Soft photon bremsstrahlung at next-to-leading power

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Abstract

The emission of soft radiation provides a fundamental probe of the consistency of the underlying quantum field theory. Correspondingly, the measurement of the soft photon bremsstrahlung, such as the one proposed with the planned future upgrade of the ALICE experiment at the LHC, is of great interest. In this letter we explore the possibility to implement analytic techniques that have been recently developed for soft gluon resummation at Next-to-Leading-Power (NLP) in the context of the soft photon spectrum. We provide a formula for the differential cross-section with shifted kinematics that is particularly suitable for numerical implementations. We also discuss the impact of loop corrections to Low’s theorem due to radiative jet functions.
1 Introduction

There are plans for a new multipurpose detector at the Large Hadron Collider (LHC) as a follow-up to the present ALICE experiment [1]. In the rich physics program that would be enabled by this upgrade, the possibility to measure ultra soft photons at very low transverse momentum is particularly attractive. In fact, measurements of soft photon spectra have been undertaken by many experiments over the last couple of decades, motivated by a clean theoretical picture of photon emissions. Yet, to this day the results of the measurements remain not understood, as yields of soft photons produced together with hadrons show significant excess above theoretical predictions [2–11]. In the light of this outstanding discrepancy, and future measurements planned at the LHC, one needs to scrutinize the theoretical predictions used to compare with data.

The theoretical foundations of soft boson emissions have a long history that has continued to attract attention until recent days [12–18]. Surprisingly, the theoretical description of soft photon emission spectra employed so far relies only on the leading-power (LP) eikonal approximation, which corresponds to terms of order $1/\omega_k$ in Low’s soft theorem [13], where $\omega_k$ is the energy of the photon with soft momentum $k^\mu$. The corresponding soft photon bremsstrahlung cross section at LP reads

$$\frac{d\sigma_{LP}}{d^3k} = \frac{\alpha}{(2\pi)^2 \omega_k} \int d^3p_3 \cdot \cdots \int d^3p_n \left( \sum_{i,j=1}^n -\eta_i \eta_j \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)} \right) d\sigma_H(p_1, \ldots, p_n), \tag{1.1}$$

where $\alpha$ is the electromagnetic constant and $d\sigma_H$ denotes the differential non-radiative cross-section depending on the $n$ hard momenta $p_i^\mu$ of the incoming and outgoing charged particles. The sign of $\eta = \pm 1$ is equal (opposite) to the sign of the electric charge of the final (initial) states.

It is one of the main purposes of this letter to show that possible improvements in the description of the photon emission spectra can be achieved by employing methods which have been already developed in the QCD resummation program. In that context, the soft boson is an unobserved gluon giving rise to large logarithms in the cross-section that need to be resummed to all orders in perturbation theory [19, 20]. In this regard, there has been a great deal of interest in the recent years in extending this framework to Next-to-Leading Power (NLP), in order to control the effects of power suppressed emissions of soft gluons and quarks [21–45]. In this letter we explore the possibility to implement these recently developed NLP techniques to the observed soft photon emissions. In particular, we discuss two distinct corrections to eq. (1.1) to which NLP effects give rise.

The first of these corrections are of order $(\omega_k)^0 \sim 1$ and have first been analyzed by Low for scalar emitters, and later extended by Burnett and Kroll to spinor emitters [15, 16]. In this letter, building on similar results derived in QCD for processes with colorless final states [23], we propose a form of the Low-Burnett-Kroll (LBK) formula in terms of kinematical shifts of external particle momenta. This formulation is particularly suitable for numerical implementations. In doing so, we clarify also an aspect
that has been recently addressed in \cite{46} regarding the validity of \((\omega_k)^0\) corrections when the non-radiative amplitude is expressed in terms of non-physical momenta that violate momentum conservation.

Furthermore, the method of shifting external particle momenta has proven invaluable while deriving soft gluon resummed expressions in Mellin space at NLP for processes with colorless final states. It is therefore justified to presume that generalizing the shifting procedure for processes with arbitrary number of charged external legs will be very useful in the extension of QCD resummation at NLP for processes with more than two colored external legs.

The second kind of corrections is due to an interplay between the soft photon emission and QCD loop effects \cite{47–50}. They have a logarithmic form in photon momentum and therefore could significantly increase yields of soft photon emissions. In particular, as they are due to QCD virtual contributions, they would only appear in calculations involving colored external lines. Hence, they might help to explain the observed discrepancies \cite{2–11} in the production rates. Although in principle one could consider loop corrections from radiative jets both in QED and QCD, it is clear that the contribution from the latter is larger because of the strong coupling constant. Therefore, one expects the effect of these corrections to be more important when soft photons are emitted from hadrons, rather than leptons.

