Prescribed-Time Control for Linear Systems in Canonical Form via Nonlinear Feedback

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Abstract—For systems in canonical form with nonvanishing uncertainties/disturbances, this work presents an approach to full-state regulation within prescribed time irrespective of initial conditions. By introducing the smooth hyperbolic-tangent-like function, a nonlinear and time-varying state-feedback control scheme is constructed, which is further extended to address output-feedback-based prescribed-time regulation by invoking the prescribed-time observer, all are applicable over the entire operational time zone. As an alternative to full-state regulation within the user-assignable time interval, the proposed method analytically bridges the divide between linear and nonlinear feedback-based prescribed-time control and is able to achieve asymptotic stability, exponential stability, and prescribed-time stability with a unified control structure.

Index Terms—Full-state regulation, nonlinear feedback, output feedback, prescribed-time stability.

I. INTRODUCTION

Finite-time convergence is highly desirable in many real-world automation applications, where the ultimate control goals are to be realized within finite time rather than infinite time, e.g., auto parts assembling, spacecraft rendezvous and docking [1], proportional navigation guidance [2], etc. Various approaches to finite-time convergence have been reported in the literature, including finite-time control, fixed-time control, time-synchronized control, predefined-time control, and prescribed-time control. The prototype of the finite-time Lyapunov theory originates from $\dot{V}(x) + kV^\alpha(x) \leq 0$, where $V(x)$ is a positive-definite function, and $\alpha \in (0, 1)$, $k \in \mathbb{R}^+$.

As an effort to achieve finite-time stabilization for high-order systems, the homogeneous method, terminal sliding-mode method, and adding a power integrator method are successively proposed (see [3]–[8]), which greatly promote the development of finite-time control theory. Since the convergence (settling) time therein depends on initial conditions and design parameters, the notion of fixed-time control is then introduced in [9] and [10], where the fractional-order plus odd-order feedback is used, leading to different closed-loop system dynamics, so that the upper boundary of the convergence time can be estimated without using initial conditions. However, neither finite-time control nor fixed-time control can actually achieve state regulation within one unified time. The time-synchronized control scheme proposed in [11] and [12], based on the norm-normalized sign function, is shown to be able to achieve output regulation simultaneously for different initial conditions with a unified control law.

To further alleviate the dependence of the settling time on design parameters, a predefined-time approach is exploited to estimate the upper bound of the convergence time by multiplying exponential signals on the basis of fractional power feedback signals in [13]–[15]. Recently, the notion of prescribed-time control is proposed in [16], which allows the user to assign the convergence time at will and irrespective of initial conditions or any other design parameter, thus offers a clear advantage over those that do not. With this concept, three different approaches have been developed, namely, states transformation approach, temporal scale transformation approach, and parametric Lyapunov equation-based approach (e.g., [16]–[21]). Based on the states transformation approach, the distributed consensus control algorithms are studied for multiagent systems in [22]–[24], a tracking algorithm based upon negative feedback is investigated for the multi-input–multi-output uncertain nonlinear system in [25], and a prescribed-time observer-based output-feedback algorithm is elegantly established for linear systems in [26]. Subsequent works further consider more complex systems, such as stochastic nonlinear systems [27], [28] and LTI systems with input delay [29], [30]. In addition, by using temporal scale transformation, a triangularly stable controller is proposed for the perturbed system in [31], a dynamic high-gain feedback algorithm is established for strict-feedback-like systems in [17], and some distributed algorithms are developed for multiagent systems in [32]–[34]. Based upon the parametric Lyapunov equation, a finite-time controller and a prescribed-time controller are studied for linear systems in [1] and [19], and then generalized to nonlinear systems in [35].

Theoretically inclined, prescribed-time control systems, under some generic design conditions, are capable of tolerating large parametric, structural, and parameterizable disturbance uncertainties on the finite-time interval, to ensure desired...
control performance, in addition to system stability. This property comes from a time-varying function which goes to $\infty$ as $t$ tends to the prescribed time. Different from [16], [21]–[24], and [26], where the time-varying function is used to scale the coordinate transformations, this article only introduces the time-varying function into the virtual/actual controller. The advantage of this approach is that a simpler controller results and the control effort are reduced. In addition, to obtain far superior transient performance, we choose a new feedback scheme that the regular feedback signal can be reconstructed into some suitable forms by a nonlinear mechanism with high design degrees of freedom.

Motivated by the above discussions, this article revisits the prescribed-time control of high-order systems via a novel nonlinear feedback approach. The main contributions of this article are as follows.

1) Both full state and partial state feedback controller are designed to achieve state regulation within prescribed time irrespective of initial conditions and any other design parameter.

2) Unlike most existing solutions that usually use the regular (direct) state feedback, this article proposes to use the “reshaped” feedback states through the hyperbolic-tangent-like function, so as to establish a nonlinear and time-varying feedback control strategy capable of addressing asymptotic, exponential, and prescribed-time control uniformly under certain conditions.

3) For high-order systems with nonparametric uncertainties/disturbances, we propose a prescribed-time sliding-mode control scheme, melting attractive stability and robustness features at the transition and steady-state stages.

The remainder of this article is organized as follows. In Section II, we study the hyperbolic-tangent-like function and a novel lemma are presented. In Section III, we study the prescribed-time control for certain LTI systems by using a nonlinear and time-varying feedback, both full state feedback and partial state feedback are considered. Section IV gives an extend prescribed-time control algorithm for uncertain LTI systems. Section V concludes this article.

Notations: $\mathbb{R}$ is the set of non-negative real numbers, $\mathbb{R}^+ = \{x \in \mathbb{R}: x > 0\}$. For nonzero integers $m$ and $n$, let $0_{m \times n}$ be the $(m, n)$-matrix with zero entries, and $J_n = ((0_{n-1}1), I_{n-1})^T$, $0_{n \times (n-1)}$, $\mathcal{L}_n(a) = ((0_{n \times (n-1)}), a)^T$, where $a = (a_1, \ldots, a_n)^T$. $\mathcal{L}_\infty(0, tp)$ denotes $\mathcal{L}_\infty$ on $[0, tp]$. Denote by $K$ the set of class-$K$ functions and denote by $KL$ the set of class $KL$-functions (see [37, Sec. 4.4]). For any vector $x$, we use $x^T$ and $|x|$ to denote its transpose and Euclidean norm, respectively. $\lim_{t \to T} f(t)$ denotes the limit of $f(t)$ as $t \to T$. We denote by $e^{q} = (q = 0, \ldots, n)$ the $q$th derivative of $e$, and denote by $\bullet^q$ the $q$th derivative of $\bullet$.

