BLOWUPS IN BPS/CFT CORRESPONDENCE, AND PAINLEVÉ VI

NIKITA NEKRASOV

ABSTRACT. We study four dimensional supersymmetric gauge theory in the presence of surface and point-like defects (blowups) and propose an identity relating partition functions at different values of Ω-deformation parameters (ε₁, ε₂). As a consequence, we obtain the formula conjectured in 2012 by O. Gamayun, N. Iorgov, and O. Lysovyy, relating the tau-function τ_{PVI} to c = 1 conformal blocks of Liouville theory and propose its generalization for the case of Garnier-Schlesinger system. To this end we clarify the notion of the quasiclassical tau-function τ_{PVI} of Painlevé VI and its generalizations. We also make some remarks about the sphere partition functions, the boundary operator product expansion in the \( \mathcal{N} = (4,4) \) sigma models related to four dimensional \( \mathcal{N} = 2 \) theories on toric manifolds, discuss crossed instantons on conifolds, elucidate some aspects of the BPZ/KZ correspondence, and applications to quantization.

1. Introduction

Since the discovery of Kramers and Wannier [96] of the duality relating different temperature regimes of the Ising model the search for strong-weak coupling transformations mapping the regime of complicated dynamics in one quantum field theory to the regime of controlled computations of, possibly, another, has been ongoing. In the case of supersymmetric field theory the conjectured dualities e.g. [116] can be tested with the help of localization techniques introduced in [127], see also [143, 89]. One instance of such duality is the classical/quantum correspondence [139], [134], relating spectrum of one quantum mechanical model to symplectic geometry of another, classical, model. For Hitchin systems [74] this correspondence is connected to Langlands duality [13], [43]. Another example of the same duality, in Sine-Gordon theory, is found in [110]. Yet another interesting duality [77] maps the correlation functions in one quantum system, e.g. spin-spin correlators in the abovementioned Ising model, to some classical dynamics, described, e.g. by the Painlevé III [12] or Painlevé VI equations [85].

In this paper, which is, in many ways a part VI of the series [132], although the number VI in the title stands for something else¹ we explore a relatively recently found connection between a classical mechanical system and c = 1 conformal blocks of a two dimensional conformal field theory. The conformal blocks behave in many respects as the wavefunctions of some quantum mechanical system. So we could loosely call this relation a classical/quantum correspondence. Actually, we shall find it useful to deform

¹So that by taking the confluent limits from PVI to PV, PIV, PIII, etc. one will not recover the previous titles in [132]
both sides of the correspondence (in some sense, quantize the classical mechanical side, while \( \beta \)-deforming the conformal block side):

\[
\Psi(a, \varepsilon_1, \varepsilon_2, m; w, q) = \sum_{n \in \Lambda} \Psi(a + \varepsilon_1 n, \varepsilon_1, \varepsilon_2 - \varepsilon_1, m; w, q) Z(a + \varepsilon_2 n, \varepsilon_1 - \varepsilon_2, \varepsilon_2, m, q).
\]

Here \( \Psi(a, \varepsilon_1, \varepsilon_2, m; w, q) \) and \( Z(a, \varepsilon_1, \varepsilon_2, m, q) \) are the conformal blocks of the current algebra and \( \mathcal{W} \)-algebra, respectively. The natural habitat for (1) is the four dimensional \( \mathcal{N} = 2 \) supersymmetric \( \Omega \)-deformed gauge theory, where it is the relation between the (unnormalized) expectation value \( \Psi \) of a surface defect located at the surface \( z_2 = 0 \) with its own couplings \( w \), and the supersymmetric partition function \( Z \) of the theory on \( \mathbb{R}^4 \), with the bulk coupling \( q \):

\[
q = e^{-\frac{8g^2}{\pi^2} \epsilon^2}. \tag{2}
\]

The relation (1) accompanies the well-known equivariant blowup formula

\[
Z(a, \varepsilon_1, \varepsilon_2, m, q) = \sum_{n \in \Lambda} Z(a + \varepsilon_1 n, \varepsilon_1, \varepsilon_2 - \varepsilon_1, m, q) Z(a + \varepsilon_2 n, \varepsilon_1 - \varepsilon_2, \varepsilon_2, m, q) \tag{3}
\]

found in [123].

We shall exploit the consequences of the Eq.(1) in the context of the BPS/CFT correspondence, which we review in the next section. Our main focus will be on the limit \( \varepsilon_1 \to 0 \). The surface defect has the thermodynamic limit

\[
\Psi(a, \varepsilon_1, \varepsilon_2, m; w, q) \sim \exp \frac{1}{\varepsilon_1} S(a, \varepsilon_2, m; w, q) \tag{4}
\]

for some complexified free energy \( S(a, \varepsilon_2, m; w, q) \), which is a twisted superpotential of the effective \( \mathcal{N} = (2, 2) \) two dimensional theory. On the CFT side \( \Psi \) is identified with a certain conformal block obeying first order differential equation with respect to \( q \), which becomes a Hamilton-Jacobi equation for \( S \) in the \( \varepsilon_1 \to 0 \) limit. We show, that the exponentiated \( \varepsilon_2 \)-derivative of the Hamilton-Jacobi potential \( S \), when expressed in appropriate variables, coincides with the Painlevé VI \( \tau \)-function (as well as formulate a conjecture about more general isomonodromy problems):

\[
\tau(\alpha, \beta; \vec{\vartheta}; q) = \exp \frac{\partial S}{\partial \varepsilon_2}, \tag{5}
\]

with the monodromy data \( \alpha, \beta, \) and \( \vec{\vartheta} \) expressed in terms of \( a/\varepsilon_2, w, \) and \( m/\varepsilon_2 \) (the identification of the monodromy data is achieved in the forthcoming companion paper [83]). For \( SU(2) \) theory with \( N_f = 4 \) hypermultiplets the relation (1) becomes, in the limit \( \varepsilon_1 \to 0 \), the one found in 2012 by O. Gamayun, N. Iorgov and O. Lysovyy (GIL for short).

The paper is organized as follows. First, we recall a few facts about the supersymmetric gauge theories, twists, \( \Omega \)-deformations, surface defects, and blowup formulas. Then we proceed with some remarks about non-stationary integrable systems, and their \( \tau \)-functions. We show, for a restricted class of such systems (which include \( SL_2 \) Schlesinger equations in genus 0), how to relate the \( \tau \)-function to the Hamilton-Jacobi potential. After that we return to the realm of supersymmetric gauge theories. We
review the non-perturbative Dyson-Schwinger equation in $\mathcal{N} = 2$ $SU(2)$ theory with $N_f = 4$ fundamental hypermultiplets, and show that the surface defect partition function obeys the BPZ \cite{14} equation, whose $\varepsilon_1 \to 0$ limit gives the Painlevé VI equation (PVI, for short). We also study more sophisticated defects and show that their partition functions can be organized into a horizontal section $\Upsilon$ of a meromorphic flat connection on a 4-punctured sphere. Then we state the blowup formula in the presence of the surface defect and establish the relation between the PVI Hamilton-Jacobi potential and the partition functions of the bulk theory taken at the special $\Omega$-background locus $\varepsilon_1 + \varepsilon_2 = 0$. In this way we recover the GIL formula.

In the appendix we recall a few facts about the geometry of the blown up $\mathbb{C}^2$, and the arguments leading to the blowup formula. We also indicate how one extends the gauge origami constructions of \cite{130, 132} to the simplest nontrivial toric Calabi-Yau fourfold $\mathcal{Y}$: the product of resolved conifold $\mathcal{Z}$ and a complex line, $\mathcal{Y} = \mathcal{Z} \times \mathbb{C}$, in other words a total space of the $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{O}$ bundle over $\mathbb{P}^1$. The key point is that the blowup $\hat{\mathbb{C}}^2$ is a hypersurface of the resolved conifold $\mathcal{Z}$.

Many crucial points, in particular the relation between the potential $\mathbf{S}$, Coulomb moduli $\mathbf{a}$, and the monodromy of the meromorphic connections, as well as the detailed relation between $\Upsilon$ and intersecting defects, will be established in the companion paper \cite{83}.

Acknowledgements. I thank M. Bershtein and A. Zamolodchikov for discussions in 2013 which helped to sharpen the formula \cite{1}, and to S. Jeong for numerous discussions and collaboration in \cite{83}. I am also grateful to C. LeBrun, I. Krichever, H. Nakajima and A. Okounkov for their comments. My interest in blowup formulas goes back to 1996-1997, thanks to the numerous discussions with A. Losev aimed at solving the constraints the electromagnetic duality of effective theory imposes on the contact terms between supersymmetric non-local observables, while the $qq$-characters, the main technical tool of our analysis, grew out of our work with V. Pestun \cite{135}.

When \cite{53} came out, we immediately pointed out the connection of their formula to the $\varepsilon_2 = -\varepsilon_1$, $SU(2)$, $N_f = 4$ supersymmetric partition function and to the blowup equations \cite{131}. Since then a few approaches to the formula \cite{53} were put forward \cite{118, 116, 119}, as well as some generalizations \cite{65}. Our approach is different and in some sense more direct as it produces the result in the GIL form. I lectured about it at the IHES, Saint Petersburg Steklov Mathematics Institute, Harvard University (CMS and Physics Department), Skoltech Center for Advanced Studies, Northeastern University, Department of Mathematics at the Higher School of Economics, at the Physics Department at Ludwig Maximilian University, at the Imperial College London, at the Department of Mathematics at Columbia University. I am grateful to these institutions for their hospitality and to the audiences for useful discussions.

The need to go to extra dimensions to make the formula look natural may sound, to algebraically minded, like proving the main theorem of algebra using topology. But we always liked that proof \cite{6}....
2. BPS/CFT correspondence

In the early days of exact computations in supersymmetric gauge theories [160, 127, 105, 128, 132], which were built on the works of H. Nakajima [119, 120, 121, 122], it was observed, that the correlation functions of selected observables, including the partition function, coincide with conformal blocks of some two dimensional conformal field theories, or, more generally, are given by the matrix elements of representations of some infinite dimensional algebras, such as Kac-Moody, Virasoro, or their $q$-deformations. In [127] this phenomenon was attributed to the chiral nature of the tensor field propagating on the worldvolume of the fivebranes. The fivebranes ($M5$ branes in $M$-theory and $NS5$ branes in $IIA$ string) were used in [92], [161] to engineer, in string theory setup, the supersymmetric systems whose low energy is described by $\mathcal{N} = 2$ supersymmetric gauge theories in four dimensions. This construction was extended and generalized in [52]. This correspondence, named the BPS/CFT correspondence in [130] has been supported by a large class of very detailed examples in [132, 2, 3], and more recently in [87, 91, 72, 73].

In this paper we shall be mostly dealing with the $\mathcal{N} = 2$ supersymmetric theory with gauge group $SU(2)$ and $N_f = 4$ matter hypermultiplets in fundamental representation. In [2] the partition function of this theory is identified with the 4-point conformal block of Liouville theory. In [8, 87] the expectation value of a regular surface defect in that theory is conjectured to be given by the $\mathfrak{sl}_2$ conformal block. Earlier, in [130], we presented less precise conjectures about the expectation values of surface defects and Knizhnik-Zamolodchikov [94] (KZ) equations. The surface defects or surface observables in Donaldson theory and its generalizations were studied in [99, 106, 24, 130, 69]. In [132], using the construction of [46] and the theory of $qq$-characters, the expectation value of certain surface defects has been rigorously shown to obey the equations, which are equivalent to Belavin-Polyakov-Zamolodchikov [14] and Knizhnik-Zamolodchikov [94] equations [138].

In this paper we utilize these results to deduce a nontrivial property of conformal blocks of two dimensional quantum field theories from a natural property of correlation functions of four dimensional supersymmetric gauge theories, adding one more piece to the puzzling complete set of constraints on a quantum field theory.

3. Gauge theory and defects

$\mathcal{N} = 2$ supersymmetric theory can be coupled to $\mathcal{N} = 2$ supergravity. In the limit $M_{Planck} \rightarrow \infty$ it becomes non-dynamical. Among the supergravity backgrounds the ones with global fermionic symmetries $Q$ are important as the correlation functions of $Q$-invariant operators, in particular, the partition function, exactly coincide with those of a potentially much simpler theory whose fields are the $Q$-invariant fields of the original theory. In other words, the superversion of Kaluza-Klein reduction is exact (recall that Kaluza-Klein reduction of gravity theory on the manifold $X$ with $G$-isometry is the approximation to the gravity theory on $B = X/G$ coupled to $G$-gauge theory and a sigma model valued in the space of $G$-invariant metrics on the generic
fiber $F$ of the projection $X \to X/G$). The simplest, in which the background $SU(2)$ $R$-symmetry connection is taken to coincide with the $SU(2)_R$-part of the $Spin(4) = SU(2)_L \times SU(2)_R$ spin connection, is Donaldson-Witten twist. The resulting $(4|8)$-dimensional supermanifold on which the theory lives is split $S_M = \Pi E_M$ where $E_M$ is the vector bundle over the ordinary four-dimensional manifold $M$, $E = T_M + \Lambda^2 + T^*_M \oplus \mathbb{R}$ is the sum of the tangent bundle, the bundle of self-dual two-forms, and a one dimensional trivial vector bundle.

There are other backgrounds of $\mathcal{N} = 2$ supergravity, admitting global fermionic symmetries, albeit for a restricted class of underlying four-manifolds $M$. These backgrounds, starting with the $\Omega$-background in [127], Pestun’s background in [143], and their generalizations [32], can be used to perform exact computations in supersymmetric theories, learning about the effective low-energy action, gravitational topological susceptibility and other interesting aspects of the theory.

In this paper we shall be dealing with the $\Omega$-backgrounds, which are associated with four-manifolds $M$ with isometries. Remarkably, the important parameters, such as the central charge $c$ or background charge $Q$, of the two dimensional conformal field theories which are BPS/CFT duals of theories subject to such backgrounds, depend on the $\Omega$-background parameters $(\varepsilon_1, \varepsilon_2)$ [2].

Moreover, the limit, where one of the parameters of the $\Omega$-background, e.g. $\varepsilon_1$, goes to zero, corresponds to the semiclassical, $c \to \infty$, limit of the corresponding CFT. The relations [1], [3] then relate this limit to the limit $\varepsilon_1 + \varepsilon_2 \to 0$, which corresponds to the “free fermion” $c = 1$ point.

3.1. Gauge theory on the blowup. Let us compare the gauge theory on the supermanifolds $S_M$ and $S_M'$ for which the underlying four-manifolds $M$ and $M'$ differ only in the neighborhood of a point $p \in M$. For example, let $M'$ be obtained by cutting a small ball $B^3_\varepsilon(p)$ centered at $p$, with the boundary $S^3_\varepsilon = \partial B^4_\varepsilon(p)$, and contracting the orbits of Hopf $U(1)$ action on the latter. In other words, in the local model of $S^3_\varepsilon$

$$|u_1|^2 + |u_2|^2 = \varepsilon^2$$

(6)

with $(u_1, u_2) \in \mathbb{C}^2 \approx \mathbb{R}^4 \approx T_p M$, we identify the point $(u_1, u_2)$ with the point $(u_1 e^{i\theta}, u_2 e^{i\theta})$ for any $\theta$. The boundary $S^3_\varepsilon$ disappears, leaving a non-contractible copy of $\mathbb{C}P^1$ called the exceptional curve (or, more generally, an exceptional set). The resulting space $M'$ is the blowup of $M$ at the point $p$. Different parameterizations of the neighborhood $B^4_\varepsilon(p)$ of $p$ lead to different spaces, but for small $\varepsilon$ they are all diffeomorphic.

Now consider a quantum field theory on the manifolds $M$ and $M'$. Since $M \setminus B^4_\varepsilon(p)$ and $M' \setminus \{\text{small, size } \sim \varepsilon\text{ neighborhood of the exceptional set}\}$ are diffeomorphic, the topological quantum field theories on $M$ and $M'$ share the same space of states, defined on a small three-sphere. On $M$ this sphere is the boundary of $B^4_\varepsilon(p)$ while on $M'$ it is a boundary of a tubular neighborhood of the exceptional curve, the total space of the bundle of small circles over $\mathbb{C}P^1$. The partition functions $Z_M$ and $Z'_M$ are, therefore, the matrix elements between the vector (out) representing the path integral over $M \setminus B^4_\varepsilon(p)$ and the vectors $|B^4\rangle$, and $|\mathbb{C}P^2 \setminus B^4\rangle$, representing the path integrals over the fillings of the three-sphere inside $M$ and $M'$, respectively.
Now let us use the state-operator correspondence, and represent these vectors by local operators. The state $|\tilde{B}^4\rangle$ corresponds, naturally, to the unit operator $1$, while the state $|\tilde{\mathbb{C}P}^2\setminus\tilde{B}^4\rangle$ is, in general, a nontrivial element of the topological ring. It was argued in $[108]$ for asymptotically free theories, and shown in $[123]$ for $SU(n)$ theory with up to $2n$ fundamental hypermultiplets, that the state $|\tilde{\mathbb{C}P}^2\setminus\tilde{B}^4\rangle$ also corresponds to the unit operator in the topological ring:

$$Z_{M'} = Z_M$$

(7)

Everything above can be generalized to the equivariant topological field theories, which are defined on four manifolds with some symmetry. The generalization, in which the topological nilpotent supercharge is deformed to the equivariant differential $Q_V$, which squares to the infinitesimal isometry $V$, is called the $\Omega$-deformation, the corresponding supergravity background the $\Omega$-background.

The computations of correlation functions of the $Q_V$-closed observables in the $\Omega$-background on four-manifold $M^4$ are facilitated by the fact that the $Q_V$-invariant field configurations tend to be concentrated near the $V$-fixed points on $M^4$, so that for noncompact $M^4$ (cf. $[129]$):

$$Z_{M^4} = \sum_{\text{fluxes}} \prod_{\text{fixed points}} Z_{\text{loc}}$$

(8)

where by fluxes we mean the magnetic fluxes (in the low-energy effective theory) passing through compact two-cycles on $M^4$. The noncompactness here is important, as the scalar $\phi$ in the vector multiplet has an expectation value fixed by the boundary conditions at infinity.

3.2. Blowup formulas. Since the work of Fintushel and Stern $[47]$ on invariants of smooth 4-manifolds, comparing the Donaldson invariants of $M$ and $M' \approx M \# \mathbb{C}P^2$ has been a fruitful exercise in Donaldson theory. In fact, it was in $[47]$ where the connection of Donaldson theory to a family of elliptic curves was found. The very same family of elliptic curves was later found in $[148]$ to govern the exact low-energy effective action of the theory. In $[108]$ and later in $[107]$ these observations were turned around into a way of testing Seiberg-Witten theory of a larger class of $\mathcal{N} = 2$ gauge theories in four dimensions. Finally, in $[123]$ the equivariant version of the blowup relations was proven, leading to the proof of the $[127]$ conjecture. We should point out that in the literature the term “blowup formula” $[47], [118], [63], [108], [123]$ usually refers to the representation of the expectation value of some observables associated to the exceptional cycle on $M'$, as a local operator in the theory on $M$. This local operator represents the point-like (codimension four) defect in supersymmetric gauge theory. In what follows we shall discuss other types of defects.

3.3. Surface defects in gauge theory. Supersymmetric surface defects in $\mathcal{N} = 2$ gauge theories are two dimensional sigma models with global $G$-symmetry coupled to four dimensional gauge theories with gauge group $G$. The surface defect as an operator depends on some continuous parameters, such as the complexified Kähler moduli of the corresponding sigma model. It was expected from the early days of studies of
dualities that these defects obey differential equations involving the parameters of the
defect, and the bulk parameters, i.e. the parameters of the gauge theory in the ambient
four dimensional space. Moreover, these equations are identical to the Ward identities
of some two dimensional conformal theories. Some of these relations are rigorously
established [132].

3.3.1. Regular defects. There are several practical (i.e. convenient for computations)
realizations of surface defects in $\mathcal{N} = 2$ gauge theories. The construction we use is
inspired by [46], [3], it can be interpreted as a $\mathbb{Z}_n$-orbifold (generalizing those studied
in [38]) of $U(n)$ gauge theory.

The regular surface defect is defined by imposing a $\mathbb{Z}_n$-orbifold projection on the
space of fields of the $\Omega$-deformed $\mathcal{N} = 2$ theory, where the group $\mathbb{Z}_n$ acts both on the
physical space by $2\pi n$-rotation in one of the two orthogonal two-planes $\mathbb{R}^2 \subset \mathbb{R}^2 \oplus \mathbb{R}^{1,1} \approx \mathbb{R}^{1,3}$ in physical spacetime and in the color and flavor spaces. The path integral over
the space of projected fields can be interpreted as the path integral in gauge theory
defined on the quotient space $\mathbb{R}^{1,3}/\mathbb{Z}_n \approx \mathbb{R}^{1,3}$ in the presence of the defect located
at the surface $0 \times \mathbb{R}^{1,1}$, where $0 \in \mathbb{R}^2$ is the fixed point of the $\mathbb{Z}_n$ action. We shall
work in the Euclidean spacetime, with the complex coordinates $(z_1, z_2)$ on the covering
space, and the coordinates $(z_1, z_2)$ on the quotient space, the defect being located at
the $\{(z_1, 0) | z_1 \in \mathbb{C}\}$ plane.

3.3.2. Parameters of the theory with defect. The regular surface defect has both con-
tinuous and discrete parameters. The continuous parameters are the fugacities of the
fractional instanton charges. The latter are identified with the dimensions of the $\mathbb{Z}_n$-
isotypical components in the space $K$ of Dirac zeromodes on the covering space:

$$\tilde{K} = \tilde{K}_0 \otimes \mathcal{R}_0 \oplus \ldots \oplus \tilde{K}_{n-1} \otimes \mathcal{R}_{n-1}$$ (9)

where $\mathcal{R}_\omega$ is the nontrivial one dimensional irreducible representations of $\mathbb{Z}_n$, in which
the generator acts as the $\omega$’th root of unity:

$$T_{\mathcal{R}_\omega}[\Omega_n] = e^{2\pi i \omega}$$ (10)

Similarly to (9), the color and flavor (Chan-Paton) spaces decompose as

$$\tilde{N} = \tilde{N}_0 \otimes \mathcal{R}_0 \oplus \ldots \oplus \tilde{N}_{n-1} \otimes \mathcal{R}_{n-1}$$

$$\tilde{M} = \tilde{M}_0 \otimes \mathcal{R}_0 \oplus \ldots \oplus \tilde{M}_{n-1} \otimes \mathcal{R}_{n-1}$$ (11)

The contribution of the moduli space of instantons on the orbifold $\mathbb{C}^2/\mathbb{Z}_n$ is weighed
with the factor

$$\prod_{\omega=0}^{n-1} q^{k_\omega}, \quad k_\omega = \dim K_\omega$$ (12)

which can be further expressed as:

$$q^{k_{n-1}} \times \prod_{\omega=1}^{n-1} e^{2\pi i \omega (k_{n-1}-k_\omega)}$$ (13)
where $q$ is the bulk instanton fugacity, and $t_\omega$, $\omega = 1, \ldots, n-1$ are the Kähler moduli (a sum of the theta angle and $i$ times a Fayet-Illiopoulos term) of the two dimensional theory on the surface of the surface defect. There is also a perturbative prefactor which we define in the next section.

On the covering space $\mathbb{C}^2$ the $U(1) \times U(1)$-torus acts with the weights $(q_1, \bar{q}_2)$, where $\bar{q}_2 = q_2$, corresponding to the standard action on the quotient $\mathbb{C} \times \mathbb{C}/\mathbb{Z}_n = Spec\mathbb{C}[z_1, z_2]$.

In the main body of this paper we shall deal with the $n = 2$ case (the Appendix and [132] present more general discussion). The tautological ADHM complex (see the appendix for more details and [132] for notations), in the $\mathbb{Z}_2$-equivariant setting, splits as

$$\tilde{S} = \left[ \tilde{K} \otimes \tilde{q}_1 \tilde{q}_2 R_1 \to \tilde{K} \otimes (q_1 R_0 \oplus \tilde{q}_2 R_1) \oplus \tilde{N} \to \tilde{K} \right] = \tilde{S}_0[1] \otimes R_0 \oplus \tilde{S}_1[1] \otimes R_1$$

$$\tilde{S}_0 = \left[ q_1 \tilde{q}_2 K_0 \to q_1 \tilde{K}_0 \oplus \tilde{q}_2 \tilde{K}_1 \oplus \tilde{N}_0 \to \tilde{K}_1 \right]$$

$$\tilde{S}_1 = \left[ q_1 \tilde{q}_2 \tilde{K}_1 \to q_1 \tilde{K}_1 \oplus \tilde{q}_2 \tilde{K}_0 \oplus \tilde{N}_1 \to \tilde{K}_0 \right]$$

with $\tilde{N}_{0,1}$ being the $\mathbb{Z}_2$-eigenspaces in the color space. We denote by $(\tilde{a}_0, \tilde{a}_1)$ the eigenvalues of the complex scalar in the vector multiplet (on the covering space) correspond to the $\mathbb{Z}_2$ eigenspaces in the color space:

$$Ch(\tilde{N}_0) = e^{\tilde{a}_0}, \ Ch(\tilde{N}_1) = e^{\tilde{a}_1}, \quad (15)$$

while the masses are encoded by the characters of the flavor spaces:

$$Ch(\tilde{M}_0) = e^{m_{0}^+} + e^{m_0^-}, \ Ch(\tilde{M}_1) = e^{m_1^+} + e^{m_1^-}. \quad (16)$$

To define the instanton contribution in the $\Omega$-deformed theory the global characters $Ch(\tilde{H}_0), Ch(\tilde{H}_1)$ are useful:

$$\tilde{H} = \frac{\tilde{S}}{(1 - q_1 R_0)(1 - \bar{q}_2 R_1)} = \tilde{H}_0 \otimes R_0 \oplus \tilde{H}_1 \otimes R_1 \quad (17)$$

3.3.3. Instanton measure and bulk parameters. The instanton measure, in the fixed point formula, has been computed in [132] (in $SU(2)$ $N = 2^*$ case it was done in [3, 87]). Using the plethystic exponent $E[\cdot]$, it can be written quite simply:

$$\mu = -q_0^{-ch_0(\tilde{H}_0)} q_1^{-ch_0(\tilde{H}_1)} E \left[ \frac{-Ch \tilde{S} Ch \tilde{S}^* + Ch \tilde{S}^* Ch \tilde{M}}{(1 - q_1^{-1} R_0)(1 - \bar{q}_2^{-1} R_1)} \right]_0$$

where the expression in $(\ldots)$ takes values in the representation ring of $\mathbb{Z}_2$, which is isomorphic to $\mathbb{C}[\mathbb{Z}_2]$ (the trivial representation $R_0$ is a unit, while $R_1$ squares to $R_0$), and $(\ldots)_0$ stands for the $R_0$ coefficient. Let us introduce the notation

$$q_0 = -u, \quad q_1 = -qu^{-1}, \quad (19)$$

for the fractional charge fugacities. The instanton with the fractional charges $(k_0, k_1)$ contributes

$$(-u)^{k_0-k_1} q_1^{k_1} \times \text{one-loop contribution} \quad (20)$$
to the partition function $\Psi$. The measure factorizes
\[ \mu = \mu_{\text{bulk}} \cdot \mu_{\text{surface}} \]  
with
\[ \mu_{\text{bulk}} = q^{\frac{a_0^2 + a_1^2 - 2a_0a_1 + a_2(\varepsilon_1 + \varepsilon_2)}{2}} \cdot k_1 \cdot \mathcal{E} \left[ \frac{Ch(S)^* (Ch(M) - Ch(S))}{(1 - q_1^{-1})(1 - q_2^{-1})} \right], \]  
being the usual instanton measure of the $A_1$ theory ($U(n)$ theory with $2n$ fundamental hypermultiplets) with
\[ Ch(S) = Ch(\tilde{S}_0) + Ch(\tilde{S}_1) q_2^{-1} = Ch(N) - (1 - q_1)(1 - q_2)Ch(\tilde{K}_1) q_2^{-1}, \]  
\[ Ch(M) = Ch(\tilde{M}_0) + q_2^{-1}Ch(\tilde{M}_1), \quad Ch(N) = e^{\tilde{a}_0} + q_2^{-1}e^{\tilde{a}_1} \]  
and
\[ \mu_{\text{surface}} = u^{\frac{a_0 - a_1}{2\xi_1} + k_0 - k_1} \cdot \mathcal{E} \left[ \frac{q_2}{1 - q_1} Ch(\tilde{S}_0 - \tilde{M}_0) Ch(\tilde{S}_1)^* \right] \]  
being the purely surface defect contribution. In (22), (25) we dropped a multiplicative constant independent of $\tilde{a}$'s and $k$'s.

3.3.4. A plethora of regular defects. By writing
\[ Ch(N) = e^a + e^{-a}, \quad Ch(M) = \sum_{f=1}^{4} e^{m_f} \]  
we identify:
\[ \tilde{a}_0 = \pm a, \quad \tilde{a}_1 = \mp a + \tilde{\varepsilon}_2, \quad \]  
\[ \tilde{m}_0^+ = m_{s(3)}, \quad \tilde{m}_0^- = m_{s(4)}, \quad \tilde{m}_1^+ = m_{s(1)} + \tilde{\varepsilon}_2, \quad \tilde{m}_1^- = m_{s(2)} + \tilde{\varepsilon}_2. \]  
for some permutations $\pm \in S(2)$, $s \in S(4)$. Obviously, permuting the masses $3 \leftrightarrow 4$ or $1 \leftrightarrow 2$ does not change anything. The exchange $a \leftrightarrow -a$ is a residual gauge symmetry of the bulk theory. However, it is convenient to distinguish the surface defects corresponding to different $u \in S(2)$, as they are permuted by the monodromy in the fractional gauge couplings.

4. Tau-function

We take a pause now. From the world of the four dimensional physics we turn our attention to classical integrable systems, albeit the ones defined over complex numbers. This is partly motivated by the well-known, by now, connection between four dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and integrable systems [61, 33, 113, 114, 78]. Actually, this connection is a $\varepsilon_1, \varepsilon_2 \to 0$ limit of the BPS/CFT correspondence. In the present discussion we shall keep some of the $\Omega$-deformation parameters finite. Let us forget for a moment about gauge theories, conformal field theories, or supersymmetry.

In this section we introduce a notion of $\tau$-function for the classical integrable system. We also formulate and prove the formula relating the $\tau$-function to the Hamilton-Jacobi potential of the same system, in the special cases. The Schlesinger isomonodromy deformation equations (in the case of $2 \times 2$-matrices) fall into that special class. In this case
the \( \tau \)-function coincides with the one introduced by [84]. It plays an important rôle in the discussions of movable singularities of the solutions to isomonodromic deformation equations [111, 21].

4.1. Classical integrability in the non-stationary case. Let \((\mathcal{P}, \omega)\) be a complex symplectic manifold, \(\dim \mathcal{P} = 2r\). Denote by \(\text{Symp}(\mathcal{P}) = \text{Diff}(\mathcal{P}, \omega)\) the group of holomorphic diffeomorphisms \(\mathcal{P}\) preserving \(\omega\). Let \((H_i(q))_{i=1}^r, q \in \mathcal{U}, H_i : \mathcal{P}^{2r} \to \mathbb{C}\) be a family of (generically) functionally-independent Poisson-commuting holomorphic functions, parameterized by a simply-connected \(r\)-dimensional complex manifold \(\mathcal{U}\):

\[
\{H_i, H_j\}_\omega = 0.
\] (28)

Specifically, in some local Darboux coordinates \((y^a, p_y^a)_{a=1}^r\),

\[
\omega = \sum_{a=1}^r dp_y^a \wedge dy^a
\] (29)
on \(\mathcal{P}\), the Eq. (28) reads:

\[
\sum_{a=1}^r \frac{\partial H_j(y, p; q)}{\partial y^a} \frac{\partial H_k(y, p; q)}{\partial p_y^a} = \sum_{a=1}^r \frac{\partial H_k(y, p; q)}{\partial y^a} \frac{\partial H_j(y, p; q)}{\partial p_y^a}
\] (30)

A priori, (28), (30) admit some transformations of the Hamiltonians \(H_i\), e.g. \(H_i \to H_i + \eta^j_{ik} H_j H_k + \ldots\).

