TRAVELLING WAVES FOR DISCRETE STOCHASTIC BISTABLE EQUATIONS

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Abstract. Many physical, chemical and biological systems have an inherent discrete spatial structure that strongly influences their dynamical behaviour. Similar remarks apply to internal or external noise, as well as to nonlocal coupling. In this paper we study the combined effect of nonlocal spatial discretization and stochastic perturbations on travelling waves in the Nagumo equation, which is a prototypical model for bistable reaction-diffusion partial differential equations (PDEs). We prove that under suitable parameter conditions, various discrete-stochastic variants of the Nagumo equation have solutions, which stay close on long time scales to the classical monotone Nagumo front with high probability if the noise level and spatial discretization are sufficiently small.

Keywords: Nagumo equation, bistability, stochastic partial differential equation, lattice differential equation, travelling wave, noise, discretization, Allen-Cahn equation, Ginzburg-Landau equation, Φ⁴ model, Schlögl equation.

1. Introduction

The Nagumo [41] partial differential equation (PDE) for \( V = V(t, x) \in \mathbb{R} \) is given by

\[
\partial_t V = \nu \partial_x^2 V + f(V), \quad (t, x) \in [0, \infty) \times \mathbb{R},
\]

where \( f(V) = f(V; a) = V(1 - V)(V - a) \), where \( a \in (0, 1/2) \) and \( b, \nu > 0 \) are parameters. The PDE (NagR) is a prototypical model of bistability in the sense that \( V \equiv 0 \) and \( V \equiv 1 \) are locally asymptotically steady states, while \( V \equiv a \) is unstable. For any \( a \in (0, 1/2) \) there exist travelling front solutions

\[
V(t, x) = V^{TW}(x - ct) = V^{TW}(\zeta), \quad \zeta := x - ct
\]

connecting the two locally stable states, i.e., \( V^{TW}(\infty) = 0 \) and \( V^{TW}(-\infty) = 1 \). The front is spatially monotone \((V^{TW})'(\zeta) > 0\), left-moving with a unique wave speed satisfying \( c = c(a) < 0 \), unique up to translation, and (locally) nonlinearly stable [17, 31]. Extensions to the standing wave for \( a = 1/2 \), and to right-moving waves for \( a \in (1/2, 1) \) are easily obtained from symmetry arguments [17, 31].

The Nagumo equation plays an important role in neuroscience [16] as the simplest toy model of signal propagation through axons. It is very actively studied also outside neuroscience applications, e.g., as an amplitude equation [12], in population dynamics modelling [8], and in material science [2]. In fact, the PDE (NagR) is also referred to as the Allen-Cahn equation in materials science, as the real Ginzburg-Landau equation in normal form theory, as the Φ⁴-model in quantum field theory and as the Schlögl model in chemistry. When modelling signal propagation in neurons, several effects are not taken into account in (NagR):

(I) The electric signal travelling through a myelinated nerve fiber do not move continuously. The signal jumps from one gap in the myeline coating of the nerve fiber to the next [30]. This suggests the use of a spatially discrete setting.
(II) The propagation of the electric signal along the axon is influenced by many internal and/or 
external biophysical processes. Since modelling every process microscopically is usually 
impossible, this leads naturally to a stochastic version of the Nagumo equation.

(III) The precise coupling distance of diffusion between myeline coating gaps is not easy to 
measure. This implies we should also allow for some form of nonlocal coupling.

(IV) The axon does not have infinite length. Hence, one should consider bounded domains 
instead. Furthermore, propagation takes place on a finite time scale.

(V) The propagation of fronts is an idealization of the electrical signal as usually we would 
expect localized pulses. This requires systems of reaction-diffusion equations.

Here we shall not cover the case (V), which is usually modelled using the Hodgkin-Huxley [23] 
or FitzHugh-Nagumo [18] PDEs but see [13]. However, all the arguments we present can be carried 
over, in principle, to these cases. Instead, we focus on a model to cover the combined 
effects (I)-(IV).

In fact, each of the individual aspects (I)-(IV) have received some attention recently. We briefly 
review some background and introduce the relevant PDEs.

The space-discrete setting will be modeled via a lattice differential equation (LDE), whose so-
lution at node $i$, called $V_i = V_i(t)$, represents the potential at the $i$-th myeline gap. The discrete 
Nagumo equation with nonlocal diffusive coupling, reads at each node $i$ for some fixed coupling 
range $R \in \mathbb{N}$ as follows

\begin{equation}
\partial_t V_i = \frac{\nu}{R h^2} \left( \sum_{j=-R}^{R} J(j) (V_j - V_i) \right) + f(V_i), \quad i \in \mathbb{Z},
\end{equation}

where $h$ is a parameter controlling the discretization and $J(j) \in \mathbb{R}$ are weights. The classical case of local diffusive coupling is given by

\begin{equation}
\partial_t V_i = \frac{\nu}{h^2} (V_{i+1} - 2V_i + V_{i-1}) + f(V_i), \quad i \in \mathbb{Z},
\end{equation}

The equation (1.2) is the nearest-neighbor discretization of the Nagumo LDE can be 
interpreted as being posed on an infinite lattice $\mathbb{Z}$ with lattice spacing $h$ so that $V_i$ corresponds to $V(ih)$. We write

\begin{equation}
V^h := (\ldots, V_{-2}, V_{-1}, V_0, V_1, V_2, \ldots)
\end{equation}

to emphasize that $V^h$ solves the discrete Nagumo equation. The LDE (1.2) also admits travelling 
wave solutions for sufficiently strong diffusion strength $\nu$, i.e., for sufficiently large coupling; for small 
coupling, propagation failure may occur [30, 23, 40]. More generally, the type of the discrete model 
may have substantial impact on the existence and uniqueness of travelling waves of the Nagumo and 
FitzHugh-Nagumo PDEs [14, 15, 26] as well as on the numerical analysis of discretization schemes 
for travelling waves [20].

Notice that in (1.1) the general difference stencil involves $2R$ nodes, and $R$ may diverge with $N$, 
so it can be viewed as nonlocal. In fact, nonlocal variants of the Nagumo equation have been studied 
in the LDE/PDE setting in several analytical and numerical works; see e.g. [11, 14, 5, 10, 14] and 
references therein. A similar difference stencil as used here was studied in [4, 25], where existence 
of travelling wave solutions was proven for unbalanced nonlinearities and under certain conditions 
on the weights.

Another important variation of the Nagumo equation is the stochastic PDE (SPDE) version for 
$U = U(t, x)$ given by

\begin{equation}
\partial_t U = \nu \partial_x^2 U + f(U) + g(U)\xi, \quad (t, x) \in [0, \infty) \times \mathbb{R}
\end{equation}

where $\xi = \xi(t, x) = \partial_t W(t, x)$ is a space-time dependent stochastic process, $W$ is a trace class 
Wiener process, and $g$ arises as a suitable mapping from modelling considerations, see [28 - 210].
Although there is a detailed existence theory for many SPDEs \cite{13 11 37} going back to at least the late 1970s, and good physical understanding of many noisy pattern phenomena going back at least to the 1990s \cite{19}, the rigorous mathematical study of noisy (Nagumo) waves has just started to develop recently; see e.g. \cite{22 27 30 18}. These studies have been driven by numerical observations \cite{39 17 49} revealing that travelling wave solutions may persist under stochastic forcing, but their speed and form may change with varying noise strength. Of course, these results are also connected to recent advances in the numerical analysis of classical numerical schemes for the Nagumo SPDE \cite{46}; see also e.g. \cite{21 28 38}. Furthermore, we refer to a recent survey of stochastic travelling wave problems for scalar reaction-diffusion equations SPDE for additional detailed background references \cite{32}.  

In this paper we are interested in the combined influence of (I)-(IV) on the finite-time evolution of travelling fronts. In this context, the key object is we are going to study is the stochastic LDE (SLDE)

$$
(d\text{SNag}N) \quad du_i = \left( \frac{\nu}{R h^2} \left( \sum_{j=-R}^{R} J(j)(u_j - u_i) \right) + f(u_i) \right) dt + g_N(u_i) \, dB_i,
$$

with \(i \in \{1, 2, \ldots, N\}, \ u_i = u_i(t)\) stochastic processes on the lattice points \(i\), independent identically distributed (iid) Brownian Motions \(B_i(t)\), a suitable matrix-valued map \(g_N\) obtained as a projection of \(g\), and \(R \leq N\) with \(R \in \mathbb{N}\). In addition to viewing the solution as a vector \(u^h := (u_1, u_2, \ldots, u_N)\) we may interpret the solution \(u^h\), say via piecewise linear interpolation, as a function on the interval \(D := [-L, L]\) with \(u_1\) and \(u_N\) corresponding to the values at the left and right endpoints. Despite its evident importance for applications, particularly in the context of neuroscience, there seems to be no study available regarding the dynamics of \(d\text{SNag}N\), although some first study without reference to dynamics is \cite{7}. One potential reason could be that physical intuition would lead us to believe that the effects coming from (I)-(IV) are somehow “small” so that we can neglect them, at least in certain parameter regimes. To make this intuition mathematically precise is a key contribution of our study. For parameters for which travelling waves to the deterministic PDE are known to exist, we prove in the stochastic setting that for sufficiently small noise and sufficiently small \(h\) that the solution to the Nagumo SLDE \(d\text{SNag}N\) is close to a phase-adapted travelling front solution of the Nagumo PDE \(\text{Nag}R\) on the interval \(D := [-L, L]\) over finite time scales. Our main result can be stated as follows:

