COMPLEXITY TEST MODULES

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Abstract. A method is provided for computing an upper bound of the complexity of a module over a local ring, in terms of vanishing of certain cohomology modules. We then specialize to complete intersections, which are precisely the rings over which all modules have finite complexity.

1. Introduction

The notion of complexity of a module was introduced by Alperin and Evens in [AlE], in order to study modular representations of finite groups. A decade later, in [Av1, Av2], Avramov introduced this concept for finitely generated modules over local rings, as a means to distinguish between modules of infinite projective dimension. The complexity of a module measures the growth of its minimal free resolution. For example, a module has complexity zero if and only if its projective dimension is finite, and complexity one precisely when its minimal free resolution is bounded.

For an arbitrary local ring, not much is known about the modules of finite complexity. For example, a characterization of these modules in terms of cohomology does not exist. Even worse, it is unclear whether every local ring has finite finitistic complexity dimension, that is, whether there is a bound on the complexities of the modules having finite complexity. In other words, the status quo for complexity is entirely different from that of projective dimension: there simply does not exist any analogue of the Auslander-Buchsbaum formula. The only class of rings for which these problems are solved are the complete intersections; over such a ring every module has finite complexity, and the complexity is bounded by the codimension of the ring.

In this paper we give a method for computing an upper bound of the complexity of classes of modules over local rings. This is done by looking at the vanishing of cohomology with certain “test” modules, using the notion of reducible complexity introduced in [Be1]. We then specialize to the complete intersection, and use the theory of support varieties, introduced in [Av1] and [AvB], both to sharpen our results and to obtain new ones.

The paper is organized as follows. In the next section we introduce the class of modules having “free reducible complexity”, a slight generalization of the notion of reducible complexity. We then study the homological behavior of such modules, proving among other things some results on the vanishing of (co)homology. In the final section we prove our main results on complexity testing, specializing at the end to complete intersections.

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Throughout we let \((A, m, k)\) be a local (meaning also commutative Noetherian) ring, and we suppose all modules are finitely generated. For an \(A\)-module \(M\) with minimal free resolution

\[
\cdots \to F_2 \to F_1 \to F_0 \to M \to 0,
\]

the rank of \(F_n\), i.e., the integer \(\dim_k \Ext^n_A(M, k)\), is the \(n\)th Betti number of \(M\), and we denote this by \(\beta_n(M)\). The \(i\)th syzygy of \(M\), denoted \(\Omega^i_A(M)\), is the cokernel of the map \(F_{i+1} \to F_i\), and it is unique up to isomorphism. Note that \(\Omega^0_A(M) = M\) and that \(\beta_n(\Omega^i_A(M)) = \beta_{n+i}(M)\) for all \(i\). The complexity of \(M\), denoted \(\text{cx}\ M\), is defined as

\[
\text{cx}\ M = \inf\{t \in \mathbb{N} \cup \{0\} | \exists a \in \mathbb{R} \text{ such that } \beta_n(M) \leq an^{t-1} \text{ for } n \gg 0\}.
\]

In general the complexity of a module may be infinite, in fact the rings for which all modules have finite complexity are precisely the complete intersections. From the definition we see that the complexity is zero if and only if the module has finite projective dimension, and that the modules of complexity one are those whose minimal free resolutions are bounded, for example the periodic modules. Moreover, the complexity of \(M\) equals that of \(\Omega^n_A(M)\) for every \(i \geq 0\).

Let \(N\) be an \(A\)-module, and consider an element \(\eta \in \Ext^n_A(M, N)\). By choosing a map \(f_\eta : \Omega^i_A(M) \to N\) representing \(\eta\), we obtain a commutative pushout diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \Omega^i_A(M) \\
\downarrow f_\eta & & \downarrow \\
0 & \longrightarrow & N
\end{array}
\]

\[
\begin{array}{ccc}
& & \Omega^{i-1}_A(M) \\
& & \downarrow \\
& & 0
\end{array}
\]
It should be commented on the fact that in the original definition of the notion of reducible complexity in [Be1], the reducing process did not necessarily reduce the complexity by exactly one. Namely, if $\eta \in \text{Ext}^*_A(M, M)$ is a cohomological element reducing the complexity of the module $M$, then the requirement was only that $\text{cx} K_\eta$ be strictly less than $\text{cx} M$, and not necessarily equal to $\text{cx} M - 1$. However, we are unaware of any example where the complexity actually drops by more than one. On the contrary, there is evidence to suggest that this cannot happen. If $\text{CI-dim}_A M$ is finite and

$$0 \rightarrow M \rightarrow K \rightarrow \Omega^n_A(M) \rightarrow 0$$

is an exact sequence, then it is not hard to see that there exists one “joint” quasi deformation $A \rightarrow R \leftarrow Q$ such that all these three modules have finite projective dimension over $Q$ when forwarded to $R$. In this situation we can apply [Jor] Theorem 1.3], a result originally stated for complete intersections, and obtain a “principal lifting” (of $R$) over which the complexities drop by exactly one. By altering this procedure, we can show that the complexity of $K$ cannot drop by more than one. Note also that this is trivial if $\text{cx} M \leq 2$.