4.2. The \( \tau \)-function. Let us restrict this freedom by the additional nontrivial requirement: there exist a coordinate system \((q_i)_{i=1}^r\) on \(\mathcal{U}\), such that

\[
\frac{\partial H_j(y, p; q)}{\partial q_i} = \frac{\partial H_i(y, p; q)}{\partial q_j}
\] (31)

Let us introduce the 1-form \(\mathfrak{h}\) on \(\mathcal{U}\), valued in the Poisson algebra of functions on \(\mathcal{P}\), viewed as Lie algebra:

\[
\mathfrak{h} = \sum_{i=1}^r H_i dq_i
\] (32)

Then (30), (31) can be neatly summarized as:

\[
d_\mathcal{U} \mathfrak{h} = 0, \ [\mathfrak{h}, \mathfrak{h}] = 0,
\] (33)
or, as the flatness of the \(\lambda\)-connection: \(\lambda d_\mathcal{U} + \mathfrak{h}\). Let us denote by \(\mathfrak{h}^\vee\) the corresponding \(\text{Vect}(\mathcal{P})\)-valued 1-form:

\[
\mathfrak{h}^\vee = \sum_{i=1}^r (\omega^{-1} d_\mathcal{P} H_i) \ dq_i
\] (34)

One can, therefore, for any \(\lambda\), look for \(g(\lambda) \in \text{Symp}(\mathcal{P})\), such that

\[
\mathfrak{h}^\vee = \lambda g(\lambda)^{-1} d_\mathcal{U} g(\lambda)
\] (35)
For $\lambda \to \infty$ one has, $g(\lambda) = \exp \lambda^{-1} V + \ldots$, with the Hamiltonian vector field $V = \omega^{-1} dS_0$, $S_0(y, p; q)$ being the obvious potential for $h$, $h = d_{iq}S_0$, or, in coordinates:

$$H_i(y, p; q) = \frac{\partial S_0(y, p; q)}{\partial q_i} \quad (36)$$

In writing (30), (31), (36) we are explicitly showing the arguments of the functions involved, so as to make it clear which arguments are kept intact while taking partial derivatives. Now, the fun fact is that for finite $\lambda \neq 0, \infty$ the Eq. (35) can be also solved. Let us set $\lambda = 1$ (otherwise rescale $H_i$’s). Fix a basepoint $q_0 \in \mathcal{U}$, and define, for $q \in \mathcal{U}$ sufficiently close to $q_0$, a symplectomorphism $g^q_{q_0} \in \text{Symp}(\mathcal{P})$, so that $g^q_{q_0} = id$, as a solution $g^q_{q_0}(y(0), p^0_y) = (y, p_y)$, $y = y \left( y(0), p^0_y; q \right)$, $p_y = p \left( y(0), p^0_y; q \right)$ of the system of Hamilton equations:

$$\frac{dy^a}{dq_i} = \frac{\partial H_i}{\partial y^a}, \quad \frac{dp^a_y}{dq_i} = -\frac{\partial H_i}{\partial p^a_y} \quad (37)$$

with the initial condition

$$y(0) = y \left( y(0), p^0_y; q_0 \right), \quad p^0_y = p \left( y(0), p^0_y; q_0 \right) \quad (38)$$

Now consider the $(1, 0)$-form on $\mathcal{U} \times \mathcal{P}$ (more precisely on a sufficiently small neighborhood of $q_0$)

$$\tilde{h} = (g^q_{q_0})^* h = \sum_{i=1}^{r} \tilde{H}_i dq_i, \quad \tilde{H}_i \left( y(0), p^0_y; q \right) = H_i \left( y \left( y(0), p^0_y; q \right), p \left( y(0), p^0_y; q \right) \right) \quad (39)$$

It is now a simple matter to verify that $d_{iq} \tilde{h} = 0$, therefore locally $\tilde{h} = d_{iq} \log \tau$.

The family $g^q_{q_0}$ of symplectomorphisms can be described, in $y$-polarization, by the generating function $S(y, y(0); q, q_0)$, via:

$$p^a_y = \frac{\partial S}{\partial y^a}, \quad p^0_y = \frac{\partial S}{\partial y^a(0)}, \quad \frac{\partial S}{\partial q_i} = -H_i \left( y, \frac{\partial S}{\partial y}; q \right), \quad \frac{\partial S}{\partial q_{0,i}} = H_i \left( y(0), \frac{\partial S}{\partial y(0)}; q_0 \right) \quad (40)$$

or, in the mixed e.g. $y - p$ polarization, by the generating function $\Sigma(y, p^0; q, q_0)$, via:

$$p^a_y = \frac{\partial \Sigma}{\partial y^a}, \quad y^a(0) = -\frac{\partial \Sigma}{\partial p^a_y}, \quad \frac{\partial \Sigma}{\partial q_i} = -H_i \left( y, \frac{\partial \Sigma}{\partial y}; q \right), \quad \frac{\partial \Sigma}{\partial q_{0,i}} = H_i \left( -\frac{\partial \Sigma}{\partial p^0_y}, p^0; q_0 \right), \quad (41)$$

with

$$\Sigma(y, p^0; q, q_0) = \text{Crit}_{y(0)} \left( S(y, y(0); q, q_0) - \sum_{a=1}^{r} p^0_y y^a(0) \right) \quad (42)$$

What is the connection between $\Sigma, S$, and $\tau$?
4.3. \( \tau \)-function for non-relativistic Hamiltonians. Let us further assume that the Hamiltonians \( H_i \)'s are quadratic functions of the momenta \( p_{y_a} \)'s and auxiliary parameters \( \vartheta_a \)'s:

\[
H_i(y, p; \vartheta) = \sum_{a,b=1}^r g^{ab}_i(y; q)p_{y_a}p_{y_b} + g^a_i(y; q)p_{y_a}\vartheta_b + g_{i|ab}(y; q)\vartheta_a\vartheta_b .
\]  

(43)

Let \( S(y, \alpha; q; \vartheta) \) be the generating function of the canonical transformation sending \((y, p)\) over \( q \in \mathcal{U} \) to the constants \((\alpha, \beta)\) of motions given by the Hamilton equations,

\[
\frac{dy^a}{dq_i} = \frac{\partial H_i}{\partial p_{y^a}}, \\
\frac{dp_{y^a}}{dq_i} = -\frac{\partial H_i}{\partial y^a},
\]

aka the Hamilton-Jacobi potential:

\[
\frac{\partial S(y, \alpha; q; \vartheta)}{\partial q_i} = -H_i\left(y, \frac{\partial S(y, \alpha; q; \vartheta)}{\partial y}; q; \vartheta\right)
\]

(45)

with

\[
p_{y^a} = \frac{\partial S(y, \alpha; q; \vartheta)}{\partial y^a}, \quad \beta_i = \frac{\partial S(y, \alpha; q; \vartheta)}{\partial \alpha_i}
\]

(46)

For example, we can take \( \alpha = y^{(0)} \), \( \beta = p_{y}^{(0)} \). In our subsequent discussion \((\alpha, \beta)\) will parameterize the monodromy data in the Riemann-Hilbert correspondence. We claim:

\[
\log \tau(\alpha, \beta; \vartheta; q) = -\sum_{i=1}^r \alpha_i\beta_i + \\
\left. \left( S(y, \alpha; q; \vartheta) - \sum_{a=1}^r \vartheta_a \frac{\partial}{\partial \alpha_a} S(y, \alpha; q; \vartheta) \right) \right|_{y: \beta = \frac{\partial S}{\partial y}, \vartheta = \frac{\partial S}{\partial \alpha}}.
\]

(47)

**Proof.** We should keep in mind that \( y^a \)'s, defined through \( \beta_i = \frac{\partial S}{\partial \alpha_i} \), depends on \( q, \alpha \) and \( \vartheta \). This dependence is easy to compute, using (45):

\[
0 = \frac{\partial^2 S}{\partial \alpha_j \partial y^a} \frac{dy^a}{dq_i} + \frac{\partial^2 S}{\partial \alpha_j \partial q_i} = \\
\frac{\partial^2 S}{\partial \alpha_j \partial y^a} \left( \frac{dy^a}{dq_i} - \frac{\partial H_i}{\partial p_{y^a}} \right) \Rightarrow \frac{dy^a}{dq_i} = \frac{\partial H_i}{\partial p_{y^a}} \bigg|_{y: \beta = \frac{\partial S}{\partial y}, \vartheta = \frac{\partial S}{\partial \alpha}}
\]

(48)

as

\[
\text{Det} \left| \frac{\partial^2 S}{\partial \alpha \partial y} \right| \neq 0
\]

(49)

for \( S \) to define a symplectomorphism in the \( y - \alpha \) polarization.
Now let us differentiate (with $\alpha, \beta, \vartheta$ fixed):
\[
\frac{\partial}{\partial q_i} \log \tau(\alpha, \beta; \vartheta) = -H_i(y, p; q; \vartheta) + \sum_{a=1}^r \frac{\partial S}{\partial y} \frac{dy^a}{dq_i} - \sum_{b=1}^r \left( \frac{\partial^2 S(y, \alpha; q; \vartheta)}{\partial \vartheta \partial q_i} \bigg|_{y: \beta = \vartheta} \right) =
\]
\[
= \left( p \frac{\partial}{\partial p} + \vartheta \frac{\partial}{\partial \vartheta} - 1 \right) H_i(y, p; q; \vartheta) \bigg|_{y: p = \frac{\alpha S}{\partial y}, \beta = \vartheta} = H_i(y, p; q; \vartheta) \bigg|_{y: p = \frac{\alpha S}{\partial y}, \beta = \vartheta} \quad (50)
\]
where we used (43), (48), and, most importantly:
\[
\vartheta \frac{\partial}{\partial \vartheta} H_i(y, p; q; \vartheta) = \left( \vartheta \frac{\partial}{\partial \vartheta} H_i(y, p; q; \vartheta) \bigg|_{p = \partial S/\partial y} \right) + \sum_{a=1}^r \vartheta \frac{\partial H_i}{\partial p} \frac{\partial^2 S(y, \alpha; q; \vartheta)}{\partial y^a \partial \vartheta} \quad (51)
\]

5. Painlevé VI, Garnier, Gaudin, and Schlesinger systems

An important case of (43) is given by the $SL(2)$ Schlesinger system, which we present in some detail.

Let $p \geq 1$ denote a positive integer, and let $z \subset \mathbb{CP}^1$ denote a finite ordered set of cardinality $p + 3$:
\[
z = \{z_{-1}, z_0, \ldots, z_p\} \quad (52)
\]
Sometimes we set $z_{p+1} = \infty$, $z_0 = 1$, $z_1 = q_1$, $z_2 = q_1 q_2$, $\ldots$, $z_p = q_1 q_2 \ldots q_p$, $z_{-1} = 0$, with $q_1, q_2, \ldots, q_p \in \mathbb{C}[\lambda < 1]$.

There are two $SL(2, \mathbb{C})$ symmetries in our story. One, which we shall denote by $H$, is the two-fold cover of the symmetry group of the $\mathbb{CP}^1$, parametrizing the points $z_i$ above. Say, an element
\[
h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H \quad (53)
\]
acts via
\[
z \mapsto z^h = \begin{pmatrix} az_{-1} + b & az_0 + b & \cdots & az_{p+1} + b \\ cz_{-1} + d & cz_0 + d & \cdots & cz_{p+1} + d \end{pmatrix}
\]
Let us denote by $C_p = \left( (\mathbb{CP}^1)^{p+3} \setminus \text{diagonals} \right)$ the space of the $z$’s. The quotient $C_p/H = \mathfrak{M}_{0, p+3}$ is the moduli space of smooth genus 0 curves with $p + 3$ punctures.
The other $SL(2, \mathbb{C})$ symmetry group, denoted by $G$ (or $G_1$ in later chapters), is the gauge group, meaning it does not act, in itself, on any physical observable. However, in some description of the system it is a symmetry acting on redundant variables. Its rôle is described momentarily.

5.1. The phase space $\mathcal{M}_p^{\text{alg}}$. To begin with, the phase space $\mathcal{M}_p^{\text{alg}}$ is the set of $p + 3$-tuples $(A_\xi)_{\xi \in \mathbb{Z}}$, obeying the equations

$$
\sum_{\xi \in \mathbb{Z}} A_\xi = 0 \tag{55}
$$

modulo the equivalence relation $(A_\xi)_{\xi \in \mathbb{Z}} \sim (g A_\xi g^{-1})_{\xi \in \mathbb{Z}}$ for $g \in G$. Each $A_\xi \in \text{Lie}(G)$ is a $2 \times 2$ traceless, $\text{Tr} A_\xi = 0$, complex matrix, obeying:

$$
\frac{i}{2} \text{Tr} A_\xi^2 = \Delta_\xi = \vartheta_\xi^2. \tag{56}
$$

5.1.1. Complexified two-sphere. Before we proceed, we need to recall a few facts about the orbit of an individual matrix.

Let $\vartheta \in \mathbb{C}$. We denote by $\mathcal{O}_\vartheta \subset \mathbb{C}^3$ the quadric surface

$$
x^2 + y^2 + z^2 = \vartheta^2 \tag{57}
$$

We identify $\mathbb{C}^3$ with the space of $2 \times 2$ traceless matrices of the form:

$$
A(x, y, z) = \begin{pmatrix} z & x + i y \\ x - i y & -z \end{pmatrix}, \tag{58}
$$

which makes $\mathcal{O}_\vartheta$ the space of $A(x, y, z)$, such that $\text{Det} A(x, y, z) = -\vartheta^2$.

It is a complex symplectic manifold with the holomorphic $(2, 0)$ symplectic form:

$$
\Omega_\vartheta = \frac{dx \wedge dy}{2z} \tag{59}
$$

We have, for $\vartheta \neq 0$, two holomorphic maps:

$$
\mathcal{O}_\vartheta \quad \xrightarrow{p_+ \vee} \quad \mathbb{C}P^1 \quad \xrightarrow{p_-} \quad \mathbb{C}P^1 \quad \text{with} \quad \gamma \in \mathbb{C} \quad \text{parameterizing} \quad U_1 \quad \text{and} \quad \tilde{\gamma} \in \mathbb{C} \quad \text{parameterizing} \quad U_\infty \quad \text{so that} \quad \gamma \tilde{\gamma} = 1 \quad \text{on} \quad \mathbb{C} \mathbb{P}^1 \quad \text{covered by} \quad \mathbb{C}P^1 \quad \text{with two coordinate charts} \quad U_+ \cup U_-, \quad \text{with} \quad \gamma \in \mathbb{C} \quad \text{parameterizing} \quad U_+ = \{ (\gamma : 1) \} \quad \text{and} \quad \tilde{\gamma} \in \mathbb{C} \quad \text{parameterizing} \quad U_- = \{ (1 : \tilde{\gamma}) \},
$$

In writing (61) we associate to the vector $\begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$ a point $(\psi_+ : \psi_-) \in \mathbb{C}P^1$ in homogeneous coordinates. This is well-defined as long as $(\psi_+ , \psi_- ) \neq 0$, which is true for either $(z \pm \vartheta , x - iy )$ or $(x + iy , -z \pm \vartheta )$ in (61), or both.

By covering $\mathbb{C}P^1$ with two coordinate charts $U_+ \cup U_-$, with $\gamma \in \mathbb{C}$ parameterizing $U_+ = \{ (\gamma : 1) \}$ and $\tilde{\gamma} \in \mathbb{C}$ parametrizing $U_- = \{ (1 : \tilde{\gamma}) \}$, so that $\gamma \tilde{\gamma} = 1$ on
the intersection $U_+ \cap U_- \cong \mathbb{C}^\times$, we can produce two Darboux coordinate systems\footnote{We choose a non-standard normalization of Darboux coordinates, with an extra $i/2$-factor in the symplectic form, for later convenience} $\{ (\gamma_+, \beta), (\tilde{\gamma}_+, \tilde{\beta}) \}$ on $\mathcal{O}_\theta$, where $\gamma_\pm = \gamma \circ p_\pm$, $\tilde{\gamma}_\pm = \tilde{\gamma} \circ p_\pm$:

\begin{equation}
( z \pm \vartheta : x - iy ) = (x + iy : -z \pm \vartheta) = (\gamma_\mp : 1) = (1 : \tilde{\gamma}_\mp),
\end{equation}

so that

\begin{equation}
\Omega_\theta = -\frac{i}{2} d\beta \wedge d\gamma_\pm = -\frac{i}{2} d\tilde{\beta} \wedge d\tilde{\gamma}_\pm,
\end{equation}

so that the global functions $x, y, z$ are expressed as:

\begin{equation}
\begin{pmatrix} z \\ x + iy \\ x - iy \end{pmatrix} = \begin{pmatrix} \pm \vartheta - \beta \gamma_\pm \\ 2\vartheta \gamma_\pm \\ -\beta \end{pmatrix} = \begin{pmatrix} \tilde{\gamma}_\mp \tilde{\beta} + \vartheta \\ -\tilde{\beta} \\ \tilde{\gamma}_\mp^2 \tilde{\beta} + 2\vartheta \gamma_\pm \end{pmatrix}
\end{equation}

The presence of the shift $\beta \mapsto \tilde{\beta} = -\beta \gamma_\pm^2 \mp 2\vartheta \gamma_\pm$ is what makes the bundle an affine line bundle, as opposed to the more familiar vector line bundle. We can solve (64) to see how “+” coordinates translate to the “−” coordinates:

\begin{equation}
\gamma_+ - \gamma_- = 2\vartheta / \beta, \quad \tilde{\gamma}_+ - \tilde{\gamma}_- = 2\vartheta / \tilde{\beta},
\end{equation}

equivalently, the map $(\beta, \gamma_+) \mapsto (\beta, \gamma_-)$ is generated by the generating function

\begin{equation}
F(\gamma_+, \gamma_-) = 2\vartheta \log(\gamma_+ - \gamma_-).
\end{equation}

When $\vartheta = 0$ something special happens. The singular quadric $x^2 + y^2 + z^2 = 0$ has a resolution $\tilde{\mathcal{O}}_0$ of singularity at $x = y = z = 0$, which is the space of pairs $\{ (A(x, y, z), \Upsilon) \mid \det A(x, y, z) = 0, \Upsilon \approx \mathbb{C} \subset \ker A(x, y, z) \}$. For $(x, y, z) \neq (0, 0, 0)$ the kernel is one-dimensional and is uniquely determined by the matrix $A(x, y, z)$. For $(x, y, z) = (0, 0, 0)$ the choices of a one-dimensional kernel span a copy of $\mathbb{C}\mathbb{P}^1$. The projection $\tilde{\mathcal{O}}_0 \to \mathbb{C}\mathbb{P}^1$ (forgetting $A(x, y, z)$) identifies $\tilde{\mathcal{O}}_0$ with the cotangent bundle $T^*\mathbb{C}\mathbb{P}^1$.

The formulae (64) become, then, the standard coordinates on $T^*\mathbb{C}\mathbb{P}^1$ with $\gamma$ and $\tilde{\gamma}$ covering the base $\mathbb{C}\mathbb{P}^1$ in the standard way, with $\gamma \tilde{\gamma} = 1$ away from the North and South pole, $\gamma = 0$, and $\tilde{\gamma} = 0$, respectively.

The group $G$ acts on $\mathcal{O}_\theta$ in the usual way:

\begin{equation}
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : A \mapsto gAg^{-1},
\end{equation}

where $g \in G$, i.e. $ad - bc = 1$, which translates to the fractional-linear transformations of $\gamma_\pm$, and affine transformations of $\beta$:

\begin{equation}
(\gamma_\pm, \beta) \mapsto \begin{pmatrix} a\gamma_\pm + b \\ c\gamma_\pm + d \end{pmatrix}, \beta(c\gamma_\pm + d)^2 \mp 2\vartheta c(c\gamma_\pm + d)
\end{equation}
and $\beta(c\gamma_+ + d)^2 - 2\vartheta c (c\gamma_+ + d) = \beta(c\gamma_- + d)^2 + 2\vartheta c (c\gamma_- + d)$ thanks to (65). Notice that the symplectic form written in the $\gamma_\pm$ coordinates has the manifestly $G$-invariant form:

$$\Omega_\vartheta = i\vartheta \frac{d\gamma_+ \wedge d\gamma_-}{(\gamma_+ - \gamma_-)^2}$$

(69)

The symplectic form $\Omega_\vartheta$ is, for $\vartheta \neq 0$, cohomologically nontrivial. Indeed, there is a compact two-cycle $S \subset O_\vartheta$, given by: $(x, y, z) = \vartheta(n_1, n_2, n_3)$, with $(n_1, n_2, n_3) \in \mathbb{R}^3$, with

$$n_1^2 + n_2^2 + n_3^2 = 1.$$  

(70)

On this cycle $\bar{\gamma}_+ = -\bar{\gamma}_-$, $\bar{\gamma}_- = -\bar{\gamma}_+$, $\bar{\gamma}_-\gamma_+ = -1$, $\bar{\gamma}_+\gamma_- = -1$. It is easy to calculate

$$\frac{1}{2\pi} \int_S \Omega_\vartheta = \pm \vartheta,$$

(71)

the sign depending on the choice of the orientation of $S$. It is not canonical, as $S$ is not a holomorphic curve. The cycle $S$ is not, of course, $G$-invariant. However, it is preserved by the maximal compact subgroup of $G$, isomorphic to $SU(2)$.

Now, let us discuss the geometry of the eigenbundles $L_\pm$ over $O_\vartheta$, first, for $\vartheta \neq 0$. The line $L_\pm$ over $(x, y, z) \in O_\vartheta$ is the $\pm \vartheta$ eigenline $\Upsilon^{\pm} \subset \mathbb{C}^2$ of $A(x, y, z)$:

$$A(x, y, z) \Upsilon^{(\pm)} = \pm \vartheta \Upsilon^{(\pm)},$$

(72)

We can trivialize $L_\pm$ over $U_{-\pm}$ by:

$$\Upsilon^{(\pm)} = \mathbb{C} \cdot \begin{pmatrix} z \pm \vartheta \\ x - iy \end{pmatrix}$$

(73)

and, over $U_{+\pm}$ by:

$$\Upsilon^{(\pm)} = \mathbb{C} \cdot \begin{pmatrix} -x - iy \\ z \mp \vartheta \end{pmatrix}.$$  

(74)

The scalar factors taking (73) to (74)

$$\frac{z \mp \vartheta}{x - iy} = \gamma^{\pm}$$

(75)

are the transition functions defining the holomorphic line bundles $L_\pm$ over $O_\vartheta$. They are well-defined on $\mathcal{U}$. Thus

$$L_\pm = p_\pm^* \mathcal{O}(-1)$$

(76)

The case $\vartheta = 0$ is special, in that $A(x, y, z)$ for $(x, y, z) \neq 0$ is not diagonalizable. It has a Jordan block form, with the eigenline $\ker A(x, y, z)$ spanned by

$$\Upsilon \propto \begin{pmatrix} z \\ x - iy \end{pmatrix}$$

(77)

away from the $x = iy$, $z = 0$ line, and by

$$\Upsilon \propto \begin{pmatrix} -x - iy \\ z \end{pmatrix}.$$  

(78)

---

3This relation is responsible for the so-called Berry phase [17]
away from the $x = -iy, z = 0$ line. The bundle of eigenlines extends to the $(x, y, z) = 0$
locus on the resolution $T^* \mathbb{CP}^1$. Of course, the matrix $A(0, 0, 0) = 0$ has the two
dimensional space of zero eigenstates, however a point $(\gamma_0 : \gamma_1)$ of the zero section
$\mathbb{CP}^1 \subset T^* \mathbb{CP}^1$ singles out a one dimensional subspace $\mathbb{C} \left( \frac{1}{\gamma} \right) \subset \mathbb{C}^2$ which is
an eigenspace of the matrix
\[
\beta \left( \begin{array}{cc}
-\gamma & \gamma^2 \\
-1 & \gamma
\end{array} \right) = \tilde{\beta} \left( \begin{array}{cc}
\tilde{\gamma} & -1 \\
\tilde{\gamma}^2 & -\tilde{\gamma}
\end{array} \right)
\] (79)
which, in the limit $\beta \to 0$ or $\tilde{\beta} \to 0$, becomes $A(0, 0, 0)$.

5.1.2. $\mathcal{M}^{\text{alg}}$ as symplectic quotient. We can now return back to the vicinity of the Eqs.
(55) and write
\[
\mathcal{M}^{\text{alg}}_p = \left( \times_{\xi \in z} O_{\theta_\xi} \right) // G
\] (80)
where $A_\xi = A(x_\xi, y_\xi, z_\xi) \in O_{\theta_\xi} \subset \mathbb{C}^3$, is its moment map, and $//$ stands for taking the
symplectic quotient, which means imposing the moment map equation (55) and then
taking the quotient (with subtleties having to do with the noncompactness of $G$
swept under a rug) with respect to the action of $G$. The product of the coadjoint orbits in
(80) has the symplectic form:
\[
\Omega = \sum_{\xi \in z} \Omega_{\theta_\xi}
\] (81)
which descends to the symplectic form $\varpi$ on $\mathcal{M}^{\text{alg}}_p$.

Let us denote by $a \in \mathcal{M}_p$ the gauge equivalence class of the $p + 3$-tuple $a = [(A_\xi) \sim (g A_\xi g^{-1})]$.

5.1.3. Symplectic and Poisson structures. Computing $\varpi$ is somewhat cumbersome, however it is easy to compute the Poisson brackets of functions of matrix elements
of $A_\xi$’s:
\[
\{x_\xi, y_\eta\} = z_\eta \delta_{\xi,\eta}, \quad \{y_\xi, z_\eta\} = x_\eta \delta_{\xi,\eta}, \quad \{z_\xi, x_\eta\} = y_\eta \delta_{\xi,\eta}.
\] (82)
Let us define algebraic functions on $\mathcal{M}^{\text{alg}}_p$:
\[
h_{\xi\eta}(a) = \text{Tr} A_\xi A_\eta = h_{\eta\xi}(a), \quad \xi, \eta \in z, \ a \in \mathcal{M}_p
\] (83)
out of which we build the functions on $\mathcal{M}^{\text{alg}}_p \times \mathbb{C}_p$:
\[
H_\xi(a, z) = \sum_{\eta \neq \xi} \frac{h_{\eta\xi}}{\xi - \eta}
\] (84)
We shall view them, up to a small subtlety, as “time”-dependent Hamiltonians on $\mathcal{M}^{\text{alg}}_p$.
It is easy to show that $H_\xi, H_\eta$ Poisson-commute, and obey $\partial H_\xi / \partial \eta = \partial H_\eta / \partial \xi$, as in
(31). Define $(1, 0)$-form on $\mathbb{C}_p$ valued in functions on $\mathcal{M}^{\text{alg}}_p$:
\[
\mathcal{F} = \sum_{\xi \in z} H_\xi d\xi = \frac{1}{2} \sum_{\xi \neq \eta \in z} h_{\xi\eta} d\log(\xi - \eta)
\] (85)
Thanks to $\sum_{\xi \in z} A_\xi = 0$, we have the relations, cf. (56):

$$
\sum_{\xi \in z} \left( \begin{array}{c}
H_\xi \\
\xi H_\xi - \Delta_\xi \\
\xi^2 H_\xi - 2\Delta_\xi \xi
\end{array} \right) = 0,
$$

(86)

which imply that (85) is covariant with respect to the $H$-action, cf. (53):

$$
h^*\Xi = \Xi + 2 \sum_{\xi \in X} \Delta_\xi d\log (c\xi + d)
$$

(87)

As in our general discussion,

$$d_z \Xi = 0, \quad D \Xi = 0
$$

(88)

where $d_z$ stands for the exterior derivative on $C_p$, and

$$D = d_z + \sum_{\xi \in X} d\xi \wedge \{H_\xi, \cdot\}_a
$$

(89)

where we put the subscript $\{\cdot, \cdot\}_a$ to stress that the Poisson brackets are on $\mathcal{M}_p^{\text{alg}}$.

5.2. 

**Isomonodromy equations.** Let us now introduce the Schlesinger system [147]. It is simply the collection of flows, generated by the Hamiltonians $H_\xi$:

$$
\frac{\partial A_\xi}{\partial \eta} = \frac{[A_\xi, A_\eta]}{\xi - \eta} (1 - \delta_{\eta, \xi}) - \delta_{\eta, \xi} \sum_{\xi \neq \zeta} \frac{[A_\xi, A_\zeta]}{\xi - \zeta}
$$

(90)

They remarkable property is that the monodromy data of

$$\nabla = dz (\partial_z + A(z)), \quad A(z) = \sum_{\xi \in z} \frac{A_\xi}{z - \xi},
$$

(91)

remains fixed along the flow. Indeed, (90) imply, for every $\xi \in z$:

$$
\frac{\partial}{\partial \xi} \nabla = [\nabla, \frac{A_\xi}{z - \xi}]
$$

(92)

In the case $r = 1$ the equations (90) read, more explicitly:

$$
\frac{DA_0}{Dq} = \frac{[A_0, A_q]}{q}, \quad \frac{DA_1}{Dq} = \frac{[A_1, A_q]}{q - 1}, \quad \frac{DA_q}{Dq} = \frac{[A_q, A_0]}{q} + \frac{[A_1, A_q]}{1 - q},
$$

(93)

where $D/Dq = d/dq + [B, \cdot]$, with $B$ a compensating gauge transformation (recall that the residues $A_\xi$ are defined up to an overall conjugation). To avoid dealing with $B$ we can consider the $q$-evolution of the gauge-invariant functions $h_{\xi q}$, so that in the $r = 1$ case:

$$
\frac{d}{dq} h_{0q} = \frac{\tilde{h}_{01q}}{1 - q}, \quad \frac{d}{dq} h_{1q} = \frac{\tilde{h}_{01q}}{q}
$$

(94)

where

$$
\tilde{h}_{01q} = \text{Tr} ([A_0, A_1] A_q)
$$

(95)
Let $\vec{\alpha} = (\alpha_1, \ldots, \alpha_r), \vec{\beta} = (\beta_1, \ldots, \beta_r)$ denote the parameters of the gauge equivalence classes of the monodromy data or, more specifically, Darboux coordinates $(\vec{\alpha}, \vec{\beta})$ on the moduli space $\mathcal{M}_p^{\text{loc}}$ of flat $G$-connections on the $p + 3$-punctured sphere. By sending the pair $(a, z) \in \mathcal{M}_p^{\text{alg}} \times C_p$ to the monodromy data of $\nabla$ we get a map

$$m : \mathcal{M}_p^{\text{alg}} \times C_p \longrightarrow \mathcal{M}_p^{\text{loc}}$$

(96)

which commute with the flows (90), in other words, the orbits of the flows belong to the fibers of $m$. Now, (88) imply, that locally on $m^{-1}(\vec{\alpha}, \vec{\beta})$ the $(1,0)$-form $\Sigma$ is exact:

$$\Sigma \bigg|_{m^{-1}(\vec{\alpha}, \vec{\beta})} = dS$$

(97)

where the potential $S$ can be viewed as a function on $C_r \times \mathcal{M}_p^{\text{loc}}, S = S(z; \vec{\alpha}, \vec{\beta})$, obeying

$$\text{D}S(z; \vec{\alpha}, \vec{\beta}) = \Sigma$$

(98)

Finally, note that the connection $\nabla$ is $H$-covariant, in that

$$h^*\nabla(a, z) = \nabla(a, z^h)$$

(99)

where we set $A_h(\xi) = A_\xi$ for all $\xi \in z$. Thus, the monodromy data is $H$-invariant (provided we transform the combinatorial data, such as the generating loops of $\pi_1(S^2 \setminus z, pt)$). In this way the map $m$ descends to the quotient

$$\tilde{m} : \mathcal{M}_p^{\text{alg}} \times \widetilde{\mathcal{M}}_{0,p+3} \longrightarrow \mathcal{M}_p^{\text{loc}}$$

(100)

where $\widetilde{\mathcal{M}}_{0,p+3} \rightarrow \mathcal{M}_{0,p+3}$ is a finite cover, keeping track of the additional combinatorics involved in describing the monodromy data.

5.2.1. Schlesinger $\tau$-function. The $\tau$-function for the $p + 3$-point Schlesinger problem is defined through the potential $S(z; \vec{\alpha}, \vec{\beta})$:

$$\tau(q_1, \ldots, q_r; \vec{\alpha}, \vec{\beta}) = e^{S(z; \vec{\alpha}, \vec{\beta})} \prod_{\xi \neq \eta} (s - \eta)^{D_{\eta \xi}},$$

(101)

where the symmetric, $D_{\eta \xi} = D_{\xi \eta}$, matrix $\| D_{\eta \xi} \|$, linear in $\partial_2^2$, obeys

$$\sum_{\eta \neq \xi} D_{\eta \xi} = 2 \partial_2^2.$$  

(102)

The $\tau$-function is defined on the product

$$\widetilde{\mathcal{M}}_{0,p+3} \times \mathcal{M}_p^{\text{loc}}$$

(103)

of the universal cover of the moduli space of complex structures of $S^2 \setminus z$ and the moduli space of $G$-flat connections on $S^2 \setminus z$ with fixed conjugacy classes of monodromy around individual punctures. To stay away from complications [21], we assume all conjugacy classes generic.

4We hope there is no confusion between the $\beta_i$ coordinates on the moduli space of local systems and the Darboux coordinates on the orbits $O_{\beta_i}$’s.

5The precise construction depends on additional combinatorial data.
6. DARBOUX COORDINATES

Let us describe the system (90) more explicitly, in some Darboux coordinates on $M_p^{\text{alg}}$. Basically, there are two classes of coordinate systems on $M_p^{\text{alg}}$: 1) the $z$-dependent action-angle [7] variables $(a_i, \varphi_i)^{p}_{i=1}$, or separated variables [154], which we shall call the $w$-coordinates $(w_i, p_{w_i})^{p}_{i=1}$, and 2) what we call the background-independent, or $y$-coordinates $(y^a, p_{y^a})^{p}_{a=1}$. For the latter there are several interesting possibilities, which we review below. See [10] for some earlier work.

