Scaling in the Neutrino Mass Matrix, $\mu - \tau$ Symmetry and the See-Saw Mechanism

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Abstract

The scaling hypothesis postulates proportionality of two columns of the Majorana neutrino mass matrix in the flavor basis. This Ansatz was shown to lead to an inverted hierarchy and $U_{e3} = 0$. We discuss theoretical and phenomenological properties of this hypothesis. We show that (i) the neutrino mass matrix with scaling follows as a consequence of a generalized $\mu - \tau$ symmetry imposed on the type-I see-saw model; (ii) there exists a unique texture for the Dirac mass matrix $m_D$ which leads to scaling for arbitrary Majorana matrix $M_R$ in the context of the type-I see-saw mechanism; (iii) unlike in the $\mu - \tau$ symmetric case, a simple model with two right-handed neutrinos and scaling can lead to successful leptogenesis both with and without the inclusion of flavor effects.

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1 Introduction

There have been many approaches to explain the peculiar (both measured and possible) features of neutrino masses and lepton mixing, for recent reviews see e.g. [1, 2]. Very often, the see-saw mechanism [3], and mostly the type I variant, is used to explain the smallness of neutrino masses. As additional motivation, the see-saw mechanism contains all necessary ingredients to produce the baryon asymmetry of the Universe via leptogenesis [4]. Unfortunately, the number of parameters in general see-saw scenarios exceeds the measurable ones, and full reconstruction of the see-saw parameters is at least very difficult [5]. Moreover, irrespective of their origin, it seems very unlikely that (ranges of) all the elements of the neutrino mass matrix can be determined purely from experiments. This has led to postulates of various Ansätze or symmetries for the neutrino mass matrix in order to have more predictivity. Texture zeros or $\mu-\tau$ symmetry are popular examples for such strategies. Another Ansatz, proposed in Refs. [6, 7], is called scaling\footnote{Similar mass matrices with this form have been obtained in specific models in [8, 9].}. In this letter we wish to show that the scaling hypothesis is deeply connected to a generalized version of the $\mu-\tau$ symmetry and can follow from such symmetry. We also study yet unexplored implications of the scaling hypothesis on the see-saw structure and at the phenomenological level on leptogenesis and lepton flavor violation.

We will first, in Section 3 discuss properties of the Dirac mass matrix $m_D$ in general see-saw frameworks incorporating scaling. We show that the scaling hypothesis uniquely determines its structure. Moreover, scaling is shown to follow from a generalized $\mu-\tau$ invariance applied to the type-I see-saw. As the possibility of leptogenesis is one important consequence of the see-saw mechanism, we will investigate leptogenesis in the context of the scaling hypothesis in Section 4. First we study a simple two right-handed neutrino case, followed by a three right-handed neutrino case generated by a $Z_{2L} \times Z_{2R}$ symmetry. In the case of two right-handed neutrinos we show that the baryon asymmetry for unflavored leptogenesis is proportional to the solar neutrino mass-squared difference. The three neutrino model is shown to lead to the same result. Details of the related case of $\mu-\tau$ symmetric see-saw with two heavy neutrinos are delegated to the Appendix, where we also summarize relevant formulae for flavored and unflavored leptogenesis. We conclude and summarize in Section 5. In the following, to set the stage, we will first summarize the properties and predictions of scaling.

2 Neutrino Mixing, Scaling and generalized $\mu-\tau$ Symmetry

In the charged lepton basis, neutrino mass and lepton mixing originates at low energies from the following neutrino mass matrix appearing in the Lagrangian

$$ \mathcal{L} = \frac{1}{2} \bar{\nu}_L^c m_D \nu_L + h.c. \quad (1) $$
Diagonalization of $m_\nu$ is achieved via $U^* m_\nu^{\text{diag}} U^\dagger = m_\nu$, where $U$ is the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, whose standard parametrization is

$$U = \begin{pmatrix}
c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
-s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i\delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i\delta} & s_{23} c_{13} \\
s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i\delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i\delta} & c_{23} c_{13}
\end{pmatrix} P .$$  \hspace{1cm} (2)

Here $c_{ij} = \cos \theta_{ij}$, $s_{ij} = \sin \theta_{ij}$ and $P = \text{diag}(1, e^{i\alpha}, e^{i\beta})$ contains the Majorana phases.

Because neutrino data implies that $\theta_{23}$ is (close to) maximal and $\theta_{13}$ (close to) zero, many works have been devoted to $\mu-\tau$ symmetry \cite{10, 11, 12}. It is defined as an interchange symmetry $\nu_\mu L \leftrightarrow \nu_\tau L$ in the diagonal charged lepton basis. The generator of this $Z_2$ is

$$S_{\mu\tau} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}.$$  \hspace{1cm} (3)

and the resulting mass matrix is obtained from the requirement $S_{\mu\tau}^{-1} m_\nu S_{\mu\tau} = m_\nu$ and reads (unless stated otherwise, all entries in the mass matrices here and in the following can be complex)

$$m_\nu = \begin{pmatrix}
a & b & b \\
\cdot & d & e \\
\cdot & \cdot & d
\end{pmatrix}.$$  \hspace{1cm} (4)

One eigenvector of this matrix is $(0, -1, 1)^T$. If the eigenvalue $|d - e|$ corresponding to this eigenvector is the largest or the smallest one, then one predicts $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ in the normal or inverted mass ordering, respectively.

