Non-linear WKB Analysis of the String Equation

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ABSTRACT

We apply non-linear WKB analysis to the study of the string equation. Even though the solutions obtained with this method are not exact, they approximate extremely well the true solutions, as we explicitly show using numerical simulations. “Physical” solutions are seen to be separatrices corresponding to degenerate Riemann surfaces. We obtain an analytic approximation in excellent agreement with the numerical solution found by Parisi et al. for the $k = 3$ case.

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1. Introduction

During the last two years, the partition function of two-dimensional gravity has been set in correspondence with the $\tau$-function of the KdV hierarchy, subjected to the constraint of the so-called string equation \cite{1}. In the case of pure gravity, described by a matrix model with criticality index $k = 2$, the string equation reduces to the well known Painlevé type I equation for the specific heat of the theory. The string equation can be seen as a perturbation of a stationary KdV equation, and thus solved in a semiclassical approximation \cite{2} around the stationary KdV solution. Novikov and Krichever \cite{3} have conjectured that such an approach lead to exact solutions already at 0-th order in perturbation theory. We have resorted to accurate numerical simulations in order to check this conjecture. On the other side, we show that “physical” solutions to the $k$-th multicritical model are obtained as separatrices corresponding to degenerate Riemann surfaces. We explicitly compare the $k = 3$ solution with the one found numerically by Parisi et al. \cite{4}. Purely asymptotic analysis seems to give in this case the same degree of precision of the semiclassical approximation in the non-degenerate case.

1.1. The String Equation

The string equation

$$-x + \sum_{j=0}^{N} (j + 1) t_{j+1} R_j[u(x)] = 0 \quad (1.1)$$

(where $R_j$ are the Gel'fand-Dikii differential polynomials) for finite $N$ gives a constraint which is compatible with the first $N$ flows of KdV. For $N \to \infty$ this gives a constraint compatible with all of the KdV flows. As a matter of fact, write the $j$-th KdV flow as $\frac{\partial u}{\partial t_j} = K_j[u(x)] \equiv \frac{\partial}{\partial \epsilon} R_j[u(x)]$, differentiate (1.1) with respect to $x$ and compute

$$\frac{\partial}{\partial t_s} (1 + \sum_{j=0}^{N} (j + 1) t_{j+1} K_j) =$$

$$s K_{s-1} + \sum_{j=0}^{N} (j + 1) t_{j+1} K_j' [K_s] =$$

$$s K_{s-1} + K_s' \sum_{j=0}^{N} (j + 1) t_{j+1} K_j =$$

$$s K_{s-1} + K_s' [-1] = 0 \quad (1.2)$$

where $K_j'[\phi](u) \equiv \frac{\partial}{\partial \epsilon} K_j(u + \epsilon \phi)|_{\epsilon=0}$ and we used known properties of the KdV
flows, in particular (i) commutativity, i.e. $K'_j[K_s] - K'_s[K_j] \equiv [K_j, K_s] = 0$; (ii) $K'_s[1] = [K_s, 1] = s K_{s-1}$, expressing the fact that $1 = \tau_{-1}$ is the first master symmetry\[5\] of the KdV hierarchy. We note by the way that (1.2) can be written in a more satisfactory invariant form\[6\] and easily generalized to arbitrary master-symmetries of the KdV equation.

1.2. The Whitham method

The Whitham method is the application to non-linear equations of the semi-classical approximation known in the realm of linear equations as WKB method. Let us consider e.g. the wave equation

$$\varphi_{xx} - \frac{1}{c^2} \varphi_{tt} = 0 \quad (1.3)$$

and the plane wave solutions $\varphi = A e^{i(kx + \omega t)}$. If we are studying propagation of light we have a “small” scale, i.e. the characteristic length of oscillation, which is about $\epsilon = 10^{-6}$ times smaller than the natural unit length used by the observer: so we can consider that solutions be locally given by plane waves, while on the “observational” scale the parameters $A, k$ are “slowly” varying:

$$\varphi(x, t) = A(\epsilon x, \epsilon t) e^{i S(\epsilon x, \epsilon t)}/\epsilon \quad (1.4)$$

where $\frac{\partial S}{\partial X} = k(X, T)$, $\frac{\partial S}{\partial T} = \omega(X, T)$, $X = \epsilon x$, $T = \epsilon t$, $\frac{\partial}{\partial x} = \epsilon \frac{\partial}{\partial X}$, $\frac{\partial}{\partial t} = \epsilon \frac{\partial}{\partial T}$. At 0-th order, we find the eikonal equation of geometrical optics: $(\frac{\partial S}{\partial X})^2 = \frac{1}{c^2} (\frac{\partial S}{\partial T})^2$. The evolution of $k = \frac{\partial S}{\partial X}$ gives the paths of the “rays”; the eikonal equation is equivalent to

$$\begin{cases} k^2 = \frac{\omega^2}{c^2} \\ \frac{\partial k}{\partial T} = \frac{\partial \omega}{\partial X} \end{cases} \quad (1.5)$$

In other words, we started with a class of exact solutions $\varphi(x, t; A, k, \omega) = A e^{i(kx + \omega t)}$ of (1.3) and passed from the precise description of the oscillation process to the approximate description of the “slow” variation of parameters $k, \omega$ given by (1.5); as a matter of fact we averaged over the rapid variation of the function $\varphi$ and chose to observe only “secular” variations. The same method could have been readily applied to non-linear equations if we had
i) a family of exact solutions depending on an adequate number of parameters $E_1, \ldots, E_{2g+1}$;

ii) a way of “averaging out” fast oscillations, in order to obtain an analog of the second equation in (1.5), which we will call “Whitham equation”.

This is the case for the KdV equation [7] and more generally for equations that can be written in the form

$$\frac{\partial L}{\partial t} - \frac{\partial A}{\partial y} + [L, A] = 0,$$  

(1.6)

that is, as the compatibility condition for the existence of a solution $\psi$ of the linear system

$$\begin{cases} 
L \psi = \frac{\partial \psi}{\partial y} \\
A \psi = \frac{\partial \psi}{\partial t}
\end{cases}$$

(1.7)

where $L = \sum_{j=0}^{n} u_j(x, y, t) \frac{\partial^j}{\partial x^j}$, $A = \sum_{k=0}^{m} v_k(x, y, t) \frac{\partial^k}{\partial x^k}$ are differential operators with scalar or matrix coefficients: as is known, these equations admit large sets of exact solutions (the so-called $g$-zone solutions) expressed in terms of the Riemann $\theta$ function.

In particular, the KdV equation

$$4u_t = u_{xxx} - 6uu_x$$

(1.8)

can be written as $L_t = [L, A]$ with

$$L = -\partial^2 + u(x, t)$$

$$A = \partial^3 - \frac{3}{2} u(x, t) \partial - \frac{3}{4} u_x(x, t), \quad \partial \equiv \frac{\partial}{\partial x}$$

(1.9)

and admits the “cnoidal wave” solution

$$u(x, t) = \wp(x - vt; g_2, g_3)$$

(1.10)

where $\wp$ is the Weierstrass elliptic function and $g_2, g_3$ are arbitrary constants. Roughly speaking, the Whitham method consists in slowly varying the constants

\[
[L, A] = LA - AL \text{ is the usual commutator of differential operators, while } \frac{\partial}{\partial y} \text{ and } \frac{\partial}{\partial t} \text{ are supposed to act on the coefficients } u_j, v_k.
\]
$g_2, g_3$, that is in finding the correct dependence $g_2 = g_2(X, T), g_3 = g_3(X, T)$, in order to approximate either (i) new solutions of (1.8) (corresponding e.g. to non-periodic initial data!) or (ii) solutions to the perturbed equation

$$4u_t = u_{xxx} - 6uu_x + \epsilon K(x).$$

(1.11)

The cnoidal wave solution is a so-called 1-zone ($g = 1$) solution; general $g$-zone solutions are given by

$$u(x) = -2\frac{\partial^2}{\partial \alpha^2} \log \theta(U_x|B) + C$$

(1.12)

where the exact form of $U = U(E_j), B = B(E_j), C = C(E_j)$ is given in the Appendix.