Throughout this section we shall use the notation

$$\vec{j} = (\vartheta_0, \vartheta_1, \vartheta_1, \vartheta_\infty) \in \mathbb{C}^4$$

(104)

6.1. Background independent coordinates.

6.1.1. The $y$-coordinates. Using the coordinates (64) on $O_\vartheta$ we can represent the symplectic quotient $M_p$ [80] as the set of $2|z| = 2(p + 3)$-uples $(\gamma_{\pm, \xi}, \beta_\xi)_{\xi \in z}$, obeying

$$\sum_{\xi \in z} \beta_\xi = \sum_{\xi \in z} \gamma_{\pm, \xi} \beta_\xi \mp \vartheta_\xi = \sum_{\xi \in z} \gamma_{\pm, \xi}^2 \beta_\xi \mp 2 \vartheta_\xi \gamma_{\pm, \xi} = 0$$

(105)

(and similarly in the $(\tilde{\gamma}_{\pm, \xi}, \tilde{\beta}_\xi)$-coordinates) modulo the diagonal $G$-action [68]:

$$(\gamma_{\pm, \xi}, \beta_\xi)_{\xi \in z} \mapsto \left( \frac{a \gamma_{\pm, \xi} + b}{c \gamma_{\pm, \xi} + d}, \beta_\xi (c \gamma_{\pm, \xi} + d)^2 \mp 2 \vartheta_\xi c(c \gamma_{\pm, \xi} + d) \right)_{\xi \in z}$$

(106)

The idea behind the $y$-parametrization is the observation that the $\gamma_{\pm, \xi}$-coordinates transform under the $G$-action independently of $\beta_\xi$'s. We thus may project an open subset $U_\epsilon$, with $\epsilon = (\epsilon_\xi)_{\xi \in z}$, $\epsilon_\xi = \pm$ of $M_p^{\text{alg}}$ (the one where the $\gamma_{\epsilon, \xi}$'s are well-defined, i.e. away from a codimension one locus, cf. the discussion below the Eq. (64), and also are different from each other) to $M_{0, p + 3}$, which, in turn, can be parameterized by

$$y^a_\epsilon = \frac{(\gamma_a - \gamma_{-1})(\gamma_0 - \gamma_{p+1})}{(\gamma_0 - \gamma_{-1})(\gamma_a - \gamma_{p+1})}, \quad a = 1, \ldots, p$$

(107)

where we use the short-hand notation $\gamma_i = \gamma_{\epsilon_i, z_i}$, $i = -1, \ldots, p + 1$, and $p_{y^a}$, defined through the linear relations:

$$\beta_i = \sum_{i \neq j} \frac{\Theta_{ij}}{\gamma_i - \gamma_j} + \sum_{a=1}^r p_{y^a} \frac{\partial y^a}{\partial \gamma_i}$$

(108)

where $\Theta_{ij} = \Theta_{ji}$ is any symmetric matrix, which we assume linear in $$(\epsilon_\xi, \vartheta_\xi)_{\xi = 1}^{r+3}$, obeying

$$\sum_{j \neq i} \Theta_{ij} = 2 \epsilon_{z_i} \vartheta_{z_i}.$$ 

(109)

in order to solve (105). There are many choices for the $\Theta$-matrix, which makes the choice of $p_{y^a}$’s non-canonical:

$$p_{y^a} \rightarrow p_{y^a} + \partial_{y^a} f(y)$$

(110)

with

$$f(y) = \sum_{i \neq j} \delta \Theta_{ij} \log(\gamma_i - \gamma_j)$$

(111)
for any symmetric matrix $\delta \Theta$ obeying $\sum_{i \neq j} \delta \Theta_{ij} = 0$ for any $j$.

For example, for $r = 1$, $\mathbf{z} = \{0, q, 1, \infty\}$, we may take, for $\epsilon = (+, +, +, +)$,

$$ y = y^1 = \frac{\gamma_{+, q} - \gamma_{+, 0} \gamma_{+, 1} - \gamma_{+, \infty}}{\gamma_{+, 1} - \gamma_{+, 0} \gamma_{+, q} - \gamma_{+, \infty}} ,$$

and

$$ \beta_0 = \frac{\vartheta_0 + \vartheta_q - \vartheta_1 + \vartheta_{\infty}}{\gamma_{+, 0} - \gamma_{+, q}} + \frac{\vartheta_0 - \vartheta_q + \vartheta_1 - \vartheta_{\infty}}{\gamma_{+, 0} - \gamma_{+, 1}} + \frac{\partial y}{\partial \gamma_{+, 0}} p_y $$

$$ \beta_q = \frac{\vartheta_0 + \vartheta_q - \vartheta_1 + \vartheta_{\infty}}{\gamma_{+, q} - \gamma_{+, 0}} + \frac{\vartheta_0 - \vartheta_q - \vartheta_1 + \vartheta_{\infty}}{\gamma_{+, 1} - \gamma_{+, q}} + \frac{\partial y}{\partial \gamma_{+, q}} p_y $$

$$ \beta_1 = \frac{\vartheta_0 - \vartheta_q + \vartheta_1 - \vartheta_{\infty}}{\gamma_{+, 1} - \gamma_{+, 0}} + \frac{\vartheta_0 - \vartheta_q - \vartheta_1 + \vartheta_{\infty}}{\gamma_{+, q} - \gamma_{+, 1}} + \frac{2\vartheta_{\infty}}{\gamma_{+, 1} - \gamma_{+, \infty}} + \frac{\partial y}{\partial \gamma_{+, 1}} p_y $$

$$ \beta_{\infty} = \frac{2\vartheta_{\infty}}{\gamma_{+, \infty} - \gamma_{+, 1}} + \frac{\partial y}{\partial \gamma_{+, \infty}} p_y ,$$

The pair $(y, p_y)$ are the Darboux coordinates, i.e. $\omega = dp_y \wedge dy$, on one patch of $\mathcal{M}^{alg}_1$. Changing $\gamma_{+, \xi}$ to $\gamma_{-, \xi}$, accompanied by the change of the corresponding $\vartheta_{\xi}$ to $(-\vartheta_{\xi})$ leads to the canonical transformation $(y, p_y) \mapsto (y_{\xi}, p_{\xi})$ connecting to another patch on $\mathcal{M}^{alg}_1$, as we show in some detail in the appendix. Also, choosing another ordering of $\mathbf{z}$ maps $y$ to $1 - y$, $y^{-1}$ etc. and produces coordinates on other patches. The conjugate momentum $p_y$ changes accordingly, albeit non-canonically, as one can shift it by $\partial_y f$ for some $f(y)$.

The functions $h_{0q}$ and $h_{1q}$, expressed in the $y$-coordinates, are simple polynomials:

$$ h_{0q} = y^2 (1 - y) p_y^2 + y (y (\vartheta_0 - \vartheta_q + \vartheta_1 - \vartheta_{\infty}) + 2(\vartheta_{\infty} - \vartheta_1)) p_y + (\vartheta_1 - \vartheta_{\infty})^2 - \vartheta_q^2 - \vartheta_1^2 ,$$

$$ h_{1q} = y (1 - y) p_y^2 - (1 - y) ((1 - y) (\vartheta_0 - \vartheta_q + \vartheta_1 - \vartheta_{\infty}) - 2(\vartheta_0 + \vartheta_{\infty})) p_y + \vartheta_q^2 - \vartheta_1^2 - \vartheta_{\infty}^2 + 2\vartheta_{\infty}(\vartheta_1 - \vartheta_q) + 2\vartheta_{\infty}y^{-1} (\vartheta_0 + \vartheta_q - \vartheta_1 + \vartheta_{\infty}) .$$

Now, the Hamiltonian $H_q = \frac{h_{0q}}{q} + \frac{h_{1q}}{q-1}$ contains both $p_y^2$ and $p_y$ terms. In order to write down the equations of motion it is convenient to shift the $p_y$ variable

$$ p_y = p_y + \frac{\widehat{\vartheta}_0 - \frac{1}{2}}{y} + \frac{\widehat{\vartheta}_1}{y(y-1)} + q \frac{\widehat{\vartheta}_q - \frac{1}{2}}{y(y-q)} ,$$

where we defined

$$ \widehat{\vartheta}_0 = \frac{1}{2} (\vartheta_0 + \vartheta_q - \vartheta_1 - \vartheta_{\infty}) ,$$

$$ \widehat{\vartheta}_1 = \frac{1}{2} (\vartheta_0 - \vartheta_q - \vartheta_1 + \vartheta_{\infty}) ,$$

$$ \widehat{\vartheta}_q - \frac{1}{2} = \frac{1}{2} (\vartheta_0 + \vartheta_q + \vartheta_1 + \vartheta_{\infty}) ,$$

$$ \widehat{\vartheta}_{\infty} - \frac{1}{2} = \frac{1}{2} (\vartheta_0 - \vartheta_q + \vartheta_1 - \vartheta_{\infty}) .$$
so as to eliminate the terms in $H_q$, linear in $p_y$. In doing so, one should be careful as the shift depends on $q$, so the Hamiltonian will also change, according to:

$$p_y dy - H_q dq = p_y dy - h^y_{PVr}(y, p_y; q; \vec{\vartheta}) dq + d\mathcal{Y},$$

$$\mathcal{Y} = \left(\vartheta_\infty - \frac{1}{2}\right) \log(y) + \left(\vartheta_q - \frac{1}{2}\right) \log(1 - q/y) + \vartheta_i \log(1 - 1/y) - \left(\vartheta_0^2 + \vartheta_q^2\right) \log(q) - \frac{1}{2} \left((\vartheta_0 - \vartheta_\infty)^2 - (\vartheta_q + \vartheta_1)^2\right) \log(1 - q), \quad (117)$$

and

$$h^y_{PVr}(y, p_y; q; \vec{\vartheta}) = \frac{y (y - 1) (y - q)}{q (q - 1)} p_y^2 + \frac{\vartheta_0^2}{y (1 - q)} + \frac{1}{y - q} + \frac{\vartheta_1^2}{q (y - 1)} + \frac{(\vartheta_\infty - \frac{1}{2})^2 y}{q (1 - q)}. \quad (118)$$

The equation of motion, which follows from the Hamiltonian (118), is

$$\frac{d^2 y}{dq^2} = \frac{1}{2} \left(\frac{dy}{dq}\right)^2 \left(1 + \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - q}\right) - \frac{dy}{dq} \left(\frac{1}{q} + \frac{1}{q - 1} + \frac{1}{q - y}\right) + \frac{2y (y - 1) (y - q)}{q (q - 1)} \left(\frac{\vartheta_0^2}{y^2 (1 - q)} + \frac{1}{y - q} + \frac{\vartheta_1^2}{q (y - 1)} + \frac{(\vartheta_\infty - \frac{1}{2})^2}{q (q - 1)}\right). \quad (119)$$

This is the celebrated Painlevé VI equation [142], or PVI, for short. However, this is not the only connection of the isomonodromy deformation to PVI.

In the $r > 1$ case the Hamiltonian system we’d get in the $y$-coordinates is known as the Garnier system [55].

6.1.2. Tau-function of Painlevé VI. In [84] the $\tau$-function $\tau_{PVr}(q | \alpha, \beta; \vec{\vartheta})$ was associated to PVI. It is a holomorphic on the universal cover $\mathbb{C}^p \setminus \{0, 1, \infty\}$ function, defined by (cf. [51])

$$\frac{\partial}{\partial q} \log \left( q^{\frac{\vartheta_0^2 + \vartheta_q^2 - \vartheta_\infty^2 - \vartheta_1^2}{2}} \left(1 - q\right)^{-\frac{\vartheta_0^2 + \vartheta_q^2 - \vartheta_\infty^2 - \vartheta_1^2}{2}} \tau_{PVr}(q | \alpha, \beta; \vec{\vartheta}) \right) = H_q \quad (120)$$

where the Schlesinger Hamiltonian $H_q$ is to be evaluated on the trajectory characterized by the conserved monodromy data $(\alpha, \beta)$ of the corresponding Schlesinger isomonodromy (= Painlevé Hamiltonian) flow.

6.1.3. Polygon variables. We can view $a$ as a complex version of the $r + 3$-gon $P_{r+3}$ in the complex three dimensional Euclidean space $\mathbb{C}^3 = Lie(G)$, with the vertices $v_1 = 0, v_2 = A_{z_1}, v_3 = A_{z_2} + A_{z_1}, \ldots, v_{r+3} = A_{z_1} + A_{z_2} + \ldots + A_{z_{r+2}}$, and the edges $e_i = v_i v_{i+1}$, $i = 1, \ldots, r + 3$ [8]. The “length” of an edge $e_i$ is the integral of the square root of the restriction of the holomorphic metric $ds^2 = \frac{1}{2} Tr dA^2$ on $e_i$, along any path connecting

\footnote{Yes, Schlesinger, not Painlevé}

\footnote{An edge $e = v'v''$ is a complex line $v' + \mathbb{C}(v'' - v') \subset \mathbb{C}^3$.}

\footnote{with $v_{r+4} = v_1$}
the points $v_i$ and $v_{i+1}$, and is clearly given by $\sqrt{\frac{1}{2} \text{Tr}(v_{i+1} - v_i)^2} = \pm \vartheta_z$, as the complex length is defined up to a sign. Let us now partition $P_{r+3}$ into $r+1$ triangle by $r$ "diagonals" $\delta_1, \ldots, \delta_r$. One choice of partitioning is

$$
\delta_1 = v_1v_3, \ \delta_2 = v_1v_4, \ldots, \ \delta_r = v_1v_{r+2}
$$

(121)

The corresponding triangles are

$$
\Delta_1 = \overline{v_1v_2v_3}, \ \Delta_2 = \overline{v_1v_3v_4}, \ldots \Delta_{r+1} = \overline{v_1v_{r+2}v_{r+3}}
$$

(122)

More generally, let $T$ be a tree with $r+3$ tails (vertices of valency 1) $e_1, \ldots, e_{r+3}$, and all internal vertices of valency 3. Euler formula $\#$ vertices $- \#$ edges $= 1$ implies that $T$ has $r+1$ internal vertex $\Delta_1, \ldots, \Delta_{r+1}$, and $r$ internal edges (i.e. edges whose both ends are internal) $\delta_1, \ldots, \delta_r$. We assign the tails of $T$ to the edges of the polygon, while the internal edges correspond to the diagonals. The internal vertices correspond to the triangles. The three sides of the triangle $\Delta_j$ are the edges of $P_{r+3}$ and diagonals corresponding to the three edges of $T$ attached to $\Delta_j$. The Darboux coordinates in the coordinate chart $U_T$ are $(\ell_i, \theta_i)_{i=1}^r$ where $\ell_i$ are the complex lengths of the diagonals $\delta_i$, and the angles $\theta_i$ between the two triangles $\Delta_j$ and $\Delta_j'$ corresponding to the two ends of the edge $\delta_i$ of $T$.

Let us describe this explicitly in the $r = 1$ case (the general case reduces to this one). Again, we compute the algebraic functions $h_{0\xi}$, defined in (83), in terms of the length $\ell$ of the diagonal and the dihedral angle $\theta$. The functions $h_{01}, h_{0q}, h_{1q}$ obey the linear relation

$$
h_{01} + h_{0q} + h_{1q} = \vartheta^2 - \vartheta_0^2 - \vartheta_q^2 - \vartheta_1^2
$$

(123)

and have nontrivial Poisson brackets in the symplectic form $\omega$ (cf. (95)):

$$
\{h_{0q}, h_{1q}\} = i \hat{h}_{01q}
$$

(124)

where

$$
\frac{1}{2} \hat{h}_{01q} = \vartheta_0^2 h_{0q}^2 + \vartheta_q^2 h_{01}^2 + \vartheta_1^2 h_{0q}^2 - h_{01} h_{0q} h_{1q} - 4 \vartheta_0^2 \vartheta_q^2 \vartheta_1^2
$$

(125)

The length $\ell$ of the diagonal $\delta_1$, connecting $v_1 = 0$ to $v_3 = A_0 + A_q$, is given by:

$$
\ell^2 = \frac{1}{2} \text{Tr}(A_0 + A_q)^2 = \vartheta_0^2 + \vartheta_q^2 + h_{0q}
$$

(126)

It is not difficult to compute

$$
\frac{1}{2} \text{Tr}[A_0, A_q]^2 = (\ell^2 - (\vartheta_0 - \vartheta_q)^2)(\ell^2 - (\vartheta_0 + \vartheta_q)^2)
$$

$$
\frac{1}{2} \text{Tr}[A_1, A_q]^2 = (\ell^2 - (\vartheta_1 - \vartheta_\infty)^2)(\ell^2 - (\vartheta_1 + \vartheta_\infty)^2)
$$

(127)
The angle $\theta$ between the two triangles $\overline{v_1v_2v_3}$ and $\overline{v_1v_3v_4}$ which are formed by the diagonal $\delta_1$ can be computed via
\[
\cos(\theta) = \frac{-2\ell^2 h_{1q} - (\ell^2 + \vartheta_0^2)(\ell^2 + \vartheta_1^2 - \vartheta_\infty^2)}{\sqrt{\prod_{\pm} \left((\ell^2 - (\vartheta_0 \pm \vartheta_q^2)(\ell^2 - (\vartheta_1 \pm \vartheta_\infty)^2)\right)}}.
\]
\[
\sin(\theta) = \frac{i\ell \tilde{h}_{01q}}{\sqrt{\prod_{\pm} \left((\ell^2 - (\vartheta_0 \pm \vartheta_q^2)(\ell^2 - (\vartheta_1 \pm \vartheta_\infty)^2)\right)}}
\]
(128)

Using (124), (125) it is easy to verify that $\{\ell, \theta\} = 1$, as in [23, 83]. See also [10].

6.2. Time-dependent coordinates. We now present two sets of coordinate systems on $\mathcal{M}_r^{alg}$ which depend explicitly on the complex structure of the underlying genus zero curve with $r + 3$ punctures.

6.2.1. Action-angle variables. It was observed long time ago that the polygon coordinates are actually the extremal versions of the action-angle coordinates on the partially compactified $\mathcal{M}_r^{alg}$, which is an algebraic integrable system, obtained by a degeneration [125] of Hitchin system [74]. Fix $z_0 \in C_r$. Define the analytic curve $\Sigma \subset T^* (\mathbb{C}P^1 \setminus z_0)$ by the equation
\[
\det (\mathcal{A}(z) - p \cdot 1) = 0,
\]
\[
p^2 = \sum_{\xi \in \Lambda_0} \frac{\Delta_\xi}{(z - \xi)^2} + \frac{H_\xi(a, z_0)}{z - \xi}
\]
(129)

It is not difficult to see that $\Sigma$ has genus $r$, e.g. by tropical limit, or by counting the number of branch points of (129) (and adding $2(r + 3)$ points $z = \xi$, $p(z - \xi) = \pm \vartheta_\xi$ to make $\Sigma$ compact). The eigenlines $L_{z,p}$ of $\mathcal{A}(z)$ corresponding to the eigenvalue $p$ found from (129) form a line bundle $\mathcal{L}$ (one needs to work a little bit at the ramification point $z = z_*$ where $p = 0$, where $\mathcal{A}(z_*)$ generically has the Jordan block form) over $\Sigma$. By appropriate non-canonical normalization one can make $\mathcal{L}$ to be a degree zero line bundle, $c_1(\mathcal{L}) = 0$, thus defining a point $[\mathcal{L}] \in Jac(\Sigma)$ in the Jacobian. A holomorphic line bundle can be equivalently described by a $\mathbb{C}^\times$-gauge equivalence class of a $(0, 1)$-part $\tilde{\partial} + \tilde{a}$ of a $\mathbb{C}^\times$-connection on a trivial bundle $\mathbb{C} \times \Sigma$ over $\Sigma$:
\[
\text{Bun}^0_{\mathbb{C}^\times}(\Sigma) = Jac(\Sigma) = H^0(\Sigma)/H^1(\Sigma, \mathbb{Z}) = \{ \tilde{a} \sim \tilde{a} + \tilde{\partial} \phi \},
\]
(130)

with gauge transformations $g = e^{i\phi} : \Sigma \rightarrow \mathbb{C}^\times$. The 1-form $id\theta = g^{-1}dg$ for such transformation is globally well-defined on $\Sigma$, but it is not, in general, exact. Its periods are in $2\pi i \mathbb{Z}$.

The action-angle coordinates are defined relative to the choice of symplectic basis $A_i, B^j \in H_1(\Sigma, \mathbb{Z}) \approx \mathbb{Z}^{2r}$, obeying $A_i \cap B^j = \delta_i^j$, $A_i \cap A_j = 0$, $B^i \cap B^j = 0$. Then the Darboux coordinates $(a_i, \phi_i)_{i=1}^r$ are defined via:
\[
a_i = \frac{1}{2\pi} \oint_{A_i} pdz,
\]
(131)
and $d\varphi_i$ are the holomorphic differentials on $Jac(\Sigma)$, defined e.g. by fixing a representative
\[
\bar{a} = \sum_{i=1}^{r} \varphi_i \varpi_i^*,
\]
(132)
where $\varpi_i, i = 1, \ldots, r$ are holomorphic differentials on $\Sigma$, which are normalized relative to the $A_i, B^j$ basis:
\[
\frac{1}{2\pi} \oint_{A_i} \varpi_i = \delta_i
\]
(133)
The coordinates $\varphi_i$ are defined up to the shifts by $2\pi \mathbb{Z}$ (this is familiar from the classical Liouville integrability [7]), and up to the shifts $\varphi_i \sim \varphi_i + 2\pi \tau_{ij} m^j$, with $m^j \in \mathbb{Z}$ and
\[
\frac{1}{2\pi} \oint_{B^j} \varpi_i = \tau_{ij}
\]
(134)
the period matrix of $\Sigma$. This feature is the general property of algebraic integrable systems [74]. In our case the algebraic integrable system is the genus zero version of the $G$-Hitchin system, often called the (classical limit of the ) Gaudin model.

Now, the polygon variables are obtained by taking the degeneration limit $[z_0] \rightarrow [z_T] \in \mathbb{M}_{0,r+3}$ in which the genus zero curve becomes a stable genus zero curve, corresponding to the tree $T$ (it is built out of the three holed spheres $S_i$, corresponding to the internal vertices $i$ of $T$). In this limit the curve $\Sigma$ also degenerates, so that its Jacobian becomes an algebraic torus $(\mathbb{C}^x)^r$, and the Hamiltonian flows generated by $H_\xi(a, z_0)$ (recall, that $z_0$ here are fixed, not to be confused with the times) become the bending flows. The periods (131) then approach $\ell_i$, the complex lengths of the polygon diagonals.

6.2.2. Separated variables. Define $w_i, i = 1, \ldots, r$ to be the zeroes of $A(z)_{21}$ (using $A(z)_{12}$ leads to equivalent theory), in the gauge where $A_\infty = \text{diag}(\vartheta_\infty, -\vartheta_\infty)$ is diagonal, i.e.
\[
\sum_{\xi \in \mathbb{Z}} \frac{\beta_\xi}{w_i - \xi} = 0, \quad i = 1, \ldots, r
\]
(135)
while setting $p_{w_i}$ to be equal to the specific eigenvalue of $A(w_i)$, namely the value of $A(w_i)_{22} = -A(w_i)_{11}$, i.e.
\[
p_{w_i} = \sum_{\xi \in \mathbb{Z}} -\vartheta_\xi + \beta_\xi \gamma_\xi
\]
\[
\frac{1}{w_i - \xi}
\]
(136)
Comparing with the Eq. (133) we understand the invariant definition of $(w_i, p_{w_i})_{i=1}^{r}$ is the locus of vanishing of $T^{+}_1$, the section of the line-bundle $L(p,z) = \ker(A(z) - p)$ over the spectral curve $\Sigma$, [154, 143, 62], which is the same eigenbundle of the Higgs field $A(z)$ as in [129], except that now one normalizes it to have degree $r$ (so that its unique, up to a constant multiple, section has $r$ zeroes and no poles).

\footnote{corresponding to $L_+$ in the discussion below (75)}
In the $r = 1$ case
\[
H_q = \frac{h_{0q}}{q} + \frac{h_{1q}}{q - 1} = \frac{w(w - 1)(w - q)}{q(q - 1)} p_w^2 + \frac{\vartheta_0^2}{w(1 - q)} + \frac{\vartheta_q^2}{q - w} + \frac{\vartheta_1^2 w}{q(w - 1)} + \frac{\vartheta_\infty^2 w}{q(1 - q)} - \bar{u}_0(q),
\]
where
\[
\bar{u}_0(q) = \frac{\vartheta_0^2 + \vartheta_q^2}{q} + \frac{\vartheta_0^2 - \vartheta_q^2 - \vartheta_1^2 + \vartheta_\infty^2}{1 - q}
\]
(137)
Here comes the trap: the equations of motion, derived from Hamiltonian $H_q$ in the $(w, p_w)$ coordinates, do not coincide with the isomonodromic flow (93). Instead, (93) is a Hamiltonian flow, generated, in the $(w, p_w)$-frame, by the $q$-dependent Hamiltonian
\[
H_q + \delta h(w, p_w; q)
\]
where
\[
\delta h(w, p_w; q) = -\frac{w(w - 1)p_w + \vartheta_\infty w}{q(q - 1)},
\]
(140)
is compensating the $q$-dependence of the coordinates $(w, p_w)$, as we explain in the next subsection. Since (139) also contains both $p_w^2$ and $p_w$ terms, we shift the momentum:
\[
p_w = p_w + \frac{1}{2(w - q)}
\]
(141)
and shift accordingly the Hamiltonian (139), arriving, finally, at the true Hamiltonian $h_{PVl}(w, p_w; q; \vec{\vartheta})$,
\[
p_w dw - (H_q + \delta h(w, p_w; q)) dq = p_w dw - h_{PVl}(w, p_w; q) dq + dW
\]
(142)
of the (93) flow 10:
\[
h_{PVl}(w, p; q; \vec{\vartheta}) = \frac{w(w - 1)(w - q)}{q(q - 1)} p^2 + \frac{\vartheta_0^2}{w(1 - q)} + \frac{\vartheta_q^2}{w - q} + \frac{w \vartheta_1^2}{q(w - 1)} + \frac{w (\vartheta_\infty - \frac{1}{2})^2}{q(1 - q)}
\]
(143)
with
\[
W = -\frac{1}{2} \log(w - q) + \text{function of } q.
\]
(144)
A glance at (143) and (118) shows that the isomonodromy flow (93) is, again, a solution of the PVI equation, but now with the exponents $(\vartheta_0, \vartheta_q, \vartheta_1, \vartheta_\infty)$ instead of $(\hat{\vartheta}_0, \hat{\vartheta}_q, \hat{\vartheta}_1, \hat{\vartheta}_\infty)$ given by (116). Note that the Hamiltonian (143) is invariant under the flips $\vec{\vartheta} \mapsto (\pm \vartheta_0, \pm \vartheta_q, \pm \vartheta_1, \vartheta_\infty)$, which is to be expected, since the eigenvalues of the residues $A_0, A_q, A_1$ are defined up to a permutation. However, the flip of the last exponent $\vartheta_\infty$ does not leave (143) invariant, instead the symmetry is $\vartheta_\infty \mapsto 1 - \vartheta_\infty$.
This is a reflection of background-dependence of the $(w, p_w)$-coordinate system.

10 cf. [158], [104] $\delta_1 = \frac{1}{2} - \vartheta_0^2$, $\delta_2 = \frac{1}{2} - \vartheta_q^2$, $\delta_3 = \frac{1}{2} - \vartheta_1^2$, $\delta_5 = \frac{1}{2} - (\vartheta_\infty - \frac{1}{2})^2$
6.3. Canonical transformations. Let us discuss the background-dependence and the canonical transformations \((y, \hat{p}_y) \to (w, \hat{p}_w)\) in more detail. So we set \(r = 1\) throughout this section.

We shall use the involutions
\[ z \mapsto z^0 = q \frac{1-z}{q-z}, \quad z \mapsto z^i = \frac{q}{z}, \quad z \mapsto z^{\bar{i}} = \frac{z-q}{z-1} \] (145)
of \(\mathbb{CP}^1\), \((z^0)^\circ = z\), which induce the three nontrivial involutions of \(z\):
\[ (0 \leftrightarrow 1, q \leftrightarrow \infty), (0 \leftrightarrow \infty, q \leftrightarrow 1), \text{ and } (0 \leftrightarrow q, 1 \leftrightarrow \infty) \] (146)
respectively. Each has two fixed points \(z^0_\pm = q \pm \sqrt{q(1-q)}\), \(z^i_\pm = \pm \sqrt{q}\), \(z^{\bar{i}}_\pm = 1 \pm \sqrt{1-q}\), respectively. The transformations \(z \mapsto z^0, z^i, z^{\bar{i}}\) together with the identity map represent the action on \(\mathbb{CP}^1\setminus z\) of the celebrated Klein 4-group \(D_2\), isomorphic to \(\mathbb{Z}_2 \times \mathbb{Z}_2\).

6.3.1. Background dependence of \(w\)-coordinates. Let us write
\[ -\mathcal{A}(z)_{21} = \frac{\beta_0}{z} + \frac{\beta_q}{z-q} + \frac{\beta_1}{z-1} = \kappa \frac{z-w}{z(z-q)(z-1)}, \] (147)
in the basis, where \(A_{\infty}\) is diagonal, from which we deduce: \(w^0 = -(\mathcal{A}_{121}/(\mathcal{A}_{q})_{21})\), or
\[ w = (-\beta_1/\beta_q)^\circ. \] (148)
In other words, the \(w^0\) variable is \(q\)-independent, while \(w\) is. Now substituting into (136), we get
\[ p_w = \frac{(w^0 - q)^2}{q(1-q)w^0} \left( \frac{(A_q)_{11}}{w^0 - 1} + (A_q)_{11} + \frac{q\beta_1}{q - w^0} \right). \] (149)
Accordingly, the functions \(h_{\xi q}\), when expressed through \((w, p_w)\), acquire an explicit \(q\)-dependence.

