We evaluate the three bulk viscosity coefficients $\zeta_1$, $\zeta_2$, and $\zeta_3$ in the color-flavor locked (CFL) superfluid phase due to phonons and kaons, which are the lightest modes in that system. We first show that the computation is rather analogous to the computation of the same coefficients in superfluid $^4$He, as due to phonons and rotons. For astrophysical applications, we also find the value of the viscosities when there is a periodic disturbance, and the viscosities also depend on the frequency of the disturbance. In a temperature regime that might be of astrophysical relevance, we find that the contributions of both the phonons and kaons should be considered, and that $\zeta_2$ is much less that the same coefficient in unpaired quark matter.

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I. INTRODUCTION

The ground state of three-flavor quark matter at asymptotically large densities and low temperature is the color-flavor locked (CFL) phase [1]. Many properties of CFL quark matter have been studied up to now [2]. The transport or kinetic coefficients of CFL quark matter are of particular interest, as they may allow us to identify whether the CFL phase is realized or not in the core of compact stars, even in the situation where the density is not so large. In the literature one can find computations of the shear viscosity [3], bulk viscosities [4–6], thermal conductivity [7, 8], mutual friction [9], as well as a general discussion on the kinetics of a CFL superfluid [10].

CFL quark matter is a superfluid. The hydrodynamical description of superfluidity differs from that corresponding to a normal fluid, as in that case one has to describe the flow corresponding to the quantum condensate, and the flow associated to the thermal quasiparticles of the system [11, 12]. Due to the existence of two different flows, more equations than those corresponding to a normal fluid are needed. In the dissipative regime of the superfluid one can also define more transport coefficients than for a normal fluid. In particular, it is known that due to the existence of these two different flows one can define up to three bulk viscosity coefficients in a superfluid system.

In this paper we study the three bulk viscosities of CFL quark matter due to superfluid phonons and kaons. These collective excitations are the Goldstone modes associated with the spontaneous breaking of baryon and of chiral symmetry, respectively. These are the lightest modes in the CFL phase, and thus, one can expect that they provide the leading contribution to its transport properties. The contribution to one of the bulk viscosity coefficients due to kaons only was computed in Ref. [4]. The most relevant process for that contribution comes from the decay of the neutral kaons into superfluid phonons. In Ref. [6] the phonon contribution to the three bulk viscosity coefficients was evaluated in a temperature regime where the kaon population was assumed to be thermally suppressed. In that case, the most relevant process that contributes to the viscosities is one that changes the phonon number density, and implies five phonon collisions. Because the kaon masses are not well-known, it is not clear at which temperatures one should consider the kaon contribution into account.

In this article we compute the three bulk viscosity coefficients of superfluid CFL quark matter due to kaons and phonons, assuming a value of the kaon mass gap within a reasonable range of values. While one could naively think that the result is the sum of the contributions of the kaons and the phonons, (roughly speaking, adding up the results of Ref. [4] and Ref. [6]) we will show that this is not the case, the reason being that kaon decay process also changes the phonon number density. We will first show that the computation of the bulk viscosities is totally analogous to that corresponding to the phonon and roton contribution to the bulk viscosities in superfluid $^4$He. Further, and having in mind possible astrophysical applications, we will compute the viscosities when there is a periodic disturbance, when the viscosities depend also on the frequency of the disturbance. Obtaining the algebraic and numerical values of the viscosities is the final goal of this manuscript.

This paper is structured as follows. In Section [11] we give a very brief review to the relativistic hydrodynamic
equations, and how the bulk viscosities can be interpreted as coefficients that parametrize the deviations of both the pressure and chemical potential from its equilibrium values. In Section III we also review the two different processes and associated decay rates which contribute the most to the computation of the bulk viscosities. These are a five phonon process, and the (electroweak mediated) kaon decay into two superfluid phonons. In Section IV we compute the three viscosity coefficients when there is a periodic disturbance, as a function of the frequency of this. In Subsection V A we present the analytical results, while the numerical results are displayed in Subsection V B. Finally, we summarize our main findings in Section VI. We will use natural units $\hbar = c = k_B = 1$ everywhere in the manuscript, except when we show the numerical values of the viscosities.

II. BULK VISCOSITY COEFFICIENTS IN A RELATIVISTIC SUPERFLUID

The hydrodynamical equations governing the bulk fluctuations of a non-relativistic superfluid are essentially different from the standard fluid equations. At non-vanishing temperature one has to employ the two-fluid description of Landau [11], which takes into account the motion of both the superfluid and of the normal component of the system. In order to describe the different dissipative processes one has to introduce more transport coefficients than in a normal fluid. In particular, one has three independent bulk viscosities, as well as the shear viscosity and the thermal conductivity. The same occurs in the relativistic version of Landau’s two fluid model, as we briefly review here (see Refs. [14, 15] for more explicit details).

In this paper we will follow the approach derived by Son [14]. This formulation takes into account that superfluidity arises after the appearance of a quantum condensate that spontaneously breaks a \( U(1) \) global symmetry. The Goldstone mode associated with this breaking then should appear in any low energy (and low momentum) effective theory. Hydrodynamics, being a long spacetime dynamical formulation, should then necessarily incorporate the Goldstone field in their equations.

The hydrodynamical equations for the superfluid take the form of conservation laws for both the current, \( n^\mu \), and energy-momentum tensor, \( T^{\mu\nu} \), of the system

$$\partial_\mu n^\mu_q = 0, \quad \partial_\mu T^{\mu\nu} = 0.$$  

One further adds the Josephson equation, which describes the dynamical evolution of the Goldstone field, \( \varphi \), or phase of the condensate

$$u^\mu \partial_\mu \varphi + \mu = 0,$$  

where \( u^\mu \) is the hydrodynamical velocity, and \( \mu \) is the chemical potential of the system. The stress-energy tensor and the current are expressed as

$$T^{\mu\nu} = (\epsilon_0 + P) u^\mu u^\nu - P g^{\mu\nu} + V^2 \partial^\mu \varphi \partial^\nu \varphi,$$

$$n^\mu_q = n_{q,0} u^\sigma - V^2 \partial^\mu \varphi,$$

where \( \epsilon_0 \) stands for the energy density, \( P \) is the pressure, and \( V \) is a variable proportional to the quantum condensate. The energy density obeys the relation

$$\epsilon_0 = ST + n_{q,0} \mu - P,$$

where \( S \) is the entropy of the system.

