CROSSED PRODUCTS OF 4-ALGEBRAS. APPLICATIONS

G. MILITARU

Abstract. A 4-algebra is a commutative algebra \( A \) over a field \( k \) such that \( (a^2)^2 = 0 \), for all \( a \in A \). We have proved recently [22] that 4-algebras play a prominent role in the classification of finite dimensional Bernstein algebras. Let \( A \) be a 4-algebra, \( E \) a vector space and \( \pi : E \to A \) a surjective linear map with \( V = \ker(\pi) \). All 4-algebra structures on \( E \) such that \( \pi : E \to A \) is an algebra map are described and classified by a global cohomological object \( \mathcal{G}H^2(A, V) \). Any such 4-algebra is isomorphic to a crossed product \( V \# A \) and \( \mathcal{G}H^2(A, V) \) is a coproduct, over all 4-algebra structures \( \cdot \) on \( V \), of all non-abelian cohomologies \( H^n_{\text{nonab}}(A, (V, \cdot V)) \), which are the classifying objects for all extensions of \( A \) by \( V \). Several applications and examples are provided: in particular, \( \mathcal{G}H^2(A, k) \) and \( \mathcal{G}H^2(k, V) \) are explicitly computed and the Galois group \( \text{Gal}(V \# A/V) \) of the extension \( V \hookrightarrow V \# A \) is described.

Introduction

Bernstein algebras were introduced independently by Lyubich [19] and Holgate [17] as an algebraic tool to answer the Bernstein problem [5] which consists in classifying all possible situations of a population that attains genetic equilibrium after one generation ([23, Section 4], [25, Chapter 9]). A Bernstein algebra is a commutative algebra \( B \) over a field \( k \) of characteristic \( \neq 2 \) such that there exists a non-zero morphism of algebras \( \omega : B \to k \) such that \( (x^2)^2 = \omega(x)^2 x^2 \), for all \( x \in B \). For algebrists, the Bernstein problem can now be rephrased as follows: for a given positive integer \( n \), describle and classify all simplicial stochastic Bernstein \( \mathbb{R}\)-algebras of dimension \( n \). For more details we refer to the work of Gutierrez Fernandez [13] which completely solved this problem.

Now, leaving aside the simplicial stochastic condition and replacing \( \mathbb{R} \) by an arbitrary field \( k \), the problem of classifying finite dimensional Bernstein algebras is still open: it was solved only up to dimension 4 and there are partial answers in dimensions 5 or 6 (see [8, 9, 14, 17, 20, 25]).

If \((B, \omega)\) is a Bernstein algebra, then its barideal \( A := \ker(\omega) \) is itself a commutative algebra satisfying the compatibilities \( (a^2)^2 = 0 \), for all \( a \in A \): we called this class of algebras 4-algebras [22] and we have proved that they play the key role in the structure and classification of Bernstein algebras. It is worth pointing out that 4-algebras are a special case of admissible cubic algebras as introduced by Elduque and Okubo [11]: these are commutative algebras satisfying the identity \( (a^2)^2 = N(a) a \), where \( N : A \to k \) is a
cubic form. In [22, Theorem 2.6] we prove that any Bernstein algebra $B$ is isomorphic to a semidirect product $A \ltimes_{\cdot, \Omega} k$, where $(A, \cdot)$ is a 4-algebra and $\Omega$ is a Bernstein operator on $A$, i.e. $\Omega = \Omega^2 \in \operatorname{End}_k(A)$ is an idempotent endomorphism of $A$ such that for any $x \in A$:

$$x^2 \cdot \Omega(x) = 0, \quad \Omega(x)^2 + \Omega(x^2) = x^2.$$ 

The classification of two such Bernstein algebras $A \ltimes_{\cdot, \Omega} k$ and $A' \ltimes_{\cdot', \Omega'} k$ is also proven in [22, Theorem 2.7 and Theorem 2.11]. Based on this, the first step we have to take in order to classify all Bernstein algebras is to classify all 4-algebras of a given finite dimension $n$. For this reason the paper is devoted to the study of this class of algebras. An efficient tool for classifying finite objects and a source for developing cohomology theories is the extension problem introduced by Hölder [16] at the level of groups and intensively studied in the last 100 years for many categories of algebras such as associative algebras [12], Lie algebras [7], Hopf algebras [4], Poisson algebras [18], Lie-Rinehart algebras [6], etc. For 4-algebras the extension problem consists of the following question: Let $A$ and $V$ be two given 4-algebras. Describe and classify all extensions of $A$ by $V$, i.e. all triples $(E, i, \pi)$ consisting of a 4-algebra $E$ and two morphisms of algebras that fit into an exact sequence of the form: $0 \longrightarrow V \overset{i}{\longrightarrow} E \overset{\pi}{\longrightarrow} A \longrightarrow 0$.

Two extensions $(E, i, \pi)$ and $(E', i', \pi')$ of $A$ by $V$ are called equivalent (or cohomologous) if there exists a morphism $\varphi : E \to E'$ of algebras that stabilizes $V$ and co-stabilizes $A$, i.e. $\varphi \circ i = i'$ and $\pi' \circ \varphi = \pi$. Any such a morphism is an isomorphism and we denote by $\operatorname{Ext}(A, V)$ the set of equivalence classes of all 4-algebras that are extensions of $A$ by $V$; an answer to the extension problem means to calculate explicitly $\operatorname{Ext}(A, V)$ for two given 4-algebras $A$ and $V$. The Schreier [24] approach to the extension problem for groups works, mutatis-mutandis, also for 4-algebras as for others varieties of algebras: the classifying object $\operatorname{Ext}(A, V)$ is parameterized by the non-abelian cohomology $\pi^2_{\text{nab}}(A, V)$ (Corollary 2.12). More general than the extension problem is what we have called [21] global extension problem (GE-problem) and was studied for Leibniz algebras, associative algebras, Poisson algebras or Jacobi-Jordan algebras [1, 2, 3]. The GE-problem, formulated for 4-algebras, is the following question:

**Global Extension Problem.** Let $A$ be a 4-algebra, $E$ a vector space and $\pi : E \to A$ a linear epimorphism of vector spaces. Describe and classify the set of all 4-algebra structures that can be defined on $E$ such that $\pi : E \to A$ becomes a morphism of algebras.

The difference between the GE-problem and the classical extension problem is explained in detail in [1]. Proposition 2.4 proves that any such a 4-algebra structure $\cdot_E$ on $E$ is isomorphic to a crossed product $V \# A = V \#_{\left(\cdot, f, \cdot' \cdot V\right)} A$, which is a 4-algebra associated to $A$ and $V := \operatorname{Ker}(\pi)$ connected by a weak action $\triangleright : A \times V \to V$, a symmetric non-abelian 2-cocycle $f : A \times A \to V$ and a 4-algebra structure $\cdot_V$ on $V$ satisfying the axioms of Proposition 2.2. The crossed product $V \# A$ is a 4-algebra containg $V$ as an ideal and the Galois group $\operatorname{Gal}(V \# A/V)$ of the extension $V \hookrightarrow V \# A$ is described in Corollary 2.9 as a subgroup of the semidirect product of groups $\operatorname{Hom}_k(A, V) \ltimes \operatorname{GL}_k(A)$. The main result of the paper is Theorem 2.11 that gives the theoretical answer to the GE-problem: the classifying object for the GE-problem is parameterized by a global non-abelian cohomological object denoted by $\mathbb{G}\mathbb{H}^2(A, V)$. Corollary 2.12 proves that
CROSSED PRODUCTS OF 4-ALGEBRAS. APPLICATIONS

$\mathbb{G}H^2(A, V)$ is the coproduct of all non-abelian cohomologies $\mathbb{H}^2(A, (V, \cdot_V))$, the latter being the classifying object for the classical extension problem at the level of 4-algebras. Several examples and applications are given in Section 3: in particular, $\mathbb{G}H^2(A, k)$ and $\mathbb{G}H^2(k, V)$ are computed and the structure of metabelian 4-algebras is given in Corollary 3.6.

1. Preliminaries

Throughout this paper all vector spaces, linear or bilinear maps are over a field $k$ of characteristic $\neq 2$. For a family of sets $(A_i)_{i \in I}$ we shall denote by $\Pi_{i \in I} A_i$ their coproduct, i.e. $\Pi_{i \in I} A_i$ is the disjoint union of all sets $A_i$. If $A$ and $V$ are two vector spaces, $\text{Hom}_k(A, V)$ denotes the vector space of all linear maps $A \to V$ and $\text{Sym}(A \times A; V)$ the set of all symmetric bilinear maps $f : A \times A \to V$; $\text{End}_k(A)$ is the usual associative and unital endomorphisms algebra of $A$ and $\text{GL}_k(A)$ is the automorphisms group of $A$.

