Black Hole and Neutron Star Binary Mergers in Triple Systems: Merger Fraction and Spin–Orbit Misalignment

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Abstract

Black hole (BH) mergers driven by gravitational perturbations of external companions constitute an important class of formation channels for merging BH binaries detected by LIGO. We have studied the orbital and spin evolution of binary BHs in triple systems, where the tertiary companion excites large eccentricity in the inner binary through Lidov–Kozai oscillations, causing the binary to merge via gravitational radiation. Using the single-averaged and double-averaged secular dynamics of triples (where the equations of motion are averaged over the inner orbit and both orbits, respectively), we perform a large set of numerical integrations to determine the merger window (the range of companion inclinations that allows the inner binary to merge within $\sim 10$ Gyr) and the merger fraction as a function of various system parameters (e.g., the binary masses $m_1, m_2$ and initial semimajor axis $a_0$, the mass, semimajor axis, and eccentricity $e_{\text{out}}$ of the outer companion). For typical BH binaries ($m_{1,2} \approx 20 M_\odot$–$30 M_\odot$ and $a_0 \gtrsim 10$ au), the merger fraction increases rapidly with $e_{\text{out}}$ because of the octupole perturbation, ranging from $\sim 1\%$ at $e_{\text{out}} = 0$ to $10\%$–$20\%$ at $e_{\text{out}} = 0.9$. We derive analytical expressions and approximate scaling relations for the merger window and merger fraction for systems with negligible octupole effect, and apply them to neutron star binary mergers in triples. We also follow the spin evolution of the BHs during the companion-induced orbital decay, where de Sitter spin precession competes with Lidov–Kozai orbital precession/nutation. Starting from aligned spin axes (relative to the orbital angular momentum axis), a wide range of final spin–orbit misalignment angles $\theta_{\text{sl}}$ can be generated when the binary enters the LIGO sensitivity band. For systems where the octupole effect is small (such as those with $m_1 \approx m_2$ or $e_{\text{out}} \sim 0$), the distribution of $\theta_{\text{sl}}$ peaks around 90°. As the octupole effect increases, a more isotropic distribution of final spin axis is produced. Overall, merging BH binaries produced by Lidov–Kozai oscillations in triples exhibit a unique distribution of the effective (mass-weighted) spin parameter $\chi_{\text{eff}}$; this may be used to distinguish this formation channel from other dynamical channels.

Key words: binaries: general – black hole physics – gravitational waves – stars: black holes – stars: kinematics and dynamics

1. Introduction

Over the last two years, several mergers of black hole (BH) and neutron star (NS) binaries have been observed in gravitational waves (GWs) by aLIGO/VIRGO (e.g., Abbott et al. 2016a, 2016b, 2017a, 2017b, 2017c, 2017d). With the estimated binary BH merger rate of 10–200 Gpc$^{-3}$ yr$^{-1}$, many hundreds of BH mergers are expected to be detected in the coming years. It is therefore important to systematically study various formation mechanisms of such compact binaries and their observable signatures.

The formation channels of merging BH binaries can be broadly divided into two categories: isolated binary evolution and dynamical formation, corresponding to different ways of bringing widely separated BHs into sufficiently close orbits to allow their coalescence to be driven by gravitational radiation. In the scenario of isolated binary evolution, massive stellar binaries formed with relatively small separations ($< 10$ au) have their orbit tightened by drag forces through common-envelope phases (e.g., Lipunov et al. 1997, 2017; Podsiadlowski et al. 2003; Belczynski et al. 2010, 2016; Dominik et al. 2012, 2013, 2015) or through chemically homogeneous evolution associated with rapid stellar rotations (e.g., Mandel & de Mink 2016; Marchant et al. 2016). The dynamical formation mechanism includes various “flavors,” all involving gravitational interactions between multiple stars/BHs. In one class of scenarios, binary BHs become bound and tighter through three-body encounters and/or secular interactions in dense star clusters (e.g., Portegies Zwart & McMillan 2000; Miller & Hamilton 2002; O’Leary et al. 2006; Miller & Lauburg 2009; Banerjee et al. 2010; Downing et al. 2010; Rodriguez et al. 2015; Chatterjee et al. 2017; Samsing et al. 2018) or galactic nuclei (e.g., O’Leary et al. 2009; Antonini & Perets 2012; Antonini & Rasio 2016; VanLandingham et al. 2016; Petrovich & Antonini 2017; Hoang et al. 2018; Leigh et al. 2018); alternatively, binary BH mergers can be induced in triples in the galactic fields (e.g., Antonini et al. 2017; Silsbee & Tremaine 2017).

Despite many studies, there are large uncertainties in the predicted event rates and binary BH properties in various formation scenarios. Some of these involve uncertainties in the physical processes (e.g., common-envelope evolution), while others are “environmental” uncertainties (e.g., BH population in clusters, orbital parameter distributions in triples). In particular, it is difficult to distinguish different formation mechanisms on the basis of event rates and mass measurements of merging binaries. Other discriminant observables would be desirable. In the dynamical channel, a BH binary could acquire substantial eccentricity through close encounters, so the detection of eccentric merging binaries would indicate certain
where \( m_{1,2} \) are the masses of BHs, \( \chi_{1,2} = c S_{1,2}/(Gm^2_{1,2}) \) are the dimensionless BH spins, and \( \vec{L} \) is the unit orbital angular momentum vector. In the isolated binary evolution channel, because of mass transfer and accretion in the common-envelope phase, the BH spin tends to be aligned with the orbital angular momentum, although a velocity kick during BH envelope phases, the BH spin tends to be aligned with the orbital angular momentum vector. In the isolated binary evolution channel, the BH spin axis has a propensity to point in any direction. Therefore, the distribution of spin tilts is of great importance and could be used as a probe to understand the formation channels of merging binaries (e.g., Rodríguez et al. 2016; Farr et al. 2017). The five BH binaries detected by aLIGO so far have relatively small \( \chi_{\text{eff}} \) (−0.06\text{+0.14}−0.14) for GW150914, 0.21\text{+0.12}−0.09 for GW151226, −0.12\text{+0.21}−0.13 for GW170104, 0.07\text{+0.23}−0.09 for GW170608, and 0.06\text{+0.12}−0.12 for GW170814). This could be the result of either slowly spinning BHs (e.g., Zaldarriaga et al. 2017) or large spin–orbit misalignments. Of particular interest is that GW170104 (e.g., Abbott et al. 2017a) has a negative \( \chi_{\text{eff}} \) value (with appreciable error bars), implying that the configurations with both component spins positively aligned with the orbital angular momentum are disfavored. Such a negative \( \chi_{\text{eff}} \) value may not be produced in the standard binary evolution channel, but would be natural if the binary is formed dynamically.

In this work, we study the orbital and spin evolution of merging BH binaries and NS binaries in the presence of an external companion. It is well known that a tertiary body on an inclined orbit can accelerate the orbital decay of an inner binary by inducing Lidov–Kozai (LK) oscillations in eccentricity/inclination (e.g., Kozai 1962; Lidov 1962). This effect was first studied in the context of supermassive BH binary mergers (e.g., Blaes et al. 2002). There have been a number of previous studies of LK-induced mergers of stellar-mass BH binaries in globular clusters or active galactic nuclei (e.g., Miller & Hamilton 2002; Wen 2003; Thompson 2011; Antonini & Perets 2012; Antonini et al. 2014; Hoang et al. 2018) and in the galactic fields (e.g., Antonini et al. 2017; Silsbee & Tremaine 2017). Many of these works involved population synthesis calculations, adopting various assumptions for the BH binary/triple parameters and distributions and accounting for the effects of cluster dynamics. Such approaches are important, but it can be difficult to know how the numerical results (such as the predicted binary merger rates) depend on the input parameters and assumptions. In this paper we focus on the “clean” problem of isolated triples. Using the secular equations of motion of hierarchical triples (both the octupole-level “double-averaged” (DA) equations and “single-averaged” (SA) equations that we develop in this paper), we systematically examine the “merger window” (i.e., the range of inclination angles between the inner binary and the outer companion that induces binary merger) and merger fraction as a function of BH and companion masses and orbital parameters. Guided by numerical integrations and analytic estimates, we identify the key parameters and scaling relations for understanding LK-induced mergers.

Another important goal of our work is to examine how misalignments between the BH spins and the orbital angular momentum in the BH binaries can be produced in LK-induced mergers. This problem was first studied in our recent paper (Liu & Lai 2017), where we focused on BH binaries with small initial orbital separations (\( \lesssim 1 \) au) such that the external companion induces zero or only modest (\( e \lesssim 0.9 \)) eccentricity excitation in the inner binary. We found that by starting from aligned BH spins, a wide range of spin–orbit misalignments (including retrograde spins) can be generated. In this paper, we consider more general, wide BH binaries (such that the binaries have no chance of merging by themselves within \( \sim 10^{10} \) yr) where an external companion induces extreme eccentricity excitation and merger of the binary. As we show in this paper, the BH spin exhibits a wide range of evolutionary paths, and different distributions of final spin–orbit misalignments can be produced depending on the system parameters.

Our paper is organized as follows. In Section 2, we present the equations for calculating the evolution of triples including gravitational radiation. These equations are based on the approximations of single averaging (for the inner orbit) and double averaging (for both inner and outer orbits) for the orbital evolution of hierarchical triples. We also present the basic properties of LK oscillations for general triple systems; these are useful for determining analytical expressions of the merger windows and merger fractions for “quadrupole” systems. In Section 3, we perform a large set of numerical integrations to determine the merger windows for LK-induced binary mergers, assuming isotropic distribution of the orientations of tertiary companions. The associated merger fractions of BH binaries and NS binaries are obtained, including various analytical/scaling relations and fitting formulae. In Section 4, we study the BH spin evolution during LK-induced binary mergers. We identify various dynamical behaviors for the spin evolution and calculate the distributions of the spin–orbit misalignment angle and the effective spin parameter \( \chi_{\text{eff}} \) when the binary enters the LIGO/VIRGO band. We summarize our main results in Section 5.

2. LK Oscillations in Triples with Gravitational Radiation

We consider a hierarchical triple system, composed of an inner BH binary of masses \( m_1, m_2 \) and a distant companion of mass \( m_3 \) that moves around the center of mass of the inner bodies. The reduced mass for the inner binary is \( \mu_{\text{in}} = m_1 m_2/(m_1 + m_2) \), with \( m_2 \equiv m_1 + m_3 \). Similarly, the outer binary has \( \mu_{\text{out}} = m_2 m_3 /(m_2 + m_3) \), with \( m_3 \equiv m_2 + m_3 \). The semimajor axes and eccentricities are denoted by \( a_{\text{in}}, a_{\text{out}} \) and \( e_{\text{in}}, e_{\text{out}} \) respectively. The orbital angular momenta of two
orbits are
\[ L_{\text{in}} = L_{\text{in}} \dot{L}_{\text{in}} = \mu_{\text{in}} \sqrt{Gm_1 a_{\text{in}} (1 - e_{\text{in}}^2)} \dot{L}_{\text{in}}, \]  
(2)
\[ L_{\text{out}} = L_{\text{out}} \dot{L}_{\text{out}} = \mu_{\text{out}} \sqrt{Gm_{12} a_{\text{out}} (1 - e_{\text{out}}^2)} \dot{L}_{\text{out}}, \]  
(3)
where \( \dot{L}_{\text{in}} \) and \( \dot{L}_{\text{out}} \) are unit vectors. Similarly, we define the eccentricity vectors as \( e_{\text{in}} = e_{\text{in}} \hat{e}_{\text{in}} \) and \( e_{\text{out}} = e_{\text{out}} \hat{e}_{\text{out}} \). Throughout the paper, for convenience of notation, we will frequently omit the subscript “in” for the inner orbit.

To study the evolution of the merging inner BH binary under the influence of the tertiary companion, we first develop the secular equations of motion in terms of the angular momentum \( L \) and eccentricity \( e \) vectors:
\[ \frac{dL}{dt} = \frac{dL}{dt}_{\text{L,K}} + \frac{dL}{dt}_{\text{GW}}, \]  
(4)
\[ \frac{de}{dt} = \frac{de}{dt}_{\text{L,K}} + \frac{de}{dt}_{\text{GR}} + \frac{de}{dt}_{\text{GW}}, \]  
(5)
where we include the contributions from the external companion that generate LK oscillations (to be discussed in Section 2.1.1), the post-Newtonian (PN) correction for general relativity (GR), and the dissipation due to GW emission.