**Theorem 1.1.** Let \( V^{TW} = V^{TW}(t, x) \) be a travelling front solution to \(\text{Nag}R\), \( u_0^n \) be deterministic lattice initial data and \( u^h \) a solution to \(d\text{SNag}N\) on the interval \(D := [-L, L]\) to the initial data \( u_0^n \equiv u^n(0) \). Suppose \( L > 0 \) is large enough, while \( \delta > 0, \ T > 0, \) and \( \varepsilon > 0 \) are given. Suppose the initial data \( u^h(0) \) satisfies

\[
||u^h(0) - V^{TW}(0, \cdot)||_{L^2(\mathbb{R})}^2 < \varepsilon.
\]

Then there exists \( \varepsilon > 0 \) and \( c \in \mathbb{R} \) such that, for sufficiently small noise and sufficiently small \( h \), we have for the solution \( u^h = u^h(t) \) of \(d\text{SNag}N\) the estimate

\[
P\left( \sup_{t \in [0, T]} ||u^h(t) - V^{TW}(\cdot - ct)||_{L^2(\mathbb{R})} > \delta \right) \leq \varepsilon.
\]

The precise formulation of the noise structure, such as the statement of “sufficiently small noise” will be discussed below, it mainly deals with a sufficiently small covariance of the underlying Wiener process, and the growth of \( g \). In summary, Theorem 1.1 confirms our intuition from biophysics/neuroscience, i.e., the wave propagation mechanism is robust against structural nonlocal and stochastic perturbations, which makes the Nagumo equation a good model.
Our proof relies on a discrete version of the monotone operator theory approach to SPDEs, as presented in [12] and described in the monographs [11, 57]. Our proof essentially decomposes the different error terms [53], e.g., the dynamical stochastic approximation error is treated separately from the discretization error in the stencil. Therefore, it is natural to consider several intermediate evolution equations, e.g., the Nagumo PDE on a bounded domain
\[(\text{NagD}) \quad \partial_t v = \nu \partial_x^2 v + f(v), \quad (t,x) \in [0,T) \times \mathcal{D},\]
with Neumann boundary conditions, and similarly the Nagumo SPDE on a bounded domain
\[(\text{SNagD}) \quad \partial_t u = \nu \partial_x^2 u + f(u) + g(u)\xi, \quad (t,x) \in [0,T) \times \mathcal{D}\]
with suitable multiplicative noise described in section 2.2.

Hopefully, our notation conventions are by now already evident to the reader but let us stress that we use \(v, V\) for the deterministic PDE solutions whereas \(u, U\) are SPDE solutions. Small letter solutions \(u, v\) are based on the bounded domain \(\mathcal{D}\) and capital letter solutions \(U, V\) on the unbounded domain \(\mathbb{R}\). Furthermore, discrete solutions will be treated as vectors \(u^h, U^h, v^h, V^h\) or indicated by subindices.

2. Notation and Setting

We discretize the interval \(\mathcal{D} \subset \mathbb{R}\) into \(N\) intervals of size \(h\) and enumerate the respective grid points with the index \(i\). The set of grid points is denoted by \(\mathcal{D}^h\). We work with the Gelfand triple of Banach spaces
\[H^1_0(\mathcal{D}) \cong W^{1,2}_0(\mathcal{D}) \subset L^2(\mathcal{D}) \subset H^{-1}(\mathcal{D}).\]
Note that functions in the Sobolev spaces such as \(L^2(\mathcal{D})\) or \(W^{1,2}_0(\mathcal{D})\) evaluated on the grid \(\mathcal{D}^h\) are \(N\)-dimensional vectors. Extending these functions in a piecewise linear manner, we can work with them also in the original Sobolev spaces. We choose an orthonormal basis \(\{e_k\} \subset L^2(\mathcal{D})\), consisting of elements in \(W^{1,2}_0(\mathcal{D})\), and span \(\mathbb{R}^N\) with the first \(N\) of these basis vectors. The projections of the Sobolev spaces on their first \(N\) basis vectors can then be identified with \(\mathbb{R}^N\), e.g. \(P_N H^1_0(\mathcal{D}) \cong \mathbb{R}^N\). To simplify notation, we denote both the scalar product on \(L^2(\mathcal{D})\) and on \(\mathbb{R}^N \cong P_N L^2(\mathcal{D})\) by \((\cdot, \cdot)\), while we denote the associated norm by \(\| \cdot \|\). By \((\cdot, \cdot)_{H^{-1}(\mathcal{D}), H^1_0(\mathcal{D})} \) we denote the dual product as well as the scalar product on \(\mathbb{R}^N \cong P_N H^1_0(\mathcal{D})\), where the projection to \(\mathbb{R}^N\) is spanned by the first \(N\) basis vectors of \(H^1_0(\mathcal{D})\). Using the representation
\[w = \sum_{k=1}^N (w, e_k)e_k \quad \text{for all elements } w \in H^{-1}(\mathcal{D}^h),\]
we can work with the same basis vectors also in the space \(H^1_0(\mathcal{D})\) and its dual.

2.1. Operations with discrete-in-space functions. Let \(u^h(\cdot, t)\) be a piecewise linear function on the grid \(\mathcal{D}^h\). There are several ways to define a (discrete) gradient. Using only two nodal values, we can identify \(\nabla^h u^h(ih, t)\) either with the backward difference \(D^- u_i(t) = \frac{1}{h} (u_i(t) - u_{i-1}(t))\) or its adjoint, the forward difference \(D^+ u_i(t) = \frac{1}{h} (u_{i+1}(t) - u_i(t))\). This choice leads to the discrete nearest-neighbour Laplacian as \(\Delta^h u_i = D^+ D^- v_i := h^{-2} (u_{i+1} - 2u_i + u_{i-1})\).

We would like to use more general discrete stencils, which involve up to \(R\) neighbours of \(u_i\) in each direction, in other words involving the nodal values \(u_{i-R} \ldots u_{i+R}\). We introduce coefficients \(J(j) \in \mathbb{R}\) to attribute a weight of the \(j\)-th right neighbouring nodal value \(u_{i+j}\). Such a general
second-order stencil then reads
\[
\Delta_h^2 u_i = \frac{1}{h^2} \sum_{j=-R}^{R} J(j)(u_j - u_i)
\]
\[= \frac{1}{h^2} \left( -2 \sum_{k=1}^{R} J(k)u_i + \sum_{k=1}^{R} J(k)u_{i+k} + \sum_{k=1}^{R} J(k)u_{i-k} \right) \tag{2.2}
\]
\[= \frac{1}{h^2} \sum_{k=1}^{R} J(k) (-2u_i + u_{i+k} + u_{i-k})
\]
\[= \frac{1}{h} \left( \frac{1}{h} \sum_{k=1}^{R} J(k)(u_{i+k} - u_i) - \frac{1}{h} \sum_{k=1}^{R} J(k)(u_i - u_{i-k}) \right). \]

Note that $J(j)$ are fixed numbers and do not change with time, therefore the central difference operator $\Delta_h^2$ it is deterministic and time-independent. It is therefore natural to define
\[
\nabla_R u_i := (\Delta_h^2)^{1/2} u_i = \frac{1}{h} \sum_{k=1}^{R} J(k)(u_i - u_{i-k})
\]
as the long-range analogues of the difference operators $D^j$. Using the adjoint operator to $\nabla_R$, denoted by $\nabla_R^*$, we can write (2.2) as
\[
\Delta_h^2 u_i = \nabla_R^* (\nabla_R u_i). \tag{2.4}
\]

We need to impose conditions on the coefficients $J(j)$ to ensure that (2.2) approximates a Laplacian. To this aim, notice first of all that by construction $J(0) = -\sum_{|j|<R} J(j)$, which is a special case of diagonal dominance, from which we immediately conclude $(\Delta_h^2 u^h, u^h) \leq 0$. We make the following assumptions throughout:

**Assumption 2.1.** The weights $J(j) \in \mathbb{R}$ satisfy

(A1) $J(j) = J(-j)$

(A2) $\sum_{j=-R}^{R} J(j) j^2 = 1$

(A3) $\sum_{j=-R}^{R} J(j) j^4 < \infty$ or at least $h^4 \sum_{j=-R}^{R} J(j) j^4 \sim o(h^3)$

The symmetry condition (A1) ensures that the operator $\Delta_h^2$ is self-adjoint in $\ell^2$. (A1) is often not strictly necessary mathematically, but it simplifies computations and is moreover very natural considering the real-world phenomena from which the model was derived. The moment conditions (A2)-(A3) guarantee that we approximate a Laplacian, as can be seen from the construction of finite difference operators via Taylor approximation at nodal distance $jh$, which gives
\[
\sum_{j=-R}^{R} J(j)(v_j - v_i) = h^2 \left( \sum_{j=-R}^{R} j^2 J(j) \frac{\partial^2 v_i}{\partial x^2} + \frac{h^4}{12} \left( \sum_{j=-R}^{R} j^4 J(j) \right) \frac{\partial^4 v_i}{\partial x^4} + O(h^6). \right) \tag{2.5}
\]

Note that, in contrast to [7], the assumptions on the moments imply a certain decay in the coefficients $J(j)$. This is because we allow for arbitrarily diverging stencil range $R$, especially also for $R = N$, while in the semigroup approach used in [7], the range was limited to $R \sim N^{1/2}$.

