On Vector ARMA Models Consistent with a Finite Matrix Covariance Sequence

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Abstract

We formulate the so called “VARMA covariance matching problem” and demonstrate the existence of a solution using the degree theory from differential topology.

I. INTRODUCTION

The vector autoregressive moving-average (VARMA) model is a general finitely parametrized stationary linear model for an \( m \)-dimensional stochastic process. Given a finite number of matrix covariances, it is an important problem to construct VARMA models that are consistent with the covariance data. The scalar version of the problem goes back to the famous rational covariance extension which has been extensively discussed in the literature with applications in signal processing, identification and control; cf. e.g., [1], [3]–[9], [12]–[16], [18]–[20] and references therein. It aims to find an infinite extension of a finite covariance sequence such that the resulted spectral density is rational and satisfies a complexity constraint. The major approach today is recasting the problem in the context of the optimization-based theory of moment problems with rational measures developed in [4], [5], [7]–[9], [18], [19]. The framework of optimization has later been extended to the circulant variation of the problem [10], [21]–[23], [25] with applications to texture image modeling [11], [26]. One feature that distinguishes this methodology from classical methods such as maximum likelihood identification of ARMA models is that: when the numerator Laurent polynomial (with both positive and negative powers) of the spectral density is fixed, the solution is unique and depends continuously on the data (well-posedness in the sense of Hadamard).

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Various attempts have been made to generalize the methodology of optimization to multidimensional spectral estimation, cf. [36]–[39] for treatments under a more general setting. For the problem of matrix-valued covariance extension as reported in e.g., [21], [23], [25], however, the solution has a very special structure, in that the numerator polynomial is scalar instead of being matrix-valued, resulting a much smaller model class. In this note, we try to cast off such restriction by starting directly from the vector ARMA model. This also connects with some recent developments of the theory that are not based on optimization [30], [31], [35], where the problem is formulated as finding a spectral density in a parameterized family which satisfies the (generalized) moment constraints, cf. also [32].

We follow the idea of fixing the MA part as was done in [12], [13], [15], [16] to study the corresponding scalar problem. The difficulty here is that, the generalized entropy criterion for scalar problems does not extend naturally to the vector case with a generic matrix MA polynomial. Hence, after the formulation of the VARMA covariance matching problem in the next section, the first question is whether or not a solution exists, and this is the main concern of the present note. We show that a solution exists using the topological degree theory. Application of the degree theory to rational covariance extension was first made by Georgiou [15]–[17] to show existence of a solution and later developed by Byrnes et al. [1] to prove the uniqueness (and much more). These theories predate the optimization framework.

The outline of the paper is as follows. In Section [II] we provide some background material, formulate the covariance matching problem, and state the main theorem. Section [III] is devoted to the existence proof. An equivalent algebraic problem is formulated which allows the application of the degree theory. At last, we pose an open question on the uniqueness of the solution and give a possible direction towards it.

**Symbols**

Throughout this note, $E$ denotes mathematical expectation and $\mathbb{Z}$ the set of integer numbers. The open complex unit disk $\{z : |z| < 1\}$ will be denoted $\mathbb{D}$ and the unit circle will be $\mathbb{T} \equiv \partial \mathbb{D}$, where $\partial$ stands for the boundary. Boldface symbols are used for block matrices. In our setting, to match Fourier-domain notations, polynomials will be written as functions of the indeterminate $z^{-1}$. In particular, the definition of Schur polynomials and their matrix counterparts, which normally can only have zeros inside $\mathbb{D}$, has been slightly modified to accommodate this convention.
II. VARMA COVARIANCE MATCHING

