On the number of bound states in QED$_3$

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October 30, 2018

Abstract

We establish the consistency conditions for two-particle non relativistic bound states for any three space-time dimensional quantum electrodynamics model (QED$_3$) exhibiting an attractive scattering potential of $K_0$-type, the Bessel-Macdonald function. Initially, we study the self-adjointness of the hamiltonian operator, then by using Setô-type estimates, an estimate number of two-quantum bound states is derived for any value of the angular momentum. In fact, this result in connection with the condition that guarantees the self-adjointness of the hamiltonian shows that there can always be a number of two-quantum bound states for any QED$_3$ model where a $K_0$-type attractive interaction emerges. To the best of our knowledge, this result has not yet been addressed in the literature.

1 Introduction

The quantum electrodynamics in three space-time dimensions (QED$_3$) has been drawn attention, since the works by Schonfeld, Deser, Jackiw and Templeton [1, 2], as a potential theoretical framework to be applied to quasi-planar condensed matter systems [3], namely high-$T_c$ superconductors [4, 5], quantum Hall [6], topological insulators [7], topological superconductors [8] and graphene [9, 10, 11]. Thenceforth, planar quantum electrodynamics models have been studied in many physical configurations: small (perturbative) and large (non perturbative) gauge transformations, abelian and non-abelian gauge groups, fermions families, even or odd under parity, compact space-times, space-times with boundaries, curved space-times, discrete (lattice) space-times, external fields and finite temperatures. In condensed matter systems, quasiparticles usually stem from two-particle (Cooper pairs), particle-quasiparticle (excitons) or two-quasiparticle (bipolarons) non relativistic bound states. Bearing in mind these issues together with the fact that there are QED$_3$ models in which, fermion-fermion, fermion-antifermion or antifermion-antifermion, scattering potentials – mediated by massive$^a$ scalars or vector mesons – can be attractive and of $K_0$-type (a Bessel-Macdonald function) [5, 13], we propose to study the Schrödinger equation in three space-time dimensions by using the modified Bessel function of the second kind, $K_0$, as the interaction radial potential. In this work we prove the “smallness” of this potential relative to the free hamiltonian operator $H_0$, in the sense of Kato, implying the self-adjointness of the hamiltonian operator $H = H_0 + V$, where $V(r) = -\alpha K_0(\beta r)$ is the attractive two-particle scattering potential, with $\text{Dom}(H) = \text{Dom}(H_0)$. Then, by using Setô-type estimates [14], we obtain an upper limit for the number of two-quantum bound states

$^a$Otherwise, if the mediated quanta were massless, the interaction potential would be a logarithm-type (confining) potential [12].
for any value of the angular momentum. Consequently, this result in conjunction with the condition that assures the self-adjointness of the Hamiltonian implies that there can ever be a nonvanishing number of two-quantum bound states for any $K_0$-type attractive interaction potential. To the best of our knowledge, this result has not yet been addressed in the literature. This corroborates the well-known fact that in two space dimensions arbitrary weak potentials always possess at least one bound state [15, 16].

2 Non-relativistic planar quantum electrodynamics

The Hamiltonian operator modelling any non-relativistic planar quantum electrodynamics models with $K_0$-type attractive scattering potential reads

$$ H = H_0 + V = -\frac{\hbar^2}{2\mu} \Delta(x) - \alpha K_0(\beta \|x\|) , \quad (2.1) $$

where $\mu$ is the reduced mass and $\alpha$ is the coupling parameter taken to be, without the loss of generality, non-negative. The constants $\alpha$ and $\beta$ shall depend on some model parameters, like coupling constants, characteristic lengths, mass parameters or vacuum expectation value of a scalar field.

