Double Affine Hecke Algebras of Rank 1 and the $\mathbb{Z}_3$-Symmetric Askey–Wilson Relations

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Abstract. We consider the double affine Hecke algebra $H = H(k_0, k_1, k_0^\vee, k_1^\vee; q)$ associated with the root system $(C^\vee_1, C_1)$. We display three elements $x$, $y$, $z$ in $H$ that satisfy essentially the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. We obtain the relations as follows. We work with an algebra $\hat{H}$ that is more general than $H$, called the universal double affine Hecke algebra of type $(C^\vee_1, C_1)$. An advantage of $\hat{H}$ over $H$ is that it is parameter free and has a larger automorphism group. We give a surjective algebra homomorphism $\hat{H} \rightarrow H$. We define some elements $x$, $y$, $z$ in $\hat{H}$ that get mapped to their counterparts in $H$ by this homomorphism.

We give an action of Artin’s braid group $B_3$ on $\hat{H}$ that acts nicely on the elements $x$, $y$, $z$; one generator sends $x \mapsto y \mapsto z \mapsto x$ and another generator interchanges $x$, $y$. Using the $B_3$ action we show that the elements $x$, $y$, $z$ in $\hat{H}$ satisfy three equations that resemble the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. Applying the homomorphism $\hat{H} \rightarrow H$ we find that the elements $x$, $y$, $z$ in $H$ satisfy similar relations.

Key words: Askey–Wilson polynomials; Askey–Wilson relations; braid group

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1 Introduction

The double affine Hecke algebra (DAHA) for a reduced root system was defined by Cherednik [2], and the definition was extended to include nonreduced root systems by Sahi [11]. The most general DAHA of rank 1 is associated with the root system $(C^\vee_1, C_1)$ [8]; this algebra involves five nonzero parameters and will be denoted by $H = H(k_0, k_1, k_0^\vee, k_1^\vee; q)$. We mention some recent results on $H$. In [12] Sahi links certain $H$-modules to the Askey–Wilson polynomials [1]. This link is given a comprehensive treatment by Noumi and Stokman [7]. In [9] Oblomkov and Stoica describe the finite-dimensional irreducible $H$-modules under the assumption that $q$ is not a root of unity. In [8] Oblomkov gives a detailed study of the algebraic structure of $H$, and finds an intimate connection to the geometry of affine cubic surfaces. His point of departure is the case $q = 1$; under that assumption he finds that the spherical subalgebra of $H$ is generated by three elements $X_1$, $X_2$, $X_3$ that mutually commute and satisfy a certain cubic equation [8, Theorem 2.1, Proposition 3.1]. In [4, 5] Koornwinder describes the spherical subalgebra of $H$ under the assumption that $q$ is not a root of unity. His main results [4, Corollary 6.3], [5, Theorem 3.2] are similar in nature to those of Oblomkov, although he formulates these results in a very different way and works with a different presentation of $H$. In Koornwinder’s formulation the spherical subalgebra of $H$ is related to the Askey–Wilson algebra $AW(3)$, which was introduced by Zhedanov in [17]. The original presentation of $AW(3)$ involves three generators and three
relations [17], lines (1.1a)–(1.1c)]. Koornwinder works with a slightly different presentation for $AW(3)$ that involves two generators and two relations [4, lines (2.1), (2.2)]. These two relations are sometimes called the Askey–Wilson relations [15]. For the algebra $AW(3)$ a third presentation is known [10, p. 101], [14], [16, Section 4.3] and described as follows. For a sequence of scalars $g_x, g_y, g_z, h_x, h_y, h_z$ the corresponding Askey–Wilson algebra is defined by generators $X, Y, Z$ and relations

\begin{align*}
qXY - q^{-1}YX &= g_z Z + h_z, \\
qYZ - q^{-1}ZY &= g_x X + h_x, \\
qZX - q^{-1}XZ &= g_y Y + h_y.
\end{align*}

We will refer to (1)–(3) as the $Z_3$-symmetric Askey–Wilson relations. Upon eliminating $Z$ in (2), (3) using (1) we obtain the Askey–Wilson relations in the variables $X, Y, Z$. Upon substituting $Z' = g_x Z + h_x$ in (1–3) we recover the original presentation for $AW(3)$ in the variables $X, Y, Z'$.

