Normal modes and the no zero mode theorem of scalar fields in BTZ black hole spacetime

Masakatsu Kenmoku¹, Maiko Kuwata¹ and Kazuyasu Shigemoto²

¹ Department of Physics, Nara Women's University, Nara 630-8506, Japan
² Tezukayama University, Nara 631-8501, Japan

E-mail: kenmoku@asuka.phys.nara-wu.ac.jp, kuwata@asuka.phys.nara-wu.ac.jp and shigemot@tezukayama-u.ac.jp

Received 21 January 2008, in final form 13 May 2008
Published 30 June 2008
Online at stacks.iop.org/CQG/25/145016

Abstract
The eigenfunctions for normal modes of scalar fields in BTZ black hole spacetime are studied. The orthonormal relations among them are derived. Quantization for scalar fields is done, and particle number, energy and angular momentum are expressed by the creation and annihilation operators. The allowed physical normal mode region is studied on the basis of the no zero mode theorem. Its implications for statistical mechanics is also studied.

PACS numbers: 04.70.Dy, 04.62.+v

1. Introduction

One of the important aspects of general relativity is black hole physics. Extensive studies in this field have been accomplished from the observational and/or theoretical viewpoints. Much evidence of black holes has been observed, including the supermassive black holes at the center of the galaxies [1, 2]. Such black holes are expected to be well described by axial symmetric solutions of the Einstein’s field equation. Recently, multi-parameter rotating black hole solutions in higher dimensions are studied theoretically from the viewpoint of string theory, M theory, braneworld and with the interest of AdS/CFT correspondence [3–5].

One of the most interesting aspects of black hole physics is the thermodynamical properties. Black holes can be considered as thermal objects [6, 7]: the entropy is proportional to the surface area on the horizon with the Hawking temperature $T_H$ [8]. The statistical understanding of the thermodynamics of black holes has been tried in the frames of field theories [9], string theory [10] and others. Among them, the brick wall model has been proposed by ‘t Hooft in order to interpret the area law of the black hole entropy by considering the freedom of scalar fields around the black hole horizon in the standard framework of statistical mechanics [11].
On the other hand, there are some problematic issues in understanding the black hole thermodynamics. One of the fundamental problems is the super-radiant instability, which occurs in the case of rotating black holes. This problem is the case that the flux intensity of scattered outgoing fields to the black hole becomes larger than that of ingoing fields under the condition of $\omega - \Omega m < 0$ [12, 13], where $\omega$ and $m$ are the frequency and azimuthal angular momentum of the field respectively and $\Omega_H$ is the angular velocity of the black hole. The Boltzmann factor and then the partition function become ill-defined due to the existence of the super-radiance [14]. In BTZ black hole spacetime [15], the super-radiance problem for the scalar fields is also discussed extensively [16–19]. In our previous papers, statistical mechanics of scalar fields in the multi-rotating black hole spacetime are studied [20, 21].

The purpose of this paper is to investigate the normal modes of scalar fields around rotation black holes to define the statistical mechanics well and to understand the super-radiant problem. In order to make the problem clear, we study the BTZ black hole model which is the anti-de Sitter (AdS) rotating black hole in (2+1) dimension. In the case of quasinormal modes in BTZ spacetime, exact analytical treatment has been done [22, 23]. Therefore we expect to obtain the exact eigenfunctions in the case of normal modes too. As the boundary conditions, we impose the Dirichlet boundary condition at infinity and the Dirichlet or Neumann boundary condition at horizon for eigenfunctions of normal modes.

The organization of this paper is as follows. In section 2, we prepare notations and definitions of BTZ black hole spacetime and scalar fields for the convenience of the subsequent sections. In section 3, we will give the explicit expression of boundary conditions for the radial direction. In section 4, we derive the orthonormal relations among eigenfunctions of normal modes. In section 5, the quantization of the scalar fields will be done. In section 6, particle number, energy and angular momentum are calculated to be represented by creation and annihilation operators. In section 7, we will show the theorem of nonexistence of zero normal mode eigenstates. The allowed physical normal mode region will be derived as its application and statistical implication will also be studied in that section. The results will be summarized in the final section.

2. Scalar fields in BTZ black hole spacetime

This section will provide the preparation for definitions and notation in the following sections.

