ON RESTRICTED TERNARY WORDS AND INSETS

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Abstract. We investigate combinatorial properties of a kind of insets we defined in an earlier paper, interpreting them now in terms of restricted ternary words. This allows us to give new combinatorial interpretations of a number of known integer sequences, namely the coefficients of Chebyshev polynomials of both kinds, Fibonacci numbers, Delannoy numbers, asymmetric Delannoy numbers, Sulanke numbers, coordinating sequences for some cubic lattices, crystal ball sequences for some cubic lattices, and others. We also obtain several new properties of said insets. In particular, we derive three generating functions when two of three variables are constant.

At the end, we state 40 combinatorial configurations counted by our words.

1. Introduction

In M. Janjić and B. Petković [2], we defined the notion of inset in the following way. Let $m, k, n$ be nonnegative integers. Let $q_1, q_2, \ldots, q_n$ be positive integers. Consider the set $X$ consisting of blocks $X_1, X_2, \ldots, X_n, Y$ such that $|X_i| = q_i, (i = 1, 2, \ldots, n), |Y| = m$. Then, $(m, n, k, Q)$ equals the number of $(n + k)$-subsets of $X$ intersecting each block $X_i, (i = 1, 2, \ldots, n)$. We call such subsets the $(n + k)$-insets of $X$.

We investigate the properties of the function $(m, n, k)$ in which $q_i = 2, (i = 1, 2, \ldots, n)$. To simplify our notation, we write $(m, n, k, 2)$ instead of $(m, n, k, Q)$.

Thus, we consider the three-dimensional array

$$A = \left\{ \binom{m, n}{k, 2} : 0 \leq m, 0 \leq n, 0 \leq k \leq m + n \right\}.
$$

The following two explicit formulae for $(m, n, k)$ are proved in [2] Equations(8) and (11)].

$$\binom{m, n}{k} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{m + 2n - 2i}{n + k}.
$$

$$\binom{m, n}{k} = 2^{n-k} \sum_{i=0}^{m} 2^i \binom{m}{i} \binom{n}{k-i}.
$$

In particular, we have $\binom{0,0}{0} = 1.$
Remark 1. We note that \( \binom{m,n}{k} \) are even if \( n > k \). In the case \( n = k \), Equation (2) becomes
\[
\binom{m,n}{k} = \sum_{i=0}^{m} 2^i \binom{m}{i} \binom{n}{i},
\]
which is the well-known formula for the Delannoy numbers \( D(m,n) \).

From [2, Eq. (11)], we obtain the following identity:
\[
\binom{m,n}{k} = \binom{m,n-1}{k} + \binom{m+1,n-1}{k},
\]
Using this identity, we easily express the Fibonacci numbers in terms of insets.

Corollary 1. For \( m \geq 0 \), we have
\[
F_{m+3} = \sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m-i,1}{i}.
\]

Proof. From (3), we obtain
\[
\sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \binom{m-i,1}{i} = \sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} \left[ \binom{m-i}{i} + \binom{m-i-1}{i} \right],
\]
and the proof follows from the well-known expression of the Fibonacci numbers in terms of the binomial coefficients. \( \square \)

From [2, Eq. (13)], we obtain the following identity:
\[
\binom{m,n}{k} = 2 \binom{m,n-1}{k} + \binom{m,n-1}{k+1}.
\]

We proceed by deriving some new properties of the function \( \binom{m,n}{k} \).

Proposition 1. For \( 0 \leq p \leq m; 0 < n \), the following formula holds:
\[
\binom{m+1,n-1}{k} = \sum_{i=0}^{p} (-1)^i \binom{p}{i} \binom{m-p+1,n+p-1-i}{k}.
\]

Proof. The formula is obviously true for \( p = 0 \). We next write (3) in the following form:
\[
\binom{m+1,n-1}{k} = \binom{m,n}{k} - \binom{m,n-1}{k}.
\]
This means that (3) holds for \( p = 1 \).

Applying (3) on the right-hand side of the same formula yields
\[
\binom{m+1,n-1}{k} = \binom{m-1,n+1}{k} - 2 \binom{m-1,n}{k} + \binom{m-1,n-1}{k},
\]
This means that (3) holds for \( p = 2 \). Repeating the same procedure, we obtain that (3) is true for each \( p \leq m \). \( \square \)

Proposition 2. The following formula holds:
\[
\binom{m+1,n}{k+1} = 2^{n-k-1} \binom{n}{k+1} + \sum_{i=0}^{m} \binom{i,n}{k}.
\]
In particular, for \( n \leq k \leq m + n \), we have

\[
\binom{m + 1, n}{k + 1} = \sum_{i=0}^{m} \binom{i, n}{k}.
\]

