Representational Power of ReLU Networks and Polynomial Kernels: Beyond Worst-Case Analysis

Frederic Koehler †, Andrej Risteski †

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Abstract

There has been a large amount of interest, both in the past and particularly recently, into the power of different families of universal approximators, e.g. ReLU networks, polynomials, rational functions. However, current research has focused almost exclusively on understanding this problem in a worst-case setting, e.g. bounding the error of the best infinity-norm approximation in a box. In this setting a high-degree polynomial is required to even approximate a single ReLU.

However, in real applications with high dimensional data we expect it is only important to approximate the desired function well on certain relevant parts of its domain. With this motivation, we analyze the ability of neural networks and polynomial kernels of bounded degree to achieve good statistical performance on a simple, natural inference problem with sparse latent structure. We give almost-tight bounds on the performance of both neural networks and low degree polynomials for this problem. Our bounds for polynomials involve new techniques which may be of independent interest and show major qualitative differences with what is known in the worst-case setting.

1 Introduction

The concept of representational power has been always of great interest in machine learning. In part the reason for this is that classes of “universal approximators” abound – e.g. polynomials, radial bases, rational functions, etc. Some of these were known to mathematicians as early as Bernstein and Lebesgue – yet it is apparent that not all such classes perform well empirically.

In recent years, the class of choice is neural networks – which have inspired a significant amount of theoretical work. Research has focus on several angles of this question, e.g. comparative power to other classes of functions (Yarotsky, 2017; Safran and Shamir, 2017; Telgarsky, 2017) the role of depth and the importance of architecture (Telgarsky, 2016; Safran and Shamir, 2017; Eldan and Shamir, 2016), and many other topics such as their generalization properties and choice of optimization procedure (Hardt et al., 2016; Zhang et al., 2017; Bartlett et al., 2017).

Our results fall in the first category: namely, comparing the relative power of polynomial kernels and ReLU networks – but with a significant twist. The flavor of existing results that compare different classes of approximators is approximately the following: every predictor in a class $C_1$ can be approximately represented as a predictor in a different class $C_2$, with some blowup in the size/complexity of the predictor (e.g. degree, number of nodes, depth).

The unsatisfying aspect of such results is the way approximation is measured: typically, one picks a domain relevant for the approximation (e.g. an interval or a box), and considers the $L_\infty, L_2, L_1, \ldots$ norm of the difference between the two predictors on this domain. This is an inherently “worst-case” measure of approximation: it’s quite conceivable that in cases like multiclass classification, it would suffice to approximate the predictor well only on some “relevant domain”, e.g. far away from the prediction boundary.

The difficulty with the above question is that it’s not always easy to formalize what the “relevant domain” is, especially without modeling the data distribution. We tackle here arguably the easiest nontrivial incarnation of this question: namely, when there is sparse latent structure.

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* Department of Mathematics, Massachusetts Institute of Technology. Email: fkoehler@mit.edu
† Department of Mathematics and IDSS, Massachusetts Institute of Technology. Email:risteski@mit.edu
1 Lebesgue made use of the universality of absolute value and hence ReLu – see the introduction of (Newman et al., 1964).
2 Overview of results

We will be considering a very simple regression task, where the data has a latent sparse structure. More precisely, our setup is the following. We wish to fit pairs of (observables, labels) \((X, Y)\) generated by a (latent-variable) process:

- First we sample a latent vector \(Z \in \mathbb{R}^m\) from \(H\), where \(H\) is a distribution over sparse vectors.
- To produce \(X \in \mathbb{R}^n\), we set \(X = AZ + \xi\), where the noise \(\xi \sim \text{subG}(\sigma^2)\) is a subgaussian random vector with variance proxy \(\sigma^2\) (e.g. \(N(0, \sigma^2 I)\)).
- To produce \(Y \in \mathbb{R}\), we set \(Y = \langle w, Z \rangle\).

We hope the reader is reminded of classical setups like sparse linear regression, compressive sensing and sparse coding: indeed, the distribution on the data distribution \(X\) is standard in all of these setups. In our setup additionally, we attach a regression task to this data distribution, wherein the labels are linearly generated by a predictor \(w\). (One could also imagine producing discrete labels by applying a softmax operation, though the proofs get more difficult in this case, so we focus on the linear case.)

Our interest however, is slightly different than usual. Typically, one is interested in the statistical/algorithms problem of inferring \(Z\), given \(X\) as input (the former studying the optimal rates of “reconstruction” for \(Z\), the latter efficient algorithms for doing so). Therefore, one does not typically care about the particular form of the predictor as long as it is efficiently computable.

In contrast, we will be interested in the representational power of different types of predictors which are commonly used in machine learning, and their relative statistical power in this model. In other words, we want to understand how close in average-case error we can get to the optimal predictor for \(Y\) given \(X\) using standard function classes in machine learning. Informally, what we will show is the following.

**Theorem 2.1 (Informal).** For the problem of predicting \(Y\) given \(X\) in the generative model for data described above, it holds that:

1. Very simple two-layer ReLU networks achieve close to the statistically optimal rate.
2. Polynomial predictors of degree lower than \(\log n\) achieve a statistical rate which is substantially worse. (In fact, in a certain sense, close to “trivial”. ) Conversely, polynomial predictors of degree \((\log n)^2\) achieve close to the statistically optimal rate.

In particular, if we consider fitting a polynomial to data points of the form \((x_i, y_i)\), we would need to search through the space of multivariate polynomials of degree \(\Omega(\log n)\) which has super-polynomial dimension \(n^{\Omega((\log(n))}\), and thus even writing down all of the variables in this optimization problem takes super-polynomial time. Practical aspects of using polynomial kernels even with much lower degree than this have been an important concern and topic of empirical research; see for example (Chang et al., 2010) and references within.

Note that our lower bound holds even though very simple ReLU networks can easily solve this regression problem. On the other hand, our upper bound shows that this analysis is essentially tight: greater than \(\text{polylog}(n)\) degree is not required to achieve good statistical performance, which is much different from the situation under worst-case analysis (see Section 4.2.2).

For formal statements of the theorems, see Section 4.

3 Prior Work

There has been a large body of work studying the ability of neural networks to approximate polynomials and various classes of well-behaved functions, such as recent work (Yarotsky, 2017; Safran and Shamir, 2017; Telgarsky, 2017; Poggio et al., 2017). As described in the introduction, these results all focus on the worst-case setting where the goal is to find a network close to some function in some norm (e.g. infinity-norm or 1-norm). There has been comparatively
little work on the problem of approximating ReLU networks by polynomials: mostly because it is well-known by classical results of approximation theory (Newman et al., 1964; DeVore and Lorentz, 1993) that polynomials of degree \( \Omega(1/\epsilon) \) are required to approximate even a single absolute value or ReLU function within error \( \epsilon \) in infinity-norm on the interval \([-1, 1]\). In contrast to this classical result, we will show that if we do not seek to achieve uniform \( \epsilon \)-error everywhere for the ReLU (in particular not near the non-smooth point at 0) we can build good approximations to ReLU using polynomials of degree only \( O(\log^2(1/\epsilon)) \) (see discussion in Section 4.2.2 and Theorem 4.1).

Due to the trivial \( \Omega(1/\epsilon) \) lower bound for worst-case approximation of ReLU networks by polynomials, (Telgarsky, 2017) studied the related problem of approximating a neural network by rational functions. (A classical result of approximation theory (Newman et al., 1964) shows that rational functions of degree \( O(\log^2(1/\epsilon)) \) can get within \( \epsilon \)-error of the absolute value function.) In particular, (Telgarsky, 2017) shows that rational functions of degree \( \text{polylog}(1/\epsilon) \) can get within \( \epsilon \) distance in \( L_\infty \)-norm of bounded depth ReLU neural networks.

