Spectral bounds for vanishing of cohomology and the neighborhood complex of a random graph

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Abstract

In this article, we derive two spectral gap bounds for the reduced Laplacian of a general simplicial complex. Our two bounds are proven by comparing a simplicial complex in two different ways with a larger complex and with the corresponding clique complex respectively. Both of these bounds lead to generalizations of the result of Aharoni et al. (2005) [1] which is valid only for clique complexes. As an application, we decrease by a logarithmic factor, the upper bound for the threshold for vanishing of cohomology of the neighborhood complex of the Erdős-Rényi random graph derived by Kahle (2007) [14]. We also increase the lower bound for the above threshold by a polynomial factor.

Keywords: spectral bounds, Laplacian, cohomology, neighborhood complex, Erdős-Rényi random graphs.

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1 Introduction

Let G be a graph with vertex set V(G) (often to be abbreviated as V) and let L(G) denotes the (unnormalized) Laplacian of G. Let λ1(G) ≤ λ2(G) ≤ ⋯ ≤ λ|V(G)| denote the eigenvalues of L(G). Here, the second smallest eigenvalue λ2(G) is called the spectral gap. The clique complex of a graph G is the simplicial complex whose simplices are all subsets σ ⊂ V which spans a complete subgraph of G. We shall denote the kth reduced cohomology of a simplicial complex X by Hk(X). In this article, we always consider the reduced cohomology with real coefficients. For more detailed definitions, see section 2.

In [1], Aharoni et al. proved the following result which guarantees the vanishing of cohomology of a clique complex, provided the spectral gap of its 1-skeleton is large enough.

Theorem 1.1. [1] Theorem 1.2] Let X be the clique complex of a graph G. If \( \lambda_2(G) > \frac{k|V|}{k+1} \), then \( \tilde{H}^k(X) = 0 \).

Aharoni et al. ([1]) used Theorem [1.1] to find a lower bound of the homological connectivity of the independence complex of a graph G (a simplicial complex whose simplices are the

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Theorem 1.1 can be viewed as a global counterpart for clique complexes of spectral gap results of Garland ([8 Theorem 5.9]) and Ballman-Świątkowski ([6 Theorem 2.5]) for vanishing of cohomology of a simplicial complex. In their simplest form, these results say that for a pure \( k \)-dimensional finite simplicial complex \( \Delta \), if the spectral gap of link \( \text{lk}_\Delta(\tau) \) is sufficiently large for every \((k - 2)\)-dimensional simplex \( \tau \), then \( \tilde{H}_{k-1}(\Delta) = 0 \). A very powerful application of the afore-mentioned result is by Kahle ([15]) to derive sharp vanishing thresholds for cohomology of random clique complexes. See [10] for more applications of this spectral gap result in random topology. Recently, Hino and Kanazawa ([9, Theorem 2.5]) generalized this result of Garland and Ballman-Świątkowski, and upper bounded the \((d-1)\)-th Betti number of a pure \( d \)-dimensional simplicial complex \( \Delta \) by the sum (taken over all \((d-2)\)-dimensional simplices \( \tau \)) of the number of ‘suitably small’ eigenvalues in the spectrum of the laplacian of the link \( \text{lk}_\Delta(\tau) \). They used this quantitative version of the spectral gap result to prove weak laws for (persistent) lifetime sums of randomly weighted clique and \( d \)-dimensional complexes.

Motivated by applications of spectral gap bounds to random complexes, we seek to generalize Theorem 1.1 to more general simplicial complexes. We achieve two different generalizations (see Corollaries 1.5 and 1.7) by comparing an arbitrary simplicial complex with a larger complex and the corresponding clique complex in two different ways. Our aim in exploring this generalization was to obtain vanishing thresholds for cohomology in other random complex models. We use one of our generalizations to improve the vanishing threshold for cohomology of a random neighbourhood complex (see Theorem 1.8) by a logarithmic factor. After the result, we also discuss why it is difficult to apply Garland’s method and hence a different spectral gap result is needed. By computing the probabilities involved more precisely than [14], we also improve the lower bound by a polynomial factor.

The paper is organized as follows. In section 1.1 we introduce some notation, which we shall use in rest of the paper. In section 1.2 we state our results, which relates the cohomology and spectral gap. In section 1.3 we state the results about the neighborhood complexes of a random graph. We also discuss our improvements in relation to the results of [14]. We give the necessary preliminaries from graph theory and topology in section 2. Section 3 is dedicated to the proofs of the results stated in section 1.2 and 1.3.

1.1 Notations

We shall use the following notations throughout this paper. Let \( X \) be a (simplicial) complex on \( n \) vertices. We denote \( G_X \) as the 1-skeleton of \( X \), i.e., \( G_X \) is the graph whose vertices are the 0-dimensional simplices and edges are the 1-dimensional simplices of \( X \). Let \( X(k) \) denotes the set of all \( k \)-dimensional oriented simplices of \( X \). \( X \) is said to be a clique complex if for all \( k \geq 0 \), \( X(k) \) is the set of \((k + 1)\)-cliques in the graph \( G_X \). For \( k \geq -1 \), let \( C^k(X; \mathbb{R}) \) denote the space of real valued \( k \)-cochains of \( X \). Let \( \delta_k(X) : C^k(X; \mathbb{R}) \to C^{k+1}(X; \mathbb{R}) \) denote the coboundary operator.

For \( k \geq 0 \), let \( \delta_k^*(X) \) denote the adjoint of \( \delta_k(X) \) and let \( \Delta_k(X) = \delta_{k-1}(X)\delta_k^*(X) + \delta_k^*(X)\delta_k(X) \) (see section 2 for details). Let \( \mu_k(X) \) denote the minimal eigenvalue of \( \Delta_k(X) \). Observe that \( \lambda_2(G_X) = \mu_0(X) \). We again emphasize that we consider reduced cohomology
with real coefficients.

We shall now define two ways to measure the difference between two complexes. The first compares a complex to its subcomplex whereas the second compares a complex \(X\) to the corresponding clique complex of \(G_X\). Denoting the indicator function by \(1[\cdot]\), define for \(k \geq 1\)
\[
S_k(X, X') := \max_{\sigma \in X^{(k)}} \left\{ \sum_{\tau \in X^{(k+1)} \setminus X'^{(k+1)}} 1[\sigma \subset \tau] \right\},
\]
where \(X'\) is a subcomplex of \(X\). Throughout the article, we shall use \(X\) to denote a complex and we shall denote a subcomplex of \(X\) by \(X'\).

