Second-order SUSY partners of the trigonometric Pöschl-Teller potentials

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Abstract. The second-order supersymmetry transformations are used to generate Hamiltonians with known spectra departing from the trigonometric Pöschl-Teller potentials.

1. Introduction
The supersymmetric quantum mechanics (SUSY QM) is the simplest technique to design systems whose Hamiltonians have prescribed energy spectra [1–13]. In this procedure, departing from an initial solvable Hamiltonian $H$, it can be constructed a new solvable one $\tilde{H}$ whose levels differ slightly from those of $H$. By iterating appropriately this method, one can construct Hamiltonians whose spectra are arbitrarily close to any desired one.

Contrasting with the standard first-order technique, in which it is possible to manipulate just the ground state energy, the second-order SUSY QM offers additional possibilities of spectral manipulation [2, 12]: i) two new levels can be ‘created’ between a pair of neighbor physical ones $E_{i-1}$, $E_{i}$ of $H$; ii) two neighbor physical energies can be deleted; iii) one level can be created at an arbitrary position; iv) one physical energy can be deleted; v) there is not modification of the initial spectrum.

The SUSY methods have been extensively applied to potentials defined on the full real line or the positive semi-axis. In order to complete the scheme, here we will work in detail the second-order technique for the trigonometric Pöschl-Teller potentials, which are defined in a finite interval [13]. One of our aims is to find additional information about the behavior of the SUSY partner potentials at the boundaries of the $x$-domain (see e.g. [14]).

2. Second-order supersymmetric quantum mechanics
In the second-order SUSY QM a differential operator of second order intertwines two hermitian Schrödinger Hamiltonians in the way:

$$\tilde{H}B^\dagger = B^\dagger H,$$

where

$$H = \frac{1}{2} \frac{d^2}{dx^2} + V(x), \quad \tilde{H} = -\frac{1}{2} \frac{d^2}{dx^2} + \tilde{V}(x),$$

$$B^\dagger = \frac{1}{2} \left( \frac{d^2}{dx^2} - \eta(x) \frac{d}{dx} + \gamma(x) \right).$$
After some manipulations, the intertwining relationship reduces to:

\[ \tilde{V} = V - \eta', \]
\[ \gamma = \frac{\eta'}{2} + \frac{\eta^2}{2} - 2V + d, \]
\[ \frac{\eta''}{2} - \frac{\eta'^2}{4} + \eta^2 \eta' + \frac{\eta^4}{4} - 2V \eta^2 + d \eta^2 + c = 0, \]

with \( c, d \in \mathbb{R} \). Given \( V(x) \), the new potential \( \tilde{V}(x) \) and \( \gamma(x) \) are obtained from (4,5) once we find a solution \( \eta(x) \) of (6), which is obtained from the Ansatz

\[ \eta' = -\eta^2 + 2\beta \eta + 2\xi, \]

where \( \beta(x) \) and \( \xi(x) \) are to be determined. After some calculations we get \( \xi^2 = c \) and:

\[ \beta'(x) + \beta^2(x) = 2[V(x) - \epsilon], \quad \epsilon = \frac{d + \xi}{2}. \]

We can work as well with a Schrödinger equation, obtained from (8) through \( \beta = u'/u \):

\[ -\frac{u''}{2} + Vu = \epsilon u. \]

Note that if \( c \neq 0 \), \( \xi \) takes two different values, \( \pm \sqrt{c} \), and thus we need to solve the Riccati equation (8) for two factorization energies \( \epsilon_{1,2} = (d \pm \sqrt{c})/2 \). Then, one constructs algebraically a common solution \( \eta(x) \) of the corresponding pair of equations (7). On the other hand, if \( c = 0 \) one has to solve (8) for \( \epsilon = d/2 \) and to find then the general solution of the Bernoulli equation resulting for \( \eta(x) \) (see (7)). There is an obvious difference between the case with real factorization constant \( (c > 0) \) and the complex case \( (c < 0) \), suggesting therefore a classification for the solutions \( \eta(x) \) based on the sign of \( c \) [15].

