GENERALIZED FRACTIONAL DIFFERINTEGRAL OPERATORS OF THE $K$-SERIES

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Abstract. In the present paper, we further study the generalized fractional differintegral (integral and differential) operators involving Appell’s function $F_3$ introduced by Saigo-Maeda [9], and are applied to the $K$-Series defined by Gehlot and Ram [3]. On account of the general nature of our main results, a large number of results obtained earlier by several authors such as Ram et al. [7], Saxena et al. [14], Saxena and Saigo [15] and many more follow as special cases.

1. Introduction and Preliminaries

The $K$-Series is defined and represented by Gehlot and Ram [3] as follows:

$$pK_q^{(\beta,\eta)m}[z] = pK_q^{(\beta,\eta)m}(a_1, \ldots, a_p; b_1, \ldots, b_q; (\beta, \eta)_m; z)$$

$$= \sum_{n=0}^{\infty} \prod_{j=1}^{p} (a_j)_n \prod_{r=1}^{q} (b_r)_n \prod_{i=1}^{m} \Gamma (\eta n + \beta_i) z^n,$$

(1.1)

where $a_j, b_r, \beta_i \in \mathbb{C}; \eta_i \in \mathbb{R}, \ (j = 1, \ldots, p; r = 1, \ldots, q; i = 1, \ldots, m)$.

The series (1.1) is valid for none of the parameter $b_r \ (r = 1, \ldots, q)$ being negative integer or zero. If any parameter $a_j \ (j = 1, \ldots, p)$ in (1.1) is zero or negative, then the series terminates into a polynomial in $z$; and

(i) if $p < q + \sum_{i=1}^{m} \eta_i$, then the power series on the right side of (1.1) is absolutely convergent for all $z \in \mathbb{C}$.

Received August 21, 2016. Accepted December 30, 2016.
2010 Mathematics Subject Classification. Primary: 26A33, 33C70; Secondary: 33E20.
Key words and phrases. Generalized fractional calculus operators, $K$-Series, Mittag-Leffler function, Special function.
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(ii) If \( p = q + \sum_{i=1}^{m} \eta_i \) and \( |z| = 1 \), then the series is absolutely convergent for all \( |z| < \prod_{i=1}^{m} (|\eta_i|^n) \), \( |z| = \prod_{i=1}^{m} (|\eta_i|^n) \) and
\[
\Re \left( \sum_{r=1}^{p} b_r + \sum_{i=1}^{m} \beta_i - \sum_{j=1}^{p} a_j \right) > \frac{2+q+m-p}{2}.
\]

Let \( \alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C} \), \( \Re (\gamma) > 0 \) and \( x > 0 \). Then the generalized (Saigo-Maeda) fractional integral operators involving Appell function \( F_3 \) [9, p. 393, Eqs. (4.12) and (4.13)] are defined as follows:

\[
\left( I^{\alpha, \alpha', \beta, \beta', \gamma}_{0+} f \right) (x) = \frac{x^{-\alpha}}{\Gamma (\gamma)} \int_{0}^{x} t^{-\alpha'} (x-t)^{\gamma-1} F_3 \left( \alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{1-x} \right) f(t) \, dt,
\]
(1.2)

and

\[
\left( I^{-\alpha, \alpha', \beta, \beta', \gamma}_{-} f \right) (x) = \frac{x^{-\alpha'}}{\Gamma (\gamma)} \int_{x}^{\infty} t^{-\alpha} (t-x)^{\gamma-1} F_3 \left( \alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{x}{t}, 1 - \frac{1}{1-x} \right) f(t) \, dt.
\]
(1.3)

Also, the corresponding Saigo-Maeda fractional differential operators [9] are given as follows:

\[
\left( D^{\alpha, \alpha', \beta, \beta', \gamma}_{0+} f \right) (x) = \left( I^{-\alpha', -\alpha, -\beta', -\beta, -\gamma}_{0+} f \right) (x) \quad (\Re (\gamma) > 0)
\]
\[
= \left( \frac{d}{dx} \right)^{k} (I^{\alpha', -\alpha, -\beta', -\beta, -\gamma+k}_{0+} f) (x) \quad (\Re (\gamma) > 0; \ k = [\Re (\gamma)] + 1)
\]
\[
= \frac{1}{\Gamma (k - \gamma)} \left( \frac{d}{dx} \right)^{k} (x)^{\alpha'} \int_{0}^{x} (x-t)^{k-\gamma-1} t^{\alpha} \times F_3 \left( -\alpha', -\alpha, k - \beta', -\beta, k - \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{1-x} \right) f(t) \, dt.
\]
(1.4)
and
\[
\begin{aligned}
\left(D_{-}^{\alpha', \beta', \gamma} f\right)(x) &= \left(\int_{-}^{\alpha', -\alpha, -\beta', -\beta, -\gamma} f\right)(x) \quad (\Re(\gamma) > 0) \\
&= \left(-\frac{d}{dx}\right)^k \left(\int_{-}^{\alpha', -\alpha, -\beta', -\beta+k, -\gamma+k} f\right)(x) \quad (\Re(\gamma) > 0; \; k = \lceil\Re(\gamma)\rceil + 1) \\
&= \frac{1}{\Gamma(k - \gamma)} \left(-\frac{d}{dx}\right)^k (x)^{\alpha + \beta - 1} k \alpha' \int_x^\infty (t - x)^{k - \gamma - 1} t^{\alpha'} \\
&\times F_3\left(-\alpha', -\alpha, -\beta', k - \beta, k - \gamma; 1 - \frac{x}{t}, 1 - \frac{t}{x}\right) f(t) dt.
\end{aligned}
\]

(1.5)

Here $F_3(\alpha, \alpha', \beta, \beta'; \gamma; z, \xi)$ is the familiar Appell hypergeometric function of two variables defined by
\[
F_3(\alpha, \alpha', \beta, \beta'; \gamma; z, \xi) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\alpha')_n (\beta)_m (\beta')_n \xi^n}{(\gamma)_{m+n} m! n!},
\]

(1.6)

\(|z| < 1 \quad \text{and} \quad |\xi| < 1\),

where $(\lambda)_n$ denotes the Pochhammer symbol defined by
\[
(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} \lambda (\lambda + 1) \ldots (\lambda + n - 1) & (n \in \mathbb{N}) \\
1 & (n = 0), \end{cases}
\]
it being understood conventionally that $(0)_0=1$ and assumed tacitly that the $\Gamma$-quotient exists (see, for details, [16, p. 21]); definitions and properties of the Appell functions are available in the book [2].

The left-hand sided and right-hand sided generalized fractional integration of the type (1.2) and (1.3) for a power function formulas are given by Saigo-Maeda [9, p. 394, Eqs. (4.18) and (4.19)], as follows:
\[
I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} x^{\rho - 1}
\]

(1.7)

\[
= \Gamma \left[ \frac{\rho + \gamma - \alpha - \alpha' - \beta, \rho + \beta' - \alpha'}{\rho + \gamma - \alpha - \alpha', \rho + \gamma - \alpha' - \beta, \rho + \beta'} \right] x^{\rho - \alpha - \alpha' + \gamma - 1},
\]

where $\Re(\gamma) > 0, \Re(\rho) > \max\{0, \Re(\alpha + \alpha' + \beta - \gamma), \Re(\alpha' - \beta')\}, \; (x > 0)$; and
\[
I_{0+}^{\alpha, \alpha', \beta, \beta', \gamma} x^{\rho - 1}
\]

(1.8)

\[
= \Gamma \left[ \frac{1 + \alpha + \alpha' - \gamma - \rho, 1 + \alpha + \beta' - \gamma - \rho, 1 - \beta - \rho}{1 - \rho, 1 + \alpha + \alpha' + \beta' - \gamma - \rho, 1 + \alpha - \beta - \rho} \right] x^{\rho - \alpha - \alpha' + \gamma - 1},
\]

where $\Re(\gamma) > 0, x > 0, \Re(\rho) < 1 + \min\{\Re(-\beta), \Re(\alpha + \alpha' - \gamma), \Re(\alpha + \beta' - \gamma)\}$. 

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The symbol occurring in (1.7) and (1.8) is given by
\[ \Gamma\left( a, b, c, d, e, f \right) = \frac{\Gamma(a) \Gamma(b) \Gamma(c)}{\Gamma(d) \Gamma(e) \Gamma(f)}. \]

2. Generalized Fractional Integration formulas of the K-Series

In this section we will establish the left-sided and right-sided Saigo-Maeda fractional integration formulas for the K-series.