For a simplex \(\eta \in X\), the link of \(\eta\) is the complex defined as
\[
\text{lk}_X(\eta) = \{ \sigma \in X \mid \sigma \cup \eta \in X \text{ and } \sigma \cap \eta = \emptyset \}.
\]

For \(k \geq 1\) and \(1 \leq j \leq k + 1\), define
\[
D_k(X, j) := \max_{\sigma \in X^{(k)}} \left\{ \sum_{u} 1[u \notin \text{lk}_X(\sigma) \text{ and } \exists \text{ exactly } j \text{ vertices } v_1, \ldots, v_j \in \sigma \text{ such that } u \in \text{lk}_X(\sigma \setminus \{v_i\}) \forall 1 \leq i \leq j] \right\}.
\]

**Remark 1.2.** If the \(k\)-skeleton of \(X\) is a clique complex of \(G_X\), i.e., for \(i \leq k\), any \((i+1)\)-tuple of vertices of \(X\) form a simplex in \(X\) if and only if they induce a complete subgraph of \(G_X\), then \(u \in \text{lk}_X(\sigma \setminus \{v\}) \cap \text{lk}_X(\sigma \setminus \{w\})\) for some \(\{v, w\} \subseteq \sigma\) implies that \(u \in \text{lk}_X(\sigma \setminus \{v\}) \forall v \in \sigma\), i.e., any \((k+1)\)-subset of \(\sigma \cup \{u\}\) will be a \(k\)-simplex. Therefore, in this case \(D_k(X, j) = 0\) for all \(2 \leq j \leq k + 1\) and
\[
D_k(X, k + 1) = \max_{\sigma \in X^{(k)}} \left\{ \sum_{w} 1[w \notin \text{lk}_X(\sigma) \text{ and any } (k+1)\text{-subset of } \sigma \cup \{w\} \text{ is a } k\text{-simplex}] \right\}.
\]

Thus, if \(X\) is a clique complex then \(D_k(X, j) = 0\) for all \(2 \leq j \leq k + 1\).

1.2 Spectral gap and cohomology

We shall now present our two spectral gap results and corollaries that generalize Theorem 1.1. We first recall the following result from [1], which formed the crux of the proof of Theorem 1.1.

**Theorem 1.3.** [1, Theorem 1.1] Let \(X\) be the clique complex of \(G_X\). For \(k \geq 1\),
\[
k \mu_k(X) \geq (k + 1) \mu_{k-1}(X) - |V(G_X)|.
\]

We prove our first main spectral gap result by directly comparing the operators \(\delta_k(X)\), \(\delta_k(X')\). Following the theorem, we state a simple corollary which generalizes Theorem 1.1.

**Theorem 1.4.** For \(k \geq 1\),
\[
\mu_k(X') \geq \mu_k(X) - (k + 2) S_k(X, X').
\]
Corollary 1.5. Let $X$ be the clique complex of $G_X$ and the 1-skeleton of $X'$ is $G_X$. If
\[ \lambda_2(G_X) > \frac{k_n}{k+1} + \frac{k+2}{k+1} s_k(X, X'), \]
then $H^k(X') = 0$.

If $X' = X$, then $S_k(X, X') = 0$ and so Corollary 1.5 implies Theorem 1.1. Now, we present our generalization of Theorem 1.3 using $D_k(X, j)$’s and as before a simple corollary for later use.

Theorem 1.6. For $k \geq 1$,
\[ k \mu_k(X) \geq (k+1) \mu_{k-1}(X) - n - (k(k+1) + 1) \sum_{j=2}^{k+1} D_k(X, j). \] 

Corollary 1.7. Let $k$-skeleton of $X$ is the clique complex of $G_X$. If $\lambda_2(G_X) > \frac{k_n}{k+1} + (k + \frac{1}{k+1}) D_k(X, k+1)$, then $H^k(X) = 0$.

Since for a clique complex $X$, $D_k(X, j) = 0 \forall 2 \leq j \leq k+1$, we see that in this case Theorem 1.6 implies Theorem 1.3 and Corollary 1.7 implies Theorem 1.1. The proof of Theorem 1.6 follows the ideas of [1] but some of the terms that cancel out in the case of clique complexes do not cancel out for a general simplicial complex. Hence, it requires more careful bounding to derive suitable bounds.

1.3 Neighborhood complex of a random graph

We shall now introduce neighborhood complex of random graphs, recall results from [14] and state our results about the cohomology of the same. For more on random graphs, we refer the reader to [11] [7] and refer to [16] for a survey on random simplicial complexes.

The neighborhood complex, $N(G)$ of a graph $G$ is the simplicial complex whose simplices are those subsets $\sigma$ of $V$ which have a common neighbor. The concept of neighborhood complex was introduced by Lovász ([17]) in his proof of the famous Kneser conjecture. We now introduce the Erdős-Rényi random graph $G(n, p)$ on $n$ vertices and with edge-probability $p$. $G(n, p)$ is constructed by deleting edges of the complete graph on $n$ vertices independently of each other with probability $1 - p$ or equivalently the edges are retained independently of each other with probability $p$. In this article, we consider $p$ as a function of $n$. A graph property $\mathcal{P}$ is a class of graphs such that for any two isomorphic graphs either both belong to the class or both do not belong to the class. For any graph property $\mathcal{P}$, we say that $G(n, p) \in \mathcal{P}$ with high probability (w.h.p.) if $\mathbb{P}(G(n, p) \in \mathcal{P}) \to 1$ as $n \to \infty$. We shall also say $\mathcal{P}$ holds for $G(n, p)$ instead of $G(n, p) \in \mathcal{P}$.

In [14], M. Kahle considered the neighborhood complex of the Erdős-Rényi random graph. He showed that (see [14] Theorem 2.1]), if \( \binom{n}{k+2}(1 - p^{k+2})^{n-(k+2)} = o(1), \) then w.h.p. $\widetilde{H}^i(N(G(n, p))) = 0$, for $i \leq k$. In particular, if $p = \left( \frac{(k+2) \log n + c_n}{n} \right)^{1/(k+2)}$ for $c_n \to \infty$, then w.h.p. $\widetilde{H}^i(N(G(n, p))) = 0$ for $i \leq k$. Using Corollary 1.7 we achieve the following improvement on Kahle’s result.

Theorem 1.8. Let $k \geq 1$. If $p = \left( \frac{(k+1) \log n + c_n}{n} \right)^{1/(k+2)}$ with $c_n \to \infty$, then $\widetilde{H}^i(N(G(n, p))) = 0$ w.h.p. for $i \leq k$.
Note that in Theorem 1.8 \( \binom{n}{k+2} (1 - p^{k+2})^{n-(k+2)} \to \infty \) and therefore we cannot apply Kahle’s result in this case. His proof involves showing that for \( p \) satisfying \( \binom{n}{k+2} (1 - p^{k+2})^{n-(k+2)} = o(1) \), \( \mathcal{N}(G(n, p)) \) has the full \((k+1)\)-skeleton, i.e., any \( t \)-tuple of vertices form a \((t-1)\)-simplex in \( \mathcal{N}(G(n, p)) \) for \( t \leq k + 2 \). This trivially yields that \( \bar{H}^i(\mathcal{N}(G(n, p))) = 0 \) for \( i \leq k \). But one would expect that this is a very strong condition for vanishing of cohomology and our theorems shows that this can be reduced a little. We expect that our bound for the threshold for vanishing of cohomology to be reduced even further.

