SUBCOHOMOLOGY FOR ZOOMING SYSTEMS

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Abstract. In the context of continuous zooming systems $f : M \to M$ on a compact metric space $M$, which include the non-uniformly expanding ones, possibly with the presence of a critical set, with the zooming set dense on $M$, we prove that any Hölder potential $\phi : M \to \mathbb{R}$ for which the integrals $\int \phi d\mu \geq 0$ with respect to any $f$-invariant probability $\mu$, admits a continuous function $\lambda_0 : M \to \mathbb{R}$ (which can be Hölder if some integral is positive) such that

$$\phi \geq \lambda_0 - \lambda_0 \circ f.$$ 

This extends a result in [9] for $C^1$-expanding maps on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ to important classes of maps as uniformly expanding, local diffeomorphisms with non-uniform expansion, Viana maps, Benedicks-Carleson maps and Rovella maps. We also give an example beyond the exponential contractions context.

1. Introduction

Given a compact metric space $M$ and a continuous map $f : M \to M$, it is a trivial task to verify that for any $f$-invariant probability $\mu$ we have the integral $\int (\alpha - \alpha \circ f) d\mu = 0$ for any continuous function $\alpha : M \to \mathbb{R}$. Also, if $x \in M$ is a periodic point with period $n \in \mathbb{N}$, then the sum $\sum_{i=0}^{n-1} (\alpha - \alpha \circ f)(f^i(x)) = 0$. It is readily obtained from the previous remark, by considering the following $f$-invariant probability:

$$\mu = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}.$$

The question about the converse of this fact leads to the well known Livsic Theorem (see [13], Theorem 19.2.1), which states:

**Theorem 1.1** (Livsic Theorem). Let $M$ be a Riemannian manifold, $U \subset M$ open, $f : U \to M$ a smooth embedding, $\Lambda \subset U$ a compact topologically transitive hyperbolic set, and $\varphi : \Lambda \to \mathbb{R}$ Hölder continuous. Suppose that for every $x \in \Lambda$ such that $f^n(x) = x$ we have $\sum_{i=0}^{n-1} \varphi(f^i(x)) = 0$. Then there exists a continuous function $\Phi : \Lambda \to \mathbb{R}$ such that $\varphi = \Phi - \Phi \circ f$. Moreover, $\Phi$ is unique up to an additive constant and Hölder with the same exponent as $\varphi$.

There are variations of this classical result, as can be seen, for example, in the Introduction of [9]. Livsic Theorem is stated as follows:

**Theorem 1.2** (Livsic Theorem). Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $T : \mathbb{T} \to \mathbb{T}$ be a $C^\omega$ expanding map. Let $f : T \to \mathbb{T}$ be $C^k$ for some $k = 1, 2, \ldots, \infty$ (resp. $\beta$-Hölder for some $0 < \beta \leq 1$), and suppose that $\int f d\mu = 0$ for all $T$-invariant probability $\mu$. Then there exists a $C^k$ (resp. $\beta$-Hölder) continuous function $\varphi$ such that $f = \varphi - \varphi \circ T$.

Also in [9], Theorem A, the following somehow related result appears:

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Theorem 1.3. Let $T : \mathbb{T} \to \mathbb{T}$ be a $C^1$ expanding map. Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous $\beta$-Hölder function for some $0 < \beta \leq 1$, and suppose that $\int f d\mu \geq 0$ for all $T$-invariant probability $\mu$. Then there exists a $\beta$-Hölder function $\varphi : \mathbb{T} \to \mathbb{R}$ such that $f \geq \varphi - \varphi \circ T$.

This theorem has been stated and proved independently by several authors. It first appears in an unpublished manuscript by Conze-Guivarc’h in [14], where it is proved using thermodynamic formalism; the same approach is used by Savchenko in [19]. More direct proofs, which do no use the Ruelle transfer operator, can be found in [7], [8] and [10].

In this work, we extend this result to the context of zooming systems, which includes the non-uniformly expanding maps, by using techniques that can be seen in [10], Proposition 11. Our approach is constructing a continuous potential on the dense zooming set and we are able to extend such a potential to the whole space. After that, by denseness of the Hölder functions, we obtain the result.

Our work is organised as follows. In section 2, we begin by giving some preliminaries and notations that will be useful for the remainder of the work and stating our main result. In section 3, we proceed with the proof of our first main result and the proof of our second main result in section 4. In section 5, we finish our paper by giving some applications. We stress that our result extends to several important classes of examples beyond expanding maps on the circle.

From now and on, we proceed with definitions and statements. We begin by defining zooming systems as can be seen in [15]. It also can be seen in [18].

2. Preliminaries and Main Results

2.1. Zooming Systems. The zooming times generalize hyperbolic times beyond the exponential context. Details can be seen in [15]. Let $f : M \to M$ be a measurable map defined on a connected, compact, metric space $M$.

Definition 1. (Zooming contractions). A zooming contraction is a sequence of functions $\alpha_n : [0, +\infty) \to [0, +\infty)$ such that

- $\alpha_n(r) < r$, for all $n \in \mathbb{N}$, for all $r > 0$.
- $\alpha_n(r) < \alpha_n(s)$, if $0 < r < s$, for all $n \in \mathbb{N}$.
- $\alpha_m \circ \alpha_n(r) \leq \alpha_{m+n}(r)$, for all $r > 0$, for all $m, n \in \mathbb{N}$.
- $\sup_{r \in (0,1)} \sum_{n=1}^{\infty} \alpha_n(r) < \infty$.