Somewhat related is also the work of (Livni et al., 2014) who considered neural networks with quadratic activations and related their expressivity to that of sigmoidal networks in the depth 2 case by building on results of (Shalev-Shwartz et al., 2011) for approximating sigmoids. The result in (Shalev-Shwartz et al., 2011) is also proved using complex-analytic tools, though the details are different (in particular, they do not use Bernstein’s theorem).

There is a vast literature on high dimensional regression and compressed sensing which we do not attempt to survey, since the main goal of our paper is not to develop new techniques for sparse regression but rather to analyze the representation power of kernel methods and neural networks. Some relevant references for sparse recovery can be found in (Vershynin, 2018; Rigollet, 2017). We emphasize that the upper bound via soft thresholding we show (Theorem 4.1), is implicit in the literature on high-dimensional statistics; we include the proofs here solely for completeness.

### 4 Main Results

In this section we will give formal statements of the results, explain their significance in detail, and give some insight into the techniques used.

First, let us state the assumptions on the parameters of our generative model:

- We require \( Z \) to be sparse; more precisely we will require that \( |\text{supp}(Z)| \leq k \) and \( \|Z\|_1 \leq M \) with high probability.\(^3\)
- We assume that \( A \) is a \( \mu \)-incoherent \( n \times m \) matrix, which means that \( \|A^\top A - I\|_\infty \leq \mu \) for some \( \mu \geq 0 \).
- We assume (without loss of generality, since changing the magnitude of \( w \) just rescales \( Y \)) that \( \|w\|_\infty = 1 \).

The assumption on \( A \) is standard in the literature on sparse recovery (see reference texts (Rigollet, 2017; Moitra, 2018)). In general one needs an assumption like this (or a stronger one, such as the RIP property) in order to guarantee that standard algorithms such as LASSO actually work for sparse recovery. Note that such an \( A \) can be produced e.g. by taking a matrix with i.i.d. entries of the form \( \pm 1/\sqrt{m} \) where the sign is picked randomly and this is possible even when \( m \gg n \); furthermore the resulting \( \mu \) is quite small \( (O(1/\sqrt{m})) \).

For notational convenience, we will denote \( \|A\|_\infty = \max_{i,j} |A_{i,j}| \).

We proceed to the results:

#### 4.1 Regression Using 2-layer ReLU Networks

Proceeding to the upper bounds, we prove the following theorem, which shows that 2-layer ReLU networks can achieve an almost optimal statistical rate. Let us denote the soft threshold function with threshold \( \tau \) as \( \rho_\tau(x) := \text{sgn}(x)\max(0, |x| - \tau) = \text{ReLU}(x - \tau) + \text{ReLU}(-x + \tau) \).

Consider the following estimator (for \( y \)), corresponding to a 2-layer neural network:

\[
\tilde{Z}_{NN} := \rho_\tau^\otimes n(A^\top X)
\]

\[
\hat{Y}_{NN} := \langle w, \tilde{Z}_{NN} \rangle
\]

\(^3\)The assumed 1-norm bound \( M \) plays a minor role in our bounds and is only used when the incoherence \( \mu > 0 \).
We can prove the following result for this estimator (see Appendix A of the supplement):

**Theorem 4.1** (2-layer ReLU). Assume $A$ is $\mu$-incoherent. With high probability, the estimator $\hat{Y}_{NN}$ satisfies

$$(\hat{Y}_{NN} - Y)^2 = O((1 + \mu)\sigma^2 k^2 \log(m) + \mu^2 k^2 M^2)$$

In order to interpret this result, recall that one typically considers incoherent matrices where $\mu$ is quite small – in particular $\mu \ll 1$. Thus the error of the estimator is essentially $O(\sigma^2 k^2 \log(m))$, i.e. $\sigma^2$ error “per-coordinate”. It can be shown that this upper bound is nearly information-theoretically optimal (see Remark 6.1).

**4.2 Regression Using Polynomials**

**4.2.1 Lower Bounds for Low-Degree Polynomials**

We first show that polynomials of degree smaller than $O(\log n)$ essentially cannot achieve a “non-trivial” statistical rate. This holds even in the simplest possible case for the dictionary $A$: when it’s the identity matrix.

More precisely, we consider the situation in which $A$ is an orthogonal matrix (i.e. $\mu = 0, m = n$), $w \in \{\pm 1\}^n$, the noise distribution is Gaussian $N(0, \sigma^2 I)$, and the entries of $Z$ are independently $0$ with probability $1 - k/n$ and $N(0, \gamma^2)$ with probability $k/n$. Then we show

**Theorem 4.2.** Suppose $k < n/2$ and $f$ is a multivariate degree $d$ polynomial. Then

$$\mathbb{E}[(f(X) - Y)^2] \geq (1/4) \frac{\gamma^2 k}{\left(1 + \sqrt{k/n} (d + 1)^{3d+2} (1 + (\gamma/\sigma)^d)\right)^2}$$

In order to parse the result, observe that the numerator is of order $\gamma^2 k$ which is the error of the trivial estimator and the denominator is close to 1 unless $d$ is sufficiently large with respect to $n$. More precisely, assuming the signal-to-noise ratio $\gamma/\sigma$ does not grow too quickly with respect to $n$, we see that the denominator is close to 1 unless $d^d = \Omega(\sqrt{n})$, i.e. unless $d$ is of size $\Omega((\log n)/\log \log n)$.

**4.2.2 Nearly Matching Upper Bounds via Novel Polynomial Approximation to ReLU**

The lower bound of the previous section leaves open the possibility that polynomials of degree $O(\log(n))$ still do not suffice to perform sparse regression and solve our inference problem; Indeed, it is a well-known fact (see e.g. [Telgarsky, 2017]) that to approximate a single ReLU to $\epsilon$-closeness in infinity norm in $[-1, 1]$ requires polynomials of degree $\text{poly}(1/\epsilon)$; this follows from standard facts in approximation theory (DeVore and Lorentz, 1993) since ReLU is not a smooth function.

Surprisingly, we show this intuition is incorrect! In fact, we show how to convert the neural network $\hat{Y}_{NN}$ to a $\text{polylog}(n)$ degree polynomial with similar statistical performance by designing a new low-degree polynomial approximation to ReLU. Formally this is summarized by the following theorem, where $\hat{Y}_{d,M}$ is the corresponding version of $\hat{Y}_{NN}$ formed by replacing each ReLU by our polynomial approximation.

**Theorem 4.3.** Suppose $\tau = \Theta(\sigma \sqrt{(1 + \mu) \log m + \mu M})$ and $d \geq d_0 = \Omega((2 + M) \log^2(Mm/\tau^2))$. With high probability, the estimator $\hat{Y}_{d,M}$ satisfies

$$(\hat{Y}_{d,M} - Y)^2 = O(k^2((1 + \mu)\sigma^2 \log(m) + \mu^2 M^2))$$
5 Techniques

We briefly survey each of the results above.

Proceeding to the upper bound using a 2-layer ReLU network, results similar to Theorem 4.1 are standard in the literature on sparse linear regression, though we include the proof for completeness in Appendix A of the supplement. The intuition is quite simple: the estimator $\hat{Z}_{NN}$ can make use of the non-linearity in the soft threshold to zero out the coordinates in the estimate $A^T X$ which are small and thus “reliably” not in the support of the true $z$. This allows the estimator to only make mistakes on the non-zero coordinates.

