Overconvergent de Rham-Witt cohomology for semistable varieties

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Abstract
We define an overconvergent version of the Hyodo-Kato complex for semistable varieties over perfect fields of positive characteristic, and prove that its hypercohomology tensored with $\mathbb{Q}$ recovers the log-rigid cohomology when $Y$ is quasi-projective. We then describe the monodromy operator using the overconvergent Hyodo-Kato complex. Finally, we show that overconvergent Hyodo-Kato cohomology agrees with log-crystalline cohomology in the projective semistable case.

1 Introduction
For a proper and smooth variety $Y$ over a perfect field of characteristic $p > 0$, the hypercohomology of the Deligne-Illusie de Rham-Witt complex tensored with $\mathbb{Q}$ computes - using the comparison isomorphism with crystalline cohomology [Ber97] - the rigid cohomology

$$H^\ast_{\mathrm{rig}}(Y/W(k)[1/p]) \cong \mathbb{H}^\ast(Y, W^\bullet \Omega^\bullet_{Y/k}) \otimes \mathbb{Q}$$

However, rigid cohomology is well-defined without any properness assumption on $Y$. In [DLZ11], Davis-Langer-Zink define an overconvergent de Rham-Witt complex $W^\dagger \Omega^\bullet_{Y/k}$, which is a subcomplex of $W^\bullet \Omega^\bullet_{Y/k}$, and show that

$$H^\ast_{\mathrm{rig}}(Y/W(k)[1/p]) \cong \mathbb{H}^\ast(Y, W^\dagger \Omega^\bullet_{Y/k}) \otimes \mathbb{Q}$$

for $Y$ quasi-projective and smooth over $k$.

On the other hand, one could instead relax the smoothness condition on $Y$. Let $S_0 = \text{(Spec } k, \mathbb{N})$ be the standard log point, and let $Y$ be a fine $S_0$-log scheme. Then Grosse-Klönne [Gro05] defines the log-rigid cohomology $H^\ast_{\log, \rig}(Y/\mathcal{S}_0)$ of $Y$ (we recall the definition in [2]). Grosse-Klönne shows that the log-rigid cohomology of $Y$ agrees with Shiho’s log-convergent cohomology of $Y$ whenever $Y$ is a semistable variety whose irreducible components are proper. In particular, by the comparison between log-convergent and log-crystalline cohomology [Shi02], there is an isomorphism

$$H^\ast_{\log, \rig}(Y/\mathcal{S}_0) \cong \mathbb{H}^\ast(Y, W^\bullet \Omega^\bullet_{Y/k}) \otimes \mathbb{Q}$$

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for proper semistable varieties $Y$ over $k$, where $W\omega^\bullet_{Y/k}$ is the Hyodo-Kato complex \cite{HK94}.

There is, however, currently no Hyodo-Kato style theory available for non-proper semistable varieties. For example, we remark that the lack of a Hyodo-Kato complex for non-proper situations caused some additional difficulties in de Shalit’s proof of the $p$-adic weight-monodromy conjecture for $p$-adically uniformisable varieties (see the comment before Theorem 2.1 in \cite{dS05}). The main result of this paper is to define a suitable overconvergent Hyodo-Kato complex $W^\dagger\omega^\bullet_{Y/k}$ which extends this comparison to non-proper situations. More precisely, we prove

**Theorem 1.1.** Let $Y$ be a quasi-projective semistable variety over $S_0$. Then there is a quasi-isomorphism

$$R\Gamma_{\log-rig}(Y/S_0) \cong R\Gamma(Y, W^\dagger\omega^\bullet_{Y/k} \otimes \mathbb{Q})$$

We then describe the monodromy operator on log-rigid cohomology in terms of the overconvergent Hyodo-Kato complex, using the method of \cite{Mok93}.

In the final part of the paper, we compare the usual and overconvergent Hyodo-Kato cohomology in the proper case:

**Theorem 1.2.** Let $Y$ be a projective semistable variety over $S_0$. Then the canonical map

$$H^*(Y, W^\dagger\omega^\bullet_{Y/k}) \to H^*(Y, W\omega^\bullet_{Y/k}) = H^*_{\log-cris}(Y/W(k))$$

induced by the inclusion $W^\dagger\omega^\bullet_{Y/k} \subset W\omega^\bullet_{Y/k}$ is an isomorphism of finite type $W(k)$-modules.

It should be noted that in fact we give two definitions of the overconvergent Hyodo-Kato complex in this paper. The first is constructed in the style of \cite{HK94}, and the second in the more modern approach of \cite{Mat17}. We prove that the two complexes are the same. Along the way, this has the serendipitous consequence that we show that Matsuue’s log de Rham-Witt complex $W\Lambda^\bullet_{Y/(\mathbb{R},\mathbb{N})}$ gives the Hyodo-Kato complex $W\omega^\bullet_{Y/k}$ in the special case that $R = k$, thus filling a gap in the literature.

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## 2 Log-rigid cohomology

Let $k$ be a field of characteristic $p > 0$, and let $W = W(k)$ be the Witt vectors of $k$ and $K := \text{Frac} W$. We will write $S_0$ for the weak formal log scheme $(\text{Spwf} W, N \to k, 1 \mapsto 0)$. The special fibre of $S_0$ is the standard log point $S_0 = (\text{Spec} k, N \to k, 1 \mapsto 0)$.

We briefly recall Grosse-Klönne’s definition of the log-rigid cohomology of a fine $S_0$-log scheme $Y$. For the details, one should consult \cite{Gro05}. Let $Y = \bigcup_{i \in I} V_i$ be an open covering, and suppose there are exact closed immersions
$V_i \hookrightarrow \mathfrak{P}_i$ into log smooth weak formal $\mathfrak{S}_0$-log schemes for each $i$. For each $H \subset I$, choose an exactification of the diagonal embedding:

$$V_H := \bigcap_{i \in H} V_i \hookrightarrow \mathfrak{P}_H \xrightarrow{f} \prod_{i \in H} \mathfrak{P}_i$$

(this means that $\iota$ is an exact closed immersion and $f$ is log-étale). Then the log de Rham complex $\omega^\bullet_{\mathfrak{P}_H/\mathfrak{S}_0}$ tensored with $\mathbb{Q}$ induces a complex of sheaves $\omega^\bullet_{\mathfrak{P}_H,K}$ on the $K$-dagger space $\mathfrak{P}_H,K$ associated to the generic fibre of $\mathfrak{P}_H$ (see [Gro00]).

Let $\text{sp} : \mathfrak{P}_H,K \to \mathfrak{P}_H$ be the specialisation map, and write $|V_H|_{\mathfrak{P}_H} := \text{sp}^{-1}(V_H)$ for the tubular neighbourhood of $V_H$ in $\mathfrak{P}_H,K$. Then $|V_H|_{\mathfrak{P}_H,K}$ and $\omega^\bullet_{\mathfrak{P}_H,K}|_{|V_H|_{\mathfrak{P}_H}}$ are independent of the choice of exactification above ([Gro05], Lemma 1.2). Now, for $H_1 \subset H_2$, the projections $p_{12} : |V_{H_2}|_{\mathfrak{P}_{H_2}} \to |V_{H_1}|_{\mathfrak{P}_{H_1}}$ induces $p_{12}^* \omega^\bullet_{|V_{H_2}|_{\mathfrak{P}_{H_2}}} \to \omega^\bullet_{|V_{H_1}|_{\mathfrak{P}_{H_1}}}$. This defines a complex of simplicial sheaves $\omega^\bullet_{|V_{H_1}|_{\mathfrak{P}_{H_1}}}$ on a simplicial dagger space $|V_{H_1}|_{\mathfrak{P}_{H_1}}$. The log-rigid cohomology of $Y$ is given by

$$R\Gamma_{\log-rig}(Y/\mathfrak{S}_0) := R\Gamma(|V_{H_1}|_{\mathfrak{P}_{H_1}}, \omega^\bullet_{|V_{H_1}|_{\mathfrak{P}_{H_1}}})$$

Then $R\Gamma_{\log-rig}(Y/\mathfrak{S}_0)$ is independent on the choice of open cover $Y = \bigcup_{i \in I} V_i$ and the choice of embeddings $V_i \hookrightarrow \mathfrak{P}_i$ ([Gro05], Lemma 1.4).

3 The overconvergent Hyodo-Kato complex

Now suppose that $k$ is a perfect field of characteristic $p > 0$. We follow closely the approach of [HK94]. Let $X$ be a regular flat $W$-scheme and write

$$Y \xleftarrow{i} X \xrightarrow{j} X_K$$

for the special and generic fibres of $X$. We suppose that $X$ has semistable reduction, that is to say we suppose that étale locally on $X$, there is a smooth morphism $X \to \text{Spec } W[T_1, \ldots, T_n]/(T_1 \cdots T_d - p)$ for some $n \geq d$. In particular, $X_K$ is smooth and $Y$ is a reduced normal crossings divisor on $X$. If we endow $X$ with the log-structure induced by the special fibre and consider $Y$ with the pullback log structure, then $Y$ is a fine log-smooth $\mathfrak{S}_0$-log scheme. Étale locally on $Y$, the structure morphism factors as

$$Y \xrightarrow{f} (\text{Spec } k[T_1, \ldots, T_n]/(T_1 \cdots T_d), N^d, e_i \mapsto T_i) \xrightarrow{\delta} \mathfrak{S}_0$$

where $f$ is exact and étale and $\delta$ is induced by the diagonal. We say that $Y$ is semistable over $\mathfrak{S}_0$.

Since $k$ is a perfect field, we can find a dense open subscheme $u : U \hookrightarrow Y$ which is smooth over $k$. We may therefore consider the pushforward of
overconvergent de Rham-Witt complex $W^\dagger \Omega^\dagger_{U/k}$ of \cite{DLZ11}. Let
\[ d\log : i^{-1} j_*(\mathcal{O}^\dagger_{X_K}) \to u_* W\Omega^\dagger_{U/k} \]
be the homomorphism considered in \cite[(§1)]{HK94}. Let $W^\dagger(\mathcal{O}_Y)$ denotes the sheaf of overconvergent Witt vectors (see \cite{DLZ12}) on $Y$.

**Definition.** Let $W^\dagger \omega^\dagger_{Y/k}$ be the weakly completed $W^\dagger(\mathcal{O}_Y)$-subalgebra of $u_* W^\dagger \Omega^\dagger_{U/k}$ generated by $dW^\dagger(\mathcal{O}_Y)$ and $d\log(i^{-1} j_*(\mathcal{O}^\dagger_{X_K}))$.

Then $W^\dagger \omega^\dagger_{Y/k}$ is a subcomplex of $W^\dagger(\mathcal{O}_Y)$-subalgebra of $u_* W^\dagger \Omega^\dagger_{U/k}$ and inherits the operators $F$ and $V$ and satisfying the usual de Rham-Witt relations, as in \cite{HK94}.

Now let $u_* W^\dagger \Omega^\dagger_{U/k}[\theta]/(\theta^2)$ be the complex given by adjoining an indeterminate $\theta$ in degree one, subject to $\theta a = (-1)^a a \theta$ for all $a \in u_* W^\dagger \Omega^\dagger_{U/k}$ and $d\theta = 0$. Let
\[ d\log : i^{-1} j_*(\mathcal{O}^\dagger_{X_K}) \to u_* W^\dagger \Omega^\dagger_{U/k}[\theta]/(\theta^2) \]
be the unique homomorphism which induces on $u^{-1} i^{-1}(\mathcal{O}^\dagger_X)$ the composite map
\[ u^{-1} i^{-1} j_*(\mathcal{O}^\dagger_{X_K}) \to \mathcal{O}^\dagger_Y \xrightarrow{d\log} W\Omega^1_{U/k} \]
and induces on $K^\times$ the map $a \mapsto \ord_K(a) \theta$ (again, see \cite[(§1)]{HK94}).

