Wavy film flows down an inclined plane. Part I: Perturbation theory and general evolution equation for the film thickness

A. L. Frenkel and K. Indireshkumar

Department of Mathematics,
University of Alabama,
Tuscaloosa, Alabama 35487-0350
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Abstract

Wavy film flow of incompressible Newtonian fluid down an inclined plane is considered. The question is posed as to the parametric conditions under which the description of evolution can be approximately reduced for all time to a single evolution equation for the film thickness. An unconventional perturbation approach yields the most general evolution equation and least restrictive conditions on its validity. The advantages of this equation for analytical and numerical studies of 3D waves in inclined films are pointed out.

47.20.Ft, 47.35.+i, 47.20.Ma, 47.20.Gv
I. INTRODUCTION

Thin liquid layers ("films") flowing along solid surfaces—such as inclined planes or vertical cylinders—occur in both natural and man-made environments. Industrial applications of film flows started as long ago as the 1800s and have been growing in their scope and importance ever since (see e.g. Refs. [4,5]).

Accordingly, the studies of film flows, in particular those down an inclined plane, have a considerable history (see e.g. Refs. [4,3]). However, the dynamics of nonlinear waves (typically present in film flows) is still far from being satisfactorily understood (see Refs. [4,5] for the most recent progress reviews). The nonlinear Navier-Stokes (NS) partial differential equations (PDEs) of this "Kapitza problem" couple together several fields (pressure and the components of velocity), each a function of time and three spatial coordinates. Furthermore, the boundary conditions (BCs) of the problem involve a free boundary (the free surface of the film) whose PDE itself is coupled to the NS equations. The full spatially 3-dimensional (3D) problem is too hard to simulate even with the most powerful modern computers.

Even simpler, 2D computations of wavy films have been undertaken only under the simplifying assumptions of short spatial intervals (e.g. Refs. [6,7] and/or time-independence (e.g. Ref. [8]). However, 2D flows are frequently unstable to 3D disturbances, and three-dimensionality can be important for many inclined films (see e.g. recent experiments in Ref. [9]).

Therefore, naturally, one looks for more manageable *approximate* descriptions of wavy-film evolution. Such simplified theories are possible in certain domains of the space of parameters (for which, typically, the slopes of the surface waves are small). Of course, the greater simplification has to be paid for by a more limited applicability of the theory. The greatest simplification is achieved when the problem reduces to a single evolution PDE approximating the thickness of the film. [From the numerical-simulation point of view, the most important simplification here is the reduction in the number of independent spatial variables, and even a lower-dimensional *system* of equations would have been not much more
difficult than a *single* evolution equation (EE). However, no such system has ever been derived consistently for an inclined single layer film, the subject of our consideration in this paper. The same is true for the *nonlocal*, i.e. integro-differential, film-thickness equations. (The only known example of such a nonlocal EE, as far as film flows are concerned, is an EE for a core-annular flow, and not for an inclined film.) In any case, in this paper, the nonlocal equations are altogether excluded from the consideration; thus, when we say “the most general EE” it should be understood as “the most general *local* EE”, etc.] Although less drastic simplifications than a single EE, and therefore having a larger parametric range of validity, are known (see e.g. Ref. 5), so far fully-dimensional simulations for sufficiently extended spatial domains have been carried out only for the theories hinged on a single evolution equation (EE). Such single-EE theories of inclined-film flows are the subject of the present paper.

Evolution equations for film thickness have been known since the pioneering work of Benney 11. The conventional perturbation approach to their derivation (e.g. Refs. 12–15) used a small (longwave) parameter, say \( \epsilon \). In particular, each of the “global” (“internal”, “basic”) parameters specifying the problem (the parameters appearing in the dimensionless NS equations and the free-surface BCs) must be ascribed, in such a single-parameter (SP) technique, a certain power of \( \epsilon \) as its order of magnitude \( (O_M) \). Therefore, an artificial dependence is forced on the—intrinsically independent—parameters. This unnecessarily restricts the domain of justified validity of the resulting EE. For example, for the vertical film, there are just two independent parameters: the “Reynolds number” \( R \) and the “Weber number” \( W \). If \( R \) is of the order of magnitude of (\( \sim \)) \( \epsilon^a \) and \( W \sim \epsilon^b \), then \( W \sim R^{b/a} \). The set of points \( W = R^{b/a} \) is just a 1D curve in the 2D space \( (R, W) \), while the complete domain for which the EE is valid is likely to have the same dimensionality as the parameter space \( (R, W) \) itself, that is it should be a 2D domain.

Furthermore, each time the powers assigned to the parameters are altered, it is in principle necessary to again go through the entire procedure or the SP derivation, and one can arrive at a different EE as a result. For example, Topper and Kawahara 16 considered two
cases of an inclined-film flow (such a flow is specified by three global parameters: in addition to $R$ and $W$, the inclination angle $\theta$ can be independently varied). In one case, they required the angle $\phi$ ($\equiv \pi/2 - \theta$) of the film plane with the vertical to be small, $\phi \sim \epsilon$, and stipulated $R \sim \epsilon^{1}$ and $W \sim \epsilon^{-1}$ (in our definitions of $R$ and $W$; see section II below). As a result, they obtained an EE containing both dissipative and dispersive terms (and with all coefficients in the EE being $\sim 1$). However, their second case, for which they chose $\cot \theta \sim 1$, $R \sim 1$ and $W \sim \epsilon^{-1}$, resulted in an equation with no dispersive terms. (In the SP framework, it is only formally that the latter equation can be obtained from the former by omitting its dispersive term, and the only way to really “justify” the nondispersive equation is to repeat the entire derivation procedure starting all the way back from before NS equations.)

If a given inclined-film system is not close to any of these two parameter-space curves, the theory \cite{16} is invalid. Logically, there are three possibilities: (i) the flow evolution cannot be (approximately) reduced to a single EE for all time; (ii) such a reduction is possible but the resulting EE is different from each of those obtained in Ref. \cite{16}; or (iii) the EE coincides with one of their two EEs, despite the different parameter curve. We are naturally led to the following questions: (i) under what (parametric) conditions is an approximate description of the film flow possible (for all time) which can be reduced to a single EE? (ii) How can the set of all such EEs be characterized? These are the questions we pose and attempt to answer, for inclined-film flow, in this paper.

We note that one must distinguish between the all-time validity of an EE and the limited in time validity. This issue arises because in the SP approach, the fixed powers of the small parameter are stipulated not only for the global parameters of the system, which do not depend on time, but also for the characteristic time- and length-scales. However, these characteristic scales can change with time as the long—dissipative—system proceeds to the attractor; they are instantaneous parameters. So, the assumption of fixed scales is incorrect after a limited time has elapsed, and therefore, the SP derivation is invalid.

The rest of the paper is organized as follows. In section II, we formulate the full NS problem. In section III, we introduce an iterative perturbation procedure. It starts with the
well-known (although typically unstable), waveless, “Nusselt” solution of the NS problem. The only principle necessary in deciding the iteration steps is the requirement that in the end, a single (and valid for all time) EE should be arrived at, with minimal simplification of exact equations. No dependencies are imposed on the internal parameters: the validity conditions (VCs) we obtain require that several quantities, which are certain products of powers of the internal parameters, be small—*independently* of one another. [So, several independent (small) parameters emerge in the derivation. Thus, the iterative procedure we introduce here is a variation of the “multiparameter” perturbation (MP) approach developed in Refs. 17–20.]

In section IV, we arrive at the most general evolution equation (GEE) valid (provided certain restrictions on parameters are satisfied) for all time: Any all-time-valid EE derivable with a conventional single-parameter (SP) approach necessarily coincides with one of just a few prototype equations which are certain truncations of the GEE. The GEE also has the least restrictive domain of validity: It is easy to obtain the domain of validity for each simplified EE, and it is a subdomain of the all-embracing (corresponding to the GEE) validity domain. Outside of the latter parametric domain, there is no single EE which could approximate the film evolution for all time. That this is the case is argued in section V by analyzing the structure of possible correction terms of the EE through all orders of the iteration procedure. (In particular, as was shown in Ref. 21 and Ref. 22—where a different derivation of the same EE was sketched—the amplitudes of inclined-film waves which can be described (for all time) by a single EE are necessarily small.) The paper is summarized in the last section. Some of the more technical considerations are relegated to Appendices.

Simulation results for different versions of the inclined-film flow EEs we obtain here will be given in a sequel to this paper. [For some of those results (in particular, those showing good agreement with experiments), see Ref. 3.]
II. THE EXACT NAVIER-STOKES PROBLEM

We consider a layer of an incompressible Newtonian liquid flowing down an inclined plane under the action of gravity. Our (cartesian) coordinates are as follows: the $\overline{x}$ axis is normal to the plane and directed into the film; the $\overline{y}$ axis is in the spanwise direction; and the $\overline{z}$ axis is directed streamwise (the overbar here and below indicates a dimensional quantity). The corresponding components of velocity are $\overline{u}$, $\overline{v}$, and $\overline{w}$. We denote by $\overline{p}$ the pressure field in the film; the pressure of the ambient passive gas is neglected for simplicity.

The system is determined by the following independent (dimensional) parameters: the average thickness of the film $\overline{h}_0$; the liquid density $\overline{\rho}$, viscosity $\overline{\mu}$, and surface tension $\overline{\sigma}$; gravity acceleration $\overline{g}$; and the angle of the plane with the horizontal $\theta$.

There is a well-known time-independent, Nusselt’s solution of the NS problem for the inclined film. The thickness of the Nusselt film is constant (hence, Nusselt’s flow is also referred to as a “flat-film” solution). The only nonzero component of velocity is the streamwise one. It only changes across the film, starting from the zero value at the solid plane. The free-surface value $\bar{U}$ of the Nusselt velocity is $\bar{U} = \bar{g}\bar{h}_0^2 \sin \theta / (2\bar{\nu})$ (where $\bar{\nu} = \overline{\mu} / \overline{\rho}$ is the kinematic viscosity). We nondimensionalize all quantities with units of measurement based on $\overline{\rho}$, $\overline{h}_0$, and $\overline{U}$ (we have, e.g., the following units: $\overline{h}_0$ for all coordinates; $\overline{U}$ for velocities; $\overline{h}_0 / \overline{U}$ for time $t$; $\overline{\rho}\overline{U}^2$ for pressure; $\overline{\rho}\overline{U}^2\overline{h}_0$ for surface tension; etc.). We will see that exactly three independent basic parameters (BPs) appear in the dimensionless equations and boundary conditions of the problem; one can choose e.g. the inclination angle $\theta$, the Reynolds number $R \equiv \bar{h}_0\bar{U} / \bar{\nu} [= \bar{g}\bar{h}_0^3 \sin \theta / (2\bar{\nu}^2)]$, and the Weber number $W \equiv \sigma R / 2 [= \overline{\sigma} / (\overline{\rho}\overline{g}\overline{h}_0^2 \sin \theta)]$, as such BPs.

We can write the Navier-Stokes momentum equations in coordinates moving relative to the solid plane with a constant velocity $V$ in the $z$-direction (i.e. introducing $\tilde{z} = z - Vt$ and omitting the tilde) in the following dimensionless form (see e.g. Ref. [23]):

$$u_t + uu_x + vu_y + wu_z - V u_z =$$
\[- p_x - \frac{2}{R} \cot \theta + \frac{1}{R}(u_{xx} + u_{yy} + u_{zz}), \quad (1)\]

\[v_t + uv_x + vv_y + wv_z - Vv_z = \]

\[- p_y + \frac{1}{R}(v_{xx} + v_{yy} + v_{zz}), \quad (2)\]

\[w_t + uw_x + vw_y + ww_z - Vw_z = \]

\[- p_z + \frac{2}{R} + \frac{1}{R}(w_{xx} + w_{yy} + w_{zz}). \quad (3)\]

(The subscripts \(x, y, z, \) and \(t\) here and below denote the corresponding partial derivatives. We will see below that it is appropriate to choose \(V = 2, \) the common phase velocity of all infinitesimally weak waves.) The continuity equation is

\[u_x + v_y + w_z = 0. \quad (4)\]

The BCs are as follows. The no-slip conditions at the solid plane are

\[u = v = w = 0 \quad (x = 0). \quad (5)\]

The tangential-stress balance conditions at the free surface \(x = h(y, z, t), \) the local film thickness, are

\[(v_x + u_y)(1 - h_y^2) + 2(u_x - v_y)h_y - (v_z + w_y)h_z \]

\[- (u_z + w_x)h_y h_z = 0 \quad (x = h) \quad (6)\]

and

\[(u_z + w_x)(1 - h_x^2) + 2(u_x - w_z)h_z - (v_z + w_y)h_y \]

\[- (u_y + v_x)h_y h_z = 0 \quad (x = h). \quad (7)\]
The normal-stress balance condition is

\[-p(1 + h_y^2 + h_z^2)^{3/2} - \frac{2}{R} \left( u_x + v_y h_y^2 + w_z h_z^2 \right) \]

\[-(u_y + v_x) h_y + (v_z + w_y) h_y h_z \]

\[-(u_z + w_x) h_z \left( 1 + h_y^2 + h_z^2 \right)^{1/2} = \sigma \left[ h_{yy}(1 + h_z^2) \right] \]

\[+ h_{zz}(1 + h_y^2) - 2h_y h_z h_{zy} \]

\[= (x = h). \quad (8)\]

Finally, the kinematic condition at the free surface is

\[h_t + v h_y + w h_z - V h_z = u \quad (x = h). \quad (9)\]

III. ITERATIVE PERTURBATION PROCEDURE

A. Minimal requirement of derivability

As was motivated in the Introduction, we are interested in the question of (approximate) reducibility of the above complicated description of inclined-film dynamics to a single EE. It appears that an iterative perturbation approach is appropriate for the general analysis of the problem.

Looking at the previously known, conventional (single-parameter expansion) derivations of EEs (valid each for its own particular curve in the parameter space; see the Introduction), it is clear that each of them in effect discards some terms of the NS momentum equations so that each of those essentially becomes an ordinary differential equation (ODE) in \(x\), linear and with constant coefficients, that is easy to solve (with similar simplifications in BCs). When these solutions (for velocities in terms of film thickness) are substituted into the kinematic condition (9), the EE for thickness \(h\) (or, equivalently, for the thickness deviation \(\eta \equiv h - 1\)) ensues. However, after all the quantities have been expanded in power of \(\epsilon\) (see
Introduction), one has no control over which NS terms to omit: this is simply dictated by the expansion scheme of the SP approach. Thus, sometimes “harmless” terms are discarded: even if they were retained, one still would be able to solve for velocities, arriving, as a result, at a clearly more general EE.

Accordingly, our main idea in this paper is to look for a derivation in which the single postulated requirement would be that a maximally general EE for film thickness be arrived at in the end; so only those terms of the exact NS equations will be discarded which are clearly in the way of obtaining linear ODEs for velocities and pressure [we call this the “minimal requirement of derivability” (MRD)]. In this way we obtain approximate solutions for the deviations of exact solutions from the “seed” Nusselt’s fields [e.g., the approximation \( w_0 \) to \( \tilde{w}_0 \equiv w - w_N \), where the Nusselt \( w_N \) is known: see (12) below]. For some parameter values, this immediately leads to an EE whose approximation of exact evolution is good for all time (here, as was mentioned in the Introduction, we are only interested in such all-time-valid EEs; see e.g. Ref. \[4\] for a further discussion of their difference from the EEs whose validity is limited in time). In other cases, however, the procedure must be repeated, with the refined solutions, such as \( w_N + w_0 \), playing the seed role that was played by the Nusselt solutions on the original stage (so that at the second iteration stage one determines the approximation \( w_1 \) to \( \tilde{w}_1 \equiv w - w_N - w_0 \), etc.). Thus, ours is an iterative perturbation approach—which is known even in general to be an alternative to the expansion method (see e.g. Ref. \[24\]).

Estimating all members of NS problem equations in terms of parameters, the requirement that the discarded members be much smaller than those retained leads to the parametric conditions for our derivation to be valid (see Appendix \[A\]). Thus, the method yields (i) the evolution equation for film thickness; (ii) explicit expressions for velocity components and pressure in terms of the film thickness; and (iii) the parametric validity conditions of the theory.

