What does an experimental test of quantum contextuality prove or disprove?

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Abstract

The possibility of experimentally testing the Bell–Kochen–Specker theorem is investigated critically, following the demonstrations by Meyer, Kent, and Clifton–Kent that the predictions of quantum mechanics are indistinguishable (up to arbitrary precision) from those of a non-contextual model, and the subsequent debate about the extent to which these models are actually classical or non-contextual. The present analysis starts from a careful consideration of these ‘finite-precision’ approximations. A stronger condition for non-contextual models, dubbed ontological faithfulness, is exhibited. It is shown that this allows us to approximately formulate the constraints in Bell–Kochen–Specker theorems, such as to render the usual proofs robust. Consequently, one can experimentally test to finite precision ontologically faithful non-contextuality, and thus experimentally refute explanations from this smaller class. We include a discussion of the relation of ontological faithfulness to other proposals to overcome the finite precision objection.

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Attempts at proving that quantum mechanics is fundamentally non-classical go back to its very beginning. They start with the Copenhagen school’s claims of the necessity of complementarity [1, 2], to von Neumann’s proof [3], via Einstein–Podolsky–Rosen’s attempt to show the incompleteness of quantum theory [4], and on to an intellectual culmination in the work of Bell [5, 6], building on Gleason [7], as well as Kochen and Specker [8, 9]. But the subject is full of vigor even today. We are currently witnessing a renewed evaluation of the foundations of quantum mechanics [10–14], which is now even reaching out to applications such as device independent quantum cryptography [15, 16].
The work of Bell marked a particular turning point for the discourse on the non-classical nature of quantum mechanics, in two ways. On the one hand, it showed the need for clear, operationally motivated criteria for the classicality of a theoretical explanation. On the other hand, it demonstrated operational differences between the quantum mechanical predictions and those of any theory based on classical hidden variable in the former sense. This opened the way for experimental tests of quantum mechanics at an unprecedented level, from quantum violations of Bell inequalities [17–19] to experimental verification of quantum contextuality [20–24].

But just as these various no-go theorems have inspired the thinking of physicists, so have their refutations, or nullifications, to use the term of Meyer [25]. Here we shall take the view of no-go theorems as actually true mathematical theorems, hence a refutation, rather than being the demonstration of a mathematical error, which consists in showing that a tacit, yet not necessarily plausible assumption was made.

The present paper focuses on the Bell–Kochen–Specker (BKS) theorems on the impossibility of a non-contextual hidden variable explanation of the predictions of quantum mechanics. We shall essentially stick to the original viewpoint of these authors, owing to Bell’s 1966 review [6] (which appeared in print later, but which actually predates the 1964 non-locality paper [5], whose semi-centenary is celebrated in the present special issue). Non-contextuality is a property of hidden variable theories, assigning ‘true’ outcomes to all or some observables in a quantum mechanical system relative to a given state. At the risk of laboring an obvious point, recall that for quantum theory it does not matter how these observations are made, as long as they lead to a well-defined POVM—and there will always be very different-looking procedures, involving separate systems, different quantum information carriers, or temporal orderings in which the outcome is generated step-by-step. However, with regard to the hypothetical hidden variable description, assumptions have to be made regarding the relation of the variables attached to different or even incompatible observables, or relating them to different measurement procedures. The locality of the hidden variables is one of these conditions [5, 17], assuming a multi-partite quantum system with space-like separated observers. Non-contextuality can be viewed as a more ‘bare bones’ and abstract condition that, in particular, does not require a multi-partite system. It is important to observe that the assumption of locality imposes non-contextuality on certain sets of hidden variables, this fact is noticed repeatedly (for a recent and exhaustive discussion see e.g. [26]). Conversely, Bell–Kochen–Specker proofs of quantum contextuality have inspired Bell inequalities for non-locality [27].

Since its inception, the notion of non-contextuality has been deeply examined in a variety of forms. In particular, the work of Spekkens [28] is noteworthy in that it identified several distinct aspects of contextuality and because it argued the necessity to have operational, theory-independent definitions of the basic terms. A recent paper [29] has taken up this challenge for the notion of ‘sharp measurement’. (Non-)contextuality is also a recent popular topic in the ongoing quest for physical axioms limiting the range of conceivable probabilistic theories. In particular, Local Orthogonality [30] and Consistent Exclusivity [31–36], were anticipated by Specker [9], cf [37].