It is possible to replace the (left) $\mu-\tau$ symmetry by a more general one which leads to the prediction $\theta_{13} = 0$ but allows arbitrary $\theta_{23}$. This generalization discussed in \cite{12} amounts to imposing the following $Z_2$ symmetry:

$$S(\theta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos 2\theta & \sin 2\theta \\
0 & \sin 2\theta & -\cos 2\theta
\end{pmatrix} ,$$  \hspace{1cm} (5)

It follows that $S^2 = 1$, independent of $\theta$. Therefore $S$ defines a $Z_2$ symmetry. Invariance of $m_\nu$ under this $Z_2$ leads to the prediction $\theta_{23} = \theta$ and $\theta_{13} = 0$. Obviously, with $\theta = \frac{\pi}{4}$ the generator $S$ coincides with the one in Eq. (3):

$$S(\theta \to \pi/4) = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} = S_{\mu\tau} .$$  \hspace{1cm} (6)

The $Z_2$ symmetry defined by Eq. (5) can therefore be considered as generalized $\mu-\tau$ symmetry \cite{12}. Note that in order to identify $\theta_{23} = \theta$ and $\theta_{13} = 0$ in the normal or inverted mass
ordering, it is necessary that the eigenvalue of the corresponding eigenvector \((0, \tan \theta, 1)^T\) is the largest or smallest one.

The scaling Ansatz, first proposed in [6] and further analyzed in [7], is more predictive than the (generalized) \(\mu-\tau\) symmetry. It postulates a particular structure of the low energy neutrino mass matrix:

\[
m_\nu = \begin{pmatrix}
a & b & b/c \\
\cdot & d & d/c \\
\cdot & \cdot & d/c^2
\end{pmatrix}.
\]

This form originates from the requirement that the ratio \((m_\nu)_{\alpha\mu}/(m_\nu)_{\alpha\tau}\) equals the “scaling factor” \(c\) for all \(\alpha = e, \mu, \tau\). We summarize here some properties of scaling:

- the mass matrix has rank 2, i.e., one vanishing mass eigenvalue. The eigenvector for this eigenvalue is \((0, -1, c)\). Hence, the matrix is only compatible with an inverted hierarchy and predicts that \(U_{e3} = 0\) and \(\tan^2 \theta_{23} = 1/c^2\). There is no CP violation in oscillation experiments and atmospheric neutrino mixing is in general non-maximal. Solar neutrino mixing is naturally large, but not specified;

- in contrast to the flavor symmetry \(L_e - L_\mu - L_\tau\) [13], which has frequently been used to obtain an inverted hierarchy, scaling requires no breaking of the symmetry in order to generate successful phenomenology. Recall that \(L_e - L_\mu - L_\tau\) predicts two degenerate eigenvalues, maximal \(\theta_{12}\), and breaking terms which are at least 30\% the magnitude of the terms allowed by the symmetry;

- there can be one physical phase, which is a low energy Majorana phase. It would appear in the effective mass governing neutrino-less double beta decay: \(\langle m \rangle \simeq \sqrt{\Delta m^2_{\alpha}} \sqrt{1 - \sin^2 2\theta_{12} \sin^2 \alpha}\). The scaling Ansatz is therefore fully reconstructible;

- interestingly, if not more than one Higgs doublet is present, the beta-functions of both \(m_3\) and \(\theta_{13}\) are proportional to \(m_3\). Thus, the prediction \(m_3 = U_{e3} = 0\) is not modified by radiative corrections, as has been noted in the first paper of Ref. [8].

We stress here that scaling leads to an inverted hierarchy, vanishing \(\theta_{13}\) and arbitrary \(\theta_{23}\). However, in contrast to generalized \(\mu-\tau\) symmetry, there is no ambiguity with regards to the identification of the mass eigenvalues, the angles \(\theta_{13} = 0\) and \(\theta_{23}\) always belong to the vanishing mass.

All features of scaling are attractive and easily testable, but rely on the low energy part of lepton physics. Let us note here that the analysis that we will present applies to the case of a real and diagonal charged lepton mass matrix, i.e., in the natural limit of negligible charged lepton corrections to the mixing angles. A model based on the flavor symmetry \(D_4 \times Z_2\) in which the charged lepton mass matrix is diagonal and \(m_\nu\) obeys scaling has been constructed in Ref. [6]. Nevertheless, in this paper we will not consider any specific model, but will continue with a discussion of scaling in the charged lepton basis and in the type I see-saw mechanism.
We will show in the next Section that the scaling is a consequence of the generalized $\mu-\tau$ symmetry imposed on the type-I see-saw model.

# 3 Scaling and the See-Saw Mechanism

It was noted in [6] that the scaling form for $m_\nu$ follows in the type-I see-saw model for an arbitrary $M_R$ if $m_D$ has the form:

$$m_D = \begin{pmatrix} a_1 & b & b/c \\ a_2 & d & d/c \\ a_3 & e & e/c \end{pmatrix}.$$  \hspace{1cm} (8)

We will prove next a theorem which states that this is the only allowed $m_D$ if $m_\nu$ is to have the form Eq. (7).