Sometimes it is better to write the hydrodynamical equations in terms of the new variable

$$w^\mu = - (\partial^\mu \varphi + \mu u^\mu),$$

which represents a counterflow momentum, see Ref. [15]. In the non-relativistic limit, the spatial component of this four vector corresponds to the counterflow momentum of Landau’s two-fluid model formulation [15].

In the absence of dissipation it is possible to show that the entropy current is conserved, \( \partial_\mu (S w^\mu) = 0 \). There is generation of entropy, otherwise. The dissipative terms associated to the above hydrodynamical equations have been constructed in Ref. [15], showing that in the non-relativistic limit they correspond to those appearing in Landau’s two-fluid model. After defining a comoving frame, where \( u_\mu = (1, 0, 0, 0) \) one can get the terms associated with all the viscosity coefficients and thermal conductivity.

In this article we will only be interested in the three bulk viscosity coefficients, which modify the equilibrium hydrodynamical equations. For purposes of interpretation of these coefficients, and for the explicit computations in the remaining part of this article, we can see them as follows. The friction forces due to the bulk viscosities can be
understood as modifications, with respect to their equilibrium values, in the main driving forces acting on the normal and superfluid components. These forces are given by the gradients of the pressure $P$ and the chemical potential $\mu$, respectively. One can write

$$P = \bar{P} + \delta P = \bar{P} - \zeta_1 \partial_\mu (V^2 w^\mu) - \zeta_2 \partial_\mu u^\mu, \quad (7)$$

$$\mu = \bar{\mu} + \delta \mu = \bar{\mu} - \zeta_3 \partial_\mu (V^2 w^\mu) - \zeta_4 \partial_\mu w^\mu, \quad (8)$$

where $\bar{P}$ and $\bar{\mu}$ are the equilibrium pressure and chemical potential, respectively. According to the Onsager symmetry principle, the coefficients should satisfy $\zeta_1 = \zeta_4$, so that in fact there are only three independent coefficients. It is also important to stress that the requirement of positive entropy production imposes that $\zeta_2, \zeta_3$ should be positive and that $\zeta_2^2 \leq \zeta_2 \zeta_3 \frac{\pi}{12}$. While $\zeta_2$ has the same meaning as the one that occurs in a normal fluid, $\zeta_1$ and $\zeta_3$ refer to dissipation that occurs when the superfluid counterpart is not incompressible.

The expressions given in Eqs. (7,8) are interesting, as they show us that in order to compute the different viscosity coefficients we will simply have to evaluate the modulations of both the pressure and chemical potential when the system leaves the equilibrium configuration. This is what we will do in the remaining part of this article for the CFL phase, using this hydrodynamical formulation, where $n_p$ is the quark current, and $\mu$ the quark chemical potential. In that case the parameter $V^2$ can be computed using a Ginzburg-Landau formalism, see Ref. 16 for the computation of $V$ for the CFL superfluid.

**III. PARTICLE NUMBER CHANGING PROCESSES TO RESTORE EQUILIBRIUM AFTER A COMPRESSION OR RAREFACTION**

After an expansion (or rarefaction) of a system in equilibrium, the pressure diminishes (or increases above) its equilibrium value. Microscopic processes that involve reactions that change the particle densities take place to restore the equilibrium configuration. These kind of processes are needed as the system cannot equilibrate to a different value of the volume without modifying the density of particles.

We consider CFL matter at relatively low temperature, where $T \ll T_c$, where $T_c$ is the critical temperature for the onset of superfluidity, which is believed to be above several tens of MeV. In this temperature regime we expect that all the transport coefficients are dominated by the lightest degrees of freedom of CFL quark matter, as the contribution of the heavier modes is Boltzmann suppressed. The lightest modes in the CFL phase are the superfluid phonon, which is the Goldstone mode associated with the spontaneous breaking of $U(1)_B$ and remains always massless, and the kaons, which are pseudo Goldstone modes associated with chiral symmetry breaking. Effective field theories associated to these degrees of freedom can be constructed from QCD 18-20.

In this Section we review the most relevant reaction rates of particle number changing processes that involve the CFL light degrees of freedom, and which are relevant for the computation of the bulk viscosity coefficients 4,13.

1. Emission, and absorption, of phonons by phonons. The superfluid phonons are massless degrees of freedom. Their dispersion law is of the form

$$E_p = c_s p + B p^3 + \cdots,$$  

(9)

where $c_s$ is the speed of sound, and $B$ is also a density dependent parameter that in the CFL phase obeys $B < 0$ 21. As realized in Ref. 6, and because $B < 0$, the first possible phonon number changing reaction involves five phonons $\varphi + \varphi \leftrightarrow \varphi + \varphi + \varphi$, as the so-called Beliaev process that describes the decay of one phonon in two is only kinematically possible for a dispersion law with positive values of $B$.

It has been also found that the decay rate associated to the five phonon reaction behaves very differently for large or small angle scatterings, being the last processes those which dominate. The decay associated to five phonon small scatterings has not been computed, but power counting techniques allows one to estimate it as behaving as

$$\Gamma_{ph} = a_{ph} g_3^2 g_4^2 T_{12}^{12} \frac{\mu^4}{c_s^4},$$  

(10)

where $g_3$ and $g_4$ are the phonon self-coupling constants associated to the three-phonon and four-phonon vertices, respectively, and $a_{ph}$ is an unknown number.

