Fractional decoding: Error correction from partial information

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Abstract—To be considered for the 2017 IEEE Jack Keil Wolf ISIT Student Paper Award. We consider error correction by maximum distance separable (MDS) codes based on a part of the received codeword. Our problem is motivated by applications in distributed storage. While efficiently correcting erasures by MDS storage codes (the “repair problem”) has been widely studied in recent literature, the problem of correcting errors in a similar setting seems to represent a new question in coding theory.

Suppose that \( k \) data symbols are encoded using an \((n, k)\) MDS code, and some of the codeword coordinates are located on faulty storage nodes that introduce errors. We want to recover the original data from the corrupted codeword under the constraint that the decoder can download only an \( \alpha \) proportion of the codeword (fractional decoding). For any \((n, k)\) code we show that the number of correctable errors under this constraint is bounded above by \((n - k/\alpha)/2\). Moreover, we present two families of MDS array codes which achieve this bound with equality under a simple decoding procedure. The decoder downloads an \( \alpha \) proportion of each of the codeword’s coordinates, and provides a much larger decoding radius compared to the naive approach of reading some \( \alpha n \) coordinates of the codeword. One of the code families is formed of Reed-Solomon (RS) codes with well-chosen evaluation points, while the other is based on folded RS codes.

Finally, we show that folded RS codes also have the optimal list decoding radius under the fractional decoding constraint.

I. INTRODUCTION

Efficient recovery of data from a part of the codeword has been studied recently in the context of applications to distributed storage. One special case of this problem, which was the subject of a large number of publications, is erasure correction by array codes and in particular by MDS array codes. The most well-studied case of the erasure correction problem is recovery of one erasure from a part of the codeword. This problem was introduced by Dimakis et al. [1], who also suggested the repair bandwidth (the proportion of the codeword downloaded for the repair of one erasure) as a useful measure of quality of the correction procedure. The authors of [1] also derived a bound on the minimum possible repair bandwidth, and MDS codes that meet this bound are called Minimum Storage Regenerating (MSR) codes. Suppose that \( k \) data symbols are encoded by an MDS code of length \( n \). Explicit constructions of MSR codes known in the literature for the case of low rate, i.e., \( k < n/2 \), were given by Rashmi et al. [2], while for the high-rate regime such codes were constructed in [3]-[5]. Guruswami and Wootters studied the repair bandwidth of Reed-Solomon (RS) codes [6]. A construction of RS codes with asymptotically optimal repair bandwidth is given in [7].

In this paper we extend the setting of erasure correction from partial information to the problem of error correction. In a distributed system, we usually face a limitation on the disk input/output operations as well as on the amount of information transmitted for the purpose of decoding (decoding bandwidth). Under no limitations on the decoding bandwidth, it is possible to recover the information from any \( [(d - 1)/2] \) errors, where \( d \) is the distance of the code. Assuming that the system permits the decoder to utilize only an \( \alpha < 1 \) proportion of the whole codeword, we face the natural question of how many errors we can guarantee to correct in this setup. In other words, how much do we give up in terms of error-correcting capability by limiting the decoding bandwidth?

Informal description of the problem. Let \( C \) be a code of length \( n \) and dimension \( k \) over a field \( F \). Let \( y = c + e \) be equal to a codeword \( c \) plus an error vector \( e \in F^n \). Suppose that the decoder attempts to recover the data encoded in \( c \) based on a part of the vector \( y \). More specifically, suppose that the input to the decoder is formed as some functions \( f_i \) of the symbols \( y_i, i = 1, \ldots, n \), each defined on an individual coordinate, and suppose that the total number of field symbols available to the decoder is \( \alpha n, \alpha \leq 1 \). Clearly we should take \( \alpha \geq k/n \) because the codeword encodes \( k \) data symbols, and even without errors to recover the data the decoder needs at least as many input symbols. If \( \alpha = 1 \), we return to the standard decoding problem, so our goal is to study error correction for \( \alpha \) in the range \( k/n \leq \alpha < 1 \).

We call the number of errors correctable from an \( \alpha \) proportion of the codeword the \( \alpha \)-decoding radius, and denote it by \( r_\alpha(n, k) \). Our first result, proved in Sec. [II] is an upper bound on \( r_\alpha(n, k) \). Then in Sections [III] and [IV] we present two code constructions that achieve this upper bound with equality. The code families that we consider belong to the class of array codes, and therefore we phrase our definitions and results in terms of such codes. At the same time, the general problem of fractional decoding as well as the upper bound on the \( \alpha \)-decoding radius in Theorem [II] apply to all codes, and do not depend on the specific setting considered below.