We have special types of zooming contractions. As defined in [10], we call the contraction $(\alpha_n)_n$ exponential if $\alpha_n(r) = e^{-\lambda n} r$ for some $\lambda > 0$ and Lipschitz if $\alpha_n(r) = a_n r$ with $0 \leq a_n < 1, a_m a_n \leq a_{m+n}$ and $\sum_{n=1}^{\infty} a_n < \infty$. In particular, every exponential contraction is Lipschitz. We can also have the example with $a_n = (n + b)^{-\alpha}, a > 1, b > 0$.

Definition 2. (Zooming times). Let $(\alpha_n)_n$ be a zooming contraction and $\delta > 0$. We say that $n \in \mathbb{N}$ is an $(\alpha, \delta)$-zooming time for $p \in X$ if there exists a neighbourhood $V_n(p)$ of $p$ such that

- $f^n$ sends $V_n(p)$ homeomorphically onto $B_\delta(f^n(p))$;
- $d(f^j(x), f^j(y)) \leq \alpha_{n-j} d(f^n(x), f^n(y))$ for every $x, y \in V_n(p)$ and every $0 \leq j < n$. 

We call $B_\delta(f^n(p))$ a \textbf{zooming ball} and $V_n(p)$ a \textbf{zooming pre-ball}.

We denote by $Z_n(\alpha, \delta, f)$ the set of points in $M$ for which $n$ is an $(\alpha, \delta)$-zooming time.

**Definition 3.** (Zooming measure) A $f$-non-singular finite measure $\mu$ defined on the Borel sets of $M$ is called a weak zooming measure if $\mu$ almost every point has infinitely many $(\alpha, \delta)$-zooming times. A weak zooming measure is called a zooming measure if

$$\limsup_{n \to \infty} \frac{1}{n} \{1 \leq j \leq n \mid x \in Z_n(\alpha, \delta, f)\} > 0,$$

$\mu$ almost every $x \in M$.

**Definition 4.** (Zooming set) A forward invariant set $\Lambda \subset M$ is called a zooming set if the above inequality holds for every $x \in \Lambda$.

**Definition 5.** (Bounded distortion) Given a measure $\mu$ with a jacobian $J_\mu f$, we say that the measure has bounded distortion if there exists $\rho > 0$ such that

$$\left| \log \frac{J_\mu f(y)}{J_\mu f(z)} \right| \leq \rho d(f^n(y), f^n(z)),$$

for every $y, z \in V_n(x)$, $\mu$-almost everywhere $x \in M$, for every hyperbolic time $n$ of $x$.

The map $f$ with an associated zooming measure is called a \textbf{zooming system}. Every non-uniformly expanding map as considered in [4], for example, is a zooming system. Theorem C in [15] guarantees the existence of invariant probabilities for a zooming system. We denote the set of all $f$-invariant probabilities as $\mathcal{M}_1^f(M)$.

Now, we state our main results. The following is similar to Theorem A in [9]. It is also somehow related to Theorem 11 in [10].

**Theorem A.** Let $f : M \to M$ be a continuous zooming system with the zooming set $\Lambda$ dense on $M$. Given a $\beta$-Hölder continuous potential $\phi : M \to \mathbb{R}$ such that

$$m(\phi, f) := \min_{\eta \in \mathcal{M}_1^f(M)} \left\{ \int \phi d\eta \right\} \geq 0,$$

then there exists a continuous function $\lambda_0 : M \to \mathbb{R}$ such that

$$\phi \geq \lambda_0 - \lambda_0 \circ f.$$

If $m(\phi, f) > 0$ then $\lambda_0$ can be taken $\gamma$-Hölder for some $\gamma$.

The following result is similar to Proposition 10 in [10]. Denote by $C^0(M, \mathbb{R})$ the set of continuous functions $\phi : M \to \mathbb{R}$ with norm $\| \cdot \|_\infty$.

**Theorem B.** Let $K$ be a compact convex subset of the set of probability measures on $M$ and $(\mathcal{H}, \| \cdot \|_\mathcal{H})$ be a dense Banach space in $C^0(M, \mathbb{R})$ which embeds continuously in $C^0(M, \mathbb{R})$. Then there exists a residual set $\mathcal{R}$ in $\mathcal{H}$ (for the $\| \cdot \|_\mathcal{H}$-topology) such that, for all $\phi \in \mathcal{R}$, if

$$m(\phi) := \max \left\{ \int \phi d\eta \mid \eta \in \mathcal{K} \right\} \text{ and } \mathcal{M}(\phi) := \left\{ \eta \in \mathcal{K} \mid \int \phi d\eta \mid \eta = m(\phi) \right\},$$

then $\mathcal{M}(\phi)$ contains a unique measure.
3. Proof of Theorem A

Let $C^\beta$ denote the space of $\beta$-Hölder continuous functions $\varphi : M \to \mathbb{R}$. We define the following norm on $C^\beta$:

$$\| \varphi \|_\beta := \sup_{x,y \in M, x \neq y} \left\{ \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\beta} \right\} < \infty.$$  

The following Lemma proves Theorem A.

