Real and complex connections for canonical gravity

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Abstract

Both real and complex connections have been used for canonical gravity: the complex connection has $SL(2,C)$ as gauge group, while the real connection has $SU(2)$ as gauge group. We show that there is an arbitrary parameter $\beta$ which enters in the definition of the real connection, in the Poisson brackets, and therefore in the scale of the discrete spectra one finds for areas and volumes in the corresponding quantum theory. A value for $\beta$ could be could be singled out in the quantum theory by the Hamiltonian constraint, or by the rotation to the complex Ashtekar connection.

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I. INTRODUCTION

A lot of recent work in canonical quantum gravity is based on using as canonical variables the pair \((E^i_a, A^i_a)\), where \(E^i_a\) is a triad of density weight one and \(A^i_a\) a real \(SU(2)\) connection; these variables were introduced by Barbero \([1]\) as an alternative to the complex Ashtekar variables \([2]\). For \(SU(2)\) to be the gauge group for general relativity we have to fix the local inertial frames, choosing what is called ”time gauge”; once this choice is made, it will be shown that there is a certain freedom in the choice of the connection, parametrized by a variable \(\beta\). No particular real real value of \(\beta\) has a geometric motivation, and any one gives an acceptable, if somewhat bizarre, \(SU(2)\) connection. The choice \(\beta = i\) gives the Ashtekar variables, requires that we impose reality conditions on the variables, but restores the full invariance of the canonical theory under local Lorentz transformations; as an additional benefit, this choice simplifies the expression of the hamiltonian constraint considerably.

The difference between these choices becomes important when we attempt to quantize the theory. It is difficult to find a scalar product if the gauge group is \(SL(2,C)\), even in simplified cases \([3]\). On the contrary, for gauge invariant functionals of a \(SU(2)\) connection, the metric being well defined, one can study the spectrum of operators that can be interpreted as representing the area of a surface and the volume of a region in space \([4]\); both these spectra turn out to be discrete \([5–8]\), indicating a discrete structure of space at the Planck scale. Unfortunately, the arbitrary parameter \(\beta\) multiplies the scale of these spectra, and therefore the results are meaningless unless we find a way to fix this constant. Since the constant \(\beta\) appears explicitly in the expression of the Hamiltonian constraint, it may be that that solutions to this constraint exist only for particular values, most likely \(\beta = 1\). An alternative, but strictly related possibility would be that \(\beta = 1\) is required because it is the value for which a rotation to the Ashtekar variables is possible \([9,10]\), restoring the gauge group \(SL(2,C)\). Both these goals have been attempted in recent work \([11]\).

II. CONNECTIONS.

A canonical formulation of general relativity may be obtained starting from the action 
\[ S = \frac{1}{4k} \int \varepsilon_{IJKL} e^I \wedge e^J \wedge R^{KL} \] 
(where \(I, J, \ldots = 0, 1, 2, 3\), \(e^I\) are vierbeins, \(R^{KL}\) is the curvature 2–form of the Levi–Civita connection \(\Omega^{I,J}\), and \(\kappa := (8\pi G_{\text{Newton}})/c^3\)), then taking space slices \(\Sigma_t : t(x) = \text{const.}\), with a vector \(t^a : t^a \partial_a t = 1\) related to the unit normal to the slice \(n^a\) by \(t^a = N n^a + N^a\), or \(n_a = -N \partial_a t\). If at this stage we partially fix the \(O(3,1)\) gauge freedom choosing ”time gauge”, i.e. setting:

\[ e^0_a = -n_a = N \partial_a t \]  

we are left with invariance under local \(O(3)\) transformations with \(e^{ia}, i = 1, 2, 3\) space–like providing a local frame on the slice and the inverse 3–metric \(q^{ab} = e^{ia} e^{ib}\). The pull–back of \(\Omega^{I,J}\) to the space slice gives the 3–d Levi–Civita connection \(\Gamma^i_a := \frac{1}{2} q_a^b e_{ijk} q_b^j\), and the extrinsic curvature \(K^i_a := q_a^b q^0_b = e^{ib} K_{ab}\). Instead of the A.D.M. variables \((q_{ab}, \pi^{ab} := \sqrt{q}(K^{ab} - q^{ab} K))\), we can take the pair \([3]\):

\[ (E^{ia} := \frac{1}{2} e^{abc} e_{ijk} e_j^a e_k^b = \sqrt{q} e^{ia}, \ K^i_a) \]  