In this paper we return to the elements $X_1, X_2, X_3$ considered by Oblomkov, although for notational convenience we will call them $x, y, z$. We show that $x, y, z$ satisfy three equations that resemble the $Z_3$-symmetric Askey–Wilson relations. The resemblance is described as follows. The equations have the form (1–3) with $h_x, h_y, h_z$ not scalars but instead rational expressions involving an element $t_1$ that commutes with each of $x, y, z$. The element $t_1$ appears earlier in the work of Koornwinder [4, Definition 6.1]; we will say more about this at the end of Section 2. Our derivation of the three equations is elementary and illuminates a role played by Artin’s braid group $B_3$.

Our proof is summarized as follows. Adapting some ideas of Ion and Sahi [3] we work with an algebra $\hat{H}$ that is more general than $H$, called the universal double affine Hecke algebra (UDAH) of type $(C_1^\vee, C_1)$. An advantage of $\hat{H}$ over $H$ is that it is parameter free and has a larger automorphism group. We give a surjective algebra homomorphism $\hat{H} \to H$. We define some elements $x, y, z$ in $\hat{H}$ that get mapped to their counterparts in $H$ by this homomorphism. Adapting [3, Theorem 2.6] we give an action of the braid group $B_3$ on $\hat{H}$ that acts nicely on the elements $x, y, z$: one generator sends $x \mapsto y \mapsto z \mapsto x$ and another generator interchanges $x, y$. Using the $B_3$ action we show that the elements $x, y, z$ in $\hat{H}$ satisfy three equations that resemble the $Z_3$-symmetric Askey–Wilson relations. Applying the homomorphism $\hat{H} \to H$ we find that the elements $x, y, z$ in $H$ satisfy similar relations.

## 2 The double affine Hecke algebra of type $(C_1^\vee, C_1)$

Throughout the paper $F$ denotes a field. An algebra is meant to be associative and have a 1.

We recall the double affine Hecke algebra of type $(C_1^\vee, C_1)$. For this algebra there are several presentations in the literature; one involves three generators [4, 5, 7] and another involves four generators [6, p. 160], [7, 8, 9]. We will use essentially the presentation of [6, p. 160], with an adjustment designed to make explicit the underlying symmetry.

**Definition 2.1.** Fix nonzero scalars $k_0, k_1, k_0^\vee, k_1^\vee$, $q$ in $F$. Let $H(k_0, k_1, k_0^\vee, k_1^\vee; q)$ denote the $F$-algebra defined by generators $t_i, t_i^\vee$ ($i = 0, 1$) and relations

\begin{align*}
(t_i - k_i)(t_i - k_i^{-1}) &= 0, \\
(t_i^\vee - k_i^\vee)(t_i^\vee - k_i^{-1}) &= 0, \\
t_0^\vee t_0^\vee t_1 &= q^{-1}.
\end{align*}

This algebra is called the double affine Hecke algebra (or DAHA) of type $(C_1^\vee, C_1)$. 
Note 2.2. In [6] p. 160] Macdonald gives a presentation of $H$ involving four generators. To go from his presentation to ours, multiply each of his generators and the corresponding parameter by $\sqrt{-1}$, and replace his $q$ by $q^2$.

The following result is well known; see for example [13, Corollary 1].

**Lemma 2.3.** Referring to Definition 2.1 for $i \in \{0, 1\}$ the elements $t_i, t_i^\vee$ are invertible and

$$ t_i + t_i^{-1} = k_i + k_i^{-1}, \quad t_i^\vee + t_i^{-1} = k_i^\vee + k_i^{\vee^{-1}}. $$

**Proof.** Define $r_i = k_i + k_i^{-1} - t_i$ and $r_i^\vee = k_i^\vee + k_i^{\vee^{-1}} - t_i^\vee$. Using (4), (5) we find $t_i r_i = r_i t_i = 1$ and $t_i^\vee r_i^\vee = r_i^\vee t_i^\vee = 1$. The result follows. □

We now state our main result. In this result part (ii) follows from [8, Theorem 2.1]; it is included here for the sake of completeness.

