Quantum fluctuations as deviation from classical dynamics of ensemble of trajectories parameterized by unbiased hidden random variable

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A quantization method based on replacement of c-number by c-number parameterized by an unbiased hidden random variable is developed. In contrast to canonical quantization, the replacement has straightforward physical interpretation as statistical modification of classical dynamics of ensemble of trajectories, and implies a unique operator ordering. We then apply the method to develop quantum measurement without wave function collapse and external observer à la pilot-wave theory.

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I. MOTIVATION

The present paper discusses three closely interrelated aspects of quantum mechanics: canonical quantization of classical system, quantum-classical correspondence and measurement problem. The discussion will be confined to system of non-relativistic particles with no spin. It is well known that even in this case, despite the astonishing pragmatical successes of quantum mechanics, its foundation with regard to the above three aspects, is not without ambiguity \[1\].

Let us denote the position of the particles as \(q = \{q_i\}\) and the corresponding conjugate momentum as \(p = \{p_i\}\) where \(i\) goes for all degree of freedom. In this paper all symbols with underline is used to denote quantities satisfying the law of classical mechanics. In canonical quantization, given a classical quantity \(F(q,p)\), then the so-called “quantum observable” is obtained by promoting \(q\) and \(p\) to Hermitian operators \(q \mapsto \hat{q}\) and \(p \mapsto \hat{p}\), and replacing the Poisson bracket by commutator \([\cdot,\cdot] \mapsto [\cdot,\cdot]/(\hbar)\). This replacement of c-number (classical number) by q-number (quantum number/Hermitian operator) consequently does not in general give a unique Hermitian operator due to operator ordering ambiguity. An even deeper difficulty is lying at the conceptual level for while classical mechanics is developed using the basic notion of conventional trajectory, the resulting quantum mechanics does not refer to the notion of trajectory, except in the so-called pilot-wave interpretation \[2\] to be discussed later. A closely related question is that despite the fact that Planck constant plays a pivotal role in connecting the quantum and classical mechanics through quantization and classical limit, its physical origin is not clear. Elaborating this issue might open the way to discuss the limitation of quantum mechanics.

On the other hand, in its standard interpretation, quantum mechanics is based on two different processes: unitary, continuous and deterministic evolution described by the Schrödinger equation when there is no measurement; and non-unitary, discontinuous and non-causal (random) process of wave function collapse in measurement \[3\]. For general model of measurement, the first process alone will give a superposition of macroscopically distinct states, which in the standard interpretation leads to the so-called paradox of Schrödinger's cat. It is then assumed that measurement reveals, randomly, only one of the term in the superposition. Accordingly, the second process mentioned above is needed. Moreover, this interpretation assumes an apparatus which must behave according to classical mechanics. The whole system must then be divided into quantum system being measured and classical apparatus of measurement. It is however well-known that such line of division can be made anywhere, thus is ambiguous. Further, since in general quantum measurement does not reveal the pre-existing value prior measurement, then there is a question whether another apparatus is needed to probe the record of the first apparatus, which immediately leads to infinite regression. In the context of quantum-classical correspondence, one can thus ask why classical mechanics does not suffer from measurement problems mentioned above, and how the probability of finding in quantum measurement becomes the probability of being in classical measurement.

Below we shall attempt to propose a solution to the above problems. Our basic idea is to first understand the physical meaning behind the formal rule of canonical quantization. We shall develop a quantization method by directly modifying classical dynamics of ensemble of trajectories parameterized by an unbiased hidden random variable. We shall show that given the classical Hamiltonians, the resulting equations for important class of physical systems can be rewritten into the Schrödinger equation with unique quantum Hamiltonians. The method is based on replacement of c-number by c-number, thus is free from operator ordering ambiguity. A couple examples where canonical quantization is ambiguous will be given. We shall further show that in all the cases to be considered, the particles posses effective velocity equal to the velocity of the particles in the pilot-wave theory and the Born’s interpretation of wave function is valid for all time by construction. This allows us to describe quantum measurement without wave function collapse and external (classical) observer à la pilot-wave theory. However unlike the latter, our model is inherently stochastic and the wave function is not physically real.
II. MODIFICATION OF CLASSICAL DYNAMICS OF ENSEMBLE OF TRAJECTORIES USING HIDDEN RANDOM VARIABLE

Let us consider the dynamics of N particles system whose classical Hamiltonian is denoted by \( \mathcal{H}(q, p; t) \). The classical dynamics of the particles is then given by the following Hamilton-Jacobi equation:

\[
\partial_t \mathcal{S}(q; t) + \mathcal{H}(q, \partial_q \mathcal{S}(q; t); t) = 0, \tag{1}
\]

where \( \mathcal{S}(q; t) \) is the Hamilton principle function (HPF) so that the momentum field is given by \( p = \partial_q \mathcal{S} \) where \( \partial_q = \{ \partial_{q_i} \} \). Hamilton-Jacobi equation thus describes the dynamics of ensemble (congruence) of trajectories in configuration space. To solve this equation, one needs to set up an initial HPF \( \mathcal{S}(q; 0) \) which implies an initial classical momentum field \( p(q; 0) = \partial_q \mathcal{S}(q; 0) \). A single trajectory in configuration space is picked up if one also fixes the initial position of the particles.

Let us consider an ensemble of classical systems so that the position of the particles are initially distributed in configuration space with probability density \( \rho(q; 0) \), \( \int dq \rho(q; 0) = 1 \). The probability density of the configuration of the particles at any time \( \rho(q; t) \) then satisfies the following continuity equation:

\[
\partial_t \rho + \partial_q \cdot (\mathcal{V}(\mathcal{S}) \rho) = 0, \tag{2}
\]

where \( \mathcal{V} = \{ \mathcal{V}_i \} \) is the classical velocity field. In the above equation, we have made explicit that in general, the classical velocity field \( \mathcal{V} \) might depend on the HPF \( \mathcal{S} \). Given a classical Hamiltonian, this relation can be obtained through (the Legendre transformation part of) the Hamilton equation:

\[
\mathcal{L} = \frac{\partial \mathcal{H}}{\partial \dot{q}} \bigg|_{(q, \dot{q}, \mathcal{S})} = f_i(\mathcal{S}), \tag{3}
\]

where \( f_i \), \( i = 1, \ldots, N \) are some functions. The dynamics and statistics of the ensemble of classical trajectories are then obtained by solving Eqs. (1), (2) and (3) in term of \( \mathcal{S}(q; t) \), \( \rho(q; t) \) and \( \mathcal{V}(\mathcal{S}) \).

Let us then proceed to develop a general scheme to modify the above dynamics of ensemble of classical trajectories. To do this, let us introduce a pair of real-valued functions, \( S(q, \lambda; t) \) and \( \Omega(q, \lambda; t) \), assumed to take over the role of \( \mathcal{S}(q; t) \) and \( \rho(q; t) \), respectively, in the modified dynamics. Here \( \lambda \) is a hidden random variable of action dimensional whose statistical properties will be specified later. Hence \( \Omega(q, \lambda; t) \) is the joint-probability density that the particles are at configuration coordinate \( q \) and the value of the hidden variable is \( \lambda \). The marginal probability densities of the fluctuations of \( q \) and \( \lambda \) are thus given by

\[
\rho(q; t) \equiv \int d\lambda \Omega(q, \lambda; t),
\]

\[
P(\lambda) \equiv \int dq \Omega(q, \lambda; t), \tag{4}
\]

where we have assumed that the statistics of \( \lambda \) is independent of time.

Let us then propose the following rule of replacement to modify the classical dynamics of ensemble of trajectories governed by Eqs. (1) and (2):

\[
\rho \mapsto \Omega, \\
\partial_q \mathcal{S} \mapsto \partial_q S + \frac{\lambda}{2} \frac{\partial_q \Omega}{\Omega}, \quad i = 1, \ldots, N, \\
\partial_t \mathcal{S} \mapsto \partial_t S + \frac{\lambda}{2} \frac{\partial_t \Omega}{\Omega} + \frac{\lambda}{2} \partial_q \cdot f(S), \tag{5}
\]

where the functional form of \( f = \{ f_i \} \) is determined by the classical Hamiltonian according to Eq. (3).

Let us first show that the replacement of Eq. (5) possesses a consistent classical correspondence if \( S \to \lambda \) so that the Hamilton-Jacobi equation of (1) is restored (notice that we have used the symbol “\( \to \)” to denote replacement and “\( \mapsto \)” to denote a limit). First, using the last two equations of (5), for sufficiently small \( \Delta t \) and \( \Delta q = \{ \Delta q_i \} \), then expanding \( \Delta F(\approx F(q + \Delta q; t + \Delta t) - F(q; t) \approx \partial_t F \partial_q + \partial_q F \partial_\lambda + \partial_q \cdot f \) for any function \( F \), one has

\[
\Delta \mathcal{S} \mapsto \Delta S + \frac{(\Delta \Omega)}{2} + \partial_q \cdot f(S) \Delta t. \tag{6}
\]

One can see that in the limit \( S \to \lambda \), in order to be consistent then the second term on the right hand side has to be vanishing. One thus has \( \partial \Omega/\partial t = -\Omega \partial \mathcal{V} \), by Eq. (3). This is just the continuity equation of (2). Hence, since \( \mathcal{V} \) is independent of \( \lambda \), in the limit \( S \to \lambda \), one has \( \rho = \int d\lambda \Omega \to \rho \). We have thus a smooth classical correspondence.

