Fidelity, entanglement, and information complementarity relation

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We investigate the dynamics of information in isolated multi-qubit systems. It is shown that information is in not only local form but also nonlocal form. We apply a measure of local information based on fidelity, and demonstrate that nonlocal information can be directly related to some appropriate well defined entanglement measures. Under general unitary transformations, local and nonlocal information will exhibit unambiguous complementary behavior with the total information conserved.

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I. INTRODUCTION

Understanding the physics of quantum many-body systems is a fundamental goal of condensed matter theory. People are interested in the study of strongly correlated quantum states which will exhibit lots of fascinating phenomena, such as quantum phase transitions [1, 2, 3] and Kondo effect [4]. In the field of quantum information processing, quantum many-body systems are also the essential ingredient. Perspective scalable quantum computation [5, 6, 7, 8, 9] and communication [10] schemes based on different many-body models of condensed matter physics have have attracted intensive interest. Moreover, unlike the situation of bipartite entanglement, the picture of multipartite entanglement in quantum many-body systems is still not very clear [11, 12, 13].

However, the fact that the number of parameters required to describe a quantum state of many particles grows exponentially with the number of particles leaves a practical obstacle to the study of quantum many-body systems. Therefore most of research focuses on the static properties of the ground states of certain types of many-body models with quantum Monte Carlo calculations [14] and the density matrix renormalization group [15]. In this paper, we adopt a measure of local information based on fidelity and reveal a heuristic connection between measures of entanglement and nonlocal information. Therefore, we establish an elegant complementarity relation between local and nonlocal information. Our results link the local information and measures of entanglement, particularly genuine multi-qubit entanglement (i.e. shared by all the involved qubits). This make it possible to propose some appropriate information-theoretic measure of such genuine multi-qubit entanglement [16], which is one of the most central issues in quantum information theory [20, 21, 22, 23, 24].

The paper is structured as follows. In Sec II, we introduce the measure of local information based on the definition of optimal fidelity. In Sec III and IV, we investigate the information dynamics and demonstrate the perfect complementary behavior through simple examples in two- and three-qubit systems. In Sec V, we formulate the information complementarity relation based on appropriate measures of local information and entanglement. In Sec VI are discussions and conclusions.

II. OPTIMAL FIDELITY AND LOCAL INFORMATION

We start by considering the information content in one qubit. Suppose Alice get one qubit in the state $|\varphi\rangle$ from a random source \{\{|$\Omega\rangle$,|$\Omega\perp\rangle$\} with the probabilities $\{1/2, 1/2\}$. Alice hold the qubit in state $|\varphi\rangle$, and can exactly eliminate the uncertainty about the state preparation, i.e. the information content in $|\varphi\rangle$ should be 1 bit. If Alice sent the qubit to Bob through quantum channels, the qubit Bob received becomes $\rho = \xi(|\varphi\rangle\langle\varphi|)$. If $\rho = \frac{1}{2}(I + \vec{s} \cdot \vec{s})$, with $\vec{s}$ the Bloch vector and the Pauli operators $\vec{s} = (\sigma_x, \sigma_y, \sigma_z)$, is a mixed state, Bob can only tell which of the two states that Alice sent to

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him with some success probability, which results from the information loss through the quantum channels. The success probability can be characterized by the fidelity $F = \langle \varphi | \rho | \varphi \rangle$ \cite{25}. However, we note that Bob can apply some kind of physical realizable local strategy to maximize the success probability. Therefore, we can naturally define the optimal fidelity as

$$F_o(\rho) = \max_{A \in SU(2)} \langle \varphi | A \rho A^\dagger | \varphi \rangle$$

Here unitary operation $A$ can be interpreted as some kind of local evolution for Bob to maximize the success probability. After simple calculations, it can be seen that $F_o(\rho) = 1$ (1 + $|\rho|^2$). The optimal fidelity ranges from $\frac{1}{2}$ to 1, i.e. $\frac{1}{2} \leq F_o(\rho) \leq 1$. Therefore, Bob hold the qubit in state $\rho$ can eliminate a part of uncertainty about the state preparation. The information content in the state $\rho$ can thus be characterized by the above success probability in Eq.(1). In principle any monotone function of the success probability can serve as a measure of information content in the state $\rho$. We introduce a measure of local information based on the optimal fidelity as

$$I_F(\rho) = [2F_o(\rho) - 1]^2$$

where $I_F(\rho)$ is normalized such that $I_F(\rho) = 0$ for $F_o(\rho) = \frac{1}{2}$ and $I_F(\rho) = 1$ for $F_o(\rho) = 1$. We will show that $I_F(\rho)$ defined above is equivalent to an operationally invariant information measure \cite{26} and is a suitable measure of local information in the situation we discuss here. For an $N$-qubit quantum system $|\psi\rangle$, the total local information is

$$I^t_{total} = \sum_{i=1}^{N} I_F(\rho_i)$$

where $\rho_i = \text{Tr}_{1,\ldots,i-1,i+1,\ldots,N}(|\psi\rangle\langle \psi |)$ is the reduced density operator of the $i$th qubit.

