The decomposition of global conformal invariants
I: On a conjecture of Deser and Schwimmer.

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Abstract

This is the first in a series of papers where we prove a conjecture of Deser and Schwimmer regarding the algebraic structure of “global conformal invariants”; these are defined to be conformally invariant integrals of geometric scalars. The conjecture asserts that the integrand of any such integral can be expressed as a linear combination of a local conformal invariant, a divergence and of the Chern-Gauss-Bonnet integrand.

In this paper we set up an iterative procedure that proves the decomposition. We then derive the iterative step in the first of two cases, subject to a purely algebraic result which is proven in [5, 7, 8].

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1 Introduction.

This is the first in a series of papers, [4, 5, 6, 7, 8], where we provide a rigorous proof of a conjecture of Deser and Schwimmer, originally formulated in [19]. This series is a continuation of the previous work of the author, [1, 2] which established the conjecture in a special case and developed some useful tools which we will use below.

The purpose of this introduction is to firstly provide a formulation of the conjecture, and then to give a very brief synopsis of some of the main ideas in the proof, followed by a more detailed outline of the present paper.

1.1 Formulation of the problem.

We start by recalling the conjecture of Deser and Schwimmer. Firstly, we recall a classical notion from Riemannian geometry, that of a “scalar Riemannian invariant”:

In brief, given a Riemannian manifold \((M,g)\), scalar Riemannian invariants are intrinsic, scalar-valued functions of the metric \(g\). More precisely:

**Definition 1.1** Let \(L(g)\) be a formal polynomial expression in the in the (formal) variables \(\partial^{(k)}_{r_1...r_k}g_{ij}, k \geq 0\) and \((\det g)^{-1}\) (here the indices \(r_1,...,r_k,i,j\) take values \(1,...,n\)). Given any coordinate neighborhood \(U \subset \mathbb{R}^n\) and any Riemannian metric \(g\) expressed in the form \(g_{ij}dx^i dx^j\) in terms of the coordinates \(\{x^1,...,x^n\} \in U\), let \(L^U_g\) stand for the function that arises by plugging in the values \(\partial^{(k)}_{r_1...r_k}g_{ij}, (\det g)^{-1}\) into the formal expression \(L(g)\). We say that \(L(g)\) is a Riemannian invariant of weight \(K\) if:

1. Let \(g, g'\) be two Riemannian metrics defined over neighborhoods \(U, U' \subset \mathbb{R}^n\), and let \(L^U_g, L^{U'}_{g'}\) be the scalar-valued functions defined over \(U, U'\) that we obtain by substituting \(g, g'\) into the formal expression \(L(g)\). Then we require that if \(g, g'\) are isometric via the map \(\Phi : U \rightarrow U'\) then \(L^U_g(x) = L^{U'}_{g'}(\Phi(x))\) for every \(x \in U\). (This property is called the intrinsicness property of \(L(g)\)).

To continue the proof of Proposition 3.1, let’s consider the case where \(s = \sigma\).

4 Proof of Proposition 3.1 in the easy case \(s = \sigma\).

4.1 Proof of Proposition 3.1 when \(s = \sigma\).
2. Let $g$ be a Riemannian metric defined over $U \subset \mathbb{R}^n$ and let $t > 0$. Let $g'$ be the Riemannian metric $t^2 \cdot g$. Let $L_g^U, L_{g'}^U$ be the scalar-valued functions defined over $U$ that we obtain by substituting $g, g'$ into the formal expression $L(g)$. We then require that $L_g^U(x) = t^K L_{g'}^U(x)$ for every $x \in U$. (We then say that $L(g)$ has weight $K$).

In view of the first property, a Riemannian invariant $L(g)$ assigns a well-defined, 1 scalar-valued function to any Riemannian manifold $(M, g)$. We next review a classical theorem which essentially goes back to Weyl, [30], which states that any scalar Riemannian invariant can be expressed in terms of complete contractions of covariant derivatives of the curvature tensor. To state this result precisely, let us recall some basic facts from Riemannian geometry:

Given a Riemannian metric $g$ defined over a manifold $M$, consider the curvature tensor $R^{ijkl}$ and its covariant derivatives $\nabla^{(m_1)}_r \cdots \nabla^{(m_s)}_r R^{ijkl}$ (these are thought of as $(0, m+4)$-tensors). This gives us a list of tensors defined over $M$.

A natural way to form intrinsic scalars out of this list of intrinsic tensors is by taking tensor products and then contracting indices using the metric $g^{ab}$:

Firstly we take a (finite) number of tensor products, say:

$$\nabla^{(m_1)}_r \cdots \nabla^{(m_s)}_r R^{ijkl} \otimes \cdots \otimes \nabla^{(m_s)}_r R^{ijkl},$$

thus obtaining a tensor of rank $(m_1 + 4) + \cdots + (m_s + 4)$. Then, we can repeatedly pick out pairs of indices in the above expression and contract them against each other using the metric $g^{ab}$. In the end we obtain a scalar. We will denote such complete contractions by $C(g)$. Observe that any such complete contraction will be a scalar Riemannian invariant of weight $-(m_1 + 2) \cdots + (m_s + 2)$.

Thus, taking linear combinations of complete contractions of a given weight $w$ we can construct local Riemannian invariants of weight $w$. We will denote such linear combinations by $\sum_{r \in \mathcal{R}} a_r C^r(g)$ (here $\mathcal{R}$ is the index set of the complete contractions, $C^r(g), r \in \mathcal{R}$ are the different complete contractions appearing and $a_r$ are their coefficients).

Now, a classical result in Riemannian geometry (essentially due to Weyl, [30]) is that the converse is also true: For any Riemannian invariant $L(g)$ there exists a (non-unique) linear combination of complete contractions in the form

$$\sum_{r \in \mathcal{R}} a_r C^r(g)$$

so that for every manifold $(M, g)$ the value of $L(g)$ is equal to the value of the linear combination $\sum_{r \in \mathcal{R}} a_r C^r(g)$. Thus from now on we will be identifying Riemannian invariants with linear combinations of the form:

$$L(g) = \sum_{l \in \mathcal{L}} a_l C^l(g),$$

where each $C^l(g)$ is a complete contraction (with respect to the metric $g$) in the form:

1 (Meaning coordinate-independent).

2 A rigorous, if somewhat abstract, definition of a complete contraction appears in the introduction of [1].
\[ C^i(g) = \text{contr}(\nabla^{(m_1)}R \otimes \cdots \otimes \nabla^{(m_n)}R). \] (1.3)

(We do not write out the indices of the tensors involved for brevity). We also remark that a complete contraction is determined by the pattern according to which different indices contract against each other. Thus, for example, the complete contraction \( R_{abcd} \otimes R^{abcd} \) is different from the complete contraction \( R_{badc} \otimes R^{sbsd} \). The notation (1.3) of course does not encode this pattern of which index is contracting against which etc.

The Deser-Schwimmer conjecture: The conjecture deals with conformally invariant integrals of Riemannian scalars:

**Definition 1.2** Consider a Riemannian invariant \( P(g) \) of weight \(-n\) (\( n \) even). We will say that the integral \( \int_{M^n} P(g) dV_g \) is a “global conformal invariant” if the value of \( \int_{M^n} P(g) dV_g \) remains invariant under conformal re-scalings of the metric \( g \).

In other words, \( \int_{M^n} P(g) dV_g \) is a “global conformal invariant” if for any \( \phi \in C^\infty(M^n) \) we have \( \int_{M^n} P(e^{2\phi}g) dV_{e^{2\phi}g} = \int_{M^n} P(g) dV_g \).

In order to state the Deser-Schwimmer conjecture, we recall that a local conformal invariant of weight \(-n\) is a Riemannian invariant \( W(g) \) for which \( W(e^{2\phi}g) = e^{-n\phi}W(g) \) for every Riemannian metric \( g \) and every function \( \phi \in C^\infty(M^n) \). Furthermore, a Riemannian vector field \( T^i(g) \) is a linear combination \( T^i(g) = \sum_{q \in Q} a_q C^{q,i}(g) \), where each \( C^{q,i}(g) \) is a partial contraction (with one free index) in the form:

\[ C^{q,i}(g) = p\text{contr}(\nabla^{(m_1)}R \otimes \cdots \otimes \nabla^{(m_n)}R) \]

with \( \sum_{i=1}^n (m_i + 2) = n - 1 \). (Notice that for each such vector field, the divergence \( \text{div}_i T^i(g) \) is a Riemannian invariant of weight \(-n\)). Finally, we recall that \( \text{Pfaff}(R_{ijkl}) \) stands for the Pfaffian of the curvature tensor.

The Deser-Schwimmer conjecture [19] asserts:

**Conjecture 1** Let \( P(g) \) be a Riemannian invariant of weight \(-n\) such that the integral \( \int_{M^n} P(g) dV_g \) is a global conformal invariant. Then there exists a local conformal invariant \( W(g) \), a Riemannian vector field \( T^i(g) \) and a constant \( (\text{Const}) \) so that \( P(g) \) can be expressed in the form:

\[ P(g) = W(g) + \text{div}_i T^i(g) + (\text{Const}) \cdot \text{Pfaff}(R_{ijkl}). \] (1.4)

We recall the theorem we proved in [1] and [2]:

\[ \text{Recall the Chern-Gauss-Bonnet theorem which says that for any compact orientable Riemannian n-manifold} (M^n, g) \text{ we must have} \int_{M^n} \text{Pfaff}(R_{ijkl}) dV_g = \frac{2^n n!}{2(n-1)!} \chi(M^n). \]
Theorem 1.1. Let $\int_M P(g) dV_g$ be a global conformal invariant, where $P(g)$ is in the special form:

$$P(g) = \sum_{l \in L} a_l \text{contr}^l(R_{i_1 j_1 k_1 l_1} \otimes \cdots \otimes R_{i_2 j_2 k_2 l_2})$$

(i.e. each of the complete contractions above has $\frac{n^2}{2}$ undifferentiated factors $R_{ijkl}$). Then $P(g)$ can be expressed in the form:

$$P(g) = W(g) + (\text{const}) \cdot \text{Pfaff}(R_{ijkl}).$$

In this series of papers we will build on the work in [1], [2] to prove the whole Deser-Schwimmer conjecture:

Theorem 1.2. Conjecture 1 is true.

Related questions: The motivation for the above theorem, along with its implications to the notions of $Q$-curvature and renormalized volume have been discussed in the introduction of [1]. We refer to that paper for that discussion. We just wish to mention the recent work of A. Juhl [28], where he obtains new remarkable insight on the significance of $Q$-curvature, from an entirely fresh point of view. For now, we remark that an analogous problem arises in the context of understanding the asymptotic expansion of the Szegő kernel of strictly pseudo-convex domains in $\mathbb{C}^n$ (or alternatively of abstract CR-manifolds). In particular, the leading term of the logarithmic singularity of the Szegő kernel exhibits a global invariance which is very similar to the one we discuss here, see [27]. A further problem related to the Deser-Schwimmer conjecture arises in Kähler geometry: The problem is to understand the algebraic structure of the coefficients in the Tian-Yau-Zelditch expansion; this is a local version of the classical Riemann-Roch theorem regarding the dimension of the space of holomorphic sections of high powers of ample line bundles over complex manifolds, see [31] for a detailed discussion. The analogy with the Deser-Schwimmer conjecture lies in the fact that these coefficients are local invariants of a Kähler metric, whose integral over the base manifold remains invariant under Kähler deformations of the metric.

Finally, we wish to point out that alternative notions of “global conformal invariants” have been introduced and studied in the context of general relativity, see [12].

Before proceeding to outline the proof of Theorem 1.2 and to synopsize the present paper, we briefly digress in order to discuss the relationship of this work [3]–[8] with the study of local invariants of geometric structures (mostly Riemannian and conformal) and with certain questions motivated by index theory.

Broad Discussion: The theory of local invariants of Riemannian structures (and indeed, of more general geometries, e.g. conformal, projective, or CR) has a long history. As stated above, the original foundations of this field were laid in the work of Hermann Weyl and Élie Cartan, see [31][18]. The task of writing out
local invariants of a given geometry is intimately connected with understanding which polynomials in a space of tensors with given symmetries remain invariant under the action of a Lie group. In particular, the problem of writing down all local Riemannian invariants reduces to understanding the invariants of the orthogonal group.

In more recent times, a major program was laid out by C. Fefferman in [21] aimed at finding all scalar local invariants in CR geometry. This was motivated by the problem of understanding the local invariants which appear in the asymptotic expansions of the Bergman and Szegő kernels of strictly pseudoconvex CR manifolds, in similar way to which Riemannian invariants appear in the asymptotic expansion of the heat kernel; the study of the local invariants in the singularities of these kernels led to important breakthroughs in [11] and more recently by Hirachi in [26]. This program was later extended to conformal geometry in [22]. Both these geometries belong to a broader class of structures, the parabolic geometries; these are structures which admit a principal bundle with structure group a parabolic subgroup \( P \) of a semi-simple Lie group \( G \), and a Cartan connection on that principle bundle (see the introduction in [16]). An important question in the study of these structures is the problem of constructing all their local invariants, which can be thought of as the natural, intrinsic scalars of these structures.

In the context of conformal geometry, the first (modern) landmark in understanding local conformal invariants was the work of Fefferman and Graham in 1985 [22], where they introduced the ambient metric. This allows one to construct local conformal invariants of any order in odd dimensions, and up to order \( \frac{\dim}{2} \) in even dimensions. The question is then whether all invariants arise via this construction.

The subsequent work of Bailey-Eastwood-Graham [11] proved that this is indeed true in odd dimensions; in even dimensions, they proved that the result holds when the weight (in absolute value) is bounded by the dimension. The ambient metric construction in even dimensions was recently extended by Graham-Hirachi, [25]; this enables them to identify in a satisfactory manner all local conformal invariants, even when the weight (in absolute value) exceeds the dimension.

An alternative construction of local conformal invariants can be obtained via the tractor calculus introduced by Bailey-Eastwood-Gover in [10]. This construction bears a strong resemblance to the Cartan conformal connection, and to the work of T.Y. Thomas in 1934, [29]. The tractor calculus has proven to be very universal; tractor bundles have been constructed [16] for an entire class of parabolic geometries. The relation between the conformal tractor calculus and the Fefferman-Graham ambient metric has been elucidated in [17].

The present work, while pertaining to the question above (given that it ultimately deals with the algebraic form of local Riemannian and conformal invariants), nonetheless addresses a different type of problem: We here con-
sider Riemannian invariants $P(g)$ for which the integral $\int_{M^n} P(g) dV_g$ remains invariant under conformal changes of the underlying metric; we then seek to understand the possible algebraic form of the integrand $P(g)$, ultimately proving that it can be decomposed in the way that Deser and Schwimmer asserted. It is thus not surprising that the prior work on the construction and understanding of local conformal invariants plays a central role in this endeavor, in [4, 5]. We will explain in [4] how some of the local conformal invariants that we identify in $P(g)$ would be expected (given the properties of the ambient metric but also the insight obtained in [14]), while others are much less obvious.

On the other hand, our resolution of the Deser-Schwimmer conjecture will also rely heavily on a deeper understanding of the algebraic properties of the classical local Riemannian invariants. The fundamental theorem of invariant theory (see Theorem B.4 in [14] and also Theorem 2 in [1]) is used extensively throughout this series of papers. However, the most important algebraic tool on which our method relies are certain “main algebraic Propositions” presented in the present paper and [4, 5]. These are purely algebraic propositions that deal with local Riemannian invariants. While the author was led to led to these Propositions out of the strategy that he felt was necessary to solve the Deser-Schwimmer conjecture, they can be thought of as results of independent interest. The proof of these Propositions, presented in [6, 7, 8] is in fact not particularly intuitive. It is the author’s sincere hope that deeper insight will be obtained in the future as to why these algebraic Propositions hold.

Index Theory: Questions similar to the Deser-Schwimmer conjecture arise naturally in index theory; a good reference for such questions is [13]. For example, in the heat kernel proof of the index theorem (for Dirac operators) by Atiyah-Bott-Patodi [9], the authors were led to consider integrals arising in the (integrated) expansion of the heat kernel over Riemannian manifolds of general Dirac operators, and sought to understand the local structure of the integrand. In that setting, however, the fact that one deals with a specific integrand which arises in the heat kernel expansion plays a key role in the understanding of its local structure. This is true both of the original proof of Patodi, Atiyah-Bott-Patodi [9] and of their subsequent simplifications and generalizations by Getzler, Berline-Getzler-Vergne, see [13].

The closest analogous problem to the one considered here is the work of Gilkey and Branson-Gilkey-Pohjanpelto, [24, 15]. In [24], Gilkey considered Riemannian invariants $P(g)$ for which the integral $\int_{M^n} P(g) dV_g$ on any given (topological) manifold $M^n$ has a given value, independent of the metric $g$. He proved that $P(g)$ must then be equal to a divergence, plus possibly a multiple of the Chern-Gauss-Bonnet integrand if the weight of $P(g)$ agrees with the dimension in absolute value. In [15] the authors considered the problem of Deser-Schwimmer for locally conformally flat metrics and derived the same

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5 A summary of these is provided in subsection 1.3 below.
6 We note that the geometric setting in [9] is more general than the one in the Deser-Schwimmer conjecture: In particular one considers vector bundles, equipped with an auxiliary connection, over compact Riemannian manifolds; the local invariants thus depend both on the curvature of the Riemannian metric and the curvature of the connection.
decomposition (for locally conformally flat metrics) as in [24]. Although these two results can be considered precursors of ours, the methods there are entirely different from the ones here; it is highly unclear whether the methods of [24, 15] could be applied to the problem at hand.

### 1.2 Outline of the argument.

**A one-page outline of the argument:** The Deser-Schwimmer conjecture is proven by a multiple induction. At the roughest level, the induction works as follows: Express $P(g)$ as a linear combination of complete contractions:

$$P(g) = \sum_{l \in L} a_l C^l(g), \quad (1.6)$$

each $C^l(g)$ in the form (1.3).

The different complete contractions $C^l(g)$ appearing above can be grouped up into “categories” according to certain algebraic features of the tensors involved. Accordingly, we divide the index set $L$ into subsets $L^1, \ldots, L^T$ so that the terms indexed in the same index set $L^i$ belong to the same category (and vice versa), and $\bigcup_{t=1}^T L^t = L$; accordingly, we write:

$$P(g) = \sum_{t=1}^T \sum_{l \in L^t} a_l C^l(g). \quad (1.7)$$

We will also introduce a grading among the set of categories: A given category of complete contractions will be “better” or “worse” than any other given category. For future reference, the “best” category of complete contractions are the ones with $\frac{n}{2}$ factors.

Assume that in (1.7), for each pair $1 \leq \alpha < \beta \leq T$ the category of complete contractions indexed in $L^\beta$ is “worse” than the category of complete contractions indexed in $L^\alpha$. (Therefore, in particular the “worst” category of complete contractions in (1.7) is the category $\sum_{l \in L^T} a_l C^l(g)$).

The main step of our induction is to prove that unless the complete contractions $C^l(g), l \in L^T$ are in the “best” category 7 there exists a local conformal invariant $W(g)$ and a divergence of a vector field $\text{div}_i T^i(g)$ so that:

$$\sum_{l \in L^T} a_l C^l(g) - W(g) - \text{div}_i T^i(g) = \sum_{l \in L^{new}} a_l C^l(g), \quad (1.8)$$

where the complete contractions in the RHS of the above belong to categories that are all “better” than the category of $\sum_{l \in L^T} a_l C^l(g)$.

Observe that once this “main step” is proven, we can iteratively apply it to derive that there exists a local conformal invariant $\tilde{W}(g)$ and a divergence $\text{div}_i \tilde{T}^i(g)$ so that:

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7See the next page for more details.

8I.e. unless $P(g)$ is in the form (1.3).
\[
P(g) - \tilde{W}(g) - \text{div}_i \tilde{T}_i(g) = \tilde{P}(g), \quad (1.9)
\]
where \(\tilde{P}(g)\) is a linear combination of terms with \(\tilde{\mathcal{H}}\) factors. Furthermore, \(\int_M \tilde{P}(g) dV_g\) is also a global conformal invariant. Therefore, invoking the main theorem of [2], we derive that \(\tilde{P}(g)\) can be written in the form:

\[
\tilde{P}(g) = W'(g) + (\text{const}) \cdot \text{Pfaff}(R_{ijkl}), \quad (1.10)
\]

where \(W'(g)\) is a local conformal invariant. Therefore, combining (1.9) and (1.10) we derive the Deser-Schwimmer conjecture.

### 1.3 A more detailed outline of the present paper.

This series of papers (the present paper and [4, 5, 6, 7, 8]) can naturally be divided into two parts: Part I (which consists of the present paper together with [4] and [5]) proves the Deser-Schwimmer conjecture \textit{subject to proving certain “main algebraic propositions”}; these are Proposition 5.2 in the present paper, and the two propositions 3.1, 3.2 in the section “The important tools” in [4].

Part II (which consists of the papers [6], [7], [8]) are devoted to proving these “main algebraic propositions”. Thus, the logical dependence of this work is that the present paper and [4, 5] depend on [6, 7, 8]. Here we present a more detailed, yet broad, outline of this entire work, putting emphasis on the results proven in the present paper. In the subsequent papers of this series, we will provide further synopses of the other main ideas that appear in this work.

\textbf{“Categories” and the notion of “better” vs. “worse” categories:}

We now explain in more detail the notion of “categories” explained above, and how one category is “better” or “worse” than another category. Firstly, recall that the curvature tensor \(R_{ijkl}\) admits a natural decomposition into its trace-free part (the Weyl tensor) and its trace parts (consisting essentially of the Ricci tensor).\(^9\) We will write out the global conformal invariant \(P(g)\) as a linear combination of complete contractions involving covariant derivatives of the Weyl tensor \(\nabla^{(m)} W\) and covariant derivatives of the Schouten tensor \(\nabla^{(p)} P\).\(^{11}\)

\[
P(g) = \sum_{l \in L} a_l \text{contr}^l(\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_a)} W \otimes \nabla^{(p_1)} P \otimes \cdots \otimes \nabla^{(p_b)} P). \quad (1.11)
\]

Then, two complete contractions in the above form belong to the same “category” if they have the same number \(a + b\) of factors (in total), and also if they have the same number \(b\) of factors \(\nabla^{(p)} P\). Furthermore, if we consider two complete contractions \(C^1(g)\) and \(C^2(g)\) in the above form, then \(C^1(g)\) is

\(^9\text{Pfaff}(R_{ijkl})\) is the Pfaffian of the curvature tensor (i.e. the Gauss-Bonnet integrand).

\(^{10}\text{See (2.4) below.}\)

\(^{11}\text{The Schouten tensor, defined in (2.3), is a trace-adjusment of the Ricci tensor. For the purpose of this brief introduction, the reader may wish to think of the Schouten tensor as “essentially” the Ricci tensor.}\)
“worse” than $C^2(g)$ if it has fewer factors in total. If $C^1(g)$, $C^2(g)$ have the same number of factors in total, then $C^1(g)$ is “worse” than $C^2(g)$ if it has more factors $\nabla^{(a)}P$.

Thus: Let $\sigma$ be the minimum total number of factors among all the complete contractions indexed in $L$ in (1.6). Among the complete contractions with $\sigma$ factors, let $s$ be the maximum number of factors $\nabla^{(p)}P$. Then the “worst” complete contractions in (1.11) are the ones with $\sigma - s$ factors $\nabla^{(m)}W$ and $s$ factors $\nabla^{(p)}P$. Denote the index set of the “worst” complete contractions by $L^*_s \subset L$. We define $P(g)_{\text{worst-piece}} := \sum_{l \in L^*_s} a_l C^l(g)$.

Our main claim is that if $\sigma < \frac{2}{2}$ then $P(g)_{\text{worst-piece}}$ can be expressed as follows:

$$P(g)_{\text{worst-piece}} = W(g) + \text{div}T^i(g) + \sum_{f \in F^1} a_g C^f_g(\phi) + \sum_{f \in F^2} a_g C^f_g(\phi), \quad (1.12)$$

where $W(g)$ is a local conformal invariant 12 $\text{div}T^i(g)$ is the divergence of a Riemannian vector field and each of the complete contractions indexed in $F^1, F^2$ are in the form:

$$\text{contr}(\nabla^{(m)}W \otimes \cdots \otimes \nabla^{(m)}W \otimes \nabla^{(p)}P \otimes \cdots \otimes \nabla^{(p)}P), \quad (1.13)$$

with the following additional properties: The terms indexed in $F^1$ have more than $\sigma$ factors in total, while the terms indexed in $F^2$ have $\sigma$ factors in total but strictly fewer than $s$ factors $\nabla^{(p)}P$. (In other words, the terms indexed in $F^1, F^2$ are “better” than the terms in $P(g)_{\text{worst-piece}}$).

The main ideas in the derivation of (1.12), and a discussion of the difficulties: The starting point in deriving (1.12) is to pass from the invariance under integration enjoyed by $P(g)$ to a local formula for its conformal variation.

The main tool we developed in [1] (in order to address the Deser-Schwimmer conjecture) is the so-called super divergence formula. In one sentence, this formula applies to the conformal variation $I_g(\phi)$ of $P(g)$ 13 and explicitly expresses $I_g(\phi)$ as a divergence of a vector-valued differential operator $X^i_g(\phi)$:

$$I_g(\phi) = \text{div}X^i_g(\phi). \quad (1.14)$$

Then, the main task in proving (1.12) is two-fold: Firstly, to identify a “piece” in $I_g(\phi)$ which is in one-to-one correspondence with the the “worst piece” of $P(g)$. Secondly, to use the fact that $I_g(\phi)$ can be expressed as a divergence 14 to derive (1.12).

