GRADIENT ESTIMATES FOR WEIGHTED HARMONIC FUNCTION WITH DIRICHLET BOUNDARY CONDITION

NGUYEN THAC DUNG AND JIA-YONG WU

Abstract. We prove a Yau’s type gradient estimate for positive $f$-harmonic functions with the Dirichlet boundary condition on smooth metric measure spaces with compact boundary when the infinite dimensional Bakry-Émery Ricci tensor and the weighted mean curvature are bounded below. As an application, we give a Liouville type result for bounded $f$-harmonic functions with the Dirichlet boundary condition. Our results do not depend on any assumption on the potential function $f$.

1. Introduction

In this paper, we will give Yau’s type gradient estimates for positive $f$-harmonic functions with the Dirichlet boundary condition (i.e., they are constant on the boundary) on smooth metric measure spaces with the compact boundary when the infinite dimensional Bakry-Émery Ricci tensor and the weighted mean curvature are bounded below. As an application, we will prove a Liouville theorem for bounded $f$-harmonic functions with the Dirichlet boundary condition.

Recall that an $n$-dimensional smooth metric measure space denoted by $(M, g, e^{-f}dv_g)$ is an $n$-dimensional smooth complete Riemannian manifold $(M, g)$ coupled with a weighted volume $e^{-f}dv_g$ for some $f \in C^\infty(M)$, where $dv_g$ is the standard Riemannian volume element on $M$ and $f$ is called the potential function. Smooth metric measure spaces are closely related to gradient Ricci solitons, the Ricci flow, probability theory, and optimal transport; see e.g. [1], [13] and [10]. On smooth metric measure space $(M, g, e^{-f}dv_g)$, for any $m > 0$, the $m$-Bakry-Émery Ricci tensor, introduced by Bakry and Émery [1], is defined by

$$\text{Ric}_f^m := \text{Ric} + \text{Hess} f - \frac{1}{m} df \otimes df,$$

where Ric is the Ricci tensor of the manifold $(M, g)$ and Hess is the Hessian with respect to the Riemannian metric $g$. Clearly, $m$-Bakry-Émery Ricci tensor is a natural generalization of Ricci curvature on Riemannian manifolds.

When $m < \infty$, $\text{Ric}_f^m$ is called the finite dimensional Bakry-Émery Ricci tensor and it often shares many similar geometric results for $(n + m)$-dimensional manifolds with the Ricci tensor; see for example [9], Appendix A in [23] and references therein. This is because the Bochner formula for $\text{Ric}_f^m$ can be considered as the Bochner formula for the Ricci tensor.
of an \((n + m)\)-dimensional manifold, i.e.,
\[
\frac{1}{2} \Delta_f |\nabla u|^2 = |\nabla^2 u|^2 + \langle \nabla \Delta_f u, \nabla u \rangle + \text{Ric}_f^m(\nabla u, \nabla u) + \frac{1}{m} |\langle \nabla f, \nabla u \rangle|^2 \\
\geq \frac{(\Delta_f u)^2}{m + n} + \langle \nabla \Delta_f u, \nabla u \rangle + \text{Ric}_f^m(\nabla u, \nabla u)
\]
for any \(u \in C^\infty(M)\), where \(\Delta_f\) is called the \(f\)-Laplacian, which is defined by
\[
\Delta_f := \Delta - \nabla f \cdot \nabla.
\]
This operator is a natural generalization of the usual Laplacian and is self-adjoint with respect to the weighted measure \(e^{-f}dv_g\). When \(m = \infty\), we have
\[
\text{Ric}_f := \lim_{m \to \infty} \text{Ric}_f^m = \text{Ric} + \nabla^2 f,
\]
which is called the infinite dimensional Bakry-Émery Ricci tensor. When \(\text{Ric}_f\) is bounded below, many geometric properties of manifolds with the Ricci tensor bounded below were also possibly generalized to smooth metric measure spaces; but some extra assumption on \(f\) is needed, see for example [23], [24], [25] and references therein. It is easy to see that \(\text{Ric}_f^m \geq c\) implies \(\text{Ric}_f \geq c\), but the opposite may be not true.