**Lemma 3.1.** Given a $\beta$-Hölder function $\phi : M \to \mathbb{R}$ and $Z$ the collection of all zooming pre-balls $V := V_n(z)$, with zooming ball $B := f^n(V) = B_\delta(f^n(z))$, there exists a continuous function $\lambda : M \to \mathbb{R}$ such that

$$\phi \geq \lambda \circ f - \lambda + m(\phi, f) \text{ i.e. } \phi \geq \lambda - \lambda \circ f,$$

where $\lambda = -\lambda$. As a consequence, if $m(\phi, f) > 0$, there exists a $\gamma$-Hölder function $\lambda_0 : M \to \mathbb{R}$ for some $\gamma$ such that $\phi \geq \lambda_0 - \lambda_0 \circ f$.

**Proof.** We are assuming the zooming set $\Lambda$ dense on $M$, let $z \in \Lambda$ and $V_n(z) \in Z$. We consider $\varphi := \phi - m(\phi, f)$ and recall that $S_n \varphi = \sum_{i=0}^{n-1} \varphi \circ f^i$, $n \geq 1$ and $S_0 \varphi = 0$. Define $\lambda : \Lambda \to \mathbb{R}$ as follows. Set $Z(x) := \{V_n(z) \in Z \mid x \in V_n(z)\}$. Define

$$\lambda(x) := \inf_{V_n(z) \in Z(x)} \{S_n \varphi(x)\}.$$

We recall that

$$\| \varphi \|_\beta := \sup_{x,y \in M, x \neq y} \left\{ \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\beta} \right\} < \infty.$$  

Given $x, y \in V_n(z)$, we have

$$d(f^i(x), f^i(y)) \leq \alpha_{n-i}(d(f^n(x), f^n(y))), 0 \leq i \leq n$$

So,

$$|S_n \varphi(x) - S_n \varphi(y)| \leq \sum_{i=0}^{n-1} |\varphi(f^i(x)) - \varphi(f^i(y))| \leq \sum_{i=0}^{n-1} \frac{|\varphi(f^i(x)) - \varphi(f^i(y))|}{d(f^i(x), f^i(y))^\beta} \leq \sum_{i=0}^{n-1} \| \varphi \|_\beta \alpha_{n-i}(d(f^n(x), f^n(y)))^\beta < \| \varphi \|_\beta \sum_{i=0}^{n-1} d(f^n(x), f^n(y))^\beta \leq n \| \varphi \|_\beta d(f^n(x), f^n(y))^\beta.$$  

We used the property of zooming contractions that $\alpha_{n-i}(r) < r, r > 0$ and obtained that

$$|S_n \varphi(x) - S_n \varphi(y)| \leq n \| \varphi \|_\beta d(f^n(x), f^n(y))^\beta,$$

and it implies that

$$|\lambda(x) - \lambda(y)| \leq n \| \varphi \|_\beta d(f^n(x), f^n(y))^\beta.$$  

With $V_n(z)$ fixed, we have $n \in \mathbb{N}$ fixed. Hence, if $y \to x$ we have $f^n(y) \to f^n(x)$ uniformly because $f^n$ is uniformly continuous. It implies that $\lambda$ is continuous. The property of infimum implies that $\lambda \circ f \leq \varphi + \lambda$, which implies that

$$\varphi = \phi - m(\phi, f) \geq \lambda \circ f - \lambda \implies \phi \geq \phi - m(\phi, f) \geq (\lambda - (\lambda) \circ f,$$

and we obtain $\phi \geq \lambda - \lambda \circ f$. The function $\lambda$ is uniformly continuous on the dense set $\Lambda$ and we can extend it to the whole compact space $M$ and still have the inequality $\phi \geq \lambda - \lambda \circ f$. Moreover, if $m(\phi, f) > 0$, by denseness of the Hölder functions, there exists a $\gamma$-Hölder function.
for some γ such that for 0 < ε small enough we have \( \| \lambda - \lambda_0 \|_\infty < \epsilon \) and
\[ \phi > \lambda - \lambda \circ f + 2\epsilon \geq \lambda_0 - \lambda_0 \circ f. \] The Lemma is proved.

We readily obtain the following corollary as a weak version of Livsic Theorem.

**Corollary 3.2.** Let \( f : M \to M \) be a continuous zooming system with the zooming set \( \Lambda \) dense on \( M \). Given a \( \beta \)-Hölder continuous potential \( \phi : M \to \mathbb{R} \) such that

\[ \int \phi d\eta = 0, \text{ for every } \eta \in \mathcal{M}_1^1(M), \]

then there exist continuous functions \( \lambda_1, \lambda_2 : M \to \mathbb{R} \) such that

\[ \lambda_1 - \lambda_1 \circ f \leq \phi \leq \lambda_2 - \lambda_2 \circ f. \]

**Proof.** In fact, we have both \( \int \phi d\eta \geq 0 \) and \( \int -\phi d\eta \geq 0 \) for all \( \eta \in \mathcal{M}_1^1(M) \). By applying Theorem \( \Delta \) for both \( \phi \) and \( -\phi \), we obtain the result.

\[ \square \]

4. **Proof of Theorem \( \mathcal{E} \)**

We follow the ideas of Section 2 in \[10\]. We begin by proving that, generically, there exists a unique maximizing measure. This comes mainly from the fact that, for a compact convex set in \( \mathbb{R}^n \), among the set of hyperplanes which support the convex set, the set of those hyperplanes having an intersection reduced to a single point is generic (intersection of countably many open and dense sets). Nonetheless, the proof has to be carried in infinite dimension and requires more details.

