IRREGULAR WAKIMOTO MODULES
AND THE CASIMIR CONNECTION

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ABSTRACT. We study some non-highest weight modules over an affine Kac–Moody algebra \( \hat{\mathfrak{g}} \) at non-critical level. Roughly speaking, these modules are non-commutative localizations of some non-highest weight “vacuum” modules. Using free field realization, we embed some rings of differential operators in endomorphism rings of our modules.

These rings of differential operators act on a localization of the space of coinvariants of any \( \hat{\mathfrak{g}} \)-module with respect to a certain level subalgebra. In a particular case this action is identified with the Casimir connection.

1. Introduction

Let \( \mathfrak{g} \) be a simple Lie algebra. Consider the affine Kac–Moody algebra \( \hat{\mathfrak{g}} \) that is a non-split 1-dimensional central extension of the loop algebra \( \mathfrak{g} \otimes \mathbb{C}((t)) \). It is well known that such an extension is unique up to an isomorphism. Denote a generator of the center by \( \mathbf{1} \). Let us set \( \hat{\mathfrak{g}} / (k-1) \cdot \hat{\mathfrak{g}} \), where \( \hat{\mathfrak{g}} \) is the universal enveloping algebra.

Representations of Kac–Moody algebras have been studied for a few decades, see for example [Kac]. However, the study was mostly concerned with highest weight representations. In particular, these representations have the property that \( t \mathfrak{g}[[t]] \) acts locally nilpotently. One of the simplest (and most important) examples of such a representation is the so-called vacuum module

\[
\mathfrak{g} \otimes \mathbb{C}((t)) = \hat{\mathfrak{g}} / (k-1) \cdot \hat{\mathfrak{g}}.
\]

It is well known that the ring of endomorphisms of this module is equal to \( \mathbb{C} \) for all but one value of \( k \), this exceptional \( k \) is called the critical level, see [Fre, Ch. 3].

In this paper we consider a more general class of representations, called smooth representations, i.e., representations in which every vector is annihilated by \( t^N \mathfrak{g}[[t]] \) for large enough \( N \). An example of such representation is

\[
\mathfrak{g} \otimes \mathbb{C}((t^N \mathfrak{g}[[t]])) = \hat{\mathfrak{g}} / (k-1) \cdot t^N \mathfrak{g}[[t]],
\]

where \( N \) is a positive integer. We would like to calculate the ring of endomorphisms, however we shall have to make a few simplifications. Firstly, instead of \( t^N \mathfrak{g}[[t]] \), we shall be using \( t^N (\mathfrak{u} - \oplus \mathfrak{h} \oplus t^{N+1} \mathfrak{g}[[t]]) \), where \( \mathfrak{g} = \mathfrak{u} - \oplus \mathfrak{h} \oplus \mathfrak{u} \) is a triangular decomposition of \( \mathfrak{g} \). Secondly, we shall consider a certain non-commutative localization of the corresponding vacuum module, see [2,2]. The endomorphisms of this localization can be promoted to those of the original module by “clearing denominators”, but we do not know how to characterize those endomorphisms that have “no denominators”, see the corollary of Proposition [2,1].

We shall provide the conjectural answer for the ring of endomorphisms: this is essentially the ring of differential operators on a certain complement of hyperplanes.
We shall use this injection to construct a natural functor $F$ from the category of representations of $\hat{\mathfrak{g}}$ to the category of $\mathcal{D}$-modules on the complement of hyperplanes.

1.1. Irregular Wakimoto modules. To construct the above injection we shall use free field realization. We shall define some irregular analogues of Wakimoto modules. Then we shall show that, after localization, our modules are isomorphic to the corresponding irregular Wakimoto modules. The sought-after endomorphisms are transparent on the Wakimoto side. This is somewhat analogous to the construction of central elements in the vertex algebra, corresponding to $\mathfrak{g}$ on the critical level, cf. [Fre].

1.2. Relation to the Casimir connection. Let $V$ be a $\mathfrak{g}$-module, $\mathfrak{h}' \subset \mathfrak{g}$ be the regular part of a Cartan subalgebra; let us view $V \times \mathfrak{h}'$ as a trivial vector bundle on $\mathfrak{h}'$. A certain connection on this bundle has been constructed independently by De Concini (unpublished), Felder–Markov–Tarasov–Varchenko [FMTV, TV], and Milson–Toledano Laredo [ML]. Following [Lar3], we shall call it the Casimir connection.

The main feature of this connection is that its monodromy gives the quantum Weyl group action, as was recently proved by Toledano Laredo in [Lar3]. It turns out that if one takes $N = 1$ and restricts our functor $F$ to certain $\hat{\mathfrak{g}}$-modules, induced from $\mathfrak{g}$-modules, then $F$ coincides with the Casimir connection up to a twist, see $\S$ 2.4.

1.3. Relation to results of D. Ben Zvi and E. Frenkel. The representation theory of $\hat{\mathfrak{g}}$ is closely related to numerous structures on moduli spaces of principal bundles on complex curves. In particular, using $\hat{\mathfrak{g}}$ and the Virasoro algebra, E. Frenkel and D. Ben–Zvi have constructed in [BZF] the following deformation-degeneration picture for a smooth projective curve:

\[
\begin{align*}
\text{KZB (or Heat) connection} & \quad \longrightarrow \quad \text{Beilinson–Drinfeld system (Oper)} \\
\downarrow & \quad \downarrow \\
\text{Isomonodromic Deformation} & \quad \longrightarrow \quad \text{Quadratic part of Hitchin System}
\end{align*}
\]

This also works in a ramified case. However, in the case when the ramification divisor has multiple points (we shall call this case irregular), a new direction arises, which was missing in [BZF]. For isomonodromic deformation this direction was put in the framework of $\hat{\mathfrak{g}}$ in [Fed].

According to the above picture, the quasi-classical limit of KZB operators gives isomonodromic hamiltonians. As will be briefly explained in $\S$ 2.9, the quasi-classical limit of operators, coming from Theorem 1, gives hamiltonians for irregular direction of isomonodromy. Thus one could think about these operators as about “irregular direction of KZB”. We hope to return to this point elsewhere. Note also, that these irregular directions are in a sense local, so that for every curve the picture has a global version (cf. $\S$ 2.7 and $\S$ 2.9).
We would like to remark, that the irregular opers were studied in [FFL]. We expect that the limits of our operators as the level tends to the critical one, should be related to irregular opers. Note also, that the quasi-classical limit was identified with isomonodromic deformation in [Boa].

1.4. Acknowledgments. The author wants to thank Dima Arinkin, David Ben-Zvi, Roman Bezruchkivnikov, Pasha Etingof, Edward Frenkel, Dennis Gaitsgory, Victor Ginzburg, Valerio Toledano Laredo, Matthew Szczesny, and Alexander Varchenko for valuable discussions. The author would like to thank the referee for valuable comments. The idea to look at the action of endomorphisms on the spaces of coinvariants belongs to Pasha Etingof. I have also learnt recently that a similar work is done in a paper in preparation by Boris Feigin, Edward Frenkel, and Valerio Toledano Laredo.

2. Main results

2.1. Notation. We are working over the field \( \mathbb{C} \) of complex numbers. Let \( G \) be a connected simple Lie group, \( \mathfrak{g} \) its Lie algebra. Fix a triangular decomposition \( \mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{h} \oplus \mathfrak{u}_+ \). Let \( U_- \) and \( U \) be the maximal unipotent subgroups of \( G \) with Lie(\( U_- \)) = \( \mathfrak{u}_- \), Lie(\( U \)) = \( \mathfrak{u} \), let \( B_- \) and \( B \) be Borel subgroups with Lie(\( B_- \)) = \( \mathfrak{b}_- := \mathfrak{h} \oplus \mathfrak{u}_- \), Lie(\( B \)) = \( \mathfrak{b} := \mathfrak{h} \oplus \mathfrak{u} \). We denote the unit element in a group by 1.

Let \( \Delta \subset \mathfrak{h}^* \) be the root system, \( \Delta_+ \subset \Delta \) be the set of positive roots. Denote by \( \alpha_1, \ldots, \alpha_k \) the simple roots. For each \( \alpha \in \Delta_+ \), fix an \( \mathfrak{sl}_2 \)-triple \( (e_\alpha, f_\alpha, h_\alpha) \). We denote \( e_\alpha \) by \( e_\alpha \) and define \( f_\alpha \) and \( h_\alpha \) similarly.

Denote by \( \mathfrak{h}^* \) the regular part of \( \mathfrak{h} \), namely,

\[
\mathfrak{h}^* := \{ h \in \mathfrak{h} : \forall \alpha \in \Delta \, \alpha(h) \neq 0 \}.
\]

Similarly, \( \mathfrak{h}^{*+} \) is the regular part of the dual space

\[
\mathfrak{h}^{*+} := \{ \chi \in \mathfrak{h}^* : \forall \alpha \in \Delta \, \chi(h_\alpha) \neq 0 \}.
\]

We define the invariant bilinear form on \( \mathfrak{g} \) by

\[
(x, y) = \frac{1}{2h^\vee} \text{tr}(\text{ad}_x \text{ad}_y),
\]

where \( x, y \in \mathfrak{g} \), \( h^\vee \) is the dual Coxeter number of \( \mathfrak{g} \), \( \text{ad} \) is the adjoint representation of \( \mathfrak{g} \).

2.1.1. The affine Kac–Moody algebra. Let \( \mathfrak{g}(t) = \mathfrak{g} \otimes \mathbb{C}((t)) \) be the formal loop algebra of \( \mathfrak{g} \). Let \( \widehat{\mathfrak{g}} = \mathfrak{g}(t) \oplus \mathbb{C} \mathbf{1} \) be the affine Kac–Moody algebra of \( \mathfrak{g} \), here \( \mathbf{1} \) is the standard generator of the center. The Lie algebra structure is given by

\[
[x_1 \otimes g_1(t), x_2 \otimes g_2(t)] = [x_1, x_2] \otimes (g_1(t)g_2(t)) + (x_1, x_2) \text{Res}_t(g_1dg_2) \mathbf{1}.
\]

We denote \( e_\alpha \otimes t^n \in \widehat{\mathfrak{g}} \) by \( e_{\alpha,n} \) and define \( f_{\alpha,n}, h_{\alpha,n} \) similarly. We also set \( e_{i,n} := e_{\alpha_i,n} \), \( f_{i,n} := f_{\alpha_i,n} \), and \( h_{i,n} := h_{\alpha_i,n} \).

For a Lie algebra \( \mathfrak{a} \) we denote its universal enveloping algebra by \( U\mathfrak{a} \). For \( k \in \mathbb{C} \) let \( U_k\widehat{\mathfrak{g}} \) be the quotient ring of \( U\widehat{\mathfrak{g}} \) modulo the ideal generated by \( k-1 \). We call \( k \) the level. The representations of \( U_k\widehat{\mathfrak{g}} \) are the same as representations of \( \widehat{\mathfrak{g}} \) on which \( \mathbf{1} \) acts by \( k \). The critical level \( k = -h^\vee \) is denoted by \( k_c \).

To simplify notation we write \( t^n u \) instead of \( u \otimes t^n \subset \widehat{\mathfrak{g}} \), \( t\mathfrak{g}[\![t]\!] \) instead of \( \mathfrak{g} \otimes (t\mathbb{C}[\![t]\!]) \) etc.

