The Faddeev–Popov trick in the presence of boundaries

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Abstract

We formulate criteria of applicability of the Faddeev–Popov trick to gauge theories on manifolds with boundaries. With the example of Euclidean Maxwell theory we demonstrate that the path integral is indeed gauge independent when these criteria are satisfied, and depends on a gauge choice whenever these criteria are violated.
1 Introduction

Modern interest to quantum field theory on manifolds with boundary is motivated by applications to quantum cosmology and the Casimir effect. Over last few years, substantial progress has been made in calculation of the heat kernel asymptotics and functional determinants. However, the situation with gauge field contribution is far from being clear (for a review, see recent monograph [1]). One of the most important problems is gauge dependence of on-shell effective action [1]. The simplest way to demonstrate gauge independence of the path integral is given by the Paddeev-Popov trick [2]. The aim of the present paper is to formulate criteria of applicability of this trick on manifolds with boundary. Two gauge conditions give equivalent path integrals if they are admissible for the same set of gauge invariant boundary conditions. Admissibility means that a gauge condition eliminates all linearized gauge transformations in a unique way. Gauge invariance may be replaced by BRST invariance of boundary conditions [3]. The criteria are formulated in the next section. In the Section 3 we consider an example of Euclidean Maxwell theory. We show that the path integral is gauge independent when the criteria of Sec. 2 are satisfied. For generic choice of boundary geometry the path integral becomes gauge dependent whenever these criteria are violated. This means that to achieve admissibility one should choose gauge dependent boundary conditions. Such boundary conditions describe different physics. In the Appendix we collect all geometric notations and expressions for the heat kernel asymptotics.

2 General gauge theories

Consider a gauge theory with classical action $S(\Phi)$ being invariant under infinitesimal gauge transformations $\delta_\xi \Phi = G \xi$. The path integral is given by the expression:

$$Z(\alpha, \chi) = \int D\Phi J(\chi) \exp(-S(\Phi) - \frac{1}{2\alpha} \chi^2),$$

where $\chi$ is a gauge fixing condition, $J(\chi)$ is the Faddeev–Popov determinant, $J = \det(-\chi G)$. We assume that $Z$ depends on external geometry of the space-time domain and on boundary conditions for the quantum field $\Phi$. We do not introduce any sources or background fields explicitly. If background field corresponding to the quantum filed $\Phi$ is present, we must assume that the background is taken "on-shell", i.e. satisfying equations of motion. It is well known that the path integral (1) can be obtained from another path integral

$$Z(\chi a) = \int D\Phi J(\chi) \delta(\chi - a) \exp(-S(\Phi))$$

(2)
after averaging over $a$ with the weight $\exp(-\frac{1}{2a}a^2)$. Hence, it is enough to study gauge-independence of the path integral

$$Z(\chi) = \int \mathcal{D}\Phi J(\chi)\delta(\chi) \exp(-S(\Phi)).$$

(3)

The equivalence of two path integrals, $Z(\chi_1)$ and $Z(\chi_2)$, can be established by using the Faddeev-Popov trick. One should use twice the following representation of unity

$$1 = \int \mathcal{D}\xi J(\chi)\delta(\chi(\Phi + G\xi)),$$

(4)

One should insert (4) with $\chi = \chi_2$ in the integrand of $Z(\chi_1)$, change integration variables to $\Phi - \delta_\xi \Phi$, and use again eq. (4) with $\chi = \chi_1$. This procedure can be done successfully if the two gauges $\chi_1$ and $\chi_2$ satisfy the following requirements.

(i) **Gauge-invariance of the boundary conditions.** Let

$$\mathcal{B}\Phi|_{\partial M} = 0$$

(5)

be a boundary condition for the fields $\Phi$ with some boundary operator $\mathcal{B}$. There should exist boundary conditions

$$\mathcal{B}_\xi \xi|_{\partial M} = 0$$

(6)

for gauge transformation parameters $\xi$ such that

$$\mathcal{B}\delta_\xi \Phi|_{\partial M} = 0.$$

(7)

The eq. (7) means that gauge transformations map the functional space defined by eq. (5) onto itself for some boundary conditions (6) imposed on gauge parameter $\xi$. It is clear that the operator $\mathcal{B}_\xi$ defines boundary conditions for the ghost fields.

