ABELIAN GAUGE THEORIES ON COMPACT MANIFOLDS AND THE GRIBOV AMBIGUITY

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Abstract. We study the quantization of abelian gauge theories of principal torus bundles over compact manifolds with and without boundary. It is shown that these gauge theories suffer from a Gribov ambiguity originating in the non-triviality of the bundle of connections whose geometrical structure will be analyzed in detail. Motivated by the stochastic quantization approach we propose a modified functional integral measure on the space of connections that takes the Gribov problem into account. This functional integral measure is used to calculate the partition function, the Green’s functions and the field strength correlating functions in any dimension using the fact that the space of inequivalent connections itself admits the structure of a bundle over a finite dimensional torus. The Green’s functions are shown to be affected by the non-trivial topology, giving rise to non-vanishing vacuum expectation values for the gauge fields.

1. Introduction

Functional integral techniques together with the Faddeev-Popov method [1] play a central role in the quantization of Yang-Mills theories. Impressive successes of this method were obtained within perturbation theory. The fundamental object in the quantized pure (non-abelian) Yang-Mills theory is the non-normalized density

\[ \Xi^{(P)}(A) = \text{vol}_{A^{(P)}} e^{-S_{\text{inv}}(A)}, \] (1.1)

where \( \text{vol}_{A^{(P)}} \) denotes the (formal) volume form on the space of all connections \( A^{(P)} \) of a certain principal \( G \)-bundle \( P \) over \( M \). \( S_{\text{inv}}(A) = \frac{1}{2} \int_M \text{tr}(F_A \wedge \star F_A) \) is the gauge invariant classical Yang-Mills action, defined by the field strength \( F_A \) and the Hodge star operator \( \star \) with respect to a fixed metric on \( M \). The trace \( \text{tr} \) is taken along the Lie-algebra of the corresponding compact symmetry group \( G \).

The vacuum expectation value (VEV) of a gauge invariant observable \( f \in C^\infty(A^{(P)}) \) may be defined by

\[ <f>_P = \frac{\int_{A^{(P)}} \Xi^{(P)}(A) \cdot f(A)}{\int_{A^{(P)}} \Xi^{(P)}(A)}. \] (1.2)

A difficulty arises because the integrands appearing in (1.2) are constant along the orbits of the corresponding gauge group \( G^{(P)} \), which have infinite measure. This implies that the non-physical degrees of freedom must be eliminated before the theory can be quantized. According to the Faddeev-Popov approach a unique representative is selected from each \( G^{(P)} \)-orbit in a smooth way giving rise to a gauge fixing submanifold. The functional integral over the total space \( A^{(P)} \) is then restricted to this submanifold by extracting the infinite volume of the gauge group, which is absorbed into an overall normalization constant in the end. As a result the quadratic part of the classical action \( S_{\text{inv}} \) becomes invertible and the resulting functional integral leads to a consistent perturbative expansion with corresponding Feynman diagrams. However, the treatment of the infinite gauge group volume is not satisfying from a mathematical point of view.
Moreover, in the non-perturbative regime it was envisaged very soon that the Faddeev-Popov formulation suffers from so called Gribov ambiguities [2]. Topologically this is related to the fact that $\mathcal{A}^{(P)}$ is a non-trivial $G^{(P)}$-bundle over the gauge orbit space $\mathcal{M}^{(P)} = \mathcal{A}^{(P)}/G^{(P)}$ preventing the definition of a smooth global gauge fixing submanifold [3-6]. Gauge fixing is thus only locally possible and the aim is to find a constructive way to take all relevant gauge fields into account. Not only the non-abelian Yang-Mills theory suffers from this obstruction but even pure Maxwell theory on the four-dimensional torus $\mathbb{T}^4$ was shown to be affected by the Gribov problem using general topological arguments [7].

The question whether or how the original Faddeev-Popov approach can be modified appropriately has generated a controversial discussion during the last years. Several proposals to overcome the Gribov problem have been published (see [8] for a recent overview). In the following outline we will focus on formulations for an appropriate functional integral in the continuum theory.

One direction follows the original suggestion of Gribov by restricting the functional integral to a submanifold of $\mathcal{A}^{(P)}$, where the Gribov problem is absent and all gauge fields are uniquely determined [9,10]. Hence the challenge is to find a systematic way to restrict the Yang-Mills density to this domain of definition and to perform the integration explicitly, see also [11].

A different formulation avoiding the Gribov problem in the Yang-Mills theory has been proposed by some authors in [12,13], where the original functional integral is modified by the introduction of a non-local gauge fixing term by hand. The modified functional integral restricts the domain of integration appropriately, yet in this approach the infinite volume of the gauge group has to be omitted.

An alternative way towards the quantization of Yang-Mills theory is to construct a functional integral directly on the gauge orbit space $\mathcal{M}^{(P)}$ instead of $\mathcal{A}^{(P)}$. The so called ”invariant integration“ [14,15] relies methodically on the reduction of an integral of invariant functions over the total space of a finite dimensional principal fibre bundle with compact structure group to an integral over the base manifold of this bundle multiplied with the finite volume of the symmetry group. If applied literally to the Yang-Mills theory, one would encounter once again the problem with the ill-defined volume of the gauge group. Hence the idea is to define the partition function of the gauge theory completely in terms of the resulting functional integral over $\mathcal{M}^{(P)}$. However, compared to the affine space $\mathcal{A}^{(P)}$ the structure of $\mathcal{M}^{(P)}$ is much more complicated so that an explicit computation of this integral over the gauge orbit space is often not possible.

In [16] the functional integral has been constructed directly on the gauge orbit space $\mathcal{M}^{(P)}$. The basic ingredient is a regularized Brownian motion governed by the Riemannian structure of $\mathcal{M}^{(P)}$, which is induced by the kinetic term of the (regularized) Yang-Mills action.

A further attempt to be mentioned is [17], where a patching construction for the locally restricted Faddeev-Popov densities - disregarding the infinite volume of the gauge group - has been investigated.

In this paper we want to present a functional integral measure on the space of all connections that resolves the Gribov problem and provides for a mathematically reasonable treatment of the gauge degrees of freedom. A novel way to treat these two problems has been proposed some time ago within the stochastic quantization scheme [18,19]. Generally, the stochastic quantization method of Parisi and Wu [20] was introduced as a new method for quantizing field theories. It is based on concepts of non-equilibrium statistical mechanics and provides novel and alternative insights into quantum field theory (see ref. [21], for a comprehensive review and referencing). Let us comment on our proposal in brief: The gauge fields are regarded as stochastic processes with respect to a fictive so-called ”stochastic time“, which are governed by an equivalence class of stochastic differential equation. The notion of equivalence refers to the fact that stochastic correlation functions of gauge invariant observables are well-defined and unique. This equivalence allows to select a distinguished representative [18,19]. The stochastic scheme can thus be recast into a formulation in terms of a normalizable probability density as functional of the gauge fields, which has to satisfy the Fokker-Planck equation [21]. In this respect, the introduction of a damping force along the gauge degrees of freedom regularizing the volume of the gauge group is one of the main aspects of this approach. The strategy in taking care of the Gribov copies is to restrict the domain of the stochastic processes to local coordinate patches in the configuration space $\mathcal{A}^{(P)}$, furthermore to construct locally defined equilibrium distributions and finally to paste them together in such a way that the physical relevant objects become independent of the
particular way this pasting is provided. Distinguished by its concept the whole field content, even the
gauge degrees of freedom, has to be taken into account within the stochastic quantization scheme to
permit a probability interpretation.

Our aim in this paper is to apply the modified functional integral [19] to abelian gauge theories
of connections of principal torus bundles over \( n \)-dimensional compact manifolds. These theories are
proved to suffer from Gribov ambiguities. So we generalize the results of [7] to a wider class of
manifolds. The motivation to study this theory is twofold: First of all we want to get a more profound
understanding of our new concept for the functional integral by analyzing a simple but non-trivial field
theoretical model. As a consequence the interrelation between the occurrence of the Gribov problem
and the necessity for a regularizing measure for the gauge group can be elucidated. However, besides
serving as a laboratory for the new concept the second reason is that abelian gauge theories gained a
strong interest during the last years by its own. Examples are the analysis of two dimensional gauge
theories, the description of the fractional quantum Hall effect (see [22] for a comprehensive review)
and questions related to the duality in field theory on three and four dimensional manifolds with and
without boundary [23-26].

The paper is structured as follows: In section 2 the concept of the modified functional integral
measure will be briefly reviewed. The abelian field theoretical model which we are going to consider
is introduced in section 3. Sections 4 and 5 are devoted to the analysis of the abelian gauge theory on
closed compact manifolds respectively on compact manifolds with a boundary. Since the calculation of
the modified functional integral relies on the knowledge of the bundle geometry of the space of gauge
fields, we will analyze its structure in detail in subsection 4.1 for closed manifolds and in subsection
5.1 for manifolds with a non-empty boundary. In both cases, the bundle of connections exhibits a non-
trivial structure which implies that it is impossible to fix the gauge globally. It should be remarked
that on closed manifolds the topology of the gauge orbit space of \( T^1 \)-connections has been studied for
several years, often in low dimensions. In this respect some of our results regarding the structure of
the gauge group and the topology of the gauge orbit space have already been displayed using different
methods (see e.g. [27]). However, to our knowledge an explicit construction of the various bundle
structures in terms of local sections has not appeared in the literature so far. Our geometrical results
will then be used to compute the partition function, the vacuum expectation value of gauge invariant
functions and the Green’s functions of the gauge fields for closed manifolds in the subsections 4.2 and
4.3. Analogous results will be displayed in subsection 5.2 for manifolds with a non-empty boundary.
Furthermore our results are compared with those obtained by the conventional covariant quantization
schemes. In section 6 the concept of the modified functional integral and its consequences are illustrated
with two examples, namely the abelian gauge theory on the circle and the abelian gauge theory on
two-dimensional closed manifolds. The paper concludes with a summary of the main results in section
7.

2. A modified functional integral measure for gauge theories

In the present publication we want to shed some new light onto the question of how a reasonable
partition function can be formulated for a gauge theory suffering from Gribov ambiguities. In order to
ascribe a probabilistic interpretation to the Yang-Mills density according to the stochastic quantization
scheme, the formal measure in (1.1) has to be modified appropriately so that it becomes integrable.
The strategy is to introduce a damping force which regularizes the gauge degrees of freedom. This will
be provided by a so called gauge fixing function \( S_{gf} \) on the gauge group \( \mathcal{G}^{(P)} \), which is assumed to
render the volume of the gauge group

3
Accordingly, the vacuum expectation value (VEV) of a gauge invariant function

\[ Vol(G^{(P)}; S_{gf}) := \int_{G^{(P)}} \text{vol}_{G^{(P)}} e^{-S_{gf}} \]  

(2.1)

finite. Here \( \text{vol}_{G^{(P)}} \) is the (formal) left-invariant volume form on \( G^{(P)} \). In the following we shall consider only gauge fixing functions \( S_{gf} \), which satisfy \( Vol(G^{(P)}; S_{gf}) = 1 \).

The (non-abelian) gauge group \( G^{(P)} \) acts freely on \( A^{(P)} \), denoted by \( (A,g) \mapsto A^g \), so that \( A^{(P)} \) admits the structure of a principal \( G^{(P)} \)-bundle over the gauge orbit space \( M^{(P)} := A^{(P)} / G^{(P)} \) with projection \( \pi_{A^{(P)}} \). (In fact, the gauge group has to be restricted appropriately to give a free action [4]). The theory is said to possess a Gribov problem if this bundle is non-trivial. There exists a family of local trivializations \( (U_a, \varphi_a) \) given by \( U_a \times G^{(P)} \xrightarrow{\varphi_a} \pi_{A^{(P)}}^{-1}(U_a) \) where \( \{U_a\} \) is a locally finite open cover of \( M^{(P)} \) and the local diffeomorphisms \( \varphi_a(\pi_A(A), g) = \sigma_a(\pi_{A^{(P)}}(A))^g \) are generated by a family of local sections \( U_a \xrightarrow{\sigma_a} \pi_{A^{(P)}}^{-1}(U_a) \). For the inverse we write \( \varphi_a^{-1}(A) = (\pi_{A^{(P)}}(A), \omega_a(A)) \).

We propose that the quantization of the Yang-Mills theory is described by the following local densities on \( A^{(P)} \)

\[ \Xi_a = \text{vol}_{A^{(P)}} |_{\pi_{A^{(P)}}^{-1}(U_a)} e^{-S_{gf} - \omega_a} \]  

(2.2)

which - if normalized - appear as equilibrium solutions of the Fokker-Planck operator on each open set \( \pi_{A^{(P)}}^{-1}(U_a) \subseteq A^{(P)} \). Due to the Gribov ambiguity these local partition functions must be pasted together using a partition of unity on the gauge orbit space.

**Definition 2.1.** Let \( \{p_a\} \) denote a partition of unity on \( M^{(P)} \) subordinate to the open cover \( \{U_a\} \). We define a global (non-perturbative) Yang-Mills density \( \Xi^{(P)} \) by

\[ \Xi^{(P)} := \sum_a (\pi_{A^{(P)}}^* p_a) \cdot \Xi_a^{(P)}. \]  

(2.3)

Accordingly, the vacuum expectation value (VEV) of a gauge invariant function \( f \) is given by

\[ < f > := \frac{I^{(P)}(f)}{I^{(P)}(1)}, \quad I^{(P)}(f) = \int_{A^{(P)}} \Xi^{(P)} \cdot f. \]  

(2.4)

For the partition function we take \( Z^{(P)} := I^{(P)}(1) \).

It has been shown in [19] that based on this constructive procedure the VEV of gauge invariant observables

1. coincides with the Faddeev-Popov result in the perturbative regime
2. is independent of the particular form of the damping force \( S_{gf} \) along the gauge group
3. is independent of the particular local trivialization
4. is independent of the special choice for the partition of unity.

This idea to patch the local Yang-Mills densities together to obtain a global functional integral in the field space takes up a suggestion raised by Singer [3] in his seminal paper.

For some applications it is necessary to consider the total configuration space, which consists of disconnected components \( A^{(P)} \) labelled by the equivalence class of bundles \( P \). The set of all \( \mathbb{T}^N \) connections over \( M \), denoted by \( A^{(M)} \), is given as disjoint union

\[ A^{(M)} = \bigsqcup_{(P)} A^{(P)}. \]  

(2.5)

Correspondingly, the partition function and the VEV of gauge invariant observables are represented by a sum over equivalence classes of principal bundles \( P \), namely

\[ Z = \sum_{(P)} Z^{(P)}, \quad < f > := \frac{\sum_{P} I^{(P)}(f)}{\sum_{P} I^{(P)}(1)}. \]  

(2.6)
3. The geometrical setting for the abelian gauge theory

In this section the abelian field theoretical model which we are going to consider in this paper is introduced. As we focus on compact abelian structure groups only, we can restrict ourselves to the $N$-dimensional torus $T^N$ as relevant symmetry group. We shall begin with a brief review of torus bundles:

Let $M$ be an $n$-dimensional connected, oriented and compact manifold with a fixed Riemannian metric. Let us now consider an arbitrary principal $T^N$-bundle $P(M, \pi_P, T^N)$ over $M$ with projection $\pi_P$. The group structure on $T^N$ is provided by point-wise multiplication and its Lie algebra $t^N$ is given by $t^N = \sqrt{-1} \mathbb{R}^N$. A $L^2$ inner product can be defined on the complex $\Omega^k(M; t^N)$ of $k$-forms on $M$ by

\[
<v_1, v_2> = \sum_{\alpha=1}^N \int_M v_1^\alpha \wedge \bar{v}_2^\alpha, \tag{3.1}\]

where $\star$ is the Hodge star operator with respect to the given metric on $M$, satisfying $\star^2 = (-1)^{k(n-k)}$ and $\bar{v}^\alpha$ denotes the complex conjugate of $v = (v^\alpha)_{\alpha=1}^N \in \Omega^k(M; t^N)$.

The $C^\infty$-Hilbert manifold of all connections on $P$ of a certain Sobolev class will be denoted by $A^P$. The gauge group $G^{(M)}$ is defined as the group of vertical bundle automorphisms on $P$ and can be identified with the Hilbert Lie-Group $C^\infty(M, T^N)$ of differentiable maps between $M$ and $T^N$. Finally, its Lie-algebra $\mathfrak{g}^{(M)}$ is given by $\mathfrak{g}^{(M)} = C^\infty(M; t^N)$.

Under an arbitrary gauge transformation $g \in G^{(M)}$, the gauge fields transform according to

\[
A \mapsto A^g = A + (\pi_P g)^* \partial, \quad g \in G^{(M)}, \tag{3.2}\]

where $\partial \in \Omega^1(T^N; t^N)$ is the Maurer Cartan form on $T^N$. (For notational convenience we shall not distinguish between $\pi_P g$ and $g$.)

How can torus bundles be classified? The topological type of $T^N$ torus bundles is expressed by the first Cech-cohomology $H^1(M; sh_M(T^N))$, where $sh_M(T^N)$ denotes the sheaf of all $T^N$ valued differentiable functions on $M$. The sheaves of $\mathbb{Z}^N$ and $\mathbb{R}^N$ valued differentiable functions on $M$, which are denoted by $sh_M(\mathbb{Z}^N)$, $sh_M(\mathbb{R}^N)$, respectively, fit into the following exact sequence of sheaves

\[
0 \rightarrow sh_M(\mathbb{Z}^N) \rightarrow sh_M(\mathbb{R}^N) \rightarrow sh_M(T^N) \rightarrow 1, \tag{3.3}\]

which induces a corresponding long exact sequence in cohomology

\[
\cdots \rightarrow \hat{H}^1(M, sh_M(\mathbb{R}^N)) \rightarrow \hat{H}^1(M, sh_M(T^N)) \rightarrow \hat{H}^2(M, sh_M(\mathbb{Z}^N)) \rightarrow \hat{H}^2(M, sh_M(\mathbb{R}^N)) \rightarrow \cdots \tag{3.4}\]

Since the sheaf $sh_M(\mathbb{R}^N)$ is fine, the set $\mathfrak{P}[M, T^N]$ of equivalence classes of principal $T^N$ bundles over $M$ is given by

\[
\mathfrak{P}[M, T^N] = \hat{H}^1(M, sh_M(T^N)) = H^2(M, \mathbb{Z}^N) = \bigoplus_{i=1}^N H^2(M, \mathbb{Z}), \tag{3.5}\]

so that any principal $T^N$-bundle is classified by an integer cohomology class $c \in H^2(M, \mathbb{Z}^N)$. Accordingly, $c = c_1 \oplus \cdots \oplus c_N$, where each component $c_\alpha \in H^2(M, \mathbb{Z})$ determines a principal circle bundle $P^{\alpha}(M, T^1)$ over $M$ having $c_\alpha$ as its first Chern class. Thus $P$ can be equivalently viewed as $N$-fold fiber product $P^1 \times_M \cdots \times_M P^N$ over $M$.

