INVISCID LIMIT FOR VORTEX PATCHES
IN A BOUNDED DOMAIN

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Abstract: In this paper, we consider the inviscid limit of the incompressible Navier-Stokes equations in a smooth, bounded and simply connected domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. We prove that for a vortex patch initial data the weak Leray solutions of the incompressible Navier-Stokes equations with Navier boundary conditions will converge (locally in time for $d = 3$ and globally in time for $d = 2$) to a vortex patch solution of the incompressible Euler equation as the viscosity vanishes. In view of the results obtained in [1] and [19] which dealt with the case of the whole space, we derive an almost optimal convergence rate $(\nu t)^{\frac{1}{2} - \varepsilon}$ for any small $\varepsilon > 0$ in $L^2$.

Keywords: inviscid limit, Navier boundary condition, vortex patches

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1. INTRODUCTION

The incompressible Navier-Stokes equations read as
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= 0, \\
\text{div} \ u &= 0,
\end{aligned}
\]  

where $u = (u_1, \cdots, u_d)(d = 2 \text{ or } 3)$ is the velocity fields, $p$ is the pressure function and $\nu$ is the kinetic viscosity.
Formally, when \( \nu = 0 \), (1.1) becomes the following incompressible Euler equations:

\[
\begin{aligned}
\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p &= 0, \\
\text{div } u &= 0.
\end{aligned}
\] (1.2)

The inviscid limit for the incompressible Navier-Stokes equations in the whole space has been well understood (see [22, 5, 19, 17, 1, 7, 8] and references therein) for both smooth and non-smooth initial data. However, in the case of a bounded domain, the inviscid limit for the Navier-Stokes equations with Dirichlet boundary conditions is still a completely open problem. This is mainly due to the difference between the Dirichlet boundary conditions of the incompressible Navier-Stokes equations (1.1) and the tangential boundary conditions of the incompressible Euler equations (1.2) and a boundary layer will appear near the boundary of the domain.

This paper is concerned with the inviscid limit problem of the incompressible Navier-Stokes equations (1.1) with the following Navier boundary conditions:

\[
\begin{aligned}
u \cdot \vec{n} &= 0, \\
[D(u)\vec{n} + \alpha u]_{\text{tan}} &= 0, \quad \text{on } \partial \Omega \times (0, +\infty),
\end{aligned}
\] (1.3)

where \( \Omega \subset \mathbb{R}^d \) is a smooth bounded domain, \( \vec{n} \) is the unit exterior normal to the boundary \( \partial \Omega \), \( D(u) = \frac{1}{2}[\nabla u + (\nabla u)^T] \) is the rate of strain tensor and \( [D(u)\vec{n} + \alpha u]_{\text{tan}} \) is the tangential component of the vector \( D(u)\vec{n} + \alpha u \). Here \( \alpha = \alpha(x, t) \) is a known function representing the friction coefficient of the material.

The Navier boundary conditions, introduced by Navier in [20], say that the tangential component of the viscous stress at the boundary is proportional to the tangential velocity. They were rigorously justified as a homogenization of the no-slip condition on a rough boundary in [14] and widely used when studying the inviscid limit of the incompressible flows in a bounded domain (see [6, 13, 15, 16, 18, 24]) in recent years.

Of particular interest of this paper is the inviscid limit for vortex patches in a bounded domain. It is known that when the initial data are vortex patch ones (see Definition 2.1 for details), there exists a unique solution to the incompressible Euler equations which preserves the vortex-patch structures globally (in time) in the whole plane ([4, 2]) and locally in three-dimensional whole space ([11]). In a smooth, bounded and simply connected domain \( \Omega \subset \mathbb{R}^d \) \((d = 2, 3)\), the vortex patch solutions of the incompressible Euler equations were derived in [10, 9]. In this paper, we will show that the weak Leray solutions of the incompressible Navier-Stokes equations with Navier boundary conditions will tend to a vortex patch solution of the incompressible Euler equations as the viscosity vanishes if the initial data are vortex patch ones. Moreover, we obtain that the convergence
rate in $L^2$ is $(\nu t)^{3/4 - \varepsilon}$ for any small $\varepsilon > 0$. In the case of the whole plane, Constantin and Wu studied the inviscid limit for the 2D vortex patches in [7,8] and obtained the convergence rate in $L^2$ is $\sqrt{\nu t}$. Abidi and Danchin improved the convergence rate to be $(\nu t)^{3/4}$ in $L^2$ which is optimal since the circular vortex patches provide a lower bound (see [1]). Later, Masmoudi extended the results to the case of three-dimensional whole space in [19]. Recently, Sueur [21] dealt with the vorticity internal transition layers for the Navier-Stokes equations and described how the smoothing effect is (micro-)localized in the case where vortex patches are prescribed as initial data, using the method of asymptotic expansion. In the case of the two-dimensional bounded domain, the inviscid limit for the incompressible Navier-Stokes equations with Navier- boundary conditions was discussed in [16] and the obtained convergence rate in $L^2$ is $\sqrt{\nu t}$ for initial vorticity in $L^\infty$.