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From the definitions it follows that for small variations:

\[ K_i^a \delta E^{ia} = \frac{1}{2\sqrt{q}} K_{ab} \delta(E^{ia}E^i) = \]
\[ = \frac{1}{2q}(\pi_{ab} - \frac{1}{2} \pi q_{ab})\delta(qq^{ab}) = -\frac{1}{2} \pi^{ab} \delta q_{ab} \]

and therefore the only non vanishing Poisson-brackets are:

\[ \{K_i^a(x), E^{jb}(y)\} = \kappa \delta_i^b \delta^j_\alpha \delta(x,y) \]

With the connection \( \Gamma_i^a \) we can compute covariant derivatives \( D_a \) and the curvature \( R^i_{ab} \), and write the constraints of the theory in the form:

\[ G_i := \epsilon_{ijk} K_j^a E^{ka} \approx 0 \]
\[ \mathcal{V}_c := q_{bc} \nabla_a \pi^{ab} = 2 E_i^a D_a K_i^a \approx 0 \]
\[ \mathcal{H} := \frac{1}{\sqrt{q}}(\pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 - q R) = \]
\[ = \frac{1}{\sqrt{\det E}} \left( 2 E_i^a E_j^b K^i_a K^j_b + \epsilon_{ijk} E^{ia} E^{jb} R^i_{ab} \right) \approx 0 \]

The first constraint has been added to the A.D.M. ones to make sure that \( K_{ab} \) is symmetrical.

The trouble with this formulation is that \( \Gamma_i^a \) is a derived quantity, that can be expressed in terms of \( E^{ia} \), its derivatives and its inverse solving the 9 linear equations it satisfies:

\[ D_a E^{ia} = 0 \quad ; \quad \epsilon_{ijk} E^a_i E^b_j D_a E^c_k = 0 \]

To obtain a connection that is in some sense a conjugate quantity to the triads we may follow \[ ]\ and change our basic variables, for some \( \beta \) to be fixed, to:

\[ (E^{ia}, A_i^{(\beta)a} := \Gamma_i^a + \beta K_i^a) \]

By the properties of the Levi–Civita connection, we find:

\[ E^{ia} \delta A_i^{(\beta)a} = \beta E^{ia} \delta K_i^a + E^{ia} \delta \Gamma_i^a = \]
\[ = \beta E^{ia} \delta K_i^a + \delta_a(\varepsilon^{abc} c_b^k \delta c^k) \]

so that the only non vanishing Poisson brackets are:

\[ \{A_i^{(\beta)a}(x), E^{jb}(y)\} = \beta \kappa \delta_i^b \delta^j_\alpha \delta(x,y) \]

Unlike \( K_i^a \), \( A_i^{(\beta)a} \) transforms like an \( SU(2) \) connection for any \( \beta \), so that one can have \( D_i^{(\beta)a} \) derivatives, a curvature \( F_{ab}^{(\beta)i} \), and for any path \( \gamma \) a ”transporters” \( g_\gamma := P \exp(\int \tau_i A_i^{(\beta)a} d\gamma^a) \).

Geometrically, all this is perhaps a bit artificial: we are using a connection with torsion, and we are smuggling the dynamics of the theory in this torsion.

With some algebra one finds that the constraints eq. \( (\mathbb{E}) \) can be rewritten in the form:
\[ D^{(\beta)}_a E^{ia} = \beta G_i \approx 0 \]
\[ E^{ia} F^{(\beta)i}_{ab} = \beta V_b + \beta^2 K^i_b \hat{G}_j \approx 0 \]
\[ \mathcal{H} = \frac{\epsilon^{ijk}}{\sqrt{\text{det}(E)}} E^{ia} E^{jb} F^{(\beta)k}_{ab} - 2 \frac{(1 + \beta^2)}{\sqrt{\text{det}(E)}} E^{[a} E^{b]} K^i_a K^j_b + \ldots \approx 0 \] (10)

where the dots are terms proportional to \( \beta G_i \) and its derivatives.

