Spectra of general hypergraphs

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Abstract

Here, we show a method to reconstruct connectivity hypermatrices of a general hypergraph (without any self loop or multiple edge) using tensor. We also study the different spectral properties of these hypermatrices and find that these properties are similar for graphs and uniform hypergraphs. The representation of a connectivity hypermatrix that is proposed here can be very useful for the further development in spectral hypergraph theory.

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1 Introduction

Spectral graph theory has a long history behind its development. In spectral graph theory, we analyse the eigenvalues of a connectivity matrix which is uniquely defined on a graph. Many researchers have had a great interest to study the eigenvalues of different connectivity matrices, such as, adjacency matrix, Laplacian matrix, signless Laplacian matrix, normalized Laplacian matrix, etc. Now, a recent trend has been developed to explore spectral hypergraph theory. Unlike in a graph, an edge of a hypergraph can be constructed with more than two vertices, i.e., the edge set of a hypergraph is the subset of the power set of the vertex set of that hypergraph\textsuperscript{22}. Now, one of the main challenges is to uniquely represent a hypergraph by a connectivity hypermatrix or by a tensor, and vice versa. It is not trivial for a non-uniform hypergraph, where the cardinalities of the edges are not the same. Recently, the study of the spectrum of uniform hypergraph becomes popular. In a \((m-)\) uniform hypergraph, each edge contains the same, \((m)\), number of vertices. Thus an \(m\)-uniform hypergraph of order \(n\) can be easily represented by an \(m\) order \(n\) dimensional connectivity hypermatrix (or tensor). In \textsuperscript{7}, the results on the spectrum of adjacency matrix of a graph are extended for uniform hypergraphs by using characteristic polynomial. Spectral properties of adjacency uniform hypermatrix are deduced from matroids in \textsuperscript{16}. In 1993, Fan Chung defined Laplacian of a uniform hypergraph by considering various homological aspects of hypergraphs and studied the eigenvalues of the same \textsuperscript{5}. In \textsuperscript{8, 9, 10, 11, 18, 19}, different spectral properties of Laplacian and signless Laplacian of a uniform hypergraph, defined by using tensor, have been studied. In 2015, Hu and Qi introduced the normalized Laplacian of a uniform hypergraph and analyzed its spectral properties \textsuperscript{8}. The important tool that has been used in spectral hypergraph theory is tensor. In 2005, Liqun Qi introduced the different eigenvalues of a real supersymmetric tensor \textsuperscript{17}. The various properties of the eigenvalues of a tensor have been studied in \textsuperscript{3, 4, 13, 14, 20, 21, 23, 24}.

But, still the challenge remains to come up with a mathematical framework to construct a connectivity hypermatrix for a non-uniform hypergraph, such that, based on this connectivity hypermatrix the spectral graph theory
for a general hypergraph can be developed. Here, we propose a unique representation of a general hypergraph (without any self loop or multiple edge) by connectivity hypermatrices, such as, adjacency hypermatrix, Laplacian hypermatrix, signless Laplacian hypermatrix, normalized Laplacian hypermatrix and analyze the different spectral properties of these matrices. These properties are very similar with the same for graphs and uniform hypergraphs. Studying the spectrum of a uniform hypergraphs could be considered as a special case of the spectral graph theory of general hypergraphs.

2 Preliminary

Let \( \mathbb{R} \) be the set of real numbers. We consider an \( m \) order \( n \) dimensional hypermatrix \( A \) having \( n^m \) elements from \( \mathbb{R} \), where

\[
A = (a_{i_1,i_2,\ldots,i_m}, a_{i_1,i_2,\ldots,i_m} \in \mathbb{R} \text{ and } 1 \leq i_1, i_2, \ldots, i_m \leq n)
\]

Let \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \). If we write \( x^m \) as an \( m \) order \( n \) dimension hypermatrix with \((i_1, i_2, \ldots, i_m)\)-th entry \( x_{i_1}x_{i_2}\ldots x_{i_m} \), then \( Ax^{m-1} \), where the multiplication is taken as tensor contraction over all indices, is an \( n \) tuple whose \( i \)-th component is

\[
\sum_{i_2,i_3,\ldots,i_m=1}^{n} a_{i_2,i_3,\ldots,i_m}x_{i_2}x_{i_3}\ldots x_{i_m}.
\]

**Definition 2.1.** Let \( A \) be a nonzero hypermatrix. A pair \((\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})\) is called eigenvalue and eigenvector (or simply an eigenpair) if they satisfy the following equation

\[
Ax^{m-1} = \lambda x^{[m-1]}.
\]

Here, \( x^{[m]} \) is a vector with \( i \)-th entry \( x_{i}^{m} \). We call \((\lambda, x)\) an \( H \)-eigenpair (i.e., \( \lambda \) and \( x \) are called \( H \)-eigenvalue and \( H \)-eigenvector, respectively) if they are both real. An \( H \)-eigenvalue \( \lambda \) is called \( H^+ (H^{++}) \)-eigenvalue if the corresponding eigenvector \( x \in \mathbb{R}_+^n \ (\mathbb{R}_+^n) \).

**Definition 2.2.** Let \( A \) be a nonzero hypermatrix. A pair \((\lambda, x) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})\) is called an \( E \)-eigenpair (where \( \lambda \) and \( x \) are called \( E \)-eigenvalue and \( E \)-eigenvector, respectively) if they satisfy the following equations

\[
Ax^{m-1} = \lambda x,
\]

\[
\sum_{i=1}^{n} x_{i}^{2} = 1.
\]

We call \((\lambda, x)\) a \( Z \)-eigenpair if both of them are real.