2. The Whitham method

Let’s start with perturbation theory for a non-linear equation (e.g. the stationary KdV equation):

$$u_{xxx} - 12 uu_x = \epsilon K$$

(2.1)

and look for solutions in the form

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \cdots$$

(2.2)

with

$$u_k = u_k(t|X) = u_k(\frac{S(X)}{\epsilon} | E_j(X)), \quad X = \epsilon x, \quad k = 0, 1, 2, \ldots$$

(2.3)

where $\frac{1}{\epsilon} S_1(X), \ldots, \frac{1}{\epsilon} S_g(X)$ are rapidly oscillating functions that will be determined in the following, and $u_k$ depends on some parameters $E_j = E_j(X)$ which on their turn are slowly varying with $x$, and will be determined in the sequel. Substitute (2.2) and (2.3) in (2.1). Note that the form of the functions $u_k$ implies $\frac{\partial}{\partial x} = \frac{\partial}{\partial \epsilon x} = \frac{\partial}{\partial \alpha}$. 

\[ \frac{\partial S}{\partial X} \cdot \frac{\partial}{\partial t} + \epsilon \frac{\partial S}{\partial X} \]. Now take the various order in \( \epsilon \): e.g. at order \( O(1) \) we get

\[ (\frac{\partial S}{\partial X} \cdot \frac{\partial}{\partial t})^3 u_0(t|X) - 12 u_0(t|X)(\frac{\partial S}{\partial X} \cdot \frac{\partial}{\partial t}) u_0(t|X) = 0 \] (2.4)

Let \( \mathcal{L} = \frac{\partial S}{\partial X} \cdot \frac{\partial}{\partial t} = \sum_{j=1}^{g} \frac{\partial S_j}{\partial X} \frac{\partial}{\partial t_j} \), and go on writing equations at all orders in the compact form:

\[
\begin{align*}
O(1) &: \quad \mathcal{L}^3 u_0 - 12 u_0 \mathcal{L} u_0 = 0 \\
O(\epsilon) &: \quad \mathcal{L}^3 u_1 - 12 \mathcal{L}(u_0 u_1) = F_1 + K \\
O(\epsilon^k) &: \quad \mathcal{L}^3 u_k - 12 \mathcal{L}(u_0 u_k) = F_k, \quad k = 2, 3, 4, \ldots
\end{align*}
\] (2.5)

Note that only the first equation is non-linear, and that equations for the order \( O(\epsilon^k) \), \( k = 1, 2, 3, \ldots \), differ only in the non-homogenous term \( F_k \). This term would not be present in na"ive perturbation theory, and comes from differentiation with respect to slow variables: for instance \( u_{0xx} = (\frac{\partial S}{\partial X} \cdot \frac{\partial}{\partial t} + \epsilon \frac{\partial S}{\partial X}) u_0 \) produces the term \( \frac{\partial}{\partial x} S^j \) coming from \( u_{0xx} \), etc. (the explicit form of the term \( F_1 \) is given in the Appendix). The idea is that at order \( O(\epsilon) \) the term \( F_1 \) should compensate for the perturbation term \( K \) in order that the “correction” \( u_1 \) be bounded, and this gives equations for the correct dependence of the parameters \( E_j = E_j(X) \) on the slow variables; otherwise, we can think of averaging the \( O(\epsilon) \) equation in (2.5) over the fast variables \( t_1, \ldots, t_g \), thus remaining with the only variable \( X \). A third point of view is that the \( O(\epsilon) \) equation has the form \( \mathbf{L} u_1 = F_1 + K \), where \( \mathbf{L} \) is a linear operator: this means that periodic solutions exist iff \( F_1 + K \) is orthogonal to \( \text{Ker}(\mathbf{L}^4) \). The three points of view are all equivalent and give the same Whitham equations for \( E_j = E_j(X) \).

Equation (2.4) has been written in explicit form in order to make clear an important point: at any order the variable \( X \) appears as a parameter, while equations are only in the differential variables \( t_1, \ldots, t_g \).\(^*\) Moreover, putting

\[ \frac{\partial S}{\partial X} = U(E_j(X)), \quad t = Ux \] (2.6)

we see that (2.4) becomes equivalent to the unperturbed \( u_{0xxx} - 12 u_0 u_{0x} = 0 \), so

\(^*\) This explains why the variables \( x \) and \( X \) are usually treated as independent variables in the two-scale method, and makes rigorous the usual argument of “freezing” the slow variable.
we immediately get the form of $u_0^\dagger$:

$$u_0(t) = -2(U \cdot \frac{\partial}{\partial t})^2 \log \theta(t|E_j) + C(E_j) \quad (2.7)$$

that is, in the genus $g = 1$ case, $u_0(t_1) = \wp(2\omega \cdot t_1; g_2, g_3)$.

Functions of the form (2.7) are periodic in $t_1, \ldots, t_g$; moreover, we have seen that the variables $t$ and $X$ must be regarded as independent: so we can average both sides of the $O(\epsilon)$ equation in (2.5) over $t_1, \ldots, t_g$, thus being left with the only variable $X$. The equations for $E_1(X), \ldots, E_{2g+1}(X)$ thus obtained in the case of the stationary KdV equation have the form

$$\frac{\partial}{\partial X} \sqrt{(E-E_1(x))(E-E_2(x))(E-E_3(x))} \, dE = -6 \frac{E+r(E_1(x), E_2(x), E_3(x))}{\sqrt{(E-E_1(x))(E-E_2(x))(E-E_3(x))}} \, dE$$

where $r(E_1, E_2, E_3) = \int_{E_3}^{E_2} \frac{EdE}{\sqrt{(E-E_1)(E-E_2)(E-E_3)}} / \int_{E_3}^{E_2} \frac{dE}{\sqrt{(E-E_1)(E-E_2)(E-E_3)}}$ and $E$ is a dummy variable.

In general, the Whitham equations for KdV are differential conditions on the functions $p = p(E, E_j(X, T))$, $\Omega = \Omega(E, E_j(X, T))$ (quasi-momentum and quasi-energy) appearing in the $E \to \infty$ leading term $\psi \approx e^{px} + \Omega t$ of the solutions to the associated linear system (1.7). For KdV they take the form

$$\frac{\partial \Omega}{\partial X} - \frac{\partial p}{\partial T} = \frac{\langle \psi^3 K \psi \rangle}{\langle \psi \psi \rangle} \frac{\partial p}{\partial E}$$

Equation (2.8) can equivalently be written as $\frac{\partial \Omega}{\partial X} = -6 \frac{\partial p}{\partial E}$. A complete derivation and explanation of these formulas is given in the Appendix.

\[\dagger\] Actually, this is the complex form of $u_0$. A rigorous treatment needs the use of real variables $t_1, \ldots, t_2g$. In the $g = 1$ case we would get $u_0(t_1, t_2) = \wp(2\omega \cdot t_1 + 2\omega' \cdot t_2; g_2, g_3)$. 
3. The Painlevé type I equation

The Painlevé equation

\[ u'' = 6(u^2 - x) \]  \hspace{1cm} (3.1)

(which is obtained from (1.1) when we leave only \( t_3 \neq 0 \)) can be differentiated once with respect to \( x \) and seen as a perturbation of the (integrable) stationary KdV equation:

\[ u''' = 12uu' + \epsilon K \]  \hspace{1cm} (3.2)

with \( \epsilon = 1, K = -6 \). An asymptotic relation between the two equations is obtained through the change of variables

\[
\begin{align*}
\left\{ 
&u(x) = \sqrt{\xi} v(\xi) \\
&\xi(x) = \frac{4}{5}x^{5/4}
\end{align*}
\]  \hspace{1cm} (3.3)

It comes out (2.9) \( \uparrow^{[8]} \) that solutions of (3.1) are asymptotic to functions of the form \( u(x) = \wp(\frac{4}{5}x^{5/4}; 12, g_3) \); nothing is said about the parameter \( g_3 \). We will now resort to a finer analysis of the problem.