General relativity (1 PN correction) introduces pericenter precession as
\[ \frac{de}{dt}_{\text{GR}} = \Omega_{\text{GR}} \hat{L} \times e, \]  
(6)
with the precession rate given by
\[ \Omega_{\text{GR}} = \frac{3Gm_1}{c^3 a (1 - e^2)}, \]  
(7)
where \( n = (Gm_1/a^3)^{1/2} \) is the mean motion of the inner binary. Gravitational radiation draws energy and angular momentum from the BH orbit. The rates of change of \( L \) and \( e \) are given by (Peters 1964)
\[ \frac{dL}{dt}_{\text{GW}} = -\frac{32}{5} \frac{G^{7/2}}{c^5} \frac{\mu^2 m_{12}^3}{a^{7/2}} (1 + \frac{7e^2}{8}) \dot{L}, \]  
(8)
\[ \frac{de}{dt}_{\text{GW}} = -\frac{304}{15} \frac{G^{3}}{c^5} \frac{\mu m_{12}^2}{a^2 (1 - e^2)^{5/2}} \left(1 + \frac{121}{304} e^2 \right) e. \]  
(9)
The associated orbital decay rate is
\[ \left(\frac{\dot{a}}{a}\right)_{\text{GW}} = -\frac{1}{T_{\text{GW}}} \]  
\[ = -\frac{64}{5} \frac{G^{3} \mu m_{12}^3}{c^5 a^4 (1 - e^2)^{5/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right). \]  
(10)
The merger time due to GW radiation of an isolated binary with initial semimajor axis \( a_0 \) and eccentricity \( e_0 = 0 \) is given by
\[ T_{m,0} = \frac{5c^2 a_0^4}{256G^{3} m_{12}^2 \mu} \]  
\[ \approx 10^{10} \left(\frac{60 M_\odot}{m_{12}}\right)^{2} \left(\frac{15 M_\odot}{\mu}\right) \left(\frac{a_0}{0.202 \text{ au}}\right)^{4} \text{ yr}. \]  
(11)
Thus, for typical BH binaries \( (m_1 \sim m_2 \sim 30 M_\odot) \), only for separations less than about 0.2 au can the isolated binary be allowed to merge within a Hubble time \( (T_{\text{Hubble}} \approx 10^{10} \text{ yr}) \). In this paper, we will consider much larger initial binary separations \( (a_0 \gtrsim 10 \text{ au}) \), so that merger is possible only when the tertiary companion induces extreme eccentricity excitations in the inner binary.

2.1. Orbital Evolution in the Secular Approximation

If we introduce the instantaneous separation between the inner bodies as \( r \equiv r_0 \), and the separation between the external perturber and the center of mass of the inner bodies as \( r_{\text{out}} \equiv r_{\text{out},0} \), then the complete Hamiltonian of the system can be written as (e.g., Hartington 1968)
\[ \mathcal{H} = \frac{1}{2} \mu |\dot{r}|^2 + \frac{1}{2} \mu_{\text{out}} |\dot{r}_{\text{out}}|^2 - \frac{G m_1 m_2}{r} - \frac{G m_{12} m_3}{r_{\text{out}}} + \Phi, \]  
(12)
where
\[ \Phi = -G m_1 m_2 m_3 \sum_{l=2}^{\infty} \left[m_{1l}^{l-1} + (-1)^l m_{2l}^{l-1}\right] \]  
\[ \times \frac{r^l}{l!} P_l (\cos \theta). \]  
(13)
Here \( P_l (x) \) is the Legendre polynomial of degree \( l \) and \( \theta \) is the angle between \( r \) and \( r_{\text{out}} \).

2.1.1. Double-averaged Secular Equations

For sufficiently hierarchical systems, the angular momenta of the inner and outer binaries exchange periodically over a long timescale (longer than the companion’s orbital period), while the exchange of energy is negligible. The orbital evolution of the triple system can be studied by expanding the Hamiltonian to the octupole order and averaging over both the inner and outer orbits (double averaging), i.e., \( \Phi = \Phi_{\text{quad}} + \Phi_{\text{oct}} \). The quadrupole \((l = 2)\) piece is given by
\[ \langle \Phi_{\text{quad}} \rangle = \frac{\mu \Phi_0}{8} [1 - 6e^2 - 3(1 - e^2)(\hat{L} \cdot \hat{L}_{\text{out}})^2] \]  
\[ + 15e^2 (\hat{e} \cdot \hat{L}_{\text{out}})^2], \]  
(14)
and the octupole \((l = 3)\) potential is
\[ \langle \Phi_{\text{oct}} \rangle = \frac{15 \mu \Phi_0 \varepsilon_{\text{oct}}}{64} [e (\hat{e} \cdot \hat{e}_{\text{oct}}) [8e^2 - 1 \]  
\[ - 35e^2 (\hat{e} \cdot \hat{L}_{\text{oct}})^2 + 5(1 - e^2)(\hat{L} \cdot \hat{L}_{\text{oct}})^2] \]  
\[ + 10e(1 - e^2) (\hat{e} \cdot \hat{L}_{\text{oct}}) (\hat{L} \cdot \hat{L}_{\text{oct}})], \]  
(15)
where
\[ \Phi_0 \equiv \frac{G m_1 a^3}{a_{\text{out}} (1 - e_{\text{out}}^2)^{3/2}}, \]  
(16)
and
\[ \varepsilon_{\text{oct}} \equiv \frac{m_1 - m_2}{m_{12}} \left(\frac{a}{a_{\text{out}}}\right) \frac{e_{\text{out}}}{1 - e_{\text{out}}^2}. \]  
(17)
The explicit expressions for \( (dL/dt)_{\text{L,K}} \), \( (de/dt)_{\text{L,K}} \) and for \( (d\phi_{\text{oct}}/d\phi)_{\text{L,K}} \), \( (d\phi_{\text{oct}}/d\phi)_{\text{L,K}} \) are provided in Liu et al. (2015). In general, \( \dot{L}_{\text{in,out}} \) and \( \dot{e}_{\text{in,out}} \) consist of two pieces: a quadrupole
term and an octupole term. The quadrupole term induces the oscillations in the eccentricity and mutual orbital inclination on the timescale of

$$t_{LK} = \frac{1}{n} \left( \frac{a}{m_3} \right)^3,$$

(18)

where the effective outer binary separation is defined as

$$a_{\text{out,eff}} \equiv a_{\text{out}} \sqrt{1 - e_{\text{out}}^2}.$$

(19)

The octupole piece is quantified by terms proportional to $\varepsilon_{\text{oct}}^2$, which measures the strength of the octupole potential relative to the quadrupole one.

For systems that can be correctly described by the DA equations, the timescale for eccentricity variation of the inner binary must be longer than the period of the companion’s orbit. Otherwise, the secular equations may break down (e.g., Seto 2013; Antonini et al. 2014). Note that when the eccentricity of the inner binary is excited to the maximum value $e_{\text{max}}$, the eccentricity vector $e$ evolves on a timescale $t_{LK} \sqrt{1 - e_{\text{max}}^2}$ (e.g., Anderson et al. 2016), much shorter than the quadrupole LK period ($\sim t_{LK}$). Thus, for the DA secular equations to be valid, we require

$$t_{LK} \sqrt{1 - e_{\text{max}}^2} \gtrsim P_{\text{out}},$$

(20)

where $P_{\text{out}}$ is the period of the outer binary.

### 2.1.2. Single-averaged Secular Equations

For moderately hierarchical systems, the change in the angular momentum of the inner binary may be significant within one period of the outer orbit, and the short-term ($\lesssim P_{\text{out}}$) oscillations of the system cannot be ignored. In this case, the DA secular equations break down, and we can use the SA secular equations (only averaging over the inner orbital period).

Averaging over the inner orbit, the quadrupole term in Equation (13) becomes

$$\langle \Phi_{\text{quad}} \rangle = \frac{\mu \Phi_0'}{4} [-1 + 6e^2 + 3(j \cdot \hat{r}_{\text{out}})^2 - 15(e \cdot \hat{r}_{\text{out}})^2],$$

(21)

and the octupole term is

$$\langle \Phi_{\text{oct}} \rangle = \frac{5\mu \Phi_0' \varepsilon_{\text{oct}}}{16} [(3 - 24e^2)(e \cdot \hat{r}_{\text{out}}) - 15(j \cdot \hat{r}_{\text{out}})^2(e \cdot \hat{r}_{\text{out}}) + 35(e \cdot \hat{r}_{\text{out}})^3],$$

(22)

where

$$j \equiv jL = \sqrt{1 - e^2} L$$

(23)

is the dimensionless angular momentum vector, and the coefficients $\Phi_0'$ and $\varepsilon_{\text{oct}}$ are given by

$$\Phi_0' = \frac{G m_3 a^2}{r_{\text{out}}^3},$$

(24)

and

$$\varepsilon_{\text{oct}} = \frac{m_1 - m_2}{m_3} \frac{a}{r_{\text{out}}}.$$

(25)

In terms of the averaged potentials, the equations of motion for the inner orbital vectors $j$ and $e$ are (e.g., Tremaine et al. 2009)

$$\frac{dj}{dt} = - \frac{1}{\mu \sqrt{G m_{\text{out}}}} (j \times \nabla_j \langle \Phi \rangle + e \times \nabla_e \langle \Phi \rangle),$$

(26)

$$\frac{de}{dt} = - \frac{1}{\mu \sqrt{G m_{\text{out}}}} (j \times \nabla_j \langle \Phi \rangle + e \times \nabla_e \langle \Phi \rangle).$$

(27)

Substituting Equation (21) into (26) and (27), the quadrupole-level equations can be obtained:

$$\frac{dj}{dt}_{\text{quad}} = \frac{3}{2t_{LK}^2}[5(e \cdot \hat{r}_{\text{out}}) e \times \hat{r}_{\text{out}} - (j \cdot \hat{r}_{\text{out}}) j \times \hat{r}_{\text{out}}].$$

(28)

$$\frac{de}{dt}_{\text{quad}} = \frac{3}{2t_{LK}^2}[5(e \cdot \hat{r}_{\text{out}}) j \times \hat{r}_{\text{out}} - (j \cdot \hat{r}_{\text{out}}) e \times \hat{r}_{\text{out}} - 2j \times e].$$

(29)

Similarly, the octupole contributions are

$$\frac{dj}{dt}_{\text{oct}} = \frac{15 \varepsilon_{\text{oct}}}{16t_{LK}^2}[10(j \cdot \hat{r}_{\text{out}}) (e \cdot \hat{r}_{\text{out}}) j \times \hat{r}_{\text{out}} - (1 - 8e^2) e \times \hat{r}_{\text{out}} + 5(j \cdot \hat{r}_{\text{out}})^2 e \times \hat{r}_{\text{out}} - 35(e \cdot \hat{r}_{\text{out}})^2 e \times \hat{r}_{\text{out}}],$$

(30)

$$\frac{de}{dt}_{\text{oct}} = \frac{15 \varepsilon_{\text{oct}}}{16t_{LK}^2}[16(e \cdot \hat{r}_{\text{out}}) j \times \hat{e} - (1 - 8e^2) j \times \hat{r}_{\text{out}} + 5(j \cdot \hat{r}_{\text{out}})^2 j \times \hat{r}_{\text{out}} - 35(e \cdot \hat{r}_{\text{out}})^2 j \times \hat{r}_{\text{out}} + 10(j \cdot \hat{r}_{\text{out}})(e \cdot \hat{r}_{\text{out}}) e \times \hat{r}_{\text{out}}].$$

(31)

In the above, we have defined the SA (quadrupole) LK timescale as

$$t_{LK} = \frac{1}{n} \left( \frac{m_2}{m_3} \frac{r_{\text{out}}}{a} \right)^3.$$

(32)

The evolution equations of $L$ and $e$ are

$$\frac{dL}{dt}_{\text{LK}} = \mu \sqrt{G m_{\text{out}}} \left( \frac{dj}{dt}_{\text{quad}} + \frac{dj}{dt}_{\text{oct}} \right),$$

(33)

$$\frac{de}{dt}_{\text{LK}} = \frac{de}{dt}_{\text{quad}} + \frac{de}{dt}_{\text{oct}}.$$

(34)

On the other hand, for the external companion, the dynamics is governed by

$$\mu_{\text{out}} \frac{d^2 r_{\text{out}}}{dt^2} = \nabla_{r_{\text{out}}} \left( \frac{G m_2 m_3}{r_{\text{out}}} \right) - \nabla_{r_{\text{out}}} (\langle \Phi_{\text{quad}} \rangle + \langle \Phi_{\text{oct}} \rangle).$$

(35)

The explicit form can be obtained by substituting Equations (21) and (22) into (35). Equations (28)–(31) and (33)–(35), together with Equations (6)–(9), completely determine the evolution of the triple system in the single averaging approximation.

