A Theory of Matrices of Complex Elements

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Summary. A concept of “Matrix of Complex” is defined here. Addition, subtraction, scalar multiplication and product are introduced using correspondent definitions of “Matrix of Field”. Many equations for such operations consist of a case of “Matrix of Field”. A calculation method of product of matrices is shown using a finite sequence of Complex in the last theorem.

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The articles [11], [14], [1], [4], [2], [15], [6], [10], [9], [3], [8], [7], [13], [12], and [5] provide the terminology and notation for this paper.

The following two propositions are true:
(1) \( 1 = 1_{\mathbb{C}_F} \).
(2) \( 0_{\mathbb{C}_F} = 0 \).

Let \( A \) be a matrix over \( \mathbb{C} \). The functor \( A_{\mathbb{C}_F} \) yields a matrix over \( \mathbb{C}_F \) and is defined by:
(Def. 1) \( A_{\mathbb{C}_F} = A \).

Let \( A \) be a matrix over \( \mathbb{C}_F \). The functor \( A_{\mathbb{C}_F} \) yielding a matrix over \( \mathbb{C} \) is defined by:
(Def. 2) \( A_{\mathbb{C}_F} = A \).

We now state four propositions:
(3) For all matrices \( A, B \) over \( \mathbb{C} \) such that \( A_{\mathbb{C}_F} = B_{\mathbb{C}_F} \) holds \( A = B \).
(4) For all matrices \( A, B \) over \( \mathbb{C}_F \) such that \( A_{\mathbb{C}_F} = B_{\mathbb{C}_F} \) holds \( A = B \).
(5) For every matrix \( A \) over \( \mathbb{C} \) holds \( A = (A_{\mathbb{C}_F})_{\mathbb{C}_F} \).
(6) For every matrix \( A \) over \( \mathbb{C}_F \) holds \( A = (A_{\mathbb{C}_F})_{\mathbb{C}_F} \).

Let \( A, B \) be matrices over \( \mathbb{C} \). The functor \( A + B \) yielding a matrix over \( \mathbb{C} \) is defined as follows:
Let $A$ be a matrix over $\mathbb{C}$. The functor $-A$ yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 4) $-A = (-A)\cdot C$.

Let $A$, $B$ be matrices over $\mathbb{C}$. The functor $A - B$ yields a matrix over $\mathbb{C}$ and is defined as follows:

(Def. 5) $A - B = (A - B)\cdot C$.

Let $A$, $B$ be matrices over $\mathbb{C}$. The functor $A \cdot B$ yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 6) $A \cdot B = (A \cdot B)\cdot C$.

Let $x$ be a complex number and let $A$ be a matrix over $\mathbb{C}$. The functor $x \cdot A$ yielding a matrix over $\mathbb{C}$ is defined as follows:

(Def. 7) For every element $e_1$ of $\mathbb{C}_F$ such that $e_1 = x$ holds $x \cdot A = (e_1 \cdot A)\cdot C$.

One can prove the following propositions:

(7) For every matrix $A$ over $\mathbb{C}$ holds $\text{len}(A) = \text{len}(A\cdot C)$ and $\text{width}(A) = \text{width}(A\cdot C)$.

(8) For every matrix $A$ over $\mathbb{C}_F$ holds $\text{len}(A) = \text{len}(A\cdot C)$ and $\text{width}(A) = \text{width}(A\cdot C)$.

(9) For every matrix $M$ over $\mathbb{C}$ such that $\text{len}(M) > 0$ holds $-\cdot M = M$.

(10) For every field $K$ and for every matrix $M$ over $K$ holds $1_K \cdot M = M$.

(11) For every matrix $M$ over $\mathbb{C}$ holds $1 \cdot M = M$.

(12) For every field $K$ and for all elements $a, b$ of $K$ and for every matrix $M$ over $K$ holds $a \cdot (b \cdot M) = (a \cdot b) \cdot M$.

(13) For every field $K$ and for all elements $a, b$ of $K$ and for every matrix $M$ over $K$ holds $(a + b) \cdot M = a \cdot M + b \cdot M$.

(14) For every matrix $M$ over $\mathbb{C}$ holds $M + M = 2 \cdot M$.

(15) For every matrix $M$ over $\mathbb{C}$ holds $M + M + M = 3 \cdot M$.