**Theorem 2.4.** In the algebra $H(k_0, k_1, k_0^\vee, k_1^\vee; q)$ from Definition 2.1 define

$$ x = t_0^\vee t_1 + (t_1^\vee t_1)^{-1}, \quad y = t_1^\vee t_1 + (t_1^\vee t_1)^{-1}, \quad z = t_0 t_1 + (t_0 t_1)^{-1}. $$

Then the following (i)–(iv) hold:

(i) $t_1$ commutes with each of $x$, $y$, $z$.

(ii) Assume $q^2 = 1$. Then $x$, $y$, $z$ mutually commute.

(iii) Assume $q^2 \neq 1$ and $q^4 = 1$. Then $\text{Char}(F) \neq 2$ and

\[
\begin{align*}
\frac{xy + yx}{2} &= (k_0^\vee + k_0^{\vee^{-1}})(k_1^\vee + k_1^{\vee^{-1}}) + (k_0 + k_0^{-1})(q^{-1}t_1 + qt_1^{-1}), \\
\frac{yz + zy}{2} &= (k_1^\vee + k_1^{\vee^{-1}})(k_0 + k_0^{-1}) + (k_0^\vee + k_0^{\vee^{-1}})(q^{-1}t_1 + qt_1^{-1}), \\
\frac{zx + xz}{2} &= (k_0 + k_0^{-1})(k_0^\vee + k_0^{\vee^{-1}}) + (k_1 + k_1^{\vee^{-1}})(q^{-1}t_1 + qt_1^{-1}).
\end{align*}
\]

(iv) Assume $q^4 \neq 1$. Then

\[
\begin{align*}
\frac{qxy - q^{-1}yx}{q^2 - q^{-2}} + z &= \frac{(k_0^\vee + k_0^{-1})(k_1^\vee + k_1^{\vee^{-1}}) + (k_0 + k_0^{-1})(q^{-1}t_1 + qt_1^{-1})}{q + q^{-1}}, \\
\frac{qyz - q^{-1}zy}{q^2 - q^{-2}} + x &= \frac{(k_1^\vee + k_1^{-1})(k_0 + k_0^{-1}) + (k_0^\vee + k_0^{\vee^{-1}})(q^{-1}t_1 + qt_1^{-1})}{q + q^{-1}}, \\
\frac{qzx - q^{-1}xz}{q^2 - q^{-2}} + y &= \frac{(k_0 + k_0^{-1})(k_0^\vee + k_0^{\vee^{-1}}) + (k_1^\vee + k_1^{\vee^{-1}})(q^{-1}t_1 + qt_1^{-1})}{q + q^{-1}}.
\end{align*}
\]

The equations in Theorem 2.4 (iv) resemble the $\mathbb{Z}_3$-symmetric Askey–Wilson relations, as we discussed in Section 1.

We will prove Theorem 2.4 in Section 5.

We comment on how Theorem 2.4 is related to the work of Koornwinder [4]. Define $x$, $y$, $z$ as in Theorem 2.4. Then that theorem describes how $x$, $y$, $z$, $t_1$ are related. If we translate [4] Definition 6.1, Corollary 6.3] into the presentation of Definition 2.1 then it describes how $x$, $y$, $t_1$ are related, assuming $q$ is not a root of unity and some constraints on $k_0$, $k_1$, $k_0^\vee$, $k_1^\vee$. Under these assumptions and modulo the translation the following coincide: (i) the main relations [4] lines (6.2), (6.3)] of [4] Definition 6.1]; (ii) the relations obtained from the last two equations of Theorem 2.4 (iv) by eliminating $z$ using the first equation.
3 The universal double affine Hecke algebra of type \((C_1^\vee, C_1)\)

