Bose–Einstein condensates in charged black–hole spacetimes

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Abstract. We analyze Bose–Einstein condensates on three types of spherically symmetric and static charged black-hole spacetimes: The Reissner-Nordström spacetime, Hoffmann’s Born-Infeld black-hole spacetime, and the regular Ayón-Beato–García spacetime. The Bose-Einstein condensate is modeled in terms of a massive scalar field that satisfies a Klein-Gordon equation with a self-interaction term. The scalar field is assumed to be uncharged and not self-gravitating. If the mass parameter of the scalar field is chosen sufficiently small, there are quasi-bound states of the scalar field that may be interpreted as dark matter clouds. We estimate the size and the total energy of such clouds around charged supermassive black holes and we investigate if their observable features can be used for discriminating between the different types of charged black holes.
1 Introduction

Scalar fields appear in many situations in physics. A particularly important idea is to use scalar fields as candidates for dark matter [1–5]. According to this idea dark matter consists of a certain type of spin-zero bosons, known as Weakly Interacting Massive Particles (WIMPs), axions or other, depending on the specific model, which have to be viewed as hypothetical because they have not been observed so far. The bosonic character of these particles, using the theory of relativistic Bose gases [10, 11], also opens the door for the existence of scalar field dark matter in the form of Bose-Einstein condensates [6, 7, 9].

As we believe that most, if not all, galaxies harbor a supermassive black hole at the center, the idea of modelling dark matter by scalar fields naturally leads to the question of whether there are bound or quasi-bound states of scalar fields near black holes. For the case of a Schwarzschild black hole and a massive scalar field without self-interaction, it was found that such quasi-bound states exist [12, 13]. More precisely, it was demonstrated that there are spherically symmetric field configurations, satisfying the boundary conditions of no flux coming in from infinity or from the horizon, that persist for a very long time. Exact solutions that satisfy these boundary conditions cannot persist forever; they have to decay in the course of time. However, this is not a problem for modelling dark matter clouds around black holes in terms of such scalar field configurations as long as the decay time is in the order of the age of the Universe. Whereas in the two quoted articles the scalar field was treated as a test field on the Schwarzschild background, numerical studies have also been carried out for the case of a self-gravitating cloud [14, 15].

If one wants to pursue the idea of modelling dark matter as a Bose-Einstein condensate, a self-interaction term has to be taken into account. In a fully relativistic setting one has to add such a self-interaction term to the massive Klein-Gordon equation which results in an equation that is very similar to the (non-relativistic) Gross-Pitaevskii equation. On Schwarzschild and Schwarzschild-de Sitter spacetimes, this equation and its potential for modelling quasi-bound scalar clouds was discussed in Refs. [16, 17]. In the present paper we want to extend this analysis to charged black holes. More precisely, we want to consider three types of charged black holes that arise from different theories of electrodynamics, and we want to investigate if one can discriminate between these black holes from the observation of quasi-bound clouds of Bose-Einstein condensates. The three types of charged black holes are the Reissner-Nordström
black hole, Hoffmann’s Born-Infeld black hole [18] and the Ayón-Beato-García black hole [19]. All of them are spherically symmetric and static charged black holes. They arise from coupling to Einstein’s field equation the standard Maxwell theory, the Born-Infeld theory [20] and another particular non-linear electrodynamical theory of the Plebański class [21], respectively.

While these three types of black holes have the same asymptotics far away from the center, they have quite different features inside the horizon. The Reissner-Nordström metric has a curvature singularity and a diverging electric field strength at the center, while Hoffmann’s black hole has a curvature singularity and a finite electric field strength there. The Ayón-Beato–García metric describes a regular black hole, i.e., the metric has no curvature singularity. The existence of regular black-hole solutions makes charged black holes coupled to nonlinear electrodynamics particularly interesting from a conceptual point of view. The first regular black hole metric was brought forward by Bardeen [26] in 1968 who, however, did not investigate the question of whether this metric was a solution to Einstein’s field equation with a reasonable matter source. The properties of regular black holes were further studied in Refs. [27–31]. The Ayón-Beato–García metric was the first regular black hole that was found as a solution to Einstein’s field equation with a nonlinear electrodynamical field as the source. Soon thereafter more such solutions were found [22, 24, 25]; in particular it was shown that the Bardeen black hole is also such a solution, with a magnetic monopole at the center [23].

The fact that different electrodynamical theories lead to charged black hole solutions with qualitatively quite different interior raises the question of whether these differences are observable from the outside. Spacetimes of charged black holes may be probed with test particles and with light rays as has been demonstrated in several articles: The motion of (charged or uncharged) test particles in the Reissner-Nordström spacetime has been studied in great detail, see [32–37]. For Hoffmann’s Born-Infeld black hole, the geodesics have been investigated in Ref. [38] and the light rays in Ref. [39]. For geodesics in the Ayón-Beato–García metric we refer to Refs. [40, 41].

In addition to using particles or light rays, one may also use scalar fields for probing different spacetime geometries. It is the purpose of this paper to investigate if the quasi-bound states of scalar fields which have been suggested for modeling dark matter clouds can be used for discriminating between different types of charged black holes.

