Developments from Programming the Partition Method for a Power Series Expansion

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Abstract

In a recent series of papers [1]-[4] a novel method based on the coding of integer partitions has been used to derive power series expansions to previously intractable problems, where the standard Taylor-/Maclaurin method fails. In this method the coefficient at each order \(k\) of the resulting power series expansion is determined by summing all the specific contributions made by each partition whose parts/elements sum to \(k\). The specific contributions are evaluated by assigning values to each element in a partition and then multiplying by a multinomial factor, which depends on the frequencies of the elements in the partitions. This work aims to present for the first time the theoretical framework behind the method, which is now known as the partition method for a power series expansion. To overcome the complexity in evaluating all the contributions from the partitions as the order \(k\) increases, a programming methodology is created, thereby allowing for far more general problems to be considered than originally envisaged. This methodology is based on an algorithm called the bi-variate recursive central partition (BRCP) algorithm, which is, in turn, developed
from a novel non-binary tree-diagram approach to scanning the integer partitions summing to a specific value. The main advantage of the BRCP algorithm over other means of generating partitions lies in the fact that the partitions are generated naturally in the multiplicity representation. By developing the theoretical framework for the partition method for a power series expansion it becomes apparent that scanning over all partitions summing to a particular value can be regarded as a discrete operation denoted here by the discrete operator $L_{P,k}$. The summand inside this operator depends on the coefficients of an inner and outer series resulting from expressing the original function as a pseudo-composite function of two power series expansions. As a consequence, simple modifications to the program for the partition operator result in programs for other operators involving specific types of partitions such as those with: (1) only odd or even elements, (2) a fixed number of elements, (3) discrete elements, (4) specific elements and (5) those with restrictions on the size of their elements. Another interesting modification results in the generation of all the conjugate partitions for their original partitions by transposing the rows and columns of their Ferrers diagrams. The operator approach is then applied to the theory of integer partitions, in particular to generalisations of the generating functions for both discrete and standard partitions. The main generalisation involves the introduction of the parameter $\omega$, whose powers indicate the total number of elements in the partitions, while the coefficients of the power series expansions become polynomials in $\omega$. Finally, power series expansions for more advanced infinite products involving quotients and products of the discrete and standard partition generating functions are derived, culminating in the multi-parameter infinite product first studied by Heine.

**Keywords:** Absolute convergence; Algorithm; Bivariate recursive central partition algorithm (BRCP); Coefficient; Conjugate partition; Conditional convergence; Discrete partition; Discrete partition number; Discrete partition polynomial; Divergent series; Divisor polynomials; Doubly-restricted partition; Equivalence; Even partition; Generating function; Infinite product; Multinomial factor; Multiplicity representation; Odd partition; Partition; Partition function; Partition function polynomial; Partition method for a power series expansion; Partition operator; Partition polynomial; Power series expansion; Programming methodology; Pseudo-composite function; Recur-
rence relation; Regularised value; Taylor/Maclaurin series; Transpose; Tree diagram

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1 Introduction

This work grew out of a desire to develop a programming methodology on the partition method for a power series expansion, which has featured prominently in a recent series of papers [1]-[4] aimed at deriving power series expansions for previously intractable mathematical functions. Although still a relatively novel method, the method for a power series expansion was first introduced in the derivation of an asymptotic expansion for the particular Kummer or confluent hypergeometric function that emerges in the response theory of magnetised quantum plasmas such as the degenerate electron and charged Bose gases [5]. Specifically, a large $|\alpha|$-expansion was obtained for $\text{1}_1\text{F}_1(\alpha, \alpha + 1; z)$, which is itself a variant of the incomplete gamma function $\gamma(\alpha, z)$. As a consequence, the physical properties of the magnetised charged Bose gas were later determined in the weak field limit in Ref. [6], the first time ever that the limit had been investigated for a plasma. Since then, the method for a power series expansion has been applied successfully to various mathematical functions such as $\frac{1}{\ln^8(1 + z)}$, $\text{sec}^8 z$, $z^2 \text{csc}^8 z$, and the three Legendre-Jacobi elliptic integrals, $F(\psi, k)$, $E(\psi, k)$ and $\Pi(\psi, n, k)$, while more recently, it has been applied to a finite sum of inverse powers of cosines [4], resulting in the development of a spectacular empirical method to solve the problem.

Initially, it was observed that in order to apply the partition method for a power series expansion the original function needed to be expressed as a composite function. In the present work, however, this condition will be relaxed to quotients of “pseudo-composite” functions. The two functions involved in the construction of the quotients of the pseudo-composite functions must themselves be expressible in terms of power series expansions, which are referred to as the inner and outer series. Neither of these series, however, is required to be absolutely convergent. When the conditions for the quotients are met, a power series expansion can be obtained in which the coefficients at each order $k$ are calculated by summing the specific contributions due to each partition in which the parts or elements sum to $k$. The contribution made by each partition is in turn determined by: (1) assigning a value to each element in the partition, (2) multiplying these values by a multinomial factor composed of the factorial of the total number of elements, $N_k!$ divided by the factorial of the number of occurrences or frequencies of each element $i$ in the partition, $n_i!$, and (3) if necessary, carrying out a further multiplication with the coefficients of the inner series set at the number of elements in...
As explained in the introduction to Ref. [3], the partition method for a power series expansion is able to produce power series where the standard technique or Taylor/Maclaurin series approach breaks down, but for those cases where a power series expansion can be obtained by the standard technique, the power series expansions are identical, although from a totally different perspective. As a consequence, the cross-fertilisation of both approaches is frequently responsible for new mathematical results and properties. A particularly fascinating property of the partition method for a power series expansion is that the discrete mathematics of partitions is being employed to derive power series expansions for continuous functions.

Whilst Ref. [3] was primarily concerned with the application of the partition method for a power series expansion to basic trigonometric and related functions, it was stated that a theoretical framework was being developed for situations involving more complicated functions where the assignment of values to the elements of the partitions is no longer specific, but quite general. This theoretical framework was necessary for the development of a programming methodology whose purpose was to facilitate the summation process over the partitions. Therefore, both the theoretical framework and development of a programming methodology represent important topics in the present work. However, in the course of developing the theoretical framework it became apparent that the theory of partitions was also affected. Moreover, with the development of the new algorithm to facilitate the calculation of the coefficients in the partition method for a power series expansion, one could tackle various problems pertaining to integer partitions such as the evaluation of: (1) doubly-restricted partitions, (2) partitions with a fixed number of elements, (3) conjugate partitions via Ferrers diagrams, and (4) discrete/distinct partitions. As we shall see, such problems, which often require implementing different algorithms if they can be solved at all, can be addressed by making minor adjustments or modifications to the algorithm presented in Sec. 2 of this work. Therefore, by developing the theoretical framework and tackling the mathematical programming issues associated with the method, we have not only been to improve the method by making it more general than first thought, but we have also been able to solve other outstanding problems in the theory of partitions. In addition, the new operator approach that evolved in the process casts partitions in a different light and it is hoped that this approach will result in new advances in the future. Consequently, the work has progressed well and truly beyond
its original conception resulting in its present size.

Throughout this work the concept of regularisation of a divergent series will be employed. This concept is defined here as the removal of the infinity in the remainder so as to make the series summable or yield a finite limit. The finite limits obtained in this process are referred to as regularised values, whilst the statements in which they appear together with the series can no longer be regarded as equations. Instead, they are referred to as equivalence statements or equivalences, for short. A necessary property of the regularised value is that it must be unique, particularly if it is identical to the value one obtains when the series is absolutely convergent within a finite radius of the complex plane such as the geometric series. Much of this will become clearer as we progress further in the present work, but for those readers wishing to seek a greater understanding of the concept and its ramifications to asymptotics, they should consult Refs. [3] and [7]-[11].

Since all the partitions summing to a particular value are required to determine the coefficients in the partition method for a power series expansion, Sec. 2 examines the current state of the art for generating partitions. Here, various algorithms are compared with the bi-variate recursive central partition algorithm or BRCP for short, which has only been sketched out in Refs. [1] and [3]. The BRCP algorithm is based on a non-binary tree diagram representation for all the partitions summing to a specific value. Whilst it is acknowledged that some of the alternative algorithms can be faster than the BRCP algorithm in scanning the partitions and occasionally, when printing them out on a screen, the BRCP algorithm is the most efficient algorithm for implementation in the partition method for a power series expansion. This is because it generates the partitions in the multiplicity representation, which, as we shall see, is both the ideal and minimum amount of information required by the partition method for a power series expansion.

Sec. 3 presents the general mathematical theory underpinning the partition method for a power series expansion, which is discussed in terms of the quotient of two pseudo-composite functions in Theorem 1. Under certain conditions it is also shown that the method can be used to derive a power series expansion for the inverse or reciprocal of the quotient. Consequently, a hybrid recurrence relation involving the coefficients from both power series expansions is obtained, while an example involving the reciprocal of a Bessel function to arbitrary order, viz. $1/J_\nu(z)$, is presented to make the preceding material clearer to the reader. Later in the section, Theorem 1 is extended in a corollary, where the quotient of the pseudo-composite functions is taken
to an arbitrary power, $\rho$. This extension only affects the multinomial factor in the method for a power series expansion by transforming it into the Pochhammer symbol involving $\rho$ and the number of elements in each partition.

Because the partition method for a power series expansion is based on summing over all the partitions summing to a specific value $k$, a discrete operator called the partition operator or $L_{P,k}[\cdot]$ is introduced. When this operator acts on unity, it gives the number of partitions or partition function, $p(k)$. On the other hand, if the operator acts upon only the multinomial factor mentioned above in Step 2, then it is found to yield a value of $2^{k-1}$, while if the phase factor of $(-1)^{N_k}$, where $N_k$ represents the number of elements, is included with the multinomial factor, then it vanishes. This means that the expressions for the coefficients given in Theorem 1 can be expressed in terms of the new operator. One interesting property of the partition operator is that frequently the arguments inside the operator can be interchanged with the result the operator yields. As a consequence, not only are many of the results derived previously via the partition method for a power series expansion in Refs. [1]-[3] expressed in terms of the operator, but also the inverse relations are presented. The section concludes by showing that if the quotient of the pseudo-composite functions yields an infinitely differentiable function, then its derivatives can also be expressed in terms of the partition operator.

Sec. 4 is devoted towards the creation of the programming methodology that enables the coefficients obtained from the partition method for a power series expansion to be calculated from the general theory in Sec. 3. This is necessary because beyond the first few orders, it becomes increasingly onerous to determine the coefficients by hand. The computational task is divided into two steps. In the first step a code utilising the BRCP algorithm of Sec. 2 is written in C/C++ so that its symbolic output can be processed by Mathematica [12] in the second step. The reason for the second step is that often the coefficients in the resulting power series expansions are either rational or algebraic in nature and that neither of these forms can be handled effectively in C/C++ with its floating point arithmetic. In fact, the coefficients often become so small that they would practically vanish in C/C++. By importing the output into Mathematica, we can exploit its integer arithmetic routines to express the coefficients in integer form or we can use the symbolic routines to express them, for example, as polynomials when required.
The material in Secs. 2 to 4 serves as a platform for studying various problems in theory of partitions. First, we consider the issue of generating specific types/classes of partitions or different operators such as: (1) those with a fixed number of elements in them, (2) doubly restricted partitions where all the elements are greater one value and less than another, (3) discrete or distinct partitions where an element occurs only once in a partition and (4) partitions with specific elements in them. Solving these problems invariably means developing new algorithms or codes, but in the case of the BRCP algorithm we shall see that they can be solved with relatively minor modifications. This is due to the power and versatility of the tree diagram approach upon which the BRCP algorithm is based. Furthermore, the BRCP code can be adapted to exploit two-dimensional dynamic memory allocation when one wishes to determine conjugate partitions by means of Ferrers diagrams. For the benefit of the reader many of the codes discussed in Secs. 4 and 5 are presented in the appendix, where it can be seen that they are surprisingly compact.

In Secs. 6-8 the partition method for a power series expansion and the operator approach are employed in the derivation of generating functions from increasingly sophisticated extensions of the infinite product defined by \( P(z) = \prod_{k=1}^{\infty} (1 - z^k) \) and its inverse \( 1/P(z) \). This famous product was found by Euler to yield the generating function whose coefficients are equal to the partition function \( p(k) \). Before the study commences, however, Sec. 6 begins by showing in Theorem 2 that the generating function of \( P(z) \) is absolutely convergent within the unit disk centred at the origin in the complex plane, but is divergent for all other values of \( z \). That is, \( P(z) \) represents the regularised value for divergent values of \( z \) as given by Equivalence (73). This is often postulated in the literature, but no formal proof of this important result has ever appeared previously. Next Theorem 1 is used to derive the generating function for \( 1/P(z) \) whose coefficients are given in terms of the partition operator acting with each element \( i \) assigned a value of \( p(i) \). In this case the coefficients \( q(k) \), which are referred to as the discrete partition numbers, are only non-zero when \( k \) is a pentagonal number, again a result that was first obtained by Euler. By inverting this method we obtain the partition function \( p(k) \) in terms of the partition operator acting with each element \( i \) assigned to \( q(i) \).

Because most of the discrete partition numbers vanish, the new result for the partition function simplifies dramatically when a program based on Secs. 4 and 5 is created. In fact, scanning over those partitions in which the ele-
ments are pentagonal numbers represents a totally different application from the examples studied in Sec. 5. Therefore, a code is developed which only determines the partitions whose elements are pentagonal numbers. However, if $P(z)$ and its inverse are expressed as exponentiated double sums, then Theorem 1 can be used to derive different expressions for the coefficients of the generating functions. In this instance the elements in the partitions are assigned values $\gamma_i$, which are obtained from summing the divisors or factors of $i$ divided by $i$. As a result, it is found that the only difference between the discrete partition numbers and the partition function via this alternative approach is that the former set of numbers possess an extra phase factor of $(-1)^{N_k}$ inside the partition operator, where again $N_k$ is the total number of elements in a partition.

Sec. 7 begins with an extension of the inverse of $P(z)$, where the coefficients of $z^k$ in the product are now equal to $C_k$ rather than -1. Theorem 3 shows that a generating function can be obtained from this product where the coefficients are determined by applying the discrete partition operator $L_{DP,k}[\cdot]$ acting with each element $i$ assigned a value of $C_i$. Conversely, this theorem implies that any power series can be expressed as an infinite product. If the $C_k$ equal the parameter $\omega$, then the coefficients of the generating function for the product $Q(z, \omega)$ not only become polynomials of degree $k$ in $\omega$ giving rise to the discrete partition polynomials $q(k, \omega)$, but also the powers of $\omega$ yield the number of elements involved in the discrete partitions. Then more identities involving the discrete partition polynomials are derived before a corollary to Theorem 3 appears, the latter dealing with the case where the factors in the product are taken to an arbitrary power $\rho_k$. Hence, one can derive generating functions for very complicated products with varying powers and/or with different factors accompanying the powers of $z$.

Sec. 7 concludes by looking at the special case in the corollary to Theorem 3 where the arbitrary powers $\rho_k$ and coefficients $C_k$ are set equal to the constant value $\rho$ and the parameter $\omega$, respectively. In this case the coefficients of the generating function become $q(k, \omega, \rho)$ and are now polynomials of degree $k$ in both $\rho$ and $\omega$. Since the special cases of $\rho=2$ and 3 feature in well-known products studied by Euler and Gauss, the properties and values of $q(k, \omega, 2)$ and $q(k, \omega, 3)$ are also examined in detail.

Sec. 8 deals with the derivation of the generating functions for even more complicated products than those appearing in Sec. 7. As a result of the success of introducing the parameter $\omega$ in the inverse of $P(z)$, the first example deals with the introduction of $\omega$ into $P(z)$. This results in a generating func-
tion for the new product \( P(z, \omega) \) whose coefficients \( p(k, \omega) \) are polynomials of degree \( k \) in \( \omega \) that reduce to the partition function \( p(k) \) when \( \omega \) is set equal to unity. These partition function polynomials are expressed in terms of the partition operator acting with each element \( i \) assigned to the discrete partition polynomials or rather \( q(i, -\omega) \). They are found to possess many interesting properties, while their coefficients represent the number of partitions in which the number of elements is given by the power of \( \omega \). Out of this analysis interesting recurrence relations are obtained for the number of discrete partitions or \( q(i, 1) \).

As was the case for \( P(k) \), \( P(k, \omega) \) and its inverse can also be expressed as an exponentiated double sum, both of which can be handled by Theorem 1. Thus, it is found that the partition function and discrete partition polynomials can be be expressed in terms of the partition operator with the main difference being that in the case of the former polynomials the elements \( i \) are assigned to the polynomials \( \gamma_i(\omega) \), while for the latter they are assigned to \( -\gamma_i(\omega) \). These new polynomials represent the extension of the \( \gamma_i \) mentioned above. Their coefficients are equal to divisors \( d \) of \( i \) divided by \( i \), while each power of \( \omega \) is equal to the reciprocal of the coefficient. Moreover, they reduce to the \( \gamma_i \) when \( \omega = 1 \). Because this is a somewhat unusual situation involving divisors, a program is presented that evaluates the partition function and discrete partition polynomials in symbolic form so that they can be processed by Mathematica. This means that the final forms for the polynomials can be obtained by evaluating the divisor polynomials via the Divisors[k] routine in Mathematica rather than having to create a separate program to solve this problem.

Sec. 8 continues with the introduction of an arbitrary power \( \rho \) into the product \( P(z, \omega) \) and determining the coefficients \( p(k, \omega, \rho) \) of the generating function for this extended product. These coefficients are given in terms of the partition operator acting with the elements \( i \) assigned to minus the partition function polynomials, viz. \( -p(i, \omega) \), multiplied by the Pochhammer symbol of \( (-\rho)^{N_k} \). Next the generating function of the product of \( Q(z, -\beta \omega) \) with \( P(z, \alpha \omega) \) is studied. This infinite product denoted by \( P(z, \beta \omega, \alpha \omega) \) combines the properties of discrete partitions with standard partitions. The coefficients of the resulting generating function, which are denoted by \( QP_k(\omega, \alpha, \beta) \), are polynomials of degree \( k \) in \( \omega \), while the powers of \( \alpha \) and \( \beta \) indicate the number of elements in the standard and discrete partitions respectively. With this result Sec. 8 concludes with the derivation of the generating function for Heine’s product, which can be represented as the product of \( P(z, \omega x, \omega) \) and
$P(z, \omega y, \omega xy)$. The coefficients of the generating function for this infinite product, which arises in q-hypergeometric function theory, are denoted by $HP_k(\omega, x, y)$ and are obtained by summing the product of $QP_j(\omega, x, 1)$ with $QP_{k-j}(\omega, y, xy)$ for $j$ ranging from 0 to $k$. Several coefficients in the last two examples are tabulated in order to display their complicated nature.

2 Generating Partitions

As indicated in the introduction the partition method for a power series expansion is composed of two major steps. The first step involves determining all the partitions summing to an integer $k$, which represents the order of the variable in the resulting power series expansion. The second and more complicated step is to calculate the contribution that each partition makes to the coefficient of the $k$-th order term in the series expansion. This step will be described extensively in the next two sections when the theoretical framework and programming methodology for the partition method for a power series expansion are presented. For now, however, this section is devoted to the problem of generating partitions in a suitable format to enable the second step of the partition method for a power series expansion to proceed. This means that we shall not only be interested in determining the parts or elements in a partition, but also with evaluating their number of occurrences or frequencies. So, whilst the generation of partitions is an interesting problem in its own right and continues to be the source for new algorithms as evidenced by Refs. [13]-[16], it is required here in order to develop a programming methodology for the partition method for a power series expansion. Hence, we need to examine the existing algorithms for generating partitions to find which, if any, is the most suitable for implementation in the partition method for a power series expansion. Ultimately, we shall find that the novel algorithm sketched out in Ref. [1] will prove to be the most suitable. Moreover, we shall see in Sec. 5 that partitions with specific properties can be determined by making modifications to the algorithm, which is often difficult to achieve, if not impossible, with the other partition-generating algorithms. That is, a completely different algorithm is usually required to solve for each specific property of partitions, whereas only minor changes to the bi-variate recursive central partition algorithm presented in this section are needed to generate partitions with these specific properties. In addition, as a result of the material in appearing Secs. 3 and 4, we shall be able to re-formulate the
partition method for a power series expansion in terms of a partition operator denoted by $L_{P,k}[\cdot]$. The modifications to the BRCP algorithm presented in Sec. 5 will mean that we are effectively programming different operators, which will, in turn, lead to the presentation of new and fascinating results when we study the various generating functions belonging to the theory of partitions in Sec. 6.

As described in Ref. [1], when applying the partition method for a power series expansion there is actually no need to generate all the partitions at each order. For example, one can write down all the partitions necessary for evaluating the first few orders on a sheet of paper. Once they have been determined, one can then proceed to the second step of determining the specific contribution made by each partition to the coefficient of the series expansion. The problem occurs when we wish to evaluate the higher order terms, especially if our ultimate aim is to derive an extremely accurate approximation to the original function. When we need to go to higher orders, the complexity increases dramatically due to the exponential increase in the number of partitions. Then it is no longer feasible to write down all the partitions and carry out the calculations to determine their contributions to the coefficient at a particular order. Consequently, a programming methodology is required for all orders despite the fact that this will ultimately become very slow for very high orders of the series expansion due to combinatorial explosion. Nevertheless, we shall see that in developing this programming methodology we shall uncover very interesting results for the first time in the theory of partitions. For example, it has already been stated that series expansion obtained via the partition method produces a power series that is identical to a Taylor/Maclaurin series when the latter can be evaluated. In these situations the development of a programming methodology for the partition method means that the higher order derivatives in such a series can be expressed in terms of a sum of the contributions from all the partitions summing to a particular order. This has profound implications in mathematics in that we now have a means of linking the continuous/differentiable property of a function with the discrete mathematics of partitions or number theory.

When it was mentioned above that there was no need to employ an algorithm to generate all the partitions in the partition method for a power series expansion, it was meant that there was no need to write them down in for each value of $k$, since only the elements in each partition and their frequencies are required as the input for the second step of the partition
method for a power series expansion. Representing a partition in this manner is known as the multiplicity representation, whereas we shall refer to the representation where each element in a partition is written down as the standard representation. As \( k \) increases, the number of partitions summing to \( k \) or \( p(k) \) increases exponentially. E.g., the number of partitions summing to 100, viz. \( p(100) \), is \( 190596292 \). As a result, it is no longer practical to write the partitions in the standard representation. Whilst the multiplicity representation is sufficient for the application of the power series for a power series expansion, if one wishes to determine high orders of the resulting power series expansion via the method, then one still needs to consider the various algorithms for generating partitions because it could turn out that an algorithm generating partitions in the standard representation may require only minor modifications to provide them in the multiplicity representation. Furthermore, the generation of integer partitions continues to attract interest up to the present time. Therefore, we shall review the existing algorithms, but ultimately our aim will be to determine the most appropriate for the partition method for a power series expansion. On the other hand, those with only an interest in generating partitions may find the other algorithms more suitable in which case they are urged to obtain more information by consulting the list of references.

Having justified the need for generating all the partitions summing to an arbitrary integer, we now turn to the issue of finding an appropriate algorithm that will expedite the process, but will do so in appropriate form for the second step of the partition method for a power series expansion. For a time there seemed to be only one useful algorithm for generating partitions. This was McKay’s algorithm \(^{[17]}\), which was basically a succession rule whereby partitions were generated in linear time. It was developed further by Knuth \(^{[18]}\), who used the fact that if the last element greater than unity is a two, then the next partition can be determined very quickly. This modification means that each partition takes almost a constant amount of time to be generated. The Knuth/McKay algorithm, which is implemented in C/C++ below, generates the partitions summed to a global integer \( n \) in a particular form of the standard representation known as reverse lexicographic order. Consequently, the elements in a partition, say \( a[1] \) up to \( a[k] \), are printed out according to \( a[1] \geq a[2] \geq \cdots \geq a[k] \), while the first element of each new partition is less than or equal to the first element of the preceding partition. For example, the partitions summing to 5 in reverse lexicographic order are: 

\[ 5, 41, 32, 23, 131, 112, 1111 \]
The rules for generating partitions in reverse lexicographic order can be obtained from Refs. [18] and [19]. Briefly, if the partition is not composed entirely of ones, then it ends with a value of $x+1$ followed by zero or more ones. The next smallest partition in lexicographic order is obtained by replacing the segment of the partition \{\ldots, x+1, 1, \ldots, 1\} by \{\ldots, x, \ldots, x, r\}, where the remainder $r$ is less than or equal to $x$.

/* This program determines partitions in reverse lexicographic order following McKay/Knuth algorithm as discussed on p. 38 of Fascicle 3 of Vol. 4 of D.E. Knuth’s The Art of Computer Programming. */

#include <stdio.h>
#include <memory.h>
#include <stdlib.h>

int main(int argc, char *argv[])
{
    int *a, i, m, n, q, x;
    if(argc != 2) printf("execution: ./knuth <partition#>\n");
    else
    {
        n=atoi(argv[1]);
        a=(int *) malloc((n+1)*sizeof(int));
    }

    P1: a[0]=0;
        m=1;

    P2: a[m]=n;
        q=m-(n==1);

    P3: for(i=1; i<=m; i++) printf("%i | ", a[i]);
        printf( " \n" );
        if(a[q] != 2) goto P5;
By current standards the above code is considered to be slow for generating the partitions at each order due to the excessive unconditional branching. From a computational point of view it is also very non-structured and hence, does not accord with modern programming practice. A significantly faster algorithm for generating partitions in reverse lexicographic order has been developed by Zoghbi and Stojmenovic in Ref. [13]. Actually, these authors present two algorithms in their paper, but the second, which generates the partitions in lexicographic order, is slower than the first. Nevertheless, if one runs the above code against a C/C++-coded version of the first algorithm, then one finds that it takes 1362 CPU seconds to print out the partitions summing to 80 on the screen of a Sony VAIO laptop with 2 GB RAM compared with 1399 CPU seconds using the Knuth/McKay code given above. On the other hand, if the partitions are directed to an output file, then it takes only 30 CPU seconds with the Knuth/McKay code compared with 28 CPU seconds with the Zoghbi/Stojmenovic code.

/* This program determines partitions in ascending order following the algorithm given at J. Kelleher's web-page */


```c
#include <stdio.h>
#include <memory.h>
#include <stdlib.h>

int main(int argc, char *argv [])
{
    int *a, i, m, n, ydummy, xdummy, count;
    if(argc!= 2) printf("execution: ./kelleher <partition#>\n");
    else{
        n=atoi(argv[1]);
        a=(int  *) malloc((n+2)*sizeof(int));

        for (i=0; i<=(n+1);i++) a[i]=0;
        a[1]=n;
        count=1;
        while (count != 0){
            xdummy=a[count]-1+1;
            ydummy=a[count]-1;
            count=count -1;
            while (xdummy <= ydummy){
                a[count]=xdummy;
                ydummy=ydummy-xdummy;
                count=count +1;
            }
            a[count]=xdummy+ydummy;
            for (i=0;i<= count;i++){
                printf("%i", a[i]);
                if (i<count) printf(" | ");
            }
            printf(" \n");
        }
        printf(" \n");
        free(a);
        return (0);
    }

    Lexicographic ordering of the partitions ordering of the partitions is not
```
the only method of generating partitions in ascending order. Kelleher and O'Sullivan in Refs. [14] and [15] have developed algorithms for generating partitions in ascending order, but not in lexicographic order, one of which has been created as the C/C++-code appearing above. When this code is run for the partitions summing to 5, the following output is obtained:

1|1|1|1|1
1|1|1|2
1|1|3
1|2|2
1|4
2|3
5

Now, by inverting the reverse lexicographic order of the partitions summing to 5 given earlier, we see that the partition \{1,1,3\} in Kelleher/O'Sullivan code appears before \{1,2,2\}, while in McKay/Knuth code it appears after the latter partition. A similar situation occurs with the \{1,4\} and \{2,3\} partitions. Yet, the partitions for both codes are arranged in ascending order. Moreover, Kelleher and O'Sullivan are able to take advantage of the different ordering between to develop an even more efficient version of the above code. As a result, they state that their optimised version is much superior to the Zoghbi/Stojmenovic code. Interestingly, if the above code is run to yield the 15 796 476 partitions summing to 80 on the same Sony laptop as the previous code, then it takes 1342 CPU seconds to output the partitions. Yet, if one does the same with the more efficient version of their code, then it takes the same amount of time to generate the partitions. On the other hand, if the partitions are directed to an output file, then it is found that the above code takes 29 CPU seconds, while the more efficient version takes 28 CPU seconds, the same time taken as the Zoghbi/Stojmenovic code.

According to Ref. [20], Kelleher has found on his computer system that partitions summing to 80 are generated at a rate of $1.30 \times 10^8$ per second using the first algorithm, while with the second algorithm the rate is $2.87 \times 10^8$ per second. For the Zoghbi/Stojmenovic code the rate is $1.26 \times 10^8$ per second, while with the Knuth/McKay code the partitions are generated at a rate of $1.73 \times 10^8$ per second. Hence, the reason why there was no marked difference in performance in the C versions of the Kelleher codes is attributed to the manner in which the partitions were printed out.

Unfortunately, the above codes do not utilise the recursive nature of partitions, which can be observed by realising that the partitions summing to
$k+1$ include all the partitions summing to $k$ with an extra element of unity in them in addition to other partitions possessing elements greater than unity. In fact, according to p. 45 of Ref. [18], the number of partitions summing to $k$ with $m$ elements, which is denoted by $\left\lfloor \frac{k}{m} \right\rfloor$, obeys the following recurrence relation:

$$\left\lfloor \frac{k}{m} \right\rfloor = \left\lfloor \frac{k-1}{m-1} \right\rfloor + \left\lfloor \frac{k-m}{m} \right\rfloor .$$

(1)

In addition, $\left\lfloor \frac{k+m}{m} \right\rfloor$ represents the number of partitions summing to $k$ with at most $m$ elements. If $P(k, m)$ represents the partitions summing to $k$ with at most $m$ parts, then with the aid of the above recurrence relation we obtain

$$P(k, m) = P(k, m-1) + P(k-m, m) .$$

(2)

This result is derived on p. 96 of Ref. [21].

To incorporate recursion into the process of generating the partitions, we need to construct an algorithm that utilises the special tree diagrams, which first appeared in Ref. [5], but have since been applied to other problems or situations in Refs. [1]-[3]. These trees are different from the binary tree approach in the recent work of Yamanaka et al [16], which seeks to generate each partition in the standard representation in a constant time rather than an average time. Their work represents a further development on Fenner and Loizou [22], who seemed to have been the first to develop a binary tree representation for partitions based on a lexicographic ordering. To construct the special tree diagrams, one begins by drawing branch lines to all pairs of numbers that can be summed to the seed number $k$, where the first number in the tuple is an integer less than or equal to $\left\lfloor \frac{k}{2} \right\rfloor$. Here $[x]$ denotes the greatest integer less than or equal to $x$. For example, in Fig. 1, which displays the tree diagram for the seed number equal to 6, we draw branch lines to (0,6), (1,5), (2,4) and (3,3). Whenever a zero appears in the first element of a tuple, the path stops, as evidenced by (0,6). For the other pairs, one draws branch lines to all pairs with integers that sum to the second number under the prescription that the first member of each new tuple is now less than or equal to half its second member. Hence, for (1,5) we get paths branching out to (0,5), (1,4) and (2,3), but not (3,2) or (4,1). This recursive approach continues until all paths are terminated with a tuple containing a zero as indicated in the figure.
Figure 1: Tree diagram of the partitions summing to 6 for the BRCP algorithm.
It is obvious that all the first members plus the second member of the final tuple in each path represents a partition for $k$. E.g., the path in the figure consisting of $(1,5)$, $(1,4)$, $(2,2)$ and $(0,2)$ represents the partition $\{1,1,2,2\}$, while that consisting of $(1,5)$, $(1,4)$, $(1,3)$ and $(0,3)$ represents the partition $\{1,1,1,3\}$. In addition, the number of branches along each path represents the number of elements in each partition, whilst those tuples with zeros in vertical columns represent the partitions with the same number of elements in them. For $k > 3$ the last path in the figure consists of $([k/2]+1,[k/2])$ and $(0,[k/2])$ when $k$ is odd and $(k/2,k/2)$ and $(0,k/2)$ when $k$ is even. In both even and odd $k$ cases the tree diagram terminates at what we shall call the central partition, which represents the partition of $\{[k/2]+1,[k/2]\}$ for odd values of $k$ and $\{k/2,k/2\}$ for even values of $k$. Unfortunately, this is not all that is required to produce the final tree diagram in the figure. Duplicated paths involving permutations of the same partition must also be removed so that each partition appears only once in the final diagram. When this removal process is carried out, one will eventually end up with the tree diagram displayed in Fig.\[1\] To determine $\binom{k}{m}$, we simply count all the tuples with zero in them $m$ branches from the seed number. Hence, $\binom{6}{3}$ equals the number of tuples with zero in them in the third column, which comes to 3. In addition, the number of partitions with exactly $m$ parts is the same number as the number of partitions whose largest element is $m$, which will be become apparent when we discuss conjugate partitions.

In Ref. \[1\] it was stated initially that duplicated partitions could be removed by the introduction of a search algorithm. Such an algorithm would result in a major increase in the complexity of the above graphical approach for generating partitions, which would, in turn, make the other algorithms presented earlier not only more appealing, but also far more efficient. To avoid the introduction of a search algorithm, it was stated later in the same reference that the generation of the partitions could be improved by viewing the tree diagrams from a different perspective. Thus, one notes first that the entire tree emanating from $(1,5)$ in the diagram is the same tree that one would obtain if the graphical method was applied to 5 instead of 6, thereby exhibiting the recursive nature of the method. Similarly, the tree emanating from $(1,4)$ is the same tree diagram one would obtain by applying the graphical method to 4 instead of 6. This continues all the way down to the last partition whose elements are only composed of ones. Moreover, only the
partitions with unity in them emanate from (1,5) in the diagram, whereas the
partitions emanating from (2,4) only possess elements greater than or equal
to 2. Similarly, the partitions emanating from (3,3) only possess elements
that are greater than or equal to 3. In fact, in the last instance since 3 is
half of 6, there will only be threes involved along the path from (3,3). Thus,
we see that the last path represents the central partition \{3, 3\}, which would
have been \{4, 3\} had we constructed a tree diagram for partitions summing
to 7.

From the tree diagram we see that the second number in a tuple decre-
ments with each rightward horizontal movement or right branch, while the
first element of each tuple increments with downward vertical movement. In
short, the trees are two-dimensional. That is, two variables are required to
construct them, a property first observed by D. Balaic. This is particularly
interesting as it means that we are describing a rare instance of bi-variate
recursion. Hence, there is no need for the introduction of a search algorithm
to remove duplicated partitions. Since it has been noted that the tree di-
agrams terminate at the central partition, we shall refer to the algorithm
that generates the partitions based on the tree diagrams as the bi-variate
recursive central partition or BRCP algorithm.

Before presenting an elementary code utilising the BRCP algorithm, let
us investigate how the recursive properties of partitions are included in the
tree diagrams such as Fig. \[\text{Fig.}\] From the figure we see that the total number
of partitions \(p(k)\) can be obtained by summing all the partitions of \(k\) that
can be separated into \(m\) elements, where \(m\) ranges from 1 to \(k\) and \(k = 6\)
in this instance. Since there is only one partition with one element and one
with only \(k\) elements, by scanning over the columns in the tree diagram we
obtain the trivial equation of

\[p(k) = 2 + \sum_{m=2}^{k-1} \binom{k}{m}, \quad (3)\]

which is valid for \(k \geq 3\). On the other hand, if we scan the rows of the tree
diagram, then we see that the total number of partitions can also be obtained
by letting \(p(k, m)\) represent the number of partitions whose elements are
greater than or equal to \(m\). As a result, we arrive at

\[p(k) = 1 + \sum_{m=1}^{[k/2]} p(k - m, m), \quad (4)\]

21
where, again, \([x]\) is the greatest integer less than or equal to \(x\). This result is given in Ref. [30] except that the variables in \(p(k, m)\) have been interchanged and will be used later in Sec. 4 when we introduce the partition operator.

It has already been stated that \(\left| \begin{array}{c} k + m \\ m \end{array} \right|\) represents the number of partitions summing to \(k\) with at most \(m\) elements, i.e. \(P(k, m)\). If we put \(m=2\) and \(k=4\), then from the diagram that there are three tuples with zeros in them in the column two branches away from the seed number, viz. \{1,5\}, \{2,4\} and \{3,3\}. Hence, \(P(4, 2)=3\). From Eq. (1) we see that \(P(4, 2)\) is also equal to the sum of \(\left| \begin{array}{c} 5 \\ 1 \end{array} \right|\) and \(\left| \begin{array}{c} 4 \\ 2 \end{array} \right|\). If we treat the five in the tuple \{1,5\} in the tree diagram as a seed number, then \(\left| \begin{array}{c} 5 \\ 1 \end{array} \right|\) is equal to one corresponding to the tuple \{0,5\}. Furthermore, if we now treat the four in the tuple \{1,4\} from \{1,5\} as a seed number, then we find that two branches further to the right \(\left| \begin{array}{c} 4 \\ 2 \end{array} \right|\) equals 2 corresponding to the tuples \{0,3\} and \{0,2\}. Note that we could not have used the four in the tuple \{2,4\} in the figure as the seed number because this tree diagram gives all the partitions summing to 4, whose elements are greater than unity. Including the first branches emanating from the seed number of 4, we see that these correspond to the partitions of \{1,3\} and \{2,2\}. Hence, summing the \(\left| \begin{array}{c} 5 \\ 1 \end{array} \right|\) and \(\left| \begin{array}{c} 4 \\ 2 \end{array} \right|\), we find once more that \(P(4, 2)\) is equal to 3, confirming that the tree diagrams do indeed possess the recursive properties of partitions.

According to Knuth [18], \(\left| \begin{array}{c} k \\ m \end{array} \right|\) also represents the number of partitions summing to \(k\), whose largest element is \(m\). This connection can be observed by using Ferrers diagrams, which are studied later in this work. As an example, let us consider \(\left| \begin{array}{c} 6 \\ 3 \end{array} \right|\), which can be determined by summing all those tuples with a zero in the vertical column three branches from the seed number (the standard approach) and is, therefore, equal to 3. Meanwhile, the largest element of a partition always appears in the final tuple ending a path in the tree diagram. Hence, the number of partitions whose largest element is 3 can be determined by summing all the paths ending with the tuple of \{0,3\}. There are three of these in the tree diagram with the first occurring at the top of the fourth column from the seed number, the second at the bottom of
the third column and the third at the bottom of the second column.