**Theorem 2.1.** Let \( \alpha, \alpha', \delta, \delta', \gamma \in \mathbb{C}, a \in \mathbb{R}, x > 0, \beta_1 \in \mathbb{C}, \eta_1 \in \mathbb{R}, \)
and the convergent conditions (i) and (ii) of K-series into the account of (1.1) be also satisfied. Then the following formula holds true:
\[
\left( I_{0+}^{\alpha, \alpha', \delta, \delta', \gamma} \left[ t^{\beta_1-1} K_q^{\beta, \eta_1} (at^n) \right] \right)(x) \\
= x^{\beta_1-\alpha-\alpha'+\gamma-1} \sum_{n=0}^{\infty} \prod_{p=1}^{q} (a_j)_n \left(a x^n \right)^n \prod_{r=1}^{\eta_1} (b_r)_n \prod_{m=1}^{\eta_1} \Gamma(\eta_1 n + \beta_1) \\
\times \frac{\Gamma(\eta_1 n + \beta_1 - \alpha - \alpha' - \delta + \gamma) \Gamma(\eta_1 n + \beta_1 - \alpha' + \delta') \Gamma(\eta_1 n + \beta_1) \Gamma(\eta_1 n + \beta_1 + \delta')}{\Gamma(\eta_1 n + \beta_1 - \alpha - \alpha' + \gamma) \Gamma(\eta_1 n + \beta_1 - \alpha' - \delta + \gamma) \Gamma(\eta_1 n + \beta_1 + \delta')}.
\]

**Proof.** By using (1.1), we have
\[
\left( I_{0+}^{\alpha, \alpha', \delta, \delta', \gamma} \left[ t^{\beta_1-1} K_q^{\beta, \eta_1} (at^n) \right] \right)(x) \\
= \left( I_{0+}^{\alpha, \alpha', \delta, \delta', \gamma} \left[ t^{\beta_1-1} \sum_{n=0}^{\infty} \prod_{p=1}^{q} (a_j)_n \left(a x^n \right)^n \prod_{r=1}^{\eta_1} (b_r)_n \prod_{m=1}^{\eta_1} \Gamma(\eta_1 n + \beta_1) \right] \right)(x),
\]
whose right-side, on interchanging the order of the integration and summation, becomes
\[
\sum_{n=0}^{\infty} \prod_{p=1}^{q} (a_j)_n \left(a x^n \right)^n \prod_{r=1}^{\eta_1} (b_r)_n \prod_{m=1}^{\eta_1} \Gamma(\eta_1 n + \beta_1) \left( I_{0+}^{\alpha, \alpha', \delta, \delta', \gamma} t^{(\eta_1 n + \beta_1)-1} \right)(x).
\]
Using (1.7) and rearranging the terms, we get
\[
= x^{\beta_1-\alpha-\alpha'+\gamma-1} \sum_{n=0}^{\infty} \prod_{p=1}^{q} (a_j)_n \left(a x^n \right)^n \prod_{r=1}^{\eta_1} (b_r)_n \prod_{m=1}^{\eta_1} \Gamma(\eta_1 n + \beta_1) \times \frac{\Gamma(\eta_1 n + \beta_1 - \alpha - \alpha' - \delta + \gamma) \Gamma(\eta_1 n + \beta_1 - \alpha' + \delta') \Gamma(\eta_1 n + \beta_1) \Gamma(\eta_1 n + \beta_1 + \delta')}{\Gamma(\eta_1 n + \beta_1 - \alpha - \alpha' + \gamma) \Gamma(\eta_1 n + \beta_1 - \alpha' - \delta + \gamma) \Gamma(\eta_1 n + \beta_1 + \delta')}.
\]
This competes the proof. \( \square \)
If we take \( \alpha = \alpha + \delta, \alpha' = \delta' = 0, \delta = -\mu \) and \( \gamma = \alpha \) in (2.1), we get a known result obtained by Ram et al. [7, p. 408, Eq. (3.1)], as in the following corollary.

