RESOLUTION OF SINGULARITIES OF PAIRS
PRESERVING SEMI-SIMPLE NORMAL CROSSINGS

EDWARD BIERSTONE AND FRANKLIN VERA PACHECO

Abstract. Let $X$ denote a reduced algebraic variety and $D$ a Weil divisor on $X$. The pair $(X, D)$ is said to be semi-simple normal crossings (semi-snc) at $a \in X$ if $X$ is simple normal crossings at $a$ (i.e., a simple normal crossings hypersurface, with respect to a local embedding in a smooth ambient variety), and $D$ is induced by the restriction to $X$ of a hypersurface that is simple normal crossings with respect to $X$. We construct a composition of blowings-up $f : \tilde{X} \to X$ such that the transformed pair $(\tilde{X}, \tilde{D})$ is everywhere semi-simple normal crossings, and $f$ is an isomorphism over the semi-simple normal crossings locus of $(X, D)$. The result answers a question of Kollár.

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1. Introduction

The subject of this article is partial resolution of singularities of a pair \((X, D)\), where \(X\) is a reduced algebraic variety defined over a field of characteristic zero and \(D\) is a Weil \(\mathbb{Q}\)-divisor on \(X\).

The purpose of partial resolution of singularities is to provide representatives of a birational equivalence class that have mild singularities — almost as good as smooth — which have to be admitted in natural situations, even if they can be eliminated by normalization. For example, in order to simultaneously resolve the singularities of curves in a parametrized family, one needs to allow special fibers that have simple normal crossings singularities. Likewise, log resolution of singularities of a divisor produces a divisor with simple normal crossings. For these reasons, it is natural to consider simple normal crossings singularities as acceptable from the start, and to seek a partial resolution which is an isomorphism over the simple normal crossings locus.

Our main theorem (Theorem 1.2) is a solution of a problem of János Kollár [12, Problem 19] on resolution of singularities of pairs \((X, D)\) except for semi-simple normal crossings (semi-snc) singularities.

Definition 1.1. Following Kollár, we say that \((X, D)\) is semi-snc at a point \(a \in X\) if \(X\) has a neighborhood \(U\) of \(a\) that can be embedded in a smooth variety \(Y\), where \(Y\) has regular local coordinates \((x_1, \ldots, x_p, y_1, \ldots, y_r)\) at \(a = 0\) in which \(U\) is defined by a monomial equation

\[
\tag{1.1} x_1 \cdots x_p = 0
\]

and

\[
\tag{1.2} D = \sum_{i=1}^{r} \alpha_i (y_i = 0)|_U, \quad \alpha_i \in \mathbb{Q}.
\]

We say that \((X, D)\) is semi-snc if it is semi-snc at every point of \(X\).

According to Definition 1.1, the support, \(\text{Supp } D|_U\), of \(D|_U\) as a subset of \(Y\) is defined by a pair of monomial equations

\[
\tag{1.3} x_1 \cdots x_p = 0, \quad y_{i_1} \cdots y_{i_q} = 0.
\]

Let \(f: \tilde{X} \to X\) be a birational mapping. Denote by \(\text{Ex}(f)\) the exceptional set of \(f\) (i.e. the set of points where \(f\) is not a local isomorphism). Assuming that \(\text{Ex}(f)\) is a divisor we define \(\tilde{D} := D' + \text{Ex}(f)\), where \(D'\) is the birational transform of \(D\) by \(f^{-1}\). We call \((\tilde{X}, \tilde{D})\) the (total) transform of \((X, D)\) by \(f\).
Theorem 1.2 (Main theorem). Let $X$ denote a reduced algebraic variety over a field of characteristic zero, and $D$ a Weil $\mathbb{Q}$-divisor on $X$. Let $U \subset X$ be the largest open subset such that $(U, D|_U)$ is semi-snc. Then there is a morphism $f : \tilde{X} \to X$ given by a composite of blowings-up with smooth (admissible) centers, such that

(1) $(\tilde{X}, \tilde{D})$ is semi-snc;
(2) $f$ is an isomorphism over $U$.

Remarks 1.3. (1) We say that a blowing-up (or its center) is admissible if its center is smooth and has simple normal crossings with respect to the exceptional divisor.

(2) In the special case that $X$ is smooth, we say that $D$ is a simple normal crossings or snc divisor on $X$ if $(X, D)$ is semi-snc (i.e., Definition 1.1 is satisfied with $p = 1$ at every point of $X$). This means that the irreducible components of $D$ are smooth and intersect transversely. Theorem 1.2 in this case, will be called snc-strict log resolution — this means log resolution of singularities of $D$ by a morphism that is an isomorphism over the snc locus (see Theorem 3.11 below). The latter is proved in [4, Thm. 3.1]. Earlier versions can be found in [14], [5, Sec. 12] and [12].

Theorem 1.2 in the special case that $D = 0$ also follows from the earlier results; see Theorem 3.10 below. Both Theorems 3.10 and 3.11 are important ingredients in the proof of Theorem 1.2. Theorem 3.10 is used to reduce Theorem 1.2 to the case that $X$ has only snc singularities. When $X$ has only snc singularities Theorem 3.11 is used to begin an induction on the number of components of $X$.

(3) The desingularization morphism of Theorem 1.2 is functorial in the category of algebraic varieties over a field of characteristic zero with a fixed ordering on the components, and with respect to étale (or smooth) morphisms that preserve the number of irreducible components of $X$ and $D$ passing through every point. See Section 9. Note that a desingularization that avoids semi-snc and in particular snc points cannot be functorial with respect to étale morphisms in general (as is the case for functorial resolution of singularities), because a normal crossings point becomes snc after an étale morphism; see Definitions 3.2 and Remark 9.1 (Non-snc are to be eliminated while snc are to be preserved.) Therefore we must restrict functoriality to a smaller class of morphisms.

(4) Theorem 1.2 holds also with the following stronger version of condition 2: The morphism $f$ is a composite $\sigma_1 \circ \ldots \circ \sigma_t$ of blowings-up $\sigma_i$, where each $\sigma_i$ is an isomorphism over the semi-snc locus of
the transform of \((X, D)\) by \(\sigma_1 \circ \ldots \circ \sigma_{i-1}\). Our proof provides this stronger statement, by using a stronger version of log resolution, where every blowing up is an isomorphism over the snc locus of the preceding transform of \(D\). The latter strong version of log resolution is proved in \([2]\) and in \([3\text{, Sect. 12}]\).

Our approach to partial resolution of singularities is based on the idea developed in \([7]\) and \([3]\) that the desingularization invariant of \([5]\) together with natural geometric information can be used to characterize and compute local normal forms of mild singularities. The local normal forms in the latter involve monomials in exceptional divisors that can be simplified or cleaned by desingularization of invariantly defined monomial marked ideals. These ideas are used in \([7]\) and \([2]\) in the proofs of log resolution by a morphism which is an isomorphism over the snc locus, and are also used in \([7, 3]\) to treat other problems stated in \([12]\), where one wants to find a class of singularities that have to be admitted if normal crossings singularities in a weaker local analytic or formal sense are to be preserved.

In \([7]\) and \([2]\), the mild singularities (for example, simple normal crossings singularities) are all singularities of a hypersurface (see definition \([5, 3.1]\)). The desingularization invariant for a hypersurface is simpler than for general varieties because it begins with the order at a point, rather than with the Hilbert-Samuel function, as in the general case. Semi-simple normal crossings singularities (Definition \([1.1]\)) cannot be described as singularities of a hypersurface in an ambient smooth variety. An essential feature of this article is our use of the Hilbert-Samuel function and the desingularization invariant based on it to characterize semi-snc singularities.

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2. Characterization of semi-snc points

The inductive characterization of semi-snc (Proposition \([2.6]\) below) will be used after reduction of the main problem to the case that \(X\) is an snc hypersurface, no component of \(D\) lies in the singular locus of \(X\), and \(D\) is reduced. (See \([3, 3.1]\) and Section \([8]\).) Under the preceding assumptions, the main theorem is proved by induction on the number of components of \(X\), and Proposition \([2.6]\) is used in the inductive step.

Proposition \([2.6]\) applies to points lying in at least two components of \(X\). The inductive criterion begins with the case of a single component.
In this case, semi-snc means snc. Snc points can be characterized using the desingularization invariant \([7, \text{Lemma 3.5}]\). We begin by recalling the latter.

**Remark 2.1 (Characterization of snc singularities).** Let \(D\) be a reduced Weil divisor on a smooth variety \(X\). Assume that \(a \in \text{Supp}(D)\) lies in exactly \(q\) irreducible components of \(D\). Then \(D\) is snc at \(a\) if and only if the value of the desingularization invariant is \((q, 0, 1, 0, \ldots, 1, 0, \infty)\), where there are \(q - 1\) pairs \((1, 0)\). (This is in “year zero” — before any blowings-up given by the desingularization algorithm.)

The first entry of the invariant at a point \(a\) of a hypersurface \(D\) in a smooth variety is the order \(q\) of \(D\) at \(a\). For a subvariety in general, the Hilbert-Samuel function is the first entry of the invariant. (In the case of a hypersurface, the order and the Hilbert-Samuel function each determine the other; see \([5, \text{Remark 1.3}]\) and Section \(4\).

**Definition 2.2.** Let \(H_{p,q} = H_{p,q,n}\) denote the Hilbert-Samuel function of the ideal \((x_1 \cdots x_p, y_1 \cdots y_q)\) in a ring of formal power series \(K[[x_1, \ldots, x_p, y_1, \ldots, y_n-p]]\), where \(p + q \leq n\). (See Section \(4\).)

The \(H_{p,q}\) are precisely the values that the Hilbert-Samuel function of \(\text{Supp } D\) can take at semi-snc points. We will omit the \(n\) since it will be fixed throughout the arguments using \(H_{p,q}\).

**Definition 2.3.** Assume that \(X\) is snc and that \(D\) has no components in the singular locus of \(X\). We define \(\Sigma_{p,q} = \Sigma_{p,q}(X, D)\) as the set of points \(a \in X\) such that \(a\) lies in exactly \(p\) components of \(X\), and \(q\) is the minimum number of components of \(D\) at \(a\) which lie in any component of \(X\).

For example, if \(X := (x_1x_2 = 0)\) and \(D = (x_1 = y_1 = 0) + (x_2 = y_1y_2 = 0)\), then the origin is in \(\Sigma_{2,1}\).

Having Hilbert-Samuel function \(= H_{p,q}\) at a point of \(\Sigma_{p,q}\) is a necessary condition for semi-snc. But it is not sufficient, even for \((p, q) = (2, 1)\), as we will see in Example \(4.7\). Additional geometric data is needed. This will be given using an ideal sheaf that is a final obstruction to semi-snc. Blowing up to remove this obstruction involves transformations analogous to the cleaning procedure of \([7, \text{Section 2}]\), see Proposition \(7.1\).

Lemma \(4.8\) below, used in the proof of Proposition \(2.6\), provides some initial control over the divisor \(D\) at a point of \(\Sigma_{p,q}\) (or \(\Sigma_{q,p}\)) where the Hilbert-Samuel function has the correct value \(H_{p,q}\), provided that \(p \geq 2\).
Definition 2.4. Consider a pair \((X, D)\), where \(X\) is snc and no component of \(D\) lies in the singular locus of \(X\). Let \(X_1, \ldots, X_m\) denote the irreducible components of \(X\), with a given ordering. Let \(X^i := X_1 \cup \ldots \cup X_i\), \(1 \leq i \leq m\). Let \(D_i\) denote the sum of all components of \(D\) lying in \(X_i\); i.e. \(D_i\) is the divisorial part of the restriction of \(D\) to \(X_i\). We will sometimes write \(D_i = D|_{X_i}\).

Definition 2.5. Consider a pair \((X, D)\) as in Definition 2.4, where \(X\) is (locally) an embedded hypersurface in a smooth variety \(Y\). Assume that \(m \geq 2\). Let \(J = J(X, D)\) denote the quotient ideal \(J = J(X, D) := \left[ I_{D_m} + I_{X_{m-1}} : I_{D_{m-1}} + I_{X_m} \right] \), where \(I_{D_m}, I_{X_{m-1}}, I_{D_{m-1}}\) and \(I_{X_m}\) are the defining ideal sheaves of \(\text{Supp} \, D_m, \text{Supp} \, D_{m-1}, \text{Supp} \, D_{m-1}\) and \(X_m\) (respectively) on \(Y\).

Proposition 2.6 (Characterization of semi-snc points.). Consider a pair \((X, D)\), where \(X\) is (locally) an embedded hypersurface in a smooth variety \(Y\). Assume that \(X\) is snc, \(D\) is reduced and none of the components of \(D\) lie in the singular locus of \(X\). Let \(a \in X\) be a point lying in at least two components of \(X\). Then \((X, D)\) is semi-snc at \(a\) if and only if

1. \((X_{m-1}, D_{m-1})\) is semi-snc at \(a\).
2. There exist \(p\) and \(q\) such that \(a \in \Sigma_{p,q}\) and \(H_{\text{Supp} \, D, a} = H_{p,q}\), where \(H_{\text{Supp} \, D, a}\) is the Hilbert-Samuel function of \(\text{Supp} \, D\) at the point \(a\) and \(H_{p,q}\) is defined as in 2.2.
3. \(J_a = \mathcal{O}_{Y,a}\).

Proposition 2.6 will be proved at the end of Section 4.

Remarks 2.7. (1) If \(a\) lies in a single component of \(X\), then Condition (1) is vacuous and \(J\) is not defined. In this case, Remark 2.1 replaces Lemma 2.6.

(2) We will use Proposition 2.6 to remove unwanted singularities at points lying in more than two components of \(X\), by first blowing up to ensure condition (2), and then applying further blowings-up to get condition (3); see Section 6. At points lying in two components of \(X\), it is simpler to control the behavior of \(J(X, D)\) after admissible blowings-up; see Section 7. In this case, condition (3) is obtained by a sequence of blowings-up that are very easily described (Proposition 7.1).

3. Basic notions and structure of the proof

Definition 3.1. We say that \(X\) is a hypersurface at a point \(a\) if, locally at \(a\), \(X\) can be defined by a principal ideal on a smooth variety.
Definitions 3.2 (cf. Remark 1.3(1)). Let $X$ be an algebraic variety over a field of characteristic zero, and $D$ a Weil $\mathbb{Q}$-divisor on $X$. The pair $(X, D)$ is said to be simple normal crossings (snc) at a closed point $a \in X$ if $X$ is smooth at $a$ and there is a regular coordinate neighborhood $U$ of $a$ with a system of coordinates $(x_1, x_2, \ldots, x_n)$ such that $\text{Supp } D|_U = (x_1x_2\ldots x_k = 0)$, for some $k \leq n$ (or perhaps $\text{Supp } D|_U = \emptyset$). Clearly, the set of snc points is open in $X$. The snc locus of $(X, D)$ is the largest subset of $X$ on which $(X, D)$ is snc. The pair $(X, D)$ is snc if it is snc at every point of $X$.

Likewise, we will say that an algebraic variety $X$ is simple normal crossings (snc) at $a \in X$ if there is a neighborhood $U$ of $a$ in $X$ and a local embedding $X|_U \hookrightarrow Y$, where $Y$ is a smooth variety, such that $(Y, X|_U)$ is simple normal crossings at $\iota(a)$. (Thus, if $X$ is snc at $a$, then $X$ is a hypersurface at $a$.)

The pair $(X, D)$ is called normal crossings (nc) at $a \in X$ if there is an étale morphism $f : U \to X$ and a point $b \in U$ such that $a = f(b)$ and $(U, f^*(D))$ is snc at $b$.