The above theorem is one of our motivations to prove spectral bounds for vanishing of cohomology for general complexes. This was inspired by the proof of a sharp threshold result for vanishing of cohomology of clique complexes of Erdős-Rényi random graphs in [15] which was proven using the spectral gap result of Garland and Ballman-Świątkowski. This required to show that the spectral gap of the normalized Laplacian of the 1-skeleton of all the links are sufficiently large. For a clique complex of an Erdős-Rényi random graph, it is easy to see that the 1-skeleton of a link is also an Erdős-Rényi random graph and hence by proving suitable spectral bounds for the normalized Laplacian of Erdős-Rényi random graphs, the result of Garland and Ballman-Świątkowski was used. But it is not easy to use the same argument to prove Theorem 1.8 as the 1-skeleton of the link of a simplex in neighborhood complex of an Erdős-Rényi random graph is not an Erdős-Rényi random graph. It has a complicated dependency structure making it harder to analyse the spectral gap of the corresponding random graph.

Kahle (see [14, Corollary 2.9]) also showed that for \( p = n^\alpha \), if \( -\frac{2}{k+1} < \alpha < -\frac{4}{3(k+1)} \), then w.h.p. \( \bar{H}^k(\mathcal{N}(G(n, p))) \neq 0 \). We derive more exact bounds for the probabilities involved but still use the same argument as that of [14] to extend this result as well.

**Proposition 1.9.** Let \( p = n^\alpha \). If \( -\frac{2}{k+1} < \alpha < -\frac{1}{k+1} \), then w.h.p. \( \bar{H}^k(\mathcal{N}(G(n, p))) \neq 0 \).

Despite the improvement of the bounds presented here, it is still an open problem to determine sharp bounds for vanishing of cohomology of neighborhood complexes. From Theorem 1.8, Proposition 1.9 and [14] Corollary 2.5, we summarize the known bounds as follows: For \( k \geq 1 \),

\[
\bar{H}^k(\mathcal{N}(G(n, p))) = 0 \quad \text{w.h.p. if } p = n^\alpha \text{ with } \alpha < \frac{1}{k+2} \text{ for } k \text{ even and } \alpha < \frac{4(k+2)}{(k+1)(k+3)} \text{ for } k \text{ odd,}
\]

\[
\bar{H}^k(\mathcal{N}(G(n, p))) \neq 0 \quad \text{w.h.p. if } p = n^\alpha \text{ with } \alpha < \frac{2}{k+1}, \quad \alpha < -\frac{1}{k+1},
\]

\[
\bar{H}^k(\mathcal{N}(G(n, p))) = 0 \quad \text{w.h.p. if } p = \left( \frac{(k+1)\log{n} + c_n}{n} \right)^{\frac{1}{k+2}} \text{ with } c_n \to \infty.
\]

**Remark 1.10.** In [14] Corollary 2.5 homology groups are given with integer coefficients, but the proof was by showing that \( \mathcal{N}(G(n, p)) \) deformation retracts onto a subcomplex of dimension \( k - 1 \). Hence, the same proof is valid irrespective of the coefficients of homology. Also, we have used that the homology and cohomology groups with real coefficients are isomorphic to each other.

## 2 Preliminaries

A **graph** \( G \) is a pair \( (V, E) \), where \( V \) is the set of vertices of \( G \) and \( E \subset V \times V \) called the set of edges. If \( (u, v) \in E \), it is also denoted by \( u \sim v \) and we say that \( u \) is adjacent to \( v \). For any
\[ A \subset V, \text{ the neighborhood of } A, N(A) := \{ u \in V \mid u \sim a \ \forall \ a \in A \}. \] The degree of a vertex \( v \) is denoted by \( \deg(v) \). For a subset \( X \subset V \), the induced subgraph \( G[X] \) is the subgraph whose set of vertices \( V(G[X]) = X \) and the set of edges \( E(G[X]) = \{(u,v) \in E \mid u,v \in X\} \).

The complete graph or a clique of order \( n \) is a graph on \( n \) vertices, where any two distinct vertices are adjacent and it is denoted by \( K_n \). All the graphs in this article are assumed to be simple i.e., \( (x,y) \in E \) implies \( x \neq y \).

The (unnormalized) Laplacian of a graph \( G \) is the \( |V| \times |V| \) matrix \( L(G) \) given by

\[
L(G)(x,y) := \begin{cases} 
\deg(x) & x = y, \\
-1 & (x,y) \in E, \\
0 & \text{otherwise}.
\end{cases}
\]

For details about Laplacian we refer the reader to [4]. We next introduce the concept of simplicial complexes, which are higher dimensional counterparts of graphs.

A finite (abstract) simplicial complex \( X \) is a family of subsets of a finite set, which is closed under the deletion of elements, i.e., if \( \alpha \subset X \) and \( \beta \subset \alpha \), then \( \beta \in X \). For \( \sigma \in X \), the dimension of \( \sigma \) is defined to be \( |\sigma| - 1 \) and denoted by \( \dim(\sigma) \). If \( \dim(\sigma) = k \), then it is said to be a \( k \)-dimensional simplex or \( k \)-simplex. The 0-dimensional simplices are called vertices of \( X \). We denote the set of vertices of \( X \) by \( V(X) \). The boundary of a \( k \)-dimensional simplex \( \sigma \) is the simplicial complex, consisting of all simplices \( \tau \subset \sigma \) of dimension \( \leq k - 1 \).

We refer the reader to book by Kozlov ([13]) for more details about simplicial complexes. Let \( X \) be a simplicial complex. Two ordering of vertices of a simplex \( \sigma = \{v_0,v_1,\ldots,v_k\} \) called equivalent if they differ from one another by an even permutation. Thus the ordering of these vertices of simplex divided into two equivalences classes. Each of these classes is called an orientation of \( \sigma \). An oriented simplex is a simplex \( \sigma \) together with an orientation and we denote it by \([v_0,\ldots,v_k]\).

Let each simplex of \( X \) having arbitrary but fixed orientation. Let \( X(k) \) denote the set of oriented \( k \)-simplices of \( X \). For \( k \geq 0 \), let \( C_k(X) \) denote the free abelian group with basis \( X(k) \), with the relation \([v_0,v_1,\ldots,v_k] = -[v_1,v_0,\ldots,v_k]\) for each \( k \)-simplex \( \sigma = \{v_0,\ldots,v_k\} \).