It is known (see Ref. \[4\] and references therein) that no single-EE description can exist globally in time for those parametric regimes of inclined-film flow which lead to the eventual
amplitude of surface waves being “large”—comparable to the average film thickness. But in the present communication, as was mentioned above, we are interested exactly in the large-time behavior, when the system is already close to the attractor, and we want a single-EE description of the wavy film dynamics. Therefore, in addition to the MRD, we can use from the very beginning—in order to simplify our derivation—the requirement that the amplitude \( A \) of the film thickness deviation, \( A(t) \equiv \max |\eta| \), be small (for all time): with

\[
h = 1 + \eta,
\]

we have

\[
\max |\eta(y, z, t)| = A \ll 1.
\]

[Note that \( A \) can depend upon time; in such cases, we say that the parameter is a local parameter, in contrast to the time-independent, “global” BPs \( \theta, R, \) and \( W \).] However, unlike the conventional derivations, we do not have to postulate that the characteristic lengthscales in the film plane are large (the longwave assumption); rather, this will follow from the MRD.

**B. Nusselt’s solution**

The above-mentioned Nusselt solution (of the NS problem) which is steady and uniform along the film (i.e. in the streamwise and spanwise directions) is as follows. The dimensionless Nusselt streamwise velocity \( w_N \) is

\[
w_N(x) = 2x - x^2.
\]

This clearly satisfies the \( z \)-NS equation

\[
w_{Nxx} = -2
\]

with the boundary conditions

\[
w_{Nx} = 0 \quad (x = 1)
\]
and

\[ w_N = 0 \quad (x = 0) . \quad (15) \]

Similarly, the Nusselt pressure

\[ p_N = \frac{2}{R} (\cot \theta)(1 - x) \quad (16) \]

is the solution of the \( x \)-NS equation

\[ p_{Nx} = -\frac{2}{R} \cot \theta \quad (17) \]

with the boundary condition

\[ p_N = 0 \quad (x = 1) . \quad (18) \]

Finally, the Nusselt normal and spanwise velocities are

\[ u_N = 0 \quad \text{and} \quad v_N = 0 . \quad (19) \]

At sufficiently large Reynolds number, the destabilizing effect of inertia overcomes the stabilizing influence of gravity so that the Nusselt solution loses its stability. (An analysis later in this section leads to \( R > \frac{5}{4} \cot \theta \), the well-known criterion for instability.) As a result, the film is not uniform any more and also changes with time.

\section*{C. First iteration step}

We represent the exact velocities and pressure in the form of sums of Nusselt’s solutions and the deviations from those:

\[ w = w_N + \tilde{w}_0 , \quad u = u_N + \tilde{u}_0 , \quad v = v_N + \tilde{v}_0 , \quad p = p_N + \tilde{p}_0 , \quad (20) \]
where tildes indicate the deviations from the Nusselt solutions. First, we consider the $z$-momentum NS equation. We substitute the expressions (20) for velocities and pressure into Eq. (3). The resulting (exact) equation can be written in the form

$$
\tilde{w}_{0xx} = R\tilde{p}_{0z} - \nabla^2 \tilde{w}_{0} + R\tilde{w}_{0t} + R\tilde{u}_{0}(w_N + \tilde{w}_{0})_x
$$

$$
+ R\tilde{v}_{0} \tilde{w}_{0y} + R(w_N + \tilde{w}_{0} - V)\tilde{w}_{0z},
$$

(21)

where $\nabla^2 = \partial^2 / \partial y^2 + \partial^2 / \partial z^2$. In accordance with the MRD, as was discussed above, in order to obtain a solvable ODE in $x$, we have to discard all the terms on the right-hand-side (RHS) of (21), since every one of those contains unknown quantities. This yields a simplified equation for the velocity $\tilde{w}_{0}$, whose solution we denote by $w_{0}$:

$$
w_{0xx} = 0.
$$

(22)

We call $\tilde{w}_{1}$ the error in the approximation of $\tilde{w}_{0}$ by $w_{0}$:

$$
\tilde{w}_{0} = w_{0} + \tilde{w}_{1}.
$$

(23)

It is clear [see (10) and (11)] that the characteristic $x$-lengthscale $X$ is of $O_{M}$ of 1 ($X \sim 1$), i.e. (since $\partial / \partial x \sim 1 / X$),

$$
\frac{\partial}{\partial x} \sim 1.
$$

(24)

We denote by $Y$ and $Z$ the characteristic lengthscales in the spanwise and streamwise directions respectively, so that

$$
\frac{\partial}{\partial y} \sim \frac{1}{Y}
$$

(25)

and

$$
\frac{\partial}{\partial z} \sim \frac{1}{Z},
$$

(26)

Similarly, denoting by $T$ the characteristic timescale, we have
\[
\frac{\partial}{\partial t} \sim \frac{1}{T}.
\] (27)

Also, clearly, \(\partial^2/\partial x^2 \sim 1\), \(\partial^2/\partial z^2 \sim 1/Z^2\), \(\partial/\partial y^2 \sim 1/Y^2\), etc.

The characteristic magnitude of each of the neglected terms on the RHS of Eq. (21)—estimated by replacing \(\tilde{w}_0\) with \(w_0\), etc.—should be smaller than that of the term on the LHS, \(w_{0xx}\): \(w_0/Z^2 \ll w_0/X^2\) and \(w_0/Y^2 \ll w_0/X^2\). Hence, we have the following restrictions on the (instantaneous) lengthscales for the validity of our theory

\[
\frac{1}{Y^2} \ll 1
\] (28)

and

\[
\frac{1}{Z^2} \ll 1.
\] (29)

Thus, as a consequence of our derivability requirement, we have obtained the longwave conditions—which are rather postulated in the conventional “lubrication” or “long-wave” derivations.

Similarly, using (26) and (27), the conditions that the advective inertial term containing \(w_0\), \(RVw_0z\), and the time-derivative \(Rw_0t\) be smaller than \(w_{0xx}\) lead to the following requirements on the parameters:

\[
\frac{R}{Z} \ll 1
\] (30)

and

\[
\frac{R}{T} \ll 1.
\] (31)

Henceforth, we confine our consideration to the film configurations which satisfy the conditions (28)-(31). Note that since the local parameters \(A, Z, Y, \) and \(T\) can change with time, they may cease to satisfy the validity conditions [such as (11) and (28)-(31)] at a certain stage of the evolution. In such cases, clearly, the single-EE description would be valid only for a limited time that these (and some other, additional conditions appearing in Appendix
still hold. Later, we will determine domains in the space of global parameters for which such a violation of VCs never happens—so that the single-EE approximation is valid globally rather than merely locally in time.

The boundary condition on $\tilde{w}_0$, at $x = h$, given by the tangential-stress balance equation (4), is written in the form

$\tilde{w}_{0x} = -w_{Nx} - \bar{u}_{0z} + [2(\tilde{w}_{0z} - \bar{u}_{0z})\eta_z + (\tilde{v}_{0z} + \tilde{w}_{0y})\eta_y$

$+ (\bar{u}_{0y} + \tilde{v}_{0x})\eta_y\eta_z](1 - \eta_z^2)^{-1} \ (x = h).$ \hspace{1cm} (32)

According to the MRD, we have to drop all the terms containing the unknown quantities [those denoted by letters with tilde] on the RHS. Also, using the smallness of the surface deviation (11), we will everywhere transfer the boundary conditions from the true boundary $x = h$ to a convenient “boundary” $x = 1$ by expanding all quantities in Taylor series around $x = 1$ (cf. Ref. 24), such as

$\tilde{w}_{0x}(x = h) = \tilde{w}_{0x}(x = 1) + \tilde{w}_{0xx}(x = 1)\eta + \cdots.$ \hspace{1cm} (33)

Noting that $w_N(x = 1) = 0$, we have the simplified boundary condition (for the approximation $w_0$ of the exact quantity $\tilde{w}_0$):

$w_{0x} = -w_{Nx}\eta = 2\eta \ (x = 1).$ \hspace{1cm} (34)

Finally, the no-slip condition requires

$w_0 = 0 \ (x = 0).$ \hspace{1cm} (35)

The solution of Eq. (22) with the boundary conditions (34) and (35) is clearly

$w_0 = 2\eta x.$ \hspace{1cm} (36)

Note that from (12) and (36)

$w_{Nx} + w_{0x} = 0 \ (x = h),$ \hspace{1cm} (37)
which we will use below. We also observe that
\[ w_0 \sim A (\ll w_N \sim 1). \]  

(38)

Next, consider the incompressibility equation [see Eq. (3)] in the form
\[ \tilde{u}_{0x} = -\tilde{v}_{0y} - w_0z - \tilde{w}_{1z}. \]

(39)

(Note that \( w_N \) does not make any contribution to this equation, since \( w_N \) does not depend on \( z \).) Dropping the terms with unknowns \( \tilde{v}_0 \) and \( \tilde{w}_1 \) on the RHS, we obtain an equation for the approximation \( u_0 \):
\[ u_{0x} = -w_0z = -2x\eta_z. \]

(40)

The no-slip BC (5) requires
\[ u_0 = 0 \quad (x = 0), \]

(41)

and one readily obtains the solution
\[ u_0 = -x^2\eta_z. \]

(42)

We note that
\[ u_0 \sim \frac{A}{Z} \ll 1. \]

(43)

We denote the deviation of the exact solution \( \tilde{u}_0 \) from the approximation \( u_0 \) by \( \tilde{u}_1 \), so that
\[ \tilde{u}_0 = u_0 + \tilde{u}_1. \]

(44)

The (\( x \)-momentum) NS equation for the deviation of pressure from the Nusselt solution \( p_N \) is
\[ \tilde{p}_{0x} = \frac{1}{R}(u_{0xx} + u_{0yy} + u_{0zz}) + \frac{1}{R}(\tilde{u}_{1xx} + \tilde{u}_{1yy} + \tilde{u}_{1zz}) \]
\[ -(u_0 + \tilde{u}_1)_t - (u_0 + \tilde{u}_1)(u_0 + \tilde{u}_1)_z - \tilde{v}_0(u_0 + \tilde{u}_1)_y \]
due to (28) and (24) on the RHS is much larger than the other terms. Namely, $u_{0yy} \sim u_0/Y^2 \ll u_0 \sim u_{0xx}$ due to (28) and $u_{0zz} \sim u_0/Z^2 \ll u_0 \sim u_{0xx}$ due to (29). Similarly, $Ru_{0t} \sim (R/T)u_0 \ll u_0 \sim u_{0xx}$ by making use of (30) and $R(w_N + w_0 - V)u_{0z} \sim Ru_{0z} \sim (R/Z)u_0 \ll u_0 \sim u_{0xx}$ due to (30). Finally, $u_0u_{0x} \ll u_{0x} \sim u_{0xx}$, since $u_0 \ll 1$ [see Eq. (43)]. Thus the simplified equation (for the approximation $p_0$) is

$$p_{0x} = \frac{1}{R}u_{0xx} = -\frac{2}{R} \eta_z. \quad (46)$$

The BC at $x = h$ [see the normal-stress balance condition, Eq. (9)] is

$$\tilde{p}_0 = -p_N + \frac{2}{R} \left\{ [(u_0 + \tilde{u}_1)_x + \tilde{v}_{0y})^2 + (w_0 + \tilde{w}_1)_z \eta_z^2 - [(u_0 + \tilde{u}_1)_y + \tilde{v}_{0z}] \eta_y + [\tilde{v}_{0z} + (w_0 + \tilde{w}_1)_y] \eta_y \eta_z \right\} \left(1 + \eta_y^2 + \eta_z^2\right)^{-1} - \sigma \left[ \eta_{yy} \left( 1 + \eta_z^2 \right) + \eta_{zz} \left( 1 + \eta_y^2 \right) - 2\eta_y \eta_z \eta_{yz} \right]$$

$$\times \left(1 + \eta_y^2 + \eta_z^2\right)^{-3/2} \quad (x = h) \quad (47)$$

[where we have used (37) to eliminate $w_{Nx}$ and $w_{0x}$]. We have to drop those terms containing $\tilde{v}_0$, $\tilde{w}_1$, or $\tilde{u}_1$. In addition, a number of known terms are estimated to be smaller than the term with $u_{0x}$. Namely, by using the estimates of $w_0$ and $u_0$ (30) and (43), $w_{0y} \eta_y \eta_z \sim (A/Z)(A/Y)^2 \ll A/Z \sim u_0 \sim u_{0x}$. Similarly, $w_{0z} \eta_z^2 \sim (A/Z)(A^2/Z^2) \ll u_{0x}$. Also, $u_{0y} \eta_y \sim u_0 A/Y^2 \ll u_{0x}$ and $u_{0z} \eta_z \sim u_0 A/Z^2 \ll u_{0x}$. Finally, one can see that each of the other known nonlinear terms can be neglected since $\eta_z^2 \sim A^2/Z^2 \ll 1$, $\eta_y^2 \sim A^2/Y^2 \ll 1$, and $\eta_y \eta_z \eta_{yz} \sim A^3/(Y^2 Z^2) \ll 1$. We also note that the entire Taylor expansion for $p_N(x = 1 + \eta)$ consists of just one term,

$$p_N(h) = p_{N,x}(1) \eta. \quad (48)$$

With the known expressions for $p_N$, $w_0$, and $u_0$ (see above), and using Taylor series about $x = 1$, truncated to the first nonzero term, for all the quantities, one obtains the BC
\[ p_0 = -p_{Nz} \eta + \frac{2}{R} u_{0x} - \sigma \nabla^2 \eta \]
\[ = \frac{2}{R} (\cot \theta) \eta - \frac{4}{R} \eta_z - \sigma \nabla^2 \eta \quad (x = 1). \quad (49) \]

The solution of Eq. (46) with the boundary condition \((49)\) is
\[ R p_0 = 2(\cot \theta) \eta - 2 \eta_z (1 + x) - 2 W \nabla^2 \eta. \quad (50) \]

We observe that
\[ R p_0 \sim \max \left( \cot \theta, \frac{1}{Z}, \frac{W}{Z^2}, \frac{W}{Y^2} \right) A. \quad (51) \]

As usual, we call \( \tilde{p}_1 \) the difference between \( \tilde{p}_0 \) and its approximation \( p_0 \):
\[ \tilde{p}_0 = p_0 + \tilde{p}_1. \quad (52) \]

Finally, consider the \( y \)-NS equation,
\[ \tilde{v}_{0xx} = R(p_0 + \tilde{p}_1)_y - \nabla^2 \tilde{v}_0 + R \tilde{v}_0 t + R(u_0 + \tilde{u}_1) \tilde{v}_{0x} \]
\[ + R \tilde{v}_0 \tilde{v}_{0y} + R(w_N + w_0 + \tilde{w}_1 - V) \tilde{v}_{0z}. \quad (53) \]

Dropping the unknown terms on the RHS, we obtain the simplified equation:
\[ v_{0xx} = R p_{0y} \]
\[ = 2(\cot \theta) \eta_y - 2(1 + x) \eta_{yz} - 2 W \nabla^2 \eta_y, \quad (54) \]

where we have used the expression \((51)\) for \( p_0 \) in terms of \( \eta \) to get the RHS in the explicit form.

The BC on \( \tilde{v}_0 \), at \( x = h \), is [see Eq. \((3)\)]
\[ \tilde{v}_{0x} = -(u_0 + \tilde{u}_1)_y + \{2 \tilde{v}_{0y} \]
\[ -(u_0 + \tilde{u}_1)_z] \eta_y + [(u_0 + \tilde{u}_1)_z + \tilde{w}_{1x}] \eta_y \eta_z \]

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\[ + [\tilde{v}_{0z} + (w_0 + \tilde{w}_1)_y] \eta_z \right) \left( 1 - \eta_y^2 \right)^{-1} \quad (x = h) \]  

where we have again used (37) to eliminate \( w_{Nz} + w_{0z} \). Omitting the terms with tildes and performing the by now familiar estimates of the known terms on the RHS, we see that the term \( u_{0y} \) is larger than any other term, so that the simplified BC at \( x = 1 \) is

\[ v_{0x} = -u_{0y} = \eta_{zy} \quad (x = 1), \]  

where the expression (42) for \( u_0 \) has been used. Finally, the no-slip condition

\[ v_0 = 0 \quad (x = 0) \]  

is to be satisfied. The solution of Eq. (54) with the BCs (56) and (57) —as can be readily checked by direct substitution—is

\[ v_0 = \left( \cot \theta \eta_y - W \nabla^2 \eta_y \right) (x^2 - 2x) \]  

\[ - \eta_{zy} \left( \frac{x^3}{3} + x^2 - 4x \right) \]  

This yields the following estimate for \( v_0 \):

\[ v_0 \sim \max \left[ \cot \theta \frac{Y}{Y^2}, \frac{W}{Y^3Z}, \frac{W}{Y^2Z}, \frac{1}{YZ} \right] A. \]  

As usual,

\[ \tilde{v}_0 = v_0 + \tilde{v}_1. \]  

Substituting (60) into (39), we observe that in obtaining the simplified equation (40) for \( u_{0x} \), we have in fact discarded \( v_{0y} \). This requires

\[ v_{0y} \sim \frac{v_0}{Y} \ll u_{0x} \sim u_0 \]  

(which will be used below). By using here the estimates (43) for \( u_0 \) and (59) for \( v_0 \), we arrive at the VCs max \([Z \cot \theta/Y^2, WZ/Y^4, W/(Y^2Z)] \ll 1\). These are a subset of the complete set of (instantaneous) VCs obtained in Appendix A [see Eq. (A10)]. It is straightforward to
verify that each discarded term in every step of our procedure is small as a consequence of those VCs.

