Gravitational waves in conformal gravity

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We consider the gravitational radiation in conformal gravity theory. We perturb the metric from flat Minkowski space and obtain the wave equation after introducing the appropriate transformation for perturbation. We derive the effective energy-momentum tensor for the gravitational radiation, which can be used to determine the energy carried by gravitational waves.

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I. INTRODUCTION

The detection of gravitational waves (GWs) by the LIGO Collaboration is a milestone in GW research and opens a new window to probe general relativity (GR) and astrophysics [1-4]. Future space-borne detectors will offer access to an unprecedented signal sensitivity [5], thus it is worthwhile to explore GWs in alternative theories of gravity. Gravitational wave were considered in \( f(R) \) theories [6-17], in scalar-tensor theories [18, 19], in \( f(T) \) theories [20] and in fourth order gravity [21]. Following the original work by Weyl [22] (for review, see [23]), conformal gravity (CG), as a possible candidate alternative to GR, attracts much attention. It can give rise to an accelerated expansion [24]. It was tested with astrophysical observations and had been confirmed that it does not suffer from an age problem [25]. It can describe the rotation curves of galaxies without dark matter [26]. Cosmological perturbations in CG were investigated in [27]. It had been shown that CG accommodates well with currently available SNIa and GRB samples [28, 29]. A series of dynamical solutions in CG were found in [30]. Mass decomposition of the lens galaxies of the Sloan Lens Advanced Camera for Surveys in CG was discussed in [31]. Recently it was indicated that conformal gravity can potentially test well against all astrophysical observations to date [32]. It has been shown that CG can also give rise to an inflationary phase [33]. It was argued that Weyl action should be added to the Einstein-Hilbert action [34].

CG is also confronted with some challenges. It has been shown that CG does not agree with the predictions of general relativity in the limit of weak fields and slow motions, and it is therefore ruled out by Solar System observations [35]. It suggested that without dark matter CG can not explain the properties of X-ray galaxy clusters [36]. It is not able to describe the phenomenology of gravitational lensing [37]. The cosmological models derived from CG are not likely to reproduce the observational properties of our Universe [38].

In this paper, we will consider gravitational radiation in CG. We aim to find the equations of gravitational radiation and the energy-momentum tensor of the gravitational waves. These results will be valuable for future observations of gravitational waves to test gravity theories alternative to GR.

This paper is organised as follows. We begin with a review of the CG theory. In Section III, we consider gravitational waves in CG. In Section IV, we will discuss the energy-momentum tensor of the gravitational waves. Finally, we will briefly summarize and discuss our results.

II. BASIC EQUATIONS FOR CONFORMAL GRAVITY

Besides of the general coordinate invariance and equivalence principle structure of general relativity, CG possesses an additional local conformal symmetry in which the action is invariant under local conformal transformations on the metric: \( g_{\mu\nu} \rightarrow e^{2\alpha(x)}g_{\mu\nu} \). This symmetry forbids the presence of any \( \Lambda\sqrt{-g}d^4x \) term in the action, so CG does not suffer from the cosmological constant problem. Under such a symmetry, the action of CG with matter is given by

\[
\mathcal{I} = -\alpha e \int C_{\mu\nu\kappa\lambda} C^{\mu\nu\kappa\lambda} \sqrt{-g}d^4x + \mathcal{I}_m,
\]  