We use twice the integral (4) over the same functional space. Hence, the operators $\mathcal{B}_\xi$ are to be the same for both gauges $\chi_1$ and $\chi_2$.

(ii) **Admissibility of $\chi_1$ and $\chi_2$.** We call a gauge condition $\chi$ admissible if for given gauge-invariant boundary conditions (5), (6) the equation

$$\chi(\Phi + G\xi) = 0$$

(8)

has unique solution $\xi$ for every $\Phi$. Again, both gauges $\chi_1$ and $\chi_2$ should be admissible for the same boundary operators $\mathcal{B}$ and $\mathcal{B}_\xi$.

If path integral in one gauge can not be transformed to another gauge by the Faddeev-Popov trick, they most probably describe different physics. More precisely, such gauges require different boundary conditions for their selfconsistent formulation. These boundary conditions may describe different physics.

\textsuperscript{1}Note that we consider only linearized gauge transformations thus avoiding the question of Gribov ambiguities. This restriction is correct at least at the one-loop approximation.
3 Two examples

Consider the action for Maxwell field on $m$-dimensional Euclidean manifold $M$:

$$S = \int_M d^m x \sqrt{g} \left( \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} [\chi(A)]^2 \right)$$  \hspace{1cm} (9)

Suppose for simplicity that the metric $g_{\mu \nu}$ is flat. We shall compare different gauge conditions to the Lorentz gauge

$$\chi_L = \nabla^\mu A_\mu.$$  \hspace{1cm} (10)

Ghost operator takes the form of ordinary Laplacian, $\chi_L(\nabla \xi) = \Delta \xi$. Note, that constant ghosts should be excluded (see, e.g. [5]). Near the boundary the gauge fixing function is $\chi = (\nabla_m - L_{\ii}) A_m + \tilde{\nabla}^i A_i$, where subscript $m$ denotes normal component of a vector, $\nabla$ is covariant derivative on $M$, $L_{\ii}$ is trace of the second fundamental form on the boundary, $\tilde{\nabla}$ is covariant derivative on the boundary.

Let us choose the so called relative boundary conditions for $A_\mu$ and Dirichlet boundary conditions for the ghosts:

$$(\nabla_m - L_{\ii}) A_m |_{\partial M} = 0, \quad A_i |_{\partial M} = 0, \quad \xi |_{\partial M} = 0.$$  \hspace{1cm} (11)

Gauge invariance of the boundary conditions (11) is equivalent to the equation $\Delta \xi |_{\partial M} = 0$, which is obvious for eigenfunctions of the ghost operator $\Delta$. The equation $\nabla^\mu A_\mu = -\Delta \xi$ has unique solutions for every $A_\mu$ because $\nabla^\mu A_\mu |_{\partial M} = 0$, and $\Delta$ is invertible on the space of Dirichlet fields without constant zero mode. Hence the gauge (10) with boundary conditions (11) is admissible.

The path integral takes the form:

$$Z_L = \det_V(-\Delta)^{-\frac{1}{2}} \det_S(-\Delta),$$  \hspace{1cm} (12)

where the first determinant is taken over vector fields and the second one is calculated for scalars.

It was demonstrated in [5] that the path integral (12) is equivalent to the Hamiltonian path integral with covariant path integral measure.

3.1 An admissible gauge

Let the manifold $M$ admits a metric such that $g_{00} = 1$, $g_{0i} = 0$. Let the boundaries correspond to $x^0 = \text{const}$ surfaces. Consider the gauge

$$\chi = \alpha^{-\frac{1}{2}} (\nabla^\mu A_\mu + f(x^0) \tilde{\nabla}^i A_i)$$  \hspace{1cm} (13)

where $\tilde{\nabla}$ is covariant derivative on $m - 1$-dimensional slices, $\alpha$ is a constant, $f(x^0)$ is an arbitrary function, $f > -1$. 


It is easy to see that the gauge (13) supplemented by the relative boundary conditions (11) satisfies both requirements (i) and (ii) of the previous section. Indeed, (i) was already demonstrated above. Equation (ii) gives

\[ \chi(A) + L\chi \xi = 0, \quad L\chi = \alpha^{-\frac{1}{2}}(\Delta + f(x^0)\tilde{\Delta}), \]  

(14)

\( \tilde{\Delta} = \tilde{\nabla}^2 \). One can check that both \( \chi(A) \) and \( L\chi \xi \) vanish on the boundary if \( A \) and \( \xi \) satisfy (11). \( L\chi \) is self-adjoint (or at least symmetric) operator. Hence, if we neglect possible topological obstructions, (14) has unique solution \( \xi \) for any \( A \).