**Proof.** From [2, Proposition 10], it follows that

\[
\binom{m + 1, n}{k + 1} = \binom{m, n}{k} + \binom{m, n}{k + 1} = \binom{m - 1, n}{k} + \binom{m - 1, n}{k + 1}
\]

\[
= \binom{m, n}{k} + \cdots + \binom{0, n}{k} + \binom{0, n}{k + 1}.
\]

Taking \( m = 0 \) in (2), we obtain \( \binom{0, n}{k+1} = 2^{n-k-1} \binom{n}{k+1} \), which proves (7). The statement (8) is obvious. \( \square \)

**Remark 2.** The equation (8) is an analog of the horizontal recurrence for the binomial coefficients.

**Proposition 3.** Let \( p \) be a positive integer such that \( 1 \leq p \leq \min\{n, k\} \). Then,

\[
\binom{m, n-p}{k-p} - \binom{m, n}{k} = 2 \sum_{i=1}^{p} \binom{m, n-i}{k-i+1}.
\]

**Proof.** From Equation (4), we obtain the following sequence of equalities:

\[
\binom{m, n}{k} - \binom{m, n-1}{k-1} = 2 \binom{m, n-1}{k},
\]

\[
\binom{m, n-1}{k-1} - \binom{m, n-2}{k-2} = 2 \binom{m, n-2}{k-1},
\]

\[\vdots\]

\[
\binom{m, n-p+1}{k-p+1} - \binom{m, n-p}{k-p} = 2 \binom{m, n-p}{k-p+1}.
\]

The sum of the expressions on the left-hand side and the sum of the expressions on the right-hand side of the preceding equations produces Equation (9). \( \square \)

**Corollary 2.** The numbers \( \binom{m,n-p}{k-p} \) and \( \binom{m,n}{k} \) are of the same parity.

**Remark 3.** Note that for \( n = 0 \), the array \( A \) is the standard Pascal triangle.

**Proposition 4.** For a fixed positive \( n \), the array \( \{ \binom{m,n}{k} : (m,k) = 0,1,\ldots \} \) is a Pascal trapeze. The elements of the leftmost diagonal equal \( 2^n \), the rightmost diagonal consists of ones, and the first row consists of \( 2^{n-k} \binom{n}{k}, (k = 0,1,\ldots,m+n) \).

**Proof.** In the case \( k = 0 \), we have \( \binom{m,n}{0} \), which is the number of insets of \( n \) elements. In such an inset, we must choose exactly one element from each main block. We have \( 2^n \) such insets. We conclude that the leftmost diagonal of the array \( A \) is a constant sequence consisting of \( 2^n \). Furthermore, if \( n+k > m+2n \), then \( \binom{m,n}{k} = 0 \), since there is no inset having more than \( m+2n \) elements. If \( k + n = m + 2n \), that is, if \( k = m + n \), then \( \binom{m,n}{k} = 1 \). Hence, the rightmost diagonal consists of ones. Since the first row is obtained for \( m = 0 \), Proposition [2] yields \( \binom{0,n}{k} = 2^{n-k} \binom{n}{k} \), (k =
Finally, from \[2\] Proposition 10, the following (Pascal) recurrence holds:

\[
\binom{m, n}{k} = \binom{m-1, n-1}{k-1} + \binom{m-1, n}{k}.
\]

\[\square\]

2. A combinatorial interpretation of \(\binom{m, n}{k}\)

In this section, we relate the array \(A\) to restricted ternary words. We firstly consider the insets counted by \(\binom{0, n}{k}\). Note that \(\binom{0, n}{k} = 0\), if \(k > n\). Also, we have \(\binom{0, n}{n} = 1\). Let \(X\) be the set \(\binom{a, n}{k}\).

**Proposition 5.** The number \(\binom{0, n}{k}\) equals the number of ternary words of length \(n\) having \(k\) letters equal to 2.

**Proof.** Assume that \(k \leq n\). From each \((n + k)\)-inset of \(X\), we form a ternary word of length \(n\) in the following way. Put 2 in the place of every main block from which both elements are in \(X\). Then, insert 1 in the place of every main block from which the only first element is in \(X\), and fill all other places with zeros. It is clear that this correspondence is bijective. \(\square\)

We now consider the case \(m > 0\).

**Proposition 6.** For \(k \leq m + n\), the number \(\binom{m, n}{k}\) is the number of ternary words of length \(m + n\) having \(k\) letters equal to 2 and none of the first \(m\) letters is equal to 0.

