A HOPF ALGEBRA ASSOCIATED TO A LIE PAIR

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In tribute to Alan Weinstein on the occasion of his seventieth birthday

Abstract. The quotient $L/A[-1]$ of a pair $A \hookrightarrow L$ of Lie algebroids is a Lie algebra object in the derived category $D^b(A)$ of the category $A$ of left $\mathcal{U}(A)$-modules, the Atiyah class $\alpha_{L/A}$ being its Lie bracket. In this note, we describe the universal enveloping algebra of the Lie algebra object $L/A[-1]$ and we prove that it is a Hopf algebra object in $D^b(A)$.

1. Introduction

Let $A$ be a Lie algebroid over a manifold $M$. Its space of smooth sections $\Gamma(A)$ is a Lie-Rinehart algebra over the commutative ring $R = \mathcal{C}^\infty(M)$. By an $A$-module, we mean a module over the Lie-Rinehart algebra corresponding to the Lie algebroid $A$, i.e. a module over the associative algebra $\mathcal{U}(A)$.

Recall that the universal enveloping algebra $\mathcal{U}(A)$ of a Lie algebroid $A$ over $M$ is simultaneously an associative algebra and an $R$-bimodule. In case the Lie algebroid $A$ is real, $\mathcal{U}(A)$ is canonically identified to the algebra of left-invariant s-fiberwise differential operators on the local Lie groupoid $\mathcal{A}$ integrating $A$. Let us recall its construction.

The vector space $g = R \oplus \Gamma(A)$ admits a natural Lie algebra structure given by the Lie bracket

$$[f + X, g + Y] = \rho(X)g - \rho(Y)f + [X, Y],$$

where $f, g \in R$ and $X, Y \in \Gamma(A)$. Here $\rho$ denotes the anchor map. Let $i$ denote the natural inclusion of $g$ into its universal enveloping algebra $\mathcal{U}(g)$. The universal enveloping algebra $\mathcal{U}(A)$ of the Lie algebroid $A$ is the quotient of the subalgebra of $\mathcal{U}(g)$ generated by $i(g)$ by the two-sided ideal generated by the elements of the form $i(f) \otimes i(g + Y) - i(fg + fY)$ with $f, g \in R$ and $Y \in \Gamma(A)$.

When $A$ is a Lie algebra, $\mathcal{U}(A)$ is indeed the usual universal enveloping algebra. On the other hand, when $A$ is the tangent bundle $TM$, $\mathcal{U}(A)$ is the algebra of differential operators on $M$.

We use the symbol $A$ to denote the abelian category of $A$-modules. Abusing terminology, we say that a vector bundle $E$ over $M$ is an $A$-module if $\Gamma(E) \in A$.

Given a Lie pair $(L, A)$ of algebroids, i.e. a Lie algebroid $L$ with a Lie subalgebroid $A$, the Atiyah class $\alpha_E$ of an $A$-module $E$ relative to the pair $(L, A)$ is defined as the obstruction to the existence of an $A$-compatible $L$-connection on the vector bundle $E$. An $L$-connection $\nabla$ on an $A$-module $E$ is said to be $A$-compatible if it extends the given flat $A$-connection on $E$ and satisfies $\nabla_a \nabla_l - \nabla_l \nabla_a = \nabla_{[a, l]}$ for all $a \in \Gamma(A)$ and $l \in \Gamma(L)$. This fairly recently defined class (see [1]) has as double

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The quotient $L/A$ of any Lie pair $(L, A)$ is an $A$-module \cite{1}. Its Atiyah class $\alpha_{L/A}$ can be described as follows. Choose an $L$-connection $\nabla$ on $L/A$ extending the $A$-action. Its curvature is the vector bundle map $R^\nabla : \wedge^2 L \to \text{End}(E)$ defined by $R^\nabla(l_1, l_2) = \nabla_{l_2} \nabla_{l_1} - \nabla_{l_1} \nabla_{l_2} - \nabla_{[l_1, l_2]}$, for all $l_1, l_2 \in \Gamma(L)$. Since $L/A$ is an $A$-module, $R^\nabla$ vanishes on $\wedge^2 A$ and, therefore, determines a section $R^\nabla_{L/A}$ of $A^* \otimes (L/A)^* \otimes \text{End}(L/A)$. It was proved in \cite{1} that $R^\nabla_{L/A}$ is a 1-cocycle for the Lie algebroid $A$ with values in the $A$-module $(L/A)^* \otimes \text{End}(L/A)$ and that its cohomology class $\alpha_{L/A} \in H^1(A; (L/A)^* \otimes \text{End}(L/A))$ is independent of the choice of the connection.