The next question is then what is the statistical properties of \( \lambda \). We shall show in the next section that to reproduce the prediction of quantum mechanics, one needs to assume that the probability density function of \( \lambda \) is given by

\[
P(\lambda) = \frac{1}{2} \delta(\lambda - h) + \frac{1}{2} \delta(\lambda + h), \tag{7}
\]

where \( h \) is the reduced Planck constant. Namely, \( \lambda \) can only take binary values \( \lambda = \pm h \) with equal probability.

What we shall do in the following sections is as follows. First, given a classical Hamiltonian, we shall generate the classical dynamics of ensemble of trajectories according to Eqs. (1), (2) and (3). We then proceed to modify Eqs. (1) and (2) by imposing Eq. (3). Averaging over the distribution of \( \lambda \) and taking into account Eq. (7), we shall show that, for a class of important physical systems, the resulting equations can be put into the Schrödinger equation with a unique Hermitian quantum Hamiltonian. Below we shall assume that the fluctuations of \( q \) and \( \lambda \) are separable \( \Omega(q, \lambda; t) = \rho(q; t) P(\lambda) \). Accordingly, Eq. (5) becomes

\[
\rho \mapsto \rho P(\lambda), \\
\partial_q \mathcal{S} \mapsto \partial_q S + \frac{\lambda}{2} \frac{\partial_q \rho}{\rho}, \quad i = 1, \ldots, N, \\
\partial_t \mathcal{S} \mapsto \partial_t S + \frac{\lambda}{2} \frac{\partial_t \rho}{\rho} + \frac{\lambda}{2} \partial_q \cdot f(S). \tag{8}
\]
Let us note before proceeding that in the present paper we shall not discuss the issue of nonlocality \[^6\]. For a review of the progress of hidden variable models in view of Bell nonlocality, see \[^2,8\]. Yet since we will claim that our model reproduces the prediction of quantum mechanics then it must violate Bell inequality. See however \[^9\] for an interesting discussion that the violation of Bell inequality does not necessarily lead to nonlocality due to the contextuality loophole.

### III. PARTICLE IN EXTERNAL POTENTIAL

#### A. Emergent deterministic Schrödinger equation

Let us apply the above modification of classical mechanics to an ensemble of particle subjected to external potentials. For simplicity, let us consider the case of single particle with mass \(m\). As will be seen, generalization to many particles is straightforward. The classical Hamiltonian is thus given by

\[
H = \frac{(p - (e/c)A)^2}{2m} + eV,
\]

where \(e\) is charge of the particle, \(c\) is the velocity of light, \(A(q, t)\) and \(V(q, t)\) are the vector and scalar electromagnetic potentials, respectively. The Hamilton-Jacobi equation of (11) thus reads

\[
\partial_t S + \frac{(\partial_q S - (e/c)A)^2}{2m} + eV = 0.
\]

On the other hand, inserting Eq. (11) into Eq. (3), the classical velocity field is related to \(\dot{S}\) as

\[
\dot{v} = (\partial_q S - (e/c)A)/m.
\]

The continuity equation of (2) thus becomes

\[
\partial_t \rho + \frac{1}{m} \partial_q \cdot (\partial_q S - (e/c)A) \rho = 0.
\]

Hence, the dynamics and statistics of classical ensemble of trajectories is governed by Eqs. (10), (11) and (12).

Next, from Eq. (11) and the definition of \(f\) given in Eq. (3), its functional form with respect to \(S\) is given by

\[
f(S) = (1/m)(\partial_q S - (e/c)A).
\]

Equation (5) then becomes

\[
\rho \mapsto \rho P(\lambda),
\]

\[
\partial_q S \mapsto \partial_q S + \frac{\lambda}{2} \frac{\partial_q \rho}{\rho},
\]

\[
\partial_t S \mapsto \partial_t S + \frac{\lambda}{2} \frac{\partial_t \rho}{\rho} + \frac{\lambda}{2m} \partial_q \cdot (\partial_q S - (e/c)A).
\]

Let us investigate how the above set of equations modify Eqs. (10) and (12). First, imposing the first two equations of (11) into Eq. (12) one gets

\[
\partial_t \rho + \frac{1}{m} \partial_q \cdot \left(\rho (\partial_q S - (e/c)A)\right) + \frac{\lambda}{2m} \partial_q^2 \rho = 0,
\]

where \(\partial_q^2 = \partial_q \cdot \partial_q\) and since \(P(\lambda)\) is independent of time and space, it can be divided out. On the other hand, imposing the last two equations of (14) into Eq. (10), one obtains

\[
\partial_t S + \frac{(\partial_q S - (e/c)A)^2}{2m} + eV - \frac{\lambda^2}{2} \frac{\partial_q^2 R}{2m} R = 0.
\]

We have thus pair of coupled equations (15) and (16) which still depends on the random variable \(\lambda\).

We shall proceed to take average of Eqs. (15) and (16) over the distribution of \(\lambda\). First, imposing the first two equations (15) and (16) over the distribution of \(\lambda\), one has

\[
\partial_t S + \frac{(\partial_q S - (e/c)A)^2}{2m} + eV - \frac{\lambda^2}{2} \frac{\partial_q^2 R}{2m} R = 0.
\]

where we have defined \(R \equiv \sqrt{\lambda}\), and used the following identity:

\[
\frac{1}{4} \partial_q \rho \partial_q^2 \rho = \frac{1}{2} \partial_q \rho \partial_q \partial_q^2 \rho - \partial_q \partial_q^2 \rho R,
\]

for the case of \(i = j\). Inserting Eq. (15) into Eq. (16), one has

\[
\partial_t S + \frac{(\partial_q S - (e/c)A)^2}{2m} + eV - \frac{\lambda^2}{2} \frac{\partial_q^2 R}{2m} R = 0.
\]

Let us now assume a specific form of \(P(\lambda)\) given by Eq. (7): \(P(\lambda) = (1/2)\delta(\lambda-h) + (1/2)\delta(\lambda+h)\). Then averaging Eqs. (15) and (16) over the distribution of \(\lambda\). First, from Eq. (18), since \(R\) is independent of \(\lambda\), one can see that \(S(q, \lambda; t)\) and \(S(q, -\lambda; t)\) satisfy the same differential equation. Assuming that initially \(S\) possesses the following symmetry \(S(q, \lambda; 0) = S(q, -\lambda; 0)\), the symmetry is then preserved for all the time

\[
S(q, \lambda; t) = S(q, -\lambda; t).
\]

Let us investigate how the above set of equations modify Eqs. (10) and (12). First, imposing the first two equations of (11) into Eq. (12) one gets

\[
\partial_t \rho + \frac{1}{m} \partial_q \cdot \left(\rho (\partial_q S - (e/c)A)\right) + \frac{\lambda}{2m} \partial_q^2 \rho = 0,
\]

where \(\partial_q^2 = \partial_q \cdot \partial_q\) and since \(P(\lambda)\) is independent of time and space, it can be divided out. On the other hand, imposing the last two equations of (14) into Eq. (10), one obtains

\[
\partial_t S + \frac{(\partial_q S - (e/c)A)^2}{2m} + eV - \frac{\lambda^2}{2} \frac{\partial_q^2 R}{2m} R = 0.
\]

We have thus pair of coupled equations (15) and (16) which still depends on the random variable \(\lambda\).

We shall proceed to take average of Eqs. (15) and (16) over the distribution of \(\lambda\). First, from Eq. (18), since \(R\) is independent of \(\lambda\), one can see that \(S(q, \lambda; t)\) and \(S(q, -\lambda; t)\) satisfy the same differential equation. Assuming that initially \(S\) possesses the following symmetry \(S(q, \lambda; 0) = S(q, -\lambda; 0)\), the symmetry is then preserved for all the time

\[
S(q, \lambda; t) = S(q, -\lambda; t).
\]

Let us now assume a specific form of \(P(\lambda)\) given by Eq. (7): \(P(\lambda) = (1/2)\delta(\lambda-h) + (1/2)\delta(\lambda+h)\). Then averaging Eqs. (15) and (16) over the distribution of \(\lambda\) and defining the following function:

\[
S_Q(q; t) \equiv \frac{1}{2} \left(S(q, h; t) + S(q, -h; t)\right)
\]

\[
= S(q, h; t) = S(q, -h; t),
\]

which is valid by Eq. (19), one obtains the following pair of equations:

\[
\partial_t S_Q + \frac{(\partial_q S_Q - (e/c)A)^2}{2m} + eV - \frac{\eta^2}{2m} \partial_q^2 R = 0.
\]

Let us further define the following complex-valued function:

\[
\Psi_Q(q; t) \equiv R \exp(iS_Q/h),
\]
so that the probability density of the position of the particle is given by

\[ \rho(q; t) = |\Psi_Q(q; t)|^2. \]  

(23)

The pair of equations in (21) can then be recast into

\[ i\hbar \partial_t \Psi_Q = \frac{1}{2m} (\psi_{q; (e/c)A})^2 \Psi_Q + eV \Psi_Q. \]  

(24)

This is just the Schrödinger equation for a particle subjected to vector and scalar potentials \( A(q; t) \) and \( V(q; t) \) with the corresponding quantum Hamiltonian given by

\[ \hat{H} = \frac{1}{2m} (\psi_{q; (e/c)A})^2 + eV. \]  

(25)

One can also see from Eq. (23) that the Born’s interpretation of wave function is valid by construction.