We distinguish two main cases in order to derive (1.12): Either $s > 0$ or $s = 0$. We prove (1.12) when $s > 0$ in the present paper. We prove (1.12) when

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12 In fact, $W(g) = 0$ unless $s = 0$.
13 We recall that $I_g(\phi) := e^{\phi}\tilde{P}(\mu e^{2\phi}g) - P(g)$; thus $I_g(\phi)$ is a differential operator, depending on an auxiliary function $\phi$.
14 Via the “super divergence formula” from [1].
That shows that there exists a linear combination of partial contractions, \( X_g^s(\phi) = \sum_{r \in R} a_r C_g^{s,r}(\phi) \), where each \( C_g^{s,r}(\phi) \) is in the form:

\[
pe_{cont}((m_1) R \otimes \cdots \otimes (m_{s-1}) R \otimes (b_1) \phi \otimes \cdots \otimes (b_r) \phi)
\]

(1.16)

(each \( \nabla^{(m')} R \) is the \( m' \)th iterated covariant derivative of the curvature tensor, and each \( \nabla^{(b')} \phi \) is the \( b' \)th iterated covariant derivative of the function \( \phi \)), so that:

\[\frac{dI_g^s(\phi)}{dt} = \int_M \text{Levi-Civita connection} \text{ of homogeneity } s \text{ in the function } \phi^{18} \text{ to } \frac{d}{dt} \frac{e^{s\phi}}{\phi} P(e^{2t\phi} g).\]

\[\text{See } (2.7) \text{ below.}\]

\[\text{See } (2.8), (2.11) \text{ respectively below.}\]

\[\text{I.e. the “super divergence formula”.}\]

\[\text{Partial contractions” with one free index, to be precise.}\]
\[ (-1)^s \sum_{l \in L_s^g} a_l \text{contr}^l (\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_{s-1})} W \otimes \nabla^{(p_1+2)} \phi \otimes \cdots \otimes \nabla^{(p_s+2)} \phi) = \]
\[ \text{div}_i \sum_{r \in R} a_r C^r, i(g) + \sum_{j \in \text{Junk-Terms}} a_j C^j(g). \]

(1.17)

Furthermore, the super divergence formula also implies that each \( b_i \geq 2 \) (apart from certain very special cases where we may have \( b_i = 1 \) for some of the vector fields \( C^r, i(g) \)–for the purposes of this introduction, we will assume that each \( b_i \geq 2 \)). We will now show how the main claim, (1.12), can be derived from (1.17) when \( s = \sigma \). We will then discuss why this direct approach fails when \( 0 < s < \sigma \).

Proof of (1.12) in the case \( s = \sigma \): Now, in the case \( \sigma = s \), we derive in subsection 4.1 below that the vector field needed for (1.12) is the vector field \( X^i(g) \) that formally arises from \( X^r, i(g) = \sum_{r \in R} a_r C^r, i(g) \) in (1.17) by replacing each factor \( \nabla^{(b_i)} \phi \) by a factor \( -\nabla^{(b_i-2)} \phi \) (observe that the condition \( b_i \geq 2 \) is necessary for this operation to be well-defined).

On the other hand, in the case \( s < \sigma \) one cannot derive (1.12) by directly applying the super divergence formula to the integral equation

\[ \int_{M^n} I^s_g(\phi) dV_g = 0 \]

and then replacing the factors \( \nabla^{(b_i)} \phi \) as above. We next discuss why this direct approach will fail in this case:

The difficulty in deriving (1.12) when \( 0 < s < \sigma \): If one were to directly apply the super divergence formula to the integral equation \( \int_{M^n} I^s_g(\phi) dV_g = 0 \), one would derive a local equation in the form (1.17). Now, if one were to pick out the terms with \( \sigma \) factors in (1.17) and then replace the factors \( \nabla^{(b_i)} \phi \) (\( b \geq 2 \)) by factors \( \nabla^{(b_i-2)} \phi P_{r_{b-1}r_i} \) (as in the case \( s = \sigma \)), one would derive an equation:

\[ (-1)^s \sum_{l \in L_s^g} a_l \text{contr}^l (\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_{s-1})} W \otimes \nabla^{(p_1+2)} \phi \otimes \cdots \otimes \nabla^{(p_s+2)} \phi) = \]
\[ \text{div}_i \sum_{r \in R} a_r C^r, i(g) + \sum_{k \in K} a_k C^k(g) + \sum_{j \in \text{Junk-Terms}} a_j C^j(g), \]

(1.18)

where the terms indexed in \( \text{Junk-Terms} \) will have at least \( \sigma + 1 \) factors, but the terms indexed in \( K \) will be in the form (1.13) with \( \sigma \) factors in total, and may have as many as \( \sigma = 1 \) factors \( \nabla^{(a)} P \). In other words, the complete contractions indexed in \( K \) do not necessarily have fewer than \( s \) factors \( \nabla^{(a)} P \). Therefore in the language of the “one-page summary”, the terms indexed in \( K \) in the RHS of (1.18) are not necessarily “better” than the terms in the LHS of (1.18).
Therefore in the case $0 < s < \sigma$ we will use the super divergence formula applied to $I_g^s(\phi)$ in a less straightforward way to derive a stronger claim than $\text{(1.17)}$:

The remedy when $0 < s < \sigma$: We will prove that there exists a linear combination of vector fields, $\sum_{y \in Y} a_y C_y^{\mu,i}(\phi)$, where each $C_y^{\mu,i}(\phi)$ is in the form:

$$p\text{constr}(\nabla^{|m_1|} W \otimes \cdots \otimes \nabla^{|m_{s-\tau}|} W \otimes \nabla^{(b_1)} \phi \otimes \cdots \otimes \nabla^{(b_\tau)} \phi)$$

(with each $b_j \geq 2$) so that:

$$(-1)^s \sum_{l \in L_s} a_l C_{y_l}^l(\phi) = \text{div}_y \sum_{y \in Y} a_y C_y^{\mu,i}(\phi) + \sum_{j \in \text{Junk}} a_j C_j^j(\phi), \quad (1.19)$$

where the terms indexed in \text{Junk} have at least $\sigma + 1$ factors in total. (A brief discussion explaining the derivation of $\text{(1.19)}$ is provided further down in this 10-page summary, in “A rough discussion of the “main algebraic Proposition””.

Then, $\text{(1.19)}$ implies $\text{(1.12)}$: For each $y \in Y$ formally construct a vector field $C_y^{\mu,i}(g)$ in the form $\text{(1.13)}$ (with one free index) by replacing the factors $\nabla_{r_1 \cdots r_{y_j}} \phi$ by factors $\nabla_{r_1 \cdots r_{y_j} - 2} \phi$, $P_{r_{y_j},r_{y_j}}$. We then derive (in section 5.4 below) that the divergence needed for $\text{(1.12)}$ is precisely $\sum_{y \in Y} a_y C_y^{\mu,i}(g)$.

Note: Observe that in this case $s > 0$, $\text{(1.12)}$ holds without a local conformal invariant $W(g)$ in the RHS.

A rough description of the “main algebraic Proposition” $5.2$ and of its use in proving equation $\text{(1.12)}$ when $s > 0$.

The “main algebraic Proposition” $5.2$.

First a little notation. We will be considering linear combinations of tensor fields, $\sum_{l \in L} a_l C^{i_1 \cdots i_r}(\Omega_1, \ldots, \Omega_p)$, where each $C^{i_1 \cdots i_r}(\Omega_1, \ldots, \Omega_p)$ is a partial contraction (with $\mu$ free indices) in the form:

$$p\text{constr}(\nabla^{(m_1)} R \otimes \cdots \nabla^{(m_r)} R \otimes \nabla^{(a_1)} \Omega_1 \otimes \cdots \otimes \nabla^{(a_p)} \Omega_p), \quad (1.20)$$

with a given number (say $\tau(= r + p)$) of factors in total; among these a given number $p$ of factors are in the form $\nabla^{(a)} \Omega_x$, $1 \leq x \leq p$ \footnote{In other words, they are $a$\textsuperscript{th} covariant derivatives of a scalar function $\Omega_x$.} and the remaining $\tau - p$ are in the form $\nabla^{(m)} R$ \footnote{The $m$\textsuperscript{th} covariant derivatives of the curvature tensor $R$.}. Notice also that there is a given number $p$ of factors $\nabla^{(a)} \Omega_x$, $1 \leq x \leq p$. We furthermore require that each $a_j \geq 2$ for each tensor field above \footnote{Tensor fields in the form with this property will be called “acceptable”.} and that each tensor field has no internal contractions \footnote{Recall from \cite{1} that in a tensor field in the form $\text{(1.20)}$, an internal contraction is a pair of two indices that belong to the same factor and are contracting against each other.}.

We also let $\sum_{l \in L'} a_l C^{i_1 \cdots i_r}(\Omega_1, \ldots, \Omega_p)$ stand for a linear combination of (acceptable) tensor fields in the form $\text{(1.20)}$, each with rank $b_l \geq \mu + 1$. Recall
that for each free index \( i_s \) in \( C_g^{i_1 \ldots i_{\tau}} \), the divergence \( \text{div}_i C_g^{i_1 \ldots i_{\tau}} \) is a sum of \( \tau \) partial contractions of rank \( b_l - 1 \): the first summand arises when we hit the first factor \( T_l \) in \( C_g^{i_1 \ldots i_{\tau}} \) by a derivative \( \nabla^{i_s} \) and contract the upper index \( i_s \) against the free index \( i_s \); the second summand arises when we hit the first factor \( T_l \) in \( C_g^{i_1 \ldots i_{\tau}} \) by a derivative \( \nabla^{i_s} \) and contract the upper index \( i_s \) against the free index \( i_s \), etc.

For tensor fields in the form (1.20) we will let \( X_{\text{div}} C_g^{i_1 \ldots i_{\tau}}(\Omega_1, \ldots, \Omega_p) \) stand for the sum of the \( \tau - 1 \) terms in \( \text{div}_i C_g^{i_1 \ldots i_{\tau}}(\Omega_1, \ldots, \Omega_p) \) where the derivative \( \nabla^{i_s} \) may hit any factor other than the one to which the free index \( i_s \) belongs. The assumption of the “main Proposition” 5.2 is that:

\[
\sum_{l \in L} a_l X_{\text{div}} \ldots X_{\text{div}} C_g^{i_1 \ldots i_{\tau}}(\Omega_1, \ldots, \Omega_p) = \sum_{l \in L'} a_l X_{\text{div}} \ldots X_{\text{div}} C_g^{i_1 \ldots i_{\tau}}(\Omega_1, \ldots, \Omega_p) + (\text{Junk} - \text{Terms}),
\]

(1.21)

where \( (\text{Junk} - \text{Terms}) \) here stands for a generic linear combination of complete contractions with at least \( \tau + 1 \) factors.

The claim of the “main algebraic Proposition” is that there exists a linear combination of acceptable \((\mu + 1)\)-tensor fields, say \( \sum_{h \in H} a_h C_h^{i_1 \ldots i_{\mu + 1}}(\Omega_1, \ldots, \Omega_p) \), with each \( C_h^{i_1 \ldots i_{\mu + 1}}(\Omega_1, \ldots, \Omega_p) \) in the form (1.20), so that:

\[
\sum_{l \in L_{\mu}} a_l C_l^{i_1 \ldots i_{\mu}}(\Omega_1, \ldots, \Omega_p) - X_{\text{div}}_{i_{\mu + 1}} \sum_{h \in H} a_h C_h^{i_1 \ldots i_{\mu + 1}}(\Omega_1, \ldots, \Omega_p) = (\text{Junk} - \text{Terms}),
\]

(1.22)

where the \( (\text{Junk} - \text{Terms}) \) in the above stand for a linear combination of complete contractions with at least \( \tau + 1 \) factors. Here the symbol \( (i_1 \ldots i_{\mu}) \) means that we are symmetrizing over the indices \( i_1, \ldots, i_{\mu} \).

We next highlight how (1.22) can be used to prove (1.12) when \( s > 0 \):

The use of the “main algebraic Proposition” in deriving (1.12) (when \( s > 0 \)): We present here the argument from section 4 in brief:

Equation (1.12) is proven by a new induction. Write out:

\[
P(g)_{\text{worst-piece}} = \sum_{l \in L} a_l \text{contr}^l (\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_{s-1})} W \otimes \nabla^{(p_1)} P \otimes \cdots \otimes \nabla^{(p_s)} P).
\]

(1.23)

\[24\] This rather strange definition fits in with the conclusion of the super divergence formula—see section 2.3 below.

\[25\] Whereas the terms in the LHS of the above each have \( \tau \) factors.
We assume that among the complete contractions in $P(g)_{\text{worst-piece}}$ the minimum number of internal contractions is $\beta \geq 0$. We denote by $L_\beta \subset L$ the index set of complete contractions with $\beta$ internal contractions. (Thus the complete contractions indexed in $L \setminus L_\beta$ will each have at least $\beta+1$ internal contractions).

We will then show in section 4 that there exists a divergence of a Riemannian vector field, $\text{div}_iT^i(g)$, as allowed in the statement of Conjecture 1 such that:

$$\sum_{l \in L_\beta} a_l \text{contr}^l(\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_{s-1})} W \otimes \nabla^{(p_1)} P \otimes \cdots \otimes \nabla^{(p_s)} P) =$$

$$\text{div}_iT^i(g) + \sum_{t \in T} a_t \text{contr}^t(g) + (\text{Allowed}),$$

(1.24)

where the complete contractions indexed in $T$ are in the form (1.13) with $\sigma$ factors in total, of which $s \geq 1$ are in the form $\nabla^{(a)} P$, and with $\beta + 1$ internal contractions in total. Furthermore (Allowed) stands for a generic linear combination of complete contractions that are allowed in the right hand side of (1.12).

Observe that if we can show then will follow by iteratively repeating this step at most $\frac{n}{2}$ times.

Mini-Outline of the proof of (1.24): We recall that

$$\text{contr}^l(\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_{s-1})} W \otimes \nabla^{(p_1+2)} \phi \otimes \cdots \otimes \nabla^{(p_s+2)} \phi)$$

stands for the complete contraction that arises from

$$\text{contr}^l(\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_{s-1})} W \otimes \nabla^{(p_1)} P \otimes \cdots \otimes \nabla^{(p_s)} P)$$

(1.25)

by replacing each factor $\nabla^{(a)}_{r_1\ldots r_m} P_{ij}$ by a factor $\nabla^{(a+2)}_{r_1\ldots r_m} \phi$. We will denote by $\text{contr}^l(\phi)$ the complete contraction in the form (1.26); we will also denote by $\text{contr}^l(\phi)$ the complete contraction:

$$\text{contr}^l(\nabla^{(m_1)} R \otimes \cdots \otimes \nabla^{(m_{s-1})} R \otimes \nabla^{(p_1+2)} \phi \otimes \cdots \otimes \nabla^{(p_s+2)} \phi)$$

(1.27)

which arises from $\text{contr}^l(\phi)$ by formally replacing each factor $\nabla^{(m)}_{r_1\ldots r_m} W_{ijkl}$ by a factor $\nabla^{(m)}_{r_1\ldots r_m} R_{ijkl}$ (possibly times a constant—but for the purposes of this introduction we will ignore this fact). Observe that the resulting tensor fields still have $\beta$ internal contractions in total, and also have each function $\phi$ differentiated at least twice. We will then prove in section 4 that the integral equation

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25Recall that the complete contractions in $P(g)_{\text{worst-piece}}$ are all in the form (1.13) with $\sigma$ factors in total, of which $s \geq 1$ are in the form $\nabla^{(a)} P$.

26In other words, the complete contractions indexed in (Allowed) are either complete contractions with more than $\sigma+1$ factors in total, or they are complete contractions in the form (1.25) with $\sigma$ factors in total, but strictly fewer than $s$ factors $\nabla^{(a)} P$.

27This is because a complete contraction in the form (1.13) with weight $-n$ can contain at most $\frac{n}{2}$ internal contractions.
\[ \int_{M^n} I_g^\beta(\phi) dV_g = 0 \] implies a new integral equation in the form:

\[ \int_{M^n} (-1)^\beta \sum_{l \in L, \beta} a_l \text{contr}^u(\phi) + \sum_{v \in V} a_v \text{contr}^v_g(\phi) + (\text{Junk} - \text{Terms}) dV_g = 0. \tag{1.28} \]

Here the complete contractions indexed in \( V \) are in the form (1.27) and have \( \sigma \) factors and at least \( \beta + 1 \) internal contractions in total\(^{29}\), and each factor \( \nabla^{(b_i)} \phi \) with \( b_i \geq 2 \). Then, applying the super divergence formula\(^{30}\), we derive a local equation:

\[ \sum_{u \in U, \beta} a_u X \text{div}_{i_1} \cdots \text{div}_{i_{\beta + 1}} \text{pcontr}^{u, i_1 \cdots i_{\beta}}(\phi) + \sum_{v \in V} a_v X \text{div}_{i_1} \cdots \text{div}_{i_v} C_g^{v, i_1 \cdots i_v}(\phi) = (\text{Junk} - \text{Terms}), \tag{1.29} \]

where the tensor fields \( \text{pcontr}^{u, i_1 \cdots i_{\beta}}(\phi), C_g^{v, i_1 \cdots i_v}(\phi) \) arise from the complete contractions \( \text{contr}^u(\phi), \text{contr}^v_g(\phi) \) by formally replacing each internal contraction\(^{31}\) by a free index\(^{32}\). Furthermore recall that \( b_v > \beta \) for each \( v \in V \).

Now, applying the “main algebraic proposition” 5.2 to the above, we derive that there exists a linear combination of \((\beta + 1)\) tensor fields \( \sum_{h \in H} a_h C_g^{h, (i_1 \cdots i_{\beta})_{\beta + 1}}(\phi) \) in the form (1.20) so that:

\[ \sum_{u \in U, \beta} a_u X \text{div}_{i_1} \cdots \text{div}_{i_{\beta + 1}} \text{pcontr}^{u, (i_1 \cdots i_{\beta})_{\beta + 1}}(\phi) = (\text{Junk} - \text{Terms}), \tag{1.30} \]

Finally, we formally replace each factor \( \nabla^{(m)}_{r_1 \cdots r_m} R_{ijkl} \) by a factor \( \nabla^{(m)}_{r_1 \cdots r_m} W_{ijkl} \) and then make all the free indices into internal contractions\(^{3}4\). Denote this formal operation by \( \text{Weylify} \)\(^{3}5\). Then \( \text{Weylify}\left[ \sum_{h \in H} a_h C_g^{h, (i_1 \cdots i_{\beta})_{\beta + 1}}(\phi) \right] \) is the divergence \( \text{div}_T^i(g) \) that is needed for (1.24).

\(^{29}\)(And none of these internal contractions involve two indices from among the indices \( i, j, k, l \) in a factor \( \nabla^{(m)} R_{ijkl} \); this detail is only relevant for the next sentence).

\(^{30}\)See section 2.3 below.

\(^{31}\)Which by hypothesis will consist of two indices in the same factor contracting against each other—i.e. two indices in the form \( (\nabla^a) \)

\(^{32}\)I.e. in the notation of the previous footnote we erase the index \( \nabla^a \) and we make the index \( a \) into a free index.

\(^{33}\)We just set \( \Omega_1 = \cdots = \Omega_p = \phi \).

\(^{34}\)(times a constant, which we ignore for these purposes)

\(^{35}\)By this we mean that for each free index \( i_v \), which belongs to a factor \( T_{a \cdots c} \), we formally add a derivative \( \nabla^{i_v} \) onto the factor \( T_{a \cdots c} \) and contract it against the index \( i_b \); thus we obtain a factor \( \nabla^{i_b} T_{a \cdots c} \).
2 Conventions, Background, and the Super divergence formula from [1].

2.1 Conventions and Remarks.

We introduce some conventions that will be used throughout this series of papers. Firstly, we recall two notions introduced in [1]:

**Definition 2.1**
Given any (formal) linear combination \( \sum_{l \in L} a_l C_l \) and any subset \( L' \subset L \), then the linear combination \( \sum_{l \in L'} a_l C_l \) will be called a sublinear combination of \( \sum_{l \in L} a_l C_l(g) \).

We also recall the notion of an “internal contraction” for any (complete or partial) contraction:

**Definition 2.2**
Consider any complete or partial contraction \( C(T_1, \ldots, T_a) \), involving the tensors \( T_1, \ldots, T_a \). Then an “internal contraction” is a pair of indices \( i, j \) that belong to the same factor \( T_h \) and are contracting against each other in \( C(T_1, \ldots, T_a) \).

Finally, as in [1] and [2] we define the “length” of a complete contraction to stand for the number of its factors.

Now, a few minor conventions:

Firstly, when we say that a local equation, say \( \sum_{t \in T} a_t C_t = \sum_{y \in Y} a_y C_y \) holds modulo terms of length \( \geq \tau + 1 \), we will mean that there exists a linear combination of complete contractions with at least \( \tau + 1 \) factors, \( \sum_{f \in F} a_f C_f \), such that \( \sum_{t \in T} a_t C_t = \sum_{y \in Y} a_y C_y + \sum_{f \in F} a_f C_f \).

Secondly, when we write \( \nabla^m \), \( m \) will stand for the number of differentiations. When we write \( \nabla^m \), \( m \) will stand for a raised index.

Thirdly, we will often be referring to factors \( \nabla^m R_{ijkl} \), \( \nabla^p \text{Ric}_{ij} \) and \( R \) (the third factor being the scalar curvature) in complete and partial contractions below. Whenever we write \( \nabla^m R_{ijkl} \), we will be assuming that no two of the indices \( i, j, k, l \) are contracting against each other (unless stated otherwise). Also, in \( \nabla^p \text{Ric}_{ij} \), no two of the indices \( i, j \) will be contracting against each other (unless stated otherwise). Moreover, for brevity we will not be explicitly writing out all the indices that belong to the different terms. For example, when we refer to factors \( \nabla^m R_{ijkl} \) we have written out the four lower indices of the curvature tensor but not the covariant derivative indices.

Furthermore, throughout this paper we will often write out complete contractions with two or more factors \( \nabla^m R_{ijkl} \) or \( \nabla^p \text{Ric}_{ij} \). When we do so, and hence have the indices \( i,j,k,l \) or \( i,j \) appearing repeatedly as lower indices, we will not be assuming that these indices are contracting against each other. I.e. if we have a factor \( T = \nabla^m R_{ijkl} \) and \( T' = \nabla^m R_{ijkl} \) appearing in the same complete contraction, the indices \( i,j,k,l \) in \( T \) and \( i,j,k,l \) in \( T' \) are not assumed to

\[ ^{36} \text{In the introduction we spoke of a “piece” of } \sum_{l \in L} a_l C_l, \text{ for simplicity.} \]

\[ ^{37} \text{The reader should note that this convention was not adopted in [1] and [2].} \]
be contracting against each other. We only use this notation to avoid writing \( \nabla^{(m)}R_{ijkl}, \nabla^{(m')}R_{i'j'k'l'}, \nabla^{(m'')}R_{i''j''k''l''} \) etc. For each factor \( \nabla^{(m)}R_{ijkl} \) the indices \( i, j, k, l \) will be called \textit{internal indices}; for each factor \( \nabla^{(p)}R_{ijkl} \) the indices \( a, b \) will be called \textit{internal indices}.

"Mini-Appendices": Throughout this series of papers, we will sometimes postpone the proof of certain claims; the reader will be referred to "Appendices" or “Mini-Appendices” further down in the paper. These Appendices often refer to very special cases of more general claims which require special proofs; the reader who is interested only in the broad ideas in these papers may wish to circumvent these sections.

### 2.2 Background: Some useful formulas

**Standard formulas**\(^{38}\) The curvature tensor \( R_{ijkl} \) of a Riemannian manifold is given by the formula:

\[
[\nabla_i \nabla_j - \nabla_j \nabla_i]X_l = R_{ijkl}X^k.
\]

Moreover, the Ricci tensor \( Ric_{ik} \) arises from \( R_{ijkl} \) by contracting the indices \( j, l \):

\[
Ric_{ik} = R_{ijkl}g^{jl}.
\]

The Schouten tensor is a trace-adjustment of Ricci curvature:

\[
P_{ij} = \frac{1}{n-2} [Ric_{ij} - \frac{R}{2(n-1)}g_{ij}].
\]

(Here \( Ric_{ij} \) stands for Ricci curvature and \( R \) stands for scalar curvature \( R_{ijkl}g^{ik}g^{jl} \)).

We also recall the Weyl tensor:

\[
W_{ijkl} = R_{ijkl} - [P_{jk}g_{il} + P_{il}g_{jk} - P_{jl}g_{ik} - P_{ik}g_{jl}],
\]

which is conformally invariant, i.e:

\[
W_{ijkl}(e^{2\phi}g) = e^{2\phi}W_{ijkl}(g).
\]

Furthermore, we recall the Cotton tensor:

\[
C_{ijk} = \nabla_kP_{ij} - \nabla_jP_{ik},
\]

which is related to the Weyl curvature in the following way:

\[
\nabla^lW_{ijkl} = (3-n)C_{jkl}.
\]

The Ricci curvature transforms as follows:

\(^{38}\) Unless mentioned otherwise, these formulas come from [1].
\[
\text{Ric}_{ab}(e^{2\phi} g) = \text{Ric}_{ab}(g) + (2-n)\nabla_{ab}^{(2)} \phi - \Delta \phi g_{ab} + (n-2)(\nabla_a \phi \nabla_b \phi - \nabla_k \phi \nabla_k g_{ab}),
\]
(2.8)

While the Schouten tensor has the following transformation law:

\[
\text{P}_{ab}(e^{2\phi} g) = \text{P}_{ab}(g) - \nabla_{ab}^{(2)} \phi + \nabla_a \phi \nabla_b \phi - \frac{1}{2} \nabla^k \phi \nabla_k \phi g_{ab}.
\]
(2.9)

The curvature tensor transforms:

\[
\text{R}_{ijkl}(e^{2\phi} g) = e^{2\phi}[\text{R}_{ijkl}(g) + \nabla_{il}^{(2)} \phi g_{jk} + \nabla_{jk}^{(2)} \phi g_{il} - \nabla_{ik}^{(2)} \phi g_{jl} - \nabla_{jl}^{(2)} \phi g_{ik} + \nabla_i \phi \nabla_k \phi g_{jl} + \nabla_j \phi \nabla_l \phi g_{ik} - \nabla_i \phi \nabla_j \phi g_{kl} - \nabla_j \phi \nabla_k \phi g_{il} + |\nabla \phi|^2 g_{il} g_{jk} - |\nabla \phi|^2 g_{ik} g_{jl}].
\]
(2.10)

We also recall following transformation law for the Levi-Civita connection under general conformal transformations \(\tilde{g}_{ij}(x) = e^{2\phi} g_{ij}(x)\):

\[
\nabla_{k} \eta_{l}(e^{2\phi} g) = \nabla_{k} \eta_{l}(g) - \nabla_{k} \phi \eta_{l} + \nabla_{l} \phi \eta_{k} + |\nabla \phi|^2 g_{kl} g_{ij}. \quad (2.11)
\]

Finally, on certain rare occasions we will be using the transformation law of the curvature tensor \(\text{R}_{ijkl}\) under variations of the metric \(g_{ij}\) by a symmetric 2-tensor \(v_{ij}\):

\[
\frac{d}{dt}|_{t=0}[\text{R}_{ijkl}(g_{ab} + tv_{ab})] = \frac{1}{2}[\nabla_{il}^{(2)} v_{jk} + \nabla_{jk}^{(2)} v_{il} - \nabla_{ik}^{(2)} v_{jl} - \nabla_{jl}^{(2)} v_{ik}] + Q(R, v),
\]
(2.12)

where \(Q(R, v)\) stands for a quadratic expression involving the curvature tensor \(R_{abcd}\) and the 2-tensor \(v_{ef}\).

