Perturbative Solution to D-terms in N=2 Models and Metrics

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Abstract

$N = 2$ gauged non-linear sigma models are examined classically and their D-terms are solved. The variation of the classical Lagrangian in order to solve for the auxiliary fields is identical to integrating these modes functionally. The latter is performed for the general quotient. The D-term solution is equivalent to solving, a coupled set of algebraic equations.
Introduction

The solution to the D-terms in $\mathcal{N} = 2$ non-linear sigma models is useful in the construction of metrics on toric varieties as well as in the construction of particle physics and string models. Their solution is problematic because a very general set of algebraic equations has to be solved.

The Kähler metrics are also required in order to generate geodesic flows on these spaces. The solutions to these flow equations are required in order to find solutions to algebraic systems [1] and systems of non-linear differential equations [2].

The solution to these D-term equations can be achieved by a counting of tree diagrams in scalar field theory models [3]. The scalar field model contains interactions of an arbitrary degree in the external lines, due to the exponential interactions in the auxiliary fields. The counting can be performed in principle by a careful examination of the auxiliary field interactions and is torically model specific. However, a deformation can be added to the Lagrangian which simplifies this count quite much; after the solution to the D-terms, the deformation parameters are taken to zero. (The same tree graphs are used in the full quantization of scalar field and gauge theories [4]-[8].)

The general auxiliary components of the $\mathcal{N} = 2$ Lagrangian is described by (see for example [9] also for several completed examples),

$$\mathcal{L} = \sum_{i=1}^{a} |\phi_i|^2 \prod_{j=1}^{b} e^{Q_i^j V_j} + r_j V_j ,$$

with an implicit sum on the auxiliary components $V_j$ in the exponential. The field equations that are derived from (1) are the set,

$$r_j + \sum_{i=1}^{a} |\phi_i|^2 Q_i^j \prod_{j=1}^{b} q_j^{Q_i^j} = 0 .$$

In some cases of these torics, field redefinitions may be used to find the Kähler potential; the general case and its algebraic system requires a different approach.

Solution

As opposed to solving a complicated system of algebraic equations, the Lagrangian and its integration over the auxiliary fields $V_j$ can be analyzed with the use of classical scattering diagrams which have no momentum structure. There is no propagator or derivatives on the auxiliary fields, and as such, the integration of
these fields is obtained by counting all of the inequivalent tree diagrams that contain external \((\phi_i, \bar{\phi}_i)\) lines and internal \(V_j\) lines.

The linear dependence on the \(V_j\) field, as well as a non-trivial but required combinatoric property, is treated by changing its form as in,

\[
\sum_{j=1}^{n} r_j V_j \to \sum_{j,k} r_j r_k \gamma_j \gamma_k (V_j + \alpha_j)(V_k + \alpha_k)
\]  

\[
= \sum r_j r_k \gamma_j \gamma_k V_j V_k + \gamma \sum r_j \gamma_j \alpha_j V_j + \gamma^2 \quad \gamma = \sum r_j \gamma_j \alpha_j
\]  

A field redefinition of \(V_j \to (V_j - \alpha_j)/r_j \gamma_j\) changes the deformation into

\[
\sum_{j=1}^{n} r_j V_j \to \sum_{i<j} V_i V_j
\]

The limit of the linear term is obtained by taking \(\gamma_j \alpha_j r_j = 1\) with \(r_j \gamma_j = 0\), for example; another removal of the deformation can be obtained by field redefining \(\phi_i \to \phi_i r_i \gamma_i\) followed by taking \(\gamma_i\) to zero. The latter is due to the counting of internal lines in the diagrams; due to the vertex structure there are always \(n\) propagators with a diagram containing \(n + 2\ \phi_i\) or \(\bar{\phi}_i\) lines.

This deformation in (4) generates a uniformity to the combinatorics associated with the interacting auxiliary fields in the tree diagrams, as the propagator is completely symmetric in the indices \(j\) and \(k\). As a result, the vertices are connected from one node to each other in the most uniform manner possible; this effectively changes all of the \(V_j\) into \(V\) in the classical graphs so that only a single scalar field need be examined (while keeping the indices on the charges). The removal of the deformation is obtained in the Kähler potential \(K(\phi_i, \bar{\phi}_i)\), after its form is deduced from the classical scattering.

