On the Well-Posedness of A High Order Convective Cahn-Hilliard Type Equations

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Abstract: High order convective Cahn-Hilliard type equations describe the faceting of a growing surface, or the dynamics of phase transitions in ternary oil-water-surfactant systems. In this paper, we prove the well-posedness of the classical solutions for the Cauchy problem, associated with this equation.

Keywords: existence; uniqueness; stability; higher order convective Cahn-Hilliard type equation; Cauchy problem

MSC: 35G25; 35K55

1. Introduction

In this paper, we investigate the well-posedness of the following Cauchy problem:

\[
\begin{aligned}
\partial_t u + \kappa \partial_x u^2 - \beta^2 \partial_x^6 u + a \partial_x^4 u + \delta^2 \partial_x^4 (u^3) &= 0, \quad t > 0, \quad x \in \mathbb{R}, \\
u(0,x) &= u_0(x), \quad x \in \mathbb{R},
\end{aligned}
\]  

(1)

with \( \kappa, \beta, a, \delta \in \mathbb{R}, \quad \beta, \delta \neq 0. \) (2)

On the initial datum, we assume

\( u_0 \in H^3(\mathbb{R}), \quad \int_\mathbb{R} u_0(x) dx = 0. \) (3)

Inspired by [1–12], thanks to (3), we can define the following function:

\( P_0(x) = \int_{-\infty}^{x} u_0(y) dy, \) (4)

on which we assume

\[ \| P_0 \|_{L^2(\mathbb{R})}^2 = \int_\mathbb{R} \left( \int_{-\infty}^{x} u_0(y) dy \right)^2 dx < \infty. \] (5)

Equation (1) is derived in [13], with \( \beta = 1, \alpha = -1, \delta = 1, \) to describe a growing crystalline surface with small slopes that undergoes faceting.

The function \( u \) represents the slope of a surface, the constant \( \kappa \) is proportional to the deposition strength of an atomic flux, and the overall convective term \( \kappa \partial_x u^2 \) stems from the normal impingement of the deposited atoms. The sixth order linear term \( \partial_x^6 u \), in (1), comes from a curvature...
dependent regularization, and all other terms represent the anisotropy of the surface energy under surface diffusion.

These equations became popular objects of theoretical studies in the last decade, because they do play an important role in material modeling. Equations based on the Mullins surface diffusion formula, such as in (1), are interesting, as they describe the morphology of crystalline surfaces. As complex interfaces and surfaces are more and more important in highly technological applications such as photovoltaics, the study of analytical aspects also gains momentum (see, [14–22]).

The existence and uniqueness of weak solutions of (1) with periodic boundary conditions is proven in [23], with $\kappa > 0$. A similar result is proven in [24], in two space dimensions. In [25], the authors derived stationary solutions of (1), while in [26], the problem for optimal control of (1) is analyzed. In [27], the problem of a global attractor is studied.

Taking $\kappa = 0$, (1) reads

$$\partial_t u - \beta^2 \partial_x^6 u + \alpha \partial_x^4 u + \delta^2 \partial_x^4 \left(u^3\right) = 0,$$

which is a Cahn-Hilliard type equation [28–30]. It describes dynamics of phase transitions in ternary oil-water-surfactant systems. One part of the surfactant is hydrophilic and the other one is lipophilic (and termed amphiphile). In the system, almost pure oil, almost pure water, and microemulsion which consist of a homogeneous, isotropic mixture of oil and water can coexist in equilibrium. The function $u$, in (6), represents the local difference between oil and water concentrations.

In [31] the existence of global attractors is analyzed and in [32] the existence of weak solutions for the initial-boundary-value problem for (6) with degenerate mobility is proven.

Assuming $\beta = \delta = 0$ and $\alpha = a^2 \neq 0$, (1) reads

$$\partial_t u + \kappa \partial_x u^2 + a^2 \partial_x^4 u = 0.$$

(7) arises in interesting physical situations, for example as a model for long waves on a viscous fluid owing down an inclined plane [33] and to derive drift waves in a plasma [34]. Equation (7) was derived also independently by Kuramoto [35–37] as a model for phase turbulence in reaction-diffusion systems and by Sivashinsky [38] as a model for plane flame propagation, describing the combined influence of diffusion and thermal conduction of the gas on the stability of a plane flame front.

Equation (7) also describes incipient instabilities in a variety of physical and chemical systems [39–41]. Moreover, (7), which is also known as the Benney-Lin equation [42,43], was derived by Kuramoto in the study of phase turbulence in the Belousov-Zhabotinsky reaction [44].

