A GENERALIZATION OF THE GAUSS-BONNET AND HOPF-POINCARÉ THEOREMS. PART II

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ABSTRACT. This paper is a continuation of [1]. Let \( \pi : E \to M \) be a locally trivial fiber bundle over a two-dimensional manifold \( M \), and \( \Sigma \subset M \) be a discrete subset. A subset \( Q \subset E \) is called an \( n \)-sheeted branched section of the bundle \( \pi \) if \( Q' = \pi^{-1}(M \setminus \Sigma) \cap Q \) is an \( n \)-sheeted covering of \( M \setminus \Sigma \). The set \( \Sigma \) is called the singularity set of the branched section \( Q \). We define the index of a singularity point of a branched section, and give examples of its calculation, in particular for branched sections of the projective tangent bundle of \( M \) determined by binary differential equations. Also we define a resolution of singularities of a branched section, and prove an analog of Hopf-Poincaré-Gauss-Bonnet theorem for the branched sections admitting a resolution.

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1. INTRODUCTION

Let us recall that a branched covering is a smooth map \( f : X \to Y \), where \( X \) and \( Y \) are compact \( n \)-dimensional manifolds, such that \( df_x : T_xX \to T_{f(x)}Y \) is an isomorphism for all points \( x \in X \setminus A \) for some subset \( A \subset X \) of dimension less or equal to \( n - 2 \). In this case, if \( X' = X \setminus f^{-1}(f(A)) \) and \( Y' = Y \setminus f(A) \), then the induced map \( f' : X' \to Y' \) is a finite-sheeted covering map. The points of the set \( f(A) \) are called the branch points of the branched covering \( f \) ([2], Section 18.3).

Now let \( \xi = \{ \pi_E : E \to M \} \) be a fiber bundle. Let \( \Sigma \) be a closed subset of \( M \), \( M' = M \setminus \Sigma \), and \( E' = \pi^{-1}(M') \).

Definition 1. An \( n \)-sheeted branched section of the bundle \( \xi \) is a subset \( Q \subset E \) such that \( Q' = Q \cap E' \) is an embedded submanifold of \( E \) and \( \pi_E|_{Q'} : Q' \to M' \) is a \( n \)-sheeted covering. The set \( \Sigma \) is called the singularity set of the branched section \( Q \).

Example 1. Let \( V \) be a section of the tangent bundle \( \pi_{TN} : TN \to N \), and \( f : N \to M \) be a \( k \)-sheeted covering, then we can construct a branched section \( df(V) \) of the tangent bundle \( \pi_{TM} : TM \to M \) in the following way. Let us consider the subset \( Q = \{ df(y)(V(y)) \mid y \in N \} \subset TM \). For each \( x \in M \), let us set \( \mathcal{V}(x) = \{ df(y)(V(y)) \mid y \in f^{-1}(x) \} \subset T_xM \). Take the subset \( \Sigma \subset M \) consisting of points \( x \in M \) such that the number of elements of the set \( \mathcal{V}(x) \) is less than \( k \). Then \( M' = M \setminus \Sigma \) is open, \( Q' = Q \cap \pi_{TM}^{-1}(M') \) is a submanifold of \( TM \) and \( f \) induces a \( k \)-sheeted covering \( f' : Q' \to M' \). Indeed, for each \( x \in M' \) there exists a neighborhood \( U \subset M' \) of \( x \) such that \( f^{-1}(U) = \bigsqcup_{j=1}^{k} \tilde{U}_j \subset N \) and, for each \( j = 1, k \), the application \( f_j = f|_{\tilde{U}_j} : \tilde{U}_j \to U \) is a diffeomorphism. Therefore \( df_j : T\tilde{U}_j \subset TN \to TU \subset TM \) is also a diffeomorphism. As \( V : \tilde{U}_j \to V(\tilde{U}_j) \subset TN \) is a diffeomorphism onto its image,

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the map \( \theta_j = df_j \circ \nu \circ f_j^{-1} : U \to df_j(V(\tilde{U}_j)) \subset Q \subset TM, j = \overline{1, k} \), is a diffeomorphism onto its image, as well. Note that, for each \( y \in U \subset M' \), we have that the set \( f^{-1}(y) = \{ p_j \in \tilde{U}_j \} \) consists of \( k \) distinct points, and the set \( \{ df_{p_j}(V(p_j)) \} \) consists of \( k \) distinct vectors, by the definition of \( M' \). Therefore, \( \theta_i(U) \cap \theta_j(U) = \emptyset \), for \( i \neq j \). Thus \( \pi^{-1}_{TM}(U) \cap Q' = \bigcup_{j=1}^{k} \theta_j(U) \), this means that \( U \) is simply covered in \( Q' \), and \( Q' \) is a \( k \)-sheeted covering of \( M' \).

The branched sections naturally appear in the theory of differential equations over manifolds. Our main example in this paper is the following one.

**Example 2.** Let \( M \) be a connected compact oriented manifold and let \( \omega \) be a symmetric tensor of order \( n \) over \( M \). Recall that such a tensor can be written locally as follows

\[
\omega(x, y) = a_0(x, y)dx^n + a_1(x, y)dx^{n-1}dy + \cdots + a_n(x, y)dy^n,
\]

where \((x, y)\) are coordinate functions on an open set \( U \subset M \), and \( a_i : U \to \mathbb{R} \) are smooth functions defined in \( U \). In what follows, we suppose that \( \omega \) has the following properties:

1. The function \( \omega(x, y) \) is identically zero if and only if \( a_i(x, y) = 0 \) for \( 0 \leq i \leq n \). We set \( \Sigma = \{ p \in M : \omega_p = 0 \} \).
2. On \( M \setminus \Sigma \), the tensor \( \omega \) has the form \( \omega = \lambda_1 \lambda_2 \cdots \lambda_n \), where \( \lambda_i \in \Omega(M \setminus \Sigma) \) pairwise linearly independent.