If $D = \sum a_iD_i$, where $D_i$ are prime divisors, then $D_{\text{red}}$ denotes $\sum D_i$, i.e. $D_{\text{red}}$ is $\text{Supp } D$ considered as a divisor.

Example 3.3. The curve $X := (y^2 + x^2 + x^3 = 0) \subset \mathbb{A}^2$ is nc but not snc at 0. It is not snc because it has only one irreducible component which is not smooth at 0. But $X$ is nc at 0 because $X$ has two analytic branches at 0 which intersect transversely.

It is important to distinguish between nc and snc. For example, the analogues for nc of log resolution preserving the nc locus or of Theorem 1.2 are false:

Example 3.4. Consider the pair $(\mathbb{C}^3, D)$, where $D = (x^2 - yz^2 = 0)$. The singularity at 0 is called a pinch point. The pair is nc at every point except the origin. The analogue of Theorem 1.2 for nc fails in this example because we cannot get rid of the pinch point without blowing up the $y$-axis, according to the following argument of Kollár [12, Ex. 8] (see also Fujino [8, Cor. 3.6.10]). The hypersurface $D$ has two sheets over every non-zero point of $(z = 0)$. Going around the origin in $(z = x = 0)$ permutes the sheets, and this phenomenon persists after any birational morphism which is an isomorphism over the generic point of $(z = x = 0)$.

Definitions 3.5. If $f : X \to Y$ is a rational mapping and $Z \subset X$ is a subvariety such that $f$ is defined in a dense subset $Z_0$, then we define the birational transform $f_*(Z)$ of $Z$ as the closure of $f(Z_0)$ in $Y$. In the case that $f$ is birational, then we have the notion of $f_*^{-1}(Z)$ for
subvarieties $Z \subset Y$ such that $f^{-1}$ is defined in a dense subset of $Z$. For a divisor $D = \sum \alpha_i D_i$, where the $D_i$ are prime divisors, we define $f^{-1}_*(D) := \sum \alpha_i f^{-1}_*(D_i)$.

If $f : X \to Y$ is a birational mapping, we let $\text{Ex}(f)$ denote the set of points $a \in X$ where $f$ is not biregular; i.e., $f^{-1}$ is not a morphism at $f(a)$. We consider $\text{Ex}(f)$ with the structure of a reduced subvariety of $X$.

As before, consider $(X, D)$, where $X$ is an algebraic variety $X$ over a field of characteristic zero and $D$ is a Weil divisor. Let $f : \tilde{X} \to X$ be a proper birational map and assume that $\text{Ex}(f)$ is a divisor. Then we define $D' := f^{-1}_*(D)$ and $\tilde{D} := D' + \text{Ex}(f)$.

We call $D'$ the strict or birational transform of $D$ by $f$, and we call $\tilde{D}$ the total transform of $D$. We also call $(\tilde{X}, \tilde{D})$ the (total) transform of $(X, D)$ by $f$.

**Remark 3.6.** It will be convenient to treat $D'$ and $\text{Ex}(f)$ separately in our proof of Theorem 1.2 — we need to count the components of $D'$ rather than of $\tilde{D}$. For this reason, we will work with data given by a triple $(X, D, E)$, where initially $(X, D)$ is the given pair and $E = \emptyset$. After a blowing-up $f : X' \to X$, we will consider the transformed data given by $(X', D', E)$, where $D' := f^{-1}_*(D)$ as above and $\tilde{E} := f^{-1}_*(E) + \text{Ex}(f)$.

We will write $f : (X', D') \to (X, D)$ to mean that $f : X' \to X$ is birational and $D'$ is the strict transform of $D$ by $f$.

**Definition 3.7.** We say that a triple $(X, D, E)$, where $D$ and $E$ are both divisors on $X$, is semi-snc if $(X, D + E)$ is semi-snc (see Definition 1.1).

For economy of notation, when there is no possibility of confusion, we will sometimes denote the transform of $(X, D, E)$ by a sequence of blowings-up still simply as $(X, D, E)$. Other constructions depending on $X$ and $D$ are also denoted by symbols that will be preserved after transformation by blowings-up. This convention is convenient for the purpose of describing an algorithm, and imitates computer programs written in imperative languages.

**Example 3.8.** Consider $(X, D)$, where $X = (x_1^2 - x_2 x_3 = 0) \subset \mathbb{A}^3$ and $D = (x_1 = x_3 = 0)$. Let $f$ denote the blowing-up of $\mathbb{A}^3$ with center the $x_3$-axis. Then, the strict transform $X' = \tilde{X}$ of $X$ by $f$ (i.e., the blowing-up of $X$ with center the $x_3$-axis) lies in one chart of $f$ (the
“$x_2$-chart”) with coordinates $(y_1, y_2, y_3)$ in which $f$ is given by

$$x_1 = y_1 y_2, \quad x_2 = y_2, \quad x_3 = y_3.$$ 

Therefore we have $\tilde{X} = (y_1^2 - y_3 = 0)$ and $\tilde{D} = f_*^{-1}(D) + E$, where $E$ is the exceptional divisor; $E = (y_1^2 - y_3 = y_2 = 0)$. Then

$$\tilde{D} = (y_1 = y_3 = 0) + (y_1^2 - y_3 = y_2 = 0);$$

We see that, at the origin in the system of coordinates $z_1 := y_1, z_2 := y_2, z_3 := y_3 - y_1^2$, the pair $(\tilde{X}, \tilde{D})$ is given by $\tilde{X} = (z_3 = 0), \tilde{D} = (z_3 = y_1 = 0) + (z_3 = y_2 = 0)$, and is therefore snc.

**Example 3.9.** If $X = (xy = 0) \subset Y := \mathbb{A}^3$ and $D = a_1 D_1 + a_2 D_2$, where $D_1 = (x = z = 0)$ and $D_2 = (y = z = 0)$, then the pair $(X, D)$ is semi-snc if and only if $a_1 = a_2$.

At a semi-snc point, the local picture is that $X$ is a snc hypersurface in a smooth variety $Y$, and $D$ is given by the intersection of $X$ with a snc divisor in $Y$ which is transverse to $X$ (in Example 3.9 $(z = 0)$). For this reason, we should have the same multiplicities when one component of this divisor intersects different components of $X$.

### 3.1. Structure of the proof.

The desingularization morphism from Theorem 1.2 is a composition of blowings-up with smooth centers. In the rest of the paper, $(X, D)$ will always denote a pair satisfying the assumptions of Theorem 1.2. Our proof of the theorem involves an algorithm for successively choosing the centers of blowings-up, that will be described precisely in section 5. We will give an idea of the main ingredients in the current subsection. As noted in Remark 1.3 (2), the following two theorems are previously known special cases of our main result that are used in its proof.

**Theorem 3.10** (snc-strict desingularization). Let $X$ denote a reduced scheme of finite type over a field of characteristic zero. Then, there is a finite sequence of blowings-up with smooth centers

$$X := X_0 \leftarrow^\sigma_1 X_1 \leftarrow^\sigma_2 \ldots \leftarrow^\sigma_t X_t =: \tilde{X},$$

such that, if $\tilde{D}$ denotes the exceptional divisor of (3.1), then $(\tilde{X}, \tilde{D})$ is semi-snc and $(X, 0) \leftarrow (\tilde{X}, \tilde{D})$ is an isomorphism over the snc-locus $X^{snc}$ of $X$.

Theorem 3.10 can be strengthened so that, not only is $\tilde{X} \to X$ an isomorphism over the snc locus of $X$ but also $\sigma_{k+1}$ is an isomorphism over the semi-snc points of $(X_k, D_k)$, where $D_k$ is the exceptional divisor of $\sigma_1 \circ \ldots \circ \sigma_k$, for every $k = 0, \ldots, t - 1$. (See [2]; cf. Remarks 1.3(4)).
Theorem 3.11 (snc-strict log resolution [7, Thm. 3.1]). Consider a pair \((X, D)\), as in Theorem 1.2. Assume that \(X\) is smooth. Then there is a finite sequence of blowings-up with smooth centers over the support of \(D\) (or its strict transforms)

\[
X := X_0 \xleftarrow{\sigma_1} X_1 \xleftarrow{\sigma_2} \ldots \xleftarrow{\sigma_t} X_t =: \tilde{X},
\]

such that the (reduced) total transform of \(D\) is snc and \(X \xleftarrow{} \tilde{X}\) is an isomorphism over the snc locus of \((X, D)\).

Remark 3.12. Theorems 3.10 and 3.11 are both functorial in the sense of Remark 1.3(3). Moreover, regarding \(D\) as a hypersurface in \(X\), the blow-up sequence for \(D\) is independent of the embedding space \(X\). Theorem 3.10 follows from functoriality in Theorem 3.11.

Proof of Theorem 3.10. We can first reduce Theorem 3.10 to the case that \(X\) is a hypersurface: If \(X\) is of pure dimension, this reduction follows simply from the strong desingularization algorithm of [5, 6]. The algorithm involves blowing up with smooth centers in the maximum strata of the Hilbert-Samuel function \(H_{X,a}\). The latter determines the local embedding dimension \(e_X(a) := H_{X,a}(1) - 1\), so the algorithm first eliminates points of embedding codimension > 1 without modifying nc points.

When \(X\) is not of pure dimension the desingularization algorithm [7, 6] may involve blowing up hypersurface singularities in higher dimensional components of \(X\) before \(X\) becomes a hypersurface everywhere. This problem can be corrected by a modification of the desingularization invariant described in [15]:

Let \(\#(a)\) denote the number of different dimensions of irreducible components of \(X\) at \(a \in X\). Let \(q(a)\) be the smallest dimension of an irreducible component of \(X\) at \(a\) and set \(d := \text{dim}(X)\). Then, instead of using the Hilbert-Samuel function as first entry of the invariant, we use the pair \(\phi(a) := (\#(a), H_{X \times \mathbb{A}^d,q(a),a,0})\).

The original and modified invariants admit the same local presentations (in the sense of [5]). This implies that every component of a constant locus of one of the invariants is also a component of a constant locus of the other. The modification ensures that the irreducible components of the maximal locus of the usual invariant are blown up in a convenient order rather that at the same time. Since the modified invariant begins with \(\#(a)\), points where there are components of different dimensions will be blown up first. Points with \(\#(a) > 1\) are not hypersurface points.
If \( \#(a) = \#(b) = 1 \) and \( q(a) < q(b) \), then the adjusted Hilbert-Samuel function guarantees that the point with larger value of

\[
H_{X \times \mathbb{A}^d - q(a)}(1) = e(\cdot) + d - q(\cdot) + 1,
\]

where \( e = e_X \), will be blown up first. In particular, non-hypersurface singularities (where \( e(\cdot) - q(\cdot) > 1 \)) will be blown up before hypersurface singularities (where \( e(\cdot) - q(\cdot) \leq 1 \)).

We can thus reduce to the case in which \( X \) is everywhere a hypersurface. Then \( X \) locally admits a codimension one embedding in a smooth variety. For each local embedding we can apply Theorem 3.11. Functoriality in Theorem 3.11 (with respect to embeddings, and étale morphisms preserving the number of components) can be used to show that the local desingularizations glue together to define global centers of blowing up for \( X \) (cf. [11, proof of Prop. 3.37]).

We now outline the proof of the main theorem. First, we can use Theorem 3.10 to reduce to the case that \( X \) is snc; see Section 5, Step 1. Moreover, there is a simple combinatorial argument to reduce to the case that \( D \) is a reduced divisor (i.e., each \( \alpha_i = 1 \) in Definition 1.1); see Step 4 in Section 5 and Section 8.

So we can assume that \( X \) is snc and \( D \) is reduced. We now argue by induction on the number of components of \( X \).

To begin the induction (Section 5, Step 3), we use Theorem 3.11 to transform the first component of \( X \) together with the components of \( D \) lying in it, into a semi-snc pair. By induction, we can assume that the pair given by \( X \) minus its last component, together with the corresponding restriction of \( D \), is semi-snc. (By restriction we mean the divisorial part of the restriction of \( D \)). To complete the inductive step, we then have to describe further blowings-up to remove the unwanted singularities in the last component of \( X \). These blowings-up are separated into blocks which resolve the non-semi-snc singularities in a sequence of strata \( \Sigma_{p,q} \) that exhaust the variety; see Definition 2.3.

Note first that, in the special case that \( X \) is snc, each component of \( D \) either lies in precisely one component of \( X \) (as, for example, if \( (X, D) \) is semi-snc) or it is a component of the intersection of a pair of components of \( X \) (e.g., if \( X := (xy = 0) \subset \mathbb{A}^2 \) and \( D = (x = y = 0) \)).

We can reduce to the case that each component of \( D \) lies in precisely one component of \( X \) by blowing up to eliminate components of \( D \) that are contained in the singular locus of \( X \) (see Section 5, Step 2). Except for this step, our algorithm never involves blowing up with centers of codimension one in \( X \).
We remove non-semi-snc singularities iteratively in the strata \( \Sigma_{p,q} \), for decreasing values of \((p,q)\). The cases \( p = 1, p = 2 \) and \( p \geq 3 \) are treated differently.

In the case \( p = 1 \) the notions of snc and semi-snc coincide, so again we use snc-strict log resolution (Theorem 3.11). The cases \( p = 2 \) and \( p \geq 3 \) will be treated in sections 7 and 6, respectively. All of these cases are part of Step 3 in Section 5.

As remarked in Section 1, our approach is based on the idea that the desingularization invariant of [5] together with natural geometric information can be used to characterize mild singularities. For snc singularities, it is enough to use the desingularization invariant for a hypersurface together with the number of irreducible components at a point, see [7, §3].

In this article, the main object is a pair \((X, D)\). If \( X \) is locally embedded as a hypersurface in a smooth variety \( Y \) (for example, if \( X \) is snc), then (the support of) \( D \) is of codimension two in \( Y \). We will need the desingularization invariant for the support of \( D \). The first entry in this invariant is the Hilbert-Samuel function of the local ring of \( \text{Supp} \, D \) at a point (see Section 4 below). Information coming from the Hilbert-Samuel function will be used to identify non-semi-snc singularities.

4. The Hilbert-Samuel function and semi-simple normal crossings

Lemma 4.8 of this section plays an important part in our use of the Hilbert-Samuel function to characterize semi-snc points. We begin with the definition of the Hilbert-Samuel function and its relationship with the diagram of initial exponents (cf. [4]). At the end of this section, we use Lemma 4.8 to prove the inductive characterization of semi-snc (Lemma 2.6).

**Definition 4.1.** Let \( A \) denote a Noetherian local ring \( A \) with maximal ideal \( m \). The Hilbert-Samuel function \( H_A \in \mathbb{N}^\mathbb{N} \) of \( A \) is defined by

\[
H_A(k) := \text{length } A/m^{k+1}, \quad k \in \mathbb{N}.
\]

If \( I \subset A \) is an ideal, we sometimes write \( H_I := H_{A/I} \). If \( X \) is an algebraic variety and \( a \in X \) is a closed point, we define \( H_{X,a} := H_{O_{X,a}} \), where \( O_{X,a} \) denotes the local ring of \( X \) at \( a \).

**Definition 4.2.** Let \( f, g \in \mathbb{N}^\mathbb{N} \). We say that \( f > g \) if \( f(n) \geq g(n) \), for every \( n \), and \( f(m) > g(m) \), for some \( m \). This relation induces a partial
order on the set of all possible values for the Hilbert-Samuel functions of Noetherian local rings.

Note that \( f \nless g \) if and only if either \( f > g \) or \( f \) is incomparable with \( g \).