In our proof of Theorem 2.4 we will initially work with a homomorphic preimage \(\hat{H}\) of \(H(k_0, k_1, k_0^\vee, k_1^\vee; q)\) called the universal double affine Hecke algebra of type \((C_1^\vee, C_1)\). Before we get into the details, we would like to acknowledge how \(\hat{H}\) is related to the work of Ion and Sahi \([3]\). Given a general DAHA (not just rank 1) Ion and Sahi construct a group \(\hat{A}\) called the double affine Artin group \([3]\) Definition 3.4, Theorem 3.10]. The given DAHA is a homomorphic image of the group \(F\)-algebra \(F\hat{A}\) \([3]\) Definition 1.13]. For the case \((C_1^\vee, C_1)\) of the present paper, their homomorphism has a factorization \(F\hat{A} \to \hat{H} \to H(k_0, k_1, k_0^\vee, k_1^\vee; q)\). In this section and the next we will obtain some facts about \(\hat{H}\). We could obtain these facts from \([3]\) by applying the homomorphism \(F\hat{A} \to \hat{H}\), but for the purpose of clarity we will prove everything from first principles.

We now define \(\hat{H}\) and describe some of its basic properties. In Section 4 we will discuss how the group \(B_3\) acts on \(\hat{H}\). In Section 5 we will use the \(B_3\) action to prove Theorem 2.4.

**Definition 3.1.** Let \(\hat{H}\) denote the \(F\)-algebra defined by generators \(t_i^\pm, (t_i^\vee)^\pm (i = 0, 1)\) and relations

\[
\begin{align*}
    t_i t_i^{-1} &= t_i^{-1} t_i = 1, & \quad t_i t_i^\vee t_i^{-1} &= t_i^\vee t_i^{-1} t_i^\vee = 1, \\
    t_i + t_i^{-1} &\text{ is central,} & \quad t_i^\vee + t_i^{\vee -1} &\text{ is central,} \\
    t_0^\vee t_0 t_1^\vee &\text{ is central.}
\end{align*}
\]

We call \(\hat{H}\) the universal double affine Hecke algebra (or UDAHA) of type \((C_1^\vee, C_1)\).

**Note 3.2.** The double affine Artin group \(\hat{A}\) of type \((C_1^\vee, C_1)\) is defined by generators \(t_i^\pm, (t_i^\vee)^\pm (i = 0, 1)\) and relations \([7], [9], [3]\) Theorem 3.11].

**Definition 3.3.** Observe that in \(\hat{H}\) the element \(t_0^\vee t_0 t_1^\vee t_1\) is invertible; let \(Q\) denote the inverse.

**Lemma 3.4.** Given nonzero scalars \(k_0, k_1, k_0^\vee, k_1^\vee, q\) in \(F\), there exists a surjective \(F\)-algebra homomorphism \(\hat{H} \to H(k_0, k_1, k_0^\vee, k_1^\vee; q)\) that sends \(Q \mapsto q\) and \(t_1 \mapsto t_1, t_i^\vee \mapsto t_i^\vee\) for \(i \in \{0, 1\}\).

**Proof.** Compare the defining relations for \(\hat{H}\) and \(H(k_0, k_1, k_0^\vee, k_1^\vee; q)\).

One advantage of \(\hat{H}\) over \(H(k_0, k_1, k_0^\vee, k_1^\vee; q)\) is that \(\hat{H}\) has more automorphisms. This is illustrated in the next lemma. By an automorphism of \(\hat{H}\) we mean an \(F\)-algebra isomorphism \(\hat{H} \to \hat{H}\).

**Lemma 3.5.** There exists an automorphism of \(\hat{H}\) that sends

\[
\begin{align*}
    t_0^\vee &\mapsto t_0, & \quad t_0 &\mapsto t_1^\vee, & \quad t_i &\mapsto t_i^\vee, & \quad t_1 &\mapsto t_0^\vee.
\end{align*}
\]

This automorphism fixes \(Q\).

**Proof.** The result follows from Definition 3.1 once we verify that \(t_0 t_1^\vee t_1 t_0^\vee = Q^{-1}\). This equation holds since each side is equal to \(t_0^\vee Q^{-1} t_0^\vee\).