The present paper is structured as follows: section 1 gives an account of the usual (infinite precision) BKS theorems in terms of non-contextual inequalities, which is followed by the objections raised by Meyer, Kent and Clifton-Kent (MKC) in section 2 due to finite precision of any realistic experiment. After this, we present some reflections on the necessity of being able to identify outcomes in different experiments (section 3) as being in a certain sense ‘the same’. The central part is section 4, where the notion of ontologically faithful non-contextuality (ONC) is introduced, which is designed to reflect the finite precision of actual
experiments in the supposed hidden variable theory. We then show that this notion gives substance to experimental tests of quantum contextuality, and concretely makes non-contextual inequalities robust to finite precision. In section 5, we discuss our results and make a comparison with other approaches, in particular one based on the sequential execution of measurements.

1. Bell–Kochen–Specker theorems and non-contextual inequalities

Traditionally, Bell–Kochen–Specker (BKS) proofs start from a collection \((P_i; i \in V)\) of projectors on a finite-dimensional Hilbert space. Each subset \(M \subset V\) of indices, with the property that \(\sum_{i \in M} P_i = 1\), describes a measurement, more precisely a von Neumann measurement, with possible outcomes labeled by \(M\). Given a quantum state \(\rho\), the Born rule then prescribes the probabilities of outcomes for each measurement \(M\) that can be formed by collecting projectors \(P_i\):

\[
\text{Pr}(i|\rho) = \text{Tr} \rho P_i.
\]  

(1)

The remarkable thing is that this probability depends only on the outcome \(i\), more precisely on the projector \(P_i\) (given the state, which we consider fixed in our discussion).

A classical non-contextual hidden variable model for this scenario is meant to reveal pre-existing values of the measurements but in such a way that the indicator \(X_i \in \{0, 1\}\) depends on the label \(i\) only and not whether \(P_i\) appears in a measurement \(M\) or in another measurement \(M'\)—the different contexts. Here, \(X_i = 1\) indicates that \(i \in M\) is the outcome if \(M\) is measured. The model reproduces the predictions of quantum theory if \(\mathbb{E}X_i = \text{Pr}(i|\rho) = \text{Tr} \rho P_i\) is in accordance with the Born rule (1).

Clearly, a necessary requirement for the possibility of such a model is that for all possible measurements \(M \subset V\), that is \(\sum_{i \in M} P_i = 1\),

\[
\sum_{i \in M} X_i = 1.
\]  

(2)

In other words: the assignment \(X_i\) picks one and only one ‘real’ outcome for each measurement \(M\). To be painfully precise, the relation (2) should hold with probability 1 (allowing for inequality on an event of probability 0), but we can ignore this detail for the present discussion. Recall that a random variable is a (measurable) function \(X_i: \Omega \to \mathbb{R}\) from a probability space \(\Omega\) with a probability distribution \(\mu\) (where there is implicit the \(\sigma\)-algebra of events; cf [38], or any other modern textbook on probability for the basic terms). Since \(X_i\) only takes values 0 and 1, it is equivalently described by the sets (events in probability jargon)

\[
E_i = X_i^{-1}(1) = \{\omega \in \Omega: X_i(\omega) = 1\},
\]

so that \(X_i = 1_{E_i}\) is the indicator function of \(E_i\). The condition (2) is then equivalent to

\[
\bigcup_{i \in M} E_i = \Omega,
\]  

(3)

where the notation on the left hand side refers to the disjoint union. In other words, \((E_i; i \in M)\) is a set partition of \(\Omega\).

Kochen and Specker [8], and later many other authors (for a selection see [39, 40] and references therein), have found sets of projectors \(P_i\) in three- and higher-dimensional Hilbert spaces, for which these conditions are contradictory: there exists no non-contextual hidden-variable model satisfying either of equations (2) or (3).
In these works, the concept of coloring is central, which is the evaluation of the $X_i$ on a single point $\omega \in \Omega$: $X_i(\omega) \in \{0, 1\}$ indicates whether $i$ is singled out or not. The coloring rule states that in every measurement $M$ one must choose one and only one element. Note that a 0-1-coloring is a special case of our above notion of a classical non-contextual hidden variable model; namely, when $X_i$ takes values 0 or 1 with unit probability. Conversely, a probability distribution over colorings gives rise to a classical non-contextual hidden variable model.