Let \( \hat{A} \) denotes the completion of \( A \) with respect to \( m \). Then \( H_A = H_{\hat{A}} \), see [13, §24.D]. If \( A \) is regular, then we can identify \( \hat{A} \) with a ring of formal power series, \( K[[x]] \), where \( x = (x_1, \ldots, x_n) \). Then \( n := (x_1, \ldots, x_n) \) is the maximal ideal of \( K[[x]] \). If \( I \subset K[[x]] \) is an ideal, then

\[
H_I(k) := \dim_K \frac{K[[x]]}{I + n^{k+1}}.
\]

If \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \), set \( |\alpha| := \alpha_1 + \ldots + \alpha_n \). The lexicographic order of \((n + 1)\)-tuples, \((|\alpha|, \alpha_1, \ldots, \alpha_n)\) induces a total ordering of \( \mathbb{N}^n \). Let \( f \in K[[x]] \) and write \( f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha \), where \( x^\alpha \) denotes \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). Define \( \text{supp}(f) = \{\alpha \in \mathbb{N}^n : f_\alpha \neq 0\} \). The \textit{initial exponent} \( \text{exp}(f) \) is defined as the smallest element of \( \text{supp}(f) \). If \( \alpha = \text{exp}(f) \), then \( f_\alpha x^\alpha \) is called the \textit{initial monomial} \( \text{mon}(f) \) of \( f \).

**Definition 4.3.** Consider an ideal \( I \subset K[[x]] \). The \textit{initial monomial ideal} \( \text{mon}(I) \) of \( I \) denotes the ideal generated by \( \{\text{mon}(f) : f \in I\} \).

The \textit{diagram of initial exponents} \( \mathcal{N}(I) \subset \mathbb{N}^n \) is defined as

\[
\mathcal{N}(I) := \{\text{exp}(f) : f \in I \setminus \{0\}\}.
\]

Clearly, \( \mathcal{N}(I) + \mathbb{N}^n = \mathcal{N}(I) \). For any \( \mathcal{N} \subset \mathbb{N}^n \) such that \( \mathcal{N} = \mathcal{N} + \mathbb{N}^n \), there is a smallest set \( \mathcal{V} \subset \mathcal{N} \) such that \( \mathcal{N} = \mathcal{V} + \mathcal{N} \); moreover, \( \mathcal{V} \) is finite. We call \( \mathcal{V} \) the set of vertices of \( \mathcal{N} \).

**Proposition 4.4.** For every \( k \in \mathbb{N} \), \( H_I(k) = H_{\text{mon}(I)}(k) \) is the number of elements \( \alpha \in \mathbb{N}^n \) such that \( \alpha \notin \mathcal{N}(I) \) and \( |\alpha| \leq k \).

**Proof.** See [5, Corollary 3.20]. \( \square \)

**Definition 4.5.** We can use the partial ordering of the set of all Hilbert-Samuel functions to also order the strata \( \Sigma_{p,q} \) (see Definition 2.3). We say that \( \Sigma_{p_1,q_1} \) \textit{precedes} \( \Sigma_{p_2,q_2} \) if \( (\delta(p_1), H_{p_1,q_1}) > (\delta(p_2), H_{p_2,q_2}) \) in the lexicographic order, where

\[
\delta(p) = \begin{cases} 
3, & \text{if } p \geq 3 \\
p, & \text{otherwise}.
\end{cases}
\]

This order corresponds to the order in which we are going to remove the non-semi-snc from these strata.

The following two examples illustrate the kind of information we can expect to get from the Hilbert-Samuel function.
Example 4.6. Let $X := X_1 \cup X_2$, where $X_1 := (x_1 = 0)$, $X_2 := (x_2 = 0) \subset \mathbb{A}^4_{(x_1,x_2,y,z)}$. Note that, if $(X, D)$ is semi-snc, then $\text{Supp } D |_{X_1} \cap \text{Supp } D |_{X_2}$ has codimension 2 in $X$. Consider $D := (x_1 = y = 0) + (x_2 = z = 0)$. Then, the origin is not semi-snc. In fact, $\text{Supp } D |_{X_1} \cap \text{Supp } D |_{X_2} = (x_1 = x_2 = y = z = 0)$, which has codimension 3 in $X$. The Hilbert-Samuel function of $\text{Supp } D$ at the origin detects such an anomaly in codimension at a point in a given stratum $\Sigma_{p,q}$ (see Remark 4.11 and Lemma 4.12). In the preceding example, the origin belongs to $\Sigma_{2,1}$ but the Hilbert-Samuel function is not equal to $H_{2,1}$. In fact, the ideal of $\text{Supp } D$ (as a subvariety of $\mathbb{A}^4$) is $(x_1, y) \cap (x_2, z) = (x_1, y) \cdot (x_2, z)$, which has order 2 while $(x_1x_2, y)$, which is the ideal of the support of $D$ at a semi-snc point in $\Sigma_{2,1}$, is of order 1. The Hilbert-Samuel function determines the order and therefore differs in these two examples.

Example 4.7. This example will show that, nevertheless, the Hilbert-Samuel function together with the number of components of $X$ and $D$ does not suffice to characterize semi-snc. Consider $X := (x_1, x_2 = 0) \subset \mathbb{A}^4_{(x_1,x_2,y,z)}$ and $D := D_1 + D_2 := (x_1 = y = 0) + (x_2 = x_1 + yz = 0)$. Again the origin is not semi-snc, since the intersection of $D_1$ with $X_2 := (x_2 = 0)$ and of $D_2$ with $X_1 := (x_1 = 0)$ are not the same (as they should be at semi-snc points). On the other hand, the Hilbert-Samuel function does not detect the non-semi-snc singularity, since it is the same for the ideals $(x_1, y) \cap (x_2, x_1 + yz)$ and $(x_1x_2, y)$. In fact, the Hilbert-Samuel function is determined by the initial monomial ideal of $\text{Supp } D$. Since $(x_1, y) \cap (x_2, x_1 + yz) = (x_1x_2, x_2y, x_1 + yz)$, we compute its initial monomial ideal as $(x_1, x_2y)$. The latter has the same Hilbert-Samuel function as $(x_1x_2, y)$. This example motivates definition 2.3 which is the final ingredient in our characterization of the semi-snc singularities (Lemma 2.6).

In Example 4.7, although the intersections of $D_1$ with $X_2$ and of $D_2$ with $X_1$ are not the same, the intersection $D_2 \cap X_1$ has the same components as $D_1 \cap X_2$ plus some extra components (precisely, plus one extra component $(x_1 = x_2 = z = 0)$). The following lemma shows that this is the worst that can happen when we have the correct value $H_{p,q}$ of the Hilbert-Samuel function in $\Sigma_{p,q}$.

Lemma 4.8. Assume that $(X, D)$ is locally embedded in a coordinate chart of a smooth variety $Y$ with a system of coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_q, w_1, \ldots, w_{n-p-q})$. Assume $X = (x_1 \cdots x_p = 0)$. Suppose that $D$ is a reduced divisor (so we view it as a subvariety), with no components in the singular locus of $X$, given by an ideal $I_D$ at $a = 0$ of the
form

$$I_D = (x_1 \cdots x_p-1, y_1 \cdots y_r) \cap (x_p, f).$$

Consider $a \in \Sigma_{p,q}$, where $p \geq 2$. (In particular $q$ is the minimum of $r$ and the number of irreducible factors of $f\vert_{(x_p=0)}$). Let $H_D$ denote the Hilbert-Samuel function of $I_D$.

Then $H_D = H_{p,q}$ if and only if we can choose $f$ so that $\text{ord}(f) = q$, $r = q$ and $f \in (x_1 \cdots x_p-1, y_1 \cdots y_r, x_p)$. Moreover, if either $f \notin (x_1 \cdots x_p-1, y_1 \cdots y_r, x_p)$, $\text{ord}(f) > q$ or $r > q$ then $H_D \not\leq H_{p,q}$ (see Definition 4.2 ff.).

Remark 4.9. It follows immediately from the conclusion of the lemma that $H_D \not< H_{p,q}$ at a point in $\Sigma_{p,q}$.

Proof of Lemma 4.8 First we will give a more precise description of the ideal $I_D$. Let $I \subset \{1, 2, \ldots, p-1\} \times \{1, 2, \ldots, r\}$ denote the set of all $(i, j)$ such that $(x_p, f) + (x_i, y_j)$ defines a subvariety of codimension 3 in the ambient variety $Y$ (i.e. a subvariety of codimension 2 in $X$). For such $(i, j)$, any element in $(x_p, f)$ belongs to the ideal $(x_p, x_i, y_j)$. Set $G := \bigcap_{(i,j) \in I} (x_i, y_j)$ and $H := \bigcap_{(i,j) \notin I} (x_i, y_j)$; note that these are the prime decompositions. Then any element of $(x_p, f)$ belongs to $\bigcap_{(i,j) \in I} (x_p, x_i, y_j) = (x_p) + G$. Therefore we can take $f \in G$. Observe that we still have $f \notin (x_i, y_j)$ for $(i, j) \notin I$.

We claim that

$$G \cap (x_p, f) = (x_p) \cdot G + (f).$$

To prove (4.2): The inclusion $G \cap (x_p, f) \supset (x_p) \cdot G + (f)$ is clear since $f \in G$. To prove the other inclusion, consider $a \in G \cap (x_p, f)$. Write $a = fg_1 + x_pg_2$. Then $x_pg_2 \in G = \bigcap_{(i,j) \in I} (x_i, y_j)$. Since $x_p \notin (x_i, y_j)$, for every $(i, j) \in G$, we have $g_2 \in G$. It follows that $a = x_pg_2 + fg_1 \in (x_p) \cdot G + (f)$, as required.

We now claim that

$$H \cap [G \cdot (x_p) + (f)] = (x_p) \cdot [G \cap H] + H \cap (f).$$

As in the previous claim, the inclusion $H \cap [G \cdot (x_p) + (f)] \supset (x_p) \cdot [G \cap H] + H \cap (f)$ is clear. To prove the other inclusion, consider $a \in H \cap [G \cdot (x_p) + (f)]$. Then $a = fg_1 + x_pg \in H$, where $g \in G$. This implies that $fg_1 \in (x_p) + H = \bigcap_{(i,j) \notin I} (x_p, x_i, y_j)$. Consider $(i, j) \notin I$. Assume that $f \in (x_p, x_i, y_j)$. Then there is an irreducible factor $f_0$ of $f$, such that $f_0 \in (x_p, x_i, y_j)$. If $f_0 = x_ph_1 + x_ih_2 + y_jh_3$ with $h_3 \neq 0$, then $(x_p, f) + (x_i, y_j) = (x_p, x_i, y_j)$, which contradicts $(i, j) \notin I$. Now, if $h_3 = 0$, then $f_0 = x_ph_1 + x_ih_2 \in (x_p, x_i)$, which implies $f \in (x_p, x_i)$, contradicting the assumption that $D$ has no component in the singular
locus of $X$. Thus $f \notin (x_p, x_i, y_j)$. Since $(x_p, x_i, y_j)$ is prime, it follows that $g_1 \in (x_p) + H$ and $g_1 = x_pg_{11} + h$, where $h \in H$. Thus $a = fh + x_p(fg_{11} + g)$ and therefore $x_p(fg_{11} + g) \in H$. Since $x_p$ is not in any of the prime factors of $H$, it follows that $fg_{11} + g \in H$. Thus $a \in (x_p) \cdot (G \cap H) + H \cap (f)$.

By (4.2) and (4.3),

$$I_D = G \cap H \cap (x_p, f)$$

$$= H \cap [G \cdot (x_p) + (f)]$$

$$= (x_p) \cdot [H \cap G] + H \cap (f).$$

We are allowed to pass to the completion of the local ring of $Y$ at $a$ with respect to its maximal ideal. So we can assume we are working in a formal power series ring where $(x_1, \ldots, x_p, y_1, \ldots, y_{n-p})$ are the indeterminates. We can pass to the completion because this doesn’t change the Hilbert-Samuel function, the order of $f$ or ideal membership.

For simplicity, we use the same notation for ideals and their generators before and after completion.

We can compute the Hilbert-Samuel function $H_D$ using the diagram of initial exponents of our ideal $I_D$. This diagram should be compared to the diagram of the ideal $(x_1 \cdots x_p, y_1 \cdots y_q)$, which has exactly two vertices, in degrees $p$ and $q$.

All elements of $H \cap (f) = H \cdot (f)$ have order strictly greater than $\text{ord}(f)$ (which is $\geq q$), unless $H = (1)$ and $\text{ord}(f) = q$. Moreover, all elements of

$$(x_p) \cdot [G \cap H] = (x_1 \cdots x_p, x_py_1 \cdots y_r)$$

of order less than $q + 1$ have initial monomial divisible by $x_1x_2 \cdots x_p$.

It follows that, if $f \notin (x_1 \cdots x_{p-1}, y_1 \cdots y_r)$ i.e. if $H \neq (1)$, then $H_D \notin H_{p,q}$. To see this, first assume that $p \geq q + 1$. Then all elements of the ideal $I_D = (x_p) \cap [H \cap G] + H \cap (f)$ have order $\geq q + 1$, but $(x_1 \cdots x_p, y_1 \cdots y_q)$ contains an element of order $q$. Therefore $H_D \notin H_{p,q}$ (obvious from the diagram of initial exponents). Now suppose that $p < q+1$. All elements of $(x_p) \cap [H \cap G]$ of order less than $q+1$ have initial monomials divisible by $x_1 \cdots x_p$, while $y_1 \cdots y_q \in (x_1 \cdots x_p, y_1 \cdots y_q)$ has order $q < q + 1$ but its initial monomial is not divisible by $x_1 \cdots x_p$. Therefore we again get $H_D \notin H_{p,q}$.

Assume that $\text{ord}(f) > q$. We have just seen that every element of $(x_p) \cap [H \cap G]$ of order $< q + 1$ has initial monomial divisible by $x_1 \cdots x_{p-1}$. Therefore every element of $I_D = (x_p) \cap [H \cap G] + H \cdot (f)$ of order $< q + 1$ has initial monomial divisible by $x_1 \cdots x_{p-1}$. But, in $(x_1 \cdots x_p, y_1 \cdots y_q)$, the element $y_1 \cdots y_q$ has order $q < q + 1$ but is not divisible by $x_1 \cdots x_{p-1}$. Therefore $H_D \notin H_{p,q}$. 

If \( f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_r) \), \( \text{ord}(f) = q \) but \( r > q \), then the initial monomial of \( f \) is divisible by \( x_1 \cdots x_{p-1} \). A simple computation shows that the ideal of initial monomials of \( I_D \) is

\[
(x_1 \cdots x_p, x_py_1 \cdots y_r, \text{mon}(f)).
\]

This follows from the fact that canceling the initial monomial of \( f \) using \( x_1 \cdots x_p \) or \( x_py_1 \cdots y_r \) leads to a function whose initial monomial is already in \( (x_1 \cdots x_p, x_py_1 \cdots y_r) \). For convenience, write \( a := x_1 \cdots x_p \), \( b := \text{mon}(f) \) and \( c := x_py_1 \cdots y_r \). From the diagram it follows that

\[
H_D \not\leq H_{p,q}
\]

because the monomials that are multiples of both \( a \) and \( b \) are not only those that are multiples of \( ab \) and therefore \( H_D(q+1) > H_{p,q}(q+1) \).