**Proof.** The following formula is proved in \[2\] Proposition 11:

\[
\binom{m, n}{k} = \sum_{i=0}^{m} \binom{m}{i} \binom{0, n}{k-i}.
\]

Take \(i \in \{0, 1, \ldots, m\}\). According to Proposition 5, \(\binom{0, n}{k-i}\) is the number of ternary words of length \(n\) having \(k - i\) letters equal to 2. On the other hand, \(\binom{m}{i}\) is the number of words of length \(m\) over \(\{1, 2\}\) having \(i\) letters equal to 2. Hence, \(\binom{m}{i} \binom{0, n}{k-i}\) is the number of ternary words of length \(m + n\), having \(k\) letters equal to 2, and beginning with a subword of length \(m\) over \(\{1, 2\}\). Summing over all \(i\), we obtain the assertion. \(\square\)

**Example 1.**

1. For \(m = 1, n = 3, \) and \(k = 2\), we have \(\binom{1, 3}{2} = 18\). The corresponding words counted by this number are

\[
1022, 1122, 1202, 1212, 1220, 1221, 2000, 2211, 2210, 2201, 2020, 2121, 2021, 2120, 2002, 2112, 2012, 2102.
\]

2. For \(m = 1, n = 3, \) and \(k = 3\), we have \(\binom{1, 3}{3} = 7\). The corresponding words are

\[
1222, 2122, 2022, 2212, 2202, 2221, 2220.
\]

3. For \(m = 2, n = 3, \) and \(k = 4\), we have \(\binom{2, 3}{4} = 8\). The corresponding numbers are

\[
1222, 2122, 2212, 2221, 2220, 2202, 2202, 2202.
\]

We derive a property of the function \(\binom{m, n}{k}\) by introducing a new parameter \(p\).
Identity 1. If $0 \leq p \leq n$, then
\begin{equation}
\binom{m, n}{k} = \sum_{i=0}^{p} \binom{p}{i} \binom{m + i, n - p}{k}.
\end{equation}

Proof. The result may be proved by induction on $p$, using Equation (3). We add a combinatorial proof.

For $0 \leq i \leq p \leq n$, we put $p - i$ zeros in some of the last $p$ places in a word. This may be done in $\binom{p}{i}$ ways. Now, we have to choose the remaining $m + n + i - p$ letters. In other words, we have to count the ternary words of length $m + n - p + i$ satisfying the following conditions:

1. A word must have $k$ letters equal to 2.
2. Zeros may stand in the positions $m + 1, m + 2, \ldots, m + n - p$.

It is clear that we have $\binom{m + i, n - p}{k}$ such words. Summing over $i$ from 0 to $p$, we obtain the equation. □

In particular, taking $p = n$, we obtain

Corollary 3. The following formula holds:
\begin{equation}
\binom{m, n}{k} = \sum_{i=0}^{n} \binom{n}{i} \binom{m + i}{k}.
\end{equation}

We derive combinatorially one more formula in which $\binom{m, n}{k}$ is expressed as a convolution of binomial coefficients.

Proposition 7. The following formula holds:
\begin{equation}
\binom{m, n}{k} = \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} \binom{m}{k - i + j}.
\end{equation}

Proof. We firstly designate positions of 0 in a word.

Since zeros may be only some of the last $n$ letters in a word, we choose $n - i$ zeros for $i = 0, 1, \ldots, n$. This may be done in $\binom{n}{i}$ ways.

For the remaining $i$ of the last $n$ letters in a word, we firstly choose $i - j$ letters equal to 2, which may be done in $\binom{i}{j}$ ways. The remaining $j$ letters are equal to 1.

Furthermore, for fixed $i$ and $j$, among the first $m$ letters in a word, we have to choose $k - i + j$ letters equal to 2, and the remaining letters must be equal to 1. This may be done in $\binom{m}{k - i + j}$ ways. Summing over all $i$ and $j$, we obtain the result. □

Corollary 4. B. Braun, W. K. Hough [5] showed that terms of the array $\{ \binom{2, n}{k} : n = 0, 1, \ldots ; k = 0, 1, \ldots , n + 2 \}$ count cells in the cellular complex $X_n^m$ defined in [5, Definition 3.4]. More precisely, let $C_n^d$ denote the number of $d$-dimensional cells in $X_n^2$. It is shown in [5, Proposition 4.6] that the following formula holds:
\begin{equation}
C_n^d = \binom{2, n - d + 2}{3d - 2n}.
\end{equation}

Corollary 5. In Hetyei’s paper [6, Definition 2.1], the numbers on the form $\binom{m, n}{m} (m = 0, 1, \ldots ; n = 0, 1, \ldots )$ are called the asymmetric Delannoy numbers.

