Elliptical invariance of distributions of the power type: the stability and extensivity issues

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Abstract. In this paper we delve into some important properties of probability distributions of the power type in order to provide some answers to questions recently raised in the literature. More precisely, we focus on the properties of maximizers of generalized information measures and give results about their stability under addition-composition processes.

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1. Introduction

Non-logarithmic information measures have become very fashionable nowadays, with multiple applications to different scientific disciplines (see, for instance, [1] and references therein). They were introduced in the cybernetic-information communities by Havrda-Charvat [2] in 1967 and Vadja [3] in 1968, and rediscovered by Daroczy in 1970 [4] with several echoes mostly in the field of image processing: see [5] for a historic summary and the pertinent references. In astronomy, physics, economics, biology etc..., these non-logarithmic information measures are often used under the form of the $q$–entropies as introduced by Tsallis since 1988 [6].

These entropies are maximized by power-type distributions. The properties of both discrete and continuous power-type distributions have been carefully reviewed recently in Ref. [7] in what respects to

(i) their behavior by convolution and
(ii) their relationships with stable Lévy distributions.

In this paper, we wish to focus attention more closely on further properties of these distributions, and answer some open questions as raised in [7]; this way, we hope, in the wake of Refs. [8, 9, 10], to positively contribute to a more complete understanding of the ensuing theoretical context.

2. Definitions and Notations

In what follows we consider some probability density $f_X(X \in \mathbb{R}^N)$ that maximize a generalized entropy, either of the Havrda-Charvat-Rényi type

$$H_q(X) = \frac{1}{1-q} \log \left( \int_{\mathbb{R}^n} f_X^q(X) \, dX \right).$$

(1)

or of the Tsallis type

$$S_q(X) = \frac{1}{q-1} \left( 1 - \int_{\mathbb{R}^n} f_X^q(X) \, dX \right).$$

(2)

where $q$ is a real parameter (called "nonextensivity parameter" in [11]). As $H_q$ can be expressed as an increasing function of $S_q$, both entropies have the same maximizers. As a consequence, all results expressed in this paper hold for both types of entropies, except in Section 6 that deals with a special property of $S_q$. To each density $f_X$, we associate its so-called escort distribution [11] defined as

$$F_X(X) = \frac{f_X^q(X)}{\int_{\mathbb{R}^n} f_X^q(X) \, dX}.$$

Note that the dependence of $F_X$ on $q$ is not explicitly stated for notational simplicity.
2.1. Power-law distributions as entropy-maximizers

The following theorem generalizes to the \( n \)-variate case the characterization given in Ref. [7, Eq. (42)] for the maximum entropy distributions with fixed \( q \)-covariance.

**Theorem 1** Under the \( q \)-covariance constraint

\[
\int XX^T F_X (X) dX = K
\]

(where the \( q \)-covariance matrix \( K \) is symmetric definite positive) and the normalization constraint \( \int f_X = 1 \), the power-law entropy (1) or (2) has a single maximizer equal to:

- if \( 1 < q < \frac{n+2}{n} \)

\[
f_X (X) = A_q \left( 1 + X^T \Lambda^{-1} X \right)^{\frac{1}{q}}
\]

with

\[
A_q = \frac{\Gamma \left( \frac{1}{q-1} \right)}{\Gamma \left( \frac{1}{q-1} - \frac{n}{2} \right) |\pi \Lambda|^{1/2}}, \quad \Lambda = mK, \quad m = \frac{2}{q-1} - n.
\]

- if \( q < 1 \)

\[
f_X (X) = A_q \left( 1 - X^T \Sigma^{-1} X \right)^{\frac{1}{q}}
\]

with

\[
A_q = \frac{\Gamma \left( \frac{2-q}{1-q} + \frac{n}{2} \right)}{\Gamma \left( \frac{2-q}{1-q} \right) |\pi \Sigma|^{1/2}}, \quad \Sigma = pK, \quad p = \frac{2}{1-q} + n
\]

and with notation \((x)_+ = \max (0, x)\).

