3, 17]).
Let $\sigma$ be a substitution over $\mathcal{A} = \{a, b, +, -\}$ defined by:

$$
\begin{align*}
+ &\mapsto +, \\
- &\mapsto -, \\
a &\mapsto -a + b - , \\
b &\mapsto +a - b + .
\end{align*}
$$

The rotation angle $\theta = \pi/4$, the initial word is $F = a$, and the step lengths for $a$ and $b$ are both 1. The re-normalization map is $f(z) = \frac{\sqrt{2}}{2}z$.

Starting from the initial word $F$, iterate $n$ times by $\sigma$ to obtain a word. For example, the word $\sigma^2(a)$ is

$$
- - a + b - + + a - - b + - .
$$

Next, we realize the word: starting from the initial angle 0 and initial position 0, if we encounter $+$ or $-$, then we adjust the angle state by adding $\theta$ or $-\theta$; if we encounter $a$ or $b$, then we adjust the position state by moving a distance 1 along the present angle. The trace of the movement gives us a broken line. Applying the re-normalization map $n$ times to the trace, we obtain the $n$th approximation of the dragon curve.

(ii) **Recurrent set method** ([9]).
Let $\sigma$ be a substitution over $\mathcal{A} = \{a, b, c, d\}$ defined by:

$$
a \mapsto ab, \quad b \mapsto cb, \quad c \mapsto cd, \quad d \mapsto ad.
$$

The vectors corresponding to $a, b, c, d$ are

$$
v_a = (1, 0), \quad v_b = (0, 1), \quad v_c = (-1, 0), \quad v_d = (0, -1).
$$

The re-normalization map is $f(z) = \frac{z}{1+i}$ (see [9]).

The third iteration of $a$ is $abc\bar{c}dcb$. To realize a word, we move a vector $v_x$ if we encounter the letter $x$. Compared to the L-system, this method is more convenient because one does not have to record the current angle.
(iii) Linear GIFS method.

The following linear GIFS completely describes the dragon curve.

\[
\begin{align*}
a &= S_1(a) + S_2(b), \\
b &= S_2(a) + S_1(b),
\end{align*}
\]  

(1.2)

where \(S_1, S_2\) is given by (1.1). Here we use ‘+’ instead of ‘∪’ to emphasize the order structure (See section 5 for details.).

How do we determine whether an ordered GIFS like (1.2) is a linear GIFS? To answer this question, in section 4, we define a chain condition which provides a simple criterion.

**Theorem 1.2.** An ordered GIFS is a linear GIFS if and only if it satisfies the chain condition.

Indeed, the Heighway dragon curve can be generated by the so-called path-on-lattice IFS, which is briefly discussed in section 5. The path-on-lattice IFS is a well-known method to construct space-filling curves. A nice collection of such space-filling curves can be found on the website [30].

### 1.2. Visualization of a space-filling curve

To ‘see’ a space-filling curve, we need to visualize or to approximate a space-filling curve. Using linear GIFS, in section 6, we give an algorithm of visualizations of a space-filling curve.

Using different initial patterns, we can obtain very different amazing visualizations of a space-filling curve. Figure 2 provides two visualizations of the dragon curve; figures 3 and 4 illustrate two visualizations of the four-tile star. The choices of the initial patterns are completely free. Hence various beautiful visualizations are obtained by simple procedures.

### 1.3. Further studies

To find the linear GIFS structure of a given self-similar set is a hard task. It is investigated in a series of papers [7] and [24]. We prove the following result.

**Theorem 1.3 ([7] and [24]).** Let \(K\) be a connected self-similar set satisfying the open set condition. If \(K\) has the finite skeleton property, then it admits optimal parameterizations. In particular, if \(K\) satisfies a finite type condition (another important condition in fractal geometry), then it possesses finite skeletons and hence admits optimal parameterizations.

Our theory gives a universal algorithm to find space-filling curves of a self-similar set of finite type (That is, as soon as the IFS is given, a computer will do everything.).

**Example 1.3 (The four-tile star).** Pictures in figure 3 are taken from [28], but there is no explanation as to how to obtain the space-filling curve. Our study will fill all the gaps from the
left picture to the right in figure 3, which is highly non-trivial ([7]). A sketch of the approach is provided in section 7.

The paper is organized as follows. In section 2, we define the optimal parametrization for general compact sets. We introduce the linear GIFS and the chain condition in sections 3 and 4, respectively. Section 5 is devoted to the path-on-lattice IFS on the plane. Visualizations of space-filling curves are discussed in section 6. Section 7 studies the four-tile star. In section 8, we prove theorem 1.1 using a measure-recording GIFS.

2. Optimal parameterizations of self-similar sets

Let \( K \subset \mathbb{R}^d \) be a non-empty compact set. We call \( K \) a self-similar set if it is a union of small copies of itself, in particular, if there exist similitudes \( S_1, \ldots, S_N : \mathbb{R}^d \to \mathbb{R}^d \) such that

\[
K = \bigcup_{j=1}^{N} S_j(K).
\]

In fractal geometry, the family \( \{S_1, \ldots, S_N\} \) is called an iterated function system (IFS); \( K \) is called the invariant set of the IFS [11, 16]. We denote by \( \mathcal{H}^s \) the \( s \)-dimensional Hausdorff measure. A set \( E \subset \mathbb{R}^d \) is called an \( s \)-set, if \( 0 < \mathcal{H}^s(E) < \infty \) for some \( s \geq 0 \).

The IFS \( \{S_1, \ldots, S_N\} \) is said to satisfy the open set condition (OSC), if there is an open set \( U \) such that \( \bigcup_{i=1}^{N} S_i(U) \subset U \) and the sets \( S_i(U) \) are disjoint. It is well-known that if a self-similar set \( K \) satisfies the OSC, then it is an \( s \)-set (see [11]).

**Remark 2.1.** If an IFS satisfies the OSC, and \( \dim_H K \) equals the space dimension, then \( K \) has a non-empty interior ([27]), and it is a self-similar tile. In particular, if the contraction ratios
of $S_i$ are all equal to $r$, then $K$ is called a reptile (In this case, we must have $r = 1/\sqrt{N}$, where $d$ is the dimension of the space.).