II. PRELIMINARIES

A. Problem Statement

We restrict our analysis to the following system in canonical form with uncertainties/disturbances:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + D(x, t) \\ y(t) = Cx(t) \end{cases}$$  \hspace{1cm} (1)$$

where $A = J_n + L_n(a)$ is the system matrix, $B = [0, \ldots, 0, 1]^T$, $C = [b_0, b_1, \ldots, b_{n-1}]$ are coefficient vectors, $D = [0, \ldots, 0, d(x, t)]^T$ with $d(x, t) : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ modeling the unknown nonvanishing uncertainties/disturbances of the system, $x(t) = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$ is the vector of system states, and $u(t) \in \mathbb{R}$ is the control input. $(A, B)$ is controllable and $(A, C)$ is observable. The control objective is to design a feedback control $u(t)$ to stabilize (1) within prescribed-time $tp$, i.e., $x(t) \in \mathcal{L}_\infty(0, tp)$ and $\lim_{t \to tp} |x(t)|_\infty = 0$. We are particularly interested in making use of the feedback information $x$ through a nonlinear way to construct the control scheme.

Definition 1 [29]: The origin of the system $\dot{x} = f(x, t)$ is said to be prescribed-time globally asymptotically stable (PT-GAS) if there exist a class $KL$ function $\beta$ and a function $\mu : [0, tp) \to \mathbb{R}^+$ such that $\mu$ tends to infinity as $t$ goes to $tp$ and, $\forall \in [0, tp)$

$$\|x(t)\| \leq \beta(\|x(0)\|, \mu(t))$$

where $tp$ is a time that can be prescribed in the design.

B. Hyperbolic-Tangent-Like Function

Instead of using $x$ directly, we process the feedback information $x$ through the following hyperbolic-tangent-like function $h(x) : (-\infty, +\infty) \to [-1/b, 1/a]$ as:

$$h(x) := \frac{e^{ax} - e^{-bx}}{a e^{ax} + b e^{-bx}}, \quad 0 < b \leq a$$

where $a$ and $b$ are design parameters, which becomes the standard hyperbolic tangent function for $a = b = 1$. Such nonlinear feedback exhibits two salient properties.

Property 1: The function $h(x)$ is $C^\infty$ on $\mathbb{R}$ and $h(x) = 0$ if and only if $x = 0$. Under $0 < b \leq a$, the inequality $0 \leq |x|h(|x|) \leq xh(x)$ holds.

Proof: Define a continuous function $F(x) = xh(x) - |x|h(|x|)$. For $\forall x \geq 0$, we have

$$F(x) = x(e^{ax} - e^{-bx}) - |x|(e^{ax} - e^{-bx}) = 0.$$

For $\forall x < 0$, it follows that:

$$F(x) = x(e^{ax} - e^{-bx}) - x(e^{-ax} - e^{bx}) = x(b - a)(e^{(a+b)x} - e^{-(a+b)x} - 2) \geq 0.$$

Thus, under $0 < b \leq a$, the inequality $F(x) \geq 0$ holds for $\forall x \in (-\infty, +\infty)$, implying that $0 \leq |x|h(|x|) \leq xh(x)$ for $\forall x \in (-\infty, +\infty)$.

Property 2: Function $h(x)$ is strictly monotonically increasing with respect to (w.r.t.) $x$, its upper and lower bounds are $1/a$ and $-1/b$, respectively. By selecting different design parameters $a$ and $b$, various functions can be obtained from $h(x)$. In particular, if choosing $a$ and $b$ small enough, then $h(x) = x$.

Proof: The upper and lower bounds of $h(x)$ are

$$\lim_{x \to +\infty} h(x) = \lim_{x \to +\infty} \frac{e^{ax} - e^{-bx}}{a e^{ax} + b e^{-bx}} = \frac{1}{a}, \quad a > 0,$$

$$\lim_{x \to -\infty} h(x) = \lim_{x \to -\infty} \frac{e^{ax} - e^{-bx}}{a e^{ax} + b e^{-bx}} = \frac{1}{b}, \quad b > 0.$$
When $a$ and $b$ are sufficiently small, for $x \in (-\infty, +\infty)$, by using L'Hôpital's rule, we have
\[
\lim_{b \to 0} \frac{e^{ax} - e^{-bx}}{ae^{ax} + be^{-bx}} = \lim_{b \to 0} \frac{xe^{ax} + xe^{-bx}}{aexe^{ax} + e^{-bx} - bexe^{-bx}} = x
\]
implying that the expanded/compressed signal $h(x)$ reverses back to the regular signal $x$.

Remark 1: The classical finite-time control adopts fractional power of $x$ (i.e., $x^{1/3}$) to expand the feedback signal $x$ on $x \in [-1, 1]$, and compress $x$ on $x \in (-\infty, -1) \cup (-1, +\infty)$; the fixed-time control uses an additional nonlinear damping term (e.g., $x^3$) on the basis of the original feedback signal to expand the feedback signal on $(-\infty, +\infty)$. Consequently, different feedback signals cause different convergence properties, which are mainly reflected in the relationship between settling time and initial conditions. Since the hyperbolic-tangent-like function $h(x)$ can expand or compress the feedback signal with different $a$ and $b$, it provides control design extra flexibility and degree of freedom. In addition, the right-hand side of (2) remains bounded within $(-1/b, 1/a)$ even if $|x|$ grows large, this special property makes $h(x)$ perfectly suitable to function as the core part of the controller, and our motivation for this work partly stems from such appealing features of $h(x)$. Fig. 1 illustrates $h(x)$ with different $a$ and $b$, and the fractional power feedback signals in terms of $x$.

The interesting feature behind this nonlinear feedback is that it includes regular (direct) feedback of $x$ as a special case, and it allows the linear regular feedback control and nonlinear feedback control to be unified through such function, providing a variety of ways to making use of $x$ for control development. By using the properties of the hyperbolic-tangent-like function, we establish the following lemma, which is crucial to our later technical development.

Lemma 1: Consider the functions $\mu(t) = 1/(t_p - t)$ and $h(x)$ be given as in (2). For $t \in [0, t_p)$, if a positive continuously differentiable function $V(t)$ satisfies
\[
\dot{V}(t) \leq -k\mu(t)h(V), \quad k > 1
\]
where $\dot{V} = -k\mu(t)h(V)$ if and only if $V = 0$, then we have $V(t) \leq \beta(V_0, \mu(t))$ and $\beta$ being of class-KL. In particular, it holds
\[
\lim_{t \to t_p} V(t) = 0, \quad \lim_{t \to t_p} \dot{V}(t) = 0.
\]