2.1. The real case \( (c > 0) \)

Here we have \( \epsilon_1, \epsilon_2 \in \mathbb{R}, \epsilon_1 \neq \epsilon_2 \), the corresponding Riccati solutions of (8) being denoted by \( \beta_1(x), \beta_2(x) \) respectively. The resulting expression for \( \eta(x) \) becomes:

\[ \eta(x) = \frac{2(\epsilon_1 - \epsilon_2)}{\beta_1(x) - \beta_2(x)} = \frac{2(\epsilon_1 - \epsilon_2)u_1u_2}{W(u_1, u_2)} = \frac{W'(u_1, u_2)}{W(u_1, u_2)}, \]

where \( W(f, g) = fg' - f'g \). It is clear from Eqs.(4,10) that the potentials \( \tilde{V}(x) \) have no new singularities if the Wronskian \( W(u_1, u_2) \) is nodeless in \( (x_l, x_r) \).

Note that the intertwining relationship (1) ensures that, if \( \psi_n(x) \) are normalized eigenfunctions of \( H \) with eigenvalues \( E_n \), then the normalized eigenfunctions \( \phi_n \) of \( \tilde{H} \) with the same eigenvalues read:

\[ \phi_n(x) = \frac{B^\dagger \psi_n(x)}{\sqrt{(E_n - \epsilon_1)(E_n - \epsilon_2)}}. \]

However, in general the set \( \{ \phi_n(x), n = 0, 1, \ldots \} \) is not complete [12], since there are two mathematical eigenstates \( \phi_{\epsilon_i}(x) \) of \( \tilde{H} \) with eigenvalues \( \epsilon_i, i = 1, 2 \), belonging as well to the kernel of \( B = (B^\dagger)^\dagger \) which could be normalized and therefore should be added, i.e.,

\[ B\phi_{\epsilon_i} = 0, \quad \tilde{H}\phi_{\epsilon_j} = \epsilon_j \phi_{\epsilon_j}, \quad j = 1, 2, \]
\[ \Rightarrow \phi_{\epsilon_1} \propto \frac{\eta}{u_1} \propto \frac{u_2}{W(u_1, u_2)}, \quad \phi_{\epsilon_2} \propto \frac{\eta}{u_2} \propto \frac{u_1}{W(u_1, u_2)}. \]
As a consequence, several interesting spectral changes can be done through the second-order SUSY QM, and some of them are worth to be mentioned [2,12].

a) For \( \epsilon_2 < \epsilon_1 < E_0 \), by taking a nodeless \( u_1 \) and \( u_2 \) with one node it turns out that \( W(u_1, u_2) \) is nodeless, \( \phi_{\epsilon_1}, \phi_{\epsilon_2} \) are normalizable and thus \( \text{Sp}(\bar{H}) = \{ \epsilon_2, \epsilon_1, E_0, E_1, \ldots \} \).

b) For \( E_{i-1} < \epsilon_2 < \epsilon_1 < E_i, \ i = 1, 2, \ldots \) with \( u_2, u_1 \) having \( i + 1 \) nodes respectively, \( W(u_1, u_2) \) becomes nodeless, \( \phi_{\epsilon_1}, \phi_{\epsilon_2} \) are normalizable and \( \text{Sp}(\bar{H}) = \{ E_0, \ldots, E_{i-1}, \epsilon_2, \epsilon_1, E_i, \ldots \} \).

c) For \( \epsilon_2 = E_{i-1}, \ \epsilon_1 = E_i, \ u_2 = \psi_{i-1}, \ u_1 = \psi_i, \) the Wronskian is nodeless but \( \phi_{\epsilon_1}, \phi_{\epsilon_2} \) are non-normalizable. Thus, \( \text{Sp}(\bar{H}) = \{ E_0, \ldots, E_{i-2}, E_{i+1}, \ldots \} \).

