ON A NEW NORM ON $B(\mathcal{H})$ AND ITS APPLICATIONS TO NUMERICAL RADIUS INEQUALITIES

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Abstract. We introduce a new norm on the space of bounded linear operators on a complex Hilbert space, which generalizes the numerical radius norm, the usual operator norm and the modified Davis-Wielandt radius. We study basic properties of this norm, including the upper and the lower bounds for it. As an application of the present study, we estimate bounds for the numerical radius of bounded linear operators. We illustrate with examples that our results improve on some of the important existing numerical radius inequalities.

1. Introduction

The purpose of the present article is to introduce a new norm, christened the $(\alpha, \beta)$-norm, on the space of bounded linear operators on a complex Hilbert space which generalizes the numerical radius norm, the usual operator norm and the recently introduced modified Davis-Wielandt radius (see [6]). We study some important properties of the $(\alpha, \beta)$-norm, and obtain upper and lower bounds for the said norm. This allows us to obtain some interesting numerical radius inequalities, that improve the existing results. Let us first introduce the following notations and terminologies.

Let $\mathcal{B}(\mathcal{H})$ denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with usual inner product $\langle \cdot, \cdot \rangle$. For $T \in \mathcal{B}(\mathcal{H})$, $T^*$ denotes the adjoint of $T$ and $|T|$ stands for the positive operator $(T^*T)^{1/2}$. The numerical range of $T$, denoted by $W(T)$, is defined as the collection of all complex scalars $\langle Tx, x \rangle$ with $\|x\| = 1$, i.e., $W(T) = \{\langle Tx, x \rangle : x \in \mathcal{H}, \|x\| = 1\}$. Let $\sigma(T)$ denote the spectrum of $T$. The usual operator norm, the numerical radius, the Crawford number and the spectral radius of $T$, denoted respectively by $\|T\|$, $w(T)$, $c(T)$ and $r(T)$, are defined as follows:

\[
\|T\| = \sup\{\|Tx\| : x \in \mathcal{H}, \|x\| = 1\},
\]

\[
w(T) = \sup\{|c| : c \in W(T)\},
\]

\[
c(T) = \inf\{|c| : c \in W(T)\},
\]

\[
r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}.
\]

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Clearly, $r(T) \leq w(T)$. Let $M_T$ and $c_T$ denote the norm attainment set and the Crawford number attainment set of $T$, respectively, i.e.,

$$M_T = \{ x \in \mathcal{H} : \|Tx\| = \|T\|, \|x\| = 1 \},$$

$$c_T = \{ x \in \mathcal{H} : c(T) = |\langle Tx, x \rangle|, \|x\| = 1 \}.$$

For $T \in \mathcal{B}(\mathcal{H})$, let $Re(T)$ and $Im(T)$ denote the real part and the imaginary part of $T$ respectively, i.e., $Re(T) = \frac{1}{2}(T + T^*)$ and $Im(T) = \frac{1}{2i}(T - T^*)$. It is well-known that

$$w(T) = \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta}T)\| = \sup_{\theta \in \mathbb{R}} \|Im(e^{i\theta}T)\|.$$

Also we know that $w(.)$ defines a norm on $\mathcal{B}(\mathcal{H})$ and is equivalent to the usual operator norm, satisfying the following inequality:

$$\frac{1}{2} \|T\| \leq w(T) \leq \|T\|, \ T \in \mathcal{B}(\mathcal{H}).$$

The above inequality is sharp. The first inequality becomes equality if $T^2 = 0$ and the second inequality becomes equality if $T$ is normal. Kittaneh \cite{9, 10} improved on the above inequality to show that

$$w(T) \leq \frac{1}{2} \|T\| + \|T^*\| \leq \frac{1}{2} \|T\| + \frac{1}{2} \|T^2\|^\frac{3}{2}$$

and

$$\frac{1}{4} \|T^*T + TT^*\| \leq w^2(T) \leq \frac{1}{2} \|T^*T + TT^*\|.$$

Abu-Omar and Kittaneh in \cite[Th. 2.4]{1} proved that

$$w^2(T) \leq \frac{1}{2} w(T^2) + \frac{1}{4} \|T^*T + TT^*\|,$$

which improves on the above upper bounds. A lot of study has been done in this direction to improve upper and lower bounds for the numerical radius, and we refer the readers to \cite{3, 4, 5, 13, 14, 15}, and the references therein, for a comprehensive idea of the current state of the art.

Let us now introduce the $(\alpha, \beta)$-norm on $\mathcal{B}(\mathcal{H})$. Throughout the paper we reserve $\alpha, \beta$ for non-negative real scalars, i.e., $\alpha, \beta \geq 0$, such that $(\alpha, \beta) \neq (0, 0)$. Let us consider a mapping $\|\cdot\|_{\alpha, \beta} : \mathcal{B}(\mathcal{H}) \to \mathbb{R}^+$, defined as follows:

$$\|T\|_{\alpha, \beta} = \sup \left\{ \sqrt{\alpha \|Tx\|^2 + \beta \|T^*x\|^2} : x \in \mathcal{H}, \|x\| = 1 \right\}.$$

We observe that $\|\cdot\|_{\alpha, \beta}$ defines a norm on $\mathcal{B}(\mathcal{H})$. We also observe that if $\alpha = 1, \beta = 0$ then $\|T\|_{\alpha, \beta} = w(T)$ and if $\alpha = 0, \beta = 1$ then $\|T\|_{\alpha, \beta} = \|T\|$. Moreover, if we consider $\alpha = \beta = 1$ then we get the modified Davis-Wielandt radius of $T$, i.e., $\|T\|_{\alpha, \beta} = d^w(T)$ (see \cite{6}).

In this paper we show that the $(\alpha, \beta)$-norm is equivalent to the numerical radius norm and the usual operator norm. We also study the equality conditions for the said bounds. We then obtain some upper and lower bounds for the $(\alpha, \beta)$-norm of bounded linear operators, and apply the results to obtain bounds for the numerical radius of bounded linear operators. Among other results obtained in this article,
we prove the following three important inequalities:

\[
\begin{align*}
    w(T) & \leq \inf_{\alpha,w} \left\{ \frac{1}{\sqrt{\alpha + \beta}} \left( \frac{\alpha}{4} |T| + \beta |T^*T| \right)^2 \right\} \leq \frac{1}{2} |T| + |T^*|, \\
    w^2(T) & \leq \inf_{\alpha,\beta} \left\{ \frac{1}{\alpha + \beta} \left( \frac{\alpha}{4} (T^*T + TT^*) + \beta |T| \right) \right\} \leq \frac{1}{2} |T^*T + TT^*| \quad \text{and} \\
    w^2(T) & \leq \inf_{\alpha,\beta} \frac{1}{\alpha + \beta} \left\{ \frac{\alpha}{2} w(T^2) + \frac{\alpha}{4} (T^*T + TT^*) + \beta |T| \right\} \\
    & \leq \frac{1}{2} w(T^2) + \frac{1}{4} |T^*T + TT^*|. 
\end{align*}
\]

We discuss conditions under which the bounds obtained here is strictly sharper than the existing bounds. We also discuss conditions under which the bounds are equal. Using a similarly motivated approach, we also improve the existing lower bound for the numerical radius of an operator, as obtained in \cite[Th. 1]{10}.

## 2. The $(\alpha, \beta)$-norm and numerical radius inequalities

We begin this section with the observation that $\|\cdot\|_{\alpha,\beta}$ defines a norm on $\mathcal{B}(\mathcal{H})$, equivalent with the numerical radius norm and the usual operator norm.

**Theorem 2.1.** $\|\cdot\|_{\alpha,\beta}$ defines a norm on $\mathcal{B}(\mathcal{H})$ and is equivalent to the numerical radius norm $w(.)$ and the usual operator norm $\|\cdot\|$, satisfying the following inequalities:

\[
\sqrt{(\alpha + \beta)} \ w(T) \leq \|T\|_{\alpha,\beta} \leq \sqrt{(\alpha + 4\beta)} \ w(T),
\]

\[
\max \left\{ \frac{\sqrt{\alpha + \beta}}{2}, \sqrt{\beta} \right\} \|T\| \leq \|T\|_{\alpha,\beta} \leq \sqrt{(\alpha + 3\beta)} \|T\|.
\]

