Online Reinforcement Learning of Optimal Threshold Policies for Markov Decision Processes

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Abstract—To overcome the curses of dimensionality and modeling of Dynamic Programming (DP) methods to solve Markov Decision Process (MDP) problems, Reinforcement Learning (RL) methods are adopted in practice. Contrary to traditional RL algorithms which do not consider the structural properties of the optimal policy, we propose a structure-aware learning algorithm to exploit the ordered multi-threshold structure of the optimal policy, if any. We prove the asymptotic convergence of the proposed algorithm to the optimal policy. Due to the reduction in the policy space, the proposed algorithm provides remarkable improvements in storage and computational complexities over classical RL algorithms. Simulation results establish that the proposed algorithm converges faster than other RL algorithms.

Index Terms—Markov Decision Process, Stochastic Approximation Algorithms, Reinforcement Learning, Stochastic Control, Online Learning of Threshold Policies.

I. INTRODUCTION

Markov Decision Process (MDP) [2] is a framework which is widely used for the optimization of stochastic systems (e.g., queue [3], inventory and production management) to make optimal temporal decisions. Dynamic Programming (DP) methods [2] to compute the optimal policy suffer from the curse of dimensionality [4, Chapter 4.1], [5] in many practical applications as they are computationally inconvenient due to extremely high dimension of the iterates. Furthermore, they suffer from the curse of modeling since the knowledge of the underlying transition probabilities (which often depend on the statistics of unknown system parameters) required by DP methods, may not be available beforehand. RL techniques [6] address the curse of modeling by learning the optimal policy iteratively. RL algorithms do not require any prior knowledge regarding the transition probabilities of the underlying model. RL being sampling based, updates only one component at a time, reducing per iterate computation at the expense of speed. Examples are: Q-learning [7] iteratively evaluates the Q-function of every state-action pair using a combination of exploration and exploitation. In [8]–[10], upper confidence bound based exploration improves the convergence speed over classical ε-greedy exploration. PDS learning algorithm [4], [11] removes the requirement of action exploration and results in faster convergence than Q-learning. Virtual Experience (VE) learning algorithm in [12] updates multiple PDSs at a time. Faster convergence is achieved at the cost of increased computational complexity.

However, popular RL techniques [4], [6], [7], [11], [13], [14] do not exploit the known structural properties of the optimal policy if any, and consider the set of all policies as the policy search space. However, in operations research and communications literature, various structural properties of the optimal policy including threshold structure, transience of certain states and index rules [15], [16] are often established using monotonicity, convexity/concavity and sub-modularity/super-modularity properties of value functions of states. If one can exploit these structural properties to reduce the search space while learning, then faster convergence can be achieved with a reduction in the computational complexity. A few works in the literature [17]–[20] focus on the exploitation of the structural properties [16] while learning the optimal policy. Q-learning based approaches in [17], [19], in every iteration, project the value functions to guarantee the monotonicity in system state. Although improved convergence speed is obtained, the per-iteration computational complexity does not improve over Q-learning. In [18], a scheme based on piecewise linear approximation of the value function, is proposed. However, as the approximation becomes better, the complexity increases. In [21], a Stochastic Approximation (SA) [22] approach based on simultaneous perturbation is proposed to compute the optimal thresholds. It uses a combination of ideas from renewal theory and Monte Carlo simulation. The low complexity Q-learning algorithm proposed in [23] exploits the threshold nature of the optimal policy. The proposed algorithm estimates the optimal policy for a subset of sets, and then policy interpolation is performed for the unvisited states. The performance of policy interpolation is further improved by a policy refinement. However, the proposed algorithm provides a near-optimal policy after a finite number of visits to a set of state-action pairs.

We consider a scenario where the optimal policy has a multi-threshold structure, and the thresholds for different events are ordered. Therefore, learning the optimal policy is equivalent to learning the value of threshold for each event. Motivated by this, we propose a Structure-Aware Learning for Multiple Thresholds (SALMUT) algorithm which considers only the set of ordered threshold policies. We consider a two timescale approach where the value functions of the states and threshold parameters are updated in faster and slower timescale, respectively. Threshold parameters are updated based on the gradients of the average reward w.r.t the threshold. The proposed scheme results in reductions in storage and computational complexities (by amortizing the computation, and hence the complexity of the original problem, over several iterations) compared to traditional RL schemes. We prove that the proposed algorithm converges to the optimal policy asymptotically. Simulation results exhibit that SALMUT converges faster than classical RL schemes due to reduction in policy search space. The techniques in this paper can be employed to learn the optimal policy in problems involving optimality of threshold policies [15], [24]–[27]. We illustrate this using an example of a finite buffer multi-server queue with multiple customer classes (similar to [3], [28]).

The proposed SALMUT algorithm can be adopted for single threshold case in [1], without any modification. In [29], a structure-aware learning algorithm learns a single parameterized threshold where the thresholds for different parameter values are independent of each other. However, in this paper, the thresholds have to
satisfy certain ordering constraints and hence, are not independent. Therefore, the threshold update scheme in the slower timescale and corresponding convergence behavior differs significantly from [29]. To the best of our knowledge, contrary to other works [17]–[20], we for the first time consider the threshold vector as a parameter while learning to reduce a non-linear iteration (involving maximization over a set of actions) into a linear iteration for a quasi-static value of threshold and thereby, achieve a significant reduction in the per-iteration computational complexity.