The kinematic condition [Eq. (5)] at \( x = h \) becomes

\[
\eta_t + (v_0 + \tilde{v}_1)\eta_y + (w_N + w_0 + \tilde{w}_1 - V)\eta_z
\]

\[
= u_0 + \tilde{u}_1. \quad (62)
\]

Dropping the unknown terms with tilde and the smaller terms \( v_0\eta_y \sim (v_0/Y)\eta \ll u_0 \) [see Eq. (61)] and \( w_0\eta_z \sim A^2/Z \ll u_0 \), we have

\[
\eta_0 + (w_N - V)\eta_z = u_0, \quad (63)
\]

where we have introduced the “fast” time \( t_0 \), such that \( \partial_t = \partial_{t_0} + \partial_{t_1} \approx \partial_{t_0} \). Using the Taylor expansions for \( w_N \) and \( u_0 \), we obtain, at \( x = 1 \),

\[
\eta_0 + (2 - V)\eta_z = 0. \quad (64)
\]

Choosing \( V = 2 \), we can eliminate the fast-time undulations (which are clearly due to the uniform translation of the wave, with no change in its shape):

\[
\eta_0 = 0. \quad (65)
\]

Thus, the leading approximation determines the velocity of a reference frame in which film thickness does not change on the fast time scale. However, it will change with the slower time \( t_1 \). In order to obtain this slower-time evolution of the film thickness, one needs to consider the next approximation for the velocities and pressure. From now on, we fix

\[
V = 2. \quad (66)
\]

Then, in view of (63), \( \partial_t = \partial_{t_1} \).
D. Second iteration

We now proceed to consider the “corrections” $\tilde{\omega}_1$, $\tilde{\upsilon}_1$, $\tilde{p}_1$, and $\tilde{\nu}_1$ for the velocities and pressure. By substituting

$$\tilde{\omega}_1 = w_0 + \tilde{w}_1, \quad \tilde{\upsilon}_1 = u_0 + \tilde{u}_1, \quad \tilde{p}_1 = p_0 + \tilde{p}_1 \quad (67)$$

into the $z$-NS equation (21), and taking into account $w_{0xx} = 0$ (22), we have the exact equation

$$\tilde{\omega}_{1xx} = -w_{0xx} + Rp_0z - \nabla^2 w_0 + Rw_0t + Ru_0(w_N + w_0)x + Rv_0 w_0y + (w_N + w_0 - 2)w_0z \quad + [\text{terms containing } \tilde{\omega}_1, \tilde{\upsilon}_1, \tilde{v}_1, \text{or } \tilde{p}_1]. \quad (68)$$

Performing our standard simplification procedure, i.e. discarding the unknown terms (containing tilde) and small terms, and also taking into account that $w_0 \ll w_N$ [see Eq. (38)] and $v_{0y} \ll u_0$ [see Eq. (61)], the simplified equation for the approximation $w_1$ is

$$w_{1xx} = Rp_0z - \nabla^2 w_0 + Ru_0w_{Nx} + Rw_N w_0z \quad -2Rw_0z + Rw_0t = 2\left(\cot \theta \eta_z - W \nabla^2 \eta_z\right) - 2(x + 1)\eta_{zz} \quad - 2x \nabla^2 \eta + (2x^2 - 4x)R\eta_z + 2xR\eta_t \quad (69)$$

where we have used the known expressions (in terms of surface deviation $\eta$; see the preceding section) for the first-iteration approximations $p_0$, $w_0$, and $u_0$. (For analogous equations of the general, $n$th iteration step, see Appendix C.)

The BC on $\tilde{\omega}_1$, at $x = h$, comes from (7):
\[
\tilde{w}_{1x} = -u_{0z} + 2w_{0z}\eta_z - 2u_{0z}\eta_z + w_{0y}\eta_y + v_{0z}\eta_y + (u_{0y} + v_{0x})\eta_y\eta_z
\]

\[+ \text{[terms containing } \tilde{w}_1, \tilde{u}_1, \tilde{v}_1, \text{or } \tilde{p}_1]\quad (x = h) \quad (70)
\]

where we have taken into account \[\text{[37]}\]. Continuing to use the simplification procedure established in the previous section, we arrive at the boundary condition

\[
w_{1x} = -u_{0z} = \eta_{zz} \quad (x = 1). \quad (71)
\]

All other terms on the RHS of Eq. (70) are readily estimated in our usual way to be smaller than \(u_{0z}\) [Eq. (71) is useful in estimating terms containing \(v\)]. The no-slip condition is

\[
w_1 = 0 \quad (x = 0). \quad (72)
\]

The solution of the problem \[\text{(69)}, (71), \text{and } (72)\] is

\[
w_1 = \left(\cot \theta \eta_z - W \nabla^2 \eta_z\right) (x^2 - 2x)
\]

\[+ \left(\frac{x^4}{6} - \frac{2}{3}x^3 + \frac{4}{3}x\right) R\eta_z + \left(5x - \frac{5}{3}x^3 - x^2\right) \eta_{zz}
\]

\[+ \left(x - \frac{x^3}{3}\right) \eta_{yy} + \left(\frac{x^3}{3} - x\right) R\eta, \quad (73)
\]

which can be verified by direct substitution into the problem equations. Note that all the terms of \(w_1\) are estimated to be quadratic in the local parameters \[\{10\}\]; we will see that, in general (see Appendix \[\{1\}\]), \(w_n\) is of the power \((n + 1)\), and similarly for \(u_n, v_n,\) and \(p_n\).

Taking into account \(u_{0x} = -w_{0z}\) [see Eq. \(\{10\}\)], the incompressibility condition yields the equation for \(\tilde{u}_1\),

\[
\tilde{u}_{1x} = -w_{1z} - \tilde{w}_{2z} - v_{0y} - \tilde{v}_{1y}, \quad (74)
\]
where we have expressed \( \tilde{w}_1 \) as \( \tilde{w}_1 = w_1 + \tilde{w}_2 \). Dropping the unknown terms, we obtain an equation for the approximation \( u_1 \):

\[
\begin{align*}
\tilde{u}_1 & = -w_{1z} - v_{0y} \\
& = -(x^2 - 2x)(\cot \theta \nabla^2 \eta - W \nabla^4 \eta) - \left( \frac{x^4}{6} - \frac{2}{3} x^3 + \frac{4}{3} x \right) R\eta_{zz} \\
& \quad - \left( 5x - \frac{2}{3} x^3 - x^2 \right) \nabla^2 \eta_z - \left( \frac{x^3}{3} - x \right) R\eta_z 
\end{align*}
\]

with the BC

\[
\begin{align*}
\tilde{u}_1 & = 0 \quad (x = 0).
\end{align*}
\]

It is easy to verify that

\[
\begin{align*}
\tilde{u}_1 & = \left( \frac{x^3}{3} - x^2 \right) \left[ W \nabla^4 \eta - \cot \theta \nabla^2 \eta \right] \\
& \quad - \left( \frac{x^5}{30} - \frac{x^4}{6} + \frac{2}{3} x^2 \right) R\eta_{zz} + \left( \frac{x^4}{6} + \frac{x^3}{3} - \frac{5}{2} x^2 \right) \nabla^2 \eta_z \\
& \quad - \left( \frac{x^4}{12} - \frac{x^2}{2} \right) R\eta_z 
\end{align*}
\]

is the solution. We note that the complete set of VCs \( \{A10\} \) obtained in Appendix A guarantees that all the terms discarded in obtaining solvable ODEs for \( w_1 \) and \( u_1 \) are small in comparison with (the biggest of) those terms that are retained.

At this point, we could proceed to solve the \( x \)-NS and \( y \)-NS equations for the pressure and velocity corrections \( \tilde{p}_1 \) and \( \tilde{v}_1 \) respectively. However, these corrections are not needed for obtaining the second-iteration EE. (We will only need the pressure and velocity corrections for later iteration stages, and we calculate those corrections in Appendix B.)
IV. THE DISPERSIVE-DISSIPATIVE EVOLUTION EQUATION

The (exact) kinematic condition (1) at $x = h$ can be written in the form

$$\eta_t + (v_0 + \tilde{v}_1)\eta_y + (w_N + w_0 + w_1 + \tilde{w}_2 - 2)\eta_z$$

$$= u_0 + u_1 + \tilde{u}_2 \quad (x = 1 + \eta) \quad (78)$$

where we have used $V = 2$ (66). Dropping the terms containing unknown velocities (those with tilde) and using the Taylor series to relate the velocity components at $x = h$ to those at $x = 1$, we have

$$\eta_t + (w_N x \eta + w_0)\eta_z = u_{0x} \eta + u_1 \quad (x = 1). \quad (79)$$

In (79), we have dropped the terms $v_0 \eta_y$ and $w_1 \eta_z$ as they are smaller than $u_1$ [see Eq. (75)]. Also, recall that $w_N x (x = 1) = 0$ [see Eq. (12)]. Using the expressions (36), (42), and (77) for $w_0, u_0, \text{ and } u_1$, we obtain

$$\left[\eta - \frac{5}{12} R\eta_z\right]_t + 4\eta\eta_z + \frac{2}{3} \delta\eta_{zz} - \frac{2}{3} \cot \theta \eta_{yy}$$

$$+ \frac{2}{3} W \nabla^4 \eta + 2 \nabla^2 \eta_z = 0 \quad (80)$$

where by definition

$$\delta \equiv \frac{4}{5} R - \cot \theta. \quad (81)$$

However, the term $\propto R\eta_z \sim (R/Z)\eta_t \ll \eta_t$ since $R/Z \ll 1$ [see (A1)]. Dropping this small term, we have

$$\eta_t + 4\eta\eta_z + \frac{2}{3} \delta\eta_{zz} - \frac{2}{3} \cot \theta \eta_{yy}$$

$$+ \frac{2}{3} W \nabla^4 \eta + 2 \nabla^2 \eta_z = 0. \quad (82)$$
Simple linear-stability analysis can reveal the dynamical role of some terms here. Assuming an infinitesimally small disturbance in the form of a normal mode, \( \eta \propto \exp(st - iwt) \exp(i(jy + kz)) \), it readily follows from the linearized version of (82) that

\[
s = \frac{2}{3}[\delta k^2 - (\cot \theta)j^2 - W(k^4 + 2j^2k^2 + j^4)]
\] (83)

and

\[
\omega = -2k(k^2 + j^2).
\] (84)

Here \( s \) is the growth (or decay) rate for the disturbance and so the third, fourth, and fifth terms in Eq. (82) which give rise to growth (or decay) are dissipative [considering the destabilizing term (the one with \( \delta \)) as a negative dissipation]. In contrast, the last term in (82) only makes a contribution to the (real) frequency \( \omega \), rather than to the growth rate \( s \), i.e. it does not lead to growth or decay of disturbances. Thus, this (third-derivative) term is dispersive.

Clearly, for instability to develop (i.e. for \( s > 0 \)), we need

\[
\delta > 0,
\] (85)

a condition we assume fulfilled from now on. This yields the so-called critical value \( R_c \) of the Reynolds number, \( R_c = \frac{4}{5} \cot \theta \), at which the instability sets in.

One can see from the above derivation of the dispersive-dissipative EE (82) that the destabilizing (third) term originates from the inertia terms of the NS equations. The (stabilizing) fourth and fifth terms are due respectively to hydrostatic and capillary (i.e. surface-tension) parts of the pressure. Finally, the last, odd-derivative term is due to the viscous part of the pressure. Such a purely dispersive term also appeared in the EE obtained by Topper and Kawahara\(^{16} \) for an almost vertical plane: they used the small angle of the plane with the vertical as their (single) perturbation parameter (see also the discussion in the Introduction of the present paper). Our derivation shows that assumption to be unnecessary. In particular, for the vertical film \( \cot \theta = 0 \), and Eq. (82) becomes...
\[ \eta_t + 4\eta\eta_z + \frac{8}{15}R\eta_{zz} + \frac{2}{3}W\nabla^4\eta + 2\nabla^2\eta_z = 0. \] 

(86)

Although an equation of this structure (but with arbitrary coefficients) was postulated as a model equation in Ref. 25, it cannot be obtained from the derivation of Topper et al. If the small parameter is proportional to \( \cot \theta \), it becomes zero for the vertical case, and the Reynolds number (also proportional to the small parameter in that SP derivation) vanishes—which, clearly, cannot correspond to any flow at all.

The (infinite-dimensional) dynamical system governed by the dissipative equation essentially forgets initial conditions as it evolves towards an attractor. There may be fluctuations on the attractor, but there is no systematic change in time. Clearly then the amplitude-decreasing, stabilizing term must balance the destabilizing one (the latter tends to increase the deviation amplitude). So the two dissipative terms are necessarily of the same order of magnitude on the attractor.

As to the magnitude of the dispersive term relative to that of the dissipative terms, there can occur, depending on location in the parameter space, each of the following three possibilities: (1) these terms are of the same \( O_M \); (ii) the dispersive term is small (and then the amplitude is determined by the balance of the nonlinear term with the dissipative terms); and (iii) the dissipative terms are small (and the nonlinear term balances the dispersive one).

In the first case, continuing the iteration process would lead to small corrections to the terms which cannot significantly change the evolution. In the second case, the small dispersive term can be omitted with a negligible effect, so the corrections would be again immaterial.

But in the third case, when dissipation is small, the situation is very different. Discarding the dissipative terms leads to a 2D Korteweg-deVries (KdV) equation which was simulated numerically in Ref. 26. The KdV equation is purely dispersive and never forgets the initial conditions. It has a one-parameter family of axisymmetric solutions which are traveling solitons, similar to the well-known 1D KdV case. Depending on the initial state, there may be solitons of different lengthscales (and therefore moving with different speeds) in the final state. However, if the small dissipative terms are present, they will slowly change the initial
soliton of an arbitrary lengthscale. It will evolve along the soliton family until the lengthscale is attained which provides for the balance between the two dissipative terms (this effect was first studied for the 1D case in Ref. 27; see also Refs. 28–30). Therefore, the dissipative terms, even when small, are important: they determine the lengthscale of the solution.

However, only the largest-magnitude terms are guaranteed to be correct in the above beginning-iteration derivation; as for the smaller terms, further iterations might yield significant corrections to them. We consider this question (of higher iterations and corrections to small dissipative terms) in the next section. It turns out (perhaps, surprisingly) that such corrections can be important locally under some parametric conditions, but that no (single) corrected EE can approximate the evolution for all time. Equation (82) is thus the most general of those EEs that can be valid globally in time—under appropriate parametric restrictions, which can be completely determined only with the analysis of higher iterations of the NS problem, as is done in the next section. For the rest of this section, we continue the consideration of the GEE (82).

From the condition $\delta > 0$, Eq. (85), it follows that

$$R > \frac{5}{4} \cot \theta > \cot \theta.$$  

(87)

From Eqs. (82) and (83), it is clear that in order for instability to develop, we need $\delta \eta_{zz} > (\cot \theta) \eta_{yy}$. Using the $y$ and $z$ lengthscales, this yields, $Y^2/Z^2 > (\cot \theta)/\delta$. Noting that either $\delta \ll R \sim R_c \sim \cot \theta$ or $\delta \sim R$, we see that $Y \gg Z$ or $Y \sim Z$, except perhaps for $R \gg \cot \theta$.