For a scheme \( Z \) we denote the set of functions on \( Z \) by \( \text{Fun}(Z) \). Thus if \( Z \) is affine, then a quasi-coherent sheaf on \( Z \) is the same as a \( \text{Fun}(Z) \)-module. We denote
by \( \mathcal{D}(Z) \) the sheaf of differential operators on \( Z \). If \( Z \) is an affine scheme, we shall abuse notation by denoting the ring of differential operators by \( \mathcal{D}(Z) \) as well.

For a vector space \( L \) we denote by \( L^* \) the dual vector space, and by \( \text{Sym}(L) \) its symmetric algebra. Thus \( \text{Sym}(L) = \text{Fun}(L^*) \).

2.2. Non-highest weight modules. Fix an integer \( N \geq 1 \) and a level \( k \neq k_c \). Let us introduce the level subalgebra

\[
\hat{g}_+ := t^N(u_+ \oplus u) \oplus t^{N+1}g[[t]] \subset \hat{g}.
\]

Let \( C_k \) be the \( \hat{g}_+ \oplus \mathbb{C}1 \)-module on which \( \hat{g}_+ \) acts by zero and \( 1 \) acts by \( k \). Consider the \( \hat{g} \)-module

\[
\hat{M}_{N,k} := \text{Ind}_{\hat{g}_+ \oplus \mathbb{C}1}^\hat{g} C_k = U_k \hat{g} \otimes U \hat{g}_+ = \mathcal{U}_k \hat{g}/(\mathcal{U}_k \hat{g} \cdot \hat{g}_+).
\]

We denote the image of \( 1 \in U_k \hat{g} \) in \( \hat{M}_{N,k} \) by \( |0\rangle \) and call it the vacuum vector. Note that \( \hat{M}_{N,k} \) has a universal property that

\[
\text{Hom}_{\mathcal{U}_k \hat{g}}(\hat{M}_{N,k}, M) = M^{\hat{g}_+}.
\]

In particular, we have an identification of sets \( \text{End}_\hat{g}(\hat{M}_{N,k}) = (\hat{M}_{N,k})^{\hat{g}_+} \), given by \( X \mapsto (|0\rangle \cdot X \) (we view endomorphisms of left modules as acting on the right). In particular we have

\[
\mathcal{U}(t\hat{g}[[t]])/\hat{g}_+^{op} \subset \text{End}_\hat{g}(\hat{M}_{N,k}),
\]

where for a ring \( A \) we denote by \( A^{op} \) the same ring with the opposite multiplication. It follows that we have a natural inclusion \( \text{Sym}(t^N\mathfrak{h}) \hookrightarrow \text{End}_\hat{g}(\hat{M}_{N,k}) \). It can be also described as a subring, generated by endomorphisms \( |0\rangle \mapsto h_{i,N}|0\rangle \).

Let \( \hat{M}_{N,k,\text{reg}} \) be the localization of \( \hat{M}_{N,k} \) given by

\[
\hat{M}_{N,k,\text{reg}} := \hat{M}_{N,k} \bigotimes_{\text{Sym}(t^N\mathfrak{h})} \text{Fun}(\mathfrak{h}^{*\mathfrak{r}}),
\]

where we view \( \text{Fun}(\mathfrak{h}^{*\mathfrak{r}}) \) as a module over \( \text{Sym}(t^N\mathfrak{h}) \approx \text{Fun}(\mathfrak{h}^*) \).

**Remark.** Another way to define \( \hat{M}_{N,k,\text{reg}} \) is as follows: the action of \( \text{Sym}(t^N\mathfrak{h}) \approx \text{Fun}(\mathfrak{h}^*) \) makes \( \hat{M}_{N,k} \) a sheaf on \( \mathfrak{h}^* \), and \( \hat{M}_{N,k,\text{reg}} \) is the restriction of \( \hat{M}_{N,k} \) to \( \mathfrak{h}^{*\mathfrak{r}} \).

**Proposition 2.1.**

\[
\text{End}_\hat{g}(\hat{M}_{N,k,\text{reg}}) = (\hat{M}_{N,k,\text{reg}})^{\hat{g}_+}.
\]

**Proof.** A map \( \text{End}_\hat{g}(\hat{M}_{N,k,\text{reg}}) \to (\hat{M}_{N,k,\text{reg}})^{\hat{g}_+} \) is given by \( X \mapsto (|0\rangle \otimes 1) \cdot X \). It follows from the PBW theorem that if \( \phi \cdot x = 0 \), where \( x \in \hat{M}_{N,k} \), \( \phi \in \text{Sym}(t^N\mathfrak{h}) \), then \( x = 0 \) or \( \phi = 0 \). This implies that the above map is injective.

It remains to show that every \( x \in (\hat{M}_{N,k,\text{reg}})^{\hat{g}_+} \) gives rise to an endomorphism of \( \hat{M}_{N,k,\text{reg}} \). This is somewhat similar to the construction of the sheaf of differential operators on a variety. First we check the following analogue of an Ore condition: for all \( x \in \hat{M}_{N,k} \), \( i = 1, \ldots, r \mathfrak{g}_i \), and all large enough \( j \) we have \( x \cdot (h_{i,N})^j \in h_{i,N}\hat{M}_{N,k} \). This, in turn, follows immediately from the fact that \( \text{ad}_{h_{i,N}} \) is a nilpotent operator on \( \hat{M}_{N,k} \). (By definition \( \text{ad}_{h_{i,N}} x := h_{i,N}x - x \cdot h_{i,N} \).)

Thus for \( x \in \hat{M}_{N,k,\text{reg}} \) the equation \( h_{i,N}y = x \) has a unique solution. Therefore we can construct a left action of \( \text{Fun}(\mathfrak{h}^{*\mathfrak{r}}) \) on \( \hat{M}_{N,k,\text{reg}} \) extending the action of \( \text{Sym}(t^N\mathfrak{h}) \). Now we can define the action of \( x \in (\hat{M}_{N,k,\text{reg}})^{\hat{g}_+} \) on \( \hat{M}_{N,k,\text{reg}} \) by \( g|0\rangle \otimes \phi \mapsto g(\phi x) \), where \( g \in U_k \hat{g} \), \( \phi \in \text{Fun}(\mathfrak{h}^{*\mathfrak{r}}) \). We need to check two things: (i) \( g(\phi x) = 0 \) if \( g \in \hat{g}_+ \), and (ii) \( g\psi(\phi x) = g(\phi \psi x) \) whenever \( \psi \in \text{Sym}(t^N\mathfrak{h}) \).
For (i), let us write \( \varphi = \varphi_1/\varphi_2 \) with \( \varphi_1 \in \text{Sym}(t^N \mathfrak{h}) \), and set \( y := (\varphi_2)^{-1}x \). Since \([t^N \mathfrak{h}, \hat{\mathfrak{g}}_+] \subset \hat{\mathfrak{g}}_+ \), it is enough to show that \( gy = 0 \) as long as \( g \in \hat{\mathfrak{g}}_+ \). We have \( g\varphi_2 y = 0 \) for all \( k \geq 1 \). Therefore \( (\varphi_2)^ny = (\text{ad}_{\varphi_2}^n) y \), and this is zero if \( n \) is large enough. Now (i) follows; (ii) is proved easily by writing \( \varphi = \varphi_1/\varphi_2 \) as before. □

**Corollary.** The multiplicative set generated by \( h_{i,N} \in \text{End}_{\mathfrak{g}}(M_{N,k}) \), where \( i = 1, \ldots, \text{rk} \mathfrak{g} \), satisfies the Ore conditions, and \( \text{End}_{\mathfrak{g}}(M_{N,k,reg}) \) is the localization with respect to this set. Also,

\[
\text{End}_{\mathfrak{g}}(M_{N,k,reg}) = \text{End}_{\mathfrak{g}}(M_{N,k}) \bigotimes_{\text{Sym}(t^N \mathfrak{h})} \text{Fun}(\mathfrak{h}^{*,r}),
\]

and \( \text{End}_{\mathfrak{g}}(M_{N,k}) \subset \text{End}_{\mathfrak{g}}(M_{N,k,reg}) \).

**Remark.** We see that \( \forall X \in \text{End}_{\mathfrak{g}}(M_{N,k,reg}) \) we have \( \prod_i h_{i,N} X \in \text{End}_{\mathfrak{g}}(M_{N,k}) \) for some non-negative integers \( \ell_i \).

**Theorem 1.** For \( k \neq k_c \) there is a natural injective homomorphism of rings

\[
\mathcal{D} \left( \prod_{i=1}^{N-1} \mathfrak{h}^* \times \mathfrak{h}^{*,r} \right) \otimes \text{Fun}(\mathfrak{h}^*) \hookrightarrow \text{End}_{\mathfrak{g}}(M_{N,k,reg}).
\]

**Remarks.** 1. We can re-write the ring of endomorphisms as

\[
\mathcal{D} \left( \prod_{i=1}^{N-1} u^r \right) \otimes \text{Fun}(\mathfrak{h}^*) \otimes \mathcal{D} \left( \prod_{i=1}^{N-1} \mathfrak{h}^* \times \mathfrak{h}^{*,r} \right).
\]

In fact the endomorphisms, corresponding to the first multiple as well as corresponding to \( \text{Fun} \left( \prod_{i=1}^{N-1} \mathfrak{h}^* \times \mathfrak{h}^{*,r} \right) \subset \mathcal{D} \left( \prod_{i=1}^{N-1} \mathfrak{h}^* \times \mathfrak{h}^{*,r} \right) \) can easily be seen: they correspond to the inclusion (2) (though the isomorphism between this subring and the LHS of (2) is not obvious). The endomorphisms, corresponding to the second multiple, are also easy to see.

The remaining endomorphisms are hidden. For example, as we shall see below in case \( N = 1 \), these endomorphisms correspond to the Casimir connection operators.

2. We expect that the homomorphism is an isomorphism for all \( k \neq k_c \).

3. If \( k = k_c \) the ring of endomorphisms is much bigger, one can show that it contains the infinite-dimensional space of functions on opers on a formal disc as a subquotient.

4. The level group \( \hat{\mathfrak{g}}_+ \) looks a little bit unnatural. It seems more reasonable to take \( \hat{\mathfrak{g}}_+ = t^N \mathfrak{g}[\mathfrak{t}] \) or \( \hat{\mathfrak{g}}_+ = u[t] + t^N \mathfrak{g}[\mathfrak{t}] \). In the second case the results and proofs are expected to be very similar, however, the interpretation of \( N = 1 \) case as the Casimir connection is not clear.

The first case seems to be more complicated and we hope to address it in future publications.

5. The above corollary shows that the inclusion (2) is strict and \( \text{End}_{\mathfrak{g}}(M_{N,k}) \) has a lot of “non-obvious” endomorphisms.