**Corollary 2.2.** Let \( \alpha, \delta, \mu \in \mathbb{C}, \Re(\alpha) > 0, a \in \mathbb{R}, \beta_1 \in \mathbb{C}, \eta_1 \in \mathbb{R}, x > 0 \), and the convergent conditions (i) and (ii) of K-series into the account of (1.1) be also satisfied. Then we obtain following result:

\[
\left( I_{a+}^{\alpha,\delta,\mu} \left[ \frac{I_{b}^{\beta} \left( \mathbb{C}_{q}^{\beta}(\eta,m) \right) (at^{m})}{\Gamma(\eta_n + \beta_1 - \delta + \mu)} \right] \right)(x)
\]

\[
= x^{-\beta_1 - \delta - 1} \sum_{n=0}^{\infty} \frac{\prod_{r=1}^{p} (a_j)_n (ax^{-\eta})^n}{\prod_{r=1}^{p} (b_r)_n \prod_{i=1}^{m} \Gamma(\eta_n + \beta_i)} \frac{\Gamma(\eta_1 n + \beta_1 - \delta + \mu)}{\Gamma(\eta_1 n + \beta_1 + \alpha + \mu)}.
\]

**Remark 2.3.** If we take \( p = q = 1, a_1 = \rho, b_1 = 1 \) and \( \delta = -\alpha \) in the above equation (2.2), we get the result for the Mittag-Leffler function \( E_{\rho}^{\alpha,\beta,\gamma} (z) \) given by Saxena et al. [14, Eq. (2.1)]. Further, if we set \( m = 1 \) then (2.2) reduces to the result for the function \( E_{\rho}^{\alpha,\beta,\gamma} (z) \) given by Saxena and Saigo [15, Eq. (14)].

**Theorem 2.4.** Let \( \alpha, \alpha', \delta, \delta', \gamma \in \mathbb{C}, a \in \mathbb{R}, \beta_1 \in \mathbb{C}, \eta_1 \in \mathbb{R}, x > 0 \), and the convergent conditions (i) and (ii) of K-series into the account of (1.1) be also satisfied. Then the following formula holds true:

\[
\left( I_{a}^{\alpha,\alpha',\delta,\delta',\gamma} \left[ t^{-\gamma-\beta_1} \frac{I_{b}^{\beta} \left( \mathbb{C}_{q}^{\beta}(\eta,m) \right) (at^{m})}{\Gamma(\eta_n + \beta_1 - \delta + \gamma)} \right] \right)(x)
\]

\[
= x^{-\beta_1 - \alpha - \alpha'} \sum_{n=0}^{\infty} \frac{\prod_{r=1}^{p} (a_j)_n (ax^{-\eta})^n}{\prod_{r=1}^{p} (b_r)_n \prod_{i=1}^{m} \Gamma(\eta_n + \beta_i)} \frac{\Gamma(\eta_1 n + \beta_1 + \alpha + \gamma)}{\Gamma(\eta_1 n + \beta_1 + \alpha - \delta + \gamma)}
\]

**Proof.** By using (1.1), we arrive at

\[
\left( I_{a}^{\alpha,\alpha',\delta,\delta',\gamma} \left[ t^{-\gamma-\beta_1} \frac{I_{b}^{\beta} \left( \mathbb{C}_{q}^{\beta}(\eta,m) \right) (at^{m})}{\Gamma(\eta_n + \beta_1 - \delta + \gamma)} \right] \right)(x)
\]

\[
= \left( I_{a}^{\alpha,\alpha',\delta,\delta',\gamma} \left[ t^{-\gamma-\beta_1} \sum_{n=0}^{\infty} \frac{\prod_{r=1}^{p} (a_j)_n (ax^{-\eta})^n}{\prod_{r=1}^{p} (b_r)_n \prod_{i=1}^{m} \Gamma(\eta_n + \beta_i)} \right] \right)(x),
\]

next, interchanging the order of the integration and summation, we have

\[
= \sum_{n=0}^{\infty} \frac{\prod_{r=1}^{p} (a_j)_n (a)^n}{\prod_{r=1}^{p} (b_r)_n \prod_{i=1}^{m} \Gamma(\eta_n + \beta_i)} \left( I_{-a}^{\alpha,\alpha',\delta,\delta',\gamma} \left[ (1-(\eta_1 n + \beta_1)\gamma)^{-1} \right] x \right) (x).
\]
Using (1.8) and rearranging the terms, we get
\[ x^{-\beta_1-\alpha-\alpha'} \sum_{n=0}^{\infty} \prod_{j=1}^{p} (a_j)_n \frac{(ax^{-m})^n}{\prod_{r=1}^{q} (b_r)_n \prod_{i=1}^{m} \Gamma (\eta_i n + \beta_i) \Gamma (\eta_1 n + \beta_1 + \alpha + \delta') \Gamma (\eta_1 n + \beta_1 + \alpha + \delta + \gamma) \Gamma (\eta_1 n + \beta_1 + \alpha + \delta + \gamma) \Gamma (\eta_1 n + \beta_1 + \alpha + \delta + \gamma)} \times \Gamma (\eta_1 n + \beta_1 + \alpha + \delta + \gamma). \]
This completes the proof.