We now define the class of modules we are going to study in this paper, a class which is a slight generalization of modules of reducible complexity. Namely, we do not require equality of depth for the modules involved.

**Definition 2.1.** Denote by $C_A$ the category of all $A$-modules having finite complexity. The full subcategory $C_A^{frc} \subseteq C_A$ consisting of the modules having free reducible complexity is defined inductively as follows.

1. Every module of finite projective dimension belongs to $C_A^{frc}$.
2. A module $X \in C_A$ of positive complexity belongs to $C_A^{frc}$ if there exists a homogeneous element $\eta \in \text{Ext}^*_A(X, X)$ of positive degree such that $\text{cx} K_\eta = \text{cx} X - 1$ and $K_\eta \in C_A^{frc}$.

Thus if the $A$-module $M$ has finite positive complexity $c$, say, then it has free reducible complexity if and only if the following hold: there exists a sequence $M = K_0, K_1, \ldots, K_c$ of $A$-modules such that for each $1 \leq i \leq c$ there is an integer $n_i \geq 0$ and an exact sequence

$$0 \rightarrow K_{i-1} \rightarrow K_i \rightarrow \Omega^n_A(K_{i-1}) \rightarrow 0$$

with $\text{cx} K_i = c - i$. Note that the only difference between this definition and that of “ordinary” reducible complexity is that we do not require that depth $K_i$ equals depth $K_{i-1}$. However, when the ring $A$ is Cohen-Macaulay, this is always the case (see the remark following [Be1] Definition 2.1). Therefore, for such a ring, the categories $C_A$ and $C_A^{frc}$ coincide, that is, a module has reducible complexity if and only if it has free reducible complexity.

It is quite clear that the inclusion $C_A^{rc} \subseteq C_A^{frc}$ holds, that is, every module of reducible complexity also has free reducible complexity. In particular, every $A$-module of finite complete intersection dimension belongs to $C_A^{frc}$. However, the converse is a priori not true, that is, a module in $C_A^{frc}$ need not have reducible complexity. Moreover, a module having finite complexity need not have free reducible complexity. We illustrate all this with an example from [GaP].

**Example.** Let $(A, m, k)$ be the local finite dimensional algebra $k[X_1, \ldots, X_5]/\mathfrak{a}$, where $\mathfrak{a} \subset k[X_1, \ldots, X_5]$ is the ideal generated by the quadratic forms

$$X_1^2, X_2^2, X_5^2, X_3X_4, X_3X_5, X_4X_5, X_1X_4 + X_2X_4$$

$$\alpha X_1X_3 + X_2X_3, X_3^2 - X_2X_5 + \alpha X_1X_5, X_1^2 - X_2X_5 + X_1X_5$$
for a nonzero element \( \alpha \in k \). By [GaP, Proposition 3.1] this ring is Gorenstein, and the complex

\[ \cdots \to A^2 \xrightarrow{d_{n+1}} A^2 \xrightarrow{d_n} A^2 \to \cdots \]

with maps given by the matrices \( d_n = \begin{pmatrix} x_1 & \alpha x_3 + x_4 \\ 0 & x_2 \end{pmatrix} \) is exact. This sequence is therefore a minimal free resolution of the module \( M := \text{Im} d_0 \), hence this module has complexity one. If the order of \( \alpha \) in \( k \) is infinite, then \( M \) cannot have reducible complexity (recall that the notion of reducible complexity coincides with that of free reducible complexity since the ring is Cohen-Macaulay); if \( M \) has reducible complexity, then there exists an exact sequence

\[ 0 \to M \to K \to \Omega^n_\mathcal{A}(M) \to 0 \]

in which \( K \) has finite projective dimension. As \( A \) is selfinjective, the module \( K \) must be free, hence \( M \) is a periodic module, a contradiction. Moreover, if the order of \( \alpha \) is finite but at least 3, then the argument in [Be1, Section 2, example] shows that \( M \) has reducible complexity but not finite complete intersection dimension.