II. SYSTEM MODEL & PROBLEM FORMULATION

Consider a continuous time MDP problem where we aim to obtain the optimal control policy in a multi-event scenario. Let the system state in the state space \((S \times I, \text{say})\) be \((s, i)\) where \(s \in \{0, 1, \ldots, W\}\), and \(i \in \{0, 1, \ldots, N\}\) denotes the event type. Since we have a finite-state regular Markov chain [30], it is sufficient to observe the system state only at the decision epochs [31]. Let the action space \(A\) consists of two actions, viz., \(A_1\) and \(A_2\). The transition probability from state \((s, i)\) to \((s', i')\) under action \(a\) \(p((s, i), (s', i'), a)\), say can be factored into two parts, viz., the deterministic and the probabilistic transitions due to the chosen action \((p(s, s', a), \text{say})\) and the next event, respectively. Let the mean transition rate from state \((s, i)\) be denoted by \(\lambda(s)\). Let the arrival times of events in state \(s\) be independent exponentially distributed with means \(\lambda(s)\) respectively, so that the next event is \(i\) with probability \(\lambda(s)/\sum_{j=0}^{\infty} \lambda(j)\).

Now, \(v(s) = \sum_{i=0}^{\infty} \lambda(s)\). Hence,

\[
p((s, i), (s', i'), a) = p(s, s', a) \frac{\lambda(s')}{v(s')},
\]

Let the non-negative reward rate \((r((s, i), a))\) obtained by choosing action \(A_2\) for event \(i\) be \(R_i\), where \(R_i > R_j\) for \(i < j\). Therefore, \(r((s, i), a) = R_i\). Let the non-negative cost rate in state \((s, i)\) be \(h(s)\) (independent of \(i\)).

Let \(Q\) be the set of stationary policies. Since the zero state is reachable from any state with positive probability, the underlying Markov chain is unichain and hence, a unique stationary distribution exists. Let the infinite horizon average reward (independent of the initial state) under policy \(Q\in Q\) be \(\rho_Q\). We aim to maximize \(\rho_Q = \lim_{t \to \infty} \frac{1}{t} \mathbb{E}_Q[R(t)],\) where \(R(t)\) is the total reward till time \(t\). The DP equation describing the necessary condition for optimality in a semi-Markov decision process \(\forall \{(s, i), (s', i')\} \in S \times I\) is

\[
\bar{V}(s, i) = \max_{a \in A} \left[ R_i \mathbb{1}_{\{a = A_2\}} + \sum_{s', i'} p((s, i), (s', i'), a) \bar{V}(s', i') - \rho \bar{\beta}((s, i), a) \right] - h(s),
\]

where \(\bar{V}(s, i)\) and \(\bar{\beta}(s, i, a)\) denote the value function of state \((s, i)\), the optimal average reward and the mean transition time from state \((s, i)\) upon choosing action \(a\), respectively. We rewrite the DP equation after substituting the values of \(r((s, i), a)\) and transition probabilities as

\[
\bar{V}(s, i) = \max_{a \in A} \left[ R_i \mathbb{1}_{\{a = A_2\}} + \sum_{s', i'} p(s, s', a) \frac{\lambda(s')}{v(s')} \bar{V}(s', i') - \rho \bar{\beta}((s, i), a) \right] - h(s),
\]

where \(\bar{V}(s, i) := \sum_{i=0}^{\infty} \lambda(s)/v(s)\bar{V}(s, i)\). Therefore, the following relations hold.

\[
\bar{V}(s, i) = \max_{a \in A} \left[ R_i \mathbb{1}_{\{a = A_2\}} + \sum_{s'} p(s, s', a) \bar{V}(s') - \rho \bar{\beta}((s, i), a) \right] - h(s),
\]

and (since \(\lambda(s)/v(s)\) is independent of \(a\))

\[
V(s) = \max_{a \in A} \left[ \sum_{i} \lambda(s)/v(s) R_i \mathbb{1}_{\{a = A_2\}} + \sum_{s'} p(s, s', a) \bar{V}(s') - \rho \bar{\beta}((s, i), a) \right] - h(s),
\]

where \(\bar{\beta}(s, a) = \sum_{i} \lambda(s)/v(s) \bar{\beta}((s, i), a)\). Using (1), instead of considering \((s, i) \in S \times I\), we can consider the system state as \(s \in S\) with value function \(V(s)\) and transition probability \(p(s, s', a)\), and the analysis remains unaffected. However, in this model, the reward rate is the weighted average of original reward rates. The sojourn times being exponentially distributed, following [2], we obtain

\[
V(s) = \max_{a \in A} \left[ \sum_{i} \lambda(s)/v(s) R_i \mathbb{1}_{\{a = A_2\}} + \sum_{s'} p(s, s', a) V(s') \right] - h(s) - \rho,
\]

where \(p(s, s,a) = 1 - \sum_{s' \neq s} p(s, s', a)\). (2) is the DP equation for an equivalent discrete time MDP having controlled transition probabilities \(p(s, s', a)\) which is used throughout the rest of the paper. This problem can be solved using Relative Value Iteration Algorithm (RVIA) as follows.

\[
V_{n+1}(s) = \max_{a \in A} \left[ \sum_{i} \lambda(s)/v(s) R_i \mathbb{1}_{\{a = A_2\}} + \sum_{s'} p(s, s', a) V_{n}(s') \right] - V_{n}(s') - h(s),
\]

where \(s^* \in S\) is a fixed state and \(V_{n}(s)\) is the estimate of value function of state \(s\) at \(n\)th iteration.