For simplicity, we assume that $Z \leq Y$, which seems to be the case in all experiments we know about. Then

$$L \equiv \min(Z, Y) = Z.$$  

(88)

With this and the condition (87), the VCs (A15) reduce to

$$\max \left[ A, \frac{1}{L^2}, \frac{R}{L}, \frac{W}{L^3} \right] \ll 1.$$  

(89)

These are conditions of instantaneous validity; they involve the local (i.e. instantaneous) parameters $A(t)$ and $L(t)$.
As was discussed above, due to the dissipativeness of the EE (82), the system evolves towards an attractor, and in the asymptotic limit of large times we have \( A(t) = \text{const} \equiv A_a \) and \( L(t) = \text{const} \equiv L_a \). Since (similar to Refs. 31, 17) the destabilizing inertia term should be of the same order as the stabilizing, capillary one, i.e. \( \delta \eta_{zz} \sim \delta A / L_a^2 \sim W \nabla^4 \eta \sim WA / L_a^4 \), the (dimensionless) characteristic lengthscale at large times \( L_a \) can be taken to be

\[
L_a = \left( \frac{W}{\delta} \right)^{1/2}.
\]  

Similarly, the asymptotic magnitude of the characteristic amplitude \( A_a \) is determined by the balance between the nonlinear “advective” term and either the dispersive term or the capillary one (whichever is larger): \( A_a = \max(W/L_a^3, 1/L_a^2) \). Using these asymptotic values of parameters, the conditions (89) can be written as \( \max(W/L_a^3, R/L_a, 1/L_a^2) \ll 1 \). Noting that in view of Eq. (90), \( W/L_a^3 = \delta/L_a \) and [see Eq. (81)] \( R = (5/4)(\delta + \cot \theta) > \delta \), so that \( W/L_a^3 < R/L_a \), we can simplify the VC to \( \max(R/L_a, L_a^{-2}) \ll 1 \); in terms of the basic parameters,

\[
\alpha \equiv \frac{1}{L_a^2} = \frac{\delta}{W} \ll 1, \quad \beta \equiv \frac{R}{L_a} = R \left( \frac{\delta}{W} \right)^{1/2} \ll 1.
\]  

In the next section, it is shown that we also need

\[
\gamma \equiv \frac{\max(R, R^3)}{W} \ll 1
\]  

(otherwise, the dissipative terms contributed by higher iterations can become significant, and the evolution cannot be all-time-describable by a single EE). All three parameters, \( \alpha \), \( \beta \), and \( \gamma \), are small if (recall that \( \delta < R \))

\[
\alpha_R \equiv \frac{R}{W} = \frac{\bar{\rho} \hat{g}^2 \hat{h}_0^5 \sin^2 \theta}{2 \bar{\sigma} \bar{v}^2} \ll 1
\]  

and

\[
\beta_R \equiv \frac{R^{3/2}}{W^{1/2}} = \left( \frac{\bar{\rho} \hat{h}_{11}}{8 \delta} \right)^{1/2} \left( \frac{\hat{g}^2 \sin^2 \theta}{\bar{v}^4} \right) \ll 1.
\]

So, if we are in the domain of the space of basic parameters which satisfies the condition \( \max(\alpha_R, \beta_R) \ll 1 \) [or a bit more general, but less simple condition \( \max(\alpha, \beta, \gamma) \ll 1 \)], then
Eq. (82) is good for all time [provided the initial amplitude and lengthscale satisfy the conditions (89)]. Therefore, we call such conditions the “global” VCs.

We can transform the GEE (82) to a “canonical” form—which contains only two “tunable” constants—by rescaling the variables with appropriate units:

\[
\eta = N\tilde{\eta}, \quad z = L_a\tilde{z},
\]

\[
y = L_a\tilde{y}, \quad \text{and} \quad t = T_a\tilde{t},
\]

(95)

where \( N = 1/(2L_a^2) \) and \( T_a = L_a^3/2 \). Dropping the tildes in the notations of variables, the resulting canonical EE is

\[
\eta_t + \eta\eta_z - \kappa\eta_{yy} + \nabla^2\eta_z + \epsilon \left( \eta_{zz} + \nabla^4\eta \right) = 0.
\]

(96)

The control parameters in this equation are

\[
\epsilon = \frac{1}{3}\sqrt{W\delta}
\]

(97)

and

\[
\kappa = \frac{L_a \cot \theta}{3} = \frac{1}{3}\sqrt{\frac{W}{\delta}}\cot \theta
\]

(98)

For the case of \( \kappa = 0 \) (e.g. flow down a vertical wall), Eq. (96) becomes essentially the model equation postulated and numerically studied in Ref. 25. If, in addition, \( \epsilon = \kappa = 0 \), we have the 2D KdV equation, whose 1D (\( \partial_y = 0 \)) version is the usual KdV equation. When \( \kappa = 0 \) but \( \epsilon \to \infty \) (so that, after an appropriate rescaling, the dispersive term disappears from the equation), Eq. (96) becomes the one obtained by Nepomnyashchy, whose 1D version is the Kuramoto-Sivashinsky equation. All these equations are thus limiting cases of EE (96).

Numerical simulations of this equation have shown that its solutions remain bounded for all time (which property always is implicitly assumed in global-validity considerations, and therefore must be directly verified after the EE has been obtained). For example, Fig. [I]
shows evolution of the surface deviation “energy” $\int \eta^2 dy dz$ from an initial small-amplitude ($\eta \sim 10^{-2}$) “white-noise” condition to a statistically steady state (for $\kappa = 0$ and $\epsilon = 1/50$).

Detailed numerical studies of Eq. (96) will be presented in a sequel to this paper.

V. ADDITIONAL DISSIPATIVE TERMS

In this section, we examine the implications of the additional dissipative terms arising from further iterations. We take into account explicitly all the linear dissipative and dispersive terms with derivatives of order four or less by going through the iterative process twice. We also estimate the effect of dissipative terms with derivatives of order six or more on the EE.

We note that for obtaining the evolution equation, we need only the successive iterates of the normal velocity $u$. This is because the nonlinear terms involving $w \eta_z$ and $v \eta_y$ in the kinematic condition make smaller contributions to the EE. (Indeed $u \sim w_z + v_y$, and e.g. $w_z \eta$ generates the same terms as the $u_z$ part of $u$, but with the extra small factor $\eta$.) We have found the next two iterative corrections for the normal velocity, $u_2$ and $u_3$, Eqs. (B22) and (B37) in Appendix B. [We need $u_3$ (in addition to $u_2$) because its dissipative terms are not guaranteed to be much smaller than those of $u_2$. However, the dissipative terms of $u_4$ are much smaller than those of $u_2$, so we do not have to consider $u_4$.] Using these in the kinematic condition (1), we obtain [see Appendix B for details]

$$
\eta_t = \left[ \frac{5}{12} R \eta_z - \frac{4}{15} R \cot \theta \nabla^2 \eta + \frac{295}{672} R^2 \eta_{zzz} \right] + 4 \eta \eta_z - \frac{2}{3} \delta \eta_{zz} + \frac{2}{3} \cot \theta \eta_{yy} - 2 \nabla^2 \eta_z - \frac{2}{3} W \nabla^4 \eta - \left[ \frac{23}{15} R - 2 \cot \theta \right] (\eta_t)
$$

$$
+ \frac{5}{14} R \cot \theta \nabla^2 \eta_z - \frac{2}{7} R^2 \eta_{zzz} + \frac{6}{5} \cot \theta \nabla^4 \eta - \frac{331}{168} R \nabla^2 \eta_{zz} - \frac{1241483}{8108100} R^3 \eta_{zzzz} + \frac{477523}{2494800} R^2 \cot \theta \nabla^2 \eta_{zzz}.
$$

(99)

We get rid of time-derivatives in RHS by twice iterating this equation, substituting (for each time-derivative in the RHS) the RHS (99) of (99) itself; the remaining time-derivatives in the final RHS are omitted, because further iterations would only lead to derivatives of an
order higher than four, with small resulting contributions. [In fact, in the first iteration, when substituting into the (mixed-derivative) terms with the second spatial derivatives, it is enough to retain from the RHS (99) only the terms with no time-derivatives, and with space-derivatives of the order two only. As for the term with the first spatial derivative, \( \propto \eta_{tz} \), the terms without time-derivatives and with space-derivatives of orders two and three are substituted into it, and also the term \((5/12)R\eta_{tz}\) itself, with the result \((5/12)^2R^2\eta_{zz}z\).

In the second iteration, it is sufficient to only substitute into the latter term, and—since it already contains two spatial differentiations—only the purely spatial second-derivative terms should be substituted into it.] As a result, we obtain the following equation:

\[
\eta_t + 4\eta_z + \frac{2}{3}\delta \eta_{zz} - \frac{2}{3} \cot \theta \eta_{yy} + 2\nabla^2 \eta_z + \frac{40}{63} R\delta \eta_{zzzz} - \frac{40}{63} R \cot \theta \eta_{zzyy} + \frac{2}{3} W \nabla^4 \eta - \frac{2}{3} \cot \theta \nabla^4 \eta + \frac{157}{56} R \nabla^2 \eta_{zz} \\
+ \frac{8}{45} R \cot^2 \theta \nabla^4 \eta + \frac{1213952}{2027025} R^3 \eta_{zzzz} - \frac{138904}{155925} R^2 \cot \theta \nabla^2 \eta_{zz} + \left( \frac{16}{5} R - 2 \cot \theta \right) (\eta_{zz})_z = 0.
\]

(100)

The 1D \((\partial_y = 0)\) limit of this equation coincides with the small-amplitude limit of the EE obtained, with the same numerical coefficients, in Ref. 14, but our 2D version is new. (We remark that there are several mistakes in the presentation of steps leading to the final equation, Eq. (27) in Ref. 14. However, Eq. (27) itself appears to be correct. The same numerical coefficients appeared in an even earlier paper in a linearized 1D context.) The (2D) terms with derivatives of order three or less agree with Ref. 15, and those plus the surface-tension \((W)\) term—with Ref. 36. However, note that some terms of (100) are always negligible. For example, the two dispersive third-derivative terms containing \(R\) are clearly smaller than the corresponding second-order dissipative terms (because of the instantaneous VCs \(R/L \ll 1 \) and \(\cot \theta/L \ll 1\)), and therefore are negligible in all cases.

As was mentioned above, since we are only interested in situations where persistent nonlinear waves are present, \(\delta > 0\), we have either \(\delta \sim R\) or \(\delta \ll R\) (but not \(\delta \gg R\); see Eq. (81)). When \(\delta \sim R\), the additional fourth-derivative terms in (82) are each smaller than the destabilizing second-derivative term (because of \(R/L \ll 1\) and/or \(1/L^2 \ll 1\)). If
max(R, R^3) \ll W \text{ [see Eq. (92)]}, those terms are much smaller than the stabilizing capillary term, and we return to the GEE (82) with global VCs (93) and (94) (this holds as well even for } \delta \ll R). But if W is not large enough, so that Eq. (92) is violated and the capillary fourth-derivative term is much smaller than (at least) one of the non-capillary fourth-derivative members, then the destabilizing term cannot be balanced, and the EE leads to the unlimited growth of amplitude. Thus, clearly, at such parameter-space locations the EE (100) can be valid locally, i.e. for a limited time only, but it is not globally-good.

Consider now (for the rest of this section) the complementary case } \delta \ll R \text{ (so } R \simeq R_c), and with } W \text{ sufficiently small, so that the condition } \max(R, R^3) \ll W \text{ is violated. We have obtained, except for numerical coefficients, all the essential terms (which turn out to be linear; see the end of Appendix C). Including these terms in (100), the EE can be written in the general form

\eta_t + 4\eta \eta_z + \frac{2}{3} \delta \eta_{zz} - \frac{8}{15} R_c \eta_{yy} + 2\eta_{zzz} + \frac{2}{3} W \eta_{zzzz} + \frac{8}{5} R_c (\eta \eta_z)_z + \left[ \frac{2581}{1400} R_c - \frac{32}{3378375} R_c^3 \right] \eta_{zzzz}

+ \text{sum of terms of type } R_c^{2k+1} \eta^{(2l)} = 0. \tag{101}

Here the superscript on } \eta, \text{ enclosed in parentheses, refers to the order of the spatial derivative; } l > k \geq 0; k = 0, 1, 2, \cdots; l = (k + 1), (k + 2), \cdots; \text{ and every } (k + l) \text{ is odd. In (101), we have used the leading Taylor-series approximation putting } R = R_c \text{ (recall also } \cot \theta \approx 4R_c/5). \text{ Also, } Y \gg Z \text{ (as a consequence of } \delta \eta_{zz} \geq R_c \eta_{yy}, \text{ which is required for instability to develop). Furthermore, we have not included additional nonlinear or dispersive terms in Eq. (101): the dissipative nonlinear terms can be shown (see Appendix C) to be smaller than the leading nonlinear dissipative term } \propto R(\eta \eta_z)_z, \text{ and all the dispersive terms are smaller than the linear one } \nabla^2 \eta_z.

Normally, the coefficients of the terms are } O_M(1). \text{ However, sometimes a coefficient is } \ll 1 \text{ because of an accidental near cancellation of terms, as e.g. is the case for the term } \propto R_c^3 \eta_{zzzz}
in (101). Then we say the term is “degenerate”. Comparing the additional dissipative terms with the term $\propto R_c^3 \eta_{zzzz}$ and taking into account the instantaneous VCs $R/L \ll 1$ and $1/L^2 \ll 1$, the only terms which may not be negligible are those of the structure $R_c^{2n-1} \eta^{(2n)}$ ($n = 2, 3, \cdots$) (even those could have been neglected in comparison with the term $\propto R_c^3 \eta_{zzzz}$ were the latter non-degenerate). Hence, the EE can be simplified:

$$\eta_t + 4 \eta \eta_z + 2 \frac{2}{3} \delta \eta_{zz} - \frac{8}{15} R_c \eta_{yy} + 2 \eta_{zzz}$$

$$+ \frac{8}{5} R_c (\eta \eta_z)_z + \left[ \frac{2581}{1400} R_c - \frac{32}{3378375} R_c^3 \right] \eta_{zzzz}$$

$$+ \sum_{n=3}^{\infty} c_n R_c^{2n-1} \eta^{(2n)} = 0,$$

(102)

where $c_n$ are numerical coefficients. Can this equation be valid for all time? For this to be the case, the destabilizing term $(2/3) \delta \eta_{zz}$ has to be balanced by a stabilizing one—by the term $\propto R_c \eta_{zzzz}$ (the other, degenerate fourth-derivative term is clearly destabilizing) or by the first non-degenerate and stabilizing higher-derivative term of (102), whichever term is dominant.

Suppose first that the fourth-derivative term is the dominant stabilizing one. Then Eq. (102) becomes

$$\eta_t + 4 \eta \eta_z + 2 \frac{2}{3} \delta \eta_{zz} - \frac{8}{15} R_c \eta_{yy} + 2 \eta_{zzz}$$

$$+ \frac{8}{5} R_c (\eta \eta_z)_z + \frac{2581}{1400} R_c \eta_{zzzz} = 0.$$ (103)

The balance of the destabilizing term $\delta \eta_{zz}$ with the stabilizing term $R \eta_{zzzz}$ yields the lengthscale $L \sim R_c/\delta$. Comparing the dissipative term with the dispersive term, we have $R_c \eta_{zzzz}/\eta_{zz} \sim R_c/L \ll 1$: the dispersive term is always dominant. Therefore, it is the dispersive term which is to balance the nonlinear one, which yields the characteristic amplitude $\sim 1/L^2$. Using this, it is easy to see that the nonlinear dissipative term $\sim R_c(\eta \eta_z)_z$ is exactly $O_M$ of the linear dissipative terms; therefore, the nonlinear term plays a significant role. It
is destabilizing, and numerical simulations indicate that the solutions of (103) blow up [see Fig. 4] which shows the blow-up of energy (in contrast to Fig. 1) governed by the equation rescaled to the canonical form similar to Eq. (96),

\[ \eta_t + \eta \eta_z + \eta_{zzz} - \kappa \eta_{yy} + \varepsilon [\eta_{zz} + \eta_{zzzz} + (1120/2581)(\eta_{zz})_z] = 0, \] (104)

with \( \kappa = 0 \) and \( \varepsilon = 1 \); we have observed similar blow-up behavior with all values of \( \varepsilon \) we tested in the range \( 10^{-2} \)-\( 10^2 \). The thickness of the real film, of course, is bounded uniformly for all time. Hence, Eq. (103) (which can be good for a limited time) is not globally-valid. Physically, we believe that the growth of amplitude will be arrested by viscosity after small lengthscales develop, which would violate the small-\( R/L \) VC for the single-EE description. So the EE (103) cannot be valid for large times.