Motivated by the nice properties enjoyed by many space-filling curves ([21]), we define optimal parameterizations of general sets (see [6]). Denote by $\mathcal{L}$ the one-dimensional Lebesgue measure:

**Definition 2.2.** Let $K \subset \mathbb{R}^d$ be an $s$-set. An onto mapping $\psi : [0, 1] \to K$ is called an optimal parametrization of $K$ if the following three conditions are fulfilled.

(i) $\psi$ is almost one-to-one; in particular, there exist $K' \subset K$ and $I' \subset [0, 1]$ such that $\mathcal{H}^s(K \setminus K') = \mathcal{L}([0, 1] \setminus I') = 0$ and $\psi : I' \to K'$ is a bijection;

(ii) $\psi$ is measure-preserving in the sense that

$$\mathcal{H}^s(\psi(F)) = c\mathcal{L}(F)$$

and

$$\mathcal{L}(\psi^{-1}(B)) = c^{-1}\mathcal{H}^s(B),$$

for any Borel set $F \subset [0, 1]$ and any Borel set $B \subset K$, where $c = \mathcal{H}^s(K)$.

(iii) $\psi$ is $1/s$-Hölder continuous, that is, there is a constant $c' > 0$ such that

$$|\psi(x) - \psi(y)| \leq c'|x - y|^\frac{1}{s} \text{ for all } x, y \in [0, 1].$$

Our main concern is: Does every connected self-similar set admit an optimal parametrization? According to the theorem of Mazurkiewicz–Hahn ([26]), a set is the image of $[0, 1]$ under a continuous mapping if and only if it is compact, connected, and locally connected. We note that a connected self-similar set fulfills these conditions, since a self-similar set is locally connected as soon as it is connected ([15]).

Concerning parameterizations of fractal sets, the previous studies focused on the Hölder continuity. De Rham [10], Hata [15], and Remes [25] showed the existence of $1/s$-Hölder continuous parameterizations for certain classes of self-similar sets. Akiyama and Loridant [1, 2] showed the existence of parameterizations when $K$ is the boundary of a class of self-affine tiles (their motivation is to provide an alternative way to show the disk-like property of some planar tiles). Martin and Mattila [19] gave some negative results when $K$ was disconnected. Figure 5 provides optimal parametrizations of Sierpinski gasket and carpet.

**3. Linear GIFS**

In this section, we introduce the notion of linear GIFS.

Let us start with the definition of GIFS. Let $G = (\mathcal{A}, \Gamma)$ be a directed graph with vertex set $\mathcal{A}$ and edge set $\Gamma$ (Sometimes we also call a vertex a state.). Let

$$\mathcal{G} = (g_\gamma : \mathbb{R}^d \to \mathbb{R}^d)_{\gamma \in \Gamma}$$

be a family of similitudes. We call the triple $(\mathcal{A}, \Gamma, \mathcal{G})$, or simply $\mathcal{G}$, a GIFS. We call $(\mathcal{A}, \Gamma)$ the base graph of the GIFS. Very often but not always, we set $\mathcal{A} = \{1, \ldots, N\}$.

Let $\Gamma_{ij}$ be the set of edges from state $i$ to $j$. It is well-known that there exist unique non-empty compact sets $\{E_i\}_{i=1}^N$ satisfying

$$E_i = \bigcup_{j=1}^N g_\gamma(E_j), \quad 1 \leq i \leq N. \quad (3.1)$$

We call $\{E_i\}_{i=1}^N$ the invariant sets of the GIFS ([20] [4]).

We say the above GIFS satisfies the OSC, if there exist open sets $U_1, \ldots, U_N$ such that
\[
\bigcup_{j=1}^{N} \bigcup_{\gamma \in E_j} g_j(U_j) \subseteq U_i, \quad 1 \leq i \leq N,
\]
and the left-hand sides are non-overlapping unions ([20] [12]).

**Remark 3.1.** It is seen that the set equations (3.1) give all the information of a GIFS, and hence provide an alternative way to define a GIFS. We shall call (3.1) the *set equation form* of a GIFS.

### 3.1. The symbolic space related to a graph \( G \)

Let \( G \) be a directed graph. A sequence of edges in \( G \), denoted by \( \omega = \omega_1 \omega_2 \ldots \omega_n \), is called a *path*, if the terminal state of \( \omega_j \) coincides with the initial state of \( \omega_{j+1} \) for \( 1 \leq i \leq n - 1 \). We will use the following notations to specify the sets of finite or infinite paths on \( G = (\mathcal{A}, \Gamma) \). For \( i \in \mathcal{A} \), let

\[
\Gamma^k_i, \quad \Gamma^*_i \quad \text{and} \quad \Gamma^\infty_i
\]

be the set of all paths with length \( k \), the set of all paths with finite length, and the set of all infinite paths, emanating from the state \( i \), respectively. Note that \( \Gamma^k_i = \bigcup_{j \geq k} \Gamma^j_i \).

For a sequence \( \omega = (\omega_k)_{k=1}^{\infty} \), set \( \omega|_n = \omega_1 \omega_2 \ldots \omega_n \) to be the prefix of \( \omega \) of length \( n \). For \( \omega_1 \ldots \omega_n \in \Gamma^*_i \), we call

\[
[\omega_1 \ldots \omega_n] := \{ \gamma \in \Gamma^\infty_i, \quad \gamma|_n = \omega_1 \ldots \omega_n \}
\]

the *cylinder* associated with \( \omega_1 \ldots \omega_n \). For a path \( \gamma = \gamma_1 \ldots \gamma_n \), we denote

\[
E_\gamma := g_{\gamma_1} \circ \ldots \circ g_{\gamma_n}(E(t(\gamma)));
\]

where \( t(\gamma) \) denotes the terminal state of the path \( \gamma \) (also \( \gamma_n \)). Iterating (3.1) \( k \)-times, we obtain

\[
E_i = \bigcup_{\gamma \in \Gamma^k_i} E_\gamma.
\]

We define a projection \( \pi : (\Gamma^\infty_1, \ldots, \Gamma^\infty_N) \to (\mathbb{R}^d, \ldots, \mathbb{R}^d) \), where \( \pi_i : \Gamma^\infty_i \to \mathbb{R}^d \) is defined by

\[
\{ \pi(\omega) \} := \bigcap_{n \geq 1} E_{\omega|_n}.
\]

For \( x \in E_i \), we call \( \omega \) a coding of \( x \) if \( \pi(\omega) = x \). It is folklore that \( \pi(\Gamma^\infty_i) = E_i \).