The outline of the paper is as follows: in Section 2 we briefly review the basic properties of the charged black hole spacetimes under consideration. In Section 3 we consider, on these spacetimes, an uncharged scalar field that satisfies the Klein-Gordon equation with a self-interaction term, we derive the corresponding Gross–Pitaevskii equation and we analyze some properties of the effective potential. In Section 4 we numerically construct quasi-bound states and we compare them for our three charged black-hole spacetimes. In Section 5 we discuss if the Thomas-Fermi approximation is viable for this kind of quasi-bound states. In Section 6, we present the conclusions and some outlook.

# 2 Spherically symmetric and static charged black holes

We consider spherically symmetric and static spacetimes where the metric, in standard spherical coordinates, is given as

\[ g_{\mu\nu}dx^\mu dx^\nu = -f(r)c^2dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]  

(2.1)
We compare three different types of such spacetimes all of which describe charged black holes, but in different theories of electrodynamics. The first is the Reissner-Nordström (RN) spacetime with the metric function \( f(r) \) equal to

\[
 f_{\text{RN}}(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2}.
\]  

(2.2)

This is the unique spherically symmetric and static solution of Einstein’s field equation coupled to standard Maxwell electrodynamics. The second one is Hoffmann’s Born-Infeld (HBI) black-hole spacetime [18],

\[
 f_{\text{HBI}}(r) = 1 - \frac{2m}{r} + \frac{2}{\sigma r} \int_{r}^{\infty} \left( \sqrt{s^4 + \sigma^2 q^2} - s^2 \right) ds
\]  

(2.3)

which is a solution of Einstein’s field equation coupled to Born-Infeld electrodynamics [20]. The third one is the Ayón-Beato–García (ABG) spacetime [19],

\[
 f_{\text{ABG}}(r) = 1 - \frac{2m r^2}{(q^2 + r^2)^{3/2}} + \frac{q^2 r^2}{(q^2 + r^2)^2}
\]  

(2.4)

which is a regular black-hole solution of Einstein’s field equation coupled to a certain nonlinear electrodynamical theory of the Plebański class [21]. In all three cases, \( m \) is the mass parameter and \( q \) is the charge parameter, both of which have the dimension of a length. They are related to the ADM mass \( M \) and the electric charge \( Q \) in SI units by

\[
 m = \frac{GM}{c^2}, \quad q = \frac{\sqrt{GQ}}{\sqrt{4\pi \varepsilon_0 c^2}}
\]  

(2.5)

where \( G \) is Newton’s gravitational constant and \( \varepsilon_0 \) is the permeability of the vacuum. The HBI metric involves a third parameter, \( \sigma \), which also has the dimension of a length. The Born-Infeld theory postulates the existence of a constant of Nature with the dimension of a magnetic field strength, \( b \), and the parameter \( \sigma \) is equal to

\[
 \sigma = \frac{cM}{bQ}.
\]  

(2.6)

For \( b \to \infty \) the Born-Infeld theory approaches the standard Maxwell theory; correspondingly, the HBI metric approaches the RN metric for \( \sigma \to 0 \). For \( \sigma \to \infty \) the HBI metric approaches the Schwarzschild metric. The fact that to date the standard Maxwell theory is in agreement with all experiments demonstrates that \( b \) must be big. If we assume that \( cb \) (which has the dimension of an electric field) is much bigger than the electric field of our black hole in the entire domain \( m \lesssim r < \infty \), we have to require that

\[
 \sigma \ll m^3/q^2.
\]  

(2.7)

The ABG metric is regular at the origin while the RN and the HBI solutions have a curvature singularity there. All three metrics describe black holes, i.e., they have one or more horizons, if the parameters \( m \), \( q \) and \( \sigma \) are chosen appropriately. Horizons are indicated by zeros of the metric function \( f(r) \). The RN metric has an inner and an outer horizon at radii \( 0 < r_{\text{Hi}} < r_{\text{Ho}} < \infty \) if \( 0 < |q| < m \). For \( |q| = m \) the two horizons merge and for \( |q| > m \) the singularity at the origin is naked. Similarly, the ABG metric features two horizons if

\[
 0 < |q| < q_c, \quad q_c \approx 0.634 m.
\]  

(2.8)
For $|q| > q_c$ the ABG metric is a regular metric without a horizon. As we want to compare different black-hole spacetimes with the same parameters $m$ and $q$ we will restrict to values of $q$ and $m$ that satisfy the inequality (2.8). It is then easy to check that the HBI metric describes a black hole (with one or two horizons) for all values of $\sigma$.

While the RN, the ABG and the HBI metrics are quite different near the origin, they have the same asymptotics far away from the center,

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} + O(r^{-3}) .$$

(2.9)

In the following we consider the region between the (outer) horizon and infinity, where $f(r) > 0$, and it is our goal to investigate if scalar field condensates in this region show significant differences for the three cases.