III. EXPLOITATION OF STRUCTURAL PROPERTIES IN RL

We assume that the optimal policy is of threshold-type where it is optimal to choose action \(A_2\) only up to a threshold \(\tau(i)\) in \(S\) which is a non-increasing function of \(i\). In this section, we propose an RL algorithm which exploits the knowledge regarding the existence of a threshold-based optimal policy.

Given that the optimal policy is threshold in nature where the optimal action changes from \(A_2\) to \(A_1\) at \(\tau(i)\) for \(i\)th event, the knowledge of \(\tau(0), \ldots, \tau(N)\) uniquely characterizes the optimal policy. However, computation of these threshold parameters requires the knowledge of the event probabilities in state \(s\) (governed by \(\lambda(s)\)). When the \(\lambda(s)\)'s are unknown, then we can learn these ordered thresholds instead of learning the optimal policy from the set of all policies including the non-threshold policies. We devise an iterative update rule for a threshold vector of dimensionality \((N + 1)\) so that the threshold vector iterate converges to the optimal threshold vector.

We consider the set of threshold policies where the thresholds for different events are ordered \((\tau(i) \geq \tau(j)\) for \(i < j)\) and represent them as policies parameterized by the threshold vector \(\tau = [\tau(0), \tau(1), \ldots, \tau(N)]^T\) where \(\tau(0) \geq \tau(1) \geq \ldots \geq \tau(N)\). In this context, we redefine the notations associated with the MDP to reflect their dependence on \(\tau\). We intend to compute the gradient of the average reward w.r.t \(\tau\) and improve the policy by updating \(\tau\) in the direction of the gradient. Let the transition probability from state \(s\) to state \(s'\) corresponding to the threshold vector \(\tau\) be \(P_{ss'}(\tau)\). Hence, \(P_{ss'}(\tau) = P(X_{n+1} = s'|X_n = s, \tau)\). Let the value function of state \(s\), the average reward of the Markov chain and the stationary probability of state \(s\) parameterized by \(\tau\) be denoted by \(V(s, \tau), \sigma(\tau)\) and \(\pi(s, \tau)\), respectively.

The optimal policy can be computed using (3) if we know the state transition probabilities and \(\lambda(s)\). When these parameters are unknown, theory of SA [22] enables us to replace the expectation
operation in (3) by averaging over time and still converge to the optimal policy. Let \(a(n)\) be a positive step-size sequence satisfying \(\sum_{n=1}^{\infty} a(n) = \infty; \sum_{n=1}^{\infty} (a(n))^2 < \infty\). Let \(b(n)\) be another step-size sequence which apart from the above properties satisfies \(\lim_{n \to \infty} \frac{b(n)}{a(n)} = 0\). We update the value function of the system state (based on the type of event) at any given iteration and keep the value functions of other states unchanged. Let \(S_n\) be the state of the system at \(n^{th}\) iteration. Let \(\gamma(s, n) = \sum_{i=1}^{\infty} \mathbb{I}(s, i)\). For a fixed threshold vector \(\tau\) (i.e., a fixed policy), max operator in (2) goes away, and the resulting system becomes a linear system. Therefore we have

\[
V(s) = \sum_{i} \frac{\lambda_i}{v(s)} \Pi_i \left( l(s, \tau(i)) = A_2 \right) + \sum_s \pi(s, s', \sigma)V(s') - h(s) - \rho,
\]

where \(l(s, \tau(i)) = A_2\) if \(s < \tau(i)\), \(l(s, \tau(i)) = A_1\) else.

The update of value function of state \(s\) (corresponding to the \(i^{th}\) event) is done using the following scheme:

\[
V_{n+1}(s, \tau) = (1 - a(\gamma(s, n)))V_n(s, \tau) + a(\gamma(s, n))[h(s) + \Pi_2 l(s, \tau(i)) = A_2 + V_n(s', \tau) - V_n(s, \tau)],
\]

where \(V_n(s, \tau)\) denotes the value function of state \(s\) at \(n^{th}\) iteration provided the threshold vector is \(\tau\). This is known as the primal RVI which is performed in the faster timescale. The scheme (4) works for a fixed value of threshold vector. To obtain the optimal value of \(\tau\), the threshold vector needs to be iterated in a slower timescale \(b(.)\). The idea is to learn the optimal threshold vector by computing \(\nabla \sigma(\tau)\) and update the value of threshold in the direction of the gradient. This scheme is similar to stochastic gradient routine:

\[
\tau_{n+1} = \tau_n + b(n) \nabla \sigma(\tau_n),
\]

where \(\tau_n\) is the threshold vector at \(n^{th}\) iteration. Conditions on step sizes ensure that the value function and threshold vector iterates are updated in different timescales. From the slower timescale, the value functions seem to be quasi-equilibrated, whereas form the faster timescale, the threshold vector appears to be quasi-static (known as the “leader-follower” behavior).

