PSEUDOSPECTRA AND SIMULTANEOUS CONTROL.

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Abstract. We prove that, under some conditions, for $N \geq 10$, $M > 0$ and two functions $f$ and $g$ holomorphic in a domain $\Omega$, we can find two $N \times N$ matrices $A$ and $B$ with identical pseudospectra such that we have simultaneously $\|f(A)\|/\|f(B)\| > M$ and $\|g(A)\|/\|g(B)\| > M$. In particular, this is the case if $f(z) = z^n$ and $g(z) = z^m$, where $n, m \geq 2$.

1. Introduction

Let $N \geq 1$ and $M_N(\mathbb{C})$ be the algebra of complex $N \times N$ matrices. Let $\|\cdot\|_2$ be the Euclidean norm on $\mathbb{C}^N$ and let $\|\cdot\|$ be the associated operator norm on $M_N(\mathbb{C})$. Given $A \in M_N(\mathbb{C})$ and $\varepsilon > 0$, the $\varepsilon$-pseudospectrum of $A$ is defined by

$$
\sigma_\varepsilon(A) := \{ z \in \mathbb{C}, \| (zI - A)^{-1} \| > 1/\varepsilon \}
$$

with the convention $\| (zI - A)^{-1} \| = \infty$ if $z \in \sigma(A)$, the spectrum of $A$. For more information on pseudospectra, see the book of Trefethen and Embree [6]. A well-known result related to the pseudospectra is the Kreiss matrix theorem. The version below is not the original version of Kreiss result in [1] but the optimal version of such an inequality (see [7]).

**Theorem** (Kreiss Matrix Theorem). Let $N \in \mathbb{N}$ and $A \in M_N(\mathbb{C})$. Let $r(A) := \sup_{|z| > 1} |z|^{-1} \| (zI - A)^{-1} \|$. Then

$$
r(A) \leq \sup_{k \geq 0} \| A^k \| \leq eN r(A).
$$

In this present paper, we are interested in matrices with same $\varepsilon$-pseudospectrum, for all $\varepsilon$. We say that two matrices $A$ and $B$ have identical pseudospectra if for all $\varepsilon > 0$, $\sigma_\varepsilon(A) = \sigma_\varepsilon(B)$, i.e. if

$$
\| (zI - A)^{-1} \| = \| (zI - B)^{-1} \| \quad (z \in \mathbb{C}).
$$

As direct consequence of the Kreiss matrix theorem, if $A, B \in M_N(\mathbb{C})$ have identical pseudospectra, we have

$$
\sup_{n \geq 0} \| A^n \| \leq eN \sup_{n \geq 0} \| B^n \|.
$$

Now the question that has been asked is: Do we have a similar result if we consider individual powers of $A$ and $B$? Consider $n = 1$, the answer

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\end{footnotesize}
is Yes. We know that if $A, B \in M_N(\mathbb{C})$ have identical pseudospectra, then $||A|| \leq 2||B||$ (See [4]). In the other hand, if we consider some powers higher than 1, the answer becomes No. Ransford proved, in [3], that for all $M > 0$, there exist $A, B \in M_n(\mathbb{C})$ such that $A$ and $B$ have identical pseudospectra and

$$||A^n|| > M||B^n|| \quad (2 \leq n \leq (N - 3)/2).$$

There is also a result for powers higher than $(N - 3)/2$, due to Ransford and Raouafi in [5]. They showed that if $f$ is an holomorphic function in a domain $\Omega \subset \mathbb{C}$, which is not a Möbius transformation, $M > 0$ and $N \geq 6$, then there exist $A, B \in M_N(\mathbb{C})$, with their spectrum in $\Omega$, such that $A$ and $B$ have identical pseudospectra and

$$||f(A)|| > M||f(B)||.$$  

However, their result only applies to a single function $f$. In the next theorem, we extend this result for two holomorphic functions.

**Theorem 1.1.** Let $f$ and $g$ be two holomorphic functions in a domain $\Omega \subset \mathbb{C}$ and let $M > 0$ be a constant and $N \geq 10$.

