A CPT anomaly

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Abstract

We consider chiral gauge theories defined over a four-dimensional spacetime manifold with a Cartesian product structure for at least one compact spatial dimension. For a simple setup, we calculate the effective gauge field action by integrating out the chiral fermions, while maintaining gauge invariance. Due to a combination of infrared and ultraviolet effects, there appears a CPT-odd term in the effective gauge field action. This CPT anomaly could occur in chiral gauge theories relevant to elementary particle physics, provided the spacetime manifold has the appropriate topology. Two possible applications for cosmology are discussed.

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1 Introduction

The CPT theorem is one of the most important results in flat-spacetime quantum field theory [1–3]. The theorem states that the combined operation (CPT) of charge conjugation (C), parity reflection (P) and time reversal (T) is an invariance of local relativistic quantum field theory, even if some of the separate invariances do not hold. Any CPT violation is, therefore, believed to require fundamentally different physics, for example quantum gravity [4, 5] or strings [6]. It may, then, come as a surprise that a particular class of local relativistic quantum field theories has given an indication of CPT violation [7].

The specific theories considered in Ref. [7] are non-Abelian chiral gauge theories with one compact spatial dimension singled out by the prescribed four-dimensional spacetime manifold \( M \). An example would be \( SU(3) \) Yang–Mills theory [8] with a single triplet of left-handed Weyl fermions [9], defined over the flat spacetime manifold \( M = \mathbb{R}^3 \times S^1 \), which corresponds to the usual Minkowski spacetime with one spatial coordinate compactified to a circle. The perturbative chiral gauge anomalies of this theory [10–13] can be cancelled by the introduction of an octet of elementary pseudoscalar fields with the standard gauged Wess-Zumino term in the action [14, 15]. There remains a nonperturbative \( SU(3) \) gauge anomaly [16], which is similar to, but not the same as, the Witten \( SU(2) \) gauge anomaly [17]. In this case, however, there exists a local counterterm for the action which restores \( SU(3) \) gauge invariance, but at the price of Lorentz noninvariance and CPT violation [7]. In other words, the remaining non-Abelian chiral gauge anomaly is transmuted into a CPT anomaly. (The situation is analogous to that of certain three-dimensional non-Abelian gauge theories with massless fermions, where gauge invariance is restored at the price of P and T violation and the non-Abelian gauge anomaly is transmuted into the so-called parity anomaly [18–23].)

The particular counterterm presented in Ref. [7] is the spacetime integral of a Chern–Simons density [24] involving three of the four gauge potentials. (The precise definition will be given later.) Such a term obviously violates local Lorentz invariance. Also, the integrand of the counterterm is CPT-odd, whereas the standard Yang–Mills action density is CPT-even. This Lorentz and CPT noninvariance would show up, to first order, as a direction-dependent, but wavelength-independent, rotation of the linear polarization of a plane wave of gauge fields traveling in \( \text{vacuo} \) [25].

We have obtained some heuristic arguments of why the counterterm must violate Lorentz and CPT invariance, but the uniqueness of the counterterm has not been established. If, on the other hand, the non-Abelian chiral gauge anomaly is really as discussed above, then the effective gauge field action due to the chiral fermions, formulated and regularized in a gauge-invariant manner, must already exhibit some sign of CPT violation (and Lorentz noninvariance). It is the goal of the present paper to establish this CPT violation. Maintaining chiral gauge invariance, we will find for spacetime manifolds with the appropriate Cartesian product structure a CPT-violating term in the effective
gauge field action which is precisely equal to the counterterm presented in Ref. [7]. This CPT-violating term can appear in anomaly-free chiral gauge theories but not in vectorlike gauge theories such as quantum electrodynamics.

The outline of this paper is as follows. In Section 2, we give the setup of the problem and establish our notation. In Section 3, we choose some simple background gauge potentials and find a Chern–Simons term in the effective gauge field action. The calculation applies to both Abelian and non-Abelian chiral gauge groups, provided the theory is free of chiral gauge anomalies. In Section 4, we show that the Chern–Simons term found violates CPT. In other words, these particular chiral gauge field theories have a CPT anomaly if gauge invariance is maintained. In Section 5, we give some generalizations of our basic result. Also, we exhibit a class of chiral gauge theories which necessarily have the CPT anomaly, as long as the spacetime manifold has the appropriate topology. Remarkably, the so-called Standard Model of elementary particle physics (with three families of quarks and leptons) can be embedded in some of these anomalous theories. In Section 6, finally, we present some remarks on how the CPT theorem is circumvented and discuss two possible applications of the CPT anomaly.