Given a threshold vector \(\tau\), it is assumed that the transition from state \(s\) for \(i^{th}\) event is driven by the rule \(P_1(s'|s)\) if \(s < \tau(i)\) and by the rule \(P_0(s'|s)\) otherwise. Under rule \(P_1(s'|s)\), the system moves from state \(s\) to state \(s' = s + 1\) following action \(A_2\) if \(s < \tau(i)\). On the other hand, rule \(P_0(s'|s)\) dictates that the system remains in state \(s\) following action \(A_1\). For a fixed \(\tau\), (4) is updated using the above rule. Let \(g_\tau(\tau(i)) = P_0 l(s, \tau(i)) = A_2 + h(s)\). The following assumption is made on \(P_{sa'}(\tau)\) and \(g_\tau(\tau)\) to embed the discrete parameter \(\tau\) into a continuous domain later.

**Assumption 1.** \(P_{sa'}(\tau)\) and \(g_\tau(\tau)\) are bounded and twice differentiable functions of \(\tau\). It has bounded first and second derivatives.

For a given event, the threshold policy chooses the rule \(P_1(\cdot|\cdot)\) up to a threshold and follows the rule \(P_0(\cdot|\cdot)\), thereafter. Therefore the threshold policy is defined at discrete points and does not satisfy Assumption 1 as the derivative is undefined. To address this issue, we propose an approximation (see interpolation to continuous domain) of the threshold policy, which resembles a step function, so that the derivative exists at every point. This results in a randomized policy which in state \(s\), choices policies \(P_0(s'|s)\) and \(P_1(s'|s)\) with probabilities \(f(s, \tau)\) and \(1 - f(s, \tau)\), respectively. In other words,

\[
P_{sa'}(\tau) \approx P_0(s'|s) f(s, \tau) + P_1(s'|s) (1 - f(s, \tau)).
\]

Intuitively, \(f(s, \tau)\) should allocate similar probabilities to \(P_0(s'|s)\) and \(P_1(s'|s)\) near the threshold. As we move away towards the left (right) direction, the probability of choosing \(P_0(s'|s)\) (\(P_1(s'|s)\)) should decrease. The following function is chosen as a convenient approximation as it is continuously differentiable and the derivative is non-zero at every point.

\[
f(s, \tau(i)) = \frac{e^{(s-\tau(i)-0.5)} + e^{(s-\tau(i)+0.5)}}{1 + e^{(s-\tau(i)+0.5)}}.
\]

Similar to (6), we approximate \(g_\tau(\tau)\) as

\[
g_\tau(\tau(i)) \approx f(s, \tau(i)) h(s) + (1 - f(s, \tau(i)))(R_i - h(s)).
\]

**Remark 1.** Note that although the state space is discrete, individual threshold vector component iterates may take values in the continuous domain. However, only an ordinal comparison dictates which action needs to be chosen in the current state.

**Remark 2.** Instead of the sigmoid function in (7), the function \(f(s, \tau(i)) = 1.3 [s > \tau(i) + 1] + (s - \tau(i)) [s < \tau(i)]\) which uses reward is almost continuous and has been chosen. This function suffers lesser approximation error than that of (7). However, it may lead to slow convergence since the derivative of the function and hence the gradient becomes zero outside \(s < \tau(i) < s + 1\). Although the derivative of sigmoid function decays exponentially fast too, we observe in simulations that the convergence behavior with (7) is better.

Under Assumption 1, the following proposition [32, Proposition 1] provides a closed form expression for the gradient of \(\sigma(\tau)\). The proposition stated next is extended to policy gradient theorem in [33].

**Proposition 1.**

\[
\nabla \sigma(\tau) = \sum_{s \in S} \pi(s, \tau) \nabla g_\tau(s, \tau) + \sum_{s' \in S} \nabla P_{sa'}(\tau)V(s', \tau).
\]

**Remark 3.** The first term on the right of (8) was dropped earlier, the reason given being that “the reward is independent of the parameter”. This is incorrect because the indicator function multiplying the reward is not so. Nevertheless, it turns out (see (Fig. 1a and 1b)) that the performance is hardly affected by the omission of this term. This is presumably because this term is significant only in a small neighborhood of the threshold. This mistake is present in our earlier works [1], [29] and similar remarks apply there. 1

Based on the proposed approximation (7), we set to devise an online update rule for the threshold vector in the slower timescale \(b(.)\), following (5). We evaluate \(\nabla P_{sa'}(\tau)\) as a representative of \(\nabla \sigma(\tau)\) since the stationary probabilities in (8) can be replaced by averaging over time. Using (6), we get

\[
\nabla P_{sa'}(\tau) = (P_0(s'|s) - P_1(s'|s)) \nabla f(s, \tau).
\]

We incorporate a multiplying factor of \(\frac{1}{2}\) in the right hand side of (9) since multiplication by a constant term does not alter the scheme.

The physical significance of this operation is that at every iteration, transitions following rules \(P_1(\cdot|\cdot)\) and \(P_0(\cdot|\cdot)\) are adopted with equal probabilities. \(\nabla f(s, \tau)\) depends on the system state and threshold vector at any given iteration. Similar online rule can be devised for \(\nabla g_\tau(\tau)\) using an identical procedure.