With respect to these colorings, we strongly suggest to take a probabilistic point of view, as explained above, since the aim of a hidden variable theory is not merely a logically consistent assignment of values (surely a necessary condition) but the explanation of observed statistical data. Indeed, a broader approach, also leaving a role for the quantum state, is to consider within the set $V$ subsets $C$ (which we shall call contexts) such that $\sum_{i \in C} P_i \leq 1$; that is, rather than demanding that $C$ describes a measurement, we only require that it can be completed to one. The set of all contexts, denoted $\Gamma$, is a collection of subsets of $V$ (aka hypergraph), and now a non-contextual hidden variable model only has to satisfy

$$\forall \text{ contexts } C \in \Gamma \quad \sum_{i \in C} X_i \leq 1. \quad (4)$$

With this small modification every set of projectors, with the associated collection $\Gamma$ of contexts, has a non-contextual hidden variable model, although there are differences in the attainable expectation values $(t_i = \Pr\{i\}; i \in V)$, be it as quantum expectations $(\text{Tr} \rho P_i; i \in V)$ or as classical expectations $(EX_i; i \in V)$.

For instance, in a traditional BKS proof, where $\sum_{i \in C} P_i = 1$ for each context $C \in \Gamma$, we know that $\sum_{i \in C} X_i \leq 1$, but we cannot reach equality in all of the $C \in \Gamma$, hence

$$\sum_{C \in \Gamma} \sum_{i \in C} EX_i \leq |\Gamma| - 1. \quad (5)$$

By contrast, in fact quantum mechanics attains $|\Gamma| = \sum_{C \in \Gamma} \sum_{i \in C} \langle P_i \rangle$, for every state! Thus, BKS proofs can be interpreted as statements concerning the (im-)possibility of realizing certain constraints among projector effects in the quantum case, as well as events in the classical case.

More generally, we may consider non-contextual inequalities of the form

$$\sum_{i \in V} \lambda_i \sum_{C \in \Gamma} \sum_{i \in C} X_i \leq \beta_{cl}, \quad (6)$$

where $\lambda_i \geq 0$ are certain coefficients and $\beta_{cl}$ is the maximum of the lhs over all non-contextual hidden variable models; that is, $X_i \in \{0, 1\}$ satisfying equation (4). By substituting quantum expectations, $\sum_{i} \lambda_i \text{Tr} \rho P_i$, may exceed the classical limit $\beta_{cl}$ up to a quantum maximum of $\beta_{\text{qu}}$. A logically consistent way to think about these structures is to start with the set $V$ of outcomes and the contexts, that is with the hypergraph $\Gamma$—this sets the scene for the possible experiments we may want to describe and their outcomes (crucially identifying outcomes in different experiments as being the same; we will return to this point later). In this way, we can divorce the logic of speaking about the experiments from the standpoint of the theory that we believe or hypothesize to underly the experiment. For instance, a classical, non-contextual model for this abstract structure is given by 0-1-variables $X_i$, satisfying (4), while a quantum model is a collection of projectors $P_i$ with
\[ \forall C \in \Gamma \quad \sum_{i \in C} P_i \leq 1. \] (7)

Note that classical models for \( \Gamma \) are a special case of quantum models, where all \( P_i \) commute. See [41], where this formalism was developed further (also including more general probabilistic theories with convex sets of states and linear functions on states as effects). Although in [33] this framework was developed even further, they did so by returning to the equality conditions (2) and \( \sum_{i \in C} P_i = 1 \). To justify the emphasis on linear inequalities, observe that

\[ E(\Gamma) = \{ (EX_i_i) : (X_i_i) \text{ non-contextual HV model} \}, \]

\[ \Theta(\Gamma) = \{ (\text{Tr } \rho P_i) : (P_i) \text{ quantum model, } \rho \text{ state} \}, \]

are convex, the first being in fact a polytope, cf [33, 41, 42].

**Remark 1.** Since we are dealing with projectors, the condition \( \sum_{i \in C} P_i \leq 1 \) is evidently equivalent to \( P_i P_j = 0 \) (i.e. orthogonality of their supports) for all \( i \neq j \) occurring jointly in some context \( C \ni i, j \).

Likewise, for a classical, non-contextual model, the condition \( \sum_{i \in C} X_i \leq 1 \) is equivalent to \( X_i X_j = 0 \) for all \( i \neq j \) occurring jointly in some context \( C \ni i, j \).