**Statement 1.** The \( n \)-form \( \omega \) determines a branched section of the bundle \( \pi : PTM \to M \).

**Proof.** Let \( Q \) be the solution on \( PTM \) of the equation (11). We will prove that \( Q \) is a branched section of \( \pi \). Let \( E' = \pi^{-1}(M \setminus \Sigma) \) and \( Q' = Q \cap E' \). It follows from the property (2) that the set \( F_p = Q \cap \pi^{-1}(p), p \in M \setminus \Sigma \) has exactly \( n \) elements, therefore each fiber of the surjective map \( \pi' := \pi|_{Q'} : Q' \to M \setminus \Sigma \) is finite with \( n \) elements. On the other hand, if \( \phi : : \pi^{-1}(U) \to U \times \mathbb{R}P^1 \) is a trivialization of \( PTM \) on \( U \), then the restriction \( \phi' := \phi|_{\pi^{-1}(U) \cap Q'} : \pi^{-1}(U) \cap Q' \to U \times \mathbb{R}P^1 \) is a homeomorphism on its image. Since \( \pi|_{\pi^{-1}(U) \cap Q'} : \pi^{-1}(U) \cap Q' \to U \cap (M \setminus \Sigma) \) has finite fiber with \( n \) elements over each point \( p \in U \cap (M \setminus \Sigma) \), from the following commutative diagram

\[
\begin{align*}
\pi'(\pi^{-1}(U \cap (M \setminus \Sigma)) \cap Q') & \quad \phi' \\
\pi^{-1}(U \cap (M \setminus \Sigma)) \cap Q' & \quad p_{\pi'1}
\end{align*}
\]

It follows that \( \pi|_{Q'} : Q' \to M \setminus \Sigma \) is a local diffeomorphism. Therefore, \( \pi|_{Q'} : Q' \to M \setminus \Sigma \) is a \( n \)-sheeted branched covering, and so \( Q \) is a branched section of \( PTM \). \( \square \)

**Example 3.** Let \( \xi = \{ \pi : \overline{P} \to M \} \) be a \( \overline{G} \)-principal bundle which reduces to a finite subgroup \( G \subset \overline{G} \) over \( M \setminus \Sigma \), where \( \Sigma \subset M \) is a closed subset. Then the corresponding \( G \)-principal bundle \( P \subset \overline{P} \) is a branched section of the bundle \( \xi \) with singularity set \( \Sigma \).

For example, let \( M \) be a two-dimensional oriented Riemannian manifold, and \( \overline{P} = SO(M) \), the \( SO(2) \)-principal bundle of orthonormal positively oriented frames of \( M \). Any finite subgroup \( G \subset SO(2) \) is a cyclic group \( G \cong \mathbb{Z}_m \) generated by the rotation \( R_{2\pi/m} \).
If $P \subset SO(M') \subset SO(M)$ is a reduction of $SO(M)$ to $G$ over $M' = M \setminus \Sigma$, then at each point $x \in M'$ we have the set $\mathcal{N}(x) = \{ e \in T_xM \mid (e, R_{x/2}e) \in P \}$, which consists of unitary vectors such that the angle between any two of them is $2\pi l/m$. The set $\mathcal{N}(x)$ defines a regular $m$-polygon $P_m \subset T_xM'$ inscribed in a unit circle centered at $0 \in T_xM$.

It is clear that, vice versa, if at any point of $M' = M \setminus \Sigma$, we are given a unitary $m$-polygon $P_m \subset T_pM'$ and the field of these polygons is smooth (these means that locally we can choose $m$ unitary vector fields whose values are the vertices of the polygons $P_m$), then the bundle $SO(M)$ reduces to the subgroup $G \cong \mathbb{Z}_m$ of the Lie group $SO(2)$.

This situation occurs, for example, when $M$ is a surface in $\mathbb{R}^3$, and $\Sigma$ is the set of umbilic points of $M$. Then at each point of $M'$ we have two orthogonal eigenspaces of the shape operator of the surface, which determine a square in $T_pm$ with vertices at points where these eigenspaces meet the unit circle centered at $0 \in T_pM$. Therefore, over $M' = M \setminus \Sigma$ the bundle $SO(M)$ reduces to the subgroup $G \cong \mathbb{Z}_4$ generated by the rotation $R_{\pi/2}$. The corresponding principal subbundle $P$, the branched section of the bundle $SO(M) \to M$, consists of oriented orthogonal frames such that the frame vectors span the eigenspaces.

Moreover, as the difference of the principal curvatures never vanish on $M'$, we can order the principal curvatures in such a way that $k_1(p) > k_2(p)$ at any $p \in M'$. Let $L_a(p)$, $a = 1, 2$ be the eigenspace corresponding to the principal curvature $k_a(p)$, $a = 1, 2$. Then we can choose the subbundle $P \subset SO(M)$ in such a way that, for $\{ e_1, e_2 \} \in P$, the vector $e_a$ spans $L_a$, $a = 1, 2$, therefore in this case the bundle $SO(M) \to M$ reduces to the group $G \cong \mathbb{Z}_2$.

Also, note that this example is related to Example 2. Indeed if we have the reduction of $P \subset SO(M)$ to the subgroup $G \cong \mathbb{Z}_m$ over $M'$, then at each point $p \in M'$ we have $m$ (or $m/2$) subspaces spanned by the vector $e_1$ from the frame $\{ e_1, e_2 \} \in P$. Then we can take the binary differential equation (I) such that these subspaces are the roots of the corresponding algebraic equation.

The paper is organized as follows. In Section 2 we define the index of an isolated singular point of a branched section of locally trivial bundle $\xi = \{ \pi_E : E \to M \}$ over a two-dimensional oriented manifold $M$ (see Definition 2), this definition generalizes the definition of the index of a singular point of a section from (I, Section 2.2, Definition 1). In Section 3 we define a resolution of a branched section (see Definition 3), and give various examples of resolutions (see Examples 3–9). And, finally, in Section 2 we prove an analogue of the Gauss-Bonnet theorem for a branched section which admits resolution (see Theorem 1).

2. The index of a singular isolated point

2.1. Local monodromy group. Let $M$ be a two-dimensional closed oriented manifold. Let $\xi = \{ \pi_E : E \to M \}$ be a fiber bundle with oriented typical fiber $F$.

Let us consider a $k$-sheeted branched section $Q$ of $\xi$ (see Definition 1 with singularity set $\Sigma$, and let $\pi_Q = \pi_E|_Q : Q \to M$. Recall that $M' = M \setminus \Sigma$, $E' = \pi^{-1}(M')$, and $Q' = Q \cap E'$.