Taking into account that the two-quantum interaction potential $K_0(\beta \|x\|)$ depends only on $\|x\|$, the distance among the two interacting quanta, in order to estimate the number of two-particle bound states (in Section 4) we shall introduce polar coordinates $(r = \|x\|, \theta)$, so that

$$ H^2(\mathbb{R}^2) = H_0^2((0, \infty); r dr) \otimes L^2(S^1, d\theta) , $$

where $S^1$ is the usual unit circle in $\mathbb{R}^2$. Let $D$ be the set of all functions that are linear combinations of products $\Psi(r) \Theta(\theta)$ with $\Psi \in L^2((0, \infty); r dr)$ and $\Theta \in L^2(S^1, d\theta)$. Then, by the separation of variables for the two-dimensional Schrödinger equation associated to the two-particle pairing states

$$ \left[ -\frac{\hbar^2}{2\mu} \Delta - \alpha K_0(\beta \|x\|) \right] \Psi(x) = E \Psi(x) , \quad x \in \mathbb{R}^2 \quad (E: \text{energy}) , $$

into radial and angular pieces, where the radial part takes the form

$$ \left[ -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) + \frac{\hbar^2 m^2}{2\mu r^2} - \alpha K_0(\beta r) \right] \Psi(r) = E \Psi(r) , \quad (2.2) $$

while the angular part reads

$$ \frac{d^2 \Theta(\theta)}{d\theta^2} = -m^2 \Theta(\theta) . \quad (2.3) $$

The operator $d^2/d\theta^2$, with domain $C_0^\infty(S^1)$, is essentially self-adjoint. Its eigenvectors $\Theta_m(\theta) = (2\pi)^{-1/2} e^{im\theta}$, with $m \in \mathbb{Z}$, constitute an orthonormal basis of $L^2(S^1, d\theta)$.

Let $\Omega_m$ denote the subspace spanned by $\Theta_m$ and $L_m = L^2((0, \infty); r dr) \otimes \Omega_m$. Then,

$$ L^2(\mathbb{R}^2) = \bigoplus_{m \in \mathbb{Z}} L_m . $$

If $1_{m}$ is the identity operator on $\Omega_m$, the restriction of Hamiltonian operator $H$ to $D_m = D \cap L_m$ is given by $H|_{D_m} = H_m \otimes 1_{m}$, with

$$ H_m = -\frac{\hbar^2}{2\mu} \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) + \frac{\hbar^2 m^2}{2\mu r^2} - \alpha K_0(\beta r) . \quad (2.4) $$

\[2\]
In terms of a function $\Phi$ defined by
\[ \Phi(r) = r^{1/2} \Psi(r) , \tag{2.4} \]
$H_m$ is expressed as
\[ H_m = -\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2} + \left( \frac{\hbar^4 (m^2 - 1/4)}{2\mu r^2} - \alpha K_0(\beta r) \right) . \tag{2.5} \]
This reduces the problem of the Hamiltonian operator, $H$, in two dimensions in a problem of the Hamiltonian operator, $H_m$, in one dimension, with an effective potential given by:
\[ V_{\text{eff}}(r) \overset{\text{def}}{=} \frac{\hbar^2 (m^2 - 1/4)}{2\mu r^2} - \alpha K_0(\beta r) , \quad V(r) = -\alpha K_0(\beta r) , \]

**Remark 1.** According to Ref.[14, Lemma 2.1], the wave function (2.4) of a bound state with negative energy, $E < 0$, behaves as
\[ \Phi(r) = \begin{cases} O(r^{m+\frac{1}{2}}) & \text{for } r \to 0 \\ O(e^{-kr}) & \text{for } r \to \infty ; \quad k = \frac{\sqrt{-2\mu E}}{\hbar} . \end{cases} \]
Therefore such a function $\Phi$ belongs $\mathcal{H} = L_2((0, \infty); |V(r)|dr)$, as well as to $L_2((0, \infty); dr)$.