Our results here applies to both 2D and 3D vortex patches in a bounded domain and the convergence rate obtained in this paper is almost optimal in view of the results in [1] and [19]. Since we consider the case of the bounded domain, estimates in Besov space in [1] [19] can not be used directly and we will use the interpolation space theory to deduce that the vorticity belongs to $L^\infty([0, T^*); H^s(\Omega))$ for some $T^* > 0$ and $s > 0$. More subtle estimates will be given in this paper. Meanwhile, whether the convergence rate can be improved to $(\nu t)^{3/4}$ is still open.

The paper is organized as follows. In Section 2, we will give some preliminaries and the main results. Section 3 is devoted to the proof of the main result.

2. Preliminaries and Main Results

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a smooth, bounded and simply connected domain. The initial-boundary problem to the incompressible Euler equations is written as

$$
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + \nabla p &= 0, \quad (x, t) \in \Omega \times (0, +\infty), \\
\text{div} \ u &= 0, \quad (x, t) \in \Omega \times [0, +\infty), \\
u x \cdot \vec{n} &= 0, \quad (x, t) \in \partial \Omega \times [0, +\infty), \\
u u(x, 0) &= u_0(x), \quad x \in \Omega.
\end{align*}
$$

Denote by $(u', p')$ the solutions of the incompressible Navier-Stokes equations with corresponding kinetic viscosity $\nu$. The initial-boundary problem to the incompressible Navier-Stokes equations with Navier boundary
conditions is written as
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p &= 0, \quad (x, t) \in \Omega \times (0, +\infty) \\
\text{div } u &= 0, \quad (x, t) \in \Omega \times [0, +\infty), \\
u u \cdot \vec{n} &= 0, \quad [D(u')\vec{n} + \alpha u']_{\text{tan}} = 0, \quad (x, t) \in \partial\Omega \times [0, +\infty), \\
\Phi(x, 0) &= \Phi_0(x), \quad x \in \Omega.
\end{aligned}
\]
(2.2)

Let \( \omega_0 = \text{curl } u_0 \) be the initial vorticity of \( u_0 \). In this paper, for any vector-valued function \( \varphi \), \( D(\varphi) \) denotes the symmetric part of \( \nabla \varphi \), i.e.,
\[\frac{\nabla \varphi + (\nabla \varphi)^T}{2}.
\]

Denote by \( C^r, C^{1+r} \) (\( 0 < r < 1 \)) the usual Hölder space. In particular, \( C^r(\mathbb{R}^d) \) consists of functions in \( C^r(\mathbb{R}^d) \) with compact support. Let \( L^p(\Omega), W^{s,p}(\Omega) \) be the usual Sobolev spaces defined in \( \Omega \), where \( 1 \leq p \leq \infty \) and \( s \) is permitted to be a real number. If \( p = 2 \), \( W^{s,2}(\Omega) \) is denoted by \( H^s(\Omega) \). \( H^s_0(\Omega) \) is the closure of \( C^\infty_0(\Omega) \) in \( H^s(\Omega) \). Define
\[
C^\infty_{0,\sigma}(\Omega) = \{ f | f \in C^\infty_0(\Omega), \text{ div } f = 0 \},
\]
\[
C^\infty_{\sigma}(\Omega) = \{ f | f \in C^\infty(\Omega), \text{ div } f = 0 \}.
\]
\( L^2_{\sigma}(\Omega) \) is the closure of \( C^\infty_{0,\sigma}(\Omega) \) in \( L^2(\Omega) \), and \( H^1_{\sigma}(\Omega) \) is the closure of \( C^\infty_{\sigma}(\Omega) \) in \( H^1(\Omega) \).

We first recall the definition of a vortex patch in a bounded domain (see [10], [19]).

