A Hardy’s Uncertainty Principle Lemma in Weak Commutation Relations of Heisenberg-Lie Algebra

Toshimitsu TAKAESU

Faculty of Mathematics, Kyushu University, Fukuoka, 812-8581, Japan

Abstract. In this article we consider linear operators satisfying a generalized commutation relation of a type of the Heisenberg-Lie algebra. It is proven that a generalized inequality of the Hardy’s uncertainty principle lemma follows. Its applications to time operators and abstract Dirac operators are also investigated.

Key words: weak commutation relations, Heisenberg-Lie algebra, time operators, Hamiltonians, time-energy uncertainty relation, Dirac operators, essential self-adjointness.

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1 Introduction and Results

In this article we investigate a norm-inequality of the linear operators which obey a generalized weak commutation relation of a type of the Heisenberg-Lie algebra, and consider its application to the theory of the time operator [7, 2], and an abstract Dirac operator. Let $X = \{X_j\}_{j=1}^{N}$, $Y = \{Y_j\}_{j=1}^{N}$, and $Z = \{Z_j\}_{j=1}^{N}$ be symmetric operators on a Hilbert space $\mathcal{H}$. The weak commutator of operators $A$ and $B$ is defined for $\psi \in \mathcal{D}(A) \cap \mathcal{D}(B)$ and $\phi \in \mathcal{D}(A^*) \cap \mathcal{D}(B^*)$ by

$$\{A, B\}_w(\phi, \psi) = (A^* \phi, B \psi) - (B^* \phi, A \psi).$$

Here the inner product has a linearity of

$$\langle \eta, \alpha \phi + \beta \psi \rangle = \alpha \langle \eta, \phi \rangle + \beta \langle \eta, \psi \rangle$$

for $\alpha, \beta \in \mathbb{C}$. We assume that $(X, Y, Z)$ satisfies the following conditions.

(A.1) $Z_j$, $1 \leq j \leq N$, is bounded operator.

(A.2) Let $\mathcal{D}_X = \bigcap_{j=1}^{N} \mathcal{D}(X_j)$ and $\mathcal{D}_Y = \bigcap_{j=1}^{N} \mathcal{D}(Y_j)$. It follows that for $\phi, \psi \in \mathcal{D}_X \cap \mathcal{D}_Y$,

$$[X_j, Y_l]_w(\phi, \psi) = \delta_{j,l} (\phi, iZ_j \psi),$$

$$[X_j, Z_l]_w(\phi, \psi) = [Y_j, Z_l]_w(\phi, \psi) = 0$$

$$[X_j, X_l]_w(\phi, \psi) = [Y_j, Y_l]_w(\phi, \psi) = [Z_j, Z_l]_w(\phi, \psi) = 0.$$

Note that $[Z_j, Z_l] \psi = 0$ follows for $\psi \in \mathcal{H}$, since $Z_j, j = 1, \cdots, N$, is bounded. In this article we consider an generalization of the inequality

$$\int_{\mathbb{R}^N} \frac{1}{|r|^2} |u(r)|^2 \, dr \leq \frac{4}{(N-2)^2} \int_{\mathbb{R}^N} |\nabla u(r)|^2 \, dr, \quad N \geq 3.$$
This inequality is a basic one of Hardy’s uncertainty principle inequalities. For Hardy’s uncertainty inequalities, refer to e.g. [5, 6, 13].

Let us introduce the additional conditions.

(A.3) $X_j$ is self-adjoint for all $1 \leq j \leq N$.

(A.4) $X_i$ and $Z_l$ strongly commutes for all $1 \leq j \leq N$ and $1 \leq l \leq N$.

Since $Z_j$, $j = 1, \cdots, N$, is bounded self-adjoint operator, we can set $\lambda_{\min}(Z)$ and $\lambda_{\max}(Z)$ by

$$\lambda_{\min}(Z) = \min_{1 \leq j \leq N} \inf \sigma(Z_j),$$

$$\lambda_{\max}(Z) = \max_{1 \leq j \leq N} \sup \sigma(Z_j),$$

where $\sigma(O)$ denotes the spectrum of the operator $O$.

