Existence and exact asymptotic behaviour of positive solutions for fractional boundary value problem with P-Laplacian operator

Bilel Khamessia, b and Adnane Hamiaz c, d

a Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia; b Université de Tunis El Manar, Faculté des Sciences de Tunis, UR 11ES22 Potentiels et Probabilités, Tunis, Tunisia

ABSTRACT

This paper deals with existence, uniqueness and global behavior of a positive solution for the fractional boundary value problem $D^\beta(\psi(x)\Phi_p(D^\alpha u(x))) = a(x)u^\alpha$ with $p > 1$, $0 < \alpha < 1$ and $\beta > 1$, where $x \in (0, 1)$, and $\psi$ and $a$ are positive and continuous in $(0, 1)$ that may are singular at $x = 0$ or $x = 1$ and satisfies some appropriate conditions.

KEYWORDS

Fractional differential equation; Dirichlet problem; positive solution; Schauder fixed point theorem

1. Introduction

In this paper, we consider the following nonlinear boundary value problem of fractional differential equation with p-Laplacian

$$D^\alpha(\psi(x)\Phi_p(D^\beta u(x))) = a(x)u^\alpha, \quad x \in (0, 1),$$

$$\lim_{x \to 0} D^\beta-1(\psi(x)\Phi_p(D^\alpha u(x))) = 0$$

where $\alpha, \beta \in (1, 2)$ and for $t \in \mathbb{R}$, $\Phi_p(t) := \frac{t^{p-2}}{p-2}$, $p > 1$ and $\sigma \in (1, p, p - 1)$. The operator $D^\alpha$ and $D^\beta$ is the Riemann–Liouville derivative of order $\alpha$ (of order $\beta$). The functions $\psi$ and $a$ are positive and continuous in $(0, 1)$ that may are singular at $x = 0$ or $x = 1$ and satisfying some conditions detailed below. Then, we will address the question of existence, uniqueness and exact asymptotic behaviour of positive solutions to problem (1).

We believe that there are few papers dedicated to the study of fractional differential equations with p-Laplacian operator, see for instance [1–11]. These types of problems arise in applied field such as turbulent flow of a gas in a porous medium, biophysics, plasma physics and material science.

This work is motivated by recent advances in the study of p-Laplacian fractional differential equations involving singular or sublinear nonlinearities with different boundary conditions. Namely, in [12], Liu considered the fractional differential equation

$$D^\beta(\Phi_p(\psi(x)D^\alpha u(x))) = f(x, u(x)), \quad x \in (0, 1),$$

where $0 < \alpha, \beta \leq 1$, $\psi \in C((0, 1))$ and $f$ is a non-negative function on $(0, 1) \times \mathbb{R}$ allowed to be singular at $t = 0$. The author proved the existence of positive solution with fractional non-local integral boundary conditions.

Recently, in [13], Mâagli et al. considered the following problem

$$D^\alpha u(x) = -a(x)u^\sigma, \quad x \in (0, 1),$$

$$\lim_{x \to 0} D^{\alpha-1} u(x) = 0, \quad u(1) = 0,$$

where $1 < \alpha < 2$, $-1 < \sigma < 1$ and the function $a$ is required to satisfy some assumptions related to $\Lambda$, the set of all Karamata functions $L$ defined on $(0, \eta]$, by

$$L(t) := c \exp\left(\int_t^\eta z(s) \, ds\right),$$

for some $\eta > 1$, where $c > 0$ and $z$ is a continuous function on $[0, \eta]$, with $z(0) = 0$. To describe the result of [13] in more details, we need some notations.

- For two non-negative functions $f$ and $g$ defined on a set $S$, the notation $f(x) \approx g(x)$, $x \in S$ means that there exists $c > 0$ such that $(1/c)f(x) \leq g(x) \leq cf(x)$, for all $x \in S$.
- $C_{2-\alpha}([0, 1]) = \{ f \mid t \to t^{2-\alpha}f(t) \in C((0, 1)) \}$. 

CONTACT Adnane Hamiaz hzadnane@gmail.com

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In [13], Mâagli et al. studied problem (3) where \( a \) verifies the assumption \((H_0)\):

\[
(H_0) : a \in C((0,1)) \text{ satisfying for each } x \in (0,1),
\]

\[
a(x) \approx x^{-\lambda} (1-x)^{-\mu} L(x) L(1-x),
\]

where \( \lambda + (2 - a) \sigma \leq 1, \mu \leq \alpha \) and \( L, \tilde{L} \in \mathcal{K} \) such that

\[
\int_0^\eta \frac{L(t)}{t^{\alpha + (2 - a) \sigma}} dt < \infty \quad \text{and} \quad \int_0^\eta \frac{L(t)}{t^{\alpha + 1}} dt < \infty.
\]  

Based on the Schauder fixed-point theorem, the authors showed in [13] the following result.