Proof: Consider the analytical expression of (3)
\[
\dot{V} \leq -k\mu(t)e^{aV} - e^{-bV}.
\]
Let $V_* = e^{aV} - e^{-bV}$, where $V_{0*} = e^{aV(0)} - e^{-bV(0)}$. Then, we have $V_* \geq 0$ and
\[
\dot{V}_* \leq \left( e^{aV} + e^{-bV} \right) - k\mu(t)e^{aV} + e^{-bV} \leq -k\mu V_*
\]
holds for $t \in [0, t_p)$. Hence, we derive $V_* \leq \beta(V_0, \mu(t))$ according to [16, Lemma 1], namely, $V_* \in C_\infty[0, t_p]$ and $\lim_{t \to t_p} V_* = 0$. The same result can be established for $(e^{aV} - e^{-bV})$. It follows from the fact $V$ tends to $+\infty$ “slower” than $(e^{aV} - e^{-bV})$ as $V \to +\infty$ that $V(t) \in C_\infty[0, t_p]$ and $\lim_{t \to t_p} V(t) = 0$. In addition, the inequality (4) can be transformed into the following form:
\[
V \leq -\frac{1}{a} \ln \left( C_1(t_p - t)^k + e^{-bV} \right), \quad t \in [0, t_p]
\]
where $C_1 = (e^{aV_0} - e^{-bV_0})/k^a$ is the integral constant. In fact, one can easily verify the following calculations:
\[
(6) \Rightarrow e^{aV} - e^{-bV} \leq C_1(t_p - t)^k
\]
\[
\Rightarrow \frac{e^{aV} - e^{-bV}}{t_p - t} \leq C_1(t_p - t)^{k-1}
\]
\[
(6.1) \Rightarrow \left( e^{aV} + e^{-bV} \right) \dot{V} \leq -kC_1(t_p - t)^{k-1}
\]
\[
(6.2) \Rightarrow \dot{V} \leq \frac{-kC_1(t_p - t)^{k-1}}{e^{aV} + e^{-bV}} \leq \frac{-k}{e^{aV} + e^{-bV}}
\]

Furthermore, from (6.3), we have $(a + b)\dot{V} \leq 0$ and
\[
\dot{V} \leq \frac{-kC_1(t_p - t)^{k-1}}{e^{aV} + e^{-bV}}, \quad \dot{V}(0) = \frac{-kC_1^{k-1}}{e^{aV_0} + e^{-bV_0}}.
\]

Indeed, notice that $\dot{V}$ is a continuous function and $\dot{V} = -kh(V)$ for $V = 0$, implying that $\dot{V} \in C_\infty[0, t_p)$ and $\dot{V}(t) \to 0$ as $t \to t_p$. This completes the proof.

Note that $k > 1$ is a necessary and sufficient condition to ensure $\lim_{t \to t_p} \dot{V} = 0$. In addition, as discussed earlier, when $a$ and $b$ are sufficiently small, we have $h_{a, b} \leq 0 h(V) = V$. Lemma 1 is therefore equivalent to [16, Corollary 1].

III. PRESCRIBED-TIME CONTROL FOR LINEAR SYSTEMS IN CANONICAL FORM WITHOUT UNCERTAINTIES

Motivated by the appealing features of the nonlinear scaling function $h(x)$, we now discuss how to introduce it into the prescribed-time control design of the $n$th order systems (1). We first design prescribed-time control schemes using nonlinear and time-varying full state feedback and partial state feedback to achieve full-state regulation for system (1) without uncertainties/disturbances (i.e., $d(x, t) \equiv 0$), then we extend the control scheme to cope with nonvanishing uncertainties/disturbances in the system.

1Note that $V_* \geq 0$ holds for $V \geq 0$ and $a \geq b$, and $V_* = 0$ if and only if $V = 0$.
A. Prescribed-Time State-Feedback Controller

By using the time-varying scaling function and the hyperbolic-tangent-like function as introduced in Section II, we construct the vectors as

\[ H(z) = [h(z_1), \ldots, h(z_n)]^T \]
\[ \Gamma(z) = k\mu^T H(z) \] *(8)*

where \( \mu = 1/(t_p - t) \) and \( h(\cdot) \) is defined in *(2)*. In addition, we introduce the following two auxiliary vectors:

\[ z = x + J_n^T \Phi \]
\[ \Phi = J_n^T (\Phi + z) + \Gamma(z) \] *(9)*

where \( z = [z_1, \ldots, z_n]^T \in \mathbb{R}^n \), \( \Phi = [\phi_1, \ldots, \phi_n]^T \in \mathbb{R}^n \). Note that \( J_n^T \) is lower triangular, thus both \( z \) and \( \Phi \) can be easily calculated recursively (see *(28)* for specific example of computing \( z_i \) and \( \phi_i \)). It is interesting to see that the error property of the closed-loop system only depends on the parameters \( a, b, k, \) and \( r, \) in vector \( \Gamma(z) \).

**Theorem 1:** Consider system *(1)* with \( d(x, t) \equiv 0 \) and the state-feedback control law

\[ u = -B^T (\mathcal{L}(a)x + \Phi) \] *(10)*

then all closed-loop signals are bounded and the origin of the closed-loop system *(1)* is as follows.

1) **Globally uniformly asymptotically stable (GUAS),** that is, \( \|x(t)\| \leq \beta(\|x(0)\|, t) \), if the controller parameters are selected as \( k > 0 \) and \( r = 1 \). In addition, exponential output regulation is achieved if \( a \) and \( b \) are chosen sufficiently small.

2) **Prescribed-time globally uniformly asymptotically stable,** that is, \( \|x(t)\| \leq \beta(\|x(0)\|, \mu(t)) \), if the controller parameters are selected as \( k > n \) and \( r = 1 \).

**Proof:** By the definition of \( J_n, B, \) and \( \mathcal{L}(a) \), it is readily verified that \( J_n^T + BB^T = I_n, \) and \( BB^T \mathcal{L}(a) = \mathcal{L}(a) \), therefore, it holds that

\[ z = Ax + Bu + J_n^T \Phi \]
\[ = (J_n + \mathcal{L}(a))z - J_nJ_n^T \Phi - \mathcal{L}(a)J_n^T \Phi \]
\[ - BB^T \mathcal{L}(a)x - BB^T \Phi + \Phi - \Gamma(z) - J_n^T z \]
\[ = (J_n + \mathcal{L}(a))z - \Phi - \mathcal{L}(a)\left(J_n^T \Phi + x\right) \]
\[ + \Phi - \Gamma(z) - J_n^T z \]
\[ = (J_n - J_n^T)z - \Gamma(z). \] *(11)*

1) **Proof of GUAS Result:** We prove that the closed-loop system under the control law *(10)* with \( r = 0 \) [in this case, \( \Gamma(z) = kH(z) \)] is GUAS. To this end, we define the error vector between \( H(z) \) and \( z(t) \) as

\[ E := H(z) - z(t) \] *(12)*

where \( E = [e_1, \ldots, e_n]^T \in \mathbb{R}^n \) is a smooth function satisfying \( \lim_{\beta \to 0} E = 0 \) and \( E \) is bounded as long as \( z(t) \) is bounded. Consider a positive-definite function \( V_1 = z^T z/2 \), the time derivative of \( V_1 \) along *(11)* is

\[ \dot{V}_1 = z^T (J_n - J_n^T)z - z^T \Gamma(z) = -kz^T H(z) \leq 0 \] *(13)*

where the fact that \( \bar{z}^T (J_n - J_n^T)z = 0 \) \( \forall z \in \mathbb{R}^n \) is used since \( J_n - J_n^T \) is a skew-symmetric matrix. It follows from *(13)* that \( \dot{V}_1 = 0 \) if and only if \( z = 0 \), thus the transformed system *(11)* is asymptotically stable on \([0, +\infty)\), establishing the same to system *(1)* according to the converging-input–converging-output property of the corresponding auxiliary vectors.

