CONNECTIVITY AND IRREDUCIBILITY OF ALGEBRAIC VARIETIES OF FINITE UNIT NORM TIGHT FRAMES

JAMESON CAHILL, DUSTIN G. MIXON, AND NATE STRAWN

ABSTRACT. In this paper, we settle a long-standing problem on the connectivity of spaces of finite unit norm tight frames (FUNTFs), essentially affirming a conjecture first appearing in [12]. Our central technique involves continuous liftings of paths from the polytope of eigensteps (see [8]) to spaces of FUNTFs. After demonstrating this connectivity result, we refine our analysis to show that the set of nonsingular points on these spaces is also connected, and we use this result to show that spaces of FUNTFs are irreducible in the algebro-geometric sense, and that generic FUNTFs are full spark.

1. Introduction

1.1. Background. Frame theory began with the definition of frames by Duffin and Schaeffer [11], and today, frames provide a rich source of redundant representations and transformations. A frame is a collection of vectors \( \{f_n\}_{n \in I} \) in a Hilbert space \( \mathcal{H} \) for which there exists strictly positive constants \( A \) and \( B \) satisfying

\[
A\|x\|_\mathcal{H}^2 \leq \sum_{n \in I} |\langle x, f_n \rangle_\mathcal{H}|^2 \leq B\|x\|_\mathcal{H}^2
\]

for all \( x \in \mathcal{H} \), where \( \langle \cdot, \cdot \rangle_\mathcal{H} \) is the inner product on \( \mathcal{H} \) which induces the norm \( \| \cdot \|_\mathcal{H} \). We call the frame tight if we can take \( A = B \). If the index set \( I \) is finite, then \( \mathcal{H} \cong \mathbb{R}^d \), where \( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \), and if \( \|f_n\|_\mathcal{H} = 1 \), we say that the frame is unit norm. If the frame is finite, unit norm, and tight, we call it a finite unit norm tight frame (FUNTF). To put it simply, a FUNTF is the collection of column vectors in a matrix whose row vectors are orthogonal with equal norm, and whose column vectors each have unit norm.

Much of the early work on frames focused on infinite-dimensional frames: Fourier frames [11], Gabor frames [4], and wavelet frames [13]. In recent years, finite frames have been studied more rigorously because of their applications (for example, in wireless telecommunications [21] and sigma-delta quantization [3]). While applications for frames abound, some of the most basic questions concerning frames remain unresolved.

1.2. The Frame Homotopy Problem. The sets of real and complex FUNTFs of \( N \) vectors in \( d \) dimensions are denoted

\[
\mathcal{F}^\mathbb{R}_{N,d} \text{ and } \mathcal{F}^\mathbb{C}_{N,d}
\]

2010 Mathematics Subject Classification. Primary 42C15, 47B99; Secondary 14M99.
Key words and phrases. frame theory, real algebraic geometry.
respectively. The Frame Homotopy Problem asks for which $N$ and $d$ these spaces are path-connected. Speculation on all the possible pairs of $N$ and $d$ for which path-connectivity holds was first formally enunciated in Conjectures 7.6 and 7.7 of [12], but the problem itself was first posed by D. R. Larson in a Research Experiences for Undergraduates summer program in 2002. Though there are a large number of degrees of freedom in the spaces of FUNTFs, it has been surprisingly difficult to analytically construct anything but the simplest paths through these spaces. Moreover, many of these spaces have singularities around which the geometry is not yet understood.

The first step forward for the homotopy problem was shown in [12]. The identification of FUNTFs in $\mathbb{R}^2$ with closed planar chains having links of length one in Theorem 3.3 of [2] made it possible to obtain the connectivity result of [12] using connectivity results for these chains. Such connectivity results are readily abundant because of the relevance with the well-studied problem of robotic motion planning. However, the analogue of the characterization in $\mathbb{R}^3$ is not so simple. This is because the identification in $\mathbb{R}^2$ is essentially obtained by the identification of the circle $S^1$ with the real projective space $\mathbb{RP}^1$ via the map $(x_1, x_2) \mapsto (x_1 x_2, x_2^2, x_1 x_2)$. In $\mathbb{R}^3$, the sphere $S^2$ is not identifiable with $\mathbb{RP}^2$. Closed chains are still relevant in this higher dimensional situation, but the configuration space is a subset of products of $\mathbb{RP}^2$ embedded into the space of $3 \times 3$ symmetric matrices. This makes the motion much more difficult to imagine and the problem is no longer relatable to a well-trod domain.

The most recent contribution to the homotopy problem came in 2009 with the work of Giol, Kovalev, Larson, Nguyen, and Tenner [14]. This work demonstrates the connectivity of certain families of projection operators which correspond to FUNTFs of $2d$ vectors in $d$ complex dimensions. Aside from this work, a lack of techniques for constructing paths has basically made it impossible to move forward on the problem over the last decade.

1.3. Main Results. In this paper, we completely resolve the Frame Homotopy Problem. In particular, we establish the following theorems:

**Theorem 1.1.** The space $\mathcal{F}_{N,d}^C$ is path-connected if $N \geq d \geq 1$.

**Theorem 1.2.** The space $\mathcal{F}_{N,d}^R$ is path-connected if $N \geq d + 2 \geq 4$.

The proofs of these results are distinct since the unitary group $U(d)$ is connected and the orthogonal group $O(d)$ is not. This makes the proof of Theorem 1.1 substantially simpler. However, it is clear that Theorem 1.2 implies Theorem 1.1 if any complex FUNTF is path-connected to some real FUNTF. While our methods may be used to produce a reduction, we establish the results independently.

Let $\mathcal{G}_{N,d}^R$ be the set of rank $d$ orthogonal projections on $\mathbb{R}^N$ with all diagonal entries equal to $d/N$. Note that there is a natural projection from $\mathcal{F}_{N,d}^R$ to $\mathcal{G}_{N,d}^R$ (see [12]), so we also solve the main problem presented in [14] as a corollary of Theorems 1.1 and 1.2.

**Corollary 1.3.** The space $\mathcal{G}_{N,d}^C$ is path-connected if $N \geq d \geq 1$ and $\mathcal{G}_{N,d}^R$ is path-connected if $N \geq d + 2 \geq 4$. 

For $N$ and $d$ relatively prime, it was shown in [12] that $\mathcal{F}_{N,d}^\mathbb{F}$ has no singularities for $\mathbb{F} = \mathbb{R}, \mathbb{C}$, and hence Theorems 1.1 and 1.2 immediately imply that such an $\mathcal{F}_{N,d}^\mathbb{F}$ is an irreducible real algebraic variety. This raises the interesting possibility that the singularities in a general space of FUNTFs might be due to the existence of numerous irreducible components in $\mathcal{F}_{N,d}^\mathbb{F}$. Indeed, for $\mathcal{F}_{4,2}^\mathbb{R}$, this is evidently the case given the extensive analysis of this space in [12]. However, by refining our connectivity result, we show that this is simply not the case in general, and that $\mathcal{F}_{4,2}^\mathbb{R}$ is exceptional in this regard:

**Theorem 1.4.** $\mathcal{F}_{N,d}^\mathbb{C}$ is an irreducible algebraic variety for all $N \geq d \geq 1$. $\mathcal{F}_{N,d}^\mathbb{R}$ is an irreducible algebraic variety for all $N \geq d + 2 > 4$.

This is an extremely surprising result and it tells us that the singularities of spaces of FUNTFs either result from the space folding in on itself or developing cusps. However, we do not believe that $\mathcal{F}_{N,d}^\mathbb{F}$ admits any cusp singularities. We are still left with the problem of understanding the local geometry around the singular points of $\mathcal{F}_{N,d}^\mathbb{F}$:

**Problem 1.5.** Describe the local geometry around singular points of $\mathcal{F}_{N,d}^\mathbb{F}$.

Finally, we use these methods to show that generic FUNTFs are full spark (see [1]), i.e., they have the property that every subcollection of $d$ frame elements is linearly independent:

**Theorem 1.6.** A generic frame in $\mathcal{F}_{N,d}^\mathbb{C}$ is full spark for every $d$ and every $N \geq d$. A generic frame in $\mathcal{F}_{N,d}^\mathbb{R}$ is full spark for every $d$ and every $N \geq d$.

1.4. **Organization.** In Section 2, we fix our notation and provide the background on necessary frame theory concepts and results from algebraic geometry. In Section 3, we establish our key technical tool, a lifting lemma for paths in spaces of eigensteps. The proofs of Theorems 1.1 and 1.2 are demonstrated in Section 4, and we refine these results in Section 5 to show that the nonsingular points of $\mathcal{F}_{N,d}^\mathbb{F}$ form a path-connected set as well. In Section 6, we conclude with the consequences of our path-connectivity results, including the proofs of Theorems 1.4 and 1.6.

2. **Prelimaries and Notation**

In general, we work with vectors in $\mathbb{F}^d$, where $\mathbb{F}$ is either the set $\mathbb{R}$ of real numbers or the set $\mathbb{C}$ of complex numbers. We assume that the inner product on $\mathbb{F}^d$ is the standard inner product. We let $I_k$ denote the $k \times k$ identity matrix, and we let $1_k$ denote the vector in $\mathbb{F}^k$ with entries all equal to 1. For any $k \times k$ matrix $A$, we use $\text{diag}(A)$ to denote the vector in $\mathbb{F}^k$ with entries equal to the diagonal entries of $A$ in order, and for any vector $v \in \mathbb{F}^k$, we let $\text{diag}(v)$ denote the $k \times k$ matrix with diagonal entries coinciding with the entries of $v$ and off-diagonal entries equal to zero. For a given collection of vectors $\{v_i\}_{i \in I} \subset \mathbb{F}^k$, we let $\text{span}\{v_i\}_{i \in I}$ denote the linear span of the collection, and we use $\text{span}^\perp\{v_i\}_{i \in I}$ to denote the orthogonal complement of the linear span in $\mathbb{F}^k$.