### 3.2. Ordered GIFS and linear GIFS

Let \( (\mathcal{A}, \Gamma, G) \) be a GIFS. To study the ‘advanced’ connectivity property of the invariant sets, we equip the edge set with a partial order. Let \( \Gamma^*_i \) be the set of edges emanating from the state \( i \).

**Definition 3.2.** We call the quadruple \( (\mathcal{A}, \Gamma, G, \prec) \) an *ordered GIFS*, if \( \prec \) is a partial order on \( \Gamma \) such that

(i) \( \prec \) is a linear order when restricted on \( \Gamma_j \) for every \( j \in \mathcal{A} \);

(ii) elements in \( \Gamma_i \) and \( \Gamma_j \) are not comparable if \( i \neq j \).

The order \( \prec \) induces a lexicographical order on each \( \Gamma^k_i \), namely, \( \gamma_1 \cdots \gamma_k \prec \omega_1 \omega_2 \cdots \omega_k \) if and only if \( \gamma_1 \cdots \gamma_{l-1} = \omega_1 \cdots \omega_{l-1} \) and \( \gamma_\ell \prec \omega_\ell \) for some \( 1 \leq \ell \leq k \). Observe that \( (\Gamma^k_i, \prec) \) is a
linear order; two paths $\gamma, \omega \in \Gamma_i^k$ are said to be adjacent if there is no path between them with respect to the order $\prec$.

**Definition 3.3.** Let $(\mathcal{A}, \Gamma, \mathcal{G}, \prec)$ be an ordered GIFS with invariant sets \( \{E_i\}_{i=1}^N \). It is termed a linear GIFS, if for all \( i \in \mathcal{A} \) and \( k \geq 1 \),

\[ E_\gamma \cap E_\omega \neq \emptyset \]

provided \( \gamma, \omega \) are adjacent paths in \( \Gamma_i^k \).

**Remark 3.4 (Linear IFS).** An IFS \( \{S_j\}_{j=1}^N \) is a special class of GIFS where the state set is a singleton and the edge set consists of \( N \) self-edges. The IFS becomes an ordered IFS if we assume the natural order. For convention, we denote the edges by \( 1, \ldots, N \); instead of calling \( i_1 \ldots i_n \in \{1, \ldots, N\}^n \) a path, we call it a word.

**Example 3.5 (Sierpiński curve).** (A nice survey of the Sierpiński curve can be found in [5].) Let \( \{S_j\}_{j=1}^4 \) be the IFS

\[ S_j(z) = \frac{z + d_j}{2}, \quad \text{where } (d_1, d_2, d_3, d_4) = (0, 1, 1 + i, i). \]

Clearly the invariant of the IFS is the unit square.

Let \( A, B, C, D \) be a partition of the unit square indicated by figure 6 (left). In figure 6 (right), each big triangle is divided into 4 small triangles. Accordingly, we define the following ordered GIFS

\[
\begin{align*}
A &= S_1(A) + S_1(B) + S_2(D) + S_2(A), \\
B &= S_2(B) + S_2(C) + S_3(A) + S_3(B), \\
C &= S_3(C) + S_3(D) + S_4(B) + S_4(C), \\
D &= S_4(D) + S_4(A) + S_1(C) + S_1(D). \\
\end{align*}
\] (3.4)

As we have mentioned before, we use ‘+’ instead of ‘∪’ to emphasize the order structure.

It is easy to check that (3.4) is a linear GIFS by the chain condition introduced in the next section. Apparently the open set condition is fulfilled.

Hence, by theorem 1.1, each triangle admits a space-filling curve, and the union of these curves is a closed space-filling curve of the unit square. Indeed, this curve is the famous Sierpińska space-filling curve. Figure 9(right) is a visualization of this curve, where the construction is discussed in section 6 (see example 6.2).
4. The chain condition, proof of theorem 1.2

Let \((A, \Gamma, \prec)\) be an ordered GIFS. Denote the invariant sets by \(\{E_i\}_{i \in A}\). For an edge \(\omega \in \Gamma\), recall that \(g_{\omega}\) is the associated similitude and \(t(\omega)\) is the terminal state.

For \(i \in A\), a path \(\omega \in \Gamma^\infty_i\) is called the lowest path, if \(\omega^n\) is the lowest path in \(\Gamma^n_i\) for all \(n\); in this case, we call \(a = \pi(\omega)\) the head of \(E_i\). Similarly, we define the highest path \(\omega'\) of \(\Gamma^\infty_i\), and we call \(b = \pi(\omega')\) the tail of \(E_i\).

**Definition 4.1.** An ordered GIFS is said to satisfy the chain condition, if for any \(a \in A\), and any two adjacent edges \(\omega, \gamma \in \Gamma_i\) with \(\omega \prec \gamma\),

\[
(\text{tail of } E_{\pi(\omega)}) = g_{\omega}(\text{head of } E_{\pi(\gamma)}).
\]

Clearly, for an ordered IFS

\[
K = S_1(K) + S_2(K) + \cdots + S_N(K),
\]

the lowest coding is \(K^n\) and the highest coding is \(N^n\). Therefore, the head of \(K\) is the fixed point of \(S_1\), denoted by \(\text{Fix}(S_1)\), and the tail of \(K\) is \(\text{Fix}(S_N)\). Consequently, the chain condition holds if and only if

\[
S_{i+1}(\text{Fix}(S_i)) = S_i(\text{Fix}(S_N)) \quad \text{for } i = 1, 2, \ldots, N - 1.
\]

Condition (4.1) first appeared in Hata [15], which dealt with the H"older continuous parametrization of self-similar sets.