2 Setup

For definiteness, we take spacetime to be the flat Euclidean manifold

\[ M = \mathbb{R}^3 \times S^1, \]

(2.1)

with Cartesian coordinates \( x^m \in \mathbb{R}^3, \) \( m = 1, 2, 3, \) and \( x^4 \in S^1. \) At the end of the calculation, we can make the Wick rotation [20] from Euclidean to Lorentzian metric signature, with \( x^4 \) corresponding to a compact spatial coordinate and \( x^1, \) say, to the time coordinate. The length of the circle in the 4-direction is denoted by \( L. \) Throughout this paper, Latin indices \( k, l, m, \) etc. run over the coordinate labels 1, 2, 3, and Greek indices \( \kappa, \lambda, \mu, \) etc. over 1, 2, 3, 4. Repeated coordinate (and internal) indices are summed over. Also, natural units are used for which \( c = \hbar = k = 1. \)

We will first consider non-Abelian chiral gauge theories with a single irreducible representation of massless left-handed fermions. Specifically, we take the standard chiral Yang–Mills theory [8, 9] with gauge group \( G = SO(10) \) and left-handed Weyl fermions in the complex representation \( R_L = 16. \) (This particular model may have some relevance for elementary particle physics, as part of a so-called grand-unified theory. See Refs. [27, 28] and references therein.) The left-handed fermion field is then \( \psi_{L\alpha}(x), \) with a spinor index \( \alpha = 1, 2, \) and an internal symmetry index \( i = 1, \ldots, 16. \) The gauge potentials are \( A_\mu(x) \equiv e A^a_\mu(x) T^a, \) with \( e \) the gauge coupling constant and \( T^a, a = 1, \ldots, 45, \) the anti-Hermitian generators of the Lie group \( SO(10) \) in the representation chosen, normalized by \( \text{tr} (T^a T^b) = -\frac{1}{2} \delta^{ab}. \) The fermion and gauge fields are periodic in \( x^4, \) with period \( L. \)
In this paper, we are interested in the effective gauge field action obtained from integrating out the chiral fermions, while maintaining gauge invariance. Formally, we have the following functional integral \[26\]:

\[
\exp\left\{-\Gamma_W[A]\right\} = \int D\psi_L^\dagger D\psi_L \exp\left\{-I_W[\psi_L^\dagger, \psi_L, A]\right\},
\]

(2.2)

for the Euclidean Weyl action

\[
I_W[\psi_L^\dagger, \psi_L, A] = \int_M d^4x \psi_L^\dagger i\sigma^\mu (\partial_\mu + A_\mu) \psi_L,
\]

(2.3)

with \(\sigma_\pm \equiv (\pm i\sigma^m, 1)\) defined in terms of the \(2 \times 2\) Pauli matrices \(\sigma^m\) and the \(2 \times 2\) identity matrix \(1\). The \(SO(10)\) chiral gauge theory is anomaly free \([12, 13, 16, 17]\) and the effective gauge field action is invariant under local gauge transformations,

\[
\Gamma_W[g (A + d) g^{-1}] = \Gamma_W[A], \quad g(x) \in G,
\]

(2.4)

with \(d\) the exterior derivative for differential forms \((dg \equiv \partial g/\partial x^\mu \, dx^\mu)\) and \(A \equiv A_\mu \, dx^\mu\) a one-form taking values in the Lie algebra (here, in the defining representation).

If the chiral gauge theory considered is not anomaly free (for example, the theory mentioned in the Introduction, with \(G = SU(3)\) and \(R_L = 3\)), then the theory has to be modified in order to make it gauge invariant. One way to restore gauge invariance is by averaging over the gauge orbits,

\[
\exp\left\{-\Gamma [A]\right\} \equiv \int Dh \, \exp\left\{-\Gamma_W[h (A + d) h^{-1}]\right\}.
\]

(2.5)

But the interpretation of the resulting theory with the dimensionless variables \(h(x) \in G\) is not entirely clear \([29]\). Another way to restore gauge invariance is by introducing further fermions, which cancel the chiral anomalies of the original fermions \([13]\). In Section 5, we will discuss some of these theories with reducible fermion representations. All of these complications are, however, not necessary for the anomaly-free chiral gauge theory considered here, which has the gauge group \(G = SO(10)\) and the fermion representation \(R_L = 16\).

At this point, there is no need to be explicit about the regularization of the effective gauge field action \(\Gamma_W[A]\). One possible regularization would be the introduction of a spacetime lattice cutoff, which (temporarily?) sacrifices Lorentz invariance but keeps the gauge and chiral invariances intact. (See Refs. \([30, 31]\) and references therein.) This last condition on the regularization method is important, since we intend to look for symmetry violations being forced upon us by maintaining exact gauge invariance in a theory with genuine chiral fermions.
3 Calculation

As discussed in the Introduction, our goal is to establish the presence of a CPT-violating term in the effective gauge field action for the theory defined in Section 2. The strategy is to simplify the calculation as much as possible. We, therefore, take the case of $x^4$-independent $SO(10)$ gauge potentials, with the one gauge potential corresponding to the special direction (here, $x^4 \in S^1$ for the Euclidean spacetime manifold $M = \mathbb{R}^3 \times S^1$) vanishing altogether,

$$A_m(x, x^4) = \tilde{A}_m(x) \, , \quad A_4(x, x^4) = \tilde{A}_4(x) = 0 \, .$$

(3.1)

Also, the gauge potentials considered vanish on the boundary of a ball $B^3$ embedded in $\mathbb{R}^3$, and outside of it,

$$\tilde{A}_m(x) = 0 \quad \text{for} \quad |\vec{x}| \geq R \, ,$$

(3.2)

with $R$ a fixed radius which can be taken to infinity at the end of the calculation.