This example also shows that, in general, a module of finite Gorenstein dimension and finite complexity need not have reducible complexity. Recall that a module \( X \) over a local ring \( R \) has finite Gorenstein dimension, denoted \( \dim_R X < \infty \), if there exists an exact sequence

\[ 0 \to G_\ell \to \cdots \to G_0 \to X \to 0 \]

of \( R \)-modules in which the modules \( G_i \) are reflexive and satisfy \( \Ext^j_R(G_i, R) = \Ext^j_R(\Hom_R(G_i, R), R) = 0 \) for \( j \geq 1 \). Every module over a Gorenstein ring has finite Gorenstein dimension, in fact this property characterizes Gorenstein rings. Using this concept, Gerko introduced in [Ger] the notion of lower complete intersection dimension; the module \( X \) has finite lower complete intersection dimension, written \( \operatorname{CI}^*_\dim R X < \infty \), if it has finite Gorenstein dimension and finite complexity (and in this case \( \operatorname{CI}^*_\dim R X = \dim_R X \)). The Gorenstein dimension, lower complete intersection dimension, complete intersection dimension and projective dimension of a module are all related via the inequalities

\[ \dim_R X \leq \operatorname{CI}^*_\dim R X \leq \operatorname{CI} \dim R X \leq \operatorname{pd} R X. \]

If one of these dimensions happen to be finite, then it is equal to those to its left. Note that the class of modules having (free) reducible complexity and finite Gorenstein dimension lies properly “between” the class of modules having finite lower complete intersection dimension and the class of modules having finite complete intersection dimension.

We now continue investigating the properties of the category of modules having free reducible complexity. The following result shows that \( \mathcal{C}^\text{fc}_A \) is closed under taking syzygies and preserved under faithfully flat extensions. We omit the proof because it is analogous to that of [Be1] Proposition 2.2.

**Proposition 2.2.** Let \( M \) be a module in \( \mathcal{C}^\text{fc}_A \).

(i) The kernel of any surjective map \( F \to M \) in which \( F \) is free also belongs to \( \mathcal{C}^\text{fc}_A \). In particular, any syzygy of \( M \) belongs to \( \mathcal{C}^\text{fc}_A \).

(ii) If \( A \to B \) is a faithfully flat local homomorphism, then the \( B \)-module \( B \otimes_A M \) belongs to \( \mathcal{C}^\text{fc}_B \).

Note that, in the first part of this result, as opposed to the corresponding result [Be1 Proposition 2.2(ii)] for modules belonging to \( \mathcal{C}^\text{rc}_A \), we do not need to assume that the ring in question is Cohen-Macaulay. The reason for this is of course that in the definition of free reducible complexity, we do not require equality of depth for the modules involved. By dropping this requirement, the main results from [Be1]...
on the vanishing of Ext and Tor do not carry over to the category $\mathcal{C}_A^{fr}$ of modules having free reducible complexity. However, as the following results show, modified versions of the mentioned results hold for modules belonging to $\mathcal{C}_A^{fr}$. We prove only the cohomology case; the homology case is totally similar.

**Proposition 2.3.** Suppose $M$ belongs to $\mathcal{C}_A^{fr}$ and has positive complexity. Choose modules $M = K_0, K_1, \ldots, K_c$, integers $n_1, \ldots, n_c$ and exact sequences

$$0 \to K_{i-1} \to K_i \to \Omega^n_{A}(K_{i-1}) \to 0$$

as in the equivalent definition following Definition 2.7. Then for any $A$-module $N$, the following are equivalent:

(i) There exists an integer $t > \max\{\text{depth } A - \text{depth } K_i\}$ such that $\text{Ext}^i_A(M, N) = 0$ for $0 \leq i \leq n_1 + \cdots + n_c$.

(ii) $\text{Ext}^i_A(M, N) = 0$ for $i \gg 0$.

(iii) $\text{Ext}^i_A(M, N) = 0$ for $i > \max\{\text{depth } A - \text{depth } K_i\}$.

**Proof.** We have to prove the implication (i) $\Rightarrow$ (iii), and we do this by induction on the complexity of $M$. If $\text{ex } M = 1$, then in the exact sequence

$$0 \to M \to K_1 \to \Omega^n_{A}(M) \to 0$$

the module $K_1$ has finite projective dimension, which by the Auslander-Buchsbaum formula equals depth $A - \text{depth } K_1$. Then for any $i > \max\{\text{depth } A - \text{depth } K_1\}$ the cohomology group $\text{Ext}^i_A(K_1, N)$ vanishes, implying an isomorphism

$$\text{Ext}^i_A(M, N) \simeq \text{Ext}^{i+n_1+1}_A(M, N).$$

By assumption, the cohomology group $\text{Ext}^i_A(M, N)$ vanishes for $t \leq i \leq t + n_1$, and so the isomorphisms just given ensure that $\text{Ext}^i_A(M, N) = 0$ for all $i > \max\{\text{depth } A - \text{depth } K_1\}$.