If we take \( \alpha = \alpha + \delta, \alpha' = \delta' = 0, \delta = -\mu \) and \( \gamma = \alpha \) in (2.3), we obtain a known result given by Ram et al. [7, p. 409, Eq. (4.1)] as follows:

**Corollary 2.5.** Let \( \alpha, \delta, \mu \in \mathbb{C}, \Re(\alpha) > 0, a \in \mathbb{R}, \) the convergent conditions (i) and (ii) of \( K \)-series into the account of (1.1) be also satisfied, and \( x > 0 \). Then we obtain
\[ (I_{-}^{\alpha, \delta, \mu} \left[ t^{-\beta_1} pK_{\eta}(\beta, \eta) m (at^{-\eta_1}) \right] (x)) = x^{-\beta_1-\alpha-\delta} \sum_{n=0}^{\infty} \prod_{j=1}^{p} (a_j)_n \frac{(ax^{-m})^n}{\prod_{r=1}^{q} (b_r)_n \prod_{i=1}^{m} \Gamma (\eta_i n + \beta_i) \Gamma (\eta_1 n + \beta_1 + \alpha + \delta) \Gamma (\eta_1 n + \beta_1 + \alpha + \mu) \Gamma (\eta_1 n + \beta_1 + 2\alpha + \delta + \mu)}. \]

**Remark 2.6.** If we take \( p = q = 1, a_1 = \rho, b_1 = 1 \) and \( \delta = -\alpha \) in (2.4), then we get the result for the Mittag-Leffler function \( E_{\rho}^\eta [(\beta, \eta)_m; z] \) given by Saxena et al. [14, Eqn. (2.4)]. Further, if we set \( m = 1 \) then (2.4) reduces to the result for the function \( E_{\rho, \beta}^{\eta, \beta} [z] \) given by Saxena and Saigo [15, Eq. (23)].

**Remark 2.7.** If we set \( \delta = -\alpha \) in Corollary 1.1 and 2.1 then we can easily obtain results concerning Riemann-Liouville fractional integral operators.

### 3. Generalized Fractional Derivative formulas of the \( K \)-Series

In this section we will establish the left- and right-sided Saigo-Maeda fractional differentiation formulas for the \( K \)-series.

**Theorem 3.1.** Let \( \alpha, \alpha', \delta, \delta', \gamma \in \mathbb{C}, a \in \mathbb{R}, \beta_1 \in \mathbb{C}, \eta_1 \in \mathbb{R}, x > 0, \) and the convergent conditions (i) and (ii) of \( K \)-series into the account
of (1.1) are also satisfied. Then the following formula holds true:
\[
\left(D_{0+}^{\alpha,\alpha',\delta',\gamma} \left[t^{\beta_1-1} \int_0^x M_r^{(\beta,\eta)n} (at^n) \right]\right) (x)
\]
\[
= x^{\beta_1+\alpha+\alpha'-\gamma-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n (ax^n)_n}{\prod_{r=1}^q (b_r)_n \prod_{i=2}^m \Gamma(\eta n + \beta_i)}
\]
(3.1)
\[
\times \frac{\Gamma(\eta n + \beta_1 + \alpha + \alpha' + \delta' - \gamma)}{\Gamma(\eta n + \beta_1 + \alpha + \alpha' - \gamma)} \frac{\Gamma(\eta n + \beta_1 + \alpha - \delta)}{\Gamma(\eta n + \beta_1 - \delta)}.
\]