We note that rows of this array are diagonals of A049600 in Sloane[7].
In our previous paper [1], we defined a class of polynomials which generalize the Tchebychev polynomials. We state a particular result concerning our present investigations.

For a fixed nonnegative integer \( m \) and nonnegative integers \( n, k \) such that \( 0 \leq k \leq n \), \( 0 \leq m \leq \frac{n+k}{2} \), we defined polynomials \( P_{m,n}(x) = \sum_{k=0}^{n} c_m(n,k) x^k \) such that

\[
c_m(n,k) = (-1)^{\frac{n-k}{2}} \binom{m, \frac{n+k}{2} - m}{\frac{n-k}{2}},
\]

if \( n \) and \( k \) are of the same parity and \( c_m(n,k) = 0 \) otherwise. Using Proposition 2, we obtain the following combinatorial interpretation of the coefficients of the polynomials \( P_{m,n}(x) \).

Example 2. Assume that \( n \) and \( k \) are of the same parity. Let \( c_m(n,k) \) denote the coefficient of \( x^k \) of the polynomial \( P_{m,n}(x) \). Then \( |c_m(n,k)| \) equals the number of ternary words of length \( \frac{n+k}{2} \) having \( \frac{n-k}{2} \) letters equal to 2 and no zeros among the first \( m \) letters in a word.

Remark 4. In this way we give a new combinatorial interpretation of the coefficients of polynomials which appear in Sloane [7] A136388, A136389, A136390, A136397, and A136398.

3. THREE GENERATING FUNCTIONS

In this part, we derive three generating functions assuming that two of parameters \( m, n, k \) are constant. Firstly, we prove an identity.

**Proposition 8.** For \( m+k \geq n \), we have

\[
\binom{m+k-n,n}{k} = \sum_{i=0}^{m} \binom{n}{m-i} \binom{k+i}{k}.
\]

**Proof.** We have to count ternary words of length \( m+k \), having \( k \) letters equal to 2, and zeros may appear only among last \( n \) letters in a word. It is clear that the maximal number of zeros is \( m \). Hence, for \( 0 \leq i \leq m \), we choose \( m-i \) zeros, which may be done in \( \binom{n}{m-i} \) ways. On the remaining \( k+i \) places, we choose \( k \) twos, which may be done in \( \binom{k+i}{k} \) ways. The remaining letters are equal to 1. Summing over all \( i \), we obtain the formula. \( \square \)

We next prove three formulas concerning ordinary generating functions for insets, when two of parameters \( m, n, k \) are constant.

**Proposition 9.** If \( m+k \geq n \), then the following expansion holds.

\[
\frac{(1+x)^n}{(1-x)^{k+1}} = \sum_{m=m_0}^{\infty} \binom{m+k-n,n}{k} x^m,
\]

where \( m_0 = \max\{0, n-k\} \).

**Proof.** In the case \( k \geq n \), the formula is a particular case of [3] Theorem 1. So, we consider the case \( n > k \) and \( m \geq n-k \). It is well-known fact that

\[
\frac{1}{(1-x)^{k+1}} = \sum_{i=0}^{\infty} \binom{k+i}{i} x^i.
\]
Multiplying by \((1 + x)^n\), we obtain
\[
(1 + x)^n = \frac{(1 + x)^n}{(1 - x)^{k+1}} = \sum_{i=0}^{\infty} \sum_{j=0}^{n} \binom{k+i}{k} \binom{n}{j} x^{i+j}.
\]

We calculate the coefficient \(a_{m+k-n}\) of \(x^{m+k-n}\) on the right-hand side of this equation. It is obtained for \(i = 0, 1, 2, \ldots, m + k - n; j = m + k - n - i\) and is equal to
\[
a_{m+k-n} = \sum_{i=0}^{p} \binom{n}{p-i} \binom{k+i}{i},
\]
where \(p = m + k - n\). Applying (14), we obtain that
\[
a_{m+k-n} = \binom{m+k-n,n}{k}.
\]
□

The preceding generating function is with respect to the parameter \(m\). The following is with respect to \(n\).

**Proposition 10.** For \(n + k \geq m\), we have
\[
\frac{(1-x)^m}{(1-2x)^{k+1}} = \sum_{n=0}^{\infty} \binom{m, n + k - m}{k} x^n.
\]

**Proof.** Similarly to the proof of the preceding proposition, we reduce the proof to the following identity:
\[
\binom{m, n + k - m}{k} = \sum_{i=0}^{n} (-1)^i \cdot 2^{n-i} \cdot \binom{m}{i} \cdot \binom{k + n - i}{k},
\]
which easily follows from (15).
□

Finally, we derive a generating function with respect to parameter \(k\).