Now suppose the complexity of $M$ is at least two. From the assumption on the vanishing of $\text{Ext}^i_A(M, N)$ and the exact sequence from the beginning of the proof, we see that $\text{Ext}^i_A(K_1, N) = 0$ for $t \leq i \leq t + n_2 + \cdots + n_c$. The complexity of $K_1$ is one less than that of $M$, hence by induction the cohomology group $\text{Ext}^i_A(K_1, N)$ vanishes for all $i > \max\{\text{depth } A - \text{depth } K_1\}$. The same argument we used in the case when the complexity of $M$ was one now shows that $\text{Ext}^i_A(M, N) = 0$ for $i > \max\{\text{depth } A - \text{depth } K_1\}$.

**Proposition 2.4.** Suppose $M$ belongs to $\mathcal{C}_A^{fr}$ and has positive complexity. Choose modules $M = K_0, K_1, \ldots, K_c$, integers $n_1, \ldots, n_c$ and exact sequences

$$0 \to K_{i-1} \to K_i \to \Omega^n_{A}(K_{i-1}) \to 0$$

as in the equivalent definition following Definition 2.7. Then for any $A$-module $N$, the following are equivalent:

(i) There exists an integer $t > \max\{\text{depth } A - \text{depth } K_i\}$ such that $\text{Tor}^i_A(M, N) = 0$ for $0 \leq i \leq n_1 + \cdots + n_c$.

(ii) $\text{Tor}^i_A(M, N) = 0$ for $i \gg 0$.

(iii) $\text{Tor}^i_A(M, N) = 0$ for $i > \max\{\text{depth } A - \text{depth } K_i\}$.

We end this section with a result from [Be1], a result which will be of use in the next section. It shows that when $A$ is Gorenstein, then symmetry holds for the vanishing of cohomology between modules of reducible complexity. This is a generalization of Jørgensen’s result [Jor 1976, Theorem 4.1], which says that symmetry in the vanishing of cohomology holds for modules of finite complete intersection dimension over a local Gorenstein ring.
Proposition 2.5. [Be1 Theorem 3.5] If $A$ is Gorenstein and $M$ and $N$ are modules with $M \in C_A^{rc}$, then the implication

$$\Ext_A^i(N, M) = 0 \text{ for } i \gg 0 \Rightarrow \Ext_A^i(M, N) = 0 \text{ for } i \gg 0$$

holds. In particular, symmetry in the vanishing of cohomology holds for modules having reducible complexity.

3. Complexity testing

In this section we introduce a method for computing an upper bound for the complexity of a given module in $C_A^{frc}$. We start with a key result which shows that the modules in $C_A^{frc}$ having infinite projective dimension also have higher self extensions. The result is just a generalization of [Be1, Corollary 3.2] from $C_A^{rc}$ to $C_A^{frc}$, but we include the proof for the convenience of the reader.

Proposition 3.1. If $M$ belongs to $C_A^{frc}$, then

$$\text{pd } M = \sup \{ i | \Ext_A^i(M, M) \neq 0 \}.$$ 

Proof. If the projective dimension of $M$ is finite, then the first part of the proof of [Be1] Theorem 3.1] shows that $\Ext_A^{\text{pd } M}(M, M) \neq 0$. Suppose therefore that the projective dimension is infinite. By definition, there exists a positive degree homogeneous element $\eta \in \Ext_A^*(M, M)$ such that $\text{cx } K_\eta = \text{cx } M - 1$. Suppose now that $\Ext_A^i(M, M)$ vanishes for $i \gg 0$. Then $\eta$ is nilpotent, that is, there is a number $t$ such that $\eta^t = 0$. The exact sequence

$$0 \rightarrow M \rightarrow K_{\eta^t} \rightarrow \hat{\Omega}_A^{i|\eta|} (M) \rightarrow 0$$

corresponding to $\eta^t$ then splits, and therefore $\text{cx } K_{\eta^t} = \text{cx } M$ since the end terms are of the same complexity. However, it follows from [Be1] Lemma 2.3] that for any numbers $m$ and $n$ the $A$-modules $K_\eta^m$ and $K_\eta^n$ are related through an exact sequence

$$0 \rightarrow \hat{\Omega}_A^{i|\eta|} (K_\eta^m) \rightarrow K_{\eta^m+n} \oplus F \rightarrow K_{\eta^n} \rightarrow 0,$$

in which $F$ is some free module. Using this and the fact that in a short exact sequence the complexity of the middle term is at most the maximum of the complexities of the end terms, an induction argument gives the inequality $\text{cx } K_{\eta^i} \leq \text{cx } K_\eta$ for every $i \geq 1$. Combining all our obtained (in)equalities on complexity, we get

$$\text{cx } M = \text{cx } K_{\eta^t} \leq \text{cx } K_\eta = \text{cx } M - 1,$$

a contradiction. Therefore $\Ext_A^i(M, M)$ cannot vanish for all $i \gg 0$ when the projective dimension of $M$ is infinite. □

We are now ready to prove the main result. For a given natural number $t$, denote by $C_A^{frc}(t)$ the full subcategory

$$C_A^{frc}(t) \overset{\text{def}}{=} \{ X \in C_A^{frc} | \text{cx } X = t \}$$

of $C_A^{frc}$ consisting of the modules of complexity $t$. The main result shows that this subcategory serves as a “complexity test category”, in the sense that if a module has no higher extensions with the modules in $C_A^{frc}(t)$, then its complexity is strictly less than $t$.