For \( k \geq 0 \), let \( C^k(X;\mathbb{R}) \) be the dual group \( \text{Hom}(C_k(X);\mathbb{R}) \). The elements of \( C^k(X;\mathbb{R}) \) are called \( k \)-cochains of \( X \). For an ordered \((i+1)\)-simplex \( \sigma = [v_0,\ldots,v_{i+1}] \) the \( j \)-face of \( \sigma \) is an ordered \( i \)-simplex \( \sigma_j = [v_0,\ldots,v_j,\ldots,v_{i+1}] \). The coboundary operator \( \delta_k(X) : C^k(X;\mathbb{R}) \to C^{k+1}(X;\mathbb{R}) \) is given by

\[
\delta_k(X)\phi(\sigma) := \sum_{j=0}^{k+1} (-1)^j \phi(\sigma_j).
\]

By letting \( C^{-1}(X;\mathbb{R}) = \mathbb{R} \), define \( \delta_{-1}(X) : C^{-1}(X;\mathbb{R}) \to C^0(X;\mathbb{R}) \) by \( \delta_{-1}(X)(x)(v) = x \) for all \( x \in \mathbb{R} \) and \( v \in X(0) \). It is well known that \( \delta_k\delta_{k-1} = 0 \) for all \( k \geq 1 \). For \( k \geq 0 \), the quotient \( \text{Ker} \delta_k(X) / \text{Im} \delta_{k-1}(X) \) is called the \( k \)-th reduced cohomology group of \( X \) with real coefficients and it is denoted by \( \tilde{H}^k(X) \). For more details about cohomology we refer the reader to [13].

For each \( k \geq -1 \) we can defined the standard inner product on \( C^k(X;\mathbb{R}) \) by \( \langle \phi,\psi \rangle := \sum_{\sigma \in X(k)} \phi(\sigma)\psi(\sigma) \) and the corresponding \( L^2 \) norm \( ||\phi|| := (\sum_{\sigma \in X(k)} \phi(\sigma)^2)^{\frac{1}{2}} \).
Let \( \delta^*_k(X) : C^{k+1}(X; \mathbb{R}) \to C^k(X; \mathbb{R}) \) denote the adjoint of \( \delta_k(X) \) with respect to these standard inner product, i.e., the unique operator satisfying \( \langle \delta_k(X) \phi, \psi \rangle = \langle \phi, \delta^*_k(X) \psi \rangle \) for all \( \phi \in C^k(X; \mathbb{R}) \) and \( \psi \in C^{k+1}(X; \mathbb{R}) \). The reduced \( k \)-Laplacian of \( X \) is the mapping

\[
\Delta_k(X) := \delta_{k-1}(X) \delta^*_{k-1}(X) + \delta^*_k(X) \delta_k(X) : C^k(X; \mathbb{R}) \to C^k(X; \mathbb{R}).
\]

It can be easily verified that if \( I \) denotes the \(|V(G_X)| \times |V(G_X)|\) matrix with all entries 1, then \( I + L(G_X) \) represents \( \Delta_0(X) \) with respect to the standard basis. In particular the minimal eigenvalue of \( \Delta_0(X) \) (i.e., \( \mu_0(X) \)) is \( \lambda_2(G_X) \). More details about the operator \( \Delta_k(X) \) can be found in [6] and [8].

We now recall the following well known simplicial Hodge theorem.

**Proposition 2.1.** For \( k \geq 0 \), \( \text{Ker} \, \Delta_k(X) \cong \tilde{H}^k(X) \).

### 3 Proofs

#### 3.1 Proofs of the results of section 1.2

Throughout this article, for any positive integer \( m \), we denote the set \( \{1, \ldots, m\} \) by \( [m] \).

Recall that, \( X \) is complex and \( X' \) is a subcomplex of \( X \). For two oriented simplices \( \eta \in X \) and \( \tau \in l\kappa(X(\eta)) \), \( \eta \tau \) denotes their oriented union, i.e., if \( \eta = [v_0, \ldots, v_k] \) and \( \tau = [u_0, \ldots, u_i] \), then \( \eta \tau = [v_0, \ldots, v_k, u_0, \ldots, u_i] \).

Throughout this article, for any \( k \)-cochain \( \phi \) of \( X' \), we also consider \( \phi \) as a cochain of \( X \) by simply taking \( \phi(\sigma) = 0 \) whenever \( \sigma \in X(k) \setminus X'(k) \) and a cochain \( \psi \) of \( X \) considered as a cochain of \( X' \) by taking restriction of \( \psi \) on \( X' \).

In the rest of the section, we shall abbreviate as follows : \( \delta_k = \delta_k(X) \), \( \delta'_k = \delta_k(X') \), \( \delta^*_k = \delta^*_k(X) \), \( \delta^*_k' = \delta^*_k(X') \), \( \Delta_k = \Delta_k(X) \) and \( \Delta'_k = \Delta'_k(X) \).

**Lemma 3.1.** For \( \phi \in C^k(X'; \mathbb{R}) \)

\[
||\delta^*_{k-1} \phi||^2 = ||\delta^*_{k-1} \phi||^2.
\]

**Proof.** For any \( \tau \in X(k-1) \) and \( \sigma \in X'(k-1) \), by the definition of \( \phi \) on \( X \), we have that

\[
\delta^*_{k-1} \phi(\tau) = \sum_{v \in l\kappa(X(\tau))} \phi(v\tau) = \sum_{v \in l\kappa(X'(\sigma))} \phi(v\sigma) = \delta^*_{k-1} \phi(\sigma).
\]

\[\square\]

We shall require the following simple inequality : For any real numbers \( x_1, x_2, \ldots, x_n \), it holds that

\[
\sum_{\{i,j\}, i \neq j} x_i x_j \leq \frac{(n-1)}{2} \sum_{i=1}^n x_i^2.
\]

**Lemma 3.2.** For \( \phi \in C^k(X'; \mathbb{R}) \), recalling \( S_k(X, X') \) as defined in (1), we have that

\[
||\delta_k \phi||^2 - ||\delta^*_k \phi||^2 \leq (k+2)S_k(X, X')||\phi||^2.
\]
Proof.

\[
||\delta_k \phi||^2 - ||\delta'_k \phi||^2 = \sum_{\tau \in X(k+1)\setminus X'(k+1)} \left( \delta_k \phi(\tau) \right)^2
\]

\[
= \sum_{\tau \in X(k+1)\setminus X'(k+1)} \sum_{i=0}^{k+2} \sum_{j=0}^{k+2} (-1)^i \phi(\tau_i) (-1)^j \phi(\tau_j)
\]

\[
= \sum_{\tau \in X(k+1)\setminus X'(k+1)} \left( \sum_{i=0}^{k+2} (-1)^{2i} \phi(\tau_i)^2 + \sum_{i \neq j} (-1)^{i+j} \phi(\tau_i) \phi(\tau_j) \right)
\]

\[
\leq \sum_{\tau \in X(k+1)\setminus X'(k+1)} \left( \sum_{i=0}^{k+2} \phi(\tau_i)^2 + (k+1) \sum_{i=0}^{k+2} \phi(\tau_i)^2 \right)
\]

where last inequality follows from (7). Hence, we derive that

\[
||\delta_k \phi||^2 - ||\delta'_k \phi||^2 \leq (k+2) \sum_{\tau \in X(k+1)\setminus X'(k+1)} \sum_{i=0}^{k+2} \phi(\tau_i)^2
\]

\[
= (k+2) \sum_{\sigma \in X(k)} \phi(\sigma)^2 \sum_{\tau \in X(k+1)\setminus X'(k+1)} 1[\sigma \subset \tau]
\]

\[
= (k+2) \sum_{\sigma \in X'(k)} \phi(\sigma)^2 \sum_{\tau \in X(k+1)\setminus X'(k+1)} 1[\sigma \subset \tau]
\]

\[
\leq (k+2)S_k(X, X')||\phi||^2.
\]

\[
\square
\]

We now recall the following well known Minmax principle.