This relation defines a graph on \( V \), with an edge \( \sim i j \) if, and only if, there is a \( C \in \Gamma \) with \( i, j \in C \). It is known as exclusivity graph [33, 43].

Rather than dwelling more on the abstract formalism, let us look at an example [44], which is indeed the one that inspired the general hypergraph approach [41]:

**Example 2.** For 5 outcomes \( V = \{0, 1, 2, 3, 4\} \) and contexts \( \Gamma = \{01, 12, 23, 34, 40\} \), there is a well-known quantum realization by 5 rank-one projectors \( P_i \) in three-dimensional Hilbert space. This means that \( P_i P_{i+1} = 0 \) for all \( i \), where \( i + 1 \) is understood mod 5. Notice that from these projectors many POVMs can be built, the simplest ones being the binary measurements \( (P_i, 1 - P_i) \), but because this and \( (P_{i+1}, 1 - P_{i+1}) \) are compatible, we also have \( (P_i, P_{i+1}, 1 - P_i - P_{i+1}) \).

However, there can be many other quantum models, including higher-rank projectors in higher dimensions. For a suitably chosen state \( \rho \) (and interestingly not any state), we can achieve \( \sum_{i} \text{Tr } \rho P_i = \sqrt{5} \) [44, 45], which is indeed the quantum maximum, \( \beta_{\text{qu}} = \sqrt{5} \) [41].

On the other hand, it is straightforward to check that the maximum of \( \sum_{i=0}^{4} EX_i \) is \( \beta_{\text{cl}} = 2 \).

The above translation, from an unsatisfiable set of logical constraints to a limitation on expectation values, all of which are in principle observable quantities, is significant. It elevates (and generalizes) BKS proofs to experimentally testable propositions, or so it would seem. Indeed, non-contextual hidden variable models impose a bound \( \beta_{\text{cl}} \) on the expectation value of \( \sum_{i} X_i \), equation (6), which is violated by the quantum expectation value \( \sum_{i} \lambda_i P_i \). There is only one catch, or rather the very reason why quantum mechanics can outperform non-contextual classical models: the quantum expectation values are not accessible in a single von Neumann measurement. What is more, it is necessary for a gap \( \beta_{\text{cl}} < \beta_{\text{qu}} \) to occur, that the same projector \( P_i \) occurs in different, incompatible measurements for various \( i \). In the next section, we shall see that this poses not only a conceptual problem but also a practical one when purportedly testing quantum contextuality (i.e. experimentally refuting classical non-contextual hidden variable explanations).
2. Meyer–Clifton–Kent’s nullification of BKS

Let us start with an easy objection against any physical relevance of BKS theorems stemming from the fact that the hidden variable theory for a set of effects is supposed to assign pre-existing values only to projective measurements but in experiments it is highly unlikely that a sharp von Neumann measurement is ever implemented. What is more, experimental evidence based on observable expectation values never allows the experimenter to distinguish conclusively between a projector (an element in an ideal measurement) and some arbitrarily close POVM element (aka effect), that is, a positive semidefinite operator upper bounded by 1, be it another projector or a genuinely non-projective POVM element. However, in this form this does not pose a concern since the quantum mechanical expectations values of two operators $A$ and $B$ cannot differ by more than $\| A - B \|$ which is the operator norm of their difference. This norm difference can be experimentally estimated via the fundamental relation

$$\| A - B \| = \max_{\rho \ \text{state}} | \text{Tr} \rho A - \text{Tr} \rho B |.$$

Thus, we can at least in principle confirm experimentally (within the rules of quantum mechanics and according to our command of the underlying physical system) that the experiments implement POVM elements close to the required projectors. This is important because by allowing general POVM elements, one can reach values of $\sum_i \lambda_i \langle P_i \rangle$ that are even larger than $\beta$, all the way to $\sum \beta \lambda \Gamma = \forall \leq \forall \in \leq \forall \in t C \max s.t. 0, 1, 1, i, i$, which is the maximum value allowable by generalized probabilistic theories [41, 46]. For instance for the pentagon (example 2), this value is $\beta = \frac{g}{2}$.

At the same time, we would naturally demand that whatever the experiment does, it should have a classical hidden variable explanation for each measurement outcome.