### 2.2. The probabilistic setting

Denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ a filtered probability space. A function $u : \mathcal{D} \times [0, T] \times \Omega \to \mathbb{R}$, which is evaluated on the grid $\mathcal{D}^h$, will be denoted by $u^h(\cdot, t, \omega)$ and at each node identified with a stochastic process $X^h_i(\omega)$, which takes values in $\mathbb{R}$.

We denote by $W : [0, T] \times \Omega \to L^2(\mathcal{D})$ a $Q$-Wiener process with values in $L^2(\mathcal{D})$. We assume that $W(t) = W(t, \omega)$ is adapted to the filtration $\mathcal{F}_t$. We construct the covariance operator $Q$ such
that is linear, bounded, self-adjoint, and positive semi-definite and that \( Q \) has a common set of eigenfunctions with \( \Delta \), so that
\[
Q e_k = \mu_k e_k.
\]
Moreover we ensure that \( Q \) is of trace class, i.e., \( M_Q := \text{Tr} Q < +\infty \), which implies that the sum of the eigenvalues of \( Q \) is bounded \( \sum_{k=1}^{\infty} \mu_k < \infty \). It is well known that a \( Q \)-Wiener process in \( L^2(\mathcal{D}) \) can be represented in \( L^2(\Omega, C([0, T], L^2(\mathcal{D}))) \) using a sequence of iid Brownian motions \( \{B_j\}_{j \in \mathbb{N}} \)
and considering the series
\[
W(t) = W(\cdot, t) = \sum_{k=1}^{\infty} \sqrt{\mu_k} e_k(\cdot) B_k(t).
\]
By means of an exponential inequality and the Borel-Cantelli Lemma, the convergence of the series can be obtained uniformly with probability one. Thus, the sample paths of \( W(t) \) belong to \( C([0, T], L^2(\mathcal{D})) \) almost surely, and we may therefore choose a continuous version.

2.3. The stochastic Nagumo equation. The stochastic Nagumo equation we are using in this work is a perturbation of the (deterministic) Nagumo equation \([\text{Nag}_D]\). As stochastic perturbation, we choose a \( Q \)-Wiener Process \( W(t) \) on \( L^2(\mathcal{D}) \) with covariance operator \( Q \) being positive semi-definite, symmetric and of trace class. Moreover, we take a multiplicative noise term called \( G(u) : L^2(\mathcal{D}) \rightarrow \mathcal{H} \), where we denote by \( \mathcal{H} \) the space of Hilbert-Schmidt operators, and assume it is Lipschitz continuous and satisfies linear growth conditions. More precisely, we assume
\[
\|G(u)\|^2_{\mathcal{H}} \leq c(1 + \|u\|^2) \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega
\]
and
\[
\|G(u) - G(v)\|^2_{\mathcal{H}} \leq c(\|u - v\|^2) \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega
\]
for all \( u, v \in L^2(\mathcal{D}) \).

For the rest of this work, we will avoid the operator notation and use the representation \((G(u)\chi)(x) := g(u(x))\chi(x)\) for \( u, \chi \in L^2(\mathcal{D}) \) and \( g : \mathbb{R} \rightarrow \mathbb{R} \). In this notation, we state the core modelling assumption
\[
g(0) = g(1) = 0,
\]
which means that the effect of the noise should be concentrated on the wave front. It is well-known that for such noise, if the initial data \( u_0(x) \in [0, 1] \) for all \( x \in \mathcal{D} \), the solution also satisfies \( u_0(x) \in [0, 1] \) for all \( x \in \mathcal{D} \) and \( t > 0 \).

To summarize, the lattice equation \([\text{SNag}_D]\) we are considering in this work should be an approximation of a stochastic Nagumo equation on a bounded interval \([-L, L]\) driven by multiplicative trace-class noise, which acts only on the front. The continuum model is given by
\[
(S\text{Nag}_D) \quad du(t) = [\nu u_t(t) + f(u(t))]dt + g(u(t)) dW(t) \quad \text{on } \mathcal{D} \times [0, T].
\]
The existence of mild solutions to \([\text{SNag}_D]\) for Lipschitz nonlinearities is classical, see e.g. \([43, 34]\). Using a localization and truncation argument, see e.g. \([20, 7]\), local-in-time results can be carried over polynomial nonlinearity \( f \) with one-sided Lipschitz condition such as in \([\text{SNag}_D]\), while global-in-time results have to exploit the sign in the cubic nonlinearity leading to dissipativity for large \( u \). \([5, 11, 35]\). Via monotone operator theory, one may see furthermore \([42]\) that \([\text{SNag}_D]\) admits a variational solution in \( L^2(\Omega, C([0, T], L^2(\mathcal{D}))) \cap L^2(\Omega \times [0, T], H^1_0(\mathcal{D})) \). In particular, we have that almost surely \( u \in L^\infty([0, T]; L^2(\mathcal{D})) \cap L^2([0, T]; H^1_0(\mathcal{D})) \). Due to Itô’s formula, the stochastic
By Sobolev embedding, we can get for some constants $C$(2.13)
$$
\|u(t)\|^2_{L^2} = \|u(0)\|^2_{L^2} + 2\nu \mathbb{E} \left[ \int_0^t (\Delta u(s), u(s)) \ ds \right] + 2 \mathbb{E} \left[ \int_0^t (f(u(s)), u(s)) \ ds \right] + \mathbb{E} \left[ \int_0^t g(u(s))^2 \ ds \right].
$$
(2.11)

Regarding the stochastic LDE version of the Nagumo SPDE, we may introduce the convenient abbreviation
$$
\Delta h \Delta_{j} u_i := \frac{1}{R h^2} \sum_{j=-R}^{R} J(j)(u_j - u_i)
$$
to indicate that the difference stencil can be regarded a (generalized) discretization of the Laplace operator, if the coefficients $J(j) \in \mathbb{R}$ satisfy Assumption 2.4. This fact will be justified in more detail in the next section. Moreover, taking advantage of the vector notation $u^h := (u_1, u_2, \ldots, u_N)$, we may write the discrete-in-space evolution \((dSNag)\) as
$$
du^h(t) = \nu \Delta h \Delta_{j} u_i(t) dt + f(u^h(t)) dt + g(u^h(t)) dW^h(t) \text{ on } \mathcal{D}^h \times [0, T],
$$
where we denoted by $W^h(t)$ a sufficiently large, yet finite, partial sum to the infinite sum in (2.7) and denoted $\mathcal{D}^h := \{1, \ldots, N\}$. Recalling $(G(u)\chi)(x) := g(u(x))\chi(x)$ for $u, \chi \in L^2(\mathcal{D})$, and that $g$ is a function $g: \mathbb{R} \to \mathbb{R}$, we may define the discrete multiplicative noise operator in the same way for $u^h$ (without changing the notation). It then obviously satisfies (2.8) and (2.9). Due to the trace class assumption, we may always select the truncation level for the Wiener process sufficiently large to guarantee that solutions of \((dSNag)\) stay close to the same equation driven by $W$.

2.4. Monotone operators. The following paragraph recalls that, due to the properties of $f$, the sum $\nu \Delta + f$ defines a monotone operator. In the continuous context, this is well-understood: a concise treatment of the theory of monotone operators can be found for example, in [50], or, including local monotonicity, in [37]. We will briefly state those precise properties, which will be used in the proofs. First, we note that the nonlinear term $f(u)$ is Lipschitz continuous w.r.t. $u$ on bounded subsets of $H^1_0(\mathcal{D})$ with Lipschitz constant independent of $t$. More precisely, we have the following standard results, which we include for completeness here:

**Lemma 2.2.** For any $M > 0$, there exists a constant $K_M > 0$ such that the local Lipschitz continuity condition holds:
$$
\|f(v_1) - f(v_2)\| \leq K_M \|v_1 - v_2\|_{H^1_0(\mathcal{D})} \quad \text{a.e. } (t, \omega) \in [0, T] \times \Omega
$$
for any $v_1, v_2 \in H^1_0(\mathcal{D})$ with $\|v_1\|_{H^1_0(\mathcal{D})} < M$ and $\|v_2\|_{H^1_0(\mathcal{D})} < M$.