From *(13)*, it can be further shown that

\[ \dot{V}_1 = -kz^T (E + z) \leq -k\|z\|^2 + k\|z\|\|E\| \]
\[ \leq -\Delta^2/2 \] *(14)*

where \( \Delta := \sup\{\|E\|\} \). By integrating both sides of the inequality *(14)*, we obtain \( V_1(t) \leq V_1(0)e^{-\Delta t^2/2} + \Delta^2/2 \). If we further choose the design parameters \( a \) and \( b \) in \( H(z) \) small enough, one can obtain \( \lim_{\beta \to 0} \Delta = 0 \), yielding \( V_1(t) \leq V_1(0)e^{-\Delta t^2/2} \), thus we have \( \|z(t)\| \leq \|z(0)\|e^{-\Delta t^2/2} \) and the transformed system *(11)* is exponentially stable. In addition, it follows from *(9)* that \( x_1(t) = z_1(t) \), therefore, the exponential output regulation to zero of *(1)* can be achieved. The word “exponential” actually means that “near exponential,” because the parameters \( a \) and \( b \) can only be selected as sufficiently small, not zero.

2) **Proof of PT-GUAS Result:** We now show that the closed-loop system under the control law *(10)* with \( r = 1 \) [in this case, \( \Gamma(z) = k\mu H(z) \)] is PT-GUAS. For \( t \in [0, t_p) \), there exists a continuous positive function \( W(z, t) = \sqrt{z^T z/n} \) such that \( W(z, t) = 0 \) as \( z = 0 \) and

\[ W^2 = z^T z/n \rightarrow W \leq \max\{|z_1|, \ldots, |z_n|\} \triangleq z^* \] *(15)*

In addition, its time derivative can be shown as

\[ \dot{W} = z^T \dot{z} nW = z^T (J_n - J_n^T)z - z^T \Gamma(z) \]
\[ = -z^T \Gamma(z) - nW = -k\mu \sum_{i=1}^{n} z_i h(z_i). \] *(16)*

Using Property 1 and *(15)*, we have

\[ \dot{W} \leq -k\mu \sum_{i=1}^{n} |z_i| h(|z_i|) \leq -k\mu \frac{n}{W} |z^* h(z^*)| \]
\[ \leq -k\mu \frac{n}{W} W h(W) = -k\mu \frac{n}{W} h(W) \leq 0. \] *(17)*

Since we select \( k > n \), then \( k/n > 1 \). By virtue of Lemma 1, one can prove that \( W(t) \in L_\infty[0, t_p) \), \( \dot{W}(t) \in L_\infty[0, t_p) \), and

\[ \lim_{t \rightarrow t_p} W(t) = 0, \quad \lim_{t \rightarrow t_p} \dot{W}(t) = 0. \] *(18)*

Hence, the closed-loop signals \( [z_i]_{i=1}^{n} \in L_\infty[0, t_p) \) and \( [\dot{z}_i]_{i=1}^{n} \in L_\infty[0, t_p) \), and meanwhile converge to zero as \( t \) tends to \( t_p \). Using the property of the hyperbolic-tangent-like function, we can proceed to prove that \( \Gamma(z) \in L_\infty[0, t_p) \) and the convergence of which to zero at the prescribed time.

By means of the auxiliary vectors as introduced in *(9)*, one can find that \( x_1 = z_1 \) and the following closed-loop \( z_1 \)-dynamics holds

\[ \dot{z}_1 = -k\mu h(z_1) + z_2 \] *(19)*
where $z_2$ is a bounded function, which also can be treated as a vanishing disturbance. When $t \to t_p$, the equivalent form of (19) is

$$
\dot{x}_1 = \frac{-k}{t_p - t} e^{ax_1} - e^{-bx_1}.
$$

(20)

It follows from (4)–(6) and (20) that:

$$
e^{ax_1} - e^{-bx_1} = C_2(t_p - t)^k
$$

where $C_2 = (e^{ax_1(t_p)} - e^{-bx_1(t_p)})/t_p$. Then taking time derivative on both sides of (21), we have

$$
(ae^{ax_1} + be^{-bx_1})x_2 = kC_1(t_p - t)^{k-1}, \quad k > n.
$$

(22)

Observe that (22) means that $x_2 \to 0$ as $t \to t_p$. Continue, using the analysis similar to that used in (21) and (22), by taking the $i$th ($i = 2, \ldots, n$) derivative of both sides of (21), we can generalize that $\{x_i\}_{i=2}^n$ and $\{x_i\}_{i=2}^n$ converge to zero as $t \to t_p$ (this is the reason for $k > n$). Therefore, $\lim_{t \to t_p} \|x(t)\| = 0$ and $\lim_{t \to t_p} \|\dot{x}(t)\| = 0$.

In addition, it follows from (9) that $J_n^t \Phi = \dot{\zeta} - \dot{x}$, then $\Phi = \dot{\zeta} \dot{x} + J_n^t \zeta + \zeta(z)$, then

$$
\|\Phi\| \leq \|\dot{\zeta}\| + \|\dot{x}\| + \|z\| + \|\zeta(z)\| \in \mathcal{L}_\infty[0, t_p).
$$

(23)

Consequently, from (10), we have

$$
|u(t)| = \left| -B^T (L_n(a)x + \Phi) \right|
\leq \|L_n(a)\| \|x\| + \|\Phi\|
\leq \|L_n(a)\| \|x\| + \|\dot{\zeta}\| + \|\dot{x}\| + \|z\| + \|\zeta(z)\|.
$$

(24)

Note that each term in the third line of (24) is bounded on $[0, t_p)$ and converges to zero as $t \to t_p$. Therefore, $u(t) \in \mathcal{L}_\infty[0, t_p)$ and $\lim_{t \to t_p} |u(t)| = 0$.

Remark 2: When we consider a scalar system $\dot{x} = u$, from Theorem 1, one can immediately obtain a prescribed-time controller as $u_p = -(k/\mu)(r - x)$. Note that according to Theorem 1, this controller is only a special case under the design parameters $a$ and $b$ are chosen as small enough, and the design parameter $k$ satisfies $k > 1$. Note that the classical finite-time controllers $u_f = -k\text{sign}(\dot{x})|\dot{x}|^{\alpha-1} (k > 0, 0 < \alpha < 1)$ (see [3], [19]) then the unique dynamic solution is

$$
x(t) = \begin{cases} 
\text{sign}(x_0)|x_0|^{1-\alpha} - k(1-\alpha)r t^{1-\alpha}, & t \in [0, t_f) \\
0, & t \geq t_f 
\end{cases}
$$

where $t_f = ((|x_0|^{1-\alpha})/(k(1-\alpha)))$. Thereafter, the finite-time controller is equivalent to

$$
u_f = -k\text{sign}(x)|x|^{\alpha-1} = -k(1-\alpha)r t^{1-\alpha} x.
$$

(25)

Inserting $|x_0|^{1-\alpha} = k(1-\alpha)t_f$ into (25), we have

$$
u_f = -k(1-\alpha) \dot{x}_f(t) - (1-\alpha) r t^{1-\alpha} x = -\frac{x}{(1-\alpha)(t_f - t)} \forall t \in [0, t_f).
$$

(26)

Let $k = (1/1-\alpha)$, we have $u_p = u_f$. Note that here $k > 1$ since $0 < \alpha < 1$. This conclusion is consistent with the parameter selection rules of (3) in Lemma 1 and (22) in Section III.