**Proof.** First we prove that $\|\cdot\|_{\alpha,\beta}$ defines a norm on $\mathcal{B}(\mathcal{H})$. Clearly, it is sufficient to show that $\|\cdot\|_{\alpha,\beta}$ satisfies the triangle inequality, as all other defining conditions for being a norm follow trivially. Let $S, T \in \mathcal{B}(\mathcal{H})$. Then we get

\[
\begin{align*}
    \|S + T\|_{\alpha,\beta}^2 & = \sup_{\|x\| = 1} \left\{ \alpha |\langle (S + T)x, x \rangle|^2 + \beta \|S + Tx\|^2 \right\} \\
    & \leq \sup_{\|x\| = 1} \left\{ \alpha \left( |\langle Sx, x \rangle| + |\langle Tx, x \rangle| \right)^2 + \beta \|Sx\|^2 + \|Tx\|^2 \right\} \\
    & \leq \sup_{\|x\| = 1} \left\{ \alpha |\langle Sx, x \rangle|^2 + \beta \|Sx\|^2 \right\} + \sup_{\|x\| = 1} \left\{ \alpha |\langle Tx, x \rangle|^2 + \beta \|Tx\|^2 \right\} \\
    & \quad + \sup_{\|x\| = 1} \{ 2(\alpha |\langle Sx, x \rangle| |\langle Tx, x \rangle| + \beta \|Sx\||\|Tx\|) \} \\
    & \leq \sup_{\|x\| = 1} \left\{ \alpha |\langle Sx, x \rangle|^2 + \beta \|Sx\|^2 \right\} + \sup_{\|x\| = 1} \left\{ \alpha |\langle Tx, x \rangle|^2 + \beta \|Tx\|^2 \right\} \\
    & \quad + 2 \sup_{\|x\| = 1} \sqrt{\alpha |\langle Sx, x \rangle|^2 + \beta \|Sx\|^2} \sqrt{\alpha |\langle Tx, x \rangle|^2 + \beta \|Tx\|^2}, \\
    & = \|S\|_{\alpha,\beta}^2 + \|T\|_{\alpha,\beta}^2 + 2 \|S\|_{\alpha,\beta} \|T\|_{\alpha,\beta} \\
    & = (\|S\|_{\alpha,\beta} + \|T\|_{\alpha,\beta})^2.
\end{align*}
\]
Therefore, \( \| S + T \|_{\alpha, \beta} \leq \| S \|_{\alpha, \beta} + \| T \|_{\alpha, \beta} \) for all \( S, T \in \mathcal{B}(\mathcal{H}) \). Hence, \( \| \cdot \|_{\alpha, \beta} \) defines a norm on \( \mathcal{B}(\mathcal{H}) \). Next we have,

\[
\| T \|_{\alpha, \beta}^2 = \sup_{\| x \| = 1} \{ \alpha |\langle T x, x \rangle|^2 + \beta \| T x \|_2^2 \} \leq \sup_{\| x \| = 1} \{ \alpha |\langle T x, x \rangle|^2 \} + \sup_{\| x \| = 1} \{ \beta \| T x \|_2^2 \} = \alpha w^2(T) + \beta \| T \|_2^2 \\
\leq (\alpha + 4\beta) w^2(T),
\]

so that \( \| T \|_{\alpha, \beta} \leq \sqrt{(\alpha + 4\beta)} \ w(T) \). Again we have,

\[
\| T \|_{\alpha, \beta} = \sup_{\| x \| = 1} \sqrt{\alpha |\langle T x, x \rangle|^2 + \beta \| T x \|_2^2} \\
\geq \sup_{\| x \| = 1} \sqrt{(\alpha + \beta) |\langle T x, x \rangle|^2} = (\alpha + \beta) w(T).
\]

Thus we get,

\[
\sqrt{(\alpha + \beta)} \ w(T) \leq \| T \|_{\alpha, \beta} \leq \sqrt{(\alpha + 4\beta)} \ w(T).
\]

Proceeding similarly we can show that,

\[
\max \left\{ \frac{\sqrt{\alpha + \beta}}{2}, \sqrt{\beta} \right\} \| T \| \leq \| T \|_{\alpha, \beta} \leq \sqrt{(\alpha + \beta)} \| T \|.
\]

This completes the proof.

**Remark 2.2.** The classical numerical radius bounds follow easily from the above inequalities by considering either \( \alpha = 1, \beta = 0 \) or \( \alpha = 0, \beta = 1 \).

In the following theorems we study the equality conditions for the bounds of the \((\alpha, \beta)\)-norm. We begin with the following theorem, which characterises operators \( T \) for which \( \| T \|_{\alpha, \beta} = \sqrt{\alpha w^2(T) + \beta \| T \|_2^2} \).

**Theorem 2.3.** Let \( T \in \mathcal{B}(\mathcal{H}) \) and let \( \alpha \beta \neq 0 \). Then the following conditions are equivalent:

(i) \( \| T \|_{\alpha, \beta} = \sqrt{\alpha w^2(T) + \beta \| T \|_2^2} \).

(ii) \( T \) is normaloid, i.e., \( w(T) = \| T \| \).

(iii) There exist a sequence of unit vectors \( \{ x_n \} \) in \( \mathcal{H} \) such that

\[
\lim_{n \to \infty} \| T x_n \| = \| T \| \text{ and } \lim_{n \to \infty} |\langle T x_n, x_n \rangle| = w(T).
\]

**Proof.** The equivalence of (ii) and (iii) is well-known. We only prove the equivalence of (i) and (iii).

We first prove (i) \( \Rightarrow \) (iii). Since \( T \in \mathcal{B}(\mathcal{H}) \), there exists a sequence of unit vectors \( \{ x_n \} \) in \( \mathcal{H} \) such that

\[
\| T \|_{\alpha, \beta} = \lim_{n \to \infty} \sqrt{\alpha |\langle T x_n, x_n \rangle|^2 + \beta \| T x_n \|_2^2}.
\]

Clearly \( \{ |\langle T x_n, x_n \rangle| \} \) and \( \{ \| T x_n \| \} \) both are bounded sequences of real numbers, so there exists a subsequence \( \{ x_{n_k} \} \) of the sequence \( \{ x_n \} \) such that both \( \{ |\langle T x_{n_k}, x_{n_k} \rangle| \} \)
implies that there exists $12$ and Theorem we get, $2.1$ $2.3$ $2.3$ $easily,$ except for property $(v)$ gously, we skip them to avoid monotonicity. Next we observe the pro perties of the $B$ $| \langle \sqrt{\alpha w}, x, x \rangle | \leq \sqrt{\alpha \|T x, x\|^2 + \beta \|T x_n\|^2}$ 
$\leq \lim_{k \to \infty} \alpha \|T x_n, x_k\|^2 + \lim_{k \to \infty} \beta \|T x_n\|^2$ 
$\leq \lim_{k \to \infty} \alpha \|T x_n, x_k\|^2 + \beta \|T\|^2$ 
$\leq \alpha w^2(T) + \beta \|T\|^2.$

This implies that 
$$\lim_{k \to \infty} \|T x_n\| = \|T\| \quad \text{and} \quad \lim_{k \to \infty} \|T x_n, x_n\| = w(T).$$

To show that $(iii) \Rightarrow (i),$ we observe that 
$$\|T\|_{\alpha, \beta} = \sup_{\|x\|=1} \sqrt{\alpha \|T x, x\|^2 + \beta \|T x\|^2}$$ 
$$\geq \lim_{n \to \infty} \sqrt{\alpha \|T x_n, x_n\|^2 + \beta \|T x_n\|^2}$$ 
$$= \sqrt{\alpha w^2(T) + \beta \|T\|^2}.$$

This completes the proof. $\square$

**Remark 2.4.** (i) In case $\mathbb{H}$ is finite-dimensional, the condition $(iii)$ in Theorem 2.3 implies that there exists $x \in \mathbb{H}, \|x\| = 1$ such that $\|T x\| = \|T\|$ and $w(T) = |\langle T x, x \rangle|.$

(ii) The equality conditions for $\|T\|_{\alpha, \beta} = \sqrt{\alpha + \beta} w(T)$ as well as $\|T\|_{\alpha, \beta} = \sqrt{\alpha + \beta} \|T\|$ follows from Theorem 2.1 and Theorem 2.3.

We next find the equality condition for $\|T\|_{\alpha, \beta} = \sqrt{\alpha + 4\beta} w(T)$.

**Theorem 2.5.** Let $T \in \mathcal{B}(\mathbb{H})$ and let $\alpha \beta \neq 0.$ Then $\|T\|_{\alpha, \beta} = \sqrt{\alpha + 4\beta} w(T)$ if and only if $T = 0.$

**Proof.** The sufficient part is trivial. We only prove the necessary part. Let $\|T\|_{\alpha, \beta} = \sqrt{\alpha + 4\beta} w(T).$ Then we have, $(\alpha + 4\beta) w^2(T) = \|T\|^2 \leq \alpha w^2(T) + \beta \|T\|^2 \leq \alpha w^2(T) + \beta (2w(T))^2 = (\alpha + 4\beta) w^2(T).$ This implies that $w(T) = \frac{\|T\|}{2}.$ On the other hand using Theorem 2.3 we get, $w(T) = \|T\|.$ Therefore, $\|T\| = 0,$ i.e., $T = 0.$ $\square$

The characterizations for the equality of other bounds can be obtained analo-gously, we skip them to avoid monotonicity. Next we observe the properties of the $(\alpha, \beta)$-norm on $\mathcal{B}(\mathbb{H})$ in the form of following proposition, the proof of which follows easily, except for property $(v),$ which follows from [12].

**Proposition 2.6.** The $(\alpha, \beta)$-norm on $\mathcal{B}(\mathbb{H})$ satisfies the following properties:

(i) $\|\cdot\|_{\alpha, \beta}$ is not an algebra norm, i.e., there exists $A, B \in \mathcal{B}(\mathbb{H})$ for which 
$$\|AB\|_{\alpha, \beta} \leq \|A\|_{\alpha, \beta} \|B\|_{\alpha, \beta}$$
does not hold.

(ii) $\|\cdot\|_{\alpha, \beta}$ does not satisfy the power inequality, i.e., there exist operators $A, B \in \mathcal{B}(\mathbb{H})$ and $n \in \mathbb{N}$ \{1\} such that 
$$\|A^n\|_{\alpha, \beta} < \|A\|_{\alpha, \beta}^n \text{ and } \|B^n\|_{\alpha, \beta} > \|B\|_{\alpha, \beta}^n.$$
(iii) If $\alpha + \beta = 1$ and $A$ is normal then $\|A^n\|_{\alpha, \beta} = \|A\|_{\alpha, \beta}^n$, for all $n \in \mathbb{N}$. 

