History effects on network growth

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Growth dynamic of real networks because of emerging complexities is an open and interesting question. Indeed it is not realistic to ignore history impact on the current events. The mystery behind that complexity could be in the role of history in some how. To regard this point, the average effect of history has been included by a kernel function in differential equation of Barabási Albert (BA) model. This approach leads to a fractional order BA differential equation as a generalization of BA model. As opposed to unlimited growth for degree of nodes, our results show that over time the memory impact will cause a decay for degrees. This gives a higher chance to younger members for turning to a hub. In fact in a real network, there are two competitive processes. On one hand, based on preferential attachment mechanism nodes with higher degree are more likely to absorb links. On the other hand, node history through aging process prevents new connections. Our findings from simulating a network grown by considering these effects also from studying a real network of collaboration between Hollywood movie actors conforms the results and significant effects of history and time on dynamic.

I. INTRODUCTION

One of the widely studied model to describe network dynamics, Barabási Albert (BA), is based on preferential attachment mechanism [1–4]. According to this model, over time new nodes join system and link to the earlier nodes, with probability proportional to the degree of preexisting nodes. Nodes with higher degree have greater chance to attract more new connections. This leads to the emergence of hubs with very large degree. Therefore, BA model characterizes one of the important features in real networks, namely, scale invariance [5–9].

BA model has been very successful in describing many properties of real networks [1], however, its unlimited growth prediction for degree of nodes is not always in line with reality [10, 11]. In fact, in real networks other factors work against a node’s growth which BA model fails to account for [12–14]. For instance, in realistic networks like citation and scientific collaboration network [15, 16], world-wide web network [17] and network of movie actors collaboration [18], there are some important phenomena such as aging, screening and censorship to be included [19–24].

In network of movie actors for example, overtime, hubs are replaced by new ones. Superstars get promoted for some time period as they get to their height of popularity and eventually lose their attractiveness in eyes of media and decay gradually. New generation of stars will replace the old generation and repeat the same cycle again. An other example is political network [25, 26], influence of people in politics grow and eventually decreases over time, political hubs are replaced by other people. No one experience unbound growth in politics.

As opposed to many realistic situations where aging process causes the hubs to be replaced by newcomers, in BA model an emergent hub will always remain powerful and prevent newcomers from becoming strong hub. In other word, we need to consider the effect of losing power for old hubs and emergence of new hubs. Number of studies have been done in this area and some strategies have been proposed [12, 27–29]. As a common issue with most of these approaches, the outcome has been requested manually from the model, i.e, it is imposed on the network by adding some new terms to BA equation. The objection against these approaches is that, new terms that they added are not capable to demonstrate the main issue; even more, it is not always possible to find a closed form solution.

According to BA model nodes can connect to each other without any restriction, hence their degree can grow boundlessly. However, in many real systems, each node has limited capacity for joining other nodes. For example number of active friends a person can have is limited as a person would denote part of his/her time to each of those friends. Since we have limited amount of time per day/week, one can not simply have infinite friends.

A scientist or actor has just enough time to collaborate with small number of partners. Therefore, having some connections makes the space and opportunity scarce for more connections. It’s current connections do not allow the node to join other nodes easily. This is similar to the phenomenon in fermi systems known as screening effect, reduction in effective electronic potential because of the cloud of electrons around one electron [30]. To extend the preferential attachment mechanism, the model has to also pay attention to these kind of effects.

A good candidate to include history effects could be fractional calculus. Fractional calculus, the generalized form of ordinary differentiation and integration to non-integer order [31–35] has unique features such as nonlocality and memory, making it highly applicable in many fields of science and engineering [36–39]. With respect to the presence of kernel for history in fractional operators, the results from this model carries the system memory. In other word, memory of the system plays a substan-
FIG. 1: Schematic diagram represents dynamical growth of nodes in a network, developed under preferential attachment mechanism by considering limitations imposed by aging process and screening effect on growth process in (1a-1c) and comparison with standard preferential attachment without considering these restrictions (1d-1f). Red to blue color shades of nodes represent their attractiveness for new links. Highest attractiveness is represented by red. The radial distance from central node shows arrival time for a node. Over the time a new node joins the network and connects to \( m \) existed nodes according to their degree. However, old members because of their age have lower chance of being selected. As (1d-1f) show, in the absence of these effects hubs are dominated in absorbing new links.