Let $\text{Ch}^b(A)$ denote the category of bounded complexes in $A$ and let $D^b(A)$ denote the corresponding derived category. We write $L/A[-1]$ to denote the quotient $L/A$ regarded as a complex in $A$ concentrated in degree 1.

The following was proved in \cite{1}.

**Proposition 1.1** (\cite{1}). Let $(L, A)$ be a Lie algebroid pair. The Atiyah class $\alpha_{L/A}$ of the quotient $L/A$ relative to the pair $(L, A)$ determines a morphism

$$L/A[-1] \otimes L/A[-1] \to L/A[-1]$$

in the derived category $D^b(A)$ making $L/A[-1]$ a Lie algebra object in $D^b(A)$.

It is well known that every ordinary Lie algebra $\mathfrak{g}$ admits a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, which is a Hopf algebra. We are thus led to the following natural questions: does there exist a universal enveloping algebra for $L/A[-1]$ in $D^b(A)$ and, if so, is it a Hopf algebra object?

In this Note, we give a positive answer to the questions above. For a complex manifold $X$, the Atiyah class of the Lie pair $(T_X \otimes \mathbb{C}, T_X^{1, 1})$ is simply the usual Atiyah class of the holomorphic tangent bundle $T_X$ recently exploited by Kapranov \cite{2}. It was proved that the universal enveloping algebra of the Lie algebra object $T_X[-1]$ in $D^b(X)$ is the Hochschild cochain complex $(\mathcal{D}^*_{\text{poly}}(X), d)$ \cite{5, 6, 7}. This result played an important role in the study of several aspects of complex geometry including the Riemann-Roch theorem \cite{5}, the Chern character \cite{6} and the Rozansky-Witten invariants \cite{7, 8}. Applications of our result will be developed elsewhere.

### 2. Hochschild-Kostant-Rosenberg map

It is known \cite{9} that the universal enveloping algebra $\mathcal{U}(L)$ of a Lie algebroid $L$ admits a cocommutative coassociative coproduct $\Delta : \mathcal{U}(L) \to \mathcal{U}(L) \otimes \mathcal{U}(L)$, which is defined on generators as follows: $\Delta(f) = f \otimes 1 = 1 \otimes f, \forall f \in R$ and $\Delta(l) = l \otimes 1 + 1 \otimes l$, $\forall l \in \Gamma(L)$. Here, and in the sequel, $\otimes$ stands for the tensor product of left $R$-modules. Moreover, $\mathcal{U}(L)$ is an $L$-module since each section $l$ of $L$ acts on $\mathcal{U}(L)$ by left multiplication: $\nabla_i u = l \cdot u$, $\forall u \in \mathcal{U}(L)$.

Now, given a Lie pair $(L, A)$, consider the quotient $\mathcal{D}^1_{\text{poly}}$ of $\mathcal{U}(L)$ by the left ideal generated by $\Gamma(A)$. It is straightforward to see that the comultiplication on $\mathcal{U}(L)$ induces a comultiplication $\Delta : \mathcal{D}^1_{\text{poly}} \to \mathcal{D}^1_{\text{poly}} \otimes \mathcal{D}^1_{\text{poly}}$ on $\mathcal{D}^1_{\text{poly}}$ and the action of $L$ on $\mathcal{U}(L)$ determines an action of $A$ on $\mathcal{D}^1_{\text{poly}}$. 