It remains to show that if \( f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_r) \) (i.e., \( H = (1) \)), \( r = q \) and that \( \text{ord}(f) = q \) then \( H_D = H_{p,q} \). Assume that \( H = (1) \) and that \( \text{ord}(f) = q \). The first assumption implies that

\[
(4.5) \quad I_D = (x_1 \cdots x_p, x_py_1 \cdots y_q, f).
\]

We consider two cases: (1) \( p \leq q \). Since \( H = (1) \), \( f \in G = I_D \). Therefore, we have one of the following options for the initial monomial of \( f \).

\[
(4.6) \quad \text{mon}(f) = \begin{cases} 
  y_1y_2 \cdots y_q \\
  x_1 \cdots x_{p-1}y \\
  x_1 \cdots x_{p-1}y z \\
  x_1 \cdots x_{p-1}z,
\end{cases}
\]

where \( \overline{y} \) is a product of some of the \( y_j \) and \( z \) is a product of some of the remaining coordinates (possibly including some of the \( x_i \)). (In every case, the degree of the monomial is \( q \).)

In each case in (4.6) the ideal of initial monomials of \( I_D \) is \( (x_1 \cdots x_p, x_py_1 \cdots y_q, \text{mon}(f)) \).

We want to prove now that, in all cases in (4.6), \( H_{\text{mon}(I_D)} = H_{p,q} \). For convenience, write \( a := x_1 \cdots x_p \), \( b := \text{mon}(f) \) and \( c := x_py_1 \cdots y_q \). In the first case of (4.6), the equality is precisely the definition of \( H_{p,q} \). Note that, in the remaining cases, the Hilbert-Samuel function of the ideal \((a, b)\) is larger than \( H_{p,q} \) because the monomials that are multiples of both \( a \) and \( b \) are not only those that are multiples of \( ab \). For example, in the second case (i.e., \( \text{mon}(f) = x_1 \cdots x_{p-1}\overline{y} \)), such monomials are those of the form \( a\overline{y}m = bx_pm \) where \( m \not\in (x_1 \cdots x_{p-1}) \). When \( \deg(m) = d \), these terms have degree \( q+d+1 \), but the monomial \( x_py_1 \cdots y_qm \in \text{mon}(I_D) \) (of the same degree) does not belong to the ideal \((a, b)\). This implies that the diagrams of initial exponents of the
ideals \( I_D \) and \((x_1 \cdots x_p, y_1 \cdots y_q)\) have the same number of points in each degree. Therefore \( H_D = H_{\text{mon}(I_D)} = H_{p,q} \).

Case (2) \( q < p \). Then (from (4.3)), the options for the initial monomial of \( f \) are:

\[
\text{mon}(f) = \begin{cases} 
y_1 \cdots y_q, & q < p - 1, 
x_1 \cdots x_{p-1}, & q = p - 1.
\end{cases}
\]

In each of these cases, we can compute the initial monomial ideal of \( I_D \). In the first case, \( \text{mon}(I_D) = (x_1 \cdots x_p, y_1 \cdots y_q) \). In the second case, \( \text{mon}(I_D) = (x_p y_1 \cdots y_q, x_1 \cdots x_{p-1}) \). In both cases, \( H_D = H_{p,q} \). This completes the proof of the lemma.

\[\square\]

**Corollary 4.10.** In the settings of Lemma 4.8 if there are \( p', q' \) such that \( H_{p',q'} \geq H_D \) at \( a \in \Sigma_{p,q} \), then \( H_{p',q'} \geq H_{p,q} \).

**Proof.** Without loss of generality we can assume that \( p' \leq q' \). As in the proof of Lemma 4.8 we pass to the completion of the local ring at \( a \) in \( Y \). We also have that

\[ I_D = (x_1 \cdots x_p, x_p y_1 \cdots y_r) + (f) \cap H. \]

Recall that \( r \geq q \) and \( \text{ord}(f) \geq q \). If \( p > q \) then \( \text{ord}(I_D) \geq q \). Since \( H_{p',q'} \geq H_D \) we must have \( p', q' \geq q \) and then \( H_{p',q'} \geq H_{p,q} \). If \( p \leq q \) then \( \text{ord}(I_D) = p \). Since \( H_{p',q'} \geq H_D \) we must have \( \min(p', q') = p' \geq p = \min(p, q) \). Any element of \( I_D \) of order \( q + 1 \) has initial monomial divisible by \( x_1 \cdots x_p \), therefore the inequality \( H_{p',q'} \geq H_D \) is not possible if \( q' < q \). Hence \( p' \geq p \) and \( q' \geq q \), i.e., \( H_{p',q'} \geq H_{p,q} \). \[\square\]

**Remark 4.11.** Lemma 4.8 is at the core of our proof of Theorem 1.2. The lemma describes the ideal of the support of \( D \) at a point \( a \in \Sigma_{p,q} \), under the following assumptions:

1. \( X \) is snc at \( a \) and, after removing its last component, the resulting pair (with \( D \)) is semi-snc at \( a \).
2. No component of \( D \) at \( a \) lies in the singular locus of \( X \).
3. \( H_{\text{supp} \ D, a} = H_{p,q} \).

Under these assumptions we see that

\[(x_p = 0) \cap (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0) \subset (x_1 \cdots x_{p-1} = 0) \cap (x_p = f = 0) \]

(as in Example 4.7), and also

\[(x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0) \cap (x_p = f = 0) = (x_p = x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0); \]

i.e., the intersection of \( D^{p-1} := (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0) \) and \( D_p := (x_p = f = 0) \) has only components of codimension 2 in \( X \).
The previous statement is in fact true without the assumption (1) of Remark 4.11. This stronger version will likely be useful to prove Theorem 1.2 without using an ordering of the components of $X$. The stronger lemma can be stated as follows.

**Lemma 4.12.** Assume that $X$ is snc and no component of $D$ lies in the singular locus of $X$. Let $X_i$, $i=1,\ldots,n$, be the irreducible components of $X$ at $a$, and let $D_i$ be the divisorial part of $D|_{X_i}$. If $a \in X$ belongs to the stratum $\Sigma_{p,q}$ and $H_{suppD,a} = H_{p,q}$, then, for every $i,j$, the irreducible components of the intersection $D_i \cap D_j$ are all of codimension 2 in $X$.

We plan to publish a proof of this lemma in a future paper.

**Proof of Proposition 2.6** The assertion is trivial at a point in $X \setminus X_m$, so we assume that $a \in X_m$.

At a semi-snc point $a$ of the pair $(X,D)$ the conditions are clearly satisfied. In fact, the ideal of $D$ is of the form $(x_1 \cdots x_p, y_1 \cdots y_q)$ in a system of coordinates for $Y$ at $a = 0$ (recall that $D$ is reduced). We can then compute

$$J_a = [(x_p, x_1 \cdots x_{p-1}, y_1 \cdots y_q) : (x_p, x_1 \cdots x_{p-1}, y_1 \cdots y_q)] = \mathcal{O}_{Y,a}.$$ 

Assume the conditions (1)–(3). By (1), there is system of coordinates $(x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_{n-p-q})$ for $Y$ at $a$, in which $X_m = (x_p = 0)$ and $D$ is of the form $D = (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0) + (x_p = f = 0)$.

By Condition (2) and Lemma 4.8 we can choose $f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_q, x_p)$ and, therefore, we can choose $f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_q)$. Write $f$ in the form $f = x_1 \cdots x_{p-1} g_1 + y_1 \cdots y_q g_2$. Then

$$J_a = [(x_p, x_1 \cdots x_{p-1}, f) : (x_p, x_1 \cdots x_{p-1}, y_1 \cdots y_q)]$$

(4.7) 

$$= [(x_p, x_1 \cdots x_{p-1}, y_1 \cdots y_q g_2) : (x_p, x_1 \cdots x_{p-1}, y_1 \cdots y_q)]$$

$$= (x_p, x_1 \cdots x_{p-1}, g_2).$$

The condition $J_a = \mathcal{O}_{Y,a}$ means that $g_2$ is a unit. Then

$$D = (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0) + (x_p = f = 0)$$

$$= (x_1 \cdots x_{p-1} = y_1 \cdots y_q g_2 = 0) + (x_p = f = 0)$$

$$= (x_1 \cdots x_{p-1} = x_1 \cdots x_{p-1} g_1 + y_1 \cdots y_q g_2 = 0) + (x_p = f = 0)$$

$$= (x_1 \cdots x_{p-1} = f = 0).$$

By Lemma 4.8 since $a \in \Sigma_{p,q}$, ord($f$) = $q$. It follows that $f|_{(x_p=0)}$ is a product $f_1 \cdots f_q$ of $q$ irreducible factors each of order one. For each $f_i$ set $I_i := \{(j,k) : f_i \in (x_j,y_k)|_{x_p=0}, \ j \leq p-1, \ k \leq q\}$ then
$f_i \in \cap_{(j,k) \in I}(x_j,y_k)|_{(x_p=0)}$, where the intersection is understood to be the whole local ring if $I_i$ is empty. Note that $\cup_i I_i = \{(j,k) : j \leq p-1, k \leq q\}$, since $f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_q)$.

We will extend each $f_i$ to a regular function on $Y$ (still denoted $f_i$) preserving this condition, i.e. such that $f_i \in \cap_{(j,k) \in I}(x_j,y_k)$. In fact, $\cap_{(j,k) \in I}(x_j,y_k)|_{(x_p=0)}$ is generated by a finite set of monomials $\{m_r\}$ in the $x_j|_{(x_p=0)}$ and $y_k|_{(x_p=0)}$. Then $f_i$ is a combination, $\sum m_r a_r$, of these monomials. So we can get an extension of $f_i$ as desired, using arbitrary extensions of the $a_r$ to regular functions on $Y$. This means we can assume that $f = f_1 \cdots f_q \in (x_1 \cdots x_{p-1}, y_1 \cdots y_q)$ (using the extended $f_i$).

Since $f|_{(x_1=\cdots=x_p=0)} = y_1 \cdots y_q g_2$ where $g_2$ is a unit, it follows that $f = y_1 \cdots y_q g_2 \mod (x_1, \ldots, x_p)$, where $g_2$ is a unit. Because $D = (x_1 \cdots x_p, f)$, it remains only to check that $x_1, \ldots, x_p, f_1, \ldots, f_q$ are part of a system of coordinates. We can pass to the completion of the ring with respect to its maximal ideal, which we can identify with a ring of formal power series in variables including $x_1, \ldots, x_p, y_1, \ldots, y_q$. It is enough to prove that the images of the $f_i$ and $x_i$ in $\hat{m}/\hat{m}^2$ are linearly independent, where $\hat{m}$ is the maximal ideal of the completion of the local ring $\mathcal{O}_{X,a}$. If we put $x_1 = \cdots = x_p = 0$ in the power series representing each $f_i$ we get

$$(f_1 \cdots f_q)|_{(x_1=\cdots=x_p)} = y_1 \cdots y_q.$$ 

This means that, after a reordering the $f_i$, each $f_i|_{(x_1=\cdots=x_p)} \in (y_i)$, and the desired conclusion follows. \qed

5. Algorithm for the main theorem

In this section we prove Theorem 1.2. We divide the proof into several steps or subroutines each of which specify certain blowings-up.

**Step 1:** Make $X$ snc. This can be done simply by applying Theorem 3.10 to $(X,0)$. The blowings-up involved preserve snc singularities of $X$ and therefore also preserve the semi-snc singularities of $(X,D)$. After Step 1 we can therefore assume that $X$ is everywhere snc.

**Step 2:** Remove components of $D$ lying inside the singular locus of $X$. Consider the union $Z$ of the supports of the components of $D$ lying in the singular locus of $X$. Blowings-up as needed can simply be given by the usual desingularization of $Z$, followed by blowing up the final strict transform.

The point is that, locally, there is a smooth ambient variety, with coordinates $(x_1, \ldots, x_p, \ldots x_n)$ in which each component of $Z$ is of the form $(x_i = x_j = 0)$, $i < j \leq p$. Let $C$ denote the set of irreducible
components of intersections of arbitrary subsets of components of $Z$. Elements of $C$ are partially ordered by inclusion. Desingularization of $Z$ involves blowing up elements of $C$ starting with the smallest, until all components of $Z$ are separated. Then blowing up the final (smooth) strict transform removes all components of $Z$.

After Step 2 we can therefore assume that no component of $D$ lies in the singular locus of $X$.

**Step 3:** Make $(X, D_{\text{red}})$ semi-snc. (I.e., transform $(X, D)$ by the blowings-up needed to make $(X, D_{\text{red}})$ semi-snc.) The algorithm for Step 3 is given following Step 4 below.

We can now therefore assume that $X$ is snc, $D$ has no components in the singular locus of $X$ and $(X, D_{\text{red}})$ is semi-snc.

**Step 4:** Make $(X, D)$ semi-snc. A simple combinatorial argument for Step 4 will be given in Section 8. This finishes the algorithm.

**Algorithm for Step 3:** The input is $(X, D)$, where $X$ is snc, $D$ is reduced and no component of $D$ lies in the singular locus of $X$. We will argue by induction on the number of components of $X$. It will be convenient to formulate the inductive assumption in terms of triples rather than pairs.

**Definition 5.1.** Consider a triple $(X, D, E)$, where $X$ is an algebraic variety, and $D$, $E$ are Weil divisors on $X$. Let $X_1, \ldots, X_m$ denote the irreducible components of $X$ with a given ordering. We use the notation of Definition 2.4. Define

$$E^i := E|_{X^i} + (X - X^i)|_{X^i},$$

$$(X, D, E)^i := (X^i, D^i, E^i),$$

where $(X - X^i)|_{X^i}$ is viewed as a divisor on $X^i$.

Recall Definitions 3.5, 3.7 and Remark 3.6.

**Definition 5.2.** Given $(X, D, E)$, we write $\Sigma_{p,q} = \Sigma_{p,q}(X, D, E)$ to denote $\Sigma_{p,q}(X, D)$ (so the strata $\Sigma_{p,q}$ depend on $X$ and $D$ but not on $E$). See Definition 2.3.

**Theorem 5.3.** Assume that $X$ is snc, $D$ is a reduced Weil divisor on $X$ with no component in the singular locus of $X$, and $E$ is a Weil divisor on $X$ such that $(X, E)$ is semi-snc. Then there is a composite of blowings-up with smooth centers $f : X' \to X$, such that:

1. Each blowing-up is an isomorphism over the semi-snc points of its target triple.
2. The transform $(X', D', \tilde{E})$ of the $(X, D, E)$ by $f$ is semi-snc.
Proof. The proof is by induction on the number of components \( m \) of \( X \).

Case \( m = 1 \). Since \( m = 1 \), then \((X, D + E)\) is semi-snc if and only if \((X, D + E)\) is snc. This case therefore follows from Theorem 3.11 applied to \((X, D + E)\).

General case. The sequence of blowings-up will depend on the ordering of the components \( X_i \) of \( X \). We will use the notation of Definitions 2.4, 5.1. Since \( X \) is snc and no component of \( D \) lies in the singular locus of \( X \), it follows that every component of \( D \) lies inside exactly one component of \( X \).

By induction, we can assume that \((X^{m-1}, D^{m-1}, E^{m-1})\) is semi-snc. We want to make \((X^m, D^m, E^m)\) semi-snc. For this purpose, we only have to remove the unwanted singularities from the last component \( X_m \) of \( X = X^m \).

Recall that \( X \) is partitioned by the sets \( \Sigma_{p,q} = \Sigma_{p,q}(X, D) \). Clearly for all \( p \) and \( q \), the closure \( \overline{\Sigma_{p,q}} \) of \( \Sigma_{p,q} \) has the property

\[
\overline{\Sigma_{p,q}} \subset \bigcup_{p' \geq p, q' \geq q} \Sigma_{p',q'}.
\]

We will construct sequences of blowings-up \( X' \rightarrow X \) such that \( X' \) is semi-snc on certain strata \( \Sigma_{p,q}(X', D') \), and then iterate the process. The following definitions are convenient to describe the process precisely.