Suppose that $f$ and $g$ do not satisfy the condition

$$\tilde{h}_{1,2}\left(\tilde{h}_{2,4} - \tilde{h}_{3,4}(2\tilde{h}_{2,3} + \tilde{h}_{1,4})\right) + \tilde{h}_{1,3}\left(\tilde{h}_{1,3}\tilde{h}_{3,4} - \tilde{h}_{2,3}\tilde{h}_{2,4}\right) + \tilde{h}_{2,3}^3 = 0, \quad (*)$$

where $\tilde{h}_{i,j} := \frac{1}{i!j!}((f^{(i)}g^{(j)} - f^{(j)}g^{(i)})$.

Then there exist $A, B \in M_N(\mathbb{C})$ with spectrum in $\Omega$ such that $A$ and $B$ have identical pseudospectra and

$$\frac{||f(A)||}{||f(B)||} > M \quad \text{and} \quad \frac{||g(A)||}{||g(B)||} > M.$$  

In Section 2, we will give a proof of this result. In Section 3, we will show that the result holds for the powers of matrices, regardless of the condition $(*)$, thereby obtaining next theorem.

**Theorem 1.2.** Let $m, n \in \mathbb{N}$ with $n, m \geq 2$. Let $M > 0$ and $N \geq 10$. Then there exist $A, B \in M_N(\mathbb{C})$ such that $A$ and $B$ have identical pseudospectra and

$$\frac{||A^n||}{||B^n||} > M \quad \text{and} \quad \frac{||A^m||}{||B^m||} > M.$$  

2. Case of two functions.

In the proof of Theorem 1.1, we will build the matrices $A$ and $B$ by blocks. The following lemma will be used to verify that the matrices thus constructed do indeed have identical pseudospectra.

**Lemma 2.1.** Let $S$ be the unilateral shift on $\mathbb{C}^8$. For $t \geq 1$, we define

$$C_t := tS + ut^3S^3 + c_4t^4S^4 + c_5t^5S^5 + c_6t^6S^6 + c_7t^7S^7.$$
There exists a polynomial condition in \( c_k \) and \( u \) (given in the proof) such that, if this condition is satisfied, then there exists \( \mu > 0 \) such that

\[
||(I - zC_t)^{-1}|| \geq 1 + \mu t^6 |z| \quad (t \geq 1, z \in \mathbb{C}).
\]

**Proof.** We will first show that there exist two polynomials \( P_1(z) \) and \( P_2(z) \) such that

\[
||(I - zC_t)^{-1}|| \geq 1 + \frac{t^6 |z|}{2} \max(|P_1(z)|, |P_2(z)|) \quad (t \geq 1, z \in \mathbb{C}).
\]

When we compute \((I - zC_t)^{-1}\), we obtain

\[
(I - zC_t)^{-1} = I + ztS + z^2 t^2 S^2 + (z^3 + uz) t^3 S^3
\]

\[
+ (z^4 + 2uz^2 + c_4)t^4 S^4 + (z^5 + 3uz^3 + 2c_4 z^2 + c_5 z)t^5 S^5
\]

\[
+ (z^6 + 4uz^4 + 3c_4 z^3 + (u^2 + 2c_5) z^2 + c_6 z)t^6 S^6
\]

\[
+ (z^7 + 5uz^5 + 4c_4 z^4 + 3(u^2 + c_5) z^3 + 2(u c_5 + c_6) z^2 + c_7 z)t^7 S^7.
\]

Then, when \( t \geq 1 \), by applying the Lemma 2.1 in [5], we have

\[
||(I - zC_t)^{-1}|| \geq 1 + \frac{1}{2} \max \left( |tz|, |z^2 t^2|, |(z^3 + uz) t^3|, \right.
\]

\[
\left. |(z^4 + 2uz^2 + c_4)t^4|, |(z^5 + 3uz^3 + 2c_4 z^2 + c_5 z)t^5|, \right.
\]

\[
\left. |(z^6 + 4uz^4 + 3c_4 z^3 + (u^2 + 2c_5) z^2 + c_6 z)t^6|, \right.
\]