The rest of this paper is organized as the following. In section (II) fractional version of BA model by use of fractional calculus operators is introduced. Section (III) devotes to solving the fractional equation and it’s results. Results of a network simulation by considering aging effects on it’s dynamic has been presented and discussed in section (IV). In section (V) the real network of Hollywood movie actor collaboration which shows bounded growth for collaboration rate is investigated. Section (VI) concludes the results.

II. FRACTIONAL APPROACH FOR PREFERENTIAL ATTACHMENT ALGORITHM

In the BA model for network growth, the governing equation system,

\[
\frac{dk_i(t)}{dt} = \frac{mk_i(t)}{\sum_j k_j(t)} \quad k_i(t_{i0}) = m,
\]

(1)
is an integer order equation for each node $i$, does not include the past history of the considered node. Only node's degree in previous step affects its present degree. In order to consider the history on the growth process, a kernel on time, $\kappa(t - t')$, will be imposed on the right hand side of the Eq. \ref{eq:1}. This kernel accounts for the memory in solving fractional differential equations (FDE), uses integer order boundary or initial conditions.

In this way, the governing equation is a fractional order differential equation guarantees the presence of memory.

\begin{equation}
\frac{dk_i(t)}{dt} = \int_{t_0}^{t} dt' \kappa(t - t') \frac{mk_i(t')}{\sum_j k_j(t)}.
\end{equation}

From fractional calculus it is apparent that the right hand side of this equation is a fractional integral of order $(\alpha - 1)$ on the interval $[t_0, t]$ \cite{33}. Therefore, it can be shown as,

\begin{equation}
\frac{dk_i(t)}{dt} = t_0 D_t^{-(\alpha-1)} \left[ \frac{mk_i(t)}{\sum_j k_j(t)} \right].
\end{equation}

Now applying a fractional Caputo derivative of order $(\alpha - 1)$ \cite{17} on both side of the above equation, we can write it in the form of a differential equation,

\begin{equation}
\frac{c}{t_0} D_t^\alpha [k_i(t)] = \frac{mk_i(t)}{\sum_j k_j(t)}.
\end{equation}

For a continuous function $f$ on the $[a, b]$ interval, left Caputo derivative of order $\alpha$ is defined as follows \cite{18}:

\begin{equation}
\frac{c}{t_0} D_t^\alpha [f(t)] = \frac{1}{\Gamma(\alpha - \eta)} \int_{t_0}^{t} (t - \xi)^{\alpha-\eta-1} \frac{d}{d\xi} f(\xi) d\xi,
\end{equation}

where $n$ is the smallest integer greater than or equal to $\alpha$, $n = \lceil \alpha \rceil + 1$. Caputo derivative has the advantage that in solving fractional differential equations (FDE), uses integer order boundary or initial conditions.

In the last step to obtain Eq. \ref{eq:5}, we applied the fact that Caputo fractional derivative and fractional integral are inverse operators \cite{19}, for $\alpha > 0$,

\begin{equation}
\frac{c}{t_0} D_t^\alpha \left[ t_0 D_t^{-\alpha} \right] f(t) = f(t).
\end{equation}

In this way, the governing equation is a fractional order differential equation guarantees the presence of memory.