**Proposition 11.** We have
\[
\frac{(2-x)^n}{(1-x)^{m+n+1}} = \sum_{k=0}^{\infty} \binom{m+k,n}{k} x^k.
\]

**Proof.** We have
\[
\frac{(2-x)^n}{1-x} = \left(1 + \frac{1}{1-x}\right)^n = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{(1-x)^i}.
\]
It follows that
\[
\frac{(2-x)^n}{(1-x)^{m+n+1}} = \sum_{i=0}^{n} \binom{n}{i} \frac{1}{(1-x)^{i+m+1}} = \sum_{k=0}^{\infty} \sum_{i=0}^{n} \binom{n}{i} \binom{k+i+m}{k} x^k.
\]
On the other hand, using (12) we obtain
\[
\sum_{i=0}^{n} \binom{n}{i} \binom{k+i+m}{k} = \binom{m+k,n}{k}.
\]
□

We finish with two sets of examples. The first concerns two dimensional arrays in Slone [7] generated by our function. The second concerns sequences in Slone [7]. Some examples are new, and some are only a new combinatorial interpretations of our earlier results.
4. Examples I

**Example 3.** It follows from Proposition 4 that the array \( A = \binom{m+1}{k} \), \((m = 0, 1, \ldots; k = 0, 1, \ldots, m + 1)\) is (2, 1) Pascal triangle A029653. It is the mirror of Lucas triangle, which is A029635.

We thus obtain the following combinatorial interpretation of entries of Lucas triangle: The \((k, m)\) entry of Lucas triangle equals the number of ternary words of length \(m + 1\) having \(k\) letters equal to 2 and 0 may only be the last letter of a word.

**Example 4.** From (2) follows \( \binom{n}{n-k} = 2^{n-k}(\binom{n}{k}) \). A038207

Hence, the number \(2^{n-k}(\binom{n}{k})\) equals the number of ternary words of length \(n\) having \(k\) letters equal to 2.

For instance, for \( n = 3, k = 2 \), we have the following 6 words:

\[
221, 212, 122, 220, 202, 022.
\]

**Example 5.** Taking \( m = 0 \) in (13), we obtain \( P_{0,n}(x) = U_n(x) \), where \( U_n(x) \) is the Tchebychev polynomial of the second kind. If \( u_{n,k} \) is the coefficients of \( x^n \) of \( U_n(x) \), then \(|u_{n,k}|\) equals the number of ternary words of length \( \frac{n+k}{2} \) having \( \frac{n-k}{2} \) letters equal to 2. A038207

For instance, we have \(|u_{4,2}| = 12\), and the corresponding words are:

\[
200, 201, 210, 211, 020, 120, 021, 121, 002, 102, 012, 112.
\]

**Example 6.** Taking \( m = 1 \) in (13), we obtain \( P_{1,n}(x) = T_n(x) \), where \( T_n(x) \) is the Tchebychev polynomial of the first kind. Hence, if \( t_{n,k} \) is the coefficients of \( x^n \) of \( T_n(x) \), then \(|t_{n,k}|\) equals the number of ternary words of length \( \frac{n+k}{2} \) having \( \frac{n-k}{2} \) letters equal to 2 and 0 is not the first letter of a word. seqnumA200139

In particular, for \(|t_{4,2}| = 8\), we obtain the following words:

\[
112, 121, 211, 120, 102, 210, 201, 200.
\]

**Example 7.** The array \( \{\binom{3, n}{k} : n = 0, 1, \ldots; k = 0, 1, \ldots, n + 3\} \) is in a way connected with array seqnumA289921.

Namely, arrays have a number terms which are the same, but not all.

**Example 8.** In [2] Proposition 45], we proved that \( \binom{m+1}{m}n-1\) equals the number of weak compositions of \(m + n\) having \(n - 1\) parts equal to zero. Hence, this number equals the number of ternary words of length \(m + n\) having \(m\) letters equal to 2 and the initial subword of length \(m + 1\) contains no zero.

In particular, we have \( \{\binom{3, n}{2} = 2^{n-3}(n^2 + 11n + 24)\}, \((n \geq 0)\). seqnumA058396

The number counts weak compositions of \(n + 1\) having exactly 2 parts equal to 0. For instance, for \( n = 1 \), we have the following 9 weak composition of 2 having two zeros:

\[
200, 020, 002, 1100, 1010, 1001, 0110, 0101, 0011.
\]

Ternary words of length 4 having 2 letters equal to 2, and the initial subword of length 3 contains no zero are:

\[
2211, 2121, 2112, 1212, 1122, 1220, 2120, 2210.
\]

In [2] Proposition 27], the following explicit formula for the Delannoy numbers \(D(m, n)\) is derived:

\[D(m, n) = \binom{m}{n}.\]
We have the following combinatorial interpretation of the Delannoy numbers.