**Proof of theorem 1.2.** Suppose \((A, \Gamma, \prec)\) satisfies the chain condition. Let \(i \in A\), and let \(\omega\) and \(\gamma\) be two adjacent paths in \(\Gamma^\infty_i\) with \(\omega \prec \gamma\). Let \(\eta = \omega \wedge \gamma\) be the largest common prefix of \(\omega\) and \(\gamma\). Let \(k = |\eta|\), then \(\omega\) and \(\gamma\) can be written as \(\omega = \eta \omega_{k+1} \cdots \omega_n\) and \(\gamma = \eta \gamma_{k+1} \cdots \gamma_m\).

The fact that \(\omega\) and \(\gamma\) are adjacent implies that

(i) \(\omega_{k+1} \ldots \omega_n\) and \(\gamma_{k+1} \ldots \gamma_m\) are adjacent edges in \(\Gamma_{t(\eta)}\) where \(j = t(\eta)\), and \(\omega_{k+1} \prec \gamma_{k+1}\);

(ii) \(\omega_{k+2} \ldots \omega_n\) is the highest path in \(\Gamma_{t(\eta)}^{n-\eta_{k+1}}\) and \(\gamma_{k+2} \ldots \gamma_m\) is the lowest path in \(\Gamma_{t(\eta)}^{n-\eta_{k+1}}\).

By item (ii), we have \(b = (\text{tail of } E_{t(\omega_{k+1})}) \in E_{\omega_{k+2} \ldots \omega_n}\) since the coding of \(b\) begins with \(\omega_{k+2} \ldots \omega_n\). Hence

\[
g_{\omega_{k+1}}(\text{tail of } E_{t(\omega_{k+1})}) \in E_{\omega_{k+2} \ldots \omega_n}.
\]

Similarly,

\[
g_{\gamma_{k+1}}(\text{head of } E_{t(\gamma_{k+1})}) \in E_{\gamma_{k+2} \ldots \gamma_m}.
\]
Therefore $E_{\omega_1,\ldots,\omega_k} \cap E_{\gamma_1,\ldots,\gamma_k} \neq \emptyset$ by the chain condition. So

$$E_\omega \cap E_\gamma = g_k(E_{\omega_1,\ldots,\omega_k} \cap E_{\gamma_1,\ldots,\gamma_k}) \neq \emptyset,$$

which proves that the GIFS is linear.

On the other hand, assume that $(A, \Gamma, G, \prec)$ is a linear GIFS. Fix $i \in A$. Let $\omega_1$ and $\gamma_1$ be adjacent edges in $\Gamma_i$ satisfying $\omega_1 \prec \gamma_1$. Let $(\omega_k)_{k=1}^\infty$ be the highest path in $\Gamma_{(\omega_1)}$ and $(\gamma_k)_{k=1}^\infty$ be the lowest path in $\Gamma_{(\gamma_1)}$. Denote $\omega[..]_k = \omega_1 \ldots \omega_k$ and $\gamma[..]_k = \gamma_1 \ldots \gamma_k$, then for all $k \geq 1$, $\omega[..]_k$ and $\gamma[..]_k$ are adjacent paths in $\Gamma_i$ and so $E_{\omega[..]_k} \cap E_{\gamma[..]_k} \neq \emptyset$. As we know that $\pi_{\omega[..]_k}$ is the tail of, and $\pi_{\gamma[..]_k}$ is the head of $t_{1k}$ for all $k \geq 1$, the distance between $g_{\omega[..]_k} (\text{tail of } E_{\omega[..]_k})$ and $g_{\gamma[..]_k} (\text{head of } E_{\gamma[..]_k})$ can be arbitrarily small. Thus

$$g_{\omega[..]_k} (\text{tail of } E_{\omega[..]_k}) = g_{\gamma[..]_k} (\text{head of } E_{\gamma[..]_k}),$$

and the chain condition is verified. The theorem is proved. □

**Corollary 4.2.** An ordered IFS $\{S_1, \ldots, S_N\}$ is a linear IFS if and only if (4.1) holds.

### 5. Path-on-lattice IFS on the plane

In this section, we study the path-on-lattice IFS on the plane (which is identified with the complex plane $\mathbb{C}$).

Let $L = \mathbb{Z} + i\mathbb{Z}$ be the square lattice or $\mathbb{L} = \mathbb{Z} + \omega \mathbb{Z}$ be the triangle lattice in the plane, where $\omega = \exp(2\pi i/3)$. We define two points in $L$ to be neighbors if their distance is 1. Then we obtain a graph and we still denote it by $L$.

Let $P$ be a path in $L$ passing through the points $0 = z_0, z_1, \ldots, z_{n-1}, z_n = d$ in this order. Let $\Phi = \{\phi_k\}_{k=1}^n$ be an ordered IFS on $\mathbb{C}$ such that

$$\phi_k((0, d)) = (z_{k-1}, z_k), \text{ for all } k = 1, \ldots, n.$$ (5.1)

We call $\Phi$ a path-on-lattice IFS with respect to the path $P$. Clearly the mapping $\phi_k$ has the form $\phi_k(z) = \alpha z + \beta$, or $\phi_k(z) = \alpha z + \beta$ with $\alpha, \beta \in \mathbb{C}$, and there are four choices of $\phi_k$ for each $k$. If we indicate the four mappings by line segments with a half-arrow, then the IFS can be described by a path consisting of marked line segments. If all $\phi_k$ are of the form $\alpha z + \beta$, then we say $\Phi$ is reflection-free.

**Theorem 5.1.** Let $K$ be the invariant set of a path-on-lattice IFS. Then $K$ can be generated by a linear GIFS with at most two states. Moreover, we can set up the linear GIFS in a way that it satisfies the OSC if the original IFS does.

**Proof.** Let $\{\phi_k\}_{k=1}^n$ be a path-on-lattice IFS defined by (5.1). Let $K$ be the invariant set. For $k = 1, \ldots, n$, define
We define an ordered GIFS with two states \(\{1, -1\}\) as follows:

\[
\begin{align*}
E_1 &= \phi_1(E_{n_1}) + \ldots + \phi_k(E_{n_k}), \\
E_{-1} &= \phi_1(E_{-n_1}) + \ldots + \phi_k(E_{-n_k}).
\end{align*}
\]  

(One may think that the first equation corresponds to the path \(P\), and the second equation corresponds to the reverse path of \(P\).) Clearly \(E_1 = E_{-1} = K\).