Further, from the lower equation of (21), it is straightforward to show that

\[ \int dq \rho(q; t) P(\lambda) (-\partial_t S) = \int dq \rho(q; t) (-\partial_t S_Q) \]

\[ = \int dq \Psi_Q \left( \psi_{q; (e/c)A})^2 + eV \right) \Psi_Q. \]  

(26)

The right hand side is just the quantum mechanical average energy which is conserved by the Schrödinger equation of (24). Hence, one should interpret \( E \equiv -\partial_t S_Q \) as the effective energy of the particle.

Note that while the Schrödinger equation is obtained when \( \lambda \) can take only binary values \( \pm \hbar \), \( \lambda \) may be a function of a set of continuous random variables, say \( \lambda = \lambda(\nu_1, \nu_2, \ldots) \). For example, one may have \( \lambda = \sqrt{\nu_1^2 + \nu_2^2 + \nu_3^2} = \pm \hbar \) where \( \nu_i, i = 1, 2, 3 \), take continuous real value. One thus has to solve

\[ \nu_1^2 + \nu_2^2 + \nu_3^2 = \hbar^2, \]  

(27)

namely the point \( \nu = \{\nu_1, \nu_2, \nu_3\} \) lies on the surface of a ball of radius \( \hbar \). Now let us divide the surface of the ball into two with equal area and attribute to each division with \( \pm \) signs. Then, if the point \( \nu \) is distributed uniformly on the surface of the ball (say it moves sufficiently chaotic on the surface), the resulting \( \lambda \) will satisfy Eq. (7).

The time-dependent Schrödinger equation is thus obtained through specific choice of \( P(\lambda) \) given by Eq. (7). This result suggests that generalization of Schrödinger equation might be attained by allowing the probability density function of \( \lambda \) to deviate from Eq. (7). This might further lead to possible correction to the prediction of quantum mechanics.

Let us mention here that there are many approaches reported in the literature to derive the Schrödinger equation with quantum Hamiltonian of the type given in Eq. (25) [11, 31]. The advantage of our derivation is three folds. First, it is derived in the scheme of quantizing a general classical Hamiltonian. Hence, taking aside that the solution exists, the method can be applied directly to other class of classical Hamiltonians. Second, it is derived by modifying the classical dynamics of ensemble of trajectories so that the quantum-classical correspondence is conceptually kept transparent. In particular, we have no problem of conceptual jump in quantum-classical transition from a quantum theory which does not refer to conventional notion of trajectories to a classical theory which is founded based on the notion of trajectories. The classical limit of Schrödinger equation is given by the dynamics of ensemble of classical trajectories. Finally, the Schrödinger equation is shown to correspond to a specific distribution of hidden random variable. Hence, it hints to a straightforward generalization [10].

Let us note before proceeding that the hidden variable \( \lambda \) is not the property of a single particle. Rather, it suggests the existence of background field which pervades all space whose detail interaction with the particle is not known, resulting in the stochastic motion of the particle. The presence of background field is also assumed in Nelson’s stochastic mechanics [12] and stochastic electrodynamics [31] approaches to explain the origin of quantum fluctuations. Next, as shown above, to get the correct time evolution, we have to first calculate the solutions (in the form of differential equation parameterized by the hidden variable) and then take average over the distribution of the hidden variable. The converse will lead to wrong time evolution. The situation is more like random walk. Namely, one has to evolve the walker (the particle) using the random step to obtain the correct time evolution, and then do the averaging over the probability of each single step. Taking the average of the single step in random walk will lead to trivial motion. In this case, to be meaningful, the fluctuations of the hidden variable \( \lambda \) has to be much faster than the fluctuations of \( q \).

### B. Effective velocity and pilot-wave theory

First, the upper equation of (21) can be regarded as a generalized continuity equation so that one can read off an effective velocity field of the ensemble of particle which is given by

\[ v(S_Q) = \frac{1}{m} (\partial_q S_Q - (e/c)A) = f(S_Q). \]  

(28)

where in the last equality we have used Eq. (13). On the other hand, if \( \lambda \) satisfies Eq. (7), then as shown in Eq. (22), \( S_Q \) is just the phase of Schrödinger wave function \( \Psi_Q \). Hence, in this case, the numerical value of the effective velocity of the particles is equal to the actual velocity of the particle in pilot-wave theory [2]. One can also see from Eq. (28) that \( \partial_q S_Q \) should be interpreted as the effective momentum field of ensemble of the particle. We have thus an effectively similar picture with pilot-wave theory in the sense that the particle always possesses definite position and momentum and it moves “as if” it is guided by the wave function so that the effective velocity is given by Eq. (28).
Our model however differs from pilot-wave theory as follows. The latter is based on the assumption that: (a) for any dynamical system, the Schrödinger equation and the corresponding guidance relation are postulated; (b) the time evolution is deterministic; (c) the wave function ΨQ(q; t) is physically real; and (d) the initial distribution of the particle is assumed to be given by ρ(q; 0) = |ΨQ(q; 0)|² to reproduce the prediction of quantum mechanics. In particular, the last two assumptions constitute one of its main critics [32, 33]. With the assumption that the wave function is physically real field, first, it is not clear how to prepare an ensemble to satisfy ρ(q; 0) = |ΨQ(q; 0)|². See however Refs. [34, 50] for argumentation against this critics. Second, why there is no back-reaction from the particle to the wave function like for example in the particle-field interaction in the theory of electromagnetic. Unlike pilot-wave theory, in our model, the (effective) deterministic time evolution governed by the Schrödinger equation and the corresponding guidance relation emerge naturally from a statistical modification of classical dynamics rather than postulated. The original dynamics is inherently stochastic. Moreover, the wave function is not physically real. It is just a mathematical tool to describe the dynamics and statistics of the ensemble of trajectories, and ρ(q; t) = |ΨQ(q; t)|² is valid for all time by construction. Nevertheless, despite the above conceptual difference, one can conclude that our model will reproduce the pilot-wave theory prediction on statistical wave-like pattern in slits experiment and tunneling of potential barrier [37].

C. Quantization: unique ordering and quantum-classical correspondence

We have mentioned in the previous subsection that the scheme to derive the Schrödinger equation presented in subsection III A can be viewed as to provide a method of quantization of classical Hamiltonian. In fact, the quantum Hamiltonian of Eq. (25) can be obtained from the classical Hamiltonian of Eq. (3) by replacing the classical momentum with the quantum momentum operator

\[ \hat{p} \rightarrow \hat{p} = -i\hbar \partial_q, \]

as prescribed by the canonical quantization in configuration space representation (see however the discussion at the end of the present subsection). Hence, for the case of particle in external potentials, our method correctly reproduces the result of canonical quantization.

As mentioned in Section I, however, given a classical Hamiltonian, the canonical quantization rule in general will give an infinite alternatives of quantum Hamiltonians due to the operator ordering ambiguity. In contrast to this, it is evident that the method of quantization proposed in the present paper will lead to unique Hermitian quantum Hamiltonian, if the solution exists. This came from the fact that while in canonical quantization one replaces c-number by q-number, in our method, c-number is replaced by another c-number parameterized by random hidden variable as prescribed by Eq. (8). To give an example where the canonical quantization rule leads to ambiguity, let us consider the dynamics of particle with position-dependent mass m(q) which has wide applications in solid state physics [35, 52]. For simplicity let us assume that the particle is free. The classical Hamiltonian then takes the form

\[ \hat{H} = B(q)\hat{p}^2, \]

where B(q) = 1/(2m) is a real-valued differentiable function of q. Using canonical quantization, one can then choose one out of infinite alternatives of quantum Hamiltonians. For example, if B(q) \sim q² up to some multiplicative constant, then one can either choose for the corresponding Hermitian quantum Hamiltonian \( \hat{p}^2 \hat{q}^2 + \hat{q}^2 \hat{p}^2 / 2 \) or \( \hat{p}^2 \hat{p} \) which are related to each other, by virtue of the canonical commutation relation \( [\hat{q}, \hat{p}] = i\hbar \), as \( \hat{p}^2 \hat{p} / 2 = \hat{p}^2 \hat{q}^2 + \hat{q}^2 \hat{p}^2 / 2 + \hbar^2 \).