**Example 9.** The Delannoy number $D(m, n)$ equals the number of ternary words of length $m + n$ having $n$ letters equal to 2 and the initial subword of length $m$ contains no zero.

The following example also concerns the Delannoy numbers.

**Example 10.** In [2] Proposition 31.1, we proved that the number of solutions of the Diophantine inequality

$$|x_1| + |x_2| + \cdots + |x_n| \leq m$$

equals \binom{m+n}{n}(= \sum_{i=0}^{n} \binom{n}{i}(\binom{m-i+n-1}{n-1})). We denote this number by $G(m), (m = 0, 1, \ldots)$.

**Remark 5.** It follows from Conway and Sloane [4, Eq. (3.3)] that $G(m)$ are the chiral ball numbers for the cubic lattice $\mathbb{Z}^n$.

**Example 11.** In [2, Proposition 31.2], we proved that the number of solutions of the Diophantine equality

$$|x_1| + |x_2| + \cdots + |x_n| = m$$

equals \binom{m+n-1}{n-1}(= \sum_{i=0}^{n} \binom{n}{i}(\binom{m-i+n-1}{n-1})). We denote this number by $S(m), (m = 0, 1, \ldots)$.

Hence, $S(m)$ equals the number of ternary words of length $m + n - 1$ having $n - 1$ letters equal to 2 and the initial subword of length $n$ contains no zero.

Some sequence in Sloane [7] concerning this case are: A005899, A008412, A008413, A008414, A008415.

**Remark 6.** It follows from Conway and Sloane [4, Eq. (3.2)] that $S(m), (m = 0, 1, \ldots)$ is the coordinating sequence for the cubic lattice $\mathbb{Z}^n$.

**Example 12.** We now consider the asymmetric Delannoy numbers $\binom{m+n}{m}, (n, k \geq 0)$. A049600

The asymmetric Delannoy number $\binom{m+n}{m}$ equals the number of ternary words of length $m + n$ having $n$ letters equal to 2 and the initial subword of length $m$ contains no zero.

For $m = n = 2$, we have $\binom{2,2}{2} = 13$, and ternary words are: six permutations of 1122, three permutations of 122 ending by 0, three permutations of 122 with 0 at the next to the last place, and 2200.

**Example 13.** From [2] Proposition 45] follows that the number of composition of $m$ in which $k$ parts are equal to zero, which equals $\binom{m+1,k-1}{k}$ is the number of ternary words of length $m + k$ having $k$ letters equal to 2 and the initial subword of length $m + 1$ contains no zero.

The rows of array \{ $\binom{m+1,k-1}{k} : m = 0, 1, \ldots; k = 1, 2, \ldots$ \} are figurate numbers based on the $k$-dimensional regular convex polytope.

Some sequences related to this case are: A005900, A014820, A069038, A069039, A099193.

**Example 14.** We next consider the Sulanke numbers, which we denote by $s(n, k)$.

It is proved in [2, Proposition 29] that $s(n, k) = \binom{\frac{n+k}{2}, k}{k}$, if $n + k$ is even. In this
case, the number \( s_{n,k} \) equals the number of ternary words of length \( n + k \) having \( k \) twos and the initial subword of length \( \frac{n+k}{2} \) contains no zero. A064861

**Example 15.** If \( n + k \) is odd, the \( s(n,k) = \left( \frac{n+k-1}{2} \right) \left( \frac{n+k+1}{2} \right) \). Now, the number \( s_{n,k} \) is the number of ternary words of length \( n + k \) having \( k \) twos and the initial subword of length \( \frac{n+k-1}{2} \) contains no zero. A064861

5. **Examples 2**

**Example 16.** Values of \( \left( \begin{array}{c} m \\ 1 \end{array} \right) \) are odd numbers \( 2m + 1, (m = 0, 1, \ldots) \). seqnumA005408

Hence, \( 2m + 1 \) equals the number of ternary words of length \( m + 1 \) having one letter equal to 2 and and the initial subword of length \( m \) contains no zero.

For instance, for \( m = 2 \), five ternary words of length 3 are 112, 121, 211, 120, 210.

**Example 17.** Values of \( \left( \begin{array}{c} m \\ 2 \end{array} \right) \) are squares \( m^2, (m = 0, 1, \ldots) \). A000290

Hence, \( m^2 \) equals the number of ternary words of length \( m + 1 \) having two letters equal to 2 and the initial subword of length \( m \) contains no zero.

For instance, for \( m = 3 \), nine ternary words of length 4 are 1122, 1212, 1221, 2121, 2211, 2112, 2210, 2120, 1220.

