AGGREGATE PREFERRED CORRESPONDENCE AND THE EXISTENCE OF A MAXIMIN REE

ANUJ BHOWMIK, JILING CAO, AND NICHOLAS C. YANNELIS

Abstract. In this paper, a general model of a pure exchange differential information economy is studied. In this economic model, the space of states of nature is a complete probability measure space, the space of agents is a measure space with a finite measure, and the commodity space is the Euclidean space. Under appropriate and standard assumptions on agents’ characteristics, results on continuity and measurability of the aggregate preferred correspondence in the sense of Aumann in [4] are established. These results together with other techniques are then employed to prove the existence of a maximin rational expectations equilibrium (maximin REE) of the economic model.

1. Introduction

When traders come to a market with different information about the items to be traded, the resulting market prices may reveal to some traders information originally available only to others. The possibility for such inferences rests upon traders having expectations of how equilibrium prices are related to initial information. This endogenous relationship was considered by Radner in his seminal paper [14], where he introduced the concept of a rational expectations equilibrium by imposing on agents the Bayesian (subjective expected utility) decision doctrine. Under the Bayesian decision making, agents maximize their subjective expected utilities conditioned on their own private information and also on the information that the equilibrium prices generate. The resulting equilibrium allocations are measurable with respect to the private information of each individual and also with respect to the information the equilibrium prices generate and clear the market for every state of nature. In both papers [1] and [14], conditions on the existence of a Bayesian rational expectations equilibrium (REE) were studied and some generic existence results were proved. However, Kreps [12] provided an example that shows that the Bayesian REE may not exist universally. In addition, a Bayesian REE may fail to be fully Pareto optimal and incentive compatible and may not be implementable as a perfect Bayesian equilibrium of an extensive form game, refer to Glycopantis et al. [9] for more details.

In a recent paper [7], de Castro et al. introduced a new notion of REE by a careful examination of Kreps’s example of the nonexistence of the Bayesian REE. In this formulation, the Bayesian decision making adopted in the papers of Allen [1] and Radner [14] was abandoned and replaced by the maximin expected utility (MEU) (see Gilboa and Schmeidler [8]). In this new setup, agents maximize their

JEL Classification Numbers: D51, D82.

Keywords. Aggregate correspondence; Budget correspondence; Differential information; Hausdorff continuous; Lower measurable; Maximin rational expectations allocation; Walrasian equilibrium.
MEU conditioned on their own private information and also on the information the equilibrium prices have generated. Contrary to the Bayesian REE, the resulting maximin REE may not be measurable with respect to the private information of each individual or the information that the equilibrium prices generate.

Although Bayesian REE and maximin REE coincide in some special cases (e.g., fully revealing Bayesian REE and maximin REE), these two concepts are in general not equivalent. Nonetheless, the introduction of the MEU into the general equilibrium modeling enables de Castro et al. to prove that the maximin REE exists universally under the standard continuity and concavity assumptions on the utility functions of agents. Furthermore, they showed that the maximin REE is incentive compatible and efficient. Note that in the economic model considered in [7], it is assumed that there are finitely many states of nature and finitely many agents, and the commodity space is finite-dimensional. Thus, one of open questions is whether their existence theorem can be extended to more general cases. The main motivation of this paper is to tackle this question. In this paper, a general model of a pure exchange differential information economy is studied. In this economic model, the space of states of nature is a complete probability measure space, and the space of agents is a measure space with a finite measure. Under appropriate and standard assumptions on agents’ characteristics, the existence of a maximin rational expectations equilibrium (maximin REE) of this general economic model is established.

This paper is organized as follows. In Section 2 the economic model, some notation and assumptions are introduced and explained. Section 3 some results on continuity and measurability of agents’ aggregate preferred correspondence in the sense of Aumann in [4] are established. These results are key techniques employed to prove the existence of a maximin rational expectations equilibrium of the economic model in Section 4. In Section 5 results and techniques appeared in this paper are compared with relevant results in the literature. Finally, some mathematical preliminaries and key facts used in this paper are presented in the appendix appeared at the end of the paper.

2. Differential information economies

In this paper, a model of a pure exchange economy $E$ with differential information is considered. The space of states of nature is a complete probability measure space $(\Omega, \mathcal{F}, \nu)$. The space of agents is a measure space $(T, \Sigma, \mu)$ with a finite measure $\mu$. The commodity space is the $\ell$-dimensional Euclidean space $\mathbb{R}^\ell$, and the positive cone $\mathbb{R}^\ell_+$ is the consumption set for each agent $t \in T$ in every state of nature $\omega \in \Omega$. Each agent $t \in T$ is associated with her/his characteristics $(\mathcal{F}_t, U(t, \cdot, \cdot), a(t, \cdot), q_t)$, where $\mathcal{F}_t$ is the $\sigma$-algebra generated by a partition $\Pi_t$ of $\Omega$ representing the private information of $t$; $U(t, \cdot, \cdot) : \Omega \times \mathbb{R}_+^\ell \to \mathbb{R}$ is a random utility function of $t$; $a(t, \cdot) : \Omega \to \mathbb{R}_+^\ell$ is the random initial endowment of $t$ and $q_t$ is a probability measure on $\Omega$ giving the prior of $t$. The economy extends over two time periods $\tau = 1, 2$. At the ex ante stage ($\tau = 0$), only the above description of the economy is a common knowledge. At the stage $\tau = 1$, agent $t$ only knows that the realized state of nature belongs to the event $\mathcal{F}_t(\omega^*)$, where $\omega^*$ is the true state of nature at $\tau = 2$. With this information (or with the information acquired through prices), agents trade. At the ex post stage ($\tau = 2$), agents execute the
trades according to the contract agreed at period \( \tau = 1 \), and consumption takes place. The coordinate-wise order on \( \mathbb{R}^\ell \) is denoted by \( \leq \) and the symbol \( x \gg 0 \)

means that \( x \) is an interior point of \( \mathbb{R}^\ell_+ \), and \( x > 0 \)

means that \( x \geq 0 \) but \( x \neq 0 \). Let \( L_1 (\mu, \mathbb{R}^\ell_+) \)

be the set of Lebesgue integrable functions from \( T \) to \( \mathbb{R}^\ell_+ \). An allocation in \( \mathcal{E} \)

is a function \( f : T \times \Omega \rightarrow \mathbb{R}^\ell_+ \) such that \( f(\cdot, \omega) \in L_1 (\mu, \mathbb{R}^\ell_+) \)

for all \( \omega \in \Omega \). An allocation \( f \) in \( \mathcal{E} \)

is feasible if

\[
\int_T f(\cdot, \omega) d\mu = \int_T a(\cdot, \omega) d\mu
\]

for all \( \omega \in \Omega \).

The following standard assumptions on agents’ characteristics shall be used:

**\( \mathbf{A}_1 \)** The initial endowment function \( a : T \times \Omega \rightarrow \mathbb{R}^\ell_+ \)

is jointly measurable such that \( \int_T a(\cdot, \omega) d\mu \geq 0 \) for each \( \omega \in \Omega \).

**\( \mathbf{A}_2 \)** \( U(\cdot, \cdot, x) \)

is jointly measurable for all \( x \in \mathbb{R}^\ell_+ \) and \( U(t, \omega, \cdot) \) is continuous for all \( (t, \omega) \in T \times \Omega \).