3 The Gross–Pitaevskii–like equation

The Klein–Gordon equation for a complex test–scalar field $\Phi$ with a scalar potential $V(\Phi)$ in a spacetime with metric $g_{\mu\nu}$ can be written as follows:

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \Phi \right) - \frac{dV(\Phi^*)}{d\Phi^*} = 0 ,$$

(3.1)

where $g$ is the determinant of the metric and an upper star means complex conjugation. We consider a potential of the form

$$V(\Phi\Phi^*) = \mu^2 \Phi\Phi^* + \frac{\lambda}{2} (\Phi\Phi^*)^2$$

(3.2)

which allows to interpret the scalar field as a Bose-Einstein condensate of some bosonic particles. Here $\mu$ is, as usual, the scalar field mass parameter which equals the inverse Compton wavelength of the particles, i.e.

$$\mu = \frac{M_\Phi c}{\hbar}$$

(3.3)

where $M_\Phi$ is the mass of the particles, and $\lambda$ is the self–interaction coupling constant which equals, up to a numerical factor, the scattering length $a_s$ of the particles, $\lambda = 16\pi a_s$. Note that $\lambda$ has the dimension of a length while $\mu$ has the dimension of an inverse length. Here we assume that the scalar field describes uncharged particles. For particles with a charge $Q_\Phi$ one would have to couple in an electromagnetic field by the usual minimal replacement rule, $\partial_\mu \Phi \mapsto \partial_\mu \Phi + i\hbar^{-1}Q_\Phi A_\mu \Phi$ where $A_\mu$ is the electromagnetic potential.

We are interested in (approximate) solutions of the Klein-Gordon equation on a spacetime of the form (2.1) that are spherically symmetric and harmonic in time,

$$\Phi(t,r) = e^{i \omega t} \frac{u(r)}{r}$$

(3.4)

with a real frequency $\omega$. In this case, after some algebraic manipulations the Klein–Gordon equation reduces to a Gross–Pitaevskii–like equation,

$$\left( - \frac{d^2}{dr^2} + V_{\text{eff}}(r) + \lambda_{\text{eff}}(r) \frac{|u(r)|^2}{r^2} \right) u(r) = \frac{\omega^2}{c^2} u(r) .$$

(3.5)
Here $r_*$ denotes the *tortoise coordinate*, which is defined by the equation

$$dr_* = \frac{dr}{f(r)}.$$  \hfill (3.6)

The tortoise coordinate runs over the entire real line when $r$ runs from the (outer) horizon to infinity. Moreover, we have introduced the effective potential

$$V_{\text{eff}}(r) = f(r) \left( \mu^2 + \frac{f'(r)}{r} \right)$$  \hfill (3.7)

and the effective self-interaction parameter

$$\lambda_{\text{eff}}(r) = \lambda f(r).$$  \hfill (3.8)

The fact that $\lambda_{\text{eff}}$ depends on $r$ via the metric function $f(r)$ reflects the influence of the spacetime geometry on the self-interaction, cf. Refs. \cite{16, 17}. By comparison with the standard Gross-Pitaevskii equation we see that $\omega^2/c^2$ may be identified with an effective chemical potential. Of course, in the relativistic case $\omega$ occurs quadratic, rather than linear as in the standard Gross–Pitaevskii equation, because the Klein-Gordon equation involves a second time derivative.

Introducing the tortoise coordinate $r_*$, viewing the radial coordinate $r$ as an implicitly given function of $r_*$, is convenient for analyzing the differential equation. However, when comparing different black-hole spacetimes we will always work with the coordinate $r$, rather than with $r_*$. The reason is that the radius coordinate $r$ always has the same geometric meaning of giving the area $A(r)$ of the sphere at radius $r$ by the usual formula $A(r) = 4\pi r^2$. By contrast, the tortoise coordinate depends on the chosen spacetime and admits no general interpretation in terms of a measurable quantity.

With every solution $\Phi$ of the Klein-Gordon equation (3.1) we associate the current

$$j^\mu = i\alpha g^{\mu\nu}(\Phi^* \partial_\nu \Phi - \Phi \partial_\nu \Phi^*)$$  \hfill (3.9)

where $\alpha$ is a real constant. It follows immediately from (3.1) that this current is conserved,

$$\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} j^\mu \right) = 0.$$  \hfill (3.10)

With $\alpha$ chosen appropriately, (3.10) is to be interpreted as the conservation law of the particle number. If $\Phi$ is of the form of (3.4), with a real frequency $\omega$, we may choose

$$\alpha = \frac{\omega^2}{2\omega}.$$  \hfill (3.11)

Then the only non-zero components of the current are

$$j^t = \frac{|u(r)|^2}{f(r)r^2},$$  \hfill (3.12)

$$j^r = \frac{i\omega^2 f(r)}{2\omega r^2} \left( u(r)^* \frac{du(r)}{dr} - u(r) \frac{du(r)^*}{dr} \right)$$  \hfill (3.13)

and the conservation law (3.10) reduces to

$$\partial_t j^t + \frac{1}{r^2} \partial_r \left( r^2 j^r \right) = 0$$  \hfill (3.14)
where the two terms on the left-hand side are separately equal to zero. If we multiply (3.10) with \( r^2 \sin \theta \, dr \, d\theta \, d\varphi \) and integrate over a spherical shell with inner radius \( r_1 \) and outer radius \( r_2 \) we get

\[
\frac{d}{dt} N_{r_1 r_2} + J_{r_2} - J_{r_1} = 0
\]

(3.15)

where

\[
N_{r_1 r_2} = 4 \pi \int_{r_1}^{r_2} \frac{|u(r)|^2}{f(r)} \, dr
\]

(3.16)

is the number of particles in the shell and

\[
J_r = \frac{2i \pi c^2}{\omega} \left( (u(r))^* f(r) \frac{d u(r)}{d r} - u(r) f(r) \frac{d (u(r))^*}{d r} \right)
\]

(3.17)

is the number flux through the sphere of radius \( r \).