Consider the following forward unilaterial ARMA model describing an \( m \)-dimensional stationary process \( \{y(t)\} \)
\[
\sum_{k=0}^{n} A_k y(t - k) = \sum_{k=0}^{n} B_k w(t - k), \quad t \in \mathbb{Z},
\] (1)
where \( \{A_k, B_k \in \mathbb{R}^{m \times m}\} \) are matrix parameters and \( \{w(t)\} \) is an \( m \)-dimensional white noise with variance \( \mathbb{E}[w(t)w(t)^\top] = I_m \). Define two matrix polynomials
\[
A(z) := \sum_{k=0}^{n} A_k z^{-k}, \quad B(z) := \sum_{k=0}^{n} B_k z^{-k}.
\] (2)
We shall require that
\[
\det A(z) \neq 0, \quad \det B(z) \neq 0, \quad \forall z \in \mathbb{T}.
\] (3)
We also need a proper choice of matrix Schur polynomials. More precisely, the set \( \mathcal{S}_{m,n} \) contains matrix polynomials
\[
M(z) = \sum_{k=0}^{n} M_k z^{-k}, \quad M_k \in \mathbb{R}^{m \times m}
\] (4)
such that
\[ \cdot \quad M_0 \text{ is lower triangular with positive diagonal elements;} \]
\[ \cdot \quad \det M(z) = 0 \text{ implies } z \in \mathbb{D}; \]
\[ \cdot \quad \int_{-\pi}^{\pi} \text{tr} [M(e^{i\theta})M(e^{i\theta})^\top] \frac{d\theta}{2\pi} = \text{tr}(MM^\top) < \mu. \] (5)
Here the matrix \( M = [M_0, \ldots, M_n] \) and \( \mu \) is a fixed positive constant which can be chosen arbitrarily large. This last restriction (5) is a technical condition to ensure boundedness of the set. Indeed, it is easy to see that the set \( \mathcal{S}_{m,n} \) is open and bounded if identified as a subset of the Euclidean space of dimension \( N := \frac{1}{2}m(m + 1) + m^2 n \). Openness follows from the continuous dependences of the determinant on matrix elements, and roots of a polynomial on its coefficients.

Remark 1. Those constraints on \( \mathcal{S}_{m,n} \) are necessary for the degree theoretic argument in the later development. Condition (5) is not need in the scalar case if we introduce some normalization, e.g., by enforcing the constant term of the polynomial equal to 1. More precisely, if a scalar polynomial \( p(z) = 1 + \sum_{k=1}^{n} p_k z^{-k} \) has all its roots in \( \mathbb{D} \), then the vector \( [p_1 \ldots p_n] \) must
have bounded norm, which can be seen after performing a factorization. However, the example below shows that this is not the case for matrix polynomials:

\[ M(z) = \begin{bmatrix} 1 + \frac{1}{2}z^{-1} & \kappa z^{-1} \\ 0 & 1 \end{bmatrix}, \]

since it is evident that the only root of \( \det M(z) = 0 \) is \( z = -1/2 \) but the number \( \kappa \) can tend to infinity. For this reason, we have to impose (5) in order to have boundedness.

A closely related concept is the set of Hermitian matrix pseudo-polynomials of order \( n \)

\[ P(z) = \sum_{k=-n}^{n} P_k z^{-k}, \quad P_{-k} = P_k^\top \in \mathbb{R}^{m \times m}, \tag{6} \]

which will be denoted \( \mathcal{P}_{m,n}^+ \) and the subset that contains positive ones on the unit circle

\[ \mathcal{P}_{m,n}^+ := \{ P(z) \in \mathcal{P}_{m,n} : P(z) > 0, \forall z \in \mathbb{T} \}. \tag{7} \]

The set \( \mathcal{P}_{m,n}^+ \) can also be viewed as a subset of \( \mathbb{R}^N \) by identifying polynomials with their matrix coefficients. It is now a classical fact that a strictly positive matrix polynomial \( P(z) \in \mathcal{P}_{m,n}^+ \) has a unique outer spectral factor that satisfies the first two constraints of \( \mathcal{S}_{m,n} \) and there are algorithms to do such factorization, e.g., [27].