Let $F_A = (F_\alpha^A)_{\alpha=1}^N \in \Omega^2(M; t^N)$ denote the field strength of the $T^N$-connection $A$ on $P$. Each component $F_\alpha^A$ can be regarded as field strength of the $\alpha$-th principal $T^1$-bundle $P^\alpha$ in the fiber product $P$. The classical gauge invariant action is defined by

\[
S_{inv}(A) = \frac{1}{2} \sum_{\alpha, \beta=1}^N \int_M \lambda_{\alpha \beta} F_\alpha^A \wedge \star F_\beta^A, \tag{3.6}\]
where \((\lambda_{\alpha\beta})_{\alpha,\beta=1}^{N}\) is a symmetric positive definite matrix with \(\det \lambda = 1\). This matrix determines the relative couplings between the components \(A^\alpha\) of the \(T^N\)-gauge fields \(A\) on \(P\). From a physical point of view some extensions of (3.6) are of particular interest: If the conventional Maxwell action is extended by a theta term the resulting partition function was shown to exhibit a non-trivial transformation behavior under electric-magnetic duality [23-26]. On the other hand, if the action (3.6) is extended by an additional Chern-Simons term in a three dimensional space-time, this model allows for a mathematical description of the fractional quantum Hall effect. The integer resulting from the evaluation of the corresponding Chern classes \(c^\alpha\) \((\alpha = 1, \ldots, N)\) along the 2-dimensional space admits the interpretation of the total number of electrons in the \(\alpha\)-th Landau level [22].

Provided by the matrix of couplings there is a second \(L^2\) inner product on the complex \(\Omega(M; t^N)\) given by

\[
< \upsilon_1, \upsilon_2 >_\lambda = \sum_{\alpha,\beta=1}^{N} \int_M \lambda_{\alpha\beta} \upsilon_1^\alpha \wedge \ast \bar{\upsilon}_2^\beta,
\]

(3.7)

where \(\upsilon = (\upsilon^1, \ldots, \upsilon^N) \in \Omega^k(M; t^N)\).

4. Abelian gauge theories on closed manifolds

In this chapter we want to construct the modified functional integral for the abelian gauge theory on closed manifolds. We begin with an analysis of the geometrical properties of the gauge group. Based on these considerations we will then derive two results regarding the bundle structure of the space of connections.

4.1. The geometry of the abelian gauge fields

The action (3.2) of the gauge group \(G(M)\) is not free possessing the non-trivial isotropy group \(T^N\), namely the subgroup of constant gauge transformations. In order to get a free action let us now choose an arbitrary but fixed reference point \(x_0 \in M\). By restricting the gauge group to the subgroup \(G^k(M) = \{g \in G | g(x_0) = 1\}\) which itself is diffeomorphic to \(G^k(M)/T^N\) by \(g \rightarrow g \cdot g(x_0)\), we finally obtain a free action of \(G^k(M)\) on \(A^{(p)}\). This gives rise to a smooth gauge orbit space \(M^k_p = A^{(p)}/G^k(M)\), which has to be regarded as the true configuration space of the theory. For the Maxwell theory \((N = 1)\) some of the results regarding the gauge group topology have been considered in [28].

Let us denote by \(Z_k(M; \mathbb{Z})\) the subcomplex of all closed smooth singular \(k\)-cycles on \(M\). We define the abelian group

\[
\Omega^k_{\mathbb{Z}}(M; \mathbb{R}^N) = \{ \alpha \in \Omega^k(M; \mathbb{R}^N) | \quad d\alpha = 0, \quad \int_\gamma \alpha \in \mathbb{Z}^N, \quad \forall \gamma \in Z_k(M; \mathbb{Z}) \}
\]

(4.1.1)

of all closed \(\mathbb{R}^N\)-valued differential \(k\)-forms with integer periods and denote by \(H^k_{\mathbb{Z}}(M; \mathbb{R}^N)\) the corresponding cohomology group.

The question of how the subgroup of constant gauge transformations is related to the gauge group is answered by the following statement:

**Proposition 4.1.** The following sequence of abelian groups is split exact

\[
0 \rightarrow T^N \rightarrow G^k(M) \xrightarrow{\kappa(M)} \Omega^1_{\mathbb{Z}}(M; \mathbb{R}^N) \rightarrow 0 \quad \kappa(M)(g) = \frac{1}{2\pi\sqrt{-1}} g^* \theta.
\]

(4.1.2)
Proof. The split is given by the isomorphism of abelian groups

\[
\tilde{\kappa}(M) \colon \Omega^2_k(M, \mathbb{R}^N) \times T^N \to G(M)
\]

\[
\tilde{\kappa}(M)(\alpha, t)(x) = t \cdot \exp 2\pi \sqrt{-1} \int_{c_x} \alpha = t \cdot \exp 2\pi \sqrt{-1} \int_0^1 c_x \alpha
\]

\[
\tilde{\kappa}^{-1}(M)(g) = (\kappa(M)(g), g(x_0)),
\]

where \(c_x \colon [0, 1] \to M\) is a path in \(M\) connecting \(x_0\) with \(x\). That this integral is already well-defined can be seen by choosing a different path \(c'_x\) connecting \(x_0\) and \(x\). Since the combined path \(c'_x \circ c_x\) can be regarded as element in \(Z_1(M; \mathbb{Z})\). The integration of any element in \(\Omega^2_k(M, \mathbb{R}^N)\) along this cycle gives an integer. \(\square\)

The co-differential \(d_k^* = (-1)^{n(k+1)+1} \star d_{n-k} \colon \Omega^k(M; \mathbb{R}) \to \Omega^{k-1}(M; \mathbb{R})\) gives rise to the Laplacian operator \(\Delta_k = d_{k+1}^* d_k + d_{k-1} d_k^*\). Let \(\text{Harm}^k(M)^\perp\) denote the orthogonal complement of the space of harmonic \(k\)-forms \(\text{Harm}^k(M)\) with values in \(\mathbb{R}\), then we can define the Greens operator [29]

\[
G_k \colon \Omega^k(M; \mathbb{R}) \to \text{Harm}^k(M)^\perp, \quad G_k = (\Delta_k |_{\text{Harm}^k(M)^\perp})^{-1} \circ \Pi^{\text{Harm}^k(M)^\perp},
\]

where \(\Pi^{\text{Harm}^k(M)^\perp}\) is the projection of \(\Omega^k(M; \mathbb{R})\) onto \(\text{Harm}^k(M)^\perp\). By construction \(\Delta_k \circ G_k = G_k \circ \Delta_k = \Pi^{\text{Harm}^k(M)^\perp}\).

It is evident that the Lie algebra \(\mathfrak{g}^{(M)}_\ast\) of the restricted gauge group \(G^{(M)}_\ast\) consists of those \(C^\infty\) maps from \(M\) to \(t^N\), which vanishes in \(x_0\). The next result shows that the pointed gauge group \(G^{(M)}_\ast\) is not connected.

**Proposition 4.2.** The following sequence of abelian groups is split exact

\[
0 \to G^{(M)}_\ast \xrightarrow{\exp} G^{(M)}_\ast \to H^1(M; \mathbb{R}^N) \to 0,
\]

where \(\kappa(M)(g) = [\kappa(M)(g)]\).

**Proof.** It is easy to see that the exponential function \(\exp\) is indeed a monomorphism. A split of (4.1.5) is given by the following isomorphism of abelian groups

\[
\tilde{\kappa}(M) \colon H^1_\ast(M; \mathbb{R}^N) \times G^{(M)}_\ast \to G^{(M)}_\ast
\]

\[
\tilde{\kappa}(M)([\alpha], \xi)(x) = \exp (2\pi \sqrt{-1} \int_{c_x} \Pi^{\text{Harm}^1(M)}(\alpha)) \cdot \exp \xi(x)
\]

\[
\tilde{\kappa}^{-1}(M)(g) = (\kappa(M)(g), G_0 d_{k_0}^* g^* \theta - (G_0 d_{k_0}^* g^* \theta)(x_0)).
\]

\(\square\)

Now we will prove that even an abelian gauge theory would admit a Gribov ambiguity if the space time manifold \(M\) is topologically non-trivial. This generalizes the previous result [7], where the existence of Gribov ambiguities has been shown for Maxwell theory on the four-torus.

**Theorem 4.3.** \(\mathcal{A}^{(P)}\) is a flat principal bundle over \(\mathcal{M}^{(P)}_\ast\) with structure group \(G^{(M)}_\ast\) and projection \(\pi_{\mathcal{A}^{(P)}}\). This bundle is trivializable if \(H^1(M; \mathbb{Z}) = 0\).

**Proof.** We are going to construct a bundle atlas explicitly. For this we have to define an open cover of the gauge orbit space and a family of local sections. For any fixed \(l, N \in \mathbb{N}\) we consider the exact sequence of abelian groups

\[
0 \to \mathbb{Z}^l \to \mathbb{R}^l \xrightarrow{\exp 2\pi \sqrt{-1}(\cdot)} \mathbb{R}^l \to 0,
\]

(4.1.7)
which gives the universal covering of the $IN$-dimensional torus $\mathbb{T}^N$. Let us view $\mathbb{T}^N$ as the product

$$\mathbb{T}^N = \mathbb{T}^N \times \cdots \times \mathbb{T}^N = (\mathbb{T}^1 \times \cdots \times \mathbb{T}^1) \times \cdots \times (\mathbb{T}^1 \times \cdots \times \mathbb{T}^1).$$

(4.1.8)

We introduce an open cover $\mathcal{V}$ of $\mathbb{T}^N$ by the following family of open sets

$$\mathcal{V} = \{ V_a \mid a := (a_1, \ldots, a_j, \ldots, a_l), \quad a_j := (a_{j1}, \ldots, a_{jN}) \in \mathbb{Z}_2 = \{1, 2\}, \}$$

(4.1.9)

where $V_a = V_{a_1} \times \cdots \times V_{a_j} \times \cdots \times V_{a_l}$ is a open set in $\mathbb{T}^N$. Each $V_{a_j}$ is itself the product of open sets $V_{a_{j1}} = V_{a_{j11}} \times \cdots \times V_{a_{j1N}}$ in the $k$-th $N$-dimensional torus $\mathbb{T}^N$ within (4.1.8). Here $V_1 = \mathbb{T}^1 \setminus \{ \text{northern pole} \}$ for $a_{j1} = 1$ and $V_2 = \mathbb{T}^1 \setminus \{ \text{southern pole} \}$ for $a_{j1} = 2$ provide an open cover of each 1-torus $\mathbb{T}^1$. Let us choose the following two local sections of the universal covering $\mathbb{R}^l \to \mathbb{T}^l$.

$$s_{a_{j1}}(z) = \left\{ \begin{array}{ll}
\frac{1}{2\pi} \arccos |(0,z)\mathbb{R}\mathbb{Z}| & \text{if } z \geq 0, \\
\frac{1}{2\pi} \arccos |(\pi,2z)\mathbb{R}\mathbb{Z}| & \text{if } z < 0, \\
\frac{1}{2\pi} \arccos |(2z)\mathbb{R}\mathbb{Z}| & \text{if } z < 0, \quad a_{j1} = 1.
\end{array} \right.$$

$$s_{a_{j2}}(z) = \left\{ \begin{array}{ll}
\frac{1}{2\pi} \arccos |(\pi,2z)\mathbb{R}\mathbb{Z}| & \text{if } z \leq 0, \\
\frac{1}{2\pi} \arccos |(2z)\mathbb{R}\mathbb{Z}| & \text{if } z > 0, \quad a_{j2} = 2.
\end{array} \right.$$

(4.1.10)

where $z = \mathbb{R}z + \sqrt{-1} \mathbb{Z}z \in \mathbb{T}^l$. The corresponding transition functions $g_{a_{j1}a_{j2}}^\alpha : V_{a_{j1}} \cap V_{a_{j2}} \to \mathbb{Z}$ are given by

$$s_{a_{j1}}(z_{j1}) = s_{a_{j2}}(z_{j2}) + g_{a_{j1}a_{j2}}^\alpha (z_{j1}).$$

(4.1.11)

Evidently a family of $2^{IN}$ local sections $s_a : V_a \subset \mathbb{T}^N \to \mathbb{R}^{IN}$ can be induced by

$$s_a = (s_{a_1}, \ldots, s_{a_l}) = ((s_{a_{11}}, \ldots, s_{a_{1N}}), \ldots, (s_{a_{l1}}, \ldots, s_{a_{lN}})),$$

(4.1.12)

where on $V_a \cap V_{a'}$ the corresponding sections $s_a$ and $s_{a'}$ are related by the locally constant transition functions $g_{aa'}^{N} : V_a \cap V_{a'} \to \mathbb{Z}^{NI}$

$$g_{aa'}^{N} (z_{11}, \ldots, z_{l1}) = (g_{a_{11}a_{11}'} (z_{11}), \ldots, g_{a_{1N}a_{1N}'} (z_{1N})), \ldots, (g_{a_{l1}a_{l1}'} (z_{l1}), \ldots, g_{a_{lN}a_{lN}'} (z_{lN})).$$

(4.1.13)

for $z_{j} = (z_{j1}, \ldots, z_{jN}) \in \mathbb{T}^N$ with $j = 1, \ldots, l$. These local sections will be the building blocks for the construction of a bundle atlas.

Let $\text{Harm}_0^k(M; \mathbb{R})$ denote the abelian group of harmonic $k$-forms with integer periods and let $D_{n-1} : H^{n-1}(M; \mathbb{Z}) \to H_1(M; \mathbb{Z})$, $D_{n-1}(\nu) = \nu \cap [M]$ be the Poincaré duality isomorphism [30]. Here $\cap$ is the cap product and $[M]$ denotes the fundamental cycle.

Since the homology of $M$ is finitely generated with rank $b_1$ (the first Betti number of $M$) we shall choose a set of 1-cycles $\gamma_i \in Z_1(M; \mathbb{Z})$, $i = 1, \ldots, b_1$, whose homology classes $[\gamma_i]$ provides a Betti basis thus generating the free part $H_1(M; \mathbb{Z})/\text{Tor} H_1(M; \mathbb{Z})$ in $H_1(M; \mathbb{Z})$. Here $\text{Tor} H_1(M; \mathbb{Z})$ denotes the torsion part of the first homology group. Then $D_{n-1}^{-1}([\gamma_i])$ provides a basis for cohomology, from which a basis of harmonic forms $(\rho_i^{(n-1)})_{i=1}^{b_1} \in \text{Harm}_Z^{n-1}(M; \mathbb{R})$ can be selected according to the following isomorphisms

$$H^{n-1}(M; \mathbb{Z})/\text{Tor} H^{n-1}(M; \mathbb{Z}) \cong H_{Z}^{n-1}(M; \mathbb{R}) \cong \text{Harm}_Z^{n-1}(M; \mathbb{R}).$$

(4.1.14)

Using the Poincaré duality and the Universal Coefficient Theorem it follows that the product