Based on this, when \(i^{th}\) event occur, the online update rule for the \(i^{th}\) component of the threshold vector is as follows.

\[
\tau_{n+1}(i) = \Omega_n[\tau_n(i) + b(n) \nabla f(s, \tau_n(i))] + (-1)^\alpha_n V_n(s, \tau_n(i)) + (-1)^\alpha_n V_n(s, \tau_n(i)) + (-1)^\alpha_n V_n(s, \tau_n(i)).
\]

1We thank Prof. Aditya Mahajan, McGill University for pointing out this error.
where $\alpha_n$ and $\beta_n$ are i.i.d. random variables, each of which can take values 0 and 1 with equal probabilities. If $\alpha_n = 1$, then the transition is governed by the rule $P_1(\cdot, \cdot)$, else by $P_0(\cdot, \cdot)$. In other words, the next state is $\hat{s}$ with probability $\alpha_n P_1(\hat{s}|s) + (1 - \alpha_n)P_0(\hat{s}|s)$. Similarly, $\hat{h}_\beta(s, i) = -h(s) + (1 - \beta)(R_i - h(s))$ where the obtained reward is $-h(s)$ and $R_i - h(s)$ with equal probabilities. The projection operator $\Omega_{i}$ ensures that $\tau(i)$ iterates remain bounded in a specific interval, as specified later. Recall that, we have assumed that the threshold for $i^{th}$ event is a non-increasing function of $i$. Therefore, the $i^{th}$ component of the threshold vector iterates should always be less than or equal to the $(i - 1)^{th}$ component. The first component of $\tau$ is considered to be a free variable which can choose any value in $[0, W]$. The projection operator $\Omega_{i}$, $i > 0$ ensures that $\tau_n(i)$ remains bounded in $[0, \tau_n(i - 1)]$. To be precise, $\Omega_0: x \mapsto 0$ or $(x \wedge W) \in [0, W]$. $\Omega_i: x \mapsto 0$ or $(x \wedge \tau(i - 1)) \in [0, \tau(i - 1)]$ for all $i > 0$.

The framework of SA enables us to obtain the effective drift in (9) by performing averaging. Therefore the online RL scheme where the value functions and the threshold vector update are faster in the slower timescale, respectively, is as follows. We suppress the parametric dependence of $V$ on $\tau$.

$$V_{n+1}(s) = (1 - \alpha)n V_n(s) + \alpha \gamma f(s, \tau_n(i))(-h(s) + R_i^{\hat{s}}((s, \tau(i)) = A_2)) + V_n(s') - V_n(s^*),$$

and

$$\tau_{n+1}(i) = \Omega_{i}^\tau(\tau_n + b(n) \nabla f(s, \tau_n(i))(-1 - \beta h(s) + (1 - \beta)h(s)),$$

for all state $s$, transition to next state $s'$ in (10) refers to a single run of a simulated chain as commonly seen in RL, and $\hat{s}$ in (11) is determined separately following the probability distribution $\alpha_n P_1(\hat{s}|s) + (1 - \alpha_n)P_0(\hat{s}|s)$. The immediate reward is determined using $-h(s) + (1 - \beta)(R_i - h(s))$. The physical significance of (11) is that when $i^{th}$ event of type occurs, the $i^{th}$ component of $\tau$ is updated. However, since the components are provably ordered, we need to update the $i^{th}$ component too where $i' < i$. We no longer need to update the components for $i' < i$ since the order is already preserved while updating the $i^{th}$ component. Also, in (10), the reward function is taken to be $R_i^{\hat{s}}((s, \tau(i)) = A_2)$ when $i^{th}$ event occurs. The expectation operation in (3) is mimicked by the averaging over time implicit in an SA scheme. Note that contrary to [29], where due to the independence among the threshold parameters, only one threshold is updated at a time, in this paper, multiple threshold parameters may need to get updated to capture the ordering constraints.

Remark 4. Instead of the two-timescale approach adopted in this paper, a multi-timescale approach where each individual threshold is updated in a separate timescale, may be chosen. However, since the updates of thresholds are coupled only through the ordering constraints, they can be updated in the same timescale. Moreover, in practice, a multi-timescale scheme may not work well since the fastest (slowest) timescale may be too fast (slow), leading to higher fluctuations and/or very slow speed.

Theorem 1. Update rules (10) and (11) converge to the optimal policy almost surely (a.s.).

Proof: Proof is given in Appendix I.

Remark 5. Unlike the value function iterates, the threshold vector iterates do not require local clocks of individual elements for convergence. However, all components of the threshold vector are required to get updated comparably often. In other words, their frequencies of update should be bounded away from zero which generally holds true for stochastic gradient approaches [22, Chapter 7].

Based on the foregoing analysis, we describe the resulting SALMUT algorithm in Algorithm 1. On a decision epoch, action $a$ is chosen based on the current value of threshold vector. Based on the event, value function of current state $s$ is updated then using (10) in the faster timescale. The threshold vector is also updated following (11) in the slower timescale. Note that the value function is updated one component at a time. However, multiple components of the threshold vector may need to be updated in a single iteration. The scheme resembles actor-critic method [13] with policy gradient for the actor part (see (11)) and post-decision framework for the critic (see (10)).

Algorithm 1 Two-timescale SALMUT algorithm

1: Initialize $n \leftarrow 1$, $V(s) \leftarrow 0, \forall s \in S$ and $\tau \leftarrow 0$.
2: while TRUE do
3: if $i^{th}$ event occurs then
4: Choose action $a$ based on current value of $\tau(i)$.
5: end if
6: Update value function of state $s$ using (10).
7: Update threshold $\tau$ using (11).
8: Update $s \leftarrow s'$ and $n \leftarrow n + 1$.
9: end while

Remark 6. Even if there does not exist an optimal threshold policy for a given MDP problem, the techniques in this paper can be applied to learn the best threshold policy (locally at least) asymptotically. Threshold policies are easy to implement and often provide comparable performances to that of the optimal policy, with a significantly lower storage complexity.