The Einstein–Hilbert action with negative cosmological constant ($\Lambda = -1/\ell^2$) in (2+1) dimension is

\[
I_G = \frac{1}{16\pi G} \int d^3x \sqrt{-g} \left( R + \frac{2}{\ell^2} \right).
\]

(2.1)

In the following, the gravitational constant is normalized as $G = 1/8\pi$. The vacuum Einstein’s equations for this action are

\[
R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R - g_{\mu\nu} \frac{\Lambda}{\ell^2} = 0.
\]

(2.2)

The Banados, Teitelboim and Zanelli (BTZ) solution is in the form [15]

\[
dx^2 = g_{tt} dt^2 + g_{\phi\phi} d\phi^2 + 2g_{t\phi} dt d\phi + g_{rr} dr^2,
\]

(2.3)

where the components of the metric are

\[
g_{tt} = M - \frac{r^2}{\ell^2}, \quad g_{t\phi} = -\frac{J}{2},
\]

(2.4)

\[
g_{\phi\phi} = r^2, \quad g_{rr} = \left( -M + \frac{J^2}{4r^2} + \frac{r^2}{\ell^2} \right)^{-1}
\]

(2.5)
with black hole mass $M$ and rotation parameter $J$. The contravariant time component of the metric is negative of covariant radial component:

$$g^{tt} = -g_{rr}. \quad (2.6)$$

Zeros of their inverse function denote the horizon of black hole:

$$r_\pm^2 = \frac{M\ell^2}{2} \left(1 \pm \sqrt{1 - \frac{J^2}{M^2\ell^2}}\right), \quad (2.7)$$

where the event horizon is $r_+$. Note that the BTZ metrics can be rewritten in a diagonal form:

$$ds^2 = \frac{1}{g^{tt}} dr^2 + g_{\phi\phi} \left(\frac{d\phi + g_{t\phi}}{g_{\phi\phi}} dr\right)^2 + g_{rr} dr^2. \quad (2.8)$$

The action and the Lagrangian density for the minimally coupled complex scalar field with dimensionless mass $\mu$ under the BTZ spacetime is

$$I_{\text{scalar}} = \int dt dr d\phi \sqrt{-g} L_{\text{scalar}}, \quad (2.9)$$

$$L_{\text{scalar}} = -\left(g^{tt}\partial_t \Phi(x)\partial_t \Phi(x) + \frac{\mu}{\ell^2} \Phi^*(x)\Phi(x)\right). \quad (2.10)$$

The Klein–Gordon equation for this action is

$$\left(\frac{1}{\sqrt{-g}} \partial_\mu \left(g^{\mu\nu}\sqrt{-g}\partial_\nu\right) - \frac{\mu}{\ell^2}\right) \Phi = 0. \quad (2.11)$$

The background BTZ metric does not depend on time and azimuthal angle variables, and then the scalar field solution is put in the form:

$$\Phi = e^{-i\omega t} e^{i m \phi} R(r), \quad (2.12)$$

where $\omega$ and $m$ ($m = 0, \pm 1, \pm 2, \ldots$) denote the frequency and azimuthal angular momentum respectively. The equation for the radial wavefunction becomes

$$\left(-g^{tt} \left(\frac{\omega - J}{2\ell^2} m\right)^2 - m^2 + \frac{1}{r} \frac{d}{dr} \frac{d}{dr} - \frac{\mu}{\ell^2}\right) R(r) = 0. \quad (2.13)$$

We note that this equation is invariant under the symmetry of $(\omega, m)$ and $(-\omega, -m)$.

3. The solution and boundary conditions

In this section, we will give the explicit expression of the solution and boundary conditions for the radial direction of the scalar fields. For details about this context, see our paper [24]. As the BTZ spacetime is asymptotically AdS spacetime, the Dirichlet boundary condition is imposed at $r = \infty$

$$R_{\omega,m}(z)|_{r=\infty} = 0, \quad (3.1)$$

and the solution is given by using the hypergeometric function in the form

$$R_{\omega,m}(z) = \frac{z^{-i\omega}(1 - z)^{\beta}}{\Gamma(1 - a + b + 1)} z^{-a-b} F(c - a, c - b, c - a - b + 1; 1 - z)$$

$$= \frac{\Gamma(1 - c)}{\Gamma(1-a)\Gamma(1-b)} z^{-ia}(1 - z)^{\beta} F(a, b, c; z)$$