2.3 The main consequence of the super divergence formula.

In this subsection we codify a consequence of the super divergence formula, which was the main result in [1]; (We recall from the 10-page outline that this formula considers Riemannian operators \(L_g(\phi)\), depending both on the metric and auxiliary functions, whose integral over any closed manifold is always zero, and expresses them as divergences of explicitly constructed vector fields). Here we codify into a Lemma a main consequence of this formula, which is what we will mostly be using in this series of papers.

We start with some notation:

We will be considering complete contractions \(C_g^l(\psi_1, \ldots, \psi_Z)\) in the normalized form:
Consider any internal contraction in \( \zeta = \langle \cdots \rangle \) by a factor \(-\nabla^i\) from each \( C \) denote by \( \text{contr} \) contracting between themselves. Note that any complete contraction in the form \( \zeta \geq \langle \cdots \rangle \) assume that for some \( r \) also none of the indices \( u \) against one of the indices \( v \) in the curvature identity and the second Bianchi identity.

Finally, for the last \( Z \) factors (which we denote by the generic notation \( \nabla w_1 \cdots w_y \nabla^a u \nabla^b \psi \)) we assume that each of the indices \( a_1, \ldots, a_t \) is contracting against one of the indices \( r_1, \ldots, r_m, i, j, k, l \). Moreover, none of the indices \( r_1, \ldots, r_m, i, j, k, l \) are contracting between themselves. Each factor \( R \) is a scalar curvature term.

For the next \( q \) factors (in the generic form \( \nabla^{e_1 \cdots e_c} \nabla^{(p)} r_1 \cdots r_p \nabla \)) each of the indices \( e_1, \ldots, e_c \) is contracting against one of the indices \( r_1, \ldots, r_p, i, j \) and also none of the indices \( r_1, \ldots, r_p, i, j \) are contracting between themselves. Each factor \( R \) is a scalar curvature term.

Finally, for the last \( Z \) factors (which we denote by the generic notation \( \nabla w_1 \cdots w_y \nabla^a u \nabla^b \psi \)) we assume that each of the indices \( a_1, \ldots, a_t \) is contracting against one of the indices \( u_1, \ldots, u_s \) and none of the indices \( u_1, \ldots, u_s \) are contracting between themselves. Note that any complete contraction in the form \( \text{contr}(\nabla^{(m)} R \cdots \nabla^{(n)}) R \cdots \nabla^{(p)} \nabla \psi \cdots \nabla^{(q)}) \psi_h \) can be expressed as a linear combination of contractions in the form \( (2.13) \), by just repeatedly applying the curvature identity and the second Bianchi identity.

Consider a set \( \{ C_g^f(\psi_1, \ldots, \psi_r) \}_{f \in L} \) of normalized complete contractions, indexed in \( L \). Let \( L_M \subset L \) stand for an index set of complete contractions \( C_g^f(\psi_1, \ldots, \psi_Z) \) with a total of \( q + p = M \) factors \( \nabla^{e_1 \cdots e_c} \nabla^{(p)} r_1 \cdots r_p \nabla \) and \( R \). We assume that for some \( M \geq 0 \), all index sets \( L_s \) with \( s > M \) are empty. We then denote by \( C_g^f(\psi_1, \ldots, \psi_Z, \Omega^M) \) the complete contraction that formally arises from each \( C_g^f(\psi_1, \ldots, \psi_Z) \), \( f \in L_M \), by replacing each factor \( \nabla^{e_1 \cdots e_c} \nabla^{(p)} r_1 \cdots r_p \nabla \) by a factor \(-\nabla^{e_1 \cdots e_c} \nabla^{(p+2)} r_1 \cdots r_p \nabla \Omega \) and each factor \( R \) by a factor \(-2\Delta \Omega \). (\( \Omega \) will be a scalar function).

(Our contract \( C_g^f(\psi_1, \ldots, \psi_Z, \Omega^M) \), \( f \in L_M \), is a complete contraction in the form:

\[
(-1)^{M-2p} \text{contr}(\nabla^{a_1 \cdots a_t} \nabla^{(m_1)} R_{ijkl} \cdots \nabla^{b_1 \cdots b_a} \nabla^{(m_a)} R_{ij'}k'l'\cdots \\
\nabla^{c_1 \cdots c_v} \nabla^{(p_1 + 2)} \Omega \cdots \nabla^{d_1 \cdots d_z} \nabla^{(p_z + 2)} \Omega \cdots \Delta \Omega \cdots \Delta \Omega \cdots \\
\n\nabla w_1 \cdots w_y \nabla^a \psi_1 \cdots \nabla^x \cdots \nabla^z \psi_z) \tag{2.14}
\]

**Definition 2.3** Consider any internal contraction in \( C_g^f(\psi_1, \ldots, \psi_Z, \Omega^M) \), say \( \zeta = \langle \cdots \rangle \) (notice that \( a \) must necessarily be a derivative index). We then say that we replace the internal contraction \( \zeta \) by a free index if we erase the index \( a \) and make the index \( a \) into a free index.

We thus obtain a 1-tensor field \( (C^f)_g^\zeta(\psi_1, \ldots, \psi_Z, \Omega^M) \) of weight \(-n + 1\) (the free index \( i \_i \) is the index \( a \) above). The same formal definition can also
be applied to \( k \) internal contractions: If we pick out \( k \) internal contractions, say \((a_1, a_1), \ldots, (a_k, a_k)\) and then erase the indices \( a_1, \ldots, a_k \) and make the indices \( a_1, \ldots, a_k \) into free indices \( i_1, \ldots, i_k \) we obtain a \( k \)-tensor field \((C^l)^{i_1 \ldots i_k}_l(\psi_1, \ldots, \psi_2, \Omega^M)\) of weight \(-n+k\).

This language convention (of making an internal contraction into a free index) will be used throughout this series of papers.

**Definition 2.4** Now for each \( l \in L_M \), we denote by \((C^l)^{i_1 \ldots i_k}_l(\psi_1, \ldots, \psi_2, \Omega^M)\) the tensor field that arises from \( C^l_g(\psi_1, \ldots, \psi_2, \Omega^M) \) by making all the internal contractions into free indices. We denote by

\[
X\text{div}_{i_1} \ldots X\text{div}_{i_k}(C^l)^{i_1 \ldots i_k}_l(\psi_1, \ldots, \psi_2, \Omega^M)
\]

the sublinear combination in \( \text{div}_{i_1} \ldots \text{div}_{i_k}(C^l)^{i_1 \ldots i_k}_l(\psi_1, \ldots, \psi_2, \Omega^M) \) that arises when each \( \nabla_{i_a} \) is allowed to hit any factor other than the one to which \( i_a \) belongs.

**The main consequence of the super divergence formula:**

**Lemma 2.1** Assume an integral equation:

\[
\int_{M^n} \sum_{l \in L} a_l C^l_g(\psi_1, \ldots, \psi_2) + \sum_{h \in h'} a_h C^h_g(\psi_1, \ldots, \psi_2) dV_g = 0, \tag{2.15}
\]

which is assumed to hold for every compact \( (M^n, g) \), and every \( \psi_1, \ldots, \psi_2 \in C^\infty(M^n) \). Here the complete contractions indexed in \( L \) have length \( \sigma \) and are in the normalized form \((2.13)\), and the complete contractions indexed in \( H \) have length \( > \sigma \). We let \( M \) stand for the maximum number of factors \( \nabla^{(p)}Ric \) and \( R \) (in total) among the complete contractions \( C^l_g(\ldots, l \in L; \text{let } L_M \subset L \) be the index set of complete contractions with \( M \) factors \( \nabla^{(p)}Ric, R \) (in total).

We claim:

\[
\sum_{l \in L_M} a_l(-1)^{b_l} X\text{div}_{i_1} \ldots X\text{div}_{i_k}(C^l)^{i_1 \ldots i_k}_l(\psi_1, \ldots, \psi_2, \Omega^M) = 0, \tag{2.16}
\]

modulo complete contractions of length \( \geq \sigma + 1 \).

**Proof:** We will show this claim in two steps. Initially, we show that for some linear combination \( \sum_{h \in h'} a_h C^h_g(\psi_1, \ldots, \psi_2, \Omega^M) \) of complete contractions with length \( \geq \sigma + 1 \) we have:

\[
\int_{M^n} \sum_{l \in L_M} a_l C^l_g(\psi_1, \ldots, \psi_2, \Omega^M) + \sum_{h \in h'} a_h C^h_g(\psi_1, \ldots, \psi_2, \Omega^M) dV_g = 0. \tag{2.17}
\]

**Proof of (2.17):** Let us denote the integrand of \((2.15)\) by \( L_g(\psi_1, \ldots, \psi_2) \).

We consider any dimension \( N \geq n \) and denote by \( L_g(\psi_1, \ldots, \psi_2) \) the rewriting of \( L_g(\psi_1, \ldots, \psi_2) \) in dimension \( N \).
Then, as shown in [1] (using the silly divergence formula) we derive that for any $N \geq n$, any $(M^N, g)$ and any $\psi_1, \ldots, \psi_Z \in C^\infty(M^N)$:

\[
\int_{M^N} L_g^N (\psi_1, \ldots, \psi_Z) dV_g^N = 0.
\] (2.18)

Now, let $L_g^M (\psi_1, \ldots, \psi_s, \Omega^M) := \frac{\partial L_g^M}{\partial \lambda}|_{\lambda=0} [e^{(N-n)\lambda}(x) L_{e^{2\lambda}(x)}g^N (\psi_1, \ldots, \psi_s)]$. It follows from (2.18) that:

\[
\int_{M^N} L_g^M (\psi_1, \ldots, \psi_Z, \Omega^M) dV_g^N = 0.
\] (2.19)

Now, using (2.10) and (2.11) and the transformation law for the volume form: $dV_{e^{2\lambda}(x)}g = e^{N\lambda}(x) dV_g$, it follows that we can re-express (2.19) as follows:

\[
N^M \int_{M^N} \sum_{l \in L^M} a_l C_g^l (\psi_1, \ldots, \psi_Z, \Omega^M) dV_g^N
\]

\[+ \sum_{x=0}^{N^M} \int_{M^N} \sum_{u \in U^x} a_u C_g^u (\psi_1, \ldots, \psi_Z, \Omega^M) dV_g^N = 0,
\] (2.20)

where the summands $C_g^u (\psi_1, \ldots, \psi_Z, \Omega^M), u \in U^x$ are independent of the dimension $N$. Also, each $C_g^u (\psi_1, \ldots, \psi_s, \Omega^M), u \in U^M$ has at least $\sigma + 1$ factors (possibly with factors $\Omega$ without derivatives). Picking $M^N = M^n \times S^1 \cdots \times S^1$ with the product metric $g^n = g^n + (dt)^2 + \cdots + (dt^{N-n})^2$ we derive an integral equation in dimension $n$, where $N$ is just a free variable:

\[
N^M \int_{M^n} \sum_{l \in L^M} a_l C_g^l (\psi_1, \ldots, \psi_Z, \Omega^M) dV_g
\]

\[+ \sum_{x=0}^{N^M} \int_{M^n} \sum_{u \in U^x} a_u C_g^u (\psi_1, \ldots, \psi_Z, \Omega^M) dV_g = 0.
\] (2.21)

Therefore, viewing the above as a polynomial in $N$ and restricting attention to the coefficient of $N^M$ we derive:

\[
\int_{M^n} \sum_{l \in L^M} a_l C_g^l (\psi_1, \ldots, \psi_Z, \Omega^M) dV_g + \int_{M^n} \sum_{u \in U^M} a_u C_g^u (\psi_1, \ldots, \psi_Z, \Omega^M) dV_g = 0,
\] (2.22)

where each $C_g^u (\psi_1, \ldots, \psi_Z, \Omega^M)$ has length $\geq \sigma + 1$. This is exactly (2.17).

Now, we denote the integrand in (2.22) by $L_g (\psi_1, \ldots, \psi_Z, \Omega^M)$ and we apply the super divergence formula to $L_g (\psi_1, \ldots, \psi_Z, \Omega^M)$. We focus on the sublinear combination $\supdiv \{L_g (\psi_1, \ldots, \psi_Z, \Omega^M)\}$ in $\supdiv \{L_g (\psi_1, \ldots, \psi_Z, \Omega^M)\}$ that consists of complete contractions of length $\sigma$ and with no internal contractions. By virtue of the super divergence formula and Lemma 8 in [1], we derive:
moduło complete contractions of length $\geq \sigma + 1$.

On the other hand, by the algorithm for the super divergence formula in [1], we derive:

\[
\text{supdiv}_+ [L_g(\psi_1, \ldots, \psi_Z, \Omega^M)] = \sum_{l \in L_M} a_l (-1)^{b_l} X\text{div}_{i_1} \cdots X\text{div}_{i_l} (C^l)_{g}^{i_1 \cdots i_l} (\psi_1, \ldots, \psi_Z, \Omega^M).
\]  

Combining the two above equations we derive (2.16).

3 From the super divergence formula for $I_g(\phi)$ back to $P(g)$: The two main claims of this series of papers.

Throughout this section, $P(g)$ will be a Riemannian scalar of weight $-n$ with the feature that $\int_M P(g) dV_g$ is a global conformal invariant (see Definition 1.2).

Let us begin by writing $P(g)$ as a linear combination:

\[
P(g) = \sum_{l \in L} a_l C^l(g),
\]

where each complete contraction $C^l(g)$ is in the form:

\[
\text{contr}(\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_s)} W \otimes \nabla^{(p_1)} P \otimes \cdots \otimes \nabla^{(p_q)} P).
\]

Our next two Propositions flesh out the claims made in the first page of our "10-page outline". We will define the "worst piece" in $P(g)$ and claim that by subtracting a divergence and a local conformal invariant we can cancel it out modulo introducing “better” correction terms. The “worst piece” will consist of terms with a given number $\sigma$ of factors in total and a given number $s$ of factors $\nabla^{(a)} P$ (see the next paragraph). The two propositions correspond to the cases $s > 0$ and $s = 0$.

Consider $P(g)$ as in (3.1). Denote by $\sigma$ the minimum number of factors among the complete contractions indexed in $L$ in (3.1). Denote by $L_\sigma \subset L$ the index set of those complete contractions. Now, denote by $\Theta_r \subset L_\sigma$ the index set of complete contractions with $r$ factors $\nabla^{(a)} P$ (and hence $\sigma - r$ factors $\nabla^{(m)} W$). We note that some of these sets may a priori be empty.

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Proposition 3.1 Suppose that \( P(g) = \sum_{l \in L} a_l C^l(g) \) is a linear combination of contractions in the form (3.2), and the minimum number of factors among the contractions \( C^l(g) \) is \( \sigma < \frac{2}{3} \). We assume that (for \( P(g) \)) the sets \( \Theta_1, \ldots, \Theta_{\sigma+1} \) are empty, where \( 1 \leq s \leq \sigma \), but \( \Theta_s \) is not empty. We claim that there is a Riemannian vector field \( T^i(g) \) so that

\[
\sum_{l \in \Theta_s} a_l C^l(g) - \text{div}_i T^i(g) = \sum_{r \in R} C^r(g),
\]

where each \( C^r(g) \) is either in the form (3.2) with length \( \sigma \) and fewer than \( s \) factors \( \nabla^{(p)} P \), or it has length \( > \sigma \).

Clearly, if we can show the above Proposition, then by iterative repetition we can derive that there is a vector field \( T^i(g) \) so that

\[
P(g) - \text{div}_i T^i(g) = \sum_{l \in L'} a_l C^l(g) + \sum_{j \in J} a_j C^j(g),
\]

with each \( C^l(g) \), \( l \in L' \) in the form:

\[
\text{contr}(\nabla^{(m_1)} W \otimes \cdots \otimes \nabla^{(m_\sigma)} W),
\]

while each \( C^j(g) \) will have at least \( \sigma + 1 \) factors. Thus, if we can show Proposition 3.1 we will be reduced to proving Theorem 1.2 in the case where all complete contractions in \( P(g) \) with \( \sigma \) factors are in the form (3.4).

Our next “main claim” applies precisely to that setting:

Proposition 3.2 Consider any \( P(g) \), \( P(g) = \sum_{l \in L} a_l C^l(g) \) where each \( C^l(g) \) has length \( \geq \sigma \), and each \( C^l(g) \) of length \( \sigma \) is in the form (3.4). Denote by \( L_\sigma \subset L \) the index set of terms with length \( \sigma \).

We claim that there is a local conformal invariant \( W(g) \) of weight \( -n \) and also a vector field \( T^i(g) \) as in the statement of Theorem 1.2 so that:

\[
\sum_{l \in L_\sigma} a_l C^l(g) - W(g) - \text{div}_i T^i(g) = 0
\]

modulo complete contractions of length \( \geq \sigma + 1 \).

We observe that if we can show the above two propositions, then by iterative repetition our Theorem will follow, in view of [2].

Now, in the remainder of the present paper we will explain how to derive Proposition 3.1 in the case \( \sigma \geq 3 \) assuming the “main algebraic Proposition” 5.2 below. Proposition 3.2 in the case \( \sigma \geq 3 \) will be proven in [4, 5] assuming another two “main algebraic Propositions” which are formulated in [4]. The cases \( \sigma < 3 \) of Propositions 3.1 and 3.2 will be proven in [5]. The three “main algebraic propositions are then proven in [6, 7, 8].
4 Proof of Proposition 3.1 in the easy case \( s = \sigma \).

We will distinguish two cases: Either \( s = \sigma \) or \( s < \sigma \). We will firstly show the claim when \( s = \sigma \). This proof is much easier than the case \( s < \sigma \), but it will contain simple forms of certain arguments that will be used throughout this series of papers. It also is instructive, in the sense that it can illustrate how the super divergence formula applied to \( I_g(\phi) \) can be used to understand the algebraic structure of \( P(\gamma) \).

**Definition 4.1** If \( P(\gamma) \) is in the form \( P(\gamma) = \sum_{l \in L} a_l C_l(\gamma) \) then for any subset \( A \subset L \), we will denote by \( P(\gamma)|_A = \sum_{l \in A} a_l C_l(\gamma) \).

Finally, for complete contractions \( C(\gamma), C_g(\phi) \) of weight \( -\gamma \), we define the operation \( \text{Image}^\phi_\gamma \) as follows:

\[
\text{Image}^\phi_\gamma[C(\gamma)] = \partial^\phi|_{\gamma=0}\{e^{n\psi(x)}C(e^{2\psi(x)})\},
\]

and

\[
\text{Image}^\phi_\gamma[C_g(\phi)] = \partial^\phi|_{\gamma=0}\{e^{n\psi(x)}C_{e^{2\psi(x)}}g(\phi)\}.
\]

4.1 Proof of Proposition 3.1 when \( s = \sigma \).

Our main tool will be to use the super divergence formula applied to \( I_g^\sigma(\phi) \) in order to show that the sublinear combination \( \sum_{s \in \Theta_\sigma} a_s C_\sigma(\gamma) \) in \( P(\gamma) \) is equal to a divergence modulo “better” correction terms.

Recall that \( I_g(\phi) := e^{n\phi}P(e^{2\phi}g) - P(\gamma) \) is the “image” of \( P(\gamma) \) under conformal variations of the metric \( g \); recall that \( I_g(\phi) \) consists of the terms in \( I_g(\phi) \) which have homogeneity \( \sigma \) in the function \( \phi \).

We have two tools at our disposal: Firstly, the “super divergence formula” for \( I_g(\phi) \). Secondly, we will momentarily show how the “worst piece” in \( P(\gamma) \) (i.e. the sublinear combination \( \sum_{s \in \Theta_\sigma} a_s C_\sigma(\gamma) \)) is in almost one-to-one correspondence with a particular sublinear combination in \( I_g^\sigma(\phi) \).

Let us flesh out the second remark: Observe that given the formula (4.1) for \( P(\gamma), I_g^\sigma(\phi) \) can be explicitly computed by applying the identities (2.9), (2.11) and (2.11). With a simple observation we can derive much more precise information:

Since \( I_g^\sigma(\phi) \) consists of terms of homogeneity \( \sigma \) in \( \phi \) and the minimum number of factors in \( P(\gamma) \) is \( \sigma \), we observe that the only complete contractions in \( P(\gamma) \) which can give rise to a term with \( \sigma \) factors in \( I_g(\phi) \) are the ones indexed in \( \Theta_\sigma \). In fact, we can derive more: For each \( s \in \Theta_\sigma \) we define \( C_{\sigma,i}^s(\phi) \) to stand for the complete contraction that arises from \( C_\sigma(\gamma) \) by replacing each of the factors \( \nabla_{r_1...r_n}P_{ij} \) by \( -\nabla_{r_1...r_n+i-j}^{|\phi|} \phi \). Then formulas (2.9), (2.11) and (2.11) imply that:

\[
I_g^\sigma(\phi) = \sum_{s \in \Theta_\sigma} a_s C_{\sigma,i}^s(\phi) + \sum_{k \in K} a_k C_k^\sigma(\phi), \tag{4.1}
\]

\(^39\)Observe that in the case \( s = \sigma \), all complete contractions in \( P(\gamma)|_{\Theta_\sigma} \) contain only factors \( \nabla_i^{|\phi|} P \).
where each $C^k_g(\psi)$ is a complete contraction in the form:

$$
\text{contr}(\nabla^{(m_1)}R_{ijkl} \otimes \cdots \otimes \nabla^{(m)}R_{ijkl'}) \otimes \nabla^{(p_1)}\text{Ric} \otimes \cdots \otimes \nabla^{(p_k)}\text{Ric} \otimes \nabla^{(p'_i)} \otimes \nabla^{(p'_j)} \otimes \cdots \otimes \nabla^{(p'_k)}),
$$

(4.2)

with length $\geq \sigma + 1$.

Now, we are ready to prove our Proposition 3.1 in this case $s = \sigma$.

Mini-outline of the proof of Proposition 3.1 when $s = \sigma$: The proof relies strongly on the super divergence formula. We will show that this formula implies that modulo terms with length $\geq \sigma + 1$:

$$
I^g(\phi) = \text{div}_i \sum_{r \in \mathcal{R}} a_r C^r, i_g(\phi),
$$

where each vector field $C^r, i_g(\phi)$ ($i$ is the free index) if a partial contraction in the form:

$$
p\text{contr}(\nabla^{(a)} \otimes \cdots \otimes \nabla^{(a_s)}),
$$

(4.3)

with one free index and $\nu_1, \ldots, \nu_s \geq 2$ so that modulo complete contractions of length $\geq \sigma + 1$:

$$
\sum_{s \in \Theta} a_s C^s, i_g(\phi) - \sum_{i \in I} a_i \text{div}_i C^i, i_g(\phi) = 0.
$$

(4.4)

Proof of (4.4): Recall the algorithm for the super divergence formula from [1]. By Lemma 20 in [1], we only need to restrict our attention to the good, hard and undecided descendants of each $C^s, i_g(\phi)$, $s \in \Theta_\sigma$. By Lemma 16 in [1], these will all be $\xi$-contractions $C^i, \xi_g(\phi, \xi)$ in the form:

$$
\text{contr}(\nabla^{(a_1 - \nu_1)} \otimes \cdots \otimes \nabla^{(a_s - \nu_s)} \otimes \xi \otimes \cdots \otimes \xi),
$$

(4.5)
where each factor $\vec{\xi}$ contracts against a factor $\nabla^{(\nu-a)}\phi$. Furthermore, since we have no factors $\Delta\phi$ in any $C^s_g(\phi)$, $s \in \Theta$, it follows that each $\nu - a \geq 2$. If a $\vec{\xi}$-contraction above has $M$ factors $\vec{\xi}$, we perform $M - 1$ integrations by parts. The correction terms that we introduce have length $\geq \sigma + 1$. So, we indeed derive (4.4). □

We then construct Riemannian vector fields $C^{s,j_{i}}(g)$ out of each Riemannian vector-valued differential operator $C^s_g(\phi)$ by substituting each factor $\nabla_i \nabla_j \phi$ by a factor $-\nabla^{(\nu-2)}_{a_1...a_{\nu-2}}P_{a_{\nu-1}a_{\nu}}$. We see that each $\text{div}_j C^{j_{i}}(g)$ is a linear combination:

$$\text{div}_j C^{j_{i}}(g) = (-1)^{\sigma} \sum_{s \in S} a_{s} C^{s}(g)$$

of complete contractions in the form:

$$\text{contr}(\nabla^{(m')_{i}} P \otimes \cdots \otimes \nabla^{(m')_{P}}).$$

**Derivation of Proposition 3.1 (with $s = \sigma$) from (4.4):** We use the fact that (4.4) holds formally (see [1] for a definition of this notion). We then repeat the sequence of permutations of indices by which we make the linearization of the left hand side of (4.4) formally zero to the linear combination:

$$P(g)|_{\Theta} - \text{div} \sum_{i \in I} C^{s,j_{i}}(g).$$

It follows that we can also make the above formally equal also, modulo introducing correction terms by virtue of the identities

$$\nabla_i \nabla_j X_l - \nabla_j \nabla_i X_l = R_{ijkl}X_k$$

and

$$\nabla_a P_{bc} - \nabla_b P_{ac} = \frac{1}{n-3} \nabla^d W_{abcd}.$$

Observe that the correction terms that we obtain by virtue of the above identities are precisely in the form allowed by our Proposition 3.1. This concludes the proof of our claim in this case. □

**Proof of Proposition 3.1 (when $s = \sigma$) in the general case (without the simplifying assumption).**

We now consider the case where the complete contractions $C^l(g)$, $l \in \Theta$, are allowed to contain factors $P_i^l$.

In this case we observe that if $C^l(g)$ contains $A$ factors $P_i^l$, then $C^{l,i}_{g}(\phi)$ will contain $A$ factors $\Delta\phi$. Recall the super divergence formula for $I^g_{\sigma}(\phi)$:

$$\Sigma_{l \in \Theta} \alpha_{l} C^{l,i}_{g}(\phi) = \Sigma_{k \in K} a_{k} \text{div}_k C^{l,i}_{g}(\phi),$$

modulo complete contractions in the form (4.2) with length $\geq \sigma + 1$.

The problem is, now, that there might be vector fields $C^{k,i\nu}_{g}(\phi)$ which are in the form (4.3) with one free index and with a factor $\nabla_i \phi$. Hence the procedure carried out for the previous simple case cannot be carried over to this case (because we can not replace the factors $\nabla \phi$, with only one derivative).
So, in this case, we claim the following:

**Lemma 4.1** Consider (4.6). There is a subset of the vector fields \( \{ C^{k,i}g(\phi) \}_{k \in K} \), indexed in \( K^\sharp \), \( \{ C^{k,i}g(\phi) \}_{k \in K^\sharp} \), in the form (4.3), with the property that each \( C^{k,i}g(\phi) \), \( k \in K^\sharp \) contains factors \( \nabla^{(l)} \phi \) with \( l \geq 2 \), so that:

\[
\sum_{l \in \Theta} a_l C^{l,i}g(\phi) - \sum_{k \in K^\sharp} a_k \text{div}_k C^{k,i}g(\phi) = 0 \tag{4.7}
\]

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modulo complete contractions of length \( \geq \sigma + 1 \).