$$\text{Harm}_Z^{n-1}(M; \mathbb{R}) \cong \bigoplus_{i=1}^{b_1} \mathbb{R} \gamma_i.$$
\[ H^1(M; \mathbb{Z})/\text{Tor} H^1(M; \mathbb{Z}) \times H^{n-1}(M; \mathbb{Z})/\text{Tor} H^{n-1}(M; \mathbb{Z}) \to \mathbb{Z} \]
\[ (\mu, \nu) \mapsto < \mu, D_{n-1}(\nu) > = < \mu \cup \nu, [M] > , \quad (4.1.15) \]
gives a perfect pairing [30], where \(<,>\) denotes the evaluation in cohomology. We remark that \(\text{Tor} H^1(M; \mathbb{Z}) = 0\). A basis \((\rho_1^{(1)})_{b_i}^{j}, \pi_1 \in \text{Harm}_0^2(M; \mathbb{R})\) can be adjusted in such a way so that
\[ \int_{\gamma_j} \rho_1^{(1)} = \int_{M} \rho_1^{(1)} \wedge \rho_j^{(n-1)} = \delta_{ij}. \quad (4.1.16) \]
Hence \(\int_{\gamma_j} \alpha = \int_{M} \alpha \wedge \rho_j^{(n-1)}\) holds for any \([\alpha] \in H^1(M; \mathbb{R})\). On \(\text{Harm}^1(M; \mathbb{R})\) there exists an induced metric
\[ h_{jk} = < \rho_j^{(1)}, \rho_k^{(1)}>. \quad (4.1.17) \]
For any choice of an arbitrary but fixed background gauge field \(A_0 \in \mathcal{A}^{(p)}\) there exists a smooth surjective map \(\pi^{A_0}_{\mathcal{M}^{(p)}} : \mathcal{M}_*^{(p)} \to \mathbb{T}^{b_1 N}\) defined by
\[ \pi^{A_0}_{\mathcal{M}^{(p)}}([A]) = (e^{\int_M (A - A_0) \wedge \rho_1^{(n-1)}}, \ldots, e^{\int_M (A - A_0) \wedge \rho_1^{(n-1)}}), \quad (4.1.18) \]
where its components can be rewritten in terms of the inner product (3.1), namely
\[ \int_{M} (A - A_0) \wedge \rho_j^{(n-1)} = (-1)^n < A - A_0, \star \rho_j^{(n-1)}>. \quad (4.1.19) \]
The family of open sets \(U_a^{A_0} = (\pi^{A_0}_{\mathcal{M}^{(p)}})^{-1}(V_a)\) provides a finite open cover \(\mathcal{U}^{A_0} = \{U_a^{A_0}\}\) of the infinite dimensional manifold \(\mathcal{M}_*^{(p)}\). Now we can construct a bundle atlas from the family of local trivializations \(\varphi_{a_0}^{A_0} : U_a^{A_0} \times G_0^{(M)} \to (\pi_{\mathcal{A}^{(p)}})^{-1}(U_a^{A_0})\), \(\varphi_{A_0}^A([A], g) = A^{a_0(A)} - g, \) and \((\varphi_{A_0}^A)^{-1}(A) = (\pi_{\mathcal{A}^{(p)}}(A), \omega_{a_0}^A(A))\), where
\[ \begin{align*}
\omega_{a_0}^A : & \pi^{-1}_{\mathcal{A}^{(p)}}(U_a^{A_0}) \to G_0^{(M)} \quad \omega_{a_0}^A(A) = \tilde{\kappa}_M(\sum_{j=1}^{b_1} \epsilon_{a_j}(A) \rho_k^{(1)} \cdot G_0 d_1^*(A - A_0) \cdot G_0 d_1^*(A - A_0)(x_0)), \\
\epsilon_{a_j} & = (\epsilon_{a_{j_1}}, \ldots, \epsilon_{a_{j_{b_1}}}, \ldots, \epsilon_{a_{j_1}}); \pi^{-1}_{\mathcal{A}^{(p)}}(U_a) \to \mathbb{Z}^N, \\
\epsilon_{a_{j_0}}^A(A) & = \frac{1}{2 \pi \sqrt{-1}} \int_M (A^\alpha - A_0^\alpha) \wedge \rho_j^{(n-1)} - s_{a_{j_0}}(\exp \int_M (A^\alpha - A_0^\alpha) \wedge \rho_j^{(n-1)}),
\end{align*} \quad (4.1.20) \]
for \(a = 1, \ldots, N\). To verify that (4.1.20) indeed gives a local trivialization of the bundle, we recognize that \(\frac{1}{2 \pi \sqrt{-1}} \int_{\gamma_j} g^* \vartheta : m_j \in \mathbb{Z}^N\). With respect to the basis \((\rho_1^{(1)})_{b_i}^{j},\) the orthogonal projector onto \(\text{Harm}^1(M)\) reads
\[ \Pi^{\text{Harm}^1(M)}(\alpha) = \sum_{j=k=1}^{b_1} h_{jk}^{-1} < \alpha, \rho_j^{(1)}> \rho_k^{(1)}, \quad (4.1.21) \]
From \(\epsilon_{a_0}^A(A) = \epsilon_{a_0}^A(A) + m_j\) and \(\Pi^{\text{Harm}^1(M)}(g^* \vartheta) = 2 \pi \sqrt{-1} \sum_{j=1}^{b_1} m_j \rho_j^{(1)}\) one gets \(\omega_{a_0}^A(A^g) = \omega_{a_0}^A(A) g\). According to the transition functions \(\varphi_{a_0}^{A_0} : U_a^{A_0} \cap U_{a'}^{A_0} \to G_0^{(M)}\),
\[ \varphi_{a_0}^{A_0}([A]) = \tilde{\kappa}_M(\sum_{j=1}^{b_1} g_{a_{j_0}}^{a_{j_0}}(e^{\int_M (A - A_0) \wedge \rho_j^{(n-1)}}) \rho_j^{(1)}), 0) \quad (4.1.22) \]
one concludes that the bundle is trivializable if $H^1(M; \mathbb{Z}) = 0$. Since the transition functions (4.1.22) are locally constant in the field space, $A^{(P)}$ is a flat principal bundle over $\mathcal{M}_{\pi}^{(P)}$.

In the next step of the proof we want to discuss the dependence on the background connection $A_0$. However, let $A'_0$ denote another background connection which generates the open cover $\mathcal{U}^{A'_0} = \{U^{A'_0}_a\}$ of the gauge orbit space. By passing to the common refinement (if necessary) $\epsilon^{A'_0}_a$ is related to $\epsilon^{A_0}_a$ by $\epsilon^{A'_0}_a(A) = \epsilon^{A_0}_a(A) + \hat{h}_a^{A_0,A'_0}(A)$, where

$$
\hat{h}_a^{A_0,A'_0}(A) = \frac{1}{2\pi \sqrt{-1}} \int_M (A_0 - A'_0) \wedge \rho_j^{(n-1)} + s_\alpha(\exp(\int_M (A - A_0) \wedge \rho_j^{(n-1)}))
- s_\alpha(\exp(\int_M (A - A'_0) \wedge \rho_j^{(n-1)})) \cdot \exp(\int_M (A - A'_0) \wedge \rho_j^{(n-1)})) \quad (4.1.23)
$$

is a locally constant gauge invariant function on $\pi^{-1}\mathcal{A}^{(P)}_a(U_a^{A_0} \cap U_a^{A'_0})$. Since this function $\hat{h}_a^{A_0,A'_0}(A) \in \mathbb{Z}^N$, there exists a map $h_a^{A_0,A'_0}: U_a^{A_0} \cap U_a^{A'_0} \rightarrow G_*(M)$ which is given by

$$
h_a^{A_0,A'_0}([A]) = \hat{\chi}^{(1)}(\sum_{j=1}^b h_a^{A_0,A'_0}(A) \rho_j^{(1)}), G_0 d^1_\alpha(A_0 - A'_0) - G_0 d^1_\alpha(A_0 - A'_0)(x_0)) \quad (4.1.24)
$$

resulting in $\sigma_a^{A'_0}([A]) = \sigma_a^{A_0}([A]) + (h_a([A]) A_0' A'_0)^* \theta$. This finally proves that any different choice of the background connection gives rise to an equivalent bundle atlas of $A^{(P)}(\mathcal{M}_{\pi}^{(P)}, \pi_{\mathcal{A}^{(P)}}, G_*(M))$. This concludes the proof of theorem 4.3. □

**Remark.** That the bundle of connections is non-trivial in general can be seen alternatively as follows: Since $A^{(P)}$ is contractible, the exact homotopy sequence

$$
\ldots \rightarrow \pi_k(A^{(P)}) \rightarrow \pi_k(M_*(P)) \rightarrow \pi_{k-1}(G_*(M)) \rightarrow \pi_{k-1}(A^{(P)}) \rightarrow \ldots
$$

implies $\pi_k(M_*(P)) \cong \pi_{k-1}(G_*(M))$. If $A^{(P)} \rightarrow M_*(P)$ was trivializable, then $A^{(P)} \cong M_*(P) \times G_*(P)$ would result in $\pi_{k-1}(G_*(M)) \times \pi_{k-1}(G_*(M)) = 0$. However, if any of the homotopy groups of $G_*(M)$ does not vanish, the premise is wrong and the bundle cannot be trivializable. In our case we have proved in proposition 4.2 that the gauge group is not connected.

The second important result which we are going to present is that the gauge orbit space $\mathcal{M}_*(P)$ itself admits the structure of a bundle over a finite dimensional manifold:

**Theorem 4.4.** For each arbitrary but fixed connection $A_0 \in A^{(P)}$, the manifold $\mathcal{M}_*(P)$ admits the structure of a trivializable vector bundle over $\mathbb{T}^N$ with projection $\pi^{A_0}_{\mathcal{M}_*(P)}$ and typical fiber $N^{(M)} := \text{im} d^*_2 \otimes \mathbb{T}^N$.

**Proof:**

A bundle atlas is provided by the following local diffeomorphisms

$$
\chi_a^{A_0}: V_a \times \mathcal{N}^{(M)} \rightarrow \mathcal{M}_*(P)
$$

$$
\chi_a^{A_0}([z_1, \ldots, z_b, \tau]) = [A_0 + 2\pi \sqrt{-1} \sum_{j=1}^b s_\alpha(z_j) \rho_j^{(1)} + \tau]
$$

$$
(\chi_a^{A_0})^{-1}([A]) = \left(\pi^{A_0}_{\mathcal{M}_*(P)}([A]), d^*_2 G_2(F_\pi - F_{A_0})\right) \quad (4.1.26)
$$

On each fiber $(\pi^{A_0}_{\mathcal{M}_*(P)})^{-1}([z_1, \ldots, z_b])$, there is a unique structure of a real vector space induced by the bundle chart $\chi_a^{A_0}$, giving rise to the vector bundle structure on $\mathcal{M}_*(P)$. Here $A_0$ represents a choice of origin in the fibers. □

This concludes the analysis of the geometrical structure of the configuration space. As a consequence, the topology of the gauge orbit space is characterized as follows:
Proposition 4.6. Let $\pi_k(M_\ast(P)) = \pi_{k-1}(G_s(M)) = 0 \quad k \geq 2$ (4.1.27)

How does the choice of the background gauge field affect the vector bundle structure of $M_\ast(P)$?

Proposition 4.6. Let $A_0$ and $A'_0$ be two arbitrary but fixed connections. Then the fiber bundles $M_\ast(P) \xrightarrow{\pi_{A_0}} T^{b_1 N}$ and $M_\ast(P) \xrightarrow{\pi_{A'_0}} T^{b_1 N}$ are isomorphic with respect to their vector bundle structures.

Proof. The invertible map $\Upsilon_{A_0,A'_0}: M_\ast(P) \rightarrow M_\ast(P)$, given by $\Upsilon_{A_0,A'_0}([A]) := [A + A'_0 - A]$ makes the following diagram of vector bundles commutative:

$$
\begin{array}{ccc}
M_\ast(P) & \xrightarrow{\Upsilon_{A_0,A'_0}} & M_\ast(P) \\
\pi_{A_0} \downarrow & & \pi_{A'_0} \downarrow \\
T^{b_1 N} & \xrightarrow{\Upsilon_{A_0,A'_0}} & T^{b_1 N}.
\end{array}
$$

4.2. The partition function, and the VEV of gauge invariant observables

In this section we are going to apply the results of the previous sections to calculate the partition function and the VEV of gauge invariant functions on a closed manifold $M$. According to the defining relations in (2.2) and (2.3), this requires first of all the choice of an appropriate gauge fixing function $S_{gf}$, which renders the gauge group volume (2.1) finite: Let $\theta = (\theta^\alpha)^N_{\alpha=1} \in \Omega^1(G_s(M), \Theta_s(M))$ denote the Maurer Cartan form on $G_s(M)$. Using the Maurer Cartan form the canonical metric (3.7) on the Lie algebra $\Theta_s(M)$ can be extended to the whole gauge group. This finally generates a left-invariant volume form $\text{vol}_{\Theta_s(M)} := \det (\text{tr})^\frac{1}{2} \text{D}g$ on $G_s(M)$. Let us now define a candidate for $S_{gf}$ by

$$
e^{-S_{gf}(g)} = \frac{e^{-S'_{gf}(g)}}{\int_{\Theta_s(M)} \text{vol}_{\Theta_s(M)} e^{-S'_{gf}}},
$$

with the auxiliary gauge fixing function

$$
S'_{gf}(g) = \frac{1}{2} \|d^* g^* \theta\|^2 + \frac{1}{2} \|\Pi^{\text{Harm}_\ast}(g^* \theta)\|^2
$$

In order to prove that $S_{gf}$ indeed gives a reasonable regularization of the gauge group we firstly recall the definition of the Riemann Theta function: Let $\Lambda$ be any symmetric complex $r \times r$ dimensional square matrix whose imaginary part is positive definite, $u \in \mathbb{C}^r$ and $\alpha, \beta \in \mathbb{Z}^r$ then the $r$-dimensional Theta function is defined by

$$
\Theta_r(u|\Lambda) = \sum_{n \in \mathbb{Z}^r} \exp \left\{ \pi \sqrt{-1} n^\dagger \cdot \Lambda \cdot n + 2\pi \sqrt{-1} n^\dagger \cdot u \right\},
$$

where the superscript $\dagger$ denotes the transpose.
Lemma 4.7. For the auxiliary gauge fixing function $S'_{gf}$ in (4.2.2), the regularized volume of the gauge group yields

$$
\int_{G^0\alpha} \text{vol}_{G^0\alpha} e^{-S'_{gf}} = (\det \Delta_{0|\text{im}d^*})^{-N} \cdot \Theta_{b_1N}(0/2\pi\sqrt{-1} \Lambda),
$$

where $\Lambda = \lambda \otimes h$ is the tensor product of the matrix $(\lambda_{\alpha,\beta})_{\alpha,\beta=1}^N$ of coupling constants and the metric on the harmonic 1-forms $(h_{jk})_{j,k=1}^{b_1}$.

Proof. The integral is calculated by using the isomorphism $\tilde{k}_{(M)}$ in (4.1.16). With respect to the fixed basis of harmonic 1-forms (4.1.16), any cohomology class $[\nu] \in H^1_{\text{Harm}}(M; \mathbb{R}^N)$ gives rise to the unique harmonic representative $\Pi^{\text{Harm}}(\nu) = \sum_{j=1}^{b_1} m_j \rho_j^{(1)}$, where $m_j \in \mathbb{Z}^N$. Hence any $g \in G^0\alpha$ is uniquely characterized by a pair $(\xi, m) \in G^0\alpha \times \mathbb{Z}^N$. As a consequence the integration over $G^0\alpha$ means integration over $\xi$ and summation over the integers $m_j$, where $j = 1, \ldots, b_1$, and $\alpha = 1, \ldots, N$.

It is easily shown that the integral over $G^0\alpha$ yields the determinant of the Laplacian whereas the sum over the harmonic forms with integer periods gives the Riemann Theta function. $\square$

At this point we would like to notice that generally all determinants of elliptic differential operators appearing in this paper are understood in terms of the zeta function regularization [15]: For any non-negative self-adjoint elliptic operator $\mathcal{B}$ its regularized determinant is defined by

$$
\det \mathcal{B} = \exp \left( -\frac{d}{ds}|_{s=0} \zeta(s|\mathcal{B}) \right),
$$

where $\zeta(s|\mathcal{B})$ is the zeta-function of the operator $\mathcal{B}$, given by

$$
\zeta(s|\mathcal{B}) = \sum_{\nu_j \neq 0} \nu_j^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-t\mathcal{B}} - \Pi^\mathcal{B}) dt,
$$

where $\nu_j$ are the non-vanishing eigenvalues of $\mathcal{B}$ and $\Pi^\mathcal{B}$ is the orthogonal projector onto the kernel of $\mathcal{B}$. Here the $\zeta$-function is analytic at the origin and possesses a meromorphic extension over $\mathbb{C}$.

What is the geometrical meaning of the regularizing gauge fixing function $S_{gf}$? Formally $\varpi = \text{vol}_{G^0\alpha} e^{-S_{gf}}$ can be regarded as a differential form of top degree on the gauge group with integral $\int_{G^0\alpha} \varpi = 1$. Hence $\varpi$ gives rise to a class in the cohomology with fast decrease of the gauge group. Since any other differential form $\varpi'$ of top degree which integrates to one belongs to the same cohomology class as $\varpi$, any two different gauge fixing functions are ambiguous up to an exact differential form on $G^0\alpha$.

Let us now introduce a specific partition of unity $\{p_a\}$ for $\mathcal{M}^{(P)}_\alpha$: We begin with a partition of unity $\{p_{a,j}\}_{j=1}^{Z_2}$ on the $j$-th 1-torus $\mathbb{T}^1$ subordinate to the open cover $\{U_{a,j}\}$. Let $q_{a,j}: \mathbb{T}^{b_{1N}} \to \mathbb{T}^1$, $q_{a,j}(z_{j,1}, \ldots, z_{j,N}) = z_{j,a}$ be the projection onto the $j$-th factor. Then $\tilde{p}_a := \prod_{a=1}^N \prod_{j=1}^{p_{a,j}} q_{a,j} \tilde{p}_{a,j}$ induces a partition of unity of $\mathbb{T}^{b_{1N}}$ subordinate to $\{V_a\}$. Finally $p_a := \pi^*_{\mathcal{M}^{(P)}_\alpha} \tilde{p}_a$ is the sought-after partition of unity subordinate to the open cover $\mathcal{U}^{A_\alpha}$ of the gauge orbit space $\mathcal{M}^{(P)}_\alpha$.

Now we are prepared to display the modified global functional integral in the field space $\mathcal{A}^{(P)}$ according to the defining relations in (2.2) and (2.3):

**Proposition 4.8.** Let us choose the gauge fixing function $S_{gf}$ (4.2.1). The partition function for the abelian gauge theory on a closed manifold $M$ with the classical action (3.6) is given by

$$
Z^{(P)}_{A_\alpha} = \int_{\mathcal{A}^{(P)}} \text{vol}_{\mathcal{A}^{(P)}} \mathcal{F}(A) \cdot e^{-\frac{1}{2} \left( |F_A|^2 + |d^*(A-A_\alpha)|^2 \right)},
$$

where
\[ \mathcal{F}(A) = (\det \Lambda_0|_{\text{ind}_0^N})^N \cdot \Theta_{b_1 N}(0) (2\pi \sqrt{-1})^{-1} \sum_{a \in \mathbb{Z}_2^{b_1 N}} (\pi_A^* \rho_a) e^{-2\pi^2 \sum_{a=1}^{b_1} \sum_{j,k=1}^N \lambda_{\alpha\beta} h_{jk} \epsilon_{\alpha \gamma}^{\beta \delta} (A) \epsilon_{\alpha \gamma}^{\beta \delta} (A)}, \]

(4.2.8)

with the multi-index \( a = (a_1, \ldots, a_{b_1 N}) \in \mathbb{Z}_2^{b_1 N} \).

**Proof.** The formula for the partition function can be verified directly by using the bundle coordinates (4.1.20) and the gauge fixing function (4.2.2). \( \square \)

Unlike the conventional Faddeev-Popov result for the partition function there has appeared an additional contribution in \( \mathcal{F}(A) \) caused by the non-triviality of the bundle of connections and the non-compactness of the gauge group. Since \( \mathcal{F}(A) \) is non-vanishing and positive, the interpretation of the integrand as a probability density remains valid. In the case of simply connected manifolds \( M \), where the Gribov problem is absent, eq. (4.2.7) does reproduce exactly the conventional Faddeev-Popov formula for the abelian gauge theory in the Lorentz gauge. In fact, \( \mathcal{F}(A) \) reduces to the field independent multiplicative constant \( (\det \Lambda_0|_{\text{ind}_0^N})^N \) which according to lemma 4.7 is nothing but the inverse of the regularized volume of the subgroup of infinitesimal gauge transformations.