(iv) $\|\cdot\|_{\alpha, \beta}$ is weakly unitarily invariant, i.e., 

$$\|U^*TU\|_{\alpha, \beta} = \|T\|_{\alpha, \beta}, \forall T \in \mathcal{B}(\mathcal{H}),$$

where $U \in \mathcal{B}(\mathcal{H})$ is an unitary operator.

(v) $\|\cdot\|_{\alpha, \beta}$ is preserved under the adjoint operation, i.e., 

$$\|T\|_{\alpha, \beta} = \|T^*\|_{\alpha, \beta}, \forall T \in \mathcal{B}(\mathcal{H}).$$

We next obtain the following lower bound for the $(\alpha, \beta)$-norm.

**Theorem 2.7.** Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$\|T\|^2_{\alpha, \beta} \geq \max \left\{ \alpha w^2(T) + \beta c(T^*T), \alpha c^2(T) + \beta \|T\|^2, \right\}$$

$$\frac{2\sqrt{\alpha\beta}w(T)}{\sqrt{c(T^*T)}}, \frac{2\sqrt{\alpha\beta}c(T)}{\|T\|}. $$

Proof. For all $x \in \mathcal{H}$ with $\|x\| = 1$, we get,

$$\|T\|^2_{\alpha, \beta} \geq \alpha \langle Tx, x \rangle^2 + \beta \|Tx\|^2$$

$$= \alpha \langle Tx, x \rangle^2 + \beta \langle T^*Tx, x \rangle$$

$$\geq \alpha \langle Tx, x \rangle^2 + \beta c(T^*T).$$

Therefore, taking supremum over all unit vectors in $\mathcal{H}$, we get,

$$\|T\|^2_{\alpha, \beta} \geq \alpha w^2(T) + \beta c(T^*T).$$

Also, $\|T\|^2_{\alpha, \beta} \geq \alpha \langle Tx, x \rangle^2 + \beta \|Tx\|^2 \geq \alpha c^2(T) + \beta \|T\|^2$. Taking supremum over all unit vectors in $\mathcal{H}$, we get, $\|T\|^2_{\alpha, \beta} \geq \alpha c^2(T) + \beta \|T\|^2$. Again, we have

$$\|T\|^2_{\alpha, \beta} \geq \alpha \langle Tx, x \rangle^2 + \beta \|Tx\|^2$$

$$\geq 2\sqrt{\alpha\beta}\langle Tx, x \rangle \|Tx\|$$

$$\geq 2\sqrt{\alpha\beta}c(T^*T).$$

Therefore,

$$\|T\|^2_{\alpha, \beta} \geq 2\sqrt{\alpha\beta}w(T) \sqrt{c(T^*T)}.$$

We also observe that $\|T\|^2_{\alpha, \beta} \geq 2\sqrt{\alpha\beta}\langle Tx, x \rangle \|Tx\| \geq 2\sqrt{\alpha\beta}c(T)\|Tx\|$. Therefore, $\|T\|^2_{\alpha, \beta} \geq 2\sqrt{\alpha\beta}c(T)\|T\|$. Combining the above inequalities we get the required inequality. □

We next obtain bounds for the $(\alpha, \beta)$-norm of the product of two bounded linear operators. We require the following known lemmas, which can be found in [8, pp. 37-39].

**Lemma 2.8.** Let $A, B \in \mathcal{B}(\mathcal{H})$. Then

(i) $w(AB) \leq 4w(A)w(B)$.

(ii) If $AB = BA$ then $w(AB) \leq 2w(A)w(B)$.

(iii) If $AB = BA$ and $A$ is an isometry, then $w(AB) \leq w(B)$.

(iv) If $AB = BA$ and $AB^* = B^*A$, then $w(AB) \leq w(B)\|A\|$.

We are now in a position to obtain upper bounds for the $(\alpha, \beta)$-norm of the product of two bounded linear operators.
Theorem 2.9. Let \( A, B \in B(\mathcal{H}) \) and let \( \beta \neq 0 \). Then we have the following inequality:

\[
\|AB\|_{\alpha, \beta} \leq \sqrt{\min\left\{ \frac{4}{\beta^2}, \frac{16}{\alpha + \beta} \right\}} \|A\|_{\alpha, \beta} \|B\|_{\alpha, \beta}.
\]

Proof. From the definition of \((\alpha, \beta)\)-norm, we have

\[
\|AB\|^2_{\alpha, \beta} = \sup_{\|x\|=1} \left\{ \alpha |\langle ABx, x \rangle|^2 + \beta \|ABx\|^2 \right\}
\]

\[
\leq \alpha w^2(AB) + \beta \|AB\|^2
\]

\[
\leq 4(\alpha + \beta)w^2(A)\|B\|^2, \quad \text{using} \quad \|A\| \leq 2w(A)
\]

\[
\leq 4(\alpha + \beta)w^2(A)(1/\beta)\|B\|_{\alpha, \beta}^2,
\]

\[
\leq \frac{4}{\beta}\|A\|_{\alpha, \beta}^2\|B\|_{\alpha, \beta}^2, \quad \text{using Theorem 2.1}.
\]

Thus we get,

\[
\|AB\|_{\alpha, \beta} \leq \sqrt{\frac{4}{\beta}\|A\|_{\alpha, \beta}\|B\|_{\alpha, \beta}}.
\]

Proceeding as above and using \(\sqrt{\beta}\|A\| \leq \|A\|_{\alpha, \beta}, \sqrt{\beta}\|B\| \leq \|B\|_{\alpha, \beta}\), we get,

\[
\|AB\|_{\alpha, \beta} \leq \sqrt{\frac{\alpha + \beta}{\beta^2}}\|A\|_{\alpha, \beta}\|B\|_{\alpha, \beta}.
\]

Finally using \(\|A\| \leq 2w(A), \|B\| \leq 2w(B)\) and Theorem 2.1 we obtain,

\[
\|AB\|_{\alpha, \beta} \leq \sqrt{\frac{16}{\alpha + \beta}}\|A\|_{\alpha, \beta}\|B\|_{\alpha, \beta}.
\]

Combining all the above inequalities we get the required inequality.

\[\Box\]

In the following theorem we obtain upper bounds for the \((\alpha, \beta)\)-norm of the product of two bounded linear operators, under the additional assumption that they commute.

Theorem 2.10. Let \( A, B \in B(\mathcal{H}) \) be such that \( AB = BA \).

(i) If \( \beta \neq 0 \), then

\[
\|AB\|_{\alpha, \beta} \leq \sqrt{\left( \frac{4\alpha}{(\alpha + \beta)^2} + \frac{1}{\beta} \right)}\|A\|_{\alpha, \beta}\|B\|_{\alpha, \beta}.
\]

(ii) If \( A \) is an isometry, then

\[
\|AB\|_{\alpha, \beta} \leq \sqrt{\left( \frac{2}{\alpha + \beta} + 1 \right)}\|B\|_{\alpha, \beta}.
\]

(iii) If \( AB^* = B^*A \) and \( \beta \neq 0 \), then

\[
\|AB\|_{\alpha, \beta} \leq \sqrt{\left( \frac{\alpha}{\alpha + \beta} + 1 \right)} \min\left\{ \frac{2}{\sqrt{\alpha + \beta}}, \frac{1}{\sqrt{\beta}} \right\}\|A\|_{\alpha, \beta}\|B\|_{\alpha, \beta}.
\]
Proof. (i). Using the definition of the \( (\alpha, \beta) \) norm and Lemma 2.8 (ii), we get,
\[
\|AB\|_{\alpha, \beta}^2 = \sup_{\|x\|=1} \left\{ \alpha |\langle ABx, x \rangle|^2 + \beta \|ABx\|^2 \right\}
\leq \alpha w^2(AB) + \beta \|AB\|^2
\leq 4\alpha w^2(A)w^2(B) + \beta \|A\|^2\|B\|^2
\leq \left( \frac{(\alpha + \beta)^2}{\alpha \beta} + 1 \right) \|A\|_{\alpha, \beta}^2\|B\|_{\alpha, \beta}^2 \text{, using Theorem 2.1.}
\]
Thus we get the inequality in (i).

(ii). As above, using the definition of \( (\alpha, \beta) \) norm and Lemma 2.8 (iii), we get,
\[
\|AB\|_{\alpha, \beta}^2 = \sup_{\|x\|=1} \left\{ \alpha |\langle ABx, x \rangle|^2 + \beta \|ABx\|^2 \right\}
\leq \alpha w^2(AB) + \beta \|AB\|^2
\leq \alpha w^2(B) + \beta \|B\|^2,
\leq \left( \frac{\alpha}{\alpha \beta} + 1 \right) \|B\|_{\alpha, \beta}^2 \text{, using Theorem 2.1.}
\]
Thus we get the inequality in (ii).

(iii). Proceeding as in the above two cases and using Lemma 2.8 (iv), we get the required inequality.

To proceed further in the estimation of upper bound for the \( (\alpha, \beta) \)-norm of product of two bounded linear operators, we need the following two lemmas.

Lemma 2.11. ([14]) (i) The Power-Mean inequality:
\[
\alpha^t \beta^{(1-t)} \leq \alpha^{\frac{nt}{n+m}} \beta^{\frac{mt}{n+m}} \leq \left( \frac{\alpha^n}{n} + \frac{\beta^m}{m} \right)^{\frac{1}{p}} \text{,}
\]
for all \( t \in [0,1] \), \( a, b \geq 0 \) and \( p \geq 1 \).

