T^2LR-Net: An Unrolling Network Learning Transformed Tensor Low-Rank Prior for Dynamic MR Image Reconstruction

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ABSTRACT

The tensor low-rank prior has attracted considerable attention in dynamic MR reconstruction. Tensor low-rank methods preserve the inherent high-dimensional structure of data, allowing for improved extraction and utilization of intrinsic low-rank characteristics. However, most current methods are still confined to utilizing low-rank structures either in the image domain or predefined transformed domains. Designing an optimal transformation adaptable to dynamic MRI reconstruction through manual efforts is inherently challenging. In this paper, we propose a deep unrolling network that utilizes the convolutional neural network (CNN) to adaptively learn the transformed domain for leveraging tensor low-rank priors. Under the supervised mechanism, the learning of the tensor low-rank domain is directly guided by the reconstruction accuracy. Specifically, we generalize the traditional t-SVD to a transformed version based on arbitrary high-dimensional unitary transformations and introduce a novel unitary transformed tensor nuclear norm (UTNN). Subsequently, we present a dynamic MRI reconstruction model based on UTNN and devise an efficient iterative optimization algorithm using ADMM, which is finally unfolded into the proposed T^2LR-Net. Experiments on two dynamic cardiac MRI datasets demonstrate that T^2LR-Net outperforms the state-of-the-art optimization-based and unrolling network-based methods.

1. Introduction

Dynamic magnetic resonance (MR) imaging is critical in several clinical applications, such as cardiac, perfusion, and vocal tract imaging. It captures more information than static MR imaging, resulting in a spatiotemporal three-dimensional MR data structure that aids in the identification of certain diseases, such as cardiovascular diseases. However, obtaining dynamic MR images with high spatiotemporal resolution within clinically acceptable scan time is still very challenging.

In clinical practice, imaging acceleration is achieved through undersampling k-space (Fourier domain). This compromises the Nyquist sampling theorem, leading to aliasing in the image domain. Therefore, effective algorithms are required for the reconstruction of undersampled data. Currently, sparsity in the wavelet [1], temporal Fourier [2], and total variation domain [3] has been employed for reconstruction, yielding satisfactory results. Additionally, due to the slow variations in the same organ, dynamic MR frames are, in fact, temporally correlated throughout the entire image sequence. Methods based on the low-rank Casorati matrix [4] can effectively utilize this temporal redundancy information for reconstruction. Furthermore, approaches combining sparsity and low rank have been proposed, demonstrating superior reconstruction performance compared to single-constraint methods. These methods primarily fall into two categories. One is methods employing joint low-rank and sparse constraints, such as k-t SLR [3]. The other involves separating low-rank and sparse components [5], decomposing dynamic MR images into a slowly varying background and a foreground part that is both small in proportion and exhibits noticeable motion. The mentioned methods above fall under the category of iterative optimization algorithms. They utilize algorithms such as ISTA [6] and ADMM [7] to solve the reconstruction model. However, these methods require manual tuning of hyperparameters and exhibit a relatively slow convergence speed.

Deep unrolling networks [8] have recently shown promising results in dynamic MR reconstruction [9–11]. These networks implement iterative optimization algorithms within the supervised framework of deep learning, eliminating the need for manual hyperparameter tuning. This approach not only leverages physics-driven priors but also harnesses the powerful learning capabilities of neural networks. ISTA-Net [12] employs the convolutional neural network (CNN) to develop a transformed sparse prior for reconstruction, while SLR-Net [13] jointly utilizes the sparsity in the CNN transformed domain and the Casorati matrix low-rank prior. DCCNN [14] uses CNN to extract and leverage implicit deep image information for reconstruction. These methods have all achieved reconstruction performance surpassing those of traditional iterative optimization algorithms.

In recent years, the tensor low-rank prior [15] has emerged as a promising approach for accelerating dynamic MR imaging due to its ability to capture the inherent high-dimensional structure of the dynamic MR data. Nevertheless, unlike the unique definition of the matrix rank, the tensor rank has different definitions under the Candecomp/Parafac (CP) and Tucker decompositions. Yaman et al. [16] successfully applied the low-CP-rank approximation for cardiac MR image reconstruction. However, determining the tensor CP rank is computationally expensive and NP-hard. In contrast, the Tucker rank has been employed by formulating a reconstruction model regularized by the sum of the nuclear
norms (SNN) of the unfolding matrices [17]. The SNN and sparsity were incorporated in [18, 19] to further improve the reconstruction performance. However, the SNN is not the exact convex envelope of the sum of the Tucker ranks [20], leading to suboptimal reconstruction performance at high acceleration factors. Some studies have attempted to address this limitation by leveraging the sparsity constraint on the core tensor [21, 22], but this approach can be computationally complex. Recently, multilayer sparsity in tensor decomposition [23, 24] has also been proposed to leverage the low-rank properties of tensors.

Compared to CP and Tucker, the tensor singular value decomposition (t-SVD) [25] offers a simpler decomposition form that can be computed by the matrix SVDs of the frontal slices. The induced tensor nuclear norm (TNN) [26] is the exact convex envelope of the corresponding tensor rank. Due to the mathematical soundness of t-SVD, it has been widely used in tensor completion [26–29] and hyperspectral image restoration [28, 30] with excellent results. Specifically, t-SVD is based on Fast Fourier Transform (FFT), indicating that tensor low-rank methods under the t-SVD framework utilize low-rank representation information in the FFT domain, which, however, may be imprecise and limited. Therefore, finding a suitable transformed domain and utilizing tensor low-rank priors in that domain may lead to superior reconstruction results. F2TNN [28] employed the framelet transform on the spatial dimensions and the FFT on the temporal dimension. FTNN [31] adopted the temporal framelet transformation. S2NTNN [32, 33] utilized fully connected layers in an unsupervised framework to approximate the required tensor low-rank transform domain. More generally, t-SVD can be extended to one-dimensional invertible [34] or unitary [35] transformed versions to provide theoretical support for tensor low-rank priors in the corresponding transformed domain.

However, several challenges remain to be addressed. Firstly, most of the existing works related to transformed tensor low-rank priors are concentrated on tensor completion. Further research is needed to explore their applicability and effectiveness in dynamic MR reconstruction tasks. The data acquisition in MRI occurs in the frequency domain, while the reconstruction target is an image. This cross-domain characteristic fundamentally distinguishes it from tensor completion tasks. Secondly, current research only focused on one-dimensional transformations, possibly due to considerations of the complexity of designing artificial transforms. However, this lack of flexibility in the design of the transform may result in suboptimal reconstruction performance. Additionally, the choice of the transform is crucial. Existing models require handcrafted and predefined transformations. However, there is no direct relationship between the transformation and the final reconstruction accuracy, making it challenging to find an appropriate transformation.

In this paper, we propose an unrolling network learning Transformed Tensor Low-Rank prior for dynamic MR imaging, termed T²LR-Net. Specifically, We employ CNN to adaptively learn the tensor low-rank transformed domain from the dynamic MR dataset, aiming to maximize the reconstruction accuracy. Due to the supervised mechanism of the deep unrolling network, the reconstruction error serves as a loss function that directly guides the selection of the transformed domain. Furthermore, given the high-dimensional transformations of CNNs, the existing one-dimensional transformation-based t-SVD framework is no longer applicable. Therefore, we extend t-SVD to arbitrary multi-dimensional unitary transformations, providing theoretical support for the proposed T²LR-Net. Our contributions can be summarized as follows:

- We propose a novel transformed t-SVD framework based on arbitrary unitary transform, which can be multi-dimensional. This distinguishes our approach from existing t-SVDs that predominantly rely on one-dimensional transforms. The corresponding unitary transformed tensor nuclear norm (UTNN) is also developed and proved to be the convex envelope of the transformed tensor sum rank.

- We introduce a novel dynamic MR reconstruction model regularized by UTNN and develop an efficient UTNN-based iterative optimization algorithm using ADMM. All subproblems of ADMM can be solved analytically, according to the mathematical derivation of the proposed UTNN.

- We establish the T²LR-Net by unrolling the iterations of the UTNN-based iterative optimization algorithm into multiple iteration modules. In each module, the convolutional neural network (CNN) is used to learn an independent transformation to explore the specific intrinsic low-rank property of the data. The supervised mechanism in the deep unrolling network directly guides the learning of the transformed domain, leading to a significant improvement in reconstruction performance.

A preliminary version of this work was presented in IEEE ISBI 2023 [36]. However, this journal paper has undergone significant changes. We have further provided the mathematical derivations for UTNN and the tensor singular value thresholding method. Experiments involving multi-coil and prospective scenarios have been validated. Additionally, in the discussion section, we performed a comprehensive analysis of the interplay between CNN and tensor low-rank prior. It is worth noting that the authors of [37] also propose leveraging the tensor low-rank properties in the CNN-transform domain. However, they focus on video synthesis, whereas our work targets dynamic MR reconstruction. The differences arise from the inherent Fourier domain sampling characteristics of MRI, distinguishing it from video applications. In their paper, the t-SVD architecture fundamentally relies on one-dimensional transformations [38], yet they directly generalize this using a three-dimensional transformation like 2D CNN, lacking theoretical assurance. In contrast, our work extends t-SVD
to multidimensional transformations, covering both one-dimensional and three-dimensional transformations. We provide detailed proofs in the Appendix, thus offering robust mathematical theoretical support. Additionally, their method utilized the ISTA algorithm, while we design an efficient ADMM algorithm to solve the proposed reconstruction model.