\[
|t^7(z^7 + 5uz^5 + 4c_4 z^4 + 3(u^2 + c_5) z^3 + 2(u c_5 + c_6) z^2 + c_7 z))|)
\]

\[
\geq 1 + \frac{1}{2} \max \left( |(z^6 + 4uz^4 + 3c_4 z^3 + (u^2 + 2c_5) z^2 + c_6 z)t^6|, \right.
\]

\[
\left. |t^7(z^7 + 5uz^5 + 4c_4 z^4 + 3(u^2 + c_5) z^3 + 2(u c_5 + c_6) z^2 + c_7 z))| \right)
\]

\[
\geq 1 + \frac{t^6 |z|}{2} \max(|P_1(z)|, |P_2(z)|)
\]

with

\[
P_1(z) := c_7 + 2z(u c_4 + c_6) + 3z^2(u^2 + c_5) + 4z^3 c_4 + 5uz^4 + z^6
\]

and

\[
P_2(z) := c_6 + z(u^2 + 2c_5) + 3z^2 c_4 + 4uz^3 + z^5.
\]

Let \( \mu := \frac{1}{2} \inf_{z \in \mathbb{C}} \max(|P_1(z)|, |P_2(z)|) \). Then

\[
||(I - zC_t)^{-1}|| \geq 1 + \mu t^6 |z| \quad (t \geq 1, z \in \mathbb{C})
\]

and \( \mu > 0 \) if and only if \( \gcd(P_1, P_2) = 1 \).

Let \( P_3(z) := uz^4 + c_4 z^3 + (2u^2 + c_5) z^2 + (2c_4 u + c_6) z + c_7 \) be the remainder of the Euclidean division of \( P_1 \) by \( P_2 \). Suppose \( u \neq 0 \) and let \( P_3 = Az^3 + Bz^2 + Cz + D \) be the remainder of the Euclidean division of \( u^2 P_2 \) by \( P_3 \).
Thus
\[ A := 2u^3 - c_5u + c_4^2 \]
\[ B := 3c_4u^2 - 6u + c_4c_5 \]
\[ C := u^4 + 2c_5u^2 + (2c_4^2 - c_7)u + c_4c_6 \]
\[ D := c_6u^2 + c_4c_7. \]

Assume \( A \neq 0 \) and let \( P_5 = Ez^2 + Fz + G \), the remainder of the Euclidean division of \( A^2P_3 \) by \( P_4 \). Then
\[ E := A^2(2u^2 + c_5) - ACu + B^2u - ABc_4 \]
\[ F := A^2(2c_4u + c_6) + C(Bu - Ac_4) - ADu \]
\[ G := D(Bu - Ac_4) + A^2c_7. \]

Suppose \( E \neq 0 \) and let \( P_6 = Hz + I \), the remainder of the Euclidean division of \( E^2P_4 \) by \( P_5 \). Then
\[ H := -AEG + AF^2 - BEF + CE^2 \]
\[ I := (AF - BE)G + DE^2. \]

Finally let \( J = EI^2 - FHI + GH^2 \). To conclude, if \( AEJ \neq 0 \) then \( gcd(P_1, P_2) = 1 \) and \( \mu > 0 \).

\[ \square \]

Proof of Theorem 1.1 Without less of generality, we can suppose that \( 0 \in \Omega \), \( f'(0) \neq 0 \), \( g'(0) \neq 0 \), \( h_{1,2}(0) \neq 0 \) and \( f \) and \( g \) fail to satisfy the equation (*) for \( z = 0 \). For \( k \in \mathbb{N} \), we define \( f_k := \frac{1}{k!}f^{(k)}(0) \) and \( g_k := \frac{1}{k!}g^{(k)}(0) \). And for \( i, j \in \mathbb{N} \), we define \( h_{i,j} := f_ig_j - f_jg_i \). Then the Taylor developments of \( f \) and \( g \) are
\[ f(z) = f_0 + f_1z + f_2z^2 + f_3z^3 + f_4z^4 + f_5z^5 + f_6z^6 + f_7z^7 + O(z^8) \]
\[ g(z) = g_0 + g_1z + g_2z^2 + g_3z^3 + g_4z^4 + g_5z^5 + g_6z^6 + g_7z^7 + O(z^8). \]