We denote by $\text{Hom}_k(A, V) \times \text{GL}_k(A) := \text{Hom}_k(A, V) \times \text{GL}_k(A)$ the semidirect product of groups having the multiplication defined for any $(r, \alpha)$ and $(r', \alpha') \in \text{Hom}_k(A, V) \times \text{GL}_k(A)$ by:

$$(r, \alpha) \cdot (r', \alpha') := (r' + r \circ \alpha', \alpha \circ \alpha').$$

A 4-algebra [22] is a vector space $A$ together with a bilinear map $\cdot : A \times A \to A$, called multiplication, such that for any $a, b \in A$ we have:

$$a \cdot b = b \cdot a, \quad (a^2)^2 = 0. \tag{2}$$

The concepts of subalgebras, ideals, morphisms of algebras, etc. for 4-algebras are defined in the obvious way. The class of 4-algebras were studied before in [15] where it was proved that any 4-algebra of dimension $\leq 7$ is solvable and it was conjectured that any finite dimensional 4-algebra is solvable. Any vector space $V$ is a 4-algebra with the trivial multiplication $x \cdot y := 0$, for all $x, y \in V$: we call this algebra abelian and it will be denoted by $V_0$. If $(A, \omega)$ is a Bernstein algebra, then the barideal $\text{Ker}(\omega)$ is a 4-algebra. Conversely, if $A$ is a 4-algebra and $\Omega = \Omega^2 \in \text{End}_k(A)$ is a Bernstein operator on $A$, then $A \times k$ has a canonical structure of Bernstein algebra with the barideal $A$ (for details see [22, Proposition 2.3]). Linearizing several times the second compatibility of (2) we obtain that in a 4-algebra over a field of characteristic $\neq 2, 3$ the following relations hold [15]:

$$a^2 \cdot (a \cdot b) = 0, \quad a^2 \cdot b^2 + 2(a \cdot b)^2 = 0, \quad a^2 \cdot (b \cdot c) + 2(a \cdot b) \cdot (a \cdot c) = 0,$$

$$(a \cdot b) \cdot (c \cdot d) + (a \cdot c) \cdot (b \cdot d) + (a \cdot d) \cdot (b \cdot c) = 0$$

for all $a, b, c, d \in A$. For a 4-algebra $A$ we denote by $A' := A \cdot A$ its derived algebra, i.e. $A'$ is the $k$-subspace of $A$ generated by all $a \cdot b$, for any $a, b \in A$. Similarly to the groups or Lie algebras theory, a 4-algebra $A$ is called metabelian if $A'$ is an abelian subalgebra of $A$, i.e. $(a \cdot b) \cdot (c \cdot d) = 0$, for all $a, b, c, d \in A$. Throughout this paper we use the following convention: the multiplication of a 4-algebra $A$ will be written on the elements of a $k$-basis $\{e_i | i \in I\}$ of $A$ and undefined multiplications are all zero.
Examples 1.1. (1) Any 1-dimensional 4-algebra is isomorphic to the abelian one: $e_1 \cdot e_1 := 0$. We can easily prove that, up to an isomorphism, there are exactly three 2-dimensional 4-algebras, namely the abelian one and the algebras with the basis $\{e_1, e_2\}$ and the multiplication given by:

$$A_1: \quad e_1^2 = e_2, \quad e_1 e_2 = e_2^2 = 0,$$

$$A_2: \quad e_1 e_2 = e_2, \quad e_1^2 = e_2^2 = 0.$$

(2) Let $n$ be a positive integer and $h(2n + 1)$ the $(2n + 1)$-dimensional algebra having $\{e_1, \cdots, e_n, f_1, \cdots, f_n, z\}$ as a basis and multiplication defined by $e_i \cdot f_i = f_i \cdot e_i := z$, for all $i = 1, \cdots, n$ and the other products of basis elements are zero. Then $h(2n + 1)$ is a 4-algebra called the commutative Heisenberg 4-algebra.

(3) Let $m$ and $n$ be two positive integers and two bilinear maps $\triangleright : k^m \times k^n \to k^n$, $f : k^m \times k^m \to k^n$ such that $f$ is symmetric. Let $\{e_i \mid i = 1, \cdots, n\}$ be a basis of $k^n$ and $\{f_j \mid j = 1, \cdots, m\}$ a basis of $k^m$. Let $A$ be the $(n + m)$-dimensional algebra with the basis $\{e_i, f_j \mid i = 1, \cdots, n, j = 1, \cdots, m\}$ and the multiplication defined by:

$$e_i \cdot f_j = f_j \cdot e_i := f_j \triangleright e_i, \quad f_j \cdot f_l := f(f_j, f_l)$$

for all $i = 1, \cdots, n, j, l = 1, \cdots, m$, and the other products of basis elements are zero. Then $A$ is a metabelian 4-algebra and will be denoted by $\text{Met}^m_n(\triangleright, f)$: Corollary 3.6 will prove that any finite dimensional metabelian 4-algebra is isomorphic to such an algebra $\text{Met}^m_n(\triangleright, f)$.

Similarly, with other classes of non-associative algebras we define the concept of modules/representations over 4-algebras as follows:

Definition 1.2. An $A$-module over a 4-algebra $A$ is a vector space $V$ with a bilinear map $\triangleright : A \times V \to V$, called action of $A$ on $V$, such that for any $a \in A$ and $x \in V$ we have:

$$a^2 \triangleright (a \triangleright x) = 0. \quad (3)$$

A representation of a 4-algebra $A$ on a vector space $V$ is a linear map $\varphi : A \to \text{End}_k(V)$ such that for any $a \in A$ we have $\varphi(a^2) \circ \varphi(a) = 0$, in the endomorphism algebra of $V$.

Remark 1.3. The axiom (3) of defining modules over a 4-algebra was influenced by the viewpoint of Eilenberg [10] of defining modules over a given object $O$ in a $k$-linear category $C$: that is, a vector space $V$ with a bilinear map $\triangleright : O \times V \to V$ such that the trivial extension $V \times O$, with the multiplication given by $(x, a) \cdot (y, b) := (a \triangleright y + b \triangleright x, a \cdot b)$, has to be an object inside the $k$-linear category $C$. For details see Example 2.3 below.

Of course, representations and modules over a 4-algebras are equivalent concepts and $A$ is a module over itself with $a \triangleright b := a \cdot b$, for all $a, b \in A$. 
2. Crossed products and the global extension problem for 4-algebras

Let $A$ be a 4-algebra, $E$ a vector space, $\pi : E \to A$ a linear epimorphism of vector spaces with $V := \text{Ker}(\pi)$ and denote by $i : V \to E$ the inclusion map. We recall that a linear map $\varphi : E \to E$ stabilizes $V$ (resp. co-stabilizes $A$) if $\varphi \circ i = i$ (resp. $\pi \circ \varphi = \pi$). Two 4-algebra structures $\cdot : E$ and $\cdot' : E$ on $E$ such that $\pi : E \to A$ is a morphism of 4-algebras are called cohomologous and we denote this by $(E, \cdot) \approx (E, \cdot')$, if there exists an algebra map $\varphi : (E, \cdot) \to (E, \cdot')$ which stabilizes $V$ and co-stabilizes $A$. Any such morphism is an isomorphism and therefore $\approx$ is an equivalence relation on the set of all 4-algebra structures on $E$ such that $\pi : E \to A$ is an algebra map. The set of all equivalence classes via the equivalence relation $\approx$ will be denoted by $\text{Gext} (A, E)$ and it is the classifying object for the $GE$-problem for 4-algebras. In this section we will prove that $\text{Gext} (A, E)$ is parameterized by a global non-abelian cohomological object $\mathbb{G}H^2 (A, V)$ which will be explicitly constructed. First, we need to introduce the crossed product of 4-algebras:

**Definition 2.1.** Let $A = (A, \cdot)$ be a 4-algebra and $V$ a vector space. A crossed data of $A$ by $V$ is a system $\mathcal{C} (A, V) = (\triangleright, f, \cdot_V)$ consisting of three bilinear maps

$$\triangleright : A \times V \to V, \quad f : A \times A \to V, \quad \cdot_V : V \times V \to V.$$ 

For a crossed data $\mathcal{C} (A, V) = (\triangleright, f, \cdot_V)$ we denote by $V\#A = V\#(\triangleright, f, \cdot_V) A$ the vector space $V \times A$ with the multiplication $\odot$ defined for any $a, b \in A$ and $x, y \in V$ by:

$$(x, a) \odot (y, b) := (x \cdot_V y + a \triangleright y + b \triangleright x + f(a, b), \; a \cdot b). \quad (4)$$