Theorem 3.2. Let $M$ be a module belonging to $C_A^{frc}$ and $t$ a natural number. If $\Ext_A^i(M, N) = 0$ for every $N \in C_A^{frc}(t)$ and $i \gg 0$, then $\text{cx } M < t$. 

Proof. We show by induction that if $\text{cx} M \geq t$, then there is a module $N \in C^{fr}_A(t)$ with the property that $\text{Ext}_A^i(M, N)$ does not vanish for all $i > 0$. If the complexity of $M$ is $t$, then by Proposition 3.1 we may take $N$ to be $M$ itself, so suppose that $\text{cx} M > t$. Choose a cohomological homogeneous element $\eta \in \text{Ext}_A^t(M, M)$ of positive degree reducing the complexity. In the corresponding exact sequence

$$0 \to M \to K_\eta \to \Omega^{[\eta]}_{A}(M) \to 0,$$

the module $K_\eta$ also belongs to $C^{rc}_A$ and has complexity one less than that of $M$, hence by induction there is a module $N \in C^{fr}_A(t)$ such that $\text{Ext}_A^i(K_\eta, N)$ does not vanish for all $i > 0$. From the long exact sequence

$$\cdots \to \text{Ext}_A^{i+|\eta|-1}(M, N) \to \text{Ext}_A^i(K_\eta, N) \to \text{Ext}_A^i(M, N) \to \text{Ext}_A^{i+|\eta|}(M, N) \to \cdots$$

resulting from $\eta$, we see that $\text{Ext}_A^i(M, N)$ cannot vanish for all $i > 0$. \hfill \Box

In particular, we can use the category $C^{fr}_A(1)$ to decide whether a given module in $C^{fr}_A$ has finite projective dimension. We record this fact in the following corollary.

**Corollary 3.3.** A module $M \in C^{fr}_A$ has finite projective dimension if and only if $\text{Ext}_A^i(M, N) = 0$ for every $N \in C^{fr}_A(1)$ and $i > 0$.

**Remark.** Let $C^{ci}_A$ denote the category of all $A$-modules of finite complete intersection dimension, and for each natural number $t$ define the two categories

$$C^{ci}_A(t) \overset{\text{def}}{=} \{ X \in C^{ci}_A | \text{cx} X = t \}$$

$$C^{rc}_A(t) \overset{\text{def}}{=} \{ X \in C^{rc}_A | \text{cx} X = t \}.$$

Then Theorem 3.2 and Corollary 3.3 remain true if we replace $C^{fr}_A$ and $C^{fr}_A(t)$ by $C^{ci}_A$ (respectively, $C^{ci}_A(t)$) and $C^{rc}_A(t)$ (respectively, $C^{rc}_A(t)$). That is, when the module we are considering has reducible complexity (respectively, finite complete intersection dimension), then we need only use modules of reducible complexity (respectively, finite complete intersection dimension) as test modules.

When the ring is Gorenstein, then it follows from Proposition 2.5 that symmetry holds for the vanishing of cohomology between modules of reducible complexity. We therefore have the following symmetric version of Theorem 3.2.

**Corollary 3.4.** Suppose $A$ is Gorenstein, let $M$ be a module belonging to $C^{rc}_A$, and let $t$ be a natural number. If $\text{Ext}_A^i(N, M) = 0$ for every $N \in C^{rc}_A(t)$ and $i > 0$, then $\text{cx} M < t$.

We now turn to the setting in which every $A$-module has reducible complexity. For the remainder of this section, we assume $A$ is a complete intersection, i.e. the $m$-adic completion $\hat{A}$ of $A$ is the residue ring of a regular local ring modulo a regular sequence. For such rings, Avramov and Buchweitz introduced in [Av1] and [AvB] a theory of cohomological support varieties, and they showed that this theory is similar to that of the cohomological support varieties for group algebras. As we will implicitly use this powerful theory in the results to come, we recall now the definitions (details can be found in [Av1] Section 1 and [AvB] Section 2]).