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where $C_{\mu\nu\lambda}$ the Weyl tensor and $\alpha_g$ is a dimensionless coupling constant which implies CG is renormalizable \cite{39}, unlike general relativity. Variation with respect to the metric generates the field equations
\begin{equation}
4\alpha_g W_{\mu\nu} = T_{\mu\nu},
\end{equation}
where
\begin{equation}
W_{\mu\nu} = -\frac{1}{6}g_{\mu\nu}R^\lambda_{;\lambda} + \frac{2}{3}R_{\mu\nu,\lambda}^\lambda - R_{\lambda\nu,\mu}^\lambda - R_{\lambda\mu,\nu}^\lambda
+ \frac{2}{3}RR_{\mu\nu} - 2R_{\lambda\mu}R_{\nu}^\lambda + \frac{1}{2}g_{\mu\nu}R_{\lambda\kappa}R_{\lambda\kappa}^\lambda - \frac{1}{6}g_{\mu\nu}R^2,
\end{equation}
and $T_{\mu\nu}$ is the energy-momentum tensor of matter. Though the equations of motion are fourth-order, CG is a ghost-free theory \cite{40}. However, CG is also confronted with some challenges as discussed in the previous section. Since $W_{\mu\nu}$ is obtained from an action that is both conformal invariant and general coordinate invariant, it is traceless and kinematically covariantly conserved: \(W_{\mu\mu} \equiv g_{\mu\nu}W_{\mu\nu} = 0\) and \(W_{\mu\nu,\mu} = 0\).

\section{III. GRAVITATIONAL WAVES IN CONFORMAL GRAVITY}

The linearized framework provides a natural way to study gravitational waves (GWs), which is a weak-field approximation that assumes small deviations from a flat background
\begin{equation}
g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},
\end{equation}
where $|h_{\mu\nu}| \sim \epsilon$ which is a small parameter. We will consider terms up to $O(\epsilon)$. Thus the inverse metric is $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ where the indices are raised by using the Minkowski metric. To the first-order, the covariant derivative of any perturbed quantity will be the same as the partial derivative, so the connection and the Riemann tensor are, respectively, given by
\begin{equation}
\Gamma^{(1)}_{\mu\nu} = \frac{1}{2} \eta^{\rho\lambda}(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}),
\end{equation}
\begin{equation}
R^{(1)}_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\mu \partial_\nu h_{\rho\sigma} + \partial_\rho \partial_\sigma h_{\mu\nu} - \partial_\mu \partial_\sigma h_{\rho\nu} - \partial_\nu \partial_\rho h_{\mu\sigma}).
\end{equation}
Contracting the Riemann tensor gives the Ricci tensor
\begin{equation}
R^{(1)}_{\mu\nu} = \frac{1}{2}(\partial_\mu \partial_\nu h_{\lambda\lambda} + \partial_\nu \partial_\lambda h_{\mu\lambda} - \partial_\mu \partial_\lambda h_{\nu\lambda} - \Box h_{\mu\nu}),
\end{equation}
where the d’Alembertian operator is $\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu$. Contracting the Ricci tensor gives the first-order Ricci scalar
\begin{equation}
R^{(1)} = \partial_\mu \partial_\nu h^{\mu\nu} - \Box h.
\end{equation}
Inserting Eqs. 5 and 6 into 5 and retaining terms to the first-order, we obtain
\begin{equation}
W^{(1)}_{\mu\nu} = -\frac{1}{6}\eta_{\mu\nu} \Box R^{(1)} + \frac{2}{3}R^{(1)}_{\mu\nu} + \Box R^{(1)}_{\mu\nu} - R^{(1)}_{\lambda\nu,\mu}^\lambda - R^{(1)}_{\lambda\mu,\nu}^\lambda.
\end{equation}
In general relativity, if we define the trace-reversed perturbation $\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$ and impose the Lorenz gauge $\partial^\mu \tilde{h}_{\mu\nu} = 0$, the linearized Einstein field equations reduce to the wave equation
\begin{equation}
\Box \tilde{h}_{\mu\nu} = -16\pi GT_{\mu\nu}.
\end{equation}
We can apply this similar standard reasoning within the CG framework and find a quantity $\tilde{h}_{\mu\nu}$ that satisfies a wave equation when linearizing the field equations \cite{38}. We look for a similar solution
\begin{equation}
\tilde{h}_{\mu\nu} = \Box h_{\mu\nu} + \alpha \eta_{\mu\nu} \Box h + \beta \eta_{\mu\nu} R,
\end{equation}
where $\alpha$ and $\beta$ are constants. Taking the trace of Eq. 5 yields
\begin{equation}
\tilde{h} = (4\alpha + 1)\Box h \Box + 4\beta R.
\end{equation}
where the index was raised or lowered by using the Minkowski metric $\eta_{\mu\nu}$. If choosing $\xi_{\mu}$ so that it satisfies the equation

