Optimal time-decay estimates for an Oldroyd-B model with zero viscosity

Jinrui Huang\textsuperscript{a}, Yinghui Wang\textsuperscript{b}, Huanyao Wen\textsuperscript{a, b} and Ruizhao Zi\textsuperscript{c}

\textsuperscript{a}School of Mathematics and Computational Science, Wayi University, Jiangmen 529020, China.
\textsuperscript{b}School of Mathematics, South China University of Technology, Guangzhou 510641, China.
\textsuperscript{c}School of Mathematics and Statistics & Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, China.

Abstract

In this work, we consider the Cauchy problem for a diffusive Oldroyd-B model in three dimensions. Some optimal time-decay rates of the solutions are derived via analysis of upper and lower time-decay estimates provided that the initial data are small and that the absolute value of Fourier transform of the initial velocity is bounded below away from zero in a low-frequency region. It is worth noticing that the optimal rates are independent of the fluid viscosity or the diffusive coefficient, which is a different phenomenon from that for incompressible Navier-Stokes equations.

keywords. Oldroyd-B model, Cauchy problem, optimal time-decay estimates, zero viscosity.

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*Corresponding author.
E-mail addresses: huangjinrui1@163.com (J. Huang), yhwangmath@163.com (Y. Wang), mahywen@scut.edu.cn (H. Wen), rzz@mail.ccnu.edu.cn (R. Zi).
1 Introduction

In this paper, we consider the Cauchy problem for a diffusive Oldroyd-B system describing the motion of viscoelastic fluids:

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p - \epsilon \Delta u &= \kappa \text{div} \tau, \\
\partial_t \tau + u \cdot \nabla \tau - \mu \Delta \tau + \beta \tau &= Q(\nabla u, \tau) + \alpha \mathbb{D} u, \\
\text{div} u &= 0, \\
(u, \tau)(x, 0) &= (u_0, \tau_0),
\end{align*}
\]

on \( \mathbb{R}^3 \times (0, \infty) \). Here we denote by \( u = (u_1, u_2, u_3)^T : \mathbb{R}^3 \to \mathbb{R}^3 \) the velocity field of fluid, \( \tau \in S_3(\mathbb{R}) \) is the purely elastic (the polymer) part of the stress tensor, \( p \in \mathbb{R} \) is the pressure function of the fluid, \( \mathbb{D} u = \frac{1}{2} \left( \nabla u + (\nabla u)^\top \right) \) is the deformation tensor, and

\[
Q(\nabla u, \tau) = \Omega \tau - \tau \Omega + b(\mathbb{D} u \tau + \tau \mathbb{D} u)
\]

admits the usual bilinear form with the skew-symmetric part of velocity gradient \( \Omega = \frac{1}{2} \left( \nabla u - (\nabla u)^\top \right) \) and some \( b \in [-1, 1] \). The parameters \( \epsilon, \kappa, \mu, \beta, \alpha \) satisfy that \( \epsilon, \mu \geq 0, \kappa, \beta, \alpha > 0 \).

Note that the diffusive term \( \mu \Delta \tau \) corresponds to a spatial diffusion of the polymeric stresses where \( \mu \) is usually called the center-of-mass diffusion coefficient. It is known that this coefficient is significantly smaller than other effects (6), and thus this term is usually ignored in the Oldroyd-B model (non-diffusive model). However, the center-of-mass diffusion effects can be physically justified to model the shear and vorticity banding phenomena (7 10 15 20 41 43). As showed by Málek et al. (43), the stress diffusion term can be interpreted either as a consequence of a nonlocal energy storage mechanism or as a consequence of a nonlocal entropy production mechanism. Thus the diffusive Oldroyd-B system has attracted much attention and been studied extensively, see 11 12 13 14 18 19 33 42 49.

Our main goal here is to investigate the optimal time-decay estimates including the lower and upper estimates, and more importantly to study dependence of the optimal rates on the fluid viscosity \( \epsilon \) and the center-of-mass diffusion coefficient \( \mu \), which is divided into two cases, namely,

- Case I: \( \mu > 0 \) and \( \epsilon \geq 0 \);
- Case II: \( \epsilon > 0 \) and \( \mu \geq 0 \).

Without loss of generality, we assume that \( \mu, \epsilon \leq 1 \) throughout the rest of the paper. In what follows, we introduce some related works.

Non-diffusive Oldroyd-B model

The non-diffusive incompressible Oldroyd-B model (i.e., 14 with \( \mu = 0 \)) was first proposed by Oldroyd in 1958 (45), which obeys an invariant constitutive law describing the general motion of an isotropic, incompressible visco-elastic liquid with significant non-Newtonian effects such as the Weissenberg climbing effect. An early existence result for the model was obtained by Guillopé and Saut (24). They obtained the global existence and uniqueness of strong solution with small initial data and a small coupling constant (like \( \alpha \) in 14 with \( \mu = 0 \)) in the Sobolev space \( H^1(\Omega) \) where \( \Omega \) is a bounded domain of \( \mathbb{R}^3 \). Fernández-Cara, Guillen and Ortega (23) extended this results to the \( L^p \)-setting. Later
on, the smallness restriction on the coupling constant was removed by Molinet and Talhouk (44). In [26], Hieber, Naito and Shibata obtained a global existence and uniqueness of the solution in exterior domains provided the initial data and the coupling constant are sufficiently small. Fang, Hieber and Zi (21) extended the work [26] to the case without any smallness assumption on the coupling constant. Note that the global existence theory for the non-diffusive model with arbitrarily large initial data is still open. However, for a special case that fluid flows do not stretch polymers but allow rotation, namely, \(Q(\nabla u, \tau) = \Omega \tau - \tau \Omega\) in the non-diffusive model, Lions and Masmoudi (40) obtained the global existence of weak solution with arbitrarily large initial data. In this case, the model is usually called the corotational model which enjoys better \textit{a priori} estimates due to the simplification of \(Q(\nabla u, \tau)\). For the upper-convected case (i.e. \(b = 1\), \(Q(\nabla u, \tau) = \nabla u \tau + \tau (\nabla u^T)\)), Hu and Lelièvre (28) derived some new entropy estimates for the Oldroyd-B model and some related models. Concretely, they study the following nondimensional system,

\[
\begin{align*}
\begin{cases}
\partial_t u + (u \cdot \nabla)u + \frac{1}{\text{Re}} \nabla p - \frac{1 - \omega}{\text{Re}} \Delta u = \frac{1}{\text{Re}} \text{div} \tau,
\tau_t + (u \cdot \nabla)\tau - \nabla u \tau - \tau (\nabla u^T)^T + \frac{1}{\text{We}} = \frac{We}{Re} (\nabla u + (\nabla u)^T),
\end{cases}
\end{align*}
\] (1.2)

where \(\omega \in (0, 1)\) is a coupling constant, \(\text{Re}\) is Reynolds number and \(\text{We}\) is Weissenberg number. Note that if we set \(e = \frac{1 - \omega}{\text{Re}}, \kappa = \frac{1}{\text{Re}}, \beta = \frac{1}{\text{Re}}, \alpha = \frac{2 \omega}{\text{We}}\) and \(\mu = 0\), system (1.1) becomes (1.2). By introducing the so-called conformation tensor, \(A = \frac{\text{We}}{\omega} \tau + I\), the authors transform (1.2) into the following system,

\[
\begin{align*}
\begin{cases}
\partial_t u + (u \cdot \nabla)u + \frac{1}{\text{Re}} \nabla p - \frac{1 - \omega}{\text{Re}} \Delta u = \frac{\omega}{\text{We}} \text{div} A,
A_t + (u \cdot \nabla)A - \nabla u A - A (\nabla u)^T + \frac{1}{\text{We}} (A - I) = 0.
\end{cases}
\end{align*}
\] (1.3)

We remark that (1.3) can also be deduced from the dumbbell model of polymeric fluids (refer to E, Li and Zhang (17) for instance). Hu and Lelièvre (28) obtained the following entropy estimate, for the Dirichlet problem in bounded domains,

\[
\frac{d}{dt} F(u, A) + (1 - \omega) \int_D |\nabla u|^2 \, dx + \frac{\omega}{2 \text{We}^2} \int_D \text{tr}(A + A^{-1} - 2I) \, dx = 0,
\] (1.4)

where

\[
F(u, A) = \frac{\text{Re}}{2} \int_D |u|^2 \, dx + \frac{\omega}{2 \text{We}} \int_D (\text{tr}A - \log(\text{det}A) - 3) \, dx
\]

is the sum of the kinetic energy and the Helmholtz free energy (see equation (3.13) in [28]). If one assumes that the initial data \(A_0\) is symmetric positive definite, then \(A(t)\) is symmetric positive definite and \(\text{tr}A - \log(\text{det}A) - 3 \geq 0\) (see Lemmas 1.1 and 2.1 in [8] for details). Boyaval, Lelièvre and Mangoubi (8) analyzed the stability of some numerical schemes for system (1.3). For the long-time behavior of the solution of system (1.1) with \(\epsilon > 0\) and \(\mu = 0\) in \(\mathbb{R}^3\), Hieber, Wen and Zi (27) obtained some upper time-decay estimates, and the decay rates are the same as those for the heat equation.

For the case without damping mechanism and the scaling invariant approach, see the works [51 and 11, 22, 52], respectively. An energetic variational approach was first introduced by Lin, Liu, and Zhang (38) to understand the physical structure for the related systems (see for instance [5, 29, 30, 44, 55, 56, 37, 39, 50] for more progress).

**Diffusive Oldroyd-B model**

As mentioned in the beginning, the physical consideration for the diffusive Oldroyd-B model can be referred to [6, 7, 20, 41, 43]. Inspired by the works of Rajagopal, Srinivasa (46) and Ziegler, Wehrli (54), Málek et al. (43) derived variants of Oldroyd-B models with stress diffusion for viscoelastic...
rate-type fluids, using the specific Helmholtz free energy, the entropy production and the concept of evolving natural configuration, in both incompressible and compressible framework. We remark that the diffusive Oldroyd-B model can also be derived as a macroscopic closure of the Fokker-Planck-Navier-Stokes systems (refer to [4, 33] and references therein for details). In the case \( \epsilon > 0 \) and \( \mu > 0 \), the existence of global-in-time weak solutions in two or three dimensions was obtained by Barrett, Boyaval, and Süli ([13]). For the regular solutions, Constantin and Kliewer ([13]) obtained the global existence and uniqueness of strong solutions to the two-dimensional model with varied polymer number density. La (22) gave a rigorous derivation of the model in two dimensions as a macroscopic closure of a micro-macro model. By using a similar free energy estimates in [28], Bathory, Bulíček and Málek (5) proved the existence of weak solutions for a generalized rate-type viscoelastic fluids with stress diffusion in three-dimensional bounded domains. Dostálik, Průša and Stein ([16]) studied the large time behaviors for the related viscoelastic models with the free energy estimates in three-dimensional vessel. For compressible case, please refer to [2, 42, 49] for global existence of weak solutions, weak-strong fusion in three-dimensional bounded domains. Dostalý, Průša and Stein ([16]) studied the large time behaviors for the related viscoelastic models with the free energy estimates in three-dimensional vessel. For compressible case, please refer to [2, 42, 49] for global existence of weak solutions, weak-strong fusion in three-dimensional bounded domains. Dostalý, Průša and Stein ([16]) studied the large time behaviors for the related viscoelastic models with the free energy estimates in three-dimensional vessel.

In the case \( \epsilon = 0 \) and \( \mu > 0 \), i.e., the fluid viscosity vanishes, the global existence and uniqueness of regular solutions was proved by Elgindi and Rousset (19) in two dimensions provided that the initial data are arbitrary when \( Q = Q(\nabla u, \tau) \) is ignored and that the initial data are small when \( Q \neq 0 \). Later on, Elgindi and Liu (18) extended the result in [19] to the three-dimensional case with small initial data in \( H^3(\mathbb{R}^3) \)-norm. Note that in this case the velocity of fluid obeys Euler system without damping whose global existence theory is unknown. ([18] and [19]) demonstrate that the coupling effect of (1.1) for \( \epsilon = 0 \) and \( \mu > 0 \) can make the solutions exist globally in time.

In summary, the previous works demonstrate what follows.

- For the non-diffusive Oldroyd-B model (i.e., \( \mu = 0 \) and \( \epsilon > 0 \)), some global well-posedness results and upper time-decay rates have been obtained where the decay estimate is given by

\[
\|(\nabla^k u, \nabla^k \tau)(t)\|_{L^2(\mathbb{R}^3)} \leq C(1 + t)^{-\frac{k}{2} - \frac{1}{4}}
\]  

(1.5)

for \( k = 0, 1, 2 \) and some known constant \( C \) independent of \( t \). However, the lower time-decay estimates are unknown.