**\( \mathbf{A}_3 \)** For each \( (t, \omega) \in T \times \Omega \), \( U(t, \omega, \cdot) \)

is monotone in the sense that if \( x, y \in \mathbb{R}^\ell_+ \)

with \( y > 0 \), then \( U(t, \omega, x + y) > U(t, \omega, x) \).

**\( \mathbf{A}_4 \)** For each \( (t, \omega) \in T \times \Omega \), \( U(t, \omega, \cdot) \)

is concave.

Let

\[
\Delta = \left\{ p \in \mathbb{R}^\ell_+ : p \gg 0 \text{ and } \sum_{h=1}^{\ell} p_h = 1 \right\}.
\]

Then, each element \( p \in \Delta \) is viewed as a (normalized) price system in the deterministic case. The budget correspondence \( B : T \times \Omega \times \Delta \rightarrow \mathbb{R}^\ell_+ \)

is defined by

\[
B(t, \omega, p) = \{ x \in \mathbb{R}^\ell_+ : \langle p, x \rangle \leq \langle p, a(t, \omega) \rangle \}
\]

for all \( (t, \omega, p) \in T \times \Omega \times \Delta \). Obviously, \( B \) is non-empty and closed-valued. For each \( \omega \in \Omega \), by Theorem 2 in [10, p.151], there are \( p(\omega) \in \Delta \)

and an allocation \( f(\cdot, \omega, p(\omega)) \)

such that \( (f(\cdot, \omega, p(\omega))) \)

is a Walrasian equilibrium of the deterministic economy \( \mathcal{E}(\omega) \), given by

\[
\mathcal{E}(\omega) = ((T, \Sigma, \mu); \mathbb{R}^\ell_+; (U(t, \omega, \cdot), a(t, \omega)) : t \in T).
\]

Define a function \( \delta : \Delta \rightarrow \mathbb{R}_+ \) by

\[
\delta(p) = \min \{ p_h : 1 \leq h \leq \ell \},
\]

where \( p = (p^1, ..., p^\ell) \in \Delta \). For any \( (t, \omega, p) \in T \times \Omega \times \Delta \), let

\[
\gamma(t, \omega, p) = \frac{1}{\delta(p)} \sum_{h=1}^{\ell} a^h(t, \omega)
\]

and

\[
b(t, \omega, p) = (\gamma(t, \omega, p), ..., \gamma(t, \omega, p)).
\]

Define \( X : T \times \Omega \times \Delta \rightarrow \mathbb{R}^\ell_+ \) by

\[
X(t, \omega, p) = \{ x \in \mathbb{R}^\ell_+ : x \leq b(t, \omega, p) \}
\]

for all \( (t, \omega, p) \in T \times \Omega \times \Delta \). Note that \( X \) is non-empty, compact- and convex-valued such that \( B(t, \omega, p) \subseteq X(t, \omega, p) \) for all \( (t, \omega, p) \in T \times \Omega \times \Delta \). It can be readily verified
that for every \((t, \omega) \in T \times \Omega\), the correspondence \(X(t, \omega, \cdot) : \Delta \rightarrow \mathbb{R}_+^3\) is Hausdorff continuous. Define two correspondences \(C, C^X : T \times \Omega \times \Delta \rightarrow \mathbb{R}_+^3\) by
\[
C(t, \omega, p) = \{ y \in \mathbb{R}_+^3 : U(t, \omega, y) \geq U(t, \omega, x) \text{ for all } x \in B(t, \omega, p) \}
\]
and
\[
C^X(t, \omega, p) = C(t, \omega, p) \cap X(t, \omega, p).
\]
Obviously,
\[
B(t, \omega, p) \cap C(t, \omega, p) = B(t, \omega, p) \cap C^X(t, \omega, p)
\]
holds for all \((t, \omega, p) \in T \times \Omega \times \Delta\). Note that under \((A_2)\), \(U(t, \omega, \cdot)\) is continuous on the non-empty compact set \(B(t, \omega, p)\). Thus, one has
\[
B(t, \omega, p) \cap C(t, \omega, p) \neq \emptyset
\]
for all \((t, \omega, p) \in T \times \Omega \times \Delta\).

**Proposition 2.1.** Let \((t, \omega, p) \in T \times \Omega \times \Delta\). Under \((A_3)\), \(\langle p, x \rangle \geq \langle p, a(t, \omega) \rangle\) for every point \(x \in C^X(t, \omega, p)\).

**Proof.** Assume that \(\langle p, x_0 \rangle < \langle p, a(t, \omega) \rangle\) for some point \(x_0 \in C^X(t, \omega, p)\). Then, one can choose some \(y \in \mathbb{R}_+^3\) such that \(y > 0\) and \(\langle p, x_0 + y \rangle < \langle p, a(t, \omega) \rangle\). Thus, \(x_0 + y \in B(t, \omega, p)\). Since \(x_0 \in C^X(t, \omega, p)\), one has \(U(t, \omega, x_0) > U(t, \omega, x_0 + y)\). However, \((A_3)\) implies \(U(t, \omega, x_0 + y) > U(t, \omega, x_0)\). This is a contradiction, which completes the proof. \(\square\)

Following Aumann in [4], \(C^X(t, \omega, p)\) is called the preferred set of agent \(t\) at the price \(p\) and state of nature \(\omega\), and \(\int_T C^X(\cdot, \omega, p) d\mu\) is called the aggregate preferred set at the price \(p\) and state of nature \(\omega\). Moreover, we shall call
\[
\int_T C^X(\cdot, \cdot, \cdot) d\mu : \Omega \times \Delta \rightarrow \mathbb{R}_+^3
\]
the aggregate preferred correspondence. In the next section, we shall discuss some descriptive properties of this object. These properties will be used to derive the existence of a maximin rational expectations equilibrium of our economic model.

### 3. Properties of the Aggregate Preferred Correspondence

In this section, some results on continuity and measurability of the aggregate preferred correspondence are presented.

**Proposition 3.1.** Under \((A_1)\), for every \((\omega, p) \in \Omega \times \Delta\), \(B(\cdot, \omega, p) : T \rightarrow \mathbb{R}_+^3\) and \(X(\cdot, \omega, p) : T \rightarrow \mathbb{R}_+^3\) are lower measurable.