By standard spectral analysis of stationary processes [24, Chapter 3], the spectral density of \( \{y(t)\} \) is obtained for all \( z \in \mathbb{T} \), as

\[ \Phi(z) = A(z)^{-1}B(z)B(z^{-1})^\top A(z^{-1})^{-\top}. \tag{8} \]

 Apparently, taking \( B(z) \in \mathcal{S}_{m,n} \) entails no loss of generality and we shall do so in the following. Now, consider the next problem.

**Problem 1 (VARMA Covariance Matching).** Suppose we are given an MA polynomial \( B(z) \in \mathcal{S}_{m,n} \) and \( n + 1 \) real \( m \times m \) matrices \( C_0, C_1, \ldots, C_n \), such that the block-Toeplitz matrix

\[
\begin{bmatrix}
C_0 & C_1 & C_2 & \cdots & C_n \\
C_1^\top & C_0 & C_1 & \cdots & C_{n-1} \\
C_2^\top & C_1^\top & C_0 & \cdots & C_{n-2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
C_n^\top & C_{n-1}^\top & C_{n-2}^\top & \cdots & C_0
\end{bmatrix}, \quad n \in \mathbb{Z}_+ \tag{9}
\]

is positive definite. Determine the AR polynomial \( A(z) \in \mathcal{S}_{m,n} \) such that the first \( n+1 \) covariance matrices of the process \( \{y(t)\} \) described by the VARMA model (1) match the sequence \( \{C_k\} \).
Remark 2. When the MA part is trivial, i.e., $B_0 = I$, $B_1 = \cdots = B_n = 0$, the problem reduces to the standard vector AR modeling subjected to a finite number of matrix covariances. In this case, the problem is linear, cf. [23] for an algorithm that generalizes the Levinson-Durbin iteration.

An equivalent formulation of Problem 1 can be stated in the language of the trigonometric moment problem. In fact, we are looking for a spectral density $\Phi(z)$ of the form (8) with the middle part $B(z)B(z^{-1})^\top$ fixed, whose Fourier coefficients satisfy the constraints

$$
\int_{-\pi}^{\pi} e^{ik\theta} \Phi(e^{i\theta}) \frac{d\theta}{2\pi} = C_k, \quad k = 0, 1, \ldots, n.
$$

A special case of this problem has been considered in [23, Section 9], where a restriction is made by taking $B(z) = b(z)I$ as a scalar polynomial times the identity matrix. However, as reported in [39], there is a difficulty in extending the methodology of [23] to accommodate a generic matrix polynomial $B(z)$.

Without loss of generality, we can let $C_0 = I$. In fact, given $T_n > 0$, we can transform the data as $\tilde{C}_k := L_{C_0}^{-1}C_kL_{C_0}^{-\top}$ ($k = 0, \ldots, n$), where the notation $L_\Sigma$ stands for the Cholesky factor of a matrix $\Sigma > 0$. Obviously, the block-Toeplitz matrix $\tilde{T}_n$ composed of the transformed data is again positive definite. If one can find a matrix polynomial $\tilde{A}(z)$ that solves Problem 1 for the data $\{\tilde{C}_k\}$, then the solution for the original data $\{C_k\}$ can be recovered as $A(z) = \tilde{A}(z)L_{C_0}^{-1}$.

According to the general theory [33], a measure $dF(e^{i\theta})$ (with $F(e^{i\theta})$ the corresponding spectral distribution) solving (10) exists if $T_n \geq 0$ and there are infinitely many if $T_n > 0$. When a structure of the spectral density is imposed as (8), however, it is not trivial that Problem 1 in fact admits a solution. We report our main result of this note in the next theorem that confirms the existence of a solution. The proof is deferred to the next section.

**Theorem 1.** Given the block Toeplitz matrix $T_n > 0$, let $\lambda_{\min} > 0$ be its smallest eigenvalue. Then for any fixed $B(z) \in \mathbb{S}_{m,n}$ such that $P(z) := B(z)B(z^{-1})^\top$ satisfies

$$
\det P_n \neq 0 \text{ and } \tr P_0 < \min\{1, \lambda_{\min}\} \mu,
$$

there exists a matrix polynomial $A(z) \in \mathbb{S}_{m,n}$ such that the first $n+1$ covariance matrices of the VARMA model (1) defined by the coefficients of $A(z), B(z)$ match the given data $C_0, \ldots, C_n$.