Let $n, m$ be natural numbers. The functor \[
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}_{n \times m}
\] yields a matrix over $\mathbb{C}$ and is defined by:

(Def. 8) \[
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}_{n \times m} = \left( \begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}_{n \times m} \cdot C \right)_{C_F}.
\]

One can prove the following propositions:
(16) For all natural numbers $n$, $m$ holds 
\[
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}
\in \mathbb{C}_{\mathbb{F}}^{n \times m} =
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}
\in \mathbb{C}_{\mathbb{F}}^{n \times m},
\]

(17) For every matrix $M$ over $\mathbb{C}$ such that $\text{len } M > 0$ holds $M + -M =
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}
\in \mathbb{C}^{(\text{len } M) \times (\text{width } M)}.
\]

(18) For every matrix $M$ over $\mathbb{C}$ such that $\text{len } M > 0$ holds $M - M =
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}
\in \mathbb{C}^{(\text{len } M) \times (\text{width } M)}.
\]

(19) For all matrices $M_1$, $M_2$, $M_3$ over $\mathbb{C}$ such that $\text{len } M_1 = \text{len } M_2$ and $\text{len } M_2 = \text{len } M_3$ and $\text{width } M_1 = \text{width } M_2$ and $\text{width } M_2 = \text{width } M_3$ and $\text{len } M_1 > 0$ and $M_1 + M_3 = M_2 + M_3$ holds $M_1 = M_2$.

(20) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\text{len } M_2 > 0$ holds $M_1 - (-M_2 = M_1 + M_2$.

(21) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\text{len } M_1 = \text{len } M_2$ and $\text{width } M_1 = \text{width } M_2$ and $\text{len } M_1 > 0$ and $M_1 = M_1 + M_2$ holds
\[
M_2 = \begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}
\in \mathbb{C}^{(\text{len } M_1) \times (\text{width } M_1)}.
\]

(22) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\text{len } M_1 = \text{len } M_2$ and $\text{width } M_1 = \text{width } M_2$ and $\text{len } M_1 > 0$ and $M_1 - M_2 =
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}
\in \mathbb{C}^{(\text{len } M_1) \times (\text{width } M_1)}
\]
holds $M_1 = M_2$.

(23) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\text{len } M_1 = \text{len } M_2$ and $\text{width } M_1 = \text{width } M_2$ and $\text{len } M_1 > 0$ and $M_1 + M_2 =
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}
\in \mathbb{C}^{(\text{len } M_1) \times (\text{width } M_1)}
\]
holds $M_2 = -M_1$.

(24) For all natural numbers $n$, $m$ such that $n > 0$ holds
\[
-\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}^{n \times m}_{\mathbb{C}} = \begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}^{n \times m}_{\mathbb{C}}.
\]

(25) For all matrices \( M_1, M_2 \) over \( \mathbb{C} \) such that \( \text{len} M_1 = \text{len} M_2 \) and width \( M_1 = \text{width} M_2 \) and \( \text{len} M_1 > 0 \) and \( M_2 - M_1 = M_2 \) holds

\[
M_1 = \begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}^{(\text{len} M_1) \times (\text{width} M_1)}_{\mathbb{C}}.
\]

(26) For all matrices \( M_1, M_2 \) over \( \mathbb{C} \) such that \( \text{len} M_1 = \text{len} M_2 \) and width \( M_1 = \text{width} M_2 \) and \( \text{len} M_1 > 0 \) holds \( M_1 = M_1 - (M_2 - M_2) \).

(27) For all matrices \( M_1, M_2 \) over \( \mathbb{C} \) such that \( \text{len} M_1 = \text{len} M_2 \) and width \( M_1 = \text{width} M_2 \) and \( \text{len} M_1 > 0 \) holds \(- (M_1 + M_2) = -M_1 + -M_2\).

(28) For all matrices \( M_1, M_2 \) over \( \mathbb{C} \) such that \( \text{len} M_1 = \text{len} M_2 \) and width \( M_1 = \text{width} M_2 \) and \( \text{len} M_1 > 0 \) holds \( M_1 - (M_1 - M_2) = M_2 \).

(29) For all matrices \( M_1, M_2, M_3 \) over \( \mathbb{C} \) such that \( \text{len} M_1 = \text{len} M_2 \) and \( \text{len} M_2 = \text{len} M_3 \) and width \( M_1 = \text{width} M_2 \) and width \( M_2 = \text{width} M_3 \) and \( \text{len} M_1 > 0 \) and \( M_1 - M_3 = M_2 - M_2 \) holds \( M_1 = M_2 \).

(30) For all matrices \( M_1, M_2, M_3 \) over \( \mathbb{C} \) such that \( \text{len} M_1 = \text{len} M_2 \) and \( \text{len} M_2 = \text{len} M_3 \) and width \( M_1 = \text{width} M_2 \) and width \( M_2 = \text{width} M_3 \) and \( \text{len} M_1 > 0 \) and \( M_3 - M_1 = M_3 - M_2 \) holds \( M_1 = M_2 \).