**Definition.** Let $W^\dagger \omega^\dagger_{Y/k}$ be the weakly completed $W^\dagger(\mathcal{O}_Y)$-subalgebra of $u_* W^\dagger \Omega^\dagger_{U/k}[\theta]/(\theta^2)$ generated by $dW^\dagger(\mathcal{O}_Y)$ and $d\log(i^{-1} j_*(\mathcal{O}^\dagger_{X_K}))$.

Then we have a short exact sequence of complexes
\[ 0 \to W^\dagger \omega^\dagger_{Y/k}[-1] \to W^\dagger \omega^\dagger_{Y/k} \to W^\dagger \omega^\dagger_{Y/k} \to 0 \]
\[ a \mapsto a \wedge \theta, \quad \theta \mapsto 0 \]
\[ (1) \]

### 3.1 An equivalent approach

In this section we shall outline another definition of the overconvergent Hyodo-Kato complex, this time in the style of \cite{Mat17}, and we will show that the two definitions are the same. This will become particularly useful in \cite{G}.

In \cite[(§3.4)]{Mat17}, Matsuue defines the log de Rham-Witt complex $W^\Lambda^\bullet_{(S,Q)/(R,P)}$ for any morphism of pre-log rings $(R, P) \to (S, Q)$, where $R$ is a $\mathbb{Z}(p)$-algebra, as the initial object in the category of log $F$-$V$-procomplexes. The construction is a logarithmic generalisation of the construction given in \cite[(§1.3)]{DLZ11}.

Fix integers $n \geq d$ and let $(B := k[T_1, \ldots, T_n], \mathbb{N}^d, e_i \mapsto T_i)$ be considered as a pre-log ring over the trivial base $(k, \{\star\})$. Then one has, in particular, the log de Rham-Witt complex $W^\Lambda^\bullet_{(B, N^d)/(k, \{\star\})}$ as a special case. Any element of $W^\Lambda^\bullet_{(B, N^d)/(k, \{\star\})}$ can be written as a convergent sum of basic log Witt differentials \cite[(Prop. 4.3)]{Mat17}. Matsuue then defines a subcomplex $W^\dagger \Lambda^\bullet_{(B, N^d)/(k, \{\star\})}$ as those elements of $W^\Lambda^\bullet_{(B, N^d)/(k, \{\star\})}$ which are overconvergent \cite[(§10.1)]{Mat17}.

Now consider $(A := k[T_1, \ldots, T_n]/(T_1 \cdots T_d), \mathbb{N}^d, e_i \mapsto T_i)$ as a pre-log ring over $(k, \{\star\})$. Then we have a surjective morphism of pre-log rings over $(k, \{\star\})$:—a
and this induces a morphism of log de Rham-Witt complexes

\[ \lambda : W^\bullet_{(B, N^d)/(k, \{+\})} \to W^\bullet_{(A, N^d)/(k, \{+\})} \]

Matsuue defines \( W^+_{\Lambda^\bullet} \) := \( \lambda \left( W^+_{\Lambda^\bullet_{(B, N^d)/(k, \{+\})}} \right) \). Notice that one could have taken any log polynomial algebra over \( k \) which surjects onto \( (A, N^d) \), but Matsuue shows that \( W^+_{\Lambda^\bullet_{(k, \{+\})}} \) is independent of the choice (see (Mat17, 10.2) and the subsequent discussion). One may then glue to define \( W^+_{\Lambda^\bullet_{Y/S_0}} \) for semistable schemes \( Y \) over \( S_0 \). Finally, we define the overconvergent Hyodo-Kato complex \( W^+_{\Lambda^\bullet_{Y/S_0}} \) to be the image of \( W^+_{\Lambda^\bullet_{Y/(k, \{+\})}} \) under the projection

\[ W^\bullet_{Y/(k, \{+\})} \to W^\bullet_{Y/(k, N)} = W^\bullet_{Y/S_0} \]

of log de Rham-Witt complexes.

**Proposition 3.1.** The overconvergent Hyodo-Kato complex \( W^+_{\omega^\bullet_{Y/k}} \) of the previous section is the same as \( W^1_{\Lambda^\bullet_{Y/S_0}} \).

**Proof.** We may assume that \( Y = \text{Spec} \, k[T_1, \ldots, T_n]/(T_1 \cdots T_d) \). It suffices to show that Matsuue’s log de Rham-Witt complex \( W^\bullet_{\Lambda^\bullet_{Y/S_0}} \) agrees with the Hyodo-Kato complex \( W^\bullet_{\omega^\bullet_{Y/k}} \) of [HK94]. This must be well-known to the experts, but the authors do not know of a proof recorded in the literature; it seems important to reconcile the two approaches, so we give a proof here.

Given a log \( F-V \) procomplex \( \{E^i_m\}_{m \in \mathbb{N}} \), define differential graded ideals

\[ \text{Fil}^s E^i_m := V^s E^i_{m-s} + dV^s E^{i-1}_{m-s} \subset E^i_m \]

Then this gives a filtration of log \( F-V \)-procomplexes which is compatible with \( F, V, d \) and the projections (Mat17, §3.5).

Now, \( W^\bullet_{\Lambda^\bullet_{Y/S_0}} = \left\{ W^\bullet_{m \Lambda^\bullet_{Y/S_0}} \right\}_{m \in \mathbb{N}} \) and \( W^\bullet_{\omega^\bullet_{Y/k}} = \left\{ W^\bullet_{m \omega^\bullet_{Y/k}} \right\}_{m \in \mathbb{N}} \) are log \( F-V \)-procomplexes, so we have a map of log \( F-V \)-procomplexes

\[ W^\bullet_{\Lambda^\bullet_{Y/S_0}} \to W^\bullet_{\omega^\bullet_{Y/k}} \]

by the universal property of \( W^\bullet_{\Lambda^\bullet_{Y/S_0}} \). This map induces diagrams

\[ \begin{array}{c}
0 \to \text{Fil}^m W^\bullet_{m+1 \Lambda^\bullet_{Y/S_0}} \to W^\bullet_{m+1 \Lambda^\bullet_{Y/S_0}} \to W^\bullet_{m \Lambda^\bullet_{Y/S_0}} \to 0 \\
0 \to \text{Fil}^m W^\bullet_{m+1 \omega^\bullet_{Y/k}} \to W^\bullet_{m+1 \omega^\bullet_{Y/k}} \to W^\bullet_{m \omega^\bullet_{Y/k}} \to 0
\end{array} \]

of short exact sequences (see (Mat17, Prop. 3.6) for the top row, and (HK94, §4.9) for the bottom row) for each \( m \in \mathbb{N} \). Now one notices that

\[ W^1_{\Lambda^\bullet_{Y/S_0}} = W^1_{\omega^\bullet_{Y/k}} = \omega^\bullet_{Y/k} \]
is the usual logarithmic de Rham complex, by definition. This gives
\[ \text{Fil}^m W_{m+1}^\bullet_{Y/S_0} = \text{Fil}^m W_{m+1}^\bullet_{Y/k} \]
and then the diagrams give \( W_m^\bullet_{Y/S_0} = W_m^\bullet_{Y/k} \) for all \( m \in \mathbb{N} \).

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4 Comparison with log-Monsky-Washnitzer cohomology

Let \( Y = \text{Spec } A \) be a semistable affine scheme over \( S_0 \). Let \( Y \hookrightarrow Z = \text{Spec } B \) be a closed embedding into a smooth affine \( k \)-scheme such that \( Y \) is a normal crossings divisor on \( Z \), in other words \( A = B/(f_1 \cdots f_r) \) and each \( B/(f_i) \) is smooth. Let \( \tilde{B} \) be a smooth \( W \)-algebra lifting \( B \) (this is always possible by [Elk73]) and set \( \tilde{A} := \tilde{B}/(\tilde{f}_1 \cdots \tilde{f}_r) \) for some liftings \( \tilde{f}_i \in \tilde{B} \) of the \( f_i \), such that \( \tilde{Y} := \text{Spec } \tilde{A} \) is a normal crossings divisor in \( \tilde{Z} \). That is, we have a diagram

\[
\begin{array}{ccc}
Y = \text{Spec } A & \hookrightarrow & Z = \text{Spec } B \\
\| & & \| \\
\tilde{Y} = \text{Spec } \tilde{A} & \hookrightarrow & \tilde{Z} = \text{Spec } \tilde{B}
\end{array}
\]

We define the complexes \( W^\dagger \omega^\bullet_{Y/k} \) and \( W^\dagger \tilde{\omega}^\bullet_{\tilde{Y}/k} \) as in §3. Indeed, \( X = \text{Spec } \tilde{B}/(\tilde{f}_1 \cdots \tilde{f}_r - p) \) is a regular scheme whose special fibre is \( Y \).

Now let \( \Omega^\bullet_{\tilde{Z}/W}^\dagger(\log \tilde{Y}) \) denote the logarithmic de Rham complex of \( \tilde{Z} \) with respect to the normal crossings divisor \( \tilde{Y} \). We set

\[
\tilde{\omega}^\bullet_{\tilde{Y}} := \mathcal{O}_{\tilde{Y}} \otimes_{\mathcal{O}_{\tilde{Z}}} \Omega^\bullet_{\tilde{Z}/W}^\dagger(\log \tilde{Y})
\]

and write

\[
\tilde{\omega}^\bullet_{\tilde{Y}^+} := \mathcal{O}_{\tilde{Y}^+} \otimes_{\mathcal{O}_{\tilde{Z}^+}} \Omega^\bullet_{\tilde{Z}^+/W}^\dagger(\log \tilde{Y}^+) \]

for the weak completion.

**Definition.** The *logarithmic Monsky-Washnitzer complex* of \( Y \) is defined to be

\[
\omega^\bullet_{Y^+} := \tilde{\omega}^\bullet_{\tilde{Y}^+}/(\tilde{\omega}^\bullet_{\tilde{Y}/k}^{-1} \wedge \theta)
\]

where \( \theta := d \log \tilde{f}_1 + \cdots + d \log \tilde{f}_r \). We define

\[
H^*_{\text{log-MW}}(Y/K) := \mathbb{H}^*(Y, \omega^\bullet_{Y^+} \otimes \mathbb{Q})
\]

This is the logarithmic Monsky-Washnitzer cohomology, as discussed in [Gro05, §5]. It is clear that

\[
H^*_{\text{log-MW}}(Y/K) \cong H^*_{\text{log-rig}}(Y/S_0)
\]

and that there is a short exact sequence of complexes

\[
0 \to \omega^\bullet_{Y^+}[-1] \to \tilde{\omega}^\bullet_{\tilde{Y}^+} \to \omega^\bullet_{Y^+} \to 0
\]

\[
a \mapsto a \wedge \theta, \quad \theta \mapsto 0
\]

(2)
In this section we shall construct a morphism of short exact sequences from (2) to (1). We will then prove that the arrows become quasi-isomorphisms after tensoring with \( \mathbb{Q} \).