**Proof.** We have
$$
\|v_1^3 - v_2^3\|^2 = \|v_1^2(v_1 - v_2) + v_2^2(v_1 - v_2)\|^2 \leq 8 \left( \|v_1^2(v_1 - v_2)\|^2 + \|v_2^2(v_1 - v_2)\|^2 \right)
$$
By Sobolev embedding, we can get for some constants $C_1, C_2 > 0$ the estimates $\|v_1\|_{L^4} \leq C_1 \|v_1\|_{H^1_0}$ and $\|v_1^2 v_2\|_{L^2} \leq C_2 \|v_1\|^4_{H^1_0} \|v_2\|^2_{H^1_0}$. Hence, there exists a constant $C_3 > 0$ such that
$$
\|v_1^3 - v_2^3\|^2 \leq C_3 \left( \|v_1\|^4_{H^1_0} + \|v_2\|^4_{H^1_0} \right) \|v_1 - v_2\|^2_{H^1_0}
$$
which satisfies (2.12). \(\square\)

Furthermore, we can derive estimates similar to Lemma 2.2, providing us growth bounds such as
$$
\|f(v)\|_{H^{-1}(\mathcal{D})} \leq c_1 \|v\|_{H^1_0(\mathcal{D})} \left( 1 + \|v\|^2_{L^2(\mathcal{D})} \right)
$$
(2.15)
as well as
\[
\|f(v_1) - f(v_2)\|_{H^{-1}(\mathcal{D})} \leq c_2 \left(1 + \|v_1\|_{H^1(\mathcal{D})}^2 + \|v_2\|_{H^1(\mathcal{D})}^2\right) \|v_1 - v_2\|_{L^2(\mathcal{D})}.
\]
Moreover, the combined operator \(A := \nu \Delta + f\) is obviously semi-continuous in \(H^1_0(\mathcal{D})\), in the sense that for all \(v_1, v_2, v_3 \in H^1_0(\mathcal{D})\) and \(t \in [0, T]\) the mapping
\[
\theta \mapsto \langle A(v_1 + \theta v_2), v_3 \rangle
\]
is continuous from \(\mathbb{R}\) into \(\mathbb{R}\). Due to \((2.8)\) and \((2.9)\), we know that the sum of operators satisfies for all \(v \in H^1_0(\mathcal{D})\), \(t \in [0, T]\) the coercivity condition
\[
\langle \nu \Delta v + f(v), v \rangle + \|g(v)\|_N \leq -\nu \|v\|_{H^1(\mathcal{D})}^2 + (c_0 + \nu)\|v\|_{L^2(\mathcal{D})}^2,
\]
with \(c_0 = \sup_{v \in \mathbb{R}} f'(v) = \frac{1}{3}(a^2 - a + 1)\), holds. Finally, the sum of operators satisfies for all \(v_1, v_2 \in H^1_0(\mathcal{D})\) the monotonicity condition
\[
\langle \nu \Delta v_1 + f(v_1) - \nu \Delta v_2 - f(v_2), v_1 - v_2 \rangle + \|g(v_1) - g(v_2)\|_{L^2}^2 \leq c_a \|v_1 - v_2\|^2
\]
on \([0, T] \times \Omega\). We will now verify similar properties hold in the discrete setting of our LDEs. The proof is elementary, yet we will provide it in detail to make the strategy transparent for stochastic LDEs.

**Lemma 2.3.** Let the conditions of Assumption 2.1 on the general stencil \(\Delta \mathbb{R}_h\) be satisfied. Then the discrete operators appearing in the ISNag algorithm satisfy the following estimates:

\section*{(L1) coercivity}
\[
\sum_{i=1}^{N} (\nu \Delta \mathbb{R}_h u_i + f(u_i)) u_i + \|g(u_i)\|^2 \leq -\nu \|\nabla u^h\|^2 + c_a \|u^h\|^2
\]

\section*{(L2) monotonicity}
\[
\sum_{i=1}^{N} \left(\nu \Delta \mathbb{R}_h u_i - \nu \Delta \mathbb{R}_h v_i + f(u_i) - f(v_i)\right) (u_i - v_i) + \|g(u_i) - g(v_i)\|^2 \leq c_a \|u^h - v^h\|^2.
\]

**Proof.** First, note that
\[
\sum_{i=1}^{N} f(u_i) \cdot u_i = \sum_{i=1}^{N} (f(u_i) - f(0)) (u_i - 0) = \sum_{i=1}^{N} \left(\frac{f(u_i) - f(0)}{u_i - 0}\right) (u_i - 0)^2 \leq c_a \sum_{i=1}^{N} u_i^2
\]
where we used the mean-value theorem in the last inequality. Second, we look at the discrete integration by parts formula, which reads in its standard form
\[
\sum_{i=1}^{N} \Delta u_i \cdot u_i = \sum_{i=1}^{N} D^+(D^- u_i) \cdot u_i = -\sum_{i=1}^{N} (D^- u_i) \cdot D^- u_i = -\sum_{i=1}^{N} (D^- u_i)^2.
\]
It can be extended to the long-range case via the operators \(\nabla \mathbb{R}^-\) and \(\nabla \mathbb{R}^+\). Using the last equation and previous inequality, we can derive easily, in the special case of the nearest-neighbour stencil, the coercivity
\[
\sum_{i=1}^{N} \left(\nu D^+(D^- u_i) + f(u_i)\right) u_i + \|g(u_i)\|^2 \leq -\nu \|D^- u^h\|^2 + c_a \|u^h\|^2
\]
where we used \((2.8)\). Using \((2.9)\), we get the monotonicity of the sum of operators
\[
\sum_{i=1}^{N} \left(\nu D^+(D^- u_i) - \nu D^+(D^- v_i) + f(u_i) - f(v_i)\right) (u_i - v_i) + \|g(u_i) - g(v_i)\|^2 \leq c_a \|u^h - v^h\|^2.
\]
For the general case, it was already noted (without proof) by Bates, Chen, Chmait [4], that general stencils of the form (2.2) satisfy the monotonicity condition with $c = 0$ instead of $c_a$. Indeed, testing with $u^h$ and using the summation by parts formula we obtain

\begin{equation}
\langle \Delta_R^h u^h, u^h \rangle = \frac{1}{h^2} \sum_{i=1}^{N} \sum_{j=-R}^{R} J(j)(u_j - u_i) \cdot u_i
\end{equation}

(2.26)

Repeating the strategy of the nearest neighbour case, using again (2.22), (2.8), (2.9) and (2.26) gives us immediately (2.20) and (2.21), which concludes the proof.

\section{Existence and stability of travelling wave fronts}

Recall that our goal is show that the stochastic LDE (dSNag) admits, for sufficiently small $h$ and sufficiently small noise, travelling front-like solutions, in the sense that its solutions are very likely to be close to classical deterministic travelling fronts; see also Theorem 1.1.

To approach this problem, we use several ingredients: first, results on the existence and properties of solutions to the LDE (dSNag); second, the convergence of the solutions of the $h$ approximations to the solution of (SNag); third, approximations of the classical deterministic front via the truncated spatial problem (Nag); fourth, small noise stability of traveling wave fronts for (SNag).

The rest of this paper is organized along these ingredients: In Section 3.1, we investigate existence and properties of solutions to the LDE (dSNag). We follow this up by a discrete-to-continuum convergence result in Section 3.2. In Section 3.3, we study the truncation error of restricting the solutions to a bounded interval and in Section 3.4 we quantify the error coming from the stochastic perturbation. In Section 3.5, we finally obtain the main result, incorporating an SPDE small-noise stability result.

\subsection{A priori estimates}

Recall again that we are going to employ the notation $\| \cdot \| := \| \cdot \|_{L^2(D)}$. Rigorous results on the existence and properties of solutions to equation (dSNag) are obtained in the classical framework of strong solutions of stochastic ordinary differential equations (SODEs). The key part of this section is the following a priori estimate:

\begin{proposition}
Let the initial data $u_0 \in C^4(D)$ be deterministic and suppose Assumption 2.1 is satisfied. Then the solution $u^h$ of (dSNag) satisfies for any $h$

\begin{equation}
\mathbb{E} \left[ \int_0^T \| \nabla_R^h u^h(t) \|^2 \, dt + \sup_{t \leq T} \| u^h(t) \|^2 \right] < \infty.
\end{equation}

(3.1)

The proof of Proposition 3.1 is split into three parts, which are Lemmata 3.2, 3.3 and 3.4 which are proved under the same assumptions as Proposition 3.1.

\begin{lemma}
For any $h > 0$, the solution $u^h$ of (dSNag) exists and satisfies an energy equality

\begin{equation}
\| u^h(t) \|^2_{L^2} = \| u^h(0) \|^2_{L^2} + 2\nu \int_0^t \langle \Delta_R^h u^h(s), u^h(s) \rangle \, ds
\end{equation}

(3.2)

\begin{align*}
+ 2 \int_0^t (f(u^h(s)), u^h(s)) \, ds + 2 \int_0^t g(u^h(s))(u^h(s), dW^h(s)) + \int_0^t g(u^h(s))^2 \, ds
\end{align*}