(31) For all matrices \( M_1, M_2, M_3 \) over \( \mathbb{C} \) such that \( \text{len} M_2 = \text{len} M_3 \) and width \( M_2 = \text{width} M_3 \) and \( \text{len} M_1 = \text{len} M_2 \) and \( \text{len} M_1 > 0 \) and \( \text{len} M_2 > 0 \) holds \( M_1 \cdot (M_2 + M_3) = M_1 \cdot M_2 + M_1 \cdot M_3 \).

(32) For all matrices \( M_1, M_2, M_3 \) over \( \mathbb{C} \) such that \( \text{len} M_2 = \text{len} M_3 \) and width \( M_2 = \text{width} M_3 \) and \( \text{len} M_1 = \text{width} M_2 \) and \( \text{len} M_2 > 0 \) and \( \text{len} M_1 > 0 \) holds \( (M_2 + M_3) \cdot M_1 = M_2 \cdot M_1 + M_3 \cdot M_1 \).

(33) For all matrices \( M_1, M_2 \) over \( \mathbb{C} \) such that \( \text{len} M_1 = \text{len} M_2 \) and width \( M_1 = \text{width} M_2 \) holds \( M_1 + M_2 = M_2 + M_1 \).

(34) For all matrices \( M_1, M_2, M_3 \) over \( \mathbb{C} \) such that \( \text{len} M_1 = \text{len} M_2 \) and \( \text{len} M_1 = \text{len} M_3 \) and width \( M_1 = \text{width} M_2 \) and width \( M_1 = \text{width} M_3 \) holds \( (M_1 + M_2) + M_3 = M_1 + (M_2 + M_3) \).

(35) For every matrix \( M \) over \( \mathbb{C} \) such that \( \text{len} M > 0 \) holds

\[
\begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \vdots \\
0 & \ldots & 0
\end{pmatrix}^{(\text{len} M) \times (\text{width} M)}_{\mathbb{C}} = M.
\]

(36) Let \( K \) be a field, \( b \) be an element of \( K \), and \( M_1, M_2 \) be matrices over \( K \). If \( \text{len} M_1 = \text{len} M_2 \) and width \( M_1 = \text{width} M_2 \) and \( \text{len} M_1 > 0 \), then \( b \cdot (M_1 + M_2) = b \cdot M_1 + b \cdot M_2 \).
(37) Let $M_1$, $M_2$ be matrices over $\mathbb{C}$ and $a$ be a complex number. If $\text{len} M_1 = \text{len} M_2$ and $\text{width} M_1 = \text{width} M_2$ and $\text{len} M_1 > 0$, then $a \cdot (M_1 + M_2) = a \cdot M_1 + a \cdot M_2$.

(38) For every field $K$ and for all matrices $M_1$, $M_2$ over $K$ such that $\text{width} M_1 = \text{len} M_2$ and $\text{len} M_1 > 0$ and $\text{len} M_2 > 0$ holds

\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}_K \cdot M_2 = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}_K.
\]

(39) For all matrices $M_1$, $M_2$ over $\mathbb{C}$ such that $\text{width} M_1 = \text{len} M_2$ and $\text{len} M_1 > 0$ and $\text{len} M_2 > 0$ holds

\[
\begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}_\mathbb{C} \cdot M_2 = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}_\mathbb{C}.
\]

(40) For every field $K$ and for every matrix $M_1$ over $K$ such that $\text{len} M_1 > 0$ holds $0_K \cdot M_1 = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}_K$.

(41) For every matrix $M_1$ over $\mathbb{C}$ such that $\text{len} M_1 > 0$ holds $0 \cdot M_1 = \begin{pmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{pmatrix}_\mathbb{C}$.

Let $s$ be a finite sequence of elements of $\mathbb{C}$ and let $k$ be a natural number. Then $s(k)$ is an element of $\mathbb{C}$.

We now state the proposition

(42) Let $i$, $j$ be natural numbers and $M_1$, $M_2$ be matrices over $\mathbb{C}$. Suppose $\text{len} M_1 > 0$ and $\text{len} M_2 > 0$ and $\text{width} M_1 = \text{len} M_2$ and $1 \leq i$ and $i \leq \text{len} M_1$ and $1 \leq j$ and $j \leq \text{width} M_2$. Then there exists a finite sequence $s$ of elements of $\mathbb{C}$ such that $\text{len} s = \text{len} M_2$ and $s(1) = (M_1 \circ (i, 1)) \cdot (M_2 \circ (1, j))$ and for every natural number $k$ such that $1 \leq k$ and $k < \text{len} M_2$ holds $s(k+1) = s(k) + (M_1 \circ (i, k+1)) \cdot (M_2 \circ (k+1, j))$.

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