Analytical approximation for the sphere-sphere Coulomb potential

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A simple analytical expression, which closely approximates the Coulomb potential between two uniformly charged spheres, is presented. This expression can be used in the optical potential semiclassical analyses which require that the interaction be analytic on and near the real $r$-axis.

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The semiclassical analyses of the optical potential cross sections play an important role in understanding the scattering mechanism of light- and heavy-ions [3]. The direct evaluation of the semiclassical amplitudes is based on the calculation of the integral actions between the turning points of the radial motion. The turning points of the radial motion are solution of the equation

$$k^2(r, \lambda) = k^2(1 - \frac{U(r)}{E} - \frac{\lambda^2}{k^2 r^2}) = 0,$$

(1)

where $k, \lambda, E,$ and $U(r)$ are, respectively, the wave number, the angular momentum in units of $\hbar$, the center of mass energy, and the complex optical potential. The integral actions are defined by

$$S_{ij}(\lambda) = \int_{r_i}^{r_j} k(r, \lambda) \, dr,$$

(2)

where $r_i$ and $r_j$ are appropriate solutions of Eq. (1).

In the semiclassical approximations both these quantities are treated as analytical functions of $\lambda$ around the real $\lambda$-axis. This is true only if the complex optical potential is an analytical function of $r$, on and near the real $r$-axis.

In the phenomenological analyses the optical potential between heavy-ions is written in the form

$$U(r) = V(r) + iW(r) + V_{C}(r),$$

(3)

where $V(r)$ and $W(r)$ are the real and imaginary part of the nuclear interaction, and $V_{C}(r)$ is the Coulomb interaction potential.

As a rule, analytic functions are used for the real and the imaginary part of the interaction. On the contrary, the Coulomb part of the interaction is described using either the Coulomb potential of one point charged particle with a uniformly charged sphere of radius $R_C$ (in the least recent works) or the Coulomb potential between two uniformly charged spheres of radii, say, $R_1$ and $R_2$ (in the more recent ones). These potentials are not analytic functions around the real $r$-axis.

This fact, rigorously, excludes the possibility of applying the semiclassical methods which assumes the analyticity of the potential. The lack of analyticity must however be considered accidental. It arises from our preference to use simple expressions for the interaction, and we do not really think that it is connected with some physical fact. Owing to this we expect that this, and similar troubles, can be cured with a little effort.

To do this we here briefly recall some techniques used in the past to treat the point-sphere case, and we present a similar method for the sphere-sphere case.

Both these Coulomb potentials can be written in the form

$$V_{C}(r) = \frac{Z_1 Z_2 e^2}{R_C} f_{C}(x),$$

(4)

where $Z_1$ and $Z_2$ are the charges of the colliding partners, $x = r/R_C$, and $f_C(x)$ is an appropriate form factor depending on the case considered. For the two sphere case, obviously, $R_C$ is the sum of $R_1$ and $R_2$.

The form factor for the point-sphere Coulomb potential is given by

$$f_{pS}^C(x) = \begin{cases} \frac{1}{2}(3 - x^2) & x \leq 1 \\ \frac{1}{4} & x > 1 \end{cases}$$

(5)

This form factor is continuous, as is its first derivative, on the real $x$-axis, but has a discontinuity at $x = 1$ in the second derivative.

To avoid problems connected with this discontinuity, Knoll and Schaeffer [3], in their semiclassical analysis, simply described the Coulomb interaction by using the point-point Coulomb potential for all the $r$-values. This choice is only justified for strongly absorptive interactions, for which the contributions from the internal region of the interaction are unimportant. In order to be free from this restriction, Brink and Takigawa [3], in their uniform semiclassical analysis, preferred to approximate the point-sphere form factor given by Eq. (5) with

$$f_{BT}^S(x) = \frac{1}{x} [1 - \exp(-a_1 x - a_2 x^2 - a_3 x^3)].$$

(6)

By imposing the condition that the values of this form factor, and of its first two derivatives, coincide with the corresponding quantities of the exact one at $x = 0$, one obtains $a_1 = 3/2, a_2 = 9/8,$ and $a_3 = 19/16$. In this way the analytical form factor closely approximates the point-sphere one, with a maximum relative error smaller than 3%.

A similar analytical form factor, with the additional constraint of being an even function of $x$, was also used in
the past. The evenness of the potential ensures that the scattering functions, of the multireflection expansion of the Brink and Takigawa approximation, have definite reflection properties in the complex $\lambda$-plane. These properties are useful in the semiclassical analysis, and to ensure their validity in Ref. [4], although not there explicitly given, the following expression was used for the point-sphere form factor

$$f_{pS}^{AR}(x) = \frac{1}{x} \tanh(a_1 x + a_3 x^3). \quad (7)$$

Using the same condition used by Brink and Takigawa one obtains for the parameters the values $a_1 = 3/2$, and $a_3 = 5/8$. With this parameter values, the maximum relative error between the approximated and the exact form factors is lower than 4%.

In Fig. 1 the behavior of the exact point-sphere form factor is compared with that of the two above approximations.

The use of the point-sphere Coulomb potential is not, however, appropriate to describe realistically the Coulomb interaction in the heavy-ion scattering processes in which the projectile and the target have similar radii. More than 25 years ago [5–8] the role of more realistic descriptions of the Coulomb interaction between two heavy-ions was investigated.