$$\Box^2 \xi_\nu = \partial^\mu \tilde{h}_{\mu\nu},$$

then we have the Lorenz gauge condition $\partial^\mu \tilde{h}'_{\mu\nu} = 0$. Using this gauge conditions and taking $\alpha = -\frac{1}{2}$ and $\beta = -2$, the first-order linear vacuum field equations (3) give

$$\frac{1}{2} \Box h + \frac{1}{4} \eta_{\mu\nu} \Box h = 0,$$

(22)

Taking trace and yields

$$-\frac{1}{2} \Box h + \Box h = 0,$$

(23)

which leads to

$$\Box h = 0,$$

(24)

Combining Eqs. (22) and (24), we finally obtain

$$\Box \tilde{h}_{\mu\nu} = 0.$$

(25)

The solution to Eq. (25) is

$$\tilde{h}_{\mu\nu} = q_{\mu\nu} \exp(ik_\lambda x^\lambda).$$

(26)

where $k_\lambda$ is a four-wavevector and satisfies $\eta_{\mu\nu} k^\mu k^\nu$ and $k^\mu q_{\mu\nu} = 0$. For a wave traveling along the $z$-axis, $k^\mu = \omega(1, 0, 0, 1)$ with $\omega$ the angular frequency. The Lorentz gauge condition can not fixed the gauge freedom completely, it leaves a residual coordinate transformation with $\Box^2 \xi_\nu = 0$. If $\xi_{\mu}$ also satisfies the equation $\tilde{h} = -2 \Box \partial_\sigma \xi^\sigma$, we then have $\tilde{h} = 0$. We may impose further constrains upon $q_{\mu\nu}$: $q_{0\nu} = 0$, and can define

$$[q_{\mu\nu}] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q_+ & q_\times & 0 \\ 0 & q_\times & -q_+ & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

(27)

where $q_+$ and $q_\times$ are constants representing the amplitudes of the two transverse polarizations of gravitational radiation.
IV. ENERGY-MOMENTUM TENSOR OF THE GRAVITATIONAL WAVES

Physically, we could expect the gravitational field to carry energy-momentum, but as it is well known that it is difficult to define an energy-momentum tensor for a gravitational field. Nevertheless, one can regard the linearised gravitational field as a simple rank-2 tensor field \( h_{\mu\nu} \) in Cartesian inertial coordinates propagating in a fixed Minkowski spacetime background, and then can assign an energy-momentum tensor to this field in Minkowski spacetime. As was discussed above, the linearised gravitational theory ignores the energy-momentum associated with the gravitational field itself. To include this contribution, and thereby go beyond the linearised theory, we must modify the linearised field equations to read

\[
W_{\mu\nu}^{(1)} = \frac{1}{4\alpha_g} (T_{\mu\nu} + t_{\mu\nu}), \tag{28}
\]

where \( t_{\mu\nu} \) is the energy-momentum tensor of the gravitational field itself. Rearranging this equation gives

\[
W_{\mu\nu}^{(1)} - \frac{1}{4\alpha_g} t_{\mu\nu} = \frac{1}{4\alpha_g} T_{\mu\nu}. \tag{29}
\]

Returning to the equations \( 3 \), we can expand beyond first order to obtain

\[
W_{\mu\nu} \equiv W_{\mu\nu}^{(1)} + W_{\mu\nu}^{(2)} + \ldots = \frac{1}{4\alpha_g} T_{\mu\nu}. \tag{30}
\]

where indexes \((i)\) indicate the order of the expansion in \( h_{\mu\nu} \). To a good approximation, we should make the identification

\[
t_{\mu\nu} = -4\alpha_g W_{\mu\nu}^{(2)}. \tag{31}
\]