Assume that $x \in \Sigma$ is an isolated point of $\Sigma$. Let us take a neighborhood $U(x)$ such that $U'(x) = U(x) \setminus \{ x \}$ is an open subset of $M'$ and there exists a diffeomorphism $\varphi : (D, 0) \to (U(x), x)$, where $D \subset \mathbb{R}^2$ is the standard open 2-disk centered at the origin $0 \in \mathbb{R}^2$. We will call $U(x)$ a disk neighborhood of $x$ and assume that $\varphi$ sends the standard orientation of $D$ to the orientation of $U(x)$ induced by the orientation of $M$.

By Definition 1 the map $\pi_Q|_{\pi_Q^{-1}(U'(x))} : \pi_Q^{-1}(U'(x)) \to U'(x)$ is a $k$-sheeted covering.
If $U(x)$ is a disk neighborhood of an isolated point $x \in \Sigma$, then for each point $y \in U'(x)$, the fundamental group $\Pi_1(y) = \pi_1(U'(x), y)$ is isomorphic to $\mathbb{Z}$. There are two generators of $\Pi_1(y)$: $[\gamma_+]$ and $[\gamma_-]$, where $\gamma_\pm = \phi(C_\pm)$ and $C_\pm$ is a circle in $D$ passing through the point $\varphi^{-1}(y)$ and enclosing the origin, and having positive (negative, respectively) orientation. We will call the element $[\gamma_\pm] \in \Pi_1(y)$ the positive (the negative, respectively) generator of $\Pi_1(y)$.

The group $\Pi_1(y)$ acts on the fiber $Q_y = \pi_{1Q}^{-1}(y)$ in the following way: for an element $[\gamma] \in \Pi_1(y)$ and $q \in Q_y$ we set $[\gamma] \cdot q = \tilde{q}$ if the lift $\tilde{\gamma}$ of $\gamma$ starting at $q$ terminates in $\tilde{q}$. This action is well defined, this means that if $\gamma_1$ and $\gamma_2$ represent the same element in $\Pi_1(y)$, then the lifts $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ starting at a same point $q$ terminate at a same point $\tilde{q}$.

This action is a homomorphism of the group $\Pi_1(y)$ to the group of permutations of the fiber $Q_y$ and its image is called the local monodromy group of $\Pi_1(y)$.

**Statement 2.** The local monodromy group does not depend on a choice of the disk neighborhood $U(x)$.

**Proof.** Let $U(x)$ and $V(x)$ be two disk neighborhoods of $x$, and $y$ lies in $U(x) \cap V(x)$. Then $\Pi_1^U(y) = \pi_1(U'(x), y) = \Pi_1^V(y) = \pi_1(V'(x), y)$ because for each class $[\gamma] \in \pi_1(V'(x), y)$ or $[\gamma] \in \pi_1(U'(x), y)$ one can find a representative $\gamma_1 \in [\gamma]$ which takes values in $U'(x) \cap V'(x)$.

**Statement 3.** Let $\gamma$ be a loop in $U'(x)$ based at a point $y \in U'(x)$ such that its homotopic class represents the positive generator of $\Pi_1(y)$. Then for each orbit $O$ of the local monodromy group action on $Q_y$ and each point $q \in O$, there exists a loop $\tilde{\gamma}$ in $\pi_{1Q}^{-1}(U'(x))$ based at $q$ which passes through each point of the orbit once and only once and such that $\pi_1(\pi_E) ([\tilde{\gamma}]) = [\gamma]^k$, where $k$ is the number $\#O$ of elements of the orbit $O$. Here $\pi_1(\pi_E) : \pi_1(\pi_{1Q}^{-1}(U'(x)), q) \to \pi_1(U(x), y)$ is the homomorphism of the fundamental groups induced by the map $\pi_E$.

**Proof.** First of all note that if we have an action of the group $\mathbb{Z}$ on a finite set, then we can enumerate elements of each orbit in such a way that the action of the group generator 1 on this orbit is represented by the cycle $\sigma = (2, 3, \cdots, 1)$. Indeed, let $O$ be an orbit of the action, and $q \in O$. The map $F : \mathbb{Z}/H_q \to O$, $[g] \to g \cdot q$, where $H_q$ is the isotropy subgroup of the action, is an equivariant bijection. The group $H_q$ is a cyclic group, this means that there exists $k \in \mathbb{Z}$, $k \geq 0$ such that $H_q = \{km \mid m \in \mathbb{Z}\}$, therefore $\mathbb{Z}/H_q = \{[0], [1], \cdots, [k - 1]\}$, and the action of the generator 1 in $\mathbb{Z}$ on $\mathbb{Z}/H_q$ is given exactly by the cycle $\sigma$.

Now, for a point $y \in U'(x)$, let $[\gamma] : [0, 1] \to U'(x)$, be the positive generator of $\Pi_1(y)$. Let us take an orbit $O$ of the local monodromy group action on $Q_y$ and a point $q \in O$. Let $k$ be the number of elements of $O$. As we have seen, the action of $[\gamma]$ on $O$ is represented by the cycle $\sigma$, this means we can enumerate the points of the orbit $O$ in such a way that $q_1 = q$, $[\gamma]q_1 = q_2$, $\cdots$, $[\gamma]q_{k-1} = q_k$, and $[\gamma]q_k = q_1 = q$. Therefore, by the construction of the action of $\Pi_1(y)$ on $Q_y$, for the lift $\tilde{\gamma}_1$ of $\gamma$ to $Q'$ such that $\tilde{\gamma}_1(0) = q_1$ we have that $\tilde{\gamma}_1(1) = q_2$, for the lift $\tilde{\gamma}_2$ of $\gamma$ to $Q'$ such that $\tilde{\gamma}_2(0) = q_2$ we have that $\tilde{\gamma}_2(1) = q_3$, $\cdots$, and finally for the lift $\tilde{\gamma}_k$ of $\gamma$ to $Q'$ such that $\tilde{\gamma}_k(0) = q_k$ we have that $\tilde{\gamma}_k(1) = q_k = q$.