2.1. **Frame Theory.** For given integers $d$ and $N$, we let $[d]$ denote the first $d$ nonzero integers and $[N]_0$ denote $[N] \cup \{0\}$. We shall use $\mathcal{F}_{N,d}^\mathbb{F}$.
to denote the space of finite unit norm tight frames (FUNTFs). Given an \( F \in \mathcal{F}_{N,d}^d \), we shall interchangeably identify \( F \) with the indexed collection \( \{f_n\}_{n \in [N]} \) and the \( d \times N \) matrix \([f_1 f_2 \cdots f_N]\). Exploiting this identification, the matrix \( FF^* \) is called the frame operator of \( F \). When \( \mathcal{F}_{N,d}^d \) is viewed as a subset of \( d \times N \) matrices \( \mathcal{M}_{d \times N}(F) \), \( \mathcal{F}_{N,d}^d \) is the zero set of a set of quadratic equations (in the real and imaginary parts of the matrix entries), and hence it may be viewed as a real algebraic variety.

Eigensteps \([8]\) are the key tool from which we are able to derive our central technical lemmas. We shall let \( \Lambda_{N,d} \) denote the space of FUNTF eigensteps, or sequences \( \lambda = \{\lambda_{n,i}\}_{i \in [d], n \in [N]} \) of nonincreasing sequences \( \lambda_n \) satisfying

(i) \( \lambda_{0,i} = 0 \) for all \( i \in [d] \)
(ii) \( \lambda_{N,i} = N/d \) for all \( i \in [d] \)
(iii) \( \lambda_n \subseteq \lambda_{n+1} \) for all \( n \in [N-1] \)
(iv) \( 1 + \sum_i \lambda_{n,i} = \sum_i \lambda_{n+1,i} \) for all \( n \in [N-1] \)

where \( a \subseteq b \) means that the interlacing inequalities

\[
a_i \leq b_i \leq a_{i+1}
\]

hold for all \( i \in [d-1] \) and that \( a_d \leq b_d \). Note that \( \Lambda_{N,d} \) is a polytope, and in particular it is convex and hence path-connected. For a given frame \( F \in \mathcal{F}_{N,d}^d \), we shall let \( \lambda(F) \) denote the eigensteps associated with \( F \); that is, \( \{\lambda_{k,i}(F)\}_{i \in [d]} \) is the set of nonincreasing eigenvalues (counting multiplicity) of the frame operator of \( \{f_n\}_{n \in [k]} \) for \( k = 0, \ldots, N \) \(([0] = \emptyset) \). Note that \( \lambda : \mathcal{F}_{N,d}^d \rightarrow \Lambda_{N,d} \) is a well defined mapping, but it is not injective so there can be many frames that have the same eigensteps.

A frame \( F = \{f_n\}_{n \in [N]} \) is said to be orthodecomposable (OD, pronounced “odd”) if there is a nontrivial disjoint partition of \([N]\) into \( S \) and \( T \) such that

\[\text{span}\{f_n\}_{n \in S} = \text{span}^\perp\{f_n\}_{n \in T} \]

The importance of the OD frames is the following characterization first discovered in \([12]\):

**Proposition 2.1.** A point \( F \in \mathcal{F}_{N,d}^d \) is a singularity if and only if \( F \) is OD.

Specifically, this proposition is a consequence of the regular value theorem and Lemmas 4.2 and 4.7 in \([12]\). Thus, all of the non-orthodecomposable (NOD) frames in \( \mathcal{F}_{N,d}^d \) constitute the nonsingular points of the variety.

The correlation network was introduced in \([13]\) to provide a simple characterization of OD frames. The correlation network of a frame \( \gamma(F) = (V,E) \) is the undirected graph with vertex set \( V = [N] \) such that \((i,j) \in E\) if and only if \( \langle f_i, f_j \rangle \neq 0 \).

**Proposition 2.2** (Lemma 3.2.5 in \([13]\)). A frame \( F \) is NOD if and only if \( \gamma(F) \) is connected.

Given a frame \( F \in \mathcal{F}_{N,d}^d \), we define the spark of \( F \) (written \( \text{spark}(F) \)) to be the size of the smallest linearly dependent subset of \( F \). Note that \( \text{spark}(F) \leq d + 1 \); if \( \text{spark}(F) = d + 1 \) then we say that \( F \) is full spark. Observe that a frame is full spark if and only if the following statement holds: If \( k < d \) and \( W \subseteq \mathbb{F}^d \) is a subspace of dimension \( k \), then \( |W \cap F| \leq k \). Therefore it follows that any orthodecomposable frame with \( N > d \) is not full spark.
Our final essential ingredient is the Naimark complement of a Parseval frame \[15\]. A frame \(F\) for \(\mathbb{F}^d\) is said to be a Parseval frame if \(FF^* = I_d\) (see the first chapter of \[10\]). This is equivalent to \(F^*F\) being an orthogonal projection. A frame \(G\) for \(\mathbb{F}^{N-d}\) is called a Naimark complement of a Parseval frame \(F\) if \(F^*F + G^*G = I_N\). Naimark complements preserve many important properties of frames:

**Proposition 2.3.** Suppose \(F\) and \(G\) are Naimark complementary Parseval frames. Then

(i) \(F\) is equal norm if and only if \(G\) is equal norm,
(ii) \(F\) is OD if and only if \(G\) is OD, and
(iii) \(F\) is full spark if and only if \(G\) is full spark.

Here, (i) follows from considering the diagonal entries of \(F^*F\) and \(G^*G\), (ii) follows from Proposition 2.2 since \(\gamma(F) = \gamma(G)\), while (iii) is far less obvious (see Theorem 4(iii) in \[1\] for the essential ingredients of the proof). We extend the notion of Naimark complements to FUNTFs in the obvious way. The following is an important consequence of the Naimark complement:

**Proposition 2.4** (Corollary 7.3 in \[12\]). When \(N > d\), \(\mathcal{F}_N^d\) is connected if and only if \(\mathcal{F}_{N,N-d}^d\) is connected.

With Proposition 2.3(ii), it is easy to imitate the argument for the above result to get the same result for NOD frames:

**Proposition 2.5.** When \(N > d\), the set of NOD frames in \(\mathcal{F}_N^d\) is connected if and only if the set of NOD frames in \(\mathcal{F}_{N,N-d}^d\) is connected.

### 2.2. Algebraic Geometry

A subset \(V \subseteq \mathbb{R}^k\) is called a real algebraic variety if there is a set of polynomials \(\{f_i\}_{i \in I} \subseteq \mathbb{R}[x_1,...,x_k]\) such that \(V = \{x \in \mathbb{R}^k : f_i(x) = 0 \text{ for every } i \in I\}\), where \(\mathbb{R}[x_1,...,x_k]\) is the ring of polynomials in the variables \(x_1,...,x_k\) with real coefficients. By Hilbert’s basis theorem, we may always assume \(|I| < \infty\), and since we are working over the real numbers, we may further assume \(|I| = 1\) by replacing \(\{f_i\}_{i \in I}\) with \(\sum_{i \in I} f_i^2\). We also call a subset \(V \subseteq \mathbb{C}^k\) a real algebraic variety if there is a polynomial \(f \in \mathbb{R}[x_1,...,x_k,y_1,...,y_k]\) such that \(V = \{z = (x_1 + iy_1,...,x_k + iy_k) \in \mathbb{C}^k : f(z) = 0\}\).

By defining closed sets to be real algebraic varieties, we get a topology on \(\mathbb{F}^k\) called the real Zariski topology (note that this is different from the usual Zariski topology on \(\mathbb{C}^k\)). For a subset \(V \subseteq \mathbb{F}^k\) we use the notation \(\mathcal{Z}(V)\) to denote the closure of \(V\) in this topology, that is, \(\mathcal{Z}(V)\) is the smallest variety containing \(V\). We will also use the real Zariski topology of a real algebraic variety \(V \subseteq \mathbb{F}^k\) to mean the subspace topology of the real Zariski topology of \(\mathbb{F}^k\). Note that any set which is closed in the real Zariski topology is also closed in the standard topology, but the converse of this is far from true.

A variety \(V \subseteq \mathbb{F}^k\) is called irreducible if we cannot write \(V = V_1 \cup V_2\) where \(V_1\) and \(V_2\) are proper subvarieties of \(V\). Another notion we will need is that of a singular point of a variety. The definition of singular point is technical (see \[16\]), and we do not need it in its precise form. Intuitively, a point \(v\) on a variety \(V\) is called singular if \(V\) fails to be a smooth manifold at \(v\); that is, there is no neighborhood of \(v\) in \(V\) which is diffeomorphic to some open subset of Euclidean space. A variety is called nonsingular if it has no singular points. If a variety is reducible and path-connected, then any point in the intersection of two irreducible components is a
singularity. Therefore, if a variety is path-connected and nonsingular, then it must be irreducible. However, by Proposition 2.1 we know that \( F^N_{K,d} \) may have singular points in general, so irreducibilty does not follow immediately from connectedness. We shall make use of the following proposition:

**Proposition 2.6.** Suppose \( V \) is an algebraic variety such that

(i) the set of nonsingular points of \( V \) is path-connected, and

(ii) the set of nonsingular points is dense in \( V \) (in the standard topology).