Taking into account the value of the strange quark mass, $m_s$, one can find the value of all the phonon self-coupling parameters to leading order in a $m^2_s/\mu^2$ expansion. In the density of interest for compact stars, this is the typical small parameter used to expand our knowledge of CFL at asymptotic densities. In particular, one finds 8

$$c_s^2 = \frac{1}{3} \left(1 - \frac{m^2_s}{3\mu^2}\right), \quad g_3 = \frac{\pi}{9\mu^2} \left(1 + \frac{m^2_s}{4\mu^2}\right), \quad g_4 = \frac{\pi^4}{108\mu^4} \left(1 + \frac{m^2_s}{3\mu^2}\right).$$  

(11)
Furthermore, a microscopic computation with vanishing quark masses allows one to find the phonon dispersion law at cubic order \[ B = \frac{-11c_s}{540\Delta^2}. \] (12)

2. Decay of neutral kaons (and antikaons) into two superfluid phonons. This is a process mediated by electroweak interactions, which allow for this flavor-changing reaction. First, a kaon is converted into a virtual phonon, which finally decays into two other superfluid phonons, so the total reaction can be seen as \( K^0(\bar{K}^0) \leftrightarrow \varphi + \varphi \). The neutral kaons/antikaons have a dispersion law of the form \[ E_{K^0/\bar{K}^0} = \mp\mu_{K^0}^{\mathrm{eff}} + \sqrt{v^2\bar{p}^2 + m_{K^0}^2}, \] (13)

where \( v = \frac{1}{3} \), and \( \mu_{K^0}^{\mathrm{eff}} \) is an effective chemical potential given by \[ \mu_{K^0}^{\mathrm{eff}} = \frac{m_s^2 - m_d^2}{2\mu}, \] (14)

where \( m_d \) is the down quark mass, that in this work we will take as negligible in front of the strange quark mass. The neutral kaon mass \( m_{K^0} \) was computed in the limit of asymptotic high density [18, 19]. However, at more moderate values instantons effects, which are not under full control, give a contribution to all the meson masses [22, 23]. As a result, the kaon mass is poorly known, and we will take it here as an unknown parameter.

The decay rate associated with this process has been computed in Ref. [4], where it has been shown that in a good range of temperatures it can be approximated by \[ \Gamma_{K,\varphi} \approx \frac{G^2 f_\pi^2 f_H^2}{18\sqrt{3}\pi} \left( 1 + \frac{m_{K^0}^2}{\mu_{K^0}^{\mathrm{eff}}^2} \right) \frac{e^{\bar{p}/T}}{(e^{\bar{p}/T} - 1)^2}, \] (15)

where the low energy effective field theory coupling constants read \[ f_H^2 = 3 \left( \frac{\mu^2}{2\pi^2} \right), \quad f_\pi^2 = \frac{21 - 8\log(2)}{18} \frac{\mu^2}{2\pi^2}, \quad G_{ds} = \sqrt{2} V_{ud} V_{us} G_F \approx 0.304 G_F \] (16)

where \( G_F \) is the Fermi constant and

\[ \bar{p} = \frac{m_{K^0}^2 - \mu_{K^0}^{\mathrm{eff}}^2}{2v\mu_{K^0}^{\mathrm{eff}}}; \] (17)

is the momentum where the virtual phonon that mediates the reaction is nearly on-shell. This expression becomes invalid for very low temperatures, \( T \ll T_a = \frac{4m_s^2}{\mu_{K^0}^{\mathrm{eff}}^2} \), or for large values of \( T \), which are outside the value of physical interest [4]. In our numerical computations for the bulk viscosities we will use this approximated form of the kaon decay rate, as in the low \( T \) regime where this expression is not valid the kaon decay rate becomes meaningless for the computation of the frequency dependent bulk viscosities (see Table 2 of Ref. [4]).

Other processes that involve the kaons are very much suppressed with respect to the one described above. In particular, it is possible to show that the rate of the reaction \( K^0 + \varphi \leftrightarrow \varphi + \varphi \) is suppressed with respect to the decay rate considered above. In Ref. [4], it has been also shown that this is the case for the reaction rates involving charged kaons and leptons.

IV. PHONON AND KAON CONTRIBUTION TO THE BULK VISCOSITY COEFFICIENTS

In this Section we will compute the phonon and kaon contribution to the bulk viscosities following the same strategy devised by Khalatnikov to compute the bulk viscosities coefficients due to phonons and rotons in He\(^4\). The computation is quite analogous, even in the CFL superfluid the kaons are collective modes with different quantum numbers than the phonons, while in He\(^4\) phonons and rotons are collective modes with the same quantum numbers; they are actually the same excitations but seen in a different momentum regime. In any case, all what matters in the computation is to have two well-defined different collective modes which interact with each other.

The superfluid phonon and the kaon are bosons. At equilibrium the number densities of these particles are entirely determined by the temperature \( T \) and by their dispersion laws, which depend on the quark chemical potential, and on
QCD parameters, such as the gauge coupling constant and quark masses. Small deviations from equilibrium can be parametrized with some “fake” phonon and kaon chemical potentials, $\delta \mu_{ph}$ and $\delta \mu_k$, respectively [23]. Equilibrium is restored by microscopic processes that modify the particle number densities. The dynamical evolution of the phonon and kaon particle densities will be governed by the following equations

$$\partial_\mu (n_{ph} u^\mu) = -\gamma_{phph} \delta \mu_{ph} - \gamma_{phk} \delta \mu_k,$$

$$\partial_\mu (n_k u^\mu) = -\gamma_{kk} \delta \mu_k - \gamma_{kph} \delta \mu_{ph},$$

where the different $\gamma_{ij}$ parameters are kinetic coefficients which have to be symmetric in their indices, and which can be related to the decay rates associated to the microscopic processes under consideration [12]. If we take into account only the scattering processes mentioned in the previous Section on then has

$$\gamma_{phph} = \frac{\Gamma_{ph}}{T} + 4 \Gamma_{K,ph}, \quad \gamma_{phk} = \gamma_{kph} = -2 \frac{\Gamma_{K,ph}}{T}, \quad \gamma_{kk} = \frac{\Gamma_{K,ph}}{T}. \tag{20}$$