**Definition 2.1** Let \( 0 < r < 1 \). The vorticity \( \omega = \text{curl } u \) of a vector field \( u \) is called a \( C^r \) vortex patch of support \( P \) if the following decomposition holds:
\[
\omega = (\omega_i \chi_P + \omega_e \chi_{\Omega \setminus P})|_{\Omega},
\]
where \( P \) is an open set of class \( C^{1+r} \), \( \omega_i, \omega_e \in C^r_c(\mathbb{R}^d) (d = 2, 3) \) and \( \chi_P, \chi_{\Omega \setminus P} \) are the characteristic functions of \( P \) and \( \Omega \setminus P \) respectively.

Notice that when \( d = 3 \), \( \text{curl } u \) is of divergence free, we need \( \omega_i \cdot \vec{n} = \omega_e \cdot \vec{n} \) on \( \partial P \).

If the initial data of the incompressible Euler equations is a \( C^r \) vortex patch, the global existence of 2-d vortex patch solutions and the local existence of 3-d vortex patch solutions have been proved (see [10], [9]). More precisely, one has

**Theorem 2.1** Let \( u_0 \) be a divergence free vector field in \( \mathbb{R}^d (d = 2, 3) \), tangent to \( \partial \Omega \), whose vorticity \( \omega_0 \) is a \( C^r \) vortex patch of support \( P \), the boundary of \( \partial P \) is a \((d - 1)\)-dimensional compact submanifold of \( \mathbb{R}^d \). If \( \bar{P} \subset \Omega \), then there exists a \( T^* > 0 \) such that the Euler equations (2.1) have
a (unique) solution \( u \in L^{\infty}([0, T^*); \text{Lip}(\Omega)) \). Moreover, \( \omega(t) = \text{curl} \ u(t) \) remains a vortex patch, whose support \( \Psi(t, P) \) is of class \( C^{1+r} \) for any \( t \in [0, T^*) \), \( \Psi \) denoting the flow of \( u \). In addition, \( T^* > 0 \) can be arbitrarily large if \( d = 2 \).

**Remark 2.1** Under assumptions of Theorem 2.1, if \( P \) is tangent to \( \partial \Omega \), a little regularity may be lost. However, local existence of 3-D vortex patch of \( C^s(0 < s < r) \) and global existence of 2-D vortex patch of \( C^s(0 < s < r) \) is proved in [10].

Now we give the definition of a Leray weak solution of the incompressible Navier-Stokes equations with Navier boundary conditions.

**Definition 2.2** We call a vector field \( u^\nu(t, x) : [0, +\infty) \times \Omega \rightarrow \Omega \), denoted by \( u(t, x) \), a weak Leray solution of (2.2) if \( u \) verifies

1. \( u \in C_w([0, \infty); L^2(\Omega)) \cap L^2_{\text{loc}}([0, \infty); \text{H}^1(\Omega)) \);
2. \( u \) verifies the system of equations (2.2) under the following weak form: for every \( \varphi \in C^\infty_0([0, \infty); \text{C}^\infty_{\sigma}(\Omega)) \) with \( \varphi \cdot \vec{n} = 0 \) on \( \partial \Omega \),

\[
2\alpha \nu \int_0^\infty \int_{\partial \Omega} u \cdot \varphi + 2\nu \int_0^\infty \int_{\Omega} D(u)D(\varphi) + \int_0^\infty \int_{\Omega} (u \cdot \nabla)u \cdot \varphi
= \int_0^\infty \int_{\Omega} u_0 \cdot \partial_t \varphi + \int_{\Omega} u(0) \cdot \varphi(0)
\]

3. \( u \) verifies the energy inequality, for all \( t \geq 0 \),

\[
\|u(t)\|^2_{L^2(\Omega)} + 4\alpha \nu \int_0^t \int_{\partial \Omega} |u|^2 + 4\nu \int_0^t \int_{\Omega} |D(u)|^2 \leq \|u(0)\|^2_{L^2(\Omega)}.
\]

We remark that the global existence of the Leray weak solution in the case of Dirichlet boundary conditions is well known for any \( u_0 \in L^2(\Omega) \). The extensions of this result to the case of Navier boundary conditions is straightforward by the Galerkin method.