- For the diffusive Oldroyd-B model with zero viscosity (i.e., \( \mu > 0 \) and \( \epsilon = 0 \)), there have been global well-posedness results and however time-decay estimates are unknown.

1.1 Main results

Denote by \( \hat{f}(\xi) \) the Fourier transform of \( f(x) \) throughout the paper. Now we are in a position to state our main results.

**Theorem 1.1.** (Global existence) Assume that \((u_0, \tau_0) \in H^3(\mathbb{R}^3)\). For any given \( \epsilon \) and \( \mu \) satisfying the Case I or Case II, then there exists a sufficiently small constant \( \epsilon_0 > 0 \) such that the Cauchy problem (1.1) admits a unique global solution \((u^{\epsilon, \mu}, \tau^{\epsilon, \mu}) \in C([0, +\infty); H^3(\mathbb{R}^3))\) satisfying

\[
\begin{align*}
\|u^{\epsilon, \mu}(t)\|_{H^3(\mathbb{R}^3)}^2 + \int_0^t \left( \|\nabla u^{\epsilon, \mu}(s)\|_{H^1(\mathbb{R}^3)}^2 + \|\nabla u^{\epsilon, \mu}(s)\|_{L^2(\mathbb{R}^3)}^2 \right) ds & \leq C_1 \epsilon_0^2, \\
\|\tau^{\epsilon, \mu}(t)\|_{H^3(\mathbb{R}^3)}^2 + \int_0^t \left( \|\nabla \tau^{\epsilon, \mu}(s)\|_{H^1(\mathbb{R}^3)}^2 + \|\tau^{\epsilon, \mu}(s)\|_{H^3(\mathbb{R}^3)}^2 \right) ds & \leq C_1 \epsilon_0^2
\end{align*}
\]  

(1.6)

for \( t \geq 0 \), provided that

\[
\|u_0\|_{H^3(\mathbb{R}^3)} + \|\tau_0\|_{H^3(\mathbb{R}^3)} \leq \epsilon_0.
\]
where \( \varepsilon_0 \) is a constant depending on \( \mu \) (or \( \varepsilon \)) and some other known constants but independent of \( \varepsilon \) (or \( \mu \)) and \( t \) for Case I (or for Case II), and the positive constant \( C_1 \) may depend on \( \mu \) (or \( \varepsilon \)) but independent of \( \varepsilon \) (or \( \mu \)) and \( t \) for Case I (or for Case II).

**Theorem 1.2.** (Optimal time-decay rates) For any given \( \varepsilon \) and \( \mu \) satisfying the Case I or Case II, letting \((u^{\varepsilon,\mu}, \tau^{\varepsilon,\mu})\) be the solution as in Theorem 1.1 then the following optimal time-decay estimates hold.

(i) Assume that \((u_0, \tau_0) \in L^1(\mathbb{R}^3)\). Then we have upper time-decay estimates of the solution as below:

\[
\|\nabla^k u^{\varepsilon,\mu}(t)\|_{L^2} \leq C_2(1 + t)^{-\frac{k}{2} - \frac{3}{2}}, \quad k = 0, 1, 2, \tag{1.7}
\]

and

\[
\|\nabla^{k_1} \tau^{\varepsilon,\mu}(t)\|_{L^2} \leq C_2(1 + t)^{-\frac{k_1}{2} - \frac{3}{2}}, \quad k_1 = 0, 1 \tag{1.8}
\]

for any \( t \geq 0 \) and generic positive constant \( C_2 \) which depends only on \( \|u_0, \tau_0\|_{H_0^1(\Omega)} \) and \( C_1 \).

(ii) Assume that \((u_0, \tau_0) \in L^1(\mathbb{R}^3)\) and in addition that \( \inf_{0 \leq r \leq R} |u_0| \geq c_0 > 0 \). Then there exist a positive time \( t_0 = t_0(\alpha, \kappa, \beta, \|u_0, \tau_0\|_{L^1(\Omega)}) \) and a positive generic constant \( c = c(\alpha, \kappa, \beta, c_0, C_2) \), such that

\[
\|\nabla^k u^{\varepsilon,\mu}(t)\|_{L^2} \geq c(1 + t)^{-\frac{3}{2} - \frac{k}{2}}, \quad k = 0, 1, 2, \tag{1.9}
\]

and

\[
\|\nabla^{k_1} \tau^{\varepsilon,\mu}(t)\|_{L^2} \geq c(1 + t)^{-\frac{3}{2} - \frac{k_1}{2}}, \quad k_1 = 0, 1 \tag{1.10}
\]

for any \( t \geq t_0 \).

**Remark 1.3.** Let \( A = \tau + I \) be the conformation tensor, and define the free energy entropic term by \( \int \text{tr}(A - \log(A) - I)dx \) (refer to [28] [3] for example). Note that

\[
\int \text{tr}(A - \log(A) - I)dx = \int \text{tr}(\tau - \log(\tau + I))dx = \int \text{tr}(\frac{\tau^2}{2}) + O(\tau^3))dx \approx \|\tau\|_{L^2}^2,
\]

for \( \|\tau\|_{L^\infty} \ll 1 \). As a consequence, the entropic term \( \int \text{tr}(A - \log(A) - I)dx \) has the same time-decay rates as \( \|\tau\|_{L^2}^2 \).

**1.2 Novelty of the main results**

(1.7)-(1.10) indicate that the time-decay rates are optimal, and that the decay speed of \( \tau \) is faster than \( u \) for the same order derivative. In particular, for the diffusive Oldroyd-B model with zero viscosity, we prove that the optimal rates are independent of the fluid viscosity \( \varepsilon \) when \( \mu > 0 \) is fixed. It demonstrates that the diffusive effect can contribute decay in the absence of viscosity, which is a different phenomenon from that for incompressible Navier-Stokes equations (cf. [47], [48] by Schonbek) and the incompressible Euler equations. For the non-diffusive Oldroyd-B model, the upper and lower time-decay rates of \( \|\nabla^3 u^{\varepsilon,\mu}\|_{L^2} \) and \( \|\nabla^2 \tau^{\varepsilon,\mu}\|_{L^2} \) can be improved to \( -\frac{\beta}{4} \) which are optimal (see Theorem 1.1) and improves Hieber-Wen-Zi’s result (see (1.5)) in [27]. In Theorem 1.2 we can not get optimal time-decay rates for the spatial derivative of the solution at the highest order for the diffusive Oldroyd-B model with zero viscosity, which is left open.

---

1 For the definition and properties of the matrix logarithm, one can refer to Section 2.3 in [25].
1.3 Ingredients of the proof

The results in this paper consist of two parts: (1) The global well-posedness, and (2) the optimal decay rates of the solution.

For part (1), the main difficulty is to pursue the dissipation of the velocity $u$. In particular for Case I, due to the lack of diffusion in $u$, the dissipation mechanism hidden in the system must be shown in a proper way. In contrast with [18, 19] where the authors detect a damping phenomenon on $\omega = \text{curl}u$ and define a new quantity $\Gamma = \omega - R\tau$ with $R(\cdot) = -(-\Delta)^{-1}\text{curl}(\text{div}(\cdot))$ to construct the energy estimates, our strategy is to first consider the auxiliary system of $(u, \sqrt{-\Delta}^{-1}\text{div}\tau)$ (see (2.2) in Section 2), where $P$ is the Larey projection operator. We are lucky to find that the auxiliary system (2.2) can handle Case I and Case II in a unified way, and the linear coupling term $\alpha \text{div}u$ plays an important role in the supplement of the dissipation of $u$. Furthermore, no additional conditions are required for any parameters. Thus we give a new proof for global existence in Case I compared with [18].

For part (2), to obtain the decay rates, we study the Green matrix of system (1.1) in Fourier variables. One of the novel part of this paper is that we find that in both Case I and Case II, the decay rate of $\tau$ is faster than that of $u$ for each derivatives up to 1. This phenomenon has not been discovered in previous works [27, 49]. Let us now explain this phenomenon more specifically in the linear level. As for part (1), the main difficulty lies in that the nonlinear term $Q$ in (1.1), the observation (4.24) enables us to offset the insufficient time decay of the nonlinear term $Q(\nabla u, \tau)$.
The rest of the paper are organized as follows. In Section 2 we give explicit expressions for the Fourier transform of Green matrix of the linearized system and an associated auxiliary system. Then the pointwise estimates on the low-frequency part of them are given, which are independent of the fluid viscosity and the center-of-mass diffusion coefficient. This will give some possibilities to achieve the desired results for the nonlinear system even if $\epsilon = 0$ or $\mu = 0$. In Section 3 we present the global existence-uniqueness theorem for strong solutions with small initial data, and sketch the proof. In Section 4, based on the energy estimates in Section 3 and the pointwise estimates on the low-frequency part of the Green matrix in Section 2 and by virtue of the Duhamel’s principle, we first get the time-decay estimates of the first and the second derivatives of the solutions in $L^2$ norm. Furthermore, we get $L^2$ decay estimates on the zero order and the third order derivatives of the solutions. Finally, we establish the lower bounds for the decay estimates by a delicate analysis on the eigenvalue. In Appendix (Section 5), for Case II we give optimal time-decay estimates for $\|\nabla^3 u_{\epsilon,\mu}\|_{L^2}$ and $\|\nabla^2 \tau_{\epsilon,\mu}\|_{L^2}$ at the same rate. However, it seems difficult to get sharper decay rate for $\|\nabla^3 \tau_{\epsilon,\mu}\|_{L^2}$, since the corresponding sharper decay for $\|\nabla^4 u_{\epsilon,\mu}\|_{L^2}$ is usually needed in our approach (see (4.21)).

2 Analysis of a linearized system

The main difficulty in exploring the optimal time-decay estimates of (1.1) with $\epsilon$ or $\mu$ vanishing is the lack of dissipation. In order to supplement it, we introduce an auxiliary system. Namely, applying the Leray projection operator $\mathbb{P}$ and the operator $\mathbb{P} \text{div}$ to (1.1) and (1.1), respectively, we obtain

$$\begin{aligned}
\partial_t u + \mathbb{P} (u \cdot \nabla u) - \epsilon \Delta u &= \mathbb{P} \text{div} \tau, \\
\partial_t (\mathbb{P} \text{div} \tau) + \mathbb{P} \text{div} (u \cdot \nabla \tau) - \mu \Delta (\mathbb{P} \text{div} \tau) + \beta \mathbb{P} \text{div} \tau &= \mathbb{P} \text{div} Q(\nabla u, \tau) + \frac{\alpha}{2} \Delta u.
\end{aligned} \quad (2.1)$$

Then, applying the operator $\Lambda^{-1} = (\sqrt{-\Delta})^{-1}$ to (2.1) and denoting by

$$\sigma \doteq \Lambda^{-1} \mathbb{P} \text{div} \tau$$

with $(\hat{\sigma})^j = i \left( \delta_{j,k} - \frac{\xi_j \xi_k}{|\xi|^2} \right) \hat{\xi}^l (\hat{r})^{l,k}$, one can rewrite (2.1) as follows

$$\begin{aligned}
\partial_t u - \epsilon \Delta u - \kappa \Lambda \sigma &= F_1, \\
\partial_t \sigma - \mu \Delta \sigma + \beta \sigma + \frac{\alpha}{2} \Lambda u &= F_2,
\end{aligned} \quad (2.2)$$

where the nonlinear terms are stated as below:

$$\begin{aligned}
F_1 &= -\mathbb{P} (u \cdot \nabla u), \\
F_2 &= -\Lambda^{-1} \mathbb{P} \text{div} (u \cdot \nabla \tau) + \Lambda^{-1} \mathbb{P} \text{div} Q(\nabla u, \tau).
\end{aligned}$$

2.1 Fourier transform of the Green matrix

We consider the linearized system of (1.1), i.e.,

$$\begin{aligned}
\partial_t u - \epsilon \Delta u &= \kappa \mathbb{P} \text{div} \tau, \\
\partial_t \tau - \mu \Delta \tau + \beta \tau &= \alpha \Delta u, \\
(u, \tau)(x, 0) &= (u_0, \tau_0),
\end{aligned} \quad (2.3)$$
and an auxiliary linear system which is the linearized equations of (2.2):

\[
\begin{align*}
\begin{cases}
\partial_t u - \epsilon \Delta u - \kappa \Lambda \sigma &= 0, \\
\partial_t \sigma - \mu \Delta \sigma + \beta \sigma + \frac{\kappa}{2} \Lambda u &= 0, \\
(u, \sigma)(x, 0) &= (u_0, \sigma_0)(x),
\end{cases}
\end{align*}
\]

(2.4)

where \( \sigma \pm \Lambda^{-1} \mathbf{P} \text{div} \tau \). Letting \( G_{u, \tau} \) and \( G_{u, \sigma} \) denote the Green matrixes of the systems (2.3) and (2.4), respectively, then we have the following expressions of Fourier transform of them.