Note that as \(q \to 0,1,\infty\), with fixed \(A_\xi^\circ\)'s, and \(w^0\), the coordinate \(w\) approaches \(0,1,w^{(\infty)} = 1 - w^0\), respectively. The coordinate \(p_w\) is singular when \(q \to 0,1\), while as \(q \to \infty\) it approaches
\[ p_w^{(\infty)} = -\frac{1}{w^0} \left( \frac{(A_q)_{11}}{w^0 - 1} + (A_q)_{11} + \beta_\infty \right). \] (150)
Thus, we can solve for \((w, p)\) in terms of \((w^{(\infty)}, p_w^{(\infty)}, q)\):
\[ w = \frac{q w^{(\infty)}}{w^{(\infty)} + q - 1}, \quad p_w = \frac{p_w^{(\infty)}(w^{(\infty)} + q - 1)^2 - \beta_\infty (w^{(\infty)} + q - 1)}{q(q - 1)}. \] (151)
The map \((w^{(\infty)}, p_w^{(\infty)}) \mapsto (w, p_w)\) preserves the symplectic form \(dp \wedge dw\):
\[ dp_w \wedge dw = dp_w^{(\infty)} \wedge dw^{(\infty)} \] (152)
and is generated, \(\delta h(w, p_w; q) = -\frac{\partial}{\partial q}\), by our friend (140), which naturally obeys \(\delta h(w, p_w; q) = \delta h(w^{(\infty)}, p_w^{(\infty)}; q)\). Here
\[ L(w, p_w^{(\infty)}; q) = p_w^{(\infty)} w \frac{1-q}{w-q} + \beta_\infty \log \left( 1 - \frac{w}{q} \right) \] (153)
is the generating function of the map \((w^{(\infty)}, p_w^{(\infty)}) \mapsto (w, p_w)\), is
\[
p_w = \frac{\partial L}{\partial w}, \quad w^{(\infty)} = \frac{\partial L}{\partial p_w^{(\infty)}}
\] (154)

6.3.2. From \(y\) to \(w\) – Okamoto transformations. It is instructive to find the explicit map between \((y, p_y)\) and \((w, p_w)\) coordinates. Let us use the notation
\[
\vartheta = \vartheta_0 + \vartheta_q + \vartheta_1 + \vartheta_\infty.
\] (155)

We diagonalize \(A_\infty\), i.e. find \(g_\infty\), such that \(g_\infty^{-1}A_\infty g_\infty = \text{diag}(\vartheta_\infty, -\vartheta_\infty)\), meaning we set \(\gamma_\infty = 0\), \(\beta_\infty = 0\), which implies, cf. [107], [113], that
\[
y = \frac{1}{1 - \gamma_0/\gamma_1}, \quad \gamma_1 = 1 + \frac{2\vartheta_\infty}{yp_y}, \quad \frac{\gamma_1}{\gamma_0} = 1 - \frac{2\vartheta_\infty}{y(y - 1)p_y}
\] (156)

Next, setting the zero of the 21-matrix element of \(g_\infty^{-1}A(z)g_\infty\) to be equal to \(w\),
\[
\left[g_\infty^{-1}\left(\frac{A_0}{w} + \frac{A_q}{w - q} + \frac{A_1}{w - 1}\right)g_\infty\right]_{21} = 0,
\] (157)
we get:
\[
\beta_0 = -\kappa \frac{w}{q}, \quad \beta_q = \kappa \frac{w - q}{q(1 - q)}, \quad \beta_1 = \kappa \frac{w - 1}{q - 1}
\] (158)
for some \(\kappa\). Finally, identifying \(p_w\) with the eigenvalue \(\left[g_\infty^{-1}\left(\frac{A_0}{w} + \frac{A_q}{w - q} + \frac{A_1}{w - 1}\right)g_\infty\right]_{11}\) of \(A(w)\), and imposing the rest of the moment map equations: \(\beta_0\gamma_0 + \beta_q\gamma_q + \beta_1\gamma_1 = \vartheta\), \(\beta_0\gamma_0^2 + \beta_q\gamma_q^2 + \beta_1\gamma_1^2 = 2\beta_0\gamma_0 + 2\beta_q\gamma_q + 2\beta_1\gamma_1\) leads to the relations
\[
p_w = \frac{\vartheta_0}{w} + \frac{\vartheta_q}{w - q} + \frac{\vartheta_1}{w - 1} - \frac{\vartheta}{w - x}, \quad p_y = \frac{2\vartheta_\infty}{y - x^o} - \frac{2\vartheta_\infty}{y},
\] (159)
which can be interpreted as a composition of the symplectomorphism: \((y, p_y) \mapsto (x, p_x)\), which is generated by
\[
S_0(x, y) = 2\vartheta_\infty \log(xy - q(x + y - 1)) - 2\vartheta_\infty \log(y) + (2\vartheta_0 - \vartheta) \log(x) + (2\vartheta_q - \vartheta) \log(x - q) + (2\vartheta_1 - \vartheta) \log(x - 1)
\] (160)
so that
\[
p_x dx - p_y dy - h^x dq = -dS_0,
\] (161)
and the symplectomorphism \((x, p_x) \mapsto (w, p_w)\), generated by
\[
S_1(x, w) = \vartheta_0 \log(w) + \vartheta_q \log(w - q) + \vartheta_1 \log(w - 1) - \vartheta \log(w - x),
\] (162)
so that
\[
p_w dw - p_x dx - h^w dq = dS_1
\] (163)
respectively. This is one of the so-called Okamoto transformations \([140]\), usually discussed in the context of Painlevé equations \([81, 155]\). See also \([57]\) for some gauge theory applications.
6.3.3. **Vierergruppe: the other coordinate changes.** Let us perform the involutions $z \mapsto z^0$, $z \mapsto z^1$, $z \mapsto z^\ii$ and see what happens with the meromorphic 1-form $\mathcal{A}(z)$, e.g.

$$\mathcal{A}^\circ = A_0 d\log(z^0) + A_q d\log(z^0 - q) + A_1 d\log(z^0 - 1) = A_1 d\log(z) + A_\infty d\log(z - q) + A_0 d\log(z - 1) \quad (164)$$

Thus, it would appear that we can permute the $\vartheta$-parameters $(\vartheta_0, \vartheta_q, \vartheta_1, \vartheta_\infty) \mapsto (\vartheta_1, \vartheta_\infty, \vartheta_0, \vartheta_q)$ simply by a change of coordinates on a base curve, assuming, $\mathcal{A}(z)$ is a connection on a trivial vector bundle $\mathcal{O} \oplus \mathcal{O}$. However, the canonical transformation

$$(w, \ p_w) \mapsto \left( w^\circ, \ p_w^\circ = \frac{(w - q)^2}{q(1 - q)}(p_w + \vartheta w f) \right) \quad (165)$$

where we tuned $f = -\frac{1}{2} \log(w - q) + a$ function of $q$, to eliminate linear in $p_w^\circ$ terms in the transformed Hamiltonian:

$$p_w dw - h_{\text{PVI}}^w(w, p_w; q; \vartheta^0) dq + df = p_w^\circ dw^\circ - h_{\text{PVI}}^w(w^\circ, p_w^\circ; q) dq, \quad (166)$$

gives us

$$h_{\text{PVI}}^w(w, p; q; \vartheta^0) = \frac{w(w - 1)(w - q)}{q(q - 1)} p^2 + \frac{\vartheta_1^2}{w(1 - q)} - \frac{\vartheta_\infty(\vartheta_\infty - 1)}{w - q} + \frac{w \vartheta_0^2}{q(w - 1)} + \frac{w \vartheta_q^2}{q(1 - q)} \quad (167)$$

which encodes a slightly different transformation of $\vartheta$-parameters:

$$(\vartheta_0, \vartheta_q, \vartheta_1, \vartheta_\infty) \mapsto (\vartheta_1, \vartheta_\infty - \frac{1}{2}, \vartheta_0, \vartheta_q + \frac{1}{2}) \quad (168)$$

The associated transformations of the PVI variables $w \mapsto w^i = q/w$ and $w \mapsto w^{\ii} = \frac{w-q}{w-1}$ induce transformations

$$(\vartheta_0, \vartheta_q, \vartheta_1, \vartheta_\infty)^i = (\vartheta_\infty - \frac{1}{2}, \vartheta_1, \vartheta_q, \vartheta_0 + \frac{1}{2}) \quad (169)$$

and

$$(\vartheta_0, \vartheta_q, \vartheta_1, \vartheta_\infty)^{\ii} = (\vartheta_1, \vartheta_\infty - \frac{1}{2}, \vartheta_0, \vartheta_q + \frac{1}{2}) \quad (170)$$

The transformations $\vartheta \mapsto \vartheta^i, \vartheta^{\ii}$, $\vartheta^0, \vartheta^1, \vartheta^\ii$, $\vartheta^0, \vartheta^1, \vartheta^\ii$ together with the identity form the celebrated Klein 4-group $D_2$, isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

6.4. **Monodromy data: coordinates on $\mathcal{M}_r^{\text{loc}}$.** The monodromy of $\nabla$ around the punctures $z$, defined through a choice of a marked point $pt \notin z$ and a basis in the fundamental group $\pi_1(\mathbb{C}P^1 \setminus z, pt)$, are the group elements $g_\xi \in G$, obeying

$$\prod_{\xi \in \mathcal{Z}} g_\xi = 1 \quad (171)$$

where the order in the product is dictated by the choice of the basis loops in $\pi_1(\mathbb{C}P^1 \setminus z, pt)$. The conjugacy class of each $g_\xi$ is fixed:

$$c_\xi := \frac{1}{2} \text{Tr} g_\xi = \cos(2\pi \vartheta_\xi), \quad (172)$$
The Darboux atlas of \( M_v \) is introduced in \([134]\).

This symplectic manifold has a system of Darboux coordinates \( (g_\alpha, g_\beta) \), where

\[
g_\alpha := \frac{1}{4\pi} \int \text{Tr} A \wedge A
\]

This symplectic manifold has a system of Darboux coordinates \( (\alpha, \beta) \) on \( M_v \), introduced in \([134]\).

For \( r = 1 \), for the specific ordering of \( z = \{0, q, 1, \infty\} \) the monodromy matrices \( g_0, g_q, g_1, g_\infty \), obey

\[
g_\infty g_1 g_q g_0 = 1,
\]

The Darboux atlas of \( M_v \) has the coordinate charts labeled by the splittings \( z = z_0 \coprod z_1 \), \( |z_0, 1| = 2 \), plus some additional discrete data, such as ordering of \( z_0, 1 \). In the case \( z_0 = \{0, q\}, z_1 = \{1, \infty\} \) the coordinates are \( (\alpha, \beta) \), where

\[
c_{0\alpha} := \frac{i}{4} \text{Tr} g_0 g_q = \cos(2\pi \alpha),
\]

while \( \beta \), the remaining monodromy parameter, can be found from

\[
c_{1q} = \frac{i}{4} \text{Tr} g_1 g_q = \cos(2\pi \bar{\alpha})
\]

via:

\[
e^{\pm \beta} (\cos 2\pi (\alpha \pm \vartheta_1) - \cos 2\pi \vartheta_\infty) (\cos 2\pi (\alpha \pm \vartheta_q) - \cos 2\pi \vartheta_0) =
\]

\[
= c_q c_1 + c_0 c_\infty \pm i c_{01} \sin(2\pi \alpha) - (c_0 c_1 + c_q c_\infty \mp i c_{1q} \sin 2\pi \alpha) e^{\pm 2\pi i \alpha},
\]

\[
c_0^2 + c_1^2 + c_q^2 + 2 c_0 c_1 c_q + c_0^2 + c_1^2 + c_q^2 + 4 c_0 c_q c_1 c_\infty =
\]

\[
1 + 2 c_0 \left( c_0 c_1 + c_q c_\infty \right) + 2 c_1 \left( c_1 c_q + c_0 c_\infty \right) + 2 c_q \left( c_q c_0 + c_1 c_\infty \right)
\]

which is equivalent to the definition of the \( \beta \)-coordinate in \([134]\) (in our notations):

\[
\cosh(\beta) = \frac{-c_{1q} \sin(2\pi \alpha)}{4 \prod_{\pm, \pm} \sqrt{\sin 2\pi (\alpha \pm \vartheta_0 \pm \vartheta_q) \sin 2\pi (\alpha \pm \vartheta_1 \pm \vartheta_\infty)}},
\]

\[
\sinh(\beta) = \frac{i \sin(2\pi \alpha) (c_{01} + c_0 c_1 c_q - c_0 c_1 - c_q c_\infty)}{4 \prod_{\pm, \pm} \sqrt{\sin 2\pi (\alpha \pm \vartheta_0 \pm \vartheta_q) \sin 2\pi (\alpha \pm \vartheta_1 \pm \vartheta_\infty)}}.
\]

The \( \alpha \)-coordinate, up to a reflection \( \alpha \mapsto -\alpha \) and integer shifts \( \alpha \sim \alpha + \mathbb{Z} \) does not depend on the ordering of \( z_0, 1 \), while \( \beta \) has a simple ordering dependence (see \([134]\) for more detail).
7. The nonperturbative Dyson-Schwinger equations

In this section we return to the study of surface defects in supersymmetric $U(n)$ gauge theory. We relate them to Painlevé VI for $n = 2$. The higher $n$ case will be studied elsewhere.

7.1. Fractional $qq$-characters. For the $U(n)$ theory with $N_f = 2n$ flavors, introduce the fractional $Y$-observables:

$$Y_\omega(x) = (x - \tilde{a}_\omega) \frac{Q_\omega(x)}{Q_{\omega-1}(x - \tilde{\varepsilon}_2)}; \quad \omega \sim \omega + n, \; \omega \in \mathbb{Z}/n\mathbb{Z} \quad (180)$$

where

$$Q_\omega(x) = \prod_{k \in \kappa_\omega} \left(1 - \frac{\varepsilon_1}{x - k}\right), \quad Ch(\tilde{K}_\omega) = \sum_{k \in \kappa_\omega} e^k \quad (181)$$

For large $x$,

$$Q_\omega(x) = 1 - \frac{\varepsilon_1 k_\omega}{x} + \ldots \quad (182)$$

7.2. Dyson-Schwinger equation for the surface defect.

7.2.1. Mass and Coulomb manipulations. Introduce the notations

$$\tilde{m}_\omega = \tilde{m}_\omega^+ + \tilde{m}_\omega^-,$$

$$\mu_\omega^+ = \frac{\tilde{m}_\omega^+ - \tilde{a}_+}{2\tilde{\varepsilon}_2},$$

$$\mu_\omega^- = \mu_\omega^+ + \mu_\omega^-; \quad \mu_\omega = \mu_\omega^+ + \mu_\omega^-; \quad \mu = \mu_0^+ + \mu_1^+ .$$

for the masses of the hypermultiplets, which transform under the $\mathcal{R}_\omega$ representation of the orbifold group, where

$$\tilde{a}_+ = \frac{1}{n} \sum_\omega \tilde{a}_\omega \quad (184)$$

For $n = 2$ we also use:

$$\tilde{a}_- = \frac{\tilde{a}_0 - \tilde{a}_1}{2}. \quad (185)$$

With $\sum$ in place, we have $\tilde{a}_+ = \frac{1}{2}\tilde{\varepsilon}_2$ and:

$$\mu_0^+ = \frac{m_{s(3)}}{\varepsilon_2} - \frac{1}{4}; \quad \mu_0^- = \frac{m_{s(4)}}{\varepsilon_2} - \frac{1}{4}; \quad \mu_1^+ = \frac{m_{s(1)}}{\varepsilon_2} + \frac{1}{4}; \quad \mu_1^- = \frac{m_{s(2)}}{\varepsilon_2} + \frac{1}{4}. \quad (186)$$

7.2.2. DS equation in the orbifold variables. Recall the expression for the $qq$-characters of the $\mathcal{N} = 2$ theory in the presence of the surface defect:

$$z_\omega(x) = Y_{\omega+1}(x + \varepsilon_1 + \tilde{\varepsilon}_2) + q_\omega P_\omega(x) Y_\omega(x)^{-1}, \quad \omega \sim \omega + n, \; \omega \in \mathbb{Z}/n\mathbb{Z}, \quad (187)$$

with

$$P_\omega(x) = (x - \tilde{m}_\omega^+)/ (x - \tilde{m}_\omega^-). \quad (188)$$

The main property of the operators $\{187\}$ is the cancellation of poles in $x$ of their expectation values, cf. $\{19\}$:

$$\langle z_\omega(x) \rangle = (1 + q_\omega) (x + \langle \delta_\omega \rangle) + \varepsilon_1 + \tilde{\varepsilon}_2 - (\tilde{a}_\omega + \tilde{a}_{\omega+1}) + q_\omega \tilde{m}_\omega, \quad \omega = 0, 1 \quad (189)$$
where
\[ \delta_{\omega} = \tilde{a}_\omega + \varepsilon_1(k_\omega - k_{\omega+1}). \] (190)

For \( n = 2 \), cf. [20], [22], [25]:
\[ \langle \delta_{0,1} \rangle = (\tilde{a}_+ + \varepsilon_1u\partial_u)\Psi, \quad \langle \delta_{0,1}^2 \rangle = (\tilde{a}_+ \pm \varepsilon_1u\partial_u)^2\Psi, \] (191)

where we used the notation [19] for the fractional couplings.

The vanishing of the \( x^{-1} \)-term in the large \( x \) expansion of the equations (189) implies the relation between the correlators:
\[ 0 = (1 - q) \left( \frac{1}{2} \langle \delta_0^2 \rangle + \frac{1}{2} \langle \delta_1^2 \rangle - \tilde{a}_+^2 - \tilde{a}_-^2 + \varepsilon_1 \tilde{e}_2(k_0 + k_1) \right) + \]
\[ + (q - u) \left( \tilde{m}_0^+ \tilde{m}_0^- - \tilde{m}_0 \langle \delta_0 \rangle + \langle \delta_0^2 \rangle \right) + q(1 - u^{-1}) \left( \tilde{m}_1^+ \tilde{m}_1^- - \tilde{m}_1 \langle \delta_1 \rangle + \langle \delta_1^2 \rangle \right). \] (192)

Now, with the help of the relation
\[ \varepsilon_1 \tilde{e}_2(u\partial_u + 2q\partial_q)\Psi = \langle \tilde{a}_+ (1 + \varepsilon_2 - \tilde{a}_+) - \tilde{a}_-^2 + \varepsilon_1 \tilde{e}_2(k_0 + k_1) \rangle, \] (193)

we can rewrite (192) as the differential equation on \( \Psi \):
\[ (1 - q) \left( \varepsilon_1 \tilde{e}_2 \left( \frac{1}{2} u\partial_u + q\partial_q \right) + \tilde{a}_+ (\tilde{a}_+ - \varepsilon_1 - \tilde{e}_2) \right) \Psi + \]
\[ \frac{(1 - u)(u - q)}{u} \left( \varepsilon_1 u\partial_u \right)^2 \Psi + \varepsilon_1 \tilde{e}_2 (\mu_0 u(u - q) + \mu_1 (u - 1)) \partial_u \Psi + \]
\[ + \frac{\varepsilon_2^2}{u} \left( (u - 1)q \mu^+_1 \mu^-_1 - u(u - q) \mu^+_0 \mu^-_0 \right) \Psi = 0 \] (194)

This is the \( n = 2 \) case of the more general fact, established in the BPS/CFT V paper in [132], that the regular surface defect \( \Psi \) of the \( U(n) \) theory with \( 2n \) fundamental hypermultiplets obeys the Knizhnik-Zamolodchikov/Belavin-Polyakov-Zamolodchikov [94, 14] equation.

7.2.3. Enters Painlevé VI. In the limit \( \varepsilon_1 \to 0 \), the surface defect partition function has the asymptotics:
\[ \Psi \sim e^{\varepsilon_1 S(u; q, \mu; \varepsilon_2)} u^{-\frac{\varepsilon_2 \mu_1}{x_1} (u - 1)^{-\frac{\varepsilon_2 (\mu_0 + \frac{1}{2})}{2x_1} - \frac{\varepsilon_2 (\mu_1 - \frac{1}{2})}{2x_1}}, \] (195)

where \( \sim \) means that we dropped a \( u \)-independent factor (it leads to a \( q \)-dependent shift in \( S \)), so that the Eq. (194) becomes the Hamilton-Jacobi equation:
\[ \partial_q S(u; q; \vartheta|\alpha) = -H(u, \partial_u S(u; q; \vartheta|\alpha); q; \vartheta) \] (196)

with the Hamiltonian
\[ H(u, p; q; \vartheta) = \frac{u(u - 1)(u - q)}{q(q - 1)} p^2 + U(u, q; \vartheta), \]
\[ U(u, q; \vartheta) = \frac{u(\vartheta^{-\frac{1}{2}} - \frac{1}{2})^2}{q(1 - q)} + \frac{\vartheta_0^2}{u(1 - q)} + \frac{u \vartheta_0^2}{q(u - 1)} - \frac{\vartheta_0^2 - \frac{1}{4}}{u - q} \] (197)

with
\[ \pm 2\vartheta_0 = \mu^+_1 - \mu^-_1, \quad \pm 2\vartheta_1 = \mu^+_1 + \mu^-_1 + \frac{1}{2}, \quad \pm 2\vartheta_\infty = \mu^+_0 + \mu^-_0 + \frac{1}{2}, \quad \pm (2\vartheta_\infty - 1) = \mu^+_0 - \mu^-_0 \] (198)
We recognize in (197) our friend Painlevé VI Hamiltonian (143). The signs ± in (198) reflect the ambiguity of the relation between the parameters of the Hamiltonian (139) and the $\vartheta$-parameters, which stems from the symmetries of (143) we discussed at the end of the section 6.2. The Hamiltonian (139) is invariant under the permutations $\tilde{m}_\vartheta^+ \leftrightarrow \tilde{m}_\vartheta^-$, which do not change the surface defect.

7.2.4. DS equation in the parameters of the bulk theory. Using (186), we get:

$$\vartheta_0 = \pm \theta_0, \quad \vartheta_q = \pm (\theta_q + \frac{i}{2}), \quad \vartheta_1 = \pm \theta_1, \quad \vartheta_\infty = \frac{i}{2} \pm \theta_\infty,$$

where

$$\tilde{\vartheta} = \frac{1}{2\varepsilon_2} (m_{s(1)} - m_{s(2)}, m_{s(1)} + m_{s(2)}, m_{s(3)} + m_{s(4)}, m_{s(3)} - m_{s(4)}).$$

(199)

7.2.5. Playing with the fugacity. Armed with our knowledge of (168) we can transform the DS equation for $\Psi$ to the form of Hamilton-Jacobi potential

$$\partial_q S^0(w; q; a/\varepsilon_2, m/\varepsilon_2) = -h_{PVI}^w \left( w, \partial_w S^0; q, \tilde{\vartheta}^{DS} \right)$$

(200)

with

$$w = u^\circ = q_1 \frac{1 + q_0}{1 + q_1}$$

(201)

and the $\vartheta$-parameters

$$\vartheta_0^{DS} = \frac{m_{s(3)} + m_{s(4)}}{2\varepsilon_2}, \quad \vartheta_q^{DS} = \frac{m_{s(3)} - m_{s(4)}}{2\varepsilon_2}, \quad \vartheta_1^{DS} = \frac{m_{s(1)} - m_{s(2)}}{2\varepsilon_2}, \quad \vartheta_\infty^{DS} = \frac{m_{s(1)} + m_{s(2)}}{2\varepsilon_2}$$

(202)

which are linear in the masses of fundamental multiplets. This property is crucial in our argument below.

The Hamilton-Jacobi potentials $S^0(w; q; a/\varepsilon_2, m/\varepsilon_2)$ and $S(u; q; a/\varepsilon_2, m/\varepsilon_2)$ are related by, cf. (165):

$$S^0(w; q; a/\varepsilon_2, m/\varepsilon_2) = S(u; q; a/\varepsilon_2, m/\varepsilon_2) - \frac{i}{2} \log(u - q) + \text{function of } q$$

(203)

We interpret $e^{S^0/\varepsilon_1}$ as the partition function of the surface defect in $SU(2)$ gauge theory, which differs from $\Psi$ by a $U(1)$-factor.

7.2.6. Initial conditions. In order to solve for the surface defect partition function we need to recover the $a$-dependence of $S$. One can do this by fixing a small $q$ asymptotics:\[11\]

First of all, it is easy to compute, for $q \to 0$ with $u$ fixed, i.e. $q_0$ is finite, while

\[11\] Technically this is similar to the analysis in [101]
\( q_1 \to 0 \), that

\[
\Psi \sim u^{\frac{\tilde{a}}{\tilde{e}_1}} \sum_{d=0}^{\infty} u^d E \left[ \left( \frac{e^{\tilde{a}_1} + \tilde{q}_2 e^{\tilde{a}_0} (1 - q_1^d)}{1 - q_1} \right)^* \tilde{q}_2 \left( e^{\tilde{a}_0} q_1^d - e^{\tilde{m}_0^+} - e^{\tilde{m}_0^-} \right) \right] =
\]

\[
\sum_{d=0}^{\infty} u^d E \left[ \left( \frac{e^{\tilde{a}_1}}{1 - q_1} \right)^* \tilde{q}_2 \left( e^{\tilde{a}_0} - e^{\tilde{m}_0^+} - e^{\tilde{m}_0^-} \right) \right] \times
\]

\[
\sum_{d=0}^{\infty} u^d E \left[ \left( \frac{\tilde{q}_2 e^{\tilde{a}_0} (1 - q_1^d)}{1 - q_1} \right)^* \tilde{q}_2 \left( e^{\tilde{a}_0} q_1^d - e^{\tilde{m}_0^+} - e^{\tilde{m}_0^-} \right) \right] E \left[ \frac{e^{\tilde{a}_0 - \tilde{a}_1 + \tilde{q}_2} (q_1^d - 1)}{1 - q_1^{-1}} \right] (204)
\]

since only partitions \( \lambda^{(0)} = 1^d \), \( \lambda^{(1)} = \emptyset \) can contribute, for which

\[
Ch(K_0) = e^{\tilde{a}_0} \frac{1 - q_1^d}{1 - q_1}, \ Ch(K_1) = 0, \ Ch(S_0) = e^{\tilde{a}_0} q_1^d, \ Ch(S_1) = e^{\tilde{a}_1} + \tilde{q}_2 e^{\tilde{a}_0} (1 - q_1^d) (205)
\]

Thus, the surface defect partition function, for \( q_1 = 0 \), is essentially a hypergeometric function

\[
\Psi = u^{\frac{\tilde{a} - 1}{\tilde{e}_1}} \Gamma \left( \frac{a - \tilde{m}_0^+}{\tilde{e}_1} \right) \Gamma \left( \frac{a - \tilde{m}_0^-}{\tilde{e}_1} \right) \frac{\Gamma \left( 1 + \frac{2a}{\tilde{e}_1} \right)}{\Gamma \left( 1 + \frac{2a}{\tilde{e}_1} \right)} 2F_1 \left( \frac{a - \tilde{m}_0^+}{\tilde{e}_1}, \frac{a - \tilde{m}_0^-}{\tilde{e}_1}; 1 + \frac{2a}{\tilde{e}_1}; u \right) (206)
\]

See [83] for more detail.

### 7.3. Two dimensional connection from the four dimensional gauge theory.

As we discussed in the chapter 5, the Painlevé VI equation describes the isomonodromic deformation of a meromorphic connection on a 4-punctured sphere, i.e. Schlesinger evolution. However, we saw that the Schlesinger-Painlevé correspondence is not one-to-one. We have the \( y \)-coordinates, the \( w \)-coordinates, the \( w^0 \)-coordinates, the \( y^0 \)-coordinates etc.

It would be useful to identify an observable in the four dimensional gauge theory which can be identified with the meromorphic connection whose isomonodromic deformation is described by [200] or [196]. Indeed, the \( U(n) \) \( N_f = 2n \) theory has the observables \( \Upsilon(z) \), which are the horizontal sections of the meromorphic connection similar to [91] (for \( r = 1, n = 2 \) in our example). Here is the construction.

#### 7.3.1. Crossed surface defect.

Recall that the \( \Upsilon(z) \)-observable represent the regularized characteristic polynomial \( \det(x - \Phi) \) of the complex scalar in the vector multiplet, so it is a local observable. In the \( \Omega \)-deformed theory the supersymmetric non-local observables can be related to the local ones and vice versa. It is sometimes useful to work with the observable \( Q_{\Sigma}(x) \) associated with the surface \( \Sigma \) in the four dimensional space. One way to think about \( Q_{\Sigma}(x) \) is that it is a partition function of a certain two dimensional theory interacting with the four dimensional one, i.e. a surface defect. For example, it can be engineered using the folded instanton construction [132]. In the theory with matter fields we have a choice of boundary conditions in the matter sector. In the
theory on \( \mathbb{R}^4 \) with \( \Omega \)-deformation the \( Q \)-observable associated with the surface \( \Sigma = \mathbb{R}^2 \) given by the equation \( z_1 = 0 \) solves the operator equation

\[
Y(x) = \prod_f (x - m_f^+) \frac{Q(x)}{Q(x - \varepsilon_2)},
\]

(207)

where \( f \) labels the fundamental hypermultiplets, which have propagating modes along the \( z_1 = 0 \) surface, \( m_f^+ \) being the corresponding masses. The analogous observable associated with the \( z_2 = 0 \) surface obeys a similar equation with \( \varepsilon_2 \) replaced by \( \varepsilon_1 \).

Now let us add the surface defect, of the regular orbifold type, that we were studying in the previous sections. The type of the boundary conditions which are imposed on the matter fields is encoded in the way the \( Q \)-observable, brought near the surface defect, fractionalizes to \( \tilde{Q}_\omega(x), \omega = 0, 1, \) cf. (180). There are four types in total. The Vieregruppe \( D_2 \) acts on the set of regular surface defects, by exchanging \( \tilde{m}_0^+ \) with \( \tilde{m}_0^- \) and/or \( \tilde{m}_1^+ \) with \( \tilde{m}_1^- \).

In what follows we use the \((\tilde{m}_0^+, \tilde{m}_1^+)\) choice (other choices will be important in the computation of the monodromy data in [33]). The \( \tilde{Q}_\omega(x) \)-observables are related to \( Y_\omega(x) \) in this case by:

\[
Y_\omega(x) = (x - \tilde{m}_0^+) \frac{\tilde{Q}_\omega(x)}{\tilde{Q}_{\omega-1}(x - \varepsilon_2)}
\]

(208)

The observables \( \tilde{Q}_\omega(x) \) can be expressed as the ratio of the Euler class of the bundle of zero modes of chiral fermions propagating along \( z_1 = 0 \) surface in the instanton background, and that of the matter fields:

\[
\tilde{Q}_\omega(x) = \mathcal{E} \left[ -e^{\frac{x}{2}} \left( \frac{\mathcal{S} - ((\mathcal{M}^+)*)}{1 - q_2^{-1} \mathcal{R}^{-1}} \right)_\omega \right]
\]

(209)

By comparing (208) to the Eq. (180) we can write:

\[
\tilde{Q}_\omega(x) = Q_\omega(x) \gamma_\omega(x)
\]

(210)

where the functions \( \gamma_\omega(x) \) solve the functional equation:

\[
\frac{\gamma_\omega(x)}{\gamma_{\omega-1}(x - \varepsilon_2)} = \frac{x - \tilde{a}_\omega}{x - \tilde{m}_0^+}.
\]

(211)

The Eq. (211) can be easily solved in \( \Gamma \)-functions, since (211) implies:

\[
\frac{\gamma_\omega(x)}{\gamma_\omega(x - \varepsilon_2)} = \prod_{j=0}^{n-1} \frac{x - \tilde{a}_{\omega-j} - j\varepsilon_2}{x - \tilde{m}_{\omega-j} - j\varepsilon_2}
\]

(212)

In the limit \( \varepsilon_1 \to 0 \) the fractional \( qq \)-character becomes the fractional \( q \)-character obeying the functional equation, for \( n = 2 \):

\[
Y_0(x + \varepsilon_2) + q_1 P_1(x) = (1 + q_1)(x - \tilde{a}_+ - \rho) + \varepsilon_2(\frac{1}{2} - q_1 \mu_1),
\]

(213)

\[
Y_1(x + \varepsilon_2) + q_0 P_0(x) = (1 + q_0)(x - \tilde{a}_+ + \rho) + \varepsilon_2(\frac{1}{2} - q_0 \mu_0),
\]
with (cf. (25), (195))
\[
\rho = a_- + \langle \varepsilon_1(k_0 - k_1) \rangle = u \partial_u S - \frac{\mu_1}{2} + u \frac{\mu_0 + \frac{1}{2}}{2(u - 1)} + u \frac{\mu_1 - \frac{1}{2}}{2(u - q)},
\] (214)
the last equality holding in the \( \varepsilon_1 \to 0 \) limit. The Eqs. (213) are equivalent to the system of linear functional-difference equations for \( \tilde{Q} \)'s. Denoting:
\[
\tilde{Q}_\omega(x) = \tilde{Q}_\omega(a_+ + x \varepsilon_2),
\] (215)
and recalling (183) we rewrite (213) as:
\[
(x + \frac{1}{2} - \mu_0^+) \tilde{Q}_0(x + \frac{1}{2}) + q_1(x - \mu_1^-) \tilde{Q}_0(x - \frac{1}{2}) = \left((1 + q_1)(x - \rho) + \frac{1}{2} - q_1 \mu_1 \right) \tilde{Q}_1(x),
\] (216)
\[
(x + \frac{1}{2} - \mu_1^+) \tilde{Q}_1(x + \frac{1}{2}) + q_0(x - \mu_0^-) \tilde{Q}_1(x - \frac{1}{2}) = \left((1 + q_0)(x + \rho) + \frac{1}{2} - q_0 \mu_0 \right) \tilde{Q}_0(x).
\]
Let us now Fourier transform the \( \tilde{Q}_\omega \)-functions:
\[
\mathcal{Y}(z) = \psi^{ab}(z) \sum_{x \in L} \left( \frac{\tilde{Q}_0(x)}{\tilde{Q}_1(x + \frac{1}{2})} \right) (z/q)^x,
\] (217)
with
\[
\psi^{ab}(z) = z^{\frac{1}{2} - \mu_0^+}(z - 1)^{-\frac{1}{2} - \mu_0^-}(z - q)^{1 + \mu_0^+},
\] (218)
and \( L = L + Z \), a \( \mathbb{Z} \)-invariant lattice \( L \subset \mathbb{C} \), such that the sum in (217) converges. There are several choices for \( L \), dictated by the asymptotics of the sets of zeroes or poles of \( \tilde{Q}_\omega \)'s, respectively. The zeroes are the Chern roots of the bundle of zero modes, while the poles are determined by the masses. The different choices of \( L \) leads to the solutions of (219) which are convergent in various domains of the 4-punctured sphere, such as \(|z| < |q|, |q| < |z| < 1\), and \( 1 < |z| \), respectively.

In terms of \( \mathcal{Y}(z) \) the Eq. (216) is the system of first order differential equations:
\[
\left( \frac{\partial}{\partial z} + A(z) \right) \mathcal{Y} = 0, \quad A(z) = \frac{A_0}{z} + \frac{A_q}{z - q} + \frac{A_1}{z - 1}
\] (219)
with the residues \( A_\xi \) in the form (64), with the parameters summarized in the Table (1). Notice that \( \gamma_{+,-} = \infty \), since \( \gamma_{-,+} \) is finite while \( \beta_\infty = 0 \). Therefore the \( y \)-variable, computed using \( \gamma_{+,\xi} \) is equal to \( u^\circ \)
\[
y = \frac{\gamma_{+,q} - \gamma_{+,0}}{\gamma_{+,1} - \gamma_{+,0}} \cdot \frac{\gamma_{+,1} - \gamma_{+,\infty}}{\gamma_{+,q} - \gamma_{+,\infty}} = u^\circ = w
\] (220)
Now, computing \( p_y \) from (113), \( \tilde{\vartheta} \) from (116), and \( p_y \) from (115) we get:
\[
p_y = \frac{(u - q)^2}{q(q - 1)} \left( \partial_u S + \frac{1}{2(q - u)} \right) = \frac{\partial S^\circ}{\partial w}
\] (221)
Table 1. The residues $A_0, A_q, A_1, A_\infty$

| $\xi$ | $2\partial_\xi$ | $\gamma_{+\xi}$ | $\gamma_{-\xi}$ | $\beta_\xi$ |
|------|-----------------|-----------------|-----------------|---------------|
| 0    | $\mu_1^+ - \mu_0^- - \frac{1}{2}$ | 0               | $\frac{1}{2} + \mu_0^+ + u\mu_0^- + \rho(1-u)$ |               |
| $q$  | $-\mu^+ - 1$    | 1               | $-\beta_1 - \beta_0$ |               |
| 1    | $\mu^-$         | $\frac{1}{\alpha}$ | $-\frac{q(1-u)}{u(1-q)}(\mu_1^- + u\mu_0^- + \rho(1-u))$ |               |
| $\infty$ | $\mu_0^- - \mu_1^- - \frac{1}{2}$ | $\infty$ | $1 - \frac{\mu_1^+ + \rho + \frac{q}{2}(\mu_0^- - \rho)}{2\vartheta_{\infty}}$ | 0              |

and

$$
\begin{align*}
\hat{\vartheta}_0 &= -\frac{\frac{1}{2} + \mu_0}{2} = -\vartheta_0^{DS}, \\
\hat{\vartheta}_q &= \frac{\mu_0^- - \mu_0^+}{2} = -\vartheta_q^{DS}, \\
\hat{\vartheta}_1 &= \frac{\mu_1^+ - \mu_1^-}{2} = \vartheta_1^{DS}, \\
\hat{\vartheta}_\infty &= \frac{\mu_1^+ + \frac{3}{2}}{2} = 1 - \vartheta_\infty^{DS}
\end{align*}
$$

We have thus figured out the meaning of the parameters of the Painlevé VI equation which is the Dyson-Schwinger equation obeyed by the $SU(2)$ surface defect partition function. The coordinate $w = q^{1+\frac{\alpha}{1+q\alpha}}$ is identified with the $y^\circ$-coordinate, so that the isomonodromy problem for the connection $[219]$ is the Painlevé VI equation governed by the Hamiltonian of the form $[43]$ with $\vartheta$'s strictly linear in the masses of the fundamental hypermultiplets.