What is the effect of \( \mathcal{F}(A) \) in the topologically non-trivial case where it modifies the gauge fixed action? Since the bundle of connections is flat, the functional integral (4.2.7) can be decomposed on each \( \pi_{\mathcal{A}(U)_a} \subset \mathcal{A}(P) \) into a disjoint union of open sets, each of them diffeomorphic to the product of the gauge fixing submanifold \( \{ A|\omega_A^0 (A) = 1 \} \) and a sufficiently small open set of those gauge transformations which are connected to the identity. On each of these slices the functionals \( \epsilon_{\alpha \gamma}^{\beta \delta} \) are constant giving rise to a regularization of gauge transformations not connected to the identity. Moreover, the partition of unity \( \rho_a \) is constant on each \( \pi_{\mathcal{A}(P)}^{-1} (U_a) \). We will see in the sequel that the VEV’s of gauge invariant observables are not affected by the explicit form of the gauge fixing function \( S_{gf} \).

In the next step we aim to find an explicit expression for the partition function in (4.2.7). The strategy is to split the gauge fields \( A \in \mathcal{A}(P) \) into components according to the bundle structures which were described by the theorems 4.2 and 4.3 and then to split the integration over \( \mathcal{A}(P) \) into an integration over the base manifold \( \pi_{\mathcal{A}(P)} (U_a) \) and an integration over the fiber \( \mathcal{G}_s^M \times \mathcal{N}^M \).

We shall begin with the decomposition of the volume form \( \text{vol}_{\mathcal{A}(P)} \). Let us define the local diffeomorphisms \( \psi_A^a = \psi_A^a \circ (\chi_0^a \times \mathcal{I}): V_a \times \mathcal{N}^M \times \mathcal{G}_s^M \rightarrow (\pi_{\mathcal{A}(P)} \circ \pi_{\mathcal{A}(P)})^{-1} (V_a) \). Then the differential of \( \psi_A^a \) can be easily calculated to yield

\[ T(\bar{z}_1, \ldots, \bar{z}_N, \mathcal{Y})^a = \frac{N}{b_1} \sum_{i=1}^{b_1} \sum_{j=1}^N \theta_{z_{i \alpha}}^{\beta \gamma} (w_{i \alpha}) \rho_{i j} (1) + u + d\theta_g (\mathcal{Y}), \]

(4.2.9)

where \( \theta_{z_{i \alpha}}^{\beta \gamma} \) is the Mauer Cartan form on \( T^1, \bar{z}_j = (z_{j1}, \ldots, z_{jN}) \in \mathbb{T}^N \) for \( j = 1, \ldots, b_1 \) and \( \bar{w}_j = (w_{j1}, \ldots, w_{jN}) \in T_{\mathbb{Z}_2} \mathbb{T}^N, u \in T_{\mathbb{Z}_2} \mathcal{N}^M \) and \( \mathcal{Y} \in T_{\mathbb{Z}_2} \mathcal{G}_s^M \). The metric (3.7) can be recast into

\[ ((\psi_A^a)^*)^* <, \lambda>(\bar{z}_1, \ldots, \bar{z}_N, \mathcal{Y}) = ((\bar{w}_1^1, \ldots, \bar{w}_N^1, u^1, \mathcal{Y}^1), (\bar{w}_1^2, \ldots, \bar{w}_N^2, u^2, \mathcal{Y}^2)) = \sum_{\alpha = 1}^{b_1} \sum_{i=1}^N \sum_{j=1}^N \lambda_{\alpha \beta} \theta_{\gamma \delta} (w_{i \alpha}^1) \theta_{\gamma \delta} (w_{j \beta}^2) h_{jk} + < u^1, u^2 > \lambda + < d\theta_g (\mathcal{Y}^1), d\theta_g (\mathcal{Y}^2) > \lambda. \]

(4.2.10)

Formally (4.2.10) can be equivalently rewritten into the following matrix form

\[ ((\psi_A^a)^*)^* <, \lambda>(\bar{z}_1, \ldots, \bar{z}_N, \mathcal{Y}) = \begin{pmatrix} h_{jk} \lambda_{\alpha \beta} \theta_{\gamma \delta} (w_{i \alpha}^1) \theta_{\gamma \delta} (w_{j \beta}^2) & 0 & 0 \\ 0 & \lambda_{\alpha \beta} & 0 \\ 0 & 0 & \lambda_{\alpha \beta} \theta_{\gamma \delta} (w_{i \alpha}^1) \theta_{\gamma \delta} (w_{j \beta}^2) \end{pmatrix} \]

(4.2.11)
with \( \bar{\psi}^T \) and \( \bar{\theta} \) denoting the complex conjugates of the Maurer Cartan forms on \( T^1 \) and \( G_s^{(M)} \) respectively. Each component in (4.2.11) displays the induced metric on the corresponding space. In terms of the local trivialization the volume form becomes

\[
(\psi^A)\ast \text{vol}_{A(P)} = (\det h)^{N/2} \text{det} (\Delta_0)_{\text{ind}^d T}^{N/2} \text{vol}_{\Psi_1 N} \land \text{vol}_{\mathcal{N}(M)} \land \text{vol}_{G_s^{(M)}}, \tag{4.2.12}
\]

where \( \text{vol}_{\Psi_1 N} = (\sqrt{\text{det} T})^{-\beta_1 N} q_{f}^T \land \ldots \land q_{b_1 N}^T \) is the induced volume form on \( \mathbb{T}^{b_1 N} \), which in (4.2.12) is restricted to the patch \( V_a \). The volume form \( \text{vol}_{\mathcal{N}(M)} \) is induced by the flat metric on \( \mathcal{N}(M) \) and can be formally written as \( \text{vol}_{\mathcal{N}(M)} = \mathcal{D} \).

**Lemma 4.49.** The background connection \( A_0 \in \mathcal{A}(P) \) can be chosen to satisfy the classical equation of motion, \( d^*_s F_{A_0} = 0 \).

*Proof.* Given any background gauge field \( A_0 \) the modified background connection \( A_0 = A'_0 - G_1 d^*_s F_{A_0} \) satisfies the requested equation. \( \square \)

**Proposition 4.10.** There exists a globally defined density \( \hat{\Xi}^{(P)} \) on the product \( \mathbb{T}^{b_1 N} \times \mathcal{N}(M) \times G_s^{(M)} \) so that \( (\psi^A_0)^\ast \hat{\Xi}^{(P)} = i_{V_a} \hat{\Xi}^{(P)}(\gamma) \), where \( i_{V_a} : V_a \rightarrow \mathbb{T}^{b_1 N} \) is the restriction to \( V_a \).

*Proof.* Using (2.2) and (4.2.12) one verifies by a direct calculation that

\[
\hat{\Xi}^{(P)} = (\det h)^{N/2}(\det \Delta_0)_{\text{ind}^d T}^{N/2} \text{vol}_{\Psi_1 N} \land \text{vol}_{\mathcal{N}(M)} \land \text{vol}_{G_s^{(M)}} e^{-\frac{1}{2} \| F_{A_0} \|_2^2 + \langle r, \mathcal{D} \rangle} \text{vol}_{\mathcal{N}(M)} \land \text{vol}_{G_s^{(M)}}. \tag{4.2.13}
\]

gives the global density with the required property. \( \square \)

Geometrically the existence of the global density \( \hat{\Xi}^{(P)} \) is related to the fact that the bundle of connections is flat in the abelian gauge theory. Since the bundle of connections is not flat in the non-abelian Yang-Mills theory [3,4], the corresponding local densities cannot be extended to a global form without use of a partition of unity.

Let \( f \) be a gauge invariant observable, i.e. a (real-valued) function on \( \mathcal{A}(P) \), satisfying \( f(A^\prime) = f(A) \). In the following we will denote the induced function on \( \mathcal{M}_s^{(P)} \) with the same symbol. Then \( \hat{f} := (\chi^A_0)^\ast f \) is a globally defined function on \( \mathbb{T}^{b_1 N} \times \mathcal{N}(M) \).

Let \( e_{z_{11}, \ldots, z_{b_1 N}} := z_{11} \ldots z_{b_1 N} \) be an orthonormal basis of \( L^2(\mathbb{T}^{b_1 N}; \mathbb{C}) \) with respect to the inner product \( \langle f_1, f_2 \rangle := \frac{1}{(2\pi)^{b_1 N}} \int_{\mathbb{T}^{b_1 N}} \text{vol}_{\Psi_1 N} \text{vol}_{\Psi_1 N} f_1 f_2 \), where \( z_{j\alpha} \in \mathbb{T}^1 \) and \( mj_{\alpha} \in \mathbb{Z} \). Then \( \hat{f}(\cdot, A_0 + \tau) \) can be rewritten in terms of a Fourier series expansion on \( \mathbb{T}^{b_1 N} \) as

\[
\hat{f}(\cdot, A_0 + \tau) = \sum_{m_{11} \in \mathbb{Z}} \cdots \sum_{m_{b_1 N} \in \mathbb{Z}} \hat{f}(m_{11}, \ldots, m_{b_1 N})(A_0 + \tau) e_{m_{11}, \ldots, m_{b_1 N}}, \tag{4.2.14}
\]

with Fourier coefficients

\[
\hat{f}(m_{11}, \ldots, m_{b_1 N})(A_0 + \tau) := \langle e_{m_{11}, \ldots, m_{b_1 N}}, \hat{f}(\cdot, A_0 + \tau) \rangle = \int_0^{1} \cdots \int_0^{1} dt_{11} \ldots dt_{b_1 N} (e^{2\pi \sqrt{-1} t_{11}} \ldots, e^{2\pi \sqrt{-1} t_{b_1 N}}, A_0 + \tau) e^{-2\pi \sqrt{-1} \sum_{a=1}^{b_1 N} m_{j\alpha} t_{j\alpha}}. \tag{4.2.15}
\]

Using (4.2.13) and (4.2.14) we get

\[
I^{(P)}(f) = \sum_{a \in \mathbb{T}^{b_1 N}} \int_{V_a \times \mathcal{N}(M) \times G_s^{(M)}} \text{pr}_{\Psi_1 N}^\ast \mathcal{D} \hat{f}(0) \cdot \text{vol}_{\Psi_1 N} \land \text{vol}_{\mathcal{N}(M)} \land \text{vol}_{G_s^{(M)}} e^{-\frac{1}{2} \| F_{A_0} \|_2^2} \mathcal{D} \hat{f}(0) (A_0 + \tau) e^{-\frac{1}{2} \langle r, \mathcal{D} \rangle} \tag{4.2.16}
\]
where \( pr_{\psi_{A}} \) and \( pr_{G(M)} \) are the projections onto the first and third factor in \( T^{\otimes N} \times N^{(M)} \times G^{(M)}_{*} \), respectively. Because of the gauge invariance, the integral \( I^{(P)}(f) \) is independent of the explicit form of the partition of unity. Using that \( \det(\Delta_{1}|_{imd_{1}}) = \det(\Delta_{0}|_{imd_{1}}) \) one gets from (4.2.16):

**Proposition 4.11.** 
For any fixed component \( A^{(P)} \) the following holds:

1. The partition function \( Z_{A_{0}}^{(P)} \) is given by

\[
Z_{A_{0}}^{(P)} = (2\pi)^{h_{2}}(\det h)^{N/2}(\det \Delta_{0}|_{imd_{1}})^{N}(\det \Delta_{1}|_{Harm^{1}(M)})^{-N/2}e^{-\frac{1}{2}\|F_{A_{0}}\|^{2}}. \quad (4.2.17)
\]

2. The VEV of any gauge invariant function \( f \) reads

\[
<f>_{(P)} = (\det \Delta_{0}|_{imd_{1}})^{-N/2}(\det \Delta_{1}|_{Harm^{1}(M)})^{N/2} \int_{N^{(M)}} D\tau \hat{f}(0,\ldots,0)(A_{0} + \tau) \cdot e^{-\frac{1}{2}<\tau,\Delta_{1}|_{\chi(M)},\tau>^{\lambda}}. \quad (4.2.18)
\]

Let us take a simply connected manifold \( M \) and consider the trivial \( T^{N} \)-bundle \( P \cong M \times T^{N} \). Then one can choose \( A_{0} = 0 \). In that case \( M^{(P)}_{*} \cong N^{(M)} \) implying that the inequivalent gauge fields are in one-to-one correspondence with the space of transversal fields. For \( N = 1 \) the partition function reduces to

\[
Z_{A_{0}=0}^{(M\times T^{N})} = (\det \Delta_{0}|_{imd_{1}})(\det \Delta_{1})^{-1/2}, \quad (4.2.19)
\]

which is the well-known covariant expression for the quantized Maxwell theory (see e.g. [15]).

Let us reflect on the difference between the modified functional integral and the Faddeev-Popov procedure once again. Usually it is called that a convergent factor is introduced by the gauge fixing in a way that does not affect the VEV of any gauge invariant observable. However, if Gribov ambiguities are present the conventional gauge fixed partition function in the Lorentz gauge

\[
\int_{A^{(P)}} vol_{A^{(P)}} e^{-S_{inv}(A) - \frac{1}{2}\|d^{*}(A - A_{0})\|^{2}}, \quad (4.2.20)
\]

would never yield a finite functional integral: In fact, rewriting (4.2.20) in terms of the local coordinates \( \{\psi^{A}_{a}\} \) and using proposition 4.2 the following divergent integral

\[
\int_{G^{M}_{2}} vol_{G^{(M)}} e^{-\frac{1}{2}\|d^{*}g^{*}\|^{2}} \quad (4.2.21)
\]

appears as a factor in the total functional integral. This infinite factor is a consequence of the fact that the conventional gauge fixing term \( \frac{1}{2}\|d^{*}(A - A_{0})\|^{2} \) does not damp the gauge transformations which are not connected to the unity. Nevertheless this term is sufficient to regularize the subgroup of infinitesimal gauge transformations.

**The partition function on \( A^{(M)} \).** The principal \( T^{N} \)-bundles \( P \) are labelled by their Chern-class \( c = (c^{a})_{a=1}^{N} \in H^{2}(M;\mathbb{Z}^{N}) \). Since the cohomology of \( M \) is finitely generated, \( c \) takes the form

\[
c^{a} = \sum_{j=1}^{b_{2}} m_{ja}c_{j}^{(2)} + \sum_{k=1}^{r} t_{ka}y_{k}^{(2)}, \quad (4.2.22)
\]

where \( (c_{j}^{(2)})_{j=1}^{b_{2}} \) denotes a Betti basis of \( H^{2}(M;\mathbb{Z}) \), \( b_{2} = dim H^{2}(M;\mathbb{R}) \) and \( m_{ja} \in \mathbb{Z} \) for \( j = 1,\ldots,b_{2}, \alpha = 1,\ldots,N \). On the other hand \( Tor H^{2}(M;\mathbb{Z}) \) is generated by a basis \( (y_{k}^{(2)})_{k=1}^{r} \) with torsion coefficients \( l_{k}, \) i.e. \( l_{k}y_{k}^{(2)} = 0 \) and \( t_{ka} \in \mathbb{Z} \) for \( k = 1,\ldots,r, \alpha = 1,\ldots,N \). According to lemma
(4.9), \( F_{\alpha_0} \in \text{Harm}_2^2(M; \mathbb{R}) \otimes \mathbb{R}^N \). Let \( \rho_j^{(2)} \in \text{Harm}_2^2(M; \mathbb{R}) \), for \( j = 1, \ldots, b_2 \), be a basis of harmonic two forms on \( M \) with integer periods, and let \( h_{jk}^{(2)} = < \rho_j^{(2)}, \rho_k^{(2)} > \) denote the induced metric on \( \text{Harm}_2^2(M; \mathbb{R}) \). Then the field strength can be rewritten into

\[
F_{\alpha_0} = 2\pi \sqrt{-1}\sum_{k=1}^{b_2} m_{k\alpha} \rho_k^{(2)}, \quad m_{k\alpha} = \sum_{j=1}^{b_2} (h_{jk}^{(2)})^{-1} < F_{\alpha_0}^{(2)}, \rho_j^{(2)} > \in \mathbb{Z}. \tag{4.2.23}
\]

Let us define the tensor product \( \Lambda^{(2)} := \lambda \otimes h^{(2)} \), then the sum over the equivalence classes of principal bundles \( (P) \) in (2.6) can be split into a sum over the free part and the torsion part of \( H^2(M; \mathbb{Z}^N) \). Hence we obtain from (2.6):

**Proposition 4.12.** The partition function \( Z \) for the abelian gauge theory on the total configuration space \( \mathcal{A}^{(M)} \) is given by

\[
Z = (2\pi)^{b_1 N} (\det h)^{N/2} (\det \Delta_0 |_{\text{indt}})^N (\det \Delta_1 |_{\text{Harm}^1(M^+)})^{-N/2} \\
\times \Theta_{N b_2}(0) [2\pi \sqrt{-1} \Lambda^{(2)} \text{ord}(\text{Tor} H^2(M; \mathbb{Z}^N))],
\]

where \( \text{ord}(\text{Tor} H^2(M; \mathbb{Z}^N)) \) is the order of the finite torsion subgroup of \( H^2(M; \mathbb{Z}^N) \).

The partition function \( Z \) does not depend on the choice of the basis \( \{ \rho_j^{(1)} \} \) and \( \{ \rho_j^{(2)} \} \) because any other basis of \( \text{Harm}_2^2(M; \mathbb{R}) \) \((k = 1, 2)\) is connected by a unimodular transformation, under which both \( \det h \) and the \( \Theta \)-function remain invariant.