## 3.1 A Theorem for $m_D$

We claim that the most general $m_D$ which can reproduce the scaling form for $m_\nu$ is given by Eq. (8) if $m_\nu$ is to possess two non-zero eigenvalues. To see this, we first note that scaling implies that the low energy mass matrix has an eigenvector of the form

$$m_\nu |\psi\rangle = 0 \text{ where } |\psi\rangle = \begin{pmatrix} 0 \\ -1 \\ c \end{pmatrix}. \hspace{1cm} (9)$$

The fact that one mass is zero implies that $\text{det}(m_\nu) = 0$ and therefore

$$\text{det}(m_D) = 0. \hspace{1cm} (10)$$

Recall that $M_R$ needs to be non-singular in order to have a valid type I see-saw. The above relation implies now for $m_D$ that

$$m_D |\chi\rangle = 0 \hspace{1cm} (11)$$

for one of its eigenvectors $|\chi\rangle$. With this expression and the definition of $m_\nu$ it also follows that

$$m_\nu |\chi\rangle = 0. \hspace{1cm} (12)$$

Equations (9) and (12) together imply that either $m_\nu$ possesses two massless states, or that $|\chi\rangle$ is proportional to $|\psi\rangle$. The first possibility cannot generate two different scales unless there are huge perturbations acting in $m_\nu$. Instead, one is lead to the second possibility, which means

$$m_D |\psi\rangle = 0. \hspace{1cm} (13)$$

With $|\psi\rangle$ defined in Eq. (9), it is straightforward to show that the most general solution of this equation is indeed given by Eq. (8). Note that we have not made any assumptions about the structure of $M_R$, other than it is non-singular.
There is another way to show that Eq. (8) is the implied form of \( m_D \): consider the following parametrization of the Dirac mass matrix [14]:

\[
m_D = i \sqrt{M_R R^\dagger} \sqrt{m_\nu \text{diag}} U^\dagger,
\]

(14)

where \( R \) is a complex and orthogonal matrix. Here, \( M_R \) need not be diagonal. Inserting \( m_3 = U e_3 = 0 \) in this parametrization will show that \( (m_D)_{12}/(m_D)_{13} = (m_D)_{22}/(m_D)_{23} = (m_D)_{32}/(m_D)_{33} = -\cot \theta_{23} \), independent of \( M_R \) and \( R \). Since \( \cot \theta_{23} = c \), the same form for \( m_D \) as in Eq. (8) is implied.

3.2 Scaling and \( Z_2 \) Symmetries

The scaling form as given in Eq. (7) has been proposed as an hypothesis which leads to \( U e_3 = 0 \) and one massless neutrino. As discussed in [12], any neutrino mass matrix which yields \( U e_3 = 0 \) must be invariant under a \( Z_2 \) symmetry. Therefore, \( m_\nu \) in Eq. (7) must also be invariant under some \( Z_2 \). This \( Z_2 \) is easily seen to be the generalized \( \mu - \tau \) symmetry defined in Eq. (5).

Indeed if we identify

\[
\cos 2\theta = \frac{c^2 - 1}{1 + c^2} \quad \text{and} \quad \sin 2\theta = \frac{2c}{1 + c^2},
\]

(15)

then the mass matrix is invariant according to

\[
S^{-1} m_\nu S = m_\nu.
\]

(16)

While the scaling form for \( m_\nu \) satisfies the above equation, it is not the most general form implied by the invariance under \( S \). It is easy to see that the most general \( m_\nu \) invariant under

\[
\nu_L \rightarrow S \nu_L
\]

(17)

is given by

\[
m_\nu = \begin{pmatrix}
    a & b & b/c \\
    B + C \cos 2\theta & C \sin 2\theta & \\
    \cdot & \cdot & B - C \cos 2\theta
\end{pmatrix}
\]

(18)

This matrix gives \( U e_3 = 0 \) but all eigenvalues are non-zero. It reduces to the scaling form in Eq. (7) if

\[
B = C = \frac{d}{c \sin 2\theta},
\]

in which case one gets the inverted hierarchy structure.

However, scaling follows from the generalized \( Z_2 \) invariance if one additionally assumes that \( m_\nu \) is obtained from the type-I see-saw mechanism. In this case, Eq. (17) is sufficient to imply scaling. In the see-saw mechanism, the mass matrix \( m_\nu \) is obtained through

\[
m_\nu = -m_D^T M_R^{-1} m_D,
\]

where \( m_D \) is the Dirac mass matrix and \( M_R \) the Majorana mass matrix for the right-handed (RH) neutrinos. The Lagrangian reads

\[
\mathcal{L} = \frac{1}{2} \overline{N_R} M_R N_R^c + \overline{N_R} m_D \nu_L + h.c.
\]

(19)
Eq. (17) implies now that \( m_D S = m_D \). This in turn implies first that \( m_\nu = -m_D^T M_R^{-1} m_D \) takes the generalized \( \mu-\tau \) invariant form given in Eq. (17). Secondly, it implies an \( m_D \) satisfying Eq. (8). As a consequence, the resulting \( m_\nu \) also possesses the scaling form with inverted hierarchy. Note that the above results follow for arbitrary non-singular \( M_R \). Our only assumption is the type-I see-saw and the symmetry defined by Eq. (17).

In the context of \( \mu-\tau \) symmetry one often \([10]\) applies an additional \( Z_2 \) symmetry which exchanges the RH neutrino fields \( \nu_{\mu R} \leftrightarrow \nu_{\tau R} \). This implies a form of \( M_R \) in analogy to Eq. (4) and in total the mass matrices read for \( c = 1 \)

\[
\begin{align*}
\text{Eq. (20)}
\end{align*}
\]

Note that this second \( Z_2 \) is neither necessary nor sufficient to obtain a \( \mu-\tau \) symmetric \( m_\nu \).

The low energy \( \mu-\tau \) symmetry follows purely from the left-handed \( Z_2 \) symmetry as we have seen.