2.2.1. The right version. There is a standard anti-involution \( \iota : \mathcal{U}_k \hat{\mathfrak{g}} \rightarrow \mathcal{U}_{-k} \hat{\mathfrak{g}} \) sending \( x \in \hat{\mathfrak{g}} \) to \( -x \). By composing the actions of \( \mathcal{U}_{-k} \hat{\mathfrak{g}} \) on \( M_{N,-k} \) or \( M_{N,-k,reg} \) with \( \iota \) we get right \( \mathcal{U}_k \hat{\mathfrak{g}} \)-modules \( \mathcal{M}'_{N,k} \) and \( \mathcal{M}'_{N,k,reg} \). It is easy to check that \( \mathcal{M}'_{N,k} \approx \mathcal{U}_k \hat{\mathfrak{g}}/(\mathfrak{g}+\mathcal{U}_k \hat{\mathfrak{g}}) \) and \( \mathcal{M}'_{N,k,reg} \approx \text{Fun}(\mathfrak{h}^{*,r}) \otimes_{\text{Sym}(t^N \mathfrak{h})} \mathcal{M}'_{N,k} \). The rings of endomorphisms are clearly the same, thus we have an inclusion

\[
\text{End}_{\mathfrak{g}}(\mathcal{M}'_{N,k,reg}) \supset \mathcal{D} \left( \prod_{i=1}^{N-1} \mathfrak{h}^* \times \mathfrak{h}^{*,r} \right) \otimes \text{Fun}(\mathfrak{h}^*).
\]
2.3. The functor of coinvariants. Let $M$ be any left $\mathfrak{g}$-module of level $k$. Note that $\text{Sym}(t^{N} h)$ acts on the space of coinvariants $M/\mathfrak{g}+ \cdot M$ because $[t^{N} h, \mathfrak{g}+] \subset \mathfrak{g}+$. Thus we can form the tensor product

$$\mathcal{F}(M) := \text{Fun}(\mathfrak{h}^{*}\cdot r) \bigotimes_{\text{Sym}(t^{N} h)} (M/\mathfrak{g}+ \cdot M).$$

**Lemma 2.1.** $\text{End}_{\mathfrak{g}}(M_{N,k,\text{reg}})$ acts on $\mathcal{F}(M)$.

**Proof.** Clearly, $\text{End}_{\mathfrak{g}}(M_{N,k}) = (\mathcal{U}_{k}\mathfrak{g}/(\mathfrak{g}+ \cdot \mathcal{U}_{k}\mathfrak{g}))\hat{+}$ acts on $M/\mathfrak{g}+ \cdot M$. Thus its localization $\text{End}_{\mathfrak{g}}(M_{N,k,\text{reg}})$ acts on

$$\text{End}_{\mathfrak{g}}(M_{N,k,\text{reg}}) \bigotimes_{\text{End}_{\mathfrak{g}}(M_{N,k})} (M/\mathfrak{g}+ \cdot M) = \mathcal{F}(M).$$

The last equality follows from the right version of \cite{1}.

Thus we get a functor $\mathcal{F}$ from $\mathcal{U}_{k}\mathfrak{g}$-mod to the category of modules over

$$\mathcal{D} \left( \prod_{i=1}^{N-1} \mathfrak{b}^{*} \times \mathfrak{h}^{*}\cdot r \right) \otimes \text{Fun}(\mathfrak{h}^{*}).$$

Such a module can be viewed as an $\mathfrak{h}^{*}$-family of $\mathcal{D}$-modules on $\prod_{i=1}^{N-1} \mathfrak{b}^{*} \times \mathfrak{h}^{*}\cdot r$ and also as a $\mathcal{D}$-module on $\prod_{i=1}^{N-1} \mathfrak{b}^{*} \times \mathfrak{h}^{*}\cdot r$ by forgetting the action of $\text{Fun}(\mathfrak{h}^{*})$. In particular, it is a quasi-coherent sheaf on $\prod_{i=1}^{N-1} \mathfrak{b}^{*} \times \mathfrak{h}^{*}\cdot r$.

2.4. Case $N = 1$ and the Casimir connection. For $N = 1$ we have

$$\mathcal{F} : \mathcal{U}_{k}\mathfrak{g}$-mod $\rightarrow (\mathcal{D}(\mathfrak{h}^{*}\cdot r) \otimes \text{Fun}(\mathfrak{h}^{*}))$-mod $= (\mathcal{D}(\mathfrak{h}^{*}) \otimes \text{Fun}(\mathfrak{h}^{*}))$-mod,

where we use the bilinear form to identify $\mathfrak{h}^{*}$ with $\mathfrak{h}$. For any $\mathfrak{g}$-module $V$ set

$$V := \text{Ind}_{\mathfrak{g}[t^{-1} \otimes \mathbb{C}]}^{\mathfrak{h}[t^{-1} \otimes \mathbb{C}]} V = \mathcal{U}_{k}\mathfrak{g} \otimes \mathcal{U}(\mathfrak{g}[t^{-1}]) V,$$

where $t^{-1} \mathfrak{g}[t^{-1}]$ acts on $V$ by zero, $1$ acts by $k$.

Consider the trivial vector bundle $V \times \mathfrak{h}^{*}$ over $\mathfrak{h}^{*}$. Recall that Casimir connection is a connection on this bundle, see \cite{Lar1, Lar2, Lar3}.

**Theorem 2.** (a) $\mathcal{F}(V)$ is naturally identified with $V \times \mathfrak{h}^{*}$ as a sheaf on $\mathfrak{h}^{r}$.

(b) Under this identification the $\mathcal{D}(\mathfrak{h}^{*})$-structure on $\mathcal{F}(V)$ is given by the Casimir connection up to a twist by a line bundle on $\mathfrak{h}^{*} \times \mathfrak{h}^{*}$ with a connection along $\mathfrak{h}^{*}$.

**Remarks.** 1. The structure of a sheaf on $\mathfrak{h}^{*}$ on $\mathcal{F}(V)$ comes from the action of $\mathfrak{h}$ on $V$, and this action commutes with both the Casimir and our connections. Assume that $V$ is finite-dimensional; then $\mathcal{F}(V) = V \times \mathfrak{h}^{r}$ is supported at finitely many points $\beta_{i} \in \mathfrak{h}^{*}$ as a sheaf on $\mathfrak{h}^{*}$ (the weight decomposition). The restrictions of two connections to $\{\beta_{i}\} \times \mathfrak{h}^{*}$ differ by a line bundle with connection. In particular the monodromies of our connection and of the Casimir connection differ by an explicit scalar factor.

2. The first part of this theorem follows easily from PBW theorem.

In concrete terms, the operators of our connection correspond to the following endomorphisms of $M_{N,k,\text{reg}}$, see \cite{3.10.1}

$$1 \otimes |0\rangle \mapsto 1 \otimes |0\rangle h_{i,-1} + \sum_{\alpha \in \Delta_{+}} \alpha(h_{i}) h_{\alpha,1}^{-1} \otimes |0\rangle f_{\alpha,0} e_{\alpha,0}.$$
The corresponding “regularized” endomorphisms of $\mathcal{M}_{N,k}$ are given by
$$|0\rangle \mapsto |0\rangle h_{i,-1} \prod_{\beta \in \Delta_+} h_{\beta,1} + \sum_{\alpha \in \Delta_+} a(h_i)|0\rangle \prod_{\beta \in \Delta_+ \beta \neq \alpha} h_{\beta,1} f_{a,0} e_{a,0}.$$

2.5. Irregular Wakimoto modules. The idea of the proof of Theorem 1 is to construct an isomorphism between $\mathcal{M}_{N,k,\text{reg}}$ and another module, which we call an irregular Wakimoto module. Its ring of endomorphisms is easier to calculate. Irregular Wakimoto modules will be defined by means of free field realization. Our notation follows [Fre].

Let $\mathcal{A}$ be the associative algebra with generators $a_{\alpha,n}, a_{\alpha,n}^*, \alpha \in \Delta_+, n \in \mathbb{Z}$ and relations
$$[a_{\alpha,n}, a_{\beta,m}^*] = \delta_{\alpha,\beta}\delta_{n,-m}, \quad [a_{\alpha,n}, a_{\beta,m}] = [a_{\alpha,n}^*, a_{\beta,m}^*] = 0.$$  
(The same algebra is defined in [Fre] §5.3.3, where it is denoted by $\mathcal{A}^q$.)

Let $\hat{\mathcal{M}}_N$ be the module over $\mathcal{A}$ generated by a vector, which we also denote by $|0\rangle$ and call a vacuum vector, and relations:

$$(6) \quad a_{\alpha,n}|0\rangle = 0, \ n \geq N, \quad a_{\alpha,n}^*|0\rangle = 0, \ n \geq 0.$$  

This module is analogous to the module $\mathcal{M}_g$ defined in [Fre] §5.4.1. Note that $a_{\alpha,n}$ with $n \geq N$ and $a_{\alpha,n}^*$ with $n \geq 0$ generate a (commutative) subalgebra in $\mathcal{A}$.

Further, for $k \in \mathbb{C}$ let $\hat{\mathcal{h}}_k$ be the Cartan part of $\hat{\mathfrak{g}}_k$, where $\hat{\mathfrak{g}}_k$ is the central extension $\hat{\mathfrak{g}}$, rescaled by $k$. More precisely, it is the Lie algebra generated by $b_{i,n}$, $i = 1, \ldots, \text{rk} \mathfrak{g}$, $n \in \mathbb{Z}$, and a central element $1$. The bracket is given by

$$(7) \quad [b_{i,n}, b_{j,m}] = -nk(h_i, h_j)\delta_{n,-m}1.$$  

Let $\hat{\pi}^k_N$ be an $\hat{\mathcal{h}}_k$-module generated by $|0\rangle$ with relations
$$b_{i,n}|0\rangle = 0, \ n > N, \ 1|0\rangle = |0\rangle.$$  

Note the similarity between $\hat{\pi}^k_N$ and $\pi^k_\lambda$ in [Fre] §6.2.1. Let $\hat{\mathfrak{h}}^+$ be the subalgebra of $\hat{\mathfrak{h}}^+$ generated by $b_{i,n}$ with $n > N$. If $1$ acts on an $\hat{\mathfrak{h}}_k$-module $M$ by $1$, then
$$\text{Hom}_{\hat{\mathfrak{h}}_k}(\hat{\pi}^k_N, M) = M_{\hat{\mathfrak{h}}^+}.$$  

Let $\hat{\pi}^k_{N,\text{reg}}$ be the localization of $\hat{\pi}^k_N$ defined as follows: we have a unique endomorphism of $\hat{\pi}^k_N$ such that $|0\rangle \mapsto b_{i,N}|0\rangle$ (follows from the universal property of $\hat{\pi}^k_N$). This gives an action of $\text{Sym}(t^N \mathfrak{h})$ on $\hat{\pi}^k_N$. Analogously to (3) we set
$$\hat{\pi}^k_{N,\text{reg}} := \hat{\pi}^k_N \otimes_{\text{Fun}(\mathfrak{h}^+)} \text{Fun}(t^N \mathfrak{h}).$$  

We call $\mathcal{M}_N \otimes \hat{\pi}^k_{N,\text{reg}}$ an irregular Wakimoto module by analogy with $W_{\lambda,k}$ from [Fre] §6.2.