In the case \( n = 1 \) we recover the results of [7, eq. (42)], namely

- if \( 1 < q < 3 \)

\[
f_X (x) = \frac{\Gamma \left( \frac{1}{q-1} \right)}{\Gamma \left( \frac{2-q}{1-q} \right) \sqrt{3 - q} \sqrt{q - 1}} \left( 1 + \frac{q-1}{3-q} \frac{x^2}{\sigma^2} \right)^{\frac{1}{1-q}}
\]

- if \( q < 1 \)

\[
f_X (x) = \frac{\Gamma \left( \frac{5-3q}{2(1-q)} \right)}{\Gamma \left( \frac{2-q}{1-q} \right) \sqrt{3 - q} \sqrt{q - 1}} \left( 1 - \frac{1-q}{3-q} \frac{x^2}{\sigma^2} \right)^{\frac{1}{1-q}}.
\]

Note the existence of a minor typo in [7] for the definition of \( A_q \) in the case \( q > 1 \) (replace \( 2/(1-q) \) by \( 2(1-q) \)). For the correct expression see also [12].
2.2. Student-t and Student-r distributions

In statistics, distribution (3) is called an \( n \)-variate Student-t with \( m \) degrees of freedom and \( q \)-covariance matrix \( K \); it will be denoted as \( \mathcal{T}(m, n, K) \) in the following. We notice that its nonextensivity parameter \( q \) is linked to the dimension \( n \) and the number of degrees of freedom \( m \) by

\[
q = \frac{m + n + 2}{m + n}.
\]

Moreover, convergence of both integrals \( \int f_X(X) \, dX \) and \( \int XX^T F_X(X) \, dX \) requires the same condition, namely \( q < \frac{n+2}{n} \), or equivalently \( m > 0 \). In the next section, we will endow parameter \( m \) with a meaning.

Accordingly, distribution (4) is an \( n \)-variate Student-r with \( p \) degrees of freedom and \( q \)-covariance matrix \( K \); it will be denoted as \( \mathcal{R}(p, n, K) \). We remark that its nonextensivity parameter \( q \) is linked to parameter \( p \) and dimension \( n \) as

\[
q = \frac{p - n - 2}{p - n}.
\]

2.3. Stochastic representations

Beck and Cohen \([11]\) have recently introduced in the literature an interesting statistical concept, baptized with the name superstatistics, that links different types of probability densities. In this vein, our distributions above can be shown to correspond to multivariate Gaussian densities whose covariance matrix fluctuates according to a certain law, as detailed in the two following theorems.

**Theorem 2** If \( X \) follows a \( \mathcal{T}(m, n, K) \) distribution then a stochastic representation of \( X \) writes \( \dagger \)

\[
X \sim \frac{\Lambda^{1/2} G}{a}
\]

where \( G \) is an \( n \)-variate Gaussian vector with unit covariance matrix, \( a \) is a random variable independent of \( G \) that follows a \( \chi \) distribution \( \S \) with number of degrees of freedom \( m = \frac{2}{q-1} - n \) and \( \Lambda = mK \).

A remarkable fact deserves here emphasizing upon: this approach can be extended to the case when \( q < 1 \), with a noticeable difference. This extension is based on the following duality result.

**Theorem 3** if \( X \sim \mathcal{T}(m, n, K) \) and \( \Lambda = mK \) then random vector \( Y \) defined as

\[
Y = \frac{X}{\sqrt{1 + X^T \Lambda^{-1} X}}
\]

\( \dagger \) in the following, sign \( \sim \) means “is distributed as”

\( \S \) a chi distribution is \( f_a(a) = \frac{a^{\frac{1}{2} - m/2}}{\Gamma(\frac{1}{2})} a^{m-1} \exp(-a^2/2) \); chi distributions are restricted to integer degrees of freedom. If \( m \notin \mathbb{N} \) then the \( \chi \) distribution should be extended to the distribution of the square-root of a gamma random variable with shape parameter equal to \( 2m \). For the sake of simplicity, we will speak of \( \chi \) distribution in this case too.
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is such that \( Y \sim \mathcal{R}(p, n, K) \) with

\[
p = m + n - 2.
\]

If \( q \) and \( q' \) denote the respective nonextensivity indices of \( X \) and \( Y \), then

\[
\frac{1}{1 - q'} = \frac{1}{q - 1} - \frac{n}{2} - 1.
\]

(9)

In Figure 1 below, values of \( q' \) as a function of \( q \) are plotted for \( n = 1, 2, 5 \) and 10 (right to left). We remark that transformation (8) induces a one-to-one relationship between \( q \in [1, \frac{n+4}{n+2}] \) and \( q' \in (-\infty, 1] \) and has the Gaussian distribution \( (q = q' = 1) \) as fixed point.