**Remark 2.** From expression $V(r) = -\alpha K_0(\beta r)$ we see that $\alpha$ has energy dimension and gives us an energy scale for the interaction among the two particles. In turn, the parameter $\beta$ has inverse length dimension, thus fixing a length scale, an interaction range, which is related to the mass of the boson-mediated quantum ($M_b$) exchanged during the two particle scattering [13, 5]. This can be verified if we consider the Compton wavelength of the boson-mediated field, $2\pi \hbar (M_b c)^{-1}$, hence $\beta \sim M_b$. Also, if we take the constant, $\hbar^2 \beta^2 (2\mu)^{-1}$, which has energy dimension, together with the relation among $\beta$ and $M_b$, an energy scale is fixed as well. Thus, we introduce a dimensionless constant $C = 2\mu \alpha (\hbar \beta)^{-2}$ that gives a notion of how strong is the two quanta interaction ($\alpha$) when compared to the energy of the boson-mediated quantum ($M_b c^2$). By taking into account this analysis, we rewrite the effective potential in a most convenient way:
\[ v_{\text{eff}}(z) = \frac{(m^2 - 1/4)}{z^2} - CK_0(z) , \quad v(z) = -CK_0(z) , \tag{2.6} \]
where we define
\[ z = \beta r , \quad C = \frac{2\mu \alpha}{\hbar^2 \beta^2} . \]
Hence,
\[ v_{\text{eff}}(z) = \frac{2\mu}{\hbar^2 \beta^2} V_{\text{eff}}(z) . \]

## 3 Self-adjointness of the Hamiltonian operator $H_0 + V$

In this section, the preservation of self-adjointness of the free particle Hamiltonian $H_0$ under small symmetric perturbations is considered. In particular, the application of Kato-Rellich Theorem to the Hamiltonian (2.1) is discussed. This theorem is a cornerstone in the theory of self-adjointness for Hamiltonian operators $H_0 + V$. As a starting point, let us remember the following
Definition 3.1 (Kato’s Criterion). Suppose $A, B$ are two densely defined linear operators in $\mathscr{H}$. $B$ is called a Kato perturbation of $A$ if, and only if, $\text{Dom}(A) \subset \text{Dom}(B)$ and there are constants $0 \leq a < 1$ and $b(a) \in (0, \infty)$ such that

$$\|B\Psi\| \leq a\|A\Psi\| + b\|\Psi\|, \quad \forall \Psi \in \text{Dom}(A).$$

In this case, $B$ is said to be $A$-bounded.

Generally, in the Definition 3.1, $b$ must be chosen larger as $a$ is chosen smaller. In other words, we can increase $b$ if we decrease $a$, but we cannot take $a = 0$ for any finite $b$ unless $B$ is a bounded operator. For this reason, we have $b = b(a)$. The infimum (greatest lower bound) of the possible $a$ is called the relative bound of $B$ with respect to $A$.

The notion of a Kato perturbation is very effective in solving the problem of self-adjointness of the sum, under natural restrictions. The next result is the celebrated

Theorem 3.2 (Kato-Rellich Theorem). Suppose $A$ is a self-adjoint and $B$ is a symmetric operator in $\mathscr{H}$. If $B$ is a Kato perturbation of $A$, then the sum $A + B$ is self-adjoint on the domain $\text{Dom}(A)$.

Now the Kato-Rellich Theorem will be applied to small perturbations of the free particle hamiltonian, $H_0$. The domain

$$\text{Dom}(H_0) = \left\{ \Psi \in L^2(\mathbb{R}^n) \mid -\Delta \Psi(x) \in L^2(\mathbb{R}^n) \text{ in the sense of distributions} \right\}, \quad (3.1)$$

is discussed in details by Reed-Simon [17, Theorem IX.27].

Remark 3. The domain (3.1) is equivalent to the condition

$$\text{Dom}(H_0) = \left\{ \Psi \in L^2(\mathbb{R}^n) \mid \|k\|^2 \tilde{\Psi}(k) \in L^2(\mathbb{R}^n) \text{ in the sense of distributions} \right\}.$$

Moreover, $\text{Dom}(H_0) = \mathcal{H}_2(\mathbb{R}^n)$, which is exactly the Sobolev space of order 2. And more, as the free hamiltonian, $H_0$ is self-adjoint, then the powers, $H_0^m$, of the free hamiltonian are also self-adjoint, since $H_0^m = \mathcal{F}^{-1}\|k\|^m\mathcal{F}$. Therefore, the domain of $H_0^m$ is the Sobolev space of order $2m$. This immediately implies that a vector $\Psi \in L^2(\mathbb{R}^n)$ it is at $C^\infty(H_0) = \bigcap_m \text{Dom}(H_0^m)$ if, and only if, $\Psi \in C^\infty(\mathbb{R}^n)$ and $D^\kappa \Psi \in L^2(\mathbb{R}^n)$ for all $\kappa$.