It remains to consider the hypothetical case when a higher-derivative term \( \propto R_c^{2n-1} \eta^{(2n)} \) is the dominant stabilizing one. [In particular, this implies that \( 3 \leq n = m \) (say), \( c_n \ll 1 \) for \( n < m \), \( c_m \sim 1 \), and \( c_m \) has an appropriate sign: \( c_m > 0 \) if \( m \) is even and \( c_m < 0 \) if \( m \) is odd.] We do not believe this actually happens: such terms are traced back to be due to the inertia terms in the momentum NS equations, the same inertia terms which give the destabilizing second-derivative term of Eq. (102), and it seems unlikely that the same physical cause can be responsible for both stabilization and destabilization. Therefore, we believe the first nondegenerate term will turn out to be destabilizing.

Based on the (linear) studies\(^3\) of the Orr-Sommerfeld equation, we have computed the next coefficient, which shows that, albeit stabilizing, the sixth-derivative term is again degenerate: \( c_3 = -16173184/1718663821875 \approx -0.941 \times 10^{-5} \). (This is also remarkably close to the fourth-derivative coefficient, \( c_2 \approx -0.947 \times 10^{-5} \). If further \( c_n \) were all degenerate too and of the same order of magnitude, the destabilizing fourth-derivative term would be dominant, and hence stabilization and all-time valid EE impossible.)

A special case to consider is when the destabilizing fourth-derivative term (which is clearly much greater than the sixth-derivative one) is nearly cancelled by the stabilizing
4th-derivative term $aR_c\eta^{(4)}$ (where $a \equiv 2581/1400$). As the term $c_3R^5_c\eta^{(6)}$ is stabilizing, one can ask whether there can be an all-time valid EE if $|c_3R^5_c\eta^{(6)}| \gg |aR_c - c_2R^3_c\eta^{(4)}|$. Our answer to this question is as follows. In this case, $aR_c\eta^{(4)} \sim |c_2|R^3_c\eta^{(4)} \gg |c_3|R^5_c\eta^{(6)}$. Since the dispersive (third-derivative) term must dominate the fourth-derivative term $aR_c\eta^{(4)}$, it dominates even stronger the sixth-derivative term $c_3R^5_c\eta^{(6)}$ [see the discussion following Eq. (103)]. As usual, the lengthscale is determined by a balance between the dominant linear dissipative terms, $\delta\eta_{zz} \sim |c_3|R^5_c\eta^{(6)} \ll aR_c\eta^{(4)} \sim R_c(\eta\eta_z)_{zz}$ [see the arguments following Eq. (103)]. Thus, the destabilizing nonlinear term is the greatest one; there is no other term which could serve as a counterbalance. We arrive at the conclusion that the hypothetical EE with the sixth-derivative term cannot be valid for all time.

We have not attempted to determine the next coefficient, $c_4$, because of the large volume of calculations which would be required. Based on what we have said above, we expect the eighth-derivative term to be nondegenerate and destabilizing, and then no single EE can be globally valid under the circumstances. It follows that the GEE (82) is the most general one. The validity condition $W \gg R^3_c$ (which would be sufficient even if $c_2$ were nondegenerate) can be relaxed a bit: it is enough to require that the capillary term dominates the (presumably nondegenerate) eighth-derivative one, $W/L^3 \gg R^7_c/L^8$. (With $L^2 \sim W/\delta$ [see Eq. (10)], we get $W^{7/2}/(\delta^{5/2}R^7_c) \gg 1$.) The EE (102) is valid locally only, under the instantaneous VCs (89).

If, however, the first nondegenerate term $\propto R^2_{c-1}\eta^{(2n)}$ turned out to be stabilizing, the EE (102) would have a domain of global validity, albeit a very limited one. It is straightforward to find the corresponding global VCs. Indeed, the balance between this term and the destabilizing one, $R^2_{c-1}\eta^{(2n)} \sim \delta\eta_{zz}$, yields the lengthscale $L$,

$$L^{2m-2} \sim \frac{R^{2m-1}_c}{\delta}. \quad (105)$$

Using this lengthscale, the "modified-R" VC takes the form

$$\frac{R_c}{L} \sim \left(\frac{\delta}{R_c}\right)^{1/(2m-2)} \ll 1, \quad (106)$$
and the small-slope VC becomes
\[
\frac{1}{L^2} \sim \left[ \frac{\delta}{R_c^{(2m-1)}} \right]^{1/(m-1)} \ll 1. \tag{107}
\]

The (relaxed) conditions of dominance are \( W \eta^{(4)} \ll R_c^{2m-1} \eta^{(2m)} \) and \( R_c \eta^{(4)} \ll R_c^{2m-1} \eta^{(2m)} \), i.e. \( W \ll R_c^{2m-1}/L^{2m-4} \) and \( R_c^2 (R_c/L)^{2m-4} \gg 1 \) (which with (105) are easy to recast in terms of the global parameters only). The ratio of the nonlinear dissipative term, \( \propto R_c (\eta \eta_z)_z \), to the stabilizing one, \( \propto R_c^{2m-1} \eta^{(2m)} \), is \( A/(R_c/L)^{2m-2} \), which is required to be small—otherwise, we can have the blow-up of solutions, similar to the case with the term \( \propto R_c \eta^{(4)} \) being dominant. The comparison of dissipative term, \( \propto R_c^{2m-1} \eta^{(2m)} \), to the dispersive term, \( \propto \eta_{zzz} \), shows that the latter can be greater or smaller than the former depending upon whether \( R_c^{2m-1}/L^{2m-3} \ll 1 \) or \( R_c^{2m-1}/L^{2m-3} \gg 1 \).

When \( R_c^{2m-1}/L^{2m-3} \ll 1 \) (dispersion is large), the characteristic amplitude, obtained by balancing the term \( \eta \eta_z \) with the dispersive term, is \( 1/L^2 \) [hence, the small-amplitude VC, \( A \ll 1 \), coincides with (107)]. Otherwise, i.e. if \( R_c^{2m-1}/L^{2m-3} \gg 1 \), the amplitude is determined by balancing the term \( \propto \eta \eta_z \) with the dominant stabilizing term, \( A \sim (R_c/L)^{2m-1} \).

Using the global-parameter estimate of the lengthscale, Eq. (105), all the above conditions are readily reduced to certain global VCs (expressed in terms of global parameters only); we do not write them here, in view of the likely nonreality of the imaginary case of dominance of the term \( \propto R_c^{2n} \eta^{(2n)} \).

VI. SUMMARY

We have considered flow of a liquid film down an inclined plane. We have posed and studied the following questions: What are the least restrictive parametric conditions for which the wavy film flow can be approximated for all time by a single (local) evolution equation, and what (if any) is the most general form of such an equation?

We have argued that the dissipative-dispersive evolution equation (82) [which we derived by an iterative perturbation method of a multiparameter type] is such a general EE. Any
all-time valid EE derived by a single-parameter technique is necessarily nothing else but essentially the general EE in which some terms have been omitted. Also, the domain of validity of such a “partial” EE is necessarily a subdomain of the “umbrella” domain of global validity given by Eqs. (93) and (94). [In particular, in such domains the amplitude of waves is necessarily much smaller than the mean film thickness.]

It is clear that any evolution equation which follows from a multiparametric approach (such as the iterative technique we have employed in this paper) can be also obtained with the conventional SP approach. However, the significant advantage of the MP derivation is that it covers at once all possible SP derivations of EEs (the number of which is, in principle, infinite in the SP approach, corresponding to the different choices of the small-parameter powers for the system parameters). Also, comparing the two derivations of even a particular EE, the MP derivation is justifiable for much less restrictive domains of the parameter space.

The theory also yields the explicit approximate expressions in terms of film thickness for the pressure and components of velocity [Eqs. (50), (12), (58), and (36)], and thus a complete description of film dynamics.

We have derived an EE (100) containing additional high-derivative terms which can be essential near the threshold of instability, \( R \approx R_c \), if the surface tension is sufficiently small. However, this EE (102) is good for a limited time only and cannot be good for all time. [The conditions of such a local (in time) validity are given by Eq. (89).] Under certain parametric conditions, the (numerical) solutions of that EE blow up due to a nonlinear (quadratic, second-derivative) destabilizing term.

The EE (82) is relatively easy for numerical simulations of the 3D waves in the inclined film. We have obtained good agreement with transient states and transitions observed in the physical experiments of Ref. 9. Under certain parametric conditions for which the dissipative terms of the EE are small, we observed self-organization (from the initial white-noise small-amplitude conditions) of unusual highly-ordered patterns of soliton-like structures on the film surface (the pattern consists of two travelling-wave subpatterns which move with different velocities). The studies of the evolution equation (82) will be published elsewhere (Ref. 39;
see also Refs. \cite{22}.

A similar analysis leads to analogous single-EE theory of a film flowing down a vertical “fiber”. We believe that similar theories can be useful for a variety of other systems.

VII. ACKNOWLEDGMENT

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APPENDIX A: VALIDITY CONDITIONS

From (73), the estimate of $w_1$, in terms of the basic parameters and length- and amplitude-scales, is

$$w_1 \sim \max \left( \frac{1}{Z^2}, \frac{1}{Y^2}, \frac{R}{Z}, \frac{\cot \theta}{Z}, \frac{R}{T}, \frac{W}{Z^3}, \frac{W}{ZY^2} \right).$$

(A1)

Estimating $u_1$ from (77), we have

$$u_1 \sim \max \left( \frac{1}{Z^2}, \frac{1}{ZY^2}, \frac{R}{Z^2}, \frac{\cot \theta}{Z^2}, \frac{\cot \theta}{Y^2}, \frac{R}{TZ}, \frac{W}{Z^4}, \frac{W}{Z^2Y^2}, \frac{W}{Y^4} \right).$$

(A2)

In obtaining a solvable equation for the boundary condition on $w_{1x}$ at $x = 1$ [Eq. (71)], we have dropped the term $u_{1z}(x = 1)$. This implies

$$u_{1z} \ll w_{1x} \quad (x = 1).$$

(A3)

Using the $O_M$ estimates for $u_1$, Eq. (A2), and $w_{1x}(x = 1)$, Eq. (71), the above requirement reduces to
\[ u_{1x}(x = 1) \sim \max \left( \frac{1}{Z^3}, \frac{1}{ZY^2}, \frac{R}{Z^2}, \cot \theta, \frac{\cot \theta}{Y^2}, \frac{R}{YZ} \right) \]

\[
\frac{W}{Z^3}, \frac{W}{Y^2Z^2}, \frac{W}{Y^4} \left( \frac{A}{Z} \right) \ll w_{1x}(x = 1) \sim \frac{A}{Z^2}. \tag{A4}
\]

This again yields the conditions (28), (29), (30), (31), and, in addition, the following conditions:

\[
\frac{\cot \theta}{Z} \ll 1, \tag{A5}
\]

\[
\left( \frac{Z}{Y} \right) \left( \frac{\cot \theta}{Y} \right) \ll 1 \tag{A6}
\]

\[
\frac{W}{Z^3} \ll 1, \tag{A7}
\]

\[
\frac{W}{ZY^2} \ll 1, \tag{A8}
\]

and

\[
\left( \frac{Z}{Y} \right) \left( \frac{W}{Y^3} \right) \ll 1. \tag{A9}
\]

These conditions, along with (11), (28), (29), (30), and (31), form the complete set of validity conditions for the present theory. They are sufficient to justify all the simplifications of the equations. Thus, the complete set of VCs is

\[
\max \left[ A, \frac{1}{Z^3}, \frac{1}{Y^2}, \frac{R}{Z}, \frac{\cot \theta}{Z^2}, \frac{Z \cot \theta}{Y^2}, \frac{R}{T} \right] \]

\[
\frac{W}{Z^3}, \frac{W}{Y^2Z^2}, \frac{W}{Y^4} \left( \frac{A}{Z} \right) \ll 1. \tag{A10}
\]

Using these conditions, it is easy to see that \( w_1 \ll w_0 \) and \( u_1 \ll u_0 \).

From (B6), we can estimate the \( O_M (Rp_1) \):

\[
Rp_1 \sim \max \left( \frac{A}{Z}, \frac{1}{Z^3}, \frac{1}{ZY^2}, \frac{R}{Z^2}, \frac{\cot \theta}{Z^2}, \frac{\cot \theta}{Y^2} \right),
\]

38
\[
\begin{align*}
\frac{R}{TZ}, \frac{W}{Z^4}, \frac{W}{Z^2Y^2}, \frac{W}{Y^4}) A. \\
(A11)
\end{align*}
\]

Using the conditions (A10), it is easy to show that \(R_{p1}/Z \ll R_{p0}/Z\) or \(p_1 \ll p_0\). From (B12), we estimate the \(O_M(v_1)\) as

\[
\begin{align*}
v_1 \sim \max \left[ \frac{A}{YZ}, \frac{1}{YZ^3}, \frac{1}{ZY^3}, \frac{R}{YZ^2}, \frac{\cot \theta}{Y^2} \right], \\
(A12)
\end{align*}
\]

Using the estimate of \(v_0\) [Eq. (59)] and the conditions (A10), it is easy to see that \(v_1 \ll v_0\). The VCs (A10) guarantee that all the terms involving \(w_0, u_0, v_0, p_0, w_1, u_1, v_1,\) and \(p_1\) that were dropped in obtaining solvable ODEs for the same quantities are small in comparison with the terms that were retained.

Estimating the \(O_M\) of various terms in Eq. (82), and noting that \(\delta \ll R\) or \(\delta \sim R\), we find that

\[
\begin{align*}
\frac{A}{T} \sim \max \left[ A, \frac{1}{Z^2}, \frac{1}{Y^2}, \frac{\delta}{Z}, \frac{\cot \theta}{Z}, \frac{Z \cot \theta}{Y^2}, \frac{W}{Z^3}, \frac{W}{ZY^2}, \frac{W}{Y^4} \right] \frac{A}{Z}, \\
(A13)
\end{align*}
\]

and, consequently, taking (A10) into account,

\[
\begin{align*}
\frac{R}{T} \sim \max \left[ A, \frac{1}{Z^2}, \frac{1}{Y^2}, \frac{\delta}{Z}, \frac{\cot \theta}{Z}, \frac{\cot \theta Z}{Y^2}, \frac{W}{Z^3}, \frac{W}{Y^4}, \frac{W}{ZY^2} \right] \frac{R}{Z} \ll 1. \\
(A14)
\end{align*}
\]
Hence, the parameter $R/T$ is small as a consequence of the smallness of other parameters in (A10), and thus can be omitted from there. Also, (87) yields $\cot \theta/Z < R/Z \ll 1$, so that the parameter $\cot \theta/Z$ can be omitted as well. The somewhat simplified validity conditions are

$$\max \left[ A, \frac{1}{Z^2}, \frac{1}{Y^2}, \frac{R}{Z}, \frac{\cot \theta Z}{Y^2}, \frac{W}{Z^3}, \frac{W}{Y^2 Z}, \frac{W Z}{Y^4} \right] \ll 1 \quad (A15)$$

(which, in turn, significantly simplify [see Eq. (89)] if $Y \geq Z$).