Denote by $c$ the codimension of $A$, that is, the integer $\dim_k (m/m^2) - \dim A$, and by $\chi$ the sequence $\chi_1, \ldots, \chi_c$ consisting of the $c$ commuting Eisenbud operators of cohomological degree two. For every $A$-module $X$ there is a homomorphism

$$\hat{A}[\chi] \overset{\text{def}}{\to} \text{Ext}_A^c(X, X)$$

of graded rings, and via this homomorphism $\text{Ext}_A^c(X, Y)$ is finitely generated over $\hat{A}[\chi]$ for every $\hat{A}$-module $Y$. Denote by $H$ the polynomial ring $k[\chi]$, and by $E(X, Y)$
the graded space \( \text{Ext}_{\hat{A}}^t(X, Y) \otimes_{\hat{A}} k \). The above homomorphism \( \phi_X \), together with the canonical isomorphism \( H \simeq \hat{A}[\chi] \otimes_{\hat{A}} k \), induce a homomorphism \( H \to E(X, X) \) of graded rings, under which \( E(X, Y) \) is a finitely generated \( H \)-module. Now let \( M \) be an \( A \)-module, and denote by \( \hat{M} \) its \( m \)-adic completion \( \hat{A} \otimes_A M \). The support variety \( V(M) \) of \( M \) is the algebraic set

\[
V(M) \overset{\text{def}}{=} \{ \alpha \in \hat{k}^e | f(\alpha) = 0 \text{ for all } f \in \text{Ann}_H E(\hat{M}, \hat{M}) \},
\]

where \( \hat{k} \) is the algebraic closure of \( k \). Finally, for an ideal \( \alpha \subseteq H \) we define the variety \( V_H(\alpha) \subseteq \hat{k}^e \) to be the zero set of \( \alpha \).

As mentioned above, this theory shares many properties with the theory of cohomological support varieties for modules over group algebras of finite groups. For instance, the dimension of the variety of a module equals the complexity of the module, in particular the variety is trivial if and only if the module has finite projective dimension. The following complexity test result relies on \([\text{Be}2, \text{Corollary 2.3}]\), which says that every homogeneous algebraic subset of \( \hat{k}^e \) is realizable as the support variety of some \( A \)-module.

**Proposition 3.5.** Let \( M \) be an \( A \)-module, let \( \eta_1, \ldots, \eta_r \in H \) be homogeneous elements of positive degrees, and choose an \( A \)-module \( T_{\eta_1, \ldots, \eta_r} \) with the property that \( V(T_{\eta_1, \ldots, \eta_r}) = V_H(\eta_1, \ldots, \eta_r) \). If \( \text{Ext}_A^t(M, T_{\eta_1, \ldots, \eta_r}) = 0 \) for \( i \gg 0 \), then \( \text{cx} M \leq t \).

**Proof.** Denote the ideal \( \text{Ann}_H E(\hat{M}, \hat{M}) \subseteq H \) by \( \alpha \). If \( \text{Ext}_A^t(M, T_{\eta_1, \ldots, \eta_r}) = 0 \) for \( i \gg 0 \), then from \([\text{AvB}, \text{Theorem 5.6}]\) we obtain

\[
\{0\} = V(M) \cap V(T_{\eta_1, \ldots, \eta_r}) = V_H(\alpha) \cap V_H(\eta_1, \ldots, \eta_r) = V_H(\alpha + (\eta_1, \ldots, \eta_r)),
\]

hence the ring \( H/(\alpha + (\eta_1, \ldots, \eta_r)) \) is zero dimensional. But then the dimension of the ring \( H/\alpha \) is at most \( t \); i.e. \( \text{cx} M \leq t \). \( \square \)

We illustrate this last result with an example.

**Example.** Let \( k \) be a field and \( Q \) the formal power series ring \( k[[x_1, \ldots, x_c]] \) in \( c \) variables. For each \( 1 \leq i \leq c \), let \( n_i \geq 2 \) be an integer, let \( \alpha \subseteq Q \) be the ideal generated by the regular sequence \( x_1^{n_1}, \ldots, x_c^{n_c} \), and denote by \( A \) the complete intersection \( Q/\alpha \). For each \( 1 \leq i \leq c \) we shall construct an \( A \)-module whose support variety equals \( V_H(\chi_i) \), by adopting the techniques used in \([\text{SnS}, \text{Section 7}]\) to give an interpretation of the Eisenbud operators.

Consider the exact sequence

\[
0 \to \mathfrak{m}_Q \to Q \to k \to 0
\]

of \( Q \)-modules. Applying \( A \otimes_Q - \) to this sequence gives the four term exact sequence

\[
0 \to \text{Tor}_1^Q(A, k) \to \mathfrak{m}_Q / \alpha \mathfrak{m}_Q \to A \to k \to 0
\]

of \( A \)-modules. Consider the first term in this sequence. By tensoring the exact sequence

\[
0 \to \alpha \to Q \to A \to 0
\]

over \( Q \) with \( k \), we obtain the exact sequence

\[
0 \to \text{Tor}_1^Q(A, k) \to \alpha \otimes_Q k \to Q \otimes_Q k \to A \otimes_Q k \to 0,
\]