**III. EXISTENCE OF A SOLUTION**

The proof of Theorem 1 relies on the construction of a certain map and the use of degree theory. We first set up some notations. Since we almost exclusively work on the unit circle,
the usual complex variable $z$ will be understood as taking values in $\mathbb{T}$ when there is no risk of confusion. The spectral density of a second-order stationary process is defined by the Fourier transform of its covariance sequence

$$\Phi(z) := \sum_{k=-\infty}^{\infty} \Sigma_k z^{-k}, \quad \Sigma_k = \Sigma_k^\top \in \mathbb{R}^{m \times m}. \quad (12)$$

Define the half series

$$F(z) := \frac{1}{2} \Sigma_0 + \sum_{k=1}^{\infty} \Sigma_k z^{-k}, \quad (13)$$

and we would have

$$\Phi(z) = F(z) + F(z^{-1})^\top \geq 0, \quad \forall z \in \mathbb{T}, \quad (14)$$

which corresponds to the notion of positive realness in the scalar case.

The following procedure is inspired by [16, Section VI]. Given covariance data organized as a block row vector

$$\mathbf{C} = \begin{bmatrix} C_0 & C_1 & \cdots & C_n \end{bmatrix}, \quad (15)$$

we define a map

$$f_{\mathbf{C}} : \mathcal{S}_{m,n} \to \mathcal{P}_{m,n}$$

$$A(z) \mapsto P(z)$$

as follows. Compute the polynomial $H(z) := \sum_{k=0}^{n} H_k z^{-k}$ of order $n$ by polynomial multiplication and truncation

$$H(z) := \left[ A(z)(C_0 + 2C_1 z^{-1} + \cdots + 2C_n z^{-n}) \right]_0^{n}, \quad (16)$$

where the operation $\left[ \cdot \right]_0^{-n}$ means retaining only the terms with powers from $-n$ up to 0. This is equivalent to the linear equation for the coefficients of $H(z)$

$$\mathbf{H} = \mathbf{A} \mathbf{U}_\mathbf{C}, \quad (17)$$

where,

$$\mathbf{H} = \begin{bmatrix} H_0 & H_1 & \cdots & H_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} A_0 & \cdots & A_n \end{bmatrix},$$

and

$$\mathbf{U}_\mathbf{C} = \begin{bmatrix} C_0 & 2C_1 & \cdots & 2C_n \\ 0 & C_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 2C_1 \\ 0 & 0 & \cdots & C_0 \end{bmatrix}.$$
Then we proceed to define
\[ P(z) := \frac{1}{2}[H(z)A(z^{-1})^T + A(z)H(z^{-1})^T]. \] (18)

In particular, we notice the matrix coefficients
\[ P_0 = AT_nA^\top, \quad P_n = \frac{1}{2}(H_n + A_nC_0)A_0^\top. \] (19)

In order to explain why we define such a map \( f_C \), let us formulate an algebraic problem.

**Problem 2.** Given \( B(z) \in \mathbb{S}_{m,n} \), let \( P(z) = B(z)B(z^{-1})^\top \). Find a solution \( A(z) \in \mathbb{S}_{m,n} \) to the system of quadratic equations
\[ f_C(A(z)) = P(z). \] (20)

Despite apparent difference, Problems 1 and 2 are in fact equivalent, i.e., they have the same solution set, which will be shown in the next proposition.