In particular, if there exists a real constant \(\lambda\) such that
\[
\text{Ric}_f = \lambda g,
\]
then \((M, g, e^{-f}dv_g)\) is called the gradient Ricci soliton. The gradient Ricci soliton is called shrinking, steady, or expanding, if \(\lambda > 0\), \(\lambda = 0\), or \(\lambda < 0\), respectively. Gradient Ricci solitons are natural generalizations of Einstein metrics. They are also self-similar solutions to the Ricci flow and play important roles in the Ricci flow and Perelman’s resolution of the Poincaré conjecture and the geometrization conjecture; see [6], [13], [14], [15] and references therein for nice details.

On smooth metric measure space \((M, g, e^{-f}dv_g)\), a smooth function \(u\) is called \(f\)-harmonic (also called weighted harmonic) if
\[
\Delta_f u = 0.
\]
On \((M, g, e^{-f}dv_g)\) with compact boundary \(\partial M\), the \(f\)-mean curvature (also called weighted mean curvature) is defined by
\[
H_f := H - \nabla f \cdot \nu,
\]
where \(\nu\) is the unit outer normal vector to \(\partial M\) and \(H\) is the mean curvature of \(\partial M\) with respect to \(\nu\). When \(f\) is constant, the above concepts all recover the manifold case.

For manifolds with the boundary, most of geometric results concentrate on the Neumann boundary condition; see for example [8], [21], [8] and [12]. Recently, Kunikawa and Sakurai [7] extended Yau’s gradient estimates and Liouville theorems for harmonic functions [26] to the Dirichlet boundary condition. Shortly after, H. Dung, N. Dung and Wu [5] further extended their results to \(f\)-harmonic functions on the smooth metric measure space with some compact boundary. In summary, we have

**Theorem A.** Let \((M, g, e^{-f}dv)\) be an \(n\)-dimensional smooth metric measure space with the compact boundary. For a fixed \(m > 0\), assume that
\[
\text{Ric}_f^m \geq -(n + m - 1)K \quad \text{and} \quad H_f \geq -L
\]
for some non-negative constants $K$ and $L$. Let $u : B_R(\partial M) \to (0, \infty)$ be a positive $f$-harmonic function with the Dirichlet boundary condition. If $u_\nu$ is non-negative over $\partial M$, then there exists a constant $c$ depending on $n + m$ such that

$$\sup_{B_{R/2}(\partial M)} \frac{|\nabla u|}{u} \leq c \left( \frac{1}{R} + L + \sqrt{K} \right).$$

Here $B_R(\partial M) := \{ x \in M | d(x, \partial M) < R \}$.

In Theorem A $\text{Ric}_f^m \geq -(n + m - 1)K$ means that the infimum of $\text{Ric}_f^m$ on the unit tangent bundle on the interior of $M$ is at least $-(n + m - 1)K$; $H_f \geq -L$ means that boundary $\partial M$ has some weak convex property. As pointed out in [5], when $\text{Ric}_f \geq -(n - 1)K$ and $H_f \geq -L$, there seems to be essential obstacles to derive Yau’s type gradient estimates by directly following their proof of Theorem A in [7] or [5]. This is because their proof depends on a refined Kato inequality, which is not suitable to the case when $\text{Ric}_f \geq -(n - 1)K$.

In this paper, we will solve the above question and give Yau’s type gradient estimates for positive $f$-harmonic functions with the Dirichlet boundary on a neighborhood of the boundary under lower bounds of $\text{Ric}_f$ and $H_f$. Our gradient estimate does not depend on any assumption on $f$.

**Theorem 1.1.** Let $(M, g, e^{-f} dv)$ be an $n$-dimensional smooth metric measure space with the compact boundary. Assume that

$$\text{Ric}_f \geq -(n - 1)K \quad \text{and} \quad H_f \geq -L$$

for some non-negative constants $K$ and $L$. Let $u$ be a positive $f$-harmonic function on $B_R(\partial M)$ with the Dirichlet boundary condition. If $u_\nu \geq 0$ over $\partial M$, then there exists a constant $c(n)$ depending on $n$ such that

$$\sup_{B_{R/2}(\partial M)} |\nabla u| \leq c(n) \left( \frac{1}{R} + L + \sqrt{K} \right) \sup_{y \in B_R(\partial M)} u(y).$$

Notice that our gradient estimate in Theorem 1.1 holds for all $R > 0$; however Brighton’s result (see Theorem 1 in [2]) in the complete non-compact case without the boundary only holds for $R > 1$. The reason is that in our case we use a new weighted Laplacian comparison on any neighborhood of the boundary; see Theorem 2.1 in Section 2. Also notice that if $d(x, \partial M)$, where $x \in M$, is finite, then all boundary balls $B_R(\partial M)$ are possibly the same if $R$ is large.