8. The GIL Conjecture

The only formulas worth working on these days can be named with one or two three-letter abbreviation(s). So here is the GIL formula$^{12}$ proposed in [53] in 2012. It expresses the $\tau$-function $[120]$ of the PVI equation through the $c = 1$ conformal blocks of Virasoro algebra:

$$
\tau(q; \alpha, \beta; \vec{\vartheta}) = q^{-\vartheta_0^2 - \vartheta_1^2} (1-q)^{2\vartheta_0 \vartheta_1} \sum_{n \in \mathbb{Z}} e^{n\beta} C_h(a+n\hbar, \vec{m}) C_h(-a-n\hbar, \vec{m}) Z_{\text{inst}}^{a, \vec{m}}(a+n\hbar, \vec{m}; q),
$$

where $a = \hbar \alpha$

$$
C_h(a, \vec{m}) = \frac{1}{G_h(h + 2a)} \prod_{f=1}^4 G_h(h + m_f + a),
$$

$^{12}$Also known as the Kiev $\tau$-function formula
\[ Z^{\text{inst}}_h(a, \vec{m}; q) = q^{\varepsilon_2^2} \sum_{\lambda, \mu \in \mathcal{P}} q^{\mid \lambda^+ | \mu^+} \frac{\prod_{f=1}^{4} P_{\lambda}(m_f + a) P_{\mu}(m_f - a)}{\left( H_\lambda \cdot H_\mu \cdot \mathcal{F}(2a)_\lambda^\mu \cdot \mathcal{F}(\bar{2}a)_\mu^\lambda \right)^2}, \]  
with \( \mathcal{P} \) denoting the set of all partitions \( \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0) \) of an integer \( |\lambda| = \lambda_1 + \lambda_2 + \ldots \), or Young diagrams, with \((i, j) \in \lambda \) meaning \( 1 \leq j \leq \lambda_i \), and

\[ H_\lambda = \prod_{(i, j) \in \lambda} (\lambda_j^1 - i + \lambda_i - j + 1), \quad P_\lambda(x) = \prod_{(i, j) \in \lambda} \left( x + h(i - j) \right), \]

\[ \mathcal{F}(x)_\lambda^\mu = \prod_{(i, j) \in \lambda} \left( x + h(\lambda_j^1 - i + \mu_i - j + 1) \right), \]

and

\[ \frac{m_1}{h} = \vartheta_0 + \vartheta_\infty, \quad \frac{m_2}{h} = \vartheta_0 - \vartheta_\infty, \quad \frac{m_3}{h} = \vartheta_1 + \vartheta_\infty, \quad \frac{m_4}{h} = \vartheta_1 - \vartheta_\infty. \]

### 8.1. Symmetries, manifest, and less so.

The Ref. [2] observed that by writing

\[ Z_{U(2)}(a, -a, \varepsilon_1, \varepsilon_2, m_1, m_2, m_3, m_4; q) = (1 - q)^{m_1+2m_2} Z_{SU(2)}(a, \varepsilon_1, \varepsilon_2, m_1, m_2, m_3, m_4; q) \]

we obtain the partition function of the \( SU(2) \) gauge theory which is invariant under the \( a \rightarrow -a \), and under the transformations \( m_1 \leftrightarrow m_2 \), or \( m_3 \leftrightarrow m_4 \) (these are manifest in the \( U(2) \) theory as well), and \( m_1 \leftrightarrow \varepsilon_1 + \varepsilon_2 - m_2 \) or \( m_3 \leftrightarrow \varepsilon_1 + \varepsilon_2 - m_4 \). The identification with the Liouville conformal blocks also suggests a symmetry \( q \rightarrow \tilde{q} = -q/(1 - q) \) (it corresponds to the global transformation \( z \mapsto (z - q)/(1 - q) \), sending \( 0, q, 1, \infty \) to \( \tilde{q}, 0, 1, \infty \)), accompanied by the exchange \( \Delta_0 \leftrightarrow \Delta_\infty \leftrightarrow \varepsilon_1 + \varepsilon_2 - m_2 \leftrightarrow m_2 \) (while keeping \( m_1, m_3, m_4 \) intact).

In the case \( \varepsilon_1 = -\varepsilon_2 = h \) the decoupling factor in (228) becomes:

\[ (1 - q)^{2\vartheta_0\vartheta_1} \]

Note that the right hand side of the Eq. (223) is, up to a simple prefactor \( q^\#(1 - q)^\# \) invariant under the permutations of \( m_1, m_2, m_3, m_4 \), as well as the overall change of sign \( \vec{m} \rightarrow -\vec{m} (\lambda \leftrightarrow \mu^t) \), whereas the Painlevé VI equation is invariant under the transformations \( (\vartheta_0, \vartheta_\infty) \rightarrow (\pm \vartheta_0, \pm \vartheta_\infty) \). The permutation \( m_2 \leftrightarrow m_3 \) corresponds to the Okamoto transformation, similar to (116):

\[ (\vartheta_0, \vartheta_\infty) \rightarrow \left( \vartheta_0, \vartheta_\infty, \vartheta_1, \vartheta_\infty \right) = \left( \vartheta_0 - \frac{1}{2} \vartheta, \vartheta_\infty - \frac{1}{2} \vartheta, \vartheta_1 - \frac{1}{2} \vartheta, \vartheta_\infty - \frac{1}{2} \vartheta \right), \]

with

\[ \vartheta = \vartheta_0 + \vartheta_\infty + \vartheta_1 + \vartheta_\infty, \]

while sending the canonical coordinates \((w, p)\) of (143) to \((\tilde{w}, \tilde{p})\), related by, cf. (159):

\[ \frac{\vartheta}{\tilde{w} - w} = p - \frac{\vartheta_0}{w - 1} - \frac{\vartheta_1}{w - q}. \]
The map is a canonical transformation, with the generating function \( \sigma(w, \tilde{w}; q) : \)

\[
p = \frac{\partial \sigma}{\partial w}, \quad \tilde{p} = -\frac{\partial \sigma}{\partial \tilde{w}}, \quad \tilde{H} - H = \frac{\partial \sigma}{\partial q},
\]

\[
e^\sigma = (1 - q)^{\frac{\vartheta(a_0 - \vartheta_0 - \vartheta_1 + \vartheta_\infty - 1)}{2}} \left( \frac{w}{\tilde{w}} \right)^{\vartheta_0} \left( \frac{1 - w}{1 - \tilde{w}} \right)^{\vartheta_1} \left( \frac{w - q}{\tilde{w} - q} \right)^{\vartheta_4 + \frac{1}{2}} \left( \frac{\tilde{w}(\tilde{w} - 1)(\tilde{w} - q)}{(w - \tilde{w})^2} \right)^{\frac{\vartheta_2}{2}}
\]

(233)

8.2. The Mystery of GIL. The mystery of the GIL formula is that the right-hand side of the Eq. (223) is a simple sum\(^{13}\) of the partition functions of \( \Omega \)-deformed \( A_1 \)-type \( \mathcal{N} = 2 \) gauge theory \( Z(\vec{a}, \varepsilon_1, \varepsilon_2; \vec{m}; q) \) on \( \mathbb{R}^4 \), with the following specifications:

- the gauge group is \( U(2) \),
- the Coulomb parameters are fixed at the \( SU(2) \) point, i.e. \( \vec{a} = (a, -a) \),
- there are precisely \( N_f = 4 \) fundamental hypermultiplets, with the masses given by

\[
(m_1, m_2, m_3, m_4) = (\vartheta_q + \vartheta_0, \vartheta_q - \vartheta_0, \vartheta_1 + \vartheta_\infty, \vartheta_1 - \vartheta_\infty) \quad \text{up to a} \ S(4) \ \text{permutation},
\]

(234)

and the \( q \)-parameter of the PVI equation is related to the complexified gauge coupling in the simplest possible way:

\[
q = \exp 2\pi i \tau_{uv}, \quad \tau_{uv} = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}
\]

(235)

Finally, the \( \Omega \)-deformation parameters correspond to the \( c = 1 \) value of the Virasoro central charge in the dictionary of [2]:

\[
\varepsilon_1 = 1 = -\varepsilon_2
\]

(236)

On the other hand, it is in the limit \( \varepsilon_1 \to 0, \varepsilon_2 \to \hbar \) that the surface defect partition function \( \Psi_{SU(2)} \) of the same \( A_1 \) theory, with \( U(1) \)-factor stripped, has the exponential asymptotics

\[
\Psi_{SU(2)}(a, \varepsilon_1, \varepsilon_2, \vec{m}; w, q) \sim \exp \frac{S^\circ(a, \varepsilon_2, \vec{m}; w, q)}{\varepsilon_1},
\]

(237)

where \( S^\circ(a, \varepsilon_2, \vec{m}; w, q) \) obeys the Painlevé VI equation in the Hamilton-Jacobi form [200].

\[
\bullet \sim \bullet \sim \bullet \sim \bullet \sim \bullet \sim \bullet
\]

So, what is going on?

\[
\bullet \sim \bullet \sim \bullet \sim \bullet \sim \bullet \sim \bullet
\]

\(^{13}\)Similar to, but not identical to the magnetic partition function of [128]
8.3. **Four dimensional perspective.** The secret of the GIL relation gets uncovered in the four dimensional picture. To relate the \((\varepsilon_1, \varepsilon_2) \to (0, \hbar)\) limit to the \((\varepsilon_1, \varepsilon_2) \to (\hbar, -\hbar)\) limit, we use the blowup technique applied to the four dimensional gauge theory.

Namely, let us study the partition function of the surface defect in gauge theory on \(\mathbb{R}^4 = \mathbb{C}^2\) and its blowup \(\hat{\mathbb{C}}^2\), in the \(\Omega\)-backgrounds which are identical at infinity.

The partition functions are expected to coincide,

\[
\Psi_{\mathbb{C}^2} = \Psi_{\hat{\mathbb{C}}^2},
\]

as we reviewed in the introduction.

The localization computation, however, gives a non-trivial identity

\[
\Psi_{\hat{\mathbb{C}}^2} = \Psi_{\mathbb{C}^2} \ast Z_{\mathbb{C}^2}. \tag{239}
\]

Here \(\ast\) stands for the convolution involving the sum over the fluxes through the single exceptional two-cycle on \(M' = \hat{\mathbb{C}}^2\) of the products of local contributions to the partition function from the two fixed points \(p_0\) and \(p_{\infty}\) of the torus \(U(1) \times U(1)\) acting on \(\hat{\mathbb{C}}^2\) (see appendix for detail). The analysis identical to \([123, 129]\) gives the formula \([1]\), which we now specialize to the case of \(SU(2)\) theory with \(N_f = 4\) hypermultiplets (and divided both sides by \(\Psi_{\mathbb{C}^2}\):

\[
1 = \sum_{n \in \mathbb{Z}} \frac{\Psi(a + \varepsilon_1 n, \varepsilon_1, \varepsilon_2 - \varepsilon_1, \vec{m}; w, q)}{\Psi(a, \varepsilon_1, \varepsilon_2, \vec{m}; w, q)} Z(a + \varepsilon_2 n, \varepsilon_1 - \varepsilon_2, \varepsilon_2, \vec{m}; q). \tag{240}
\]

Recall that the difference of the effective Coulomb parameters in the contributions of the two fixed points in \([240]\) is due to the contribution of the gauge curvature to the localization equation on the scalar in the vector multiplet:

\[
D_A \sigma + \imath_V F_A = 0 \tag{241}
\]

which in the limit of abelian theory becomes:

\[
d \sigma + \imath_V F = 0. \tag{242}
\]

The \(U(n)\) gauge theory on \(M' = \hat{\mathbb{C}}^2\) is dominated, at low energy, by the gauge field configurations, where \(F = \text{diag}(n_1, \ldots, n_n) F_\hat{A}\), with \(F_\hat{A}\) the curvature of a \(U(1)\)-instanton on \(M'\) (see the Appendix), with small instanton modifications near the zeroes of \(V\). The reason is that the fixed points of the combined \(U(1) \times U(1)\) and \(U(1)^n\) action (the latter being the group of constant \(U(n)\) transformations, whose equivariant parameter is the asymptotic value \(a\) of the scalar \(\sigma\) in the vector multiplet) are the point-like instantons sitting at the fixed points of \(U(1) \times U(1)\) action on \(M'\) (we assume the gauge bundle is endowed with some lift of the \(U(1) \times U(1)\) action), grafted onto the \(U(1) \times U(1)\)-invariant abelian instantons. This rough picture is made more precise using the algebraic-geometric description, where the holomorphic rank \(n\) vector bundle (the structure defined by the instanton gauge field thanks to the \(F^{0,2} = 0\) equation) is extended to the rank \(n\) torsion free sheaf, which in turns splits, thanks to the \(U(1)^n\)-invariance, as a sum of \(n\) ideal sheaves twisted by holomorphic line bundles \(L_i\) (see \([129]\) for more details). On \(M\) such bundles are classified by their first Chern classes.
\( n_i = \int_{S^2} c_1(L_i) \). The Eq. (242) can be now solved by observing that (see the Appendix A for detail)

\[ \tau_V F_\mathcal{A} = \varepsilon_1 \sigma_1 + \varepsilon_2 \sigma_2 \]  
(243)

with the functions \((\sigma_1, \sigma_2)\) taking values \((1, 0)\) and \((0, 1)\) at the fixed points \(p_0\) and \(p_\infty\), respectively. At infinity both \(\Sigma_1, \Sigma_2\) approach zero.

In this way the scalar \(\sigma\) equals

\[ \sigma = a + (\varepsilon_1 \sigma_1 + \varepsilon_2 \sigma_2) \text{diag} (n_1, \ldots, n_n) \]  
(244)

with \(a\) being the asymptotic value at \(r \to \infty\).

Now we are in business – let us take the \(\varepsilon_1 \to 0\) limit of both sides of (240):

\[ 1 = \sum_{n \in \mathbb{Z}} e^{S^0(a+\varepsilon_1 n, \varepsilon_1 m; w, q) - S^0(a, \varepsilon_2 n, \varepsilon_2 m; w, q) + \ldots} \times (Z(a+\varepsilon_2 n, -\varepsilon_2, \varepsilon_2, \bar{m}; q) + \ldots) = \]

\[ e^{-\frac{\partial S^0(a, \varepsilon_2 n, \varepsilon_2 m; w)}{\partial \varepsilon_2}} \sum_{n \in \mathbb{Z}} e^{n \beta} \cdot Z(a+\varepsilon_2 n, -\varepsilon_2, \varepsilon_2, \bar{m}; q) + \ldots, \]  
(245)

where

\[ \beta = \frac{\partial S^0}{\partial a}. \]  
(246)

Now, let us use the homogeneity of \(Z\) and \(\Psi\) under the simultaneous rescaling of \(a, \varepsilon_1, \varepsilon_2, m_1, m_2, m_3, m_4\) to bring (245) to the form:

\[ e^{\frac{1}{\varepsilon_2} \left( 1 - \frac{\sum_{f=1}^{4} m_f}{\bar{m}} \right)} S^0 = \sum_{n \in \mathbb{Z}} e^{n \beta} Z(a+\hat{h} n, \hat{h}, -h, \bar{m}; q) \]  
(247)

Now, recall that \(S^0\) solves the Hamilton-Jacobi equation whose Hamiltonian is quadratic in momenta and masses, and use (47) to obtain (223)\(^{14}\).

9. Conclusions and open questions

In this paper we have been discussing the applications of the four dimensional side of the BPS/CFT correspondence to the “CFT” side. Specifically, we applied the blowup equations, expressing the behavior of the correlation functions of four dimensional supersymmetric (twisted, \(\Omega\)-deformed) gauge theories under the simplest surgeries of the underlying four-manifold, to produce quite unusual identities among the conformal blocks of two dimensional conformal field theories. To fully appreciate these identities the conformal blocks must be analytically continued in their parameters, such as the central charges and spins. However, certain limits make perfect sense within the realm of two dimensional unitary theories.

We mostly concentrated on the example of the four-point conformal block of the \(SL(2)\)-current algebra and the five point conformal block of Virasoro algebra involving a \((1, 2)\) degenerate field. The classical, by now, relation [42], which in the large \(k\), large \(c\) limit becomes the relation between the Painlevé VI equation and the isomonodromy problem, does not require much sophisticated reasoning. However, the recently discovered [53] unexpected connection between the same isomonodromy problem, Painlevé

\(^{14}\) Also, recall that \(Z(a) = \mathcal{C}(a)\mathcal{C}(-a) Z^{\text{inst}}(a)\)
VI and the four-point $c = 1$ Virasoro conformal blocks becomes explained using the simple blowup formula applied to the $SU(2)$ $N_f = 4$ gauge theory, in the presence of a surface defect.

Below we list a few other potential applications of these ideas.

9.1. **Quantization and extremal correlators.** The recent work \[66\] on extremal correlators in $\mathcal{N} = 2$ superconformal field theories makes an extensive use of the $S^4$ partition function of the theory, which was originally computed by V. Pestun in \[143\] in terms of the $Z$-function \[127\] of the $\Omega$-deformed theory with $\varepsilon_1 = \varepsilon_2 = \frac{1}{R}$, $R$ being the sphere radius. Althought the analysis of \[66\] mostly relies on the perturbative, one-loop, part of the $Z$-function, it is conceivable a full non-perturbative treatment hides an interesting story. As we reviewed in \[132\], the instanton partition function $Z_{\text{inst}}(a,\varepsilon_1,\varepsilon_2, m; q)$ for $\varepsilon_1/\varepsilon_2$ a positive rational number is not given by the sum over $N$-tuples of Young diagrams. The reason is that the fixed points of the rotational symmetry on the moduli space of instantons are not isolated, albeit the fixed points sets are compact. The blowup formula gives a way to partly circumvent this difficulty. Let us use the homogeneity of the partition function of the asymptotically conformal theory, and the symmetry $\varepsilon_1 \leftrightarrow \varepsilon_2$ to write it as:

$$Z(a, \varepsilon_1, \varepsilon_2, m; q) = Z(a/\varepsilon_1, \varepsilon_1, m/\varepsilon_1; q) = Z(a/\varepsilon_2, \varepsilon_2, m/\varepsilon_2; q)$$

(248)

The small $\varepsilon_2$ expansion of $Z$ defines the effective twisted superpotential and the effective dilaton coupling:

$$Z(a, \varepsilon_1, \varepsilon_2, m; q) = \exp\frac{\varepsilon_1}{\varepsilon_2} w_0(a/\varepsilon_1, m/\varepsilon_1; q) + w_1(a/\varepsilon_1, m/\varepsilon_1; q) + \ldots$$

(249)

The function $\varepsilon_1 w_0$ is the effective twisted superpotential of an effective two dimensional $\mathcal{N} = 2$ theory living on the fixed plane of the remaining $\varepsilon_1$-rotation. The genus one term $w_1$ is the dilaton coupling, computed for a class of two dimensional theories in \[137\], and presumably computable along the lines of the recent discussion in \[112\].

Then the chiral block of $S^4$ partition function, with the radius of the sphere $R$, and some prescription for the masses $m$ of the matter fields (see \[141\] for the relevant discussion),

$$z(a R, 1, m R; q) =$$

$$\operatorname{Lim}_{\varepsilon_1, \varepsilon_2 \to \frac{1}{R}} \sum_{a \in \mathbb{A}} e^{\varepsilon_1 \varepsilon_2^{-1} w_0(\frac{a}{\varepsilon_1} + n m e_1) + w_1(\frac{a}{\varepsilon_1} + n m e_1)} + \varepsilon_1 \varepsilon_2^{-1} w_0(\frac{a}{\varepsilon_2} + n m e_2) + w_1(\frac{a}{\varepsilon_2} + n m e_2) + \ldots =$$

$$\sum_{a \in \mathbb{A}} e^{-w_0(\alpha, \mu; q) + 2 w_1(\alpha, \mu; q) + R m \partial_{\alpha} w_0(\alpha, \mu; q) + R m \partial_{\mu} w_0(\alpha, \mu; q)} \bigg|_{\alpha = a R + n, \ \mu = m R}.$$  (250)

In this way the problems of computing the extremal correlators, the study of $c = 25$ Liouville theory etc. are connected to the quasiclassical conformal blocks, geometry of the variety of opers, and quantization of algebraic integrable systems.
9.2. Blowups and boundary operator product expansion. Recall that the supersymmetric $\mathcal{N} = 2$ theories in four dimensions have an intimate connection with hyper-Kähler geometry. More specifically, the moduli space $\mathcal{M}$ of vacua contains branches (the Higgs branches) which are hyper-Kähler manifolds, perhaps with singularities, which connect to other branches (the Coulomb branches), which are special Kähler manifolds. Once the theory is compactified on a circle, these other branches get enhanced to hyper-Kähler manifolds \[ M_H \].

The four dimensional $\mathcal{N} = 2$ theory on a manifold $M^4$ with $T^2 = U(1) \times U(1)$-isometry, subject to the $\Omega$-deformation associated with this isometry, can be viewed, at low energy, as a two dimensional sigma model with the worldsheet $\Sigma = M^4/T^2$ and $\mathcal{M}_H$ as a target space. The fixed points or irregular orbits of $T^2$ give rise to the boundaries and corners of $\Sigma$. The study of boundary conditions in the sigma model, or branes, descending from the torus fixed loci, and surface defects added at these loci, is a rich and fruitful subject \[ 139 \].

In this note we would like to point out that the blow-up of a fixed point of $T^2$ action modifies $\Sigma$ in a simple way, by cutting a little piece of the corner (and creating two corners instead, see Fig. in the appendix A.1.2). The values of the $\Omega$-deformation parameters correspond to the type of the brane on $\mathcal{M}_H$ a piece of the boundary of $\Sigma$ supports. The corners are, typically, boundary-changing vertex operators (such as the c.c.-brane/brane of opers in the case of $\mathcal{M}_H$ being the Hitchin moduli space). The transformation $(\varepsilon_1, \varepsilon_2) \mapsto (\varepsilon_1 - \varepsilon_2, \varepsilon_2)$ which we see at one of the corners after the blowup can be viewed as turning on a unit flux of a $U(1)$ Chan-Paton gauge field on a c.c.-brane. It would be nice to work out this correspondence in full detail.

The blowup formulae \[ 1, \] then teach us something about the operator product expansion of these special (geometric) vertex operators.

9.3. Parts I, II, III, IV, and V. An asymptotically conformal field theory can be made asymptotically free by sending some masses to infinity while keeping some relevant energy scales finite. For example, by sending $m_4 \to \infty$ while keeping $\Lambda^2 = m_3^2 e^{2\pi i \tau} \infty$ finite we get the $SU(2)$ gauge theory with $N_f = 3$ massive hypermultiplets in the fundamental representation. Similarly, we can furthermore send $m_3, m_2$, and, finally, $m_1 \to \infty$, leaving us with the pure $SU(2)$ $\mathcal{N} = 2$ theory, the original model studied in \[ 138 \].

The corresponding limits can be applied to the partition function $Z(a, \bar{m}, \varepsilon_1, \varepsilon_2, \tau)$, as was done in \[ 127, 128 \]. The blowup formulas hold for the asymptotically free theories (in fact, they can be proven with less pain by ghost number anomaly considerations, \[ 108 \]). One expects therefore the formulas similar to the GIL formula for the Painlevé V, IV, etc. $\tau$-functions, as indeed proposed in \[ 54 \]. The stumbling blocks in these identifications are the good Darboux coordinates on the moduli spaces of flat connections with irregular singularities (which is what the asymptotically free limits amount to). Fortunately a nice theory seems to be emerging \[ 115 \].

9.4. Quasinormal modes. Another application of the blowup equations might be the study of the black hole quasinormal modes. It was observed in \[ 133 \], that the Schrödinger equation obtained by separating variables in the wave equation describing
particles in the black hole background, has an interesting WKB geometry, related to Seiberg-Witten integrable systems. In recent papers [1] this analogy was shown to hold on a much more detailed level, with the relevant $N = 2$ four dimensional theory being an $SU(2)$ gauge theory with $N_f = 2, 3$ fundamental hypermultiplets. The quantization problem, which emerges in the $\varepsilon_2 \to 0, \varepsilon_1 = \hbar$ NS limit [136] of that theory is related in [4] to the $\varepsilon_1 = -\varepsilon_2$ limit of the same theory. This is surely another manifestation of the blowup relations we explored in this paper, adapted to the asymptotically free theory case. It is remarkable that [4] also connects this story to the resurgent expansions. We were reminded that an earlier work [30] also used the Painlevé transcendents and the GIL formula in the studies of scattering of black holes and their quasinormal modes.

This is a peculiar example of an intricate relation between gauge theory and gravity.

9.5. Strong/weak coupling dualities. The Painlevé equations are traditionally used as a testing ground for the resurgence ideas, attempting in reconstructing the non-perturbative description of a physical system by resumming its perturbation expansion in terms of trans-series. Our blowup formulas are, in many respects, a way of relating weak and strong coupling expansions of the same theory, and should probably be recast in the form of the resurgent expansion. See [40] for the recent discussion of closely related problems.

9.6. Whitham hierarchies. It is tempting to conjecture, based on the possibility of subjecting $\mathcal{N} = 2$ theory to a two dimensional $\Omega$-deformation, that the analogues of the isomonodromic deformations are universal companions of the algebraic integrable systems, even outside the domain of Hitchin systems. Perhaps the universal Whitham hierarchies [97] are the answer (see [156] for some discussion to that effect). The first interesting examples would be the moduli spaces of ADE instantons on $\mathbb{R}^2 \times \mathbb{T}^2$ and the moduli spaces of ADE monopoles on $\mathbb{R}^{d+1} \times \mathbb{T}^{2-d}$, which are the moduli spaces of vacua of quiver gauge theories, as was rigorously shown in [135] using gauge theory analysis, and, in the $A$-type case in [31], using string dualities.

9.7. Higher genus, more points... In the appendix we discuss some generalizations to the case of higher rank gauge theories in four dimensions, higher rank current algebras in two dimensions, more punctures, higher genus.

9.8. Five dimensions, quantizations and topological strings. The uplift of the blowup relations to the case of five dimensional theories [126] compactified on a circle, is very interesting from many perspectives. Mathematically the story is much more complicated (ref. 3 in [123]).

In [1], [64], [68], and, more recently [67] various features of topological strings on local Calabi-Yau manifolds has been observed to connect to quantum integrable systems, with the string coupling $g_s$ playing the role of the Planck constant. This is in contrast with the conjectures in [127] (mostly proven today) relating the topological string to the $\varepsilon_1 = -\varepsilon_2 = g_s$ case of the $\Omega$-deformed four dimensional theory, while quantum integrability is found in the $\varepsilon_2 \to 0, \varepsilon_1 = \hbar$ NS limit [136].

Now we know how to relate the $(\varepsilon_1, \varepsilon_2) \to (\hbar, -\hbar)$ limit to the $(\varepsilon_1, \varepsilon_2) \to (\hbar, 0)$ limit, using the blowup formulas. In the five dimensional version, we would be relating
the theories at \((q_1, q_2)\) to \((q_1, q_2/q_1)\) and \((q_1/q_2, q_2)\), so that the topological string limit \((e^{g_s}, e^{-g_s})\) would be related to the NS limit. Hopefully the blowup formula will also shed some light on the mysteriously beautiful proposal of \([41]\) to add the effective twisted superpotentials of theories with S-dual \(q\)-parameters.

Another possible extension of our story might be the search for the analogue of the \(y\)-coordinates in the topological \(B\) model, as a way to parametrize the holomorphic anomaly equation \([159]\).

**Appendix A. Blowup, Conifold, and Origami**

A.1. Blowup of a plane. From now on we are working with the local model of \(M\), namely a copy of Euclidean space \(\mathbb{R}^4\) which we shall identify with \(\mathbb{C}^2\) by fixing one of its complex structures. We shall place the point \(p\) at the origin \(0 \in \mathbb{C}^2\). The blowup \(M'\) in this case is denoted by \(\hat{\mathbb{C}}^2\), and can be defined as the space of pairs \((u, l)\), where \(u \in \mathbb{C}^2\), and \(l \approx \mathbb{C}^1\) is a complex line passing through \(u\) and \(0\). Forgetting \(l\) defines the map:

\[
p : \hat{\mathbb{C}}^2 \longrightarrow \mathbb{C}^2, \quad p(u, l) = u
\]

(251)

For \(u \neq 0\), \(p^{-1}(u)\) consists of exactly one point, as there is only one line passing through two distinct points on a plane. For \(u = 0\) the preimage \(E\) is the space of all lines passing through a point on a plane, also known as the projective line \(\mathbb{C}P^1 = p^{-1}(0)\). This is the exceptional set \(E\) we mentioned previously. We can cover \(\hat{\mathbb{C}}^2\) with two coordinate charts \(U_0\) and \(U_\infty\), both isomorphic to \(\mathbb{C}^2\), with the transition function:

\[
(z_1, l_1) \in U_0 \mapsto (l_2, z_2) \in U_\infty, \quad z_1l_1 = z_2, \quad z_2l_2 = z_1
\]

(252)

which is invertible when both \(l_1, l_2 \neq 0\) (and are mutual inverses, \(l_1l_2 = 1\)), so that \(U_0 \cap U_\infty = \mathbb{C} \times \mathbb{C}^\times\).

A.1.1. Kähler quotient construction. One description of \(\hat{\mathbb{C}}^2\) is via the Kähler quotient of \(\mathbb{C}^3\) by the \(U(1)\) action:

\[
(w_1, w_2, w_3) \approx (e^{i\ell}w_1, e^{i\ell}w_2, e^{-i\ell}w_3)
\]

(253)

as \(\mathbb{C}^3//U(1)\):

\[
|w_1|^2 + |w_2|^2 - |w_3|^2 = \zeta > 0.
\]

(254)

The coordinate patches \(U_0\) and \(U_\infty\) correspond to the loci where \(w_1 \neq 0\) and \(w_2 \neq 0\), respectively. Indeed, \([254]\) implies that, for \(\zeta > 0\), \(w_1\) and \(w_2\) cannot vanish simultaneously. The holomorphic coordinates on \(U_0\) are the holomorphic gauge invariants \(z_1 = w_3w_1\) and \(l_1 = w_2/w_1\), while the holomorphic coordinates on \(U_\infty\) are the gauge invariants \(l_2 = w_1/w_2\) and \(z_2 = w_3w_2\). Restricting the flat metric on \(\mathbb{C}^3\) onto the surface \(S\) defined by the Eq. \((254)\) and projecting along the \(U(1)\) action gives a metric on \(M' = \hat{\mathbb{C}}^2\), which is Kähler with the Kähler potential, which we express in terms of \(z_1 = w_3w_1\) and \(z_2 = w_3w_2\):

\[
K_{kq}(z, \bar{z}) = \text{Crit}_V \left(e^{-2V}|z_1|^2 + e^{-2V}|z_2|^2 + e^{2V} + 2\zeta V\right) = 2\ell + \zeta \log (\ell),
\]

\[
\ell = \sqrt{|z_1|^2 + |z_2|^2 + \zeta^2/4} - \zeta/2.
\]

(255)
For small $|z_1|, |z_2|$ we can approximate $K_{kq}$ by the sum of the Euclidean space Kähler potential $\propto |z_1|^2 + |z_2|^2$ and that of a two-sphere $\propto \log(|z_1|^2 + |z_2|^2)$. The Kähler form $\omega_{kq} = d\bar{z}dK_{kq}$ has a non-trivial period over the two cycle $E$ given by the equation $w_3 = 0$.

Let $w_a = |w_a|e^{i\varphi_a}, \ a = 1, 2, 3$. Another structure we get from the restriction of the flat $\mathbb{C}^3$ metric onto the surface $S$ is the Riemannian connection on the $U(1)$-bundle

$$\pi : S \to M'$$

The corresponding connection form $A_{kq}$ is also obtained by minimizing:

$$\partial_{\bar{z}} \bigg|_{A=A_{kq}} \left( |dw_3 - iA_{kq}w_3|^2 + \sum_{\alpha=1}^2 |dw_\alpha + iA_{kq}w_\alpha|^2 \right) = 0,$$

$$A_{kq} = \frac{|w_3|^2 d\varphi_3 - |w_1|^2 d\varphi_1 - |w_2|^2 d\varphi_2}{w_1^2 + |w_2|^2 + |w_3|^2} = d\varphi_3 - \frac{r_1^2 d\theta_1 + r_2^2 d\theta_2}{r_1^2 + r_2^2 + r^2}. \quad (257)$$

Naturally, it obeys: $\nu A_{kq} = 1$, where $\nu = -\partial_{\bar{z}} + \partial_{\varphi_1} + \partial_{\varphi_2} = \partial/\partial t$ is the generator of the (253) action, and $dA_{kq} = \pi^* F_{kq}$, with (256):

$$F_{kq} = -d \left( \frac{(r_1/r)^2}{1 + (\ell/r)^2} \right) \wedge d\theta_1 - d \left( \frac{(r_2/r)^2}{1 + (\ell/r)^2} \right) \wedge d\theta_2$$

(258)

where $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$. The formula for the curvature $F_{kq}$ gets a bit tricky when both $r_1$ and $r_2$ approach zero. On the patch $U_0$ the good coordinates are $z_1 = r_1 e^{i\theta_1}$, and $z_2/z_1 = r e^{i\theta}$, so we need to rewrite (258) in terms of $r = r_2/r_1$, $\theta = \theta_2 - \theta_1$:

$$F_{kq} = d \left( \frac{\lambda_1^2}{1 + r^2 + \lambda_1^2} \right) \wedge d\theta_1 - d \left( \frac{r^2}{1 + r^2 + \lambda_1^2} \right) \wedge d\theta,$$

(259)

where

$$\lambda_1 = -\frac{\zeta}{2r_1} + \sqrt{1 + r^2 + \frac{\zeta^2}{4r_1^2}} \approx \frac{r_1}{\zeta} (1 + r^2) + \ldots, \ r_1 \to 0$$

(260)

Analogous expression holds on $U_\infty$ with $r_1 \leftrightarrow r_2$, $r \leftrightarrow 1/r$, $\theta \leftrightarrow -\theta$, etc.