**Proof.** Here, only the proof of lower measurability of \(B(\cdot, \omega, p)\) is provided. The other case can be done analogously. Fix \((\omega, p) \in \Omega \times \Delta\). Define a function \(h : T \times \mathbb{R}_+^3 \rightarrow \mathbb{R}\) by letting
\[
h(t, x) = \langle p, x \rangle - \langle p, a(t, \omega) \rangle
\]
for all \((t, x) \in T \times \mathbb{R}_+^3\). Then, \(h(\cdot, x)\) is measurable for all \(x \in \mathbb{R}_+^3\). Note that
\[
B(t, \omega, p) = h(t, \cdot)^{-1}((\infty, 0])
\]
Let \(V \subseteq \mathbb{R}_+^3\) be a non-empty open subset, and put
\[
V \cap Q_+^3 = \{ x_k : k \geq 1 \}.
\]
It is worth to point out that if \( x \in B(t, \omega, p) \cap V \), then \( x_k \in B(t, \omega, p) \) for some \( k \geq 1 \). Since \( h(\cdot, x_k) \) is measurable, one has
\[
\{ t \in T : h(t, x_k) \in (-\infty, 0] \} \in \Sigma
\]
for all \( k \geq 1 \). Thus,
\[
B(\cdot, \omega, p)^{-1}(V) = \bigcup_{k \geq 1} \{ t \in T : x_k \in B(t, \omega, p) \} = \bigcup_{k \geq 1} \{ t \in T : h(t, x_k) \in (-\infty, 0] \}
\]
belongs to \( \Sigma \). It follows that \( B(\cdot, \omega, p) \) is lower measurable. \( \square \)

**Proposition 3.2.** Under (A\(_1\))-(A\(_3\)), \( \int_T C^X(\cdot, \cdot, \cdot) d\mu \) is non-empty compact-valued.

**Proof.** Fix \( (\omega, p) \in \Omega \times \Delta \). By (A\(_2\)), \( C^X(t, \omega, p) \) is non-empty closed for all \( t \in T \). By the lower measurability of \( B(\cdot, \omega, p) \), there exists a sequence \( \{ f_n : n \geq 1 \} \) of measurable functions from \( T \) to \( \mathbb{R}_+^d \) such that
\[
B(t, \omega, p) = \{ f_n(t) : n \geq 1 \}
\]
for all \( t \in T \). For each \( n \geq 1 \), define a correspondence \( C_n : T \rightrightarrows \mathbb{R}_+^d \) by letting
\[
C_n(t) = \{ x \in \mathbb{R}_+^d : U(t, \omega, x) \geq U(t, \omega, f_n(t)) \}
\]
for all \( t \in T \). Obviously, one has
\[
C(t, \omega, p) \subseteq \bigcap_{n \geq 1} C_n(t)
\]
for all \( t \in T \). If \( x \in \mathbb{R}_+^d \setminus C(t, \omega, p) \) for some \( t \in T \), there exists a point \( y \in B(t, \omega, p) \) such that \( U(t, \omega, y) > U(t, \omega, x) \). By (A\(_2\)), there exists an \( n_0 \geq 1 \) such that \( U(t, \omega, f_n(t)) > U(t, \omega, x) \). This implies \( x \notin C_{n_0}(t) \). Thus, it is verified that
\[
C(t, \omega, p) = \bigcap_{n \geq 1} C_n(t)
\]
for all \( t \in T \). Fix \( n \geq 1 \), and define a function \( h : T \times \mathbb{R}_+^d \to \mathbb{R} \) by
\[
h(t, x) = U(t, \omega, f_n(t)) - U(t, \omega, x).
\]
Clearly, \( h \) is Carathéodory. Similar to Proposition 3.1 one can show that \( C_n \) is lower measurable. Since \( X(\cdot, \omega, p) \) is compact-valued and
\[
C^X(\cdot, \omega, p) = \bigcap_{n \geq 1} C_n(\cdot) \cap X(\cdot, \omega, p),
\]
then \( C^X(\cdot, \omega, p) \) is lower measurable. By the Kuratowski-Ryll-Nardzewski measurable selection theorem in [9], \( C^X(\cdot, \omega, p) \) has a measurable selection which is also integrable, as \( b(\cdot, \omega, p) \) is so. Since \( C^X(\cdot, \omega, p) \) is closed-valued and integrably bounded, \( \int_T C^X(\cdot, \omega, \cdot) d\mu \) is compact-valued. \( \square \)

Next, we establish Hausdorff continuity of the aggregate preferred correspondence with respect to the variable \( p \in \Delta \).

**Theorem 3.3.** Assume (A\(_1\))-(A\(_3\)). For each \( \omega \in \Omega \), \( \int_T C^X(\cdot, \cdot, \cdot) d\mu : \Delta \rightrightarrows \mathbb{R}_+^d \) is Hausdorff continuous.
The verification of the above equation can be split into two steps. First, one verifies 

\[ x \to t \]

To do this, it is enough to verify that for any \( k \in K \) \( u \in U \) \( t, \omega, x \to 0 \) such that

\[ \varepsilon > 0 \quad \text{and} \quad \varepsilon < \delta(p_n) \quad \text{for all} \quad n \geq N. \]

Let

\[ c = \min \{ \delta(p_n), \varepsilon : n = 1, 2, \ldots, N - 1 \}, \]

\[ d(t, \omega) = \frac{1}{c} \sum_{h=1}^{t} a^h(t, \omega), \]

and

\[ \xi(t, \omega) = (d(t, \omega), \ldots, d(t, \omega)). \]

Define \( M(\omega) \) by

\[ M(\omega) = \left\{ x \in \mathbb{R}_+^t : x \leq \int_{T} \xi(\cdot, \omega) d\mu \right\}. \]

Since \( X(\cdot, \omega, p_n) \) and \( X(\cdot, \omega, p) \) are upper bounded by \( \xi(\cdot, \omega) \), then \( \int_{T} C^X(\cdot, \omega, p_n) d\mu \) and \( \int_{T} C^X(\cdot, \omega, p) d\mu \) are contained in the compact subset \( M(\omega) \) of \( \mathbb{R}_+^t \). Thus, one only needs to show that \( \left\{ \int_{T} C^X(\cdot, \omega, p_n) d\mu : n \geq 1 \right\} \) converges to \( \int_{T} C^X(\cdot, \omega, p) d\mu \) in the Hausdorff metric topology on \( \mathcal{H}(M(\omega)) \), which is equivalent to

\[ \lim \int_{T} C^X(\cdot, \omega, p_n) d\mu = \lim \int_{T} C^X(\cdot, \omega, p) d\mu. \]

The verification of the above equation can be split into two steps. First, one verifies

\[ \lim \int_{T} C^X(\cdot, \omega, p_n) d\mu \subseteq \int_{T} C^X(\cdot, \omega, p) d\mu. \]

To do this, it is enough to verify that for any \( t \in T \),

\[ \lim_{n \to \infty} C^X(t, \omega, p_n) \subseteq C^X(t, \omega, p). \]

Pick \( t \in T \) and \( x \in \lim_{n \to \infty} C^X(t, \omega, p_n) \). Then, there exist positive integers \( n_1 < n_2 < n_3 < \cdots \) and for each \( k \) a point \( x_k \in C^X(t, \omega, p_{n_k}) \) such that \( \{ x_k : k \geq 1 \} \) converges to \( x \). It is obvious that \( x \in X(t, \omega, p) \). If \( x \notin C^X(t, \omega, p) \), by the continuity of \( U(t, \omega, \cdot) \), one can choose some \( y \in \mathbb{R}_+^t \) such that \( \langle p, y \rangle < \langle p, a(t, \omega) \rangle \) and \( U(t, \omega, y) > U(t, \omega, x) \). By the Hausdorff continuity of \( X(t, \omega, \cdot) \), \( \{ X(t, \omega, p_{n_k}) : k \geq 1 \} \) converges to \( X(t, \omega, p) \) in the Hausdorff metric topology. Since \( y \in X(t, \omega, p) \), there exists a sequence \( \{ y_k : k \geq 1 \} \) such that \( y_k \in X(t, \omega, p_n) \) for all \( k \geq 1 \) and \( \{ y_k : k \geq 1 \} \) converges to \( y \). It follows that \( U(t, \omega, y_k) > U(t, \omega, x_k) \) and \( \langle p_{n_k}, y_k \rangle < \langle p_{n_k}, a(t, \omega) \rangle \) for all sufficiently large \( k \), which is a contradiction with \( x_k \in C^X(t, \omega, p_{n_k}) \) for all \( k \geq 1 \). Therefore, one must have \( x \in C^X(t, \omega, p) \). Secondly, one needs to verify

\[ \int_{T} C^X(\cdot, \omega, p) d\mu \subseteq \lim \int_{T} C^X(\cdot, \omega, p_n) d\mu. \]