We first recall some definitions. We say that a point \( p \) is an extremal point of a compact convex set \( C \) of \( \mathbb{R}^n \) if \( p \) is not the mid point of a segment totally included in \( C \). We say that \( p \) is strictly extremal if there exists a linear form which attains its maximum at the point \( p \) only. A classical result (see \[17\]) states that \( C \) is equal to the closed convex hull of its strictly extremal points. We reproduce now the proof of Proposition 10 in \[10\].

**Proof.** Let \( \{\phi_n\}_{n \geq 1} \) be a dense subset of the unit ball of \( \mathcal{H} \). Since \( \mathcal{H} \) is dense,

\[ d(\eta, \eta') := \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \int \phi_n d\eta - \int \phi_n d\eta' \right| \]

defines a metric on \( \mathcal{K} \) compatible with the weak* topology. Let us call

\[ \mathcal{R}_\epsilon := \{ \phi \in C^0(M, \mathbb{R}) \mid \text{diam}(\mathcal{M}(\phi)) < \epsilon \}. \]

We claim that \( \mathcal{R}_\epsilon \) is open in \( C^0(M, \mathbb{R}) \) and \( \mathcal{R}_\epsilon \cap \mathcal{H} \) is dense in \( \mathcal{H} \) for the \( \| \cdot \|_{\mathcal{H}} \)-topology. The desired residual set will be \( \mathcal{R} = \cap_{\epsilon \geq 1} \mathcal{R}_{1/\epsilon} \cap \mathcal{H} \).

We show by contradiction that \( \mathcal{R}_\epsilon \) is open. If not, one can find \( \phi \in \mathcal{R}_\epsilon \), \( \varphi_n \in C^0(M, \mathbb{R}) \) and \( \mu_n, \nu_n \in \mathcal{M}(\phi + \varphi_n) \) such that \( \| \varphi_n \|_\infty \) converges to zero and \( d(\mu_n, \nu_n) \geq \epsilon \) for all \( n \). We may assume by taking a subsequence that \( \mu_n \to \mu_0 \) and \( \nu_n \to \nu_0 \). Let us prove that \( \mu_0 \in \mathcal{M}(\phi) \):

indeed for every \( \mu \in \mathcal{K} \),

\[ \int (\phi + \varphi_n) d\mu \leq \int (\phi + \varphi_n) d\mu_n \leq \int \phi d\mu_n + \| \varphi_n \|_\infty \]

and \( \int \phi d\mu \leq \int \phi d\mu_0 \), by taking the limit on \( n \). For the same reason we have that \( \nu_0 \in \mathcal{M}(\phi) \).

We have obtained the contradiction once we have \( d(\mu_0, \nu_0) \geq \epsilon \).

We now show that \( \mathcal{R}_\epsilon \cap \mathcal{H} \) is dense in \( \mathcal{H} \). Let \( \phi_0 \in \mathcal{H} \) and \( \mathcal{K}_0 = \mathcal{M}(\phi_0) \). The continuous projection \( \pi_n : \mathcal{K} \to \mathbb{R}^n \), \( \pi_n(\mu) = (\int \phi_1 d\mu, \ldots, \int \phi_n d\mu) \) sends \( \mathcal{K}_0 \) to a compact convex set.
Let \( n \) be large enough so that \( k^{-n} < \epsilon \). By definition of \( n = (p^1, \ldots, p^n) \) there exists \((a^1, \ldots, a^n) \in R^n\) such that

\[
\sum_{i=1}^{n} a^i p^i > \sum_{i=1}^{n} a^i q^i \quad \text{for all } q = (q^1, \ldots, q^n) \in \pi_n(K_0), q \neq p.
\]

In particular, if \( \psi = \sum_{i=1}^{n} a^i \phi_i \),

\[
m_0(\psi) := \max \left\{ \int \psi d\eta \mid \eta \in K_0 \right\} \text{ and } M_0(\psi) := \left\{ \eta \in K_0 \mid \int \psi d\eta = m(\psi) \right\},
\]

then \( M_0(\psi) = \pi_n^{-1}(p_n) \) has diameter less than \( \epsilon \). We show that for small enough \( \delta > 0 \), \( \psi_\delta = (1 - \delta)\phi + \delta \psi \in R_\epsilon \). More precisely we show that, for any open set \( U \supset M_0(\psi) \), for any sufficiently small \( \delta > 0 \) we have \( U \supset M(\psi_\delta) \). By contradiction, there exists a sequence \( \mu_n \in M(\psi_\delta) \setminus U \) for some \( \delta_n \to 0 \). We may assume that \( \mu_n \to \mu_0 \in K \setminus U \). We first show that \( \mu_0 \in K_0 = M(\phi) \) for every \( \mu \in K \),

\[
\int \psi_\delta \, d\mu_n \leq \int \psi_\delta \, d\mu_n + \delta_n \| \psi - \varphi \|_\infty,
\]

and by taking limit in \( n \), \( \int \varphi \, d\mu \leq \int \varphi \, d\mu_0 \). We then show that \( \mu_0 \in M_0(\psi) \) for every \( \mu \in K_0 \),

\[
\int \psi_\delta \, d\mu = (1 - \delta_n) \int \psi_\delta \, d\mu + \delta_n \int \psi \, d\mu \leq (1 - \delta_n) \int \varphi \, d\mu_n + \delta_n \int \varphi \, d\mu_n.
\]

Since \( \int \varphi \, d\mu_n \leq \int \varphi \, d\mu \), we have obtained \( \int \psi_\delta \, d\mu \leq \int \psi_\delta \, d\mu_n \) and at the limit \( \int \psi \, d\mu \leq \int \psi \, d\mu_0 \).