**Proposition 1.** A solution (if it exists) to Problem 2 solves Problem 1 and vice versa.

**Proof.** (First half of the statement) Let \( A(z) \) be a solution to (20), compute \( H(z) \) as in (16), then define \( F(z) := \frac{1}{2}A(z)^{-1}H(z) \) and consider its power series expansion. Since we have chosen \( A(z) \in \mathbb{S}_{m,n} \), the expansion must contain only the terms with non-positive powers exactly of form (13). More precisely, the first \( n + 1 \) coefficients of the expansion satisfy the relation
\[ H_{col} = L_A \Sigma_{col}, \] (21)

where,
\[
H_{col} = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_n \end{bmatrix}, \quad \Sigma_{col} = \begin{bmatrix} \Sigma_0 \\ \Sigma_1 \\ \vdots \\ \Sigma_n \end{bmatrix},
\]

and
\[
L_A = \begin{bmatrix} A_0 \\ A_1 & 2A_0 \\ \vdots & \vdots & \ddots \\ A_n & 2A_{n-1} & \cdots & 2A_0 \end{bmatrix}.
\]
If $\Sigma_k$ is replaced with $C_k$, the above relation (21) is equivalent to (17). By the definition of $\hat{S}_{m,n}$, $A_0$ is nonsingular, which means $L_A$ is invertible. From here we can conclude that the first $n + 1$ Fourier coefficients of $F(z)$ must match the given data, that is,

$$\Sigma_k = C_k, \quad k = 0, 1, \ldots, n.$$  

It is simple algebra to compute

$$\Phi(z) := F(z) + F(z^{-1})^\top = A(z)^{-1}f_C(A(z))A(z^{-1})^{-\top}. \quad (22)$$

which is a bona fide spectral density (8) since $A(z)$ solves (20), and thus the covariance matching problem would be solved.

(Vice versa) Assume $C_0 = I$. Given $A(z)$ that solves Problem 1, consider the matrix equation

$$H(z)A(z^{-1})^\top + A(z)H(z^{-1})^\top = 2P(z) \quad (23)$$

in the unknown $H(z)$, where $P(z) = B(z)B(z^{-1})^\top$ is also given. According to [40, Theorem MP3], (23) has a unique solution $\hat{H}(z)$ with $\hat{H}_0$ being lower-triangular. Similar to the previous argument, define $\hat{F}(z) := \frac{1}{2}A(z)^{-1}\hat{H}(z) = \frac{1}{2}\hat{\Sigma}_0 + \sum_{k=1}^{\infty} \hat{\Sigma}_k z^{-k}$ and form the corresponding spectral density (22). Since the first $n + 1$ covariances are matched, we have $\hat{\Sigma}_k = C_k$ for $k = 1, \ldots, n$. For $k = 0$, we have

$$\hat{\Sigma}_0 + \hat{\Sigma}_0^\top = 2I,$$

whose general solution is $\hat{\Sigma}_0 = I + Q$, where $Q$ is an arbitrary skew-symmetric matrix. Since (21) also holds, $\hat{\Sigma}_0 = A_0^{-1}\hat{H}_0$ is again lower-triangular, and so is $Q$. Therefore, we must have $Q = 0$ and $\hat{\Sigma}_0 = I$. From here, we recover the relations (17) and (16) for $\hat{H}(z)$, and see that $A(z)$ is a solution to the quadratic equation (20).

\[ \square \]

We can now focus on the equivalent Problem 2 and study the solution set of (20). The degree theory is a powerful tool to prove existence of a solution to a set of nonlinear equations. Some basic facts of the theory are reviewed next for our purpose, cf. [29] for reference.

A. Review of the degree theory in Euclidean spaces

Assume $D \subset \mathbb{R}^n$ is bounded open and $f : \overline{D} \to \mathbb{R}^n$ is smooth (in $C^\infty$). Our major concern is solvability of the equation

$$f(x) = y. \quad (24)$$
We call $y \in \mathbb{R}^n$ a regular value of $f$ if either

(i) for any $x \in f^{-1}(y)$, $\det f'(x) \neq 0$ or

(ii) $f^{-1}(y)$ is empty.