An \((n, k, l)\) array code \( C \) is formed of \( l \times n \) matrices \( C = (C_1, \ldots, C_n) \in (F^l)^n \), where \( F \) is a finite field. Each column \( C_i \) of the matrix is a codeword coordinate, and the parameter \( l \) that determines the dimension of the column vector \( C_i \) is called sub-packetization. We may also consider \( C \) as a code over the alphabet \( F^l \), and then one error amounts to an incorrect column \( C_i \). Accordingly, correcting up to \( l \) errors means correcting any combination of

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errors \( E = (E_1, E_2, \ldots, E_n) \in (F^l)^n \) of Hamming weight \( w(E) := |\{i : E_i \neq 0\}| \leq t \), where the received codeword is the matrix \( C + E = (C_i + E, i = 1, \ldots, n) \).

**Definition I.1** (\( \alpha \)-decoding radius). Let \( \mathcal{C} \) be an \((n, k, l)\) array code over the field \( F \). We say that \( \mathcal{C} \) corrects up to \( t \) errors by downloading \( \alpha n \) symbols of \( F \) if there exist functions \( f_i : F^l \to F^{\alpha n l}, i = 1, \ldots, n \) and \( g : F^{(\sum_{i=1}^n \alpha_i) l} \to F^{\alpha n l} \)

\[
f_i(C_1 + E_1), f_2(C_2 + E_2), \ldots, f_n(C_n + E_n)) = (C_1, C_2, \ldots, C_n). \tag{2}
\]

For \( \alpha \geq k/n \), we define the \( \alpha \)-decoding radius of \( \mathcal{C} \) as the maximum number of errors that \( \mathcal{C} \) can correct by downloading \( \alpha n \) symbols of \( F \), and denote it as \( r_\alpha(\mathcal{C}) \).

Finally, define the \( \alpha \)-decoding radius \( r_\alpha(n, k) \) as follows:

\[
r_\alpha(n, k) = \max_{\mathcal{C} \in \mathcal{M}_{n,k}} r_\alpha(\mathcal{C}),
\]

where \( \mathcal{M}_{n,k} \) is the set of all \((n, k)\) codes.

**Remark 1.1.** Note that the part of the codeword that we access may be larger than \( \alpha n \), and we only require that the decoding function \( g \) use no more than \( \alpha n l \) symbols of \( F \). The motivation comes from distributed storage where what matters is the amount of information transmitted over the network. In the context of regenerating codes and erasure correction, this quantity is usually called the repair bandwidth. If the portion of the codeword accessed for the repair purposes is the same as the minimum possible repair bandwidth, then the codes are referred to as optimal-access regenerating codes (see [5], [8]). In this paper we resort to the same language for the problem of error correction.

For any code \( \mathcal{C} \) we have \( r_1(\mathcal{C}) \leq \lfloor (n - k)/2 \rfloor \), with equality if \( \mathcal{C} \) is an MDS code. Thus, \( r_1(n, k) = \lfloor (n - k)/2 \rfloor \). We also have the following obvious lower bound on \( r_\alpha(n, k) \).

**Lemma I.2.** For any \( k \leq n \) and \( k/n \leq \alpha \leq 1 \)

\[
r_\alpha(n, k) \geq \lfloor (\alpha n - k)/2 \rfloor. \tag{3}
\]

To see this, take an \((n, k)\) MDS code and pick \( \alpha n \) coordinates to form a punctured code \( \mathcal{C} \). The punctured code \( \mathcal{C} \) is an MDS code of length \( \alpha n \) and dimension \( k \), so (3) follows by definition.

In this paper we show that

\[
r_\alpha(n, k) = \lfloor (n - k/\alpha)/2 \rfloor \tag{4}
\]

for any \( n, k \) and \( \alpha \) and we give two families of explicit constructions of MDS codes together with decoding schemes for which the optimal value in (4) is achieved. The optimal \( \alpha \)-decoding radius in (4) improves upon the lower bound (3) obtained from the naive decoding strategy by a factor of \( 1/\alpha \) (see Fig. 1 which shows \( r_\alpha(n, k)/n \) vs \( \alpha \)).