The rest of this paper is organized as follows. Section 2 introduces the preliminaries and the generic dynamic MR image reconstruction models. Section 3 describes the proposed method, and the experiments and results are shown in Section 4. Discussion and conclusion are provided in Section 5 and 6, respectively.

2. Background

2.1. Notations and Preliminaries

In this paper, we denote tensors by Euler script letters, e.g., $\mathcal{X}$, matrices by bold capital letters, e.g., $\mathbf{X}$, vectors by bold lowercase letters, e.g., $\mathbf{x}$, and scalars by lowercase letters, e.g., $x$. For a 3-way tensor $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, we denote $\mathcal{X}^{(i)}$ as the $i$-th frontal slice, $\mathcal{X}(:, :, i), i = 1, 2, ..., n_3$. Operators or transformations are denoted by sans serif capital letters, e.g., $\mathbf{T}$, $\mathbf{F}$.

Lemma 1 (Unitary transform). A transform $\mathbf{T}$ is the unitary transform only if it preserves the Frobenius norm and inner product [39], i.e.,

$$\|\mathcal{X}\|_F = \|\tilde{\mathcal{X}}\|_F \text{ and } \langle\mathcal{X}, \mathcal{B}\rangle = \langle\tilde{\mathcal{X}}, \tilde{\mathcal{B}}\rangle,$$

where $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, $\mathcal{B} \in \mathbb{C}^{n_2 \times n_3 \times n_1}$, $\mathbf{T} : \mathbb{C}^{n_1 \times n_2 \times n_3} \rightarrow \mathbb{C}^{m_1 \times m_2 \times m_3}$, and $\tilde{\mathcal{X}} = \mathbf{T}(\mathcal{X})$.

The Frobenius norm of 3-way tensor is defined by $\|\mathcal{X}\|_F = \sqrt{\sum_{ijk} |a_{ijk}|^2}$. The inner product between $\mathcal{X}$ and $\mathcal{B}$ is calculated by the sum of inner products of all frontal slices, i.e., $\langle\mathcal{X}, \mathcal{B}\rangle = \sum_{i=1}^{n_3} \langle \mathcal{X}^{(i)}, \mathcal{B}^{(i)} \rangle$, and for two matrices $\mathbf{X}$ and $\mathbf{Y}$, $\langle\mathbf{X}, \mathbf{Y}\rangle = \text{trace}(\mathbf{X}^H\mathbf{Y})$. Due to the unitary property, $\mathcal{X}$ can also be obtained by applying the Hermitean transpose transform $\mathbf{T}^H : \mathbb{C}^{m_1 \times m_2 \times m_3} \rightarrow \mathbb{C}^{n_1 \times n_2 \times n_3}$ on $\tilde{\mathcal{X}}$, i.e., $\mathcal{X} = \mathbf{T}^H(\tilde{\mathcal{X}})$.

The block diagonal matrix based on the frontal slices of $\tilde{\mathcal{X}}$ is denoted as follows,

$$\tilde{\mathcal{X}} = \begin{pmatrix} \mathcal{X}_1^{(1)} & & \\ & \ddots & \\ & & \mathcal{X}_{n_3}^{(n_3)} \end{pmatrix},$$

which can be converted back into a tensor by the following fold operator,

$$\text{fold}(\tilde{\mathcal{X}}) = \tilde{\mathcal{X}}.$$ 

Thus, we can obtain the following relationship,

$$\|\mathcal{X}\|_F = \|\tilde{\mathcal{X}}\|_F = \|\tilde{\mathcal{X}}\|_F,$$

$$\langle\mathcal{X}, \mathcal{B}\rangle = \langle\tilde{\mathcal{X}}, \tilde{\mathcal{B}}\rangle >.$$ 

Definition 1 (T-product). The $T$-product of $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ and $\mathcal{B} \in \mathbb{C}^{n_2 \times n_3 \times n_1}$ based on a unitary transform $\mathbf{T}$ is a tensor $\mathcal{C} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, which can be expressed as

$$\mathcal{C} = \mathcal{X} \circ \mathbf{T} = \mathbf{T}^H \circ \text{fold}(\mathcal{X} \times \mathbf{T}),$$

where ‘$\circ$’ denotes the standard matrix product and $\circ$ is the composition operator.

Note that the $T$-product can be expressed as:

$$\mathcal{C}_T = \mathcal{X}_T \times \mathbf{B}_T,$$

which means that the $T$-product, in the transformed domain, is conducted by slice-wise matrix multiplication of each frontal slice of $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{B}}$ [34, 35].

Given $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$, the identity tensor is defined by $(\mathbf{I}_T)_{n_1} \circ \mathcal{X} = \mathcal{X} \circ (\mathbf{I}_T)_{n_2}$, where $(\mathbf{I}_T)_{n_1} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ is the identity tensor with the first two dimensions equal to $n_1$. The Hermitian transpose is denoted as $\mathbf{HH} = \mathbf{T}^H[\text{fold}(\mathcal{X}_T^H)] \in \mathbb{C}^{n_1 \times n_1 \times n_3}$, and the unit tensor $\mathcal{Q} \in \mathbb{C}^{n_1 \times n_1 \times n_3}$ is denoted as $\mathcal{Q}^H \circ \mathcal{Q} = \mathcal{Q} \circ \mathcal{Q}^H = \mathbf{I}_T$.

2.2. Problem Formulation

The data acquisition of dynamic MR imaging can be modeled as

$$\mathbf{b} = \mathbf{A}(\mathcal{X}) + \mathbf{n},$$

where $\mathcal{X} \in \mathbb{C}^{n_1 \times n_2 \times n_3}$ denotes distortion-free dynamic MR data, $n_x$, $n_y$ denote the spatial coordinates, $n_t$ is the temporal coordinate, $\mathbf{b} \in \mathbb{C}^m$ is the observed undersampled $k$-space data, $\mathbf{A} : \mathbb{C}^{n_1 \times n_2 \times n_3} \rightarrow \mathbb{C}^m$ is the encoding operator, and $\mathbf{n} \in \mathbb{C}^m$ is the Gaussian distributed white noise.

In the single-coil scenario, $\mathbf{A}$ is the Fourier sampling operator, i.e., $\mathbf{A} = \mathbf{F}_u$. In the multi-coil scenario, $\mathbf{A} = \mathbf{F}_u \circ \mathbf{C}$, where $\mathbf{C}$ denotes the coil sensitivity map and $\circ$ denotes the composition operator. Specifically in the Cartesian sampling cases, $\mathbf{F}_u = \text{SoF}$, where $\text{S} : \mathbb{C}^{n_2 \times n_3 \times n_1} \rightarrow \mathbb{C}^m$ denotes the sampling operator and $\mathbf{F} : \mathbb{C}^{n_1 \times n_2 \times n_3 \times n_4} \rightarrow \mathbb{C}^{n_2 \times n_3 \times n_4 \times n_1}(\mathbf{F} \circ \mathbf{F}(\mathcal{X}) = \mathbf{F} \circ \mathbf{F}(\mathcal{X} = \mathcal{X})$ is the unitary two-dimensional spatial Fourier transform on $x$ and $y$ axes.

The dynamic MR image reconstruction model is commonly formulated as the following optimization problem:

$$\mathcal{X}^* = \arg \min_{\mathcal{X}} \frac{1}{2} \|\mathbf{A}(\mathcal{X}) - \mathbf{b}\|_2^2 + \lambda \mathcal{R}(\mathcal{X}),$$

where $\|\mathbf{A}(\mathcal{X}) - \mathbf{b}\|_2^2$ is the data fidelity term that guarantees the consistency between the $k$-space of the reconstruction and the observation, $\mathcal{R}(\mathcal{X})$ is the regularization term, and $\lambda$ is the balancing parameter.

3. Methodology

3.1. Transformed t-SVD and UTNN Based on Arbitrary Unitary Transform

In this part, we propose a novel transformed t-SVD framework based on arbitrary unitary transform. We introduce the unitary transformed tensor nuclear norm (UTNN)
that corresponds to this framework. The UTNN is proved to be the convex envelope of the transformed tensor sum rank, providing a theoretical guarantee for utilizing UTNN to exploit the tensor low-rank prior.

**Theorem 1 (Unitary Transformed t-SVD).** The transformed t-SVD of \( X \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) based on arbitrary unitary transform \( T \) can be factorized as follows:

\[
X = U^* \ast_T S \ast_T Y^H, \tag{10}
\]

where \( U^* \in \mathbb{C}^{n_1 \times n_3} \) and \( Y \in \mathbb{C}^{n_2 \times n_3} \) are unitary tensors with respect to \( T \)-product, and \( S \) is the core tensor of the transformed t-SVD.