What do we do in fact is that we take a point $q_1 = q \in Q_y$, then construct the points $q_2 = [\gamma]q_1$, $q_3 = [\gamma]q_2$, $\ldots$, up to $[\gamma]q_k = q_1$. Then the set $\{q_1, q_2, \cdots, q_k\}$ is the orbit $O$ of the point $q$.

It is clear that $\tilde{\gamma} = \tilde{\gamma}_k \cdot \tilde{\gamma}_{k-1} \cdots \tilde{\gamma}_1$, where $\cdot$ is the path composition, is a loop in $\pi_{1Q}^{-1}(U'(x))$ at the point $q_1 = q$, $\tilde{\gamma}$ passes once and only once through each point of $O$, and $\pi_1(\pi_E) ([\tilde{\gamma}]) = [\gamma]^k$. Thus $\tilde{\gamma}$ is the required loop.
2.2. **The index of isolated singular point.** Let $M$ be an oriented two-dimensional manifold. Let $\xi = \{\pi_E : E \to M\}$ be a locally trivial fiber bundle with standard fiber $F$ and a connected structure Lie group $G$.

Let $Q$ be a branched section of $\xi$ with singularity set $\Sigma$, and $x$ be an isolated point of $\Sigma$. Take a disk neighborhood $U(x)$, and for a point $y \in U'(x)$, let $\mathcal{O}_y$ be the set of orbits of local monodromy group action on $Q_y$. Take an orbit $O \in \mathcal{O}(y)$ and a point $q \in O$. Let $[\gamma]$ be a positive generator of the group $\Pi_1(y)$, and $\tilde{\gamma}$ the loop at $q$ constructed in Statement 3.

Let $\psi : \pi_E^{-1}(U(x)) \to U(x) \times F$ be a trivialization of the bundle $\xi$, and $p : \pi_E^{-1}(U(x)) \to F$ be the corresponding projection. Then the element $[p \circ \tilde{\gamma}] \in \pi_1(F)$ is called the index of the branched section $Q$ at the singular point $x$ corresponding to the orbit $O \in \mathcal{O}_y$, call it $\text{ind}_x(Q; y, O)$.

**Statement 4.**

a) The index $\text{ind}_x(Q; y, O)$ does not depend on a choice of the loop $\gamma : [0, 1] \to U(x')$ representing the positive generator of the group $\Pi_1(y)$.

b) The index $\text{ind}_x(Q; y, O)$ does not depend on a trivialization.

c) The index $\text{ind}_x(Q; y, O)$ does not depend on a choice of the disk neighborhood $U(x)$, this means that, if $U(x)$ and $V(x)$ are two disk neighborhoods of $x$, and $y \in U(x) \cap V(x)$, then the constructions of $\text{ind}_x(Q; y, O)$ performed for $U(x)$ and for $V(x)$ result in the same element in $\pi_1(F)$.

**Proof.** a) If $\gamma$ and $\mu$ are two representatives of the positive generator of $\Pi_1(y)$, then $\gamma$ is homotopic to $\mu$, therefore $\gamma^k$ is homotopic to $\mu^k$, therefore the lift $\tilde{\mu}$ of $\mu^k$ is homotopic to the lift $\tilde{\gamma}$ of $\gamma^k$, hence $p\tilde{\gamma}$ is homotopic to $p\tilde{\mu}$.

b) This is because the gluing functions are homotopic to the identity as the structure group is connected.

c) This follows directly from the fact that $\Pi_1^U(y) = \Pi_1^V(y)$ (see the proof of Statement 3) and from a).

**Example 4.** Let us consider the trivial bundle $\xi = \{\pi_E : E = \mathbb{C} \times \mathbb{C}^* \to M = \mathbb{C}\}$, where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and $\pi_E(z, w) = z$. Let us take the subset $Q = \{(z, w) \mid z^2 = w^3\} \subset E$.

As $\pi_E|_Q : Q \to M \setminus \{z = 0\}$ is a 3-sheeted covering, we see that $Q$ is a 3-sheeted branched section of the bundle $\xi$.

It is clear that the singularity set of $Q$ is $\Sigma = \{0\}$, so $Q$ has only one singular point $z = 0$ and this point is isolated. For the disk neighborhood of the isolated singular point $z = 0$ we take the entire $M = \mathbb{C}$.

Let us take $y = 1$, then $Q_y = \{(a, b, c) \mid a = (1, 1), b = (1, \varepsilon), c = (1, \varepsilon^2)\}$, where $\varepsilon = \exp(2\pi i/3)$. The loop $\gamma(t) = \exp(2\pi it), t \in [0, 1]$, represents the positive generator of the group $\Pi(y = 1)$, and the lift $\tilde{\gamma}_a$ of $\gamma$ which starts at the point $a = (1, 1)$ is given by $\tilde{\gamma}_a(t) = (\exp(2\pi it), \exp(\frac{4}{3}\pi it))$, $t \in [0, 1]$. Therefore $[\gamma]a = c$. In the same manner one can prove that $[\gamma]b = a, [\gamma]c = b$.

Thus, the orbit $O$ of the point $a$ is $Q_{y=1} = \{(a, b, c)\}$, and for a representative of the class $[\gamma]$ constructed in Statement 3 we can take the loop $\tilde{\gamma} = (\exp(6\pi it), \exp(4\pi it))$ for $t \in [0, 1]$.

Therefore, the loop $p\tilde{\gamma} : [0, 1] \to \mathbb{C}^*$ is given by $p\tilde{\gamma} = \exp(4\pi it)$ for $t \in [0, 1]$. Hence

$$\text{ind}_0(Q; y = 1, Q_{y=1}) = 2 \in \mathbb{Z} = \pi_1(\mathbb{C}^*)$$. 

(3)
Let us consider the finite set of elements of $\pi_1(F)$:

\begin{equation}
\ind_x(Q; y) = \{\ind_x(Q; y, O) \mid O \in O(y)\}.
\end{equation}

**Statement 5.** The set $\ind_x(Q; y)$ does not depend on $y \in U'(x)$.

**Proof.** Let $y, \bar{y}$ be two points in $U'(x)$. Take a curve $\gamma : [0, 1] \to U'(x)$ such that $\gamma(0) = y, \gamma(1) = \bar{y}$.