The main result of the paper is stated as

**Theorem 2.2** Suppose that the assumptions of Theorem 2.1 hold and \( u \in L^{\infty}([0, T^*); \text{Lip}(\Omega)) \) is the vortex patch solution of the incompressible Euler equations with initial data \( u_0 \). Suppose that \( u^\nu \) are Leray weak solutions of the incompressible Navier-Stokes equations with Navier boundary conditions (2.2). The corresponding initial data \( u^\nu(0) \) is uniformly bounded in \( L^2(\Omega) \), and \( \alpha \in L^{\infty}(\partial \Omega) \). Then for all \( 0 < T < T^* \) and any small \( \epsilon > 0 \), one has

\[
\|(u^\nu - u)(t)\|_{L^2(\Omega)} \leq C((\nu t)^{\frac{1+\beta}{2}} + \|u^\nu(0) - u_0\|_{L^2(\Omega)}),
\]

where \( \beta = \min\left(\frac{1}{2}, r\right) \) and \( C \) is a constant depending only on \( \epsilon, u, T, \|\alpha\|_{L^\infty(\partial \Omega)} \) and \( M \equiv \sup_{\nu} \|u^\nu(0)\|_{L^2(\Omega)} \).
3. Proof of Main Result

Since we are concerned with the case of the bounded domain, the estimates in Besov space as in [11] and [19] can not be used directly. However, we have

**Lemma 3.1** Suppose that \( \omega = \text{curl} \ u \) is the vortex solution to the incompressible Euler system, derived in Theorem 2.1. Then, for any \( s < \beta = \min(r, \frac{1}{2}) \), one has \( \omega \in L^\infty(\Omega; H^s(\Omega)) \).

**Proof.** It is proved in [10] that the vortex patch solution has the following structures:

\[
\omega(x, t) = \omega_i(x, t)\chi_{P(t)}(x) + \omega_e(x, t)\chi_{\Omega \setminus P(t)}(x), \quad t \in [0, T^*),
\]

where \( \omega_i, \omega_e \in L^\infty([0, T^*); C^{\tilde{r}}(\mathbb{R}^d)), \quad P(t) \in L^\infty([0, T^*); C^{1+\tilde{r}}(\mathbb{R}^d)) \)

for any \( \tilde{r} < r \), which means that for any \( t \in [0, T^*) \), \( P(t) \) is a \( C^{1+\tilde{r}} \)-domain, and the \( C^{1+\tilde{r}} \)-norm of the boundary \( \partial P(t) \) is locally bounded. Hence \( \mathcal{H}^{d-1}(\partial P(t)) \), the \((d-1)\)-dimensional Hausdorff measure of \( \partial P(t) \) is locally bounded which induces that

\[
\chi_{P(t)}(x), \chi_{\Omega \setminus P(t)}(x) \in L^\infty([0, T^*); L^\infty(\mathbb{R}^d) \cap BV(\mathbb{R}^d)).
\]

Following Lemma 4.2 and Lemma 4.3 in [19], after extending \( \omega \) to the whole space by zero extension, we derive

\[
\omega(x, t) \in L^\infty([0, T^*); \dot{B}^{\tilde{s}}_{2,\infty}(\mathbb{R}^d)),
\]

where \( \tilde{s} = \min(\tilde{r}, \frac{1}{2}) \) and \( \dot{B}^{\tilde{s}}_{2,\infty} \) is the classical homogeneous Besov space (see [23] for definition).

Using the fact that \( \omega(x, t) \in L^\infty([0, T^*); L^2(\mathbb{R}^d)) \), one has

\[
\omega(x, t) \in L^\infty([0, T^*); \dot{B}^{\tilde{s}}_{2,\infty}(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)). \tag{3.3}
\]

Moreover, for any \( s < \beta = \min(r, \frac{1}{2}) \), there exists a \( \tilde{s} > s \) such that (3.3) holds. Thus using standard interpolation theory (see [23]) yields

\[
\omega(x, t) \in L^\infty([0, T^*); H^{\tilde{s}}(\Omega)).
\]

The proof of Lemma 3.1 is finished.

The following are some known facts, of which the proofs are omitted here.

**Lemma 3.2** (see [3]) For any \( s \geq 1 \) and \( 1 < p < \infty \), there exists a positive constant \( C, \) depending only on \( \Omega, s, p, \) such that for any vector-valued function \( w, \) one has

\[
\|w\|_{W^{s,p}(\Omega)} \leq C\left\| \text{div} \ w \right\|_{W^{s-1,p}(\Omega)} + \|\text{curl} \ w\|_{W^{s-1,p}(\Omega)} + \|w \cdot \bar{n}\|_{W^{s-1/p,p}(\partial\Omega)} + \|w\|_{W^{s-1,p}(\Omega)}.
\]
Here \( \vec{n} \) is the exterior normal vector on \( \partial \Omega \).