We are here concerned (2.9) \( \uparrow^{[9]} \) with the case of the torus of equation \( (2\omega)^2 = 4E^3 - g_2E - g_3 = 4(E - E_1)(E - E_2)(E - E_3) \); we distinguish two cases; for \( \Delta = g_2^3 - 27g_3^2 \geq 0 \) we have real roots \( E_1, E_2, E_3 \) and two periods \( 2\omega \) and \( 2\omega' \) which are respectively real and pure imaginary (2.9) \( \uparrow^{[10]} \):

\[
\begin{align*}
2\omega &= 2 \int_{E_3}^{E_2} \frac{dE}{\sqrt{4E^3 - g_2E - g_3}} = \frac{2K(m)}{\sqrt{E_1 - E_3}}, \\
2\omega' &= 2 \int_{E_2}^{E_3} \frac{dE}{\sqrt{4E^3 - g_2E - g_3}} = \frac{2iK(1-m)}{\sqrt{E_1 - E_3}},
\end{align*}
\]  \hspace{1cm} (3.4)

where

\[
\begin{align*}
K(m) &= \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - m \sin^2 \phi}}, & E(m) &= \int_0^{\frac{\pi}{2}} \sqrt{1 - m \sin^2 \phi} d\phi, \\
m &= \frac{E_2 - E_3}{E_1 - E_3}
\end{align*}
\]  \hspace{1cm} (3.5)

are standard elliptic integrals of the 1st and 2nd kind, respectively, and \( m \) is called the Jacobi modulus. We remind that the solutions of (3.2) with \( \epsilon = 0 \) can be
expressed in terms of the Weierstrass elliptic function (see Ref. 10)

\[ \varphi(x, g_2, g_3) = -\frac{\partial^2}{\partial x^2} \log \theta_1 \left( \frac{x}{2\omega} \right) - \frac{\eta}{\omega}. \]  \hfill (3.6)

The Whitham equations for (3.2) take the form

\[ \frac{\partial w}{\partial X} dE = -6p \]  \hfill (3.7)

or, explicitly,

\[ \frac{\partial}{\partial X} \sqrt{4E^3 - g_2 E - g_3} dE \equiv \frac{-\frac{\partial g_2}{\partial X} E - \frac{\partial g_3}{\partial X}}{2\sqrt{4E^3 - g_2 E - g_3}} dE = -6 \frac{E + r(X)}{\sqrt{4E^3 - g_2 E - g_3}} dE \]  \hfill (3.8)

giving

\[ \frac{\partial g_2}{\partial X} = 12, \quad \frac{\partial g_3}{\partial X} = 12 r(X). \]  \hfill (3.9)

The first equation is readily integrated, giving \( g_2 = 12 \cdot X + \text{const}; \) moreover, the second too is integrable by quadratures, because \( \frac{\partial w}{\partial X} dE \) already has the same behaviour as \(-6p\) for \( E \to \infty \), so we need only to impose the normalization condition (compare with (5.4))

\[ \text{Im} \left( \oint w dE \right) = \text{Re} \left( \frac{1}{i} \int_{E_1}^{E_2} \sqrt{4E^3 - 12XE - g_3(X)} dE \right) = \text{const} = h \]  \hfill (3.10)

in order to get \( \text{Im} \int \frac{\partial w}{\partial X} dE = \frac{\partial}{\partial X} \text{Im} \int w dE = 0 \) and \( \frac{\partial}{\partial X} w dE = -6p \). But (3.10) can be solved explicitly for \( g_3 \), yielding the correct dependence \( g_3 = g_3(X) \).

In order to do this it is convenient to introduce new parameters \((\lambda, m)\) in place of \((g_2, g_3)\):

\[ \lambda = \frac{(E_1 - E_3)^2}{9}, \quad m = \frac{E_2 - E_3}{E_1 - E_3}, \]

\[ g_2 = 12\lambda (1 - m + m^2), \quad g_3 = 4\lambda^{3/2}(2 - 3m - 3m^2 + 2m^3), \]

\[ E_1 = \sqrt{\lambda}(2 - m), \quad E_2 = \sqrt{\lambda}(-1 + 2m), \quad E_3 = \sqrt{\lambda}(-1 - m), \]

\[ \int_{E_2}^{E_1} \sqrt{4E^3 - g_2 E - g_3} dE \equiv i \frac{\lambda^{5/4}}{5\sqrt{3}} \Phi(m); \]  \hfill (3.11)
giving (for $\Delta > 0$) the solution

$$\lambda(m) = \left( \frac{5\sqrt{3}}{\Phi(m)} \right)^{\frac{4}{5}}.$$

(3.12)

In Fig. 1 we show the form of the resulting function $x = x(m)$, both for the cases $\Delta > 0$ and $\Delta < 0$ (for graphic convenience we plotted $x$ versus $\frac{1}{m}$ instead of $m$). The pole in the $\Delta < 0$ region gives rise to two distinct curves in the space of parameters ($g_2, g_3$), which we plotted in Fig. 2. Knowing the properties (2.9) \cite{11} of the Weierstrass elliptic function $\wp$ we see that for $x = \frac{g_2}{12} > 0$ the Whitham method gives us either small oscillations (curve in the $\Delta > 0$ region) or a sequence of poles (lower part of the curve, lying in the $\Delta < 0$ region). If we start with oscillatory behaviour at $+\infty$ and go in the direction of decreasing $x$, we ultimately reach $\Delta = 0$ where a transition to polar behaviour occurs. Note that for $x < 0$ we can have only polar behaviour. The two parts of the curve correspond to solutions having or not having poles at $+\infty$. The “physical” solutions will be seen to be separatrices lying between these two kinds of solutions.

The function $\Phi(m)$ is computed reducing the integral in (3.10) to standard elliptic integrals (2.9) \cite{12}. We get

$$\int_{E_3}^{E_2} \frac{dE}{2w} = \frac{K(m)}{\sqrt{3} \lambda^{1/4}}, \quad \int_{E_2}^{E_1} \frac{dE}{2w} = K(1-m)\sqrt{3} \lambda^{1/4},$$

$$\int_{E_3}^{E_2} \frac{E dE}{2w} = \frac{\lambda^{1/4}}{\sqrt{3}} K(m) - \sqrt{\lambda^{1/4}} E(m), \quad \int_{E_2}^{E_1} \frac{E dE}{2w} = \frac{\lambda^{1/4}}{\sqrt{3}} iK(1-m) + \sqrt{\lambda^{1/4}} iE(1-m),$$

and finally

$$\int_{E_3}^{E_2} \sqrt{4E^3 - g_2 E - g_3} \, dE = \frac{2 \lambda^{5/4}}{5 \sqrt{3}} (18(2-3m+m^2)K(m) - 36(1-m+m^2)E(m)),$$

$$\int_{E_2}^{E_1} \sqrt{4E^3 - g_2 E - g_3} \, dE = \frac{2 i \lambda^{5/4}}{5 \sqrt{3}} (-18(m+m^2)K(1-m) + 36(1-m+m^2)E(1-m)),$$

$$\Phi(m) = -18(m+m^2)K(1-m) + 36(1-m+m^2)E(1-m).$$

(3.13)

This procedure gives us also the “actions” $S_1, S_2$:

$$S_1 = -\frac{1}{3} \int_{E_3}^{E_2} \sqrt{4E^3 - g_2 E - g_3} \, dE, \quad S_2 = -\frac{1}{3} \int_{E_3}^{E_2} \sqrt{4E^3 - g_2 E - g_3} \, dE,$$

(3.14)
because from $\frac{\partial \omega}{\partial X} dE = -6dp$ it follows

$$\frac{\partial S_1}{\partial X} = U_1 = 2 \int_{E_2}^{E_3} dp, \quad \frac{\partial S_2}{\partial X} = U_2 = -2 \int_{E_2}^{E_3} dp.$$ (3.15)

Take $S = S_1 + \tau S_2$ as prescribed by (2.6), (5.15) and (5.16), where $\tau = i \frac{K(1-m)}{K(m)}$ is the 1-dimensional analog of the period matrix $B$, and find from (3.13) and the Legendre relation $EK' + E'K - KK' = \pi^2$ (see Ref. 10):

$$S = i \frac{\sqrt{3}}{5} \frac{5^{1/4} \lambda}{K(m)} = \frac{4}{5} \frac{x}{2\omega}. \quad \text{(3.16)}$$

The Whitham solution has the form

$$u_0(x) = -\varphi(\frac{4}{5}ix + \omega; 12x, 4I(x)x^{3/2}) = -\sqrt{x} \varphi(\frac{4}{5}ix^{5/4} + \omega; 12, 4I(x)); \quad \text{(3.17)}$$

(We have used here the homogeneity property $\varphi(tx; \frac{g_2}{t^3}, \frac{g_3}{t^6}) = \frac{1}{t^2} \varphi(x; g_2, g_3)$, $t = x^{-1/4}$; the $\omega$-shift is needed in order to get non-singular solutions). We thus recover the “Boutroux” asymptotic form (3.3), but with a more precise

$$I(x) = I(m(x)) = \frac{2 - 3m - 3m^2 + 2m^3}{(1 - m + m^2)^{3/2}}. \quad \text{(3.18)}$$