**Theorem 1** Assume (A.1)-(A.4). Let $\Psi \in \mathcal{D}(\|X\|^{-1}) \cap \mathcal{D}_X \cap \mathcal{D}_Y$. Then the following (1) and (2) hold

(1) If $N\lambda_{\min}(Z) - 2\lambda_{\max}(Z) > 0$, it follows that

$$\|X^{-1}\Psi\|^2 \leq \frac{4}{(N\lambda_{\min}(Z) - 2\lambda_{\max}(Z))^2} \sum_{j=1}^{N} \|Y_j \Psi\|^2.$$  \hspace{1cm} (1)

(2) If $2\lambda_{\min}(Z) - N\lambda_{\max}(Z) > 0$, it follows that

$$\|X^{-1}\Psi\|^2 \leq \frac{4}{(2\lambda_{\min}(Z) - N\lambda_{\max}(Z))^2} \sum_{j=1}^{N} \|Y_j \Psi\|^2.$$  \hspace{1cm} (2)

Before proving Theorem 1 let us consider the replacement of $X$ and $Y$ in Theorem 1. Let us introduce the following conditions substitute for (A.3) and (A.4).

(A.5) $Y_j$ is self-adjoint for all $1 \leq j \leq N$.

(A.6) $Y_i$ and $Z_l$ strongly commutes for all $1 \leq j \leq N$ and $1 \leq l \leq N$.

It is seen from (A.2), that

$$[Y_j, X_i]^w(\phi, \psi) = \delta_{j,i}(\phi, i(-Z_j)\psi), \hspace{1cm} \phi, \psi \in \mathcal{D}_X \cap \mathcal{D}_Y.$$  \hspace{1cm} (3)

Note that $\inf \sigma(-Z_j) = -\sup(Z_j)$ and $\sup(-Z_j) = -\inf \sigma(Z_j)$ follow. Then we obtain a following corollary:
Corollary 2 Assume (A.1)-(A.2) and (A.5)-(A.6). Let $\Psi \in D(|Y|^{-1}) \cap D_X \cap D_Y$. Then the following (1) and (2) hold.
(1) If $2\lambda_{\min}(Z) - N\lambda_{\max}(Z) > 0$, it follows that
\[
\left\| |Y|^{-1}\Psi \right\|^2 \leq \frac{4}{(2\lambda_{\min}(Z) - N\lambda_{\max}(Z))^2} \sum_{j=1}^{N} \left\| X_j\Psi \right\|^2. \tag{4}
\]
(2) If $N\lambda_{\min}(Z) - 2\lambda_{\max}(Z) > 0$, it follows that
\[
\left\| |Y|^{-1}\Psi \right\|^2 \leq \frac{4}{(N\lambda_{\min}(Z) - 2\lambda_{\max}(Z))^2} \sum_{j=1}^{N} \left\| X_j\Psi \right\|^2. \tag{5}
\]