$V\#A$ is called the crossed product associated to $\mathcal{C} (A, V)$ if it is a 4-algebra with the multiplication given by $(4)$. In this case $\mathcal{C} (A, V) = (\triangleright, f, \cdot_V)$ is called a crossed system of $A$ by $V$. The set of all crossed systems of $A$ by $V$ will be denoted by $\mathcal{CS} (A, V)$. Our first result provides the necessary and sufficient conditions for $V\#A$ to be a crossed product:

**Proposition 2.2.** Let $A = (A, \cdot)$ be a 4-algebra, $V$ a vector space and $\mathcal{C} (A, V) = (\triangleright, f, \cdot_V)$ a crossed data of $A$ by $V$. Then $V\#A$ is a crossed product if and only if the following compatibility conditions hold:

1. $(V, \cdot_V)$ is a 4-algebra and the bilinear map $f : A \times A \to V$ is symmetric;
2. $f(a^2, a^2) + f(a, a) \cdot_V f(a, a) + 2 a^2 \triangleright f(a, a) = 0$;
3. $a^2 \triangleright x^2 + x^2 \cdot_V f(a, a) + 2 x^2 \cdot_V (a \triangleright x) + 2 (a \triangleright x)^2 + 2 (a \triangleright x) \cdot_V f(a, a) + 2 a^2 \triangleright (a \triangleright x) = 0$, for all $a \in A$ and $x \in V$.

Borrowing the terminology from crossed products of Hopf algebras [4], the symmetric bilinear map $f : A \times A \to V$ satisfying (CS2) is called a non-abelian 2-cocycle while the axiom (CS3) we called the twisted module condition for $\triangleright : A \times V \to V$.

**Proof.** First of all we can easily prove that the multiplication $\odot$ defined by $(4)$ is commutative if and only if $f : A \times A \to V$ is symmetric and $\cdot_V : V \times V \to V$ is commutative algebra structure on $V$. Assume that $\odot$ is commutative. Then for any $x \in V$ and $a \in A$
we have that \((x,a)^2 = (x^2 + 2a \triangleright x + f(a,a), a^2)\). Based on this and taking into account that \(A\) is a 4-algebra we obtain that \(((x,a)^2)^2 = (0,0)\) if and only if
\[
(x^2)^2 + 4(a \triangleright x)^2 + f(a,a)^2 + 4 x^2 \cdot V (a \triangleright x) + 2 x^2 \cdot V f(a,a) + \\
+ 4(a \triangleright x) \cdot V f(a,a) + 2a^2 \triangleright x^2 + 4a^2 \triangleright (a \triangleright x) + 2a^2 \triangleright f(a,a) + f(a^2,a^2) = 0.
\]
This equation holds for \(a := 0\) if and only if \((x^2)^2 = 0\), for all \(x \in V\) (i.e. \((V, \cdot V)\) is a 4-algebra) and it holds for \(x := 0\) if and only if (CS2) holds. Assuming these conditions, we obtain that the above equation holds if and only if (CS3) holds and this finishes the proof.

From now on a crossed system of a 4-algebra \(A\) by a vector space \(V\) will be seen as a system of bilinear maps \(C(A,V) = (\triangleright, f, \cdot V)\) satisfying axioms (CS1)-(CS3) of Proposition 2.2.

**Examples 2.3.** (1) Let \(C(A,V) = (\triangleright, f, \cdot V)\) be a crossed data of a 4-algebra \(A\) by \(V\) such that \(f\) is the trivial map, i.e. \(f(a,b) := 0\), for all \(a, b \in A\). Then, applying Proposition 2.2 we obtain that \((\triangleright, f := 0, \cdot V)\) is a crossed system if and only if \((V, \cdot)\) is a 4-algebra and for all \(a \in A, x \in V\) we have:
\[
a^2 \triangleright x^2 + 2 x^2 \cdot V (a \triangleright x) + 2(a \triangleright x)^2 + 2a^2 \triangleright (a \triangleright x) = 0. \tag{5}
\]
The associated crossed product will be called a *semidirect product* of 4-algebras \(A\) and \(V\) and will be denoted by \(V \ltimes A := V \ltimes (\triangleright, f, \cdot V)\). The terminology is motivated below in Corollary 2.7: exactly as in the case of groups or Lie algebras, this construction describes split epimorphisms in the category of 4-algebras.

Assume, in addition, that the multiplication \(\cdot V\) is also the trivial map. Then, the compatibility condition (5) becomes \(a^2 \triangleright (a \triangleright x) = 0\), for all \(a \in A\) and \(x \in V\), i.e. \((V, \triangleright)\) is a \(A\)-module as we introduced in Definition 1.2. In this case, the semidirect product \(V \ltimes A\) is called the *trivial extension* of the 4-algebra \(A\) by the \(A\)-module \(V\).

(2) The following special case of crossed products play an important role in the classification of finite dimensional 4-algebras. Assume that \(A\) is the abelian 4-algebra. Then a crossed data \(C(A,V) = (\triangleright, f, \cdot V)\) of \(A\) by \(V\) is a crossed system if and only if \((V, \cdot)\) is a 4-algebra, \(f : A \times A \to V\) is a symmetric bilinear map such that:
\[
f(a,a)^2 = 0, \quad x^2 \cdot V f(a,a) + 2 x^2 \cdot V (a \triangleright x) + 2(a \triangleright x)^2 + 2(a \triangleright x) \cdot V f(a,a) = 0 \tag{6}
\]
for all \(a \in A\) and \(x \in V\). In this case, the associated crossed product \(V \# A\) will be called the *twisted product* of \(V\) and \(A\). If \(\{e_i \mid i \in I\}\) is a basis of \(V\) and \(\{f_j \mid j \in J\}\) is a basis of \(A\), then the twisted product \(V \# A\) is the 4-algebra having \(\{e_i, f_j \mid i \in I, j \in J\}\) as a basis and the multiplication given by:
\[
e_i \circ e_j := e_j \cdot V e_i, \quad e_i \circ f_l = f_l \circ e_i := f_l \triangleright e_i, \quad f_l \circ f_m := f(f_l, f_m) \tag{7}
\]
for all \(i, j \in I\) and \(l, m \in J\). In Corollary 2.5 we shall prove that any 4-algebra is isomorphic to a such a twisted product.

(3) Let \(C(A,V) = (\triangleright, f, \cdot V)\) be a crossed data of a 4-algebra \(A\) by \(V\) such that \(\cdot V\) is the trivial map. By applying Proposition 2.2 we obtain that \((\triangleright, f, \cdot V := 0)\) is a crossed
system if and only if \((V, \triangleright)\) is an \(A\)-module and \(f : A \times A \to V\) is symmetric abelian 2-cocycle, i.e.
\[
f(a^2, a^2) + 2a^2 \triangleright f(a, a) = 0
\]
for all \(a \in A\). This case will appear in the study of the classical extension problem for 4-algebras, namely those with an abelian kernel (see Corollary 2.13 below).

(4) Let \(\mathcal{C}(A, V) = (\triangleright, f, \cdot_V)\) be a crossed data of a 4-algebra \(A\) by \(V\) such that \(\triangleright\) is the trivial action, i.e. \(a \triangleright x := 0\), for all \(a \in A\) and \(x \in V\). Then \((\triangleright := 0, f, \cdot_V)\) is a crossed system if and only if \((V, \cdot)\) is a 4-algebra, \(f : A \times A \to V\) is symmetric and
\[
f(a^2, a^2) + f(a, a)^2 = 0, \quad x^2 \cdot_V f(a, a) = 0
\]
for all \(a \in A\) and \(x \in V\).

Let now \(\mathcal{C}(A, V) = (\triangleright, f, \cdot_V)\) be a crossed system of a 4-algebra \(A\) by \(V\). Then, the canonical projection \(\pi_A : V \# A \to A\), \(\pi_A(x, a) := a\) is a surjective algebra map and \(\text{Ker}(\pi_A) = V \times \{0\} \cong V\) is an ideal in the 4-algebra \(V \# A\). Thus, we obtain that the 4-algebra \(V \# A\) is an extension of the 4-algebra \(A\) by the 4-algebra \((V, \cdot_V)\) via
\[
0 \longrightarrow V \overset{i_V}{\longrightarrow} V \# A \overset{\pi_A}{\longrightarrow} A \longrightarrow 0
\]
where \(i_V(x) = (x, 0)\), for all \(x \in V\). Conversely, we have:

**Proposition 2.4.** Let \(A\) be a 4-algebra, \(E\) a vector space and \(\pi : E \to A\) an epimorphism of vector spaces with \(V = \text{Ker}(\pi)\). Then any 4-algebra structure \((E, \cdot_E)\) which can be defined on \(E\) such that \(\pi : (E, \cdot_E) \to A\) is a morphism of 4-algebras is isomorphic to a crossed product \(V \# A\). Furthermore, the isomorphism of 4-algebras \((E, \cdot_E) \cong V \# A\) can be chosen such that it stabilizes \(V\) and co-stabilizes \(A\).