### III. TWO-QUBIT SYSTEM

We start out by investigating the information dynamics through simple examples in a system of two qubits with interaction between qubits. The interaction between qubits can be described as the following Hamiltonian in the canonical form with three parameters $c_1, c_2, c_3$ \cite{27}:

$$H = c_1 \sigma_x^1 \otimes \sigma_x^2 + c_2 \sigma_y^1 \otimes \sigma_y^2 + c_3 \sigma_z^1 \otimes \sigma_z^2$$

Here, for simplicity, the local evolutions have been neglected. The initial state is set as $|\psi(0)\rangle = |\Omega\rangle \otimes |0\rangle$, where $|\Omega\rangle = \alpha |0\rangle + \beta |1\rangle$ is some prescribed pure state. Due to the interaction between two qubits, the information will be transferred between two qubits and can also be converted into the form of nonlocal information.

**Ising coupling.** We first consider the Ising interaction, i.e. the coupling parameter $c_1 = c, c_2 = c_3 = 0$. After some time $t$, the system becomes $|\psi(t)\rangle = e^{i H t} |\psi(0)\rangle = \alpha \cos ct |0\rangle - i \beta \sin ct |1\rangle + \beta \cos ct |0\rangle - i \alpha \sin ct |1\rangle$. Therefore, we can calculate the local optimal fidelity defined in Eq.(1) $F_o(\rho_i, t) = \{1 + [1 - |\alpha^2 - \beta^2| \sin^2(2ct)]^{1/2}\}/2$, which will lead to the total local information as

$$I^t_{total}(t) = 2[1 - |\alpha^2 - \beta^2| \sin^2(2ct)]$$

On the other hand, we can obtain the entanglement contained in the state $|\psi(t)\rangle$ measured by 2-tangle, which is the square of concurrence \cite{28}. After some simple calculations, we can get

$$\tau_{12}(t) = |\alpha^2 - \beta^2| \sin^2(2ct)$$

We depict the dynamics behavior of local information and entanglement in Fig.(a). It can be seen that these two quantities exhibit perfect complementary behavior during the time evolution. In fact, this can be easily verified from the above Eq.(5-6) by deriving the complementarity relation that $I^t_{total}(t) + 2\tau_{12}(t) = 2$. It is well known that entanglement is some kind of nonlocal information. Our results in this simple example demonstrate that based on some suitable definition of local information (e.g. $I_F$ here), the nonlocal information is directly related to some appropriate measure of entanglement. In the following section, we can see that this viewpoint of entanglement and nonlocal information is also applicable in three-qubit systems and for arbitrary two-qubit system Hamiltonian $H$.

![FIG. 1: (Color online) Information dynamics via Ising interaction (a) and XY interaction (b). Total local information $I^t_{total}(t)$ vs. $t$ (Solid) and nonlocal information $2\tau_{12}(t)$ vs. $t$ (Dashed). The coupling parameter $c = 1$, the initial state is $|\psi(0)\rangle = (\sqrt{1/3}|0\rangle + \sqrt{2/3}|1\rangle) \otimes |0\rangle$.](image-url)

**XY coupling.** In addition, we also demonstrate the quantum state information dynamics for XY interaction, when the coupling parameters of the Hamiltonian in Eq.(4) are $c_1 = c_2 = c, c_3 = 0$. The result is depicted in Fig.1 (b), which exhibit the same perfect complementary behavior as the situation of Ising coupling.