\end{lemma}
Due to (2.7), we can interpret the stochastic integral term as Lemma 3.3. The discrete solution and relate the notation of the stochastic integral term in (3.5) to the weak form (3.6). Hence, we may write the stochastic Nagumo equation in weak form, using the scalar product that (SNagD has a solution, which is an adapted process. Moreover, as \( u_i \) are the coefficients of \( u^h \) in the basis of \( \mathbb{R}^N \cong P_N H^1_0(D) \), (3.3) is a finite-dimensional Itô equation, which therefore has a solution as an adapted process. This adapted process also has a continuous version. To derive the energy equation, for some fixed \( h \) itô equation, which therefore has a solution as an adapted process. This adapted process also has a solution, using an orthonormal basis \( e_i \) of \( V^h := P_N H^1_0 \), in the following form

\[
(u^h(t), e_i) = (u^h_0, e_i) + \nu \int_0^t \langle \Delta_R u^h, e_i \rangle \, ds + \int_0^t (f(u^h), e_i) \, ds
\]

Next, in (3.4) we sum up from \( i = 1 \) to \( i = N \) and apply Itô’s formula in finite dimensions to get

\[
\|u^h(t)\|^2 = \|u^h(0)\|^2 + 2\nu \int_0^t \langle \Delta_R u^h(s), u^h(s) \rangle \, ds + 2 \int_0^t (f(u^h(s)), u^h(s)) \, ds
\]

\[
+ 2 \int_0^t g(u^h(s)) (u^h(s), dW^h(s)) + \int_0^t g(u^h(s))^2 \, ds
\]

which is exactly (3.2).

With the way of writing (3.4) we already point to the fact that we consider the stochastic LDE as a generalized Galerkin approximation of the stochastic Nagumo equation. Indeed, recall that (SNagD has a solution whose trajectory is in \( L^2([0, T], H^1_0(D)) \), \( \Delta u(\cdot) \in L^2([0, T], H^{-1}(D)) \). Hence, we may write the stochastic Nagumo equation in weak form, using the scalar product \( \langle \cdot, \cdot \rangle_{L^2(D)} = (\cdot, \cdot) \) and the dual product \( \langle \cdot, \cdot \rangle \) as follows

\[
(u(t), \varphi) = (u^h_0, \varphi) + \nu \int_0^t \langle \Delta u(s), \varphi \rangle \, ds + \int_0^t (f(u(s)), \varphi) \, ds
\]

Due to (3.7), we can interpret the stochastic integral term as

\[
\int_0^t (g(u(s)), \varphi) \, dW(s) = \sum_{k=1}^\infty \int_0^t (g_k(u(s)), \varphi) \, dW_k(s) = \int_0^t (\varphi, g(u(s)) \, dW(s)
\]

and relate the notation of the stochastic integral term in (3.5) to the weak form (3.6).

**Lemma 3.3.** The discrete solution \( u^h \) of (SNagD is uniformly bounded in \( L^2(\Omega \times [0, T], H^1_0(D)) \), i.e.

\[
\sup_h \mathbb{E} \left[ \int_0^T \|\nabla_R u^h(t)\|_{L^2(D)}^2 \, dt \right] < \infty.
\]
Proof. We start with the energy equation (3.2). Taking the expectation, the stochastic integral is zero and we arrive at

\[ E[\|u^h(t)\|^2] - E[\|u^h(0)\|^2] = 2\nu E\left[ \int_0^t (\Delta^h R u^h(s), u^h(s)) \, ds \right] + 2 E\left[ \int_0^t (f(u^h(s), u^h(s)) \, ds \right] + E\left[ \int_0^t g(u^h(s))^2 \, ds \right]. \tag{3.9} \]

Abbreviate now the right hand side of (3.9) by \( \text{RHS} := 2\nu E\left[ \int_0^t (\Delta^h R u^h(s), u^h(s)) \, ds \right] + 2 E\left[ \int_0^t (f(u^h(s), u^h(s)) \, ds \right] + E\left[ \int_0^t g(u^h(s))^2 \, ds \right]. \) We use the coercivity estimate (2.20) to get that the right hand side of (3.9) satisfies

\[ \text{RHS} \leq - 2\nu E\left[ \int_0^t \|\nabla^h R u^h(s)\|^2 \, ds \right] + 2(c_a + \nu)E\left[ \int_0^t \|u^h(s)\|^2 \, ds \right]. \tag{3.10} \]

which is, as the initial data is deterministic,

\[ E[\|u^h(t)\|^2] + 2\nu E\left[ \int_0^t \|\nabla^h R u^h(s)\|^2 \, ds \right] \leq \|u_0\|^2 \]

\[ + 2(c_a + \nu)E\left[ \int_0^t \|u^h(s)\|^2 \, ds \right]. \tag{3.11} \]

Now we apply Gronwall’s Lemma to \( E[\|u^h(t)\|^2] \) to get

\[ E[\|u^h(t)\|^2] \leq e^{2(c_a + \nu)T} \|u_0\|^2. \tag{3.12} \]

Furthermore, as the RHS is independent of \( h \) for \( t \in [0, T] \) we get

\[ \sup_{0 \leq t \leq T} E[\|u^h(t)\|^2] \leq c(a, \nu, T) \|u_0\|^2 \tag{3.13} \]

with a constant \( c \) which is independent of \( h \). Going back to (3.11), we see that the term \( E[\|u^h(t)\|^2] \) on the left-hand side (LHS) is estimated against a constant by (3.13), so we remain with the second term and get therefore its boundedness

\[ E\left[ \int_0^T \|\nabla^h R u^h(t)\|^2 \, dt \right] \leq c(a, \nu, T, u_0), \tag{3.14} \]

which is the desired estimate.

Note that the constant in the Gronwall estimate grows exponentially with \( t \), therefore (3.13) diverges for \( T \to \infty \) but since we are only going to work on finite time scales, this will not be relevant here. Moreover, we did not directly estimate the discrete gradient \( \|\nabla^h R u^h(t)\|^2 \), but we made use of the energy equation, which is a consequence of Itô’s formula. Therefore, the exact range \( R \) of the discrete stencil does not directly affect the a-priori estimates.

**Lemma 3.4.** The discrete solution \( u^h \) of (3SNag) is \( L^2(\Omega, C([0, T], L^2(D))) \) independently of \( h \), i.e.,

\[ E\left[ \sup_{t \leq T} \|u^h(t)\|_{L^2(D)}^2 \right] \leq c(a, \nu, T, u_0) \]
Proof. We start with the energy equation (3.2) over which we take the supremum in \( t \) and the expectation
\[
E \left[ \sup_{t \leq T} \| u^h(t) \|^2 \right] = \| u^h(0) \|^2 + 2\nu E \left[ \int_0^T (\Delta_R u^h(s), u^h(s)) \, ds \right]
+ 2 E \left[ \int_0^T (f(u^h(s)), u^h(s)) \, ds \right] + 2 E \left[ \sup_{t \leq T} \int_0^T g(u^h(s)) (u^h(s), dW^h(s)) \right]
+ E \left[ \int_0^T g(u^h(s))^2 \, ds \right].
\]
The noise term can be analyzed using the Burkholder-Davis-Gundy inequality
\[
E \left[ \sup_{t \leq T} \left| \int_0^t g(u^h(s)) (u^h(s), dW^h(s)) \right| \right] \leq c E \left[ \left( \int_0^T (g(u^h(s)), u^h(s))^2_{L^2(D)} \, ds \right)^{1/2} \right]
\leq c E \left[ \sup_{t \leq T} \| u^h(t) \| \left( \int_0^T g(u^h(s))^2 \, ds \right)^{1/2} \right]
\leq \frac{1}{2} E \left[ \sup_{t \leq T} \| u^h(t) \|^2 \right] + \frac{c^2}{2} E \left[ \int_0^T g(u^h(s))^2 \, ds \right].
\]
Estimating the other terms by coercivity, we get
\[
\frac{1}{2} E \left[ \sup_{t \leq T} \| u^h(t) \|^2 \right] \leq \| u_0 \|^2 - 2\nu E \left[ \int_0^T \| \nabla_R u^h(s) \|^2 \, ds \right] + c(c_0, \nu) E \left[ \int_0^T \| u^h(s) \|^2 \, ds \right].
\]
Notice now that we can estimate, thanks to the last lemma, in particular (3.14),
\[
(3.15) \quad E \left[ \int_0^T \| u^h(s) \|^2 \, ds \right] \leq E \left[ \int_0^T \| \nabla_R u^h(s) \|^2 \, ds \right] \leq c(a, \nu, T)
\]
and as \( \| u_0 \|^2 \leq c \) by assumption, so
\[
E \left[ \sup_{t \leq T} \| u^h(t) \|_{L^2(D)}^2 \right] \leq c(a, \nu, T, u_0)
\]
which means that \( u^h \) is bounded in \( L^2(\Omega, C([0, T], L^2(D))) \) independently of \( h \).

Note that we used here again the estimate (3.13), which comes from Gronwall’s inequality, so this result holds only for finite \( t \).