**Proof.** Given a three-dimensional tensor \( X \in \mathbb{C}^{n_1 \times n_2 \times n_3} \), to calculate the transformed t-SVD based on arbitrary unitary transform \( T \), we first apply the unitary transformation and obtain transformed tensor \( \hat{X}_T \). Then, for each frontal slice of \( \hat{X}_T \), we compute the matrix SVD,

\[
\hat{X}_T^{(i)} = \hat{U}_T^{(i)} \hat{S}_T^{(i)} \hat{Y}_T^{(i)H}, \tag{11}
\]

where \( \hat{U}_T^{(i)} \in \mathbb{C}^{n_1 \times n_1} \) and \( \hat{Y}_T^{(i)} \in \mathbb{C}^{n_2 \times n_2} \) are unitary matrices, and \( \hat{S}_T^{(i)} \in \mathbb{C}^{n_1 \times n_2} \) is a diagonal matrix with the singular values of \( \hat{X}_T^{(i)} \) on the diagonal. Then, by rearranging all the unitary matrices and diagonal matrices into three block diagonal matrices, we can obtain \( \hat{U}_T, \hat{V}_T, \) and \( \hat{S}_T \), respectively. Therefore, the above equation can be rewritten as,

\[
\hat{X}_T = \hat{U}_T \times \hat{S}_T \times \hat{V}_T^H. \tag{12}
\]

From equation (6) and (7), we can further convert this formula and obtain equation (10). The unitary properties of \( U^* \) and \( Y \) can be easily proved by verifying the definition of unitary tensor.

An illustration of the proposed transformed t-SVD factorization is shown in Figure 1, which has the similar structure as the other t-SVDs. Nevertheless, our main distinction from existing t-SVDs [25, 26, 34, 35] is that current t-SVDs are established based on one-dimensional invertible or unitary transformations, while our transformation is an arbitrary unitary transformation that can be multidimensional. If we set the transformation to be one-dimensional, it can degrade to existing t-SVDs. This generalization aligns with our intention to leverage CNN for learning transformation because CNN operates as a multi-dimensional transformation. The t-SVDs based on one-dimensional transformations are no longer applicable in this context.

The transformed multirank [35] of \( X \) can be defined as a vector \( r \in \mathbb{R}^{n_3} \) with its \( i \)-th entry being the rank of \( \hat{X}_T^{(i)} \). Then, we propose the transformed tensor sum rank as follows.

**Definition 2 (Transformed tensor sum rank).** The transformed sum rank, \( \text{rank}_{\text{sum}}(X) \), is defined as the sum of the tensor multirank, i.e.,

\[
\text{rank}_{\text{sum}}(X) = \sum_i r_i = \text{rank}(\hat{X}_T). \tag{13}
\]

![Figure 1: An illustration of the proposed transformed t-SVD factorization of a tensor with dimensions \( n_1 \times n_2 \times n_3 \).](image)

**Definition 3 (UTNN).** The unitary transformed tensor nuclear norm (UTNN) of \( X \in \mathbb{C}^{n_1 \times n_2 \times n_3} \) is defined as the nuclear norm of the block diagonal matrix in the transformed domain, i.e.,

\[
\|X\|_{T^*} = \|\hat{X}_T\|_* = \text{Tr}(S_T^2).
\]

Note that UTNN is derived from the dual norm of the transformed tensor spectral norm, and we have proved that UTNN is the convex envelope of the transformed tensor sum rank (13) on a unit ball of the transformed tensor spectral norm. See Appendix A for more detail.

In addition, it is worth noting that our proposed transformed t-SVD can be extended to tensors of order greater than three via a recursive approach similar to how [40] extended the t-product to tensors of order higher than three.

### 3.2. UTNN-based Iterative Optimization Algorithm

Based on UTNN of the tensor, we propose the UTNN regularized dynamic MR reconstruction model as

\[
\min \frac{1}{2} \|A(X) - b\|_2^2 + \lambda \|X\|_{T^*}. \tag{15}
\]

The above optimization problem can be converted into the following constraint problem,

\[
\min \frac{1}{2} \|A(X) - b\|_2^2 + \lambda \|Z\|_{T^*}, \quad \text{s.t.} \quad Z = X. \tag{16}
\]

The augmented Lagrangian function of the above optimization problem is formulated as,

\[
L(X, Z, \mathcal{W}) = \frac{1}{2} \|A(X) - b\|_2^2 + \lambda \|Z\|_{T^*} + \frac{\mu}{2} \|Z - X\|_F^2 - \mathcal{W}, \quad \text{s.t.} \quad Z = X. \tag{17}
\]

where \( \mathcal{W} \) is the Lagrangian multiplier and \( \mu > 0 \) is the penalty parameter. After a straightforward complete-the-squares procedure for the last two terms of (17), we have

\[
L(X, Z, \mathcal{L}) = \frac{1}{2} \|A(X) - b\|_2^2 + \lambda \|Z\|_{T^*} + \frac{\mu}{2} \|Z - X\|_F^2 - \mathcal{W} - \frac{\mu}{2} \|\mathcal{L}\|_F^2, \tag{18}
\]

where \( \mathcal{L} = \frac{\mathcal{W}}{\mu} \). The above can be efficiently solved with ADMM [41], which yields in solving the following subproblems:

\[
Z_n = \min \lambda \|Z\|_{T^*} + \frac{\mu}{2} \|Z - X_{n-1} - L_{n-1}\|_F^2. \tag{19}
\]
Figure 2: The proposed T²LR-Net framework. The T²LR-Net is an unrolling neural network that unrolls N (fixed) iteration of the algorithm (27) into N iteration modules. Each iteration module contains three blocks: the transformed tensor low-rank prior block $Z_n$, the reconstruction block $X_n$, and the multiplier update block $L_n$. The transformed tensor low-rank prior block is incorporated with the CNN and the hyperparameters are learned through the training process. The number above each color block represents the current channel count. The first row of the figure shows the T²LR-Net framework, and the second row shows the detail of the three blocks.

$Z_n = T^H \circ \text{SVT}_{\lambda/\mu} \circ T(X_{n-1} + L_{n-1})$

where $\lambda$ denotes an all-one tensor and the division is an element-wise operation. Note that $S^H : \mathbb{C}^m \rightarrow \mathbb{C}^n$ denotes the Hermitian transpose of $S$, and $S^H(x)$ puts the element of the sampled vector $x$ into the sampling location of the Cartesian grid and fills the rest with zeros. Meanwhile, if the acquisition is non-Cartesian and in the multi-coil scenario, the solution can be replaced by a gradient descent step [11, 13], i.e.,

$X_n = Y_{n-1} - \beta \left[ (A^H \circ A + \mu) T(X_{n-1} + L_{n-1}) - (A^H(b) + \mu Z_n - \mu L_{n-1}) \right].$

Finally, we obtain the following iterative procedures:

$Z_n = T^H \circ \text{SVT}_{\lambda/\mu} \circ T(X_{n-1} + L_{n-1})$

$X_n = (A^H \circ A + \mu)^{-1} (A^H(b) + \mu Z_n - \mu L_{n-1})$

$L_n = L_{n-1} - \eta (Z_n - X_n)$

3.3. The Proposed Unrolling Network: T²LR-Net

In the traditional optimization-based methods, the reconstruction result is obtained by iteratively solving (26), and the hyperparameters $\lambda, \mu, \eta$ need to be selected empirically, which is usually time-consuming and un-robust. In addition, it is challenging to choose a suitable predefined unitary transform $T$ in a simple formation.

To address the aforementioned issues, we adopt the deep unrolling strategy and introduce the T²LR-Net. First, we generalize the iterative solution (26) into the following

$X_n = \begin{cases} 
1 \frac{1}{2} ||A(X) - b||^2 + \frac{\mu}{2} ||X_n - X_{n-1}||_F^2, \\
L_n = L_{n-1} - \eta (Z_n - X_n), 
\end{cases}$

where $\lambda$ denotes an all-one tensor and the division is an element-wise operation. Note that $S^H : \mathbb{C}^m \rightarrow \mathbb{C}^n$ denotes the Hermitian transpose of $S$, and $S^H(x)$ puts the element of the sampled vector $x$ into the sampling location of the Cartesian grid and fills the rest with zeros. Meanwhile, if the acquisition is non-Cartesian and in the multi-coil scenario, the solution can be replaced by a gradient descent step [11, 13], i.e.,

$X_n = X_{n-1} - \beta \left[ (A^H \circ A + \mu) T(X_{n-1} + L_{n-1}) - (A^H(b) + \mu Z_n - \mu L_{n-1}) \right].$

Finally, we obtain the following iterative procedures:
scheme:

\[
\begin{aligned}
Z_n & : Z_n = T_n^H \circ SVT_{\gamma_n} \circ T_n(X_{n-1} + L_{n-1}) \\
X_n & : X_n = (\gamma_n A^H A + 1)^{-1}(\gamma_n A^H b + Z_n - L_{n-1}) \\
L_n & : L_n = L_{n-1} - \eta_n(Z_n - X_n)
\end{aligned}
\]

(27)

The \( \tau \geq 0 \in \mathbb{R}_+ \) replaces the single real number \( \lambda/\mu \in \mathbb{R}_+ \) with a tensor singular value threshold vector. The \( \tau \)-th element \( \tau_i \) thresholds the \( i \)-th frontal slice of the tensor, making the thresholding more flexible. We replace \( 1/\mu \) with \( \gamma \) to avoid numerical instability. This is because, in the Cartesian sampling cases (24), if the parameter \( \mu \) is learned to be zero, the unsampled k-space position would result in \( 0 \), which is numerically unstable. The hyperparameters are denoted by a subscript \( n \) (e.g., \( \tau_n \)), indicating that these hyperparameters vary from iteration to iteration. Note that the unitary transform \( T_n \) differs across iterations to leverage different transformed tensor low-rank priors to enhance reconstruction performance.