**Proposition 3.3.** (Minmax principle ; Corollary III.1.2 & Exercise III.1.3) Let \(A\) be the self-adjoint operator on inner product space \((V, \langle \cdot, \cdot \rangle)\). Let \(\lambda_{\min}\) be the minimum eigenvalue of \(A\). For \(0 \neq x \in V\),

\[
\lambda_{\min} \leq \frac{\langle Ax, x \rangle}{\langle x, x \rangle}.
\]

**Proof of Theorem** Let \(0 \neq \phi \in C^k(X'; \mathbb{R})\) be an eigenvector of \(\Delta'_k = \Delta_k(X')\) with eigenvalue \(\mu'_k = \mu_k(X')\). Using (3) and (5) for the first inequality below along with definition of Laplacian and min-max principle (Proposition 3.3), we derive that

\[
\mu'_k||\phi||^2 = \langle \Delta'_k \phi, \phi \rangle = ||\delta'_k \phi||^2 + ||\delta''_k \phi||^2
\]

\[
\geq ||\delta_k \phi||^2 + ||\delta''_k \phi||^2 - (k+2)S_k(X, X')||\phi||^2
\]

\[
= \langle \Delta_k \phi, \phi \rangle - (k+2)S_k(X, X')||\phi||^2
\]

\[
\geq \mu_k||\phi||^2 - (k+2)S_k(X, X')||\phi||^2.
\]

\[
\square
\]

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Proof of Corollary \cite{1.3} By applying induction on \( k \) in Theorem \cite{1.3} we derive that \( \mu_k(X) \geq (k+1)\mu_0(X) - kn \). Now, substituting the above bound in Theorem \cite{1.4} and using the fact that \( \mu_0(X) = \lambda_2(G_X) \), we obtain

\[
\mu_k(X') \geq (k+1)\lambda_2(G_X) - kn - (k+2)S_k(X, X').
\]

Hence, if \( \lambda_2(G_X) > \frac{kn}{k+1} + \frac{k+2}{k+1}S_k(X, X') \), then we have that \( \mu_k(X') > 0 \) and Proposition \cite{2.1} implies that \( \tilde{H}^k(X') = 0 \).

For an \( i \)-simplex \( \eta \in X \) let \( \deg(\eta) \) denote the number of \( (i+1) \)-simplices in \( X \) which contain \( \eta \). For \( \phi \in C^k(X) \) and a vertex \( u \in V(X) \) define \( \phi_u \in C^{k-1}(X; \mathbb{R}) \) by

\[
\phi_u(\tau) = \begin{cases} 
\phi(\tau) & \text{if } u \in lk_X(\tau), \\
0 & \text{otherwise.}
\end{cases}
\]

We now recall some results from \cite{1}, which was stated and proved for a clique complex but the same proof is also valid for any general simplicial complex.

Claim 3.4. \cite{1} Claim 3.1] For \( \phi \in C^k(X; \mathbb{R}) \)

\[
||\delta_k \phi||^2 = \sum_{\sigma \in X(k)} \deg(\sigma)\phi(\sigma)^2 - 2 \sum_{\eta \in X(k-1)} \sum_{vw \in lk_X(\eta)} \phi(v\eta)\phi(w\eta).
\]

Claim 3.5. \cite{1} Claim 3.3] For \( \phi \in C^k(X; \mathbb{R}) \)

\[
\sum_{u \in V(X)} ||\delta_{k-2} \phi_u||^2 = k||\delta_{k-1} \phi||^2.
\]

Claim 3.6. \cite{1} page 7, upto second equality in the proof of Claim 3.2]

\[
\sum_{u \in V(X)} ||\delta_{k-1} \phi_u||^2 = \sum_{\sigma \in X(k)} \left( \sum_{\tau \in \sigma(k-1)} \deg(\tau) \right) \phi(\sigma)^2 - 2 \sum_{\eta \in X(k-2)} \sum_{vw \in lk_X(\eta)} \sum_{u \in lk_X(v\eta) \cap lk_X(w\eta)} \phi(vu\eta)\phi(wu\eta).
\]

Proof of Theorem \cite{1.6} Let \( 0 \neq \psi \in C^k(X; \mathbb{R}) \) be an eigenvector of \( \Delta_k \) with eigenvalue \( \mu_k(X) \). By double counting

\[
\sum_{v \in V(X)} ||\psi_v||^2 = (k+1)||\psi||^2.
\]

We first derive the expression for \( \sum_{u \in V(X)} \langle \Delta_{k-1} \psi_u, \psi_u \rangle \). We shall use \cite{1.1} in the second equality below.

\[
\sum_{u \in V(X)} \langle \Delta_{k-1} \psi_u, \psi_u \rangle = \sum_{u \in V(X)} (||\delta_{k-1} \psi_u||^2 + ||\delta_{k-2} \psi_u||^2)
\]

\[
= \sum_{u \in V(X)} ||\delta_{k-2} \psi_u||^2 + \sum_{\sigma \in X(k)} \left( \sum_{\tau \in \sigma(k-1)} \deg(\tau) \right) \psi(\sigma)^2 - 2 \sum_{\eta \in X(k-2)} \sum_{vw \in lk_X(\eta)} \sum_{u \in lk_X(v\eta) \cap lk_X(w\eta)} \psi(vu\eta)\psi(wu\eta).
\]
Now, we relate \( \sum_{u \in V(X)} \langle \Delta_{k-1} \psi_u, \psi_u \rangle \) to \( k \langle \Delta_k \psi, \psi \rangle \). In the following derivation, we shall use (9) and (10) for the second equality and the third equality will follow from (13).

\[
k \langle \Delta_k \psi, \psi \rangle = k(||\delta_k \psi||^2 + ||\delta_{k-1} \psi||^2)
\]
\[
= k \left( \sum_{\sigma \in X(k)} \deg(\sigma) |\psi(\sigma)|^2 - 2 \sum_{\eta \in X(k-1)} \sum_{vw \in lk_X(\eta)} \psi(\eta) \psi(\eta) \right) \\
+ \sum_{u \in V(X)} ||\delta_{k-2} \psi_u||^2 \\
= k \sum_{\sigma \in X(k)} \deg(\sigma) |\psi(\sigma)|^2 - 2k \sum_{\eta \in X(k-1)} \sum_{vw \in lk_X(\eta)} \psi(\eta) \psi(\eta) \\
+ \sum_{u \in V(X)} \langle \Delta_{k-1} \psi_u, \psi_u \rangle - \sum_{\sigma \in X(k)} \left( \sum_{\tau \in \sigma(k-1)} \deg(\tau) \right) |\psi(\sigma)|^2 \\
+ 2 \sum_{\eta \in X(k-2)} \sum_{vw \in lk_X(\eta)} \sum_{u \in lk_X(\eta) \cap lk_X(\eta)} \psi(\nu \eta) \psi(\nu \eta).
\]