The SA equations are applicable to a wider range of system parameters than the DA equations. Nevertheless, their validity still requires that the timescale for eccentricity evolution at $e \sim e_{\text{max}}$ be longer than the orbital period of the inner binary, i.e.,

$$t_{LK} \sqrt{1 - e_{\text{max}}^2} \gtrsim P_{\text{in}}.$$

(36)
2.2. LK Eccentricity Excitation: Analytical Results

Before exploring the LK-induced mergers systematically (Section 3), we summarize some key analytical results for LK eccentricity excitations. It is well known that the effects of short-range forces (such as GR-induced apsidal precession; see Equation (6)) play an important role in determining the maximum eccentricity $e_{\text{max}}$ in LK oscillations (e.g., Holman et al. 1997; Fabrycky & Tremaine 2007). Analytical expression for $e_{\text{max}}$ for general hierarchical triples (arbitrary masses) can be obtained in the DA secular approximation when the disturbing potential is truncated to the quadrupole order (Liu et al. 2015; Anderson et al. 2016, 2017).

In the absence of energy dissipation, the evolution of the triple is governed by two conservation laws. The first is the total orbital angular momentum of the system, $L_{\text{tot}} = L + L_{\text{out}}$. In the quadrupole approximation, $e_{\text{out}}$ is constant, and the conservation of $L_{\text{tot}}$ implies

$$K = j \cos I - \frac{\eta}{2} e^2 = \text{constant},$$

(37)

where $j = \sqrt{1 - e^2}$, $I$ is the angle between $\hat{L}$ and $L_{\text{out}}$, and we have defined

$$\eta \equiv \frac{L_{\text{tot}}}{L_{\text{tot}}} = e_{\text{in}}^2 = \mu \left[ \frac{m_{12} a}{m_{12} a_{\text{out}} (1 - e_{\text{out}}^2)} \right]^{1/2}. \tag{38}$$

In the limit $L \ll L_{\text{out}}$, Equation (37) reduces to the usual “Kozai constant,” $\sqrt{1 - e^2} \cos I = \text{constant.}$

The second conserved quantity is the total energy. In the double averaging approximation, it is given by (to the quadrupole order)

$$\Phi = \langle \Phi_{\text{quad}} \rangle + \langle \Phi_{\text{GR}} \rangle,$$

(39)

where $\langle \Phi_{\text{quad}} \rangle$ is given by Equation (14), and $\langle \Phi_{\text{GR}} \rangle$ is given by

$$\langle \Phi_{\text{GR}} \rangle = -\frac{3G^2 m_3^2 a_{\text{out}}}{a^2 \Delta^2 j} = -\varepsilon_{\text{GR}} \frac{\mu_0}{j}, \tag{40}$$

with

$$\varepsilon_{\text{GR}} = \frac{3G m_3^2 a_{\text{out}}}{a^2 \Delta^2 m_3}$$

$$\simeq 3.6 \times 10^{-5} \left( \frac{m_{12}}{60 M_\odot} \right)^2 \left( \frac{m_3}{30 M_\odot} \right)^{-1} \times \left( \frac{a_{\text{out}}}{10^3 \text{au}} \right)^3 \left( \frac{a}{10^3 \text{au}} \right)^{-4}. \tag{41}$$

Using Equations (37) and (39), the maximum eccentricity $e_{\text{max}}$ attained in the LK oscillation (starting from an initial $I_0$ and $e_0 \approx 0$) can be calculated analytically (Liu et al. 2015; Anderson et al. 2017):

$$\frac{3}{8} j_{\text{min}}^2 - \frac{1}{j_{\text{min}}^2} \left[ 5 \left( \cos I_0 + \frac{\eta}{2} \right)^2 - \left( 3 + 4\eta \cos I_0 + \frac{9}{4} \eta^2 \right)^2_{\text{min}} + \eta^2 j_{\text{min}}^4 \right] + \varepsilon_{\text{GR}}(1 - j_{\text{min}}^{-1}) = 0,$$

where $j_{\text{min}} = \sqrt{1 - \frac{e_{\text{max}}^2}{2}}$. Note that in the limit $\eta \to 0$ and $\varepsilon_{\text{GR}} \to 0$, Equation (42) yields the well-known relation $e_{\text{max}} = \sqrt{1 - (5/3)\cos^2 I_0}$. For general $\eta$, the maximum possible $e_{\text{max}}$ for all values of $I_0$, called $e_{\text{lim}}$, is achieved at $I_0,\lim$ that satisfies $de_{\text{max}}/dI_0 = 0$, i.e.,

$$\cos I_0,\lim = \frac{\eta (4/5)^2 - 1}{2/5}. \tag{43}$$

Substituting Equation (43) into (42), we find that the limiting eccentricity $e_{\text{lim}}$, the maximum of the $e_{\text{max}}(I_0)$ curve, is determined by

$$\frac{3}{8} (j_{\text{lim}}^2 - 1) \left[ -3 + \frac{\eta^2}{4} \left( \frac{4}{5} j_{\text{lim}}^2 - 1 \right) \right] + \varepsilon_{\text{GR}}(1 - j_{\text{lim}}^{-1}) = 0. \tag{44}$$

On the other hand, eccentricity excitation ($e_{\text{max}} \geq 0$) occurs only within a window of inclinations ($\cos I_0,\pm \leq \cos I_0 \leq \cos I_0,\pm$), where (Anderson et al. 2017)

$$(\cos I_0,\pm) = \frac{1}{10} \left( -\frac{\eta}{10} \pm \sqrt{\eta^2 + 60 - \frac{80}{3} \varepsilon_{\text{GR}}} \right). \tag{45}$$

This window exists only when

$$\varepsilon_{\text{GR}} \leq \frac{9}{4} + \frac{3}{8} \eta^2. \tag{46}$$

In other words, no eccentricity excitation is possible when relation (46) is not satisfied.

Figure 1 shows some examples of the $e_{\text{max}}(I_0)$ curves. For $\eta \approx 1$, these curves depend mainly on $m_3/a_{\text{out}}$ (for given inner binary parameters). We see that the excitation of eccentricity can only happen within a finite range of $I_0$, and the achieved maximum $e$ cannot exceed $e_{\text{lim}}$ for any values of $\eta$.

For systems with $m_1 = m_2$ and $e_{\text{out}} = 0$, $e_{\text{oct}}$ is non-negligible, the octupole effect may become important (e.g., Ford et al. 2000; Blaes et al. 2002; Katz et al. 2011; Naoz et al. 2011, 2013; Naoz 2016). This tends to widen the inclination window for large eccentricity excitation. However, the analytic expression for $e_{\text{lim}}$ given by Equation (44) remains valid even for $e_{\text{oct}} = 0$ (Liu et al. 2015; Anderson & Lai 2017). In other words, because of the effect of short-range forces due to GR, the maximum eccentricity cannot exceed $e_{\text{lim}}$ even when the octupole potential is significant. Higher eccentricity may be achieved when the double averaging approximation breaks down (see Section 3.2).

2.3. Summary of Parameter Regimes

Figure 2 summarizes the parameter regimes of BH triples in terms of the mass ($m_3$) and semimajor axis ($a_{\text{out}}$) of the tertiary companion. For concreteness, we consider a fixed set of inner binary parameters ($m_1 = 30 M_\odot$, $m_2 = 20 M_\odot$, and $a_0 = 100$ au), with $e_{\text{out}} = 0$ (upper panel) and $e_{\text{out}} = 0.6$ (lower panel). The stability of the triple requires (e.g., Mardling & Aarseth 2001)

$$\frac{a_{\text{out}}}{a} > 2.8 \left( \frac{m_3}{m_{12}} \right)^{2/5} \left( \frac{1 + e_{\text{out}}^{2/5}}{1 - e_{\text{out}}^{2/5}} \right)^{1 - \frac{3}{180} \cos I_0}. \tag{47}$$
The Astrophysical Journal, 863:68 (23pp), 2018 August 10
Liu & Lai

parameters of the companion are given by Equations (42). In Figure 2, several regions have been identified using Equation (42). The inner binary has $m_1 = m_2 = 30M_\odot$, $a = 100$ au, and initial $e_0 = 0$. The parameters of the companion are $m_3 = 30M_\odot$, $a_{\text{out}} = 6000$ au, and $e_{\text{out}} = 0.001$ (blue); $m_3 = 20M_\odot$, $a_{\text{out}} = 5241$ au, and $e_{\text{out}} = 0.001$ (cyan); $m_3 = 20M_\odot$, $a_{\text{out}} = 6000$ au, and $e_{\text{out}} = 0.487$ (brown). The $e_{\text{max}}(I_0)$ curve depends mainly on $m_3/a_{\text{out},\text{eff}}$. The horizontal ($e_{\text{lim}}$) and vertical ($I_\pm$) lines are given by Equations (44) and (45), respectively.

In Figure 2, several regions have been identified (color-coded) and the boundaries are given by the various criteria (relations (20), (36), and (47); see the dotted curves). We see that, in the rightmost region, the perturber is so far away that no LK oscillations occur (no e-excitation). In the “DA region,” the dynamics of the system can be well described by the DA secular equations. The numbers shown on the dashed curves indicate the values of $\log_{10}(1 - e_{\text{lim}})$, suggesting the extent of the excitation of eccentricity (Equation (44)). In the “SA region,” the outer averaging fails, but the SA secular equations are valid.

3. Merger Window and Merger Fraction

In this section, we use numerical integrations to determine the “merger window” of BH binaries, i.e., the range of inclination angles of the tertiary companion such that the inner binary can attain a sufficiently large eccentricity and merge within a critical timescale $T_{\text{crit}}$ (chosen to be the Hubble time, $10^{10}$ yr, throughout this paper; but see Section 3.3). For an isotropic distribution of the tertiary inclinations, the merger window then determines the “merger fraction.” Our main goal is to determine how the merger window and merger fraction depend on the parameters of the triples.

3.1. Binary Mergers Induced by the Quadrupole LK Effect

We first consider the cases when the octupole effect is negligible ($e_{\text{oct}} \approx 0$; see Equation (17)). These apply when the tertiary companion has a circular orbit ($e_{\text{out}} = 0$) or when the inner BHs have equal masses ($m_1 = m_2$). Figure 3 summarizes our results for a given set of binary and companion parameters ($m_1 = m_3 = 30M_\odot$, $m_2 = 20M_\odot$, $a_0 = 100$ au, and $a_{\text{out}} = 4500$ au) as a function of the initial mutual inclination angle $I_0$. All initial systems satisfy the criterion of double averaging for triples (relation (20)).

The top panel of Figure 3 shows $e_{\text{max}}$ for a grid of inclinations (uniformly distributed in $\cos I_0$) in the absence of GW emission (this panel is similar to Figure 1, but is an enlarged version). We see that the eccentricity can be driven to be as large as $e_{\text{max}} \approx 1 - 10^{-6}$ at $I_0 = I_{\text{0,lim}} \approx 92^\circ$ (see Equation (43)). In the second panel, we plot $I_{\text{max}}$, which is the instantaneous inclination at $e = e_{\text{max}}$, as a function of $I_0$. When $e_{\text{max}}(I_0)$ achieves the maximum, $I_{\text{max}}$ becomes very close to $I_{\text{0,lim}}$, implying that the range of oscillation in $I$ is small (i.e., $L$ exhibits negligible nutation).