Let \( C_t \) be defined as Lemma 2.1. We begin by finding the \( c_k \), rational functions in \( u \), such that \( ||f(C_t)|| = O(t^5) \) and \( ||g(C_t)|| = O(t^5) \). We have
\[ f(C_t) = f_0I + f_1tS + f_2t^2S^2 + (f_3 + uf_1)t^3S^3 \]
\[ + (f_4 + 2uf_2 + c_4f_1)t^4S^4 + (f_5 + 3uf_3 + 2c_4f_2 + c_5f_1)t^5S^5 \]
\[ + (f_6 + 4uf_4 + 3c_4f_3 + (u^2 + 2c_5)f_2 + c_6f_1)t^6S^6 \]
\[ + (f_7 + 5uf_5 + 4c_4f_4 + 3(u^2 + c_5)f_3 + 2(u^3 + c_6)f_2 + c_7f_1)t^7S^7 \]
Therefore we can choose \( u \) such that \( U \neq 0 \) and then the system has a unique solution. Moreover the solutions \( c_k \) satisfy

\[
g(C_t) = g_0 I + g_1 t S + g_2 t^2 S^2 + (g_3 + u g_1) t^3 S^3
\]
\[
+ (g_4 + 2 u g_2 + c_1 g_1) t^4 S^4 + (g_5 + 3 u g_3 + 2 c_4 g_2 + c_5 g_1) t^5 S^5
\]
\[
+ (g_6 + 4 u g_4 + 3 c_4 g_3 + (u^2 + 2 c_5) g_2 + c_6 g_1) t^6 S^6
\]
\[
+ (g_7 + 5 u g_5 + 4 c_4 g_4 + 3 (u^2 + c_5) g_3 + 2 (u c_5 + c_6) g_2 + c_7 g_1) t^7 S^7.
\]

Thus the \( c_k \) are the solutions of

\[
\begin{cases}
    f_1 c_7 + 2 f_2 (u c_4 + c_6) + 3 f_3 (u^2 + c_5) + 4 f_4 c_4 + 5 u f_5 + f_7 = 0 \\
    f_1 c_6 + f_2 (u^2 + 2 c_5) + 3 f_3 c_4 + 4 u f_4 + f_6 = 0 \\
    g_1 c_6 + g_2 (u^2 + 2 c_5) + 3 g_3 c_4 + 4 u g_4 + g_6 = 0 \\
    g_1 c_7 + 2 g_2 (u c_4 + c_6) + 3 g_3 (u^2 + c_5) + 4 g_4 c_4 + 5 u g_5 + g_7 = 0
\end{cases}
\]

If we define

\[
U = \det \begin{pmatrix} f_1 & 2 f_2 & 3 f_3 & 4 f_4 + 2 u f_2 \\ 0 & f_1 & 2 f_2 & 3 f_3 \\ g_1 & 2 g_2 & 3 g_3 & 4 g_4 + 2 u g_2 \\ 0 & g_1 & 2 g_2 & 3 g_3 \end{pmatrix},
\]

then

\[
U = (4 f_1^2 g_2^2 - 8 f_1 f_2 g_1 g_2 + 4 f_2^2 g_1^2) u
\]
\[
+ f_1^3 (8 g_2 g_4 - 9 g_3^2) + f_1 f_2 (12 g_2 g_3 - 8 g_1 g_4) + f_3 (f_1 (18 g_1 g_3 - 12 g_2^2)
\]
\[
+ 12 f_2 g_1 g_2) - 12 f_2^2 g_1 g_3 + f_4 (8 f_2 g_1^2 - 8 f_1 g_1 g_2) - 9 f_3 g_1^2
\]
\[
= 4 h_{1,2}^2 u + O(1).
\]