This brings us to Meyer’s [25] objection, which was greatly refined and extended by Kent [47], and as well as Clifton and Kent [48]. These authors show that in each dimension $d$ of the underlying Hilbert space of a quantum system, there exists a dense set $\mathcal{M} = \{ M^{(1)}, M^{(2)}, \ldots \}$ (w.l.o.g. countable) of complete von Neumann measurements $M^{(j)} = \{ P_{jk}, \ldots, P_{kd} \}$, consisting of rank-one projectors, with the property that every $P_{jk}$ occurs in only one measurement, namely $M^{(j)}$. Here, ‘dense’ refers to the set of all von Neumann measurements: for every von Neumann measurement $(Q_1, \ldots, Q_d)$ and every $\epsilon > 0$, there exists an $M^{(j)} \in \mathcal{M}$ such that $\| Q_k - P_{jk} \| \leq \epsilon$ for all $k$.

This set $\mathcal{M}$ of measurements clearly has a non-contextual hidden variable model reproducing the correct statistics for any given state $\rho$. Any random variables $X_{jk} \in \{0, 1\}$ such that

$$\text{Pr} \{ X_{jk} = 1, \ X_{\hat{k} \neq k} = 0 \forall \hat{k} \neq \hat{k} \} = \text{Tr} \rho P_{jk}$$

will do. [In fact, this can even be extended to POVMs with a bounded number of outcomes.] Such sets are not hard to come by, either by existence proofs or constructively.

Now, if the experimenter needs to implement a measurement $Q = (Q_1, \ldots, Q_d)$, she can only ensure (and demonstrate by experimental verification) that she has done so up to a finite accuracy $\epsilon$. In particular, she cannot distinguish her experimental observations from those of a suitably close measurement $M^{(j)} \in \mathcal{M}$—which, however, has a genuinely non-contextual hidden variable explanation! Hence, in practice, where one can never be sure which one of the infinitely many measurements arbitrarily close to $Q$ was responsible for the observations, the
experimenter cannot rule out a fully non-contextual hidden variable theory nor the con-
comitant requirement that ‘really’ only the measurements in the set $\mathcal{M}$ are implemented. Note
that this has nothing to do with the correctness of the mathematical reasoning of Gleason,
Bell, Kochen–Specker, and so on (of which the prerequisites simply do not apply), but only
concerns its relevance to the observable world; see Appleby’s analysis [49].

Not surprisingly, perhaps, this argument, although simple and in our opinion irrefutable,
has sparked a considerable debate, which continues in some form until today, although a
review of this is beyond the scope of this paper. For more detail, see the excellent discussion
and extensive references in the article of Barrett and Kent [50].

In a nutshell, it is clear to physicists that something quantum is demonstrated in the
experiments. An inequality is violated and incompatible measurements are performed in
successive runs. Yet, the Meyer–Clifton–Kent (MKC) argument clearly shows that some-
ingthing is lacking to be able to claim experimental confirmation of quantum contextuality. So,
what can we salvage from this intolerable situation?

3. What does ‘same outcome’ mean?

All of the discussions of hidden variables in quantum mechanics have to labor one point, in
one way or another: the identification of the quantum theoretic entities, as given by the
formalism, with counterparts in the hypothetical hidden variable model. This is where
invariably unproven assumptions are made—have to be made, indeed, as we know next to
nothing about this alternative description, which after all may not even be real, and rather only
want to talk about the general form of that hypothetical theory, which in addition is usually
only meant to ‘explain’ some small section of the range of observable phenomena.

In non-contextual hidden variable models this has to be done on the level of the mea-
surement outcomes by postulating 0-1-indicator random variables, singling out at most one of
the possible outcomes of each measurement (one and only one in the original BKS theorems,
but as explained in section 1 we can relax this to ‘at most one’). So far, this is nothing
peculiar, and is already the end of the story in the MKC models (see the previous section).
The curious thing, however, is that we also demand that certain outcomes ‘i’ in one mea-
surement and outcome ‘j’ in another measurement (which may require completely different
experiments one from the other) are to be identified. For the hidden variable theory this is
taken to mean that the two associated random variables take the same value. For the quantum
model it means that these outcomes are represented by the same projector, a more general but
same effect, in different measurements.