In the third panel of Figure 3, we turn on orbital decay due to gravitational radiation. The eccentricity of the inner binary is achieved within the Hubble time ($T_{\text{m}} \lesssim 10^{10}$ yr) for a range of inclinations around $I_{\text{0,lim}}$. The eccentricity excitation leads to a shorter binary merger time $T_{\text{m}}$ compared to the “circular” merger time $T_{\text{m,0}}$ (see Equation (11)). In Liu & Lai (2017), we found that the merger timescale in LK-induced mergers can be described by the fitting formula $T_{\text{m}} = T_{\text{m,0}}(1 - e_{\text{max}}^2)^3$; the coefficient $\alpha$ depends on $e_{\text{max}}$ (from Equation (42)), with $\alpha \approx 1.5, 2,$ and $2.5$ for $e_{\text{max}} = (0, 0.6), (0.6, 0.8),$ and $(0.8, 0.95)$, respectively. Here we consider the regime where $e_{\text{max}}$ is much closer to unity, and we find that $T_{\text{m}}$ can be best fitted by $\alpha = 3$, i.e.,

$$T_{\text{m}} \simeq T_{\text{m,0}}(1 - e_{\text{max}}^2)^3. \tag{48}$$

This scaling can be understood as follows: the intrinsic GW-induced orbital decay rate $|\dot{a}/a|_{\text{GW}}$ is proportional to $(1 - e^2)^{-7/2}$ (Equation (10)). In an LK-induced merger, the orbital decay mainly occurs at $e \approx e_{\text{max}}$. During the LK cycle, the binary only spreads a fraction $\sim \sqrt{1 - e_{\text{max}}^2}$ of the time near $e \approx e_{\text{max}}$. Thus, the LK-averaged orbital decay rate is of order $T_{\text{m,0}}^{-1}(1 - e_{\text{max}}^2)^3$, as indicated by Equation (48). Using
rather small. Also note that $T_m$ shows a constant distribution around $I_0 \sim I_{0,\text{lim}}$. This is the result of a “one-shot” merger, where the system only undergoes the first LK cycle, then “suddenly” merges during the high-$e$ phase.

Figures 4–6 show a few examples of the orbital evolution for systems inside the merger window, for which the initial inclination is $I_0 = 92.5^\circ, 92.3^\circ$, and $92.18^\circ$, respectively. The evolution of BH spin is also shown, and this will be discussed in Section 4. In the three upper panels of Figure 4, we see that the inner binary undergoes cyclic excitations to the maximum eccentricity $e_{\text{max}}$, with accompanying oscillations in the inclination $I$. As the binary decays, the range of eccentricity oscillations becomes smaller, and the eccentricity “freezes” at a large value. In the final phase, GW dissipation causes the orbit to circularize and its semimajor axis to shrink.

In Figures 5 and 6, $I_0$ is closer to $I_{0,\text{lim}}$, so that $e_{\text{max}}$ achieved during LK cycles is closer to $e_{\text{lim}}$. The GW-induced orbital decay is more efficient (Equations (8) and (9)), so the binary only experiences a few or even less than one LK cycles before merging. In Figure 5, the orbit undergoes the usual freezing of eccentricity oscillations as in Figure 4. In Figure 6, $a$ decays abruptly, and the binary merges in the first high-eccentricity episode (“one-shot merger”).

Figure 7 shows another example for a system with a more distant companion ($a_{\text{out}} = 6700$ au). Even though $I_0 \simeq I_{0,\text{lim}}$ for this example, the inner BH binary does not attain sufficiently large $e_{\text{max}}$ to enable a “one-shot” merger.

For all the examples considered in Figures 3–7, we find that the merging BH binaries have a negligible eccentricity ($e \lesssim 0.01$) when entering the aLIGO band.

The lower panel of Figure 8 shows the merger window (in terms of $\cos I_0$) as a function of the effective semimajor axis of the tertiary companion. From Section 2.2 (see Figure 1), we have found that in the quadrupole approximation, the eccentricity excitation depends on $m_3, e_{\text{out}}$, and $e_{\text{out}}$ through the ratio $m_3/a_{\text{out,eff}}^3$ (where $a_{\text{out,eff}}$ is given by Equation (19)). We therefore introduce the dimensionless scaled semimajor axis

$$
\tilde{a}_{\text{out,eff}} \equiv \frac{a_{\text{out,eff}}}{1000 \text{ au}} \left( \frac{m_3}{30 M_\odot} \right)^{-1/3}
$$

$$
= \frac{a_{\text{out}} \sqrt{1 - e_{\text{out}}^2}}{1000 \text{ au}} \left( \frac{m_3}{30 M_\odot} \right)^{-1/3}
$$

(50)

to characterize the “strength” of the outer perturber (note that Figure 8 neglects the octupole effect, which can complicate the single dependence of the merger window on $\tilde{a}_{\text{out,eff}}$; see Section 3.2). For a given BH binary ($m_1 = 30 M_\odot$, $m_2 = 20 M_\odot$, $a_0 = 100$ au), we fix $e_{\text{out}} = 0$ and $m_3 = 30 M_\odot$, or $10 M_\odot$, but vary $a_{\text{out}}$. For each $\tilde{a}_{\text{out,eff}}$, we consider 3000 values of $I_0$ spaced equally in $\cos I_0 \in (-1, 1)$, evolve the systems numerically, and record every merger event. The results obtained from the double- and single-averaged secular equations are marked by dark and light colors, respectively.

The upper panel of Figure 8 shows the merger fraction from the mergers shown in the lower panel, which can also be characterized by the analyzed expression

$$
f_{\text{merger}}(a_0, a_{\text{out}}, e_{\text{out}}) = \frac{1}{2} \left| \cos I_0^{\text{merger}} - \cos I_0^{\text{out}} \right|.
$$

(51)
In the upper panel, the merger fraction is around $\sim 1\%$, and gradually decreases as $\bar{a}_{\text{out},\text{eff}}$ increases. The merger window is closed when $\bar{a}_{\text{out},\text{eff}}$ exceeds a certain critical value. Figure 8 also shows results for a different value of $m_3 (10 M_{\odot})$. This gives different values of $\eta$ and $I_{0,\text{lim}}$ (Equation (43)), but the merger window is qualitatively similar to the $m_3 = 30 M_{\odot}$ case, except shifted to relatively large values of $I_0$. Both merger windows (for the two values of $m_3$) are well described by Equations (42) and (49). The merger fractions, $f_{\text{merger}}$, are essentially identical (see the dashed curve in the upper panel). These indicate that the fitting formula (49), together with Equation (42), can be used to predict what types of systems will undergo a merger in less than $10^{10}$ yr, at least in the quadrupole order. For “pure” quadrupole systems, simple scaling relations for $f_{\text{merger}}$ (as a function of $a_0$, $m_1$, $m_2$, and $T_{\text{cri}}$) can be obtained (see Section 3.3).

### 3.2. Eccentric Companions: Mergers Induced by the Octupole LK Effect

For $m_1 \neq m_2$ and eccentric companions ($e_{\text{out}} \neq 0$), the octupole effect becomes important when $\varepsilon_{\text{oct}}$ (Equation (17)) is appreciable, and some of the analytical expressions given in Section 2.2 break down. Previous works (Liu et al. 2015; Anderson et al. 2017) have shown that the main effect of the octupole potential is to broaden the range of the initial $I_0$ for extreme eccentricity excitations ($e_{\text{max}} = e_{\text{lim}}$), while the quadrupole expression for limiting eccentricity $e_{\text{lim}}$ (Equation (44)) remains valid.

Figure 9 shows some examples of the merger windows for different values of $\varepsilon_{\text{oct}}$. To illustrate the effect of octupole perturbation, we consider four cases with the same $\bar{a}_{\text{out},\text{eff}}$ (Equation (50)), but different $e_{\text{out}} (= 0, 0.3, 0.6, 0.9)$ and thus different $a_{\text{out}}$. All parameters in these examples satisfy the double averaging approximation (relation (20)). The initial longitude of the periapse $\omega_{\text{out}}$ is randomly chosen in the range $(0, 2\pi)$.

We find that, when the octupole effect gets stronger, the eccentricity excitation becomes increasingly erratic as a function of $I_0$, and more systems have the maximum eccentricity driven to be $1 - e_{\text{max}} \lesssim 10^{-4}$. Consequently, more mergers over the Hubble timescale can be generated, and the merger window becomes noticeably broader. Because of the erratic variation of $e_{\text{max}}$, the merger events are not uniformly spaced in $\cos I_0$. In this situation, the merger window cannot be described by the fitting formula (Equation (48); see the orange dashed curves in Figure 9).

Figure 10 depicts an example of the time evolution of binary merger for $\varepsilon_{\text{oct}} = 0$. Because of the octupole effect, the maximum eccentricities reached in successive (quadrupole) LK cycles increase. Eventually $e_{\text{max}}$ becomes sufficiently large and the binary merges quickly.

When $\varepsilon_{\text{oct}}$ is sufficiently large, the orbital evolution of the inner binary becomes chaotic, and the evolution shows a strong dependence on the initial conditions (e.g., Lithwick & Naoz 2011; Li et al. 2014). Figure 11 illustrates this chaotic behavior. We see that the octupole-induced extreme eccentricity excitation occurs in an irregular manner, shortening or extending the time for mergers. As a result, $T_{\text{in}}$ exhibits an irregular dependence on $I_0$, as seen in Figure 9.

The merger windows shown in Figure 9 are based on the DA secular equations. For close and more eccentric companions, these DA equations break down, and we can use SA equations (see Section 2.1.2). Figure 12 shows sample numerical results for the merger windows computed using SA equations and DA...
equations. Here, $a_{\text{out,eff}}$ is chosen to be relatively small ($\approx 5.6$), where the system lies near the boundary of parameter regions between single and double averaging. The upper panel shows the eccentricity excitation for a grid of $I_0$ values when the system is non-dissipative (i.e., GW emission is turned off). We find that a portion of the systems computed from SA equations can reach higher $e_{\text{max}}$, even beyond $e_{\text{crit}}$ (which is derived from the double averaging approximation). In particular, when gravitational radiation is included, a larger number of mergers with $T_{\text{m}} \lesssim (\text{a few}) \times 10^9$ yr occur due to the extreme $e_{\text{max}}$ as depicted in the lower panel; such rapid mergers are relatively rare in the calculations based on the DA equations.

Figure 5 shows the merger windows and merger fractions as a function of $a_{\text{out,eff}}$ (see Equation (50)) for different values of $e_{\text{out}}$ in our calculations, the orientation of the initial $e_{\text{out}}$ (for a given $I_0$) is random (i.e., $\omega_{\text{out,0}}$ is uniformly distributed in $0-2\pi$). We see that, for a given $e_{\text{out}}$, the merger window shows a general trend of widening as $a_{\text{out,eff}}$ decreases, resulting in an increase in $f_{\text{merger}}$. Moreover, for the same value of $a_{\text{out,eff}}$ (thus the same quadrupole effect), the merger window and merger fraction can be very different for different $e_{\text{out}}$. In general, the larger the eccentricity $e_{\text{out}}$, the stronger the octupole effect, and therefore the wider the window. Compared to the analytical expressions based on the quadrupole approximation (see Section 3.1), $f_{\text{merger}}$ can be enhanced by a factor of a few. Note that for some values of $\bar{a}_{\text{out,eff}}$, an irregular distribution of merger events inside the merger window appears; this results from the chaotic behaviors of the octupole-order LK oscillations (see also the examples in Figure 9, particularly the $e_{\text{out}} = 0.6$ case).

3.3. Scaling Relations for Quadrupole Systems and Application to NS Binaries

Although in this paper we focus on BH binaries, a similar analysis can be done for NS binary mergers induced by tertiary companions. A double NS merger event (GW170817) has recently been detected through GWs and electromagnetic radiation (e.g., Abbott et al. 2017d). We can expect more such detections in the future.