Let us notice that if we can prove the above, we can then repeat the argument from the previous case, by using (4.7). Hence, we will have proven our Proposition 3.1 (when \( s = \sigma \)) in full generality.

**Proof of Lemma 4.1** We will construct the set \( K^\sharp \subset K \).

We consider the set of good, hard or undecided descendants (see the last definition in subsection 5.1 in [1] for a description of these notions) of the complete contractions \( C^{l,i}g(\phi) \), \( l \in \Theta, \xi \) -length \( \sigma \), and proceed to integrate by parts as explained in the algorithm for the super divergence formula in [1]. We impose the restriction that any factor \( \xi \) which contracts against a factor \( \nabla \phi \) will not be integrated by parts, provided there is another factor \( \xi \) does not contract against a factor \( \nabla \phi \). Furthermore, whenever along the iterative integration by parts we obtain a \( \xi \)-contraction of \( \xi \)-length \( \sigma \) whose only factors \( \xi \) contract against a factor \( \nabla \phi \), we cross it out and index it in the set \( H \). The \( \xi \)-contractions that are not crossed out give rise to the divergences \( \{ a_k \text{div}_k C^{k,i}g(\phi) \}_{k \in K^\sharp} \).

Thus, we derive the equation:

\[
\sum_{l \in \Theta} a_l C^{l,i}g(\phi) + \sum_{k \in K^\sharp} a_k \text{div}_k C^{k,i}g(\phi) + PO[\sum_{h \in H} a_h C^{h,i}g(\phi, \xi)] = 0, \tag{4.8}
\]

which holds modulo complete contractions of length \( \geq \sigma + 1 \). By construction, each vector field \( C^{k,i}g(\phi) \) has length \( \sigma \) and is in the form (4.3) with each \( \nu_i \geq 2 \).

Therefore, it suffices to show:

\[
PO[\sum_{h \in H} a_h C^{h,i}g(\phi, \xi)] = 0, \tag{4.9}
\]

modulo complete contractions of length \( \geq \sigma + 1 \). Hence it would suffice to show:

\[
\sum_{h \in H} a_h C^{h,i}g(\phi, \xi) = 0, \tag{4.10}
\]

modulo \( \xi \)-contractions of \( \xi \)-length \( \geq \sigma + 1 \).

We do this as follows: Notice that in any \( \xi \)-contraction \( C^{h,i}g(\phi, \xi), h \in H \), the function \( \phi \) appears only in expressions \( \nabla_{\xi} \xi \phi \xi^2 \), or in factors \( \nabla^{(a)} \phi \) with \( a \geq 2 \). Let us consider the \( \xi \)-contraction with the maximum number \( M \) of factors \( \nabla_{\xi} \phi \xi^2 \).
Suppose they are indexed in $H^M \subset H$. Notice that $M < \sigma$, otherwise we would have $\sigma = \frac{n}{2}$ (we see this by considering the weight). If we can show that:

$$\sum_{h \in H^M} a_h C^h_g (\phi, \bar{\xi}) = 0 \quad (4.11)$$

modulo $\bar{\xi}$-contractions of $\bar{\xi}$-length $\geq \sigma + 1$, then (4.11) will follow by induction.

We write each $C^h_g (\phi, \bar{\xi})$ with $h \in H^M$ as follows:

$$C^h_g (\phi, \bar{\xi}) = C^*_h g (\phi) \cdot (\nabla_k \phi \bar{\xi}^k)^M.$$  

For any $h \in H^M$, we then define $PO^*[C^h_g (\phi, \bar{\xi})]$ to stand for the sublinear combination in $PO[C^h_g (\phi, \bar{\xi})]$ which arises as follows: We integrate by parts with respect to each factor $\bar{\xi}^k$ and then force each derivative $\nabla_k$ to hit a factor $\nabla^{(a)}_1 ... r_a \phi$ ($a \geq 2$) in $C^*_h g (\phi)$. We define

$$PO^- [C^h_g (\phi, \bar{\xi})] := PO[C^h_g (\phi, \bar{\xi})] - PO^*[C^h_g (\phi, \bar{\xi})].$$

Notice that by definition, each complete contraction of length $\sigma$ in $PO^- [C^h_g (\phi, \bar{\xi})]$ will have strictly fewer than $M$ factors $\nabla \phi$.

We write out the super divergence formula as follows:

$$I^g_\sigma (\phi) + \sum_{k \in K} a_h \text{div} v_{k} C^k_{g,i} (\phi) + \sum_{h \in H \setminus H^M} a_h PO[C^h_g (\phi, \bar{\xi})] + \sum_{h \in H^M} a_h \{PO^*[C^h_g (\phi, \bar{\xi})] + PO^- [C^h_g (\phi, \bar{\xi})]\} = 0,$$  

modulo complete contractions of length $\geq \sigma + 1$.

Now, let us observe: Each complete contraction in (4.12) that does not belong to $\sum_{h \in H^M} a_h \{PO^*[C^h_g (\phi, \bar{\xi})] + PO^- [C^h_g (\phi, \bar{\xi})]\}$ will have fewer than $M$ factors $\nabla \phi$. This follows from the fact that $M$ is the maximum number of factors $\nabla^k \phi \bar{\xi}^k$ among the $\bar{\xi}$-contractions $C^h_g (\phi, \bar{\xi})$, and since each complete contraction $C^l_{g,i} (\phi)$, $l \in \Theta_{\sigma}$ and each vector field $C^k_{g,i} (\phi)$ have only factors $\nabla^{(a)} \phi$, $a \geq 2$, by construction.

We now claim that

$$\sum_{h \in H^M} a_h PO^*[C^h_g (\phi, \bar{\xi})] = 0,$$  

modulo complete contractions of length $\geq \sigma + 1$. This holds because (4.12) holds formally, and since (4.13) is the sublinear combination in (4.12) of complete contractions of length $\sigma$ with $M$ factors $\nabla \phi$.

Now, (4.13) also holds formally. Write out:

$$\sum_{h \in H^M} a_h PO^*[C^h_g (\phi, \bar{\xi})] = \sum_{t \in T} a_t C^t_{g} (\phi),$$

where each complete contraction $C^t_{g} (\phi)$ is in the form:

$$\text{contr}(\nabla^{(m_1)}_{r_1} \phi \otimes \cdots \otimes \nabla^{(m_{\sigma - M})}_{1 \cdots m_{\sigma - M}} \phi \otimes \nabla_{y_1} \phi \otimes \cdots \otimes \nabla_{y_M} \phi). \quad (4.14)$$
We observe that the linear combination \( \Sigma_{t \in T} a_t C_g^t(\phi) \) arises from the linear combination \( \Sigma_{h \in H} a_t C^h_g(\phi) \cdot (\nabla_k \phi \xi^k)^M \) by making each factor \( \xi^k \) into a derivative \( \nabla_k \), then allowing the derivative \( \nabla_k \) to hit any of the factors \( \nabla^{(A)} \phi \) in \( C^h_g(\phi) \) and adding all the complete contractions we thus obtain.

In particular, each factor \( \nabla_k \phi \) in any \( C_g^t(\phi) \) contracts against a factor \( \nabla^{(a)} \phi \), \( a \geq 3 \).

Now, for each \( C_g^t(\phi) \) let \( \tilde{C}_g^t(\phi) \) stand for the complete contraction of weight \( -n+2M \) which arises from \( C_g^t(\phi) \) by erasing each factor \( \nabla_i \phi \) and also erasing the index against which \( i \) contracts. Since (4.13) holds formally, it follows that:

\[
\Sigma_{t \in T} a_t \tilde{C}_g^t(\phi) = 0. \tag{4.15}
\]

But (4.15) just tells us that:

\[
\sum_{h \in H} a_h (\sigma - M)^M C^h_g(\phi) = 0.
\]

Therefore, we have shown (4.11). \( \square \)

We have fully proven the Proposition 3.1 when \( s = \sigma \).

5 Proposition 3.1 in the hard case (where \( s < \sigma \)).

5.1 Technical Tools:

Useful identities: Now, we will put down a few identities that will prove useful later on.

**Decomposition of the Weyl tensor:** Recall the Weyl tensor \( W_{ijkl} \), see (2.4). Consider the tensor \( T = \nabla^{r_1 \cdots r_m} \nabla_{r_1 \cdots r_m} W_{ijkl} \) where each index \( r_m \) is contracting against the (derivative) index \( r_m \), and all the other indices are free. We have then introduced the language convention that the tensor \( T \) has \( x \) internal contractions.

We will decompose the tensor \( T \) into a linear combination of tensors in the form \( \nabla^{(m)} R_{ijkl} \). By just applying formula (2.4) we find:

\[
\nabla^{r_1 \cdots r_x} \nabla_{r_1 \cdots r_m} W_{ijkl} = \nabla^{r_1 \cdots r_x} \nabla_{r_1 \cdots r_m} R_{ijkl} + \sum_{z \in Z^{x+1}} a_z T^z(g) + \sum_{z \in Z^{x+2}} a_z T^z(g), \tag{5.1}
\]

where \( \sum_{z \in Z^{x+1}} a_z T^z(g) \) stands for a linear combination of tensor products of the form \( \nabla^{r_1 \cdots r_x} \nabla_{r_1 \cdots r_m} Ri_{sq} \otimes g_{ab} \) in the same free indices as \( T \), with the feature that there are a total of \( x+1 \) internal contractions in the tensor \( \nabla^{(m)} Ric_{sq} \) (including the one in the tensor \( Ric_{ab} = R^a_{rub} \) itself). \( \sum_{z \in Z^{x+2}} a_z T^z(g) \) stands

\[40\text{A rigorous proof of this fact can be found in the Appendix below—see the operation \textit{Erase}.}\]
for a linear combination of tensor products of the form $\nabla^{r_{a_1} \cdots r_{a_x}} \nabla^{(m)\,r_{1} \cdots r_{m}} R \otimes g_{b \otimes h_{ij}}$ ($R$ stands for the scalar curvature) in the same free indices as $T$, with the feature that there are a total of $x + 2$ internal contractions in the tensor $\nabla^{(m)\,R}$ (including the two in the factor $R = R_{st}^{tt}$ itself). If $m > 0$ we will use the contracted second Bianchi identity to think of $\nabla^{r_{a_1} \cdots r_{a_x}} \nabla^{(m)\,r_{1} \cdots r_{m}} R$ as a factor $2\nabla^{r_{a_1} \cdots r_{a_x} r_{x+1}} \nabla^{(m-1)\,r_{1} \cdots r_{m-1}} \text{Ric}_{r_{m} r_{x+1}}$, modulo introducing quadratic correction terms.

Next useful identity: We consider a factor $T$ in the form

$$T = \nabla^{r_{a_1} \cdots r_{a_x}} \nabla^{(m)\,r_{1} \cdots r_{m}} W_{m+1 \cdots m+2 \cdots m+4}$$

where again each of the indices $r_{a_{u}}$ is contracting against the index $r_{a_{u}}$, and moreover now at least one of the indices $r_{a_{v}}$ is contracting against one of the internal indices $r_{m+1}, \ldots, r_{m+4}$. Then we calculate:

$$\nabla^{r_{a_1} \cdots r_{a_x}} \nabla^{(m)\,r_{1} \cdots r_{m}} W_{m+1 \cdots m+2 \cdots m+4} = \frac{n-3}{n-2} \nabla^{r_{a_1} \cdots r_{a_x}} \nabla^{(m)\,r_{1} \cdots r_{m}} R_{m+1 \cdots m+2 \cdots m+4}$$

$$+ \sum_{z \in \mathbb{Z}^{x+1}} a_{z} T^{z}(g) + \sum_{z \in \mathbb{Z}^{x+1}} a_{z} T^{z}(g),$$

(5.2)

where $\sum_{z \in \mathbb{Z}^{x+1}} a_{z} T^{z}(g)$ stands for the same generic linear combination as before. $\sum_{z \in \mathbb{Z}^{x+1}} a_{z} T^{z}(g)$ only appears in the case where there are two indices $r_{a_{u}}$, $r_{a_{v}}$ contracting against two internal indices in $W_{ijkl}$ (and moreover the indices $r_{a_{b}}, r_{a_{c}}$ do not belong to the same block $[ij], [kl]$). $\sum_{z \in \mathbb{Z}^{x+1}} a_{z} T^{z}(g)$ stands for a linear combination of tensors $\nabla^{r_{a_1} \cdots r_{a_{x-1}}} \nabla^{(m)\, \text{Ric}_{ab}}$ with $x$ internal contractions (also counting the internal contraction in the factor $\text{Ric}_{ab}$ itself), and with the extra feature that one of the indices $r_{a_{1}}, \ldots, r_{a_{x-1}}$ is contracting against one of the internal indices $a_{b}$ in $\text{Ric}_{ab}$.

The “fake” second Bianchi identities for the derivatives of the Weyl tensor: We recall that the Weyl tensor $W_{ijkl}$ is antisymmetric in the indices $i, j$ and $k, l$, and also $W_{ijkl} = W_{kl ij}$. It also satisfies the first Bianchi identity. Nevertheless, it does not satisfy the second Bianchi identity. We now present certain substitutes for the second Bianchi identity.

Firstly, if the indices $r, i, j, k, l$ are all free then:

$$\nabla_{r} W_{ijkl} + \nabla_{j} W_{rikl} + \nabla_{i} W_{jkl} = \sum (\nabla^{s} W_{aryt} \otimes g),$$

(5.3)

where the symbol $\sum (\nabla^{s} W_{aryt} \otimes g)$ stands for a linear combination of a tensor product of the three-tensor $\nabla^{s} W_{aryt}$ (i.e. essentially the Cotton tensor) with an un-contracted metric tensor. The exact form of $\sum (\nabla^{s} W_{aryt} \otimes g)$ is not important for our study so we do not write it down.

On the other hand, if the indices $i, j, k, l$ are free we then have:

$$\nabla_{i} W_{ijkl} + \frac{n-2}{n-3} \nabla_{j} W_{ikl} + \frac{n-2}{n-3} \nabla_{k} W_{jkl} = \sum (W_{i} g) + \sum Q(R),$$

(5.4)
where the symbol $\sum (W, g)$ stands for a linear combination of tensor products: $\nabla^{ik} W_{iakb} \otimes g_{cd}$ (where $g_{cd}$ is an un-contracted metric tensor—note that there are two internal contractions in the factor $\nabla^{ik} W_{ijkl}$) and the symbol $\sum Q(R)$ stands for some linear combination of quadratic expressions in the curvature tensor. Again the exact form of these expressions is not important so we do not write them down.

On the other hand, if the indices $r, i, j, l$ are free then:

$$\nabla^k \nabla_r W_{ijkl} + \nabla^k \nabla_j W_{rikl} + \nabla^k \nabla_i W_{jrkl} = \sum Q(R). \quad (5.5)$$

Furthermore, we have that the analogue of the second Bianchi identity clearly holds if both the index $r$ and one of the indices $i, j$ are involved in an internal contraction:

$$\nabla^r \nabla_r W_{ijkl} + \nabla^r \nabla_j W_{rikl} + \nabla^r \nabla_i W_{jrkl} = \sum Q(R). \quad (5.6)$$

Lastly, we also note the identity:

$$\Delta \nabla^k W_{ijkl} + \nabla^k \nabla_r W_{ijkl} + \nabla^k \nabla_i W_{jrkl} = \sum Q(R). \quad (5.7)$$

Let us also recall the identity:

$$\nabla_a P_{bc} - \nabla_b P_{ac} = \frac{1}{n-3} \nabla^d W_{abcd}. \quad (5.8)$$

These identities will be useful in the context of the next formal constructions.

The operations “Weylify” and “Ricciify”: These two operations are formal operations that act on complete contractions in the forms (5.9), (5.14) and produce complete contractions in the forms (5.10), (5.15), respectively. We will show two important technical Lemmas concerning these two formal operations, Lemmas 5.1 and 5.2.

For the first construction, we will be considering complete and partial contractions (with no internal contractions) in the form:

$$\text{contr}(\nabla^{(m_1)} R_{ijkl} \otimes \cdots \otimes \nabla^{(m_s)} R_{ijkl} \otimes \nabla^{(p_1)} \psi \otimes \cdots \otimes \nabla^{(p_q)} \psi \otimes \nabla \psi \otimes \cdots \otimes \nabla \psi) \quad (5.9)$$

($\psi$ is a scalar) with the following restrictions: In each complete contraction and vector field there are $a \geq 0$ factors $\nabla \psi$ ($a$ is fixed) and $q$ factors $\nabla^{(p)} \psi$ ($q$ also fixed). We require that none of the factors $\nabla \psi$ are contracting between themselves and none of them contains a free index. Furthermore, we require that for any factor $\nabla^{(p)} \psi$ which is not contracting against a factor $\nabla \psi$, $p \geq 2$.

**Definition 5.1** We consider a collection of complete contractions, $\{C_r^a(\psi^q, \psi^a)\}_{r \in L}$ and a collection of such vector fields $\{C_r^q(\psi^q, \psi^a)\}_{r \in R}$ in the form (5.9). Assume that the complete contractions and vector fields above all have a given length $\tau + a$. 

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We define an operation \( Weylify'[\ldots] \) that acts on the contractions and vector fields above by performing the following operations: Each factor \( \nabla_{r_1,\ldots,r_m} R_{ijkl} \) that is not contracting against a factor \( \nabla v \) is replaced by a factor \( \nabla_{r_1,\ldots,r_m} W_{ijkl} \). Also, each factor \( \nabla_{r_1,\ldots,p} \psi \) that is not contracting against any factor \( \nabla v \) is replaced by a factor \( \nabla_{r_1,\ldots,p-2} P_{r_{p-1}r_p} \).

Now, any factor \( T = \nabla_{r_1,\ldots,m} R_{ijkl} = \nabla_{r_1,\ldots,m} W_{ijkl} \) that is contracting against \( s > 0 \) factors \( \nabla v \), with the restriction that all these \( s \) factors \( \nabla v \) are contracting against the factors \( \nabla v \). Then, we replace \( T \) by \( \nabla_{r_1,\ldots,r_x} \nabla_{r_1,\ldots,r_m} W_{ijkl} = \nabla_{r_1,\ldots,r_m} W_{ijkl} \).

On the other hand, if there are internal indices also contracting against factors \( \nabla v \), we replace \( T \) by \( \nabla_{r_1,\ldots,r_x} \nabla_{r_1,\ldots,r_m} W_{ijkl} = \nabla_{r_1,\ldots,r_m} W_{ijkl} \). Now, each factor \( \nabla_{r_1,\ldots,r_p} \psi \) with \( p \geq 2 \) that is contracting against \( w \) factors \( \nabla v \) (say the indices \( r_{a_1},\ldots,r_{a_w} \)) is replaced by \( \nabla_{r_1,\ldots,r_w} \nabla_{r_1,\ldots,r_p-2} P_{r_{p-1}r_p} \). Finally, every expression \( \nabla_{r_1,\ldots,r_p} \psi \nabla v \) is replaced by a factor \( P_{\alpha} \). In the end, we also erase all the factors \( \nabla v \) (they were left uncontracted).

Thus, by acting on the complete contractions and vector fields in the form \ref{5.9}, with the operation \( Weylify'[\ldots] \), we obtain complete contractions and vector fields of length \( \tau \) in the form:

\[
\begin{align*}
\text{contr}(\nabla_{f_1,\ldots,f_s} \nabla_{r_1,\ldots,r_m} W_{ijkl} \otimes \cdots \otimes \nabla_{u_1,\ldots,u_p} \nabla_{r_1,\ldots,r_m} W_{ijkl} \otimes \cdots) \\
\nabla_{a_1,\ldots,a_1} \psi_{ij} \otimes \cdots \otimes \nabla_{a_1,\ldots,a_1} \psi_{ij} \otimes P_{ij} \otimes P_{\alpha} \otimes \cdots \otimes P_{\psi} \psi_{ij} \otimes \cdots,
\end{align*}
\tag{5.10}
\]

where we are making the following conventions: In each factor \( \nabla_{f_1,\ldots,f_s} \nabla_{r_1,\ldots,r_m} W_{ijkl} \) each of the the indices \( f_1,\ldots,f_s \) contracts against one of the indices \( r_1,\ldots,l \), while no two of the indices \( r_1,\ldots,l \) contract between themselves. On the other hand, for each factor \( \nabla_{y_1,\ldots,y_r} \nabla_{a_1,\ldots,a_\alpha} P_{ij} \), each of the upper indices \( y_1,\ldots,y_r \) contracts against one of the indices \( a_1,\ldots,a_\alpha, i, j \). Moreover, none of the indices \( a_1,\ldots,a_\alpha, i, j \) contract between themselves.

**Definition 5.2** Consider any complete contraction (or tensor field) of the form \ref{5.11}, with the properties described above. We will let \( \delta_W \) stand for the number of internal contractions in all the factors \( \nabla_{r_1,\ldots,r_m} W_{ijkl} \), and the number \( \delta_P \) to stand for the number of internal contractions among all the factors \( \nabla_{(\psi)} P_{ij} \).

We see that for a contraction or vector field \( C_g(\psi^q, \nu^\alpha) \) in the form \ref{5.9}, the complete contraction or vector field \( Weylify[\nabla_{r_1,\ldots,r_m} W_{ijkl}, \nabla_{a_1,\ldots,a_\alpha} P_{ij}] \) will have length \( \tau \) and a total of \( q \) factors in the form \( \nabla_{r_1,\ldots,r_m} W_{ijkl} \), and \( \delta_W + \delta_P = \alpha \). This operation extends to linear combinations of contractions. One last definition prior to stating our Lemma:

**Definition 5.3** For any vector field \( C_{a}^i \) in the form \ref{5.9}, \( X \psi C_{a}^i \) will stand for the sublinear combination in \( \psi C_{a}^i \) where \( \nabla_i \) is not allowed to hit the factor to which the free index \( i \) belongs, nor any of the factors \( \nabla v \).
Now, our claim regarding the operation \textit{Weylify} is the following:

\textbf{Lemma 5.1} \ Assume an equation:

\[
\sum_{l \in L} a_l C_g^l(\psi^q, \nu^a) - \text{div}_i \sum_{r \in R} a_r C_g^{r,i}(\psi^q, \nu^a) = 0, \tag{5.11}
\]

that holds modulo complete contractions of length \( \geq \tau + a + 1 \). Here the contractions and tensor fields are in the form \((5.9)\) with length \( \tau + a \). We claim:

\[
\sum_{l \in L} a_l \text{Weylify}[C_g^l(\psi^q, \nu^a)] - \text{div}_i \sum_{r \in R} a_r \text{Weylify}[C_g^{r,i}(\psi^q, \nu^a)]
= \sum_{d \in D_1} a_d C^d(g) + \sum_{d \in D_2} a_d C^d(g), \tag{5.12}
\]

where each \( C^d(g) \) is in the form \((5.11)\) (with length \( \tau \)) and moreover if \( d \in D_1 \) then \( C^d(g) \) has less than \( q \) factors \( \nabla^i P \), while if \( d \in D_2 \) then \( C^d(g) \) has \( q \) factors \( \nabla^i P \) but also \( \delta_W + \delta_P \geq a + 1 \). This equation holds modulo complete contractions of length \( \geq \tau + 1 \).

\textbf{Proof:} We will use the fact that \((5.11)\) holds formally to \textit{repeat} the formal applications of identities that make the LHS of \((5.11)\) formally zero to the LHS of \((5.12)\); the RHS of \((5.12)\) will then arise as correction terms in this process. Now, we first observe that it would be sufficient to show that

\[
\text{Weylify}\left\{ \sum_{l \in L} a_l C_g^l(\psi^q, \nu^a) \right\} - \text{div}_i \sum_{r \in R} a_r \text{Weylify}[C_g^{r,i}(\psi^q, \nu^a)]
\]

is equal to the right hand side of \((5.12)\). That this is sufficient is clear because the contraction that arises in each

\[
\text{div}_i \text{Weylify}[C_g^{r,i}(\psi^q, \nu^a)]
\]

when \( \nabla^i \) hits the factor to which \( i \) belongs is clearly in the form \( C^d(g), d \in D_2 \), and moreover because for each \( r \in R \)

\[
\text{Weylify}\{X \text{div}_i C_g^{r,i}(\psi^q, \nu^a)\} = X \text{div}_i \text{Weylify}[C_g^{r,i}(\psi^q, \nu^a)],
\]

modulo contractions of length \( \geq \tau + 1 \).

Next, we use the fact that \((5.11)\) holds exactly (with no correction terms) at the linearized level (i.e. if we replace each complete contraction \( C_g(\psi^q, \nu^a) \) by \( \text{lin} C_g(\psi^q, \nu^a) \)). We “memorize” the sequence of permutations of indices (and applications of the distributive rule) by which we can make the linearization of \((5.11)\) formally zero. We may then repeat the same sequence of permutations to the left hand side of \((5.12)\), to make it vanish, modulo introducing correction terms, as follows:

\footnote{See the introduction of \cite{1} for a definition of linearization.}
1. We introduce correction terms of length $\geq \tau + 1$ by virtue of (2.1) when we permute derivative indices in a factor $\nabla^{(m)}W_{ijkl}$ or when we permute the first $p-2$ derivative indices in a factor $\nabla^{(p-2)}P_{ij}$.

2. We introduce correction terms of the form $\sum_{d \in D^2} a_d C^d(g)$ by virtue of (5.9) when we apply the “fake” second Bianchi identity to the indices $r_m, i, j$ in a factor $\nabla^{(m)}_{\tau_{r_{m}}...r_{m}}W_{ijkl}$ with no internal contractions involving internal indices.

3. We introduce correction terms of length $> \tau$ or of the form $\sum_{d \in D^2} a_d C^d(g)$, by virtue of the identities (5.3), (5.7) when we apply the “fake” second Bianchi identity to the indices $r_m, i, j$ in a factor $\nabla^{(m)}_{\tau_{r_{m}}...r_{m}}W_{ijkl}$ with one or two internal contractions respectively.

4. We introduce correction terms of the form $\sum_{d \in D^1} a_d C^d(g)$ from the right hand side of (5.8) when we want to switch the indices $r_{m-2}, r_{m-1}$ in a factor $\nabla^{(p)}_{\tau_{r_{m-2}}...r_{m-1}}P_{r_{m-1}r_{m}}$.