The above analysis proves that the prescribed-time controller is equivalent to the finite-time controller under choosing some special design parameters.

Remark 3: If we utilize the PT-GUAS controller (as defined in (10) with $k > n, r = 1$) and the GUAS controller ($k > 0, r = 0$) through the following way:

$$
u(t) = \begin{cases} 
-B^T (L_n(a)x + J_n^t (\Phi + z) + (n + 1)\mu H(z)) & 0 \leq t < t_p \\
-B^T (L_n(a)x + J_n^t (\Phi + z) + H(z)), & t \geq t_p
\end{cases}
$$

then the system states converge to zero as $t \to t_p$ and then remain zero for $t \geq t_p$. In fact, this switching method means that the prescribed-time controller guarantees that the closed-loop system is PT-GUAS on $[0, t_p)$ and the GUAS controller guarantees that the closed-loop system is ISS in the presence of some external disturbance on $[t_p, +\infty)$.

Remark 4: Various methods of finite-time control have been reported in the literature during the past few years, among which the most typical ones include adding a power integral (AAPI), linear matrix inequalities (LMIs), and implicit Lyapunov function (ILF), where the key element utilized is the fractional power state feedback (e.g., [8] and [9]). In pursuit of an alternative solution, we exploit a unified control law such that the closed-loop system (11) can be regulated asymptotically, exponentially, or within prescribed time by choosing the design parameters $k, r, a$, and $b$ in (12) properly. One salient feature with this method is that it analytically bridges the divide between prescribed-time control and traditional asymptotic control. Furthermore, different design parameters ($a$ and $b$ in $h(x)$) allow different reshaped feedback signals to be utilized in the control scheme. Such treatment provides extra design flexibility and degree of freedom in tuning regulation performance.

Remark 5: Compared with the existing prescribed-time control results (see [16], [26], [38]), the proposed NTV feedback scheme, making use of the reshaped (compressed/expanded) feedback signal, is applicable over the entire operational process. In addition, this scheme has a numerical advantage over the aforementioned methods, this is because here only $1/(t_p - t)$ rather than $1/(t_p - t)^2$ is involved for state scaling. Furthermore, with the proposed hyperbolic tangent function, the magnitude of initial control input can be adjusted through the parameters $a$ and $b$.

Example 1: To verify the effectiveness and benefits of the control scheme as presented in Theorem 1, we conduct a comparative simulation study through a third-order system

$$
\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = x_3(t), \quad \dot{x}_3(t) = u(t).
$$

According to Theorem 1-GUAS, the asymptotically stabilization controller under $a, b \to 0$ is $u_{\text{GUAS}} = -Kx(t)$ with $K = [k^3 + 2k; 3k^2 + 2, 3k]^T$ and $x(t) = [x_1, x_2, x_3]^T$, the control parameters are selected as $a = b = 0.01, k = 0.5$; and the initial condition are selected as $x_1(0); x_2(0); x_3(0) = [−1; 0; 1]$. According to Theorem 1-PT-GUAS, the following NTV feedback prescribed-time stabilization controller can be obtained:

$$
u_{\text{PT-GUAS}} = -z_2 - \dot{\phi}_2 - k\mu h(z_3), \quad t \in [0, t_p)
$$

(27)
with
\[
\begin{align*}
\mu(t, tp) &= 1/(tp - t) \\
\phi_1(x_1, t) &= k\mu h(x_1) \\
\phi_2(x_1, x_2, t) &= k\mu h(z_2) + x_1 + \phi_1 \\
h(z_2) &= e^{az_i} - e^{-az_i}, \quad i = 1, 2, 3 \\
\phi_i &= \sum_{k=1}^{i} \frac{\partial\phi_i}{\partial x_k} x_{k+1} + \sum_{k=0}^{i-1} \frac{\partial\phi_i}{\partial (\mu(k))} (\mu(k + 1)) \\
z_1 &= x_1, \quad z_2 = x_2 + \phi_1, \quad z_3 = x_3 + \phi_2 \quad (28)
\end{align*}
\]

where the corresponding design parameters are selected as \( a = b = 1, k = 6, \) and \( tp = 6 \) s. In addition, to compare the control performance, we adopt our previous result (a linear feedback scheme in [16]) for simulation. The corresponding control law is given by \( u_5 = -\sum_{i=1}^{3} (6/[k!(3-k)!])((\mu(0))/\mu)x_{4-k} - \nu^k(\bar{w}_1 + \bar{w}_2\hat{w}_1) - k(\bar{w}_1 + \bar{w}_2\hat{w}_1), \) where \( \mu_0 = (tp/(tp - t))^4, \) \( \nu = tp/(tp - t), \) and \( \bar{w}_1 = \mu_0 x_1. \) The design parameters in \( u_5 \) are selected as \( tp = 6.2 \) s, \( k_1 = k_2 = 0.6, \) and \( k = 1. \)

For comparison, simulation results obtained with the three different control schemes are shown in Fig. 2, from which asymptotic stabilization and prescribed-time stabilization are observed. Furthermore, it is seen that: 1) the settling time with the proposed prescribed-time control is indeed irrespective of the initial condition and any other design parameters; 2) the proposed scheme works within and after the prescribed-time interval; and 3) compared with the linear feedback scheme (black dotted line), it can be seen that the NTV feedback schemes (red dotted line and blue solid line) have a superior transient performance with a smaller initial control effort, verifying the effectiveness and benefits of the proposed algorithms.

B. Prescribed-Time Observer

When only partial state is measurable, we employ the prescribed-time observer proposed in [38], to construct the prescribed-time control using output feedback. As in [38] and [36], our solution is based on the separation principle, namely, the controller is derived by designing a prescribed-time observer and an NTV output-feedback control separately.

Considering that \( d(x, t) \equiv 0 \) and only output is available for feedback. The system (1) can be transformed into the following observer canonical form by a linear nonsingular transformation \( \xi(t) = Mx(t): \)
\[
\begin{align*}
\dot{\xi}(t) &= \mathcal{A}\xi(t) + \mathcal{B}u(t) + \mathcal{D}y(t) \\
y(t) &= \xi_1(t)
\end{align*}
\]
where \( \mathcal{A} = J_n, \quad \mathcal{B} = [b_{n-1}, \ldots, b_0]^\top, \quad \mathcal{D} = [a_n, \ldots, a_1]^\top, \) and \( a_is, b_is \) are the same as those in (1).