Li and Yau [8] proved gradient estimates for the heat equation with the Neumann boundary condition on compact manifolds of the convex boundary. From the course of our proof, it is easy to see that our gradient estimate also holds for $f$-harmonic functions with the Neumann boundary condition.

The main trick of proving Theorem 1.1 stems from the arguments of Brighton [2] and Kunikawa and Sakurai [7]. Far away from the boundary of space, we will apply Yau’s gradient estimate technique to function $u^{7/8}$ instead of $\ln u$. On the boundary of space, we will apply a derivative equality (Proposition 2.2 in Section 2) to prove the desired estimate.

Taking a limit as $R$ tends to infinity, we immediately get a Liouville type theorem for $f$-harmonic functions with the Dirichlet boundary condition.
Corollary 1.2. Let \((M, g, e^{-f}dv)\) be an \(n\)-dimensional smooth metric measure space with compact boundary \(\partial M\). If \(\text{Ric}_f \geq 0\) and \(H_f \geq 0\), then any bounded \(f\)-harmonic function \(u\) with the Dirichlet boundary condition and \(u_\nu \geq 0\) over \(\partial M\) must be constant.

Remark 1.3. The assumption \(u_\nu \geq 0\) over \(\partial M\) in Corollary 1.2 is necessary. For example, let \(u(x) = x\) and \(f = \text{constant}\) in \(M = [1, 2]\) with compact boundary \(\{1, 2\}\). Then,

\[
\text{Ric}_f = H_f = \Delta_f u = 0
\]

However, \(u_\nu|_{x=1} = -1\) and \(u_\nu|_{x=2} = 1\). Thus \(u(x)\) is a non-constant bounded \(f\)-harmonic function with the Dirichlet boundary condition.

Remark 1.4. The boundness of \(f\)-harmonic function in Corollary 1.2 is necessary. Indeed, let \(u(x) = -e^{-x}\) and \(f(x) = -x\) in \(M = (-\infty, 0]\) with compact boundary \(\{0\}\). Then the unit outer normal vector \(\nu = 1\),

\[
\text{Ric}_f = 0, \quad H_f = 1, \quad \Delta_f u = u'' + u' = 0 \quad \text{and} \quad u_\nu|_{x=0} = 1.
\]

However, \(u(x) = -e^{-x}\) is unbounded in \((-\infty, 0]\) and it is a non-constant \(f\)-harmonic function with the Dirichlet boundary condition.

Remark 1.5. The assumption \(H_f \geq 0\) in Corollary 1.2 is necessary. We provide two examples to illustrate it. One example is that, for any real number \(\alpha > 0\), let \(u(x) = e^{\alpha x}\) and \(f(x) = \alpha x\) in \(M = (-\infty, 0]\) with compact boundary \(\{0\}\). Then \(\nu = 1\),

\[
\text{Ric}_f = 0, \quad \Delta_f u = u'' - \alpha u' = 0 \quad \text{and} \quad u_\nu|_{x=0} = \alpha.
\]

However, \(H_f = -\alpha < 0\) and \(u(x)\) is a non-constant bounded \(f\)-harmonic function with the Dirichlet boundary condition.

Another example is that \(u(x) = x^{-1}\) and \(f(x) = -2 \ln x\) in \(M = [1, \infty)\) with compact boundary \(\{1\}\). Then \(\nu = -1\),

\[
\text{Ric}_f = 2x^{-2} \geq 0, \quad \Delta_f u = 0 \quad \text{and} \quad u_\nu|_{x=1} = 1.
\]

However, \(H_f = -2x^{-1} < 0\) and \(u(x)\) is a non-constant bounded \(f\)-harmonic function with the Dirichlet boundary condition.