Thus, from the previous two derivations, we obtain that

\[
k \langle \Delta_k \psi, \psi \rangle = \sum_{u \in V(X)} \langle \Delta_{k-1} \psi_u, \psi_u \rangle + I_1 - I_2 - T,
\]

where

\[
T := \sum_{\sigma \in X(k)} \left( \sum_{\tau \in \sigma(k-1)} \deg(\tau) - k \deg(\sigma) \right) |\psi(\sigma)|^2, \quad \text{(14)}
\]

\[
I_1 := 2 \sum_{\eta \in X(k-2)} \sum_{vw \in lk_X(\eta)} \sum_{u \in lk_X(\eta) \cap lk_X(\eta)} \psi(\nu \eta) \psi(\nu \eta), \quad \text{(15)}
\]

and

\[
I_2 := 2k \sum_{\eta \in X(k-1)} \sum_{vw \in lk_X(\eta)} \psi(\nu \eta) \psi(\nu \eta). \quad \text{(16)}
\]

We now use the bounds for \( |I_1 - I_2| \) and \( T \) given in Claims 3.7 and 3.8 which we prove later at the end of this section. Combining Claims 3.7 and 3.8 we have the following.

\[
k \langle \Delta_k \psi, \psi \rangle \geq \sum_{v \in V(X)} \langle \Delta_{k-1} \psi_v, \psi_v \rangle - (|V(X)| + (k(k + 1) + 1) \sum_{j=2}^{k+1} D_k(X, j)) ||\psi||^2. \quad \text{(17)}
\]

From (17) and (12) we have

\[
k \mu_k(X)||\psi||^2 = k \langle \Delta_k \psi, \psi \rangle \geq \sum_{v \in V(X)} \langle \Delta_{k-1} \psi_v, \psi_v \rangle - (n + (k(k + 1) + 1) \sum_{j=2}^{k+1} D_k(X, j)) ||\psi||^2 \\
\geq \mu_{k-1}(X) \sum_{v \in V(X)} ||\psi_v||^2 - (n + (k(k + 1) + 1) \sum_{j=2}^{k+1} D_k(X, j)) ||\psi||^2
\]

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\[ ((k + 1)\mu_{k-1}(X) - n - (k(k + 1) + 1) \sum_{j=2}^{k+1} D_k(X, j))||\psi||^2. \]

**Proof of Corollary 1.7** Since the \(k\)-skeleton of \(X\) is a clique complex, \(D_i(X, i) = 0\) for all \(2 \leq i \leq k\). Hence, Theorem 1.6 implies that \(\mu_k(X) \geq (k + 1)\mu_0(X) - kn - (k(k + 1) + 1)D_k(X, k + 1)\). Therefore, if \(\mu_0(X) = \lambda_2(G) > \frac{k}{k+1} + (k + \frac{1}{k+1})D_k(X, k+1)\), then \(\mu_k(X) > 0\) and result follows from Proposition 2.1.

Now, we shall give proofs of Claims 3.7 and 3.8. For a \(k\)-simplex \(\sigma\) and \(v_1, \ldots, v_l \in \sigma\), \(\hat{\sigma}_{v_1, \ldots, v_l} := \sigma \setminus \{v_1, \ldots, v_l\}\) is a \((k - l)\)-simplex. Recalling the definition of \(D_i(X, j), i \geq 1\) and \(1 \leq j \leq i + 1\) from (2) and the definitions of \(T, I_1\) and \(I_2\) from (14), (15) and (16) respectively.

**Claim 3.7.**

\[ |I_1 - I_2| \leq k(k + 1) \sum_{j=2}^{k+1} D_k(X, j)||\psi||^2. \]  

**(Proof.)** In this proof, we use the convention that \(\psi(\tau) = 0\), whenever \(\tau \notin X(k)\). Observe that the expression for \(I_2\) given in (16) can be rewritten as,

\[ I_2 = 2 \sum_{\eta \in X(k-2)} \sum_{\{v, w\} \subseteq \eta} \sum_{u \in lK_X(vu)} \psi(vu)\psi(wu). \]

By recalling the definition of \(I_1\) from (15), we obtain,

\[ I_1 - I_2 = 2 \sum_{\eta \in X(k-2)} \sum_{\{v, w\} \subseteq \eta} \sum_{u \in lK_X(vu)} \psi(vu)\psi(wu) \]

\[ = 2 \sum_{\sigma \in X(k)} \sum_{\{v, w\} \subseteq \sigma} \sum_{u \in lK_X(vu) \cap lK_X(wu)} \psi(uv)\psi(wu) \]

\[ = 2 \sum_{\sigma \in X(k)} \sum_{\{v, w\} \subseteq \sigma} \sum_{u \notin lK_X(\sigma)} 1[u \in lK_X(\hat{\sigma}_v) \cap lK_X(\hat{\sigma}_w)]\psi(uv)\psi(wu) \]

\[ = 2 \sum_{\sigma \in X(k)} \sum_{i=1}^{j} \sum_{\{v, w\} \subseteq \sigma} 1[u \in lK_X(\hat{\sigma}_v) \cap lK_X(\hat{\sigma}_w)]\psi(uv)\psi(wu) \]

\[ = 2 \sum_{\{v_1, \ldots, v_{k+2}\} \notin X(k+1)} \sum_{i=1}^{k+1} 1[\gamma^i = \{v_1, \ldots, \hat{v}_i, \ldots, v_{k+2}\} \in X(k)] \]

\[ \sum_{j=2}^{k+1} 1[v_i \in lK_X(\hat{\gamma}^{i}_{v_{i+1}}) \text{ for exactly } j \text{ vertices } v_{i+1}, \ldots, v_j \in \gamma^i] \]
\[
\sum_{i \notin \{p, q\} \subseteq [k+2]} 1[v_i \in lk_X(\hat{\gamma} v_p) \cap lk_X(\hat{\gamma} v_q)] \psi(v_i v_p \hat{\gamma} v_p v_q) \psi(v_i v_q \hat{\gamma} v_q v_q)
\]
\[
= 2 \sum_{\{v_1, \ldots, v_{k+2}\} \notin X(k+1)} 1[\gamma^i = \{v_1, \ldots, v_{k+2}\} \subseteq X(k)]
\]
\[
\sum_{j=2}^{k+1} \sum_{i \in [k+2]} \sum_{p \in [k+2] \setminus \{i\}} \psi(v_i v_p \hat{\gamma} v_p v_q) \psi(v_i v_q \hat{\gamma} v_q v_q).
\]
Hence, using \((7)\) we obtain
\[
|I_1 - I_2| \leq 2 \cdot k \frac{k}{2} \sum_{\{v_1, \ldots, v_{k+2}\} \notin X(k+1)} 1[\gamma^i = \{v_1, \ldots, v_{k+2}\} \subseteq X(k)]
\]
\[
\sum_{j=2}^{k+1} \sum_{i \in [k+2]} \sum_{p \in [k+2] \setminus \{i\}} \psi(\gamma^p)^2
\]
\[
= k(k+1) \sum_{\sigma \in X(k)} \psi(\sigma)^2 \sum_{j=2}^{k+1} D_k(X, j)
\]
\[
= k(k+1) \sum_{\sigma \in X(k)} \psi(\sigma)^2 \sum_{j=2}^{k+1} D_k(X, j) ||\psi||^2.
\]