From the above definitions it is clear that, a constant multiplication of an eigenvector is also an eigenvector corresponding to an \( H \)-eigenvalue, but, this is not always true for \( E \)-eigenvalue and \( Z \)-eigenvalue. Now, we recall some results that are used in the next section.

**Theorem 2.1 ([7]).** The eigenvalues of \( A \) lie in the union of \( n \) disks in \( \mathbb{C} \). These \( n \) disks have the diagonal elements of the supersymmetric tensor as their centers, and the sums of the absolute values of the off-diagonal elements as their radii.

The above theorem helps us to bound the eigenvalues of a tensor.

**Lemma 2.1.** Let \( A \) be an \( m \) order and \( n \) dimensional tensor and \( D = \text{diag}(d_1, \ldots, d_n) \) be a positive diagonal matrix. Define a new tensor

\[
B = A.D^{-(m-1)}.
\]

with the entries

\[
B_{i_1i_2\ldots i_m} = A_{i_1i_2\ldots i_m}d_{i_1}^{-(m-1)}d_{i_2}\ldots d_{i_m}.
\]

Then \( A \) and \( B \) have the same \( H \)-eigenvalues.
Proof. From the remarks of lemma (3.2) in [23].

Some results of spectral graph theory also hold for general hypergraphs. If λ is any eigenvalue of an adjacency matrix of a graph G with the maximal degree Δ, then λ ≤ Δ. For a k-regular graph k is the maximum eigenvalue with a constant eigenvector of the adjacency matrix of that graph. If λ and μ are the eigenvalues of the adjacency matrices, represent the graphs G and H, respectively, then λ + μ is also an eigenvalue of the same for G × H, the cartesian product of G and H. All the eigenvalues of a Laplacian matrix of a graph are nonnegative and a very rough upper bound of these eigenvalues is 2Δ, whereas, any eigenvalue of a normalized Laplacian matrix of a graph lies in the interval [0, 2]. Zero is always an eigenvalue for both, Laplacian and normalized Laplacian matrices, of a graph, with a constant eigenvector. If L is a connectivity matrix of a graph with r connected components then σ(M) = σ(M1) ∪ σ(M2) · · · ∪ σ(Mr), where Mi is the same connectivity matrix corresponding to the component i.

3 Spectral properties of general hypergraphs

Definition 3.1. A (general) hypergraph G is a pair G = (V, E) where V is a set of elements called vertices, and E is a set of non-empty subsets of V called edges. Therefore, E is a subset of \(\mathcal{P}(V)\) \(\setminus\{\emptyset\}\), where \(\mathcal{P}(V)\) is the power set of V.

Example 3.1. Let \(G = (V, E)\), where \(V = \{1, 2, 3, 4, 5\}\) and \(E = \{\{1\}, \{2, 3\}, \{1, 4, 5\}\}\). Here, G is a hypergraph of 5 vertices and 3 edges.

3.1 Adjacency hypermatrix and eigenvalues

Definition 3.2. Let \(G = (V, E)\) be the hypergraph where \(V = \{v_1, v_2, \ldots, v_n\}\) and \(E = \{e_1, e_2, \ldots, e_k\}\). Let \(m = \max\{|e_i| : e_i \in E\}\) be the maximum cardinality of edges, m.c.e(G), of G. Define the adjacency hypermatrix of G as

\[
\mathcal{A}_G = (a_{i_1i_2\cdots i_m}), \quad 1 \leq i_1, i_2, \ldots, i_m \leq n.
\]

For all edges \(e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_s}\} \in E\) of cardinality \(s \leq m\),

\[
ap_{p_1p_2\cdots p_m} = \frac{s}{\alpha}, \quad \text{where} \quad \alpha = \sum_{k_1, k_2, \ldots, k_s \geq 1, \sum k_i = m} \frac{m!}{k_1!k_2!\cdots k_s!},
\]

and \(p_1, p_2, \ldots, p_m\) chosen in all possible way from \(\{l_1, l_2, \ldots, l_s\}\) with at least once for each element of the set. The other positions of the hypermatrix are zero.

Example 3.2. Let \(G = (V, E)\) be a hypergraph in example 3.1. Here, the maximum cardinality of edges is 3. The adjacency hypermatrix of G is

\[
\mathcal{A}_G = (a_{i_1i_2i_3}), \quad 1 \leq i_1, i_2, i_3 \leq 5.
\]

Here, \(a_{111} = 1, a_{233} = a_{232} = a_{223} = a_{323} = a_{332} = a_{322} = \frac{1}{3}, a_{145} = a_{154} = a_{451} = a_{415} = a_{541} = a_{514} = \frac{1}{2}\), and the other elements of \(\mathcal{A}_G\) are zero.

Definition 3.3. Let \(G = (V, E)\) be a hypergraph. The degree, \(d(v)\), of a vertex \(v \in V\) is the number of edges consist of v.

Let \(G = (V, E)\) be a hypergraph, where \(V = \{v_1, v_2, \ldots, v_n\}\) and \(E = \{e_1, e_2, \ldots, e_k\}\). Then, the degree of a vertex \(v_i\) is given by

\[
d(v_i) = \sum_{i_2, i_3, \ldots, i_m=1}^{n} a_{i_1i_2\cdots i_m}.
\]

\footnote{For different spectral properties of a graph see [2, 6].}

\footnote{For a similar construction on uniform multi-hypergraph see [13].}
Definition 3.4. A hypergraph is called $k$-regular if every vertex has the same degree $k$.

Now, we discuss some spectral properties of $A_G$ of a hypergraph $G$. Some of these properties are very similar as in general graph (i.e. for a 2-uniform hypergraph).

Theorem 3.1. Let $\mu$ be an $H$-eigenvalue of $A_G$. Then $|\mu| \leq \Delta$, where $\Delta$ is the maximum degree of $G$.