Let us see how the Kato’s Criterion and the Kato-Rellich Theorem allows us to establish the self-adjointness of the operator of Schrödinger (2.1). First of all, taking into account Remark 2, we will rewrite the hamiltonian operator (2.1) as follows

$$H = \frac{\hbar^2 \beta^2}{2\mu}H',$$

where

$$H' = -\Delta(x) - CK_0(\|x\|), \quad (3.2)$$

Next, we shall apply the Kato’s Criterion and the Kato-Rellich Theorem with $A = H_0' = -\Delta$ and $B = -CK_0(\|x\|)$. We will prove the following

Theorem 3.3. The hamiltonian operator (3.2) in $L^2(\mathbb{R}^2)$ is self-adjoint on the domain $\text{Dom}(H_0') = \text{Dom}(-\Delta)$.

\(^b\)Throughout this article $\mathscr{H}$ is assumed to be a complex Hilbert space.
we obtain the estimate
\[ \| \Psi \|_\infty \leq \frac{1}{4\pi^{3/2}\lambda} (\lambda^2 \| \Psi \|_2 + \| \mathcal{H}_0' \Psi \|_2) . \]

**Proof.** Technically, the basic idea of the argument is to take an arbitrary constant \( \lambda > 0 \) and consider that for \( n \leq 3 \), for every \( \Psi \in \mathcal{D}(H_n) \subset L_2(\mathbb{R}^n) \), the function \( k \mapsto (\lambda^2 + \| \mathbf{k} \|^2)^{-1} \in L_2(\mathbb{R}^n) \). Moreover \( L_2(\mathbb{R}^n) \ni (\lambda^2 + \| \mathbf{k} \|^2) \hat{\Psi}(\mathbf{k}) = \mathcal{F}(\lambda^2 \Psi + \mathcal{H}_n' \Psi) \). Therefore, by the Hölder inequality,
\[ (\lambda^2 + \| \mathbf{k} \|^2)^{-1}(\lambda^2 + \| \mathbf{k} \|^2) \hat{\Psi}(\mathbf{k}) \in L_1(\mathbb{R}^2), \]
and
\[ \| \hat{\Psi} \|_1 = \int_{\mathbb{R}^2} d^2 \mathbf{k} \ (\lambda^2 + \| \mathbf{k} \|^2)^{-1}(\lambda^2 + \| \mathbf{k} \|^2)|\hat{\Psi}(\mathbf{k})| \leq \left( \int_{\mathbb{R}^2} d^2 \mathbf{k} \ (\lambda^2 + \| \mathbf{k} \|^2)^{-2} \right)^{1/2} \left( \int_{\mathbb{R}^2} d^2 \mathbf{k} \ (\lambda^2 + \| \mathbf{k} \|^2)^2 |\hat{\Psi}(\mathbf{k})|^2 \right)^{1/2}. \]

Note that, by using polar coordinates, we obtain
\[ \int_{\mathbb{R}^2} d^2 \mathbf{k} \ (\lambda^2 + \| \mathbf{k} \|^2)^{-2} = \frac{2\pi}{\lambda^2} \int_0^\infty dk \ k (1 + k^2)^{-2} = \frac{\pi}{\lambda^2} . \]

Now, using Minkowski inequality, we have
\[ \| \hat{\Psi} \|_1 \leq \frac{\pi^{1/2}}{\lambda} \| (\lambda^2 + \| \mathbf{k} \|^2) \hat{\Psi} \|_2 \leq \frac{\pi^{1/2}}{\lambda} (\lambda^2 \| \Psi \|_2 + \| \mathbf{k} \|^2 \| \hat{\Psi} \|_2) . \]