III. NUMERICAL RESULTS OF FRACTIONAL ORDER GROWTH EQUATION

In attempting to deploy fractional calculus in BA model of growing networks, we start by dynamic equation in form of,

\begin{equation}
\frac{c}{t_0} D_t^\alpha k_i(t) = m \sum_j k_j(t),
\end{equation}

\begin{equation}
k_i(t_{00}) = m.
\end{equation}

Where $0 < \alpha \leq 1$. For $\alpha = 1$ the above equation becomes the well-known dynamic equation in BA model. We have a system of fractional order differential equations coupled by the summation in denominator. From here the problem becomes an initial value problem for FDE,

\begin{equation}
\frac{c}{t_0} D_t^\alpha y(t) = f(t, y(t)),
\end{equation}

\begin{equation}
y(t_0) = y_0,
\end{equation}

which can be solved numerically by the predictor-corrector algorithm \cite{50,51}. Hence, we can reformulated it to equivalent Volterra integral equation in the form,

\begin{equation}
y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - s)^{\alpha-1} f(s, y(s)) ds.
\end{equation}

To deal with this integral, we use product rectangle method, which divides the domain into $n$ fragments, $t_j = t_0 + jh$, with equal space $h$. The right hand side function, in terms of numerical approximation $y_j$ for $y(t_j)$, is denoted by $f(t_j, y_j)$. Finally, we have the discrete form as follows,

\begin{equation}
y_n = y_0 + \frac{h^{\alpha}}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} b_{n-j-1} f_j,
\end{equation}

with $b_n$ coefficients as,

\begin{equation}
b_n = \frac{(n + 1)^\alpha - (n)^\alpha}{\Gamma(\alpha + 1)}.
\end{equation}

As a result, the discrete form of fractional order differential equation \ref{eq:8} becomes,

\begin{equation}
k_n = k_0 + \frac{h^{\alpha}}{\Gamma(\alpha + 1)} \sum_{j=0}^{n-1} b_{n-j-1} \frac{mk_j}{\sum_j k_j(t)}.
\end{equation}

Here, $b_n$’s are time dependent coefficients which indicate the aging effect. This factor shows contribution of the previous degree of $i$th node on its present value. With increasing the lifetime (increasing $n$), $b_n$ becomes smaller. In other words, over time old links of a node will lose their effect on its growth process based on $b_n$’s coefficients. Therefore, the effective degree of a node (Eq. \ref{eq:13}) will decrease. For $\alpha = 1$ the $b_n$ converges to unity and we get back to the standard BA model in which old degrees all have the same weight. By numerically solving equation system \ref{eq:13}, we found interesting results confirming the effect of history on network evolution. The results in Fig. \ref{fig:2} for various amounts of $\alpha$, show time dependence of $k_i(t)$ (effective degree has been left for $i$th
node at time $t$). It can be seen that for all values of $\alpha < 1$, effective degree of node $i$ increases at first. It reaches the maximum value, then starts to decay.

This behavior could be as a result of competition between two mechanisms governing the network dynamic. On one hand, nodes absorb new links according to preferential attachment algorithm, on the other hands, aging process will reduce the probability of being selected by new members by reducing its effective degree at that moment. The older the member, the smaller is its portion from past history. However, on the same time period, node with higher degree would receive less impact from aging than node with smaller degree.

Many real world networks exhibit the above mentioned behavior. Here as a well known example which will be studied in particular in section V, network of movie actors collaboration could be mentioned. In this network, we can see that superstars do collaborate with many actors as they become popular. However, because of screening effect one can just have the ability to cooperate with finite number of actors. There is not enough room for collaboration with every one. The rate of collaboration will reach a peak and then decline gradually over time as a result of aging process.

The growth and decaying rates depend on the order of fractional equation. Smaller $\alpha$ reaches the peak quickly but larger $\alpha$ takes more time. The smaller the exponent is, the more aging can reduce growth rate. Getting closer to exponent 1 aging effect is reduced and nodes have longer growth time. For $\alpha = 1$ which results in preferential attachment without aging, it increases boundlessly. Exponential increase in time distance that effective degree takes to reach its maximum for different $\alpha$ has been shown in Fig. 3.