Proof. By using (1.1) and (1.4), we have
\[
\left(D_{0+}^{\alpha,\alpha',\delta',\gamma} \left[t^{\beta_1-1} \int_0^x M_r^{(\beta,\eta)n} (at^n) \right]\right) (x)
\]
\[
= \left(D_{0+}^{\alpha,\alpha',\delta',\gamma} \left[t^{\beta_1-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n (at^n)_n}{\prod_{r=1}^q (b_r)_n \prod_{i=1}^m \Gamma(\eta n + \beta_i)} \right]\right) (x),
\]

now, interchanging the order of the differentiation and summation, we have
\[
= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n (a)^n}{\prod_{r=1}^q (b_r)_n \prod_{i=1}^m \Gamma(\eta n + \beta_i)} \left(D_{0+}^{\alpha,\alpha',\delta',\gamma} t^{(\eta n + \beta_1)-1} \right) (x).
\]

Using the relation (1.4) and taking (1.7) into account, then after rearranging the terms and little simplification, we get the expression as in the right-hand side of (3.1). This completes the proof. \(\square\)

If we take \(\alpha = \alpha + \delta, \alpha' = \delta' = 0, \delta = -\mu\) and \(\gamma = \alpha\) in (3.1), we get known result obtained by Ram et al. [7, p. 410, eqn. (5.1)], as given by

**Corollary 3.2.** Let \(\alpha, \delta, \mu \in \mathbb{C}, \Re(\alpha) > 0, a \in \mathbb{R}, \beta_1 \in \mathbb{C}, \eta_1 \in \mathbb{R}, x > 0, \) and the convergent conditions (i) and (ii) of K-series into the account of (1.1) be also satisfied. Then we obtain the following formula:
\[
\left(D_{0+}^{\alpha,\delta,\mu} \left[t^{\beta_1-1} \int_0^x M_r^{(\beta,\eta)n} (at^n) \right]\right) (x)
\]
\[
= x^{\beta_1+\delta-1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^p (a_j)_n (ax^n)_n}{\prod_{r=1}^q (b_r)_n \prod_{i=2}^m \Gamma(\eta n + \beta_i)}
\]
\[
\times \frac{\Gamma(\eta n + \beta_1 + \alpha + \delta + \mu)}{\Gamma(\eta n + \beta_1 + \mu \Gamma(\eta n + \beta_1 + \delta)}.
\]
(3.2)

**Remark 3.3.** If we take \(p = q = 1, a_1 = \rho, b_1 = 1\) and \(\delta = -\alpha\) in the above corollary, then we get the result for the Mittag-Leffler function \(E_\rho \left[(\beta,\eta)n; z\right]\) given by Saxena et al. [14, Eq. (2.6)]. Further, if we set
Let $\alpha, \alpha', \delta, \delta', \gamma \in \mathbb{C}$, $a \in \mathbb{R}$, $x > 0$, $\beta_1 \in \mathbb{C}$, $\eta_1 \in \mathbb{R}$, and the convergent conditions (i) and (ii) of $K$-series into the account of (1.1) be also satisfied. Then the following result holds true:

\[
\begin{align*}
&\left( D_{-}^{\alpha, \alpha', \delta', \gamma} \left[ t^{\gamma - \beta_1} \left( pK_q^{(\beta, \eta)m} (at^{-\eta}) \right) \right] \right) (x) \\
&= x^{-\beta_1 + \alpha + \alpha'} \sum_{n=0}^{\infty} \prod_{r=1}^{p}(a_r)_{n} \left( ax^{-\eta_1} \right)^n \prod_{i=1}^{n} \Gamma (\eta_n + \beta_i) \\
&\times \frac{\Gamma (\eta_1 n + \beta_1 - \alpha - \alpha') \Gamma (\eta_1 n + \beta_1 - \alpha' - \delta) \Gamma (\eta_1 n + \beta_1 + \delta' - \gamma)}{\Gamma (\eta_1 n + \beta_1 - \gamma) \Gamma (\eta_1 n + \beta_1 - \alpha - \alpha' - \delta) \Gamma (\eta_1 n + \beta_1 - \alpha' + \delta' - \gamma)} .
\end{align*}
\]

(3.3)