For lower bound on low-degree polynomial kernels, we first make a probabilistic argument which lets us reduce to studying the coordinate-wise error in reconstructing $Z$, and subsequently use Fourier analysis on orthogonal polynomials to get a bias-variance tradeoff we can lower bound.  

We prove the upper bound for polylogarithmic degree kernels by constructing a new polynomial approximation to ReLU (Theorem 7.1). The key insight here is that approximating the ReLU in infinity-norm is difficult in large part because it is hard to approximate ReLU at 0, its point of non-smoothness; however, in our case the precise behavior of ReLU very close to 0 is not important for getting a good regression rate. Instead, the polynomial approximation to ReLU we design uses only $O(\log^2 n)$ degree polynomials and sacrifices optimizing accuracy in approximation near the point of non-smoothness in favor of optimizing closeness to 0 in the negative region. The reason that closeness to 0 is so important is that it captures the ability of ReLU to denoise, since the 0 region of ReLU is insensitive to small changes in the input; essentially all commonly used neural network activations have such a region of insensitivity (e.g. for a sigmoid, the region far away from 0).

Our polynomial approximation to ReLU is built using powerful complex-analytic tools from approximation theory and may be of independent interest; we are not aware of a way to get this result using only generic techniques such as FT-Mollification (Diakonikolas et al., 2010).

6 Part 1: Lower Bounds for Polynomial Kernels

In this section, we fill flesh out the lower bound results somewhat more.

The lower bound instance is extremely simple: the noise distribution is $N(0, \sigma^2 I d)$ and the distribution for $Z$ is s.t. every coordinate is first chosen to be non-zero with probability $k/n$, and if it is non-zero, it’s set as an independent sample from $N(0, \gamma^2)$.

This construction makes $Z$ approximately $k$-sparse with high probability while making its coordinates independent. We choose $A$ as an arbitrary orthogonal matrix, so $m = n$. We choose $w$ to be an arbitrary $\pm 1$ sign vector, so $w_i^2 = 1$ for every $i$.

We will show first that linear predictors, and then fixed low degree polynomials cannot achieve the information-theoretic rate of $O(\sigma^2 k \log n)$ – in fact, we will show that they achieve a “trivial” rate. Furthermore, we will show that even if the degree of our polynomials is growing with $n$, if $d = o(\log n / \log \log n)$ the state of affairs is similar.

6.1 Warmup: Linear Predictors

As a warmup, and to illustrate the main ideas of the proof techniques, we first consider the case of linear predictors. (i.e. kernels of degree 1.)

The main idea is to use a bias-variance trade-off: namely, we show that the linear predictor we use, say $f(x) = \langle \hat{w}, x \rangle$ either has to have too high of a variance (when $\|\hat{w}\|$ is large), or otherwise has too high of a bias. (Recall, the bias captures how well the predictor captures the expectation.)

We prove:

**Theorem 6.1.** For any $\hat{w} \in \mathbb{R}^n$,

$$\mathbb{E}[((\hat{w}, X) - Y)^2] \geq \gamma^2 k \frac{\sigma^2}{\gamma^2 (k/n) + \sigma^2}$$

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4 On a technical note we observe that this statement is given with respect to expectation but a similar one can be made with high probability, see Remark 6.2.

5 See Remark 6.1
Before giving the proof, let us see how the theorem should be interpreted.

The trivial estimator which always returns 0 makes error of order $\gamma^2 K$ and a good estimator (such as thresholding) should instead make error of order $\sigma^2 K \log n$ when $\gamma >> \sigma \sqrt{\log n}$. The next theorem shows that as long as the signal to noise ratio is not too high, more specifically as long as $\gamma^2 (k/n) = o(\sigma^2)$, any linear estimator must make square loss of $\Omega(\gamma^2 k)$, i.e. not significantly better than the trivial 0 estimator.

Note that the most interesting (and difficult) regime is when the signal is not too much larger than the noise, e.g. $\gamma^2 = \sigma^2 \text{polylog}(n)$ in which case it is definitely true that $\gamma^2 (k/n) << \sigma^2$.

**Proof.** Note that

$$\langle \tilde{w}, x \rangle - y = \langle \tilde{w}, Az + \xi \rangle - \langle w, z \rangle = \langle A^\top \tilde{w} - w, z \rangle + \langle \tilde{w}, \xi \rangle$$

which gives the following *bias-variance decomposition* for the square loss:

$$\mathbb{E}[(\langle \tilde{w}, x \rangle - y)^2] = \mathbb{E}[(\langle A^\top \tilde{w} - w, z \rangle + \langle \tilde{w}, \xi \rangle)^2]$$

$$= \mathbb{E}[(\langle A^\top \tilde{w} - w, z \rangle)^2 + \langle \tilde{w}, \xi \rangle^2]$$

$$= \frac{k}{n} \gamma^2 \| A^\top \tilde{w} - w \|^2 + \sigma^2 \| \tilde{w} \|^2$$

$$= \frac{k}{n} \gamma^2 \| \tilde{w} - Aw \|^2 + \sigma^2 \| \tilde{w} \|^2$$

where in the second-to-last step we used that the covariance matrix of $\gamma$ in the second-to-last step we used that the covariance matrix of $Z$ is $\gamma^2 (k/n) I$, and in the last step we used that $A$ is orthogonal. Now observe that if we fix $R = \| \tilde{w} \|_2$, then by the Pythagorean theorem the minimizer of the square loss is given by the projection of $Aw$ onto the $R$-dilated unit sphere, so $\tilde{w} = \sqrt{R^2 / m} (Aw)$ since $\| Aw \|_2 = \| w \|_2 = \sqrt{m}$. In this case the square loss is then of the form

$$\frac{k}{n} \gamma^2 \sqrt{R^2 / m} (Aw) - Aw \|^2 + \sigma^2 \| \tilde{w} \|^2 = \frac{k}{n} \gamma^2 (R - \sqrt{m})^2 + \sigma^2 R^2$$

and the risk is minimized when

$$0 = 2 \frac{k}{n} \gamma^2 (R - \sqrt{m}) + 2 \sigma^2 R$$

i.e. when

$$R = \frac{\gamma^2 (k/n)}{\gamma^2 (k/n) + \sigma^2 \sqrt{m}}$$

so the minimum square loss is

$$(\sqrt{m} - R) \sigma^2 R + \sigma^2 R^2 = \sigma^2 \frac{\gamma^2 k}{\gamma^2 (k/n) + \sigma^2}$$

since $m = n$. \(\square\)

### 6.2 Main Technique for General Case: Structure of the Optimal Estimator

Before proceeding to the proof of the lower bound for general low-degree polynomials, we observe that the optimal estimator for $Y = \langle w, Z \rangle$ given $X$ has a particularly simple structure. More precisely the optimal estimator in the squared loss is the conditional expectation, $\mathbb{E}[\langle w, Z \rangle | X] = \sum_i w_i \mathbb{E}[Z_i | X]$ so the optimal estimator for $Y$ simply reconstructs $Z$ as well as possible coordinate-wise and then takes an inner product with $w$.

In our setup the coordinates of $Z$ are independent, which allows us to show that the optimal polynomial of degree $d$ to estimate $Y$ has no “mixed monomials” when we choose the appropriate basis. This is the content of the next lemma.
Lemma 6.1. Suppose $X = AZ + \xi$ where $A$ is an orthogonal $m \times m$ matrix, $Z$ has independent entries and $\xi \sim N(0, \sigma^2 I_d)$. Then there exists a unique minimizer $f_d^*$ over all degree $d$ polynomials $f_d$ of the square-loss,

$$\mathbb{E}[(f_d(A^T X) - \langle w, Z \rangle)^2]$$

and furthermore $f_d^*$ has no mixed monomials. In other words, we can write $f_d^*(A^T X) = \sum_i f_{d,i}^*((A^T)X)_i$ where each of the $f_{d,i}^*$ are univariate degree $d$ polynomials.