Let us show that our method of quantization leads to unique Hermitian quantum Hamiltonian with a specific ordering of operators. First, given the classical Hamiltonian of Eq. (30), the Hamilton-Jacobi equation of (11) reads

\[ \partial_t \hat{S} + B(\partial_q \hat{S})^2 = 0. \]

Substituting Eq. (30) into Eq. (3), the classical velocity field is given by

\[ \bar{v} = 2B \partial_q \hat{S}. \]

Hence, the continuity equation of (2) becomes

\[ \partial_t \rho + 2 \partial_q \cdot (B \rho \partial_q \hat{S}) = 0. \]

Next, from Eq. (32), f defined in Eq. (32) is given by

\[ f(\hat{S}) = 2B \partial_q \hat{S}, \]

so that Eq. (6) becomes

\[ \rho \rightarrow \rho P(\lambda), \]

\[ \partial_t \hat{S} \rightarrow \partial_t \hat{S} + \frac{\lambda}{2} \rho \]

\[ \partial_t \hat{S} \rightarrow \partial_t \hat{S} + \frac{\lambda}{2} \rho + \lambda \partial_q (B \partial_q S). \]

Let us see how the above set of equations modify the pair of Eqs. (34) and (35). Imposing the first two equations of (34) into Eq. (35), one gets

\[ \partial_t \rho + 2 \partial_q (B \rho \partial_q S) + \lambda \partial_q (B \partial_q \rho) = 0. \]

On the other hand, imposing the last two equations of (34) into Eq. (31), one obtains

\[ \partial_t \hat{S} + B(\partial_q \hat{S})^2 - \lambda^2 \left( B \frac{\partial^2 R}{R} + \partial_q B \frac{\partial_q R}{R} \right) + \frac{\lambda}{2} \rho = 0. \]
where we have again defined $R = \sqrt{p}$ and used the identity of Eq. (17) for $i = j$. Substituting Eq. (35) into Eq. (36) one gets

$$\partial_t S + B(\partial_q S)^2 - \lambda^2 \left( \frac{\partial^2 R}{R} + \partial_q B \frac{\partial_q R}{R} \right) = 0. \quad (37)$$

We have thus pair of coupled equations (35) and (37) which still depend on the hidden variable $\lambda$.

One can then see that $S(q; h; t) = S(q; -h; t) = S_Q(q; t)$ satisfies the same equation (37) where $\lambda^2$ is replaced by $\hbar^2$. Hence, averaging over the fluctuations of the parameter $\lambda = \pm \hbar$ which is assumed to be equally probable, Eqs. (36) and (37) become

$$\partial_t \rho + 2 \partial_q (\rho \partial_q S) = 0, \quad \partial_t S_Q + B(\partial_q S_Q)^2 - \hbar^2 \left( \frac{\partial^2 R}{R} + \partial_q B \frac{\partial_q R}{R} \right) = 0. \quad (38)$$

Finally, recalling Eq. (22) that $\Psi_Q = R \exp(iS_Q/\hbar)$, the above pair of equations can be recast into

$$i\hbar \partial_t \Psi_Q = \frac{-\hbar^2}{2} \left( B \partial_q^2 + \partial_q^2 B \right) \Psi_Q + \frac{\hbar^2}{2} \left( \partial_q B \right) \Psi_Q. \quad (39)$$

This is just the Schrödinger equation with quantum Hamiltonian given by $\hat{H} = \hat{p}B(q)\hat{p}$. One thus obtains the following quantization mapping

$$B(q)\rho \rightarrow \hat{B}\rho \hat{B}. \quad (40)$$

In particular for constant $B$, one has

$$p^2 \rightarrow \hat{p}^2. \quad (41)$$

Let us mention that the same result as in Eq. (40) is also reported in the derivation of Schrödinger equation using the principle of exact-uncertainty and a principle of extremization of ensemble of Hamiltonian density $S$.

However, in contrast to the latter which can only be applied to quantize a specific type of classical Hamiltonian with a quadratic momentum dependence, our method formally applies as well, as will be shown below and in the next section, to classical Hamiltonian which contains a term that is linear in momentum. One might then expect that Eq. (41) can be extended to any power in momentum, namely $p^n \rightarrow \hat{p}^n$, where $n$ is integer. The answer is however negative. A straightforward calculation to quantize a classical Hamiltonian which is proportional to $p^2$, regardless of its physical meaning, will not lead to a Schrödinger equation with quantum Hamiltonian proportional to $\hat{p}^2$.

Next, let us assume that the classical Hamiltonian under interest is decomposable as $H = a \hat{H}_1 + b \hat{H}_2$, where $a$ and $b$ are real numbers. Then from Hamilton equation, the classical velocity field is also decomposable into $\nu(S) = (\partial H/\partial p)|_{p=\partial_q S} = af_1(S) + bf_2(S)$, where the function $f_i$ corresponds to $\hat{H}_i$, $i = 1, 2$. Hence, $f$ defined in Eq. (3) is also decomposable into $f(S) = af_1(S) + bf_2(S)$. Further let us assume that each term of the decomposition of classical Hamiltonian is mapped into a quantum Hamiltonian as $\hat{H}_1 \rightarrow H_1$ and $\hat{H}_2 \rightarrow H_2$. Keeping all of these in mind and recalling that $\hat{H}$ and $\nu$ appear linearly in Eqs. (11) and (12) and also the linearity of the Schrödinger equation, one can conclude that applying the quantization method to the total classical Hamiltonian one will get

$$\hat{H} = a\hat{H}_1 + b\hat{H}_2 \rightarrow \hat{H} = a\hat{H}_1 + b\hat{H}_2. \quad (42)$$

The quantization mapping induced by our hidden variable model is thus linear.

To apply the above property, let us first formally quantize the following classical Hamiltonian

$$\hat{H} = B(q)p. \quad (43)$$

which is assumed to be one of the term of a physically sensible Hamiltonian, say a term that appears in Eq. (9). Here $B(q)$ is a differentiable function of $q$. Applying the method of quantization developed in this paper one formally has the following quantization mapping:

$$B(q)p \rightarrow \frac{\hat{B}\hat{p} + \hat{p}\hat{B}}{2}. \quad (44)$$

Detail of the calculation is given in Appendix A. In particular, for $B(q) = 1$ one has $p \rightarrow \hat{p} = -i\hbar \partial_q$ and putting $p = 1$, one has $B(q) \rightarrow B(q)$, or formally $q \rightarrow \hat{q} = q$.

Let us apply the above results to quantize the classical Hamiltonian of a particle in electromagnetic field given in Eq. (9). First, Eq. (9) can be expanded into

$$\hat{H} = \frac{p^2}{2m} - \frac{eAeV}{mc} + \frac{e^2A^2}{2mc^2} + eV. \quad (45)$$

Applying the quantization mapping of Eqs. (41) and (44), recalling the linearity of the quantization mapping of Eq. (42), one then obtains

$$\hat{H} = \frac{\hat{p}^2}{2m} - \frac{e}{2mc} (A\hat{p} + \hat{p}A) + \frac{e^2A^2}{2mc^2} + eV,$$

which is equal to Eq. (25), as expected. Hence, in developing Eq. (25) from Eq. (9) using canonical quantization by directly promoting the classical momentum into quantum momentum operator $p \rightarrow \hat{p}$, one is implicitly assuming the ordering given in Eq. (41). In contrast to this, in our hidden variable model for quantization, Eq. (44) is derived rather than assumed.

**IV. MEASUREMENT OF MOMENTUM, POSITION, ANGULAR MOMENTUM WITHOUT WAVE FUNCTION COLLAPSE**

In the present section, we shall apply the modification of classical dynamics developed in the previous section using the type of hidden random variable with probability density given by Eq. (7) to a class of classical model of measurement of momentum, position and angular momentum.
A. Classical measurement

Let us first discuss a class of measurement model in classical mechanics. Let us consider the dynamics of two interacting particles, the first particle with coordinate \( q_1 \) represents the system to be measured and the other with coordinate \( q_2 \) represents the measuring apparatus. Let us suppose that one wants to measure a physical quantity \( A_1 \) of the system. It is a function of the position \( q_1 \) and classical momentum \( p_1 \), \( A_1 = A_1(q_1, p_1) \). To do this, let us choose the following classical measurement-interaction Hamiltonian:

\[
H = gA_1(q_1, p_1)p_2, \tag{46}
\]

where \( g \) is a coupling constant. Let us further assume that the interaction is impulsive so that the individual free Hamiltonians of the particles are ignorable.

\( A_1 \) is thus conserved \( dA_1/dt = \{A_1, H\} = 0 \). The idea is then to correlate the value of \( A_1(q_1, p_1) \) with the classical momentum of the apparatus \( p_2 \) while keeping the value of \( A_1(q_1, p_1) \) unchanged. On the other hand, one also has \( dq_2/dt = \{q_2, H\} = gA_1 \), which can be integrated to give \( q_2(T) = q_2(0) + gA_1 T \), where \( T \) is time span of the measurement-interaction. The value of \( A_1 \) prior to the measurement can thus be inferred from the observation of the initial and final values of \( q_2 \) (the pointer of the apparatus). Since each measurement reveals the value of \( A_1 \) prior-measurement then there is no need to introduce a second apparatus (third particle) to observe the position of the second particle (the pointer of the first apparatus).