With regard to the BRCP algorithm, an elementary version in C/C++, which first appeared in Ref. [1], is

```c
void idx(int k, int j) {
    printf("%d",k);
    k=k-j;
    while (k >= j){
        printf(",%d(",j);
        idx(k--,j++);
        printf(" ) ");
    }
}
```

Note that the order of the variables in the above code is interchanged compared with the tuples in the tree diagram. For $k = 4$ the output from this code is 4,1(3,1(2,1(1))), 2(2). By processing the commas and parentheses, we obtain the partitions in the order they appear in the tree diagram, viz. {4}, {1,3}, {1,1,2}, {1,1,1,1} and {2,2}. Although the output is very compact, the code in this form is not suitable for the implementation into the second step of the partition method for a power series expansion. In fact, the above code has to be adapted in order to solve various problems connected with the theory of partitions studied later in Sec. 5. Even if we want to list the partitions on separate lines in a similar manner to the other codes, modifications are required. Nevertheless, the above code does represent the simplest implementation of the BRCP algorithm. It is not only more structured and hence, more elegant than the reverse lexicographic algorithm of McKay and Knuth, but it is also more powerful or versatile. For example, one single call to `idx(6,1)` results in all the other calls to the routine as shown in the tree diagram. In fact, the total number of calls to `idx` yields the total number of partitions $p(k)$, which is an important quantity in its own right. By introducing a counter for the number of calls to `idx` in the above code, we obtain the total number of partitions required to construct a tree diagram without the need to create new routine. We shall observe later in this work, especially in Sec. 5, how making only a few changes can result in a host of the special properties being determined from partitions.

Let us consider the generation of partitions summing to a particular value as we have done for the other codes presented in this section. To demonstrate the versatility of the BRCP code, we shall not print the partitions as
in the same manner as these codes. Instead, we shall generate the partitions in the multiplicity representation, which is required for the second step of the partition method for a power series expansion. Consequently, a function prototype called `termgen` needs to be introduced into the BRCP code called `partgen` as displayed below. When it is called, it will compute the frequencies of the elements by counting the same elements in each partition, the latter being represented by the array called `part`. In order to facilitate the call to `termgen`, `idx` has undergone minor modification so that in processing the partitions the ones are counted first, the twos next and so on. As a result, the following output is obtained for $k=5$:

1 : 1(5)
2 : 1(1)1(4)
3 : 2(1)1(3)
4 : 3(1)1(2)
5 : 5(1)
6 : 1(1)2(2)
7 : 1(2)1(3)

In the code given below the variable `term` represents a rolling count of the partitions as they are being determined by `idx`. In the above output the first value printed out on each line is the value of `term`, which is followed by a colon. Hence, the final value of `term` represents the total number of partitions for each value of $k$ or the variable `tot` in the code. Each line of output only gives the nonzero values of the frequencies of the elements accompanied by the values of the elements presented in parentheses. For example, 1(5) denotes the partition \{5\} where $n_5=1$ and all the other $n_i$ equal zero, while 3(1) 1(2) represents \{1,1,1,2\}, in which case $n_1=3$ and $n_2=1$. As expected, the final partition is the central partition for $k=5$, viz. \{2,3\}.

As a comparison, the code given below was run on the same Sony laptop as the other codes, where it was found that it took 1561 CPU seconds to compute all the partitions summing to 80 according to the above format. Hence, the execution time compared with the other codes has increased, primarily due to the extra processing of the partitions. However, if the output is directed to a file, which can, in turn, be used as input to the partition method for a power series expansion, then it only takes 26 CPU seconds to execute, which makes it the best performing code in this mode.

Whilst the other algorithms/codes discussed in this section may prove to be faster than even an optimised version of the BRCP algorithm in other situations, they do not possess the versatility or flexibility of the latter. We shall
use this versatility when we embark on programming the partition method for a power series expansion in Sec. 4. It should also be noted that Refs. [18] and [16] present extra algorithms for generating partitions according to a specific number of parts, whilst the latter reference present another algorithm which outputs doubly-restricted partitions or where the elements lie in a specified range. In Sec. 5 we shall see that only minor modifications to the BRCP algorithm are required to solve these problems. That is, there is no need to create an entirely different algorithm to solve such problems.

```c
#include <stdio.h>
#include <memory.h>
#include <stdlib.h>

int tot,*part;
long unsigned int term=1;

void termgen()
{
    int freq,i;
    printf("%ld: ",term++);
    for(i=0;i<tot;i++)
    {
        freq=part[i];
        if(freq) printf("%i (%i) ",freq,i+1);
    }
    printf("\n");
}

void idx(int p,int q)
{
    part[p-1]++;
    termgen();
    part[p-1]--;
    p -= q;
    while(p >= q){
        part[q-1]++;
        idx(p--, q);
        part[q++] -=1;--;
```
int main(int argc, char *argv[])
{
    int i;
    if(argc !=2) printf("partgen <sum of the partitions>\n");
    else{
        tot=atoi(argv[1]);
        part=(int *) malloc(tot*sizeof(int));
        if(part==NULL) printf("unable to allocate array\n\n");
        else{
            for(i=0;i<tot;i++) part[i]=0;
            idx(tot,1);
            free(part);
        }
    }
    printf("\n");
    return(0);
}

Another interpretation of the tree diagram in Fig. 1 is to realise that the first branch from the seed number is the only single element partition, viz. \{6\}. The partitions summing to \(n-1\) or 5 in this case appear along the second branch to \{1,5\} in the tree diagram. If the partitions summing to \(n-1\) have already been stored in an array, then all we need to do is increment the number of ones by one in all these partitions to get the partitions for \(n\). The next branch emanating from the seed number represents the tree diagram for \(n-2\), but now all the elements in the partitions are greater than unity. So, if the partitions summing to \(n-2\) have been stored previously, then we disregard those partitions where the number of ones is non-zero and increment the number of twos in the remaining partitions by one to get the partitions summing to \(n\). The next branch from the seed number represents the tree diagram for \(n-3\) except that those partitions, where the number of ones and twos are non-zero, are now neglected. To get the partitions that sum to \(n\), we increment the number of threes by one in the remaining partitions. This process continues until \([n/2]\) tree diagrams have been processed. Although the new interpretation may lead to less processing
of the partitions, it comes at the expense of having to store all previous partitions in memory. Nevertheless, we shall return to this interpretation in the next section.

3 The Partition Method for a Power Series Expansion

As mentioned earlier, the second step in the partition method for a power series expansion is the more important step. It involves coding partitions so that each makes a distinct contribution to the coefficient in the resulting power series expansion. As discussed in Refs. [2] and [3] these contributions need not be numerical in nature as was the case in the computation of the reciprocal logarithm numbers in Ref. [1]. They can also be functions or polynomials, which will become evident later in this work.

The contribution to a coefficient made by each partition in a tree diagram is not only dependent upon the total number of elements or parts in the partitions, \( N \), but also on the elements in the partitions. Originally, when the method was devised, the elements were set equal to \( l_i \), while \( n_i \) represented the number of occasions or frequency each \( l_i \) appeared in a particular partition. Therefore, if there were \( j \) elements in a partition, then \( \sum_{i=1}^{j} n_i = N \), while \( \sum_{i=1}^{j} n_i l_i = k \), which represents the order of the power series expansion. Later, it was decided to let \( n_1 \) represent the number of ones in a partition, \( n_2 \) represent the number of twos, \( n_3 \), the number of threes and so on. Then \( \sum_{i=1}^{k} n_i = N \). The reason why the former approach was adopted initially was that it eliminated the redundancy caused by the fact that often, many of the \( n_i \) were equal to zero in the partitions. For example, \( n_6 \) equals unity in only one partition in the tree diagram, while for this partition \( n_1 \) to \( n_5 \) equal zero. On the other hand, the problem with the first approach is that \( N \) varies and this means that writing down general formulae or expressions involving all the partitions is far more awkward. Furthermore, the redundancy due to the fact that many of the \( n_i \) are equal to zero in the partitions has no effect on the tree diagram and consequently, on the the BRCP algorithm. Hence, we shall adopt the second approach when discussing partitions, which seems to be the generally accepted approach used by mathematicians [18].

In Refs. [1]-[3] the partition method was applied to specific instances where the standard method of deriving Taylor/Maclaurin series expansions
breaks down. However, Taylor/Maclaurin series expansions are simpler to
derive when the original function is differentiable. In cases where both the
partition method can be applied and a Taylor/Maclaurin series can be de-

erived, they yield identical results even if the resulting expansion is divergent.

In addition, it was described extensively in both Refs. [2] and [3] how the
method can be extended to situations where the coefficients may be depen-
dent upon a variable rather than provide a pure number. Therefore, it should
be possible to develop a general theorem describing the partition method for a
power series expansion as is the case with Taylor/Maclaurin series. Of course,
such a theorem will need to indicate under what conditions the method is
valid. Moreover, in Refs. [2] and [3] it was described how the method could
be inverted to yield the power series expansion of the reciprocal function.

That is, if the method can be applied to a function $f(x)$, then it could often
be applied to $1/f(x)$. A theorem on the method would also need to indicate
under what conditions it can be inverted.

Before we present the theorem describing the partition method for a power
series expansion, we need to introduce some preliminaries. First, in the lit-
erature a composite function $g \circ f(x)$ is defined as being equal to $g(f(x))$.

Here we shall define a pseudo-composite function $g_a \circ f$ as being equal to

$g(af(x))$, where $a$ need not necessarily be a number. Next we need to explain
the concept of regularisation. For more detailed descriptions of this process
the reader is referred to Refs. [2]-[3] and [7]-[10]. According to this concept,
when a power series representation for a function is divergent, particularly
as happens with an asymptotic expansion, it needs to be regularised in order
to yield meaningful values that are representative of the original function.

As a consequence, when it is uncertain that a power series representation is
convergent or when it is known to be divergent, we cannot use the equals
sign in a mathematical statement. Instead, we introduce the less stringent
equivalence symbol and refer to the resulting expression as an equivalence
statement. For example, in Refs. [2] and [7]-[8] it is shown that the geometric
series, i.e. $\sum_{k=0}^{\infty} z^k$ is absolutely convergent for $|z|<1$, conditionally con-
vergent for $\Re z<1$ and $|z|\geq 1$, undefined for $\Re z=1$ and divergent for $\Re z>1$.

In the last two instances it is simply invalid to say that the series is equal
to anything. However, through the process of regularisation it is found that
the regularised value of the geometric series is the same value of $1/(1-z)$
that one obtains as the limit value when the series is either conditionally or
absolutely convergent. Furthermore, this value is bijective and hence, unique
for $z$ lying in the principal branch of the complex plane. Therefore, when
the series is divergent, we can only say that it is equivalent to its regularised value. Because the regularised value is equal to the limit value of the series when it is convergent, which is not always the case as discussed in Ch. 4 of Ref. [7], we replace the equals sign by the equivalence symbol for all values of $z$. That is, we can express the geometric series as an equivalence statement or equivalence for short by the following statement:

$$\sum_{k=N}^{\infty} z^k \equiv \frac{z^N}{1 - z} , \quad \forall z .$$  \hspace{1cm} (5)

Now we introduce the main theorem in this work.

**Theorem 1.** Given that the function $f(z)$ can be expressed in terms of a power series referred to here as the inner power series, in which $f(z) \equiv \sum_{k=0}^{\infty} p_k y^k$ and $y = z^\nu$, and that the function $g(z)$ can be expressed in terms of another or outer power series, i.e. $g(z) \equiv h(z) \sum_{k=0}^{\infty} g_k z^k$, where $h(z)$ is an arbitrary function or number, then for non-zero values of $p_0$ there exists a power series expansion for the quotient of the pseudo-composite functions $g_a \circ f$ and $h_a \circ f$ given by

$$\frac{g_a \circ f}{h_a \circ f} \equiv \sum_{k=0}^{\infty} D_k y^k .$$  \hspace{1cm} (6)

In the above equivalence the first few coefficients $D_k$ are given by

$$D_0 = F(ap_0) , \quad D_1 = aF^{(1)}(ap_0) p_1 ,$$  \hspace{1cm} (7)

and

$$D_2 = \frac{a^2}{2} F^{(2)}(ap_0) p_1^2 + aF^{(1)}(ap_0) p_2 .$$  \hspace{1cm} (8)

A general formula for the coefficients can be derived by analysing the partitions summing to $k$. This yields

$$D_k = \sum_{n_1, n_2, n_3, \ldots, n_k=0}^{k, [k/2], [k/3], \ldots, 1} a^N F^{(N)}(ap_0) \prod_{i=1}^{k} \frac{p_i^{n_i}}{n_i!} .$$  \hspace{1cm} (9)

In this equation $\sum_{i=1}^{k} n_i = N$, while $F^{(N)}(ap_0)$ represents the $N$-th derivative of the function $F(ap_0)$, which, in turn, represents the regularised value of the
power series expansion \( \sum_{j=0}^{\infty} q_j (a p_0)^j \). That is, \( F(a p_0) \equiv \sum_{j=0}^{\infty} q_j (a p_0)^j \). For the important case where \( p_0 = 0 \), the coefficients are given by

\[
D_k = \sum_{n_1,n_2,n_3,...,n_k =0 \atop \sum_{i=1}^k in_i = k} \frac{k \cdot [k/2] \cdot [k/3] \cdots 1}{n_1! n_2! n_3! \cdots n_k!} \cdot q_N a^N N! \prod_{i=1}^k \frac{p_i^{n_i}}{n_i!} .
\]

Moreover, for \( D_0 \neq 0 \), the inverted quotient of the pseudo-composite functions can also be expressed in terms of a power series and is given by

\[
\frac{h_a \circ f}{g_a \circ f} \equiv \frac{1}{D_0} \sum_{k=0}^{\infty} E_k y^k .
\]

Here the coefficients \( E_k \) are found to be

\[
E_k = \sum_{n_1,n_2,n_3,...,n_k =0 \atop \sum_{i=1}^k in_i = k} (-1)^N N! \prod_{i=1}^k \frac{D_i^{n_i}}{n_i!} .
\]

Finally, the coefficients \( D_k \) and \( E_k \) satisfy the following recurrence relation:

\[
\sum_{j=1}^{k} D_j E_{k-j} = 0 .
\]

**Remark 1.** As in the case of a Taylor/Maclaurin series the power series given by Equivalences (6) and (11) can be either (1) convergent for all values of the variable, (2) absolutely convergent within a finite radius of convergence or (3) asymptotic, which is defined here as a power series expansion with zero radius of absolute convergence.

**Remark 2.** The second result for the \( D_k \), viz. Eq. (10), is similar in form to the definition on p. 134 of Ref. [21] for the partial or second type of Bell polynomial. In fact, the latter are a special case of the above theorem, which can be obtained by setting \( q_k = a = 1 \) and \( p_k = x_k/k! \).

**Proof.** Since \( g(z) \) can be expressed in terms of a power series expansion and the function \( h(z) \), we have

\[
\frac{g_a \circ f}{h_a \circ f} \equiv \left( q_0 + \sum_{k=1}^{\infty} q_k a^k f(z)^k \right) .
\]
Introducing the power series expansion for \( f(z) \) into the above result yields
\[
\frac{g_a \circ f}{h_a \circ f} \equiv \left( a_0 + a_1 \sum_{k=0}^{\infty} p_k y^k + q_2 a^2 \left( \sum_{k=0}^{\infty} p_k y^k \right)^2 + \cdots \right) .
\] (15)

Isolating the zeroth order term of the power series expansion for \( f(z) \) in the above result yields
\[
\frac{g_a \circ f}{h_a \circ f} \equiv \left( a_0 + a_1 \left( p_0 + \sum_{k=1}^{\infty} p_k y^k \right) + q_2 a^2 \left( p_0 + \sum_{k=1}^{\infty} p_k y^k \right)^2 + \cdots \right) .
\] (16)

Expanding in descending powers of \( p_0 \) yields
\[
\frac{g_a \circ f}{h_a \circ f} \equiv \sum_{j=0}^{\infty} \left( \sum_{k=1}^{\infty} q_k (a p_0)^k \right) + \sum_{j=1}^{\infty} \sum_{j=1}^{\infty} q_j a^j p_0^{j-1} \sum_{k=1}^{\infty} p_k y^k + \sum_{j=2}^{\infty} \sum_{j=2}^{j} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} q_j a^j p_0^{j-2} \sum_{k=1}^{\infty} p_k y^k + \cdots .
\] (17)

Let us now represent the regularised value of the series \( \sum_{k=0}^{\infty} q_k (a p_0)^k \) by \( F(a p_0) \). That is,
\[
\sum_{k=0}^{\infty} q_k (a p_0)^k \equiv F(a p_0) .
\] (18)

We can use this definition to simplify the sums on the rhs of the Equivalence (17), but first we note that
\[
\sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} \left( \begin{array}{c} j \\ i \end{array} \right) q_j a^i p_0^{j-i} \right) = \left( \frac{a^i}{i!} \sum_{j=0}^{\infty} q_j (a p_0)^j - 1 \prod_{l=0}^{j-1} (j - l) \right) = \left( \frac{a^i}{i!} \frac{d^i}{dz^i} \sum_{j=0}^{\infty} q_j z^j \right)_{z=a p_0} .
\] (19)

Introducing Equivalence (18) into the above result yields
\[
\sum_{j=0}^{\infty} \left( \sum_{i=0}^{\infty} \left( \begin{array}{c} j \\ i \end{array} \right) q_j a^i p_0^{j-i} \right) \equiv \left( \frac{a^i}{i!} \frac{d^i}{dz^i} F(z) \right)_{z=a p_0} .
\] (20)
Consequently, Equivalence (17) can be expressed as
\[ \frac{g \circ h}{h \circ f} \equiv \sum_{k=0}^{\infty} \frac{a^k}{k!} F^{(k)}(a p_0) \left( \sum_{j=1}^{\infty} p_j y^j \right)^k. \] (21)

If Equivalence (21) is expanded in powers of \( y \), then we obtain
\[
\frac{g \circ h}{h \circ f} \equiv F(a p_0) + a F^{(1)}(a p_0) p_1 y + \left( \frac{a^2}{2} F^{(2)}(a p_0) p_1^2 + a F^{(1)}(a p_0) p_2 \right) y^2 \\
+ \left( \frac{a^3}{3!} F^{(3)}(a p_0) + a^2 F^{(2)}(a p_0) p_1 p_2 + a F^{(1)}(a p_0) p_3 \right) y^3 + O(y^4). \] (22)

From this result we see that the coefficients of the zeroth, first and second order terms in \( y \) correspond to the results for \( D_0, D_1 \) and \( D_2 \) given in the theorem. Moreover, we see that
\[ D_3 = \frac{a^3}{3!} F^{(3)}(a p_0) p_1^3 + a^2 F^{(2)}(a p_0) p_1 p_2 + a F^{(1)}(a p_0) p_3. \] (23)

The first few coefficients are relatively easy to write down, but beyond that it becomes progressively more difficult. It is at this stage we need to introduce partitions into the analysis to facilitate the derivation of a general expression for the coefficients of the powers of \( y \) in Equivalence (22).

If we look closely at the result for \( D_3 \), then we see that it is the sum of three separate contributions, which is due to the fact that there are only three partitions summing to 3, namely \( \{1,1,1\} \), \( \{1,2\} \) and \( \{3\} \). As stated in Ref. [1]-[3], in order to evaluate the specific contribution due to each partition, a value must be assigned to the elements appearing in the partitions. In particular, Ref. [2] states that these assigned values depend upon the coefficients of the inner series, i.e. the power series expansion for \( f(x) \), which becomes the variable in the outer power series for \( g(z) \). Hence, each element \( i \) in a partition is assigned a value of \( p_i \). Furthermore, there is also a multinomial factor associated with each partition. This factor is not only dependent upon the frequencies \( n_i \) of the elements in a partition, but also on their sum, \( N = \sum_{i=1}^{k} n_i \). It arises from the fact that the value of \( k \) on the rhs of Equivalence (21) is used to render each partition. That is, \( k \) in Equivalence (21) corresponds to \( N \) in the partition method for a power series expansion. Often the multinomial factor simply becomes \( N! / n_1! n_2! \cdots n_k! \), but in Equivalence (21) there is an extra factor of \( a^k F^{(k)}(a p_0) / k! \) in the terms. Again, as \( k \) in
Equivalence (21) plays the role of $N$ in the partition method for a power series expansion, this means that the standard multinomial factor must be multiplied by a factor of $a^N F^{(N)}(ap_0)/N!$ for each partition. Therefore, the contribution from a partition to the overall coefficient is given by

$$C(n_1, n_2, \ldots, n_k) = a^N F^{(N)}(ap_0) \prod_{i=1}^{k} \frac{p^n_i}{n_i!}.$$ 

(24)

We see from this result that we do not actually require the partitions themselves, but their frequencies in order to evaluate the coefficients in the power series for the quotient of the pseudo-composite functions. Nevertheless, it should be noted that each set of frequencies identifies a distinct partition.

To make the preceding material more understandable, let us evaluate the fourth order term in $y$ in Equivalence (21) or $D_4$, which is determined by considering all the partitions summing to 4. There are 5 of these: \{1,1,1,1\}, \{1,1,2\}, \{2,2\}, \{1,3\} and \{4\}. For the first partition, $n_1 = 4$, while the other $n_i$ are zero. Therefore, from Eq. (24) we have

$$C(4,0,0,0) = a^4 F^{(4)}(ap_0) \frac{p_4^4}{4!}.$$ 

(25)

In the case of the second partition $n_1 = 2$, $n_2 = 1$ and the other $n_i$ vanish, while for the third partition, only $n_2 (=2)$ does not vanish. According to Eq. (24), the contributions due to both partitions are

$$C(2,1,0,0) = a^3 F^{(3)}(ap_0) \frac{p_2^2}{3!}.$$ 

(26)

and

$$C(0,2,0,0) = a^2 F^{(2)}(ap_0) \frac{p_2^2}{2!}.$$ 

(27)

For the fourth partition $n_1 = n_3 = 1$ with $n_2$ and $n_4$ equal to zero, while in the final partition only $n_4 (=1)$ is non-zero. Hence, these partitions yield

$$C(1,0,1,0) = a^2 F^{(2)}(ap_0) \frac{p_3}{2!}.$$ 

(28)

and

$$C(0,0,0,1) = a F^{(1)}(ap_0) p_4.$$ 

(29)

Hence, $D_4$ is given by the sum of five or $p(4)$ contributions.
To derive a general formula for the coefficients $D_k$, we need to sum over all the partitions summing to $k$. This entails summing over all values that the frequencies of the elements can take. Each frequency, $n_i$, can only range from 0 to $[k/i]$, since this is the maximum number that the element $i$ can appear in a partition summing to $k$. In addition, the partitions are constrained by the condition that
\[
\sum_{i=1}^{k} n_i i = n_1 + 2n_2 + 3n_3 + \cdots + kn_k = k .
\] (30)

From this result we see that $n_k$ equals zero for all partitions except $\{k\}$, in which case it will equal unity, while the other $n_i$ will equal zero for this partition. That is, the $n_i$ are more often zero than non-zero. Consequently, much redundancy occurs when summing over the allowed values of $n_i$. More succinctly, the sum over all partitions summing to $k$ can be expressed as
\[
D_k = C(n_1, n_2, \ldots, n_k) .
\] (31)

where $N = \sum_{i=1}^{k} n_i$. If Eq. (24) is introduced into the above equation, then we obtain the general formula given by Eq. (11).

Now we consider the case of $p_0 = 0$. This means that Equivalence (16) reduces to
\[
\frac{g_a \circ f}{h_a \circ f} = q_0 + \sum_{k=1}^{\infty} q_k a^k \left( \sum_{j=1}^{\infty} p_j y^j \right)^k .
\] (32)

Expanding the first few powers in $y$ yields
\[
\frac{g_a \circ f}{h_a \circ f} = q_0 + q_1 a p_0 y + \left( q_1 a p_2 + q_2 a^2 p_1^2 \right) y^2 + \left( q_1 a p_3 + 2q_2 a^2 p_1 p_2 + q_3 a^3 p_1^3 \right) y^3 + O(y^4) .
\] (33)

Hence, we obtain a power series expansion in $y$ except now the coefficients $D_k$ take a different form. In particular, we see that the first few coefficients are given by
\[
D_0 = q_0 , \quad D_1 = q_1 a p_0 , \quad D_2 = q_1 a p_2 + q_2 a^2 p_1^2 ,
\] (34)
and

\[ D_3 = q_1 a p_3 + 2q_2 a^2 p_1 p_2 + q_3 a^3 p_1^3. \]  

(35)

Furthermore, we can introduce the partition method to derive a general formula for coefficients since Equivalences (21) and (32) are isomorphic. In fact, the only difference between the two equivalences is that the terms in the outer series in the latter equivalence are multiplied by \( q_k a^k \) as opposed to \( a^k F^{(k)}(a p_0)/k! \) in Equivalence (21). This means that the only change to the partition method will occur in the multinomial factor, which will now be \( q_N a^N N!/n_1! n_2! \cdots n_k! \). Consequently, the contribution by each partition to the \( D_k \) is given by

\[ C(n_1, n_2, \ldots, n_k) = q^N a^N N! \prod_{i=1}^k \frac{p_i^{n_i}}{n_i!}. \]  

(36)

Introducing this result into Eq. (31) gives Eq. (9).

Although it has been stated that the expansion given by Equivalence (6) can be asymptotic, this does not mean that it will be divergent for all values of \( y \). For if it were, then inverting the equivalence implies that the inverted quotient of the pseudo-composite functions vanishes for all values of \( y \). Therefore, there must be a region in the complex plane where the equivalence symbol can be replaced by an equals sign. In Ref. [2] it was found that an asymptotic series possesses zero radius of absolute convergence, but it is either conditionally convergent or divergent depending upon which sector in the complex plane the variable is situated. For the values of \( y \) where Equivalence (6) is convergent, we can invert the quotient of the pseudo-composite functions \( g_a \circ f \) and \( h_a \circ f \). Therefore, provided \( D_0 \neq 0 \), we find that

\[ \frac{h_a \circ f}{g_a \circ f} = \frac{1}{D_0} \frac{1}{(1 + \sum_{k=1}^{\infty} (D_k / D_0) y^k)}. \]  

(37)

The rhs of this equation can now be regarded as the regularised value of the geometric series where the variable equals to \(- \sum_{k=1}^{\infty} (D_k / D_0) y^k \). According to Refs. [2] and [7]-[9], the geometric series or \( 1 + z + z^2 + \cdots \) is either conditionally or absolutely convergent for \( \Re z < 1 \). For all other values of \( z \) in the principal branch of the complex plane, it is either divergent or undefined and must be regularised. Hence, treating the rhs of the above equation as
the limit of the geometric series means that \( \Re \sum_{k=1}^{\infty} (D_k/D_0)y^k > -1 \). For all other values of \( y \) the series will be either divergent or undefined, the latter case occurring when \( \Re \sum_{k=1}^{\infty} (D_k/D_0)y^k = -1 \). Therefore, Eq. (37) can be expressed as

\[
\frac{h_a \circ f}{g_a \circ f} \equiv \frac{1}{D_0} \sum_{k=0}^{\infty} \left( -\sum_{j=1}^{\infty} (D_j/D_0) y^j \right)^k .
\] (38)

Equivalence (38) is isomorphic to Equivalence (21), which means in turn that we can apply the partition method again. In this instance the coefficients of \( y^j \) in the inner series or rather the values to be assigned to the elements \( j \) in the partitions are equal to \(-D_j/D_0\) instead of \( p_j \). In addition, the multinomial factor becomes the standard value of \( N! / n_1! n_2! \cdots n_k! \) for the partitions summing to \( k \). Hence, the contribution from each partition is given by

\[
C(n_1, n_2, \ldots, n_k) = (-1)^N N! D_0^N \prod_{i=1}^{k} \frac{D_i^{n_i}}{n_i!} .
\] (39)

Introducing the above into Eq. (31) yields the result given by Eq. (12).

To derive the final result, we use Equivalences (6) and (11) to obtain

\[
1 = \frac{g_a \circ f}{h_a \circ f} \frac{h_a \circ f}{g_a \circ f} \equiv \sum_{k=0}^{\infty} D_k y^k \left( \frac{1}{D_0} \sum_{k=0}^{\infty} E_k y^k \right) .
\] (40)

Since \( E_0 = 1 \) from Eq. (12), we can separate the zeroth order term in \( y \), which cancels the term of unity on the lhs of the equivalence. After multiplying the series together, we are left with

\[
\sum_{k=1}^{\infty} y^k \sum_{j=0}^{k} D_j E_{k-j} \equiv 0 .
\] (41)

As indicated earlier in the proof, there will be some values of \( y \), actually a region in the complex plane, where the equivalence symbol can be replaced by an equals sign. Otherwise, either Equivalence (6) or (11) would be zero for all values of \( y \). Given that there will be an infinite number of values of \( y \) where an equals sign applies, the lhs of the above result will also vanish for these values of \( y \). For this to occur, it means that the inner series, which
is independent of $y$, must also vanish. Hence, we arrive at Eq. (13), thereby completing the proof of the theorem.

It should be noted that Theorem 1 has avoided the issue of determining the radius of absolute convergence for the power series on the rhs of Equivalence (6). This is because although one can determine a value for the radius of absolute convergence when deriving the resulting power series, it is often only an estimate, not the supremum as demonstrated by various examples in Ref. [3]. Furthermore, one can introduce a divergent power series as the inner series and despite the fact that the outer series may also possess a finite radius of absolute convergence, the resulting power series appearing in Equivalence (6) may be convergent over the entire complex plane. For example, the power series expansion for cosecant derived via the partition method for a power series expansion in Ref. [3] has a radius of absolute convergence equal to $\pi$. This inner series became the variable for the outer series, which was given by the geometric power series, whose radius of absolute convergence is unity. Yet, the resulting power series in Equivalence (6) was merely another representation of the standard Taylor/Maclaurin series for sine, which is convergent for all values of the variable. So, this is an example where both the inner and outer series were both absolutely convergent within different radii in the complex plane, but the resulting series expansion obtained via the partition method for a power series expansion was convergent for all values of the variable.

To make the preceding material clearer, let us consider an example. Power series expansions for $\csc z$ and $\sec z$ have already been obtained in Ref. [3] via the partition method for a power series expansion. Because $\cos z$ and $\sin z$ can be expressed as Bessel functions of half-integer order, namely $J_{-1/2}(z)$ and $J_{1/2}(z)$, we can also say that power series expansions have been developed for the reciprocal or inversion of these special functions for $\nu = -1/2$ and $\nu = 1/2$. Whilst various extensions of these results are presented in Ref. [3], one which has been overlooked is the derivation of a power series expansion for the reciprocal of a Bessel function to arbitrary order $\nu$. Therefore, if we introduce the standard power series expansion for Bessel functions given by No. 8.440 in Ref. [23] into the denominator, then we obtain

$$J_\nu(z)^{-1} = \frac{(2/z)^\nu}{1 - z^2/4(\nu + 1) + z^4/16 \cdot 2!(\nu + 2)(\nu + 1) - \cdots}.$$  (42)

The above result is not valid for $\Re \nu = -1$ since in this case the leading term in the series expansion for $J_\nu(z)$ vanishes. Then we would need to examine
this case by itself, which is left for the reader to consider. From Theorem 1
we see that the inner power series is simply the Taylor/Maclaurin power
series expansion for Bessel functions. Hence, \( y = z^2 \), and \( p_0 = 0 \), while for
\( k \geq 1 \), \( p_k = (-1)^k/2^{2k}(\nu + 1)_k! \), where \( (\nu + 1)_k \) is the Pochhammer notation
for \( \Gamma(k + \nu + 1)/\Gamma(\nu + 1) \). Moreover, we can regard the denominator on the
rhs of Eq. (42) as the regularised value of the geometric series, which means
that coefficients of the outer series, viz. \( q_k \), are equal to \((-1)^k\) for \( k \geq 1 \),
while the function \( h(z) = (2/z)^{\nu}\Gamma(\nu + 1) \). Since \( a = 1 \), the pseudo-composite
functions become composite functions with the quotient in Theorem 1 equal
to \((z/2)^{\nu}/\Gamma(\nu + 1) J_{\nu}(z)\). By using Equivalence (6) we arrive at

\[
\frac{(z/2)^{\nu}}{\Gamma(\nu + 1) J_{\nu}(z)} \equiv \sum_{k=0}^{\infty} h_k(\nu) \left( \frac{z}{2} \right)^{2k}, \tag{43}
\]

where the coefficients \( h_k(\nu) \) are determined from Eq. (10) and are given by

\[
h_k(\nu) = (-1)^k \frac{k_{k/2,k/3,...,1}}{\sum_{i=1}^{N} \prod_{i=1}^{k} \frac{1}{(\nu + 1)_i i!} \frac{1}{n_i!}}. \tag{44}
\]

From this result we see that \( h_0(\nu) = 1 \), \( h_1(\nu) = 1/(\nu + 1) \), and \( h_2(\nu) = (\nu + 3)/2(\nu + 1)^2(\nu + 2) \), while for \( k = 3 \), Eq. (43) yields

\[
h_3(\nu) = \left( \frac{(\Gamma(\nu + 1))^3}{\Gamma(\nu + 2)} - 2! \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 2)} \frac{\Gamma(\nu + 1)}{2! \cdot \Gamma(\nu + 3)} + \frac{\Gamma(\nu + 1)}{3! \cdot \Gamma(\nu + 4)} \right)
= \left( \frac{\nu^2 + 8\nu + 19}{3! \cdot (\nu + 1)^3(\nu + 2)(\nu + 3)} \right). \tag{45}
\]

Since \( J_{1/2}(z) \) is related to \( \sin z \), the power series expansion for \( 1/J_{1/2}(z) \)
is identical to the one derived in Ref. [3] for cosecant, whose coefficients were expressed in terms of the cosecant numbers denoted by \( c_k \). These numbers were later found to be related to the Riemann zeta function. Hence, we obtain

\[
h_k(1/2) = 2^{2k} c_k = 2 \left( 2^{2k} - 2 \right) \frac{\zeta(2k)}{\pi^{2k}}. \tag{46}
\]

Similarly, because \( J_{-1/2}(z) \) is related to \( \cos z \), the expansion for secant, which is derived in terms of special numbers called the secant numbers or \( d_k \) in Ref.
is related to $\nu = -1/2$ in Equivalence (12). It was then found that the secant numbers could be expressed as the difference of specific values of Hurwitz zeta function. Thus, we arrive at

$$h_k(-1/2) = 2^{2k} d_k = \frac{1}{\pi^{2k+1}} \left( \zeta(2k + 1, 1/4) - \zeta(2k + 1, 3/4) \right).$$

(47)

It was also observed in Ref. [3] that the absolute convergence of the power series expansions for both cosecant and secant were determined by the distance from the origin to the first zeros of these functions. In the case of cosecant the expansion is absolutely convergent for $|z| < \pi$, while in the case of secant it is absolutely convergent for $|z| < \pi/2$. Therefore, the expansion given in Equivalence (43) will be absolutely convergent for $|z|$ less than the magnitude of the first zero for $J_\nu(z)$. For these values of $z$ we can replace the equivalence symbol by an equals sign, thereby producing an equation. Consequently, we can use the “equation form” of Equivalence (43) to demonstrate that the coefficients $h_k(\nu)$ can also be evaluated by recurrence relations. Because there is an infinite number of values of $z$ where the “equation form” is valid, a recurrence relation can be obtained simply by multiplying Equivalence (43) by the convergent power series expansion for $J_\nu(z)$ and setting the resulting product equivalent to unity. Since $z$ is still fairly arbitrary within the radius of absolute convergence, we can equate like powers of $z$ of the resulting equation. This yields

$$h_k(\nu) = \sum_{j=0}^{k-1} \frac{(-1)^{k-j+1} \Gamma(\nu + 1) h_j(\nu)}{(k-j)! \Gamma(k-j+\nu+1)}. \quad (48)$$

Other recurrence relations can be developed by using integral results involving Bessel functions. For example, one can express No. 3.768(9) in Ref. [23] as

$$\int_0^1 dx \, x^{\mu-1} (1-x)^{\nu-1} \frac{\cos(ax)}{J_{\nu-1/2}(a/2)} = \sqrt{\pi} a^{1/2-\mu} \Gamma(\mu) \cos\left(\frac{a}{2}\right). \quad (49)$$

We have already seen that there exists a finite radius of absolute convergence for $z$, where Equivalence (11) becomes an equation. This means that there is a region in the complex plane where the expansion for the reciprocal of the Bessel function can be introduced into the above result without the necessity to replace the equals sign by an equivalence symbol. Furthermore, by
introducing the power series expansion for cosine into Eq. (49) we arrive at

\[
\int_0^1 dx \ x^{\mu-1} (1-x)^{\nu-1} \sum_{k=0}^{\infty} a^{2k} \sum_{j=0}^{k} \frac{(-1)^{k-j}}{2^{4j}} \frac{x^{2k-2j}}{(2k-2j)!} h_j(\mu - 1/2)
\]

\[
\equiv \frac{\sqrt{\pi}}{2^{2\mu-1}\Gamma(\mu + 1/2)} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \left(\frac{a}{2}\right)^{2k}.
\]

(50)

The integral on the lhs of the above equivalence is merely the integral representation for the beta function. Hence, we can introduce the gamma function product for the beta function. Since \(\mu\) is fairly arbitrary, we can equate like powers of \(a\) on both sides of the equation, thereby obtaining

\[
\sum_{j=0}^{k} \frac{(-1)^{k-j}}{2^{4j} (2k-2j)!} \frac{\Gamma(2k-2j+\nu+1/2)}{\Gamma(2k-2j+2\nu+1)} h_j(\nu)
\]

\[
= \frac{\sqrt{\pi}}{2^{2k+2\nu} (2k)!} \frac{(-1)^k}{\Gamma(\nu+1)}.
\]

(51)

where \(\mu - 1/2\) has been replaced by \(\nu\).