**Lemma 3.3** (Korn’s Inequality) (see [12]) Let \( \omega \in H^1(\Omega) \). Then there exists a constant \( C \) depending only on the domain \( \Omega \), such that

\[
\|w\|_{H^1(\Omega)} \leq C(\|D(w)\|_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}).
\]

Now we are ready to prove our main result.

**Proof of Theorem 2.2** Let \( v^\nu = u^\nu - u \). For any fixed \( T < T^* \), one has for every \( 0 < t \leq T \),

\[
\|u^\nu(t)\|^2_{L^2(\Omega)} + 4\alpha \nu \int_0^t \int_{\partial \Omega} |u^\nu|^2 + 4\nu \int_0^t \int_{\Omega} |D(u^\nu)|^2 \\
\leq \|u^\nu(0)\|^2_{L^2(\Omega)}, \tag{3.4}
\]

\[
\|u(t)\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega)}. \tag{3.5}
\]

Here (3.4) is the energy inequality for the Leray weak solution \( u^\nu \) and (3.5) is the energy equality for the vortex patch solution \( u \). Using \( u \) as a test function in the weak form satisfying by the Leray weak solution \( u^\nu \) (see Definition 2.2), we obtain

\[
\int_{\Omega} u^\nu \cdot u(t) dx + 2\alpha \nu \int_0^t \int_{\partial \Omega} u^\nu \cdot udSd\tau + 2\nu \int_0^t \int_{\Omega} D(u^\nu) : D(u) dx d\tau \\
+ \int_0^t \int_{\Omega} (u^\nu \cdot \nabla) u^\nu \cdot udxd\tau = \int_{\Omega} u^\nu(0) \cdot u_0 dx. \tag{3.6}
\]

Adding (3.4) and (3.5) and then subtracting (3.6), one deduces

\[
\frac{1}{2} \|u^\nu(t)\|^2_{L^2(\Omega)} + 2\nu \int_0^t \int_{\Omega} |D(u^\nu)|^2 dx d\tau \\
\leq \frac{1}{2} \|u^\nu(0)\|^2_{L^2(\Omega)} - \int_0^t \int_{\Omega} (u^\nu \cdot \nabla) u \cdot u^\nu dx d\tau - 2\nu \alpha \int_0^t \int_{\partial \Omega} u^\nu \cdot u^\nu dSd\tau \\
- 2\nu \int_0^t \int_{\Omega} D(u) : D(u^\nu) dx d\tau \equiv \sum_{i=1}^{4} I_i. \tag{3.7}
\]

Now we estimate the terms on the right hand of (3.7). By Hölder’s inequality,

\[
|I_2| = \left| \int_0^t \int_{\Omega} (u^\nu \cdot \nabla) u \cdot u^\nu dx d\tau \right| \leq \|\nabla u\|_{L^1(0,T;L^\infty(\Omega))} \|v^\nu\|^2_{L^\infty(0,T;L^2(\Omega))} \tag{3.8}
\]

For any \( 0 < \epsilon < \beta \), there exists \( 0 < s = \beta - \epsilon \) such that \( u \in L^\infty(0, T; H^s(\Omega)) \). Using the duality between \( H^s(\Omega) \) and \( H^{-s}(\Omega) \) (note that when \( s < \frac{1}{2}, H^s(\Omega) = \)).
where the second inequality is the result of Lemma 3.2, the third one is from an interpolation inequality, and the fourth one is due to Lemma 3.3.

Since $v' = u' - u$, one has

$$|I_3| = 2\nu \left| \alpha \int_{\partial \Omega} v' \cdot v' dS \right| \leq 2\nu \left| \alpha \int_{\partial \Omega} u \cdot v' dS \right| + 2\nu \left| \alpha \int_{\partial \Omega} |v'|^2 dS \right|.$$ 

Note that

$$\nu \left| \alpha \int_{\partial \Omega} u \cdot v' dS \right| \leq \nu \left| \alpha \right|_{L^\infty(\partial \Omega)} \left| u \right|_{L^2(\partial \Omega)} \left| v' \right|_{L^2(\partial \Omega)},$$

$$\leq C\nu \left| \alpha \right|_{L^\infty(\partial \Omega)} \left| u \right|_{H^{1/2}(\Omega)} \left| v' \right|_{H^{1/2-}(\Omega)},$$

$$\leq C\nu \left| \alpha \right|_{L^\infty(\partial \Omega)} \left( \left| \omega \right|_{H^s(\Omega)} + \left| u \right|_{L^2(\Omega)} \right) \left| v' \right|_{H^{1/2-}(\Omega)},$$