Lemma 2.1. The quotient \( D^1_{\text{poly}} = \frac{\mathcal{W}(L)}{\mathcal{W}(L)[1]} \) is simultaneously a cocommutative coassociative \( R \)-coalgebra and an \( A \)-module. Moreover, its comultiplication is compatible with its \( A \)-action:

\[ \nabla_X(\Delta p) = \Delta(\nabla_X p), \quad \forall X \in \Gamma(A), p \in D^1_{\text{poly}}. \]

Let \( D^n_{\text{poly}} \) denote the \( n \)-th tensorial power \( D^1_{\text{poly}} \otimes \cdots \otimes D^1_{\text{poly}} \) of \( D^1_{\text{poly}} \) and, for \( n = 0 \), set \( D^0_{\text{poly}} = R \). We define a coboundary operator \( d : D^\bullet_{\text{poly}} \to D^\bullet+1_{\text{poly}} \) on \( D^\bullet_{\text{poly}} = \bigoplus_{n=0}^{\infty} D^n_{\text{poly}} \) by

\[ d(p_1 \otimes \cdots \otimes p_n) = 1 \otimes p_1 \otimes \cdots \otimes p_n - (\Delta p_1) \otimes \cdots \otimes p_n + p_1 \otimes (\Delta p_2) \otimes \cdots \otimes p_n - \cdots + (-1)^n p_1 \otimes \cdots \otimes p_{n-1} \otimes (\Delta p_n) + (-1)^{n+1} p_1 \otimes \cdots \otimes p_n \otimes 1, \quad (1) \]

for any \( p_1, p_2, \ldots, p_n \in D^n_{\text{poly}} \). Since the comultiplication \( \Delta \) is compatible with the action of \( A \), the operator \( d \) is a morphism of \( A \)-modules. Moreover, \( \Delta \) being coassociative, \( d \) satisfies \( d^2 = 0 \). Thus \( (D^\bullet_{\text{poly}}, d) \) is an object of \( \text{Ch}^b(A) \).

When endowed with the trivial coboundary operator, the space of sections of

\[ S^\bullet(L/A[-1]) = \bigoplus_{k=0}^{\infty} S^k(L/A)[-1] = \bigoplus_{k=0}^{\infty} \left( \wedge^k L/A \right)[-k] \]

is a complex of \( A \)-modules:

\[ 0 \to R \xrightarrow{0} \Gamma(L/A) \xrightarrow{0} \Gamma(\wedge^2(L/A)) \xrightarrow{0} \Gamma(\wedge^3(L/A)) \xrightarrow{0} \cdots \]

The natural inclusion \( \Gamma(L/A) \hookrightarrow D^1_{\text{poly}} \) extends naturally to the Hochschild-Kostant-Rosenberg map

\[ \text{HKR} : \Gamma(S^\bullet(L/A[-1])) \to D^\bullet_{\text{poly}} \]

by skew-symmetrization:

\[ \text{HKR}(b_1 \wedge \cdots \wedge b_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma) b_{\sigma(1)} \otimes b_{\sigma(2)} \otimes \cdots \otimes b_{\sigma(n)}, \quad \forall b_1, \cdots, b_n \in \Gamma(L/A). \quad (2) \]

Proposition 2.2. In \( \text{Ch}^b(A) \), the Hochschild-Kostant-Rosenberg map is a quasi-isomorphism from \( (\Gamma(S^\bullet(L/A[-1]))), 0) \) to \( (D^\bullet_{\text{poly}}, d) \).

Sketch of proof. Assuming \( L \) and \( A \) are real Lie algebroids, let \( \mathcal{L} \) and \( \mathcal{A} \) be local Lie groupoids integrating \( L \) and \( A \) respectively. The source map \( s : \mathcal{L} \to M \) induces a surjective submersion \( J : \mathcal{L}/\mathcal{A} \to M \). The right quotient \( \mathcal{L}/\mathcal{A} \) is a left \( \mathcal{L} \)-homogeneous space with momentum map \( J^3 \). Therefore, it admits an infinitesimal \( L \)-action, and hence an infinitesimal \( A \)-action. The coalgebra \( D^1_{\text{poly}} \) may be regarded as the space of distributions on the \( J \)-fibers of \( \mathcal{L}/\mathcal{A} \) supported on \( M \). Its \( A \)-module structure then stems from the infinitesimal \( A \)-action on \( \mathcal{L}/\mathcal{A} \). The \( n \)-th tensorial power \( D^n_{\text{poly}} \) may be viewed as the space of \( n \)-differential operators on the \( J \)-fibers of \( \mathcal{L}/\mathcal{A} \) evaluated along \( M \) and the differential \( d \) as the Hochschild coboundary. The conclusion follows from the classical Hochschild-Kostant-Rosenberg theorem. To prove the proposition for complex Lie algebroids, it suffices to consider formal groupoids instead of local Lie groupoids [3].
3. Universal enveloping algebra of $L/A[-1]$ in $D^b(A)$