On the other hand, by the inverse Fourier transform [19],
\[ \Psi(x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} d^2 \mathbf{k} \ e^{-i\mathbf{k} \cdot \mathbf{x}} \hat{\Psi}(\mathbf{k}) , \]
we obtain the estimate \( \| \Psi \|_\infty \leq (2\pi)^{-2} \| \hat{\Psi} \|_1 \) well known. This implies that,
\[ \| \Psi \|_\infty \leq \frac{1}{4\pi^{3/2}\lambda} \left( \lambda^2 \| \hat{\Psi} \|_2 + \| \mathbf{k} \|^2 \| \hat{\Psi} \|_2 \right) = \frac{1}{4\pi^{3/2}\lambda} \left( \lambda^2 \| \Psi \|_2 + \| \mathcal{H}_0' \Psi \|_2 \right) , \]
since the Fourier transform is a unitary operator. This completes the prove. \( \square \)

**Proof of Theorem 3.3.** Firstly, we shall show that \( V(x) = -CK_0(\| \mathbf{x} \|) \in L_2(\mathbb{R}^2) \). Indeed, by using the Table of Integrals of Gradshteyn-Ryzhik [20], pg. 665, we obtain
\[ \| V \|_2 = \left( C^2 \int_{\mathbb{R}^2} d^2 \mathbf{x} \ K_0^2(\| \mathbf{x} \|) \right)^{1/2} = (2\pi)^{1/2}C \left( \int_0^\infty ds s K_0^2(s) \right)^{1/2} = \pi^{1/2}C . \]
Therefore, $V(x) = -\alpha K_0(\|x\|) \in L_2(\mathbb{R}^2)$. Obviously, this result also implies that if $\Psi \in L_2(\mathbb{R}^2)$, then $V\Psi \in L_2(\mathbb{R}^2)$.

Now, again, using the Hölder inequality, it follows that $\|V\Psi\|_2 \leq \|V\|_2 \|\Psi\|_\infty$. Thus, by Lemma 3.4, we obtain

$$\|V\Psi\|_2 \leq \frac{1}{4\pi^{3/2}} \lambda (\lambda^2 \|\Psi\|_2 + \|H_0'\Psi\|_2) = \frac{C}{4\pi \lambda} (\lambda^2 \|\Psi\|_2 + \|H_0'\Psi\|_2).$$

We define

$$a(C, \lambda) = \frac{C}{4\pi \lambda} \quad \text{and} \quad b(C, \lambda) = \frac{C\lambda}{4\pi}.$$

Since $\lambda$ is an arbitrary positive constant, then for sufficiently large $\lambda$ the factor $a(C, \lambda)$ is smaller than 1, since the constant $C$ is fixed by model. The latter proves that the potential $V(x) = -CK_0(\|x\|)$ is $H_0'$-bounded, so that Theorem 3.2 applies and proves the self-adjointness of the hamiltonian operators (3.2) – and, consequently, the self-adjointness of the hamiltonian operator (2.1). It should be stressed that whatever the value of $C$ – a dimensionless constant which is model parameter dependent – it is always possible to choose $\lambda$ such that $\lambda > \frac{C}{4\pi}$. As a conclusion, the hamiltonian operator (2.1) is self-adjoint for any model which presents a two-particle scattering potential of the type $V(x) = -CK_0(\|x\|)$.

4 Bounds on the number of two-quasiparticle bound states

In this section, we obtain an estimate for the number of two-quantum bound states. Firstly, note that for the case $m = 0$ (zero angular momentum), it is not difficult to see that the potential will be uniquely attractive, as we can see through a direct inspection of Eq.(2.6) (see also Fig.1).

The potential (2.6) behaves, qualitatively, as the Coulomb potential in three dimensional space, and if there are bound states in the model, these will probably appear for vanishing angular momentum ($m = 0$) state (or $s$-wave state as it is most known) since the potential is uniquely attractive. Note that if we keep the parameter $C$ fixed and increase the angular momentum in a unit, the centrifugal term which is positive gives a repulsive contribution to the effective potential and may even exceed the attractive term (see Fig.2).

On the other hand, a somewhat more careful analysis of the effective potential ($v_{eff}(z)$) function (2.6) suggests that by increasing the value of $C$, for a given value of $m$, the attractive contribution may be greater than the repulsive one (this is shown in Fig.3 for $C > 1$).
Figure 2: The effective potential $v_{\text{eff}}(z)$ (2.6) for $m = 1$ with the same values of $C$ as in Fig.1.