Operationally, we can recognize identical effects by their identical response to all dif-
ferent state preparations. Turning this around, this is how a theory tells us what its effects are:
concise descriptions of the different responses (as probabilities of a ‘click’) of experiments.
Cf. the theory of Ludwig [51], in which states are equivalence classes of state preparations
with respect to the statistics under all possible measurements and, vice versa, measurements
(and indeed effect) equivalence classes of experiments with respect to the statistics under all
possible state preparations. In particular, coarse graining the outcomes of any procedure to
only two yields a binary measurement; for example, assigning ‘yes’ if a particular click
happens, and ‘no’ otherwise. If starting from two experiments and singling out an outcome in
each of them, we end up with equivalent binary observables, which allows us to identify the
effects corresponding to these outcomes. (Recall that a priori these outcomes are simply
defined as clicks of a certain kind in two potentially completely different experimental
procedures.)
Following Spekkens’ operational approach [28], a non-contextual classical hidden variable model has to assign random variables to each effect; in particular, the same value to all appearances of that effect in different contexts. There is probably no compelling reason to believe a priori that the hidden variables should have this property, but it is undoubtedly a very intuitive idea that is rooted in our classical intuition, cf Bell’s discussion [6].

What the MKC constructions exploit states is that our access to the quantum mechanical effects is governed by a topology of closeness, rather than identity. On the other hand, the previous requirement of non-contextuality does not put any conditions on the classical entities assigned to two distinct but infinitesimally close effects. Towards the end of Spekkens’ article [28], it is suggested that, to overcome the MKC critique based on finite precision, we need an extended version of the above identification principle, requiring that elements which represent similar effects in the quantum theory are in some sense similar. In the next section, we present a simple proposal to formulate such a similarity principle.

4. Finite precision: ontological faithfulness

The effects of quantum theory have a geometry thanks to the operator norm $\| A - B \|$. As explained in section 2, this is the largest difference between the expectation values of the two operators on the same state. Consequently, experiments can pin down an effect only up to statistical error bars in this norm. In particular, let us consider a set $E$ of effects sufficiently close to projectors. We call this an $\epsilon$-precise quantum model of a set $V$ of outcomes $i$ and a collection $\Gamma$ of contexts $C \subset V$, if every $Q_i \in E$ is $\epsilon$-close to a projector $P_i$, and furthermore for every $C \in \Gamma$, there is a collection of $Q^C_i \in E$, such that $\sum_{i \in C} Q^C_i \leq 1$ and $\| Q^C_i - P_i \| \leq \epsilon$ for all $i \in C$. This encapsulates the notion of contexts $C$, each outcome $i$ of which can be approximately identified with a projector $P_i$.

The corresponding classical hidden variable models are captured by the following definition.

Definition 3. An $\epsilon$-ontologically faithful non-contextual ($\epsilon$-ONC) model for a hypergraph $\Gamma$ of contexts $C \subset V$ consists of a family of random variables $X^C_i \in \{0, 1\}$, $i \in C \in \Gamma$, such that

$$\forall C \in \Gamma \quad \sum_{i \in C} X^C_i \leq 1,$$

and

$$\forall C, C' \in \Gamma \quad \forall i \in C \cap C' \quad \Pr\{X^C_i \neq X^{C'}_i\} \leq \epsilon.$$

In other words, for each context $C \in \Gamma$, the family $(X^C_i : i \in C)$ is a classical hidden variable model of a measurement containing the outcomes $C$, and the models for the ‘same’ outcome $i$ occurring in different contexts $C$ and $C'$ almost coincide.

The spirit of this definition is that it imposes a distance on the random variables $X^C_i$ representing the different ‘incarnations’ of the outcome $i$. Since we are dealing with 0-1-variables, the probability of disagreeing is a natural measure, but in more complex situations other distances may be employed.

Logically, both definitions, of an approximate quantum model and of an ontologically faithful non-contextual classical model, are independent, hinging directly on the combinatorial structure of the permissible contexts $\Gamma$. However, it may be helpful to think of the
From approximate to exact non-contextual hidden variables. We will now show how the \( \epsilon \)-approximations introduced above can be eliminated at the expense of having, with some small probability, no outcome among the set \( V \). Indeed, we can easily build a non-contextual hidden variable model from any \( \epsilon \)-ONC model by letting

\[
Y_i := \prod_{C \in C} X^C_i.
\]

(8)

Note that \( Y_i^C := Y_i \) for \( i \in C \) defines a 0-ONC model and that it approximates the original one well:

**Proposition 4.** Consider an \( \epsilon \)-ONC model \( (X^C_i)_{C \in C} \) and associated \( Y_i \) as per equation (8). Let \( k_i \) be the number of times an outcome \( i \) occurs in some context \( C \in \Gamma \). Then, the collection \( (Y_i) \) is a non-contextual hidden variable model, and for every \( i \in V \),

\[
\Pr \{ \exists C \ni i \text{ s.t. } X^C_i \neq Y_i \} \leq (k_i - 1)\epsilon.
\]

**Proof.** To check that \( (Y_i) \) defines a non-contextual hidden variable model, we observe for every \( i \in C \in \Gamma \), \( Y_i \leq X^C_i \), hence

\[
\sum_{i \in C} Y_i \leq \sum_{i \in C} X^C_i \leq 1.
\]

On the other hand, for each \( i \),

\[
\Pr \{ \exists C \ni i \text{ s.t. } X^C_i \neq Y_i \} = \Pr \{ \big( X^C_i : i \in C \in \Gamma \big) \text{ not all equal} \} \leq (k_i - 1)\epsilon,
\]

because it is enough to compare a single \( X^C_i \) against each other \( X^C_i \). \( \square \)
We especially record the implication of the preceding result for (linear) non-contextual inequalities.

**Proposition 5.** Consider a non-contextual inequality \( \sum \lambda_i \mathcal{E} \leq \beta_\text{cl} \) for a set \( \Gamma \) of contexts \( C \subset V \), and \( k_i \) as in proposition 4.

Then, for any \( \epsilon \)-ONC \( (\mathcal{E}_i^C) \) and any assignment of a context \( i \mapsto C \exists i \), and letting \( t_i := \mathcal{E}_i^C \) we have

\[
\sum_{i} \lambda_i t_i \leq \beta_\text{cl} + \epsilon \sum_{i} \lambda_i (k_i - 1).
\]

**Proof.** Let \( Y_i := \prod_{i \in C \subseteq \Gamma} X_i^C \) as in equation (8). Then, proposition 4 implies

\[
t_i = \mathcal{E}_i^C \leq \mathcal{E}_i + (k_i - 1) \epsilon,
\]

and summing over \( i \) we are done. \( \square \)

**Example 6.** Returning to the pentagon inequality of Klyachko et al [44] (example 2), we have \( k_i = 2 \) for all outcomes \( i = 0, 1, 2, 3, 4 \). The original inequality as \( \mathcal{E}_i \leq 2 = \beta_\text{cl} \), that is, all \( \lambda_i = 1 \). Thus, for any

\[
\epsilon < \frac{\sqrt{5} - 2}{5} \approx 0.047,
\]

the observation of a value sufficiently close to \( \beta_\text{qs} = \sqrt{5} \) in a quantum setup with projectors rules out \( \epsilon \)-ONC hidden variables.

**Example 7.** Another example is the contextuality proof via the Mermin-Peres square [39], which is also the subject of the experiment in [23]. The set of outcomes one has to consider here consists of 24 rank-one projectors in \( \mathbb{C}^4 \), forming 24 contexts of four elements each, each of which corresponds to a complete orthonormal basis (cf [52, figure 1]).

It can be shown that \( \sum \mathcal{E}_i \leq 5 = \beta_\text{cl} \), whereas the maximum quantum values is \( \beta_\text{qs} = 6 \) (once more see [52]). Since each outcome occurs in exactly \( k_i = 4 \) contexts, a value of

\[
\epsilon < \frac{6 - 5}{72} \approx 0.0138
\]

suffices to rule out \( \epsilon \)-ONC hidden variables for this experiment.

Clearly, in the same way, every non-contextual inequality is made robust to finite precision by restricting it to \( \epsilon \)-ONC models with suitably small \( \epsilon > 0 \). In particular, this means that the MKC hidden variable models cannot be ontologically faithful since they are consistent with measuring an arbitrary quantum observable to any finite precision.

Consequently, MKC-like hidden variable models necessarily have to have parameters inaccessible in quantum mechanics since they have to assign very different classical random variables to arbitrarily close projectors.

5. Discussion

Any proof that classical hidden variables cannot reproduce the predictions of quantum mechanics is invariably achieved only under some other assumption, be it non-contextuality...
or locality of the classical variables. The assumption itself is not testable, so it has to be chosen on other, ‘reasonable’, grounds. For instance, in Bell tests, one would argue that no-signaling is a well-established fact (which follows from special relativity) and that non-local classical variables would have to come with some incredible mechanism to remain absolutely hidden to prevent some eventual faster-than-light signal getting out.