$$+ \frac{\Gamma(c - 1)}{\Gamma(c - a)\Gamma(c - b)} z^{i\omega}(1 - z)^{\beta} F(1 + b - c, 1 + a - c, 2 - c; z).$$
where we use the notation
\[
\begin{align*}
  z &= r^2 - r_+^2, \\
  \alpha &= \frac{\ell^2 r_+}{2(r^2_+ - r_-^2)} (\omega - \Omega q m), \\
  \beta &= \frac{1 - \sqrt{1 + \mu}}{2}, \\
  a &= \beta - i \frac{\ell^2}{2(r_+ + r_-)} \left( \omega + \frac{m}{\ell} \right), \\
  b &= \beta - i \frac{\ell^2}{2(r_+ - r_-)} \left( \omega - \frac{m}{\ell} \right), \\
  c &= 1 - 2i \alpha.
\end{align*}
\]

We put the Dirichlet/Neumann boundary condition at \( r = r_+ + \epsilon \), where we take the brick wall regularization according to 't Hooft [11],
\[
\left. \frac{\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} z^{-ia} (1-z)^b F(a,b,c;z) \right|_{r=r_++\epsilon}
\]
which gives the eigenvalue of \( \omega \). Radial solutions of normal modes \( R_{\omega,m} \) are specified with each eigenvalue of \( \omega \).

For the radial direction, the boundary term of the inner product becomes zero in the form
\[
\sqrt{-g} g_{rr} (\partial_r R_{\omega,m}(z) \partial_r R_{\omega,m}(z) - \partial_r R_{\omega,m}(z) \partial_r R_{\omega,m}(z)) \right|_{r=r_++\epsilon} = 0,
\]
by using equations (3.1) and (3.2). For the azimuthal direction, the boundary term of the inner product becomes trivially zero for integer values of \( m \). With these boundary conditions, the inner products are self-adjoint. We will include the above argument in the paper. The explicit analysis on the boundary condition is given by [24].

### 4. Normal modes and orthonormal relations

We consider the radial wave equation as the eigenvalue equation for the eigenfunction \( R_{\omega,m} \) with eigenvalues \( \omega \) and \( m \):
\[
\Delta_r R(r)_{\omega,m} = \left( g^{rr} (\omega - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right) R(r)_{\omega,m},
\]
where Laplacian and the angular velocity at radial position \( r \) are defined by:
\[
\Delta_r : = \frac{1}{r} \frac{d}{dr} r \frac{d}{dr}, \quad \Omega_r : = - \frac{g_{\phi\phi}}{g^{rr}} = \frac{g_{\phi r}}{g^{rr}} = \frac{J}{2r^2}.
\]
The following two integrations are considered
\[
\int dr \sqrt{-g} R_{\omega,m}^* \Delta_r R_{\omega,m} = \int dr \sqrt{-g} \left( g^{rr} (\omega - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right) R_{\omega,m}^* R_{\omega,m}.
\]
\[
\int dr \sqrt{-g} \Delta_r R_{\omega,m}^* \Delta_r R_{\omega,m} = \int dr \sqrt{-g} \left( g^{rr} (\omega - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right) R_{\omega,m}^* R_{\omega,m}.
\]
In cases of vanishing the boundary terms, the difference of these integrations becomes
\[
0 = \int dr \sqrt{-g} \left( - \left( g^{rr} (\omega - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right) + \left( g^{rr} (\omega' - \Omega_r m)^2 + \frac{m^2}{r^2} + \frac{\mu}{\ell^2} \right) \right) \times R_{\omega,m}^* R_{\omega,m}
\]
\[
= \int dr \sqrt{-g} g_{rr} (\omega' - \omega) (\omega' + \omega) (2\Omega_r m) R_{\omega,m}^* R_{\omega,m}.
\]
For the case of $\omega = \omega'$, this relation shows the reality of eigenvalue $\omega$. For the case of $\omega \neq \omega'$, this relation shows the orthonormal relations among eigenfunctions:

$$\int dr \sqrt{-g}(-g'^a)(\omega + \omega' - 2\Omega_m)R^a_{w,m}R_{a',m} = \delta_{\omega,\omega'}.$$  (4.6)