The curve $\delta$ defines the group isomorphism $\psi_\delta : \Pi_1(y) \to \Pi_1(\bar{y}), [\gamma] \mapsto [\delta^{-1} \cdot \gamma \cdot \delta]$, where $\delta^{-1}(t) = \delta(1-t)$. Also, $\delta$ defines the bijection $\psi_\delta : Q_y \to Q_{\bar{y}}, q \in Q_y \mapsto \bar{q} \in Q_{\bar{y}}$, such that for the lift $\tilde{\delta}$ of $\delta$ to $Q$ with $\tilde{\delta}(0) = q$ we have that $\tilde{\delta}(1) = \bar{q}$. In addition, the bijection $\psi_\delta$ is equivariant in the sense that $\tilde{\psi}_\delta([\gamma]q) = \psi_\delta([\gamma])\tilde{\psi}_\delta(q)$.

Therefore $\tilde{\delta}$ induces a bijection $\alpha_\delta : O(y) \to O(\bar{y}), O_q \mapsto O_{\bar{\psi}_\delta(q)}$, where $O_q$ is the $\Pi_1(y)$-orbit of the point $q \in Q_y$ and $O_{\bar{\psi}_\delta(q)}$ is the $\Pi_1(\bar{y})$-orbit of the point $\tilde{\psi}_\delta(q) \in Q_{\bar{y}}$.

Let us prove that the loop $\bar{\gamma}$ which passes through the points of the orbit $O_q \in O(y)$ constructed in Statement 3 is homotopic in $\pi^{-1}(U'(x))$ to the corresponding loop of the orbit $O_{\bar{\psi}_\delta(q)} \in O(\bar{y})$.

Let $\gamma$ be a loop at $y \in U'(x)$ which represents the positive generator of $\Pi_1(y)$. The loop $\bar{\gamma}$ constructed in Statement 3 is homotopic to the lift of the loop $\gamma^k$ starting at a point $q \in Q_y$. As the loop $\gamma^k$ is freely homotopic to the loop $\bar{\gamma}^k$, where $\bar{\gamma} = \delta^{-1}\gamma\delta$, the lift $\bar{\gamma}$ is freely homotopic to the lift of $\bar{\gamma}^k$ starting at the point $\tilde{\psi}_\delta(q)$, but this lift is in turn homotopic to the loop $\bar{\gamma}$.

Therefore the loops $\bar{p}\bar{\gamma}$ and $\bar{p}\bar{\gamma}_\delta$ are freely homotopic in $F$, therefore define the same element in $\pi_1(F)$. Thus we have that $\ind_x(Q; y, O) = \ind_x(Q; \bar{y}, \alpha_\delta(O))$ for all $O \in O(y)$, hence $\ind_x(Q; y) = \ind_x(Q; \bar{y})$. □

**Corollary 1.** The set $\ind_x(Q)$ does not depend on the disk neighborhood $U(x)$, this means if $U_1(x)$ and $U_2(x)$ are disk neighborhoods of an isolated singular point $x \in \Sigma$, and $y_1 \in U_1(x)$ and $y_2 \in U_2(x)$, then the set $\ind_x(Q; y_1)$ constructed via $U_1(x)$ and the set $\ind_x(Q; y_2)$ constructed via $U_2(x)$ coincide.

**Proof.** Follows from Statement 6 □

From Statement 5 it follows that we can give the following definition.

**Definition 2.** Let $Q$ be a branched section of the bundle $\xi$. The *index* of $Q$ at $x \in M$ is

\begin{equation}
\ind_x(Q) = \ind_x(Q; y),
\end{equation}

where $y$ is a point of $U'(x)$, where $U(x)$ is a disk neighborhood of $x$.

Let us fix an element $a \in H^1(F)$. The index of $Q$ at a point $x$ with respect to $a$ is

\begin{equation}
\ind_x(Q; a) = \sum_{O \in O(y)} \frac{1}{\#O} \langle a, \ind_x(Q; y, O) \rangle = \sum_{O \in O(y)} \frac{1}{\#O} \int_{\gamma(Q; y, O)} \alpha,
\end{equation}

where $\alpha \in \Omega^1(F)$ represents $a \in H^1(F)$ and $\gamma(Q; y, O)$ represents the class $\ind_x(Q; y, O) \in \pi_1(F)$.

**Example 5.** Let $M$ be a connected compact oriented manifold and let $\omega$ be a symmetric tensor of order $n$ over $M$. In Example 2 we have constructed a branched section $Q \subset PTM$ determined by the binary differential equation (11).

If we consider the covering $q : \mathbb{S}^1TM \to PTM$ given by $q((p, \vec{v})) = [\vec{v}]$, we see that $q \circ \pi : \mathbb{S}^1TM \to M$ is a fiber bundle and $q^{-1}(Q)$ is a $2n$-sheeted branched covering of the bundle $\mathbb{S}^1TM \to M$. Let
\( p \in \Sigma \) be a singular point, \( U'(p) \) be a neighborhood disk of \( p \), and \( \mathcal{O}_p = \{O_1, \ldots, O_r\} \) the set of the orbits of the action of \( \pi_1(U'(p)) \) on \( \pi^{-1}(p) \). From the equation \([3]\) it follows that the index of \( q^{-1}(Q) \) at the singular point \( p \in \Sigma \) with respect the cohomology class \( a = \left[ \frac{1}{2\pi} d\theta \right] \in H^1(S^1) \), where \( d\theta \) is the angular form on \( S^1 \) is given by

\[
\text{ind}_p(Q; a) = \sum_{i=1}^{r} \frac{1}{2\pi k_i} \int_{\gamma_i} d\theta,
\]

where \( k_i \) is the number of elements of the orbit \( O_i \), and \( \gamma_i \) is the index of the point \( p \) corresponding to the orbit \( O_i \). Let us choose a frame \((e_1, e_2)\) along the curve \( \gamma \), and we consider a unit vector field \( X(t), 0 \leq t \leq 1 \) such that \( \omega_{\gamma(t)}(X(t)) \) around the curve \( \gamma : I \to U'(p) \). If \( \tilde{\theta} \) is the angle between \( e_1 \) and \( X(0) \), we obtain that the index of \( Q \) at the point \( p \) with respect to the form \( a \) can be also calculated in terms of this rotation angle by the formula

\[
\text{ind}_p(Q, O_i, a) = \frac{\tilde{\theta}(2k_i) - \tilde{\theta}(0)}{2\pi k_i}.
\]