Clearly, complications will come from the last constraint, in particular from its overall factor \( 1/\sqrt{\text{det}(E)} \) and from the messy term proportional to \((1 + \beta^2)\). The quickest way to get rid of it would be to take \( \beta = \imath \), which is Ashtekar’s original choice \[2\] (together with the idea of absorbing the factor \( 1/\sqrt{\text{det}(E)} \) in the Lagrange multiplier). But this choice can be motivated by much more than the simplification of the constraints: in fact for this value of \( \beta \) the variables eq.(7) can be defined quite generally, without the gauge choice eq. (1), as the pull–back to the space slice of the self–dual connection, and of the self–dual product of two vierbeins:

\[ A^{(i)i} := q_a^b C^{i}_{IJ} \Omega^I_b := q_a^b (-\frac{1}{2} \epsilon^{ijk} \Omega^j_b + i \Omega^b_i) \] (11)
\[ E^{ia} := -\epsilon^{abc} C^{i}_{IJ} e^I_a e^J_c = -2i \frac{\text{det}(e^I_a)}{N} C^{i}_{IJ} n_b e^{lb} e^{Ja} \] (12)

here \( C^{i}_{IJ} \) projects antisymmetric tensors to the \((1, 0)\) representation of \( SL(2, C) \), i.e. selects the self–dual part, so that \( \epsilon^{iJ} K^i_{KL} C^{i}_{KL} = 2i C^{i}_{IJ} \). This definition of the basic variables coincides with eq. (\[2\]) if we assume ”time gauge”, eq. (\[1\]), but we have to add to the constraints eq. (10) the reality conditions:

\[ E^{ia} E^{ib} = \text{real} \quad ; \quad \epsilon^{ijk} E^{a} E^{(b} D^{(i)}_a E^{c)} = \text{imaginary} \] (13)

On the other hand, adopting this connection we retain the Lorentz group as gauge group of the canonical theory\[4\].

Because of the obvious complications of dealing with complex variables and with the non-compact gauge group \( SL(2, C) \), the alternative choice \( \beta = 1 \), the ”Barbero connection”, has been used in all the more recent work. In Euclidean \((++++)\) gravity the second term of \( \mathcal{H} \) is proportional to \((1 - \beta^2)\), and the choice \( \beta = 1 \) is natural, just like \( \beta = \imath \) was ”natural” for the \((-+++)\) signature, but that is not particularly relevant, unless somehow we learn to ”Wick–rotate” the theory. Otherwise, there is no obvious geometric meaning, and nothing special about the value \( \beta = 1 \), and and we can leave \( \beta \) arbitrary (but real > 0) in the following.

One can also introduce a Poisson bracket preserving map on the algebra of functions on phase space which, in a certain sense, allow us to change \( \beta \). Following Thiemann \[10\], one notices that

\[ 1\text{It is not often emphasized that the derivation of the Ashtekar variables does not require any gauge fixing, but it is obvious from their original definitions; for ex. the alternative definition given for } E^{ia} \text{ in eq.} (\[12\]) \text{ is a direct transcription of eq.1.13b of } \[12\].} \]
\[ T := \int_{\Sigma} K_a^i E^{ia} d^3x \quad (14) \]

from eq. (4) and the properties of Levi–Civita connections has the following Poisson brackets:

\[
\{ T, E^{ia} \} = E^{ia} ; \quad \{ T, \Gamma_a^i \} = 0 ; \quad \{ T, K_a^i \} = -K_a^i
\quad (15) \]

It follows that if we indicate the Hamiltonian evolution induced by some \( H \) on a function \( f \) on phase space by the map:

\[
W_H(t) \circ f := f + t\{ f, H \} + \frac{t^2}{2!}\{\{ f, H \}, H \} + \ldots
\quad (16) \]

we shall have:

\[
W_T(t) \circ E^{ia} = e^{-t} E^{ia} ; \quad W_T(t) \circ A^{(\beta) i}_a = \Gamma_a^i + \beta e^i K_a^i
\quad (17) \]

At a more speculative level, we can think of choosing \( t = i\pi/2 \) and using this map to perform a rotation from the \( \beta = 1 \) to the Ashtekar connection \([10,9]\), similar to a "Wick rotation", a possibility that is particularly exciting in the quantum case.

**III. QUANTIZATION.**

To quantize the theory we may use the connection representation, in which states are functionals of \( A^{(\beta) i}_a(x) \), and

\[
E^{ia} \rightarrow \hat{E}^{ia} := \frac{\beta \hbar}{i} \frac{\delta}{\delta A^{(\beta) i}_a}
\quad (18) \]