Then \( V \) is an irreducible algebraic variety.

**Proof.** Let \( V_0 \) denote the set of nonsingular points of \( V \). We first claim that (i) implies \( Z(V_0) \) is irreducible. To see this, suppose to the contrary that there are two subvarieties of \( Z(V_0) \), say \( V_1 \) and \( V_2 \), such that \( V_1 \cap V_2 = Z(V_0) \) and there exists \( x \in V_0 \cap V_1 \) and \( y \in V_0 \cap V_2 \). Then the path connecting \( x \) and \( y \) must pass through \( V_1 \cap V_2 \) (since \( Z(V_0) \setminus (V_1 \cap V_2) = (Z(V_0) \setminus V_1) \cup (Z(V_0) \setminus V_2) \) is disconnected). Overall, we have that \( V_1 \cap V_2 \) intersects \( V_0 \) nontrivially, but this contradicts the fact that components must intersect at singular points; this fact follows from Theorem I.5.1 in [10] (as written, Hartshorne’s proof only considers varieties over an algebraically closed field, unlike \( \mathbb{R} \), but this assumption can be removed without affecting the result).

Next, we apply (ii) to get \( \overline{V_0} = V \), where bar denotes closure in the standard topology. Since any Zariski closed set is also closed in the standard topology, we further have \( V = \overline{V_0} \subseteq Z(V_0) \). Moreover, since \( V_0 \subseteq V \) and \( V \) is Zariski closed, the reverse containment also holds: \( Z(V_0) \subseteq V \). As such, \( V = Z(V_0) \) and so \( V \) is irreducible by the previous paragraph. \( \square \)

The converse of Proposition 2.6 is false (see Figure 2.2 for counterexamples).

Given a subset of Euclidean space \( V \subseteq \mathbb{R}^k \), we say \( W \subseteq V \) is *generic* (in \( V \)) if \( W \) contains an open and dense set in the topology induced on \( V \) by the standard topology on \( \mathbb{R}^k \). If \( V \) is an irreducible variety over the complex numbers then any Zariski-open subset of \( V \) is generic, however this is not necessarily the case for irreducible varieties over the real numbers. Indeed, the variety shown in Figure 2.2(b) is an example of an irreducible variety that contains a Zariski-open set that is not dense in the standard topology, namely the complement of the origin. Nonetheless, using the same hypotheses from Proposition 2.6 we can demonstrate the following:

**Proposition 2.7.** Let \( V \) be a real algebraic variety such that the nonsingular points of \( V \) form a dense connected subset. If \( U \) is another real algebraic variety, then either \( V \setminus U \) is either empty or generic in \( V \).

**Proof.** Equivalently, we show that either \( V \subseteq U \) or \( V \cap U \) is nowhere dense in \( V \) in the topology on \( V \) induced by the standard topology. If \( V \cap U \) is nowhere dense in \( V \) in the relative topology, then the statement holds, so suppose that \( V \cap U \) is not nowhere dense in \( V \) in the relative topology. Because \( V \cap U \) is closed and not nowhere dense, this means that \( V \cap U \) contains an open subset \( Q \subseteq V \) in this relative topology. Now, let \( U \) denote an open cover of the nonsingular points of \( V \) so that for each \( W \in U \), there is an analytic coordinate patch on \( W \). If \( Q \) does not intersect some member \( W \) of \( U \), then \( Q \) is entirely contained in the singular points of \( V \), which contradicts the hypothesis that the nonsingular points of \( V \) form a
Figure 1. Real algebraic counterexamples to the converse of Proposition 2.6. (a) An irreducible variety which violates (i); removing the singularity at the origin would disconnect the nonsingular points into three pieces. (b) An irreducible variety which violates (ii); the nonsingular points form a curve which is not dense in the variety, since the variety also contains an isolated point (a singularity).

dense subset of $V$. Thus, $Q$ intersects some $W \in \mathcal{U}$, and hence there is a connected open subset $R \subset Q \cap W$.

Now, there are real polynomials $f$ and $g$ such that

$$V = \{x : f(x) = 0\} \text{ and } U = \{x : g(x) = 0\}$$

since $V$ and $U$ are real algebraic varieties. Thus, $f$ and $g$ must coincide on $R$. Because $W$ admits an analytic coordinate system $\phi$, we have that

$$f \circ \phi - g \circ \phi = 0$$

on $\phi(R)$ as a multivariate analytic function. We claim that $f \circ \phi - g \circ \phi = 0$ on all of $\phi(W)$. This is simply a consequence of the Identity Principle for single-variable analytic functions. That is, suppose $h : \mathbb{C}^k \to \mathbb{C}$ is analytic and that $h(x) = 0$ for all $x \in \mathbb{R}^k$ with $\|x\| < r$. Then by fixing $x_2, \ldots, x_k$ with $x_2^2 + \cdots + x_k^2 < r^2$, we have that

$$h_1(x_1) = h(x_1, x_2, \ldots, x_k)$$

is a one-dimensional analytic function that vanishes on a sequence of points having an accumulation point in $\mathbb{C}$. Consequently, $h_1(x_1) = 0$ for all $x_1 \in \mathbb{C}$, and hence

$$h(x_1, \ldots, x_k) = h(x_2, \ldots, x_k).$$

Now, since $h(x_2, \ldots, x_k)$ still vanishes on a ball contained in $\mathbb{R}^{k-1}$, we may use induction to see that $h = 0$ on all of $\mathbb{C}^k$.

We then have that $f = g$ on all of $W$. If $W' \in \mathcal{U}$ intersects $W$ nontrivially, then the same reasoning as above shows that $f = g$ on $W'$. Now, let $A$ denote the union of all $W' \in \mathcal{U}$ such that $f = g$ on $W'$ and let $B$ denote the union of all open sets $W' \in \mathcal{U}$ such that $f \neq g$ on $W'$. Then $A$ and $B$ are both open, and if $A$ and $B$
intersect nontrivially, there is a nontrivial intersection between some \( W' \) and \( W'' \) in \( \mathcal{U} \) such that \( f = g \) on \( W' \) and \( f \neq g \) on \( W'' \). This is a contradiction by our above observation, so we see that the nonsingular points of \( V \) coincides with the disjoint union of open \( A \) and \( B \), which, by connectedness of \( V \) and nonemptiness of \( A \), implies that \( B \) is empty and hence \( f = g \) on all nonsingular points of \( V \).

Since \( U \) is closed, the above reasoning immediately implies that \( V \subset U \). \( \Box \)

Overall, if the nonsingular points of a real algebraic variety form a dense connected subset, then the nonempty Zariski-open subsets of that variety are generic, as desired. It should be noted that the above proposition employs the topological definition of connectivity, but Corollaries 5.10 and 6.11 say that our set of nonsingular points form a path-connected set. However, it is clear that the nonsingular points form an analytic manifold, and hence these two definitions of connectivity coincide.

3. LIFTING PATHS IN \( \Lambda_{N,d} \) TO \( \mathcal{F}^\mathbb{R}_{N,d} \)

This section provides the technical lemmas involving eigensteps that we will exploit throughout the remainder of the paper. The main idea behind these lemmas is that paths in the eigensteps polytope \( \Lambda_{N,d} \) can be lifted to paths of frames in \( \mathcal{F}^\mathbb{R}_{N,d} \) in such a way that the eigensteps of each frame in the frame path are given by the corresponding point in the eigensteps path.

**Lemma 3.1.** Given any \( F \in \mathcal{F}^\mathbb{R}_{N,d} \) such that \( \lambda = \lambda(F) \) is in the interior of \( \Lambda_{N,d} \) (\( \text{int}(\Lambda_{N,d}) \)), there is a continuous map \( \theta: \text{int}(\Lambda_{N,d}) \to \mathcal{F}^\mathbb{R}_{N,d} \) such that \( \theta(\lambda(F)) = F \) and \( \lambda(\theta(\mu)) = \mu \) for all \( \mu \in \Lambda_{N,d} \).

**Proof.** For \( n \in [N-1] \), let \( \gamma_n(x) = \sum_{i=1}^{d} 1_{\{\lambda_{n,i}\}}(x) \) and note that \( g_n(x) = \gamma_n(x) - \gamma_{n+1}(x) \) enjoys the following properties:

(i) \( g_n(x) = 0 \) for all but a finite set of \( x \) and \( \sum_{x: g_n(x) \neq 0} g_n(x) = 0 \)

(ii) \( g_n(x) \) only takes values in the set \( \{-1, 0, 1\} \) because of the interlacing condition

We let

\[
\mathcal{I}_n = \left\{ 1 + \sum_{x < y; \gamma_n(x) \neq 0} \gamma_n(x) : g_n(x) = 1 \right\}
\]

and

\[
\mathcal{J}_n = \left\{ 1 + \sum_{x < y; \gamma_{n+1}(x) \neq 0} \gamma_{n+1}(x) : g_n(x) = -1 \right\}.
\]

We clearly have that both \( \mathcal{I}_n \) and \( \mathcal{J}_n \) are contained in \([d]\) and \( K_n := |\mathcal{I}_n| = |\mathcal{J}_n| \) by the properties of \( g(x) \). Let \( \sigma_n \) denote the unique permutation that is increasing on \( \mathcal{I}_n \) and \( \mathcal{J}_n \) and such that \( \sigma_n(\mathcal{I}_n) = \mathcal{J}_n \). Define the permutation \( \tau_n \) similarly but using \( \mathcal{J}_n \). Finally, let \( P_n \) and \( Q_n \) denote the permutation matrices corresponding to \( \sigma_n \) and \( \tau_n \) respectively.