Note that if the action of \( \pi_1(U'(p)) \) on \( \pi^{-1}(p) \) is transitive, then the equation \([7]\) reduces to the following

\[
\text{ind}_p(Q; a) = \frac{1}{4\pi n} \int_\gamma d\theta,
\]

where \( \gamma \) is the index of \( p \) in \( \pi^{-1}(p) \), and it is also true that

\[
\text{ind}_p(Q, \pi^{-1}(p), a) = \frac{\tilde{\theta}(2k_i) - \tilde{\theta}(0)}{4n\pi}.
\]

The equation \([10]\) coincides with the index of a binary differential \( n \)-form given in \([3]\).

Now, we note that the index of \( Q \) at a singular point \( x \) seen as a singularity of the bundle \( \pi : PTM \to M \) is twice the index of the same point as a singular point of the bundle \( \pi \circ q : S^1TM \to M \).

**Remark 1.** This construction can be used to calculate an index of singular points of singular distributions over a two dimensional manifold \( M \). In \([3]\), pages 218-223, the author gives another constructions of indexes of singular points of 1-dimensional singular distributions and branched covering of two sheets defined by such a distributions.

### 3. Resolution of a Branched Section

Let \( M \) be a two-dimensional oriented manifold, and \( \xi = \{\pi_E : E \to M\} \) be a fiber bundle. Let \( \Sigma \) be a discrete subset of the manifold \( M \).

**Definition 3.** Let \( Q \) be an \( n \)-sheeted branched section of the bundle \( \xi \) with singularity set \( \Sigma, M' = M \setminus \Sigma, E' = \pi^{-1}(M'), \) and \( Q' = Q \cap E' \). A resolution of \( Q \) is a map \( \iota : S \to E \), where \( S \) is an oriented two-dimensional manifold with boundary, such that

1. \( \iota(S) = Q \);
2. \( \pi = \pi_E \circ \iota : S \to M \) is surjective;
3. the map \( \iota \) is an embedding of \( S' = S \setminus \partial S \) onto \( Q' \).

In case \( M \) is compact, we assume \( S \) to be compact, too.

**Remark 2.** From Definition \([3]\) it follows that \( \pi_E(Q) = M \) and \( \pi_E(\partial S) = \Sigma \).
Example 6. Let $M = \mathbb{R}^2$, $E = \mathbb{P}T\mathbb{R}^2$ and a branched section is the solution of the differential equation
\[ xydx^2 - (x^2 - y^2)dx dy - xydy^2 = 0. \]
As the discriminant of this equation is $(x^2 - y^2)^2 - 4(xy)^2 = (x^2 + y^2)^2$, this differential equation is a binary differential equation (see Example 2). This differential equation is represented in the form $(xdx + ydy)(ydx - xdy) = 0$, therefore its solution $Q$ consists of two 1-dimensional distributions $L_1$ and $L_2$ on $\mathbb{R}^2$ given respectively by the equations $xdx + ydy = 0$ and $ydx - xdy = 0$. One can easily see that these equations determine sections with singularities $s_1$ and $s_2$ of the bundle $E$, which admit resolutions (see [1]), call them $S_1$ and $S_2$, so the manifold $S_1 \cup S_2$ is a resolution of the branched section $Q$.

Example 7. Let $M = \mathbb{R}^2$, $E = \mathbb{P}T\mathbb{R}^2$ and the branched section $Q$ is the solution of the binary differential equation
\[ ydx^2 - 2xdxdy - ydy^2 = 0. \]
The discriminant of equation (11) is $4(x^2 + y^2)$, therefore this equation has two real roots for all $(x, y)$ different from the origin, and at the origin all the coefficients vanish. That is why, equation (11) is a binary differential equation (see Example 2).

The standard coordinates $(x, y)$ on $\mathbb{R}^2$ induce a trivialization of the bundle $\pi_E = E = \mathbb{P}T\mathbb{R}^2 \to M = \mathbb{R}^2$, namely for the one-dimensional subspace $l \in PT(x,y)\mathbb{R}^2$ spanned by a vector $p\partial_x + q\partial_y$, we assign the point $(x, y, [p : q]) \in \mathbb{R}^2 \times \mathbb{R}P^1$. Thus, $\mathbb{P}T\mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{R}P^1$, and
\[ Q = \{(x, y, [p : q]) \in \mathbb{R}^2 \times \mathbb{R}P^1 \mid yp^2 - 2xpq - yq^2 = 0\} \]
In this case
\[ \Sigma = (0, 0), \quad Q' = \{(x, y, [p : q]) \in Q \mid x^2 + y^2 > 0\}, \quad M' = \mathbb{R}^2 \setminus \{(0, 0)\}. \]
The projection $\pi_{\mathbb{P}T\mathbb{R}^2} : \mathbb{P}T\mathbb{R}^2 \to \mathbb{R}^2$ restricted to $Q'$ is a trivial (as a fiber bundle) double covering of $M'$. Indeed, take the following open sets $U_1$ and $U_2$:
\[ U_1 = M' \setminus (-\infty, 0) \times \{0\}, \quad U_2 = M' \setminus (0, \infty) \times \{0\} \]
It is clear that $M' = U_1 \cup U_2$. Also, at the points of $U_1$ we have $x + \sqrt{x^2 + y^2} > 0$, and at the points of $U_2$ we have $x - \sqrt{x^2 + y^2} > 0$.