### IV. THREE-QUBIT SYSTEM

In this section, we will extend the above discussions to three-qubit systems. In such a system, not only two-qubit entanglement but also genuine three-qubit entan-
plingment, which is shared by all the three qubits. Therefore, the corresponding relation between nonlocal information and entanglement should be dealt with carefully. We consider a simple example for demonstration, the system Hamiltonian is

\[
H = \sum_{ij} (c_1 \sigma_z^i \otimes \sigma_z^j + c_2 \sigma_y^i \otimes \sigma_y^j + c_3 \sigma_z^i \otimes \sigma_z^j) \tag{7}
\]

The initial state is set as \( |\psi(0)\rangle = |\Omega\rangle \otimes |0\rangle \otimes |0\rangle \), where \( |\Omega\rangle = \alpha |0\rangle + \beta |1\rangle \). We first calculate the spectrum of the above Hamiltonian \( H \), the eigen energy is denoted as \( \epsilon_k \) \( (k = 1, 2, \ldots, 8) \) and the corresponding eigen state is \( |\phi_k\rangle \). The initial state can be expressed in the eigen basis as \( |\psi(0)\rangle = \sum_{k=1}^{8} \gamma_k |\phi_k\rangle \). Therefore, the system state at time \( t \) becomes \( |\psi(t)\rangle = \exp(-itH) |\psi(0)\rangle = \sum_{k=1}^{8} \gamma_k e^{-i \epsilon_k t} |\phi_k\rangle \). According to Eq.(1-3), we can obtain the total local information \( I_{total}(t) \) easily. The two-qubit entanglement between qubit \( i, j \) \( (ij = 12, 23, 13) \) measured by 2–tangle \( \tau_{ij}(t) \) can also be calculated from the reduced density matrices \( \rho_{ij}(t) \) \cite{28}. As we have mentioned above, there will be another form of entanglement besides pairwise entanglement, i.e. genuine three-qubit entanglement, in systems of three qubits. The genuine three-qubit entanglement can be measured by the 3–tangle \( \tau_{123}(t) \) proposed in \cite{29}. In order to observe the perfect complementary behavior between the total local information and entanglement, we should adopt some suitable function which combine the contributions of both two-qubit and three-qubit entanglement. Here we choose the function as \( \mathcal{E}(t) = 2 |\tau_{12}(t) + \tau_{23}(t) + \tau_{13}(t)|^2 + 3 |\tau_{123}(t) \rangle \rangle \) and the information dynamics exhibit perfect complementary behavior in Fig.(2). In the following section, we can see that the coefficients before \( \tau_{ij} \) and \( \tau_{123} \) are not arbitrary. In fact, this function has clear information-theoretic meaning.

**FIG. 2:** (Color online) Information dynamics via Ising interaction (a) and XY interaction (b). Total local information \( I_{total}(t) \) vs. \( t \) (Solid) and nonlocal information \( \mathcal{E}(t) \) vs. \( t \) (Dashed). The coupling parameter \( c = 1 \), the initial state is \( |\psi(0)\rangle = (\sqrt{1/3}|0\rangle + \sqrt{2/3}|1\rangle) \otimes |0\rangle \otimes |0\rangle \).

Similar to the situation of two qubits, we can also easily derive the complementarity relation as \( I_{total}(t) + \mathcal{E}(t) = 3 \). In three-qubit systems, both two-qubit and three-qubit entanglement are nonlocal form of information. The above complementary behavior imply that if we choose suitable measures of local information and different levels of entanglement (e.g. two-qubit and genuine three-qubit entanglement here), the nonlocal information is just contributed by the entanglement linearly with appropriate weights. Though we only demonstrate this result via simple examples, we can see that it is applicable for any Hamiltonian \( H \) of three-qubit systems in the following section.

V. INFORMATION COMPLEMENTARITY RELATION

In this section, we will formalize the basic information complementarity relation underlie the above information dynamics. We adopt the operationally invariant information measure proposed by Brukner and Zeilinger \cite{26} as the measure of local information, which is defined as the sum of one-shot information gained over a complete set of mutually complementary observables (MCO). Consider a pure \( n \)-qubit state, if measured by operationally invariant information measure, the total information content is \( n \) bit and is completely contained in the system. For a spin-1/2 system with the density matrix \( \rho \), the operationally invariant information content is

\[
I_{BZ}(\rho) = 2Tr\rho^2 - 1. \tag{8}
\]

Therefore, for an \( n \)-qubit quantum system in pure state \( |\Omega\rangle \), the amount of information in local form is

\[
I_{local}(|\Omega\rangle) = \sum_{i=1}^{n} I_i, \quad \text{where} \quad I_i \text{ is the local information measured by } I_i = I_{BZ}(\rho_i) = \sum_{i=1}^{n} (2Tr\rho_i^2 - 1), \quad \text{where} \quad \rho_i = Tr_{i-1,i+1,...,n}(|\Omega\rangle \langle \Omega|) \text{ is the reduced density operator of the } i\text{th qubit. The non-local information is } I_{non-local} = n - I_{local}. \text{ We will show that such non-local information is related to entanglement. In other words, entanglement can be viewed as non-local form of information.}