**Lemma 3.6.** In the algebra \(\hat{H}\) the element \(Q^{-1}\) is equal to each of the following:

\[
\begin{align*}
    t_0^\vee t_0 t_1^\vee t_1, & \quad t_0 t_1^\vee t_1 t_0^\vee, & \quad t_1^\vee t_1 t_0^\vee t_0, & \quad t_1 t_0^\vee t_0 t_1^\vee.
\end{align*}
\]

**Proof.** To each side of the equation \(t_0^\vee t_0 t_1^\vee t_1 = Q^{-1}\) apply three times the automorphism from Lemma 3.5.
Definition 3.7. We define elements \( x, y, z \) in \( \hat{H} \) as follows.
\[
x = t_0^\vee t_1 + (t_1^\vee t_1)^{-1}, \quad y = t_1^\vee t_1 + (t_1^\vee t_1)^{-1}, \quad z = t_0 t_1 + (t_0 t_1)^{-1}.
\]

The following result suggests why \( x, y, z \) are of interest.

Lemma 3.8. Let \( u, v \) denote invertible elements in any algebra such that each of \( u + u^{-1}, v + v^{-1} \) is central. Then

(i) \( uv + (uv)^{-1} = vu + (vu)^{-1} \);

(ii) \( uv + (uv)^{-1} \) commutes with each of \( u, v \).

Proof. (i) Observe that
\[
uv + (uv)^{-1} = uv + vu - (v + v^{-1})u - v(u + u^{-1}) + (v + v^{-1})(u + u^{-1}),
\]
\[
vu + (vu)^{-1} = uv + vu - u(v + v^{-1}) - (u + u^{-1})v + (u + u^{-1})(v + v^{-1}).
\]
In these equations the expressions on the right are equal since \( u + u^{-1} \) and \( v + v^{-1} \) are central. The result follows.

(ii) We have
\[
(u^{-1}(uv + (uv)^{-1})u = uv + (uv)^{-1}
\]
since each side is equal to \( vu + (vu)^{-1} \). Therefore \( uv + (uv)^{-1} \) commutes with \( u \). One similarly shows that \( uv + (uv)^{-1} \) commutes with \( v \).

Corollary 3.9. In the algebra \( \hat{H} \) the element \( t_1 \) commutes with each of \( x, y, z \).

Proof. Use Definition 3.7 and Lemma 3.8(ii).

4 The braid group \( B_3 \)

In this section we display an action of the braid group \( B_3 \) on the algebra \( \hat{H} \) from Definition 3.1. This \( B_3 \) action will be used to prove Theorem 2.4.

Definition 4.1. Artin’s braid group \( B_3 \) is defined by generators \( b, c \) and the relation \( b^3 = c^2 \).

For notational convenience define \( a = b^3 = c^2 \).

The following result is a variation on [3, Theorem 2.6].

Lemma 4.2. The braid group \( B_3 \) acts on \( \hat{H} \) as a group of automorphisms such that \( a(h) = t_1^{-1}ht_1 \) for all \( h \in \hat{H} \) and \( b, c \) do the following:

\[
\begin{array}{c|cccc}
  h & t_0^\vee & t_0 & t_1^\vee & t_1 \\
  \hline
  b(h) & t_1^{-1} t_0^\vee t_1 & t_0^\vee & t_0 & t_1 \\
  c(h) & t_1^{-1} t_1^\vee t_1 & t_0^\vee t_0^{-1} t_1 & t_0 & t_1 \\
\end{array}
\]

Proof. There exists an automorphism \( A \) of \( \hat{H} \) that sends \( h \mapsto t_1^{-1}ht_1 \) for all \( h \in \hat{H} \). Define
\[
T_0^\vee = t_1^{-1} t_1^\vee t_1, \quad T_0 = t_0^\vee, \quad T_1^\vee = t_0, \quad T_1 = t_1.
\]