If we put \(k=4\) into either recurrence relation and introduce the values of \(h_k(\nu)\) for \(k=1\) to 3 given earlier, then we find that

\[
h_4(\nu) = \frac{\nu^4 + 17\nu^3 + 117\nu^2 + 379\nu + 422}{4! (\nu + 1)^4 (\nu + 2)^2 (\nu + 3)(\nu + 4)}.
\]

(52)

This agrees with the value given in Table 1, which displays the values of \(h_k(\nu)\) up to \(k = 7\). These values have been evaluated by introducing the recurrence relation into the Sum routine in Mathematica [12]. In the next section we shall develop a programming methodology where summing the contributions from the partitions in Eq. (44) will be just as expedient as using the recurrence relations.

From the table it can be seen that the \(h_k(\nu)\) possess common properties. For example, there is always a factor of \(k!(1+\nu)^k\) in the denominator. In fact, if the denominator of a function \(f(x)\) is denoted by \(DN(f(x))\), then we have is given by

\[
DN(h_k(\nu)) = k! (\nu + 1)^k (\nu + 2)^{[k/2]} \cdots (\nu + k - 1)^{[k/(k-1)]} (\nu + k),
\]

(53)

where \([x]\) denotes the greatest integer less than or equal to \(x\). This result also follows from Eq. (44) when we examine the upper limits in the summations.
On the other hand, the highest order term in $\nu$ in the numerator is always $k$ less than the highest order term in the denominator. Therefore, for large $|\nu|$, $h_k(\nu) \approx \nu^{-k}/k!$.

By applying d’Alembert’s ratio test as described on p. 24 of Ref. [24] to the series on the rhs of Equivalence (43), we find that for all $k$ the series is only absolutely convergent when $|z| < 2 \sqrt{\frac{h_k(\nu)}{h_k(\nu+1)}}[\nu+1]$. Therefore, if there is a supremum for this ratio that applies to all values of $k$, then it represents the radius of absolute convergence. Fig. 2 presents a graph of $h_k(\nu)/h_k(\nu+1)$ versus $\nu$ for $-2 < \nu < 10$ and various values of $k$. The figure shows that the larger $k$ is, the greater the value of the ratio $h_k(\nu)/h_k(\nu+1)$. Therefore, the limit as $k \to \infty$ represents the supremum for the ratio when $\nu > -2$ with the nearest singularities to the origin occurring at

$$z_{0,\nu} = \pm 2 \lim_{k \to \infty} \sqrt{\frac{h_k(\nu)}{h_k(\nu+1)}}. \quad (54)$$

The figure also shows that as $k$ increases, $h_k(\nu)/h_k(\nu+1)$ becomes a better approximation to $\lim_{k \to \infty} h_k(\nu)/h_k(\nu+1)$ over an ever-increasing range of values for $\nu$. E.g., we see that $h_5(\nu)/h_5(\nu)$ is an accurate approximation for $-2 < \nu < 4.5$, while $h_8(\nu)/h_8(\nu)$ is an even better approximation for $-2 < \nu < 7$.

According to p. 372 of Ref. [25], for $\nu > -1$, the first zero for $J_\nu(z)$ is real, but for $\nu < -1$, excluding negative integers, or $\nu$ complex, the first zero is complex. Since the first zero represents the singularity of $1/J_\nu(z)$

| $k$ | $h_k(\nu)$ |
|-----|-------------|
| 0   | 1           |
| 1   | $\frac{1}{\nu+1}$ |
| 2   | $\frac{2(\nu+1)^2(\nu+2)}{\nu^2+8\nu+19}$ |
| 3   | $\frac{3(\nu+1)^4(\nu+2)(\nu+3)}{\nu^4+17\nu^3+117\nu^2+379\nu+422}$ |
| 4   | $\frac{4(\nu+1)^6(\nu+2)^2(\nu+3)(\nu+4)}{\nu^6+26\nu^5+294\nu^4+1816\nu^3+5969\nu^2+1091052}$ |
| 5   | $\frac{5(\nu+1)^8(\nu+2)^3(\nu+3)(\nu+4)(\nu+5)}{\nu^8+42\nu^7+81\nu^6+9412\nu^5+71155\nu^4+349786\nu^3+1043637\nu^2+1674616\nu+1091052}$ |
| 6   | $\frac{6(\nu+1)^{10}(\nu+2)^4(\nu+3)^2(\nu+4)(\nu+5)(\nu+6)}{\nu^{10}+55\nu^9+1417\nu^8+22535\nu^7+243311\nu^6+1837401\nu^5+9292435\nu^4+29539597\nu^3+51572980\nu^2+36978156\nu+1091052}$ |

Table 1: Coefficients for the power series expansion of the reciprocal of the Bessel function of order $\nu$. 

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Figure 2: $h_k(\nu)/h_{k+1}(\nu)$ versus $\nu$ for $k = 1, 5, 8, 12$ and 18
nearest to the origin, we expect the rhs to be real only for \( \nu > -1 \). Table 2 presents a sample of the values for the first zero of \( J_\nu(x) \) for various values of \( \nu \) obtained by using the special numerical routine known as BesselJZero in Mathematica and by setting \( k = 17 \) in Eq. (54). For \( \nu > -1 \) we see that the values obtained via both approaches agree. In particular, for \(-1 < \nu < 1\) the values obtained from Eq. (54) were found to be accurate to at least 10 decimal places. As expected, as \( \nu \) becomes greater than unity, the accuracy of Eq. (54) begins to wane compared with the values from BesselJZero. The values presented in the second and third columns for these values of \( \nu \) only agree to the first decimal place. Mathematica is able to evaluate the zeros to arbitrary precision, as evidenced by the values for \( \nu \) equal to -1 and \(-3/2\) in the second column of the table. On the other hand, if one wishes to obtain more accurate values of the zeros via Eq. (54), then one will have to determine the \( h_k(\nu) \) for much greater values of \( k \).

For \( \nu \leq -1 \), BesselJZero[\( \nu, 1 \)] gives the first zero of \( J_\nu(z) \), but it is the first zero situated on the positive real axis, not necessarily the closest to the origin. To see this more clearly, for \( \nu = -3/2 \) Mathematica gives as its first zero \( z = 2.798386 \cdots \), but this is not the first zero in the complex plane. According to Eq. (54), the closest zero is given by \( z = 1.199687 \cdots i \) as displayed in the adjacent column of the table. The latter value can be confirmed as a zero by

| \( \nu \) | BesselJZero | \( \pm 2\sqrt{h_{17}(\nu)/h_{18}(\nu)} \) |
|------|-------------|-----------------|
| 0    | 2.4048255576957 | \( \pm 2.4048255576954 \) |
| 1    | 3.83170597    | \( \pm 3.83170596 \) |
| 2    | 5.1356223     | \( \pm 5.1356221 \) |
| 5    | 8.7714        | 8.7713          |
| 10   | 14.47         | 14.46           |
| -1/3 | 1.8663508588734 | 1.8663508588738 |
| -4/5 | 0.936806664511 | 0.936806664510 |
| -3/2 | 2.798386045783878 | \( \pm 1.199678640257655i \) |
| -1/3 + i | Unable to evaluate | \( \pm 2.076341434394476 \) |
| -3/2 + i | Unable to evaluate | \( \pm 1.556637759994043i \) |
| 3/2 - i | Unable to evaluate | \( \pm 4.529756943967303 \) |
|      |              | \( \pm 1.293935107111323i \) |

Table 2: Evaluation of the first zero of \( J_\nu(x) \) using BesselJZero in Mathematica and Eq. (54) with \( k = 17 \).
introducing it into the BesselJ routine in Mathematica, whereupon one finds that it yields the tiny value of \((7.35 \cdots + 7.35 \cdots i) \times 10^{-14}\).

It also appears that for \(\Re \nu > -2\), Eq. (54) still gives the nearest zero to the origin of the complex plane. For example, the remaining values in the table represent the zeros obtained for complex orders of Bessel functions. Zeros for such functions cannot be obtained via BesselJZero since the routine can only handle real numbers. However, Eq. (54) can yield them. When the value obtained from Eq. (54) for \(J_{-1/3+i}(z)\) is introduced into the BesselJ routine in Mathematica, a complex value with a magnitude of the order of \(10^{-12}\) is obtained, while a complex value with a magnitude of the order of \(10^{-8}\) is obtained when the final value in the third column of the table is introduced into the routine. This last example demonstrates again that the accuracy of Eq. (54) wanes as \(|\nu|\) increases when \(k\) is fixed, which we have already observed in Fig. 2.

The reader may well ask if it is possible to adapt the preceding method to evaluate the next Bessel zero or even higher order zeros. This does not seem to be possible at this stage unless the specific value for \(z_{0,\nu}\) can be determined. For example, by taking the logarithm of No. 8.544 in Ref. [23] and differentiating, we eventually arrive at

\[
\frac{J'_\nu(z)}{J_\nu(z)} - \frac{\nu}{z} = -2z \sum_{k=0}^{\infty} \frac{1}{z^2_{k,\nu} - z^2} .
\]  

(55)

In the above equation, \(J'_\nu(z)\) can be written as a series in powers of \(z/2\) with coefficients \(a_k = 1/k! (\nu+1)_k\), while the rhs of Equivalence (55) can be used to replace the reciprocal of the Bessel function. Multiplying both series yields

\[
\frac{J'_\nu(z)}{J_\nu(z)} - \frac{\nu}{z} \equiv \frac{\nu}{z} \sum_{k=0}^{\infty} \left( \frac{z}{2} \right)^k e_k(\nu) ,
\]

(56)

where \(e_k(\nu) = \sum_{j=0}^{k} a_{k-j} h_k(\nu)\). To carry out the singularity analysis for the next zero, we need to remove the \(k = 0\) term from the rhs of Eq. (55). Hence, the power series for this term would need to be introduced into the lhs with the above power series, yielding another power series with coefficients expressed in terms of \(e_k(\nu)\) and powers of \(z_{0,\nu}\). If an approximation for the latter is introduced, then the coefficients would only be approximate. For large values of \(k\) they would still be affected by \(z_{0,\nu}\) and thus, it would not be possible to isolate the next zero. On the other hand, if the exact result
for the \( z_{0,\nu} \), which is known for \( \nu = \pm 1/2 \), then the series would represent the power series for the rhs of Eq. (55) with the sum beginning at \( k = 1 \) or with the singularity at the second zero. This situation is discussed immediately below Eq. (83) in Ref. [3]. It should be noted that there are typographical errors there since the power of \( 1/(2\pi) \) in the expression for \( c_k^* \) should be \( k \), not \( k + 1 \) and the limit in the next line below should refer to \( c_{k+1}^*/c_k^* \), not \( c_{k+1}/c_k \).

Before investigating how the BRCP algorithm can be implemented to evaluate the coefficients \( D_k \) and \( E_k \) in Theorem 1, we now prove an interesting corollary to the theorem, but before we can do this, the following lemma is required:

**Lemma 1.** For \( \alpha \) complex, regularisation of the binomial series yields

\[
\binom{\alpha}{0}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + \alpha)}{\Gamma(\alpha) k!} z^k \begin{cases} = (1 - z)^{-\alpha}, & \Re z < 1, \\ \equiv (1 - z)^{-\alpha}, & \Re z \geq 1. \end{cases}
\tag{57}
\]

**Remark.** This lemma represents the generalisation of the regularisation of the geometric series, which reduces to the latter for \( \alpha = 1 \).

**Proof.** The proof of Lemma 1 appears immediately below Proposition 1 in Ch. 4 of Ref. [7]. Although the proof is concerned with real values of \( \alpha \), there is no reason why \( \alpha \) cannot be complex. This is due to the fact that the proof involves differentiating and integrating the integral representation for the beta function, viz.

\[
B(k + \alpha, 1 - \alpha) = \int_0^1 dt \frac{t^{k+\alpha-1}}{(1 - t)^{\alpha}}.
\tag{58}
\]

This integral is not only defined for real values of \( \alpha \), but also for complex values of \( \alpha \). Its convergence is limited by the values for the real part of \( \alpha \), not its imaginary part. Since differentiation and integration only affect the real part of \( \alpha \), the proof of Proposition 1 in Ref. [7] can be extended to complex values of \( \alpha \). Moreover, the result in Lemma 1 can be simplified by replacing the equals sign with an equivalence symbol. Then we have the one equivalence statement, which is valid for all values of \( z \) and \( \alpha \).

One final remark is in order. The reader should note that a similar notation to the standard notation for generalised hypergeometric functions has been introduced by expressing the binomial series as \( \binom{\alpha}{0} \). The reason for the slight variation in notation is that the generalised hypergeometric function notation of \( F_p_q(\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z) \) is only valid when the series is
absolutely convergent, i.e. when $p \leq q$ or for $|z| < 1$ when $p = q$ [20]. The conditions in the lemma are more general. That is, the binomial series has been taken into the regions of the complex plane where it becomes either conditionally convergent or divergent. Hence, it has been necessary to alter the notation to avoid any confusion with the standard notation. This completes the proof of the lemma.

We are now in a position to generalise the derivation of the coefficients in Theorem 1 by considering the following corollary.

**Corollary 1 to Theorem 1.** Given the same conditions on the pseudo-composite functions $g_a \circ f$ and $h_a \circ f$ as in Theorem 1, there exists a power series expansion for the quotient of the pseudo-composite functions raised to an arbitrary power $\rho$ which is given by

$$
\left( \frac{g_a \circ f}{h_a \circ f} \right)^\rho \equiv \sum_{k=0}^\infty D_k(\rho) y^k ,
$$

(59)

For $k \geq 1$, the generalised coefficients for the $D_k$ of Theorem 1 or $D_k(\rho)$ in the above result are given by

$$
D_k(\rho) = \sum_{n_1,n_2,n_3,...,n_k=0}^{k,[k/2],[k/3],...,1} (-\rho)_N D_0^{-N} \prod_{i=1}^{k} \frac{(-D_i)^{n_i}}{n_i!} ,
$$

(60)

and

$$
D_k(-\rho) = \sum_{n_1,n_2,n_3,...,n_k=0}^{k,[k/2],[k/3],...,1} N! D_0(\rho)^{-N-1} \prod_{i=1}^{k} \frac{(-D_i(\rho))^{n_i}}{n_i!} ,
$$

(61)

where $(\rho)_N$ denotes the Pochhammer notation for $\Gamma(N+\rho)/\Gamma(\rho)$. In addition, for $\rho = \mu + \nu$ the coefficients satisfy the following recurrence relation:

$$
D_k(\rho) = \sum_{j=0}^{k} D_j(\mu) D_{k-j}(\nu) .
$$

(62)

**Remark 1.** By putting $\rho = -1$ in Eq. (60) we see immediately that the $D_k(-1)$ become the $E_k$ in Theorem 1 given by Eq. (12). Furthermore, the $D_k(1)$ represent another representation for the $D_k$ given by either Eq. (9) or (10), the latter being valid when $p_0 = 0$. 

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Remark 2. Note that when $\rho$ equals a non-negative integer $j$, Eq. (60) simplifies dramatically because $(-\rho)_N$ vanishes for $N > j$, reflecting the fact that we are dealing with a finite polynomial of order $j$. For all other values of $j$ the coefficients $D_k(\rho)$ represent polynomials in $\rho$ of order $k$.

Proof. Since the pseudo-composite functions $g_a \circ f$ and $h_a \circ f$ are subject to the conditions in Theorem 1, we know that there exists a power series expansion where

$$
\left( \frac{g_a \circ f}{h_a \circ f} \right)^\rho \equiv \left( \sum_{k=0}^\infty D_k y^k \right)^\rho .
$$

(63)

We also know that there will be a region in the complex plane where the equivalence symbol can be replaced by an equals sign. Separating the zeroth order term in above result yields

$$
\left( \frac{g_a \circ f}{h_a \circ f} \right)^\rho \equiv D_0^\rho \left( 1 + \sum_{k=1}^\infty (D_k/D_0) y^k \right)^\rho .
$$

(64)

We can treat the series on the rhs as the variable in the regularised value of the binomial series. Then according to Lemma 1 we have

$$
\left( \frac{g_a \circ f}{h_a \circ f} \right)^\rho \equiv D_0^\rho \sum_{k=0}^\infty \frac{\Gamma(k-\rho)}{\Gamma(-\rho) k!} \left( -\sum_{j=1}^\infty (D_j/D_0) y^j \right)^k .
$$

(65)

Now the above equivalence is isomorphic to Equivalence (21), which means that it is in the form where the partition method can be applied. As stated in the proof to Theorem 1, the values to be assigned to elements $i$ in the partitions are given by coefficients of $y^i$ in the inner series. Therefore, in this case each element $i$ is assigned a value of $-D_i/D_0$. To evaluate the contribution from each partition, we need to multiply the product of all the assigned values by a factor consisting of the multinomial factor $N!/n_1!n_2!\cdots n_k!$ and $\Gamma(N-\rho)/N!\Gamma(-\rho)$. The latter term arises from the fact that $k$ in the coefficient of the outer series plays the role of $N$ in the partition method. Therefore, by introducing the Pochhammer notation of $(\rho)_k$ for $\Gamma(N+\rho)/\Gamma(\rho)$, we find that the contribution due to each partition is

$$
C\left(n_1, n_2, \ldots, n_k\right) = (-1)^N \left( -\rho \right)_N D_0^{-N} \prod_{i=1}^k \frac{D_i^{n_i}}{n_i!} ,
$$

(66)

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As expected, for $\rho = -1$ the above result reduces to Eq. (49). Furthermore, by summing over all partitions summing to $k$, we obtain the total coefficient in the resulting power series expansion, which is given by Eq. (60).

As a result of establishing Equivalence (59), we have

$$
(\frac{g_a \circ f}{h_a \circ f})^{-\rho} \equiv \sum_{k=0}^\infty D_k(-\rho) y^k ,
$$

(67)

We also know that there will be a region in the complex plane where the equivalence symbol can be replaced by an equals sign in Equivalence (59). For these values of $y$ we can invert the equivalence, thereby obtaining

$$
\left(\frac{h_a \circ f}{g_a \circ f}\right)^\rho = \frac{1}{D_0(\rho)} \left(1 + \sum_{k=1}^\infty D_k(\rho) y^k \right) ,
$$

(68)

Theorem 1 can again be applied to this result by treating the rhs as the regularised value of the geometric series with the variable equal to the series in the denominator. In this case $h(z) = 1$, $q_k = (-1)^k$, $a = 1$ and $p_k = D_k(\rho)/D_0(\rho)$ for $k \geq 0$, while $p_0 = 0$. The coefficients of the resulting power series expansion can be determined by Eq. (10). Moreover, the resulting power series expansion will be equal to the power series expansion on the rhs of Equivalence (67) since both have the same regularised value. Hence, we can equate like powers of both expansions, which yields

$$
D_k(-\rho) = D_0(\rho)^{-1} \sum_{n_1, n_2, n_3, \ldots, n_k=0}^{k-1} N! \prod_{i=1}^{k} \frac{(D_i(\rho)/D_0(\rho))^{n_i}}{n_i!} .
$$

(69)

By taking the factor of $(-1)^N$ inside the product and $1/D_0(\rho)$ outside of it, we obtain Eq. (61).

Since $\rho = \mu + \nu$, we have

$$
\left(\frac{g_a \circ f}{h_a \circ f}\right)^\rho = \left(\frac{g_a \circ f}{h_a \circ f}\right)^\mu \left(\frac{g_a \circ f}{h_a \circ f}\right)^\nu .
$$

(70)

We also note that there will be a region of complex plane for $y$ in Equivalence (59) where we can replace the equivalence symbol by an equals sign. Therefore, for these values of $y$, Eq. (70) yields

$$
\sum_{k=0}^\infty D_k(\rho) y^k = \sum_{k=0}^\infty D_k(\nu) y^k \sum_{k=0}^\infty D_k(\mu) y^k = \sum_{k=0}^\infty y^k \sum_{j=0}^{k} D_j(\nu) D_{k-j}(\mu) .
$$

(71)
Since there is an infinite number of values of $y$ for which the above equation holds, we can equate like powers of $y$. As a consequence, we obtain Eq. (62), which completes the proof of the corollary.

To make the preceding material more concrete, let us consider a couple of examples. In order to simplify these examples, we now regard the sum over partitions as a discrete operator, which will be denoted by $L_P[\cdot]$. We shall refer to this form for the sum over partitions summing to $k$ as the partition operator. That is, the partition operator is defined as

$$L_{P,k}[\cdot] = \sum_{n_1, n_2, n_3, \ldots, n_k=0, \sum_{i=1}^{k} n_i = k}^{k,[k/2],[k/3],\ldots,1} \cdot$$ (72)

For the situation where the partition operator acts on unity, it yields $p(k)$ or the number of partitions summing to $k$. Hence, $L_{P,k}[1] = p(k)$. The number of partitions summing to $k$ can also be obtained from the following generating function [18]:

$$P(z) = \prod_{m=1}^{\infty} \frac{1}{1 - z^m} \equiv \sum_{k=0}^{\infty} p(k)z^k.$$ (73)

Note the appearance of the equivalence symbol in the above result since the lhs can become divergent. This is because the lhs has been treated as an infinite product of regularised values of the geometric series in obtaining the power series on the rhs. This will become clearer in Sec. 6, where we shall also extend the above result.

By applying Theorem 1 to the simple case where $p_0 = 0$, $p_k = b^k$, $h(z) = 1$, $a = 1$, $q_k = 1$ and $y = z$, we find that the quotient of the composite functions becomes

$$g(f(z)) = 1 + \frac{bz}{1 - 2bz} \equiv 1 + \frac{1}{2}\sum_{k=0}^{\infty} (2bz)^{k+1}.$$ (74)

Hence, via Eq. (10) we arrive at

$$L_{P,k}\left[N_k! \prod_{i=1}^{k} \frac{1}{n_i!}\right] = 2^{k-1},$$ (75)

where, as before, $N_k = \sum_{i=1}^{k} n_i$. The $k$ subscript has been introduced here for the first time as it will become apparent that we shall need to sum the
$n_i$ to different limits shortly. If we choose $p_k = (-b)^k$ and $q_k = (-1)^k$ instead, then following the same procedure we obtain

$$L_{P,k} \left[ (-1)^{N_k} N_k! \prod_{i=1}^{k} \frac{1}{n_i!} \right] = 0 ,$$

(76)

where $k \geq 2$. The above is an interesting result where the partitions with an even number of elements are cancelled by those with an odd number of elements according to the frequencies of the elements.

Eq. (76) is not the only instance where the sum over all partitions vanishes. For example, consider the application of Theorem 1 to the function $f(z) = \exp(a \ln(1 + z))$ or $f(z) = (1 + z)^a$. Here, the coefficients of the inner series are given by $p_0 = 0$ and $p_k = (-1)^{k+1}/k$ for $k \geq 1$, while the coefficients of the outer series are given by $q_k = 1/k!$. Then from Theorem 1 we obtain

$$D_k = (-1)^k L_{P,k} \left[ (-1)^{N_k} a^{N_k} \prod_{i=1}^{k} \frac{1}{i^{n_i} n_i!} \right] .$$

(77)

In Ref. [7] it is shown that $f(z)$ represents the regularised value of the binomial series. That is, for all values of $a$ and $z$, we have

$$\sum_{k=0}^{\infty} \frac{\Gamma(k - a)}{\Gamma(-a) k!} (-z)^k \equiv (1 + z)^a .$$

(78)

For $|z| < 1$, the series is absolutely convergent and we can replace the equivalence symbol by an equals sign. Since the $D_k$ are the coefficients of the power series expansion in $z$, they are equal to the coefficients in the above result. Then we find that

$$L_{P,k} \left[ (-1)^{N_k} a^{N_k} \prod_{i=1}^{k} \frac{1}{i^{m_i} n_i!} \right] = \frac{\Gamma(k - a)}{\Gamma(-a) k!} .$$

(79)

The results in Theorem 1 are actually more general than the above result. As a consequence, for $p_0 = 0$ we arrive at

$$D_k = L_{P,k} \left[ q_N a^{N_k} N_k! D_0^{-N_k} \prod_{i=1}^{k} \frac{p_i^{m_i}}{n_i!} \right] ,$$

(80)
and

\[ E_k = L_{P,k} \left[ (-1)^{N_k} N_k! D_0^{-N_k} \prod_{i=1}^{k} \frac{D_i^{n_i}}{i^{n_i} n_i!} \right]. \quad (81) \]

When \( a = 1 \), we have \( f(z) = 1 + z \), which, in turn, means that \( D_k = 0 \) for \( k > 1 \). Then we find that

\[ L_{P,k} \left[ (-1)^{N_k} \prod_{i=1}^{k} \frac{1}{i^{n_i} n_i!} \right] = 0, \quad (82) \]

for \( k > 1 \). On the other hand, when \( a = -1 \), we have \( f(z) = 1 / (1 + z) \), which represents the regularised value for the geometric series. Since the coefficients of the latter series are equal to \( (-1)^k \), Eq. (77) reduces to

\[ L_{P,k} \left[ \prod_{i=1}^{k} \frac{1}{i^{n_i} n_i!} \right] = 1. \quad (83) \]

More importantly, the above results can first be generalised by letting \( a = l \), where \( l \) is an arbitrary integer. Then \( f(z) = (1+z)^l \), whose coefficients courtesy of the binomial theorem equal \( \binom{l}{k} \) for \( k \leq l \) and vanish for the remaining values of \( l \). As a result, Eq. (77) yields

\[ L_{P,k} \left[ (-l)^{N_k} \prod_{i=1}^{k} \frac{1}{i^{n_i} n_i!} \right] = \begin{cases} 0, & k > l, \\ (-1)^k \binom{l}{k}, & k \leq l. \end{cases} \quad (84) \]

If \( -a \) is replaced by \( \alpha \) in Eq. (79), then the equation reduces to

\[ k! L_{P,k} \left[ \alpha^{N_k} \prod_{i=1}^{k} \frac{1}{i^{n_i} n_i!} \right] = (\alpha)_k. \quad (85) \]

According to Chs. 24 and 18 of Refs. [25] and [27] respectively, the Pochhammer polynomials can be written as

\[ (\alpha)_k = (-1)^k \sum_{j=0}^{k} (-1)^j S_k^{(j)} \alpha^j, \quad (86) \]

where \( S_k^{(j)} \) are known as the Stirling numbers of the first kind and satisfy

\[ S_{k+1}^{(j)} = S_k^{(j-1)} - k S_k^{(j)} \quad (87) \]
We can proceed further with Eq. (85) by introducing a new operator that only considers a fixed number of elements in the partitions. By setting this number equal to \( j \), we can define the operator for a fixed number of elements as

\[
L_{p,k}^j[] = \sum_{n_1, n_2, \ldots, n_k = 0}^{n_1 + n_2 + \cdots + n_k = k, \sum_i n_i = j} \text{k,}[k/2],[k/3],\ldots,1.
\] (88)

Moreover, the above operator is related to the partition operator by

\[
L_{p,k}[] = \sum_{j=1}^{k} L_{p,k}^j[]. \quad (89)
\]

As mentioned in Sec. 2, the number of partitions of \( k \) with exactly \( j \) parts is denoted by \(|k|_j\). This means that

\[
L_{p,k}^j[1] = |k|_j, \quad (90)
\]

while the recurrence relation given by Eq. (1) becomes

\[
L_{p,k}^j[1] = L_{p,k-1}^{j-1}[1] + L_{p,k-j}^j[1]. \quad (91)
\]

Furthermore, introducing the new operator into Eq. (85) with the rhs replaced with the aid of Eq. (86) yields

\[
L_{p,k}^j \left[ \prod_{i=1}^{k} \frac{1}{i^{n_i} n_i!} \right] = (\frac{-1}{k})^{j+k} S_k^{(j)}. \quad (92)
\]

From the recurrence relation given by Eq. (87), we obtain

\[
(k + 1)L_{p,k+1}^j \left[ \prod_{i=1}^{k+1} \frac{1}{i^{n_i} n_i!} \right] = L_{p,k}^{j-1} \left[ \prod_{i=1}^{k} \frac{1}{i^{n_i} n_i!} \right] + kL_{p,k}^j \left[ \prod_{i=1}^{k} \frac{1}{i^{n_i} i!} \right]. \quad (93)
\]

Specific results for the Stirling numbers of the first kind when \( j \) is relatively small have been derived in Refs. [1] and [21]. For example, these
references give $S^{(2)}_k = (-1)^k \Gamma(k) H_1(k)$, where $H_1(k) = \sum_{j=1}^{k-1} 1/j$. Therefore, we find that

$$L^2_{P,k} \left[ \prod_{i=1}^{k} \frac{1}{i^{m_i} n_i!} \right] = \frac{1}{k} H_1(k). \quad (94)$$

Values for the Stirling numbers of the first kind are presented for $j$ close to $k$ in the appendix of Ref. [2]. For example, when $j = k - 1$, $S^{(k-1)}_k = -\binom{k}{2}$.

This, in turn, leads to

$$L^{k-1}_{P,k} \left[ \prod_{i=1}^{k} \frac{1}{i^{m_i} n_i!} \right] = \frac{(-1)^k}{k!} \binom{k}{2}. \quad (95)$$

Another fundamental result can be obtained by applying Theorem 1 to $\exp(-x)$. By writing the function as $1/\exp(x)$, we see that the coefficients of the inner series, viz. $p_k$, are equal to $1/k!$ for $k \geq 1$, while $p_0 = 0$. Meanwhile, the outer series is given by the geometric series so that $q_k = (-1)^k$. The coefficients of the power series for $\exp(-x)$ are $(-1)^k/k!$, which are also equal to the $D_k$. Therefore, according to Theorem 1 we have

$$L^k_{P,k} \left[ N_k! \prod_{i=1}^{k} \frac{1}{(i!)^{m_i} n_i!} \right] = \frac{(-1)^k}{k!}. \quad (96)$$

In the above result the $i = k$ term in the product is simply the result on the rhs. Therefore, it can be simplified to

$$L^k_{P,k} \left[ N_k! \prod_{i=1}^{k-1} \frac{1}{(i!)^{m_i} n_i!} \right] = \frac{2(-1)^k}{k!}. \quad (97)$$

Note in the above result that even though $n_k = 0$, the constraint in the partition operator still applies to $k$.

In Ref. [3] we found that the cosecant numbers denoted by $c_k$ were the coefficients generated when the partition method for a power series expansion was applied to $s \csc s$. This means that the method was basically applied to

$$s \csc s = \frac{1}{1 - s^2/3! + s^4/5! - s^6/7! + \cdots}. \quad (98)$$
By applying Theorem 1 to this example, in which $h(z) = z$, we see that $y = s^2$, $a = 1$, $p_k = (-1)^{k+1}/(2k + 1)!$ with $p_0 = 0$ and $q_k = 1$ since the outer series corresponds to the geometric series. Therefore, from Theorem 1 we obtain

$$s \csc s \equiv \sum_{k=0}^{\infty} c_k s^{2k}, \quad (99)$$

where according to Eq. (10),

$$(-1)^k c_k = L_{P,k} \left[ N_k! \prod_{i=1}^{k} \left( \frac{-1}{(2i + 1)!} \right)^{n_i} \frac{1}{n_i!} \right], \quad (100)$$

and $N_k = \sum_{i=1}^{k} n_i$. It should be noted that $\prod_{i=1}^{k} (-1)^{m_i+n_i} = (-1)^{k+N_k}$, although we shall retain the phase factor of $(-1)^{n_i}$ in order to observe a remarkable correspondence arising from the inversion of Equivalence (99). It was also found in Ref. [3] that the radius of absolute convergence for the power series expansion in Equivalence (99) was $\pi$, while the cosecant numbers were seen to be rapidly decreasing positive fractions given by Eq. (46). Hence, Eq. (100) represents a means of determining even integer values of the Riemann zeta function.

Now we invert the power series expansion in Equivalence (99) and apply Theorem 1 again, but in the case we have $p_k = -c_k$ with $p_0 = 0$. The resulting power series expansion was found to yield the standard Taylor/Maclaurin power series for $\sin(s)/s$ or $\sum_{k=0}^{\infty} (-1)^k s^{2k}/(2k + 1)!$, which is convergent for all values of $s$, despite the fact that Equivalence (99) has a radius of absolute convergence equal to $\pi$. This confirms the earlier remark concerning the fact that the resulting power series expansion obtained via Theorem 1 can turn out to be convergent even though the inner or the outer series may in fact be divergent. Furthermore, from Eq. (10) we obtain

$$\frac{(-1)^k}{(2k + 1)!} = L_{P,k} \left[ N_k! \prod_{i=1}^{k} (-c_i)^{n_i} \frac{1}{n_i!} \right], \quad (101)$$

It should also be mentioned that in Ref. [3] numerous recurrence relations were derived for the cosecant numbers. One of these is

$$\sum_{j=0}^{k-1} \frac{(-1)^{k-j-1}}{(2k - 2j + 1)!} c_j = c_k, \quad (102)$$

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where \( c_0 = 1 \). If we introduce Eq. (100) into the Eq. (102), then we obtain the interesting result of

\[
\sum_{j=0}^{k-1} \frac{1}{(2k-2j+1)!} L_{P,j} \left[ (-1)^{N_j - 1} N_j! \prod_{i=1}^{j} \left( \frac{1}{(2i+1)!} \right)^{n_i} \frac{1}{n_i!} \right] = L_{P,k} \left[ (-1)^{N_k} N_k! \prod_{i=1}^{k} \left( \frac{1}{(2i+1)!} \right)^{n_i} \frac{1}{n_i!} \right],
\]

(103)

Similar results to the above can be obtained by considering the other recurrence relations. Furthermore, in the same reference it was found that just as the Bernoulli numbers give rise to Bernoulli polynomials, the cosecant numbers give rise to their own polynomials, which are related to the former. In particular, the value at unity was found to be given by

\[
c_k(1) = (-1)^k 2^{2k} L_{P,2k} \left[ (-1)^{N_{2k}} N_{2k}! \prod_{i=1}^{2k} \left( \frac{1}{(i+1)!} \right)^{n_i} \frac{1}{n_i!} \right],
\]

(104)

where \( N_{2k} = \sum_{i=1}^{2k} n_i \) in the above result. Consequently, we see the reason for the introduction of the \( k \)-subscript to \( N \) in Theorem 1. The value of the cosecant polynomials at unity was found to equal

\[
c_k(1) = \frac{c_k}{2^{2k} - 1}.
\]

(105)

Hence, by introducing Eq. (100) into the above result and then equating it to Eq. (104), we arrive at

\[
L_{P,2k} \left[ (-1)^{N_{2k} - 1} N_{2k}! \prod_{i=1}^{2k} \left( \frac{1}{(i+1)!} \right)^{n_i} \frac{1}{n_i!} \right] = \frac{1}{(2^{2k} - 2)} L_{P,k} \left[ (-1)^{N_k} N_k! \prod_{i=1}^{k} \left( \frac{1}{(2i+1)!} \right)^{n_i} \frac{1}{n_i!} \right].
\]

(106)

Consequently, we do not need to consider all the partitions up to \( 2k \) to determine the sum on the lhs, which is a significant reduction in computational effort.

In Ref. [3] another infinite set of related numbers denoted by \( d_k \) and known as the secant numbers were obtained when the partition method for
a power series expansion was applied to \( \sec s \). Specifically, the method was applied to

\[
\sec s = \frac{1}{1 - s^2/2! + s^4/4! - s^6/6! + \ldots}.
\]  

(107)

By applying Theorem 1 to the above result we have \( y = s^2, p_0 = 0, a = 1 \) and \( q_k = 1 \) as before, but on this occasion, \( p_k = (-1)^{k+1}/(2k)! \). The resulting power series expansion, which can be expressed as

\[
\sec s \equiv \sum_{k=0}^{\infty} d_k s^{2k},
\]

(108)

where the coefficients \( d_k \) from Eq. (10) are given by

\[
(-1)^k d_k = L_{P,k} \left[ N_k \prod_{i=1}^{k} \left( -\frac{1}{(2i)!} \right) \frac{1}{n_i!} \right],
\]

(109)

and \( N_k \) is the same sum over the frequencies as before. Equivalence (107) was found to possess a narrower radius of absolute convergence compared with the power series expansion for cosecant, viz. \( \pi/2 \) as opposed to \( \pi \), while the \( d_k \) or secant numbers were found to be not as rapidly decreasing fractions as their cosecant counterparts. In addition, instead of being related to the Riemann zeta function as the cosecant numbers are, they were found to be related to the Hurwitz zeta function by

\[
d_k = \frac{2^{2k+2}}{\pi^{2k+1}} \left( \zeta(2k+1, 1/4) - \zeta(2k+1, 3/4) \right).
\]

(110)

The bracketed expression can also be written as \( \sum_{j=1}^{\infty} (-1)^{j+1}/(2j - 1)^{2k+1} \). To invert the analysis, either we can apply Theorem 1 to the power series expansion in Equivalence (108) or we can go directly to Equivalence (11). In the latter case we replace \( D_i \) by \( d_i \) in Eq. (12), while the lhs of Equivalence (11) equals \( \cos s \), which we replace by its power series expansion. Then we find that the coefficients \( E_k \) in Eq. (12) equal \( (-1)^k/(2k)! \). Hence, we arrive at

\[
\frac{(-1)^k}{(2k)!} = L_{P,k} \left[ N_k \prod_{i=1}^{k} (-d_i)^{n_i} \frac{1}{n_i!} \right].
\]

(111)
The secant numbers were also found to obey recurrence relations, although not as many as their cosecant counterparts. Nevertheless, an analogue of Eq. (102) was obtained, which is given by

\[ \sum_{j=0}^{k-1} \frac{(-1)^{k-j-1}}{(2k-2j)!} d_j = d_k , \tag{112} \]

with \( d_0 = 1 \). Introducing Eq. (111) into the above result yields

\[ \sum_{j=0}^{k-1} \frac{1}{(2k-2j)!} L_{P,j} \left[ (-1)^{N_j-1} N_j! \prod_{i=1}^{j} \left( \frac{1}{(i!)!} \right)^{n_i} \frac{1}{n_i!} \right] = L_{P,k} \left[ (-1)^{N_k} N_k! \prod_{i=1}^{k} \left( \frac{1}{(i!)!} \right)^{n_i} \frac{1}{n_i!} \right] . \tag{113} \]

This result is virtually identical to Eq. (103) except that there are no “+1’s” in the denominators of the above equation.