Invoking the observer proposed in [38], as follows:
\[
\begin{align*}
\dot{\hat{\xi}}(t) &= \mathcal{A}\hat{\xi}(t) + \mathcal{B}u(t) + \mathcal{D}y(t) \\
&\quad + \left[ g_1(t, T), \ldots, g_n(t, T) \right]^\top (y - \hat{\xi}_1) \quad (30)
\end{align*}
\]

where the time-varying observer gains \( \{g_i(t, T)\}_{i=1}^n \) satisfy
\[
\begin{align*}
g_i(t, T) &= \left( \frac{n + m_0 + i - 1}{T} \right) p_{0,i+1} - \tilde{p}_{0,i+1} \\
&\quad - \sum_{j=1}^{i-1} \frac{\partial g_i}{\partial \mu_1} p_{0,i-j} + r_i \\
&\quad + \left[ g_1(t, T), \ldots, g_n(t, T) \right]^\top (y - \hat{\xi}_1) (30)
\end{align*}
\]

where \( \mu_1(t, T) = T/(T - t) \) and
\[
\begin{align*}
\tilde{p}_{0,0,j} &= 1, \quad \tilde{p}_{0,j} = 0, \quad j \geq 1 \\
\tilde{p}_{0,0-j} &= -\frac{n + m_0 + i - j}{T} \tilde{p}_{0,j} + \tilde{p}_{0,i+1,j} \\
n - 1 \geq i \geq j &\geq 2 \\
\tilde{p}_{0,n-j} &= -\frac{2n + m_0 - j}{T} \tilde{p}_{0,j} 
\end{align*}
\]

and \( m_0 \geq 1 \) is an integer and \( \mathbf{r} = [r_1, \ldots, r_n]^\top \) is selected to make the n-dimensional matrix \( \Lambda = [\mathbf{r}, [\mathbf{r}, \ldots, 0_1, (n-1)]^\top]^\top \) Hurwitz. With (29) and (30) and observer error state \( \hat{\xi} = \xi - \hat{\xi}, \) we get the observer error dynamics
\[
\dot{\hat{\xi}}(t) = J_n \tilde{\xi}(t) - \left[ g_1(t, T), \ldots, g_n(t, T) \right]^\top \tilde{\xi}_1. \quad (32)
\]

Lemma 2 [38]: For the dynamic system (1), consider the observer (30) having error dynamic (32) and observer gains \( \{g_i(t, T)\}_{i=1}^n \), \( \{r_i\}_{i=1}^n \) are constants to be selected such that the companion matrix \( \Lambda \) is Hurwitz, then the closed-loop observer error system (32) is prescribed-time stable, and there exist two positive constants \( c_1 \) and \( c_2 \) such that
\[
\begin{align*}
\xi(t) &\leq \mu_1(t, T)^{-m_0-1} c_1 \exp(-c_2t) \xi(0) 
\end{align*}
\]
for all \( t \in [0, T). \) In addition, the output estimation error injection terms \( \{g_i(t, T)\}_{i=1}^n \) remain uniformly bounded over \( [0, T), \) and converge to zero as \( t \to T. \) Also, \( \hat{\xi}(t) \) has the same dynamic properties as \( \xi(t) \) since \( \hat{\xi}(t) = M^{-1}\xi(t) \) with \( M \) being a nonsingular constant matrix.
C. Prescribed-Time Output-Feedback Controller

The output-feedback prescribed-time control law for system (1) is constructed by replacing \( x_1, x_2, \ldots, x_n \) with \( \hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n \) in (10) as follows:

\[
 u = -B^T(L_n(a)\hat{x} + \Phi(\hat{x})) \tag{34}
\]

where only \( x_1 \) is measurable, \( \Phi(\hat{x}) = J_n^T(\Phi(\hat{x}) + \hat{z}) + \Gamma(\hat{z}) \) with \( \Gamma(\hat{z}) = k\mu'[h_1(\hat{z}_1), \ldots, h_n(\hat{z}_n)]^T \) and \( \hat{z} = \hat{x} + J_n^T(\Phi(x)) \).

The control law (34) involves the design parameters \( k \) and \( r \) as defined in Theorem 1. It can be verified that different \( k \) and \( r \) lead to different convergence rate. For instance, we can find that \( k > r \) and \( r = 0 \), and by invoking the classical high-gain observer, to achieve asymptotic or exponential output regulation. Here in this section, our ambitious goal is to achieve state regulation with output feedback with the aid of the prescribed-time observer developed in [38].

Theorem 2: For the dynamic system (1) with \( d(x, t) \equiv 0 \), consider the output-feedback control law (34) with the prescribed-time observer (30), the closed-loop system is prescribed-time stable if the controller and the observer parameters are selected according to Theorem 1-PT-GUAS and Lemma 2, and \( T \leq t_p \).

Proof: The proof consists of two steps. The first step is to prove that the closed-loop system with the observer and the output-feedback control scheme does not escape during \( [0, T] \), and the second step is to show that all closed-loop trajectories converge to zero as \( t \) tends to \( t_p \) and remain zero thereafter.

Step 1: We consider the Lyapunov function \( V = z^Tz/2 \). Using \( J_n^T + BB^T = L_n \) and \( BB^T\tilde{L}_n(a) = L_n(a) \), the derivative of \( V \) over \( [0, T] \) along (1) under the output-feedback control law (34) becomes

\[
 V = z^T\left[(J_n + \tilde{L}_n(a))z - J_n^T(\Phi(x) - \tilde{L}_n(a)J_n^T(\Phi(x)) \right] \\
 - z^T(2BB^T\tilde{L}_n(a)(x - \hat{x}) + BB^T(\Phi(x) - \Phi(\hat{x})) \\
 + z^T(\Phi(x) - \Gamma(z) - J_n^Tz) \\
 = z^T(J_n + \tilde{L}_n(a) - J_n^T)z - z^T(\Phi(x) + z^TBB^T(\Phi(\hat{x}) \\
 - z^T\tilde{L}_n(a)(x - \hat{x}) + z^T(\Phi(x) - \Gamma(z)) \\
 = -z^T\Gamma(z) + z^T(L_n(a)\hat{x} + BB^T(\Phi(x))
\]

where \( \hat{x} = x - \hat{x} \), and \( \Phi(\hat{x}) = \Phi(x) - \Phi(\hat{x}) \). It is seen from Lemma 2 that \( \hat{x} \) remains uniformly bounded over \( [0, T] \), and converges to zero as \( t \to T \). The boundedness of \( \Phi(\hat{x}) \) is also guaranteed by the bounded \( \hat{x} \), and \( \Phi(\hat{x}) \) also converges to zero as \( t \) tends to \( T \). Therefore, there exist a positive constant \( \gamma < \infty \) such that \( \dot{V} \leq \gamma \) holds for \( \forall t \in [0, T] \). It follows that \( V \geq z \) cannot escape during the interval \( [0, T] \).

Step 2: From Lemma 2, we know that there exists a prescribed-time \( T \), such that \( \{x(t)\}_{t=0}^{t_p} \) for \( t \geq T \). In consequence, the output-feedback control law \( u = -B^T(L_n(a)\hat{x} + \Phi(\hat{x})) \) coincides with the state-feedback control law \( u = -B^T(L_n(a)x + \Phi(x)) \) for \( \forall t \geq T \). In other words, this output-feedback law can be used to establish prescribed-time stability and performance recovery (see [26], [36], [37]). Note that the closed-loop trajectory under (34) does not escape during \( t \in [0, T] \), it follows from Theorem 1-PT-GUAS that under the proposed output-feedback control law, there exists another preset time \( t_p \geq T \) to steer the system from an arbitrary bounded state to zero as \( t \to t_p \). The boundedness of \( u(t) \) can also be easily established according to Theorem 1. This completes the proof.