Now let us take two sections of the bundle $\pi_{\mathbb{P}T\mathbb{R}^2} : \mathbb{P}T\mathbb{R}^2 \to \mathbb{R}^2$ defined on $M'$:
\[ s_1 : (x, y) \mapsto \begin{cases} (x, y, [x + \sqrt{x^2 + y^2} : y]), & (x, y) \in U_1, \\ (x, y, [-y : x - \sqrt{x^2 + y^2}]), & (x, y) \in U_2, \end{cases} \]
and
\[ s_2 : (x, y) \mapsto \begin{cases} (x, y, [-y : x + \sqrt{x^2 + y^2}]), & (x, y) \in U_1, \\ (x, y, [x - \sqrt{x^2 + y^2} : y]), & (x, y) \in U_2, \end{cases} \]
Note that over $U_1 \cap U_2$ there holds
\[ [x + \sqrt{x^2 + y^2} : y] = [-y : x - \sqrt{x^2 + y^2}] \quad \text{and} \quad [-y : x + \sqrt{x^2 + y^2}] = [x - \sqrt{x^2 + y^2} : y], \]
therefore the sections $s_1$ and $s_2$ are well defined. One can easily prove that $s_i(M') \subset Q'$, $i = 1, 2$, and $s_1(M') \cap s_2(M') = \emptyset$. Therefore $Q'$ is a trivial double covering of $M'$.
Now let us construct a resolution of the branched section $Q$. Recall that $S^1 = \{(u, v) \mid u^2 + v^2 = 1\}$, then let take the diffeomorphism

$$f : S^1 \to \mathbb{R}P^1, (u, v) \mapsto \begin{cases} [u + 1 : v], & u > -1, \\ [-v : u - 1], & u < 1, \end{cases}$$

and then the diffeomorphism $f$ “rotated” at the angle $\pi/2$ gives the diffeomorphism,

$$g : S^1 \to \mathbb{R}P^1, (u, v) \mapsto \begin{cases} [-v : u + 1], & u > -1, \\ [u - 1 : v], & u < 1. \end{cases}$$

We take $S_1 = S_2 = \mathbb{R}_+ \times S^1 = [0, \infty) \times S^1$, and $S'_1 = S'_2 = (0, \infty)$. We set $S = S_1 \sqcup S_2$, then $S' = S'_1 \sqcup S'_2$. Then $\iota : S \to \mathbb{R}^2 \times \mathbb{R}P^1$ is given by

$$\iota|_{S_i} (r, (u, v)) = (ru, rv, f(u, v)), \quad \iota|_{S'_i} (r, (x, y)) = (ru, rv, g(u, v)).$$

One can easy see that $\iota|_{S'_i} : S'_i \to Q'$, $i = 1, 2$ is a diffeomorphism. For example, any point $(x, y, [p : q]) \in V_1$, is the image of the point $(r, (u, v))$ under the map $\iota|_{S_1}$, where

$$u = \frac{x}{\sqrt{x^2 + y^2}}, \quad v = \frac{y}{\sqrt{x^2 + y^2}}, \quad r = \sqrt{x^2 + y^2}.$$

**Example 8.** As a generalization of Examples 6 and 7 one can take $n$ sections with singularities $\frac{1}{1}$ of a bundle $\xi = \pi_E : E \to M$ which have the same set of singularities $\Sigma$, call them $s_i$, $i = 1, n$. These sections define a branched section $Q$ of the bundle $\xi$: $Q = \{s_i(x) \mid x \in M \setminus \Sigma\}$. If $S_i$ is a resolution of $s_i$, then $S = \sqcup S_i$ is a resolution of $Q$.

**Example 9.** Let us present an example of branched section, where the covering $\pi_Q|_{Q'} : Q' \to M'$ is not trivial. Take $M = \mathbb{R}^2 = \mathbb{C}$, $E = S^1(\mathbb{C}) = \mathbb{C} \times S^1$, the bundle of unit vectors over $M$, and let

$$Q = \{(z, w) \in \mathbb{C} \times S^1 \mid |z| w^2 = z\}.$$  

Then $M' = \mathbb{C} \setminus \{0\}$, $Q' = \{(z, w) \mid w^2 = z/|z|\}$, and it is well known that $\pi_Q|_{Q'} : Q' \to M'$ is a non trivial double covering. Now let us take

$$S = [0, \infty) \times S^1,$$

and $\iota : S \to E$, $\quad (r, e^{i\varphi}) \mapsto (re^{2i\varphi}, e^{i\varphi})$

Then $S' = (0, \infty)$, and it is clear that the properties (1)–(3) of Definition 3 hold true for $\iota$.

**Example 10.** Let us present another example of branched section, where the covering $\pi_Q|_{Q'} : Q' \to M'$ is not trivial. Take $M = \mathbb{R}^2 = \mathbb{C}$, $E = PT\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}P^1 = \mathbb{C} \times \mathbb{R}P^1$, and let

$$Q = \{(z, [w] \mid |w| = 1 \text{ and } |z|^2 w^4 = z^2\}.$$  

Then $M' = \mathbb{C} \setminus \{0\}$, $Q' = \{(z, w) \mid w^4 = z^2/|z|^2\}$, and it is clear that $\pi_Q|_{Q'} : Q' \to M'$ is a non trivial double covering. Now let us take

$$S = [0, \infty) \times \mathbb{R}P^1,$$

and $\iota : S \to E$, $\quad (r, [w]) \mapsto (rw^2, [w])$, where $|w| = 1$. Then $S' = (0, \infty)$, and it is clear that the properties (1)–(3) of Definition 3 hold true for $\iota$.

**Remark 3.** In Examples 9, 10, for each $x \in M$, the set $S_x$ is a discrete set if $x \in M \setminus \Sigma$, or is diffeomorphic to a circle $S^1$ if $x \in \Sigma$. 

Now let us consider a point \( x \in \Sigma \). Then, according to Definition 3, \( S_x = \pi^{-1}(x) \) consists of the connected components of the boundary \( \partial S \). Let us denote by \( C(S_x) \) the set of connected components of \( S_x \). As \( S_x \) is compact, the set \( C(S_x) \) is finite, and each element of this set is diffeomorphic to a circle \( S^1 \).

**Statement 6.** Let \( C \) be a connected component of a boundary. Then there exists a neighborhood \( N(C) \) of \( C \) and a diffeomorphism \( f_C : N(C) \to S^1 \times [0,1] \) such that \( f_C(C) = S^1 \times \{0\} \) and \( U(x) = \pi(N(C)) \) is a disk neighborhood of \( x \). For each \( y \in U'(x) \), the set of orbits \( \mathcal{O}_y \) consists of only one element. In this case the curve \( \tilde{\gamma} \) corresponding to the orbit by Statement 3 is a generator of the group \( \pi_1(N(C)) \cong \mathbb{Z} \).