Similarly, we have an orthogonal relation for a couple of positive and negative azimuthal angular momentum $(m, -m)$ as

$$\int dr \sqrt{-g} g'^a(\omega - \omega' - 2\Omega_m)R_{a,m}R_{a',-m} = 0.$$  (4.7)

Defining the full eigenfunction with normalization factor $N_{w,m}$

$$f_{w,m} := N_{w,m}e^{-i\omega t}e^{im\phi}R_{w,m},$$  (4.8)

we have the full orthonormal relations:

$$\int d\phi dr \sqrt{-g} (-g'^a)(\omega + \omega' - 2\Omega_m) f^*_{w,m}f_{w',m'} = \delta_{\omega,\omega'}\delta_{m,m'},$$  (4.9)

where the integration region $\Sigma$ denotes $0 \leq \phi < 2\pi$ and $r_s \leq r < \infty$. For more compact notation, the inner products are introduced:

$$(A, B) := \int \int d\phi dr \sqrt{-g} (-g'^a)(A^*(x)\delta_vB(x) - \delta_vA^*(x)B(x)),$$  (5.1)

and the general form of orthonormal relations is obtained:

$$(f_{w,m}, f_{w',m'}) = -(f^*_{w,m}, f^*_{w',m'}) = \delta_{\omega,\omega'}\delta_{m,m'},$$  (5.2)

$$(f^*_{w,m}, f_{w',m'}) = (f_{w,m}, f^*_{w',m'}) = 0.$$  (5.3)

It should be noted that the orthonormal relations hold for the allowed normal mode region $0 < \omega - \Omega_Hm$, where $\Omega_H = J/2r_s^2$ is the angular velocity on the horizon.

### 5. Quantization

We can define the canonical momentum conjugate to the scalar field $\Phi$ as

$$\Pi := -\frac{\partial L_{\text{scalar}}}{\partial \dot{\Phi}} = -g'^aa_\Phi = -g'^a(\delta_v\Phi^\dagger + \Omega_v\delta_\phi\Phi^\dagger),$$  (5.4)

where $\dagger$ denotes the Hermitian conjugate operation for the quantized fields. We impose the equal time commutation relations among fields and their momenta:

$$[\Phi(t, r, \phi), \Pi(t, r', \phi')] = \frac{i}{\sqrt{-g}}\delta(r - r')\delta(\phi - \phi'),$$  (5.5)

and others are zero. We make the normal mode expansion for the scalar fields and the conjugate momenta as

$$\Phi = \sum_{w,m} (a_{w,m}f_{w,m} + b_{w,m}^\dagger f^*_{w,m}),$$

$$\Pi = -ig'^a\sum_{w,m} (\omega - \Omega_m)(a^\dagger_{w,m}f^*_{w,m} - b_{w,m}f_{w,m}).$$  (5.6)

In order to regularize the divergence on the horizon, the cutoff parameter is introduced in explicit construction of normal mode solutions [24]. The cutoff parameter plays a similar role in the brick wall model [11].
Expansion coefficients are inversely expressed by fields and their momenta using the orthonormal relations:

\[ a_{\omega,m} = (f_{\omega,m}, \Phi) \]

\[ = \int_{\Sigma} d\phi dr \sqrt{-g}(if^{*}_{\omega,m}(t, r, \phi)\Pi^{\dagger}(t, r, \phi) \]

\[ - g^{ij}(\omega - \Omega, m)f^{*}_{\omega,m}(t, r, \phi)\Phi(t, r, \phi)), \]

\[ b^{\dagger}_{\omega,m} = -(f^{*}_{\omega,m}, \Phi) \]

\[ = -\int_{\Sigma} d\phi dr \sqrt{-g}(if_{\omega,m}(t, r, \phi)\Pi(t, r, \phi) \]

\[ + g^{ij}(\omega - \Omega, m)f_{\omega,m}(t, r, \phi)\Phi(t, r, \phi)). \]

Completeness relations are derived from the consistency of normal mode expansions of fields:

\[ \sum_{\omega,m} (-g^{ij})(\omega - \Omega, m)(f_{\omega,m}(t, r, \phi)f^{*}_{\omega,m}(t, r, \phi') + f^{*}_{\omega,m}(t, r, \phi)f_{\omega,m}(t, r, \phi')) \]

\[ = \frac{1}{\sqrt{-g}}\delta(r - r')\delta(\phi - \phi'), \] (5.5)

\[ \sum_{\omega,m} (f_{\omega,m}(t, r, \phi)f^{*}_{\omega,m}(t, r', \phi') - f^{*}_{\omega,m}(t, r, \phi)f_{\omega,m}(t, r', \phi')) = 0. \]