Here $f'(x)$ denotes the Jacobian matrix of $f$ evaluated at $x$. Let $y$ be a regular value of type (i) and $y \notin f(\partial D)$, we define the degree of $f$ at $y$ as

$$\deg(f, y, D) := \sum_{f(x) = y} \text{sign} \det f'(x),$$

(25)

where the sign function

$$\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 \\
\text{not defined at } 0.
\end{cases}$$

and not defined at 0. Following the lines in [29, pp. 61], one can show that the set

$$\{x \in D : f(x) = y\}$$

is of finite cardinality and hence the sum above is well defined. For regular values of type (ii), we set $\deg(f, y, D) = 0$. Moreover, the set of regular values is dense in $\mathbb{R}^n$ by Sard’s theorem.

When $y$ is not a regular value (also called a critical value), the degree can also be defined by means of limit. Further properties of the degree related to our problem are listed below:

- If $y_1$ and $y_2$ belong to the same component of $\mathbb{R}^n - f(\partial D)$, then $\deg(f, y_1, D) = \deg(f, y_2, D)$.
- If $\deg(f, y, D) \neq 0$, there exists $x \in D$ such that $f(x) = y$.
- Homotopy invariance. If $F : \overline{D} \times [a, b] \to \mathbb{R}^n$ is continuous and if $F(x, t) \neq y$ for $x \in \partial D$ and $t \in [a, b]$, then $\deg(F_t, y, D)$ is defined and independent of $t \in [a, b]$. Here $F_t$ is the map of $D$ into $\mathbb{R}^n$ defined by $F_t(x) = F(x, t)$.

### B. Proof of existence

Define the subset

$$\mathcal{D}_{m,n}^+ := \{P(z) \in \mathcal{P}_{m,n}^+ \text{ that satisfies } (\Pi)\}.$$  

(26)

Our target is to show for any $P(z) \in \mathcal{D}_{m,n}^+$, there exists $A(z) \in \mathcal{S}_{m,n}$ such that (20) holds. Given the second listed property of the degree, it is sufficient to show that $\deg(f_C, P(z), \mathcal{S}_{m,n}) \neq 0$ for $P(z) \in \mathcal{D}_{m,n}^+$. The homotopy invariance property will be useful to simplify the computation of the degree. Consider the data $O = [I_m, 0, \ldots, 0]$, and the corresponding map becomes

$$P(z) = f_O(A(z)) := A(z)A(z^{-1})^\top,$$

(27)
and the problem is reduced to the matrix spectral factorization.

**Lemma 1.** Define $\Sigma(t) := tC + (1-t)O$, $t \in [0,1]$, and the corresponding function $f_{\Sigma(t)}$ in the same way as $f_C$. Then the function

$$f : \mathcal{S}_{m,n} \times [0,1] \to \mathcal{P}_{m,n}$$

$$(A(z), t) \mapsto f_{\Sigma(t)}(A(z))$$

is a smooth homotopy between $f_C$ and $f_O$.

*Proof.* One can easily observe that $f(\cdot, t) = tf_C + (1-t)f_O$. Then smoothness comes from the fact that the function $f_C$ is quadratic.

$$\square$$

**Lemma 2.** For any $P(z) \in \mathcal{D}^+_{m,n}$,

$$|\deg (f_O, P(z), \mathcal{S}_{m,n})| = 1.$$  \hspace{1cm} (29)

*Proof.* We just compute the degree of $f_O$ at $P(z) \in \mathcal{D}^+_{m,n}$ using the definition \[25\]. By the matrix version of the Fejér-Riesz theorem \[34\], a solution to \[27\] exists. Moreover, such a spectral factor $A(z)$ is unique in $\mathcal{S}_{m,n}$. One can easily see that the Jacobian of $f_O$ is the linear map

$$f'_O (A(z)) : \mathbb{R}^N \to \mathbb{R}^N$$

$$V(z) \mapsto A(z)V(z^{-1})^\top + V(z)A(z^{-1})^\top.$$  \hspace{1cm} (30)

An important fact is that $f'_O$ evaluated at a matrix Schur polynomial is always nonsingular, which is a simple consequence of Theorem MP1 in \[40\]. Hence every $P(z) \in \mathcal{D}^+_{m,n}$ is a regular value of $f_O$ and we have

$$\deg (f_O, P(z), \mathcal{S}_{m,n}) = \text{sign} \det f'_O(A(z)).$$

Clearly, this implies (29).

$$\square$$

We are now ready to prove Theorem 1.