Now, because \( \lambda(F) \) is in the interior of \( \Lambda_{N,d} \), we have that \( \mathcal{I}_n = [n+1] \) if \( n < d \), \( \mathcal{I}_n = [d] \) if \( d \leq n < N-d \), and \( \mathcal{I}_n = \{k+1, \ldots, d\} \) for \( n = N-d+k \) where \( 1 \leq k \leq d-1 \). Similarly, we have that \( \mathcal{J}_n = \mathcal{I}_n \) if \( n \leq N-d \) and \( \mathcal{J}_n = \{1\} \cup \{k+2, \ldots, d\} \) for \( n = N-d+k \) if \( 1 \leq k \leq d-2 \) and \( \mathcal{J}_{N-1} = \{1\} \).
For all $\mu$ in the interior of $\Lambda_{N,d}$, we note that the sequences

$$v_{n;\sigma_n(i)}(\mu) = \left[ \frac{\prod_{j \in J_n} (\mu_{n;i} - \mu_{n+1;j})}{\prod_{j \in J_n, j \neq i} (\mu_{n;i} - \mu_{nj})} \right]^{1/2},$$

and

$$w_{n;\tau_n(i)}(\mu) = \left[ \frac{\prod_{j \in J_n} (\mu_{n+1;i} - \mu_{nj})}{\prod_{j \in J_n, j \neq m} (\mu_{n+1;i} - \mu_{n+1;j})} \right]^{1/2},$$

are well-defined rational functions, and hence these sequences are continuous functions of $\mu$ on the interior of $\Lambda_{N,d}$. Letting $F = \{ f_n \}_{n \in [N]}$, by Theorem 7 of [8] there is a unitary (or orthogonal if $F = \mathbb{R}$) $U_1$ and a sequence of block diagonal unitary matrices $V_n$ such that

(i) $f_1 = u_{1;1}$ where $u_{1;1}$ is the first column of $U_1$

(ii) $f_{n+1} = U_n V_n P_n \begin{bmatrix} v_n(\lambda) & 0 \\ 0 & I \end{bmatrix}$

(iii) $U_{n+1} = U_n V_n Q_n \begin{bmatrix} W_n(\lambda) & 0 \\ 0 & I \end{bmatrix}$.

We define $\theta(\mu) = \{ \theta_n(\mu) \}_{n \in [N]}$ by setting $U_1(\mu) = U_1$ and then recursively setting

(i) $\theta_1(\mu) = u_{1;1}$ where $u_{1;1}$ is the first column of $U_1$

(ii) $\theta_{n+1}(\mu) = U_n(\mu) V_n P_n \begin{bmatrix} v_n(\mu) & 0 \\ 0 & I \end{bmatrix}$

(iii) $U_{n+1}(\mu) = U_n(\mu) V_n Q_n \begin{bmatrix} W_n(\mu) & 0 \\ 0 & I \end{bmatrix}$.

By construction, we have that $\theta(\lambda) = F$ and that $\theta$ is continuous on the interior of $\Lambda_{N,d}$. Moreover, the converse part of Theorem 7 of [8] ensures that $\theta(\mu) \in F_{N,d}$ satisfies $\lambda(\theta(\mu)) = \mu$.

**Lemma 3.2.** Suppose $F \in F_{N,d}$ with $N \geq d + 2$, assume that $\lambda = \lambda(F)$ is in the interior of $\Lambda_{M,d}$, and let $\ell : [0, 1] \to \Lambda_{M,d}$ be the linear path $\ell(t) = (1 - t)\lambda + t\mu$ where $\mu \in \Lambda_{N,d}$. Then there is a continuous lifting (or frame path), $\tilde{\ell} : [0, 1] \to F_{N,d}$ such that $(\lambda \circ \tilde{\ell})(t) = \ell(t)$.

**Proof.** It is clear that we may lift $\ell$ to $F_{N,M}$ on $[0, 1)$ using $\theta$ from Lemma 3.1. That is, we set $\tilde{\ell}(t) = \theta(\ell(t))$ for $t \in [0, 1)$. If $\mu$ is also on the interior, it is clear that this path can be extended continuously to all of $[0, 1]$. On the other hand, on the boundary some interlacing inequalities become equalities, and so $\mu_{n;m} = \mu_{n+1;j}$ or $\mu_{nj} = \mu_{n+1;j}$ and the definitions of $v_n$, $w_n$, and $W_n$ involve undefined quantities. We shall now show that these potential discontinuities at the endpoint are removable. Noting that

$$(1 - t)\lambda_{n;j} + t\mu_{n;i} - (1 - t)\lambda_{n+1;j} - t\mu_{n+1;j} = (1 - t)(\lambda_{n;i} - \lambda_{n+1;j})$$

and

$$(1 - t)\lambda_{n;i} + t\mu_{n;i} - (1 - t)\lambda_{nj} - t\mu_{nj} = (1 - t)(\lambda_{n;i} - \lambda_{nj})$$

If $\mu_{n;i} = \mu_{n+1;j}$ and $\mu_{nj} = \mu_{n+1;i}$, we see that any singularities in $v_n(\ell(t))$, $w_n(\ell(t))$, and $W_n(\ell(t))$ at $t = 0$ are removable. Consequently, $\tilde{\ell}(t)$ can be defined
to have a continuous limit at \( t = 1 \) and so that \( \lambda(\tilde{\ell}(1)) = \mu \). This completes the construction. \( \square \)

4. CONNECTIVITY OF \( F_{N,d}^C \)

Perhaps the most obvious motion that a FUNTF may undergo is the “spinning” of an orthogonal pair of vectors in the frame inside of their span. The following lemma simply states that this continuous motion exists in a more general setting:

**Lemma 4.1.** Let \( G = \{g_k\}_{k=1}^{K_1} \) be a tight frame for \( W = \text{span}\{g_k\}_{k=1}^{K_1} \) and suppose \( H = \{h_k\}_{k=1}^{K_2} \) is such that \( F = G \cup H \) is a frame with frame operator \( S \). Let \( U \) be a unitary operator such that \( UW = W \) and take \( G' = \{Ug_k\}_{k=1}^{K_1} \). Then \( F' = G' \cup H \) is a frame with frame operator \( S \).

**Proof.** Because \( G \) is a tight frame for \( W \), the frame operator of \( G \) is a scalar multiple of the orthogonal projection onto \( W \). Now, the frame operator of \( G' \) is this scaled projection conjugated by the unitary \( U \). Since \( U \) leaves \( W \) fixed, it commutes with the orthogonal projection onto \( W \) and hence the frame operator of \( G' \) is the same as the frame operator for \( G \). The frame operator of \( F' \) is the sum of the frame operators of \( G' \) and \( H \), which is also the sum of the frame operators of \( G \) and \( H \), which is the frame operator for \( F \). \( \square \)

We now prove the connectivity of algebraic varieties of complex FUNTFs using the path lifting argument of the previous section.

**Proof of Theorem 1.1.** The connectivity result is well known if \( N = d \) or \( N = d + 1 \), so we only consider the cases \( N \geq d + 2 \) where the interior of \( \Lambda_{N,d} \) is not empty. Let \( F \) and \( G \) belong to \( F_{N,d}^C \). We consider two cases. First, suppose that \( \lambda(F) \) belongs to the interior of \( \Lambda_{N,d} \). By the above lemma, we may connect \( F \) to \( G' \) with \( \lambda(G') = \lambda(F) \). By Theorem 7 of [8], we have that the only difference between \( G \) and \( G' \) is the choice of \( U_1 \) and the \( V_n \). However, since the unitary matrices are connected and products of connected sets are connected, there are continuous paths connecting \( U_1' \) to \( U_1 \) and each \( V_n' \) to \( V_n \). These then induce a continuous path from \( G' \) to \( G \). By traversing this path after the path provided by the lemma, we successfully connect \( F \) to \( G \). On the other hand, if \( F \) and \( G \) both belong to the boundary of \( \Lambda_{N,d} \), we may choose \( H \) from the interior of \( \Lambda_{N,d} \) and connect \( F \) to \( H \) and then \( H \) to \( G \). Traversing these paths in order produces the desired path. \( \square \)

In the complex case, the fiber above a point of \( \Lambda_{N,d} \) is connected because \( V_n \)'s are block diagonal with unitary blocks and this set is connected (it is a product of unitary groups). In the real case, this fiber has at least two components. This makes the proof much more delicate.

**Proof of Theorem 1.2.** We shall show the result using induction on \( N \) inside of an induction on \( d \), with the following induction structure:

(i) For \( d = 2 \), we prove connectivity independently for \( N = 4 \), \( N = 5 \), and \( N = 6 \), and then show that connectivity in the case \( N' \) implies connectivity in the case \( N' + 3 \).

(ii) For \( d > 2 \), we have that connectivity of \( F_{N,d}^R \) for all \( N' \geq d' + 2 \) when \( d' < d \) implies connectivity of \( F_{N,d}^R \) when \( d + 2 \leq N < 2d \).
(iii) For $d > 2$, we prove connectivity independently for $N = 2d, 2d + 1,$ and $2d + 2$, and then show that connectivity in the case of $N'$ implies connectivity in the case of $N = (d + 2) + d + 1 = 2d + 3$ and beyond.