More sophisticated results involving both the secant and cosecant numbers can also be derived. From No. 1.518(2) of Ref. [23] we have

\[ \ln \sec(\pi z) = \sum_{k=1}^{\infty} \frac{(2^{2k} - 1)}{k (2k)!} 2^{2k-1} B_{2k} (\pi z)^{2k} , \tag{114} \]

which is absolutely convergent for \(|z| < 1/2\). In Ref. [3] it is shown that the cosecant numbers are related to the Bernoulli numbers by

\[ c_k = \frac{(-1)^{k+1}}{(2k)!} \frac{(2^{2k} - 2)}{B_{2k}} . \tag{115} \]

Moreover, we can write the lhs of Eq. (114) as

\[ \ln \sec(\pi z) = \ln \left( 1 + \sum_{k=1}^{\infty} d_k (\pi z)^{2k} \right) . \tag{116} \]

We now apply Theorem 1 to the rhs of the above result. This means that we expand the logarithm in terms of its Maclaurin series expansion, in which case \( q_k = (-1)^{k+1}/k \), while \( y = z^2 \) and \( p_k = d_k \pi^{2k} \). We then equate the resulting power series expansion to like powers of \( z^2 \) or \( y \) in Eq. (114). By substituting Eq. (115) we replace the Bernoulli numbers by the secant numbers, thereby obtaining

\[ L_{P,k} \left[ (-1)^{N+1} (N - 1)! \prod_{i=1}^{k} \frac{d_{n_i}^{n_i}}{n_i!} \right] = \frac{1}{2k} \frac{(2^{2k} - 1)}{\left( 1 - 2^{1-2k} \right) c_k} . \tag{117} \]
Once again, we see the partition operator acting on another strange argument to yield an interesting finite quantity for all values of $k$.

In Ref. [1] the partition method for a power series expansion is applied to the reciprocal of the logarithmic function $\ln(1+z)$. There a power series expansion is obtained in terms of special coefficients $A_k$, which are referred to as the reciprocal logarithm numbers. On p. 138 of Ref. [28] these numbers are referred to as the Gregory or Cauchy numbers when their modulus is taken. That is, the following result is obtained:

$$\frac{1}{\ln(1+z)} \equiv \sum_{k=0}^{\infty} A_k z^{k-1}.$$  

(118)

The reciprocal logarithm numbers are found to be oscillating fractions, which are more slowly converging to zero than either the cosecant or secant numbers. Moreover, they are given by

$$A_k = \frac{(-1)^k}{k!} \int_0^1 dt \frac{\Gamma(k+t-1)}{\Gamma(t-1)}.$$  

(119)

Expressing $\ln(1+z)$ in terms of its Maclaurin series as in the previous example, which is absolutely convergent only for $|z| < 1$, we are in a position to apply Theorem 1 to $z/\ln(1+z)$. In this case, $h(z) = 1/z$ and $f(z) = \ln(1+z)$. Then the coefficients of the inner series, viz. $p_k$, are equal to $(-1)^{k+1}/(k+1)$ for $k > 0$, while for $k = 0$, $p_0 = 0$. The resulting denominator can be regarded as the regularised value of the geometric series, which means that the $q_k$ are equal to $(-1)^k$ as in the preceding examples. Again, $a = 1$. Hence, according to Eq. (10), the reciprocal logarithm numbers can be written as

$$(-1)^k A_k = L_{P,k} \left[ N! \prod_{i=1}^{k} \left(-\frac{1}{i+1}\right)^{n_i} \frac{1}{n_i!} \right].$$  

(120)

The inverse of this result is obtained by putting $D_0 = 1$ and $D_k = A_k$ in Eq. (12), while the $E_k$ equal the coefficients in the Maclaurin series for $\ln(1+z)$, i.e. $(-1)^{k+1}/(k+1)$. Then we find that

$$\frac{(-1)^{k+1}}{k+1} = L_{P,k} \left[ N! \prod_{i=1}^{k} \frac{(-A_i)^{n_i}}{n_i!} \right].$$  

(121)

As an aside, in Ref. [1], Euler’s constant is derived in terms of an infinite series involving the reciprocal logarithm numbers, where it is also referred
to as Hurst’s formula. Since the publication of Ref. [1], it has been revealed that the formula was independently discovered by Kluyver [29]. By using Eq. (120), we can express Euler’s constant as

$$\gamma = -\sum_{k=1}^{\infty} \frac{1}{k} L_{P,k} \left[ (-1)^N N! \prod_{i=1}^{k} \left( \frac{1}{i+1} \right)^{n_i} \frac{1}{n_i!} \right].$$  \hspace{1cm} (122)

Alternatively, we can introduce Eq. (121) into Hurst’s formula, which yields

$$\gamma = -\sum_{k=1}^{\infty} A_k L_{P,k} \left[ (-1)^N N! \prod_{i=1}^{k} \left( \frac{A_{n_i}}{n_i!} \right) \right].$$  \hspace{1cm} (123)

Moreover, Euler’s constant is not the only result found in Ref. [1] that can be expressed as an infinite sum over the reciprocal logarithm numbers. For example, \(\ln 2\) can be expressed as a similar sum to Hurst’s formula. Therefore, with the aid of Eq. (120) we find that

$$\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k+1} L_{P,k} \left[ (-1)^N N! \prod_{i=1}^{k} \left( \frac{1}{i+1} \right)^{n_i} \frac{1}{n_i!} \right].$$  \hspace{1cm} (124)

With the aid of Corollary 1 to Theorem 1 we can generalise the preceding examples to where the generating functions are raised to an arbitrary power \(\rho\). For example, the quotient in Eq. (74) raised to an arbitrary power \(\rho\) becomes

$$g(f(z))^\rho = \left( \frac{1 - bz}{1 - 2bz} \right)^\rho \equiv \sum_{k=0}^{\infty} (bz)^k \sum_{j=0}^{k} \frac{\Gamma(j - \rho)}{\Gamma(-\rho) j!} \frac{2^{k-j} \Gamma(k - j + \rho)}{\Gamma(\rho) (k - j)!},$$  \hspace{1cm} (125)

where we have used the regularised value for the binomial series in Lemma 1. On the other hand, according to Equivalence (63), the lhs of the above result can be written as

$$g(f(z))^\rho \equiv 1 + \frac{1}{2} \sum_{k=0}^{\infty} (2bz)^k.$$  \hspace{1cm} (126)

Hence, \(D_0 = 1\) and \(D_k = 2^{k-1}b^k\) for \(k \geq 1\), while \(y = z\). From Eq. (60) we obtain

$$D_k(\rho) = (2b)^k L_{P,k} \left[ (-1/2)^N (-\rho)_N \prod_{i=1}^{k} \frac{1}{n_i!} \right].$$  \hspace{1cm} (127)
Equating like powers of $z$ on the rhs’s of the preceding equivalences yields

$$L_{P,k} \left[ \left(-1/2\right)^N (-\rho)_N \prod_{i=1}^{k} \frac{1}{n_i!} \right] = \sum_{j=0}^{k} \frac{\Gamma(j - \rho)}{\Gamma(-\rho)_j j!} \frac{\Gamma(k - j + \rho)}{2j \Gamma(\rho)(k - j)!} .$$ (128)

If Equivalence (99) is taken to the arbitrary power of $\rho$, then the $D_k$ of Equivalence (63) become the cosecant numbers, i.e. $D_i = c_i$. If we denote $D_k(\rho)$ by $c_{\rho,k}$, then we find that according to Eq. (60), these generalised cosecant numbers are given by

$$c_{\rho,k} = L_{P,k} \left[ (-1)^N (-\rho)_N \prod_{i=1}^{k} \frac{c_{n_i}}{n_i!} \right] .$$ (129)

Alternatively, the original equation can be expressed as

$$s^\rho \csc^\rho s = \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k + 1)!}\right)^{-\rho} .$$ (130)

Now the $D_k$ in Equivalence (63) are equal to $(-1)^k/(2k + 1)!$, while $\rho$ has changed sign. Thus, the generalised cosecant numbers can be written as

$$c_{\rho,k} = (-1)^k L_{P,k} \left[ (-1)^N (\rho)_N \prod_{i=1}^{k} \left(\frac{1}{(2i+1)!}\right)^{n_i} \frac{1}{n_i!} \right] .$$ (131)

This form for the generalised cosecant numbers has been recently been employed in evaluating a finite sum of inverse powers of cosines given as $S_{m,v} = \frac{\pi}{2m} \sum_{k=1}^{m-1} \cos(k\pi/m)^{-2v}$ in Ref. [4].

In a similar manner we can generalise the secant numbers. By taking the $\rho$-th power of Equivalence (108), we see that the $D_k$ in Equivalence (63) are equal to $d_i$. By denoting the generalised secant numbers as $d_{\rho,k}$, we find via Eq. (60) that

$$d_{\rho,k} = L_{P,k} \left[ (-1)^N (\rho)_N \prod_{i=1}^{k} \frac{d_{n_i}}{n_i!} \right] ,$$ (132)

while taking the $\rho$-th power of Eq. (107) yields

$$d_{\rho,k} = (-1)^k L_{P,k} \left[ (-1)^N (\rho)_N \prod_{i=1}^{k} \left(\frac{1}{(2i)!}\right)^{n_i} \frac{1}{n_i!} \right] .$$ (133)
The above result appears as Eq. (290) in Ref. 3.

To generalise the reciprocal logarithm numbers, we take the $\rho$-th power of Equivalence (119). Then the $D_k$ in Equivalence (63) are equal to $A_k$. Denoting the generalised reciprocal numbers by $A_k(-\rho)$, we find via Eq. (60) that they are given by

$$A_k(-\rho) = L_{P,k} \left[ (-1)^N (-\rho)^N \prod_{i=1}^{k} \frac{A_i^{n_i}}{n_i!} \right].$$

(134)

Alternatively, the generalised reciprocal logarithm numbers can be determined by taking the $\rho$-th power of the Maclaurin series for $\ln(1 + z)$, which is

$$\ln(1 + z) \equiv \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} z^k.$$

(135)

As explained in Ref. 2, the above result is absolutely convergent for $\Re z < -1$, in which case the equivalence symbol can be replaced by an equals sign. With regard to Equivalence (63) the $D_k$ are now equal to $(-1)^{k+1}/k$, while $y = z$. Consequently, Eq. (60) yields

$$A_k(\rho) = (-1)^k L_{P,k} \left[ (-1)^N (\rho)^N \prod_{i=1}^{k} \left( \frac{1}{i + 1} \right)^{n_i} \frac{1}{n_i!} \right].$$

(136)

Eq. (136) appears as Eq. (118) in Ref. 2.

At the end of Sec. 2 an alternative algorithm was given for accessing the partitions via the tree diagram in Fig. 1. As a result, we can present a new formulation of the partition method for a power series expansion. Before doing so, however, we need to amend the definition of the partition operator $L_{P,k}$. The amendment is necessary so that we can sum along each of the branches emanating from the seed number. For example, the entire sub-tree from $\{1,5\}$ represents the tree diagram for partitions summing to 5, but to obtain those summing to 6, we need to increment $n_1$ in all the partitions summing to 5 by one. This, of course, affects all the contributions to the coefficients. In addition, when we consider the partitions emanating from $\{2,4\}$, we need to ensure that no partitions with unity will appear, while there should be no ones or twos for the partitions emanating from $\{3,3\}$ and
so on. Therefore, we define the restricted partition operator as follows:

\[
L_{RP,k,i}[\cdot] = \sum_{n_i=1, n_{i+1}, \ldots, n_{k-i}=0}^1 \sum_{\sum_{j=1}^{k-i} jn_j=k} 1+[(k-i)/i],[(k-i)/(i+1)],\ldots,1.
\] (137)

There are major differences between the above operator and the partition operator as defined by Eq. (72). The first is that the sum begins at \(n_i\) rather than at \(n_1\). This is due to the fact that we need to exclude elements less than \(i\) when summing along the branches emanating from the seed number. The next difference is that \(n_i\) begins at unity rather than zero as for the other elements, which accounts for the fact already a one element of \(i\) has occurred in moving from the seed number. Furthermore, as a result of separating the element \(i\), the maximum element in the resulting partitions becomes \(k-i\). Hence, the elements range from \(i\) to \(k-i\), which not only affects the number of summations, but also their upper limits. In addition, whilst the number of summations is restricted in the constraint, the value remains invariant, viz. \(k\).

We are now in a position to implement the algorithm described at the end of Sec. 2. Basically, this entails expressing a result like Eq. (10) in terms of the partition operator on the lhs and the sum of restricted partition operators on the rhs. Therefore, we arrive at

\[
L_{P,k}[q_{N_1,k} a^{N_1,k} N_1,k! \prod_{i=1}^{k} \frac{p_i^{n_i}}{n_i!}] = q_1 a p_k
\]

\[
+ \sum_{j=1}^{[k/2]} L_{RP,k,j}[q_{N_j,k-j} a^{N_j,k-j} N_j,k-j! \prod_{i=j}^{k-j} \frac{p_i^{n_i}}{n_i!}],
\] (138)

where \(N_{i,k-i} = \sum_{j=i}^{k-i} n_j\) and \(N_{1,k} = N\). For \(p_0 \neq 0\) in Theorem 1, we use Eq. (9) instead, which amounts to replacing \(q_1 a p_k\) and \(q_{N_j,k-j} a^{N_j,k-j} N_{j,k-j}!\) in the above result by \(a F^{(k)}(ap_0)\) and \(a^{N_j,k-j} F^{N_j,k-j}(ap_0)\) respectively.

To complete this section, we consider the situation where the quotient of the pseudo-composite functions in Theorem 1 yields a function \(r(y)\), which is infinitely differentiable. This results in the following corollary.

**Corollary 2 to Theorem 1.** If the functions \(f(z)\) and \(g(z)\) obey the same conditions as in Theorem 1 and the quotient of the pseudo-composite functions \(g_a \circ f\) and \(h_a \circ f\) yields an infinitely differentiable function, \(r(y)\), then
for $p_0 \neq 0$,
\[
    r^{(k)}(0) = k! \, L_{P,k} \left[ a^N F^{(N)}(ap_0) \prod_{i=1}^{k} \frac{p_i^{n_i}}{n_i!} \right],
\]
(139)
while for $p_0 = 0$,
\[
    r^{(k)}(0) = k! \, L_{P,k} \left[ q_N a^N N! \prod_{i=1}^{k} \frac{p_i^{n_i}}{n_i!} \right].
\]
(140)
Furthermore, if $r(0) \neq 0$, then inversion of the quotient yields
\[
    \left( \frac{1}{r(y)} \right)^{(k)} \bigg|_{y=0} = k! \, \frac{E_k}{D_0},
\]
(141)
where the $E_k$ are given by Eq. (12).

**Proof.** From the proof of Theorem 1, we know that the ratio of $g_a \circ f$ over $h_a \circ f$ is equivalent to the power series $\sum_{k=0}^{\infty} D_k y^k$, where the coefficients $D_k$ are given by either Eq. (9) for $p_0$ non-zero, or Eq. (10) when $p_0 = 0$. Since the ratio of the pseudo-composite functions yields an infinitely differentiable function $r(y)$ according to the corollary, the ratio can also be expressed as a Taylor/Maclaurin series given by
\[
    \frac{g_a \circ f}{h_a \circ f} \equiv \sum_{k=0}^{\infty} r^{(k)}(0) \frac{y^k}{k!},
\]
(142)
where the superscript $(k)$ now denotes the $k$ times differentiation of $r(y)$ w.r.t. $y$. From Theorem 1 we also know that the above quotient can be expressed as
\[
    \frac{g_a \circ f}{h_a \circ f} \equiv \sum_{k=0}^{\infty} D_k y^k,
\]
(143)
where the coefficients $D_k$ are given by either Eq. (9) when $p_0 \neq 0$ or Eq. (10) for $p_0 = 0$. Since the regularised value is unique as described in Refs. [2] and [7]-[10], the rhs’s of the two preceding equivalence statements are equal to one another. Since $y$ is arbitrary in the resulting equation, we can equate like powers or the coefficients of both power series, which yields Eq. (139) or (140) depending upon the the value of $p_0$. 

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If the quotient of the pseudo-composite functions is inverted, then it will equal \(1/r(y)\). If \(r(0) \neq 0\), then the function \(1/r(y)\) can also be expressed as a Taylor/Maclaurin series since \(r(y)\) is infinitely differentiable. Therefore, we have

\[
\frac{h_a \circ f}{g_a \circ f} \equiv \sum_{k=0}^{\infty} \left( \frac{d^k}{dy^k} \frac{1}{r(y)} \right) \bigg|_{y=0} \frac{y^k}{k!}.
\]

(144)

From Theorem 1 we know that the above quotient also represents the regularised value of a power series in \(y\) whose coefficients are equal to \(E_k/D_0\), while the \(E_k\) are given by Eq. (12). Moreover, since \(r(0) \neq 0\), \(D_0\) does not vanish. Since both power series have the same regularised value, they are equal to one another for the same reason as in the first part of the proof. Again, as \(y\) is arbitrary, we can equate like powers of \(y\), which results in Eq. (141). This completes the proof of the corollary.

So far, we have described the partition method for a power series expansion in terms of a novel discrete operator, which has been referred to as the partition operator and is denoted by \(L_{P,k}[:]\). At this stage the operator has been used to derive general results for the coefficients of the power series expansions obtained via Theorem 1. However, it is not always possible to derive general results dependent only upon \(k\) for the coefficients \(D_k\) and \(E_k\) in Theorem 1. Often the coefficients become polynomials dependent upon another variable or even a function. For these cases we need to develop a program that can automatically evaluate specific coefficients, which is addressed in the following section.

4 Programming the Partition Method for a Power Series Expansion

At the end of the previous section we indicated the need for developing a computer program to enable the evaluation of the coefficients via the partition method for a power series expansion. As indicated earlier, such an approach can be developed by employing the BRCP algorithm of Sec. 2, but before doing so here, some remarks are necessary. Because the partition method for a power series expansion relies on evaluating the contribution due to each partition and the number of partitions \(p(k)\) according to the Hardy-Rademacher-Ramanujan formula is \(O(\exp(\pi \sqrt{2k/3})/4k\sqrt{3})\) \cite{16, 18, 30}, any
computer program based on partitions as its input will ultimately become very slow. In fact, since all the partitions summing to the order of each power are involved, such a program represents a brute-force approach to deriving power series expansions. Nevertheless, determining power series expansions for orders up to 40 \((p(50) = 37338)\) should be achievable with most number-crunching computers around today. So, at least for intermediate values of the order \(k\), programming the partition method is still of great benefit, particularly for intractable functions where it represents the only method we have of deriving a power series expansion.

With regard to very high orders it should be noted that the partition method does not actually use the partitions themselves. What the method requires is each element appearing in each partition and their frequency, which is referred to as the multiplicity representation in Sec. 2. This information can be stored in external arrays which can be called upon when one wishes to determine the series expansion for different situations. Therefore, there is no need to repeat the process of generating the partitions when dealing with different problems. In addition, the contributions due to many of the partitions will often be negligible even by today’s computing standards. In those cases the calculation of the coefficients can be simplified yielding extremely accurate approximations. In other cases it is possible to sum the contributions in classes or groups, thereby avoiding the necessity of processing each partition separately. This issue will be addressed later in this work. Finally, by developing a programming approach to the partition method, we will be in a position to consider different problems in the theory of partitions such as the evaluation of partitions with specific elements including those with discrete elements, doubly-restricted partitions and the transposes of partitions. The significant issue here is that such problems only require small changes to the BRCP algorithm, whereas separate programs are required when other codes such as those presented in Sec. 2 are used to generate partitions. E.g., in order to determine the partitions with a fixed number of elements in them, Knuth presents another algorithm based on the 18-th century dissertation by C.F. Hindenburg on p. 38 of Ref. \([18]\). As we shall see in Sec. 5, this problem can be solved by inserting a few lines into the BRCP algorithm.

In the previous section the partition method for a power series expansion was described in terms of the partition operator, \(L_{P,k}[\cdot]\). There many general results involving this operator were given without requiring to evaluate the sum over partitions for specific values of \(k\). Consequently, the partition operator can be regarded as an intricate abstract operator when compared with
more well-known operators such as the differential operator. However, whilst
taking the derivative of a function is also viewed as an abstract operation, we
at least have an understanding of the process because we can always calculate
the limit of Newton’s difference quotient provided, of course, it exists. As a
result of this understanding, general shorthand rules such as \( \frac{dx^k}{dx} = kx^{k-1} \)
have evolved. Yet, the opposite situation applies to the partition operator—
we have a few general results in the previous section and in Refs. [1]- [3], but
we do not even have a means of applying the operator outside of those cases
to evaluate the first few coefficients of the expansions given in Theorem 1.
Therefore, we require an approach that will allow us to apply the partition
operator for any value of \( k \) to any situation that obeys the conditions in
Theorem 1, even if it is no longer feasible to evaluate the coefficients for very
large values of \( k \).

Now that we have indicated why it is necessary to program the partition
method for a power series expansion, we turn to the issue of the program-
ing languages required for the task. The first point to be noted is that if we
choose a standard high-level programming language like C/C++ or Fortran,
then our results for the coefficients will inevitably become decimal num-
ders when they could be rational. Moreover, they will invariably be rounded off
or worse still, may only equal zero if significantly smaller than the precision
allowed by the computing system. In addition, the coefficients need not
be numerical as exemplified by the examples appearing after Corollary 1 to
Theorem 1. All this means is that we require a mathematical software pack-
age such as Mathematica to retain either the rationality of the coefficients
or when applicable, their symbolic form. However, programming the BRCP
algorithm in Mathematica with its bi-variate recursion is also formidable. In-
stead, the issue can be overcome by using the material in Sec. 2. There-
fore, the best option is to combine the strengths of both C/C++ and Mathem-
atica. Basically, this means that the initial program is to be written in C/C++
so that the coefficients can be printed out in a symbolic form. Then these
forms can be introduced into Mathematica where we can use either the in-
teger arithmetic routines to evaluate the coefficients, thereby avoiding the
round-off that occurs with floating point numbers or its symbolic routines to
reduce all the terms generated by the C/C++ code into simple mathematical
statements such as polynomials.

The appendix presents the C/C++ program called partmeth, which
outputs in symbolic form the coefficients \( D_k \) and \( E_k \) given in Theorem 1.
Here, we are only concerned with the case of \( p_0 = 0 \) or Eq. (9) for the \( D_k \),
while the $E_k$ are given by Eq. (12). The case of $p_0 \neq 0$ or Eq. (6) is left as an exercise for the reader. If we compare the code with the final code in Sec. 2, then we see that the overall structure remains the same. That is, there is a main section with the same two function prototypes termgen and idx. In fact, idx or the BRCP algorithm has not been altered at all, but termgen and the main routine have been changed to produce the symbolic forms for the coefficients in the partition method for a power series expansion. Besides evaluating the execution time, main carries out the calculation of the coefficients in one for loop, which is limited by the variable $dim$, representing the maximum value of $k$ or the coefficient of the highest order term which the user must input. Within this for loop there are two calls to idx, one of which applies to the calculation of the $D_k$ and the other to the calculation of the $E_k$. Therefore, it is termgen that is doing the heavy work in the programme. In fact, we shall see in the next section that by modifying termgen we can determine many of the properties of partitions, which often require separate programs.

Within termgen we see that the $D_k$ and $E_k$, which are represented by the variables $DS[k,n]$ and $ES[k,n]$ respectively, are evaluated depending upon the value of the variable inv_case. If it equals zero, then the $D_k$ are evaluated, while if it equals unity, then the $E_k$ are evaluated. In evaluating the latter there is also an extra complication due to the phase factor of $(-1)^N$ in Eq. (12). Consequently, for this case termgen must determine the number of distinct elements in each partition. When $dim=4$, partmeth prints out the first four values of the $E_k$ and $D_k$ in symbolic form. E.g., the $k=4$ values that it prints out are:

\[
DS[4,n\_]:= p[4,n\_] q[1\_] a + p[1,n\_] p[3,n\_] q[2\_] a^{\wedge}(2) 2! + p[1,n\_]{\wedge}(2) p[2,n\_] q[3\_] a^{\wedge}(3) 3!/2! + p[1,n\_]{\wedge}(4) q[4\_] a^{\wedge}(4) + p[2,n\_]{\wedge}(2) q[2\_] a^{\wedge}(2) \\
ES[4,n\_]:= -DS[0,0]{\wedge}(-2) DS[4,n\_] + DS[0,0]{\wedge}(-3) DS[1,n\_] DS[3,n\_] 2! - DS[0,0]{\wedge}(-4) DS[1,n\_]{\wedge}(2) DS[2,n\_] 3!/2! + DS[0,0]{\wedge}(-5) DS[1,n\_]{\wedge}(4) + DS[0,0]{\wedge}(-3) DS[2,n\_]{\wedge}(2)
\]

From these results we see that each coefficient is composed of five distinct terms corresponding to the fact that the number partitions summing to 4, i.e. $p(4)$, is equal to 5. These results allow for the situation where the $p[k,n]$ may be dependent upon another variable, viz. $n$, even though it may not be necessary.

The first code presented in the appendix is suitable for values up to
and around \( k = 20 \). In fact, all the values of \( D_k \) and \( E_k \) for \( k \leq 20 \) are computed within one CPU second. For \( k \geq 20 \), however, the expressions become unwieldy and thus, it is better to evaluate them separately so that each can be introduced directly into Mathematica. This amounts to removing the for loop in main and computing only for the value of \( k \) or \( \text{dim} \). The code for computing the \( D_k \), which is called mathpm, appears immediately after partmeth in the appendix. Furthermore, in order that the coefficients can be introduced directly in Mathematica, only three terms appear on each line of output, while a plus sign now appears as the last character on each line except, of course, on the final line.

Let us now consider the evaluation of \( D_{30} \) via mathpm. Since \( p(30) = 5604 \), this is the number of distinct terms when mathpm prints out DS[30, n]. Even though the output file for DS[30, n] is very large, it can still be imported into Mathematica. If we now set \( p_k = (-1)^k / (2k + 1)! \), \( q_k = 1(-1)^k \) and \( a = 1 \), which represent the inner and outer series for the cosecant numbers, then it takes 0.15 CPU sec to evaluate \( c_{30} \) in integer form on the Sony VAIO laptop mentioned in Sec. 2. In this instance the numerator is given by a 60 digit integer, while the denominator is given by a 90 digit integer. In decimal notation the value of \( c_{30} \) is approximately \( 2.965 \times 10^{-30} \). If we use Eq. (100) to evaluate \( c_{30} \), then we find that it takes almost zero CPU sec to evaluate the same result. On the other hand, if we set \( p_k = (-1)^k / (2k)! \), i.e. the situation for the secant numbers \( d_k \), then we find that it takes 0.14 CPU sec to evaluate \( d_{30} \) in integer form on the Sony VAIO laptop. In this case the numerator and denominator are respectively 67 and 78 digit numbers, while in decimal form \( d_{30} \) is approximately equal to \( 2.176 \times 10^{-12} \). Unfortunately, if Eq. (110) is implemented in Mathematica, then we only obtain approximate values in decimal form for the secant numbers. Hence, we need to implement a recurrence relation such as Eq. (112) in order to obtain them in integer form. When this is done, it is found that Mathematica takes 6548 CPU sec to compute \( d_{30} \).

If we set \( p_k = (-1)^k / (k + 1) \), which represents the situation for the reciprocal logarithm numbers \( A_k \), then we find that it takes only 0.1 CPU sec to determine \( A_{30} \) in integer form. In this instance the numerator and denominator are 35 and 38 digit numbers yielding an approximate decimal value of \( 1.474 \times 10^{-3} \), the slowest converging of the numbers considered so far. The reciprocal logarithm numbers can be evaluated by either relating them to the
Stirling numbers of the first kind \([1]\) via

\[
A_k = \frac{(-1)^k}{k!} \sum_{j=0}^{k-1} \frac{S_k^{(j)}}{j + 1} .
\]  

or by the recurrence relation of

\[
A_k = \sum_{j=0}^{k-1} \frac{(-1)^{k-j+1}}{(k-j+1)} A_j .
\]  

If the first form is implemented in Mathematica, then it takes 0.1 CPU sec to compute \(A_{30}\), while with the second form it takes 5719 CPU sec. Therefore, we see that the evaluating the coefficients of power series expansions via the partition method for a power series expansion can be vastly superior to using recurrence relations and is almost on a par with the cases where intrinsic forms have already been implemented within a mathematical software package. Furthermore, by altering the relations for the coefficients of the inner and outer series in addition to \(a\), we obtain results for other mathematical quantities with different power series expansions.

As discussed previously, the coefficients of the inner and outer series do not need to be numbers as has been the case so far. If we set \(q_k = (\rho) - k/k!\), \(p_k = (-1)^{k+1}/(2k+1)!\) and \(a = 1\), then \(DS[y,n]\) yields the generalised cosecant number \(c_{\rho,30}\) as given by Eqs. (129) and (131). Then we find that it only takes 0.36 CPU sec to compute the resulting polynomial, which is thirtieth order in \(\rho\). By invoking the Simplify routine in Mathematica the polynomial can be arranged in increasing order within another 0.36 CPU sec. Before this calculation was performed, the results produced by \(DS[5,n]\) and \(DS[8,n]\) had been found to agree with the generalised cosecant numbers, \(c_{\rho,5}\) and \(c_{\rho,8}\), obtained in Ref. [4].

As stated earlier, the calculation of the coefficients via the forms generated by either of the first two programs in the appendix can be continued beyond the thirtieth order, but eventually problems arise due to the combinatorial explosion occurring in the number of partitions. It has already been stated that there are 190 569 272 partitions summing to 100, which means that this number of terms will be present in \(DS[100,n]\). If \texttt{mathpm} is run for this case, then it takes around 600 CPU sec to compute \(DS[100,n]\). Whilst this is not an overly long time of computation compared with the earlier results obtained via recurrence relations, it produces a file whose size is over 16 GB. Files of
this size are going to present a problem when imported into mathematical software packages. For example, it appears that Mathematica 8.0.1 is only able to import files with 2Gb of data. One method of circumventing this problem would be to divide the file into smaller files so that could be handled by the different processors on a supercomputer. Then the results generated by each processor could combined to yield the final answer.

Another method of overcoming this problem is to introduce the values for $p_k$, $q_k$ and $a$ first and then evaluate a specific number or limit of terms via \texttt{mathpm}. Once the limit point is reached, the values where this occurs would need to be stored. In terms of Fig. this amounts to storing the values of both arguments in \texttt{idx}. Then the partial value of the coefficient could be evaluated and stored, while all the terms outputted in running \texttt{mathpm} can be either deleted or overwritten in a re-run of \texttt{mathpm}. In the re-run of \texttt{mathpm} the code would not print out any terms until \texttt{idx} reaches the values of the arguments of \texttt{idx} stored from the first run. Then the code would either continue to print out the next limit of terms stopping at two new values of \texttt{idx} or would terminate on reaching the central partition. Then the terms stored in the second run could be evaluated and combined with the result obtained from the first run. If the central partition has not been reached in the second run, then the process can be continued until the central partition is eventually reached. Of course, the disadvantage in this approach is that we have lost the ability to evaluate a new coefficient by altering the $p_k$, $q_k$ and $a$ as we were able to do with $DS/30,n$ above. This second method of solving the problem of very large data files produced by running \texttt{mathpm} is contingent on whether we can stop and re-start the program at specific points in the tree diagrams for the partitions. Hence, we need to be able to adapt the BRCP algorithm so that specific partitions can be determined, which represents the topic of the following section.

5 Specific Types of Partitions

In this section we aim to investigate how the BRCP algorithm presented in Sec. 2 can be modified to determine specific types of partitions. By specific partitions we mean such general problems in partitions as the determination of: (1) partitions with a fixed number of elements, (2) doubly-restricted partitions, (3) discrete partitions, (4) conjugate partitions and (5) partitions with specific elements in them. Solving such problems invariably means
creating a different algorithm or program for each problem as can be seen in Refs. [16], [18] and [22]. However, we shall see here that such problems and others can be solved with relatively minor modifications to the BRCP algorithm presented in Sec. 2, once again highlighting its versatility. As we shall be modifying partgen in Sec. 2 when solving these problems, we shall generate the partitions in the compact multiplicity representation, although it should be mentioned that the various algorithms presented in this section would only require minor modification to termgen to generate partitions in the standard representation.

Of all the problems mentioned in the previous paragraph perhaps the simplest one to consider is the determination of those partitions with a specific element or elements in them. E.g., suppose we wish to determine all those partitions summing to 15 with the element of \{5\} in them? From the material presented in Sec. 2, it is obvious that the total number of partitions for this problem must equal the number of partitions summing to 10 or \(p(10)\). Moreover, the partitions generated by the new code should yield the same partitions obtained by running the various codes presented in Sec. 2 except that each partition generated by the new code will have an additional element of \{5\} in them. We shall see that this is indeed the case, although the order in which the partitions are generated is different from those discussed in Sec. 2.

In order to modify partgen in Sec. 2 so that it generates partitions with a specific element in them, we first need to alter the main prototype of the program. This is required to enable users to input the specific values of the elements that they wish to appear in the partitions printed out by the new program. Consequently, main becomes

```c
int main( int argc , char *argv [] )
{
    int i ;
    if(argc!=3) printf(" usage: specpart <partition sum> <sp_val> \n") ;
    else{
        tot=atoi(argv [1]) ;
        sp_val=atoi(argv [2]) ;
        part=(int *) malloc(tot*sizeof(int)) ;
        if(part == NULL) printf(" unable to allocate array\n") ;
}
```

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else {
    for (i = 0; i < tot; i++) part[i] = 0;
    idx(tot, 1);
    free(part);
}

printf(
"\n"
); return (0);
}

Here we see that the new program called specpart has a global variable called sp_val, which represents the specific element that is to appear at least once in each partition printed out by the program.

As discussed in Sec. 2, idx(tot, 1) scans over all the partitions summing to tot, while termgen, which is called in idx, is responsible for generating or printing out partitions. Thus, in order to determine those partitions with a specific element or elements in them, we need to modify termgen. That is, we still need to scan over all partitions by calling idx(tot, 1). In fact, aside from making minor modifications to main, we shall find that to solve all the problems mentioned in the introduction to this section, we need only modify termgen.

In Sec. 2 the termgen function prototype in partgen was responsible for printing out all the partitions summing to tot in the multiplicity representation. This was achieved by processing the array part, which stored the frequencies of the elements in each partition. That is, the variable freq was used to represent the frequency of the element $i + 1$ in the partition with $i$ ranging from 0 to $tot - 1$. Now that we wish to determine those partitions with a specific element or elements in them, we need to restrict the partitions that are printed out by termgen. This is accomplished simply by introducing a local variable called freq_spval, which evaluates the frequency of sp_val in each partition. If this value is non-zero, then we know that there is at least one occurrence of sp_val in the partition and the partition is then printed out in the same manner as in Sec. 2. If freq_spval is zero, then the partition is ignored. Therefore, the termgen function prototype for specpart becomes

```c
void termgen()
{
    int freq, i, freq_spval;
```
freq_spval=part[sp_val−1];
if(freq_spval){
    printf("%ld: ",term++);
    for (i=0;i<tot;i++){
        freq=part[i];
        if(freq) printf("%i(%i) ",freq,i+1);
    }
    printf("
" );
}
}

In Sec. 2 the partitions summing to 5, which amounted to 7, were printed out by running partgen. The output produced by running specpart with tot and sp_val set equal to 11 and 6, respectively, is:
1: 5(1) 1(6)
2: 3(1) 1(2) 1(6)
3: 2(1) 1(3) 1(6)
4: 1(1) 2(2) 1(6)
5: 1(1) 1(4) 1(6)
6: 1(2) 1(3) 1(6)
7: 1(5) 1(6)
Hence, we observe that the number of partitions is once again 7 or p(5). If we remove one six from each partition, then we obtain the same partitions as those generated in Sec. 2 by partgen except that the order in which they appear is now different. If we define the specific element partition operator $L_{SEP,k,j}[\cdot]$ as

$$L_{SEP,k,j}[\cdot] = \sum_{n_1,...,n_{j−1}=0,n_j=1,n_{j+1},...n_k=0}^{k,...,k/(j−1),[k/j],k/(j+1)} \ldots 1,$$  \quad (147)

then it follows that

$$L_{SEP,k,j}[1] = L_{P,k−j}[1] = p(k−j) \quad .$$  \quad (148)

As a result of the preceding analysis, it becomes a simple matter to consider partitions with more than one specific element in them. All we need to do is introduce more values in main and then create local variables like freq_spval to represent the number of occurrences or frequencies for each of
these values. As before, each frequency is set equal to \( \text{part}[\text{sp\_val} - 1] \) in `termgen`. Next, we modify the if statement to include all the frequencies. For example, if we wish to determine all the partitions with two specific elements, viz. \( \text{sp\_val} \) and \( \text{sp\_val2} \), then the if statement becomes

\[
\text{if} \ (\text{freq\_spval} \&\& \text{freq\_spval2}) \{ \text{etc.} \}
\]

On the other hand, we may want those partitions with at least one occurrence of either \( \text{sp\_val} \) or \( \text{sp\_val2} \). In this instance all we need to do is replace the logical AND operator by the logical OR operator or \(&\&\) in the if statement.

By making a few minor modifications, mostly to `termgen`, we have been able to solve several problems involving specific partitions. Now we consider evaluating the partitions with a fixed number of elements in them, which has already been considered when we modified the partition operator into the form given by Eq. (88). Previously, it was stated that to solve this problem, often a completely different approach is employed. For example, Knuth presents a different algorithm on p. 38 of Ref. [18] compared with the algorithm used to generate partitions in reverse lexicographic form. In the case of the BRCP algorithm, however, the number of elements in a partition is determined by the number of branches along its path prior to termination in the tree diagram. E.g., the partitions with only two elements in them in Fig. 1 are obtained by searching for the terminating tuples appearing in the third column, viz. (0,5), (0,4) and (0,3). In other words, these are the terminating tuples two branches away from the seed number situated in the first column. When we include the elements from the second column, they yield the partitions \{1,5\}, \{2,4\} and \{3,3\}. Since partitions with a fixed number of elements can be determined from the tree diagram, it means that only minor modifications to the program in Sec. 2 are again required in order to generate these partitions.