We shall show that the head and tail of \(E_1\) are 0 and \(d\), respectively, and the head and tail of \(E_{-1}\) are \(d\) and 0, respectively. Then using (5.2), we deduce that the GIFS (5.3) satisfies the chain condition, and hence is a linear GIFS by theorem 1.2.

According to \(v_1 = \pm 1\) and \(v_n = \pm 1\), we consider four cases. Suppose \(v_1 = 1\) and \(v_n = 1\). Let us denote the \(k\)th edge emanating from \(E_i\) by \(\lambda_{ik}\), then the first edge emanating from the vertex 1 is a self-edge, and hence \((\lambda_{11})^\infty\) is the lowest coding. It follows that the head of \(E_1\) is 0. Similarly, the highest coding emanating from vertex 1 is \((\lambda_{1n})^\infty\), and so the tail of \(E_1\) is \(d\). By the same argument, the head of \(E_{-1}\) is \(d\) and the tail of \(E_{-1}\) is 0. We get the desired result.

The other three cases can be proved in the same manner. Moreover, if all \(v_k\) are equal to 1, or all \(v_k\) are equal to \(-1\), then GIFS (5.3) degenerates to a linear IFS.

As for the open set condition, if the original IFS satisfies the OSC with an open set \(U\), then GIFS (5.3) satisfies the OSC with open sets \(\{U, U\}\). The theorem is proved. \(\Box\)

Example 5.1. The Peano curve, the Hilbert curve and the Heighway dragon curve are all generated by (reflection-free) path-on-lattice IFS’s, where the paths are given in figure 7.

Let us take the Hilbert curve as an example. Clearly \(v_1 = v_3 = -1, v_2 = v_5 = 1\), and the corresponding linear GIFS is:

\[
\begin{align*}
E_1 &= \phi_1(E_{-1}) + \phi_2(E_1) + \phi_3(E_1) + \phi_4(E_{-1}), \\
E_{-1} &= \phi_1(E_1) + \phi_2(E_{-1}) + \phi_3(E_{-1}) + \phi_4(E_1),
\end{align*}
\]

where \(\phi_1(z) = \frac{z - i}{2} + i, \phi_2(z) = \frac{z + i}{2} + i, \phi_3(z) = \frac{z}{2} + (1 + i), \phi_4(z) = \frac{z - i}{2} + 2.\)

Example 5.2 (Gosper curve and anti-Gosper curve). The IFS of the Gosper curve is given by the path in figure 8 (top-left), where \(v_1 = v_4 = v_5 = 1, v_2 = v_3 = v_7 = -1\). If we change the orientations of some arrows in the path, we obtain other path-on-lattice IFS’s. Indeed, there are 128 of them, and it is shown in [29] that only two of them satisfy the open set condition, the Gosper curve and anti-Gosper curve (For the anti-Gosper curve, \(v_1 = v_2 = v_5 = 1\) and \(v_3 = v_4 = v_6 = v_7 = -1\); see figure 8, bottom.).

![Figure 7. Paths for the Heighway dragon curve, the Peano curve and the Hilbert curve.](image-url)
6. Visualizations of space-filling curves

We consider the visualizations of linear GIFS in this section.

Let us start with linear IFS. Let \( \{ S_i \} = \{ \mathcal{S} \} \) be a linear IFS satisfying the open set condition. According to theorem 1.1, an optimal parametrization \( \varphi \) of \( K \) can be constructed accordingly. To visualize the limit curve \( \varphi \), we need to choose an initial pattern. Indeed, the initial pattern can be any curve, but a suitable choice will make the visualization beautiful.

Let us denote by \( L_0 \) the initial pattern, and let \( a \) and \( b \) be its initial and termination point, respectively. For an \( \omega \in \{1, \ldots, N\}^n \), we set \( S_\omega(L_0) = \omega \omega x \mathcal{S} x \), and denote by \( \omega + \omega \) the follower of \( \omega \). Connecting \( \omega \omega S a \) and \( \omega S b \) by \( \omega \omega L_0 \), and connecting \( \omega S b \) and \( \omega + \omega S a \) by a line segment, we obtain the curve \( \sum_{\omega \in \Gamma_n} ( S_\omega(L_0) + [b_\omega, a_\omega] ) \). Here we use \( \sum \) to indicate that \( L_n \) is the joining of small curves, where the order is given by \( \omega \).

We call \( L_n \) the \( n \)th approximation of the space-filling curve \( \varphi \). Different choices of initial patterns may give very different approximations in appearance, though the limit curve is the same.

**Example 6.1 (Peano curve).** The linear GIFS structure is given in example 5.1. Figure 9(left) shows the second approximation (but with a dilation and a rotation), where \( L_0 \) is the diagonal of the square multiplied by a factor 0.87.

As for a linear GIFS, we have to choose an initial pattern for each \( E_j \), which we denote by \( L_0^j, \ldots, L_N^j \). Denote the initial point of the pattern \( L_0^j \) by \( a_j \) and the end point by \( b_j \). We define the \( n \)th approximation of \( E_j \) to be

\[
L_n^j = \sum_{\omega \in \Gamma_n^j} ( g_\omega(L_0^{j(\omega)}) + [b_\omega, a_\omega] ),
\]

where \( b_\omega = g_\omega(b_{\omega \omega}) \) and \( a_\omega = g_\omega(a_{\omega \omega}) \).
Example 6.2. For the space-filling curves in this example, the linear GIFS structures are given in examples 3.5, 5.1 and 5.2. The visualizations and initial patterns are listed in the table below.

| Curve          | Visualization | Iteration | Visualization | Initial Pattern |
|----------------|---------------|-----------|---------------|-----------------|
| Hilbert curve  | Figure 9 (middle) | 3rd       | $L_0^1 = L_0^2 = \{\text{center of the square}\}$ |
| Heighway dragon| Figure 2 (left) | 8th       | $L_0^3 = [0.15, 0.85], L_0^4 = [0.85, 0.15]$ |
|                | Figure 2 (right) | 8th       | $L_0^5 = [0.15, 0.85](1+i), L_0^6 = [0.85, 0.15](1+i)$ |
| Gosper curve   | Figure 10 (right) | 3rd       | $L_0^7 = [0,d], L_0^8 = [d,0]$ where $d = 2 + e^{i\pi/3}$ |
| Sierpiński curve | Figure 9 (right) | 3rd       | Figure 11 |

7. The four-tile star

The four-tile star is a 4-reptile generated by the IFS

$$S_1(z) = -\frac{z}{2}, \quad S_2(z) = -\frac{z}{2} - i, \quad S_3(z) = -\frac{z}{2} + \exp\left(\frac{5\pi}{6}\right), \quad S_4(z) = -\frac{z}{2} + \exp\left(\frac{\pi}{6}\right).$$

Actually, the $S_j$ map the big dotted triangle to the four small triangles in figure 13, respectively. Teachout [28] gave a visualization of the four-tile star without any explanation (see figure 3). The mathematical theory behind it is provided in [7], and here we give a sketch of it.