**Theorem 1.1:** Assume that \( a \) satisfies \((H_0)\). Then problem (3) has a unique positive solution \( u \in C_{2-\alpha}([0,1]) \) satisfying for \( x \in (0,1) \),

\[
u(x) \approx x^{\alpha-2} (1-x)^{h(1,\alpha-\mu)/(1-\sigma))} \Pi_{\Lambda_{\rho,\sigma},\alpha}(1-x),
\]

where \( \Pi_{\Lambda_{\rho,\sigma},\alpha} \) defined on \( (0,1) \) by

\[
\Pi_{\Lambda_{\rho,\sigma},\alpha}(t) = \begin{cases} 
1, & \text{if } \mu < \sigma + \alpha - 1, \\
\left( \int_t^\eta \frac{\nu(s)}{s} ds \right)^{1/(1-\sigma)}, & \text{if } \sigma + \alpha - 1 < \mu < \alpha,
\end{cases}
\]

\[
\left( \int_0^\eta \frac{\nu(s)}{s} ds \right)^{1/(1-\sigma)}, \quad \text{if } \mu = \alpha.
\]  

The main goal of this paper is to improve and extend the above results on the boundary behaviour of solutions to problem (1). More precisely, we consider the following assumption: \((H_1)\) : \( a \) and \( \psi \) are non-negative functions in \( C((0,1)) \) satisfying for each \( x \in (0,1) \),

\[
a(x) \approx x^{-\lambda} (1-x)^{-\mu} L_1(x) L_2(1-x) \quad \text{and} \quad \psi(x) \approx (1-x)^{\beta \cdot (p-1) \mu + \beta - \mu},
\]

where \( \lambda + (2 - a) \sigma \leq 1, \beta - 1 < \mu < \beta, \beta < \mu + p \) and \( L_1, L_2 \in \mathcal{K} \) such that

\[
\int_0^\eta \frac{L_1(t)}{t^{\alpha + (2 - a) \sigma}} dt < \infty \quad \text{and} \quad \int_0^\eta \frac{L_2(t)}{t^{\alpha + 1}} dt < \infty.
\]  

**Theorem 1.2:** Assume \((H_1)\), then problem (1) has a unique positive solution \( u \in C_{2-\alpha}([0,1]) \) satisfying for \( x \in (0,1) \),

\[
u(x) \approx x^{\alpha-2} \left( \int_0^{1-x} \frac{L_2(t)}{t^{\alpha + 1}} dt \right)^{1/(p-1)/(p-1-\sigma)}.
\]  

The rest of the paper is as follows. In Section 2, we state some already known results on functions in \( \mathcal{K} \) and some definitions and lemmas from fractional calculus theory. In Section 3, we present some necessary conditions to existence result and we prove our main result.

### 2. Preliminaries: Karamata properties and fractional calculus

In what follows, we are quoting without proof some fundamental properties of functions belonging to the class \( \mathcal{K} \) collected from [14, 15]. We recall that a function \( L \) defined on \( [0, \eta] \) belongs to the class \( \mathcal{K} \), if

\[
L(t) := c \exp \left( \int_0^t \frac{z(s)}{s} ds \right),
\]

for some \( \eta > 1, c > 0 \) and \( z \) is a continuous function on \([0, \eta]\) with \( z(0) = 0 \).

**Proposition 2.1:**

(i) A function \( L \in \mathcal{K} \) if and only if \( L \) is a positive function in \( \mathcal{C}^1((0,\eta)) \) such that

\[
\lim_{t \to 0^+} \frac{L(t)}{t} = 0.
\]

(ii) Let \( L_1, L_2 \in \mathcal{K} \) and \( \alpha \in \mathbb{R} \). Then we have

\[
L_1 + L_2 \in \mathcal{K}, L_1 L_2 \in \mathcal{K \text{ and } } L_1^\alpha \in \mathcal{K}.
\]

(iii) Let \( L \in \mathcal{K} \) and \( \beta > 0 \). Then we have

\[
\lim_{t \to 0^+} t^\beta L(t) = 0.
\]

As a standard example of functions belonging to the class \( \mathcal{K} \) (see [15]), we give

**Example 2.2:** Let \( m \in \mathbb{N}^* \) and \( \eta > 0 \). Let \( (\mu_1, \mu_2, \ldots, \mu_m) \in \mathbb{R}^m \) and \( w \in \mathbb{R}^m \) be a sufficiently large positif real number such that the function

\[
L(t) = \prod_{1 \leq i \leq m} \left( \log \left( \frac{W}{t} \right) \right)^{\mu_i}
\]

is defined and positive on \([0, \eta]\), where \( \log_t t = \log \circ \cdots \circ \log \circ (t) \). Then we have \( L \in \mathcal{K} \).