Below we shall modify the classical dynamics of ensemble of trajectories generated by classical Hamiltonian of Eq. (46) for measurement of momentum, position and angular momentum following the method developed in the previous section. Momentum and position represent physical quantities with continuous quantum mechanical spectrum. They are also important in view of canonical commutation relation and Heisenberg uncertainty relation. On the other hand, angular momentum represents physical quantity with discrete quantum mechanical spectrum. The crucial problem in this model is that whether one needs further “quantum apparatus” to observe \( q_2(t) \), the position of the apparatus pointer. We shall show that this is not the case. Namely the model with interacting two particles will be shown to be sufficient for this purpose.

We have to mention that in reality, however, the above model with one dimensional apparatus is oversimplified. To this end let us emphasize that the aim of the discussion is only to show that in principle, the method of quantization proposed in the present paper can lead to measurement without wave function collapse and necessitating no external (classical) observer. In this respect, we believe that if it does not work for one degree of freedom then it will be more difficult to expect that it will work for realistic measurement model. Especially, our model excludes the irreversibility of the registration process which can only be done by realistic apparatus plus bath using large degree of freedom. See Ref. 58 for an elaborated discussion of quantum measurement using realistic model of apparatus.

B. Quantum Hamiltonian for the measurement of momentum, position and angular momentum

1. Quantum Hamiltonian for the measurement of momentum

Let us first discuss the case of momentum measurement. One thus puts \( A_1 = p_2 \) into Eq. (46) to have the following measurement-interaction classical Hamiltonian:

\[
H = gP_2 \tag{47}
\]

In impulsive measurement, the Hamilton-Jacobi equation of (1) then reads

\[
\partial_t S + g\partial_{q_1} S \partial_{q_2} S = 0. \tag{48}
\]

On the other hand, inserting Eq. (17) into Eq. (3), one obtains the following classical velocity field for the two particles

\[
\mathbf{v}_1 = g\partial_{q_1} S, \quad \mathbf{v}_2 = g\partial_{q_1} S. \tag{49}
\]

The continuity equation of (2) thus becomes

\[
\partial_t \rho + g\partial_{q_1}(\rho \partial_{q_2} S) + g\partial_{q_2}(\rho \partial_{q_1} S) = 0. \tag{50}
\]

From Eq. (19), \( f \) defined in Eq. (3) takes the form

\[
f_1(S) = g\partial_{q_1} S, \quad f_2(S) = g\partial_{q_2} S. \tag{51}
\]

so that \( \partial_q \cdot f(S) = 2g\partial_{q_1}\partial_{q_2} S \). Hence, Eq. (8) becomes

\[
\rho \rightarrow \rho P(\lambda),
\]

\[
\partial_q S \rightarrow \partial_q S + \frac{\lambda}{2} \frac{\partial_q \rho}{\rho}, \quad i = 1, 2,
\]

\[
\partial_t S \rightarrow \partial_t S + \frac{\lambda}{2} \frac{\partial_t \rho}{\rho} + g\lambda \partial_{q_1}\partial_{q_2} S. \tag{52}
\]

Let us proceed to investigate the change brought by Eq. (52) onto the Hamilton-Jacobi equation of (19) and the corresponding continuity equation of (50) describing ensembles of classical trajectories. First, imposing the first two equations of (52) into Eq. (50) one obtains

\[
\partial_t \rho + g\partial_{q_1}(\rho \partial_{q_2} S) + g\partial_{q_2}(\rho \partial_{q_1} S) + g\lambda \partial_{q_1}\partial_{q_2} \rho = 0. \tag{53}
\]

On the other hand, imposing the last two equations of (52) into Eq. (19) one gets

\[
\partial_t S + g\partial_{q_1} S \partial_{q_2} S - g\lambda \frac{\partial_{q_1}\partial_{q_2} R}{R} + \frac{\lambda}{2}\partial_q \rho
\]

\[
+ g\partial_{q_1}(\rho \partial_{q_2} S) + g\partial_{q_2}(\rho \partial_{q_1} S) + g\lambda \partial_{q_1}\partial_{q_2} \rho = 0, \tag{54}
\]
where we have defined $R = \sqrt{S}$ and used Eq. \eqref{eq:7}. Substituting Eq. \eqref{eq:53} into Eq. \eqref{eq:54} one thus has

$$\partial_t S + g\partial_{q_1} S \partial_{q_2} S - g\lambda^2 \frac{\partial_{q_1} \partial_{q_2} R}{R} = 0.$$  \hspace{1cm} \text{(55)}$$

We have thus pair of coupled equations \eqref{eq:53} and \eqref{eq:55} which still depend on the random variable $\lambda$ whose probability density function is assumed to be given by Eq. \eqref{eq:7}.

One can then see that $S(q, h; t) = S(q, -h; t) = S_Q(q; t)$ satisfies the same differential equation of \eqref{eq:55} with $\lambda$ replaced by $h^2$. Then averaging Eqs. \eqref{eq:53} and \eqref{eq:55} over the distribution of $\lambda = \pm h$ with equal probability one obtains the following pair of equations:

$$\partial_t \rho + g\partial_{q_1} (\rho \partial_{q_2} S_Q) + g\partial_{q_2} (\rho \partial_{q_1} S_Q) = 0,$$

$$\partial_t S_Q + g\partial_{q_1} S_Q \partial_{q_2} S_Q - g h^2 \frac{\partial_{q_1} \partial_{q_2} R}{R} = 0.$$  \hspace{1cm} \text{(56)}$$

Finally, recalling Eq. \eqref{eq:22} that $\Psi_Q = \sqrt{\rho} \exp(i S_Q / \hbar) = R \exp(i S_Q / \hbar)$, the above pair of equations can be combined into the Schrödinger equation $i\hbar \partial_t \Psi_Q = H\Psi_Q$ with measurement-interaction quantum Hamiltonian

$$\hat{H}_p = g p_1 p_2,$$  \hspace{1cm} \text{(57)}$$

where $\hat{p}_i = -i \hbar \partial_{q_i}, \ i = 1, 2$. Again, by construction one has $\rho(q; t) = |\Psi_Q(q; t)|^2$. Moreover, from the upper equation of \eqref{eq:56}, the effective velocity is $f(S_Q)$ where $f$ is given by Eq. \eqref{eq:51} so that it is equal to the actual velocity field of the particles in pilot-wave theory.

2. Quantum Hamiltonian for the measurement of position

Next let us consider the measurement of position. One thus put $A_1 = q_1$ into Eq. \eqref{eq:46} to have the following classical measurement-interaction Hamiltonian:

$$H_q = g q_1 p_2.$$  \hspace{1cm} \text{(58)}$$

The Hamilton-Jacobi equation of \eqref{eq:11} thus reads

$$\partial_t \mathbf{S} + g q_1 \partial_{q_2} \mathbf{S} = 0.$$  \hspace{1cm} \text{(59)}$$

On the other hand, inserting Eq. \eqref{eq:53} into Eq. \eqref{eq:4}, the classical velocity field is given by

$$u_1 = 0, \ u_2 = g q_1.$$  \hspace{1cm} \text{(60)}$$

The above pair of equations provide constraint to the dynamics of the particles. Hence, the continuity equation of \eqref{eq:2} becomes

$$\partial_t \rho + g q_1 \partial_{q_2} \rho = 0.$$  \hspace{1cm} \text{(61)}$$

From Eq. \eqref{eq:60} and the definition of $f$ given in Eq. \eqref{eq:3} one has

$$f_1 = 0, \ f_2 = g q_1,$$  \hspace{1cm} \text{(62)}$$

so that $\partial_q \cdot f = 0$. Equation \eqref{eq:8} thus becomes

$$\rho \mapsto \rho P(\lambda),$$

$$\partial_q S \mapsto \partial_q S + \frac{\lambda \partial_q \rho}{\rho}, \ i = 1, 2,$$

$$\partial_q S \mapsto \partial_q S + \frac{\lambda \partial_q \rho}{\rho}.$$  \hspace{1cm} \text{(63)}$$

Now let us apply the above set of equations to Eqs. \eqref{eq:59} and \eqref{eq:61}. First, imposing the first equation of \eqref{eq:63} into Eq. \eqref{eq:61} one obtains

$$\partial_t \rho + g q_1 \partial_{q_2} \rho = 0,$$  \hspace{1cm} \text{(64)}$$

namely Eq. \eqref{eq:61} is kept unchanged. Next, imposing the last two equations of \eqref{eq:63} into Eq. \eqref{eq:59} one obtains

$$\partial_t S + g q_1 \partial_{q_2} S + \frac{\lambda}{2 \rho} (\partial_t \rho + g q_1 \partial_{q_2} \rho) = 0.$$  \hspace{1cm} \text{(65)}$$

Substituting Eq. \eqref{eq:61} one thus gets

$$\partial_t S + g q_1 \partial_{q_2} S = 0.$$  \hspace{1cm} \text{(66)}$$

Again, Eq. \eqref{eq:59} remains unchanged. We have thus pair of decoupled equations \eqref{eq:64} and \eqref{eq:66}.