It is enough to verify that for all \( t \in T \),

\[ C^X(t, \omega, p) \subseteq \lim C^X(t, \omega, p_n). \]

Fix \( t \in T \) and pick \( d \in C^X(t, \omega, p) \). If \( d = b(t, \omega, p) \), then \( b(t, \omega, p_n) \in C^X(t, \omega, p_n) \) and the sequence \( \{ b(t, \omega, p_n) : n \geq 1 \} \) converges to \( d \). Assume \( d < b(t, \omega, p) \). Select \( \delta > 0 \) such that

\[ d + (0, \ldots, \delta, \ldots, 0) \leq b(t, \omega, p). \]

Further, choose a sequence \( \{ \delta_i : i \geq 1 \} \) in \( (0, \delta] \) converging to \( 0 \). For each \( i \geq 1 \), let

\[ d^i = d + (0, \ldots, \delta_i, \ldots, 0), \]
and choose a sequence \( \{d_{1}^{n} : n \geq 1 \} \) such that for each \( n, d_{1}^{n} \in X(t, \omega, p_n) \) and \( \{d_{1}^{n} : n \geq 1 \} \) converges to \( d^0 \). It is claimed that for each \( i \geq 1, d_{i}^{n} \in C^{X}(t, \omega, p_n) \) for sufficiently large \( n \). Otherwise, there must exist an \( i_0 \) and a subsequence \( \{d_{i_k}^{n_k} : k \geq 1 \} \) of \( \{d_{i}^{n} : n \geq 1 \} \) such that \( d_{i_k}^{n_k} \notin C^{X}(t, \omega, p_{n_k}) \). Let \( b_k \in B(t, \omega, p_{n_k}) \) and \( U(t, \omega, b_k) > U(t, \omega, d_{i_k}^{n_k}) \) for all \( k \geq 1 \). Then \( \{b_k : k \geq 1 \} \) has a subsequence converging to some \( b \in B(t, \omega, p) \). Applying (A2) and (A3), one can obtain

\[
U(t, \omega, b) \geq U(t, \omega, d^0) > U(t, \omega, d),
\]

which contradicts with the fact \( d \in C^{X}(t, \omega, p) \). To complete the proof, note that the previous claim implies that for each \( i, \{\text{dist}(d^i, C^{X}(t, \omega, p_n)) : n \geq 1 \} \) converges to 0. Since \( \{d^i : i \geq 1 \} \) converges to \( d \), one concludes that \( \{\text{dist}(d, C^{X}(t, \omega, p_n)) : n \geq 1 \} \) converges to 0. This means that \( d \in LiC^{X}(t, \omega, p_n) \).

The next result is crucial for the existence theorem in Section 4. In its proof, the following characterization of lower measurability of correspondences in [2] is used:

A correspondence \( F : (\Omega, \mathcal{F}, \nu) \rightarrow \mathbb{R}_+^\ell \) is lower measurable if and only if for all \( y \in \mathbb{R}_+^\ell, \text{dist}(y, F(\cdot)) : (\Omega, \mathcal{F}, \nu) \rightarrow \mathbb{R}_+^\ell \) is a measurable function.

**Theorem 3.4.** Assume (A1)-(A3). For each \( p \in \Delta \), \( \int_{\Delta} C^{X}(\cdot, \cdot, p)d\mu : \Omega \rightarrow \mathbb{R}_+^\ell \) is lower measurable.

**Proof.** Fix \( p \in \Delta \). Since \( a \) and \( U \) are \( \Sigma \otimes \mathcal{F} \)-measurable and \( \Sigma \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+^\ell) \)-measurable respectively, by the argument of a result in [16, 131], there exist two sequences \( \{a_n : n \geq 1\} \) and \( \{\psi_n : n \geq 1\} \) of simple \( \Sigma \otimes \mathcal{F} \)-measurable and simple \( \Sigma \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+^\ell) \)-measurable functions respectively such that \( \{a_n : n \geq 1\} \) uniformly converges to \( a \) on \( T \times \Omega \) and \( \{\psi_n : n \geq 1\} \) uniformly converges to \( U \) on \( T \times \Omega \times \mathbb{R}_+^\ell \). For each \( n \geq 1 \), write \( a_n \) and \( \psi_n \) as

\[
a_n = \sum_{i \geq 1} e_i \chi_{T_1^n \times \Omega_1^n},
\]

and

\[
\psi_n = \sum_{i \geq 1} v_i \chi_{T_1^n \times \Omega_1^n \times B_1^n},
\]

where \( e_i \in \mathbb{R}_+^\ell, v_i \in \mathbb{R}_+ \), and \( \{T_1^n \times \Omega_1^n \times B_1^n : i \geq 1\} \) is a partition of \( T \times \Omega \times \mathbb{R}_+^\ell \) for all \( n \geq 1 \). Choose \( N \geq 1 \) such that \( \|a_n - a\|_{\infty} < 1 \) for all \( n \geq N \). By the measurability of \( a_n(\cdot, \omega), a_n(\cdot, \omega) \in L_1(\mu, \mathbb{R}_+^\ell) \) for all \( \omega \in \Omega \) and all \( n \geq 1 \) (replacing \( a_n \) for all \( 1 \leq n < N \) by some constant functions, if necessary). Let

\[
\gamma_n(t, \omega) = \frac{1}{\delta(p)} \sum_{b=1}^{l} a_n^{b}(t, \omega)
\]

and

\[
b_n(t, \omega) = (\gamma_n(t, \omega), \ldots, \gamma_n(t, \omega)).
\]

Define \( X_n, B_n, C_n : T \times \Omega \rightarrow \mathbb{R}_+^\ell \) such that

\[
X_n(t, \omega) = \{x \in \mathbb{R}_+^\ell : x \leq b_n(t, \omega)\},
\]

\[
B_n(t, \omega) = \{x \in \mathbb{R}_+^\ell : \langle p, x \rangle \leq \langle p, a_n(t, \omega) \rangle\}
\]

and

\[
C_n(t, \omega) = \{y \in \mathbb{R}_+^\ell : \psi_n(t, \omega, y) \geq \psi_n(t, \omega, x) \text{ for all } x \in B_n(t, \omega)\}.
\]
In addition, define $C_n^X : T \times \Omega \to \mathbb{R}_+^\ell$ such that for all $(t, \omega) \in T \times \Omega$,

$$C_n^X(t, \omega) = (C_n(t, \omega) \cup \{b_n(t, \omega)\}) \cap X_n(t, \omega).$$

For every $n \geq 1$, define the correspondence $H_n : (\Omega, \mathcal{F}, \nu) \to L_1(\mu, \mathbb{R}_+^\ell)$ by letting $H_n(\omega) = \mathcal{C}_{n^{\mathcal{F}}}^X(\cdot, \omega)$. Obviously, $H_n(\omega) \neq \emptyset$ for all $\omega \in \Omega$.