We start by considering the simplest case of a two-qubit system in the pure state \( |\Omega\rangle_{12} = \sum_{i,j=0,1} a_{ij}|ij\rangle \). The local information contained in qubit 1 and 2 is

\[
I_1 = I_2 = 1 - 4|a_{00}a_{11} - a_{01}a_{10}|^2. \quad \text{Therefore, the non-local information is } I_{non-local} = 2 - I_1 - I_2 = 8|a_{00}a_{11} - a_{01}a_{10}|^2. \quad \text{If measured by 2-tangle, which is the square of concurrence \cite{28}, the pairwise entanglement is } \tau_{12} = 4|a_{00}a_{11} - a_{01}a_{10}|^2. \quad \text{Thus we can write local and non-local information as } I_1 = I_2 = 1 - \tau_{12} \text{ and } I_{non-local} = 2\tau_{12}. \quad \text{The relation between local information and non-local entanglement is depicted in Fig. 3 (A)}.

\[
I_1 + I_2 + 2\tau_{12} = 2. \tag{9}
\]

If we focus on one qubit, say qubit 1, then the 1 bit information of this qubit is partly contained in itself, i.e.,
I1. The residual information is contained in its entanglement with its environment, in this case qubit 2. The amount of this kind of information is \( \tau_{12} \). It can be seen that \( I_{1(2)} + \tau_{12} = 1 \). If the system is not isolated, in general it will be in a mixed state \( \rho_{12} \). In this case, the 2 bit information is not only contained in the system but also in its correlations with the outside environment. Therefore, \( I_1 + I_2 + \tau_{12} < 2 \) for a mixed state \( \rho_{12} \). This result can be proved through the convexity of \( I_1 \), \( I_2 \) and \( \tau_{12} \).

Now we will extend our discussions to the case of three-qubit systems in a pure state. The total information content in this system is 3 bit. The local bit information contained in each individual qubit is \( I_m = 2Tr\rho_m^2 - 1 \), \( m = 1, 2, 3 \), where \( \rho_m \) is the reduced density matrix for each qubit. Different from the two-qubit system, in this case the non-local information exists not only in 2-qubit entanglement, but also in genuine 3-qubit entanglement. It can be written as \( I_{\text{nonlocal}} = 3 - (I_1 + I_2 + I_3) \). Similar to the case of two-qubit, the nonlocal information in two-qubit entanglement is \( I_{\text{nonlocal}}(2) = I_{12} + I_{13} + I_{23} = 2(\tau_{12} + \tau_{13} + \tau_{23}) \), where \( \tau_{ij} \) is the 2-tangle between qubit \( i \) and \( j \). Therefore, the residual non-local information in the form of genuine three-qubit entanglement should be \( I_{\text{nonlocal}}(3) = 3 - (I_1 + I_2 + I_3) - 2(\tau_{12} + \tau_{13} + \tau_{23}) \). We note that \( 2\tau_{12} = 2(\lambda_{12}^1 - \lambda_{12}^2)^2 = (1 - I_{12} + I_2) - 4\lambda_{12}^1\lambda_{12}^2 \), where \( \lambda_{12}^1 \geq \lambda_{12}^2 \) are the squared roots of the eigenvalues of \( \rho_{12} \).

\[
\hat{\rho}_{12} = (\hat{\sigma}_y \otimes \hat{\sigma}_y)\rho_{12}^{-1}(\hat{\sigma}_y \otimes \hat{\sigma}_y)
\]

is the time-reversed density matrix of \( \rho_{12} \). Similarly, \( 2\tau_{13} = (1 - I_1 - I_3 + I_2) - 4\lambda_{13}^1\lambda_{13}^2 \) and \( 2\tau_{23} = (1 - I_2 - I_3 + I_1) - 4\lambda_{23}^1\lambda_{23}^2 \). Then the residual non-local information in the form of genuine three-qubit entanglement is

\[
I_{\text{nonlocal}}(3) = 4(\lambda_{12}^1\lambda_{12}^2 + \lambda_{13}^1\lambda_{13}^2 + \lambda_{23}^1\lambda_{23}^2). 
\]