2.2. The complex case \((c < 0)\) [16,17]

Now \( \epsilon \equiv \epsilon_1 \in \mathbb{C}, \ \epsilon_2 = \bar{\epsilon} \), and the real \( \eta(x) \) resulting of (7) becomes:

\[
\eta(x) = \frac{2\text{Im}(\epsilon)}{\text{Im}[\beta(x)]} = \frac{w'(x)}{w(x)}, \quad w(x) = \frac{W(u, \bar{u})}{2(\epsilon - \bar{\epsilon})}, \quad (14)
\]

where \( \beta(x) \) is the Riccati solution of (8) associated to \( \epsilon \). Once again, \( w(x) \) must be nodeless in \((x_l, x_r)\) to avoid extra singularities in \( \tilde{V}(x) \). Since \( w(x) \) is non-decreasing monotonic \((w'(x) = |u(x)|^2)\), a sufficient condition ensuring the lack of zeros is

\[
\lim_{x \to x_l} u(x) = 0 \quad \text{or} \quad \lim_{x \to x_r} u(x) = 0. \quad (15)
\]

Transformation functions obeying (15), associated to complex values of \( \epsilon \), generate real potentials \( \tilde{V}(x) \) isospectral to \( V(x) \). The corresponding normalized eigenfunctions \( \phi_n(x) \) can be obtained from (11) by the substitutions \( \epsilon_1 = \epsilon, \ \epsilon_2 = \bar{\epsilon} \).

2.3. The confluent case \((c = 0)\) [18-20]

Here we have \( \xi = 0, \ \epsilon \equiv \epsilon_1 = \epsilon_2 \in \mathbb{R} \). The general solution of the Bernoulli equation resulting from (7) reads:

\[
\eta(x) = \frac{w'(x)}{w(x)}, \quad (16)
\]

\[
w(x) = w_0 + \int e^{2 \int \beta(x) dx} dx = w_0 + \int_{x_0}^{x} [u(y)]^2 dy, \quad (17)
\]

where \( x_0 \) is a fixed point in \([x_l, x_r]\). Once again, \( w(x) \) must be nodeless in order that \( \tilde{V}(x) \) has no singularities in \((x_l, x_r)\). Since \( w(x) \) is non-decreasing monotonic \((w'(x) = u^2(x))\), the simplest choice ensuring a nodeless \( w(x) \) is to take \( u(x) \) satisfying either

\[
\lim_{x \to x_l} u(x) = 0, \quad I_- = \int_{x_l}^{x_0} [u(y)]^2 dy < \infty, \quad (18)
\]

or

\[
\lim_{x \to x_r} u(x) = 0, \quad I_+ = \int_{x_0}^{x_r} [u(y)]^2 dy < \infty. \quad (19)
\]

In both cases one can determine a \( w_0 \)-domain for which \( w(x) \) is nodeless.

The expressions for the eigenfunctions \( \phi_n(x) \) of \( \bar{H} \) associated to the initial eigenvalues \( E_n \) can be obtained from equation (11) by taking \( \epsilon = \epsilon_1 = \epsilon_2 \). However, the spectrum of \( \bar{H} \) depends
once again on the normalizability of the eigenfunction $\phi_\epsilon$ of $\widetilde{H}$ associated to $\epsilon$ which is also annihilated by $B$, with explicit expression given by:

$$\phi_\epsilon(x) \propto \frac{\eta(x)}{u(x)} = \frac{u(x)}{w(x)}. \quad (20)$$

Note that, for $\epsilon > E_0$, one can find solutions satisfying either (18) or (19) and such that the $\phi_\epsilon(x)$ given in (20) is normalized, i.e., the confluent algorithm allows to embed single energy levels above the ground state energy of $H$.