Claim 3.8.
\[
T \leq (|V(X)| + \sum_{j=2}^{k+1} D_k(X, j)) ||\psi||^2. \tag{19}
\]
Proof.
\[
T = \sum_{\sigma \in X(k)} \left( \sum_{\tau \in \sigma(k-1)} \deg(\tau) - k \deg(\sigma) \right) \psi(\sigma)^2
\]
\[
= \sum_{\sigma \in X(k)} \left( \sum_{u \in \sigma} \sum_{v \in lk_X(\hat{\sigma} v)} \psi(\sigma)^2 - k \sum_{v \in lk_X(\sigma)} \psi(\sigma)^2 \right)
\]

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Proof of Theorem 1.8. Let \( p = \left( \frac{(k+1) \log n + c_n}{n} \right)^{\frac{1}{k+1}} \), where \( c_n \to \infty \). The two main proof steps are (i) \( \mathcal{N}(G(n,p)) \) has full \( k \)-skeleton and (ii) \( n^{-1} \mathbb{E}(D_k(\mathcal{N}(G(n,p)), k+1)) \to 0 \).

From (ii) and Markov’s inequality, we have that for all \( \epsilon > 0 \), \( \mathbb{P}(D_k(\mathcal{N}(G(n,p)), k+1) \geq \epsilon \frac{n}{k(k+1)+1}) \to 0 \) and hence w.h.p. \( D_k(\mathcal{N}(G(n,p)), k+1) < \frac{n}{k(k+1)+1} \). Let \( G \) be the 1-skeleton of \( \mathcal{N}(G(n,p)) \). Since \( G \) is a complete graph w.h.p. (because of (i)), the spectral gap \( \lambda_2(G) = n \) and therefore Corollary 1.7 and (i) imply that \( \tilde{H}^k(\mathcal{N}(G(n,p))) = 0 \). This completes the proof provided we establish (i) and (ii).

First, we show (i). The expected number of \((k+1)\)-tuples of vertices in \( G(n,p) \) with no neighbor is

\[
\binom{n}{k+1} (1 - p^{k+1})^{n-k-1} \leq \binom{n}{k+1} e^{-(n-k-1)p^{k+1}} = \binom{n}{k+1} e^{-n \left( \frac{(k+1) \log n + c_n}{n} \right)^{\frac{k+1}{k+2}}} e^{(k+1) \left( \frac{(k+1) \log n + c_n}{n} \right)^{\frac{k+1}{k+2}}} = \binom{n}{k+1} e^{-n^{\frac{k+1}{k+2}} \left( (k+1) \log n + c_n \right)^{\frac{k+1}{k+2}}} e^{(k+1) \left( \frac{(k+1) \log n + c_n}{n} \right)^{\frac{k+1}{k+2}}}.
\]

\[3.2 \quad \text{Proofs of the results of section 1.3}\]

Proof of Theorem 1.8
= o(1).

Hence, w.h.p. \( \mathcal{N}(G(n,p)) \) has full k-skeleton. In particular, w.h.p. k-skeleton of \( \mathcal{N}(G(n,p)) \) is a clique complex. It is well known that, if a complex has full k-skeleton then it has trivial cohomology in all dimensions less than \( k \). Therefore \( \tilde{H}^i(\mathcal{N}(G(n,p))) = 0 \) for \( i < k \).

Now, we shall establish (ii). Let \( B_k \) be the number of subcomplexes of \( \mathcal{N}(G(n,p)) \), which are isomorphic to the simplicial boundary of a \((k + 1)\)-simplex. Since, k-skeleton of \( \mathcal{N}(G(n,p)) \) is a clique complex, from [14, Theorem 3.9] we observe that \( D_k(\mathcal{N}(G(n,p)), k + 1) \leq B_k \). Thus it suffices to show that \( n^{-1}\mathbb{E}(B_k) \to 0 \) to prove (ii).

The rest of the proof is to compute \( \mathbb{E}(B_k) \) and show the above. For \( 0 \leq i \leq \binom{k+2}{2} \), let \( C_i \) denotes the number of graphs on \( k + 2 \) vertices with \( i \) edges. For any \( \{v_1, \ldots, v_{k+2}\} \subset [n] \), the probability that the induced subcomplex of \( \mathcal{N}(G(n,p)) \) on \( \{v_1, \ldots, v_{k+2}\} \) is isomorphic to the simplicial boundary of a \((k + 1)\)-simplex, is bounded above by \( (1 - p^{k+2})^{n-k-2} \). Therefore

\[
\frac{\mathbb{E}B_k}{n} \leq \frac{1}{n} \binom{n}{k+2} \sum_{i=0}^{\binom{k+2}{2}} C_i p^i (1-p)^{\binom{k+2}{2}-i}(1-p^{k+2})^{n-k-2} \\
\leq \frac{1}{n} \binom{n}{k+2} \sum_{i=0}^{\binom{k+2}{2}} C_i p^i (1-p)^{\binom{k+2}{2}-i} e^{-(n-k-2)p^{k+2}} \\
= \frac{1}{n} \binom{n}{k+2} \sum_{i=0}^{\binom{k+2}{2}} C_i p^i (1-p)^{\binom{k+2}{2}-i} e^{-(n-k-2)\left(\frac{(k+1)\log n+c}{n}\right)} \\
= \frac{1}{n} \binom{n}{k+2} n^{-(k+2)} e^{-cn} e^{(k+2)\left(\frac{(k+1)\log n+c}{n}\right)} \sum_{i=0}^{\binom{k+2}{2}} C_i (1-p)^{\binom{k+2}{2}-i} p^i.
\]

Now using that \( c_n \to \infty \) and \( p \to 0 \), we derive that \( n^{-1}\mathbb{E}(B_k) \to 0 \) completing the proof of (ii) as well as that of the theorem.

Let \( U = \{u_1, \ldots, u_r\}, V = \{v_1, \ldots, v_r\} \) be subsets of \([n]\) such that \( U \cap V = \emptyset \). The graph \( X_{U,V} \) is defined as the graph with vertex set \( U \cup V \) and edges \( u_i \sim u_j \) and \( u_i \sim v_j \) for all \( 1 \leq i \neq j \leq r \). To prove Proposition 1.3 we need the following result relating \( X_{U,V} \) graphs to cohomology. For two sequences of real numbers \( a_n, b_n, n \geq 1 \), we shall use the notation \( a_n = o(b_n) \) to denote that \( a_n/b_n \to 0 \) as \( n \to \infty \).