Let \( \tilde{A}^! \) and \( \tilde{B}^! \) denote the weak completion of \( \tilde{A} \) and \( \tilde{B} \), respectively. Then we have an induced diagram

\[
\begin{array}{c}
\tilde{B}^! \\
\downarrow t_F \\
W(B)
\end{array}
\quad \rightarrow 
\begin{array}{c}
\tilde{A}^! \\
\downarrow t_F \\
W(A)
\end{array}
\]

where the vertical arrows are the Lazard morphisms ([Ill79] 0 1.3.6). Since \( B \) is a smooth finitely generated \( k \)-algebra, \( t_F : \tilde{B}^! \rightarrow W(B) \) has image contained in the overconvergent Witt vectors \( W^!(B) \) ([DLZ11], Prop. 3.2). By functoriality of the Lazard morphisms and that \( W(B) \rightarrow W(A) \) sends \( W^!(B) \) to \( W^!(A) \), we deduce that we in fact have a diagram

\[
\begin{array}{c}
\tilde{B}^! \\
\downarrow t_F \\
W^!(B)
\end{array}
\quad \rightarrow 
\begin{array}{c}
\tilde{A}^! \\
\downarrow t_F \\
W^!(A)
\end{array}
\]

Let \( U \) and \( \tilde{U} \) denote the respective open immersions \( U := Z \setminus Y \hookrightarrow Z \) and \( \tilde{U} := \tilde{Z} \setminus \tilde{Y} \hookrightarrow \tilde{Z} \). Then the map

\[
\tilde{u}_* \Omega^\bullet_{Z/W}(\log \tilde{Y}) \rightarrow u_* \Omega^\bullet_{U/W}(\log Y)
\]

sends logarithmic differentials along \( \tilde{Y} \) to logarithmic differentials along \( Y \). (Here \( [f_i] \in W(\tilde{B}) \) denotes the Teichmüller lift.) In particular, it induces a map

\[
\Omega^\bullet_{Z/W}(\log \tilde{Y}) \rightarrow W^\bullet_{Z/k}(\log Y)
\]

and by the above discussion, this induces a map

\[
\Omega^\bullet_{Z/W}(\log \tilde{Y}) \rightarrow W^\bullet_{Z/k}(\log Y)
\]

These maps were considered in ([Mat17], §10), and become quasi-isomorphisms after tensoring with \( \mathbb{Q} \), by ([Mat17], Lemma 10.9). In any case, this gives a map

\[
\tilde{\omega}^\bullet_{Y^!} \rightarrow W^!(A) \otimes_{W^!(B)} W^\bullet_{Z/k}(\log Y)
\]

(3)

Notice that the logarithmic differentials \( d\log [f_i] \) along \( Y \) coincide with the logarithmic differentials as defined by Hyodo-Kato as the image of \( d\log \) using the regular \( W \)-scheme \( X = \text{Spec } B/(f_1 \cdots f_r - p) \). Indeed \( \tilde{f}_i \) is mapped to \( d\log [f_i] \) and \( p \) is mapped to \( \theta = d\log [f_1] + \cdots + d\log [f_r] \).

Now we recall that one can express the sheaves \( W_n \tilde{\omega}^\bullet_{Y^!} \) as quotients of the \( W_n \Omega^\bullet_{Z/k}(\log Y) \), as in ([Hyo91], §1.6). Indeed, we may assume that \( \tilde{Z} \) is an admissible lifting of \( Y \) (see [Mok93], §2.4 for the definition). Set \( \tilde{Z}_n := \tilde{Z} \times_{W_n} W_n \)
and $\tilde{Y}_n := \tilde{Y} \times_W W_n$, so that $Z = \tilde{Z}_1$ and $Y = \tilde{Y}_1$. Let $I_n \subset \mathcal{O}_{\tilde{Z}_n}$ be the ideal sheaf of the closed immersion $\tilde{Y}_n \hookrightarrow \tilde{Z}_n$. Then

$$W_n \omega_{\tilde{Y}/k} = W_n \Omega_{\tilde{Z}/k}(\log Y)/(I_n \otimes_{\mathcal{O}_{\tilde{Z}_n}} \Omega_{\tilde{Z}_n}/W_n(\log \tilde{Y}_n))$$

where we have identified $W_n \Omega_{\tilde{Z}/k}(\log Y) = \Omega_{\tilde{Z}_n}/W_n(\log \tilde{Y}_n)$. The fact that the right-hand side of (4) gives the same sheaf $W_n \omega_{\tilde{Y}/k}$ as defined in [HK94] is discussed in ([Mok93], §2.6). Passing to the projective limit gives $W \omega_{\tilde{Y}/k}$ as a quotient of $W \Omega_{\tilde{Z}/k}(\log \tilde{Y})$. We therefore deduce a map

$$W(A) \otimes W(B) \Omega_{\tilde{Z}/k}(\log Y) \to W \omega_{\tilde{Y}/k}$$

Since the $d \log [f_i]$ are overconvergent, we may pass to the overconvergent sub-sheaves and get a map

$$W(A) \otimes W(B) \Omega_{\tilde{Z}/k}(\log Y) \to W \omega_{\tilde{Y}/k}$$

Composing with (3) defines a comparison morphism

$$\omega_{\tilde{Y}/k} \to W \omega_{\tilde{Y}/k}$$

(5)

which sends $\theta = d \log \tilde{f}_1 + \cdots d \log \tilde{f}_r$ to $\theta = d \log [f_1] + \cdots + d \log [f_r]$. Since the “divide by $\theta$” projection $W \omega_{\tilde{Y}/k} \to W \omega_{\tilde{Y}/k}$ sends $\theta$ to 0, we get an induced comparison morphism

$$\omega_{\tilde{Y}/k} \to W \omega_{\tilde{Y}/k}$$

(6)

between the logarithmic Monsky-Washnitzer and overconvergent Hyodo-Kato complexes. Moreover, (5) and (6) give a diagram of exact rows

$$0 \to \omega_{\tilde{Y}/k} \to W \omega_{\tilde{Y}/k} \to 0$$

$$0 \to W \omega_{\tilde{Y}/k} \to W \omega_{\tilde{Y}/k} \to 0$$

We will use the weight filtration of Steenbrink to show that the vertical arrows (5) and (6) become quasi-isomorphisms after tensoring with $Q$.

**Theorem 4.1.** The comparison morphisms (5) and (6) induce quasi-isomorphisms

$$\omega_{\tilde{Y}/k} \otimes Q \to W \omega_{\tilde{Y}/k} \otimes Q$$

and

$$\omega_{\tilde{Y}/k} \otimes Q \to W \omega_{\tilde{Y}/k} \otimes Q$$

**Proof.** Recall that the weight filtration $P_j \omega_{\tilde{Y}/k}$ of $\omega_{\tilde{Y}/k}$ (see [Gro05], §5) for it in this context) is defined as

$$P_j \omega_{\tilde{Y}/k} := \text{image} \left( \Omega_{\tilde{Z}_1/W}(\log \tilde{Y}) \otimes \Omega_{\tilde{Z}_1/W}^{\leq j} \to \Omega_{\tilde{Z}_1/W}(\log \tilde{Y}) \right) \otimes_{\mathcal{O}_{\tilde{Z}_1}} \mathcal{O}_{\tilde{Y}_1}$$

Via the Poincaré residue maps, the graded pieces of the filtration are identified as

$$\text{Gr}_j(\omega_{\tilde{Y}/k} \otimes Q) \to \bigoplus_{Y_1 \in \mathcal{M}_j} \Omega_{Y_1}^{\leq j} \otimes [-j]$$

8
where $\mathcal{M}_j$ denotes the collection of all (smooth) intersections of $j$ different components of $Y$ which lift to a smooth intersection of $j$ different liftings in $\tilde{Y}$, and where $\Omega^{\bullet}_{[Y_j]_{\mathbb{Z}^\dagger}}$ denotes the usual de Rham complex on the smooth affinoid dagger space $[Y_j]_{\mathbb{Z}^\dagger}$. By ([Gro05], §5.2), one has an isomorphism of exact sequences

$$0 \longrightarrow \text{Gr}_0(\mathbb{Z}^{\bullet}_{Y_j} \otimes \mathbb{Q}) \xrightarrow{\wedge \theta} \text{Gr}_1(\mathbb{Z}^{\bullet}_{Y_j} \otimes \mathbb{Q})[1] \xrightarrow{\wedge \theta} \text{Gr}_2(\mathbb{Z}^{\bullet}_{Y_j} \otimes \mathbb{Q})[2] \xrightarrow{\wedge \theta} \cdots$$

and

$$0 \longrightarrow \Omega^{\bullet}_{Y_j} \otimes \mathbb{Q} \longrightarrow \bigoplus_{Y_j \in \mathcal{M}_3} \Omega^{\bullet}_{[Y_j]_{\mathbb{Z}^\dagger}} \otimes \mathbb{Q} \longrightarrow \bigoplus_{Y_j \in \mathcal{M}_2} \Omega^{\bullet}_{[Y_j]_{\mathbb{Z}^\dagger}} \otimes \mathbb{Q} \longrightarrow \cdots$$

(the bottom row is exact because the $Y_j$'s are normal crossings intersections).

Similarly, consider the weight filtration of Mokrane ([Mok93]) on $W\tilde{\omega}^{\bullet}_{Y/k}$

$$P_j W\tilde{\omega}^i_{Y/k} := \text{image} \left( W\tilde{\omega}^i_{Y/k} \otimes W\Omega^{i-j}_{Y/k} \to W\tilde{\omega}^i_{Y/k} \right)$$

and set

$$P_j W^j \tilde{\omega}^i_{Y/k} := \text{image} \left( W^j \tilde{\omega}^i_{Y/k} \otimes W^j \Omega^{i-j}_{Y/k} \to W^j \tilde{\omega}^i_{Y/k} \right)$$

By construction, the comparison morphism ([53]) induces maps $P_j \tilde{\omega}^{\bullet}_{Y_j} \to P_j W^j \tilde{\omega}^{\bullet}_{Y/k}$ for each $j$, and therefore respects the weight filtrations. Moreover, we have $P_j W^j \tilde{\omega}^i_{Y/k} = W^j \tilde{\omega}^i_{Y/k} \cap P_j W\tilde{\omega}^i_{Y/k}$ for each $j$. By ([Mok93], 3.7), the graded pieces of the weight filtration are identified, via the Poincaré residue maps, as

$$\text{Gr}_j \tilde{\omega}^i_{Y/k} \simeq \bigoplus_{Y_j \in \mathcal{M}_j} W\Omega^{j-i}_{Y_j/k}[-j]$$

and therefore

$$\text{Gr}_j W^j \tilde{\omega}^i_{Y/k} \simeq \bigoplus_{Y_j \in \mathcal{M}_j} W^j \Omega^{j-i}_{Y_j/k}[-j]$$

Since each $Y_j$ is smooth over $k$, we know by [DLZ11] that $W^j \Omega^{i}_{Y_j/k} \otimes \mathbb{Q}$ is quasi-isomorphic to $\Omega^{i}_{[Y_j]_{\mathbb{Z}^\dagger}}$, and therefore conclude that the comparison morphism ([53]) is a quasi-isomorphism when tensored with $\mathbb{Q}$.

To show that the second comparison morphism induces a quasi-isomorphism after tensoring with $\mathbb{Q}$, define a double complex

$$A^{i,j}_Q := \frac{\tilde{\omega}^{i+j+1}_{Y/k} \otimes \mathbb{Q}}{P_j \tilde{\omega}^{i+j+1}_{Y/k} \otimes \mathbb{Q}}$$

with differential $A^{i,j}_Q \to A^{i+1,j}_Q$ induced by $(-1)^i d$, and the other differential $A^{i,j}_Q \to A^{i,j+1}_Q$ induced by $\omega \mapsto \omega \wedge \theta$. Let $A^{\bullet,\bullet}_Q$ be the total complex of $A^{i,j}_Q$.