More specifically, virtual collinear contributions modify the structure of the LBK theorem \cite{1} though these effects are absent if the soft photon energy \(\omega_k \ll \frac{m^2}{Q}\), where \(m\) is the mass of the lightest charged particle and \(Q\) is the typical energy of the process. However, this condition is not fulfilled for massless particles, which are needed in the standard partonic framework of scattering amplitudes. Similarly, it would not be fulfilled for high-energy processes with leptons in the final state. In order to extend the soft theorem at the loop level to the larger region \(\frac{m^2}{Q} \leq \omega_k \sim m\) (which includes the massless limit) one needs to take into account so-called radiative jets \cite{52–56}. We point out that they give rise to corrections of the form \(\log(\mu^2/p_i \cdot k)\) to eq. (1.1), where \(\mu\) is the renormalization scale. Therefore, they are enhanced for very small photon energies, and in particular for soft photons with very low transverse momentum. At the amplitude level, our calculation with radiative jets clarifies the massless limit of logarithmic corrections to soft photon theorems derived with a somewhat unconventional regularization of infrared divergences in \cite{57, 58}.

The structure of this letter is as follows. We begin in Section 2 with a review of the LBK theorem at NLP. In Section 3 we discuss how to implement the LBK theorem with kinematical shifts. In Section 4 we discuss the impact of QCD corrections with radiative jet functions. Finally, we discuss our results in Section 5.

\footnote{An extension of the LBK theorem accounting for soft virtual contributions has been recently discussed in \cite{51}.}
2 Review of the LBK theorem

In this section we briefly review the original theorem by Low, Burnett and Kroll in the language used in current literature. Without loss of generality, we consider the emission of a soft photon of momentum $k$ in a scattering amplitude $A(p_1, p_2, k)$, where for simplicity the only charged particles are the incoming particle-antiparticle pair of spin $1/2$, mass $m$ and momenta $p_1$ and $p_2$. The extension to more general processes is straightforward.

We start at the amplitude level. Diagrammatically, the external lines of these amplitudes are connected via a hard subdiagram $H(p_1, p_2)$, representing an unspecified hard dynamics. As depicted in fig. 1, the photon can couple either to one of the external line or to the internal hard blob $H$. We define the corresponding external and internal diagrams after stripping off the photon polarization vector $\epsilon_\mu(k)$ as $A^\mu_{\text{ext}}$ and $A^\mu_{\text{int}}$, respectively. Let us compute them separately.

The external emission from the particle of momentum $p_1$ reads

$$A^\mu_{\text{ext},1} = -\eta_1 \bar{v}(p_2) H(p_1 - k, p_2) \left( \frac{p_1^\mu - k^\mu + m^2}{(p_1 - k)^2 - m^2} - \gamma^\mu u(p_1) \right)$$

$$= \eta_1 \bar{v}(p_2) H(p_1, p_2) \left( \frac{p_1^\mu}{p_1 \cdot k} - \frac{k^\mu}{2p_1 \cdot k} + \frac{k^2 p_1^\mu}{2(p_1 \cdot k)^2} - \frac{i k^\nu S^{\mu\nu}}{p_1 \cdot k} \right) u(p_1)$$

$$- \eta_1 \frac{p_1^\mu}{p_1 \cdot k} \bar{v}(p_2) \frac{\partial H(p_1, p_2)}{\partial p_1^\nu} u(p_1) + \mathcal{O}(k). \tag{2.1}$$

where we introduced the Lorentz generator $S^{\mu\nu} = \frac{i}{4} [\gamma^\mu, \gamma^\nu]$. In eq. (2.1) we expanded the amplitude at NLP in the soft photon momentum $k$ and exploited the functional dependence of $H$ to set

$$\frac{\partial H(p_1 - k, p_2)}{\partial k^{\nu}} \bigg|_{k=0} = -\frac{\partial H(p_1, p_2)}{\partial p_1^{\nu}}. \tag{2.2}$$

The first term in the second line of eq. (2.1) is of order $1/k$ and gives rise to the LP cross-section in eq. (1.1). The remaining terms are suppressed by a power of $k$ and therefore are NLP corrections. The emission for the incoming antiparticle follows analogously.

\footnote{For simplicity, we set the electric charge $e$ to unity.}
The internal emission diagram cannot be naively evaluated since we do not know how the photon couples with the short distance dynamics of the hard function. However, by exploiting the gauge invariance of the full amplitude, one can use the following Ward identity

\[ k_\mu \left( \sum_{i=1}^{2} A_{\text{ext},i}^\mu + A_{\text{int}}^\mu \right) = 0 \]  

(2.3)

to get

\[ A_{\text{int}}^\mu = \sum_{i=1}^{2} \eta_i \bar{v}(p_2) \frac{\partial \mathcal{H}(p_1, p_2)}{\partial p_i^\mu} u(p_1) \].