![Figure 1. \( q' \) as a function of \( q \) as in (9) for \( n = 1, 2, 5 \) and 10 (right to left)](image)

An important consequence of the above is the following dual result of theorem (2).

**Theorem 4** If \( Y \) follows a \( \mathcal{R}(p, n, K) \) distribution then a stochastic representation of \( Y \) writes

\[
Y \sim \frac{\Sigma^{1/2} G}{\sqrt{G^T G + b^2}}
\]

(10)

where \( G \) is an \( n \)-variate Gaussian vector with unit covariance matrix, where \( \Sigma = (p - n + 2) K \) and \( b \) is a random variable independent of \( G \) that follows a \( \chi \) distribution with \( p - n + 2 \) degrees of freedom.

Here, the important difference, as compared to the case \( q > 1 \), is to be found in the fact that the fluctuations, represented by the denominator of (10), are now dependent of the values of the Gaussian system through the presence of term \( G^T G \).

2.4. Covariance matrices

The covariance matrices \( R = EXX^T \) of both distributions are related to their \( q \)-covariance matrices as follows.
Theorem 5. Distribution $T(m, n, K)$ has covariance matrix

$$R = \frac{m}{m-2}K$$

provided $m > 2$, that is $q < \frac{n+4}{n+2}$. For example, a finite covariance matrix exists in the case $n = 1$ only if $1 < q < \frac{5}{3}$.

Proof

Using the stochastic representation (7), we deduce

$$EXX^T = EGG^T E \frac{1}{a^2},$$

with $Ea^{-2} = \frac{1}{m-2}$ and $EGG^T = \Lambda = mK$. □

Theorem 6. Distribution $R(p, n, K)$ has covariance matrix

$$R = \frac{p-n+2}{p+2}K.$$  (12)

Proof

The proof uses the polar factorization property [15] of stochastic representation (10), namely the fact that $G\sqrt{G^TG+b^2}$ and $\sqrt{G^TG+b^2}$ are independent. As a consequence

$$EYY^T = \frac{\Sigma_{1/2} EGG\Sigma_{1/2}}{E(G^TG+b^2)} = \frac{p-n+2}{p+2}K.$$ □

We note that in the Gaussian case ($p \to +\infty$ in (12) or $m \to +\infty$ in (11)), the $q$-covariance and the variance matrices coincide.

2.5. Geometric characterization

Geometric characterizations of both distributions (3) and (4) in terms of projections of the uniform distribution on the sphere in $\mathbb{R}^n$ are detailed in [14]. According to the stochastic representation (10), $\Sigma^{-1/2}Y$ can be interpreted, if $p \in \mathbb{N}$, as the marginal vector of a $(p+2)$-variate random vector uniformly distributed on the sphere in $\mathbb{R}^{p+2}$. A link between this observation and the role of extended information measures in the microcanonical framework can be found in [14].

3. The stability issue

As noted in [7], distributions (3) and (4) are not stable by convolution since they do not belong to the Lévy class: the sum of two independent random variables following either distribution (3) or distribution (4) does not follow any of these distributions again, as opposed to the Maxwellian-Gaussian case. It is then suggested in [7] that, in order to recover the original distribution after summation, a certain kind of dependence should be introduced between the components of the sum.

It is the aim of the next paragraph to show that such dependence can be accurately characterized in the case of power-law distributions.
3.1. A first example: case $q > 1$

Let us assume, for instance, that $q > 1$ and choose $X$ to be a random vector of dimension $n$ distributed according to (3). We extract from it two scalar components, say $X_1$ and $X_2$; according to (7), these two components can be expressed as

$$X_1 \sim \left( \frac{\Lambda^{1/2}G}{a} \right)_1, \quad X_2 \sim \left( \frac{\Lambda^{1/2}G}{a} \right)_2$$

(13)

where $(.)_1$ denotes the first vector component.