For a monomial $A$ in $a_{\alpha,n}$ and $a_{\alpha,m}^*$ we define its normal ordering $:\Lambda A: \text{by moving all } a_{\alpha,n} \text{ with } n \geq 0 \text{ and all } a_{\alpha,m}^* \text{ with } m > 0 \text{ to the right.}$

Define the generating functions
$$a_\alpha(z) := \sum_{n \in \mathbb{Z}} a_{\alpha,n} z^{-n-1}, \quad a_\alpha^*(z) := \sum_{n \in \mathbb{Z}} a_{\alpha,n}^* z^{-n}, \quad b_i(z) := \sum_{n \in \mathbb{Z}} b_{i,n} z^{-n-1},$$  
$$e_\alpha(z) := \sum_{n \in \mathbb{Z}} e_{\alpha,n} z^{-n-1}, \quad h_\alpha(z) := \sum_{n \in \mathbb{Z}} h_{\alpha,n} z^{-n-1}, \quad f_\alpha(z) := \sum_{n \in \mathbb{Z}} f_{\alpha,n} z^{-n-1}.$$  

Also, set \( e_i(z) := e_{\alpha_i}(z), \) \( f_i(z) := f_{\alpha_i}(z). \)

**Theorem 3.** (a) For certain polynomials \( P^i_\beta \) and \( Q^i_\beta \) without constant terms, and certain constants \( c_i \), the following formulae give a level \( k \) action of \( \hat{\mathfrak{g}} \) on \( M_N \otimes \hat{\pi}^{k-k_c}_{N,reg} \):

\[
e_i(z) \mapsto e_{\alpha_i}(z) + \sum_{\beta \in \Delta^+} :P^i_\beta(a^*_\alpha(z))a_\beta(z):,
\]

\[
h_i(z) \mapsto - \sum_{\beta \in \Delta^+} \beta(h_i):a^*_\beta(z)a_\beta(z) + h_i(z),
\]

\[
f_i(z) \mapsto \sum_{\beta \in \Delta^+} :Q^i_\beta(a^*_\alpha(z))a_\beta(z): - (c_i + (k - k_c)(e_i, f_i))\partial_z a^*_\alpha(z) + b_i(z)a^*_\alpha(z).
\]

(b) This action gives rise to an action of \( \hat{\mathfrak{g}} \) on \( M_N \otimes \hat{\pi}^{k-k_c}_{N,reg} \).

(c) There is a unique isomorphism \( \psi : M_{N,k,reg} \to M_N \otimes \hat{\pi}^{k-k_c}_{N,reg} \) of \( \hat{\mathfrak{g}} \)-modules such that \( |0\rangle \otimes 1 \mapsto |0\rangle \otimes (|0\rangle \otimes 1) \). (In the RHS the first multiple is in \( M_N \), the second multiple \( |0\rangle \otimes 1 \in \hat{\pi}^{k-k_c}_{N,reg} \).)

**Remark.** Our sign convention (1) does not agree with [Fre, (1.3.3)]. So our sign in front of the second term in the last formula in (8) is different, as well as the sign in (7).

In fact, part (a) is an easy consequence of [Fre, Theorem 6.2.1], part (b) is an easy consequence of part (a), the proof of part (c) will occupy a substantial part of this paper, it will be done by constructing certain filtrations.

Theorem 1 follows from the above Theorem: the required endomorphisms of \( M_{N,k,reg} \approx M_N \otimes \hat{\pi}^{k-k_c}_{N,reg} \) are given by \( |0\rangle \otimes (|0\rangle \otimes 1) \mapsto -a_{\alpha,n}|0\rangle \otimes (|0\rangle \otimes 1), \) where \( 0 < n < N, \) \( 0 \otimes (|0\rangle \otimes 1) \mapsto a^*_\alpha|0\rangle \otimes (|0\rangle \otimes 1), \) where \( -N < n < 0, \) and \( |0\rangle \otimes (|0\rangle \otimes 1) \mapsto |0\rangle \otimes (b_{i,n}|0\rangle \otimes 1), \) with \( -N \leq n \leq N. \) We shall give more details in (3.9)

### 2.6. The quasi-classical limit.

Let \( \hat{\mathfrak{g}}^* := \mathfrak{g}^* \otimes \mathbb{C}((t))dt \oplus \mathbb{C}d \) be the restricted dual to \( \hat{\mathfrak{g}} \), where \( d = 1^* \). Its subset \( \text{Conn} := \mathfrak{g}^* \otimes \mathbb{C}((t))dt + d \) is identified with the set of connections on the trivial \( G \)-bundle over the punctured formal disc. Set

\[
\text{Conn}_{N,h} := \left\{ \nabla \in \text{Conn} : \nabla = d + \sum_{n \geq -N-1} A_nt^n dt, A_{-N-1} \in \mathfrak{h}^{*,r} \right\}.
\]

We can view \( \mathcal{U}_k \hat{\mathfrak{g}} \) and \( M_{N,k,reg} \) as \( \mathbb{C}[k]-\)modules. Choosing appropriate \( \mathbb{C}[k^{-1}]-\)lattices in \( \mathcal{U}_k \hat{\mathfrak{g}} \otimes \mathbb{C}[k, k^{-1}] \) and \( M_{N,k,reg} \otimes \mathbb{C}[k, k^{-1}] \), we extend \( \mathcal{U}_k \hat{\mathfrak{g}} \) and \( M_{N,k,reg} \) to \( k = \infty \) (this is similar to [FBZ, §16.3.2]). Then one identifies \( \mathcal{U}_\infty \hat{\mathfrak{g}} \) with the algebra of functions on \( \text{Conn} \), and \( M_{N,\infty,reg} \) with the algebra of functions on \( \text{Conn}_{N,h} \). Note that \( \text{Conn} \) is a Poisson subspace of \( \hat{\mathfrak{g}}^* \). Set \( G_+ := \exp(\hat{\mathfrak{g}}^*_+); \) then \( \text{Conn}_{N,h} / G_+ \) is an open subspace of the hamiltonian reduction \( \text{Conn} // G_+ \). The following statement is very close to [Fed, Proposition 5] and to standard theorems about normal forms of connections.

**Lemma 2.2.** Every \( G_+\)-orbit in \( \text{Conn}_{N,h} \) contains a unique element of the form

\[
d + \sum_{n=-N-1}^{N-1} A_n t^n dt,
\]

where \( A_{-N-1} \in \mathfrak{h}^{*,r}, A_n \in \mathfrak{h} \) for \( n \geq -1. \)
At $k = \infty$ the action of $\hat{g}_+$ on $M_{N,k,\text{reg}}$ becomes a hamiltonian action of $\hat{g}_+$ on $M_{N,\infty,\text{reg}}$. Thus $(M_{N,\infty,\text{reg}})^{\hat{g}_+} = \text{Fun}(\text{Conn}_{N,h}/G_+) + \mathcal{B}_k = \text{Fun}(\text{Conn}_{N,h}/G_+) + (M_{N,\infty,\text{reg}})^{\hat{g}_+}$. Thus the above lemma shows that our (semi-conjectural) answer for $\text{End}_{\hat{g}}(M_{N,k,\text{reg}}) = (M_{N,k,\text{reg}})^{\hat{g}_+}$ is the quasi-classical limit of $\text{End}_{\hat{g}}(M_{N,k,\text{reg}}) = (M_{N,k,\text{reg}})^{\hat{g}_+}$. Thus the above lemma shows that our (semi-conjectural) answer for $\text{End}_{\hat{g}}(M_{N,k,\text{reg}})$ has “the right size”. One can show that the “easy” endomorphisms (see remark after Theorem 1) correspond to $A_n$ with $n \leq -1$.

Note also that the injection of Theorem 1 provides Darboux coordinates on $\text{Conn}_{N,h}/G_+$.  

2.7. Global version. Let $G((t))$ be the loop group of $G$, let $G_- \subset G$ denote the subgroup of loops that can be extended to a $G$-valued function on $\mathbb{P}^1 \setminus 0$ that is equal to 1 at infinity.

Set $B_{\text{un}}^h_{N(0) + (\infty)} := G_- \setminus G((t))/G_+$. Then $B_{\text{un}}^h_{N(0) + (\infty)}$ is the moduli space of $G$-bundles on $\mathbb{P}^1$ with the fiber at $\infty$ trivialized, and with a certain higher level structure at zero. More precisely, it is a principal $\mathfrak{h}$-bundle over the moduli space of $G$-bundles with the fiber at $\infty$ trivialized and with the fiber at zero trivialized to order $N - 1$ (we view $\mathfrak{h}$ as an abelian Lie group).

Remark. The space $B_{\text{un}}^h_{N(0) + (\infty)}$ looks a little bit unnatural. It would be more reasonable to look at the moduli space of bundles trivialized to order $N$ at zero, which would correspond to $\hat{g}_+ = t^{N+1}\mathfrak{g}[[t]]$, or having some higher order unipotent structure, which would correspond to $\hat{g}_+ = \mathfrak{u}[[t]] + t^{N}\mathfrak{g}[[t]]$ (cf. the last remark after Theorem 1).

The determinant line bundle on $B_{\text{un}}^h_{N(0) + (\infty)}$ yields a 1-parametric family of TDO, which we denote by $D_k(B_{\text{un}}^h_{N(0) + (\infty)})$. It is easy to see that each element of $\text{End}_{\hat{g}}(M_{N,k})$ gives rise to a global section of $D_k(B_{\text{un}}^h_{N(0) + (\infty)})$ (similar to [BZF, Corollary 2.4.3]). We conjecture that this construction gives all global level $k$ differential operators on $B_{\text{un}}^h_{N(0) + (\infty)}$ for $k \neq k_c$. This conjecture is related to a known fact that there are no non-constant differential operators on the moduli space of bundles for $k \neq k_c$.

We expect similar statements to hold for other level groups.

2.8. Recollection on isomonodromic deformation. Let $\mathcal{X} \to S$ be a family of smooth projective curves, $(\mathcal{E}, \nabla)$ be a family of bundles with connections. Precisely, $\mathcal{E}$ is a principal $G$-bundle on $\mathcal{X}$, $\nabla$ is a relative connection on $\mathcal{E}$ along the fibers of $\mathcal{X} \to S$. This family is called isomonodromic, if $\nabla$ can be extended to an absolute flat connection $\tilde{\nabla}$ on $\mathcal{X}$.

It is an easy exercise that this condition is equivalent to the monodromy of $\nabla$ being constant over $S$.

In the case of a connection with singularities the above definition still makes sense. In the case of regular singularities, the condition is still equivalent to the monodromy being constant; in the case of irregular singularities one has to require that both monodromy and the so-called Stokes structures at the singularities do not change. In this case a new direction of deformation arises: one can deform formal normal forms of connections as well. (We assume the singularities to be generic.) We refer the reader to [Fed] for more details.
In the simplest case of connections on \( \mathbb{P}^1 \) with a pole of order 2 at zero, and a pole of order 1 at infinity, the isomonodromic deformation amounts to deforming the conjugacy class of the leading term at zero. This isomonodromic deformation is especially important because it is closely related to Frobenius manifolds.