The rest of this paper is organized as follows. In Section 2 we will recall some results about smooth metric measure spaces with the compact boundary, including the weighted Laplacian comparison, the derivative equality, the Bochner type formula and the cut-off function. These results will be used in the proof of our gradient estimate. In Section 3 we will apply the Brighton’s trick [2] and the Kunikawa-Sakurai’s argument [7] to prove Theorem 1.1.

Acknowledgement. The authors would like to thank the referee for valuable comments and useful suggestions for this work. The first author is supported by the research project QG.21.01 “Geometric operators on Riemannian manifolds” of Vietnam National University, Hanoi. The second author is supported by the Natural Science Foundation of Shanghai (17ZR1412800).
2. Background

In this section, we list some known results about smooth metric measure spaces with the boundary. These results will be used in the proof of our result. For more properties, the interested reader are referred to [18]. On a smooth metric measure space \((M^n, g, e^{-f} dv)\) with the boundary \(\partial M\), the distance function from the boundary is denoted by
\[
\rho(x) = \rho_{\partial M}(x) = d(x, \partial M),
\]
where \(x \in M\). By [17], we may assume that \(\rho\) is smooth outside of the cut locus for the boundary. In [22], Wang, Zhang and Zhou obtained weighted Laplacian comparisons for the distance function on smooth metric measure spaces with the boundary under some assumptions on \(\rho(x)\) (see also [19]). Later, Sakurai [18] proved the following general comparison result without any assumption on \(\rho(x)\), which is a key step in our proof of Theorem 1.1.

**Theorem 2.1.** Let \((M^n, g, e^{-f} dv)\) be an \(n\)-dimensional complete smooth metric measure space with compact boundary \(\partial M\). Assume that
\[
\text{Ric}_f \geq -(n - 1)K \quad \text{and} \quad H_f \geq -L
\]
for some constants \(K \geq 0\) and \(L \in \mathbb{R}\). Then
\[
\Delta_f \rho(x) \leq (n - 1)KR + L
\]
for all \(x \in B_R(\partial M)\).

*Proof of Theorem 2.1.* We will give a quick explanation of the result based on Sakurai’s result [18]. Assume that \(x \in B_R(\partial M)\). Let \(B(x, d(x, \partial M)) \subset M\) be the largest geodesic ball with center \(x\) such that \(\partial B(x, d(x, \partial M)) \cap \partial M = z\). We also let \(\gamma_{z,x}(s)\) be a geodesic line with the arc-length parameter \(s\) which starts from point \(z\) to \(x\). It is easy to see that \(\gamma'_{z,x}(0)\) is the unit inner normal vector for \(\partial M\) at \(z\). Since \(\gamma_{z,x}(d(x, \partial M)) = x\) and \(d(x, \partial M) \leq R\), by Lemma 6.1 in [18], we easily get Theorem 2.1. We remark that the above statement can be described below; see Figure 1.

![Figure 1. Comparison on weighted manifolds with boundary](image)

Next, we recall the following derivative equality, which was ever used in the proof of the weighted Reilly formula in [11]; see also (24) in Appendix of [4]. In fact it is a slight generalization of the classical case [16]. This formula will be used in our gradient estimate for the boundary of the weighted manifold.

□
Proposition 2.2. Let \((M, g, e^{-f}dv)\) be a complete smooth metric measure space with compact boundary \(\partial M\). For any \(u \in C^\infty(M)\), we have
\[
\frac{1}{2} (|\nabla u|^2)_\nu = u_\nu [\Delta_f u - \Delta_{\partial M} u] + g_{\partial M}(\nabla_{\partial M} u, \nabla_{\partial M} u_\nu) - \Pi(\nabla_{\partial M} u, \nabla_{\partial M} u_\nu),
\]
where \(\nu\) is the outer unit normal vector to \(\partial M\), and \(\Pi\) is the second fundamental form of \(\partial M\) with respect to \(\nu\).