Proof. Let $G$ be a hypergraph with $n$ vertices and $m.c.e(G) = m$. Let $\mu$ be an $H$-eigenvalue of $A_G = (a_{i_1i_2...i_m})$ with an eigenvector $x = (x_1, x_2, ..., x_n)$. Let $x_p = \max\{|x_1|, |x_2|, ..., |x_n|\}$. Without loss of any generality we can assume that $x_p = 1$. Now,

$$|\mu| = |\mu x_p^{m-1}| = \left| \sum_{i_2, i_3, ..., i_m = 1}^n a_{i_2i_3...i_m}x_1x_2x_3...x_m \right|$$

$$\leq \sum_{i_2, i_3, ..., i_m = 1}^n |a_{i_2i_3...i_m}|x_p^{m-1} = d(v_p) \leq \Delta.$$

Thus, for a $k$-regular hypergraph the theorem (3.1) implies $|\mu| \leq k$.

Theorem 3.2. Let $G = (V, E)$ be a $k$-regular hypergraph with $n$ vertices. Then, $A_G = (a_{i_1i_2...i_m})$ has an $H$-eigenvalue $k$.

Proof. Since, $G$ is $k$-regular, then $d(v_i) = k$ for all $v_i \in V, i \in \{1, 2, 3, ..., n\}$. Now, for a vector $x = (1, 1, 1, ..., 1) \in \mathbb{R}^n$, we have

$$A_Gx^{m-1} = \sum_{i_2, i_3, ..., i_m = 1}^n a_{i_1i_2...i_m} = k.$$ 

Thus the proof.

Theorem 3.3. Let $G = (V, E)$ be a $k$-regular hypergraph with $n$ vertices. Then, $A_G = (a_{i_1i_2...i_m})$ has a $Z$-eigenvalue $k(\frac{1}{\sqrt{n}})^{m-2}$.

Proof. The vector $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, ..., \frac{1}{\sqrt{n}}) \in \mathbb{R}^n$ satisfies the $Z$-eigenvalue equations for $\lambda = k(\frac{1}{\sqrt{n}})^{m-2}$. 

Theorem 3.4. Let $G$ be a hypergraph with $n$ vertices and maximum degree $\Delta$. Let $x = (x_1, x_2, ..., x_n)$ be a $Z$-eigenvector of $A_G = (a_{i_1i_2...i_m})$ corresponding to an eigenvalue $\mu$. If $x_p = \max\{|x_1|, |x_2|, ..., |x_n|\}$, then $|\mu| \leq \frac{\Delta}{x_p}$.

Proof. The $Z$-eigenvalue equations of $A_G$ for $\mu$ and $x$ are $Ax^{m-1} = \mu x$, and $\sum x_i^2 = 1$. Therefore, $|x_i| \leq 1$, for all $i = 1, 2, 3, ..., n$. Now,

$$|\mu||x_j| = \left| \sum_{i_2, i_3, ..., i_m = 1}^n a_{i_1i_2...i_m}x_{i_2}x_{i_3}...x_{i_m} \right|,$$

which implies $|\mu||x_j| \leq d(j) \leq \Delta, \forall j = 1, 2, 3, ..., n$. Therefore, $|\mu| \leq \frac{\Delta}{x_p}$.

Definition 3.5. A hypergraph $H = (V_1, E_1)$ is said to be a spanning subhypergraph of a hypergraph $G = (V, E)$, if $V = V_1$ and $E_1 \subseteq E$.

Theorem 3.5. Let $G = (V, E)$ be hypergraph. Let $H = (V', E')$ be a subhypergraph of $G$, such that, $m.c.e(G) = m.c.e(H)$ be even. Then, $\mu_{\text{max}}(H) \leq \mu_{\text{max}}(G)$, where $\mu_{\text{max}}$ is the highest $Z$-eigenvalue of the corresponding adjacency hypermatrix.
Proof. Let $|V| = n$, $|V'| = n'$ ($\leq n$) and $m.c.e(G) = m.c.e(H) = m$. Now,

$$
\mu_{\max}(H) = \max_{|x| = 1} x^t A_H x^{m-1} \quad \text{(by using lemma (3.1) in [13])}
$$

$$
= \max_{|x| = 1} \left( \sum_{i_1, i_2, \ldots, i_m = 1}^n a_{i_1 i_2 \cdots i_m}^H x_{i_1} x_{i_2} \cdots x_{i_m} \right)
$$

$$
= \max_{|x| = 1} \left( \sum_{i_1, i_2, \ldots, i_m = 1}^n a_{i_1 i_2 \cdots i_m}^H x_{i_1} x_{i_2} \cdots x_{i_m} \right), \quad \text{where } a_{i_1 i_2 \cdots i_m}^H = x_{i_r} = 0 \text{ when } i_r > n'
$$

$$
\leq \left( \sum_{i_1, i_2, \ldots, i_m = 1}^n a_{i_1 i_2 \cdots i_m}^G x_{i_1} x_{i_2} \cdots x_{i_m} \right)
$$

$$
\leq \mu_{\max}(G),
$$

since each component of $x$ is nonnegative (by Perron-Frobenious theorem [3]) and the number of edges of $G$ is greater than or equal to the number of edges of $H$. Hence the proof. \(\square\)

**Definition 3.6.** Let $G = (V, E)$ be a hypergraph with $V = \{v_1, v_2, \ldots, v_n\}$, $E = \{e_1, e_2, \ldots, e_k\}$, and $m.c.e(G) = m$. Let $x = (x_1, x_2, \ldots, x_n)$ be a vector in $\mathbb{R}^n$ and $p \geq s - 1$ be an integer. For an edge $e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_s}\}$ and a vertex $v_{l_i}$, we define

$$
x_{e/v_{l_i}} := \sum x_{r_1} x_{r_2} \cdots x_{r_p},
$$

where the sum is over $r_1, r_2, \ldots, r_p$ chosen in all possible way from $\{l_1, l_2, \ldots, l_s\}$, such that, all $l_j (j \neq i)$ occur at least once. Whereas,

$$
x_{e} := \sum x_{r_1} x_{r_2} \cdots x_{r_p},
$$

where the sum is over $r_1, r_2, \ldots, r_p$ chosen in all possible way from $\{l_1, l_2, \ldots, l_s\}$ with at least once for each element of the set.