IV. Complexity Analysis

In this section, we compare the storage and computational complexities of SALMUT algorithm with those of existing learning schemes and summarize in Table I. Q-learning and PDS learning need to store the value function of every state-action pair and every PDS, respectively. While updating the value function, both choose the best one after evaluating $|A|$ functions. Since only one state-action pair is updated at a time, the computational complexity associated with remaining operations is constant. Thus, the storage complexities of Q-learning and PDS learning are $O(|S| \times |A|)$ and $O(|S|)$, respectively. The per-iteration computational complexity of

| Algorithm       | Computational complexity | Storage complexity |
|-----------------|-------------------------|--------------------|
| Q-learning [6], [7] | $O(|A|)$                | $O(|S| \times |A|)$ |
| Monotone Q-learning [17], [19] | $O(|A|)$ | $O(|S| \times |A|)$ |
| MH-Q-learning    | $O(\log |A|)$            | $O(|S| \times |A|)$ |
| OQL [9]          | $O(|A|)$                | $O(|S| \times |A|)$ |
| EQL [10]         | $O(\log |A|)$            | $O(|S| \times |A|)$ |
| PDS learning [4], [11] | $O(|A|)$ | $O(|S|)$ |
| MH-PDS learning  | $O(\log |A|)$            | $O(|S|)$ |
| VE learning [12] | $O(|V| \times |A|)$      | $O(|S|)$ |
| Grid learning [20] | $O(|V| \times |A|)$      | $O(|S|)$ |
| Adaptive appx. learning [18] | $O(q(|S|)|A|)$ | $O(|S|)$ |
| SALMUT           | $O(1)$                  | $O(|S|)$ |

TABLE I: Complexities of RL algorithms.
Q-learning and PDS learning is $O(|S|)$. Since at each iteration, the monotone Q-learning algorithm [17], [19] projects the policy obtained using Q-learning within the set of monotone policies, the complexities are identical to those of Q-learning. VE learning [12] updates multiple PDSs at a time. Therefore, the computational complexity contains an additional term $|\mathcal{V}|$ which signifies the cardinality of the VE tuple. Similarly, grid learning [20] and adaptive approximation learning [18] are associated with additional factors $|\mathcal{W}_i|$ and $q(\delta)$, respectively. $|\mathcal{W}_i|$ and $q(\delta)$ depend on the depth of a quadratree used for value function approximation and the approximation error threshold $\delta$, respectively. Note that computational complexities of Q-learning and PDS learning can be reduced to $O(\log |\mathcal{A}|)$ using a max-heap implementation (MH-Q-learning and MH-PDS learning in Table I) where the complexities of obtaining the best action and updating a value are $O(1)$ and $O(\log |\mathcal{A}|)$, respectively. The computational complexity of a max-heap implementation of Optimistic Q-learning (OQL) [9] and Exploitation Enhanced Q-learning (EEQL) [10] is $O(\log |\mathcal{A}|)$. The storage complexity of OQL and EEQL is $O(|S| \times |\mathcal{A}|)$. It is not clear whether the knowledge of structural properties can be encoded in Q-learning easily since the range of the threshold is large, viz., the entire state space. The storage complexity of SALMUT algorithm is $O(|S|)$ as we need to store the value functions of states. SALMUT may require to update all components of the threshold vector at a time (See (11)). Furthermore, the update of value function involves the computation of a single function based on the current threshold (See (10)). Therefore, the per-iteration computational complexity is $O(1)$. Thus SALMUT provides significant improvements in storage and per-iteration computational complexities compared to other schemes. Note that the computational complexity of SALMUT does not depend on $|\mathcal{A}|$ and depends only on the number of events.

V. ILLUSTRATIVE EXAMPLE

We consider a queuing system with $m$ identical servers, $N$ customer classes and a finite buffer of size $B$. We investigate an optimal admission control problem. It is assumed that the arrival of class-$i$ customers is a Poisson process with mean $\lambda_i$. Let the service time be exponentially distributed with mean $\frac{1}{\mu}$, irrespective of the customer class. Note that presence of $m$ identical servers does not change the nature of the model. Similar analysis holds for the single server case also. The state of the system is $(s, i)$ where $s \in \{0, 1, \ldots, m + B\}$ denotes the total number of customers in the system and $i \in \{0, 1, \ldots, N\}$ denotes the class type. Arrival and departures of customers are taken as decision epochs. We take $i = 0$ as the departure event and $i = 1, \ldots, N$ as an arrival of a $1, \ldots, N$th class of customer. For example, states $(2, 0)$ and $(2, 1)$ correspond to a departure and an arrival of class 1 customer while there are 2 users in the system, respectively. $A$ consists of three actions, viz., blocking of an arriving user (A1), admission of an arriving user (A2) and continue/do nothing (A0, say) for departures. When $s = m + B$, then the only feasible action for an arrival (i.e., $i > 0$) is A1. We have $p(s, s', A_2) = 1$ (s'=s+1), $p(s, s', A_1) = 1$ (s'=s) and $p(s, s', A_0) = 1$ (s'=s-1). Let $x = \max(x, 0)$ and $v(s) = \sum_{i=1}^{N} \lambda_i + \min(s, m) \mu$. Let $\lambda(s) = \min(s, m) \mu$ and $\lambda_i(s) = \lambda_i(s)$, $\forall i \neq 0$. Note that $\lambda(s)$ and $\lambda_i(s)$ are non-decreasing in $s$.