**Lemma 2.3:**

(i) Let \( L \) be a function in \( \mathcal{K} \). Then we have

\[
\lim_{t \to 0^+} \frac{L(t)}{t} = 0.
\]

In particular,

\[
\frac{\int_0^t L(s) ds}{s} \in \mathcal{K}.
\]

(ii) If \( \int_0^\eta \frac{L(s)}{s} ds \) converges, then

\[
\lim_{t \to 0^+} \frac{\int_0^t L(s) ds}{s} = 0.
\]

In particular,

\[
\frac{\int_0^\eta L(s) ds}{s} \in \mathcal{K}.
\]
Applying Karamata’s theorem, we get the following.

**Lemma 2.4:** Let \( y \in \mathbb{R} \) and \( L \) be a function in \( K \) defined on \((0, \eta)\) for some \( \eta > 1 \). We have

(i) If \( y > -1 \), then \( \int_0^t s^y L(s)ds \) converges and

\[
\int_0^t s^y L(s)ds \sim \frac{t^{1+y}L(t)}{1 + y}.
\]

(ii) If \( y < -1 \), then \( \int_0^n s^y L(s)ds \) diverges and

\[
\int_n^\infty s^y L(s)ds \sim -\frac{t^{1+y}L(t)}{1 + y}.
\]

Next, we give some definitions and fundamental facts of fractional calculus theory, which can be found in [16, 17].

**Definition 2.5:** Let \( y > 0 \), the Riemann–Liouville fractional integral of order \( y \) of a measurable function \( f : (0, \infty) \rightarrow \mathbb{R} \) is given by

\[
I^y f(x) = \frac{1}{\Gamma(y)} \int_0^x (x - t)^{y-1} f(t)dt,
\]

provided that the right-hand side is pointwise defined on \((0, \infty)\).

Here \( \Gamma \) is the Euler Gamma function.

**Definition 2.6:** The Riemann–Liouville derivative of order \( y > 0 \) of a measurable function \( f : (0, \infty) \rightarrow \mathbb{R} \) is given by

\[
D^y f(x) = \left( \frac{d}{dx} \right)^n \frac{1}{\Gamma(n-y)} \int_0^x (x-t)^{y-n-1} f(t)dt
\]

provided that the right-hand side is pointwise defined on \((0, \infty)\).

Here \( n = \lfloor y \rfloor + 1 \), where \( \lfloor y \rfloor \) means the integer part of the number \( y \).

**Lemma 2.7:** Let \( \alpha, \gamma > 0 \), \( n \) be the smallest integer greater or equal to \( \alpha \) and \( f \in L^1((0, 1]) \). Then, we have

(i) \( I^{\alpha + \gamma} f(x) = I^{\alpha} I^{\gamma} f(x) \) for \( x \in [0, 1] \) and \( \alpha + \gamma \geq 1 \).

(ii) \( D^{\alpha + \gamma} f(x) = D^{\alpha} D^{\gamma} f(x) \) for a.e. \( x \in [0, 1] \).

(iii) \( D^{\alpha} f(x) = 0 \) if and only if \( f(x) = \sum_{i=1}^n c_i x^{a_i} \), where \( (c_1, \ldots, c_n) \in \mathbb{R}^n \).

Since our approach is based on potential theory, we recall in the following some basic tools. For \( y \in (1, 2) \), we denote by \( G_y(x, t) \) the Green’s function for the boundary value problem (3). From [13], we have

\[
G_y(x, t) = \frac{1}{\Gamma(\gamma)} [x^{\gamma-2}(1-t)^{\gamma-1} - ((x-t)^{+})^{\gamma-1}],
\]

where \( x^+ = \max(x, 0) \).

**Proposition 2.8 (see [13]):** Let \( 1 < \gamma \leq 2 \) and \( f \) be a non-negative measurable function on \((0, 1]\). Then we have

(i) For \( x, t \in (0, 1) \),

\[
G_y(x, t) \approx x^{\gamma-2}(1-t)^{\gamma-2}(1 - \max(x, t)).
\]

(ii) For \( x \in (0, 1) \), \( G_y f(x) := \int_0^1 G_y(x, t)f(t)dt \approx \infty \) if and only if \( \int_0^1 (1-t)^{\gamma-1} f(t)dt \approx \infty \).

(iii) If the map \( t \mapsto (1-t)^{\gamma-1} f(t) \) is continuous and integrable on \((0, 1)\), then \( G_y f \) is the unique solution in \( C_{2 - \gamma}([0, 1]) \) of the boundary value problem

\[
D^{\gamma} u(x) = -f(x), \quad x \in (0, 1),
\]

\[
\lim_{x \to 0^+} D^{\gamma-1} u(x) = u(1) = 0.
\]