Proof. Let $X' = A^T X$, so by orthogonality $X' = Z + \xi'$ where $\xi' \sim N(0, \sigma^2 I_d)$. Observe that if we look at the optimum over all functions $f$, we see that

$$\min_f \mathbb{E}[(f(X') - \langle w, Z \rangle)^2] = \mathbb{E}[(\mathbb{E}[f(X')] - \langle w, Z \rangle)^2]$$

$$= \mathbb{E}[(\sum_i w_i \mathbb{E}[Z_i|X'] - \langle w, Z \rangle)^2]$$

$$= \mathbb{E}[(\sum_i w_i \mathbb{E}[Z_i|X'] - \langle w, Z \rangle)^2].$$

where where in the first step we used that the conditional expectation minimizes the squared loss, in the second step we used linearity of conditional expectation, and in the last step we used that $Z_i$ is independent of $X_i'$.

By the Pythagorean theorem, the optimal degree $d$ polynomial $f_d^*$ is just the projection of $\sum_i w_i \mathbb{E}[Z_i|X']$ onto the space of degree $d$ polynomials. On the other hand observe that

$$\mathbb{E}[(\sum_i w_i \mathbb{E}[Z_i|X'] - \langle w, Z \rangle)^2] = \sum_i w_i^2 \mathbb{E}[(\mathbb{E}[Z_i|X'] - Z_i)^2]$$

so the optimal projection $f_d^*$ is just $\sum_i w_i f_{i,d}^*(X_i')$ where $f_{i,d}^*$ is just the projection of each of the $\mathbb{E}[Z_i|X_i']$. Therefore $f_d^*$ has no mixed monomials. \qed

Remark 6.1. The previous calculation shows that the problems of minimizing the squared loss for predicting $Y$ is equivalent to that of minimizing the squared loss for the sparse regression problem of recovering $Z$. It is a well-known fact that the information theoretic rate for sparse regression (with our normalization convention) is $\Theta(\sigma^2 k)$ (see for example (Rigollet, 2017)), and so the information-theoretic rate for predicting $Y$ is the same, and is matched by Theorem A.2. In our particular model it is also possible to compute this directly, since one can find an explicit formula for $\mathbb{E}[Z_i|X_i']$ using Bayes rule.

6.3 Lower Bounds for Polynomial Kernels

The lower bound for polynomials combines the observation of Lemma 6.1 with a more general analysis of bias-variance tradeoff using Fourier analysis on orthogonal polynomials. Concretely, since the noise we chose for the lower bound instance is Gaussian, the most convenient basis will be the Hermite polynomials.

Recall that the probabilist’s Hermite polynomial $H_{n}(x)$ can be defined by the recurrence relation

$$H_{n+1}(x) = x H_{n}(x) - n H_{n-1}(x).$$

where $H_0(x) = 1$, $H_1(x) = x$. In terms of this, the normalized Hermite polynomial $H_n(x)$ is

$$H_n(x) = \frac{1}{\sqrt{n!}} H_{n}(x).$$

Let $H_{n}(x)$ for a vector of indices $n \in \mathbb{N}_0^m$ denote the multivariate polynomial $\prod_{i=1}^m H_{n_i}(x_i)$. It’s easy to see the polynomials $H_{n}(x)$ form an orthogonal basis with respect to the standard m-variate Gaussian distribution. As a consequence, we get

$$\mathbb{E}_{X \sim \mathcal{N}(0, \sigma^2 I_d)} H_{n}(X/\sigma) H_{n'}(X/\sigma) = \begin{cases} 0, & \text{if } n \neq n' \\ 1, & \text{otherwise} \end{cases}$$

which gives us Plancherel’s theorem:
Theorem 6.2 (Plancherel in Hermite basis). Let $f(x) = \sum_n \hat{f}(n)H_n(x/\sigma)$, then
\[ \mathbb{E}_{X \sim \mathcal{N}(0,\sigma^2I)}[(f(X))^2] = \sum_n |\hat{f}(n)|^2 \]

We can use Plancherel’s theorem to get lower bounds on the noise sensitivity of degree $d$ polynomials. This will be an analogue of the variance.

Lemma 6.2. [Variance analogue in Hermite basis] Let $f(x) = \sum_n \hat{f}(n)H_n(x/\sigma)$ and let $f \neq 0 := f - \hat{f}(0)$. Then
\[ \mathbb{E}[(f(A^T X) - Y)^2] \geq (1 - k/n)\|\hat{f}\|_2^2 \]

Proof. First suppose $Z$, and thus $y$, is fixed. Let $S$ denote the support of $Z$. Recall that $A^T x = Z + \xi'$ where $\xi' \sim \mathcal{N}(0,\sigma^2I_{n \times n})$. Define $f_Z(\xi) := f(Z + \xi) - y$, then by Plancherel
\[ \mathbb{E}_\xi[(f(A^T x) - y)^2] = \mathbb{E}_\xi[f_Z(\xi)^2] = \sum_n |\hat{f}_Z(n)|^2 \]

Furthermore
\[ \sum_n |\hat{f}_Z(n)|^2 \geq \sum_{n: \text{supp}(n) \notin S} |\hat{f}(n)|^2 \]
because $(\xi' + Z)|_{S^c} = \xi'|_{S^c}$ so by expanding out $f_Z$ in terms of the fourier expansion of $f$, we see $\hat{f}_Z(n) = \hat{f}(n)$ for $n$ such that $\text{supp}(n) \notin S$. Finally the probability $n \subset S$ for $n \neq 0$ is upper bounded by the probability a single element of its support is in $S$, which is $k/n$. \hfill \square

Next we give a lower bound for the bias, showing that if $\|\hat{f}(0)\|_2^2$ is small for a low-degree polynomial, it cannot accurately predict $y$. Here we will assume $f$ is of the form given by Lemma 6.1.

Lemma 6.3 (Low variance implies high bias). Suppose $f$ is a multivariate polynomial of degree $d$ with no mixed monomials, i.e. $f(x) = \sum_i f_i(x_i)$ where $f_i$ is a univariate polynomial of degree $d$. Expand $f$ in terms of Hermite polynomials as $f(x) = \sum_n \hat{f}(n)H_n(x/\sigma)$. Then
\[ \mathbb{E}[(f(A^T X) - Y)^2] \geq (k/n) \sum_{i=1}^n w_i^2 \max(0, \gamma - \sqrt{\sum_{i=1}^n |\hat{f}(ke_i)|^2(d + 1)^{3d+2}(1 + (\gamma/\sigma)^d)})^2 \]

Before proving the lemma, let us see how it proves the main theorem:

Proof of Theorem 4.2. By Lemma 6.1, Lemma 6.2, and Lemma 6.3 we have that for the $f$ which minimizes the square loss among degree $d$ polynomials, we have a variance-type lower bound
\[ \mathbb{E}[(f(A^T X) - Y)^2] \geq (1 - k/n) \sum_{i=1}^n \sum_{k=1}^d |\hat{f}(ke_i)|^2 \]

and (using that $w_i^2 = 1$ to simplify) a bias-type lower bound
\[ \mathbb{E}[(f(A^T X) - Y)^2] \geq (k/n) \sum_{i=1}^n \max(0, \gamma - \sqrt{\sum_{i=1}^n |\hat{f}(ke_i)|^2(d + 1)^{3d+2}(1 + (\gamma/\sigma)^d)})^2. \]
Let \( \| \hat{f}_i \|_2 := \sqrt{\sum_{i=1}^{n} |\hat{f}(ke_i)|^2} \). Then averaging these lower bounds and simplifying using \( k < n/2 \) gives

\[
\mathbb{E}[(f(A^T X) - Y)^2] \geq (1/4) \sum_{i=1}^{n} \max(\|\hat{f}_i\|_2, \sqrt{k/n} (\gamma - \|\hat{f}_i\|_2 (d + 1)^{3d+2} (1 + (\gamma/\sigma)^d)))^2 \\
\geq (1/4) \frac{\gamma^2 (k/n)}{1 + \sqrt{k/n} (d + 1)^{3d+2} (1 + (\gamma/\sigma)^d)^2} \\
\geq (1/4) \frac{\gamma^2 k}{(1 + \sqrt{k/n} (d + 1)^{3d+2} (1 + (\gamma/\sigma)^d))^2}
\]

Returning to the proof of Lemma 6.3, we have:

**Proof of Lemma 6.3.** Since \( f \) has no mixed monomials, we get for the Hermite expansion that \( \hat{f}(n) = 0 \) unless \( |\text{supp}(n)| \leq 1 \). Let \( X' := A^T X = Z + \xi' \) where \( \xi' \sim N(0, \sigma^2 I) \). Next observe by independence that

\[
\mathbb{E}[(f(X') - Y)^2] = \sum_i w_i^2 \mathbb{E}[(f_i(X'_i) - Z_i)^2] \geq (k/n) \sum_i w_i^2 \mathbb{E}[(f_i(X'_i)|Z_i - Z_i)^2|Z_i \neq 0]
\]

where the last inequality follows since there is a \( k/n \) chance that \( Z_i \sim N(0, \sigma^2 I) \), equivalently that \( Z_i \neq 0 \). By the conditional Jensen’s inequality we have

\[
(k/n) \sum_i w_i^2 \mathbb{E}[(f_i(X'_i) - Z_i)^2|Z_i \neq 0] \geq (k/n) \sum_i w_i^2 \mathbb{E}[(f_i(X'_i)|Z_i - Z_i)^2|Z_i \neq 0].
\]

Observe that \( f_i(X'_i) = \sum_{k=0}^{d} \hat{f}(ke_i) H_k(Z_i/\sigma + \xi'/\sigma) \) and let \( g_i(Z_i) := \mathbb{E}[f_i(X'_i)|Z_i - Z_i, \xi'/\sigma] \), then \( g_i(Z_i) \) is a polynomial of degree \( d \) in \( Z_i \). Write the Hermite polynomial expansion of \( g_i \) in terms of \( H_k(Z_i/\gamma) \) as

\[
g_i(x) = \sum_{k=0}^{d} \hat{g}_i(k) H_k(Z_i/\gamma),
\]

then by Plancherel’s formula

\[
(k/n) \sum_i w_i^2 \mathbb{E}[(\mathbb{E}[g(Z_i) - Z_i)]^2|Z_i \neq 0] = (k/n) \sum_i w_i^2 \sum_{k=0}^{d} |\hat{g}_i(k)|^2 \geq (k/n) \sum_i w_i^2 |\hat{g}_1(1)|^2
\]

and it remains to lower bound \( |\hat{g}_1(1)| \). By orthogonality and direct computation,

\[
\hat{g}_1(1) = \mathbb{E}_{Z_i \sim N(0, \gamma)}[(\mathbb{E}[f_i(X'_i)|Z_i] - Z_i) H_1(Z_i/\gamma)] = -\gamma + \mathbb{E}_{Z_i \sim N(0, \gamma)}[\mathbb{E}[f_i(X'_i)|Z_i](Z_i/\gamma)].
\]

Now we upper bound the last term

\[
\mathbb{E}_{Z_i \sim N(0,\gamma)}[\mathbb{E}[f_i(X'_i)|Z_i](Z_i/\gamma)] = \hat{f}(0) \mathbb{E}[Z_i/\gamma] + \sum_{k=1}^{d} \hat{f}(ke_i) \mathbb{E}_{Z_i \sim N(0, \gamma)}[\mathbb{E}[H_k(Z_i/\sigma + \xi'/\sigma)|Z_i](Z_i/\gamma)]
\]

\[
= \sum_{k=1}^{d} \hat{f}(ke_i) \mathbb{E}_{Z_i \sim N(0,\gamma)}[H_k(Z_i/\sigma + \xi'/\sigma)(Z_i/\gamma)]
\]

\[
\leq \left( \sum_{k=1}^{d} |\hat{f}(ke_i)|^2 \right)^{1/2} \left( \sum_{k=1}^{d} \mathbb{E}_{Z_i \sim N(0,\gamma)}[H_k(Z_i/\sigma + \xi'/\sigma)(Z_i/\gamma)]^2 \right)^{1/2}
\]

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where the second equality is by the law of total expectation and the last inequality is Cauchy-Schwarz. Using the recurrence relation (1), we can bound the sum of the absolute value of the coefficients of \( H_k(x) \) by \( k^k/\sqrt{k!} \leq k^k \). We can also bound the moments of the absolute value of a Gaussian by \( \mathbb{E}_{\xi \sim \mathcal{N}(0,1)}[|\xi|^k] \leq k^k \). Therefore by Holder’s inequality

\[
\mathbb{E}_{Z_i \sim \mathcal{N}(0,\gamma)}[H_k(Z_i/\sigma + \xi'/\sigma)(Z_i/\gamma)] \leq k^k \sup_{\ell=1}^{k} \mathbb{E}_{Z_i \sim \mathcal{N}(0,\gamma)}[(Z_i/\sigma + \xi'/\sigma)^\ell(Z_i/\gamma)] \\
\leq k^k \sup_{\ell=1}^{k} \mathbb{E}_{Z_i \sim \mathcal{N}(0,\gamma)}[(Z_i/\sigma)|\xi'|/\sigma]^\ell(Z_i/\gamma)] \\
\leq 2^k k^k \sup_{\ell=1}^{k} (\mathbb{E}_{Z_i \sim \mathcal{N}(0,\gamma)}[|Z_i|/(\sigma|\xi'|)^\ell(Z_i/\gamma)] + \mathbb{E}_{Z_i \sim \mathcal{N}(0,\gamma)}[|\xi'|Z_i/(\sigma|\xi'|)]) \\
\leq 2^k k^k \max(1,(\gamma/\sigma)^k(k+1)^{k+1} + k^k) \\
\leq (k+1)^{3k+1}(1+(\gamma/\sigma)^k).
\]

Therefore by reverse triangle inequality

\[
||\hat{g}_i(1)||^2 \geq \max(0,\gamma - \sqrt{\sum_{i=1}^{n} |\hat{f}^2(ke_i)|^2(d+1)^{3d+2}(1+(\gamma/\sigma)^d)^2}).
\]

\[
\square
\]

\textbf{Remark 6.2. Remarks on results:}

We make a few remarks regarding the results in this section. Recall that \( \gamma^2 k \) is the square loss of the trivial zero-estimator. Suppose as before that \( \gamma = \Theta(\sigma^2 \text{polylog}(n)) \), then we see that if \( d = o(\log n/\log \log n) \) then the denominator of the lower bound tends to 1, hence any such polynomial estimator has a rate no better than that of the trivial zero-estimate.