Asymptotic analysis(2.9) $\uparrow^{[13]}$ shows that the amplitude of the oscillatory solutions of (3.1) decreases at $+\infty$ as $\frac{1}{x^{1/8}}$. This feature, not recovered by the simple Boutroux-type solution, is obtained from the fine tuning realized by the term $I(x)$. Just use the expansion

$$K(m) = \frac{\pi}{2} (1 + \frac{m}{4} + \frac{9}{64} m^2 + \cdots), \quad E(m) = \frac{\pi}{2} (1 - \frac{m}{4} - \frac{3}{64} m^2 + \cdots), \quad \text{(3.19)}$$

for $m \to 0$ (see Ref. 10). From (3.11), in the limit $m \to 1$, we get $x = \frac{9}{12} \simeq \lambda(m), \quad \Phi(m) \simeq \frac{135\pi}{8} (1-m)^2, \quad \lambda(m) \simeq \frac{(8\sqrt{3}\pi)^{4/5}}{2\pi} (1-m)^{-8/5}, -E_1, -E_2 \simeq \sqrt{x}, E_1 - E_2 \simeq \sqrt{\frac{8h}{\sqrt{3}\pi}} x^{-1/8}$ (Notice that the function $\varphi(ix + \omega)$ oscillates between the extremal values $-E_1$ and $-E_2$, see Ref. 11).
If we want to examine the case $x < 0$, corresponding to $\Delta < 0$, formulas (3.11) are no longer convenient, and we resort to the following real parameterization

$$m_* = \frac{1}{m} = \frac{1}{2} + i\sigma, \quad \lambda_* = m^2 \lambda = -\rho^2, \quad i\lambda^{5/4} \Phi(m) = -i\lambda_*^{5/4} \Phi(m_*)$$ (3.20)

The solution corresponding to (3.12) in the complex case becomes

$$\lambda_*(m_*) = -\left(\frac{5\sqrt{3}h}{\text{Re}(\sqrt{i\Phi(m_*))})}\right)^{4/5}.$$ (3.21)

A numerical computation shows that the function $\Phi(m_*) \equiv \Phi(\frac{1}{2} + i\sigma) \equiv \Phi(\sigma)$ has a zero for $\sigma = \sigma_0 = -0.231026398427\ldots$

In the case $\Delta < 0$ real and pure imaginary combinations of periods are given by $\omega = \frac{K(m)}{\sqrt{H}}$, $\omega' = \frac{iK(1-m)}{\sqrt{H}}$, where $H^2 = 3E_1^2 - \frac{g_2}{4} = |\frac{3-4\sigma^2}{1+4\sigma^2}|$, $m = \frac{1}{2} + \frac{\sigma}{\sqrt{1+4\sigma^2}}$ (see Ref. 10). With this parameterization we recover the numerical result of Ref. 13, asserting that the distance of poles goes asymptotically as $\tilde{c} x^{1/4}$, with $\tilde{c} = 7.276726\ldots$; as a matter of fact we find $c = \frac{294}{7} = 2\frac{2}{\sqrt{3}} K(\frac{1}{2} - \frac{\sigma_0}{\sqrt{1+4\sigma_0^2}})(\frac{3-4\sigma_0^2}{1+4\sigma_0^2})^{1/4} = 2.970711275212\ldots$, which exactly coincides with the result of Ref. 13 after the rescaling $\tilde{c} = \sqrt{6} c$ (due to our factor 6 in (3.1)). An analogous reasoning for $x > 0$ gives the period of the oscillatory solutions going as $\frac{c}{x^{1/4}}$, with $c = \frac{294}{7} = \frac{\pi}{\sqrt{3}}$.

In Fig. 3 we show the approximate solution (with $h = 1$), together with an exact numerical solution obtained with the Runge-Kutta method (the Painlevé equation is satisfied with an error of $10^{-14}$).Fig. 4 is a magnification of the region around the zero where the approximation seems to be less effective. We see that the solution we are considering seems to be out of phase with respect to the exact numerical one. This is no surprise because we really have neglected a phase: the Whitham method made variating the “constants” $g_2$ and $g_3$, but in $\varphi(x+c; g_2, g_3)$ we have also a third integration constant $c$, that we assumed to be zero. We do believe that an equation for this phase can be deduced(2.9)↑[14], and its behaviour will be discussed in a forthcoming paper. Here we have just made a fit of such a phase on the side of the positive $x$, getting a correction $\delta\phi(x) \simeq \frac{175}{(x-0.0098)^{3.18}}$. We have also done the same on the negative $x$ side but we got a really tiny correction that we chose to neglect. The fit of the “experimental” data is shown in Fig. 5, while Fig. 6 shows the effect of putting in the correction by hand (for $x \simeq 0$ we let $\delta\phi(x)$ die smoothly).
4. Degenerate solutions

The solutions \( u(x) \) of Painlevé-like equations represent specific heats of the random matrix models, which in the planar limit must satisfy the boundary condition \( u(x) \simeq x^{1/k} \ (x \to \infty) \) for scaling arguments of the partition function: \( Z \sim x^{-\gamma+2} \), where \( \gamma \) is the string susceptibility. Thus \( u(x) \equiv \frac{\partial^2 F}{\partial x^2} \sim \frac{\partial}{\partial x} x^{-\gamma+1} \sim x^{-\gamma} \), and the result comes from the fact that \( \gamma = \frac{1}{k} \) in the proximity of the critical point. The situation is common to other non-linear physical models in the critical regime. It is well-known that the problems of mathematical physics must be complemented by boundary conditions, and that the boundary conditions contain in some sense the physics of the problem. In our particular case we come to the request that the solutions to the string equation for the \( k \)-th multicritical model (which is obtained from (1.1) putting all \( t_j = 0 \), except \( t_{k+1} \)) must satisfy the physical constraint \( u(x) \simeq x^{1/k} \) for \( x \to +\infty \). However, the Whitham method gave us either small oscillations modulating over \(-\sqrt{x}\) (see Fig. 3), or solutions with poles. The only possibility to get non-periodic solutions is to consider degenerate Riemann surfaces, where the length of the bands is sent to 0. These solutions are degenerate cases of the periodic solutions and are themselves unstable separatrices, lying between the two sets of solutions with poles and without poles for \( x \to +\infty \).

All we have to do is to compute the spectral curve corresponding to the given \( k \)-th stationary KdV equation and imposing the coincidence of the pairs of branch points; \( g = k - 1 \) conditions are found by requesting that \( dw = dE^{g+\frac{1}{2}} + \mathcal{O}(1) \); one more condition comes from fixing the periods of the solutions at \( x = \pm \infty \); the last \( g \) conditions come from the request that the branch points coalesce in pairs.

For the \( k = 2 \) case we require \( dw \simeq dE^{3/2} + \mathcal{O}(1) \), giving

\[
\begin{align*}
E_1 + E_2 + E_3 &= 0 \\
E_1 E_2 + E_2 E_3 + E_3 E_1 &= c.
\end{align*}
\]

The asymptotic condition fixes \( c = 3 \), and we ask for \( E_2 = E_3 \). This gives \( E_2 = E_3 = -\frac{E_1}{2} = 1 \). The \( \theta \) function degenerate to a combination of hyperbolic functions and we finally get

\[
u_0(x) = \begin{cases} 
\sqrt{x} - \frac{3}{\cosh(\sqrt{3} \frac{\pi}{k} x^{\frac{3}{2}})}, & x \geq 0, \\
\sqrt{-x} (\varphi(\frac{1}{2})(-x)^{\frac{3}{2}}; -12, -I(\frac{1}{2} + i\sigma_0)), & x < 0.
\end{cases}
\]

(In the \( x \leq 0 \) case the surface does not degenerate, but \( I(x) \to I(\frac{1}{2} + i\sigma_0) = \text{const} \), where \( \sigma_0 \) is the constant introduced in the previous section). The function \( u_0(x) \) is plotted in Fig. 7. In \( x = 0 \) we get a cusp as we are trying to connect at finite \( x \) two asymptotic solutions: a smoother curve would probably require a phase-type correction as suggested in the previous section.
In the $k = 3$ case the condition $dw = dE^{5/2} + \mathcal{O}(1)$ gives

$$
\begin{cases}
\sum_{i=1}^{5} E_i = 0 \\
\sum_{i<j} E_i E_j = 0 \\
\sum_{i<j<k} E_i E_j E_k = c.
\end{cases}
$$

Asymptotic conditions fix $c = \pm \frac{5}{8}$ for $x \to \pm \infty$; we put $E_2 = E_3 = s$, $E_3 = E_4 = t$, and find $s = \pm \left(\frac{1 + i \sqrt{5}}{4}\right)$, $t = \bar{s}$, $E_1 = \pm 1$ ($x \to \pm \infty$). Again the $\theta$ function factorizes in products of trigonometric and hyperbolic functions (for details see (2.9) \textsuperscript{[15]} and (2.9) \textsuperscript{[16]}) giving

$$
v_0(x) = \begin{cases}
1 - 2a^2 \frac{b - \frac{a}{a}}{b - \frac{a}{a}} \frac{\sinh(ax + 2 \ln \frac{a}{b}) \sin(bx) + 2 \cosh(ax + 2 \ln \frac{a}{b}) \cos(bx) - 2}{(\sinh(ax + 2 \ln \frac{a}{b}) - \frac{b}{b} \sin(bx))^2}, & x < 0; \\
-1 - 2b^2 \frac{b - \frac{a}{a}}{b - \frac{a}{a}} \frac{\sinh(bx + 2 \ln \frac{b}{a}) \sin(ax) + 2 \cosh(bx + 2 \ln \frac{b}{a}) \cos(ax) - 2}{(\sinh(bx + 2 \ln \frac{b}{a}) - \frac{a}{b} \sin(ax))^2}, & x \geq 0.
\end{cases}
$$