**Definitions 5.4.** Consider the partial order on \( \mathbb{N}^2 \) induced by the order on the set \( \{\Sigma_{p,q}\} \), see Definition 4.5. For \( I \subset \mathbb{N}^2 \), define the monotone closure \( \overline{I} \) of \( I \) as \( \overline{I} := \{x \in \mathbb{N}^2 : \exists y \in I, \ x \geq y\} \). We say that \( I \subset \mathbb{N}^2 \) is monotone if \( \overline{I} = I \). The set of monotone subsets of \( \mathbb{N}^2 \) is partially ordered by inclusion, and has the property that any increasing sequence stabilizes. Given a monotone \( I \) and a pair \((X, D)\), set

\[
\Sigma_I(X, D) = \bigcup_{(p,q) \in I} \Sigma_{p,q}(X, D).
\]

Then \( \Sigma_I(X, D) \) is closed. In fact, if \( I \) is monotone then \( \Sigma_I(X, D) = \bigcup_{(p,q) \in I} \overline{\Sigma_{p,q}} \).

**Definition 5.5.** Given \((X, D)\) and monotone \( I \), let \( K(X, D, I) \) denote the set of maximal elements of \( \{(p,q) \in \mathbb{N}^2 \setminus I : \Sigma_{p,q}(X, D) \neq \emptyset\} \). Also set \( K(X, D) := K(X, D, \emptyset) \). Note that \( K(X, D, I) \) consists only of incomparable pairs \((p,q)\) and that it does not simultaneously contain strata \( \Sigma_{p,q} \) with \( p \geq 3, p = 2 \) and \( p = 1 \).
Case A: We first deal with the case in which \( K(X, D) \) contains strata \( \Sigma_{p,q} \) with \( p \geq 3 \). We can apply Proposition 6.1 to reduce to the case in which \( (X, D, E) \) is semi-snc at every point lying in at least 3 components of \( X \).

Case B: Assume that \( (X, D, E) \) is semi-snc at every point lying in at least \( 3 \) components of \( X \). Let \( I := \{(p, q) \in \mathbb{N}^2 : p \geq 3\} \) and \( U \) the complement of \( \Sigma_{I_k}(X, D) \). Assume that \( K(U, D|_U) \) contains a stratum \( \Sigma_{2,q} \), for some \( q \). In particular this means that \( K(U, D|_U) \) doesn’t contain any stratum \( \Sigma_{1,q} \). We can apply Proposition 7.1 to \( (X, D) \) to reduce to the case in which \( (X, D, E) \) is semi-snc at every point lying in at least \( 2 \) components. Observe that the centers involved never intersect a stratum \( \Sigma_{p,q} \) with \( p \neq 2 \).

Case C: Finally, assume that \( (X, D, E) \) is semi-snc at every point in \( \Sigma_{p,q} \) for \( p \geq 2 \). Recall that if \( X \) has only one component (and is therefore smooth), then semi-snc is the same as snc. Hence this case follows from Theorem 3.11 applied to the pair \( (X^m, D^m + E^m)|_U \), where \( U \) is the complement of the union of all \( \Sigma_{p,q} \) with \( p \geq 2 \).

Remark 5.6. The centers of blowing up used in Proposition 7.1 (Case B) and also in Theorem 3.11 (Case C) are closed in \( U \) and contain only non-semi-snc points. Since \( (X, D, E) \) is semi-snc on \( \Sigma_{I_k}(W^k_0, F^k_0) \), and therefore in a neighborhood of the latter, we see that these centers are also closed in \( W^k_0 \).

6. The case of more than 2 components

In this section, we show how to remove the unwanted singularities in the strata \( \Sigma_{p,q}(X, D) \), with \( p \geq 3 \).

Throughout the section, \( (X, D, E) \) denotes a triple as in Definition 5.1 and we use the notation of the latter. As in Theorem 5.3 we assume that \( X \) is snc, \( D \) is reduced and has no component in the singular locus of \( X \), and \( (X, E) \) is semi-snc. We consider \( K(X, D) \) as in Definition 5.5.

Proposition 6.1. With the hypothesis of Theorem 5.3, assume that \( (X, D, E) \) is such that \( K(X, D) \) contains a stratum \( \Sigma_{p,q} \) with \( p \geq 3 \). Then, there is a composition of admissible blowings-up \( X' \to X \) such that \( (X', D', E) \) is semi-snc at every point lying in at least \( 3 \) components of \( X \).

Proof. We start with the variety \( W_0 := X \) and the divisors \( F_0 := D, G_0 = E \), and we define \( I_0 \) as the monotone closure of

\[ \{\text{maximal elements of } (p, q) \in \mathbb{N}^2 : \Sigma_{p,q}(W_0, F_0) \neq \emptyset\} \]
Put \( j_0 = 0 \). Inductively, for \( k \geq 0 \), we will construct admissible blowings-up

\[
W_{j_k} \leftarrow \cdots \leftarrow W_{j'_k} \leftarrow \cdots \leftarrow W_{j_{k+1}}
\]

such that, if \((W_{j_{k+1}}, F_{j_{k+1}}, G_{j_{k+1}})\) denotes the transform of the triple \((W_{j_k}, F_{j_k}, G_{j_k})\), then \((W_{j_{k+1}}, F_{j_{k+1}})\) semi-snc on \( \Sigma_{I_k}(W_{j_{k+1}}, F_{j_{k+1}}) \). Then we define

\[
I_{k+1} := I_k \cup K(W_{j_{k+1}}, F_{j_{k+1}}, I_k).
\]

We have \( I_{k+1} \supset I_k \), with equality only if \( \Sigma_{I_k}(W_{j_k}, F_{j_k}) = W_{j_k} \).

In this way we define a sequence \( I_0 \subset I_1 \subset \ldots \). Since this sequence stabilizes, there is \( t \) such that \( \Sigma_{I_t}(W_{j_t}, F_{j_t}) = W_{j_t} \). By construction, \( W_{j_t} \) is semi-snc on \( \Sigma_{I_t}(W_{j_t}, F_{j_t}) \), so that \( (W_{j_t}, F_{j_t}) \) is everywhere semi-snc.

The blowing-up sequence (6.1) will be described in two steps. The first provides a sequence of admissible blowings-up \( W_{j_k} \leftarrow \ldots \leftarrow W_{j'_k} \) for the purpose of making the Hilbert-Samuel function equal to \( H_{p,q} \) on \( \Sigma_{p,q} \), for each \((p, q) \in K(W_{j'_k}, F_{j'_k})\). The second step provides a sequence of admissible blowings-up \( W_{j'_k} \leftarrow \ldots \leftarrow W_{j_{k+1}} \) that finally removes the non-semi-snc points from the \( \Sigma_{p,q} \), where \((p, q) \in K(W_{j_{k+1}}, F_{j_{k+1}})\).

**Step 1:** We can assume that, locally, \( X + E \) is embedded as an snc hypersurface in a smooth variety \( Z \). We consider the embedded desingularization algorithm applied to Supp \( D \) with the divisor \( X + E \) in \( Z \). We will blow up certain components of the centers of blowing up involved. These centers are the maximum loci of the desingularization invariant, which decreases after each blowing-up. Our purpose is to decrease the Hilbert-Samuel function, which is the first entry of the invariant. During the desingularization process, some components of \( X + E \) may be moved away from Supp \( D \) before Supp \( D \) becomes smooth. We will only use centers from the desingularization algorithm that contain no semi-snc points. By assumption, all non-semi-snc points lie in \( X_m \), so that all centers we will consider are inside \( D_m \). Therefore \( X_m \) (which is a component of \( X + E \)) is not moved away before \( D_m \) becomes smooth.

We are interested in the maximum locus of the invariant on the complement \( U_k \) of \( \Sigma_{I_k}(W_{j_k}, F_{j_k}) \) in \( W_{j_k} \). The corresponding blowings-up are used to decrease the maximal values of the Hilbert-Samuel function.

**Lemma 6.2.** Let \( C \) be an irreducible smooth subvariety of Supp \( D \). Assume that the Hilbert-Samuel function equals \( H_{p,q} \) (for given \( p, q \)) at every point of \( C \). If \( C \cap \Sigma_{p,q} \neq \emptyset \), then \( C \subset \Sigma_{p,q} \).

**Proof.** Let \( a \in C \cap \Sigma_{p,q} \). Since the Hilbert-Samuel function of Supp \( D \) is constant on \( C \), then \( a \) has a neighborhood \( U \subset C \), each point of which lies in precisely those components of \( D \) at \( a \). Therefore, \( U \subset \Sigma_{p,q} \).
Since the closure of $\Sigma_{p,q}$ lies in the union of the $\Sigma_{p',q'}$ with $p' \geq p$, $q' \geq q$, any $b \in C \setminus U$ belongs to $\Sigma_{p',q'}$, for some $p' \geq p$, $q' \geq q$. Thus $H_{\text{Supp}, D, b} = H_{p,q} \leq H_{p',q'}$. But, by Lemma 4.8, the Hilbert-Samuel function cannot be $< H_{p',q'}$ on $\Sigma_{p',q'}$. Therefore $b \in \Sigma_{p,q}$.

We write the maximum locus of the invariant in $U_k$ as a disjoint union $A \cup B$ in the following way: $A$ is the union of those components of the maximum locus containing no semi-snc points, and $B$ is the union of the remaining components. Thus $B$ is the union of those components of the maximum locus of the invariant with generic point semi-snc. Each component of $B$ has Hilbert-Samuel function $H_{p,q}$, for some $p, q$, and lies in the corresponding $\Sigma_{p,q}$ by Lemma 6.2. On the other hand, any component $C$ of the maximum locus of the invariant where either the invariant does not begin with $H_{p,q}$, for some $p, q$, or the invariant begins with some $H_{p,q}$ but no point of $C$ belongs to $\Sigma_{p,q}$, is a component of $A$.

Both $A$ and $B$ are closed in the open set $U_k \subset W_{j_k}$. But, all points in the complement of $U_k$ are semi-snc, and the semi-snc points are open. Since no points of $A$ are semi-snc, $A$ has no limit points in the complement of $U_k$. Thus $A$ is closed in $W_{j_k}$.

We blow up with center $A$. Then the invariant decreases in the preimage of $A$. Recall that $A$ and $B$ depend on $(X, D)$. We use the same notation $A$ and $B$ to denote the sets with the same meaning as above, after blowing up. So we can continue to blow up until $A = \emptyset$. Say we are now in year $j_k'$.

**Claim 6.3.** If $(p, q) \in K(W_{j_k}', F_{j_k}')$ (so that $A = \emptyset$), then the Hilbert-Samuel function equals $H_{p,q}$ at every point of $\Sigma_{p,q}$.

**Proof.** Let $a \in \Sigma_{p,q}$, where $(p, q) \in K(W_{j_k}', F_{j_k}')$. Assume that the Hilbert-Samuel function $H$ at $a$ is not equal to $H_{p,q}$. Recall that every point of $B$ has Hilbert-Samuel function of the form $H_{p',q'}$ for some $p', q'$, and belongs to $\Sigma_{p',q'}$. Therefore $a \notin B$, so the invariant at $a$ is not maximal. Thus there is $b \in B$ where the Hilbert-Samuel function is $H_{p',q'} > H$ for some $p', q'$ and $b \in \Sigma_{p',q'}$. By Corollary 4.10, $H_{p',q'} > H_{p,q}$. This means that $\Sigma_{p',q'} > \Sigma_{p,q}$. Since $(p, q) \in K(W_{j_k}', F_{j_k}')$ then $(p', q') \in I_k$. We have reached a contradiction because $b \in B$ and $B$ lies in the complement of $\Sigma_{I_k}(W_{j_k}', F_{j_k}')$.

The claim 6.3 shows that when $A = \emptyset$ we have achieved the goal of Step 1, i.e., the Hilbert-Samuel function equals $H_{p,q}$ at every point of $\Sigma_{p,q}$, where $(p, q) \in K(W_{j_k}', F_{j_k}')$. 
Step 2: We now describe blowings-up that eliminate non-semi-snc points from the strata $\Sigma_{p,q}$, with $(p, q) \in K(W^j_k, F^j_k)$. Note this does not mean that all the points in the preimage of these strata will be semi-snc. Only the points of the strata $\Sigma_{p,q}$, for the transformed $(X, D, E)$, for $(p, q) \in K(W^j_k, F^j_k)$, will be made semi-snc. The remaining points of the preimages will belong to new strata $\Sigma_{p',q'}$, where $p' < p$ or $q' < q$ and therefore will be treated in further iterations of Steps 1 and 2.

We are assuming that $K(W^j_k, F^j_k)$ contains some stratum $\Sigma_{p,q}$ with $p \geq 3$. Hence, by Definition 4.5, all strata in $K(W^j_k, F^j_k)$ is of the form $\Sigma_{p,q}$ with $p \geq 3$. Therefore this case follows from Proposition 6.7 below applied to $(X, D, E)|_U$, where $U$ is the complement of $\Sigma_I^j_k(W^j_k, F^j_k)$ in $W^j_k$. Observe that the center of the blowing-up involved never intersects a stratum $\Sigma_{p,q}$ with $p \leq 2$. □

The following lemmas are needed to state Proposition 6.7.

Lemma 6.4. Assume that $(X^{m-1}, D^{m-1}, E^{m-1})$ is semi-snc and let $(p, q) \in K(X, D)$. Define

\[(6.2) \quad C_{p,q} := X_m \cap \Sigma_{p-1,q}(X^{m-1}, D^{m-1}).\]

Then:

1. $C_{p,q}$ is smooth;
2. $\Sigma_{p,q}(X, D) \subset C_{p,q} \subset \bigcup_{q' \leq q} \Sigma_{p,q'}(X, D)$.

Lemma 6.5. Assume that $(X^{m-1}, D^{m-1}, E^{m-1})$ is semi-snc and let $(p, q) \in K(X, D)$. Assume that $p \geq 3$ and that the Hilbert-Samuel function equals $H_{p,q}$, at every point of $\Sigma_{p,q} = \Sigma_{p,q}(X, D)$. Then:

1. Every irreducible component of $C_{p,q}$ which contains a non-semi-snc point of $\Sigma_{p,q}$ consists entirely of non-semi-snc points.
2. Every irreducible component of $\Sigma_{p,q}$ consists entirely either of semi-snc points or non-semi-snc points.

Definition 6.6. Assume that $(X^{m-1}, D^{m-1}, E^{m-1})$ is semi-snc and that, for all $(p, q) \in K(X, D)$, where $p \geq 3$, the Hilbert-Samuel function equals $H_{p,q}$, at every point of $\Sigma_{p,q}$. Let $C$ denote the union over all $(p, q) \in K(X, D)$, $p \geq 3$, of the union of all components of $C_{p,q}$ which contain non-semi-snc points of $\Sigma_{p,q}$.

Proposition 6.7. Under the assumptions of Definition 6.6, let $\sigma : X' \to X$ denote the blowing-up with center $C$ defined above. Then:

1. The transform $(X', D', \tilde{E})$ of $(X, D, E)$ is semi-snc on the strata $\Sigma_{p,q}(X', D')$, for all $(p, q) \in K(X, D)$ with $p \geq 3$.\]
(2) Let \( a \in \Sigma_{p,q} \), where \( (p, q) \in K(X, D) \) and \( p \geq 3 \). If \( a \in C \) and \( a' \in \sigma^{-1}(a) \), then \( a' \in \Sigma_{p',q'}(X', D') \), where \( p' \leq p, q' \leq q \), and at least one of these inequalities is strict.