A.1.2. Torus action on $M'$. The two torus $U(1) \times U(1)$ acts of $\mathbf{C}^2$ by $(w_1, w_2, w_3) \mapsto (e^{i\theta_1}, e^{i\theta_2} w_1, w_2, w_3)$, which, in coordinate patches, looks like the standard $U(1) \times U(1)$ action on $\mathbf{C}^2$ albeit in different bases of generators: on $U_0$ it is given by $(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, l_1) = (e^{i\theta_1} z_1, e^{i(\theta_2-\theta_1)} l_1)$, while on $U_\infty$ it is given by: $(e^{i\theta_1}, e^{i\theta_2}) \cdot (l_2, z_2) = (e^{i(\theta_1-\theta_2)} l_2, e^{i\theta_2} z_2)$. The fixed points of the $U(1) \times U(1)$ action on $\mathbf{C}^2$, therefore, are: $p_0 = (0, 0) \in U_0$, and $p_\infty = (0, 0) \in U_\infty$. Recall, that since $l_{1, 2} \neq 0$ on $U_0 \cap U_\infty$, the points $p_0$ and $p_\infty$ are distinct.
It is not difficult to compute the generators of the $U(1) \times U(1)$ action. In the $z_1, z_2$ coordinates, the transformation $(z_1 e^{i\varepsilon_1}, z_2 e^{i\varepsilon_2})$ is generated by the vector field

$$V = \varepsilon_1 \partial_{\theta_1} + \varepsilon_2 \partial_{\theta_2},$$

which is a Hamiltonian vector field preserving the symplectic form $\varpi_{kq} = \partial \bar{\partial} K_{kq}$, with the Hamiltonian function

$$h = \varepsilon_1 h_1 + \varepsilon_2 h_2 = \frac{\varepsilon_1 |z_1|^2 + \varepsilon_2 |z_2|^2}{|z_1|^2 + |z_2|^2} \zeta + \ell.$$  

(262)

At the fixed point $p_0$ we need to use the $U_0$ parameterization:

$$h = \frac{\varepsilon_1 + \varepsilon_2 r^2 \zeta + r_1 \lambda_1}{1 + \frac{r^2}{2}},$$

(263)

so that $h(p_0) = \varepsilon_1 \zeta / 2$. Similarly, $h(p_\infty) = \varepsilon_2 \zeta / 2$.

A.1.3. Another metric and another gauge field. Canonical as it is the Kähler quotient construction does not always produce nice results when the amount of underlying supersymmetry is too low (in plain English, things are not hyper-Kähler). In particular, the gauge field $A_{kq}$ is not anti-self-dual in the Kähler metric defined by (255). Fortunately, there exists a simple nudge to the formulas (255), (258), making everything fall in place. We endow $M'$ with the Kähler metric $\varpi_{k\zeta} = \frac{dd^c K_{k\zeta}}{\zeta}$, with

$$K_{\zeta} = \frac{|z_1|^2 + |z_2|^2}{\zeta} + \zeta \log \left(\frac{|z_1|^2 + |z_2|^2}{\zeta}\right)$$

(264)

so that the cohomology class of $\varpi_{k\zeta}$ equals to that of $\varpi_{kq}$. The metric corresponding to (264) is found in [28] and is remarkable in that it is Weyl anti-self-dual, $W^+ = 0$ and scalar free. A larger class of metrics containing those on multi-blowups is in [102]. In [23] a charge 1 instanton was found:

$$A_\zeta = \frac{(r_1/r)^2 d\theta_1 + (r_2/r)^2 d\theta_2}{1 + (r/\zeta)^2}.$$  

(265)

This is the gauge field we use in the low energy description of gauge theory on $M'$. We have:

$$- \iota_\nu F_{A} = \varepsilon_1 d\Sigma_1 + \varepsilon_2 d\Sigma_2$$

(266)
with
\[ \Sigma_\alpha = \frac{(r_\alpha / r)^2}{1 + (r/\zeta)^2} \]  

(267)

A.2. Blowup and conifold. It is often useful to view the four dimensional \( \mathcal{N} = 2 \) gauge theory as the theory of low energy modes of open strings attached to \( D3 \) branes of \( IIB \) string theory, in some supersymmetric ten dimensional background. The ten dimensional geometry \( \mathcal{X} \) must contain a \( \mathbb{C} \approx \mathbb{R}^2 \) factor, transverse to the worldvolume of the branes, representing the complex line of eigenvalues of the scalars in the vector multiplets, i.e. \( \mathcal{X} = \mathbb{C} \times \mathcal{Y} \). The simplest way to achieve supersymmetry is to demand \( \mathcal{Y} \) to be a Calabi-Yau fourfold. In [132] we took \( \mathcal{Y} = \mathbb{C}^4 \). The construction below corresponds to \( \mathcal{Y} = \mathcal{Z} \times \mathbb{C} \), with \( \mathcal{Z} \) the total space of the \( \mathcal{O}(-1) \oplus \mathcal{O}(-1) \) bundle over \( \mathbb{C} \mathbb{P}^1 \), also known as a small resolution of the conifold singularity

\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \quad x_i \in \mathbb{C} \]  

(268)

where \( x_1 + ix_2 = w_1 w_3 \), \( x_1 - ix_2 = w_2 w_4 \), \( x_3 + ix_4 = w_2 w_3 \), \( x_3 - ix_4 = -w_1 w_4 \) are the gauge invariant holomorphic coordinates on the Kähler quotient \( \mathbb{C}^4 / U(1) \) of \( \mathbb{C}^4 \) with the coordinates \( w_1, w_2, w_3, w_4 \) by the \( U(1) \) action

\[ (w_1, w_2, w_3, w_4) \mapsto (e^{it} w_1, e^{it} w_2, e^{-it} w_3, e^{-it} w_4) , \]  

(269)

where we choose the positive level \( \zeta > 0 \) of the moment map

\[ |w_1|^2 + |w_2|^2 - |w_3|^2 - |w_4|^2 = \zeta \]  

(270)

We observe that \( M' = \hat{\mathbb{C}}^2 \subset \mathcal{Z} \) is a hypersurface, defined by the equation homogeneous equation \( w_4 = 0 \). This equation has a charge \( -1 \) under the gauge group of the Kähler quotient, so the normal bundle to \( M' \) inside \( \mathcal{Z} \) is topologically nontrivial, without global holomorphic sections, meaning a stack of \( D \)-branes wrapping \( M' \) inside \( \mathcal{Z} \) will not have infinitesimal deformations within \( \mathcal{Z} \).

However, if one uses a gauge-invariant function, e.g. \( x_1 - ix_2 \), or \( x_3 + ix_4 \), its zeroes define a two-component surface, \( M' \cup M'' \), with \( M'' \) being a copy of \( \mathbb{C}^2 \), a vanishing locus of \( w_2 \).

Recall the crossed instanton construction of [132]. It looks at the generalized gauge theory whose worldvolume is a union of two surfaces, each isomorphic to \( \mathbb{C}^2 \) with some multiplicity, intersecting transversely inside \( \mathcal{Y} \approx \mathbb{C}^4 \). Such singular surface is invariant under the action of the Calabi-Yau torus \( U(1)^3 \subset SU(4) \). In the context of the present paper we will be interested in the union of the surface defined by the equation \( w = w_4 = 0 \) (with \( w \) being the coordinate on the \( \mathbb{C} \)-factor in \( \mathcal{Y} = \mathcal{Z} \times \mathbb{C} \), again with some multiplicity, corresponding to the rank of the gauge group, and the (multiple) surfaces located at \( w_1 = w_3 = 0 \), and at \( w_2 = w_3 = 0 \). In other words, there are two fundamental \( gg \)-characters in the \( \Omega \)-deformed theory on \( \hat{\mathbb{C}}^2 \), one inserted at the fixed point \( p_0 \) and another at the fixed point \( p_\infty \).

**Appendix B. Instantons on \( \mathbb{R}^4 \) and on \( \hat{\mathbb{C}}^2 \)**

The instantons, i.e. minimal Euclidean action solutions to the Yang-Mills equations, were first found in [15]. It was soon found that these solutions come in families, so
that on a compact manifold $X$ the charge $k$ $G$-instantons have
\[ 4h^*c_k - \frac{\chi + \sigma}{2}\dim G \]
moduli. The complete description of the moduli space of charge $k$ instantons on four-sphere $M^4 = S^4$ was given in [8]. We shall need a version of that construction, in which $M^4 = \mathbb{R}^4$, so that a priori one may get an infinite dimensional space of solutions. However, with appropriate boundary conditions imposed at infinity the corresponding moduli space is finite dimensional. The construction for gauge group $U(n)$ can be summarized as follows. Fix two complex vector spaces, $K$ and $N$, of dimensions $k$ and $n$, respectively, endowed with Hermitian metrics. Consider the space of quadruples $(B_1, B_2, I, J)$ where $B_{1,2} \in \text{End}(K, K)$, $I \in \text{Hom}(N, K)$, $J \in \text{Hom}(K, N)$, obeying the equations $\mu^c = 0$ and $\mu^R = 0$:
\[ \mu^c = [B_1, B_2] + IJ, \quad \mu^R = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J \]  
(272)
The moduli space $\mathcal{M}(n, k)$ of framed charge $k$ instantons with gauge group $SU(n)$ is isomorphic to
\[ \mathcal{M}$^c(n, k) = (\mu^{c - 1}(0) \cap \mu^{r - 1}(0)) / U(k) \]  
(273)
where the term framed means that $\mathcal{M}(n, k)$ parameterizes the solutions to the equation
\[ F_A + *F_A = 0, \quad F_A = dA + A \wedge A \]  
(274)
where $A$ is an $\mathfrak{su}(n)$-valued (i.e. $A = ia_\mu dx^\mu$, with $a_\mu, \mu = 1, 2, 3, 4$ being traceless $n \times n$ Hermitian matrices) one-form on $\mathbb{R}^4$, obeying
\[ A(x) \rightarrow h^{-1}_x dh, \quad x \rightarrow \infty, \quad h : S^3 \rightarrow SU(n) \]  
(275)
modulo the equivalence relation
\[ A \sim h^{-1} Ah + h^{-1} dh, \quad h \rightarrow 1 + O(1/r^2) \]  
(276)
The charge $k$ is the degree of the map $g_\infty$, which can be computed as the integral:
\[ k = \frac{1}{8\pi^2} \int_{S^3} \text{Tr}(g^{-1}_\infty d g_\infty)^3 \]  
(277)
The group $U(k)$ in (273) acts via: $g \cdot (B_1, B_2, I, J) = (g^{-1}B_1g, g^{-1}B_2g, g^{-1}I, Jg)$, and the symbol $\circ$ in (273) means to keep only those $(B_1, B_2, I, J)$ solving the $\mu^c = \mu^R = 0$ equations, whose the stabilizer is trivial, i.e. $g \cdot (B_1, B_2, I, J) = (B_1, B_2, I, J)$ iff $g = 1$. Differential geometers often work with the partially compactified Uhlenbeck space
\[ \mathcal{M}(n, k) = (\mu^{c - 1}(0) \cap \mu^{r - 1}(0)) / U(k) \]  
(278)
which parametrizes the so-called ideal framed instantons, i.e. the collections: $[A']$ where $[A'] \in \mathcal{M}(k - l, n)$, $x \in \text{Sym}^l(\mathbb{R}^4) = (\mathbb{R}^4)^l / \text{Sym}(l)$. However, $\mathcal{M}(n, k)$ is not very useful for doing computations since it is a singular contractible space, and we want to use topology to circumvent doing direct integration. The so-called Gieseker-Nakajima space [122]
\[ \widetilde{\mathcal{M}}(n, k) = (\mu^{c - 1}(0) \cap \mu^{r - 1}(\zeta \cdot 1_K)) / U(k) \]  
(279)
comes in handy. It parametrizes the torsion free sheaves $\mathcal{N}$ on $\mathbb{CP}^2 = \mathbb{C}^2 \cap \mathbb{CP}^1_\infty$ with the trivialization at $\mathbb{CP}^1_\infty$, $\mathcal{N}|_{\mathbb{CP}^1_\infty} \approx N \otimes \mathcal{O}_{\mathbb{CP}^1_\infty}$. Physically it is the moduli space of $U(n)$
instantons (note the change of the gauge group) on non-commutative $\mathbb{R}^4$ \cite{3} which is (very) roughly the algebra generated by $z_1, z_2, \bar{z}_1, \bar{z}_2$ obeying $[z_1, z_2] = 0, [\bar{z}_1, \bar{z}_2] = 0, [z_1, \bar{z}_1] = \theta_1 \cdot 1, [z_2, \bar{z}_2] = \theta_2 \cdot 1$ with $\theta_1 + \theta_2 = -\zeta$.

B.1. **Instantons on blowup.** The ADHM construction extends to the blowup $\hat{\mathbb{C}}^2$ \cite{123}. One fixes now three Hermitian spaces $N, K', K''$ of dimensions $n, k', k''$, respectively. Consider the space of 5-tuples: $(b_1, b_2, b, i, j)$, where $b_{1,2} \in \text{Hom}(K', K'')$, $b \in \text{Hom}(K'', K')$, $i \in \text{Hom}(N, K'')$, $j \in \text{Hom}(K', N)$, obeying $m_C = 0, m'_R = \zeta' \cdot 1_{K'}$, $m''_R = \zeta'' \cdot 1_{K''}$, where $m_C = b_1 b b_2 - b_2 b b_1 + i j : K' \rightarrow K''$, and

$$
m'_R = -b_1^\dagger b_1 - b_2^\dagger b_2 - j^\dagger j + b b^\dagger, \\
m''_R = b_1 b_1^\dagger + b_2 b_2^\dagger + i i^\dagger - b^\dagger b,
$$

(280)

The formula, which is (for $n = 2$) equivalent to the main theorem in \cite{123}, is

$$Z(\tilde{a}, \varepsilon_1, \varepsilon_2, \bar{m}; q) = \sum_{\tilde{n} \in \mathbb{Z}^n \cap C Z_n} Z(\tilde{a} + \tilde{n} \varepsilon_1, \varepsilon_1 - \varepsilon_1, \bar{m} + \bar{\rho} \varepsilon_1; q) Z(\tilde{a} + \tilde{n} \varepsilon_2, \varepsilon_2 - \varepsilon_2, \bar{m} + \bar{\rho} \varepsilon_2; q)
$$

(281)

where $\tilde{n} = (n_1, \ldots, n_n) \in \mathbb{Z}^n$, $n_1 + \ldots + n_n = 0$, and $\bar{\rho}$ is determined by the choice of the matter fields twists.

B.1.1. **Perturbative check.** The classical and one-loop piece of $Z$ is given by \cite{5}

$$Z^{\text{pert}}(\tilde{a}, \varepsilon_1, \varepsilon_2, \bar{m}; q) = q^{-c_{\text{ch}}(\mathcal{H}_0)} \mathcal{E} \left[ \frac{-N N^* + N^* M}{(1 - q_1^{-1})(1 - q_2^{-1})} \right],
$$

(283)

where

$$\mathcal{H}_0 = \frac{N}{(1 - q_1)(1 - q_2)}, \quad c_{\text{ch}}(\mathcal{H}_0) = \sum_{\alpha = 1}^{n} a_\alpha (a_\alpha - \varepsilon_1 - \varepsilon_2) + \frac{n (1 + Q^2)}{12},
$$

(284)

with

$$Q^2 = \frac{(\varepsilon_1 + \varepsilon_2)^2}{\varepsilon_1 \varepsilon_2}
$$

(285)

The identity

$$\frac{1}{(1 - q_1^{-1})(1 - q_1 q_2^{-1})} + \frac{1}{(1 - q_2 q_1^{-1})(1 - q_2^{-1})} = \frac{1}{(1 - q_1^{-1})(1 - q_2^{-1})}
$$

(286)

implies the first check of \cite{3}:

$$Z^{\text{pert}}(\tilde{a}, \varepsilon_1, \varepsilon_2, \bar{m}; q) = Z^{\text{pert}}(\tilde{a}, \varepsilon_1 - \varepsilon_1, \bar{m}; q) Z^{\text{pert}}(\tilde{a}, \varepsilon_2 - \varepsilon_2, \bar{m}; q) Z^{\text{pert}}(\tilde{a}, \varepsilon_1 - \varepsilon_2, \bar{m}; q)
$$

(287)

\textsuperscript{15}Note in passing, that we insist on using our normalizations, which differ from those in \cite{2}:

$$\mathcal{E} \left[ - \sum_{\alpha < \beta} \frac{N_\alpha^* N_\beta q_1 + N_\alpha^* N_\beta q_2}{(1 - q_1^{-1})(1 - q_2^{-1})} \right]
$$

(282)
B.2. ADHM construction for \( \mathcal{N} = 2^* \) theory. In [132] we proposed a generalization of the ADHM construction for instantons on \( \mathbb{R}^4 \) which geometrizes the instanton counting in \( \mathcal{N} = 2 \) gauge theory with massive adjoint hypermultiplet. Namely, in addition to the \((B_1, B_2, I, J)\) matrices one introduces the operators \( Z_{1,2} \in \text{End}(K_{16}) \) so that the sextuple \((B_1, B_2, Z_1, Z_2, I, J)\) obeys:

\[
\begin{align*}
[B_1, B_2] + IJ + [Z_1, Z_2]^\dagger & = 0, \\
[B_1, Z_1] - [B_2, Z_2]^\dagger & = 0, \\
[B_1, Z_2] + [B_2, Z_1]^\dagger & = 0, \\
II^\dagger - J^\dagger J + \sum_{a=1}^2 ([B_a, B_a^\dagger] + [Z_a, Z_a^\dagger]) & = \zeta \cdot 1_K, \\
Z_1 I + (JZ_2)^\dagger & = 0, \\
Z_2 I - (JZ_1)^\dagger & = 0,
\end{align*}
\]

modulo \( U(K) \) action \((B_a, I, J) \mapsto (g^{-1}B_a g^{-1}Z_a g^{-1}I, J g)\).

B.3. ADHM construction for \( \mathcal{N} = 2^* \) theory on the blowup. We now propose the analogous construction for the \( \mathcal{N} = 2^* \) theory on the blown up plane \( \hat{\mathbb{C}}^2 \). The linear algebra data consists now of the maps:

\[
i : N \to K'', \ j : K' \to N, \ b_{1,2} : K' \to K'', \ b_3, b_4 : K'' \to K', \ b' \in \text{End}(K'), \ b'' \in \text{End}(K''),
\]

as in Fig. 1 below

![Fig 1. Quiver description of instantons in \( \mathcal{N} = 2^* \) theory on \( \hat{\mathbb{C}}^2 \)](image)

\(^{16}\)In [132] we called them \( B_3, B_4 \)
while (288) generalize to
\[
\begin{align*}
    b_1b_3b_2 - b_2b_3b_1 + (b'b_4 - b_4b'')^\dagger &= -ij , \\
    b_1b_4b_2 - b_2b_4b_1 - (b'b_3 - b_3b'')^\dagger &= 0 , \\
    b_1b' - b'b_1 - (b_3b_2b_4 - b_4b_2b_3)^\dagger &= 0 , \\
    b_2b' - b''b_2 + (b_3b_1b_4 - b_4b_1b_3)^\dagger &= 0 , \\
    m''_e &= ii^\dagger + b_1b_1^\dagger + b_2b_2^\dagger - b_3b_3 - b_4^\dagger b_4 - [b'', b'''] = \zeta'' \cdot 1_{K''} , \\
    m'_{\mathbb{R}} &= -j^\dagger j - b_1b_1 - b_2b_2 + b_3b_3 + b_4b_4^\dagger + [b', b'] = \zeta' \cdot 1_{K'} , \\
    b''i + (jb_4)^\dagger &= 0 , \\
    b'^i - (jb')^\dagger &= 0 ,
\end{align*}
\]

(290)
The matrices $b_1, b_2, b_3, b_4$ and $b', b''$ correspond, roughly, to the coordinates $w_1, w_2, w_3, w_4$ and $w$ of the Kähler quotient construction of the local Calabi-Yau fourfold $\mathcal{Y} = \mathcal{Z} \times \mathbb{C}$, with the resolved conifold $\mathcal{Z}$:
\[
(w_1, w_2, w_3, w_4, w) \mapsto (e^{i\theta}w_1, e^{i\theta}w_2, e^{-i\theta}w_3, e^{-i\theta}w_4, w),
\]
\[(291)\]
where (270) is imposed. The assignments of spaces $K', K''$ and maps between them can be interpreted as the cross-product construction of instantons on the quotient $\mathbb{C}^5 / \mathbb{C}^\times$, with the specific choice of $\mathbb{C}^\times$-representation in the framing space and the special choice of fractional charges, namely, $N$ corresponds to the trivial representation, while $K'$ and $K''$ correspond to the charge $-1$ and charge 0 representations, respectively. The operators $b_1, b_2$ raise the charge by $+1$, as $w_1, w_2$ have charge $+1$ in (291), $b', b''$ do not change the charge, as $w$ has the charge 0, and $b_3, b_4$ lower the charge by 1, as both $w_3$ and $w_4$ have the charge $-1$.

The equations (290) have the $SU(2)$ symmetry
\[
(b_1, b_2) \mapsto (\alpha b_1 + \beta b_2, -\bar{\beta}b_1 + \bar{\alpha}b_2),
\]
\[(292)\]
with $|\alpha|^2 + |\beta|^2 = 1$. The second $SU(2)$ symmetry
\[
(b_3, b_4) \mapsto (\alpha'b_3 + \beta'b_4, -\bar{\beta}'b_3 + \bar{\alpha}'b_4),
\]
\[(293)\]
with $|\alpha'|^2 + |\beta'|^2 = 1$, is broken by the contribution of $ij$. Geometrically, these equations describe the supersymmetric gauge field configurations on the brane located at the $w_4 = 0, w = 0$ surface inside $\mathcal{Y}$, as can be seen from the holomorphic equations
\[
\begin{align*}
    b_1b_3b_2 - b_2b_3b_1 + ij &= 0 , \\
    b_1b_4b_2 - b_2b_4b_1 &= 0 , \\
    b_4b' - b''b_a &= 0 , a = 1, 2 , \\
    b'a - b''b &= 0 , a = 3, 4 \\
    b_2b_4b_1 - b_1b_4b_3 &= 0 , a = 1, 2 \\
    b''i &= 0 , \\
    jb_4 &= 0 , \\
    b_4i &= 0 , \\
    jb' &= 0 ,
\end{align*}
\]
\[(294)\]
which, as in [132] follow from the real (as opposed to $\mathbb{C}$-linear) Eqs. (290).

In addition, imposing the equations $m'_{\mathbb{R}} = \zeta' \cdot 1_{K'}, m''_{\mathbb{R}} = \zeta'' \cdot 1_{K''}$ and dividing by $U(K') \times U(K'')$ can be replaced by imposing the stability conditions and dividing by the complex group $GL(K') \times GL(K'')$. For $\zeta' > 0, \zeta'' > 0$ the stability condition reads:
Any pair \((S', S'')\) of subspaces \(S' \subset K', S'' \subset K''\), such that \(i(N) \subset S''\), \(b'(S') \subset S'\), \(b''(S'') \subset S''\), \(b_{1,2}(S') \subset S'', b_4, b_3(S'') \subset S'\) coincides with \((K', K'')\). A simple consequence of this condition, and the equations \((290)\) is \(K'' = \mathbb{C} [b_1 b_3, b_2 b_3] i(N)\), \(K' = \mathbb{C} [b_2 b_1, b_3 b_2] b_3 i(N)\), \(b_4 = 0, b' = 0, b'' = 0\), therefore set-theoretically the moduli space of solutions to \((290)\) coincides with the moduli space of perverse coherent sheaves on \(\mathbb{C}^2\) as constructed in [123].

**B.4. Crossed instantons on the conifold and qq-characters on the blowup.**

It is now easy to generalize \((290)\) to the case of crossed instantons. It suffices to add another framing space \(W\), the maps \(\tilde{i} : W \to K', \tilde{j} : K'' \to W\), and modify the Eqs. \((290)\) to:

\[
\begin{align*}
    b_1 b_2 b_3 - b_2 b_3 b_1 + (b' b_4 - b_3 b'')^\dagger &= -i \tilde{j} + \tilde{j}^\dagger i, \\
    b_1 b_2 b_3 - b_2 b_3 b_1 - (b' b_3 - b_3 b'')^\dagger &= 0, \\
    b_1 b' - b'' b_1 - (b_3 b_2 b_4 - b_4 b_2 b_3)^\dagger &= 0, \\
    b_2 b' - b'' b_2 + (b_3 b_1 b_4 - b_3 b_4 b_1)^\dagger &= 0, \\
    m'''_{\mathbb{R}} &= i i^\dagger + \tilde{j} j^\dagger - b_1 b_1^\dagger + b_2 b_2^\dagger + b_3 b_3^\dagger - b_4 b_4^\dagger + [b', b''^\dagger] = \zeta'' \cdot 1_{K''}, \\
    m'_{\mathbb{R}} &= -j^\dagger j - j^\dagger j - b_1 b_1^\dagger - b_2 b_2^\dagger - b_3 b_3^\dagger + b_4 b_4^\dagger + [b', b^\dagger] = \zeta' \cdot 1_{K'}, \\
    b'' i + (j b_4)^\dagger &= 0, \\
    b_3 i - (j b^\dagger)^\dagger &= 0, \\
    b_3 b_2 i + (\tilde{j} b_1)^\dagger &= 0, \\
    b_3 b_2 i - (\tilde{j} b_2)^\dagger &= 0,
\end{align*}
\]

which again have the symmetry \((292)\), and again imply the \(\mathbb{C}\)-linear equations.

We leave the analysis of the moduli space of solutions to \((295)\) to further study. We conjecture an analogue of the compactness theorem of part II in [132] holds, opening another possibility for the proof of blowup formulas with and without surface defects.

**APPENDIX C. A LITTLE BIT OF GEOMETRY AND REPRESENTATION THEORY**

**C.1. Another face of \(G\) and \(H\).** The Lie algebra of \(G\) (and \(H\), of course), can be realized in the space of densities on a (real) line. These are formal expressions \(f(\gamma) d\gamma^{-s}\), for some complex number \(s \in \mathbb{C}\). Let us apply the fractional-linear transformation

\[
\gamma \mapsto \frac{a \gamma + b}{c \gamma + d},
\]

which acts on \(f\) by

\[
f(\gamma) d\gamma^{-s} \mapsto f \left( \frac{a \gamma + b}{c \gamma + d} \right) (c \gamma + d)^{2s} d\gamma^{-s}.
\]

Now, for complex \(s\) the expression \((c \gamma + d)^{2s}\) is not well-defined. However, for infinitesimal \(a \approx 1, b \approx 0, c \approx 0, d \approx 1\) there is a well-defined branch, which means that the generators \(J^+, J^0, J^-\) of \(\text{Lie}(G)\) are well-defined. They act on the coefficient function \(f(\gamma)\) via:

\[
J^+ = -\gamma^2 \partial_\gamma + 2s \gamma, \quad J^0 = \gamma \partial_\gamma - s, \quad J^- = \partial_\gamma
\]
The quadratic Casimir
\[ \tilde{C} = J^- J^+ + J^0 (J^0 + 1) \]  
(299)
is a central element, acting by multiplication by \( s(s + 1) \).

The well-known Verma \( \text{Lie}(G) \)-modules \( V_s^\pm \) consist of \( f \) of the form \( \gamma^n d\gamma^{-s} \) and \( \gamma^{2s-n} d\gamma^{-s} \), with \( n \geq 0 \), respectively. Another module \( M_s \) consists of \( f \) of the form \( \gamma^{s+a+n} d\gamma^{-s} \), with \( n \in \mathbb{Z} \). These three types of \( \text{Lie}(G) \)-representations are characterized by the existence of the vector, annihilated by \( J^\pm \), or \( J^0 \), respectively.

Finally, a twisted version of the module \( M_s \), which we denote by \( M_s^a \), with \( s, a \in \mathbb{C} \), consists of the densities of the form \( \gamma^{s+a+n} d\gamma^{-s} \), with the same generators \((298)\). For special values of \( s \) and \( a \) these representations are not irreducible. For example, \( V_s^+ \subset M_s^-, V_s^- \subset M_s^+ \).

In \([138]\), in the higher rank generalizations of our story, we encounter all these representations (see also the Appendix D).

C.1.1. Integrable representations. By restricting \( s \) and the type of functions \( f(\gamma) \) one can build the unitary representations of some real forms of \( G \). For example, for \( 2s \in \mathbb{Z}_{\geq 0} \), and \( f(\gamma) \) polynomials of degree \( 2s \), we get a spin \( s \) representation of \( SU(2) \subset G \). The invariant Hermitian product is given by:
\[ \langle f, g \rangle = \int \bar{f}(\gamma) d\gamma^{-s} g(\gamma) d\gamma^{-s} \left( \frac{d\gamma d\bar{\gamma}}{(1 + \gamma \bar{\gamma})^2} \right)^{s+1} \]  
(300)
where the integral is over \( \mathbb{CP}^1 = \mathbb{C} \cap \infty \). As another example, taking \( 2s + 1 \in i\mathbb{R} \), and requiring \( f \in L^2(\mathbb{R}) \) makes up a unitary \( SL(2, \mathbb{R}) \) representation with the Hermitian inner product
\[ \langle f, g \rangle = \int_{\mathbb{R}} \bar{f}(\gamma) d\gamma^{-s} g(\gamma) d\gamma^{-s} \]  
(301)
where we use \( s + \bar{s} = -1 \).

C.1.2. Invariants and intertwiners. Much of the theory below is built using the \( G \)-invariance of the cross-ratio
\[ \frac{(\gamma_2 - \gamma_1)(\gamma_3 - \gamma_4)}{(\gamma_3 - \gamma_1)(\gamma_2 - \gamma_4)} \]  
(302)
and its infinitesimal version:
\[ \frac{d\gamma_1 \otimes d\gamma_2}{(\gamma_1 - \gamma_2)^2} \]  
(303)
For example, \((300)\) is built using \((303)\) with \( \gamma_1 = \gamma, \gamma_2 = -\bar{\gamma}^{-1} \).

Given three spins \( s_1, s_2, s_3 \), we build the \( \text{Lie}(G) \)-invariant
\[ I_{123} = (\gamma_1 - \gamma_2)^{s_1+s_2-s_3} (\gamma_1 - \gamma_3)^{s_1+s_3-s_2} (\gamma_2 - \gamma_3)^{s_2+s_3-s_1} d\gamma_1^{-s_1} d\gamma_2^{-s_2} d\gamma_3^{-s_3} \]  
(304)
Depending on the relative positions of the three points \( \gamma_1, \gamma_2, \gamma_3 \), the expression in \((304)\) can be viewed as an element of the tensor product of three representations of \( \text{Lie}(G) \). For example, in the domain \(|\gamma_1| \ll |\gamma_2| \ll |\gamma_3| \) (this condition is \( \text{Lie}(G) \)-invariant, not \( G \)-invariant, obviously),
\[ I_{123} \in V_{s_1}^+ \otimes M_{s_2}^{s_3-s_1} \otimes V_{s_3}^- \]  
(305)
C.2. **Antipodal maps and flipping spins.** The map $\gamma_+ \mapsto \gamma_-$ we discussed in (65) is in fact a semiclassical version of the transformation $V_{s-1}^\pm \to V_s^\pm$

$$f(\gamma)d\gamma^{-s} \mapsto f(\tilde{\gamma})d\tilde{\gamma}^{s+1} = \oint f(\gamma)d\gamma^{-s} \left( \frac{d\gamma d\tilde{\gamma}}{(\gamma - \tilde{\gamma})^2} \right)^{s+1}$$

(306)

In the limit $\kappa \to \infty$, $s = \vartheta \kappa$ with $\vartheta$ finite, the integral (306) is dominated by a saddle point, giving rise to the canonical transformation generated by (66).