Fig. 1: Normalized \( \alpha \)-decoding radius: Trivial bound (3) and improved bound (4) for codes of rate \( R = 0.4 \).

The underlying idea behind our code constructions is as follows. Let \( \epsilon \) be the proportion of erroneous coordinates in the codeword. The naive decoding procedure suggests to choose some \( \alpha n \) coordinates and decode based on their entire contents. However, in the worst case the number of errors in the chosen coordinates can be as large as \( \epsilon n \), and we will have only \((\alpha - \epsilon)n\) error-free coordinates in the chosen subset. At the same time, by symmetrizing the decoding procedure and reading exactly an \( \alpha \) proportion of each coordinate’s contents, we can increase the proportion of error-free downloaded contents regardless of the location of the erroneous coordinates and improve the chances for successful decoding.

One code family that we construct is formed of Reed-Solomon (RS) codes with carefully chosen evaluation points, and the other one is based on Folded RS codes of Guruswami and Rudra [9].

In Sect. IV we also introduce the notion of \( \alpha \)-list decoding capacity, and show that Folded Reed-Solomon codes can achieve the \( \alpha \)-list decoding capacity.

**II. UPPER BOUND ON THE \( \alpha \)-DECODING RADIUS**

In this section we prove the following result.

**Theorem II.1.**

\[
r_\alpha(n, k) \leq \lfloor (n - k/\alpha)/2 \rfloor. \tag{5}
\]

**Proof:** We need to show that if an \((n, k, l)\) code \( \mathcal{C} \) over the field \( F \) can correct up to \( t \) errors by downloading \( \alpha n \) symbols of \( F \) (see Def. [1]), then \( t \leq \lfloor (n - k/\alpha)/2 \rfloor \). By assumption, for any codeword \( \mathcal{C} = (C_1, \ldots, C_n) \in \mathcal{C} \) and any error vector \( E = (E_1, E_2, \ldots, E_n) \) of weight \( \leq t \) there exist \( n + 1 \) functions \( f_i : F^l \to F^{\alpha n l}, i = 1, 2, \ldots, n \) and \( g : F^{(\sum_{i=1}^n \alpha_i) l} \to F^{\alpha n l} \) that satisfy (1)-(2).

We claim that \( \sum_{i \in I} \alpha_i \geq k \) for any set \( I \subseteq \{1, 2, \ldots, n\} \) with cardinality \( |I| = n - 2t \). Assume toward a contradiction that there is a set \( \mathcal{I}_0 \subseteq \{1, 2, \ldots, n\}, |\mathcal{I}_0| = n - 2t \) such that \( \sum_{i \in \mathcal{I}_0} \alpha_i < k \). Without loss of generality, assume that \( \mathcal{I}_0 = \{1, 2, \ldots, n - 2t\} \). Let us partition the set \( \{1, 2, \ldots, n\} \setminus \mathcal{I}_0 \) into two disjoint sets \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \) such that \( |\mathcal{I}_1| = |\mathcal{I}_2| = t \). Since the dimension of \( \mathcal{C} \) is \( k \), there are a total of \( |F|^k \) codewords. At the same time, the vector \( (f_i(C_i), i = 1, \ldots, n - 2t) \) takes at most \( \prod_{i \in \mathcal{I}_0} |F|^{\alpha_i l} = |F|^{(\sum_{i \in \mathcal{I}_0} \alpha_i) l} < |F|^k \) different values, so there exist two distinct codeword \( \tilde{\mathcal{C}} \) and \( \bar{\mathcal{C}} \) for which these vectors coincide.
\[
(f_1(\tilde{C}_1), f_2(\tilde{C}_2), \ldots, f_{n-2t}(\tilde{C}_{n-2t}))
= (f_1(\tilde{C}_1), f_2(\tilde{C}_2), \ldots, f_{n-2t}(\tilde{C}_{n-2t})).
\]

(6)

Define error vectors \( \hat{E} \) and \( \tilde{E} \) by setting

\[
\hat{E}_i = \begin{cases} 
\tilde{C}_i - \hat{C}_i & \text{if } i \in J_1, \\
0 & \text{if } i \notin J_1,
\end{cases}
E_i = \begin{cases} 
\tilde{C}_i - \hat{C}_i & \text{if } i \in J_2, \\
0 & \text{if } i \notin J_2.
\end{cases}
\]

(7)

Clearly, the weight of both \( \hat{E} \) and \( \tilde{E} \) is at most \( t \). By (2), we have

\[
g(f_1(\hat{C}_1 + \hat{E}_1), f_2(\hat{C}_2 + \hat{E}_2), \ldots, f_n(\hat{C}_n + \hat{E}_n)) = \hat{C}
\]

\[
g(f_1(\tilde{C}_1 + \tilde{E}_1), f_2(\tilde{C}_2 + \tilde{E}_2), \ldots, f_n(\tilde{C}_n + \tilde{E}_n)) = \tilde{C}.
\]

According to (6)-(7), \( f_j(\hat{C} + \hat{E}) = f_j(\tilde{C} + \tilde{E}), j = 1, 2, \ldots, n \).