**The correlation functions for the field strength.** As an example we shall apply our results to the determination of the VEV of a polynomial in the field strength \( F_A \). This VEV is understood in the following distributional sense

\[
\mathcal{W}_q^{(F_A)}(\eta_1, \ldots, \eta_q) := < \prod_{j=1}^{q} < F_A, \eta_j > >_p, \tag{4.2.25}
\]

for \( \eta_1, \ldots, \eta_q \in \Omega^2(M; \mathbb{R}^N) \). To calculate (4.2.25) let us consider the gauge invariant observable \( f(A, \hat{J}) = e^{<F_A, \hat{J}>} \) with source \( \hat{J} \in \Omega^2(M; \mathbb{R}^N) \). Its VEV follows from (4.2.18)

\[
< e^{<F_A, \hat{J}>} >_p = (2\pi)^{b_1 N} (\det h)^{N/2} (\det \Delta_0 |_{\text{indt}})^N (\det \Delta_1 |_{\text{Harm}^1(M^+)})^{-N/2} \\
\times e^{-\frac{(2\pi)^2}{\alpha,\beta=1, \ldots, b_2} \sum_{k=1}^{b_2} \lambda_{\alpha\beta} h_{jk}^{(2)} m_{jk} + 2\pi \sum_{\alpha=1}^{b_2} \sum_{j=1}^{b_2} m_{j\alpha} < \sqrt{-1} \rho_j^{(2)}, \hat{J}_\alpha >} \\
\times \exp \left\{ \frac{\pi}{\alpha, \beta=1} \sum_{\alpha, \beta=1}^{b_2} (\lambda^{-1})_{\alpha\beta} < J^\alpha, (\Pi \text{Harm}^2(M)^+) - d^2 G_3 d_2 > J^\beta > \right\}. \tag{4.2.26}
\]

Finally \( \mathcal{W}_q \) can be obtained from (4.2.26) by differentiation, namely

\[
\mathcal{W}_q^{(F_A)}(\eta_1, \ldots, \eta_q) = \frac{\partial^q}{\partial t_1 \cdots \partial t_q} |_{t_1 = . . . = t_q = 0} < e^{<F_A, \sum_{i=1}^{q} t_i \eta_i >} > (p). \tag{4.2.27}
\]

Let us now introduce the following abbreviations

\[
\mu_{ij} := < \eta_i, (\Pi \text{Harm}^2(M)^+) - d^2 G_3 d_2 > \eta_j >_{\lambda^{-1}}
\]

\[
\nu_i := < F_{\alpha_0}, \eta_i >, \tag{4.2.28}
\]

and let \( \varsigma \) denote a permutation of the indices \( \{1, \ldots, q\} \) then a lengthy calculation gives the following:
Proposition 4.13. For any fixed component $A^{(P)}$, the VEVs for the field strength of degree $q$ are given by

1) $q = 2k$

\[
W_{2k}^{(P)}(\eta_1 \ldots, \eta_{2k}) = \frac{1}{(2k)!} \sum_\zeta \nu_{\zeta(1)} \cdots \nu_{\zeta(2k)} + \sum_{l=1}^{k-1} \frac{1}{2^l l!(2-k-l)!} \sum_\zeta \mu_{\zeta(1)\zeta(2)} \cdots \mu_{\zeta(2l-1)\zeta(2l)} \nu_{\zeta(2l+1)} \cdots \nu_{\zeta(2k)} + \frac{1}{2^k k!} \sum_\zeta \mu_{\zeta(1)\zeta(2)} \cdots \mu_{\zeta(2k-1)\zeta(2k)}
\]

(4.2.29)

2) For $q = 2k + 1$

\[
W_{2k+1}^{(P)}(\eta_1 \ldots, \eta_{2k+1}) = \frac{1}{(2k+1)!} \sum_\zeta \nu_{\zeta(1)} \cdots \nu_{\zeta(2k+1)} + \sum_{l=1}^{k} \frac{1}{2^l l!(2-k-l+1)!} \sum_\zeta \mu_{\zeta(1)\zeta(2)} \cdots \mu_{\zeta(2l-1)\zeta(2l)} \nu_{\zeta(2l+1)} \cdots \nu_{\zeta(2k+1)}
\]

(4.2.30)

\[ \square \]

By construction (4.2.29) and (4.2.30) are independent of the actual choice for the gauge fixing function.

4.3. The Green’s functions for the gauge fields

Let $J \in \Omega^1(M; t^N)$ be the source for the gauge fields then we define the generating functional by

\[
Z_{A_0}^{(P)}[J] = \langle \pi_{A_0}^{(P)} p_{a} \rangle \cdot \Xi_{A_0}^{(P)} \cdot e^{<J,A_0>},
\]

which is a generalization of the definition for the generating functional around classical solutions [31]. From (4.3.1) the $q$-point Green’s functions $S_q$ are constructed as follows:

\[
S_q^{(P)}(v_1 \ldots, v_q) := \frac{\partial^q}{\partial t_1 \cdots \partial t_q} |_{t_1 = \ldots = t_q = 0} \frac{Z_{A_0}^{(P)}[\sum_{i=1}^{q} t_i v_i; A_0]}{Z_{A_0}^{(P)}[0]},
\]

(4.3.2)

for $v_1, \ldots, v_q \in \Omega^1(M; t^N)$. In order to find an explicit expression for the Green’s functions we shall recast $Z^{(P)}[J; A_0]$ in terms of the local trivialization $\{\psi_a^{A_0}\}$. Let $\Pi^{imd} := d_0 G_0 d_1^*$ be the projector onto the space of exact one-forms on $M$ then the integration over the gauge group yields:

Lemma 4.13. Let us choose the auxiliary gauge fixing function $S_{gf}$ in (4.2.2). Then the integration over the gauge group gives

\[
\int_{G^1(M)} \text{vol}_{G^1(M)} e^{-S_{gf}(g) + <J,g^*\vartheta>} = (\text{det} \Delta_0 |_{imd_1})^{-N} \cdot e^{1/2 <\Pi^{imd}(J),G_1 \Pi^{md}(J)>} \cdot \Theta_{b_1N}(K(J)|2\pi \sqrt{-1} \Lambda),
\]

(4.3.3)

where $K_\alpha^a(J) = -\sqrt{-1} < J^\alpha, \sqrt{-1} \Pi_{j}^{(1)}> \text{ with } j = 1, \ldots, b_1 \text{ and } \alpha = 1, \ldots, N$ is regarded as $b_1N$-dimensional complex vector $K(J)$. \[ \square \]
Using (4.2.9) and (4.3.3) one obtains for the generating functional in (4.3.1)

\[
Z_{A_0}^{(P)}[J] = (\det h)^{N/2} (\det \Delta_{0|\text{Im}d^*})^N \cdot (\det \Delta_1|_{Harm^1(M)\perp}^{-N/2} \cdot e^{-\frac{b}{2}P_{A_0}^2} \cdot e^{\frac{1}{2}g_{J,G_1}J} \cdot \Theta_{b_1N}(K[J]2\pi\sqrt{-\Lambda})^{-1} \\
\times \int_{\mathbb{T}^k} vol_{T^k} N \sum_{a_{11}=1}^{2} \cdots \sum_{a_{b_1 N}=1}^{2} q_{11}^{*} \hat{p}_{a_{11}} \cdots q_{b_1 N}^{*} \hat{p}_{a_{b_1 N}} e^{2\pi \sum_{\alpha=1}^{N} \sum_{j=1}^{b_1} q_{j a}^{*} s_{a j} <J^{\alpha}, \sqrt{-\Lambda} P_{j}^{(1)}>}. 
\]

(4.3.4)

Defining the two field independent factors

\[
\varepsilon^{(1)}_{j a} := \sum_{a_{j a}=1}^{2} \int_{V_{a j a}} vol_{V_{a j a}} \hat{p}_{a_{j a}} s_{a j a}, \\
\varepsilon^{(2)}_{j a, k \beta} := \int_{\mathbb{T}^k} vol_{T^k} N \sum_{a_{11}=1}^{2} \cdots \sum_{a_{b_1 N}=1}^{2} q_{11}^{*} \hat{p}_{a_{11}} \cdots q_{b_1 N}^{*} \hat{p}_{a_{b_1 N}} e^{2\pi \sum_{\alpha=1}^{N} \sum_{j=1}^{b_1} q_{j a}^{*} s_{a j a} <J^{\alpha}, \sqrt{-\Lambda} P_{j}^{(1)}>}. 
\]

(4.3.5)

one finally ends up with the following result:

**Proposition 4.14.** For any fixed component \( A^{(P)} \) and chosen gauge fixing function (4.2.1), the Green’s functions are given by

1) One-point function:

\[
S_1^{(P)}(v) = \sum_{\alpha=1}^{N} \sum_{j=1}^{b_1} \varepsilon^{(1)}_{j a} <v^{\alpha}, \sqrt{-\Lambda} P_{j}^{(1)}> + \frac{d}{dt}_{t=0} \ln \Theta_{b_1N}(K(tv)|2\pi\sqrt{-\Lambda}). 
\]

(4.3.6)

2) Two-point function:

\[
S_2^{(P)}(v_1, v_2) = <v_1, G_1 v_2 >_{\lambda^{-1}} \\
+ \sum_{\alpha=1}^{N} \sum_{j=1}^{b_1} \varepsilon^{(1)}_{j a} <v_1^{\alpha}, \sqrt{-\Lambda} P_{j}^{(1)}> \left( \frac{d}{dt}_{t=0} \ln \Theta_{b_1N}(K(tv)|2\pi\sqrt{-\Lambda}) \right) \\
+ \sum_{\alpha=1}^{N} \sum_{j=1}^{b_1} \varepsilon^{(1)}_{j a} <v_2^{\alpha}, \sqrt{-\Lambda} P_{j}^{(1)}> \left( \frac{d}{dt}_{t=0} \ln \Theta_{b_1N}(K(tv)|2\pi\sqrt{-\Lambda}) \right) \\
+ (2\pi)^{2-b_1 N} \sum_{\alpha,\beta=1}^{N} \sum_{j,k=1}^{b_1} \varepsilon^{(2)}_{j a, k \beta} <v_1^{\alpha}, \sqrt{-\Lambda} P_{j}^{(1)}> <v_2^{\beta}, \sqrt{-\Lambda} P_{k}^{(1)}> \\
+ \Theta_{b_1N}(0|2\pi\sqrt{-\Lambda})^{-1} \frac{\partial^2}{\partial t_1 \partial t_2} |_{t_1=t_2=0} \Theta_{b_1N}(K(\sum_{l=1}^{2} t_l v_l)|2\pi\sqrt{-\Lambda}). 
\]

(4.3.7)

\[\square\]

On manifolds with vanishing first Betti number, the equations (4.3.6) and (4.3.7) yield the well-known result: The one-point function vanishes and the two-point function reduces to the Greens operator \( G_1 \).

Before closing this section we want to display explicit results for the factors in (4.3.5). In the following local coordinate system of \( \mathbb{T}^1 \)

\[
v_1: V_1 \to (0,1) \quad v_1^{-1}(t) = (\cos 2\pi t, \sin 2\pi t) \\
v_2: V_2 \to (-\frac{1}{2}, \frac{1}{2}) \quad v_2^{-1}(t) = (\cos 2\pi t, \sin 2\pi t),
\]

(4.3.8)
a partition of unity subordinate to \( V_{j \alpha} \subset T^1 \) (see subsection 4.2) can be given by \( \tilde{p}_1(t) = \sin^2(\pi t) \) and \( \tilde{p}_2(t) = \cos^2(\pi t) \). Then

\[
\varepsilon_{j \alpha}^{(1)} = \frac{3\pi}{2}, \quad \varepsilon_{j \alpha, k \beta}^{(2)} = \begin{cases} (2\pi)^{b_1N-1} \left( \frac{17\pi}{12} - \frac{1}{\pi} \right), & \text{for } j = k \text{ and } \alpha = \beta \\ (2\pi)^{b_1N-2} \left( \frac{17\pi}{12} \right)^2, & \text{for } j \neq k \text{ or } \alpha \neq \beta \text{ or both.} \end{cases} \tag{4.3.9}
\]

Any other local section \( s'_a \) of (4.1.7) is connected with the section \( s_a \), which was defined in (4.1.10), by \( s'_a := (s'_{a_1}, \ldots, s'_{a_{b_1}}) = (s_{a_1} + m^{11}_{a_1}, \ldots, s_{a_{b_1}} + m^{b_1}_{a_{b_1}}) \) with \( m^{j \alpha}_{a \alpha} \in \mathbb{Z} \) for \( j = 1, \ldots, b_1 \) and \( \alpha = 1, \ldots, N \). In terms of these new sections, the factors in (4.3.5) become

\[
\varepsilon_{j \alpha}^{(1)} = \pi(m^{j \alpha}_1 + m^{j \alpha}_2 + \frac{3}{2}), \\
\varepsilon_{j \alpha, k \beta}^{(2)} = \begin{cases} (2\pi) \left( \frac{17\pi}{12} - \frac{1}{\pi} + \pi(m^{11}_\alpha(m^{11}_\alpha + 1) + m^{j \alpha}_2(m^{j \alpha}_2 + 2)) \right), & \text{for } j = k \text{ and } \alpha = \beta \\ \pi^2(m^{j \alpha}_1 + m^{j \alpha}_2 + \frac{3}{2})(m^{k \beta}_1 + m^{k \beta}_2 + \frac{3}{2}), & \text{for } j \neq k \text{ or } \alpha \neq \beta \text{ or both.} \end{cases} \tag{4.3.10}
\]

Thus it is not possible to arrange a local trivialization of \( A^{(P)} \) in such a way that the additional contributions in the Green’s functions would vanish. The novel feature of the modified functional integral is that the existence of Gribov ambiguities affects the contributions in the Green’s functions would vanish. The novel feature of the modified functional integral is that the existence of Gribov ambiguities affects the \( q \)-th point Green’s functions of the \( \lambda \)-theory of \( \hat{A} \).

## 5. Abelian gauge theories on manifolds with boundary

In this chapter we want to address the construction of the modified functional integral for the abelian gauge theory with the classical action (3.6) over a manifold without non-empty boundary. The functional integral on such manifolds requires boundary conditions to be imposed on the fields on \( \partial M \). In consequence the functional integral will become a functional of the fields on the boundary.

### 5.1. The geometry of gauge fields

Let \( M \) denote a \( n \)-dimensional compact, connected and oriented manifold with a non-empty boundary \( \partial M \), where \( i_{\partial M} : \partial M \rightarrow M \) is the inclusion. We choose an arbitrary but fixed principal \( T^N \)-bundle \( Q(\partial M, \pi_Q, T^N) \) over \( \partial M \). Let us consider the set \( \mathfrak{P}_Q[M; T^N] \) of those principal \( T^N \)-bundles \( P(M, \pi_P, T^N) \) over \( M \) which, if pulled back to the boundary \( \partial M \), are isomorphic to \( Q \) via a map \( \phi \) such that the following diagram of bundles commutes:

\[
\begin{array}{ccc}
Q & \xrightarrow{\phi} & \partial P := i^\partial M P \xrightarrow{i^\partial M} P \\
\pi_Q \downarrow & & \pi_{\partial P} \downarrow \\
\partial M & \xrightarrow{id} & \partial M \\
\pi_P \downarrow & & \pi_P \downarrow \\
M & \xrightarrow{i_{\partial M}} & M.
\end{array}
\tag{5.1.1}
\]

Boundary conditions on the gauge fields are imposed by choosing a fixed but arbitrary connection \( B \in A^Q \). For a given \( P \in \mathfrak{P}_Q[M; T^N] \) the role of the relevant configuration space for the abelian gauge theory is taken by the affine space...
\[ \mathcal{A}_B^{(P,Q)} := \{ A \in \mathcal{A}^{(P)} | \ i_{\partial M}^* A = \phi^{-1} B \} \]  

(5.1.2)
of those gauge potentials in \( \mathcal{A}^{(P)} \) whose restrictions to \( \partial P \) equals the fixed connection \( B \) under the bundle isomorphism \( \phi \). The tangent bundle of the configuration space is \( T\mathcal{A}_B^{(P,Q)} \cong \mathcal{A}^{(P)} \times \Omega^1(M, \partial M; \mathbb{T}^N) \).

Let us consider the subgroup \( \mathcal{G}^{(M, \partial M)} = \{ g \in \mathcal{G}^{(P)} | \ i_{\partial M}^* g = 1 \} \) of gauge transformations approaching the unity on the boundary. This group gives a free action on \( \mathcal{A}_B^{(P,Q)} \) and correspondingly induces a smooth gauge orbit space \( \mathcal{M}_B^{(P,Q)} = \mathcal{A}_B^{(P,Q)}/\mathcal{G}^{(M, \partial M)} \). In the following we will analyze the structure of the restricted gauge group and exhibit the bundle structure of the space of connections and of the corresponding gauge orbit space. However, in the case \( N = 1 \) some results regarding the geometry of the gauge group have been already presented in [32].

In order to characterize \( \mathfrak{P}_Q[M, \mathbb{T}^N] \) we consider the following long exact sequence in relative cohomology,

\[ \cdots \rightarrow H^1(\partial M; \mathbb{Z}^N) \xrightarrow{\delta_1} H^2(M, \partial M; \mathbb{Z}^N) \rightarrow H^2(M; \mathbb{Z}^N) \xrightarrow{i_{\partial M}^*} H^2(\partial M; \mathbb{Z}^N) \xrightarrow{\delta_2} H^3(M, \partial M; \mathbb{Z}^N) \rightarrow \cdots, \]  

(5.1.3)
where \( \delta_i \) are the connecting homomorphisms. Any two bundles \( P_1, P_2 \in \mathfrak{P}_Q[M, \mathbb{T}^N] \) are related to each other by a principal \( \mathbb{T}^N \)-bundle over \( M \) whose Chern-class belongs to \( H^2(M, \partial M; \mathbb{Z}^N) \). On the other hand the principal bundle \( Q \) over \( \partial M \) can be extended to an principal bundle \( P \in \mathfrak{P}_Q[M, \mathbb{T}^N] \) if and only if \( \delta_2(Q) = 0 \). Generally the obstruction belongs to \( H^3(M, \partial M; \mathbb{Z}^N) \). If \( Q \) is chosen to be the trivial bundle, then there is an isomorphism \( \mathfrak{P}_{Q=0}[M, \mathbb{T}^N] \cong H^2(M, \partial M; \mathbb{Z}^N) \).