Claim 1. For each $n \geq 1$, $H_n$ is lower measurable. For convenience, let $\Theta : L_1(\mu, \mathbb{R}_+^\ell) \times \Omega \to \mathbb{R}_+$ be the function such that

$$\Theta(g, \omega) = \text{dist}(g, H_n(\omega))$$

for all $g \in L_1(\mu, \mathbb{R}_+^\ell)$ and $\omega \in \Omega$. To verify the claim, one needs to verify that for all $g \in L_1(\mu, \mathbb{R}_+^\ell)$, $\Theta(g, \cdot)$ is measurable. Since $\Theta(\cdot, \omega) : L_1(\mu, \mathbb{R}_+^\ell) \to \mathbb{R}_+$ is norm-continuous, it suffices to show that $\Theta(g, \cdot) : \Omega \to \mathbb{R}_+$ is measurable for every simple function $g = \sum_{j=1}^{r} x_j \chi_{T_j}$, where $x_j \in \mathbb{R}_+^\ell$. To this end, consider the function $\Gamma : T \times \Omega \to \mathbb{R}_+$ such that

$$\Gamma(t, \omega) = \text{dist}(g(t), C_n^X(t, \omega))$$

for all $(t, \omega) \in T \times \Omega$. Since $\Gamma$ is constant on each $(T_i \cap T_j) \times \Omega^n$, it is jointly measurable. Note that

$$\Gamma(t, \omega) \leq \|g(t) - b_n(t, \omega)\|$$

for all $(t, \omega) \in T \times \Omega$. This implies for all $\omega \in \Omega$, $\Gamma(\cdot, \omega)$ is integrable. Thus, $\Theta(g, \cdot)$ is measurable and the claim is verified if one can show for all $\omega \in \Omega$,

$$\int_T \Gamma(\cdot, \omega) d\mu = \Theta(g, \omega).$$

Assume that

$$\int_T \Gamma(\cdot, \omega_0) d\mu < \Theta(g, \omega_0)$$

for some $\omega_0 \in \Omega$. Pick $\varepsilon > 0$ such that

$$\int_T \Gamma(\cdot, \omega_0) d\mu + \varepsilon \mu(T) < \Theta(g, \omega_0).$$

Further, pick $t \in T_i \cap T_j$ and $y(i,j) \in C_n^X(t, \omega_0)$ so that

$$\|x_j - y(i,j)\| < \Gamma(t, \omega_0) + \varepsilon.$$ 

Then the function $\zeta : T \to \mathbb{R}_+^\ell$, defined by $\zeta(t) = y(i,j)$ if $t \in T_i \cap T_j$, belongs to $H_n(\omega_0)$ and

$$\|g - \zeta\|_1 < \int_T \Gamma(\cdot, \omega_0) d\mu + \varepsilon \mu(T),$$

which is a contradiction.

Claim 2. The correspondence $\int_T C_n^X(\cdot, \cdot) d\mu : (\Omega, \mathcal{F}, \nu) \to \mathbb{R}_+^\ell$ is lower measurable. Consider the function $\xi : L_1(\mu, \mathbb{R}_+^\ell) \to \mathbb{R}_+$ defined by $\xi(f) = f d\mu$ for all $f \in L_1(\mu, \mathbb{R}_+^\ell)$. Let $V$ be an open subset of $\mathbb{R}_+^\ell$. Note that

$$\xi \circ H_n(\omega) = \int_T C_n^X(\cdot, \omega) d\mu$$

for all $\omega \in \Omega$, and

$$(\xi \circ H_n)^{-1}(V) = \{\omega \in \Omega : H_n(\omega) \cap \xi^{-1}(V) \neq \emptyset\}.$$ 

Since $\xi$ is norm-continuous, by Claim 1, $(\xi \circ H_n)^{-1}(V) \in \mathcal{F}$. This verifies the claim.
Claim 3. For each $\omega \in \Omega$, 
\[
\mathcal{L} \int_T C_n^X(\cdot, \omega) d\mu = \mathcal{L} \int_T C_n^X(\cdot, \omega) d\mu = \int_T C_n^X(\cdot, \omega, p) d\mu.
\]
To see this, for each $\omega \in \Omega$, put
\[
\alpha(\cdot, \omega) = \sup \left\{ b_1(\cdot, \omega), \ldots, b_{N-1}(\cdot, \omega), b(\cdot, \omega, p) + \left( \frac{\ell}{\delta(p)}, \ldots, \frac{\ell}{\delta(p)} \right) \right\}.
\]
Then, $C_n^X(\cdot, \omega, p)$ and all $C_n^X(\cdot, \omega)$ are integrably bounded by $\alpha(\cdot, \omega)$. Now, it suffices to verify that for all $t \in T$, 
\[
\mathcal{L} \int C_n^X(t, \omega, p) \subseteq C_n^X(t, \omega, p),
\]
and
\[
C_n^X(t, \omega, p) \subseteq \mathcal{L} C_n^X(t, \omega).
\]
First, let $x \in \mathcal{L} C_n^X(t, \omega)$. If $x = b(t, \omega, p)$, then \{\(b_n(t, \omega) : n \geq 1\}\} converges to $x$ and $b_n(t, \omega) \in C_n^X(\cdot, \omega)$ for all $n \geq 1$. Otherwise, there exist positive integers $n_1 < n_2 < \cdots$ and for each $k$ a point $x_k \in C_n^X(t, \omega)$ such that \{\(x_k : k \geq 1\}\} converges to $x$. Obviously, $x_k \neq b_{n_k}(t, \omega)$ for all sufficiently large $k$, and $x \in X(t, \omega, p)$. If $x \notin C^X(t, \omega, p)$, there exists some $y \in B(t, \omega, p)$ such that $U(t, \omega, y) > U(t, \omega, x)$. By the continuity of $U(t, \omega, \cdot)$, $y$ can be chosen so that $\langle p, y \rangle < \langle p, a(t, \omega) \rangle$. Since \{\(X_{n_k}(t, \omega) : k \geq 1\}\} converges to $X(t, \omega, p)$ in the Hausdorff metric topology, there exists a sequence \{\(y_k : k \geq 1\}\} such that $y_k \in X_{n_k}(t, \omega)$ for all $k \geq 1$ and \{\(y_k : k \geq 1\}\} converges to $y$. By the inequality
\[
|U(t, \omega, x) - \psi_{n_k}(t, \omega, x_k)| < |U(t, \omega, x) - U(t, \omega, x_k)| + |U(t, \omega, x_k) - \psi_{n_k}(t, \omega, x_k)|,
\]
the continuity of $U(t, \omega, \cdot)$ and the uniform convergence of $\psi_{n_k}(t, \omega, \cdot)$ to $U(t, \omega, \cdot)$, one concludes that $\psi_{n_k}(t, \omega, y_k) > \psi_{n_k}(t, \omega, x_k)$ and $\langle p, y_k \rangle < \langle p, a_{n_k}(t, \omega) \rangle$ for sufficiently large $k$, which contradicts the fact that $x_k \in C_n^X(\cdot, \omega)$ for all $k \geq 1$.