**Lemma 2.1.** Fourier transform of the solution to the auxiliary system (2.4) can be solved as follows:

\[
\begin{align*}
\left( \hat{u} \right) &= \hat{G}_{u, \sigma}(\xi, t) \left( \hat{u}_0 \right),
\end{align*}
\]

with

\[
\hat{G}_{u, \sigma}(\xi, t) = \begin{pmatrix}
G_3(\xi, t) - \epsilon \| \xi \|^2 G_1(\xi, t)|_{t=0} & \kappa \| \xi \|^2 G_1(\xi, t)|_{t=0} \\
-\frac{\kappa}{2} \| \xi \|^2 G_1(\xi, t)|_{t=0} & G_2(t, \xi) + \epsilon \| \xi \|^2 G_1(\xi, t)|_{t=0}
\end{pmatrix},
\]

(2.5)

and

\[
\begin{align*}
G_1(\xi, t) &= \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-},
G_2(\xi, t) &= \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-},
G_3(\xi, t) &= \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-},
\end{align*}
\]

(2.6)

where \( \| \) is a 3 \times 3 unit matrix.

**Proof.** Applying Fourier transform to the linearized system (2.4), we arrive at

\[
\begin{align*}
\begin{cases}
\partial_t \hat{u} - \kappa \| \xi \|^2 \hat{\sigma} + \epsilon \| \xi \|^2 \hat{u} &= 0, \\
\partial_t \hat{\sigma} + (\mu \| \xi \|^2 + \beta) \hat{\sigma} + \frac{\kappa}{2} \| \xi \|^2 \hat{u} &= 0.
\end{cases}
\end{align*}
\]

(2.7)

On one hand, it follows from the first equation of (2.7) that

\[
\kappa \| \xi \|^2 \hat{\sigma} = \partial_t \hat{u} + \epsilon \| \xi \|^2 \hat{u}.
\]

(2.8)

On the other hand, multiplying (2.7) by \( \kappa |\xi|^2 \), one has

\[
\partial_t \left( \kappa |\xi|^2 \hat{\sigma} \right) + (\mu |\xi|^2 + \beta) \kappa |\xi|^2 \hat{\sigma} + \frac{\alpha \kappa}{2} |\xi|^2 \hat{u} = 0.
\]

(2.9)

Then, substituting (2.8) into (2.9), one obtains

\[
\partial_t \hat{u} + \left( \mu + \epsilon \right) |\xi|^2 \partial_t \hat{u} + |\xi|^2 \left[ \epsilon \left( \mu |\xi|^2 + \beta \right) + \frac{\alpha \kappa}{2} \right] \hat{u} = 0,
\]

(2.10)

which directly implies the corresponding characteristic equation:

\[
\lambda^2 + \left( \mu + \epsilon \right) |\xi|^2 \lambda + |\xi|^2 \left[ \epsilon \left( \mu |\xi|^2 + \beta \right) + \frac{\alpha \kappa}{2} \right] = 0,
\]

with roots \( \lambda_{\pm} \) satisfying

\[
\begin{align*}
\lambda_+ &= -\frac{(\mu + \epsilon) |\xi|^2 + \beta}{2} \pm \sqrt{\left( \mu + \epsilon \right) |\xi|^2 + \beta + \frac{\alpha \kappa}{2}}, \\
\lambda_+ + \lambda_- &= -\left( \mu + \epsilon \right) |\xi|^2 + \beta, \\
\lambda_+ \lambda_- &= |\xi|^2 \left[ \epsilon \left( \mu |\xi|^2 + \beta \right) + \frac{\alpha \kappa}{2} \right].
\end{align*}
\]

(2.11)
Therefore, (2.10) can be solved in the following form.

\[
\begin{align*}
    \hat{u}^j(t) &= c_1^j e^{\lambda_+^j t} + c_2^j e^{\lambda_-^j t}, \\
    \hat{u}^j(0) &= \hat{u}_0^j, \\
    \partial_t \hat{u}^j(0) &= \kappa |\xi|^2 \hat{\sigma}_0^j - \kappa |\xi|^2 \hat{\nu}_0^j,
\end{align*}
\]  

(2.12)

where

\[
c_1^j = \frac{(\lambda_+ + \eta |\xi|^2) \hat{u}_0^j - \kappa |\xi|^2 \hat{\sigma}_0^j}{\lambda_+ - \lambda_-}, \quad c_2^j = \frac{-(\lambda_+ + \eta |\xi|^2) \hat{u}_0^j + \kappa |\xi|^2 \hat{\sigma}_0^j}{\lambda_+ - \lambda_-}.
\]

In conclusion, we obtain

\[
\hat{u}^j(t) = \left( \frac{\lambda_+ e^{\lambda_+^j t} - \lambda_- e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} - \eta |\xi|^2 \frac{e^{\lambda_+^j t} - e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} \right) \hat{u}_0^j + \kappa |\xi|^2 \frac{e^{\lambda_+^j t} - e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} \hat{\sigma}_0^j.
\]  

(2.13)

Similar calculations conducted on (2.7) yield that

\[
\hat{\sigma}^j(t) = -\frac{\alpha}{2} |\xi|^2 \frac{e^{\lambda_+^j t} - e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} \hat{u}_0^j + \left[ \frac{\lambda_+ e^{\lambda_+^j t} - \lambda_- e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} + \eta |\xi|^2 \frac{e^{\lambda_+^j t} - e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} \right] \hat{\sigma}_0^j.
\]  

(2.14)

The proof of the lemma is complete. \(\square\)

From (2.13) and (2.14), we deduce the following corollary immediately.

**Corollary 2.2.** The explicit expression of \(\hat{G}_{\alpha,\tau}(\xi, t)\) is determined by

\[
\hat{u}^j(\xi, t) = \left( \frac{\lambda_+ e^{\lambda_+^j t} - \lambda_- e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} - \eta |\xi|^2 \frac{e^{\lambda_+^j t} - e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} \right) \hat{u}_0^j(\xi) \\
+ i \kappa \frac{e^{\lambda_+^j t} - e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} \left( \delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2} \right) \xi_i \hat{\tau}_{ij}^k(\xi),
\]

and

\[
\hat{\tau}_{ij}^k(\xi, t) = e^{-i(\beta + \mu |\xi|^2) t} \hat{\tau}_{ij}^k(\xi) + \frac{\alpha}{2} \left( \frac{e^{\lambda_+^j t} - e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} \right) \left( \xi_k \hat{u}_0^j + \xi_j \hat{u}_0^k \right) \\
- \left( e^{-i(\beta + \mu |\xi|^2) t} \frac{\lambda_+ e^{\lambda_+^j t} - \lambda_- e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} - \eta |\xi|^2 \frac{e^{\lambda_+^j t} - e^{\lambda_-^j t}}{\lambda_+ - \lambda_-} \right) \\
\times \frac{\xi_k \left( \delta_{jp} - \frac{\xi_j \xi_p}{|\xi|^2} \right) + \xi_j \left( \delta_{kp} - \frac{\xi_k \xi_p}{|\xi|^2} \right)}{|\xi|^2} \xi_i \hat{\tau}_{ij}^k(\xi).
\]

**Proof.** Conducting Fourier transform on both sides of (2.8) \textsuperscript{2}, we have

\[
\partial_t \hat{u}^j + (\beta + \mu |\xi|^2) \hat{u}^j = i \frac{\alpha}{2} \left( \xi_k \hat{u}^i + \xi_i \hat{u}^k \right),
\]

which implies

\[
\hat{\tau}_{ij}^k = e^{-i(\beta + \mu |\xi|^2) t} \hat{\tau}_{ij}^k(\xi) + \frac{\alpha}{2} e^{-i(\beta + \mu |\xi|^2) t} \int_0^t e^{i(\beta + \mu |\xi|^2) s} \left( \xi_k \hat{u}^i + \xi_i \hat{u}^k \right) ds.
\]  

(2.15)
Indeed, substituting (2.13) into the second term on the right-hand side of (2.15), and using (2.11), we have

\[
\int_0^t e^{(\beta + \mu k_t^2)s} \left( \xi_k \hat{u}^j + \xi_j \hat{\mu}^k \right) ds \\
= \left( \xi_k \hat{u}^j + \xi_j \hat{\mu}^k \right) \int_0^t \frac{\left( \lambda_+ + e|\xi|^2 \right) e^{\lambda_+ s} - \left( \lambda_- + e|\xi|^2 \right) e^{\lambda_- s}}{\lambda_+ - \lambda_-} d s \\
+ \kappa |\xi| \left( \xi_k \hat{\sigma}^j_0 + \xi_j \hat{\sigma}^k_0 \right) \int_0^t \frac{e^{(\beta + \mu k_t^2)s} e^{\lambda_+ s} - e^{\lambda_- s}}{\lambda_+ - \lambda_-} d s, 
\]

(2.16)

where

\[
I_1 = \left( \xi_k \hat{u}^j + \xi_j \hat{\mu}^k \right) \int_0^t \frac{\left( \lambda_+ + e|\xi|^2 \right) e^{\lambda_+ s} - \left( \lambda_- + e|\xi|^2 \right) e^{\lambda_- s}}{\lambda_+ - \lambda_-} e^{-(\lambda_+ + \lambda_- + e|\xi|^2)s} d s \\
= \left( \xi_k \hat{u}^j + \xi_j \hat{\mu}^k \right) \int_0^t \frac{\left( \lambda_+ + e|\xi|^2 \right) e^{-\lambda_+ s} - \left( \lambda_- + e|\xi|^2 \right) e^{-\lambda_- s}}{\lambda_+ - \lambda_-} d s \\
= \left( \xi_k \hat{u}^j + \xi_j \hat{\mu}^k \right) \left[ -e^{-(\lambda_+ + e|\xi|^2)s} + e^{-(\lambda_- + e|\xi|^2)s} \right]_0^t \lambda_+ - \lambda_- \\
= -e^{-(\lambda_+ s)} + e^{-(\lambda_- s)} \left( \xi_k \hat{\sigma}^j_0 + \xi_j \hat{\sigma}^k_0 \right),
\]

and thus

\[
i \frac{\alpha}{2} e^{-(\beta + \mu k_t^2)t} I_1 = i \frac{\alpha}{2} e^{-(\beta + \mu e|\xi|^2)t} \frac{e^{-\lambda_+ t} - e^{-\lambda_- t}}{\lambda_+ - \lambda_-} \left( \xi_k \hat{\sigma}^j_0 + \xi_j \hat{\sigma}^k_0 \right) \\
= i \frac{\alpha}{2} e^{(\lambda_+ + \lambda_-)t} \frac{e^{-\lambda_+ t} - e^{-\lambda_- t}}{\lambda_+ - \lambda_-} \left( \xi_k \hat{\sigma}^j_0 + \xi_j \hat{\sigma}^k_0 \right) \\
= i \frac{\alpha}{2} e^{\lambda_+ t} - e^{\lambda_- t} \left( \xi_k \hat{\sigma}^j_0 + \xi_j \hat{\sigma}^k_0 \right). 
\]

(2.17)

The further calculations about the term \( I_2 \) are based on a key observation that

\[
\left( \lambda_+ + e|\xi|^2 \right) \left( \lambda_- + e|\xi|^2 \right) = \frac{\alpha k}{2} |\xi|^2.
\]

Indeed,

\[
I_2 = \kappa |\xi| \left( \xi_k \hat{\sigma}^j_0 + \xi_j \hat{\sigma}^k_0 \right) \int_0^t e^{-(\lambda_+ + \lambda_- + e|\xi|^2)s} \frac{e^{\lambda_+ s} - e^{\lambda_- s}}{\lambda_+ - \lambda_-} d s \\
= 2 \frac{\xi_k \hat{\sigma}^j_0 + \xi_j \hat{\sigma}^k_0}{|\xi|} \left( \lambda_+ + e|\xi|^2 \right) \left( \lambda_- + e|\xi|^2 \right) \int_0^t e^{-(\lambda_+ + \lambda_- + e|\xi|^2)s} \frac{e^{\lambda_+ s} - e^{\lambda_- s}}{\lambda_+ - \lambda_-} d s \\
= 2 \frac{\xi_k \hat{\sigma}^j_0 + \xi_j \hat{\sigma}^k_0}{|\xi|} \lambda_+ - \lambda_- \left( \lambda_+ + e|\xi|^2 \right) e^{-(\lambda_+ + \lambda_- + e|\xi|^2)s} \frac{e^{\lambda_+ s} - e^{\lambda_- s}}{\lambda_+ - \lambda_-} \\
= 2 \frac{\xi_k \hat{\sigma}^j_0 + \xi_j \hat{\sigma}^k_0}{|\xi|} \lambda_+ - \lambda_- \left( \lambda_+ + e|\xi|^2 \right) e^{-(\lambda_+ + \lambda_- + e|\xi|^2)s} + \frac{e^{\lambda_+ s} - e^{\lambda_- s}}{\lambda_+ - \lambda_-} + e|\xi|^2 e^{\lambda_+ s} e^{\lambda_- s} e^{-\lambda_+ t} e^{-\lambda_- t}.
\]
Accordingly, 

$$\frac{\alpha}{2} e^{-((\beta + \mu + \epsilon) \xi^2)} I_2 = \frac{\xi_j \hat{\sigma}_j^i + \xi_i \hat{\sigma}_i^j}{|\xi|} \left( e^{-(\beta + \mu + \epsilon) \xi^2} + e^{-(\beta + \mu + \epsilon) |\xi|^2} \right) \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-} + e|\xi|^2 e^{\lambda_+ t} - e^{\lambda_- t}.$$  

Collecting (2.13), (2.15), (2.16), (2.17) and (2.18), and noticing the fact that

we finish the proof the corollary.

\[\square\]

2.2 The low-frequency part

Proposition 2.3. There exist positive constants \( R = R(\alpha, \kappa, \beta) \), \( \theta = \theta(\alpha, \kappa, \beta) \) and \( K = K(\alpha, \kappa, \beta) \) such that

\[
|\mathcal{G}_1(\xi, t)|, |\mathcal{G}_2(\xi, t)|, |\mathcal{G}_3(\xi, t)|, |\hat{\mathcal{G}}_{u,0}(\xi, t)|, |\hat{\mathcal{G}}_{u,1}(\xi, t)| \leq K e^{-\theta|\xi|^2 t}
\]  

(2.19)

hold for any \( |\xi| \leq R \) and \( t > 0 \).