The symmetric (adjacency) hypermatrix $A_G$ of order $m$ and dimension $n$ uniquely defines a homogeneous polynomial of degree $m$ and in $n$ variables by

$$
F_{A_G}(x) = \sum_{i_1, i_2, \ldots, i_m = 1}^n a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.
$$

We rewrite the above polynomial as:

$$
F_{A_G}(x) = \sum_{e \in E} a_e^G x_{e/m},
$$

where $a_e^G = \frac{s}{\alpha}$, $\alpha = \sum_{k_1, k_2, \ldots, k_s \geq 1, \sum k_i = m} \frac{m!}{k_1! k_2! \cdots k_s!}$, and $s$ is the cardinality of the edge $e$.

**Definition 3.7.** Let $G$ and $H$ be two hypergraphs. The Cartesian product, $G \times H$, of $G$ and $H$ is defined by the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(v) \times e : v \in V(G), e \in E(H)\} \cup \{e \times \{v\} : e \in E(G), v \in V(H)\}$.

**Definition 3.8.** Let $G$ be a hypergraph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and $m.c.e(G) = m$. For an edge $e = \{v_{l_1}, v_{l_2}, \ldots, v_{l_s}\}$ and an integer $r \geq m$, the arrangement $(v_{p_1} v_{p_2} \cdots v_{p_r})$ (where $p_1, p_2, \ldots, p_r$ are chosen in all possible way from $\{l_1, l_2, \ldots, l_s\}$ with at least once for each element of the set) represents the edge $e$ in order $r$.

**Example 3.3.** Let $G = (V, E)$ where $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 2, 3\}, \{2, 3, 5\}, \{1, 3, 4, 5\}\}$, then the arrangement (12233) represents the edge $1, 2, 3$ in order 5. (12123) is also a representation of the edge $\{1, 2, 3\}$ in order five, whereas, (111123) represents the edge $\{1, 2, 3\}$ in 6 order.
Let $G = (V,E)$ be a hypergraph with $m.c.e(G) = m$ and $E_i = \{e \in E : v_i \in e\}$. Now, the $H$-eigenvalue equation for $A_G$ becomes

$$\sum_{e \in E_i} a^e_{Ax_{m-1}} = \lambda x^{(m-1)}_i, \text{ for all } i.$$ 

**Theorem 3.6.** Let $G$ and $H$ be two hypergraphs with $m.c.e(G) = m.c.e(H)$. If $\lambda$ and $\mu$ are $H$-eigenvalue for $G$ and $H$, respectively, then $\lambda + \mu$ is an $H$-eigenvalue for $G \times H$.

**Proof.** Let $n_1$ and $n_2$ be the number of vertices in $G$ and $H$, respectively, and $m.c.e(G) = m.c.e(H) = m$. Let $(\lambda, u)$ and $(\mu, v)$ be $H$-eigenpairs of $A_G$ and $A_H$, respectively. Let $w \in \mathbb{C}^{n_1 \times n_2}$ be a vector with the entries indexed by the pairs $(a, b) \in [n_1] \times [n_2]$, such that, $w(a,b) = u(a)v(b)$. Now, we show that $(\lambda + \mu, w)$ is an $H$-eigenpair of $A_{G \times H}$.

$$\sum_{e \in E(a,b)} a^e_{G \times H} w_{m-1} = \sum_{\{(a)\times e \in E(a,b)\} \text{ with } e \in E_h} a^e_{G \times H} w_{m-1} + \sum_{\{(b)\times e \in E(a,b)\} \text{ with } e \in E_a} a^e_{G \times H} w_{m-1}$$

$$= u^{m-1}(a) \sum_{e \in H_h} a^e_{G \times H} b^{m-1} + \sum_{e \in G_a} a^e_{G \times H} b^{m-1}(b)$$

$$= u^{m-1}(a) \mu v^{m-1}(b) + v^{m-1}(b)\lambda u^{m-1}(a)$$

$$= (\lambda + \mu) w^{m-1}(a,b).$$

Hence the proof. □

**Lemma 3.1.** Let $A$ and $B$ be two symmetric hypermatrices of order $m$ and dimension $n$, where $m$ is even. Then $\lambda_{\max}(A + B) \leq \lambda_{\max}(A) + \lambda_{\max}(B)$, where $\lambda_{\max}(A)$ denotes the largest $Z$-eigenvalue of $A$.

**Proof.**

$$\lambda_{\max}(A + B) = \max_{||x||=1} x^t(A + B)x^{m-1} \quad \text{(by using lemma (3.1) in [13])}$$

$$\leq \max_{||x||=1} x^tAx^{m-1} + \max_{||x||=1} x^tBx^{m-1}$$

$$= \lambda_{\max}(A) + \lambda_{\max}(B).$$

□

Let $G = (V,E)$ be a hypergraph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and $m = \max\{|e| : e \in E\}$. We partition the edge set $E$ as $E = E_1 \cup E_2 \cup \cdots \cup E_m$, where $E_i$ contains all the edges of the cardinality $i$ and construct a hypergraph $G_i = (V,E_i)$, for a nonempty $E_i$.