One application of this model is a make-to-stock production system that produces $m$ items with $N$ demand classes [34] and buffer size $B$. Satisfaction of a demand (requesting a single unit of the product) from class-$i$ gives rise to reward rate $R_i$. The production time is exponentially distributed with mean $\frac{1}{\mu}$. The inventory holding cost rate is $b(s)$.

Now, we derive that there exists a threshold based optimal policy which admits a class-$i$ customer only up to a threshold $\tau(i)$ which is a non-increasing function of $i$. We prove these properties using the following lemma. The proof follows from [35, Theorem 3.1]. Detailed proof is given in [36, Lemma 1].

**Lemma 1.** $V(s + 1) - V(s)$ is decreasing in $s$.

**Theorem 2.** The optimal policy is of threshold-type where it is optimal to admit class-$i$ customers only up to a threshold $\tau(i)$ which is a non-increasing function of $i$.

**Proof:** For class-$i$ customers, if $A_1$ is optimal in state $s$, then $R_i + v(s + 1) \leq V(s)$ (Using (2)). From Lemma 1, $V(s + 1) - V(s)$ is decreasing in $s$. This proves the existence of a threshold $\tau(i)$ for class-$i$ customers. Since $R_i > R_j$ for $i < j$, $R_i + v(s + 1) \leq V(s)$ implies $R_j + v(s + 1) \leq V(s)$. Therefore, $\tau(i)$ is a non-increasing function of $i$.

The subsequent lemmas establish the unimodality of the average reward with respect to $\tau$. Hence, Theorem 1 holds. Proofs are presented in Appendix II.

**Lemma 2.** $v_n(s + 1) - v_n(s)$ is decreasing in $n$.

**Lemma 3.** $\sigma(\tau)$ is unimodal in $\tau$.

For this particular problem, the improvement in computational complexity offered by SALMUT may not be significant since $|\mathcal{A}| = 3$. However, in general, for a large $|\mathcal{A}|$, the improvement in computational complexity may be remarkable.

VI. SIMULATION RESULTS

In this section, we compare the convergence speed of SALMUT algorithm with traditional RL algorithms. We simulate the finite buffer multi-server system with two customer classes. We take $\lambda_1 = \lambda_2 = 1 \text{ s}^{-1}$, $m = B = 5$, $R_1 = 20$, $R_2 = 10$, $b(s) = 0.1s^2$, $a(n) = \frac{1}{(1+2n)^{0.5}}$ and $b(n) = \frac{10}{n}$. We exclude initial 10 burn-in period values of the iterates.

In practical cases, when the average reward of the system does not change much over a suitable window, we may conclude that stopping condition is met as the obtained policy is close to the optimal policy with a high probability. The choice of window size $\sum_{k=p}^{p+n} a(k)$ over $n$ is to eliminate the effect of diminishing step size affecting the convergence behavior. We choose a window size of $50$ and stop when the ratio of maximum and minimum average rewards exceeds $0.95$. Fig. 1a (1b) reveals that practical for Q-learning, PDS learning and SALMUT algorithms are achieved in 1180 (1180), 580 (1180) and 426 (580) iterations, respectively. Since unlike PDS learning, Q-learning is associated with exploration mechanism, PDS learning converges faster than Q-learning. However, SALMUT converges faster than both Q-learning and PDS learning algorithms since it operates on a smaller policy space (threshold policies only). Due to efficient exploration, the performances of OQL [9] and EEQL [10] are better than that of Q-learning and slightly worse than that of SALMUT. Since for a given sample path, the sequence of actions are same, the performance of MH-Q-learning (MH-PDS learning) is identical to that of Q-learning (PDS learning). The performance of adaptive approximation learning algorithm [18] is identical to that of PDS learning as batch update of PDSs is not possible. Therefore, we do not show their performances in the plot. The convergence behavior of Renewal Monte Carlo (RMC) [21] algorithm (with discount factor=0.999) is slightly worse than that of SALMUT. Since the sampled Q-learning algorithm [23] provides a
near-optimal policy after a finite number of visits to a set of state-action pairs, the convergence behavior may not be comparable.

VII. CONCLUSIONS & FUTURE DIRECTIONS

In this paper, we have proposed an RL algorithm which exploits the ordered multi-threshold nature of the optimal policy. The convergence of the proposed algorithm to the globally optimal threshold vector is established. The proposed scheme provides improvements in storage and computational complexities over traditional RL algorithms. Simulation results establish the improvement in convergence behavior with respect to state-of-the-art RL schemes. In future, this can be extended to develop RL algorithms for Constrained MDP problems by updating the Lagrange Multiplier (LM) in a slower timescale than that of value functions. The LM and the threshold parameter can be updated to develop RL algorithms for restless bandits such as [37] since they are independent of each other. Another possible future direction is to consider the ODE

\[ \dot{\tau}(t) = \sum_{s} P_{ss'}(\tau) x(s') - x(s^*), x \in \mathbb{R}^{|S|}. \]

Clearly, if the reward and cost functions are zero, (12) is same as (13). Also, \( \lim_{n \to \infty} \frac{M_1(n \tau)}{n} = M_0(V) \). The globally asymptotically stable equilibrium of the ODE \( \dot{V}(t) = M_0(V(t)) - V(t) \) which is a scaled limit of ODE \( \dot{V}(t) = M_1(V(t)) - V(t) \), is the origin. Boundedness of value functions and threshold vector iterates follow from [40] and (11), respectively.