3. Trigonometric Pöschl-Teller potentials

Let us apply the previous techniques to the trigonometric Pöschl-Teller potentials [13, 21]:

$$V(x) = \frac{(\lambda - 1)\lambda}{2\sin^2(x)} + \frac{(\nu - 1)\nu}{2\cos^2(x)}, \quad \lambda, \nu > 1, \quad (21)$$

defined on $(0, \pi/2)$, which are illustrated in gray in Figure 1. The eigenfunctions $\psi_n(x)$ of $H$, satisfying the boundary conditions $\psi_n(0) = \psi_n(\pi/2) = 0$, and the corresponding eigenvalues are given by:

$$\psi_n(x) = c_n \sin^\lambda(x) \cos^\nu(x) \ {}_2F_1(-n, n + \mu; \lambda + 1/2; \sin^2(x)), \quad (22)$$

$$E_n = \frac{(\mu + 2n)^2}{2}, \quad n = 0, 1, 2, \ldots, \quad \mu = \lambda + \nu, \quad (23)$$

where $c_n$ are the normalization factors.

3.1. Second-order SUSY transformations

Let us explore briefly the three cases of the classification discussed at section 2 (for illustrations of the new potentials see the dashed, dotted and continuous black curves in Figure 1).

3.1.1. Real case. For $\epsilon_1, \epsilon_2 \in \mathbb{R}$, several possibilities of modifying the initial spectrum of $H$ are available.

a) Deletion of two neighbor physical levels. Let us take first $\epsilon_1 = E_i$, $\epsilon_2 = E_{i-1}$, $u_1(x) = \psi_i(x)$, $u_2(x) = \psi_{i-1}(x)$ (see equation (22)). The second-order SUSY partner of $V(x)$ thus takes the form:

$$\widetilde{V}(x) = \frac{\lambda(\lambda + 3)}{2\sin^2(x)} + \frac{\nu(\nu + 3)}{2\cos^2(x)} - [\ln(W_{i,i-1})]'', \quad (24)$$

where

$$W_{i,i-1} = W(2F_1(-i, i + \mu; \lambda + 1/2; \sin^2(x)), 2F_1(-i + 1, i - 1 + \mu; \lambda + 1/2; \sin^2(x))) \quad (25)$$

is nodeless in $(0, \pi/2)$.

b) Creation of two new levels. First we find the general solution of the Schrödinger equation (9) with the potential (21) for an arbitrary factorization energy $\epsilon$:

$$u(x) = \sin^\lambda(x) \cos^\nu(x) \left[ \frac{A}{2} F_1 \left( \frac{\lambda}{2} + \lambda + \frac{1}{2}; \frac{1}{2}; \lambda + \frac{1}{2} \sin^2(x) \right) \right]$$

$$+ C \sin^{1-2\lambda}(x) \ {}_2F_1 \left( \frac{1+\nu-\lambda}{2} + \lambda + \frac{1}{2} \sin^2(x) \right), \quad (26)$$

Here, $A$, $B$, and $C$ are constants that can be determined from the normalization condition.
with asymptotic behavior

\[ u(x) \sim C \sin^{1-\lambda}(x), \quad u(x) \sim (Aa + Cb) \cos^{1-\nu}(x), \]

where

\[ a = \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(\nu-\frac{1}{2})}{\Gamma(\lambda+\nu+1)}, \quad b = \frac{\Gamma(\nu+\frac{1}{2})\Gamma(\nu-\frac{1}{2})}{\Gamma(\lambda+\nu+1)}, \]

Without losing generality, let us take \( C = 1 \) and \( A = -b/a + D \) so that, for \( E_{i-1} < \epsilon < E_i \), the solution \( u(x) \) will have \( i + 1 \) nodes for \( D < 0 \) while it will have \( i \) nodes for \( D > 0 \).