NS binaries differ from BH binaries in that the NS mass is much smaller than the BH mass, and thus for the same initial $a_{\text{in}} = a_0$ ($\gtrsim 1$ au), a larger eccentricity excitation is required to induce NS binary merger. Moreover, since the masses of the two members of NS binaries are typically quite similar, the octupole LK effect is negligible ($e_{\text{crit}} \approx 0$). Therefore, the mergers of NS binaries in the presence of distant companions can be well described in the quadrupole approximation (see Section 3.1). Thus, the maximum eccentricity required for mergers (within time $T_{\text{crit}}$) can be obtained from Equation (49), and the required initially mutual inclination can be calculated using Equation (42) by replacing $e_{\text{max}}$ with $e_{\text{crit}}$. In other words, the merger window and merger fraction for NS binaries can be calculated analytically (Equation (51)), without the need for numerical integrations of the single- or double-averaged equations.

Figure 14 presents the results for the merger window and merger fraction for equal-mass NS binaries ($m_1 = m_2 = 1.4M_\odot$). All systems shown satisfy the stability criterion. We choose three different initial semimajor axes ($a_0 = 1, 10$, and 100 au). For each NS binary, we consider a variety of tertiary bodies (different $m_3$ and $e_{\text{out}}$ as labeled). We find that, for a given $a_0$, different $m_3$ and $e_{\text{out}}$ (with the same $\bar{a}_{\text{out,eff}}$) affect the position of the merger window (i.e., the range of cos $I_0$) but not the value of $f_{\text{merger}}$. On the other hand, the merger windows and fractions have a strong dependence on the initial semimajor axis (e.g., $f_{\text{merger}}$ for $a_0 = 1$ au is about 100 times larger than that for $a_0 = 100$ au). This is because, for the small $a_0$, the induced eccentricity in the LK oscillations does not have to be too large to produce mergers within $10^{10}$ yr (e.g., $1 - e_{\text{m}} \approx 10^{-3}$ for $a_0 = 1$ au, $10^{-4}$ for $a_0 = 10$ au, and $10^{-6}$ for $a_0 = 100$ au). In addition, the range of $\bar{a}_{\text{out,eff}}$ producing a merger is different for different $a_0$.

Note that for $a_0 \lesssim 1$ au, binary interactions, such as mass transfer and common-envelope phase may be important. We include the $a_0 = 1$ au case to illustrate the dependence of our results on $a_0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{Same as Figure 4, except for $I_0 = 92\,\text{hr}$.}
\end{figure}
The result of Figure 14 (upper panel) for the merger fraction can be summarized by the fitting formula for $f_{\text{merger}}$, the maximum value of $f_{\text{merger}}$ (for a given $a_0$), and $\dot{a}_{\text{out,eff}}$, the maximum value of $\dot{a}_{\text{out,eff}}$ for a merger to be possible. Figure 15 shows that for the parameters of Figure 14 (with $m_1 = m_2 = 1.4 M_\odot$, $T_{\text{crit}} = 10^{10}$ yr), we have

$$f_{\text{merger}}^{\text{max}} \approx 5.8\% \left( \frac{a_0}{\text{au}} \right)^{-0.67}, \quad \dot{a}_{\text{out,eff}}^{\text{max}} \approx 0.23 \left( \frac{a_0}{\text{au}} \right)^{1.1}. \quad (52)$$

Scaling relations for general quadrupole systems, Equation (52) can be generalized to other types of systems (with different $m_1$, $m_2$) and different value of merger time $T_{\text{crit}}$. From Equations (11) and (49), we see that the critical eccentricity $e_{\text{crit}}$ required for merger within time $T_{\text{crit}}$ depends on $\mu T_{\text{crit}}(m_2/a_0^2)^2$. From Equation (42) we see that for $\eta \ll 1$ (a good approximation), the critical inclinations ($i_{\text{crit}}^\pm$; see Equation (51)) for a given $e_{\text{max}} = e_{\text{crit}}$ depend only on $e_{\text{GR}}$, or the combination ($m_1/a_0^2)^2(a_{\text{out,eff}}/m_3)$. Thus the merger fraction $f_{\text{merger}}$ depends on $m_1$, $m_2$, $a_0$, and $T_{\text{crit}}$ only through $m_2/a_0^2$ and $\mu T_{\text{crit}}$. We therefore expect from Equation (52) that

$$f_{\text{merger}}^{\text{max}} \propto (a_0/m_1)^{(5\beta/3)}(\mu T_{\text{crit}})^{\alpha}$$

and

$$\dot{a}_{\text{out,eff}}^{\text{max}} \propto (a_0/m_1)^{(5\beta/3)}(\mu T_{\text{crit}})^{\alpha}, \quad (53)$$

where $\alpha$, $\beta$ are fitting parameters. Figure 16 shows the fitting. We find

These fitting formulae are valid for any type of LK-induced BH/NS merger in the quadrupole order.

### 4. Evolution of BH Spin and Spin–Orbit Misalignment

#### 4.1. Spin–Orbit Coupling

We now study how the BH spin evolves during LK-induced binary mergers. We present the evolution equation for $S_1 = S_0 S_1$ (where $S_1$ is the magnetic spin angular momentum of $m_1$ and $S_1$ is the unit vector). The de Sitter precession of $S_1$ around $\hat{L}$ (1.5 PN effect) is governed by (e.g., Barker & O’Connell 1975)

$$\frac{d\hat{S}_1}{dt} = \Omega_{\text{SL}} \times \hat{S}_1, \quad (55)$$

with the orbital-averaged spin precession rate

$$\Omega_{\text{SL}} = \frac{3 G \eta (m_2 + \mu/3)}{2 c^2 a (1 - e^2)} \quad (56)$$

A similar equation applies to the spinning body 2. Note that $\Omega_{\text{SL}}$ is of the same order as $\Omega_{\text{GR}}$ (Equation (7)) for $m_1 \sim m_2$. There are also back-reaction torques from $\hat{S}_1$ on $\hat{L}$ and $\hat{e}$:

$$\frac{d\hat{L}}{dt}_{\text{LS}} = \Omega_{\text{LS}} \hat{L} \times \hat{S}_1, \quad (57)$$

$$\frac{d\hat{e}}{dt}_{\text{LS}} = \Omega_{\text{LS}} \hat{S}_1 \times \hat{e} - 3 \Omega_{\text{LS}}(\hat{L} \cdot \hat{S}_1) \hat{L} \times \hat{e}, \quad (57)$$

where

$$\Omega_{\text{LS}} = \Omega_{\text{SL}} \frac{S_1}{L} = \frac{G S_1 (4 + 3 m_2/m_1)}{2 c^2 a^2 (1 - e^2)^{3/2}}. \quad (58)$$
by the tertiary companion is neglected as well (since we consider \( m_1 \sim m_2 \), and \( m_3 \) is not much larger, but \( a_{\text{out}} \gg a \). In all our calculations, we use \( \chi_1 = \chi_2 = 0.1 \) for concreteness. But note that the values of \( \chi_1 \) and \( \chi_2 \) do not affect the results of our paper (except Figure 20, which assumes \( \chi_1 = \chi_2 \)). This is because (i) the de Sitter precession frequency \( \Omega_{\text{SL}} \) (Equation (56)) is independent of \( \chi_1, \chi_2 \), and (ii) the inequality \( S_1, S_2 \ll L \) is well satisfied (see footnote 7).

In terms of the inner BH binary axis \( \hat{L} \), the effect of the companion is to induce precession of \( \hat{L} \) around \( \hat{L}_{\text{out}} \) with nutation (when \( e \neq 0 \)). In the quadrupole order, we have

\[
\frac{d\hat{L}}{dt}_{\text{LK,quad}} = \frac{-3L}{4\hat{L}_{\text{K,quad}} \sqrt{1 - e^2}} \times (J \cdot \hat{L}_{\text{out}}) \hat{L}_{\text{out}} \times j + 5(e \cdot \hat{L}_{\text{out}}) e \times \hat{L}_{\text{out}}.
\]

(59)

An approximate expression for the rate of change of \( \hat{L} \) is given by (Anderson et al. 2016)

\[
\Omega_L = \left| \frac{d\hat{L}}{dt}_{\text{LK,quad}} \right| \approx \frac{3(1 + 4e^2)}{8\hat{L}_{\text{K,quad}} \sqrt{1 - e^2}} |\sin 2I|.
\]

(60)

The spin evolution is determined by two competing processes: \( \dot{\hat{S}}_i \) precesses around \( \hat{L} \) at the rate \( \Omega_{\text{SL}} \), and \( \hat{L} \) varies at the rate \( \Omega_L \). There are three possible spin behaviors depending on the ratio \( \Omega_{\text{SL}}/\Omega_L \):

(i) For \( \Omega_{\text{SL}} \gg \Omega_L \) (“nonadiabatic” regime), the spin axis \( \dot{\hat{S}}_i \) cannot “keep up” with the rapidly changing \( \hat{L} \), which precesses around a fixed \( \hat{L}_{\text{out}} \) (for \( L_{\text{out}} \gg L \)). Thus \( \dot{\hat{S}}_i \) effectively precesses around \( \hat{L}_{\text{out}} \), keeping the misalignment angle between \( \hat{S}_i \) and \( \hat{L}_{\text{out}} \), \( \theta_{\text{db}} = \cos^{-1}(\hat{S}_i \cdot \hat{L}_{\text{out}}) \), approximately constant.

(ii) For \( \Omega_{\text{SL}} \gg \Omega_L \) ("adiabatic" regime), \( \dot{\hat{S}}_i \) is strongly coupled to \( \hat{L} \). The spin axis \( \dot{\hat{S}}_i \) closely “follows” \( \hat{L} \), maintaining an approximately constant spin–orbit misalignment angle \( \theta_{\text{sl}} = \cos^{-1}(\hat{S}_i \cdot \hat{L}) \).

---

Footnote 7: With \( S_1 = \chi_1 Gm_1^2/c^2 \), the ratio \( S_1/L \) is \( \chi_1 (m_1/m_2)(Gm_2/c^2)(1 - e^2) \), which is \( \ll 1 \) for \( a \gg Gm_2/c^2 \) (i.e., the inner binary pericenter distance is much larger than the gravitational radius). The inequality \( S_1 \ll L \) is well satisfied for our calculations since the spin–orbit misalignment angle \( \theta_{\text{db}} \) is frozen well before the inner binary reaches the separation \( Gm_2/c^2 \) (see Equations (61) and (62)).
For $\Omega_{SL} \sim \Omega_L$ ("trans-adiabatic" regime), the spin evolution can be complex, potentially generating large spin–orbit misalignment $\theta_{sl}$. Since both $\Omega_{SL}$ and $\Omega_L$ depend on $e$ during the LK cycles, the precise transitions between these regimes can be fuzzy.
To help characterize the spin dynamics, we introduce an “adiabaticity parameter” as

$$A \equiv \frac{\Omega_{\text{SL}}}{\Omega_L} = A_0 \frac{1}{1 + 4e^2} \sqrt{1 - e^2 \sin 2I},$$

where

$$A_0 \equiv \left| \frac{\Omega_{\text{SL}}}{\Omega_L} \sin 2I \right|_{e=0} = \frac{4G(m_2 + \mu/3)m_1a_{\text{out}}^3}{c^2m_3a^4} \left( \frac{m_2}{35 M_\odot} \right) \left( \frac{m_1}{60 M_\odot} \right) \times \left( \frac{m_3}{30 M_\odot} \right)^{-1} \left( \frac{a_{\text{out}}}{10^2 \text{ au}} \right)^{-4}.$$

Note that $A$ has a steep dependence on the eccentricity $e$ and inclination $I$, and it is time-varying, while $A_0$ is an intrinsic indicator for identifying which system may undergo potentially complicated spin evolution. Since $A_0$ depends sensitively on $a$, during the orbital decay a system may transit from “non-adiabatic” at large $a$ to “adiabatic” at small $a$, where the final spin–orbit misalignment angle $\theta_{\text{si}}$ is “frozen.” Note that $A_0$ is directly related to $\varepsilon_{\text{GR}}$ (see Equation (41)) by

$$A_0 \equiv \frac{\varepsilon_{\text{GR}}}{3} \frac{4m_2 + \mu/3}{m_1a^2}.$$

Thus, when the initial value of $\varepsilon_{\text{GR}}$ (at $a = a_0$) satisfies $\varepsilon_{\text{GR},0} \lesssim 9/4$ (a necessary condition for LK eccentricity excitation; see relation (46)), we also have the initial $A_0 \lesssim (3m_2 + \mu)/m_1 \sim 1$. This implies that any system that experiences enhanced orbital decay due to LK oscillations must go through the “trans-adiabatic” regime and therefore possibly complicated spin evolution (Liu & Lai 2017).

