TRACE HOMOMORPHISM FOR SMOOTH MANIFOLDS

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Abstract. Let $M$ be a closed connected smooth manifold and $G = \text{Diff}_0(M)$ denote the connected component of the diffeomorphism group of $M$ containing the identity. The natural action of $G$ on $M$ induces the trace homomorphism on homology. We show that the image of trace homomorphism is annihilated by the subalgebra of the cohomology ring of $M$, generated by the characteristic classes of $M$. Analogously, if $J$ is an almost complex structure on $M$ and $G$ denotes the identity component of the group of diffeomorphisms of $M$ preserving $J$ then the image of the corresponding trace homomorphism is annihilated by subalgebra generated by the Chern classes of $(M, J)$.

1. Introduction and the results

Let $G$ be any topological group acting on a topological space $X$ and $R$ any commutative ring. We define the trace homomorphism,

$$H_k(G, R) \times H_l(X, R) \xrightarrow{\text{tr}} H_{k+l}(X, R),$$

corresponding to this action as follows: if $\phi : U \to G$ and $\sigma : A \to X$ are cycles in $G$ and $X$ of degrees $k$ and $l$ representing classes $\upsilon$, $\alpha$, respectively, let $\text{tr}_{\sigma}(\upsilon, \alpha)$ be the class represented by the homology cycle $(u, a) \mapsto \phi(u)(\alpha)$, $(u, a) \in U \times A$. In 2003, it is proved in [1, 2] that the trace homomorphism of the Hamiltonian group of a closed symplectic manifold $(M, \omega)$ on the rational homology of $M$,

$$H_k(\text{Ham}(M, \omega), \mathbb{Q}) \times H_l(M, \mathbb{Q}) \xrightarrow{\text{tr}} H_{k+l}(M, \mathbb{Q}),$$

is trivial, for $k \geq 1$. Inspired by this result we prove the following smooth analog:

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Theorem 1.1. Let $M$ be a closed connected smooth manifold and $R$ denote the either field $\mathbb{Z}_2$ or $\mathbb{Q}$. Also let $P$ denote the subalgebra of the cohomology algebra $H^*(M, R)$, generated by the Stiefel-Whitney classes $w_i(M)$, if $R = \mathbb{Z}_2$, and the subalgebra generated by the Pontryagin classes $p_i(M)$ and the Euler class $e(M)$, if $R = \mathbb{Q}$. If $\text{Diff}_0(M)$ denotes the connected component of the diffeomorphism group of $M$ containing the identity then the image of the trace homomorphism

$$H_k(\text{Diff}_0(M), R) \times H_l(M, R) \xrightarrow{\text{tr}_u} H_{k+l}(M, R)$$

is in the annihilator of $P$, provided that $k \geq 1$.

The proof of the above result yields immediately the following almost complex analog:

Theorem 1.2. Assume that $M$ is a closed connected smooth manifold and $J$ is an almost complex structure on $M$. Let $P$ denote the subalgebra of the cohomology algebra $H^*(M, \mathbb{Q})$, generated by the Chern classes $c_i(M)$. If $\text{Diff}_0(M, J)$ denotes the identity component of the group of diffeomorphisms of $M$ preserving $J$, then the image of the trace homomorphism

$$H_k(\text{Diff}_0(M, J), \mathbb{Q}) \times H_l(M, \mathbb{Q}) \xrightarrow{\text{tr}_u} H_{k+l}(M, \mathbb{Q})$$

is in the annihilator of $P$, provided that $k \geq 1$.

1.1. Trace homomorphism on cohomology. For $R = \mathbb{Z}_2$ or $\mathbb{Q}$ we have $H^p(M, R) = \text{Hom}(H_p(M, R), R)$ and using this duality we may define trace homomorphism in cohomology: Let $u \in H_k(\text{Diff}_0(M), R)$ and define

$$\text{tr}_u^*: H^p(M, R) \to H^{p-k}(M, R)$$

by the formula $a \mapsto \text{tr}_u^*(a)$, $a \in H^p(M, R)$, where

$$\text{tr}_u^*(a) : H^{p-k}(M, R) \to R; \quad \text{tr}_u^*(a)(\alpha) = a(\text{tr}_u(u, \alpha)), \quad \alpha \in H^{p-k}(M, R).$$

Hence, the conclusions of Theorem 1.1 and of Theorem 1.2 can be written as $\text{tr}_u^*(P) = 0$, for all $u \in H_k(\text{Diff}_0(M), R)$, $k \geq 1$.

