\textbf{L}^2 \text{ Extension of } \overline{\partial} \text{-Closed Forms on Weakly Pseudoconvex Kähler Manifolds}

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\section*{Abstract}
Combining V. Koziarz’s observation about the regularity of some modified section related to the initial extension with J. McNeal–D. Varolin’s regularity argument, we generalize two theorems of McNeal–Varolin for the \(L^2\) extension of \(\overline{\partial}\)-closed high-degree forms on a Stein manifold to the weakly pseudoconvex Kähler case under mixed positivity conditions.

\textbf{Keywords} Continuation of analytic objects in several complex variables \cdot Sheaves and cohomology of sections of holomorphic vector bundles \cdot General results \cdot Kähler manifolds \cdot Exhaustion functions

\textbf{Mathematics Subject Classification} Primary 32D15 \cdot Secondary 32L10, 32Q15, 32T35

\section{1 Introduction: Main Results and Applications}
As is well known, the task of the classical Ohsawa–Takegoshi theorem \cite{24} is to extend a holomorphic object from some lower or same dimensional analytic subvariety...
to the ambient space with some $L^2$ estimate involved. The problem of extension of holomorphic sections has received a lot of attention and been solved in many situations. So it is natural to ask whether these extensions are feasible for $\bar{\partial}$-closed forms of high degree, which is a natural broadening of the classical Ohsawa–Takegoshi–Manivel extension theorem for holomorphic sections of line bundles.

Many interesting works appear along this line, such as [2, 4, 9, 16, 19, 22, 29, 30], etc. The biggest difficulty of this problem is the regularity issue for the solutions of related $\bar{\partial}$-equation for high degree because $\bar{\partial}$ operator for high degree is no longer hypoelliptic. For solving the regularity issue, two main methods were adopted: minimizer method and Leray’s isomorphism method.

Manivel [19] firstly considered this problem, but his proof has a gap due to the use of a singular weight and the failure of regularity for the solution of the related $\bar{\partial}$-equation. Then Demailly [9] suggested an approach to overcome this difficulty, but no one seems to have implemented his program yet completely. Koziarz [16] used the Leray’s isomorphism to reduce the extension of high-degree forms to the classical zero-degree case and thereby deduced extensions of cohomology classes over compact manifolds. Berndtsson [4] applied the minimizer method by solving a $\bar{\partial}$-equation for a current to get the related extension theorem also on compact manifolds. In [22], McNeal–Varolin made use of the Kohn solution, to handle the well-known regularity issues on a Stein or essentially Stein manifold (i.e., a Kähler manifold that becomes Stein after a hypersurface is removed from it). Furthermore, Zhu et al. [30] and Baracco et al. [2] also got some results in some special cases.

It is natural to ask whether we can establish extension theorems of $\bar{\partial}$-closed forms of high degree on general weakly pseudoconvex Kähler manifolds. Recall that a complex manifold is weakly pseudoconvex if it admits a smooth plurisubharmonic exhaustion function. As is well known, there is a richer cohomology theory on weakly pseudoconvex Kähler manifolds than that on Stein manifolds. For example, it is interesting to find out the conditions for a Kähler family (the total space of which is a weakly pseudoconvex Kähler manifold, more precisely, a holomorphic convex manifold) to admit the deformation invariance of higher cohomology of the pluricanonical bundle. In fact, this question is our initial motivation to study the problem of $L^2$ extension of $\bar{\partial}$-closed forms on weakly pseudoconvex Kähler manifolds. By the way, even for a (fiberwise) projective family, higher cohomology of the pluricanonical bundle may not be deformation invariant [14].

Combining Koziarz’s observation [16] about the regularity of some modified section related to the initial ambient extension with McNeal–Varolin’s regularity argument [22] for the extension of high-degree forms, we generalize the $L^2$ extension theorems (cf. Theorems 2.3 and 2.5) of McNeal–Varolin [22] on a Stein manifold to a weakly pseudoconvex Kähler manifold under mixed positivity conditions.

**Theorem 1.1** (Ambient $L^2$ extension) On a weakly pseudoconvex $n$-dimensional Kähler manifold $(X, \omega)$, let the smooth hypersurface $Y \subseteq X$ be the zero set of a holomorphic section $s \in H^0(X, E)$ of a smooth Hermitian holomorphic line bundle $(E, e^{-\lambda})$ and $(L, e^{-\phi})$ a smooth Hermitian holomorphic line bundle. Assume that for any $0 \leq q \leq n - 1$, the inequalities hold on $X$. 

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\[
\sqrt{-1} \bar{\partial} (\varphi - \lambda) \wedge \omega^q \geq \sigma \omega \wedge \omega^q, \tag{1.1}
\]
\[
\sqrt{-1} \bar{\partial} (\varphi - (1 + \delta) \lambda) \wedge \omega^q \geq 0, \tag{1.2}
\]
\[
|s|^2 e^{-\lambda} \leq 1, \tag{1.3}
\]

where \(\sigma\) is some positive lower semi-continuous function and \(0 < \delta \leq 1\) is some constant. Then there is a universal constant \(C > 0\) such that for any smooth section \(f\) of the bundle \((K_X \otimes L \otimes \Lambda^{0,q} T^*_X)|_Y \to Y\), satisfying

\[
\bar{\partial} (\iota^* f) = 0 \quad \text{and} \quad \int_Y \frac{|f|^2_\omega e^{-\varphi}}{|ds|^2_\omega e^{-\lambda}} dV_{Y,\omega} < \infty,
\]

there exists a smooth \(\bar{\partial}\)-closed \(K_X \otimes L\)-valued \((0,q)\)-form \(F\) on \(X\) with

\[
F|_Y = f \quad \text{and} \quad \int_X |F|^2_\omega e^{-\varphi} dV_{X,\omega} \leq \frac{C}{\delta} \int_Y \frac{|f|^2_\omega e^{-\varphi}}{|ds|^2_\omega e^{-\lambda}} dV_{Y,\omega} < \infty.
\]

Here we denote by \(\Lambda^{r,s} T^*_X\) the bundle of differential forms of bidegree \((r,s)\) on \(X\) and similarly for others.

Note that since \((K_X \otimes L \otimes \Lambda^{0,q} T^*_X)|_Y\) does not admit a natural notion of \(\bar{\partial}\), the above \(\iota^* f\) is not the usual pullback of differential forms, but induced as Definition 2.2. And the positivity conditions (1.1) (1.2) hold in the sense of \((q + 1, q + 1)\)-forms as follows: a real \((1,1)\)-form \(\theta\) on \(X\) satisfies the positivities on \(X\)

\[
\theta \wedge \omega^q > ( \text{resp.} \geq ) 0 \tag{1.4}
\]

if and only if

\[
\lambda_1 + \cdots + \lambda_{q+1} > ( \text{resp.} \geq ) 0,
\]

where \(\lambda_1 \leq \cdots \leq \lambda_n\) are the eigenvalues of \(\theta\) with respect to \(\omega\) at any point \(x \in X\). Moreover, they are both equivalent to that \([\theta, \Lambda_\omega]_\beta, \beta\)_\(\omega\) is positive (resp. semipositive) at any point \(x \in X\) for any \((n, q + 1)\)-form \(\beta(x) \neq 0\) (see Lemmata 2.9 and 2.10 for a better understanding).

We give an example satisfying the conditions of Theorem 1.1 on a weakly pseudo-convex Kähler manifold, which is neither compact nor Stein nor essentially Stein.

**Example 1.2** Let \((\mathbb{B}^m, \omega_1)\) be the unit ball in \(\mathbb{C}^m\) equipped with the Euclidean metric \(\omega_1 = \sqrt{-1} \bar{\partial} (|z_1|^2 + \cdots + |z_m|^2)\), and \((Y, \omega_2)\) a \(k\)-dimensional compact Kähler manifold which does not have any closed complex hypersurfaces. By the heredity

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1 In [22, Theorem 1.1], the \(\sigma\) is taken as zero. So the theorem here cannot cover [22, Theorem 1.1] completely. We will discuss more on the condition (1.1) in Appendix B.
property of weakly pseudoconvexity, \( X := \mathbb{B}^m \times Y \) is a weakly pseudoconvex Kähler manifold equipped with the natural Kähler metric \( \omega := \pi_X^* \omega_1 + \pi_Y^* \omega_2 \). As \( X \) admits a compact submanifold \( Y \) which contains no hypersurfaces, it is neither a Stein nor an essentially Stein manifold. Apparently, \( X \) is not a compact manifold, either. Let

\[
L = E = \mathcal{O}_X, \quad s = z_1, \quad \varphi = |z|^2, \quad \sigma = \frac{q + 1 - k}{2(q + 1)}, \quad \text{and} \quad \lambda \equiv 0,
\]

where \( z_1 \) is the first global coordinate function on \( \mathbb{B}^m \), \( |z|^2 := |z_1|^2 + \cdots + |z_m|^2 \). Then the above setting satisfies (1.1), (1.2) and (1.3) as \( q \geq k \).

Varolin suggested that one can take \( Y \) as a generic torus of dimension \( \geq 2 \). In fact, a torus admits no divisors if and only if it has algebraic dimension zero, i.e., the only meromorphic functions are constant (e.g., [11, p. 31]). Of course, any compact complex manifold with algebraic dimension zero admits no divisors, since \( H^0(X, \mathcal{K}_X^*/\mathcal{O}_X^*) \cong \text{Div}(X) \). On the other hand, for a very general lattice \( \mathcal{K} \subset \mathbb{C}^n / \mathbb{Z} \), the meromorphic function field \( \mathcal{K} (\mathbb{C}^n / \Gamma) \) is trivial (e.g., [15, p. 58]). By the way, one can also take \( Y \) as any simple manifold (e.g., \( K3 \) surfaces or the general member of the deformation families of hyperkähler manifolds), since the algebraic dimension of a simple manifold always vanishes (e.g., [6, p. 132]).

Just as [22, p. 423], if \( \eta \) is an \( L \)-valued \((0, q)\)-form on \( Y \), the orthogonal projection

\[
P : T_{X,Y}^{0,1} \rightarrow T_Y^{0,1}\n\]

induced by the Kähler metric \( \omega \) maps \( \eta \) to the ambient \( L \)-valued \((0, q)\)-form \( P^* \eta \), given by

\[
\{ P^* \eta, \bar{v}_1 \wedge \cdots \wedge \bar{v}_q \} := \{ \eta, (P\bar{v}_1) \wedge \cdots \wedge (P\bar{v}_q) \}
\]

in \( L_y \) for all \( y \in Y \hookleftarrow X \) and \( v_1, \ldots, v_q \in T_{X,Y}^{1,0} \). Around \( y \), we choose a local coordinate and frame \((U, \{z_1, \ldots, z_n\}, \sigma)\) such that \( Z \cap U = \{z_1 = 0\} \) and \( \{d\bar{z}_1, d\bar{z}_2, \ldots, d\bar{z}_n\} \) is an \( \omega(y) \)-orthonormal basis of \( \wedge^{0,1}T_{X,Y}^* \). At \( y \), set \( \eta = \sum_{1 \neq J} a_J d\bar{z}_J \circ \iota \otimes \sigma \) and then \( P^* \eta = \sum_{1 \neq J} a_J d\bar{z}_J \otimes \sigma \). So \( P^* \) is an isometry for the pointwise norm of \( L \)-valued \((0, q)\)-forms induced by \( \omega \) and the metric of \( L \). As \( \iota^* P^* \eta = \eta \), we can apply Theorem 1.1 to \( f = P^* u \) and obtain Theorem 1.3, while a sketch of a direct proof for it is also given in Appendix A.

**Theorem 1.3** (Intrinsic \( L^2 \) extension) With the setting of Theorem 1.1, there is a universal constant \( C > 0 \) such that for any smooth \( \bar{\partial} \)-closed \((K_X \otimes L)|_Y \)-valued \((0, q)\)-form \( u \) on \( Y \) satisfying

\[
\int_Y \left\| u \right\|^2_\omega e^{-\varphi} dV_{Y,\omega} < \infty,
\]

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there exists a smooth \( \bar{\partial} \)-closed \( K_X \otimes L \)-valued \((0, q)\)-form \( U \) on \( X \) with

\[
t^* U = u \quad \text{and} \quad \int_X |U|^2 e^{-\varphi} dV_{X, \omega} \leq \frac{C}{\delta} \int_Y |u|^2 e^{-\varphi} ds_\omega^2 e^{-\lambda} dV_{Y, \omega} < \infty.
\]

We now present two applications of the main theorems. The first one is a surjectivity theorem for the restriction maps in Dolbeault cohomology. It is similar to the extension theorems for cohomology classes (without \( L^2 \) estimate) recently obtained by Cao–Demailly–Matsumura [7, Theorem 1.1] and Zhou-Zhu [29, Theorem 1.1, Remark 1.1] on a holomorphically convex manifold with the more general curvature conditions, respectively, while our method is rather different from theirs.