Notice then that $\lambda$ does not appear explicitly as parameter. Identifying $S_Q = S$, and defining $\Psi_Q = \sqrt{\rho} \exp(i S_Q / \hbar)$ so that $|\Psi_Q(q; t)|^2 = \rho(q; t)$, the pair of Eqs. \eqref{eq:64} and \eqref{eq:66} can then be combined together into the Schrödinger equation $i\hbar \partial_t \Psi_Q = H\Psi_Q$ with quantum Hamiltonian

$$\hat{H}_q = g q_1 p_2.$$  \hspace{1cm} \text{(67)}$$

Again, one can see from Eq. \eqref{eq:66} that the effective velocity of the particles is $f(S_Q)$ where $f$ is given by Eq. \eqref{eq:22}. Hence, it is again equal to the velocity of the particles in pilot-wave theory.

3. Quantum Hamiltonian for the measurement of angular momentum

Let us proceed to develop the quantum Hamiltonian for the measurement of angular momentum. To make explicit the three dimensional nature of the problem, let us put $q_1 = (x_1, y_1, z_1)$. For simplicity let us first consider the measurement of $z$-part of angular momentum. In this case $A_1$ in Eq. \eqref{eq:46} takes the form

$$A_1 = x_1 \mathbf{p}_{y_1} - y_1 \mathbf{p}_{x_1},$$

where $\mathbf{p}_{x_1}$ is the conjugate momentum of $x_1$ and so on, so that the measurement-interaction classical Hamiltonian of Eq. \eqref{eq:46} reads

$$H_q = g (x_1 \mathbf{p}_{y_1} - y_1 \mathbf{p}_{x_1}) p_2.$$  \hspace{1cm} \text{(68)}$$

The Hamilton-Jacobi equation of \eqref{eq:11} thus becomes

$$\partial_t \mathbf{S} + g (x_1 \partial_{y_1} \mathbf{S} - y_1 \partial_{x_1} \mathbf{S}) \partial_{q_2} \mathbf{S} = 0.$$  \hspace{1cm} \text{(69)}$$
On the other hand, substituting Eq. (68) into Eq. (3), the classical velocity field is given by
\[
\mathbf{v}_x = -gy_1 \partial_y S, \quad \mathbf{v}_y = gx_1 \partial_y S, \quad \mathbf{v}_z = 0,
\]
\[
\mathbf{v}_2 = g(x_1 \partial_y S - y_1 \partial_x S).
\]  
(70)

Hence, the continuity equation of (2) becomes
\[
\partial_t \rho = -gy_1 \partial_y \rho + gx_1 \partial_y \rho + gy_1 \partial_q \rho - y_1 \partial_x \rho = 0.
\]  
(71)

Next, from Eq. (70), f defined in Eq. (3) takes the form
\[
f_x(S) = -gy_1 \partial_y S, \quad f_y(S) = gx_1 \partial_y S, \quad f_z(S) = 0,
\]
\[
f_2(S) = g(x_1 \partial_y S - y_1 \partial_x S).
\]  
(72)

One thus has \( \partial_q \cdot f(S) = 2g(x_1 \partial_q \partial_y S - y_1 \partial_q \partial_x S) \). Substituting this into Eq. (5), one then obtains
\[
\mathbf{v}_x = -gy_1 \partial_y \mathbf{S}, \quad \mathbf{v}_y = gx_1 \partial_y \mathbf{S}, \quad \mathbf{v}_z = 0,
\]
\[
\mathbf{v}_2 = g(x_1 \partial_y \mathbf{S} - y_1 \partial_x \mathbf{S}).
\]  
(73)

Let us proceed to see how the above set of equations modify Eqs. (60) and (71). Imposing the first four equations of (73) into Eq. (71) one obtains, after a simple calculation
\[
\partial_t \rho = -gy_1 \partial_z (\rho \partial_z S) + gx_1 \partial_y (\rho \partial_y S) + gy_1 \partial_q (\rho \partial_q S) - gy_1 \partial_q (\rho \partial_x S) - g\lambda(y_1 \partial_x \partial_q \rho - x_1 \partial_y \partial_q \rho) = 0.
\]  
(74)

On the other hand, imposing the last four equations of (73) into Eq. (60), one has, after an arrangement

\[
\partial_t S + g(x_1 \partial_y S - y_1 \partial_x S) \partial_q S - g\lambda^2(x_1 \partial_y \partial_q R \frac{R}{R} - y_1 \partial_x \partial_q R \frac{R}{R}) + \frac{\lambda}{2\rho} \left( \partial_t \rho - gy_1 \partial_z \rho \right) + gx_1 \partial_y \rho + y_1 \partial_q \rho - gy_1 \partial_q \rho - x_1 \partial_y \partial_q \rho = 0.
\]  
(75)

where \( R = \sqrt{\rho} \) and we have used Eq. (17). Substituting Eq. (74), the last term in the bracket vanishes to give
\[
\partial_t S + g(x_1 \partial_y S - y_1 \partial_x S) \partial_q S - g\lambda^2(x_1 \partial_y \partial_q R \frac{R}{R} - y_1 \partial_x \partial_q R \frac{R}{R}) = 0.
\]  
(76)

One thus has pair of coupled equations (74) and (76) which are parameterized by the random variable \( \lambda \).

Again, one can see that \( S(q, h; t) = S(q, -h; t) = S_Q(q; t) \) satisfies the same differential equation of (76) where \( \lambda^2 \) is replaced by \( h^2 \). Hence, taking the average of Eqs. (74) and (76) over the distribution of \( \lambda = \pm h \) with equal probability as in Eq. (7) gives the following pair of equations:

\[
\partial_t \rho - gy_1 \partial_z (\rho \partial_z S_Q) + gx_1 \partial_y (\rho \partial_y S_Q) + gy_1 \partial_q (\rho \partial_q S_Q) - gy_1 \partial_q (\rho \partial_x S_Q) - y_1 \partial_x \partial_q R = 0.
\]  
(77)

Finally, recalling Eq. (22) that \( \Psi_Q = R \exp(iS_Q/h) \) so that \( \Psi_Q(q; t)^2 = R(q; t)^2 = \rho(q; t) \), the above pair of equations can be recast into the Schrödinger equation

\[
h\partial_t \Psi_Q = H_{PQ} \Psi_Q \quad \text{with quantum Hamiltonian}
\]
\[
H_{PQ} = g\hat{L}_z \hat{p}_2.
\]  
(78)

where \( \hat{L}_z = x_1 \hat{p}_y - y_1 \hat{p}_x = -i(h(x_1 \partial_y - y_1 \partial_x)) \). As expected, \( \hat{L}_z \) is just the \( z \)-component of the (quantum mechanical) angular momentum operator. Moreover, one can again see from the upper equation of (77) that the effective velocity of the particles are \( f(S_Q) \) where \( f \) is given by Eq. (22) so that it is equal to the actual velocity of the particles in pilot-wave theory. The above results can be extended to measurement of angular momentum along \( x \) and \( y \) directions straightforwardly using cyclic permutation among the coordinates \( (x_1, y_1, z_1) \). One will then obtain the Schrödinger equation with quantum Hamiltonian of Eq. (78) where \( \hat{L}_z \) is replaced by the quantum mechanical angular momentum operators along the \( x \) and \( y \) directions, respectively. Moreover, since in principle one can take any direction as \( z \)-axis, then the above result applies as well to angular momentum measurement along any direction.

C. Measurement without wave function collapse and external observer

In the previous section, starting from a class of classical Hamiltonian for the measurement of momentum, position and angular momentum, \( \hat{H} = gA_1 \hat{p}_q \), where \( A_1 \) is the physical quantities being measured, we have arrived at the following Schrödinger equation:

\[
i\hbar \partial_t \Psi_Q = \hat{H} \Psi_Q = g\hat{A}_1 \hat{p}_2 \Psi_Q.
\]  
(79)

where \( \hat{A}_1 \) is a Hermitian operator given by \( \hat{A}_1 = A_1(q, \hat{p}_1) \). Our hidden variable model of quantization thus reproduces the results of canonical quantization. However, unlike the latter, in all of the cases considered, one can identify an effective velocity of the particles which turns out to be equal to the actual velocity of the particles in pilot-wave theory, and the Born’s interpretation of wave function, \( |\Psi_Q(q; t)|^2 = \rho(q; t) \), is valid for
all time, by construction. We can then follow all the argumentation of the pilot-wave theory in describing the process of measurement without wave function collapse 2.

To do this, let us first assume that \( \hat{A}_1 \) has discrete spectrum as the case for measurement of angular momentum. Hence, one has \( \hat{A}_1 \psi_n(q_1) = a_n \psi_n(q_1), n = 0, 1, 2, \ldots \), where \( a_n \) is the real-valued eigenvalue of \( \hat{A}_1 \) and \( \psi_n \) is the corresponding eigenfunction. \( \{ \psi_n \} \) thus makes a complete set of orthonormal functions. Then, ignoring the free Hamiltonian for impulsive measurement-interaction, the Schrödinger equation of (79) has the following general solution:

\[
\Psi_Q(q_1, q_2; t) = \sum_n c_n \psi_n(q_1) \varphi(q_2 - ga_n t),
\]

where \( \varphi(q_2) \) is the initial wave function of the apparatus which is assumed to be sufficiently localized, \( c_n \) is complex number, and \( \varphi(q_1) \doteq \sum_n c_n \psi_n(q_1) \) is the initial wave function of the system. In other words, \( c_n \) is the coefficient of expansion of the initial wave function of the system in term of the orthonormal set of the eigenfunctions of \( \hat{A}_1 \).