**Example 18.** For \( m \geq 2 \) values of \( \left( \begin{array}{c} m \\ 3 \end{array} \right) \) are square pyramidal numbers A000330

Hence, the square pyramidal number \( \frac{m-1}{6}m(2m-1) \) equals the number of ternary words of length \( m + 1 \) having three letters equal to 2 and the initial subword of length \( m \) contains no zero.

For instance, for \( m = 4 \), 14 ternary words of length 5 are: 15 permutations of 11222 and 4 permutation of 1222 ending by 0.

**Example 19.** For \( m \geq 2 \) values of \( \left( \begin{array}{c} m \\ 4 \end{array} \right) \) are four dimensional pyramidal numbers: \( \frac{(m+1)^2((m+1)^2-1)}{12} \). A002415

Hence, \( \frac{(m+1)^2((m+1)^2-1)}{12} \) equals the number of ternary words of length \( m + 2 \) having four letters equal to 2 and the initial subword of length \( m \) contains no zero.

For instance, for \( m = 4 \), 20 ternary words of length 6 are: 15 permutations of 11222 and 5 permutation of 1222 ending by 0.

**Example 20.** We have \( \left( \begin{array}{c} m+2 \\ 2 \end{array} \right) = m^2 + (m + 1)^2 \). It is seqnumA001844 in [7]. Hence, the sum \( m^2 + (m + 1)^2 \) equals the number of ternary words of length \( m + 2 \) having 2 letters equal to 2 and the initial subword of length \( m \) contains no zero.

For \( m = 2 \) this number equals 13. The corresponding ternary words are 2211, 2121, 2121, 2112, 2112, 2112, 2110, 2110, 2120, 2201, 2120, 2120, 2200.

**Example 21.** We have \( \left( \begin{array}{c} m+2 \\ 3 \end{array} \right) = \frac{m(2m^2+1)}{3} \), which are the octahedral numbers. A005900

Hence, the octahedral number \( \frac{m(2m^2+1)}{3} \) equals the number of ternary words of length \( m + 2 \) having three letters equal to 2 and the initial subword of length \( m \) contains no zero.

In particular, we have \( \left( \begin{array}{c} 3 \\ 2 \end{array} \right) = 19 \) and the corresponding ternary words have length 5 with three letters equal to 2 and zero may stand only at the last two position. The word consists of 10 permutation of 22211, 8 permutation of 2221 ending with either 1 or 0, and 22200.
Example 22. The sequence \( \binom{m}{3}, (m = 0, 1, \ldots) \) consists of the centered octahedral numbers \( \frac{(2m+1)(2m^2+2m+3)}{3} \). A001845

Hence, the centered octahedral number \( \frac{(2m+1)(2m^2+2m+3)}{3} \) equals the number of ternary words of length \( m+3 \) having three letters equal to 2 and the initial subword of length \( m \) contains no zero.

Example 23. Sequence \( \binom{m}{4}, (m = 0, 1, 2, \ldots) \) consists of 4-dimensional analog of centered polygonal numbers \( m \frac{(m-1)(m^2-m+1)}{6} \). A006325

Example 24. We have \( \binom{1+n}{2} = n(n+3) \cdot 2^n, (n = 0, 1, \ldots) \). A001793

The number \( \binom{1+n}{2} \) equals the number of ternary words of length \( n+1 \) having two letters equal to 2 and not beginning by 0.

Example 25. In [2, Proposition 50], we proved that for \( n > 2 \), the possible bishop moves on \( n \times n \) chessboard equals \( \binom{1+n}{2} \). A002492

Hence, for \( n > 2 \) the possible bishop moves on \( n \times n \) chessboard equals the number of ternary words of length \( n+1 \) having \( n-2 \) letters equal to 2, and not beginning by 0.

Example 26. For \( m \geq 2 \), the number \( \binom{m}{2}, \binom{m}{5}, (m = 0, 1, \ldots) \) is the convolution of nonzero squares with themselves, that is, \( \binom{m}{2} = \sum_{i=2}^{m} i^2(m-i)^2 \), and is equal to the total area of all square regions from an \( m \times m \) grid. A033455

Example 27. The central Delannoy number \( D(n, n) \) is the number of ternary words of length \( 2n \) having \( n \) letters equal to 2 and 0 can not appear in the initial subword of length \( n \). A001850

Example 28. We derive a result concerning the Catalan numbers. A000108

It is proved in [2, Proposition 25] that for the \( k \)th Catalan number \( C_k \) holds

\[
C_k = \frac{1}{3k+2} \binom{2k,1}{k}.
\]

Hence, \( (3k+2) \cdot C_k \) equals the number of ternary words of length \( 2k+1 \) having \( k \) letters equal to 2 and 0 may appear only as the last letter in a word. A051960

In particular, for \( k = 2 \) we have \( (3k+2)C_2 = 16 \). These 16 word of length 5 are 10 permutation of 11122, and 6 permutation of 1122 ending by zero.