The above iterative procedure (27) is then unrolled into the proposed \( T^2 \)LR-Net. The pseudocode of the \( T^2 \)LR-Net is shown in Alg. 1. The framework of the network is shown in Figure 2. \( T^2 \)LR-Net unrolls the iterative steps of the UTNN-based optimization algorithm (27) into \( N \) iteration modules. Each module contains three blocks corresponding to the three subproblems in (27), i.e., the transformed tensor low-rank prior block \( Z_n \), the reconstruction block \( X_n \), and the multiplier update block \( L_n \). The CNN is utilized in the \( Z_n \) block to adaptively learn the transformation from the dynamic MR image datasets, and the hyperparameters \( (\tau, \gamma, \eta) \) are automatically learned through the training process. The three blocks are described in detail as follows:

**Algorithm 1: \( T^2 \)LR-Net**

**Require:** \( \{T_n, \gamma_n, \tau_n, \eta_n : n = 1, \ldots, N\} \)

**Initialize:** \( \lambda_n = A^H b, \ Z_0 = 0, \ L_0 = 0 \)

for \( n = 1, \ldots, N \) do

**Transformed Tensor Low-rank Prior Block:**

\( Z_n = T_n^H \circ SVT_{\gamma_n} \circ T_n(X_{n-1} + L_{n-1}) \);

**Reconstruction Block:**

\( X_n = (\gamma_n A^H A + 1)^{-1}(\gamma_n A^H b + Z_n - L_{n-1}) \);

**Multiplier Update Block:**

\( L_n = L_{n-1} - \eta_n(Z_n - X_n) \)

end

**Return:** \( X_N \)

### 3.3.1. Transformed Tensor Low-rank Prior Block \( Z_n \)

In this block, a learned T-TSVT related to the \( Z_n \) subproblem in (27) (22) is embedded into the neural network.

\[
Z_n = T_n^H \circ SVT_{\gamma_n} \circ T_n(X_{n-1} + L_{n-1}) \\
\approx \bar{T}_n \circ SVT_{\gamma_n} \circ T_n(X_{n-1} + L_{n-1}).
\]

(28)

Specifically, the transform \( T_n \) and the Hermitian transpose transform \( T_n^H \) are learned by two different CNNs. Instead of the traditional SVT, a learned matrix SVT operator \([13]\) is incorporated to adaptively determine the threshold \( \tau_n \) of every frontal slice of the transformed image, i.e., the threshold vector \( \tau_n \). For a certain frontal slice \( X_T^{(i)} = T_n(X_{n-1} + L_{n-1}) = USV^H \), the learned matrix SVT is based on the following scheme:

\[
\begin{align*}
SVT_{\gamma_n}(X_T^{(i)}) &= USV^H \\
S &= ReLU(S - \tau_n) \\
\tau_n &= \text{sigmoid}(a_n) \cdot \text{max}(S)
\end{align*}
\]

In the above equation, \( \text{ReLU} \) denotes the rectifier linear units \([43]\), \( \text{sigmoid} \) denotes the sigmoid activate function, and \( a_n \) is set as a learnable network parameter with an initial value of -2.

Due to the complexity of determining the Hermitian transpose of a CNN-learned transform, we avoid strictly using the Hermitian transpose transform \( T_n^H \). Instead, we build another CNN with the same structure, denoted as \( \bar{T}_n \), to approximate \( T_n^H \). In addition, instead of imposing the unitary or inverse constraints on the two CNNs in the loss function \([12, 13]\), we allow them to adaptively learn priors from dynamic MR datasets. In this way, the network can utilize both explicit low-rank properties and implicit deep image priors to enhance reconstruction performance. We will provide a more detailed discussion in Section 5.1.

#### 3.3.2. Reconstruction Block \( X_n \)

This block corresponds to the \( X_n \) subproblem in (27) and generates the reconstruction results. We set \( \gamma_n = \text{ReLU}(\hat{\gamma}_n) \), where \( \hat{\gamma}_n \) is learnable with an initial value of 0.1.

#### 3.3.3. Multiplier Update Block \( L_n \)

This block corresponds to the \( L_n \) subproblem in (27) and is used to update the Lagrange multiplier. We define \( \eta_n = \text{ReLU}(\hat{\eta}_n) \), where \( \hat{\eta}_n \) is learnable with an initial value of 1.

### 3.4. Loss Function

We use the mean squared error (MSE) as the loss function of \( T^2 \)LR-Net, which is defined as

\[
\text{Loss} = \sum_{(\hat{X}, b)\notin \Omega} \|\hat{X} - \text{f}_\text{net}(b|\theta)\|_F^2.
\]

(30)

In the above equation, \( \Omega \) denotes the given training data, \( \hat{X} \) is a fully sampled ground-truth data, \( b \) is the undersampled k-space data, \( \text{f}_\text{net} \) denotes the output of the network, and \( \theta \) is the learnable parameters of the network, which include \( \tau_n, \gamma_n, \eta_n \), as well as the CNN-learned transformations \( T_n \) and \( \bar{T}_n \).
3.5. Implementation Details

Taking into account the trade-off between computational burden and reconstruction performance, we choose 15 iteration modules to compose the proposed $T^2$LR-Net. To facilitate training of the embedded CNN in $Z_n$ blocks, we split each input complex-valued data into two real-valued channels. The CNNs for learning $T_n$ and $\overline{T}_n$ consist of three convolutional layers. The first two layers have 16 channels followed by a ReLU operator. The third layer comprises 2 channels and does not use a ReLU activation function to avoid truncating the negative part of the output. The convolution kernels of all the convolutional layers are of the size $3 \times 3 \times 3$, and the stride is 1. He initialization [44] is used for the convolutional layers.

During training, we adopt a pseudo-random mask strategy instead of using a fixed sampling mask throughout the training process, as seen in previous training schemes [13, 14]. At each training step, we generate a random sampling mask of a particular type and acceleration (e.g., a radial sampling mask with 16 randomly selected lines). This approach allows the network to learn deep features more adaptively and avoid overfitting to a fixed sampling mask.

The model is implemented in the framework of TensorFlow [45]. The batch size is set as 1. The Adam optimizer [46] with parameters $\beta_1 = 0.9$, $\beta_2 = 0.999$ and $\epsilon = 10^{-8}$ is adopted, and the exponential decay learning rate [47] is used with an initial learning rate of 0.001 and a decay of 0.95. All the experiments are performed on a workstation with Intel Xeon W-2123 CPU and NVIDIA Tesla GV100 GPU (32 GB memory). The source code of our method is available at https://github.com/yhao-z/T2LR-Net.

3.6. Complexity and Convergence Analysis

Regarding the computational complexity of the model, assuming the size of the input tensor image $X$ is $H \times W \times T$ with $H > W$. For the $Z_n$ subproblem in (27), the total complexity of the CNNs (the transformation $T$ and its transpose) is $HWT \times 3^3 \times 16 \times (16 + 2 + 2) \times 2$, i.e., $O(HWT)$. Here, we substitute the kernel size and the number of channels from Subsec.3.5 to simplify the calculation. The cost of the SVT operator is dominated by tensor SVD, by linear operations and fast Fourier transforms, resulting in $O(HW \log(HW)T)$ computational complexity. The $L_n$ subproblem only includes linear operations, resulting in the cost of $O(HWT)$. Therefore, for $N$ iteration modules, the total computational complexity should be $O(NW^2HT)$.

Regarding the convergence of (27), since this algorithm shares the same form as the fundamental 2-block ADMM, its convergence can be guaranteed by Section 3.1 of reference [7]. Furthermore, in the case of single-coil Cartesian sampling, as each substep of ADMM has an exact analytical solution, assurance of the convergence rate of this algorithm to $O(1/K)$ can be obtained from references [48, 49], where $K$ denotes the number of iterations.