The commutation relations between annihilation and creation operators are derived using the completeness relations

\[ [a_{\omega,m}, a_{\omega',m}^{\dagger}] = \delta_{\omega,\omega'}\delta_{m,m'}, \quad [b_{\omega,m}, b_{\omega',m}^{\dagger}] = \delta_{\omega,\omega'}\delta_{m,m'}, \] (5.6)

and others are zero.

6. Particle number, energy and angular momentum

From the symmetry of the action for the scalar field, conserved particle number, energy and angular momentum are derived and expressed by the creation and annihilation operators.

1) Particle number

From the phase translation invariance of the action, the particle number current is defined as

\[ j^{\mu} = -ig^{\mu\nu}(\Phi^{\dagger}\partial_{\nu}\Phi - \partial_{\nu}\Phi^{\dagger}), \] (6.1)

which is shown to satisfy the current conservation

\[ j^{\mu} ;_{\mu} = 0. \] (6.2)

The corresponding particle number is

\[ N := \int_{\Sigma} d\phi dr \sqrt{-g} j^{0} \]

\[ = \int_{\Sigma} d\phi dr \sqrt{-g}(\Pi\Phi - \Phi^{\dagger}\Pi^{\dagger}). \] (6.3)

The particle number is also expressed by the creation and annihilation operators by using normal mode expansion of fields and their momenta

\[ N = \sum_{\omega,m} (a^{\dagger}_{\omega,m}a_{\omega,m} - b_{\omega,m}b_{\omega,m}^{\dagger}). \] (6.4)

In the derivation, we have used orthonormal relations (4.11) and completeness relations (5.5).
(2) **Energy and angular momentum**

Because the metrics do not depend on time and azimuthal angle, two Killing vectors exist

\[ \xi^{(t)} = (1, 0, 0), \quad \xi^{(\phi)} = (0, 0, 1). \]  

(6.5)

Defining the energy–momentum tensor

\[ T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta I_M}{\delta g_{\mu\nu}} \]

\[ = \partial_\phi \Phi^\dagger \partial_\phi \Phi + \partial_\phi \Phi^\dagger \partial_\phi \Phi - g_{\mu\nu} \left( g^{\alpha\beta} \partial_\alpha \Phi^\dagger \partial_\beta \Phi + \frac{\mu}{\ell^2} \Phi^\dagger \Phi \right). \]  

(6.6)

local conservation laws hold for two Killing vectors

\[ (\xi^{(i)} T^i_{\mu\nu})_{;\nu} = 0, \quad \text{for} \quad i = t, \phi. \]  

(6.7)

Corresponding conservative quantities are energy and angular momentum

\[ E = -\int_\Sigma d\phi dr \sqrt{-g} (\xi^{(t)} T^t_{\mu}) \]

\[ = \int_\Sigma d\phi dr \sqrt{-g} \left( g^{tt} (\Phi^\dagger \partial_\phi \Phi + \Phi \partial_\phi \Phi^\dagger) + 2 g^{t\phi} \Phi^\dagger \partial_\phi \Phi \right). \]  

(6.8)

\[ L = \int_\Sigma d\phi dr \sqrt{-g} (\xi^{(\phi)} T^\phi_{\mu}) \]

\[ = \int_\Sigma d\phi dr \sqrt{-g} \left( g^{\phi\phi} (\Phi^\dagger \partial_\phi \Phi + \Phi \partial_\phi \Phi^\dagger) + 2 g^{\phi t} \Phi^\dagger \partial_\phi \Phi \right). \]  

(6.9)

The energy and angular momentum are expressed by the creation and annihilation operators by using normal mode expansion of fields and their momenta

\[ E = \sum_{\omega, m} \omega (a^\dagger_{\omega, m} a_{\omega, m} + b_{\omega, m} b^\dagger_{\omega, m}). \]  

(6.10)

\[ L = \sum_{\omega, m} m (a^\dagger_{\omega, m} a_{\omega, m} + b_{\omega, m} b^\dagger_{\omega, m}). \]  

(6.11)

The effective energy, which is the energy taking the rotation effect on the horizon, is expressed as

\[ E - \Omega H L = \sum_{\omega, m} (\omega - \Omega H m) (a^\dagger_{\omega, m} a_{\omega, m} + b_{\omega, m} b^\dagger_{\omega, m}). \]  

(6.12)

The effective energy is positive definite for the allowed normal mode region \( 0 < \omega - \Omega H m \) except for the zero point energy.