**Corollary 1.4** Let \( X, Y, E, L \) be as in Theorem 1.3 and also \( Y \) compact. Then the restriction morphism

\[
H^{0, q}(X, K_X \otimes L) \longrightarrow H^{0, q}(Y, (K_X \otimes L)|_Y)
\]

is surjective for any \( 0 \leq q \leq n - 1 \).

Much inspired by [9, Corollary 4.11], we consider the extension behavior of \( \bar{\partial} \)-closed forms on bounded pseudoconvex domains as the second application of the main theorems. It tells us some information about the relationship between smooth “generalized quasi-plurisubharmonic” functions and \( \bar{\partial} \)-closed forms.

**Corollary 1.5** Let \( \Omega \subset \mathbb{C}^n \) be a bounded pseudoconvex domain equipped with the complex Euclidean metric, and \( Y \hookrightarrow \Omega \) a smooth hypersurface defined by a section \( s \) of some Hermitian holomorphic line bundle \( (E, e^{-\lambda}) \) over \( \Omega \). Assume that \( s \) is everywhere transverse to the zero section and that \( |s|^2 e^{-\lambda} \leq 1 \) on \( \Omega \). Let \( \varphi \) be a smooth function such that for \( 0 \leq q \leq n - 1 \), some \( \delta \in (0, 1] \) and the Chern curvature form \( \Theta_E \) of \( (E, e^{-\lambda}) \),

\[
(\sqrt{-1} \partial \bar{\partial} \varphi - \Theta_E) \wedge \omega^q > \sigma_1 \omega \wedge \omega^q
\]

and

\[
(\sqrt{-1} \partial \bar{\partial} \varphi - (1 + \delta) \Theta_E) \wedge \omega^q \geq \sigma_2 \omega \wedge \omega^q
\]

on \( \Omega \) with \( \sigma_1 \) and \( \sigma_2 \) two real constants. Then there is a constant \( \widehat{C} > 0 \) (depending on \( \sup_{\Omega} |z|^2 \), \( \sigma_1 \) and \( \sigma_2 \); exactly equal to the universal \( C \) in Theorem 1.1 when \( \sigma_1 = \sigma_2 = 0 \)) with the following properties:

1. For any smooth section \( f \) of the bundle \( K_\Omega|_Y \otimes \Lambda^{0, q} T^*_\Omega|_Y \) satisfying \( \bar{\partial}(t^* f) = 0 \) and

\[
\int_Y |f|^2 |ds|^2 e^{-\lambda - \varphi} dV_Y < +\infty,
\]

\[\text{Of course, applying [22, Theorems 1, 2] can also get the similar result to Corollary 1.5.}\]
there exists a smooth $\bar{\partial}$-closed $(n, q)$-form $F$ on $\Omega$ satisfying $F|_Y = f$ and
\[ \int_{\Omega} |F|^2 e^{-\varphi} dV \leq \frac{\hat{\mathcal{C}}}{\delta} \int_{Y} \frac{|f|^2}{|ds|^2 e^{-\lambda}} e^{-\varphi} dV_Y. \]

(2) For any smooth $\bar{\partial}$-closed $K|_Y$-valued $(0, q)$-form $f$ on $Y$ with
\[ \int_{Y} |f|^2 |ds|^{-2} e^{\lambda-\varphi} dV_Y < +\infty, \]
there exists a smooth $\bar{\partial}$-closed $(n, q)$ form $F$ on $\Omega$ satisfying $\iota^* F = f$ and
\[ \int_{\Omega} |F|^2 e^{-\varphi} dV \leq \frac{\hat{\mathcal{C}}}{\delta} \int_{Y} \frac{|f|^2}{|ds|^2 e^{-\lambda}} e^{-\varphi} dV_Y. \]

**Proof** We only prove (1) here. Let $L := \Omega \times \mathbb{C}$ be the trivial line bundle equipped with a metric $e^{-\varphi - A|z|^2}$ on $\Omega$. We can choose a sufficiently large constant $A > 0$ which depends on $\sigma_1$ and $\sigma_2$ such that the curvature assumptions (1.1) and (1.2) are satisfied. Then there exists a smooth ambient extension $F$ of $f$ to $\Omega$ such that
\[ \int_{\Omega} |F|^2 e^{-\varphi} e^{-A|z|^2} dV \leq \frac{C}{\delta} \int_{Y} \frac{|f|^2}{|ds|^2 e^{-\lambda}} e^{-\varphi} e^{-A|z|^2} dV_Y \]
according to Theorem 1.1. Note that $e^{-A|z|^2}$ has lower bound which depends on $\sup_{\Omega} |z|^2$. So there exists $\hat{\mathcal{C}}$ which depends on $\sup_{\Omega} |z|^2$, $\sigma_1$ and $\sigma_2$, such that
\[ \int_{\Omega} |F|^2 e^{-\varphi} dV \leq \frac{\hat{\mathcal{C}}}{\delta} \int_{Y} \frac{|f|^2}{|ds|^2 e^{-\lambda}} e^{-\varphi} dV_Y. \]

\[ \square \]

The special case of Corollary 1.5 that $Y = (z_1 = 0) \cap \Omega$ with $z_1$ the first global coordinate function of $\mathbb{C}^n$, may be interesting for someone, when all terms involved can be expressed by the basis $\{dz_i, d\bar{z}_i\}_{i=1,...,n}$ globally and their pointwise norms can thereby be expressed by their coefficients. Then the complicated relationships of their coefficients can be demonstrated from Corollary 1.5.

One could ask whether it is possible to weaken the pseudoconvexity assumption of $X$ in Theorem 1.1 as the existence of an upper semi-continuous exhaustion on $X$.

However, Varolin provided us with a counterexample to the expectation.

**Example 1.6** Set the domain $\Omega := \{z \in \mathbb{C}^2; \frac{1}{2} < |z| < 1\}$ and the subspace
\[ Y := \left\{ z = (z^1, z^2) \in \Omega; z^2 = 0 \right\}. \]

Then it is easy to construct a continuous exhaustion, such as $\rho = \frac{1}{(|z|-1/2)(1-|z|)}$. Take the function $f(\zeta, 0) := \zeta^{-n}$ for any $n \in \mathbb{N}^+$. Then $\int_Y |f|^2 |dz^2|^{-2} dV_Y < +\infty$. So
if one assumes the above expectation, then there exists some \( F \in \mathcal{O}(\Omega) \) such that \( F|_Y = f \). By Hartogs theorem and the identity theorem, \( F \) has a unique extension to the unit ball. The restriction of this extension to \( Y \) agrees with \( f \), and this means that \( f \) itself has a holomorphic extension to the unit disk \( \mathbb{D} \times \{0\} \). This is impossible by the identity theorem and that \( \zeta^{-n} \) blows up near the origin in \( \mathbb{D} \).

A further interesting topic about the extension of \( \bar{\partial} \)-closed forms is the singular metric version of the main results here, which is very attractive and full of application prospects, e.g., the deformation invariance problem of higher cohomology of the pluri-canonical bundle of a Kähler family. However, it seems very difficult for extensions of general \( \bar{\partial} \)-closed forms. In [22, Remark 1.2], McNeal–Varolin told us that the routine method—taking a regularization of the singular weight first and then passing to some kind of limit—cannot get the singular version of their extension theorems at least on a Stein manifold due to that the minimal extension operator may not exist.

The other difficulty of dealing with the singular metric version is that the operator \( \bar{\partial} \)-Laplacian with respect to a singular metric may lose the ellipticity in general. Demailly [9, p. 17] hoped that the Laplacian with respect to a singular metric may have a little “ellipticity” when the singularity of the metric involved is mild. The expectation of Demailly seems to be a difficult problem in PDE. All in all, it seems very difficult to obtain the singular metric version of an \( L^2 \) extension theorem of general \( \bar{\partial} \)-closed forms.

The paper is organized as follows. We list some results in Sect. 2 to be used in the proof of Theorem 1.1. Then, we prove Theorem 1.1 in Sect. 3. At last, we give a sketch of a direct proof of Theorem 1.3 in Appendix A, and some explanations on the condition (1.1) in Appendix B.

**Notation 1.7** Unless otherwise stated, we will always adopt the notations in Sect. 1 in the latter sections and in particular, use \(|s|\) or \(|s|e^{-\lambda/2}\) to denote the pointwise norm of \( s \).

### 2 Some Results Used in the Proofs

In this section, we collect several results to be used in the proofs of our main results. Let \( \iota : Y \hookrightarrow X \) be the natural inclusion of a smooth complex hypersurface \( Y \) in a complex manifold \( X \) and \( L \) a line bundle on \( X \).

It is noteworthy that when \( q \geq 1 \), there are two natural choices for the restriction to \( Y \) of an \( L \)-valued \((0, q)\)-form on \( X \):

(i) one can pull back the \( L \)-valued differential form on \( X \) via the natural inclusion \( Y \hookrightarrow X \) to produce an \( L \)-valued \((0, q)\)-form on \( Y \), i.e., a section of \( L|_Y \otimes \Lambda^0, q T^*_Y \rightarrow Y \), which we call the **intrinsic restriction**, or

(ii) one can view an \( L \)-valued \((0, q)\)-form as an abstract section of an abstract bundle \( L \otimes \Lambda^0, q T^*_X \) and pullback the section. That is to say, the restriction is a section of the restricted vector bundle \( (L \otimes \Lambda^0, q T^*_X)|_Y \rightarrow Y \). We call a section of the vector bundle \( (L \otimes \Lambda^0, q T^*_X)|_Y \rightarrow Y \) an **ambient form**, and the restriction of an \( L \)-valued \((0, q)\)-form on \( X \) to \( Y \) an **ambient restriction**.
More precisely, one has the following definitions.

**Definition 2.1** [22, Definition 3.1]

(i) An $L|_Y$-valued $(0, q)$-form $\eta$ on $Y$ is called the *intrinsic restriction* of an $L$-valued $(0, q)$-form $\theta$ on $X$ if

$$\iota^* \theta = \eta.$$ 

(ii) A section $\xi$ of the vector bundle $(L \otimes \Lambda^{0, q} T^*_X)|_Y$ is called the *ambient restriction* of an $L$-valued $(0, q)$-form $\theta$ on $X$ if

$$\theta(y) = \xi(y)$$

for all $y \in Y$, that is $\theta|_Y = \xi$. Note that $|\theta(y)| = |\xi(y)|$ on $Y$ when $X$ and $L$ are equipped with some Hermitian metrics in this case.

Since $(L \otimes \Lambda^{0, q} T^*_X)|_Y$ is not a holomorphic vector bundle, it does not admit a natural notion of $\overline{\partial}$. Then one needs:

**Definition 2.2** [22, p. 422] Let $\iota : Y \hookrightarrow X$ be the natural inclusion of a smooth complex hypersurface $Y$ in a complex manifold $X$ and $L$ a line bundle on $X$. Then for any ambient form $\xi$, $\iota^* \xi$ is defined to be an $L|_Y$-valued $(0, q)$-form on $Y$ by

$$\langle \iota^* \xi, \bar{v}_1, \ldots, \bar{v}_q \rangle := \langle \xi, d\iota(y) \bar{v}_1, \ldots, d\iota(y) \bar{v}_q \rangle$$

in $L_y$ ($y \in Y$), $v_1, \ldots, v_q \in T^{1,0}_{Y, y}$.

Note that $\overline{\partial}(\iota^* \xi)$ is now well defined naturally.

In [22], McNeal–Varolin established the following two extension theorems and we will use them to construct some smooth extensions locally on a weakly pseudoconvex Kähler manifold in Sect. 3.1 and Appendix A.