That completes the proof of our claim. □

The operation Riccify: We now define the operation $Riccify$ that acts on complete contractions $C(\Omega^i, \psi^a, \nu^a)$ and vector fields $C'(\Omega^i, \psi^a, \nu^a)$ in the form:

$$
\text{contr}(\nabla^{(m)} R_{ijkl} \otimes \cdots \otimes \nabla^{(m)} R_{ijkl} \otimes \nabla^{(a_1)} \psi_1 \otimes \nabla^{(a_2)} \psi_2) \nabla^{(p)} \Omega \otimes \cdots \otimes \nabla^{(p)} \Omega \otimes \nabla \psi \otimes \cdots \otimes \nabla \psi
$$

(5.14)

with length $\tau + a$ (and with $a$ factors $\nabla \psi$), where both the $s$ factors $\nabla^{(u)} \psi_h$ and the $q$ factors $\nabla^{(p)} \Omega$ are subject to the same restrictions as for the factors $\nabla^{(p)} \psi$ in the contractions in the form (5.9). In particular: In each complete contraction and vector field in the above form there are $a \geq 0$ factors $\nabla \psi$ (a is fixed) and $q$ factors $\nabla^{(p)} \psi$ (q also fixed). Also, none of the factors $\nabla \psi$ are contracting between themselves and none of them contains a free index. Also, we require that any factor $\nabla^{(p)} \Omega$ or $\nabla^{(p)} \psi$ which is not contracting against a factor $\nabla \psi$ must have $p \geq 2$. Moreover, we assume that the complete contractions and vector fields above all have a fixed length $\tau + a$.

Definition 5.4 We define an operation $Ricci\mathit{fy}[\ldots]$ that acts on complete and partial contractions in the form (5.14) as follows: We replace each factor $\nabla^{(m)}_{\tau_{r_{m}}...r_{m}}R_{ijkl} R_{r_{m+1}r_{m+2}r_{m+3}r_{m+4}}$ for which the indices $r_{a_1}, \ldots, r_{a_d}$ are contracting against factors $\nabla \psi$ by a factor $\nabla^{r_{a_1}...r_{a_d}} \nabla^{(m)}_{\tau_{r_{m}}...r_{m}}R_{ijkl} R_{r_{m+1}r_{m+2}r_{m+3}r_{m+4}}$. We also replace each factor $\nabla^{(p)}_{\tau_{r_{p}}...r_{p}} \Omega$ for which the indices $r_{a_1}, \ldots, r_{a_d}$ are contracting against factors $\nabla \psi$ by a factor $\nabla^{r_{a_1}...r_{a_d}} \nabla^{(p)}_{\tau_{r_{p}}...r_{p}} \Omega$. Then, we replace all expressions $\nabla \psi \nabla \psi$ by a factor $\nabla \psi$. Finally, we erase each factor $\nabla^{r_{r_1}...r_a} \psi_h$ (for which the indices $r_{a_1}, \ldots, r_{a_d}$ are contracting against factors $\nabla \psi$) by an expression $\nabla^{r_{r_1}...r_a} \psi_h$. In the end we also erase all the factors $\nabla \psi$ (they have been left uncontracted).
Thus acting by the operation \textit{Riccify} on complete and partial contractions in the form (5.14) we obtain complete and partial contractions in the form:

\[
\text{contr}(\nabla f_1 \cdots \nabla f_m R_{ijkl} \otimes \cdots \otimes \nabla u_1 \cdots u_r ) \nabla (m_1) R_{ijkl} \otimes \nabla v_1 \cdots v_s \nabla (a_1) \psi_1
\]

\[
\otimes \cdots \otimes \nabla q_1 \cdots q_w \nabla (a_s) \psi_s \otimes \nabla c_1 \cdots c_t \nabla (u_z) \text{Ric}_{ij} \otimes \cdots \otimes \nabla a_1 \cdots a_t \nabla (a_t) \text{Ric}_{ij}.
\]

(5.15)

\textbf{Definition 5.5} For contractions in the form (5.15) we define \(\delta_R\) to stand for the total number of internal contractions in the factors \(\nabla (m) R_{ijkl}\) and \(\delta_{\text{Ric}}\) to stand for the total number of internal contractions in the factors \(\nabla (p) \text{Ric}\) (including the one in the factors \(\text{Ric}\) themselves) and also \(\delta_\psi\) to stand for the total number of internal contractions in the factors \(\nabla (a) \psi\).

(Note: In the future we will sometimes denote this operation \textit{Riccify} by \(\Omega_{\text{toRic}}\)).

\textbf{Note:} In (5.14), we may have \(s = 0\). Furthermore, we recall from Definition 5.3 that if \(C^i_g\) is a vector field in the form (5.14) then \(X_{\text{div}} i\) will stand for the sublinear combination in \(\text{div}_i C^i_g\) where \(\nabla_i\) is not allowed to hit the factor to which the free index \(i\) belongs, nor any of the factors \(\nabla v\).

Our Lemma is then the following:

\textbf{Lemma 5.2} Assume an equation:

\[
\sum_{l \in L} a_l C^l_g(\Omega^g, \psi^a) - X_{\text{div}} i \sum_{r \in R} a_r C^r_g(\Omega^g, \psi^a) = 0,
\]

which holds modulo complete contractions of length \(\geq \tau + a + 1\).

We claim:

\[
\sum_{l \in L} a_l \text{Ric} f(y)[C^l_g(\Omega^g, \phi^a)] - X_{\text{div}} i \sum_{r \in R} a_r \text{Ric} f(y)[C^r_g(\Omega^g, \psi^a)]
\]

\[
= \sum_{d \in D^1} a_d C^d_g(\phi^a) + \sum_{d \in D^2} a_d C^d_g(\psi^a),
\]

(5.17)

where each \(C^d_g(\phi^a)\) is in the form (5.15) (with length \(\tau\)) and moreover if \(d \in D^1\) we will have that \(C^d_g(\phi^a)\) has less than \(q\) factors \(\nabla (p) \text{Ric}\) but will also have \(\delta_R + \delta_{\text{Ric}} + \delta_\psi \geq a\), while if \(d \in D^2\) then \(C^d_g(\phi^a)\) has \(q\) factors \(\nabla (p) \text{Ric}\) but also \(\delta_R + \delta_{\text{Ric}} + \delta_\psi \geq a + 1\). This equation holds modulo complete contractions of length \(\geq \tau + 1\).

\textbf{Proof:} The proof is an easier version of the proof of the previous Lemma. We use the fact that (5.16) holds formally and we repeat the applications of the formal identities and the distributive rule that make (5.16) formally zero to the LHS of (5.17).

Now, in (5.17), we use the identity \(\nabla_i R = 2 \nabla^k \text{Ric}_{ik}\) (\(R\) here is the scalar curvature) once if needed, and we may assume that all the complete contractions
in the LHS of (5.17) have any factors $\nabla^{(p)}Ric$ (i.e. factors involving the Ricci curvature) being in the form $\nabla f_1 \ldots f_b \nabla_{r_1 \ldots r_p}Ric_{a b}$ (where each of the indices $f_1, \ldots, f_b$ is contracting against one of the indices $r_1, \ldots, r_p, a, b$, and none of the lower indices are contracting between themselves), or in the form $R$ (scalar curvature).

Furthermore, when we repeat the permutations by which the LHS of (5.16) is made formally zero to the LHS of (5.17), we may assume wlog that the upper indices in each factor $\nabla f_1 \ldots f_b \nabla_{r_1 \ldots r_p}Ric_{a b}$ are not permuted (since they correspond to factors $\nabla \upsilon$ in the LHS of (5.16)).

Therefore, the RHS in (5.17) can arise either when the divergence index $\nabla_i$ in $\text{div}_{\nabla}Riccify[C^r_i(\Omega^m, \psi^s, \nu^a)]$ hits the factor to which $i$ belongs, or by virtue of the identity:

$$\nabla a Ric_{bc} - \nabla b Ric_{ac} = \nabla d R_{abcd}$$

(where by the observation above the indices $a, b, c$ will not be contracting against each other). □

### 5.2 Proof of Proposition 3.1 when $s < \sigma$: Reduction to an inductive statement.

In the rest of this section we will explain how to derive Proposition 3.1 in the case where $\sigma \geq 3$. The cases $\sigma = 1, \sigma = 2$ will be covered in the paper [5] in this series.

Recall (see the discussion above Definition 5.2) that we are assuming that for $P(g), \Theta_s \neq \emptyset$ and $\Theta_h = \emptyset$ for each $h > s$. We write $P(g)|_{\Theta_s}$ as a linear combination:

$$P(g)|_{\Theta_s} = \sum_{l \in L} a_l C^l(g)$$

(modulo longer complete contractions), where each $C^l(g)$ is of the form:

$$\text{contr}(\nabla f_1 \ldots f_b \nabla_{r_1 \ldots r_p} W_{ijkl} \otimes \ldots \otimes \nabla s_1 \ldots s_z \nabla_{k_1 \ldots k_m} W_{i' j' k' l'} \otimes \nabla y_1 \ldots y_t \nabla_{a_1 \ldots a_u} P_{ij} \otimes \ldots \otimes \nabla u_1 \ldots u_v \nabla_{b_1 \ldots b_o} P_{ij'} \otimes (P.a)^K),$$

with the usual conventions: In each factor $\nabla f_1 \ldots f_b \nabla_{r_1 \ldots r_p} W_{ijkl}$ each of the the raised indices $f_1, \ldots, f_b$ contracts against one of the indices $r_1, \ldots, r_p$, while no two of the indices $r_1, \ldots, r_p$ contract between themselves. On the other hand, in each factor $\nabla y_1 \ldots y_t \nabla_{a_1 \ldots a_u} P_{ij}$ each of the raised indices $y_1, \ldots, y_t$ contracts against one of the indices $a_1, \ldots, a_u$, and $i, j$. Moreover, none of the indices $a_1, \ldots, a_u, i, j$ contract between themselves. We call such complete contractions $W$-normalized.

By virtue of the curvature identity it is clear that modulo introducing correction terms of length $\geq \sigma + 1$, we can write $P(g)|_{\Theta_s}$ as a linear combination of $W$-normalized complete contractions $C^l(g)$.
Definition 5.6 Now, for each complete contraction $C^l(g)$ in the form (5.19), we define $\delta_W$ to stand for the number of internal contractions among the factors $\nabla^{(n)}W_{ijkl}$. We defined $\delta_P$ to stand for the number of internal contractions among the factors $\nabla^{(p)}P_{ij}$ plus the number $K$ of factors $P_{\alpha}^n$. In order to distinguish these numbers among the various complete contractions $C^l(g)$, $l \in L$, we will write $\delta_W(l), \delta_P(l)$. We also define $\delta(l) = \delta_W(l) + \delta_P(l)$ (sometimes we will write $\delta$ instead of $\delta(l)$). This notation trivially extends to vector fields in the form (5.19) with one free index.

Furthermore, in the cases $s = \sigma - 1$ and $\sigma - 2$ we will introduce an extra piece of notation purely for technical reasons:

Special definition: If $s = \sigma - 2$ then $P(g)|_{(s)}$ is “good” if the only complete contraction in $P(g)|_{(s)}$, with $\sigma - 2$ factors $P_{\alpha}^n$ is of the form $(\text{Const}) \cdot \nabla^{\sigma-2} \nabla_{ijkl} \otimes \nabla_{ijkl} \otimes (P_{\alpha}^n)^{\sigma-2}$ (when $\sigma < \frac{n}{2} - 1$) or $(\text{Const}) \cdot \text{contr}(\nabla^{\sigma}W_{ijkl} \otimes \nabla_{ijkl} \otimes (P_{\alpha}^n)^{\sigma-2})$ when $\sigma = \frac{n}{2} - 1$. If $s = \sigma - 1$, then $P(g)|_{(s)}$ is “good” if all complete contractions in $P(g)|_{(s)}$ have $\delta_W + \delta_P = \frac{n}{2} - 1$.

We will prove in the paper [5] in this series the following Lemma:

Lemma 5.3 There exists a divergence $\text{div}_iT^i(g)$ so that

$$P(g)|_{(s)} - \text{div}_iT^i(g) = \sum_{l \in \Theta'_s} a_l C^l(g) + \sum_{t \in T} a_t C^t(g).$$

Here each $C^l(g)$ is in the form (5.19) and has fewer than $s$ factors $\nabla^{(p)}P$. The complete contractions indexed in $\Theta'_s$ are in the form (5.19) with $s$ factors $\nabla^{(p)}P$ and moreover this linear combination is good.

Lemma 5.3 will be proven in [5], by explicitly constructing the divergence $\text{div}_iT^i(g)$.

(There is no recourse to the “main algebraic Proposition”). Therefore, for the rest of this section when $s = \sigma - 1$ or $s = \sigma - 2$ we will be assuming that $P(g)|_{(s)}$ is good.

We consider $\mu = \min_{l \in L} \delta(l)$ (recall that $L$ is the index set on the right hand side of (5.18)). We denote by $L_{\mu} \subset L$ to be the set for which $l \in L_{\mu}$ if and only if $\delta(l) = \mu$. We claim the following:

Proposition 5.1 Under the assumptions of Proposition 5.1 (and assuming the Lemma 5.3) we claim that there is a linear combination $T^i(g) = \sum_{r \in \mathcal{R}} a_r C^{r,i}(g)$, where each $C^{r,i}(g)$ is in the form (5.19) with length $\sigma$, weight $-n + 1$ and $\delta = \mu$, so that modulo complete contractions of length $\geq \sigma + 1$:

$$\sum_{l \in L_{\mu}} a_l C^l(g) - \text{div}_t \sum_{r \in \mathcal{R}} a_r C^{r,i}(g) = \sum_{u \in \mathcal{U}} a_u C^u(g) + \sum_{x \in \mathcal{X}} a_x C^x(g),$$

(5.20)

\footnote{In other words, if there are complete contractions in $P(g)|_{(s)}$ with $\delta_W + \delta_P < \frac{n}{2} - 1$ then $P(g)|_{(s)}$ is “good” if no complete contractions in $P(g)|_{(s)}$ have $\sigma - 2$ factors $P_{\alpha}^n$.}

\footnote{Recall in particular the definition of the index set $\Theta_s$, and that we have written out $P(g)|_{(s)} = \sum_{l \in L} C^l(g)$ (modulo longer complete contractions); recall also that if $s = \sigma - 1$ or $s = \sigma - 2$ then $P(g)|_{(s)}$ is assumed to be good.}
where each \( C^\nu (g) \) is in the form (5.19) with \( s \) factors \( \nabla^{(p)} P \) and \( \delta = \mu + 1 \). Each \( C^\nu (g) \) is in the form (5.19) with \( s - 1 \) factors \( \nabla^{(p)} P \).

The remainder of this paper is devoted to proving the above (subject to the “main algebraic Proposition” 5.2). For now, we note that Proposition 5.2 implies Proposition 3.1, by iterative repetition: After a finite number of applications of the above, we will be left with correction terms that are of the form

\[
\sum_{x \in X} a_x \ldots
\]

This is because we are dealing with complete contractions of a fixed weight \( -n \), thus there can be at most \( \frac{n}{2} \) internal contractions in any such complete contraction.

**Proof of Proposition 5.1**

We firstly wish to understand explicitly how the terms of length \( \sigma \) in \( I_s^\nu (g) \) arise from \( P(g)|\Theta_s \). Then, we reduce Proposition 5.1 to the Lemmas 5.4, 5.5.

We consider \( I_s^\nu (g)(:= \frac{d}{dt}|_{t=0}[e^{t \cdot n} P(e^{t \cdot 2 \psi} g)] - P(g)] \). It follows straightforwardly from the transformation law of the Schouten tensor that:

\[
I_s^\nu (g) = (-1)^s \sum_{l \in L} a_l C_l^\nu (g),
\]

(5.21)

where each \( C_l^\nu (g) \) arises from \( C_l^\nu (g) \) (which is in the form (5.19)) by replacing each factor \( \nabla_{a_1 \ldots a_t} \nabla_{t_1 \ldots t_p} P_{ij} \) by \( \nabla_{a_1 \ldots a_t} \nabla_{t_1 \ldots t_p} \psi \). Explicitly, it will be in the form:

\[
\text{contr} (\nabla_{a_1 \ldots a_t} \nabla^{m_1} W_{ijkl} \otimes \ldots \otimes \nabla_{b_1 \ldots b_s} \nabla^{m_s} W_{ij'k'l'},
\]

\[
\otimes \nabla_{v_1 \ldots v_c} \nabla^{(p+2)} \psi \otimes \ldots \otimes \nabla_{w_1 \ldots w_d} \nabla^{(p+2)} \psi),
\]

(5.22)

and will have \( \delta_W + \delta_\psi \geq \mu \) (\( \delta_\psi \) here stands for the total number of internal contractions among the factors \( \nabla^{(k)} \psi \)). \( \text{(Junk)} \) stands for a generic linear combination of terms with at least \( \sigma + 1 \) factors in the form \( \nabla^{(m)} R, \nabla^{(n)} \psi \). Furthermore, \( \int_{M^n} I_s^\nu (g) dV_g = 0 \); hence we may apply the super divergence formula to this integral equation. Now, for convenience, we polarize the function \( \psi \) and thus we will be considering \( I_s^\nu (\psi_1, \ldots, \psi_s) \).

We will now re-write \( I_s^\nu (\psi) \) as a linear combination of complete contractions involving curvature, rather than Weyl, tensors:

By decomposing the Weyl tensor as in (2.4) and applying the curvature and Bianchi identities, we re-write \( I_s^\nu (\psi_1, \ldots, \psi_s) \) as a linear combination:

\[
I_s^\nu (\psi_1, \ldots, \psi_s) = \sum_{b \in B} a_b C_b^\nu (\psi_1, \ldots, \psi_s),
\]

(5.23)

where each \( C_b^\nu (\psi_1, \ldots, \psi_s) \) is in the form:
$$\text{contr}(\nabla f_1 \cdots f_p \nabla^{(m)} R_{ijkl} \otimes \cdots \otimes \nabla g_{q_1 \cdots q_p} \nabla^{(n)} R_{ijkl})$$

$$\otimes \nabla_{g_{1 \cdots 1}} \nabla^{(d_1)} \text{Ric}_{ij} \otimes \cdots \otimes \nabla_{x_1 \cdots x_p} \nabla^{(d_q)} \text{Ric}_{ij} \otimes R^n \otimes$$

$$\nabla_{a_1 \cdots a_{t_1}} \nabla^{(a_1)} \psi_1 \otimes \cdots \otimes \nabla_{r_1 \cdots r_{t_2}} \nabla^{(a_u)} \psi_u),$$

with the usual conventions: In each factor $\nabla f_1 \cdots f_p \nabla^{(m)} R_{ijkl}$, each of the the indices $f_1, \ldots, f_p$ contracts against one of the indices $r_1, \ldots, r_t$, while no two of the indices $r_1, \ldots, r_t$ contract between themselves. On the other hand, for each factor $\nabla g_{q_1 \cdots q_p} \nabla^{(n)} R_{ijkl}$, each of the upper indices $y_1, \ldots, y_t$ contracts against one of the indices $a_1, \ldots, a_u$. Moreover, none of the indices $a_1, \ldots, a_u$ contract between themselves. For the factors $\nabla_{g_{1 \cdots 1}} \nabla^{(d_1)} \text{Ric}_{ij} \otimes \cdots \otimes \nabla_{x_1 \cdots x_p} \nabla^{(d_q)} \text{Ric}_{ij} \otimes R^n \otimes$, we impose the condition that each of the indices $x_1, \ldots, x_p$ must contract against one of the indices $t_1, \ldots, t_{s_1}, i, j$. Moreover, we impose the restriction that none of the indices $t_1, \ldots, t_{s_1}, i, j$ contract between themselves (this assumption can be made by virtue of the contracted second Bianchi identity).

**Definition 5.7** A contraction in the form \([5.24]\) with all the features described above, and with the additional requirement that each factor $\nabla^{a_1 \cdots a_t} \nabla^{(a_u)} \psi_h$ has $t + u \geq 2$ (i.e. $\psi_h$ is differentiated at least twice) will be called normal.

For any complete contraction in the form \((5.24)\), $\delta_R$ will stand for the number of internal contractions in factors $\nabla^{(m)} R_{ijkl}$. $\delta_{\text{Ric}}$ will stand for the number of internal contractions in factors $\nabla^{(p)} \text{Ric}_{ij}$, where we also count the internal contraction in $\text{Ric}_{ij} = R_{ikj}$, plus $2\alpha$, where $\alpha$ stands for the number of factors $R$ (scalar curvature). Lastly, $\delta_{\psi}$ will stand for the total number of internal contractions in the factors of the form $\nabla^{a_1 \cdots a_t} \nabla^{(b)} \psi_h$.

By the formula \([5.22]\), we see that the sublinear combination of length $\sigma$ in $I^a_{\psi}(\psi_1, \ldots, \psi_s)$ consists of complete contractions with at least two derivatives on each function $\psi_h$.

Let us now understand more concretely how a given term in the form \([5.22]\) gives rise to terms of the form \([5.24]\). We first introduce some definitions:

**Definition 5.8** For each complete contraction $C^l_{\psi}(\psi)$, $l \in L_\mu$, let us denote by $C^{l, \psi}_{\psi}(\psi)$ the complete contraction (times a constant) that arises from $C^l_{\psi}(\psi)$ by replacing the factors $\nabla^{(m)} W_{ijkl}$ according to the following rule: If $\nabla^{(m)} W_{ijkl}$ does not have an internal contraction involving one of the indices $i, j, k, l$, we replace it by $\nabla^{(m)} R_{ijkl}$. If it has at least one internal contraction involving one of the indices $i, j, k, l$, we replace it by $\frac{n-3}{n-2} \nabla^{(m)} R_{ijkl}$.

Observe that by construction, if $C^l_{\psi}(\psi)$ has $\delta_W + \delta_R = b$, then $C^{l, \psi}_{\psi}(\psi_1, \ldots, \psi_s)$ has $\delta_R + \delta_{\psi} = b$, and no factors $\nabla^{(p)} \text{Ric}$ or $R$.

In particular, $C^{l, \psi}_{\psi}(\psi_1, \ldots, \psi_s)$ will be in the form:
\[(\text{Const}) \cdot \text{contr}(\nabla_{a_1} \cdots \nabla_{a_l} R_{ijkl} \otimes \cdots \otimes \nabla_{b_1} \cdots \nabla_{b_p} \nabla_{(m_1 \cdots s_p)} R_{tj'k'l'}) \otimes \nabla_{v_1} \cdots \nabla_{v_q} \nabla_{(p_1+2)} \nabla_{(p_2+2)} \psi_s)\].

**Definition 5.9** Consider any $C_g^i(\psi_1, \ldots, \psi_s)$ in the form \((5.22)\) with $\sigma$ factors. If $C_g^i(\psi_1, \ldots, \psi_s)$ has $q = 0$ and $\delta = \mu$ it will be called a target. If $C_g^i(\psi_1, \ldots, \psi_s)$ has $q = 0$ and $\delta > \mu$, it will be called a contributor.

If $C_g^i(\psi_1, \ldots, \psi_s)$ has $q > 0$ and $\delta > \mu$ we call it 1-cumbersome. We call $C_g^i(\psi_1, \ldots, \psi_s)$ 2-cumbersome if it has $q > 0$ and $\delta = \mu$ and the feature that each factor $\nabla_{a_1} \cdots \nabla_{b_p} \nabla_{r_1 \cdots r_p} \text{Ric}_{ij}$ has $t > 0$ and the index $j$ is contracting against one of the indices $a_1, \ldots, a_t$.

Finally, when we say $C_g^i(\psi_1, \ldots, \psi_s)$ is “cumbersome”, we will mean it is either 1-cumbersome or 2-cumbersome.

We make the convention that when $\sum_{j \in J} a_j C_g^j(\psi_1, \ldots, \psi_s)$, $\sum_{F} a_f C_g^f(\psi_1, \ldots, \psi_s)$ appear on the right hand sides of equations below, they will stand for generic linear combinations of contributors and cumbersome complete contractions, respectively.

Then using the decomposition of the Weyl curvature \((2.4)\), we explicitly write each $C_g^i(\psi_1, \ldots, \psi_s)$ as a linear combination of terms in the above forms:

For each $l \in L_\mu$, it follows that:

$$C_g^i(\psi_1, \ldots, \psi_s) = C_g^{i, t}(\psi_1, \ldots, \psi_s) + \sum_{j \in J} a_j C_g^j(\psi_1, \ldots, \psi_s) + \sum_{f \in F} a_f C_g^f(\psi_1, \ldots, \psi_s),$$

(5.26)

while for each $l \in L \setminus L_\mu$:

$$C_g^i(\psi_1, \ldots, \psi_s) = \sum_{j \in J} a_j C_g^j(\psi_1, \ldots, \psi_s) + \sum_{f \in F} a_f C_g^f(\psi_1, \ldots, \psi_s),$$

(5.27)

where each $C_g^j(\psi_1, \ldots, \psi_s)$ has $\delta_R + \delta_\phi + \delta_{\text{Ric}} \geq \mu + 1$ (and hence is 1-cumbersome).

This follows since $C_g^j(\psi_1, \ldots, \psi_s)$, $l \in L \setminus L_\mu$ has $\delta_W + \delta_P \geq \mu + 1$.

**Remark:** We observe that for each complete contraction $C_g^j(\psi_1, \ldots, \psi_s)$ in the RHSs of \((5.20)\), \((5.27)\) with $\alpha > 0$ factors $R$ (the scalar curvature) will respectively satisfy $\delta \geq \mu + 2\alpha$, $\delta \geq \mu + 1 + 2\alpha$. This is because a factor $R$ in the RHS can only arise from an (undifferentiated) factor $W_{ijkl}$ in the LHS of \((5.20)\), \((5.27)\); thus a factor with no internal contractions in the LHS gives rise to a factor $R = R^{ab}_{\cdots}$ with two internal contractions. (This remark will be useful in \([6]\)).

In view of the form \((5.22)\) where each complete contraction has $\delta_W + \delta_P \geq \mu$, we derive that:
\[ I_g^s(\psi_1, \ldots, \psi_s) = \sum_{i \in L_\mu} a_i C_g^{i,1}(\psi_1, \ldots, \psi_s) + \sum_{j \in J} a_j C_g^j(\psi_1, \ldots, \psi_s) \]
\[ + \sum_{f \in F} a_f C_g^f(\psi_1, \ldots, \psi_s) + (\text{Junk}); \quad (5.28) \]

$L_\mu$ here is the same index set as in Proposition 5.1. The linear combinations $\sum_{j \in J} \ldots, \sum_{f \in F} \ldots$ are generic linear combinations of contributors and cumbersome complete contractions (see definition 5.9). (Junk) stands for a generic linear combination of terms with at least $\sigma + 1$ factors in total.