7.1. Skeleton

We introduce the notion of a skeleton of a self-similar set in [7] (a skeleton is a kind of vertex set of a fractal). It is shown that $\{a_1, \ldots, a_6\}$ is a skeleton of the four-tile star, where $a_1 = \frac{4}{3}\exp\left(\frac{5\pi}{6}\right), a_2 = \frac{2}{3}i, a_3 = \frac{a_1}{\omega}, a_4 = \frac{a_2}{\omega}, a_5 = \frac{a_3}{\omega}, a_6 = \frac{a_4}{\omega}$, and $\omega = \exp(2i\pi/3)$ (See figure 12 and 13.).

7.2. Substitution rule

Let $P$ be the directed graph with edges $\{a_1a_2, a_2a_3, \ldots, a_6a_5, a_5a_1\}$. (See figure 13(left).) Figure 13(right) indicates the graph $S_1(P) \cup S_2(P) \cup S_3(P) \cup S_4(P)$. 

Figure 9. Space-filling curves of Peano, Hilbert (1891) and Sierpiński(1912).
which consists of 24 directed edges. An Eulerian path of the graph is indicated by figure 13(right), and the path is divided into 6 parts indicated by different colors. Replacing a segment in figure 13(left) by the broken lines in figure 13(right) with the same color, we obtain a substitution rule.

7.3. Linear GIFS

According to the substitution rule, we introduce the following linear GIFS:

\[
\begin{align*}
X &= S_3(U) + S_3(V) + S_3(W) + S_3(U), \\
Y &= S_3(V) + S_3(Z) + S_3(U) + S_3(V), \\
Z &= S_3(W) + S_3(X) + S_3(Y) + S_3(W), \\
U &= S_3(X) + S_3(V) + S_3(W) + S_3(X), \\
V &= S_3(Y) + S_3(Z) + S_3(U) + S_3(Y), \\
W &= S_3(Z) + S_3(X) + S_3(Y) + S_3(Z).
\end{align*}
\]

One can show that the four-tile star coincides with \(X \cup Y \cup Z \cup U \cup V \cup W\), which is a non-overlapping union (w.r.t. Lebesgue measure). Hence, according to our theory, a space-filling curve of the four-tile star is obtained.

7.4. Visualizations

Figure 14(left) provides the initial patterns of the third visualization in figure 3. If we choose the initial patterns in figure 14(right), we obtain the visualization in figure 4(right).
In this section, we show that the invariant sets of a linear GIFS with the open set condition admit optimal parameterizations. An auxiliary GIFS, called the measure-recording GIFS, will play an important role.

8. Preliminaries to dimensions and measures of graph-directed sets

Let \((\mathcal{A}, \Gamma, \mathcal{G})\) be the GIFS given by (3.1). We say a directed-graph \((\mathcal{A}, \Gamma)\) is strongly connected, if there exists a path from \(i\) to \(j\) for any pair \(i, j \in \mathcal{A}\).
Let $r_e$ denote the contraction ratio of the similitude $g_e$ associated with $e \in \Gamma$. Define a matrix $M(t)$, $t > 0$, as

$$M(t) = \left( \sum_{e \in E_0} r_e \right) \left( \sum_{i \notin j \in N} \right) . \tag{8.1}$$

Then there exists a unique positive number $\delta$, called the similarity dimension of the GIFS, such that

$$\rho(M(\delta)) = 1,$$

where $\rho(M)$ denotes the spectral radius of a matrix $M$ (see [20, 12]).

**Theorem 8.1.** ([20]) Let $(\mathcal{A}, \Gamma, \mathcal{G})$ be a GIFS satisfying the OSC and let $\delta$ be the similarity dimension. Then

(i) $\dim_H E_i = \delta$; and $0 < H^\delta(E_i) < \infty$ for all $i = 1, 2, \ldots, N$ if $\Gamma$ is strongly connected.

(ii) $(H^\delta(E_1), \ldots, H^\delta(E_N))^T$ is an eigenvector of $M(\delta)$ corresponding to eigenvalue 1.

(iii) $H^\delta(E_i \cap E_\gamma) = 0$ for any incomparable $\omega, \gamma \in \Gamma_i^*$ (two paths are said to be comparable if one of them is a prefix of the other).

In the rest of the section, we will always assume that $\mathcal{G}$ satisfies the OSC, and that $0 < H^\delta(E_i) < \infty$ for all $i = 1, \ldots, N$. Let us denote

$$h_i = H^\delta(E_i) \text{ and } \mu_i = H^\delta|_{E_i}, \quad i = 1, \ldots, N.$$

Now, we define Markov measures on the symbolic spaces $\Gamma_i^\infty, i \in \mathcal{A}$. For an edge $e \in \Gamma$ such that $e \in \Gamma_0$, set

$$P_e = \frac{h_j}{h_i} r_e. \tag{8.2}$$

Using theorem 8.1(ii), it is easy to verify that $(P_e)_{e \in \Gamma}$ satisfies

$$\sum_{j \in \mathcal{A}} \sum_{e \in \Gamma_0} P_e = 1, \text{ for all } i \in \mathcal{A}. \tag{8.3}$$

We call $(P_e)_{e \in \Gamma}$ a probability weight vector. Let $P_i$ be a Borel measure on $\Gamma_i^\infty$ satisfying the relations

$$P_i([\omega_1 \ldots \omega_n]) = h_i p_{\omega_1} \ldots p_{\omega_n} \tag{8.4}$$

for all cylinder $[\omega_1 \ldots \omega_n]$. The existence of such measures are guaranteed by (8.3). We call $\{P_i\}_{i=1}^N$ the Markov measures induced by the GIFS $\mathcal{G}$. The following result is folklore, see for instance [18, 20].