As emphasized before, the equilibrium phonon and kaon densities are functions of $\mu$ and $T$, or equivalently, $n_q$ and $S$. For small departures from equilibrium, one can follow the change in the phonon and kaon number densities by following the dynamical evolution of $n_q$ and $S$. To this end, we split all thermodynamical and hydrodynamical variables into their equilibrium values, denoted with a bar, and a small fluctuation, denoted with a $\delta$. Thus, we write

$$n_{ph} = \bar{n}_{ph} + \delta n_{ph},$$

and similarly for the remaining thermodynamic and hydrodynamical variables. We then write

$$\delta n_{ph} = \frac{\partial n_{ph}}{\partial n_q} \delta n_q + \frac{\partial n_{ph}}{\partial S} \delta S, \quad \delta n_k = \frac{\partial n_k}{\partial n_q} \delta n_q + \frac{\partial n_k}{\partial S} \delta S, \tag{22}$$

where the derivatives are taken at the equilibrium values. Because the deviation out of equilibrium is assumed to be sufficiently slow we can use the linearized hydrodynamical equations to evaluate the dynamical evolution of $\delta n_q$ and $\delta S$ [20]

$$\bar{u}^\mu \partial_\mu \delta n_q = -\bar{n}_q \partial_\mu u^\mu - \partial_\mu (V^2 w^\mu),$$

$$\bar{u}^\mu \partial_\mu \delta S = -\bar{S} \partial_\mu w^\mu,$$ \tag{24}

to extract the value of the phonon and kaon chemical potentials. One finds

$$\delta \mu_{ph} = \frac{\gamma_{kk}}{\gamma_{kph} - \gamma_{kk} \gamma_{phph}} \left( I_{2}^{ph} \partial_\mu u^\mu - I_{1}^{ph} \partial_\mu (V^2 w^\mu) \right) - \frac{\gamma_{phph}^k}{\gamma_{phph}^k - \gamma_{kk} \gamma_{phph}} \left( I_{2}^{K} \partial_\mu u^\mu - I_{1}^{K} \partial_\mu (V^2 w^\mu) \right),$$

$$\delta \mu_k = -\frac{\gamma_{kk}}{\gamma_{kph} - \gamma_{kk} \gamma_{phph}} \left( I_{2}^{ph} \partial_\mu u^\mu - I_{1}^{ph} \partial_\mu (V^2 w^\mu) \right) + \frac{\gamma_{phph}^k}{\gamma_{phph}^k - \gamma_{kk} \gamma_{phph}} \left( I_{2}^{K} \partial_\mu u^\mu - I_{1}^{K} \partial_\mu (V^2 w^\mu) \right), \tag{25}$$

where we have defined the functions

$$I_{1}^i = \frac{\partial n_i}{\partial n_q}, \quad I_{2}^i = \bar{n}_i - S \frac{\partial n_i}{\partial S} - \bar{n}_q \frac{\partial n_i}{\partial n_q}, \quad i = ph, k. \tag{27}$$

Finally, we compute the change in both the pressure and quark chemical potential caused by the presence of a non-vanishing $\delta \mu_k$ and $\delta \mu_{ph}$

$$\delta P = \frac{\partial P}{\partial \mu_{ph}} \delta \mu_{ph} + \frac{\partial P}{\partial \mu_k} \delta \mu_k,$$ \tag{28}

$$\delta \mu = \frac{\partial \mu}{\partial \mu_{ph}} \delta \mu_{ph} + \frac{\partial \mu}{\partial \mu_k} \delta \mu_k.$$

Using the thermodynamical relation Eq. 4 and the identity [12]

$$d\epsilon_0 = T dS + \mu d n_0 - n_{ph} d \mu_{ph} - n_k d \mu_k,$$ \tag{30}

one finds

$$I_{1}^{i} = \frac{\partial \mu}{\partial \mu_i}, \quad I_{2}^{i} = \frac{\partial P}{\partial \mu_i}, \quad i = ph, k. \tag{31}$$
The bulk viscosity coefficients can then be obtained by identifying the different coefficients in the variations of the pressure and chemical potential that are proportional to the divergences of the two velocity vectors, using Eqs (7,8). One then finds

\begin{align}
\zeta_1 &= -\frac{T}{\Gamma_{\text{ph}}} \left( I_{1}^{\text{ph}} + 2I_{1}^{k} \right) \left( I_{2}^{\text{ph}} + 2I_{2}^{k} \right) - \frac{T}{\Gamma_{K,\text{ph}}} I_{1}^{k} I_{2}^{k}, \\
\zeta_2 &= \frac{T}{\Gamma_{\text{ph}}} \left( I_{2}^{\text{ph}} + 2I_{2}^{k} \right) \left( I_{1}^{\text{ph}} + 2I_{1}^{k} \right) + \frac{T}{\Gamma_{K,\text{ph}}} (I_{2}^{k})^2, \\
\zeta_3 &= \frac{T}{\Gamma_{\text{ph}}} \left( I_{1}^{\text{ph}} + 2I_{1}^{k} \right) \left( I_{2}^{\text{ph}} + 2I_{2}^{k} \right) + \frac{T}{\Gamma_{K,\text{ph}}} (I_{1}^{k})^2, \\
\zeta_4 &= \zeta_1. 
\end{align}

We note that the last equality is related to the symmetry properties of the kinetic coefficient \( \gamma_{kph} \). If the symmetry on the indices of \( \gamma_{ij} \) is violated, then the Onsager symmetry principle is not fulfilled, and then \( \zeta_1 \neq \zeta_4 \).

As emphasized at the beginning of this Section, we want to stress that the expressions we found for the viscosities are very similar to those found for the phonon and roton contributions to the bulk viscosities to \(^4\text{He} \) (see Ref. [12]).

In the situation where the temperature is much smaller than the kaon gap \( T \ll \delta m = m_{k}^0 - \mu_{K}^{\text{eff}} \), then the contribution of the kaons is thermally suppressed. In Ref. [6] the three bulk viscosity coefficients due to the phonons were evaluated, that is, in the situation where \( I_{k}^{k} = 0 \). Because the kaon gaps are poorly known, it is not clear at which values of \( T \) the kaons might become relevant or not for transport in the CFL phase.

While the phonons are never thermally suppressed because they are massless degrees of freedom, one can still find a regime where the kaons dominate the contribution to the bulk viscosities. This may happen when \( \Gamma_{\text{ph}} \gg \Gamma_{K,\text{ph}} \), as then the first terms of Eqs. (32-34) would be suppressed. Because of the high power dependence of the phonon decay rate, one may expect that this is the situation that one may encounter at high temperatures.