In particular, any 4-algebra structure on \(E\) such that \(\pi : E \to A\) is an algebra map is cohomologous to an extension of the form \((8)\).

**Proof.** Let \(\cdot_E\) be a 4-algebra structure of \(E\) such that \(\pi : (E, \cdot_E) \to A\) is an algebra map and let \(s : A \to E\) be a \(k\)-linear section of \(\pi\), i.e. \(\pi \circ s = \text{Id}_A\). Then \(\varphi : V \times A \to E\), \(\varphi(x, a) := x + s(a)\) is an isomorphism of vector spaces with the inverse \(\varphi^{-1}(y) = (y - s(\pi(y)), \pi(y))\), for all \(y \in E\). Using the section \(s\) we define the following bilinear maps:
\[
\triangleright = \triangleright_s : A \times V \to V, \quad a \triangleright x := s(a) \cdot_E x
\]
\[
f = f_s : A \times A \to V, \quad f(a, b) := s(a) \cdot_E s(b) - s(a \cdot b)\]
\[
\cdot_V : V \times V \to V, \quad x \cdot_V y := x \cdot_E y
\]
for all \(a, b \in A\) and \(x, y \in V\). These are well-defined maps since \(\pi\) is an algebra map and \(s\) a section of \(\pi\). Using this crossed data \((\triangleright, f, \cdot_V)\) connecting \(A\) and \(V\) we can prove that the unique 4-algebra structure \(\circ\) that can be defined on the direct product of vector spaces \(V \times A\) such that \(\varphi : V \times E \to (E, \cdot_E)\) is an isomorphism of 4-algebras is given by:
\[
(x, a) \circ (y, b) := (x \cdot_V y + a \triangleright y + b \triangleright x + f(a, b), a \cdot b)
\]
for all $a, b \in A$, $x, y \in V$. Indeed, let $\circ$ be such a 4-algebra structure on $V \times A$. Then we have:

\[
(x, a) \circ (y, b) = \varphi^{-1}(\varphi(x, a) \cdot \varphi(y, b)) = \varphi^{-1}((x + s(a)) \cdot (y + s(b))) = \varphi^{-1}(x \cdot \varphi E y + s(a) \cdot \varphi E y + x \cdot \varphi E s(b) + s(a) \cdot \varphi E s(b)) = (x \cdot \varphi E y + s(a) \cdot \varphi E y + s(b) \cdot \varphi E x + s(a) \cdot \varphi E s(b) - s(a \cdot b), a \cdot b)
\]

as desired. Thus, $\varphi : V \# A \rightarrow (E, \cdot \varphi E)$ is an isomorphism of 4-algebras which stabilizes $V$ and co-stabilizes $A$ since the diagram

\[
\begin{array}{ccc}
V & \overset{i}{\rightarrow} & V \# A \\
\downarrow{Id} & & \downarrow{\varphi} \\
V & \overset{i}{\rightarrow} & E \\
\downarrow{Id} & & \downarrow{\pi} \\
& A & \end{array}
\]

is obviously commutative. □

As a first consequence we obtain that any 4-algebra is isomorphic to a twisted product as constructed in (2) of Example 2.3.

**Corollary 2.5.** Any 4-algebra $E = (E, \cdot \varphi E)$ is isomorphic to a twisted product $V \# A$ associated to a 4-algebra $V$ with $\dim_k(V) = \dim_k(E^2)$ and an abelian algebra $A$.

**Proof.** Indeed, let $(E, \cdot \varphi E)$ be a 4-algebra and $V := E'$ its derived algebra. Then $A := E/E'$ is an abelian algebra and the canonical projection $\pi : E \rightarrow A$ is a surjective 4-algebra map. Now, we apply Proposition 2.4. □

Corollary 2.5 is a tool that can be used for the classification of 4-algebras of a given finite dimension. Let us explain briefly how it works: let $E$ be an $(m+n)$-dimensional 4-algebra. An invariant of such algebras is the dimension $m$ of the derived algebra. Thus, we have to fix a positive integer $m$. Based on Corollary 2.5 we obtain that $E$ is isomorphic to a twisted product $V \# A$, where $V$ is an $m$-dimensional 4-algebra and $A$ is a vector space (viewed with the abelian 4-algebra structure) of dimension $n$ (explicitly, the multiplication on $V \# A$ is given by (7) in Example 2.3).

**Example 2.6.** Let $V = (V, \cdot V)$ be an $m$-dimensional 4-algebra having $\{e_1, \ldots, e_m\}$ as a basis. Then any $(m+1)$-dimensional 4-algebra having the derived algebra $V$ is isomorphic to the 4-algebra having $\{e_i, f_0 \mid i = 1, \ldots, m\}$ as a basis and the multiplication given for any $i, j = 1, \ldots, m$ by:

\[
e_i \circ e_j := e_j \cdot V e_j, \quad e_i \circ f_0 := \xi(e_i), \quad f_0 \circ f_0 := F
\]

for some pair $(F, \xi) \in V \times \text{End}_k(V)$ satisfying the following compatibility conditions

\[
F^2 = 0, \quad x \cdot V F + 2 x^2 \cdot V \xi(x) + 2 \xi(x)^2 + 2 \xi(x) \cdot V F = 0
\]

for all $x \in V$.

The semidirect product of 4-algebras characterizes split epimorphisms in the category on 4-algebras, as we mentioned in Example 2.3:
Corollary 2.7. An algebra map $\pi : B \to A$ between two 4-algebras $A$ and $B$ is a split epimorphism in the category of 4-algebras if and only if there exists an isomorphism of 4-algebras $B \cong V \times A$, where $V = \text{Ker}(\pi)$ and $V \times A$ is a semidirect product of 4-algebras as constructed in Example 2.3.

Proof. For a semidirect product $V \times A$, the canonical projection $p_A : V \times A \to A$, $p_A(x, a) = a$ has a section that is an algebra map defined by $s_A(a) = (0, a)$, for all $a \in A$. Conversely, let $s : A \to B$ be an algebra map with $\pi \circ s = \text{Id}_A$. Then, the bilinear map $f_s$ given by (10) from Proposition 2.4 is the trivial map and hence the corresponding crossed product $V\#A$ reduces to a semidirect product $V \times A$. $\square$

Now we shall describe the morphisms that stabilize $V$ between two arbitrary crossed products $V\#A$ and $V\#'A$, for two reasons: to compute the Galois group $\text{Gal}(V\#A/A)$ of the extension $V \hookrightarrow V\#A$ and then to answer the classification part of the GE problem for 4-algebras.

Lemma 2.8. Let $C(A, V) = (\triangleright, f, \cdot_V)$ and $C'(A, V) = (\triangleright', f', \cdot_V')$ be two crossed systems of a 4-algebra $A$ by $V$, with $V\#A$, respectively $V\#'A$, the corresponding crossed products. If $\cdot_V = \cdot_V'$, then there exists a bijection between the set of all 4-algebra morphisms $\psi : V\#A \to V\#'A$ which stabilize $V$ and the set of all pairs $(r, \alpha) \in \text{Hom}_k(A, V) \times \text{End}_{\text{alg}}(A)$, where $r : A \to V$ is a linear map, $\alpha : A \to A$ is an algebra morphism satisfying the following compatibilities for all $a, b \in A$ and $x \in V$:

(M1) $a \triangleright x = \alpha(a) \triangleright' x + r(a) \cdot_V' x$;
(M2) $f(a, b) = f'(\alpha(a), \alpha(b)) + \alpha(a) \triangleright' r(b) + \alpha(b) \triangleright' r(a) + r(a) \cdot_V' r(b) - r(a \cdot b)$

Under the above bijection the 4-algebra morphism $\psi = \psi_{(r, \alpha)} : V\#A \to V\#'A$ corresponding to the pair $(r, \alpha)$ is given by $\psi(x, a) = (x + r(a), \alpha(a))$, for all $a \in A$, $x \in V$. Furthermore, $\psi = \psi_{(r, \alpha)}$ is an isomorphism of 4-algebras if and only if $\alpha$ is bijective and $\psi = \psi_{(r, \alpha)}$ co-stabilize $A$ if and only if $\alpha = \text{Id}_A$.