In our previous study (Liu & Lai 2017), we considered initially compact BH binaries (with $a_0 \sim 0.2 \text{ au}$), which can merge by themselves without the aid of a tertiary companion. We focused on systems with initial $A_0$ not much less than unity, and showed that such systems can experience complex/chaotic spin evolution during the LK-enhanced mergers. In this paper, we consider the inner BH binaries with large initial semimajor axis ($a_0 = 20$ or $100 \text{ au}$) and initial $A_0 \ll 1$. As we shall see, such systems exhibit a variety of different spin evolutionary behaviors during the LK-induced mergers.

The bottom panel of Figure 4 shows a representative example of the spin evolution during the LK-induced orbital decay. The system begins with $A_0 \sim 10^{-3} \ll 1$ (Equation (62)). At the early stage of the evolution, $\Omega_{\text{SL}} \ll \Omega_L$, leading $\theta_{\text{si}}$ to be nearly constant. Because of the large variation of $\vec{L}$, the spin–orbit angle $\theta_{\text{si}}$ oscillates with a large amplitude. As the orbit decays and circularizes, $\vec{L}$ becomes frozen relative to $\vec{L}_\text{out}$ (with final inclination $I \simeq 125^\circ$), while $\vec{S}$ precesses rapidly around $\vec{L}$, with $\theta_{\text{si}}$ settling down to the final value ($\approx 90^\circ$). A non-zero final spin–orbit misalignment has been produced from the originally aligned configuration. This is only one example of the complex BH spin evolutionary paths during LK-induced mergers (Section 4.2).
Equation (59)), whereas the precession of $\mathbf{S}$ around $\dot{\mathbf{L}}$ is prograde, so secular resonance does not usually happen when the orbital evolution is regular.

In the case of NS binaries, a Newtonian effect due to the oblateness of the NS ($m_1$) also contributes to the spin precession. Equation (55) is changed to

$$\frac{d\mathbf{S}_1}{dt} = \Omega_{\text{SL}}^{(\text{Newtonian})} \mathbf{L} \times \mathbf{S}_1,$$

(64)

where

$$\Omega_{\text{SL}}^{(\text{Newtonian})} = \frac{3Gm_2(I_3 - I_1)}{2a^3(1 - e^2)^{3/2}} \cos \theta_{\text{nl}},$$

(65)

Here, $I_1$ and $I_3$ are principal moments of inertia of the NS. For $I_3 - I_1 \equiv k_{q*}m_1R_1^2\Omega_1^2$ and $S_1 = I_1\Omega_1 = k_{q*}m_1R_1^2\Omega_1$, where $R_1$ is the NS radius and $\Omega_1$ is the rotation rate of the NS in units of $(Gm/R_1)^{1/2}$, we have

$$\Omega_{\text{SL}}^{(\text{Newtonian})} = -\frac{3k_{q*}}{2k_*} \left( \frac{m_2}{m_1} \right) \left( \frac{R_1}{a} \right)^3 (1 - e^2)^{3/2} \cos \theta_{\text{nl}}.$$

(66)

Thus the ratio $\Omega_{\text{SL}}^{(\text{Newtonian})}/\Omega_{\text{SL}}^{(\text{NS})}$ is

$$\left| \frac{\Omega_{\text{SL}}^{(\text{Newtonian})}}{\Omega_{\text{SL}}^{(\text{NS})}} \right| = \left| \frac{k_{q*}}{k_*} \left( \frac{m_2}{m_1} \right) \frac{R_1}{a(1 - e^2)} \right|^{1/2}$$

$$\times \frac{R_1c^2}{G(m_2 + \mu/3)} \hat{\Omega}_1 |\cos \theta_{\text{nl}}|.$$

(67)

For a typical NS, $m_1 \approx 1.4M_\odot$, $R_1 \approx 10$ km, $k_{q*} \approx 0.26$, and $\hat{\Omega}_1 \approx 0.023(P_1/20 \text{ ms})^{-1}$ (where $P_1$ is the rotation period of the NS). Since $a(1 - e^2) > R_1$, we see that $|\Omega_{\text{SL}}^{(\text{Newtonian})}/\Omega_{\text{SL}}^{(\text{NS})}|$ is always $\ll 1$.

For polytropic stellar models (with index $n$), $k_{q*}$ is approximately related to $k_*$ via the relations $k_* = 2k_q/5$ and $k_{q*} \approx k_q^*/2(1 - n/5)$. For $n = 1$, $k_q^* \approx 0.65$ (see Table 1 of Lai et al. 1993).

Figure 11. Same as Figure 10, but for $e_{\text{out}} = 0.9$. The initial parameters $I_0$ and $\omega_{\text{out},0}$ are as labeled. Because of the chaotic nature of the octupole LK effect, small changes in $I_0$ or $\omega_{\text{out},0}$ lead to very different merger times.

Figure 12. Eccentricity excitation (no GW emission; upper panel) and merger time (with GW emission; lower panel) as functions of $\cos I_0$. The parameters are the same as in the $e_{\text{out}} = 0.9$ case of Figure 9, except for a closer companion ($a_{\text{out}} = 12,800$ au). The cyan and purple dots are obtained from calculations based on the single- and double-averaged equations, respectively.

4.2. Complex BH Spin Evolution Paths

We have seen from Section 3 that LK-induced BH binaries can have a variety of orbital evolution paths toward the final merger. Correspondingly, the evolution of BH spin in these binaries also exhibits a rich set of evolutionary behaviors. They can be roughly divided into four cases (see Figure 17).

Case I (see Figures 4 and 5). This usually occurs when the initial inclination $I_0$ is sufficiently different from $I_{0,\text{lim}}$ (see Equation (43)) that $(1 - e_{\text{max}})$ is much larger than $(1 - e_{\text{lim}})$. In this case, the inner binary experiences multiple LK oscillations; the amplitude of the eccentricity oscillations shrinks gradually as the orbit decays; eventually the binary circularizes and merges quickly. As shown in the lower panels of Figures 4 and 5, during
the early stage, the angle $\theta_{sb}$ is approximately constant (since $A_0 \ll 1$), while $\theta_d$ exhibits a larger variation due to the rapid precession of $\hat{L}$ around $L_{\text{out}}$; at the later stage, as the orbit decays, $\hat{L}$ becomes fixed relative to $L_{\text{out}}$, while $\hat{S}$ precesses rapidly around $\hat{L}$ with a fixed final $\theta_d$ close to 90°.

Case II (Figures 10 and 11). This occurs when $I_0$ is not close to $I_{0,\text{lim}}$, but $e_{\text{max}}$ is driven to a value close to $e_{\text{lim}}$ due to the octupole effect. As seen from Figures 10 and 11, the inner binary experiences multiple LK cycles, each with increasing $e_{\text{max}}$ driven by the octupole potential; eventually $e_{\text{max}}$ becomes sufficiently large and the orbit decays rapidly. Unlike case I, the spin evolution transitions from the “nonadiabatic” regime to the “adiabatic” regime quickly. Because of the extremely rapid orbital decay, the oscillation of $\theta_d$ continues to the end (by contrast, in Figures 4 and 5, the $\theta_d$ oscillation freezes as the orbit decays), and the final $\theta_d$ lies in the range $0 < \theta_d < 90°$.

Case III (Figure 7). This occurs when $I_0$ is close to $I_{0,\text{lim}}$. Similarly to Case I, the orbit goes through eccentricity oscillations, suppression of the oscillations, and circularization. However, since $I_0 \approx I_{0,\text{lim}}$, the orbital inclination oscillates with a small amplitude and passes through 90°. This implies that $\hat{S}$ stays fairly close to $\hat{L}$ at the early stage (see Figure 17) and $\theta_d$ does not experience large-amplitude (0–$\pi$) oscillations. Eventually, $\theta_d$ settles down to a value below 90°.

Case IV (see Figure 6). This also occurs when $I_0$ is close to $I_{0,\text{lim}}$. Similarly to Case II, the inner binary experiences extreme eccentricity excitation and merges within one LK cycle (one-shot merger). Because $\hat{L}$ basically does not evolve in time (see Figure 17), a small ($<90°$) $\theta_d$ is produced.

It is clear that the spin evolution is complicated and depends on various parameters and timescales. Our understanding of the spin behaviors is based largely on the numerical integrations. The four cases discussed above are representative, and do not capture the complete sets of spin evolutionary behaviors.

Figures 3 and 9 (bottom panels) show the final distribution of $\theta_d$ as a function of $e_0$ in the merger window for several different systems. When $e_0 = 0$, $\theta_d$ has a regular distribution, and most of the values are found around $\approx 90°$; the spin
evolution follows the examples in Case I ($\theta_0^f \approx 90^\circ$), Case III, and Case IV (one-shot merger) discussed above. When $e_{\text{out}} = 0$, $\theta_0^f$ shows a much wider range of values from $0^\circ$ to $180^\circ$, due to the octupole effect (as in Case II discussed above).

Note that, for small $e_{\text{out}}$, the final spin–orbit misalignment angles $\theta_{0,\text{s}}^f$ and $\theta_{0,\text{sl}}^f$ are strongly correlated; this correlation is particularly strong for the $e_{\text{out}} = 0$ case (see Figure 9). This arises because the orbital evolution is regular for small $e_{\text{out}}$. The spin vectors $\hat{S}_1$ and $\hat{S}_2$ evolve independently during the orbital decay (since spin–spin coupling is negligible). Although the de Sitter precession rates of $\hat{S}_1$ and $\hat{S}_2$ are different (since $m_1 \neq m_2$), the spin evolution is regular as long as $A_0 \ll 1$ (corresponding to Cases I, III, and IV discussed above in this section). In particular, the “90$^\circ$ attractor” is a generic feature independent of the precise value of $\Omega_{\text{SL}}$ (see Section 4.3).

In contrast, for high $e_{\text{out}}$ (see the $e_{\text{out}} = 0.9$ case in Figure 9), the octupole effect makes the orbital evolution chaotic, which also induces chaotic evolution of the spin–orbit misalignment (see Figure 11). Therefore, $\theta_{0,\text{s}}^f$ and $\theta_{0,\text{sl}}^f$ become largely uncorrelated.

4.3. Understanding the Spin Evolution: The 90$^\circ$ Attractor?

We see from the previous subsections that when the octupole effect is negligible ($e_{\text{out}} \ll 1$), the spin–orbit misalignment angle (starting from initial $\theta_0^f = 0^\circ$) often evolves toward $\theta_0^f \approx 90^\circ$ as the binary orbit decays. What is the origin of this 90$^\circ$ “attractor”?

In Liu & Lai (2017), we used the principle of adiabatic invariance to derive an analytical expression for $\theta_0^f$ for the case where the inner BH binary remains circular in the presence of an inclined tertiary companion (i.e., the inner binary merges by itself without eccentricity excitation, although the binary orbital angular momentum axis $\hat{L}$ does vary and precess around $\hat{L}_{\text{out}}$).

We can use a similar idea to understand qualitatively the origin of the 90$^\circ$ attractor in quadrupole LK-induced mergers. In the quadrupole order, the angular momentum axis $\hat{L}$ varies at the rate given by Equation (60). This variation involves precession around $\hat{L}_{\text{out}}$ and nutation (change in $I$). If we neglect nutation, we have

$$\frac{d\hat{L}}{dt}_{\text{quad}} = -\Omega_{\text{pl}} \hat{L}_{\text{out}} \times \hat{L} = -\Omega_{\text{pl}} \frac{L_{\text{out}}}{L_{\text{tot}}} \times \hat{L},$$

(68)

where $L_{\text{tot}} = L + L_{\text{out}}$, and

$$\Omega_{\text{pl}} = \frac{3\hat{L} \cdot \hat{L}_{\text{out}}}{4\Omega_{\text{tot}} \sqrt{1 - e^2}} (1 + 4e^2).$$

(69)

Even when nutation is neglected, Equation (68) is approximate since a fast-varying term in Equation (69) has been neglected.
Equation (68) shows that $\hat{L}$ rotates around the $L_{\text{rot}}$ axis. In this rotating frame, the spin evolution Equation (55) transforms to

$$\left(\frac{d\hat{S}_i}{dt}\right)_{\text{rot}} = \Omega_{\text{eff}} \times \hat{S}_i,$$

where

$$\Omega_{\text{eff}} = \Omega_{\text{SL}} \hat{L} + \Omega_{\text{pl}} \frac{L_{\text{rot}}}{L_{\text{out}}}.$$  \hspace{1cm} (71)\hspace{1cm}

Note that the ratio between $\Omega_{\text{SL}}$ and $\Omega_{\text{pl}}$ is

$$\left| \frac{\Omega_{\text{SL}}}{\Omega_{\text{pl}}} \right| = \mathcal{A} \sin \theta = -\frac{\mathcal{A}_0}{2(1 + 4e^2) \sqrt{1 - e^2|\cos \theta|}},$$  \hspace{1cm} (72)\hspace{1cm}

where $\mathcal{A}, \mathcal{A}_0$ are given by Equations (61) and (62).