Hence, $x \in C^X(t, \omega, p)$. Now, let $d \in C^X(t, \omega, p)$. If $d = b(t, \omega, p)$, there is nothing to verify. Thus, $d \in \mathcal{L} C_n^X(t, \omega)$. Assume $d < b(t, \omega, p)$. Similar to that in the proof of Theorem 3.3, one can show that $d \in \mathcal{L} C_n^X(t, \omega)$.

To complete the proof, for each $\omega \in \Omega$, put
\[
M(\omega) = \left\{ x \in \mathbb{R}_+^\ell : x \leq \int_T \alpha(\cdot, \omega) d\mu \right\}.
\]
Clearly, $\int_T C_n^X(\cdot, \omega) d\mu$ and $\int_T C^X(\cdot, \omega, p) d\mu$ are contained in the compact set $M(\omega)$. It follows from Claim 3 that \{\(\int_T C_n^X(\cdot, \omega) d\mu : n \geq 1\}\} converges to $\int_T C^X(\cdot, \omega, p) d\mu$ in $M(\omega)$ in the Hausdorff metric topology. It is well known that a nonempty compact-valued correspondence is lower measurable if and only if it is measurable when viewed as a single-valued function whose range space is the space of nonempty compact sets endowed with the Hausdorff metric topology. By Claim 2, $\int_T C^X(\cdot, \omega, p) d\mu$ is lower measurable. $\square$

Corollary 3.5. Assume (A1)-(A3). Then $\int_T C^X(\cdot, \cdot, \cdot) d\mu : \Omega \times \Delta \to \mathcal{K}_0(\mathbb{R}_+^\ell)$ is a jointly measurable function, where $\mathcal{K}_0(\mathbb{R}_+^\ell)$ is endowed with the Hausdorff metric topology.
for almost all $t$, by the Aumann-Saint-Beuve measurable selection theorem, there exists a measurable function $f : \Omega \times \Delta \to \mathcal{X}_0(\mathbb{R}^\ell_+)$.

By Proposition 3.2, and $(A)$, the correspondence $f : \Omega \mapsto \mathbb{R}^\ell_+$ is lower measurable. Hence, for every $p \in \Delta$, the function $\int_T C^X(\cdot,\omega,p)d\mu : \Omega \mapsto \mathbb{R}^\ell_+$ is measurable. This means that $\int_T C^X(\cdot,\cdot,p)d\mu : \Omega \times \Delta \to \mathcal{X}_0(\mathbb{R}^\ell_+)$ is measurable and therefore is jointly measurable. 

4. The existence of a maximin REE

A price system $\pi : (\Omega,\mathcal{F},\nu) \to \Delta$. Let $\sigma(\pi)$ be the smallest sub-algebra of $\mathcal{F}$ such that $\pi$ is measurable and let $\mathcal{G}_i = \mathcal{F} \cap \sigma(\pi)$.

For each $\omega \in \Omega$, let $\mathcal{G}_i(\omega)$ denote the smallest element of $\mathcal{G}_i$ containing $\omega$. Given $t \in T$, let $\mathcal{G}_i(\omega)$ be measurable with respect to $\mathcal{G}_i$ at an allocation $f : T \times \Omega \to \mathbb{R}^\ell_+$.

The maximin utility of each agent $t \in T$ with respect to $\mathcal{G}_i$ is denoted by $U^\text{REE}(t,\omega,f(t,\cdot))$, is defined by

$$U^\text{REE}(t,\omega,f(t,\cdot)) = \inf_{\omega' \in \mathcal{G}_i(\omega)} U(t,\omega',f(t,\omega')).$$

**Definition 4.1.** Given a feasible allocation $f$ and a price system $\pi$, the pair $(f,\pi)$ is called a maximin rational expectations equilibrium (REE) if $f(t,\omega) \in B(t,\omega,\pi(\omega))$ for almost all $(t,\omega) \in T \times \Omega$.

**Theorem 4.2.** Under $(A_1)$-$(A_4)$, $MREE(\mathcal{E}) \neq \emptyset$.

**Proof.** Consider the correspondence $Z : \Omega \times \Delta \rightrightarrows \mathbb{R}^\ell$ defined by

$$Z(\omega,p) = \int_T C^X(\cdot,\omega,p)d\mu - \int_T a(\cdot,\omega)d\mu.$$

By Proposition 3.2, $Z$ is non-empty compact-valued. In addition, by Corollary 3.5 and $(A_1)$, $Z : \Omega \times \Delta \to \mathcal{X}_0(\mathbb{R}^\ell)$ is jointly measurable. Define another correspondence $F : \Omega \rightrightarrows \Delta$ such that

$$F(\omega) = \{p \in \Delta : Z(\omega,p) \cap \{0\} \neq \emptyset\}.$$

By Theorem 2 in [10], p.151, $F$ is non-empty valued. Since $\text{Gr}_F = Z^{-1}(\{0\})$, $\text{Gr}_F \in \mathcal{F} \otimes \mathcal{B}(\Delta)$, by the Aumann-Saint-Beuve measurable selection theorem, there exists a measurable function $\hat{\pi} : \Omega \to \Delta$ such that $\hat{\pi}(\omega) \in F(\omega)$ for all $\omega \in \Omega$. By the definition of $Z$, there exists an allocation $f$ such that $f(t,\omega) \in C^X(t,\omega,\hat{\pi}(\omega))$ and

$$\int_T f(\cdot,\omega)d\mu = \int_T a(\cdot,\omega)d\mu$$

for almost all $t \in T$ and all $\omega \in \Omega$. By Proposition 2.1, one has

$$\langle \hat{\pi}(\omega), f(t,\omega) \rangle \geq \langle \hat{\pi}(\omega), a(t,\omega) \rangle$$
for almost all \( t \in T \) and all \( \omega \in \Omega \). Then, the previous equation implies
\[
\langle \hat{\pi}(\omega), f(t, \omega) \rangle = \langle \hat{\pi}(\omega), a(t, \omega) \rangle
\]
for almost all \( t \in T \) and all \( \omega \in \Omega \). Thus, \( f(t, \omega) \in B(t, \omega, \hat{\pi}(\omega)) \) for almost all \( t \in T \) and all \( \omega \in \Omega \). For every \( \omega \in \Omega \), define \( T_\omega \subseteq T \) by
\[
T_\omega = \{ t \in T : f(t, \omega) \in B(t, \omega, \hat{\pi}(\omega)) \cap C(t, \omega, \hat{\pi}(\omega)) \}.
\]
Then, \( \mu(T_\omega) = \mu(T) \) for all \( \omega \in \Omega \). Next, for every \( \omega \in \Omega \) and every \( t \in T \setminus T_\omega \), as \( B(t, \omega, \hat{\pi}(\omega)) \cap C(t, \omega, \hat{\pi}(\omega)) \neq \emptyset \), one can pick a point
\[
h(t, \omega) \in B(t, \omega, \hat{\pi}(\omega)) \cap C(t, \omega, \hat{\pi}(\omega))
\]
and then define a function \( \hat{f} : T \times \Omega \to \mathbb{R}_+^T \) such that
\[
\hat{f}(t, \omega) = \begin{cases} 
 f(t, \omega), & \text{if } t \in T_\omega; \\
 h(t, \omega), & \text{if } t \in T \setminus T_\omega.
\end{cases}
\]
It is obvious that \( \hat{f}(t, \omega) \in B(t, \omega, \hat{\pi}(\omega)) \cap C(t, \omega, \hat{\pi}(\omega)) \) for all \( (t, \omega) \in T \times \Omega \). Assume that there are an agent \( t_0 \in T \), a state of nature \( \omega_{t_0} \in \Omega \) and an element \( y(t_0, \cdot) \in B^{REE}(t_0, \omega_{t_0}, \hat{\pi}) \) such that
\[
U^{REE}(t_0, \omega_{t_0}, g(t_0, \cdot)) > U^{REE}(t_0, \omega_{t_0}, \hat{f}(t_0, \cdot)).
\]
Then, one obtains
\[
U(t_0, \omega_{t_0}', g(t_0, \omega_{t_0}')) > U(t_0, \omega_{t_0}', \hat{f}(t_0, \omega_{t_0}'))
\]
for some \( \omega_{t_0}' \in \mathcal{G}_{t_0}(\omega_{t_0}) \), which contradicts with \( \hat{f}(t_0, \omega_{t_0}') \in C(t_0, \omega_{t_0}', \hat{\pi}(\omega_{t_0}')) \). This verifies that \( (\hat{f}, \hat{\pi}) \) is a maximin rational expectations equilibrium of \( \mathcal{E} \). \( \square \)