Suppose that $u \in H_k(\text{Diff}_0(M), R)$ is a spherical class. Using any cycle representing $u$ we can build a fiber bundle $M \to E \to S^{k+1}$, such that the connecting homomorphism in the Wang sequence corresponding to this bundle is nothing but the trace homomorphism:

$$\to H^{p-1}(E, R) \to H^{p-1}(M, R) \xrightarrow{\text{tr}_u^*} H^{p-k-1}(M, R) \to H^p(E, R) \to$$

It is well known that the connecting homomorphism in the Wang sequence is a derivation of degree $k$ (3). In other words, for any $x, y \in H^*(M, R)$,

$$\text{tr}_u^*(xy) = \text{tr}_u^*(x) \cdot y + (-1)^{\deg(x)} \cdot x \cdot \text{tr}_u^*(y).$$
On the other hand, for general $u$, since $\text{Diff}_0(M)$ is an $H$-space any rational homology class $u$ is a product of spherical classes (cf. see Section 5 of [1]) and therefore $tr_u$ is the composition of the trace homomorphisms corresponding to the spherical factors of $u$. Hence, we obtain the following result:

**Proposition 1.3.** Let $u \in H_k(\text{Diff}_0(M), \mathbb{Q})$, $k > 0$. For cohomology classes $x, y \in H^*(M, \mathbb{Q})$ such that $y \in P$ (hence $tr_u^*(y) = 0$) we have $tr_u^*(xy) = tr_u^*(x)y$. Moreover, if $\deg(x) < k$ then $tr_u^*(xy) = 0$.

The above proposition yields the following corollary:

**Corollary 1.4.** The natural map

$$tr^*: H_k(\text{Diff}_0(M), \mathbb{Q}) \to \text{hom}_P(H^*(M, \mathbb{Q}), H^{*-k}(M, \mathbb{Q}))$$

is a homomorphism, where we regard $H^*(M, \mathbb{Q})$ as a right module over its subalgebra $P$ and $\text{hom}_P(H^*(M, \mathbb{Q}), H^{*-k}(M, \mathbb{Q})$ denotes the group of $P$-modulo homomorphisms.

**Example 1.5.** Let $u$ be as in the complex analog of the above proposition, where $P$ is generated by the Chern classes of the almost complex manifold $(M, J)$ and $u$ belongs to $H_k(\text{Diff}_0(M, J), R)$. Assume that $(M, \omega)$ is a monotone closed symplectic manifold of dimension $2n$. So $[\omega]$ is a multiple of $c_1(M)$ and hence it is in $P$. Assume further that $M$ has the Hard Lefschetz Property, i.e.,

$$\cup [\omega]^r : H^{n-r}(M, \mathbb{C}) \to H^{n+r}(M, \mathbb{C})$$

is an isomorphism for any $r \geq 0$. So, if $b \in H^{n+r}(M, \mathbb{C})$ then $b = a [\omega]^r$ for some $a \in H^{n-r}(M, \mathbb{C})$ and hence

$$tr_u^*(b) = tr_u^*(a [\omega]^r) = tr_u^*(a) [\omega]^r.$$ 

In particular, $tr_u^*([\omega]^r) = 0$. It follows that, if $k > n$ then $tr_u^* = 0$.

2. Proof of the Theorem

To prove the above results we need to recall the definition and some basic properties of equivariant bundles: Let $G$ be any Lie group and $F \to E \xrightarrow{\pi} B$ a fiber bundle. If $G$ acts on both $E$ and $B$ such that the projection map $\pi$ is $G$-equivariant; i.e., $\pi(v \cdot g) = \pi(v) \cdot g$, for all $g \in G$ and $v \in E$, we say that the bundle is $G$-equivariant. Note that if $X$ is also a $G$-space and $f : X \to B$ is a $G$-equivariant map then the pullback bundle has an induced $G$-equivariant structure.