If we assume that $\Omega_{\text{eff}} \equiv \Omega_{\text{eff}} / |\Omega_{\text{eff}}|$ varies slowly (much more slowly than $|\Omega_{\text{eff}}|$; see below), then $\theta_{\text{eff},S}$, the angle between $\hat{S}_i$ and $\Omega_{\text{eff}}$, is an adiabatic invariant. Suppose $\hat{S}_i$ and $\hat{L}$ are aligned initially ($\theta_{\text{sl}} = 0^\circ$), then the initial $\theta_{\text{eff},S}$ equals the

Figure 17. Trajectories of the spin and angular momentum axes of the inner BH binary. The left panels show the projection of $\hat{L}$ in the $x$-$y$ plane (where the $z$-axis is along the initial total angular momentum of the triple, which is also approximately aligned with $L_{\text{out}}$). The middle panels show the similar projection of $\hat{S}$. The right panels show the projection of $\hat{S}$ in the plane perpendicular to $\hat{L}$ (with $\theta_{sl}$ the angle between $\hat{S}$ and $\hat{L}$, and $\phi_{sl}$ the rotational phase of $\hat{S}$ around $\hat{L}$). In each panel, the filled circle (square) denotes the initial (final) position. The four cases shown here correspond to Figures 5, 10, 7, and 6, respectively.
the inner binary, and the adiabatic evolution correctly predicts the 90° attractor in the spin–orbit misalignment. For the other cases (Cases II–IV), \( \theta_{\text{eff,S}} \) undergoes significant change during the inner binary’s evolution, especially near the final orbital decay phase; in these cases, Equation (73) does not predict the correct \( \theta_0 \).

The validity of adiabatic evolution requires that the rate of change of \( \Omega_{\text{eff}} = \Omega_{\text{eff}}/|\Omega_{\text{eff}}| \) be much slower than \( |\Omega_{\text{eff}}| \), i.e., \( |d\Omega_{\text{eff}}/dt| \ll |\Omega_{\text{eff}}| \). In order of magnitude, we have \( |d\Omega_{\text{eff}}/dt| \sim T_{\text{GW}}^{-1} \) (see Equation (10)).\(^\text{11}\) In Case I, \( \epsilon_{\text{max}} \) induced by the tertiary companion is not too extreme. So the orbital decay is “gentle” and the adiabatic condition is satisfied. In Cases II, III, and IV, the rapid orbital decay at high eccentricity implies \( T_{\text{GW}}^{-1} \gg |\Omega_{\text{eff}}| \), so the adiabatic evolution breaks down.

We reiterate that the above analysis cannot be considered rigorous, since the precession rate in Equation (68) is approximate and the nutation of \( \hat{L} \) has been neglected. Nevertheless, this analysis (especially Figure 18) provides a qualitative understanding as to why \( \theta_0 \) evolves toward 90° under some conditions.

4.4. Final Distribution of Spin–Orbit Misalignment Angles

Having studied the various spin evolutionary paths in the previous subsections, we now calculate the distribution of final spin–orbit misalignment angle for the merging systems studied in Figure 13. We consider the spins of both BHs, and assume that both \( S_1 \) and \( S_2 \) are initially aligned with respect to the binary orbital angular momentum axis.

Figure 19 summarizes our results for \( a_0 = 100 \) au (the results for \( a_0 = 20 \) au are very similar). The range of \( \theta_{\text{out,eff}} \) values considered in this figure all lie in the regime where the double averaging approximation is valid (see Section 2.3). As in Figure 13, four different values of \( \epsilon_{\text{out}} \) are considered. When \( \epsilon_{\text{out}} = 0 \) (the top left panels), the spin evolution is regular, following the examples of Case I, Case III, and Case IV (see Section 4.2). For \( \theta_{\text{out,eff}} \in \{4.2, 6.5\} \), the tertiary companion is relatively close, and many BH binaries inside the merger window pass through successive stages of LK oscillations, LK suppression, and orbital circularization (Case I), producing a large number of systems with \( \theta_0 \) around 90°. For \( \theta_{\text{out,eff}} \in \{6.6, 8.8\} \), the tertiary companion is relatively distant; the eccentricity cannot grow to be in the case of small \( \theta_{\text{out,eff}} \), even when \( L_0 \sim L_{\text{lim}} \). Thus, the spin mainly evolves as described in Case III, and \( \theta_0 \) lies in the range 0°–90°.

When the companions are eccentric (\( \epsilon_{\text{out}} \neq 0 \)), the octupole effect comes into play and the spin may follow the dynamics of Case II. For a given \( \epsilon_{\text{out}} \), when the companion is relatively close (\( \theta_{\text{out,eff}} \) is small), the octupole effect becomes more prominent. In the case of \( \epsilon_{\text{out}} = 0.9 \), the orbital evolution is dominated by the octupole effect, and the distribution of \( \theta_0 \) is close to being isotropic (i.e., uniform distribution in \( \cos \theta_0 \)), as shown in the bottom-right panel of Figure 19.

Since the two components of the BH binary have comparable masses, the de Sitter precession rates are similar. Thus it is not surprising that the distributions of \( \theta_0 \) for both

\(^{11}\) Since \( \Omega_{\text{SL}} \) and \( \Omega_{\text{d}} \) both depend on \( r \), vector \( \Omega_{\text{eff}} \) also varies on the timescale \( r_{KX} \sqrt{1 - e^2} \), which can be comparable to \( |\Omega_{\text{eff}}|^{-1} \). However, in the early phase, \( |\Omega_{\text{eff}}| \gg \Omega_{\text{SL}} \) (this breaks down when \( L \) crosses 90°, as in Case III; see Figure 7), we have \( \Omega_{\text{eff}} \sim \Omega_{\text{d}} \sim \Omega_{\text{sl}}, \) which is nearly constant. As the orbit decays, \( \Omega_{\text{SL}} \) becomes large relative to \( \Omega_{\text{d}}, \) and \( \Omega_{\text{eff}} \) transitions to \( \Omega_{\text{d}}. \)
Figure 19. Final spin–orbit misalignment angles for both $S_1$ and $S_2$ as a function of $a_{\text{out,eff}}$ (see Equation (50)), and the associated distribution. The four cases ($e_{\text{out}} = 0, 0.3, 0.6, 0.9$) shown here are from the mergers achieved by the double-averaged secular equations as depicted in the left panels of Figure 13. The system parameters are $m_1 = 30 M_\odot$, $m_2 = 20 M_\odot$, $m_3 = 30 M_\odot$, $a_0 = 100$ au. The parameter $\theta_0$ (Equation (62)) here corresponds to the spinning body $m_1$. In the distribution ($N/N_{\text{max}}$ vs. $\cos \theta_0$), the range of $a_{\text{out,eff}}$ is specified, and $N_{\text{max}}$ is the number of merger events for the corresponding range of $a_{\text{out,eff}}$. In all panels, orange corresponds to $S_1$, green corresponds to $S_2$, and brown corresponds to the overlapped region.
spins are similar. Note that $\theta_{s1}^L$ and $\theta_{s2}^L$ are strongly correlated for $e_{\text{out}} = 0$, and this correlation becomes much weaker as the octupole effect becomes stronger (see Figure 9).

Having obtained the distributions of $\cos \theta_{s1}^L$ and $\cos \theta_{s2}^L$ for a range of systems with different parameters, we can compute the distribution of the effective spin parameter for the merging binaries (see Equation (1)):  

$$\chi_{\text{eff}} = \frac{m_1 \chi_1 \cos \theta_{s1}^L + m_2 \chi_2 \cos \theta_{s2}^L}{m_2},$$

(75)

where $\chi_{1,2}$ are the dimensionless BH spins (we set $\chi_1 = \chi_2 = 0.1$ in our calculations, although our results for $\theta_{s1}^L$ and $\theta_{s2}^L$ are not affected by this choice since $S_1, S_2 << L$ for all the systems considered in this paper). Figure 20 shows two examples (for $a_0 = 100$ au and 20 au; see Figures 13 and 19), assuming $\chi_1 = \chi_2$. To obtain the $\chi_{\text{eff}}$ distribution, we consider systems with $\tilde{a}_{\text{out},\text{eff}} \in (5.6, 8.8)$ for the $a_0 = 100$ au case and $\tilde{a}_{\text{out},\text{eff}} \in (0.8, 1.4)$ for the $a_0 = 20$ au case, and assume that the eccentricity of the tertiary companion has a uniform distribution in $e_{\text{out}}$ (i.e., $e_{\text{out}} = 0, 0.3, 0.6, 0.9$ are equally probable), and that the initial mutual inclination is randomly distributed (uniform in $\cos \theta_0$). We see that although the systems with the most eccentric companion ($e_{\text{out}} = 0.9$) contribute a substantial fraction of mergers, the overall distribution of $\chi_{\text{eff}}$ has a peak around 0. Importantly, our result indicates that LK-induced BH binary mergers can easily have $\chi_{\text{eff}} < 0$ (see also Liu & Lai 2017). This is quite different from the standard isolated binary evolution channel, where we typically expect spin–orbit alignment and $\chi_{\text{eff}} > 0$.

If the distributions of $\cos \theta_{s1}^L$ and $\cos \theta_{s2}^L$ are uncorrelated, as we may expect to be the case for $m_1 = m_2$, when the octupole effect is significant (see Figure 9 and the discussion in the last paragraph of Section 4.2), the distribution of $\chi_{\text{eff}}$ can be derived directly from $P_1(\cos \theta_{s1}^L)$ and $P_2(\cos \theta_{s2}^L)$, the distribution functions of $\cos \theta_{s1}^L$ and $\cos \theta_{s2}^L$. Define

$$\tilde{\chi}_1 = \frac{m_1 \chi_1}{m_1 \chi_1 + m_2 \chi_2},$$

(76)

$$\tilde{\chi}_2 = \frac{m_2 \chi_2}{m_1 \chi_1 + m_2 \chi_2},$$

(77)

$$\tilde{\chi}_{\text{eff}} = \frac{\chi_{\text{eff}}}{\chi_{\text{eff}}^{\max}} = \frac{m_2 \chi_{\text{eff}}}{m_1 \chi_1 + m_2 \chi_2},$$

(78)

where $\chi_{\text{eff}}^{\max} = (m_1 \chi_1 + m_2 \chi_2)/m_2$ is the maximum possible value of $\chi_{\text{eff}}$ for given $m_1 \chi_1$ and $m_2 \chi_2$ (this maximum is achieved at $\cos \theta_{s1}^L = \cos \theta_{s2}^L = 1$). Then Equation (75) becomes

$$\tilde{\chi}_{\text{eff}} = \tilde{\chi}_1 \mu_1 + \tilde{\chi}_2 \mu_2,$$

(79)

where $\mu_1 \equiv \cos \theta_{s1}^L$ and $\mu_2 \equiv \cos \theta_{s2}^L$. Note that $\tilde{\chi}_1 + \tilde{\chi}_2 = 1$ and $\tilde{\chi}_{\text{eff}} \in [-1, 1]$. Given $P_1(\mu_1)$ and $P_2(\mu_2)$, the distribution function of $\tilde{\chi}_{\text{eff}}$ is

$$P(\tilde{\chi}_{\text{eff}}) = \int_{-1}^{1} d\mu_1 P_1(\mu_1) \int_{-1}^{1} d\mu_2 P_2(\mu_2) \times \delta(\tilde{\chi}_{\text{eff}} - \tilde{\chi}_1 \mu_1 - \tilde{\chi}_2 \mu_2).$$

(80)

In the special case when $\mu_1$ and $\mu_2$ are uniformly distributed, we have $P_1 = P_2 = 1/2$, and Equation (80) gives

$$P(\tilde{\chi}_{\text{eff}}) = \left\{ \begin{array}{ll} (1 - \tilde{\chi}_{\text{eff}})(4\tilde{\chi}_1), & \tilde{\chi}_{\text{eff}} > \tilde{\chi}_1 - \tilde{\chi}_2 \\ 1/(2\tilde{\chi}_1), & \tilde{\chi}_2 - \tilde{\chi}_1 \leq \tilde{\chi}_{\text{eff}} \leq \tilde{\chi}_1 - \tilde{\chi}_2 \\ (1 + \tilde{\chi}_{\text{eff}})(4\tilde{\chi}_1), & \tilde{\chi}_{\text{eff}} \leq \tilde{\chi}_2 - \tilde{\chi}_1. \end{array} \right.$$ 

(81)

where we have assumed $\tilde{\chi}_1 > \tilde{\chi}_2$ without loss of generality. Thus, even for uniform distributions of $\cos \theta_{s1}^L$ and $\cos \theta_{s2}^L$ (see Figures 13 and 19), the effective spin parameter $\chi_{\text{eff}}$ is preferentially distributed around $\chi_{\text{eff}} = 0$ (see Figure 20).