4 Scaling and Leptogenesis

Scaling with \( c = 1 \) leads to the same mixing pattern as \( \mu-\tau \) symmetry. However, their implications at high energy can differ. Thus both leptogenesis and the lepton flavor violation pattern can be different. Leptogenesis in the presence of \( \mu-\tau \) symmetry was considered in \([10]\) and it was shown that the exact \( \mu-\tau \) symmetry implies vanishing lepton asymmetry if there are only two RH neutrinos. If three RH neutrinos are assumed then the lepton asymmetry is related to the solar neutrino mass-squared difference. We will show here that in the case of scaling even for 2 RH neutrinos the lepton asymmetry can be non-zero. Interestingly, the lepton asymmetry in this simple case is related to the solar scale and coincides with the one obtained in case of three generations by Mohapatra and Nasri in Ref. \([10]\). We trace this coincidence by considering the case of three RH neutrinos with scaling and show that one of the neutrinos decouples, leading essentially to the two RH neutrino result. In addition, we also evaluate various flavor lepton asymmetries in these cases.

4.1 Scaling with two heavy Neutrinos and Leptogenesis

With only two RH neutrinos present, the Dirac and Majorana mass matrices can be written as

\[
\begin{align*}
\text{Eq. (21)}
\end{align*}
\]

Here, we have imposed the scaling form of \( m_D \) while \( M_R \) is kept general. Thus \( V_R \) is a general unitary \( 2 \times 2 \) matrix and \( D_R = \text{diag}(M_1, M_2) \) with \( M_2 > M_1 \). The \( 3 \times 3 \) mass matrix of the light neutrinos reads

\[
\begin{align*}
\text{Eq. (22)}
\end{align*}
\]
We have defined here $\tilde{m}_D = V_R^T m_D$. The most general matrix diagonalizing $m_\nu$ will be called $U$, and is defined by

$$U^T m_\nu U = D_\nu = \text{diag}(m_1 e^{i\alpha_1}, m_2 e^{i\alpha_2}, 0), \text{ where } U = R_{23} U_{12}. \quad (23)$$

We have kept the phases of the eigenvalues of $m_\nu$. Here $R_{23}$ is a rotation in 23-space

$$R_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & -s_{23} \\ 0 & s_{23} & c_{23} \end{pmatrix} \text{ with } c_{23} = \frac{c}{\sqrt{1+c^2}}, \ s_{23} = \frac{1}{\sqrt{1+c^2}} \quad (24)$$

and

$$U_{12} = \begin{pmatrix} u & 0 & 0 \\ 0 & c_{12} & s_{12} \\ 0 & -s_{12} & c_{12} \end{pmatrix} P(\alpha) \quad (25)$$

The phase matrices are defined via $P(\gamma) = \text{diag}(e^{i\gamma}, e^{-i\gamma})$. Note that $\theta_{12}$ is the solar neutrino mixing angle and $\theta_{23}$ the atmospheric neutrino mixing angle with $\tan^2 \theta_{23} = 1/c^2$.

Knowing the matrix $U$ allows us to go into the neutrino mass basis:

$$D_\nu = -U_{12}^T R_{23} \tilde{m}_D^T D_R^{-1} \tilde{m}_D R_{23} U_{12} \equiv -Z^T D_R^{-1} Z, \quad (26)$$

where $Z$ is a $3 \times 2$ matrix defined by

$$Z = \tilde{m}_D R_{23} U_{12} = \begin{pmatrix} Z_{11} & Z_{12} & 0 \\ Z_{21} & Z_{22} & 0 \end{pmatrix} \text{ and } Z_{ij} = (V_R^T m'_D u)_{ij}. \quad (27)$$

The definition of $m'_D$ is

$$m'_D = \begin{pmatrix} A_1 & B/c_{23} \\ A_2 & D/c_{23} \end{pmatrix}. \quad (28)$$

With Eq. (26) it follows

$$Z^T D_R^{-1} Z = -\begin{pmatrix} m_1 e^{i\alpha_1} & 0 \\ 0 & m_2 e^{i\alpha_2} \end{pmatrix}. \quad (29)$$

From this relation one obtains constraints on the $Z_{ij}$:

$$m_1 e^{i\alpha_1} = -\frac{Z_{11}^2}{M_1} \left( 1 + \frac{Z_{22}^2}{Z_{11}^2} M_1 \right) \equiv -\frac{Z_{11}^2}{M_1} (1 + r e^{i\rho}),$$

$$m_2 e^{i\alpha_2} = -\frac{Z_{22}^2}{M_2} \left( 1 + \frac{Z_{11}^2}{Z_{22}^2} M_2 \right) \equiv -\frac{Z_{22}^2}{M_2} (1 + r e^{i\rho}),$$

$$\frac{Z_{12}}{Z_{22}} = -\frac{Z_{21} M_1}{Z_{11} M_2} = -\sqrt{\frac{M_1}{M_2}} e^{i\rho/2}. \quad (30)$$

Since in the inverted hierarchy with $m_3 = 0$ it holds that $m_2 \simeq m_1$, we see that we need to fulfill the requirement $|Z_{11}^2/M_1| \simeq |Z_{22}^2/M_2|$. The above equations imply

$$\Delta m^2_{\odot} = |1 + r e^{i\rho}|^2 \left( \frac{|Z_{22}|^4}{M_2^2} - \frac{|Z_{11}|^4}{M_1^2} \right). \quad (31)$$
The decay asymmetries of the heavy neutrinos are determined in the flavor basis and in terms of $\tilde{m}_D$ defined in Eq. (22). We can write

$$\tilde{m}_D = Z U_{12}^T R_{23}^T,$$

(32)

from which it follows that

$$\tilde{m}_D \tilde{m}_D^\dagger = ZZ^\dagger = \begin{pmatrix} Z Z^\dagger & 0 \\ 0 & 0 \end{pmatrix}.$$  

(33)

With the relations for the $Z_{ij}$ in Eq. (30) it follows for the effective mass governing the wash-out:

$$\tilde{m}_1 = \frac{(Z Z^\dagger)_{11}}{M_1} = \frac{m_1 + r m_2}{|1 + r e^{i \rho}|}.$$  