**Theorem 8.2.** Suppose the GIFS $\mathcal{G}$ satisfies the OSC and $0 < h_i < +\infty$ for all $i$. Let $\pi_i : \Gamma_i^\infty \to E_i$ be the projections defined by (3.3). Then

$$\mu_i = P_i \circ \pi_i^{-1}.$$
Fix a state \( i \). We list the edges in \( \Gamma_i \) in the ascending order with respect to \( \prec \):
\[
\gamma_1, \ldots, \gamma_{\ell_i}.
\]

Recall that \( t(\gamma) \) denotes the terminal state of an edge \( \gamma \). Then according to the set equation of \( G \), \( E_i \) can be written as
\[
E_i = g_{\gamma_i}(E(t(\gamma_i))) + \cdots + g_{\gamma_1}(E(t(\gamma_1))).
\]

Since \( |F_i| = h_i \) (the length of \( F_i \)) is the Hausdorff measure of \( E_i \), we have
\[
|F_i| = r^i_{\gamma_i}|F(t(\gamma_i))| + \cdots + r^i_{\gamma_1}|F(t(\gamma_1))|,
\]
which means that \( F_i \) is a non-overlapping union of small intervals. Let
\[
g^*_\gamma(x) = r^\delta_{\gamma}x + b_k : \mathbb{R} \rightarrow \mathbb{R}, \quad 1 \leq k \leq \ell_i
\]
be similitudes such that
\[
F_i = g^*_\gamma(F(t(\gamma_i))) + \cdots + g^*_\gamma(F(t(\gamma_1))),
\]
where the right hand side is a non-overlapping union of consecutive intervals from left to right; indeed, \( b_k = \sum_{j=1}^{k-1} h_{t(\gamma_j)} r^\delta_{\gamma_j} \). Doing this for all \( i \in A \), then (8.5) give us an ordered GIFS, which we denote by
\[
(A, \Gamma, G^*, \prec),
\]
and call it the measure-recording GIFS of \((A, \Gamma, G, \prec)\). In other words, the measure-recoding GIFS has the same basic graph as the original GIFS, but the mappings \( g_e \) are replaced by \( g^*_e \).

Clearly, the measure-recording GIFS inherits the graph structure and the order structure of the original GIFS. Moreover, it records the Hausdorff measure information of the original GIFS. The following facts are obvious.

- \( \{F_i\}_{i=1}^N \) are the invariant sets of the measure-recording GIFS.
- For an edge \( e \in \Gamma \), the contraction ratio of \( g^*_e \) is \( r^\delta_{e} \), and the similarity dimension \( \delta^* \) of \( G^* \) is 1.
- \( G^* \) satisfies the OSC.
- The measure-recording GIFS shares the same symbolic spaces with the original GIFS.

Let
\[
\pi : \Gamma^\infty_i \rightarrow E_i \quad \text{and} \quad \rho_i : \Gamma^\infty_i \rightarrow F_i, \quad i = 1, \ldots, N,
\]
be projections w.r.t. the GIFS \( G \) and \( G^* \), respectively (see (3.3)). Then the following lemma holds.

**Lemma 8.1.** The Markov measure induced by the measure-recording GIFS coincides with the one induced by the original GIFS.

**Proof.** Let \( (p_e)_{e \in \Gamma} \) and \( (p^*_e)_{e \in \Gamma} \) be the probability weights corresponding to \( G \) and \( G^* \), respectively. Since
\[
p_e = \frac{h_j}{h_i}(r^\delta_{ij})^e = \frac{\mathcal{L}(F_i)}{\mathcal{L}(F_j)}(r^\delta_{ij})^e = p^*_e,
\]
the two systems define the same probability weight vector and hence define the same Markov measure.

Define

$$\psi_i := \pi_i \circ \rho_i^{-1}.$$  \hspace{1cm} (8.6)

The following lemma verifies that $\psi_i$ is a well-defined mapping from $F_i$ to $E_i$.

**Lemma 8.2.** Suppose $x \in F_i$ has two $\rho_i$-codings, say $\rho_i^{-1}x = \{\omega, \gamma\}$. Then $\pi_i(\omega) = \pi_i(\gamma)$.

**Proof.** Write $\omega = (\omega_k)_{k=1}^\infty$ and $\gamma = (\gamma_k)_{k=1}^\infty$. We claim that $\omega_1 \ldots \omega_n$ and $\gamma_1 \ldots \gamma_n$ are adjacent for all $n \geq 1$, otherwise there exists $\eta_1 \ldots \eta_n$ such that

$$\omega_1 \ldots \omega_n \prec \eta_1 \ldots \eta_n \prec \gamma_1 \ldots \gamma_n,$$

and the interval $F_{\eta_1 \ldots \eta_n}$ separates $E_{\omega_1 \ldots \omega_n}$ and $E_{\gamma_1 \ldots \gamma_n}$, contradicting to $\rho_i(\omega) = \rho_i(\gamma) = x$. Our claim is proved. It follows that $E_{\omega_1 \ldots \omega_n} \cap E_{\gamma_1 \ldots \gamma_n} = \emptyset$, since $(A, \Gamma, \mathcal{G}, \prec)$ is a linear GIFS. Hence $|\pi_i(\omega) - \pi_i(\gamma)| \leq \text{diam } E_{\omega_1} + \text{diam } E_{\gamma_1}$, so the distance between $\pi_i(\omega)$ and $\pi_i(\gamma)$ can be arbitrarily small, which implies that $\pi_i(\omega) = \pi_i(\gamma)$. \qed

Now, we prove theorem 1.1 by showing that the mapping $\psi_i$ is an optimal parametrization of $E_i$.