4. Experiments and Results

4.1. Dataset

We evaluate the $T^2$LR-Net using two cardiac cine MRI datasets. The first one is the open-access OCMR dataset [50], while the other, named TCMR, is established based on the work of Tsotsos et al. [51] and previously evaluated in [52].

The OCMR dataset contains 78 fully sampled raw data collected on 3T Siemens MAGNETOM Prisma machine and 126 fully sampled raw data on 1.5T Siemens Avanto and Sola machine. We choose 68 of the 3T data for training, and select the rest 10 of the 3T data and the other 12 data from the 1.5T Siemens Sola machine for testing to evaluate the robustness and generalization ability of the proposed network. We crop the training data into the size of $144 \times 112 \times 16 \times (x \times y \times t)$, and the strides along three dimensions are 15, 15 and 7, respectively. Finally, we obtain 1099 training data and 22 uncropped test data. The coil sensitivity maps are computed by ESPiRiT [53]. Both single-coil and multi-coil data were used for the experiments. The raw multi-coil data of each frame were combined using the coil sensitivity maps estimated by ESPiRiT to produce single-coil complex-valued data.

The TCMR dataset is comprised of short-axis cardiac cine MR images from 33 subjects. Although the images may not match a realistic experiment as the raw data, they are still useful as an auxiliary dataset to evaluate the efficacy of methods. The images are scanned with a GE Genesis Signa MR scanner using the FIESTA scan protocol. A total of 399 slices with each slice of $256\times256\times20$ are collected. We crop the images into the size of $128 \times 128 \times 16$ with the strides of 15, 15, and 7 along $x$, $y$, and $t$ dimensions, respectively. Finally, we obtain an augmented dataset consisting of 3782 cardiac cine MR images. We use 3392 images from 30 subjects for training and 390 images from the rest 3 subjects for testing.

4.2. Experimental settings

To demonstrate the efficacy of the $T^2$LR-Net in dynamic MR cine imaging, we conduct comprehensive experiments involving single-coil/multi-coil scenarios, retrospective/prospective reconstruction, different acceleration factors, and different sampling patterns, as shown in Table 1. In retrospective experiments, two commonly used sampling patterns are considered, i.e., the Cartesian pseudo-radial sampling pattern [3] and the variable density random sampling (Vds) pattern. Three different sampling cases are considered for each sampling pattern, where the radial sampling pattern involves 8, 16, and 30 lines, and the Vds pattern involves 8, 10, and 12 acceleration rates (acc = the total number of pixels / the number of the sampled pixels). In prospective experiments, we evaluate the proposed $T^2$LR-Net using the VISTA sampling pattern [54] since the undersampled raw data from OCMR are acquired by VISTA.

We use signal-to-noise ratio (SNR) and structural similarity (SSIM) [55] to evaluate the results. In each scenario, we have carefully tuned the parameters of all the compared
optimization-based methods to ensure optimal performance. The unrolling network-based methods are retrained properly using the corresponding network for 50 epochs to guarantee fair comparisons. Figure 3 shows the training and test loss curves, as well as the SNR curves, for the proposed T^2LR-Net under the 10-fold Vds pattern on the OCMR dataset. The training and test loss curves under other sampling patterns are similar and are omitted here for brevity. It can be observed that the training has converged after 50 epochs, and there is no degradation in the reconstructed SNR for the test set, indicating that there is no overfitting.

![Figure 3: The training and test loss curves of the proposed T^2LR-Net.](image)

### 4.3. Retrospective Experiments in single-coil scenario

We compare the reconstruction results of the proposed T^2LR-Net with four state-of-the-art (SOTA) optimization-based methods (TNN [26], F2TNN [28], k-t SLR [3] and SNNTV [18]) and two unrolling network-based methods (DCCNN [14], SLR-Net [13]). Note that TNN and F2TNN are based on the t-SVD framework with predefined transforms, and these two methods are originally applied in tensor completion tasks. We adapt them to the dynamic MR image reconstruction by ourselves.

Figure 4 presents the reconstruction results for a test image from the OCMR dataset using the pseudo-radial sampling pattern [3] with 16 lines, and Figure 5 shows the results of OCMR under the variable density random sampling pattern with the 8-fold acceleration. We report the quantitative metrics of different methods under six different sampling scenarios on the OCMR and TCMR datasets in Table 2 and Table 3, respectively. The visualization results demonstrate that our network can provide clearer edges, finer textures, and lower errors. Quantitative analysis results indicate that the T^2LR-Net achieves the highest SNR and SSIM metrics. Moreover, compared to SOTA unrolling networks, it possesses the fewest parameters and comparable training and inference time.

The TNN and F2TNN methods utilize the tensor low-rank priors in predefined transformed domains. However, artificially designed transformations may not be suitable for dynamic MR images, lacking flexibility and consequently leading to suboptimal reconstruction results. Both k-t SLR and SNNTV methods are composite approaches that combine low-rank characteristics and total variation priors, achieving better results than sole low-rank methods. However, their computational complexity significantly increases, and the required convergence time significantly increases. This poses challenges for practical clinical applications. DCCNN and SLR-Net, as SOTA deep unfolding structures, have demonstrated satisfying reconstruction results. However, DCCNN relies on CNNs to extract implicit image priors, lacking physical motivation and thus facing challenges in achieving efficient and accurate training. While SLR-Net combines low-rank and sparse composite priors and utilizes them in the deep unrolling framework, its utilization of matrix low-rank prior disrupts the inherent tensor structure of dynamic MR images. Our proposed network, starting from tensor theory and leveraging CNNs to extract the tensor low-rank features, fully exploits the inherent high-dimensional structure of dynamic MR images, resulting in optimal reconstruction performance.

Additionally, we compared three SOTA transform-based TNN methods: FTNN [31], S2NTNN [32], and TNN-data [35]. These methods employ one-dimensional transformations on the temporal dimension: FTNN uses a framelet transformation, S2NTNN employs a fully connected network to learn the transformation in an unsupervised manner, and TNN-data first obtains the reconstruction result using the FFT-TNN method, then utilizes the SVD of the reconstructed image to obtain a data-driven transformation. Simultaneously, we compared the results between the same transformation (T^2LR-Net-shared, achieved by sharing the parameters of CNNs between different iteration modules) and different transformations (proposed) within multiple iteration modules in T^2LR-Net. The reconstruction results for the five mentioned methods under radial-16 sampling on the OCMR test data (same as in Fig.4) are presented in Tab.4. The table also includes the average and standard deviation of SNR and SSIM for both T^2LR-Net-shared and T^2LR-Net across the entire test dataset. The comparison reveals that our proposed CNN-learned transform-based TNN, with its advantages of multi-dimensional transformation and supervised learning, effectively adapts to the task of dynamic MR reconstruction. In contrast, other transform-domain TNN methods, while achieving good results in tensor completion, struggle to accommodate the demands of MRI reconstruction. This challenge arises due to the frequency-domain data acquisition in MRI, introducing difficulties in reconstruction across different domains, whereas tensor completion techniques often only consider the image domain. The decreased reconstruction accuracy in T^2LR-Net-shared suggests that
### Figure 4:
The OCMR reconstruction results of different methods under the pseudo-radial sampling pattern [3] with 16 lines in the single-coil scenario. The first row shows the reconstruction images of the different methods, and the second row shows the enlarged view of the heart regions marked by the orange box. The first image in the third row displays the sampling mask, while the other images show the reconstruction error maps w.r.t. the different methods. The fourth row and the fifth row show the x-t images indicated by the blue dot line and their reconstruction error maps. The reconstruction SNRs of different methods are listed in parentheses.

### Figure 5:
The OCMR reconstruction results of different methods under the variable density random sampling pattern with the 8-fold acceleration in the single-coil scenario.
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Table 5

The quantitative results in noisy cases under radial-16 sampling on the OCMR test data (same as in Fig.4). The results are shown in SNR/SSIM. The table also includes the average and standard deviation of SNR and SSIM for both SLR-Net and $T^2$LR-Net across the entire test dataset.

|            | TNN | k-t SLR | SLR-Net | $T^2$LR-Net |
|------------|-----|---------|---------|-------------|
| noisy-30dB | 15.24/0.818 | 17.43/0.927 | 18.60/0.936 | 21.00/0.967 |
|            |      |         | 18.22±1.46/0.930±0.019 | 20.47±1.80/0.956±0.020 |
| noisy-20dB | 11.07/0.654 | 12.94/0.757 | 15.57/0.890 | 17.43±1.34/0.928±0.021 |

Table 6

The quantitative metrics of different methods under two sampling cases in multi-coil scenario on the OCMR dataset (Mean ± Standard deviation).

|            | DCCNN | SLR-Net | $T^2$LR-Net |
|------------|-------|---------|-------------|
| Radial-16  |       |         |             |
| SNR        | 15.258±1.131 | 17.676±1.840 | 20.941±2.271 |
| SSIM       | 0.938±0.027  | 0.952±0.015  | 0.969±0.013  |
| Vds-10     |       |         |             |
| SNR        | 13.509±1.146 | 15.456±1.671 | 16.597±1.580 |
| SSIM       | 0.925±0.025  | 0.942±0.019  | 0.947±0.019  |

sensitivity maps by ESPIRiT, which increases the difficulty of achieving accurate reconstructions [56].