Distribution of the components

We first remark from stochastic representation (13) that $X_1$ and $X_2$ are again distributed according to a Student-t distribution with dimension $n = 1$; moreover, the extraction of components keeps the fluctuation variable $a$ unchanged, so that both $X_1$ and $X_2$ have unchanged number of degrees of freedom $m' = m = \frac{2}{q-1} - n$. Both have thus a new nonextensivity parameter $q'$ that verifies

$$2 \frac{q' - 1}{q' - 1 - n} = \frac{2}{q - 1} - n$$

or equivalently

$$q' = 1 + \frac{2(q - 1)}{2 + (1 - q)(n - 1)}.$$  

(14)

Moreover, it is easy to check that their respective $q$—variances are $K_{11}$ and $K_{22}$, the two first diagonal entries of $q$—covariance matrix $K$. The three curves in Figure 2 represent $q'$ as a function of $q$ for $n = 2, 5$ and 10 (from right to left).

![Figure 2](image-url)

**Figure 2.** $q'$ as a function of $q$ as in (14) for $n = 2, 5$ and 10 (right to left)

We note that

- $q = 1 \Rightarrow q' = 1$, since any component of a Gaussian vector is Gaussian
- the nonextensivity parameter $q'$ of a single component is larger than the nonextensivity parameter $q$ of the system it is extracted from
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- moreover, $q'$ is all the larger since the dimension $n$ is large.

**Distribution of the convolution**

The distribution of a linear combination $Z$ of $X_1$ and $X_2$ can be computed as

$$Z = \alpha X_1 + \beta X_2 \sim \frac{1}{a} \left( \alpha \left( \Lambda^{1/2} G \right)_1 + \beta \left( \Lambda^{1/2} G \right)_2 \right)$$

$$\sim \sqrt{m \sqrt{\alpha^2 K_{11} + \beta^2 K_{22} + 2\alpha\beta K_{12}} a},$$

(15)

so that $Z$ is again distributed as a Student-t distribution with same parameter $m$ and $q-$variance $\alpha^2 K_{11} + \beta^2 K_{22} + 2\alpha\beta K_{12}$. We underline the fact that stability under convolution originates from the special type of dependence that exists between the components $X_1$ and $X_2$, namely from the fact that they belong to a same (larger) system: in more physical terms, $X_1$ and $X_2$ are components that have experienced the same random source of fluctuations.

**3.2. A second example: case $q < 1$**

We assume now that we extract two components $Y_1$ and $Y_2$ from a vector $Y \sim \mathcal{R}(p, n, K)$. Then a stochastic representation of $Y_1$ and $Y_2$ is

$$Y_1 \sim \left( \Sigma^{1/2} G \right)_1 \sqrt{G^T G + b^2}, \quad Y_2 \sim \left( \Sigma^{1/2} G \right)_2 \sqrt{G^T G + b^2}$$

so that $Y_1$ (resp. $Y_2$) follows a distribution $\mathcal{R}(p', 1, K')$ with $p' = p$, $K' = K_{11}$ (resp. $K' = K_{22}$) and its new index of nonextensivity verifies

$$\frac{2}{1 - q'} + 1 = \frac{2}{1 - q' + n}$$

or

$$q' = 1 - \frac{2 (1 - q)}{2 + (n - 1) (1 - q)}.$$  \hspace{1cm} (16)

We remark that (16) coincides with (14) since conservation of degrees of freedom $m$ in the Student-t case and $p$ in the Student-r case is expressed by the same condition.

In Figure below, $q'$ is represented as a function of $q$ for $n = 2, 5$ and $10$ (bottom to top).