**Proof of Lemma 6.4.** This is immediate from the definitions of \( \Sigma_{p,q} = \Sigma_{p,q}(X, D) \), \( K(X, D) \) and \( C_{p,q} \). \( \square \)

**Proof of Lemma 6.5.** Let \( a \in \Sigma_{p,q} \) be a non-semi-snc point, and let \( S \) be the irreducible component of \( \Sigma_{p,q} \) containing \( a \). Let \( C_0 \) denote the component of \( C_{p,q} \) containing \( S \). We will prove that all points of \( C_0 \) are non-semi-snc, as required for (1). In particular, all points in \( S \) are non-semi-snc and (2) follows.

By Lemma 4.8, \( X \) is embedded locally at \( a \) in a smooth variety \( Y \) with a system of coordinates \( x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_{n-p-q} \) in a neighborhood \( U \) of \( a = 0 \), in which we can write:

\[
X_m = (x_p = 0),
X = (x_1 \cdots x_p = 0),
D = D^{m-1} + D_m,
\]

where

\[
D^{m-1} := (x_1 \cdots x_{p-1} = y_1 \cdots y_q = 0),
D_m := (x_p = x_1 \cdots x_{p-1}g_1 + y_1 \cdots y_qg_2 = 0).
\]

Since \((X, D, E)\) is not semi-snc at \( a \) then \( g_2 \) is not a unit (see Lemma 2.6(3) and (4.7)). In fact, by Lemma 6.8 following, the ideal \( J(X, D) \) (see Definition 2.5) is given at \( a \) by \((x_p, x_1 \cdots x_{p-1}, g_2)\); the latter coincides with the local ring of \( Y \) at \( a \) if and only if \( g_2 \) is a unit. In the given coordinates,

\[
C_0 = (x_1 = \ldots = x_p = y_1 = \ldots = y_q = 0).
\]

To show that all the points in \( C_0 \) are non-semi-snc, it is enough to show that \( g_2 \) is in the ideal \((x_1, \ldots, x_p, y_1, \ldots, y_q)\). In fact, the latter implies that \( g_2 \) is not a unit, and therefore that \( J(X, D) = (x_p, x_1 \cdots x_{p-1}, g_2) \) is a proper ideal at every point of \( C_0 \cap U \). Since \( C_0 \) is irreducible, \( C_0 \cap U \) is dense in \( C_0 \). But the set of semi-snc is open, so it follows that all points in \( C_0 \) are non-semi-snc.

Proposition 6.9 below shows that if \( g_2 \) is not a unit, then \( g_2 \in (x_1, \ldots, x_p, y_1, \ldots, y_q) \), concluding the proof of Lemma 6.5. \( \square \)

**Lemma 6.8.** Let \( R \) denote a regular local ring, and suppose that \( x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_{n-p-q} \) is a regular system of parameters of \( R \). If
\( \text{Proof.} \) We have
\[
[(x_p, x_1 \cdots x_{p-1}, y_1 \cdots y_q g_2) : (x_p, x_1 \cdots x_{p-1}, y_1 \cdots y_q)] = (x_p, x_1 \cdots x_{p-1}, g_2).
\]

Of course,
\[
(x_p, x_1 \cdots x_{p-1}, y_1 \cdots y_q g_2) \cap (y_1 \cdots y_q) \supseteq y_1 \cdots y_q \cdot (x_p, x_1 \cdots x_{p-1}, g_2).
\]

To prove the reverse inclusion, assume that \( h \) belongs to the left hand side. Then we can write \( h = x_p a + x_1 \cdots x_{p-1} b + y_1 \cdots y_q g_2 c \). Since \( h \in (y_1 \cdots y_q) \), then \( x_p a + x_1 \cdots x_{p-1} b \in (y_1 \cdots y_q) \). This implies that \( x_1 \cdots x_{p-1} b \in (x_p, y_1 \cdots y_q) \). Since \( (x_p, y_1 \cdots y_q) \) is an intersection of primes, none of which contains \( x_k, k = 1, \ldots, p-1, \) then \( b \in (x_p, y_1 \cdots y_q) \). Therefore, we can write \( h = x_p a' + x_1 \cdots x_{p-1} b' + y_1 \cdots y_q g_2 c \), where \( b' \in (y_1 \cdots y_q) \). This implies that \( x_p a' \in (y_1 \cdots y_q) \), and therefore that \( a' \in (y_1 \cdots y_q) \). Hence \( h \in y_1 \cdots y_q \cdot (x_p, x_1 \cdots x_{p-1}, g_2) \).

This gives the reverse inclusion required to complete the proof. \( \square \)

**Proposition 6.9.** Let \( f \) denote an element of a regular local ring. Assume that \( f \) has \( q \) irreducible factors, each of order 1, that \( f \in (x_1 \cdots x_{p-1}, y_1 \cdots y_q) \), where \( p \geq 3 \) and the \( x_i, y_i \) form part of a regular system of parameters, and that \( f = x_1 \cdots x_{p-1} g_1 + y_1 \cdots y_q g_2 \), where \( g_2 \) is not a unit. Then \( g_2 \in (x_1, \ldots, x_{p-1}, y_1, \ldots, y_q) \).

**Remark 6.10.** The condition \( p \geq 3 \) is crucial, as can be seen from Example 4.7. In the latter, we have \( D = (x_1 = y_1 = 0) + (x_2 = x_1 + y_1 z = 0) \), so that \( f = x_1 + y_1 z \) and \( g_2 = z \notin (x_1, x_2, y_1) \).

To prove Proposition 6.9 we will use the following lemma.

**Lemma 6.11.** Let \( p \geq 3 \) and \( s \geq 0 \) be integers. Consider
\[
f = (x_1 m_1 + a_1) \cdots (x_{p-1} m_{p-1} + a_{p-1})(y_r n_1 + b_1) \cdots (y_r n_s + b_s) g,
\]
where the $x_i, y_i, a_i, b_i, m_i, n_i$ and $g$ are elements of a regular local ring with $x_1, \ldots, x_{p-1}, y_1, \ldots, y_q$ part of a regular system of parameters, and $1 \leq r_1 < \cdots < r_s \leq q$. Assume that, for every $i = 1, \ldots, p-1$ and $j = 1, \ldots, q$,

\[(6.5) \quad \text{if } a_i \notin (y_j), \text{ then } y_j = y_{r_k} \text{ and } b_k \in (x_i), \text{ for some } k.\]

Then, after expanding the right hand side of \((6.4)\), all the monomials (in the elements above) appearing in the expression are in either the ideal $\langle x_1 \cdots x_{p-1} \rangle$ or the ideal $\langle y_1 \cdots y_q \rangle \cdot \langle x_1, \ldots, x_{p-1}, y_1, \ldots, y_q \rangle$.

Remark 6.12. The conclusion of the lemma implies that $f$ can be written as $x_1 \cdots x_{p-1} g_1 + y_1 \cdots y_q g_2$ with $g_2 \in \langle x_1, \ldots, x_{p-1}, y_1, \ldots, y_q \rangle$. This is precisely what we need for Proposition 6.9.

Proof of Lemma 6.11. First consider $s = 0$. Then \((6.5)\) implies that each $a_i$ is in the ideal $\langle y_1 \cdots y_q \rangle$. The expansion of

\[(x_1 m_1 + a_1) \cdots (x_{p-1} m_{p-1} + a_{p-1}),\]

includes the monomial $x_1 \cdots x_{p-1} m_1 \cdots m_{p-1}$, which belongs to the ideal $\langle x_1 \cdots x_{p-1} \rangle$. Each of the remaining monomials is a multiple of some $x_i a_j$ or of some $a_i a_j$, and therefore belongs to $\langle y_1 \cdots y_q \rangle \cdot \langle x_1, \ldots, x_{p-1}, y_1, \ldots, y_q \rangle$.

By induction, assume the lemma for $p, s-1$, where $s \geq 1$. Consider $f$ as in the lemma (for $p, s$). Then $f/(y_{r_s} n_s + b_s)$ satisfies the hypothesis of the lemma (with $s-1$) when $y_{r_s}$ is deleted from the given elements of the ring. (Note that the lemma also depends on $q$. Here we are using it for $s-1$ and $q-1$.) Then, by induction, all the terms appearing after expanding $f/(y_{r_s} n_s + b_s)$ are either in the ideal $\langle x_1 \cdots x_{p-1} \rangle$ or in the ideal

\[(6.6) \quad \left( \frac{y_1 \cdots y_q}{y_{r_s}} \right) \cdot \langle x_1, \ldots, x_{p-1}, y_1, \ldots, y_q \rangle.\]

Assume there is a term $\xi$ appearing after expanding \((6.4)\) which is not in $\langle x_1 \cdots x_{p-1} \rangle$. Then there is $x_k$ such that $\xi \notin (x_k)$. Then $\xi$ is divisible by $a_k$, according to \((6.4)\), and $\xi$ belongs to the ideal \((6.6)\).

If $a_k \in (y_{r_s})$, we are done. By \((6.4)\), $\xi$ is a multiple either of $y_{r_s} n_s$ or $b_s$. If $a_k \notin (y_{r_s})$, and if we assume that $\xi$ was obtained by multiplying by $b_s$ rather than by $y_{r_s} n_s$, then $\xi$ is divisible by $x_k$, which is a contradiction. \(\square\)

Proof of Proposition 6.9. To prove this proposition it is enough to show that $f$ can be written as a product as in the previous lemma. To begin with, $f = h_1 \cdots h_q \in \langle x_1 \cdots x_{p-1}, y_1 \cdots y_q \rangle = \cap (x_i, y_j)$. Since each $(x_i, y_j)$ is prime, it follows that, for each $i = 1, \ldots, p-1$ and
j = 1, \ldots, q$, there is a $k$ such that $h_k \in (x_i, y_j)$. If there is a unit $u$ such that $h_k = y_j u + a$, where $\ord(a) \geq 2$, then we say that $h_k$ is associated to $y_j$; otherwise we say that $h_k$ is associated to $x_i$. There may be $h_k$ that belong to no $(x_i, y_j)$ and are, therefore, not associated to any $x_i$ or $y_j$.

By definition, any $h = h_k$ cannot be associated to some $x_i$ and $y_j$ at the same time. Let us prove that $h$ can be associated to at most one $x_i$. Assume that $h$ is associated to $x_{i_1}$ and $x_{i_2}$, where $i_1 \neq i_2$. Then $h \in (x_{i_1}, y_{j_1}) \cap (x_{i_2}, y_{j_2})$, for some $j_1$ and $j_2$. If $j_1 \neq j_2$, then $h$ cannot be of order 1, since $(x_{i_1}, y_{j_1}) \cap (x_{i_2}, y_{j_2})$ only contains elements of order $\geq 2$. If $j_1 = j_2$ then $(x_{i_1}, y_{j_1}) \cap (x_{i_2}, y_{j_2}) = (x_{i_1}, x_{i_2}, y_{j_1})$, but this would mean that $h$ is associated to $y_{j_1}$, and therefore not to $x_{i_1}$ or $x_{i_2}$.

An analogous argument shows that an $h$ cannot be associated to two different $y_j$. Therefore, the collection of $h_k$ is partitioned into those associated to a unique $x_i$, those associated to a unique $y_j$ and those associated to neither some $x_i$ nor some $y_j$.

We now show that, for each $i = 1, \ldots, p - 1$, there exists $h = h_k$ associated to $x_i$. Assume there is an $x_i$ (say $x_1$) with no associated $h$. For each $j = 1, \ldots, q$, there exists $k_j$ such that $h_{k_j} \in (x_1, y_j)$. Then $h_{k_j}$ is associated to $y_j$. It follows that each $k_j$ corresponds to a unique $j$. Thus, after reordering the $h_k$, we have $h_i$ is associated to $y_i$, for each $i = 1, \ldots, q$. This means that $h_i = y_i u_i + a_i$, where $u_i$ is a unit and $\ord(a_i) \geq 2$. This contradicts the assumption that $g_2$ is not a unit. Therefore, for each $i = 1, \ldots, p - 1$, there exists $h_k$ associated to $x_i$.

We take the product of all members of each set in the partition above. The product of all $h_k$ associated to $x_i$ can be written as $x_i m_i + a_i$, and it satisfies the property that

\begin{equation}
(6.7) \quad x_i m_i + a_i \notin (x_\alpha, y_\beta) \quad \text{unless} \quad \alpha = i.
\end{equation}

In fact, if $x_i m_i + a_i \in (x_\alpha, y_\beta)$ then there exists $h = h_k$ associated to $x_i$ such that $h \in (x_\alpha, y_\beta)$. But then $h$ is associated to either $y_\beta$ or $x_\alpha$, which contradicts the condition that $h$ is associated to $x_i$, where $i \neq \alpha$.

In the same way, write the product of all $h_k$ associated to $y_{r_i}$ as $y_{r_i} m_i + b_i$. Then

\begin{equation}
(6.8) \quad y_{r_i} m_i + b_i \notin (x_\alpha, y_\beta) \quad \text{unless} \quad \beta = r_i.
\end{equation}

Also write the product of all $h_k$ not associated to any $x_i$ or $y_j$ as $g$. We get the expression

\begin{equation}
(6.9) \quad f = (x_1 m_1 + a_1) \cdots (x_{p-1} m_{p-1} + a_{p-1})(y_{r_1} n_1 + b_1) \cdots (y_{r_s} n_s + b_s) g,
\end{equation}

but (6.9) does not a priori satisfy the hypotheses of Lemma 6.11.
We will use the properties (6.7) and (6.8) above to modify the elements $m, a, n$ and $b$ in (6.9) to get the hypotheses of the lemma.

We will check whether (6.5) is satisfied, for all $i = 1, \ldots, p - 1$ and $j = 1, \ldots, q$. Order the pairs $(i, j)$ reverse-lexicographically (or, in fact, in any way). Given $(i, j)$, assume, by induction, that (6.5) is satisfied for all $(i', j') < (i, j)$. Suppose that (6.5) is not satisfied for $(i, j)$. Then we will modify $m, a, b$ and $n$ so that (6.5) will be satisfied for all $(i', j') \leq (i, j)$. We consider the following cases.

Case (1): $j \neq r_k$, for any $k$. Then, if $a_i \in (y_j)$, there is nothing to do. If $a_i \notin (y_j)$, we can modify $a_i$ and $m_i$ so that the new $a_i$ will satisfy $a_i \in (y_j)$, and (6.5) will still be satisfied for $(i', j') < (i, j)$: Since $f \in (x_i, y_j)$ and, for every $k$, $y_j \neq y_k$, then $a_i \in (x_i, y_j)$. Write $a_i = ya$, where $y$ is a monomial in the $y_k$ and $a$ is divisible by no $y_k$. Then $a \in (x_i, y_j)$ and we can write $a = x_i g_1 + y_j g_2, x_i m_i + a_i = x_i (m_i + y g_1) + y y_j g_2$. Relabel $m_i + y g_1$ and $y y_j g_2$ as our new $m_i$ and $a_i$, respectively. Then $a_i \in (y_j)$, and clearly (6.5) is still satisfied for $(i', j') < (i, j)$.