$$\leq C\nu \left| \alpha \right|_{L^\infty(\partial \Omega)} \left| v' \right|_{L^2(\Omega)}^2 + \frac{\nu}{2} \left| D(v') \right|_{L^2(\Omega)}^2,$$

and

$$\nu \left| \alpha \int_{\partial \Omega} |v'|^2 dS \right| \leq C\nu \left| \alpha \right|_{L^\infty(\partial \Omega)} \left| v' \right|_{H^{1/2}(\Omega)}^2,$$

$$\leq C\nu \left| \alpha \right|_{L^\infty(\partial \Omega)} \left| v' \right|_{L^2(\Omega)} \left( \left| v' \right|_{L^2(\Omega)} + \left| D(v') \right|_{L^2(\Omega)} \right),$$

$$\leq C\nu \left| \alpha \right|_{L^\infty(\partial \Omega)} \left| v' \right|_{L^2(\Omega)}^2 + \frac{\nu}{2} \left| D(v') \right|_{L^2(\Omega)}^2.$$

The term $I_3$ is estimated as

$$I_3 \leq C\nu \left| \alpha \right|_{L^\infty(\partial \Omega)} \left( \left| v' \right|_{L^2(\Omega)}^2 + \left| v' \right|_{L^2(\Omega)} \frac{\nu}{2} \right) + \frac{\nu}{2} \left| D(v') \right|_{L^2(\Omega)}^2.$$  \hfill (3.9)

Moreover, from (3.4), one deduces

$$\|u'(t)\|_{L^2(\Omega)}^2 + 4\nu \int_0^t \|D(u')\|_{L^2(\Omega)}^2 d\tau,$$

$$\leq \|u'(0)\|_{L^2(\Omega)}^2 + \nu \left| \alpha \right|_{L^\infty(\partial \Omega)} \int_0^t \left( \|u'\|_{L^2(\Omega)} + \|D(u')\|_{L^2(\Omega)} \right)^2 \frac{1}{2} \|u'\|_{L^2(\Omega)}^2 d\tau,$$

$$\leq \|u'(0)\|_{L^2(\Omega)}^2 + C\nu \int_0^t \|u'\|_{L^2(\Omega)}^2 d\tau + 2\nu \int_0^t \|D(u')\|_{L^2(\Omega)}^2 d\tau.$$
which implies that \( \|u^\nu\|_{L^\infty(0,T;L^2(\Omega))} \) and \( \|v^\nu\|_{L^\infty(0,T;L^2(\Omega))} \) are uniformly bounded by some constant \( C \) depending on \( M, T \) and \( \|\alpha\|_{L^\infty(\partial\Omega)} \). Hence by the Young’s inequality, since \( \frac{s}{s+1} < 1 \),

\[
|I_4| = \nu \left| \int_\Omega D(u) : D(v^\nu) \, dx \, d\tau \right| \\
\leq C \nu \|v^\nu\|_{L^2(\Omega)} + C \nu \|v^\nu\|^s_{L^2(\Omega)} \|D(v^\nu)\|^{1-s}_{L^2(\Omega)} \\
\leq C \nu \frac{\|v^\nu\|_{L^2(\Omega)}}{\nu^{s+1}} + \frac{\nu}{4} \|D(v^\nu)\|_{L^2(\Omega)}^2, 
\]

where \( C \) is a constant depending on \( \Omega, s, M, T \), \( \|\alpha\|_{L^\infty(\partial\Omega)} \).

Putting (3.8)-(3.10) into (3.7), we get

\[
\frac{1}{2} \|v^\nu(t)\|^2_{L^2(\Omega)} \\
\leq \frac{1}{2} \|v^\nu(0)\|^2_{L^2(\Omega)} + \int_0^t (\|\nabla u\|_{L^\infty(\Omega)} + C) \|v^\nu\|^s_{L^2(\Omega)} \, d\tau + C \nu \int_0^t \|v^\nu\|_{L^2(\Omega)} \, d\tau.
\]

By the Gronwall lemma, we deduce that

\[
\|v^\nu(t)\|^2_{L^2(\Omega)} \leq C \|v^\nu(0)\|^2_{L^2(\Omega)} + C \nu t,
\]

where \( C \) is a constant depending on \( \epsilon, u, T, \|\alpha\|_{L^\infty(\partial\Omega)} \) and \( M \equiv \sup_{\nu} \|u^\nu(0)\|_{L^2(\Omega)} \).

The proof of the theorem is finished.

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