We have obtained a contradiction since \( \mu_0 \notin U \).

Analogously to Theorem 6 in [10] we obtain the next corollary.

**Corollary 4.1.** Let \( f : M \to M \) be a continuous zooming system with the zooming set \( \Lambda \) dense on \( M \). Then the set of \( \beta \)-Hölder functions \( \phi \) admitting a unique maximizing measure is generic in \( C^\beta \). For such functions \( \phi \), the map \( f \) is strictly ergodic on the support of its unique maximizing measure.

**Proof.** This is a direct consequence of Theorem 3 by taking \( K = M^1(M) \) and \( H = C^\beta \), once \( K = M^1(M) \) is a compact and convex set of probability measures and \( C^\beta \) is dense in \( C^0(M, \mathbb{R}) \) for any compact metric space \( M \).

5. Applications

In this section, we give examples of zooming systems. Most of them are nonuniformly expanding maps in the sense of [1] or [4]. The reference [15] contains a more general approach. We begin by the hyperbolic times as a particular case of zooming times.

5.1. **Hyperbolic Times.** The idea of hyperbolic times is a key notion on the study of non-uniform hyperbolic dynamics and it was introduced by Alves et al. This is powerful to get expansion in the context of non-uniform expansion. Here, we recall the basic definitions and results on hyperbolic times that we will use later on. In the following, we give definitions taken from [4] and [15].
Definition 6. Let $M$ be a compact Riemannian manifold of dimension $d \geq 1$ and $f : M \to M$ a continuous map defined on $M$. The map $f$ is called non-flat if it is a local $C^{1+\alpha}_f(x)$, $(\alpha > 0)$ diffeomorphism in the whole manifold except in a non-degenerate set $C \subset M$. We say that $C \subset M$ is a non-degenerate set if there exist $\beta, B > 0$ such that the following two conditions hold.

- $\frac{1}{B}d(x,C)^\beta \leq \frac{\|Df(x)\|}{\|v\|} \leq Bd(x,C)^{-\beta}$ for all $v \in T_xM$.
- For every $x, y \in M \setminus C$ with $d(x, y) < d(x, C)/2$ we have $| \log \| Df(x)^{-1} \| - \log \| Df(y)^{-1} \| | \leq \frac{B}{d(x,y)}d(x,y)$.

In the following, we give the definition of a hyperbolic time $[1], [15]$.

Definition 7. (Hyperbolic times). Let us fix $0 < b = \frac{1}{4}\min\{1, 1/\beta\} < \frac{1}{4}\min\{1, 1/\beta\}$. Given $0 < \sigma < 1$ and $\epsilon > 0$, we will say that $n$ is a $(\sigma, \epsilon)$-hyperbolic time for a point $x \in M$ (with respect to the non-flat map $f$ with a $\beta$-non-degenerate critical/singular set $C$) if for all $1 \leq k \leq n$ we have

$$\prod_{j=n-k}^{n-1} \| (Df \circ f^j(x)^{-1} \| \leq \sigma^k$$

and $\text{dist}_c(f^{n-k}(x), C) \geq \sigma^k$. We denote de set of points of $M$ such that $n \in \mathbb{N}$ is a $(\sigma, \epsilon)$-hyperbolic time by $H_n(\sigma, \epsilon, f)$.

Proposition 5.1. (Positive frequency). Given $\lambda > 0$ there exist $\theta > 0$ and $\epsilon_0 > 0$ such that, for every $x \in U$ and $\epsilon \in (0, \epsilon_0]$,

$$\#\{1 \leq j \leq n; x \in H_j(e^{-\lambda/4}, \epsilon, f)\} \geq \theta n,$$

whenever $\frac{1}{n} \sum_{i=0}^{n-1} \log \| (Df(f^i(x)))^{-1} \|^{-1} \geq \lambda$ and $\frac{1}{n} \sum_{i=0}^{n-1} - \log \text{dist}_c(x, C) \leq \frac{1}{16}\theta$.

If $f$ is non-uniformly expanding, it follows from the proposition that the points of $U$ have infinitely many moments with positive frequency of hyperbolic times. In particular, they have infinitely many hyperbolic times.

The following proposition shows that the hyperbolic times are indeed zooming times, where the zooming contraction is $\alpha_k(r) = \sigma^{k/2}r$.

Proposition 5.2. Given $\sigma \in (0, 1)$ and $\epsilon > 0$, there is $\delta, \rho > 0$, depending only on $\sigma$ and $\epsilon$ and on the map $f$, such that if $x \in H_n(\sigma, \epsilon, f)$ then there exists a neighbourhood $V_n(x)$ of $x$ with the following properties for all $y, z \in V_n(x)$:

1. $f^n$ maps $V_n(x)$ diffeomorphically onto the ball $B_\delta(f^n(x))$;
2. $\text{dist}(f^n(y), f^n(z)) \leq \sigma^{j/2}\text{dist}(f^n(y), f^n(z))$, for all $y, z \in V_n(x)$ and $1 \leq j \leq n$.
3. $\log \frac{\|\text{det}Df^n(y)\|}{\|\text{det}Df^n(z)\|} \leq pd(f^n(y), f^n(z))$.