Remark 6: To close this section, it is worth making the following comments.

1. The output-feedback controller inherits the properties and advantages of the state-feedback controller as stated in Theorem 1, that is, elegant parameter tuning, one-step design process, and simple controller structure.

2. Prescribed-time stabilization is the result of employing the scaling function \( \mu(t, t_p) \) and the nonlinear feedback function \( h(x) \) inside the control scheme (10). All observer errors \( \{\tilde{x}_i\}_{i=1}^n \) are regulated to zero within the preset time \( T \), and all system states \( \{x_i\}_{i=1}^n \) are regulated to zero within preset time \( t_p \), where \( t_p \geq T \).

3. Only the output state \( x_1 \) is required in constructing the output-feedback prescribed-time control (34). Such control scheme has an obvious numerical advantage because its maximum implemented gain is \( 1/(t_p - t)^n \), while the standard linear control scheme is to use fractional power state feedback or sign functions switching the controller to zero when the system is regulated to the equilibrium point. The other implementation solutions for prescribed-time control are given in [16], [18], and [26]. One typical way to address this issue is to let the system operate in a finite-time interval, i.e., adjusting the operational time slightly shorter than the prescribed convergence time. Another typical way is to let the system operate in an infinite time interval by making suitable saturation on the control gains. Here in this work, we use the method as described in Remark 3 to implement the prescribed-time controller for \( t \in [0, t_p] \) and initiate the asymptotically stable controller for \( t \geq t_p \).

Example 2: We illustrate the performance of the observer and the output-feedback controller through the following model:

\[
 \dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t), \quad y(t) = x_1(t)
\]

the observer is

\[
 \begin{cases}
 \dot{x}_1(t) = \dot{x}_2(t) + g_1(t, T)(y(t) - \dot{x}_1(t)) \\
 \dot{x}_2(t) = u(t) + g_2(t, T)(y(t) - \dot{x}_1(t))
\end{cases} \tag{35}
\]

with \( g_1(t, T) = r_1 + ((2(m_0 + 2)/T)\mu_1 + (z(m_0 + 1)(m_0 + 2))/T^2)\mu_2 \). For observer parameters, we select \( r_1 = r_2 = 1, m_0 = 3, T = 4 \) s, and \( \{x_1(0); x_2(0)\} = [0, 0] \). For output-feedback controller parameters, we select \( k = 5, a = b = 1 \), and \( t_p = 6 \) s. The initial condition is \( \{x_1(0); x_2(0)\} = [4, -3] \). The control law \( u \) is implemented similar to (28), just replacing \( x_2 \) in (28) with \( \dot{x}_2 \). Furthermore, the control performance in a
The norm of system state \( \|x(t)\| \) is studied by considering the output signal \( y(t) \) corrupted with an uncertain measurement noise \( \eta(t) \), namely, \( y(t) = x_1(t) + \eta(t) \) with \( \eta(t) = 0.001 \sin(3t) \). The closed-loop state \( \{x_i(t)\}_{i=1}^n \) trajectories, state estimate \( \{\hat{x}_i(t)\}_{i=1}^n \) trajectories, the norm of observer estimation error \( \|\hat{x}(t)\| \), the norm of system state \( \|x(t)\| \), and the control input signal \( u(t) \) are shown in Figs. 3 and 4.

It is observed from Fig. 3 that the controller remains operational after \( t_p \), and all closed-loop signals are bounded on the whole time domain, in particular, the observer estimation errors converge to zero as \( t \to T \). The theoretical prediction and analysis. Fig. 4 shows that the proposed controller remains its performance even in the presence of measurement noise. Although a slightly chattering phenomenon, caused by noise and controller switching, occurs near \( t_p \), the control input remains bounded on the whole time domain. In addition, the numerical advantage leading to friendly implementation has been verified in simulation.

![Fig. 3. Simulation results with the prescribed-time observer (T = 4 s) and the proposed output-feedback control (\( t_p = 6 \) s).](image1)

![Fig. 4. Simulations results with measurement noise.](image2)

where \( \phi_{n-1} \) is defined in (9). Some other sliding surface selection can be referred to [39] and [40]. The derivative of the auxiliary variable along the trajectories of (36) is

\[
\dot{s} = u + \sum_{i=1}^{n} a_{i} x_{i} + d(x, t) + \phi_{n-1}
\]

(38)

where \( \phi_{n-1} = \sum_{k=1}^{n-1} (\partial \phi_{n-1}) / \partial x_{k} x_{k+1} + \sum_{k=0}^{n-2} (\partial \phi_{n-1}) / (\partial \mu^{(k)}) \mu^{(k+1)} \) belongs to a computable function.

**Theorem 3:** Consider system (1) and the transformed system (36), the closed-loop signals \( \{x_i\}_{i=1}^n \) and \( s(t) \) are PT-GUAS, if the control law is designed as

\[
u = -\tilde{d} \text{sign}(s) - \phi_{n-1} - k \mu h(s) - \sum_{i=1}^{n} a_{i} x_{i}
\]

(39)

where \( k > n, \tilde{d} \geq d(\cdot), \mu = 1/(t_p - t) \), and \( \phi_{n-1} \) is a computable function as described after (38), and \( h(\cdot) \) is the hyperbolic-tangent-like function as defined in (2).

**Proof:** For \( t \in [0, t_p) \), let \( V = |s| \). With the control scheme (39), for \( s \neq 0 \), the upper right-hand derivative of \( V \) along the trajectory of the closed-loop system (36) becomes

\[
D^+ V = \frac{\dot{s}}{|s|} \leq - (\tilde{d} - d \text{sign}(s)) - k \mu h(|s|) \leq -k \mu h(V).
\]

(40)

By using Lemma 1, it is easy to get that \( V \in \mathcal{L}_\infty[0, t_p) \) and \( \lim_{t \to t_p} V = 0 \), establishing the same for \( s(t) \) and \( \dot{s}(t) \). At the same time, the closed-loop \( \{x_i\}_{i=1}^{n-1} \) dynamics become

\[
\dot{x}_{i} = x_{i+1}; \quad i = 1, \ldots, n - 2
\]

\[
\dot{x}_{n-1} = -\phi_{n-1}
\]

(41)

where \( \phi_{n-1} \) is the virtual control input. It is seen that the control law (39) reduces the perturbed nth-order system to

Fig. 3. Simulation results with the prescribed-time observer (\( T = 4 \) s) and the proposed output-feedback control (\( t_p = 6 \) s).

IV. PRESCRIBED-TIME CONTROL FOR LINEAR SYSTEMS IN CANONICAL FORM WITH UNCERTAINTIES

In the presence of nonvanishing uncertain term \( d(x, t) \), system (1) can be rewritten as

\[
\begin{align*}
\dot{x}_i &= x_{i+1}, \quad i = 1, \ldots, n - 1 \\
\dot{x}_n &= u + \sum_{i=1}^{n} a_{i} x_{i} + d(x, t)
\end{align*}
\]

(36)

where \( d(x, t) \) is an unknown smooth function and satisfies \( |d(\cdot)| \leq \tilde{d}(x) \) with \( \tilde{d}(\cdot) \) being a known scalar real-valued function.