**Definition 3.9.** Define the adjacency hypermatrix of $G_i$ in $m (> i)$-order by an $n$ dimensional $m$ order hypermatrix

$$A_{G_i}^m = (a^{m}_{G_i})_{p_1p_2 \cdots p_m}, \quad 1 \leq p_1, p_2, \ldots, p_m \leq n,$$

such that, for any $e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_1}\} \in E_i$,

$$(a^{m}_{G_i})_{p_1p_2 \cdots p_m} = \frac{i}{\alpha}, \quad \text{where } \alpha = \sum_{k_1, k_2, \ldots, k_i \geq 1, \sum k_j = m} \frac{m!}{k_1!k_2! \ldots k_i!}$$

and $p_1, p_2, \ldots, p_m$ are chosen in all possible way from $\{l_1, l_2, \ldots, l_i\}$ with at least once for each element of the set. The other positions of $A_{G_i}^m$ are zero.

\(^3\)For similar proof on uniform hypergraph see [7].
Thus, we can represent a hypergraph $G$, with $m.c.e(G) = s$, in higher order $m > s$ by the hypermatrix $A_G^m$. Clearly, all the eigenvalue equations show that the eigenvalues of $A_G^{m_1}$ and $A_G^{m_2}$ are not equal for $m_1 \neq m_2$.

**Theorem 3.7.** Let $G = (V, E)$ be a hypergraph and $m.c.e(G) = m$ be even. Then $\lambda_{\max}(A_G) \leq \sum_{i=1}^m \lambda_{\max}(A_{G_i}^m)$, where $\lambda_{\max}(A)$ is the largest Z-eigenvalue of $A$.

**Proof.** Since $A_G = \sum_{i=1}^m A_{G_i}^m$, the proof follows from the lemma (3.1).

Moreover, the theorem (3.7) implies $\lambda_{\max}(A_G) \leq \sum_{i=1}^m n_i \lambda_{\max}(A_{G_i}^m)$, where $n_i$ is the number of edges of cardinality $i$ and $A_i^m$ is the adjacency hypermatrix in $m$-order of a hypergraph contains a single edge of cardinality $i$.

### 3.2 Laplacian hypermatrix and eigenvalues

**Definition 3.10.** Let $G = (V, E)$ be a (general) hypergraph without any isolated vertex where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_k\}$. Let $m.c.e(G) = m$. We define the Laplacian hypermatrix, $L_G$, of $G = (V, E)$ as $L_G = D_G - A_G = (l_{i_1i_2\ldots i_m}), 1 \leq i_1, i_2, \ldots, i_m \leq n$, where $D_G = (d_{i_1i_2\ldots i_m})$ is the $m$ order $n$ dimensional diagonal hypermatrix with $d_{i_1i_2\ldots i_m} = d(v_i)$ and others are zero. The signless Laplacian of $G$ is defined as $L_G = D_G + A_G$.

Let $G = (V, E)$ be a hypergraph with $m.c.e(G) = m$. For any edge $e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\}$, we define a homogeneous polynomial of degree $m$ and in $n$ variables by

$$L(e)x^m = \sum_{j=1}^s x_{i_j}^m - \frac{s}{\alpha} x_m^e \ (s \leq m).$$

**Proposition 3.1.** $\sum_{j=1}^s x_{ij}^m \geq \frac{s}{\alpha} x_m^e \ (x_{ij} \in \mathbb{R}_+)$.

**Proof.** $x_m^e$ is the sum of all possible terms, $x_{i_1}^{k_1}x_{i_2}^{k_2}\ldots x_{i_s}^{k_s}$ (where $\sum k_i = m$ and $k_i \geq 1$)

$$\text{where } \alpha = \sum_{k_1,k_2,\ldots,k_s \geq 1, \sum k_i = m} \frac{m!}{k_1!k_2!\ldots k_s!},$$

with some natural coefficient. Now, by applying AM-GM inequality on $k_1x_{i_1}^m, k_2x_{i_2}^m, \ldots, k_sx_{i_s}^m$ we get

$$\frac{1}{m} \sum_{j=1}^s k_jx_{ij}^m \geq x_{i_1}^{k_1}x_{i_2}^{k_2}\ldots x_{i_s}^{k_s}.$$ (1)

If we apply (1) for each term of $x_m^e$ and take the sum, we get

$$\frac{\alpha}{s} \sum_{j=1}^s x_{ij}^m \geq x_m^e.$$ 

□

Many properties of Laplacian and signless Laplacian tensors are discussed in [18]. Now we show that some of the results in general graph are also true for non-uniform (general) hypergraph. Note that, here, $L$ is co-positive tensor since, $Lx^m = \sum_{e \in E} L(e)x^m \geq 0$ for all $x \in \mathbb{R}_+^n$.

**Theorem 3.8.** Let $G = (V, E)$ be a general hypergraph. Let $L = (l_{i_1i_2\ldots i_m})$ where $1 \leq i_1, i_2, \ldots, i_m \leq n$, be the Laplacian hypermatrix of $G$. Then $0 \leq \lambda \leq 2\Delta$, where $\lambda$ is an $H$-eigenvalue of $L$. 


Proof. For a vector \( y = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}) \), \( Ly = 0 \). Since \( L \) is a co-positive tensor, thus \( \min \{ Lx^m : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^m = 1 \} = 0 \). Therefore \( \lambda \geq 0 \).