As in the previous example, we need to modify `main` so that the user can input the number of parts, which will be represented by the global variable `numparts`. Then we can proceed to the modification of `termgen`, which is presented below. There we see that a new local variable called `sumparts` has been introduced. This variable determines the number of elements in each partition. When this number equals `numparts`, the partition is printed out. Otherwise, the partition is discarded.

```c
void termgen ()
{
int freq , i , sumparts=0;
```
for (i=0; i<tot; i++) {
    sumparts = sumparts + part[i];
}
if (sumparts == numparts) {
    printf("%d: ", term++);
    for (i=0; i<tot; i++){
        freq = part[i];
        if (freq) printf("%d(%d) ", freq, i+1);
    }
    printf("\n");
}

When the code is run for the case where the partitions sum to 10 and possess 5 elements, i.e. for \( \text{tot} = 10 \) and \( \text{numparts} = 5 \), the following output is produced:

1: 4(1) 1(6)
2: 3(1) 1(2) 1(5)
3: 3(1) 1(3) 1(4)
4: 2(1) 2(2) 1(4)
5: 2(1) 1(2) 2(3)
6: 1(1) 3(2) 1(3)
7: 5(2)

Therefore, we see that the number of partitions in this instance is 7, which can also be expressed as either \( \binom{10}{5} \) according to Sec. 2 or as \( L_{n,10}^5[1] \) according to Eq. (90).

It should also be mentioned that if the condition,

\[
\text{if (sumparts==numparts)}
\]

in the above program is replaced by

\[
\text{if (sumparts<=numparts)}
\]

then the resulting code generates all those partitions summing to \( \text{tot} \) with at most \( \text{numparts} \) elements. E.g., the number of partitions summing to 10
with at most 5 elements, which according to Sec. 2 is equal to 30. As stated in Sec. 2, this is also equivalent to \( \binom{15}{5} \). However, the partitions generated by the code given above for \( \text{tot} \) and \( \text{numparts} \) equal to 15 and 5 respectively, are different from those generated by the version of the program with the modified \textit{if} statement.

Doubly-restricted partitions are those partitions where all the elements are greater than a particular value and less than another value. Since this is a combination of two separate conditions, first we need to be able to modify \texttt{partgen} so that it generates the partitions where the elements are either greater than or lower than a specified value. Therefore, let us consider the situation where all the elements in the partitions are less than or equal to a value, which will be represented by the global variable \texttt{largest_elt}. As in the previous examples, this value must be introduced into \texttt{main}. Then we proceed to the modification of \texttt{termgen}.

To modify \texttt{termgen} so that only partitions with elements less than or equal to \texttt{largest\_elt} are generated, we need to introduce an extra for loop. This extra loop is required so that if a partition is encountered where an element is greater than \texttt{largest\_elt}, it is discarded via a \texttt{goto} statement as demonstrated by the modified version of \texttt{termgen} given below. Although \texttt{goto} statements are generally frowned upon by programmers, it is being used here to abandon processing in a nested structure of two loops. In fact, the code behaves much like \texttt{partgen} in Sec. 2 when all the elements in the partitions are less than or equal to \texttt{largest\_elt}. However, when an element is greater than \texttt{largest\_elt}, the \texttt{goto} statement discards the partition by diverting to the \texttt{end} statement label.

```c
void termgen ()
{
    int f, i;
    for (i=largest_elt; i<dim; i++)
    {
        f=part[i];
        if (f)
            goto end;
    }
    printf("%ld: ", term++);
    for (i=0; i<dim; i++)
        f=part[i];
```
When the above code is run for partitions summing to 14 in which the elements are greater than 3, the following output is produced:

1: 1(14)
2: 1(4) 1(10)
3: 2(4) 1(6)
4: 1(4) 2(5)
5: 1(5) 1(9)
6: 1(6) 1(8)
7: 2(7)

Hence, we see that there only seven partitions summing to 14, in which all the elements are greater than 3.

As a result of the above code, it is now a simple matter to consider the case where all the elements are greater than or equal to another value specified by the user. In this instance we simply replace `largest_elt` by another global variable called `smallest_elt` and alter the condition in the first for loop of the previous version of `termgen`. That is, the first for loop in the preceding version of `termgen` simply becomes

```c
for (i = 0; i < smallest_part - 1; i++) {
    f = part[i];
    if (f)
        printf("%i (%i ) ", f, i + 1);
}
```

Note also that if the upper value in the for loop had been `smallest_elt` instead of `smallest_elt - 1`, then all the elements generated by the code would only be greater than the value specified by the user.

For doubly-restricted partitions, where the elements are greater or equal to value and less than or equal to another (larger) value, all we need to do is incorporate two for loops that divert to the `end` statement label. For example, `termgen` for this situation would become

```c
void termgen ()
{
    int f, i;
    for (i = 0; i < smallest_part - 1; i++)
        f = part[i];
        if (f)
goto end; }

for(i=largest part;i<dim;i++){
  f=part[i];
  if(f)
    goto end;
  }

printf("%ld: ",term++);
for(i=0;i<dim;i++){
  f=part[i];
  if(f) printf("%i (%i) ", f, i+1);
}

printf("\n");
}

When the above code is run for partitions summing to 13 with the elements greater than or equal to 3 and less than or equal to 9, the following output is produced:
1: 2(3) 1(7)
2: 3(3) 1(4)
3: 1(3) 1(4) 1(6)
4: 1(3) 2(5)
5: 1(4) 1(9)
6: 2(4) 1(5)
7: 1(5) 1(8)
8: 1(6) 1(7)
Thus, we see that there are 8 partitions with all elements lying in the interval [3,9].

In Ch. 3 of Ref. [31] Andrews defines restricted partitions differently from the earlier definition given by Eq. (137). There, the partitions represent those in which the elements are less than a value, say $el_{-max}$, while the number of elements is less than or equal to another value, which we take again to be $numparts$ as in the preceding examples. Although there is now a condition pertaining to the number of parts, evaluating these partitions is again similar to the doubly-restricted case studied above. First, we must introduce $el_{-max}$ into $main$ in addition to $numparts$. Then we need to insert an extra for loop into $termgen$ so that it can make use of the different condition. The new
loop appears first since if it is true, we immediately by-pass any action to process the current partition. Therefore, this modified version of termgen becomes

```c
void termgen()
{
  int freq, i, sumparts = 0;
  for (i = 0; i < tot; i++){
    if (i > el_max - 1 && part[i] > 0) goto end;
  }
  /*(1) sumparts is the number of elements in a partition */
  /*(2) all elements are now less than or equal to el_max*/
  for (i = 0; i < tot; i++){
    sumparts = sumparts + part[i];
  }
  if (sumparts <= numparts){
    printf("%ld: ", term++);
    for (i = 0; i < tot; i++){
      freq = part[i];
      if (freq) printf("%i (%i) ", freq, i + 1);
    }
    printf("\n");
  }
end: ;
}
```

When the code is run with `tot`, `numparts` and `el_max` set equal to 10, 3 and 5 respectively, the following output is produced:

1: 1(1) 1(4) 1(5)
2: 1(2) 1(3) 1(5)
3: 1(2) 2(4)
4: 2(3) 1(4)
5: 2(5)

Hence, we see that there are 5 partitions summing to 10 with at most 3 elements and all elements less than or equal to 5.

The number of partitions summing to \( k \) with at most \( M \) parts and each element less than or equal to \( N \) is represented as \( p_G(N, M, k) \) in Ref. [31].
The subscript $G$ has been introduced here so that the reader is not confused with similar notation in the next section. From the above example we have $p_G(5,3,10) = 5$. If $k > MN$, then $p_G(N,M,k)$ vanishes, while $p_G(N,M,NM) = 1$. These numbers also appear as the coefficients in the generating function for Gaussian polynomials, which are given by

$$G(N,M;q) = \prod_{i=1}^{N} \frac{1 - q^{M+i}}{1 - q^i} = 1 + \sum_{k=1}^{NM} p_G(N,M,k) q^k .$$

Hence, Gaussian polynomials are polynomials in $q$ of degree $NM$. Moreover, to avoid confusion with the restricted partitions studied earlier, we shall refer to the above partitions as Gaussian partitions. As a consequence, we define the Gaussian partition operator $L_{GP,k,N,M}[\cdot]$ as

$$L_{GP,k,N,M}[\cdot] = \sum_{\sum_{i=1}^{N} n_i = k, \sum_{i=1}^{N} n_i \leq M} \min\{[k/i],M\} .$$

As a result, we have $L_{GP,k,N,M}[1] = p_G(N,M,k)$.

We now turn our attention to a more complicated example— the problem of determining discrete or distinct partitions. By discrete partitions, we mean those partitions in which the elements appear at most once, if at all. Like all the preceding examples, they too represent a subset or class of the set of integer partitions. As we shall see in the next section, such partitions figure prominently in the theory of partitions. Because of this, we define the discrete partition operator, $L_{DP,k}[\cdot]$, by

$$L_{DP,k}[\cdot] = \sum_{\sum_{i=1}^{k} n_i = 0, \sum_{i=1}^{k} n_i = k, \sum_{i=1}^{k} n_i \leq M} 1^{\cdot} .$$

Therefore, the only difference between this operator and the partition operator introduced in Sec. 3 is that the upper limits of the summations are restricted to unity, whereas for the latter the upper limits were set to $[k/i]$ for each element $i$.

A computer program that generates discrete partitions is presented in its entirety in the appendix. As stated previously, the number of partitions summing to 100 or $p(100)$ equals 190,569,272, which is the reason why it is
Figure 3: The number of discrete partitions to the total number of partitions versus $k$

cumbersome to evaluate the 100-th coefficient via the partition method for a power series expansion. On the other hand, if we run the program in the appendix called dispart to determine the discrete partitions summing to 100, then we find that it only takes 5 CPU seconds to generate the 444793 partitions, which represent 0.2 per cent of $p(100)$. Moreover, Fig. 3 displays the ratio of the number of discrete partitions or $L_{DP,k}[1]$ to the number of standard partitions versus $k$ for $k \leq 50$. From the figure we see that this ratio decreases monotonically for $k \geq 10$, reaching a value of 0.0179 \ldots when $k = 50$.

The final example we shall consider in this section is the evaluation of the conjugate partition as a partition is generated. According to Ref. [18], the conjugate partition is obtained by transposing the rows and columns of the corresponding Ferrers diagram for the original partition. In a Ferrers diagram a partition is represented by an array of dots in which the first element, say $a_1$, is allocated a row of $a_1$ dots, while the next element ($a_2$) is represented by another row of $a_2$ dots immediately below the first row of dots. The process of allocating rows of dots for each element in a partition is continued up to the final element in the partition. The conjugate partition or $\alpha^T$ of the partition $\{a_1, a_2, \ldots, a_k\}$ is obtained by transposing the rows and
columns of the Ferrers diagram for the partition. For example, the Ferrers
diagram for the partition \( \{1, 1, 2, 3, 3, 4\} \) has two rows with one dot, a row
with two dots followed by two rows with 3 dots and finally a row with 4 dots.
The transpose is obtained by counting the dots in the vertical columns of
the Ferrers diagram. Hence, \( \alpha^T \) for our example is found to be \( \{6, 4, 3, 1\} \).
Note also that the conjugate partition does not necessarily possess the same
number of elements as the original partition.

A code that determines the conjugate partition for each partition gener-
ated by \texttt{partgen} is presented in the appendix. As with all the other examples
in this section, it is \texttt{termgen} that has been modified. The major complica-
tion in this code compared with the others is that one is required to create
and allocate space for a two-dimensional array \texttt{ferrers} of size \texttt{tot x tot} in addi-
tion to creating a one-dimensional array of pointers called \texttt{rptr} to it. In both
cases the arrays are of integer type. In reality they should be of type char
since a true Ferrers diagram consists of dots. That is, instead of allocating
dots the program allocates ones in creating a Ferrers diagram. Nevertheless,
by adding the ones vertically, one obtains the conjugate partition. For the
purists it should be a simple matter to alter the \texttt{termgen} to count dots
rather than ones.

When the program called \texttt{transp} for the partitions summing to 4, the
following output is produced:
Partition 1 is: 1(4) and its conjugate is: 4(1)
Partition 2 is: 1(1) 1(3) and its conjugate is: 1(2) 2(1)
Partition 3 is: 2(1) 1(2) and its conjugate is: 1(3) 1(1)
Partition 4 is: 4(1) and its conjugate is: 1(4)
Partition 5 is: 2(2) and its conjugate is: 2(2)
As can be seen from the output, the conjugate partitions are printed out in
a different order to those produced by the BRCP algorithm. For example,
the conjugate partition to Partition 2 or \( \{1,3\} \) is printed out as 1(2) 2(1)
or \( \{2,1,1\} \), whereas the form produced by the BRCP algorithm, viz. Parti-
tion 3, is 2(1) 1(2) or \( \{1,1,2\} \). Throughout this work we have not been
concerned with the order of the elements in the partitions. When the order
is important, partitions become compositions \footnote{31}. For example, there are
three compositions of the partition \( \{1,1,2\} \) are: \( \{112\} \), \( \{121\} \) and \( \{211\} \). We
shall not be concerned with compositions here.

There are some interesting properties concerning conjugate partitions.
For example, there is at least one partition whose conjugate is itself, although
it can turn out to be a different composition. Such self-conjugate partitions
arise when the partitions sum to a square of an integer \( k \) since the partition \( k(k) \) in the multiplicity representation is a self-conjugate. On the other hand, if the partitions sum to \( k(k+1)/2 \), then the partition \( \{1,2,\ldots,k\} \) is a self-conjugate, although again the composition is different. When partitions sum to \( 2k \) for \( k > 2 \), the partition given by \((k-2)(1)(2)(1)(k)\) is also a self-conjugate, while if they sum to \( 2k+1 \), where \( k \geq 7 \), then the partition given by \((k-6)(1)(2)(1)(3)(1)(k-2)\) is another self-conjugate. In fact, according to Ref. [30], the number of self-conjugate partitions is the same as the number of partitions with distinct odd elements.

There are other problems that can be solved by modifying \texttt{termgen} given in Sec. 2. One such problem is determining those partitions with only odd elements in them. As a consequence, we can also define an odd-element partition (OEP) operator as

\[
L_{OEP,2k+1}[\cdot] = \sum_{\substack{n_1,n_3,\ldots,n_{2k+1}=0 \\sum_{i=1}^{k}(2i+1)n_{2i+1}=2k+1}}^{2k+1,[(2k+1)/3],\ldots,1}, \quad (152)
\]

which is only valid for partitions summing \((2k+1)\), while for those summing to \( 2k \), the operator becomes

\[
L_{OEP,2k}[\cdot] = \sum_{\substack{n_1,n_3,\ldots,n_{2k}=0 \\sum_{i=1}^{k}(2i-1)n_{2i-1}=2k}}^{2k,[2k/3],\ldots,1}. \quad (153)
\]

To obtain partitions with only odd elements in them, all we need to do is insert the following for loop at the beginning of \texttt{termgen} just after the type declarations:

```c
for ( i=0; i<tot; i++){
    freq=part[i];
    if ( i % 2 && freq > 0) goto end;
}
printf("%ld: ",term++);
```

It has already been mentioned that the number of partitions with discrete odd elements equals the number of self-conjugate partitions. To obtain the former, the upper limits in the definitions for \( L_{OEP,2k+1}[\cdot] \) and \( L_{OEP,2k}[\cdot] \) must be set equal to unity. In addition, if we wish to generate partitions with
discrete odd elements in them, all we need to do is introduce the above for loop at the beginning of termgen in the program dispart that is presented in the appendix. With regard to self-conjugate partitions, transp in the appendix would need to be modified so that the original partition is stored in a temporary array before it undergoes conjugation. Then a test would need to be introduced to see if both partitions are identical to one another. If they are, then the partition is printed out. Otherwise, it is discarded. This problem is left as an exercise for the reader.

In a similar manner we can define an even-element partition (EEP) operator that only applies to those partitions summing to $2k$ with even elements in them. This is defined as

$$L_{EEP,2k}[\cdot] = \sum_{n_2, n_4, \ldots, n_{2k}=0}^{k \cdot \lfloor k/2 \rfloor, \ldots, 1} \cdot .$$

(154)

The number of partitions generated by this operator is equal to the number of partitions summing to $k$. Hence, we find that

$$L_{EE,2k}[1] = L_{P,k}[1] .$$

(155)

In Ref. [3] the cosecant numbers or $c_k$ are first derived in terms of the partitions summing to $2k$ with even elements in them before the form in terms of the partition operator given by Eq. (100) is derived. Consequently, we arrive at the following identity:

$$L_{P,k} \left[ (-1)^{N_k} N_k! \prod_{i=1}^{k} \left( \frac{1}{(2i+1)!} \right)^{n_i} \frac{1}{n_i!} \right]$$

$$= L_{EEP,2k} \left[ (-1)^{N_k^*} N_k^*! \prod_{i=1}^{k} \left( \frac{1}{(2i+1)!} \right)^{n_{2i}} \frac{1}{n_{2i}!} \right] ,$$

(156)

where $N_k^* = \sum_{i=1}^{k} n_{2i}$.

In this section various programs have been presented for obtaining specific types/classes of partitions, which can be regarded as subsets of the total number of partitions summing to a particular value. Consequently, the various operators given in this section represent restricted forms of the partition operator $L_{P,k}[\cdot]$. In the previous section we showed how the partition method for a power series expansion could be developed into a computer program.
scanning the entire set of partitions. From the material presented in this section, it should, therefore, be possible to determine the contributions that the specific partitions contribute to the partition method for a power series expansion. In particular, in the next section we shall see that the partition function \( p(k) \) can be obtained via the partition method for a power series expansion involving a restricted set of the partitions summing to \( k \), although this restricted set is more difficult to evaluate than many of the examples considered in this section. We shall not only present the program that can generate this restricted set of partitions, but also present the program that outputs the partition function in symbolic form so that it can be handled by Mathematica. As for the more important programs discussed in this section, the new program will appear in the appendix. Finally, it should be noted that by possessing the capacity to adapt the partition method for a power series expansion to handle specific partitions, we may be able to determine which partitions make the largest contribution to the coefficients in Theorem 1. As a result, accurate approximations to the coefficients can be considered, which may avoid the combinatorial explosion that occurs for large orders as described at the beginning of the previous section.

6 Generating Function for \( P(k) \)

An important topic in the theory of partitions are the generating functions whose power series expansions possess coefficients that are dependent on the properties of partitions. One of the greatest achievements in this context is the derivation of the asymptotic formula for the number of partitions \( p(k) \). The first step that led to this formula was the derivation of a remarkable formula for the product \( P(z) \) in Equivalence (73) by Dedekind. As explained in Ref. [18], this can be derived by the application of standard analytic techniques, namely Poisson’s summation formula, to the logarithm of \( P(z) \). Then by studying the behaviour of Dedekind’s formula for \( \ln P(\exp(-t)) \) with \( \Re t > 0 \), Hardy and Ramanujan [32] were able with amazing insight [31] to deduce the asymptotic behaviour of the partition function \( p(k) \) for large \( k \). Eventually, the asymptotic behaviour of \( p(k) \) was completely evaluated by Rademacher [33], which culminated in the now famous Hardy-Rademacher-Ramanujan formula mentioned in the introduction to Sec. 4.

Although we shall not reach such lofty heights, we shall, nevertheless, turn our attention in the remainder of this work to how the partition method for
a power series expansion can be applied in the analysis of the various generating functions that occur in the theory of partitions and their extensions or generalisations. We begin in this section by applying the partition method for a power series expansion to $P(z)$, but before we can embark upon this task, we need to determine when the power series expansion or generating function is convergent or in another words, for what values of $z$ Equivalence (73) becomes an equation. According to Knuth [18], it was Euler who noticed that the coefficient of $z^n$ in the infinite product of

\[(1 + z + z^2 + z^3 + \cdots)(1 + z^2 + z^4 + \cdots + z^{2k} + \cdots)(1 + z^3 + z^6 + \cdots + z^{3k} + \cdots)\cdots\]

is the number of non-negative solutions to $k + 2k + 3k + \cdots = n$ or the partition function $p(n)$ and that $1 + z^m + z^{2m} + \cdots$ equals $1/(1 - z^m)$. As a result, he arrived at Equivalence (73) except that the equivalence symbol was replaced by an equals sign, which is not entirely correct as can be seen by the following theorem.

**Theorem 2.** The equivalence statement relating $P(z)$ or $\prod_{m=1}^{\infty} 1/(1 - z^m)$ to the generating function given by the power series expansion with coefficients equal to $p(k)$, viz. Equivalence (73), is absolutely convergent for $|z| < 1$, in which case the equivalence symbol can be replaced by an equals sign. On the other hand, it is divergent for all other values of $z$. Then $P(z)$ represents the regularised value of the series on the rhs. For $|z| = 1$, the generating function is singular.

**Proof.** The reason why an equivalence symbol has had to be introduced is due to Euler’s second observation concerned with the geometric series. Replacing the series by its limit value of $1/(1 - z^m)$ is strictly not valid for all values of $z$ as described in Refs. [2], [7], [8] and [11]. There it is shown that the standard geometric series, i.e. $\sum_{k=0}^{\infty} z^k$, is divergent for $\Re z > 1$, absolutely convergent for $|z| < 1$ and conditionally convergent for $|z| > 1$ and $\Re z < 1$. Regardless of the type of convergence, the limit value of the series is found to equal $1/(1 - z)$. For $\Re z > 1$, however, summing the series yields an infinity. In this case, if the infinity is removed after the series is summed, which is the essence of regularisation, then the remaining finite part is found to equal $1/(1 - z)$ again. Hence, for $z \geq 1$, the regularised value of the geometric series is equal to $1/(1 - z)$. Moreover, along the line $\Re z = 1$, the limit of the series is undefined or indeterminate, while at the point $z = 1$, where the line is tangent to the unit disk of absolute convergence, it is singular. This type of behaviour is expected since $\Re z = 1$, represents the border between the convergence to the left and divergence to the right. Since the limit value is the same on both sides after regularisation, we set the regularised value to
$1/(1 - z)$ along $\Re z = 1$.

With regard to Equivalence (73) we are dealing with an infinite product of geometric series involving different powers of $z$ in the limit value. Nevertheless, we can use the above knowledge of the geometric series to determine where the series on the rhs of Equivalence (73) is convergent and where it is divergent. For the product on the lhs to equal the series on the rhs of Equivalence (73), we must have for all positive integer values of $l$, $\Re z^l < 1$.

For $l = 1$ we end up with the standard geometric series, but for $l = 2$, the series will now only be convergent for $\Re z^2 < 1$ or $-1 < \Re z < 1$. Thus, the range of values of $z$ has changed, which means that the convergence of the series of the series on the rhs of Equivalence (73) will be affected by each value of $l$ or each series in the product.

Let us examine the third series in the product, whose limit is $1/(1 - z^3)$. In order to analyse this version of the geometric series, we write the limit value as

$$
\frac{1}{1 - z^3} = \frac{1}{(1 - z)(1 - ze^{2\pi i/3})(1 - z^{-2\pi i/3})}.
$$

(157)

Decomposing the rhs into partial fractions, we see that this version of geometric series is actually the sum of three geometric series, each with a different limit. The first yields the standard geometric series discussed above. The second series has a limit of $1/(1 - z \exp(2i\pi/3))$. In this case we replace $z$ by $z \exp(2i\pi/3)$ and continue with the same analysis. Then the second series is convergent for $\Re (z \exp(2i\pi/3)) < 1$ or $y < (2 - x)/\sqrt{3}$ when $z = x + iy$, while it is divergent for $y > (2 - x)/\sqrt{3}$. That is, the line $\Re z = 1$ separating the regions or planes of convergence and divergence has been rotated by $2\pi/3$ in a clockwise direction. The “left side” of the line representing where the series is convergent is now given by $y < (2 - x)/\sqrt{3}$. On the other hand, it is divergent for $y > (2 - x)/\sqrt{3}$ in which case the limit becomes the regularised value of the second series. The third series, whose limit is $1/(1 - z \exp(-2i\pi/3))$, represents the opposite of the previous series. That is, the singularity at $z = 1$ in the standard geometric series has now been rotated by $2\pi/3$ in an anti-clockwise direction. Hence, the third series is convergent for $\Re (z \exp(-2i\pi/3)) < 1$ or $y > -(x + 2)/\sqrt{3}$ when $z = x + iy$. This represents the “left” side, while the “right” side or where it is divergent is given by $y < -(x + 2)/\sqrt{3}$. For these values of $z$ the limit represents the regularised value of the series.

It is the intersection of the “left” sides for the three series that represents
the region of the complex plane for which the third series in the product or \( \sum_{k=0}^{\infty} z^{3k} \) is convergent. Outside this region the series is divergent. The intersection of the tangent lines yields an equilateral triangle, where the midpoints of the edges coincide with the three singular points of the component geometric series on the circle \(|z| = 1\). Moreover, the unit disk of absolute convergence is circumscribed by this triangle. Those parts of the triangle not in the unit disk of absolute convergence represent the regions of the complex plane for which the third series in the product is conditionally convergent. In total they are significantly less than the corresponding region of the complex plane for the second series in the product, which we found is given by the region outside the unit disk of absolute convergence in the plane \(-1 < \Re z < 1\).

If we consider the fourth series in the product, i.e. \( \sum_{k=0}^{\infty} z^{4k} \), then decomposing its limit into partial fractions yields four distinct geometric series with the singularities situated again on the unit circle, but at \( \pm 1 \) and \( \exp(\pm i\pi/2) \). If we draw tangent lines through each of these singularities, then we find the common region or the intersection of their “left” sides is now a square circumscribing the unit disk of absolute convergence. The singularities in the component geometric series appear at the mid-points of the square’s sides. Outside the square, the fourth series in the product is divergent. Then the regularised value of the series is obtained by combining the limits of the component series. The regions inside the square, but outside the unit disk, represent the values of \( z \) for which the fourth series in the product is conditionally convergent. As expected, these regions in total are less than either of the regions of conditional convergence for the second and third series in the product.

If we continue this analysis to the \( l \)-th series in the product, i.e. for \( \sum_{k=0}^{\infty} z^{lk} \), then we find that the intersection of tangent lines yields an \( l \)-sided polygon that circumscribes the unit disk of absolute convergence. For the values of \( z \) outside the polygon the series will be divergent, while for those values of \( z \) within the polygon, but outside of the unit disk, the series will be conditionally convergent. Furthermore, as higher values of \( l \) are considered, the number of tangent lines not only increases, but also the total region of conditional convergence contracts. In the limit as \( l \to \infty \) we will be left with the unit disk as the sole region where the series is (absolutely) convergent, while outside the disk the series is divergent. Since the product in Equivalence (73) includes all values of \( l \), the \( l = \infty \) limit by virtue of the fact that it possesses the smallest region of convergence in the complex plane.
determines the values of \( z \), where the series on the rhs of Equivalence (73) is convergent. This means that the series or generating function, which we shall call from here on the partition number series, is equal to the product on the lhs only for \( z \) situated in the unit disk. Outside the unit disk, the lhs represents the regularised value of the series and an equivalence symbol must be used instead of an equals sign. Finally, the circle \(| z | = 1\) represents a ring of singularity separating the divergent values of the partition number series from the absolutely convergent values. In the case of the standard geometric series, there is only one point where the absolutely convergent region is separated from the divergent region, namely the singularity situated at \( z = 1 \). Elsewhere, the line \( \Re z = 1 \) separates conditionally convergent values from divergent values. As mentioned above, the limit for the geometric series is indeterminate along the line, but is assigned a regularised value of \( 1/(1-z) \). However, for \( z = 1 \) the regularised value yields infinity. Hence, it can be seen that there is a difference between separating absolutely convergent values from divergent values and separating conditionally convergent values from divergent values. This completes the proof of the theorem.

For \(| z | < 1\), we can invert Equivalence (153), thereby obtaining

\[
\frac{1}{P(z)} = \prod_{k=1}^{\infty} (1 - z^k) = \frac{1}{1 + \sum_{k=1}^{\infty} b(k) z^k} . \tag{158}
\]

Since the above product produces a power series that is valid for all values of \( z \), we can write it as

\[
(z; z)_{\infty} = \prod_{k=1}^{\infty} (1 - z^k) = 1 + \sum_{k=1}^{\infty} q(k) z^k . \tag{159}
\]

The leftmost expression is a special case of the q-Pochhammer symbol \([35]\), which is defined as

\[
(a; z)_n = \prod_{k=0}^{n-1} (1 - az^k) . \tag{160}
\]

According to Knuth \([18]\), it was Euler, who was the first to discover that much cancellation occurs when multiplying the various terms in the infinite product given in Eq. (160). Specifically, he found that

\[
\prod_{m=1}^{\infty} (1 - z^m) = 1 + \sum_{k=1}^{\infty} (-1)^k \left( z^{(3k^2-k)/2} + z^{(3k^2+k)/2} \right) . \tag{161}
\]
Therefore, comparing the above result with rhs of Eq. (159) we th as the \( q(k) \)
are frequently equal to zero and when non-zero are either equal to 1 or -1. The values of \( k \) for which the \( q(k) \) vanish are known today as the pentagonal numbers \[36\]. They are themselves a particular case of a broader class of numbers known as the figurate or figural numbers \[37, 38\]. By applying Theorem 1 to Eq. (158), we see that the coefficients of the outer series, viz. \( q_k \), are equal to \((-1)^k\), while those for the inner series or the \( p_k \) are equal to the partition function \( p(k) \) for \( k \geq 1 \) and zero for \( k = 0 \). Therefore, with the aid of Eq. (160) we arrive at

\[
q(k) = L_{P,k} \left[ (-1)^{N_k} N_k! \prod_{i=1}^{k} \frac{p(i)^{n_i}}{n_i!} \right]. \tag{162}
\]

As a consequence of the fact that the \( q(k) \) are non-zero when \( k = (3j^2 \pm j)/2 \), Eq. (162) can also be written as

\[
L_{P,k} \left[ (-1)^{N_k} N_k! \prod_{i=1}^{k} \frac{p(i)^{n_i}}{n_i!} \right] = \begin{cases} (-1)^j, & k = (3j^2 \pm j)/2, \\ 0, & \text{otherwise}. \end{cases} \tag{163}
\]

Although they do not give the actual number of discrete partitions, we shall refer to the \( q(k) \) as the discrete partition numbers. Shortly, we shall see how these coefficients are related to the number of discrete partitions. They do, however, have an interesting connection with the partition function, which follows when both power series on the rhs’s of Eqs. (158) and (159) are multiplied by one another. Then we find that

\[
\sum_{k=0}^{\infty} z^k \sum_{j=0}^{k} p(j) q(k - j) = 1, \tag{164}
\]

where \( p(0) = q(0) = 1 \). Again, since \( z \) is fairly arbitrary, like powers of \( z \) can be equated on both sides of the above equation. For \( k \geq 1 \), we obtain the following recurrence relation:

\[
\sum_{j=0}^{k} p(j) q(k - j) = 0. \tag{165}
\]

This is simply Euler’s recurrence relation for the partition function, which is given by Eq. (20) in Ref. \[18\]. Occasionally, it is referred to as MacMahon’s
recurrence relation as in Eq. (20) of Ref. [39]. Because most of the discrete partition numbers or $q(k)$ vanish, it means that only a few of the previous values of the partition function are required to evaluate the latest value of the partition function.

Eq. (162) is an interesting result, but perhaps, not very practical for determining the discrete partition numbers, when it is realised that the partition function or $p(k)$ grows exponentially. We can, however, use the partition method for a power series expansion to derive another result for the discrete partition numbers. First, assuming that $|z| < 1$, we write $1/P(z)$ as

$$\frac{1}{P(z)} = \exp\left(\sum_{m=1}^{\infty} \ln(1 - z^m)\right) = \exp\left(-\sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \frac{z^m}{j} \right).$$  \hspace{1cm} (166)$$

The Taylor/Maclaurin series for the logarithm has been introduced in this result, since it is absolutely convergent for $|z| < 1$ [1]. The double sum can be expressed as a single sum by realising that the coefficients of $z$ can be expressed as a sum over divisors or factors of the power [34]. Then we arrive at

$$P(z)^{-1} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(z + \frac{3z^2}{2} + \frac{4z^3}{3} + \frac{7z^4}{4} + \cdots + \gamma_j z^j + \cdots\right)^k. \hspace{1cm} (167)$$

where $\gamma_j = \sum_{d|j} d/j$ and $d$ represents a divisor of $j$. That is, the sum is only over the divisors of $j$. Some values of the $\gamma_j$ are: $\gamma_1 = 1$, $\gamma_2 = 3/2$, $\gamma_3 = 4/3$, $\gamma_4 = 7/4$, and $\gamma_5 = 6/5$. More explicitly, we find that $\gamma_6 = 1/6 + 1/3 + 1/2 + 1 = 2$, while for the case of $j = \rho^m$, where $\rho$ is a prime number, the sum yields

$$\gamma_{\rho^m} = \frac{(1 - 1/\rho^{m+1})}{(1 - 1/\rho)}. \hspace{1cm} (168)$$

Eq. (168) can be derived simply by using the limit for the geometric series. Furthermore, Eq. (167) also means that

$$P(z) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left(z + \frac{3z^2}{2} + \frac{4z^3}{3} + \frac{7z^4}{4} + \cdots + \gamma_j z^j + \cdots\right)^k. \hspace{1cm} (169)$$

Therefore, we have obtained alternative representations for the generating functions of both $P(z)$ and its inverse.
Now we are in a position to apply Theorem 1 to Eq. (167), whereupon we see that the coefficients of the inner series $p_k$ equal $\gamma_k$ for $k \geq 1$, while $p_0 = 0$. On the other hand, the coefficients of the outer series $q_k$ are equal to $(-1)^k/k!$. Hence, from Eq. (110) the discrete partition numbers are given by

$$q(k) = L_{P,k} \left[ (-1)^N \prod_{i=1}^{k} \frac{\gamma_i^{n_i}}{n_i!} \right].$$  \hspace{1cm} (170)$$

Hence, we have derived an alternative version of Eq. (162). Moreover, whenever $k = (3j^2 \pm j)/2$, for $j$ an integer, this result can also be written as

$$L_{P,(3j^2 \pm j)/2} \left[ (-1)^{N(3j^2 \pm j)/2} \prod_{i=1}^{(3k^2 \pm j)/2} \frac{\gamma_i^{n_i}}{n_i!} \right] = (-1)^j.$$  \hspace{1cm} (171)$$

For all other values of $k$, the sum over all partitions in Eq. (170) vanishes. Therefore, to evaluate $q(6)$, we require all the $\gamma_j$ ranging from $j = 1$ to $6$, which have already been given above. Summing over the eleven partitions summing to 6 in Eq. (170) yields

$$q(6) = -2 + 6/5 + 21/8 - 7/8 + 16/18 - 2 + 4/18$$
$$- 27/48 + 9/16 - 3/48 + 1/6! = 0,$$

which is indeed the value of this discrete partition number. By using these results, the reader can readily verify that $q(0) = 1$, $q(1) = -1$, $q(2) = -1$, $q(5) = 1$ and $q(3) = q(4) = 0$. Moreover, since $D_0$ in Theorem 1 is non-zero in this case due to the fact that $q(0) = 1$, we can use Eq. (12) to determine the coefficients of the inverted power series expansion or the generating function for $P(k)$. Hence, we find that

$$p(k) = L_{P,k} \left[ (-1)^N \prod_{i=1}^{k} \frac{q(i)^{n_i}}{n_i!} \right].$$  \hspace{1cm} (173)$$

This result, which represents the inverse of Eq. (162), incorporates much redundancy since the $q(i)$ are only non-zero when for specific values of $i$. Consequently, both the sum over the partitions and the product are only non-zero for those values of $i$, which are of the form of $(3j^2 - j)/2$ or $(3j^2 + j)/2$, where $j$ is an integer ranging from 1 to $j_m = [(1 + \sqrt{1 + 24k})/6]$. In addition, when the $q(i)$ are non-zero, they are only equal to unity in magnitude. Therefore,
it is the factor of \( N! \) that is responsible for the exponential increase in the partition function as \( k \) increases, although this factor will often be countered by the \( 1/n_i! \) terms in the denominator of the product. For example, if we wish to determine \( p(6) \), there will not be any sums over \( n_3, n_4 \) and \( n_6 \) in the above equation since \( q(3), q(4) \) and \( q(6) \) vanish, while in the product, \( n_3!, n_4! \) and \( n_6! \) will equal 0! or unity. Furthermore, since \( q(1) = q(2) = -1 \) and \( q(5) = 1 \), Eq. (173) yields

\[
p(6) = 1 + (-1)^5(-1)^4(-1) 5!/4! + (-1)^4(-1)^2(-1)^2 4!/(2! \cdot 2!) + (-1)^3(-1)^3 3!/3! + (-1)^2(-1)^2 2! = 11 \ ,
\]

(174)

Hence, the contributions from the five relevant partitions are positive except for the last one, which represents the contribution due to the partition \( \{1,5\} \).

On the other hand, if we apply Theorem 1 to Eq. (169), then the only difference to the previous evaluation of the discrete partition numbers is that the coefficients of the outer series \( q_k \) are now equal to \( 1/k! \). That is, the coefficients of the outer series are still equal to \( \gamma_k \). Hence, we arrive at

\[
p(k) = L_{P,k} \left[ \prod_{i=1}^{k} \frac{\gamma_{n_i}}{n_i!} \right] \ .
\]

(175)

Therefore, we have an entirely different means of evaluating the partition function with the sum of the reciprocals of the divisors of each element being the assigned values in the partition method for a power series expansion. Consequently, Eq. (165) becomes

\[
\sum_{j=0}^{k} L_{P,k} \left[ \prod_{i=1}^{j} \frac{\gamma_{n_i}}{n_i!} \right] L_{P,k} \left[ (-1)^{N_k-j} \prod_{i=1}^{k-j} \frac{\gamma_{n_i}}{n_i!} \right] = 0 \ .
\]

(176)

In deriving these new results for the partition function \( p(k) \) we have encountered a more complex example involving a specific class of partitions than any of those studied in the previous section. In this case we only require partitions whose elements are pentagonal numbers or of the form of \( (3j \pm 1)j/2 \), where \( j \) is any integer lying between zero and \( j_m \). In view of the importance of the Eq. (173), let us consider introducing the modifications to the program \texttt{partgen} presented in Sec. 2. As in the examples of the previous section, most of these modifications will be made to the function prototype \texttt{termgen}.  

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The first modification is that we need to make is introduce the math library with the other header files. This is needed so that the floor function can be called to evaluate the maximum value of \( j \) or \( i_m \) when the value of \( k \) or \( tot \) is typed in by the user. Since \( j_m \) is called in both the \texttt{main} and \texttt{termgen} prototypes, it must be declared as a global variable. Once these modifications are carried out, we can concentrate on the changes that are required for \texttt{termgen}, which is displayed below:

```c
void termgen()
{
    int freq, i, j, jval;

    for (i = 0; i < tot; i++){
        jval = 0;
        freq = part[i];
        if (freq > 0){
            for (j = 1; j < j_m; j++){
                if (i == ((3*j - 1)*j - 2)/2) jval = j;
                if (i == ((3*j + 1)*j - 2)/2) jval = j;
            }
            if ((jval == 0) && (freq > 0)) goto end;
        }
    }

    printf("%ld: ", term++);
    for (i = 0; i < tot; i++){
        freq = part[i];
        if (freq) printf("%i (%i) ", freq, i + 1);
    }
    printf("\n");
}
```

Basically, this version of \texttt{termgen} is similar to those that appeared in the previous section. That is, before a partition is printed out, testing is done in the function prototype to see if each partition belongs or conforms to the particular set of partitions, which the user desires. In this case our aim is to print out only those partitions whose elements can be written in the pentagonal number forms of \((3j^2 - j)/2\) and \((3j^2 + j)/2\), where \( j \) is an
integer. This is accomplished by introducing another for loop in the function prototype, which evaluates the value of \( j \) called \( jval \), for the appropriate elements. If an element is not of the required form, then \( jval \) remains zero. Otherwise, it is non-zero. If \( jval \) is zero for an element, then the partition is examined to see if there any occurrences of the element by checking the variable \( freq \). If it is greater than zero, then the entire partition is discarded by the goto statement. This test is carried out on all elements in the partition. The procedure is then applied to all partitions summing to \( tot \). Only those partitions whose elements are of the required form are printed out by the code. As an example, when \( k = 6 \), the output for this program called \texttt{partfn} is:

1: 1(1) 1(5)  
2: 4(1) 1(2)  
3: 6(1)  
4: 2(1) 2(2)  
5: 3(2)

The above partitions represent the five that contributed to the calculation of \( p(6) \) given above. Interestingly, when the code is run for partitions summing to 100, it only prints out 42205 partitions which represents about 0.02 percent of the total number of partitions or \( p(100) \). Moreover, the number of partitions summing to \( k \) whose elements are pentagonal numbers or \( L_{\text{Pentel},k}[1] \) starts off greater than the number of discrete partitions for the same value of \( k \), but then drops away when \( k > 17 \). Fig. 4 displays the ratio of \( L_{\text{Pentel},k}[1] \) to the number of partitions summing to \( k \) for \( k \leq 50 \). Once again, it is monotonically decreasing for \( k \) beyond a certain value, which in this case is about 8.

It was also stated at the end of the previous section that the partition method for a power series expansion can be adapted to handle situations where only a subset of the partitions summing to a particular value are required as in the above example. We shall demonstrate this by modifying the code in Sec. 4 so that the partition function \( p(k) \) can be evaluated via Eq. (173). The resulting code called \texttt{pfm} is presented in its entirety in the appendix. Basically, it expresses the partition function in a symbolic form where the final values can be evaluated by introducing the output into Mathematica. As expected, \texttt{termgen} has the same for loop presented above. The interesting feature about the code is that the for loop appears twice at the beginning of \texttt{termgen}. This is necessary because the first step prints out \( p(k) := \) before considering the single element partition of \( \{k\} \) in the first
branch of the if statement. Clearly, the single element partition may not be of the required form—hence, the first appearance of the for loop. All the other partitions are processed by the second branch of the if statement, which means that we require the for loop in this part of the program in order to determine which are of the required form.

The code is also capable of expressing the final symbolic form for the partition function in two different forms. In the first form a printf statement prints the final form for \( p(k) \) in terms of the \( q(i) \) as they appear in Eq. \((173)\). This statement has been commented out in favour of the version of the code appearing in the appendix. In the second version of the code the value of each \( q(i) \) is evaluated. That is, the code prints out \(-1\) to the power of \( jval \) for the elements in a suitable or candidate partition. In order to accomplish this, the for loop mentioned in the previous paragraph has had to be introduced into the latter part of \texttt{termgen} again. Finally, the quantity \( q[N]a^N \) appearing in the output of the code discussed in Sec. 4 has been replaced \((-1)^N(N)\) in \texttt{pfn}. Therefore, if the version of \texttt{pfn} in the appendix is run for \( tot = 6 \), then the following output is produced:

\[
p[6]:= ((-1)^{(1)}) ((-1)^{(2)}) (-1)^{(2)} 2! + ((-1)^{(1)})^{(4)} ((-1)^{(1)}) (-1)^{(5)} 5!/4! + ((-1)^{(1)})^{(6)} (-1)^{(6)} + ((-1)^{(1)})^{(2)} ((-1)^{(1)})^{(2)} (-1)^{(4)} 4!/2! \]

Figure 4: The ratio of the number of partitions whose elements are pentagonal numbers to the total number of partitions versus \( k \)
\[2! + ((-1)^{(1)})^3 \cdot (-1)^3\]

Time taken to compute the coefficient is 0.000000 seconds.

This output can be imported into Mathematica whereupon it gives the correct value of 11 for \(p(6)\). When the code is run for \(tot\) equal to 100, it takes 34 seconds to compute the symbolic form for \(p(100)\) or \(L_{p,100}[1]\) and only 0.27 seconds to produce the value of 190,569,292 in Mathematica on the same Sony VAIO laptop mentioned in previous sections. However, whilst this is not too bad, the fastest method of obtaining \(p(k)\) or \(L_{p,k}[1]\) from the various programs considered in this work is to comment out \texttt{termgen} and introduce the statement

\[\text{term++;}\]

immediately below in \texttt{partgen} of Sec. 2. Then \texttt{term} needs to be initialised to zero and a printf statement introduced into \texttt{main}. When the resulting program is run for \(tot\) equal to 100, it only takes 3 seconds to compute \(p(100)\). Nevertheless, both methods for computing the partition function will slow down dramatically as \(tot\) continues to increase due to combinatorial explosion. This is where either Eq. (165) or even the Hardy-Ramanujan-Rademacher formula should be used. In fact, we may write

\[
\lim_{k \to \infty} L_{P,k}[1] \rightarrow \frac{1}{4\sqrt{3}k} \exp(\sqrt{2k/3 \pi}), \tag{177}
\]

and

\[
\lim_{k \to \infty} L_{P,k} \left[(-1)^N N! \prod_{i=1}^{k} \frac{q(i)^{n_i}}{n_i!} \right] \rightarrow \frac{1}{4\sqrt{3}k} \exp(\sqrt{2k/3 \pi}). \tag{178}
\]

In addition, the arrow symbols in the above results can be replaced by equals signs.

7 Generalisation of the Inverse of \(P(z)\)

In the previous section we demonstrated how the partition method for a power series expansion can be used to derive different forms for the generating functions of the product \(P(z)\) and its inverse or reciprocal. In the case of \(P(z)\) two different forms for the coefficients of the generating function or the partition function \(p(k)\) were obtained in terms of the partition operator acting
on different arguments. The first given by Eq. (173) involved the discrete partition numbers \( q(k) \), which represent the coefficients of the generating function for the inverse of \( P(z) \), while the second given by Eq. (175) involved a sum over the inverses of the divisors for each element \( i \) in a partition. For the inverse of \( P(z) \) two different forms for the discrete partition numbers were also obtained, but now the partition operator was found to act on either the partition function as in the first form given by Eq. (162) or in the second form given by Eq. (170) the same sum over the inverses of the divisors with an extra phase factor of \((-1)^{N_k}\), where \( N_k \) represents the number of elements in each partition summing to \( k \). This extra phase factor is responsible for ensuring that the discrete partition numbers equal either \( \pm 1 \) when \( k \) is a pentagonal number or zero, otherwise. Because the phase factor does not appear in Eqs. (173) and (175), the partition function experiences exponential growth as can be seen from the Hardy-Ramanujan-Rademacher formula given above. Nevertheless, it was possible to derive all these results because the coefficients of the powers of \( z \) in the product \( P(z) \) are simple, namely equal to -1. In this section we aim to generalise \( P(z) \) to the situation where the coefficients of \( z^k \) are now equal to \( C_k \). We begin by studying the inverse of \( P(z) \). As a consequence, we arrive at the following theorem.

**Theorem 3.** The infinite product defined by

\[
H(z) = \prod_{i=1}^{\infty} \left( 1 + C_i z^i \right) ,
\]

(179)
can be written as a power series expansion or generating function of the form:

\[
H(z) = 1 + \sum_{k=1}^{\infty} h_k z^k ,
\]

(180)
where the coefficients \( h_k \) can be expressed in terms of the discrete partition operator defined by Eq. (151) and are given by

\[
h_k = L_{DP,k} \left[ \prod_{i=1}^{k} C_i^n \right] .
\]

(181)

**Proof.** If we multiply out the lowest order terms in \( z \) in the product given in Eq. (179), then we obtain a power series expansion for \( H(z) \) in a similar manner to the proof of Theorem 1. The zeroth order term in the
resulting power series is unity, while the first order term becomes \( C_1 z \). We then find that the second order term in the power series becomes \( C_2 z^2 \). Once we go beyond second order, however, the coefficients become more complex to evaluate due to an ever-increasing number of terms appearing in them. For example, the third, fourth and fifth order coefficients are respectively equal to \( C_3 + C_1 C_2 \), \( C_4 + C_1 C_3 \), and \( C_5 + C_1 C_4 + C_2 C_3 \). In fact, on close inspection we find that the coefficient of \( z^k \) in the power series depends on the number of discrete partitions summing to \( k \). For example, the third and fourth order coefficients are composed of only two terms because there are only two discrete partitions for \( k = 3 \) and \( k = 4 \), while the fifth order term is composed of three terms due to the three discrete partitions summing to 5, viz. \{5\}, \{1,4\} and \{2,3\}. Therefore, instead of summing over all the partitions summing to \( k \) as we did in Theorem 1, we need only sum over the subset comprising the discrete partitions summing to \( k \), which we have already seen is significantly less than \( p(k) \). Hence, the sum over all partitions in Eq. (10) simplifies drastically with all the frequencies lying between zero and unity, not between zero and \([k/i]\) for each element \( i \). Furthermore, since there is no outer series in the expansion of the product, we can drop \( q^N \) and \( a^N \) in Eq. (10). In addition, there is no multinomial factor. Consequently, we find that according to Eq. (10) the coefficients \( h_k \) in the power series for \( H(z) \) are given by

\[
h_k = \sum_{n_1, n_2, n_3, \ldots, n_k = 0}^{1,1,1,\ldots,1} \sum_{\sum_{i=1}^k n_i = k} \prod_{i=1}^k C_{n_i}^{n_i}.
\] (182)

The sum over partitions in the above result is simply the discrete partition operator as defined by Eq. (151). When this is introduced into Eq. (182), we arrive at Eq. (183). This completes the proof of the theorem.

It should also be mentioned that we can invert the rhs of Eq. (180) and apply Theorem 1 again. Then we obtain a power series expansion for \( 1/H(z) \). As a result, we find that

\[
H_k = L_{P,k} \left[ (-1)^N_k N_k! \prod_{i=1}^k \frac{h_i^{n_i}}{n_i!} \right],
\] (183)

while from Eq. (12), we obtain

\[
h_k = L_{DP,k} \left[ \prod_{i=1}^k C_i^{N_i} \right] = L_{P,k} \left[ (-1)^N_k N_k! \prod_{i=1}^k \frac{H_i^{n_i}}{n_i!} \right].
\] (184)
Since Theorem 3 is quite general, it means conversely that any power series expansion can be expressed as an infinite product of the form given by the rhs of Eq. (179). For example, as discussed on p. 111 of Ref. [18] the geometric series can be represented by the following product:

\[ \sum_{k=0}^{\infty} z^k = \prod_{i=0}^{\infty} \left(1 + z^{2^i}\right) \quad . \tag{185} \]

In this example \( C_{2i} = 1 \), while the other \( C_i \) simply vanish. This means that the product of \( \prod_{i=1}^{k} C_i^{m_i} \) in the \( h_k \) is either zero or unity. In addition, since the coefficients of the geometric series are equal to unity, we have \( L_{DP,k}[\prod_{i=1}^{k} C_i^{m_i}] = 1 \). Since all the terms in the discrete partition operator are either equal to zero and unity, this means that there can only be one discrete partition where all the elements are of the form \( 2^j \) with each value of \( j \) lying between zero and \( \log k / \log 2 \). For \( k = 5 \) and \( k = 7 \), these discrete partitions are respectively \{1,4\} and \{1,2,4\}, while for \( k = 11 \) it is \{1,2,8\}.

Another example of a well-known power series that can be expressed as an infinite product is the exponential function, which can be expressed as

\[ e^y = (1 + y)(1 + y^2/2)(1 - y^3/3)(1 + 3y^4/8)(1 - y^5/5)(1 + 13y^6/72) \times (1 - y^7/7)(1 + 27y^8/128)(1 - 8y^9/81)(1 + 91y^{10}/800) \cdots . \tag{186} \]

From Eq. (186) we find that \( C_1 = 1 \), while the other \( C_i \) can be determined by the following recurrence relation involving the divisors of \( i \):

\[ \sum_{d|i} \frac{(-1)^{d+1}}{d} C_{i/d}^d = 0 \quad . \tag{187} \]

This means that whenever \( i \) is a prime number greater than 2, say \( p \), we find that \( C_p = -1/p \). Eq. (187) is left as an exercise for the reader.

We can derive other interesting results by taking the logarithm of both Eqs. (179) and (180), which yields

\[ \ln \left(1 + \sum_{k=1}^{\infty} h_k z^k\right) = \sum_{i=1}^{\infty} \ln \left(1 + C_i z^i\right) \quad . \tag{188} \]

By introducing the Taylor/Maclaurin series expansion for logarithm into the rhs, we obtain

\[ \ln \left(1 + \sum_{k=1}^{\infty} h_k z^k\right) = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} C_i^j z^{ij} \quad , \tag{189} \]
where we have now assumed that $|z| < 1$. We now apply Theorem 1 to the left hand side of the above result. The inner series coefficients are given by $p_k = h_k$, while the outer series coefficients are given by $q_k = (-1)^{k+1}/k$. By equating like powers of $z$ in the resulting expression, we arrive at

$$L_{P,k} \left[ \left( -1 \right)^{N_k} (N_k - 1)! \prod_{i=1}^{k} \frac{h_i^{n_i}}{n_i!} \right] = \sum_{i|k} \frac{(-1)^i}{i} C_{k/i}^i .$$

(190)

For the particular case of the geometric series, viz. Eq. (185), $h_k = 1$ and $C_{2j} = 1$, while the other values of the $C_k$ vanish. Introducing these results into the above equation produces

$$L_{P,2j} \left[ \left( -1 \right)^{N_{2j}} (N_{2j} - 1)! \prod_{i=1}^{2j} \frac{1}{n_i!} \right] = \left( -\frac{1}{2} \right)^j .$$

(191)

If we put the $C_i$ equal to unity in Eq. (179), then we find that

$$\prod_{k=1}^{\infty} (1 + z^k) = 1 + \sum_{k=1}^{\infty} L_{DP,k} \left[ 1 \right] z^k .$$

(192)

In another words, the coefficients of the power series expansion for the product are given by the number of discrete partitions summing to $k$. Therefore, from Eq. (181) we have

$$h_k = L_{DP,k} \left[ 1 \right] .$$

(193)

On the other hand, if we put the $C_i = -1$, then we obtain

$$q(k) = L_{DP,k} \left[ (-1)^{N_k} \right] .$$

(194)

Prior to Eq. (162) it was mentioned that the $q(k)$ or discrete partition numbers are equal to $(-1)^j$ whenever $k$ is a pentagonal number [36] or is equal to $j(3j \pm 1)/2$. Hence, Eq. (194) can be expressed as

$$L_{DP,k} \left[ (-1)^{N_k} \right] = \begin{cases} (-1)^j, & \text{for } k = j(3j \pm 1)/2, \\ 0, & \text{otherwise} . \end{cases}$$

(195)

The above result tells us that the number of discrete partitions with an odd number of elements is equal to the number of discrete partitions with an
even number of elements when \( k \) is not a pentagonal number. It also tells us that the discrete partition numbers \( q(i) \) come in pairs of either 1 or -1, the former corresponding to a pentagonal number derived from an even number, i.e. by setting \( j \) equal an even number and the latter to a pentagonal number derived by an odd number. A value of 1 in the above result or for \( j \) an even number, means that the number of discrete partitions with an even number of elements is one more than the number of discrete partitions with an odd number of elements. A value of -1 or \( j \) odd, represents the opposite situation. A proof of this result based on Ferrers diagrams appears in Ch. 1 of Ref. [31].

Now if we put \( C_i = -\omega \) in Eq. (180), then the power series for the ensuing product becomes

\[
\prod_{k=1}^{\infty} \left( 1 - \omega z^k \right) = \frac{(-\omega; z)_\infty}{(1 - \omega)} = 1 + \sum_{k=1}^{\infty} q(k, -\omega) z^k ,
\]

(196)

where \((-\omega; z)_\infty\) is again the q-Pochhammer symbol presented in Eq. (160) and

\[
q(k, -\omega) = L_{DP,k} \left[ (-\omega)^{N_k} \right].
\]

(197)

Thus we see that \( q(k, -1) = q(k) \) or the discrete partition numbers. Note also that the power of \( \omega \) gives the number of elements in each discrete partition. By summing over all partitions each power of \( \omega \), say \( \omega^n \), in \( q(k, \omega) \) gives the total number of discrete partitions with \( n \) elements in them. A similar situation will arise when we study the inverse of Eq. (196). Moreover, by putting \( C_i = \omega \) in Eq. (180), we obtain

\[
Q(z, \omega) = \prod_{k=1}^{\infty} \left( 1 + \omega z^k \right) = \frac{\omega; z)_\infty}{(1 + \omega)} = 1 + \sum_{k=1}^{\infty} q(k, \omega) z^k ,
\]

(198)

where

\[
q(k, \omega) = L_{DP,k} \left[ \omega^{N_k} \right].
\]

(199)

The above polynomials will be referred to as the discrete partition polynomials. From Eq. (192) we have already seen that the \( \omega = 1 \) case gives the coefficients that represent the total number of discrete partitions summing to \( k \), viz. \( L_{DP,k}[1], \) while from the above the discrete partition numbers are given by \( q(k) = L_{DP,k}((-1)^{N_k}) \). Hence, we see that the difference between
Table 3: Discrete partition polynomials \( q(k, \omega) \) and partition function polynomials \( p(k, \omega) \).

| \( k \) | \( q(k, \omega) \) | \( p(k, \omega) \) |
|---|---|---|
| 0  | 1  | 1  |
| 1  | \( \omega \) | \( \omega \) |
| 2  | \( \omega \) | \( \omega^2 + \omega \) |
| 3  | \( \omega^2 + \omega \) | \( \omega^3 + \omega^2 + \omega \) |
| 4  | \( \omega^2 + \omega \) | \( \omega^4 + \omega^3 + 2\omega^2 + \omega \) |
| 5  | \( 2\omega^2 + \omega \) | \( \omega^5 + \omega^4 + 2\omega^3 + 2\omega^2 + \omega \) |
| 6  | \( \omega^3 + 2\omega^2 + \omega \) | \( \omega^6 + \omega^5 + 2\omega^4 + 3\omega^3 + 3\omega^2 + \omega \) |
| 7  | \( \omega^3 + 3\omega^2 + \omega \) | \( \omega^7 + \omega^6 + 2\omega^5 + 3\omega^4 + 4\omega^3 + 3\omega^2 + \omega \) |
| 8  | \( 2\omega^3 + 3\omega^2 + \omega \) | \( \omega^8 + \omega^7 + 2\omega^6 + 3\omega^5 + 5\omega^4 + 5\omega^3 + 4\omega^2 + \omega \) |
| 9  | \( 3\omega^3 + 4\omega^2 + \omega \) | \( \omega^9 + \omega^8 + 2\omega^7 + 3\omega^6 + 5\omega^5 + 6\omega^4 + 7\omega^3 + 4\omega^2 + \omega \) |
| 10 | \( \omega^4 + 4\omega^3 + 4\omega^2 + \omega \) | \( \omega^{10} + \omega^9 + 2\omega^8 + 3\omega^7 + 5\omega^6 + 7\omega^5 + 9\omega^4 + 8\omega^3 + 5\omega^2 + \omega \) |

The discrete partition numbers and the number of discrete partitions for a particular value of \( k \) is that for the latter the discrete partition operator acts on unity when scanning the discrete partitions summing to \( k \), while for the former it acts on \((-1)^{N_k}\), where \( N_k \) represents the number of elements in each discrete partition. That is, the difference is caused again by a phase difference in the number of elements in the discrete partitions summing to \( k \).

The discrete partition polynomials up to \( k = 10 \) are displayed in the second column of Table 3. As expected, for \( \omega = -1 \) these results reduce to the discrete partition numbers \( q(k, \omega) \), while for \( \omega = 1 \) they yield the number of discrete partitions. From the table it can be seen that the discrete partition polynomials are polynomials of degree \( n \), where \( n(n+1)/2 \leq k < (n+1)(n+2)/2 \) since the partition with the most discrete elements when \( k = n(n+1)/2 \) is \{1,2,3,..,n\}. The lowest order term in \( \omega \) corresponds to the only one-element partition, viz. \{k\}. If we run the program \texttt{dispart}, which is displayed in the appendix, then we find that there are 4 discrete partitions summing to 6, which are \{6\}, \{1,5\}, \{2,4\} and the self-conjugate \{1,2,3\}. By referring to Fig. 1 we see that the first partition is just one branch from the seed number, the second and third partitions are two branches away and the third is three branches away. Hence, we arrive at \( q(6, \omega) = \omega + 2\omega^2 + \omega^3 \), where the
magnitude of the coefficients of $\omega^k$ represent the number of distinct partitions with $k$ elements. That is, if the number of distinct elements summing to $k$ with $i$ elements in them are given by $q_i(k)$, then the polynomials can be expressed as

$$q(k, \omega) = \sum_{i=1}^{n} q_i(k) \omega^i ,$$

where $n(n+1)/2 \leq k < (n+1)(n+2)/2$. Hence, if we let $q(k, 1) = L_{DP,k}[1]$ or the number of distinct partitions summing to $k$, then we obtain the trivial equation of $q(k, 1) = \sum_{i=1}^{n} q_i(k)$. Moreover, the lower bound on $k$ gives us a limit as to the maximum number of elements that can appear in a discrete partition, which is given by

$$i_{\text{max}} = \left[ \frac{\sqrt{8k+1} - 1}{2} \right].$$

By fixing the number of elements to $i$ in the discrete partition operator so that it becomes

$$L_{DP,k,i}[\cdot] = \sum_{n_1,n_2,...,n_k=0 \atop \sum_{j=1}^{k} n_j = k \atop \sum_{j=1}^{k} n_j = i} ^{1,1,...,1} ,$$

we arrive at $L_{DP,k,i}[1] = q_i(k)$. This means that we need to input a second value into the program, which represents the number of elements the user desires. If this is set equal to a global variable called $\text{numparts}$, then the only changes to be made to $\text{dispart}$ are: (1) introduce a local variable $\text{sumpart}$ which adds all the values of $freq$ in the first for loop and (2) insert the following if statement before anything is printed out:

```python
if (sumparts != numparts) goto end;
```

When these modifications are implemented and the resulting code run for the discrete partitions summing to 100 with the number of elements, i.e. $\text{numparts}$, set equal to 5, one finds that there are 25,337 discrete partitions beginning with $\{1,2,3,4,90\}$ and ending with $\{18,19,20,21,22\}$. That is, $q_5(100)$ or $L_{DP,100,5}[1]$ is equal to 25337. According to Eq. (201), the maximum number of elements in the discrete partitions summing to 100 is 13. When the code is run for $\text{numparts}$ set equal to 13, 30($= q_{13}(100)$) partitions are printed out beginning with $\{1,2,3,4,...,12,22\}$ and ending with
Running the code for higher values of numparts with \( k = 100 \) does not result in any partitions being printed out. Hence, \( q_i(100) = 0 \) for \( i > 13 \), which is consistent with Eq. (201).

From Eq. (196) we have

\[
\prod_{k=1}^{\infty} (1 - \omega^2 z^{2k}) = \frac{(-\omega; z^2)_{\infty}}{(1 - \omega^2)} = 1 + \sum_{k=1}^{\infty} q(k, -\omega^2) z^{2k} .
\] (203)

The product on the lhs of this result can also be written as

\[
\prod_{k=1}^{\infty} (1 - \omega^2 z^{2k}) = \prod_{k=1}^{\infty} (1 - \omega z^k) (1 + \omega z^k) .
\] (204)

Introducing the rhs of Eq. (196) into the rhs of the above equation yields

\[
\prod_{k=1}^{\infty} (1 - \omega^2 z^{2k}) = \sum_{k=1}^{\infty} z^k \sum_{j=0}^{k} q(j, \omega) q(k - j, -\omega) .
\] (205)

The equals sign is only valid in Eqs. (203) and (205) for \( |\omega| < 1 \) and \( |z| < 1 \). By equating like powers of \( z \) on the rhs’s of both these equations, we find that for \( k \), an odd number equal to \( 2m+1 \),

\[
\sum_{j=0}^{2m+1} q(j, \omega) q(2m + 1 - j, -\omega) = 0 .
\] (206)

On the other hand, for \( k=2m \), we obtain

\[
\sum_{j=0}^{2m} q(j, \omega) q(2m - j, -\omega) = q(m, -\omega^2) .
\] (207)

When \( \omega = 1 \), Eq. (207) reduces to

\[
\sum_{j=0}^{2m} L_{DP, j} [1] L_{DP, 2m-j} [-1]^{N_{2m-j}} = L_{DP, m} [-1]^{N_m} .
\] (208)

From Eq. (195) we see that the lhs of the above equation is effectively a sum over the pentagonal numbers less than \( 2m \), while the rhs is non-zero if and only if \( m \) is a pentagonal number. Furthermore, if we let \( \omega \) equal the complex
number $i$, then Eq. (207) gives the number of discrete partitions summing to $m$ or the number of discrete partitions summing to $2m$ with even elements.

The foregoing analysis can also be extended by raising Eq. (196) to an arbitrary power $\rho$ and applying Corollary 1 to Theorem 1. Then we obtain a power series expansion like Equivalence (59), but in powers of $z$ with the coefficients depending upon $\rho$. Hence, we arrive at

$$\prod_{k=1}^{\infty} (1 - \omega z^k)^{\rho} \equiv 1 + \sum_{k=1}^{\infty} q(k, -\omega, \rho) z^k,$$  \hspace{1cm} (209)

where the coefficients $q(k, \omega, \rho)$ can determined from Eq. (60) with $D_i = q(i, \omega)$ and are given by

$$q(k, \omega, \rho) = L_{P,k} \left[ (-1)^{N_k} (-\rho)^{N_k} \prod_{i=1}^{k} q(i, \omega)^{n_i} \right].$$  \hspace{1cm} (210)

It should be emphasised that in the above results $\rho$ can be any value including a complex number. For integer values of $\rho$ greater than zero the equivalence symbol can be replaced by an equals sign. As we shall see shortly, when $\rho = -1$ and $\omega = 1$ in Equivalence (209), the $q(k, -1, -1)$ equal the partition function or $p(k)$, while if $\rho$ is equal to a positive integer, say $j$, then Eq. (210) simplifies drastically due to the fact that for $k > j$, the factor $(-\rho)^{N_k}$ vanishes for $N_k > j$. That is, the partitions with more than $j$ elements in them do not contribute to the $q(k, \omega, \rho)$. Moreover, if $\omega = -1$, further redundancy occurs in Eq. (210) since the $q(i, \omega)$ become the discrete partition numbers or $q(i)$, which we have seen are only non-zero when $i$ is a pentagonal number [36].

There is also another approach to developing a power series expansion to the generating function given on the lhs of Equivalence (209). This too involves the partition operator, but rather than consider the generating function in Equivalence (209), we shall investigate the more general case of Eq. (179) raised to an arbitrary power $\rho$ as set out in the following theorem.

**Corollary to Theorem 3.** The generalised version of the product in Equivalence (209) whereby the coefficients of $z^k$ are set equal to $C_k$ can be expressed as a power series or generating form given by

$$\prod_{k=1}^{\infty} (1 + C_k z^k)^{\rho_k} \equiv 1 + \sum_{k=1}^{\infty} B_k(\rho) z^k,$$  \hspace{1cm} (211)
while the coefficients in the series are found to be

\[ B_k(\rho) = L_{P,k} \left[ (-1)^N_k \prod_{i=1}^{k} \frac{(-\rho_i)_n_i}{n_i!} C_i^{n_i} \right] . \tag{212} \]

In the above results \((\rho)\) denotes \((\rho_1, \rho_2, \ldots, \rho_k)\). If \(S = \inf |C_i|^{-1/i} > 0\), then for \(|z| < S\) the equivalence symbol can be replaced by an equals sign. In addition, the coefficients \(B_k(\rho)\) satisfy the following relations:

\[ L_{P,k} \left[ (-1)^N_k N_k! \prod_{i=1}^{k} \frac{B_i(\rho)^{n_i}}{n_i!} \right] = L_{P,k} \left[ (-1)^N_k \prod_{i=1}^{k} \frac{(\rho_i)_n_i}{n_i!} C_i^{n_i} \right] , \tag{213} \]

and

\[ B_k(\mu + \nu) = \sum_{j=0}^{k} B_j(\mu) B_{k-j}(\nu) . \tag{214} \]

**Remark.** The reader should observe that in previous cases involving a constant power of \(\rho\) the Pochhammer symbol appeared outside the product in the partition operator as in Corollary 1 to Theorem 1. Thus, the \(\rho\)-dependence of the coefficients in the resulting power series was only affected by the total number of elements in each partition. In the above case each element \(i\) is assigned a value that is dependent upon \(\rho_i\) and consequently, \((\rho_i)_n_i\) appears inside the product being acted upon by the partition operator.

**Proof.** In order to prove the theorem, we shall use Lemma 1 again. Then the generating function can be expressed as

\[
\prod_{k=1}^{\infty} (1 + C_k z^k)^{\rho_k} \equiv \left(1 + \sum_{j=1}^{\infty} \frac{(-\rho_1)_j}{j!} (-C_1 z)^j\right) \left(1 + \sum_{j=1}^{\infty} \frac{(-\rho_2)_j}{j!} (-C_2 z^2)^j\right) \times \left(1 + \sum_{j=1}^{\infty} \frac{(-\rho_3)_j}{j!} (-C_3 z^3)^j\right) \left(1 + \sum_{j=1}^{\infty} \frac{(-\rho_4)_j}{j!} (-C_4 z^4)^j\right) \cdots \tag{215}\]

According to Lemma 1, each binomial series in the above product is absolutely convergent in the disk given by \(|z| < |C_i|^{-1/i}\). Each series is also conditionally convergent in a specific region of the complex plane. From the proof of Theorem 2, which deals with the \(\rho_i = -1\) and \(C_i = -1\) case, we
found that these regions can cancel each other when all the series appear in a product as in the above generating function. As a result, we were left with the region in which all the series are only absolutely convergent. Therefore, if \( S = \inf |C_i|^{-1/i} > 0 \), then all the series in Equivalence (215) are absolutely convergent whenever \(|z| < S\) and the equivalence symbol can be replaced by an equals sign. Furthermore, when the \( \rho_i \) all equal a positive integer, say \( k \), the binomial series become polynomials of degree \( k \). Then the equivalence symbol can be replaced by an equals sign for any value of \( z \).

Expanding the product of all the series in Equivalence (215) in powers of \( z \) yields

\[
\prod_{k=1}^{\infty} \left(1 + C_k z^k\right)^{\rho_k} \equiv 1 - (\rho_1)_1 C_1 z + \left(\frac{(-\rho_1)_2}{2!} C_1^2 - (\rho_2)_1 C_2\right) z^2
\]

\[
+ \left(-\frac{(-\rho_1)_3}{3!}\right) C_1^3 + (\rho_1)_1 (-\rho_2)_1 C_1 C_2 - (\rho_3)_1 C_3\right) z^3 + \cdots . \quad (216)
\]

From this result we see that there is one term appearing in the first order coefficient in \( z \), while there are two terms appear in the second order coefficient. The third order coefficient is composed of three terms. Had the fourth order term been displayed, there would have been five terms in the coefficient. In fact, the number of terms in the \( k \)-th order coefficient is the number of partitions summing to \( k \) or \( p(k) \). Therefore, we need to develop a means of coding the partitions as was done for the partition method for a power series expansion in Theorem 1.

Since the first order term corresponds to \( \{1\} \), we assign a value of \(-C_1\) to each occurrence of a one in a partition. The first term in the second order coefficient possesses a factor of \( C_1^2 \). Therefore, it must correspond to \( \{1,1\} \), while the other term must correspond to the other partition summing to 2, viz. \( \{2\} \). This term possesses a factor of \(-C_2\). Hence, each occurrence of a two in a partition yields a factor of \(-C_2\). If we continue this process indefinitely for the higher orders, then we find that each occurrence of an element \( i \) yields a factor \(-C_i\).

This, however, is not the complete story. Accompanying the factor of \(-C_1\) in the first order term is the factor of \((-\rho_1)_1\), while the first and second terms in the second order coefficient possess factors of \((-\rho_1)_2/2!\) and \((-\rho_2)_1\), respectively. That is, when there is one occurrence of an element \( i \) in a partition, its assigned value must be multiplied by \((-\rho_i)_1\), but if there are two occurrences of an element in a partition, then the assigned value must
be multiplied by \((-\rho_i)^2/2!\). Therefore, if an element \(i\) occurs \(n_i\) times in a partition, then it contributes a factor of \((-\rho_i)^2n_i / n_i!\) to the coefficient. This can be checked with the various terms comprising the third order term. The total contribution made by a partition is then given by the product over all elements, viz. \(\prod_{i=1}^{k} (-\rho_i)^2 n_i / n_i!\). Finally, the coefficient \(B_k(\rho)\) is evaluated by summing over all partitions summing to \(k\). Hence, we arrive at Eq. (212).

If we assume that \(z < S\), which is actually not necessary, we can invert the equation form of Equivalence (211), thereby obtaining

\[
\prod_{k=1}^{\infty} \left( 1 + C_k z^k \right)^{-\rho_k} = \frac{1}{1 + \sum_{k=1}^{\infty} B_k(\rho) z^k} .
\]  

Because the rhs can be regarded as the regularised value of the geometric series, we can apply the method for a power series expansion where the coefficients of the inner series, viz. \(p_k\) in Theorem 1, are equal to \(-B_k(\rho)\), while the coefficients of the outer series \(q_k\) are once again equal to \((-1)^k\). Then we have

\[
\prod_{k=1}^{\infty} \left( 1 + C_k z^k \right)^{-\rho_k} \equiv 1 + \sum_{k=1}^{\infty} D_k z^k ,
\]  

where according to Eq. (10), the coefficients of this expansion are given by

\[
D_k = L_{P,k} \left[ (-1)^N k ! \prod_{i=1}^{k} \frac{B_i(\rho) n_i}{n_i!} \right] .
\]  

We also know from earlier in the proof that the above product can be expressed in terms of a power series of the form given on the rhs of Equivalence (211) except that now the \(\rho_k\) are replaced by \(-\rho_k\). Because the resulting power series expansions possess the same regularised value, they are equal to one another. Moreover, since \(z\) is arbitrary, we can equate like powers of \(z\), thereby yielding Eq. (213).

The final identity is easily proved by noting that

\[
\prod_{i=1}^{\infty} \left( 1 + C_k z^k \right)^{\mu_k + \nu_k} = \prod_{i=1}^{\infty} \left( 1 + C_k z^k \right)^{\mu_k} \left( 1 + C_k z^k \right)^{\nu_k} .
\]  

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The lhs represents the regularised value for the series on the rhs of Equivalence (211) with coefficients $B_k(\mu + \nu)$, while the rhs represents the regularised value of the product of two series, one with coefficients $B_k(\mu)$ and the other with coefficients $B_k(\nu)$. As the regularised value is the same in both situations, we have

$$\begin{align*}
1 + \sum_{k=1}^{\infty} B_k(\mu + \nu) z^k &= \left(1 + \sum_{j=1}^{\infty} B_k(\mu) z^k\right) \left(1 + \sum_{k=1}^{\infty} B_k(\nu) z^k\right) \\
&= 1 + \sum_{k=1}^{\infty} z^k \sum_{j=0}^{k} B_j(\mu) B_{k-j}(\nu).
\end{align*}$$

(221)

Since $z$ is arbitrary, we can equate like powers yet again. Therefore, we obtain Eq. (214), which completes the proof.

In order to make the foregoing material clearer, we now consider a few examples. We have already generalised the discrete partition polynomials $q(k, \omega)$ by introducing the power of $\rho$ into their associated product as demonstrated by Equivalence (209). It was found that the new polynomials $q(k, \omega, \rho)$ could be expressed in terms of the partition operator acting on the discrete partition polynomials given by Eq. (210). Now we apply the corollary to Theorem 3 to the product with the $C_k$ and $\rho_k$ set equal to $\omega$ and $\rho$ respectively. Consequently, we find that

$$q(k, \omega, \rho) = L_{P,k} \left[ (-\omega)^N \prod_{i=1}^{k} \left( -\rho \right) \frac{n_i}{n_i!} \right].$$

(222)

So, let us evaluate $q(3, \omega, \rho)$ via Eqs. (210) and (222). In the case of Eq. (210) we require $q(1, \omega)$, $q(2, \omega)$ and $q(3, \omega)$, which have already been evaluated. Therefore, we obtain

$$q(3, \omega, \rho) = \left( -\rho \right)^3 \frac{1}{3!} \omega^3 + \left( -\rho \right)^2 \frac{1}{2} \omega^2 - \left( -\rho \right) \frac{1}{1} \omega + \omega.$$  

(223)

After a little algebra, the above becomes

$$q(3, \omega, \rho) = \left( \frac{\rho^3}{6} - \frac{\rho^2}{2} + \frac{\rho}{3} \right) \omega^3 + \rho^2 \omega^2 + \rho \omega.$$  

(224)

By setting $k=3$ in Eq. (222), we obtain

$$q(3, \omega, \rho) = \left( -\rho \right)^3 \frac{1}{3!} \omega^3 + \left( -\rho \right) \left( -\rho \right) \frac{1}{1} \omega + \left( -\rho \right) \omega.$$  

(225)
Again after a little algebra, we end up with Eq. (224), although we can see that Eqs. (223) and (225) are composed of different quantities. This demonstrates that while Eqs. (210) and (222) both possess the same sums over partitions, they are in fact different representations for $q(k, \omega, \rho)$.

The next example, which is a result attributed to Euler, appears on p. 56 of Ref. [18]. This is

$$\prod_{k=1}^{\infty} (1 - z^k)^3 = 1 - 3z + 5z^3 - 7z^6 + \cdots = \sum_{k=0}^{\infty} (-1)^k (2k + 1) z^{(k+1)/2} . \quad (226)$$

From Equivalence (209), we have

$$\prod_{k=1}^{\infty} (1 - z^k)^3 = 1 + \sum_{k=1}^{\infty} q(k, -1, 3) z^k . \quad (227)$$

Note that the equivalence symbol has been replaced by an equals sign because $\rho = 3$ in this case. Euler’s result shows that only when the power of $z$ is another type of figurate number called a triangular number [37, 38], are the coefficients of the power series or generating function non-zero. If we equate like powers of $z$ in both power series expansions given above, then we arrive at

$$L_{P,k} \left[ (-1)^{N_k} (-3)^{N_k} \prod_{i=1}^{k} q(i)^{m_i} / n_i! \right] = \begin{cases} (-1)^j (2j + 1), & \text{if } k = \binom{j+1}{2}, \\ 0, & \text{otherwise}. \end{cases} \quad (228)$$

On the other hand, putting $C_i = -1$ and $\rho_i = 3$ in Eq. (212) yields

$$q(k, -1, 3) = L_{P,k} \left[ \prod_{i=1}^{k} \frac{(-3)^{n_i}}{n_i!} \right] . \quad (229)$$

In the above result $(-3)^{n_i}$ is only non-zero for $n_i$ equal to 1, 2 and 3, in which case it equals -3, 6 and -6, respectively. This is a different type of restricted partition from those we have encountered previously since it means that partitions in which an element appears more than three times are excluded. To generate such partitions all that needs to be done is to scan each partition twice by introducing another for loop in termgen of the partition generating program presented towards the end of Sec. 2. If in the first scan the $n_i$ or the variable $freq$ is greater than 3, then a goto statement is required so that the
program avoids the next for loop, which is responsible for printing out the specific partitions. When such a code is constructed, one will find that out of the total of 627 partitions summing to 20, there are only 320 partitions in which all the elements occur at most three times. As a result of Eq. (229), we also arrive at

\[ L_{P,k} \left[ \prod_{i=1}^{k} \frac{(-3)^{n_i}}{n_i!} \right] = \begin{cases} (-1)^{i} (2j + 1), & \text{if } k = \left( \frac{j+1}{2} \right), \\ 0, & \text{otherwise.} \end{cases} \]  

Moreover, it should be noted that if \( \rho = 1 \) such that \( n_i \leq 1 \) for all elements \( i \) in the partition, then the sum over all partitions in Eq. (212) reduces to the discrete partition operator, viz. \( L_{DP,k}[\cdot] \), irrespective of the values for the coefficients \( C_i \).

If we put \( \rho = 1 \), take the cube power of the series on the rhs of Eq. (209) and equate like powers of the resulting series with those on the rhs of Eq. (227), then we obtain

\[ q(k,-1,3) = \sum_{j_1=0}^{k} \sum_{j_2=0}^{j_1} q(k - j_1)q(j_1 - j_2)q(j_2) , \]  

while multiplying the \( \rho = 1 \) and \( \rho = 2 \) versions of the series on the rhs of Eq. (209) yields

\[ q(k,-1,3) = \sum_{j=0}^{k} q(j)q(k - j,-1,2) = \begin{cases} (-1)^{i} (2i + 1), & k = \left( \frac{i+1}{2} \right), \\ 0, & \text{otherwise.} \end{cases} \]  

In the above result \( q(2,\omega,2) \) can be evaluated by putting \( \rho = 2 \) in Eq. (210). They can also be determined by equating like powers of \( z \) when taking the square of the series on the rhs of Equivalence (209) with the \( \rho = 2 \) series. This gives

\[ q(k,\omega,2) = \sum_{i=0}^{k} q(i,\omega)q(k - i,\omega) . \]  

For \( \omega = -1 \), Eq. (233) reduces to

\[ q(k,-1,2) = \sum_{i=0}^{k} q(i)q(k - i) . \]
### Table 4: Coefficients \( q(k, \omega, 2) \) and \( q(k, \omega, 3) \) in the power series expansions of the \( \rho = 2 \) and \( \rho = 3 \) cases of Equivalence (209).

| \( k \) | \( q(k, \omega, 2) \) | \( q(k, \omega, 3) \) |
|-------|-----------------|-----------------|
| 0     | 1               | 1               |
| 1     | \( 2\omega \)   | \( 3\omega \)   |
| 2     | \( 2\omega + \omega^2 \) | \( 3\omega + 3\omega^2 \) |
| 3     | \( 2\omega + 4\omega^2 \) | \( 3\omega + 9\omega^2 + \omega^3 \) |
| 4     | \( 2\omega + 5\omega^2 + 2\omega^3 \) | \( 3\omega + 12\omega^2 + 9\omega^3 \) |
| 5     | \( 2\omega + 8\omega^2 + 4\omega^3 \) | \( 3\omega + 18\omega^2 + 18\omega^3 + 3\omega^4 \) |
| 6     | \( 2\omega + 9\omega^2 + 10\omega^3 + \omega^4 \) | \( 3\omega + 21\omega^2 + 37\omega^3 + 12\omega^4 \) |
| 7     | \( 2\omega + 12\omega^2 + 14\omega^3 + 4\omega^4 \) | \( 3\omega + 27\omega^2 + 54\omega^3 + 33\omega^4 + 3\omega^5 \) |
| 8     | \( 2\omega + 13\omega^2 + 22\omega^3 + 9\omega^4 \) | \( -3\omega + 30\omega^2 + 81\omega^3 + 66\omega^4 + 12\omega^5 \) |
| 9     | \( 2\omega + 16\omega^2 + 30\omega^3 + 16\omega^4 \) | \( 3\omega + 36\omega^2 + 109\omega^3 + 114\omega^4 + 39\omega^5 \) |
| 10    | \( 2\omega^5 \) | \( \omega^6 + \omega^6 \) |
| 11    | \( 2\omega + 17\omega^2 + 40\omega^3 + 30\omega^4 \) | \( 3\omega + 39\omega^2 + 144\omega^3 + 189\omega^4 + 81\omega^5 + 9\omega^6 \) |

On p. 23 of Ref. [31] it is stated that Gauss derived the following result:

\[
\sum_{k=0}^{\infty} z^{(k^2+k)/2} = \prod_{k=1}^{\infty} \frac{1 - z^{2k}}{1 - z^{2k-1}} = \prod_{k=1}^{\infty} \left(1 - z^k\right) \left(1 + z^k\right)^2 . \tag{235}
\]

Once again, the equals sign is only valid for \(|z| < 1\). From this result we see that only the powers of \( z \) equal to \((j^2 + j)/2\), where \( j \) is any non-negative integer, possess a non-zero coefficient. For any power \( k \) we obtain contributions from all the partitions summing to \( k \) with only odd elements in them and from the discrete partitions summing to \( k \) consisting of only even elements. If the number of even elements in these discrete partitions is even, then the partition will yield a value of unity. Otherwise, it will yield a value of -1. In addition, the partition function possesses mixed partitions composed of discrete partitions with only even elements and standard partitions with only odd elements. The values contributed by the mixed partitions to the coefficients of the power series depend upon the number of elements in the discrete partitions. For example, if we consider \( z^{10} \) or \( k = 4 \) on the lhs of Eq. (235), then it will be composed of the contributions due to the partitions...
summing to 10 with only odd elements. There are 10 of these beginning with \( \{1,9\} \) and ending with \( \{5,5\} \). Hence, these partitions contribute a value of 10 to the coefficient. On the other hand, there are only 3 discrete partitions with even numbers summing to 10. These are \( \{10\} \), \( \{2,8\} \) and \( \{4,6\} \). Since the last two possess an even number of elements, they each contribute a value of unity, while the single element partition gives a value of -1. Overall, the discrete partitions summing to 10 contribute a value of unity to the the coefficient, which now becomes 11 when the contribution from the standard partitions summing to 10 with only odd elements is included. However, when the discrete partition is \( \{2\} \), we need to consider the partitions summing to 8 with odd elements. There are six of these, beginning with \( \{1,7\} \) and ending with \( \{3,5\} \). Because there is only one element in the discrete partition, these mixed partitions contribute a value of -6 to the coefficient, which now drops to 5. However, there are still more mixed partitions. We need to consider the discrete partition of \( \{4\} \). In this case we need the partitions summing to 6 with odd elements. There are four of these, beginning with \( \{1,5\} \) and ending with \( \{3,3\} \). In this instance the mixed partitions contribute a value of -4, yielding a value of 1 for the coefficient as indicated above. We do not need to consider the contributions where the discrete partitions sum to either 6 or 8 because in these cases there are only 2 partitions, one of which has two elements and one with one element. Hence, they cancel each other yielding a value of 0.

If we introduce Eq. (209) into the Eq. (235), then we find that

\[
\sum_{k=0}^{\infty} z^{k^2+k/2} = 1 + \sum_{k=0}^{\infty} z^k \sum_{j=0}^{k} q(j)q(k-j,1,2) .
\] (236)

By equating like powers of \( z \), we obtain for \( i \), a positive integer,

\[
\sum_{j=0}^{k} q(j)q(k-j,1,2) = \begin{cases} 
1, & k = (i^2 + i)/2, \\
0, & \text{otherwise.}
\end{cases}
\] (237)

Therefore, when \( k \) is not a triangular number \( \text{[37]} \), by combining the above result with Eq. (232) we arrive at

\[
\sum_{j=0}^{k} q(j) \left( q(k-j,1,2) \pm q(k-j,-1,2) \right) = 0 .
\] (238)
8 Other Products

By using the material of the previous section we are in a position to study more advanced products. We begin by introducing the variable \( \omega \) next to the power of \( z \) in the denominator of \( P(z) \). Based on the similar extension of the product yielding discrete partitions in the previous section, we expect to obtain polynomials as the coefficients of the resulting generating function. Therefore, we define the new product as

\[
P(z, \omega) = \prod_{k=1}^{\infty} \frac{1}{(1 - \omega z^k)}.
\]  

(239)

Obviously, when \( \omega = 1 \), this reverts to the generating function of the partition function given by Equivalence (73), which we have seen becomes an equation when \( |z| < 1 \). According to p. 112 of Ref. [18], the product in the above result can be written alternatively as

\[
P(z, \omega) = (1 - \omega) \sum_{k=0}^{\infty} \frac{\omega^k z^{k^2}}{(z; z)_k (\omega; z)_{k+1}},
\]  

while inversion of the rhs of Eq. (196) yields

\[
P(z, \omega) = \frac{1}{1 + \sum_{k=1}^{\infty} q(k, -\omega) z^k}.
\]  

(241)

We have seen that \( P(z) \) or \( \omega = 1 \) in Eq. (239) yields a power series expansion whose coefficients are given by the partition function or \( p(k) \). This expansion is obtained by expanding each term in the generating function into the geometric series for each value of \( i \). It is this value in the generating function, which is responsible for yielding the specific elements in a partition, while the power of \( z^i \) in each geometric series represents the frequency or number of occurrences of the element in the partition. For example, multiplying \((z^2)^3\) in the expansion of \(1/(1 - z^2)\) by \((z^3)^4\) in the expansion of \(1/(1 - z^3)\) means that the partition has three twos and four threes in it. By introducing \( \omega \) into the generating function as indicated above, we see that the overall power of \( \omega \) yields the total number of elements in a partition. In the example just mentioned we now obtain \((\omega z^2)^3\) multiplied by \((\omega z^3)^4\), which yields \(\omega^7 z^{18}\). The power of 7 on \( \omega \) represents the total number of twos and threes in the partition. Therefore, we expect the coefficient of each power of
\( \omega \) in the coefficients of the resulting generating function to indicate the total number of partitions where the number of elements equals the power of \( \omega \).

By regarding the rhs as the regularised value of the geometric series and the \( q(k, -\omega) \) as the inner series whose coefficients are \( p_k \), we can apply Theorem 1 to the last form for \( P(z, \omega) \) with \( y = z \). If we express the generating function as a power series expansion with coefficients, \( p(k, \omega) \), i.e.

\[
P(z, \omega) = \prod_{k=1}^{\infty} \frac{1}{(1 - \omega^k z^k)} \equiv 1 + \sum_{k=1}^{\infty} p(k, \omega) z^k ,
\]

then according to Eq. (10) the coefficients of the resulting power series expansion are given by

\[
p(k, \omega) = L_{P,k} \left[ (-1)^{N_k} N_k! \prod_{i=1}^{k} \frac{q(i, -\omega) n_i}{n_i!} \right] .
\]

Furthermore, if we put \( \rho = -1 \) and \( C_i = -\omega \) in the corollary to Theorem 3, then we observe that the \( p(k, \omega) \) become the coefficients \( B_k(-1) \) given by Eq. (212). Thus, we arrive at

\[
p(k, \omega) = L_{P,k} \left[ \omega^{N_k} \right] .
\]

This tells us that the coefficients in the \( p(k, \omega) \) will be the number of partitions summing to \( k \) where each power of \( \omega \) corresponds to the number of elements in the partitions. That is,

\[
p(k, \omega) = \sum_{i=1}^{k} p_i(k) \omega^i ,
\]

where from Sec. 2, \( p_i(k) = \binom{k}{i} \). It has already been stated the sub-partition numbers obey the recurrence relation given by Eq. (1). In terms of the fixed number of elements operator defined by Eq. (88) we also find that

\[
p_i(k) = L_{P,k}^i \left[ 1 \right] .
\]

The first few partition function polynomials are found to be: \( p(0, \omega) = 1 \), \( p(1, \omega) = \omega \), \( p(2, \omega) = \omega^2 + \omega \), \( p(3, \omega) = \omega^3 + \omega^2 + \omega \) and \( p(4, \omega) = \omega^4 + \omega^3 + \omega^2 + \omega \).
In fact, those up to $k = 10$ are displayed in the third column of Table 3. Since the coefficients of these polynomials represent the number of partitions in which the number of elements is given by the power of $\omega$, the highest order term of these polynomials is $k$, which arises from the $k$-element partition of $\{1, 1, \ldots, 1_k\}$. The other partitions are unable to provide an $\omega^k$ term because the highest order of all other $q(k, \omega)$ is less than $k$. The partition $\{1, 1, \ldots, 1_k-1, 2\}$ only produces an $\omega^{k-1}$ term as its highest order term because the highest order term of $q(2, \omega)$ is 1. Hence, $\deg p(k, \omega) = k$.

Conversely, the lowest order term in the $p(k, \omega)$ is the lowest order term in $q(k, \omega)$ stemming from the single element partition $\{k\}$, which is unity. Therefore, $p_k(k) = p_{k-1}(k) = p_1(k) = 1$, $p_{k-2}(k) = 2$ and $p(k, 1) = \sum_{i=1}^{k} p_i(k) = p(k)$. Moreover, the total number of partitions with even elements is given by $\sum_{i=1}^{\lfloor k/2 \rfloor} p_{2i}(k)$, while the total number of partitions with odd elements is equal to $\sum_{i=1}^{m} p_{2i-1}(k)$, where $m = \lfloor k/2 \rfloor$ when $k$ is even and $m = \lfloor k/2 \rfloor + 1$ when $k$ is odd. On the other hand, the coefficient $p_2(k)$ can be evaluated by noting that it represents the product of the two lowest order terms in each partition, namely $\{j, k - j\}$, where $j$ ranges from 1 to $\lfloor k/2 \rfloor$. According to Ref. [39], the number of two-element partitions summing to $k$ is given by $p_2(k) = [k/2]$, while the number of three-element partitions is given by $p_3(k) = [k^2/12]$ for $k > 3$. Moreover, a table of the sub-partition numbers for $k$ and $i$ ranging from 0 to 11 is presented on p. 46 of Ref. [18]. All these results agree with those obtained via Eq. (181), confirming that the latter result does yield the number of partitions summing to $k$ with $i$ elements in them.

An interesting property of the partition number polynomials can be derived by setting $z = z^2$ and $\omega = \omega^2$ in Equivalence (240). Then we obtain

$$P(z^2, \omega^2) = \prod_{k=1}^{\infty} \frac{1}{(1 - \omega^2 z^{2k})} \equiv 1 + \sum_{k=1}^{\infty} p(k, \omega^2) z^{2k}. \quad (247)$$

The quantity on the lhs of the above equivalence can also be written

$$P(z^2, \omega^2) = P(z, \omega)P(z, \omega) \quad (248)$$

Introducing Equivalence (242) into the above equation yields

$$P(z, \omega)P(z, -\omega) \equiv \left(1 + \sum_{k=1}^{\infty} p(k, \omega) z^k\right) \left(1 + \sum_{k=1}^{\infty} p(k, -\omega) z^k\right). \quad (249)$$

Since the series on the rhs’s of Equivalences (247) and (249) are derived from the identity given by Eq. (248), they are equal to one another in accordance
with the concept of regularisation [3, 7, 8, 9]. Because \( z \) is arbitrary, once again we can equate like powers of \( z \). Therefore, we obtain

\[
\sum_{j=0}^{2k+1} p(j, \omega) p(2k + 1 - j, -\omega) = 0 , \quad (250)
\]

and

\[
\sum_{j=0}^{2k} p(j, \omega) p(2k - j, -\omega) = p(2k, \omega^2) . \quad (251)
\]

As in the previous section we can generalise the foregoing analysis by replacing \(-\omega\) in \( P(z, \omega) \) by \( C_i \), which effectively represents the inversion of Eq. (179). From Theorem 2 we know that

\[
H(z)^{-1} = \prod_{i=1}^{\infty} \frac{1}{(1 + C_i z^i)} \equiv (1 - C_1 z + C_1^2 z^2 + \cdots)(1 - C_2 z^2 + C_2^2 z^4 + \cdots) \\
\times (1 - C_3 z^3 + C_3^2 z^6 + \cdots)(1 - C_4 z^4 + C_4^2 z^8 + \cdots) \cdots , \quad (252)
\]

where the equivalence symbol can be replaced by an equals sign provided \(|C_i z^i| < 1\) for all \( i \). Expanding the above yields

\[
H(z)^{-1} \equiv 1 - C_1 z + (C_1^2 - C_2) z^2 + (-C_1^3 + C_2 C_1 - C_3) z^3 + \cdots . \quad (253)
\]

Each coefficient of \( z^k \) in the above power series is composed of contributions that can be related to the partitions summing to \( k \) as was the case in the proof of Theorem 1. The major difference between the above situation and that in the proof of Theorem 1 is that there is no multinomial factor associated with each contribution made by a partition as we also found in the proof to the corollary to Theorem 3. Furthermore, each element \( i \) in a partition is now assigned a value of \(-C_i\) so that the above result becomes

\[
H(z)^{-1} \equiv \sum_{k=0}^{\infty} H_k z^k , \quad (254)
\]

where \( H_0 = 1 \), and

\[
H_k = L_{P,k} \left[ (-1)^{N_k} \prod_{i=1}^{k} C_i^{n_i} \right] . \quad (255)
\]
The above result represents the case when the $\rho_k$ in the corollary to Theorem 3 are set equal to -1.

If the $C_i$ are set equal to unity in Equivalence (252), then we find that
\[
\prod_{i=1}^{\infty} \frac{1}{1 + z^i} = 1 + \sum_{k=1}^{\infty} L_{P,k} \left[ (-1)^N_k \right] z^k ,
\]
(256)
where the equivalence symbol can be replaced by an equals sign for $|z| < 1$.

Therefore, the coefficients on the power series expansion on the rhs represent the difference between the number of even- and odd-element partitions summing to $k$. In addition, the above equivalence is analogous to putting the $C_i$ equal to unity in Eq. (179) and applying Theorem 1 to its inverted form. In this case the coefficients of the inner series are given by $p_k = L_{DP,k}[1]$, the number of discrete partitions, while the coefficients of the outer series are given by $q_k = (-1)^k$. Then the coefficients of the power series expansion on the rhs of Equivalence (256) can be expressed in terms of the number of discrete partitions as
\[
L_{P,k} \left[ (-1)^N_k \right] = L_{P,k} \left[ N_k! \prod_{i=1}^{k} L_{DP,i} \left[ 1 \right]^{n_i!} / n_i! \right] .
\]
(257)

If we multiply the product on the lhs of Equivalence (256) by $(1 - z^i)$ in both the numerator and denominator, then we find that
\[
\prod_{i=1}^{\infty} \frac{1}{1 + z^i} = \prod_{i=0}^{\infty} (1 - z^{2i+1}) .
\]
(258)
When the product on the rhs of Eq. (258) appeared in powers of $z^i$ rather than $z^{2i+1}$, we saw that the resulting power series possessed coefficients which were equal to the number of discrete partitions summing to $k$. In the above result all the even powers are now missing. This means that the coefficients of the resulting power series will be the number of discrete partitions with only odd elements in them. That is,
\[
\prod_{i=0}^{\infty} (1 - z^{2i+1}) = 1 + \sum_{k=0}^{\infty} (-1)^k L_{ODP,k} \left[ 1 \right] z^k ,
\]
(259)
where ODP denotes that only partitions with odd elements are to be considered in the sum over partitions. That is, $n_{2i} = 0$ for all values of $i$. The
phase factor of \((-1)^k\) in the series expansion arises from the fact that only an even number of odd discrete elements yields an even power of \(z\), while only an odd number of discrete elements yields an odd power of \(z\). That is,

\[ (-1)^k L_{ODP,k}[1] = L_{DP,k} \left[ \prod_{i=1}^{[k/2]} (-1)^{n_{2i}-1} \right]. \]  

(260)

In a similar fashion we arrive at

\[ \prod_{i=0}^{\infty} (1 + z^{2i}) = 1 + \sum_{k=1}^{\infty} L_{EDP,2k}[1] z^{2k}, \]  

(261)

where only even elements are to be considered in the even discrete partition operator, i.e \(n_{2i+1}=0\) for all \(i\). Alternatively, we can replace the even discrete partition operator by the discrete partition operator since

\[ L_{EDP,2k}[1] = L_{DP,k}[1]. \]  

(262)

Moreover, by equating like powers of \(z\) in the power series on both rhs’s of Equivalences (256) and (259), we arrive at

\[ L_{P,k} \left[ (-1)^{N_k} \right] = (-1)^k L_{ODP,k}[1]. \]  

(263)

From this result we see that when \(k\) is even, the number of even partitions is greater than the number of odd partitions, while for odd values of \(k\), the opposite applies. Multiplying both sides by \((-1)^k\) results in taking the absolute value or modulus of the lhs. Thus, the above statement tells us that absolute value of the difference between the number of even and odd partitions is equal to the number of discrete partitions with only odd elements in them or the number of partitions with distinct odd parts, a result first proved by Euler according to p. 14 of Ref. [31].

As a consequence of the previous section, it is a relatively simple exercise to produce a code that evaluates the difference between the number of partitions with an even number of elements and those with an odd number of elements. Two new global variables are required, one for evaluating the difference as each partition is scanned and another that is either equal to 1 or -1 depending on whether there is an even number of elements or an odd number of elements. Once the second value is determined, it needs to be
Figure 5: The difference between even and odd partitions to the total number of partitions summing to \( k \) versus \( k \) added to the first global variable in the main function prototype. The second global variable must be evaluated in the `termgen` function prototype after the for loop has been altered to calculate the total number elements in the partition, which is determined by summing all components of `part`. As a consequence, one finds after running the code for several values of \( k \) that

\[
\left| L_{P,k} \left[ (-1)^{N_k} \right] \right| \geq \left| L_{P,j} \left[ (-1)^{N_j} \right] \right| ,
\]

for \( k \geq j \). Fig. 5 presents the graph of the ratio of the absolute value of the difference between odd and even partitions summing to \( k \) to the total number of partitions or \( p(k) \) for \( k \leq 50 \). Whilst the absolute value of the \( L_{P,k} \left[ (-1)^{N_k} \right] \) increases with \( k \), we see that in relation to the total number of partitions the ratio decreases monotonically, once \( k \) exceeds 15.

From Eqs. (192) and (194) we see that the infinite products, \( \prod_{k=1}^{\infty} (1 + z^k) \) and \( \prod_{k=1}^{\infty} (1 - z^k) \) can be expressed in terms of a series in successive powers of \( z \), where the coefficients are equal to \( L_{DP,k}[1] \) and \( L_{DP,k} \left[ (-1)^{N_k} \right] \) respectively. The first product can also be written as

\[
\prod_{k=1}^{\infty} (1 + z^k) = \prod_{k=1}^{\infty} \frac{1}{(1 - z^{2k-1})} .
\]
Eq. (265) is easily obtained by manipulating the rhs after multiplying it by \( (1 - z^{2k}) / (1 - z^{2k}) \). If we put \( C_i = -1 \) in Equivalence (252), then according to Equivalence (254) and Eq. (255) we arrive at

\[
\prod_{k=1}^{\infty} \left( 1 + z^{k} \right) = 1 + \sum_{k=1}^{\infty} L_{OEP,k} \left[ 1 \right] z^{k} ,
\] (266)

where \( L_{OEP,k}[\cdot] \) represents the odd element partition operator, which we have seen has two different forms given by Eqs. (152) and (153) depending upon whether \( k \) is an even or odd number. Because the generating function on the lhs also yields a power series whose coefficients represent the number of discrete partitions summing to \( k \), we see immediately that

\[
L_{DP,k} \left[ 1 \right] = L_{OEP,k} \left[ 1 \right] .
\] (267)

Hence, the number of discrete partitions is equal to the number of partitions composed only of odd elements, another result attributed to Euler according to p. 5 of Ref. [31].

By putting \( C_i = -1 \) in Eq. (179), we obtained Eq. (194). If we put \( C_i = -1 \) and \( z = z^2 \), then the product on the lhs of Eq. (194) yields a series expansion in powers of \( z^2 \), but now the coefficients represent the number of distinct partitions summing to \( 2k \) with only even elements operating on \((-1)^{N_{2k}}\). Alternatively, this is equivalent to the number of distinct partitions summing to \( k \) operating on \((-1)^{N_{k}}\). Therefore, we can write

\[
\prod_{k=1}^{\infty} \left( 1 - z^{2k} \right) = 1 + \sum_{k=1}^{\infty} L_{EDP,2k} \left[ (-1)^{N_{2k}} \right] z^{2k}
\]

\[
= 1 + \sum_{k=1}^{\infty} L_{DP,k} \left[ (-1)^{N_{k}} \right] z^{2k} .
\] (268)

In Eq. (268) \( L_{EDP,2k}[\cdot] \) denotes the even discrete partition operator, which acts on the number of discrete partitions summing to \( 2k \) where the elements are only even integers. This is opposed to the odd discrete partition operator, \( L_{ODP,k}[\cdot] \), where the elements are odd numbers and discrete, but can sum to both even and odd integers. Since the infinite product of \((1 - z^{2k})\) is the product of two separate infinite products involving \((1 - z^k)\) and \((1 + z^k)\), we have

\[
\prod_{k=1}^{\infty} \left( 1 - z^{2k} \right) = \left( 1 + \sum_{k=1}^{\infty} L_{DP,k} \left[ (-1)^{N_{k}} \right] z^{k} \right) \left( 1 + \sum_{k=1}^{\infty} L_{DP,k} \left[ 1 \right] z^{k} \right) .
\] (269)
Equating like powers of $z$ in Eq. (269) with those in Eq. (268) yields

\[ L_{DP,k} \left[ (-1)^{N_k} \right] = \sum_{j=0}^{2k} L_{DP,j} \left[ (-1)^{N_j} \right] L_{DP,2k-j} \left[ 1 \right], \quad (270) \]

and

\[ \sum_{j=0}^{2k+1} L_{DP,j} \left[ (-1)^{N_j} \right] L_{DP,2k+1-j} \left[ 1 \right] = 0. \quad (271) \]

Since $L_{DP,k} \left[ (-1)^{N_k} \right]$ and $L_{DP,k}[1]$ are equal to $q(k)$ and $q(k,1)$ respectively, Eqs. (270) and (271) can also be written as

\[ q(k) \left( 1 - q(k,1) \right) = \sum_{j=0}^{k-1} \left( q(j)q(2k - j, 1) + q(2k - j)q(j, 1) \right), \quad (272) \]

and

\[ \sum_{j=0}^{k} \left( q(j)q(2k + 1 - j, 1) + q(2k + 1 - j)q(j, 1) \right) = 0. \quad (273) \]

In these results it should be borne in mind that $q(0) = q(0,1) = 1$. Isolating the $j=0$ terms in the above equations yields

\[ q(2k,1) + q(2k) = q(k) \left( 1 - q(k,1) \right) - \sum_{j=1}^{k-1} \left( q(j)q(2k - j, 1) \right. \]

\[ \left. + q(2k - j)q(j, 1) \right), \quad (274) \]

and

\[ q(2k + 1, 1) + q(2k + 1) = \sum_{j=1}^{k} \left( q(j)q(2k + 1 - j, 1) \right. \]

\[ \left. + q(2k + 1 - j)q(j, 1) \right). \quad (275) \]

Eqs. (274) and (275) represent the recurrence relations for determining the number of discrete partitions or $q(k,1)$. Like the Euler/MacMahon recurrence relation given by Eq. (165), they utilise the discrete partition numbers
or \( q(k) \) and consequently, most of the terms in the sums vanish when the summation index \( j \) is not equal to a pentagonal number.

It should also be noted that the analysis resulting in Eq. (169) can be adapted to provide another representation for the partition number polynomials or \( p(k, \omega) \). First, we re-write the generalised product in Equivalence (242) as

\[
P(z, \omega) = \exp \left( \sum_{m=1}^{\infty} \sum_{j=1}^{\infty} \omega^j z^{mj/j} \right),
\]

where now it is assumed that \(|\omega z| < 1\). Consequently, the modified version of Eq. (169) becomes

\[
P(z, \omega) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \left( \omega z + (\omega + \omega^2/2) z^2 + (\omega + \omega^3/3) z^3 + (\omega + \omega^2/2 + \omega^4/4) z^4 + \cdots + \gamma_j(\omega) z^j + \cdots \right)^k,
\]

where \( \gamma_j(\omega) = \sum_{d|j}(d/j) \omega^{j/d} \) and \( d \) represents a divisor of \( j \) as before. The first few divisor polynomials are: \( \gamma_0(\omega) = 1 \), \( \gamma_1(\omega) = \omega \), \( \gamma_2(\omega) = \omega + \omega^2/2 \), and \( \gamma_3(\omega) = \omega + \omega^3/3 \). As expected, for \( \omega = 1 \) the \( \gamma_j(\omega) \) reduce to the \( \gamma_j \) below Eq. (169). This means that Eq. (170) can be generalised to

\[
q(k, -\omega) = L_{P,k} \left[ (-1)^{N_k} \prod_{i=1}^{k} \frac{\gamma_i(\omega)^{n_i}}{n_i!} \right].
\]

Furthermore, the \( \gamma_j(\omega) \) are polynomials in \( \omega \) whose highest and lowest orders are respectively \( j \) and unity. That is, deg \( \gamma_j(\omega) = j \). We shall refer to these unusual polynomials as the divisor polynomials. By applying Theorem 1 to Eq. (277) with the coefficients of the inner and outer series set equal to \( \gamma_k(\omega) \) and \( 1/k! \) respectively, we arrive at

\[
p(k, \omega) = L_{P,k} \left[ \prod_{i=1}^{k} \frac{\gamma_i(\omega)^{n_i}}{n_i!} \right].
\]

Since deg \( \gamma_i(\omega) = i \) and \( \prod_{i=1}^{k} \omega^{n_i} = \omega^k \), the highest order term in the \( p(k, \omega) \) is \( k \). On the other hand, since the lowest order term in the \( \gamma_i(\omega) \) is \( \omega \),
\[ \prod_{i=1}^{k} \omega_i^{n_i} = \omega^{N_k} \text{ and } N_k \text{ ranges from unity to } k, \text{ the lowest order term in the } p(k, \omega) \text{ is unity, again confirming that the partition number polynomials are polynomials in } \omega \text{ with } \deg p(k, \omega) = k. \]

The sixth and last program presented in the appendix is called \texttt{dispfnpoly}. It prints out both \( q(k, -\omega) \) and \( p(k, \omega) \) in symbolic form so that it can be handled by Mathematica \cite{12}. To run this program, the user must specify the order \( k \) of the polynomials. The program is different from the other programs in the appendix since it is not required to determine the factorial of the total number of distinct parts, i.e. \( N_k! \). When the global variable \texttt{polytype} is equal to unity in the for loop in \texttt{main}, the program determines the discrete partition polynomial for the order specified by the user. When it becomes equal to 2, the program determines the corresponding partition function polynomial. E.g., for \( k = 6 \), the following output is generated:

\[
Q[6, -w_] := DP[6, w] (-1) + DP[1, w] DP[5, w] (-1)^2 + DP[1, w]^2 DP[4, w] (-1)^3/2! + DP[1, w]^3 DP[3, w] (-1)^4/3! + DP[1, w]^4 DP[2, w] (-1)^5/4! + DP[1, w]^5 DP[2, w] (-1)^6/6! + DP[1, w]^6 DP[2, w] DP[4, w] (2) (-1)^4/(2! 2!) + DP[1, w] DP[2, w] DP[3, w] (-1)^3 + DP[2, w] DP[4, w] DP[3, w] (-1)^2 + DP[4, w]^2 (-1)^2/3! + DP[3, w]^2 (-1)^2/2!
\]

\[
P[6, w_] := DP[6, w] + DP[1, w] DP[5, w] + DP[1, w]^2 DP[4, w] /2! + DP[1, w]^3 DP[3, w] /3! + DP[1, w]^4 DP[2, w] /4! + DP[1, w]^5 DP[3, w] /5! + DP[1, w]^6 DP[2, w] /6! + DP[1, w]^7 DP[3, w] /7! + DP[1, w]^8 DP[2, w] DP[4, w] /8! + DP[1, w]^9 DP[2, w] DP[3, w] /9! + DP[1, w]^10 DP[2, w] DP[4, w] /10! + DP[1, w]^11 DP[3, w] /11!
\]

Time taken to compute the coefficients is 0.000000 seconds.

The terms on the rhs denoted by \( DP[k,w] \) represent the divisor polynomials of order \( k \). These can be obtained by typing in the following line in Mathematica:

\[
DP[k, -w_] := \text{Sum}[w^d/d, \{d, \text{Divisors}[k] \}].
\]

From Eqs. (278) and (279) we see that the discrete partition polynomials or rather the \( q(k, -\omega) \) are almost identical to the partition function polynomials or \( p(k, \omega) \) when they are both expressed in terms of the divisor polynomials, the only difference being a phase factor that appears in the former. This factor is positive when the number of elements in a partition is even, but is negative when there is an odd number of elements in a partition. Hence, the only difference between the two sets of polynomials occurs for
odd partitions or those with an odd number of elements.

If we multiply Eq. (196) with Eq. (242), then by equating like powers on both sides of the resulting equation we obtain

\[ \sum_{j=0}^{k} q(j, -\omega) p(k - j, \omega) = 0 \]  
(280)

Alternatively, Eq. (280) can be expressed as

\[ \sum_{j=0}^{k} L_{P,j} \left[ \left( -1 \right)^{N_{j}} \prod_{i=1}^{j} \frac{\gamma_{i}(\omega)^{n_{i}}}{n_{i}!} \right] L_{P,k-j} \left[ \prod_{i=1}^{k-j} \frac{\gamma_{i}(\omega)^{n_{i}}}{n_{i}!} \right] = 0 \]  
(281)

These results represent the generalisation of the Euler/MacMahon recurrence relation given by Eq. (165).

We can also generalise the product in Equivalence (242) by introducing the arbitrary power of \( \rho \) as we did in the discrete partition case of Equivalence (209). Thus, the generalised product becomes

\[ P_{\rho}(z, \omega) = \prod_{k=1}^{\infty} \frac{1}{(1 - \omega z^{k})^{\rho}} \equiv 1 + \sum_{k=1}^{\infty} p(k, \omega, \rho) z^{k} \]  
(282)

In this case the coefficients \( p(k, \omega, \rho) \) in the generating function can be determined by setting the \( D_{i} \) equal to \( p(i, \omega) \) in Eq. (60). Then one finds that

\[ p(k, \omega, \rho) = L_{P,k} \left[ \left( -1 \right)^{N_{k}} (-\rho)^{N_{k}} \prod_{i=1}^{k} \frac{\gamma_{i}(\omega)^{n_{i}}}{n_{i}!} \right] \]  
(283)

As a consequence of the preceding analysis, we are now in a position to study more advanced products. In particular, let us consider the following quotient:

\[ P(z, \beta \omega, \alpha \omega) = Q(z, -\beta \omega) P(z, \alpha \omega) = \prod_{k=1}^{\infty} \frac{(1 - \beta \omega z^{k})}{(1 - \alpha \omega z^{k})} \]  
(284)

In deriving a power series expansion or generating function for the above product we expect the power of \( \omega \) in the coefficients to yield the total number of elements in the partitions summing to the power of \( z \). Furthermore, the power of \( \beta \) in the coefficients should represent the number of elements due
to the discrete partitions, while the power of $\alpha$ should give the number of elements due to the standard partitions. By adopting the same approach as for the other infinite products that have already been presented in this section, we can express Eq. (284) as

$$P(z, \beta \omega, \alpha \omega) = \sum_{k=0}^{\infty} q(k, -\beta \omega) z^k \sum_{k=0}^{\infty} p(k, \alpha \omega) z^k = \sum_{k=0}^{\infty} z^k QP_k(\omega, \beta, \alpha)$$

(285)

where

$$QP_k(\omega, \beta, \alpha) = \sum_{j=0}^{k} q(j, -\beta \omega) p(k-j, \alpha \omega) .$$

(286)

From Eq. (286) we see that $QP_0(\omega, \beta, \alpha) = 1$. Furthermore, since $P(z, \alpha \omega, \alpha \omega)$ equals unity, it follows that $QP_k(\omega, \alpha, \alpha) = 0$ for $k > 0$.

Table 5 presents the coefficients $QP_k(\omega, \beta, \alpha)$ up till $k = 8$. As can be seen from the table they are polynomials of $O(k)$ in $\omega$. The power of $\omega$ in these polynomials gives the number of elements in the final partitions,
which combine the elements from standard partitions with those from discrete partitions. As expected, the polynomials vanish when $\alpha = \beta$ since the product $P(z, \beta \omega, \alpha \omega)$ equals unity in this case. Furthermore, the highest power of $\alpha$ is $k$, which is also the highest power of $\omega$. This corresponds to the fact that the power of $\alpha$ represents the number of elements in the standard partitions. Therefore, the greatest number of elements in the final partitions will be due to the partition $\{1,1,\ldots,1_k\}$ with no elements coming from a discrete partition. The highest power of $\beta$, however, is considerably lower since it is determined by the partition with the most number of discrete elements summing to $k$. In this instance the power of $\alpha$ will be zero. E.g., for $k = 8$, the highest power of $\beta$ is three, which is in accordance with Eq. (201). When $\alpha = 0$, the polynomials reduce to the polynomials arising from the generating function for discrete partitions, i.e. $q(k, -\beta \omega)$, while for $\beta = 0$, they reduce to the partition function polynomials or $p(k, \alpha \omega)$. In addition, for $\alpha = 0$ the coefficients in the resulting polynomials give the number of discrete partitions where the number of elements is equal to the power of $\beta$. For $\beta = 0$ the coefficients of the resulting polynomials become the sub-partition numbers or $p_i(k)$, where $i$ represents the power of $\alpha$.

The interesting terms in the polynomials displayed in Table 5 are the cross-terms involving $\alpha$ and $\beta$, which represent the mixture of the discrete partitions and standard partitions with the total number of the elements equal to the power of $\omega$. For example, in $QP_3(\omega, \alpha, \beta)$ the coefficient of $\omega^2$ has a term equal to $-2\alpha \beta$, which tells us that one element in the partition has come from the generating function for standard partitions and the other has come from the generating function for discrete partitions. There are two instances where this can occur: either the one has come from the discrete part or numerator on the rhs of Eq. (280) and the two from the denominator or vice-versa. On the other hand, there is only one instance of the partition $\{1,2\}$ emanating only from either the numerator or the denominator. Thus, the coefficients of $\omega^2$ in $QP_3(\omega, \alpha, \beta)$ for only standard and discrete partitions are respectively $\alpha^2$ and $\beta^2$. In this instance the partition maintains its discreteness when accepting an element from the discrete partitions and one from the standard partitions, but this will not always be the case. In addition, the power of $\alpha$ can be much higher than the power of $\beta$ reflecting the fact that the greatest number of elements in a discrete partition summing to a particular value is significantly less than the greatest number of elements in a standard partition summing to the same value.
According to p. 23 of Ref. [31], Gauss derived the following result:

\[ 1 + 2 \sum_{k=1}^{\infty} (-1)^k z^k = \prod_{k=1}^{\infty} \frac{1 - z^k}{1 + z^k}. \tag{287} \]

The rhs of the above result is a special case of \( P(z, \beta \omega, \alpha \omega) \), namely \( P(z, 1, -1) \).

If the values for \( \alpha \omega \) and \( \beta \omega \) are introduced into the rhs of Eq. (285), then we can equate like powers of \( z \) with the lhs of the Eq. (287). Consequently, for \( i \) equal to a positive integer we arrive at

\[ QP_k(1, 1, -1) = \sum_{j=0}^{k} q(j) p(k - j, -1) = \begin{cases} 2(-1)^i, & k = i^2, \\ 0, & \text{otherwise}. \end{cases} \tag{288} \]

Eq. (288) can be checked with the results appearing in Table 5.

We can also use the preceding analysis to derive a power series expansion or generating function for the product of two specific forms of \( P(z, x, y) \) involving the three parameters, \( \omega, x, \) and \( y \), and the variable, \( z \). This product was first studied by Heine. According to p. 55 of Ref. [18] he found that

\[
\prod_{k=1}^{\infty} \frac{(1 - \omega x z^k)}{(1 - z^k)} \frac{(1 - \omega y z^k)}{(1 - \omega y z^k)} = \sum_{k=0}^{\infty} \frac{(1/x; z)_k (1/y; z)_k}{(z; z)_k (\omega z; z)_{k+1}} (\omega xy z)_k. \tag{289}
\]

If we set \( a = 1/x, b = 1/y, c = \omega z \) and \( q = z \) with \( |c/ab| < 1 \) and \( |q| < 1 \), which are the conditions for guaranteeing absolute convergence, then the above result can be expressed as a q-hypergeometric series. This is perhaps the more familiar form for the product, where it is written as

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k (c; q)_k}{(q; q)_k (c; q)_k (c/ab)_k} = \prod_{k=0}^{\infty} \frac{(1 - (c/ab)q^k)}{(1 - c q^k)} \frac{(1 - (c/b)q^k)}{(1 - (c/a)q^k)}. \tag{290}
\]

This result appears as Corollary 2.4 on p. 20 of Ref. [31].

The lhs of Eq. (289) represents the product of \( P(z, \omega x, \omega) \) and \( P(z, \omega y, \omega xy) \).

If we denote the product of \( P(z, x, y) \) and \( P(z, s, t) \) by \( P_2(z, x, y, s, t) \) and introduce Eq. (285), then we find that

\[
P_2(z, \omega x, \omega, \omega y, \omega xy) = \prod_{k=1}^{\infty} \frac{(1 - \omega x z^k)}{(1 - z^k)} \frac{(1 - \omega y z^k)}{(1 - \omega y z^k)} \frac{(1 - \omega xy z^k)}{(1 - \omega x yz^k)} = \sum_{k=0}^{\infty} z^k HP_k(\omega, x, y), \tag{291}
\]
Table 6: Coefficients of the polynomials \( HP_k(\omega, x, y) \) arising from the three-parameter one variable generating function given in Eq. (291).

where

\[
HP_k(\omega, x, y) = \sum_{j=0}^{k} QP_j(\omega, x, 1) QP_{k-j}(\omega, y, xy) .
\]  

(292)

The equals sign appears here because of the conditions given below Eq. (289).

Table 6 presents the coefficients \( HP_k(\omega, x, y) \) up to \( k = 6 \). They have been obtained by implementing Eq. (292) in Mathematica. From the table it can be seen that the \( HP_k(\omega, x, y) \) are polynomials in \( \omega \) of degree \( k \). The coefficient of the leading order term is

\[
C_{HP}^k = \frac{(1 - x^k y^k)}{(1 - xy)} (x - 1)(y - 1) ,
\]  

(293)

while that for the penultimate leading order term is found to be

\[
C_{k-1}^{HP} = \frac{(1 - x^{k-2} y^{k-2})}{(1 - xy)} (x - 1)^2(y - 1)^2 .
\]  

(294)

In the above equation \( k > 2 \). The lowest order term in \( \omega \) for these polynomials is linear and its coefficient is equal to

\[
C_1^{HP} = (x - 1)(y - 1) .
\]  

(295)
As expected, the polynomials are zero when either $x = 1$ or $y = 1$ since in these cases $P(z, \omega x, \omega y, \omega xy)$ is equal to unity. In addition, they are symmetrical in $x$ and $y$ in the sense that any power of $\omega$ with a term of $ax^i y^j$, where $i \neq j$, in its coefficient will also possess the term of $ax^j y^i$ with the same power of $\omega$.

9 Conclusion

Originally, this work set out to devise a programming methodology on the partition method for a power series expansion, which has been used recently to solve important and intractable problems in applied mathematics [1]-[4]. In this method the coefficients of the resulting power series for a function are obtained by summing the contributions made by each partition summing to the order $k$. These contributions are evaluated by: (1) assigning values $p_i$ to each element $i$ in a partition, (2) multiplying by a multinomial factor composed of the factorial of the total number of elements in the partition, $N_k!$ divided by the factorial of each element’s frequency, $n_i!$ and (3) multiplying by the coefficient of an outer series for the total number of elements in the partition, viz. $q_{N_k}$. In order to apply the method, it means that one needs to know the composition of all the partitions summing to $k$, which includes the frequencies or numbers of occurrences of all the elements in each partition. Therefore, an algorithm is required that is capable of scanning all these partitions, the number of which increase exponentially with $k$. Whilst Sec. 2 discusses various methods of generating partitions, it turns out that the novel bi-variate recursive central partition or BRCP algorithm is the most suitable method for implementation in the partition method for a power series expansion because it is based on the graphical representation of the partitions in the form of a non-binary tree diagram as depicted for $k = 6$ in Fig. [1]. As a consequence, the BRCP algorithm is able to print out partitions in the multiplicity representation more efficiently than the other algorithms discussed in Sec. 2, while the multiplicity representation turns out to be the minimum amount of information required for carrying out the method for a power series expansion.

The theory behind the partition method for a power series expansion is presented in Sec. 3 as Theorem 1, which shows how power series expansions can be derived from a quotient of pseudo-composite functions. It also represents the lynchpin of this work. Next the regularisation of the binomial
series is presented in Lemma 1. With this result a corollary to Theorem 1 is developed, whereby the partition method is adapted to the situation in which the quotient of the pseudo-composite functions can be taken to an arbitrary power. As a result of Theorem 1 and its corollary, we observe that the process of evaluating the contributions made by each partition can be viewed as a discrete operation, giving rise to a partition operator denoted by \( L_{P,k}[:]. \) While \( L_{P,k}[1] = p(k) \) or the number of partitions summing to \( k \), varying the argument inside the operator yields completely different identities. Moreover, the partition operator can be modified so that it only applies to specific types of partitions such as discrete or odd/even partitions, again resulting in further new and fascinating identities when the arguments are altered.

Because the number of partitions increases exponentially, it becomes rather onerous to apply the partition method for a power series expansion when the order \( k \) is greater than 10. This problem is overcome by modifying the BRCP algorithm to calculate the contribution due to each partition in symbolic form. Sec. 4 presents two such programs. The first \texttt{partmeth} calculates all the coefficients \( D_k \) and \( E_k \) in Theorem 1 up to and including the value of \( k \) specified by the user. Unfortunately, for \( k > 20 \) the output files generated by this program become too large. Then we require a code that only evaluates the coefficients \( D_k \) and \( E_k \) for a particular value of \( k \), which is accomplished by the second code \texttt{mathpm}. For much larger values of \( k \), say for \( k > 100 \), storing the coefficients is no longer a viable option. In these cases the output files need to be divided into smaller files. Then each file can be evaluated separately in Mathematica and the result stored. Once a result is stored, the file can be deleted and another file can be imported into the software package. Once it is evaluated and the result stored, it too can be deleted. Finally, the stored results can be summed to yield the value for the coefficient.

As a result of the success in developing a programming methodology for the partition method for a power series expansion, we have been able to create programs that can determine various types of integer partitions such as those with either a fixed number of elements or specific elements, doubly-restricted partitions and discrete/distinct partitions. Normally, different programming approaches are required to solve each of these problems, but as explained in Sec. 5, they can all be solved by introducing minor modifications to the program \texttt{partgen} in Sec. 2. In the process new operators such as the discrete partition operator \( L_{DP,k}[:]. \) and the odd- and even-element partition opera-
tors, \(L_{OEPEk}[\cdot] \) and \(L_{EEPk}[\cdot] \) are defined. In particular, the number of discrete partitions summing to \(k \) or \(q(k, 1) \) is equal to \(L_{DPk}[1] \). Another interesting application in this section is the development of the program transp, which determines conjugate partitions by means of Ferrers diagrams. These are created by the dynamic memory allocation of two-dimensional arrays in the C/C++ code appearing in the appendix.

In Secs. 6-8 the operator approach of Secs. 3 and 5 is employed in the derivation of new power series expansions or generating functions for numerous infinite products of increasing complexity that arise in the theory of partitions. Sec. 6 begins by studying the product \(P(z) \) defined by Equivalence (73), which produces a generating function or power series expansion whose coefficients are the partition function or \(p(k) \). Theorem 2 shows that the generating function of this important product is absolutely convergent for \(|z| < 1 \) and divergent elsewhere. That is, unlike the geometric series, there is no region in the complex plane where the generating function is conditionally convergent. Instead, there is a ring of singularity separating the absolutely convergent unit disk from the rest of the divergent complex plane. As a result of Theorem 2, Equivalence (73) is only an equation when \(|z| < 1 \). For these values of \(z \), the generating function can be inverted and Theorem 1 can then be applied to the ensuing result. The coefficients of the resulting power series expansion or generating function are referred to as the discrete partition numbers \(q(k) \), which equal \((-1)^j \) when \(k \) is a pentagonal number or equal to \((3j^2 \pm j)/2 \) and zero, otherwise. We also find that they can be expressed in terms of the partition operator with each element \(i \) assigned to the partition function value of \(p(i) \) as given by Eq. (162). On the other hand, when Theorem 1 is applied to the inverse of \(P(z) \), we find that the discrete partition numbers can be expressed in terms of the partition operator acting with each element \(i \) assigned to a value of \(\gamma_i \) as given by Eq. (173).

Sec. 6 continues with the derivation of alternative representations for the generating functions of \(P(z) \) and its inverse, which are given by Eqs. (169) and (167), respectively. In these results the coefficients for each power \(j \) of the inner series are expressed in terms of a sum over the divisors \(d \) of \(j \) divided by \(j \) and are denoted by \(\gamma_j \). When Theorem 1 is applied to these new forms for the products, the coefficients of the generating functions now have the partition operator acting with the elements \(i \) are assigned to values of \(\gamma_i \) as in Eqs. (179) and (175). The difference between these results is the appearance of the phase factor \((-1)^{N_k} \) in the case of the discrete partition numbers.
Because a result like Eq. (173) gives the partition numbers in terms of the partition operator acting with the elements \( i \) assigned to the discrete partition numbers or \( q(i) \), it becomes necessary to develop a program that excludes those partitions in which the \( q(i) \) vanish. This represents a completely different type or class of partition studied in Sec. 5. Thus, Sec. 6 presents the program called \texttt{partfn} , which describes the modifications that need to be made to \texttt{partgen} in Sec. 2 so that only those partitions in which all the elements are pentagonal numbers are printed out.

Although the programs in Sec. 4 were developed with the partition operator acting on all partitions, it was stated at the end of Sec. 5 that they could be adapted to handle situations where only a subset of the total number of partitions is required. That is, the programming methodology in Sec. 4 is not restricted to the partition operator, but as a result of the material in Sec. 5, it can be adapted to handle situations in which the coefficients are expressed in terms of different operators or specific types of partitions. Such a situation occurs with Eq. (173). Consequently, Sec. 6 concludes by presenting the program \texttt{pfn} , which determines the contributions made by the partitions whose elements yield non-zero discrete partition numbers. This program expresses the partition function or \( p(k) \) in two symbolic forms, both of which must be imported into Mathematica to obtain the actual values for \( p(k) \). In the first form the partition function is expressed directly in terms of the discrete partition numbers or \( q(i) \). This means that in order to obtain the values for the partition function, another module for calculating the discrete partition numbers must be created in Mathematica. In the second form the non-zero values of the \( q(i) \) are replaced by their symbolic form of \((-1)^j \text{ for } i=(3j^2 \pm j)/2 \). When the second form is entered as input into Mathematica, it immediately computes the partition function in integer form.

Sec. 7 begins with the presentation of Theorem 3, which generalises the infinite product of \( 1/P(z) \). Instead of the coefficients of the powers of \( z \) being equal to -1, they are now assumed to be equal to general values \( C_k \) as in Eq. (179). The theorem is proved by adapting the proof of Theorem 1 and means that the coefficients of the generating series for this product are given in terms of the discrete partition operator acting with each element \( i \) assigned the value of \( C_i \) as in Eq. (181). Conversely, the theorem implies that any power series expansion for a function can be expressed as an infinite product. After some elementary examples are studied, viz. the geometric series and the exponential power series, the \( C_i \) are set equal to \( \omega \), whereupon we find that the coefficients of the resulting generating function become the polyno-
mials $q(k, \omega)$, which are referred to as the discrete partition polynomials. As expected, for $\omega = -1$ they reduce to the discrete partition numbers, while for $\omega = 1$, they yield the number of discrete partitions summing to $k$. In fact, the coefficients of the discrete partition polynomials represent the number of discrete partitions where the number of elements is equal to the power of $\omega$.

Then a brief description appears on how the program `dispart` presented in Sec. 5 can be adapted to evaluate these numbers, which can also be written as $L_{DP,k,i}[1]$.

The infinite product of Theorem 3 is further generalised by the introduction of an arbitrary power $\rho_k$ as given by Eq. (211) in the corollary to Theorem 3. In this case the coefficients of the resulting generating function are expressed in terms of the partition operator acting with the elements $i$ assigned a value of $-C_i$ multiplied by $(-\rho_i)^{n_i}$, where $n_i$ is the number of occurrence of $n_i$ and $(\rho)_k$ denotes the Pochhammer symbol. With the aid of other results appearing in the corollary, we are able to study the generating functions when $1/P(z)$ is squared or cubed. For these cases $\rho_i$ has been set equal to the uniform value of $\rho$ and the discrete partition polynomials $q(k, \omega)$ are extended to become $q(k, \omega, \rho)$ as defined by Eq. (222). Several identities involving the $q(k, \omega, \rho)$ are also derived.

Sec. 8 is also devoted to deriving generating functions for other forms of infinite products using the material from the previous two sections. First of all, the coefficients of the powers of $z$ in $P(z)$ are set equal to $-\omega$ instead of $-1$. Consequently, the coefficients of the generating function obtained after applying Theorem 1 to the modified version of $P(z)$ or $P(z, \omega)$ become polynomials in $\omega$, which are given by the partition operator acting with the elements $i$ assigned to the value of $\omega$ as in Eq. (244). As expected, these polynomials denoted by $p(k, \omega)$ reduce to the partition function $p(k)$ for $\omega = 1$. Their coefficients are referred to as the sub-partition numbers $p_i(k)$, but are also equal to $\left| k \atop m \right|$, where the latter notation was introduced in Sec. 2 to denote the number of partitions summing to $k$ with $m$ elements.

Sec. 8 continues with further generalisations of $P(z)$, where the coefficients of the powers of $z$ are set equal to $C_k$. This represents not only the inversion of Theorem 2, but is also a special case of the corollary to Theorem 3. The specific case of the $C_k$ equalling unity has the interesting property that the coefficients of the generating function represent the difference between the number of even- and odd-element partitions summing to $k$. The absolute values of these coefficients are shown to equal the number of dis-
crete partitions with only odd elements, a result first obtained by Euler. Then recurrence relations are developed for the number of discrete partitions summing to $2k$ and $2k + 1$ given by Eqs. (274) and (275), respectively. Like the Euler/MacMahon recurrence relation or Eq. (165), which evaluates the values of $p(k)$, the new recurrence relations also require the discrete partition numbers. This means that not many of the previous values of $q(k,1)$ are required to determine the highest successive value.

The approach that resulted in the alternative representations for $P(z)$ and its inverse given by Eqs. (167) and (169) is then applied to $P(z,ω)$. Replacing the $γ_i$ are the divisor polynomials $γ_i(ω)$, where each coefficient represents a divisor $d$ of $i$ divided by $i$ while the power of $ω$ equals the inverse or reciprocal of this value. As a result, the discrete partition and partition function polynomials can be expressed in terms of the partition operator acting with the elements $i$ assigned to $−γ_i(ω)$ and $γ_i(ω)$, respectively. Next, the program dispfnpoly is presented. The purpose of this code is to express both $q(k,−ω)$ and $p(k,ω)$ in symbolic form so that they can be imported into Mathematica, where the Divisors routine can be exploited to yield the final forms as polynomials in $ω$ of degree $k$.

Sec. 8 concludes by applying the preceding material to derive the generating functions for more advanced infinite products. First, the product $P_ρ(z,ω)$ is studied, in which an arbitrary power of $ρ$ is applied to $P(z,ω)$. The generating function for the new product has coefficients $p(k,ω,ρ)$, which are expressed in terms of the partition operator acting on elements $i$ assigned values of $p(i,ω)$, but now the multinomial factor is altered as described in the proof of Corollary 1 to Theorem 1. The next example is the product of $Q(z,−βω)$ as defined by Eq. (198) and $P(z,αω)$. The coefficients of the generating function become special polynomials $QP_k(ω,β,α)$ as defined by Eq. (286) and are tabulated in Table 5. In these coefficients the power of $ω$ represents the number of elements in the final partitions, which are composed of the elements from both discrete and standard partitions, while the powers of $α$ and $β$ represent the numbers of elements emanating from the standard and discrete partitions, respectively. The final example in the section is the derivation of the generating function for the famous three-parameter plus one variable product first studied by Heine and given by Eq. (291). This product, which represents the product of $P(z,ωx,ω)$ and $P(z,ωy,ωxy)$, is found to possess a generating function whose coefficients are polynomials denoted by $HP_k(ω,x,y)$. Like the $QP_k(ω,β,α)$, they are of degree $k$ in $ω$. General expressions for some of the coefficients are derived, while the first
seven polynomials are presented in Table 6.

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11 Appendix

The first code presented in this appendix is the program `partmeth` discussed in detail at the beginning of Sec. 4. This code employs the BRCP algorithm of Sec. 2 to generate the coefficients arising from the partition method for a power series expansion according to Theorem 1 in symbolic form so that the values of the coefficients can be evaluated either in integer form or as algebraic expressions in Mathematica.

/* This code deals with the application of the partition method for a power series expansion to the pseudo-composite function $g(af(z))$. Here it is assumed that $g(z) = h(z)(1 + q_1 z + q_2 z^2 + \ldots + q_k z^k + \ldots)$, where $h(z)$ can be any function, but is usually equal to unity or some factor multiplied by a non-integer power of $z$. In addition, $f(z)$ is assumed to be a power series expansion of the form $(p_0 + p_1 y + p_2 y^2 + \ldots + p_k y^k + \ldots)$ with $y = z^\alpha$. This code is only valid for the case of $p_0 = 0$. The coefficients $DS[k,n]$ and $ES[k,n]$ are those in Theorem 1 and are computed in a format suitable for processing in Mathematica. */

#include <stdio.h>
#include <memory.h>
#include <stdlib.h>
#include <math.h>
#include <time.h>

int dim,sum,*part,inv_case;

int freq,i,num_parts=0,1,spacing=0,num_dis_parts=0,dis_parts=0;
double sign, dnum_parts;

/* num_parts is the total # of parts in the partition, while num_dis_parts is the number of distinct parts with greater than unity frequency and is required for the multinomial factor. */

if (p==sum) printf((inv_case >0)?"ES[\%i, n_]: = ":"DS[\%i, n_]: = ",p);
if (inv_case==0 && p!=sum) printf("+");
for (i=1; i<=dim; i++)
  freq=part[i];
  if (freq){
    dis_parts++;
    if (inv_case==0){
      printf("p[\%i, n]\",i);
      if (freq >1) printf("\^(%i) ",freq);
      else printf(" ");
      num_dis_parts += (freq >1);
      num_parts=num_parts+freq;
    }
  }
if (inv_case >0){
  dnum_parts=(double) num_parts;
  sign=pow(-1.0, dnum_parts);
  printf((sign >0.0)?"+":"- ");

}
if (inv_case >0){
  printf("DS[0,0]^(-\%i) ",num_parts+1);
}
else{
  printf("q[\%i] a",num_parts);
  printf((num_parts >1)?"\^(\%i) ":" ",num_parts);
}
if (inv_case >0){
for (i = 1; i <= dim; i++){
    freq = part[i];
    if (freq == 1) printf("DS[%i, n]", i);
    else if (freq > 1) printf("DS[%i, n]^(%i)", i, freq);
}

if (num_parts > 1 && dis_parts > 1){
    printf("%i!", num_parts);
    if (num_dis_parts > 1){
        printf((num_dis_parts > 1)? "/(" : "/") ;
        for (i = 1; i <= dim; i++){
            freq = part[i];
            if (freq > 1){
                if (spacing++) printf(" ");
                printf("%i!", freq);
            }
        }
        if (num_dis_parts > 1) printf("") ;
    }
    printf(" ");
}
printf("\n");

void idx(int p, int q)
{
    part[p]++;
    termgen(p);
    part[p]--;
    p = q;
    while (p >= q){
        part[q]++;
        idx(p--, q);
        part[q++]--;
    }
}

int main(int argc, char *argv[])

The above code represents the implementation of Theorem 1 for the case where $p_0$ vanishes. Once the order of coefficients becomes sufficiently large, viz. for $k \geq 20$, the code needs to be adapted so that only specific values for one of the two types of coefficient are evaluated in symbolic form. This requires: (1) separating the inverse case or the $E_k$ from the $D_k$, and (2) removing the first for loop in main so that only $i = \text{dim}$ is computed. The modified code called mathpm, which determines only the $D_k$ in symbolic form, is presented below.
The code mathpm determines the coefficients of the power series for the pseudo-composite function $g(af(z))$, where $g(z) = h(z) (1 + q_1 z + q_2 z^2 + \ldots + q_k z^k + \ldots)$ and $h(z)$ can be any function. The function $f(z)$ must be expressed as $(p_0 + p_1 y + p_2 y^2 + \ldots + p_k y^k + \ldots)$ where $y = z^\alpha$ and $p_0 = 0$.