In addition to the conservation law for the particle number, there is also a conservation law of energy associated with our scalar fields. This follows from the fact that, as our spacetime is static, the Klein-Gordon equation (3.1) can be equivalently rewritten as

\[
\frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} T^\mu \right) = 0
\]

(3.18)

with the energy-momentum tensor

\[
T_{\rho \sigma} = \frac{\hbar c}{\mu} \left( \partial_\rho \Phi \partial_\sigma \Phi^* + \partial_\rho \Phi^* \partial_\sigma \Phi - g_{\rho \sigma} \left( \gamma \omega \partial_\lambda \Phi \partial_\lambda \Phi^* + V(\Phi^* \Phi) \right) \right).
\]

(3.19)

For fields of the form of (3.4) this conservation law simplifies to

\[
\partial_t \left( - T^t t \right) + \frac{1}{r^2} \partial_r \left( - r^2 T^r t \right) = 0
\]

(3.20)

with both terms on the left-hand side separately equal to zero, where

\[
- T^t t = \frac{\hbar c}{\mu} \left( \left( \frac{\omega^2}{c^2 f(r)} + \mu^2 + \frac{\lambda |u(r)|^2}{2 r^2} \right) \frac{|u(r)|^2}{r^2} + f(r) \left\{ \frac{d}{d r} \left( \frac{u(r)}{r} \right) \right\}^2 \right)
\]

(3.21)

is the energy density and

\[
- T^r t = \frac{i \hbar \omega}{\mu r^2} \left( (u(r))^* f(r) \frac{d u(r)}{d r} - u(r) f(r) \frac{d (u(r))^*}{d r} \right)
\]

(3.22)

is the energy flux. Note that the energy flux is zero if \( u(r) \) is real. In analogy to (3.16), the total energy in a shell between radii \( r_1 \) and \( r_2 \) is

\[
E_{r_1 r_2} = -4 \pi \int_{r_1}^{r_2} T^t t \, r^2 \, dr
\]

(3.23)

\[
= 4 \pi \frac{\hbar c}{\mu} \int_{r_1}^{r_2} \left( \left( \frac{\omega^2}{c^2 f(r)} + \mu^2 + \frac{\lambda |u(r)|^2}{2 r^2} \right) |u(r)|^2 + r^2 f(r) \left\{ \frac{d}{d r} \left( \frac{u(r)}{r} \right) \right\}^2 \right) \, dr.
\]

When experimenting with Bose-Einstein condensates in the laboratory one chooses an oscillator potential, or some similar potential that admits a minimum and increases to infinity
Figure 1. The effective potential $V_{\text{eff}}(r)$ for the HBI black hole for different values of the scalar field mass parameter $\mu$. The charge is chosen as $q = 0.634 m$ because we want to compare with an extremal ABG black hole. We give $r$ in units of $m$ and $V_{\text{eff}}$ in units of $m^{-2}$. For each value of $\mu$ we have plotted the limiting cases $\sigma = 0$ (dashed) and $\sigma = \infty$ (solid) which correspond to the RN black hole and the Schwarzschild black hole, respectively. For any other value of $\sigma$ we get a curve that lies between the dashed and the solid one. For realistic values of $\sigma$ that satisfy (2.7), the HBI case is practically undistinguishable from the RN case.

in all spatial directions, for trapping the condensate. Our potential $V_{\text{eff}}(r)$ has a different shape. However, for an appropriate choice of the mass parameter $\mu$ it features a local minimum which can provide some partial trapping. We speak of a partial trapping because the potential has a maximum of finite height; so the scalar field is not perfectly trapped near the local minimum but it may tunnel through the potential barrier and, actually, decay in the course of time. Therefore, as long as we assume that there is no flux of the scalar field coming in from infinity or from the horizon, we cannot expect the existence of solutions that are strictly of the form of (3.4) with a real $\omega$. However, approximate solutions of this kind may exist. We will refer to them as to quasi-bound states.

For investigating the existence of quasi-bound states it will be of crucial importance to determine for which values of $\mu$ the potential $V_{\text{eff}}$ admits a minimum. We have plotted $V_{\text{eff}}$ in Fig.1 for the HBI metric and in Fig.2 for the ABG metric, both in comparison to the RN metric, for different values of the scalar field mass parameter $\mu$. The potential goes to zero at the horizon and it approaches the value $\mu^2$ for $r \to \infty$. There is a critical value $\mu_c$ of the parameter $\mu$ such that for $0 < \mu < \mu_c$ the potential has a maximum and a minimum while for bigger values of $\mu$ it is monotonically increasing. As the existence of a minimum is necessary for the existence of quasi-bound states, for $\mu = 0$ and for $\mu > \mu_c$ such states do not exist. The value of $\mu_c$ depends on $q$ and it is different for RN, HBI and ABG black holes, but it is always near $0.25 m^{-1}$. For a supermassive black hole, $m > 10^6 \text{km}$, this value of $\mu$ corresponds to a particle mass $M_\Phi$ of not more than $10^{-14} \text{eV}/c^2$. So we need very light bosonic particles for producing condensates that may be (partially) trapped by our potential.