5. Conclusion

The first application of maximin expected utility functions to the general equilibrium theory with differential information appeared in Correia da Silva and Hervés Beloso [6], where an existence theorem for a Walrasian equilibrium in an economy was established. However, their MEU formulation is in the ex-ante sense, and REE notion was not considered.

In our paper, an existence theorem on a maximin rational expectations equilibrium (maximin REE) for an exchange differential information economy is proved. Comparing with the existence result on maximin REE in [7], our theorem applies to a more general economic model with an arbitrary finite measure space of agents and an arbitrary complete probability measure space as the space of states of nature, while the later applies only to an economic model which has finitely many agents and finitely many states of nature. Assumptions in our paper are similar to those in [7], except the joint measurability and continuity of utility functions, and the joint measurability of the initial endowment function. The proof techniques in this paper are quite different from those in [7]. Since there are only finitely many agents and states of nature in the model considered in [7], neither measurability nor continuity of utility functions and the initial endowment function plays any role in the proof of the existence of a maximin REE. Instead, the existence of a competitive equilibrium for complete information economies is applied. In contrast, both measurability and continuity of utility functions and the initial endowment function play key roles in this paper. To establish the existence theorem, Aumann’s
techniques in [4] are adopted, measurability and continuity of the aggregate preferred correspondence are investigated. However, for special cases, the techniques can be simplified. For instance, if there are finitely many states of nature, one can still apply the approach employed in [7] and obtains an existence theorem. On the other hand, if there are finitely many agents, then one can show that the demand of each agent is $\mathcal{F} \otimes \mathcal{B}(\Delta)$-measurable and so is the aggregate demand. Then, an approach similar to that in the proof of Theorem 4.2 can be applied to establish the existence theorem.

Appendix

Let $G$ be a non-empty set and $\mathbb{R}^\ell$ be the $\ell$-dimensional Euclidean space. On $\mathbb{R}^\ell$, two different but equivalent standard norms $\| \cdot \|_\infty$ and $\| \cdot \|_1$ are used in this paper, where for each point $x = (x_1, x_2, \cdots, x_\ell) \in \mathbb{R}^\ell$, 
\[
\|x\|_\infty = \max\{|x_i| : 1 \leq i \leq \ell\},
\]
and
\[
\|x\|_1 = \sum_{1 \leq i \leq \ell} |x_i|.
\]
A correspondence $F : G \rightrightarrows \mathbb{R}^\ell$ from $G$ to $\mathbb{R}^\ell$ assigns to each $x \in G$ a subset $F(x)$ of $\mathbb{R}^\ell$. Meanwhile, $F$ can also be viewed as a function $F : G \to 2^{\mathbb{R}^\ell}$, where $2^{\mathbb{R}^\ell}$ denotes the power set of $\mathbb{R}^\ell$. Further, $F$ is called non-empty valued (resp. closed-valued, compact-valued, convex-valued) if $F(x)$ is a non-empty (resp. closed, compact, convex) subset of $\mathbb{R}^\ell$ for all $x \in G$. The graph of $F$, denoted by $\text{Gr}_F$, is defined by
\[
\text{Gr}_F = \{(x, y) \in G \times \mathbb{R}^\ell : y \in F(x) \text{ and } x \in G\}.
\]
For each point $x \in \mathbb{R}^\ell$ and a subset $A \in 2^{\mathbb{R}^\ell} \setminus \{\emptyset\}$, define
\[
dist(x, A) = \inf\{d(x, y) : y \in A\},
\]
where $d$ is the Euclidean metric on $\mathbb{R}^\ell$. Let $\mathcal{K}_0(\mathbb{R}^\ell)$ be the family of non-empty compact subsets of $\mathbb{R}^\ell$. Recall that the Hausdorff metric $H$ on $\mathcal{K}_0(\mathbb{R}^\ell)$ is defined such that for any two $A, B \in \mathcal{K}_0(\mathbb{R}^\ell)$,
\[
H(A, B) = \max\left\{\sup_{a \in A} \dist(a, B), \sup_{b \in B} \dist(b, A)\right\}.
\]
For equivalent definitions of $H$, refer to [2]. The topology $\mathcal{T}_H$ on $\mathcal{K}_0(\mathbb{R}^\ell)$, generated by $H$, is called the Hausdorff metric topology. For a closed subset $M$ of $\mathbb{R}^\ell$, $\mathcal{K}_0(M)$ and the Hausdorff metric $H$ on $\mathcal{K}_0(M)$ can be defined similarly. When $G$ is a topological space, a non-empty compact-valued correspondence $F : G \rightrightarrows \mathbb{R}^\ell$ is called Hausdorff continuous if $F : G \to (\mathcal{K}_0(\mathbb{R}^\ell), \mathcal{T}_H)$ is continuous. This statement also holds when $\mathbb{R}^\ell$ is replaced by a closed subset $M$ of $\mathbb{R}^\ell$.

Let $\{A_n : n \geq 1\}$ be a sequence of non-empty subsets of $\mathbb{R}^\ell$. A point $x \in \mathbb{R}^\ell$ is called a limit point of $\{A_n : n \geq 1\}$ if there exist $N \geq 1$ and points $x_n \in A_n$ for each $n \geq N$ such that $\{x_n : n \geq N\}$ converges to $x$. The set of all limit points of $\{A_n : n \geq 1\}$ is denoted by $\text{Li} A_n$. Similarly, a point $x \in \mathbb{R}^\ell$ is called a cluster point of $\{A_n : n \geq 1\}$ if there exist positive integers $n_1 < n_2 < \cdots$ and for each $k$ a point
integrably bounded by the same function, then
\[ \int_s x \, ds \]
sequence of measurable selections
\[ x \in S, \]
If \((T, \Sigma, \mu) \Rightarrow \mathbb{R}^d \) correspondences. Recall that \( F \) Kuratowski-Ryll-Nardzewski Measurable Selection Theorem (\cite{11, 13}). It is clear that \( LiA_n \subseteq LsA_n \).

\[ \text{Li} A_n = \text{Ls} A_n = A \]
is called the limit of the sequence \( \{A_n : n \geq 1\} \). If \( A \) and all \( A_n \)'s are closed and contained in a compact subset \( M \subseteq \mathbb{R}^d \), then it is well known that
\[ \text{Li} A_n = \text{Ls} A_n = A \]
if and only if \( \{A_n : n \geq 1\} \) converges to \( A \) in Hausdorff metric topology on \( \mathcal{K}_0(M) \), refer to \( \cite{2} \).