**Lemma 2.9 ([13]):** Let \( \tilde{L}_1, \tilde{L}_2 \in K \) and let for \( x \in (0, 1) \)

\[
b(t) = \frac{1}{\Gamma(1 - \gamma)} (1-t)^{\gamma-1} \tilde{L}_1(t) \tilde{L}_2(1-t),
\]

with \( \lambda_1 \leq 1 \) and \( \mu_1 \leq \beta \). Assume that

\[
\int_0^n t^{-\lambda_1} \tilde{L}_1(t)dt < \infty \quad \text{and} \quad \int_0^n t^{\beta-1-\mu_1} \tilde{L}_2(t)dt < \infty.
\]

Then we have for \( x \in (0, 1) \),

\[
x^{2-\beta} G_{y, \beta} b(x) \approx (1 - x)^{\min(1, \beta - \mu_1)} \Pi_{2, \mu_1, 0, \beta}(1 - x).
\]

**Remark 2.1:** We need to verify condition \( \int_0^n t^{-\lambda_1} \tilde{L}_1(t)dt < \infty \) and \( \int_0^n t^{\beta-1-\mu_1} \tilde{L}_2(t)dt < \infty \) in Lemma 6, only if \( \lambda_1 = 1 \) and \( \mu_1 = \beta \). This is due to Lemma 2.

### 3. Proof of main result

We begin this section by announcing and proving some propositions and lemmas, that will play a crucial role in the proof of our main result.

Let \( p > 1 \) and let \( q = p/(p-1) \). Then \( \Phi_q(t) = |t|^{2-p}/(p-1) \) is the function inverse of \( \Phi_p \). We say easily that \( \Phi_q \) is increasing monotone and multiplicative on \((0, \infty)\).

**Lemma 3.1:** Let \( \alpha, \beta \in (1, 2) \). Let \( f \) be a non-negative function in \( B((0, 1)) \) such that \( x \mapsto (1-x)^{\beta-1} f(x) \) and \( x \mapsto (1-x)^{\alpha-1} \Phi_q((1/\gamma) G_{y, \gamma} f(x)) \) are continuous and integrable on \((0, 1]\), then the boundary value problem

\[
D^\beta (\psi(x) \Phi_p(D^\alpha u(x))) = f(x), \quad x \in (0, 1),
\]

\[
\lim_{x \to 0^+} D^{\beta-1} (\psi(x) \Phi_p(D^\alpha u(x)))
\]

\[
= \lim_{x \to 0^+} \psi(x) \Phi_p(D^\alpha u(x)) = 0,
\]

\[
\lim_{x \to 0^+} D^{\beta-1} u(x) = u(1) = 0.
\]
has a unique positive solution in $C_{2−α}([0, 1])$ given by
\[
u(x) = G_α\left(\Phi_α\left(\frac{1}{ψ}G_βf\right)\right)(x)
\]
\[\begin{equation}
\int_0^1 G_α(x, t)\Phi_α\left(\frac{1}{ψ(t)}\int_0^t G_β(t, s)f(s)ds\right)dt.
\end{equation}
\]

**Proof:** Since $x \mapsto (1−x)^α−1f(x)$ is continuous and integrable on $(0, 1)$, we deduce by Proposition 2 (iii) that for $x \in (0, 1)$, we have
\[\psi(x)\Phi_α(D^αu)(x) = −G_βf(x).
\]

Thus
\[
D^αu(x) = −\Phi_α\left(\frac{1}{ψ(x)}G_βf(x)\right).
\]

In addition, using the fact that $x \mapsto (1−x)^α−1\Phi_α((1/ψ(x))G_βf(x) )$ is continuous and integrable on $(0, 1)$, we conclude again by Proposition 2 (iii) that problem (29) has a unique solution $u$ in $C_{2−α}([0, 1])$ given by
\[\begin{equation}
u(x) = G_α\left(\Phi_α\left(\frac{1}{ψ}G_βf\right)\right)(x), 0 < x < 1.
\end{equation}
\]

Here, below we provide a crucial property concerning continuity.

**Lemma 3.2:** Let $ψ$ and $ψ$ be tow non-negative functions in $B((0, 1))$ such that
\[\int_0^1 (1−t)^α−1\Phi_α\left(\frac{1}{ψ(t)}G_βψ(t)\right)dt < ∞.
\]

Then the family
\[\mathcal{F} = \{Sf : x \mapsto x^{2−α}G_α\left(\Phi_α\left(\frac{1}{ψ}G_βf\right)\right)(x); |f| ≤ ψ\}
\]
is relatively compact in $C([0, 1])$.

**Proof:** Let $f \in B((0, 1))$ such that $|f(x)| ≤ ψ(x)$ for all $x \in (0, 1)$. Let $x \in (0, 1)$, we have
\[Sf(x) = x^{2−α}G_α\left(\Phi_α\left(\frac{1}{ψ}G_βf\right)\right)(x)
\]
\[= x^{2−α}\int_0^1 G_α(x, t)\Phi_α\left(\frac{1}{ψ(t)}G_βf(t)\right)dt.
\]

Using Proposition 2 (i), we obtain that
\[|Sf(x)| ≤ \frac{1}{Γ(α)}\int_0^1 (1−t)^α−1\Phi_α\left(\frac{1}{ψ(t)}G_βψ(t)\right)dt < ∞.
\]

Thus $\mathcal{F}$ is uniformly bounded. Now, let us prove that $\mathcal{F}$ is equicontinuous in $[0, 1]$.