For determining the maxima and minima of $V_{\text{eff}}$ we need to analyze the roots of the
The effective potential $V_{\text{eff}}(r)$ for the ABG black hole (solid) and the RN black hole (dashed) for different values of the scalar field mass parameter $\mu$. The charge of the black hole is $q = 0.634 m$, as in Figure 1. One sees a difference between the ABG and the RN case, in particular as to the position of the horizon; however, even for this highly charged black hole the minima and maxima of the ABG potential are very close to those of the RN potential.

In the RN scenario, this yields a fifth order equation for $r$,

$$6q^4 - 15mq^2r + 8m^2r^2 + 4q^2r^2 - 3mr^3 - \mu^2q^2r^4 + m\mu^2r^5 = 0,$$  \hspace{1cm} \text{Eqn. (3.25)}

For the ABG metric it gives a 14th order equation and for the HBI metric it gives an equation in terms of elliptic functions. In Figs. 3 and 4 we display the equation $\frac{d}{dr}V_{\text{eff}} = 0$ in the $\mu$-$r$ plane between the horizon and infinity, where we give $\mu$ in units of $m^{-1}$ and $r$ in units of $m$.

If a vertical line, corresponding to a particular value of $\mu$, intersects this curve twice, there are real solutions, $r_{\text{max}}$ and $r_{\text{min}}$, for equation (3.24) outside of the horizon, indicating that the function $V_{\text{eff}}$ has a maximum and a minimum. We see from the figures that this is true for $0 < \mu < \mu_c$, where the value of $\mu = \mu_c$ is indicated by a dotted vertical line in the figure. At this value the maximum and the minimum merge, forming a saddle. For $\mu > \mu_c$ the potential has no extrema.

In this section we have seen that the effective potential in the ABG and in the HBI case is very similar to that of the RN metric. Correspondingly, we expect that these different types of charged black holes admit very similar scalar field configurations. In the next section we will see that this is, indeed, true.

4 Scalar quasi-bound states

We have already emphasized that for a partial trapping $\mu$ must be chosen such that the potential $V_{\text{eff}}$ admits a local minimum, $0 < \mu < \mu_c$. For constructing quasi-bound states we
Figure 3. Plot of the equation $\frac{d}{dr} V_{\text{eff}} = 0$ for the HBI metric with $q = 0.634 \, m$. We have plotted the limiting cases $\sigma \to 0$ (dashed) and $\sigma \to \infty$ (solid) which correspond to the RN metric and to the Schwarzschild metric, respectively. For any other values of $\sigma$ the curve lies between these two ones. For realistic values of $\sigma$ satisfying (2.7), the curve for the HBI black hole is practically indistinguishable from the curve for the RN black hole.

then have to choose $\omega$ such that

$$V_{\text{eff}}(r_{\text{min}}) < \frac{\omega^2}{c^2} < \min(\mu^2, V_{\text{eff}}(r_{\text{max}})).$$

(4.1)

For the following it will be convenient to rewrite the Gross-Pitaevskii-like equation (3.5) in the form

$$\left( - \frac{d^2}{dr^2} + V_{\text{eff}}(r) + f(r) \frac{|v(r)|^2}{r^2} \right) v(r) = \frac{\omega^2}{c^2} v(r)$$

(4.2)

where

$$v(r) = \sqrt{\lambda} u(r).$$

(4.3)

Note that in terms of the function $v(r)$ the particle number (3.16) and the flux (3.17) are given by the equations

$$\lambda N_{r_1 r_2} = 4 \pi \int_{r_1}^{r_2} \frac{|v(r)|^2}{f(r)} \, dr = 4 \pi \int_{r_1}^{r_{*2}} |v(r)|^2 \, dr_*$$

(4.4)
Figure 4. Plot of the equation $\frac{d}{dr}V_{eff} = 0$ for the ABG metric (solid) and for the RN metric (dashed). As in Fig. 3, the charge parameter is chosen as $q = 0.634\, m$.

and

$$\lambda J_r = \frac{2i\pi c^2}{\omega} \left( v(r)^* f(r) \frac{dv(r)}{dr} - v(r) f(r) \frac{dv(r)}{dr} \right) = \frac{2i\pi c^2}{\omega} \left( v(r)^* \frac{dv(r)}{dr} - v(r) \frac{dv(r)}{dr} \right), \quad (4.5)$$

respectively. In analogy to (4.4), the expression (3.23) for the energy becomes

$$\lambda E_{r_1 r_2} = \frac{4\pi \hbar c}{\mu} \int_{r_1}^{r_2} \left( \frac{\omega^2}{c^2} + f(r)\mu^2 + \frac{f(r)|v(r)|^2}{2r^2} \right) |v(r)|^2 + v^2 \left| \frac{d}{dr_*} \left( \frac{v(r)}{r} \right) \right|^2 dr_*.$$

(4.6)

With $\mu$ and $\omega$ chosen appropriately, the partial trapping is reflected by the asymptotic behavior of solutions to (4.2): For big $r$, this equation can be approximated by

$$\frac{d^2 v(r)}{dr_*^2} \approx \left( \mu^2 - \frac{\omega^2}{c^2} \right) v(r), \quad r_* \to \infty$$

(4.7)

while near the horizon we have

$$\frac{d^2 v(r)}{dr_*^2} \approx - \frac{\omega^2}{c^2} v(r), \quad r_* \to -\infty.$$

(4.8)