Therefore we can choose \( u \) such that \( U \neq 0 \) and then the system has a unique solution. Moreover the solutions \( c_k \) satisfy

\[
b_4 := U c_4 = - ((3 f_1^2 g_2 g_3 - 3 f_1 f_2 g_1 g_3 + f_3 (3 f_2 g_1^2 - 3 f_1 g_1 g_2)) u^2 + O(u)
\]
\[
b_5 := U c_5 = - ((2 f_1^2 g_2^2 - 4 f_1 f_2 g_1 g_2 + 2 f_2^2 g_1^2) u^3 + O(u^2)
\]
\[
b_6 := U c_6 = ((f_1 f_2 (16 g_2 g_4 - 9 g_3^2) - 16 f_2^2 g_1 g_4 + f_3 (9 f_1 g_2 g_3 + 9 f_2 g_1 g_3)
\]
\[
+ f_4 (16 f_2 g_1 g_2 - 16 f_1 g_2^2) - 9 f_3 g_1 g_2) u^2 + O(u)
\]
\[
b_7 := U c_7 = ((6 f_1 f_2 g_2 g_3 - 6 f_2^2 g_1 g_3 + f_3 (6 f_2 g_1 g_2 - 6 f_1 g_2^2)) u^3 + O(u^2).
\]
Now, we verify the condition on the resolvent. We have
\[ A' := U^2 A = 2U^2 u^3 - Ub_5 u + b_4^2 \]
\[ = 32(f_1 g_2 - f_2 g_1)^4 u^5 + O(u^4) \]
\[ = 32h_{1,2}^4 u^5 + O(u^4) \]
\[ E' := U^5 E = U^4 (A^2 (2U^2 u + b_5) - ACU u + B^2 U u - ABB_4) \]
\[ = a (f_1 g_2 - f_2 g_1)^{10} u^{13} + O(u^{12}) \]
\[ = ah_{1,2}^{10} u^{13} + O(u^{12}) \] (with \( a = 6144 \)).
\[ J' := U^{29} J = U^{29} (EI^2 - FHI + GH^2) \]
\[ = a b^2 h_{1,2}^{55} [h_{2,4}^2 - h_{3,4}^2 (2h_{2,3} + h_{1,3}) + h_{1,3} (h_{1,3} h_{3,4} - h_{2,3} h_{2,4}) + h_{2,3}^3] u^{75} + O(u^{74}) \] (with \( b = 301989888 \)).

Thus we can finally choose \( u \) such that we also have \( A \neq 0, E \neq 0 \) and \( J \neq 0 \). By Lemma 2.1 there exists \( \mu > 0 \) such that
\[ ||(I - zC_t)^{-1}|| \geq 1 + \mu t^6 |z| \quad (t \geq 1, z \in \mathbb{C}). \]

We finish the proof with the construction of \( A \) and \( B \).
We define \( A_t := C_t + \left( \begin{array}{cc} 0 & \mu t^6 \\ 0 & 0 \end{array} \right) \in M_{10}(\mathbb{C}) \) and \( B_t := C_t + \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \in M_{10}(\mathbb{C}) \).
We have, for all \( t \geq 1 \) and \( z \in \mathbb{C} \),
\[ \left\| \left( I - z \left( \begin{array}{cc} 0 & \mu t^6 \\ 0 & 0 \end{array} \right) \right)^{-1} \right\| = \left\| \begin{array}{cc} 1 & z \mu t^6 \\ 0 & 1 \end{array} \right\| \leq 1 + \mu t^6 |z|. \]

Then
\[ ||(I - zA_t)^{-1}|| = ||(I - zC_t)^{-1}|| = ||(I - zB_t)^{-1}||. \]

Now, if \( t \geq 1 \), we see that
\[ ||f(A_t)|| \geq \left\| f \left( \begin{array}{cc} 0 & \mu t^6 \\ 0 & 0 \end{array} \right) \right\| = \left\| \begin{array}{cc} f_1 & f_1 \mu t^6 \\ 0 & f_0 \end{array} \right\| \geq |f_1| \mu t^6 \]
\[ ||f(B_t)|| = \max(||f(C_t)||, |f(0)||) = ||f(C_t)|| = O(t^5) \]
\[ ||g(A_t)|| \geq \left\| g \left( \begin{array}{cc} 0 & \mu t^6 \\ 0 & 0 \end{array} \right) \right\| = \left\| \begin{array}{cc} g_0 & g_1 \mu t^6 \\ 0 & g_0 \end{array} \right\| \geq |g_1| \mu t^6 \]
\[ ||g(B_t)|| = \max(||g(C_t)||, |g(0)||) = ||g(C_t)|| = O(t^5). \]