**Case ($N = 2d$):** Let $F$ and $G$ belong to $\mathcal{F}_{N,d}^\mathbb{R}$. Let $\mu$ denote the eigensteps for the frame consisting of two successive copies of the standard orthonormal basis. Note that any other frame with eigensteps $\mu$ consists of a union of two orthonormal bases. We shall now argue that the set of frames consisting of the union of two orthonormal bases is path-connected.

The main idea is to show that pairs of frame vectors may be swapped using a continuous path. If the pair contains a member from both orthonormal systems, then we may simply align those vectors using simultaneous continuous rotations, and then swap the roles of the vectors in the pair while we run this rotation backwards. If the pairs are from the same collection, we first choose a third chaperone vector from the other orthonormal collection, and then run three continuous paths swapping vectors from opposite collections. The first path swaps the first vector of the pair with the chaperone. After this first swap, the second and the first vector are in opposite collections, so we may swap them using a continuous path. Thus, the first and second vector have been swapped and the chaperone vector has returned to its original position.

Since we may perform arbitrary transpositions, we may order the vectors so that the first and last $d$ vectors form an orthonormal basis. Since we may swap vectors within orthonormal collections, we may force each collection to have positive orientation via a continuous path. Once the collections have positive orientation, we may continuously rotate both collections to the standard orthonormal basis and hence all such frames are path-connected to the same point. We conclude that unions of orthonormal bases are connected.

Using Lemma 3.2, we connect both $F$ and $G$ (if not in the interior, first connect to something in the interior) to $H$ and $H'$ with $\lambda(H) = \lambda(H') = \mu$. As we noted earlier, this implies that $H$ and $H'$ are both a union of two orthonormal bases. By the above argument, $H$ may be continuously connected to $H'$, and thus a proper ordering and concatenation of the paths produces a path from $F$ to $G$.

**Case ($N = 2d + 1, 2d + 2$):** Let $\mu$ denote any eigensteps for a frame $F$ such that $\{f_n\}_{n=1}^{d+1} \in \mathcal{F}_{d+1,d}^\mathbb{R}$ and $\{f_n\}_{n=d+1}^{N} \in \mathcal{F}_{N-d-1,d}^\mathbb{R}$. That is, $F$ is a union of two tight subframes. In particular, if $N = 2d + 1$, then $F$ is the union of a simplex and an orthonormal basis. If $N = 2d + 2$, then $F$ is a union of two simplices. First, we show that we may connect any two members of $\mathcal{F}_{d+1,d}^\mathbb{R} \times \mathcal{F}_{N-d-1,d}^\mathbb{R}$ by a path in $\mathcal{F}_{N,d}^\mathbb{R}$.

The central obstruction to connecting $F$ and $G$ in $\mathcal{F}_{d+1,d}^\mathbb{R} \times \mathcal{F}_{N-d-1,d}^\mathbb{R}$ is that $\mathcal{F}_{d+1,d}^\mathbb{R}$ has $2^d$ connected components as shown in [12]. In order to access each of the connected components, we must be able to negate any vector in $F$. To do this, we first align the target vector so that it is orthogonal to a chaperone vector in the other tight subframe (this is accomplished using Lemma 4.1). Now that the target (denoted $a$) and chaperone (denoted $b$) form an orthonormal pair for their span, $W$, we use Lemma 4.1 again to continuously rotate in $W$ until the target vector becomes $b$ and the chaperone becomes $-a$. Lemma 4.1 is now applied to continuously
rotate the entirety of the chaperone’s original subframe to align the target along the chaperone at \( -a \). Rotating the chaperone and its original subframe back to the starting position \( b \), we have successfully negated the target vector without changing the position of the other vectors in the frame. Combining this with the permutation paths (described above in case \( N = 2d \)), we can connect members in this product if \( N = 2d + 1 \) and \( N = 2d + 2 \).

Just as in the \( N = 2d \) case, we now use Lemma 3.2 to connect any given \( F \) and \( G \) to \( H \) and \( H' \) with eigensteps \( \mu \) and then we may continuously connect \( H \) and \( H' \) using the motions described in the previous paragraph.

**Case** (\( N \) implies \( N + d + 1 \)): We simply connect the frame with \( N + d + 1 \) members to a frame such that the first \( d + 1 \) vectors to a particular fixed simplex using the reasoning in the above case. The remaining \( N \) vectors form a member of \( \mathcal{F}^R_{N,d} \) and hence we have that connectivity of \( \mathcal{F}^R_{N,d} \) will imply connectivity of \( \mathcal{F}^R_{N+d+1,d} \).

**Case** (\( N' = d' + 2 \geq 4 \) for all \( d' < d \) implies \( d + 2 \leq N < 2d \)): Suppose \( N \) and \( d \) satisfy \( d + 2 \leq N < 2d \), and let \( N' = N \) and \( d' = N - d \). Then \( N' > d' + 2 \) and our induction step implies \( \mathcal{F}^R_{N',d'} \) is connected. Thus, by Proposition 2.4, \( \mathcal{F}^R_{N,d} \) is connected.

## 5. Connectivity of the nonsingular points of \( \mathcal{F}^R_{N,d} \)

In this section, we refine the results of the previous section to show that, given two NOD frames \( F, G \in \mathcal{F}^R_{N,d} \), there is a continuous path connecting \( F \) and \( G \) in \( \mathcal{F}^R_{N,d} \) that does not pass through the OD frames.

To begin with, we need some lemmas that reveal the structure of the OD frames. Our first lemma essentially allows us to know that a frame is NOD if we can extract a NOD basis from the frame.

**Lemma 5.1** (Proposition 4.2 in [20]). A frame \( F \) is OD if and only if every basis contained in \( F \) is OD.

This next lemma tells us that the eigensteps of an OD frame must be on the boundary of \( \Lambda_{N,d} \). Thus, when we use Lemma 3.2 to lift paths through the interior of \( \Lambda_{N,d} \), we know that the path never crosses an OD frame.

**Lemma 5.2.** Suppose \( F \in \mathcal{F}^R_{N,d} \) is OD then \( \lambda(F) \in \partial \Lambda_{N,d} \).

**Proof.** Since \( F \) is OD, there is an index \( k > 1 \) such that \( f_k \) is orthogonal to \( f_i \) for each \( i < k \). Thus, the nonzero values of \( \lambda_k(F) \) consists of a 1 and all of the nonzero values of \( \lambda_{k-1}(F) \). Since the largest nonzero value of \( \lambda_{k-1} \) is at least 1, this means that \( \lambda_{k-1}(F) = \lambda_{k-1}(F) \) and hence \( \lambda(F) \) is on the boundary of \( \Lambda_{N,d} \).

Our final lemma for this section tells us that if we reorder any NOD frame so that the first \( d \) vectors of the frame form a NOD basis, then there are no OD frames that map to the same eigensteps. This means that when the \( V_n \)'s are being continuously diagonalized in Lemma 3.2 the path avoids the OD frames.

**Lemma 5.3.** Let \( F = \{ f_n \}_{n=1}^N \in \mathcal{F}^R_{N,d} \).

(i) If the first \( d \) vectors in \( F \), \( \{ f_n \}_{n=1}^d \), form a NOD basis, then \( \lambda(F) \) is not in the image of the OD frames under the eigensteps map.
(ii) If $F$ is NOD, then there is a permutation $\sigma$ such that $\{f_{\sigma(n)}\}_{n=1}^d$ is a NOD basis and hence $F' = \{f_{\sigma(n)}\}_{n=1}^N$ has eigensteps which are not in the image of the OD frames under the eigensteps map.

Proof. First, we prove part (i). For the eigensteps of an OD basis, there will be a $k > \lambda$ eigensteps $b_f$ now briefly justify why there is always such an $F$ such that since neither of these spaces will be empty, this implies that $F$ under the eigensteps map. Therefore we may use the reasoning in the preceding paragraph to connect the path $F$ such that $\{h_n\}_{n=1}^d$ is simple in such a way that no frame along this path is OD. Let $\ell(t) = \{f_{\sigma(t)}(n)\}_{n=1}^N$ denote this path and note that $\{f_{\sigma^{-1}(n)}(t)\}_{n=1}^N$ is also a path through NOD frames which takes our original $F$ to a frame which contains a simplex. Let $H$ denote this frame.

Now, we use the exact same chaperone argument for the case of connecting orthonormal bases to rearrange $H$ so that the first $d + 1$ vectors in $H$ form a...
simplex. Note that, while transposing the order of vectors using chaperones, the intermediate frames will always contain a simplex and hence remain NOD.