(ii) The Power-Young inequality:
\[
ab \leq \frac{a^n}{n} + \frac{b^m}{m} \leq \left( \frac{a^n}{n} + \frac{b^m}{m} \right)^{\frac{1}{p}} \text{,}
\]
for all \( a, b \geq 0 \) and \( n, m > 1 \) with \( \frac{1}{n} + \frac{1}{m} = 1 \) and \( p \geq 1 \).

Lemma 2.12. ([11]) Let \( A \in \mathcal{B}(\mathcal{H}) \) be a positive operator i.e., \( A \geq 0 \). Then for any unit vector \( x \in \mathcal{H} \), we have the following inequality:
\[
\langle Ax, x \rangle^p \leq \langle A^p x, x \rangle \text{,}
\]
for all \( p \geq 1 \).

Theorem 2.13. Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be such that \( AB = BA \) and \( A^*B = BA^* \). Also let \( \alpha + \beta = 1 \). Then
\[
\|AB\|_{\alpha, \beta} \leq \left( 2w \left( \frac{\alpha(AA^*)^pn}{m} + \beta(A^*A)^pn \right) \right)^{\frac{1}{p}},
\]
where \( n, m > 1 \) with \( \frac{1}{n} + \frac{1}{m} = 1 \), \( p \geq 1 \), \( pn \geq 2 \), \( pm \geq 2 \).
Proof. Let $x$ be a unit vector in $\mathcal{H}$. Then by the convexity of the function $t^p$, we get
\[
(\alpha|\langle ABx, x\rangle|^2 + \beta\|ABx\|^2)^p \leq \alpha|\langle ABx, x\rangle|^{2p} + \beta\|ABx\|^{2p}.
\]
Using Lemma 2.11 (ii) and Lemma 2.12, we get
\[
\alpha|\langle ABx, x\rangle|^{2p} + \beta\|ABx\|^{2p} = \alpha|\langle Bx, A^*x\rangle|^{2p} + \beta|\langle A^*Ax, B^*Bx\rangle|^p
\leq \alpha\|Bx\|^{2p}\|A^*x\|^2 + \beta\|A^*Ax\|^p\|B^*Bx\|^p
= \alpha|\langle AA^*x, x\rangle|^p(B^*Bx, x)^p + \beta((A^*A)^2x, x)^\frac{p}{2}((B^*B)^2x, x)^\frac{p}{2}
\leq \alpha\left(\frac{1}{n}(AA^*x, x)^{pm} + \frac{1}{m}(B^*Bx, x)^{pm}\right)
+ \beta\left(\frac{1}{n}((A^*A)^{pm}x, x)\right) + \frac{1}{m}((B^*B)^{pm}x, x)
\leq \alpha\left(\frac{1}{n}(AA^*x, x) + \frac{1}{m}(B^*Bx, x)\right)
+ \beta\left(\frac{1}{n}((A^*A)^{pm}x, x) + \frac{1}{m}((B^*B)^{pm}x, x)\right)
= \left(\frac{1}{n}(\alpha(\alpha A^*)^{pm} + \beta(A^*A)^{pm}) + \frac{1}{m}(B^*B)^{pm}\right)x, x
\leq \left(\frac{1}{n}(\alpha(\alpha A^*)^{pm} + \beta(A^*A)^{pm}) + \frac{1}{m}(B^*B)^{pm}\right)
= 2\omega\left(\frac{0}{n}(\alpha(\alpha A^*)^{pm} + \beta(A^*A)^{pm})\right)\right).
\]
Therefore,
\[
(\alpha|\langle ABx, x\rangle|^2 + \beta\|ABx\|^2)^p \leq 2\omega\left(\frac{0}{n}(\alpha(\alpha A^*)^{pm} + \beta(A^*A)^{pm})\right)\right).
\]
Taking supremum over all unit vectors in $\mathcal{H}$, we get the required inequality. \qed

Next we obtain an inequality for the $(\alpha, \beta)$-norm of product of two bounded linear operators in terms of non-negative continuous functions on $[0, \infty)$. We need the following lemma.

Lemma 2.14. [11, Th. 5] Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $|A|B = B^*|A|$. If $f$ and $g$ are two non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, \forall $t \geq 0$ then
\[
|\langle ABx, y\rangle| \leq r(B)|f(|A|)x||g(|A^*|)y|,
\]
for any vectors $x, y \in \mathcal{H}$.

Theorem 2.15. Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $|A|B = B^*|A|$ and let $\alpha + \beta = 1$. If $f$ and $g$ are two non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, \forall $t \geq 0$ then
\[
(i) \quad \|AB\|_{\alpha, \beta} \leq \left\|\alpha^{2p}(B)\left(\frac{1}{p}f^{mp}(|A|) + \frac{1}{m}g^{mp}(|A^*|)\right)^2 + \beta((AB)^*(AB))^p\right\|^{\frac{1}{p}}
\]
and
\[
(ii) \quad w(AB) \leq \inf_{\alpha + \beta = 1} \left\|\alpha^{2p}(B)\left(\frac{1}{p}f^{mp}(|A|) + \frac{1}{m}g^{mp}(|A^*|)\right)^2 + \beta((AB)^*(AB))^p\right\|^{\frac{1}{p}}\right.,
\]
where $n, m > 1$ with $\frac{1}{n} + \frac{1}{m} = 1$, $p \geq 1$, $pn \geq 2$ and $pm \geq 2$. 
Proof. Let \( x \) be a unit vector in \( H \). Then using Lemma 2.14 and Lemma 2.11 (ii), we get

\[
|\langle ABx, x \rangle| \leq r(B)\|f(|A|)x\||g(|A^*|)x\|
\]

\[
\Rightarrow |\langle ABx, x \rangle| \leq r(B) \left( \frac{1}{n}\|f(|A|)x\|^n + \frac{1}{m}\|g(|A^*|)x\|^m \right)
\]

\[
\Rightarrow |\langle ABx, x \rangle| \leq r(B) \left( \frac{1}{n} \left| f^2(|A|)x, x \right|^{\frac{2}{2}} + \frac{1}{m} \left| g^2(|A^*|)x, x \right|^{\frac{2}{2}} \right)
\]

\[
\Rightarrow |\langle ABx, x \rangle|^p \leq r^p(B) \left( \frac{1}{n} \left| f^2(|A|)x, x \right|^{\frac{2p}{2}} + \frac{1}{m} \left| g^2(|A^*|)x, x \right|^{\frac{2p}{2}} \right)^p
\]

\[
\Rightarrow |\langle ABx, x \rangle|^p \leq r^p(B) \left( \frac{1}{n} \left| f^{np}(|A|) + \frac{1}{m}g^{np}(|A^*|) \right) x, x \right|^2
\]

\[
\Rightarrow |\langle ABx, x \rangle|^{2p} \leq r^{2p}(B) \left( \frac{1}{n} f^{np}(|A|) + \frac{1}{m}g^{np}(|A^*|) \right) x, x \right.
\]

Now using Lemma 2.12, we get

\[
\|ABx\|^{2p} = \langle ABx, ABx \rangle^p = \langle (AB)^*(AB)x, x \rangle^p \leq \langle ((AB)^*(AB))^p x, x \rangle.
\]

Then using Lemma 2.11 (ii), we get

\[
\|AB\|_{\alpha, \beta}^{2p} = \sup_{\|x\|=1} \left\{ \alpha |\langle ABx, x \rangle|^2 + \beta \|ABx\|^2 \right\}^p
\]

\[
\leq \sup_{\|x\|=1} \left( \alpha |\langle ABx, x \rangle|^{2p} + \beta \|ABx\|^{2p} \right)
\]

\[
\leq \sup_{\|x\|=1} \left( \alpha |\langle ABx, x \rangle|^{2p} \left( \frac{1}{n} f^{np}(|A|) + \frac{1}{m}g^{np}(|A^*|) \right) + \beta ((AB)^*(AB))^p \right) x, x \right)
\]

\[
= \left\| \alpha |\langle ABx, x \rangle|^{2p} \left( \frac{1}{n} f^{np}(|A|) + \frac{1}{m}g^{np}(|A^*|) \right) + \beta ((AB)^*(AB))^p \right) \right\|.
\]

Thus we obtain the inequality in (i). Next using Theorem 2.1, namely, \( w(AB) \leq \frac{1}{\sqrt{\alpha+\beta}} \|AB\|_{\alpha, \beta} \), we get,

\[
w(AB) \leq \frac{1}{\sqrt{\alpha+\beta}} \left\| \alpha |\langle ABx, x \rangle|^{2p} \left( \frac{1}{n} f^{np}(|A|) + \frac{1}{m}g^{np}(|A^*|) \right) + \beta ((AB)^*(AB))^p \right) \right\|^{\frac{1}{2p}}.
\]

Taking infimum over \( \alpha, \beta \), with \( \alpha + \beta = 1 \) we get the inequality in (ii). \( \square \)

**Remark 2.16.** It is clear that the inequality obtained in Theorem 2.15(ii) improves on the existing inequalities [3, Th. 2.5] and [2, Th. 3.1].

Next we prove the following inequality:
Theorem 2.17. Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $AB = BA, AB^* = B^*A$ and $|A|B = B^*|A|$ and let $\alpha + \beta = 1$. If $f$ and $g$ are two non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, $\forall \ t \geq 0$, then

(i) $\|AB\|_{\alpha, \beta} \leq \|\alpha r^{2p}(B^*)X^2 + \beta r^p(B^*B)Y\|^{\frac{1}{2p}}$

(ii) $w(AB) \leq \inf_{\alpha + \beta = 1} \|\alpha r^{2p}(B^*)X^2 + \beta r^p(B^*B)Y\|^{\frac{1}{2p}}$

where $X = \frac{1}{n}f^{np}(|A|) + \frac{1}{m}g^{mp}(|A^*|)$, $Y = \frac{1}{n}f^{np}(|A^*A|) + \frac{1}{m}g^{mp}(|A^*A|)$, and $n, m > 1$ with $\frac{1}{n} + \frac{1}{m} = 1$, $p \geq 1$, $pm \geq 2$, $pm \geq 2$.