The sets $V_n(x)$ are called hyperbolic pre-balls and their images $f^n(V_n(x)) = B_\delta(f^n(x))$, hyperbolic balls.

5.2. Uniformly Expanding Maps. As can be seen in [14] Chapter 11, we have the so-called uniformly expanding maps which is defined on a compact differentiable manifold $M$ as a $C^1$ map $f : M \to M$ (with no critical set) for which there exists $\sigma > 1$ such that

$$\|Df(x)v\| \geq \sigma \|v\|, \text{ for every } x \in M, v \in T_xM.$$
For compact metric spaces \((M,d)\) we define it as a continuous map \(f : M \to M\), for which there exists \(\sigma > 1, \delta > 0\) such that for every \(x \in M\) we have that the image of the ball \(B(x,\delta)\) contains a neighbourhood of the ball \(B(f(x),\delta)\) and
\[
d(f(a), f(b)) \geq \sigma d(a, b), \text{ for every } a, b \in B(x, \delta).
\]
We observe that the uniformly expanding maps on differentiable manifolds satisfy the conditions for the definition on compact metric spaces, when they are seen as Riemannian manifolds. If the metric space \(M\) is connected, uniform expansion implies topological exactness. The zooming set \(\Lambda\) is the whole space \(M\).

5.3. Local Diffeomorphisms. As can be seen in details in [1], we will briefly describe a class of non-uniformly expanding maps.

Here we present a robust (\(C^1\) open) classes of local diffeomorphisms (with no critical set) that are non-uniformly expanding. Such classes of maps can be obtained, e.g., through deformation of a uniformly expanding map by isotopy inside some small region. In general, these maps are not uniformly expanding: deformation can be made in such way that the new map has periodic saddles.

Let \(M\) be a compact manifold supporting some uniformly expanding map \(f_0\). \(M\) could be the \(d\)-dimensional torus \(\mathbb{T}^d\), for instance. Let \(V \subset M\) be some small compact domain, so that the restriction of \(f_0\) to \(V\) is injective. Let \(f\) be any map in a sufficiently small \(C^1\)-neighbourhood \(N\) of \(f_0\) so that:

- \(f\) is \textit{volume expanding everywhere}: there exists \(\sigma_1 > 1\) such that 
  \(\det Df(x) > \sigma_1\) for every \(x \in M\);
- \(f\) is \textit{expanding outside \(V\)}: there exists \(\sigma_0 > 1\) such that 
  \(\|DF(x)^{-1}\| < \sigma_0\) for every \(x \in M\setminus V\);
- \(f\) is \textit{not too contracting on \(V\)}: there is some small \(\delta > 0\) such that 
  \(\|DF(x)^{-1}\| < 1 + \delta\) for every \(x \in V\).

In [1] it is shown that this class satisfy the condition for non-uniform expansion. We have here the zooming set \(\Lambda\) dense on \(M\).

Now, we recall examples given in [4], where the expanding set is dense on \(M\) and there exist critical points.

5.4. Viana maps. We recall the definition of the open class of maps with critical sets in dimension 2, introduced by M. Viana in [21]. We skip the technical points. It can be generalized for any dimension (See [1]).

Let \(a_0 \in (1,2)\) be such that the critical point \(x = 0\) is pre-periodic for the quadratic map \(Q(x) = a_0 - x^2\). Let \(S^1 = \mathbb{R}/\mathbb{Z}\) and \(b : S^1 \to \mathbb{R}\) a Morse function, for instance \(b(\theta) = \sin(2\pi \theta)\). For fixed small \(\alpha > 0\), consider the map
\[
f_0 : S^1 \times \mathbb{R} \to S^1 \times \mathbb{R}, \quad (\theta, x) \mapsto (g(\theta), q(\theta, x))
\]
where \(g\) is the uniformly expanding map of the circle defined by \(g(\theta) = d\theta (mod \mathbb{Z})\) for some \(d \geq 16\), and \(q(\theta, x) = a(\theta) - x^2\) with \(a(\theta) = a_0 + \alpha b(\theta)\). It is easy to check that for \(\alpha > 0\) small enough there is an interval \(I \subset (-2,2)\) for which \(f_0(S^1 \times I)\) is contained in the interior of \(S^1 \times I\), as can be seen in [21]. Thus, any map \(f\) sufficiently close to \(f_0\) in the \(C^0\) topology has \(S^1 \times I\) as a forward invariant region. We consider from here on these maps \(f\) close to \(f_0\).
restricted to $S^1 \times I$. Taking into account the expression of $f_0$ it is not difficult to check that for $f_0$ (and any map $f$ close to $f_0$ in the $C^2$ topology) the critical set is non-degenerate. Also can be seen in [24].

The main properties of $f$ in a $C^3$ neighbourhood of $f$ that we will use here are summarized below (See [1], [6], [15]):

1. $f$ is non-uniformly expanding, that is, there exist $\lambda > 0$ and a Lebesgue full measure set $H \subset S^1 \times I$ such that for all point $p = (\theta, x) \in H$, the following holds

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \| Df(f^i(p))^{-1} \| < -\lambda.$$ 

2. Its orbits have slow approximation to the critical set, that is, for every $\epsilon > 0$ the exists $\delta > 0$ such that for every point $p = (\theta, x) \in H \subset S^1 \times I$, the following holds

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} - \log \text{dist}_\delta(p, C) < \epsilon.$$ 

where

$$\text{dist}_\delta(p, C) = \begin{cases} 
\text{dist}(p, C), & \text{if } \text{dist}(p, C) < \delta \\
1 & \text{if } \text{dist}(p, C) \geq \delta
\end{cases}$$

3. $f$ is topologically exact;
4. $f$ is strongly topologically transitive;
5. it has a unique ergodic absolutely continuous invariant (thus SRB) measure;
6. the density of the SRB measure varies continuously in the $L^1$ norm with $f$.