Since the energy-momentum of a gravitational field at a point in spacetime has no real meaning, this suggests that in order to probe the physical curvature of the spacetime one should average \( W_{\mu\nu}^{(2)} \) over a small region at each point in spacetime, which gives a gauge-invariant measure of the gravitational field strength. Therefore we should replace Eq. \( 31 \) by

\[
t_{\mu\nu} = -4\alpha_g \langle W_{\mu\nu}^{(2)} \rangle. \tag{32}
\]

where \( \langle \cdots \rangle \) denotes the averaging process. To calculate the energy-momentum of a gravitational wave, we must expand each team in \( W_{\mu\nu} \) up to the second order in \( h_{\mu\nu} \). The second-order Ricci tensor is

\[
R_{\mu\nu}^{(2)} = \partial_\sigma \Gamma^{(2)}_{\mu\nu\sigma} - \partial_\mu \Gamma^{(2)}_{\sigma\nu\sigma} + \Gamma^{(1)}_{\rho\mu} \Gamma^{(1)}_{\sigma\nu\rho} - \Gamma^{(1)}_{\rho\nu} \Gamma^{(1)}_{\sigma\mu\rho}, \tag{33}
\]

where \( \Gamma^{(1)}_{\mu\nu} \) is presented in equation \( 4 \) and \( \Gamma^{(2)}_{\mu\nu} \) is given by

\[
\Gamma^{(2)}_{\mu\nu} = -\frac{1}{2} h^{\sigma\lambda} (\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}). \tag{34}
\]

Since we average over all directions at each point, first derivatives average to zero: \( \langle \partial_\mu f \rangle = 0 \) for any function of position \( f \). This has the important consequence that \( \langle \partial_\mu f g \rangle = -\langle \partial_\mu g f \rangle \). Repeated application of this, we obtain the terms in \( W_{\mu\nu}^{(2)} \) as follow, respectively

\[
\langle (g_{\alpha\beta} \nabla^\alpha \nabla_\alpha R)(2) \rangle = \langle -\frac{1}{3} \eta_{\mu\nu} \partial^\sigma \partial_\beta h_{\alpha\beta} R^{(1)} + \frac{2}{3} \eta_{\mu\nu} \square h R^{(1)} + \square h_{\mu\nu} R^{(1)} + 2 \eta_{\mu\nu} \partial^\sigma \Gamma^{(1)}_{\mu\lambda\nu} \Gamma^{(1)}_{\nu\alpha\lambda} R^{(1)} \rangle, \tag{35}
\]

\[
\langle (\nabla^\alpha \nabla_\alpha R_{\mu\nu})(2) \rangle = \langle -\square h R^{(1)}_{\mu\nu} + \partial^\sigma \Gamma^{(1)}_{\mu\lambda\nu} \Gamma^{(1)}_{\alpha\sigma\lambda} R^{(1)} + \partial^\sigma \Gamma^{(1)}_{\nu\lambda\mu} R^{(1)} \rangle, \tag{36}
\]

\[
\langle (\nabla^\alpha \nabla_\mu R_{\alpha\nu})(2) \rangle = \langle -\partial_\nu \partial_\mu h^{\alpha\beta} R_{\nu\beta}^{(1)} + \partial_\lambda \Gamma^{(1)}_{\alpha\mu\nu} R_{\lambda\nu}^{(1)} + \partial_\mu \Gamma^{(1)}_{\alpha\nu\lambda} R_{\lambda\nu}^{(1)} + \eta^{\alpha\sigma} \partial_\mu \Gamma^{(1)}_{\nu\lambda} R_{\nu\lambda}^{(1)} \rangle, \tag{37}
\]

\[
\langle (\nabla_\mu \nabla_\nu R)(2) \rangle = \langle -\partial_\mu \partial_\nu h^{\alpha\beta} R_{\alpha\beta}^{(1)} + \partial_\lambda \Gamma^{(1)}_{\mu\nu\lambda} R_{\mu\nu}^{(1)} + 2 \partial_\mu \Gamma^{(1)}_{\alpha\sigma} R_{\alpha\nu}^{(1)} \rangle, \tag{38}
\]
\[ \langle (R_{\mu\alpha} R_{\nu}^\alpha)^{(2)} \rangle = \langle \eta^{\alpha\beta} R_{\mu\alpha}^{(1)} R_{\nu\beta}^{(1)} \rangle, \quad (39) \]