In order to proceed we need a brief digression on some results of Hodge theory on manifolds with a boundary [33-35]. Let us define the following vector spaces of normal and tangential forms on \( M \).

\[ \begin{align*}
\Omega^k(M, \partial M; \mathbb{R}) &= \Omega^k_{\text{nor}}(M; \mathbb{R}) = \{ \alpha \in \Omega^k(M, \mathbb{R}) | i_{\partial M}^* \alpha = 0 \} \\
\Omega^k_{\tan}(M; \mathbb{R}) &= \{ \alpha \in \Omega^k(M, \mathbb{R}) | i_{\partial M}^* \alpha = 0 \} \\
\Omega^k_{\text{abs}}(M; \mathbb{R}) &= \{ \alpha \in \Omega^k(M, \mathbb{R}) | i_{\partial M}^* \alpha = 0 \} \\
\Omega^k_{\text{rel}}(M; \mathbb{R}) &= \{ \alpha \in \Omega^k(M, \mathbb{R}) | \ i_{\partial M}^* \alpha = 0 \}.
\end{align*} \]  

(5.1.4)
For a compact, connected and oriented manifold \( M \) with boundary \( \partial M \) one obtains

\[ \begin{align*}
\langle d\alpha, \beta \rangle &= \langle \alpha, d^* \beta \rangle = \int_{\partial M} i_{\partial M}^* (\alpha \wedge * \beta) \\
\langle \Delta \alpha, \beta \rangle &= \langle d\alpha, d\beta \rangle = \langle \alpha, d^* \beta \rangle = \int_{\partial M} i_{\partial M}^* (d^* \alpha \wedge * \beta - \beta \wedge * d\alpha) \\
\langle \Delta \alpha, \beta \rangle &= \langle \Delta \alpha, \beta \rangle = \int_{\partial M} i_{\partial M}^* (d^* \alpha \wedge * \beta - d^* \beta \wedge * \alpha + \alpha \wedge * d\beta - \beta \wedge * d\alpha).
\end{align*} \]  

(5.1.5)
Let us define differential operators subjected to the different boundary conditions in (5.1.4), namely \( d_{k, \text{nor}} = d_k |_{\Omega^k(M, \partial M; \mathbb{R})} \) and \( d_k^{\text{rel}} = (-1)^{k(n+1)} * d_{n-k, \text{nor}} : \Omega^k_{\text{rel}}(M; \mathbb{R}) \rightarrow \Omega^{k-1}(M; \mathbb{R}) \). Accordingly, two different Laplacian operators can be distinguished, namely

\[ \begin{align*}
\Delta_k^{\text{abs}} &= d_{k-1} d_k^{\text{rel}} + d_{k+1}^{\text{rel}} d_k : \Omega^k_{\text{abs}}(M; \mathbb{R}) \rightarrow \Omega^k(M; \mathbb{R}) \\
\Delta_k^{\text{rel}} &= d_{k-1, \text{nor}} d_k^* + d_{k+1, \text{nor}}^* : \Omega^k_{\text{rel}}(M; \mathbb{R}) \rightarrow \Omega^k(M; \mathbb{R}),
\end{align*} \]  

(5.1.6)
which are elliptic and self adjoint on their respective domains of definition. On manifolds with a boundary there exist the following three kinds of Hodge decompositions

\[
\begin{align*}
\Omega^k(M; \mathbb{R}) &= d\Omega^{k-1}_{\text{nor}}(M; \mathbb{R}) \oplus d^*\Omega^{k+1}_{\text{tan}}(M; \mathbb{R}) \oplus H^k(M) \\
\Omega^k(M; \mathbb{R}) &= d\Omega^{k-1}(M; \mathbb{R}) \oplus d^*\Omega^{k+1}(M; \mathbb{R}) \oplus \text{Harm}_{\text{abs}}(M; \mathbb{R}) \\
\Omega^k(M; \mathbb{R}) &= d\Omega^{k-1}_{\text{nor}}(M; \mathbb{R}) \oplus d^*\Omega^{k+1}_{\text{tan}}(M; \mathbb{R}) \oplus \text{Harm}_{\text{rel}}(M; \mathbb{R}),
\end{align*}
\]

where \( H^k(M) = \{ \varpi \in \Omega^k(M; \mathbb{R}) \mid d\varpi = d^*\varpi = 0 \} \) is called the space of harmonic \( k \)-form fields. The cohomology can be equivalently characterized by the kernel of the Laplace operators

\[
\begin{align*}
\text{Harm}_{\text{abs}}(M; \mathbb{R}) := \ker \Delta^k_{\text{abs}} \cong H^k(M; \mathbb{R}) \\
\text{Harm}_{\text{rel}}(M; \mathbb{R}) := \ker \Delta^k_{\text{rel}} \cong H^k(M, \partial M; \mathbb{R}).
\end{align*}
\]

Furthermore one can define the relative Green’s operator \([32,33]\)

\[
\begin{align*}
G^k_{\text{rel}}: \Omega^k(M; \mathbb{R}) &\to \text{Harm}_{\text{rel}}(M)^\perp \cap \Omega^k_{\text{rel}}(M; \mathbb{R}), \\
G^k_{\text{rel}} &= (\Delta^k_{\text{rel}}|_{\text{Harm}_{\text{rel}}(M)^\perp})^{-1} \cdot \Pi^{\text{Harm}_{\text{rel}}(M)^\perp},
\end{align*}
\]

satisfying \( \Delta^k_{\text{rel}}G^k_{\text{rel}} = \Pi^{\text{Harm}_{\text{rel}}(M)^\perp} \), where \( \Pi^{\text{Harm}_{\text{rel}}(M)^\perp} \) is the projector onto the orthogonal complement of \( \text{Harm}_{\text{rel}}(M) \). The relative Green’s operator \( G^k_{\text{rel}} \) commutes with both the differential \( d \) and the co-differential \( d^* \). Analogously, it is possible to define the Green’s operator \( G^k_{\text{abs}} \) for absolute boundary conditions.

**Proposition 5.1.** There exists an isomorphism between the abelian groups

\[
G^{(M, \partial M)} \cong \Omega^1_{\text{rel}}(M, \partial M; \mathbb{R}^N).
\]

**Proof.** Let \( C_*(M; \mathbb{Z}) \) (\( C_*(M, \partial M; \mathbb{Z}) \)) denote the complex of smooth (relative) singular chains and let \( Z_*(M, \partial M; \mathbb{Z}) \) be the subcomplex of relative cycles. Let \( x_0 \in \partial M \) be a fixed point at the boundary. Given any \( x \in M \) we choose a path \( c_{x_0,x} \) in \( M \) connecting \( x_0 \) with \( x \). This path can be viewed as 1-chain in \( C_1(M; \mathbb{Z}) \). Then the isomorphism \( \kappa_{(M, \partial M)}: G^{(M, \partial M)} \to \Omega^1_{\text{rel}}(M, \partial M; \mathbb{R}^N) \) is provided by

\[
\kappa_{(M, \partial M)}(g) = \frac{1}{2\pi \sqrt{-1}} g^* \partial, \quad \kappa^{-1}_{(M, \partial M)}(\alpha)(x) = \exp(2\pi \sqrt{-1} \int_{c_{x_0,x}} \alpha).
\]

In order to prove that (5.1.11) is indeed well-defined, one chooses a different base point \( x_0' \) and a corresponding path \( c'_{x_0',x} \) connecting \( x_0' \) and \( x \). Then the combined path \( c_{x_0',x} \circ c_{x_0,x}^{-1} \in Z_1(M, \partial M; \mathbb{Z}) \), since \( \partial(c_{x_0,x} \circ c_{x_0,x}^{-1}) \in Z_0(\partial M; \mathbb{Z}) \). The integral of any differential 1-form in \( \Omega^1_{\text{rel}}(M, \partial M; \mathbb{R}^N) \) over this 1-cycle gives an integer. Analogously, the value of \( \kappa^{-1}_{(M, \partial M)} \) is independent of the actual path connecting \( x_0 \) and \( x \). \( \Box \)

Like in the case of empty boundary, the gauge group \( G^{(M, \partial M)} \) is not connected, which is displayed in the next statement.

**Proposition 5.2.** Let \( \mathfrak{g}^{(M, \partial M)} \) denote the Lie algebra of \( G^{(M, \partial M)} \). Then the following sequence of abelian groups is split exact:

\[
0 \to \mathfrak{g}^{(M, \partial M)} \xrightarrow{\exp} G^{(M, \partial M)} \xrightarrow{\hat{\kappa}_{(M, \partial M)}} H^1_{\text{rel}}(M, \partial M; \mathbb{R}^N) \to 0,
\]

where \( \hat{\kappa}_{(M, \partial M)}(g) = [\kappa_{(M, \partial M)}(g)] \).
**Proof.** It is easy to show that (5.1.12) is exact. Let $\Pi^{\text{Harm}_{rel}(M)}$ be the projector onto $\text{Harm}_{rel}(M)$ then the split of the exact sequence is provided by the following isomorphism of abelian groups

$$
\hat{\kappa}(M,\partial M) : H_1^1(M, \partial M; \mathbb{R}^N) \times \mathfrak{g}^{(M, \partial M)} \rightarrow \mathfrak{g}^{(M, \partial M)}
$$

where $c_{x_0, x}$ denotes a path in $M$ connecting $x_0$ and $x$. The independence of (5.1.13) of the selected path can be proved analogously than in the proof of proposition 5.1. Hence any $g \in \mathfrak{g}^{(M, \partial M)}$ admits the following realization

$$
g(x) = \exp((G_0^\text{rel} \cdot d_1^* g^* \vartheta)(x) \cdot \exp(2\pi \sqrt{-1} \int_{c_{x_0, x}} \Pi^{\text{Harm}_{rel}(M)}(x)) \right)
$$

From (5.1.12) one finally obtains $\Pi_0(\mathfrak{g}^{(M, \partial M)}) \cong H^1_2(M, \partial M; \mathbb{R}^N)$. □

The Lefschetz duality [31] states that the following isomorphisms exist

$$
D_k^{(M, \partial M)} : H^k(M, \mathbb{Z}) \rightarrow H_{n-k}(M, \partial M; \mathbb{Z}), \quad \hat{D}_k^{(M, \partial M)} : H^k(M, \partial M; \mathbb{Z}) \rightarrow H_{n-k}(M; \mathbb{Z}),
$$

implying $b^k_{\text{rel}} = b_{\text{abs}}^{k-n}$, where $b^k_{\text{rel}} = \dim H^k(M, \partial M; \mathbb{R})$ and $b^{k}_{\text{abs}} = \dim H^k(M; \mathbb{R})$. We choose a set of relative 1-cycles $\gamma_i^{rel} \in Z_1(M, \partial M; \mathbb{Z})$, for $i = 1, \ldots, b_1^{rel}$, whose homology classes $[\gamma_i^{rel}]$ provides a Betti basis for $H_1(M, \partial M; \mathbb{Z})$. Based on the following isomorphisms $\forall k = 1, \ldots, n$, namely

$$
H^k(M, \partial M; \mathbb{Z})/\text{Tor}H^k(M, \partial M; \mathbb{R}) \cong \text{Harm}^k_2(M, \partial M; \mathbb{R})
$$

$$
H^k(M; \mathbb{Z})/\text{Tor}H^k(M; \mathbb{R}) \cong \text{Harm}^k_2(M, \partial M; \mathbb{R})
$$

(5.1.16)

a basis of harmonic forms $(\varphi_i^{(\text{rel}, 1)})_{i=1}^{b_1^{rel}} \in \text{Harm}^{n-1}_{\text{abs}, \mathbb{Z}}(M; \mathbb{R})$ can be selected from the cohomology basis $D_1^{-1}([\gamma_i^{rel}])$ of $H^{n-1}(M; \mathbb{Z})$. The product

$$
H^1(M, \mathbb{Z})/\text{Tor}H^1(M, \mathbb{Z}) \cong H^{n-1}(M, \partial M; \mathbb{Z})/\text{Tor}H^{n-1}(M, \partial M; \mathbb{Z}) \rightarrow \mathbb{Z}
$$

$$(\mu, \nu) \mapsto \langle \mu, D_1^{(M, \partial M)}(\nu) \rangle = \langle \mu \cup H, [\nu] \rangle
$$

(5.1.17)

gives a perfect pairing like in the boundary-less case. Thus a basis $(\varphi_i^{(\text{rel}, 1)})_{i=1}^{b_1^{rel}} \in \text{Harm}^1(M, \partial M; \mathbb{R})$ can be adjusted in such a way so that

$$
\int_{\gamma_i^{rel}} \varphi_j^{(\text{rel}, 1)} = \int_M \varphi_i^{(\text{rel}, 1)} \wedge \varphi_j^{(\text{abs}, n-1)} = \delta_{ij}.
$$

(5.1.18)

This basis induces a metric

$$
h_{jk}^{\text{rel}} = \langle \varphi_j^{(\text{rel}, 1)}, \varphi_k^{(\text{rel}, 1)} \rangle
$$

(5.1.19)

on $\text{Harm}_{rel}(M; \mathbb{R}^N)$. Moreover for any $[\alpha] \in H^1(M, \partial M; \mathbb{R})$ the following relation holds $\int_{\gamma_j^{rel}} \alpha = \int_M \alpha \wedge \varphi_j^{(\text{abs}, n-1)}$.

Let us now choose an arbitrary but fixed background gauge field $A_0 \in \mathcal{A}_{B}^{(P, Q)}$ and define the smooth surjective map $\pi^{A_0}_{\mathcal{M}^B_{\mathcal{A}_B^{(P, Q)}}} : \mathcal{M}^B_{\mathcal{A}_B^{(P, Q)}} \rightarrow \mathcal{T}^{b_1 \times N^*}_{\text{rel}}$ by

22
\[ \pi_{A_{B}^{(P,Q)}}^A([A]) = (e_j^M(A-A_0)\wedge\phi_j^{(abs,n-1)}), \ldots, e_j^M(A-A_0)\wedge\phi_j^{(abs,n-1)} ) . \]  

(5.1.20)

In terms of the inner product (3.1) the components in (5.1.20) can be rewritten in the form

\[ \int_M (A - A_0) \wedge \phi_j^{(abs,n-1)} = (-1)^n < A - A_0, \ast \phi_j^{(abs,n-1)} > . \]  

(5.1.21)

Let us remark that \( \ast \phi_j^{(abs,n-1)} \in H^{1,rel}(M; \mathbb{R}) \) since the Hodge operator provides an isomorphism between the relative and absolute harmonic forms.

By (5.1.20) we are able to construct a finite open cover \( \tilde{U} \) of the infinite dimensional manifold \( M_{B}^{(P,Q)} \) by defining \( U_a = (\pi^{-1}_{A_{B}^{(P,Q)}})^{-1}(V_a) \), with \( a = 1, \ldots, 2^n_{rel,N} \). Then the geometrical structure of the bundle of gauge potentials is displayed by the following:

**Theorem 5.3.** \( A_{B}^{(P,Q)} \) is a flat principal bundle over \( M_{B}^{(P,Q)} \) with structure group \( G^{(M,\partial M)} \) and projection \( \pi_{A_{B}^{(P,Q)}}^A \). This bundle is trivializable if \( H^1(M, \partial M, \mathbb{Z}) \cong H_{n-1}(M; \mathbb{Z}) = 0 \).

**Proof.** A bundle atlas is provided by the following family of local trivializations \( \tilde{\varphi}^A_a : \tilde{U}_a \times G^{(M,\partial M)} \rightarrow \pi_{A_{B}^{(P,Q)}}^{-1}(\tilde{U}_a) \), \( \tilde{\varphi}^A_a ([A], g) = A_a^\alpha (A)^{-1} g \), where

\[ \tilde{\varphi}^A_a : \pi_{A_{B}^{(P,Q)}}^{-1}(\tilde{U}_a) \rightarrow G^{(M,\partial M)} \]

\[ \tilde{\varphi}^A_a (A) = \hat{\kappa}(M,\partial M)([\sum_{j=1}^{b_{rel}} \hat{\epsilon}^A_{a_j}(A) \phi_j^{(r,rel)}], \exp G_{0,rel}^{a_1} d_1 (A - A_0)) \]

\[ \hat{\epsilon}^A_{a_j} (A) = (\hat{\epsilon}^A_{a_j}(A), \ldots, \hat{\epsilon}^A_{a_{j+N}}(A)) : \pi_{A_{B}^{(P,Q)}}^{-1}(\tilde{U}_a) \rightarrow \mathbb{Z}^N \]

\[ \hat{\epsilon}^A_{a_j} (A) = \frac{1}{2 \pi \sqrt{-1}} \int_M ((A^\alpha - A_0^\alpha) \wedge \phi_j^{(abs,n-1)}) - s_{a_j} (e_j^M (A^\alpha - A_0^\alpha) \wedge \phi_j^{(abs,n-1)}) \]  

(5.1.22)

Since \( \frac{1}{2 \pi \sqrt{-1}} g^* \vartheta \) has integer periods its projection onto the space of relative harmonic one-forms with integer periods is given by

\[ \Pi_{Harm_{rel}(M)}(\frac{1}{2 \pi \sqrt{-1}} g^* \vartheta) = \sum_{j=1}^{\tilde{b}_{rel}} (k_{rel})^{-1} \left< \frac{1}{2 \pi \sqrt{-1}} g^* \vartheta, \phi_j^{(r,rel)} \right> > \phi_j^{(r,rel)} = \sum_{j=1}^{\tilde{b}_{rel}} m_j \phi_j^{(r,rel)} , \]  

(5.1.23)

where \( m_k \in \mathbb{Z}^N \). Using that \( \hat{\epsilon}^A_{a_j}(A) = \hat{\epsilon}^A_{a_j}(A) + \hat{m}_j \) and (5.1.15) one derives \( \tilde{\varphi}^A_a (A) = \tilde{\varphi}^A_a (A) g \).

The transition functions \( \tilde{\varphi}^A_a : \tilde{U}_a \cap \tilde{U}_{a'} \rightarrow G^{(M,\partial M)} \) yield

\[ \tilde{\varphi}^A_a ([A]) = \hat{\kappa}(M,\partial M)([\sum_{j=1}^{b_{rel}} g_{a_j,a_j'} (e_j^M (A - A_0) \wedge \phi_j^{(abs,n-1)})] \phi_j^{(r,rel)}, 0) \]  

(5.1.24)

and are locally constant. Together with the universal coefficient theorem one concludes that the bundle is trivializable if \( H^1(M, \partial M, \mathbb{Z}) = 0 \). Moreover the transition functions are locally constant. Like in the proof of theorem 4.3 on can easily verify that a different choice for the background gauge field \( A_0 \) would lead to an equivalent bundle atlas. \( \Box \)
where we are going to skip the details and to present the results only. Much of the results which have been elaborated for closed manifolds can be directly generalized, if the boundary conditions are appropriately specified. Thus we consider the case of manifolds with a boundary. For each arbitrary but fixed connection \( A_0 \in \mathcal{A}_B^{(P,Q)} \) and typical fibre \( \mathcal{M}^{(M,\partial M)} : = (imd_2^\ast \cap \Omega_{rel}^1(M;\mathbb{R})) \otimes t^N \).