Again using theorem (6(a)) of \cite{17} we have

\[
|\lambda - l_{ii\ldots i}| \leq \sum_{i_2, i_3, \ldots, i_m = 1, \delta_{i_2, i_3, \ldots, i_m} = 0}^{n} |l_{i_2 i_3 \ldots i_m}| = \Delta,
\]

i.e., \( |\lambda| \leq 2\Delta \). Thus \( 0 \leq \lambda \leq 2\Delta \). \( \square \)

**Theorem 3.9.** Let \( G = (V, E) \) be a general hypergraph with \( m.c.e(G) = m \geq 3 \). Let \( L \) be the Laplacian hypermatrix of \( G \). Then

(i) \( L \) has an \( H \)-eigenvalue 0 with eigenvector \( (1, 1, \ldots, 1) \in \mathbb{R}^n \) and an \( Z \)-eigenvalue 0 with eigenvector \( x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n \). Moreover, 0 is the unique \( H^{++} \)-eigenvalue of \( L \).

(ii) \( \Delta \) is the largest \( H^+ \)-eigenvalue of \( L \).

(iii) \( (d(i), e^{(j)}) \) is an \( H \)-eigenpair, where \( e^{(j)} \in \mathbb{R}^n \) and \( e^{(j)}_i = 1 \) if \( i = j \), otherwise 0.

(iv) For a nonzero \( x \in \mathbb{R}^n \) \( d(v_i, x) \) is an eigenpair if \( \sum_{e \in E_i} a^e_{e^{(i)}} e^{(i)} = 0 \).

**Proof.**

(i) It is easy to check that 0 is an \( H \)-eigenvalue with the eigenvector \( (1, 1, \ldots, 1) \in \mathbb{R}^n \) and 0 is an \( Z \)-eigenvalue with the eigenvector \( x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n \). Let \( x \) be an \( H^{++} \)-eigenvector of \( L \) with eigenvalue \( \lambda \). By theorem \( \cite{18} \), \( \lambda \geq 0 \). Suppose \( x_j = \min_i \{ x_i \} \). Therefore \( x_j \) is positive. Now,

\[
\lambda x_j^{m-1} = d(v_j)x_j^{m-1} - \sum_{e \in E, |e| = s} \frac{s}{\alpha} \sum_{e \in \{i, j, \ldots, i_m\} \text{ as set, } i, j, \ldots, i_m = 1} x_{i_2} x_{i_3} \ldots x_{i_m},
\]

which implies that

\[
\lambda = d(v_j) - \sum_{e \in E, |e| = s} \frac{s}{\alpha} \sum_{e \in \{i, j, \ldots, i_m\} \text{ as set, } i, j, \ldots, i_m = 1} x_{i_2} x_{i_3} \ldots x_{i_m}.
\]

Thus, \( \lambda \leq d(v_j) - d(v_j) = 0 \). Hence \( \lambda = 0 \).

(ii) Suppose \( \lambda \) is an \( H^+ \)-eigenvalue with non-negative \( H^+ \)-eigenvector, \( x \) of \( L \). Assume that \( x_j > 0 \). Now, we have

\[
\lambda x_j^{m-1} = d(v_j)x_j^{m-1} - \sum_{e \in E, |e| = s} \frac{s}{\alpha} \sum_{e \in \{i, j, \ldots, i_m\} \text{ as set, } i, j, \ldots, i_m = 1} x_{i_2} x_{i_3} \ldots x_{i_m} \leq d(v_j)x_j^{m-1}.
\]

Therefore \( \lambda \leq d(v_j) \leq \Delta \). Thus, \( \Delta \) is the largest \( H^+ \)-eigenvalue of \( L \).

(iii) Proof is obvious.

(iv) It is clear from the eigenvalue equation. \( \square \)

Let \( G = (V, E) \) be a general hypergraph and \( m.c.e(G) = m \). The analytic connectivity, \( \alpha(G) \), of \( G \) is defined as \( \alpha(G) = \min \min \{ Lx^m : x \in \mathbb{R}_+^n, \sum_{i=1}^n x_i^m = 1, x_j = 0 \} \).

**Theorem 3.10.** The general hypergraph \( G = (V, E) \) with \( m.c.e(G) \geq 3 \) is connected if and only if \( \alpha(G) > 0 \).
Proof. Suppose \( G = (V, E) \) is not connected. Let \( G_1 = (V_1, E_1) \) be a component of \( G \). Then there exists \( j \in V \setminus V_1 \). Let \( x = \frac{1}{|V_1|^m} \sum_{i \in V_1} e^{(i)} \). Then \( x \) is a feasible point. Therefore \( \min \{ Lx^m | x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^m = 1, x_j = 0 \} = 0 \), which implies \( \alpha(G) = 0 \).

Let \( \alpha(G) = 0 \). Thus there exists \( j \) such that \( \min \{ Lx^m | x \in \mathbb{R}^n_+, \sum_{i=1}^n x_i^m = 1, x_j = 0 \} = 0 \). Suppose that \( y \) is a minimizer of this minimization problem. Therefore \( y_j = 0, Ly^m = 0 \). By optimization theory, there exists a Lagrange multiplier \( \mu \) such that for \( i = 1, 2, \ldots, n \) and \( i \neq j \), either, \( y_i = 0 \) and \( \frac{\partial}{\partial y_i}(Ly^m) = 0 \) or, \( y_i > 0 \) and

\[
\frac{\partial}{\partial y_i}(Ly^m) = \mu \frac{\partial}{\partial y_i} \left( \sum_{i=1}^n y_i^m - 1 \right).
\]

In (2) and (3) \( y \in \mathbb{R}^n_+, \sum_{i=1}^n y_i^m = 1, y_j = 0 \). Now, multiplying (2) and (3) by \( y_i \) and summing them for \( i = 1, \ldots, n \), we have \( Ly^m = \mu (\sum_{i=1}^n y_i^m) \). Thus \( Ly^m = \mu \). Hence \( \mu = 0 \). Therefore, for \( i = 1, 2, \ldots, n \) and \( i \neq j \), either \( y_i = 0 \) or \( \frac{\partial}{\partial y_i}(Ly^m) = 0 \). Hence, either \( y_i = 0 \) or \( d_i(y_i)^{m-1} - \sum_{i=2,i=3,\ldots,i=m} a_{i_2i_3\ldotsi_m} y_{i_2}y_{i_3} \ldots y_{i_m} = 0 \). Let \( y_k = \max \{ y_i : i = 1, 2, \ldots, n \} \). Hence, we have

\[
d_k = \sum_{i=2,i=3,\ldots,i=m=1}^n a_{i_2i_3\ldotsi_m} y_{i_2} y_{i_3} \ldots y_{i_m} = 0.
\]