Let $x, y ∈ (0, 1)$, then we have
\[|Sf(x) − Sf(y)| = |x^{2−α}G_α\left(\Phi_α\left(\frac{1}{ψ}G_βf\right)\right)(x)
\]
\[− y^{2−α}G_α\left(\Phi_α\left(\frac{1}{ψ}G_βf\right)\right)(y)|
\]
\[= \frac{1}{Γ(α)}\int_0^1 (y^{2−α}(y−t)^α−1−x^{2−α}(x−t)^α−1)|
\]
\[= \frac{1}{Γ(α)}\int_0^1 (y^{2−α}(y−t)^α−1−x^{2−α}(x−t)^α−1)|
\]
\[≤ \frac{1}{Γ(α)}\int_0^1 |y^{2−α}(y−t)^α−1−x^{2−α}(x−t)^α−1|\]
\[≤ 2(1−t)^α−1.
\]

Then we obtain by Lesbegue’s theorem that
\[|Sf(x) − Sf(y)| → 0 as |x−y| → 0.
\]

Now, let $x \in (0, 1)$, we have
\[|Sf(x) − \frac{1}{Γ(α)}\int_0^1 (1−t)^α−1\Phi_α\left(\frac{1}{ψ(t)}G_βf(t)\right)dt|
\]
\[= \frac{1}{Γ(α)}\int_0^1 x^{2−α}(x−t)^α−1\Phi_α\left(\frac{1}{ψ(t)}G_βf(t)\right)dt|
\]
\[≤ \frac{1}{Γ(α)}\int_0^1 x^{2−α}(x−t)^α−1\Phi_α\left(\frac{1}{ψ(t)}G_βf(t)\right)dt.
\]

Using the fact that for $t \in (0, 1)$,
\[x^{2−α}(x−t)^α−1 → 0 as x → 0
\]

and
\[0 ≤ x^{2−α}(x−t)^α−1 ≤ (1−t)^α−1,
\]

we get again by Lesbegue’s theorem that
\[|Sf(x) − \frac{1}{Γ(α)}\int_0^1 (1−t)^α−1\Phi_α\left(\frac{1}{ψ(t)}G_βf(t)\right)dt|
\]
\[→ 0 as x → 0.
\]

Furthermore, for $x \in (0, 1)$, we have
\[|Sf(x)| ≤ \frac{1}{Γ(α)}\int_0^1 ((1−t)^α−1−x^{2−α}(x−t)^α−1)\Phi_α
\]
\[\left(\frac{1}{ψ(t)}G_βf(t)\right)dt.
\]
By similar arguments as above, we deduce that
\[ |Sf(x)| \longrightarrow 0 \text{ as } x \longrightarrow 1. \] (44)

Finally, we conclude that the family \( \mathcal{F} \) is equicontinuous in \([0, 1]\). Hence, by Ascoli’s theorem, we deduce that \( \mathcal{F} \) is relatively compact in \( C([0, 1]) \).

**Proposition 3.3:** Assume \((H_1)\) and suppose that there exist a non-negative function \( \theta \) in \( C([0, 1]) \) such that
\[ \int_0^1 (1 - t)^{\beta - 1} w(t) \, dt < \infty \] (45)
and
\[ x^{2-a} G_{\alpha} \left( \Phi_q \left( \frac{1}{\psi} G_\beta w \right) \right)(x) \approx \theta(1 - x), \] (46)
where \( w(t) = a(t)t^{(a - 2)\theta} \alpha(t) \) for \( t \in (0, 1) \).

Then problem \((1)\) has a unique solution \( u \in C_{2-a}([0, 1]) \) satisfying for each \( x \in (0, 1) \)
\[ u(x) \approx x^{\alpha - 2}\theta(1 - x). \] (47)

**Proof:** Let \( m \geq 1 \) and \( \theta \) be a non-negative function satisfying for each \( x \in (0, 1) \)
\[ \frac{1}{m} \theta(1 - x) \leq x^{2-a} G_{\alpha} \left( \Phi_q \left( \frac{1}{\psi} G_\beta w \right) \right)(x) \leq m \theta(1 - x). \] (48)

Put \( c_0 := m^{(p-1)/(p-1-|\alpha|)} \). We consider the closed convex set given by
\[ Y := \{ v \in C([0, 1]) : \frac{1}{c_0} \theta(1 - x) \leq v(x) \leq c_0 \theta(1 - x) \}. \] (49)