Proof. Let $x \in \mathcal{H}$ with $\|x\| = 1$. Proceeding similarly as in the proof of Theorem 2.15, we get

$$|\langle ABx, x\rangle|^{2p} \leq r^{2p}(B)\left(\frac{1}{n}f^{np}(|A|) + \frac{1}{m}g^{mp}(|A^*|)\right)^2 x, x.$$ 

Noting that $(AB)^*(AB) = A^*AB^*B$, $|A^*A|B^*B = B^*B|A^*A|$ and using Lemma 2.14 we get,

$$\|ABx\|^2 = \langle A^*AB^*Bx, x \rangle \Rightarrow \|ABx\|^2 \leq r(B^*B)\|f(|A^*A|)\|\|g(|A^*A|)\|$$

$$\Rightarrow \|ABx\|^{2p} \leq r^p(B^*B)\left(\frac{1}{n}f(|A^*A|)\|g(|A^*A|)\|^n + \frac{1}{m}f(|A^*A|)\|g(|A^*A|)\|^m\right)^p$$

$$\Rightarrow \|ABx\|^{2p} \leq r^p(B^*B)\left(\frac{1}{n}\langle f^2(|A^*A|)x, x \rangle^{\frac{mp}{2}} + \frac{1}{m}\langle g^2(|A^*A|)x, x \rangle^{\frac{mp}{2}}\right)$$

$$\Rightarrow \|ABx\|^{2p} \leq r^p(B^*B)\left(\frac{1}{n}\langle f^{np}(|A^*A|)x, x \rangle + \frac{1}{m}\langle g^{mp}(|A^*A|)x, x \rangle\right)$$

$$\Rightarrow \|ABx\|^{2p} \leq r^p(B^*B)\left(\frac{1}{n}\langle f^{np}(|A^*A|)x, x \rangle + \frac{1}{m}\langle g^{mp}(|A^*A|)x, x \rangle\right)$$

Next using Lemma 2.11, we get

$$\|AB\|_{\alpha, \beta}^{2p} = \sup_{\|x\|=1} \{\alpha|\langle ABx, x \rangle|^2 + \beta\|ABx\|^2\}^{\frac{1}{2p}}$$

$$\leq \sup_{\|x\|=1} \{\alpha|\langle ABx, x \rangle|^2 + \beta\|ABx\|^2\}$$

$$\leq \sup_{\|x\|=1} \{\langle (\alpha r^{2p}(B^*)X^2 + \beta r^p(B^*B)Y) x, x \rangle\}$$

$$= \|\alpha r^{2p}(B^*)X^2 + \beta r^p(B^*B)Y\|$$

where $X = \frac{1}{n}f^{np}(|A|) + \frac{1}{m}g^{mp}(|A^*|)$ and $Y = \frac{1}{n}f^{np}(|A^*A|) + \frac{1}{m}g^{mp}(|A^*A|)$. Thus we obtain the inequality in (i). Next using Theorem 2.1, namely, $w(AB) \leq \frac{1}{\sqrt{\alpha + \beta}}\|AB\|_{\alpha, \beta}$, we get

$$w(AB) \leq \frac{1}{\sqrt{\alpha + \beta}}\|\alpha r^{2p}(B^*)X^2 + \beta r^p(B^*B)Y\|^{\frac{1}{2p}}.$$ 

Taking infimum over $\alpha, \beta$, with $\alpha + \beta = 1$, we get the inequality in (ii). □
In particular, if we consider \( n = m = 2, \ p = 1 \) and \( f(t) = g(t) = t^{\frac{1}{2}} \) in Theorem 2.17(i), then we get the following inequality:

**Corollary 2.18.** Let \( A, B \in \mathcal{B}(\mathcal{H}) \) such that \( AB = BA, AB^* = B^*A \) and \( |A|B = B^*|A| \). Also let \( \alpha + \beta = 1 \). Then

\[
\|AB\|_{\alpha, \beta}^2 \leq \left\| \frac{\alpha}{4} r^2(B) (|A| + |A^*|)^2 + \beta r(B^* B) A^* A \right\|.
\]

In the following two theorems we obtain inequalities for the \((\alpha, \beta)\)-norm of a bounded linear operator which generalize and improve some existing numerical radius inequalities.

**Theorem 2.19.** Let \( T \in \mathcal{B}(\mathcal{H}) \). If \( f \) and \( g \) are two non-negative continuous functions on \([0, \infty)\) satisfying \( f(t)g(t) = t, \ \forall \ t \geq 0 \) then

(i) \[ \|T\|_{\alpha, \beta}^2 \leq \left\| \frac{\alpha}{4} (f^2(|T|) + g^2(|T^*|))^2 \right\| + \beta T^* T \]

(ii) \[ w(T) \leq \inf_{\alpha, \beta} \frac{1}{\sqrt{\alpha + \beta}} \left\| \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta T^* T \right\|^2 \leq \frac{1}{2} \|T\| + \|T^*\| \]

**Proof.** Let \( x \in \mathcal{H} \) with \( \|x\| = 1 \). Then using Lemma 2.14 and Lemma 2.12, we get

\[
|\langle T x, x \rangle| \leq \|f(|T|)x\| \|g(|T^*|)x\|
\]

\[ \Rightarrow |\langle T x, x \rangle| \leq \left( f^2(|T|)x, x \right)^{\frac{1}{2}} \left( g^2(|T^*|)x, x \right)^{\frac{1}{2}} \]

\[ \Rightarrow |\langle T x, x \rangle| \leq \frac{1}{2} \left( f^2(|T|)x, x \right) + \left( g^2(|T^*|)x, x \right) \]

\[ \Rightarrow |\langle T x, x \rangle|^2 \leq \left( \frac{1}{4} \left( f^2(|T|) + g^2(|T^*|) \right), x, x \right) \]

\[ \Rightarrow |\langle T x, x \rangle|^2 \leq \left( \frac{1}{4} \left( f^2(|T|) + g^2(|T^*|) \right), x, x \right) \]

Next, from the definition of the \((\alpha, \beta)\)-norm, we get,

\[
\|T\|_{\alpha, \beta}^2 = \sup_{\|x\|=1} \left\{ \alpha |\langle TX, x \rangle|^2 + \beta \|TX\|^2 \right\}
\]

\[ \leq \sup_{\|x\|=1} \left\{ \alpha \left( \frac{1}{4} \left( f^2(|T|) + g^2(|T^*|) \right), x, x \right) + \beta \langle T^* TX, x \rangle \right\}
\]

\[ = \sup_{\|x\|=1} \left\{ \frac{\alpha}{4} \left( f^2(|T|) + g^2(|T^*|) \right)^2 + \beta T^* T \right\}
\]

Thus we obtain the inequality in (i). In particular, if we take \( f(t) = g(t) = t^{\frac{1}{2}} \) in (i) and apply Theorem 2.1, namely, \( w(T) \leq \frac{1}{\sqrt{\alpha + \beta}} \|T\|_{\alpha, \beta} \), we get

\[
w(T) \leq \frac{1}{\sqrt{\alpha + \beta}} \left\| \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta T^* T \right\|^\frac{1}{2}.
\]

Taking infimum over all \( \alpha, \beta \), we get the inequality

\[
w(T) \leq \inf_{\alpha, \beta} \left\{ \frac{1}{\sqrt{\alpha + \beta}} \left\| \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta T^* T \right\|^\frac{1}{2} \right\}.
\]

The remaining inequality follows from the case \( \alpha = 1, \beta = 0 \). \( \square \)
In the following theorem, we provide a condition on the operator $T$, for which the bound obtained in Theorem 2.19(ii) is strictly sharper than the inequality given in [9, Th. 1].

**Theorem 2.20.** Let $T \in \mathcal{B}(\mathcal{H})$.

(i) Suppose there exists $\alpha, \beta$ with $\alpha \beta \neq 0$ such that $M_{\alpha A + \beta B} \neq \emptyset$, where $A = \frac{1}{4}(|T| + |T^*|)^2$ and $B = T^* T$. Let $L = \text{span}\{M_{\alpha A + \beta B}\}$. If $\|B\|_L < \|A\|$, then

$$w(T) \leq \frac{1}{\sqrt{\alpha + \beta}} \left\| \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta T^* T \right\|^{\frac{3}{2}} < \frac{1}{2} \| |T| + |T^*| \|.$$

(ii) If $4|T|^2 - (|T| + |T^*|)^2 < 0$ and $\beta \neq 0$, then

$$w(T) \leq \frac{1}{\sqrt{\alpha + \beta}} \left\| \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta T^* T \right\|^{\frac{3}{2}} < \frac{1}{2} \| |T| + |T^*| \|.$$

**Proof.** (i) Suppose there exist $\alpha, \beta$ with $\alpha \beta \neq 0$ such that $\|B\|_L < \|A\|$. Let $x \in M_{\alpha A + \beta B}$. Then we have,

$$w^2(T) \leq \frac{1}{\alpha + \beta} \|\alpha A + \beta B\| = \frac{1}{\alpha + \beta} \|\alpha A + \beta B\| x \|
\leq \frac{\alpha}{\alpha + \beta} \|Ax\| + \frac{\beta}{\alpha + \beta} \|Bx\|
< \frac{\alpha}{\alpha + \beta} \|A\| + \frac{\beta}{\alpha + \beta} \|A\|
= \|A\|.$$