Figure 3: The effective potential $v_{\text{eff}}(z)$ (2.6) for $m = 1$ with a value of $C$ greater than the previous one for the case $C > 1$ in Fig.2.

Next we obtain the upper limit number of two-quasiparticle bound states for any value of the angular momentum, $m$. We start with the angular momentum $m > 0$ and use the following

**Theorem 4.1** (N. Setô [14], Theorem 3.2). For $2m + d - 2 \geq 1$, the number of bound states, $N^m_d$, produced by the potential $V$, satisfying the condition

$$\int_0^\infty dr r|V(r)| < \infty,$$

in the $m$-th wave state in the $d$-dimensional space satisfies the inequality

$$N^m_d < \frac{1}{2m + d - 2} \int_0^\infty dr r|V(r)| .$$

(4.1)

First of all, we have to be careful in the above estimate calculation due to dimensional consistency condition fulfilment, therefore we shall carry out the following substitution:

$$\int_0^\infty dr r|V(r)| \rightarrow \frac{2\mu}{\hbar^2} \int_0^\infty dr r|V(r)| .$$

Thus, we can evaluate the expression (4.1) appropriately for $d = 2$ and $m > 0$ (non vanishing angular momentum), obtaining:

$$N^m_2 < \frac{1}{2m} \frac{2\mu}{\hbar^2} \int_0^\infty dr r|V(r)| = \frac{1}{2m} \frac{2\mu\alpha}{\hbar^2 \beta^2} \int_0^\infty dz zK_0(z) = \frac{C}{2m} .$$
The solution of the last integral is displayed in [20], moreover, by analysing the result above we conclude that if \( C/2m < 1 \), which implies that \( C/2 < m \), there will be no bound states at all, as already conjectured through graphical analysis (see Fig.3) – for a fixed value of \( C \), provided \( m \leq C/2 \), the greater the angular momentum \( (m) \) the less the number of two-quantum bound states.

Let us now consider the vanishing angular momentum case, where \( m = 0 \). It shall be pointed out that for zero angular momentum the Theorem 4.1 does not apply. Therefore, we have to resort to the following

**Theorem 4.2** (N. Setô [14], Theorem 5.1). The number of bound states, \( N_2^0 \), produced by the potential \( V \), satisfying the condition

\[
\int_0^\infty dr \left( 1 + \left| \ln \frac{r}{R} \right| \right) |V(r)| < \infty ,
\]

in the 0-th wave state in the 2-dimensional space satisfies the inequality

\[
N_2^0 < 1 + \frac{1}{2} \int_0^\infty dr |V(r)| \left( \int_0^\infty ds s(1 + |\ln r/s|)|V(s)| \right) \int_0^\infty dr |V(r)| .
\]

We shall verify if the potential \( K_0 \) satisfies the condition (4.2). For this, again, based on dimensional consistency, we make the following substitution:

\[
\int_0^\infty dr \left( 1 + \left| \ln \frac{r}{R} \right| \right) |V(r)| \longrightarrow \frac{2\mu\alpha}{\hbar^2\beta^2} \int_0^\infty dz z(1 + |\ln z|)|V(z)| .
\]

Now we rewrite the last integral as follows:

\[
\frac{2\mu\alpha}{\hbar^2\beta^2} \int_0^\infty dz z(1 + |\ln z|)|V(z)| = C \left( \int_0^\infty dz z K_0(z) + \int_0^1 dz z \ln z K_0(z) + \int_1^\infty dz z K_0(z) \right) .
\]

Let us analyze each of the three integrals above separately:

1. The first integral is presented in [20], and it is equal to 1.

2. In the case of the third integral, we consider that

\[
\int_1^\infty dz z K_0(z) < \int_1^\infty dz z^2 K_0(z) = \frac{1}{2} ,
\]

since for \( z > 1 \), we have \( \ln z < z \). The integral above at the right-hand side was obtained from Ref.[20].