(2.4)

Here we assumed that the terms in \( A_{\text{int}}^\mu \) which are separately transverse to \( k^\mu \) can be ignored at NLP. While this follows straightforwardly at tree-level from power counting, special care must be taken when loops of collinear lines are present \[55\].

One can then combine external and internal emissions. Enforcing the on-shell condition on the radiated photon yields the following LBK theorem at the amplitude level:

\[ A^\mu(p_1, p_2, k) = \sum_{i=1}^{2} \eta_i \frac{p_i^\mu}{p_i \cdot k} \bar{v}(p_2) \mathcal{H}(p_1, p_2) u(p_1) + \sum_{i=1}^{2} \eta_i \bar{v}(p_2) G_{i}^{\mu\nu} \frac{\partial \mathcal{H}(p_1, p_2)}{\partial p_i^\nu} u(p_1) \]

\[ + \eta_2 \bar{v}(p_2) \frac{i k_\nu J_{i}^{\mu\nu}}{p_2 \cdot k} \mathcal{H}(p_1, p_2) u(p_1) - \eta_1 \bar{v}(p_2) \mathcal{H}(p_1, p_2) \frac{i k_\nu S_{i}^{\mu\nu}}{p_1 \cdot k} u(p_1), \]

(2.5)

where we defined

\[ G_{i}^{\mu\nu} = g^{\mu\nu} - \frac{(2p_i - k)\mu k^\nu}{2p_i \cdot k} = g^{\mu\nu} - \frac{p_i^\mu k^\nu}{p_i \cdot k} + O(k) \].

(2.6)

The first term on the r.h.s. of eq. (2.5) corresponds to the LP soft theorem, while the other terms corresponds to NLP corrections.

It is common in the contemporary literature to introduce the orbital angular momentum generator \( L_i^{\mu\nu} = i \left( p_i^\mu \frac{\partial}{\partial p_i^\nu} - p_i^\nu \frac{\partial}{\partial p_i^\mu} \right) \). Indeed, by observing that \( G_{i}^{\mu\nu} \frac{\partial}{\partial p_i^\nu} = i \frac{k_\nu}{p_i \cdot k} L_i^{\mu\nu} \), one can rewrite eq. (2.5) in terms of the total angular momentum \( J_i^{\mu\nu} = S_i^{\mu\nu} + L_i^{\mu\nu} \) as

\[ \epsilon_\mu(k) \mathcal{A}(p_1, p_2, k) = (S_{\text{LP}} + S_{\text{NLP}}) \mathcal{A}(p_1, p_2) \]

(2.7)

where \( \mathcal{A}(p_1, p_2) \) is the non-radiative amplitude and

\[ S_{\text{LP}} = \sum_{i=1}^{2} \eta_i \frac{p_i \cdot \epsilon(k)}{p_i \cdot k}, \quad S_{\text{NLP}} = \sum_{i=1}^{2} \eta_i \frac{i k_\nu J_i^{\mu\nu} \epsilon_\mu(k)}{p_i \cdot k}. \]

(2.8)

This factorized form of the theorem emphasizes that the NLP term is coupled to the total angular momentum of the hard emitter. One should be careful though since \( S_{\text{NLP}} \) in eq. (2.7) is not a multiplicative factor, but rather an operator acting on the hard function only, as made explicit in eq. (2.5).
After squaring eq. (2.5), summing over the polarizations and neglecting NNLP terms, we get the LBK theorem for the unpolarized squared amplitude, which reads

\[ |A(p_1, p_2, k)|^2 = \sum_{ij} (-\eta_i \eta_j) \frac{p_i \cdot p_j}{p_i \cdot k} |A(p_1, p_2)|^2 + \sum_{ij} (-\eta_i \eta_j) \frac{p_i \mu}{p_i \cdot k} G_{\mu\nu}^j \frac{\partial}{\partial p_j^\nu} |A(p_1, p_2)|^2 . \] (2.9)

Eq. (2.9) is valid both for scalar and spinning particles. This is straightforward in the former case, since after setting the spin generator to zero in eq. (2.5) and replacing all spinor wave functions with unity, we are left only with the orbital angular momentum contribution. Things are subtler for spinning particles, as we now discuss.