Figure 6: The OCMR reconstruction results of different methods under the Vds pattern with 10 acceleration rate in multi-coil scenario.

4.5. Prospective Experiments

We use an 18-coil real undersampled data from the OCMR dataset, acquired on a 1.5T Siemens Avanto scanner, with the VISTA [54] sampling pattern at 8-fold acceleration. To facilitate the prospective experiments, we retrain unrolling networks using the same sampling pattern and acceleration rate on OCMR dataset for 50 epochs. We show the reconstruction results of different methods in Figure 7. DCCNN produces blurry results and the reconstruction results of SLR-Net show some artifacts, while our proposed $T^2$LR-Net produces sharper edges and clearer texture details. It is noticed that the reconstruction performance of all networks seems to degrade compared to the retrospective experiments. This is because VISTA only undersamples the phase encoding lines while keeping the frequency encoding lines fully sampled. The incoherence between samples is less pronounced compared to pseudo-radial or Vds sampling, which poses challenges in reconstruction.

Figure 7: Prospective results of different methods. We use the 18-coil undersampled raw data with the VISTA sampling pattern of the OCMR dataset. The x-t images indicated by the blue dot line are shown in the second row.

5. Discussion

In this subsection, we discuss the efficacy of the transformed tensor low-rank prior and also study the necessity of the symmetric constraint on the CNN-learned transforms. All the networks in this section are trained using a pseudo-radial sampling mask with 16 lines for 50 epochs on the OCMR dataset.

5.1. The Efficacy of The Proposed Transformed Tensor Low-rank Prior: A Study of Two Important Indicators

To evaluate the effectiveness of the transformed tensor low-rank ($T^2$LR) prior, we analyze two key indicators on the $T^2$LR-Net. The first indicator, $\Phi_n$, represents the number of singular values that fall below the thresholds of the $n$-th iteration module. It is defined as follows:

$$\Phi_n = \sum_{i=1}^{n} \sum_{m=1}^{n_{\text{slice}}} \mathbb{I}(\sigma_{n,i}^m < \tau_{n,i})$$

where $\sigma_{n,i}^m$ is the $m$-th singular value of the $i$-th frontal slice $T_n(\mathcal{X}_{n-1} + \mathcal{L}_{n-1})^{(i)}$ of the $n$-th iteration module, $\tau_{n,i}$ is the corresponding threshold, and $\mathbb{I}$ is the indicator function.
The second indicator, $\text{INV}_n$, measures the invertibility of the two transforms learned by the CNNs in the $n$-th iteration module, and it is expressed as

$$\text{INV}_n = \| \hat{T}_n \circ T_n(\mathcal{X}_{n-1}) - \mathcal{X}_{n-1} \|_F. \quad (32)$$

Note that we use this indicator as a relaxed measurement of the Hermitian symmetry of the transforms.

We evaluate these indicators using a test image from the OCMR dataset and plot their values across iteration modules in Figure 8. Note that the blue y-axis regarding $\Phi_n$ in the figure is reversed and the value of indicator $\Phi_n$ is labeled on the corresponding curve. The two curves exhibit a remarkable degree of similarity. The results show that the T$^2$LR prior is given more emphasis in the first four iteration modules because of the significantly high $\Phi_n$ values. In contrast, $\text{INV}_n$ values are relatively small, which is consistent with the Hermitian symmetric requirement of the transforms $T_n$ and $\hat{T}_n$ in the Equation (28). In the last eight iteration modules, the low-rank prior is no longer used since $\Phi_n$ becomes zero, which renders the singular value thresholding (SVT) operator in (28) invalid. Consequently, the transformed tensor low-rank block degenerates into a cascade of two CNNs that learn implicit and complex priors beyond low rank to enhance the reconstruction performance. The increasing $\text{INV}_n$ values imply that the symmetric relation between the two CNN-learned transforms is broken. The T$^2$LR-Net exploits the powerful learning capability of CNNs in the last iteration blocks. The iteration modules in the middle combine information from both the low-rank prior and the CNN-learned implicit prior to achieve the reconstruction.

We also display the reconstruction results from the iteration modules in Fig.9, where $\mathcal{X}_0$ is the input aliased image, and $\mathcal{X}_i$ related to (27) is the output of the $i$th iteration module. The results reveal that the first four iteration modules effectively remove aliasing by utilizing the transformed tensor low-rank. The middle modules further preserve the edges and texture details, while the last eight modules adaptively suppress noise and smooth the tissues through the flexible learning CNN.

In summary, our proposed T$^2$LR-Net not only utilizes the transformed tensor low-rank prior but also incorporates the CNN-learned flexible and implicit prior. This is achieved by adaptively changing $\Phi_n$. The balance between the transformed low-rank and the implicit CNN-learned deep image prior is adjusted through the training process and adapted to the dataset.

5.2. The Necessity of the Symmetric Constraint on the CNN-learned Transforms

In line with the approaches of SLR-Net [13] and ISTA-Net [12], we explore the impact of incorporating an inverse constraint (as a relaxed version of the unitary constraint) for the two CNNs in the loss function. The combined loss function is reformulated as:

$$\text{Loss} = \sum_{(\mathcal{X},b)\in\Omega} \| \hat{\mathcal{X}} - f_{\text{cnn}}(b|\theta) \|_F^2 + \zeta \sum_{n=1}^{N} \| \hat{T}_n \circ T_n(\mathcal{X}_{n-1}) - \mathcal{X}_{n-1} \|_F^2. \quad (33)$$

We retrain the network with varying $\zeta$ values and report the reconstruction SNRs on the OCMR test dataset. The T$^2$LR-Net that we proposed is a specific case where $\zeta = 0$. The results, presented in Figure 10, show that the average SNR increases from 17.17 dB to 22.27 dB as $\zeta$ decreases from 1E-1 to 1E-4. The best average SNR of 22.44 dB is achieved with $\zeta = 0$.

We observe that despite adopting the relaxed invertibility constraint instead of Hermitian symmetry, the performance worsens with an increasing weight of the invertibility constraint in the loss function. As discussed in Section 5.1, if $T_n \circ \hat{T}_n(\mathcal{X}) = \mathcal{X}$ strictly holds, the CNNs in the iteration modules where $\Phi_n$ equals 0 (i.e., the ones where the SVT...
Effect of the number of iteration modules. The SNR values are shown in the form of Mean ± Standard deviation.

| num | 5  | 10 | 15  | 20  |
|-----|----|----|-----|-----|
| SNR(dB) | 19.38 ± 1.78 | 21.70 ± 2.38 | 22.44 ± 2.58 | 22.71 ± 2.66 |

Table 8

Effect of different SVT strategies. The quantitative metrics are shown in the form of Mean ± Standard deviation.

|                | SNR   | SSIM  |
|----------------|-------|-------|
| HardSVT-10% thres | 21.41±2.18 | 0.960±0.020 |
| SoftSVT-10% thres  | 22.29±2.56 | 0.965±0.020 |
| SoftSVT-learned thres | 22.44±2.60 | 0.968±0.020 |

5.3. Effect of the number of iteration modules

We retrain the networks with 5, 10, 15 and 20 iteration modules and compare the reconstruction results on the test dataset of OCMR to investigate the effect of the number of iteration modules. The results are shown in Tab.7. From the results, it is observed that the improvement in SNR slows down after more than 15 iterations. Therefore, considering the trade-off between efficiency and accuracy, we chose 15 iteration modules. However, it is important to note that without considering efficiency, a higher number of iteration modules would lead to higher reconstruction accuracy.

5.4. Effect of different SVT strategies

We compared the impact of different SVT strategies on our proposed unrolling network. We reiterate that our network utilizes a learnable SVT strategy, specifically applying soft thresholding to all singular values of the tensor with the learnable thresholds. Here, we contrast it with two other SVT strategies: fixed truncation with Soft/Hard thresholding on the last 10% of singular values. The comparison results are presented in Table 8. From the results, it is evident that the soft thresholding strategy better leverages low-rank information, achieving superior reconstruction compared to the hard thresholding strategy. Additionally, the learned soft thresholding further enhances reconstruction performance by increasing the flexibility and better utilizing low-rank information.

6. Conclusion and Future Work

In this paper, we introduced T²LR-Net, a novel unrolling reconstruction network that exploits the transformed tensor low-rank prior for dynamic MR imaging. By extending the conventional t-SVD to a transformed version based on an arbitrary unitary transform, we outlined the mathematical definition of the UTNN and proposed a novel dynamic MR reconstruction model regularized by UTNN. The ADMM-based iterative algorithm was adopted to ensure efficient model solutions. By unfolding the iterative steps of the ADMM algorithm, we proposed a deep unrolling network that comprised numerous iteration modules, each employing a CNN to extract and exploit the tensor low-rank representation within the learned transformed domain. Both retrospective and prospective experimental results illustrated that T²LR-Net surpassed existing optimization-based methods and unrolling network-based methods, thus proving its superior performance.