It is possible to derive a similar statement to Theorem 4.2 that holds with high probability instead of in expectation for polynomials of degree \( o(\log n/\log \log n) \). All that is needed is to bound the contribution to the expectation from very rare tail events when the realization of the noise \( \xi \) is atypically large. Since the polynomials we consider are very low degree \( o(\log n/\log \log n) \), they can only grow at a rate of \( x^d = x^{o(\log(n)/\log \log n)} \); thus standard growth rate estimates (e.g. the Remez inequality) combined with the Gaussian tails of the noise can be used to show that a polynomial which behaves reasonably in the high-probability region (e.g. which has small w.h.p. error) cannot contribute a large amount to the expectation in the tail region.

7 Part 2: A Nearly Optimal Polynomial Construction

We previously showed that for polynomials to match the statistical performance of a 2-Layer ReLu network, the degree needs to be \( \Omega(\log n) \). In this section, we show that this is almost tight by constructing polynomials of degree \( O(\log^2 m) \).

Our strategy is to plug in a good approximation to ReLU in the 2-layer ReLU network construction. One might hope that simply taking the “best polynomial approximation” of degree \( d \) in the typical approximation theory sense to ReLU in the interval \([-1, 1]\) would suffice, but in fact this is extremely inefficient; because ReLU is not smooth, standard results in approximation theory (see Chapters 7.8 of (DeVore and Lorentz, 1993)) show that a degree \( d \) polynomial cannot get closer than \( O(1/\text{poly}(d)) \) in infinity-norm. (And as noted before, we don’t need to approximate the ReLU particularly well near the kink.)

Instead we will carefully design an approximation to ReLu: in particular, the polynomial we take will be extremely close to 0 in the threshold region of the ReLu. We prove the following theorem, in which the parameter \( \tau \) in our theorem controls the trade-off between the polynomial \( p_d \) being close to 0 for \( x < 0 \) and being close to \( x \) for \( x > 0 \).
**Theorem 7.1.** Suppose \( R > 0, 0 < \tau < 1/2 \) and \( d \geq 7 \). Then there exists a polynomial \( p_d = p_d,\tau,R \) of degree \( d \) such that for \( x \in [-R, 0] \)

\[
|p_d(x) - \text{ReLU}(x)| \leq 14R \sqrt{\frac{d}{\tau \pi}} e^{-\sqrt{\pi d/4}}
\]

and for \( x \in [0, R] \),

\[
|p_d(x) - \text{ReLU}(x)| \leq 2R \tau + 2R \sqrt{\frac{4\tau}{\pi d}} + 12R \sqrt{\frac{d}{\tau \pi}} e^{-\sqrt{\pi d/4}}.
\]

The proof proceeds by combining a mollification of ReLU with complex analytic machinery from approximation theory. Before presenting it, let us see how it can be used to imply the main result, Theorem 4.3.

Toward that, we substitute our polynomial construction for \( \rho_\tau \) in Lemma A.2. Namely, define \( M_\tau = M + 2\tau \) and

\[
\tilde{p}_{d,\tau,M} = p_{d,\tau/M,\tau,M}(x - \tau) + p_{d,\tau/M,\tau,M}(-x + \tau)
\]

where \( p \) is the polynomial constructed in Theorem 7.1. We then have:

**Lemma 7.1.** Suppose \( \epsilon, \tau > 0 \) and \( M \geq 1 \). Then for all \( d \geq d_0 = \Omega((M_\tau^2 \log^2 (M_\tau / \epsilon \tau))) \), for \( |x| \in (\tau, M_\tau) \) we have

\[
|\tilde{p}_{d,\tau,M}(x) - x| \leq 3\tau + \epsilon
\]

and for \( |x| \leq \tau \) we have

\[
|\tilde{p}_{d,\tau,M}(x)| \leq \epsilon.
\]

**Proof.** By the guarantee of Theorem 7.1, we see that for \( |x| \leq \tau \) that

\[
|\tilde{p}_{d,\tau,M}(x)| \leq 28M_\tau \sqrt{\frac{dM_\tau}{\pi \tau}} e^{-\sqrt{\pi d/4M_\tau}}.
\]

Thus we see that taking \( d = \Omega((M_\tau^2 \log^2 (M_\tau / \epsilon \tau))) \) suffices to make the latter expression at most \( \epsilon \). Similarly for \( |x| > \tau \) we know that

\[
|\tilde{p}_{d,\tau,M}(x)| \leq 2\tau + 2M_\tau \sqrt{\frac{4\tau}{M_\tau \pi d}} + 26M_\tau \sqrt{\frac{dM_\tau}{\pi \tau}} e^{-\sqrt{\pi d/4M_\tau}}
\]

and taking \( d = \Omega((M_\tau^2 \log^2 (M_\tau / \epsilon \tau))) \) with sufficiently large constant guarantees the middle term is at most \( \tau \) and the last term is at most \( \epsilon \).

Using this, we can show that if we use a polynomial of degree \( \Omega((M/\sigma \sqrt{\log n}) \log^2 m) \) we can achieve similar statistical performance to the ReLU network. Namely, we can show:

**Lemma 7.2.** Suppose \( A \) is \( \mu \)-incoherent i.e. \( \|A^T A - I\|_{\infty} \leq \mu \). Let \( z \) be an arbitrary fixed vector such that \( \|z\|_1 \leq M \) and \( |\text{supp}(z)| \leq k \). Suppose \( x = Az + \xi \) where \( \xi \sim N(0, \sigma^2 I_{n \times n}) \). Then for some \( \tau = \Theta(\sqrt{1 + \mu} \log m + \mu M) \), for any \( d \geq d_0 = \Omega((M_\tau^2 \log^2 (M_\tau m / \tau^2))) \), if we take \( \hat{z} := \tilde{p}_{d,\tau,M}^\circ (A^T x) \), then with high probability we have \( \|\hat{z} - z\|_1 \leq 6k\tau \).

**Proof.** Apply Lemma 7.1 with \( \epsilon = \tau / m \). Then we see for \( |x| \in (\tau, M_\tau) \) we havn

\[
|\tilde{p}_{d,\tau,M}(x) - x| \leq (3 + 1/m)\tau \leq 4\tau
\]

and for \( |x| \leq \tau \) we have

\[
|\tilde{p}_{d,\tau,M}(x)| \leq \tau / m
\]

Observe that

\[
A^T x = z + (A^T A - I_d)z + A^T \xi.
\]

Note that entry \( i \) of \( A^T \xi \) is \( \langle A_i, \xi \rangle \) where \( \|A_i\|_2^2 \leq (1 + \mu) \) so \( A^T \xi \) is Gaussian with variance at most \( \sigma^2 (1 + \mu) \).

By choosing \( \tau \) with sufficiently large constant, then applying the sub-Gaussian tail bound and union bound, with high probability all coordinates not in the true support are thresholded to at most \( \tau / m \). Similarly we see that for each of the coordinates in the support, an error of at most \( 5\tau \) is made. Therefore \( \|\hat{z} - z\|_1 \leq 5k\tau + m(\tau / m) \leq 6k\tau \).
Now we have all the ingredients to prove Theorem 4.3:

**Proof of Theorem 4.3.** Define an estimate for $Y$ by taking $\hat{Y}_{d,M} := \hat{\rho}^{\otimes n}_{d,M}(A^\top X)$ where $\tau$ is defined as in the Lemma, and then taking $\hat{Y}_{d,M} := \langle \hat{w}, \hat{Z}_{d,M} \rangle$. Applying the previous Lemma, we get analogous versions of Theorem A.1 by the same argument as in that theorem.