(4.3)

where $a = (30)^{\frac{1}{4}} \cos \frac{\vartheta}{2}$, $b = (30)^{\frac{1}{4}} \sin \frac{\vartheta}{2}$, $\vartheta = \arctan \frac{1}{\sqrt{5}}$. (We checked that this solution satisfies the higher order stationary KdV equation). Fig. 8 shows that $u_0(x) = 3 \sqrt{x} \cdot v_0(\frac{6}{x} \frac{x}{\pi})$ approximate very well the form of the solution found numerically in Ref. 4, except that in the proximity of $x = 0$. (In the graph we have shifted $z_1 \mapsto z_1 + \frac{1}{4}$, $z_2 \mapsto z_2 - \frac{1}{4}$ for $x < 0$ to match the phase of Ref 4. The resulting function still satisfies the KdV equation).

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5. Appendix

In sections 1, 2, 3 we review some facts about algebraic geometry and KdV equations, mainly for notational convenience. In section 4 we report the proof of Krichever’s theorem, following Ref. 2.
### 5.1. Complex Curves

The algebraic equation

\[ w^2 = E^{2g+1} + a_1 E^{2g-1} + a_2 E^{2g-2} + \ldots + a_{2g-1} E + a_{2g} \equiv p_{2g+1}(E) \tag{5.1} \]

defines a curve \( \Gamma \) in the complex plane of the variables \((E, w)\). The curve is compactified at \( \infty \) and is known to be topologically equivalent to a compact surface with \( g \) holes. If the polynomial \( p_{2g+1}(E) \) has \( 2g+1 \) real roots \( E_1, \ldots, E_{2g+1} \), we can draw them on the complex plane and use solid lines for the segments \((E_{2k-1}, E_{2k})\), where the square root \( w = \pm \sqrt{p_{2g+1}(E)} = \pm \sqrt{(E - E_1)(E - E_2) \cdots (E - E_{2g+1})} \) takes real values. In spectral theory these are the forbidden zones of the spectrum.

Coordinates on \( \Gamma \) are given by

\[
\begin{cases}
    u = E & \text{almost everywhere}, \\
    u = \sqrt{E - E_j} & \text{in the neighborhood of } E_j, \\
    u = \frac{1}{\sqrt{E}} & \text{in the neighborhood of } \infty.
\end{cases}
\tag{5.2}
\]

Consider integrals of the form \( \int \Omega_k = \int \frac{E^h dE}{2\sqrt{(E - E_1) \cdots (E - E_{2g+1})}} \), \( k = 0, 1, 2, \ldots \). Using (5.2) it is easy to see that \( \Omega_0, \ldots, \Omega_{g-1} \) are everywhere non-singular, while \( \Omega_g, \Omega_{g+1}, \ldots \) have poles at the infinity of order 2, 4, \ldots, etc.

To fix a basis of differentials we chose first a canonical \cite{17} basis of paths \( a_1, \ldots, a_g \) and \( b_1, \ldots, b_g \) on \( \Gamma \) and take \( \omega_1, \ldots, \omega_g \) as linear combinations of \( \Omega_0, \ldots, \Omega_{g-1} \) satisfying the normalization condition

\[
\oint_{a_k} \omega_j \equiv 2 \int_{E_{2k-1}}^{E_{2k}} \omega_k = \delta_{jk}, \quad j, k = 1, \ldots, g \tag{5.3}
\]

We are left with the \( b \)-periods, forming a \( g \times g \) matrix \( B_{jk} = B_{kj} = \oint_{b_k} \omega_j \equiv 2 \int_{E_{2k}}^{E_{2k+1}} \omega_j \). Differentials with poles of order 2\( j \) will be indicated by \( \omega^{(2j-1)} \) for future commodity, and can be fixed by requiring that they go at the infinity as \( \omega^{(2j-1)} \sim dE^{j-\frac{1}{2}} + \mathcal{O}(1) \). the arbitrariness on the holomorphic tail can be eliminated by imposing the \( 2g \) real conditions

\[
\text{Im} \oint_{a_k} \omega^{(2j-1)} = 0, \quad \text{Im} \oint_{b_k} \omega^{(2j-1)} = 0, \quad k = 1, \ldots, g. \tag{5.4}
\]

(Another standard choice of the normalization is to impose instead of (5.4) the \( g \)
complex conditions $\int_{a_k} \omega^{(2j-1)} = 0, k = 1, \ldots, g$. We will also use the notation

$$dp = \omega^{(1)} \simeq d\sqrt{E}, \quad d\Omega = \omega^{(3)} \simeq dE^{3/2}, \quad E \to \infty; \quad (5.5)$$

these are the differentials of the quasi-momentum $p(E)$ and quasi-energy $\Omega(E)$, fundamental in the theory of the KdV equation (see 5.11); $p(E) = \int_{-\infty}^{E} dp$ and $\Omega(E) = \int_{-\infty}^{E} d\Omega$ are multivalued functions with periods $U_j = \oint_{b_j} dp$, $W_j = \oint_{b_j} d\Omega$, $j = 1, \ldots, g$. Note that $p(E)$ and $\Omega(E)$ are uniquely determined by the asymptotic behaviour and the normalization conditions.

5.2. Functions on the surface $\Gamma$

The Abel map $P \mapsto A(P) \equiv t(\int_{-\infty}^{P} \omega_1, \ldots, \int_{-\infty}^{P} \omega_g)$ maps any point $P$ on $\Gamma$ on the $g$-dimensional torus $C^g/\{\text{period lattice}\}$. The Abel map can be inverted by means of the Fourier series

$$\theta(z|B) = \sum_{n \in \mathbb{Z}^g} e^{\pi i n B \cdot n + 2\pi i n \cdot z} \quad (5.6)$$

defining (for positive definite $\text{Im}B$) the Riemann $\theta$ function, which has the periodicity properties $\theta(z + e_j) = \theta(z)$, $\theta(z + Be_j) = e^{-\pi i B_{jj} - 2\pi i z_j} \theta(z)$ (see Ref. 17). As a matter of fact, a theorem of Jacobi asserts that the function

$$f(P; P_1, \ldots, P_g) = \theta(\int_{-\infty}^{P} \omega_k - \sum_{j=1}^{g} \int_{-\infty}^{P_j} \omega_k + K_k) \quad (5.7)$$

(where $K = (K_k)$ is a certain constant vector) has exactly $g$ zeroes $P_1, \ldots, P_g$. This theorem gives an analog of the development of a rational function in simple fractions.