Appendix

A. Derivation of UTNN

According to the derivation in [26], the transformed tensor spectral norm w.r.t. \( T \), termed \( \| \mathbf{X} \|_T \), induced by the Frobenius-normed operator norm, is defined by the matrix spectral norm of \( \mathbf{X}_T \), i.e.,

\[
\| \mathbf{X} \|_T = \| \mathbf{X}_T \|.
\]

Then, according to the truth that the nuclear norm is the dual norm of the spectral norm, we define the UTNN as the dual norm of the transformed tensor spectral norm [26]. For any \( B \in \mathbb{C}^{n_1 \times n_2 \times n_3} \), by (5) and (A.1), we have

\[
\| \mathbf{X} \|_{T^*} = \sup_{\|B\|_T \leq 1} \langle \mathbf{X}, B \rangle = \sup_{\|\mathbf{B}_T\| \leq 1} < \mathbf{X}_T, \mathbf{B}_T > \leq \sup_{\|\mathbf{B}_T\| \leq 1} < \mathbf{X}_T, \mathbf{\hat{B}} > = \| \mathbf{X}_T \|_*.
\]

where equation (A.3) is from (5) and (A.1), equation (A.4) is because we relax the block diagonal matrix \( \mathbf{B}_T \) into an arbitrary matrix \( \tilde{B} \in \mathbb{C}^{n_1 \times n_2 \times n_3} \), equation (A.5) holds due to that the matrix nuclear norm is the dual norm of the matrix spectral norm. Then, we show the equality (A.4) holds and thus \( \| \mathbf{X} \|_{T^*} = \| \mathbf{X}_T \|_* \). If \( \mathbf{X} = \mathbf{U} *_{\tau} \mathbf{S} *_{\tau} \mathbf{Y}^H \), we find an \( \mathbf{B} = \mathbf{U} *_{\tau} \mathbf{Y}^H \), then we have,

\[
\langle \mathbf{X}, B \rangle = \langle \mathbf{U} *_{\tau} \mathbf{S} *_{\tau} \mathbf{Y}^H, \mathbf{U} *_{\tau} \mathbf{Y}^H \rangle = \langle \mathbf{X}_T, \mathbf{\bar{B}}_T \rangle = \langle \mathbf{U}_T \times \mathbf{S}_T \times \mathbf{\bar{V}}_T^H, \mathbf{U}_T \times \mathbf{\bar{V}}_T^H \rangle = \text{Tr}(\mathbf{S}_T^*) = \| \mathbf{X}_T \|_*
\]
\[ T^2 \text{LR-Net} \]

where equation (A.7) is from (7) (12), and equation (A.8) is from the unitary property of \( U' \) and \( Y \). (A.9) holds due to the fact that the matrix nuclear norm is calculated by the sum of singular values. Thus, we have the definition of the UTNN (Definition.3).

Please note that the matrix nuclear norm \( \| X \|_* \) is the convex envelope of the matrix rank \( \text{rank}(X) \) within the set \( \{ X \mid \| X \|_* \leq 1 \} \). Thus, from (13), (14) and (A.1), the UTNN \( \| X \|_{\text{UTNN}} \) is the convex envelope of the transformed tensor sum rank \( \text{ran}_{\text{sum}}(X) \) on a unit ball of transformed tensor spectral norm \( \| X \|_{T} \leq 1 \).

B. Derivation of Transformed Tensor Singular Value Thresholding

Based on the definition of UTNN, the transformed tensor singular value thresholding (T-TSVT) [26, 35] is derived as follows.

For any \( \tau > 0 \) and \( Y = U^* \tau S \tau V^H \in \mathbb{C}^{n_1 \times n_2 \times n_3} \), the closed solution of the following optimization problem is given by,

\[
\text{T-TSVT}_\tau(Y) = \arg \min_{X} \tau \| X \|_{T, \tau} + \frac{1}{2} \| X - Y \|_F^2. \tag{B.1}
\]

where \( \tau \) becomes the threshold of T-TSVT operator.

From (14) and (4), (B.1) can be converted into the following matrix optimization problem,

\[
T^{H} \circ \text{fold} \left( \arg \min_{X} \tau \| X \|_{T, \tau} + \frac{1}{2} \| X - Y \|_F^2 \right), \tag{B.2}
\]

where \( \sum = U_T \times S_T \times V_T^{H} \), and the closed solution of the inside matrix optimization problem is given by the matrix SVT operator [42] with the threshold \( \tau \).

\[
\text{SVT}_\tau(Y_T) = U_T \times S_T \tau \times V_T^{H}, \tag{B.3}
\]

For clearness, we omit the fold operator; and thus obtain,

\[
\text{T-TSVT}_\tau(Y) = T^{H} \circ \text{SVT}_\tau \circ T(Y). \tag{B.4}
\]

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References

[1] M. Lustig, D. Donoho, J. M. Pauly, Sparse MRI: The application of compressed sensing for rapid MR imaging. Magnetic Resonance in Medicine 58 (2007) 1182–1195.

[2] H. Jung, K. Sung, K. S. Nayak, E. Y. Kim, J. C. Ye, k-t FOCUSS: a general compressed sensing framework for high resolution dynamic MRI. Magnetic Resonance in Medicine 61 (2009) 103–116.

[3] S. G. Lingala, Y. Hu, E. DiBella, M. Jacob, Accelerated dynamic MRI exploiting sparsity and low-rank structure: k-t SLR, IEEE transactions on medical imaging 30 (2011) 1042–1054.

[4] Z.-P. Liang. Spatiotemporal imaging with partially separable functions, in: 2007 4th IEEE international symposium on biomedical imaging: from nano to macro, IEEE, 2007, pp. 988–991.

[5] B. Trémouilhac, N. Dikaios, D. Atkinson, S. R. Arridge, Dynamic MR image reconstruction—separation from undersampled (k, t)-space via low-rank plus sparse prior. IEEE transactions on medical imaging 33 (2014) 1689–1701.

[6] A. Beck, M. Teboulle. A fast iterative shrinkage-thresholding algorithm for linear inverse problems, SIAM journal on imaging sciences 2 (2009) 183–202.

[7] S. Boyd, N. Parikh, E. Chu, B. Peleato, J. Eckstein, et al., Distributed optimization and statistical learning via the alternating direction method of multipliers, Foundations and Trends® in Machine learning 3 (2011) 1–122.

[8] K. Gregor, Y. LeCun, Learning fast approximations of sparse coding, in: Proceedings of the 27th international conference on machine learning, 2010, pp. 399–406.

[9] H. K. Aggarwal, M. P. Mani, M. Jacob, MoDL: Model-based deep learning architecture for inverse problems, IEEE transactions on medical imaging 38 (2019) 394–405.

[10] C. Qin, J. Schlemper, J. Caballero, A. N. Price, J. V. Hajnal, D. Rueckert. Convolutional recurrent neural networks for dynamic MR image reconstruction, IEEE transactions on medical imaging 38 (2018) 280–290.

[11] W. Huang, Z. Ke, Z.-X. Cui, J. Cheng, Z. Qiu, S. Jia, L. Ying, Y. Zhu, D. Liang. Deep low-rank plus sparse network for dynamic MR imaging, Medical Image Analysis 73 (2021) 102190.

[12] J. Zhang, B. Ghanem, ISTA-Net: Interpretative optimization-inspired deep network for image compressive sensing, in: Proceedings of the IEEE conference on computer vision and pattern recognition, 2018, pp. 1828–1837.

[13] Z. Ke, W. Huang, Z.-X. Cui, J. Cheng, S. Jia, H. Wang, X. Liu, H. Zheng, L. Ying, Y. Zhu, et al., Learned low-rank priors in dynamic MR imaging, IEEE Transactions on Medical Imaging 40 (2021) 3698–3710.

[14] J. Schlemper, J. Caballero, J. V. Hajnal, A. N. Price, D. Rueckert. A deep cascade of convolutional neural networks for dynamic MR image reconstruction, IEEE Transactions on Medical Imaging 37 (2017) 491–503.

[15] T. G. Kolda, B. W. Bader, Tensor decompositions and applications, SIAM review 51 (2009) 455–500.

[16] B. Yaman, S. Weingärtner, N. Kargas, N. D. Sidirooulos, M. Akçakaya, Low-rank tensor models for improved multidimensional MRI: Application to dynamic cardiac \( r_1 \) mapping, IEEE transactions on computational imaging 6 (2019) 194–207.

[17] Y. Liu, T. Liu, J. Liu, C. Zhu, Smooth robust tensor principal component analysis for compressed sensing of dynamic MRI, Pattern Recognition 102 (2020) 107252.

[18] S. F. Roohi, D. Zonobi, F. A. Kassim, J. L. Jaremko, Multidimensional low rank plus sparse decomposition for reconstruction of under-sampled dynamic MRI, Pattern Recognition 63 (2017) 667–679.