The left-handed fermion field $\psi_L$ in the complex representation $R_L = 16$ of $SO(10)$ and the independent fermion field $\psi_L^\dagger$ in the conjugate representation can be expanded in Fourier modes

$$\psi_L(x, x^4) = \sum_{n=-\infty}^{\infty} e^{+ i 2\pi n x^4/L} \xi_n(x) \, ,$$

$$\psi_L^\dagger(x, x^4) = \sum_{n=-\infty}^{\infty} e^{- i 2\pi n x^4/L} \xi_n^\dagger(x) \, .$$

(3.3)

The Weyl action (2.3) for the gauge potentials (3.1) then becomes

$$I_W = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^3} d^3 x \, L \, \xi_n^\dagger \left( \sigma^m (\partial_m + \tilde{A}_m) - 2\pi n / L \right) \xi_n \, .$$

(3.4)

Redefining the two independent sets of spinor fields

$$\chi_n(x) \equiv i L \, \xi_n(x) \, , \quad \chi_n^\dagger(x) \equiv \xi_n^\dagger(x) \, ,$$

(3.5)

the action reads

$$I_W = \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}^3} d^3 x \, \chi_n^\dagger \left( -i \sigma^m (\partial_m + \tilde{A}_m) + i 2\pi n / L \right) \chi_n$$

$$\equiv \sum_{n=-\infty}^{\infty} I_3 \left[ \chi_n^\dagger, \chi_n, \tilde{A} \right] \, .$$

(3.6)

We have thus obtained an infinite set of three-dimensional Euclidean Dirac fields $\chi_n(x)$ with masses $2\pi n / L$, all of which interact with the same three-dimensional gauge potentials $\tilde{A}_m(x)$. (This is, of course, reminiscent of Kaluza-Klein theory, which reduces five spacetime dimensions to four. See Refs. [28, 32] and references therein.)
For the special gauge potentials (3.1), the effective action (2.2) now factorizes to
\[
\exp\left\{ -\Gamma_W(\tilde{A}) \right\} \propto \prod_{n=-\infty}^{\infty} \left( \int D\chi^\dagger_n D\chi_n \exp\left\{ -I_3[\chi^\dagger_n, \chi_n, \tilde{A}] \right\} \right),
\]
with the three-dimensional action \(I_3\) as defined in (3.6). Each factor in (3.7) can be regularized separately by the introduction of appropriate three-dimensional Pauli–Villars fields [26, 33]. This ultraviolet regularization preserves the restricted gauge invariance
\[
\chi_n \to U_r(\tilde{g}) \chi_n, \quad \tilde{A}_m^{(r)} \to U_r(\tilde{g}) (\tilde{A}_m^{(r)} + \partial_m) U_r^{-1}(\tilde{g}), \quad \tilde{g}(\vec{x}) \in G,
\]
with the appropriate unitary representation for the fermions (here, \(r = 16\) and \(G = SO(10)\)) and gauge functions \(\tilde{g}(\vec{x}) = 1\) for \(|\vec{x}| \geq R\). Even though this is not the full gauge invariance (2.4) of the theory, it turns out to be sufficient for our purpose (see Section 4).

In addition to the ultraviolet divergences in the separate factors of (3.7), which are regularized by the corresponding three-dimensional Pauli–Villars fields, there are also infrared divergences in the \(n = 0\) factor. These infrared divergences can be regularized by imposing antiperiodic boundary conditions for the Dirac (and Pauli–Villars) fields on the surface of the ball \(B^3\), where the gauge potentials (3.2) vanish.

The massive Pauli–Villars regulator fields for the \(n = 0\) factor of (3.7), viewed as \(x^4\)-independent four-dimensional fields, introduce a breaking of Lorentz and CPT invariance in the four-dimensional context. This breaking will show up later as a finite remnant in the effective gauge field action. (Preliminary results seem to indicate that this is also the case for the lattice regularization mentioned in the last paragraph of Section 2.)