**8.3. Proof of theorem 1.1**

Let $\mathcal{G}$ be the measure-recording GIFS of $\mathcal{G}$. Let $\psi_i = \pi_i \circ \rho_i^{-1}$. Let $\nu_i = L_i |_{F_i}$ be the restriction of the Lebesgue measure on $F_i$, $\mu_i = H_i |_{E_i}$, and $P_i$ be the common Markov measure of $\mathcal{G}$ and $\mathcal{G}^*$. Then $\nu_i = P_i \circ \rho_i^{-1}$, $\mu_i = P_i \circ \pi_i^{-1}$ by theorem 8.2.

(i) First, we prove that $\psi_i$ is almost one-to-one.

Let $Q_i$ be the set of points in $E_i$ possessing more than one $\pi_i$-coding. Since

$$Q_i = \bigcup_{n \geq 1} \bigcup_{\gamma \neq \omega \in \Gamma_i} E_\gamma \cap E_\omega,$$

and $\mu_i(E_\gamma \cap E_\omega) = 0$ by theorem 8.1(iii), we obtain $\mu_i(Q_i) = 0$. Denote $\Delta_i = \pi_i^{-1}(Q_i)$, then $\pi_i$ is injective when restricted to $\Gamma_i \setminus \Delta_i$, and $P_i(\Delta_i) = \mu_i(Q_i) = 0$. 


Similarly, let \( Q'_i \) be the set of points in \( F_i \) possessing more than one \( \rho_i \)-coding, then \( \nu(Q'_i) = 0 \). Let \( \Delta'_i = \rho_i^{-1}(Q'_i) \), then \( \rho_i \) is injective when restricted to \( \Gamma_i^\infty \setminus \Delta'_i \) and \( \nu(\Delta'_i) = \nu(Q'_i) = 0 \).

Let \( \nu = \nu_i \). Let \( \rho \Delta = \rho_i - \Delta \), then \( \rho_i \) is injective when restricted to \( \Gamma_i^\infty \setminus (\Delta_i \cup \Delta'_i) \), and

\[
\nu(\Delta_i \cup \Delta'_i) = \nu(\Delta'_i || \nu_i \Delta,) = \nu_i(B),
\]

where the third equality holds since \( \rho_i \) is a bijection when restricted to \( \Gamma_i^\infty \setminus (\Delta_i \cup \Delta'_i) \). Similarly, for any Borel set \( B \subset \mathbb{R} \), one can show that \( \mu_i(B) = \nu_i(B) \).

(iii) Finally, we prove the \( 1/\delta \)-Hölder continuity of \( \psi_i \).

Let \( x_1, x_2 \) be two points in \( F_i = [0, h_i] \). Let \( k \) be the smallest integer such that \( x_1 \) and \( x_2 \) belong to different cylinders of rank \( k \), say, \( \omega \rho \in \omega \) and \( \gamma \rho \in \gamma \), where \( \omega \neq \gamma \in \Gamma_i \). It is seen that \( \omega = \omega_1 \ldots \omega_k \) and \( \gamma \) differ only at the last edge, that is,

\[
\gamma = \omega_1 \ldots \omega_{k-1} \gamma_k.
\]

We consider two cases according to whether \( \omega \) and \( \gamma \) are adjacent or not.

**Case 1.** \( \omega \) and \( \gamma \) are not adjacent (see figure 15).

Then there is a cylinder \( \eta = \omega_1 \ldots \omega_{k-1} \eta_k \) between \( \omega \) and \( \gamma \), so

\[
|x_1 - x_2| \geq \text{diam } F_\eta \geq h \cdot r_\eta \geq h \cdot r_\omega \cdot r_{\text{min}},
\]

where \( \omega^* = \omega_1 \ldots \omega_{k-1} \) is the path obtained by deleting the last edge in \( \omega \), and

\[
h = \min(h_i; \ i = 1, \ldots, N), \quad r_{\text{min}} = \min(r_i; \ e \in \Gamma_i).
\]

Since \( x_1, x_2 \) belong to \( \rho_i(\omega^*) \), the images of \( x_1 \) and \( x_2 \) under \( \pi_i \circ \rho_i^{-1} \), which we denote by \( y_1 \) and \( y_2 \) respectively, belong to \( \pi_i(\omega^*) = E_\omega^* \). It follows that

\[
|y_1 - y_2| \leq \text{diam } E_\omega^* \leq D \cdot r_\omega \leq D \cdot r_{\text{min}}^{-1} \cdot h^{-1/\delta} \cdot |x_1 - x_2|^{1/\delta},
\]

where

\[
D = \max_{1 \leq i \leq N} \text{diam } E_i.
\]

**Case 2.** \( \omega \) and \( \gamma \) are adjacent (see figure 16(left)).

Let \( x_3 \) be the intersection of \( F_\omega \) and \( F_\eta \). Let \( k' \) be the smallest integer such that \( x_1 \) and \( x_3 \) belong to different cylinders of rank \( k' \), say, \( x_1 \in \rho_i(\omega^*) \) and \( x_3 \in \rho_i(\omega'^*) \) (see figure 16(right)), then \( |x_1 - x_3| \geq \text{diam } F_{\omega^*} \) since \( x_3 \) is an endpoint.
Let $y_3 = \psi_i(x_3)$. Similar to Case 1, we have

$$|y_1 - y_3| \leq D \cdot r_{\min}^{-1} \cdot h^{-1/\delta} |x_1 - x_3|^{1/\delta}. \quad (8.7)$$

By the same argument, we have

$$|y_2 - y_3| \leq D \cdot r_{\min}^{-1} \cdot h^{-1/\delta} |x_2 - x_3|^{1/\delta}. \quad (8.8)$$

Hence, by the fact $x_3$ is located between $x_1$ and $x_2$,

$$|y_1 - y_2| \leq 2D \cdot r_{\min}^{-1} \cdot h^{-1/\delta} |x_1 - x_2|^{1/\delta}. \quad (8.8)$$

Therefore, (8.7) and (8.8) imply the $1/\delta$-Hölder continuity of $\psi_i$. □

**Remark 8.3.** We note that the initial point $\psi_i(0)$ is the head of $E_i$, and the terminate point $\psi_i(h_i)$ is the tail of $E_i$.

**Remark 8.4.** In some text books, space-filling curves are concerned; for example, *Real Analysis* by Stein and Shakarchi (2005), *Topology* by Munkres (2000), *Basic Topology* by Armstrong (1997). The above proof extends the arguments applied in these books.

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