Example 29. In this example, we express the Fibonacci numbers in terms of ternary words. A000045

Let \( m \) be a fixed positive integer. From [1] follows that \( F_{m+3} \) equals the number of ternary words of length \( m-i+1 \), \( (0 \leq i \leq m) \) having \( i \) letters equal to 2 and 0 may only be the last letter in a word. In particular, \( F_6 = 8 \) and the corresponding ternary words are 1111, 1110, 112, 121, 211, 210, 120, 22.

Example 30. The number of triangles in the Turan graph \( T(m, m-2) \), for \( m > 3 \), equals the number of ternary words of length \( m+3 \) having \( m \) letters equal to 2 and 0 can not appear in the initial subword of length \( m+1 \).

The number is \( \binom{m+1,2}{m} \). A000297.
Example 31. The number of points on surface of octahedron equals the number of ternary words of length \( m + 2 \) having two letters equal to 2 and 0 cannot appear in the initial subword of length \( m - 1 \).

For \( m > 1 \), the number is \( \binom{m-1,3}{2} (= 4m^2 + 2) \). A005899

Example 32. The number of maximal and maximum cliques in the \( n \)-cube-connected cycles graph equals the number of ternary words of length \( 2n \) having one letter equal to 2 and 0 cannot appear in the initial subword of length \( n \).

The number is \( 3n \cdot 2^{n-1} (= \binom{n}{1}) \). A167667

Example 33. The number of peaks in all Schroeder paths from \((0,0)\) to \((2n,0)\) equals the number of ternary words of length \( 2m + 1 \) having \( m + 1 \) letters equal to 2 and 0 can not appear in the initial subword of length \( m \).

The number is \( \sum_{i=1}^{m+1} \binom{m+1}{i} \binom{m+i}{i} (= \binom{m,m+1}{m+1}) \). A002002

Example 34. The number of order-preserving partial self maps of \( \{1, \ldots, m\} \) equals the number of ternary words of length \( 2m + 1 \) having \( m \) letters equal to 2 and 0 can not appear in the initial subword of length \( m \).

The number is \( \sum_{i=0}^{m+1} \binom{m+1}{i} \binom{m+i}{i} (= \binom{m,m+1}{m+1}) \). A002003

Example 35. The number of Dyck paths with semilength \( m + 4 \), and an odd number of peaks, and the central peak has height \( m - 2 \) equals the number of ternary words of length \( m + 2 \) having \( m \) letters equal to 2 and a word can not begin with 0.

The number is \( 2(m+1)^2 (= \binom{1,m+1}{m}) \). A001105

Example 36. The sum of the first \( m+1 \) even squares equals the number of ternary words of length \( m + 3 \) having \( m \) letters equal to 2 and 0 can not appear in the initial subword of length \( m \).

The number is \( \frac{2(m+1)(m+2)(2m+3)}{3} (= \binom{m,m+2}{m+1}) \). A002492

Example 37. The variance of time for a random walk starting at 0 to reach one of the boundaries at \(+m\) or \(-m\) for the first time equals the number of ternary words of length \( m + 4 \) having \( m \) letters equal to 2 and not beginning by 0.

The number is \( \frac{2(m+1)(m+2)^2(m+1)}{3} (= \binom{1,m+3}{m+1}) \). A072819

Example 38. The maximal number or regions in the plane that can be formed with \( m \) hyperbolas equals the number of ternary words of length \( m + 3 \) having \( m + 1 \) letters equal to 2 and 0 can not appear in the initial subword of length \( 3 \).

The number is \( 2(m+1)^2 + 1 (= \binom{3,m}{m}) \). A058331

Example 39. The number \( \binom{n+1,n-1}{n} \) equals the number of Dyck paths having exactly \( n \) peaks in level 1 and \( n \) peaks in level 2 and no other peaks.

It is also the number of ternary words of length \( 2n \) having \( n - 1 \) letters equal to 2 and 0 can not appear in the initial subword of length \( n + 1 \). A176479

Example 40. The number \( \binom{n^2,n}{n} \) equals the number of integer points in an \( n \)-dimensional sphere of Lee-radius \( n^2 \) centered at the origin. It is also the number of ternary words of length \( n^2 + n \) having \( n \) twos and 0 can not appear in the initial subword of length \( n^2 \). A181675

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