Entirely similarly, define another double complex $B^{i,j}_Q$ by

$$B^{i,j}_Q := \frac{W^j \tilde{\omega}^{i+j+1}_{Y/k} \otimes \mathbb{Q}}{P_j W^j \tilde{\omega}^{i+j+1}_{Y/k} \otimes \mathbb{Q}}$$

with the differential $B^{i,j}_Q \to B^{i+1,j}_Q$ induced by $(-1)^i d$ and the differential $B^{i,j}_Q \to B^{i,j+1}_Q$ induced by $\omega \mapsto \omega \wedge \theta$, and let $B^{\bullet,\bullet}_Q$ be the total complex of
$B_Q^{\bullet \bullet}$. Then $A_Q^{\bullet \bullet}$ is quasi-isomorphic to $B_Q^{\bullet \bullet}$, because the graded quotients $\Omega_{\overline{Y}_{/k}}^\bullet$ and $W^+\Omega_{\overline{Y}_{/k}}^\bullet \otimes \mathbb{Q}$ are quasi-isomorphic by the comparison theorem in the smooth case [DLZ11].

Now, the map $\tilde{\omega}_{\overline{Y}_{/k}} \otimes \mathbb{Q} \rightarrow A_Q^{\bullet \bullet}$ induces a quasi-isomorphism $\omega_{\overline{Y}_{/k}} \otimes \mathbb{Q} \simeq A_Q^{\bullet \bullet}$. This is ([Gro05], §5) in this context, but the argument goes back to [Ste76]. The same argument shows that the map $W^+\omega_{\overline{Y}_{/k}} \otimes \mathbb{Q} \rightarrow B_Q^{\bullet \bullet}$ induces a quasi-isomorphism $W^+\omega_{\overline{Y}_{/k}} \otimes \mathbb{Q} \simeq B_Q^{\bullet \bullet}$. As we already noted that $A_Q^{\bullet \bullet} \simeq B_Q^{\bullet \bullet}$, we conclude that the comparison morphism (6) is a quasi-isomorphism when tensored with $\mathbb{Q}$.

**Corollary 4.2.** Let $Y$ be a semistable affine scheme over $S_0$. Then there is a canonical isomorphism

$$H^*_\text{log-rig}(Y/\mathcal{S}_0) \cong H^*(Y, W^+\omega^\bullet_{\overline{Y}_{/k}} \otimes \mathbb{Q})$$

**Proof.** We showed that $H^*(Y, W^+\omega^\bullet_{\overline{Y}_{/k}} \otimes \mathbb{Q}) \cong H^*_{\text{log-MW}}(Y/K)$. The comparison between log-Monsky-Washnitzer cohomology and log-rigid cohomology is more or less by definition (see [Gro05], §5.2)).

## 5 Comparison with log-rigid cohomology

Our aim in this section is to globalise the comparison isomorphism between log-rigid and overconvergent Hyodo-Kato cohomology. We note here that given a $W$-scheme $X$, we shall always write $\hat{X}$ for the formal completion of $X$ along the special fibre, and $\hat{X}_K$ for the associated rigid analytic generic fibre.

**Definition.** Let $Y = \text{Spec } A$ be a semistable affine scheme over $S_0$. A semistable frame for $Y$ is the data of a normal crossings divisor

$$G = \text{Spec } C \hookrightarrow F = \text{Spec } B$$

of affine $W$-schemes where $F$ is smooth over $W$ and $Y \hookrightarrow G_k := G \times_W k$ is a closed $k$-immersion. Note that if $(F,G)$ is a semistable frame for $Y$, then $F$ is a special frame for $Y$ in the sense of [DLZ11] Definition 4.1.

**Definition.** An overconvergent semistable frame for $Y = \text{Spec } A$ is the data of a semistable frame $(F = \text{Spec } B, G = \text{Spec } C)$ for $Y$ along with a homomorphism $\varphi : C \rightarrow W^+(A)$ which lifts the comorphism $C \rightarrow A$ of the closed $W$-immersion $Y \hookrightarrow G$. 

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Let $Y = \text{Spec } A$ be a semistable affine scheme over $S_0$ and suppose that $(F, G, \omega)$ is an overconvergent semistable frame for $Y$. Choose an embedding $F \hookrightarrow \mathbb{P}$ into a proper smooth $W$-scheme and write $\overline{G}$ and $\overline{F}$ for the respective closures of $G$ and $F$ inside $\mathbb{P}$. Let $\overline{G}_k$ and $\overline{F}_k$ for the special fibres of $\overline{G}$ and $\overline{F}$, and let $\overline{V}$ be the closure of $V$ inside $\overline{G}_k$. Since $\overline{G}_K$ is defined in $\overline{F}_K$ by overconvergent functions, we can extend the normal crossings divisor $\overline{G}_K \hookrightarrow \overline{F}_K$ to a normal crossings divisor $V' \hookrightarrow V$ where $V$ is a strict neighbourhood of $\overline{F}_K$ in $\overline{F}_k$ and $V'$ is a strict neighbourhood of $\overline{G}_K$ in $\overline{G}_k$. We get the following diagram of strict neighbourhoods:

\[
\begin{array}{ccc}
\hat{F}_K & \subseteq & V \\
\downarrow & & \downarrow \\
\hat{G}_K & \subseteq & V' \\
\downarrow & & \downarrow \\
|Y|_{\hat{G}} & \subseteq & \hat{V} := V' \cap \overline{V}_|φ \subseteq |\overline{V}|_φ
\end{array}
\]

In order to define the comparison morphism, we will find it useful to have a rigid analytic description of log-rigid cohomology in terms of sheaves on strict neighbourhoods, in the style of Berthelot. Let $\hat{\omega}_Y^\bullet$ be the complex given by the restriction of $\Omega^\bullet(\log V') \otimes \mathcal{O}_{Y'}$ to $\hat{V}$, and $\omega_{\hat{V}}^\bullet := \hat{\omega}_Y^\bullet/((\hat{\omega}_Y^\bullet)^{-1} \wedge \theta)$ where $\theta = d\log f_1 + \cdots + d\log f_s$ for the functions $f_i$ cutting out the normal crossings divisor $\hat{V}$ in $V$. We claim that we have the following Berthelot-style interpretation of log-rigid cohomology:

**Lemma 5.1.**

\[ R\Gamma(|Y|_{\hat{G}}^{|G}, \hat{\omega}_{Y}^\bullet|_{|G}) = R\Gamma(\hat{V}, j^!\hat{\omega}_{\hat{V}}^\bullet) \]

and

\[ R\Gamma_{\text{log-rig}}(Y/\mathbb{S}_0) := R\Gamma(|Y|_{\hat{G}}^{|G}, \omega_{\hat{V}}^\bullet) = R\Gamma(\hat{V}, j^!\omega_{\hat{V}}^\bullet) \]

**Proof.** We shall only prove the first statement since the second is proved using exactly the same argument.

In order to prove the lemma it suffices to prove that

\[ R\Gamma(|Y|_{\hat{G}}^{|G}, \hat{\omega}_{Y}^\bullet|_{|G}) \cong R\Gamma(|\overline{V}|_{\overline{G}}^{|G}, j^!\hat{\omega}_{\hat{V}}^\bullet|_{|\overline{V}|}) \]

Let $\hat{\omega}_{Y}^\bullet|_{|G}$ be the log de Rham complex of the log morphism $|\overline{V}|_{\overline{G}}^{|G} \to (\text{Sp}^1 K, \{\ast\})$. Since $|\overline{V}|_{\overline{G}}^{|G}$ is a partially proper rigid space and $\omega_{\overline{V}}^\bullet|_{|\overline{V}|}$ is a coherent $\mathcal{O}_{\overline{V}}$-module, we have canonical isomorphisms

\[ H^j(|\overline{V}|_{\overline{G}}^{|G}, j^!\hat{\omega}_{\hat{V}}^\bullet|_{|\overline{V}|}) \cong H^j(|\overline{Y}|_{\overline{G}}^{|G}, \hat{\omega}_{Y}^\bullet|_{|\overline{Y}|}) \]

for all $i, j$ by [Gro00] Theorem 5.1(a). But $|\overline{Y}|_{\overline{G}}^{|G} = Y|_{\overline{G}}$ and therefore $|\overline{Y}|_{\overline{G}}^{|G} = Y|_{\overline{G}}$ too. Hence

\[ H^3(|\overline{Y}|_{\overline{G}}^{|G}, j^!\hat{\omega}_{\hat{V}}^\bullet|_{|\overline{Y}|}) \cong H^3(|\overline{Y}|_{\overline{G}}^{|G}, \hat{\omega}_{Y}^\bullet|_{|\overline{Y}|}) \]

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for all $i,j$. We conclude from the first hypercohomology spectral sequence that
\[ \mathrm{R}^\dagger \Gamma(Y, j^! \bar{\omega}^\bullet_{\nu}) \cong \mathrm{R}^\dagger \Gamma(Y, \bar{\omega}^\bullet_{\nu}) \]
as required.

The second statement is proved with exactly the same argument, but where one instead considers the log de Rham complex with respect to the base $(K, \log)$. \[\square\]

In [DLZ11] §4, there is constructed an explicit fundamental system of strict neighbourhoods $V_{\lambda, \eta}$ of $|Y|_G$ in $|\hat{Y}|_{\hat{G}}$ (here $0 < \lambda, \eta < 1$), canonical morphisms ([DLZ11], pp 251)
\[ \Gamma(V_{\lambda, \eta}, j^! \mathcal{O}_{V_{\lambda, \eta}}) \to W^\dagger(A) \otimes \mathbb{Q} \]
and therefore morphisms
\[ \Gamma(V, j^! \mathcal{O}_V) \to W^\dagger(A) \otimes \mathbb{Q} \]
for any strict neighbourhood $V$ of $|Y|_G$ in $|\hat{Y}|_{\hat{G}}$. The universal property of the de Rham complex then gives a map
\[ \Gamma(V, j^! \bar{\omega}^\bullet_{\nu}) \to u_* W^\dagger \Omega^\bullet_{U/k}(\theta)/\theta^2 \otimes \mathbb{Q} \]
where $u : U \hookrightarrow Y$ is the smooth locus of $Y$, and this clearly factors through
\[ \Gamma(V, j^! \bar{\omega}^\bullet_{\nu}) \to W^\dagger \bar{\omega}^\bullet_{U/k} \otimes \mathbb{Q} \]  \tag{7}
The argument used after [DLZ11] (4.28) can be used verbatim to show that this factors through a morphism
\[ \mathrm{R}^\dagger \Gamma(\bar{V}, j^! \bar{\omega}^\bullet_{\nu}) \to W^\dagger \bar{\omega}^\bullet_{U/k} \otimes \mathbb{Q} \]  \tag{8}
Indeed, given another strict neighbourhood $\bar{V}' \subset \bar{V}$ write $\alpha_{\bar{V}'} : \bar{V}' \cap V_{\lambda, \eta} \hookrightarrow V_{\lambda, \eta}$ for the inclusion. Then by the definition of $j^!$ we have
\[ j^! \bar{\omega}^\bullet_{V_{\lambda, \eta}} = \varprojlim_{\bar{V}'} \alpha_{\bar{V}'}^* \bar{\omega}^\bullet_{\nu} \cap V_{\lambda, \eta} \]
where the direct limit runs over all strict neighbourhoods $\bar{V}' \subset \bar{V}$. Therefore
\[ \mathrm{R}^\dagger \Gamma(V_{\lambda, \eta}, j^! \bar{\omega}^\bullet_{V_{\lambda, \eta}}) = \mathrm{R}^\dagger \Gamma(V_{\lambda, \eta}, \lim_{\bar{V}'} \alpha_{\bar{V}'}^* \bar{\omega}^\bullet_{\nu} \cap V_{\lambda, \eta}) \cong \lim_{\bar{V}'} \mathrm{R}^\dagger \Gamma(V_{\lambda, \eta}, \alpha_{\bar{V}'}^* \bar{\omega}^\bullet_{\nu} \cap V_{\lambda, \eta}) \]
where the isomorphism is by the quasicompactness of $V_{\lambda, \eta}$. Now for each $\bar{V}'$ find $V_{\lambda', \eta}$ such that $V_{\lambda', \eta} \subset \bar{V}' \cap V_{\lambda, \eta}$. Then since each $V_{\lambda', \eta}$ is affinoid, we have the morphism
\[ \mathrm{R}^\dagger \Gamma(V_{\lambda, \eta}, j^! \bar{\omega}^\bullet_{V_{\lambda, \eta}}) \cong \varprojlim_{\lambda'} \mathrm{R}^\dagger \Gamma(V_{\lambda', \eta}, \bar{\omega}^\bullet_{V_{\lambda', \eta}}) \cong \varprojlim_{\lambda'} \mathrm{R}^\dagger \Gamma(V_{\lambda', \eta}, \bar{\omega}^\bullet_{V_{\lambda', \eta}}) \to W^\dagger \bar{\omega}^\bullet_{U/k} \otimes \mathbb{Q} \]
Precomposing with the restriction $\mathrm{R}^\dagger \Gamma(\bar{V}, j^! \bar{\omega}^\bullet_{\nu}) \to \mathrm{R}^\dagger \Gamma(V_{\lambda, \eta}, j^! \bar{\omega}^\bullet_{V_{\lambda, \eta}})$ then gives the desired morphism.