3.2. Convergence and identification of the limit. We begin with the proof of a simple lower semicontinuity statement on \( E \left[ \| u(T) \|^2_{L^2} \right] \), which we will use in the proof of the convergence theorem, precisely in equation (3.25).

Lemma 3.5. Let \( u \in L^2(\Omega \times [0, T]; H^1_0(D)) \cap L^2(\Omega; H^\infty([0, T], L^2(D))) \). For \( u^h(t) \longrightarrow u(t) \) weakly in \( L^2(\Omega, L^2(D)) \) it holds that
\[
(3.16) \quad E \left[ \| u(T) \|^2_{L^2} - \| u_0 \|^2_{L^2} \right] \leq \liminf_{h \to 0} E \left[ \| u^h(T) \|^2_{L^2} - \| u^h_0 \|^2_{L^2} \right]
\]
Proof. First, as \( u^h \to u \) in \( L^2(D) \), the lower semicontinuity of the \( L^2 \)-norm gives \( \| u(t) \|^2_{L^2} \leq \liminf_{h \to 0} \| u^h(t) \|^2_{L^2} \). As the mapping \( u(t) \to E \left[ \| u(t) \|^2_{L^2} \right] \) is convex as a map from \( L^2(\Omega, L^2(D)) \)
to $\mathbb{R}$, and by the same argument, for any $t \in [0, T]$ also convex as a map from $L^2(\Omega \times [0, T]; L^2(D))$ to $\mathbb{R}$, we get furthermore,

\begin{equation}
\mathbb{E} \left[ \|u(T)\|^2_{L^2} \right] \leq \liminf_{h \to 0} \mathbb{E} \left[ \|u^h(T)\|^2_{L^2} \right].
\end{equation}

By strong convergence of the initial condition in $L^2(D)$, we have $u^h(0) = \sum_{i=1}^{\infty} (u_0, e_i) e_i \to u_0$ and so

\begin{align*}
\mathbb{E} \left[ \|u(T)\|^2_{L^2} - \|u_0\|^2_{L^2} \right] &= \mathbb{E} \left[ \|u(T)\|^2_{L^2} \right] - \mathbb{E} \left[ \|u_0\|^2_{L^2} \right] \\
&\leq \liminf_{h \to 0} \mathbb{E} \left[ \|u^h(T)\|^2_{L^2} \right] - \liminf_{h \to 0} \|u^h_0\|^2_{L^2}
\end{align*}

which finally leads to

\begin{equation}
\mathbb{E} \left[ \|u(T)\|^2_{L^2} - \|u_0\|^2_{L^2} \right] \leq \liminf_{h \to 0} \mathbb{E} \left[ \|u^h(T)\|^2_{L^2} - \|u^h_0\|^2_{L^2} \right]
\end{equation}

finishing the proof.

\begin{theorem}
Let the initial data $u_0 \in C^4(D)$ be deterministic and let Assumption \ref{assumption:boundedness} and the conditions \ref{condition:ellipticity} and \ref{condition:regularity} be satisfied. Then the solution $u^h$ of \ref{SNag} converges in $L^2(\Omega; L^2([0, T]; H^1(D)))$ to the solution $u$ of \ref{SNag} as $h \to 0$.
\end{theorem}

\begin{proof}
Recall that we can write the discrete problem in integral form \ref{discrete_integral_form} in a suggestive way, using an orthonormal basis $\{e_i\}_{i=1,N}$ of $P_N H^1_0$, as

\begin{align}
(u^h(T), e_i) = (u^h_0, e_i) + \nu \int_0^T \langle \Delta^h R u^h, e_i \rangle \, dt + \int_0^T \langle f(u^h), e_i \rangle \, dt \\
+ \int_0^T g(u^h) (e_i, dW^h(t)) & \quad i = 1, \ldots, N.
\end{align}

\begin{align}
\text{The above priori estimates in Proposition \ref{prop:priori_estimates} imply the boundedness of the sequence } & \Delta^h R u^h \text{ in } L^2(\Omega \times [0, T]; H^{-1}(D)) \text{ and the boundedness of the sequence } g(u^h) \text{ in } L^2(\Omega \times [0, T]; \mathcal{H}). \text{ Hence there exists a subsequence, which we do not relabel, such that } \\
& u^h \to u \text{ in } L^2(\Omega; L^2([0, T]; H^1(D))) \cap L^2(\Omega; L^\infty([0, T], L^2(D))) \\
& \Delta^h R u^h \to \zeta_1 \text{ in } L^2(\Omega \times [0, T]; H^{-1}(D)) \\
& f(u^h) \to \zeta_2 \text{ in } L^2(\Omega \times [0, T]; L^2(D)) \\
& g(u^h) \to \tilde{g} \text{ in } L^2(\Omega \times [0, T]; \mathcal{H})
\end{align}

We pass to the weak limit in \ref{discrete_integral_form} and get that for all $t \geq 0$

\begin{align}
(u(T), e_i) = (u_0, e_i) + \nu \int_0^T \langle \zeta_1(t), e_i \rangle \, dt \\
+ \int_0^T \langle \zeta_2(t), e_i \rangle \, dt + \int_0^T \tilde{g}(t) (e_i, dW(t)) & \quad i = 1, \ldots, N.
\end{align}

\begin{equation}
\int_0^T \langle \Delta^h R u^h + f(u^h) - \Delta^h R \varphi - f(\varphi), u^h - \varphi \rangle \, dt + \mathbb{E} \left[ \int_0^T \|g(u^h) - g(\varphi)\|^2_{\mathcal{H}} \, dt \right] \leq 0
\end{equation}

It remains to identify the weak limit objects in \ref{discrete_integral_form} with the objects in the stochastic Nagumo equation. We set in the rest of the proof $\nu = 1$ for convenience as it does not change the argument.

We start with identifying $\tilde{g} = g(u)$. For this, note first that, by using the monotonicity property \ref{monotonicity} with $c = 0$, we infer that for any $\varphi \in L^2(\Omega \times [0, T]; H^1_0(D))$ holds

\begin{equation}
\mathbb{E} \left[ \int_0^T \langle \Delta^h R u^h + f(u^h) - \Delta^h R \varphi - f(\varphi), u^h - \varphi \rangle \, dt \right] + \mathbb{E} \left[ \int_0^T \|g(u^h) - g(\varphi)\|^2_{\mathcal{H}} \, dt \right] \leq 0
\end{equation}
We can split the first term into four terms and use the positivity of $E \left[ \int_0^T \|g(u^h) - g(\varphi)\|_H^2 \, dt \right]$ to get

\[
E \left[ \int_0^T \langle \Delta_R^h \varphi + f(\varphi), \varphi \rangle \, dt \right] = E \left[ \int_0^T \langle \Delta_R^h \varphi + f(\varphi), \varphi \rangle \, dt \right] + E \left[ \int_0^T \langle \Delta_R^h \varphi + f(\varphi), \varphi \rangle \, dt \right] - E \left[ \int_0^T \langle \Delta_R^h \varphi + f(\varphi), u^h \rangle \, dt \right] - E \left[ \int_0^T \langle \Delta_R^h \varphi + f(\varphi), u^h \rangle \, dt \right] \leq 0.
\]

By weak convergence,

\[
\int_0^T \langle \Delta_R^h \varphi + f(\varphi), \varphi \rangle \, dt \rightarrow \int_0^T \langle \Delta \varphi + f(\varphi), \varphi \rangle \, dt
\]

(3.24)

so the last three terms in (3.23) pass to the limit and preserve the sign in (3.22). For the first term of (3.23), we employ that by semicontinuity of the norm and Lemma 3.5, we can relate solutions $u^h$ to (dSNag) with solutions $u$ to (SNag) as follows

\[
E \left[ \int_0^T \langle \Delta \varphi + f(\varphi), \varphi \rangle \, dt \right] + E \left[ \int_0^T \langle \Delta \varphi + f(\varphi), \varphi \rangle \, dt \right] + E \left[ \int_0^T \langle \Delta \varphi + f(\varphi), u^h \rangle \, dt \right] + E \left[ \int_0^T \langle \Delta \varphi + f(\varphi), u^h \rangle \, dt \right] \leq 0.
\]

Consequently also for the first term in (3.23), the sign is preserved in the limit. Passing to the limit in (3.22), we get

\[
E \left[ \int_0^T \langle \zeta_1 + \zeta_2 - \Delta \varphi - f(\varphi), u - \varphi \rangle \, dt \right] + E \left[ \int_0^T \|\tilde{g} - g(u)\|_H^2 \, dt \right] \leq 0.
\]

Choosing $u = \varphi$ in (3.26), we deduce $\tilde{g} = g(u)$.