Considering again the case of a pair of spin 1/2 emitters of momenta \( p_1 \) and \( p_2 \), the non-radiative amplitude reads

\[ |A(p_1, p_2)|^2 = \text{Tr} \left[ \left( \gamma^0 (p_1 + m) \right) \gamma^0 (p_1, p_2) (\gamma^0 (p_2 - m) \gamma^0 (p_1, p_2)) \right] . \] (2.10)

When the derivatives in eq. (2.9) act on \( \mathcal{H} \) and \( \mathcal{H}^\dagger \) one recovers the orbital angular momentum contribution. On the other hand, the derivatives acting on \( (\gamma^0 (p_1 + m)) \) and \( (\gamma^0 (p_2 - m)) \) correspond to the spin contribution. This can be seen by noting that

\[ -G_{\mu\nu}^1 \frac{\partial}{\partial p_1^\nu} (\gamma^0 (p_1 + m)) = -\gamma^\mu + \frac{p_1^\mu}{p_1 \cdot k} = \frac{k^\mu}{2p_1 \cdot k} (\gamma^0 (p_1 + m)) + (\gamma^0 (p_1 + m)) \frac{\gamma^\mu k}{2p_1 \cdot k} , \] (2.11)

where in the second equality we neglected terms proportional to \( k^\mu \) which do not contribute at the squared amplitude level. The r.h.s. of eq. (2.11) is exactly the term that one considers in the interference between the first and the third terms in eq. (2.5). In Feynman gauge, this interference term reads

\[ \text{Tr} \left[ (\gamma^0 (p_2 - m) \gamma^0 (p_1, p_2) \left( \frac{k^\mu}{p_1 \cdot k} (\gamma^0 (p_1 + m)) + (\gamma^0 (p_1 + m)) \frac{\gamma^\mu k}{p_1 \cdot k} \right) \gamma^0 (p_1, p_2)) \right] \sum_i \eta_i \frac{p_i^\mu}{p_i \cdot k} . \] (2.12)

The remaining interference terms resulting from eq. (2.5) follow analogously. The spin and the orbital contributions in eq. (2.5) combine then into a derivative of the non-radiative squared amplitude. Hence, eq. (2.9) holds also for spinning particles.

### 3 LBK theorem with shifted kinematics

In the previous section we focused on the case with two charged particles. We note however that the above arguments can be straightforwardly generalized to an arbitrary number of external charged particles. The amplitude level expression in eq. (2.5) becomes then quite cumbersome when many external charged particles (possibly of different spin) are present. However, after squaring the amplitude and summing over the polarizations one obtains an expression analogous to eq. (2.9) which is relatively compact when compared to eq. (2.5). The resulting formula for \( |A(p_1, \ldots, p_n, k)|^2 \) will contain multiple
derivatives acting on the non-radiative amplitude. This has some drawbacks, as we are going to discuss.

First of all, both the non-radiative amplitude $A(p_1, \ldots, p_n)$ and the radiative amplitude $A(p_1, \ldots, p_n, k)$ in eq. (2.9) depend on the hard momenta $p_1, \ldots, p_n$ which satisfy $\sum_i p_i = -k$. Hence, momentum conservation is violated in $A(p_1, \ldots, p_n)$ for finite $k$, as originally observed by Burnett and Kroll \cite{15}. Of course, this violation is under control, since $A(p_1, \ldots, p_n)$ multiplies an NLP term. Thus it is an NNLP effect, beyond the range of validity of the LBK theorem. Still, this is a feature that one would like to avoid in a numerical implementation of eq. (2.9), since the non-radiative cross-section is typically generated separately from the radiative one by Monte Carlo codes. Also, derivatives in eq. (2.9) act on the squared amplitude, and not on the full cross-section. Moreover, when the number of external particles grows one is forced to take many derivatives of the non-radiative amplitude. This aspect might lead to difficulties in numerical implementations.

For these reasons, it is desirable to have another analytic formula for eq. (2.9). One possibility has been investigated in \cite{23} in the context of soft gluon factorization for processes with at most two radiative particles. There, it is shown that for $m = 0$ one can recast the amplitude squared of eq. (2.9) in terms of shifted momenta as

$$|A(p_1, p_2, k)|^2 = \left( \sum_{i,j=1}^2 -\eta_i \eta_j \frac{p_i \cdot p_j}{p_i \cdot k p_j \cdot k} \right) |A(p_1 + \delta p_1, p_2 + \delta p_2)|^2,$$

(3.1)

where

$$\delta p_1^\mu = \frac{1}{2} \left( -\frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\mu - \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\mu - k^\mu \right), \quad \delta p_2^\mu = \frac{1}{2} \left( \frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\mu - \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\mu - k^\mu \right).$$

(3.2)

Note that the shifts $\delta p_i^\mu$ satisfy $\delta p_i \cdot k = -k^2/2 = 0$. Also, $\delta p_i \cdot p_i = 0$. Moreover, $\delta p_1^\mu + \delta p_2^\mu = -k^\mu$ and therefore, unlike eq. (2.9), the non-radiative amplitude in eq. (3.1) is expressed in terms of conserved momenta. Finally, note that the shifts in eq. (3.2) imply that the Mandelstam variable $s = (p_1 + p_2)^2$ is shifted according to $s \to Q^2$, where $Q$ is the invariant mass of the final-state particles in the non-radiative process.

