Asymptotically free theories based on discrete subgroups * †

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We study the critical behavior of discrete spin models related to the 2d O(3) non-linear sigma model. Precise numerical results suggest that models with sufficiently large discrete subgroups are in the same universality class as the original sigma model. We observe that at least up to correlation lengths $\xi \approx 300$ the cut-off effects follow effectively an $\propto a$ behaviour both in the O(3) and in the dodecahedron model.

1. Introduction

Here we summarize some results of a recent paper [1] where discretized version of 2d O(3) sigma model and of 4d SU(N) (N=2,3) gauge models were studied near their critical points. Here we discuss only the 2d sigma model. For further results and a more extensive list of references we also refer to that paper.

We consider models with discretized spin variables taking values from the vertices of regular polyhedra. The symmetry group is accordingly reduced from O(3) to the corresponding discrete subgroup. At strong coupling (i.e. large temperature), the fluctuations are large, hence the effects of restricting the spins to a discrete set are expected to be small. However, at sufficiently small coupling the discrete system has to be frozen. Decreasing $g$ from the strong coupling regime one expects a second order phase transition at some value $g_c$. We investigate the quantum field theory obtained in the limit $g \to g_c$ and suggest that for sufficiently large subgroups (icosahedron and dodecahedron) the resulting theory is the same as the one given by the original O(3) model in the $g \to 0$ limit.

The latter being asymptotically free (AF) this sounds surprising. Our conjecture that the discrete model is AF should be understood in the sense that the physical running coupling goes to zero as the corresponding momentum scale goes to infinity. Here we study only the question whether the discrete model is equivalent to the original O(3) model (which we can establish only numerically, within our numerical precision, of course). The title refers to the standard wisdom that the O(3) model is AF – a belief debated for long time by Patrascioiu and Seiler (for references see [4]). Here we do not study this question.

The discrete subgroups were introduced [2] in the SU(2) Yang-Mills theory as an approximation thought to be valid until the Wilson loops measured at the same bare coupling start to disagree. In contrast to this we compare the theories with discrete and continuous groups at the same (large) correlation length. That a discrete symmetry of the action can be enhanced to the continuous group at the criticality is known for the case of the 2d XY-model [3].

The investigations reported in ref. [1] and here, were inspired by the works of Patrascioiu and Seiler on the dodecahedron model [4] where they observed that it behaves as the O(3) model. We investigated the conjecture that the discrete and continuous models are in the same universality class to a high precision, down to O(0.1%).

We consider here discretized spin models when the spin vectors point to the vertices of a regular polyhedron embedded into the sphere $S_2$. If not stated otherwise, we consider the standard nearest neighbour (nn) action. The spin models with small subgroups of O(3) (tetrahedron, octahedron and cube, with 4,6 and 8 directions, respectively) are not equivalent to the original O(3) model. The tetrahedron is equivalent to a $q = 4$
Potts model, while the cube to 3 independent Ising models. Besides the standard O(3) action we also compared some of the results to those for the O(3) fixed point action [9].

We have chosen physical quantities which can be measured with high precision: the finite size scaling function (FSSF), the renormalized zero momentum 4-point function $g_R$ and the same coupling $g_R(z)$ in a finite physical volume. It is the latter quantity which can be measured with the highest precision.

2. The finite-size scaling function

We consider an $L \times L$ periodic box and measure the second moment correlation length $\xi(L)$. At the same inverse coupling $\beta$ we also measure the quantity $\xi(2L)$ on a $2L \times 2L$ lattice, and study the ratio $\xi(2L)/\xi(L)$ as a function of $\xi(L)/L$. The technique of finite size scaling (FSS) was used very effectively by Lüscher, Weisz and Wolff [5] to obtain the running coupling in the O(3) model. Instead of the (exponential) correlation length defined in a strip $L \times \infty$ used in [5] we use the second moment correlation length defined in a square box by

$$\xi(L) = \frac{1}{2 \sin(\pi/L)} \sqrt{\frac{G_2(0)}{G_2(k_0)}} - 1 , \quad (1)$$

where $G_2(k)$ is the 2-point spin correlation function in Fourier space and $k_0 = (2\pi/L, 0)$. The finite-size scaling function for this quantity has been measured by Caracciolo et al. [6] in the same model for a large set of $\xi(L)/L$ values. We have chosen to measure around the value $\xi(L)/L \approx 0.4$.

Fig. 1 shows the FSSF for O(3) and different subgroups. Points with symbols of the same shape belong to the same subgroup at different values of $L$, with the largest symbol corresponding to the largest $L$. While the tetrahedron, octahedron and cube deviate strongly from the O(3) curve and move away with increasing $L$, the icosahedron and dodecahedron results are close to O(3), and at largest $L$ lie on the curve within the small statistical errors.