Proof. First of all we observe that for any linear map $\psi : V\#A \to V\#'A$ that makes the following diagram commutative

\[
\begin{array}{ccc}
V & \xrightarrow{i_V} & V\#A \\
\downarrow{\text{Id}_V} & & \downarrow{\psi} \\
V & \xrightarrow{i_V} & V\#'A \\
\end{array}
\]

there exists a unique pair of linear maps $(r, \alpha) \in \text{Hom}_k(A, V) \times \text{End}_k(A)$ such that $\psi(x, a) = (x + r(a), \alpha(a))$, for all $a \in A$, and $x \in V$. Let $\psi = \psi_{(r, \alpha)}$ be such a linear map, i.e. $\psi(x, a) = (x + r(a), \alpha(a))$, for some linear maps $r : A \to V$, $\alpha : A \to A$. We will prove that $\psi$ is a morphism of 4-algebras if and only if $\cdot_V = \cdot_V'$, $\alpha : A \to A$ is an algebra map and the compatibility conditions (M1)-(M2) hold. To this end, it is enough to prove that the compatibility

\[
\psi((x, a) \circ (y, b)) = \psi(x, a) \circ' \psi(y, b)
\] (13)
holds on all generators of $V \# A$. We leave out the detailed computations and only indicate the main steps of this verification. First, it is easy to see that (13) holds for the pair $(x, 0)$, $(y, 0)$ if and only if $\cdot_V = \cdot'_V$. Secondly, we can prove that (13) holds for the pair $(a, 0)$, $(0, b)$ if and only if $\alpha : A \to A$ is an algebra maps and (M2) hold. Finally, (13) holds for the pair $(x, 0)$, $(0, a)$ if and only if (M1) holds. The last two statements are elementary: we just note that if $\alpha : A \to A$ is bijective, then $\psi_{(r, \alpha)}$ is an isomorphism of 4-algebras with the inverse given for any $a \in A$ and $x \in V$ by:

$$\psi_{(r, \alpha)}^{-1}(x, a) = (x - r(\alpha^{-1}(a)), \alpha^{-1}(a)).$$

This finishes the proof. 

It can be easily observed from the proof of Lemma 2.8 that if $\cdot_V \neq \cdot'_V$, then there are no 4-algebra morphisms $\psi : V \# A \to V \#' A$ which stabilize $V$. As a first application of Lemma 2.8 we can describe the Galois group $\text{Gal}(V \# A/V)$ of the extension $V \hookrightarrow V \# A$, consisting of all automorphisms of the crossed product 4-algebra $V \# A$ acting as identity on $V$. Let $\mathcal{C}(A, V) = \langle \cdot, f, \cdot_V \rangle$ be a crossed system of a 4-algebra $A$ by $V$. Let $\mathcal{G}_A^V(\cdot, f, \cdot_V)$ be the set of all pairs $(r, \alpha) \in \text{Hom}_k(A, V) \times \text{Aut}_\text{Alg}(A)$ consisting of a linear map $r : A \to V$ and an automorphism $\alpha : A \to A$ of the 4-algebra $(A, \cdot)$ such that:

$$a \triangleright x = \alpha(a) \triangleright x + r(a) \cdot_V x$$

$$f(a, b) = f(\alpha(a), \alpha(b)) + \alpha(a) \triangleright r(b) + \alpha(b) \triangleright r(a) + r(a) \cdot_V r(b) - r(a \cdot b)$$

for all $x \in V$, $a, b \in A$. Then, we can easily prove that $\mathcal{G}_A^V(\cdot, f, \cdot_V)$ has a group structure via the multiplication given by:

$$(r, \alpha) \bullet (r', \alpha') := (r + r \circ \alpha', \alpha \circ \alpha')$$

(14)

for all $(r, \alpha)$, $(r', \alpha') \in \mathcal{G}_A^V(\cdot, f, \cdot_V)$ and, moreover, $\mathcal{G}_A^V(\cdot, f, \cdot_V)$ is a subgroup in the semidirect product of groups $\text{Hom}_k(A, V) \rtimes \text{GL}_k(A)$ as defined by (1). Applying Lemma 2.8 for $(\cdot, f', \cdot'_V)$ we obtain:

**Corollary 2.9.** Let $\mathcal{C}(A, V) = \langle \cdot, f, \cdot_V \rangle$ be a crossed system of a 4-algebra $A$ by $V$. Then the map defined by:

$$\vartheta : (\mathcal{G}_A^V(\cdot, f, \cdot_V), \bullet) \to \text{Gal}(V \# A/V), \quad \vartheta(r, \alpha)(x, a) := (x + r(a), \alpha(a))$$

for all $(r, \alpha) \in \mathcal{G}_A^V(\cdot, f, \cdot_V)$ and $(x, a) \in V \times A$ is an isomorphism of groups.

Arising from Lemma 2.8 we introduce the following concept needed for the classification part of the GE-problem:

**Definition 2.10.** Let $A$ be a 4-algebra and $V$ a vector space. Two crossed systems $\mathcal{C}(A, V) = \langle \cdot, f, \cdot_V \rangle$ and $\mathcal{C}'(A, V) = \langle \cdot', f', \cdot'_V \rangle$ of $A$ by $V$ are called **cohomologous**, and we denote this by $\mathcal{C}(A, V) \approx \mathcal{C}'(A, V)$, if and only if $\cdot_V = \cdot'_V$ and there exists a linear map $r : A \to V$ such that for any $a, b \in A$, $x \in V$ we have:

$$a \triangleright x = a \triangleright' x + r(a) \cdot'_V x$$

$$f(a, b) = f'(a, b) + a \triangleright' r(b) + b \triangleright' r(a) + r(a) \cdot'_V r(b) - r(a \cdot b).$$

(15)

(16)

As a conclusion of the results obtained so far we obtain the theoretical answer to the GE-problem for 4-algebras:
Theorem 2.11. Let $A$ be a 4-algebra, $E$ a vector space and $\pi : E \to A$ a linear epimorphism of vector spaces with $V = \text{Ker}(\pi)$. Then $\approx$ defined in Definition 2.10 is an equivalence relation on the set $\mathcal{CS}(A, V)$ of all crossed systems of $A$ by $V$. If we denote by $\mathcal{GH}^2(A, V) := \mathcal{CS}(A, V)/\approx$, then the map

$$\mathcal{GH}^2(A, V) \to \text{GExt}(A, E), \quad (\triangleright, f, \cdot \cdot \cdot ) \mapsto V\#(\triangleright, f, \cdot \cdot \cdot )A$$

is bijective, where $(\triangleright, f, \cdot \cdot \cdot )$ denotes the equivalence class of $(\triangleright, f, \cdot \cdot \cdot )$ via $\approx$.

Proof. Using the last statement of Lemma 2.8 we obtain that $\mathcal{C}(A, V) \approx \mathcal{C}'(A, V)$ if and only if there exists an isomorphism of 4-algebras $V\#A \cong V\#'A$ that stabilize $V$ and co-stabilize $A$. Thus, $\approx$ is an equivalence relation on the set $\mathcal{CS}(A, V)$. The rest of the proof follows from Proposition 2.2 and Proposition 2.4.