If $f_1, \ldots, f_s$ define the normal crossings divisor $\bar{V}$ in $V$ then $f_1 \cdots f_s = 0$, and hence $f_1 \cdots f_s = 0$. Therefore $d \log[f_1] + \cdots + d \log[f_s] = 0$ and the morphism \[\square\] induces a morphism
\[ \mathrm{R}^\dagger \Gamma_{\log, \mathcal{O}_0}(Y, \mathcal{O}) = \mathrm{R}^\dagger \Gamma(\bar{V}, j^! \omega^\bullet_{\nu}) \to W^\dagger \omega^\bullet_{U/k} \otimes \mathbb{Q} \]  \tag{9}
**Proposition 5.2.** The morphisms (8) and (9) for overconvergent semistable frames are isomorphisms in the derived category and do not depend on the choice of overconvergent semistable frame for $Y$.

**Proof.** We first prove the independence assertion. Let $(F, G, \kappa)$ and $(F', G', \kappa')$ be two overconvergent semistable frames for $Y$. Let

$$F \leftarrow^{\text{pr}_1} F \times_W F' \rightarrow^{\text{pr}_2} F'$$

be the projections. Then the product

$$(F'', G'', \kappa'') := (F \times_W F', \text{pr}_1^{-1}(G) + \text{pr}_2^{-1}(G'), \kappa \otimes \kappa')$$

is another overconvergent semistable frame for $Y$. Choose strict neighbourhoods $\tilde{V}, \tilde{V}'$ and $\tilde{V}''$ such that $\tilde{V}''$ is sent to $\tilde{V}$ and $\tilde{V}'$ by the respective projections. By functoriality, the projections induce diagrams

$$\begin{array}{ccc}
\text{R} \Gamma(\tilde{V}, j^!\omega_{Y/k}) & \xrightarrow{\text{pr}_1^*} & \text{R} \Gamma(\tilde{V}'', j^!\omega_{Y/k}) \\
& & \downarrow \quad \text{pr}_2^* \\
W^!\omega_{Y/k} & \rightarrow & \text{R} \Gamma(\tilde{V}', j^!\omega_{Y/k})
\end{array}$$

and

$$\begin{array}{ccc}
\text{R} \Gamma(\tilde{V}, j^!\omega_{Y/k}) & \xrightarrow{\text{pr}_1^*} & \text{R} \Gamma(\tilde{V}'', j^!\omega_{Y/k}) \\
& & \downarrow \quad \text{pr}_2^* \\
W^!\omega_{Y/k} & \rightarrow & \text{R} \Gamma(\tilde{V}', j^!\omega_{Y/k})
\end{array}$$

and we see therefore that the morphisms (8) and (9) do not depend on the choice of overconvergent semistable frame for $Y$.

To prove that the morphisms are isomorphisms, since we have already shown independence, we may as well work with the log-Monsky-Washnitzer frame $(\tilde{Z}, Y, \kappa)$ as in §4 for the overconvergent semistable frame for $Y$ (remember that $Y$ is affine). We then conclude by Theorem 4.1.

**Theorem 5.3.** Let $Y$ be a quasi-projective semistable scheme over $S_0$. Then the overconvergent Hyodo-Kato complex computes the log-rigid cohomology of $Y$:

$$\text{R} \Gamma_{\text{log-rig}}(Y/S_0) \cong \text{R} \Gamma(Y, W^!\omega_{Y/k} \otimes Q)$$

**Proof.** Let $Y = \bigcup_{i \in I} Y_i$ be an open covering, and for $J = \{i_0, \ldots, i_t\} \subset I$, let $Y_J := Y_{i_0} \cap \cdots \cap Y_{i_t}$. By choosing a possibly finer covering, we may assume that $Y_J = \text{Spec} A_J$ is affine and that $A_J = (A_{i_0})_f$ for some element $\bar{g} \in A_{i_0}$, where $Y_{i_0} = \text{Spec} A_{i_0}$ (compare the argument in [DLZ11, Def. 4.33]). For each $i \in J$, choose a smooth affine $k$-scheme $X_i = \text{Spec} B_i$ such that $Y_i$ is a normal crossings divisor in $X_i$. We may assume that each $X_i$ is standard smooth in the sense of [DLZ11, Def. 4.33]). We may also assume that $X_J := \bigcap_{i \in J} X_i = \text{Spec} B_J$ with $B_J = (B_{i_0})_f$ for some $f$ lifting $\bar{g}$ and such that $Y_J$ is a normal crossings
divisor in $X_j$. Let $F_i$ be a smooth affine $W$-scheme lifting $X_i$, which is again standard smooth, and let $Z_i$ be a lifting over $W$ of $Y_i$ which is a normal crossings divisor in $F_i$ (compare with [Kat96], Prop. 11.3)).

Now let $F_{i_0} = \text{Spec} \  \tilde{B}_{i_0}$ and $F'_{i_0} = \text{Spec} \ (\tilde{B}_{i_0})_f$ for some lifting $f$ of $\tilde{f}$, and similarly let $Z_{i_0} = \text{Spec} \ \tilde{A}_{i_0}$ and $Z'_{i_0} = \text{Spec} \ (\tilde{A}_{i_0})_g$ for some lifting $g$ of $\tilde{g}$. Set

$$E := \prod_{i \neq i_0} F_i$$

Then, by the strong fibration theorem, the special frames $(X_J, F_{i_0} \times E)$ and $(X_J, F'_{i_0} \times E)$, resp. $(Y_J, F_{i_0} \times E)$ and $(Y_J, F'_{i_0} \times E)$, have isomorphic dagger spaces. See the argument in the proof of ([DLZ11], Prop. 4.35). Since $E$ is standard smooth, we can choose an étale covering $E \to \mathbb{A}_W^n$ for some $n$. Again by the strong fibration theorem, the dagger spaces associated to $(X_J, F_{i_0} \times E)$ and $(X_J, F'_{i_0} \times \mathbb{A}_W^n)$, resp. $(Y_J, F_{i_0} \times E)$ and $(Y_J, F'_{i_0} \times \mathbb{A}_W^n)$, are isomorphic. By the coordinate change argument in the proof of ([DLZ11], Prop. 4.35), we may assume that the map $X_J \to \mathbb{A}_W^n$ factors through the zero section $\text{Spec} \ k \to \mathbb{A}_W^n$. Hence the dagger space associated to $(X_J, F_{i_0} \times \mathbb{A}_W^n)$ is isomorphic to $\bar{Q} \times \bar{D}^n$, where $\bar{D}$ is the open unit dagger disk and $\bar{Q}$ is the dagger space associated to the special frame $(X_J, F'_{i_0})$. It contains the dagger space $Q$ associated to the special frame $(Y_J, F'_{i_0})$ as an open subspace, and such that the dagger space associated to $(Y_J, F'_{i_0} \times \mathbb{A}_W^n)$ is $Q \times \bar{D}^n =: \left[ Y_J \right]_{F'_{i_0} \times \mathbb{A}_W^n}$, where $F'_{i_0} \times \mathbb{A}_W^n$ denotes the weak formal completion of $F'_{i_0} \times \mathbb{A}_W^n$ along $p$.

Now consider the embeddings

$$Y_J \hookrightarrow \prod_{i \in J} Z_i \hookrightarrow \prod_{i \in J} (Z_i \times \prod_{j \neq i} F_j) \hookrightarrow F_{i_0} \times E$$

and

$$Y_J \hookrightarrow Z'_{i_0} \times \prod_{i \neq i_0} Z_i \hookrightarrow \sum_{i \in J} (Z'_i \times \prod_{j \neq i} F_j) \hookrightarrow F'_{i_0} \times E$$

where

$$Z'_i := \begin{cases} Z_i & \text{if } i \neq i_0 \\ Z'_{i_0} & \text{if } i = i_0 \end{cases}$$

and likewise for $F'_i$. Note that

$$D_J := \sum_{i \in J} (Z'_i \times \prod_{j \neq i} F_j) = (Z'_i \times \prod_{j \neq i} F_j) + \sum (F'_{i_0} \times Z_i \times \prod_{j \neq i_0} F_j)$$

is a normal crossings divisor in $F'_{i_0} \times E$, and $\left[ Y_J \right]_{D_J}$ is a normal crossings divisor in $\left[ Y_J \right]_{F'_{i_0} \times E}$. Applying the strong fibration theorem and coordinate change argument as above, we get a commutative diagram of dagger spaces

$$\begin{array}{ccc} \left[ Y_J \right]_{D_J} & \longrightarrow & \left[ Y_J \right]_{F'_{i_0} \times E} \\ \downarrow & & \downarrow \\ M_J & \longrightarrow & Q \times \bar{D}^n \end{array}$$

$$\begin{array}{ccc} \left[ X_J \right]_{F'_{i_0} \times E} \quad \bigg| & \bigg| & \bigg| \\ \downarrow & & \downarrow \\ \bar{Q} \times \bar{D}^n \quad \bigg| & \bigg| & \bigg| \\ \downarrow & & \downarrow \\ \bar{Q} \times \bar{D}^n \end{array}$$
where the dagger space $M_J$, which is a normal crossings divisor in $Q \times \hat{D}^n$, is a sum of normal crossings divisors of the following form:

(a) $[Y_J]_{\hat{D}_1}^! \times \hat{D}^n$, where $[Y_J]_{\hat{D}_1}^! \times \hat{D}^n$ is a normal crossings divisor in $Q$

(b) $Q \times \hat{D}^n(m)$, where $\hat{D}^n(m)$ is the divisor in $\hat{D}^n$ corresponding to $\text{Spec } K(T_1, \ldots, T_n)/(T_1 \cdots T_m)$