Using Lemma 3.2 and Proposition 2 (ii), we easily see that the function \( x \mapsto x^{2-a} G_{\alpha} \left( \Phi_q \left( \frac{1}{\psi} G_\beta w \right) \right)(x) \) belongs to \( C([0, 1]) \) and satisfies \((48)\). So \( Y \) is not empty.
In order to use the Schauder’s fixed point theorem, we denote \( \tilde{a}(x) = x^{(a-2)\theta} a(x) \) and we define the operator \( T \) on \( Y \) by
\[ T v(x) = x^{2-a} G_{\alpha} \left( \Phi_q \left( \frac{1}{\psi} G_\beta v(\tilde{a}) \right) \right)(x). \] (50)

We need to check that the operator \( T \) has a fixed point \( v \) in \( Y \). For this choice of \( c_0 \), we will prove that \( T \) maps \( Y \) into itself. Indeed, let \( v \in Y \), by using \((48)\), we have
\[ T v(x) \leq \Phi_q(c_0 |\alpha|) x^{2-a} G_{\alpha} \left( \Phi_q \left( \frac{1}{\psi} G_\beta w \right) \right)(x) \leq c_0 |\alpha|^{1/p-1} m^{\theta(1 - x)} = c_0 \theta(1 - x) \]
and
\[ T v(x) \geq \Phi_q(c_0 |\alpha|) x^{2-a} G_{\alpha} \left( \Phi_q \left( \frac{1}{\psi} G_\beta w \right) \right)(x) \geq c_0 |\alpha|^{1/p-1} \theta(1 - x) = \frac{1}{c_0} \theta(1 - x). \]

Furthermore, using \((48)\), we have for each \( x \in (0, 1) \)
\[ G_{\alpha} \left( \Phi_q \left( \frac{1}{\psi} G_\beta w \right) \right)(x) < \infty. \] (51)
This implies by Proposition 2 (ii) that
\[ \int_0^1 (1 - t)^{\beta - 1} \Phi_q \left( \frac{1}{\psi} G_\beta w(t) \right) \, dt < \infty. \] (52)
Hence, it follows from Lemma 3.2 that the family \( TY \) is relatively compact in \( C([0, 1]) \). So \( Y \) is invariant under \( T \).
Next, we shall prove the continuity of \( T \). Let \( \{ v_k \} \) be a sequence in \( Y \) which converges uniformly to \( v \) in \( Y \).
For \( x \in (0, 1) \), we have
\[ |T v_k(x) - T v(x)| = x^{2-a} \left| G_{\alpha} \left( \Phi_q \left( \frac{1}{\psi} G_\beta (\tilde{a} v_k) \right) \right) \right| \]
\[ - G_{\alpha} \left( \Phi_q \left( \frac{1}{\psi} G_\beta (\tilde{a} v) \right) \right)(x) \]
\[ \leq x^{2-a} \int_0^1 G_{\alpha}(x, t) |\Phi_q \left( \frac{1}{\psi} G_\beta (\tilde{a} v_k) \right)(t) - \Phi_q \left( \frac{1}{\psi} G_\beta (\tilde{a} v) \right)(t) | \, dt. \]
For \( t \in (0, 1) \), we have
\[ |G_\beta (\tilde{a} v_k)(t) - G_\beta (\tilde{a} v)(t)| \]
\[ \leq \int_0^1 G_\beta(t, s) |(\tilde{a} v_k)(s) - (\tilde{a} v)(s)| \, ds \] (53)
and for every \( s \in (0, 1) \),
\[ |(\tilde{a} v_k)(s) - (\tilde{a} v)(s)| \leq 2c_0 |\alpha|^{1/p} w(s). \] (54)
Using Proposition 2 (ii) and \((45)\), we obtain by Lebesgue’s theorem that
\[ |G_\beta (\tilde{a} v_k)(t) - G_\beta (\tilde{a} v)(t)| \longrightarrow 0 \text{ as } k \longrightarrow \infty. \] (55)
Since \( \Phi_q \) is continuous, we deduce that
\[ \left| \Phi_q \left( \frac{1}{\psi} G_\beta (\tilde{a} v_k)(t) \right) - \Phi_q \left( \frac{1}{\psi} G_\beta (\tilde{a} v)(t) \right) \right| \]
\[ \longrightarrow 0 \text{ as } k \longrightarrow \infty. \] (56)
We have
\[ \left| \Phi_q \left( \frac{1}{\psi} G_\beta (\tilde{a} v_k)(t) \right) - \Phi_q \left( \frac{1}{\psi} G_\beta (\tilde{a} v)(t) \right) \right| \]
\[ \leq 2 \Phi_q(c_0 |\alpha|) \Phi_q \left( \frac{1}{\psi} G_\beta w(t) \right). \] (57)
Using \((48)\), we obtain by Lebesgue’s theorem that for \( x \in [0, 1] \)
\[ T v_k(x) \longrightarrow T v(x), \text{ as } k \longrightarrow \infty. \] (58)
Since \( TY \) is a relatively compact in \( C([0, 1]) \), we have the uniform convergence, namely
\[ \| T v_k - T v \|_\infty \longrightarrow 0, \text{ as } k \longrightarrow \infty. \] (59)
Thus, we have proved that \( T \) is a compact mapping from \( Y \) into itself. It follows by the Schauder fixed point
Theorem that there exists $v \in Y$ such that $Tv = v$. Put $u(x) = x^\alpha v(x)$, \hspace{1cm} (60)

then $u \in C_{2-\alpha}([0,1])$ and $u$ satisfies the equation
\[
u(x) = G\alpha \left( \frac{1}{\psi} G\beta (au^\varphi) \right)(x).
\hspace{1cm} (61)
\]