Define a sliding surface \( s(t) \) on \([0, t_p)\) as follows:

\[
s(t) = x_n + \phi_{n-1}(x_1, \ldots, x_{n-1}, t)
\]

(37)

where \( \phi_{n-1} \) is defined in (9). Some other sliding surface selection can be referred to [39] and [40]. The derivative of the auxiliary variable along the trajectories of (36) is

\[
\dot{s} = u + \sum_{i=1}^{n} a_{i} x_{i} + d(x, t) + \phi_{n-1}
\]

(38)

where \( \phi_{n-1} = \sum_{k=1}^{n-1} (\partial \phi_{n-1}) / \partial x_{k} x_{k+1} + \sum_{k=0}^{n-2} (\partial \phi_{n-1}) / (\partial \mu^{(k)}) \mu^{(k+1)} \) belongs to a computable function.

**Theorem 3:** Consider system (1) and the transformed system (36), the closed-loop signals \( \{x_i\}_{i=1}^{n} \) and \( s(t) \) are PT-GUAS, if the control law is designed as

\[
u = -\tilde{d} \text{sign}(s) - \phi_{n-1} - k \mu h(s) - \sum_{i=1}^{n} a_{i} x_{i}
\]

(39)

where \( k > n, \tilde{d} \geq d(\cdot), \mu = 1/(t_p - t) \), and \( \phi_{n-1} \) is a computable function as described after (38), and \( h(\cdot) \) is the hyperbolic-tangent-like function as defined in (2).

**Proof:** For \( t \in [0, t_p) \), let \( V = |s| \). With the control scheme (39), for \( s \neq 0 \), the upper right-hand derivative of \( V \) along the trajectory of the closed-loop system (36) becomes

\[
D^+ V = \frac{\dot{s}}{|s|} \leq - (\tilde{d} - d \text{sign}(s)) - k \mu h(|s|) \leq -k \mu h(V).
\]

(40)

By using Lemma 1, it is easy to get that \( V \in \mathcal{L}_\infty[0, t_p) \) and \( \lim_{t \to t_p} V = 0 \), establishing the same for \( s(t) \) and \( \dot{s}(t) \). At the same time, the closed-loop \( \{x_i\}_{i=1}^{n-1} \) dynamics become

\[
\dot{x}_i = x_{i+1}; \quad i = 1, \ldots, n - 2
\]

\[
\dot{x}_{n-1} = -\phi_{n-1}
\]

(41)

where \( \phi_{n-1} \) is the virtual control input. It is seen that the control law (39) reduces the perturbed nth-order system to

Fig. 4. Simulations results with measurement noise.
an unperturbed \((n-1)\)th-order system. Therefore, by using Theorem 1-PT-GUAS, we can prove that the closed-loop signals \(\{x_i\}_{i=1}^{n}, \{\dot{x}_i\}_{i=1}^{n}, \phi_{n-1}\) and \(\phi_{n-1}\) are bounded and converge to zero as \(t \to t_p\). From (40) and the analysis process in Section III, it is not difficult to verify that \(\mu(s)\) is also bounded. Therefore, it follows from (39) that the control input \(u(t)\) is bounded for \(t \in [0, t_p]\).

**Remark 7:** For \(t \in [t_p, +\infty)\), we specifically design \(s(t) = x + \sum_{i=1}^{n-1} l_i x_i\), where \(\{l_i\}_{i=1}^{n-1}\) are assigned such that the polynomial \(l_1 + l_2 s + \cdots + l_{n-1} s^{n-2} + s^{n-1}\) is Hurwitz, and design the corresponding control law as \(u = -\bar{d} \text{sign}(s) - \sum_{i=1}^{n-1} l_i x_i + \sum_{i=1}^{n} a_i x_i\). As a result, \(D^2|s(t)| \leq 0\), we therefore obtain that \(s(t) = 0 \forall t \in [t_p, +\infty)\) by recalling that \(s(t_p) = 0\). Furthermore, it is not difficult to get \(u \in L_{\infty}[t_p, +\infty)\). As the disturbances do not disappear, the control action for \(t \geq t_p\) is no longer zero but bounded, a necessary effort to fight against the ever-lasting (nonvanishing) uncertainties/disturbances, which is comprehensible in order to maintain each state at the equilibrium (zero) after the prescribed settling time.

**Example 3:** To verify the effectiveness of the prescribed-time sliding-mode controller, we consider the following system:

\[
\dot{x}_1 = x_2, \quad \dot{x}_2 = u + 0.03 x_1 + d(x_1, x_2, t)
\]

where

\[
d(x_1, x_2, t) = 0.01 \sin(x_2) + 0.02 \sin(2t).
\]

Here, \(\bar{d}\) can be selected as \(\bar{d} = 0.03\). According to Theorem 3, the controller is given by

\[
u = -\bar{d} \text{sign}(s) - 0.03 x_1 - \phi_1 - k \mu h(s), \quad t \in [0, t_p]
\]

with

\[
\phi_1 = k \mu h(x_1), \quad \mu = 1/(t_p - t)
\]

\[
\phi_1 = \frac{\partial \phi_1}{\partial x_1} x_2 + \frac{\partial \phi_1}{\partial \mu} \mu
\]

\[
h(x) = \left( e^{ax} - e^{-bx} \right) \left( e^{ax} + be^{-bx} \right)
\]

\[
s = x_2 + \phi_1, \quad t \in [0, t_p].
\]

In addition, according to Remark 7, we design \(u = -\bar{d} \text{sign}(s) - l_1 x_2 - 0.03 x_1 \forall t \geq t_p\) with \(s = x_2 + l_1 x_1\). For simulation, the design parameters are chosen as \(a = b = l_1 = 1\) and \(k = 3\). To verify the property of prescribed-time convergence w.r.t. the initial conditions, three different initial values \([x_1(0); x_2(0)] = [1; -1], [x_1(0); x_2(0)] = [2; -1],\) and \([x_1(0); x_2(0)] = [3; -1]\) are considered in Fig. 5. To confirm the property of prescribed-time convergence w.r.t. \(t_p\), we choose \(t_p = 2, 3,\) and \(4\) s, respectively in Fig. 6.

**V. CONCLUSION**

A unified nonlinear and time-varying-feedback control scheme is developed to achieve prescribed-time regulation of high-order uncertain systems. The proposed control is able to achieve asymptotic, exponential, or prescribed-time regulation by selecting the design parameters properly. The rule of parameter selection has been given through the Lyapunov theory. Furthermore, prescribed-time output-feedback control and prescribed-time sliding-mode control for high-order systems are developed, where the advantages of simplicity (elegancy) yet superiority are retained. Extension of the proposed method to more general nonlinear systems with controllability relaxation or intermittent state feedback [41]–[44] represents an interesting future research topic.

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