The classifying object $\mathcal{GH}^2(A, V)$ constructed in Theorem 2.11 will be called the global non-abelian cohomology of $A$ by $V$ and its explicit computation for a given 4-algebra $A$ and a vector space $V$ is a very difficult problem. The first step in its calculation is inspired by the way the equivalence relation $\approx$ is expressed in Definition 2.10: two different 4-algebra structures $\cdot \cdot \cdot$ and $\cdot \cdot \cdot'$ on $V$ give rise to two different equivalence classes with respect to the relation $\approx$ on $\mathcal{CS}(A, V)$. Hence we can fix $\cdot \cdot \cdot$ a 4-algebra structure on $V$ and denote by $\mathcal{CS}_{\cdot \cdot \cdot}(A, V)$ the set of pairs $(\triangleright, f)$ such that $(\triangleright, f, \cdot \cdot \cdot ) \in \mathcal{CS}(A, V)$. Two such pairs $(\triangleright, f)$ and $(\triangleright', f') \in \mathcal{CS}_{\cdot \cdot \cdot}(A, V)$ will be called $\cdot \cdot \cdot$-cohomologous and will be denoted by $(\triangleright, f) \approx_{\cdot \cdot \cdot} (\triangleright', f')$ if there exists a linear map $r : A \to V$ such that for any $a, b \in A, x \in V$ we have:

$$a \triangleright x = a \triangleright' x + r(a) \cdot \cdot \cdot x$$
$$f(a, b) = f'(a, b) + a \triangleright' r(b) + b \triangleright' r(a) + r(a) \cdot \cdot \cdot r(b) - r(a \cdot b).$$

Then $\approx_{\cdot \cdot \cdot}$ is an equivalence relation on $\mathcal{CS}_{\cdot \cdot \cdot}(A, V)$ and we denote by $\mathbb{H}^2_{\text{nab}}(A, (V, \cdot \cdot \cdot ))$ the quotient set $\mathcal{CS}_{\cdot \cdot \cdot}(A, V)/\approx_{\cdot \cdot \cdot}$ and call it the non-abelian cohomology of the 4-algebras $(A, \cdot \cdot \cdot )$ and $(V, \cdot \cdot \cdot )$. The object $\mathbb{H}^2_{\text{nab}}(A, (V, \cdot \cdot \cdot ))$ classifies all extensions of the given 4-algebra $A$ by the given 4-algebra $(V, \cdot \cdot \cdot )$ and gives the theoretical answer to the classical extension problem for 4-algebras. We record all these observations in the following result:

Corollary 2.12. Let $A$ be a 4-algebra and $V$ a vector space. Then

$$\mathcal{GH}^2(A, V) = \Pi_{\cdot \cdot \cdot} \mathbb{H}^2_{\text{nab}}(A, (V, \cdot \cdot \cdot ))$$

where the coproduct on the right hand side is in the category of sets over all possible 4-algebra structures $\cdot \cdot \cdot$ on the vector space $V$. Furthermore, for a given 4-algebra structure on $V$ the map

$$\mathbb{H}^2_{\text{nab}}(A, (V, \cdot \cdot \cdot )) \to \text{Ext}(A, (V, \cdot \cdot \cdot )), \quad (\triangleright, f) \mapsto V\#(\triangleright, f, \cdot \cdot \cdot )A$$

is bijective, where Ext $(A, (V, \cdot \cdot \cdot ))$ is the set of equivalence classes of all 4-algebras that are extensions of the $A$ by $(V, \cdot \cdot \cdot )$ and $(\triangleright, f)$ denotes the equivalence class of $(\triangleright, f)$ via $\approx_{\cdot \cdot \cdot}$.
We continue our investigation of the object $\mathbb{G}H^2(A, V)$ observing that among all components of the coproduct in (20) the simplest one is that corresponding to the trivial 4-algebra structure on $V$, i.e. $x \cdot_V y := 0$, for all $x, y \in V$ which we denoted by $V_0$. Let $\mathcal{CS}_0(A, V_0)$ be the set of all pairs $(\triangleright, f)$ such that $(\triangleright, f, \cdot_V := 0) \in \mathcal{CS}(A, V)$. As shown in (3) of Example 2.3, a pair $(\triangleright, f) \in \mathcal{CS}_0(A, V_0)$ if and only if $(V, \triangleright)$ is an $A$-module and $f : A \times A \to V$ is symmetric abelian 2-cocycles, i.e.

$$a^2 \triangleright (a \triangleright x) = 0, \quad f(a^2, a^2) + 2a^2 \triangleright f(a, a) = 0$$

(22)

for all $a \in A$ and $x \in V$. Applying now Definition 2.10 for the trivial multiplication $\cdot_V := 0$ we obtain that two pairs $(\triangleright, f)$ and $(\triangleright', f') \in \mathcal{CS}_0(A, V_0)$ are 0-cohomologous $(\triangleright, f) \approx_0 (\triangleright', f')$ if and only if $\triangleright = \triangleright'$ and there exists a linear map $r : A \to V$ such that for all $a, b \in A$ we have:

$$f(a, b) = f'(a, b) + a \triangleright r(b) + b \triangleright r(a) - r(a \cdot b).$$

(23)

The equality $\triangleright = \triangleright'$ shows that two different $A$-module structures on $V$ give different equivalence classes in the classifying object $\mathbb{H}^2_{\text{nab}}(A, V_0)$. Thus, we can apply the same strategy as before for computing $\mathbb{H}^2_{\text{nab}}(A, V_0)$: we fix $(V, \triangleright)$ a $A$-module structure on $V$ and consider the set $Z^2_{\text{a}}(A, V_0)$ of all symmetric abelian 2-cocycle, i.e. symmetric bilinear maps $f : A \times A \to V$ such that

$$f(a^2, a^2) + 2a^2 \triangleright f(a, a) = 0$$

for all $a, b \in A$. Two symmetric abelian 2-cocycles $f$ and $f'$ are 0-cohomologous $f \approx_0 f'$ if and only if there exists a linear map $r : A \to V$ such that (23) holds. Then $\approx_0$ is an equivalence relation on $Z^2_{\text{a}}(A, V_0)$ and the quotient set $Z^2_{\text{a}}(A, V_0)/ \approx_0$, which we will denote by $H^2_{\text{a}}(A, V_0)$, plays the role of the second cohomological group from the theory of groups or Lie algebras. All in all, we have obtained the following results which classifies all extensions of a 4-algebra $A$ by an abelian algebra $V_0$.

**Corollary 2.13.** Let $A$ be a 4-algebra and $V$ a vector space viewed with the trivial 4-algebra structure $V_0$. Then:

$$\mathbb{H}^2_{\text{nab}}(A, V_0) = \mathbb{I}L, H^2_{\text{a}}(A, V_0)$$

(24)

where the coproduct on the right hand side is in the category of sets over all possible $A$-module structures $\triangleright$ on the vector space $V$.

### 3. Applications and examples

The computation of the classifying object $\mathbb{G}H^2(A, V)$ as constructed in Theorem 2.11 and Corollary 2.12 is a very challenging problem. In the following we shall compute $\mathbb{G}H^2(A, k)$ and $\mathbb{G}H^2(k, V)$. First we need the following:

**Proposition 3.1.** Let $A$ be a 4-algebra. Then there exists a bijection between the set $\mathcal{CS}(A, k)$ of all crossed systems of $A$ by $k$ and the set $\mathcal{CF}(A)$ consisting of all pairs $(\lambda, f)$, where $\lambda : A \to k$ is linear map, $f : A \times A \to k$ a symmetric bilinear form on $A$ satisfying the following compatibilities conditions for any $a \in A$:

$$f(a^2, a^2) + 2\lambda(a^2)f(a, a) = 0, \quad \lambda(a)\lambda(a^2) = 0.$$
The correspondence is given such that the crossed crossed product $k\# A$ associated to the pair $(\lambda, f) \in CF(A)$ is the 4-algebra denoted by $A(\lambda, f) := k \times A$ with the multiplication given for any $a, b \in A$ and $x, y \in k$ by:

$$(x, a) \circ (y, b) = (\lambda(a)y + \lambda(b)x + f(a, b), a \cdot b).$$

Furthermore, a 4-algebra $B$ has a surjective algebra map $B \to A \to 0$ whose kernel is 1-dimensional if and only if $B$ is isomorphic to $A(\lambda, f)$, for some $(\lambda, f) \in CF(A)$.

**Proof.** Since $k$ has dimension 1, any 4-algebra structure $\cdot_k$ on $k$ is the abelian one, $x \cdot_k y = 0$, for all $x, y \in k$. Hence, there exists a bijection between the set of all crossed datums $(\cdot, f)$ of $A$ by $k$ and the set of pairs $(\lambda, f)$ consisting of a linear map $\lambda : A \to k$ and a bilinear map $f : A \times A \to k$. The bijection is given such that $(\cdot, f)$ corresponding to $(\lambda, f)$ is defined by $a \cdot x := \lambda(a)x$ for all $a \in A$ and $x \in k$. Now the axiom (CS1) of Proposition 2.2 is equivalent to $f$ being symmetric, while the axioms (CS2) and (CS3) are equivalent to (25). The algebra $A(\lambda, f)$ is just the crossed product $k\# A$ associated to this context and the last statement follows from Proposition 2.4. □

Let $(\lambda, f) \in CF(A)$ be a pair as in Proposition 3.1. We shall explicitly describe the multiplication of the 4-algebra $A(\lambda, f)$. We will see the elements of $A$ as elements in $k \times A$ through the identification $a = (0, a)$ and denote by $g := (1, 0_A) \in k \times A$. Let $\{e_i | i \in I\}$ be a basis of $A$ as a vector space over $k$. Then, the 4-algebra $A(\lambda, f)$ is the vector space having $\{g, e_i | i \in I\}$ as a basis and the multiplication $\circ$ is given for any $i, j \in I$ by:

$$g^2 = 0, \quad e_i \circ g = g \circ e_i = \lambda(e_i) g, \quad e_i \circ e_j = e_i \cdot e_j + f(e_i, e_j) g. \quad (27)$$