Note that the spin–orbit misalignment distribution and $\chi_{\text{eff}}$ distribution obtained above refer to relatively wide BH binary systems ($a_0 \gtrsim 10–100$ au) that experience a merger due to large LK eccentricity excitation. Such systems necessarily have $A_0 \ll 1$. For BH binaries with smaller separations ($a_0 \lesssim 1$ au) and $A_0$ not much less than unity, the spin–orbit misalignment distribution can be quite different (see Liu & Lai 2017).

Antonini et al. (2018; see also Rodriguez & Antonini 2018) have carried population studies of BH mergers in triple systems (based on DA secular equations) and have found a similar peak around $\chi_{\text{eff}} = 0$ in the $\chi_{\text{eff}}$ distribution. Rodriguez & Antonini (2018) also showed an example of the final spin–orbit misalignment distribution with a peak around $90^\circ$, in qualitative agreement with our result. They did not distinguish the difference in the spin–orbit misalignment distributions between largely quadrupole systems (small $e_{\text{out}}$) and strong octupole systems (large $e_{\text{out}}$). We do not agree with the reason(s) they gave for the peak in the $\chi_{\text{eff}}$ distribution. In particular, it is important to recognize that, during the orbital decay, the adiabaticity parameter $A$ (Equation (61)) transitions from $\ll 1$ to $\gg 1$, and this transition determines $\theta_{s1}^L$.

5. Summary and Discussion

In this paper we have studied BH binary mergers in triple systems. A sufficiently inclined tertiary companion excites large eccentricity in the BH binary orbit through gravitational perturbations (the LK mechanism), significantly shortening its merger timescale due to GW emission. We focus on binaries with initial separations sufficiently large ($\gtrsim 10$ au) that a merger is not possible without large eccentricity excitations. While this problem has been studied before in various contexts (see references in Section 1), we make progress by (1) systematically determining the merger fractions for various system parameters (e.g., the masses and orbital properties of the binary and perturber) and deriving the relevant scaling relations, (2) examining the spin evolution of the BHs to predict the final spin–orbit misalignments of the merging binaries. Although our numerical examples focus on BH binaries with stellar-mass companions, our results (with appropriate rescalings) can be applied to NS binaries (see Section 3.3) and other types of perturbers (e.g., supermassive BHs).
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Liu & Lai

5.1. Summary of Key Results

1. For BH binaries with a given initial separation \(a_0 \gtrsim 10\) au, the merger window (i.e., the range of initial inclination angles \(I_0\) between the inner binary and the outer companion that induces binary merger within \(\sim 10^{10}\) yr) and merger fraction depend on the effective semimajor axis \(a_{\text{out}}/a_0\) and \(e_{\text{out}}/e_{\text{out},\text{max}}\) (Equation (50)) and eccentricity \(e_{\text{out}}\) of the companion. The results are summarized in Figure 13. Assuming that the inclination of the companion is randomly distributed, we find that the merger fraction (for typical BH masses \(m_1 = 30 M_\odot, m_2 = 20 M_\odot\)) increases rapidly with increasing \(e_{\text{out}}\), from \(~1\%\) at \(e_{\text{out}} = 0\) to \(10\%\) at \(e_{\text{out}} = 0.9\). This is because, as the octupole potential \((\propto e_{\text{out}} \propto e_{\text{out}}^2);\) see Equation (17)) of the tertiary companion increases, extreme eccentricity excitation of the inner binaries becomes possible for a wide range of \(I_0\) (see Figure 9). Regardless of the importance of the octupole effect, the maximum \(a_{\text{out},\text{eff}}\) value for which the inner binary has a chance to merge within \(10^{10}\) yr (or any other values) can be determined analytically (using Equations (44) and (49), setting \(a_m\) to \(a_{\text{lim}}\); see also Equation (54)).

2. For systems where the octupole effect is negligible (such as those with \(m_1 = m_2\) or \(e_{\text{out}} = 0\)), the merger window and merger fraction can be determined analytically (see Figure 8). In particular, these analytical results can be applied to NS–NS binaries with external companions (see Section 3.3, Figure 14). We have also obtained fitting formulae relevant to the merger fractions of various systems (Equations (53) and (54)).

3. On the technical side, we have developed new dynamical equations for the evolution of triples (Section 2.1.2) in the single averaging approximation (i.e., the equations of motion are only averaged over the inner orbit). These SA equations have a wider regime of validity in the parameter space than the usual DA secular equations (see Section 2.3 and Figure 2). For systems where the octupole effect is negligible, we find that the DA equations accurately predict the merger window and merger fractions even in the regime where the equations formally break down (see Figure 8). However, when the octupole effect is strong (large \(e_{\text{oct}}\)), using the SA equations leads to wider merger windows and larger merger fractions (see Figures 12 and 13).

4. During the tertiary-induced binary decay, the spin axes of the BHs exhibit a variety of evolutionary behaviors due to the combined effects of spin–orbit coupling (de Sitter precession), LK orbital precession/nutation, and GW emission. These spin behaviors are correlated with the orbital evolution of the BH binary (Section 4.2). Starting from aligned spin axes (relative to the orbital angular momentum axis), a wide range of spin–orbit misalignments can be generated when the binary enters the LIGO/VIRGO band:

   a. For systems where the octupole effect is negligible (such as those with \(m_1 \simeq m_2\) or \(e_{\text{out}} \sim 0\)), the BH spin axis evolves regularly, with the final spin–orbit misalignment angle \(\theta_{\text{fi}}\) depending on the initial companion inclination angle \(I_0\) in a well-defined manner (see Figure 3 and the top left panels of Figure 9).\(^{13}\) We find that when \(I_0\) is not too close to \(I_{\text{lim}}\) (the initial inclination angle for maximum/

\(^{13}\) This conclusion applies to the parameter regime studied in this paper, where the inner binary has a large initial separation \(a_0\) and thus is capable of merging only because of the extreme eccentricity excitation induced by the companion; this requires that the initial \(e_{\text{oct}} \ll 1\) (Equation (41)) or the initial adiabaticity parameter \(A_0 \ll 1\) (see Equations (61)–(63)). By contrast, for systems that have smaller \(a_0\) and experience only modest eccentricity excitations, \(A_0\) is not much smaller than unity, the BH spin may evolve chaotically even when \(e_{\text{out}} = 0\) (Liu & Lai 2017).
limiting eccentricity excitation; see Equations (43) and (44)), the spin–orbit misalignment evolves into a 90° “attractor” (Figures 4 and 5), a feature that can be qualitatively understood using adiabatic invariance (see Section 4.3). When $I_0$ is close to $I_{\text{lim}}$, a qualitatively different spin evolution leads to smaller $\theta_0^s$ (Figures 6 and 7).

b. For systems with a stronger octupole effect (larger $\varepsilon_0$), the BH spin evolution becomes increasingly chaotic, with the final spin–orbit misalignment angle depending sensitively on the initial conditions (see Figures 10 and 11). As a result, a wide range of $\theta_0^s$ values are produced, including retrograde configurations (see Figure 9). The final spin–orbit misalignment distribution typically peaks around 90°, but becomes isotropic (uniform in $\cos \theta_0^s$) for systems with sufficiently large $\varepsilon_0$ (Figure 19).

5. We have computed the distribution of the mass-weighted spin parameter $\chi_{\text{eff}}$ (Equation (75)) of merging BH binaries in triples (Figure 20). While details of this distribution depend on various parameters (e.g., distribution of the companion eccentricities), it has a characteristic shape with peak around $\chi_{\text{eff}} \approx 0$, extending to the maximum possible positive and negative values (see Equation (78)).

5.2. Discussion

The merger fraction $f_{\text{merge}}$ computed in this paper (and particularly the dependence of $f_{\text{merge}}$ on various parameters) can be used to obtain an estimate of the rate of LK-induced BH binary mergers in the galactic field, provided that one makes certain assumptions about the BH populations in triples and their properties. We do not present such an estimate here since such a calculation necessarily contains large uncertainties (see Antonini et al. 2017; Silsbee & Tremaine 2017), like all other scenarios of producing merging BH binaries. Suffice it to say that with our computed $f_{\text{merge}}$ of a few to 10 per cent, it is possible to produce (with large error bars) the observed BH binary merger rate (10–200 Gpc$^{-3}$ yr$^{-1}$).

As noted in Section 1, the mass-weighted spin parameter $\chi_{\text{eff}}$ may serve as a useful indicator of the binary BH formation mechanism. The five discovered BH binaries all have low values of $\chi_{\text{eff}}$, which could be the result of either slowly spinning BHs (e.g., Zaldarriaga et al. 2017) or large spin–orbit misalignments. The event GW170104 has $\chi_{\text{eff}} = -0.12^{+0.21}_{-0.12}$, which may require retrograde spinning BHs, especially if low individual spins ($\chi_{1,2} \lesssim 0.2$) can be ruled out. Such a retrograde spin–orbit misalignment would challenge the isolated binary BH formation channel, and point to the importance of some flavors of dynamical formation mechanisms. We note that the LK-induced BH mergers lead to a unique shape of $\chi_{\text{eff}}$ distribution (Figure 20) that may be used to distinguish it from other types of dynamical interactions. For example, a completely random distribution of spin–orbit misalignments, as expected from the mechanisms involving multiple close encounters and exchange interactions in dense clusters (e.g., Rodriguez et al. 2015; Chatterjee et al. 2017), would lead to a specific distribution given by Equation (81). As the number of detected BH merger events increases in the coming years, the distribution of $\chi_{\text{eff}}$ will be measured experimentally, therefore providing valuable constraints on the binary BH formation mechanisms.

Although we have focused on isolated BH triples in this paper, many aspects of our results (with proper rescalings) can be applied to triples that dynamically form in globular clusters or BH binaries moving around a supermassive BH (e.g., Miller & Hamilton 2002; Wen 2003; Thompson 2011; Antonini & Perets 2012; Antonini et al. 2014; Petrovich & Antonini 2017; Hoang et al. 2018). In a dense cluster, the orbits of a triple system can be perturbed or even disrupted by close fly-bys of other objects. Therefore the survival timescale of the triple may not be as long as $10^{10}$ yr, depending on the mean density of the surroundings. In this case, the merger window and merger fraction may be reduced (see Equations (53) and (54)), and the remaining systems can lead to extremely large eccentricities and shorter merger times. Also, our conclusion on the distribution of near-merger spin–orbit misalignments depends on the initial BH spin orientations. We have assumed initial spin–orbit alignment throughout this paper, but this may not be valid for dynamically formed binaries and triples in dense clusters. We plan to address some of these issues in a future paper.

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