C.2.1. **Flipping spins in the $y$-coordinates.** For completeness we list the generating functions $\sigma_\xi(y, y_\xi)$ of the canonical transformations $(y, p_y) \mapsto (y_\xi, p_{y_\xi})$, for $\xi = 0, q, 1, \infty$, corresponding to the changes of one sign $\vartheta_\xi \mapsto -\vartheta_\xi$ and $\gamma_+ \mapsto \gamma_-$. 

$$\sigma_0(y, y_0) = \vartheta_0 \log \left( \frac{y y_0}{(y - y_0)^2} \right) + (\vartheta_q - \vartheta_1 + \vartheta_\infty) \log \left( \frac{y}{y_0} \right),$$

$$\sigma_q(y, y_q) = \vartheta_q \log \left( \frac{y y_q (y - 1) (y_q - 1)}{(y - y_q)^2} \right) + (\vartheta_0 - \vartheta_1 + \vartheta_\infty) \log \left( \frac{(y_q - 1) y}{(y - 1) y_q} \right),$$

$$\sigma_1(y, y_1) = \vartheta_1 \log \left( \frac{(1 - y) (1 - y_1)}{(y - y_1)^2} \right) + (\vartheta_0 - \vartheta_1 + \vartheta_\infty) \log \left( \frac{1 - y_1}{1 - y} + 2 \vartheta_\infty \log \left( \frac{y y_1}{y_1 y} \right) \right),$$

$$\sigma_\infty(y, y_\infty) = 2 \vartheta_\infty \log \left( \frac{y y_\infty}{y - y_\infty} \right),$$

(307)

so that

$$p_{y_\xi} = -\frac{\partial \sigma_\xi}{\partial y}, \quad p_y = \frac{\partial \sigma_\xi}{\partial y}.$$  

(308)

**APPENDIX D. Higher rank generalizations I: orbits and modules**

The orbit $O_\vartheta$ of $G$ has two obvious generalizations to the higher rank case $G_r = SL(r + 1, \mathbb{C})$ with $r > 1$.

D.1. **Minimal orbit** $\mathcal{O}_{\text{min}}^{r, \nu}$. Let $\mathcal{V} = \mathbb{C}^{r+1}$ be the defining representation of $G_r$. The first generalization of $O_\vartheta$, which we shall call $\mathcal{O}_{\text{min}}^{r, \nu}$ (the subscript $\text{min}$ will be explained later), consists of the pairs $(u, v)$, with $v \in \mathcal{V}$, $u \in \mathcal{V}^*$ such that

$$u(v) = v \in \mathbb{C},$$

(309)

modulo the $\mathbb{C}^*$-action

$$(u, v) \to (t v, t^{-1} u), \quad t \in \mathbb{C}^*$$

(310)

The $u - v$ description allows for a very simple description of the holomorphic symplectic forms on $\mathcal{O}_{\text{min}}^{r, \nu}$. It is obtained from the 2-form

$$du \wedge dv := \sum_{a=1}^{r+1} du_a \wedge dv^a$$

(311)

by the symplectic reduction with respect to the $\mathbb{C}^*$-action (310), for which (309) is the moment map equation. The space $\mathcal{O}_{\text{min}}^{r, \nu}$ is an orbit of $G_r$, which acts in the obvious

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17Recall that this is a complexification of the antipodal map of a two dimensional sphere
way: \[ g : (u, v) \mapsto (u g^{-1}, g v) \] for \( g \in G_r \). This action is generated by the moment map \( \mu_{\text{min}} : \mathcal{O}_{\text{min}}^{r,\nu} \to \text{Lie}(G_r)^* \) which reads as follows:

\[
\mu_{\text{min}}(u, v) = \frac{\nu}{r+1} 1_V - v \otimes u \tag{312}
\]

The map (312) makes \( \mathcal{O}_{\text{min}}^{r,\nu} \) a coadjoint orbit. For \( \nu \neq 0 \) we have two maps \( p_{\pm} : \mathcal{O}_{\text{min}}^{r,\nu} \to \mathbb{CP}^r \), generalizing the \( p_{\pm} \) maps (61) of the \( r = 1 \) case, \( p_-(u, v) = [u] \in \mathbb{P}(V^*) \), \( p_+(u, v) = [v] \in \mathbb{P}(V) \). In fact, we can identify:

\[
\mathcal{O}_{\text{min}}^{r,\nu} \approx \mathbb{P}(V) \times \mathbb{P}(V^*) \backslash \mathcal{C} \tag{313}
\]

where \( \mathcal{C} \) is the incidence subvariety, consisting of the pairs \( (h, l) \), \( l \subset V \), \( \dim(l) = 1 \), \( h \subset V \), \( \text{codim}(h) = 1 \), such that \( l \subset h \).

For \( \nu = 0 \) one supplements the equation \( u(v) = 0 \) with one of the two stability conditions: either \( u \neq 0 \), or \( v \neq 0 \). For either stability condition the orbit \( \mathcal{O}_0^r \) is isomorphic to \( T^* \mathbb{CP}^r \).

D.1.1. \( \text{Lie}(G_r) \) module \( M_\nu \). The following \( \text{Lie}(G_r) \)-module (representation) is associated to \( \mathcal{O}_{\text{min}}^{r,\nu} \)\(^{18} \). Consider the space of formal expressions

\[
f(u_1, u_2, \ldots, u_{r+1}) = (u_1 \ldots u_{r+1})^\frac{\nu}{r+1} \psi_1(u_1/u_{r+1}, \ldots, u_r/u_{r+1}) \tag{314}
\]

As in the discussion around (297) we apply to (314) an \( G_r \)-transformation

\[
a : (u_i) \mapsto \left( \sum_{j=1}^{r+1} a_{ij} u_j \right), \quad \text{Det}(a) = 1 \tag{315}
\]

acting by: \( \psi \mapsto \psi^a \)

\[
\psi^a(\gamma) = \psi \left( \frac{a_{i1} \gamma_1 + \ldots + a_{ir} \gamma_r + a_{ir+1}}{a_{r+1,1} \gamma_1 + \ldots + a_{r+1,r} \gamma_r + a_{r+1,r+1}} \right)^{\frac{\nu}{r+1}} \prod_{i=1}^{r+1} \left( a_{i1} \gamma_1 + \ldots + a_{ir} \gamma_r + a_{ir+1} \right)^{\frac{\nu}{r+1}} \tag{316}
\]

where \( \gamma_i = u_i/u_{r+1}, \gamma_{r+1} = 1 \). Again, (316) is not well-defined for general \( \nu \), for general \( a \in G_r \). Again, for infinitesimal \( a_{ij} = \delta_{ij} + \varepsilon \xi_{ij}, \varepsilon \to 0 \) we have a preferred branch, leading to the action of \( \text{Lie}(G_r) \) via first order differential operators

\[
J_\xi \psi = \sum_{i,j=1}^{r+1} \xi_{ij} u_i \left( \frac{\partial}{\partial u_j} \psi + \frac{\nu \psi}{(r+1)u_j} \right) \tag{317}
\]

where \( \xi \) is a traceless \( r + 1 \times r + 1 \) matrix. The representation (317) is realized using the minimal number \( r \) of degrees of freedom, hence the term “minimal”. The generators (317) correspond to (312) under the naive quantization rule:

\[
v_i \mapsto \frac{\partial}{\partial u_i}, \quad (-\mu_{\text{min}})_{ij} \mapsto J_{ij} \tag{318}
\]

with the ordering prescription which pushes \( v \)'s to the right of \( u \). In comparing with the formulae (298), with \( s = \nu/2 \), one should remember the normalization \( \psi(\gamma) = \gamma^s \psi(\gamma) \). The representation \( M_\nu \) is important in applications, as its zero-weight subspace

\(^{18}\)The precise meaning of the word “associated” is worth another paper. Prospective graduate students seeking a project on topological strings are welcome to apply their effort.
$M_r[0]$, i.e. the subspace of vectors, annihilated by the generators, corresponding to the diagonal matrices $J_{E_{i+1}-E_{i+1},i+1}$, is one-dimensional.

D.2. Regular orbit. The second generalization, which we shall call $\mathcal{O}^{\vec{v},r}_{\text{reg}}$ (the subscript $\text{reg}$ will be explained later), with $\vec{v} = (\nu_1, \ldots, \nu_r) \in \mathbb{C}^r$, is the quotient of the space of $r$-tuples $(u, v) = (u^i, v_i)_{i=1}^r$, $v_i \in \mathcal{V}$, $u^i \in \mathcal{V}^*$, obeying

$$u^i(v_j) - \nu_i \delta^i_j = 0,$$

modulo the $(\mathbb{C}^*)^r$ symmetry, acting by

$$(u^i, v_i)_{i=1}^r \mapsto (t_i u^i, t_i^{-1} v_i)_{i=1}^r, \quad t_i \in \mathbb{C}^*.$$

Note that $(u, v)$ defines the $(r+1)$st pair $(u^0 \in \mathcal{V}^*, v_0 \in \mathcal{V})$, by $u^i(v_0) = 0$, $u^0(v_i) = 0$ for all $i = 1, \ldots, r$.

We would like to have a symplectic description of $\mathcal{O}^{\vec{v},r}_{\text{reg}}$. Let us start with the symplectic form

$$du^i (\wedge dv) = \sum_{i=1}^r du^i (\wedge dv_i) := \sum_{i=1}^r \sum_{a=1}^{r+1} du^a_i \wedge dv^a_i$$

on the space of all $r$-tuples $(u, v)$. We would like to interpret (319) and taking the quotient (320) as the symplectic quotient. The fact that the Poisson brackets of the equations (319) are non-zero, for $\vec{v} \neq 0$, on the surface of solutions to (319), is an obstacle. However, there is a simple well-known fix for that. Consider the group $B_r$ of invertible upper-triangular $r \times r$ matrices $\|b^j_{ij}\|_{i,j=1}^r$, i.e. $b^j_{ij} = 0$ for $i < j$ acting on the space of $r$-tuples:

$$b : (u, v) \mapsto (u^b, v^b) = \left( \sum_{k \geq i} b^i_k u^k, \sum_{k \leq i} (b^{-1})_k^i v_k \right)_{i=1}^r$$

The $B_r$-action is generated by the upper-triangular matrices-valued moment map

$$\mu_B = \left\|u^i(v_j)\right\|_{i \geq j}.$$ 

Now, since the space of diagonal matrices is preserved by the conjugation by the upper-triangular matrices (in other words, the eigenvalues of an upper-triangular matrix are read off its diagonal), we can perform the Hamiltonian reduction at the non-zero level of $\mu_B$ set to be a diagonal matrix $\nu = \text{diag}(\nu_1, \ldots, \nu_r)$, meaning we impose the equations

$$u^i(v_j) - \nu_i \delta^i_j = 0, \quad r \geq i \geq j \geq 1$$

We can define two more complex lines: $\mathbb{C}u^{r+1} \subset \mathcal{V}^*$ and $\mathbb{C}v_{r+1} \subset \mathcal{V}$, by:

$$u^i(v_{r+1}) = 0, \quad u^{r+1}(v_i) = 0, \quad i = 1, \ldots, r$$

\begin{equation}
\sum_{j \leq k \leq i} b^j_k (b^{-1})_i^k = \delta^j_i
\end{equation}
The group $B_r$ preserves two flags:

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots \subset \mathcal{V}_r \subset \mathcal{V},$$

$$\mathcal{V}_i = \sum_{k \leq i} \mathbb{C}v_k,$$

and

$$0 = \mathcal{U}^0 \subset \mathcal{U}^1 \subset \ldots \subset \mathcal{U}^r \subset \mathcal{V}^*,$$

$$\mathcal{U}^i = \sum_{k \geq i} \mathbb{C}u^k.$$

These flags are compatible in the sense that $\mathcal{U}^{i+1} / \mathcal{V}_i = 0$, $L_i = \ker \mathcal{U}^i / \mathcal{V}_{i-1}$ is one-dimensional, spanned by $v_i$, for $i = 1, \ldots, r + 1$, as well as $L^i = \mathcal{U}^i / \ker \mathcal{V}_{i-1}$ is one-dimensional, spanned by $u^i$, for $i = 1, \ldots, r + 1$. Thus

$$\mathcal{O}_{\text{reg}}^{\vec{u},r} = \mu^{-1}(\vec{u}) / B_r$$

(330)

The moment map $\mu : \mathcal{O}_{\text{reg}}^{\vec{u},r} \rightarrow \text{Lie}(G_r)^*$ is given by:

$$\mu_{\text{reg}}(\mathbf{u}, \mathbf{v}) = \vartheta_{r+1} \mathbf{1}_{\mathcal{V}} - \sum_{i=1}^{r} v_i \otimes u^i = \sum_{i=1}^{r+1} \vartheta_i v_i \otimes v^i,$$

$$\vartheta_i = \vartheta_{r+1} - \nu_i, \ i = 1, \ldots, r, \ \vartheta_{r+1} = \frac{\nu_1 + \ldots + \nu_r}{r + 1},$$

$$v^i(v_j) = \delta^i_j, \ i, j = 1, \ldots, r + 1$$

(331)

Another well-known presentation of $\mathcal{O}_{\text{reg}}^{\vec{u},r}$ is the $A_r$-type quiver variety. Namely, consider the sequence of complex vector spaces $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_r$, such that $\dim_{\mathbb{C}} \mathcal{V}_i = i$, and linear maps:

$$B_i : \mathcal{V}_i \rightarrow \mathcal{V}_{i+1}, \ \tilde{B}_i : \mathcal{V}_{i+1} \rightarrow \mathcal{V}_i$$

(332)

**20**If the collection of $r$ pairs $(u^i, v_i)_{i=1}^r$ of vectors and covectors obeys $u^i(v_j) = 0$ for $i > j$ and $u^i(v_i) = \nu_i$, then by the strictly upper-triangular transformation:

$$u^i \mapsto u^i + \sum_{k>i} \xi_{i}^k u^k,$$

$$v_i \mapsto v_i - \sum_{i<k} \tilde{\xi}_{i}^k v_k$$

(327)

with $\xi - \tilde{\xi} - \xi \tilde{\xi} = 0$, we can achieve $u^i(v_j) = 0$ for $i < j$. Indeed,

$$u^i(v_j) \mapsto u^i(v_j) + \sum_{j \geq k > i} \xi_{i}^k u^k(v_j) - \sum_{j > k \geq i} \tilde{\xi}_{i}^k u^i(v_k) - \sum_{j \geq l > k > i} \xi_{i}^l \xi_{j}^l u^k(v_l) = 0,$$

(328)

is solved recursively:

$$\xi_{i+1} = \frac{\xi_{i+1}}{\nu_i - \nu_{i+1}}, \ i = 1, \ldots, r - 1,$$

$$\tilde{\xi}_{i+2} = \xi_{i+2} - \xi_{i+1} \xi_{i+2} = \frac{u^i(v_{i+2}) - \frac{1}{\nu_{i+2} - \nu_{i+1}} u^i(v_{i+1}) u^i+1(v_{i+1})}{\nu_i - \nu_{i+2}}$, \ i = 1, \ldots, r - 2$$

(329)

...,

assuming $\nu_i \neq \nu_j, i \neq j$. 

obeying:
\[
\tilde{B}_i B_i - B_{i-1} \tilde{B}_{i-1} = \zeta_i \cdot 1_{\mathbb{V}_i}, \quad i = 1, \ldots, r,
\]
where \( \zeta_i \in \mathbb{C} \), \( \mathbb{V}_{r+1} = \mathbb{V} \), and we identify:
\[
\left( B_i, \tilde{B}_i \right)_{i=1}^r \sim \left( g_{i+1}^{-1} B_i g_i, g_i \tilde{B}_{i+1}^{-1} \right)_{i=1}^r,
\]
\( (g_i)_{i=1}^r \in \times_{i=1}^r GL(\mathbb{V}_i) \) \( (334) \)

The previous description is recovered quite simply:
\[
\mu_{\text{reg}}(u, v) = \partial_{r+1} 1_{\mathbb{V}} - B_r \tilde{B}_r
\]
so that \( v_{r+1} = \ker \tilde{B}_r \in \mathbb{V} \), \( u_i^{r+1} = \im B_i \in \mathbb{V}^* \), and assuming \( \zeta_i \neq 0 \) for all \( i = 1, \ldots, r \), \( \mathbb{C} v_1 = B_r B_{r-1} \ldots B_1 \psi_0 \),
\[
0 \neq \psi_0 \in \mathbb{V}_1
\]
\[
\mathbb{C} v_i = B_r B_{r-1} \ldots B_i \psi_{i-1},
\]
\[
0 \neq \psi_{i-1} \in \mathbb{V}_i, \quad B_{i-1} \psi_{i-1} = 0, \quad i = 2, \ldots, r
\]

Analogously,
\[
\mathbb{C} u^i = \chi_0 \tilde{B}_1 \tilde{B}_2 \ldots \tilde{B}_r,
\]
\[
0 \neq \chi_0 \in \mathbb{V}_1^*
\]
\[
\mathbb{C} u^i = \chi_{i-1} \tilde{B}_i \tilde{B}_{i+1} \ldots \tilde{B}_r,
\]
\[
0 \neq \chi_{i-1} \in \mathbb{V}_i^*, \quad \chi_{i-1} B_{i-1} = 0, \quad i = 2, \ldots, r
\]
so that
\[
\mu_{\text{reg}}(u, v) v_i = \partial_i v_i, \quad u^i \mu_{\text{reg}}(u, v) = \partial_i u^i
\]
with
\[
\partial_i = \partial_{r+1} - \zeta_r - \ldots - \zeta_i.
\]

The maps \( p_+: \mathcal{O}_{\text{reg}}^{\xi,r} \to Fl(1, 2, 3, \ldots, r, \mathbb{V}) \), \( p_-: \mathcal{O}_{\text{reg}}^{\xi,r} \to Fl(1, 2, 3, \ldots, r, \mathbb{V}^*) \), generalizing \( (61) \), send the set \( (B_i, \tilde{B}_i)_{i=1}^r \) obeying \( (333) \) modulo \( (334) \), to the flags
\[
V_1 = B_r B_{r-1} \ldots B_1 \mathbb{V}_1 \subset V_2 = B_r B_{r-1} \ldots B_2 \mathbb{V}_2 \subset \ldots \subset V_r = B_r \mathbb{V}_r \subset \mathbb{V}
\]
and
\[
U_1 = \mathbb{V}_1^* \tilde{B}_1 \tilde{B}_2 \ldots \tilde{B}_r \subset U_2 = \mathbb{V}_2^* \tilde{B}_2 \ldots \tilde{B}_r \subset \ldots \subset U_r = \mathbb{V}_r^* \tilde{B}_r \subset \mathbb{V}^*
\]

D.3. Verma module. To the regular orbit \( \mathcal{O}_{\text{reg}}^{\xi,r} \) one associates the highest weight Verma module \( V_\nu^+ \) (and, dually, the lowest weight Verma module \( V_\nu^- \)), which is generated by the vector \( v_\nu^+ \), annihilated by the \( \text{Lie}(G_r) \) generators \( J_\xi \) corresponding to the strictly upper (lower)-triangular matrices, which is the eigenvector of the diagonal traceless matrices \( J_{E_{i,i-E_{i+1,i+1}}} v_\nu = \nu_i v_\nu, \ i = 1, \ldots, r \).

Appendix E. KZ, BPZ and Okamoto

For completeness, we remind some basic details about the sl₂ Knizhnik-Zamolodchikov equation. We also explain that Okamoto transformations are the classical limit of the KZ-BPZ correspondence \( [42] \).
E.1. Conformal blocks. The genus 0, $p + 3$-point conformal block in the theory with level $k \widehat{\mathfrak{sl}}_2$ current algebra is a vector

$$\Psi \in \left( \bigotimes_{i=-1}^{p+1} V_{s_i}^\gamma \otimes V_{s_i}^z \right)^{\text{Lie}(G) \times \text{Lie}(H)}$$

(342)

where $V_{s_i}^\gamma$ stands for $\text{Lie}(G)$-representation by differential operators (298) in the $\gamma$-variable, and $V_{s_i}^z$ stands for the $\text{Lie}(H)$-representation by differential operators (298) in the $z$-variable with $s = \Delta$, the conformal dimension. The conformal dimension $\Delta$ is related to $s$ via the celebrated [94] relation

$$\Delta_i = \frac{s_i(s_i + 1)}{\kappa}, \ i = -1, \ldots, p + 1$$

(343)

with $\kappa = k + 2$. In the analysis below we don’t specify what kind of representations we mean by $V_s$, what kind of completion is used in the tensor product (342). We want $\Psi$ to be both $\text{Lie}(H)$ and $\text{Lie}(G)$-invariant.

Using (303) and (302) we build the formal invariants

$$\Psi = \prod_{-1 \leq i < j \leq p+1} \left\{ \frac{dz_idz_j}{z_{ij}^2} \right\}^{D_{ij}} \left( \frac{d\gamma_id\gamma_j}{\gamma_{ij}^2} \right)^{S_{ij}} \times \Psi (y; q)$$

(344)

with $y = (y_a)_{a=1}^p$, $q = (q_a)_{a=1}^p$,

$$\gamma_{ij} = \gamma_i - \gamma_j, \quad y_a = \frac{\gamma_{a-1,0,p+1}}{\gamma_{0,1,a,0,p+1}}$$

(345)

and similarly

$$z_{ij} \equiv z_{i,j} = z_i - z_j, \quad q_a = \frac{z_{a-1,0,p+1}}{z_{0,1,a,p+1}}$$

(346)

provided the symmetric matrices $S = S^t$, $D = D^t$, obey

$$\sum_{j \neq i} S_{ij} = -s_i, \quad \sum_{j \neq i} D_{ij} = -\Delta_i$$

(347)

We have much freedom in the choice of these symmetric matrices.\footnote{For example, kinematic invariants $S_{ij} = \vec{p}_i \cdot \vec{p}_j$ of a set of $p + 3$ momenta $\vec{p}_i \in \mathbb{C}^D$ in a $D \geq p + 3$ dimensional scattering problem, obeying}

$$\vec{p}_i \cdot \vec{p}_i = s_i, \quad \sum_i \vec{p}_i = 0$$

(348)

solve (347). Similarly, for the $\|D_{ij}\|$ matrix.
E.2. Knizhnik-Zamolodchikov connection. The KZ equation (in genus zero) is

$$\nabla \Psi = 0$$  \hspace{1cm} (350)

where $\nabla$ is a (projective) flat connection over $\mathcal{M}_{0,p+3}$ on the infinite rank vector bundle with the fibers $\mathcal{H} = (V_{s_{-1}}^\gamma \otimes \ldots \otimes V_{s_{p+1}}^\gamma)^{Lie(G)}$, explicitly:

$$\nabla_i = \kappa \frac{\partial}{\partial z_i} + \sum_{j \neq i} \frac{J_i^+ J_j^- + J_j^+ J_i^- + 2J_i^0 J_j^0}{z_i - z_j}, \ i = -1, \ldots, p + 1$$  \hspace{1cm} (351)

The equations (351) are compatible, $[\nabla_i, \nabla_j] = 0$, for any value of $\kappa$, which means that the two sets of equations, cf. (31), (28) hold:

$$\partial_{z_i} \hat{H}_j - \partial_{z_j} \hat{H}_i = 0, \quad [\hat{H}_i, \hat{H}_j] = 0$$  \hspace{1cm} (352)

with

$$\hat{H}_i = \sum_{j \neq i} \frac{1}{z_{ij}} \left( \gamma_{ij}^2 \partial_{\gamma_{ij}} - 2\gamma_{ij}(s_i \partial_{s_j} - s_j \partial_{s_i}) - 2s_i s_j \right).$$  \hspace{1cm} (353)

The Eqs. (352) follow from the celebrated relations (3)

$$d\log(z_i - z_j) \wedge d\log(z_i - z_k) + d\log(z_j - z_i) \wedge d\log(z_j - z_k) + d\log(z_k - z_i) \wedge d\log(z_k - z_j) = 0$$  \hspace{1cm} (354)

The general higher genus generalization of the KZ connection [16 [9] [104] and its quasiclassical limits $\kappa \to \infty$ [79 [80] are an involved story.

E.2.1. Degenerate cases. Let us take $p = 2$, with $s_3 = \frac{1}{2}$. The algebra $\mathfrak{s}_3$ has a two-dimensional representation $W$, which is a subrepresentation of $V_2^\gamma$:

$$W = \mathbb{C}d\gamma^{-\frac{1}{2}} \oplus \mathbb{C}\gamma d\gamma^{-\frac{1}{2}}$$  \hspace{1cm} (355)

So we restrict the conformal block $\Psi$ onto this subrepresentation. In other words we assume the linear dependence of the conformal block $\Psi d\gamma_3^{-\frac{1}{2}} dz_3^\frac{1}{2} \prod_{i=1}^2 dz_i^{-\Delta_i}$ on $\gamma_3$:

$$\Psi = \left( \frac{\gamma_{3,2}}{\gamma_{1,2}} \Psi_0(y, z; q) + \frac{\gamma_{3,-1}}{\gamma_{1,-1}} \Psi_1(y, z; q) \right) \times z_{2,1}^{-s_1 - s_2 + \frac{1}{2}} \frac{z_1 - z_{2,1}}{z_{0,-1}} \left( \frac{\gamma_{1,0} \gamma_{1,-1}}{\gamma_{0,-1}} \right)^{s_1 - s_2 + \frac{1}{2}} \left( \frac{z_1 - z_{0,-1}}{z_{1,-1}} \right)^{\Delta_0 - \Delta_1} \left( \frac{\gamma_{1,0} \gamma_{1,-1}}{\gamma_{1,-1}} \right)^{s_0 - s_1},$$  \hspace{1cm} (356)

where we recall (343), and

$$y = \frac{\gamma_{0,-1} \gamma_{1,2}}{\gamma_{1,-1} \gamma_{0,2}}, \quad q = \frac{z_{0,-1} z_{1,2}}{z_{0,-1} z_{1,2}}, \quad z = \frac{z_{3,1} z_{1,2}}{z_{2,1} z_{3,2}},$$  \hspace{1cm} (357)

are the $G \times H$ invariants (the remaining invariant, $\zeta = (\gamma_{3,-1}/\gamma_{3,2})(\gamma_{1,2}/\gamma_{1,-1})$, enters only linearly). The $G \times H$ invariance allows to set $\gamma_{-1} = 0, \gamma_1 = 1, \gamma_2 = \infty, z_{-1} =$
0, \ z_1 = 1, \ z_2 = \infty$, giving us the equation
\[ \kappa \frac{\partial}{\partial z} \Psi = \left( \frac{\hat{A}_0}{z} + \frac{\hat{A}_q}{z - q} + \frac{\hat{A}_1}{z - 1} \right) \Psi \] (358)
with $\Psi = \left( \begin{array}{c} \Psi_0 \\ \Psi_1 \end{array} \right)$, and $\hat{A}_{zi}$ are two by two matrices of differential operators in the $y$-variable. The generalization to arbitrary $p$ is straightforward.

E.2.2. From KZ to BPZ in $y$ variables. Let us consider the accompanying case $p = 1$, with the same spins $s_{-1}, s_0, s_1, s_2$, as above. The conformal block $\Psi \prod_{i=1}^{2} d\gamma_i^{-s_i} dz_i^{-\Delta_i}$ can be now written as:
\[ \Psi(\gamma_1, \gamma_0, \gamma_2; z_1, z_0, z_1, z_2) = y^a(y-1)^b(y-q)^c y(\gamma_1, \gamma_0, \gamma_2; z_1, z_0, z_1, z_2) \]
\[ \left( \frac{z_{1,0} z_{1,-1}}{z_{0,-1}} \right)^{\Delta_1 - \Delta_2} \left( \frac{z_{1,0}}{z_{0,-1} z_{1,-1}} \right)^{\Delta_0 - \Delta_1} \left( \frac{\gamma_1 \gamma_0, \gamma_2; z_1, z_0, z_1, z_2}{} \right)^{s_{1,-2}} \left( \frac{\gamma_1 \gamma_0, \gamma_2; z_1, z_0, z_1, z_2}{} \right)^{s_{0,-1}} \] (359)
with
\[ y = \frac{\gamma_0, \gamma_1, \gamma_2; z_1, z_0, z_1, z_2}{\gamma_0, \gamma_1, \gamma_2; z_1, z_0, z_1, z_2}, \quad q = \frac{z_{0,-1} z_{2,1}}{z_{1,-2} z_{2,0}}. \]
\[ 2a = -1 - s_{-1} - s_0 + s_1 \quad 2b = -1 - s_{-1} - s_0 + s_1 + s_2, \quad 2c = 1 + s_{-1} + s_0 + s_1 + s_2. \] (360)
The KZ equation assumes a form of the simple partial differential equation:
\[ \frac{1}{\kappa} \frac{\partial\psi}{\partial q} = -\frac{1}{\kappa^2} \frac{y(y-1)(y-q)}{q(q-1)} \frac{\partial^2 \psi}{\partial y^2} - \left( \frac{\hat{\vartheta}_0^2 - \frac{\kappa}{4\kappa^2}}{y(y-1)} + \frac{1}{2} \left( \frac{1 + \frac{1}{\kappa}}{y} \right)^2 + \left( \frac{\hat{\vartheta}_q^2 + \frac{1}{\kappa}}{y} \right)^2 \right)y^{y(\hat{\vartheta}_\infty - \frac{1}{2})} \frac{1}{q(1-q)} \] (362)
where $\hat{\vartheta}$'s are given by the Eqs. (116), with $s_{-1} = \kappa \hat{\vartheta}_0, s_0 = \kappa \hat{\vartheta}_q, s_1 = \kappa \hat{\vartheta}_1, s_2 = \kappa \hat{\vartheta}_\infty$.

The Eq. (362) is none other than the BPZ equation (11) for the Liouville conformal block with one degenerate field and four generic primary fields.

E.2.3. From KZ to BPZ in the separated variables. Now let us now consider the general $p$ case. What follows goes, in a way, back to [42, 52, 53] and has been rediscovered and discussed in various publications, e.g. in [51, 43], was connected to free field representations [57, 53, 56, 11, 14], to $T$-duality and Fourier-Mukai transform [62], RG flows [50] etc.\footnote{We should warn the reader, that in the absence of the microscopic derivation, the conjectures based on plausible arguments of holomorphy, classical limits, or analogies with the two dimensional CFT, may prove wrong. For example, the two realizations of surface defects, using quiver gauge theory with fine tuned masses, i.e. the vortex strings, and the regular defects, which could be realized using orbifolds, are not different points on the renormalization group trajectory, but rather are different points in the Kähler moduli space of the surface theory. In other words they are related by the generalized flop transitions [82, 83], and both are needed to fully package the geometry of the infrared theory, including the effective twisted superpotential.}
The reason we present it here is to elucidate a few subtleties usually not presented, and to resolve an apparent contradiction with the classical case.

Let us present the solution \( \Psi(\gamma_{-1}, \gamma_0, \ldots, \gamma_{p+1}; z_{-1}, z_0, \ldots, z_{p+1}) \) as an integral:

\[
\Psi(\vec{\gamma}, \vec{z}) = \mathcal{F} \left[ \frac{1}{\prod_{i=-1}^{p+1} (z - z_i)^{s_i + 1}} \right]_{\vec{z}, \vec{\beta}} = \int_{\Gamma} d^{p+2} \beta \hat{\Psi}(\vec{\beta}; \vec{z}) \exp \left( \sum_{i=-1}^{p} \beta_i (\gamma_i - \gamma_{p+1}) \right)
\]  

(363)

over an appropriate \( p+2 \)-dimensional contour \( \Gamma \). In order to manipulate the generators \( J_i^a \) acting on \( \Psi \) we form the generating functions \( J_i^a(z) \), operator-valued meromorphic functions of an auxiliary variable \( z \):

\[
J_i^a(z) = \sum_{i=-1}^{p+1} \frac{1}{z - z_i} J_i^a
\]  

(364)

We also form the generating function of KZ connections (351):

\[
D(z) = \sum_{i=-1}^{p+1} \frac{1}{z - z_i} \nabla_i
\]  

(365)

The following relations are easy to verify:

\[
D(z) \Psi = \mathcal{F} \left[ \left( D(z) - \sum_{i=-1}^{p+1} \frac{s_i(s_i + 1)}{(z - z_i)^2} + j^0(z)^2 - \partial_z j^0(z) + j^-(z) j^+(z) \right) \hat{\Psi} \right]
\]  

(366)

with

\[
D(z) = \sum_{i=-1}^{p+1} \frac{\kappa}{z - z_i} \frac{\partial}{\partial z_i},
\]

\[
j^0(z) = \mathcal{V}(z) - \sum_{i=-1}^{p+1} \frac{s_i + 1}{z - z_i}
\]

\[
\mathcal{V}(z) = -\sum_{i=-1}^{p} \frac{1}{z - z_i} \beta_i \frac{\partial}{\partial \beta_i}.
\]

(367)

The first order differential operators \( D(z) \), \( \mathcal{V}(z) \) are vector fields on the space \( \mathbb{C}^{2(p+2)} \) with the coordinates

\[
(\beta_{-1}, \ldots, \beta_{p}; z_{-1}, \ldots, z_{p})
\]

The operator \( j^-(z) \) acts by multiplication by

\[
\beta(z) = \sum_{i=-1}^{p} \left( \frac{\beta_i}{z - z_i} - \frac{\beta_i}{z - z_{p+1}} \right).
\]

(368)

Finally, \( j^+(z) \) is a second order differential operator in \( \beta \)'s whose explicit form we don’t need since we are going to use a clever trick \([154]\). Namely, we pass to the new
variables: \((\beta_1, \ldots, \beta_p) \mapsto (b; w_0, \ldots, w_p)\), where \(\beta(w_a) = 0\), i.e.

\[
\beta(z) = b \frac{W(z)}{Z(z)}, \quad \text{with } W(z) = \prod_{a=0}^{p} (z - w_a), \quad Z(z) = \prod_{i=1}^{p+1} (z - z_i),
\]

(369)

and \(b = \sum_{i=-1}^{p} (z_i - z_{p+1}) \beta_i\). It is a simple matter to express the vector fields \(D(z), V(z)\) in the new coordinate system. We observe:

\[
\mathcal{L}_{D(u)} \beta(v) = \frac{\kappa}{b} \frac{\partial}{\partial v} \frac{\beta(u) - \beta(v)}{u - v} = \frac{\beta(v)}{b} (\mathcal{L}_{D(u)} b) + \sum_{a=0}^{p} \frac{\beta(v)}{w_a - v} (\mathcal{L}_{D(u)} w_a) + \sum_{i=-1}^{p+1} \frac{\beta(v)}{v - z_i} (\mathcal{L}_{D(u)} z_i),
\]

\[
\mathcal{L}_{V(u)} \beta(v) = \frac{\beta(u) - \beta(v)}{u - v} + \frac{\beta(u)}{v - z_{p+1}} = \frac{\beta(v)}{b} (\mathcal{L}_{V(u)} b) + \sum_{a=0}^{p} \frac{\beta(v)}{w_a - v} (\mathcal{L}_{V(u)} w_a)
\]

(370)

with \(u, v\) auxiliary parameters, hence

\[
V(z) = -\beta(z)(z - z_{p+1}) \left[ \frac{\partial}{\partial b} + \sum_{a=0}^{p} \frac{1}{(z - w_a)(w_a - z_{p+1})} \frac{\partial}{\partial w_a} \right]
\]

(371)

and

\[
\frac{1}{\kappa} D(z) = \beta(z) \frac{\partial}{\partial b} + \sum_{i=-1}^{p+1} \frac{1}{z - z_i} \frac{\partial}{\partial z_i} + \sum_{a=0}^{p} \frac{1}{z - w_a} \left( 1 - \frac{\beta(z)}{\beta'(w_a)(z - w_a)} \right) \frac{\partial}{\partial w_a}.
\]

(372)

Note that the presence of the \(w_a\)-derivatives in the \(D(z)\) vector field is the reflection of the background dependence of the \(w\)-coordinates we alluded to in the section 6.