2.9. Deformation to isomonodromy. The sheaf \( \mathcal{D}_h(Bun^b_{N(0)+\infty}) \) has a natural deformation to a certain twisted cotangent bundle on \( Bun^b_{N(0)+\infty} \). Precisely, this twisted cotangent bundle is the moduli space of pairs \( (E, \nabla) \), where \( E \in Bun^b_{N(0)+\infty} \), \( \nabla \) is a connection on \( E \) with a simple pole at \( \infty \) such that \( \nabla \) has the following form at zero (in a trivialization compatible with the level structure):

\[
d + dt \left( \frac{h}{t^{N+1}} + \text{higher order terms} \right), \quad h \in \mathfrak{h}^*.
\]

Denote this moduli space by \( Conn^h_{N(0)+\infty} \) (this is a moduli space of extended connections), denote by \( Conn^h_{N(0)+\infty} \) the open subspace of \( Conn^h_{N(0)+\infty} \) corresponding to \( h \in \mathfrak{h}^{*,r} \). Trivializing a bundle in the formal neighborhood of zero (this trivialization is well defined up to the action of \( G_+ \)), we get a natural map \( Conn^h_{N(0)+\infty} \to Conn_{N,\mathfrak{h}} / G_+ \). The following statements are very close to the results of [Fed]:

1. \( Conn^h_{N(0)+\infty} \) is a Poisson space.
2. The map \( Conn^h_{N(0)+\infty} \to Conn_{N,\mathfrak{h}} / G_+ \) is Poisson.
3. The pullbacks of functions along this map are hamiltonians of isomonodromic deformation (compare with [Fed, Theorem 2]).

In §2.7 we have constructed some global sections of \( \mathcal{D}_h(Bun^b_{N(0)+\infty}) \). Comparing with §2.6 we see that these sections are quantizations of isomonodromic hamiltonians.

Note, that similar results are valid for any smooth projective curve \( X \) and for any level structure divisor.

3. Proofs of main results

3.1. Finite dimensional preliminaries. Recall first the definition of polynomials \( P_i^\beta \) and \( Q_i^\beta \) from (8). Recall that \( U \) is identified with an open subset of \( G/B_- \), thus \( g \) acts on \( U \). Also, \( U \) is isomorphic to \( u \) as a variety. Fix a homogeneous coordinate system \( y_\alpha, \alpha \in \Delta_+ \) on \( U \). Here homogeneous means that

\[
h \cdot y_\alpha = -\alpha(h)y_\alpha \forall h \in \mathfrak{h}, \alpha \in \Delta_+.
\]

Then (possibly after multiplying coordinates \( y_\alpha \) by some constants) the action is given by (cf. [Fed §5.2.5])

\[
e_i \mapsto \frac{\partial}{\partial y_\alpha} + \sum_{\beta \in \Delta_+} P_i^\beta(y_\alpha) \frac{\partial}{\partial y_\beta},
\]

\[
h_i \mapsto -\sum_{\beta \in \Delta_+} \beta(h_i)y_\beta \frac{\partial}{\partial y_\beta},
\]

\[
f_i \mapsto \sum_{\beta \in \Delta_+} Q_i^\beta(y_\alpha) \frac{\partial}{\partial y_\beta}.
\]
The Cartan subalgebra acts on $\mathfrak{g}$ and on $\mathcal{D}(U)$ diagonally, so we can define the weight $w_\mathfrak{h}$ with respect to the Cartan subalgebra. Clearly $w_\mathfrak{h}(e_\alpha) = \alpha$, $w_\mathfrak{h}(h_i) = 0$, $w_\mathfrak{h}(f_\alpha) = -\alpha$. Also, (1) implies that $w(y_\alpha) = -\alpha$, $w(\partial/\partial y_\alpha) = \alpha$. The action preserves $\mathfrak{h}$-weight. Later we shall need the following

**Lemma 3.1.** Multiplying the coordinates $y_\alpha$ by certain non-zero constants, we may assume that for all $\alpha \in \Delta_+$

$$e_\alpha \mapsto \frac{\partial}{\partial y_\alpha} + \sum_{\beta \in \Delta_+, \beta > \alpha} P^\beta_\alpha(y_\gamma) \frac{\partial}{\partial y_\beta},$$

where $P^\beta_\alpha$ are certain polynomials without constant terms.

**Proof.** We can write

$$e_\alpha \mapsto \sum_{\beta \in \Delta_+} c^\beta_\alpha \frac{\partial}{\partial y_\beta} + \sum_{\beta \in \Delta_+} P^\beta_\alpha(y_\gamma) \frac{\partial}{\partial y_\beta},$$

where $c^\beta_\alpha$ are certain constants, $P^\beta_\alpha$ are polynomials without constant terms. The $\mathfrak{h}$-weight considerations show that $c^\beta_\alpha$ is a diagonal matrix and that $P^\beta_\alpha = 0$ unless $\beta > \alpha$; $c^\beta_\alpha$ is non-degenerate because the action of $u$ on $U$ is transitive, so in particular it is transitive at the unit. \qed

**3.2. Proof of part (a) of Theorem** Let $V_k(\mathfrak{g})$ be the level $k$ vacuum representation of $\hat{\mathfrak{g}}$ with its usual structure of vertex algebra (see [Fre] §2.2.4). Let $M_\mathfrak{g}$ be the Fock representation of $\mathfrak{g}$, see [Fre] §5.4.1. Let $\pi^0_0$ be the vertex algebra associated to $\hat{\mathfrak{h}}_k$. Then by [Fre] Theorem 6.2.1 there is a vertex algebra homomorphism $V_k(\mathfrak{g}) \to M_\mathfrak{g} \otimes \pi^k_{-k}$ given by formulae (8).

It is clear that $M_N$ is a smooth representation of $\mathcal{A}$, thus $M_N$ is a module over the vertex algebra $M_\mathfrak{g}$ with the $M_\mathfrak{g}$-module structure induced from its $\mathcal{A}$-module structure (see for example [PBZ] §5.1.8). Similarly, $\hat{\pi}_N^{k-k}$ is a module over the vertex algebra $\pi^0_{-k}$ with $\pi^0_{-k}$-module structure induced from $\hat{\mathfrak{h}}^{k-k}$-module structure. It follows that (8) give a $V_k(\mathfrak{g})$-module structure on $M_N \otimes \hat{\pi}_N^{k-k}$. But this is equivalent to our statement. \qed

**Proof of part (b) of Theorem** Since $\text{Sym}(t^N \mathfrak{h})$ acts on $\hat{\mathfrak{h}}^{k-k}$-module $\hat{\pi}_N^{k-k}$, it also acts on $M_N \otimes \hat{\pi}_N^{k-k}$ by endomorphisms of $\mathcal{A} \otimes \hat{\mathfrak{h}}^{k-k}$-module structure, thus it acts by endomorphisms of the $\mathfrak{g}$-module structure and the statement follows. \qed

**3.3. Action of $\hat{\mathfrak{g}}$ revisited.** Let us extend the $\mathfrak{h}$-weight to loop algebras, by setting

$$w_\mathfrak{h}(e_{\alpha,n}) = w_\mathfrak{h}(a_{\alpha,n}) = \alpha,$$

$$w_\mathfrak{h}(f_{\alpha,n}) = w_\mathfrak{h}(a^*_{\alpha,n}) = -\alpha,$$

$$w_\mathfrak{h}(h_{i,n}) = w_\mathfrak{h}(h_{i,n}) = 0.$$

Then (8) shows that the above action of $\hat{\mathfrak{g}}$ is compatible with the $w_\mathfrak{h}$ gradation.

We would like to extend the formulae (8) to $e_\alpha(z)$ and $f_\alpha(z)$, where $\alpha$ is not necessarily a simple root. Let $b_\alpha(z)$ be the field corresponding to $h_\alpha \in \mathfrak{h}$ (i.e. $b_\alpha(z)$ is a linear combination of $b_i(z)$ with the same coefficients as in the expression of $h_\alpha$ through $h_i$).
Proposition 3.1. The action of $\hat{\mathfrak{g}}$ on $M_N \otimes \hat{\pi}_N^{k-k_c}$ is given by

$$
eq \alpha(z) + \sum_{\beta \in \Delta_+} :P^\alpha_\beta(a^*(z))a_\beta(z):,$$

$$ h_i(z) \mapsto - \sum_{\beta \in \Delta_+} \beta(h_i) a_\beta(z) a_\beta(z) + b_i(z),$$

$$ f_\alpha(z) \mapsto \sum_{\beta \in \Delta_+} :Q^\alpha_\beta(a^*(z))a_\beta(z): + \sum_{\beta \in \Delta_+} \tilde{Q}^\alpha_\beta(a^*(z))\partial_z a_\beta(z) + b_\alpha(z)a^*_\alpha(z) + \sum_i b_i(z) R^\alpha_i(a^*(z)),$$

where:

- $P^\alpha_\beta$ are polynomials in $a^*_\gamma$ without constant terms;
- $Q^\alpha_\beta$ are polynomials in $a^*_\gamma$ without constant terms;
- $R^\alpha_i$ are polynomials in $a^*_\gamma$ without constant and linear terms.

Proof. Recall from [Fre] how the homomorphism (3) is constructed. One starts with (10) and extends it to a map $\mathfrak{g} \otimes \mathbb{C}((t)) \to \text{Vect}(U((t)))$, where $U((t))$ is the ind-scheme of loops in $U$, [Fre] §5.3.2. The next step is to lift this map to a map $V_\kappa(\mathfrak{g}) \to M_\mathfrak{g}$, see [Fre], Theorem 5.6.8 and Theorem 6.1.3.

It follows from Lemma 3.1 and discussion before Theorem 6.1.3 in [Fre] that under this map

$$ e_\alpha(z) \mapsto \sum_{\beta \in \Delta_+} a_\alpha(z) + \sum_{\beta \in \Delta_+} :P^\alpha_\beta(a^*(z))a_\beta(z):.$$  

The map $V_\kappa(\mathfrak{g}) \to M_\mathfrak{g}$ can be deformed by any element of $\mathfrak{h}^*((t))$, giving rise to a map $V_\kappa(\mathfrak{g}) \to M_\mathfrak{g} \otimes \pi_0$ but the images of $e_\alpha(z)$ are not modified (see proof of [Fre] Lemma 6.1.4). The last step is to deform the level to $k \neq k_c$. However, we claim that the images of $e_{\alpha,n}$ do not depend on $k$. Indeed, for $\alpha = \alpha_i$ it is clear from (8). All other $e_{\alpha,n}$ can be obtained by commuting $e_{\alpha_i,n}$. Finally, the fields $b_1(z)$ are absent in the RHS of (12), so the result of commuting does not depend on $k$. This proves the first formula in the proposition.

To calculate the images of $f_\alpha(z)$ we shall take a different approach. First we show that

$$ f_\alpha(z) \mapsto \sum_{\beta \in \Delta_+} :Q^\alpha_\beta(a^*(z))a_\beta(z): + \sum_{\beta \in \Delta_+} \tilde{Q}^\alpha_\beta(a^*(z))\partial_z a_\beta(z) + \sum_i b_i(z) R^\alpha_i(a^*(z)).$$

Indeed, the vertex algebra formalism shows that RHS has to be a field, corresponding to some element of $M_\mathfrak{g} \otimes \pi_0^{k-k_c}$ via the vertex operation. It also has to be a field of conformal dimension one. But it is easy to see that this is the most general form of such a field. The considerations of $\mathfrak{h}$-weight show that $R^\alpha_i$ and $Q^\alpha_\beta$ have no constant terms. The same considerations show that the linear term of $R^\alpha_i$ is of the form $\lambda_i a^*_\alpha(z)$.