The same conclusions as in the case $q > 1$ hold, namely:

- if $q = 1$ then $q' = 1$ (Gaussian case)
- the nonextensivity parameter $q'$ of an extracted component is always larger than the nonextensivity parameter of the original system; it is all the larger since the dimension $n$ is large

The distribution of a linear combination can be evaluated as

$$Z = \alpha Y_1 + \beta Y_2 \sim \frac{\alpha \left( \Sigma^{1/2} G \right)_1 + \beta \left( \Sigma^{1/2} G \right)_2 \sqrt{G^T G + b^2}}{\sqrt{G^T G + b^2}}$$

$$\sim \sqrt{p} \sqrt{\alpha^2 K_{11} + \beta^2 K_{22} + 2\alpha\beta K_{12}} \sqrt{G^T G + b^2}.$$

(17)

so that $Z \sim \mathcal{R}(p, 1, \alpha^2 K_{11} + \beta^2 K_{22} + 2\alpha\beta K_{12})$. 

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3.3. Orthogonal invariance

These results can be generalized using the notion of elliptical distribution \[13\].

**Definition 7** A distribution \( f_X \) is elliptical (or elliptically invariant) if it writes

\[
f_X(X) = \phi\left(X^T C_X^{-1} X\right)
\]

for some positive definite matrix \( C_X \) called the characteristic matrix of \( f_X \) and some function \( \phi \) that may depend of \( n \).

From (3) and (4), we check immediately that Student-t and Student-r distributions are elliptically invariant. This special property can be justified as follows: up to application of the mapping \( X \rightarrow \Lambda^{1/2} X \) or \( Y \rightarrow \Sigma^{1/2} Y \), it may be assumed in (3) and (4) that \( \Lambda = I_n \) or \( \Sigma = I_n \): this special case of elliptical invariance is called spherical invariance. An equivalent definition of spherical invariance reads as follows: for all orthogonal matrices \( O \), the distribution of \( X \) coincides with the distribution of \( X \):

\[
f_{OX}(X) = f_X(X).
\]

Now, the \( H_q \) or \( S_q \)-entropy remains unchanged by orthogonal transformation since, for example

\[
S_q(OX) = \frac{1}{q-1} \left( 1 - \int f_{OX}^q \right) = \frac{1}{q-1} \left( 1 - \frac{1}{|O|^{q-1}} \int f_X^q \right) = S_q(X)
\]

(18)

where we have used the fact that for any orthogonal matrix, \(|O| = 1\). Moreover, the constraints under which the \( S_q \)-entropy is maximized, that is

\[
\int XX^T F_q(X) \, dX = I_n, \quad \int f(X) \, dX = 1
\]

are themselves spherically invariant as well. Thus, it is not surprising that the maximizer of \( S_q \) under these constraints is spherically invariant - and elliptically invariant in the more general case \( C_X \neq I_n \).
3.4. Properties of elliptical distributions and consequences

The stability property exposed in parts 3.1 and 3.2 appears as a particular case of the more general property of elliptical distributions that we cast here as follows:

**Theorem 8** If \( X \) is distributed according to an elliptical distribution
\[
f_X(X) = \phi \left( X^T C_X^{-1} X \right)
\]
and if \( A \) is a \((p \times n)\) full-rank matrix with \( p \leq n \) then \( \tilde{X} = AX \) is again elliptically invariant with characteristic matrix
\[
C_{\tilde{X}} = AC_X A^T.
\]

(19)

As a consequence, one can characterize the precise way in which power-law random vectors behave under linear transformation as follows.

**Case of components’ extraction**

Suppose we extract the \( k < n \) first components \( X' = (X_1, \ldots, X_k) \) from a vector of the power-law type \( X \sim T(m, n, K) \). This process corresponds to applying the matrix
\[
A = \begin{bmatrix} I_{k \times k} : O_{(n-k) \times k} \end{bmatrix}
\]
to vector \( X \), and we conclude that \( X' = AX \sim T(m', k, K') \) where \( K' = AKA^T \) coincides with the principal \((k \times k)\) block of \( K \) and \( m' = m \), corresponding to a new index of extensivity
\[
q' = 1 + \frac{2 (q - 1)}{2 - (n - k) (q - 1)} > 1.
\]

(20)

For a power law vector \( Y \sim R(p, n, K) \), as remarked in part 3.2, conservation of the number \( p \) of degrees of freedom yields the same condition as conservation of the number \( m \) of degrees of freedom in the Student-t case, that is
\[
\frac{2}{1 - q'} + k = \frac{2}{1 - q} + n
\]
or
\[
q' = 1 - \frac{2 (1 - q)}{2 + (1 - q) (n - k)} < 1.
\]

(21)

In both (20) and (21), \( q = 1 \Rightarrow q' = 1 \) in (21), yielding the classical property of Boltzmann systems, any subsystem of which is still of the Boltzmann type.