**Theorem 2.3** (Ambient $L^2$ extension) Let $\iota : Y \hookrightarrow X$ be the natural inclusion of a smooth complex hypersurface $Y$ in a complex manifold $X$ and $L$ a line bundle on $X$. Then for any ambient form $\xi$, $\iota^* \xi$ is defined to be an $L|_Y$-valued $(0, q)$-form on $Y$ by

$$\langle \iota^* \xi, \bar{v}_1, \ldots, \bar{v}_q \rangle := \langle \xi, d\iota(y) \bar{v}_1, \ldots, d\iota(y) \bar{v}_q \rangle$$

in $L_y$ ($y \in Y$), $v_1, \ldots, v_q \in T^{1,0}_{Y, y}$.

Assume also that for any $0 \leq q \leq n - 1$,

$$\sqrt{-1} \left( \partial \overline{\partial} (\varphi - \lambda) + \text{Ricci}(\omega) \right) \wedge \omega^q \geq 0$$

and

$$\sqrt{-1} \left( \partial \overline{\partial} (\varphi - (1 + \delta)\lambda) + \text{Ricci}(\omega) \right) \wedge \omega^q \geq 0$$

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for some constant \(0 < \delta \leq 1\). Then there is a constant \(C > 0\) such that for any smooth section \(\xi\) of the vector bundle \((L \otimes \Lambda^{0,q} T^*_X)|_Y \to Y\) satisfying

\[
\overline{\partial} (i^*\xi) = 0 \quad \text{and} \quad \int_Y \frac{|\xi|_{\omega}^2 e^{-\varphi}}{|s|_{\omega}^2} e^{-\lambda} dV_{Y,\omega} < +\infty,
\]

there exists a smooth \(\overline{\partial}\)-closed \(L\)-valued \((0,q)\)-form \(\Xi\) on \(X\) with

\[
\Xi|_Y = \xi \quad \text{and} \quad \int_X |\Xi|_{\omega}^2 e^{-\varphi} dV_{\omega} \leq \frac{C}{\delta} \int_Y \frac{|\xi|_{\omega}^2 e^{-\varphi}}{|s|_{\omega}^2} e^{-\lambda} dV_{Y,\omega}.
\]

The constant \(C\) is universal, i.e., it is independent of all the data.

**Remark 2.4** On the above theorem, there is a typo on [22, Theorem 1] which only requires \(\delta > 0\). In fact, we can conclude that the first two lines in [22, p. 438] cannot be true if \(\delta > 1\). Denote \(e^v\) in [22, p. 438] by \(|s|^2\) here, and then

\[
e^v(\tau + A) = |s|^2 (2 + 2e^{a-1} + \log(2e^{a-1} - 1)) \geq |s|^2 \cdot 2e^{a-1} = \frac{2e^{\gamma-1}|s|^2}{(\epsilon^2 + |s|^2)^{\delta}}.
\]

As \(\delta > 1\), the above is

\[
2e^{\gamma-1} \cdot \frac{|s|^2}{\epsilon^2 + |s|^2} \cdot \frac{1}{(\epsilon^2 + |s|^2)^{\delta-1}},
\]

which cannot be bounded when \(\epsilon \to 0\) and \(|s| \to 0\) (take \(|s| = \epsilon \to 0\) for example).

**Theorem 2.5** (Intrinsic \(L^2\) extension) With the hypotheses of Theorem 2.3, there is a universal constant \(C > 0\) such that for any smooth \(\overline{\partial}\)-closed \(L\)-valued \((0,q)\)-form \(\eta\) on \(Y\) satisfying

\[
\int_Y \frac{|\eta|_{\omega}^2 e^{-\varphi}}{|s|_{\omega}^2} e^{-\lambda} dV_{Y,\omega} < +\infty,
\]

there exists a smooth \(\overline{\partial}\)-closed \(L\)-valued \((0,q)\)-form \(\Pi\) on \(X\) such that

\[
i^*\Pi = \eta \quad \text{and} \quad \int_X |\Pi|_{\omega}^2 e^{-\varphi} dV_{\omega} \leq \frac{C}{\delta} \int_Y \frac{|\eta|_{\omega}^2 e^{-\varphi}}{|s|_{\omega}^2} e^{-\lambda} dV_{Y,\omega}.
\]

Consider the modified \(\overline{\partial}\)-operators

\[
T := \overline{\partial} \circ \sqrt{\tau + A} \quad \text{and} \quad S := \sqrt{\tau} \cdot \overline{\partial}
\]

acting on \((n,q)\)-forms with values in a vector bundle, where \(\tau, A\) are positive smooth functions.
Then \( S \circ T = 0 \). In solving the \( \bar{\partial} \)-equation, the basic estimate about the modified \( \bar{\partial} \) operator is always needed to construct some bounded linear functionals. On different occasions, several classical basic estimates have been established in [3, 9, 20, 23, 26], for example. Here we adopt the following one.

**Lemma 2.6** (Twisted basic estimate) Let \((X, \omega)\) be a Kähler manifold and \( E \) a holomorphic line bundle with a smooth Hermitian metric \( e^{-\varphi} \) over \( X \). Assume that \( \tau \) and \( A \) are smooth and positive functions on \( X \). Fix a smoothly bounded domain \( \Omega_1 \subset X \) such that its boundary \( \partial \Omega \) is pseudoconvex (e.g., [28, Sect. 1.5] for this notion and its effect). Then for any smooth \( E \)-valued \((n, q)\)-form \( u \) in the domain of \( \bar{\partial}^* \), one has the estimate

\[
\int_{\Omega_1} (\tau + A) |\bar{\partial}^* u|^2_{\omega} e^{-\varphi} dV_\omega + \int_{\Omega_1} |\bar{\partial} u|^2_{\omega} e^{-\varphi} dV_\omega \geq \int_{\Omega_1} \left( \sqrt{-1} \left( \frac{\tau \partial \bar{\partial} \varphi - \partial \bar{\partial} \tau - \frac{\partial \tau \wedge \bar{\partial} \tau}{A} \right), \Lambda_{\omega} \right) u, u \right)_{\omega} e^{-\varphi} dV_\omega,
\]

where \( \Lambda_\omega \) is the dual Lefschetz operator.

**Proof** The proof is the same as that of [22, Lemma 2.2] which is in the context of the Stein setting. However, the estimate attributes essentially to the usual twisted Bochner–Kodaira–Morrey–Kohn identity and the pseudoconvexity of \( \partial \Omega \).

Furthermore, the ellipticity of the twisted \( \bar{\partial} \)-Laplacian \( \square := TT^* + S^* S \) is needed.

**Lemma 2.7** [22, Proposition 2.3] Assume that the functions \( \tau \), \( A \) and the Hermitian metric \( e^{-\varphi} \) of the holomorphic line bundle \( E \) are smooth, and that \( \tau \) and \( \tau + A \) are positive, then the operator \( \square \) is second order (interior) elliptic with smooth coefficients.

**Lemma 2.8** [8, Lemma 6.9] Let \( \Omega \) be an open subset of \( \mathbb{C}^n \) and \( Y \) an analytic subset of \( \Omega \). Assume that \( v \) is a \((p, q-1)\)-form with \( L^2_{\text{loc}} \) coefficients and \( w \) is a \((p, q)\)-form with \( L^1_{\text{loc}} \) coefficients such that \( \bar{\partial} v = w \) on \( \Omega \setminus Y \) (in the sense of distribution theory). Then \( \bar{\partial} v = w \) on \( \Omega \). (A more general version for the first order differential operator can be found in [5, Proposition 4.8].)

**Lemma 2.9** [10, Chap. VI-(5.8) Proposition] Let \((X, \omega)\) be an \( n \)-dimensional Hermitian manifold and \( \gamma \) a real \((1, 1)\)-form. Then there exists an \( \omega \)-orthogonal basis \((\xi_1, \xi_2, \ldots, \xi_n)\) in \( T^{1,0}_X \) which diagonalizes both forms \( \omega \) and \( \gamma \):

\[
\omega = \sqrt{-1} \sum_{1 \leq j \leq n} \xi_j^* \wedge \bar{\xi}_j^*, \quad \gamma = \sqrt{-1} \sum_{1 \leq j \leq n} \gamma_j \xi_j^* \wedge \bar{\xi}_j^*, \quad \gamma_j \in \mathbb{R}.
\]

For every form \( u = \sum u_{J,K} \xi_j^* \wedge \bar{\xi}_K^* \), one has

\[
[\gamma, \Lambda_\omega] u = \sum_{J,K} \left( \sum_{j \in J} \gamma_j + \sum_{j \in K} \gamma_j - \sum_{1 \leq j \leq n} \gamma_j \right) u_{J,K} \xi_j^* \wedge \bar{\xi}_K^*.
\]

\( \square \) Springer
Lemma 2.10 [10, Chap. VIII-(6.4)] Let $\theta$ be a smooth real $(1, 1)$-form on an $n$-dimensional Hermitian manifold $(X, \omega)$. If $\lambda_1(x) \leq \cdots \leq \lambda_n(x)$ are the eigenvalues of $\theta$ with respect to $\omega$ for all $x \in X$ and $\lambda_1 + \cdots + \lambda_q > 0$, then for arbitrary $(n, q)$-form $g$ on $X$,

$$\langle [\theta, \Lambda_\omega]g, g \rangle_\omega \geq (\lambda_1 + \cdots + \lambda_q) |g|^2_\omega$$

and

$$\int_X \left\langle \left( [\theta, \Lambda_\omega]^{-1} g, g \right) \right\rangle_\omega dV_\omega \leq \int_X \frac{1}{\lambda_1 + \cdots + \lambda_q} |g|^2_\omega dV_\omega.$$

Lemma 2.11 Let $(X, \omega)$ be an $n$-dimensional Hermitian manifold and $\theta$ a continuous $(1, 0)$-form. Then for any $(n, q)$-form $\alpha$, we have

$$\left[ \sqrt{-1} \theta \wedge \bar{\theta}, \Lambda_\omega \right]^{-1} \alpha = T_{\bar{\theta}} T_{\bar{\theta}}^* \alpha,$$

where $T_{\bar{\theta}}$ denotes $\bar{\theta} \wedge \bullet$.

**Proof** The proof is the same as that of [12, Lemma 4.2].

From classical knowledge about matrices (e.g., [17]), we can easily get the following result.

**Lemma 2.12** Assume that $A$ and $B$ are Hermitian matrices of $n \times n$ and both of them are positive definite. Then $A - B > 0$ implies that $B^{-1} - A^{-1} > 0$.

From Lemma 2.12, we can easily conclude the following comparison theorem.

**Lemma 2.13** Let $\theta_1$ and $\theta_2$ be smooth real $(1, 1)$-forms on an $n$-dimensional Hermitian manifold $(X, \omega)$. Assume that $(\theta_1 - \theta_2) \wedge \omega^r > 0$, $\theta_1 \wedge \omega^r > 0$ and $\theta_2 \wedge \omega^r > 0$. Then

$$\left[ \theta_2, \Lambda_\omega \right]^{-1} - \left[ \theta_1, \Lambda_\omega \right]^{-1}$$

is positive definite on $(n, r + 1)$-forms.

**Remark 2.14** The above lemma is a little bit different from and stronger than the classical result about the non-increasing of $[\theta, \Lambda_\omega]^{-1}$ with respect to $\theta$ (see [8, Lemma 3.2] or [5, Proposition 5.2]). Here we know nothing about the comparison information between $\theta_1$ and $\theta_2$ and only know the comparison data between $\theta_1 \wedge \omega^r$ and $\theta_2 \wedge \omega^r$. However, from a point of view of positive definite transformation, we can easily get the above lemma.

The following observation is useful to give a direct proof of Proposition 3.3.

**Lemma 2.15** [13, p. 609], or [5, Lemma 9.20] If $g$ is an integrable function near $0 \in \mathbb{R}^d$, then there exists a sequence $x_j \rightarrow 0$ in $\mathbb{R}^d$ such that $|g(x_j)| = o \left( |x_j|^{-d} \right)$.
3 Proof of Theorem 1.1

We split the proof of Theorem 1.1 in several steps.

