Comment on: “On the Klein-Gordon oscillator subject to a Coulomb-type potential”. Ann. Phys. 355 (2015) 48

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Abstract

We analyze the conclusions of the influence of a Coulomb-type potential on the Klein-Gordon oscillator. We show that the truncation method proposed by the authors do not yield all the eigenvalues of the radial equation but just one of them for a particular value of a model parameter. Besides, the existence of allowed oscillator frequencies that depend on the quantum numbers is an artifact of the truncation method.

Key words: Klein-Gordon equation; conditionally-solvable quantum-mechanical model; Frobenius method; three-term recurrence relation; allowed oscillator frequencies

In a paper published in this journal Bakke and Furtado study the influence of a Coulomb-type potential on the Klein-Gordon oscillator. By a series of transformations of the Klein-Gordon equation in cylindrical coordinates these authors derive an eigenvalue equation for the radial part of the wavefunction.

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By means of the Frobenius method they obtain a three-term recurrence relation for the coefficients of the power series. A suitable truncation of the series enables them to derive an analytical expression for the energy levels and a most striking result: the existence of allowed oscillator frequencies. The purpose of this Comment is the analysis of the truncation method used by Bakke and Furtado and its effect on the conclusions drawn in their paper.

The starting point is the Klein-Gordon equation

\[-\frac{\partial \phi}{\partial t^2} + \frac{\partial^2 \phi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \phi}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \varphi^2} + m\omega \phi - m^2 \omega^2 \rho^2 \phi - m^2 \phi - \frac{2mf}{\rho} \phi - \frac{f^2}{\rho^2} \phi = 0,\]

(1)

where the meaning of each parameter is given in the authors’ paper [1]. Units are chosen so that \(c = \hbar = 1\) (in a recent paper [2] we have criticized this unclear choice of suitable units). The authors look for a particular solution of the form

\[\phi(t, \rho, \varphi) = e^{-i\mathcal{E}t} e^{il\varphi} R(\rho),\]

(2)

where \(l = 0, \pm 1, \pm 2, \ldots\). In this way they obtain the following equation for the radial function \(R(\rho)\)

\[\frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} - \frac{\gamma^2}{\rho^2} R - \frac{2mf}{\rho} R - m^2 \omega^2 \rho^2 R + \beta^2 R = 0,\]

(3)

where \(\beta^2 = \mathcal{E}^2 - m^2 + m\omega\) and \(\gamma^2 = l^2 + f^2\).

By means of the change of variables \(\xi = \sqrt{m\omega} \rho\) one obtains

\[\frac{d^2 R}{d\xi^2} + \frac{1}{\xi} \frac{dR}{d\xi} - \frac{\gamma^2}{\xi^2} R - \frac{\delta}{\xi} R - \xi^2 R + \frac{\beta^2}{m\omega} R = 0,\]

(4)

where \(\delta = \frac{2mf}{\sqrt{m\omega}}\). If we write the solution to this equation as

\[R(\xi) = \xi^{\gamma/2} e^{-\xi^2/2} \sum_{j=0}^{\infty} a_j \xi^j,\]

(5)
we obtain the three-term recurrence relation

\[ a_{j+2} = \frac{\delta}{(j+2)(j+1+\alpha)} a_{j+1} - \frac{\theta - 2j}{(j+2)(j+1+\alpha)} a_j, \]

\[ j = -1, 0, 1, 2, \ldots, a_{-1} = 0, a_0 = 1, \]

(6)

The authors state that “It is well-known that the quantum theory requires that the wave function (6) must be normalizable. Therefore, we assume that the function \( R(\xi) \) vanishes at \( \xi \to 0 \) and \( \xi \to \infty \). This means that we have a finite wave function everywhere, that is, there is no divergence of the wave function at \( \xi \to 0 \) and \( \xi \to \infty \), then, bound state solutions can be obtained. However, we have written the function \( H(\xi) \) as a power series expansion around the origin in Eq. (14). Thereby, bound state solutions can be achieved by imposing that the power series expansion (14) or the Heun biconfluent series becomes a polynomial of degree \( n \). In this way, we guarantee that \( R(\xi) \) behaves as \( \xi^{\gamma} \) at the origin and vanishes at \( \xi \to \infty \). Through the recurrence relation (15), we can see that the power series expansion (14) becomes a polynomial of degree \( n \) by imposing two conditions:

\[ g = 2n \text{ and } a_{n+1} = 0, \]

(7)

where \( n = 1, 2, \ldots \).”

It clearly follows from these two conditions that \( a_j = 0 \) for all \( j > n \). However, the authors’ statement is a gross conceptual error because a bound state simply requires that \( R(\xi) \) is square integrable:

\[ \int_0^\infty |R(\xi)|^2 \xi \, d\xi < \infty. \]

(8)

Consequently, by means of the truncation condition (7) the authors obtain just some particular bound states and only for particular values of a model parameter. For example, when \( n = 1 \) then \( \theta = 2 \) and we obtain a simple
analytical expression for $E_{1,l}$. The second condition $a_2 = 0$ yields $\delta = \delta_{1,l} = \pm \sqrt{2\alpha}$ and the oscillator frequency $\omega_{1,l} = 4mf^2/\delta_{1,l}^2$.