Then due to Lemma 3.1, $u$ is a positive continuous solution of problem (1).

Finally, let us prove that $u$ is the unique positive continuous solution satisfying (47). To this aim, we assume that (1, 1) has two positive solutions $u$ and $v$ satisfying (47). Then, there exists a constant $m > 1$ such that
\[
\frac{1}{m} v \leq u \leq mv.
\hspace{1cm} (62)
\]

This implies that the set
\[
J := \left\{ t \in (1, \infty) : \frac{1}{t} v \leq u \leq tv \right\}
\hspace{1cm} (63)
\]
is not empty. Now, put $c := \inf J$, then we aim to show that $c = 1$. Suppose that $c > 1$. Then by simple calculus, we obtain that
\[

    D\beta \left( \psi(x) (\Phi_p(D\beta (c^{(\alpha-1)/p-1} v))) - \Phi_p(D\beta u) \right)
    = a(c^{(\alpha-1)/p-1} - u^\varphi) \geq 0,

\]

\[
\lim_{x \to 0} D\beta^{-1} \left( \psi(x) (\Phi_p(D\beta (c^{(\alpha-1)/p-1} v))) - \Phi_p(D\beta u) \right)(x) = 0,
\]

\[
\lim_{x \to 1} \psi(x) \left( \Phi_p(D\beta (c^{(\alpha-1)/p-1} v))(x) - \Phi_p(D\beta u(x)) \right)
    = 0.
\hspace{1cm} (64)
\]

We conclude by Proposition 2 (iii) that
\[

    \psi \left( \Phi_p(D\beta (c^{(\alpha-1)/p-1} v))) - \Phi_p(D\beta u) \right)
    = -G\beta \left( a(c^{(\alpha-1)/p-1} - u^\varphi) \right) \leq 0.
\hspace{1cm} (65)
\]

Then we have
\[

    \Phi_p(D\beta (c^{(\alpha-1)/p-1} v))) \leq \Phi_p(D\beta u).
\hspace{1cm} (66)
\]

Which implies with the fact that $\Phi_p$ is increasing monotone that
\[

    D\alpha (c^{(\alpha-1)/p-1} v - u) \leq 0,
\]

\[
\lim_{x \to 0} D\alpha^{-1} (c^{(\alpha-1)/p-1} v - u)(x) = 0,
\]

\[
c^{(\alpha-1)/p-1} v(1) - u(1) = 0.
\hspace{1cm} (67)
\]

Using again Proposition 2 (iii), we conclude that
\[

    c^{(\alpha-1)/p-1} v - u \geq 0.
\hspace{1cm} (68)
\]

By symmetry, we obtain that $v \leq c^{(\alpha-1)/p-1} u$. So $c^{(\alpha-1)/p-1} \in J$. Since $(\alpha/(p-1)) < 1$ and $c > 1$, we have $c^{(\alpha-1)/p} < c$. This yields to a contradiction with the fact that $c = \inf J$. Hence $c = 1$ and consequently $u = v$.

\section{Proof of Theorem 2}

Suppose that $\beta - 1 < \mu < \beta$ and $L_2$ satisfy
\[

    \int_0^1 (L_2(t))(1/(p-1)) \, dt < \infty.
\hspace{1cm} (69)
\]

Let $\vartheta$ be the function defined on $[0,1]$ by
\[

    \vartheta(x) \equiv \left( \int_0^x (L_2(t))(1/(p-1)) \, dt \right)^{(\alpha-1)/(\beta-1)}.
\hspace{1cm} (70)
\]

Then, for $x \in (0,1)$
\[

    w(x) \equiv a(x)x^{(\alpha-2)/\varphi} \vartheta^\varphi (1 - x)
\]

\[
\approx x^{2-\alpha} (1 - x)^{\beta-1} L_2(1 - x)
\]

\[
\left( \int_0^{1-x} (L_2(t))(1/(p-1)) \, dt \right)^{(\alpha-1)/(\beta-1)}.
\hspace{1cm} (71)
\]