(34)

Since $m_3 = 0$ and an inverted hierarchy is present, we have, $m_2 \simeq m_1 \simeq \sqrt{\Delta m_A^2}$. Consequently, $\tilde{m}_1$ is of order $\sqrt{\Delta m_A^2}$ (this value is in fact the limit of $\tilde{m}_1$ for both $r \to 0$ and $r \to \infty$). We are thus in the “strong wash-out” regime, in which case there is little dependence on the initial conditions, and the efficiency factor is of order $10^{-2}$ or $10^{-3}$. More importantly, we find with Eq. (31)

$$\text{Im} \left\{ (Z Z^\dagger)_{12} \right\} = M_1 M_2 r \frac{\Delta m_\odot^2}{|1 + r e^{i \rho}|^2} \sin \rho,$$

(35)

and with the natural and usual assumption $M_2 \gg M_1$ the decay asymmetry (see the Appendix for the relevant formulae) for unflavored leptogenesis reads

$$\varepsilon_1 = -\frac{3}{16 \pi v^2 \tilde{m}_1} r \frac{\Delta m_\odot^2}{|1 + r e^{i \rho}|^2} \sin \rho.$$  

(36)

Numerically, with $\tilde{m}_1 \simeq \sqrt{\Delta m_A^2}$, we have

$$\varepsilon_1 \simeq -3.1 \times 10^{-6} \left( \frac{M_1}{10^{12} \text{ GeV}} \right) \frac{r}{|1 + r e^{i \rho}|^2} \sin \rho,$$

(37)

where we have inserted the current best-fit values $[15]$ for $\Delta m_\odot^2 = 7.67 \times 10^{-5} \text{ eV}^2$ and for $\Delta m_A^2 = 2.39 \times 10^{-3} \text{ eV}^2$. With an efficiency factor $\eta$ of order $10^{-2}$ or $10^{-3}$ the baryon asymmetry of order $Y_B \simeq 10^{-2} \varepsilon_1 \eta \simeq 10^{-10}$ can easily be generated.

Unlike in the $\mu-\tau$ symmetric case, one obtains a non-zero asymmetry with two RH neutrinos independent of the value of $c$. More interestingly, the lepton asymmetry obtained in Eq. (36) coincides with Eq. (20) in the first paper of $[10]$. Note that that analysis corresponds to a quite different situation than the one considered here namely, three RH neutrinos and $\mu-\tau$ symmetry for $m_D$ and $M_R$. We trace the origin of this result by considering a full three generation case in the next Subsection. Prior to this we discuss leptogenesis
flavor effects [10] which can be important for $M_1 \lessim 10^{12}$ GeV. These effects were not considered in [10]. We show in the Appendix that individual flavor asymmetries vanish if $\mu$-\(\tau\) symmetry is exact and only two RH neutrinos are present. This is not the case with scaling. The individual flavored decay asymmetries for our case read

$$\varepsilon_1^\mu = c_{23}^2 (\varepsilon_1 - \varepsilon_1^e) , \quad \varepsilon_1^e = s_{23}^2 (\varepsilon_1 - \varepsilon_1^e) \quad (38)$$

and

$$\varepsilon_1 = \frac{3 M_1}{16 \pi v^2 \tilde{m}_1 |1 + r e^{i\rho}|^2} \left( r (m_2^2 s_{12}^2 - m_1^2 c_{12}^2) \sin \rho + c_{12} s_{12} \sqrt{m_1 m_2} r ((m_1 - m_2) \sin(\alpha_1 - \alpha_2 - 4\beta - \rho)/2 + (m_1 - m_2) \sin(\alpha_1 - \alpha_2 - 4\beta + \rho)/2) \right). \quad (39)$$

The individual wash-out parameters are

$$\tilde{m}_1^e = \frac{1}{|1 + r e^{i\rho}|} (m_1 c_{12}^2 + r m_2 s_{12}^2 - 2 c_{12} s_{12} \sqrt{m_1 m_2} r \cos(\alpha_1 - \alpha_2 - 4\beta - \rho)/2) ,$$

$$\tilde{m}_1^\mu = \frac{2 c_{23}}{|1 + r e^{i\rho}|} (r m_2 c_{12}^2 + m_1 s_{12}^2 + 2 c_{12} s_{12} \sqrt{m_1 m_2} r \cos(\alpha_1 - \alpha_2 - 4\beta - \rho)/2) ,$$

$$\tilde{m}_1^\tau = \tan^2 \theta_{23} \tilde{m}_1^\mu . \quad (40)$$

One can explicitly check that the sum over the individual decay asymmetries and wash-out parameters results in the unflavored $\varepsilon_1$ and $\tilde{m}_1$ from Eqs. (36) and (34), respectively. Note that the individual flavor symmetries are generically proportional to the atmospheric mass scale although the combined sum depends on the solar scale.

Consider the limit $r = 1$. The term proportional to $\sin \rho$ dominates in $\varepsilon_1^e$, and

$$\varepsilon_1 - \varepsilon_1^e \sim - \frac{3}{16 \pi v^2 \tilde{m}_1} \frac{M_1}{|1 + e^{i\rho}|^2} \Delta m_A^2 \cos 2\theta_{12} \sin \rho . \quad (41)$$

We have used here again the fact that $m_2 \sim m_1 \sim \sqrt{\Delta m_A^2}$. Note that $\Delta m_A^2 \cos 2\theta_{12}$ is for the inverted hierarchy the minimal value of the effective mass governing neutrino-less double beta decay.