\[ \langle (g_{\mu\nu} R_{\mu\alpha} R_{\nu}^\alpha)^{(2)} \rangle = \langle \eta_{\mu\nu} \eta^{\alpha\beta} R_{\alpha\beta}^{(1)} R_{\mu\beta}^{(1)} \rangle, \quad (40) \]

\[ \langle (R R_{\mu\nu})^{(2)} \rangle = \langle R^{(1)} R_{\mu\nu}^{(1)} \rangle, \quad (41) \]

\[ \langle (g_{\mu\nu} R^2)^{(2)} \rangle = \langle \eta_{\mu\nu} R^{(1)2} \rangle, \quad (42) \]

Inserting the \( R^{(1)}, R_{\mu\nu}^{(1)}, \) and \( \Gamma^{(1)}_{\mu\nu} \) into these equations, and using the wave equations, the vanishing trace \( \bar{h} = 0 \), and the Lorenz gauge \( \partial^\mu \bar{h}^{\mu \nu} = 0 \), we obtain after a rather cumbersome calculation

\[ t_{\mu\nu} = -4a \langle W_{\mu\nu}^{(2)} \rangle \]

\[ = -4a \left\{ \left( -3 \partial_{\mu} \partial_{\nu} h_{\lambda\nu} - 3 \partial_{\nu} \partial_{\lambda} h_{\mu\nu} + \Box h_{\mu\nu} + \frac{5}{2} \partial_{\mu} \partial_{\nu} h \right) \partial^\alpha \partial^\beta h_{\alpha\beta} \right. \]

\[ \left. + \frac{1}{4} \partial_{\mu} \partial_{\alpha} h^{\alpha\lambda} \Box h_{\lambda\nu} + \frac{1}{4} \partial_{\nu} \partial_{\alpha} h^{\alpha\lambda} \Box h_{\lambda\mu} + \frac{1}{2} \partial_{\mu} \partial_{\nu} h^{\alpha\beta} \Box h_{\alpha\beta} - \frac{3}{2} \partial_{\mu} \partial_{\nu} h_{\alpha\lambda} \partial_{\alpha} \partial_{\beta} h^{\beta\lambda} \right\} \quad (43) \]

\[ = -4a \left\{ \left( -3 \partial_{\mu} \partial_{\nu} h_{\lambda\nu} - 3 \partial_{\nu} \partial_{\lambda} h_{\mu\nu} + \Box h_{\mu\nu} + \frac{5}{2} \partial_{\mu} \partial_{\nu} h \right) \partial^\alpha \partial^\beta h_{\alpha\beta} \right. \]

\[ \left. + \frac{1}{4} \partial_{\mu} \partial_{\alpha} h^{\alpha\lambda} \Box h_{\lambda\nu} + \frac{1}{4} \partial_{\nu} \partial_{\alpha} h^{\alpha\lambda} \Box h_{\lambda\mu} + \frac{1}{2} \partial_{\mu} \partial_{\nu} h^{\alpha\beta} \Box h_{\alpha\beta} - \frac{3}{2} \partial_{\mu} \partial_{\nu} h_{\alpha\lambda} \partial_{\alpha} \partial_{\beta} h^{\beta\lambda} \right\} \]

With this tensor, we can effectively determine the energy carried by gravitational waves.

**V. CONCLUSIONS AND DISCUSSIONS**

We have discussed the gravitational radiation in conformal gravity theory. We have linearized the field equations and obtain the wave equation after introducing the appropriate transformation for \( h_{\mu\nu} \). We have also derived the effective energy-momentum tensor for the gravitational radiation, which can be used to determine the energy carried by gravitational waves. The methods presented here can be applied to investigate other alternative gravity theories.

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