As a result, \( C = \tilde{C} \) in contradiction to our assumption. We conclude that \( \sum_{\alpha \in \mathcal{I}} \alpha_i \geq k \) for any set \( \mathcal{I} \subseteq \{1, 2, \ldots, n\} \) of size \( |\mathcal{I}| = n - 2t \).

Let \( \mathcal{I} \) be an \((n - 2t)\)-subset of \{1, \ldots, n\} such that the quantity \( \sum_{\alpha \in \mathcal{I}} \alpha_i \) is the smallest among all \((n - 2t)\)-subsets. By the above argument the average proportion of information transmitted from a coordinate in the set \( \mathcal{I} \) is at least \( k/(n - 2t) \). On the other hand, by definition, the average proportion from all the coordinates is at most \( \alpha \), which is at least \( k/(n - 2t) \) because of the property the set \( \mathcal{I} \) satisfies. Hence we get \( k/(n - 2t) \leq \alpha \), and this concludes the proof.

III. A REED-SOLOMON CODE CONSTRUCTION

All the constructions in this paper are built upon RS codes. We begin with recalling their definition.

**Definition 3.1.** A Reed-Solomon code \( RS_F(n, k, \Omega) \subseteq F^n \) of dimension \( k \) over \( F \) with evaluation points \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \subseteq F \) is the set of vectors

\[
\{(h(\omega_1), \ldots, h(\omega_n)) : h \in F[x], \text{deg } h \leq k - 1\}.
\]

In this section we construct an evaluation point set \( \Omega \) such that the corresponding RS code achieves the optimal \( \alpha \)-decoding radius \( \lfloor \alpha \rfloor \). Let \( F = GF(q^l) \) be an \( l \)-degree extension of \( B = GF(q) \) and let

\[
tr_{F/B}(\beta) = \beta + \beta^q + \beta^{q^2} + \cdots + \beta^{q^{l-1}}
\]

be the trace function. Let \( \zeta_0, \zeta_1, \ldots, \zeta_{l-1} \) be a basis of \( F \) over \( B \) and let \( \nu_0, \nu_1, \ldots, \nu_{l-1} \) be the trace-dual basis (i.e., \( tr_{F/B}(\nu_i) = \delta_{ij} \) for all \( i, j \)). Then every element \( \beta \in F \) can be represented as its \( l \) projections \( \{tr_{F/B}(\zeta_i \beta)\}_{i=0}^{l-1} \) on \( B \) as follows:

\[
\beta = \sum_{i=0}^{l-1} tr_{F/B}(\zeta_i \beta) \nu_i.
\]

Let \( \alpha = m/l < 1 \), where \( m \) and \( l \) are positive integers. Let \( q \geq n \) be the cardinality of the alphabet \( B \) and suppose that \( m|k \). We show that an \((n, k)\) RS code \( RS_F(n, k, \Omega) \subseteq F^n \) with all the evaluation points \( \Omega = \{\omega_1, \omega_2, \ldots, \omega_n\} \subseteq B \) has the optimal \( \alpha \)-decoding radius.

To link our construction with the definitions made earlier, we view each codeword coordinate as a vector of dimension \( l \) over \( B \). Thus \( RS_F(n, k, \Omega) \) can be viewed as an \((n, k, l)\) MDS array code over the base field \( B \). Our decoding scheme will be based on downloading \( m \) symbols of \( B \) from each of the codeword coordinates.