**Proof of Theorem 1.** By Lemma 1 the function $f_C$ is (smoothly) homotopic to $f_O$. In order to use the homotopy invariance of the degree, we need to ensure that

$$f_{\Sigma(t)}(A(z)) \neq P(z) \text{ for any } A(z) \in \partial \mathcal{S}_{m,n} \text{ and } t \in [0,1].$$

By the definition of $\mathcal{S}_{m,n}$, there are three boundaries. The first one is to let $A_0$ have zero diagonal elements, but by (19) this would result a singular $P_n$ which means $P(z) \notin \mathcal{D}^+_{m,n}$ by definition.
The second case is to let \( A(e^{i\theta}) \) be singular for some \( \theta \), i.e., there exists a nonzero vector \( v \in \mathbb{C}^m \) such that \( A(e^{i\theta})^*v = 0 \). Then by definition of \( f_{\Sigma(t)} \) we have
\[
v^*P(e^{i\theta})v = \frac{1}{2}v^*[H(e^{i\theta})A(e^{i\theta})^* + A(e^{i\theta})H(e^{i\theta})^*]v = 0,
\]
which also implies that \( P(z) \notin \mathcal{D}_{m,n}^+ \). The last case is to let \( \text{tr}(AA^*) = \mu \), and this can also be ruled out since by (19) it would imply that
\[
\text{tr} P_0 = \text{tr} \left( A[tT_n + (1 - t)I]A^\top \right)
\geq [t\lambda_{\text{min}} + (1 - t)]\text{tr}(AA^\top)
\geq \min\{1, \lambda_{\text{min}}\} \mu.
\]
In this way, we conclude that for \( P(z) \in \mathcal{D}_{m,n}^+ \),
\[
\deg (f_C, P(z), \mathcal{G}_{m,n}) = \deg (f_O, P(z), \mathcal{G}_{m,n}).
\]
By Lemma 2 the right-hand side is nonzero, which completes the proof.

\[\Box\]

Remark 3. Theorem 1 can be seen as a special case of Corollary 1 in [35], which follows from a more general existence result in that paper. However, the map considered in [35] is an integral while the map \( f_C \) here is algebraic (constructed via polynomial multiplication).

**IV. CONCLUDING REMARKS**

We have shown that for a fixed (but arbitrarily specified in accordance with Theorem 1) MA polynomial, there exists a vector ARMA model that is consistent with the given covariance data.

An open question: Is the solution \( A(z) \) unique in the set \( \mathcal{G}_{m,n} \)? To attack such a problem of uniqueness, the Jacobian of the map \( f_C \) seems to be the key object to investigate. Actually, it is not difficult to obtain an explicit expression for the Jacobian:

\[
f'_C(A(z)) : \mathbb{R}^N \to \mathbb{R}^N
\]
\[
V(z) \mapsto W(z)
\]
with
\[
W(z) := \frac{1}{2} \left[ R(z)A(z^{-1})^\top + H(z)V(z^{-1})^\top \right.
\]
\[
+ V(z)H(z^{-1})^\top + A(z)R(z^{-1})^\top \big],
\]
where
\[
R(z) := \left[ V(z)(C_0 + 2C_1 z^{-1} + \cdots + 2C_n z^{-n}) \right]_{-n}^0,
\]
and $H(z)$ defined in (16). If one could prove that $f_C'$ is always nonsingular at any $A(z) \in \mathcal{G}_{m,n}$, then uniqueness can be achieved following the lines of reasoning in [1], [2]. Moreover, according to [41], Section 2], injectivity is an important step towards possible well-posedness.

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