This approach can be extended to include an arbitrary number $n$ of (massless or massive) external charged particles. To do so, we note that the derivation in \cite{23} computed separately the contributions from the orbital and spin angular momentum at the amplitude level (hence specifying the nature of the hard emitter). In this way the correspondence with the modern language of soft theorems of eq. (2.7) and eq. (2.8) is made manifest. However, as the discussion in the previous section made clear, the form of the LBK theorem in eq. (2.9) does not depend on the spin of the external particles. Thus, the shifts can be determined by looking at the orbital angular momentum only in the

\footnote{See also \cite{55} and the recent Refs. \cite{46, 51}.}
scalar case. Then, a simple calculation reveals that one can generalize eq. (3.1) with

$$|A(p_1, \ldots, p_n, k)|^2 = \left( \sum_{i,j=1}^{n} -\eta_i \eta_j \frac{p_i \cdot p_j}{p_i \cdot k p_j \cdot k} \right) |A(p_1 + \delta p_1, \ldots, p_n + \delta p_n)|^2 ,$$

(3.3)

where the shift on the $\ell$-th momentum $p_{\ell}$ is defined as

$$\delta p_{\ell}^{\mu} = \left( \sum_{a,b=1}^{n} \eta_a \eta_b \frac{p_a \cdot p_b}{p_a \cdot k p_b \cdot k} \right)^{-1} \sum_{m=1}^{n} \left( \eta_m \eta_\ell \frac{(p_m)_\mu G^{\mu \nu}_{\ell}}{p_m \cdot k} \right) .$$

(3.4)

Once again, note that $\sum_{i=1}^{n} \delta p_i^{\mu} = -k^\mu$ and therefore momentum is conserved in the non-radiative amplitude. Consequently, the kinematical invariants $s_{ij} = (p_i + p_j)^2$ are shifted according to

$$s_{ij} \rightarrow s_{ij} \left( 1 - \frac{2(p_i + p_j) \cdot k}{s_{ij} R_{ij}} \right) .$$

(3.5)

Here we defined

$$R_{ij} = \left( \sum_{a,b=1}^{n} \eta_a \eta_b \frac{p_a \cdot p_b}{p_a \cdot k p_b \cdot k} \right)^{-1} \left( \eta_i (p_i)_\mu + \eta_j (p_j)_\mu \right) \sum_{c=1}^{n} \eta_c \frac{p_c^{\mu}}{p_c \cdot k} .$$

(3.6)

Note that eq. (3.5) for the variable $s = (p_1 + p_2)^2$ reads

$$s \rightarrow s \left( 1 - (1 - z) R_{12} \right) ,$$

(3.7)

where $z = Q^2 / s$ and $Q$ is the invariant mass of all final states in the non-radiative process. In the case of two legs, the shift in eq. (3.7) reduces to the much simpler $s \rightarrow sz$, which has proven to be crucial in the Mellin space approach to soft gluon resummation at NLP for processes with no colored final states [26]. In this regard, eq. (3.7) might pave the way to the extension of this program to processes with colored final states.

The squared radiative amplitude in eq. (3.3) can be implemented in fully differential cross-sections. To do so, it is convenient to express the non-radiative cross-section $d\sigma_H$ in terms of the shifted kinematics. However, in order to compensate for shifting momenta in the flux factor one has to rescale the overall cross-section by the factor in eq. (3.7). The generalization of eq. (1.1) including NLP effects of order $(\omega k)^0$ due to the LBK theorem then reads

$$\frac{d\sigma_{\text{NLP-tree}}}{d^3k} = \frac{\alpha}{(2\pi)^2} \int d^3 p_3 \cdots d^3 p_n \left( \sum_{i,j=1}^{n} -\eta_i \eta_j \frac{p_i \cdot p_j}{(p_i \cdot k)(p_j \cdot k)} \right) \left( 1 - (1 - z) R_{12} \right) d\sigma_H(p_1 + \delta p_1, \ldots, p_n + \delta p_n) ,$$

(3.8)

where the label “tree” is meant to stress that there are loop corrections (both in QCD and QED) to this formula, as we discuss in the next section. Note that the soft function in eq. (3.8) is the same LP soft function appearing in eq. (1.1). Finally, note that eq. (3.8) can be extended to include also neutral particles in the final state. To this end, it suffices to shift only the momenta of the charged particles, while $Q^2 = sz$ becomes the squared invariant mass of all (charged and neutral) final states in the non-radiative process.
Figure 2: Sample diagrams contributing to the process considered in this section, i.e. the production of a soft photon from the annihilation of a quark-antiquark pair into a neutral state of invariant mass $Q^2$ via the unspecified hard subdiagram $\mathcal{H}(p_1,p_2)$. The diagram on the left corresponds to the tree-level contribution. Virtual and real QCD corrections are shown in the middle and the right diagrams, respectively. The radiative jet function $J^\mu(1)$ of eq. (4.3) captures the contribution of the middle diagram when the momentum of the virtual gluon becomes collinear to the momentum $p_1$ of the incoming quark.