Correspondingly, there are solutions which exponentially decay for big values of $r$,

$$v(r) \approx \gamma e^{-\sqrt{\mu^2 - \omega^2/c^2} r_*}, \quad r_* \to \infty$$

(4.9)

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with a real constant \( \gamma \) while real solutions have an oscillatory behavior near the horizon,
\[
v(r) \approx \beta e^{-i\omega r/c} + \beta^* e^{i\omega r/c}, \quad r \to -\infty
\]  
(4.10)

with a complex constant \( \beta \). (4.10) is a superposition of an ingoing and an outgoing particle flux. The ingoing one can be interpreted as a current of particles that have tunneled through the potential barrier and are falling towards the black hole. The outgoing one is a hypothetical counter-current that is necessary for providing a stationary solution. As no particles can come out of a black hole, this counter-current cannot be expected to exist in Nature, so the solution that will actually be realised in Nature is not stationary but rather decaying in the course of time because of the particle flow that goes towards the horizon. However, if the amplitude \( |\beta| \) is small the solution may be considered as stationary over a long period of time. It is this kind of approximately stationary, i.e. quasi-bound, solutions that we want to discuss in this section for the various types of black holes under consideration.

As a formal means for switching off the tunneling we replace the potential \( V_{\text{eff}} \) by a modified potential
\[
V_{\text{eff}}(r) = \begin{cases} 
V_{\text{eff}}(r_{\text{max}}) & \text{if } r \leq r_{\text{max}} \\
V_{\text{eff}}(r) & \text{if } r_{\text{max}} < r < \infty
\end{cases}
\]  
(4.11)

where \( r_{\text{max}} \) is the \( r \) value where the potential takes its local maximum, see Fig. 5. In the potential \( \tilde{V}_{\text{eff}} \) we have stationary solutions in the strict sense, i.e., solutions \( \tilde{v}(r) \) that fall off towards infinity and towards the horizon and are, thus, square-integrable. These solutions coincide on the interval \( r_{\text{max}} < r < \infty \) with solutions \( v(r) \) in the potential \( V_{\text{eff}} \); in the region between the horizon and \( r_{\text{max}} \), however, the solutions \( v(r) \) need the above-mentioned unphysical counter-current for being stationary. We may view a stationary solution \( \tilde{v}(r) \) as a good approximation for a solution to our physical problem if the particle number flux (4.5) of this counter-current is sufficiently small. From the asymptotic formula (4.10) we read that this flux is given by
\[
\lambda J_{r \to -\infty} = 4\pi c|\beta|^2.
\]  
(4.12)

We will now construct such a quasi-bound state for a Reissner-Nordström black hole with \( q = 0.634 \) and compare it to the other types of charged black holes afterwards. We choose \( \mu = 0.16 \) and \( \omega/c = 0.15997 \) \( m^{-1} \). Choosing \( \mu \) bigger (i.e., closer to \( \mu_c \approx 0.25 \) \( m^{-1} \)) gives a lower potential barrier, i.e., it allows for more tunnelling; choosing \( \omega/c \) bigger (i.e., closer to \( \mu \)) gives a condensate with a higher particle number.

We consider (4.2) with the cut-off potential \( \tilde{V}_{\text{eff}} \) instead of \( V_{\text{eff}} \). With \( \mu \) and \( \omega \) given, there is a unique real and positive solution \( \tilde{v}(r) \) of this equation that exponentially falls off towards infinity and towards the horizon and has no zeros. We have determined this solution numerically, see Fig. 6. As this solution is square-integrable, it yields a finite particle number \( N \). By inserting the numerical solution into (4.4) we find
\[
\lambda N \approx 3 \times 10^6 \ m
\]  
(4.13)

where \( m = GM/c^2 \) is the mass parameter of the black hole. On the interval \( r_{\text{max}} < r < \infty \) the function \( \tilde{v}(r) \) coincides with a real solution \( v(r) \) of (4.2), with the original potential \( V_{\text{eff}} \). If extended beyond \( r_{\text{max}} \), this solution \( v(r) \) approaches the horizon in an oscillatory fashion according to (4.10), see again Fig. 6, so \( v(r) \) is not square-integrable. From the asymptotic behavior of the numerical solution we can read the value of \( \beta \), according to (4.10); inserting
Figure 5. The effective potential $V_{\text{eff}}(r)$ (solid) and the cut-off potential $\tilde{V}_{\text{eff}}(r)$ (dashed) plotted against the tortoise coordinate $r_*$ for a RN black hole with $q = 0.634\, m$ and $\mu = 0.16\, m^{-1}$.

Figure 6. Scalar field distributions $v(r)^2$ (dashed) and $\tilde{v}(r)^2$ (solid) for a RN black hole with $q = 0.634\, m$, $\mu = 0.16\, m^{-1}$ and $\omega/c = 0.15997\, m^{-1}$, plotted against the tortoise coordinate. $v(r)$ is a solution of (4.2) with the potential $V_{\text{eff}}(r)$ while $\tilde{v}(r)$ is a solution of the same equation with the cut-off potential $\tilde{V}_{\text{eff}}(r)$. The two solutions coincide on the interval $r_{\text{max}} < r < \infty$. Towards the horizon, $\tilde{v}(r)$ falls off exponentially whereas $v(r)$ oscillates according to (4.10), see the enlarged part of the plot on the right.