In order to compute $GH^2(A, k)$, we observe first that the equivalence relation given by (15) and (16) from Definition 2.10, written for the set $CF(A)$, via the bijection $CS(A, k) \equiv CF(A)$ proven in Proposition 3.1 takes the following form: $(\lambda, f) \approx (\lambda', f')$ if and only if $\lambda = \lambda'$ and there exist a linear map $r : A \to k$ such that

$$f(a, b) = f'(a, b) + \lambda'(a)r(b) + \lambda'(b)r(a) - r(a \cdot b) \quad (28)$$

for all $a, b \in A$. Thus, we obtain that $GH^2(A, k) \cong CF(A)/\approx$. We continue our investigation since the equality $\lambda = \lambda'$ in the above equivalence relation shows that two different $\lambda$ and $\lambda'$ give different equivalence classes in the classifying object $CF(A)/\approx$. Hence, we can pick a linear map $\lambda : A \to k$ such that $\lambda(a)\lambda(a^2) = 0$, for all $a \in A$. We denote by $Z^2_\lambda(A, k)$ the set of all $\lambda$-cocycles: that is, the set of all symmetric bilinear maps $f : A \times A \to k$ satisfying the first compatibility condition of (25). Two $\lambda$-cocycles $f, f' : A \times A \to k$ are equivalent $f \approx^\lambda f'$ if and only if there exists a linear map $r : A \to k$ such that

$$f(a, b) = f'(a, b) + \lambda(a)r(b) + \lambda(b)r(a) - r(a \cdot b)$$

for all $a, b \in A$. We denote $H^2_\lambda(A, k) := Z^2_\lambda(A, k)/\approx^\lambda$ and we record all the above results in the following decomposition of $GH^2(A, k)$, which is a special case of Corollary 2.13 applied for $V := k$:

**Corollary 3.2.** Let $A$ be a 4-algebra. Then,

$$GH^2(A, k) \cong CF(A)/\approx \cong H^2_\lambda(A, k) \quad (29)$$
where the coproduct on the right hand side is in the category of sets over all possible linear maps $\lambda : A \to k$ satisfying $\lambda(a)\lambda(a^2) = 0$, for all $a \in A$.

**Example 3.3.** Let $n$ be a positive integer and $A$ the 4-algebra having $\{e_1, \cdots, e_{n+1}\}$ as a basis and the multiplication given by $e_1 \cdot e_2 := e_{n+1}$ and the other products of basis elements are zero. Then we can prove that

$$\mathcal{CF}(A) \cong k^n \times \text{Sym}_{n+1}^0(k)$$

where we denoted by $\text{Sym}_{n+1}^0(k)$ the vector space of all $(n+1) \times (n+1)$ symmetric matrices $(f_{ij})$ such that $f_{n+1,n+1} := 0$. The bijection is given such that $(\lambda, f) \in \mathcal{CF}(A)$ associated to a pair $(\lambda_i, (f_{ij})) \in k^n \times \text{Sym}_{n+1}^0(k)$ is given by:

$$\lambda(e_t) := \lambda_t, \quad \lambda(e_{n+1}) := 0, \quad f(e_i, e_j) := f_{ij}$$

for all $t = 1, \cdots, n$ and $i, j = 1, \cdots, n + 1$. In particular, we obtain that

$$\mathbb{G}^2 (A, k) \cong \Pi_\lambda \text{Sym}_{n+1}^0(k)/ \approx_\lambda$$

where the coproduct on the right hand side is taken over all $\lambda = (\lambda_1, \cdots, \lambda_n) \in k^n$ and $\text{Sym}_{n+1}^0(k)/ \approx_\lambda$ is the quotient set of $\text{Sym}_{n+1}^0(k)$ via the following equivalence relation:

$$(f_{ij}) \approx_\lambda (f'_{ij})$$

if and only if there exists a linear map $r : A \to k$ such that

$$f_{ij} = f'_{ij} + \lambda_i r(e_j) + \lambda_j r(e_i) - r(e_i \cdot e_j)$$

for all $i, j = 1, \cdots, n + 1$ (with $\lambda_{n+1} := 0$).

Now we describe the opposite case, namely $\mathbb{G}^2 (k, V)$.

**Proposition 3.4.** Let $V$ be a vector space. Then there exists a bijection between the set $\mathcal{CS}(k, V)$ of all crossed systems of $k$ by $V$ and the set $\mathcal{CT}(V)$ consisting of all triples $(\theta, F, \cdot_V)$, where $\theta : V \to V$ is linear map, $F \in V$ and $\cdot_V : V \times V$ is a 4-algebra structure on $V$ satisfying the following compatibility conditions for any $x \in V$:

$$F^2 = 0, \quad \theta(x) \cdot_V x^2 = \theta(x) \cdot_V F = 0, \quad 2\theta(x)^2 + x^2 \cdot_V F = 0. \quad (30)$$

The correspondence is given such that the crossed product $V \# k$ associated to the triple $(\theta, F, \cdot_V) \in \mathcal{CT}(V)$ is the 4-algebra denoted by $V_{(\theta, F, \cdot_V)} := V \times k$ with the multiplication given for any $a, b \in k$ and $x, y \in V$ by:

$$(x, a) \circ (y, b) = (x \cdot_V y + a\theta(y) + b\theta(x) + abF, 0). \quad (31)$$

**Proof.** We leave it to the reader since it is similar to the one of Proposition 3.1, taking into account that the only 4-algebra structure on $A := k$ is the abelian one. $\Box$

We fix now $\cdot_V$ a 4-algebra structure on the vector space $V$ and denote by $\mathcal{T}_V(V)$ the set of all pairs $(\theta, F) \in \text{End}_k(V) \times V$ satisfying the compatibility conditions (30). Two pairs $(\theta, F)$ and $(\theta', F') \in \mathcal{T}_V(V)$ are $\cdot_V$-cohomologous $(\theta, F) \approx_{\cdot_V} (\theta', F')$ if and only if there exists an element $r \in V$ such that

$$\theta(x) = \theta'(x) + r \cdot_V x, \quad F = F' + 2\theta'(r) + r^2$$

for all $x \in V$. Applying Corollary 2.12 we obtain:
Corollary 3.5. Let $V$ be a vector space. Then
\[ \mathcal{G}H^2(k, V) = \Pi_V T_V(V)/\sim_V \]  
(32)
where the coproduct on the right hand side is in the category of sets over all possible 4-algebra structures $\cdot$ on $V$.

In the last part we shall apply our results to metabelian 4-algebras. We recall that a 4-algebra $E$ is called metabelian if the derived algebra $E'$ is an abelian subalgebra of $E$, i.e. $(a \cdot b) \cdot (c \cdot d) = 0$, for all $a, b, c, d \in E$. Let $I$ be an ideal of a 4-algebra $E$: then the quotient algebra $E/I$ is an abelian algebra if and only if $E' \subseteq I$. Thus, $E$ is a metabelian 4-algebra if and only if it fits into an exact sequence of 4-algebras
\[ 0 \longrightarrow B \longrightarrow E \longrightarrow A \longrightarrow 0 \]  
(33)
where $B$ and $A$ are abelian 4-algebras. Indeed, if $E$ is metabelian we can take $B := E'$ and $A := E/E'$ with the obvious morphisms. Conversely, assume that a 4-algebra $E$ is an extension of an abelian algebra $A$ by an abelian algebra $B$. Since $E/i(B) = E/\text{Ker}(\pi) \cong A$ is an abelian algebra, we obtain that $E' \subseteq i(B) \cong B$. Hence, $E'$ is abelian as a subalgebra in an abelian algebra, i.e. $E$ is metabelian. Using this observation, Proposition 2.2 and Proposition 2.4 we obtain the structure of metabelian 4-algebras:

Corollary 3.6. A 4-algebra $E$ is metabelian if and only if there exists an isomorphism of 4-algebras $E \cong V \#_{(\cdot, f)} A$, where $A$ and $V$ are two vector spaces and $\triangleright : A \times A \rightarrow V$, $f : A \times A \rightarrow V$ are two bilinear maps such that $f$ is symmetric. The crossed product $V \#_{(\cdot, f)} A$ is the vector space $V \times A$ with the multiplication given for any $a, b \in A$ and $x, y \in V$ by:
\[ (x, a) \circ (y, b) := (a \triangleleft y + b \triangleright x + f(a, b), 0). \]  
(34)