For the present calculation, it is sufficient to introduce for each (anticommuting) field \(\chi_n(\vec{x})\) with mass \(M_n \equiv 2\pi n/L\) a single (commuting) Pauli–Villars field \(\phi_n(\vec{x})\) with mass
\[
\tilde{\Lambda}_0 \equiv M_n + \text{sign}(n) \Lambda \quad \text{for} \quad n \neq 0,
\]
where \(\Lambda\) is taken to be positive. Formally, this gives for (3.7) the following product:
\[
\exp\left\{ -\Gamma_W(\tilde{A}) \right\} \propto \prod_k \frac{\lambda_k}{\lambda_k + i\tilde{\Lambda}_0} \left( \prod_{l=1}^{\infty} \frac{\lambda_k^2 + M_l^2}{\lambda_k^2 + (M_l + \Lambda)^2} \right),
\]
in terms of the real eigenvalues \(\lambda_k\) of the massless three-dimensional Dirac operator \(-i\sigma^m (\partial_m + \tilde{A}_m)\). The factors in (3.7) with \(n = \pm l\), for \(l > 0\), thus combine to give a real contribution to the effective gauge field action \(\Gamma_W[\tilde{A}]\). Moreover, it is clear that the spectral flow [18, 21] of the full three-dimensional Dirac operator as given in (3.6) can occur only in the \(n = 0\) sector (there is a mass gap for \(n \neq 0\)), and that the potential non-Abelian gauge anomaly [10], which shows up in the imaginary part of \(\Gamma_W[\tilde{A}]\), resides there.

The imaginary part of the effective gauge field action for massless three-dimensional Dirac fermions, with Pauli–Villars regularization to maintain gauge invariance, has already been calculated [18, 21]. Revisiting the perturbative calculation, we have for the \(n = 0\) sector of our non-Abelian \(SO(10)\) gauge theory the one-loop result
\[
\Gamma_W^{n=0}[\tilde{A}] \supseteq i \int_{B^3} d^3x \ s_0 \pi \omega_{CS}[\tilde{A}_1, \tilde{A}_2, \tilde{A}_3],
\]
(3.10)
in terms of a sign factor $s_0 = \pm 1$ whose origin will be explained shortly and the Chern–Simons density \[\omega_{CS}[A_1, A_2, A_3] \equiv \frac{1}{16 \pi^2} \epsilon^{klm} \text{tr} \left( A_{kl} A_{m} - \frac{2}{3} A_{k} A_{l} A_{m} \right), \tag{3.11}\]

with indices $k, l, m$, running over 1, 2, 3. Here, $\epsilon^{klm}$ is the completely antisymmetric Levi-Civita symbol, normalized to $\epsilon^{123} = +1$, and $A_{kl} \equiv \partial_k A_l - \partial_l A_k + [A_k, A_l]$ is the field strength tensor for the gauge potential $A_{m} \equiv e A_{m}^a T^a$, with gauge coupling constant $e$ and anti-Hermitian Lie group generators $T^a$, normalized by $\text{tr} (T^a T^b) = -\frac{1}{2} \delta^{ab}$.

The sign ambiguity $s_0$ in (3.10) traces back to the parity-violating Pauli–Villars mass $\Lambda_0$ used to regularize the ultraviolet divergences of the three-dimensional Feynman diagrams. (Here, parity violation is meant in the three-dimensional sense. As will become clear in the next section, three-dimensional parity corresponds effectively to CPT in the four-dimensional context \[7\].) The factor $s_0$ in (3.10) comes, in fact, from a factor $\Lambda_0 / |\Lambda_0|$ out of the momentum integrals. The triangle diagram, for example, gives in the limit $|\Lambda_0| \to \infty$

$$
\pi^{-2} \int_0^\infty dq \, 4\pi q^2 \Lambda_0 (q^2 + \Lambda_0^2) (q^2 + \Lambda_0^2)^{-3} = \Lambda_0 / |\Lambda_0| \equiv s_0, \tag{3.12}\]

with the explicit factor $\Lambda_0 (q^2 + \Lambda_0^2)$ from the spinor trace in the integrand on the left-hand side. It is also important that the infrared divergences of the three-dimensional Feynman diagrams without Pauli–Villars fields are not regularized by the introduction of a small Dirac mass $\lambda_0$, which would again violate parity invariance, but that they are kept under control by the antiperiodic boundary conditions imposed on the fermions (turning the momentum integrals into sums).

The essential conditions for the derivation of (3.10) are thus the requirement of gauge invariance (3.8) and the control of infrared divergences in the $n = 0$ factor of (3.7). For non-Abelian gauge groups, there is, in addition to the local term (3.10) obtained in perturbation theory, also a nonlocal term in $\tilde{A}_m(\vec{x})$ which restores the full three-dimensional gauge invariance (3.8), not just its infinitesimal version. This nonlocal term vanishes, however, for gauge potentials $\tilde{A}_m(\vec{x})$ sufficiently close to zero. See Refs. \[21, 23\] and references therein.