Before proceeding we need to introduce some notation. We write the encoding polynomial as

\[
h(x) = a_{k-1}x^{k-1} + a_{k-2}x^{k-2} + \cdots + a_0,
\]

then the \( i \)-th coordinate of the codeword \( i\cdot \)

\[
c_i = h(\omega_i) = a_{k-1}\omega_i^{k-1} + a_{k-2}\omega_i^{k-2} + \cdots + a_0.
\]

For \( j = 0, 1, \ldots, l - 1 \), we further define

\[
h_j(x) = \sum_{i=0}^{k-1} \text{tr}_{F/B}(\zeta_i a_i)x^i.
\]

Since the coefficients of \( \{h_j(x)\}_{j=0}^{l-1} \) contain all the projections of the coefficients of \( h(x) \) on the elements of the basis \( \zeta_j, j = 0, \ldots, l - 1 \), the coefficients of \( h(x) \) can be calculated from the coefficients of \( \{h_j(x)\}_{j=0}^{l-1} \). In other words, to recover the codeword, we only need to recover the \( l \) polynomials \( h_j(x) \).

Let \( A_0, A_1, \ldots, A_{m-1} \subseteq B \) be \( m \) pairwise disjoint subsets of the field \( B \), each of size \( k/m \). For \( j = 0, 1, \ldots, m - 1 \), define the annihilator polynomial of the set \( A_j \) to be

\[
p_j(x) = \prod_{\omega \in A_j} (x - \omega).
\]

The \( m \) symbols we download from the \( i \)-th coordinate are

\[
d_i^{(j)} = \text{tr}_{F/B}(\zeta_i \cdot j \cdot c_i)(p_j(\omega_i))^{l-m} + \sum_{u=0}^{l-m-1} \text{tr}_{F/B}(\zeta_u c_i)(p_j(\omega_i))^u, j = 0, 1, \ldots, m - 1.
\]

(10)

Clearly, \( d_i^{(j)} \in B \) for all \( j = 0, 1, \ldots, m - 1 \). Substituting (9) into (10) and noting that \( \omega_i \in B \), we see that \( d_i^{(j)} = g_j(\omega_i), j = 0, 1, \ldots, m - 1 \), where

\[
g_j(x) = h_{l-m-j}(x)(p_j(x))^{l-m} + \sum_{u=0}^{l-m-1} h_u(x)(p_j(x))^u.
\]

Since \( \deg(p_j) = k/m \), we have \( \deg(g_j) < lk/m \). Thus \( \{d_i^{(j)}, d_j^{(j)}, \ldots, d_i^{(j)}\} \subseteq RS_B(n, lk/m, \Omega) \) for every \( j = 0, 1, \ldots, m - 1 \). As a result, we can recover all the coefficients of polynomials \( \{g_j(x)\}_{j=0}^{m-1} \) as long as there are no more than \( \lfloor (lk/m)/2 \rfloor \) errors in the original codeword \( (c_1, c_2, \ldots, c_n) \). Now we only need to show that given polynomials \( \{g_j(x)\}_{j=0}^{m-1} \), we can recover the polynomials \( \{h_j(x)\}_{j=0}^{l-1} \). To see this, we notice that for \( j = 0, 1, \ldots, m - 1 \),

\[
g_j(\omega) = h_0(\omega) \text{ for all } \omega \in A_j.
\]

Consequently, we know the evaluations of \( h_0(x) \) at all the points in \( \cup_{j=0}^{m-1} A_j \). There are \( k \) distinct points in the set \( \cup_{j=0}^{m-1} A_j \) and the degree of \( h_0(x) \) is less than \( k \), so we can

\footnote{For the time being we view the codeword coordinates as scalars in \( F \) rather than vectors over \( B \), and use lowercase \( c_i \) instead of \( C_i \) in Def. [4] to denote them.}
Since \( t \) also recover the original codeword. Let \( \text{FRS} \) codes.

\( \gamma_{i,j} \) is an integer.

We shall define \( n,k,l \) functions \( f_i, i = 1,2, \ldots, n \) and \( g \) that satisfy (11). Define \( f : F^l \to F^\alpha \) as follows: For any \( (d_1, d_2, \ldots, d_l) \in F^l \),

\[
 f((d_1, d_2, \ldots, d_l)) = (d_1, d_2, \ldots, d_{\alpha l}). \tag{11}
\]

Let \( f_i = f \) for 1 \( \leq \) \( i \leq n \). Define a new code

\[
 C^\alpha = \{(C_1^\alpha, C_2^\alpha, \ldots, C_n^\alpha) : (C_1, C_2, \ldots, C_n) \in \text{FRS}(n, k, l)\} \tag{12}
\]