Let now $A$ and $V$ be two fixed vector spaces viewed with the abelian 4-algebra structure $A_0$ and $V_0$: in the next step we shall classify all metabelian 4-algebras that are extensions of $A$ by $V$, that is classify all crossed products $V \#_{(\cdot, f)} A$ up to an isomorphism that stabilizes $V$ and co-stabilizes $A$. For this purpose, we have to compute the classifying object $\mathbb{H}^2_{\text{nab}}(A, (V, \cdot_V))$ from Corollary 2.12 and Corollary 2.13 in the case that both $\cdot_V$ and $\cdot_A$ are trivial multiplication. Let $\triangleright : A \times V \rightarrow V$ be a fixed bilinear map and $\text{Sym}(A \times A; V)$ the set of all symmetric bilinear maps $f : A \times A \rightarrow V$. Two elements $f$ and $f' \in \text{Sym}(A \times A; V)$ are called $\triangleright$-cohomologous $f \approx_{\triangleright} f'$ if and only if there exists a linear map $r : A \rightarrow V$ such that:
\[ f(a, b) = f'(a, b) + a \triangleright r(b) + b \triangleright r(a) \]  
(35)
for all $a, b \in A$ (i.e. (23) holds for the trivial multiplication on $A$). We denote by $H^2_{\triangleright}(A_0, V_0) := \text{Sym}(A \times A; V)/\approx_{\triangleright}$. We have obtained the following:

Corollary 3.7. Let $A$ and $V$ be two vector spaces viewed with the abelian algebra structure $A_0$ and $V_0$. Then there exists a bijection:
\[ \mathbb{H}^2_{\text{nab}}(A_0, V_0) \cong \Pi_{\triangleright} H^2_{\triangleright}(A_0, V_0) \]  
(36)
where the coproduct on the right hand side is taken over all bilinear maps $\triangleright : A \times V \rightarrow V$. 

\[ \]
Examples 3.8. 1. Let $V$ be a vector space with a basis $\{e_i \mid i \in I\}$ viewed with the abelian 4-algebra structure $V_0$. Then

$$\mathbb{H}^2_{\text{nab}}(k_0, V_0) \cong \coprod_{g \in \text{End}_k(V)} V/\text{Im}(g).$$

In particular, any 4-algebra $E$ containing $V$ as an abelian ideal of codimension 1 is isomorphic to the 4-algebra with the basis $\{e, e_i \mid i \in I\}$ and the multiplication given for any $i \in I$ by:

$$e \circ e_i := g(e_i), \quad e^2 := f_0$$

for some $g \in \text{End}_k(V)$ and $f_0 \in V$. Indeed, since $A := k$ any bilinear map $\triangleright : k \times V \to V$ has the form $\alpha \triangleright x = \alpha g(x)$, for a linear map $g \in \text{End}_k(V)$ and the correspondence is bijective. Moreover, the set of all symmetric bilinear maps $f : k \times k \to V$ is in bijection with the set of all elements of $V$ (the bijection maps $f$ to $f_0 := f(1, 1)$). The conclusion follows from Corollary 3.7 once we observe that the equivalent relation (35) written for the set of all elements $f_0 \in V$ comes down to $f_0 \approx f_0'$ if and only if $f_0 - f_0' \in \text{Im}(g)$. For the last part we use (34) of Corollary 3.6 since any such 4-algebra is metabelian.

2. The other way around, let $A$ be a vector space with a basis $\{f_j \mid j \in J\}$ viewed with the abelian 4-algebra structure $A_0$. Then,

$$\mathbb{H}^2_{\text{nab}}(A_0, k_0) \cong \coprod_{\lambda \in A^*} \text{Sym}(A \times A; k)/\approx_\lambda$$

where, for any linear map $\lambda \in A^* = \text{Hom}_k(A, k)$, $\approx_\lambda$ is the following equivalent relation of the set of all symmetric bilinear forms on $A$: $f \approx_\lambda f'$ if and only if there exists a linear map $r \in A^*$ such that for any $a, b \in A$:

$$f(a, b) = f'(a, b) + r(a)\lambda(b) + r(b)\lambda(a).$$

Furthermore, any 4-algebra $E$ having the derived subalgebra of dimension 1 is isomorphic to the 4-algebra with the basis $\{f, f_j \mid j \in J\}$ and the multiplication given for any $j, l \in J$ by:

$$f \circ f_j := \lambda(f_j) f, \quad f_j \circ f_l := f(f_j, f_l) f$$

for some $\lambda \in A^*$ and $f \in \text{Sym}(A \times A; k)$, where $A$ is the abelian 4-algebra $A := E/E'$.

Acknowledgment: The author warmly thanks the referee for his/her suggestions which improved the first version of the paper.

4. Competing interests declaration

The author declares none.

References

[1] A.L. Agore, G. Militaru, Hochschild products and global non-abelian cohomology for algebras. Applications, J. Pure Appl. Algebra 221 (2017) 366–392.
[2] A.L. Agore, G. Militaru, The global extension problem, crossed products and co-flag non-commutative Poisson algebras, J. Algebra 426 (2015) 1–31.
[3] A.L. Agore, G. Militaru, On a type of commutative algebras, Linear Algebra Appl. 485 (2015) 222–249.
[4] N. Andruskiewitsch, J. Devoto, Extensions of Hopf algebras, Algebra i Analiz 7 (1995) 22–61.
CROSSED PRODUCTS OF 4-ALGEBRAS. APPLICATIONS

[5] S.N. Bernstein, Mathematical problems in modern biology, Science Ukraine 1 (1922) 14–19 (in Russian).

[6] J.L. Castiglioni, X. García-Martínez, M. Ladra, Universal central extensions of Lie-Rinehart algebras, J. Algebra App. 17 (2018), No. 07, 1850134.

[7] C. Chevalley, S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc. 63(1948) 85–124.

[8] T. Cortés, Classification of 4-dimensional Bernstein algebras, Comm. in Algebra, 19 (1991) 1429–1443.

[9] T. Cortés, F. Montaner, Low dimensional Bernstein-Jordan algebras, J. London Math. Soc. 51 (1995) 53–61.

[10] S. Eilenberg, Extensions of general algebras, Ann. Soc. Math. Pol. 21 (1948) 125–134.

[11] A. Elduque, S. Okubo, On algebras satisfying $x^2y^2 = N(x)y$, Math. Z. 235 (2000) 275–314.

[12] C.J. Jr. Everett, An extension theory for rings, Amer. J. Math. 64 (1942), 363–370.

[13] J.C. Gutierrez Fernandez, Solution of the Bernstein problem in the non-regular case, J. Algebra 223 (2000) 109–132.

[14] J.C. Gutierrez Fernandez, The Bernstein Problem in Dimension 6, J. Algebra 185 (1996) 420–439.

[15] H. Jr. Guzzo, A. Benh, Solvability of a commutative algebra which atisfies $(x^2)^2 = 0$, Comm. in Algebra 42 (2014) 417–422.

[16] O. Hölder, Bildung zusammengesetzter Gruppen, Math. Ann. 46 (1895) 321–422.

[17] P. Holgate, Genetic algebras satisfying Bernstein’s stationarity principle, J. London Math. Soc. 9 (1975) 613–623.

[18] J. Huebschmann, Poisson cohomology and quantization, J. Reine Angew. Math. 408 (1990) 57–113.

[19] Yu.I. Lyubich, Two-level Bernstein populations, Math. USSR Sb. 24 (1974) 593–615.

[20] Yu.I. Lyubich, A classification of some types of Bernstein algebras, Selecta Mathematica Sovietica 6 (1987) 1–14.

[21] G. Militaru, The global extension problem, co-flag and metabelian Leibniz algebras, Linear Multilinear Algebra 63 (2015) 601–621.

[22] G. Militaru, On the structure and classification of Bernstein algebras, arXiv:2203.13627.

[23] M.L. Reed, Algebras structure of genetic inheritance, Bull. Amer. Math. Soc. 34 (1997) 107–130.

[24] O. Schreier, Über die Erweiterung von Gruppen, I, Monatshefte für Mathematik und Physik 34 (1926) 165–180.

[25] A. WGrz-Busekros, Algebras in genetics, Lecture Notes in Biomathematics, Vol. 36, Springer-Verlag, 1980.

Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania and Simion Stoilow Institute of Mathematics of the Romanian Academy, P.O. Box 1-764, 014700 Bucharest, Romania

Email address: gigel.militaru@fmi.unibuc.ro and gigel.militaru@gmail.com