Next, we have

$$ \text{Res}_w[e_\alpha(z), f_\alpha(w)]dw = h_\alpha(z).$$

The operator corresponding to LHS has the form

$$ \sum_i \lambda_i b_i(z) + \sum_i b_i(z) R^\alpha_i(a^*(z)) + \text{Terms without } b_i(z),$$
where $R''$ have no constant terms. Comparing with the left hand side we conclude that $\sum_i \lambda_i b_i(z) = b_0(z)$. \hfill $\square$

3.4. Constructing a map $\varphi : M_{N,k,\text{reg}} \to M_N \otimes \hat{\pi}_{N,\text{reg}}^{k-k_c}$. We want to introduce some notation. We define the degree of an element of $A \otimes \mathfrak{h}^\mathbb{C}$ by setting

$$\deg a_{\alpha,n} = \deg a_{\gamma,\nu}^* = \deg b_{\gamma,\nu} = -n.$$ 

Similarly, we define a degree of an element of $A \otimes \hat{\mathfrak{g}}$. Note that the commutation relations preserve the degree, so we get gradations on $A \otimes \mathfrak{h}^\mathbb{C}$ and $A \otimes \hat{\mathfrak{g}}$. A field $x(z) = \sum_n x_n z^{-n}$ is called conformal if $\deg x_n + n$ does not depend on $n$. For a conformal field $x(z)$ we define $x(z)_{(n)}$ to be a unique $x_m$ with $\deg x_m = -n$. In particular $a_{\alpha}(z)_{(n)} = a_{\alpha,n}, a^*_{\beta}(z)_{(n)} = a^*_{\beta,n}, e_{\alpha}(z)_{(n)} = e_{\alpha,n}, \text{etc.}$.

For notational simplicity we denote $|0\rangle \otimes (|0\rangle \otimes 1) \in M_N \otimes \hat{\pi}_N^{k-k_c}$ by $|0\rangle'$. We also denote $|0\rangle \otimes (|0\rangle \otimes 1) \in M_N \otimes \hat{\pi}_N^{k-k_c}$ by $|0\rangle'$.

**Lemma 3.2.** $|0\rangle' \in M_N \otimes \hat{\pi}_N^{k-k_c}$ is annihilated by $\hat{\mathfrak{g}}_\mathbb{C}$.

**Proof.** Let $A$ be the product of any number of generators $a_{\alpha,n}^*$, with at least one $n > 0$, then for any $\gamma$ and $m$ we have $\langle A a_{\gamma,m} | 0 \rangle' = 0$ by definition of normal ordering.

We show first that $e_{\alpha,n} | 0 \rangle' = 0$ if $n \geq N$. Indeed, by Proposition 3.1 we have

$$e_{\alpha,n} | 0 \rangle' = a_{\alpha,n} | 0 \rangle' + \sum_{\beta \in \Delta_+} \sum_{\mu + \nu = n} :P_\beta^N(a^*(z))_{(\mu)} a_{\beta,\nu} | 0 \rangle'.$$

The first term is clearly zero; the terms with $\nu \geq N$ are also zero. If $\nu < N$, then $\mu > 0$, and the term is zero by the remark in the beginning of the proof.

Finally, we have

$$f_{\alpha,n} | 0 \rangle' = \sum_{\beta \in \Delta_+} (Q_\beta^N(a^*(z)) a_{\beta}(z))_{(\mu)} | 0 \rangle' + \sum_{\beta \in \Delta_+} (\tilde{Q}_\beta^N(a^*(z)) \partial_z a^*_\beta(z))_{(\mu)} | 0 \rangle'$$

$$+ \sum_{\mu + \nu = n} b_{\alpha,\mu} a_{\beta,n}^* | 0 \rangle' + \sum_{\mu + \nu = n} b_{\gamma,\nu} R_{\gamma\mu}^\nu(a^*(z))_{(\nu)} | 0 \rangle'.$$

For the first two terms we note that every monomial has degree $-n$. Thus, it has either a multiple $a_{\gamma,m}$ with $m \geq N$ and the term is zero, or $a_{\beta,n}^*$ with $m > 0$, and the term is again zero. The last two terms are clearly zero if $\nu \geq 0$. Otherwise, $\mu > N$, and we use the fact that $b's$ and $a^*'s$ commute. \hfill $\square$

Thus there is a unique homomorphism $M_{N,k} \to M_N \otimes \hat{\pi}_N^{k-k_c}$ sending $|0\rangle$ to $|0\rangle'$.

**Lemma 3.3.** For all $i$ we have

$$h_{i,N} | 0 \rangle' = b_{i,N} | 0 \rangle'.$$

**Proof.** By previous computations

$$h_{i,N} | 0 \rangle' = \sum_{\mu + \nu = N} \sum_{\beta \in \Delta_+} \beta(h_{i}) a_{\beta,\mu}^* a_{\beta,\nu} | 0 \rangle' + b_{i,N} | 0 \rangle'$$

(we can remove normal ordering because $a_{\beta,\mu}^*$ and $a_{\beta,\nu}$ commute, since $\mu + \nu = N \neq 0$). All terms $a_{\beta,\mu}^* a_{\beta,\nu} | 0 \rangle'$ are zero because $\mu \geq 0$ or $\nu > N$. \hfill $\square$

Thus our homomorphism intertwines the action of $t^N \mathfrak{h}$ on the modules, so it gives rise to a homomorphism $\varphi : M_{N,k,\text{reg}} \to M_N \otimes \hat{\pi}_N^{k-k_c}$. 
3.5. PBW bases and filtrations. Choosing some order of positive roots, we get a basis of $\text{Fun}(h^{\ast,r})$-module $M_{N,k,\text{reg}}$ given by lexicographically ordered monomials

$$J := \prod_{i<n} (h_{i,n})^{i_{i,n}} \prod_{\alpha<n} (f_{\alpha,n})^{\alpha_{\alpha,n}} \prod_{\beta<n} (e_{\beta,n})^{\ell_{\beta,n}} |0\rangle \otimes 1,$$

where all exponents $i_{i,n}$, $\alpha_{\alpha,n}$, and $\ell_{\beta,n}$ are nonnegative integers. Also, we require that all $h_{i,n}$ with $n > 0$ are to the left from all $h_{i,n}$ with $n < 0$. Denote the set of such monomials by $\mathcal{J}$. Every $x \in M_{N,k,\text{reg}}$ can be uniquely written as

$$\sum_{J \in \mathcal{J}} J \cdot \varphi_J,$$

where $\varphi_J$ are functions on $h^{\ast,r}$.

Further, a basis of $\text{Fun}(h^{\ast,r})$-module $M_N \otimes \hat{\pi}_{N,\text{reg}}^k$ is given by monomials

$$A := \prod_{i<n} (b_{i,n})^{i_{i,n}} \prod_{\alpha<n} (a_{\alpha,n}^\ast)^{\alpha_{\alpha,n}} \prod_{\beta<n} (a_{\beta,n}^\ast)^{\ell_{\beta,n}} |0\rangle,'$$

where all exponents $i_{i,n}$, $\alpha_{\alpha,n}$, and $\ell_{\beta,n}$ are nonnegative integers. We order the multiples by a convention that all $b_{i,n}$ with $n > 0$ are to the left from all $b_{i,n}$ with $n < 0$. We denote the set of such monomials by $\mathcal{B}$.

Set

$$I(h_{i,n}) = I(f_{\alpha,n}) = I(e_{\alpha,n}) = N - n.$$

Define the weight $I(J)$ of a monomial $J \in \mathcal{J}$ by extending the above assignment to monomials. Setting $I(J \cdot \varphi) = I(J)$ for $\varphi \in \text{Fun}(h^{\ast,r})$, we get a filtration on $M_{N,k,\text{reg}}$. Note that $(M_{N,k,\text{reg}})_{<0} = 0$.

Similarly, by assigning

$$I(b_{i,n}) = I(a_{\alpha,n}) = N - n, \quad I(a_{\alpha,n}^\ast) = -n$$

we get a filtration on $M_N \otimes \hat{\pi}_{N,\text{reg}}^k$, again $(M_N \otimes \hat{\pi}_{N,\text{reg}}^k)_{<0} = 0$.

To complete the proof of Theorem 3, we shall (i) note that the map $\varphi$ preserves the filtrations; (ii) identify $\text{gr} M_{N,k,\text{reg}}$ and $\text{gr}(M_N \otimes \hat{\pi}_{N,\text{reg}}^k)$ with rings of functions on certain schemes and (iii) identify the induced morphism of the schemes.

3.6. $\text{gr} \mathcal{U}_{h,\mathfrak{g}}$ and $\text{gr} M_{N,k,\text{reg}}$ as functions on schemes. Recall that for a vector space $V$ we have the loop space $V \otimes \mathbb{C}(t)$ and its restricted dual $V^*((t)) dt$. These spaces are ind-schemes. Further, $I$ gives a filtration on $\mathcal{U}_{h,\mathfrak{g}}$ and we have

$$\text{gr} \mathcal{U}_{h,\mathfrak{g}} = \text{Sym} \mathfrak{g}((t)) = \text{Fun}(\mathfrak{g}^*((t))) dt).$$

(More precisely, $\mathfrak{g}^*((t)) dt$ is an ind-scheme, so we should view $\text{Sym} \mathfrak{g}((t))$ as an algebra filtered by ideals.)

The filtration on $\mathcal{U}_{h,\mathfrak{g}}$ gives rise to a filtration on its quotient module $M_{N,k}$. We see that $\text{gr} M_{N,k}$ is a quotient algebra of $\text{Sym} \mathfrak{g}((t))$ equal to

$$\mathbb{C}[h_{i,n}, \bar{e}_{\alpha,m}, \bar{f}_{\alpha,m} | n \leq N, m < N],$$

where we denote the image of $h_{i,n}$ in $\text{gr} M_{N,k}$ by $\bar{h}_{i,n}$ etc. Next,

$$\text{gr} M_{N,k,\text{reg}} = \mathbb{C}[\bar{h}_{i,n}, \bar{h}^{-1}_{i,n}, \bar{e}_{\alpha,m}, \bar{f}_{\alpha,m} | n \leq N, m < N].$$
It follows that \( \text{gr} \mathcal{M}_{N,k,\text{reg}} \) can be identified with the ring of functions on the scheme of infinite type

\[
\hat{g}^* := \left\{ \sum_{n \geq -N-1} A_n t^n dt, \forall n A_n \in \mathfrak{g}^*, A_{-N-1} \in \mathfrak{h}^{* r} \right\}.
\]

The action of \( \text{gr} \mathcal{U}_k \hat{g} \) on \( \text{gr} \mathcal{M}_{N,k,\text{reg}} \) is given by the inclusion \( \hat{g}^* \mapsto g^*((t)) dt. \)

3.7. \( \text{gr}(\mathcal{A} \otimes \mathfrak{h}_k) \) and \( \text{gr}(\mathcal{M}_N \otimes \mathfrak{h}^k_{N,\text{reg}}) \) as functions on schemes. Set

\[
\hat{U} := \{ u \in U[[t]] : u(0) = 1 \}.
\]

\[
\hat{b}^* := \left\{ \sum_{n \geq -N-1} A_n t^n dt, \forall n A_n \in \mathfrak{b}^*, A_{-N-1} \in \mathfrak{h}^{* r} \right\}.
\]

\[
\hat{u}^* := \left\{ \sum_{n \geq -N} A_n t^n dt, \forall n A_n \in \mathfrak{u}^* \right\}.
\]

\[
\hat{h}^* := \left\{ \sum_{n \geq -N-1} A_n t^n dt, \forall n A_n \in \mathfrak{h}^*, A_{-N-1} \in \mathfrak{h}^{* r} \right\}.
\]

Define

\[
W := \hat{U} \times \hat{b}^* = \hat{U} \times \hat{u}^* \times \hat{h}^*.
\]

Note that all the spaces introduced are schemes (of infinite type). Recall that \( U((t)) \) stands for the ind-scheme of loops with values in \( U \).