**Proof.** Indeed, \( N(C) \setminus C \) is homeomorphic to a ring and \( U'(x) \) is homeomorphic to a ring as well. The map \( N(C) \setminus C \to U'(x) \) induced by \( \pi \) is an \( n \)-fold covering therefore \( \pi_* : \pi_1(N(C)) \cong \mathbb{Z} \to \pi_1(U'(x)) \) has the form \( m \to km \). At the same time \( \pi_*(\tilde{\gamma}) = \gamma^k \), thus \( \tilde{\gamma} \) is a generator of the group \( \pi_1(N(C)) \). \( \square \)

**Corollary 2.** The curve \( \tilde{\gamma} \) is homotopic in \( N(C) \subset S \) to the curve \( C \subset E_x \). Therefore the curve \( C \) represents \( \text{ind}_x(Q,O) \).

### 4. Connection and the Gauss-Bonnet theorem

Let \( \xi = (\pi_E : E \to M) \) be a locally trivial fiber bundle with standard fiber \( F \) and structure group \( G \). Assume that \( G \) is a connected Lie group.

Let \( (U, \psi : \pi^{-1}(U) \to U \times F) \) be a chart of the atlas of \( \xi \). Let

\[
\eta = p_F \circ \psi : \pi^{-1}(U) \to F,
\]

where \( p_F : U \times F \to F \) is the canonical projection onto \( F \). For each \( x \in U \) the map \( \eta \) restricted to \( F_x = \pi^{-1}(x) \) induces a diffeomorphism \( \eta_x : F_x \to F \), and let \( i_x : F \to F_x \) be the inverse of \( \eta_x \). Note that if we take another chart \( (U', \psi' : \pi^{-1}(U') \to U' \times F) \), and \( \eta' : \pi^{-1}(U') \to F \) is the corresponding map, then on \( \pi^{-1}(U \cap U') \) we have that

\[
\psi' \circ \psi^{-1} : (U \cap U') \times F \to (U \cap U') \times F, \quad (x, y) \mapsto (x, g(x)y),
\]

where \( g : U \cap U' \to G \) is the gluing map of the charts. Now, for any \( x \in U \cap U' \), we have \( \eta_x' \circ \psi^{-1}(y) = g(x)y \), and, as \( G \) is connected, \( \eta_x' \circ \psi^{-1} : F \to F \) is homotopic to the identity map. This means that for any \( x \in m \) we have well defined isomorphisms of the homotopy and (co)homology groups:

\[
\pi_*(\eta_x) : \pi_*(F_x) \to \pi_*(F),
\]

\[
H_*(\eta_x) : H_*(F_x) \to H_*(F), \quad H^*(\eta_x) : H^*(F) \to H^*(F_x),
\]

which do not depend on the chart.

In [1], for a locally trivial bundle with standard fiber \( F \) and structure Lie group \( G \), we have proved the following statement ([1], Statement 1):

**Statement 7.** Let \( a \in H^1(F) \) and \( H \) be a connection in \( E \). There exists a 1-form \( \alpha \in \Omega^1(E) \) such that

1. \( \alpha|_H = 0 \);
2. for each \( x \in M \), \( di^*_x \alpha = 0 \) and \( [i^*_x \alpha] = H^1(\eta_x)a \).
The decomposition $TE = H \oplus V$ gives a bicomplex representation of the complex $\Omega(E)$, then the form $\alpha$ lies in $\Omega^{(0,1)}(E)$ and $d\alpha = \theta_{(1,1)} + \theta_{(2,0)}$, where $\theta_{(1,1)} \in \Omega^{(1,1)}$ and $\theta_{(2,0)} \in \Omega^{(2,0)}$, and

$$\theta_{(1,1)}(X,Y) = (L_X \alpha)(Y), \quad \theta_{(2,0)} = \tilde{\alpha}(\Omega).$$

where $L_X$ is the Lie derivative with respect to the vector field $X$, and $\Omega$ is the curvature form of the connection $H$ (for details see [1], Section 3).

Now let $Q$ be a branched section of the bundle $\xi$ which admits a resolution $\iota: S \to E$ (see Definition 3). Let us fix an element $a \in H^1(F)$, and let $\alpha \in \Omega^1(E)$ be the corresponding 1-form (see Statement 7). Then, by the Stokes theorem we have

$$\int_{\partial S} \iota^* \alpha = \int_S \iota^* d\alpha.$$\hspace{1cm} (30)

By Remark 2 we have that $\pi_E(\partial S) = \Sigma$. For $x \in \Sigma$, let $C(S_x)$ be the set of connected components of $\pi_E^{-1}(x)$.

From Corollary 2 it follows that, for $C \in C(S_x)$, we have

$$\int_C \alpha = \int_{\gamma(Q;y,O(C))} \iota_x^* \alpha,$$

where $\gamma(Q;y,O(C))$ represents the class $ind_x(Q;y,O(C)) \in \pi_1(F)$, and $O(C)$ is the orbit of the local monodromy group corresponding to $C$. Therefore, from (31) we have that

$$ind_x(Q;a) = \sum_{C \in C(S_x)} \frac{1}{\#O(C)} \int_C \alpha.$$\hspace{1cm} (32)

If all the orbits of the local monodromy group corresponding to the components $C \in C(S_x)$ have the same number of elements $N(x)$, then

$$\int_{\partial S} \iota^* \alpha = \sum_{x \in \Sigma} \sum_{C \in C(S_x)} \int_C \alpha = \sum_{x \in \Sigma} N(x) \cdot ind_x(Q;a)$$\hspace{1cm} (33)

Thus we get the following theorem

**Theorem 1** (Gauss-Bonnet-Hopf-Poincaré for branched sections). If, for any $x \in \Sigma$, all the orbits of the local monodromy group corresponding to the components $C \in C(S_x)$ have the same number of elements $N(x)$, then

$$\int_S \iota^* \theta_{(1,1)} + \iota^* \theta_{(2,0)} = \sum_{x \in \pi(\partial S)} N(x) \cdot ind_x(Q).$$

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