3.1 Construction of a Smooth Extension $\tilde{f}_\infty$

Let $\{W_\alpha\}$ be the Stein coordinate patches of $X$, biholomorphic to polydiscs, and admit the following property: if we denote the corresponding coordinates by $(z_\alpha, w_\alpha) \in \Delta \times \Delta^{n-1}$, where $w_\alpha = (w^1_\alpha, \ldots, w^{n-1}_\alpha)$, then $W_\alpha \cap Y = \{z_\alpha = 0\}$. On each $W_\alpha$, we fix some holomorphic $\sigma_\alpha \in \Gamma (W_\alpha, K_X \otimes L)$ to trivialize $K_X \otimes L$. Let $\{\theta_\alpha\}$ be a partition of unity subordinate to $\{W_\alpha\}$. On every Stein domain $W_\alpha$, one can always find a smooth $\bar{\partial}$-closed ambient extension $f_\alpha$ of $f$ by the Stein theory (without use of any curvature assumption).

Set $\tilde{f}_\infty := \sum_\alpha \theta_\alpha \cdot f_\alpha$. Then

$$\bar{\partial} \tilde{f}_\infty = \bar{\partial} \sum_\alpha \theta_\alpha \cdot (f_\alpha - f_\beta) = \sum_\alpha \bar{\partial} \theta_\alpha \cdot (f_\alpha - f_\beta)$$

on $W_\beta$.

So $\bar{\partial} \tilde{f}_\infty = 0$ along $Y$.

For the nice regularity of $f_\varepsilon$ in Sect. 3.3, we need some regularity of $s^{-1} \bar{\partial} \tilde{f}_\infty$ twisted by $s^{-1}$. On this, Koziarz [16, Lemma 3.1] had provided us with a useful trick to raise the regularity of $s^{-1} \bar{\partial} \tilde{f}_\infty$ and still preserve the ambient extension property of $\tilde{f}_\infty$ as follows.

We proceed by induction to get the regularity of $s^{-1} \bar{\partial} \tilde{f}_\infty$. One says that $\tilde{f}_\infty$ enjoys the property $(P_k)$ for $k \geq 1$, if on each $W_\alpha$

$$\bar{\partial} \tilde{f}_\infty = \bar{\partial} \sum_\alpha \theta_\alpha \cdot (f_\alpha - f_\beta) = \sum_\alpha \bar{\partial} \theta_\alpha \cdot (f_\alpha - f_\beta)$$

for some $f_\alpha, h_\alpha \in \mathcal{E}^\infty (W_\alpha, \Lambda^{n,q+1} T_X^* \otimes L)$ and $a_I, b_{I'} \in \mathcal{E}^\infty (\Delta^{n-1}, \mathbb{C})$ with the increasing multi-indices $I, I'$.

Note that for $k \geq 2$, $\bar{\partial} \tilde{f}_\infty$ is of class $\mathcal{E}^{k-2}$ on a complex plane, where $z$ is the complex analytic coordinate. Then $\tilde{f}_\infty$ enjoying the property $(P_k)$ implies that

$$s^{-1} \bar{\partial} \tilde{f}_\infty \in \mathcal{E}^{k-2} (X, \Lambda^{n,q+1} T_X^* \otimes L \otimes \mathcal{O}_X(-Y)), \quad \text{for } k \geq 2.$$
\[ Y \cap W_\alpha = \{ z_\alpha = 0 \} \text{ and } \partial f_\infty \text{ is smooth, one can expand } \partial f_\infty \text{ along the coordinate function } z_\alpha \text{ such that } \]
\[ \partial f_\infty = z_\alpha \cdot g_1 + z_\alpha \cdot g_2, \]
where \( g_1 \) and \( g_2 \) are the corresponding smooth forms. So expanding \( g_2 \) along the coordinate function \( z_\alpha \) again similarly can conclude that \( f_\infty \) enjoys the property \((P_1)\).

A direct calculation shows
\[ \partial (\partial f_\infty) = z_\alpha \partial f_\alpha (z_\alpha, w_\alpha) + k z_\alpha^{-1} d z_\alpha \wedge \left[ \sum_{|I'|=q+1} b_{I'} (w_\alpha) \sigma_\alpha d \tilde{w}_\alpha^{I'} \right] \]
\[ + z_\alpha^k h'_\alpha (z_\alpha, w_\alpha) \]
for some \( h'_\alpha \in \mathcal{E}^{\infty} (W_\alpha, \Lambda^{n,q+2} T^*_X \otimes L) \). Then all the above \( b_{I'} \) vanish identically as \( \partial (\partial f_\infty) = 0 \). So we take
\[ \tilde{f}'_\infty = f_\infty - \sum_\alpha \theta_\alpha z_\alpha^{k+1} \sum_{|I|=q} a_I (w_\alpha) \sigma_\alpha d \tilde{w}_\alpha^{I} \]
to get
\[ \partial \tilde{f}'_\infty = \partial f_\infty - \sum_\alpha \theta_\alpha z_\alpha^{k+1} \left( \sum_{|I|=q} a_I (w_\alpha) \sigma_\alpha d \tilde{w}_\alpha^{I} \right) \]
\[ - \sum_\alpha \theta_\alpha z_\alpha^{k+1} \partial \left( \sum_{|I|=q} a_I (w_\alpha) \sigma_\alpha d \tilde{w}_\alpha^{I} \right) \]
\[ = \sum_\alpha \theta_\alpha \left( \partial f_\infty - z_\alpha^{k+1} d z_\alpha \wedge \sum_{|I|=q} a_I (w_\alpha) \sigma_\alpha d \tilde{w}_\alpha^{I} \right) + \sum_\alpha z_\alpha^{k+1} h''_\alpha (z_\alpha, w_\alpha) \]
\[ = \sum_\alpha z_\alpha \theta_\alpha f_\alpha (z_\alpha, w_\alpha) + \sum_\alpha z_\alpha^{k+1} \left( \theta_\alpha h_\alpha (z_\alpha, w_\alpha) + h''_\alpha (z_\alpha, w_\alpha) \right) \]
for some \( h''_\alpha \in \mathcal{E}^{\infty}_0 (W_\alpha, \Lambda^{n,q+1} T^*_X \otimes L) \). Then \( \partial \tilde{f}'_\infty \) satisfies (3.1) for \( k + 1 \) on each \( W_\alpha \), possibly after some transformations of coordinates. Then \( \tilde{f}'_\infty \) enjoys the property \((P_{k+1})\). Apparently, \( \tilde{f}'_\infty \) is still the ambient extension of \( f \) while \( \partial \tilde{f}'_\infty \) still vanishes along \( Y \).

In conclusion, we have proved the following results.

**Proposition 3.1** For any \( k \geq 0 \), there exists a smooth section
\[ \tilde{f}'_\infty \in \mathcal{E}^{\infty} (X, \Lambda^{n,q} T^*_X \otimes L) \]
such that
(a) $\tilde{f}_\infty$ is the ambient extension of $f$,
(b) $\bar{\partial}\tilde{f}_\infty \equiv 0$ at every point of $Y$,
(c) $s^{-1}\bar{\partial}\tilde{f}_\infty \in \mathcal{E}^k(X, \Lambda^{n,q+1}T^*_X \otimes L \otimes \mathcal{O}_X(-Y)) = \mathcal{E}^k(X, \Lambda^{n,q+1}T^*_X \otimes L \otimes E^*)$.

From now on, we fix $k \geq n + 6$, where $n$ is the complex dimension of $X$.

### 3.2 Construction of Special Weights and Twist Factors

For the sake of completeness, we write the following constructions concretely, which is almost the same as [22, Sect. 4.1].

Set

$$e^{-\psi} := e^{-\psi + \lambda}.$$ 

Next we turn to the choices of the functions $A$ and $\tau$ as in [21, 27] and more originally [21].

Let

$$h(x) := 2 - x + \log \left(2e^{x-1} - 1\right), \quad v := \log |s|^2 \quad \text{and} \quad a := \gamma - \delta \log \left(|s|^2 + \varepsilon^2\right),$$

where $0 < \delta \leq 1$ is as in the main theorems, $x > 1$, and $\gamma > 1$ is a real number such that $a > 1$. Note that

$$a \geq \gamma - \delta \log(1 + \varepsilon^2) \geq \gamma - \delta \varepsilon^2 \quad (3.2)$$

due to (1.3). That is to say, for the fixed $\gamma > 1$, there exists $\varepsilon_0 > 0$ such that $a - 1$ has a positive lower bound which is independent of $\varepsilon$ as $\varepsilon < \varepsilon_0$. It is easy to see that

$$h'(x) = (2e^{x-1} - 1)^{-1} \in (0, 1) \quad \text{and} \quad h''(x) = \frac{-2e^{x-1}}{(2e^{x-1} - 1)^2} < 0.$$

Define

$$\tau := a + h(a) \quad \text{and} \quad A := \frac{(1 + h'(a))^2}{-h''(a)}.$$ 

Then $A = 2e^{a-1}$. Furthermore, (3.2) gives

$$\tau - (1 + h'(a)) > \sigma' \quad (3.3)$$

for some constant $\sigma' > 0$ which is independent of $\varepsilon$ when $\varepsilon < \varepsilon_0$. Moreover, these choices guarantee that

$$-\partial \bar{\partial} \tau - A^{-1}\partial \tau \wedge \bar{\partial} \tau = \left(1 + h'(a)\right)(-\partial \bar{\partial}a).$$
Finally, a straightforward calculation yields

\[-\partial\bar{\partial}a = \delta\partial\bar{\partial}\log(e^v + \varepsilon^2)\]

\[= \frac{\delta|s|^2}{|s|^2 + \varepsilon^2} \partial\bar{\partial}v + \frac{4\delta\varepsilon^2 \partial|s| \wedge \bar{\partial}|s|}{(|s|^2 + \varepsilon^2)^2}\]

\[= -\delta \frac{|s|^2}{|s|^2 + \varepsilon^2} \partial\bar{\partial}\lambda + \frac{4\delta\varepsilon^2 \partial|s| \wedge \bar{\partial}|s|}{(|s|^2 + \varepsilon^2)^2},\]

where the last equality follows from the Lelong–Poincaré equation and \(|s|^2[Y] = 0\) due to Supp\([Y] = Y\).

A direct calculation together with (1.1), (1.2), and (3.3) yields

\[
\sqrt{-1} \left( \tau(\partial\bar{\partial}\psi) - \partial\bar{\partial}\tau - A^{-1}_\tau \partial\bar{\partial}\tau \wedge \bar{\partial}\tau \right) \wedge \omega^g
\]

\[= \sqrt{-1} \left( \tau \partial\bar{\partial}(\varphi - \lambda) + (1 + h'(a)) (-\partial\bar{\partial}a) \right) \wedge \omega^g
\]

\[= \left( \tau - (1 + h'(a)) \left( \frac{|s|^2}{|s|^2 + \varepsilon^2} \right) \right) \cdot \sqrt{-1} \partial\bar{\partial}(\varphi - \lambda) \wedge \omega^g
\]

\[+ \sqrt{-1} \left( 1 + h'(a) \right) \frac{|s|^2}{|s|^2 + \varepsilon^2} \left( \partial\bar{\partial}(\varphi - \lambda) - \delta\partial\bar{\partial}\lambda \right) \wedge \omega^g
\]

\[> \sqrt{-1} \delta \left( \frac{4\varepsilon^2 \partial|s| \wedge \bar{\partial}|s|}{(|s|^2 + \varepsilon^2)^2} \right) \wedge \omega^g + \sigma' \sigma \omega \wedge \omega^g\]

(3.4)

Set

\[B_\varepsilon := \sqrt{-1} \left( \tau(\partial\bar{\partial}\psi) - \partial\bar{\partial}\tau - A^{-1}_\tau \partial\bar{\partial}\tau \right).\]

Then (3.4) tells us that

\[B_\varepsilon \wedge \omega^g > \sqrt{-1} \delta \left( \frac{4\varepsilon^2 \partial|s| \wedge \bar{\partial}|s|}{(|s|^2 + \varepsilon^2)^2} \right) \wedge \omega^g\]

(3.5)

and

\[B_\varepsilon \wedge \omega^g \geq \sigma' \sigma \omega \wedge \omega^g\]

(3.6)

hold on \(X\).
3.3 Solving Twisted $\bar{\partial}$-Laplace Equations with Estimates

Recall that in Theorem 1.1, $f$ is a smooth section of the vector bundle $(K_X \otimes L \otimes \wedge^{0,q} T_X^*)|_Y \to Y$ for any $0 \leq q \leq n - 1$ satisfying

$$\bar{\partial} (i^* f) = 0 \quad \text{and} \quad \int_Y |f|^2 \omega e^{-\varphi} dV_{Y,\omega} < \infty.$$ 

And one has obtained an ambient extension $\tilde{f}_\infty$ of $f$ in Proposition 3.1. Let $0 < c \ll 1$ and $\theta \in \mathcal{E}^{(\epsilon)}_\infty ((0, +\infty))$ a cutoff function with values in $[0, 1]$ such that $\theta|_{[0,c]} \equiv 1$ and $\theta|_{[1,\infty]} \equiv 0$ and $|\theta'| \leq 1 + 2c$. For $\epsilon > 0$, define

$$g_\epsilon := s^{-1}\bar{\partial}(\theta (\epsilon^{-2}|s|^2) \tilde{f}_\infty) = s^{-1}\bar{\partial}(\theta (\epsilon^{-2}|s|^2)) \wedge \tilde{f}_\infty + s^{-1}\theta (\epsilon^{-2}|s|^2) \bar{\partial} \tilde{f}_\infty.$$

The first term in $g_\epsilon$ can be easily written as

$$g_\epsilon^{(1)} = \bar{\partial}|s| \wedge \frac{2|s|(\epsilon^{-2}|s|^2 + 1)\theta'(\epsilon^{-2}|s|^2)}{|s|^2 + \epsilon^2}s^{-1}\tilde{f}_\infty.$$ (3.7)

We also denote the second term $s^{-1}\theta (\epsilon^{-2}|s|^2) \bar{\partial} \tilde{f}_\infty$ in the above expression of $g_\epsilon$ by $g_\epsilon^{(2)}$.