One important lesson deduced from the expressions written above is that in the domain where the kaons are not thermally suppressed the pure phonon and kaon contributions cannot be disentangled, as the bulk viscosities contain products of both phonon and kaon thermodynamical functions. That is, the total bulk viscosity in the presence of phonons and kaons is not simply the sum of the phonon and kaon contributions.

V. PHONON AND KAON CONTRIBUTIONS TO THE FREQUENCY-DEPENDENT BULK VISCOSITIES

For astrophysical applications it is important to find the value of the bulk viscosities when there is perturbation which is periodic in time, with an angular frequency \( \omega_c \). Then, one also assumes that the time evolution of all the physical variables is also periodic. The bulk viscosities then turn out to be also a function of \( \omega_c \), and in this Section we will compute the frequency-dependent value of these transport coefficients. In Subsec. V A we present the algebraic expressions of the viscosities, and we leave the presentation of the numerical results for Subsec. V B.

A. Analytical results

The evaluation of the frequency dependent bulk viscosities when only phonons are considered was done in Ref. [3]. In Ref. [4], \( \zeta_2(\omega_c) \) due to kaons was also computed. Here we want to get the values of the frequency dependent viscosities when both the phonons and the kaons are taken into account. It is not a priori obvious how the coefficients will behave, as we have already seen that in a non-periodic situation one cannot simply sum the phonon and kaon contributions to the bulk viscosities.

We will work in the comoving frame, where \( \bar{u}^\mu = (1,0,0,0) \). Furthermore, as in Ref. [4] we will compute the viscosities assuming that the temperature is constant. In this regard, our computation here differs from that carried out for the phonons in Ref. [3], where such an assumption was not considered.

For purposes of comparison with the results of a non-periodic case presented in the previous Section, and the results obtained in Ref. [4], we will first find the viscosities neglecting the variation of the phonon number, that is, we will assume momentarily that \( \delta n_{\text{ph}} = \delta \mu_{\text{ph}} = 0 \). We thus use only the kaon and quark density equations. In the comoving frame, and taking into account the explicit time dependence of the variables, one has

\begin{align}
\delta n_k \omega_c + \bar{n}_k \text{div}u &= -\gamma_{kk} \delta \mu_k, \\
\delta n_q \omega_c + \bar{n}_q \text{div}u + \text{div}(V^2w) &= 0.
\end{align}
We now express the fluctuations of the particle densities in terms of a change in both the quark and kaon chemical potentials

\[
\delta n_q = \frac{\partial n_q}{\partial \mu} \delta \mu + \frac{\partial n_q}{\partial \mu_k} \delta \mu_k ,
\]

\[
\delta n_k = \frac{\partial n_k}{\partial \mu} \delta \mu + \frac{\partial n_k}{\partial \mu_k} \delta \mu_k ,
\]

where the derivatives of the particle number densities are evaluated in equilibrium, where one has \( \delta \mu_k = 0 \), see Appendix A for explicit details. In this way, we can solve for \( \delta \mu \) and \( \delta \mu_k \) as a function of the different thermodynamical variables and the angular frequency. The solutions, and therefore also the bulk viscosities, are complex quantities. However, for the computation of the dissipation in energy only the real part of these coefficients is needed, so we will only extract the real part of the viscosities.

Using Eqs. (7,8) and Eqs. (28,29), we find that the real part of the viscosities can be expressed as

\[
\zeta_i(\omega_c) = \frac{\gamma_{eff}}{\omega_c^2 + \gamma_{eff}^2} C_i ,
\]

where

\[
C_1 = C_4 = -\frac{\partial n_k}{\partial \mu} \frac{1}{L} \left( \tilde{n}_k \frac{\partial n_q}{\partial \mu} - \tilde{n}_q \frac{\partial n_k}{\partial \mu} \right) ,
\]

\[
C_2 = \frac{1}{\partial n_q} \frac{1}{\partial \mu} \frac{1}{L} \left( \tilde{n}_k \frac{\partial n_q}{\partial \mu} - \tilde{n}_q \frac{\partial n_k}{\partial \mu} \right)^2 ,
\]

\[
C_3 = \frac{1}{\partial n_q} \frac{1}{\partial \mu} \frac{1}{L} \left( \frac{\partial n_k}{\partial \mu} \right)^2 ,
\]

and

\[
\gamma_{eff} = \frac{\Gamma_{Kph}}{T} \frac{\partial n_q}{\partial \mu} \frac{1}{L} ,
\]

and we have defined the variable

\[
L = \frac{\partial n_q}{\partial \mu} \frac{\partial n_k}{\partial \mu_k} - \frac{\partial n_q}{\partial \mu} \frac{\partial n_k}{\partial \mu_k} .
\]

It is interesting to note that the condition for positive entropy production is saturated, and we have \( \zeta_1^2 = \zeta_2 \zeta_3 \).

With the results presented above, we can check that the formal expression of \( \zeta_2 \) agrees with the bulk viscosity computed in Ref. [4], as one has \( \frac{\partial n_k}{\partial \mu} = \frac{\partial n_k}{\partial \mu_k} \) (see Appendix A). We can also check that in the limit \( \omega_c \to 0 \) we obtain the values of the bulk viscosities obtained in the previous Section, in the situation where one assumes no thermal or entropy variation, and for \( I_{ph}^1 = I_{ph}^2 = 0 \). In order to check this limit, one simply has to realize that at constant temperature

\[
\frac{\partial n_k}{\partial \mu} \left( \frac{\partial n_q}{\partial \mu} \right)^{-1} = \frac{\partial n_k}{\partial \mu_k} = I_{ph}^k ,
\]

and write down all the pieces \( C_i / \gamma_{eff} \) in terms of the decay rate and also the \( I_{ph}^k \) and \( I_{ph}^2 \) functions introduced in Eq. (27).