(Proof of Theorem)
(1) Let $\Psi \in D(|X|^{-1}) \cap D_X \cap D_Y$. For $\varepsilon > 0$ and $t > 0$, it is seen that
\[
\left\| (Y_j - itX_j(X^2 + \varepsilon)^{-1})\Psi \right\|^2 = \left\| Y_j\Psi \right\|^2 - it \left[ Y_j, X_j(X^2 + \varepsilon)^{-1}\Psi, \Psi \right] + t^2 \left\| X_j(X^2 + \varepsilon)^{-1}\Psi \right\|^2. \tag{6}
\]
We see that
\[
[Y_j, X_j(X^2 + \varepsilon)^{-1}]\Psi = [Y_j, X_j](\Psi, (X^2 + \varepsilon)^{-1}\Psi) + [Y_j, (X^2 + \varepsilon)^{-1}]\Psi(X_j\Psi, \Psi). \tag{7}
\]
From (A.2) and (A.4), we obtain that
\[
[Y_j, X_j](\Psi, (X^2 + \varepsilon)^{-1}\Psi) = -i ((X^2 + \varepsilon)^{-1/2}\Psi, Z_j(X^2 + \varepsilon)^{-1/2}\Psi). \tag{8}
\]
Note that for a symmetric operator $A$ and the non-negative symmetric operator $B$, the resolvent formula $[A, (B + \lambda)^{-1}]\Psi(v,u) = [B, A]^w((B + \lambda)^{-1}v, (B + \lambda)^{-1}u)$ for $\lambda > 0$ follows. Then by using this formula, (A.2) and (A.4) yield that
\[
[Y_j, (X^2 + \varepsilon)^{-1}]\Psi(X_j\Psi, \Psi) = 2i(X_j(X^2 + \varepsilon)^{-1}u, Z_jX_j(X^2 + \varepsilon)^{-1}u) \tag{9}
\]
Since $\left\| (Y_j - itX_j(X^2 + \varepsilon)^{-1})\Psi \right\|^2 \geq 0$ and $t > 0$, we see from (7), (8) and (9) that
\[
\left\| Y_j\Psi \right\|^2 \geq -t^2 \left\| X_j(X^2 + \varepsilon)^{-1} \right\|^2 + t \left[ ((X^2 + \varepsilon)^{-1/2}\Psi, Z_j(X^2 + \varepsilon)^{-1/2}u) - 2t(X_j(X^2 + \varepsilon)^{-1}u, Z_jX_j(X^2 + \varepsilon)^{-1}\Psi) \right] \\
\geq \left( -t^2 - 2t\lambda_{\max}(Z) \right) \left\| X_j(X^2 + \varepsilon)^{-1} \right\|^2 + t\lambda_{\min}(Z) \left\| (X^2 + \varepsilon)^{-1/2}\Psi \right\|^2. \tag{10}
\]
Then we have that
\[
\sum_{j=1}^{N} \left\| Y_j\Psi \right\|^2 \geq \left( -t^2 - 2t\lambda_{\max}(Z) \right) \left\| X(X^2 + \varepsilon)^{-1}\Psi \right\|^2 + tN\lambda_{\min}(Z) \left\| (X^2 + \varepsilon)^{-1/2}\Psi \right\|^2. \tag{11}
\]
Note that $\lim_{\varepsilon \to 0} \left\| X(X^2 + \varepsilon)^{-1}\Psi \right\|^2 = \left\| X^{-1}\Psi \right\|^2$ and $\lim_{\varepsilon \to 0} \left\| (X^2 + \varepsilon)^{-1/2}\Psi \right\|^2 = \left\| X^{-1}\Psi \right\| = 0$ follow from the spectral decomposition theorem. Then we have
\[
\sum_{j=1}^{N} \left\| Y_j\Psi \right\|^2 \geq \left( -t^2 + (N\lambda_{\min}(Z) - 2\lambda_{\max}(Z))t \right) \left\| X^{-1}\Psi \right\|^2. \tag{12}
\]
By taking \( t = \frac{N\lambda_{\text{min}}(Z) - 2\lambda_{\text{max}}(Z)}{2} > 0 \) in the right side of (12), we obtain (1).

(2) By computing \( \| (Y_j + iX_j(X^2 + \varepsilon)^{-1/2}) \Psi \|^2 \) for \( t > 0 \) and \( \varepsilon > 0 \), in a similar way of (1), we see that

\[
\| Y_j \Psi \|^2 \geq -t^2 \| X_j(X^2 + \varepsilon)^{-1}u \|^2 - t \lambda_{\text{min}}(Z) \| (X^2 + \varepsilon)^{-1/2} \Psi \|.
\]

Then by taking \( \varepsilon \to 0 \) in the right side of (13), it follows that

\[
\sum_{j=1}^{N} \| Y_j \Psi \|^2 \geq \left( -t^2 + (2\lambda_{\text{min}}(Z) - N\lambda_{\text{max}}(Z))t \right) \| X^{-1} \Psi \|.
\]

By taking \( t = \frac{(2\lambda_{\text{min}}(Z) - N\lambda_{\text{max}}(Z))}{2} > 0 \) in (14), we obtain (2). ■.

2 Applications

2.1 Time-Energy Uncertainty inequality

In this subsection we consider an application to the theory of time operators [2, 7]. Let \( H, T, \) and \( C \) be linear operators on a Hilbert space \( H \). It is said that \( H \) has the weak time operator \( T \) with the uncommutative factor \( C \) if \((H, T, C)\) satisfy the following conditions.

(T.1) \( H \) and \( T \) are symmetric.

(T.2) \( C \) is bounded and self-adjoint.

(T.3) It follows that for \( \phi, \psi \in D(H) \cap D(T) \),

\[ [T, H]^{w}(\phi, \psi) = (\phi, C\psi). \]

(T.4)

\[ \delta_{C} := \inf_{\psi \in \ker C \setminus \{0\}} \frac{|\langle \psi, C\psi \rangle|}{\| \psi \|^2} > 0. \]