Proof. Denote

\[
\Delta(|\xi|) = \left( \mu + \epsilon \right) |\xi|^2 + \beta \right)^2 - 4|\xi|^2 \left[ \epsilon \left( |\mu| |\xi|^2 + \beta \right) + \frac{\alpha \kappa}{2} \right].
\]

By a simple calculation, we obtain

\[
\begin{align*}
\Delta(|\xi|) &\leq \left( \mu + \epsilon \right) |\xi|^2 + \beta \right)^2 \leq (2R^2 + \beta)^2, \\
\Delta(|\xi|) &\geq \beta^2 - 4|\xi|^2 (1 + \beta + \frac{\alpha \kappa}{2}) \geq \frac{\beta^2}{2}
\end{align*}
\]

(2.20)

for \( |\xi| \leq R = \min \left\{ 1, \frac{\beta}{2 \sqrt{2 + 2\beta + \kappa \alpha}} \right\} \), where we use the assumption \( \epsilon, \mu \leq 1 \).

Next we rewrite \( \lambda_+ \) as

\[
\lambda_+ = -\frac{2|\xi|^2 \left[ \epsilon \left( |\mu| |\xi|^2 + \beta \right) + \frac{\alpha \kappa}{2} \right]}{(\mu + \epsilon)|\xi|^2 + \beta + \sqrt{\Delta(|\xi|)}}.
\]

(2.21)

Since (2.20) yields

\[
(\mu + \epsilon)|\xi|^2 + \beta + \sqrt{\Delta(|\xi|)} \leq 2R^2 + \beta + \sqrt{(2R^2 + \beta)^2} \leq 4R^2 + 2\beta,
\]

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for $|\xi| \leq R$, then we have
\[
\lambda_+ \leq -2 \left[ \frac{4R^2 + 2\beta}{\epsilon (\mu|\xi|^2 + \beta) + \frac{\alpha\kappa}{4R^2 + 2\beta}} \right] |\xi|^2 \leq -\frac{\alpha\kappa}{4R^2 + 2\beta} |\xi|^2.
\] (2.22)

Denote
\[
\theta = \frac{\alpha\kappa}{2R^2 + 2\beta}.
\]
Then by virtue of (2.22), we obtain the upper bound for $\lambda_+$ and $\lambda_-$, i.e.,
\[
\lambda_- \leq \lambda_+ \leq -\theta |\xi|^2, \quad \text{and} \quad |e^{\lambda t}| \leq e^{-\theta |\xi|^2 t}
\] (2.23)
for all $|\xi| \leq R$.

By virtue of (2.20), we have
\[
|\lambda_+ - \lambda_-| = \sqrt{\mathcal{E}(|\xi|)} \geq \frac{\sqrt{2}}{2} \beta,
\]
for $|\xi| \leq R$, which, together with (2.23), implies that
\[
|\mathcal{G}_i(\xi, t)| \leq C e^{-\theta |\xi|^2 t}, \quad \text{for all} \quad |\xi| \leq R.
\] (2.24)

Now we are in a position to estimate $\mathcal{G}_i(\xi, t)$ for $i = 2, 3$. We observe that
\[
\mathcal{G}_2(\xi, t) = \lambda_+ \mathcal{G}_1(\xi, t) + e^{\lambda_- t}, \quad \mathcal{G}_3(\xi, t) = -\lambda_+ \mathcal{G}_1(\xi, t) + e^{\lambda_+ t},
\]
and that
\[
|\lambda_+| \leq \frac{2 |\xi|^2 \left[ \epsilon \left( \mu |\xi|^2 + \beta \right) + \frac{\alpha\kappa}{2} \right]}{\beta} \leq \frac{2R^2 \left( \frac{R^2 + \beta + \frac{\alpha\kappa}{2}}{\beta} \right)}{\beta}.
\]
In addition, in view of (2.23) and (2.24), we immediately obtain the estimates of $\mathcal{G}_i(\xi, t)$ for $i = 2, 3$. Then the upper bound of $|\hat{\mathcal{G}}_{u,\sigma}(\xi, t)|$ and $|\hat{\mathcal{G}}_{u,\tau}(\xi, t)|$ follows. The proof of Proposition 2.3 is complete. \[\Box\]

3 Global existence and uniqueness

The global existence and uniqueness of strong solutions for either the case ($\epsilon > 0, \mu = 0$) or the case ($\mu > 0, \epsilon = 0$) has been achieved in [27] and [18], respectively. In fact, it is not difficult to obtain the global existence and uniqueness of strong solution to the system (1.1) for the last case ($\epsilon > 0, \mu > 0$). For the sake of completeness and that some a priori estimates will be used next sections, we sketch the proof for all the cases (i.e., $\epsilon > 0, \mu \geq 0$ and $\mu > 0, \epsilon \geq 0$). To begin with, we have the following local existence and uniqueness result.

**Proposition 3.1** (Local existence). Suppose that $(u_0, \tau_0) \in H^3(\mathbb{R}^3)$, then there exists a positive constant $T_0 = T_0(\kappa, \alpha, \beta, \mu)$ for Case I ($\mu > 0, \epsilon \geq 0$), or $T_0 = T_0(\kappa, \alpha, \beta, \epsilon)$ for Case II ($\epsilon > 0, \mu \geq 0$), such that, the system (1.1) exists a unique local solution $(u^{\epsilon, \mu}, \tau^{\epsilon, \mu}) \in C([0, T_0]; H^3(\mathbb{R}^3))$.

**Proof.** The proof can be done by using the standard contracting map theorem. Please refer for instance to [32]. \[\Box\]
Next, we give some \textit{a priori} estimates of system (1.1). The solutions usually should depend on $\varepsilon$ and $\mu$. For brevity, we omit the superscripts throughout the rest of the paper.

Assume for the moment that
\[
\sup_{0 \leq t \leq T} \|(u, \tau)\|_{H^3}(t) \leq \delta, \tag{3.1}
\]
for some $T \in (0, T^*)$ where $T^*$ is the maximal time of existence of the solutions as in Proposition 3.1 and the small constant $\delta \geq 4\varepsilon_0 > 0$ is determined by (3.8). The rest of this section tends to prove that
\[
\sup_{0 \leq t \leq T} \|(u, \tau)\|_{H^3}(t) \leq \frac{1}{2}\delta. \tag{3.2}
\]
Then, the global existence for system (1.1) can be deduced by using (3.2) and a standard continuity argument. With the regularity, the solution is unique.

Throughout the rest of the paper, we use the notation \(\cdot \lesssim \cdot\) to represent that \(\cdot \leq C\cdot\) for some generic known constants \(C > 0\) such as the constants yielded from the Sobolev’s inequality and the Young’s inequality. The notation \(\langle \cdot, \cdot \rangle\) represents the inner product in $L^2$ space.

\section{3.1 A priori estimate on the basic energy}

Firstly, applying $\nabla^k$ ($k = 0, 1, 2, 3$) to (1.1), and then multiplying (1.1)$_1$ by $\alpha \nabla^k u$ and (1.1)$_2$ by $\kappa \nabla^k \tau$, we have from integration by parts that
\[
\frac{1}{2} \frac{d}{dt} \left( \alpha \|u\|_{H^3}^2 + \kappa \|	au\|_{H^3}^2 \right) + \alpha \varepsilon \|\nabla u\|_{H^3}^2 + \kappa \mu \|\nabla \tau\|_{H^3}^2 + \kappa \beta \|	au\|_{H^3}^2
\]
\[
= -\alpha \sum_{k=0}^{3} \langle \nabla^k(u \cdot \nabla u), \nabla^k u \rangle - \kappa \sum_{k=0}^{3} \langle \nabla^k(u \cdot \nabla \tau), \nabla^k \tau \rangle + \kappa \sum_{k=0}^{3} \langle \nabla^k Q(\nabla u, \tau), \nabla^k \tau \rangle = \sum_{p=1}^{3} N_p, \tag{3.3}
\]
where we have used the following cancellation structure due to the symmetry of $\tau$ and integration by parts,
\[
\alpha \kappa \langle \nabla^k \Delta u, \nabla^k \tau \rangle = \alpha \kappa \langle \nabla^k u_{i,j}, \nabla^k \tau_{i,j} \rangle = -\alpha \kappa \langle \nabla^k u, \nabla^k \text{div} \tau \rangle.
\]

Now we are in a position to estimate $N_k$ term by term. By using the Hölder inequality, estimates on the commutator and the Sobolev imbedding inequality that $\|f\|_{L^{\infty}} \lesssim \|\nabla f\|_{H^1}$, one has
\[
N_1 = -\alpha \sum_{k=0}^{3} \left( \langle \nabla^k(u \cdot \nabla u), \nabla^k u \rangle - \langle (u \cdot \nabla)\nabla^k u, \nabla^k u \rangle \right)
\]
\[
\leq \alpha \sum_{k=0}^{3} \|\nabla^k(u \cdot \nabla u) - (u \cdot \nabla)\nabla^k u\|_{L^2} \|u\|_{H^3} \lesssim \alpha \|\nabla u\|_{L^{\infty}} \|\nabla u\|_{H^3} \|u\|_{H^3}
\]
\[
\leq \alpha \delta \|\nabla u\|_{H^2}^2.
\]
Similarly,

\[ N_2 = -\kappa \sum_{k=0}^{3} \left( < \nabla^k (u \cdot \nabla \tau), \nabla^k \tau > - < (u \cdot \nabla) \nabla^k \tau, \nabla^k \tau > \right) \]

\[ \leq \kappa \sum_{k=0}^{3} \left( ||\nabla^k (u \cdot \nabla \tau) - (u \cdot \nabla) \nabla^k \tau||_{L^2}^2 ||\tau||_{H^3} \right) \]

\[ \leq \kappa \left( ||\nabla u||_{L^\infty} ||\nabla \tau||_{H^2} + ||\nabla \tau||_{L^\infty} ||\nabla u||_{H^2} \right) ||\tau||_{H^3} \]

\[ \leq \kappa \delta \left( ||\nabla u||_{H^2}^2 + ||\nabla \tau||_{H^3}^2 \right). \]

We remark that the estimates on \( N_1 \) and \( N_2 \) imply that we need some supplement of dissipation independent of \( \epsilon \) or \( \mu \).

The discussion on \( N_3 \) will be more complicated, which is divided into two cases.