Proof of Proposition 2.2. Its proof follows by a direct computation and we include it for the sake of completeness. We compute that
\[
\frac{1}{2} (|\nabla u|^2)_\nu = g(\nabla_{\nu} \nabla u, \nabla u) = g(\nabla u, \nabla_{\nu} u) = g(\nabla_{\nu} \nabla u, \nabla u) + g(\nabla_{\partial M} u, \nabla_{\partial M} u_\nu) = g(\nabla_{\nu} \nabla u, \nabla u) + g_{\partial M}(\nabla_{\partial M} u, \nabla_{\partial M} u_\nu) - g(\nabla u, \nabla_{\partial M} u_\nu).
\]

Thus,
\[
\frac{1}{2} (|\nabla u|^2)_\nu - (\Delta_f u) u_\nu = \left[ g(\nabla_{\nu} \nabla u, \nabla u) - \Delta_f u + g(\nabla f, \nabla u) u_\nu + g_{\partial M}(\nabla_{\partial M} u, \nabla_{\partial M} u_\nu) - g(\nabla_{\partial M} u, \nabla_{\partial M} u_\nu) \right] = \left[ - \Delta_{\partial M} u - H u_\nu + g_{\partial M}(\nabla_{\partial M} f, \nabla_{\partial M} u) + g(\nabla f, \nu) u_\nu \right] u_\nu + g_{\partial M}(\nabla_{\partial M} u, \nabla_{\partial M} u_\nu) - \Pi(\nabla_{\partial M} u, \nabla_{\partial M} u_\nu)
\]
and the desired result follows. \(\square\)

Meanwhile, we recall an important Bochner type formula in the proof of our result, which was proved by Brighton; see (2.10) in [2]. One important step of his proof is examining two cases depending on the relative magnitudes between \(\langle \nabla h, \nabla f \rangle\) and \(\frac{\langle \nabla h, \nabla h \rangle^2}{2h}\), where \(h := u^{7/8}\) and \(u\) is a positive \(f\)-harmonic function. We mention that this formula always holds without any assumption on \(f\).

Lemma 2.3. Let \((M^n, g, e^{-f}dv)\) be an \(n\)-dimensional smooth metric measure space with
\[
\text{Ric}_f \geq -(n - 1)K
\]
for some constant \(K \geq 0\). If \(u\) is a positive \(f\)-harmonic function on \(M\), then function \(h := u^{7/8}\) satisfies
\[
\frac{1}{2} \Delta_f |\nabla h|^2 \geq \frac{7n - 6}{49nh^2} |\nabla h|^4 - \frac{1}{7h} \langle \nabla h, \nabla |\nabla h|^2 \rangle - (n - 1)K |\nabla h|^2.
\]

In the end, we introduce a well-known cut-off function originated by Li and Yau [8]. Here we may adopt the statements of [20] and [7]. The cut-off function is an important tool in our proof of Yau’s type gradient estimates.

Lemma 2.4. Let \((M^n, g, e^{-f}dv)\) be an \(n\)-dimensional complete smooth metric measure space with compact boundary \(\partial M\). There exists a smooth cut-off function \(\phi = \phi(x)\) supported in \(B_R(\partial M)\) such that
(i) \(\phi = \phi(\rho_{\partial M}(x)) \equiv \phi(\rho)\); \(\phi(\rho) = 1\) in \(B_{R/2}(\partial M)\), \(0 \leq \phi \leq 1\).
(ii) \( \phi \) is decreasing as a radial function of parameter \( \rho \).

(iii) \( \left| \frac{\partial \phi}{\partial \rho} \right| \leq \frac{C \varepsilon}{R} \) and \( \left| \frac{\partial^2 \phi}{\partial \rho^2} \right| \leq \frac{C \varepsilon}{R^2}, \quad 0 < \varepsilon < 1. \)

3. Gradient estimate

In this section, we will combine the arguments of Kunikawa-Sakurai \cite{7} and Brighton \cite{2} to prove our result.