The final baryon asymmetry is given by [16]

$$Y_B \simeq \left\{ \begin{array}{ccc} -0.01 \varepsilon_1 \eta(\tilde{m}_1) & \text{one-flavor}, \\ -0.003 \left( (\varepsilon_1^e + \varepsilon_1^\mu) \eta \left( \frac{417}{589} (\tilde{m}_1^e + \tilde{m}_1^\mu) \right) + \varepsilon_1^\tau \eta \left( \frac{390}{589} \tilde{m}_1^\tau \right) \right) & \text{two-flavor}, \\ -0.003 \left( \varepsilon_1^e \eta \left( \frac{151}{179} \tilde{m}_1^e \right) + \varepsilon_1^\mu \eta \left( \frac{344}{537} \tilde{m}_1^\mu \right) + \varepsilon_1^\tau \eta \left( \frac{344}{537} \tilde{m}_1^\tau \right) \right) & \text{three-flavor}. \end{array} \right. \quad (42)$$

Here we have given separate expressions for one-, two- and three-flavored leptogenesis [16].

The three-flavor case occurs for $M_1 \sim 10^9$ GeV and corresponds to the situation when both $\mu$ and $\tau$ Yukawa coupling induced processes dominate over the Dirac neutrino couplings. The one-flavor case occurs $M_1 \sim 10^{12}$ GeV, and the two-flavor case (with the tau-flavor
decoupling first and the sum of electron- and muon-flavors, which act indistinguishably) applies in between. In case of the MSSM we need to multiply these mass values with $1 + \tan^2 \beta$. The efficiency $\eta$ is a function of the wash-out parameter $\tilde{m}$ and its form is given in the Appendix. Let us give one example to show that the correct baryon asymmetry can be generated: it follows from Eq. (40) that there can be range of parameters for which $\tilde{m}^\mu_1 \simeq \tilde{m}^\tau_1$ dominates over $\tilde{m}^\mu_1$ or vice versa. If $\tilde{m}^\mu_1$ is suppressed compared to the other two then one finds from Eq. (40) $\tilde{m}^\mu_1 \simeq \tilde{m}^\tau_1 \sim \frac{m_1^{\mu} + m_1^e}{1 + \rho}$.

In this case, Eq. (42) leads to

$$Y_B \simeq 10^{-10},$$

where we have considered the three flavor case and chosen $M_1 = 10^9$ GeV, $r = 1$ and $\rho = \frac{\pi}{4}$. It follows that the scaling case considered here can generate the required baryon asymmetry.

### 4.2 Scaling with 3 right-handed Neutrinos and $\mu - \tau$ symmetry

An effective two heavy neutrino framework discussed in the previous Subsection can be obtained quite easily. We have seen that in the framework of the see-saw mechanism invariance under the transformation $\nu_L \rightarrow S \nu_L$, where $S$ is the $Z_2$ generator defined in Eq. (3), leads to a low energy mass matrix obeying scaling. Consider now in addition to this $Z_{2L}$ symmetry an additional $Z_{2R}$ under which the heavy neutrinos are invariant:

$$N_R \rightarrow S N_R.$$  \hspace{1cm} (43)

This invariance implies

$$S m_D = m_D \text{ and } S M_R S = M_R.$$  \hspace{1cm} (44)

Together with the $Z_{2L}$ symmetry from above we have in total a $Z_{2L} \times Z_{2R}$ symmetry. The Dirac mass matrix $m_D$ which satisfies simultaneously $m_D S = m_D$ and $S m_D = m_D$ can be written as

$$m_D = \begin{pmatrix} A_1 & B & B s_{23}/c_{23} \\ A_2 c_{23} & D c_{23} & D s_{23} \\ A_2 s_{23} & D s_{23} & D s_{23}^2/c_{23} \end{pmatrix}. $$  \hspace{1cm} (45)

The most general solution of the $Z_{2R}$-invariance of the heavy neutrinos in Eq. (44) is (see also [12])

$$M_R = \begin{pmatrix} A & B & B/c \\ F (c - 1/c) + G & F \\ G \end{pmatrix}. $$  \hspace{1cm} (46)

The eigenvector corresponding to the eigenvalue $G - F/c$ of this matrix is the “scaling vector” $(0, -1, c)^T$. The matrix $M_R$ is diagonalized as

$$\tilde{V}_R^T R_{23}^T M_R R_{23} \tilde{V}_R = D_R \text{ with } \tilde{V}_R = \begin{pmatrix} V_R & 0 \\ 0 & e^{-i\phi_3/2} \end{pmatrix}, $$  \hspace{1cm} (47)
where $V_R$ is a general unitary $2 \times 2$ matrix and $\phi_3$ the phase of the third eigenvalue of $M_R$. The light neutrino mass matrix can be written as

$$m_\nu = -m_D^T R_{23} \bar{V}_R D_R^{-1} \bar{V}_R^T R_{23}^T m_D \equiv \tilde{m}_D^T D_R^{-1} \tilde{m}_D .$$

(48)

Obviously, $m_\nu$ is diagonalized by the matrix $U$ from Eq. (23). Consequently, in the neutrino mass basis we have

$$D_\nu = -\hat{Z}^T D_R^{-1} \hat{Z},$$

(49)

where

$$\hat{Z} \equiv \tilde{m}_D R_{23} U_{12} = V_R^T R_{23} m_D R_{23} U_{12} = \left( \begin{array}{ccc} Z & 0 \\ 0 & 0 \end{array} \right),$$

(50)

and we have used the relation

$$R_{23}^T m_D R_{23} = \left( \begin{array}{ccc} m'_D & 0 \\ 0 & 0 \end{array} \right)$$

(51)

with the definition of $m'_D$ in Eq. (28). Note that the matrix $Z$ appearing in Eq. (50) exactly coincides with the $Z$ defined in Eq. (27) in case of two heavy right-handed neutrinos. Likewise $\tilde{m}_D m_D^\dagger$ also coincides with the two right-handed neutrino case. Thus the three right-handed neutrino case with $Z_{2L} \times Z_{2R}$ symmetry defined in this Subsection gives the same decay and baryon asymmetry and neutrino masses as the two right-handed neutrino case treated in Section 4.1.