Case I (given \( \mu > 0 \)): Note that

\[ N_3 = \kappa < Q(\nabla u, \tau), \tau > -\kappa \sum_{k=1}^{3} < \nabla^{k-1} Q(\nabla u, \tau), \nabla^{k-1} \Delta \tau > . \]

Then one has

\[ N_3 \leq \kappa \left( ||\nabla u||_{L^\infty} ||\nabla \tau||_{L^2}^2 + ||Q(\nabla u, \tau)||_{H^2} ||\nabla \tau||_{H^3} \right) \]

\[ \leq \kappa \delta ||\nabla \tau||_{L^2}^2 + \left( ||\nabla u||_{L^\infty} ||\nabla \tau||_{H^2} + ||\nabla \tau||_{L^\infty} ||\nabla u||_{H^2} \right) ||\nabla \tau||_{H^3} \]

\[ \leq C \kappa \delta ||\nabla \tau||_{L^2}^2 + \frac{1}{4} \kappa \mu ||\nabla \tau||_{H^3}^2 + C \mu^{-1} \delta^2 ||\nabla u||_{H^2}^2. \]

Case II (given \( \epsilon > 0 \)): By using the Hölder inequality and estimates on the commutator, one has

\[ N_3 \leq \kappa \left( ||\nabla u||_{L^\infty} ||\nabla \tau||_{H^3} + ||\nabla \tau||_{L^\infty} ||\nabla u||_{H^2} \right) ||\nabla \tau||_{H^3} \]

\[ \leq \kappa ||\tau||_{H^2}^2 \delta ||\nabla u||_{H^3} \leq \kappa \delta ||\nabla u||_{H^3} \leq \frac{1}{4} \alpha \epsilon ||\nabla u||_{H^2}^2 + C \alpha^{-1} \epsilon^{-1} \kappa^2 \delta^2 ||\nabla \tau||_{H^3}^2. \]

In conclusion, we have

\[ \frac{1}{2} \frac{d}{dt} \left( \alpha ||u||_{H^2}^2 + \kappa ||\tau||_{H^3}^2 \right) + \alpha \epsilon ||u||_{H^2}^2 + \kappa \mu ||\nabla \tau||_{H^3}^2 + \kappa \beta ||\nabla \tau||_{H^3}^2 \]

\[ \leq \begin{cases} C \delta (\alpha + \kappa + \mu^{-1} \delta) ||\nabla (u, \tau)||_{H^2}^2 + \frac{1}{4} \kappa \mu ||\nabla \tau||_{H^3}^2 + C \delta ||\nabla \tau||_{L^2}^2, \text{ for given } \mu > 0; \\ C \delta (\alpha + \kappa) ||\nabla (u, \tau)||_{H^2}^2 + \frac{1}{4} \alpha \epsilon ||u||_{H^2}^2 + C \alpha^{-1} \epsilon^{-1} \kappa^2 \delta ||\nabla \tau||_{H^3}^2, \text{ for given } \epsilon > 0. \end{cases} \]  

(3.4)

### 3.2 Supplement of dissipation

Applying the operator \( \Lambda^k \) \((k = 1, 2, 3)\) to the (2.2)_1, and multiplying the resulting equation by \( \Lambda^{k-1} \sigma \), meanwhile, applying the operator \( \Lambda^{k-1} \) to the (2.2)_2 and multiplying the resulting equation by \( \Lambda^{k} u \), and
then, summing up all the results, one has

$$\partial_t \sum_{k=1}^{3} < \Lambda^{k-1} \sigma, \Lambda^k u > + \kappa \sum_{k=1}^{3} \|\Lambda^k \sigma\|_{L^2}^2 + \frac{\alpha}{2} \sum_{k=1}^{3} \|\Lambda^k u\|_{L^2}^2$$

$$= - \sum_{k=1}^{3} < \Lambda^{k-1} \mathcal{P} (u \cdot \nabla u), \Lambda^k \sigma > + \epsilon \sum_{k=1}^{3} < \Delta \Lambda^k u, \Lambda^{k-1} \sigma >$$

$$= \sum_{k=1}^{3} < \Lambda^{k-1} \mathcal{P} (u \cdot \nabla u), \Lambda^k \sigma > - \epsilon \sum_{k=1}^{3} < \Delta \Lambda^k u, \Lambda^{k-1} \sigma >$$

$$- \sum_{k=1}^{3} < \Lambda^{k-2} \mathcal{P} \text{div} (u \cdot \nabla \tau), \Lambda^k u > + \mu \sum_{k=1}^{3} < \Delta \Lambda^k u, \Lambda^{k-1} \sigma >$$

$$- \beta \sum_{k=1}^{3} < \Lambda^{k-1} \sigma, \Lambda^k u > + \sum_{k=1}^{3} < \Lambda^{k-2} \mathcal{P} \text{div} Q(\nabla u, \tau), \Lambda^k u >$$

$$= \sum_{i=1}^{6} J_i.$$

By using the Hölder inequality and estimates on commutator, one has

$$|J_1| + |J_3| + |J_6| \leq (\|u\|_{L^3} \|\nabla u\|_{L^6} + \|\nabla u\|_{L^3} \|u\|_{L^6}) \|\nabla \sigma\|_{L^2}$$

$$+ (\|u\|_{L^3} \|\nabla \tau\|_{H^2} + \|\nabla \tau\|_{L^3} \|u\|_{H^2}) \|\nabla u\|_{L^2}$$

$$+ (\|\nabla u\|_{L^2} \|\tau\|_{H^2} + \|\tau\|_{L^2} \|\nabla u\|_{H^2}) \|\nabla u\|_{H^2}$$

$$\leq \|u\|_{H^2} \|\nabla \sigma\|_{H^2} \|\nabla u\|_{H^2} + \|u\|_{H^2} \|\nabla \tau\|_{H^2} \|\nabla u\|_{H^2}$$

$$+ \|\nabla u\|_{H^2} \|\tau\|_{H^2} \|\nabla u\|_{H^2}$$

$$\leq \delta \left( \|\nabla u\|_{H^2}^2 + \|\nabla \tau\|_{H^2}^2 \right),$$

and

$$|J_2| + |J_4| + |J_5|$$

$$\leq \epsilon \|\nabla u\|_{H^3} \|\nabla \sigma\|_{H^2} + \mu \|\nabla \sigma\|_{H^1} \|\nabla u\|_{H^2} + \beta \|\sigma\|_{H^1} \|\nabla u\|_{H^2}$$

$$\leq \epsilon_1 \kappa \|\nabla \sigma\|_{H^2} + C \epsilon_1^{-1} \kappa^{-1} \epsilon \|\nabla u\|_{H^2}^2 + \epsilon_1 \alpha \|\nabla u\|_{H^2}^2 + C \epsilon_1^{-1} \alpha^{-1} \mu \|\nabla \sigma\|_{H^2}^2$$

$$+ \epsilon_1 \alpha \|\nabla u\|_{H^2}^2 + C \epsilon_1^{-1} \alpha^{-1} \beta^2 \|\sigma\|_{H^2}^2.$$

In conclusion, by choosing $\epsilon_1$ sufficiently small, one has

$$\partial_t \sum_{k=1}^{3} < \Lambda^{k-1} \sigma, \Lambda^k u > + \kappa \sum_{k=1}^{3} \|\Lambda^k \sigma\|_{L^2}^2 + \frac{\alpha}{4} \sum_{k=1}^{3} \|\Lambda^k u\|_{L^2}^2$$

$$\leq \kappa^{-1} \epsilon \|\nabla u\|_{H^3}^2 + \alpha^{-1} \mu \|\nabla \tau\|_{H^2}^2 + \alpha^{-1} \beta^2 \|\tau\|_{H^2}^2 + \delta \|\nabla (u, \tau)\|_{H^2}^2.$$
3.3 Total energy

Combining the discussions in the previous two parts, one obtains the following total energy estimates from zero-order to third-order terms,

\[
\frac{d}{dt} E(t) + \alpha \epsilon \|\nabla u\|_{H^1}^2 + \kappa \mu \|\nabla \tau\|_{H^1}^2 + \kappa \beta \|\tau\|_{H^1}^2 + \epsilon_2 \sum_{i=1}^3 \left( \frac{k_i}{2} \|\Lambda^i \sigma\|_{L^2}^2 + \frac{\alpha}{4} \|\Lambda^i u\|_{L^2}^2 \right) \\
\leq C \epsilon_2 \left( \kappa^{-1} \epsilon \|\nabla u\|_{H^2}^2 + \alpha^{-1} \mu \|\nabla \tau\|_{H^2}^2 + \alpha^{-1} \beta^2 \|\tau\|_{H^2}^2 \right) \\
+ \left\{ C \delta (\alpha + \kappa + \mu^{-1} \delta + \epsilon_2 ) \|\nabla (u, \tau)\|_{H^2}^2 + \frac{1}{2} \kappa \mu \|\nabla \tau\|_{H^2}^2 + C \kappa \delta \|\tau\|_{L^2}^2, \text{ for given } \mu > 0, \right. \\
\left. C \delta (\alpha + \kappa + \epsilon_2 ) \|\nabla (u, \tau)\|_{H^2}^2 + \frac{1}{4} \alpha \epsilon \|\nabla u\|_{H^3}^2 + C \alpha^{-1} \epsilon^{-1} \kappa^2 \delta^2 \|\tau\|_{H^3}^2, \text{ for given } \epsilon > 0, \right. \\
\right\}
\]

where

\[
E(t) = \frac{1}{2} \left( \alpha \|u\|_{H^1}^2 + \kappa \|\tau\|_{H^1}^2 \right) + \epsilon_2 \sum_{k=1}^3 < \Lambda^{k-1} \sigma, \Lambda^k u >= O(\tilde{E}(t)),
\]

by using the Young’s inequality for some small positive \( \epsilon_2 = \epsilon_2(\alpha, \kappa, \beta) \), and \( \tilde{E}(t) = \|(u, \tau)\|_{H^3}^2 \).

By further choosing suitable \( \epsilon_2 \) and \( \delta \) which depend only on \( \alpha, \kappa, \beta \) and \( \epsilon \) for the case given \( \epsilon > 0 \), and on \( \alpha, \kappa, \beta \) and \( \mu \) for the case given \( \mu > 0 \), we have

\[
\frac{d}{dt} E(t) \leq 0.
\]

Then, by integrating the above inequality over \([0, t]\) and omitting some discussion on the generic constant for simplicity, one can get

\[
(\alpha \|u\|_{H^1}^2 + \kappa \|\tau\|_{H^1}^2) (t) \leq 2 (\alpha \|u_0\|_{H^1}^2 + \kappa \|\tau_0\|_{H^1}^2) \leq 2(\alpha + \kappa) \epsilon_0^2.
\]

Note also that

\[
(\alpha \|u\|_{H^1}^2 + \kappa \|\tau\|_{H^1}^2) (t) \geq \min(\alpha, \kappa)\|(u, \tau)\|_{H^1}(t).
\]

Therefore, choosing

\[
\frac{2(\alpha + \kappa) \epsilon_0^2}{\min(\alpha, \kappa)} < \frac{\delta}{2}.
\]

The proof of (3.2) is complete. \( \square \)

With (3.2) and (3.7), it is easy to obtain (1.6).

4 Decay estimates for the nonlinear system

The aim of this section is to establish the upper and lower decay rates of the solution to the system (1.1). For simplicity, let us denote \( U = (u, \tau)^T \).

4.1 Upper time-decay estimates

The proof will be achieved in the following several steps.

**Step 1: First order decay**

We first give the decay estimates for the low-frequency part of \( U \).
Lemma 4.1. For the nonlinear terms of (1.1), we directly have the following estimates.

\begin{align}
\|M_1\|_{L^1} &\leq \|u\|_{L^2}\|\nabla u\|_{L^2}, \\
\|M_2\|_{L^1} &\leq \|u\|_{L^2}\|\nabla \tau\|_{L^2} + \|\tau\|_{L^2}\|\nabla u\|_{L^2}, \\
\|M_1\|_{L^2} &\leq \|\nabla u\|_{H^1}\|\nabla u\|_{L^2}, \\
\|M_2\|_{L^2} &\leq \|\nabla u\|_{H^1}\|\nabla \tau\|_{L^2} + \|\nabla \tau\|_{H^1}\|\nabla u\|_{L^2},
\end{align}

where

\[ M_1 = -F(u \cdot \nabla u), \]
\[ M_2 = -u \cdot \nabla \tau + Q(\nabla u, \tau). \]

Moreover, the following time-decay estimate for the low-frequency part of the solution to the nonlinear system (1.1), i.e.,

\[ \left( \int_{|\xi| \leq R} |\xi|^{2k} |\hat{U}_0|^2 \, d\xi \right)^{\frac{1}{2}} \leq (1 + t)^{-\frac{3}{4} - \frac{1}{q} - \frac{1}{2}} \|U_0\|_{L^p} \]
\[ + \int_0^t (1 + t - s)^{-\frac{3}{8} + \frac{1}{2} - \frac{1}{2}} \| (M_1, M_2)^\top (s) \|_{L^p} \, ds, \]

holds for all \( t > 0 \), where \( q \geq 2, \frac{1}{p} + \frac{1}{q} = 1 \) and \( k \in [0, 3] \).