$\beta$ into (4.12) gives us the particle number flux $J_{r_*\to-\infty}$ that is necessary for compensating the loss by particles that tunnel towards the horizon,

$$\lambda J_{r_*\to-\infty} \approx 0.09\, c.$$  

(4.14)

The quotient

$$T = \frac{N}{J_{r_*\to-\infty}} = \frac{\lambda N}{\lambda J_{r_*\to-\infty}} \approx 2.6 \times 10^6 \frac{m}{c}$$  

(4.15)

is a measure for the lifetime of the cloud. Note that $T$ is independent of $\lambda$, as long as $\lambda > 0$. (Our solutions do not have a finite limit for $\lambda \to 0$ because, by (4.3), $u(r)$ goes to infinity
in this limit.) Even for a supermassive black hole with \( m \approx 10^{10} \text{ km} \), such as the one at the center of M87, (4.15) gives a lifetime of only \( T \approx 35,000 \) years. By astrophysical standards, this is not a very long lifetime. Bigger clouds with longer lifetime may be constructed by choosing \( \omega/c \) even closer to \( \mu \). In analogy to (4.13) we find the total energy \( E \) of the cloud by inserting the numerical solution into (4.6),

\[
\lambda E \approx 2.2 \hbar c.
\] (4.16)

For specifying numerical values for the particle number and for the energy it is necessary to specify \( \lambda \). As an example, we choose \( \lambda \approx 10^{-35} \text{ m} \) which, for the above-mentioned supermassive black hole with \( m = GM/c^2 \approx 10^{10} \text{ km} \), is equivalent to \( \lambda \approx 10^{-25} \text{ km} \). This is not an unrealistic scattering length for light dark matter candidates, cf. [44]. With this choice of \( \lambda \), (4.13) yields \( N \approx 10^{35} \) and (4.16) yields \( E \approx 10^{-60} \text{ MeV}^2 \). So we see that the energy content of the cloud is tiny in comparison with the energy of the black hole. On the one hand, this confirms that the test-field approximation which we have used throughout is justified. On the other hand, it demonstrates that the gravitational effect of a cloud with the chosen parameters would be practically unobservable. Nonetheless, the chosen parameters are appropriate for our main purpose: We will demonstrate that, with these parameters, the resulting clouds in the BHI and ABG spacetimes are virtually indistinguishable from the RN case, and that the differences are even smaller for a more realistic choice of the parameters.

![Figure 7](image.png)

**Figure 7.** Density function \( v(r)^2/f(r) \) for solutions of (4.2) with \( q = 0.634 m, \mu = 0.16 m^{-1} \) and \( \omega/c = 0.15997 m^{-1} \), for the RN black hole (solid) which is the limit of the HBI black hole for \( \sigma \to 0 \), for the limit of the HBI black hole for \( \sigma \to \infty \) (dashed) and for the ABG black hole (dotted). The three graphs are lying on top of each other.

With the chosen values of \( q = 0.634 m, \mu = 0.16 m^{-1} \) and \( \omega/c = 0.15997 m^{-1} \) we repeat the calculation that we have carried through for the RN black hole now for the HBI and for the ABG black holes. As the tortoise coordinate has a different geometric meaning in different spacetimes, we plot the density distribution against the area radius function \( r \) rather than against \( r_* \), see Fig. 7. With respect to \( r \), the number density is given up to a factor of \( \lambda \) by the function \( v(r)^2/f(r) \), recall (3.16), so it is this function that we plot. We see that the solutions in the three different black hole spacetimes are virtually indistinguishable.

We have found this result for the case that \( q = 0.634 m \) which is the highest value of the charge for which a comparison is possible. It is widely believed that the black holes that exist
in Nature have a considerably lower charge. Then the differences between the three types of black holes are even smaller. Also, we have seen that, although we have chosen \( \omega/c \) rather close to \( \mu \), the lifetime of the constructed cloud is not very long and its energy content is tiny in comparison to that of the black hole. By choosing \( \omega/c \) even closer to \( \mu \) we can make the cloud more long-lived and more energetic. Again, then the differences between the three types of black hole spacetimes are even smaller than in our example. So we may conclude that, for all cases of possible astrophysical relevance, it is not possible to discriminate between the three different types of charged black holes with the help of quasi-bound states of uncharged scalar fields.

5 The Thomas–Fermi approximation for scalar quasi-bound states

Finally, we want to investigate to what extent the Thomas-Fermi approximation can be used for modeling the quasi-bound scalar field configurations we have constructed in the preceding section. The Thomas-Fermi approximation is often used for describing the behavior of Bose–Einstein condensates, see for instance Ref. [42]. Approximate solutions for scalar field distributions in a curved spacetime have been obtained with this method in Refs. [16, 17] for the cases of Schwarzschild and Schwarzschild–de Sitter background spacetimes. We also refer to Ref. [8] where the validity of the Thomas-Fermi approximation was demonstrated for dark matter halos using the non-relativistic Gross-Pitaevskii equation.