From the smoothness of $g_\epsilon^{(1)}$ and the regularity information of $g_\epsilon^{(2)}$ by Proposition 3.1.(c), we know that $g_\epsilon \in \mathcal{E}^k (X, \Lambda^{n,q+1} T_X^* \otimes L \otimes E^*)$. Moreover, $g_\epsilon$ is $\bar{\partial}$-closed outside $Y$ according to the definition of $g_\epsilon$. So $g_\epsilon$ is $\bar{\partial}$-closed on $X$ due to the continuity of $\bar{\partial} g_\epsilon$.

Assume that $\phi$ is a smooth plurisubharmonic exhaustion function of the weakly pseudoconvex Kähler manifold $X$. Due to the Sard’s theorem, we can always assume that $\Omega_j := \{ \phi < j \}, j = 1, 2, \ldots$, satisfy

$$\Omega_j \subset \subset \Omega_{j+1} \quad \text{and} \quad \lim_{j \to \infty} \Omega_j = \bigcup_{j \geq 1} \Omega_j = X,$$

and every $\partial \Omega_j$ is smooth and pseudoconvex. From now on, we will work on $\Omega_j$ instead of $X$ until the end of the penultimate step.

For any smooth $K_X \otimes L \otimes E^*$-valued $(0, q + 1)$-form $\beta$ in the domain of $T^*$ and $S$, we infer the inequality

$$\langle [\beta, g_\epsilon] \rangle_{L^2}^2 \leq \langle [B_\epsilon, \Lambda_\omega]^{-1} g_\epsilon, g_\epsilon \rangle_{L^2} \langle [B_\epsilon, \Lambda_\omega] \beta, \beta \rangle_{L^2} \leq \langle [B_\epsilon, \Lambda_\omega]^{-1} g_\epsilon, g_\epsilon \rangle_{L^2} (||T^* \beta||^2 + ||S\beta||^2)$$ (3.8)

from $B_\epsilon \wedge \omega^q > 0$, Cauchy–Schwarz inequality and Lemma 2.6. By the variant of Cauchy–Schwarz inequality

$$\langle \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 \rangle \leq \langle \alpha_1, \alpha_1 \rangle + \langle \alpha_2, \alpha_2 \rangle + c \langle \alpha_1, \alpha_1 \rangle + \frac{1}{c} \langle \alpha_2, \alpha_2 \rangle$$
we have

\[ ([B_\varepsilon, \Lambda_\omega]^{-1} g_\varepsilon, g_\varepsilon)_{L^2} \leq (1 + c) ([B_\varepsilon, \Lambda_\omega]^{-1} g_\varepsilon^{(1)}, g_\varepsilon^{(1)})_{L^2} + (1 + 1/c) ([B_\varepsilon, \Lambda_\omega]^{-1} g_\varepsilon^{(2)}, g_\varepsilon^{(2)})_{L^2}, \]  

(3.9)

where \( c \) is taken, for convenience, to be the same \( c \) as that in the definition of the cutoff function \( \theta \). Then according to Lemmata 2.13 and 2.11, (3.5) and (3.7) give the estimate

\[
(1 + c) ([B_\varepsilon, \Lambda_\omega]^{-1} g_\varepsilon^{(1)}, g_\varepsilon^{(1)})_{L^2} \leq (1 + c) \int_{\Omega_j} \frac{(|s|^2 + \varepsilon^2)^2}{4\varepsilon^2\delta} (T_{\bar{\partial}|s|} T_{\bar{\partial}|s|}^{-1} g_\varepsilon^{(1)}, g_\varepsilon^{(1)})_{\omega} e^{-\psi} dV_\omega \\
= (1 + c) \int_{\Omega_j} \frac{(|s|^2 + \varepsilon^2)^2}{4\varepsilon^2\delta} (T_{\bar{\partial}|s|}^{-1} g_\varepsilon^{(1)}, T_{\bar{\partial}|s|} g_\varepsilon^{(1)})_{\omega} e^{-\psi} dV_\omega \\
= \frac{1 + c}{\delta} \int_{\Omega_j} \frac{(\varepsilon^{-2}|s|^2 + 1)^2 \theta'(\varepsilon^{-2}|s|^2)^2}{\varepsilon^2} |f_\infty^1|_{\omega}^2 e^{-\psi} dV_\omega \\
\leq 4(1 + c)(1 + 2c)^2 \int_{\Omega_j \cap \{|s|^2 \leq 1\}} \varepsilon^{-2} |f_\infty^1|_{\omega}^2 e^{-\psi} dV_\omega.
\]  

(3.10)

We denote the right-hand side of the above inequality by \( C_{c, \varepsilon} \), whose limit is

\[
\frac{8\pi (1 + c)(1 + 2c)^2}{\delta} \int_{\Omega_j \cap Y} \frac{|f|^2_{\omega}}{s^2} e^{-\psi} dV_{Y, \omega} \quad \text{as} \quad \varepsilon \to 0,
\]  

(3.11)

by the Fubini theorem and Proposition 3.1.(a).

It’s turn to estimate the term involving \( g_\varepsilon^{(2)} \). The relative compactness of \( \Omega_j \) and the lower semi-continuity of \( \sigma \) imply that \( (q + 1)\sigma' \sigma \) has a positive lower bound \( \lambda_j \) which is independent of \( \varepsilon \). Then Lemma 2.10, (3.6) and the boundedness of \( \theta \) imply

\[
(1 + \frac{1}{c}) ([B_\varepsilon, \Lambda_\omega]^{-1} g_\varepsilon^{(2)}, g_\varepsilon^{(2)})_{L^2} \leq (1 + \frac{1}{c}) \int_{\Omega_j} \frac{1}{(q + 1)\sigma' \sigma} \left(s^{-1} \theta (\varepsilon^{-2}|s|^2) \tilde{\partial} \tilde{f}_\infty, s^{-1} \theta (\varepsilon^{-2}|s|^2) \tilde{\partial} \tilde{f}_\infty \right)_{\omega} e^{-\psi} dV_\omega \\
\leq (1 + \frac{1}{c}) \frac{1}{\lambda_j} \int_{\Omega_j \cap \{|s|^2 \leq 1\}} s^{-1} \tilde{\partial} \tilde{f}_\infty^2_{\omega} e^{-\psi} dV_\omega.
\]  

(3.12)

As \( s^{-1} \tilde{\partial} \tilde{f}_\infty \) is of class \( \mathcal{E}^k \) on \( X \) and the volume of the integral region of above is \( \sim O(\varepsilon^2) \) due to the relative compactness of \( \Omega_j \),

\[
(1 + \frac{1}{c}) ([B_\varepsilon, \Lambda_\omega]^{-1} g_\varepsilon^{(2)}, g_\varepsilon^{(2)})_{L^2} \sim O(\varepsilon^2),
\]  

(3.13)

which depends on \( c \) and \( j \).
Denote $C_{c,e} + O(\varepsilon^2)$ by $\frac{C_{c,j,c}}{\delta}$. Then it follows

$$\left| \langle \beta, g_\varepsilon \rangle \right|^2_{L^2} \leq \frac{C_{c,j,c}}{\delta} \left( \| T^* \beta \|^2 + \| S\beta \|^2 \right) \quad (3.14)$$

from (3.8),(3.9),(3.10) and (3.13).

Set $\Box := TT^* + S^*S$. Then we will solve the equation

$$\Box V_\varepsilon = g_\varepsilon$$

in the standard way on the basis of the above estimate (3.14).

The Dirichlet semi-norm is defined as

$$\| \beta \|^2_{\mathcal{H}} := \| T^* \beta \|^2 + \| S\beta \|^2$$

for any smooth $K_X \otimes L \otimes E^*$-valued $(0, q + 1)$-form in the domain of $T^*$ and $S$. Since $\sigma$ is positive lower semi-continuous and $\sigma' > 0$, the original norm is dominated by the Dirichlet norm multiplied by some constant on the relatively compact domain $\Omega_j$ due to Lemmata 2.6 and 2.10 and the strict positivity (3.6) of $B_\varepsilon$ on $X$. So the Dirichlet semi-norm is a norm on $\Omega_j$.

Let $\mathcal{H}$ denote the Hilbert space closure of the set of all smooth $K_X \otimes L \otimes E^*$-valued $(0, q + 1)$-forms in the domain of $T^*$ and $S$. Consider the functional $\ell : \mathcal{H} \rightarrow \mathbb{C}$, defined by

$$\ell(\beta) := \langle \beta, g_\varepsilon \rangle = \int_{\Omega_j} \langle \beta, g_\varepsilon \rangle_\omega e^{-\varphi + \lambda} dV_\omega.$$ 

Recall that the original $L^2$-norm is dominated by the Dirichlet norm for smooth $K_X \otimes L \otimes E^*$-valued $(0, q + 1)$-forms in the domain of $T^*$ and $S$. So it is easy to see that (3.14) holds for every element of $\mathcal{H}$ by taking limits with respect to the graph norm by the density of $D_{T^*} \cap D_S \cap \mathcal{E}_\infty(\bar{\Omega})$ in $D_{T^*} \cap D_S$. So $\ell$ is a bounded linear functional on $\mathcal{H}$, and the $\mathcal{H}^*$-norm of $\ell$ is no more than $\delta^{-1}C_{c,j,c}$. Then there exists $V_{\varepsilon,j} \in \mathcal{H}$ (here we omit the subscript $c$ for $V_{\varepsilon,j}$ due to that the $c << 1$ is fixed until Remark 3.6) such that

$$\| V_{\varepsilon,j} \|^2_{\mathcal{H}} = \| \ell \|^2_{\mathcal{H}^*} \leq \delta^{-1}C_{c,j,c} \quad \text{and} \quad (g_\varepsilon, \beta) = \langle T^*V_{\varepsilon,j}, T^*\beta \rangle + \langle SV_{\varepsilon,j}, S\beta \rangle$$

by the Riesz representation theorem. The latter defines the meaning of

$$\Box V_{\varepsilon,j} = g_\varepsilon$$

in the weak sense, as $\beta$ goes through all the smooth compactly supported forms. Moreover, Lemma 2.7 and $g_\varepsilon \in \mathcal{E}^k$ tell us that

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$V_{\varepsilon,j}$ is of class $E^{k_1+1}$ on $\Omega_j$ for some $k_1 \geq 5$ (e.g., [1, Chap. 3, Theorem 3.54]). As $S \circ T = 0$ and $Sg_{\varepsilon} = \sqrt{\tau} \bar{\partial} g_{\varepsilon} = 0$, we find that

$$0 = (S \Box V_{\varepsilon,j}, SV_{\varepsilon,j}) = \|S^* SV_{\varepsilon,j}\|^2$$

and thus $\|SV_{\varepsilon,j}\|^2 = (S^* SV_{\varepsilon,j}, V_{\varepsilon,j}) = 0$.