Consequently, the fractional order BA equation for growing networks can demonstrate the existence of history. Because fractional order derivatives by applying a kernel over time involve the effect of elapsed time, it indicates the fact that every member has an end and will be isolated. In many real networks, members do not remain in the system for ever. As time passes, they will be set aside from the community. The results from fractional equation [13] remarks this fact.

**IV. SIMULATING AN AGED NETWORK**

Along side solving fractional order dynamic equation governing the network, we have simulated a network with

![FIG. 2](image)

**FIG. 2:** Figure (a) is numerical solution of Eq. (13) and (b) is mean degree of nodes from simulating a network that it’s growth dynamic is affected by aging process. It is clear from results that $k(t)$ behaves differently from unlimited growth predicted by BA model. Due to aging process $k(t)$ reaches a peak and declines gradually afterwards. Simulation has been done for 5000 time step with initial $m_0 = 10$ nodes and every new node connects to $m = 10$ earlier nodes.

![FIG. 3](image)

**FIG. 3:** The time to reach maximum for different $0 < \alpha < 1$ values with the fitted curve. This time decreases exponentially with increasing the exponent. By considering even a very small effect of memory on evolution, which means any deviation from $\alpha = 1$, there would be a peak after that degree of nodes will experience a decay. However, for closer values to 1 this time tends to infinity.
 Aging process. In this simulation we start by a fully connected network with \( m + 1 \) members, hence each node has initial degree of \( m \). In each time step \( i \), node \( i \) is introduced to the network, which is linked to \( m \) previous nodes proportional to their degree. To include aging process in the growth dynamic, the effectiveness of old links for a node is lessened. A link between two nodes, is less effective for old node, however it preserves its influence for younger one; similar to a connection between two nodes in a directed weighted network. As a consequence, the probability of being selected by newly arrived members will be reduced for members with more old links. This reduced effectiveness of older links in growth dynamic has been applied by using \( b_n \) coefficients which are proportional to link’s life time as weight factors in calculating probability of receiving new links.

Figure (2b) displays time dependence of effective degree on time steps resulted from simulation. The observed behavior is in good agreement with numerical results, (Fig. (2a)), obtained from fractional order differential equation \( f(x) = a \cdot \log(x) + b \). Just like numerical solution, initially, effective degree per time of a node increases, after reaching its maximum value it decays gradually.

In figure (4), we have shown some statistical proprieties of the network simulation with aging concept along side BA model. Small \( \alpha \) represents severe aging effects, therefore behavior of quantities with smaller \( \alpha \) shows greater deviation from BA model. Degree distribution reveals major characteristics of a model. For preferential attachment nodes with high degree absorb most of new links, however, getting older over the time reduces probability of high degree nodes to be selected \( \alpha = 1 \), since aging reduces the effective degree of nodes. This reduces growth rate of older nodes which gives nodes with smaller degree a higher chance to receive new links.

**FIG. 4:** (a) shows probability distribution function for different orders of fractional derivative. While for \( \alpha = 1 \), BA model, the expected power law behaviour is found, for fractional orders less than unity, there is deviation from power law. This deviation gets severe for smaller \( \alpha \). It could be caused by two competitive processes, namely preferential attachment mechanism and screening effect. (b) is closeness centrality averaged over all nodes; comparison between three exponents \( \alpha = 0.3, 0.7, 1 \). Horizontal axis is nodes according to their arrival time. For higher values of \( \alpha \), this measure has higher amounts. Elders have shortest path to others. (c) indicates clustering coefficient averaged over all nodes for three exponents \( \alpha = 0.3, 0.7, 1 \). Horizontal axis shows nodes according to their arrival time. As it is clear, it has certain behavior. Older nodes have more communities around. (d-f) show degree-degree correlation averaged over all nodes for three exponents \( \alpha = 0.3, 0.7, 1 \). Tendency to bind with similar members have been increased by including history effects. The fact is that the inclination to powerfuls is reduced. Powerful sites would not be in power for all the time. There is a competition by new generations. Hence, it is not the priority anymore to connect to the elders.
The average shortest path to all nodes in a network known as closeness centrality is a criteria to measure power of centrality for nodes in network. This quantity has been studied for different orders of fractional BA equation. It can be seen from Fig. 4(b) that closeness centrality for nodes has raised for aged system compared to BA model. This figure shows that older nodes in BA model have more central position in network respect to olders in aged system. Moreover, in an aged system, older members are more close to others than younger nodes.