Proof. We only prove (4.5). From the Duhamel’s principle, we have

\[ U(t) = \mathcal{G}_{u, \tau}(t) * U_0 + \int_0^t \mathcal{G}_{u, \tau}(t - s) * (M_1, M_2)^\top (s) \, ds. \]

It follows from Proposition 2.3 and Hausdorff-Young inequality that

\[ \left( \int_{|\xi| \leq R} |\xi|^{2k} |\hat{U}_0|^2 \, d\xi \right)^{\frac{1}{2}} \leq \|\hat{U}_0\|_{L^q} \left( \int_{|\xi| \leq R} (|\xi|^{2k} e^{-\theta |\xi|^2}) \frac{d\xi}{|\xi|^2} \right)^{\frac{1}{2 \theta}} \]
\[ \leq \|U_0\|_{L^p} \left( \int_{|\xi| \leq R} (|\xi|^{2k} e^{-\theta |\xi|^2}) \right)^{\frac{1}{2 \theta}} \leq \|U_0\|_{L^p} (1 + t)^{-\frac{3}{4} - \frac{1}{q} - \frac{1}{2}}, \]

where \( q \geq 2, \frac{1}{p} + \frac{1}{q} = 1 \). Then, by similar calculations, we get

\[ \left( \int_{|\xi| \leq R} |\xi|^{2k} |\hat{U}(t)|^2 \, d\xi \right)^{\frac{1}{2}} \leq (1 + t)^{-\frac{3}{4} - \frac{1}{q} - \frac{1}{2}} \|U_0\|_{L^p} + \int_0^t (1 + t - s)^{-\frac{3}{8} + \frac{1}{2} - \frac{1}{2}} \| (M_1, M_2)^\top (s) \|_{L^p} \, ds, \]

where \( q \geq 2, \frac{1}{p} + \frac{1}{q} = 1 \). The proof is complete. \( \Box \)
Lemma 4.2. Under the assumptions of Part (i) in Theorem 1.2, it holds that

$$\sum_{1 \leq k \leq 3} \| \nabla^k (u, \tau)(t) \|_{L^2} \leq C(1 + t)^{-\frac{1}{2}}$$

for any $t \geq 0$.

Proof. It follows from a similar argument as (3.3) for the case $k = 1, 2, 3$ and (3.5) for the case $k = 2, 3$, one has,

$$\frac{d}{dt} \mathcal{H}(t) + \alpha \varepsilon \| \nabla^2 u \|^2_{H^2} + \kappa \mu \| \nabla^2 \tau \|^2_{H^2} + \kappa \beta \| \nabla \tau \|^2_{H^2} + \varepsilon_3 \sum_{i=2}^{3} \left( \frac{\kappa}{2} \| \Lambda^i \|_{L^2}^2 + \frac{\alpha}{4} \| \Lambda^i u \|^2_{L^2} \right) \leq C \varepsilon_3 \left( \kappa^2 \| \nabla^2 u \|^2_{H^2} + \alpha^{-1} \varepsilon \| \nabla^2 \tau \|^2_{H^2} + \alpha^{-1} \beta^2 \| \nabla \tau \|^2_{H^2} \right) + \left( C \delta \| \nabla (u, \tau) \|^2_{H^2} + \frac{1}{4} \kappa \mu \| \nabla^2 \tau \|^2_{H^2}, \text{ given } \mu > 0, \right)$$

where

$$\mathcal{H}(t) = \frac{1}{2} \left( \alpha \| \nabla^3 u \|^2_{H^2} + \kappa \| \nabla \tau \|^2_{H^2} \right) + \varepsilon_3 \sum_{i=2}^{3} < \Lambda^k \| \sigma, \Lambda^k u >= O(\mathcal{H}(t)), \right.$$

by using the Young’s inequality for some small positive $\varepsilon_3 = \varepsilon_3(\alpha, \kappa, \beta)$, and $\mathcal{H}(t) = \| \nabla (u, \tau) \|^2_{H^2}$.

By further choosing suitable $\varepsilon_3$ and $\delta$ which depend only on $\alpha, \kappa, \beta$ and $\varepsilon$ for the case given $\varepsilon > 0$, and on $\alpha, \kappa, \beta$ and $\mu$ for the case given $\mu > 0$, we have

$$\frac{d}{dt} \mathcal{H}(t) + \frac{1}{2} \left( \kappa \beta \| \nabla \tau \|^2_{H^2} + \frac{\alpha \varepsilon_3}{4} \sum_{i=2}^{3} \| \Lambda^i u \|^2_{L^2} \right) \leq \delta \| \nabla u \|^2_{L^2}. \quad (4.8)$$

Using the Plancherel’s theorem, and splitting the integral into two parts, we have

$$\| \Lambda^2 u \|^2_{L^2} = \| \nabla^2 u \|^2_{L^2} = \int_{|\xi| \lesssim R} |\xi|^4 |\hat{u}|^2 d\xi + \int_{|\xi| \gg R} |\xi|^4 |\hat{u}|^2 d\xi \geq C \int_{|\xi| \gg R} |\xi|^2 |\hat{u}|^2 d\xi$$

for some given radius $R > 0$. Similarly, we have

$$\| \Lambda^3 u \|^2_{L^2} \geq C \int_{|\xi| \gg R} |\xi|^2 |\hat{u}|^2 d\xi.$$
Adding $\frac{4\alpha}{16} \int_{|\xi| \leq R} |\xi|^2 |\hat{u}|^2 d\xi$ on both sides of (4.9), one has
\[
\frac{d}{dt} \tilde{H}(t) + D \tilde{\mathcal{H}}(t) \lesssim \int_{|\xi| \leq R} |\xi|^2 |\hat{u}(t)|^2 d\xi
\tag{4.10}
\]
for some positive constant $D = \min \left\{ \frac{\kappa \beta}{\alpha}, \frac{\alpha}{16} \right\}$. Define
\[
\mathcal{M}(t) = \sup_{0 \leq s \leq t} (1 + s)^{\frac{5}{4}} \sqrt{\tilde{H}(s)}.
\]
Notice that $\mathcal{M}(t)$ is non-decreasing and
\[
\sqrt{\tilde{H}(s)} \lesssim (1 + s)^{-\frac{1}{2}} \sqrt{\mathcal{M}(t)}, \quad 0 \leq s \leq t.
\]
Then, we get from Lemma 4.1 and the definition of $\tilde{H}(t)$ that
\[
\left( \int_{|\xi| \leq R} |\xi|^2 |\hat{U}(t)|^2 d\xi \right)^{\frac{1}{2}} \lesssim (1 + t)^{-\frac{1}{4}} \|U_0\|_{L^1} + \delta \int_0^t (1 + t - \tau)^{-\frac{1}{4}} (1 + \tau)^{-\frac{1}{2}} d\tau \sqrt{\mathcal{M}(t)}
\tag{4.11}
\]
From (4.10) and (4.11), we have
\[
\tilde{H}(t) \lesssim e^{-Dt} \tilde{H}(0) + \int_0^t e^{-D(t-\tau)} \left( \int_{|\xi| \leq R} |\xi|^2 |\hat{U}(\tau)|^2 d\xi \right) d\tau
\lesssim e^{-Dt} \tilde{H}(0) + \int_0^t e^{-D(t-\tau)} (1 + \tau)^{-\frac{1}{2}} \left[ \|U_0\|_{L^1}^2 + \delta^2 \mathcal{M}(t) \right] d\tau
\tag{4.12}
\lesssim (1 + t)^{-\frac{1}{4}} \left[ \tilde{H}(0) + \|U_0\|_{L^1}^2 + \delta^2 \mathcal{M}(t) \right].
\]
Using the definition of $\mathcal{M}(t)$ and the smallness of $\delta$, we have
\[
\mathcal{M}(t) \leq C \left( \tilde{H}(0) + \|U_0\|_{L^1}^2 \right),
\]
which implies that
\[
\tilde{H}^2(t) \lesssim (1 + t)^{-\frac{1}{2}} \left( \tilde{H}(0) + \|U_0\|_{L^1}^2 \right) \leq C_2 (1 + t)^{-\frac{1}{2}}.
\tag{4.13}
\]
The proof of Lemma 4.2 is complete. \( \Box \)

**Step 2: Zero order and the second order decay**

**Lemma 4.3.** Under the assumptions of Part (i) in Theorem 1.2, it holds that
\[
\| (u, \tau)(t) \|_{L^2} \leq C (1 + t)^{-\frac{1}{4}},
\tag{4.14}
\]
and
\[
\sum_{2 \leq k \leq 3} \| \nabla^k (u, \tau)(t) \|_{L^2} \leq C (1 + t)^{-\frac{3}{4}},
\tag{4.15}
\]
for any $t \geq 0$. 

In conclusion, we have for some positive constant \(D_0 = D_0(C_2)\) and

\[
\mathcal{H}_0(t) = \frac{1}{2} \left( \alpha \| u \|_{H^3}^2 + k \| \tau \|_{H^3}^2 \right) + \epsilon_4 \sum_{k=1}^{3} < \Lambda^{k-1} \sigma, \Lambda^k u > = O(\| U(t) \|_{H^3}^2),
\]

by using the Young’s inequality for some small positive \(\epsilon_4 = \epsilon_4(C_2)\). Then, it follows from Lemma 4.1 and (4.13) that

\[
\| U(t) \|_{H^3}^2 \leq e^{-D_0 t} \| U_0 \|_{H^3}^2 + \int_0^t e^{-D_0(t-\tau)} \left( \int_{|\xi| \leq R} |\hat{U}(\tau)|^2 d\xi \right) d\tau
\]

\[
\leq e^{-D_0 t} \| U_0 \|_{H^3}^2 + \int_0^t e^{-D_0(t-\tau)} (1 + \tau)^{-\frac{3}{2}} \| U_0 \|_{L^2}^2 d\tau
\]

\[
+ \int_0^t e^{-D_0(t-\tau)} \left( \int_0^\tau (1 + \tau - s)^{-\frac{3}{2}} (1 + s)^{-\frac{3}{2}} ds \right)^2 d\tau
\]

\[
\leq C_2(1 + t)^{-\frac{3}{4}}.
\]

In conclusion, we have

\[
\| U(t) \|_{L^2} \leq C_2(1 + t)^{-\frac{3}{4}}.
\]

For the second order decay, by a similar discussion as (4.10), one obtains

\[
\frac{d}{dt} \mathcal{H}_1(t) + D_1 \| \nabla^2 (u, \tau) \|_{H^3}^2 \leq \int_{|\xi| \leq R} |\xi|^4 |\hat{u}(t)|^2 d\xi,
\]

for some positive constant \(D_1 = D_1(C_2)\) and

\[
\mathcal{H}_1(t) = \alpha \| \nabla^2 u \|_{H^3}^2 + k \| \nabla^2 \tau \|_{H^3}^2 + \epsilon_5 < \Lambda^2 \sigma, \Lambda^3 u > = O(\| \nabla^2 (u, \tau) \|_{H^3}^2),
\]

where we have used the Young’s inequality for some small positive \(\epsilon_5 = \epsilon_5(C_2)\). Then, it follows from Lemma 4.1 and (4.18) that

\[
\| \nabla^2 (u, \tau)(t) \|_{H^3}^2 \leq e^{-D_1 t} \| U_0 \|_{H^3}^2 + \int_0^t e^{-D_1(t-s)} \left( \int_{|\xi| \leq R} |\xi|^4 |\hat{u}(s)|^2 d\xi \right) ds
\]

\[
\leq e^{-D_1 t} \| U_0 \|_{H^3}^2 + \int_0^t e^{-D_1(t-s)} (1 + s)^{-\frac{3}{2}} \| U_0 \|_{L^2}^2 ds
\]

\[
+ \int_0^t e^{-D_1(t-s)} \int_0^s (1 + s - s')^{-\frac{3}{2}} (1 + s')^{-\frac{3}{2}} ds' ds
\]

\[
\leq C_2(1 + t)^{-\frac{3}{4}}.
\]

Combining (4.17) with (4.19), we complete the proof of Lemma 4.3.

\[\square\]

**Step 3: Further decay estimates**
Lemma 4.4. Under the assumptions of Part (i) in Theorem 1.2, it holds that
\[ \|\nabla^k \tau\|_{L^2} \leq C(1 + t)^{-\frac{k}{2} + \frac{\beta}{2}}, \quad k = 0, 1, \] (4.20)
for any \( t \geq 0 \).

Proof. Applying \( \nabla^k \) \((k = 0, 1)\) to \((4.1)\), multiplying the result by \( \nabla^k \tau \), and integrating with respect to \( x \), we have from Lemmas 4.2 and 4.3 that
\[ \frac{1}{2} \frac{d}{dt} \|\nabla^k \tau\|_{L^2}^2 + \mu \|\nabla^{k+1} \tau\|_{L^2}^2 + \frac{\beta}{2} \|\nabla^k \tau\|_{L^2}^2 \leq \alpha \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^k Q(\nabla u, \tau)\|_{L^2}^2 + \|\nabla^k (u \cdot \nabla \tau)\|_{L^2}^2 \]
\[ \leq \alpha \|\nabla^{k+1} u\|_{L^2}^2 + \|\nabla^k Q(\nabla u, \tau)\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\tau\|_{L^\infty}^2 \]
\[ + \|\nabla^{k+1} \tau\|_{L^1}^2 + \|\tau\|_{L^2}^2 + \|\nabla^k u\|_{L^\infty}^2, \]
(4.21)
which implies that
\[ \frac{d}{dt} \|\nabla^k \tau\|_{L^2}^2 + \beta \|\nabla^k \tau\|_{L^2}^2 \leq (1 + t)^{-(\frac{k}{2} + \beta)} \]
for \( k = 0, 1 \). This completes the proof of Lemma 4.4 \( \square \)

Now the proof of Part (i) in Theorem 1.2 is accomplished by Lemmas 4.2-4.4.