The structures of the $Z_{2L} \times Z_{2R}$ symmetric $m_D$ and $M_R$ reduce to the $\mu-\tau$ symmetric structures when $c = 1$. The decoupling obtained here is the same as obtained in the case of 3 RH neutrinos and $\mu-\tau$ symmetric see-saw considered in [10], except that $c$ needs not to be 1 here. Because of the specific structures of the mass matrices, one combination of $\nu_\mu$ and $\nu_\tau$ decouples. The matrix $M_R$ for the remaining two neutrinos does not obey $\mu-\tau$ symmetry. Thus the decoupling limit of this $Z_{2L} \times Z_{2R}$ invariant case corresponds to a scaling like situation with only two RH neutrinos for arbitrary $M_R$, and the lepton asymmetries in these two cases coincide.

### 4.3 Scaling and Lepton Flavor Violation

Another place in which the predictions of scaling and $\mu-\tau$ symmetry can differ is lepton flavor violation in supersymmetric see-saw models. In such supersymmetric type I see-saw frameworks with universal (mSUGRA) boundary conditions at the GUT scale $M_X$, the branching ratios for lepton flavor violating (LFV) charged lepton decays $\ell_i \to \ell_j \gamma$ are proportional to [17]

$$\text{BR}(\ell_i \to \ell_j \gamma) \propto |\text{BR}(\ell_i \to \ell_j \nu \bar{\nu})| \left| (\tilde{m}_D^\dagger L \tilde{m}_D)_{ij} \right|^2$$

(52)

where $L_{ij} = \delta_{ij} \log M_i / M_X$ and $M_i$ are the individual heavy neutrino masses. Here we have rotated $m_D$ to $\tilde{m}_D = V_R^T m_D$, where $M_R = V_R^* M_R^{\text{diag}} V_R^\dagger$. One can check that as long
as \( m_D \) is given by Eq. (8) the relation
\[
\frac{|(\tilde{m}_D^\dagger L\tilde{m}_D)|_{12}^2}{|(\tilde{m}_D^\dagger L\tilde{m}_D)|_{13}^2} = \epsilon^2 = \cot^2 \theta_{23}
\] (53)
holds for arbitrary \( M_R \). Hence, the branching ratios of \( \mu \to e\gamma \) and \( \tau \to e\gamma \) are close to each other (up to a normalization factor \( \text{BR}(\tau \to e\nu\nu) \simeq 0.178 \)). Note that this implies that \( \tau \to e\gamma \) will be too rare to be observable. We can compare this with \( \mu-\tau \) symmetric see-saw, i.e., the matrices in Eq. (20) with which it follows:
\[
\frac{|(\tilde{m}_D^\dagger \tilde{m}_D)|_{12}^2}{|(\tilde{m}_D^\dagger \tilde{m}_D)|_{13}^2} = 1.
\] (54)
Note that we have not included here the diagonal matrix \( L \), i.e., the unity of this ratio holds only up to potentially significant logarithmic corrections [2]. This is in contrast to see-saw and scaling, cf. Eq. (53).

5 Conclusions and Summary

Maximal atmospheric mixing and vanishing \( U_{e3} = 0 \) are two important predictions generally attributed to \( \mu-\tau \) symmetry. The scaling Ansatz proposed in Ref. [6, 7] is another possibility which leads to \( U_{e3} = 0 \) and potentially also to maximal \( \theta_{23} \). In a see-saw framework, the usually applied \( \mu-\tau \) symmetry heavily restricts the form of the mass matrices \( m_D \) and \( M_R \), whereas the scaling Ansatz does not. This paper was devoted to understand relationships between scaling and \( \mu-\tau \) symmetry and to discuss some phenomenological implications of scaling. In particular, we have shown here that:

• the scaling hypothesis is a special case of the generalized \( \mu-\tau \) symmetry leading to \( U_{e3} = 0 \) and an inverted hierarchy;

• the only way to derive scaling in the type-I see-saw mechanism is to have a Dirac neutrino mass matrix satisfying Eq. (8) as long as two neutrinos are required to have non-zero masses;

• scaling follows from a generalized \( \mu-\tau \) symmetry in the type-I see-saw model;

• the scaling hypothesis in the limit \( c = 1 \) gives the same lepton mixing as \( \mu-\tau \) symmetry, but the predictions for leptogenesis and lepton flavor violation differ. Unlike the \( \mu-\tau \) symmetry, a simple model with scaling and two RH neutrino leads to non-vanishing lepton asymmetry, which is proportional to the solar neutrino mass-squared difference;
• in the simple example with scaling and two RH neutrinos we have shown that it is possible to obtain the required baryon asymmetry through both flavored and unflavored leptogenesis. The same should be true in a more general example with three RH neutrinos which contains more parameters;

• the model with two RH neutrinos and $m_D$ in the scaling form may be regarded as a limit of the 3 RH neutrino model with $\mu-\tau$ symmetry. We have explicitly shown that the latter model reduces to the former after decoupling and both lead to the same lepton asymmetry.