**APPENDIX B: SECOND AND THIRD ITERATIVE CORRECTIONS**

To find the second and third corrections for the velocity components and pressure, we need first to determine $p_1$ and $v_1$, the approximations to the exact solutions $\tilde{p}_1$ and $\tilde{v}_1$. Considering the $x$–NS equation,

$$\tilde{p}_{1x} = -p_{0x} + \frac{1}{R}(u_0 + u_1)_{xx} + \frac{1}{R} \nabla^2(u_0 + u_1)$$

$$-(u_0 + u_1)_t - (u_0 + u_1)(u_0 + u_1)_x - v_0(u_0 + u_1)_y$$

$$-(w_N + w_0 + w_1)(u_0 + u_1)_z + 2(u_0 + u_1)_z$$

$$+ \text{[terms containing } \tilde{v}_1, \tilde{w}_2, \text{ or } \tilde{u}_2] \quad (B1)$$

where

$$\tilde{u}_2 = \tilde{u}_1 - u_1. \quad (B2)$$

Using our standard procedure, the simplified equation is

$$p_{1x} = \frac{1}{R} u_{1xx} + \frac{1}{R} \nabla^2 u_0 - w_N u_{0z} + 2u_{0z} - u_{0t}$$

40
\[
\frac{2}{R}(x-1)(W\nabla^4\eta - \cot \theta \nabla^2\eta) + \eta_{tz}
\]

\[
- \left( x^4 - \frac{4}{3} x^3 + \frac{4}{3} \right) \eta_{zz} + (x^2 + 2x - 5) \frac{\nabla^2\eta_z}{R}
\]

(B3)

where Eqs. (16) and (50) for \(p_N\) and \(p_0\) have been utilized. The normal-stress balance condition at \(x = h\) yields

\[
\tilde{p}_1 = -(p_N + p_0) + \frac{2}{R} \left\{ (u_0 + u_1)_x + v_{0y}\eta_y^2 + (w_0 + w_1)_z\eta_z^2 - \left[ (u_0 + u_1)_y + v_{0x} \right] \eta_y + \left[ v_{0z} + (w_0 + w_1)_y \right] \eta_y \eta_z 
- [(u_0 + u_1)_z + w_{1z}] \eta_z \right\} \{1 + \eta_y^2 + \eta_z^2\}^{-1} \left[ \eta_{yy} \left( 1 + \eta_z^2 \right) + \eta_{zz} \left( 1 + \eta_y^2 \right) - 2\eta_y \eta_z \eta_{yz} \right] (1 + \eta_y^2 + \eta_z^2)^{-3/2} + \left[ \text{terms containing } \tilde{v}_1, \tilde{w}_2, \text{ or } \tilde{u}_2 \right] \quad (x = h) \]  

(B4)

which simplifies to the BC

\[
p_1 = -p_0 + \frac{2}{R} u_{0xx} \eta + \frac{2}{R} u_{1x} 
= -\frac{2}{R} (W\nabla^4\eta - \cot \theta \nabla^2\eta) - \frac{5}{3} \eta_{zz} 
- \frac{20}{3R} \nabla^2\eta_z - \frac{2}{R} \eta_{zz} + \frac{4}{3} \eta_{tz} \quad (x = 1). \]  

(B5)

The solution is

\[
Rp_1 = (x^2 - 2x - 1) \left[ W\nabla^4\eta - \cot \theta \nabla^2\eta \right] 
+ \left( \frac{x^4}{3} - \frac{x^5}{5} - \frac{4}{3} x - \frac{7}{15} \right) R\eta_{zz} 
+ \left( \frac{x^3}{3} + x^2 - 5x - 3 \right) \nabla^2\eta_z 
+ \left( x + \frac{1}{3} \right) R\eta_{tz} - 2\eta_{tz} \]  

(B6)
The $y$–NS equation is

$$\tilde{v}_{1xx} = -v_{0xx} + R(p_N + p_0 + p_1)_y - \nabla^2 v_0 + Rv_{0t}$$

$$+ R(u_0 + u_1)v_{0x} + Rv_0v_{0y} + R(w_N + w_0 + w_1 - 2)v_{0z}$$

$$+ [\text{terms containing } \tilde{w}_2, \tilde{u}_2, \tilde{p}_2, \text{or } \tilde{v}_1], \quad (B7)$$

where $\tilde{p}_1 = p_1 + \tilde{p}_2$, and $v_0$ is given by (58). The simplified equation (for the approximation $v_1$) is

$$v_{1xx} = Rp_{1y} - \nabla^2 v_0 + R(w_N - 2)v_{0z} + Rv_{0t}$$

or, in terms of $\eta$,

$$v_{1xx} = -(2x^2 - 4x - 1) \cot \theta \nabla^2 \eta_y + \left( \frac{2}{15} x^5 + \frac{2}{3} x^4 - \frac{16}{3} x^3 + 10x^2 - \frac{28}{3} x - \frac{7}{15} \right) R\eta_{zz}$$

$$- (x^4 - 4x^3 + 6x^2 - 4x) R \cot \theta \eta_{yz} + (x^2 - 2x) R \cot \theta \eta_{yt} - \left( \frac{x^3}{3} + x^2 - 5x - \frac{1}{3} \right) R\eta_{zyt}$$

$$+ P_1(\eta\eta_y)_y + P_2W\nabla^4 \eta_y + P_3RW\nabla^2 \eta_{yz} + P_4RW\nabla^2 \eta_{yz} + P_5RW\nabla^2 \eta_{yt}, \quad (B8)$$

where $v_1$ is the approximation to the exact solution $\tilde{v}_1$. The coefficients of terms that are likely to contribute to spatial derivatives of order five or more, as well as the coefficients of nonlinear terms which are small or dispersive in nature, are represented by $P_1$, $\cdots$, $P_5$, which are polynomials in $x$; we will not need their explicit form, just the fact that they are $\sim 1$. Henceforth, coefficients of type $P_m$ will always refer to such polynomials in $x$. The boundary condition at $x = h$ is:

$$\tilde{v}_{1x} = -v_{0x} - (u_0 + u_1)_y + \{2[v_0y - (u_0 + u_1)x]\eta_y$$

$$+ [(u_0 + u_1)_z + w_{1z}]\eta_y \eta_z$$
\[ v_{0z} + (w_0 + w_1)\eta_z \] \[ \left\{ [1 - \eta_y^2]^{-1} + \text{terms containing } \tilde{v}_1, \tilde{w}_2, \text{or } \tilde{u}_2 \right\} (x = h), \] \] (B9)

where we have taken into account (37). The simplified BCs are

\[ v_{1x} = -v_{0xx}\eta - u_{0yx}\eta - u_{1y} - 2u_{0x}\eta_y + w_0\eta_z \]

\[ = -\frac{2}{3}W \nabla^4 \eta_y - \frac{5}{12}R\eta_{tzy} + \frac{8}{15}R\eta_{zz} \]

\[ + C_1(\eta\eta_z)_y + C_2(\cot \theta)\eta\eta_y + C_3W\eta\nabla^2 \eta_y \]

\[ + C_4 W \nabla^4 \eta_y + C_5 \nabla^2 \eta_{yz} \quad (x = 1) \] (B10)

and

\[ v_1 = 0 \quad (x = 0) \] (B11)

where \( C_1, \cdots, C_5 \) are constants. (Henceforth, coefficients of type \( C_m \) are constants.) The solution is

\[ v_1 = -\left( \frac{x^4}{6} - \frac{2}{3}x^3 - \frac{x^2}{2} + 3x \right) \cot \theta \nabla^2 \eta_y + \left( \frac{x^7}{315} + \frac{x^6}{45} - \frac{4x^5}{15} + \frac{5x^4}{6} - \frac{14x^3}{9} - \frac{7x^2}{30} + \frac{158x}{45} \right) R\eta_{tzy} \]

\[ + \left( -\frac{x^6}{30} + \frac{x^5}{5} - \frac{x^4}{2} + \frac{2}{3}x^3 - \frac{4}{5}x \right) R\cot \theta \eta_{yz} + \left( \frac{x^4}{12} - \frac{x^3}{3} + \frac{2x}{3} \right) R\cot \theta \eta_{yt} \]

\[ - \left( \frac{x^5}{60} + \frac{x^4}{12} - \frac{5}{6}x^3 - \frac{x^2}{6} + 2x \right) R\eta_{tzy} + P_6(\eta\eta_z)_y + P_7(\cot \theta)\eta\eta_y + P_8 W\eta\nabla^2 \eta_y \]

\[ + P_9 W \nabla^4 \eta_y + P_{10} \nabla^2 \eta_{yz} + P_{11} RW \nabla^2 \eta_{yz} + P_{12} RW \nabla^2 \eta_{yt}. \] (B12)

By using the same procedure as above, we proceed to calculate \( \tilde{w}_2, \tilde{u}_2, \tilde{p}_2, \) and \( \tilde{v}_2. \) The equation for \( \tilde{w}_2 \) is
\begin{align*}
\ddot{w}_{2xx} &= -(w_0 + w_1)_{xx} + R(p_0 + p_1)_z - \nabla^2(w_0 + w_1) + R(w_0 + w_1)_t \\
 &\quad + R(u_0 + u_1)(w_N + w_0 + w_1)_z + R(v_0 + v_1)(w_0 + w_1)_y \\
 &\quad + R(w_N + w_0 + w_1 - 2)(w_0 + w_1)_z \\
 &\quad + \text{[terms containing } \dot{w}_2, \dot{u}_2, \dot{v}_2, \text{ or } \dot{p}_2\text{].} \\
\end{align*}

This yields
\begin{align*}
w_{2xx} &= R p_{1z} - \nabla^2 w_1 + R w_{1t} + R u_{1w_Nx} + R(w_N - 2)w_{1z} \\
 &\quad + R u_0 w_{0x} + R w_0 w_{0z}.
\end{align*}

Hence,
\begin{align*}
w_{2xx} &= (1 + 4x - 2x^2) \cot \theta \nabla^2 \eta_z - \left( \frac{x^5}{3} + \frac{x^4}{2} - \frac{19x^3}{3} + 5x^2 + \frac{4x}{3} \right) R \nabla^2 \eta_z \\
 &\quad + \left( \frac{7x^5}{15} - \frac{17x^3}{3} + 12x^2 - \frac{34x}{3} - \frac{7}{15} \right) R \eta_{zzz} - (x^4 - 4x^3 + 6x^2 - 4x) \cot \theta \eta_{zz} \\
 &\quad + \left( -\frac{x^6}{10} + \frac{3x^5}{5} - \frac{4x^4}{3} + \frac{4x^3}{3} + \frac{4x^2}{3} - \frac{8x}{3} \right) R^2 \eta_{zz} + \left( \frac{x^5}{3} - \frac{2x^4}{3} - \frac{x^3}{3} + 2x^2 - 2x \right) R \eta_{yyz} \\
 &\quad + \left( \frac{2x^4}{3} - \frac{8x^3}{3} + 2x^2 \right) R \cot \theta \nabla^2 \eta - \left( \frac{x^5}{6} - \frac{2x^4}{3} + \frac{4x^3}{3} + x^2 - \frac{10x}{3} \right) R^2 \eta_z + (x^2 - 2x) R \cot \theta \eta_z + 2x^2 R \eta_z \\
 &\quad + P_{13}(\eta_z)_z + P_{14} W \nabla^4 \eta_z + P_{15} \nabla^2 \eta_{zz} + P_{16} \eta_{zzzz} + P_{17} \eta_{yyzz} + P_{18} R W \nabla^2 \eta_{zz} + P_{19} R W \nabla^4 \eta + P_{20} R \nabla^2 \eta_t \\
 &\quad + P_{21} R W \nabla^2 \eta_{tt} + P_{22} R^2 \eta_{tt} + P_{23} R \eta_{zzt} + P_{24} R \eta_{yzt} \tag{B14}
\end{align*}

The boundary condition at \( x = h \) is
\begin{align*}
\ddot{w}_{2x} &= -(w_0 + w_1)_x - (u_0 + u_1)_x + 2[(w_0 + w_1)_z - (u_0 + u_1)_z] \eta_z
\end{align*}
\[ +[(u_0 + u_1)_y + (v_0 + v_1)_x] \eta_y \eta_z \]
\[ +[(v_0 + v_1)_z + (w_0 + w_1)_y] \eta_y \{1 - \eta_z^2 \}^{-1} \]
\[ + \text{[terms containing } \tilde{w}_2, \tilde{u}_2, \tilde{v}_2, \text{ or } \tilde{p}_2] \quad (x = h). \quad \text{(B15)} \]

Dropping all the unknown and smaller terms, we have, at \( x = 1, \)
\[ w_{2x} = -w_{1xx} \eta - u_{0zx} \eta - u_{1z} + 2(w_{0z} - u_{0z}) \eta_z + w_{0y} \eta_y \]
\[ = \frac{8}{15} R \eta_{zzz} - \frac{2}{3} \cot \theta \nabla^2 \eta_z + 2R \eta \eta_z - 2 \cot \theta \eta \eta_z + C_6 W \eta \nabla^2 \eta_z \]
\[ + C_7 (\eta \eta_z)_z + C_8 (\eta \eta_y)_y + C_9 W \nabla^4 \eta_z + C_{10} \nabla^2 \eta_{zz} \]
\[ + C_{11} R \eta_{tzz} + C_{12} R \eta \eta \quad (x = 1). \quad \text{(B16)} \]

Finally, the no-slip condition is
\[ w_2 = 0 \quad (x = 0). \quad \text{(B17)} \]

Integrating (B14) with the boundary conditions (B16) and (B17), we obtain
\[ w_2 = \left( -\frac{x^4}{6} + \frac{2x^3}{3} + \frac{x^2}{2} - 3x \right) \cot \theta \nabla^2 \eta_z + \left( -\frac{x^7}{126} + \frac{x^6}{60} + \frac{19x^5}{60} - \frac{5x^4}{12} - \frac{2x^3}{9} + \frac{163x}{180} \right) R \nabla^2 \eta_z \]
\[ + \left( \frac{x^7}{90} - \frac{17x^5}{60} + x^4 - \frac{17x^3}{9} - \frac{7x^2}{30} + \frac{721x}{180} \right) R \eta_{zzz} + \left( -\frac{x^8}{560} + \frac{x^7}{70} - \frac{2x^6}{45} + \frac{x^5}{15} + \frac{x^4}{9} - \frac{4x^3}{12} + \frac{232x}{315} \right) R^2 \eta_{zz} \]
\[ + \left( -\frac{x^6}{30} + \frac{x^5}{5} - \frac{x^4}{2} + \frac{2x^3}{3} - \frac{4x}{5} \right) R \cot \theta \eta_{zz} + \left( \frac{x^7}{126} - \frac{x^6}{45} + \frac{x^5}{6} - \frac{x^4}{3} + \frac{89x}{180} \right) R \eta_{yyz} \]
\[ + \left( \frac{x^6}{45} - \frac{2x^5}{15} + \frac{x^4}{6} - \frac{2x}{15} \right) R \cot \theta \nabla^2 \eta + \left( -\frac{x^7}{252} + \frac{x^6}{45} - \frac{x^5}{12} + \frac{x^4}{9} + \frac{199x}{180} \right) R^2 \eta_{tz} \]
\[ + \left( \frac{x^4}{12} - \frac{x^3}{3} + \frac{2x}{3} \right) R \cot \theta \eta_{tz} + \left( \frac{x^4}{6} + \frac{4x}{3} \right) R \eta_{tz} - 2x \cot \theta \eta \eta_z + P_{25}(\eta \eta_z)_z + P_{26} W \eta \nabla^2 \eta_z \]
\[ + P_{27}(\eta_y)_y + P_{28}W\nabla^4 \eta_z + P_{29}\nabla^2 \eta_{zz} + P_{30}\eta_{zzzz} + P_{31}\eta_{yyzz} + P_{32}RW\nabla^2 \eta_z + P_{33}RW\nabla^4 \eta + P_{34}RW\nabla^2 \eta_{tz} \]

\[ + P_{35}R^2 \eta_t + P_{36}R\eta_{zzt} + P_{37}R\eta_{yyt} + P_{38}R\nabla^2 \eta_t + P_{39}R\eta_t. \quad \text{(B18)} \]