4.2 Lower time-decay estimates

To prove Part (ii) in Theorem 1.2, we begin this subsection with giving some refined estimates for the Fourier transform of the Green function.

Lemma 4.5. Let \( R \) and \( \theta \) be the constants chosen in Proposition 2.3. There exist three positive constants \( \eta = \eta(\alpha, \beta, \kappa) \), \( c_1 = c_1(\alpha, \kappa, \beta) \) and \( \tilde{c}_1 = c_1(\alpha, \kappa, \beta) \), and a positive time \( t_1 = t_1(\alpha, \kappa, \beta) \), such that
\[ |G_1(\xi, t)| \geq c_1 e^{-\eta |\xi|^2 t}, \quad |G_3(\xi, t)| \geq c_1 e^{-\eta |\xi|^2 t}, \quad \text{for all } |\xi| \leq R \text{ and } t \geq t_1. \] (4.23)
Moreover, \( G_2 \) admits the following refined estimate
\[ |G_2(\xi, t)| \leq \tilde{c}_1 \left( \xi^2 e^{-\eta |\xi|^2 t} + e^{-\frac{\beta}{2}} \right), \quad \text{for all } |\xi| \leq R \text{ and } t \geq t_1. \] (4.24)

Proof. Noting first that \( \lambda_+ - \lambda_- = \sqrt{\tilde{c}(|\xi|)} \), based on the analysis in the proof of Proposition 2.3 for all \( |\xi| \leq R \), there holds
\[ \frac{\sqrt{2}}{2} \beta \leq \lambda_+ - \lambda_- \leq 2R^2 + \beta. \] (4.25)
Therefore, it follows from (2.21) that
\[ \lambda_+ \geq \frac{2}{\beta |\xi|^2} \left[ \epsilon \left( \mu |\xi|^2 + \beta \right) + \frac{\alpha \kappa}{2} \right] \geq -\frac{2}{\beta} \left( R^2 + \beta + \frac{\alpha \kappa}{2} \right) |\xi|^2 =: -\eta |\xi|^2, \]
Then it follows from (4.25) and (4.29) that
\[ |e^{\lambda_+ t} - e^{\lambda_- t}| \leq |e^{\lambda_+ t}(1 - e^{-(\lambda_+ - \lambda_-) t})| \geq \frac{1}{2} e^{-\eta |\xi|^2 t}, \quad (4.26) \]
and
\[ |\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}| \geq |\lambda_+ - \lambda_-| e^{-\theta |\xi|^2 t}. \quad (4.27) \]
Accordingly, thanks to (2.6), (4.25), (4.26) and (4.27), choosing a positive generic constant $c_1$ such that
\[ c_1 = \min \left\{ \frac{1}{2(2R^2 + \beta)}, 1 \right\}, \quad (4.28) \]
we finish the proof of (4.23).

Next, we infer from (2.11) and (4.25) that, for $|\xi| \leq R$
\[ \lambda_- \leq \frac{-\beta}{2}, \quad \lambda_- \leq (\mu + e)|\xi|^2/2 + \beta \leq R^2 + \beta, \]
\[ \lambda_+ \leq -\frac{\alpha \kappa}{2R^2 + 2\beta}|\xi|^2, \quad |\lambda_+| \leq \frac{2|\xi|^2(R^2 + \beta + \alpha \kappa)}{\beta}. \quad (4.29) \]

Then it follows from (4.25) and (4.29) that
\[ |G_2| \leq \lambda_+ \int e^{\lambda_+ t} + \frac{\lambda_-}{\lambda_+ - \lambda_-} e^{\lambda_- t} \leq \tilde{c}_1 \left( |\xi|^2 e^{-\theta |\xi|^2 t} + e^{-\frac{\beta}{2}} \right), \]
where
\[ \tilde{c}_1 = \max \left\{ \frac{2 \sqrt{\beta}(2R^2 + \beta + \alpha \kappa)}{\beta^2}, \frac{\sqrt{2}(R^2 + \beta)}{\beta} \right\}. \]
The proof of (4.24) is complete. \qed

Then, we have the lower decay bounds for the linear system. In the following, denote by
\[ G_{u,\sigma}(t) := (G_{1,\sigma}^1(t), G_{1,\sigma}^2(t))^T. \]

Lemma 4.6. Under the assumptions of Part (ii) in Theorem 4.2 there exist a positive time $t_2$ and a positive generic constant $c_2$, such that
\[ \|\nabla^k \left( G_{1,\sigma}^1(t) * U_{\sigma}^0 \right) \|_{L^2} \geq c_2(1 + t)^{-\frac{3}{2} - \frac{k}{2}}. \quad (4.30) \]
\[ \|\nabla^k \left( G_{1,\sigma}^2(t) * U_{\sigma}^0 \right) \|_{L^2} \geq c_2(1 + t)^{-\frac{3}{2} - \frac{k}{2}}. \quad (4.31) \]
for all $t \geq t_2$, and $k = 0, 1, 2, \cdots$, where $t_2 = t_2(\alpha, \kappa, \beta, ||(u_0, \sigma_0)||_{L^1}), c_2 = c_2(\alpha, \kappa, \beta, c_0, ||(u_0, \sigma_0)||_{L^1}), U_{\sigma} = (u, \sigma)^T$ and $U_{\sigma}^0 = (u_0, \sigma_0)^T$.

Proof. First of all, by Plancherel’s theorem, we have
\[ \|\nabla^k \left( G_{1,\sigma}^1(t) * U_{\sigma}^0 \right) \|_{L^2} = \left\| [e^{i|\xi|^2 \theta t} G_3(\xi, t) - e^{i|\xi|^2 \theta t} G_1(\xi, t) \hat{u}_0(\xi) + \kappa |\xi|^{1+k} G_1(\xi, t) \hat{\sigma}_0(\xi)] \right\|_{L^2} \]
\[ \geq \left( \int_{|\xi| \leq R} |\xi|^{2k} |G_3(\xi, t) \hat{u}_0(\xi)|^2 \, d\xi \right)^{\frac{1}{2}} \]
\[ - \left( \int_{|\xi| \leq R} (e^{i|\xi|^2 \theta t} |G_1(\xi, t)||\hat{u}_0(\xi)| + \kappa |\xi|^{1+k} |G_1(\xi, t)||\hat{\sigma}_0(\xi)|)^2 \, d\xi \right)^{\frac{1}{2}} \]
\[ = I_1 - I_2. \]
Using the Lemma 4.5 we have, for $t \geq t_1$,

$$I_1 \geq c_1 \inf_{|\xi| \leq R} |\hat{u}_0(\xi)| \left( \int_{|\xi| \leq R} |\xi|^{2k} e^{-2\eta|\xi|^2 t} d\xi \right)^{\frac{1}{2}} \geq c_1 c_0 \left( \frac{1}{\sqrt{\pi t}} \right) \geq (1 + t)^{-\frac{3}{2} - \frac{1}{2}}. \quad (4.33)$$

On the other hand, applying Proposition 2.3 yields

$$I_2 \leq ||(\hat{u}_0, \hat{\sigma}_0)||_{L^0} \left( \int_{|\xi| \leq R} \left( |\xi|^{4+2k} + \kappa^2 |\xi|^{2+2k} \right) |G_1(\xi, t)|^2 d\xi \right)^{\frac{1}{2}} \leq ||(u_0, \sigma_0)||_{L^1} \left( \int_{|\xi| \leq R} |\xi|^{2+2k} e^{-2\eta|\xi|^2 t} d\xi \right)^{\frac{1}{2}} \leq (1 + t)^{-\frac{3}{2} - \frac{1}{2}}. \quad (4.34)$$

Next, using Plancherel’s theorem again, we are led to

$$\|\nabla^k \left( \mathcal{G}_{\mu,\sigma}(t) \ast U_0^\sigma \right) \|_{L^2} = \left\| -\frac{\alpha}{2} |.|^{1+k} G_1(., t) \hat{u}_0(\cdot) + [.||^{1+k} G_2(., t) + \varepsilon |.|^{2+k} G_1(., t)] \hat{\sigma}_0(\cdot) \right\|_{L^2} \geq \frac{\alpha}{2} \left( \int_{|\xi| \leq R} |\xi|^{2+2k} |G_1(\xi, t)|^2 |\hat{u}_0(\xi)|^2 d\xi \right)^{\frac{1}{2}} \geq \frac{\alpha}{2} \left( \int_{|\xi| \leq R} \left( |\xi|^{4+k} |G_2(\xi, t)| + \varepsilon |\xi|^{2+k} |G_1(\xi, t)| \right)^2 |\hat{\sigma}_0(\xi)|^2 d\xi \right)^{\frac{1}{2}} \geq I_3 - I_4.$$

Similar to the analysis of $I_1$ and $I_2$, we have

$$I_3 \geq (1 + t)^{-\frac{3}{2} - \frac{1}{2}},$$

and

$$I_4 \leq ||(\hat{u}_0, \hat{\sigma}_0)||_{L^0} \left( \int_{|\xi| \leq R} \left( |\xi|^{4+2k} |G_1(\xi, t)|^2 + |\xi|^{2+k} |G_2(\xi, t)|^2 \right) d\xi \right)^{\frac{1}{2}} \leq ||(u_0, \sigma_0)||_{L^1} \left( \int_{|\xi| \leq R} |\xi|^{4+2k} e^{-2\eta|\xi|^2 t} + e^{-\beta|\xi|^2 t} d\xi \right)^{\frac{1}{2}} \leq (1 + t)^{-\frac{3}{2} - \frac{1}{2}}. \quad (4.34)$$

It follows from (4.32)-(4.34) that (4.30) and (4.31) hold. The proof of Lemma 4.6 is complete. \hfill \Box

Based on the above analysis, we can get the lower decay bounds for the nonlinear system (1.1) in the following lemma.

**Lemma 4.7.** Under the assumptions of Part (ii) in Theorem 1.2 there exist a positive time $t_0$ and a positive generic constant $c$, such that

$$\|\nabla^k u(t)\|_{L^2} \geq c(1 + t)^{-\frac{3}{2} - \frac{k}{2}}, \quad k = 0, 1, 2, \quad (4.35)$$

and

$$\|\nabla^k \tau(t)\|_{L^2} \geq c(1 + t)^{-\frac{3}{2} - \frac{k}{2}}, \quad k = 0, 1, \quad (4.36)$$

for all $t \geq t_0$ where $t_0 = t_0(\alpha, \kappa, \beta, ||(u_0, \tau_0)||_{L^2})$ and $c = c(c_2, C_2)$. 