Let us consider now the presence of fluctuations in the phonon sector. The set of linear equations that we have to solve should now also involve the phonon number density equation. The system of equations to solve is

\[
i \omega_c \delta n_{ph} + \tilde{n}_{ph} \text{div} \mathbf{u} = -\gamma_{phph} \delta \mu_{ph} - \gamma_{phk} \delta \mu_k ,
\]

\[
i \omega_c \delta n_k + \tilde{n}_k \text{div} \mathbf{u} = -\gamma_{kk} \delta \mu_k - \gamma_{kph} \delta \mu_{ph} ,
\]

\[
i \omega_c \delta n_q + \tilde{n}_q \text{div} \mathbf{u} + \text{div} (V^2 \mathbf{w}) = 0 .
\]

One can find solutions for \( \delta \mu, \delta \mu_k \) and \( \delta \mu_{ph} \) if we express the fluctuations of the particle densities in terms of the fluctuations of the chemical potentials

\[
\delta n_q = \frac{\partial n_q}{\partial \mu} \delta \mu + \frac{\partial n_q}{\partial \mu_k} \delta \mu_k + \frac{\partial n_q}{\partial \mu_{ph}} \delta \mu_{ph} ,
\]
The coefficients for the second and third bulk viscosities are

\[ \delta n_k = \frac{\partial n_k}{\partial \mu} \delta \mu + \frac{\partial n_k}{\partial \mu_k} \delta \mu_k, \]

\[ \delta n_{ph} = \frac{\partial n_{ph}}{\partial \mu} \delta \mu + \frac{\partial n_{ph}}{\partial \mu_{ph}} \delta \mu_{ph}. \]

Then, using Eqs. (18) and Eqs. (28,29), one can find the real part of the viscosities. In this case we find that the frequency dependence of the coefficients is different from the case where only the kaons (or phonons) are considered. The real part of the three coefficients can now be written as

\[ \zeta_i(\omega_c) = \frac{\tilde{D}_i(\omega_c)}{\omega_c^2 + \left(\frac{O + R_i^2}{Q_i^2}\right)^2}, \]

where

\[ O = \frac{\Gamma_{ph} \Gamma_{ph}}{T^2} \left( \frac{\partial n_q}{\partial \mu} \right)^2, \]

\[ R = \frac{\partial n_q}{\partial \mu} \left( \frac{\partial n_{ph}}{\partial \mu_{ph}} L + \left( \frac{\partial n_{ph}}{\partial \mu} \right)^2 \frac{\partial n_k}{\partial \mu_k} \right), \]

\[ Q = \frac{\partial n_q}{\partial \mu} \left\{ - \left( \frac{\partial n_q}{\partial \mu} \left( \frac{4}{\partial \mu_k} \frac{\partial n_{ph}}{\partial \mu_{ph}} \right) - \left( \frac{2}{\partial \mu_k} \frac{\partial n_{ph}}{\partial \mu_{ph}} + \frac{\partial n_{ph}}{\partial \mu} \right) \left( \frac{2}{\partial \mu_k} \frac{\partial n_{ph}}{\partial \mu_{ph}} + \frac{\partial n_{ph}}{\partial \mu} \right) \frac{\Gamma_{ph}}{\Gamma_{ph}} \frac{\Gamma_{ph}}{T^2} + \frac{\Gamma_{ph}}{T^2} \frac{\Gamma_{ph}}{T^2} \right\}, \]

and \( \tilde{D}_i \) are polynomials of second order in \( \omega_c \), expressed as

\[ \tilde{D}_i(\omega_c) = \frac{1}{Q^2} (b_i O + \omega_c^2 (b_i R + a_i Q)). \]

The coefficients \( a_i \) and \( b_i \) are different for each bulk viscosity, and depend on both the phonon and kaon thermodynamical variables, and on the the decay rates. More explicitly, the coefficients for the first bulk viscosity are

\[ a_1 = - \frac{\partial n_{ph}}{\partial \mu_{ph}} \frac{\partial n_k}{\partial \mu} \left( \tilde{n}_k \frac{\partial n_k}{\partial \mu_{ph}} - \tilde{n}_q \frac{\partial n_k}{\partial \mu} \right) - I_{2}^{ph} \frac{\partial n_{ph}}{\partial \mu_{ph}} \frac{\partial n_k}{\partial \mu_{ph}} \frac{\partial n_k}{\partial \mu}, \]

\[ b_1 = - \left( \tilde{n}_k \frac{\partial n_k}{\partial \mu_{ph}} - \tilde{n}_q \frac{\partial n_k}{\partial \mu} \right) \left\{ \frac{\Gamma_{ph}}{T} \frac{\partial n_k}{\partial \mu} + \frac{\Gamma_{ph}}{T} \left( \frac{2}{\partial \mu_k} \frac{\partial n_{ph}}{\partial \mu_{ph}} + \frac{\partial n_{ph}}{\partial \mu} \right) \right\}, \]

\[ \tilde{D}_i(\omega_c) = \frac{\Gamma_{ph}}{T} \frac{\Gamma_{ph}}{T} \left( \frac{\partial n_q}{\partial \mu} \left( \frac{2}{\partial \mu_k} \frac{\partial n_{ph}}{\partial \mu_{ph}} - \tilde{n}_k \frac{\partial n_k}{\partial \mu_{ph}} \right) \right) + \frac{\Gamma_{ph}}{T} \left( \frac{2}{\partial \mu_k} \frac{\partial n_{ph}}{\partial \mu_{ph}} - \tilde{n}_q \frac{\partial n_k}{\partial \mu} \right)^2, \]

and

\[ a_3 = \left( \frac{\partial n_{ph}}{\partial \mu} \right)^2 \frac{\partial n_k}{\partial \mu_k} + \left( \frac{\partial n_{ph}}{\partial \mu} \right)^2 \frac{\partial n_{ph}}{\partial \mu_{ph}}, \]

\[ b_3 = \frac{\Gamma_{ph}}{T} \left( \frac{4}{\partial \mu_k} \frac{\partial n_k}{\partial \mu_{ph}} + \frac{\partial n_k}{\partial \mu} + \frac{\partial n_{ph}}{\partial \mu_{ph}} + \frac{\partial n_{ph}}{\partial \mu} \frac{\partial n_q}{\partial \mu_{ph}} \right) + \frac{\Gamma_{ph}}{T} \left( \frac{\partial n_k}{\partial \mu} \right)^2, \]
respectively. Further, we have found $\zeta_1 = \zeta_4$.