**Case of convolution**

Choosing \( A = [a_1, a_2, \ldots, a_n] \) in (19) yields the following results:
\[
X \sim T(m, n, K) \Rightarrow \sum_{i=1}^{n} a_i X_i \sim T(m, 1, AK A^T)
\]
\[
Y \sim R(p, n, K) \Rightarrow \sum_{i=1}^{n} a_i Y_i \sim R(p, 1, AK A^T)
\]

We note again that this stability result requires a special type of dependence between the components \( \{X_i\} \) or \( \{Y_i\} \) namely the fact that they are extracted from the same system.
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4. The stability issue for independent vectors

Few results exist about the convolution of two independent Student-t or Student-r vectors. In Ref. [16], Oliveira et al. remark that if \( X_1 \) and \( X_2 \) are independent and \( T(m, 1, \sigma) \) distributed, their sum

\[
Z = X_1 + X_2
\]

can be very accurately approximated as a \( T(m', 1, \sigma') \) for some \( (m', \sigma') \) depending on \( (m, \sigma) \). However, they provide only an approximation to the map \( (m, \sigma) \rightarrow (m', \sigma') \).

An important result can be stated when \( q > 1 \), in the special case for which the number \( m \) of degrees of freedom is an odd integer \( m = 2l + 1 \).

**Theorem 9** [8] If \( X_1 \) and \( X_2 \) are two independent vectors following a distribution \( T(2l + 1, n, I_n/(2l+1)) \) and if \( \alpha \) is such that \( 0 \leq \alpha \leq 1 \) and \( \beta = 1 - \alpha \), then the distribution of

\[
Z = \alpha X_1 + \beta X_2
\]

can be expressed as

\[
f_Z(Z) = \sum_{k=l}^{2l} \gamma_k^{(2l+1)}(\alpha) T(2k + 1, n, I_n/(2k+1))
\]  

(22)

with

\[
\gamma_k^{(l)}(\alpha) = (4\alpha (1 - \alpha))^k \left( \frac{l!}{(2l)!} \right)^2 2^{-2l} (2l - 2k)! (2l + 2k)! \\
\times \sum_{j=0}^{l-k} \left( \frac{2l + 1}{2j} \right) \left( \frac{l-j}{k} \right) (2\alpha - 1)^{2j}, \ 0 \leq k \leq l
\]

Since coefficients \( \gamma_k^{(l)} \) are positive and sum to 1 (see [8] for a proof), this result can be interpreted as follows: the convolution of \( T \) distributions with odd degrees of freedom follows a \( T \) distribution whose degrees of freedom are randomized:

\[
f_Z(Z) = T(2l + 2K + 1, n, I_n/(2l+2K+1))
\]  

(23)

where \( K \) is a random variable defined as

\[
\Pr \{ K = k \} = \gamma_k^{(l)}(\alpha), \ 0 \leq k \leq l
\]

As an example, if \( n = 1 \) and \( m = 3 \Rightarrow l = 1 \), we have

\[
f_Z(z) = \gamma_1^{(3)} T(3, 1, 1/3) + \gamma_2^{(3)} T(5, 1, 1/5)
\]

with

\[
\gamma_1^{(3)} = 1 - 3\alpha (1 - \alpha), \ \gamma_2^{(3)} = 3\alpha (1 - \alpha).
\]

We note that conditions \( 0 \leq \alpha \leq 1 \) and \( \alpha + \beta = 1 \) are not restrictive since

- if \( \alpha < 0 \), then by parity of \( T(n, m, K) \), \( \alpha Y_1 \sim (-\alpha) Y_1 \)
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- if \( \alpha + \beta \neq 1 \), then \( \alpha Y_1 + \beta Y_2 \sim (\alpha + \beta) \left( \frac{\alpha}{\alpha + \beta} Y_1 + \frac{\beta}{\alpha + \beta} Y_2 \right) \).