Let \((T, \Sigma, \mu)\) be a measure space and \( \{F_n : n \geq 1\} \), \( F : (T, \Sigma, \mu) \Rightarrow \mathbb{R}^d \) be correspondences. Recall that \( F : (T, \Sigma, \mu) \Rightarrow \mathbb{R}^d \) is said to be lower measurable if
\[ F^{-1}(V) = \{t \in T : F(t) \cap V \neq \emptyset\} \in \Sigma \]
for every open subset \( V \) of \( \mathbb{R}^d \). It is well known that a non-empty closed-valued correspondence \( F : (T, \Sigma, \mu) \Rightarrow \mathbb{R}^d \) is lower measurable if and only if there exists a sequence of measurable selections \( \{f_n : n \geq 1\} \) of \( F \) such that for all \( t \in T \),
\[ F(t) = \{f_n(t) : n \geq 1\} \]
If all \( F_n \)'s are non-empty closed-valued and lower measurable and at least one of \( F_n \)'s is compact-valued, \( \bigcap_{n \geq 1} F_n \) is lower measurable, refer to \( \cite{11} \). If all \( F_n \)'s are integrably bounded by the same function, then
\[ \text{Ls} \int_T F_n d\mu \subseteq \int_T \text{Ls} F_n d\mu, \]
and
\[ \int_T \text{Li} F_n d\mu \subseteq \text{Li} \int_T F_n d\mu. \]
If \((S, \mathcal{S}, \nu)\) is another measure space and \( f : T \times S \rightarrow \mathbb{R}^d \) is a jointly measurable function, then it is well known that \( \int_T : f(\cdot, \cdot) d\nu : S \rightarrow \mathbb{R}^d \) is a measurable function. Let \( M \subseteq \mathbb{R}^d \) be endowed with the relative Euclidean topology, and \((Y, g)\) be a metric space. A function \( f : T \times M \rightarrow (Y, g) \) is called Carathéodory if \( f(\cdot, x) \) is measurable for all \( x \in M \), and \( f(t, \cdot) \) is continuous for all \( t \in T \). It is known that any Carathéodory function is jointly measurable with respect to the Borel structure on \( M \). A selection of \( F \) is a single-valued function \( f : (T, \Sigma, \mu) \rightarrow \mathbb{R}^d \) such that \( f(t) \in F(t) \) for almost all \( t \in T \). If a selection \( f \) of \( F \) is measurable (resp. integrable), then it is called a measurable (resp. an integrable) selection. Let \( \mathcal{F}_F \) denote the set of all integrable selections of \( F \). The integration of \( F \) over \( T \) in the sense of Aumann in \( \cite{3} \) is a subset of \( \mathbb{R}^d \), defined as
\[ \int_T F d\mu = \left\{ \int_T f d\mu : f \in \mathcal{F}_F \right\}. \]
If \( F \) is non-empty closed-valued and integrably bounded, then \( \int_T F d\mu \) is compact, refer to \( \cite{10} \). The following two theorems on measurable selections have been employed in this paper.

Kuratowski-Ryll-Nardzewski Measurable Selection Theorem (\cite{11, 13}). If \( F : (T, \Sigma, \mu) \Rightarrow \mathbb{R}^d \) is a closed-valued and lower measurable correspondence, then it has a measurable selection.
Aumann-Saint-Beuve Measurable Selection Theorem ([5][15]). Let \((T, \Sigma, \mu)\) be a complete finite measure space, \(B \subseteq \mathbb{R}^l\) be a Borel subset. If \(F : T \rightrightarrows B\) has a measurable graph, there exists a measurable function \(f : T \rightarrow B\) such that \(f(t) \in F(t)\) for all \(t \in T\).

Of course, the above two theorems were stated in more general forms in the literature. But here, they are adapted and presented in particular and simpler forms to fit in this paper.

ACKNOWLEDGEMENT. The authors are very grateful to He Wei for his valuable comments and suggestions on the early draft of the paper.

REFERENCES

[1] Allen, B.: Generic existence of completely revealing equilibria with uncertainty, when prices convey information, Econometrica 49, 1173–1199 (1981)
[2] Aubin, J.P., Frankowska, H.: Set-valued analysis. Springer (2008).
[3] Aumann, Robert J.: Integrals of set-valued functions, J. Math. Anal. Appl. 12, 1-12 (1965).
[4] Aumann, R. J.: Existence of competitive equilibria in markets with a continuum of traders, Econometrica 34, 1–17 (1966)
[5] Aumann, R. J.: Measurable utility and the measurable choice theorem, La Décision Colloque Internationaux du C.N.R.S., Paris 171, 15–26 (1969)
[6] Correia-da-Silva, J., Hervés-Beloso, C.: Prudent expectations equilibrium in economies with uncertain delivery, Econ. Theory 39. 67–92 (2009)
[7] de Castro, L. I., Pesce, M., Yannelis, N. C.: A new perspective to rational expectations: maximin rational expectation equilibrium, under preparation
[8] Gilboa, I., Schmeidler, D.: Maxmin expected utility with a non-unique prior, J. Math. Econ. 18, 141–153 (1989)
[9] Glycopantis, D., Muir, A., Yannelis, N.C.: Non-implementation of rational expectations as a perfect Bayesian equilibrium, Econ. Theory 26, 765–791 (2005)
[10] Hildenbrand, W.: Core and equilibria in large economies, Princeton University Press, 1974
[11] Himmelberg, C. J.: Measurable relations, Fund. Math. 87, 53-72 (1975)
[12] Kreps, D. M.: A note on ‘fulfilled expectations’ equilibrium, J. Econ. Theory 14(1), 32–43 (1977)
[13] Kuratowski, K., Ryll-Nardzewski, C.: A general theorem on selectors, Bull. Acad. Polon. Sci. 13, 397–403 (1965)
[14] Radner, R.: Rational expectation equilibrium: generic existence and information revealed by prices, Econometrica 47, 655-678 (1979)
[15] Saint-Beuve, M.-F.: On the extension of von Neumann-Aumann’s theorem, J. Funct. Anal. 17, 112–129 (1974)
[16] Yosida, K.: Functional analysis, Sixth edition. Springer (1980).

School of Computing and Mathematical Sciences, Auckland University of Technology, Private Bag 92006, Auckland 1142, New Zealand
E-mail address: anuj.bhowmik@aut.ac.nz

School of Computing and Mathematical Sciences, Auckland University of Technology, Private Bag 92006, Auckland 1142, New Zealand
E-mail address: jiling.cao@aut.ac.nz

Department of Economics, Henry B. Tippie College of Business, The University of Iowa, IA 52242-1994, USA, and Economics - School of Social Sciences, The University of Manchester, Manchester M13 9PL, UK
E-mail address: nicholasyannelis@gmail.com