The Thomas-Fermi approximation assumes that the kinetic energy is negligibly small in comparison to the potential energy and the self-interaction energy. Then the first term in (3.5) can be neglected and (3.5) can be algebraically solved for \( |u(r)|^2 \),

\[
|u(r)|^2 = \left( \frac{\omega^2}{c^2} - V_{\text{eff}}(r) \right) \frac{r^2}{\lambda_{\text{eff}}(r)}.
\]  

(5.1)

Of course, this equation is meaningful only as long as the right-hand side is positive. If \( \mu \) has been chosen such that \( V_{\text{eff}} \) admits a minimum, this is true on a finite interval \( r_1 < r < r_2 \) around the minimum if (4.1) holds. Outside of this interval one sets \( u(r) \) equal to zero. So in the Thomas-Fermi approximation the condensate occupies a spherical shell of inner radius \( r_1 \) and outer radius \( r_2 \), where \( r_1 \) and \( r_2 \) are the solutions of the equation \( V_{\text{eff}}(r) = \omega^2/c^2 \). If one plugs the function

\[
u(r) = \begin{cases} 
\left( \frac{\omega^2}{c^2} - V_{\text{eff}}(r) \right) \frac{r^2}{\lambda_{\text{eff}}(r)} & \text{if } \frac{\omega^2}{c^2} > V_{\text{eff}}(r) \\
0 & \text{otherwise}
\end{cases}
\]

(5.2)

into the Gross-Pitaevskii–type equation (3.5), one finds that the first term diverges if one of the boundary values, \( r_1 \) or \( r_2 \), is approached. Similarly, the energy density (3.21) diverges at \( r_1 \) and at \( r_2 \). The total energy, on the other hand, remains finite and the Thomas-Fermi approximation has proven very useful for estimating the size and the total energy of (quasi-)bound states. For a class of potentials that include the oscillator potential, but not our potential \( V_{\text{eff}} \), it has been rigorously proven [45] that the Thomas-Fermi approximation becomes arbitrarily good if \( \lambda N \) becomes sufficiently big.

As, in our case, \( \lambda N \) becomes big if \( \omega/c \) is chosen close to \( \mu \), one may expect that the Thomas-Fermi approximation is good if we choose \( \mu \) correspondingly. Our numerical studies have shown that one has to choose \( \omega/c \) very close to \( \mu \) for getting a good agreement between
the Thomas-Fermi approximation and the exact (numerical) solution. Fig. 8 shows the result for the parameters that have already been used in the preceding section. One sees that the approximation gives the correct order of magnitude, but that there is a considerable deviation in the outer part of the cloud. The difference becomes smaller if $\omega/c$ is chosen closer to $\mu$, but the difference between the Thomas-Fermi approximation and the exact (numerical) solution is, in any case, bigger than the differences between the three charged black hole spacetimes, compare Fig. 8 with Fig. 7.

![Figure 8](image.png)

**Figure 8.** Density function $v(r)^2 = \lambda u(r)^2$ in a RN black hole, plotted against the tortoise coordinate, for a numerical solution of (4.2) with $q = 0.634m, \mu = 0.16m^{-1}$ and $\omega/c = 0.15997m^{-1}$ (solid) and for the Thomas-Fermi approximation (dashed).

### 6 Conclusions and outlook

We have analyzed uncharged scalar test fields that satisfy a Klein-Gordon equation with a self-interaction term on spacetimes of different charged black hole models. We have chosen the mass parameter $\mu$ of the scalar field such that the effective potential admits a local minimum which allows for approximate solutions that depend on time only via a factor $e^{i\omega t}$ with a real frequency $\omega$. These solutions may be viewed as quasi-bound clouds of a Bose-Einstein condensate around the charged black hole. Mathematically, they come about by replacing the effective potential with a cut-off potential that prevents particles from tunneling through the potential barrier towards the horizon. The (stationary) solutions in the cut-off potential may be viewed as good approximations of (non-stationary, decaying) solutions in the original potential if the tunnel current is sufficiently small. It was our main goal to find out whether or not the density distribution of such a quasi-bound cloud is different for different charged black-hole models. We have found that for the three types of black holes considered here – the Reissner-Nordström black hole, Hoffmann’s Born-Infeld black hole and the regular Ayón-Beato–García black hole – the differences are tiny.

The type of clouds we have considered here exists only for very light bosonic particles. We have seen that, for a supermassive black hole with mass parameter $m = GM/c^2 > 10^6$km, the particle mass $M_\Phi$ cannot be bigger than $10^{-14}$eV/c$^2$ because otherwise the effective
potential does not have a minimum. We have also seen that we need a fine-tuning between the frequency $\omega$ and the mass parameter $\mu$ of the scalar field if we want to get a cloud with a lifetime that is long enough for being astrophysically relevant and with an energy content that is not completely negligible in comparison to the energy of the black hole. If the energy content of the cloud is comparable to the energy of the black hole, or even bigger, the cloud may be actually observed, e.g., with the help of lensing. Of course, for such heavy clouds we cannot use the test-field approximation anymore.

We have considered an uncharged scalar field because in this paper we wanted to concentrate on gravitational effects. For a charged scalar field the situation is different, because of the electromagnetic interaction between the black hole and the cloud. It is possible that this electromagnetic interaction gives rise to observable effects that may be used for discriminating between different types of charged black holes.

Acknowledgments

This work was supported by DFG-CONACyT Grant Nos. B330/418/11 and 211183, by CONACyT Grant No. 166041F3. E.C. acknowledges MCTP for financial support. C.L. and V.P. acknowledge support from the DFG within the Research Training Group 1620 Models of Gravity and C.L. also within the QUEST Center of Excellence.

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