The integral in (3.10) can be extended over the whole of 3-space, because the gauge potentials $\tilde{A}_m$ of (3.1), (3.2) vanish outside the ball $B^3$. The gauge potentials $\tilde{A}_m$ are also $x^4$-independent. Insisting upon translation invariance, the expression (3.10) can then be written as the following four-dimensional integral:

$$
\Gamma_{W}^{n=0} [\tilde{A}] \supseteq i \int_{\mathbb{R}^3} d^3 x \int_0^L dx^4 \, s_0 \frac{(1+a) \pi}{L} \omega_{CS}[\tilde{A}_1(\vec{x}), \tilde{A}_2(\vec{x}), \tilde{A}_3(\vec{x})], \tag{3.13}\]

with $s_0 = \pm 1$ as defined in (3.12) and parameter $a = 0$ for the simple non-Abelian gauge group considered up till now. The one-loop calculation for three-dimensional Abelian
U(1) gauge potentials gives essentially the same result [18, 22], with the factor \( \pi \) in (3.10) replaced by \( 2\pi \) and the parameter \( a \) in (3.13) set equal to 1.

The Chern–Simons term (3.13) is the main result of this paper. The result was obtained for the particular chiral gauge theory with the gauge group \( G = SO(10) \) and the fermion representation \( R_L = 16 \), but holds for an arbitrary simple compact Lie group (or Abelian \( U(1) \) group) and an arbitrary nonsinglet irreducible fermion representation, as long as the fermion representation is normalized appropriately and the complete theory is free of chiral gauge anomalies (see Section 5). In the next two sections, we will take a closer look at this result and present some generalizations.

## 4 Lorentz and CPT noninvariance

For the special gauge potentials (3.1), (3.2) and the Euclidean spacetime manifold \( M = \mathbb{R}^3 \times S^1 \), we have found in the previous section the emergence of a Chern–Simons term (3.13) in the effective gauge field action. The calculation, which relies on earlier results for the three-dimensional parity anomaly, applies to both Abelian and non-Abelian gauge groups, provided the chiral gauge anomalies cancel in the complete theory (see Section 5).

For arbitrary gauge potentials \( A_\mu(\vec{x}, x^4) \) which drop to zero faster than \( r^{-1} \) as \( r \equiv |\vec{x}| \to \infty \) and which have trivial holonomies (see below), the effective action term (3.13) can be written as the following local expression:

\[
\Gamma_{\text{CS-like}}[A] = i \int_{\mathbb{R}^3} d^3x \int_0^L dx^4 \frac{s_0(1 + a)\pi}{L} \omega_{\text{CS}}[A_1(\vec{x}, x^4), A_2(\vec{x}, x^4), A_3(\vec{x}, x^4)],
\]

with the Chern–Simons density \( \omega_{\text{CS}} \) given by (3.11), an integer factor \( s_0 = \pm 1 \) defined in (3.12), and an integer parameter \( a = 0 \) or 1 for a simple non-Abelian gauge group or an Abelian \( U(1) \) gauge group, respectively. Eq. (4.1), for simple non-Abelian gauge groups, is precisely equal to the counterterm presented in Ref. [7]. The expression (4.1) is called Chern–Simons-like, because a genuine topological Chern–Simons term exists only in an odd number of dimensions [23]. Remark that this Chern–Simons-like term (4.1) comes from a combination of infrared \((1/L)\) and ultraviolet \( (s_0 \equiv \Lambda_0/|\Lambda_0|) \) effects.

The local Chern–Simons-like term (4.1) has the important property of invariance under infinitesimal four-dimensional gauge transformations. (This property would not hold if the particular Chern–Simons density \( \omega_{\text{CS}} \) as given by (3.11) were replaced by, for example, \( \omega_{\text{CS}}[A_1(\vec{x}), A_2(\vec{x}), A_3(\vec{x})] \), with the averaged gauge potentials \( A_m(\vec{x}) \equiv L^{-1} \int_0^L dx^4 A_m(\vec{x}, x^4) \). Of course, such an effective action term using the averaged gauge potentials \( A_m \) would not be local either.) For simple compact connected Lie groups \( G \), there are also large gauge transformations with a gauge function \( g = g(\vec{x}) \in G \) corresponding to a nontrivial element of the homotopy group \( \pi_3(G) = \mathbb{Z} \). As mentioned in Section 3,
there is a nonlocal term in the effective gauge field action which restores invariance under these finite gauge transformations, but this nonlocal term vanishes for gauge potentials $A_m(\vec{x}, x^4)$ sufficiently close to zero. In addition, there are, for Lie groups $G = SO(N \geq 3)$ or $U(1)$ with homotopy group $\pi_1(G) \neq 0$, large gauge transformations with gauge function $g = g(x^4) \in G$, but the Chern–Simons-like term (4.1) is obviously invariant under these particular finite gauge transformations.

Turning to spacetime transformations, the effective action term (4.1) is clearly invariant under translations. The Chern–Simons density in its integrand, though, involves only three of the four gauge potentials $A_\mu(x)$ and three of the six components of the field strength tensor $A_{\mu\nu}(x) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, which makes the effective gauge field action manifestly Lorentz noninvariant. Physically, this would, for example, lead to anisotropic propagation (birefringence) of the gauge boson fields [25].