Consider the path integral

\[ Z = \int \mathcal{D}A_{\mu} \det(-L\chi) \exp \left( -\int d^4x \sqrt{g} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \chi^2 \right) \right) \]  

(15)

Let us change variables in (15):

\[ A_{\mu} = A^T_{\mu} + \partial_{\mu} \phi, \quad \nabla^\mu A^T_{\mu} = 0 \]

\[ \mathcal{D}A_{\mu} = \mathcal{D}A^T_{\mu} \mathcal{D}\phi \det S(-\Delta) \]  

(16)

This change is consistent with boundary conditions in question. Mixing between \( \phi \) and \( A^T \) can be removed by a shift of \( \phi \), \( \phi \rightarrow \phi' = \phi + \alpha^{\frac{1}{2}} L^{-1}_{\chi} \nabla^i A^T_i \), which does not change boundary conditions for \( \phi \) and produces unit Jacobian factor. Integration over \( \phi' \) is immediately performed giving \( \det(-L\chi)^{-1} \). As before, integration over \( A^T \) gives \( \det_T(-\Delta)^{-\frac{1}{2}} \). Collecting all contributions together, we arrive at the path integral

\[ Z = \det_T(-\Delta)^{-\frac{1}{2}} \det S(-\Delta) \]  

(17)

This coincides with the result (12) in the Lorentz gauge after taking into account factorization property of the vector determinant \( \det_V(-\Delta) = \det_T(-\Delta) \det_S(-\Delta) \) which holds for relative boundary conditions.

There are two important particular cases of the gauge condition (13). \( f(x^0) \rightarrow \infty \) corresponds to the Coulomb gauge, while \( f(x^0) \rightarrow -1 \) gives axial gauge. On manifolds with boundaries such gauges were studied in [7] for the case of Euclidean Maxwell theory and in [8] for quantum gravity. For \( f = 1 \) (or \( f = \infty \)) the ghost operator \( L\chi \) is not elliptic. Spectrum of \( L\chi \) becomes infinitely degenerate and the heat kernel technique is not applicable. A more careful way to introduce such gauges is to consider a limiting procedure from (13).

### 3.2 Esposito gauges

Consider the gauge fixing condition depending on an arbitrary vector field \( B^\mu \):

\[ \chi_E = (\nabla^\mu + B^\mu)A_{\mu} \]  

(18)
Suppose that on a boundary $B^\mu$ is parallel to normal vector $e_m$, $B^a|_{\partial M} = B(x)\delta_m^a$. $a, b, c$ will denote flat tangential indices on $M$, $B^a = e^a_\mu B^\mu$. Near the boundary the gauge (18) reads: $\chi_E = (\nabla_m - L_{ii} + B)A_m + \tilde{\nabla}iA_i$.

The boundary conditions

$$(\nabla_m - L_{ii} + B)A_m|_{\partial M} = 0, \quad A_i|_{\partial M} = 0, \quad \xi|_{\partial M} = 0$$ (19)

ensure admissibility of the gauge (18) in the sense of the previous section. These boundary conditions depend on $B$. Hence the Faddeev–Popov trick cannot be used to demonstrate gauge independence of the path integral. One could choose relative boundary conditions which do not depend on $B$. In this case, however, the gauge (18) will no longer be admissible.

The gauge (18) generalises a family of gauges considered by Esposito, Kamenshchik and co-workers [4] on manifolds with spherical boundaries. Namely, these authors calculated the one–loop conformal anomaly $A$ on a ball (“one–boundary problem”) and in a region between two concentric spheres (“two–boundary problem”) for $m = 4$ and $B = const. \times \frac{1}{r}$, where $r$ is radial coordinate. They found that $A$ depends on $B$ for the one–boundary problem and is $B$–independent for the two–boundary problem. According to the authors [4], gauge dependence in the former case is due to a singularity of the $3 + 1$ decomposition at $r = 0$. According to the present author [6], gauge independence in the latter case is totally due to special choice of geometry which allows for cancellation of contribution of the two boundaries.