Note that \( T_0^\vee, T_0, T_1^\vee, T_1 \) are invertible and that
\[
T_0^\vee + T_0^{-1} = t_1^\vee + t_1^{-1}, \quad T_0 + T_0^{-1} = t_0^\vee + t_0^{-1},
\]
\[
T_1^\vee + T_1^{-1} = t_0 + t_0^{-1}, \quad T_1 + T_1^{-1} = t_1 + t_1^{-1}.
\]
In each of these four equations the expression on the right is central so the expression on the left is central. Using (11) and Lemma 3.6
\[ T_0^\vee T_0 T_1^\vee T_1 = t_1^{-1} t_1^\vee t_1 t_0 t_1 = t_1^{-1} Q^{-1} t_1 = Q^{-1} \]
so \( T_0^\vee T_0 T_1^\vee T_1 \) is central. By these comments there exists an \( \mathbb{F} \)-algebra homomorphism \( B : \hat{H} \to \hat{H} \) that sends
\[ t_0^\vee \mapsto T_0^\vee, \quad t_0 \mapsto T_0, \quad t_1^\vee \mapsto T_1^\vee, \quad t_1 \mapsto T_1. \]
We claim that \( B^3 = A \). To prove the claim we show that \( B^3, A \) agree at each of \( t_0^\vee, t_0, t_1^\vee, t_1 \).
Note that \( A \) fixes \( t_1 \). Note also that \( t_1 \) is fixed by \( B \) and hence \( B^3 \); therefore \( B^3 \) and \( A \) agree at \( t_1 \). The map \( B \) sends
\[ t_0^\vee \mapsto t_0 \mapsto t_0^\vee \mapsto t_1^{-1} t_1^\vee t_1 \mapsto t_1^{-1} t_0 t_1 \mapsto t_1^{-1} t_0^\vee t_1. \]
Therefore \( B^3 \) sends
\[ t_0^\vee \mapsto t_1^{-1} t_1^\vee t_1, \quad t_0 \mapsto t_1^{-1} t_0 t_1, \quad t_0^\vee \mapsto t_1^{-1} t_0^\vee t_1, \]
so \( B^3 \), \( A \) agree at each of \( t_1^\vee, t_0, t_0^\vee \). We have shown \( B^3 = A \). By this and since \( A \) is invertible, we see that \( B \) is invertible and hence an automorphism of \( \hat{H} \). Define
\[ S_0^\vee = t_1^{-1} t_1^\vee t_1, \quad S_0 = t_0^\vee t_0 t_0^\vee t_0, \quad S_1^\vee = t_1^\vee, \quad S_1 = t_1. \tag{12} \]
Note that \( S_0^\vee, S_0, S_1^\vee, S_1 \) are invertible and
\[ S_0^\vee + S_0^{-1} = t_0^\vee + t_0^{-1}, \quad S_0 + S_0^{-1} = t_0 + t_0^{-1}, \]
\[ S_1^\vee + S_1^{-1} = t_0^\vee + t_0^{-1}, \quad S_1 + S_1^{-1} = t_1 + t_1^{-1}. \]
In each of these four equations the expression on the right is central so the expression on the left is central. Using (12) and Lemma 3.6
\[ S_0^\vee S_0 S_1 S_1 = t_0^{-1} t_1^\vee t_1 t_0 t_0 t_1 = t_1^{-1} Q^{-1} t_1 = Q^{-1} \]
so \( S_0^\vee S_0 S_1 S_1 \) is central. By these comments there exists an \( \mathbb{F} \)-algebra homomorphism \( C : \hat{H} \to \hat{H} \) that sends
\[ t_0^\vee \mapsto S_0^\vee, \quad t_0 \mapsto S_0, \quad t_0^\vee \mapsto S_1^\vee, \quad t_1 \mapsto S_1. \]
We claim that \( C^2 = A \). To prove the claim we show that \( C^2, A \) agree at each of \( t_0^\vee, t_0, t_1^\vee, t_1 \).
Both \( C^2 \) and \( A \) fix \( t_1 \). The map \( C \) sends \( t_0^\vee \mapsto t_1^{-1} t_1^\vee t_1 \mapsto t_1^{-1} t_0^\vee t_1 \) so \( C^2 \), \( A \) agree at \( t_0^\vee \). The map \( C \) sends \( t_1^\vee \mapsto t_0^\vee \mapsto t_1^{-1} t_1^\vee t_1 \) so \( C^2 \), \( A \) agree at \( t_1^\vee \). The map \( C \) sends
\[ t_0 \mapsto t_0^\vee t_0^\vee t_0^{-1} \mapsto t_1^{-1} t_1^\vee t_1 t_0^\vee t_0^{-1} t_1^{-1} t_1^\vee t_1 \]
In the above line the expression on the right equals \( t_1^{-1} t_0 t_1 \). To see this, note that \( t_0^\vee t_1^\vee t_0 = t_0 t_1^\vee t_1 t_0^\vee \) since each side equals \( Q^{-1} \) by Lemma 3.6. We have shown that \( C^2, A \) agree at \( t_0 \). By the above comments \( C^2, A \) agree at each of \( t_0^\vee, t_0, t_1^\vee, t_1 \) so \( C^2 = A \). Therefore \( C \) is invertible and hence an automorphism of \( \hat{H} \). We have shown that the desired \( B_3 \) action exists.