Following Markarian [5], Ramadoss [6], and Roberts-Willerton [7], we introduce the following:

**Definition 3.1.** If it exists, the universal enveloping algebra of a Lie algebra object $G$ in $D^b(A)$ is an associative algebra object $H$ in $D^b(A)$ together with a morphism of Lie algebras $i : G \rightarrow H$ satisfying the following universal property: given any associative algebra object $K$ and any morphism of Lie algebras $f : G \rightarrow K$ in $D^b(A)$, there exists a unique morphism of associative algebras $f' : H \rightarrow K$ in $D^b(A)$ such that $f = f' \circ i$.

In view of the similarity between $(D^\bullet_{\text{poly}}, d)$ and the Hochschild cochain complex, we define a cup product $\cup$ on $D^\bullet_{\text{poly}}$ by setting $P \cup Q = P \otimes Q$, for all $P, Q \in D^\bullet_{\text{poly}}$. Is is simple to check that

$$d(P \cup Q) = dP \cup Q + (-1)^{|P|}P \cup dQ,$$

for all homogeneous $P, Q \in D^\bullet_{\text{poly}}$.

**Proposition 3.2.** For any Lie pair $(L, A)$ of algebroids, $(D^\bullet_{\text{poly}}, d, \cup)$ is an associative algebra object in $D^b(A)$, which is in fact the universal enveloping algebra of the Lie algebra $L/A[-1]$ in $D^b(A)$.

Consider the inclusion $\eta : R \hookrightarrow D^n_{\text{poly}}$, the projection $\varepsilon : D^n_{\text{poly}} \rightarrow R$, and the maps $t : D^\bullet_{\text{poly}} \rightarrow D^\bullet_{\text{poly}}$ and $\hat{\Delta} : D^\bullet_{\text{poly}} \rightarrow D^\bullet_{\text{poly}} \otimes D^\bullet_{\text{poly}}$ defined, respectively, by

$$t(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = (-1)^{\frac{n(n+1)}{2}} p_n \otimes p_{n-1} \otimes \cdots \otimes p_1$$

and

$$\hat{\Delta}(p_1 \otimes p_2 \otimes \cdots \otimes p_n) = \sum_{i+j=n} \sum_{\sigma \in \mathfrak{S}_i^j} \text{sgn}(\sigma) (p_{\sigma(1)} \otimes \cdots \otimes p_{\sigma(i)}) \otimes (p_{\sigma(i+1)} \otimes \cdots \otimes p_{\sigma(n)}),$$

where $\mathfrak{S}_i^j$ denotes the set of $(i, j)$-shuffles.\footnote{An $(i, j)$-shuffle is a permutation $\sigma$ of the set $\{1, 2, \cdots, i+j\}$ such that $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(i)$ and $\sigma(i+1) \leq \sigma(i+2) \leq \cdots \leq \sigma(i+j)$.}

**Theorem 3.3.** For any Lie pair $(L, A)$ of algebroids, $(D^\bullet_{\text{poly}}, d)$ with the multiplication $\cup$, the comultiplication $\Delta$, the unit $\varepsilon$, and the antipode $t$, is a Hopf algebra object in $D^b(A)$.