**Example 2.1.** i) Let $F \to E \xrightarrow{\pi} B$ be a $G$-equivariant fiber bundle, where the action of $G$ on $B$, and hence on $E$, is free. Taking quotients of both the total space and the base by $G$, we get another fiber bundle
\( F \to E/G \xrightarrow{\pi} B/G \), whose pullback via the quotient map \( p : B \to B/G \) is isomorphic to the bundle \( F \to E \xrightarrow{\pi} B \).

ii) Let \( M \) be a smooth manifold. Since any diffeomorphism of \( M \), \( \phi : M \to M \), extends to the tangent bundle \( \phi^*: T_\ast M \to T_\ast M \), we see that the tangent bundle is \( \text{Diff}(M) \)-equivariant, where \( \text{Diff}(M) \) is the group of all diffeomorphisms of \( M \).

Proof of Theorem 1.1. Let \( G \) denote the group \( \text{Diff}_0(M) \), the group of diffeomorphism of \( M \) isotopic to the identity, and \( \sigma : A \to M \) be a cycle in \( M \) representing any given class \( \alpha \) of degree \( l \). Since the base field is either \( \mathbb{Z}_2 \) or \( \mathbb{Q} \) we may assume that \( A \) is a closed smooth manifold and \( \sigma : A \to M \) is a smooth map. Consider the trace map

\( tr : A \times G \to M, \ (a, g) \mapsto \sigma(a) \cdot g, \ 	ext{for all} \ (a, g) \in A \times G. \)

To prove the theorem it suffices to show that \( tr^*(v) = 0 \), for any \( v \in P \) of degree \( l + k \).

Note that \( G \) acts on \( A \times G \) by right multiplication on the second factor, which makes the trace map \( G \)-equivariant. By the above example the tangent bundle \( T_\ast M \to M \) is \( G \)-equivariant and hence the pullback bundle \( tr^*(T_\ast M) \to A \times G \) is \( G \)-equivariant. Since the \( G \)-action on the base space \( A \times G \) is free this bundle is induced from the quotient bundle \( tr^*(T_\ast M)/G \to (A \times G)/G \), which is isomorphic to \( \sigma^*(T_\ast M) \to A \). In particular, by the naturality of characteristic classes \( tr^*(v) \) is the pullback of a class in \( H^{k+l}(A, R) \). However, \( H^{k+l}(A, R) = 0 \), because \( A \) is \( l \)-dimensional and \( k \geq 1 \). Hence, \( tr^*(v) = 0 \) and the proof finishes. \( \square \)

Remark 2.2. i) Note that the above proof works also for Theorem 1.2. Indeed more is true: Let \( G \) be a subgroup of \( \text{Diff}_0(M) \) and \( E \to M \) be a \( G \)-equivariant real or complex vector bundle. Then the analogous result to Theorem 1.1 holds for \( G \) and the subalgebra \( P \) of the cohomology algebra of \( M \), generated by the characteristic classes of \( E \).

ii) Another extension of the main theorem, suggested by Dieter Kotschick, to foliations is as follows: Assume that the smooth manifold \( M \) is foliated and let \( G \) be the subgroup of \( \text{Diff}_0(M) \) preserving the foliation. Then \( G \) acts as isomorphisms of both the tangent bundle \( \tau \) and the normal bundle \( \eta \) of the foliation, where we have \( T_\ast M = \tau \oplus \eta \). Then for this \( G \) we can replace the subalgebra \( P \) in Theorem 1.1 with the subalgebra generated by the characteristic classes of the two summands of the tangent bundle to \( M \). Note that this subalgebra is clearly bigger than the subalgebra generated by the characteristic classes of \( T_\ast M \) only. Of course, this is no surprise since \( G \) is generally much smaller than \( \text{Diff}_0(M) \).
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