It is easy to see that \( C^\alpha \) defined above has the following equivalent description:

\[
 C^\alpha = \{(C_1^\alpha, C_2^\alpha, \ldots, C_n^\alpha) : C_i^\alpha = (h(\gamma^{(i-1)l}), h(\gamma^{(i-1)l+1}), \ldots, h(\gamma^{il-1})) \in F^l, \\
 1 \leq i \leq n, h \in F[x], \deg h \leq kl - 1 \}. \tag{13}
\]

Since any \( k/\alpha \) coordinates of \( C^\alpha \) contain \( (k/\alpha)(\alpha l) = kl \) evaluations of the encoding polynomial \( h \) of degree less than \( kl \), we can recover \( h \) and thus the whole codeword from any \( k/\alpha \) coordinates of \( C^\alpha \). We thus conclude that \( C^\alpha \) is an \((n, k/\alpha, \alpha l)\) MDS array code, and so it can correct up to \( [(n - k/\alpha)/2] \) errors.

**Remark IV.1.** Suppose that there are several values \( \alpha_1, \alpha_2, \ldots, \alpha_m \) such that \( \alpha_i l \) are integer for all \( i = 1, \ldots, m \). Then the code \( \text{FRS}(n, k, l) \) achieves the optimal \( \alpha_i \)-decoding radius for all \( i = 1, \ldots, m \) simultaneously.

We can use the decoding method described above to give more general code constructions achieving the optimal \( \alpha \)-decoding radius. In fact, we can take any \((nl, kl)\) scalar MDS code \( C^\alpha \) over finite field \( F \) and bundle together every \( l \) coordinates of \( C^\alpha \) into a vector in \( F^l \). It is clear that we obtain an \((n, k/l, l)\) MDS array code \( C^{(nl)} \) in this way. Moreover, by reading \( \alpha l \) symbols of \( F \) from each of the coordinates of \( C^{(nl)} \) we obtain an \((n, k/\alpha, \alpha l)\) MDS array code \( C^\alpha \) which can correct up to \( [(n - k/\alpha)/2] \) errors. Thus the code \( C^{(nl)} \) can correct up to \( [(n - k/\alpha)/2] \) errors by downloading \( \alpha l \) symbols of \( F \).

In conclusion note that the FRS codes accomplish fractional decoding with the optimal access property discussed briefly in Remark IV.1.

**B. Fractional list decoding and \( \alpha \)-list decoding capacity**

So far we have focused on the unique decoding problem under the constraint of fractional decoding. In this section we consider list decoding from partial information.

We say that a code \( C \) of length \( n \) corrects \( pn \) errors under list-of-\( L \) decoding (has normalized list decoding radius \( \rho \)) if every sphere of radius \( pn \) in the space \( F^n \) contains at most \( L \) codewords of \( C \). In other words, there exists a decoder of \( C \) that outputs a list of \( \leq L \) codewords including the transmitted one as long as the channel introduced no more that \( pn \) errors. To maintain low complexity of decoding, we require that \( L \) be a polynomial function of \( n \).

Let \( R = k/n \) denote the rate of the code. It is clear that for any sequence of codes with fixed rate \( R \) and growing length \( n \), the list decoding radius does not exceed \( 1 - R \). As shown in [9], FRS codes of growing length have list decoding radius that for \( n \to \infty \) approaches the optimal value of \( 1 - R \) (as [9] puts it, FRS codes achieve the list decoding capacity).

In this section we show that the conclusion about the optimal list decoding radius of FRS codes extends to the
case of fractional decoding. We begin with defining the \( \alpha \)-list decoding radius of the code.

**Definition IV.3** \((\alpha\text{-list decoding capacity})\). Let \( C \) be an \((n,k,l)\) array code, where each codeword \( C = (C_1,\ldots,C_n) \) is an \( l \times n \) matrix with \( C_i \in F^l \), \( i = 1,\ldots,n \). We say that \( C \) corrects up to \( t \) errors under list-of-\( L \) decoding by downloading \( \alpha \) symbols of \( F \) if there exist functions
\[
\begin{align*}
  f_1 : F^l &\rightarrow F^{\alpha,l}, \; i = 1,2,\ldots,n, \\
  g_i : F^{\sum_{j=1}^n \alpha_j l} &\rightarrow F^{nl}, \; i = 1,2,\ldots,L
\end{align*}
\]
\(\text{such that } \sum_{i=1}^n \alpha_i \leq n\alpha, \text{ and for any codeword } C \in C \) and any error \( E \in F^{ln}, w(E) \leq t \) the decoding list
\[
\{g_i(f_1(C_1 + E_1),f_2(C_2 + E_2),\ldots,f_n(C_n + E_n))\}_{i=1}^L
\]
contains the codeword \( C \).