Since integration by part does not introduce any surface terms, we can represent (18) in the following form

$$S = \frac{1}{2} \int d^mx \sqrt{g} A^a (D^\mu D_\mu + E^b) A_b$$ (20)

where $a, b$ are flat tangential indices, $A^a = A^\mu e^a_\mu$. New covariant derivative $D_\mu = \nabla_\mu + \omega_\mu$ contains an auxiliary connection field

$$\omega^a_\mu = \frac{1}{2} (e^a_\mu B^b - e^b_\mu B^a)$$ (21)

The matrix $E$ has the form

$$E_{ab} = \frac{1}{2} (\nabla_a B_b + \nabla_b B_a) + \frac{1}{4}(\delta_{ab} B^2 + (m - 6) B_a B_b)$$ (22)

The ghost operator corresponding to the gauge fixing term (18) is

$$L^{gh} = -(\nabla^\mu \nabla_\mu + B^\mu \nabla_\mu)$$ (23)

First order derivative term can be removed again by introducing a new connection:

$$L^{gh} = -(D^{[gh]} D^{[gh]} + E^{[gh]}), \quad \omega^{[gh]}_\mu = \frac{1}{2} B_\mu, \quad E^{[gh]} = -\frac{1}{2} \nabla^\mu B_\mu - \frac{1}{4} B^2$$ (24)
The path integral is given by a product of two determinants:

$$Z_E = \det(-(D^\mu D_\mu + E))^{-\frac{1}{2}} \det(L^{gh})$$

(25)

To study the problem of gauge dependence of $Z_E$ let us use gauge–invariant zeta-function regularization and evaluate scaling behaviour (conformal anomaly), which is given by

$$\mathcal{A} = \zeta_{ph}(0) - 2\zeta_{gh}(0)$$

(26)

where two terms represent individual contributions of the photon and ghost operators in (24). Right hand side of (26) can be calculated by using the heat kernel expansion and the relation $\zeta_L(0) = a_m(L)$. Using expressions for $a_m$ from the Appendix, one can calculate $\mathcal{A}$ for $m = 2, 3, 4$ and arbitrary $B(x)$ and boundary geometry. We observe the following properties:

1. Gauge dependence in the volume integrals is cancelled.

2. For generic boundary geometry the boundary terms are gauge dependent ($m = 3, 4$).

3. For the two–boundary problem and $B = \text{const.} \times \frac{1}{r}$ contributions of the two boundaries cancel each other.

4. For the one–boundary problem $E$ and $E^{[gh]}$ are singular at $r = 0$. Individual contributions of ghosts and photons can not be calculated by using formulas from the Appendix.

Both explanations [4, 6] to gauge dependence of the conformal anomaly in Esposito gauge are true. One–boundary case really contains a dangerous singularity. Gauge independence in two–boundary problem is really due to a very special choice of geometry. In general, Esposito gauge gives gauge dependence of the conformal anomaly in complete agreement with statements of Sec. 2.

Though the problem of gauge dependence has received an explanation from the mathematical point of view, physical consequences are not clear. In Lorentzian signature of space-time both relative and Esposito boundary conditions correspond to electromagnetic field in a conducting cavity. However, some details of interaction of photons with material of a boundary must be changed. A useful test would be to evaluate vacuum expectation value of $J^i \bar{J}^i$ with boundary values of the current $J^\mu = \nabla_\nu F^{\nu\mu}$. This can be done, in principle, after extension of the results [4] to mixed boundary conditions.
4 Conclusions

In the present paper we formulated some simple criteria of applicability of the Faddeev–Popov trick on manifolds with boundaries. Namely, if two gauges are admissible for the same set of gauge invariant boundary conditions imposed on ghosts and gauge fields, they give identical path integrals. As an example, Euclidean Maxwell theory was considered. We demonstrated that for the family of gauges (13) the above criteria are satisfied and the path integral is indeed gauge independent. Violation of these criteria for the Esposito gauges leads to gauge dependence of the path integral. Physical consequences of this effect are still to be clarified.