4 QCD corrections with radiative jet functions

It is well-known that the LP soft theorem in QED with massive particles is universal and does not receive loop corrections. At NLP, on the other hand, the LBK theorem of eq. (2.9) and eq. (3.8) is modified at higher orders in perturbation theory, both in QED and in QCD. Nonetheless, virtual collinear effects do not contribute if the photon energy $\omega_k \ll m^2/Q$, where $m$ is the mass of the lightest charged particle and $Q$ is the typical energy of the process. However, this region is very narrow at high energies and does not include the massless region, which is needed in perturbative calculations involving hadronic external states. The larger region $m^2/Q \leq \omega_k \sim m$ (which includes the massless limit) was first considered by Del Duca [52]. In this kinematical region, the coupling of the soft photon with collinear virtual particles generates new small scales (i.e. $p_i \cdot k$) that prevent a Taylor expansion in the soft photon energy $\omega_k$ and generate logarithmic corrections. The composite operators describing these emissions at all-orders are called radiative jets, and have been extensively studied in the recent years [32,53,55,56].

In contrast to the soft gluon resummation program, the intricate all-order structure of scattering amplitudes is not necessary for the one-loop analysis of the photon bremsstrahlung. In particular, in order to study QCD corrections to the soft photon spectrum in processes such as the $q\bar{q}\gamma$ production at LEP relevant for the analysis in [8,10] or the leptonic decay of the Z boson in hadronic collisions, it suffices to consider the contribution from the simplest of these radiative jets, denoted at one-loop as $J^\mu(1)$.

Specifically, let us consider a process with two charged massless quarks annihilating via a generic hard vertex $\mathcal{H}(p_1,p_2)$, as shown in fig. 2. In this case, the NLP soft theorems
for the tree-level and the one-loop amplitudes read
\[ \epsilon_{\mu}(k) A^{0}(p_1, p_2, k) = (S_{\text{LP}} + S_{\text{NLP}}) A^{0}(p_1, p_2), \]  
(4.1)  
\[ \epsilon_{\mu}(k) A^{(1)}(p_1, p_2, k) = (S_{\text{LP}} + S_{\text{NLP}}) A^{(1)}(p_1, p_2) + \sum_{i=1}^{2} \epsilon_{\mu}(k) q_i J^{\mu(1)}(p_i, n, k) A^{0}(p_1, p_2), \]  
(4.2)  
respectively. Here, \( S_{\text{LP}} \) and \( S_{\text{NLP}} \) are the same as in eq. (2.8) and \( q_i \) is the electric charge of the quark. Note that the upper indices 0, 1 relate to the expansion in \( \alpha_s \) according to \( A = \sum_i (\frac{\alpha_s}{\pi})^i A^{(i)} \), and correspondingly for \( J^{\mu} \). In particular, the one-loop radiative jet function \( J^{\mu(1)} \) in \( d = 4 - 2\epsilon \) dimensions reads [53]

\[ J^{\mu(1)}(p, n, k) = \left( \frac{\bar{\mu}^2}{2p \cdot k} \right)^\epsilon \left[ \left( \frac{2}{\epsilon} + 4 + 8\epsilon \right) \frac{n \cdot k}{p \cdot k} \frac{p_{\mu}}{p \cdot n} - \frac{n_{\mu}}{p \cdot n} \right] \left( 1 + 2\epsilon \right) \frac{i k_\alpha S^{\alpha\mu}}{p \cdot k} \]

\[ + \left( \frac{1}{\epsilon} - \frac{1}{2} - 3\epsilon \right) \frac{k_{\mu}}{p \cdot k} + (1 + 3\epsilon) \left( \frac{\gamma_{\mu} k}{p \cdot n} - \frac{p_{\mu}}{p \cdot k} \frac{k \cdot n}{p \cdot n} \right) \right] + O(\epsilon^2, k), \]  
(4.3)  
where \( \bar{\mu}^2 = 4\pi \mu^2 e^{-\gamma_E} \) is the \( \overline{\text{MS}} \) scale and \( n_{\mu} \) is a light-like reference vector that in the following will be set to \( p_1 \) or \( p_2 \), respectively.