Case (2): $j = r_k$, for some $k$. Since $f \in (x_i, y_j)$, then $a_i b_k \in (x_i, y_j)$. Since $(x_i, y_j)$ is prime, either $a_i \in (x_i, y_j)$ (in which case we proceed as before), or $b_k \in (x_i, y_j)$. Consider the latter case. If $b_k \in (x_i)$, there is nothing to do. Assume $b_k \notin (x_i)$. Write $b_k = x b$, where $x$ is a monomial in the $x_\ell$ and $b$ is divisible by no $x_\ell$. Then $b \in (x_i, y_j)$. Thus we can write $b = x_i g_1 + y_j g_2$ and $y_j m_k + b_k = y_j (m_k + x g_2) + x_i g_1$. Relabel $m_k + x g_2$ and $x_i g_1$ as our new $n_k$ and $b_k$, respectively. Then $b_k \in (x_i)$, and (6.5) is still satisfied for $(i', j') < (i, j)$.

We thus modify the $m, n, a, b$ in (6.9) to get the hypotheses of Lemma 6.11.

7. The case of two components

In this section, we show how to eliminate non-semi-snc singularities from the strata $\Sigma_{2,q}$.

Again, $(X, D, E)$ denotes a triple as in Definition 5.1 and we use the notation of the latter. As in Theorem 5.3, we assume that $X$ is snc, $D$ is reduced and has no component in the singular locus of $X$, and $(X, E)$ is semi-snc.

Proposition 7.1. Assume that every point of $X$ lies in at most two components of $X$ and that $(X^1, D^1, E^1)$ is semi-snc. Then there is a sequence of blowings-up with smooth admissible centers such that:

1. Each center of blowing-up consists of only non-semi-snc points.
2. For each blowing-up, the preimage of $\Sigma_{2,q}$, for any $q$, lies in the union of the $\Sigma_{2,r}$ ($r \leq q$) and the $\Sigma_{1,s}$.
In the final transform of $$(X, D, E)$$, all points of $$\Sigma_{2,q}$$ are semi-snc, for every $$q$$.

The proof will involve several lemmas. First we show how to blow-up to make $$J_a = \mathcal{O}_{X,a}$$ at every point $$a$$. We will use the assumptions of Proposition 7.1 throughout the section. Consider $$a \in X$$. Then $$X$$ is embedded locally at $$a$$ in a smooth variety $$Y$$ with a system of coordinates $$x_1, x_2, y_1, \ldots, y_q, z_1, \ldots, z_{n-q-2}$$ in a neighborhood $$U$$ of $$a = 0$$, in which we can write:

$$X = X_1 \cup X_2,$$
$$D = D_1 + D_2,$$

where $$X_1 = (x_1 = 0), X_2 = (x_2 = 0), D_1 = (x_1 = y_1 \cdots y_q = 0)$$ and $$D_2 = (x_2 = f = 0)$$, for some $$f \in \mathcal{O}_{Y,a}$$. This notation will be used in Lemmas 7.3 and in the proof of Proposition 7.1 below.

Recall the ideal $$J = J(X, D)$$ (Definition 2.5) that captures an important obstruction to semi-snc, see Lemma 2.6; $$J$$ is the quotient of the ideal of $$D_2 \cap X_1$$ by that of $$D_1 \cap X_2$$ in $$\mathcal{O}_Y$$. Consider $$V(J)$$ as a hypersurface in $$X_1 \cap X_2$$, and the divisor $$D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2}$$. We will blow up to get $$J = \mathcal{O}_Y$$ using desingularization of $$(V(J), D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2})$$; i.e., using the desingularization algorithm for the hypersurface $$V(J)$$ embedded in the smooth variety $$X_1 \cap X_2$$, with exceptional divisor $$D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2}$$. The resolution algorithm gives a sequence of blowings-up that makes the strict transform of $$V(J)$$ smooth and snc with respect to the exceptional divisor; we include a final blowing-up of the smooth hypersurface $$V(J)$$ to make the strict transform empty (“principalization” of the ideal $$J$$). It is not necessarily true, however, that $$J(X, D)' = J(X', D')$$. Therefore, after the preceding blowings-up, we do not necessarily have $$J(X', D') = \mathcal{O}_Y$$. Additional “cleaning” blowings-up will be needed.

Example 4.7 gives a simple illustration of the problem we resolve in this section. In the example, $$V(J) = (x_1 = x_2 = z = 0)$$, and our plan is to blow-up with the latter as center $$C$$ to resolve $$J$$. In the example, this blowing-up is enough to make $$(X, D)$$ semi-snc.

**Lemma 7.2.** Let $$R$$ denote a regular local ring, and suppose that $$x_1, x_2, y_1, \ldots, y_q$$ are part of regular system of parameters of $$R$$. Let $$f \in R$$. Then there exists a maximum subset $$\{i_1 < \ldots < i_t\}$$ of $$\{1, \ldots, q\}$$ (with respect to inclusion), such that $$f$$ can be written in the form $$f = x_1 g_1 + x_2 g_2 + y_{i_1} \cdots y_{i_t} g_3$$. Moreover,

$$[(x_1, x_2, f) : (y_1 \cdots y_q)] = (x_1, x_2, g_3).$$
Proof. Let $f = x_1g_1 + x_2g_2 + y_i \cdots y_i g_3$ with $\{i_1, \ldots, i_t\}$ maximal by inclusion, among the subsets of $\{1, \ldots, q\}$. Assume that $f = x_1h_1 + x_2h_2 + y_jh_3$ with $j \notin \{i_1, \ldots, i_t\}$. Then $y_{i_1} \cdots y_{i_t}g_3 \in (x_1, x_3, y_j)$. Since $(x_1, x_2, y_j)$ is prime and $j \notin \{i_1, \ldots, i_t\}$, we have $g_3 \in (x_1, x_2, y_j)$. It follows that there are $g'_1, g'_2, g'_3$ such that $f = x_1g'_1 + x_2g'_2 + y_jy_1 \cdots y_3g'_3$, contradicting the maximality of $\{i_1, \ldots, i_t\}$. Therefore $\{i_1, \ldots, i_t\}$ is actually maximum.

For the second part of the lemma: Clearly, $[(x_1, x_2, f) : (y_1 \cdots y_q)] \supset (x_1, x_2, g_3)$. Assume that $h \in [(x_1, x_2, f) : (y_1 \cdots y_q)]$; i.e., $y_1 \cdots y_q h \in (x_1, x_2, f)$. It follows that there is $c \in R$ such that $y_1 \cdots y_q h - y_1 \cdots y_q g_3 c \in (x_1, x_2)$. Since $(x_1, x_2)$ is prime, we must have $y_i j_1 \cdots j_{q-1} h - g_3 c \in (x_1, x_2)$, where $\{j_1, \ldots, j_{q-1}\} = 1, \ldots, q \setminus \{i_1, \ldots, i_t\}$. Then $g_3 c \in (x_1, x_2, y_j k)$, for every $k = 1, \ldots, q - t$.

We claim that $g_3 \notin (x_1, x_2, y_j k)$. In fact, if $g_3 \in (x_1, x_2, y_j k)$, then there is $\tilde{g}_3$ such that $f - y_j y_i \cdots y_i g_3 \in (x_1, x_2)$, contradicting the maximality of $\{i_1, \ldots, i_t\}$.

Therefore, $c \in (x_1, x_2, y_j k)$. So there is $\tilde{c}$ such that $\tilde{h}y_j \cdots \tilde{y}_k \cdots y_{j_{q-1}} - g_3 \tilde{c} \in (x_1, x_2)$, where $\tilde{y}_k$ means that the term is omitted. By iterating this argument for $k \in \{j_1, \ldots, j_{q-1}\}$ we get $h - g_3 \tilde{c} \in (x_1, x_2)$, for some $\tilde{c}$. This implies that $h \in (x_1, x_2, g_3)$, proving that $[(x_1, x_2, f) : (y_1 \cdots y_q)] \supset (x_1, x_2, g_3)$.

Given a smooth variety $W$ and a blowing-up $\sigma : W' \to W$ with smooth center $C \subset W$, we denote by $I'$ the strict transform by $\sigma$ of an ideal $I \subset O_W$, and by $Z'$ the strict transform of a subvariety $Z \subset W$. (We sometimes use the same notation for the strict transform by a sequence of blowings-up.) We also denote by $f'$ the “strict transform” of a function $f \in O_{W,a}$, where $a \in W$. The latter is defined up to an invertible factor at a point $a' \in \sigma^{-1}(a)$; $f' := u^{-d} f \circ \sigma$, where $(u = 0)$ defines $\sigma^{-1}(C)$ at $a'$ and $d$ is the maximum such that $f \circ \sigma \in (u^d)$ at $a'$.

Lemma 7.3. Let $\sigma : Y' \to Y$ denote a blowing-up (or a sequence of blowings-up) which is (or are) admissible for $(V(J), D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2})$; i.e., with center(s) in $\text{Supp} \ O/J$ and snc with respect to $D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2}$. (For simplicity, we maintain the same notation for the transforms.) Then

$$J(X', D') \subset J(X, D').$$

Moreover, if $J(X, D') = O_{Y'}$ and $a' \in X_1 \cap X_2$, then $J(X', D')_{a'} = (x_1, x_2, u^a)$ (in coordinates as above), where $u^a$ is a monomial in generators of the ideals of the components of the exceptional divisor of $\sigma$. 
Remark 7.4. By [7.1], if \( J(X, D') \neq \mathcal{O}_Y' \), then \( J(X', D') \neq \mathcal{O}_Y' \). Therefore, by Lemma 2.6 we never blow-up semi-snc points of the transforms of \((X, D)\) while desingularizing \(J(X, D)\).

Proof. Let \( I_{X_1}, I_{X_2}, I_{D_1} \) and \( I_{D_2} \) denote the ideals in \( \mathcal{O}_Y \) of \( X_1, X_2, D_1 \) and \( D_2 \) respectively. Locally at \( a \in X_1 \cap X_2 \), we have \( I_{X_1} = (x_1), I_{X_2} = (x_2), I_{D_1} = (x_1, y_1 \cdots y_q) \) and \( I_{D_2} = (x_2, f) \). Then

\[ J(X, D) = [I_{X_1} + I_{D_2} : I_{X_2} + I_{D_1}] = [(x_1, x_2, f) : (x_1, x_2, y_1 \cdots y_q)] = [(x_1, x_2, f) : (y_1 \cdots y_q)], \]

where the last equality follows from the definition of quotient of ideals and the fact that \( x_1, x_2 \in (x_1, x_2, f) \).

At a point \( a' \in \sigma^{-1}(a) \) with \( a' \in X_1' \cap X_2' \),

\[ J(X', D') = [I'_{X_1} + I'_{D_2} : I'_{X_2} + I'_{D_1}] = [(x_1') + (x_2', f)' : (x_1', x_2', y_1' \cdots y_q')] = [(x_1') + (x_2', f)' : (y_1' \cdots y_q')]. \]

In general, \((I + K)' \supset I' + K'\) and, if \( I \supset K \), then \([I : L] \supset [K : L]\), where \( I, K, L \) are ideals. Then

\[ J(X, D') = \left( \frac{1}{y_1' \cdots y_q'} ((x_1, x_2, f) \cap (y_1' \cdots y_q')) \right)' = \frac{1}{y_1' \cdots y_q'} ((x_1, x_2, f)' \cap (y_1' \cdots y_q') \supset [(x_1)' + (x_2', f)' : (y_1' \cdots y_q')]. \]

Now assume that \( J(X, D') = \mathcal{O}_Y' \). Write \( f = x_1g_1 + x_2g_2 + y_1 \cdots y_i g_3 \) as in Lemma 7.2. The center of the blowing-up lies in \( \text{Supp} \mathcal{O}_Y/J \subset X_1 \cap X_2 \) and has normal crossings with respect to \( D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2} \). It follows that \( I'_{X_1} + I'_{D_2} = (x_1', x_2', u^a y_1' \cdots y_i' g_3) \) for \( u^a = u_1^{a_1} \cdots u_i^{a_i} \) a monomial in generators of the ideals of the components of the exceptional divisor. We can then compute

\[ J(X', D') = [I'_{X_1} + I'_{D_2} : (y_1' \cdots y_q')] = [(x_1', x_2', u^a y_1' \cdots y_i' g_3) : (y_1' \cdots y_q')]. \]
But \( J(X, D) = (x_1, x_2, g_3) \), from (7.2). Since \( J(X, D)' = \mathcal{O}_{Y',a'} \) and \( a' \in X_1 \cap X_2 \), it follows that \( g_3' \) is a unit. The second assertion of the lemma follows by applying Lemma (7.2) to (7.3).

\[ \square \]

**Lemma 7.5.** Consider the transform \((X', D', \tilde{E})\) of \((X, D, E)\) by the desingularization of \((V(J), D_1|_{X_1 \cap X_2} + E|_{X_1 \cap X_2})\). Then:

1. For every \( q \), \( \Sigma_{2,q}(X', D') \) lies in the inverse image of \( \Sigma_{2,q}(X, D) \).
2. Let \( a' \in X' \). Then the ideal \( J(X', D')_{a'} \) is of the form \((x_1, x_2, u^a)\), where \( X_1' = (x_1 = 0) \), \( X_2' = (x_2 = 0) \) and \( u = u_1^{\alpha_1} \cdots u_t^{\alpha_t} \) is a monomial in the generators \( u_i \) of the ideals of the components of \( \tilde{E} \). Thus \( V(J(X', D')) \) consists of some components of \( X_1 \cap X_2 \cap E \).
3. After a finitely many blowings-up of components of \( V(J(X', D')) \) (and its successive transforms), the transform \((X'', D'')\) of \((X, D)\) satisfies \( J(X'', D'') = \mathcal{O}_{Y''} \). (For functoriality, the components to be blown up can be chosen according to the order on the components of \( E \).

**Proof.** (1) is clear (and is independent of the hypothesis). (2) follows from the second assertion of Lemma (7.3).

For (3), let us suppose (to simplify notation) that \( J(X, D) \) already satisfies the conclusion of (2). Consider the intersection of \( X_1, X_2 \) and the component \( H_1 \) of the exceptional divisor defined by \((u_1 = 0)\). We blow-up the irreducible components of this intersection lying inside \( \text{Supp} \mathcal{O}/J \). Locally, \( X_1 \cap X_2 \cap H_1 \) is defined by \((x_1 = x_2 = u_1 = 0)\). In the \( u_1 \)-chart, \( D_2' = (x_2' = f' = 0) \). Since \((x_1, x_2, u^a) = J(X, D) = [(x_1, x_2, f) : (y_1 \cdots y_9)]\), we can write \( f = x_1 g_0 + x_2 g_1 + y u^a \) with \( y = y_1 \cdots y_t \), as in Lemma (7.2). Therefore, after the blowing-up, \( J(X', D') = (x_1', x_2', u_1^{\beta_1} u_2^{\alpha_2} \cdots u_t^{\alpha_t}) \) with \( \beta_1 < \alpha_1 \) in the \( u_1 \)-chart. In the \( x_1 \) and \( x_2 \)-charts, \( X_1 \) and \( X_2 \) are moved apart; i.e. we have only strata \( \Sigma_{1,k} \) (for certain \( k \)). After a finite number of such blowings-up, we get \( J(X', D') = \mathcal{O}_{Y'} \) as wanted.

\[ \square \]

**Proof of Proposition (7.2).** The proof has three steps:

1. We use Lemma (7.5) to reduce to the case \( J = \mathcal{O}_Y \).
2. Let \( r = r(X, D) \) denote the maximum number of components of \( D_1 \) passing through a non-semi-snc point in \( X_1 \cap X_2 \). We make a single blowing-up to reduce \( r \). The result will be that \( J \) becomes a monomial ideal, as in Lemma (7.5)(2).
3. We proceed as in Lemma (7.5)(3) to reduce again to \( J = \mathcal{O}_Y \) (without increasing \( r \)).
Steps (2) and (3) are repeated until the set of non-semi-snc points in \( X_1 \cap X_2 \) is empty. This occurs after finitely many iterations, since \( r \) can not decrease indefinitely.