**Lemma 3.4.** There is a natural identification

\[
\text{gr}(\mathcal{A} \otimes \mathfrak{h}_k) = \text{Fun}(U((t)) \times (\mathfrak{b}^* ((t)) dt)),
\]

\[
\text{gr}(\mathcal{M}_N \otimes \mathfrak{h}^k_{N,\text{reg}}) = \text{Fun}(W),
\]

so that the action of \( \text{gr}(\mathcal{A} \otimes \mathfrak{h}_k) \) on \( \text{gr}(\mathcal{M}_N \otimes \mathfrak{h}^k_{N,\text{reg}}) \) is given by the natural inclusion \( W \rightarrow U((t)) \times (\mathfrak{b}^* ((t)) dt) \).

**Proof.** Let us identify \( T^* U = U \times \mathfrak{u}^* \) using the left trivialization. Let \( w_\alpha \) be the vector field \( \partial/\partial y_\alpha \), viewed as a function on \( T^* U \). Then \( (y_\alpha, w_\alpha) \) is a system of coordinates on \( U \times \mathfrak{u}^* \). Let \( y_{\alpha,n} \) and \( w_{\alpha,n} \) be the corresponding jet coordinates on \( U((t)) \times (\mathfrak{u}^*((t)) dt) \). That is, viewing \( (u(t), A(t) dt) \in U((t)) \times (\mathfrak{u}^*((t)) dt) \) as a \( \mathbb{C}((t)) \)-valued point of \( T^* U \), we have:

\[
\sum y_{\alpha,n} t^n = y_\alpha(u(t)), \quad \sum w_{\alpha,n} t^n = w_\alpha(u(t), A(t)).
\]

Identifying \( \tilde{a}_{\alpha,n} \mapsto w_{\alpha,-n-1}, \tilde{a}_{\alpha,n} \mapsto y_{\alpha,-n} \) we obtain

\[
\text{gr} \mathcal{A} = \text{Fun}(U((t)) \times (\mathfrak{u}^*((t)) dt)).
\]

Further, \( \text{gr} \mathfrak{h}_k = \mathbb{C}[\hat{b}_{i,n}] = \text{Fun}(\mathfrak{b}^* ((t)) dt) \) and the first statement of the lemma follows.

Clearly \( \{ y_{\alpha,n}, w_{\alpha,n} | n \geq 0 \} \) is a coordinate system on the jet scheme \( U[[t]] \times (\mathfrak{u}^*[[t]] dt) \). Note that \( w_\alpha \) is linear on the fibers of \( T^* U \rightarrow U \), so we have

\[
w_\alpha(u(t), t^{-N} A(t)) = t^{-N} w_\alpha(u(t), A(t)),
\]

therefore

\[
\{ y_{\alpha,n}, n \geq 0, \ w_{\alpha,m}, m \geq -N \}
\]
is a coordinate system on $U[[t]] \times \hat{u}^*$. Note that $u(t) \in U[[t]]$ satisfies $u(0) = 1$ iff $y_0, u = 0$ for all $\alpha$. Hence $gr M_N = Fun(\hat{U} \times \hat{u}^*)$. Now the second statement of the lemma follows. The rest of the lemma is now obvious. \qed

3.8. End of proof of Theorem 3 In part (b) of Theorem 3 we have constructed an action of $\hat{g}$ on $M_N \otimes \hat{\pi}_{N,reg}^{k-k_c}$. It gives rise to an action of $gr \mathcal{U}_k \hat{g}$ on $gr(M_N \otimes \hat{\pi}_{N,reg}^{k-k_c})$, which actually comes from a homomorphism of algebras.

**Proposition 3.2.** The action of $gr \mathcal{U}_k \hat{g}$ on $gr(M_N \otimes \hat{\pi}_{N,reg}^{k-k_c})$ is induced by $\Phi : W \to \hat{g}^*((t)) dt$ given by

$$\Phi(u, A) = Ad_u A, \quad u \in \hat{U}, A \in \hat{b}^*.$$

**Proof.** Step 1. From the proof of part (a) of Theorem 3 recall the vertex algebras $V_k(g)$ and $M_g \otimes \pi_0^{k-k_c}$. The weight $I$ defined above gives a filtration on these algebras as well, and we claim that $gr V_k(g) = Fun(g^*[t] \ dt) = Fun(g[[t]])$, where we identify $g$ and $g^*$ via a non-degenerate pairing. This statement would follow immediately from the proof of Proposition 7.1.1 in [Fed] if we take the usual Poincaré–Birkhoff–Witt filtration. However, $V_k(g)$ has a grading by degree, and on $j$-th graded piece we have $I(J) = N \cdot PBW(J) - j$, where we denote by $PBW(J)$ the PBW-weight of a monomial. It follows that the associated graded space is the same for both filtrations.

Similarly,

$$gr(M_g \otimes \pi_0^{k-k_c}) = Fun(U[[t]] \times (b^*[t] \ dt)) = Fun(U[[t]] \times b^-[[t]]).$$

Finally, the map on associated graded vertex algebras, induced from the free field realization map $V_k(g) \to M_g \otimes \pi_0^{k-k_c}$ is induced from the map $U[[t]] \times (b^*[t]) \to g^*[t]$ given by $(u, A) \mapsto Ad_u A$. (cf. [Fed] (7.1.5)).

**Step 2.** The map of vertex algebras induces a map of their enveloping algebras

$$\text{Sym} \ g((t)) \to gr(A \otimes \hat{b}_{k-k_c}).$$

It is easy to see that it is induced by the map $U((t)) \times (b^*((t)) dt) \to g^*((t)) dt$ given by the same formula $(u, A) \mapsto Ad_u A$.

Since the action of $gr(A \otimes \hat{b}_{k-k_c})$ on $gr(M_N \otimes \hat{\pi}_{N,reg}^{k-k_c})$ is given by the natural inclusion $W \hookrightarrow U((t)) \times (b^*((t)) dt)$, the proposition follows. \qed

**Corollary.** (a) The map $gr \varphi : gr M_{N,k,reg} \to gr(M_{N,k} \otimes \hat{\pi}_{N,reg}^{k-k_c})$ is induced by $\Phi' : W \to \hat{g}^*$ given by

$$\Phi'(u, A) = Ad_u A.$$

(b) $\Phi'$ is an isomorphism.

**Proof.** (a) Follows obviously from the proposition.

(b) We need to check that every $A \in \hat{g}^*$ can be conjugated to an element of $\hat{b}^*$ by a unique element of $\hat{U}$. We look for such an element of $\hat{U}$ in the form $\exp(u_1 t + u_2 t^2 + \ldots)$. We can find $u_1$'s one-by-one, using the fact that the adjoint action of any element of $\hat{b}^*$ on $u$ is invertible, cf. [Fed] Proposition 5]. \qed

We see that $gr \varphi$ is an isomorphism. So $\varphi$ is an isomorphism. This completes the proof of Theorem 3.
3.9. Proof of Theorem. Let $B_0 \subset B$ be the set of PBW monomials, which are products of $a_{\alpha,n}$ with $0 < n < N$, $a_{\alpha,n}^*$, where $-N < n < 0$ and $b_{i,n}$, with $-N \leq n < N$. Set $U_1 \hat{h}_k := U_1 \hat{h}_k / U_1 \hat{h}_k \cdot (1 - 1)$. Let $(A \otimes U_1 \hat{h}_k)_+$ be the left ideal in $A \otimes U_1 \hat{h}_k$ generated by $a_{\alpha,n} \otimes 1$ with $n \geq N$, $a_{\alpha,n}^* \otimes 1$ with $n \geq 0$ and $1 \otimes b_{i,n}$ with $n > N$.

It is easy to see that every $A \in B_0$ is annihilated by $(A \otimes U_1 \hat{h}_k)_+$. Let $\Lambda$ be the Fun($\hat{h}^{*,-r}$)-submodule of $M_N \otimes \hat{\pi}_N,reg$ generated by $B_0$, it is easy to see that

$$
\Lambda = (M_N \otimes \hat{\pi}_N,reg)_+ \otimes (A \otimes U_1 \hat{h}_k,reg)_+ = \text{End}_{A \otimes U_1 \hat{h}_k,reg} (M_N \otimes \hat{\pi}_N,reg).
$$

The last identity is proved in the same way as Proposition 2.3.

Thus $\Lambda \subset \text{End}_{\mathfrak{g}} (M_N \otimes \hat{\pi}_N,reg)$. Let us calculate the ring structure on $\Lambda$. Viewing $a_{\alpha,m}, a_{\alpha,m}^* \in B_0$ as endomorphisms and recalling that by our convention they act on the right, we get

$$
|0\rangle \langle 0|'[a_{\alpha,m}, a_{\alpha,m}^*] = (a_{\alpha,m}^* a_{\alpha,m} - a_{\alpha,m} a_{\alpha,m}^*) = a_{\alpha,m} a_{\alpha,m}^* a_{\alpha,m} + a_{\alpha,m} a_{\alpha,m}^* a_{\alpha,m} = -\delta_{\alpha,\beta} \delta_{n,-m} |0\rangle \langle 0|',
$$

where $[.,.]_A$ is the commutator in the ring $\Lambda$. Thus

$$
[a_{\alpha,m}, a_{\alpha,m}^*] = -\delta_{\alpha,\beta} \delta_{n,-m}.
$$

Similarly,

$$
(b_{i,n}, b_{j,m})_A = n(k - k_c)(h_i, h_j) \delta_{n,-m},
$$

and all the other commutators are zero. We want to identify $\Lambda$ with the LHS of (14). To this end we identify $a_{\alpha,m}$ with a coordinate system on $n$-th copy of $\mathfrak{u}^*$, we identify $a_{\alpha,m}$ with $-\partial / \partial a_{\alpha,m}$.\[52\]

Next, for $0 < n < N$, we identify $b_{i,n}$ with the function $h_i$ on $n$-th copy of $\mathfrak{h}^*$, we identify $b_{i,N}$ with the function $h_i$ on $\mathfrak{h}^{*,-r}$, hence $b_{i,N}$ can occur in negative powers. We want to identify $b_{i,-n}$ with a differential operator on $n$-th copy of $\mathfrak{h}^*$ corresponding to a constant vector field. The commutation relations (14) show that

$$
b_{i,-n} \mapsto -\frac{2n(k - k_c)}{(\alpha_i, \alpha_i)} \partial_{\alpha_i}.
$$

Finally we identify $b_{i,0}$ with the function $h_i$ on the “extra” copy of $\mathfrak{h}^*$. This gives the required isomorphism.

3.10. Case $N = 1$. According to the proof of Theorem 1, the ring $\Lambda$ in the case $N = 1$ is generated by $|0\rangle \otimes 1 \mapsto \varphi^{-1}(b_{i,-1}|0\rangle)$, $|0\rangle \otimes 1 \mapsto \varphi^{-1}(b_{i,0}|0\rangle)$, and $|0\rangle \otimes 1 \mapsto \varphi^{-1}(b_{i,1}|0\rangle)$, where $i = 1, \ldots, \text{rk} \mathfrak{g}$. Let us calculate these $\varphi$-preimages. Let $\rho$ be the half-sum of positive roots.