Now in conclusion, we obtain an $L \otimes E^*$-valued $(n, q + 1)$-form $V_{\varepsilon,j}$ of class $E^{k_1+1}$ such that

$$\Box V_{\varepsilon,j} = g_{\varepsilon} \quad \text{and} \quad \|V_{\varepsilon,j}\|_{\mathcal{H}}^2 \leq \frac{C_{\varepsilon,j,\varepsilon}}{\delta}.$$ 

Set $v_{\varepsilon,j} := T^* V_{\varepsilon,j}$. It follows that

$$Tv_{\varepsilon,j} = \Box V_{\varepsilon,j} = g_{\varepsilon}.$$ 

Then we have proved the following theorem.

**Theorem 3.2** The equation $Tv_{\varepsilon,j} = g_{\varepsilon}$ has a solution $v_{\varepsilon,j} \in E^{k_1}(\Omega_j, L \otimes E^* \otimes \Lambda^{n,q} T_X^*)$ satisfying the $L^2$-estimate

$$\int_{\Omega_j} |v_{\varepsilon,j}|^2_{\omega} e^{-\psi + \lambda} dV_\omega \leq \frac{C_{\varepsilon,j,\varepsilon}}{\delta}.$$ 

### 3.4 Construction of an $E^{k_1}$ Extension on $\Omega_j$ with Uniform $L^2$ Bound

Set

$$u_{\varepsilon,j} := \theta \left( e^{-2|s|^2} \right) \tilde{f}_\infty - \sqrt{\tau + A} v_{\varepsilon,j} \otimes s.$$ 

Then

$$u_{\varepsilon,j} \in E^{k_1}(\Omega_j, L \otimes \Lambda^{n,q} T_X^*), \quad u_{\varepsilon,j} |_Y = f \quad \text{and} \quad \bar{\partial} u_{\varepsilon,j} = s \otimes (g_{\varepsilon} - Tv_{\varepsilon,j}) = 0.$$ 

Since $\theta (e^{-2|s|^2})$ is bounded and supported on a set whose measure tends to 0 with $\varepsilon \to 0$ and $\Omega_j$ is relatively compact, there exists $\varepsilon_j > 0$ sufficiently small so that whenever $\varepsilon \leq \varepsilon_j$, one has

$$\int_{\Omega_j} |u_{\varepsilon,j}|^2_{\omega} e^{-\psi} dV_\omega = (1 + o(1)) \int_{\Omega_j} (\tau + A) |v_{\varepsilon,j}|^2_{\omega} |s|^2 e^{-\psi} dV_\omega$$

$$= (1 + o(1)) \int_{\Omega_j} (e^v (\tau + A)) |v_{\varepsilon,j}|^2_{\omega} e^{-\psi + \lambda} dV_\omega.$$ 

---

3 On Stein manifolds, McNeal–Varolin [22, p. 437] can take the $g_{\varepsilon}$ to be smooth and thereby get a smooth $V_{\varepsilon,j}$. Springer
where the infinitesimal above is as $\varepsilon \to 0$.

Now

$$e^v(\tau + A) = |s|^2(2e^{a_1} + 2 + \log(2e^{a_1} - 1))$$

$$\leq |s|^2 \cdot 4e^{a_1} \leq 4e^{\gamma_1},$$

where $0 < \delta \leq 1$. It follows that for some sufficiently small $\varepsilon_j$, the estimate

$$\int_{\Omega_j} |u_{\varepsilon,j}|_\omega^2 e^{-\varphi} dV_\omega \leq (1 + o(1))4e^{\gamma_1 - 1}\frac{C_{\varepsilon,j,c}}{\delta}$$

holds for some universal $C > 0$, as soon as $0 < \varepsilon \leq \varepsilon_j$, due to (3.11) and (3.13). Thus for any such $\varepsilon > 0$, $u_{\varepsilon,j}$ gives the extension with the desired $L^2$ estimate in $\Omega_j$.

Write

$$u_j := u_{\varepsilon_j,j}.$$

In conclusion, for each $j$, we have found an $L \otimes K_X$-valued $(0, q)$-form $u_j$ of class $\mathcal{E}^{k_1}$ on $\Omega_j$ such that

$$\bar{\partial}u_j = 0, \quad u_j|_{Y \cap \Omega_j} = f, \quad \text{and} \quad \int_{\Omega_j} |u_j|_\omega^2 e^{-\varphi} dV_\omega \leq \frac{C}{\delta} \int_{\gamma} |f|_\omega^2 e^{-\lambda} dV_{Y,\omega}.$$  

(3.16)

In particular, the right-hand side is independent of $j$.

### 3.5 A Kind of Minimization Problem for Some Continuous Extensions

First, we give an overview of this subsection. From the above process, we conclude that as $j \to \infty$, $\varepsilon$ is necessarily constrained to be smaller and smaller. In accordance with the previous practice in $L^2$ extension theory, we would like to take the limit as $\varepsilon \to 0$ of the extensions obtained in the last paragraph. The trouble is this limit cannot be directly taken due to the singularity of $\tau$ as $\varepsilon \to 0$. That is to say, the twisted $\bar{\partial}$-operators $T$ and $S$ become singular as $\varepsilon \to 0$, and thereby create a loss of control on the constant $C$ in the statement Theorem 1.1. Furthermore, as the weak limit as $\varepsilon \to 0$ may not be a smooth extension of $f$, we must look for better ambient extensions on $\Omega_j$ to ensure that the weak limit of the sequence of ambient extensions is a smooth extension.

Here, we adopt the method of McNeal–Varolin [22, Sect. 4.4] to reduce the problem involving the twisted operators to the untwisted operator, thereby eliminating the dependence on $\tau$.  

\(\square\) Springer
To attack this problem, we define a subspace which is introduced in [22, Sect. 4.4] and in which we perform our minimization procedure.

Let \( \mathcal{V}_q^2(\Omega_j) \) denote the Hilbert space closure of the set of all smooth \( K_X \otimes L \otimes E^\ast \)-valued \( \bar{\partial} \)-closed \((0, q)\)-forms \( \beta \) on \( \Omega_j \) satisfying

\[
\int_{\Omega_j} |\beta|_{\omega}^2 e^{-\varphi+\lambda} dV_{\omega} < +\infty,\
\]

and then \( \mathcal{V}_q^2(\Omega_j) \) consists precisely of all those forms in \( L^2(\omega, e^{-\varphi+\lambda}) \) that are \( \bar{\partial} \)-closed in the weak sense. Let \( \mathcal{B}_q^2(\Omega_j) \) denote the closed unit ball in \( \mathcal{V}_q^2(\Omega_j) \), i.e.,

\[
\beta \in \mathcal{B}_q^2(\Omega_j) \iff \beta \in \mathcal{V}_q^2(\Omega_j) \text{ and } \int_{\Omega_j} |\beta|_{\omega}^2 e^{-\varphi+\lambda} dV_{\omega} \leq 1.
\]

Let us define the affine ball

\[
\mathcal{B}_j := u_j + s \mathcal{B}_q^2(\Omega_j) := \left\{ u_j + s \beta; \beta \in \mathcal{B}_q^2(\Omega_j) \right\} \subset L^2(\omega, e^{-\varphi}).
\]

Now there are three problems to solve:

(i) Every continuous form in \( \mathcal{B}_j \) is an ambient extension of \( f \).

(ii) There exists a minimizer in \( \mathcal{B}_j \).

(iii) The minimizer is smooth.

Note that we can of course adopt the method of McNeal–Varolin [22, (12), (13)] to grasp the data along \( Y \) of any continuous bundle-valued form by taking wedge with the current of integration \([Y] \) due to \( \text{Supp}[Y] = Y \). Here we use a simpler direct method but not the current method to verify the ambient extension relationship.

**Proposition 3.3** Every continuous form in \( \mathcal{B}_j \) is an ambient extension of \( f \).

**Proof** For any continuous form \( g \in \mathcal{B}_j \), there exists a continuous form \( g_1 \) (in the classical usual sense but not as an element in some \( L^2 \) space) such that \( g = g_1 \) a.e. on \( \Omega_j \). We will prove that \( g_1 \) is an ambient extension of \( f \). Let \( g_1 = u_j + s \beta \in \mathcal{B}_j \), then \( \beta \) is a continuous (in the classical usual sense) form divided by \( s \). Then we just need to prove

that \((u_j + s \beta)(x) = f(x)\) for any \( x \in \Omega_j \cap Y \). For any fixed \( x \in \Omega_j \cap Y \), take any local coordinate \((U, z_1, z_2, \ldots, z_n)\) of \( X \) around \( x \). Fix holomorphic local frames \( \sigma \) of \( K_X \otimes L \) and \( \theta \) of \( E \) over \( U \), respectively, such that \( s = z_1 \otimes \theta \). Note that

\[
Y \cap U = \{z_1 = 0\}, \quad \beta|_U = \sum_{|K|=q} \lambda_K d\bar{z}_K \otimes \sigma \otimes \theta^\ast,
\]

where the multi-index \( K \) is increasing. Then it suffices to prove

\[
\lim_{z_1 \to 0} z_1 \lambda_K(z_1, z_2, \ldots, z_n) = 0 \quad \text{for any increasing multi-index } K,
\]
since \( u_j \) is the ambient extension of \( f \).

According to the definition of \( \mathcal{B}_q^2 \), we know that \( \beta \) and thereby \( \lambda_K(z_1, z_2, \ldots, z_n) \) is \( L^2 \) integrable (possibly after shrinking the domain). Then, by the integrability part of Fubini theorem (e.g., [25, 8.8.(c) Theorem]), \( |\lambda_K(z_1, z_2, \ldots, z_n)|^2 \) is \( L^1 \) integrable with respect to \( z_1 \) for \((z_2, \ldots, z_n)\) a.e.. Due to Lemma 2.15, there exists a sequence \( \{z_1, \nu\} \) such that \( \lambda_K(z_1, \nu, z_2, \ldots, z_n) \sim o(|z_1, \nu|^{-1}) \) for any fixed \((z_2, \ldots, z_n)\) a.e.. Then

\[
\lim_{z_1 \to 0} z_1 \lambda_K(z_1, z_2, \ldots, z_n) = 0
\]

for \((z_2, \ldots, z_n)\) a.e. due to the continuity of \( z_1 \lambda_K(z_1, z_2, \ldots, z_n) \). The continuity of \( z_1 \lambda_K(z_1, z_2, \ldots, z_n) \) again implies that

\[
\lim_{z_1 \to 0} z_1 \lambda_K(z_1, z_2, \ldots, z_n) = 0
\]

for any \((z_2, \ldots, z_n)\).

For the sake of completeness, we will give the following two propositions, which can be found in [22, Sect. 4.4]. They solve the second and third problems listed above, i.e., the existence (ii) and the regularity (iii) of the minimizer.

**Proposition 3.4** There exists an element of minimal norm \( U_j \in \mathcal{B}_j \) and \( U_j \) is orthogonal to \( s \mathcal{B}_q^2(\Omega_j) \).

**Proof** By Fatou’s lemma and Lemma 2.8, \( \mathcal{B}_j \) is a closed subset of the Hilbert space \( L^2(\omega, e^{-\phi}) \). Furthermore, it is apparently convex. So \( \mathcal{B}_j \) has an element of minimal norm \( U_j \).

Suppose that there exists \( \beta_0 \in s \mathcal{B}_q^2(\Omega_j) \) such that \( (U_j, \beta_0) = c \neq 0 \). Consider the form

\[
\alpha := \frac{c \beta_0}{\|\beta_0\|^2} \in s \mathcal{B}_q^2
\]

and set \( \tilde{U}_j = U_j - \alpha \). Then \( \tilde{U}_j \in \mathcal{B}_j \), but \( \|\tilde{U}_j\|^2 = \|U_j\|^2 - \frac{|c|^2}{\|\beta_0\|^2} \). This contradicts the minimality of \( \|U_j\| \).