We have checked that our expressions in the limit $\omega_c \to 0$ reproduce the results of the bulk viscosities in the limit when there are no thermal or entropy fluctuations, obtained in the previous Section. While it is not easy to verify whether the expressions for the three bulk viscosities are consistent with the constraint of positive entropy production, we have checked numerically that this is the case, as one always gets $\zeta_1^2 \leq \zeta_2 \zeta_3$.

Because of the rather non-trivial dependence of the expressions of every bulk viscosity coefficient on the phonon and kaon thermodynamical variables it is not easy to infer how the viscosities scale with the temperature, or with other relevant scales of the problem, such as $\mu$. It is also not easy to see when the phonons or the kaons give a subleading contribution. A numerical analysis is then mandatory. We present the numerical results of the viscosities in the following Subsection.

### B. Numerical results

We have numerically computed the values of the three bulk viscosities, and we present the results in different logarithmic plots for fixed values of the frequency, and as a function of the temperature. We consider temperatures in the range of $0.01 \text{ MeV} < T < 10 \text{ MeV}$. Lower temperatures might be relevant in astrophysical scenarios. However, it was found out in Ref. [3] that for lower $T$ the phonon mean free path is much larger than the radius of a compact star, and a hydrodynamical description of the phonon fluid ceases to be valid.

In our numerical computations we have chosen the value of the quark chemical as $\mu = 400 \text{ MeV}$, the quark gap $\Delta = 25 \text{ MeV}$ and the strange quark mass $m_s = 120 \text{ MeV}$. We leave as free parameters the kaon gap $\delta m = m_{K^0} - \mu_{K^0}$, and $a_{ph}$, the numerical factor present in the phonon decay rate, see Eq. (10). Notice that while we worked using natural units in the main part of the manuscript, we use SI units to show the numerical values of the viscosities. It is important to realize that the three bulk viscosities have different scalings in length (see the Appendix B).

The phonon and kaon thermodynamical variables that are needed for the viscosities can be computed from the free energy of the system (see Appendix A for details), and we use the decay rates shown in Section III.

In Fig. 1 and Fig. 2 left we present the numerical values of $\zeta_1, \zeta_3$ and $\zeta_2$, respectively, for a fixed value of the frequency, $\omega_c/2\pi = 1 \text{ KHz}$, and for five different values of the kaon mass gap, ranging from $\delta m = 0.1 \text{ MeV}$ to $10 \text{ MeV}$. We have fixed in these plots the value of $a_{ph} = 1$. Roughly speaking, the plots for the three viscosities present a similar shape, that is, a similar dependence on $T$, although the numerical values for each viscosity are very different due to the different scaling dimensions of each coefficient.

In Fig. 2 right we also present the value of $\zeta_2$ when only the kaon contribution is taken into account, that is, with the results of Ref. [4], or when only the phonon contribution is taken into account (the results here differ from those
FIG. 2: Left: Temperature dependence of second bulk viscosity for the CFL quark matter phase due to thermal phonons and kaons for a frequency $\omega_\text{c}/2\pi = 1$ Khz, with the phonon decay factor is fixed at $a_{\text{ph}} = 1$, and for five five different values of the kaon mass gap. Right: The same plot but for $\zeta_2$ due to kaons only, or to phonons only.

of Ref. [6] because here we are doing the computation assuming that the temperature remains constant). The same plots for the kaon contribution only for $\zeta_1$ and $\zeta_3$ present a rather similar shape, so we won’t show them here.

In Fig. [5] we show the sensitivity of one of the bulk viscosities to the value of $a_{\text{ph}}$, the constant in the phonon decay rate Eq. (10) and for two fixed values of the kaon mass gap. We can observe that the numerical values of the viscosities are rather insensitive to the values of $a_{\text{ph}}$ ranging form $10^{-2}$ to $10^{2}$, except in the high $T$ limit. This is due to the high power dependence on $T$ of the phonon decay rate, Eq. (10). For this reason, we have fixed the value of $a_{\text{ph}} = 1$ in most of the plots we are showing.

In Fig. 3 we show the value of $\zeta_2$ varying the value of the frequency, for $\omega_\text{c} = 1, 1/3, 0.1$ Khz, and we have shown also in the same plot the same results for the kaon contribution only.

Some generic comments can be formulated from our numerical studies. At very small temperatures the phonons dominate the contribution to the bulk viscosities, as the kaons are then thermally suppressed. This fact is not always shown clearly in the plots as this tends to occur at $T \ll 0.01$ MeV, which is outside the temperature regime where one may expect the phonons to be in a hydrodynamical regime inside the star. At very large temperature $T \geq 10$ MeV the kaon contribution seems to dominate the behavior of the viscosities. The reason comes from noting that in that regime $\Gamma_{\text{ph}}$ becomes very large, and suppresses the phonon contribution (see comments at the end of Sec. IV). In the intermediate temperature regime we always find an interval where first the kaons dominate, followed by a generic two-peak/hump structure, where both phonon and kaon contributions are relevant. The temperatures where this happens depends on the value of the kaon mass gap.

One of the main conclusions derived from our numerical studies is that in the regime where the hydrodynamical description might be valid for the superfluid phonons, which should be in the range $T \geq 0.01$ MeV, and for a range of values of kaon gaps that do not exceed the few MeV, both kaons and phonons contribute to the value of the viscosities. The superfluid phonon would only be highly dominant for $T \ll 0.01$ MeV, but as already mentioned, in this regime in a compact star the phonons do not collide enough times to maintain a fluid description in that case. Then the phonons rather behave as a gas, and one should consider other dissipative effects in the system, such as that coming from the collisions of phonons with superfluid vortices [9].

In Ref. [4] it was realized that the value of $\zeta_2(\omega_\text{c})$ as due to kaons only is much less than the value of the same coefficient for unpaired quark matter. From our numerical results, we can say that if we include both the phonon and kaon contribution (see Fig. 2 and Fig 3) the reduction is still more severe, although this depends on the actual value of $\delta m$ and on the values of $T$. 

VI. OUTLOOK

We have presented a computation of the frequency dependent bulk viscosities in the low temperature regime of CFL quark matter, extending previous results already obtained in the literature. The main contributions to the viscosities come from both kaons and superfluid phonons. The main results of this manuscript are presented in Eqs. (53-64) and Figs. 1 to 4.