Assume that \((H, T, C)\) satisfies (T.1)-(T.4). Then by using \( \| Au \| \| Bu \| \geq |\text{Im}(Au, Bu)| \geq \frac{1}{2} |A, B|^{w}(u, u)| \), it is seen that \((H, T, C)\) satisfies the time-energy uncertainty inequality ([2], Proposition 4.1):

\[
\| (H - < H \gamma > \psi) \Psi \| \| (T - < T \gamma > \psi) \Psi \| \geq \frac{\delta_{C}}{2}, \quad \psi \in D(H) \cap D(T),
\]

where \( < O \gamma > = (\gamma, O\psi) \). From (2) in Theorem 1 and (1) in Corollary 2, we obtain another type of the inequality between \( T \) and \( H \) :
Corollary 3 (Time-Energy Uncertainty Inequalities)
Assume (T.1)-(T.3). Then the following (i) and (ii) hold.
(i) If $T$ is self-adjoint, $C$ and $T$ strongly commute, and $\sup \sigma(C) < 2 \inf \sigma(C)$, it follows that for $\psi \in \mathcal{D}(|T|^{-1}) \cap \mathcal{D}(T) \cap \mathcal{D}(H)$,

$$
\| |T|^{-1} \psi \| \leq \frac{2}{2 \inf \sigma(C) - \sup \sigma(C)} \| H \psi \|. 
$$

(ii) If $H$ is self-adjoint, $C$ and $H$ strongly commute, and $\sup \sigma(C) < 2 \inf \sigma(C)$, it follows that for $\psi \in \mathcal{D}(|H|^{-1}) \cap \mathcal{D}(H) \cap \mathcal{D}(T)$,

$$
\| |H|^{-1} \psi \| \leq \frac{2}{2 \inf \sigma(C) - \sup \sigma(C)} \| T \psi \|. 
$$

2.2 Abstract Dirac Operators with Coulomb Potential

Next let us consider the application to abstract Dirac operators. We consider the self-adjoint operators $P = \{P_j\}_{j=1}^N$ and $Q = \{Q_j\}_{j=1}^N$ on a Hilbert space $\mathcal{H}$. Let us set a subspace $\mathcal{D} \subset \bigcap_{j,l}(\mathcal{D}(P_j) \cap \mathcal{D}(Q_l))$. It is said that $(\mathcal{H}, \mathcal{D}, P, Q)_N$ is the weak representation of the CCR with degree $N$, if $\mathcal{D}$ is dense in $\mathcal{H}$ and it follows that for $\phi, \psi \in \mathcal{D}$,

$$
[P_j, Q_l]^{w}(\phi, \psi) = i \delta_{j,l}(\phi, \psi),
[P_j, P_l]^{w}(\phi, \psi) = [Q_j, Q_l]^{w}(\phi, \psi) = 0.
$$

Let us define an abstract Dirac operator as follows. Let $(\mathcal{H}, \mathcal{D}, P, Q)_3$ be the weak representation of the CCR with degree three. Let $A = \{A_j\}_{j=1}^3$ and $B$ be the bounded self-adjoint operators on a Hilbert space $\mathcal{K}$. Here $A = \{A_j\}_{j=1}^3$ and $B$ satisfy the canonical anti-commutation relations $\{A_j, A_l\} = 2 \delta_{j,l}$, $\{A_j, B\} = 0$, $B^2 = I_\mathcal{K}$ where $I_\mathcal{K}$ is the identity operator on $\mathcal{K}$. The state Hilbert space space is defined by $\mathcal{H}_{\text{Dirac}} = \mathcal{K} \otimes \mathcal{H}$. The free abstract Dirac operator is defined by

$$
H_0 = \sum_{j=1}^3 A_j \otimes P_j + B \otimes M.
$$

Here we assume the following condition.

(D.1) $P_j$ and $P_l$ strongly commute for $1 \leq j \leq 3$, $1 \leq l \leq 3$. $P_j$, $1 \leq j \leq 3$, and $M$ strongly commute.

Then it is seen that $H_0^2 \Psi = (P^2 + M^2) \Psi$ for $\Psi \in \mathcal{D}$. The abstract Dirac Operator with the Coulomb potential is defined by

$$
H(\kappa) = H_0 + \kappa I_\mathcal{K} \otimes |Q|^{-1},
$$

where $\kappa \in \mathbb{R}$ is a parameter called the coupling constant. We assume that the following condition holds.

(D.2) It follows that $\mathcal{D} \subset \mathcal{D}(Q|^{-1})$.

Then it follows from (1) in Theorem[1] that for $\psi \in \mathcal{D}$,

$$
\|I_\mathcal{K} \otimes |Q|^{-1} \psi\|^2 \leq 4 \sum_{j=1}^3 \|P_j \psi\|^2 \leq 4 \|H_0 \psi\|^2.
$$

Hence by the Kato-Rellich theorem, we obtain the following corollary.
Corollary 4 Assume (D.1) and (D.2). Then for $|\kappa| < \frac{1}{2}$, $H(\kappa)$ is essentially self-adjoint on $\mathcal{D}$.

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