7. **No zero mode theorem**

In this section, we consider the zero mode eigenstates, which are defined as the states of \( 0 = \omega - \Omega H m \) with \( -\infty < m < \infty \). We will show that they do not exist and the allowed normal mode region is \( 0 < \omega - \Omega H m \). We also show that the statistical mechanics for the Hartle–Hawking state is defined well for the BTZ black hole spacetime. We impose the Dirichlet or Neumann boundary condition on the horizon and the Dirichlet boundary condition at infinity because spacetime is anti-de Sitter.
Statement 1. The eigenfunction of the normal mode for $\omega = m = 0$ does not exist.

Proof. The general radial eigenfunction of the normal mode for $\omega = m = 0$ is obtained by solving the radial wave equation in the case of massless scalar fields $\mu = 0$ as

$$R_{\omega, \mu} = c_1 \ln \left( \frac{r^2 - r_+^2}{r^2 - r_-^2} \right) + c_2,$$  
(7.1)

where $c_1, c_2$ are integration constants. This solution cannot satisfy both the boundary conditions at horizon and infinity.

Statement 2 (no zero mode theorem). Eigenfunctions of normal modes for $0 = \omega - \Omega H m$ do not exist.

Proof. The radial eigenfunctions of the normal mode for $0 = \omega - \Omega H m$ satisfying the boundary condition at infinity are obtained by the hypergeometric function as

$$R_{\omega(=\Omega H m), m} = \left( \frac{r^2 - r_+^2}{r^2 - r_-^2} \right)^b \frac{1}{\Gamma(2b)} F \left( b - ic, b + ic, 2b ; \frac{r_+^2 - r_-^2}{r^2 - r_-^2} \right),$$  
(7.2)

where the parameters are

$$b = \frac{1 + \sqrt{1 + \mu}}{2}, \quad c = \frac{\ell m}{2r_+}.$$  
(7.3)

This solution also cannot satisfy the boundary condition at horizon. Note that the solution reduces the first term in equation (7.1) for $\omega = m = \mu = 0$.

Statement 3. The allowed physical normal mode region is $0 < \omega - \Omega H m$ with $-\infty < m < \infty$.

Proof. First we remark that the allowed physical normal mode region is $0 < \omega$ with $-\infty < m < \infty$ for the case of no rotation $J = 0$. We assume that the normal mode is analytic for the rotation parameter $J$. After switching on the rotation $J \neq 0$, the allowed physical normal mode region shifts from $0 < \omega$ to $0 < \omega - \Omega H m$ because normal modes cannot cross each other.

It should be noted that the mode of $0 = \omega - \Omega H m$ is a special mode in the sense that this is the unique solution which satisfies the Dirichlet boundary condition at infinity but diverges at horizon. As a consequence, normal modes are divided into two regions: one region is $0 < \omega - \Omega H m$ with $-\infty < m < \infty$ and the other region is $0 > \omega - \Omega H m$ with $-\infty < m < \infty$ divided by the zero mode. We can confirm this result by the explicit construction of the normal mode eigenfunctions [24].

Statement 4. Statistical mechanics for the scalar fields around BTZ black hole spacetime is well defined.