Again, we know that

\[
d(v_k) = \sum_{i=2,i=3,\ldots,i=m=1}^n a_{i_2i_3\ldotsi_m}.
\]

Therefore, \( x_i = x_k \) as long as \( i \) and \( k \) belong to same edge. Thus, \( x_i = x_k \) as long as \( i \) and \( k \) are in different components of \( G \). Since \( y_j = 0 \), we have, \( j \) and \( k \) are in different components of \( G \). Hence, \( G \) is not connected. This proves the theorem.

\( \square \)

3.3 Normalized Laplacian hypermatrix and eigenvalues

Now, we define normalized Laplacian hypermatrix for a general hypergraph. For any graph, there are two ways to construct normalized Laplacian matrix (see [1] and [6] for details). Motivated by these two similar constructions, here, we also define the normalized Laplacian hypermatrix in two different ways and show that they are cospectral. The first definition is similar to the normalized Laplacian matrix defined in [1].

Definition 3.11. Let \( G = (V, E) \) be a general hypergraph without any isolated vertex where \( V = \{ v_1, v_2, \ldots, v_n \} \) and \( E = \{ e_1, e_2, \ldots, e_k \} \). Let \( m.c.e(G) = m \). The normalized Laplacian hypermatrix \( L = (l_{i_1i_2\ldotsi_m}) \), which is an \( n \)-dimensional \( m \)-th order hypermatrix, is defined as: for any edge \( e = \{ v_{l_1}, v_{l_2}, \ldots, v_{l_s} \} \in E \) of cardinality \( s \leq m \),

\[
l_{p_1p_2\ldots p_m} = -\frac{s/\alpha}{d(v_{p_1})}, \quad \text{where } \alpha = \sum_{k_1,k_2,\ldots,k_s \geq 1}^{m!} \frac{m!}{k_1!k_2!\ldots k_s!}
\]

and \( p_1, p_2, \ldots, p_m \) are chosen in all possible way from \( \{ l_1, l_2, \ldots, l_s \} \), such that, all \( l_j \) occur at least once. All the diagonal entries are 1 and the rest are zero.

Clearly, the hypermatrix \( A = I - L \), which is known as normalized adjacency hypermatrix, is a stochastic tensor, that is, \( A \) is non-negative and \( \sum_{i=2,\ldots,i_m=1}^n a_{i_2\ldots i_m} = 1 \), where \( a_{i_2\ldots i_m} \) is the \( (i_1, i_2, \ldots, i_m) \)-th entry of \( A \). The different properties of a stochastic tensor are discussed in [24] and which can be used to study the hypermatrices \( A \) and \( L \). Now, we define the normalized Laplacian hypermatrix of a general hypergraph as it is defined for a graph in [6].

These two matrices are similar, i.e., they have same eigenvalues.
Definition 3.12. Let $G = (V, E)$ be a general hypergraph without any isolated vertex, where $V = \{v_1, v_2, \ldots, v_n\}$ and $E = \{e_1, e_2, \ldots, e_k\}$. Let $m.c.e(G) = m$. The normalized Laplacian hypermatrix $L = (l_{i_1i_2\ldots i_m})$, which is an $n$-dimension $m$-th order symmetric hypermatrix, is defined as: for any edge $e = \{v_{i_1}, v_{i_2}, \ldots, v_{i_s}\} \in E$ of cardinality $s \leq m$,

$$l_{p_1p_2\ldots p_m} = -s \prod_{j=1}^{m} \frac{1}{1/v_{p_j}}, \text{ where } \alpha = \sum_{k_1,k_2,\ldots,k_s \geq 1, \sum k_i = m} m! \frac{1}{k_1!k_2!\ldots k_s!}$$

and $p_1, p_2, \ldots, p_m$ chosen in all possible way from $\{l_1, l_2, \ldots, l_s\}$ with at least once for each element of the set. The diagonal entries of $L$ are 1 and the rest of the positions are zero.

Theorem 3.11. $L$ and $L$ are co-spectral.

Proof. In the lemma (2.1) choose a diagonal matrix $D = (d_{ij})_{n \times n}$ where $d_{ii} = (d(v_i))^{1/m}$.

Theorem 3.12. Let $G = (V, E)$ be a general hypergraph. Let $L$, A be the normalized Laplacian and normalized adjacency hyper matrices of $G$, respectively. If $G$ has at least one edge, then $\lambda \in \sigma(A)$ if and only if $(1-\lambda) \in \sigma(L)$, otherwise, $\sigma(A) = \sigma(L) = 0$, where $\sigma(L)$ denotes the spectrum of $L$.

Proof. Since, $L = I - A$ and $\lambda$ is the eigenvalue of $A$ iff $\det(A - \lambda I) = 0$, thus, $\det(L - (1-\lambda)I) = 0$ implies $(1-\lambda) \in \sigma(L)$.

Theorem 3.13. Let $G = (V, E)$ be a general hypergraph. Let $L = (l_{i_1i_2\ldots i_m})$ where $1 \leq i_1, i_2, \ldots, i_m \leq n$, and $A$ be the normalized Laplacian and normalized adjacency hyper matrices of $G$, respectively, then

(i) $\rho(A) = 1$.