We are not able to determine the imaginary part of the effective gauge field action exactly. (The effective action might, for example, have some dependence on the trace of the path-ordered exponential integral (holonomy) $\text{tr} \ h(\vec{x}) \equiv \text{tr} \ P \exp \left\{ \int_0^L dx^4 \ A_4(\vec{x}, x^4) \right\}$, which could not have been detected by the gauge potentials (3.1) used in Section 3. The effective action term (4.1) holds, most likely, only for trivial holonomies $h(\vec{x}) = 1$.) But the partial result (3.13) suffices for the main purpose of this paper. The appropriate CPT transformation [1, 26] for an anti-Hermitian gauge potential is, namely,

$$A_\mu(x) \rightarrow A_\mu^T(-x),$$

with the suffix T indicating the transpose of the matrix. For a Hermitian electromagnetic vector potential $a_\mu(x)$, this corresponds to the usual transformation $a_\mu(x) \rightarrow -a_\mu(-x)$. Using (4.2), one then readily verifies that the Yang–Mills action density $\text{tr} \ (A_{\mu\nu}A^{\mu\nu})$ is CPT-even and that the integrand of the effective action term (3.13), or (4.1) for that matter, is CPT-odd. (The overall factor $i$ in (3.13), or (4.1), is absent for spacetime metrics with Lorentzian signature and need not be complex conjugated.) This establishes the CPT anomaly for chiral gauge theories with left-handed fermions in an arbitrary nonsinglet irreducible representation (provided the chiral gauge anomalies cancel in the complete theory) and spacetime manifold $M = \mathbb{R}^3 \times S^1$.

The Abelian Chern–Simons density has no cubic term and the integrand of the Abelian version of (4.1) is odd under both CPT and T (and even under both C and P), provided $x^4$ corresponds to a spatial coordinate after the Wick rotation from Euclidean to Lorentzian metric signature. The T and CPT violation would, for example, show up in the anisotropic propagation of the circular polarization modes of these Abelian gauge fields (a given circular polarization mode would, generically, have different phase velocity for propagation in opposite directions [23]).

Note, finally, that the terms (4.1) applied to the gauge groups $SU(3)$, $SU(2)$, and $U(1)$, with undetermined coefficients replacing $s_0 (1 + a) \pi / L$, have also been considered in a Standard-Model extension with Lorentz and CPT violation [14]. As discussed above,
these $SU(3)$ and $SU(2)$ Chern–Simons-like terms are noninvariant under certain large
gauge transformations \[7\] and the $U(1)$ Chern–Simons-like term may also become gauge
dependent if magnetic flux from monopoles is allowed for \[22\]. This suggests that either
the corresponding coefficients must be zero or that additional nonlocal terms restoring
gauge invariance must be included in the theory. The anomaly calculation of the present
paper follows the second path, with nonlocal terms restoring gauge invariance.\[6\]

5 Generalizations

The effective gauge field action for certain chiral gauge theories defined over a fixed four-
dimensional Euclidean spacetime manifold $M$ with Cartesian product structure $\mathbb{R}^3 \times S^1$
has been found to contain a CPT-violating term if gauge invariance is maintained. For
the special gauge potentials (3.1), this CPT-violating term is given by (3.13), which can
be written as (4.1) for arbitrary localized gauge potentials with trivial holonomies. As
mentioned in the previous section, the overall factor $i$ in (3.13) and (4.1) would be absent
for a Lorentzian signature of the metric.

Essentially the same result holds for other orientable spacetime manifolds $M$, as long
as at least one compact spatial dimension can be factored out. The crucial point is that
the Weyl operator should be separable with respect to this compact coordinate. Also,
the spin structure over the compact spatial dimension must be such as to allow for zero
momentum of the fermions, cf. Eq. (3.6). One example would be the flat Euclidean
spacetime manifold $M = \mathbb{R}^3 \times I$, with the closed interval $I \equiv [0, L] \subset \mathbb{R}$ replacing the
circle $S^1$ considered before. Here, the chiral fermions are taken to have free boundary
conditions over $I$. (There would be no CPT anomaly for strictly antiperiodic boundary
conditions. This would be the case for finite-temperature field theory in the Euclidean
path integral formulation \[26, 36\], which uses the same manifold $\mathbb{R}^3 \times I$ with antiperiodic
boundary conditions for the fermions over the interval $I \equiv [0, \beta]$, where $\beta$ stands for the
inverse temperature.) Another example would be the flat Minkowski-like manifold $M = \mathbb{R} \times S^1 \times S^1 \times S^1$, with time $t \in \mathbb{R}$ and a compact space manifold, which would have
three possible terms of the form (4.1) in the effective gauge field action. Similar effects
may occur in higher- and lower-dimensional chiral gauge theories, but for the rest of this
section we concentrate on the four-dimensional case, again with the spacetime manifold
$M = \mathbb{R}^3 \times S^1$.