Let $\omega_{M_J}^\bullet$ denote the logarithmic de Rham complex on the normal crossings divisor $M_J$ in $Q \times \hat{D}^n$, as defined in [Gro05]. Then for the case (a) we have a map

$$\omega_{[Y_J]_{\hat{D}_1}^{\ldots} \times D^n} = \omega_{[Y_J]_{\hat{D}_1}^{\ldots}} \otimes \Omega^\bullet_{\hat{D}^n} \to W^+ \omega_{Y_J/k} \otimes Q$$

where $\Omega^\bullet_{\hat{D}^n}$ is the usual (non-logarithmic) de Rham complex on $\hat{D}^n$, and where the first map is the projection and the second comes from the comparison between the log-Monsky-Washnitzer cohomology and overconvergent Hyodo-Kato cohomology. For the case (b) we have a map

$$\omega_{Q \times \hat{D}^n(m)} = \omega_{[Y_J]_{\hat{D}_1}^{\ldots} \times \hat{D}^n(m)} = \Omega^\bullet_{[Y_J]_{\hat{D}_1}^{\ldots}} \otimes \omega^\bullet_{\hat{D}^n(m)}$$

$$\to \Omega^\bullet_{[Y_J]_{\hat{D}_1}^{\ldots}} \to W^+ \Omega^\bullet_{Y_J/k} \otimes Q \to W^+ \omega^\bullet_{Y_J/k} \otimes Q$$

where $\Omega^\bullet_{Y_J}$ is the usual de Rham complex on $[Y_J]_{\hat{D}_1}^{\ldots}$ and the first map is again the projection. Let $s_p : [Y_J]_{\hat{D}_1}^{\ldots} = M_J \to Y_J$ be the specialisation map. Then by the argument ([DLZ11], 4.32), we have morphisms of complexes of Zariski sheaves on $Y_J$

$$s_p^\ast \omega_{[Y_J]_{\hat{D}_1}^{\ldots} \times D^n} \to W^+ \omega_{Y_J/k} \otimes Q$$

and

$$s_p^\ast \omega_{Q \times \hat{D}^n(m)} \to W^+ \omega_{Y_J/k} \otimes Q$$

which give rise to a morphism

$$s_p^\ast \omega_{[Y_J]_{\hat{D}_1}^{\ldots}} = s_p^\ast \omega_{M_J} \to W^+ \omega_{Y_J/k} \otimes Q \quad (10)$$

into the overconvergent Hyodo-Kato complex (tensored with $Q$) of $Y_J$.

We claim, in analogy to ([DLZ11], Cor. 4.38) that the canonical morphisms

$$s_p^\ast \omega_{M_J} \to R s_p^\ast \omega_{M_J}$$

are quasi-isomorphisms. For now we will assume this claim; the proof is postponed until Proposition 5.3. Then (10) together with the claim gives a morphism

$$R s_p^\ast \omega_{M_J} \to W^+ \omega_{Y_J/k} \otimes Q \quad (11)$$

which is an isomorphism by Proposition 5.2.
Now, as we range through the subsets $J \subset I$, we get an augmented simplicial $k$-scheme $\theta : Y_\bullet := \{Y_J\}_{J \subset I} \to Y$. We also get a simplicial object of special frames $\{(Y_J, D_J)\}_{J \subset I}$, and this gives rise to a simplicial object of dagger spaces

$$M_\bullet := \left\{Y_{J[I]} \right\}_{J \subset I} = \{M_J\}_{J \subset I}$$

The quasi-isomorphisms glue to give a quasi-isomorphism of simplicial complexes on $Y_\bullet$

$$\text{Rsp}_* \omega_{M_\bullet}^\bullet \cong W^\dagger \omega_{Y^/k}^\bullet \otimes \mathbb{Q}$$

(12)

Therefore

$$\text{R}_\theta_* \text{Rsp}_* \omega_{M_\bullet}^\bullet \cong \text{R}_\theta_* W^\dagger \omega_{Y^/k}^\bullet \otimes \mathbb{Q} \cong W^\dagger \omega_{Y^/k}^\bullet \otimes \mathbb{Q}$$

and we deduce that

$$R\Gamma_{\log-rig}(Y/S_0) = R\Gamma(Y, \text{R}_\theta_* \text{Rsp}_* \omega_{M_\bullet}^\bullet) \cong R\Gamma(Y, W^\dagger \omega_{Y^/k}^\bullet \otimes \mathbb{Q})$$

as desired.

It therefore remains to prove the following proposition:

**Proposition 5.4.** Let $M_J$ be the dagger space considered in the proof of Theorem 5.3. Then the canonical morphism

$$\text{sp}_* \omega_{M_J}^\bullet \to \text{Rsp}_* \omega_{M_J}^\bullet$$

is a quasi-isomorphism.

The proof will occupy us for the rest of the section. By using the Mayer-Vietoris exact sequence, it is easy to see that it suffices to prove the proposition separately for the two cases (a) and (b) above. That is, it suffices to prove that

$$\text{sp}_* \omega_{M_J}^\bullet \to \text{Rsp}_* \omega_{M_J}^\bullet$$

are quasi-isomorphisms. We recall from the proof of Theorem 5.3 that we have

$$\omega_{Y_J[D_\infty]}^\bullet \otimes \Omega_{D_\infty}^n$$

and

$$\omega_{Q[D_\infty]}^\bullet \otimes \Omega_{D_\infty}^n$$

are quasi-isomorphisms. The proof for case (a) is easy. Indeed, in Proposition 4.37 and Corollary 4.38 of [DLZ11], it is not needed that $Q$ is a smooth affinoid dagger space. What is needed is that $\Omega_Q^p$ is a locally free $O_Q$-module and that $H^i(Q, \Omega_Q^p)$ vanishes for $i > 0$ (Tate-acyclicity). Both properties hold for the locally free $(\tilde{A}_0)_{D_\infty}$-module $\omega_{Y_J[D_\infty]}^\bullet$ as well, indeed $H^i(Y_J[D_\infty], \omega_{Y_J[D_\infty]}^\bullet) = 0$ for $i > 0$ because $Y_J[D_\infty]$ is affinoid. Hence we can replace $Q$ by $Y_J[D_\infty]$ and $\Omega_Q^p$ by $\omega_{Y_J[D_\infty]}^p$, in the
proofs of Proposition 4.37, Corollary 4.38 and Lemmas 4.44−4.47 in [DLZ11] to obtain the desired quasi-isomorphism

\[ \mathbb{R}sp_\ast \omega_{Y_j}[n] \times D^n \cong sp_\ast \omega_{Y_j}[n] \times D^n \]

Now we will treat case (b), which is more subtle. Since \( Q \) is an open subspace in the smooth affinoid dagger space \( \tilde{Q} \), it is enough to show that

\[ \mathbb{R}sp_\ast \omega_{\tilde{Q} \times D^n(m)} = sp_\ast \omega_{\tilde{Q} \times D^n(m)} \]

Note that we have

\[ \omega_{\tilde{Q} \times D^n(m)} = \Omega_{\tilde{Q}} \otimes \omega_{\tilde{D}^n(m)} \]

We have analogues of Lemma 4.45 and Lemma 4.47 of [DLZ11]:

**Lemma 5.5.** Let \( Q = Sp^\dagger A \) be a smooth affinoid dagger space and \( D^n(m) \) the normal crossings divisor on the closed unit dagger \( n \)-ball \( \tilde{D}^n \) associated to \( \text{Spec} K^{\dagger}⟨T_1, \ldots, T_n⟩/⟨T_1 \cdots T_m⟩ \). Let

\[ \Lambda_n := (A \otimes K \omega_{D^n(m)}^0 \rightarrow A \otimes K \omega_{D^n(m)}^1 \rightarrow A \otimes K \omega_{D^n(m)}^2 \rightarrow \cdots) \]

be the complex with obvious differential. Let

\[ d_t := \dim H^t(\omega_{K^{\dagger}}(T_1, \ldots, T_n)/⟨T_1 \cdots T_m⟩) \]

be the dimension of the log-Monsky-Washnitzer cohomology of \( k[T_1, \ldots, T_n]/⟨T_1 \cdots T_m⟩ \). Then \( \Lambda_n \) is quasi-isomorphic to the complex (with zero differentials)

\[ A \rightarrow A^{d_1} \rightarrow A^{d_2} \rightarrow \cdots \]

**Lemma 5.6.** With the same notation as above, let

\[ \hat{\Lambda}_n := (A \otimes K \omega_{\tilde{D}^n(m)}^0 \rightarrow A \otimes K \omega_{\tilde{D}^n(m)}^1 \rightarrow A \otimes K \omega_{\tilde{D}^n(m)}^2 \rightarrow \cdots) \]

be the analogous complex for \( \tilde{D}^n \) and its closed normal crossings divisor \( \tilde{D}^n(m) \). Then \( \hat{\Lambda}_n \) is quasi-isomorphic to the complex

\[ A \rightarrow A^{d_1} \rightarrow A^{d_2} \rightarrow \cdots \]

We can now follow the proof of ([DLZ11], Prop. 4.37). Let \( \tilde{D}^n = \bigcup_{i=1}^{\infty} U_i \) be a union of dagger balls of ascending radius, and let \( \tilde{D}^n(m) = \bigcup_{i=1}^{\infty} U_i(m) \) be the corresponding normal crossings divisors. For notational brevity, write \( \omega^\dagger := \omega_{\tilde{Q} \times U_i(m)}^0 \). Since \( \tilde{Q} \times U_i(m) \) is affinoid, \( H^p(\tilde{Q} \times U_i(m), \omega^\dagger) \) vanishes for \( p \geq 1 \) and \( R\Gamma(\tilde{Q} \times \tilde{D}^n(m), \omega^\dagger) \) is quasi-isomorphic to the global sections of the complex

\[ \prod_{i=1}^{\infty} \omega^\dagger(\tilde{Q} \times U_i(m)) \rightarrow \prod_{i=1}^{\infty} \omega^\dagger(\tilde{Q} \times U_i(m)) \prod s_i \rightarrow \prod (s_i - s_{i+1}) \]
Note that $\omega_{Q \times U_i(m)}^q = \bigoplus \Omega^p_{Q} \otimes_K \omega_{U_i(m)}^{-p}$. Then the complex $H^0(\tilde{Q} \times U_i(m), \omega_{Q \times U_i(m)}^*)$ is represented by the double complex with components

$$C^p,q(U_i(m)) = H^0(\tilde{Q}, \Omega^p_Q) \otimes_K H^0(U_i(m), \omega^q)$$

Therefore the morphism of double complexes

$$\prod_{i=1}^{\infty} C^{\bullet \bullet}(U_i(m)) \to \prod_{i=1}^{\infty} C^{\bullet \bullet}(U_i(m))$$

given on the $(p, q)$-entry by

$$\prod_{i=1}^{\infty} C^{p,q}(U_i(m)) \to \prod_{i=1}^{\infty} C^{p,q}(U_i(m))$$

$$\prod_{i} s_i \mapsto \prod_{i} (s_i - s_{i+1})$$

induces a map of total complexes with kernel and cokernel $H^0(\tilde{Q} \times \tilde{D}^n(m), \omega_{Q \times D^n(m)}^*)$ and $H^1(\tilde{Q} \times \tilde{D}^n(m), \omega_{Q \times D^n(m)}^*)$, respectively. It follows from Lemma 5.3 that the total complex associated to $C^{\bullet \bullet}(U_i(m))$ is quasi-isomorphic to

$$\bigoplus_{t} (H^0(\tilde{Q}, \Omega_{Q}^*))^{d_t}$$

with the correction $d_0 = 1$. Analogously, it follows from Lemma 5.6 that $H^0(\tilde{Q} \times \tilde{D}^n(m), \omega_{Q \times D^n(m)}^*)$ is quasi-isomorphic to

$$\bigoplus_{t} (H^0(\tilde{Q}, \Omega_{Q}^*))^{d_t} = \left( \bigoplus_{t} (H^0(\tilde{Q}, \Omega_{Q}^*))^{d_t} \to \bigoplus_{t} (H^0(\tilde{Q}, \Omega_{Q}^1))^{d_t} \to \cdots \right)$$