The scaling hypothesis and $\mu-\tau$ symmetry are thus related, though some interesting differences exist. Finally, it is worth mentioning that the scaling hypothesis is a highly attractive possibility to generate an inverted neutrino mass hierarchy. In contrast to the usually considered $L_e - L_\mu - L_\tau$ flavor symmetry, no breaking is required in order to reach agreement with data. The many interesting features of scaling and/or the inverted hierarchy in neutrino oscillations or neutrino-less double beta decay are an easy and soon to be performed test of this Ansatz.

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A $\mu-\tau$ symmetric See-Saw and Leptogenesis

An interesting point of $\mu-\tau$ symmetric see-saw scenarios is that in case of two heavy neutrinos there is no successful leptogenesis [10]. We repeat here for completeness the derivation of this result and add its invariance under the presence or absence of flavor effects: in a $\mu-\tau$ symmetric see-saw framework with two heavy neutrinos the mass matrices read (writing here explicitly all phases)

$$m_D = \begin{pmatrix} a & e^{i\alpha} & b & e^{i\beta} & d & e^{i\delta} \end{pmatrix} \quad \text{and} \quad M_R = \begin{pmatrix} M_{11} e^{i\phi_{11}} & M_{12} e^{i\phi_{12}} \\ M_{12} e^{i\phi_{12}} & M_{11} e^{i\phi_{11}} \end{pmatrix}. \quad (A1)$$

Diagonalizing $M_R$ via $M_R = V^*_R M^\text{diag}_R V^*_R$ gives

$$V_R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & e^{-i\phi_1} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & e^{-i\phi_2} \end{pmatrix} \quad \text{and} \quad M^\text{diag}_R = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} = \begin{pmatrix} |M_{11} e^{i\phi_{11}} - M_{12} e^{i\phi_{12}}| & 0 \\ 0 & |M_{11} e^{i\phi_{11}} + M_{12} e^{i\phi_{12}}| \end{pmatrix}. \quad (A2)$$
where $\phi_{1,2}$ are the phases of $M_1$ and $M_2$. The decay asymmetries of the heavy neutrinos, to which the baryon asymmetry is proportional are in general [4]

$$
\epsilon_1^\alpha = \frac{1}{8\pi v^2} \frac{M_1}{\bar{m}_1} \sum_j \left[ \text{Im} \left\{ (\tilde{m}_D)_{1\alpha} (\tilde{m}_D^\dagger)_{\alpha j} (\tilde{m}_D \tilde{m}_D^\dagger)_{1j} \right\} f(x_j) 
+ \text{Im} \left\{ (\tilde{m}_D)_{1\alpha} (\tilde{m}_D^\dagger)_{\alpha j} (\tilde{m}_D \tilde{m}_D^\dagger)_{j1} \right\} g(x_j) \right],
$$

where the wash-out is governed by

$$
\tilde{m}_1 = \frac{(\tilde{m}_D \tilde{m}_D^\dagger)_{11}}{M_1},
$$

which is a sum over the individual wash-out parameters

$$
\tilde{m}_1^\alpha = \frac{(\tilde{m}_D \tilde{m}_D^\dagger)_{1\alpha}}{M_1}.
$$

The wash-out parameters need to be inserted in the approximate formula

$$
\eta(x) \simeq \left( \frac{8.25 \times 10^{-3} \text{eV}}{x} + \left( \frac{x}{2 \times 10^{-4} \text{eV}} \right)^{1.16} \right)^{-1}.
$$

The functions $f$ and $g$ depend on $x_j = M_2^2/M_1^2$ and are given by

$$
f(x) = \sqrt{x} \left( \frac{2 - x}{1 - x} - (1 + x) \ln \left( 1 + \frac{1}{x} \right) \right) \simeq -\frac{3}{2\sqrt{x}},
$$

where the approximate expression holds for $x \gg 1$ and

$$
g(x) = \frac{1}{1 - x}.
$$

Note that the terms proportional to $g(x_j)$ drop out when summed over $\alpha$ (i.e., when flavor effects play no role) and that they are suppressed by a factor of $M_1/M_j$ with respect to the terms proportional to $f(x_j)$. With $m_D$ given as above and $\tilde{m}_D = V_R^T m_D$ it follows

$$
\tilde{m}_D \tilde{m}_D^\dagger = \begin{pmatrix} b^2 + d^2 - 2bd \cos(\beta - \delta) & 0 \\ 0 & 2a^2 + b^2 + d^2 + 2bd \cos(\beta - \delta) \end{pmatrix},
$$

and consequently $\epsilon_1^\tau = \epsilon_1^\mu = \epsilon_1^\nu = 0$. Hence, the baryon asymmetry is non-zero in the case of a $\mu-\tau$ symmetric two heavy neutrino framework [10]. We add here to the result from Ref. [10] that flavor effects do not change the situation. We furthermore note that regarding LFV in SUSY see-saw scenarios the relation

$$
\left| (\tilde{m}_D^\dagger L \tilde{m}_D)_{12} \right|^2 = \left| (\tilde{m}_D^\dagger L \tilde{m}_D)_{13} \right|^2 = 1
$$

15
holds. The branching ratios of $\mu \to e\gamma$ and $\tau \to e\gamma$ are identical (up to a normalization factor $\text{BR}(\tau \to e\nu\bar{\nu}) \simeq 0.178$) even when the logarithmic factors in $L$ are taken into account, cf. Eq. (54). Note that this implies that $\tau \to e\gamma$ will be too rare to be observable. For completeness, we find for the expression relevant to $\tau \to \mu\gamma$ that

$$\left|(\tilde{m}_D^T L \tilde{m}_D)_{12}\right|^2 = a^2 L_2^2 (b^2 + d^2 + 2bd \cos(\beta - \delta)).$$

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