In analogy with eq. (2.7), both \( S_{\text{NLP}} \) and \( J^{\mu(1)} \) in eq. (4.2) contain spinor indices and therefore represent operators acting on the hard function as in eq. (2.5) rather than multiplicative factors of the non-radiative amplitude. However, note that the dominant contribution in eq. (4.2) originates from the logarithmic terms arising from eq. (4.3). Thus, by isolating this logarithmic contribution, the radiative jets become a scalar factor multiplying the non-radiative amplitude. More precisely, we obtain

\[ \sum_{i} \epsilon_{\mu}(k) q_i J^{\mu(1)} = \frac{2}{p_1 \cdot p_2} \sum_{ij} \left( \frac{1}{\epsilon} + \log \left( \frac{\bar{\mu}^2}{2p_1 \cdot k} \right) \right) q_j p_i \cdot k \frac{p_j \cdot \epsilon}{p_j \cdot k} + O(\epsilon, k^0). \]  
(4.4)  

When squaring the amplitude in eq. (4.2) and summing over the polarizations, the radiative jet contributes via the interference with the LP soft factor, yielding

\[ 2 \text{Re} \left[ \left( \frac{p_1^\mu}{p_1 \cdot k} - \frac{p_2^\mu}{p_2 \cdot k} \right) \left( T_r[\gamma_{\mu}^{(0)} J^{(1)}(p_1, p_2, k) \gamma_{\mu}^{(0)}] - T_r[\gamma_{\mu}^{(1)}(p_2, p_1, k) \gamma_{\mu}^{(0)}] \right) \right]. \]  
(4.5)  
Plugging eq. (4.3) into eq. (4.5) one obtains a divergent expression with a quite intricate structure involving various momenta and gamma matrices. This is the radiative jet contribution to the virtual cross-section, which must be then combined with the real diagrams containing an unobserved real gluon (together with the observed real photon, as shown in fig. [2]) in order to cancel the soft divergences. After the remaining collinear
singularities are absorbed in the parton distribution functions, one is left with a finite expression where the dominant term is given by \( O(\alpha_s) \) logarithmic corrections to the soft photon bremsstrahlung. The coefficient of these logarithms can be read directly from eq. (4.4). In fact, the poles present in the other parts of the factorized cross-section cannot give any \( \log(\omega_k) \), since this type of logarithms can only be generated by the collinear scale \( p_i \cdot k \) present in the radiative jet.

Therefore, by combining eq. (4.2), eq. (4.4) and eq. (4.5) we can replace eq. (1.1) with:

\[
\frac{d\sigma}{d^3k} = \frac{d\sigma_{NLP-tree}}{d^3k} + \frac{\alpha_s}{4\pi} \frac{d\sigma_{NLP,J(1)}}{d^3k},
\]

where \( \frac{d\sigma_{NLP-tree}}{d^3k} \) defined in eq. (3.8) contains both the standard LP contribution and the tree-level NLP correction and

\[
\frac{d\sigma_{NLP,J(1)}}{d^3k} = \frac{\alpha_s}{2(2\pi)^2} \frac{1}{\omega_k} \int d^3p_3 \cdots \int d^3p_n \left( \sum_{i=1}^{2n} \eta_i \frac{8 \log \left( \frac{\mu^2}{2p_i \cdot k} \right)}{p_i \cdot k} \right) d\sigma_H(p_1, \ldots, p_n).
\]

The result in eq. (4.7) gives the complete logarithmic dependence (at order \( \alpha_s \) and in the massless limit) for processes with a single quark pair such as the leptonic decays of the \( Z \) boson via quark-antiquark annihilation or \( e^+e^- \to Z \to q\bar{q} \). The latter in particular is the process studied in detail by the DELPHI collaboration in [8, 10] where large deviations w.r.t. the tree-level soft spectrum have been observed. In this regard, it is interesting to note that after setting the scale \( \mu \) to the typical high energy of the process, the QCD corrections in eq. (4.7) might be large when the photon has very small energy and/or very low transverse momentum w.r.t. the direction of the emitting particle.

It is tempting to generalize eq. (4.7) to more generic processes, by simply letting the index \( i \) in eq. (4.7) run over all external quarks. We note however that in this case there might be additional logarithmic corrections to the NLP cross-section due to soft photon emission from multiple collinear particles connected to the hard vertex, whose computation is beyond the scope of this letter.