Finally, we return to the proof of Theorem 7.1:

**Proof of Theorem 7.1.** We start with the case where $R = 1/2$. We build the approximation in two steps. First we approximate ReLu by the following “annealed” version of ReLu, for parameters $\beta > \pi, \tau > 0$ to be optimized later:

$$g_\beta(x) = \frac{1}{\beta} \log(1 + e^{\beta x})$$

$$f_{\beta, \tau}(x) = g_\beta(x - \tau).$$

Observe that when we look at negative inputs, $g_\beta(-x) = \frac{1}{\beta} \log(1 + e^{-\beta x}) \leq \frac{1}{\beta} e^{-\beta x}$. Therefore when $x < 0$, $f_\beta(x) \leq \frac{1}{\beta} e^{-\beta \tau}$.

For the second step, we need to show $f_\beta$ can be well-approximated by low-degree polynomials. In fact, because $f_\beta$ is analytic in a neighborhood of the origin, it turns out that its optimal rate of approximation is determined exactly by its complex-analytic properties. More precisely, define $D_\rho$ to be the region bounded by the ellipse in $\mathbb{C} = \mathbb{R}^2$ centered at the origin with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

with semi-axes $a = \frac{1}{\beta}(\rho + \rho^{-1})$ and $b = \frac{1}{\beta}|\rho - \rho^{-1}|$; the focii of the ellipse are $\pm 1$. For an arbitrary function $f : [-1, 1] \to \mathbb{R}$, let $E_d(f)$ denote the error of the best polynomial approximation of degree $d$ in infinity norm on the interval $[-1, 1]$ of $f$. Then the following theorem of Bernstein exactly characterizes the growth rate of $E_d(f)$:

**Theorem 7.2 (Theorem 7.8.1, (DeVore and Lorentz, 1993)).** Let $f$ be a function defined on $[-1, 1]$. Let $\rho_0$ be the supremum of all $\rho$ such that $f$ has an analytic extension on $D_\rho$. Then

$$\limsup_{d \to \infty} \sqrt[4]{E_d(f)} = \frac{1}{\rho_0}$$

For our application we need only the upper bound and we need a quantitative estimate for finite $n$. Following the proof of the upper bound in (DeVore and Lorentz, 1993), we get the following result:

**Theorem 7.3.** Suppose $f$ is analytic on the interior of $D_{\rho_1}$ and $|f(z)| \leq M$ on the closure of $D_{\rho_1}$. Then

$$E_d(f) \leq \frac{2M}{\rho_1 - \rho_1^{-n}}$$

We will now apply this theorem to $g_\beta$. First, we claim that $g_\beta$ is analytic on $D_{\rho_1}$ where $\rho_1$ is the solution to this equation for the semi-axis of the ellipse:

$$\frac{1}{2}(\rho - \rho^{-1}) = \frac{\pi}{2\beta}$$

which is

$$\rho_1 = \frac{\sqrt{4\beta^2 + \pi^2} + \pi}{2\beta} > 1 + \pi/2\beta.$$

To see this, first extend log to the complex plane by taking a branch cut at $(-\infty, 0]$. To prove $g_\beta$ is analytic on $D_{\rho_1}$, we just need to prove that $1 + e^{\beta z}$ avoids $(-\infty, 0]$ for $z \in D_{\rho_1}$. This follows because by the definition of $\rho_1$, for every $z \in D_{\rho_1}$, $\Re(z) < \frac{\pi}{2\beta}$ hence $\Re(1 + e^{\beta z}) \geq 1$. We also see that for $z \in D_{\rho_1}$,

$$|g_\beta(z)| = \frac{1}{\beta} |\log(1 + e^{\beta z})| \leq \frac{1}{\beta} \sup_{w \in D_{\rho_1}} |\log(1 + e^w)| \leq \frac{1}{\beta} (\log(1 + e^{\beta}) + \pi) < 6.$$
Therefore by Theorem 7.3 we have
\[ E_d(g_\beta) \leq \frac{12\beta}{\pi} (1 + \pi/2\beta)^{-n} \leq \frac{12\beta}{\pi} e^{-\pi n/4\beta} \]

where in the last step we used that \( 1 + x \geq \exp(x/2) \) for \( x < 1/2 \) and that \( \beta > \pi \). Let \( \tilde{g}_{\beta,d} \) denote the best polynomial approximation to \( g_\beta \) of degree \( d \) and let \( f_{\beta,\tau,d} = \tilde{g}_{\beta,d}(x - \tau) \)

Thus for \( x \in [-1 + \tau, 0] \),
\[ |\text{ReLu}(x) - \tilde{f}_{\beta,\tau,d}(x)| \leq |f_{\beta,\tau}(x)| + |\tilde{g}_{\beta,d}(x - \tau) - g_{\beta,\tau}(x - \tau)| \leq \frac{1}{\beta} e^{-\beta \tau} + \frac{12\beta}{\pi} e^{-\pi d/4\beta} \]

Take \( \beta = \sqrt{\pi d/4\tau} \) and require \( d > 7 \) so that \( \beta > 1 \), then for \( x \in [-1 + \tau, 0] \),
\[ |\text{ReLu}(x) - \tilde{f}_{\beta,\tau,d}(x)| \leq 7 \sqrt{\frac{d}{\tau \pi}} e^{-\sqrt{\pi d/4}} \]

For \( x \in (0, 1 - \tau) \) we have by the 1-Lipschitz property of \( g_\beta \) and calculus that
\[ |x - f_{\beta,\tau}(x)| \leq |x - g_\beta(x)| \leq \tau + \frac{\log 2}{\beta} \]

so
\[ |\text{ReLu}(x) - \tilde{f}_{\beta,\tau,d}(x)| \leq |x - f_{\beta,\tau}(x)| + |\tilde{g}_{\beta,d}(x - \tau) - g_{\beta,\tau}(x - \tau)| \leq \tau + \frac{\log 2}{\beta} + \frac{12\beta}{\pi} e^{-\pi d/4\beta}. \]

Plugging in our value of \( \beta \) and using \( \log 2 \leq 1 \) gives
\[ |\text{ReLu}(x) - \tilde{f}_{\beta,\tau,d}(x)| \leq \tau + \sqrt{\frac{4\tau}{\pi d}} + \sqrt{\frac{d}{\tau \pi}} e^{-\sqrt{\pi d/4}} \]

Now the result for general \( R \) follows by taking \( p_d(x) = 2R \tilde{f}_{\beta,\tau,d}(x/2R) \), since \( 2R \cdot \text{ReLu}(x/2R) = \text{ReLu}(x) \) and \([-1/2, 1/2] \subset [-1 + \tau, 1 - \tau] \).

8 Conclusions

We’ve attacked the problem of providing representation lower and upper bounds for different classes of universal approximators in a natural statistical setup. We hope this will inspire researches to move beyond the worst-case setup when considering the representational power of different predictors.

The techniques we develop are interesting in their own right: unlike standard approximation theory setups, we need to design polynomials which may only need to be accurate in certain regions. Conceivably, in classification setups, similar wisdom may be helpful: the approximator needs to only be accurate near the decision boundary.

Finally, we conclude with a tantalizing open problem: In general it is possible to obtain non-trivial sparse recovery guarantees for LASSO even when the sparsity \( k \) is nearly of the same order as \( n \) under assumptions such as RIP. Since LASSO can be computed quickly using iterated soft thresholding (ISTA and FISTA, see (Beck and Teboulle, 2009)), we see that sufficiently deep neural networks can compute a near-optimal solution in this setting as well. It would be interesting to determine whether shallower networks and polynomials of degree polylog(\( n \)) can achieve a similar guarantee.