Since \( |f_1| \mu > 0 \) and \( |g_1| \mu > 0 \), we have
\[ \lim_{t \to \infty} \frac{||f(A_t)||}{||f(B_t)||} = +\infty \quad \text{and} \quad \lim_{t \to \infty} \frac{||g(A_t)||}{||g(B_t)||} = +\infty. \]

So, finally, we can choose \( t \geq 1 \) such that
\[ \frac{||f(A_t)||}{||f(B_t)||} > M \quad \text{and} \quad \frac{||g(A_t)||}{||g(B_t)||} > M. \]
3. Case of Two Powers.

To prove the result for powers of matrices, we cannot just apply Theorem 1.1. Indeed, there exist an infinite number of \( n, m \geq 2 \) such that the functions \( f(z) := (1 + z)^n \) and \( g(z) := (1 + z)^m \) fail to satisfy the hypothesis of this theorem. But, as we will see in the proof, we can override this difficulty and show the result for any powers \( n \) and \( m \).

**Proof of Theorem 1.2.** If \( n = m \), then we can apply the result in [5]. We now suppose that \( n \neq m \). Let \( f(z) := (1 + z)^n \) and \( g(z) := (1 + z)^m \). We use the same notation as the proof of Theorem 1.1. We construct \( A_t \) and \( B_t \) as before. Remember that the condition in the theorem comes from the constraint to have \( \mu > 0 \): It was the condition to have \( \text{deg}(J') = 75 \) as a polynomial in \( u \). As we remarked, we can not conclude the result for all \( n \) and \( m \). So, we need to look the next coefficient and see that \( \text{deg}(J') \geq 74 \). It will imply that we can choose \( t \geq 1 \) such that \( \|f(A_t)\| > M \) and \( \|g(B_t)\| > M \).

And, finally, we just choose \( A = I + A_t \) and \( B = I + B_t \) to conclude.

By computation, we have

\[
J' = (m - 1)m^{58}(m + 1)(n - 1)n^{58}(n + 1)(n - m)^{58} \left[ \begin{array}{c}
-29296875(n^2 - 4mn + m^2 + 3) u^{75} \\
131072 \\
+ 1953125 p(n, m) u^{74} + O(u^{73})
\end{array} \right]
\]

with

\[
p(n, m) = 446n^4 - 2649mn^3 - 60n^3 + 4412m^2n^2 + 180mn^2 + 982n^2 - 2649m^3n + 180m^2n - 1411mn - 180n + 446m^4 - 60m^3 + 982m^2 - 180m - 828
\]

To conclude, we need to see that, with our conditions on \( n \) and \( m \), we cannot have \( n^2 - 4mn + m^2 + 3 = 0 \) and \( p(n, m) = 0 \) at the same time.

Assume the opposite. We have

\[
p(n, m) = q(n, m)(n^2 - 4mn + m^2 + 3) + r(n, m)
\]

with \( q(n, m) = 446n^2 + (-865m - 60)n + 506m^2 - 60m - 356 \) and \( r(n, m) = 60(m - 1)(m + 1)(4mn - m^2 - 4) \). Then, we deduce that \( r(n, m) = 0 \).

Moreover, we have

\[
n^2 - 4mn + m^2 + 3 = \frac{4mn - 15m^2 + 4}{16m^2} r(n, m) + \frac{m^4 - 8m^2 + 16}{16m^2}
\]

Then we conclude that \( m = \pm 2 \), but this implies that \( n = 1 \) and we finish with a contradiction.

\[\square\]
4. Final remarks and questions

• The condition (*) is quite mysterious and one might wonder which functions satisfy this condition. In the spirit of [5], Theorem 1.2, we shall show that the condition (*) precludes $f$ and $g$ from being Möbius transformations. More precisely, we show that, if $\lambda f + \mu g$ is a Möbius transformation for some $\lambda, \mu \in \mathbb{C}$, then $f$ and $g$ satisfy (*).