The equation for \( \tilde{u}_2 \), obtained from the incompressibility condition, is

\[ \tilde{u}_{2x} = -(v_1 + \tilde{v}_2)_y - (w_2 + \tilde{w}_3)_z \quad \text{(B19)} \]

where \( \tilde{w}_2 = w_2 + \tilde{w}_3 \). Dropping the unknown terms, we obtain

\[ u_{2x} = -v_{1y} - w_2 \]

or

\[ u_{2x} = \left( \frac{x^6}{90} - \frac{x^5}{15} + \frac{x^4}{3} - \frac{2x^3}{15} + \frac{14x}{15} \right) R \cot \theta \nabla^2 \eta_z + \left( \frac{x^8}{560} - \frac{x^7}{70} + \frac{2x^6}{45} - \frac{x^5}{15} - \frac{x^4}{9} + \frac{4x^3}{9} - \frac{232x}{315} \right) R^2 \eta_{zzz} \]

\[ + \left( \frac{x^4}{6} - \frac{2x^3}{3} - \frac{x^2}{2} + 3x \right) \cot \theta \nabla^4 \eta + \left( -\frac{x^7}{315} + \frac{x^6}{60} - \frac{x^5}{30} - \frac{7x^4}{12} + \frac{19x^3}{9} + \frac{7x^2}{30} - \frac{221x}{45} \right) R \nabla^2 \eta_z \]

\[ - \left( \frac{x^4}{12} - \frac{x^3}{3} + \frac{2x}{3} \right) R \cot \theta \nabla^2 \eta_t + \left( \frac{x^7}{252} - \frac{x^6}{45} + \frac{x^5}{15} + \frac{x^4}{12} - \frac{5x^3}{9} + \frac{199x}{180} \right) R^2 \eta_{tzz} - \left( \frac{x^4}{6} + \frac{4x}{3} \right) R(\eta_\eta)_z \]

\[ + 2x \cot \theta(\eta_z)_z + P_{39}(\eta_z)_{zz} + P_{40}W(\eta\nabla^2 \eta)_z + P_{41}(\eta_\eta)_y + P_{42}(\eta_z)_y + P_{43}(\eta_\eta)_y + P_{44}RW\nabla^4 \eta_{zz} \]

\[ + P_{45}\nabla^2 \eta_{zzzz} + P_{46}\eta_{zzzzzz} + P_{47}\eta_{yyzz} + P_{48}RW\nabla^2 \eta_{zzzz} + P_{49}RW\nabla^4 \eta_z + P_{50}RW\nabla^4 \eta_{yy} + P_{51}\nabla^2 \eta_{yy} \]

\[ + P_{52}RW\nabla^2 \eta_{zyy} + P_{53}RW\nabla^2 \eta_{tyy} + P_{54}R\eta_{tzy} + P_{55}RW\nabla^2 \eta_{tzz} + P_{56}R^2 \eta_{tzz} + P_{57}R\eta_{tzzz} \]

\[ + P_{58}R\eta_{tzy} + P_{59}R\nabla^2 \eta_{tz} + P_{60}R(\eta_\eta)_z \quad \text{(B20)} \]

where \( u_2 \) is the approximation to \( \tilde{u}_2 \). The no-slip boundary condition is

\[ u_2 = 0 \quad (x = 0). \quad \text{(B21)} \]

Integrating Eq. (B20) with the no-slip BC, we obtain
$$u_2 = \left( \frac{x^9}{5040} - \frac{x^8}{560} + \frac{2x^7}{315} - \frac{x^6}{90} - \frac{x^5}{45} + \frac{x^4}{9} - \frac{116x^2}{315} \right) R^2 \eta_{zzz}$$

$$+ \left( \frac{x^7}{630} - \frac{x^6}{90} + \frac{x^5}{15} - \frac{x^4}{6} + \frac{7x^2}{15} \right) R \cot \theta \nabla^2 \eta_z + \left( \frac{x^5}{30} - \frac{x^4}{6} - \frac{x^3}{6} + \frac{3x^2}{2} \right) \cot \theta \nabla^4 \eta$$

$$+ \left( -\frac{x^8}{2520} + \frac{x^7}{420} - \frac{x^6}{180} - \frac{7x^5}{90} + \frac{19x^4}{36} + \frac{7x^3}{90} - \frac{221x^2}{90} \right) R \nabla^2 \eta_{zz}$$

$$+ \left( \frac{x^8}{2016} - \frac{x^7}{315} + \frac{x^6}{90} + \frac{x^5}{60} - \frac{5x^4}{36} - \frac{199x^2}{360} \right) R^2 \eta_{zzz} - \left( \frac{x^5}{60} - \frac{x^4}{12} + \frac{x^2}{3} \right) R \cot \eta_{zzz} - \left( \frac{x^5}{30} + \frac{2x^2}{3} \right) R (\eta \eta_z)_z$$

$$+ x^2 \cot \theta (\eta \eta_z)_z + P_{61}(\eta \eta_z)_{zz} + P_{62}(\eta \eta_y)_{yz} + P_{63}(\eta \eta_z)_{yy} + P_{64}(\eta \eta_y)_y + P_{65}W (\eta \nabla^2 \eta_z)_z + P_{66}W \nabla^4 \eta_{zz}$$

$$+ P_{67} \nabla^2 \eta_{zzz} + P_{68} \eta_{zzzz} + P_{69} \eta_{yyzz} + P_{70}RW \nabla^2 \eta_{zzz} + P_{71}RW \nabla^4 \eta_z + P_{72}W \nabla^4 \eta_{yy}$$

$$+ P_{73} \nabla^2 \eta_{zgg} + P_{74}RW \nabla^2 \eta_{zgg} + P_{75}RW \nabla^2 \eta_{tyy} + P_{76}R \eta_{lzg} + P_{77}RW \nabla^2 \eta_{lzz} + P_{78}R^2 \eta_{lz} + P_{79}R \eta_{lzzz}$$

$$+ P_{80}R \eta_{ygg} + P_{81}RW \nabla^2 \eta_z + P_{82}R (\eta \eta_z)_z.$$  \hfill (B22)

In order to obtain all the fourth-order linear dissipative terms of the EE, we have to proceed to the next approximation [since (see Appendix C) the dissipative terms of $u_3$ are not guaranteed to be much smaller than those of $u_2$; however, the dissipative terms of $u_4$ are much smaller than those of $u_2$]. Expressing $\tilde{u}_2 = u_2 + \tilde{u}_3$, the incompressibility condition leads to an equation for the approximation $u_3$:

$$u_{3x} = -v_{2y} - w_{3z}, \hfill (B23)$$

(where $v_2$ and $w_3$ are the approximations to $\tilde{v}_2$ and $\tilde{w}_3$ respectively). The solvable ODEs for $v_2$ and $w_3$ will contain $p_2$, the approximation to $\tilde{p}_2$. Terms from $p_2$ will not make any contribution to linear terms with derivatives of order four or less and the nonlinear terms in the final EE. However, in Appendix C, we will obtain an equation for the general iterate $p_n$. 

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and the form of the \(x-NS\) equation for \(p_2\) will guide us in formulating the general equation for \(p_n\).

The \(x-NS\) equation for pressure \(\tilde{p}_2\) can be written in the form

\[
\tilde{p}_{2x} = -(p_0+p_1) + 1 \frac{1}{R} \left( u_0 + u_1 + u_2 \right)_x + \frac{1}{R} \nabla^2 \left( u_0 + u_1 + u_2 \right) - (u_0 + u_1 + u_2)_t - (u_0 + u_1 + u_2) (u_0 + u_1 + u_2)_x
\]

\[-(v_0 + v_1)(u_0 + u_1 + u_2)_y - (w_N + w_0 + w_1 + w_2 - 2)(u_0 + u_1 + u_2)_z
\]

\[+ \text{[terms containing } \tilde{v}_2, \, \tilde{w}_3, \, \text{or } \tilde{u}_3 \text{]} \quad \text{(B24)}\]

which yields

\[p_{2x} = -p_{1x} \eta + \frac{u_{2xx}}{R} + \frac{\nabla^2 u_1}{R} - u_{1t} - 2u_{1z}
\]

\[-w_N u_{1z} - u_0 u_{0x} - w_0 u_{0z}. \quad \text{(B25)}\]

The boundary condition on pressure at \(x = h\) is

\[
\tilde{p} = -(p_N + p_0 + p_1) + \frac{2}{R} \left( (u_0 + u_1 + u_2)_x + (v_0 + v_1)_y \eta_y^2 + (w_0 + w_1 + w_2)_z \eta_z^2 - [(u_0 + u_1 + u_2)_y + (v_0 + v_1)_x] \eta_y \right)
\]

\[+ [(v_0 + v_1)_z + (w_0 + w_1 + w_2)_y] \eta_y \eta_z - [(u_0 + u_1 + u_2)_z + (w_1 + w_2)_x] \eta_z \right) (1 + \eta_y^2 + \eta_z^2)^{-1}
\]

\[-\sigma \left[ \eta_{yy} \left( 1 + \eta_z^2 \right) + \eta_{zz} \left( 1 + \eta_y^2 \right) - 2\eta_y \eta_z \eta_{yz} \right] (1 + \eta_y^2 + \eta_z^2)^{-3/2}
\]

\[+ \text{[terms containing } \tilde{v}_2, \, \tilde{w}_3, \, \text{or } \tilde{u}_3 \text{]} \quad (x = h). \quad \text{(B26)}\]

This yields, at \(x = 1\),

\[p_2 = -p_{1x} \eta + \frac{2}{R} (u_{1xx} \eta + u_{2x} - (u_0 + v_0 x) \eta_y
\]

\[-(u_{0z} + w_{1x}) \eta_z - \sigma [\eta_{yy} \eta_z^2 + \eta_{zz} \eta_y^2 - 2\eta_y \eta_z \eta_{yz}]\]
\[-(3/2)\sigma \nabla^2 \eta (\eta_y^2 + \eta_z^2) \quad (x = 1). \quad \text{(B27)}\]

As mentioned before, \(p_2\) will not contribute to linear terms with derivatives of order 4 or less and the nonlinear terms. Hence, we will not solve for \(p_2\).

Representing \(\tilde{p}_2\) as \(\tilde{p}_2 = p_2 + \tilde{p}_3\), the \(y\)-NS equation for \(\tilde{v}_2\) is

\[
\tilde{v}_{2xx} = -(v_0 + v_1)_{yy} + R(p_0 + p_1 + p_2)_y \nabla^2 (v_0 + v_1) + R(v_0 + v_1)_t + R(u_0 + u_1 + u_2)(v_0 + v_1)_z + R(v_0 + v_1)(v_0 + v_1)_y \\
+ R(w_N + w_0 + w_1 + w_2 - 2)(v_0 + v_1)_z + \text{[terms containing } \tilde{v}_2, \tilde{w}_3, \tilde{u}_3, \text{ or } \tilde{p}_3] \quad \text{(B28)}
\]

The simplified equation for the approximation \(v_2\) is

\[
v_{2xx} = Rp_2y - \nabla^2 v_1 + Rv_{1t} \\
+ Ru_0v_{0x} + R(w_N - 2)v_{1z} + Rw_0v_{0z}. \quad \text{(B29)}
\]

The boundary condition on \(\tilde{v}_2\) at \(x = h\) is

\[
\tilde{v}_{2x} = -(v_0 + v_1)_x - (u_0 + u_1 + u_2)_y + \{2[(v_0 + v_1)_y - (u_0 + u_1 + u_2)_x] \eta_y + [(u_0 + u_1 + u_2)_z + (w_1 + w_2)_x] \eta_y \eta_z \\
+ [(v_0 + v_1)_z + (w_0 + w_1 + w_2)_y] \eta_z\} (1 - \eta_y^2)^{-1} + \text{[terms containing } \tilde{v}_2, \tilde{w}_3, \text{ or } \tilde{a}_3] \quad (x = h). \quad \text{(B30)}
\]

The simplified BC on \(v_2\) at \(x = 1\) is

\[
v_2 = -\frac{1}{2}v_{0xxx} \eta^2 - v_{1xx} \eta - \frac{1}{2}u_{0yxx} \eta^2 + u_{1yx} \eta + u_{2y} \\
+ [2(v_0 - u_{0xx} \eta - u_{1x}) \eta_y + \\
+ (v_0z + w_{0yx} \eta + w_{1y}) \eta_z] \quad (x = 1). \quad \text{(B31)}
\]

The no-slip condition on \(v_2\) is

\[
v_2(x = 0) = 0. \quad \text{(B32)}
\]
As mentioned before, we will not solve the equation for $v_2$ to find all the terms in it. We are only interested in the terms that make linear dissipative contributions of order four or less to the EE. Only the term involving $R(w_N - 2)v_1z$ makes such a contribution. There is no contribution from the tangential stress and the no-slip conditions. Thus,

$$v_{2xx}^{(\leq 3)} = R(w_N - 2)v_1z$$

$$= \left( \frac{x^8}{30} - \frac{4x^7}{15} + \frac{29x^6}{30} - \frac{31x^5}{15} + \frac{7x^4}{3} \right. \left. - \frac{8x^3}{15} - \frac{8x^2}{5} + \frac{8x}{5} \right) R^2 \cot \theta \eta_{zzz} \quad (B33)$$

where the superscript on $v_{2xx}$ indicates that RHS of the above expression contains only linear terms with spatial derivatives of order 3 or less with all other terms omitted.

Solving for $v_2^{(\leq 3)}$, we obtain

$$v_2^{(\leq 3)} = \left( \frac{x^{10}}{2700} - \frac{x^9}{270} + \frac{29x^8}{1680} - \frac{31x^7}{630} + \frac{7x^6}{90} \right. \left. - \frac{2x^5}{75} - \frac{2x^4}{15} + \frac{4x^3}{15} - \frac{344x}{945} \right) R^2 \cot \theta \eta_{zyy} \quad (B34)$$

Similarly, proceeding to $w_3$, we find that only two terms in the $z-$NS equation make contributions to fourth derivative terms in $u_3$, i.e.

$$w_{3xx}^{(\leq 3)} = Rw_{Nx}u_2 + R(w_N - 2)w_2z,$$

or

$$w_{3xx}^{(\leq 3)} = \left( \frac{x^{10}}{720} - \frac{x^9}{72} + \frac{19x^8}{315} - \frac{47x^7}{315} + \frac{2x^6}{15} + \frac{4x^5}{15} - \frac{8x^4}{9} + \frac{8x^3}{9} + \frac{232x^2}{315} - \frac{464x}{315} \right) R^3 \eta_{zzz}$$

$$+ \left( \frac{8x^8}{315} + \frac{64x^7}{315} - \frac{19x^6}{30} + \frac{16x^5}{15} - \frac{2x^4}{3} - \frac{4x^3}{15} + \frac{4x^2}{3} + \frac{4x}{15} \right) R^2 \cot \theta \nabla^2 \eta_z$$

$$+ \left( \frac{x^8}{30} - \frac{4x^7}{15} + \frac{29x^6}{30} - \frac{31x^5}{15} + \frac{7x^4}{3} - \frac{8x^3}{5} - \frac{8x^2}{5} + \frac{8x}{5} \right) R^2 \cot \eta_{zzz}. \quad (B35)$$
We note that only the terms with \( R \cot \theta \nabla^2 \eta \) and \( R^2 \eta_{zzz} \) in \( u_2 \) [see Eq. (B22)] make a contribution. Similarly, only the terms with \( R \cot \theta \eta_{zz}, R^2 \eta_{zzz}, \) and \( R \cot \theta \nabla^2 \eta \) in \( w_2 \) [see Eq. (B18)] make the relevant contribution. Again, there are no contributions from the tangential and no-slip BCs.