Acknowledgments

The author is grateful to Ivan Avramidi, Giampiero Esposito and Alexander Kamentschik for discussions, and especially to Andrei Barvinsky for valuable comments. This work was supported by the Russian Foundation for Fundamental Research, grant 97-01-01186.

Appendix: The heat kernel coefficients

In this appendix we give general expressions for the heat kernel asymptotics with mixed boundary conditions [10, 11]. Let $M$ be a compact smooth manifold of dimension $m$ with smooth boundary $\partial M$. Let $L$ be an operator of Laplace type on the space of smooth sections $C^\infty(V)$ of certain vector bundle over $M$. This means that by introducing suitable metric and connection fields it can be represented as

$$ L = -(g^{\mu\nu} D_\mu D_\nu + E) \quad (27) $$

where $E$ is an endomorphism.

We must impose suitable boundary conditions. Let $\Phi \in C^\infty(V)$. Dirichlet boundary conditions are

$$ B\Phi = \Phi|_{\partial M} = 0 \quad (28) $$

Choose an orthonormal frame on $M$ such that $e_m$ is inward pointing unit vector, $\{e_i\}$ is orthonormal frame on $\partial M$. Let $S$ be an endomorphism of $V$ defined on $\partial M$. Neumann boundary conditions are

$$ B\Phi = (D_m + S)\Phi|_{\partial M} = 0 \quad (29) $$

One can also introduce mixed boundary conditions. We assume given a decomposition $V = V_N \oplus V_D$ near $\partial M$. We take Neumann boundary conditions on $V_N$ and Dirichlet boundary conditions on $V_D$. $S$ acts only on $V_N$ and zero on $V_D$. Let $\Pi_N$ and $\Pi_D$ be the
corresponding projection operators and let $\psi = \Pi_N - \Pi_D$. Such boundary conditions are elliptic.

As $t \to +0$, there is an asymptotic expansion

$$\text{Tr}_{L^2}(e^{-tL}) \sim \sum_{n=0}^{\infty} a_n(D, B) t^{(n-m)/2}$$

(30)

where the coefficients depend on the boundary operator $B$. Suppose that the metric $g$ is flat. Then

$$a_0 = (4\pi)^{-m/2} \text{tr}(1)[M]$$

$$a_1 = (4\pi)^{-(m-1)/2} \frac{1}{4} \text{tr}(\psi)[\partial M]$$

$$a_2 = (4\pi)^{-(m-2)/2} \frac{1}{6} \text{tr}\{6E[M] + (2L_{ii} + 12S)[\partial M]\}$$

$$a_3 = (4\pi)^{-(m-1)/2} \frac{1}{384} \text{tr}\{96\psi E + (13\Pi_N - 7\Pi_D)L_{ii}L_{jj}$$

$$+ (2\Pi_N + 10\Pi_D)L_{ij}L_{ij} + 96SL_{jj} + 192S^2 - 12\psi;i\psi;j)[\partial M]\}$$

$$a_4 = (4\pi)^{-m/2} \frac{1}{360} \text{tr}\{(60E_{;\mu\nu} + 180E^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu})[M]$$

$$+ ((240\Pi_N - 120\Pi_D)E_{;im} + 120EL_{ii}L_{jj}L_{kk} + \frac{1}{21}((280\Pi_N + 40\Pi_D)L_{ii}L_{jj}L_{kk}$$

$$+ (168\Pi_N - 264\Pi_D)L_{ij}L_{ij}L_{kk} + (224\Pi_N + 320\Pi_D)L_{ij}L_{jk}L_{ki})$$

$$+ 720SE + 144SL_{ii}L_{jj} + 48SL_{ij}L_{ij} + 480S^2L_{ii} + 480S^3$$

$$+ 60\psi;i\psi;j\Omega_{im} - 12\psi;i\psi;jL_{jj} - 24\psi;i\psi;jL_{ij} - 120\psi;i\psi;jS)[\partial M]\}$$

(31)

Here ";;" and ";:" denote covariant differentiation on $M$ and $\partial M$ respectively. Note, that $E_{;ij}$ and $E_{;ij}$ do not coincide. Their difference is proportional to the second fundamental form of the boundary $L_{ij}$. $\Omega_{\mu\nu} = [D_\mu, D_\nu]$. $[M]$ and $[\partial M]$ denote integration over $M$ and $\partial M$ respectively with proper volume elements.

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