From the general result the authors argue as follows “... we have that the influence of the Coulomb-like potential makes that the ground state to be defined by the quantum number $n = 1$ instead of the quantum number $n = 0$. Note that we have written the angular frequency $\omega$ in terms of the quantum numbers $\{n,l\}$ in Eq. (18). From the mathematical point of view, this dependence of the angular frequency of this relativistic oscillator on the quantum numbers $\{n,l\}$ results from the fact that the exact solutions to Eq. (12) are achieved for some values of the Klein-Gordon oscillator frequency. From the quantum mechanics point of view, this is an effect which arises from the influence of the Coulomb-type potential on the Klein-Gordon oscillator.” And also “The meaning of achieving this relation is that not all values of the angular frequency $\omega$ are allowed, but some specific values of $\omega$ which depend on the quantum numbers $\{n,l\}$. For this reason, we label $\omega = \omega_{n,l}$.”

In order to put the authors’ conclusions to the test, we solve the eigenvalue equation (4) for $W = \frac{\alpha^2}{m\omega}$ so that

$$
E^2 = m\omega W + m^2 - m\omega,
W = \theta + 2(|\gamma| + 1).
$$

(9)

Anybody familiar with eigenvalue equations realizes that for a given set of values of $f$, $\delta$ and $l$ we obtain an infinite set of eigenvalues $W_{\nu,l}(f, \delta)$, $\nu = 0, 1, \ldots$, for those solutions $R_{\nu,l}(\xi)$ that satisfy equation (8). A little thinking reveals that the dependence of the angular frequency on the quantum numbers is an artifact of the truncation of the power series by means of the condition (7) and it is expected to occur when looking for exact solutions to conditionally solvable problems [4][5] (and references therein). If one solves the eigenvalue equation (4) in a proper way such dependence does not take place.

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Since the eigenvalue equation (4) is not exactly solvable (contrary to what the authors appear to believe) we resort to the reliable Rayleigh-Ritz variational method that is well known to yield increasingly accurate upper bounds to all the eigenvalues of the Schrödinger equation [6] (and references therein). For simplicity, we choose the basis set of (unnormalized) functions \( \{ u_j(\xi) = \xi^{\gamma j+j} e^{-\xi^2/2}, \ j = 0, 1, \ldots \} \).

In order to test the accuracy of the variational results we also apply the powerful Riccati-Padé method [7]. As a suitable particular case we arbitrarily choose \( l = 0 \) and \( f^2 = 1 \) so that \( \gamma^2 = 1 \) and \( \alpha = 3 \). From the truncation condition (7) with \( n = 1 \) we obtain \( \theta = 2 \), \( W_{BF} = 6 \) and \( \delta = \pm \sqrt{6} \).

When \( \delta = \sqrt{6} \) (\( f = 1 \)) the two methods mentioned above yield \( W_{0,0} = 6 \), \( W_{1,0} = 9.805784090 \), and \( W_{2,0} = 13.66928892 \) for the three lowest eigenvalues. It is clear that the truncation method only yields the lowest eigenvalue and misses all the other ones. On the other hand, when \( \delta = -\sqrt{6} \) (\( f = -1 \)) the two methods yield \( W_{0,0} = 1.600357154 \), \( W_{1,0} = 6 \), \( W_{2,0} = 10.21072810 \). In this case the truncation method only gives us the second lowest eigenvalue and misses all the other ones. It is clear that the value of \( n \) in the truncation condition (7) is not related with the quantum number \( \nu \) used to arrange the eigenvalues in increasing order of magnitude.

The authors’ equation for the energy levels suggest that \( E_{n,l}(-f) = E_{n,l}(f) \); however, such reflection symmetry only applies to the particular energy levels stemming from the truncation condition (7). Present numerical calculations show that in general \( W_{\nu,l}(-f) \neq W_{\nu,l}(f) \).

Summarizing: the truncation condition (7) only provides one energy level \( W_{n,l} \) (or \( E_{n,l}^2 \)) for a particular value of \( \delta = \delta_{n,l} \) and misses all the other energy levels \( W_{\nu,l} \) for that value of \( \delta \). Besides, it fails to provide the energy levels for other values of that parameter. Consequently, the dependence of the oscillator frequency on the quantum numbers is a mere artifact of the truncation condition (7). This artificial condition is not necessary for the existence of bound
states that should satisfy the well-known (and widely more general) condition (8) [3].

References

[1] K. Bakke and C. Furtado, On the Klein-Gordon oscillator subject to a Coulomb-type potential, Ann. Phys. 355 (2015) 48-54.

[2] F. M. Fernández, Dimensionless equations in non-relativistic quantum mechanics, 2020. arXiv:2005.05377 [quant-ph].

[3] C. Cohen-Tannoudji, B. Diu, and F. Laloë, Quantum Mechanics, John Wiley & Sons, New York, (1977).

[4] A. DeSousa Dutra, Exact solutions of the Schrödinger equation for Coulombian atoms in the presence of some anharmonic potentials, Phys. Lett. A 131 (1988) 319-321.

[5] S. Bera, B. Chakrabarti, and T. K. Das, Application of conditional shape invariance symmetry to obtain the eigen-spectrum of the mixed potential \( V(r) = ar + br^2 + \frac{c}{r} + \frac{l(l+1)}{r^2} \), Phys. Lett. A 381 (2017) 1356-1361.

[6] F. L. Pilar, Elementary Quantum Chemistry, McGraw-Hill, New York, (1968).

[7] F. M. Fernández, Q. Ma, and R. H. Tipping, Tight upper and lower bounds for energy eigenvalues of the Schrödinger equation, Phys. Rev. A 39 (1989) 1605-1609.