The next result is immediate from Lemma 4.2 and its proof.

**Lemma 4.3.** The \( B_3 \) action from Lemma 4.2 does the following to the central elements \( S_8, S_9 \).
The generators \( b, c \) fix every central element. The generators \( b, c \) fix \( Q \) and satisfy the table below.

| \( h \) | \( t_0^\vee + t_0^{-1} \) | \( t_0 + t_0^{-1} \) | \( t_1^\vee + t_1^{-1} \) | \( t_1 + t_1^{-1} \) |
|-------|------------------|------------------|------------------|------------------|
| \( b(h) \) | \( t_0^\vee + t_0^{-1} \) | \( t_0 + t_0^{-1} \) | \( t_1^\vee + t_1^{-1} \) | \( t_1 + t_1^{-1} \) |
| \( c(h) \) | \( t_0^\vee + t_0^{-1} \) | \( t_0 + t_0^{-1} \) | \( t_1^\vee + t_1^{-1} \) | \( t_1 + t_1^{-1} \) |
5 The proof of Theorem 2.4

Recall the elements $x, y, z$ of $\hat{H}$ from Definition 3.7. In this section we describe how the group $B_3$ acts on these elements. Using this information we show that $x, y, z$ satisfy three equations that resemble the $\mathbb{Z}_3$-symmetric Askey–Wilson relations. Using these equations we obtain Theorem 2.4.

Theorem 5.1. The $B_3$ action from Lemma 4.2 does the following to the elements $x, y, z$ from Definition 3.7. The generator $a$ fixes each of $x, y, z$. The generator $b$ sends $x \mapsto y \mapsto z \mapsto x$. The generator $c$ swaps $x, y$ and sends $z \mapsto z'$ where

$$Qz + Q^{-1}z' + xy = Q^{-1}z + Qz' + yx$$

$$= (t_0^\psi + t_0^{-1})(t_1^\psi + t_1^{-1}) + (t_0 + t_0^{-1})(Q^{-1}t_1 + Qt_1^{-1}).$$

Proof. The generator $a$ fixes each of $x, y, z$ by Corollary 3.9 and since $a(h) = t_1^{-1}ht_1$ for all $h \in \hat{H}$. The generator $b$ sends $x \mapsto y \mapsto z \mapsto x$ by Definition 3.7 Corollary 3.9 and Lemma 4.2. Similarly the generator $c$ swaps $x, y$. Define $z' = c(z)$. We show that $z'$ satisfies the equations in the theorem statement. We first show that

$$Q^{-1}t_0 + Qc(t_0) + yt_0^\psi = (t_0^\psi t_1)^{-1}(t_0^\psi + t_0^{-1}) + Q^{-1}(t_0 + t_0^{-1}).$$

By Lemma 4.2 $c(t_0) = t_0^\psi t_0t_0^{-1}$. By this and Definition 3.3,

$$Qc(t_0) = (t_0^\psi t_1)^{-1}t_0^{-1}.$$  

By Lemma 3.6

$$t_0^\psi t_1t_0^{-1} = Q^{-1}t_0^{-1}.$$  

Using (14), (15) and $y = t_0^\psi t_1 + (t_0^\psi t_1)^{-1}$ we obtain (13). Next we show that

$$Q^{-1}t_0^{-1} + Qc(t_0^{-1}) + yt_0^{-\psi} = t_0^\psi t_1(t_0^\psi + t_0^{-1}) + Q(t_0 + t_0^{-1}).$$