For \( \alpha \geq k/n \), we define the \((\alpha,L)\)-list decoding radius \( r_{\alpha,L}(C) \) of \( C \) as the maximum number of errors that can be corrected under list-of-\( L \) decoding by downloading \( \alpha \) symbols of \( F \). Let \( r_{\alpha,L}(n,k) = \max_C r_{\alpha,L}(C) \), where the maximum is taken over all codes of length \( n \) and dimension \( k \).

For \( \alpha \geq R \), we further define the \( \alpha \)-list decoding capacity of codes of rate \( R \) as
\[
\rho_{\alpha}(R) = \sup_{m \in \mathbb{N}} \lim_{n \to \infty} \frac{r_{\alpha,n}(n,Rn)}{n}.
\]

Using an argument that closely follows the proof of Theorem [1], we can show that \( \rho_{\alpha}(R) \leq 1 - R/\alpha \). On the other hand, there exists a family of FRS codes of increasing length \( n \) and sub-packetization \( l \) that are \((\alpha,L)\)-list decodable from a fraction arbitrarily close to \( 1 - R/\alpha \) of errors with list size \( L \) polynomial in \( n \). This leads to the following statement.

**Theorem IV.4.** We have
\[
\rho_{\alpha}(R) = 1 - R/\alpha,
\]
and this bound is achievable by a family of FRS codes.

To show that the FRS codes achieve the \( \alpha \)-list decoding capacity, we need to define the functions \( f_1,i = 1,2,\ldots,n \) and \( g_i, i = 1,2,\ldots,L \) that satisfy \([14]-[15]\). We again use the functions \( f_1 = f_2 = \cdots = f_n = f \) defined in \([11]\), i.e., we still download an \( \alpha \) symbols of \( F \) from each of the codeword coordinates. Then we obtain the code \( C^\alpha \) defined in \([13]\) whose rate is \( R/\alpha \). When the code length \( n \) and sub-packetization \( l \) of the FRS code become large enough, we can use the list decoding algorithm introduced in \([9]\) to decode \( C^\alpha \) up to a fraction arbitrarily close to \( 1 - R/\alpha \) of errors with list size \( L \) polynomial in \( n \).

Note that for sufficiently large \( n \) and \( l \), the FRS codes achieve the \( \alpha \)-list decoding capacity uniformly for all values of \( \alpha \).

**Remark IV.2.** The code \( C^\alpha \) differs from FRS codes in the sense that the evaluation points in two consecutive coordinates are not consecutive powers of the primitive element. However, the list decoding algorithm introduced in \([9]\) only requires that within each codeword coordinate, the evaluation points are consecutive powers of the primitive element. The code \( C^\alpha \) satisfies this constraint, so the list decoding algorithm in \([9]\) still applies to it.

**Remark IV.3.** Although FRS codes achieve the \( \alpha \)-list decoding capacity, they require larger node (codeword coordinate) size than the RS construction given in Sect. III. The base field in RS construction (the field \( \mathbb{F} \) in Sect. III) has to be only of size \( q \geq n \), while the base field in this section has to be of size at least \( nl \). As a result, the node size in the RS construction is \( l \log n \) bits, while the codeword coordinates of the FRS codes contain \( l \log(nl) \) bits.

**V. Conclusion**

In this paper we studied the problem of decoding from errors under the constraint of downloading only a part of the received codeword. We gave two families of optimal code constructions and their error correction (decoding) procedures. The idea behind the constructions and recovery schemes is rather similar to regenerating codes: in regenerating codes, we can repair the same number of erasures by downloading a smaller proportion of the codeword if we connect to more helper nodes; here we can correct the same number of errors by downloading a smaller proportion of the codeword if we connect to all codeword coordinates. Equivalently, for the same amount for downloaded information we can decode from a larger number of errors. Thus, a natural question to ask is whether it is possible to construct an MDS code that corrects both erasures and errors (nearly) optimally in terms of the bandwidth, where the optimality that corresponds to erasures is measured by the cut-set bound given in \([1]\), and the optimality of correcting errors is measured by the \( \alpha \)-decoding radius defined in this paper.

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