**Remark 1.** We observe that this definition of non-uniformly expanding map implies positive frequency of hyperbolic times as can be seen in [1]. So, it is included in our definition of a zooming system. Also, the zooming set is dense on $M = S^1 \times I$.

5.5. **Benedicks-Carleson Maps.** We study a class of non-hyperbolic maps of the interval with the condition of exponential growth of the derivative at critical values, called Collet-Eckmann Condition. We also ask the map to be $C^2$ and topologically exact and the critical points to have critical order $2 \leq \alpha < \infty$.

Given a critical point $c \in I$, the critical order of $c$ is a number $\alpha_c > 0$ such that $f(x) = f(c) + \mid g_c(x) \mid^{\alpha_c}$, for all $x \in U_c$ where $g_c$ is a diffeomorphism $g_c : U_c \to g(U_c)$ and $U_c$ is a neighbourhood of $c$.

Let $\delta > 0$ and denote $C$ the set of critical points and $B_\delta = \cup_{c \in C} (c - \delta, c + \delta)$. Given $x \in I$, we suppose that

- **(Expansion outside $B_\delta$).** There exists $\kappa > 1$ and $\beta > 0$ such that, if $x_k = f^k(x) \not\in B_\delta$, $0 \leq k \leq n - 1$ then $|Df^n(x)| \geq \kappa \delta^{(\alpha_{\max} - 1)} e^{\beta n}$, where $\alpha_{\max} = \max\{|\alpha_c, c \in C\}$.

Moreover, if $x_0 \in f(B_\delta)$ or $x_n \in B_\delta$ then $|Df^n(x)| \geq \kappa e^{\beta n}$.

- **(Collet-Eckmann Condition).** There exists $\lambda > 0$ such that $|Df^n(f(c))| \geq e^{\lambda n}$.

- **(Slow Recurrence to $C$).** There exists $\sigma \in (0, \lambda/5)$ such that $\text{dist}(f^k(x), C) \geq e^{-\sigma k}$.

The above conditions has an important contribution by Freitas in [12]. This is an important class of non-uniformly expanding maps.
5.6. **Rovella Maps.** There is a class of non-uniformly expanding maps known as **Rovella Maps.** They are derived from the so-called **Rovella Attractor**, a variation of the **Lorenz Attractor.** We proceed with a brief presentation. See [5] for details.

5.6.1. **Contracting Lorenz Attractor.** The geometric Lorenz attractor is the first example of a robust attractor for a flow containing a hyperbolic singularity. The attractor is a transitive maximal invariant set for a flow in three-dimensional space induced by a vector field having a singularity at the origin for which the derivative of the vector field at the singularity has real eigenvalues \( \lambda_2 < \lambda_3 < 0 < \lambda_1 \) with \( \lambda_1 + \lambda_3 > 0 \). The singularity is accumulated by regular orbits which prevent the attractor from being hyperbolic.

The geometric construction of the contracting Lorenz attractor (Rovella attractor) is the same as the geometric Lorenz attractor. The only difference is the condition (A1)(i) below that gives in particular \( \lambda_1 + \lambda_3 < 0 \). The initial smooth vector field \( X_0 \) in \( \mathbb{R}^3 \) has the following properties:

(A1) \( X_0 \) has a singularity at 0 for which the eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R} \) of \( DX_0(0) \) satisfy:

(i) \( 0 < \lambda_1 < -\lambda_3 < 0 < -\lambda_2 \),

(ii) \( r > s + 3 \), where \( r = -\lambda_3/\lambda_1, s = -\lambda_3/\lambda_1 \);

(A2) there is an open set \( U \subset \mathbb{R}^3 \), which is positively invariant under the flow, containing the cube \( \{(x, y, z) : |x| \leq 1, |y| \leq 1, |x| \leq 1\} \) and supporting the **Rovella attractor**

\[
\Lambda_0 = \bigcap_{t \geq 0} X_t^0(U).
\]

The top of the cube is a Poincaré section foliated by stable lines \( \{x = \text{const}\} \cap \Sigma \) which are invariant under Poincaré first return map \( P_0 \). The invariance of this foliation uniquely defines a one-dimensional map \( f_0 : I \setminus \{0\} \to I \) for which

\[
f_0 \circ \pi = \pi \circ P_0,
\]

where \( I \) is the interval \([-1, 1]\) and \( \pi \) is the canonical projection \( (x, y, z) \mapsto x \);

(A3) there is a small number \( \rho > 0 \) such that the contraction along the invariant foliation of lines \( x = \text{const} \) in \( U \) is stronger than \( \rho \).

See [5] for properties of the map \( f_0 \).