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Proof. First of all, by using the Duhamel’s principle, we have

\[ u(t) = G_{1,2}(t) + \int_0^t \mathcal{G}_1(t-s) * (F_1, F_2)^T(s) ds, \]

\[ \sigma(t) = G_{2,2}(t) + \int_0^t \mathcal{G}_2(t-s) * (F_1, F_2)^T(s) ds. \]

It is easy to see that

\[
\| \nabla^k u(t) \|_{L^2} \geq \left\| \int_0^t \mathcal{G}_1(t-s) * (F_1, F_2)^T(s) ds \right\|_{L^2_{R \times R}}.
\]

For (4.37) and (4.38), we obtain

\[
\| \nabla^k \sigma(t) \|_{L^2} \geq \left\| \int_0^t \mathcal{G}_2(t-s) * (F_1, F_2)^T(s) ds \right\|_{L^2_{R \times R}}.
\]

Similar to the proof of Lemma 4.6, we have

\[
\| \int_0^t \mathcal{G}_1(t-s) * (F_1, F_2)^T(s) ds \|_{L^2_{R \times R}} \geq (1 + t)^{-\frac{k}{2} - \frac{3}{4}} \quad \text{and} \quad \| \int_0^t \mathcal{G}_2(t-s) * (F_1, F_2)^T(s) ds \|_{L^2_{R \times R}} \geq (1 + t)^{-\frac{k}{2} - \frac{3}{4}}.
\]

Now we bound the nonlinear term in (4.37) for $k = 0, 1, 2$. In fact, using Proposition 2.3 Part (i) in Theorem 1.2 and Lemma 4.1, we obtain

\[
\int_0^t \left\| \int_0^t \mathcal{G}_1(t-s) * (F_1, F_2)^T(s) ds \right\|_{L^2_{R \times R}} ds
\]

\[
= \int_0^t \left( \left\| \mathcal{G}_1(t-s) \right\|_{L^2_{R \times R}} + \left\| (F_1, F_2)^T(s) \right\|_{L^2_{R \times R}} \right) \left( \int_0^t \left\| \mathcal{G}_1 \right\|_{L^2_{R \times R}} ds \right)^{\frac{1}{2}} ds
\]

\[
\leq \int_0^t \left( 1 + s \right)^{-\frac{k}{2}} \left( 1 + t - s \right)^{-\frac{k}{2}} ds + \int_0^t \left( 1 + s \right)^{-\frac{k}{2}} \left( 1 + t - s \right)^{-\frac{k}{2}} ds \leq (1 + t)^{-\frac{k}{2} - \frac{3}{4}}.
\]

Next, we turn to bound the nonlinear term in (4.38) for $k = 0, 1$. Similarly to (4.40), using Proposition 2.3 Part (i) in Theorem 1.2, Lemma 4.1 and (4.24), one deduces that

\[
\int_0^t \left\| \int_0^t \mathcal{G}_2(t-s) * (F_1, F_2)^T(s) ds \right\|_{L^2_{R \times R}} ds
\]

\[
= \int_0^t \left( \left\| \mathcal{G}_2(t-s) \right\|_{L^2_{R \times R}} + \left\| (F_1, F_2)^T(s) \right\|_{L^2_{R \times R}} \right) \left( \int_0^t \left\| \mathcal{G}_2 \right\|_{L^2_{R \times R}} ds \right)^{\frac{1}{2}} ds
\]

\[
\leq \int_0^t \left( 1 + s \right)^{-\frac{k}{2}} \left( 1 + t - s \right)^{-\frac{k}{2}} ds + \int_0^t \left( 1 + s \right)^{-\frac{k}{2}} \left( 1 + t - s \right)^{-\frac{k}{2}} ds \leq (1 + t)^{-\frac{k}{2} - \frac{3}{4}}.
\]
Collecting (4.37), (4.38), (4.39), (4.40) and (4.41), and using the fact that \( \| \nabla^k \sigma \|_{L^2} \leq \| \nabla^k \tau \|_{L^2} \), we conclude that (4.35) and (4.36) hold.

With Lemma 4.7, the proof of Part (ii) in Theorem 1.2 is finished. Thus the proof of Theorem 1.2 is complete.

5 Appendix

5.1 More time-decay estimates for vanishing center-of-mass diffusion

**Theorem 5.1.** Under the conditions of Theorem 1.2 for Case II (i.e., \( \epsilon > 0, \mu \geq 0 \)), we get the optimal time-decay estimate for the third-order derivative of the solution, namely,

\[
\sum_{2 \leq k \leq 3} \| \nabla^3 u, \nabla^k \tau \| (t)_{L^2} \leq C(1 + t)^{-\frac{9}{4}},
\]

and

\[
\| \nabla^3 u(t) \| \geq c(1 + t)^{-\frac{9}{4}}, \| \nabla^2 \tau (t) \| \geq c(1 + t)^{-\frac{3}{4}},
\]

for all \( t \geq t_0 \), where \( C \) and \( c \) are positive constants independent of \( t \).

**Proof.** To get (5.1), applying \( \nabla^3 \) to (1.1), and then multiplying (1.1)_1 by \( \alpha \nabla^3 u \) and (1.1)_2 by \( \kappa \nabla^3 \tau \), we have from integration by parts and the cancellation relation that

\[
\frac{1}{2} \frac{d}{dt} \left( \alpha \| \nabla^3 u \|^2_{L^2} + \kappa \| \nabla^3 \tau \|^2_{L^2} \right) + \alpha \epsilon \| \nabla^4 u \|^2_{L^2} + \kappa \mu \| \nabla^4 \tau \|^2_{L^2} = - \alpha < \nabla^3 (u \cdot \nabla u), \nabla^3 u > - \kappa < \nabla^3 (u \cdot \nabla \tau), \nabla^3 \tau > + \kappa < \nabla^3 Q(\nabla u, \tau), \nabla^3 \tau > = \sum_{p=1}^3 \mathcal{K}_p,
\]

where the incompressible condition, the Hölder’s inequality and estimates on the commutator imply that

\[
\mathcal{K}_1 = - \alpha \left( < \nabla^3 (u \cdot \nabla u), \nabla^3 u > - < (u \cdot \nabla) \nabla^3 u, \nabla^3 u > \right)
\leq \alpha \| \nabla^3 (u \cdot \nabla u) - (u \cdot \nabla) \nabla^3 u \|_{L^2} \| \nabla^3 u \|_{L^2} \leq \alpha \| \nabla u \|_{L^\infty} \| \nabla^3 u \|_{L^2}^2
\leq \alpha \delta \| \nabla^3 u \|_{L^2}^2.
\]

\[
\mathcal{K}_2 = - \kappa \left( < \nabla^3 (u \cdot \nabla \tau), \nabla^3 \tau > - < (u \cdot \nabla) \nabla^3 \tau, \nabla^3 \tau > \right)
\leq \| \nabla^3 (u \cdot \nabla \tau) - (u \cdot \nabla) \nabla^3 \tau \|_{L^2} \| \nabla^3 \tau \|_{L^2}
\leq \kappa \left( \| \nabla u \|_{L^\infty} \| \nabla^3 \tau \|_{L^2} + \| \nabla \tau \|_{L^\infty} \| \nabla^3 u \|_{L^2} \right) \| \nabla^3 \tau \|_{L^2}
\leq \kappa \delta \| \nabla^3 \tau \|_{L^2}^2 + \kappa \delta \| \nabla^4 u \|_{L^2} \| \nabla^3 \tau \|_{L^2}
\leq \frac{1}{2} \alpha \epsilon \| \nabla^4 u \|_{L^2}^2 + C \left( \kappa \delta + \alpha^{-1} \epsilon^{-1} \kappa^2 \delta^2 \right) \| \nabla^3 \tau \|_{L^2}^2.
\]

In conclusion, by choosing \( \delta = \delta(\kappa, \alpha, \beta, \epsilon) \) small enough, we have

\[
\frac{1}{2} \frac{d}{dt} \left( \alpha \| \nabla^3 u \|_{L^2}^2 + \kappa \| \nabla^3 \tau \|_{L^2}^2 \right) + \frac{1}{2} \alpha \epsilon \| \nabla^4 u \|_{L^2}^2 + \kappa \mu \| \nabla^4 \tau \|_{L^2}^2 + \frac{1}{2} \kappa \beta \| \nabla^3 \tau \|_{L^2}^2
\leq C \delta(\alpha, \kappa) \| \nabla^3 u \|_{L^2}^2.
\]
Then we have from a similar discussion as (4.10) that
\[
\frac{d}{dt} \| \nabla^3(u, \tau) \|^2_{L^2} + D_2 \| \nabla^3(u, \tau) \|^2_{L^2} \leq \int_{|\xi| \leq R} |\xi|^6 |\hat{u}(t)|^2 d\xi,
\]
for some positive constant \(D_2 = D_2(\alpha, \kappa, \beta, \epsilon)\). Then, it follows from Lemma 4.1 that
\[
\| \nabla^3(u, \tau)(t) \|^2_{L^2} \leq e^{-D_2 t} \| \nabla^3 U_0 \|^2_{L^2} + \int_0^t e^{-D_2(t-s)} \left( \int_{|\xi| \leq R} |\xi|^6 |\hat{u}(s)|^2 d\xi \right) ds
\]
\[
\leq e^{-D_2 t} \| \nabla^3 U_0 \|^2_{L^2} + \int_0^t e^{-D_2(t-s)} (1 + s)^{-\frac{9}{2}} \| U_0 \|_{L^2}^2 ds
\]
\[
+ \int_0^t e^{-D_2(t-s)} \left( \int_0^{t^2} (1 + s - s')^{-\frac{9}{2}} \left\| (M_1, M_2)^T (s) \right\|_{L^1} ds' \right) ds
\]
\[
\leq C_2 (1 + t)^{-\frac{9}{2}},
\]
which implies that
\[
\| \nabla^3(u, \tau)(t) \|^2_{L^2} \leq C_2 (1 + t)^{-\frac{9}{2}}.
\]

Finally, (5.5) enables us to improve the decay rate of \( \| \nabla^2 \tau \|_{L^2} \) in Case II. In fact, similar to (4.21), and using (5.5), we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla^2 \tau \|^2_{L^2} + \mu \| \nabla^3 u \|^2_{L^2} + \frac{1}{2} \| \nabla^2 \tau \|^2_{L^2}
\]
\[
\leq \alpha \| \nabla^3 u \|^2_{L^2} + \| \nabla^2 \nabla Q(u, \tau) \|^2_{L^2} + \| \nabla^2 (u \cdot \nabla \tau) \|^2_{L^2}
\]
\[
\leq \alpha \| \nabla^3 u \|^2_{L^2} + \| \nabla^3 u \|^2_{L^2} + \| \nabla^2 \nabla \tau \|^2_{L^2}
\]
\[
\leq (1 + t)^{-\frac{9}{2}}.
\]

Consequently,
\[
\| \nabla^2 \tau \|^2_{L^2} \leq C_2 (1 + t)^{-\frac{9}{2}}.
\]

Next, we will justify (5.2). Taking \(k = 3\) in the nonlinear term of (4.37), we have
\[
\int_0^t \| \xi^3 \hat{g}_{1,\nu}(t-s) (\hat{f}_1, \hat{f}_2)^T (s) \|_{L^2(\xi \leq R)} ds
\]
\[
= \int_0^t \| (1 + \xi)^3 G_3(t-s) - \epsilon^3 G_1(t-s) \hat{F}_1(s) + \kappa |\xi|^6 G_1(t-s) \hat{F}_2(s) \|_{L^2(\xi \leq R)} ds
\]
\[
\leq \int_0^t \left( \| \hat{F}_1(s) \|_{L^\infty} + \| \hat{F}_2(s) \|_{L^\infty} \right) \left( \int_{|\xi| \leq R} |\xi|^8 e^{-2|\xi|^2(t-s)} d\xi \right) ds
\]
\[
+ \int_0^t \left( \| \hat{F}_1(s) \|_{L^\infty} + \| \hat{F}_2(s) \|_{L^\infty} \right) (1 + t - s)^{-\frac{9}{2}} ds
\]
\[
\leq \int_0^t (1 + s)^{-\frac{9}{2}} (1 + t - s)^{-\frac{9}{2}} ds + \int_0^t (1 + s)^{-\frac{9}{2}} (1 + t - s)^{-\frac{9}{2}} ds \leq (1 + t)^{-\frac{9}{2}}.
\]
Then taking $k = 2$ in the nonlinear term of (4.38), we have

\[
\int_0^t \left\| \partial_\sigma^2 \hat{G}_{2,\sigma}(t-s) \left( \hat{F}_1, \hat{F}_2 \right)(s) \right\|^2_{L^2_{|\xi|^2}} \, ds
\]

\[
= \int_0^t \left\| \frac{\alpha}{2} \cdot |\cdot|^3 \hat{G}_1(t-s) \hat{F}_1(s) + \left( |\cdot|^2 \hat{G}_2(t-s) + \epsilon |\cdot|^4 \hat{G}_1(t-s) \right) \hat{F}_2(s) \right\|^2_{L^2_{|\xi|^2}} \, ds
\]

\[
\leq \int_0^t \left( \left\| \frac{1}{|\cdot|^3} \hat{F}_1(s) \right\|_{L^\infty_{|\xi|}} + \| \hat{F}_2(s) \|_{L^\infty_{|\xi|}} \right) \left( \int_{|\xi| \leq R} \left( |\xi|^6 e^{-2\beta|\xi|^2(t-s)} + e^{-\beta(t-s)} \right) d\xi \right)^{\frac{1}{2}} ds
\]

\[
+ \int_0^t \left( \| (\hat{F}_1, \hat{F}_2)(\cdot, s) \|_{L^\infty_{|\xi|}} \right)(1 + t - s)^{-1} ds
\]

\[
\leq \int_0^t (1 + s)^{-\frac{1}{2}} (1 + t - s)^{-\frac{11}{4}} ds + \int_0^t (1 + s)^{-\frac{1}{2}} \left( (1 + t - s)^{-1} + e^{-\beta(t-s)} \right) ds
\]

\[
\leq (1 + t)^{-\frac{5}{2}}.
\]

Combining (4.37), (4.38), (4.39), (5.7) and (5.8), using the fact that $\| \nabla^2 \sigma \|_{L^2} \leq \| \nabla^2 \tau \|_{L^2}$, we conclude that (5.2) holds.

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