in which the map \( g \) must be the zero map since \( \alpha k = 0 \). This gives isomorphisms

\[
\text{Tor}_1^Q(A, k) \simeq \alpha \otimes_Q k \simeq \alpha \otimes_Q (A \otimes_A k) \simeq \alpha / \alpha^2 \otimes_A k
\]
of $A$-modules. Since $a$ is generated by a regular sequence of length $c$, the $A$-module $a/a^2$ is free of rank $c$, and therefore $\text{Tor}_i^A(A, k)$ is isomorphic to $k^c$. We may now rewrite the four term exact sequence (1) as

$$0 \to k^c \xrightarrow{f} m_Q/a m_Q \to A \to k \to 0,$$

and it is not hard to show that the map $f$ is defined by

$$(\alpha_1, \ldots, \alpha_c) \mapsto \sum \alpha_i x_i^{n_i} + a m_Q.$$

The image of the Eisenbud operator $\chi_j$ under the homomorphism $\hat{A}[\chi_j] \xrightarrow{\phi_\chi} \text{Ext}_A^2(k, k)$ is the bottom row in the pushout diagram

$$\begin{array}{ccc}
0 & \xrightarrow{f} & m_Q/a m_Q \\
\downarrow{\pi_j} & & \downarrow{\pi_j} \\
0 & \to & k \\
& \to & K_{\chi_j} \\
& \to & A \\
& \to & k \\
& \to & 0
\end{array}$$

of $A$-modules, in which the map $\pi_j$ is projection onto the $j$th summand. The pushout module $K_{\chi_j}$ can be described explicitly as

$$K_{\chi_j} = \frac{k \oplus m_Q/a m_Q}{\{(\alpha_j, -\sum \alpha_i x_i^{n_i} + a m_Q) \mid (\alpha_1, \ldots, \alpha_c) \in k^c\}},$$

and by [Be2] Theorem 2.2 its support variety is given by $V(K_{\chi_j}) = \hat{V}(k) \cap \hat{V}_H(\chi_j)$.

Before proving the final result, we need a lemma showing that every maximal Cohen-Macaulay module over a complete intersection has reducible complexity by a cohomological element of degree two. This improves [Be3, Lemma 2.1(i)], which states that such a cohomological element exists after passing to some suitable faithfully flat extension of the ring.

**Lemma 3.6.** If $M$ is a maximal Cohen-Macaulay $A$-module of infinite projective dimension, then there exists an element $\eta \in \text{Ext}_A^2(M, M)$ reducing its complexity.

**Proof.** Since the dimension of $V(M)$ is nonzero, the radical $\sqrt{\text{Ann}_H E(\hat{M}, \hat{M})}$ of $\text{Ann}_H E(\hat{M}, \hat{M})$ is properly contained in the graded maximal ideal of $H$. Therefore one of the Eisenbud operators, say $\chi_j$, is not contained in $\sqrt{\text{Ann}_H E(\hat{M}, \hat{M})}$. We now follow the arguments given prior to [Be2, Corollary 2.3]. Viewing $\chi_j$ as an element of $\hat{A}[\chi]$, we can apply the homomorphism $\phi_{\hat{M}}$ and obtain the element $\phi_{\hat{M}}(\chi_j) \otimes 1$ in $\text{Ext}_A^2(\hat{M}, \hat{M}) \otimes \hat{A} k$. Now $\text{Ext}_A^2(\hat{M}, \hat{M})$ is isomorphic to $\text{Ext}_A^2(M, M) \otimes_A \hat{A}$, and there is an isomorphism

$$\text{Ext}_A^2(M, M) \otimes_A k \xrightarrow{\sim} \text{Ext}_A^2(\hat{M}, \hat{M}) \otimes \hat{A} k$$

mapping an element $\theta \otimes 1 \in \text{Ext}_A^2(M, M) \otimes_A k$ to $\hat{\theta} \otimes 1$. Therefore there exists an element $\eta \in \text{Ext}_A^2(M, M)$ such that $\hat{\eta} \otimes 1$ equals $\phi_{\hat{M}}(\chi_j) \otimes 1$ in $\text{Ext}_A^2(\hat{M}, \hat{M}) \otimes \hat{A} k$. If the exact sequence

$$0 \to M \to K_\eta \to \Omega^1_A(M) \to 0$$

corresponds to $\eta$, then its completion

$$0 \to \hat{M} \to \hat{K}_\eta \to \Omega^1_A(\hat{M}) \to 0$$
corresponds to $\hat{\eta}$, and so from [Be2, Theorem 2.2] we see that

$$V(K_\eta) = V(M) \cap V_H(\chi_j).$$

Since $\chi_j$ was chosen so that it “cuts down” the variety of $M$, we must have $\dim V(K_\eta) = \dim V(M) - 1$, i.e. $c\chi M = c\chi M - 1$.