By Lemma 4.3

$$c(t_0) + c(t_0^{-1}) = t_0 + t_0^{-1}.$$  

Combining this with (13) we obtain (16) after a brief calculation. In (13) we multiply each term on the right by $t_1$ and use $c(t_1) = t_1$ to get

$$Q^{-1}t_0t_1 + Qc(t_0t_1) + yt_0^\psi t_1 = (t_0^\psi t_1)^{-1}t_1(t_0^\psi + t_0^{-1}) + Q^{-1}t_1(t_0 + t_0^{-1}).$$

In (16) we multiply each term on the left by $t_1^{-1}$ and use $c(t_1^{-1}) = t_1^{-1}$ together with the fact that $y$ commutes with $t_1$ to get

$$Q^{-1}(t_0t_1)^{-1} + Qc((t_0t_1)^{-1}) + y(t_0^\psi t_1)^{-1} = t_1^{-1}t_0^\psi t_1(t_0^\psi + t_0^{-1}) + Qt_1^{-1}(t_0 + t_0^{-1}).$$

We have

$$(t_0^\psi t_1)^{-1}t_1 + t_1^{-1}t_0^\psi t_1 = t_0^\psi + t_0^{-1}$$

since both sides equal $t_1^{-1}(t_0^\psi + t_0^{-1})t_1$. We now add (17), (18) and simplify the result using (19) to obtain

$$Q^{-1}z + Qz' + yx = (t_0^\psi + t_0^{-1})(t_1^\psi + t_1^{-1}) + (t_0 + t_0^{-1})(Q^{-1}t_1 + Qt_1^{-1}).$$

We now apply $c$ to each side of (20) and evaluate the result. To aid in this evaluation we recall that $c$ swaps $x, y$; also $c$ swaps $z, z'$ since $c^2 = a$ and $a(z) = z$. By these comments and Lemma 4.3 we obtain

$$Qz + Q^{-1}z' + xy = (t_0^\psi + t_0^{-1})(t_1^\psi + t_1^{-1}) + (t_0 + t_0^{-1})(Q^{-1}t_1 + Qt_1^{-1}).$$

□
Theorem 5.2. In the algebra $\hat{H}$ the elements $x, y, z$ are related as follows:

\[
Qxy - Q^{-1}yx + (Q^2 - Q^{-2})z \\
= (Q - Q^{-1})((t_0^\vee + t_1^\vee)(t_1^\vee + t_0^\vee) + (t_0 + t_0^{-1})(Q^{-1}t_1 + Qt_1^{-1})), \\
Qyz - Q^{-1}zy + (Q^2 - Q^{-2})x \\
= (Q - Q^{-1})((t_1^\vee + t_1^\vee)(t_0 + t_0^{-1}) + (t_0^\vee + t_0^\vee)(Q^{-1}t_1 + Qt_1^{-1})), \\
Qzx - Q^{-1}zx + (Q^2 - Q^{-2})y \\
= (Q - Q^{-1})((t_0 + t_0^{-1})(t_0^\vee + t_0^\vee) + (t_1^\vee + t_1^\vee)(Q^{-1}t_1 + Qt_1^{-1})).
\]

Proof. To get the first equation, eliminate $z'$ from the equations of Theorem 5.1. To get the other two equations use the $B_3$ action from Lemma 4.2. Specifically, apply $b$ twice to the first equation and use the data in Lemma 4.3 together with the fact that $b$ cyclically permutes $x, y, z$.

\[\blacksquare\]

Proof of Theorem 2.4. Apply the homomorphism $\hat{H} \to H(k_0, k_1, k_0^\vee, k_1^\vee)$ from Lemma 3.4. Part (i) follows via Corollary 3.9 and parts (ii)–(iv) follow from Theorem 5.2 together with Lemma 2.3.

\[\blacksquare\]

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