(ii) $0 \leq \lambda(L) \leq 2$.

(iii) 1 is the largest $H^+$-eigenvalue of $L$.

(iv) 0 is an eigenvalue of $L$ with the eigenvector $(1, 1, \ldots, 1)$ and 0 is an $Z$-eigenvalue with eigenvector $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n$.

(v) 0 is the unique $H^+$-eigenvalue of $L$.

Proof. (i) Since $A$ is stochastic tensor, it is obvious that the spectral radius of $A$ is 1. Moreover, $(1, 1, \ldots, 1)$ is an eigenvector with eigenvalue 1.

(ii) We know that spectral radius of $A$ is 1 and $L = I - A$. By theorem (3.12), $\lambda \in \sigma(A)$ if and only if $(1-\lambda) \in \sigma(L)$. Since 1 is an eigenvalue of $A$, thus, $\lambda \geq 0$. Again, using theorem (6(a)) of [17] we have

$$|\lambda(L) - 1| \leq \sum_{i_2,i_3,\ldots,i_m=1, \delta_{i_1,i_2,\ldots,i_m}=0}^{n} |l_{i_1i_2i_3\ldots i_m}| = 1.$$

This implies $|\lambda(L)| \leq 2$. Thus, we have $0 \leq \lambda(L) \leq 2$.

(iii) Suppose that $\lambda$ is an $H^+$-eigenvalue with non-negative $H^+$-eigenvector, $x$ of $L$. Assume that $x_j > 0$. Now, we have

$$\lambda x_j^{m-1} = x_j^{m-1} - \frac{1}{d(v_j)} \sum_{e \in E, j \in e, |e|=s} s \sum_{i_2,i_3,\ldots,i_m} x_{i_2}x_{i_3}\ldots x_{i_m}.$$

Hence, $\lambda x_j^{m-1} \leq x_j^{m-1}$ implies $\lambda \leq 1$. Thus, 1 is the largest $H^+$-eigenvalue of $L$. 

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(iv) It is easy to check that 0 is an $H$-eigenvalue corresponding an eigenvector $(1, 1, \ldots, 1) \in \mathbb{R}^n$ and 0 is an $Z$-eigenvalue with the eigenvector $x = (\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}) \in \mathbb{R}^n$.

(v) Let $x$ is an $H^{++}$-eigenvector of $L$ with eigenvalue $\lambda$. From the part (ii) of this theorem we have $\lambda \geq 0$.

Suppose $x_j = \min_i \{x_i\}$. Now, 

$$\lambda x_j^{m-1} = x_j^{m-1} - \frac{1}{d(v_j)} \sum_{\substack{e \in E, j \in e, |e| = s \alpha \in \{i, i_2, \ldots, i_m\} \text{ as set, } i, i_2, \ldots, i_m = 1}} s x_i x_i \ldots x_i,$$

which implies that

$$\lambda = 1 - \frac{1}{d(v_j)} \sum_{\substack{e \in E, j \in e, |e| = s \alpha \in \{i, i_2, \ldots, i_m\} \text{ as set, } i, i_2, \ldots, i_m = 1}} s x_i x_i \ldots x_i x_j \ldots x_j.$$

Thus $\lambda \leq 1 - 1 = 0$. Hence $\lambda = 0$.

\[\Box\]

**Theorem 3.14.** Let $G = (V, E)$ be a general hypergraph and $m.c.e(G) = m$. Let $L$ be the normalized Laplacian hypermatrix of $G$ of order $m$ and dimension $n$. Let $m(\lambda)$ be the algebraic multiplicity of $\lambda \in \sigma(L)$, then $\sum_{\lambda \in \sigma(L)} m(\lambda) \lambda = n(m - 1)^{n-1}$.

**Proof.** Since, for any tensor $\mathcal{T} = (t_{i_1 i_2 \ldots i_m}), t_{i_1 i_2 \ldots i_m} \in \mathbb{C}, \ 1 \leq i_1, i_2, \ldots, i_m \leq n,$

$$\sum_{\lambda \in \sigma(\mathcal{T})} m(\lambda) \lambda = (m - 1)^{(n-1)} \sum_{i=1}^n t_{\alpha \ldots \alpha} \ (\text{see [12]}).$$

Hence, we have $\sum_{\lambda \in \sigma(L)} m(\lambda) \lambda = n(m - 1)^{n-1}$. \[\Box\]

**Theorem 3.15.** Let $G = (V, E)$ be a general hypergraph and $A$ be any connectivity hypermatrix of $G$. If $G$ has $r \geq 1$ connected components, $G_1, G_2, \ldots, G_r$, such that, $|V(G_i)| = n_i > 1$ and $m.c.e(G_i) = m.c.e(G)$ for each $i \in \{1, 2, \ldots, r\}$. Then, as sets, $\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \cup \cdots \cup \sigma(A_r)$, where $A_i$ is the connectivity hypermatrix of $G_i$.

**Proof.** Using corollary (4.2) of [21] we get

$$\phi_A(\lambda) = \prod_{i=1}^r (\phi_{A_i}(\lambda))^{(m-1)^{n-n_i}},$$

where $\phi_{A_i}(\lambda)$ is the characteristic polynomial of the tensor $A_i$. Therefore, $\sigma(A) = \sigma(A_1) \cup \sigma(A_2) \cup \cdots \cup \sigma(A_r)$. \[\Box\]

4 Discussion and conclusion

Here, we propose a mathematical framework to construct connectivity matrices for a general hypergraph and also study the eigenvalues of adjacency hypermatrix, Laplacian hypermatrix, normalized Laplacian hypermatrix. This connectivity hypermatrix reconstruction can be used for further development of spectral hypergraph theory in many aspects, but, this may not be quite useful to study dynamics on hypergraphs.

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