5.3. The KdV equation

The KdV equation $4u_t = u_{xxx} - 6uu_x$ admits the Lax representation $L_t =$
\[ L, A \], with \( L, A \), given by (1.9), and has exact solutions of the form
\[
  u(x, t) = -2 \frac{\partial^2}{\partial x^2} \log \theta(Ux + Wt + z_0|B) + C(B) \tag{5.8}
\]
where (see (5.5))
\[
  U_j = \oint_{b_j} dp, \quad W_j = \oint_{b_j} d\Omega, \quad j = 1, \ldots, g. \tag{5.9}
\]
This can be seen as follows [2.9] \( \uparrow \). The equation \( L_t = [L, A] \) is the compatibility condition for the system of linear equations
\[
\begin{align*}
  L\psi &= E\psi \\
  A\psi &= \frac{\partial \psi}{\partial t}.
\end{align*} \tag{5.10}
\]
For \( E \approx \infty \) we get \( L \approx d^2, A \approx \partial^3 \), so the asymptotic form of the common eigenvectors \( \psi \) will be
\[
  \psi(x, t; E) = e^{p(E)x + \Omega(E)t} \cdot \phi(x, t; E), \quad \text{with} \quad p(E) \approx \sqrt{E}, \quad \Omega(E) \approx E^{3/2} \quad \text{as} \quad E \rightarrow \infty. \tag{5.11}
\]
The exact form of \( \phi(x, t) \) is (compare with (5.7); see Ref. 9)
\[
  \phi(x, t; E) = \frac{\theta(\int_\omega x \omega_k - \sum_{j=1}^g \int_\omega \omega_k + U_k x + V_k t + K_k)}{\theta(\int_\omega x \omega_k - \sum_{j=1}^g \int_\omega \omega_k + K_k)} \tag{5.12}
\]
for given \( \Gamma \) and \( P_1, \ldots, P_g \) on \( \Gamma \). It is easy to verify that \( \phi = e^{px + \Omega t} \cdot \phi \) is a one-valued function of \( E \). The function \( \psi \) is uniquely determined by the behaviour at infinity \( \psi \approx e^{E^{3/2}x + H^{3/2}t} \) and the position of the \( g \) poles \( P_1, \ldots, P_g \), as can be easily seen with the help of the Riemann-Roch theorem. For \( E \equiv k^2 \rightarrow \infty \) we get
\[
  \psi(x, t; E) = c e^{kx + k^3 t} \cdot \left( 1 + \frac{\xi_1(x, t)}{k} + \frac{\xi_2(x, t)}{k^2} + \cdots \right) \tag{5.13}
\]
If \( \psi \) satisfies \( L\psi = E\psi, A\psi = \frac{\partial \psi}{\partial t} \), then we can collect terms of the same order in \( \frac{1}{k} \) and \( \xi_1, \xi_2, \ldots \) should satisfy some equation at any order. It is easily seen that the first of these equation gives \( u(x, t) = 2 \frac{\partial k}{k^2} (x, t) \), so for this choice of the potential \( u \) we get (5.10) verified at order \( \mathcal{O}(1/k) \). But note that \( (L - E)\psi, (A - \frac{\partial}{\partial t})\psi \) again have the behaviour \( e^{kx + k^3 t} \) and poles at \( P_1, \ldots, P_g \), so they must again have the form (5.13) with no \( \mathcal{O}(1) \) term: this means that they are identically zero; so (5.11) and (5.12) give an exact solution to (5.10). Developing (5.12) at 1st order in \( \frac{1}{k} \) we get precisely formula (5.8) for \( u = \frac{2}{k^2} \xi_1(x, t) \).
5.4. Krichever’s theorem

We will here report the form found by Krichever for the Whitham equations of systems of KP type, following Ref. 2. It is convenient to consider directly the general case of equations of the form (1.6), as for instance the KP equation

\[ 3u_{yy} + \partial_x (4u_t - 6uu_x + u_{xxx}) = 0. \]

(1.6) is the compatibility condition for the existence of a solution \( \psi \) of the linear system \( L\psi = \frac{\partial \psi}{\partial y}, A\psi = \frac{\partial \psi}{\partial t} \). The common eigenvector \( \psi \) will be given here using a particular real normalization, necessary for the successive averaging procedure:

\[
\psi(x, y, t; P) = e^{px + Ey + \Omega t + s \cdot t} \cdot \phi(Ux + Vy + Wt + t, P) \tag{5.14}
\]

here the spectral curve \( \Gamma \) is no more hyperelliptic, and consequently the spectral parameter \( E \) becomes itself a multi-valued function \( E(P) \) of the point \( P \) on the surface; the functions \( p(P), E(P), \Omega(P) \) are normalized by requiring that they have pure imaginary periods along all cycles \( a_1, \ldots, a_g, b_1, \ldots, b_g; U, V, W \) are the real \( 2g \)-dimensional vectors of periods of the multi-valued functions \( p(P), E(P), \Omega(P) \):

\[
U = t \left( \oint_{b_1} dp, \ldots, \oint_{b_g} dp, - \oint_{a_1} dp, \ldots, - \oint_{a_g} dp \right), \quad \text{etc.}; \tag{5.15}
\]

\( t_1, \ldots, t_{2g} \) are auxiliary “times” needed for further procedure of averaging; \( s_1, \ldots, s_{2g} \) are the corresponding “momenta”, not needed in what follows; \( \phi \) is a periodic function with period 1 with respect to all of the \( 2g \) variables \( x_1, \ldots, x_g, y_1, \ldots, y_g \) (\( A \) is the Abel map):

\[
\phi \left( \begin{pmatrix} x \\ y \end{pmatrix} \right); P) = ce^{2\pi i A(P) \cdot y} \cdot \frac{\theta(A(P) + x + By + z_0)\theta(z_0)}{\theta(A(P) + z_0)\theta(x + By + z_0)} \tag{5.16}
\]

We will also need the solutions to the adjoint system \( \psi^\dagger L = -\frac{\partial \psi^\dagger}{\partial y}, \psi^\dagger A = -\frac{\partial \psi^\dagger}{\partial t} \)

where differential operators written on the left should be intended according to \( \psi^\dagger (u \frac{\partial}{\partial x}) \equiv (-\frac{\partial}{\partial x})^j (\psi^\dagger u) \). (formal integration by parts). The left and right action differ only for a complete derivative:

\[
(\psi^\dagger L)\psi = \psi^\dagger L\psi + \frac{\partial}{\partial x} (\psi^\dagger L^{(1)} \psi) + \frac{\partial^2}{\partial x^2} (\psi^\dagger L^{(2)} \psi) + \cdots \tag{5.17}
\]

as can readily be seen by repeated applications of the Leibnitz rule. Here \( L^{(r)} \equiv (-1)^r \frac{d^r}{d(\partial)^r} L \) (formal derivation with respect to the symbol \( \partial \): for instance, \( A^{(1)} = -3\partial^2 + \frac{3}{2} u \)).
Solutions to the adjoint system can be written in the form
\[ \psi^\dagger(x, y; t; P) = e^{-px-Ey-\Omega t-s^t} \cdot \phi^\dagger(-Ux-Vy-Wt-t; P) \]

The application of the Whitham method to equations of the KP type is allowed by the possibility of averaging identities by means of some "ergodic theorem": the average of periodic functions \( \phi(t_1, \ldots, t_{2g}; I_k) \) with period 1 with respect to all arguments is given by \( \langle \phi \rangle \equiv \int_0^1 \phi(t) dt \), and for a generic vector \( U \) it coincides with the limit \( \langle \phi \rangle_x \equiv \lim_{x_0 \to +\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(Ux) dx \), because the line \( Ux \) winds densely on a \( 2g \)-dimensional torus. Note that the derivative of \( \phi \) along any direction has zero average value:

\[ \langle \frac{\partial}{\partial x} \phi(Ux) \rangle = U \cdot \langle \frac{\partial \phi}{\partial t} \rangle = 2g \sum_{j=1}^{2g} U_j \int_0^1 \frac{\partial \phi}{\partial t_j} dt_1 \ldots dt_{2g} = 0 \quad (5.18) \]

Note also that the functions \( \frac{\partial \phi}{\partial I_k} \) are again periodic, because in our real normalization the periods have the fixed value 1 (not depending on the \( I_k \)).

Solutions of the KP equation have the form
\[ u(x, y, t) = -2\frac{\partial^2}{\partial x^2} \log \theta(Ux + Vy + Wt + z_0|B) + C(B) \quad (5.19) \]

Thus, we look for a semiclassical approximation in the form

\[ u_0(x, y, t) = -2\frac{\partial^2}{\partial x^2} \log \theta(\frac{S(X, Y, T)}{\epsilon}|I(X, Y, T)) + C(X, Y, T), \]

where \( \frac{\partial S}{\partial X} = U, \frac{\partial S}{\partial Y} = V, \frac{\partial S}{\partial T} = W, \psi_0 = e^{\frac{1}{\epsilon}S}\phi(S) \). The operators \( L_0, A_0 \) obtained substituting \( u \mapsto u_0 \) are taken as first terms of the asymptotic series

\[ A = A_0 + \epsilon A_1 + \cdots, \quad L = L_0 + \epsilon L_1 + \cdots \quad (5.20) \]

Let us introduce the notation

\[ \hat{\frac{\partial}{\partial \tau}} \equiv \frac{\partial}{\partial \tau} \cdot \frac{\partial}{\partial I} \equiv \sum_j \frac{\partial I_j}{\partial \tau} \frac{\partial}{\partial I_j}; \quad (5.21) \]

then the substitution \( u \mapsto u_0 \) implies \( \hat{\frac{\partial}{\partial x}} \mapsto \hat{\frac{\partial}{\partial x}} + \epsilon \frac{\partial}{\partial x} \); taking terms of order \( O(\epsilon) \)

\[ \hat{\frac{\partial}{\partial x}} \mapsto \hat{\frac{\partial}{\partial x}} + \epsilon \frac{\partial}{\partial x}, \quad \hat{\frac{\partial}{\partial y}} \mapsto \hat{\frac{\partial}{\partial y}} + \epsilon \frac{\partial}{\partial y}, \quad \hat{\frac{\partial}{\partial t}} \mapsto \hat{\frac{\partial}{\partial t}} + \epsilon \frac{\partial}{\partial t}, \quad \hat{\frac{\partial}{\partial \tau}} \mapsto \hat{\frac{\partial}{\partial \tau}} + \epsilon \frac{\partial}{\partial \tau} \]
in (1.6) we get the linearized equation

\[
\frac{\partial L_1}{\partial t} - \frac{\partial A_1}{\partial y} + [L_0, A_1] + [L_1, A_0] = K - F, \tag{5.22}
\]

where \( F \) is the term due to derivation with respect to slow variables: this term must be adjusted in order to compensate for \( K \). We readily find for \( F \) the form

\[
F = \hat{\frac{\partial L}{\partial T}} - \hat{\frac{\partial A}{\partial Y}} + (L(1)\hat{\frac{\partial A}{\partial X}} - A(1)\hat{\frac{\partial L}{\partial X}}). \tag{5.23}
\]