For non-contextuality the case is a priori weaker because we are not taking recourse to another physical principle (compare, however, the ideas formulated in [53, 54] regarding memory bounds). Instead, we make assumptions on how to describe experiments. That there are such things as experiments can hardly be denied: the setup subjecting a system in some reproducible preparation to measurement, yielding a result. In addition, quantum mechanics already comes with the identification between outcomes in experiments and effect operators, even up to finite precision and statistical error bars. However, it cannot, by definition, tell us anything about the hypothetical hidden-variable theory—especially if the no-go theorem involves exhibiting an operational difference between the two.

Here, we have shown how to include finite precision into the reasoning about non-contextual hidden variable theories. Our analysis revolves around the idea that, in both quantum and in possible classical models of any given structure of contexts on an abstract set of outcomes, this requires the introduction of a metric on the entities of the model reflecting the degree of approximation. The definition of ontological faithfulness is one way, presumably, however, not the only way, to formalize such a notion.

Since the MKC argument has been put forward, and following the subsequent debate, other attempts to address the finite precision objection have appeared. Ignoring the ones that aimed at finding a flaw of some kind in MKC (see [50] for an extensive review), we find rather more interesting those introducing some other additional property of the hidden variables that should guarantee experimental testability. The most developed of these is in the papers of Cabello et al [55], Kirchmair et al [23] and Gühne et al [56] (cf also [57]). In these papers, the new element of sequential measurements was introduced and the assumption concerns the behaviour of the hidden parameter, and the observable consequences thereof, under sequences of (almost) compatible measurements. Some such assumption is necessary, as one can see from a suitable extension of the MKC models, to include a simulation of the state change due to measurement, which is then able to reproduce to arbitrary precision the statistics of sequences of measurements and the effects of the projection postulate.

It may not be possible to make a full comparison between this proposal and the one presented here because ontological faithfulness is a minimally invasive change of the BKS approach, in which any experiment is an indivisible whole. In particular, it does not imply anything about the history or dynamics of the hidden parameter, as [55, 23, 56, 57] and related approaches necessarily do. However, it is worth noting that by simply assuming no back-action on the hidden parameter, in an $\epsilon$-ONC, we have for $i \in C \cap C'$,\[
\Pr\left\{X_i^C \neq \xi \mid X_i^C = \xi\right\} \leq \frac{\epsilon}{\Pr\{X_i^C = \xi\}}.\tag{9}
\]
That is, unless the probability of observing an outcome $i$ in a context $C$ is very small, the subsequent consultation of the ‘same’ outcome in a different context $C'$ yields the same value, with high probability. Thus, ontological faithfulness of the hidden variable theory implies an approximate version of repeatability of measurements, even on the level of the same outcome in different contexts.

Conversely, it seems likely that the approach of [56, 57] always implies an $\epsilon$-ONC hidden variable model because the assumptions in these papers imply that one can define
hidden variables corresponding to measuring some observable jointly with other, commuting, ones, and these different variables turn out to be $\varepsilon$-close in the sense of definition 3. For this purpose, we stress that sequential application of several measurement devices yields just—another measurement, as does another ordering of the same devices, or the unitary transfer of the state into multiple qubits, which are subsequently measured in space-like separation, or for that matter any magical mystery machine as long as it is governed by the rules of quantum mechanics. What counts are the observations that are actually going to be made, and once that is decided then these observations can be reflected in a suitable compatibility structure of outcomes and contexts. The quantum mechanics of these devices and their sequential application is taken care of by quantum theory itself; this is not so for the hypothetical hidden variable theory, which has to be augmented by additional assumptions to allow for any meaningful comparison with quantum mechanics.

We shall refrain from arguing the plausibility of ontological faithfulness, which does not seem to make a lot of sense to the present author, unless one actually believes in hidden variables (contextual or non-contextual) as a viable description of quantum reality. What after all is the point of no-go theorems? In the best case, they reveal the incompatibility between a set of preconceptions, on the one hand, and a certain description of nature, on the other hand. This surely is a worthy enterprise, especially since we keep struggling with that description in the case of quantum mechanics. In this vein, no-go theorems, rather than demonstrating a lack of imagination, really are indispensable tools for a fruitful use of it [58].

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