The calculation of the CPT-violating term in Section 3 was performed for a single
irreducible representation of left-handed Weyl fermions. The particular theory considered,
\[3\] The same is to be expected for the effective gauge field action from fermions with explicit CPT-
violating, but gauge-invariant, terms in the action \[14\]. For recent results on the induced Abelian
Chern–Simons-like term from a massive Dirac fermion with a CPT-violating axial-vector term in the
action, see Ref. \[32\] and references therein. Chiral fermions with a real chemical potential $\mu$ (and a
corresponding CPT-odd term in the action) also give rise to an induced Chern–Simons-like term, which
is now proportional to $\mu$, see Ref. \[36\] and the last equation therein.
with the gauge group $G = SO(10)$ and the fermion representation $R_L = 16$, is free of chiral gauge anomalies \cite{12, 13, 16, 17}. It is, of course, possible to have more than one nonsinglet irreducible fermion representation $r$ for the left-handed fermions, provided the chiral anomalies cancel. The reducible fermion representation is then $R_L = \sum_f r_f$, with the label $f$ running over 1, \ldots, $N_F$. Here, and in the following, the gauge group $G$ is taken to be either a simple compact Lie group or an Abelian $U(1)$ group.

Vectorlike gauge theories with, for example, one nonsinglet irreducible representation $r$ have $R_L = r + \bar{r}$ and corresponding three-dimensional Pauli–Villars masses $\Lambda_{0r}$ and $\Lambda_{0\bar{r}}$, where $\bar{r}$ denotes the conjugate representation of $r$. Four-dimensional parity invariance gives $\Lambda_{0r} = -\Lambda_{0\bar{r}}$. Recalling (3.12), this implies that the CPT anomaly (4.1) cancels for this particular vectorlike gauge theory. The same cancellation occurs, in fact, for any vectorlike gauge theory.

Chiral gauge theories with $R_L = \sum_f r_f$ (and $R_L \neq \bar{R}_L$) may or may not have a CPT-violating term (4.1) left over in the effective gauge field action, depending on the relative signs of the corresponding three-dimensional Pauli–Villars masses $\Lambda_{0f}$. Of course, attention must be paid to the normalization of the different irreducible representations $r_f$. Also, the situation can be complicated further by having more three-dimensional Pauli–Villars fields than the ones used in Section 3. The factor $s_0 = \pm 1$ in (3.10), (3.13), and (4.1), is then replaced by $(2k_{0f} + 1)$, for $k_{0f} \in \mathbb{Z}$. The same odd integer prefactors of the induced Chern–Simons density also appear for other three-dimensional ultraviolet regularization methods and are, in fact, to be expected on general grounds \cite{23}.

For a chiral gauge theory with an odd number $N_F$ of equal irreducible left-handed fermion representations, there necessarily appears a CPT-violating term proportional to (4.1) in the effective gauge field action. The reason is that the sum of an odd number of odd numbers does not vanish, $\sum_f (2k_{0f} + 1) \neq 0$ for $f$ summed over 1 to $N_F$. An example for $N_F = 3$ would be the $SO(10)$ chiral gauge theory with the reducible fermion representation $R_L = 16 + 16 + 16$, which necessarily has a CPT-violating term proportional to (4.1) for the $SO(10)$ gauge fields in the effective action.

This particular $SO(10)$ model contains, as is well known, the $SU(3) \times SU(2) \times U(1)$ Standard Model \cite{26, 28} with $N_F = 3$ families of 15 left-handed Weyl fermions (quarks and leptons), together with $N_F = 3$ left-handed Weyl fermion singlets (conjugates of the hypothetical right-handed neutrinos). The Standard Model has thus a CPT-violating term proportional to (4.1) for the hypercharge $U(1)$ gauge fields in the effective action, together with similar terms for the weak $SU(2)$ and color $SU(3)$ gauge fields, as long as the spacetime manifold has the appropriate Cartesian product structure and the Standard Model is embedded in this particular $SO(10)$ chiral gauge theory.\footnote{It is not clear to what extent the 33 remaining gauge bosons from $SO(10)$ need to be physical, but they could always be given large masses by the Higgs mechanism \cite{27, 28}.} Other simple compact Lie groups instead of $SO(10)$ may also be used for the embedding of the Standard Model fermions, as long as they have an odd number of equal irreducible representations.
This embedding condition for the Standard Model fermions guarantees the presence of the CPT-violating Chern–Simons-like terms for the $SU(3) \times SU(2) \times U(1)$ gauge fields in the effective action, otherwise these terms may or may not appear, depending on the regularization scheme.