Now use \( \frac{\partial \psi}{\partial y} = L\psi, \frac{\partial \psi^\dagger}{\partial y} = -\psi^\dagger A, \frac{\partial \psi}{\partial t} = A\psi, \frac{\partial \psi^\dagger}{\partial t} = -\psi^\dagger A, \) and (5.17):

\[
\frac{\partial}{\partial t}(\psi^\dagger L_1 \psi) - \frac{\partial}{\partial y}(\psi^\dagger A_1 \psi) = \psi^\dagger \left( \frac{\partial L_1}{\partial t} - \frac{\partial A_1}{\partial y} + [L_0, A_1] + [L_1, A_0] \right) \psi + \frac{\partial}{\partial x}(...) \tag{5.24}
\]

Thus, the average of the left hand side of (5.22) comes out to be zero (being the average of a total derivative) and we obtain the Whitham equations in the implicit form

\[
\langle \psi^\dagger K \psi \rangle = \langle \psi^\dagger F \psi \rangle \tag{5.25}
\]

Explicit computing of the Whitham term \( \langle \psi^\dagger F \psi \rangle \) will give us the final form. Take respectively

(i) a curve \( I = I(\tau) \) in the space of parameters \( (P = \text{const}, t = \text{const}) \);

(ii) a curve \( P = P(\tau) \) moving the point \( P \) on the surface \( \Gamma \) \( (I = \text{const}, t = \text{const}) \);

(iii) a curve \( t = t(\tau) \) moving only the “times” \( t_i \) \( (I = \text{const}, P = \text{const}) \).

Correspondingly, we get \( L(\tau), A(\tau), \) etc., and

\[
\psi(\tau) = e^{p(\tau)x} + E(\tau)y + \Omega(\tau)t + s \cdot t \cdot \phi(U(\tau)x + V(\tau)y + W(\tau)t + t)
\]

\[
\psi^\dagger = e^{-px - Ey - \Omega t - s \cdot t \cdot \phi(-Ux - Vy - Wt - t)} \tag{5.26}
\]

Now compute \( \frac{\partial}{\partial \tau} \bigg|_{\tau=0} \psi^\dagger \psi(\tau) \) in all the three cases and use a point to denote deriva-
tion with respect to $\tau$:

\begin{align}
(i) \quad & \frac{\partial}{\partial \tau} \left|_{0} \right. \psi^\dagger \psi(\tau) = (\dot{p}x + \dot{E} y + \dot{\Omega} t) \psi^\dagger \psi + (\dot{\mathbf{U}} x + \dot{\mathbf{V}} y + \dot{\mathbf{W}} t) \cdot \psi^\dagger \frac{\partial \psi}{\partial \mathbf{t}} + \hat{\mathbf{I}} \cdot \psi^\dagger \frac{\partial \psi}{\partial \mathbf{I}} \\
(ii) \quad & \frac{\partial}{\partial \tau} \left|_{0} \right. \psi^\dagger \psi(\tau) = (d p x + d E y + d \Omega t) \phi^\dagger \phi, \\
(iii) \quad & \frac{\partial}{\partial \tau} \left|_{0} \right. \psi^\dagger \psi(\tau) = s \cdot \dot{\mathbf{t}} \phi^\dagger \phi + \phi^\dagger \frac{\partial \phi}{\partial \mathbf{t}} \cdot \mathbf{t};
\end{align}

Using $\frac{\partial \psi}{\partial t} = A \psi$, $\frac{\partial \psi^\dagger}{\partial \mathbf{t}} = - \psi^\dagger A$ and (5.17), see that

\begin{equation}
\frac{\partial}{\partial \tau} \left( \psi^\dagger \psi(\tau) \right) = \psi^\dagger (A(\tau) - A) \psi(\tau) - \frac{\partial}{\partial x} (\psi^\dagger A^{(1)} \psi(\tau)) + \frac{\partial^2}{\partial x^2} (\ldots)
\end{equation}

Deriving the left-hand side with respect to $\tau$ and using (i) we get for instance

\begin{equation}
\frac{\partial}{\partial \tau} \left( \psi^\dagger \psi(\tau) \right) = \Omega \psi^\dagger \psi + \mathbf{W} \cdot \psi^\dagger \frac{\partial \psi}{\partial \mathbf{t}} + \{ (\dot{p}x + \dot{E} y + \dot{\Omega} t) \frac{\partial}{\partial \mathbf{t}} (\psi^\dagger \psi) \\
+ (\dot{\mathbf{U}} x + \dot{\mathbf{V}} y + \dot{\mathbf{W}} t) \cdot \frac{\partial}{\partial \mathbf{t}} (\phi^\dagger \frac{\partial \phi}{\partial \mathbf{t}}) + \hat{\mathbf{I}} \cdot \frac{\partial}{\partial \mathbf{t}} (\phi^\dagger \frac{\partial \phi}{\partial \mathbf{I}}) \}
\end{equation}

Now fix $x, y, t$ and average upon $d^2g t$: the terms in braces are linear combinations of total derivatives with constant coefficients and thus vanish, giving

\begin{equation}
\langle \frac{\partial}{\partial \tau} \left|_{0} \right. \frac{\partial}{\partial \mathbf{t}} (\psi^\dagger \psi(\tau)) \rangle = \hat{\Omega} \langle \psi^\dagger \psi \rangle + \mathbf{W} \cdot \langle \psi^\dagger \frac{\partial \psi}{\partial \mathbf{t}} \rangle
\end{equation}

this passage contains the essence of the method of averaging. We can now go on deriving both sides of (5.28) with respect to $\tau$, applying (i), (ii), (iii) and finding after averaging the following identities:

\begin{align}
(i) : \quad & \hat{\Omega} \langle \psi^\dagger \psi \rangle + \mathbf{W} \cdot \langle \psi^\dagger \frac{\partial \psi}{\partial \mathbf{t}} \rangle = \langle \psi^\dagger \frac{\partial A}{\partial \tau} \psi \rangle - \dot{p} \langle \psi^\dagger A^{(1)} \psi \rangle - \hat{\mathbf{U}} \cdot \langle \phi^\dagger A^{(1)} \frac{\partial \phi}{\partial \mathbf{t}} \rangle \\
(ii) : \quad & d \Omega \langle \psi^\dagger \psi \rangle = -d p \langle \psi^\dagger A^{(1)} \psi \rangle \\
(iii) : \quad & 0 = \langle \psi^\dagger \frac{\partial A}{\partial t_j} \rangle
\end{align}

Note that $\psi^\dagger \psi = \phi^\dagger \phi$, that we posed $\hat{A}^{(1)} = e(-p x - \cdots) A^{(1)} e(p x + \cdots)$, and that
(iii) implies

$$\langle \psi^\dagger \frac{\partial A}{\partial \tau} \psi \rangle = \langle \psi^\dagger \frac{\partial A}{\partial \tau} A + (\hat{U} x + \hat{V} y + \hat{W} t) \cdot \frac{\partial A}{\partial t} \psi \rangle = \langle \psi^\dagger \hat{A} \psi \rangle$$

(5.32)

We get analogous identities for $E, V, L$ if we derive instead of (5.28) the identity

$$\frac{\partial}{\partial y} (\psi^\dagger \psi(\tau)) = \psi^\dagger (L(\tau) - L) \psi(\tau) - \frac{\partial}{\partial x} (\psi^\dagger L(1)(\psi(\tau))) + \frac{\partial^2}{\partial x^2} (...)$$

(5.33)