Finally, it is interesting to compare eq. (4.2) with a parallel body of work that discussed logarithmic corrections to soft theorems in QED and gravity in scattering amplitudes [57, 58]. Specifically, after identifying \( \log \left( \frac{\omega_k^2}{2p_i \cdot k} \right) \to \log(\omega_k) \) in eq. (4.4) one recovers a similar structure to the massless limit of the results in [58]. However, note that [58] used an unconventional prescription for the regularization of infrared divergences, while eq. (4.4) is expressed in dimensional regularization, which is more common in perturbative QCD calculations. Note also that in contrast to [58], where the analysis was completely carried in QED, here we considered a photon emission with a QCD loop correction.

\[4\] While a field theoretical definition for these additional radiative jets has been provided in the effective field theory language, only the relevant diagrams have been identified in the full-QCD factorization approach [49, 55, 56].

\[5\] See eq. (2.4) and eq. (2.5) there.
5 Discussion

Measurements of soft photon spectra remain not understood for processes with final state hadrons, as the measured spectrum shows significant excess above the theoretical predictions based on the leading-power (LP) formula in (eq. (1.1)). In this letter we have considered two different corrections to this formula at next-to-leading power (NLP).

The first type of these corrections is of order $(\omega k)^0 \sim 1$ and it is due to the LBK theorem. We have reviewed the theorem, providing a formula (eq. (3.8)) with shifted kinematics which is particularly suitable for numerical implementations. Besides, by expressing the non-radiative amplitude in terms of conserved momenta, eq. (3.8) underlines the validity of the LBK theorem at tree-level, which has been recently questioned in [46].

Although here we considered the photon bremsstrahlung, it is also worth noting that shifting the incoming momenta at NLP has turned out to be an efficient method for the derivation of threshold resummation formulae at NLP in the hadroproduction of colorless final states [26]. Therefore, the generalization of the shifts to processes with an arbitrary number of charged legs presented here in eq. (3.5) and eq. (3.8) might provide an efficient tool in the extension of the NLP resummation to processes with colored final states.

The second type of corrections is of order $\log(\mu^2/p_i \cdot k)$ and it is due to QCD loop effects originating from radiative jet functions. Although radiative jets have been so far investigated mainly in the context of soft gluon resummation, they can be naturally applied also for the soft photon spectrum. In particular, in this work we have shown for the first time how mixed QED-QCD effects can be studied with radiative jets. More specifically, we have provided a formula (eq. (4.7)) which gives $O(\alpha_s)$ logarithmic corrections in the massless limit to the soft photon spectrum in processes with a single quark-antiquark pair in the external states, such as the high energy limit of $e^+e^- \rightarrow q\bar{q}\gamma$ investigated in [8, 10].

The appearance of logarithmic corrections to soft theorems is not surprising [47–50]. In fact, one-loop logarithmic corrections of the form $\log(\omega_k)$ have been already computed in scalar QED at the amplitude level in [57, 58], although with a somewhat unconventional regularization of infrared divergences. The approach followed here with a photon emission from QCD radiative jets not only exploits the more common dimensional regularization to regularize both collinear and soft divergences (hence making the result directly implementable in partonic cross-sections such as eq. (4.7)), but emphasizes that in the massless limit (which is relevant at high energies) the origin of these logarithmic corrections is linked to the collinear scale $p_i \cdot k$, which might be small not only for small energies $\omega_k$ but also for small transverse momenta w.r.t. the direction of the radiating particles.

Summarizing, QCD corrections due to the radiative jets are a very good candidate to provide sizable contributions to soft photon yields. Hence, they are important for theoretical description of the photon radiation in processes involving hadrons. Specifically, they are very interesting to consider in the context of the observed discrepancies between theoretical predictions and experimental measurements for photon spectra in hadronic Z
decays, as observed in \cite{8,10}. Indeed, eq. (4.7) seems to go in the right direction by providing QCD corrections that (although suppressed in $\alpha_s$) might be particularly enhanced for very small photon energies and/or very low transverse momentum w.r.t. the jet axis. However, this issue can be resolved only with a complete numerical study. Work in this regard is ongoing.

Finally, we note that one could reverse the problem and exploit QCD corrections to the LBK theorem given by radiative jet functions as a tool to obtain information about the jet structure. A precise measurement of the soft photon spectra probing these corrections would be invaluable for such studies. More generally, in the light of the future plans to measure ultra soft photons at LHC, we believe that there is much potential in the recently developed techniques for NLP corrections in QCD to shed new light on infrared photon physics.

Acknowledgments

We thank Anton Andronic, Christian Klein-Bösing, Peter Braun-Munzinger, Stefan Flörchinger, Klaus Reygers and Johanna Stachel for stimulating discussions that led to this project. We are also grateful to Tim Engel, Adrian Signer, Yannick Ulrich and Leonardo Vernazza for communications regarding loop contributions to the LBK theorem.

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