(1) We begin by applying Lemma 7.5 to make \( J = \mathcal{O}_Y \).

(2) Assume that \( J = \mathcal{O}_Y \). Let \( a \in X \). We use a local embedding of \( X \) to write

\[
X = X_1 \cup X_2 \\
D = D_1 + D_2,
\]

where \( X_1 = (x_1 = 0) \), \( X_2 = (x_2 = 0) \), \( D_1 = (x_1 = y_1 \cdots y_q = 0) \), \( D_2 = (x_2 = f = 0) \) for some \( f \in \mathcal{O}_Y \) (in the notation at the beginning of the section). By Lemma 7.2, since \( J = \mathcal{O}_Y \), we have \( f = x_1g_0 + x_2g_1 + y_1 \cdots y_s \), for some \( s \leq q \). Write \( f|_{(x_2=0)} = f_1 \cdots f_\ell \), where each \( f_i \) is irreducible. We must have \( \ell \leq \text{ord}_a(f) \leq s \leq q \); therefore \( a \in \Sigma_{2,\ell} \).

By Lemma 4.8, \( H_{\text{Supp}_{D,a}} = H_{p,\ell} \) if and only if \( \ell = q \). Therefore, by Lemma 2.6 \( (X, D, E) \) is semi-snc at \( a \) if and only if \( \ell = q \). The idea is to blow-up with center given locally by \( (x_1 = x_2 = y_1 = \ldots = y_q = 0) \).

Define

\[
C_r := \Sigma_{1,r}(X_1, D_1) \cap X_2,
\]

where \( r = r(X, D) \). Consider a component \( Q \) of \( C_r \) which includes a non-semi-snc point of \( (X, D, E) \) in \( X_1 \cap X_2 \). We will prove that \( Q \) is closed and consists only of non-semi-snc points of \( (X, D, E) \). We will blow-up the union \( C \) of all such components of \( C_r \).

The set of semi-snc points is open, so the set of semi-snc points in \( Q \) is open in \( Q \). At a non-semi-snc point \( a \) in \( Q \), we have a local embedding and coordinates as above in which we can write

\[
D_1 = (x_1 = y_1 \cdots y_r = 0) \\
D_2 = (x_2 = f = 0),
\]

where \( f|_{(x_2=0)} \) factors into \( \ell < r \) irreducible factors; i.e., \( D_2 \) has \( \ell \) irreducible components passing through \( a \). In this neighborhood of \( a \) in \( Q \), all points of \( Q \) are non-semi-snc. Thus the set of non-semi-snc points is also open in \( Q \). Since \( Q \) is irreducible, it only contains non-semi-snc points. At a point \( b \) of \( \overline{Q} \setminus Q \), the number of components of \( D_1 \) can be only \( > r \). Since \( r \) is maximum over the non-semi-snc points, \( b \) is a semi-snc point. This contradicts the fact that \( Q \) contains only non-semi-snc points. Therefore \( Q \) is closed.

Thus \( C \) is closed and consists of only non-semi-snc points. We can compute locally the effect of blowing up \( C \). In the \( x_1 \)- and \( x_2 \)-charts, the preimages of a point \( a \in C \) lie in only one component of \( X \). In the
y_i-chart, we compute
\[ D_1' = (x_1 = y_1 \cdots \hat{y}_i \cdots y_r = 0) \]
\[ D_2' = (x_2 = x_1y_1^{j_1}g'_0 + x_2y_1^{j_2}g'_1 + y_1 \cdots \hat{y}_i \cdots y_s = 0), \]
where \( y_i \) is now a generator of the ideal of a component of the exceptional divisor, \( \hat{y}_i \) means that the factor is missing from the product, and at least one of \( j_1, j_2, j_3 \) equals zero. As a result, \( r(X', D') < r(X, D) \).

It may happen that \( J(X', D') \) is no longer equal to \( \mathcal{O}_{Y'} \), but we can calculate that \( J(X', D') = (x_1, x_2, y_3) \) in the \( y_i \)-chart.

(3) We apply again Lemma 7.5(3). The centers of blowing up are given locally by \( X_1 \cap X_2 \cap (y_i = 0) \). These blowings-up do not increase \( r(X, D) \).

Therefore, after a finite number of iterations, every point lying in two components of \( X \) is semi-snc. \( \square \)

8. The non-reduced case

The previous sections establish Theorem 1.2 in the case that \( D \) is reduced. In this section we describe the blowings-up necessary to deduce the non-reduced case. In other words, we assume that \( (X, D_{red}) \) is semi-snc, and we will prove Theorem 1.2 under this assumption.

The assumption implies that, for every \( a \in X \), there is a local embedding in a smooth variety \( Y \) with coordinates \( x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_{n-p-q} \) in which \( a = 0 \) and

\[ X = (x_1 \cdots x_p = 0), \]
\[ D = \sum_{(i,j)} a_{ij}(x_i = y_j = 0), \]

for some \( a_{ij} \in \mathbb{Q} \). Since the reduced pair is semi-snc, we can assume that \( a_{ij} \neq 0 \), for every \( (i, j) \) in the index set. Nevertheless, the following argument if valid even if we allow the possibility that some \( a_{ij} = 0 \).

The pair \( (X, D) \) is semi-snc at \( a \) if and only if \( a_{ij} = a_{i'j'} \) for all \( i, i', j; \) see Example 3.9. In this section, we transform \( D \) by taking only its strict transform \( D' \); see Definition 3.5. We can neglect the exceptional divisor because, if \( \sigma : X' \to X \) is a blowing-up with smooth center simultaneously normal crossings with respect to \( X \) and \( \text{Supp} \ D \), then \( (X', D'_{red} + \text{Ex(\sigma)}) \) is semi-snc provided that \( (X, D_{red}) \) is semi-snc. Since all components of \( \text{Ex(f)} \) appear with multiplicity one, if we make \( (X', D') \) semi-snc then \( (X', D' + \text{Ex(\sigma)}) \) will be semi-snc as well.

We define an equivalence relation on components of \( D \) passing through a point of \( X \).
Definition 8.1. Let \( a \in X \) and let \( D_1, D_2 \) denote components of \( D \) passing through \( a \). We say that \( D_1 \) and \( D_2 \) are equivalent (at \( a \)) if either \( D_1 = D_2 \) or the irreducible component of \( D_1 \cap D_2 \) containing \( a \) has codimension 2 in \( X \).

Clearly, if \( D_1 \) and \( D_2 \) are equivalent at \( a \), then \( D_1 \cap D_2 \) is of the form \( (x_1 = x_2 = y_j = 0) \), for some \( j \), at \( a \), so the irreducible component of \( D_1 \cap D_2 \) containing \( a \) is smooth and \( D_1, D_2 \) are equivalent at each of its points.

To check that the preceding relation is transitive, let \( D_1 = (x_{i_1} = y_{j_1} = 0) \), \( D_2 = (x_{i_2} = y_{j_2} = 0) \) and \( D_3 = (x_{i_3} = y_{j_3} = 0) \) in coordinates as before. If \( D_1 \) is equivalent to \( D_2 \) (at \( a = 0 \)), then \( j_1 = j_2 \). If \( D_2 \) is equivalent to \( D_3 \), then \( j_2 = j_3 \). Therefore \( D_3 \) is equivalent to \( D_1 \). Reflexivity and symmetry are clear.

Given \( a \in X \), let \( p(a) \) denote the number of components of \( X \) passing through \( a \), and let \( q(a) \) denote the number of equivalence classes represented by the set of components of \( D \) passing through \( a \). In local coordinates as before, \( q(a) \) is the total number of \( j \) for which there exists \( a_{ij} \neq 0 \). Define \( \iota : X \to \mathbb{N}^2 \) by \( \iota(a) := (p(a), q(a)) \). We give \( \mathbb{N}^2 \) the partial order in which \( (p_1, q_1) \geq (p_2, q_2) \) if and only if \( p_1 \geq p_2 \) and \( q_1 \geq q_2 \). Then \( \iota \) is upper semi-continuous. Therefore, the maximal locus of \( \iota \) is a closed set.

Observe that \( (X, D) \) is semi-snc at \( a \) if and only if \( a_{ij} \) is constant on each equivalence class of the set of components of \( D \) passing through \( a \). Consider the maximal locus of \( \iota \). Each irreducible component of the maximal locus of \( \iota \) consists only of semi-snc points or only of non-semi-snc points, because all points in one of these irreducible components are contained in the same irreducible components of \( D \). We blow up with center the union of those components of the maximal locus of \( \iota \) that contain only non-semi-snc points. In the preimage of the center, \( \iota \) decreases. In fact, in local coordinates at a point, as before, we are simply blowing up with center

\[
C = (x_1 = \ldots = x_{p(a)} = y_1 = \ldots = y_{q(a)} = 0).
\]

Therefore, either one component of \( X \) is moved away or all components of \( D \) in one equivalence class are moved away.

Let \( W \) be the union of those components of the maximal locus consisting of semi-snc points. The previous blowing-up is an isomorphism on \( W \). So \( (X', D') \) is semi-snc on \( W' = W \), and therefore in a neighborhood of \( W' \). For this reason, the union of the components of the maximal locus of \( \iota \) on \( X' \setminus W' \) which contain only non-semi-snc points, is a closed set in \( X' \). Therefore, we can repeat the procedure on \( X' \setminus W' \).
Clearly, $\mathbb{N}^2$ has no infinite decreasing sequences with respect to the given order. After the previous blowing-up, the maximal values of $\iota$ on the set of non-semi-snc points of $(X, D)$ decrease. Therefore, after a finite number of iterations of the procedure above, the set of non-semi-snc becomes empty.

Remark 8.2. Suppose that $(X, D_{\text{red}})$ is semi-snc (i.e., all $a_{ij} \neq 0$ in (8.1), at every point). Then the blowing-up sequence in this section is given simply by the desingularization algorithm for $\text{Supp} D$, but blowing up only those components of the maximal locus of the invariant on the non-semi-snc points.

9. Functoriality

In this final section, we make precise and prove the functoriality assertion of Remark 1.3.(3).

We say that a morphism $f : Y \to X$ preserves the number of irreducible components at every point if, for every $b \in Y$, the number of irreducible components of $Y$ at $b$ equals the number of components of $X$ at $f(b)$.

The Hilbert-Samuel function, and in fact the desingularization invariant (beginning with the Hilbert-Samuel function), is invariant with respect to étale morphisms; see [5, Remark 1.5]. A smooth morphism $f : Y \to X$ factors locally as an étale morphism and a projection from a product with an affine space $\mathbb{A}^n$. Therefore, if $f(b) = a$, then $H_{Y,b} = H_{X \times \mathbb{A}^n, (a,0)}$ and the remaining terms of the invariant are the same at $a$ and $b$. To show that the desingularization sequence of Theorem 1.2 is functorial with respect to étale (or smooth) morphisms that preserve the number of irreducible components, we just need to show that each blowing-up involved is defined using only the desingularization invariant and the number of components of $X$ and $D$ passing through a point. We can recapitulate each step of the algorithm in Section 5.

Step 1 is an application of Theorem 3.11. Functoriality of the blowing-up sequence in the latter is proved in [7] and [2]. Step 2 is obtained from the desingularization algorithm of [5] applied to the components of $D$ lying in the intersection of pairs of components of $X$. The blowing-up sequence involved is functorial with respect to étale (or smooth) morphisms in general.

Step 3, Case A provides a blowing-up sequence completely determined by the Hilbert-Samuel function and the strata $\Sigma_{p,q}$, for $p \geq 3$. The strata $\Sigma_{p,q}$ are defined in terms of the number of components of $X$ and $D$ passing through a point. Step 3, Case B gives a sequence of
blowing-up determined by desingularization of the hypersurface $V(J)$ and the number $r(X, D)$ defined in terms of number of components of $D$; see Proposition 7.1. Step 3, Case C, is again a use of Theorem 3.11.

Finally, the blowings-up of Step 4 are determined by the number of components of $D$ passing through a point and the equivalence relation on the components of $D$ passing through a point, of Definition 8.1. This equivalence relation is preserved by étale (or smooth) morphisms.

Remark 9.1. It is not possible to drop the condition on preservation of the number of components in the functoriality statement, for any desingularization that preserves precisely the class of snc singularities. In fact, assume that $X$ is nc but not snc at $a$ (see Example 3.3). Then there is an étale morphism $f : Y \rightarrow X$ such that $Y$ is snc at $b$ and $f(b) = a$. The desingularization must modify $X$ at $a$. It is impossible to pull back this desingularization to $Y$ and still get a desingularization preserving snc because the latter must be an isomorphism at $b$.

References

[1] B.M. Bennett, On the characteristic function of a local ring, Ann. of Math. (2) 91 (1970), 25–87.
[2] E. Bierstone, S. Da Silva and F. Vera Pacheco, Desingularization by blowing up avoiding simple normal crossings, preprint (2011).
[3] E. Bierstone, P. Lairez and P.D. Milman, Resolution except for minimal singularities II. The case of four variables, preprint arXiv:1107.5598 v2 [math.AG] (2011), 23 pages.
[4] E. Bierstone and P.D. Milman, Uniformization of analytic spaces, J. Amer. Math. Soc. 2 (1989), 801–836.
[5] E. Bierstone and P.D. Milman, Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant, Invent. Math. 128 (1997), 297–302.
[6] E. Bierstone and P.D. Milman, Functoriality in resolution of singularities, Publ. R.I.M.S. Kyoto Univ. 44 (2008), 609–639.
[7] E. Bierstone and P.D. Milman, Resolution except for minimal singularities I, preprint arXiv:1107.5595 v2 [math.AG] (2011), 41 pages.
[8] O. Fujino, What is log terminal?, Flips for 3-folds and 4-folds, Oxford Lecture Ser. Math. Appl., vol. 35, Oxford Univ. Press, Oxford, 2007, pp. 49–62.
[9] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, I, II, Ann. of Math. (2) 79 (1964), 109–326.
[10] H. Hironaka, Idealistic exponents of singularity, Algebraic geometry, J.J. Sylvester Sympos., Johns Hopkins Univ., Baltimore 1976, Johns Hopkins Univ. Press, Baltimore, 1977, pp. 52–125.
[11] J. Kollár, Lectures on resolution of singularities, Ann. of Math. Studies, no. 166, Princeton Univ. Press, Princeton, 2007.
[12] J. Kollár, Semi log resolutions, preprint arXiv:0812.3592v1 [math.AG] (2008), 10 pages.
[13] H. Matsumura, Commutative algebra, Benjamin, New York, 1980.
[14] E. Szabó, *Divisorial log terminal singularities*, J. Math. Sci. Univ. Tokyo 1 (1994), 631–639.
[15] E. Bierstone, P. Milman and M. Temkin, *\(\mathbb{Q}\)-universal desingularization*, Asian J. Math. 15 (2011), 229–250.

The Fields Institute, 222 College Street, Toronto, ON, Canada M5T 3J1, and University of Toronto, Department of Mathematics, 40 St. George Street, Toronto, Ontario, Canada M5S 2E4

E-mail address: bierston@fields.utoronto.ca

University of Toronto, Department of Mathematics, 40 St. George Street, Toronto, Ontario, Canada M5S 2E4

E-mail address: franklin.vp@gmail.com