Finally, we finish the proof by recalling that frames containing a simplex as the first \( d + 1 \) vectors may be connected using a path through NOD frames, and hence we can construct a path between the NOD frames \( F \) and \( G \). \( \square \)

Using the following lemma, it is also not difficult to show that \( \mathcal{F}_{2d,d}^C \) is also path connected:

**Lemma 5.5** (Lemma 2.2 in [7]). Two generic orthonormal bases are full spark.

**Theorem 5.6.** The NOD members of \( \mathcal{F}_{2d,d}^C \) are path-connected.

**Proof.** As in the proof of Theorem 5.4 if both \( F \) and \( G \) have eigensteps which are not in the image of the OD frames, then the paths constructed in the previous section will not pass through OD frames. Without loss of generality, we can now assume that there is an OD frame that has the same eigensteps as \( F \). Since \( F \) is not OD, Lemma 5.3 provides us with a permutation \( \sigma \) such that \( F' = \{f_{\sigma(n)}\}_{n=1}^N \) has that \( \lambda(F') \) is not in the image of the OD frames under the eigensteps map and note that this permutation satisfies \( \sigma(1) = 1 \). By Lemma 5.5, we can choose two orthonormal bases, say \( \{e_i\}_{i=1}^d \) and \( \{u_i\}_{i=1}^d \) such that \( \{e_i\}_{i=1}^d \cup \{u_i\}_{i=1}^d \) is full spark. Reorder these vectors into a frame \( H = \{h_n\}_{n=1}^{2d} \) so that \( h_1 = e_1, h_2 = u_1, h_\sigma^{-1}(2) = u_2 \). Observe that \( \{h_n\}_{n=1}^d \) is a NOD basis so \( \lambda(H) \) is not in the image of the OD frames by Lemma 5.3. Now we can connect \( F' \) to \( H \) so that the path does not go through any OD frames. Next apply \( \sigma^{-1} \) to this path to get a path from \( F \) to \( \sigma^{-1}(H) \) which does not pass through any OD frame. Note that \( \lambda(\sigma^{-1}(H)) \) is not in the image of the OD frames since \( h_{\sigma(1)} = e_1 \) and \( h_{\sigma(2)} = u_2 \) and \( H \) is full spark so \( \{h_{\sigma(n)}\}_{n=1}^d \) is a NOD basis. We have shown that any NOD frame in \( \mathcal{F}_{2d,d}^C \) can be connected to a frame whose eigensteps are not in the image of the OD frames. It follows that we can connect any two NOD frames in this set without passing through an OD frame. \( \square \)

The above result does not apply to the real case when \( N = 2d \). In the process of connecting our \( F' \) to \( H \), in the real case we may only ensure that we can connect \( F' \) to a frame sharing the same eigensteps as \( H \). Additionally, the path constructed in Theorem 1.2 for the case \( N = 2d \) requires that we align frame vectors directly on top of each other. Once this alignment occurs, we have reached an OD frame. This makes the proof for \( \mathcal{F}_{2d,d}^R \) much more involved.

In order to show that we can connect two NOD frames in \( \mathcal{F}_{2d,d}^R \) without passing through an OD frame, we shall still use the unions of two orthonormal bases as the central nexus of our paths. As in Theorem 1.2 once we get to a single union of two orthonormal bases which constitutes a NOD frame (a nontrivial task), we just need to show that we can permute the vectors via continuous paths while remaining NOD. The next lemma demonstrates that continuous permutations can be performed for a particular NOD frame consisting of the union of two orthonormal bases.

**Lemma 5.7.** Suppose \( d \geq 3 \), \( N = 2d \), and fix \( F \in \mathcal{F}_{d-2,d-3}^R \) (in the case \( d = 3 \), this is vacuous). Then the following are true:
(i) There is a \( \xi \in \{-1, 1\}^{d-2} \) such that

\[
U = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1_{d-2} \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 1_{d-2} \\
0 & 0 & \sqrt{\frac{d-3}{d-2}} & F \\
0_{d-3,1} & 0_{d-3,1} & \sqrt{\frac{d-3}{d-2}} & F
\end{pmatrix}, \quad V = \begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1_{d-2} \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & 1_{d-2} \\
0 & 0 & \frac{1}{\sqrt{d-2}} & \xi \\
0_{d-3,1} & 0_{d-3,1} & \frac{1}{\sqrt{d-2}} & \xi
\end{pmatrix}
\]

are both positively oriented orthonormal bases and

\[
F_* = (u_1, v_1, v_2, \ldots, v_d, u_3, \ldots, u_d)
\]

is not OD, where \( u_i \) and \( v_i \) are the \( i \)th columns of \( U \) and \( V \) respectively.

(ii) There is a continuous path through the NOD members of \( F_{2d,d}^\mathbb{R} \) which connects \( F_* \) to

\[
G_* = (v_1, u_1, v_2, \ldots, v_d, u_3, \ldots, u_d).
\]

Proof. First, we let \( \xi \) be a member of \( F_{d-2,1}^\mathbb{R} \) which corresponds to the Naimark complement of \( F \). Now, \( \xi \) is unique up to a global sign factor, so we choose the sign so that

\[
\begin{pmatrix}
\frac{1}{\sqrt{d-2}} \\
\sqrt{\frac{d-3}{d-2}} F
\end{pmatrix}
\]

is a negatively oriented orthonormal basis. Thus the resulting \( U \) and \( V \) are positively oriented orthonormal bases. Now, note that \( u_1 \) has a nonzero inner product with each member of \( v_i \) and hence \( F_* \) is not OD.

We now describe the continuous path connecting \( F_* \) to \( G_* \). First, we note that the frame operator of the collection \( F_*^1 = \{u_1, v_1, u_2, v_2\} \) is \( \text{diag}(2, 1, 1, 0, \ldots, 0) \) and thus the frame operator of \( F_*^2 = \{v_i\}_{i=3} \cup \{v_i\}_{i=3} \) is \( \text{diag}(0, 1, 1, 2, \ldots, 2) \). Consequently, we may independently rotate the members of both \( F_*^1 \) and \( F_*^2 \) in the plane spanned by the standard orthonormal vectors \( e_2 \) and \( e_3 \), and the resulting frame is still always in \( F_{2d,d}^\mathbb{R} \) and is NOD. Our first action is to continuously rotate \( F_*^2 \) in the span of \( e_2 \) and \( e_3 \) to arrive at the frame

\[
\begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & 1_{d-2} & -\frac{1}{\sqrt{2(d-2)}} \xi & -\frac{1}{\sqrt{2(d-2)}} \xi \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 & 1_{d-2} & -\frac{1}{\sqrt{2(d-2)}} \xi & -\frac{1}{\sqrt{2(d-2)}} \xi \\
0 & 0 & \sqrt{\frac{d-3}{d-2}} & F & 0 & 0 & \sqrt{\frac{d-3}{d-2}} & F \\
0_{d-3,1} & 0_{d-3,1} & 0_{d-3,1} & 0_{d-3,1} & 0_{d-3,1} & \sqrt{\frac{d-3}{d-2}} & F & 0_{d-3,1} & 0_{d-3,1}
\end{pmatrix}
\]

The reason for doing this is to avoid a potentially OD frame during a motion that will only involve the first four vectors. At the end of the motion involving only the first four vectors, we shall undo this rotation.

We now restrict our attention to the frame

\[
\begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \\
0 & 0 & \frac{1}{\sqrt{d-2}} & \xi & 0
\end{pmatrix}
\]
By a continuous rotation of $\pi/2$ radians in the plane spanned by $e_2$ and $e_3$, we may move to the frame
\[
\begin{pmatrix}
\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \\
\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0
\end{pmatrix}.
\]
This has permuted the columns by a 4-cycle. We now produce a 3-cycle on the last three columns. First, we may continuously rotate the orthonormal pair consisting of the second and fourth columns by Lemma 4.1 until we arrive at the frame
\[
\begin{pmatrix}
\frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{2}}{2} & 0 \\
0 & 0 & 0 & 1 \\
\frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0
\end{pmatrix}.
\]
Now, the third and fourth columns are orthonormal and we use Lemma 4.1 again to continuously rotate to
\[
\begin{pmatrix}
\frac{\sqrt{2}}{2} & 1 & 0 & \frac{\sqrt{2}}{2} \\
0 & 0 & -1 & 0 \\
\frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2}
\end{pmatrix}.
\]
One final application of Lemma 4.1 on the second and third columns yields
\[
\begin{pmatrix}
\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\
0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\
\frac{\sqrt{2}}{2} & 0 & 0 & -\frac{\sqrt{2}}{2}
\end{pmatrix}.
\]
Clearly, this path lifts to a continuous path taking the collection \{u_1, v_1, u_2, v_2\} to the collection \{v_1, u_1, u_2, v_2\} such that all intervening frames have unit-norm members and frame operator $\text{diag}(2, 1, 1, 0, \ldots, 0)$.

In this paragraph, we justify why this path never meets an OD frame. First, we note that the projection of $F_1^*$ into the span of $\{e_2, e_3\}$ contains an orthogonal pair and the inner product of any vector in the rotation of $F_2^*$ with any vector in $F_1^*$ is equal to the inner products of the vectors if they are first projected into the span of $\{e_2, e_3\}$. Now, simply rotating in the span of $\{e_2, e_3\}$ by some $Q$, it is easy to see that $QF_1^*$ still contains an orthogonal pair and thus the rotated vectors from $F_2^*$ shall have nonzero inner product with at least one vector from this pair. Using this fact and the fact that $QF_1^*$ always has a connected correlation network, we shall have that the entire frame has a connected correlation network and is hence NOD. Furthermore, note that $v_1$ persists as a vector throughout the remaining operations. Since $v_1$ has nonzero correlation with all the rotations of the vectors in $F_2^*$ and $e_1$, $v_1$ together with the rotations of the vectors in $F_2^*$ has a connected correlation network and forms a frame for $\mathbb{R}^d$. Consequently, the full frame is never OD throughout the continuous motions described above.