Aging process also has impacts on clustering coefficient of nodes, Fig. 4(c). It is an indicator that displays tendency of a node’s neighbors to connect to each other [18]. Senior members are associated with communities which have high degree of connectivity as opposed to recently added members. Although aged nodes lose their importance, they still preserve their strategic position among others. However, in BA model older nodes have smaller clustering coefficient than olders in system by including aging effect.

One significant consequence of considering aging is that new members can have the opportunity to grow and become a hub, as opposed to BA model where only early members have a chance of fast growth. Hence tendency of nodes to connect to their similar nodes, what is called assortativity [54] will increase. This can be seen in degree-degree correlation palette in Figs. 4d,4h. Despite BA model that considerable connections are with hubs or older nodes, in aged networks remarkable correlation can be found between similar nodes in degree.

V. NETWORK OF OSCAR WINNERS COLLABORATION

In order to verify the above results for a real network, collaboration network of Hollywood movie actors have been studied. We extracted list of actors from the Internet Movie Database available at www.imdb.com. The table of all movies and star actors of those movies have been collected. TV series and TV movies are not included in our data. Nodes in the intended network are actors and they have a common link if they have acted in the same movie. As a case study in this network, total number of collaboration of Oscar winners with other actors per year has been derived as shown in figure 5 for some of them. It is the total number of staring actors that an actor/actress played with in each year. It can be seen that almost all the actors presented in the above network have the same pattern in their career. At start of acting career, collaboration rate is increased till it reaches a maximum. After that, collaboration rate will decrease gradually. Despite some exceptions, it is a general behavior that collaboration rate will decrease ultimately. The average collaboration rate of 75 actors and 74 actresses show the mentioned pattern in the network, Fig. 6. Although the general behavior is similar for both curves, the mean value for number of collaboration for actresses reaches the maximum sooner. It behaves as if its dynamic equation has smaller order of differentiation, α, than the curve for actors collaboration. In other words, history effects and aging process have more severe impacts on actresses growth. Besides, studying the mean rate of collaboration on first and second half of the whole period shows no noticeable difference in dynamical behavior. The above results are in remarkable
VI. CONCLUSION

Complexity and collective behavior are key characteristics of many natural and social systems. The past of a realistic systems can not be ignored in its dynamics and what is happening now. Even it could be said, since future expectations of a system are reselections of system goals, a complex system dynamic can be characterized by its goals too. Some mathematical models have been proposed to expand BA model by adding new terms to it’s differential equation, however, since these additional terms are not unique, there is no clear method to prefer one over the other. More over often these additional terms do not often have analytical solution.

In attempt to describe how the history play role in the current events and growth dynamic of networks, we apply a kernel function as an average of the history on the standard equation in BA model. This choice for the kernel shows interwoven of past events that leads to dynamic of the system. This approach leads to a fractional order BA differential equation governing network dynamic.

According to results, this approach predicts a boundary for growth of members. Just by generalizing governing equation in BA model to an equation with fractional order derivative, a more realistic dynamic for systems is achieved. As it can be seen in many real systems such as movie actors collaboration, networks of scientific papers, friendship networks and etc. degree of nodes progress for a certain period of time and then decay gradually after that. This change in topology of the system causes redistribution in the members power.

Moreover, even future goals of a system be it minimizing system energy or adaptability with nature (in evolution) could shape its dynamic too. Future projection of kernel could be tough of as the way that future goal might effect one’s present decisions. Birds for example migrate seasonally to survive their life in next seasons. Therefore, considering the future goals could be an interesting study in network dynamics.

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