(ii) $w^2(T) \leq \frac{1}{\alpha + \beta} \left\| \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta T^* T \right\|
= \left\| \frac{1}{4} (|T| + |T^*|)^2 + \frac{\beta}{\alpha + \beta} \left(|T|^2 - \frac{1}{4}(|T| + |T^*|)^2\right) \right\|
< \frac{1}{4} \| |T| + |T^*| \|^2$
for all $\alpha, \beta$ with $\beta \neq 0$. This completes the proof of the theorem. \qed

**Remark 2.21.** We give an example to show that the condition mentioned in Theorem 2.20 is not necessary. Consider $T = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Then it is easy to see that there exists $\alpha = \frac{12}{5}, \beta = 1$ such that $4 = \|B\|_L > \|A\| = \frac{2}{3}$ but

$$w(T) \leq \frac{1}{\sqrt{\alpha + \beta}} \left\| \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta T^* T \right\|^{\frac{3}{2}} = \sqrt{\frac{32}{17}} \approx 1.37198868114
< \frac{1}{2} \| |T| + |T^*| \| = \frac{3}{2} = 1.5
< \frac{1}{2} \left(\|T\| + \|T^2\|^{\frac{1}{2}}\right) = \frac{2 + \sqrt{2}}{2} \approx 1.70710678119.$$

Next we provide a necessary and sufficient condition for the operator $T$ for which the bounds obtained in Theorem 2.19 (ii) are equal.
Theorem 2.22. Let $T \in B(H)$ and let $A = \frac{1}{2}(|T| + |T^*|)$, $B = |T|$, $C = |T| - |T^*|$. If $M_A \cap M_B \cap \ker C \neq \emptyset$ then

$$
\frac{1}{\alpha + \beta} \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta |T|^2 = \frac{1}{4} |T| + |T^*|^2 = |T|^2, \text{ for all } \alpha, \beta.
$$

Moreover, the converse is also true if $T$ is compact and $\alpha \beta \neq 0$.

Proof. Let $x \in M_A \cap M_B \cap \ker C$. Then for all $\alpha, \beta$, we get

$$
\frac{1}{\alpha + \beta} \frac{\alpha}{4} (\alpha A^2 + \beta B^2) \geq \frac{1}{\alpha + \beta} \frac{\alpha}{4} \langle \alpha A^2 + \beta B^2 \rangle x
$$

$$
\geq \frac{1}{\alpha + \beta} \frac{\alpha}{4} \langle (\alpha A^2 + \beta B^2) x, x \rangle
$$

$$
= \frac{1}{\alpha + \beta} \frac{\alpha}{4} \langle \alpha \|Ax\|^2 + \beta \|Bx\|^2 \rangle
$$

$$
= \|T\| \|x\|^2
$$

$$
= \|T\|^2.
$$

Also, $\frac{1}{\alpha + \beta} \frac{\alpha}{4} \|\alpha A^2 + \beta B^2\| \leq \|T\|^2$ for all $\alpha, \beta$. Therefore, $\frac{1}{\alpha + \beta} \frac{\alpha}{4} \|\alpha A^2 + \beta B^2\| = \|T\|^2$ for all $\alpha, \beta$. Thus,

$$
\frac{1}{\alpha + \beta} \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta |T|^2 = \frac{1}{4} |T| + |T^*|^2 = |T|^2, \text{ for all } \alpha, \beta.
$$

Now we prove the converse part. Let $T$ be compact and $\frac{1}{\alpha + \beta} \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta |T|^2 = \frac{1}{4} |T| + |T^*|^2 = |T|^2$, for all $\alpha \beta \neq 0$. Let $x \in M_{\alpha A^2 + \beta B^2}$. Then we have,

$$
\|T\|^2 = \frac{1}{\alpha + \beta} \frac{\alpha}{4} \|\alpha A^2 + \beta B^2\| = \frac{1}{\alpha + \beta} \frac{\alpha}{4} \|(\alpha A^2 + \beta B^2) x\|
$$

$$
= \frac{1}{\alpha + \beta} \frac{\alpha}{4} \langle (\alpha A^2 + \beta B^2) x, x \rangle
$$

$$
= \frac{1}{\alpha + \beta} \frac{\alpha}{4} \langle \alpha \|Ax\|^2 + \beta \|Bx\|^2 \rangle
$$

$$
= \frac{\alpha}{\alpha + \beta} \|Ax\|^2 + \frac{\beta}{\alpha + \beta} \|Bx\|^2
$$

$$
\leq \frac{\alpha}{\alpha + \beta} \|A\|^2 + \frac{\beta}{\alpha + \beta} \|B\|^2
$$

$$
\leq \|T\|^2.
$$

This implies that $\|Ax\| = \|A\|$ and $\|Bx\| = \|B\|$. Also from above it follows that $\|A\| = \|B\| = \|T\|$ and so $Ax = \|T\| x$, $Bx = \|T\| x$. Therefore, $Ax = Bx$, i.e., $x \in \ker C$. Hence, $x \in M_A \cap M_B \cap \ker C$. \qed

Remark 2.23. (i) If $T \in B(H)$ is a compact normal operator then it is easy to see that $M_A \cap M_B \cap \ker C \neq \emptyset$, $(A, B, C$ are in Theorem 2.22), and so the inequalities in Theorem 2.19 (ii) becomes equalities, i.e.,

$$
w^2(T) = \inf_{\alpha, \beta} \frac{1}{\alpha + \beta} \frac{\alpha}{4} (|T| + |T^*|)^2 + \beta |T|^2 = \frac{1}{4} |T| + |T^*|^2.
$$
(ii) We give an example of a matrix which is not normal but $M_A \cap M_B \cap \ker C \neq \emptyset$, $(A, B, C$ are in Theorem 2.22). Consider $T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. Then it is easy to see that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in M_A \cap M_B \cap \ker C$, but $2 \neq \|T\| = \sqrt{2}$, and so $T$ is not normal.

We next obtain the following lower bounds and upper bounds for the $(\alpha, \beta)$-norm, and apply it to the study of numerical radius inequalities.

**Theorem 2.24.** Let $T \in \mathcal{B}(\mathcal{H})$. Then

(i) $\|T\|_{\alpha, \beta}^2 \geq \max \left\{ \frac{1}{2} \left\| \frac{\alpha}{2} (T^*T + TT^*) + \beta T^*T \right\| : \frac{1}{3} \left\| \frac{\alpha}{2} (T^*T + TT^*) + \beta T^*T \right\| \right\}$,

(ii) $\|T\|_{\alpha, \beta}^2 \leq \left\| \frac{\alpha}{2} (T^*T + TT^*) + \beta T^*T \right\|$ and

(iii) $\omega^2(T) \leq \inf_{\alpha, \beta} \left\| \frac{\alpha}{2} (T^*T + TT^*) + \beta T^*T \right\| \leq \frac{1}{2} \|T^*T + TT^*\|.$

**Proof.** (i). Let $x$ be a unit vector in $\mathcal{H}$. Then,

$$|\langle Tx, x \rangle|^2 = (\langle \Re(T)x, x \rangle)^2 + (\langle \Im(T)x, x \rangle)^2 \geq \frac{1}{2} (\|\langle \Re(T)x, x \rangle\| + \|\langle \Im(T)x, x \rangle\|)^2 \geq \frac{1}{2} |\langle \Re(T) \pm \Im(T)x, x \rangle|^2.$$ 

Therefore, taking supremum over all unit vectors in $\mathcal{H}$, we get

$$\sup_{\|x\|=1} \alpha |\langle Tx, x \rangle|^2 \geq \frac{\alpha}{2} \sup_{\|x\|=1} |\langle \Re(T) \pm \Im(T)x, x \rangle|^2 = \sup_{\|x\|=1} \left\langle \frac{\alpha}{2} (\Re(T) \pm \Im(T))^2, x \right\rangle x \rangle \geq \sup_{\|x\|=1} \left\langle \frac{\alpha}{4} \left( (\Re(T) + \Im(T))^2 + (\Re(T) - \Im(T))^2 \right), x \right\rangle x \rangle.$$

Also $\|T\|_{\alpha, \beta}^2 \geq \sup_{\|x\|=1} |\langle \beta T^*T x, x \rangle|.$ Therefore,

$$2\|T\|_{\alpha, \beta}^2 \geq \sup_{\|x\|=1} \left\langle \left\langle \frac{\alpha}{4} \left( (\Re(T) + \Im(T))^2 + (\Re(T) - \Im(T))^2 \right), x \right\rangle x \right\rangle \Rightarrow |\langle T, T \rangle|^2 \geq \frac{1}{2} \left\| \frac{\alpha}{2} \left( (\Re(T))^2 + (\Im(T))^2 \right) + \beta T^*T \right\| \Rightarrow \|T\|_{\alpha, \beta}^2 \geq \frac{1}{2} \left\| \frac{\alpha}{2} (T^*T + TT^*) + \beta T^*T \right\|.$$