Finally, we undo the starting rotation on $F_2^*$ to arrive at $G_*$. □

**Theorem 5.8.** The set of NOD frames in $\mathcal{F}_{2d,d}^\mathbb{R}$ is path-connected for $d \geq 3$.

**Proof.** The proof occurs in three parts. First, we show that any NOD frame connects to a frame in $\mathcal{F}^\mathbb{R}_{2d,d}$ containing a simplex for a $(d - 1)$-dimensional subspace of $\mathbb{R}^d$, which is necessarily NOD. In the next step, we show that there is a relatively
simple motion that takes any frame in $\mathcal{F}_{2d,d}^\mathbb{R}$ containing a simplex for a $(d-1)$-dimensional subspace to a NOD union of two orthonormal bases. Finally, we show that the NOD unions of two orthonormal bases form a path-connected set.

Suppose $F \in \mathcal{F}_{2d,d}^\mathbb{R}$ is NOD. Let $\mu$ denote the eigensteps of a frame in $\mathcal{F}_{2d,d}^\mathbb{R}$ such that the first $d$ vectors of the frame form a simplex in a $(d-1)$-dimensional subspace of $\mathbb{R}^d$. Using the same permutation argument as in Theorem 1.2, we may essentially employ Lemma 3.2 to connect $F$ to a frame containing a simplex in $(d-1)$-dimensional subspace of $\mathbb{R}^d$ without crossing an OD frame.

Now, let $\{u_1, \ldots, u_d\}$ denote the frame vectors forming the simplex in a $(d-1)$-dimensional subspace and let $\{v_1, \ldots, v_d\}$ denote the remaining vectors in the frame. By continuous rotation, we may assume (without loss of generality) that $\text{span}\{u_1, \ldots, u_d\} = \text{span}\{e_2, \ldots, e_d\}$. Thus, the frame operator of $\{v_1, \ldots, v_d\}$ is

$$\text{diag} \left( 2, \frac{d - 2}{d - 1}, \ldots, \frac{d - 2}{d - 1} \right),$$

and thus there is a $\xi \in \{1, -1\}^d$ and a simplex $H \in \mathcal{F}_{d,d-1}^\mathbb{R}$ such that the coordinate representation of $\{v_1, \ldots, v_d\}$ is given by

$$\left( \frac{\sqrt{d-2} \xi}{\sqrt{d-2}^d H}, \frac{\sqrt{d-2} \xi}{\sqrt{d-2}^d H} \right).$$

Let us identify $\{u_1, \ldots, u_d\}$ with $H' \in \mathcal{F}_{d,d-1}^\mathbb{R}$, and let $\zeta \in \{-1, 1\}^d$ be a Naimark complement of $H'$. Thus, the coordinate representation of $\{u_1, \ldots, u_d\}$ is given by

$$\left( 0 \cdot \zeta, H' \right),$$

when viewed as a simplex in $\text{span}\{e_2, \ldots, e_d\}$. By continuous rotation, we may assume that

$$\frac{\xi_i}{d} + \frac{d - 1}{d} \langle h'_1, h_i \rangle \neq 0$$

where $h'_1$ is the first vector of $H'$ and $h_i$ is $i$th vector of $H$ for all $i = 1, \ldots, d$. This is because $h'_1$ may be rotated to any point on the unit sphere in $(d-1)$ dimensions, and the intersection of the sphere with the set-theoretic complement of any finite number of hyperplanes is always dense in the sphere. We now use the following path:

$$V(t) = \left( \frac{\sqrt{d-2} \xi}{\sqrt{d-2}^d H}, \frac{\sqrt{d-2} \xi}{\sqrt{d-2}^d H} \right) \quad \text{and} \quad U(t) = \left( \frac{\sqrt{d-2} \xi}{\sqrt{d-2}^d H'}, \frac{\sqrt{d-2} \xi}{\sqrt{d-2}^d H'} \right).$$

By construction, $V(0) = \{v_1, \ldots, v_d\}$ and $U(0) = \{u_1, \ldots, u_d\}$, and $V(1)$ and $U(1)$ are both orthonormal bases. Moreover, the frame operators of $V(t)$ and $U(t)$ are

$$\text{diag} \left( 2 - t, \frac{d - 2 + t}{d - 1}, \ldots, \frac{d - 2 + t}{d - 1} \right) \quad \text{and} \quad \text{diag} \left( t, \frac{d - t}{d - 1}, \ldots, \frac{d - t}{d - 1} \right),$$

so the union of $V(t)$ and $U(t)$ always forms a member of $\mathcal{F}_{2d,d}^\mathbb{R}$. By construction, we also have that $V(t)$ is an equiangular set with nonzero mutual inner products until $t = 1$. Thus, $V(t)$ is NOD for $t \in [0, 1)$ and hence the union of $V(t)$ and $U(t)$ is NOD for $t \in [0, 1)$. By our choice of $h'_1$, $h'_1(1)$ has nonzero inner product with all of the vectors in $V(1)$, and hence we also get that the union of $V(1)$ and $U(1)$ is also NOD. This completes the second step of our proof.
Our final goal is to demonstrate that the set of all NOD unions of two orthonormal bases is connected. This task is divided into two parts. First, we show that if \( \{i_k\}_{k=1}^{d} \) and \( \{j_k\}_{k=1}^{d} \) form a partition of \([2d]\) and \((a, b) \in \{-1, 1\}^{2}\) and if
\[
\mathcal{G}(\{i_k\}_{k=1}^{d}, \{j_k\}_{k=1}^{d}, a, b)
\]
is the set of all NOD frames \( G = \{g_i\}_{i=1}^{2d} \) with \( \{g_{i_k}\}_{k=1}^{d} \) and \( \{g_{j_k}\}_{k=1}^{d} \) both orthonormal bases having orientations \( a \) and \( b \) respectively, then \( \mathcal{G}(\{i_k\}_{k=1}^{d}, \{j_k\}_{k=1}^{d}, a, b) \) is path-connected. The final step is to then use Lemma 5.7 to permute the frame vectors, thus arriving at a NOD frame for which the first \( d \) vectors are the standard orthonormal basis and the last \( d \) vectors form another orthonormal basis.

Let \( G \in \mathcal{G}(\{i_k\}_{k=1}^{d}, \{j_k\}_{k=1}^{d}, a, b) \). For convenience, we first permute so that \( \{i_k\}_{k=1}^{d} = \{k\}_{k=1}^{d} \) and \( \{j_k\}_{k=1}^{d} = \{k\}_{k=d+1}^{2d} \) and \( \{g_i\}_{i=1}^{d} \) and \( \{g_j\}_{j=d+1}^{2d} \) are both positively oriented orthonormal bases. If we construct a path for this configuration, then we may invert the index permutation over the entire path to get a path through \( \mathcal{G}(\{i_k\}_{k=1}^{d}, \{j_k\}_{k=1}^{d}, a, b) \). Additionally, by continuous rotation, we may assume that \( \{g_i\}_{i=1}^{d} \) is the standard orthonormal basis. Now, consider \( g_{d+1} \). There is a continuous rotation \( U(t) \) and an \( \varepsilon > 0 \) such that \( (U(t)g_{d+1})_j \neq 0 \) for all \( j = 1, \ldots, d \), and \( g_{ji} \neq 0 \) implies \( (U(t)g_i)_j \neq 0 \) for all \( i = d + 1, \ldots, 2d \) and all \( t \in (0, \varepsilon) \). Thus, without loss of generality, we may assume that \( g_{j,d+1} \) are all nonzero. We now show that we may assume that all of \( g_{j,d+1} \) are strictly positive. Suppose \( g_{j,d+1} \) is negative, and choose another coordinate index \( i \neq j \). Without moving \( g_{d+1} \), rotate \( g_{d+2} \) so that its nonzero projection onto the span of \( e_i \) and \( e_j \) is not orthogonal or parallel to the projection of \( g_{d+1} \) onto the span of \( e_i \) and \( e_j \). This is possible because \( d > 3 \) and all of the entries of \( g_{d+1} \) are nonzero, and since \( g_{d+1} \) stays fixed and has all nonzero entries, the full frame remains NOD while \( g_{d+2} \) rotates. At this point, we continuously rotate \( \{g_i\}_{i=d+1}^{2d} \) in the span of \( e_1 \) and \( e_2 \) until the \( i \) and \( j \) entries of \( g_{d+1} \) are strictly positive. Since the \( j \)-th entry was negative, the intermediate value theorem tells us that this entry becomes zero at some point during this rotation. Thus, \( e_i \) or \( e_j \) may become orthogonal to \( g_{d+1} \) at some point. However, our positioning of \( g_{d+2} \) ensures that \( g_{d+2} \) will have nonzero inner product with both \( e_i \) and \( e_j \) at these points. Thus, the entire frame remains NOD during this procedure. Once all of the entries of \( g_{d+1} \) are all strictly positive, we continuously rotate \( g_{d+1} \) to the vector \( \frac{1}{\sqrt{d}} \mathbf{1}_d \) while keeping all of the entries of \( g_{d+1} \) strictly positive and hence the full frame remains NOD during this procedure. We are now done if we have path-connectivity of the set of frames such that \( g_i = e_i \) for \( i = 1, \ldots, d, g_{d+1} = \frac{1}{\sqrt{d}} \mathbf{1}_d \), and such that \( \{g_i\}_{i=d+1}^{2d} \) is a positively oriented orthonormal basis. All of these frames are NOD, and the path-connectivity follows from the connectivity of \( SO(d) \).