**Proof.** A bundle atlas is provided by the local diffeomorphism

\[
\tilde{\chi}_A^B : V_a \times \mathcal{N}^{(M,\partial M)} \rightarrow \mathcal{M}^{(P,Q)}
\]

\[
\tilde{\chi}_A^B(z_1, \ldots, z_{b^\text{rel}}, \tau) = [A_0 + 2\pi \sqrt{-1} \sum_{j=1}^{b_{rel}} s_{a_j}(z_j) \theta_j^{(rel,1)} + \tau]
\]

\[
(\tilde{\chi}_A^B)^{-1}([A]) = (\pi_{\mathcal{M}_B^{(P,Q)}}([A]), d_2^\ast G_2^\text{rel} (F_A - F_{A_0})).
\]

(5.1.25)

There exists a unique vector bundle structure induced by the bundle chart \( \tilde{\chi}_A^B \). In analogy with theorem 4.4 we can easily prove that the choice of a different background gauge field induces an isomorphic vector bundle structure on \( \mathcal{M}_B^{(P,Q)} \).

The topological structure of the true configuration space is characterized by the following:

**Corollary 5.5.** There exist the following isomorphisms

\[
H^k(\mathcal{M}_B^{(P,Q)} ; \mathbb{Z}) \cong H^k(\mathbb{T}^{b_{rel}} ; \mathbb{Z}) = \mathbb{Z}^{(b_{rel})}
\]

\[
\pi_k(\mathcal{M}_B^{(P,Q)}) \cong \pi_k(\mathbb{T}^{b_{rel}}) = \delta_{k1} \mathbb{Z}^{b_{rel}}.
\]

(5.2.1)

### 5.2. The partition function, VEV of gauge invariant observables and the Green’s functions

Now we are ready to apply the previous results to the construction of the partition function, the VEV of gauge invariant observables, Green’s functions and the field strength correlation functions in the case of manifolds with a boundary. Much of the results which have been elaborated for closed manifolds can be directly generalized, if the boundary conditions are appropriately specified. Thus we are going to skip the details and to present the results only.

We introduce a partition of unity \( \{\tilde{\rho}_a\} \) for \( \mathcal{M}_B^{(P,Q)} \) subordinate to \( \tilde{U} \). Thereby \( \tilde{\rho}_a := \pi_{\mathcal{M}_B^{(P,Q)}}^\ast \tilde{\rho}_a^\prime \), where \( \tilde{\rho}_a^\prime := \prod_{i=1}^N q^{\alpha_i} \tilde{a}_{\alpha_i} \), with the multi-index \( \alpha = (a_1, \ldots, a_{b_{rel}}) \in \mathbb{Z}_{\geq 0}^{b_{rel}} \).

For each \( \Phi_i \) we choose an associated source \( J_i \in \Omega_{rel}^1(M;\mathbb{R}) \otimes t^N \), \( i = 1, 2 \) and introduce a generating functional by

\[
Z_{A_0}^{(P,Q)}[J; B, \Phi_i] = \int_{\mathcal{A}_B^{(P,Q)}} \text{vol} \mathcal{A}_B^{(P,Q)} \sum_{a \in \mathbb{Z}_{\geq 1}^{b_{rel}}} (\pi_{\mathcal{A}_B^{(P,Q)}}^\ast \tilde{\rho}_a)(\pi_{\mathcal{M}_B^{(P,Q)}}^\ast \tilde{\rho}_a) e^{-S_{inv} - (\tilde{z}_A^B)^\ast \tilde{s}_{ij} + \langle \Phi_i, J_i \rangle}.
\]

(5.2.1)

This gives rise to the following correlation functions

\[
\psi^{(P,Q)}_{q}(v_1, \ldots, v_q; \Phi_i) := \frac{\partial^n}{\partial t_1 \cdots \partial t_q} |_{t_1=\ldots=t_q=0} \frac{Z_{A_0}^{(P,Q)}[\sum_{i=1}^q t_i v_i; B, \Phi_i]}{Z_{A_0}^{(P,Q)}[0; B, \Phi_i]},
\]

(5.2.2)
where \( v_1, \ldots, v_q \in \Omega^*_{rel}(M; \mathbb{R}) \otimes t^N \). The form degree depends on which of the following three cases is considered:

1. \( i = 0 \): If \( \Phi_0(A) = 0 \), eq. (5.2.1) reduces to the partition function of the theory, denoted by \( Z^{(P,Q)}_{A_0}(B) \).

2. \( i = 1 \): If \( \Phi_1(A) = A - A_0 \) and \( J_1 \in \Omega^1_{rel}(M; \mathbb{R}) \otimes t^N \), eq. (5.2.2) gives the generating functional for the \( q \)-th point Green’s functions, denoted by \( S^q_{im}(P,Q) \).

3. \( i = 2 \): If \( \Phi_2(A) = F_A \) and \( J_2 \in \Omega^2_{rel}(M; \mathbb{R}) \otimes t^N \), eq. (5.2.2) gives the VEV of the field strength polynomial of degree \( q \), denoted by \( \mathcal{W}^q_{im}(P,Q) \).

The next step is to find an appropriate choice for the regularizing function in (2.1): Let \( g(M,\partial M) \) be the Maurer Cartan form on \( g(M,\partial M) \). The induced left-invariant volume form is given by

\[
\text{vol}_{G(M,\partial M)} = \left( \det (\partial \phi)^{1/2} Dg \right) .
\]

A regularization of the volume of the gauge group \( G(M,\partial M) \) will be provided by the gauge fixing function \( \tilde{S}_{gf} \)

\[
e^{-\tilde{S}_{gf}(g)} = \frac{e^{-\tilde{S}_{gf}(g)}}{\int_{G(M,\partial M)} \text{vol}_{G(M,\partial M)} e^{-\tilde{S}_{gf}(g)}},
\]

with an auxiliary gauge fixing function

\[
\tilde{S}'_{gf}(g) = \frac{1}{2} \| d^* g^* \theta \|_\lambda^2 + \frac{1}{2} \| \Pi^H_{arm,\alpha}(M)(g^* \theta) \|_\lambda^2.
\]

Let \( \Pi^{imd, nor} := d_{0, nor} G_0^{rel} d_1 \) denote the projector onto the space of exact 1-forms on \( M \) with normal boundary conditions and let \( \tilde{K}^\alpha(J_1) = -\sqrt{-1} J_1^\alpha, \sqrt{-1} g^{\alpha,1} > \) with \( j = 1, \ldots, b_1^{rel} \) and \( \alpha = 1, \ldots, N \) be regarded as \( b_1^{rel} N \)-dimensional complex vector \( \tilde{K}(J_1) \). Using the results of proposition 5.2 a straightforward calculation leads to:

**Lemma 5.7.** For the auxiliary gauge fixing function \( \tilde{S}'_{gf} (5.2.4) \) one gets

\[
\int_{G(M,\partial M)} \text{vol}_{G(M,\partial M)} e^{-\tilde{S}'_{gf}(g) + < J_1, g^* \theta >} =
\]

\[
= (\det \Delta_0^{rel})^{-N} \cdot \Theta_{b_1^{rel} N^{rel}}(\tilde{K}(J_1)) 2 \pi \sqrt{-1} \tilde{\Lambda} \pi^{-1/2} < \Pi^{imd, nor}(J_1), G_1^{rel} \Pi^{imd, nor}(J_1) > ,
\]

where \( \tilde{\Lambda} = \lambda \otimes h^{rel} \) is the tensor product of the matrix \((\lambda_{\alpha, \beta})^{N}_{\alpha, \beta} = 1\) of coupling constants and the metric on the harmonic relative 1-forms \((h^{rel})^{b_1^{rel}}_{j,k})_{j,k=1}^{N}. \□

Based on the choice of the gauge fixing function (5.2.3), the partition function can be displayed in the gauge field space as follows:

**Proposition 5.8.** Let \( P \in \mathcal{P}_Q[M; T^N] \) and \( B \in \mathcal{A}^{(Q)} \) be an arbitrary but fixed connection. The partition function for the abelian gauge theory with the classical action (3.6) on a manifold with a non-empty boundary is given by

\[
Z^{(P,Q)}_{A_0}(B) = \int_{\mathcal{A}^{(P,Q)}_B} \text{vol}_{\mathcal{A}^{(P,Q)}_B} \tilde{\mathcal{F}}(A) e^{-\frac{1}{4} (\| F_A \|_\lambda^2 + \| d^* (A - A_0) \|_\lambda^2)},
\]

with the positive definite functional

\[
\tilde{\mathcal{F}}(A) =
\]

\[
= (\det \Delta_0^{rel})^{-N} \Theta_{b_1^{rel} N}(0) 2 \pi \sqrt{-1} \tilde{A}^{rel} \sum_{\alpha, \beta = 1, j,k=1}^{b_1^{rel}} \lambda_{\alpha, \beta} h^{rel}_{j,k} J_{\alpha, \beta}(A) J_{\alpha, \beta}(A) e^{-2 \pi^2 \sum_{\alpha, \beta = 1, j,k=1}^{b_1^{rel}} \lambda_{\alpha, \beta} h^{rel}_{j,k} J_{\alpha, \beta}(A) J_{\alpha, \beta}(A)},
\]

(5.2.7)
where the multi-index reads $a = (a_1, \ldots, a_{b^r_1 N}) \in \mathbb{Z}_{\geq 2}^{b^r_1 N}$.

With respect to the local bundle trivializations $\tilde{\psi}^A_\alpha = \tilde{\phi}^A_\alpha \circ (\tilde{\chi}^{A_0} \times \mathbb{I})$ the volume form becomes

$$(\tilde{\psi}^A_\alpha)^* \text{vol}_{A_B^{(P,Q)}} = (\det h^{rel})^{N/2} \text{det} (\Delta_0^{rel} |_{\text{imd}^*_2})^{N/2} \text{vol}_{\varphi_{V_\alpha}} \wedge \text{vol}_{A^{(M,\partial M)} \wedge \text{vol}_{G^{(M,\partial M)}},}$$  *(5.2.8)*

where $\text{vol}_{\varphi_{V_\alpha}}$ is the induced volume form on $\mathbb{T}^{b^r_1 N}$ restricted to a single patch $V_\alpha$. Like in the boundary-less case, the flat metric on $N^{(M,\partial M)}$ induces a volume form $\text{vol}_{A^{(M,\partial M)}}$ which formally is just $D\tau$.

Given any background connection $A_0 \in A^{(P,Q)}_B$, then the gauge field $A'_0 := A_0 - G^{rel} d^*_2 F_0$, fulfills $d^*_2 F_0 = 0$. Hence we can restrict ourselves to the class of background gauge fields which satisfy the classical field equation. Using *(5.2.8)* and that det $(\Delta_0^{rel} |_{\text{imd}^*_2}) = \text{det} (\Delta_1^{rel} |_{\text{imd}_0,\text{nor}})$ a direct calculation finally gives:

**Proposition 5.9.** Let $P \in \mathbb{P}_Q[M; \mathbb{T}^N]$ and let $B \in A^{(Q)}$ be an arbitrary but fixed connection. The generating functional corresponding to the three cases is given by

1. $i = 0:
   $$Z^{(P,Q)}_{A_0}[J; B, \Phi_1] = (2\pi)^{b^r_1 N} (\det h^{rel})^{N/2} (\det \Delta_0^{rel} |_{\text{imd}^*_2})^N (\det \Delta_1^{rel} |_{\text{Harm}_1^{rel}(M)\perp})^{-N/2} e^{-\frac{1}{2} \|F_0\|^2},$$  *(5.2.9)*

2. $i = 1:
   $$Z^{(P,Q)}_{A_0}[J_1; B, \Phi_1] = (2\pi)^{b^r_1 N} (\det h^{rel})^{N/2} (\det \Delta_0^{rel} |_{\text{imd}^*_2})^N (\det \Delta_1^{rel} |_{\text{Harm}_1^{rel}(M)\perp})^{-N/2} \frac{\Theta_{h^{rel}}(\tilde{K}(J_1))|2\pi\sqrt{-1}A|}{\Theta_{b^{rel}}(0)|2\pi\sqrt{-1}A|} \times \int \text{vol}_{\varphi_{V_\alpha}} \sum_{a_1=1}^2 \cdots \sum_{a_{b^r_1 N}=1}^2 q_{11}^a \bar{p}_{a_1} \cdots \bar{q}_{b^r_1 N}^* \bar{p}_a, (5.2.10)$$

3. $i = 2:
   $$Z^{(P,Q)}_{A_0}[J_2; B, \Phi_2] = (2\pi)^{b^r_1 N} (\det h^{rel})^{N/2} (\det \Delta_0 |_{\text{imd}^*_2})^N (\det \Delta_1 |_{\text{Harm}_1^{rel}(M)\perp})^{-N/2} e^{-\frac{1}{2} \|F_0\|^2 + \frac{1}{2} <\tilde{J}_2, (\Theta_{h^{rel}})^* \cdot \tilde{J}_2 > \Delta_0^{rel} |_{\text{imd}^*_2} > \lambda_1 + <F_0, J_2 >},$$  *(5.2.11)*

**Proposition 5.10.** Let $P \in \mathbb{P}_Q[M; \mathbb{T}^N]$ and let $B \in A^{(Q)}$ be an arbitrary but fixed connection. The VEV of any gauge invariant function $f$ takes the form

$$< f >_{(P,Q)} = (\det \Delta_0^{rel} |_{\text{imd}^*_2})^{-N/2} (\det \Delta_1^{rel} |_{\text{Harm}_1^{rel}(M)\perp})^{N/2} \int_{N^{(M,\partial M)}} D\tau f_{(0,\ldots,0)}(A_0 + \tau) \cdot e^{-\frac{1}{2} \|\Delta_1^{rel} |_{\text{imd}^*_2} \tau > \lambda_1},$$  *(5.2.12)*

with the Fourier components

$$f_{(0,\ldots,0)}(A_0 + \tau) = \int_0^1 \cdots \int_0^1 dt_{11} \cdots dt_{b^r_1 N} f(e^{2\pi\sqrt{-1}1_1}, \ldots, e^{2\pi\sqrt{-1}b^r_1 N}, A_0 + \tau).$$  *(5.2.13)*
Proof. Eq. (5.2.12) follows from (2.4) by a direct calculation where we follow the lines of section 4.2. Here any gauge invariant function \( f \) on \( \mathcal{A}_B^{(P,Q)} \) projects to a global function \( \hat{f} = (\chi_A^*) f \) on \( \mathbb{T}^{b_{rel}^N} \times \mathcal{N}(M,\partial M) \), which admits a Fourier series expansion. \( \square \)

Let us define the field independent factor

\[
\hat{\varepsilon}^{(2)}_{j,\alpha,k,\beta} = \int_{\epsilon_1^{rel} N} \Psi_{a_{11}}^{rel} \cdots \Psi_{a_{b_{rel},N}}^{rel} q^*_s \hat{F}_a^{rel} \hat{F}_{a_{b_{rel},N}}^{rel} s_{a_{1}} s_{a_{2}} s_{a_{3}} s_{a_{b_{rel},N}} = \begin{cases} (2\pi)^{b_{rel}^N - 1}(\frac{2\pi}{N^2} - \frac{1}{2}), & \text{for } j = k \text{ and } \alpha = \beta \\ (2\pi)^{b_{rel}^N - 2}(\frac{2\pi}{N^2})^2, & \text{for } j \neq k \text{ or } \alpha \neq \beta \text{ or both}, \end{cases}
\]

then a lengthy calculation gives:

**Proposition 5.11.** Let \( P \in \mathfrak{P}_Q[M;T^N] \) and let \( B \in \mathcal{A}^{(Q)} \) be an arbitrary but fixed connection. The \( q \)-point Green’s functions of the gauge fields read:

1) One-point function:

\[
S_1^{(P,Q)}(v) = \sum_{\alpha=1}^{N} \sum_{j=1}^{b_{rel}^N} \varepsilon^{(1)}_{j,\alpha} < v^\alpha, \sqrt{-1} \theta_j^{(rel,1)} > + \frac{d}{dt} \frac{1}{|t=0} \Theta_{b_{rel}^N}(\tilde{K}(t)) |2\pi \sqrt{-1} \tilde{\Lambda}|.
\]

2) Two-point function:

\[
S_2^{(P,Q)}(v_1, v_2) = < v_1, G^{rel}_1 v_2 >_{\lambda^{-1}} + \sum_{\alpha=1}^{N} \sum_{j=1}^{b_{rel}^N} \varepsilon^{(1)}_{j,\alpha} < v_1^\alpha, \sqrt{-1} \theta_j^{(rel,1)} > + \frac{d}{dt} \frac{1}{|t=0} \Theta_{b_{rel}^N}(\tilde{K}(tv_2)) |2\pi \sqrt{-1} \tilde{\Lambda}| \\
+ \sum_{\alpha=1}^{N} \sum_{j=1}^{b_{rel}^N} \varepsilon^{(1)}_{j,\alpha} < v_2^\alpha, \sqrt{-1} \theta_j^{(rel,1)} > + \frac{d}{dt} \frac{1}{|t=0} \Theta_{b_{rel}^N}(\tilde{K}(tv_1)) |2\pi \sqrt{-1} \tilde{\Lambda}| + (2\pi)^{2-b_{rel}^N} N \sum_{\alpha,\beta=1}^{N} \sum_{k=1}^{b_{rel}^N} \varepsilon^{(2)}_{j,\alpha,k,\beta} < v_1^\alpha, \sqrt{-1} \theta_k^{(rel,1)} > < v_2^\beta, \sqrt{-1} \theta_j^{(rel,1)} > + \Theta_{b_{rel}^N}(0) |2\pi \sqrt{-1} \tilde{\Lambda}|^{-1} \frac{\partial^2}{\partial t_1 \partial t_2} |t_1=t_2=0 \Theta_{b_{rel}^N}(|\tilde{K}(\sum_{l=1}^{2} t_l v_l)| |2\pi \sqrt{-1} \tilde{\Lambda}|).
\]

\( \square \)

We remark that the Green’s functions are already independent of the fixed component \( \mathcal{A}_B^{(P,Q)} \). Let us define the following abbreviations

\[
\mu_{ij}^{rel} = < v_i, (\Pi^{Harm^2_{rel}(M)^+} - d^*_3 G^{rel}_3 d^*_2) v_j >_{\lambda^{-1}}
\]

\[
\nu_{i}^{rel} = < F_{A_0}, v_i >.
\]

for \( v_i, v_j \in \Omega^2(M;\mathfrak{t}^N) \) and let \( \varsigma \) denote a permutation of the indices \( \{1, \ldots, q\} \) then a lengthy calculation gives the following:
Proposition 5.12. Let $P \in \Psi_Q[M;\mathbb{T}^N]$ and let $B \in \mathcal{A}^{(Q)}$ be an arbitrary but fixed connection. The correlation functions $W_q^{(P,Q)}$ of the field strength are given by

1) $q = 2k$

$$W_{2k}^{(P,Q)}(v_1 \ldots v_{2k}) = \frac{1}{(2k)!} \sum_c \mu_c^{rel(1)} \cdots \mu_c^{rel(2k)} + \sum_{l=1}^{k-1} \frac{1}{2^l! (2(k-l))!} \sum_c \mu_c^{rel(1)/(2)} \cdots \mu_c^{rel(2(l-1)/(2))} \mu_c^{rel(2l/(2))} \nu_c^{rel(1)/(2)} \cdots \nu_c^{rel(2(k-l)/(2))}.$$  

2) $q = 2k + 1$

$$W_{2k+1}^{(P,Q)}(v_1 \ldots v_{2k+1}) = \frac{1}{(2k+1)!} \sum_c \mu_c^{rel(1)} \cdots \mu_c^{rel(2k+1)} + \sum_{l=1}^{k} \frac{1}{2^l! (2(k-l)+1)!} \sum_c \mu_c^{rel(1)/(2)} \cdots \mu_c^{rel(2(l-1)/(2))} \mu_c^{rel(2l/(2))} \nu_c^{rel(1)/(2)} \cdots \nu_c^{rel(2(k-l)/(2))}.$$  

(5.2.18)

(5.2.19)

where $v_1, \ldots v_q \in \Omega^2(M;\mathbb{t}^N)$. □

For $H^1(M, \partial M; \mathbb{Z}) = 0$ and the trivial $\mathbb{T}^N$-bundle $M \times \mathbb{T}^N$ over $M$ one can choose $A_0 = 0$. For consistency, $Q$ must be also trivializable with trivial connection $B = 0$. In that case the bundle structure reduces to $\mathcal{M}_B^{(P,Q)} \simeq N^{(M,\partial M)}$ implying that the orbit space consists of transversal fields with relative boundary conditions only. Under these assumptions the partition function reduces to

$$Z_{A_0=0}^{(P,Q)}[B = 0] = (\det \Delta_0^{rel})^N (\det \Delta_1^{rel})^{-N/2},$$  

(5.2.20)

which for $N = 1$ gives the covariant expression for the functional integral of quantized Maxwell theory (e.g. see [36]).