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Chiyuan Zhang, Samy Bengio, Moritz Hardt, Benjamin Recht, and Oriol Vinyals. Understanding deep learning requires rethinking generalization. *ICLR*, 2017.
A Upper bound via 2-Layer ReLu Network

First we describe the 2-layer ReLu network that we will analyze. Define the soft threshold function with threshold $\tau$: $\rho_\tau(x) = \text{sgn}(x) \tau \min(0, |x| - \tau) = \text{ReLU}(x - \tau) + \text{ReLU}(-x + \tau)$. Then we consider the following estimate for $y$, which corresponds to a 2-layer neural network:

$$\hat{Z}_{NN} := \rho_\tau^{\otimes n}(A^T X)$$

$$\hat{Y}_{NN} := \langle w, \hat{Z}_{NN} \rangle$$

We will first prove a bound on the error of the soft-thresholding estimator $\hat{Z}_{NN}$ (Lemma A.2), which corresponds to the hidden layer of the neural network and is essentially a standard fact in high-dimensional statistics (see reference text (Rigollet, 2017)). The idea is that the soft thresholding will correctly zero-out most of the coordinates in the support while adding only a small additional error to the coordinates outside the support.

From the recovery guarantee for $\hat{Z}_{NN}$, we will then deduce the following theorem for the estimator $\hat{Y}_{NN}$:

**Theorem A.1.** With high probability, the estimator $\hat{Y}_{NN}$ satisfies

$$|\hat{Y}_{NN} - Y|^2 = O((1 + \mu)\sigma^2 k^2 \log(m) + \mu^2 k^2 M^2)$$

In order to interpret this, note that standard constructions of incoherent matrices, such as a matrix with independent random $\pm 1/\sqrt{n}$ entries, give incoherence of $\mu = O(\sqrt{\log m}/n)$ (this follows from concentration inequalities and a union bound. See e.g. reference text (Rigollet, 2017)). Therefore for such an incoherent matrix, and a choice of $k$ which is not too large with respect to $n$, the effect of $\mu$ in the bound is small and can be disregarded. Then this bound is intuitive because if we think of $k$ as small and fixed, it says the error is on the order of the noise $\sigma^2$ with an additional log factor for not knowing where the true support lies.

Towards proving the above result, we first need an estimate on the bias of $A^T x$, i.e. the error without noise:

**Lemma A.1.** Suppose $A$ is $\mu$-incoherent i.e. $\|A^T A - I\|_\infty \leq \mu$. Then for any $z$, $\|A^T A z - z\|_\infty \leq \mu \|z\|_1$.

**Proof.**

$$\langle A^T A z \rangle_i = \langle A_i, \sum_j z_j A_j \rangle = z_i \langle A_i, A_i \rangle + \sum_{j \neq i} z_j \langle A_i, A_j \rangle$$

so applying the incoherence assumption we have $|(A^T A z)_i - z_i| \leq \mu \|z\|_1$. \hfill \qed

Using this we can analyze the error in thresholding.

**Lemma A.2.** Suppose $A$ is $\mu$-incoherent i.e. $\|A^T A - I\|_\infty \leq \mu$. Let $z$ be an arbitrary fixed vector such that $\|z\|_1 \leq M$ and $|\text{supp}(z)| \leq k$. Suppose $x = Az + \xi$ where $\xi \sim N(0, \sigma^2 I_{kn})$. Then for some $\tau = \Theta(\sigma \sqrt{(1 + \mu) \log m + \mu M})$ and $\hat{z} = \rho_\tau^{\otimes n}(A^T x)$, with high probability we have $\|\hat{z} - z\|_\infty \leq 2\tau$ and $\text{supp}(\hat{z}) \subset \text{supp}(z)$.

**Proof.** Observe that

$$A^T x = z + (A^T A - I)z + A^T \xi.$$ 

Note that entry $i$ of $A^T \xi$ is $\langle A_i, \xi \rangle$ where $\|A_i\|_2^2 \leq (1 + \mu)$ so $(A^T \xi)_i$ is subgaussian with variance proxy at most $\sigma^2(1 + \mu)$.

By concentration and union bound, with high probability all coordinates not in the true support are thresholded to 0. Similarly we see that for each of the coordinates in the support, an error of at most $2\tau$ is made. \hfill \qed

From the above lemma, we can easily prove the main theorem of this section:

**Proof of Theorem A.1.** When the high probability above event happens, we have the following upper bound by Holder’s inequality:

$$|\hat{Y}_{NN} - Y|^2 = \langle w |_{\text{supp}(h)}, (\hat{Z}_{NN} - Z) |_{\text{supp}(h)} \rangle^2 \leq k^2 \|\hat{Z}_{NN} - Z\|_\infty^2 = O(k^2((1 + \mu)\sigma^2 \log(m) + \mu^2 M^2))$$

\hfill \qed
For the lower bounds we will be interested mostly in the case when $\mu = 0$, i.e. $A$ is orthogonal and so $m = n$, the coordinates of $Z$ are independent and each is nonzero with probability at most $k/n$, and the noise is Gaussian. Then the error estimate we had in the previous theorem specializes to $O(\sigma^2 k^2 \log(n))$, but under these assumptions we know that the information-theoretic optimal is actually $\sigma^2 k \log(n)$. We can redo the analysis to eliminate the extra factor of $k$, without changing the algorithm:

**Theorem A.2.** Suppose $A$ is orthogonal (hence $m = n$), the coordinates of $Z$ are independent, and $\xi \sim N(0, \sigma^2 I)$. Then

$$E|\hat{Y}_{NN} - Y|^2 = O(k\sigma^2 \log(m))$$

**Proof.** In this case, we have $A^\top X = Z + \xi'$ where $\xi' \sim N(0, \sigma^2 I)$. Therefore the coordinates of $\hat{Z}$ are independent of each other, and so we see

$$E|\hat{Y}_{NN} - Y|^2 = \sum_i w_i^2 E[(\hat{Z}_{NN} - Z)^2_i] \leq \sum_i E[(\hat{Z}_{NN} - Z)^2_i].$$

Let $E_i$ denote the event that $|\xi'|_i > \tau$. Then

$$\sum_i E[(\hat{Z}_{NN} - Z)^2_i] = \sum_i E[(\mathbb{1}_{E_i} + \mathbb{1}_{E^C_i})(\hat{Z}_{NN} - Z)^2_i] \leq 4k\tau^2 + \sum_i E[\mathbb{1}_{E_i}^C(\hat{Z}_{NN} - Z)^2_i]$$

$$= 4k\tau^2 + \sum_i \Pr(\mathbb{1}_{E_i}^C)E[(\hat{Z}_{NN} - Z)^2_i | \mathbb{1}_{E_i}^C = 1]$$

$$\leq 4k\tau^2 + \sum_i \Pr(\mathbb{1}_{E_i}^C)(2\tau^2 + 2E[|\xi'_i|^2 | \mathbb{1}_{E_i}^C = 1])$$

$$\leq 4k\tau^2 + \sum_i Cm(2\tau^2 + 2C'\tau^2)$$

where the first inequality follows as in Lemma A.2, the second inequality uses that $|\rho_\tau(x) - x| \leq \tau$, the third uses that $(a + b)^2 = a^2 + 2ab + b^2 \leq 2a^2 + 2b^2$ by Young’s inequality, and the last inequality follows from standard tail bounds on Gaussians. We see the last expression is $O(k\sigma^2 \log(m))$ so we have proved the result.  \[\square\]