Let us now solve the KZ equations

\[
D(z) \Psi = 0
\]

(373)

together with the global Lie\((G)\) constraints. The \(\sum_{i=-1}^{p+1} J_i^\Psi = 0\) is obeyed trivially in the \(\beta\)-representation, the \(\sum_{i=-1}^{p+1} J_0^\Psi = 0\) equation translates to \([z^{-1}] j^0(z) \hat{\Psi} = -\hat{\Psi}\).

Now comes the clever trick [154]. The left hand side of (373), as a function of \(z\), has only the first order poles at \(z = z_i\), being equal to (365). Moreover, with (380) in place, at large \(z\) it goes to zero faster than \(z^{-2}\), therefore the right hand side of (373) has the form:

\[
D(z) \Psi = \mathcal{F} \left[ \frac{W(z)}{Z(z)} \sum_{a=0}^{p} \frac{Z(w_a)}{W'(w_a)(z - w_a)} D(w_a) \Upsilon \right]
\]

(374)

It remains to compute \(D(w_a)\). First note that, as \(z \to w_a\), the vector field \(V(z)\) approaches \(-\partial_{w_a}\). Next, in computing \(j^0(z) j^0(z)\) for \(z \to w_a\) one should be careful in
that the variable $z$ is not acted upon by the derivatives in $w_a$’s and $z_i$’s. Fortunately the term $\partial_j j^i(z)$ in (366) precisely cancels this discrepancy leading to

$$D_a := -D(z \to w_a) = \nabla^2_a + \sum_{b \neq a} \frac{\kappa}{w_a - w_b} \left( \frac{\partial}{\partial w_a} - \frac{\partial}{\partial w_b} \right) -$$

$$- \sum_{i=-1}^{p+1} \left( \frac{s_i(s_i + 1)}{(w_a - z_i)^2} + \frac{\kappa}{w_a - z_i} \left( \frac{\partial}{\partial z_i} + \frac{\partial}{\partial w_a} \right) \right)$$

(375)

with

$$\nabla_a = \frac{\partial}{\partial w_a} + \sum_{i=-1}^{p+1} \frac{s_i + 1}{w_a - z_i}$$

(376)

The equation $\sum_{i=-1}^{p+1} J^+_i \Psi = 0$ is, in fact, contained in the equations $D_a \Upsilon = 0$, since

$$[z^{-3}]D(z) \Psi = F \left[ b^2 - s \left( [z^{-1}]j^+(z) \right) \Upsilon \right]$$

(377)

assuming we already imposed the global Lie($H$)-constraints:

$$L_{-1} \Psi = [z^{-1}]D(z) \Psi = 0, \quad L_0 \Psi = \frac{1}{\kappa} [z^{-2}]D(z) \Psi - \Delta \Psi = 0$$

(378)

with

$$\Delta = \sum_{i=-1}^{p+1} \Delta_i,$$

(379)

implying

$$\bar{\Psi}(\bar{z}; z) = b^{1-S} \Upsilon(\bar{w}; z),$$

$$\Upsilon(\bar{w}; z) = (z_p - z_{-1})^{S-1+\Delta} \Xi \left( \frac{w_0 - z_{-1}}{z_p - z_{-1}}, \ldots, \frac{w_p - z_{-1}}{z_p - z_{-1}}, z_0 - z_{-1}, \ldots, \frac{z_{p+1} - z_{-1}}{z_p - z_{-1}} \right)$$

(380)

with

$$S = \sum_{i=-1}^{p+1} (s_i + 1),$$

(381)

and

$$L_+ \Upsilon =$$

$$\sum_{i=-1}^{p+1} \left( z_i^2 \frac{\partial \Upsilon}{\partial z_i} - (2\Delta_i + S - 1) z_i \Upsilon \right) + \left( \sum_{a=0}^{p} w_a^2 \frac{\partial \Upsilon}{\partial w_a} + (S - 1) w_a \Upsilon \right) = 0$$

(382)

is the remaining global Lie($H$)-invariance constraint. Together, (378) and (382) signify that the $w$-variables transform under the $H$-action together with the $z$-parameters, with some effective conformal dimensions assigned both to $z_i$’s and $w_a$’s.
One can further define
\[
\Upsilon(w_0, \ldots, w_p; z_{-1}, \ldots, z_{p+1}) = \prod_{-1 \leq i < j \leq p+1} (z_i - z_j)^{s_i + s_j + 2 - \frac{\kappa}{2}} \prod_{a,i} (w_a - z_i)^{\frac{1}{2} - s_i - 1} \prod_{a < b} (w_a - w_b)^{-\frac{1}{2}} \Upsilon(w; \vec{z})
\] (383)
to map (373) precisely to the BPZ equations [14]
\[
\frac{\partial^2 \tilde{\Upsilon}}{\partial w_a^2} = \sum_{b \neq a} \left( \frac{\kappa}{w_a - w_b} \frac{\partial}{\partial w_b} + \frac{\kappa(\frac{3}{2} \kappa - \frac{1}{2})}{(w_a - w_b)^2} \right) \tilde{\Upsilon} + \sum_{i=-1}^{p+1} \left( \frac{\kappa}{w_a - z_i} \frac{\partial}{\partial z_i} + \frac{s_i(s_i + 1) + \frac{\kappa}{2} \left(1 - \frac{s_i}{2}\right)}{(w_a - z_i)^2} \right) \tilde{\Upsilon}
\] (384)
This relation is usually interpreted in some kind of coset realization of minimal models from the WZNW theory, using free fields [56]. Since we established this relation for complex values of all parameters, spins, levels, it cannot be explained in this way. Instead, it means that the analytic continuation of the WZNW theory, being actually a four dimensional theory, is a Liouville/Toda theory, coupled to a sigma model on the flag variety. The Liouville/Toda theory is the BPS/CFT dual of the supersymmetric gauge theory we used in our microscopic computations of the partition functions and expectation values of the surface defect, so that, in particular, \(\kappa = n \varepsilon_2 / \varepsilon_1\) for \(SU(n)\) theory. We plan to return to this elsewhere.

E.2.4. NS limits. The two limits, \(\varepsilon_2 \to 0\) and \(\varepsilon_1 \to 0\), are of special interest. In the \(\varepsilon_2 \to 0\) limit, \(\kappa \to 0\) and (384) becomes truly separated, so that \(w_a\)'s don’t talk to each other. In this way one obtains the \(SL_2\)-opers [13] (in this case simply second order differential operators with regular singularities [51]) as a way to package the low-energy information of the gauge theory, as in [134, 82]. The KZ/BPZ relation becomes, in this limit, a relation between the Gaudin quantum integrable system (a genus zero version of the Hitchin system), and the monodromy data of the \(SL_2\)-opers. For integral \(s_i\) this relation was explored in [43]. In [134] a proposal was made for general \(s_i\)'s.

The opposite, \(\varepsilon_1 \to 0\), limit corresponds to the quasiclassical limit \(\kappa \to \infty\) [144].

In this limit the Eq. (362), with the ansatz \(\psi = e^{\kappa S}\) becomes the Hamilton-Jacobi equation of the Painlevé VI, as in [12] [101]. More generally, let \(s_i = \kappa \theta_i\), \(i = -1, \ldots, p\), \(s_{p+1} = \frac{1}{2}\), and let us look for the solutions of the KZ equation in the WKB form:
\[
\Psi = e^{\kappa S(\gamma_1, \ldots, \gamma_p; z_{-1}, \ldots, z_{p+1})} \chi(\gamma_{-1}, \ldots, \gamma_p; s_{p+1}, z_{-1}, \ldots, z_{p+1})
\] (385)
where
\[
\chi(\gamma_{-1}, \ldots, \gamma_p; \gamma_{p+1}; z) = \chi(\gamma_{-1}, \ldots, \gamma_p; z) + \gamma_{p+1} \chi(\gamma_{-1}, \ldots, \gamma_p; z) + O(1/\kappa)
\] (386)
Then the leading term in \(\kappa\), the equations (350) become:
\[
\frac{\partial S}{\partial z_i} = \sum_{j \neq i} \frac{\text{Tr} A_i A_j}{z_i - z_j}, \quad i = -1, \ldots, p
\] (387)
with \( A_i = A(\beta_i \gamma_i - \partial_i \gamma_i, \beta_i \gamma_i - \partial_i \gamma_i + i \partial_i \gamma_i, \partial_i - \beta_i \gamma_i) \), cf. (58), \( \beta_i = \partial S/\partial \gamma_i \), and

\[
\frac{\partial}{\partial z_{p+1}} \left( \frac{\chi_+}{\chi_-} \right) = \sum_i \frac{A_i}{z_{p+1} - z_i} \left( \frac{\chi_+}{\chi_-} \right). \tag{388}
\]

The Eq. (387) is none other than our friend the Hamilton-Jacobi form of the Schlesinger system, while (388) shows \( \mathcal{Y} \) is the horizontal section of the meromorphic connection

\[
\partial_z + \sum_i \frac{A_i}{z - z_i}, \tag{389}
\]

A much more thorough account of these matters can be found in [144, 70].

On the other hand, setting \( \tilde{\mathcal{Y}} = e^{\kappa \mathcal{S}(\vec{w}; \vec{z})} \) in (384) and sending \( \kappa \to \infty \), we arrive at:

\[
\left( \frac{\partial \tilde{\mathcal{S}}}{\partial w_a} \right)^2 = \sum_{b \neq a} \left( \frac{1}{w_a - w_b} \frac{\partial \tilde{\mathcal{S}}}{\partial w_b} + \frac{3}{4(w_a - w_b)^2} \right) + \sum_{i=1}^{p+1} \left( \frac{1}{w_a - z_i} \frac{\partial \tilde{\mathcal{S}}}{\partial z_i} + \frac{\partial^2 - \frac{1}{4}}{(w_a - z_i)^2} \right), \tag{390}
\]

Clearly, (390) do not look like the Painlevé-Schlesinger equations, as we have \( p + 1 \) variables \( w_a \), as opposed to \( p \) variables in the \( y \)-representation. Indeed, unlike the classical case where we could fix a gauge where \( \beta_\infty = 0 \), i.e. \( w_p = z_p + 1 \), in the quantum case this is not possible, the \( J^+ \) generator of the \( \text{Lie}(G) \) symmetry being represented, in the \( \beta \)-representation, by a second order differential operator. However, the solution to the equation \( \mathcal{D}_0 \mathcal{Y} = 0 \) at fixed \( w_1, \ldots, w_p \), as a function of \( w_0 \), can be fixed once the initial conditions are chosen. Physically it means bringing one of the degenerate fields \( \mathcal{V}_{(2,1)} \) to one of the primaries or another degenerate field, and extracting the asymptotics following from the operator product expansion. Since the equation is of the second order, one needs two initial conditions, which corresponds to the fact that the fusion of the generic primary field with \( \mathcal{V}_{(2,1)} \) contains exactly two primaries.

Thus, the PVI equation is recovered in the limit e.g. \( w_0 \to \infty \) with the shift of \( \beta_\infty \), as in [101]. In this way the Okamoto transformation (159) is recovered asymptotically.

**Appendix F. Higher rank generalizations II: flat connections**

We shall now describe the moduli space \( \mathcal{M}_{alg}^{G_r} \) of meromorphic \( G_r \)-connections on the \( p + 3 \)-punctured sphere with the residues \( A_0 \) and \( A_\infty \) belonging to the orbits \( O^{(0)}_{reg} \) and \( O^{(\infty)}_{reg} \) with some generic vectors \( \vec{\rho}^{(0)}, \vec{\rho}^{(\infty)} \in \mathbb{C}^r \), and \( A_{z_0}, A_{z_1}, \ldots, A_{z_p} \) belonging to \( O_{\text{min}}^{\rho_0}, O_{\text{min}}^{\rho_1}, \ldots, O_{\text{min}}^{\rho_p} \), respectively, with \( \nu_0, \nu_1, \ldots, \nu_p \in \mathbb{C} \):

\[
\mathcal{A}(z) = \frac{\mu_{\text{reg}}(\nu_{(0)}; V_{(0)})}{z} + \sum_{b=0}^{p} \frac{\mu_{\text{min}}(u(z_b); v(z_b))}{z - z_b} \tag{391}
\]

\footnotetext{24}{It is useful to perform this exercise even in the case \( p = 0 \), where this two-fold degeneracy is exhibited in the monodromy of the hypergeometric function \( _2F_1 \).}
so that the sum of the residues vanishes:
\[
\mu_{\text{reg}}(u^{(0)}, v^{(0)}) + \mu_{\text{reg}}(u^{(\infty)}, v^{(\infty)}) + \sum_{b=0}^{p} \mu_{\text{min}}(u_{zb}, v_{zb}) = 0 \tag{392}
\]

As before, there are several coordinate systems we can put on the moduli space
\[
\mathcal{M}_{G_{r,p}}^{\text{alg}} = \left( \mathcal{O}_{\text{reg}}^{(0),r} \times \mathcal{O}_{\text{min}}^{\nu_0,r} \times \mathcal{O}_{\text{min}}^{\nu_1,r} \times \ldots \times \mathcal{O}_{\text{min}}^{\nu_p,r} \times \mathcal{O}_{\text{reg}}^{(\infty),r} \right) / G_r \tag{393}
\]
The analogues of the $\gamma$, $y$-coordinates are the invariants built out of $u$'s only. In the basis $(v_i)$, where $A_0$ is lower-triangular, while $A_\infty$ is upper-triangular, and $u^{(1)} = \sum_{i=1}^{r+1} v_i$ (this is the higher rank analogue of the $G$-gauge where $\gamma_0 = 0$, $\gamma_1 = 1$, and $\gamma_\infty = \infty$) we define simply
\[
y_a^m = v_{(za)}^m \tag{394}
\]
where $1 \leq m \leq r$, $1 \leq a \leq p$. More invariantly, define the ‘tau-functions’
\[
\tau_{0,\xi} = \psi_{\xi}^{(\infty)} \wedge \ldots \wedge \psi_{1}^{(\infty)} \wedge v_{(\xi)},
\tau_{i,\xi} = \psi_{i}^{(0)} \wedge \psi_{i-1}^{(0)} \wedge \ldots \wedge \psi_{1}^{(0)} \wedge \psi_{r-i}^{(\infty)} \wedge \ldots \wedge \psi_{1}^{(\infty)} \wedge v_{(\xi)}, \quad i = 1, \ldots, r - 1
\tau_{r,\xi} = \psi_{r}^{(0)} \wedge \psi_{r-1}^{(0)} \wedge \ldots \wedge \psi_{1}^{(0)} \wedge v_{(\xi)} \tag{395}
\]
where, in terms of the $(B^{(0)}, \tilde{B}^{(0)}), (B^{(\infty)}, \tilde{B}^{(\infty)})$ data attached to the residues $A_0$ and $A_\infty$ respectively,
\[
\text{Span}(\psi_{1}^{(\xi)}, \ldots, \psi_{i}^{(\xi)}) = \mathcal{Y}_{i}^{(\xi)}, \quad \xi = 0, \infty \tag{396}
\]
The tau-functions are not gauge-invariant, they are defined up to a scaling. However, the ratios
\[
y_a^m = \frac{\tau_{m-1,za} \tau_{m,za}}{\tau_{m,za} \tau_{m-1,za}}, \quad m = 1, \ldots, r, \quad a = 1, \ldots, p \tag{397}
\]
are well-defined meromorphic functions on $\mathcal{M}_{G_{r,p}}^{\text{alg}}$.

Let us now specify to the $p = 1$ case. The analogue of the polygon length coordinates $\pm \ell$ are the eigenvalues $\ell_i, \ell_{i+1}, \ldots, \ell_{i+r+1}$ of the sum $A_0 + A_q$, which are equal to the eigenvalues of $A_1 + A_\infty$. Let us package the eigenvalues of $A_0, A_0 + A_q = -A_1 - A_\infty, A_\infty$ with the help of the characteristic polynomials:
\[
\mathcal{L}(z) = \text{Det}(A_0 + A_q - z) = \prod_{l=1}^{r+1} (\ell_l - z),
\mathcal{A}_0(z) = \text{Det}(A_0 - z) = \prod_{l=1}^{r+1} (\psi_{l}^{(0)} - z), \tag{398}
\mathcal{A}_\infty(z) = \text{Det}(A_\infty - z) = \prod_{l=1}^{r+1} (\psi_{l}^{(\infty)} - z).
The simple identities follow from the special form of $A_q, A_1$:

$$\frac{\mathcal{L}(z)}{\mathcal{A}_0(z - \nu_q)} = 1 - u(q) \frac{1}{A_0 + \nu_q - z} v(q), \quad (-)^N \frac{\mathcal{L}(-z)}{\mathcal{A}_\infty(z - \nu_1)} = 1 - u(1) \frac{1}{A_\infty + \nu_1 - z} v(1),$$

$$\frac{\mathcal{A}_0(z)}{\mathcal{L}(z + \nu_q)} = 1 + u(q) \frac{1}{A_0 + A_q - \nu_q - z} v(q), \quad \frac{\mathcal{A}_\infty(z)}{\mathcal{L}(z + \nu_1)} = 1 + u(1) \frac{1}{A_1 + A_\infty - \nu_1 - z} v(1),$$

implying the residues:

$$\sum_i u(q)(v_i)u^i = u(q)$$

$$w^i(v^{(1)})u(1)(v_i) = -\frac{A_\infty(\ell_i - \nu_1)}{\mathcal{L}'(\ell_i)},$$

$$w^i(v^{(q)})u(q)(v_i) = -\frac{A_0(\ell_i - \nu_q)}{\mathcal{L}'(\ell_i)},$$

where $v_i \in \mathcal{V}$, $w^i \in \mathcal{V}^*$ are the eigenvectors of $A_0 + A_q = -A_1 - A_\infty$, with the eigenvalue $\ell_i$.

The analogue of the angles $\pm \theta$ are the complex angular variables $\Theta_i$ defined by:

$$e^{\Theta_i} = \frac{w^i(v^{(1)})}{w^i(v^{(q)})} \left( \frac{u^{i+1}(v^{(q)})}{u^{i+1}(v^{(1)})} \right), \quad i = 1, \ldots, r \quad (401)$$

from which we compute the Hamiltonian

$$H_q = \frac{h_{0q}}{q} + \frac{h_{1q}}{q - 1} \quad (402)$$

with

$$h_{0q} = \frac{1}{2} \left( -\ell_0^2 - \frac{r}{q} \nu_2^2 + \sum_{i=1}^{r+1} \ell_i^2 \right)$$

and

$$h_{1q} = -(r + 1)\nu_q \nu_1 + \sum_{m,n=1}^{r+1} \frac{A_\infty(\ell_n - \nu_1) A_0(\ell_m - \nu_q)}{\mathcal{L}'(\ell_n) \mathcal{L}'(\ell_m)} e^{\theta_m - \theta_n} \quad (403)$$

where $\theta_i - \theta_{i+1} = \Theta_i$.

This construction can be inductively generalized to the case of general $p$. At the $a'$th step, take $r$ of the eigenvalues $\ell^{(a)}_i$, $i = 1, \ldots, r$, $\ell^{(a)}_{r+1} = -\ell^{(a)}_1 - \ldots - \ell^{(a)}_r$ of

$$\mathcal{L}_a = A_0 + A_{z_0} + \ldots + A_{z_{a-1}} = -\left( A_{z_a} + A_{z_{a+1}} + \ldots + A_{z_p} + A_\infty \right),$$

and the angles

$$e^{\Theta_i^{(a)}} = \frac{u^i(v^{z_a})}{u^i(v^{(z_a-1)})} \frac{u^{i+1}(v^{z_{a-1}})}{u^{i+1}(v^{z_a})}, \quad i = 1, \ldots, r \quad (405)$$

where $u^i \mathcal{L}_a = \ell^{(a)}_i \mathcal{L}_a$ are the eigenvectors corresponding to the chosen eigenvalues.
Finally, the analogues of $\alpha$, $\beta$-coordinates on the moduli space $M_{r,p}^{\text{loc}}$ of local systems, i.e. the monodromy data, i.e. representations of the fundamental group of the $p+3$-punctured sphere into $G_r$, such that the conjugacy class of the monodromy around $z_0$ is that of $\exp 2\pi i A_{z_0}$, while those around $0, \infty$ are equal to the conjugacy class of $\exp 2\pi i A_{0,\infty}$, respectively, are described in [32]. Another coordinate system, based on the spectral networks, was proposed earlier in [75].

We have therefore all the necessary ingredients to formulate the higher rank, general $p$ analogue of the GIL formula. It relates the tau-functions of the higher rank Schlesinger deformations to the $c=1$ conformal blocks of Toda conformal theories, in agreement with the conjectures [162]:

$$\tau_{0,p}(q_1, \ldots, q_p; \alpha, \beta; \varphi^{(0)}, \nu_0, \ldots, \nu_p, \nu^{(\infty)}) = \sum_{n_1, \ldots, n_p \in \mathbb{Z}^r \subset \mathbb{Z}^{r+1}} e^{\sum_{a=1}^n n_a \bar{\nu}_a} Z_{A_p}((\bar{\alpha}_a + n_a)\bar{h}; \bar{m}_1, \ldots, \mu_{p-1}, \bar{m}_p, q_1, \ldots, q_p; \bar{h}, -\bar{h}) \tag{407}$$

where on the right-hand side we have the partition functions of the linear quiver gauge theories with the $A_p$-type quiver [135] with the gauge groups $U(n_i), n_i = r + 1$ at each node, with $r+1$ fundamental hypermultiplets of masses $\bar{m}_1$, and $\bar{m}_p$ at the nodes $U(n_1)$ and $U(n_p)$, respectively, and the bi-fundamental hypermultiplets charged under $U(n_i) \times U(n_{i+1})$, of mass $\mu_i$. The relations between the masses and the monodromy data $\bar{\nu}^{(0,\infty)}$, $\nu_a$ can be found in [162] [135] [132].

To support the conjecture (407) we mention that in Ref. [138] it is shown that the quantum version of (402) is a) equivalent to the Knizhnik-Zamolodchikov equation for $sl(r+1)$ four-point conformal block; b) equivalent to the non-perturbative Dyson-Schwinger equation obeyed by the surface defect in the $SU(r+1)$ four dimensional $\mathcal{N} = 2$ super-Yang-Mills theory with $2(r+1)$ fundamental hypermultiplets.

**Appendix G. Genus one and $\mathcal{N} = 2^*$ theory**

It was demonstrated in [132] that the partition function of the regular surface defect in the $SU(r+1)$ gauge theory with adjoint hypermultiplet, also known as the $\mathcal{N} = 2^*$ theory, obeys the Knizhnik-Zamolodchikov-Bernard [16] equation for the torus one-point conformal block with the minimal representation $\mathcal{V}_\nu$:

$$(r+1)\varepsilon_1 \varepsilon_2 \frac{d}{d\tau} \Psi = \left(\sum_{i=1}^{r+1} \frac{\varepsilon_1^2}{2} \frac{\partial^2}{\partial q_i^2} + m(m+\varepsilon_1) \sum_{1 \leq i < j \leq r+1} \varphi(q_i - q_j; \tau)\right) \Psi \tag{408}$$

In the limit $\varepsilon_1 \to 0$, with the rest of the parameters kept fixed the equation (408) degenerates, via the familiar ansatz $\Psi \sim e^{\varepsilon_2 S(q_0/\varepsilon_2, m/\varepsilon_2, \tau)/\varepsilon_1^2 + \cdots}$, to the Hamilton-Jacobi equation

$$(r+1)\varepsilon_2 \frac{dS}{d\tau} = H_{\text{eCM}} \left(\frac{\partial S}{\partial q_i}, q_i; m, \tau\right) \tag{409}$$

with $H_{\text{eCM}}$ the Hamiltonian of $r+1$-particle elliptic Calogero-Moser model [29]. Let us explain the meaning of (409) in the context of the isomonodromic deformation (cf. [79] [80] [103] [157] [98]).
We work on the elliptic curve $E_{\tau} = \mathbb{C}/\mathbb{Z} \oplus \tau \mathbb{Z}$, with the coordinate $z \sim z + a + b\tau$, for $a, b \in \mathbb{Z}$, which is isomorphic to the standard two-torus $T^2 = \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$, with the coordinates $(x, y)$, $x \sim x + 1$, $y \sim y + 1$. The relation between the complex and the real descriptions is simply:

$$z = x + y\tau,$$

We are going to study a family of flat connections on $T^2 \setminus p$, where the point $p$ is at $z = 0$. Moreover we assume the conjugacy class of the holonomy of the flat connection around $p$ is fixed, with the eigenvalues of multiplicity $(N - 1, 1)$, respectively. In other words, we allow for a $\delta$-function singularity in the curvature $F_{xy}$,

$$F_{xy} = \partial_x A_y - \partial_y A_x + [A_x, A_y] = \mu_{\min}(u, v)\delta(x, y)$$

for some $u, v, \nu$, as in (312), and $r = N - 1$. There are two ways of solving (411). We can fix a gauge $[59]$ in which $A_x$ (equivalently, $A_y$) is diagonal $x$-(y-)independent matrix. But we would like to fix a gauge in a $\tau$-dependent way, and explore the $\tau$-dependence of what follows. So, we fix the gauge, as in $[58]$:

$$A_{\bar{z}} = \frac{2\pi i}{\tau - \bar{\tau}} \text{diag} (q_1, \ldots, q_{r+1})$$

meaning that $\bar{\partial} - A_{\bar{z}} d\bar{z}$ defines a holomorphic vector bundle on $E_{\tau}$, isomorphic to a direct sum of $r + 1$ line bundles. Then, for $A_z$ we get the equation

$$\bar{\partial} A_z - [A_z, A_z] \propto \mu_{\min}(u, v)\delta(2)(z, \bar{z})$$

which is identical to the equation for a Higgs field in the punctured analogue of Hitchin’s equation $[58, 125]$. The solution is unique up to a shift of $A_z$ by the constant diagonal matrix (cf. (410)):

$$A_z = 2\pi i \sum_{i=1}^{r+1} \left( p_i - \frac{q_i}{\tau - \bar{\tau}} \right) E_{ii} + \nu \sum_{i \neq j} \frac{\vartheta_{11}(z + q_j; \tau)\vartheta'_{11}(0; \tau)}{\vartheta_{11}(z; \tau)\vartheta_{11}(q_j; \tau)} e^{2\pi i q_{ij}} E_{ij}$$

where $q_{ij} = q_i - q_j$, and we used the odd theta function

$$\vartheta_{11}(z; \tau) = \sum_{r \in \mathbb{Z} + \frac{1}{2}} e^{2\pi i (z + \frac{1}{2})^2 r + \pi i r^2}$$

obeying the heat equation

$$\partial_{\tau} \vartheta_{11} = \frac{1}{4\pi i} \partial_z^2 \vartheta_{11}$$

We use $[25]$:

$$\xi(z; \tau) = \partial_z \log \vartheta_{11}(z; \tau), \quad \varphi(z) = -\partial_z \xi(z)$$

The family $[112], (414)$ obeys:

$$\delta A_x = D_x \epsilon, \quad \delta A_y = D_y \epsilon$$

Note that our definition of $\varphi$ differs by a $\tau$-dependent constant from the standard one.
where the $\delta$-variation means varying $\tau$ and $\bar{\tau}$ while keeping $x,y$ fixed. Then (418) holds, with
\[
\epsilon = \nu \delta \tau \sum_{i \neq j} \left[ \varphi(q_{ij}) E_{ii} + (\xi(z + q_{ij}) - \xi(q_{ij}) + 2\pi i y) \frac{\vartheta'_{11}(z + q_{ij})}{\vartheta_{11}(z) \vartheta_{11}(q_{ij})} e^{2\pi i q_{ij} y} E_{ij} \right]
\] (419)
and the variations of the $q_i$ and $p_i$ parameters according to
\[
\delta q_i = p_i \delta \tau, \quad \delta p_i = \nu^2 \sum_{j \neq i} \varphi'(q_{ij}) \delta \tau
\] (420)
This is nothing but the Hamiltonian flow with the $\tau$-dependent Hamiltonian
\[
H_{ecM} = \frac{1}{2} \sum_{i=1}^{r+1} p_i^2 - \nu^2 \sum_{1 \leq i < j \leq r+1} \varphi(q_{ij}; \tau)
\] (421)
Note that in verifying (418), (419) the relations
\[
\varphi(z_1) + \varphi(z_2) + \varphi(z_3) - (\xi(z_1) + \xi(z_2) + \xi(z_3))^2 = -\frac{\vartheta''_{11}(0)}{\vartheta_{11}'(0)}
\] (422)
for $z_1 + z_2 + z_3 = 0 \in E_\tau$, and
\[
-\frac{\vartheta_{11}(z_1 + z_2) \vartheta_{11}(z_1 + z_3) \vartheta_{11}(z_2 + z_3) \vartheta_{11}'(0)}{\vartheta_{11}(z_1) \vartheta_{11}(z_2) \vartheta_{11}(z_3) \vartheta_{11}(z_4)} = \xi(z_1) + \xi(z_2) + \xi(z_3) + \xi(z_4)
\] (423)
for $z_1 + z_2 + z_3 + z_4 = 0 \in E_\tau$, are used. For the general theory of the related functional equations and their rôle in the theory of Calogero-Moser systems see [29, 22]. The genus one isomonodromy problem is analyzed, in a different gauge, in [98, 103]. For the relation to the KZB equations, see [104, 79, 80]. It is straightforward to generalize this analysis to the case of $p$ punctures, specifically for $p$ minimal ones. The formulas for $A_z$ are very similar to the formulas for the genus one Higgs field [125, 98, 103].

Naturally we expect the isomonodromic $\tau$-function for the genus one with $p$ punctures $z_1, \ldots, z_p$, at least in the case of the residues $A_i = \text{res}_{z_i} A_i dz$ of the poles of the meromorphic connection belonging to the minimal orbits $O^{\nu_{i,x}}_{\text{min}}$ to be related to the partition function of the $\varepsilon_1 = -\varepsilon_2 = 2\pi i$ affine $\hat{A}_{p-1}$ quiver theory (i.e. circular quiver with $p$ nodes, with the gauge group $SU(n_0) \times SU(n_1) \times \ldots \times SU(n_{p-1})$, with $n_i = r+1$ for all $i$),
\[
\tau_{x,p}^{\varepsilon_1}(\alpha_1, \beta_1, \ldots, \alpha_p, \beta_p; \tau, z_1, \ldots, z_p) \sim \sum_{n_1, \ldots, n_p \in \mathbb{Z}^r} e^{\bar{\beta}_i \cdot \bar{n}_i} Z_{A_{p-1}}((\alpha_i + \bar{n}_i)h; m_1, \ldots, m_p; h, -h; q_1, \ldots, q_p)
\] (424)
where $m_i$'s are the masses of the bi-fundamental hypermultiplets charged under the $SU(n_i) \times SU(n_{i+1})$, related to the eigenvalues $(\nu_i, -\nu_i, \ldots, -\nu_i)$ of the residues $A_i$ and the monodromy parameters $\alpha_i, \beta_i$ are defined analogously to the genus zero case studied in [82], $q_i$ are the instanton factor associated with the $SU(n_i)$ gauge group,
\[
q_1 \ldots q_p = e^{2\pi i r}, \quad q_i = e^{2\pi i (z_i - z_{i+1})}, \quad i = 1, \ldots, p - 1
\] (425)
and \( \sim \) stands for the \( U(1) \)-factors (\( \vec{\alpha}, \vec{\beta} \)-independent functions of \( q_i \)'s).

\[ \bullet \sim \bullet \sim \bullet \sim \bullet \sim \bullet \sim \bullet \]

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E-mail: niki-tastring@gmail.com

Simons Center for Geometry and Physics,, C. N. Yang Institute for Theoretical Physics., Stony Brook University, Stony Brook NY 11794-3636, USA. E-mail: niki-tastring@gmail.com

26on leave of absence from: Center for Advanced Studies, Skoltech and IITP RAS, Moscow, Russia