The couplings in the Lagrangian (1) are deduced by expanding the exponentials in the \(V_j\) fields. Each graph composed of the various \(i\) vertices containing \(p_i\) lines (two of which are external \(\phi\) fields) are weighted with these couplings. The Kähler potential is found from,

\[
K(\phi_i, \bar{\phi}_i) = \sum_{\text{perms}} \prod_{\sigma} |\phi_{\sigma}|^2 \alpha_{n,\{p_i\}} \prod_{\lambda \sigma,\{p_i\}} \lambda_{\sigma,\{p_i\}} \frac{1}{2^{n/2} \prod p_i!}
\]  

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The counting number \( a_{n,m} \) is used to count the number of diagrams containing the sum \( \sum p_i = m \) and at \( n_\phi \)-point. This counting function can be obtained by solving an associated function \([1]\). The potential is found in a patch around \( \phi_i = \bar{\phi}_i = 0 \) and as an expansion which is not necessarily Taylor.

The expansion of the exponentials in (11) generates,

\[
\sum_{i=1}^{a} |\phi_i|^2 \prod_{j=1}^{b} e^{Q_i V_j} = \sum_{i=1}^{a} \sum_{n=0}^{\infty} |\phi_i|^2 \frac{1}{n!} (\sum_{j=1}^{b} Q_i^j V_j)^n
\]

(7)

\[
= \sum_{i=1}^{a} \sum_{n=0}^{\infty} \sum_{q_i=0}^{\infty} |\phi_i|^2 \frac{1}{\prod_{j=1}^{b} q_j!} \prod_{j=1}^{b} (Q_i^j)^{n-q_i} (V_j)^{q_i} |_{\sum q_i=n}
\]

(8)

The field redefinition of \( V_i \rightarrow (V_j - \alpha_j)/r_j \gamma_j \) changes the form to,

\[
= \sum_{i=1}^{a} \sum_{n=0}^{\infty} \sum_{q_i=0}^{\infty} |\phi_i|^2 \prod_{j=1}^{b} \frac{1}{p_j!(q_j - p_j)!} (Q_i^j)^{n-q_j} (V_j)^{p_j} (-\alpha_j)^{q_j-p_j} \frac{1!}{(\gamma_j r_j)^{q_j}}.
\]

(9)

The expansion (9) leads to the vertex couplings,

\[
\lambda_{\sigma_i \{p_i\}} = \sum_{q_i=0}^{\infty} \prod_{j=1}^{b} \frac{1}{(q_j - p_j)!} (Q_i^j)^{n-q_j} (-\alpha_j)^{q_j-p_j} \frac{1!}{(\gamma_j r_j)^{q_j}}.
\]

(10)

These couplings are used to weight the classical graphs, and find the Kähler potential. The sum on \( q_j \) extends at fixed \( p_j \), with \( \sum q_j = n \), with the sum on \( n \). These couplings together with the counting function \( a_{2n_\phi,m} \) are used to find the Kähler potential with (6); recall that \( m = \sum (2 + p_j) \), that is the count of lines at each vertex.

After the computation, the deformation parameter is removed; without a field redefinition on the \( \phi, \bar{\phi} \) coordinates, there could be a potential singularity and requires some summations to be performed; however the scaling of the coordinates removes this. As \( q_i \) becomes large, with the large order scaling \( n \sim q_j \), the right hand side in (10) tends to,

\[
\sum_{q_i=0}^{\infty} \frac{1}{q_i!} (Q_i^j)^{n-q_j} \left(\frac{-\alpha_j}{Q_i^j \gamma_j r_j}\right)^{q_j} (-\alpha_j)^{p_j} = e^{Q_i^j e^{-\alpha_j/Q_i^j \gamma_j r_j}} (-\alpha_j)^{p_j}
\]

(11)

\[
\sim e^{Q_i^j (-\alpha_j)^{p_j}},
\]

(12)
This is a heuristic explanation of how the removal of deformation parameter can be explained by performing the sum in $\sum_{n,m}^\infty$, and taking $\gamma_j r_j \to 0$. The sum, and its asymptotic values has to be examined. A field redefinition of the coordinates can handle the removable singularity in the $\alpha_j$. The above is one means to remove the deformation, but a direct scaling of the coordinates by the $r_i \gamma_i$, with the limit $r_i \gamma_i \to 0$, can also eliminate the removable singularity.