**Proposition 3.5** \( \tilde{\partial} U_j = 0 \) in the sense of currents.

**Proof** For any \( \alpha \in \mathcal{E}_0^\infty(\Omega_j - Y, L^\ast \otimes \Lambda^{n,q - 1} T_X^\ast) \), \( s^{-1} \tilde{\partial} \alpha \in \mathcal{B}_q^2(\Omega_j) \) possibly after shrinking \( \alpha \) by some constant due to the smoothness of \( s^{-1} \) on \( \Omega_j - Y \). By Proposition 3.4,

\[
(U_j, \tilde{\partial} \alpha) = (U_j, s s^{-1} \tilde{\partial} \alpha) = 0.
\]

So \( \tilde{\partial} U_j = 0 \), in the sense of currents, on \( \Omega_j - Y \). The minimality of \( U_j \) implies easily \( U_j \in L^2_{loc} \). An adaptation of the proof of Lemma 2.8 (= [8, Lemme 6.9]) to
the $\bar{\partial}^*\text{-equation}$ or a direct application of [5, Proposition 4.8] yields that the above equation can extend across $Y$, i.e., $\bar{\partial}^*U_j = 0$ on $\Omega_j$. □

It follows from Proposition 3.5 that

$$\Box_0 U_j = 0$$

in the sense of currents, where $\Box_0 = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ denotes the untwisted $\bar{\partial}$-Laplacian unrelated with $\tau$. So it follows that $U_j$ is smooth on $\Omega_j$ from the ellipticity of the Laplacian $\Box_0$ due to the smoothness of the metric $e^{-\varphi}$. Thus, Proposition 3.3 gives that $U_j$ is an extension of $f$. Moreover, by the estimate (3.16) for $u_j$ and the minimality of $U_j$, we have

$$\int_{\Omega_j} |U_j|^2 \omega e^{-\varphi} dV = \frac{C}{\delta} \int_{Y} |f|^2 \omega e^{-\varphi} dV_{Y,\omega}. \quad (3.17)$$

3.6 The End of the Proof of Theorem 1.1

Now we construct the desired extension. Since there is no something like the Montel property for holomorphic objects now, we cannot derive any pointwise convergence information. Then we use the same method as [22, Sect. 4.5] since we have obtained the extension sequence $\{U_j\}$ naturally satisfying the current equations [22, (16)(17)] of the ambient extension. For the sake of completeness, we show the specific procedure (see [22, p. 439, § 4.5] for more explanations).

As $U_j$ is an ambient extension of $f$, it satisfies the distribution equations

$$U_j \wedge \sqrt{-1} \bar{\partial}\bar{\partial} \log |s|^2 = f \wedge \sqrt{-1} \bar{\partial}\bar{\partial} \log |s|^2 \quad (3.18)$$

and

$$\left( \frac{\partial}{\partial \varphi} \circ U_j \right) \wedge \sqrt{-1} \bar{\partial}\bar{\partial} \log |s|^2 = \left( \frac{\partial}{\partial \varphi} \circ f \right) \wedge \sqrt{-1} \bar{\partial}\bar{\partial} \log |s|^2, \quad (3.19)$$

which have been derived in [22, (12)(13)] due essentially to the idea that we can grasp the data along $Y$ of any continuous bundle-valued form by taking wedge with the current of integration $[Y]$ since $\text{Supp}[Y] = Y$. Note that $s$ here is considered as the holomorphic function coefficient with respect to some corresponding local frame of $E$. Of course, $f$ is only defined on $Y$. However, we can extend it smoothly in an arbitrary way to the object of the same type on $\Omega_j$. Then the support of $[Y]$ forces the above current equation to be well defined on $\Omega_j$, which does not depend on the choice of the extension of $f$. ❃ Springer
According to (3.17), Alaoglu’s Theorem shows that (possibly a subsequence of) \(\{U_j\}\) converges weakly to some \(U\) on \(X\) with the \(L^2\) estimate

\[
\int_X |U|^2 e^{-\varphi} dV \leq \liminf \|U_j\|_{L^2} \leq \frac{C}{\delta} \int_Y \frac{|f|^2 e^{-\varphi}}{|ds|^2} dV_{Y,\omega}. \tag{3.20}
\]

Then \(\square_0 U = 0\) due to that \(\square_0 U_j = 0\), and thus \(U\) is smooth.

Now we claim that \(U\) is our desired extension, i.e., \(U\) satisfies the distribution equations (3.18) and (3.19). Note that

\[
U_j \wedge \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |s|^2 = -\bar{\partial} (U_j \wedge \frac{\sqrt{-1}}{2\pi} \frac{ds}{s})
\]

and

\[
(\frac{\partial}{\partial \bar{s}} \ast U_j) \wedge \frac{\sqrt{-1}}{2\pi} \partial \bar{\partial} \log |s|^2 = \pm \left( \bar{\partial} (\frac{\partial}{\partial \bar{s}} \ast U_j) \wedge \frac{ds}{s} - \bar{\partial}((\frac{\partial}{\partial \bar{s}} \ast U_j) \wedge \frac{ds}{s}) \right).
\]

Then it suffices to show that

\[
U_j, \quad \bar{\partial} (\frac{\partial}{\partial \bar{s}} \ast U_j) \quad \text{and} \quad \frac{\partial}{\partial \bar{s}} \ast U_j \tag{3.21}
\]

are locally uniformly bounded in \(j\), for showing that \(U\) satisfies equations (3.18) and (3.19), since \(\frac{ds}{s}\) is locally integrable. In fact, take (3.18) for example, in the spirit of the argument (e.g., [10, p. 383]) of Lemma 2.8, noting that \(s^{-1}\) multiplied by any smooth function which is compactly supported outside \(Y\) is smoothly compactly supported on \(X\), we can imply the convergence of \(\{U_j \wedge \frac{\sqrt{-1}}{2\pi} \frac{ds}{s}\}\) to the current \(U \wedge \frac{\sqrt{-1}}{2\pi} \frac{ds}{s}\) due to the locally uniformly boundedness of \(U_j\). Then the weak continuity of \(\bar{\partial}\) helps us to get (3.18).

The ellipticity of \(\square_0\), basic elliptic estimate (e.g., [18, Theorem A.3.2]), Sobolev embedding theorem (e.g., [18, Theorem A.3.1-(b)]) and (3.17) together tell us that for every compact set \(K\), there exists some \(j_0 = j_0(K) > 0\) and \(C_K > 0\) such that

\[
\left\| \left| U_j \right| e^{-\varphi/2} \right\|_{C^1(K)} \leq C_K
\]

for all \(j \geq j_0\). The index \(j_0\) is large enough to make sure that \(U_j\) is defined on \(\Omega_j\).

The similar argument shows that the other terms in (3.21) are locally uniformly bounded. Now we can take limits in the distribution equations (3.18) and (3.19) to conclude that \(U\) also satisfies these equations. Thus \(U\) is our desired extension with the estimate (3.20).

**Remark 3.6** Note that we can make a better control on our extension \(U\). In fact, from (3.14), Theorem 3.2 and (3.15), we know that the universal constant \(C\) in (3.16) and (3.20) depends on \(c\). If we take a smaller \(c\) in (3.9), we can get a smaller \(C_{c,j,c}\) in (3.14) according to (3.10), (3.11) and (3.13). Then we can get a smaller universal constant.
C in (3.15) and thereby a better control in (3.20) for our extension $U$. However, the constant in (3.20) is not a sharp one and then taking a smaller $c$ may not give more useful information.

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**Appendix A: Alternative Proof of Theorem 1.3**

Theorem 1.1 can easily imply Theorem 1.3 as stated in Sect. 1. Now we present a sketch of a direct proof of Theorem 1.3.

Just as in Sect. 3.1, we can glue local intrinsic extensions which can be obtained by the Stein theory on each Stein coordinate ball, to get the initial global ambient extension.

Then we can use the same method of raising the regularity of $s^{-1}\bar{\partial}\bar{f}_\infty$, to obtain:

**Proposition A.1** [16, Lemma 3.1] For any $k \geq 0$, there exists a smooth section $\bar{f}_\infty \in \mathcal{E}^\infty(X, \Lambda^{n,q} T_X^* \otimes L)$ such that

(a) $\bar{f}_\infty$ is the intrinsic extension of $u$,
(b) $|\bar{f}_\infty|_{\omega,L} = |u|_{\omega,L}$ at every point of $Y$,
(c) $\bar{\partial}\bar{f}_\infty = 0$ at every point of $Y$,
(d) $s^{-1}\bar{\partial}\bar{f}_\infty \in \mathcal{E}^k(X, \Lambda^{n,q+1} T_X^* \otimes L \otimes \mathcal{O}_X(-Y))$.

From now on we fix $k \geq n + 6$, where $n$ is the complex dimension of $X$.

Using exactly the same constructions and calculations as in Sects. 3.2–3.4, for each $j$, we can obtain an $L \otimes K_X$-valued $(0, q)$-form $u_j$ of class $\mathcal{E}^{k_1}$ on $\Omega_j$ such that

$$\bar{\partial}u_j = 0, \quad \iota^* u_j = u, \quad \text{and} \int_{\Omega_j} |u_j|^2_{\omega} e^{-\varphi} d\mathcal{V}_\omega \leq C \int_Y \frac{|u|^2_{\omega} e^{-\varphi}}{|ds|_{\omega}^2} e^{-\lambda} d\mathcal{V}_{Y,\omega}$$

with $k_1 \geq 5$. In particular, the right-hand side is independent of $j$.

Let $\mathcal{H}^q_\Omega(\Omega_j)$ denote the Hilbert space closure of the set of all smooth $K_X \otimes L \otimes E^*$-valued $\bar{\partial}$-closed $(0, q)$-forms $\beta$ on $\Omega_j$ satisfying

$$\int_{\Omega_j} \beta^2_{\omega} e^{-\varphi + \lambda} d\mathcal{V}_\omega < +\infty.$$
and then $\mathcal{V}_q^2(\Omega_j)$ consists precisely of all forms in $L^2(\omega, e^{-\varphi+\lambda})$ that are $\bar{\partial}$-closed in the weak sense. Let $\mathcal{B}_q^2(\Omega_j)$ denote the closed unit ball in $\mathcal{V}_q^2(\Omega_j)$, i.e.,

$$\beta \in \mathcal{B}_q^2(\Omega_j) \iff \beta \in \mathcal{V}_q^2(\Omega_j) \quad \text{and} \quad \int_{\Omega_j} |\beta|^2 e^{-\varphi+\lambda} dV_\omega \leq 1.$$ 

Let us define the affine ball

$$\mathcal{B}_j := u_j + s\mathcal{B}_q^2(\Omega_j) := \left\{u_j + s\beta; \beta \in \mathcal{B}_q^2(\Omega_j)\right\} \subset L^2(\omega, e^{-\varphi}).$$

Then we have the following proposition.

**Proposition A.2** Every continuous form in $\mathcal{B}_j$ is an intrinsic extension of $u$.

**Proof** The proof is quite similar with that of Proposition 3.3. It suffices to prove that $\iota^*(u_j + s\beta)(x) = u(x)$ for any $x \in \Omega_j \cap Y$ and any continuous $u_j + s\beta$ in $\mathcal{B}_j$. For any fixed $x \in \Omega_j \cap Y$, take any local coordinate $(U, z_1, z_2, \ldots, z_n)$ of $X$ around $x$. Fix holomorphic local frames $\sigma$ of $K_X \otimes L$ and $\theta$ of $E$ over $U$, respectively, such that $s = z_1 \otimes \theta$. Note that $Y \cap U = \{z_1 = 0\}$, $\beta|_U = \sum_{|J|=q, 1 \notin J} \lambda_J^{(1)}(z_1, z_2, \ldots, z_n) d\bar{z} J \otimes \sigma \otimes \theta^*$

$$+ \sum_{|K|=q, 1 \notin K} \lambda_K^{(2)}(z_1, z_2, \ldots, z_n) d\bar{z} K \otimes \sigma \otimes \theta^*,$$

where the multi-indices $J$ and $K$ are increasing. Then

$$\iota^* \beta = \sum_{|K|=q, 1 \notin K} \lambda_K^{(2)}(0, z_2, \ldots, z_n) d\bar{z} K \otimes \sigma \otimes \iota \otimes \theta^* \otimes \iota.$$ 

So it suffices to prove

$$\lim_{z_1 \to 0} z_1 \lambda_K^{(2)}(z_1, z_2, \ldots, z_n) = 0 \quad \text{for any multi-index} \ K,$$

since $u_j$ is the intrinsic extension of $u$. The leftover argument proceeds just with $\lambda_K(z_1, z_2, \ldots, z_n)$ in the proof of Proposition 3.3 replaced by $\lambda_K^{(2)}(z_1, z_2, \ldots, z_n)$ here.