An important result is the following one: formula \((22)\) can be extended to the case where \( X_1 \sim \mathcal{T}(m, n, K) \) and \( X_2 \sim \mathcal{T}(m, n, K) \) provided \( X_1 \) and \( X_2 \) have the same \( q \)-covariance matrix \( K \): in that case, \( K^{-1/2} X_1 \) and \( K^{-1/2} X_2 \) have identity \( q \)-covariance, so that the distribution of \( K^{-1/2} (X_1 + X_2) \) can be computed using formula \((22)\) and distribution of \( X_1 + X_2 \) can be obtained by a simple change of variable.

5. Another approach to the stability issue: random convolution

A radically different approach to the problem we are discussing here, namely, the conditions of stability for power-type distributions, can be followed in the case \( q < 1 \) by considering the polar factorization property of the stochastic representation \((10)\).

**Theorem 10** If \( Y \) has stochastic representation

\[
Y = \frac{\Sigma^{1/2} G}{\sqrt{G^T G + b^2}}
\]

where \( G \) is a Gaussian vector with unit covariance matrix and \( b \) is \( \chi \) distributed, independent of \( G \), then \( Y \) is independent of \( \sqrt{G^T G + b^2} \); we remark that the later is \( \chi \) distributed with \( p + 2 = \frac{2}{1-q} + n + 2 \) degrees of freedom.

An important consequence of this property is that it allows to derive a new kind of convolution of random type, as expressed by the next theorem \((11)\).

**Theorem 11** If \( Y_1 \sim \mathcal{R}(p, n, K_1) \) and \( Y_2 \sim \mathcal{R}(p, n, K_2) \) are two independent vectors, if \( \alpha_1 \) and \( \alpha_2 \) are two real scalars and \( \alpha_1 \) and \( \alpha_2 \) are two independent \( \chi \) random variables with \( d = p + 2 = \frac{2}{1-q} + n + 2 \) degrees of freedom, and if \( \beta_1 = \frac{\alpha_1}{\sqrt{p+2}} \alpha_1, \beta_2 = \frac{\alpha_2}{\sqrt{p+2}} \alpha_2 \) then vector

\[
Y = \beta_1 Y_1 + \beta_2 Y_2
\]

is Gaussian with \( q \)-covariance matrix \( R = \frac{p-n+2}{p+2} (\alpha_1^2 K_1 + \alpha_2^2 K_2) \). Moreover, if \( c \) is \( \chi \) distributed with \( p-n+2 \) degrees of freedom and independent of \( Y \), then

\[
Z = \frac{Y}{\sqrt{Y^T R^{-1} Y + c^2}} = \frac{\beta_1 Y_1 + \beta_2 Y_2}{\sqrt{(\beta_1 Y_1 + \beta_2 Y_2)^T R^{-1} (\beta_1 Y_1 + \beta_2 Y_2) + c^2}}
\]

is again \( \mathcal{R}(p, n, K) \) distributed with \( q \)-covariance matrix

\[
K = \frac{1}{p+2} (\alpha_1^2 K_1 + \alpha_2^2 K_2) .
\]

In Figure 4 below, the distribution of \( \beta_1 = \alpha_1 \sqrt{\frac{q}{p+2}} \) is represented for \( p + 2 = 10, 20 \) and 50, and \( \alpha_1 = 2 \).

It is clearly seen that \( \beta_1 \) is a "fluctuating" version of the deterministic value \( \alpha_1 = 2 \); since

\[
E \beta_1 = \alpha_1 \sqrt{\frac{2}{p+2} \Gamma \left( \frac{p+3}{2} \right) ,}
\]

we have \( \lim_{p \to +\infty} E \beta_1 = \alpha_1 \); moreover, the variance of \( \beta_1 \) is

\[
\text{var} (\beta_1) = \alpha_1^2 \left( 1 - \frac{2}{p+2} \Gamma^2 \left( \frac{p+3}{2} \right) \right),
\]

(25)
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Figure 4. distribution of $\beta_1 = \alpha_1 \frac{a_1}{\sqrt{p+2}}$ for $p + 2 = 10, 20$ and $50$, and $\alpha_1 = 2$

so that $\lim_{p \to +\infty} \text{var} (\beta_1) = 0$: thus the number of degrees of freedom $p + 2$ - imposed by the value of $q$ that characterizes $Y_1$ and $Y_2$ through $p = n + \frac{2}{1-q}$ - rules the fluctuation intensity of $\beta_1$ around the deterministic value $\alpha_1$.