Proposition 3.3. (a) $\varphi(h_{i,1}|0\rangle \otimes 1) = b_{i,1}|0\rangle$;
(b) $\varphi((h_{i,0} - 2\rho(h_i)|0\rangle \otimes 1) = b_{i,0}|0\rangle$;
(c) $\varphi(h_{i,-1}|0\rangle \otimes 1 + \sum_{\alpha \in \Delta_+} a(h_i) e_{\alpha,0} |0\rangle \otimes h^{-1}_{\alpha,1}) = b_{i,-1}|0\rangle$.

Proof. Part (a) follows from Lemma 3.3. For part (b) we have

$$
h_{i,0}|0\rangle = -\sum_{\alpha \in \Delta_+} a(h_i) a_{\alpha,0} a_{\alpha,0}^* |0\rangle + b_{i,0}|0\rangle.
$$

But by our definition of normal ordering we have:

$$
a_{\alpha,0} a_{\alpha,0}^* |0\rangle = a_{\alpha,0}^* a_{\alpha,0} |0\rangle = -|0\rangle + a_{\alpha,0} a_{\alpha,0}^* |0\rangle = -|0\rangle.
$$
and the statement follows.

Let us prove part (c). We have
\[(15)\quad h_{i,-1}(0)^\prime = -\sum_{\alpha \in \Delta_+} \alpha(h_\alpha) a_{\alpha,-1}^* a_{\alpha,0}(0)^\prime + b_{i,-1}(0)^\prime.\]

Further,
\[(16)\quad f_{\alpha,0}(0)^\prime = \sum_{\beta \in \Delta_+} (Q_\beta^0(a^*(z))a_\beta(z))(0)|0\rangle^\prime + \sum_{\beta \in \Delta_+} (Q_\beta^0(a^*(z))\partial_2 a_\beta^*(z))(0)|0\rangle^\prime + (b_\alpha(z)a_\alpha^*(z))(0)|0\rangle^\prime + \sum_i (b_i(z)R_\alpha^0(a^*(z))(0)|0\rangle^\prime.\]

The second terms vanishes: indeed, for a monomial \(A\) a monomial in \(a_{\gamma,\mu}^*\) of degree zero it is clear that \(A(0)^\prime = 0\).

Let us show that the first term vanishes. Consider the expression \(AA_{\beta,\nu}(0)^\prime\), where \(A\) is a monomial in \(a_{\gamma,\mu}^*\) of degree \(\nu\). If \(\nu > 0\), we clearly get zero. If \(\nu < 0\), then \(A\) contains \(a_{\beta,\nu}^*\) with \(\mu > 0\) and the term vanishes. If \(\nu = 0\), and \(A\) contains \(a_{\gamma,\mu}^*\) with \(\mu > 0\), we again get zero. Thus the only interesting case is when \(A = \prod a_{\gamma,\alpha}^*\).

If there are at least two multiples in \(A\) we still get zero. In the remaining case:
\[a_{\gamma,0}^*a_{\beta,\nu}^*(0)^\prime = -\delta_{\gamma,\beta}(0)^\prime = 0,\]

since \(\gamma = \beta\) is impossible due to \(h\)-weight considerations.

The term \(b_{i,\mu}R_\alpha^0(a^*(z))(0)^\prime\) vanishes if \(\mu > 1\) and if \(\mu \leq 0\). It also vanishes when \(\mu = 1\) because any monomial in \(R_\alpha^0(a^*(z))(0)^\prime\) has at least two terms, and one of them has to kill \(|0\rangle^\prime\). The term \(b_{\alpha,\mu}a_{\alpha,-\mu}^*\) vanishes unless \(\mu = 1\). Thus \(f_{\alpha,0}(0)^\prime = b_{\alpha,1}a_{\alpha,-1}(0)^\prime\) and \(\varphi(f_{\alpha,0}(0) \otimes h_{\alpha,1}^{-1}) = a_{\alpha,-1}(0)^\prime\).

Let us calculate
\[(17)\quad \varphi(e_{\alpha,0}f_{\alpha,0}(0) \otimes h_{\alpha,1}^{-1}) = e_{\alpha,0}a_{\alpha,-1}^*(0)^\prime = a_{\alpha,0}a_{\alpha,-1}^*(0)^\prime + \sum_{\beta \in \Delta_+} \sum_{\beta > \alpha} (Q_\beta^0(a^*(z))(0)a_{\beta,\nu}^*a_{\alpha,-1}(0)^\prime).\]

For \(\nu \neq 1\), the last term vanishes for the same reasons as the first term in (16). For \(\nu = 1\) it vanishes because \(\beta > \alpha\), so that \(a_{\beta,1}\) and \(a_{\alpha,-1}^*\) commute.

Thus \(\varphi(e_{\alpha,0}f_{\alpha,0}(0) \otimes h_{\alpha,1}^{-1}) = a_{\alpha,0}a_{\alpha,-1}^*(0)^\prime\). Combining with (16) we get the proposition. □

3.10.1. Right version. Applying the involution \(\iota\) we see that \(\Lambda^r \subset \text{End}_g(M_{1,k,\text{reg}}^\ast)\) is generated by
\[
\hat{b}_{i,1} : 1 \otimes |0\rangle \mapsto 1 \otimes |0\rangle h_{i,1},
\hat{b}_{i,0} : 1 \otimes |0\rangle \mapsto 1 \otimes |0\rangle h_{i,0},
\hat{b}_{i,-1} : 1 \otimes |0\rangle \mapsto 1 \otimes |0\rangle h_{i,-1} + \sum_{\alpha \in \Delta_+} \alpha(h_\alpha) h_{\alpha,1}^{-1} \otimes |0\rangle f_{\alpha,0} e_{\alpha,0}.
\]

The ring structure is given by
\[
[\hat{b}_{i,-1}, \hat{b}_{j,1}] = (-k - k_c)(h_i, h_j),
\]
the other commutators are zero. The isomorphism \(\text{End}_g(M_{1,k,\text{reg}}^\ast) \approx g(\mathfrak{h}^\ast)^r \otimes \text{Fun}(\mathfrak{h}^\ast)\) is given by \(\hat{b}_{i,1} \mapsto h_i, \hat{b}_{i,-1} \mapsto -\frac{2(k + k_c)}{(\alpha_i, \alpha_i)} \partial_\alpha\).
Identifying \( \mathfrak{h}^* \) with \( \mathfrak{h} \) via the bilinear form, we get
\[
(18) \quad \hat{b}_{i,1} \mapsto 2\alpha_i/(\alpha_i, \alpha_i), \quad \hat{b}_{i,-1} \mapsto (-k_c - k)\partial_{h_i}.
\]

3.11. **Proof of Theorem** \[2\] By PBW Theorem the space of coinvariants \( V/\hat{\mathfrak{g}}^+ \cdot V \) is naturally isomorphic to \( \text{Sym}(\mathfrak{t}\mathfrak{h}) \otimes \mathcal{V} = \text{Fun}(\mathfrak{h}) \otimes \mathcal{V} \), where we use the bilinear form to identify \( \mathfrak{t}\mathfrak{h} \) and \( \mathfrak{h}^* \). Note that under this identification \( h_{\alpha,1} \) is identified with \( 2\alpha/(\alpha, \alpha) \). Localizing we get \( \mathcal{F}(V) = \text{Fun}(\mathfrak{h}^r) \otimes \mathcal{V} \).

Denote the connection given by the \( \mathcal{D} \)-module structure on \( \mathcal{F}(V) \) by \( \nabla \). Set \( \hbar := \frac{1}{2(k + k_c)} \). By (18) we have
\[
\nabla_{h_i}(\varphi \otimes v) = 2\hbar \hat{b}_{i,-1}(\varphi \otimes v),
\]
where \( \varphi \otimes v \in \text{Fun}(\mathfrak{h}^r) \otimes \mathcal{V} \). To calculate the connection it is enough to calculate \( \hat{b}_{i,-1}(1 \otimes v) = h_{i,-1}(1 \otimes v) + \sum_{\alpha \in \Delta^+} \alpha(h_i)(\alpha, \alpha) f_{\alpha,0} e_{\alpha,0} v \).

Note that \( h_{i,-1} v = 0 \). Using the Leibnitz rule we get
\[
\nabla_{h_i}(\varphi \otimes v) = \partial_{h_i} \varphi \otimes v + 2\hbar \sum_{\alpha \in \Delta^+} \frac{\alpha(h_i)(\alpha, \alpha)}{\alpha} \varphi \otimes f_{\alpha,0} e_{\alpha,0} v.
\]

Switching to differential forms we re-write:
\[
\nabla = d + 2\hbar \sum_{\alpha \in \Delta^+} \frac{d\alpha}{\alpha} \left( \frac{\alpha(\alpha)}{2} f_{\alpha,0} e_{\alpha} \right).
\]

Let us compare it with the Casimir connection \[\text{Lar2}, \S2\]
\[
\nabla_{\text{Casimir}} = d + \hbar \sum_{\alpha \in \Delta^+} \frac{d\alpha}{\alpha} \left( \frac{\alpha(\alpha)}{2} (e_{\alpha} f_{\alpha} + f_{\alpha} e_{\alpha}) \right) = \nabla + \hbar \sum_{\alpha \in \Delta^+} \frac{d\alpha}{\alpha} \left( \frac{\alpha(\alpha)}{2} h_{\alpha} \right).
\]

Now recall that \( \mathcal{F}(V) \) is a sheaf on \( \mathfrak{h}^r \times \mathfrak{h}^r \) with a connection along \( \mathfrak{h}^r \). Let \( \mathcal{L} \) be the trivial line bundle on \( \mathfrak{h}^r \times \mathfrak{h}^r \) with a connection along \( \mathfrak{h}^r \) given by
\[
\nabla_0 = d + \hbar \sum_{\alpha \in \Delta^+} \frac{d\alpha}{\alpha} \left( \frac{\alpha(\alpha)}{2} h_{\alpha} \right),
\]
where \( h_{\alpha} \) is now viewed as a function on \( \mathfrak{h}^r \). We see that
\[
(\mathcal{F}(V), \nabla_{\text{Casimir}}) = (\mathcal{F}(V), \nabla) \otimes (\mathcal{L}, \nabla_0).
\]

and the Theorem is proved.

**Remark.** We have used a version of the Casimir connection with truncated \( \mathfrak{sl}_2 \) Casimir operators \( \frac{\alpha(\alpha)}{2} (e_{\alpha} f_{\alpha} + f_{\alpha} e_{\alpha}) \). If we use the usual Casimir operators \( \frac{\alpha(\alpha)}{2} (e_{\alpha} f_{\alpha} + f_{\alpha} e_{\alpha} + \frac{1}{2} h_{\alpha}^2) \) (cf. \[\text{Lar1}, \S5\] and \[\text{Lar3}\]) instead, then we get similar results with
\[
\nabla_0 = d + \hbar \sum_{\alpha \in \Delta^+} \frac{d\alpha}{\alpha} \left( \frac{\alpha(\alpha)}{2} \left( h_{\alpha} + \frac{1}{2} h_{\alpha}^2 \right) \right).
\]
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