Now, for the next Lemma, we will let $Z_g(\psi_1, \ldots, \psi_s)$ stand for any linear combination in the form above, where $\sum_{i \in L_\mu} \ldots$ is the same linear combination as in $I_g^s(\psi_1, \ldots, \psi_s)$, while $\sum_{f \in F} \ldots \sum_{j \in J} \ldots$ (Junk) are allowed to be generic linear combinations of the forms described above. In these generic linear combinations $Z_g(\psi_1, \ldots, \psi_s)$ we will still be assuming that $Z_g(\psi_1, \ldots, \psi_s)$ is symmetric in the functions $\psi_1, \ldots, \psi_s$.

We partition the index set $F$ into subsets: We let $f \in F^{q,z}$ if and only if $C_g^f(\psi_1, \ldots, \psi_s)$ has $q$ factors of the form $\nabla a_1 \ldots a_1 \nabla^{(q)} \text{Ric}$ or $R$ and also has $\delta_R + \delta_{\text{Ric}} + \delta_\psi = z$. We also define $F^q = \bigcup_{z \geq q} F^{q,z}$. One last language convention before stating our claims: We will say that the index set $F^q$ (or more generally $F$) is bad if there are complete contractions $C_g^f(\psi_1, \ldots, \psi_s)$, $f \in F^q$ with at least $\sigma - 2$ factors in the form $\Delta \psi_h$ or $R$ (scalar curvature).

The main Claims:

**Lemma 5.4** Consider any $Z_g(\psi_1, \ldots, \psi_s)$, written out as a linear combination in the form (5.23). Assume that $\int_{M^n} Z_g(\psi_1, \ldots, \psi_s) dv_g = 0$ for every $(M^n, g)$ and every function $\psi_1 = \cdots = \psi_\sigma = \psi \in C^\infty(M^n)$. Assume also that for a given $q_1 > 0$, $F^q = \emptyset$ for every $q > q_1$. Moreover, we assume that for a given $z_1 \geq \mu, F^{q_1,z_1} = \emptyset$ for every $z < z_1$. We make different claims for the two cases $z_1 > \mu$ and $z_1 = \mu$.

If $z_1 > \mu$ and $F^{q_1}$ is not bad, we claim that there is a linear combination of vector fields $\sum_{h \in H^{q_1,z_1}} a_h C_g^{h,i}(\psi_1, \ldots, \psi_s)$ where each $C_g^{h,i}(\psi_1, \ldots, \psi_s)$ is in the form (5.22) with length $\sigma, q + \alpha = q_1, \delta_R + \delta_{\text{Ric}} + \delta_\psi = z_1$ and with one free index, so that:

\[ \sum_{f \in F^{q_1,z_1}} a_f C_g^f(\psi_1, \ldots, \psi_s) - \text{div}_i \sum_{h \in H^{q_1,z_1}} a_h C_g^{h,i}(\psi_1, \ldots, \psi_s) = \]
\[ \sum_{s \in S} a_s C_g^s(\psi_1, \ldots, \psi_s) + \sum_{t \in T} a_t C_g^t(\psi_1, \ldots, \psi_s), \quad (5.29) \]

44 This is just a re-statement of the fact that $Z_g(\psi_1, \ldots, \psi_\sigma)$ is symmetric in the functions $\psi_1, \ldots, \psi_\sigma$.

45 See the language convention above.
where each \( C_g^r(\psi_1, \ldots, \psi_s) \) is in the form (5.24), has length \( \sigma \) and is not bad, and has \( q + \alpha = q_1 - 1 \), \( \delta_R + \delta_{\text{Ric}} + \alpha = z_1 \). On the other hand, each \( C_g^s(\psi_1, \ldots, \psi_s) \) is of the form (5.24) with length \( \sigma \) and \( q + \alpha = q_1 \) factors \( \nabla^{(p)} \text{Ric} \) or \( R \), \( \delta_R + \delta_{\text{Ric}} + \delta_{\psi} = z_1 + 1 \). The above holds modulo complete contractions of length \( \sigma \).

In the case where \( z_1 = \mu \) and \( F^{q_1} \) is not bad, we claim that there is a linear combination of vector fields \( \sum_{h \in H^{q_1, s_1}} a_h C_g^{h,i}(\psi_1, \ldots, \psi_s) \), where each \( C_g^{h,i}(\psi_1, \ldots, \psi_s) \) is in the form (5.24) with length \( \sigma \), \( q + \alpha = q_1 \), and with one free index, so that:

\[
\sum_{f \in F_{q_1, s_2}} a_f C_g^f(\psi_1, \ldots, \psi_s) - \text{div} \sum_{h \in H_{q_1, s_1}} a_h C_g^{h,i}(\psi_1, \ldots, \psi_s) = \sum_{i \in T} a_i C_g^i(\psi_1, \ldots, \psi_s)
\]  

(5.30)

where each \( C_g^i(\psi_1, \ldots, \psi_s) \) is in the form (5.24) (not bad) with length \( \sigma \) and \( q + \alpha = q_1 \) factors \( \nabla^{(p)} \text{Ric} \) or \( R \), \( \delta = \mu + 1 \). The above holds modulo complete contractions of length \( > \sigma \).

Claim 2: Consider \( I_g^s(\psi_1, \ldots, \psi_s) \), in the form (5.25), and suppose \( F \) is bad. We claim that there is a linear combination of vector fields, \( \sum_{h \in H^{q_1}} a_h C_g^{h,i}(\psi_1, \ldots, \psi_s) \), where each \( C_g^{h,i}(\psi_1, \ldots, \psi_s) \) is in the form (5.24) with length \( \sigma \), \( q + \alpha = q_1 \), and with one free index, so that:

\[
\sum_{f \in F_{q_1, s_2}} a_f C_g^f(\psi_1, \ldots, \psi_s) - \text{div} \sum_{h \in H_{q_1}} a_h C_g^{h,i}(\psi_1, \ldots, \psi_s) = \sum_{x \in X} a_x C_g^x(\psi_1, \ldots, \psi_s)
\]  

(5.31)

where each of the complete contractions \( C_g^x(\psi_1, \ldots, \psi_s) \) is of the form (5.24) with length \( \sigma \) and \( q + \alpha \leq q_1 \) and \( \delta \geq \mu + 1 \), and is not bad. The above holds modulo complete contractions of length \( > \sigma \).

Note: Claim 2 will be proven in [5].

Observe that the Lemma 5.4 implies that there is a linear combination of vector fields \( \sum_{h \in H} a_h C_g^{h,i}(\psi_1, \ldots, \psi_s) \), where each \( C_g^{h,i}(\psi_1, \ldots, \psi_s) \) is a partial contraction of length \( \sigma \) in the form (5.24) and with one free index, so that:

\[
\sum_{f \in F} a_f C_g^f(\psi_1, \ldots, \psi_s) - \text{div} \sum_{h \in H} a_h C_g^{h,i}(\psi_1, \ldots, \psi_s) = \sum_{j \in J} a_j C_g^j(\psi_1, \ldots, \psi_s)
\]  

(5.32)

Here the first sublinear combination is not generic, but stands for the sublinear combination in (5.28). The above holds modulo complete contractions of length
> \sigma$. Therefore, assuming we can prove Lemma 5.4 we can then apply it to the integral equation $\int_{M^n} I^g_\sigma(\psi_1, \ldots, \psi_s) dV_g = 0$ (recall that $I^g_\sigma(\psi_1, \ldots, \psi_s)$ is in the form $\langle 5.28 \rangle$), to derive a new integral equation:

$$\int_{M^n} \sum_{l \in L_\mu} a_l C^{d,i}_{g}^\delta(\psi_1, \ldots, \psi_s) + \sum_{j \in J} a_j C^{j}_{g}^\delta(\psi_1, \ldots, \psi_s) + \sum_{\zeta \in Z} a_\zeta C^{\zeta}_{g}^\delta(\psi_1, \ldots, \psi_s) dV_g = 0,$$

which holds for every $(M^n, g)$ and every $\psi_1 = \ldots = \psi_s = \psi \in C^\infty(M^n)$ (recall that our complete contractions are assumed to be symmetric in the functions $\psi_1, \ldots, \psi_s$), where the complete contractions $C^\delta_{g}$ have length $\geq \sigma + 1$ and the complete contractions $C^\delta_{g}, C^\delta_{j}$ are as described below equation $\langle 5.28 \rangle$.

Our next Lemma will then apply to the new integral equation $\langle 5.33 \rangle$. In order to state it, we will need one extra definition:

**Definition 5.10** For each complete contraction $C^\delta_{g}(\phi)$ or vector field $C^\delta_{j}(\phi)$ in the form $\langle 5.24 \rangle$, with no factors $\nabla^{(p)} \text{Ric}$ or $R$ and with $\delta = \mu$ (in other words there are $\mu$ internal contractions and all of them involve a derivative index), we denote by $C^\delta_{g,i_1,\ldots,i_p}(\phi)\nabla_{i_1} v \ldots \nabla_{i_p} v$, $C^\delta_{j,i_1,\ldots,i_p}(\phi)\nabla_{i_1} v \ldots \nabla_{i_p} v$, the complete contraction or vector field that arises from it by replacing each internal contraction $(\nabla^a v, a)$ by an expression $(\nabla^a v, a)$.

**Lemma 5.5** Assume an equation:

$$\int_{M^n} \sum_{l \in L_\mu} a_l C^{d,i}_{g}^\delta(\psi_1, \ldots, \psi_s) + \sum_{j \in J} a_j C^{j}_{g}^\delta(\psi_1, \ldots, \psi_s)$$

$$+ \sum_{\zeta \in Z} a_\zeta C^{\zeta}_{g}^\delta(\psi_1, \ldots, \psi_s) dV_g = 0,$$

which holds for every compact $(M^n, g)$ and every $\psi_1, \ldots, \psi_s \in C^\infty(M^n)$, and where each $C^\delta_{g}$ has length $\geq \sigma + 1$. We then claim that there is a linear combination of normalized vector fields $\sum_{d \in D} a_d C^{d,i}_{g}(\psi_1, \ldots, \psi_s)$, where each $C^{d,i}_{g}(\psi_1, \ldots, \psi_s)$ is in the form $\langle 5.24 \rangle$ with no factors $\nabla^{(p)} \text{Ric}$ or $R$ and with $\delta = \mu$, so that:

$$\sum_{l \in L_\mu} a_l C^{d,i}_{g}(\psi_1, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_p} v$$

$$- \sum_{d \in D} a_d X \div C^{d,i_1,\ldots,i_p}_{g}(\psi_1, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_p} v = 0,$$

modulo complete contractions of length $\geq \sigma + \mu + 1$.

46In particular, the linear combination $\sum_{l \in L_\mu} a_l C^{d,i}_{g}(\psi_1, \ldots, \psi_s)$ is the same linear combination that appears in Proposition 5.4 while the linear combination $\sum_{j \in J} \ldots$ is a generic linear combination of complete contractions as explained below equation $\langle 5.28 \rangle$.

47We thus obtain complete contractions and vector fields of length $\sigma + \mu$. 44
Let us check how Proposition 5.1 follows from Lemma 5.5.

Just observe that Lemma 5.1 and (5.35) imply that the vector field
\[ T^i := Weylify \left( \sum_{d \in D} a_d \Pi^d C_{g}^{d,i}(\psi_1, \ldots, \psi_s) \right) \]
fulfils the requirements of Proposition 5.1.

Thus, if we can show Lemmas 5.4, 5.5, Proposition 3.1 will follow.

5.3 The main algebraic Proposition.

The Proposition that we state in this section will imply Lemmas 5.4 and 5.5.

In order to state and prove the main algebraic proposition we will need some more terminology. We will be considering tensor fields \( C^\alpha_{\Omega_1, \ldots, \Omega_p} \) of length \( \sigma \) (with no internal contractions) in the form:

\[ \text{pcontr}(\nabla^{(m_1)} R_{ijkl} \otimes \ldots \otimes \nabla^{(m_s)} R_{ijkl} \otimes \nabla^{(b_1)} \Omega_1 \otimes \ldots \otimes \nabla^{(b_p)} \Omega_p) \] \hspace{1cm} (5.36)

Here \( \sigma = s + p \) and \( i_1, \ldots, i_\alpha \) are the free indices. Such a complete contraction will be called acceptable if each \( b_i \geq 2 \). Recall the operation \( X\text{div} \) from Definition 5.3.

Proposition 5.2 Consider two linear combinations of acceptable tensor fields in the form (5.36),

\[ \sum_{l \in L_1} a_l C^l_{\Omega_1, \ldots, \Omega_p}, \quad \sum_{l \in L_2} a_l C^l, \quad (i_1, \ldots, i_\alpha) \]

where each \( C^l \) above has length \( \sigma \geq 3 \) and a given number \( \sigma_1 = \sigma - p \) of factors in the form \( \nabla^{(m)} R_{ijkl} \). Assume that for each \( l \in L_2, \beta_l \geq \alpha + 1 \). Assume that modulo complete contractions of length \( \geq \sigma + 1 \):

\[ \sum_{l \in L_1} a_l X\text{div}_{i_1} \ldots X\text{div}_{i_\alpha} C^l_{\Omega_1, \ldots, \Omega_p} \quad + \\
\sum_{l \in L_2} a_l X\text{div}_{i_1} \ldots X\text{div}_{i_\alpha} C^l, \quad (i_1, \ldots, i_\alpha) = 0. \] \hspace{1cm} (5.37)

We claim that there is a linear combination of acceptable \((\alpha + 1)\)-tensor fields in the form (5.36), \( \sum_{r \in R} a_r C_{g}^{r, \Omega_1, \ldots, \Omega_p} \), with length \( \sigma \) so that

\[ \sum_{l \in L_1} a_l C^l_{\Omega_1, \ldots, \Omega_p} = \sum_{r \in R} a_r X\text{div}_{i_1} \ldots X\text{div}_{i_\alpha} C^r_{\Omega_1, \ldots, \Omega_p}, \] \hspace{1cm} (5.38)

modulo terms of length \( \geq \sigma + 1 \).

\[ \text{Recall that given a } \beta\text{-tensor field } T^{i_1, \ldots, i_\beta}, \text{ } T^{(i_1, \ldots, i_\alpha)} \text{ stands for a new tensor field that arises from } T^{i_1, \ldots, i_\alpha} \text{ by symmetrizing over the indices } i_1, \ldots, i_\alpha. \]
Note: Observe that the conclusion (5.38) of this Proposition is equivalent to the equation:

\[
\sum_{l \in L_1} a_l C_g^{l,i_1\ldots i_\alpha}(\Omega_1, \ldots, \Omega_p) \nabla_{i_1} v \ldots \nabla_{i_\alpha} v = \\
\sum_{r \in R} a_r X \text{div}_{i_{\alpha+1}} [C_g^{r,i_1\ldots i_\alpha i_{\alpha+1}}(\Omega_1, \ldots, \Omega_p) \nabla_{i_1} v \ldots \nabla_{i_\alpha} v],
\]

which holds modulo complete contractions of length \( \geq \sigma + \alpha + 1 \). (Recall that \( X \text{div}_{i_{\alpha+1}} \ldots \) in the RHS of the above stands for the sublinear combination of terms in \( \text{div}_{i_{\alpha+1}} \ldots \) where the derivative \( \nabla_{i_{\alpha+1}} \) is not allowed to hit the factor to which the free index \( i_{\alpha+1} \) nor any of the factors \( \nabla_{i_\nu} \)).

In the next subsection we show how the main algebraic proposition 5.2 implies Lemmas 5.4, 5.5, and hence also Proposition 3.1.

5.4 Lemmas 5.4 and 5.5 follow from Proposition 5.2.

We first check that Lemma 5.5 indeed follows from Proposition 5.2.

Our starting point will be to apply the super divergence formula to the integral equation (5.34).

Definition 5.11 For each \( l \in L_\mu \) and each \( j \in J \), we denote by \( C_g^{l,i_1\ldots i_\mu} (\psi_1, \ldots, \psi_s) \), \( C_g^{j,i_1\ldots i_m} (\psi_1, \ldots, \psi_s) \) the tensor fields that arise from \( C_g^{l,i} (\psi_1, \ldots, \psi_s) \), \( C_g^{j} (\psi_1, \ldots, \psi_s) \), respectively, by making all the internal contractions into free indices (recall the definition 2.3).

The super divergence formula applied to (5.5) gives the local equation:

\[
(-1)^{\mu} \sum_{i \in L_\mu} a_i X \text{div}_{i_1} \ldots X \text{div}_{i_\mu} C_g^{l,i_1\ldots i_\mu} (\psi_1, \ldots, \psi_s) + \\
\sum_{j \in J} a_j (-1)^{m_j - 1} X \text{div}_{i_1} \ldots X \text{div}_{i_{m_j}} C_g^{j,i_1\ldots i_{m_j}} (\psi_1, \ldots, \psi_s) = 0,
\]

which holds modulo complete contractions of length \( \geq \sigma + 1 \).

Clearly, each of the complete contractions \( C_g^{l,i} \), \( C_g^{j} \) has factors \( \nabla^{(b)} \psi_h \) with \( b \geq 2 \). Therefore, each of the tensor fields in (5.40) has factors \( \nabla^{(c)} \psi_h \) with \( c \geq 1 \), and moreover the factors \( \nabla \psi_h \) can only arise from factors \( \Delta \psi_h \) by replacing \( \Delta \psi_h \) by \( \nabla_a \psi_h \) (\( a \) is a free index).

For each \( C_g^{l,i} \), \( C_g^{j} \) appearing in (5.5) let \( |\Delta| (l) \), \( |\Delta| (j) \) stand for the number of factors \( \Delta \psi_h \). We define \( |\Delta|_{\text{Max}} \) to stand for \( \max_{f \in L_\mu, f \in J} |\Delta| (f) \).

We observe that if \( |\Delta|_{\text{Max}} = 0 \), then Lemma 5.5 follows by just applying Proposition 5.2 to (5.40). In the case \( |\Delta|_{\text{Max}} > 0 \) we cannot directly apply
Proposition 5.2 to (5.40) due to the presence of factors $\nabla \psi$ among certain tensor fields in (5.40). We will treat the case $|\Delta_{\max}| > 0$ further down.

**Lemma 5.4 follows from Proposition 5.2 (general discussion):** (Refer to the notation of Lemma 2.1). We apply Lemma 2.1 to the equation $\int_M Z_g(\psi_1, \ldots, \psi_\sigma) dV_g$ (see the hypothesis of Lemma 5.4) and deduce that modulo complete contractions of length $\geq \sigma + 1$:

\[
\sum_{f \in F_{q_1, z_1}} a_f X_{div_{i_1}} \ldots X_{div_{i_z}} C^{q_1, z_1}_g (\psi_1, \ldots, \psi_\sigma, \Omega^{p_1}) + \\
\sum_{z > z_1} \sum_{f \in F_{q_1, z}} a_f X_{div_{i_1}} \ldots X_{div_{i_z}} C^{q_1, z}_g (\psi_1, \ldots, \psi_\sigma, \Omega^{p_1}) = 0. 
\] (5.41)

We now define $|\Delta|(f)$ for each complete contraction $C^f$ in Lemma 5.4 to stand for the number of factors $\Delta \psi$ or $\Delta \Omega$. (Observe that by construction a factor $\Delta \Omega$ can only arise in $C^{q_1, z_1}_g (\psi_1, \ldots, \psi_\sigma, \Omega^{p_1})$ by replacing some factor $R$ in $C^{q_1, z_1}_g (\psi_1, \ldots, \psi_\sigma)$ by $-2\Delta \Omega$). We define $|\Delta|_{\max}$ to stand for $\max_{f \in F_{q_1, z}} |\Delta|(f)$. We write $|\Delta|_{\max} = M$, for short.

Lemma 5.4 in the case where $|\Delta|_{\max} = 0$ can be shown by applying Proposition 5.2 and the operation Ricci to (5.41). The details of this will be provided below, in the cases where $M := |\Delta|_{\max} > 0$. That proof, if we set $M = 0$ applies to show how Lemma 5.4 follows from (5.41), in the both the case $z_1 = \mu$ and $z_1 > \mu$.

**Proof of Lemmas 5.4 and 5.5:** We now consider equations (5.40) and (5.41) where $|\Delta|_{\max} = 0$. Our strategy will then be to reduce ourselves to the case where $|\Delta|_{\max} = 0$ by a downward induction on $|\Delta|_{\max}$ (see below). In this general situation, we will not show Lemmas 5.4 and 5.5 all in one piece, but rather we will distinguish cases. We distinguish three cases: Either $|\Delta|_{\max} = \sigma - 1$ or it is $\sigma - 2$ or it is $\leq \sigma - 3$. Here we consider only the case $|\Delta|_{\max} \leq \sigma - 3$. The cases $|\Delta|_{\max} = \sigma - 1$, $|\Delta|_{\max} = \sigma - 2$ will be treated in 7. (For reference purposes, we codify the claim of Lemmas 5.4, 5.5 when $|\Delta|_{\max} = \sigma - 1$, $|\Delta|_{\max} = \sigma - 2$ in the end of this subsection.)

**Proof of Lemmas 5.4 and 5.5 in the case $M = |\Delta|_{\max} \leq \sigma - 3$.**

**Outline:** We will claim the equations (5.42), (5.43), (5.44), (5.45), (5.46) below, and will show how Lemmas 5.4 and 5.5 will follow from these four equations. We then prove these four equations (using Proposition 5.2).

**Lemma 5.5:** In (5.40), we let $L^K_{\mu}$, $K = 1, \ldots, M$ to the index set of the complete contractions $C^{q_1, z}_g$ with $K$ factors $\Delta \psi$. Accordingly, we let $J^K$, $K = 1 \ldots, M$ be the index set of complete contractions $C^{q_1, z}_g$ with $K$ factors $\Delta \psi$.

Consider (5.40). We claim that there exists a linear combination of vector
fields, $\sum_{v \in V^M} a_v C^\sigma_g(\psi_1, \ldots, \psi_s)$, with each $C^\sigma_g$ in the form (5.24) with length $\sigma$, $\delta = \mu$ and with no factors $\nabla^{(p)} \text{Ric}$ or $R$, but with $M$ factors $\Delta \psi_h$, so that

$$
\sum_{i \in L^M_\mu} a_i C^{i_1 \ldots i_\mu}_g(\psi_1, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_\mu} v = [X \text{div}_i \sum_{v \in V^M} a_v C^{\sigma_1 \ldots \sigma_\delta}_g(\psi_1, \ldots, \psi_s) + \sum_{r \in R^{M-1}} a_r C^\sigma_g(\psi_1, \ldots, \psi_s)] \nabla_{i_1} v \ldots \nabla_{i_\mu} v,
$$

(5.42)

where each $C^{i_1 \ldots i_\mu}_g(\psi_1, \ldots, \psi_s)$ on the RHS is a partial contraction in the form (5.24) (with $\mu$ free indices) with no factors $\nabla^{(p)} \text{Ric}, R$, with $\delta = \mu$ but $M - 1$ factors $\Delta \psi_h$.

If we can prove the above, then we will be reduced to proving Lemma 5.5 under the extra assumption that for every $C^\delta_g$ in (5.34) will have at most $M - 1$ factors $\Delta \psi_h$.

In this setting, we define $\delta_{\min}(M)$ to stand for the minimum number of internal contractions among the complete contractions $C^j, j \in J_M$ in (5.34). By definition, $\delta_{\min}(M) \geq \mu + 1$. We then claim that there exists a linear combination of vector fields, $\sum_{h \in H^M_{\delta_{\min}(M)}} a_h C^h_g(\psi_1, \ldots, \psi_s)$, where each $C^h_g$ is in the form (5.24) with length $\sigma$, $\delta = \delta_{\min}(M)$ and with no factors $\nabla^{(p)} \text{Ric}$ or $R$ but with $M$ factors $\Delta \psi_h$, so that:

$$
\sum_{j \in J^M_{\delta_{\min}(M)}} a_j C^{i_1 \ldots i_{\delta_{\min}(M)}}_g(\psi_1, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_{\delta_{\min}(M)}} v - X \text{div}_i \sum_{h \in H^M_{\delta_{\min}(M)}} a_h C^{i_1 \ldots i_{\delta_{\min}(M)}}_g(\psi_1, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_{\delta_{\min}(M)}} v = \sum_{r \in R^M} a_r C^{i_1 \ldots i_{\delta_{\min}(M)}}_g(\psi_1, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_{\delta_{\min}(M)}} v,
$$

(5.43)

where each $C^{i_1 \ldots i_{\delta_{\min}(M)}}_g(\psi_1, \ldots, \psi_s)$ is a partial contraction in the form (5.24) (with $\mu$ free indices) and with $\delta_R = \delta_{\min}(M)$ and $M - 1$ factors $\Delta \psi_h$.

Observe that (5.42), (5.43), imply Lemma 5.5. Iteratively applying them we reduce ourselves to proving Lemma 5.5 under the additional assumption that each $C^j$ has no factors $\Delta \psi_h$, and also each $C^j$ has no factors $\Delta \psi_h$. In that case we have already shown how Lemma 5.5 directly follows from Proposition 5.2.

**Lemma 5.4.** We make analogous claims regarding Lemma 5.3. Consider (5.41). We denote by $F^\mu_\kappa, \kappa = 1, \ldots, M$ the index set of complete contractions with $K$ factors $\Delta \psi_h$ or $\Delta \Omega$. We initially consider the sublinear combination

Note: We will be writing $\Pi(\ldots)$ instead of $\Pi(\ldots)$ to avoid confusion.
indexed in $F_{M}^{q_1,z_1}$. We then make two different claims, for the two cases $z_1 = \mu$ and $z_1 > \mu$. If $z_1 > \mu$, then for some complete contraction $C_{g}^{\delta}$, $f \in F_{M}^{q_1,z_1}$, we may have factors $R$ (of the scalar curvature); if $z_1 = \mu$ there can be no such factors (by definition). We further subdivide $F_{M}^{q_1,z_1}$ into subsets $F_{M,\alpha}^{q_1,z_1}$, $\alpha = 0, \ldots, M$, where $f \in F_{M,\alpha}^{q_1,z_1}$ if and only if $C^f$ has $\alpha$ factors $R$ (and hence $M - \alpha$ factors $\Delta \psi$).