The detailed behavior of the Coulomb interaction depends, obviously, on the properties of the two charge distributions. In any case the potentials obtained using realistic charge distributions can be rather well approximated by the Coulomb potential of two uniformly charged spheres. This is the reason why this potential is used even in the more recent and detailed analyses of the elastic scattering between light heavy-ions [10–13].

The sphere-sphere Coulomb form factor is given by [7]

$$f_{C}^{SS}(x) = \begin{cases} \frac{1}{x} b(3 - c - b^2 x^2) & x \leq x_0 \\ \frac{1}{x} \left\{ 1 - 3d(1-x)^4 \right\} x_0 < x \leq 1 \\ \frac{1}{x} [1 - \frac{d}{c} d(1-x)(5+x)] & x > 1, \end{cases}$$

where $b = 1 + 1/a$, $c = 3/5a^2$, $d = (a + 1/a + 2)/4$, $x_0 = (a - 1)/(a + 1)$, with $a = R_1/R_2$ and $R_1 \geq R_2$.

This formula is less popular than the equivalent ones given in Refs. [3,5,5], but is here preferred for its simplicity.

The sphere-sphere Coulomb form factor is continuous for real $x$-values up to the third derivative. The first discontinuities appear in the fourth derivative at $x = x_0$ and $x = 1$.

A very good approximation for the sphere-sphere form factor can be obtained by using a slight generalization of Eq. (6)

$$f_{C}^{SS}(r) = \frac{1}{x} \tanh(a_1 x + a_3 x^3 + a_5 x^5), \quad (9)$$

where the parameters $a_1$, $a_3$, and $a_5$ depends only on the ratio $a$ between the larger and the smaller of the two radii. With respect to Eq. (6), only the fifth power term was added in order to preserve the evenness of the form factor. To give more flexibility to this form factor the parameters are allowed to be determined, for each value of $a$, by fitting Eq. (6) to points calculated using Eq. (8).

This technique was used to obtain the sphere-sphere form factor parameters in a recent semiclassical analysis [14] of the optical potential scattering of $^{16}$O on $^{12}$C at $E_{Lab} = 132$ MeV. In the optical potential considered the Coulomb interaction was originally described using the potential of two uniform charge distributions of radii $R_1 = 3.54$ fm and $R_2 = 3.17$ fm. With these radii, the values obtained for the parameters are $a_1 = 2.387$, $a_3 = 1.071$, and $a_5 = 1.683$. Using our optical code, the differences between the cross sections calculated with the exact and the approximated Coulomb potential are completely negligible. In fact, calculating the cross sections from $1^\circ$ to $180^\circ$ with a step of $0.25^\circ$, the maximum relative error is lower than 0.09%.

The need to use a fitting procedure to obtain the best estimate of the parameters may be considered a drawback of the approximation (6). This complication is however unnecessary because the dependence on $\alpha = 1/a$ of the best fit parameters is rather smooth and, for practical purposes, it is sufficient to know the values of the best fit parameters on a rather sparse grid of $\alpha$ values.

The reliability of the fit procedure and the possibility of interpolating the best fit parameter values was tested using 101 $x$ values, equally spaced between 0 and 1.5, for 101 values of $\alpha = 1/a$, equally spaced between 0 and 1. In the worst case ($\alpha = 0$, which corresponds to the point-sphere case) the maximum relative error of the best fit form factor is less than 0.8%.
FIG. 2. Values of the best fit parameters of the sphere-sphere approximated form factor as a function of $\alpha$.

FIG. 3. Sphere-sphere form factors for $\alpha$ from 0.0 to 1.0, in steps of 0.1 (curves from below to above). In the inset the relative errors for the $\alpha = 0.0, 0.5$, and 1.0 cases are shown by the solid, dashed, and dotted lines, respectively.

In all the tests that we have performed, the substitution of the sphere-sphere form factor with its analytical approximation given by Eq. (9), does not produce observable effects with the thickness of the lines normally used to plot an optical cross section.

In conclusion we can say that the lack of analyticity of the Coulomb potentials, traditionally used in the optical model analyses, can be easily cured without any physically appreciable effect. The same is expected to be true also for the, more or less, similar lacks of analyticity in other term of the optical interactions, which are sometimes used. Using appropriate analytical functions it should be possible to closely approximate these interactions too. These analytical approximations could be used in the semiclassical analyses which requires that the interaction is expressed in terms of analytical functions.

| $\alpha$ | $a_1$ | $a_3$ | $a_5$ | Max. Rel. Err. (%) |
|----------|-------|-------|-------|-------------------|
| 0.0      | 1.503 | 0.501 | 0.944 | 0.76              |
| 0.1      | 1.650 | 0.674 | 1.420 | 0.60              |
| 0.2      | 1.788 | 0.899 | 1.765 | 0.30              |
| 0.3      | 1.916 | 1.162 | 1.788 | 0.13              |
| 0.4      | 2.033 | 1.408 | 1.487 | 0.11              |
| 0.5      | 2.138 | 1.552 | 1.122 | 0.14              |
| 0.6      | 2.229 | 1.548 | 0.958 | 0.16              |
| 0.7      | 2.302 | 1.416 | 1.077 | 0.15              |
| 0.8      | 2.356 | 1.229 | 1.376 | 0.12              |
| 0.9      | 2.388 | 1.069 | 1.682 | 0.08              |
| 1.0      | 2.399 | 1.008 | 1.808 | 0.07              |

TABLE I. Best fit parameters and maximum relative error for the analytical approximation (9) to the sphere-sphere Coulomb form factor with different $\alpha$ values.
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