Indeed, without loss of generality, by replacing $f$ or $g$ by $\lambda f + \mu g$, we can assume that $f$ or $g$ is a Möbius transformation. We have:

$$
\tilde{h}_{1,2}(\tilde{h}_{2,4} - \tilde{h}_{3,4}(2\tilde{h}_{2,3} + \tilde{h}_{1,4})) + \tilde{h}_{1,3}(\tilde{h}_{1,3}\tilde{h}_{3,4} - \tilde{h}_{2,3}\tilde{h}_{2,4}) + \tilde{h}_{2,3}^3 \\
= \frac{1}{12}(\tilde{h}_{1,4} - \tilde{h}_{2,3})(Sf.G + F.Sg) - \frac{\tilde{h}_{2,4}^2}{576}(Sf.(Sg)' + (Sf)'Sg) \\
- \frac{\tilde{h}_{1,3}}{48}((Sf)'G + F.(Sg)') + \frac{\tilde{h}_{3,4}}{72}Sf.Sg + \frac{\tilde{h}_{2,3}}{1152}(Sf)'(Sg)' + 2\tilde{h}_{1,2}F.G
$$

with $F = \frac{1}{48}f''f^{(4)} - \frac{1}{36}(f^{(3)})^2$ and $Sf := 2f'f^{(3)} - 3(f'')^2$, the Schwarzian derivative of $f$ (likewise for $G$ and $Sg$).

If $f$ is a Möbius transformation, then its Schwarzian derivative satisfies $Sf = 0$ and then $(Sf)' = 0$. Moreover, $f$ is a Möbius transformation if and only if the function $f_w$, defined by $f_w(z) := \frac{f(z)-f(w)}{z-w}$, is a Möbius transformation, for all $w$ in the domain of the function $f$. Then $Sf_w = 0$, and we deduce that $F(w) = \frac{1}{12}Sf_w(w) = 0$. The same applies if $g$ is a Möbius transformation.

Since some other functions satisfy the condition (*), this raises the question of whether (*) has an interpretation analogous to the condition “$f$ is not a Möbius transformation” in [5].

• The condition “$f$ and $g$ fail to satisfy (*)” is not necessary, in Theorem 1.1. Indeed, Section 3 provides plenty of counterexamples (for example $f(z) = z^2$ and $g(z) = z^7$ verify (*)). What might a necessary and sufficient condition look like?

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References

[1] Heinz-Otto Kreiss. Über die Stabilitätsdefinition für Differenzengleichungen die partielle Differentialgleichungen approximieren. *Nordisk Tidskr. Informationsbehandling (BIT)*, 2:153–181, 1962.

[2] Randall J. LeVeque and Lloyd N. Trefethen. On the resolvent condition in the Kreiss matrix theorem. *BIT*, 24(4):584–591, 1984.

[3] Thomas Ransford. On pseudospectra and power growth. *SIAM J. Matrix Anal. Appl.*, 29(3):699–711, 2007.
[4] Thomas Ransford. Pseudospectra and matrix behaviour. In Banach algebras 2009, volume 91 of Banach Center Publ., pages 327–338. Polish Acad. Sci. Inst. Math., Warsaw, 2010.

[5] Thomas Ransford and Samir Raouafi. Pseudospectra and holomorphic functions of matrices. Bull. Lond. Math. Soc., 45(4):693–699, 2013.

[6] Lloyd N. Trefethen and Mark Embree. Spectra and pseudospectra. Princeton University Press, Princeton, NJ, 2005. The behavior of nonnormal matrices and operators.

[7] Elias Wegert and Lloyd N. Trefethen. From the Buffon needle problem to the Kreiss matrix theorem. Amer. Math. Monthly, 101(2):132–139, 1994.

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