We have now arrived at the final result, which improves Theorem 3.2 when the ring is a complete intersection. Namely, for such rings it suffices to check the vanishing of finitely many cohomology groups “separated” by an odd number. The number of cohomology groups we need to check depends on the complexity value we are testing. Recall that we have denoted the codimension of the complete intersection $A$ by $c$.

**Theorem 3.7.** Let $M$ be an $A$-module and $t \in \{1, \ldots, c\}$ an integer. If for every $A$-module $N$ of complexity $t$ there is an odd number $q$ such that

$$\text{Ext}_A^n(M, N) = \text{Ext}_A^{n+q}(M, N) = \cdots = \text{Ext}_A^{n+(c-t)q}(M, N) = 0$$

for some even number $n > \dim A - \text{depth} M$, then $c\chi M < t$.

**Proof.** Since $c\chi M = c\chi \Omega_A^{\dim A - \text{depth} M}(M)$, we may without loss of generality assume that $M$ is maximal Cohen-Macaulay and that $n > 0$. We prove by induction that if $c\chi M > t$, then for any odd number $q$ and any even integer $n > 0$, the groups

$$\text{Ext}_A^n(M, N), \text{Ext}_A^{n+q}(M, N), \ldots, \text{Ext}_A^{n+(c\chi M - t)q}(M, N)$$

cannot all vanish for every module $N$ of complexity $t$. When the complexity of $M$ is $t$, take $N$ to be $M$ itself. In this case it follows from [AvB, Theorem 4.2] that $\text{Ext}_A^n(M, N)$ is nonzero, because $t \geq 1$. Now assume $c\chi M > t$, and write $q$ as $2s - 1$ where $s \geq 1$ is an integer. By Lemma 3.6 there is an element $\eta \in \text{Ext}_A^n(M, M)$ reducing the complexity of $M$, and it follows from [Be1, Proposition 2.4(i)] that the element $\eta^s \in \text{Ext}_A^{2s}(M, M)$ also reduces the complexity. The latter element corresponds to an exact sequence

$$0 \to M \to K \to \Omega_A^1(M) \to 0,$$

in which the complexity of $K$ is one less than that of $M$. By induction there exists a module $N$, of complexity $t$, such that the groups

$$\text{Ext}_A^n(K, N), \text{Ext}_A^{n+q}(K, N), \ldots, \text{Ext}_A^{n+(c\chi K - t)q}(K, N)$$

do not all vanish. Then from the exact sequence we see that the groups

$$\text{Ext}_A^n(M, N), \text{Ext}_A^{n+q}(M, N), \ldots, \text{Ext}_A^{n+(c\chi M - t)q}(M, N)$$

cannot all vanish. Since the complexity of any $A$-module is at most $c$, the proof is complete. \hfill \square

**References**

[AIE] J. Alperin, L. Evens, *Representations, resolutions and Quillen’s dimension theorem*, J. Pure Appl. Algebra 22 (1981), 1-9.

[Av1] L. Avramov, *Modules of finite virtual projective dimension*, Invent. Math. 96 (1989), 71-101.

[Av2] L. Avramov, *Homological asymptotics of modules over local rings*, Commutative algebra; Berkeley, 1987 (M. Hochster, C. Huneke, J. Sally, eds.), MSRI Publ. 15, Springer, New York 1989, pp. 33-62.

[AvB] L. Avramov, R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Invent. Math. 142 (2000), 285-318.

[AGP] L. Avramov, V. Gasharov, I. Peeva, *Complete intersection dimension*, Publ. Math. I.H.E.S. 86 (1997), 67-114.

[Be1] P. Bergh, *Modules with reducible complexity*, J. Algebra 310 (2007), 132-147.
[Be2] P. Bergh, *On support varieties for modules over complete intersections*, to appear in Proc. Amer. Math. Soc.

[Be3] P. Bergh, *On the vanishing of (co)homology over local rings*, to appear in J. Pure Appl. Algebra.

[GaP] V. Gasharov, I. Peeva, *Boundedness versus periodicity over commutative local rings*, Trans. Amer. Math. Soc. 320 (1990), no. 2, 569-580.

[Ger] A. Gerko, *On homological dimensions*, Mat. Sb. 192 (8) (2001), 79-94, Sb. Math. 192 (7-8) (2001), 1165-1179 (English translation).

[Jor] D. Jorgensen, *Complexity and Tor on a complete intersection*, J. Algebra 211 (1999), 578-598.

[Jør] P. Jørgensen, *Symmetry theorems for Ext vanishing*, J. Algebra 301 (2006), 224-239.

[SnS] N. Snashall, Ø. Solberg, *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. 88 (2004), no. 3, 705-732.

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