Finally, $H^1(\tilde{Q} \times \tilde{D}^n(m), \omega_{Q \times D^n(m)}^*)$ is quasi-isomorphic to the total complex of the triple complex

$$H^0(\tilde{Q} \times \tilde{D}^n(m), \Omega_{Q}^* \otimes \omega_{D^n(m)}^*) \to \prod_{i=1}^{\infty} C^{\bullet \bullet}(U_i(m)) \to \prod_{i=1}^{\infty} C^{\bullet \bullet}(U_i(m))$$

which is quasi-isomorphic to the total complex of the double complex

$$\bigoplus_{t} (H^0(\tilde{Q}, \Omega_{Q}^*))^{d_t} \to \prod_{i=1}^{\infty} \bigoplus_{t} H^0(\tilde{Q}, \Omega_{Q}^*)^{d_t} \to \prod_{i=1}^{\infty} \bigoplus_{t} H^0(\tilde{Q}, \Omega_{Q}^*)^{d_t}$$

$$\prod_{i} s_i \mapsto \prod_{i} (s_i - s_{i+1})$$

(we note that the direct sums are finite since the $d_t = 0$ for $t$ greater than twice the dimension).

Since the double complex is acyclic with regard to the horizontal differential, the total complex is acyclic too, and hence $H^1(\tilde{Q} \times \tilde{D}^n(m), \omega_{Q \times D^n(m)}^*)$ is also acyclic. This proves Proposition 4.37 and Corollary 4.38 in [DLZ11] for $\omega_{Q \times D^n(m)}^*$, and hence we conclude that Proposition 5.4 holds.
6 The monodromy operator

We follow the argument in [Mok93] but in the more general setting that Y need not be proper.

Let Y be a quasi-projective semistable scheme over $S_0$. Define a double complex

$$B^{i\cdot\cdot} := W^i \check{\omega}_{Y/k}^{i+j+1} / P_j W^i \check{\omega}_{Y/k}^{i+j+1}$$

with the differential $B^{i\cdot\cdot} \rightarrow B^{i+1\cdot\cdot}$ given by $(-1)^j d$ and the differential $B^{i\cdot\cdot} \rightarrow B^{i+1\cdot\cdot}$ given by $\omega \mapsto \omega \wedge \theta$. Let $B^{i\cdot\cdot}$ be the total complex of $B^{i\cdot\cdot}$. Then $B^{i\cdot\cdot} \otimes \mathbb{Q}$ is the complex $B^{i\cdot\cdot}$ considered in the proof of Theorem 4.1 in the log-Monsky-Washnitzer setting. Let $\Phi$ denote the map induced by $p_{i+1} F$ on $B^{i\cdot\cdot}$. Define also a map $\nu$ by requiring that $(-1)^{i+j+1} \nu : B^{i\cdot\cdot} \rightarrow B^{i-1\cdot\cdot}$ is the projection.

This induces a map on $B^{i\cdot\cdot}$, which we also call $\nu$. The same argument as in the proof of Theorem 4.1 shows that the natural map $W^i \check{\omega}_{Y/k} \rightarrow B^{i\cdot\cdot}$ factors through $\Theta : W^i \check{\omega}_{Y/k} \rightarrow B^{i\cdot\cdot}$, and $\Theta \Phi = \Phi \Theta$. One also has that $\Theta \otimes \mathbb{Q}$ is a quasi-isomorphism. Indeed, this is a local question on Y, so we may reduce to the case that Y is a semistable affine scheme over $S_0$, and this case was already shown in the proof of Theorem 4.1.

Proposition 6.1. The map $\nu : B^{i\cdot\cdot} \rightarrow B^{i\cdot\cdot}$ induces a nilpotent operator $N$ on

$$H^* (Y, B^{i\cdot\cdot}_0) \cong H^* (Y, W^i \check{\omega}_{Y/k} \otimes \mathbb{Q}) \cong H^*_{\log\text{-rig}} (Y / \mathcal{O}_S)$$

which coincides with the monodromy operator

$$N : H^*_{\log\text{-rig}} (Y / \mathcal{O}_S) \rightarrow H^*_{\log\text{-rig}} (Y / \mathcal{O}_S)$$

defined in ([Gro05], §5.4).

Proof. Let us define another double complex by

$$C^{i\cdot\cdot} := B^{i-1\cdot\cdot} \oplus B^{i\cdot\cdot}$$

for $i, j \geq 0$, with the differential $C^{i\cdot\cdot} \rightarrow C^{i+1\cdot\cdot}$ given by

$$(\omega_1, \omega_2) \mapsto ((-1)^j d \omega_1, (-1)^j d \omega_2)$$

and the differential $C^{i\cdot\cdot} \rightarrow C^{i, j+1} \cdot \cdot$ given by

$$(\omega_1, \omega_2) \mapsto (\omega_1 \wedge \theta + \nu \omega_2, \omega_2 \wedge \theta)$$

Let $C^{i\cdot\cdot}$ be the total complex of $C^{i\cdot\cdot}$. Then we get a natural morphism

$$\Psi : W^i \check{\omega}_{Y/k} \rightarrow C^{i\cdot\cdot}$$

fitting into the following diagram of short exact sequences

$$0 \longrightarrow W^i \check{\omega}_{Y/k}[-1] \oplus \Theta \longrightarrow W^i \check{\omega}_{Y/k} \longrightarrow W^i \check{\omega}_{Y/k} \longrightarrow 0$$

$$0 \longrightarrow B^{i\cdot\cdot}[-1] \longrightarrow C^{i\cdot\cdot} \longrightarrow B^{i\cdot\cdot} \longrightarrow 0$$
Tensoring by $Q$ gives the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & W^\dagger \omega_{Y/k} \otimes Q[-1] & \overset{-\wedge \theta}{\longrightarrow} & W^\dagger \tilde{\omega}_{Y/k} \otimes Q & \longrightarrow & W^\dagger \omega_{Y/k} \otimes Q & \longrightarrow & 0 \\
& & i & \otimes Q[-1] & \psi & \otimes Q & \otimes Q & \otimes Q & \\
0 & \longrightarrow & B^\dagger [-1] & \overset{-\wedge \theta}{\longrightarrow} & C^\dagger \otimes Q & \longrightarrow & B^\dagger \otimes Q & \longrightarrow & 0
\end{array}
\]

where the outermost vertical arrows are quasi-isomorphisms by the local argument given in the proof of Theorem 4.1, and hence we conclude that $\psi \otimes Q$ is also a quasi-isomorphism. By construction, this shows that the map

\[N : H^*(Y, W^\dagger \omega_{Y/k} \otimes Q) \rightarrow H^*(Y, W^\dagger \tilde{\omega}_{Y/k} \otimes Q)\]

induced by $\nu : B^\dagger \otimes B^\dagger$ is exactly the connecting homomorphism on cohomology associated to the top short exact sequence.

It therefore suffices to prove that the connecting homomorphism gives the monodromy operator on $H^*_{\log-rig}(Y/\mathcal{S}_0)$. Let $Y_\bullet$ be the simplicial scheme and $M_\bullet := Y_\bullet /P_\bullet$, the simplicial dagger space as constructed in the proof of Theorem 5.3. Then we have a diagram of short exact sequences of complexes of simplicial sheaves

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{R}sp_\cdot \omega_{M_\bullet} [-1] & \overset{-\wedge \theta}{\longrightarrow} & \mathbb{R}sp_\cdot \tilde{\omega}_{M_\bullet} & \longrightarrow & \mathbb{R}sp_\cdot \omega_{M_\bullet} & \longrightarrow & 0 \\
& & i & \otimes Q[-1] & \psi & \otimes Q & \otimes Q & \otimes Q & \\
0 & \longrightarrow & W^\dagger \omega_{Y/k} \otimes Q[-1] & \overset{-\wedge \theta}{\longrightarrow} & W^\dagger \tilde{\omega}_{Y/k} \otimes Q & \longrightarrow & W^\dagger \omega_{Y/k} \otimes Q & \longrightarrow & 0
\end{array}
\]

where the outermost vertical arrows are the quasi-isomorphisms (12), and the middle arrow is constructed as follows:

We rewrite the morphism (7) in terms of dagger spaces for each $J$

\[\Gamma(M_J, \tilde{\omega}_{M_J}) \rightarrow W^\dagger \tilde{\omega}_{Y_J/k} \otimes Q\]

Then applying the argument after ([DLZ11], 4.32) gives a local version

\[\mathbb{R}sp_\cdot \tilde{\omega}_{M_J} \rightarrow W^\dagger \tilde{\omega}_{Y_J/k} \otimes Q\]

The proof of Proposition 5.4 holds verbatim with $\mathbb{R}sp_\cdot \omega_{M_J}$ replaced by $\mathbb{R}sp_\cdot \tilde{\omega}_{M_J}$ to show that the canonical morphism $\mathbb{R}sp_\cdot \tilde{\omega}_{M_J} \rightarrow \mathbb{R}sp_\cdot \tilde{\omega}_{M_J}$ is a quasi-isomorphism. This then defines the middle arrow in the diagram, which is therefore a quasi-isomorphism. The monodromy operator on the log-rigid cohomology of $Y$ is, by definition, the connecting homomorphism on cohomology associated to the top short exact sequence.

\[\square\]

7 Comparison with log-crystalline cohomology in the projective case

We prove a semistable analogue of a comparison, obtained for smooth projective varieties in [LZ15] between overconvergent and usual de Rham-Witt cohomology, for Hyodo-Kato cohomology:

\[\]
**Theorem 7.1.** Let $Y$ be a projective semistable scheme over $S_0$. Then the canonical map
\[ H^\bullet(Y, W^\dagger \omega^\bullet_{Y/k}) \rightarrow H^\bullet(Y, W \omega^\bullet_{Y/k}) \]
is an isomorphism of $W(k)$-modules of finite type.

First we need a lemma:

**Lemma 7.2.** Under the assumptions of Theorem 7.1, there is a commutative diagram
\[
\begin{array}{ccc}
H^\bullet(Y, W^\dagger \omega^\bullet_{Y/k}) & \longrightarrow & H^\bullet(Y, W \omega^\bullet_{Y/k}) \\
\downarrow & & \downarrow \\
H^\bullet(Y, W^\dagger \omega^\bullet_{Y/k} \otimes \mathbb{Q}) & \longrightarrow & H^\bullet(Y, W \omega^\bullet_{Y/k} \otimes \mathbb{Q}) \\
\end{array}
\]
\[
\begin{array}{ccc}
H^\bullet_{\text{log-rig}}(Y/S_0) & \sim & H^\bullet_{\text{log-cris}}((Y, M)/(W(k), W(L))) \otimes \mathbb{Q} \\
\end{array}
\]

**Proof.** We need to show that the lower square commutes. The isomorphism on the left and right are the comparisons between log-rigid and overconvergent Hyodo-Kato, resp. between log-crystalline and Hyodo-Kato cohomology ([HK94], 4.19). These isomorphisms also hold if $Y$ is only quasi-projective. The lower horizontal isomorphism is the logarithmic analogue of a comparison between rigid and crystalline cohomology defined in [Ber97] in the proof of Theorem 1.9, and is also defined for any quasi-projective semistable variety. Hence all maps in the lower square are defined for quasi-projective varieties as well. Using the Mayer-Vietoris sequence for cohomology, we may assume that $Y$ is affine. Since the lower horizontal map in the diagram is independent from the choice of embeddings into log-smooth (weak-) formal scheme, we may assume that $H^\bullet_{\text{log-rig}}(Y/S_0)$ is given by logarithmic Monsky-Washnitzer cohomology $H^\bullet_{\text{log-MW}}(Y/K)$. In this case the map is given by a morphism of complexes
\[ \omega^\bullet_{Y/k} \rightarrow \omega^\bullet_{Y/k} \]
i.e. by taking $p$-adic completion of the logarithmic Monsky-Washnitzer complex. The comparison maps to the overconvergent and usual Hyodo-Kato complexes evidently commute with taking $p$-adic completions. This proves the lemma.