We conclude by Lemma 2.9 that for $x \in (0,1)$
\[

    G_p w(x) \approx x^{2-\alpha} (1 - x)^{\beta-1} L_2(1 - x)
\]

\[
\left( \int_0^{1-x} (L_2(t))(1/(p-1)) \, dt \right)^{(\alpha-1)/(\beta-1)}.
\hspace{1cm} (72)
\]

Using again (H$_1$) and by Lemma 2.9, we deduce that for $x \in (0,1)$
\[

    x^{2-\alpha} G_p \left( \frac{1}{\psi} G\beta w \right)(x) \approx \vartheta (1 - x).
\hspace{1cm} (73)
\]

Hence it follows from Proposition 3 that problem (1) has a unique positive solution $u$ in $C_{2-\alpha}([0,1])$ satisfying for $x \in (0,1)$,
\[

    u(x) \approx x^{2-\alpha} \vartheta (1 - x).
\hspace{1cm} (74)
\]

As an application of our main result, we give the following example.

\subsection{Example 1}

Let $\beta, \alpha \in (1,2), p > 1$ and $\sigma \in (1 - p, p - 1)$. Let $\alpha$ and $\psi$ be two positive continuous functions on $[0,1]$ such that
\[

    a_1(x) \approx (1-x)^{-\mu} \left( \log \left( \frac{3}{1-x} \right) \right)^{-p}
\hspace{1cm} (75)
\]

and
\[

    \psi(x) \approx (1-x)^{(p-1)\sigma + (p-1)}
\hspace{1cm} (76)
\]

where $\beta - 1 < \mu < \beta$ and $\beta + p > 3$. Therefore by Theorem 2, problem (1) has a unique positive solution.
\[ u \in C_{2-\alpha}([0,1]) \text{ satisfying for each } x \in (0,1) \]
\[ u(x) \approx x^{\alpha-2} \left( \log\left( \frac{3}{1-x} \right) \right)^{-1/(p-1-\sigma)}. \tag{77} \]

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ORCID

Adnane Hamiaz  http://orcid.org/0000-0001-5605-8728

References

[1] Chai G. Positive solutions for boundary value problem of fractional differential equation with p-Laplacian operator. Bound Value Probl. 2012;2012(18):20.
[2] Chen T, Liu W, Hu Z. A boundary value problem for fractional differential equation with p-Laplacian operator at resonance. Nonlinear Anal. 2012;75(6):3210–3217.
[3] Liu X, Jia M, Ge W. Multiple solutions of a p-Laplacian model involving a fractional derivative. Adv Differ Equ. 2013;2013(126):12.
[4] Liu X, Jia M, Geb W. The method of lower and upper solutions for mixed fractional four-point boundary value problem with p-Laplacian operator. Appl Math Lett. 2017;65:56–62.
[5] Liu X, Jia M, Xiang X. On the solvability of a fractional differential equation model involving the p-Laplacian operator. Comput Math Appl. 2012;64:3267–3275.
[6] Lu H, Han Z, Sun S, et al. Existence of positive solutions for boundary value problems of nonlinear fractional differential equations with p-Laplacian. Adv Differ Equ. 2013;2013(30):16.
[7] Mahmudov N, Unul S. Existence of solutions of fractional boundary value problems with p-Laplacian operator. Bound Value Probl. 2015;2015(99):16.
[8] Prasad KR, Krushna BMB. Existence of multiple positive solutions for p-Laplacian fractional order boundary value problems. Int J Anal Appl. 2014;6(1):63–81.
[9] Wang J, Xiang H, Liu Z. Positive solutions for three-point boundary value problems of nonlinear fractional differential equations with p-Laplacian operator. Far East J Appl Math. 2009;37:33–47.
[10] Wang J, Xiang H. Upper and lower solutions method for a class of singular fractional boundary value problems with p-Laplacian. Abstr Appl Anal Article ID. 2010;971824:12.
[11] Bahaa GM, Hamiaz A. Optimal control problem for coupled time-fractional diffusion systems with final observations. J Taibah University Sci. 2019;13(1):124–135.
[12] Liu Y. Existence of solutions of periodic-type boundary value problems for multi-term fractional differential equations. Math Method Appl Sci. 2013;36:2187–2207.
[13] Màagli H, Mhadhebi N, Zeddini N. Existence and exact asymptotic behavior of positive solutions for a fractional boundary value problem. Abstr Appl Anal Article ID. 2013;420514:6.
[14] Chemmam R, Màagli H, Masmoudi S, et al. Combined effects in nonlinear singular elliptic problems in a bounded domain. Adv Nonlinear Anal. 2012;1:301–318.
[15] Seneta R. Regular varying functions, Lecture notes in math. Vol. 508. Berlin: Springer-Verlag; 1976.
[16] Kilbas AA, Srivastava HM, Trujillo JJ. Theory and applications of fractional differential equations. Amsterdam: Elsevier; 2006.
[17] Podlubny I. Geometric and physical interpretation of fractional integration and fractional differentiation. Fract Calc Appl Anal. 2002;5:367–386.