3. Finally, let us evaluate the second integral. Note that the function \( f(z) = z \ln z K_0(z) \), defined in the open interval \((0, 1)\) and is well behaved in this interval, in the sense that there are no singularities of any kind. For \( z = 1 \), \( f(1) = 0 \) due to the logarithmic term, letting us verify the behaviour of \( f \) when \( z \to 0 \). To do so, we will need the following fact:

\[
|K_0(z)| \sim \ln \frac{z}{2} \quad \text{as} \quad z \to 0 .
\]

Then,

\[
\lim_{z \to 0^+} f(z) \sim \lim_{z \to 0^+} z \ln z \ln \frac{z}{2} = 0 .
\]

Since all integrals are finite, it is proved that the potential \( V(r) = -\alpha K_0(\beta r) \) respects the condition (4.2).
Now, we have the endorsement to determine the limit for the number of bound states for the case when \( m = 0 \). Again, moving to a dimensionally consistent form, we rewrite the second piece of (4.3):

\[
C \begin{bmatrix} \frac{1}{2} \int_0^\infty dr K_0(r) \left( \int_0^\infty ds s \left( 1 + |\ln r/s| \right) K_0(s) \right) \\ \int_0^\infty dr K_0(r) \end{bmatrix}.
\]

From the integration in the variable \( s \), we obtain

\[
\int_0^\infty ds s \left( 1 + |\ln r/s| \right) K_0(s) = \gamma + 2K_0(r) + \ln \frac{r}{2},
\]

where \( \gamma \) is the Euler-Mascheroni constant.

By integrating the variable \( r \) (see Ref.[20]), we obtain

\[
\int_0^\infty dr K_0(r) \left( \gamma + 2K_0(r) + \ln \frac{r}{2} \right) = 1.
\]

Consequently

\[
C \begin{bmatrix} \frac{1}{2} \int_0^\infty dr K_0(r) \left( \int_0^\infty ds s \left( 1 + |\ln r/s| \right) K_0(s) \right) \\ \int_0^\infty dr K_0(r) \end{bmatrix} = \frac{C}{2}.
\]

In short, we have

\[
N_m^2 < \begin{cases} 1 + \frac{C}{2} & \text{for } m = 0 \\ \frac{C}{2m} & \text{for } m \geq 1 \end{cases}.
\]

Remark 4. Note that the constant \( C \) has a direct effect on the number of bound states. For \( C \in (0, 2) \) there will be a single bound state and this will be found only in the \( s \)-wave state, \( i.e. \), for vanishing angular momentum, \( m = 0 \) (see Ref.[16] on the discussion about \( s \)-wave bound states for free particle in planar systems).

5 Summary and perspectives

In this work we have: (i) demonstrated the self-adjointness of the non relativistic hamiltonian operator for any three space-time dimensional quantum electrodynamics model \( (\text{QED}_3) \) exhibiting an attractive scattering potential of the type \( V(r) = -\alpha K_0(\beta r) \); (ii) proved the existence of two-quantum bound states and; (iii) computed the upper limit number of two-quasiparticle bound states for any value of the angular momentum, \( m \). For future investigations, we aim: (i) pursue possible applications to two-dimensional materials such as high-\( T_c \) superconductors, graphene and topological insulators; (ii) to study computationally and analytically the dynamics and thermodynamics of two-dimensional electron gas interacting via the scattering potential \( V(r) = -\alpha K_0(\beta r) \), as well as verify possible phase transitions and compute critical parameters. On the other hand, from the theoretical point of view, the stability of such two-particle bound states is an issue to be analyzed, including: (iii) the relativistic kinematics, where \( \sqrt{-\Delta} \) replaces \( -\Delta \) in the kinetic energy; (iv) presence of applied magnetic fields and their interaction with the quanta spin; (v) models with kinetic energy described by the Dirac operator. All of these issues are in progress.
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[19] We are adopting the following convention for the Fourier transform:

\[
\mathcal{F}\Psi(k) = \hat{\Psi}(k) = \int_{\mathbb{R}^n} d^n k \, e^{i k \cdot x} \hat{\Psi}(k) ,
\]

\[
\mathcal{F}^{-1}\Psi(x) = \Psi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d^n x \, e^{-i k \cdot x} \hat{\Psi}(k) .
\]

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