6. The Extensivity Issue

Still a different and important question was raised in [7], namely the extensivity issue: assuming that a system $A$ is composed of two independent subsystems $A_1$ and $A_2$, the total $q$–entropy

$$S_q (A) = S_q (A_1 \times A_2) = S_q (A_1) + S_q (A_2) + (1 - q) S_q (A_1) S_q (A_2)$$

is nonextensive (i.e. nonadditive) unless $q = 1$, which characterizes the Shannon entropy $\parallel$. A natural question arises then: what kind of dependence should exist between subsystems $A_1$ and $A_2$ so that $S_q$ becomes extensive?

An answer has been given to this question in the case of Gaussian systems, as follows [17].

Theorem 12 If $0 \leq Q \leq 1$ and $n \in \mathbb{N}$ then there exists a positive definite matrix $K$ and an $n$–variate Gaussian vector $X$ with covariance matrix $K$ such that $X$ verifies the extensivity condition

$$S_Q (X) = \sum_{i=1}^{n} S_Q (X_i) .$$

(26)

Trying to extend this result to the distributions (3) and (4), one should be careful about the following fact: if $X$ is an $n$–variate random vector with probability density (3) or (4) and non-extensivity parameter $q$ then any single component, say $X_1$, of $X$ is again of the power type, but with a different nonextensivity parameter, say $q_1$, related to $q$ via (20) or equivalently (21):

$$q_1 = 1 + \frac{2(q - 1)}{2 - (q - 1) (n - 1)}$$

|| this paragraph concerns only Tsallis entropy $S_q$ since the Harvda-Charvat-Rényi entropy $H_q$ is extensive
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Thus, the choice of $Q$ as related to $q$ and $q_1$ should be decided. The choice $Q = 2 - q$ has a long history in the nonextensive literature and already appeared in the paper [18] - for a thorough discussion of the issue and its physical interpretation see [19]. This choice yields the following result.

**Theorem 13** \( \forall m > 1 \) and \( n \in \mathbb{N} \), there exists a positive definite matrix $K$ and an $n$-variate Student-t vector $X$ with $m$ degrees of freedom and $q$-covariance matrix $K$ such that

\[
H_{Q_n}(X) = \sum_{i=1}^{n} H_{Q_1}(X_i)
\]

with \( q_1 = \frac{2q+(1-q)(n-1)}{2+(1-q)(n-1)} \), \( Q_n = 2 - q \) and \( Q_1 = 2 - q_1 \).

This result can be extended to the case $q < 1$ as follows.

**Theorem 14** \( \forall p > 1 \) and \( n \in \mathbb{N} \), there exists a positive definite matrix $K$ and an $n$-variate Student-r vector $Y$ with $p$ degrees of freedom and $q$-covariance matrix $K$ such that

\[
H_{Q_n}(Y) = \sum_{i=1}^{n} H_{Q_1}(Y_i)
\]

with \( q_1 = \frac{2+(n-3)(1-q)}{2+(n-1)(1-q)} \), \( Q_n = 2 - q \) and \( Q_1 = 2 - q_1 \).

7. Conclusion

In this communication we have presented several results concerning (i) the stability and (ii) the extensivity of power-law random vectors. We have shown that a certain kind of dependence between the components of these vectors, namely the fact that they belong to a larger system that is itself distributed à la power-law, ensures stability of these variables. This property is a direct consequence of the elliptical invariance of the associated $S_q$ or $H_q$ entropy.

In the case of independent components, we have introduced a random-type convolution that ensures stability for the power law distributions.

Finally, we have shown that $S_q$ can be additive if a proper kind of correlation is introduced between the components of the pertinent system, whose properties are to be described by power-law vectors. Further work in progress concerns the extension of this last result to the larger family of elliptically invariant distributions.

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