**Proof of Theorem 1.1.** Let \( u \) be a positive \( f \)-harmonic function on \( M \) and let \( h := \frac{u^7}{8} \). We then consider function

\[ G := \phi |\nabla h|^2, \]

where \( \phi \) is a smooth cut-off function supported in \( B_R(\partial M) \) introduced in Lemma 2.4. We would like to point out that if \( \rho \) is finite, then all boundary balls \( B_R(\partial M) \) are possibly the same if \( R \) is large enough. We now compute that

\[ \nabla G = |\nabla h|^2 \cdot \nabla \phi + \phi \cdot \nabla |\nabla h|^2. \]

Hence

\[ |\nabla h|^2 = \frac{G}{\phi} \quad \text{and} \quad |\nabla h|^2 = \frac{\nabla G}{\phi} - \frac{\nabla \phi}{\phi^2} G. \]

We further compute that

\[ \Delta_f G = \Delta \phi |\nabla h|^2 + 2 \langle \nabla \phi, \nabla |\nabla h|^2 \rangle + \phi \Delta |\nabla h|^2 - \langle \nabla f, \nabla \phi \rangle |\nabla h|^2 - \phi \langle \nabla f, \nabla |\nabla h|^2 \rangle \]

\[ = \Delta \phi |\nabla h|^2 + 2 \langle \nabla \phi, \nabla |\nabla h|^2 \rangle - \langle \nabla f, \nabla \phi \rangle |\nabla h|^2 + \phi \Delta_f |\nabla h|^2. \]

From this, we get that

\[ \Delta_f |\nabla h|^2 = \frac{1}{\phi} \Delta_f G - \frac{|\nabla h|^2}{\phi} \Delta_f \phi - 2 \left( \frac{\nabla \phi}{\phi} \cdot \nabla |\nabla h|^2 \right) \]

\[ = \frac{1}{\phi} \Delta_f G - \frac{G}{\phi^2} \Delta_f \phi - 2 \left( \frac{\nabla \phi}{\phi} \cdot \frac{\nabla G}{\phi} - \frac{\nabla \phi}{\phi^2} G \right). \]

Then substituting (3.1) and (3.2) into (2.1) and solving for \( \frac{1}{\phi} \Delta_f G \), we obtain

\[ \frac{1}{\phi} \Delta_f G \geq \frac{G}{\phi^2} \Delta_f \phi + 2 \left( \frac{\nabla \phi}{\phi} \cdot \frac{\nabla G}{\phi} \right) - 2 \left( \frac{\nabla \phi}{\phi} \cdot \frac{\nabla \phi}{\phi^2} G \right) + \left( \frac{14n - 12}{49nh^2} \right) \frac{G^2}{\phi^2} \]

\[ - \frac{2}{7h} \left( \nabla h, \frac{\nabla G}{\phi} - \frac{\nabla \phi}{\phi^2} G \right) - 2(n - 1)K \frac{G}{\phi}. \]

Now we assume that \( G \) obtains its maximal value at \( x_1 \in B_R(\partial M) \). We will prove the desired estimate according to two cases: \( x_1 \in B_R(\partial M) \setminus \partial M \) and \( x_1 \in \partial M \).

**Case 1:** If \( x_1 \in B_R(\partial M) \setminus \partial M \), we may assume that \( x_1 \notin \text{Cut}(\partial M) \) by the Calabi’s argument. At \( x_1 \), we have

\[ \nabla G = 0 \quad \text{and} \quad \Delta_f G \leq 0. \]