Lemma 5. \( V_n - V^\tau n \to 0 \), where \( V^\tau \) is the value function of states for \( \tau = \tau_n \) a.s.

Proof: Since the threshold vector iterates are updated in a slower timescale, value function iterates in the faster timescale treat the threshold vector iterates as fixed. Therefore, iterations for \( \tau \) are \( \tau_{n+1} = \tau_n + \gamma(n) \), where \( \gamma(n) = O(b(n)) = o(a(n)) \). Hence, the limiting ODEs for value function and threshold vector iterates are \( \dot{V}(t) = M_1(V(t)) - V(t) \) and \( \dot{\tau}(t) = 0 \), respectively. It is sufficient to consider the ODE \( \dot{V}(t) = M_1(V(t)) - V(t) \) alone for a fixed \( \tau \) because \( \dot{\tau}(t) = 0 \). The rest of the proof is analogous to that of [13].

For the time being assume that \( \sigma(\tau) \) is unimodal in \( \tau \). This is proved later. The lemmas presented next establish that threshold vector iterates \( \tau_n \) converge to the optimal threshold vector \( \tau^* \) and hence, \( (V_n, \tau_n) \) converges to the optimal pair \( (V^*, \tau^*) \).

Lemma 6. The threshold vector iterates \( \tau_n \to \tau^* \) a.s.

Proof: The limiting ODE for (11) is the gradient ascent scheme \( \dot{\tau} = \nabla \sigma(\tau) \). Note that the gradient points inwards at \( \tau(\cdot) = 0 \) and \( \tau(\cdot) = W \). Since \( \sigma(\tau) \) is unimodal in \( \tau \), there does not exist any local maximum except \( \tau^* \) which is the global maximum. This concludes the proof of the lemma.

Remark 7. If the unimodality of the average reward with respect to
the threshold vector does not hold, then convergence to only a local maximum can be guaranteed.

**APPENDIX II**

**A. Proof of Lemma 2:**

Proof: Proof methodologies are similar to [1].

\[ v_{n+1}(s) = \sum_{i=1}^{N} \lambda_i \max\{R_i + v_n(s+1), v_n(s)\} + \min\{s, m\} \mu v_n((s-1)^\top) + (1 - v(s))v_n(s) - h(s), \]

and \( \hat{v}_{i,n+1}(s) = \max\{v_n(s), R_i + v_n(s+1)\} \).

We know, \( Dv_n(s) = v_n(s + 1) - v_n(s) \). We use induction to prove that \( Dv_n(s) \) is decreasing in \( n \). If \( n = 0 \), \( v_0(s) = 0 \) and \( Dv_0(s) = 0 \). Using (14), it is easy to see that \( Dv_1(s) < Dv_0(s) \) as \( h(s+1) > h(s) \). Assuming that the claim holds for any \( n \), i.e., \( Dv_{n+1}(s) < Dv_n(s) \), we need to prove that \( Dv_{n+2}(s) < Dv_{n+1}(s) \). To analyze the second and third terms in (14), we consider two cases:

(a) \( s \geq m: m \mu Dv_n((s-1)^\top) + (1 - \sum_{i=1}^{N} \lambda_i - m\mu) Dv_{n+1}(s) \) is less than \( m \mu Dv_n((s-1)^\top) + (1 - \sum_{i=1}^{N} \lambda_i) Dv_n(s) \).

(b) \( s < m: sp Dv_n((s-1)^\top) + (1 - v(s+1)) Dv_{n+1}(s) \) is less than \( sp Dv_n((s-1)^\top) + (1 - v(s)) Dv_n(s) \).

Using Lemma 2, this inequality holds \( \forall n \geq \tau(i) \). Therefore, in the re-designed problem, \( U_{i,n} \) converges to \( \tau(i) \). Thus, the threshold policy with \( \tau(i) \) is superior that with \( \tau(i) + 1 \). Since this holds for arbitrary choice of \( \tau(i) \), average reward monotonically decreases with \( \tau(i) \), \( \forall \tau(i) > \tau(i)^* \).

If we have \( \sigma(\tau) \geq \sigma(\tau + e_i) \) (where \( e_i \in \mathbb{R}^N \) is a vector with all zeros except the \( i^{th} \) element being ‘1’), then we must have \( \tau(i) \geq \tau(i)^* \). Therefore, \( \tau(\tau + e_i) \leq \sigma(\tau + 2e_i) \). Hence, the average reward is unimodal in \( \tau(i) \). Since the proof holds for any \( i \), this concludes the proof of the lemma.

**ACKNOWLEDGMENTS**

Works of Arghyadip Roy, Abhay Karandikar and Prassana Chakrorka are supported by the Ministry of Electronics and Information Technology (MeitY), Government of India as part of “5G Research and Building Next Gen Solutions for Indian Market” project. Work of Vivek Borkar is supported by the CEFIPRA grant for “Machine Learning for Network Analytics” and a J. C. Bose Fellowship.

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