Let us implement now the second-order SUSY transformation for \( E_{i-1} < \epsilon_2 < \epsilon_1 < E_i \), the corresponding seed solutions are given by (26) with \( D_2 < 0, D_1 > 0 \), which ensures that the Wronskian is nodeless. It is convenient to express the \( u \) solutions in the way:

\[ u_1(x) = \sin^{1-\lambda}(x) \cos^{1-\nu}(x)v_1(x), \quad u_2(x) = \sin^{1-\lambda}(x) \cos^{1-\nu}(x)v_2(x), \]

where \( v_i(x), \ i = 1, 2 \) are non-divergent for \( x \in [0, \pi/2] \). Thus

\[ \tilde{V}(x) = \frac{(\lambda - 1)(\lambda - 4)}{2 \sin^2(x)} + \frac{(\nu - 1)(\nu - 4)}{2 \cos^2(x)} - \ln[W(v_1, v_2)]'' \]

For an illustration of these potentials, see the dotted curve in Figure 1.

### 3.1.2. Complex case

For \( \epsilon \in \mathbb{C} \) the expression given in (26) is still valid. The condition (15), required to avoid the zeros in the Wronskian, can be satisfied in two different ways. In the first place we select \( A = 1, \ C = 0 \). The two singularities induced in the new potential can be managed by expressing

\[ u(x) = \sin^{\lambda}(x) \cos^{1-\nu}(x)v(x). \]

Therefore:

\[ \tilde{V}(x) = \frac{\lambda(\lambda + 3)}{2 \sin^2(x)} + \frac{(\nu - 1)(\nu - 4)}{2 \cos^2(x)} - \ln[w(x)]'' \]

\[ w(x) = \frac{W(v, \bar{v})}{2(\epsilon - \bar{\epsilon})} \]

The second solution satisfying (15) arises by taking \( C = 1, \ A = -b/a \), and now it is convenient to express

\[ u(x) = \sin^{1-\lambda}(x) \cos^{\nu}(x)v(x). \]

The second-order SUSY partner of the trigonometric Pöschl-Teller potential becomes now:

\[ \tilde{V}(x) = \frac{(\lambda - 1)(\lambda - 4)}{2 \sin^2(x)} + \frac{\nu(\nu + 3)}{2 \cos^2(x)} - \ln[w(x)]'' \]

An example of these potentials is drawn in dashed in Figure 1.
3.1.3. Confluent case. For $\epsilon = \epsilon_1 = \epsilon_2$ let us use once again the expression given in (26). There are two adequate solutions, the first one arises by taking $A = 1$, $C = 0$:

$$u(x) = \sin^\lambda(x) \cos^\nu(x) \, 2F_1 \left( \frac{\nu}{2} + \sqrt{\frac{\nu}{2}}; \frac{\lambda}{2} - \sqrt{\frac{\nu}{2}}; \lambda + 1; \sin^2(x) \right).$$

(35)

A straightforward calculation leads to:

$$w(x) = w_0 + \sum_{m=0}^{\infty} \frac{\left(\frac{\nu}{2} + \sqrt{\frac{\nu}{2}}\right)_m \left(\frac{\nu}{2} - \sqrt{\frac{\nu}{2}}\right)_m}{(\lambda + \frac{\nu}{2})_m \, m! \, (2\lambda + 2m + 1)} \sin^{2\lambda + 2m + 1}(x)$$

$$\times 3F_2 \left( \frac{1 + \lambda - \nu}{2} - \sqrt{\frac{\nu}{2}}; \frac{1 + \lambda - \nu}{2} + \sqrt{\frac{\nu}{2}}; \lambda + m + \frac{1}{2}; \lambda + \frac{1}{2}, \lambda + m + \frac{\nu}{2}; \sin^2(x) \right).$$

(36)

Note that $w(x)$ is nodeless in $[0, \pi/2]$ for $w_0 > 0$. The SUSY partners of the Pöschl-Teller potentials are thus given by (4) with $\eta(x)$ given by (16,36).