6 Discussion

In the previous sections, we have established for certain chiral gauge theories defined over the spacetime manifold $M = \mathbb{R}^3 \times S^1$ the necessary presence of a CPT-violating term in the gauge-invariant effective action. The question, now, is what happened to the CPT theorem? It appears that the CPT theorem is circumvented by a breakdown of local Lorentz invariance at the quantum level. (See also the paragraph above (3.9), which discusses the breaking of Lorentz invariance by the ultraviolet regularization used.) More specifically, the second-quantized vacuum seems to play a role in connecting the global spacetime structure to the local physics. The next two paragraphs elaborate this point but may be skipped in a first reading.

For non-Abelian chiral gauge groups, there is the condition of gauge invariance to deal with in these particular quantum field theories which are potentially afflicted by the non-perturbative chiral gauge anomaly discovered earlier [16]. This nonperturbative chiral gauge anomaly depends on the global spacetime structure in a Lorentz noninvariant way, one spatial direction being singled out by the so-called $Z$-string configuration responsible for the gauge anomaly in the Hamiltonian formulation. If the theory has indeed this $Z$-string chiral gauge anomaly, then the restoration of gauge invariance obviously requires interactions which are themselves Lorentz noninvariant [7]. But, even if the theory does not have a net $Z$-string chiral gauge anomaly, there still occurs, in first-quantization, the spectral flow which treats one spatial dimension differently from the others [16]. This implies that a tentative second-quantized vacuum state varies along the corresponding loop of gauge transformations. Imposing gauge invariance throughout then leads to Lorentz noninvariance of the theory. In both cases, the invariance under the proper orthochronous Lorentz group is lost and the CPT theorem no longer applies [1]. It is then possible to have a non-Abelian CPT-odd term (4.1) in the effective gauge field action.

For Abelian chiral gauge groups, nonzero magnetic flux from monopoles can also give rise to spectral flow, as discussed for the three-dimensional case in Ref. [22]. For the four-dimensional case, this setup again breaks Lorentz invariance (just as the $Z$-string does for the non-Abelian chiral gauge anomaly) and the CPT theorem no longer applies, with the possibility of having an Abelian CPT-odd term (4.1) in the effective gauge field action.

For both Abelian and non-Abelian chiral gauge groups, it remains to be seen whether or not the gauge-invariant, but CPT-violating, theory is consistent. In particular, the properties of microcausality and positivity of the energy need to be established, cf. Refs.
If the theory in question turns out to be inconsistent, then perhaps it could be interpreted as part of a more fundamental theory, possibly involving curvature and torsion.

In this paper, we have primarily been concerned with the mechanism of the CPT anomaly, not potential applications. Let us, however, mention two possibilities. First, there may be the “optical activity” discussed in the Introduction, where the linear polarization of a plane wave of gauge fields gets rotated \textit{in vacuo} (in our case, through the quantum effects of the chiral fermions encoded in the effective gauge field action). As mentioned in Section 5, the phenomenon could occur for the photon field of the Standard Model, as long as there is the \textit{SO}(10)-like embedding of the Standard Model fermions and the appropriate Cartesian product structure of the spacetime manifold. The laboratory measurement of this optical activity of the vacuum could, in principle, provide information about the global structure and size of the universe. More realistically, the mass scale of the CPT-violating term (4.1) for the photon field is of the order of

$$\alpha L^{-1} \sim 10^{-35} \text{eV} \left(\frac{\alpha}{1/137}\right) \left(\frac{1.5 \times 10^{10} \text{lyr}}{L}\right),$$

(6.1)

with $\alpha \equiv e^2/4 \pi$ the fine-structure constant and $L$ the range of the compact spatial coordinate. This mass is, of course, very small on the scale of the known elementary particles (the present universe being very large), but, remarkably, it is only a factor 100 below the current upper bound of $\sim 10^{-33}$ eV obtained from observations on distant radio galaxies, see Refs. [25, 37] and references therein. (The “laboratory” has now been expanded to a significant part of the visible universe.) A dedicated observation program to map the linear polarization in a large number of distant radio sources [37], or future satellite experiments to measure the polarization of the cosmic microwave background [38], can perhaps reach the sensitivity level set by (6.1).

Second, the CPT anomaly may have been important in the very early universe. In the present paper, we have considered a fixed spacetime manifold with given topology. With gravity, spacetime becomes dynamic. For an inverse size $1/L(t)$ and typical scattering energies of the order of the gravitational scale (Planck mass), the CPT-violating effects of the effective action term (4.1) are relatively unsuppressed compared to gravity, that is suppressed by the square of the gauge coupling constant only. Of course, the fundamental theory of gravity remains to be determined if there is indeed Lorentz noninvariance in certain inertial frames. Still, it is conceivable that the CPT anomaly plays a role in defining a “fundamental arrow-of-time” [4, 39], as a quantum mechanical effect coming from the interplay of chiral fermions, gauge field interactions and the topology of spacetime.
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