We claim that for each of the index sets $F_{M,\alpha}^{q_1,z_1}$ there is a linear combination of vector fields, $\sum_{r \in R_{M,\alpha}^{q_1,z_1}} a_{r}C_{g}^{\sigma_{r}}(\psi_{1}, \ldots, \psi_{s})$ where each $C_{g}^{\sigma_{r}}$ is in the form (5.24) and has $q_1 - \alpha$ factors $\nabla^{(p)}Ric$, $\delta = z_1$ and $\alpha$ factors $R$ and $M - \alpha$ factors $\Delta \psi_{h}$, so that modulo complete contractions of length $\geq \sigma + 1$:

\[
\sum_{f \in F_{M}^{q_1,z_1}} a_{f}C_{g}^{\delta}(\psi_{1}, \ldots, \psi_{s}) - \text{div}_{i} \sum_{r \in R_{M,\alpha}^{q_1,z_1}} a_{r}C_{g}^{\sigma_{r}}(\psi_{1}, \ldots, \psi_{s}) = \\
\sum_{d \in D_1} a_{d}C_{g}^{\delta}(\psi_{1}, \ldots, \psi_{s}) + \sum_{d \in D_2} a_{d}C_{g}^{\delta}(\psi_{1}, \ldots, \psi_{s}) + \sum_{d \in D_3} a_{d}C_{g}^{\delta}(\psi_{1}, \ldots, \psi_{s}),
\]

(5.44)

where each $C_{g}^{d}, d \in D_1$ has $q_1$ factors $\nabla^{(p)}Ric$ and $M$ factors $\Delta \psi_{h}$ and $\delta = z_1 - 1$. Each $C_{g}^{d}, d \in D_2$ has $q_1 - 1$ factors $\nabla^{(p)}Ric$ and $M$ factors $\Delta \psi_{h}$ and $\delta = z_1$. Finally, each $C_{g}^{d}, d \in D_3$ has $q_1$ factors $\nabla^{(p)}Ric$ and $M - 1$ factors $\Delta \psi_{h}$ and $\delta = z_1$.

In the case where $z_1 = \mu$, we have noted that no $C^f$, $f \in F_{M}^{q_1,\mu}$ has a factor $R$. We then claim that there is a linear combination of vector fields, $\sum_{t \in T} a_{t}C_{g}^{t}(\psi_{1}, \ldots, \psi_{s})$ in the form (5.24) with $q_1$ factors $\nabla^{(p)}Ric$ and with $\delta = z_1 = \mu$ and with $M$ factors $\Delta \psi_{h}$, so that modulo complete contractions of length $\geq \sigma + 1$:

\[
\sum_{f \in F_{M}^{q_1,z_1}} a_{f}C_{g}^{\delta}(\psi_{1}, \ldots, \psi_{s}) - \text{div}_{i} \sum_{t \in T} a_{t}C_{g}^{t}(\psi_{1}, \ldots, \psi_{s}) = \\
\sum_{d \in D_1} a_{d}C_{g}^{\delta}(\psi_{1}, \ldots, \psi_{s}) + \sum_{d \in D_3} a_{d}C_{g}^{\delta}(\psi_{1}, \ldots, \psi_{s}),
\]

(5.45)

where the complete contractions on the right hand side are as in the notation under (5.44).

Assuming (for a moment) (5.44), (5.45), we are reduced to proving Lemma 5.3 under the additional assumption that $F_{M}^{q_1,z_1} = \emptyset$. In that setting, we define $z_{\text{min}}(M)$ to stand for the minimum $z$ for which $F_{M}^{q_1,z_1} \neq \emptyset$. By our hypothesis, $z_{\text{min}}(M) > z_1 = \mu$. On the other hand, some contractions $C_{g}^{f}, f \in F_{M}^{q_1,z_{\text{min}}(M)}$, might have factors $R$ (of the scalar curvature). We further subdivide $F_{M}^{q_1,z_{\text{min}}(M)}$ into subsets $F_{M,\alpha}^{q_1,z_{\text{min}}(M)}$, $\alpha = 0, \ldots, M$, where $f \in F_{M,\alpha}^{q_1,z_{\text{min}}(M)}$ if and only if $C^f$ has $\alpha$ factors $R$ and $M - \alpha$ factors $\Delta \psi$. 49
We claim that for each of the index sets \( F_{M,\alpha}^{q_1, z_{\min}(M)} \), there is a linear combination of vector fields, \( \sum_{r \in R_{M,\alpha}^{q_1, z_{\min}(M)}} a_r C_{\gamma}^{r,i}(\psi_1, \ldots, \psi_s) \) where each \( C_{\gamma}^{r,i} \) is in the form (5.24) and has \( \delta = z_{\min}(M) \) and \( \alpha \) factors \( R \) and \( M - \alpha \) factors \( \Delta \psi_h \), so that modulo complete contractions of length \( \geq \sigma + 1 \):

\[
\sum_{f \in F_{M,\alpha}^{q_1, z_{\min}(M)}} a_f C_{\gamma}^{f}(\psi_1, \ldots, \psi_s) - \text{div}_i \sum_{r \in R_{M,\alpha}^{q_1, z_{\min}(M)}} a_r C_{\gamma}^{r,i}(\psi_1, \ldots, \psi_s) = \\
\sum_{d \in D_1} a_d C_{\gamma}^{d}(\psi_1, \ldots, \psi_s) + \sum_{d \in D_2} a_d C_{\gamma}^{d}(\psi_1, \ldots, \psi_s) + \sum_{d \in D_3} a_d C_{\gamma}^{d}(\psi_1, \ldots, \psi_s).
\]

(5.46)

In the above, each \( C_{\gamma}^{d}, d \in D_1 \) is a complete contraction with \( \delta = z_{\min}(M) + 1 \) and all the other features being the same as the contractions \( C_{\gamma}^{f} \) indexed in \( F_{q_1 M}^{M} \) (in particular they have \( \delta = z_{\min} \)). Each \( C_{\gamma}^{d}, d \in D_2 \) is a complete contraction with \( q = q_1 - 1 \) and all the other features being the same as the contractions \( C_{\gamma}^{f} \) indexed in \( F_{q_1 M}^{M} \) (in particular they have \( \delta = z_{\min} \)). Finally, each \( C_{\gamma}^{d}, d \in D_3 \) is a complete contraction with a total of \( M - 1 \) factors \( R \) or \( \Delta \psi_h \), and all the other features being the same as the contractions \( C_{\gamma}^{f} \) indexed in \( F_{q_1 M}^{M} \) (in particular they have \( \delta = z_{\min} \)).

We remark that (5.46) implies that modulo complete contractions of length \( \geq \sigma + 1 \):

\[
\sum_{f \in F_{M,\alpha}^{q_1, z_{\min}(M)}} a_f C_{\gamma}^{f}(\psi_1, \ldots, \psi_s) - \text{div}_i \sum_{r \in R_{M,\alpha}^{q_1, z_{\min}(M)}} a_r C_{\gamma}^{r,i}(\psi_1, \ldots, \psi_s) = \\
\sum_{d \in D_1} a_d C_{\gamma}^{d}(\psi_1, \ldots, \psi_s) + \sum_{d \in D_2} a_d C_{\gamma}^{d}(\psi_1, \ldots, \psi_s) + \sum_{d \in D_3} a_d C_{\gamma}^{d}(\psi_1, \ldots, \psi_s).
\]

(5.47)

The terms in the RHS of the above have the same properties as the terms in the RHS of (5.46).

Thus, in order to derive Lemma 5.5 we need to show (5.42), (5.43), and to derive Lemma 5.4 we need to show (5.44), (5.45), (5.46).

Proof of equations (5.42) and (5.43).

Our aim is to apply Proposition 5.2 to equation (5.40). Since (5.11) is symmetric in the functions \( \psi_1, \ldots, \psi_s \), we can just set \( \psi_1 = \ldots \psi_s = \psi \) and we lose no information. For notational convenience, we will still write \( \psi_1, \ldots, \psi_s \) but the functions \( \psi_1, \ldots, \psi_s \) will in fact all be equal to \( \psi \). Now, by factoring out the factors \( \Delta \psi \) we write:
\[
\sum_{l \in L_M} C^{l,i}_g(\psi_1, \ldots, \psi_s) + \sum_{j \in J_M} C^j_g(\psi_1, \ldots, \psi_s) = \\
\sum_{l \in L_M} C^{l,i}_g(\psi_{M+1}, \ldots, \psi_s) \Delta \psi_1 \ldots \Delta \psi_M + \sum_{j \in J_M} C^j_g(\psi_{M+1}, \ldots, \psi_s) \Delta \psi_1 \ldots \Delta \psi_M.
\]

(5.48)

In view of (5.40), we claim:

\[
\sum_{l \in L_M} a_l X \text{div}_{i_1} \ldots X \text{div}_{i_{\mu-M}} C^{l,i_1 \ldots i_{\mu-M}}_g(\psi_{M+1}, \ldots, \psi_s) + \\
\sum_{j \in J_M} a_j X \text{div}_{i_1} \ldots X \text{div}_{i_{m_j-M}} C^{j,i_1 \ldots i_{m_j-M}}_g(\psi_{M+1}, \ldots, \psi_s) = 0,
\]

(5.49)

modulo complete contractions of length \(\geq \sigma - M + 1\). (5.40) follows by focusing on the sublinear combination in (5.40) that has \(M\) factors \(\nabla \psi_1, \ldots, \nabla \psi_M\) (notice that this sublinear combination vanishes separately and all \(\nabla \psi_h\)'s are contracting against derivative indices), and the formally erasing the factors \(\nabla \psi_h\) and the (derivative) indices against which they contract. This produces a new true equation\(^{50}\) which is precisely (5.49).

We now directly apply Proposition 5.2 to (5.49)\(^{51}\) (since by the hypothesis that \(|\Delta|_{\text{max}} \leq \sigma - 3\) the real length of the tensor fields in (5.49) is \(\geq 3\). In the case where \(L^M_{\mu} \neq \emptyset\), we deduce that there is a linear combination of acceptable \(\mu - M + 1\)-tensor fields,

\[
\sum_{h \in H} a_h C_{g}^{h,i_1 \ldots i_{\mu-M+1}}(\psi_{M+1}, \ldots, \psi_s),
\]

so that:

\[
\sum_{l \in L_M} a_l X \text{div}_{i_1} \ldots X \text{div}_{i_{\mu-M}}(\psi_{M+1}, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_{\mu-M}} v = \\
\sum_{h \in H} a_h X \text{div}_{i_{\mu-M+1}} C_{g}^{h,i_1 \ldots i_{\mu-M+1}}(\psi_{M+1}, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_{\mu-M}} v.
\]

(5.50)

Therefore, since the above holds formally, we observe that the linear combination of vector fields needed for (5.42) is precisely

\[
[C_{g}^{h,i_1 \ldots i_{\mu-M+1}}(\psi_{M+1}, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_{\mu-M}} v]
\]

\[
\cdot \nabla_{k_1} \psi_1 \ldots \nabla_{k_{\mu-M}} \psi_{M}.
\]

(5.51)

We prove (5.43) by a very similar argument. We again use the notation (5.48), only now \(L^M_{\mu} = \emptyset\). We claim:

\(^{50}\)This fact can be rigorously checked by applying the operation Erase—see the Appendix below.

\(^{51}\)After first re-writing (5.49) in dimension \(n - 2M\).
\[
\sum_{j \in J_{j_{\text{min}}(M)}^{\delta} \cap J^\delta} a_j \text{div}_i \ldots \text{div}_{j_{\text{min}}(M) - M} C_g^{j_{\text{i1}} \ldots j_{\text{im_{\text{min}}(M) - M}}} \left( \psi_{M+1}, \ldots, \psi_s \right) +
\sum_{\delta > \delta_{\text{min}}(M)} \sum_{j \in J_{j_{\text{min}}(M)}^{\delta} \cap J^\delta} a_j \text{div}_i \ldots \text{div}_{j_{\text{min}}(M) - M} C_g^{j_{\text{i1}} \ldots j_{\text{im_{\text{min}}(M) - M}}} \left( \psi_{M+1}, \ldots, \psi_s \right) = 0,
\]

(5.52)

modulo complete contractions of length \(\geq \sigma - M + 1\). This again follows by focusing on the sublinear combination in (5.40) that has \(M\) factors \(\nabla \psi_1, \ldots, \nabla \psi_M\) (notice this sublinear combination vanishes separately) and then applying the eraser to \(\nabla \psi_1, \ldots, \nabla \psi_M\).

We now apply Proposition 5.2 to (5.52). We deduce that there is a linear combination of acceptable \((\delta_{\text{min}}(M) - M + 1)\)-tensor fields,

\[
\sum_{h \in H} a_h C_g^{h_{i1} \ldots h_{i_{\text{min}}(M) - M + 1}} \left( \psi_{M+1}, \ldots, \psi_s \right) \nabla_{i_1} u \ldots \nabla_{i_{\text{min}}(M) - M} u - \sum_{h \in H} a_h \nabla_{k_1} u \nabla_{k_2} \psi_1 \ldots \nabla_{k_{M/2}} u \nabla_{k_{M/2+1}} \psi_M.
\]

(5.54)

Therefore, since the above holds formally, we derive that the linear combination of vector fields needed for (5.43) is precisely:

\[
\sum_{h \in H} a_h C_g^{h_{i1} \ldots h_{i_{\text{min}}(M) - M + 1}} \left( \psi_{M+1}, \ldots, \psi_s \right) \nabla_{i_1} u \ldots \nabla_{i_{\text{min}}(M) - M} u = 0.
\]

(5.53)

The proof of (5.44), (5.45), (5.46):

We start with (5.44) and (5.46). We will prove (5.46); this proof applies to show (5.44) by just setting \(z_{\text{min}}(M) = z_1\). We begin by noting an equation analogous to (5.48): Let \(\alpha_1 \geq 0\) be the smallest value of \(\alpha\) for which each \(F_{\text{min}(M), \alpha}\) with \(\alpha > \alpha_1\) is empty. We will then show (5.46) for \(\alpha = \alpha_1\). Clearly, if we can prove this, then (5.46) follows for every \(\alpha\), by induction. We observe that each of the other complete contractions appearing in the equation of Lemma 5.4 must either have less than \(M\) factors \(\Delta \psi_h, R\) (in total), or will have \(M\) such factors in total but less than \(\alpha_1\) factors \(R\). This just follows from the definition of \(M\) and \(\alpha_1\).

Again, using the fact that the complete contractions are symmetric in the functions \(\psi_1, \ldots, \psi_s\), we may assume with no loss of generality that these functions are all equal to \(\psi\). We factor out the factors \(\Delta \psi_h, R\) to write out:
\[
\sum_{f \in F_{1}^{q_{1}, z_{\min}(M)}} a_{f} C_{g}^{f}(\psi_{1}, \ldots, \psi_{s}) + \sum_{z > z_{\min}(M)} \sum_{f \in F_{1}^{q_{1}, z_{\min}(M)}} a_{f} C_{g}^{f}(\psi_{M - \alpha_{1} + 1}, \ldots, \psi_{s}) + \sum_{z > z_{\min}(M)} \sum_{f \in F_{1}^{q_{1}, z_{\min}(M)}} a_{f} C_{g}^{f}(\psi_{M - \alpha_{1} + 1}, \ldots, \psi_{s}) = \text{(5.55)}
\]

We now claim that modulo complete contractions of length \(\geq \sigma - M + 1\):

\[
X \text{div}_{i_{1}} \ldots \text{div}_{i_{z_{\min}(M) - M}} \sum_{f \in F_{1}^{q_{1}, z_{\min}(M)}} a_{f} C_{g}^{f,i_{1} \ldots i_{z_{\min}(M) - M}}(\psi_{M - \alpha_{1} + 1}, \ldots, \psi_{s}, \Omega^{q_{1} - \alpha_{1}}) + X \text{div}_{i_{1}} \ldots \text{div}_{i_{z - M}} \sum_{z > z_{\min}(M)} \sum_{f \in F_{1}^{q_{1}, z_{\min}(M)}} a_{f} C_{g}^{f,i_{1} \ldots i_{z - M}}(\psi_{M - \alpha_{1} + 1}, \ldots, \psi_{s}, \Omega^{q_{1} - \alpha_{1}}) = 0.
\text{(5.56)}
\]

This follows by picking out the terms in \([5.41]\) with \(\sigma_{1}\) factors \(\nabla \Omega\), \(\sigma_{1}\) factors \(\nabla \psi\) (this sublinear combination must vanish separately) and then formally erasing these factors and the indices against which they contract.\(^{52}\)

We may now apply Proposition \([5.2]\) to \([5.56]\). We derive that there is a linear combination of \(z_{\min}(M) - M + 1\)-tensor fields, \(\sum_{z \in Z} a_{z} C_{g}^{z,i_{1} \ldots i_{z_{\min}(M) - M + 1}}\) (written in dimension \(-n + 2M\)), as stated in Proposition \([5.2]\) so that modulo complete contractions of length \(\geq \sigma - M + z_{\min}(M) + 1\):

\[
\sum_{f \in F_{1}^{q_{1}, z_{\min}(M)}} a_{f} C_{g}^{f,i_{1} \ldots i_{z_{\min}(M) - M + 1}}(\psi_{M - \alpha_{1} + 1}, \ldots, \psi_{s}, \Omega^{q_{1} - \alpha_{1}}) \nabla_{i_{1}} v \ldots \nabla_{i_{z_{\min}(M) - M + 1}} v - X \text{div}_{i_{z_{\min}(M) - M + 1}} \sum_{t \in T} a_{t} C_{g}^{t,i_{1} \ldots i_{z_{\min}(M) - M + 1}}(\psi_{M - \alpha_{1} + 1}, \ldots, \psi_{s}, \Omega^{q_{1} - \alpha_{1}}) \nabla_{i_{1}} v \ldots \nabla_{i_{z_{\min}(M) - M + 1}} v = 0.
\text{(5.57)}
\]

We act on the above equation with the operation \textit{Riccify}. Observe that:

\(^{52}\)A rigorous proof that this formal operation produces a true equation can be derived by virtue of the operation \textit{Erase}–see the Appendix below.
\[
\sum_{f \in F^{q_1, \text{min}}_{M, \alpha_1}} a_f \text{Ricci} y[C_g^{f, z_{\text{min}}(M) - M}(\psi_{M-\alpha_1+1}, \ldots, \psi_s, \Omega)] \nabla_{\tilde{z}_{\text{min}} M} \nabla_{z_{\text{min}}(M - M)} v \\
\ldots \nabla_{z_{\text{min}}(M - M)} v \cdot R^{\alpha_1} \Delta \psi_1 \ldots \Delta \psi_{M-\alpha_1} = \sum_{f \in F^{q_1, \text{min}}_{M, \alpha_1}} a_f C_g^f (\psi_1, \ldots, \psi_s).
\]

(5.58)

Thus, by virtue of the above equation and Lemma 5.2, we see that the vector field required for equations (5.44), (5.46) is precisely:

\[
\sum_{t \in T} a_t \text{Ricci} y[C_g^{t, z_{\text{min}}(M) - M + 1}(\psi_{M-\alpha_1+1}, \ldots, \psi_s, \Omega^{q_1 - \alpha_1})] \nabla_{t_{\text{min}} M} \nabla_{z_{\text{min}}(M - M)} v \cdot R^{\alpha_1} \Delta \psi_1 \ldots \Delta \psi_{M-\alpha_1}.
\]

(5.59)

Proof of (5.45):

The proof in this case is very slightly different from the proof of (5.44), (5.46).

We recall that for each \( f \in F^{q_1, \mu}_M \), \( C_g^f(\psi_1, \ldots, \psi_s) \) must have \( M \) factors \( \Delta \psi_h \) (i.e. there are no factors \( R \) by definition) and will furthermore contain factors \( \nabla_{a_1 \ldots a_t} \nabla^{(p)} \nabla_{\tilde{r}_1 \ldots \tilde{r}_p} \text{Ric}_{ab} \) and for each such factor one of the indices \( a_1, \ldots, a_t \) is contracting against the index \( b \) (this implies that for each such factor we have \( t > 0 \)). With no loss of generality, we assume that the index \( a_1 \) is contracting against the index \( b \). Thus, applying the contracted second Bianchi identity, we may replace the factor \( \nabla_{a_1 \ldots a_t} \nabla^{(p)} \nabla_{\tilde{r}_1 \ldots \tilde{r}_p} \text{Ric}_{ab} \) by a factor \( \frac{1}{2} \nabla_{a_1 \ldots a_t} \nabla^{(p+1)} \nabla_{\tilde{r}_1 \ldots \tilde{r}_p} R_{ab} \), modulo introducing complete contractions with more than \( \sigma \) factors. Moreover, as in the previous case we set \( \psi_1, \ldots, \psi_s = \psi \), although we will still write \( \psi_1, \ldots, \psi_s \) for notational convenience.

We pick out the complete contractions indexed in \( \bigcup_{z > \mu} F^{q_1, z}_M \) that have exactly \( M \) factors \( \Delta \psi_h \). By our notational conventions, they will be indexed in \( \bigcup_{z > \mu} F^{q_1, z}_M \). We again write out:

\[
\sum_{f \in F^{q_1, \mu}_M} a_f C^f_g(\psi_1, \ldots, \psi_s) + \sum_{f \in \bigcup_{z > \mu} F^{q_1, z}_M} a_f C^f_g(\psi_1, \ldots, \psi_s) = \\
\left( \frac{1}{2} \right)^{q_1} \sum_{f \in F^{q_1, \mu}_M} a_f C^f_g(\psi_{M+1}, \ldots, \psi_s) \Delta \psi_1 \ldots \Delta \psi_M + \\
\sum_{f \in \bigcup_{z > \mu} F^{q_1, z}_M} a_f C^f_g(\psi_{M+1}, \ldots, \psi_s) \Delta \psi_1 \ldots \Delta \psi_M,
\]

(5.60)

where each \( C^f_g(\psi_{M+1}, \ldots, \psi_s), f \in F^{q_1, \mu}_M \) is now in the form:
by construction, each complete contraction 

\[ \text{contr}(\nabla f_1 \ldots f_\mu \nabla^{(m_1)} R_{ijkl} \otimes \cdots \otimes \nabla g_1 \ldots g_p \nabla^{(m_p)} R_{ijkl} \]

\[ \otimes \nabla v_1 \ldots v_u \nabla^{(d_1)} R_{a_1 \ldots a_{d_1}} \otimes \cdots \otimes \nabla x_1 \ldots x_p \nabla^{(d_q)} \]

\[ R_{b_1 \ldots b_{d_q}} \]

\[ \nabla^{a_1 \ldots a_{l_1}} \nabla^{(u_1)} \psi_1 \otimes \cdots \otimes \nabla^{c_1 \ldots c_{l_0}} \nabla^{(u_0)} \psi_s), \]

while each \( C^f_g (\psi_{M+1}, \ldots, \psi_s, \Omega^{\mu_1}) \), \( f \in F_{M,0}^{\mu, z} \), \( z > \mu \) is still in the form:

\[ \text{contr}(\nabla f_1 \ldots f_\mu \nabla^{(m_1)} R_{ijkl} \otimes \cdots \otimes \nabla g_1 \ldots g_p \nabla^{(m_p)} R_{ijkl} \]

\[ \otimes \nabla v_1 \ldots v_u \nabla^{(d_1)} R_{C} \otimes \cdots \otimes \nabla x_1 \ldots x_p \nabla^{(d_q)} \]

\[ R_{C}, \]

\[ \nabla^{a_1 \ldots a_{l_1}} \nabla^{(u_1)} \psi_1 \otimes \cdots \otimes \nabla^{c_1 \ldots c_{l_0}} \nabla^{(u_0)} \psi_s), \]

(with \( \delta > \mu \)).

Now, for each \( f \in F_{M,0}^{\mu} \) we denote by \( C^f_g (\psi_{M+1}, \ldots, \psi_s, \Omega^{\mu_1}) \) the complete contraction that arises from \( C^f_g (\psi_{M+1}, \ldots, \psi_s) \) by replacing the factors \( \nabla^{a_1 \ldots a_{l_1}} \nabla^{(p)}_{r_1 \ldots r_p} R \) by \( \nabla^{a_1 \ldots a_{l_1}} \nabla^{(p)}_{r_1 \ldots r_p} \Omega \). For each \( f \in F_{M,0}^{\mu, z} \), \( z > \mu \),

\( C^f_g (\psi_{M+1}, \ldots, \psi_s, \Omega^{\mu_1}) \) is the same as before. Applying Lemma [4.1] to the equation \( \int_M I_g dV_g = 0 \) and then the erasure to the \( M \) factors \( \nabla \psi \), we derive that modulo complete contractions of length \( \geq \sigma - M + 1 \):

\[ X \text{div}_{i_1} \ldots \text{Xdiv}_{\mu-M} \sum_{f \in F_{M,0}^{\mu, \mu}} a_f C^f_g (\psi_{M+1}, \ldots, \psi_s, \Omega^{\mu_1}) \]

\[ + X \text{div}_{i_1} \ldots \text{Xdiv}_{\mu-M} \sum_{f \in \bigcup_{\mu > \mu} F_{M,0}^{\mu, z}} a_f C^f_g (\psi_{M+1}, \ldots, \psi_s, \Omega^{\mu_1}) = 0. \]

(5.63)

Now, applying Proposition [5.2] to the above, we deduce that there is a linear combination of acceptable \((\mu-M+1)\)-tensor fields, \( \sum_{t \in T} a_t C_{g}^{f, i_1 \ldots i_{\mu-1-M}} (\psi_1, \ldots, \psi_s, \Omega) \), so that modulo complete contractions of length \( \geq \sigma - M + \mu + 1 \):

\[ \sum_{f \in F_{M,0}^{\mu, \mu}} a_f C^f_g (\psi_{M+1}, \ldots, \psi_s, \Omega^{\mu_1}) \nabla v_{i_1} \ldots \nabla v_{\mu-M} u \]

\[ - X \text{div}_{i_{\mu-M+1}} \sum_{t \in T} a_t C_{g}^{f, i_1 \ldots i_{\mu-1-M+1}} (\psi_1, \ldots, \psi_s, \Omega^{\mu_1}) \nabla v_{i_1} \ldots \nabla v_{i_{\mu-M}} u = 0. \]

(5.64)

Finally, an observation: The above equation holds formally. We observe that by construction, each complete contraction

\[ C_{g}^{f, i_1 \ldots i_{\mu-1-M}} (\psi_{M+1}, \ldots, \psi_s, \Omega^{\mu_1}) \nabla v_{i_1} \ldots \nabla v_{i_{\mu-M}} u \]
has at least one factor $\nabla v$ contracting against each factor $\nabla^{(p)} \Omega$. Therefore, since the equation holds formally we may assume with no loss of generality that the same must be true of each vector field

$$C_{g}^{\ell, i_{1} \ldots i_{\mu - M + 1}} (\psi_{1}, \ldots, \psi_{s}, \Omega) \nabla_{i_{1}} v \ldots \nabla_{i_{\mu - M}} v.$$

Moreover, we know that for each $C^{\ell}$ and each of its factors $\nabla^{(p)} \Omega$, one factor $\nabla v$ is contracting against the last index $r_{p}$. Modulo introducing complete contractions of length $\geq \sigma + \mu - M + 1$, we may assume that the same is true of each of the vector fields $C_{g}^{\ell, i_{1} \ldots i_{\mu - M + 1}}$. But then, since the above holds formally, we may assume that when we apply the permutations to make the left hand side of (5.64) formally zero, the index $r_{p}$ in each factor $\nabla^{(p)} \Omega$ is not permuted. (One can prove this by applying the operation $Erase$ repeatedly).