The important states turn out to be "spin net states", which are an obvious generalization of Wilson loops. Given a net \( n \) in 3–space with \( V \) vertices joined by (analytic) paths \( \gamma_1, \ldots, \gamma_L \), we assign: to each line an orientation, hence a "transporter" \( g_l \), and a spin \( s_l = 0, \frac{1}{2}, 1, \ldots, \)
so that \( \gamma_l \rightarrow \mathcal{D}^{\eta_l}_{mm'}(g_l) \); to each vertex \( v \) an \( SU(2) \) invariant tensor \( C^w_{m m' \ldots m'} \), in such a way that:

\[
\psi_n(\{g_l\}) = \sum_{\{m\}} \prod_v C^w_{..} \prod_l \mathcal{D}^{s_l}_{..}(g_l)
\quad (19) \]

is gauge invariant. If \( g_l \in SU(2) \), and we indicate by \( dg_l \) the Haar measure, the \( \psi_n \) that we can associate to a given net form a Hilbert space with the scalar product:

\[
< \psi_\mu | \psi_\nu > = \int \overline{\psi_\nu} \psi_\mu \prod dg_l
\quad (20) \]

Ashtekar and collaborators have discovered (see e.g. [3]) that if one considers all possible nets, then the set of these states is dense in the Hilbert space of gauge invariant functionals of \( A^{(\beta)} \). In this sense eq.(20) induces a measure \( DA^{(\beta)} \) in this space. One can therefore define a Rovelli–Smolin "loop transform" \([13]\), and focus on states which have support on "weaves" \([4]\), huge nets with mesh sizes of the order of the Planck length. In recent work these states have been used to find the spectrum of the operators that correspond to the
area of a surface $S$ and to the volume of a region $R$, and to investigate the operator form of the constraints.

Very briefly: if the surface $S$ intersects a subset $\mathcal{L}$ of lines of the net, does not touch the vertices, has no line lying on it, carefully regularizing the operator (first taking the square root, then removing the regulator), one finds:

$$\hat{A}(S)\psi_n = : \int_S d^2\sigma \sqrt{n_a n_b \hat{E}^{ia} \hat{E}^{jb}} : \psi_n = (\beta \bar{\hbar} \kappa) \sum_{l \in \mathcal{L}} \sqrt{s_l (s_l + 1)} \psi_n$$

(21)

This basic result has been extended in various directions, in particular to the volume operator $\hat{V}$. On the other hand, there is no known way to extend it to the theory based on the Ashtekar connection: all operators seem to have the wrong hermiticity, whatever scalar product we use.

However, it is clear from eq. (21) that the discrete spectra one gets for areas and volumes cannot at this stage be interpreted as evidence for a discrete structure of space, because of the arbitrariness of $\beta$. The key ingredients we have used are the gauge invariance and the commutation relations that correspond to the Poisson brackets eq. (1), and we must conclude that by themselves they are not enough to fix the scale of the theory.

The quantization program requires therefore that we find some good reason to fix $\beta$. I can imagine two ways in which this could happen. One could argue that a larger group than $SU(2)$ is necessary, as suggested for instance by the experience of current algebra, and that one should insist on using the Ashtekar connection, overcoming the difficulties that have been found; or that the states considered should be restricted by more than just gauge invariance, in particular by the Hamiltonian constraint. In fact, it may well be that given the way in which $\beta$ enters eq. (10), the Wheeler De Witt equation $\hat{\mathcal{H}} \cdot \Psi = 0$ will turn out to have solutions only for particular values of $\beta$, a possibility that could be investigated in Thiemann’s proposed realisation of $\hat{\mathcal{H}}$.

That these two ways may be related, or be the same, is indicated by another remarkable Poisson bracket identity deduced by Thiemann:

$$\left\{ \frac{\epsilon^{ijk}}{\sqrt{\det E}} E^{ia} E^{jb} F_{ab}^{(\beta)k}, \sqrt{\det E} \right\} = 2\beta^2 \kappa E^{ia} K^i_\alpha \delta(x, y)$$

(22)

If at the quantum level we could handle the Hamiltonian constraint and the volume, and translate this equation, we would be able to implement the "Wick rotation" from the Barbero to the Ashtekar connection representation.

At the same time, it is also important to establish whether the discreteness of areas and volumes follows from the kinematics, which embody the equivalence principle, or requires the Hamiltonian constraint, i.e. the Einstein equations: because the discreteness of areas may be used to establish the proportionality between area and entropy [14], and this relation can be used to derive the Einstein equations themselves, understood as equations of state [15].
IV. CONCLUSION

We have seen that one can formulate canonical gravity as either a $SL(2, C)$ or an $SU(2)$ connection theory, but that in the latter case an arbitrary parameter $\beta$ occurs in the basic Poisson brackets; at the same time, present mathematical technics can only cope with a quantum theory based on the group $SU(2)$. However, no meaningful result on the spectra of operators can be obtained unless we either fix this parameter, or learn how to handle a theory based on the group $SL(2, C)$.

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