Proof. The allowed region for the total effective energy becomes $0 < E - \Omega H L$ from the expression in equation (6.12). Then the Boltzmann factors and then the partition function taking account of rotating effect by Hartle and Hawking [25] become well defined:

$$Z = \text{Tr} \exp(-\beta H(E - \Omega H L)),$$  
(7.4)

where the trace is taken for occupation numbers of scalar particles and $\beta H$ denotes the inverse of Hawking temperature [8].
8. Summary

We have investigated normal modes of scalar fields around BTZ black hole spacetime with the Dirichlet or Neumann boundary condition at horizon and convergence to zero at infinity. Orthonormal relations and completeness relations for normal mode eigenfunctions are derived. Using these relations, conserved charge, energy and angular momentum are expressed by creation and annihilation operators of scalar particles. Zero modes of \( \omega = \omega - \Omega_{Hm} \) are shown not to exist and the allowed normal mode region is shown to be \( 0 < \omega - \Omega_{Hm} \). From the allowed normal mode region, the total effective energy should be positive; \( 0 < E - \Omega_{HL} \), which guarantees that the Boltzmann factors for the rotating black holes be well defined. This fact shows that the super-radiance does not occur for BTZ black hole spacetime and is consistent with the negative value of the imaginary part for quasinormal frequency [22].

We can extend our result of (2+1)-dimensional BTZ black hole spacetime to (3+1) or more higher dimensional rotating black hole spacetime, which will help for the deeper understanding of scalar field entropy around the rotating black holes. As for the higher dimensional case, we are now preparing a new paper. Then we will discuss how to extend our result to the higher dimensional case in the forthcoming paper [26]. We also try to study the relation between our method and the treatment of the conformal field theory, which is developed extensively in view of exact AdS/CFT correspondence in BTZ spacetime [27–30].

Acknowledgments

One of the authors (MK) would like to thank Professor Gupta Kumar for comments about the boundary condition for orthogonality of normal modes.

References

[1] Eckart A and Genzel R 1996 Nature 383 415
[2] Harrenstein J R et al 1999 Nature 400 539
[3] Myers R C and Perry M J 1986 Ann. Phys. (Berlin) 172 304
[4] Gibbons G W, Lu H, Page D N and Pope C N 2004 J. Geom. Phys. 54 49
Gibbons G W, Perry M J and Pope C N 2005 Class. Quantum Grav. 22 1503
[5] Chen W, Lu H and Pope C N 2006 Class. Quantum Grav. 23 5323
[6] Bekenstein J D 1973 Phys. Rev. D 7 2333
[7] Bardeen M, Carter B and Hawking S W 1973 Commun. Math. Phys. 31 161
[8] Hawking S W 1975 Commun. Math. Phys. 43 199
[9] Birrell N D and Davies P C W 1982 Quantum Fields in Curved Space (Cambridge: Cambridge University Press)
[10] Strominger A and Vafa C 1996 Phys. Lett. B 379 99
[11] ’t Hooft G 1985 Nucl. Phys. B 256 727
[12] Bardeen J M, Press W H and Teukolsky S A 1972 Astrophys. J. 178 347
Teukolsky S A and Press W H 1974 Astrophys. J. 193 443
[13] Cardoso V, Dias O J C, Lemos J P S and Yoshida S 2004 Phys. Rev. D 70 044039
[14] Mukohyama S 2000 Phys. Rev. D 61 124021
[15] Bahados M, Teitelboim C and Zanelli J 1992 Phys. Rev. Lett. 69 1849
[16] Ichinose I and Satoh Y 1995 Nucl. Phys. 447 340
[17] Kim S-W, Kim W T, Park Y-J and Shin H 1997 Phys. Lett. B 392 311
[18] Fattore L, Ferraris M, Fracaviglia M and Raiteri M 1999 Phys. Rev. D 60 124012
[19] Ho J and Kang G 1998 Phys. Lett. B 445 27
[20] Kenmoku M, Ishimoto K, Nandi K K and Shigemoto K 2006 Phys. Rev. D 73 064004
[21] Kenmoku M and Kobayashi Y 2006 Class. Quantum Grav. 23 6257–74
[22] Birmingham D 2001 Phys. Rev. D 64 064024
[23] Cardoso V and Lemos J P S 2001 Phys. Rev. D 63 124015
[24] Kowata M, Kenmoku M and Shigemoto K 2008 Prog. Theor. Phys. 119 939
[25] Hartle J B and Hawking S W 1976 Phys. Rev. D 13 2188
[26] Kenmoku M and Shigemoto K 2008 in preparation
[27] Carlip S 2005 Class. Quantum Grav. 22 R85
[28] Gupta K S and Sen S 2007 Phys. Lett. B 646 265
[29] Witten E 2007 Three-dimensional gravity reconsidered Preprint arXiv:0706.3359
[30] Gupta R K and Sen A 2008 J. High Energy Phys. JHEP03(2008)015