**Remark A.3** The current equation method in the proof of [22, Proposition 4.4] can also work to prove Proposition A.2. In fact, the local expressions of $\beta$ and $\iota^* \beta$ in the above proof can be used to deduce a current equation characterizing the intrinsic restriction due to that $\text{Supp}[Y] = Y$. That is, a continuous $K_X \otimes L$-valued $(0, q)$-form $U$ on $X$ is the intrinsic extension of a smooth $K_X \otimes L|_Y$-valued $(0, q)$-form $u$ on $Y$ if and only if locally

$$U \wedge \sqrt{-1} \bar{\partial} \bar{\partial} \log |s|^2 = u \wedge \sqrt{-1} \bar{\partial} \bar{\partial} \log |s|^2 \quad (A.1)$$

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as $K_X \otimes L$-valued $(1, q + 1)$-currents of order 0. Here we adopt the definition of order of currents in [10, Sect. 2 of Chap. 1]. Note that $s$ here is considered as the holomorphic function with respect to the corresponding local trivialization. Despite $u$ is only defined on $Y$, the above current equation is well defined on $X$ due to the same reason as (3.18) and (3.19). The proof of (A.1) is similar to the proof of [22, (12),(13)] characterizing the ambient restriction. Then we can also use this characterization (A.1) to verify Proposition A.2 as the proof of [22, Proposition 4.4].

Using Proposition A.2 and the similar results to Propositions 3.4 and 3.5, we can obtain an element $U_j$ of minimal norm in $B_j$ satisfying $\square_0 U_j = 0$. Then $U_j$ is smooth and thereby the intrinsic extension of $u$ on $\Omega_j$ by Proposition A.2. Of course, $U_j$ satisfies (A.1). At last, by the same method of extracting weak limits of some subsequence of $U_j$ as $j \to \infty$ as in Sect. 3.6, we can obtain our desired extension.

**Appendix B: On the Strict Positivity Condition (1.1)**

Inspired by the referee’s suggestion, we discuss further on the strict positivity condition (1.1). For more potential applications, we might expect that (1.1) can be replaced by the semi-positivity condition

$$\sqrt{-1} \partial \bar{\partial} (\varphi - \lambda) \wedge \omega^q \geq 0,$$

in Theorem 1.1. In this appendix, we will show the difficulties caused by the (B.1) in solving the twisted $\bar{\partial}$-equation in Sect. 3.3. From now on, we assume (B.1) but not (1.1).

**B.1: Direct Estimate Cannot Work Generally**

It seems difficult to get the desired estimate similar to (3.14), when we use directly the original curvature term $B_\varepsilon$ to do estimate.

One may want to use the estimating process similar to (3.10), (3.12) and (3.13). But it is easy to see that those cannot go through, since the non-strictly positive $B_\varepsilon$ can actually exist. For example, take the same setting as Example 1.2 except that $\varphi = |z|^2$ is replaced by $\varphi \equiv 0$. Then

$$B_\varepsilon = \sqrt{-1} \delta (1 + h'(a)) \left( \frac{\varepsilon^2 dz_1 \wedge d\bar{z}_1}{(|z_1|^2 + \varepsilon^2)^2} \right),$$

which is non-strictly positive on the whole of $X$. Then one cannot invert the $[B_\varepsilon, \Lambda_\omega]$ to estimate the involved terms.

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One may also try to estimate pointwisely first:

\[
|\langle \beta, g_\varepsilon^{(1)} \rangle_\omega|^2 \leq \left( \int_{\Omega_j} |\langle \beta, g_\varepsilon \rangle_\omega| e^{-\varphi + \lambda} dV_\omega \right)^2
\]

\[
\leq \left( \int_{\Omega_j} \left( |\langle \beta, g_\varepsilon^{(1)} \rangle_\omega| + |\langle \beta, g_\varepsilon^{(2)} \rangle_\omega| \right) e^{-\varphi + \lambda} dV_\omega \right)^2
\]

\[
= \left( \int_{\Omega_j} |\langle \beta, g_\varepsilon^{(1)} \rangle_\omega| e^{-\varphi + \lambda} dV_\omega + \int_{\Omega_j} |\langle \beta, g_\varepsilon^{(2)} \rangle_\omega| e^{-\varphi + \lambda} dV_\omega \right)^2
\]

\[
=: (I + II)^2
\]

\[
= I^2 + 2II + II^2.
\]

(B.3)

Note that \( \tilde{\partial}|s| \cdot u, v \rangle_\omega = \langle u, \tilde{\partial}|s| \cdot v \rangle_\omega \) for any \( x \in X \) and \( u, v \in \Lambda^* T^*_{X, x} \), where the \( \tilde{\partial}|s| \) is a (0, 1)-vector field defined by \( \tilde{\partial}|s| = \langle \bullet, \tilde{\partial}|s| \rangle_\omega \), and that

\[
\langle [\sqrt{-1} \tilde{\partial}|s| \cdot \tilde{\partial}|s|, \Lambda_\omega]u, u \rangle_\omega = |\tilde{\partial}|s| \cdot u|^2_\omega
\]

also pointwisely. Then \( I^2 \) can be well controlled by

\[
\langle \beta, g_\varepsilon^{(1)} \rangle_\omega = \langle \tilde{\partial}|s| \cdot \beta, g_\varepsilon^{(1)} \rangle_\omega,
\]

where

\[
g_\varepsilon^{(1)} = \tilde{\partial}|s| \cdot 2|s|(\varepsilon^{-2}|s|^2 + 1)\theta'(\varepsilon^{-2}|s|^2)|s|^{-1}f_\infty =: \tilde{\partial}|s| \cdot g_\varepsilon^{(1)}
\]

despite the fact that \( \sqrt{-1} \tilde{\partial}|s| \cdot \tilde{\partial}|s| \) may be non-strictly positive on some subset of positive measures.

But one can see that we cannot have a desired estimate for \( II \), despite the fact that \( ||g_\varepsilon^{(2)}||_{L^2(\Omega_j)} = o(1) \) with \( \varepsilon \). The reason of this failure is that when the condition (1.1) is replaced by (B.1), the original \( L^2(\Omega_j) \)-norm \( ||\beta||^2 \) may not be dominated by the Dirichlet semi-norm \( ||\beta||^2_{\mathcal{H}} := ||T^* \beta||^2 + ||S \beta||^2 \). Since the \( \partial(\Omega_j) \) is just (non-strictly) pseudoconvex (or without boundary), one can see from the twisted Bochner–Kodaira–Morrey–Kohn identity that the Dirichlet semi-norm \( ||\beta||^2_{\mathcal{H}} := ||T^* \beta||^2 + ||S \beta||^2 \) may be equal to \( \langle [B_\varepsilon, \Lambda_\omega] \beta, \beta \rangle_{L^2} \) and thereby be equal to zero for some nonzero \( \beta \), by the second paragraph of Sect. 1. Actually, it is also easy to see this fact from other view of point. For example, the holomorphic trivial line bundle on a compact (automatically weakly pseudoconvex) connected manifold \( X \), equipped with the trivial metric, is semi-positively curved, but \( \mathcal{H}^{n, n}(X, \mathcal{O}_X) \cong \mathbb{C} \), i.e., the (twisted) harmonic forms may not be identically 0. In other words, we cannot find some constant \( C \) such that

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\[ ||\beta||_{L^2(\Omega_j)}^2 \leq C (||T^*\beta||^2 + ||S\beta||^2) \] for any smooth \( K \otimes L \otimes E^* \)-valued \((0, q + 1)\)-form \( \beta \) in the domain of \( T^* \) and \( S \).

Since the Dirichlet semi-norm is no longer a norm, the method of Hilbert completion of a space equipped with the Dirichlet semi-norm cannot go through. One may want to use another method to solve the twisted \( \bar{\partial} \)-equations. For example, one may want to study the usual functional \( T^*\beta \mapsto \langle \beta, g_\epsilon \rangle_{L^2} \) by considering the decomposition of \( \beta \) in \( L^2 = (\ker S) \bigoplus (\ker S)^\perp \) to solve the twisted \( \bar{\partial} \)-equation and then take the minimizer to get the desired regularity for the solution. But this method also needs a good estimate similar to (3.14).

**B.2: Error Term Method**

Another possible method is to add some positive term to \( B_\delta \) to get a good estimate, just as done in extending holomorphic objects. To overcome the lack of sufficient positivity, Demailly [9, (3.2) Remark and p. 11, line-11–line-3] introduced the error term method. The basic idea is to "add a series of positive terms" to \( B_\delta \) to produce a series of good estimates, and thereby to solve a series of equations of the form \( \bar{\partial}u_{\delta'} + \delta' h_{\delta'} = g \) (sometimes \( \delta' \) has to be carefully selected for some specific purpose). Then one removes the effect induced by the extra error term \( \delta' h_{\delta'} \), by taking some (weak) limits as \( \delta' \to 0 \). We give a sketch of the obstructions of this approach in our setting.

Recall now that our positivity conditions are (1.2), (1.3) and (B.1).

For a small \( \delta' \), we have

\[
(B_\delta + \delta' \omega) \wedge \omega^q \geq \delta \omega \wedge \omega^q
\]

and

\[
(B_\delta + \delta' \omega) \wedge \omega^q \geq \sqrt{-1} \delta \left( \frac{4\epsilon^2 \partial |s| \wedge \bar{\partial} |s|}{(|s|^2 + \epsilon^2)^2} \right) \wedge \omega^q,
\]

from the process of (3.4).

Similarly to Sect. 3.3, one can solve the equation

\[
(\Box + q\delta' I) V_{\epsilon, \delta'} = g_\epsilon
\]

with \( SV_{\epsilon, \delta'} = 0 \) and \( S^* SV_{\epsilon, \delta'} = 0 \). Then we can get the error term version of Theorem 3.2,

\[
Tv_{\epsilon, \delta'} + q\delta' V_{\epsilon, \delta'} = g_\epsilon
\]

(for ease of notation, we omit other subscripts) with the estimate of the error term being \( ||q\delta' V_{\epsilon, \delta'}||_{L^2}^2 \sim O(\delta') \), where \( v_{\epsilon, \delta'} = T^* V_{\epsilon, \delta'} \), and both of \( v_{\epsilon, \delta'} \) and \( V_{\epsilon, \delta'} \) admit some good regularity.
Now we have some troubles in constructing $u_{\varepsilon, \delta'}$ like $u_{\varepsilon, j}$ in Sect. 3.4 such that $u_{\varepsilon, \delta'}|_Y = f$, $\bar{\partial}u_{\varepsilon, \delta'} = 0$ and $u_{\varepsilon, \delta'}$ admits some regularity. For $u_{\varepsilon, \delta'}|_Y = f$, one might define

$$u_{\varepsilon, \delta'} := \theta \left( \varepsilon^{-2}|s|^2 \right) \tilde{f}_{\infty} - \sqrt{\tau + A(v_{\varepsilon, \delta'} + \tilde{v}_{\varepsilon, \delta'})} \otimes s$$

with some continuous $\tilde{v}_{\varepsilon, \delta'}$. But then we have to solve $T \tilde{v}_{\varepsilon, \delta'} = q \delta' V_{\varepsilon, \delta'}$ for $\bar{\partial}u_{\varepsilon, \delta'} = 0$. However, the semi-positivity conditions (B.1) and (1.2) cannot produce such a desired solution, as one can easily see from the process of solving the $T$-equation in Sect. 3.3 and Appendix B.1, since $\langle [B_{\varepsilon, \Lambda_{\omega}}]^{-1} q \delta' V_{\varepsilon, \delta'}, q \delta' V_{\varepsilon, \delta'} \rangle_{L^2}$ may not be controlled.

Recall another way of dealing with $q = 0$ case. Roughly speaking, one can construct a sequence of objects, based on $v_{\varepsilon, \delta'}$ and $V_{\varepsilon, \delta'}$. These objects are neither necessarily $\bar{\partial}$-closed, nor necessarily the ambient extension of $f$. But some weak limit of this sequence is a $\bar{\partial}$-closed extension. However, for $q \geq 1$ case, we know nothing about the regularity of the $\bar{\partial}$-closed weak limit of some sequence. In brief, it seems difficult to apply the error term method to get the desired $u_{\varepsilon, \delta'}$ as $q \geq 1$.

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