Also, we have

$$3\|T\|_{\alpha, \beta}^2 \geq \sup_{\|x\|=1} \left\langle \left\langle \frac{\alpha}{2} \left( (\Re(T) + \Im(T))^2 + (\Re(T) - \Im(T))^2 \right), x \right\rangle x \right\rangle \Rightarrow |\langle T, T \rangle|^2 \geq \frac{1}{2} \|\alpha \left( (\Re(T))^2 + (\Im(T))^2 \right) + \beta T^*T \| \Rightarrow \|T\|_{\alpha, \beta}^2 \geq \frac{1}{3} \left\| \frac{\alpha}{2} (T^*T + TT^*) + \beta T^*T \right\|.$$
Combining (1) and (2), we get the inequality (i). Next we prove the second inequality. We have,
\[
\alpha \|Tx\|^2 + \beta \|Tx\|^2 = \alpha \left(\langle \Re(T)x, x \rangle + \langle \Im(T)x, x \rangle \right) + \beta \langle T^*Tx, x \rangle \\
\leq \alpha \left(\|\Re(T)x\|^2 + \|\Im(T)x\|^2 \right) + \beta \langle T^*Tx, x \rangle \\
= \alpha \left(\langle (\Re(T))^2, x \rangle + \langle (\Im(T))^2, x \rangle \right) + \beta \langle T^*Tx, x \rangle \\
= \left(\langle \alpha (\Re(T))^2 + (\Im(T))^2 \rangle + \beta \langle T^*Tx, x \rangle \right) \\
= \left(\frac{\alpha}{2} \langle (T^*T + TT^*) + \beta T^*T \rangle, x \rangle. \right.
\]
Therefore, taking supremum over all unit vectors in \(H\), we get the inequality (ii). It follows from the inequality in (ii), by using Theorem 2.25.

\[
w^2(T) \leq \frac{1}{\alpha + \beta} \left\| \frac{\alpha}{2} \langle T^*T + TT^* \rangle + \beta T^*T \right\|
\]

Taking infimum over all \(\alpha, \beta\) we get the inequality
\[
w^2(T) \leq \inf_{\alpha, \beta} \frac{1}{\alpha + \beta} \left\| \frac{\alpha}{2} \langle T^*T + TT^* \rangle + \beta T^*T \right\|.
\]

The remaining inequality follows from the case \(\alpha = 1, \beta = 0\). \(\square\)

We would like to note that the first inequality in Theorem 2.24(iii) improves on the existing inequality in \([10, \text{Th. 1}]\), namely,

\[
w^2(T) \leq \frac{1}{2} \|T^*T + TT^*\|.
\]

Following the ideas involved in Theorem 2.20 and Theorem 2.22, we can prove the following two theorems which provide conditions under which our bound is strictly sharper or equal to the existing bound.

**Theorem 2.25.** Let \(T \in B(H)\). Suppose there exists \(\alpha, \beta\) with \(\alpha \beta \neq 0\) such that \(\mathcal{M}_{\alpha A + \beta B} \neq 0\), where \(A = \frac{1}{2}(T^*T + TT^*)\) and \(B = T^*T\). If \(\|B\|_L < \|A\|\), where \(L = \text{span}\{\mathcal{M}_{\alpha A + \beta B}\}\), then

\[
w^2(T) \leq \frac{1}{\alpha + \beta} \left\| \frac{\alpha}{2} \langle T^*T + TT^* \rangle + \beta T^*T \right\| < \frac{1}{2} \|T^*T + TT^*\|.
\]

**Theorem 2.26.** Let \(T \in B(H)\) and let \(A = \frac{1}{2}(T^*T + TT^*)\), \(B = T^*T\), \(C = T^*T - TT^*\). If \(M_A \cap M_B \cap \ker C \neq 0\) then

\[
\frac{1}{\alpha + \beta} \left\| \frac{\alpha}{2} \langle T^*T + TT^* \rangle + \beta T^*T \right\| = \frac{1}{2} \|T^*T + TT^*\| = \|T\|^2, \text{ for all } \alpha, \beta.
\]

Moreover, the converse is also true if \(T\) is compact and \(\alpha \beta \neq 0\).

**Remark 2.27.** (i) We give an example to show that the condition mentioned in \(\text{Theorem 2.25}\) is not necessary. Consider \(T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}\). Then it is easy to see that there exists \(\alpha = 6, \beta = 1\) such that \(4 = \|B\|_L > \|A\| = \frac{5}{2}\) but

\[
w^2(T) \leq \frac{1}{\alpha + \beta} \left\| \frac{\alpha}{2} \langle T^*T + TT^* \rangle + \beta T^*T \right\| = \frac{16}{7} \approx 2.2857 < \frac{1}{2} \|T^*T + TT^*\| = 2.5.
\]
(ii) If \( T \in \mathcal{B}(\mathcal{H}) \) is normaloid operator and \( \mathcal{H} \) is finite-dimensional then it is easy to see that \( M_A \cap M_B \cap \ker C \neq \emptyset \) and so the inequalities in Theorem 2.24 (iii) becomes equalities, i.e.,

\[
\omega^2(T) = \inf_{\alpha, \beta} \frac{1}{\alpha \beta} \left\| \frac{\alpha}{2} (T^* T + TT^*) + \beta T^* T \right\| = \frac{1}{2}\left\| T^* T + TT^* \right\|.
\]

(iii) We give an example of a matrix which is not normaloid but \( M_A \cap M_B \cap \ker C \neq \emptyset \). Consider \( T = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \). Then it is easy to see that \( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \) is normaloid operator and but \( 2 = \| T \| \neq \omega(T) = \sqrt{2} \), i.e., \( T \) is not normaloid.

To prove the next result, we need the Buzano’s inequality (see [7]), which is a generalization of the Cauchy-Schwarz inequality.

**Lemma 2.28.** Let \( a, b, e \in \mathcal{H} \) with \( \| e \| = 1 \). Then

\[
|\langle a, e \rangle \langle e, b \rangle| \leq \frac{1}{2} (\|a\| \|b\| + |\langle a, b \rangle|).
\]

**Theorem 2.29.** Let \( T \in \mathcal{B}(\mathcal{H}) \). Then

\[
\begin{align*}
\text{(i)} \quad & \| T \|_{2, \beta}^2 \leq \frac{\alpha}{2} w(T^2) + \left\| \frac{\alpha}{4} (T^* T + TT^*) + \beta T^* T \right\| \quad \text{and} \\
\text{(ii)} \quad & \omega^2(T) \leq \inf_{\alpha, \beta} \frac{1}{\alpha \beta} \left\{ \frac{\alpha}{2} w(T^2) + \left\| \frac{\alpha}{4} (T^* T + TT^*) + \beta T^* T \right\| \right\} \\
& \quad \leq \frac{1}{2} w(T^2) + \frac{1}{4} \left\| T^* T + TT^* \right\|.
\end{align*}
\]

**Proof.** Let \( x \in \mathcal{H} \) with \( \| x \| = 1 \). Taking \( a = Tx, b = T^* x \) and \( e = x \) in Lemma 2.28 and using the arithmetic-geometric mean inequality, we get,

\[
\langle Tx, x \rangle^2 \quad \leq \quad \frac{1}{2} \left( \| Tx \| \| T^* x \| + |\langle T^2 x, x \rangle| \right) \\
\quad = \quad \frac{1}{2} |\langle T^2 x, x \rangle| + \frac{1}{2} \| T^* T x, x \|^{1/2} \langle T^* T x, x \rangle^{1/2} \\
\quad \leq \quad \frac{1}{2} |\langle T^2 x, x \rangle| + \frac{1}{4} (\| T^* T x \| + \langle T^* T x, x \rangle) \\
\quad = \quad \frac{1}{2} |\langle T^2 x, x \rangle| + \frac{1}{4} (\| T^* T + TT^* \| x, x).
\]

Therefore,

\[
\alpha |\langle Tx, x \rangle|^2 + \beta \| Tx \|^2 \quad \leq \quad \frac{\alpha}{2} |\langle T^2 x, x \rangle| + \left( \frac{\alpha}{4} \| T^* T + TT^* \| T^* T \right) x, x \right\}. \\
\quad \leq \quad \frac{\alpha}{2} w(T^2) + \left\| \frac{\alpha}{4} (T^* T + TT^*) + \beta T^* T \right\|.
\]

Taking supremum over all unit vectors in \( \mathcal{H} \), we get the desired inequality in (i).

Using Theorem 2.1 and taking supremum over \( \alpha, \beta \), we get the inequality

\[
\omega^2(T) \leq \inf_{\alpha, \beta} \frac{1}{\alpha \beta} \left\{ \frac{\alpha}{2} w(T^2) + \left\| \frac{\alpha}{4} (T^* T + TT^*) + \beta T^* T \right\| \right\}.
\]

The remaining inequality follows from the case \( \alpha = 1, \beta = 0 \). \(\square\)
We next state the following theorem which provide a condition on the operator $T$ for which the bound obtained by us in Theorem 2.29 (ii) is sharper than the existing bound $\frac{1}{4}w(T^2) + \frac{1}{2}\|T^* T + TT^*\|$. The proof is omitted as it follows using previous arguments.