If measured by 3-tangle proposed in [20], the genuine 3-qubit entanglement is \( \tau_{23} = 4\lambda_{12}^1\lambda_{12}^2 + 4\lambda_{13}^1\lambda_{13}^2 + 4\lambda_{23}^1\lambda_{23}^2 \). Therefore, we can establish a direct relation between nonlocal information and some appropriate measure of genuine three-qubit entanglement, i.e. \( I_{\text{nonlocal}}(3) = 3\tau_{23} \). The complementarity relation between local information and entanglement is as follows

\[
I_1 + I_2 + I_3 + 2(\tau_{12} + \tau_{13} + \tau_{23}) + 3\tau_{23} = 3. 
\]

If we focus on qubit 1, then the 1 bit information of this qubit is partly contained in itself \( (I_1) \). The residual information is contained in its entanglement with its environment, i.e., qubit 2 and 3. In fact the relation \( I_1 + \tau_{12} + \tau_{13} + \tau_{23} = 1 \) is satisfied for qubit 1. Similar results hold for qubit 2 and 3. The above results are depicted in Fig. 3 (B).

If we write the local reduced density matrix as \( \rho = \frac{1}{2}(I + \hat{r} \cdot \hat{\sigma}) \), with \( \hat{r} \) the Bloch vector, the measure of local information proposed by Brukner and Zeilinger as in Eq.(8) \( I_{BZ}(\rho) = 2Tr\rho^2 - 1 = |\hat{r}|^2 \). It can be easily verified that this measure of local information is equivalent to the measure based on optimal fidelity in Eq.(2), i.e.,

\[
I_{BZ}(\rho) = I_F(\rho) 
\]

Therefore, the complementarity relations between local information quantified by \( I_F(\rho) \) and entanglement and that between local information based on \( I_{BZ}(\rho) \) and entanglement are equivalent in nature. However, it can be seen from their definitions that \( I_F(\rho) \) and \( I_{BZ}(\rho) \) have different physical meanings. The local information based on fidelity \( I_F(\rho) \) is defined from the viewpoint of quantum communications, while \( I_{BZ}(\rho) \) is an operationally information measure from the measurement viewpoint [20]. Since the above two complementarity relations are equivalent, we could obtain that the complementarity relation between local information quantified by \( I_F(\rho) \) and entanglement is hold for arbitrary initial pure states. In particularly, the initial states could be entangled states, which means that the isolated system contains nonlocal information initially. Due to the interactions, the entanglement will change, which results in the change of nonlocal information.

VI. CONCLUSIONS AND DISCUSSIONS

In summary, we adopt a measure of local information based on optimal fidelity to investigate the information dynamics in two- and three-qubit systems with interactions between qubits. Through simple examples, we demonstrate the perfect complementary behavior between local information and entanglement. We also show that the measure of local information based on optimal fidelity is equivalent to the operationally information measure proposed by Brukner and Zeilinger. Furthermore, we establish a direct relation between nonlocal information and different levels of entanglement, and formalize the information complementarity relation by some appropriate measures of local information and entanglement.

For two-qubit pure states, using von Neumann entropy as a measure of local information, there has been a similar complementarity relation between local information and entanglement, which is measured by entanglement of formation [16]. Here we adopt a measure of local information by using linear entropy rather than von Neumann entropy. This is based on the following two considerations. One is that linearity always implies additivity, which is
simple and suitable for establishing complementarity relations. The other point is that using nonlinear entropy we demonstrate that nonlinear information can be directly related to the polynomial measures of entanglement, i.e. k-tangle. For the situation of two qubits, 2-tangle is just the square of concurrence, which is a function of entanglement of formation. However, there are no straightforward generalizations of entanglement of formation to quantum states of more than two qubits. Therefore, using linear entropy, it is more simple for us to generalize the information complementarity relations to the situations of more two qubits straightforwardly.

Though the relation between nonlinear information and entanglement is demonstrated for two- and three-qubit pure states. It is possible to generalize the information complementarity relation to arbitrary n-qubit pure states naturally as the following conjecture

\[ \sum_{i} I_i + 2 \sum_{i_1 < i_2} \tau_{i_1 i_2} + \cdots + n \sum_{i_1 < i_2 < \cdots < i_n} \tau_{i_1 \cdots i_n} = n \tag{12} \]

where \( \tau_{i_1 \cdots i_k} \) \((k = 2, 3, \ldots, n)\) are some appropriate measures of genuine k-qubit entanglement. Since nonlinear information is contributed by different levels of entanglement as can be seen from the above discussions, conversely we can characterize entanglement through nonlinear information. In our recent work \[12\], we have proposed such an information-theoretic measure of genuine multi-qubit entanglement, and utilize it to explore the genuine multi-qubit entanglement in spin systems.

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