Now letting $\tau = Y, T$ (remember that $Y, T$ are independent of $y, t$), we can rewrite (5.31) and the analogous identities for $L$ as

$$-\langle \psi^\dagger \hat{A} \psi \rangle = -\partial \Omega \partial Y \langle \psi^\dagger \psi \rangle - \partial W \partial Y \langle \phi^\dagger \phi \rangle - \partial p \partial T \langle \psi^\dagger L(1) \psi \rangle - \partial U \partial T \langle \phi^\dagger \phi \rangle$$

(5.34)

The last term we need comes from the identity

$$\frac{\partial}{\partial t} (\psi^\dagger L(1) \psi(\tau)) - \frac{\partial}{\partial y} (\psi^\dagger A(1) \psi(\tau)) = \psi^\dagger [L(1)(A(\tau) - A(1)(L(\tau) - L)] \psi(\tau) + \frac{\partial}{\partial x} (...)$$

(5.35)

which, after putting $\tau = X$ and averaging, gives

$$\langle \psi^\dagger L(1) \frac{\partial A}{\partial X} - A(1) \frac{\partial L}{\partial X} \rangle \psi \rangle = \frac{\partial \Omega}{\partial X} \langle \psi^\dagger L(1) \psi \rangle - \frac{\partial E}{\partial X} \langle \psi^\dagger A(1) \psi \rangle + \frac{\partial W}{\partial X} \langle \psi^\dagger L(1) \frac{\partial \phi}{\partial t} \rangle - \frac{\partial W}{\partial X} \langle \phi^\dagger A(1) \frac{\partial \phi}{\partial t} \rangle$$

(5.36)

Summing up (5.34) and (5.36), and using the compatibility conditions

$$\frac{\partial U}{\partial Y} = \frac{\partial W}{\partial X}, \quad \frac{\partial U}{\partial T} = \frac{\partial W}{\partial Y}, \quad \frac{\partial V}{\partial T} = \frac{\partial W}{\partial Y}$$

(5.37)

we get

$$\langle \psi^\dagger F \psi \rangle = (\frac{\partial \Omega}{\partial Y} - \frac{\partial E}{\partial T}) \langle \psi^\dagger \psi \rangle + (\frac{\partial \Omega}{\partial X} - \frac{\partial p}{\partial T}) \langle \psi^\dagger L(1) \psi \rangle + (\frac{\partial p}{\partial Y} - \frac{\partial E}{\partial X}) \langle \psi^\dagger A(1) \psi \rangle$$

(5.38)

using (ii) from (5.31) we can rewrite the Whitham equations in the final form

$$\left(\frac{\partial \Omega}{\partial Y} - \frac{\partial E}{\partial T}\right) dp + \left(\frac{\partial p}{\partial T} - \frac{\partial \Omega}{\partial X}\right) dE + \left(\frac{\partial E}{\partial X} - \frac{\partial \Omega}{\partial Y}\right) d\Omega = \frac{\langle \psi^\dagger K \psi \rangle}{\langle \psi^\dagger \psi \rangle} dp.$$  

(5.39)
5.5. WHITHAM EQUATIONS FOR THE STATIONARY AND EVOLUTIVE KdV

The KdV is a particular case of the KP equation. The solutions of the KP equation correspond to generic (non-hyperelliptic, that is not of the form $w^2 = p(E)$) Riemann surfaces $\Gamma$: in this case the function $E(P), P \in \Gamma$ itself is no more one-valued and its differential $dE$ has non-zero periods $V_j = \frac{1}{2\pi i} \int_{b_j} dE$. The solutions have the form (5.19). When $\Gamma$ is hyperelliptic the differential $dE$ is exact, $V = 0$ and the dependence on $y$ disappears, giving solutions to the KdV equation. For KdV the Whitham equations have the form

$$\left( \frac{\partial p}{\partial T} - \frac{\partial \Omega}{\partial X} \right) = \frac{\langle \psi^\dagger K \psi \rangle}{\langle \psi^\dagger \psi \rangle} \frac{dp}{dE} \quad (5.40)$$

Solutions to the stationary KdV equation, which is equivalent to the linear system

$L\psi = E\psi, A\psi = w(E)\psi$, come out when $d\Omega$ too is exact, and this is true for $\Omega(E) = 2w(E)dE = \sqrt{4E^3 - g_2E - g_3}dE$; the corresponding Whitham equation gives

$$\frac{\partial w}{\partial X} dE = \frac{\langle \psi^\dagger K \psi \rangle}{\langle \psi^\dagger \psi \rangle} dp \quad (5.41)$$

where $dp$ is normalized with $\text{Im} \int dp = 0$.

5.6. INTEGRABILITY OF THE WHITHAM EQUATIONS FOR K=0

For KdV the Whitham equations have the form (5.40). In this case, the parameters $I_j$ of the preceding section are simply the branch points $E = \{E_1, \ldots, E_{2g+1}\}$ of the spectral curve. For $K = 0$ this comes out as

$$\frac{\partial p}{\partial T} = \frac{\partial \Omega}{\partial X} \quad (5.42)$$

Krichever has shown that (5.42) has solutions $E_1(X,T), \ldots, E_{2g+1}(X,T)$ given implicitly by the conditions

$$\frac{d\Lambda}{dp}(E(X,T))|_{E=E_j(X,T)} + X + T \frac{d\Omega}{dp}(E(X,T))|_{E=E_j(X,T)} = 0, \quad j = 1, \ldots, 2g+1, \quad (5.43)$$

where $d\Lambda$ is an arbitrary differential with possibly discontinuities and singularities not depending on $X, T$. Analogous solutions exist for the KP case (see Ref. 2).
In order to see it, consider that if the function \( S(X,T) = \int_{\infty}^{P} d\Lambda(X,T) + X dp(X,T) + T d\Omega(X,T) \) is such that \( \frac{\partial S}{\partial X} = p, \frac{\partial S}{\partial T} = \Omega \), then (5.42) is automatically satisfied. Now, (5.43) is equivalent to \( dS|_{E=E_j} = 0 \) for \( j = 1, \ldots, 2g+1 \); the form \( \frac{\partial}{\partial X} dS = dp + (\frac{\partial}{\partial X} d\Lambda + X \frac{\partial}{\partial X} dp + T \frac{\partial}{\partial X} d\Omega) \) has the same normalization as \( dp \), the same singularity at the infinity and is holomorphic everywhere else, except, generally speaking, in the points \( E_j \), because e.g. \( \frac{\partial}{\partial X} dE = \frac{1}{2} \left( E-E_1 \right)^{3/2} + \cdots \) cease to be holomorphic in these points. However, thanks to \( dS|_{E=E_j} = 0 \) we have \( dS = \sqrt{E-E_j} dE + O(E-E_j)^{3/2} \) in a neighborhood of \( E_j \), and \( \frac{\partial}{\partial X} dS = \frac{\partial E_j}{\partial X} + \frac{dE}{2\sqrt{E-E_j}} + \cdots = \frac{\partial E_j}{\partial X} \frac{2 du}{2u+\cdots} \) comes out to be non-singular in \( E_j \) too: thus \( \frac{\partial}{\partial X} dS \) coincides with \( dp \), and similarly \( \frac{\partial}{\partial T} dS = d\Omega \).

5.7. Integrability of the stationary Whitham equation for \( K = 1 \)

The equation \( \frac{\partial w}{\partial X} dE = dp \) can be integrated in the following way (see Ref. 3): take \( k = \sqrt{E}, dE = 2k dk \), \( w = \sqrt{E^{2g+1} + c_1 E_{2g-1} + \cdots + c_{2g} = k^{2g+1} + \frac{2g+1}{2} T_{2g+1} k^{2g-1} + \frac{2g-1}{2} T_{2g-1} k^{2g-3} + \cdots + \frac{3}{2} T_3 k + \frac{1}{2} T_1 + O(\frac{1}{k^2})} \) and require \( \text{Im} \int_{h_i} w dE = h_i \) = const, \( \text{Im} \int_{h_i} w dE = h_i' \) = const, \( i = 1, \ldots, g \). The first condition gives \( c_1, \ldots, c_g \) as algebraic functions of \( T_{2g+1}, \ldots, T_3 \); the second condition fixes also \( c_{g+1}, \ldots, c_{2g} \) as transcendental functions of \( T_{2g+1}, \ldots, T_3, X \). Moreover, the first condition gives \( \frac{\partial}{\partial X} w dE \simeq d\sqrt{E} \simeq dp \), while the second assures that both sides have the same normalization: this means that \( \frac{\partial}{\partial X} w dE = dp \), as desired. (Introducing the functions \( \Omega_j \simeq k^j + O(1), E \to \infty \), it is easy to see that \( \frac{\partial}{\partial j} w dE = d\Omega_j \), thus giving also \( \frac{\partial \Omega_j}{\partial k} = \frac{\partial \Omega_j}{\partial k} \).)
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