```c
#include <stdio.h>
#include <memory.h>
#include <stdlib.h>
#include <time.h>

int dim, *part;
long unsigned int term = 1;
time_t init_time, end_time;

void termgen(int p)
{
    int f, i, num_parts = 0, dis_part_cnt = 0,1, num_dis_parts = 0;
    /* num_parts is the total # of parts in the partition,
     * while num_dis_parts is the number of distinct parts.
     * The latter is required for the multinomial factor. */

    if (p==dim) printf("DS[%i,n_]:= ", p);
    else {
        printf("+ ");
        if (term % 3 == 0) printf("\n");
        term++;
    }
    for (i = 1; i <= dim; i++) {
        f = part[i];
        if (f) {
            printf("p[%i]”, i);
            if (f > 1) printf(”(%i) “, f);
            else printf(” “);
            num_parts += l = f;
            num_dis_parts += (f > 1);
```
```c
void idx(int p, int q)
{
    part[p]++;  
    termgen(p); 
    part[p]--;  
    p = q;      
    while (p >= q) {
        part[q]++;  
        idx(p--;q); 
        part[q++]--; 
    }
```
int main(int argc, char *argv[])
{
    int i;
    double delta_t;
    FILE *ptr;
    char filename[10]="times";

time(&init_time);
if(argc != 2) printf("usage: ./mathpm <#partitions>\n ");
else{
    dim=atoi(argv[1]);
    part=(int *) malloc((dim+1)*sizeof(int));
    if(part==NULL) printf("unable to allocate array\n\n ");
    else{
        free(part);
    }
}
printf("\n ");
time(&end_time);
delta_t= difftime(end_time,init_time);
ptr=fopen(filename,"a");
fprintf(ptr,"Time to compute p[%i,n] is %f seconds\n",dim,delta_t);
fclose(ptr);

In Sec. 5 we discussed the problem of generating discrete partitions or partitions in which the elements only appear once. Since discrete partitions are important in the theory of partitions as can be seen from Secs. 6-8, the entire code called dispart.cpp is presented below.

#include <stdio.h>
#include <memory.h>
#include <stdlib.h>
#include <time.h>

int tot, *part;
long unsigned int term=1;
time_t init_time, end_time;
void termgen()
{
    int freq, i;

    for(i = 0; i < tot; i++) {
        freq = part[i];
        if(freq > 1) goto end;
    }
    printf("%lld: ", term++);
    for(i = 0; i < tot; i++) {
        freq = part[i];
        if(freq) printf("%i (%i) ", freq, i + 1);
    }
    printf("\n");
end: ;
} 

void idx(int p, int q)
{
    part[p-1]++;
    termgen();
    part[p-1]--;
    p -= q;
    while(p >= q) {
        part[q-1]++;
        idx(p-- , q);
        part[q++ - 1]--;
    }
}

int main(int argc, char *argv[])
{
    int i;
    double delta_t;
    FILE *ptr;
    char filename[10] = "times";

    time(&init_time);
if(argc != 2) printf("usage: ./dispart <#partitions>\n");
else{
    tot=atoi(argv[1]);
    part= (int *) malloc(tot*sizeof(int));
    if(part == NULL) printf("unable to allocate array\n\n");
    else{
        for(i=0; i<tot; i++) part[i]=0;
        idx(tot,1);
        free(part);
    }
}
printf("\n");
time(&end_time);
delta_t= difftime(end_time, init_time);
ptr=fopen(filename, "a");
fprintf(ptr,"Time taken to compute discrete partitions
    summing to %i is %f secs.\n", tot, delta_t);
fclose(ptr);
return(0);

The next code presented here determines the conjugate partition as described in Sec. 5. In evaluating conjugate partitions, the programme called transp casts the original partition in the form of a Ferrers diagram, but, instead of being composed of dots, the Ferrers diagram is composed of ones. Thus, the conjugate partition is determined by summing the ones in each column of the Ferrers diagram.

/* This program evaluates the partitions and their conjugates for any integer greater than or equal to 2. Conjugates are determined by summing the columns in the Ferrers diagram for each partition. */

#include <stdio.h>
#include <memory.h>
#include <stdlib.h>

int tot, *part;
long unsigned int term=1;
void termgen()
{
    int freq, i, j, k, next_el, index, prev_el, *part2,
        *sum_col, row_cnt = 0;
    int *ferrers, **rptr;
    /* The Ferrers array and its array of pointers */

    printf("Partition %ld is: ", term++);
    ferrers=(int *) malloc(tot*tot*sizeof(int));
    rptr=(int **) malloc(tot*sizeof(int *));

    /* Get the pointers to the rows of ferrers */
    for (i=0; i<tot; i++) rptr[i] = ferrers+(i*tot);
    /* Here tot refers to number of columns */

    for (j=0; j<tot; j++)
    {
        for (i=0; i<tot; i++)
        {
            rptr[j][i]=0;
        }
    }
    /* Creation of the Ferrers diagram. Rather than being composed of dots the Ferrers diagram is composed of unit values since these will be used to determine the conjugate partition. */

    for (j=0; j<tot; j++)
    {
        freq=part[j];
        if (freq){
            for (i=row_cnt; i<row_cnt+freq; i++)
            {
                for (k=0; k<=j; k++)
                {
                    rptr[i][k]=1;
                }
            }
            row_cnt=row_cnt+freq;
        }
    }
    /* Summation of the columns in the Ferrers diagram yielding a new array called sum_col */
sum_col=(int *) malloc(tot*sizeof(int));
/* Initialising the array elements to zero */

for (i=0; i<tot; i++) sum_col[i]=0;
/* Summation of the columns now occurs */

for (j=0; j<tot; j++) {
  for (i=0; i<tot; i++) sum_col[j]=sum_col[j]+rptr[i][j];
}
/* The array sum_col is reduced to the conjugate of the original partition through another array called part2 */

part2=(int *) malloc(tot*sizeof(int));
/* Initialisation of the array elements to zero */

for (i=0; i<tot; i++) part2[i]=0;
/* Now the conjugate partition is arranged as in the order of part */

prev_el=sum_col[0];
/* the highest part is the value of sum_col[0] */
index=prev_el-1;
/* the index in the partition array is one less */
part2[index]=1;
for (i=1; i<=tot; i++) {
  next_el=sum_col[i];
  if (next_el==0) goto out;
  if (next_el==prev_el) part2[index]=part2[index]+1;
  else {
    index=next_el-1;
    part2[index]=1;
    prev_el=next_el;
  }
}

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out: for (i=0; i<tot; i++){
    freq=part[i];
    if(freq) printf("%i(%i) ",freq,i+1);
}

/* The conjugate is actually the reverse order of part2 */
printf(" and its conjugate is: ");
for (i=0; i<tot; i++){
    freq=part2[tot-i-1];
    if(freq) printf("%i(%i) ",freq,tot-i);
}
free(sum_col);
free(part2);
free(ferrers);
free(rptr);
printf("\n");
}

void idx(int p, int q)
{
    part[p-1]++;
termgen();
    part[p-1]--;
    p -= q;
    while (p >= q){
        part[q-1]++;
        idx(p--,q);
        part[q++-1]--;
    }
}

int main(int argc, char *argv[])
{
    int i;
    if(argc != 2) printf("usage: ./transp <#partitions>\n");
    else{
        tot=atoi(argv[1]);
        part=(int *) malloc(tot*sizeof(int));
    }
if (part == NULL) printf("unable to allocate array\n\n");
else {
    for (i=0;i<tot;i++) part[i]=0;
    idx(tot,1);
    free(part);
}
printf("\n");
return(0);
}

In Sec. 6 the partition method for a power series expansion was applied to an exponential form of the generating function of the partition function \( p(k) \), which yielded Eq. (173). Although this result represents a sum over partitions involving the special numbers \( q(i) \) called the discrete partition numbers, it incorporates much redundancy because these numbers are often zero except when \( i \) can be written as a pentagonal number or as \( (3j^2 \pm j)/2 \), where \( j \) is a non-negative integer. Then they are equal to \((-1)^j\). Appearing below is the program \texttt{pfn} which gives the partition function based on the properties of the discrete partition numbers (represented by \( Q[i] \) in the code) in symbolic form. The actual values of the partition function can be evaluated by importing the output into Mathematica.

```c
#include <stdio.h>
#include <memory.h>
#include <stdlib.h>
#include <time.h>
#include <math.h>

int dim,*part,limit,freq,first_term=0;
long unsigned int term=1;
time_t init_time, end_time;

void termgen(int p)
{
    int f,i,num_parts=0,j,jval,dis_part_cnt=0,1,
    num_dis_parts=0;
    /* num_parts is the total # of parts in the partition
```
while num_dis_parts is the number of distinct parts.
The latter is required for the multinomial factor. */ */ jval=0; */

if(p==dim){
    printf("p[%i]: ",p);
    for (i=1;i<=dim;i++){
        jval=0;
        freq=part[i];
        if (freq >0){
            for (j=1;j<=limit ;j++){
                if(i == (3*j-1)*j/2) jval=j;
                if(i == (3*j+1)*j/2) jval=j;
            }
            if((jval==0) && (freq >0)) goto end;
            first_term++;
        }
    }
    if(first_term !=0) printf("+ ");
    if(first_term ==0) first_term++;
    if (term %3 == 0) printf("\n");
    term++;
}
for (i = 1; i <= dim; i++) {
    f = part[i];
    if (f) {
        /*
         * printf("Q[%i]", i);
         */
        for (j = 1; j <= limit; j++) {
            if (i == (3 * j - 1) * j / 2) jval = j;
            if (i == (3 * j + 1) * j / 2) jval = j;
        }
        printf("(-1)\(^{%i}\)", jval);
        if (f > 1) printf("^{%i}", f);
        else printf(" ");
        num_parts += 1 = f;
        num_dis_parts += (f > 1);
    }
}
printf("(-1)\n");
printf((num_parts > 1)? "\(^{%i}\)" : " ", num_parts);
if (num_parts > 1) {
    printf("%i!", num_parts);
    if (num_dis_parts) {
        printf((num_dis_parts > 1)? "/( " : "/");
        for (i = 1; i <= dim; i++) {
            f = part[i];
            if (f > 1) {
                if (dis_part_cnt++) printf(" ");
            }
        }
    }
    printf(") ");
}
printf(" ");
}
end: ;
void idx(int p, int q)
{
    part[p]++;
    termgen(p);
    part[p]--;
    p -= q;
    while(p >= q){
        part[q]++;
        idx(p--,q);
        part[q++]--;
    }
}

int main(int argc, char *argv[])
{
    int i;
    double delta_t;

    time(&init_time);
    if(argc != 2) printf("usage: ./pfn <#partitions>n");
    else{
        dim=atoi(argv[1]);
        limit= floor(1+sqrt((1+ 24 * dim))/6);

        part=(int*) malloc((dim+1)*sizeof(int));
        if(part==NULL) printf("unable to allocate array\n\n");
        else{
            /*
            for(sum=1; sum <=dim; sum++){
                for(i=1; i<=dim; i++) part[i] = 0;
                idx(dim,1);
            }
            */
            printf("\n");
            free(part);
        }
        printf("\n");
        time(&end_time);
        delta_t = difftime(end_time, init_time);
printf("Time taken to compute the coefficient is \\
%f seconds\n", delta_t);
return(0);
}

Also in Sec. 6, the discrete partition polynomials \(q(k, \omega)\) were found to be the coefficients of the generating function for by the product \(Q(z, \omega) = \prod_{k=1}^{\infty} 1/(1 + \omega z^k)\) (see Eq. [196]). Later, it was found that these polynomials of degree \(i\) in \(\omega\) could be expressed in terms of the partition operator acting with the elements \(i\) equal to special polynomials \(\gamma_i(\omega)\). The latter were referred to as divisor polynomials since their coefficients are divisors or factors of \(i\). The relationship between both types of polynomials is given by Eq. [278], while another result below it relates the partition function polynomials \(p(k, \omega)\) to another sum over partitions acting on the divisor polynomials. Appearing below is the program called \texttt{dispfnpoly}, which prints out both the discrete partition and partition function polynomials for a specified order in symbolic form so that they can be handled by Mathematica [12].

```c
#include <stdio.h>
#include <memory.h>
#include <stdlib.h>
#include <time.h>

int dim,*part,polytype;
long unsigned int term=1;
time_t init_time, end_time;

void termgen(int p)
{
    int f,i,num_parts=0,dis_part_cnt=0,l,num_dis_parts=0;
    /* num_parts is the total # of parts in the partition 
       while num_dis_parts is the number of distinct parts.
       The latter is required for the multinomial factor */
    if(p==dim) printf((polytype==1)?"Q[%i,-w_]:= ":
                        "P[%i,w_]:= ",p);
    for(i=1; i<=dim; i++)
    {
        f=part [i];
```
if(f){
    num_parts += 1 = f;
    num_dis_parts += (f>1);
}
{  
    part[0]++;  
    termgen(p);  
    part[0]--;  
    p -= q;  
    while(p >= q){  
        part[q]++;  
        idx(p--,q);  
        part[q++]--;  
    }  
}  

int main(int argc, char *argv[])
{
    int i;
    double delta_t;

    time(&init_time);
    if(argc != 2) printf("usage: ./dispfnpoly\n" );
    else{
        dim=atoi(argv[1]);
        part=(int *) malloc((dim+1)*sizeof(int));
        if(part==NULL) printf("unable to allocate array\n\n" );
        else{
            for(polytype=1; polytype<=2; polytype++){
                for(i=1; i<=dim; i++) part[i] = 0;
                idx(dim,1);
                if(polytype==1) printf("\n\n" );
                term=1;
            }
            free(part);
        }
    }
    printf("\n");
    time(&end_time);
    delta_t= difftime(end_time,init_time);
printf("Time taken to compute the coefficients is \
%f seconds\n", delta_t);
return(0);
}