There are two contributions to $a_{n,m}$ [1]. The first one is,

$$\lambda \sum_{j=3}^\infty \sum_{a=1}^{j-1} \left(\frac{(-1)^a}{a! (j-a)!} \frac{a!}{\alpha_2!(a-\alpha_2)!}\right) \sum_{\alpha_1,\alpha_2;\beta_i} \prod_{i=1}^{\alpha_2} (\beta_i - \gamma_i) (\beta_i - \gamma_i)! p_i^{p_i} (-1)^p_i \sum_{\beta_i=m+a-\alpha_2}^{\alpha_2} \prod_{i=1}^{\alpha_2} (\beta_i - \gamma_i)! (\beta_i - \gamma_i)!$$

$$\frac{(\alpha_2 + j - 2a)!}{(\alpha_2 + j - 2a - \alpha_1)!} \delta_{\alpha_2+j-2a,\alpha_1} \prod_{i=1}^{\alpha_2} 2^{-p_i} \sum_{q=0}^{p_i} \frac{p_i^q}{q!(p_i - q)!} (-1)^{p_i-q}$$

$$\sum_{r=0}^m \frac{m!}{(m-r)!} \sum_{\alpha_1,\alpha_2;\beta_i} \prod_{i=1}^{\alpha_2} (\beta_i - \gamma_i)! q^\beta_i (-p)^\gamma_i \prod_{i=1}^{\alpha_2} (\beta_i - \gamma_i)! (\beta_i - \gamma_i)! \sum_{\gamma_i=b-a+\beta-b-r}.$$  

The second one is found from setting the numbers to $j = 0$, $a = 0$, $b = 2$ and $j = 1$, $a = 0$, $b = 1$ and $j = 2$, $a = 0$, $b = 0$ in the formulae of (21)-(25) of [1]. This results in $\sum_{n,m}^\infty$ by changing $\alpha_2 \to \alpha_2 + b$ and $\tilde{p} \to \tilde{p} + b$ with the values of $a$.

These expansions are generally not Taylor, as the infinite number of derivatives about the origin of the Kähler potential might not converge. There are expansions of $x^\delta$ involving Laguerre polynomials which are polynomials in $x$; these derivatives of $x$ for small $\delta$ do not exist except by analytic continuation. This is an example of the multiple sums might converge to radicals in the lowest order terms of the $\phi_i, \bar{\phi}_i$ expansion.

The sums required to describe the Kähler potential might possess a hidden symmetry leading at a simplified form. Mirror symmetry could indicate this. However, even for low dimensional examples of torics such as simple Hirzebruch surfaces the metric forms are somewhat complicated.

The full form of the expansion of the Kähler expansion about zero is akin to solving for the roots of a coupled set of polynomials. It is interesting that the solution
can be expressed in closed form, as there are theorems stating under certain functional forms that this is not possible. Performing the sums in \([16]\) is very relevant; modular functions at specific values could play a role in summing these partitions also with values of \(n\) and \(m\).

**Discussion**

A generating function for the toric Kähler potentials is given, pertaining to both finite and infinite dimensional quotients. The explicit form of the potential, and metrics, is produced in terms of sums over rational numbers. This derivation is equivalent to solving the D-term constraints for a generic \(N = 2\) gauged non-linear sigma model, which involves solving a system of specific coupled algebraic equations following from the charges and gaugings of the model. Instead of transcendentally solving these equations, the counting of classical scalar field diagrams enables the computations of the solutions.

On the mathematical side, this derivation is equivalent to solving some complicated systems of algebraic equations. A closed form is given to these solutions. It appears that a more modular construction would simplify the results.

The explicit form of the Kähler potentials should reveal more structure in the duality of the string models, and quantum field models, that give rise to these non-linear sigma models. The specific form of the Kähler potential has clear applications to particle physics models.

The explicit form of the toric metrics also leads to further information in the construction of solutions to generic algebraic systems \([10]-[14]\). Possible simplifications in the handling of transcendental information are evident.

In addition, the finding of integer solutions to polynomial solutions is relevant to the solutions of statistical mechanics problems \([11]\).

*Note added:* The metric form may be found at, and around, integer solutions \(\phi_i = n_i\) by shifting of \(\tilde{\phi}_i = \tilde{\phi}_i + n_i\) coordinates and redoing the classical graph count. If these integer points exist in the manifold, then there should be a locally flat space representation, which also results in the appropriate monodromy in the Kähler potential, as at \(\phi_i = 0\). Counting the integer points appears possible due to the Kähler potential's expansion about these points. The explicit this count, of the algebraic system modeled by the toric, requires some simplifications of the summations in order to find the appropriate flat space limit.
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