One of the main conclusions of our study is that in the regime where the hydrodynamical description might be valid for the superfluid phonons, which should be in the range $T \geq 0.01$ MeV, and for a range of values of kaon gaps do not exceed the few MeV, both kaons and phonons contribute to the value of the viscosities, and one cannot discard...
any of the two quasiparticle contributions. Our results also agree with the main finding of Ref. \[4\], that is, that the coefficient $\zeta_2$ in the CFL phase is several order of magnitude smaller than the same coefficient in unpaired quark matter.

The studies of transport coefficients are important in order to find possible signatures of quark matter in astrophysical scenarios. The fate of different set of oscillations modes of compact stars are governed by the value of the transport coefficients in the star. In particular, the fate of the r-modes of the star \[21\] depend on the dissipative effects in the star. Previous analysis of the damping of the r-modes have been carried out in the literature \[13 \[23 \[25 \[27\]. The results of our study should be taken into account in a more refined study of the the damping of r-modes, where the value of the three bulk viscosity coefficients should be relevant.

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Appendix A: Phonon and kaon thermodynamical variables

The phonon and kaon thermodynamical variables needed for the computation of the bulk viscosities can be extracted from the free energy of the CFL system, which reads

$$\Omega(\mu, \delta \mu_k, \delta \mu_{ph}, T) = \Omega_q(\mu, T) + \Omega_k(\mu, \delta \mu_k, T) + \Omega_{ph}(\mu, \delta \mu_{ph}, T) ,$$  \hspace{1cm} (A1)

where $\Omega_q$ is the quark contribution, while $\Omega_k$ and $\Omega_{ph}$ are the kaon and phonon contributions, respectively. The free energy of the phonons and kaons in the presence of the “fake” chemical potentials that characterize the out of equilibrium state is

$$\Omega_i(\mu, \delta \mu_i, T) = \frac{T}{2\pi^2} \int_0^\infty dp p^2 \ln (1 - \exp(- (E_i - \delta \mu_i) / T)) , \hspace{1cm} i = k, ph .$$  \hspace{1cm} (A2)

The $\mu$-dependence in the contribution to the free energy of the phonons and kaons appears because the energy of the quasiparticle depends on $\mu$. In particular, for the phonons both the speed of sound $c_s$ and the parameter $B$ that enters in their dispersion relation, see Eq. (12), depend on $\mu$. For the kaons, both their velocity $v$, mass $m_{K^0}$, and chemical potential $\mu_{K^0}^{eff}$ depend on $\mu$, although here we will only consider the dependence of $\mu_{K^0}^{eff}$, as the dependence of the other variables is unknown.

The different particle densities are obtained from the free energy of the system

$$n_i = -\frac{\partial \Omega}{\partial \mu_i} , \hspace{1cm} i = q, k, ph .$$  \hspace{1cm} (A3)

From the free energy of the system we can also compute all the derivatives that appear in the expressions of the bulk viscosities. These are computed from Eq. (A1), putting $\delta \mu_i = 0$ at the end of the computation.

The phonon thermodynamical variables can be computed analytically. For the computation of the bulk viscosities that was done in Ref. \[6\] it was realized that the phonon dispersion law at non-linear order was needed. The explicit expressions of the thermodynamical functions needed in our computation can be found in that reference. The kaon thermodynamical variables have to be computed numerically.

It is interesting to note that the following conditions are satisfied

$$\frac{\partial n_k}{\partial \mu} = \frac{\partial n_q}{\partial \mu_k} , \hspace{1cm} \frac{\partial n_{ph}}{\partial \mu} = \frac{\partial n_q}{\partial \mu_{ph}} , \hspace{1cm} \frac{\partial n_{ph}}{\partial \mu_k} = \frac{\partial n_k}{\partial \mu_{ph}} = 0 .$$  \hspace{1cm} (A4)

Further, we will use the relation \[4\]

$$\frac{\partial n_k}{\partial \mu} = -\frac{m_s^2}{2\mu^2} \frac{\partial n_k}{\partial \mu_k} .$$  \hspace{1cm} (A5)
Appendix B: Dimensions of the bulk viscosity coefficients

The bulk viscosity coefficients introduced by Gusakov in Son’s formulation of the relativistic superfluid hydrodynamics are related to those introduced by Khalatnikov by mass factors. More explicitly, one has

\[ \zeta_1^{K_h} = \frac{\zeta_1}{m}, \quad \zeta_2^{K_h} = \zeta_2, \quad \zeta_3^{K_h} = \frac{\zeta_3}{m^2}. \]  

(B1)

If \( M, L, T \) refer to scales of mass, length and time, respectively, then the three bulk viscosity coefficients in Gusakov’s theory have the following dimensions

\[ [\zeta_1] = M L^2 T^{-1}, \quad [\zeta_2] = M L^{-1} T^{-1}, \quad [\zeta_3] = M L^5 T^{-1}. \]  

(B2)

In all the main part of the paper we worked using natural units. The plots are given in units of SI. For the conversion of the values of the bulk viscosities from natural units to SI units it is useful to remember that

\[ 1 \text{ MeV} = 1.78 \cdot 10^{-27} \text{ g}, \quad 1 \text{ MeV} = 1.97 \cdot 10^{-11} \text{ cm}, \quad 1 \text{ MeV} = 6.58 \cdot 10^{-22} \text{ sec}. \]  

(B3)

When the bulk coefficients are expressed in natural units they seem to behave almost similarly, that is, one can see that they have a similar \( T \) dependence. In the SI system, and due to the different scalings in length of the coefficients, the coefficients show a numerical behavior rather different, as shown in the plots of Fig. 1 and Fig. 2 the ultimate reason being the different scaling in \( L \) of every coefficient.

We note here that in the analysis of the r-modes of the CFL phase done in Ref. [27], it was claimed that only \( \zeta_3 \) is important for the computation of the damping of the r-modes. However, the dimensions of \( \zeta_3 \) in that manuscript have been wrongly assumed to be the same as those of \( \zeta_2 \), which probably affects the final numerical results of the whole analysis when converting the different magnitudes from natural units to SI units.

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