Since the union of the two orthonormal bases in Lemma 5.7 is NOD, we may swap two vectors between the orthonormal pairs by properly permuting the order of the collections, connecting to the frame exhibited in Lemma 5.7 using our above connectivity result, and then reperforming the continuous swapping in Lemma 5.7. Undoing the wisely chosen permutation produces the desired swapping of frame vectors without passing through OD frames. We then swap until the first \( d \) and last \( d \) vectors form two positively oriented orthonormal bases, and then connect this to a frame with the standard orthonormal basis as the first \( d \) vectors and the constant vectors as the \((d+1)\)st vector without passing through the OD frames. Since this set of frames is path-connected and contains no OD frames, and we can
continuously connect any NOD unions of orthonormal bases by a path through NOD unions of orthonormal bases. This completes the proof.

□

Example 5.9. Here we give an example for the motion in $\mathcal{F}^6_{0.3}$. In Figure 2 we see the motion from a frame consisting of a simplex in $x$-$y$ plane and the subframe with frame operator $\text{diag}(1, 1, 2)$. Without passing through the OD frames, we pull the simplex vectors up and the subframe vectors down to get the second frame, which is a union of two orthonormal bases.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Subfigure (a) illustrates the starting point: a union of a simplex for a two-dimensional subspace of $\mathbb{R}^3$ and a subframe with frame operator $\text{diag}(1, 1, 2)$. Subfigure (b) shows the frame obtained by moving the simplex vectors towards the top pole of the sphere and pushing the other vectors away.}
\end{figure}

For the swapping phase of the motion, we first align the orthonormal basis which complements the subframe with frame operator $\text{diag}(1, 1, 2)$. If we did not do this, the ensuing motion would pass through an OD frame. This is illustrated in Figure 3. Finally, Figure 4 shows how the swapping motion uses successive spins of subsets to swap the position of the vectors labeled $\bigcirc$ and $\triangle$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Subfigure (a) indicates the starting position. Subfigure (b) shows how the orthonormal pair spins so that the remaining motions never pass through an OD frame.}
\end{figure}
Figure 4. Subfigure (a) shows the spinning of the vectors labeled \((\triangle \square \bigcirc \bigtriangleup)\) in Figure 3(b) to the ordering \((\bigtriangleup \triangle \square \bigcirc)\). The remaining figures illustrate how to perform a cycle of \((\triangle \square \bigcirc \bigtriangleup)\) to end at the ordering \((\bigtriangleup \bigcirc \square \triangle)\). In Subfigures (b) through (d), we spin \((\triangle \square),\) \((\bigcirc \square),\) and finally \((\square \triangle)\).

These last corollaries summarize all of the results of this section.

**Corollary 5.10.** If \(N \geq d \geq 1\), then the NOD frames in \(\mathcal{F}_{N,d}^C\) form a path-connected set.

**Corollary 5.11.** If \(N \geq d + 2 > 4\), then the NOD frames in \(\mathcal{F}_{N,d}^R\) form a path-connected set.

6. **Irreducibility of \(\mathcal{F}_{N,d}^R\) and Consequences**

In this section we use the results of the Section 5 to prove Theorem 1.4. Specifically, Corollaries 5.10 and 5.11 give that the nonsingular points of \(\mathcal{F}_{N,d}^C\) are connected, so by Proposition 2.6 it remains to show that the nonsingular points are dense:
Lemma 6.1. The set of nonsingular points of $\mathcal{F}_{N,d}$ is dense in $\mathcal{F}_{N,d}$.

Proof. We show that there are nonsingular points that are arbitrarily close to a given singular point in $\mathcal{F}_{N,d}$. Recall that a singular point corresponds to an OD frame by Proposition 2.1. Given an OD frame, partition it into maximal NOD subsets. If there are only two such parts in this partition, we can pick a vector from each part and rotate this pair in their plane without changing the frame operator by Lemma 4.1. Because of maximal non-orthodecomposability, the frames obtained after a slight rotation of this pair will not be OD. If there are multiple parts, then we can perform this perturbation procedure inductively. Thus, the nonsingular points of $\mathcal{F}_{N,d}$ are arbitrarily close to any singular point of $\mathcal{F}_{N,d}$.

Proof of Theorem 1.4. Using Corollaries 5.10 and 5.11, and Lemma 6.1, the result now follows from Proposition 2.6.

Proof of Theorem 1.6. By Theorem 6.2, the set of full spark frames in $\mathcal{F}_{N,d}$ is nonempty, and hence the real generic property holds by Proposition 2.7, since the nonsingular points of $\mathcal{F}_{N,d}$ form a connected dense subset (Corollaries 5.10 and 5.11 and Lemma 6.1).

7. Discussion

Proposition 2.7 is often identified with the additional property that $V \cap U$ is either a null set or has full measure in $V$. This additional property presupposes the existence of a uniform measure on an irreducible algebraic variety. If the algebraic variety happens to also be a manifold, we can be sure that this uniform distribution exists (see [22], for example), but we have been unable to find a reference in the literature that constructs uniform measures on arbitrary compact real algebraic varieties. Because of this, we may only definitively say that the full spark frames have full measure in the uniform measure of $\mathcal{F}_{N,d}$ when $N$ and $d > 2$ are relatively prime.

Another interesting question is whether these results can be extended to the infinite-dimensional setting. Since the eigensteps construction is our primary tool, the first step of this process would involve a generalization of this construction. One could also study whether similar results hold for sets of frames with different frame operators and different norms of the frame vectors.

Acknowledgements

We would like to thank Bernhard Bodmann, Gitta Kutyniok, and Tim Roemer for organizing the American Institute of Mathematics workshop “Frame Theory intersects Geometry” where this work began. We also thank the American Institute of Mathematics for their great generosity. J. Cahill was supported by NSF Grant No. ATD-1321779. D. G. Mixon was supported by NSF Grant No. DMS-1321779. N. Strawn was supported by NSF Grant No. DMS-10-45153. The views expressed in
this article are those of the authors and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government.

References

1. B. Alexeev, J. Cahill, D. G. Mixon, Full spark frames, J. Fourier Anal. Appl. 18 (2012) 1167–1194.
2. J. J. Benedetto, M. Fickus, Finite normalized tight frames, Adv. Comput. Math. 18 (2003) 357–385.
3. J. J. Benedetto, A. M. Powell, O. Yılmaz, Sigma-Delta ($\Sigma\Delta$) quantization and finite frames, IEEE Trans. Inf. Theory 52 (2006) 1990–2005.
4. J. J. Benedetto, D. F. Walnut, Gabor frames for $L^2$ and related spaces, In: Wavelets: Mathematics and Applications, 1994, pp. 97–162.
5. J. Cahill, Flags, Frames, and Bergman Spaces, Master’s Thesis, San Francisco State University, 2009.
6. J. Cahill, P. G. Casazza, A. Heinecke, A notion of redundancy for infinite frames, Proc. Sampl. Theory Appl. (2011).
7. J. Cahill, P. G. Casazza, J. Peterson, L. Woodland, Phase retrieval by projections, Available online: arXiv:1305.0226
8. J. Cahill, M. Fickus, D. G. Mixon, M. J. Poteet, N. Strawn, Constructing finite frames of a given spectrum and set of lengths, Appl. Comput. Harmon. Anal. 35 (2013) 52–73.
9. J. Cahill, P. G. Casazza, The Paulsen problem in operator theory, Available online: arXiv:1102.2344
10. P. G. Casazza, G. Kutyniok, Finite frames: Theory and applications, Springer, 2013.
11. R. J. Duffin, A. C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952) 341–366.
12. K. Dykema, N. Strawn, Manifold structure of spaces of spherical tight frames, Available online: arXiv:math/0307367
13. I. Daubechies, B. Han, A. Ron, Z. Shen, Framelets: MRA-based constructions of wavelet frames, Appl. Comput. Harmon. Anal. 14 (2003) 1–46.
14. J. Giol, L. V. Kovalev, D. Larson, N. Nguyen, J. E. Tener, Projections and idempotents with fixed diagonal and the homotopy problem for unit tight frames, Available online: arXiv:0906.0139
15. D. Han, D. R. Larson, Frames, bases, and group representations, Mem. Amer. Math. Soc. 174, 2009.
16. R. Hartshorne, Algebraic geometry, Springer, 1977.
17. M. Püschel, J. Kovačević, Real, tight frames with maximal robustness to erasures, Proc. Data Compr. Conf. (2005) 63–72.
18. N. Strawn, Geometry and constructions of finite frames, Master’s Thesis, Texas A&M University, 2007.
19. N. Strawn, Finite frame varieties: nonsingular points, tangent spaces, and explicit local parameterizations, J. Fourier Anal. Appl. 17 (2011) 821–853.
20. N. Strawn, Optimization over finite frame varieties and structured dictionary design, Appl. Comput. Harmon. Anal. 32 (2012) 413–434.
21. T. Strohmer, R. W. Heath, Grassmannian frames with applications to coding and communication, Appl. Comput. Harmon. Anal. 14 (2003) 257–275.
22. D. W. Stroock, Essentials of integration theory for analysis, Springer, 2011.

Department of Mathematics, Duke University, Durham, North Carolina 27708
E-mail address: jcahill@math.duke.edu

Department of Mathematics and Statistics, Air Force Institute of Technology, Wright-Patterson AFB, Ohio 45433
E-mail address: dustin.mixon@afit.edu

Department of Mathematics, Duke University, Durham, North Carolina 27708
E-mail address: nstrawn@math.duke.edu