4. Ramadoss’s approach: $L(D^1_{\text{poly}})$

To prove Proposition 3.2 and Theorem 3.3, we essentially follow Ramadoss’s approach [6]. Let $L(D^1_{\text{poly}})$ be the (graded) free Lie algebra generated over $R$ by $D^1_{\text{poly}}$ concentrated in degree 1. In other words, $L(D^1_{\text{poly}})$ is the smallest Lie subalgebra of $D^1_{\text{poly}}$ containing $D^1_{\text{poly}}$. The Lie bracket of two vectors $u \in D^i_{\text{poly}}$ and $v \in D^j_{\text{poly}}$ is the vector $[u, v] = u \otimes v - (-1)^{ij} v \otimes u \in D^{i+j}_{\text{poly}}$. Actually, $L(D^1_{\text{poly}})$ is made of all linear combinations of elements of the form $[p_1, [p_2, \cdots, [p_{n-1}, p_n]]]$. One checks that $L(D^1_{\text{poly}})$ is a $d$-stable $A$-submodule of $D^1_{\text{poly}}$ and that its Lie bracket is a chain map with respect to thecoboundary operator $d$. Therefore $(L(D^1_{\text{poly}}), d)$ is a Lie algebra object in $\text{Ch}^b(A)$. 
Let $S^\bullet(L(D^1_{\text{poly}}))$ be the symmetric algebra of $L(D^1_{\text{poly}})$ and let

$$I : S^\bullet(L(D^1_{\text{poly}})) \to D^\bullet_{\text{poly}}$$

be the symmetrization map:

$$I(z_1 \odot \cdots \odot z_n) = \frac{1}{n!} \sum_{\sigma \in S_n} \text{sgn}(\sigma; z_1, \cdots, z_n) z_{\sigma(1)} \tilde{\otimes} z_{\sigma(2)} \tilde{\otimes} \cdots \tilde{\otimes} z_{\sigma(n)}.$$

The Koszul sign $\text{sgn}(\sigma; z_1, \cdots, z_n)$ of a permutation $\sigma$ of the (homogeneous) vectors $z_1, z_2, \ldots, z_n \in S^\bullet(L(D^1_{\text{poly}}))$ is determined by the relation

$$z_{\sigma(1)} \odot z_{\sigma(2)} \odot \cdots \odot z_{\sigma(n)} = \text{sgn}(\sigma; z_1, \cdots, z_n) z_1 \odot z_2 \odot \cdots \odot z_n.$$

**Lemma 4.1.** The symmetrization $I : S^\bullet(L(D^1_{\text{poly}})) \to D^\bullet_{\text{poly}}$ is an isomorphism in $\text{Ch}^b(\mathcal{A})$.

Using Lemma 4.1 and the HKR quasi-isomorphism, one can prove that the composition $\beta : \Gamma(L/A[-1]) \to L(D^1_{\text{poly}})$ of the inclusions

$$\Gamma(L/A[-1]) \subset D^1_{\text{poly}} \subset L(D^1_{\text{poly}})$$

is a quasi-isomorphism in $\text{Ch}^b(\mathcal{A})$, which intertwines the Lie brackets on $\Gamma(L/A[-1])$ and $L(D^1_{\text{poly}})$.

**Proposition 4.2.**

1. The inclusion $\beta : \Gamma(L/A[-1]) \to L(D^1_{\text{poly}})$ is a quasi-isomorphism in $\text{Ch}^b(\mathcal{A})$.
2. The inclusion $\beta : \Gamma(L/A[-1]) \to L(D^1_{\text{poly}})$ is an isomorphism of Lie algebra objects in $D^b(\mathcal{A})$ as the diagram

$$
\begin{array}{ccc}
\Gamma(L/A[-1]) \otimes \Gamma(L/A[-1]) & \xrightarrow{\beta \otimes \beta} & L(D^1_{\text{poly}}) \otimes L(D^1_{\text{poly}}) \\
\big/ \alpha_{L/A} & & \big/ [\cdot] \big/ \\
\Gamma(L/A[-1]) & \xrightarrow{\beta} & L(D^1_{\text{poly}})
\end{array}
$$

commutes in $D^b(\mathcal{A})$.

Proposition 3.2 and Theorem 3.3 now follow immediately.

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