It remains to identify the limit objects $\zeta_1$ and $\zeta_2$ to prove that $u := \lim_{h \to 0} u^h$ is indeed a solution to the stochastic Nagumo equation. To this aim, notice first that (3.26) implies

\[
E \left[ \int_0^T \langle \zeta_1(t) + \zeta_2(t) - \Delta \varphi(t) - f(\varphi(t)), u(t) - \varphi(t) \rangle \, dt \right] \leq 0.
\]

Now we take $\theta > 0$ and define another test function $w$ via

\[
\theta w(t) = u(t) - \varphi(t)
\]
with \( \varphi(t) \) the test function used in (3.27). As \( w \) is an admissible test function in \( L^2(\Omega \times [0,T]; H^1_0(D)) \), we can employ it in (3.27) instead of \( \varphi \). We obtain

\[
(3.29) \quad \mathbb{E} \left[ \int_0^T (\zeta_1(t) + \zeta_2(t) - \Delta (u(t) - \theta w(t)) - f(u(t) - \theta w(t)) \, w(t)) \, dt \right] \leq 0
\]

As \( \theta \mapsto \langle \Delta (u - \theta w), w \rangle \) and \( \theta \mapsto \langle f(u - \theta w), w \rangle \) are continuous from \( \mathbb{R} \to \mathbb{R} \), it is admissible to pass to the limit \( \theta \to 0 \) and we reach

\[
(3.30) \quad \mathbb{E} \left[ \int_0^T (\zeta_1(t) + \zeta_2(t) - \Delta u(t) - f(u(t)), w(t)) \, dt \right] \leq 0 \quad \text{for any} \quad w \in L^2(\Omega \times [0,T]; H^1_0(D)).
\]

Since \( w \) is arbitrary, the left hand side must vanish, hence \( \zeta_1 + \zeta_2 = \Delta u + f(u) \). Setting now \( w = u \) we identify \( \zeta_2 = f(u) \). Plugging this result into (3.27) gives \( \zeta_1 = \Delta u \). \( \square \)

### 3.3. The cut-off error

The above Theorem 3.6 dealt with the error between the solutions to the lattice and the continuum model in a bounded interval \( D = [-L, L] \). In this section we investigate the error that we make when truncating a solution living on the whole real line. The idea is to select \( L \) large enough so that a travelling front is contained in \( D \) for the time of interest. Then the error outside of \( D \) is small as any classical Nagumo front decays exponentially near the two endstates.

We start with some notation. To compare a solution to the Nagumo PDE on the real line (\( \text{(Nag}) \)) with a solution \( v \) of the finite-domain Nagumo PDE (\( \text{(NagD)} \)), we set, as always in this work, the domain to the symmetric interval \([-L, L]\) and extend \( v \) on \( \mathbb{R} \setminus D \) by extending suitable Dirichlet boundary conditions, i.e., we denote by \( v \) now the solution to

\[
\begin{align*}
\partial_t v &= \nu \partial_x^2 v + f(v) \quad (t, x) \in \mathbb{R}_+ \times D \\
v(x, t) &= 0 \quad \text{for all} \ x \in (-\infty, -L] \\
v(x, t) &= 1 \quad \text{for all} \ x \in (-\infty, -L] 
\end{align*}
\tag{3.31}
\]

In the same way, we extend a solution \( u \) to (\( \text{SNagD} \)) to \( \mathbb{R} \setminus D \) via Dirichlet boundary conditions

\[
\begin{align*}
du(t) &= [\nu \partial_x^2 u(t) + f(u(t))] dt + g(u(t)) \, dW(t) \quad \text{on} \ D \times [0, T] \\
u(-x, t) &= 0 \quad \text{for all} \ x \in (-\infty, -L] \\
u(x, t) &= 1 \quad \text{for all} \ x \in (-\infty, -L] 
\end{align*}
\tag{SNagD}
\]

Analogously, we extend the solution \( u^h \) to (\( \text{dsnag} \)), which is defined per definition only on \( D = [-L, L] \), by a constant \( u^h \equiv 1 \) on \( [L, \infty) \) and \( u^h \equiv 0 \) on \( [-\infty, -L] \).

**Proposition 3.7.** Assume \( u_0 = V^{TW}(: \cdot - ct) = V_0 \). Let \( u^h \) solve (\( \text{dsnag} \)) and \( V \) solve (\( \text{Nag} \)). Then, given any fixed \( T > 0 \) and \( \epsilon > 0 \), there exists sufficiently small non-vanishing noise (in the sense that \( \|Q\|_\ell^2 \leq \epsilon \)), \( h > 0 \), and \( L > 0 \) such that for all \( t \leq T \)

\[
\mathbb{E} \left[ \|u^h(t) - V(t)\|_{L^2(\mathbb{R})}^2 \right] \leq \epsilon.
\tag{3.32}
\]

**Proof.** We want to control the cut-off error for terms of the form

\[
\mathbb{E} \left[ \|u^h(t) - V(t)\|_{L^2(\mathbb{R})}^2 \right].
\]
The strategy is to split the cut-off error into several parts, making use of the deterministic solutions \( v \) and \( V \).

Applying the triangle inequality yields

\[
\mathbb{E} \left[ \| u^h(t) - u(t) \|_{L^2(\mathbb{R})}^2 \right] \leq \mathbb{E} \left[ \| u^h(t) - u(t) \|_{L^2(D)}^2 \right] + \mathbb{E} \left[ \| u(t) - v(t) \|_{L^2(D)}^2 \right] + \| v(t) - V(t) \|_{L^2(\mathbb{R})}^2,
\]

where employed that, by definition of the constant continuations in (3.31) and (SNagD) satisfy

\[
\mathbb{E} \left[ \| u^h(t) - u(t) \|_{L^2(\mathbb{R}\setminus D)}^2 \right] = \mathbb{E} \left[ \| u(t) - v(t) \|_{L^2(D)}^2 \right] = 0.
\]

The control of \( \| v(t) - V(t) \|_{L^2(\mathbb{R})} \) is established in Lemma 3.8 below. Lemma 3.9 estimates \( \mathbb{E} \left[ \| u(t) - v(t) \|_{L^2(D)}^2 \right] \leq \epsilon \). This yields the desired result, equation (3.32).

To establish the relevant auxiliary results, we start with the truncation error for the deterministic equation.

Lemma 3.8. Let \( V \) solve the Nagumo PDE on the real line (NagR) and let \( V^{TW} \) be the deterministic travelling front solution to (NagR). Let \( v \) be the solution to (3.31). Let the initial data \( v_0 \) be the truncation of \( V^{TW} \). Then, for all \( t < T \), where \( T \) depends on the size of the domain and the speed of the wave, we have

\[
\| V(t) - v(t) \|_{L^2(\mathbb{R})} \leq \epsilon.
\]

Proof. We focus on such \( v \) which are close to travelling front solutions, i.e., which satisfy suitable boundary conditions. So we always formally extend \( v \) to \( \mathbb{R} \) by setting \( v(x, t) = 1 \) for all \( x \in [L, \infty) \) and \( v(x, t) = 0 \) for all \( x \in (-\infty, -L] \). See equation (3.35) for the choice of \( L \).

We start both equations with the same initial data \( v_0 \), which forms a traveling wave with a front near zero. Due to the boundary conditions at infinity, we know that \( V(-\infty, t) = 0 \) and \( V(\infty, t) = 1 \). By classical theory, see e.g. [3], 0 and 1 are hyperbolic stable steady endstates of (NagR). Therefore, the decay of the traveling waves is exponential as it approaches the steady states, in other words, there exists \( \epsilon \) such that \( e^{\epsilon xx} \cdot V(x, t) \to 0 \) for \( x \to -\infty \), and analogously for \( x \to +\infty \) and decay towards \( V \equiv 1 \). Therefore, as long as the transition part of the traveling wave is far away from the boundary of \( D \),

\[
\exists L_0 \in \mathbb{R} : \forall L \geq L_0 : \| V(x, t) - 1 \|_{L^2((-\infty, L))} \leq \epsilon, \quad \| V(x, t) - 0 \|_{L^2((-\infty, -L))} \leq \epsilon.
\]

Consequently, by the boundary conditions of the Nagumo PDE (3.31) on the interval, we conclude that up to a time \( T_0 \), when the transition part of the traveling waves has not yet reached the boundary of the interval \( D \),

\[
\| V(x, t) - v(x, t) \|_{L^2(\mathbb{R}\setminus[-L-\delta, L+\delta])} \leq 2\epsilon.
\]