**Proposition 3.9.** [14] Theorem 2.7| If \( H \) is any graph containing a maximal clique of order \( r \) that cannot be extended to an \( X_{U,V} \) subgraph for some \( U, V \), then \( \mathcal{N}(H) \) retract onto a sphere \( S^{r-2} \).

**Proof of Proposition 3.9** Let \( p = n^\alpha \), where \( \frac{2}{r-1} < \alpha < \frac{1}{r-1} \), \( r \geq 2 \). We shall set \( G_n = G(n,p) \). Let

\[
\Lambda_r := \{A \subset [n] \mid |A| = r, G_n[A] \text{ is a maximal clique and } G_n[A] \not\subseteq X_{A,A'} \text{ for all } A' \text{ disjoint and } |A'| = r\}.
\]
For a \( A \subset [n] \) with \( |A| = r \), let

\[
I_A := 1[\text{\( G_n[A] \) is a \( r \)-clique}], \quad J_A := \prod_{A': |A'| = r, A' \not\subseteq A} 1[\text{\( G_n[A'] \) is not a clique}],
\]

\[
K_A := \prod_{A': |A'| = r, A' \cap A = \emptyset} 1[\text{\( G_n[A] \) \( \not\subseteq \) \( X_{A,A'} \)].
\]

Then, \( \Lambda_r = \sum_{A \subset [n], |A| = r} I_A J_A K_A \). By Proposition 3.9, the proof is complete provided we show that \( \Lambda_r \geq 1 \) w.h.p.. To do so, we shall use the second moment bound, i.e.,

\[
\mathbb{P}(\Lambda_r \geq 1) \geq \frac{(\mathbb{E} \Lambda_r)^2}{\mathbb{E} \Lambda_r^2}.
\]

To use the second moment bound, we first derive a lower bound for \( \mathbb{E}(\Lambda_r) \). Fix \( A = [r] \) in the below derivation.

\[
\mathbb{E}(\Lambda_r) = \binom{n}{r} \mathbb{E}(I_A J_A K_A) = \binom{n}{r} \mathbb{E}\left( I_A(J_A - J_A 1_{\bigcup_{A': |A'| = r, A' \cap A = \emptyset} \{ G_n[A] \subset X_{A,A'} \}}) \right)
\]

\[
\geq \binom{n}{r} \mathbb{E}\left( I_A\left( J_A - \sum_{A' = \{u_1, \ldots, u_r\}, A' \cap A = \emptyset} 1[G_n[A] \subset X_{A,A'}] 1[i \sim u_i \ \forall i] \prod_{v \not\in A' \cup A} 1[i \sim v \ \text{for some} \ i \in A] \right) \right)
\]

\[
\geq \binom{n}{r} \mathbb{E}\left( 1[i \sim j \ \forall 1 \leq i \neq j \leq r] \mathbb{E}\left( \prod_{v \not\in A} 1[i \sim v \ \text{for some} \ i \in A] \right) - \right)
\]

\[
\sum_{A' = \{u_1, \ldots, u_r\}, A' \cap A = \emptyset} 1[u_i \sim j \ \forall 1 \leq i \neq j \leq r] 1[i \sim u_i \ \forall i] \prod_{v \not\in A' \cup A} 1[i \sim v \ \text{for some} \ i \in A] \right) \right)
\]

\[
= \binom{n}{r} p^{(\binom{r}{2})} (1 - p^r)^{n-r} - \binom{n-r}{r} p^{(r-1)} (1 - p)^r (1 - p^r)^{n-2r}
\]

\[
\geq \binom{n}{r} p^{(\binom{r}{2})} (1 - p^r)^{n-r} (1 - n^r p^{(r-1)} (1 - p)^r (1 - p^r)^r - r)
\]

where in the equality in the penultimate line we have used the independence between the corresponding indicator random variables as they depend on disjoint sets of edges. Since \( p = n^\alpha \) for \( \alpha < \frac{1}{r-1} \), we have that \( n^r p^{(r-1)} (1 - p)^r (1 - p^r)^r \to 0 \) and hence for large enough \( n \), we derive that

\[
\mathbb{E}(\Lambda_r) \geq \binom{n}{r} p^{(\binom{r}{2})} (1 - p^r)^{n-r} (1 - o(1)). \tag{20}
\]
Now, we proceed to derive upper bounds for the second moment.

$$\Lambda_r^2 = \sum_{i=0}^{r} \sum_{|A_1|=|A_2|=r \atop |A_1 \cap A_2|=i} I_{A_1}J_{A_1}K_{A_1}I_{A_2}J_{A_2}K_{A_2} \leq \sum_{i=0}^{r} Y_i,$$

where $Y_i := \sum_{|A_1|=|A_2|=r \atop |A_1 \cap A_2|=i} I_{A_1}I_{A_2}$ for $0 \leq i \leq r$. Hence, using (20), we derive that for large enough $n$

$$\frac{\mathbb{E} \Lambda_r^2}{(\mathbb{E} \Lambda_r)^2} \leq \frac{1}{(\mathbb{E} \Lambda_r)^2} \sum_{i=0}^{r} \mathbb{E} Y_i$$

$$= \frac{1}{\left(\binom{n}{r} p^{1/2}(1-p)^{n-r}(1-o(1))\right)^2} \sum_{i=0}^{r} \left(\frac{n}{2r}\right)^{r-1-i} \frac{p^{r(1-\alpha)(i-1)}}{2r-i}$$

$$= \frac{\binom{n}{2r}}{(n-r)^2(1-p)^{2(n-r)}(1-o(1))^2} + \sum_{i=1}^{r} \left(\frac{n}{2r}\right)^{r-i} \frac{p^{r(1-\alpha)(i-1)}}{2r-i} \frac{n^{2(n-r)}(1-p)^{2(n-r)}(1-o(1))^2}{n^{2r}}$$

$$\leq \frac{\binom{n}{2r}}{(n-r)^2(1-p)^{2(n-r)}(1-o(1))^2} + C \sum_{i=1}^{r} \left(\frac{n}{2r}\right)^{r-i} n^{2(n-r)}(1-p)^{2(n-r)}(1-o(1))^2$$

$$= 1 + o(1), \quad \text{(21)}$$

where $C$ is a constant and (21) follows as $1 + \alpha(r - 1) < 0$, $i + \alpha\frac{i(i-1)}{2} > 0$ (because $\alpha > \frac{-2}{r-1}$) and further $(1 - p^{r})^{2(n-r)} = e^{-2np^{r}}(1 + o(1)) = 1 + o(1)$ for large $n$.

From (21), we conclude that $\lim \inf_{n \to \infty} \frac{\mathbb{E} \Lambda_r^2}{(\mathbb{E} \Lambda_r)^2} = 1$. Therefore by the second moment bound, we derive that $\mathbb{P}(\Lambda_r \geq 1) \to 1$ as $n \to \infty$ as required. \(\blacksquare\)

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