**Theorem 2.30.** Let $T \in \mathcal{B}(\mathcal{H})$. Suppose there exists $\alpha, \beta$ with $\alpha \beta \neq 0$ such that $M_{\alpha A + \beta B} \neq \emptyset$, where $A = \frac{1}{4}(T^* T + TT^*)$ and $B = T^* T$. If $\|B\|_L < \|A\|$, where $L = \text{span}\{M_{\alpha A + \beta B}\}$, then

$$w^2(T) \leq \frac{1}{\alpha + \beta} \left\{ \frac{\alpha}{2} w(T^2) + \left\| \frac{\alpha}{4}(T^* T + TT^*) + \beta T^* T \right\| \right\} < \frac{1}{2} w(T^2) + \frac{1}{4}\|T^* T + TT^*\|.$$ 

**Remark 2.31.** We give an example to show that the condition mentioned in Theorem 2.30 is not necessary. Consider $T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then it is easy to see that there exists $\alpha = 12, \beta = 1$ such that $4 = \|B\|_L > \|A\| = \frac{5}{4}$ but

$$w^2(T) \leq \frac{1}{\alpha + \beta} \left\{ \frac{\alpha}{2} w(T^2) + \left\| \frac{\alpha}{4}(T^* T + TT^*) + \beta T^* T \right\| \right\} = \frac{16.5}{13} \approx 1.26923077$$ 

Our penultimate result is on the estimation of the upper bound for the $(\alpha, \beta)$-norm of bounded linear operator $T$ in terms of $\text{Re}(T)$, $\text{Im}(T)$. For this we first need the following lemma.

**Lemma 2.32.** [11] Let $T \in \mathcal{B}(\mathcal{H})$ be self-adjoint and let $x \in \mathcal{H}$. Then

$$|\langle Tx, x \rangle| \leq \|T\| \|x\|.$$ 

**Theorem 2.33.** Let $T \in \mathcal{B}(\mathcal{H})$. Then

(i) $\|T\|_{\alpha, \beta}^2 \leq \|\text{Re}(T)\| + \|\text{Im}(T)\|^2 + \beta T^* T$ and

(ii) $w^2(T) \leq \inf_{\alpha, \beta} \frac{1}{\alpha + \beta} \left\| \text{Re}(T) \right\|^2 + \beta T^* T.$

**Proof.** Let $x$ be a unit vector in $\mathcal{H}$. Then by using Lemma 2.32 and Lemma 2.12, we get,

$$\alpha |\langle Tx, x \rangle|^2 + \beta \|Tx\|^2 = \alpha |\langle (\text{Re}(T)x, x \rangle + i \langle \text{Im}(T)x, x \rangle|^2 + \beta (T^* T)x, x \rangle$$ 

$$\leq \alpha (|\langle (\text{Re}(T)x, x \rangle| + |\langle \text{Im}(T)x, x \rangle|^2 + \beta (T^* T)x, x \rangle$$ 

$$\leq \alpha (\|\text{Re}(T)x\| + \|\text{Im}(T)|x, x \rangle)^2 + \beta (T^* T)x, x \rangle$$ 

$$= \alpha (\langle \text{Re}(T)\rangle^2 + \|\text{Im}(T)|x, x \rangle)^2 + \beta (T^* T)x, x \rangle$$ 

$$\leq \left\langle \alpha (\|\text{Re}(T)\| + \|\text{Im}(T)\|^2 + \beta T^* T \right\rangle x, x \rangle$$ 

Therefore, taking supremum over all unit vectors in $\mathcal{H}$, we get the required inequality in (i). Using Theorem 2.1 in (i) and taking infimum over $\alpha, \beta$, we get the inequality in (ii), i.e.,

$$w^2(T) \leq \inf_{\alpha, \beta} \frac{1}{\alpha + \beta} \left\| \alpha (\|\text{Re}(T)\| + \|\text{Im}(T)\|^2 + \beta T^* T \right\|.$$ 

$\Box$
Remark 2.34. In particular, if we take $\alpha = 1, \beta = 0$ in Theorem 2.33 (ii), then we get the following inequality

$$w(T) \leq \left| |Re(T)| + |Im(T)| \right|.$$ 

The existing upper bound of numerical radius in terms of $Re(T)$ and $Im(T)$ is

$$w^2(T) \leq \| Re(T) \|^2 + \| Im(T) \|^2.$$ 

We give an example to show that the upper bound obtained here is better than the existing one. Considering the matrix $T = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, it is easy to see that the inequality $w^2(T) \leq \| Re(T) \|^2 + \| Im(T) \|^2$ gives $w(T) \leq \sqrt{2}$, whereas our inequality gives $w(T) \leq 1$.

Before presenting the final result of this article, we would like to note that the introduction of the $(\alpha, \beta)$-norm on $B(\mathcal{H})$ has played a crucial role in improving the upper bounds of the existing numerical radius inequalities. Therefore, a natural query in this context would be regarding the possible improvements of the lower bounds of the existing numerical radius inequalities, using similar techniques. Our final result is dedicated to answering this query in a fruitful way. Given an arbitrary but fixed $T \in B(\mathcal{H})$, let us consider an expression $f_{\alpha, \beta}(x) = \alpha |\langle Tx, x \rangle|^2 + \frac{\beta}{4} \| (T^*T + TT^*)x \|$, where $x \in \mathcal{H}$. It is quite straightforward to check that the above expression does not induce a norm on $B(\mathcal{H})$ by taking the supremum over all unit vectors. Nevertheless, we show in the following theorem that it is possible to improve the existing lower bound for the numerical radius of bounded linear operators, obtained in [10, Th. 1], namely, $w^2(T) \geq \frac{1}{4} \| T^*T + TT^* \|$, by using the above expression.

**Theorem 2.35.** Let $T \in B(\mathcal{H})$ and let $\mathcal{H}_0 = \{ x \in \mathcal{H} : \| (T^*T + TT^*)x \| = \| T^*T + TT^* \| \| x \| \}$. Let $\mathcal{H}_0 \not= \emptyset$ and let $T|_{\mathcal{H}_0} = T_0$. Then

$$w^2(T) \geq \sup_{\alpha, \beta} \sup_{\| x \| = 1} \left\{ \alpha \sup_{\| x \| = 1} |\langle T_0 x, x \rangle|^2 + \frac{\beta}{4 (\alpha + \beta)} \| T^*T + TT^* \| \right\} \geq \frac{1}{4} \| T^*T + TT^* \| .$$

**Proof.** We have,

$$\sup_{\| x \| = 1} f_{\alpha, \beta}(x) = \sup_{\| x \| = 1} \left\{ \alpha |\langle Tx, x \rangle|^2 + \frac{\beta}{4} \| (T^*T + TT^*)x \| \right\} \leq \alpha w^2(T) + \frac{\beta}{4} \| T^*T + TT^* \| \leq (\alpha + \beta) w^2(T).$$

Also,

$$\sup_{\| x \| = 1, x \in \mathcal{H}_0} f_{\alpha, \beta}(x) = \alpha \sup_{\| x \| = 1} |\langle T_0 x, x \rangle|^2 + \frac{\beta}{4} \| T^*T + TT^* \| .$$

Therefore, we get,

$$w^2(T) \geq \frac{1}{\alpha + \beta} \left\{ \alpha \sup_{\| x \| = 1} |\langle T_0 x, x \rangle|^2 + \frac{\beta}{4} \| T^*T + TT^* \| \right\}.$$
Since this holds for all admissible values of $\alpha, \beta$, the first inequality follows. The second inequality follows trivially by considering the particular case $\alpha = 0, \beta = 1$. □

We end this article with the following remark that justifies that Theorem 2.35 is a proper refinement of the existing lower bound for the numerical radius as obtained in [10, Th. 1], for a large class of operators.

**Remark 2.36.** (i) We note that, if $\mathcal{H}_0$ is invariant under $T$, then $\sup_{\|x\| = 1} |\langle T_0 x, x \rangle|^2 = w^2(T_0) \geq \frac{1}{4} \|T^*T + TT^*\|$. Therefore, for all $\alpha, \beta$,

$$
\frac{\alpha}{\alpha + \beta} w^2(T_0) + \frac{\beta}{4(\alpha + \beta)} \|T^*T + TT^*\| \geq \frac{1}{4} \|T^*T + TT^*\|.
$$

Moreover, if $w^2(T_0) > \frac{1}{4} \|T^*T + TT^*\|$, then the first inequality in Theorem 2.35 is strictly sharper than the first inequality in [10, Th. 1], obtained by Kittaneh.

(ii) Even if $\mathcal{H}_0$ is not invariant under $T$ then also the lower bound obtained here may give a better bound than that in [10, Th. 1]. As an illustrative example, let us consider $T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Then it is easy to check that $\mathcal{H}_0$ is not invariant under $T$ but $\sup_{\|x\| = 1} |\langle T_0 x, x \rangle|^2 = \left(\frac{4 + 3\sqrt{2}}{4 + 2\sqrt{2}}\right)^2 > \frac{1}{4} \|T^*T + TT^*\| = \frac{2 + \sqrt{2}}{4}$ so that for all $\alpha, \beta$ with $\alpha \neq 0$ we get,

$$
w^2(T) \geq \left(\frac{\alpha}{\alpha + \beta}\right) \frac{4 + 3\sqrt{2}}{4 + 2\sqrt{2}} + \left(\frac{\beta}{\alpha + \beta}\right) \frac{2 + \sqrt{2}}{4} > \frac{2 + \sqrt{2}}{4}.
$$

Therefore, for this matrix $T$, the first inequality in Theorem 2.35 gives a better bound than that in [10, Th. 1].

(iii) We also note that if $\sup_{\|x\| = 1} |\langle T_0 x, x \rangle|^2 = \frac{1}{4} \|T^*T + TT^*\|$ , then for all $\alpha, \beta$,

$$
\frac{\alpha}{\alpha + \beta} \sup_{\|x\| = 1} |\langle T_0 x, x \rangle|^2 + \frac{\beta}{4(\alpha + \beta)} \|T^*T + TT^*\| = \frac{1}{4} \|T^*T + TT^*\|,
$$

i.e., the first inequality in Theorem 2.35 and the first inequality in [10, Th. 1] give the same bound.

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