Second, we investigate the error at the boundary of \( D \). First, we look at the positive boundary point of \( D \), i.e. the point \( L \in \mathbb{R} \). In a neighborhood \( B_L(\delta) \) we have \( f(1-\delta) = (1-\delta-a)(1-\delta)\delta \leq \delta \) and \( V(x, t) \) is almost constant so by spatial regularity, we have \( \max_x \partial^2_x V(x, t) \leq \tilde{\epsilon} \), and as \( V \equiv 1 \)
in $[L, L + \delta]$. This gives
\begin{equation}
\|V(x, t) - v(x, t)\|_{L^2(\mathbb{B}_\delta(L))}^2 = \int_L^{L+\delta} \left( \nu \partial_x^2 V(x, t) + f(V(x, t)) - 0 \right)^2 \, dx \\
+ \int_{L-\delta}^L \left( \nu \partial_x^2 V(x, t) + f(V(x, t)) - \nu \partial_x^2 v(x, t) - f(v(x, t)) \right)^2 \, dx \\
\leq \int_L^{L+\delta} \left( \nu \max_{x \in [L, L+\delta]} \partial_x^2 V(x, t) + \max_{x \in [L-\delta, L]} f(V(x, t)) - 0 \right)^2 \, dx \\
+ \delta \cdot \left( \nu \max_{x \in [L-\delta, L]} (\partial_x^2 V(x, t) - \partial_x^2 v(x, t)) + \max_{x \in [L-\delta, L]} (f(V) - f(v)) \right)^2 \\
\leq \delta \nu^2 \epsilon^2 + \delta^3 + 3 \delta \nu^2 \left( \max_{x \in [L-\delta, L]} (\partial_x^2 V(x, t) - \partial_x^2 v(x, t)) \right)^2 + 3 \delta^2 \\
\leq \epsilon
\end{equation}
where we employed the regularity in space of solutions to \cite{Nag} and (3.31) again to infer that
\[ \max_{x \in [-\delta, L]} (\partial_x^2 V(x, t) - \partial_x^2 v(x, t)) \leq c \text{ is a finite quantity.} \]
Analogous reasoning holds for the $\delta$-neighbourhood around $-L$. Therefore
\begin{equation}
\|V(x, t) - v(x, t)\|_{L^2(\mathbb{B}_\delta(L))} \leq \epsilon, \quad \|V(x, t) - v(x, t)\|_{L^2(\mathbb{B}_\delta(-L))} \leq \epsilon.
\end{equation}

Third, we look at the difference between the two solutions in the interior of the interval $\mathcal{D}$, $\|V(x, t) - v(x, t)\|_{L^2([-L+\delta, L-\delta])}$. By construction, this difference is zero at time $t = 0$, as we started with the same wave as initial condition. Moreover, as
\begin{equation}
\|\partial_t V(x, t) - \partial_t v(x, t)\|_{L^2([-L+\delta, L-\delta])}^2 = \int_{-L+\delta}^{-L-\delta} \left( \nu \partial_x^2 V(x, t) + f(V(x, t)) - \nu \partial_x^2 v(x, t) - f(v(x, t)) \right)^2 \, dx \\
\leq 3 \nu \int_{-L+\delta}^{-L-\delta} (\partial_x^2 V(x, t) - v(x, t))^2 \, dx \\
+ 3 \int_{-L+\delta}^{-L-\delta} (f(V(x, t)) - f(v(x, t)))^2 \, dx
\end{equation}
so $\|\partial_t (V(x, t) - v(x, t))\|_{L^2([-L+\delta, L-\delta])} = 0$ for times $t \leq T_0$, i.e., for all $t$ small enough. As we have already controlled the error at the boundary by $\epsilon$, by continuity of the solutions $V$ and $v$, we can conclude $\|\partial_t (V(x, t) - v(x, t))\|_{L^2([-L+\delta, L-\delta])} \leq \epsilon$. Using (3.36), (3.38) and (3.39) finishes the proof.

3.4 Small noise estimate. To complete the proof of Proposition 3.7, it remains to estimate the remaining term in (3.33), namely $\mathbb{E} \left[ \|u(t) - v(t)\|_{L^2(\mathcal{D})} \right]$. This term quantifies the error between the deterministic solution on a bounded interval for (3.31) and the stochastic solution on a bounded interval for (SNagD).

**Lemma 3.9.** Let $v$ be a solution to (3.31) and $u$ a solution to (SNagD). Then, given any $\epsilon > 0$ and any fixed finite $T > 0$, there exists a sufficiently small (non-vanishing) noise with $\|Q\|_{H}^2 \leq \epsilon$, such that
\begin{equation}
\sup_{t \in [0, T]} \mathbb{E} \left[ \|v(t) - u(t)\|_{L^2(\mathcal{D})}^2 \right] \leq \epsilon
\end{equation}

**Proof.** We assume that both equations start with the same deterministic initial data $u_0$, therefore at time $t = 0$, both equations satisfy the same boundary conditions and the error is zero. At time
the error between the deterministic and the stochastic solution via the mild solution expression as

\[ u(t) - v(t) = S(t)(u_0 - u_0) + \int_0^t S(t-s) (f(u(s)) - f(v(s))) \, ds \]

(3.41)

with \( \tilde{W} \) a cylindrical Wiener Process on \( L^2(\mathcal{D}) \)

(3.42)

where \( \sigma \) is the positive definite square root of the covariance operator \( Q \) of the Wiener Process. We use the boundedness of the heat semigroup, the local Lipschitz continuity of \( f \) from Lemma \ref{lem:lip} and Itô’s Isometry and Gronwall’s equality to get

(3.43)

where we used the notation \( \sigma := \|Q\|^2_H \) and \( c = c \left( t, c_0, \|u\|_{H^1_0(\Omega)}, \|v\|_{H^1_0(\Omega)}, \|\cdot\|_{L^\infty(\mathcal{D})} \right) \). Under the assumption \( g(u(s)) \in L^2(\Omega, L^2([0, T] \times \mathcal{D})) \), we can conclude that for finite times \( t \leq T \),

(3.44) \( \mathbb{E} \left[ \|u(t) - v(t)\|^2_{L^2(\mathcal{D})} \right] \leq c \cdot c \left( T, b, c_0, \|u\|_{H^1_0(\mathcal{D})}, \|v\|_{H^1_0(\mathcal{D})}, \|\cdot\|_{L^\infty(\mathcal{D})} \right) \)

The choice of \( \|Q\|^2_H \leq \frac{\epsilon}{c} \) with the \( c \) from \( \ref{lem:lip} \) finally gives \( \mathbb{E} \left[ \|u(T) - v(T)\|^2_{L^2(\mathcal{D})} \right] \leq \epsilon \), finishing the proof.

3.5. Proof of the main theorem. We can now finally prove that the solution of the stochastic LDE (3.41) from the introduction, with the additional specifications on what sufficiently small noise means.

Theorem 3.10. Let \( V^{TW} = V^{TW}(t, x) \) be a travelling front solution to (3.41), \( u_0^h \) be deterministic lattice initial data and \( u^h \) a solution to (3.41) on the interval \( \mathcal{D} := [-L, L] \) to the initial data \( u_0^h \equiv u^h(0) \). Suppose \( L > 0 \) large enough, \( \delta > 0 \), \( T > 0 \), and \( \epsilon > 0 \) are given, and the initial data \( u^h(0) \) satisfies

(3.45)

\[ \|u^h(0) - V^{TW}(0, \cdot)\|^2_{L^2(\mathcal{R})} < \epsilon. \]
Then there exists $\varepsilon > 0$ and $c \in \mathbb{R}$ such that, for sufficiently small $h > 0$, and $\|Q\|_H^2 \leq \varepsilon$, the solution $u^h$ to (dSNag) satisfies

$$
P \left( \sup_{t \in [0,T]} \|u^h(t) - V^TW(\cdot - ct)\|_{L^2(\mathbb{R})} > \delta \right) \leq \varepsilon.
$$

Proof. As we have outlined in the beginning of this section, there exists a solution to equation (dSNag), which is an adapted process with a continuous version. The question is whether this (discrete-in-space) solution is likely to be close to the deterministic travelling wave. We will show that this is indeed true by comparing $u^h$ to solutions to several intermediate problems.

To set up notation, let us now denote by $u^h(t)$ be the piecewise linear extension of the solution to the stochastic LDE (dSNag) to the whole interval $\mathcal{D}$. Moreover we extend the solution as $u^h(t) \equiv 1$ and $u^h(t) \equiv 0$ from the two boundary points of $\mathcal{D}$ to $\pm \infty$. Consider the adapted stochastic process $e_t := \|u^h(t) - V^TW(\cdot - ct)\|_{L^2(\mathcal{D})}$. By the properties of $u^h$, $e_t$ defines a martingale whose trajectories are continuous almost surely. We can then estimate by Doob’s inequality

$$
P \left( \sup_{t \in [0,T]} \|u^h(t) - V^TW(\cdot - ct)\|_{L^2(\mathbb{R})} > \delta \right) \leq \frac{1}{\delta^2} \mathbb{E} \left[ \|u^h(T) - V^TW(\cdot - cT)\|_{L^2(\mathbb{R})}^2 \right].
$$

We split the error between the stochastic LDE and the travelling wave front into three parts, using the linearity of the expectation, i.e., we use

$$
\|u^h - V^TW\|_{L^2} = \|u^h - V + V - V^TW\|_{L^2} \leq \|u^h - V\|_{L^2} + \|V - V^TW\|_{L^2},
$$

and then take expectations. By Proposition 3.4, the first term goes to zero as $h \to 0$, while the second term is small by the standard deterministic local asymptotic $L^2$-stability of the travelling wave front in the deterministic setting, i.e., the front is known to be deterministically stable for the Nagumo equation [10] [31] [45].

Acknowledgment. CK was supported by a Lichtenberg Professorship of the VolkswagenStiftung as well as by the Deutsche Forschungsgemeinschaft (DFG) via the CRC/TR109 “Discretization in Geometry and Dynamics”.

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