For sufficiently large \( g \), \( \varphi_n(q_2) \doteq \varphi(q_2 - ga_n t) \) is not overlapping for different \( n \) and each is correlated to a distinct \( \psi_n(q_1) \). One then argues, following pilot-wave theory 2, that the outcome of single measurement event corresponds to the packet \( \varphi_n(q_2) \) which is actually entered by the apparatus particle. Namely, if \( q_2(t) \) belongs to the support of \( \varphi_n(q_2) \), then we admit that the result of measurement is given by \( a_n \). This can be generalized to \( \hat{A}_1 \) with continuous spectrum, as the general case for the measurement of momentum or position, by replacing the sum in Eq. (80) with integration. As in pilot-wave theory, the probability to find the outcome \( a_n \) is given by \( P(a_n) = |c_n|^2 \), that is the experimentally well-verified Born’s statistical rule. This can be shown as direct implication of \( \rho(q, t) = |\Psi_Q(q; t)|^2 \). The prediction of quantum mechanics is thus reproduced without invoking wave function collapse induced by external (classical) observer.

Notice that the linearity of the Schrödinger equation plays a very pivotal role in the discussion of measurement. The superposition of solution in Eq. (80) is made possible by the linearity of the Schrödinger equation of (79). Since \( \varphi_n(q_2) = \varphi(q_2 - ga_n t) \) refers to the wave function of pointer of the apparatus, then it has been argued within the standard quantum mechanics that Eq. (80) is a superposition of macroscopically distinct states. This leads to the paradox of Schrödinger’s cat suggesting an indefiniteness of the state of macroscopic body which is against our everyday experience (recall that in the standard quantum mechanics, an observable possesses definite value only when the state is an eigenfunction of the observable which is not the case for Eq (80)). It is to save this situation that in the standard quantum mechanics one needs to invoke a wave function collapse to get one of the term in the superposition of Eq. (80).

This paradox however is based on the assumption that the superposition of states of Eq. (80) refers to an individual system (and apparatus) and that the description of an individual system by the wave function is complete 59. In contrast to this, in our dynamical model, the superposition of state, or in general any wave function, describes an ensemble of identically prepared system rather than individual system. Moreover, the description of an individual system by the wave function is not complete: a single system is always described by definite values of position and momentum of the particles and an unbiased random variable \( \lambda = \pm \hbar \). In this respect, the superposition of state in Eq. (80) does not mean macroscopic indefiniteness since at any time, the pointer always possesses definite position. Hence, there is no paradox of Schrödinger’s cat and accordingly there is no need to invoke the wave function collapse to get one term of the superposition as required by the standard quantum mechanics.

Further, one can see in the discussion of the previous subsection that the measurement of position is different from the measurement of momentum and angular momentum. Namely, unlike in the two latter cases, in the case of position measurement, Eq. (63) does not change the classical Hamilton-Jacobi and continuity equations of (60) and (61). Both pair of functions \( (S, p) \) and \( (S, \rho) \) satisfy the same pair of equations, that of Eqs. (60) and (61). Hence, the classical results of measurement is preserved by Eq. (63): there is no quantum correction. Conversely, the Schrödinger equation with quantum Hamiltonian of Eq. (64) can be rewritten into the classical Hamilton-Jacobi equation of (60) and the continuity equation of (61) describing classical dynamics of ensemble of trajectories. One can thus conclude that, as in the case of classical measurement, it is possible to reveal the pre-existing value of the position immediately prior to the measurement. On the other hand, for the cases of measurement of momentum and angular momentum, the results of the measurement are not equal to the pre-existing values possessed by the systems. In this regards, the measurement of position is special. The derivation of the quantum Hamiltonian of measurement of position also shows that the ability to write the dynamics of ensemble of trajectories into the Schrödinger equation is not sufficient to distinguish quantum from classical mechanics.

The above results on position measurement further leads to an important implication. Recall that the results of the measurement of momentum and angular momentum are inferred from the position of the second particle (apparatus pointer). Then one might argue that one needs another, the third particle, as the second apparatus to probe the position of the second particle (the first apparatus). Proceeding in this way thus will lead to infinite regression: one will further need the forth particle (the third apparatus) to probe the position of the third particle (the second apparatus) and so on. In our model, however, since the quantum treatment of the position
measurement is equivalent to the classical treatment revealing the position of the particle prior-measurement, then the second measurement on the position of the second particle (the first apparatus) is in principle not necessary. Namely, the results of position measurement by the second, third, forth apparatuses and so on are all equal to each other.

D. Quantum mechanical observable and quantum-classical correspondence

First, the development of quantum Hamiltonian with measurement-interaction \( H = g \hat{A}_1 \hat{p}_2 \) from the corresponding classical Hamiltonian \( H = q \hat{A}_1 (q_1, p_1) p_2 \) in subsection [V.B] can be formally summarized into the following mapping

\[
\hat{p}_2 \mapsto \hat{p}_2, \quad \hat{A}_1 \mapsto \hat{A}_1.
\]

Hence, it can be regarded as the quantization of classical quantity \( A_1 \) to get the corresponding Hermitian operator \( \hat{A}_1 \) in the context of measurement. \( \hat{A}_1 \) is called as “quantum observable” in the standard formalism of quantum mechanics. As shown in subsection [V.B] for the case where \( \hat{A}_1 \) is momentum, position and angular momentum, the corresponding Hermitian operator \( \hat{A}_1 \) can be obtained formally by the following substitution rule: \( \hat{p}_2 \mapsto \hat{p}_2 = -i\hbar \delta_2 \) and \( \hat{q}_1 \mapsto \hat{q}_1 = q_1 \). For these specific but fundamental dynamical variables, our method thus reproduces the results of canonical quantization. In contrast to the latter, however, the quantization method reported in the present paper is developed by directly modifying classical dynamics of ensemble of measurement parameterized by an unbiased binary random variable \( \lambda = \pm \hbar \). We have thus a continuous and transparent transition from quantum to classical measurement.

Further, recall that \( \{\hat{q}_i, \hat{p}_j\} = i\hbar \delta_{ij} \) leads to the Heisenberg uncertainty relation \( \sigma_q \sigma_p \geq \hbar / 2 \), where \( \sigma_q \) and \( \sigma_p \) are the standard deviation of results of measurement of position and the corresponding conjugate momentum in ensemble of identically prepared systems. Our dynamical model thus shows that the Heisenberg uncertainty relation is a direct implication of modification of classical dynamics for ensemble of trajectories as prescribed by Eq. [S] being applied to measurement. In particular, in the limit where \( S \to \beta \) one smoothly regains the classical dynamics so that \( \sigma_q, \sigma_p \geq 0 \).

An immediate question then arises whether the method of quantization of classical quantity in the context of measurement developed in the present paper can be applied to any classical quantities, namely any function of position and classical momentum \( F = F(q, p) \). To discuss this matter, first, it is not clear even in the classical mechanics whether any arbitrary function \( F(q, p) \) is physically meaningful at all. In reality, hitherto, for spin-less particle, only position, momentum, angular momentum and energy have unambiguous physical meaning. Second, even if \( F(q, p) \) is physically meaningful, it is not clear whether it can be measured directly. This is due to the fact that in reality measurement is done by mapping the properties of the system being measured to non-overlapping subsets of the configuration space. Hence, measurement-interaction is a special type of interaction. This gives a physical limitation to the kind of classical quantities which can be directly measured.

Taking all the above physical aspects aside, in contrast to canonical quantization which in general leads to infinite alternative of Hermitian operators for a given general classical quantity which is the direct implication of replacing c-number by q-number, it is evident that the method of quantization in the context of measurement model with classical Hamiltonian of Eq. [40] presented in this paper, which is based on replacement of c-number by q-number, will give a unique Hermitian observable, if a solution exists. An example of the quantization of classical quantity of the type \( \hat{B}(q)p \) in the context of measurement, where canonical quantization leads to ambiguity, is given in appendix [B].

V. CONCLUSION AND DISCUSSION

We have proposed a quantization method by modifying the classical dynamics of ensemble of trajectories. The deviation from the classical mechanics is characterized by pair of real-valued functions \( S(q, \lambda; t) \) and \( \Omega(q, \lambda; t) \) parameterized by a hidden random variable \( \lambda \) with specific statistical property following the rule of Eq. [S]. In the classical limit, \( S(q, \lambda; t) \) and \( \Omega(q, \lambda; t) \) reduce into the Hamilton principle function \( S(q; t) \) and the classical probability density of the position \( \rho(q; t) \). Given a classical Hamiltonian, the model is applied to system of particles in external potentials, with position-dependent mass, and to a class of classical measurement of momentum, position and angular momentum. We showed that the resulting equations can be put into the Schrödinger equation with unique Hermitian quantum Hamiltonian. The wave function refers to ensemble of system rather than to an individual system. In contrast to the canonical quantization which replaces c-number by q-number implying operator ordering ambiguity, our method is based on replacement of c-number by q-number, thus is free from the problem of operator ordering ambiguity. The canonical commutation relation \( \{\hat{q}_i, \hat{p}_j\} = i\hbar \delta_{ij} \), which lies at the bottom of the canonical quantization, is thus given statistical and dynamical meaning as a modification of classical dynamics of ensemble of trajectories in configuration space parameterized by an unbiased hidden random variable. This offers a conceptually smooth and physically transparent quantum-classical correspondence.