The relevant part of \( w_3 \) is

\[
\begin{align*}
\left( w^{(3)}_3 \right) & = - \left( \frac{x^{12}}{95040} - \frac{x^{11}}{7920} + \frac{19x^{10}}{28350} - \frac{47x^9}{22680} + \frac{x^8}{420} + \frac{2x^7}{315} - \frac{4x^6}{135} \\
& \quad + \frac{2x^5}{45} + \frac{58x^4}{945} - \frac{232x^3}{945} + \frac{12358x}{31185} \right) R^3 \eta_{zzz} \\
& \quad + \left( \frac{x^{10}}{2700} - \frac{x^9}{270} + \frac{29x^8}{1680} - \frac{31x^7}{630} + \frac{7x^6}{90} - \frac{2x^5}{75} + \frac{2x^4}{15} + \frac{4x^3}{15} - \frac{344x}{945} \right) R^2 \cot \theta \eta_{zzz} \\
& \quad - \left( \frac{-4x^{10}}{14175} + \frac{8x^9}{2835} - \frac{19x^8}{1680} + \frac{8x^7}{315} - \frac{x^6}{45} - \frac{x^5}{25} + \frac{x^4}{18} + \frac{2x^3}{45} - \frac{107x}{810} \right) R^2 \cot \theta \nabla^2 \eta_z.
\end{align*}
\]

Using the expressions for \( v_2 \) and \( w_3 \) in the incompressibility condition (B23), we obtain

\[
\begin{align*}
\left( u^{(4)}_3 \right) & = - \left( \frac{x^{13}}{1235520} - \frac{x^{12}}{95040} + \frac{19x^{11}}{311850} - \frac{47x^{10}}{226800} \\
& \quad + \frac{x^9}{3780} + \frac{x^8}{1260} - \frac{4x^7}{945} + \frac{x^6}{135} + \frac{58x^5}{4725} - \frac{58x^4}{945} + \frac{6179x^2}{31185} \right) R^3 \eta_{zzzz} \\
& \quad - \left( \frac{x^{11}}{124740} - \frac{x^{10}}{11340} + \frac{x^9}{1512} - \frac{x^8}{336} + \frac{x^7}{126} - \frac{x^6}{90} - \frac{7x^5}{450} + \frac{7x^4}{90} - \frac{2813x^2}{11340} \right) R^2 \cot \theta \nabla^2 \eta_{zz}.
\end{align*}
\]

Following our standard procedure [see Eqs. (78) and (79)], the kinematic condition at \( x = 1 \) is written as

\[
\eta_t + (w_0 + w_1) \eta_z = u_{0x} \eta + u_1 + u_{1x} \eta + u_2 + u_3 \quad (x = 1)
\]
where we have taken into account Eq. (B4). Also, terms \( w_1 \eta_z \) and \( u_{1z} \eta \) which could normally be discarded in comparison with \( u_1 (x = 1) \) contains \( \eta_{xx} \) with the factor \( \delta \) which is allowed to be small. Note that we have not shown terms of type \( v_0 \eta_y, v_1 \eta_y, v_2 \eta_y \) and \( w_2 \eta_z \) as they make smaller contributions. Also, \( w_{Nz} \eta = 0 \) at \( x = 1 \). Using the expressions for the velocities \((36), (73), (42), (77), (B22), \) and \((B37)\) in Eq. \((B38)\), we obtain

\[
\left[ \eta - \frac{5}{12} R \eta_z + \frac{4}{15} R \cot \theta \nabla^2 \eta - \frac{295}{672} R^2 \eta_{zz} \right]_t + 4 \eta \eta_z + \frac{8}{15} R \eta_{zz} - \frac{2}{3} \cot \theta \nabla^2 \eta
+ 2 \nabla^2 \eta_z + \frac{2}{3} W \nabla^4 \eta + \left[ \frac{23}{15} R - 2 \cot \theta \right] (\eta \eta_z)_z - \frac{5}{14} R \cot \theta \nabla^2 \eta_z + \frac{2}{7} R^2 \eta_{zzz} - \frac{6}{5} \cot \theta \nabla^4 \eta

+ \frac{331}{168} R^2 \eta_{zzz} + \frac{1241483}{8108100} R^2 \eta_{zzzz} - \frac{477523}{2494800} R^2 \cot \theta \nabla^2 \eta_{zz} = 0.
\]

(B39)

In this equation, we do not shown the linear terms with derivatives of order greater than four. Also, we have not shown the nonlinear terms with the derivatives of odd orders, of the type \( \eta_z \eta_{yy}, \eta_y \eta_{zy}, (\eta \eta_{zz})_z, \) etc. Using a procedure identical to the one of Refs. 28,29 (see also Ref. 30), it can be shown that the nonlinear terms with odd-order derivatives are dispersive in nature, (i.e. they do not give rise to any change in the amplitude). In comparison with the linear dispersive term \( \nabla^2 \eta_z \), the nonlinear dispersive terms are small.

Similarly, nonlinear terms with even-order derivatives can be shown to be dissipative. Among the nonlinear dissipative terms, the terms involving \( W \) are of the type \( W \eta \nabla^4 \eta, W \eta_y \nabla^2 \eta_y, \) etc. These terms are smaller than the linear dissipative term \( \sim W \nabla^4 \eta, \cot \theta \eta_{yy}, \) and \( \delta \eta_{zz} \) by the factor \( A \). However, the nonlinear dissipative term \( R(\eta \eta_z)_z \) may be \( \sim \delta \eta_{zz} \) when \( \delta \ll R \).

**APPENDIX C: GENERAL ANALYSIS OF ITERATION PROCEDURE**

We want to analyze the general \( n \)th step of the iteration procedure. Since, as was pointed out in the main text, this is of interest mainly for the near-critical conditions, \( R \approx R_c = (5/4) \cot \theta \), where the flow is essentially 2D, \( Y \gg Z \), for simplicity we will consider from the outset a 2D flow \( (\partial_y = v = 0) \) with \( \cot \theta = 0 \).
From the preceding iteration steps, the corrections are known up to (including) \( u_{n-1} \), \( w_{n-1} \), and \( p_{n-2} \). We substitute into the exact NS problem the expansion

\[
w = w_N + w_0 + w_1 + \cdots + w_{n-1} + \tilde{w}_n
\]

where \( \tilde{w}_n \) is an unknown correction and all the other members of the sum are known from the preceding iteration steps, and similar expansions for \( u \) and \( p \). The exact equations and boundary conditions for \( \tilde{p}_{n-1} \) are

\[
R\tilde{p}_{(n-1)x} = -\sum_{k=0}^{n-2} Rp_{kx} + \sum_{k=0}^{n-1} [u_{kxx} + u_{kzz} - Ru_k - R(w_N - 2)u_k] - R \left( \sum_{k=1}^{n-1} u_k \right) \left( \sum_{k=1}^{n-1} u_{kx} \right)
\]

\[- R \left( \sum_{k=1}^{n-1} w_k \right) \left( \sum_{k=1}^{n-1} u_{kz} \right) + \text{terms containing } \tilde{w}_n \text{ or } \tilde{u}_n \tag{C1} \]

and

\[
R\tilde{p}_{n-1} = -Rp_N - \sum_{k=0}^{n-2} Rp_k - 2 \left( 1 - \eta_z^2 + \eta_z^4 + \cdots + (-1)^p \eta_z^{2p} \right) \sum_{k=0}^{n-1} u_{kx}
\]

\[+ 2 \left( \eta_z - \eta_z^3 + \eta_z^5 - \cdots \right) \sum_{k=0}^{n-1} (u_{kz} + w_{kx}) + 2 \left( \eta_z - \eta_z^3 + \eta_z^5 - \cdots \right) w_N \]

\[- 2 \left( \eta_z^2 - \eta_z^4 + \eta_z^6 - \cdots \right) \sum_{k=0}^{n-1} u_{kx} + \text{terms containing } \tilde{w}_n \text{ or } \tilde{u}_n \quad (x = h), \tag{C2} \]

and the corresponding equations for the approximate \( p_{n-1} \) are written as

\[
Rp_{(n-1)x} = u_{(n-1)xx} + u_{(n-2)zz} - Ru_{(n-2)t} - R(w_N - 2)u_{(n-2)z}
\]

\[- R\sum_{k=0}^{n-3} \left[ u_k u_{(n-3-k)x} + w_k u_{(n-3-k)z} \right] \tag{C3} \]

and

\[
 Rp_{n-1} = -Rp_N \eta^n - \sum_{k=0}^{n-2} Rp_k \frac{\eta^{n-k-1}}{(n-k-1)!}
\]

\[- \sum_{k=0}^{[\frac{(n-k-1)/3}]} \left[ 2(-1)^p \eta_z^{2p} \sum_{k=0}^{n-3p-1} u_{kx} \frac{\eta^{n-k-3p-1}}{(n-k-3p-1)!} \right] \]
\[
\begin{align*}
&+ \sum_{p=0}^{[n-k-3]/3} \left[ 2(-1)^p \eta_z^{2p+1} \sum_{k=0}^{n-3p-3} u_{kz}^{(n-k-3p-3)} \frac{\eta^{n-k-3p-3}}{(n-k-3p-3)!} \right] \\
&+ \sum_{p=0}^{[n-k-2]/3} \left[ 2(-1)^p \eta_z^{2p+1} \sum_{k=0}^{n-3p-2} w_{kx}^{(n-k-3p-2)} \frac{\eta^{n-k-3p-2}}{(n-k-3p-2)!} \right] \\
&- \sum_{k=0}^{[n-k-4]/3} \left[ 2(-1)^p \eta_z^{2p+2} \sum_{k=0}^{n-3p-4} w_{kz}^{(n-k-3p-4)} \frac{\eta^{n-k-3p-4}}{(n-k-3p-4)!} \right] \\
&+ 2 \left( \eta_z - \eta_z^3 + \eta_z^5 - \cdots \right) w_{Nx} \quad (x = 1) \tag{C4}
\end{align*}
\]

where the notation \([\cdots]\) above the summation sign \(\sum\) indicates that only the integer part of the expression inside the brackets needs to be considered.

After the solution \(p_{n-1}\) has been found, the exact equations

\[
\tilde{w}_{nx} = \sum_{k=0}^{n-1} \left[ -w_{kxx} - w_{kzz} + R(w_N - 2)w_{kz} + Rw_{kt} \right. \\
\left. + Ru_kw_{N}w + Rp_{kz} \right]
\]

\[
+ R \left( \sum_{k=0}^{n-1} u_k \right) \left( \sum_{k=0}^{n-1} w_{kx} \right) + R \left( \sum_{k=0}^{n-1} w_k \right) \left( \sum_{k=0}^{n-1} w_{kz} \right) \\
+ \text{[terms containing \(\tilde{w}_n, \tilde{u}_n, \text{or} \, \tilde{p}_n\)]} \tag{C5}
\]

\[
\tilde{w}_{nx} = -w_{Nx} - \sum_{k=0}^{n-1} (w_{kx} + u_{kz}) \\
+ 2 \sum_{k=0}^{n-1} \left( w_{kz} \eta_z - u_{kx} \eta_z \right) \left( 1 - \eta_z^2 \right)^{-1} \\
+ \text{[terms containing \(\tilde{w}_n, \tilde{u}_n, \tilde{v}_n \text{or} \, \tilde{p}_n\)]} \quad (x = h) \tag{C6}
\]

and
\[ \tilde{w}_n = 0 \quad (x = 0) \quad (C7) \]

lead to the system for \( w_n \),

\[
\begin{align*}
  w_{nxx} &= -w_{(n-1)zz} + R(w_N - 2)w_{(n-1)z} + Rw_{(n-1)t} \\
  + Ru_{(n-1)}w_N + Rp_{(n-1)}z + R\sum_{k=0}^{n-2} u_k w_{(n-2-k)x} \\
  + w_k w_{(n-2-k)z},
\end{align*}
\]

\[ w_{nx} = -w_{N_x} \frac{\eta^{n+1}}{(n+1)!} - \sum_{k=0}^{n-1} \left( w_{kx} \frac{\eta^{n-k}}{(n-k)!} + u_{kz} \frac{\eta^{n-k-1}}{(n-k-1)!} \right) + \sum_{p=0}^{[(n-k-2)/3]} 2(-1)^p \eta^{2p+1} \\
\times \sum_{k=0}^{n-3p-2} \left[ \left( w_{kz} - u_{kx} \right) \frac{\eta^{n-k-3p-2}}{(n-k-3p-2)!} \right] (x = 1), \quad (C8)
\]

and

\[ w_n = 0 \quad (x = 0). \quad (C10) \]

Finally, after the solution \( w_n \) is known, the system

\[ \tilde{u}_{nx} = -\sum_{k=0}^{n-1} (u_{kx} + w_{kz}) - w_{nz} - \tilde{w}_{(n+1)z} \]

\[ (C11) \]

and

\[ \tilde{u}_n = 0 \quad (x = 0) \quad (C12) \]

leads to the problem for \( u_n \),

\[ u_{nx} = -w_{nz} \quad (C13) \]

and

\[ u_n = 0 \quad (x = 0), \quad (C14) \]
solution of which completes the \( n \)th iteration step. In view of the exact equations for \( \tilde{p}_{n-1} \), \( \tilde{w}_n \), and \( \tilde{u}_n \), it is easy to justify by mathematical induction the equations for the approximates \( p_{n-1}, w_n, \) and which are the principal objects of our consideration here.

Since each of these problems is a linear (nonhomogeneous) ODE with linear boundary condition, its solution is the sum of contributions from each term of the RHS of the equation (which contribution is the solution of the ODE with all other RHS terms put to zero and with homogeneous BC) and of each term of the free-surface BC (the latter contribution is the solution of the homogeneous ODE with the BC truncated to that single term). With this, it can be seen, also inductively, that the order of magnitude of each term of \( w_n \) is the product of certain powers of our four independent small parameters—\( A, 1/L^2, R/L, \) and \( R/T \)—with the sum of the exponents being \( (n + 1) \), the same for all terms of \( u_n \) (we will say that the term has the “order of smallness” \( n + 1 \)); and, moreover, all possible types of terms of order \( (n + 1) \) are present in general, except for the terms of the type (i.e. order of magnitude) \( A^{n+1} \). Essentially the same orders of magnitude are found in \( u_n \), since, from (C13), it is the sum of contributions of each term of \( w_n \) (differentiated in \( z \), which gives the additional factor \( 1/L \) in the order of magnitude, and integrated in \( x \), which does not affect the order of magnitude at all). Similarly, \( Rp_{n-1} \) is made of all possible terms of the order \( n \) times the factor \( 1/L \) (in this case, without any exceptions, since the terms of type \( A^n/L \) are present). One can note that the first three terms of the RHS (C8) give contributions which are smaller than \( w_{n-1} \) by the factors \( 1/L^2, R/L, \) and \( R/T \), respectively, while the BC term \( w_{(n-1)xx}\eta \) contributes terms which are smaller by the factor \( A \) (including the terms whose types contain \( A^n \), such as \( A^nR/L \)). A similar role for \( Rp_{n-1} \) is played by the BC term \( Rp_{n-2}\eta \).

With the above order-of-magnitude structure of \( u_n \) being known, it is easy to see that each term of it is much smaller than some term of \( u_{n-1} \), by a small parameter \( 1/L^2, R/L, R/T, \) or \( A \). [It may seem at first sight that terms of type, say, \( A^n(R/L) \) present a difficulty: there is no \( A^n \) in \( u_{n-1} \) (we omit the common \( 1/L \)). However, \( A^n(R/L) = A[A^{n-1}(R/L)] \), and the type \( A^{n-1}(R/L) \) is present in \( u_{n-1} \).] It follows that
Thus, the iterative corrections of \( u \) are monotonously decreasing. (The same can be seen for the sequences \( \{ w_n \} \) and \( \{ p_n \} \).)

All \( u_n \) are substituted into the kinematic equation, and the time-derivatives are eliminated by iterating the EE (similar to sec. \( \ssection{V} \)). Then we have, in addition to the terms of the GEE (82) (which are of order two), also linear and nonlinear terms of order three and more (not counting the factor \( 1/L \)). Of these, the nonlinear dissipative (i.e. with the total number of \( z \)-differentiations being even) terms are of a type containing \( A \) to the power at least two, and containing \( R/L \) to the power not less than one (since there are at least two differentiations). Thus, the \( O_M \)-type of each of these terms contains all the type-factors of the (nondegenerate) nonlinear dissipative term \( \propto R(\eta^2)_{zz} \), \( \sim (1/L)A^2(R/L) \), and, except for this principal nonlinear dissipative term, also some additional small factor, the product of powers of the small parameters \( A \), \( 1/L^2 \), and \( R/L \), which makes that term negligible. Therefore, we do not have to retain any nonlinear dissipative term other than the principal one (which is of order three). Similarly, of all the dispersive terms (both linear and nonlinear), only the (nondegenerate, second-order) term \( \propto \eta_{zz} \) is retained.

As we noted in the main text, all the linear dissipative terms could be similarly neglected as compared with the linear second-derivative one, but the latter is degenerate if \( \delta \ll R \). Therefore, we should consider these linear terms. If the type of a linear term does not contain a factor which is a power of \( R/L \), that term is just a \( (2n+1) \)-derivative one, clearly a dispersive term. Thus, the dissipative linear terms we are interested in must have the \((lth-derivative)\) form \( R^k \eta^{(l)} \), which leads immediately to Eq. (101).
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FIGURES

FIG. 1. Evolution of energy illustrating that the solutions of Eq. (96) remain bounded.

FIG. 2. Evolution of energy indicates that the solutions of Eq. (104) blow up.
FIG. 1  A. L. Frenkel
FIG. 2    A. L. Frenkel