Proof. By using (1.1) and (1.5), we have

\[
\begin{align*}
&\left( D_{-}^{\alpha, \alpha', \delta', \gamma} \left[ t^{\gamma - \beta_1} \left( pK_q^{(\beta, \eta)m} (at^{-\eta}) \right) \right] \right) (x) \\
&= \left( D_{-}^{\alpha, \alpha', \delta', \gamma} \left[ t^{\gamma - \beta_1} \sum_{n=0}^{\infty} \prod_{r=1}^{p}(a_r)_{n} \left( at^{-\eta_1} \right)^n \prod_{i=1}^{n} \Gamma (\eta_n + \beta_i) \right] \right) (x) ,
\end{align*}
\]

whose right-side, interchanging the order of the differentiation and summation, becomes

\[
\begin{align*}
&\sum_{n=0}^{\infty} \prod_{r=1}^{p}(a_r)_{n} \left( a_n \right)^n \prod_{i=1}^{n} \Gamma (\eta_n + \beta_i) \left( D_{-}^{\alpha, \alpha', \delta', \gamma} t^{\gamma - (\eta_1 n + \beta_1)} \right) (x) ,
\end{align*}
\]

by using the relation (1.5), and taking into (1.8), we arrive at

\[
\begin{align*}
&= x^{-\beta_1 + \alpha + \alpha'} \sum_{n=0}^{\infty} \prod_{r=1}^{p}(a_r)_{n} \left( ax^{-\eta_1} \right)^n \prod_{i=1}^{n} \Gamma (\eta_n + \beta_i) \\
&\times \frac{\Gamma (\eta_1 n + \beta_1 - \alpha - \alpha') \Gamma (\eta_1 n + \beta_1 - \alpha' - \delta) \Gamma (\eta_1 n + \beta_1 + \delta' - \gamma)}{\Gamma (\eta_1 n + \beta_1 - \gamma) \Gamma (\eta_1 n + \beta_1 - \alpha - \alpha' - \delta) \Gamma (\eta_1 n + \beta_1 - \alpha' + \delta' - \gamma)} .
\end{align*}
\]

This competes the proof. \( \square \)

If we take $\alpha = \alpha + \delta$, $\alpha' = \delta' = 0$, $\delta = -\mu$ and $\gamma = \alpha$ in (3.3), we obtain known result given by Ram et al. [7, p. 412, Eq. (6.1)], as given by

Corollary 3.5. Let $\alpha, \delta, \mu \in \mathbb{C}$, $\Re (\alpha) > 0$, $a \in \mathbb{R}$, $\beta_1 \in \mathbb{C}$, $\eta_1 \in \mathbb{R}$ and the convergent conditions (i) and (ii) of $K$-series into the account of
(1.1) be also satisfied, and \( x > 0 \). Then we obtain the following formula:

\[
D_{-}^{\alpha,\delta,\mu} \left[ i^{\alpha-\beta_{1}} pK_{q}^{(\beta,\eta)} m (at^{-\eta}) \right] (x) \\
= x^{-\beta_{1}+\alpha+\delta} \sum_{n=0}^{\infty} \prod_{j=1}^{p}(a_{j})_{n} \frac{(ax^{-\eta})^{n}}{\prod_{r=1}^{q}(b_{r})_{n} \prod_{i=1}^{m} \Gamma (\eta_{i}n + \beta_{i})} \\
\frac{\Gamma (\eta_{1}n + \beta_{1} - \alpha - \delta) \Gamma (\eta_{1}n + \beta_{1} + \mu)}{\Gamma (\eta_{1}n + \beta_{1} - \alpha - \delta + \mu) \Gamma (\eta_{1}n + \beta_{1} - \alpha)}.
\]

\textbf{Remark 3.6.} If we take \( p = q = 1, a_{1} = \rho, b_{1} = 1 \) and \( \delta = -\alpha \) in (3.4), then we get the result given by Saxena et al. [14, Eq. (2.8)]. Further, if we set \( m = 1 \) then (3.4) reduces to the known result given by Saxena and Saigo [15, Eq. (35)].

\textbf{Remark 3.7.} If we set \( \delta = -\alpha \) in Corollary 3.1 and 4.1 then we can easily obtain results concerning Riemann-Liouville fractional derivative operators.

4. Concluding Remarks

In the present paper, we have studied and given new unified fractional calculus (differintegral) formulas associated with the \( K \)-Series. The theorems have been developed in terms of series form with the help of Saigo-Maeda power function formulas. Certain special cases of our main results are also pointed out to be related to some earlier works of many authors.

Acknowledgment

The third author (Dinesh Kumar) is grateful to the National Board of Higher Mathematics (NBHM) India, for granting a Post-Doctoral Fellowship (sanction no.2/40(37)/2014/R&D-II/14131).

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