[19] D. Liu, J. Zhou, M. Meng, F. Zhang, M. Zhang, Q. Liu. Highly undersampling dynamic cardiac MRI based on low-rank tensor coding, Magnetic Resonance Imaging 89 (2022) 12–23.

[20] B. Romera-Paredes, M. Pontil, A new convex relaxation for tensor completion, Advances in neural information processing systems 26 (2013).

[21] J. He, Q. Liu, A. G. Christodoulou, C. Ma, F. Lam, Z.-P. Liang, Accelerated high-dimensional MR imaging with sparse sampling using low-rank tensors, IEEE transactions on medical imaging 35 (2016) 2119–2129.
121. A. G. Christodoulou, J. L. Shaw, C. Nguyen, Q. Yang, Y. Xie, N. Wang, D. Li, Magnetic resonance multitasking for motion-resolved quantitative cardiovascular imaging, Nature biomedical engineering 2 (2018) 215–226.

122. J. Xue, Y. Zhao, S. Huang, W. Liao, J. C.-W. Chan, S. G. Kong, Multilayer sparsity-based tensor decomposition for low-rank tensor completion, IEEE Transactions on Neural Networks and Learning Systems 33 (2021) 6916–6930.

123. J. Xue, Y. Zhao, Y. Bu, J. C.-W. Chan, S. G. Kong, When laplacian scale mixture meets three-layer transform: A parametric tensor sparsity for tensor completion, IEEE Transactions on Cybernetics 52 (2022) 13887–13901.

124. M. E. Kilmer, C. D. Martin, Factorization strategies for third-order tensors, Linear Algebra and its Applications 435 (2011) 641–658.

125. C. Lu, J. Feng, Y. Chen, W. Liu, Z. Lin, S. Yan, Tensor robust principal component analysis with a new tensor nuclear norm, IEEE transactions on pattern analysis and machine intelligence 42 (2019) 925–938.

126. Z. Zhang, S. Aeron, Exact tensor completion using t-SVD, IEEE Transactions on Signal Processing 65 (2016) 1511–1526.

127. J.-L. Wang, T.-Z. Huang, X.-L. Zhao, T.-X. Jiang, M. K. Ng, Multidimensional visual data completion via low-rank tensor representation under coupled transform, IEEE Transactions on Image Processing 30 (2021) 3581–3596.

128. C. Lu, J. Feng, Y. Chen, W. Liu, Z. Lin, S. Yan, Tensor robust principal component analysis: Exact recovery of corrupted low-rank tensors via convex optimization, in: Proceedings of the IEEE conference on computer vision and pattern recognition, 2016, pp. 5249–5257.

129. H. Zeng, X. Xie, H. Cui, H. Yin, J. Ning, Hyperspectral image restoration via global L1−2, spatial–spectral total variation regularized local low-rank tensor recovery, IEEE Transactions on Geoscience and Remote Sensing 59 (2020) 3309–3325.

130. T.-X. Jiang, M. K. Ng, X.-L. Zhao, T.-Z. Huang, Framelet representation of tensor nuclear norm for third-order tensor completion, IEEE Transactions on Image Processing 29 (2020) 7233–7244.

131. Y.-S. Luo, X.-L. Zhao, T.-X. Jiang, Y. Chang, M. K. Ng, C. Li, Self-supervised nonlinear transform-based tensor nuclear norm for multidimensional image recovery, IEEE Transactions on Image Processing (2022).

132. Y. Luo, X.-L. Zhao, D. Meng, T.-X. Jiang, HLRTF: Hierarchical low-rank tensor factorization for inverse problems in multi-dimensional imaging, in: Proceedings of the IEEE/ICVF Conference on Computer Vision and Pattern Recognition, 2022, pp. 19303–19312.

133. E. Kernfeld, M. Kilmer, S. Aeron, Tensor-tensor products with invertible linear transforms, Linear Algebra and its Applications 485 (2015) 545–570.

134. G. Song, M. K. Ng, X. Zhang, Robust tensor completion using transformed tensor singular value decomposition, Numerical Linear Algebra with Applications 27 (2020) e2299.

135. Y. Zhang, P. Li, Y. Hu, Dynamic mri using learned transform-based tensor low-rank network (LT2LR-NET), in: 2023 IEEE 20th International Symposium on Biomedical Imaging (ISBI), 2023, pp. 1–4.

136. Y. Zhang, X.-Y. Liu, B. Wu, A. Wald, Video synthesis via transformed-based tensor neural network, in: Proceedings of the 28th ACM International Conference on Multimedia, 2020, pp. 2454–2462.

137. C. Lu, X. Peng, Y. Wei, Low-rank tensor completion with a new tensor nuclear norm induced by invertible linear transforms, in: Proceedings of the IEEE/ICVF Conference on Computer Vision and Pattern Recognition, 2019, pp. 5996–6004.

138. R. A. Horn, C. R. Johnson, Matrix analysis, Cambridge university press, 2012.

139. C. D. Martin, R. Shafir, B. LaRue, An order-p tensor factorization with applications in imaging, SIAM Journal on Scientific Computing 35 (2013) A474–A490.

140. M. V. Afonso, J. M. Bioucas-Dias, M. A. Figueiredo, Fast image recovery using variable splitting and constrained optimization, IEEE transactions on image processing 19 (2010) 2345–2356.

141. J.-F. Cai, E. J. Candès, Z. Shen, A singular value thresholding algorithm for matrix completion, SIAM Journal on optimization 20 (2010) 1956–1982.

142. X. Glorot, A. Bordes, Y. Bengio, Deep sparse rectifier neural networks, in: Proceedings of the fourteenth international conference on artificial intelligence and statistics, JMLR Workshop and Conference Proceedings, 2011, pp. 315–323.

143. K. He, X. Zhang, S. Ren, J. Sun, Delving deep into rectifiers: Surpassing human-level performance on imagenet classification, in: Proceedings of the IEEE international conference on computer vision, 2015, pp. 1026–1034.

144. M. Abadi, P. Barham, J. Chen, Z. Chen, A. Davis, J. Dean, M. Devin, S. Ghemawat, G. Irving, M. Isard, et al., Tensorflow: A system for Large-Scale machine learning, in: 12th USENIX symposium on operating systems design and implementation (OSDI 16), 2016, pp. 265–283.

145. D. P. Kingma, J. Ba, Adam: A method for stochastic optimization, arXiv preprint arXiv:1412.6980 (2014).

146. M. D. Zeiler, Adadelta: an adaptive learning rate method, arXiv preprint arXiv:1212.5701 (2012).

147. B. He, X. Yuan, On the o(1/n) convergence rate of the Douglas–rachford alternating direction method, SIAM Journal on Numerical Analysis 50 (2012) 700–709.

148. R. D. Monteiro, B. F. Svaiter, Iteration-complexity of block-decomposition algorithms and the alternating direction method of multipliers, SIAM Journal on Optimization 23 (2013) 475–507.

149. C. Chen, Y. Liu, P. Schniter, M. Tong, K. Zareba, O. Simonetti, L. Potter, R. Ahmad. OCMR (v1.0)-open-access multi-coil k-space dataset for cardiovascular magnetic resonance imaging, arXiv preprint arXiv:2008.03410 (2020).

150. A.Andreopoulos, J. K. Tsotsos, Efficient and generalizable statistical models of shape and appearance for analysis of cardiac MRI, Medical image analysis 12 (2008) 335–357.

151. H. Zheng, F. Fang, G. Zhang, Cascaded dilated dense network with two-step data consistency for MRI reconstruction, Advances in Neural Information Processing Systems 32 (2019).

152. M. Uecker, P. Lai, M. J. Murphy, P. Virtue, M. Elad, J. M. Pauly, S. S. Vasanawala, M. Lustig, ESPRIT—An eigenvalue approach to autocalibrating parallel MRI: Where sense meets grappa, Magnetic resonance in medicine 71 (2014) 990–1001.

153. R. Ahmad, H. Xue, S. Giri, Y. Ding, J. Craft, O. P. Simonetti, Variable density incoherent spatiotemporal acquisition (VISTA) for highly accelerated cardiac MRI, Magnetic resonance in medicine 74 (2015) 1266–1278.

154. Z. Wang, A. C. Bovik, H. R. Sheikh, E. P. Simoncelli, Image quality assessment: from error visibility to structural similarity, IEEE transactions on image processing 13 (2004) 600–612.

155. M. J. Muckley, B. Riemenschneider, A. Radmanesh, S. Kim, G. Jeong, J. Ko, Y. Jun, H. Shin, D. Hwang, M. Mostapha, et al., Results of the 2020 fastMRI challenge for machine learning mr image reconstruction, IEEE transactions on medical imaging 40 (2021) 2306–2317.

156. Z. Chen, J. Yao, G. Xiao, S. Wang, Efficient and differentiable low-rank matrix completion with back propagation, IEEE Transactions on Multimedia (2021).