The second solution, appropriate to implement the confluent algorithm, is obtained from (26) with $C = 1$, $A = -b/a$ which, up to a constant factor, reduces to:

$$u(x) = \sin^\lambda(x) \cos^\nu(x) \, 2F_1 \left( \frac{\nu}{2} + \sqrt{\frac{\nu}{2}}; \frac{\nu}{2} - \sqrt{\frac{\nu}{2}}; \nu + \frac{1}{2}; \cos^2(x) \right).$$

(37)

This expression leads straightforwardly to:

$$w(x) = w_0 - \sum_{m=0}^{\infty} \frac{\left(\frac{\nu}{2} + \sqrt{\frac{\nu}{2}}\right)_m \left(\frac{\nu}{2} - \sqrt{\frac{\nu}{2}}\right)_m}{(\nu + \frac{1}{2})_m \, m! \, (2\nu + 2m + 1)} \cos^{2\nu + 2m + 1}(x)$$

$$\times 3F_2 \left( \frac{1 + \nu - \lambda}{2} - \sqrt{\frac{\nu}{2}}; \frac{1 + \nu - \lambda}{2} + \sqrt{\frac{\nu}{2}}; \nu + m + \frac{1}{2}; \nu + \frac{1}{2}, \nu + m + \frac{\lambda}{2}; \cos^2(x) \right).$$

(38)

Now $w(x)$ is nodeless in $[0, \pi/2]$ for $w_0 < 0$, and by choosing such a nodeless $w(x)$ one can built up straightforwardly the confluent second-order SUSY partner potentials of $V(x)$.

It is important to observe that, when $\epsilon = E_n$ for some given $n$, the physical solution (22) and the one given in equation (35) just differ by the factor $c_n$. Moreover, the infinite summation.

**Figure 1.** Trigonometric Pöschl-Teller potential for $\lambda = 5$, $\nu = 8$ (gray curve) and its second order SUSY partners arising from the deletion of $E_4 = 220.5$ using the confluent algorithm (black continuous curve), the creation of two levels at $\epsilon_1 = 160.265$ and $\epsilon_2 = 146.265$ (dotted curve), and an isoespectral transformation with $\epsilon = 120.125 + 1.5i$ (dashed curve).
in (36) truncates in this case at \( m = n + 1 \); by combining appropriately all this information, it turns out that the \( w \)-expression when \( u(x) = \psi_n(x) \) becomes:

\[
\begin{align*}
    w(x) &= w_0 + \epsilon^2 \sum_{m=0}^{n} \frac{(\mu+n)_m (\nu+n)_m}{(\lambda + \frac{1}{2})_m (2\lambda + 2n + 1)_m} \sin^{2\lambda + 2m + 1}(x) \\
    &\quad \times F_2 \left( \frac{1}{2} - \nu - n, \frac{1}{2} + \lambda + n, \lambda + m + \frac{1}{2}; \lambda + \frac{1}{2}, \lambda + m + \frac{3}{2}; \sin^2(x) \right). \quad (39)
\end{align*}
\]

This function is nodeless in \([0, \pi/2]\) for \( w_0 > 0 \) or \( w_0 < -1 \), and in this domain \( \tilde{V}(x) \) becomes isospectral to the initial Pöschl-Teller potential. However, if we choose either \( w_0 = 0 \) or \( w_0 = -1 \), the level \( \epsilon = E_n \) will not be longer an eigenvalue of \( \tilde{H} \), since the corresponding eigenfunction is non-normalizable. Thus \( \text{Sp}(\tilde{H}) = \{ E_0, \ldots, E_{n-1}, E_{n+1}, \ldots \} \), i.e. we have suppressed the initial \( n \)-th level (for an illustration, see the black continuous curve in Figure 1).

4. Conclusions

In this paper we have applied successfully the second-order SUSY QM to generate exactly solvable Hamiltonians departing from the trigonometric Pöschl-Teller potentials. Through this technique, it is possible to produce interesting deformations of the quadratic spectrum of \( H \), namely, we can embed pairs of (or single) energy levels between two neighbor initial energies. It has been possible as well to delete two neighbor (or a single) energy levels, as well as to obtain Hamiltonians isospectral to \( H \). Thus, we have shown that a simple mathematical tool is available for designing potentials with an arbitrarily prescribed spectrum.

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