We then identified an effective velocity of the particles which turns out to be equal to the velocity of the particles in pilot-wave theory. However, unlike pilot-wave theory, our model is strictly stochastic, the wave function is not physically real and the Born’s interpretation
of wave function is valid by construction. This allows us to conclude that our model will reproduce the prediction of pilot-wave theory on statistical wave-like pattern in single and double slits experiments and also in tunneling of potential barrier. Moreover, following the argumentation of pilot-wave theory, we then developed the process of measurement without wave function collapse and external observer, reproducing the statistical prediction of quantum mechanics. Since our dynamical model of measurement reduces into the classical dynamics of measurement when \( S \to \sum \), one can conclude that in this limit, the probability of finding of quantum measurement reduces into the probability of being of classical measurement. In this sense, we have thus argued that quantum mechanics is an emergence statistical phenomena [60].

A common pragmatical question against any alternative approaches to quantum mechanics is that whether it offers new testable predictions which can not be calculated using the standard formalism of quantum mechanics. This is a very hard wall to tunnel in view of the pragmatical successes of the quantum mechanics. In our approach, however, since the Schrödinger equation is shown to be emergent corresponding to a specific type of distribution of hidden random variable \( P(\lambda) \) given by Eq. (7), then we may expect that it will lead to new prediction if \( P(\lambda) \) is allowed to deviate from Eq. (7). This, for example can be attained by allowing |\( \lambda \)| to fluctuate around \( \hbar \) with very small yet finite width. We shall elaborate the detail implications of this idea in separate work [10].

Acknowledgments

Appendix A:

Let us quantize a classical Hamiltonian which takes the following form:

\[
\mathcal{H} = B(q)p.
\]  

which is assumed to be part of a physically sensible Hamiltonian, and \( B(q) \) is a differentiable function of \( q \). First, the Hamilton-Jacobi equation of (11) becomes

\[
\partial_t S + B\partial_q S = 0. \tag{A2}
\]

Further, inserting Eq. (A1) into Eq. (3), the classical velocity field is given by

\[
\nu = B. \tag{A3}
\]

This provides a constraint to the motion of the particle. Thus, the continuity equation of (2) reads

\[
\partial_t \rho + \partial_q (B\rho) = 0. \tag{A4} 
\]

Next, from Eq. (A3), \( f \) defined in Eq. (3) is given by \( f = B \), so that Eq. (8) becomes

\[
\rho \mapsto \rho P(\lambda),
\]

\[
\partial_t S \mapsto \partial_q S + \frac{\lambda}{2} \partial^2 \rho \rho,
\]

\[
\partial_t S \mapsto \partial_q S + \frac{\lambda}{2} \partial^2 \rho \rho + \frac{\lambda}{2} \partial_q B. \tag{A5}
\]

Now let us apply the above set of equations to modify Eqs. (A2) and (A3). First, imposing the upper equation of (A5), Eq. (A3) becomes

\[
\partial_t \rho + \partial_q (B\rho) = 0. \tag{A6}
\]

Hence, Eq. (A3) is kept unchanged. Further, imposing the last two equations of (A5) into Eq. (A2) one obtains

\[
\partial_t S + B\partial_q S + \frac{\lambda}{2} (\partial_t \rho + \partial_q (B\rho)) = 0. \tag{A7}
\]

Inserting Eq. (A6) one thus has

\[
\partial_t S + B\partial_q S = 0. \tag{A8}
\]

Namely, Eq. (A2) is also kept unchanged. We have thus pair of decoupled equations (A6) and (A8).

Notice then that \( \lambda \) does not appear explicitly as a parameter of the resulting differential equations. Identifying \( S_Q = S \), and defining \( \Psi_Q = \sqrt{\rho} \exp(iS_Q/\hbar) \) so that \( |\Psi_Q(q; t)|^2 = \rho(q; t) \), the pair of Eqs. (A6) and (A8) can then be combined together into the following Schrödinger equation:

\[
i\hbar \partial_t \Psi_Q = -\frac{i\hbar}{2} (B\partial_q + \partial_q B)\Psi_Q, \tag{A9}
\]

from which one can extract a Hermitian quantum Hamiltonian as

\[
\hat{H} = \frac{B(-i\hbar\partial_q) + (-i\hbar\partial_q)B}{2} = B\hat{p} + \hat{p}B. \tag{A10}
\]

Appendix B:

Let us quantize the classical quantity of the type \( F = B(q)p \) in the context of measurement discussed in Section IV, where \( B(q) \) is a differentiable function of \( q \). One thus put \( A_1 = B(q_1)p_1 \) into Eq. (10) so that the classical Hamiltonian for the interaction-measurement is given by

\[
\mathcal{H} = gB(q_1)p_1p_\perp. \tag{B1}
\]

Notice that \( B \) does not depend on \( q_2 \), the coordinate of the apparatus. The Hamilton-Jacobi equation of (11) thus reads

\[
\partial_t S + gB\partial_q S \partial_q q_2 S = 0. \tag{B2}
\]

Next, inserting Eq. (B1) into Eq. (3) one has

\[
\nu_1 = gB\partial_{q_2} S, \quad \nu_2 = gB\partial_{q_1} S. \tag{B3}
\]
The continuity equation of (2) thus becomes
\[ \partial_t \rho + g \partial_{\rho_1} \left( \rho B \partial_{q_2} \hat{S} \right) + g \partial_{\rho_2} \left( \rho B \partial_{q_1} \hat{S} \right) = 0. \] (B4)

On the other hand, from Eq. (B3), \( f \) defined in Eq. (3) is given by
\[ f_1(\hat{S}) = g B \partial_{q_2} \hat{S}, \quad f_2(\hat{S}) = g B \partial_{q_1} \hat{S}. \] (B5)

Hence, Eq. (8) becomes
\[ \partial_t S + g B \partial_{q_1} S \partial_{q_2} S - \frac{g \lambda}{2} \left( \partial_{q_1} \left( B \partial_{q_2} S \right) + \partial_{q_2} \left( B \partial_{q_1} S \right) \right) + \frac{g \lambda}{2} \partial_{q_1} \left( B \partial_{q_2} S \right) = 0. \] (B6)

Next, imposing the last two equations of Eq. (B6) into Eq. (B4) one has
\[ \partial_t \rho + g \partial_{\rho_1} \left( \rho B \partial_{q_2} S \right) + g \partial_{\rho_2} \left( \rho B \partial_{q_1} S \right) + \frac{g \lambda}{2} \partial_1 \left( B \partial_{q_2} \rho \right) + \frac{g \lambda}{2} \partial_2 \left( B \partial_{q_1} \rho \right) = 0. \] (B7)

We have thus pair of coupled equations (B7) and (B9) which are parameterized by the random variable \( \lambda = \pm \hbar \).

One can again see that \( S(q, h; t) = S(q, -h; t) = S_Q(q; t) \) satisfies the same differential equation of (B9) where \( \lambda^2 \) is replaced by \( \hbar^2 \). Hence, averaging over the fluctuations of \( \lambda = \pm \hbar \) which is assumed to be equally probable, Eqs. (B7) and (B9) become
\[ \partial_t \rho + g \partial_{\rho_1} \left( \rho B \partial_{q_2} S_Q \right) + g \partial_{\rho_2} \left( \rho B \partial_{q_1} S_Q \right) = 0, \]
\[ \partial_t S_Q + g B \partial_{q_1} S_Q \partial_{q_2} S_Q - \frac{g \hbar^2}{2} \left( \partial_{q_1} \left( B \partial_{q_2} R \right) + \partial_{q_2} \left( B \partial_{q_1} R \right) \right) = 0. \] (B10)

Recalling Eq. (22) that \( \Psi_Q = R \exp(i S_Q / \hbar) \), the above pair of coupled equations can be written into the Schrödinger equation \( i \hbar \partial_t \Psi_Q = H \Psi_Q \) with quantum Hamiltonian given by
\[ \hat{H} = \frac{g}{2} \left( B(q_1) \hat{p}_1 + \hat{p}_1 B(q_1) \right) \hat{p}_2. \] (B11)

Hence, comparing the above equation with Eq. (B11), we have the following quantization mapping in the context of measurement:
\[ \frac{p_2}{2} \mapsto \hat{p}_2, \quad B(q_1) \frac{p_1}{2} \mapsto \frac{1}{2} \left( B(q_1) \hat{p}_1 + \hat{p}_1 B(q_1) \right). \] (B12)
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