Consequently, at \( x_1 \), (3.3) becomes

\[ \left( \frac{14n - 12}{49nh^2} \right) G \leq -\Delta_f \phi + \frac{2}{\phi} |\nabla \phi|^2 - \frac{2}{7h} \langle \nabla h, \nabla \phi \rangle + 2(n - 1) \phi K. \]
Since $\text{Ric}_f \geq -(n-1)K$ and $H_f \geq -L$, by the weighted Laplacian comparison of Theorem 2.1, we have that
\begin{equation}
\Delta_f \phi \geq -c \frac{\Delta_f (\rho)}{R} - \frac{c}{R^2} \geq -c(n-1)K - \frac{cL}{R} - \frac{c}{R^2}.
\end{equation}
For any $\delta \geq 0$, the term $\langle \nabla h, \nabla \phi \rangle$ can be estimated by
\begin{equation}
2|\langle \nabla h, \nabla \phi \rangle| \leq 2|\nabla h||\nabla \phi| \leq \frac{\delta \phi}{h} |\nabla h|^2 + \frac{h}{\delta \phi} |\nabla \phi|^2.
\end{equation}
Substituting (3.5) and (3.6) into (3.4), we have that at $x_1$,
\begin{equation}
\left( \frac{14n - 12}{49nh^2} - \frac{\delta}{\vartheta h^2} \right) G(x_1) \leq \frac{cL}{R} + \frac{c}{R^2} + \left( 2 + \frac{1}{7\delta} \right) \frac{|\nabla \phi|^2}{\phi} + 2(n-1)K(\phi + c).
\end{equation}
Now choosing $\delta = \frac{1}{7}$, and using $\frac{|\nabla \phi|^2}{\phi} \leq \frac{c^2}{R^2}$, we get that
\begin{equation}
\frac{13n - 12}{49nh(x_1)^2} G(x_1) \leq \frac{cL}{R} + \frac{c + 3c^2}{R^2} + 2(n-1)K(\phi(x_1) + c)
\leq \frac{cL}{R} + \frac{c + 3c^2}{R^2} + 2(n-1)K(1 + c).
\end{equation}
Notice that
\[\sup_{B_{R/2}(\partial M)} G(x) \leq G(x_1).\]
Combining this with (3.3) and using the definition of $h$, we conclude that
\[\sup_{B_{R/2}(\partial M)} |\nabla u| \leq \sqrt{\frac{64n(cLR + c + 3c^2)}{13n - 12} \frac{1}{R^2} + \frac{128n(n-1)(1 + c)}{13n - 12} K \sup_{y \in B_R(\partial M)} u(y)}.\]
So the desired result follows by using the Cauchy-Schwarz inequality
\[2\sqrt{\frac{L}{R}} \leq \frac{1}{R} + L.\]

**Case 2:** If maximal point $x_1 \in \partial M$, our gradient estimate still holds by adapting the argument of [7]. Indeed, at $x_1$, we have $G_\nu \geq 0$. Since $\phi(x_1) = 1$, then $|\nabla h|^2 \geq 0$.

Here we notice that the Dirichlet boundary condition for $u$ and the assumption $u_\nu \geq 0$ imply that
\[|\nabla u| = u_\nu.\]
Since $h = u^{7/8}$ and $u$ satisfies the Dirichlet boundary condition, by Proposition 2.2, we have

$$0 \leq \frac{64}{49} (|\nabla h|^2)_\nu = (u^{-1/4}|\nabla u|^2)_\nu$$

$$= u^{-1/4}(|\nabla u|^2)_\nu - \frac{1}{4} u^{-5/4}|\nabla u|^2 u_\nu$$

$$= 2u^{-1/4}u_\nu(\Delta_f u - H_f u_\nu) - \frac{1}{4} u^{-5/4}u_\nu^3,$$

where we used $|\nabla u| = u_\nu$ in the last line. Since $u$ is a positive $f$-harmonic function, the above inequality reduces to

$$\frac{u_\nu}{8u} + H_f \leq 0.$$

By our theorem assumption, this implies

$$\frac{u_\nu}{8u} \leq -H_f \leq L.$$

Hence,

$$G(x_1) = |\nabla h|^2(x_1) = \frac{49}{64} u^{-1/4}(x_1)u_\nu^2$$

$$= 49u^{7/4}(x_1) \left(\frac{u_\nu}{8u}\right)^2$$

$$\leq 49u^{7/4}(x_1)L^2.$$

By the definition of $G$ and $\phi(x) \equiv 1$ for $x \in B_{R/2}(\partial M)$, we indeed have

$$\sup_{B_{R/2}(\partial M)} |\nabla h|^2(x) = \sup_{B_{R/2}(\partial M)} G(x)$$

$$\leq G(x_1)$$

$$\leq 49L^2u^{7/4}(x_1).$$

Since $h = u^{7/8}$, the above estimates gives that

$$\sup_{B_{R/2}(\partial M)} \frac{49}{64} u^{-1/4}(x)|\nabla u|^2 \leq 49L^2 \sup_{y \in B_R(\partial M)} u^{7/4}(y).$$

Namely,

$$\sup_{B_{R/2}(\partial M)} |\nabla u| \leq 8L \sup_{y \in B_R(\partial M)} u(y)$$

and the theorem follows. \qed

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Department of Mathematics, Vietnam National University (VNU), University of Science, Hanoi, Vietnam

Email address: dungmath@vnu.edu.vn

Department of Mathematics, Shanghai University, Shanghai 200444, China

Email address: wujiayong@shu.edu.cn