3. The renormalized zero momentum 4-point coupling $g_R$

Define the quantity $g_R(z)$ as

$$g_R(z) = \left( \frac{L}{\xi(L)} \right)^2 \left( 1 + \frac{2}{N} - \frac{\langle (M^2)^2 \rangle}{\langle M^2 \rangle^2} \right) , \quad (2)$$

where $\xi(L)$ is the second moment correlation length, eq. (1), $z = L/\xi(L)$, $N = 3$ for O(3) and $M$ is the magnetization,

$$M^a = \sum_x S^a(x) . \quad (3)$$

The coupling $g_R = g_R(\infty)$ is defined as the infinite volume limit $z \to \infty$, where first the continuum limit $\xi \to \infty$ is taken for fixed $z$. For the O(3) $\sigma$-model $g_R$ has been calculated using the form-factor bootstrap method: $g_R = 6.770(17)$ in agreement with the MC result $g_R = 6.77(2)$ [7].

We have measured $g_R(z)$ for the icosahedron and dodecahedron models for $z \sim 6$ and extrapolated to infinite volume using the finite size formula of ref. [6]. Figure 2 gives the deviation from the O(3) result as the function of $\xi$. For comparison, the coupling from the cubic group is
$g_{R}^{\text{cubic}} = 1/3 g_{R}^{\text{Ising}}$ with $g_{R}^{\text{Ising}} = 14.6975(1)$ from ref.\cite{8}, i.e. for this case the deviation from O(3) is 1.87(2).

Interpolating in $z \approx z_0$ gives $g_R(z_0)$. By measuring this quantity we could compare results for different models to a high precision.

Table 1 summarizes the results for the three actions considered.

| $L$  | $\beta$ | $z$     | $g_R(z_0)$       |
|------|---------|---------|------------------|
| 10   | 1.3637  | 2.3199(2)| 3.0105(1)       |
| 28   | 1.5785  | 2.3200(3)| 3.0765(2)       |
| 56   | 1.697   | 2.3201(4)| 3.0979(2)       |
| 112  | 1.8074  | 2.3230(4)| 3.1095(2)       |
| 224  | 1.9176  | 2.3178(5)| 3.1145(3)       |
| 448  | 2.028   | 2.3204(10)| 3.1175(6)      |
| 10   | 0.8565  | 2.3202(4)| 3.1064(2)       |
| 20   | 1.000   | 2.3197(5)| 3.1140(3)       |
| 40   | 1.1347  | 2.3235(7)| 3.1163(4)       |
| 80   | 1.2659  | 2.3199(9)| 3.1180(5)       |
| 28   | 1.5636  | 2.3177(7)| 3.0683(5)       |
| 56   | 1.670   | 2.3190(9)| 3.0923(7)       |
| 112  | 1.7626  | 2.3211(6)| 3.1051(5)       |
| 224  | 1.8465  | 2.3185(7)| 3.1106(5)       |

Table 1 Measurements of $z = L/\xi(L)$ and $g_R(z_0)$ for the O(3) nn action, the FP action and dodecahedron action, respectively. The last column was obtained by interpolating $g_R(z) - c(z-z_0)$ with $c = 2$ and $z_0 = 2.32$.

Figure 3 shows $g_R(z_0)$ as the function of $a/L$ for the three actions considered.

The striking feature here is that even the standard O(3) action shows a linear, $O(a/L)$, cut-off effect, while the perturbative approach predicts an $O((a/L)^2)$ behaviour (up to log($L/a$) factors).

Below we present for the O(3) nn action fits with different functional forms, together with the corresponding value of $\chi^2$/dof:

$$3.1201(2) - 1.224(9) \frac{a}{L},$$
$$3.1205(4) - 1.28(5) \frac{a}{L} + 1.1(1.1) \left( \frac{a}{L} \right)^2,$$
$$3.1207(6) - 1.1(1) \frac{a}{L} - 0.04(4) \frac{a}{L} \log \frac{L}{a},$$
Figure 3. Cut-off dependence of $g_R(z_0)$ (at $z_0 = 2.32$) for the standard, FP and dodecahedron actions. The data points are connected by straight lines to guide the eye.

$$3.1168(3) + 103(5)(\frac{a}{L})^2 - 40.4(1.4)(\frac{a}{L})^2 \log \frac{L}{a},$$

The corresponding values of $\chi^2$/dof are 2.1, 2.2, 2.5 and 4.1, respectively. (The fits for the FP action and dodecahedron are given in [1].)

One concludes that an $a/L$ term is needed to describe the cut-off effects. (Note also the large coefficients in the $a^2$ fit!)

5. On the cut-off effects

For bosonic models in any order of perturbation theory the cut-off effects are given by $O(a^2)$ terms, up to logarithmic corrections [10]. Although not proven beyond perturbation theory, this form of cut-off effects is generally assumed when one extrapolates to the continuum limit. Therefore an $O(a)$ is rather surprising. Note that already in ref. [7] such behaviour for $g_R$ at $z \geq 5$ was preferred. However, due to larger errors this form was not compelling.

Because the Ansatz for the cut-off corrections can significantly modify the predictions in the continuum limit, and because a non-standard behaviour of the lattice artifacts can have other theoretical implications, it is important to investigate the related questions further. In particular, it will be interesting to study other non-perturbative quantities as the LWW coupling $\tilde{R}$ in the $O(3)$ sigma model. Obviously, the same questions in $d=4$ gauge theories are even more relevant.

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