Next we show the analogue of ([LZ15], 2.2).

**Proposition 7.3.** Under the assumptions of Theorem 7.1, we have quasi-isomorphisms
\[ W^\dagger \omega^\bullet_{Y/k}/p^n \simeq W \omega^\bullet_{Y/k}/p^n \simeq W_n \omega^\bullet_{Y/k} \]
for all $n \in \mathbb{N}$.

Only the first quasi-isomorphism requires a proof, the second quasi-isomorphism follows from ([HK94], Cor. 4.5).

This is a Zariski-local question, so we may assume that $Y$ is affine. Moreover, by a result of Kedlaya ([Ked05], Thm. 2), we may assume that $Y = \text{Spec } B$ is finite étale and free over $\text{Spec } A = \text{Spec } k[T_1, \ldots, T_d]/(T_1 \cdots T_r)$ for some $r$. 

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We note that ([LZ15], Prop. 2.3) is based on ([DLZ12], Cor. 2.46) and does not need that $A$ is a smooth $k$-algebra, hence we conclude that $W^\dagger(B)$ is a finite étale $W^\dagger(A)$-algebra and free as a $W^\dagger(A)$-module.

The proof of ([DLZ11], Prop. 1.9) transfers verbatim to the Hyodo-Kato complexes and extends the étale base change for the Hyodo-Kato complexes in ([Mat17], Prop. 3.7) to the overconvergent setting, hence we have

$$W^\dagger \omega^\dagger_{A/k} \otimes_{W^\dagger(A)} W^\dagger(B) \cong W^\dagger \omega^\dagger_{B/k}$$

Let $\kappa_A : \tilde{A}^\dagger = W(k)^\dagger/(T_1 \cdots T_d) \rightarrow W^\dagger(A)$ be the canonical map obtained by sending $T_i$ to $[T_i]$ for $i = 1, \ldots, d$. Note that $[T_1 \cdots T_r] = [T_1] \cdots [T_r]$ is zero in $W(A)$, hence $\kappa_A$ is well-defined. By reproducing the argument before ([LZ15], Prop. 2.5), we conclude that the above map extends uniquely to

$$\kappa_B : \tilde{B}^\dagger \rightarrow W^\dagger(B)$$

(note that this map is used to construct the comparison morphisms (5) and (6)). Then we have

**Proposition 7.4.** Let $B$ be finite étale over $A = k[T_1, \ldots, T_d]/(T_1 \cdots T_r)$. Then there is a decomposition of $W^\dagger \omega^\dagger_{B/k}$ into subcomplexes

$$W^\dagger \omega^\dagger_{B/k} = W^\dagger \omega^\dagger_{B/k}^{\text{int}} \oplus W^\dagger \omega^\dagger_{B/k}^{\text{frac}}$$

where $W^\dagger \omega^\dagger_{B/k}^{\text{frac}}$ is acyclic and $W^\dagger \omega^\dagger_{B/k}^{\text{int}}$ is isomorphic to $\omega^\dagger_B$ via the morphism induced by

$$\kappa_B : \omega^\dagger_B \rightarrow W^\dagger \omega^\dagger_{B/k}$$

**Proof.** It is enough to treat the case $A = B$. The extension to the finite étale $B/A$ is identical to the proof of ([LZ15], Prop. 2.5) and hence omitted. For the case $A = B$, we use the description of the de Rham-Witt complex of a (Laurent-) polynomial algebra given in ([BMS16], §10.6):

For a $\mathbb{Z}_{(p)}$-algebra $R$, any element $\omega$ in $W^\dagger \Omega^\dagger_{R[T_1^{-1}, \ldots, T_d^{-1}]/R}$ can be uniquely written as a finite sum

$$\omega = \sum_{k, \mathcal{P}} e(\xi_{k, \mathcal{P}, k, \mathcal{P}_{\leq \rho}}) \prod_{j = \rho + 1}^\ell d \log(\prod_{i \in I_j} [T_i])$$

(13)

where $\rho$ ranges over the weight functions $\rho : [1, d] \rightarrow \mathbb{Z}_{(p)}$ satisfying properties i), ii), iii) in ([BMS16], §10.6), and $\mathcal{P} = \{I_0, I_1, \ldots, I_\rho, I_{\rho + 1}, \ldots, I_\ell\}$ is a disjoint partition of $I = \text{supp } k$, such that $\mathcal{P}_{< \rho} = \{I_0, I_1, \ldots, I_\rho\}$, $I_0$ is possibly empty and $e(\xi_{k, \mathcal{P}, k, \mathcal{P}_{\leq \rho}})$ is a basic Witt differential of type Case 1, Case 2, Case 3 given in ([LZ04], 2.15-2.17) (but where the exponents of the $T_i$ for $i$ occuring in $I_0$ can be negative).

Consider now the log-scheme $\text{Spec}(A, \mathbb{N}^r)$ where $\mathbb{N}^r \ni e_i \mapsto T_i, 1 \leq i \leq r$, over the trivial base $\text{Spec}(k, \ast)$. Then the complex $W^\dagger_{(A, \mathbb{N}^r)/(k, \ast)}$, defined in ([Mat17], can be described as follows (our description differs from the description in [Mat17] but is equivalent): any $\omega$ in $W^\dagger_{(A, \mathbb{N}^r)/(k, \ast)}$ has a unique expression as a convergent sum

$$\omega = \sum_{k, \mathcal{P}} e(\xi_{k, \mathcal{P}, k, \mathcal{P}_{\leq \rho}}) \prod_{j = \rho + 1}^\ell d \log(\prod_{i \in I_j} [T_i])$$
as in [13], where for any given \( m \) we have
- \( \xi_k, \rho \in V^m W(k) = \rho^m W(k) \) for all but finitely many \( k \).
- All weight functions take non-negative values, i.e. on \( I_0 \cup \cdots \cup I_\rho \) they take values in \( \mathbb{Z}_{\geq 0}[\frac{1}{m}] \).
- \( [1, r] \not\subset I_j \) for any \( j = 0, \ldots, \rho \), and for any \( i \) occuring in \( I_j \) for \( j = \rho + 1, \ldots, \ell \) we have \( i \in [1, r] \).

It is clear from this description that we get a decomposition

\[
W \Lambda^\bullet_{(A,N')}((k,*)) = W \Lambda^\bullet_{(A,N')}((k,*)) = W \Lambda^\bullet_{(A,N')}((k,*) \oplus W \Lambda^\bullet_{(A,N')}((k,*)
\]

given by integral and purely fractional weights, and that the fractional part is acyclic, as in the case of (Laurent-) polynomial algebras.

Now we apply ([Mat17], §7.2). Let \( W_m \Lambda^\bullet := W_m \Lambda^\bullet_{(A,N')}((k,*)) \) and \( W_m \Lambda^\bullet := W_m \Lambda^\bullet_{(A,N')}((k,N)) = W_m \Lambda^\bullet_{(A,N')}((k,N)) \), which is isomorphic to the Hyodo-Kato complex \( W_m \omega_{Y/k} \) by the proof of Proposition [7.1] Then we have a short exact sequence ([Mat17], Lemma 7.4)

\[
0 \rightarrow W_m \Lambda^{\bullet - 1} \xrightarrow{\cdot \theta_m} W_m \Lambda^\bullet \rightarrow W_m \Lambda^\bullet \rightarrow 0
\]

where \( \theta_m := d \log[T_1] + \cdots + d \log[T_\ell] \). This implies that any element \( \omega \in W_m \Lambda^\ell \) can be written uniquely as a sum

\[
\omega = \sum_{k, \rho} e(\xi_k, \rho, k, P_{\leq \rho}) \prod_{j=\rho+1}^\ell d \log(\prod_{i \not\subset \ell} [T_i])
\]

with the following properties:
- \( [1, r] \not\subset I_j \) for any \( j = 0, \ldots, \rho \).
- If \( \rho + 1 = \ell \) then \( [1, r] \not\subset I_{\rho+1} = I_\ell \).
- \( e(\xi_k, \rho, k, P_{\leq \rho}) \) as before.

From the definitions it is clear that we again have a decomposition

\[
W_m \Lambda^\bullet = W_m \Lambda^{\bullet \cdot} \oplus W_m \Lambda^{\bullet \frac{\cdot}{\cdot}}
\]

and the fractional part is acyclic. Passing to the projective limit and overconvergent subcomplexes, we obtain decompositions

\[
W \Lambda^\bullet = W \Lambda^{\bullet \cdot} \oplus W \Lambda^{\bullet \frac{\cdot}{\cdot}}
\]

and

\[
W^\dagger \Lambda^\bullet = W^\dagger \Lambda^{\bullet \cdot} \oplus W^\dagger \Lambda^{\bullet \frac{\cdot}{\cdot}}
\]

and the fractional parts are acyclic subcomplexes (the acyclicity is inherited from the case of polynomial algebras). Hence we have the desired decomposition

\[
W^\dagger \omega_{Y/k} = W^\dagger \omega_{Y/k} \oplus W^\dagger \omega_{Y/k}
\]

in the case that \( Y = \text{Spec} k[T_1, \ldots, T_\ell]/(T_1 \cdots T_\ell) \). It is evident that \( W^\dagger \omega_{Y/k} \) is isomorphic to \( \omega_{Y/k} \).

Since the \( W^\dagger \omega_{Y/k} \) and \( \omega_{Y/k} \) are \( p \)-torsion free ([HK94], Cor. 4.5), we conclude that \( W^\dagger \omega_{Y/k} \oplus \mathbb{Z}/p^n \) is acyclic too. It is clear that \( \omega_{Y/k} \otimes \mathbb{Z}/p^n \) is isomorphic to \( \omega_{Y/k} \otimes \mathbb{Z}/p^n \). This concludes the proof of Proposition [7.3].
Finally, since
\[
\lim_{n} H^i(Y, W^t\omega_{Y/k}^*/p^n) = \lim_{n} H^i(Y, W_n\omega_{Y/k}^*) = H^i(Y, W\omega_{Y/k}^*)
\]
where the last equality holds because all $H^i(Y, W_n\omega_{Y/k}^*)$ are of finite length if $Y$ is proper ([HK94], §3.2) and $H^i(Y, W\omega_{Y/k}^*)$ are $W(k)$-modules of finite type, we can apply the arguments in ([LZ15], page 1392) to conclude that
\[
H^*(Y, W^t\omega_{Y/k}^*) \cong H^*(Y, W\omega_{Y/k}^*)
\]
This proves Theorem 7.1.

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