The partition function on $\mathcal{A}_B^{(M,Q)}$. The set of all connections on $M$, denoted by $\mathcal{A}_B^{(M,Q)}$, is the disjoint union

$$\mathcal{A}_B^{(M,Q)} = \bigsqcup_{P \in \Psi_Q[M;\mathbb{T}^N]} \mathcal{A}_B^{(P,Q)}.$$  

(5.2.21)

Let us now discuss the special case if $Q$ is the trivial principal bundle over $\partial M$ with trivial flat connection $B = 0$. Any $P \in \Psi_{Q=0}[M;\mathbb{T}^N]$ is uniquely characterized by a Chern class $c^\alpha = (c^\alpha)_{\alpha=1}^{N} \in H^2(M, \partial M; \mathbb{Z}^N)$ which can be written in the form

$$c^\alpha = \sum_{j=1}^{b_2^{rel}} m_j c_j^{(rel,2)} + \sum_{k=1}^{r} t_{ka} y_k^{rel(2)},$$  

(5.2.22)

where $(c_j^{(rel,2)})_{j=1}^{b_2^{rel}}$ denotes a Betti basis of $H^2(M, \partial M; \mathbb{Z})$ with rank $b_2^{rel}$ and $m_j, \alpha \in \mathbb{Z}$ for $j = 1, \ldots, b_2^{rel}$, $\alpha = 1, \ldots, N$. Furthermore $(y_k^{rel(2)})_{k=1}^{r}$ is assumed to generate $Tor H^2(M, \partial M; \mathbb{Z})$ with torsion coefficients $l_k$, i.e. $l_k y_k^{rel(2)} = 0$ and $l_{ka} \in \mathbb{Z}_{l_k}$ for $k = 1, \ldots, r$, $\alpha = 1, \ldots, N$. Accordingly $F_{A_0} \in \text{Harm}_{rel,\mathbb{Z}}(M)$ denotes the space of harmonic relative differential 2-forms with integer periods. Let $\theta_j^{rel(2)}$ be a basis of $\text{Harm}_{rel,\mathbb{Z}}^2(M)$ then
\[ F_{A_0} = 2\pi \sqrt{-1} \sum_{j=1}^{b_2^{rel}} m_j \rho_j^{(rel,2)}, \quad m_j \in \mathbb{Z}. \] (5.2.23)

Let \( h_{jk}^{(rel,2)} = \langle \rho_j^{(rel,2)}, \rho_k^{(rel,2)} \rangle \) be the induced metric on the harmonic 2-forms then one finds:

**Proposition 5.13.** Let \( Q \) be the trivial \( \mathbb{T}^N \)-bundle over \( \partial M \). The partition function, denoted by \( Z^{(M,Q)[B=0]} \), on the total configuration space \( A^{(M,Q)}_{B} \) is then given by

\[
Z^{(M,Q)[B=0]} = (2\pi)^{b_2^{rel}} \Theta_{b_2^{rel}}(0|2\pi \sqrt{-1} \Lambda^{(rel,2)}|) \cdot \text{ord}(\text{Tor} \mathcal{H}^2(\mathcal{M}, \partial \mathcal{M}; \mathbb{Z}^N)),
\] (5.2.24)

where \( \Lambda^{(rel,2)} = \lambda \otimes h_{jk}^{(rel,2)}. \) □

The partition function \( Z^{(M,Q)[B=0]} \) does not depend on the explicit choice of the basis of \( \text{Harm}^k_{rel,Z}(M), k = 1, 2 \), since any different basis is connected by a unimodular transformation leaving the partition function invariant.

### 6. Two examples

In this chapter we want to illustrate the general results obtained in the previous sections with two simple examples.

#### 6.1. The Maxwell theory on the circle \( \mathbb{T}^1 \)

Since any \( \mathbb{T}^1 \)-bundle over \( \mathbb{T}^1 \) is trivial it is possible to choose a global section such that \( A_0 = 0 \). Any gauge potential \( A \in \mathcal{A}(\mathbb{T}^1) \) can be regarded as \( \mathbb{T} \)-valued 1-form on the base manifold \( \mathbb{T}^1 \). Physically this model does not possess any dynamics and describes the behavior of time-independent gauge fields on the circle. Furthermore, \( \mathcal{A}(\mathbb{T}^1) \) appears as the configuration space for the gauge fields in the Hamiltonian formulation of electrodynamics on the manifold \( M = \mathbb{T}^1 \times \mathbb{R}^1 \) [37].

According to the theorems 4.3 and 4.4 the geometry of the configuration space simplifies as it will be stated in the following:

**Corollary 6.1.** There exists a principal bundle isomorphism

\[
\mathcal{A}(\mathbb{T}^1) \xrightarrow{\cong} \Omega^1(\mathbb{T}^1; \mathbb{R}) \otimes \mathbb{T}^1
\]

\[
\mathcal{A}(\mathbb{T}^1)/\mathfrak{G}_s(\mathbb{T}^1) \xrightarrow{\hat{x}} \mathbb{T}^1
\]

\[
\mathcal{M}_s(\mathbb{T}^1) \xrightarrow{(x_s)^{-1}} \mathbb{T}^1
\]

with \( \hat{x}([A]) = \int_{\mathbb{T}^1} A \), where \( \{A\} \) is the equivalence class with respect to the infinitesimal gauge transformations. Furthermore \( (x_s)^{-1}([A]) = e^{\int_{\mathbb{T}^1} A} \) according to (4.1.26). □

The 1-dimensional basis of the harmonic differential one forms is provided by the volume form \( \rho^{(1)} = vol_{\mathbb{T}^1} = -\sqrt{-1} \theta \) induced by the canonical flat metric on \( \mathbb{T}^1 \). The metric \( h \) on \( \text{Harm}^1(\mathbb{T}^1; \mathbb{R}) \) is just \( h = 2\pi \). Any \( \mathbb{T} \)-valued 0-form \( \xi \), respectively any \( \mathbb{T} \)-valued 1-form \( \tau \) on \( \mathbb{T}^1 \) can be rewritten in terms of a Fourier series expansion...
\[ \xi = \sum_{k \in \mathbb{Z}} \xi_k e^{2\pi \sqrt{-1} kt}, \quad \tau = \sum_{k \in \mathbb{Z}} \tau_k e^{2\pi \sqrt{-1} kt} dt. \]  

(6.1.2)

Accordingly, all nonzero modes of the gauge field \( A \) may always be gauged away completely by a suitable infinitesimal gauge transformation, whereas its zero mode is affected by topologically non-trivial gauge transformations, only.

The operators \( \Delta_0|_{imd_1} \) and \( \Delta_1|_{Harm}(M) \) possess the same spectrum \( \{ k^2 : k \in \mathbb{Z} \} \). Let \( \zeta_R(s) := \sum_{k=1}^{\infty} k^{-s} \) be the Riemann \( \zeta \)-function with the well-known properties \( \zeta_R(0) = -\frac{1}{2} \) and \( \frac{d}{ds}|_{s=0} \zeta_R(s) = \ln (2\pi)^{-1/2} \), then from (4.2.6) one obtains

\[ \zeta(s|\Delta_0|_{imd_1}) = 2\zeta_R(2s), \]  

(6.1.3)

which finally yields

\[ \det (\Delta_0|_{imd_1}) = (2\pi)^2. \]  

(6.1.4)

By inserting (6.1.4) into (4.3.4) one obtains for the generating functional

\[ Z[J] = (2\pi)^{5/2} \frac{\Theta_1(-J_0)(2\pi)^2}{\Theta_1(0)(2\pi)^2} \cdot e^{-\frac{\pi}{4} \sum_{k \in \mathbb{Z}} \frac{|J_k|^2}{k^2} \sqrt{-1}} \times \frac{\sqrt{-1}}{4\pi J_0(1-J_0^2)} \cdot \left( e^{-3\pi \sqrt{-1} J_0} + e^{-2\pi \sqrt{-1} J_0} - e^{-\pi \sqrt{-1} J_0} - 1 \right), \]  

(6.1.5)

where \( J_k (k \in \mathbb{Z}) \) are the Fourier coefficients of the source \( J \) according to the decomposition in (6.1.2).

Let us introduce the one-point Green’s operator in momentum space

\[ S^{(1)} = \sum_{k \in \mathbb{Z}} S_k^{(1)} e^{2\pi \sqrt{-1} k t} dt, \]  

(6.1.6)

which is related to the defining relation (4.3.2) by

\[ S_1(v) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} S_k^{(1)} \bar{v}_k. \]  

(6.1.7)

Then (4.3.6) reduces to

\[ S_k^{(1)} = \begin{cases} 0, & \text{for } k \neq 0, \\ 6\pi^3 \sqrt{-1}, & \text{for } k = 0, \end{cases} \]  

(6.1.8)

showing that the one-point Greens function does not vanish in the topologically nontrivial case. Analogously, we define the two-point function \( S_{kl}^{(2)} \) in momentum space by

\[ S_2(v^{(1)}, v^{(2)}) = \frac{1}{2\pi} \sum_{k,l \in \mathbb{Z}} S_k^{(2)} \bar{v}_k v_l^{(2)}. \]  

(6.1.9)

Then (4.3.7) leads to

\[ S_k^{(2)} = \begin{cases} \frac{1}{2\pi k} \delta_{kl}, & \text{for } k, l \neq 0, \\ -2\pi \left( \frac{17\pi}{12} - \frac{1}{2} \right) + 2\pi \sum_{m \in \mathbb{Z}} m^2 e^{-\frac{(2\pi)^2}{4} m^2}, & \text{for } k = l = 0. \end{cases} \]  

(6.1.10)

Using (4.1.26) the VEV (4.2.18) of a gauge invariant function \( f \) on \( A^{(T^1)} \) simplifies to
\[
< f > = \hat{f}_0 = \int_0^1 dt (\chi_a f)(e^{2\pi \sqrt{-1} t}),
\]
(6.1.12)
showing that the zero mode of the gauge field determines the VEV completely. For instance, the Wilson loop observable \( f(A) = e^{\int_1^1 A} \) admits a vanishing VEV, i.e. \( < e^{\int_1^1 A} > = 0 \).

### 6.2. Abelian gauge theory on two-dimensional manifolds

On a closed two-dimensional manifold \( M \) all principal \( \mathbb{T}^N \)-bundles \( P \) are uniquely characterized by a \( N \)-dimensional vector of integers \( m = (m^\alpha)^N_{\alpha=1} \in \mathbb{Z}^N \). On each connected component of the space of connections, denoted by \( A^m \), let us choose a background connection \( A^m_0 \in A^m \) such that
\[
\int_M F_{A^m_0} = 2\pi \sqrt{-1} m,
\]
yields the Chern number of the corresponding torus bundle \( P \). Any two-form on \( M \) is proportional to the volume form \( \text{vol}_M \). In particular, \( F_{A^m_0} = \mu \text{vol}_M \). Because \( d^* F_{A^m_0} = 0 \) it follows that \( \mu = \frac{2\pi \sqrt{-1}}{V_0(M)} \). Using (4.2.17) the partition function for a single topological sector reads
\[
Z^{(m)}_{A^m_0} = (2\pi)^{b_1} (\det h)^{N/2} (\det \Delta_0 |_{\text{im} d_1^*})^{N} (\det \Delta_1|_{\text{Har}_1(M)^+})^{-N/2} e^{-\frac{1}{2}(2\pi)^2 \sum_{\alpha, \beta=1}^N \lambda_{\alpha \beta} m^\alpha m^\beta}, \tag{6.2.2}
\]
which sums up to the total partition function on the total configurations space \( A^{(M)} \)
\[
Z = (2\pi)^{b_1} (\det h)^{N/2} (\det \Delta_0 |_{\text{im} d_1^*})^{N} (\det \Delta_1|_{\text{Har}_1(M)^+})^{-N/2} \Theta_N(0) \frac{2\pi \sqrt{-1}}{V_0(M)} \Lambda). \tag{6.2.3}
\]
We conclude with some results regarding the VEV of a polynomial of the field strength on the total configuration space \( A^{(M)} \). The VEV - called \( W_q \) - can be derived from
\[
W_q(\eta_1 \ldots, \eta_q) = \left. \frac{\partial^q}{\partial t_1 \ldots \partial t_q} \right|_{t_1 = \ldots = t_q = 0} \sum_{m \in \mathbb{Z}^N} I^{(m)}(e^{< F_A, \sum_{i=1}^q t_i \eta_i >}) \sum_{m \in \mathbb{Z}^N} Z^{(m)}_{A^m_0}.
\tag{6.2.4}
\]

**Corollary 6.2.** If the matrix \( \lambda \) of couplings between the gauge fields is diagonal, the correlation functions of odd degree vanish, i.e. \( W_q^{(F)} = 0 \).

**Proof.** This result follows directly from the fact that only an odd number of \( F_{A_0} \) appears in (6.2.4).

For the Maxwell theory \( (N=1) \) the correlation functions of odd degree henceforth vanish whereas the correlation functions of even degree \( q = 2k \) yield
\[ W_{2k}(\eta_1, \ldots, \eta_{2k}) = \frac{(2\pi)^{2k}}{Vol(M)^{2k}} \sum_{m \in \mathbb{Z}} e^{-\frac{4\pi^2}{Vol(M)} m^2} \int_M \eta_1 \cdots \eta_{2k} \]
\[ + \sum_{l=1}^{k-1} \frac{(2\pi)^{2(k-l)}}{2^l l!(2(k-l))! Vol(M)^{(2(k-l)-1)}} \sum_{m \in \mathbb{Z}} e^{-\frac{4\pi^2}{Vol(M)} m^2} m^{2(k-l)} \Theta_1(0) \frac{2\pi \sqrt{-1}}{Vol(M)} \]
\[ \times \sum_s <\eta_s(1), \Pi^{Harm^2(M)}_s \eta_s(2) > \cdots <\eta_s(2l-1), \Pi^{Harm^2(M)}_s \eta_s(2l) > \]
\[ \times \int_M \eta_s(2l+1) \cdots \int_M \eta_s(2k) \]
\[ + \frac{1}{2^k k!} \sum_s <\eta_s(1), \Pi^{Harm^2(M)}_s \eta_s(2) > \cdots <\eta_s(2k-1), \Pi^{Harm^2(M)}_s \eta_s(2k) >, \] (6.2.5)

where \( \Pi^{Harm^2(M)}_{\eta_s(j)} = \eta_s(j) - \frac{1}{Vol(M)} (\int_M \eta_s(j)) \) \( vol_M \) is the projection onto the subspace \( imd_2 \otimes \mathbb{R} \).

The first term in (6.2.5) is topological as it depends only on the volume of \( M \). (For sake of completeness we refer to [38,39] where the field strength correlation functions in two dimensions have been calculated by a different method).

7. Concluding Remarks

In this paper we have tried to elaborate on the definition of a measure for gauge theories, which are affected by the Gribov problem. The starting point has been a modified functional integral, which we have applied to abelian gauge theories residing on \( n \)-dimensional compact manifolds \( M \) with and without a boundary. In both cases the non-triviality of the bundle of gauge fields and thus the existence of Gribov ambiguities have been proved. A patching prescription has been developed for the functional integral. We have used the description of the gauge orbit space as a bundle over a multi-dimensional torus to calculate the partition function, the vacuum expectation value (VEV) of gauge invariant observables and the Green’s functions of the theory. This explicit analysis of the underlying bundle structure may be also useful for further discussions on abelian gauge theories.

In the particular case of Maxwell theory, our results for both the partition function and the VEV of gauge invariant observables are in exact agreement with calculations based on the conventional Faddeev-Popov treatment: The volume of the gauge group can be indeed absorbed into a finite normalization constant.

On the other hand, the Green’s functions have been shown to be affected by the non-triviality of the bundle of gauge fields resulting in a non-vanishing vacuum expectation value of the gauge field. This phenomenon has been related to the first (relative) cohomology of the manifold \( M \).

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