Now, we define an operation $Riccify''$ which is slightly different from the standard $Riccify$: We replace each of the expressions of the form

$$\nabla^{(r)} \psi^{t_{1} v} \ldots \nabla^{t_{s} v}, \nabla^{(m)} R_{ijkl}^{t_{1} v} \ldots \nabla^{t_{s} v}$$

(where each of the factors $\nabla v$ is contracting against the factor $\nabla^{(r)} \psi_{h}$, $\nabla^{(m)} R_{ijkl}$ respectively) as in the operation $Riccify$. But we also replace each of the expressions

$$\nabla^{(p)} \Omega^{r_{1} a_{1} v} \ldots \nabla^{r_{s} a_{1} v} \nabla^{r_{p}} v$$

by a factor

$$\nabla^{r_{1} a_{1} v} \ldots \nabla^{r_{p} a_{1} v} \nabla^{(p-1)}_{r_{1} \ldots r_{p-1}} R$$

We then observe that since (5.64) holds formally without permuting the last index $r_{p}$ in each factor $\nabla^{(p)} \Omega$ (and that index is contracting against a factor $\nabla v$), we then have that:

\[
\sum_{f \in F_{\mu + M, \mu - M}} a_{f} Riccify''[C_{g}^{\ell, i_{1} \ldots i_{\mu - M}} (\psi_{M+1}, \ldots, \psi_{s}, \Omega^{q_{1}}) \nabla_{i_{1}} v \ldots \nabla_{i_{\mu - M}} v] \\
- Xdiv_{\mu - M + 1} \sum_{t \in T} Riccify''[a_{t} C_{g}^{\ell, i_{1} \ldots i_{\mu - M + 1}} (\psi_{1}, \ldots, \psi_{s}, \Omega^{q_{1}}) \nabla_{i_{1}} v \ldots \nabla_{i_{\mu - M}} v] = 0, \tag{5.65}
\]

modulo complete contractions of length $\geq \sigma - M + 1$.

Hence, the vector field needed for (5.45) is precisely:

$$\sum_{t \in T} Riccify'[a_{t} C_{g}^{\ell, i_{1} \ldots i_{\mu - M + 1}} (\psi_{1}, \ldots, \psi_{s}, \Omega^{q_{1}}) \nabla_{i_{1}} v \ldots \nabla_{i_{\mu - M}} v] \Delta \psi_{1} \ldots \Delta \psi_{M}.$$

Note: Codification of the remaining cases of Lemmas 5.4 and 5.5.

\[54^{56}\text{See the appendix for the strict definition of the operation } Erase[\ldots].\]
Here we codify what remains to be proven for Lemmas 5.4 and 5.5. We will then prove these claims in the paper [5] in this series.

Lemma 5.4 What remains to be proven is the second claim in that Lemma: Recall the index set $F$ in Lemma 5.4 (this indexes the complete contractions in $I^g(\psi_1, \ldots, \psi_s)$, in the form (5.24), with at least one factor $\nabla_p R_{ic}$ or $R$).

Recall that for each $q, 1 \leq q \leq \sigma - s$, $F_q \subset F$ stands for the index set of complete contractions with precisely $q$ factors $\nabla_p R_{ic}$ or $R$.

For each index set $F_q$ above, let us denote by $F_{q,\ast} \subset F_q$ the index set of complete contractions with $|\Delta| \geq \sigma - 2$, and $F_{\ast} = \bigcup_{q>0} F_{q,\ast}$.

Claim: There exists a linear combinations of vector fields (indexed in $H$ below), each in the form (5.24) with $\sigma$ factors, so that modulo complete contractions of length $> \sigma$:

$$
\sum_{f \in F_{\ast}} a_f C^f_g(\psi_1, \ldots, \psi_s) - \text{div}_i \sum_{h \in H} a_h C^h_{g,i}(\psi_1, \ldots, \psi_s) = \sum_{y \in Y} a_y C^y_g(\psi_1, \ldots, \psi_s),
$$

where the complete contractions indexed in $Y$ are in the form (5.24) with length $\sigma$, and satisfy all the properties of the sublinear combination $\sum_{f \in F} \ldots$ but in addition have $|\Delta| \leq \sigma - 3$.

The remaining for Lemma 5.6

Lemma 5.6 Denote by $L^\ast_{\mu} \subset L_{\mu}, J^\ast \subset J$ the index sets of complete contractions in the hypothesis of Lemma 5.5 with $|\Delta| \geq \sigma - 2$, among the complete contractions indexed in $L_{\mu}, J$ respectively. We claim that there exists a linear combination of $(\mu + 1)$-tensor fields fields (indexed in $H$ below), with length $\sigma$, in the form (5.24) without factors $\nabla^{(p)} R_{ic}, R$ and with $\delta = \mu$ so that:

$$
\sum_{l \in L^\ast_{\mu}} a_l C^l_{g,i_1 \ldots i_\mu}(\psi_1, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_\mu} v - \text{div}_i \sum_{h \in H} a_h C^h_{g,i_1 \ldots i_\mu}(\psi_1, \ldots, \psi_s) \cdot \nabla_{i_1} v \ldots \nabla_{i_\mu} v = \sum_{l \in L^\ast_{\mu}} a_l C^l_{g,i_1 \ldots i_\mu}(\psi_1, \ldots, \psi_s) \nabla_{i_1} v \ldots \nabla_{i_\mu} v,
$$

where the $\mu$-tensor fields indexed in $L$ are in the form (5.24) without factors $\nabla^{(p)} R_{ic}, R$ and with $|\Delta| \leq \sigma - 3$.

(Notice that if we can show the above, then in proving Lemma 5.5 we can assume with no loss of generality that $L^\ast_{\mu} = \emptyset$). In the setting $L^\ast_{\mu} = \emptyset$, what remains to be shown to complete the proof of Lemma 5.6 is the following:

Lemma 5.7 Assume that $L^\ast_{\mu} = \emptyset$ in the hypothesis of Lemma 5.5. Denote by $J^\ast \subset J$ the index set of the complete contractions $C^I_{g}(\psi_1, \ldots, \psi_s)$ with $|\Delta| \geq \sigma$,
σ − 2. We then claim that there exists a linear combination of vector fields (indexed in \( H \) below) so that:

\[
\sum_{j \in J} a_j C^j_g(\psi_1, \ldots, \psi_s) - \text{div} \sum_{h \in H} a_h C^h_g(\psi_1, \ldots, \psi_s) = \sum_{y \in Y'} a_y C^y_g(\psi_1, \ldots, \psi_s),
\]

where the complete contractions indexed in \( Y' \) are in the form (5.24) with length \( \sigma \), with no factors \( \nabla(p) \text{Ric} \) or \( R \) and have \( \delta \geq \mu + 1 \) and in addition \( |\Delta| \leq \sigma - 3 \).

If we can show the above claims, then we will have completely shown Lemmas 5.4 and 5.5, and hence also Proposition 5.1, which implies Proposition 3.1.

6 Appendix: Some Technical Tools.

We prove here some technical claims, which will be useful in this series of papers.

The Eraser: We consider complete contractions \( C^h_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \) in the form:

\[
p\text{contr}(\nabla^{(m_1)} R_{ijkl} \otimes \cdots \otimes \nabla^{(m_s)} R_{ijkl} \otimes \\
\nabla^{(b_1)} \Omega_1 \otimes \cdots \otimes \nabla^{(b_p)} \Omega_p \otimes \nabla \phi_1 \otimes \cdots \otimes \nabla \phi_u),
\]

with length \( \sigma + u \). We firstly define a formal operation on such complete contractions, which we call the eraser operation:

**Definition 6.1** Consider a set of complete contractions \( C^h_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \), \( h \in H \), each in the form (6.1). Assume that for each \( h \in H \), some particular factor \( \nabla \phi_b \) (\( b \) is fixed, i.e. \( b \) is independent of \( h \in H \)) is contracting against a factor \( \nabla^{(m)} R_{ijkl} \) and moreover against a derivative index in that factor.

We then define \( \text{Erase}_{\nabla \phi_b}[C^h_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)] \) to stand for the complete contraction (of weight \( -n + 2 \)) that formally arises from \( C^h_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \) by erasing the factor \( \nabla \phi_b \) and also erasing the derivative index that it contracts against in (5.24).

**Lemma 6.1** Consider a set of complete contractions \( \{C^h_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)\}_{h \in H} \) as in the above definition and assume that modulo complete contractions of length \( \geq \sigma + u + 1 \):

\[
U_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = \sum_{h \in H} a_h C^h_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = 0.
\]

We claim that modulo complete contractions of length \( \geq \sigma + u \):

\[54\text{Note that we thus obtain a complete contraction of length } \sigma + u - 1.\]
\[
\sum_{h \in H} a_h \text{Erase}_{\phi_b}[C_g^h(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)] = 0. \quad (6.3)
\]

Proof: We call the factor \(\nabla^{(m)} R_{ijkl}\) against which \(\nabla \phi_b\) contracts the special factor. We break the index set \(H\) into subsets \(H_\mu\), where \(h \in H_\mu\) if and only if \(C^h\) has \(m = \mu\) derivatives on the special factor. Observe that since (6.2) holds formally, it follows that for each different \(\mu\):

\[
\sum_{h \in H_\mu} a_h C_g^h(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = \sum_{t \in T} a_t C_g^t(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u), \quad (6.4)
\]

where each \(C^t\) has length \(\geq \sigma + u + 1\). This holds because the linearized version\(^{55}\) of (6.2) must hold formally (for the linearized complete contractions), and also because under any of the linearized permutations by which we can make the linearized version of (6.2) formally zero, the number of derivatives on the special factor remains invariant. Now, it would suffice to show that for each \(\mu\):

\[
\sum_{h \in H_\mu} a_h \text{Erase}_{\phi_b}[C_g^h(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)] = 0, \quad (6.5)
\]

modulo complete contractions of length \(\geq \sigma + u\).

In order to show this we write:

\[
U_\mu^g := \sum_{h \in H} a_h C_g^h(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) - \sum_{t \in T} a_t C_g^t(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)(= 0).
\]

Now, consider Image\(^1\)\(_{r_\phi}\)[\(U_\mu^g\)]. We denote by Image\(^1\)\(^A\)[\(U_\mu^g\)] the sublinear combination that arises in Image\(^1\)\(_{r_\phi}\)[\(U_\mu^g\)] by replacing one of the factors of the form \(\nabla^{(m)} R_{ijkl}\) by one of the four linear terms on the right hand side of (2.10). Now, let us denote by \(a\) the index in the special factor that contracts against the factor \(\nabla \phi_b\). We denote by Image\(^1\)\(_{r_\phi}\)[\(U_\mu^g\)] \(B\) the linear combination that arises from Image\(^1\)\(_{r_\phi}\)[\(U_\mu^g\)] by applying the transformation law (2.11) to the special factor and bringing out a factor \(\nabla_a \phi'\) (observe that for every contraction in Image\(^1\)\(_{r_\phi}\)[\(U_\mu^g\)] the two factors \(\nabla \phi_b, \nabla \phi'\) contract against each other). We denote by Image\(^1\)\(_{r_\phi}\)[\(U_\mu^g\)] \(C\) the sublinear combination that arises in Image\(^1\)\(_{r_\phi}\)[\(U_\mu^g\)] when we apply the transformation law (2.11) to any complete contraction \(C_g^b(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)\) and bring out a factor \(\nabla_f \phi'\), where \(f \neq a\). We thus have that each complete contraction in Image\(^1\)\(_{r_\phi}\)[\(U_\mu^g\)] has length \(\sigma + u + 1\) and a factor \(\nabla \phi'\) but it does not contract against a factor \(\nabla \phi_b\).

Finally, we denote by \(\sum_{w \in \mathcal{W}} a_w C_g^w(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi')\) a generic linear combination of complete contractions with either length \(\sigma + u + 1\) and a factor \(\nabla^{(q)} \phi'\) \((q \geq 2)\) or with length \(\geq \sigma + u + 1\).

\(^{55}\)See the introduction in [1] for a discussion of linearized complete contractions.
By virtue of (6.4), we derive that
\[ \text{Image}_{\phi'}^{1}[U^\mu_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi')] = 0. \tag{6.6} \]
In addition, we deduce that:
\[ \text{lin}\{\text{Image}_{\phi'}^{1,A}[U^\mu_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi')]\} = 0. \tag{6.7} \]
Hence, since the above holds formally, we may repeat the permutations by which we make the above formally zero to the linear combination \( \text{Image}_{\phi'}^{1,A}[U^\mu_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi')] \); we deduce that:
\[ \text{Image}_{\phi'}^{1,A}[U^\mu_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi')] = \sum_{y \in Y} a_y C^y_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi') + \sum_{w \in W} a_w C^w_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi'), \tag{6.8} \]
where each \( C^y_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \phi') \) has length \( \sigma + u + 1 \) and a factor \( \nabla \phi \), but that factor contracts against a factor \( \nabla^{(m)} R_{ijkl} \). This follows by virtue of the formula (2.1).
Hence we deduce that, modulo complete contractions of length \( \geq \sigma + u + 2 \):
\[ \text{Image}_{\phi'}^{1,B}[U^\mu_g] + \text{Image}_{\phi'}^{1,C}[U^\mu_g] + \sum_{y \in Y} a_y C^y_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \psi_u, \phi') \]
\[ + \sum_{w \in W} a_w C^w_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \psi_u, \phi') = 0. \tag{6.9} \]
Then, since the above must hold formally, it follows that, modulo complete contractions of length \( \geq \sigma + u + 1 \):
\[ \text{Image}_{\phi'}^{1,B}[U^\mu_g] + \text{Image}_{\phi'}^{1,C}[U^\mu_g] + \sum_{y \in Y} a_y C^y_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \psi_u, \phi') = 0. \tag{6.10} \]
Now, since the above must hold formally, and since each complete contraction in \( \text{Image}_{\phi'}^{1,C}[U^\mu_g] \) has the factor \( \nabla \phi' \) not contracting against the factor \( \nabla \phi \), we derive:
\[ \text{Image}_{\phi'}^{1,B}[U^\mu_g] = 0, \tag{6.11} \]
(modulo contractions of length \( \geq \sigma + u + 2 \)).
Lastly we observe, by virtue of the formula (2.11) and by virtue of the factor \( e^{2\phi(x)} \) in (2.10), that:
\[ \text{Image}_{\phi'}^{1,B}[U^\mu_g] = -(\mu + 1)\nabla^c \phi \nabla_c \phi' \sum_{h \in H} a_h \text{Erase} \nabla \phi \{C^h_g\} \tag{6.12} \]
contraction is between internal indices in a factor $\nabla$ with a factor $C$ where each $D$ index set $C$ while $d \in D$.

If $d \in C$, we assume an equation: $\nabla^w f = 0$ holds modulo complete contractions of length $\sigma + u + 1$. We will divide the index set $D$ into subsets $D_1, D_2 \subset D$. We will say that $d \in D_1$ if the internal contraction is between internal indices in a factor $\nabla^{(m)} R_{ijkl}$ and $D_2 = D \setminus D_1$.

In other words, $C_g^d(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)$, $d \in D_1$ will have a factor $\nabla^{(m)} Ric_{ik}$, while $C_g^d(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)$, $d \in D_2$ will have an internal contraction between indices $(\nabla^s, s)$.

We then define an operation $Sub_w$ that acts on the complete contractions $C_g^d$ $d \in D$ as follows: For $d \in D_1$ $Sub_w[C_g^d]$ will stand for the complete contraction that arises from $C_g^d$ by replacing the factor $\nabla^{(p+2)} R_{ijkl} Ric_{ik}$ by a factor $-\nabla^{(p+2)} R_{ijkl}$. If $d \in D_2$, $Sub_w[C_g^d]$ will stand for the complete contraction that arises from $C_g^d$ by picking out the internal contraction $(\nabla^s, s)$, then erasing the derivative index $\nabla^s$ and then adding a factor $\nabla^s \omega$ and contracting it against the index $s$ that has been left hanging.

**Lemma 6.2** Assuming (6.14) we claim, in the notation above:

\[ \sum_{d \in D} a_d C_g^d(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = 0, \]

where each $C_g^d$ is a contraction of length $\sigma + u + 1$ in the form

\[ \text{contr}(\nabla^{(m)} R \otimes \cdots \otimes \nabla^{(m_a)} R \otimes \nabla^{(r_1)} \omega \otimes \cdots \otimes \nabla^{(r_p)} \Omega_{p} \otimes \nabla \phi_1 \otimes \cdots \otimes \nabla \phi_u), \]

with a factor $\nabla^{(a)} \omega$, $a \geq 2$.\[\square\]
Proof: The proof goes as follows: We re-write (6.14) in the form:

\[ S_g = \sum_{d \in D} a_d C_d^g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) + \sum_{h \in H} a_h C_h^g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = 0, \]  

(6.17)

where each \( C_h^g \) has length \( \geq \sigma + u + 1 \). We then re-write this in a high dimension \( N \) (we can do this since the equation holds formally—see the discussion in the section on “Trans-dimensional isomorphisms” in [1]) and take \( \text{Image}_\omega[S_g] \). We of course have \( \text{Image}_\omega[S_g] = 0 \). By virtue of the transformation laws (2.10), (2.11), we derive:

\[ (0 =) \text{Image}_\omega[S_g] = \sum_{d \in D} a_d N \cdot \text{Sub}_\omega[C_d^g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)] + \]  

\[ N \sum_{v \in V} a_v C_v^g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \omega) + N \sum_{j \in J_1} a_j(N) C_j^g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u, \omega), \]  

(6.18)

where here the contractions \( C_v^g \) are in the form (6.16), have length \( \sigma + u + 1 \) and a factor \( \nabla^{(a)} \omega, a \geq 2 \), while the contractions \( C_j^g, j \in J_1 \) have length \( \geq \sigma + u + 2 \); each coefficient \( a_j(N) \) is a polynomial in \( N \), of degree 0 or 1. Now, re-writing the above in dimension \( n \) and picking out the sublinear combination of terms that are multiplied by \( N \) (notice this sublinear combination must vanish separately) gives us our claim. □

\( \nabla \)'s into \( X\text{div}'s: \) We finally present a final technical Lemma which will be used on numerous occasions in this series of papers.

First some notation: We let \( \sum_{f \in F} a_f C_f^{j_1 \ldots j_s}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \) stand for a linear combination of \( \alpha \)-tensor fields, with each \( C_f^{j_1 \ldots j_s}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \) being a partial contraction in the form:

\( p\text{contr}(\nabla^{(m_1)} R_{ijkl} \otimes \cdots \otimes \nabla^{(m_s)} R_{ijkl} \otimes \nabla^{(b_1)} \Omega_1 \otimes \cdots \otimes \nabla^{(b_p)} \Omega_p \otimes \nabla \phi_1 \otimes \cdots \otimes \nabla \phi_u), \)  

(6.19)

each having a given number \( \sigma_1 \) of factors \( \nabla^{(m_i)} R_{ijkl} \), a given number \( p \) of factors \( \nabla^{(b_i)} \Omega_h, 1 \leq h \leq p \) and a given number \( u \) of factors \( \nabla \phi_y, 1 \leq y \leq u \). We are also assuming that each \( b_i \geq 2, 1 \leq i \leq p \) and that each factors \( \nabla \phi_h \) is contracting against one of the factors \( \nabla^{(m_i)} R_{ijkl}, \nabla^{(b_i)} \Omega_h \). Furthermore, we assume that none of these tensor fields has an internal contraction.

We assume an equation:

\[ {\text{56}} \text{In particular, no free index belongs to one of the factors } \nabla \phi_y. \]
\[
\sum_{f \in F} a_f C_{g}^{j_1 \cdots j_\alpha} (\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} v \cdots \nabla_{i_\alpha} v = \\
\sum_{y \in Y} a_y C_{g}^{y_1 \cdots y_\nu} (\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} v \cdots \nabla_{i_\nu} v \\
+ \sum_{z \in Z} a_z C_{g}^{z_1 \cdots z_\mu} (\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \nabla_{i_1} v \cdots \nabla_{i_\mu} v,
\]

(6.20)

where the tensor fields in the RHS have length \(\sigma + u + 1\); furthermore, the ones indexed in \(Y\) have a factor \(\nabla \Omega_p\) (with only one derivative), while the ones indexed in \(Z\) have a factor \(\nabla (b)\Omega_p\) with \(b \geq 2\).

We recall that for the tensor fields indexed in \(F\), \(X\text{div}_{i_1} \cdots \text{div}_{i_\alpha} [C_{g}^{j_1 \cdots j_\alpha}]\) stands for the sublinear combination in \(\text{div}_{i_1} \cdots \text{div}_{i_\alpha} [C_{g}^{j_1 \cdots j_\alpha}]\) where neither of the derivatives \(\nabla^{i_r}\) is allowed to hit any factor \(\nabla \phi_t\), nor the factor \(T\) to which \(i_h\) belongs. For the tensor fields indexed in \(Y\), \(X\text{div}_{i_1} \cdots \text{div}_{i_\alpha} [C_{g}^{y_1 \cdots y_\nu}]\) stands for the sublinear combination in \(\text{div}_{i_1} \cdots \text{div}_{i_\alpha} [C_{g}^{y_1 \cdots y_\nu}]\) where neither of the derivatives \(\nabla^{i_r}\) is allowed to hit any factor \(\nabla \phi_t\), nor the factor \(T\) to which \(i_h\) belongs, nor the factor \(\nabla \Omega_p\).

Our claim is the following:

**Lemma 6.3** Assume the equation (6.20). We then claim that:

\[
\sum_{f \in F} a_f X\text{div}_{i_1} \cdots X\text{div}_{i_\alpha} C_{g}^{j_1 \cdots j_\alpha} (\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = \\
\sum_{y \in Y} a_y X\text{div}_{i_1} \cdots X\text{div}_{i_\alpha} C_{g}^{y_1 \cdots y_\nu} (\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \\
+ \sum_{z \in Z} a_z C_{g}^{z_1 \cdots z_\mu} (\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u),
\]

(6.21)

here \(\sum_{z \in Z} \cdots\) stands for a generic linear combination of complete contractions in the form (6.19) with length \(\sigma + u + 1\) and with a factor \(\nabla^{(b)} \Omega_p, A \geq 2\).

**Proof of Lemma 6.3.** We consider (6.20) and immediately derive an equation:

\[
\sum_{f \in F} a_f \text{div}_{i_1} \cdots \text{div}_{i_\alpha} C_{g}^{j_1 \cdots j_\alpha} (\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = \\
\sum_{y \in Y} a_y \text{div}_{i_1} \cdots \text{div}_{i_\alpha} C_{g}^{y_1 \cdots y_\nu} (\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \\
+ \sum_{z \in Z} a_z \text{div}_{i_1} \cdots \text{div}_{i_\alpha} C_{g}^{z_1 \cdots z_\mu} (\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u).
\]

(6.22)

Now, we divide the LHS of the above into three linear combinations: \(L^1\) is the sublinear combination which consists of terms with no internal contractions.
and with one derivative on each function \( \phi_h \); \( L^2 \) is the sublinear combination which consists of terms with at least one internal contraction in some factor and with one derivative on each function \( \phi_h \); \( L^3 \) stands for the sublinear combination of terms with at least one function \( \phi_h \) differentiated more than once (and \( \nabla^{(B)} \Omega_p \) still satisfies \( B \geq 2 \) by construction).

It easily follows that each of these three sublinear combinations must vanish separately at the linearized level. We denote by \( \text{lin}\{L^1\}, \text{lin}\{L^2\}, \text{lin}\{L^3\} \) the linear combinations of linearized complete contractions that arise from \( L^1, L^2, L^3 \) by replacing each complete contraction \( C_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) \) by its linearization \( \text{lin}\{C_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)\} \).

Then, repeating the permutations by which we make the equation \( \text{lin}\{L^2\} = 0 \) and \( \text{lin}\{L^3\} = 0 \) formally zero to the non-linear setting, we derive that:

\[
L^2 = \sum_{z \in Z \cup Z'} a_z C^z_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u),
\]

where the terms indexed in \( Z \) above are a generic linear combinations with the properties described above. The terms indexed in \( Z' \) have length \( \sigma + u + 1 \) and have only factors \( \nabla \phi_h, \nabla \Omega_p \) but also have at least one internal contraction. By the same reasoning we derive an equation:

\[
L^3 = \sum_{z \in Z \cup Z' \cup Z''} a_z C^z_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)
\]

where the tensor fields indexed in \( Z'' \) have length \( \sigma + u + 1 \) and at least one factor \( \nabla^{(B)} \phi_h, B \geq 2 \).

Thus, replacing the above into (6.22) we derive:

\[
\sum_{f \in F} a_f X_{\text{div}i_1} \ldots X_{\text{div}i_\mu} C^f_{g^{i_1 \ldots i_\mu}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = \sum_{y \in Y} a_y X_{\text{div}i_1} \ldots X_{\text{div}i_\mu} C^y_{g^{i_1 \ldots i_\mu}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u)
+ \sum_{z \in Z \cup Z' \cup Z''} a_z C^z_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u),
\]

(6.23)

(and the above holds modulo terms with length \( \geq \sigma + u + 2 \)).

Now, using the above we derive that we can write:

\[
\sum_{f \in F} a_f X_{\text{div}i_1} \ldots X_{\text{div}i_\mu} C^f_{g^{i_1 \ldots i_\mu}}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = \sum_{m \in M} a_m C^m_g(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u),
\]

(6.24)

---

57 Observe that \( L^1 = \sum_{f \in F} a_f X_{\text{div}i_1} \ldots X_{\text{div}i_\mu} C^f_{g^{i_1 \ldots i_\mu}}(\Omega_1, \ldots, \Omega_p) \).

58 See the section “Background material” in [1] for a strict definition of linearized complete contractions.
where the terms indexed in $M$ have length $\sigma+u+1$ and no internal contractions, and also have one derivative on each function $\phi_k$.

Therefore, substituting the above into (6.23) (and using the fact that (6.23) holds modulo complete contractions of length $\geq \sigma + u + 2$), we derive that in (6.23):

$$\sum_{z \in \mathbb{Z}' \cup \mathbb{Z}''} a_z C^g_\mathbb{Z}(\Omega_1, \ldots, \Omega_p, \phi_1, \ldots, \phi_u) = 0,$$

modulo complete contractions of length $\geq \sigma + u + 2$. This completes the proof of our claim. □

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