5.6.2. **Rovella Parameters.** The Rovella attractor is not robust. However, the chaotic attractor persists in a measure theoretical sense: there exists a one-parameter family of positive Lebesgue measure of \( C^3 \) close vector fields to \( X_0 \) which have a transitive non-hyperbolic attractor. In the proof of that result, Rovella showed that there is a set of parameters \( E \subset (0, a_0) \) (that we call **Rovella parameters**) with \( a_0 \) close to 0 and 0 a full density point of \( E \), i.e.

\[
\lim_{a \to 0} \frac{|E \cap (0, a)|}{a} = 1,
\]

such that:

(C1) there is \( K_1, K_2 > 0 \) such that for all \( a \in E \) and \( x \in I \)

\[
K_2 |x|^{s-1} \leq f'_a(x) \leq K_1 |x|^{s-1},
\]

where \( s = s(a) \). To simplify, we shall assume \( s \) fixed.

(C2) there is \( \lambda_c > 1 \) such that for all \( a \in E \), the points 1 and \(-1\) have **Lyapunov exponents** greater than \( \lambda_c \):

\[
(f^n_a)'(\pm 1) > \lambda_c^n, \text{ for all } n \geq 0;
\]
(C3) there is $\alpha > 0$ such that for all $a \in E$ the basic assumption holds:

$$\left| f_a^{n-1}(\pm 1) \right| e^{-\alpha n}, \text{ for all } n \geq 1;$$

(C4) the forward orbits of the points $\pm 1$ under $f_a$ are dense in $[-1, 1]$ for all $a \in E$.

**Definition 8.** We say that a map $f_a$ with $a \in E$ is a **Rovella Map**.

**Theorem 5.3.** (Alves-Soufi [5]) Every Rovella map is non-uniformly expanding and has slow recurrence to the critical set.

In the following, we give definitions for a map on a metric space to have similar behaviour to maps with hyperbolic times and which can be found in [15].

Given $M$ a metric spaces and $f : M \to M$, we define for $p \in M$:

$$D^-(p) = \lim \inf d(f(x), f(p))$$

Define also,

$$D^+(p) = \lim \sup d(f(x), f(p))$$

We will consider points $x \in M$ such that

$$\lim \sup_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log D^- \circ f^i(x) > 0.$$  

The critical set is the set of points $x \in M$ such that $D^-(x) = 0$ or $D^+(x) = \infty$. For the non-degnerateness we ask that there exist $B, \beta > 0$ such that

- $\frac{1}{2}d(x, C)\beta \leq D^-(x) \leq D^+(x) \leq Bd(x, C)^{-\beta}$.

For every $x, y \in M \setminus C$ with $d(x, y) < d(x, C)/2$ we have

- $| \log D^-(x) - \log D^-(y) | \leq \frac{B}{d(x, C)}d(x, y)$.

With these conditions we can see that all the consequences for hyperbolic times are valid here and the expanding sets and measures are zooming sets and measures.

**Definition 9.** We say that a map is conformal at $p$ if $D^-(p) = D^+(p)$. So, we define

$$D(p) = \lim \frac{d(f(x), f(p))}{d(x, p)}$$

Now, we give an example of such an open non-uniformly expanding map.

**5.7. Expanding sets on a metric space.** Let $\sigma : \Sigma^+ \to \Sigma^+$ be the one-sided shift, with the usual metric:

$$d(x, y) = \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n},$$

where $x = \{x_n\}, y = \{y_n\}$. We have that $\sigma$ is a conformal map such that $D^-(x) = 2$, for all $x \in \Sigma^+$. Also, every forward invariant set (in particular the whole $\Sigma^+$) and all invariant measure for the shift $\sigma$ are expanding.
5.8. Zooming sets on a metric space (not expanding). Let $\sigma : \Sigma^+_2 \to \Sigma^+_2$ be the one-sided shift, with the following metric for $\sum_{n=1}^{\infty} b_n < \infty$:

$$d(x, y) = \sum_{n=1}^{\infty} b_n |x_n - y_n|,$$

where $x = \{x_n\}$, $y = \{y_n\}$ and $b_{n+k} \leq b_n b_k$ for all $n, k \geq 1$. By induction, it means that $b_n \leq b^n_1$. Let us suppose that $b_n \leq a_n := (n+b)^{-a}$, $a > 1$, $b > 0$ for all $n \geq 1$.

We claim that $a_n$ defines a Lipschitz contraction for the shift map. We require that there exists $n_0 > 1$ such that $b_n > a^n_1 \geq b^n_1$ for $n \leq n_0$. So, the contraction is not exponential. In fact, if $x, y$ belongs to the cylinder $C_k$ we have

$$d(x, y) = \sum_{n=1}^{\infty} b_n |x_n - y_n| = \sum_{n=k+1}^{\infty} b_n |x_n - y_n| = \sum_{n=1}^{\infty} b_{n+k} |x_{n+k} - y_{n+k}| \leq b_k \sum_{n=1}^{\infty} b_n |x_{n+k} - y_{n+k}| = b_k d(\sigma^k(x), \sigma^k(y)) \leq a_k d(\sigma^k(x), \sigma^k(y)).$$

It implies that

$$d(\sigma^i(x), \sigma^i(y)) \leq a_{k-i} d(\sigma^{k-i}(\sigma^i(x)), \sigma^{k-i}(\sigma^i(y))) = a_{k-i} d(\sigma^k(x), \sigma^k(y)), i \leq k.$$  

It means that the sequence $a_n$ defines a Lipschitz contraction, as we claimed.

Every forward invariant set (in particular the whole $\Sigma^+_2$) and all invariant measure for the shift $\sigma$ are zooming.

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