The dynamical properties and the existence of exact solutions for (7) were investigated in [45–50]. In [51–53], the control problem for (7) with periodic boundary conditions, and on a bounded interval are studied, respectively. In [54], the problem of global exponential stabilization of (7) with periodic boundary conditions is analyzed. In [55], it is proposed a generalization of optimal control theory for (7), while in [56] the problem of global boundary control of (7) is considered. In [57], the existence of solitonic solutions for (7) is proven. In [58–60], the well-posedness of the Cauchy problem for (7) is proven, using the energy space technique, a priori estimates together with an application of the Cauchy-Kovalevskaya and the fixed point method, respectively. Finally, following [61–63], in [64], the convergence of the solution of (7) to the unique entropy one of the Burgers equation is proven.

The main result of this paper is the following theorem.

**Theorem 1.** Assume (2), (3), (4) and (5). There exists a unique solution $u$ of (1), such that

$$u \in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^3(\mathbb{R})),$$

$$\int_{\mathbb{R}} u(t, x)dx = 0, \quad t > 0.$$
Moreover, if \( u_1 \) and \( u_2 \) are two solutions of (1), we have that
\[
\|u_1(t, \cdot) - u_2(t, \cdot)\|_{L^2(\mathbb{R})} \leq e^{C(T)t} \|u_{1,0} - u_{2,0}\|_{L^2(\mathbb{R})},
\]
for some suitable \( C(T) > 0 \), and every \( 0 \leq t \leq T \).

Compared to [23,31,65], we do not have additional assumption on the constants. Theorem 1 gives the global well–posedness of the classical solution of the Cauchy problem of (1). The argument of Theorem 1 relies on deriving suitable a priori estimates together with an application of the Fixed Point Theorem. Indeed, to make the argument completely rigorous we should approximate the initial datum with an analytic one. Considering the semigroup \( S_t \) generated by the operator \(-\beta^2 \partial_x^2 u\) we rewrite (1) using the Duhamel formula
\[
u(t, \cdot) = S_t u_0 - \int_0^t S_{t-s} \left( \kappa \partial_x u^2 + \alpha \partial_x^4 u + \delta \partial_x^4 u^3 \right) (s, \cdot) ds.
\]

Then standard arguments and the Fixed Point Theorem would give us the local in time existence of analytic solutions. The estimates will guarantee that these approximate solutions are indeed global in time and allow us to come back to the original initial datum.

The paper is organized as follows. In Section 2, we prove some a priori estimates of (1). Those play a key role in the proof of our main result, which is given in Section 3.

2. A Priori Estimates

In this section, we prove some a priori estimates on \( u \). We denote with \( C_0 \) the constants which depend only on the initial data, and with \( C(T) \), the constants which depend also on \( T \).

We begin by proving the following lemma.

Lemma 1. For each \( t > 0 \), we have that (9).

Proof. Integrating (1) on \( \mathbb{R} \), we have that
\[
\int_\mathbb{R} \partial_x u(t, x) dx = \frac{d}{dt} \int_\mathbb{R} u(t, x) dx = 0
\]
(11) follows from (3) and (11). \(\Box\)

Remark 1. In light of (9), we can consider the following function:
\[
P(t, x) = \int_{-\infty}^x u(t, y) dy.
\]
Moreover, again by (9), we have that
\[
P(t, -\infty) = P(t, \infty) = 0.
\]

Lemma 2. Let \( T > 0 \). There exists a constant \( C(T) > 0 \), such that
\[
\|P(t, \cdot)\|_{L^2(\mathbb{R})}^2 + 2\beta^2 e^t \int_0^t e^{-s} \|\partial_x^2 u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds
\]
\[
+ 2\delta^2 e^t \int_0^t e^{-s} \|u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds + C(T) \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds,
\]
\[
\leq C(T) + C(T) \int_0^t \|u(s, \cdot)\|_{L^4(\mathbb{R})}^4 ds + C(T) \int_0^t \|\partial_x u(s, \cdot)\|_{L^2(\mathbb{R})}^2 ds,
\]
\[
(14)
for every \(0 \leq t \leq T\).

**Proof.** Let \(0 \leq t \leq T\). We begin by observing that

\[
\partial_x u^3 = 3u^2\partial_x u. \tag{15}
\]

Integrating (1) on \((-\infty, x)\), we have that

\[
\int_{-\infty}^{x} \partial_x u \, dy + \kappa u^2 - \beta^2 \partial_x^3 u + a \partial_x^3 u + \delta^2 \partial_x^3 \left( u^3 \right) = 0. \tag{16}
\]

Differentiating (12) with respect to \(t\), we obtain that

\[
\partial_t P(t, x) = \frac{d}{dt} \int_{-\infty}^{x} u(t, y) \, dy = \int_{-\infty}^{x} \partial_t u(t, y) \, dy. \tag{17}
\]

From (16) and (17)

\[
\partial_t P + \kappa u^2 - \beta^2 \partial_x^3 u + a \partial_x^3 u + \delta^2 \partial_x^3 \left( u^3 \right) = 0. \tag{18}
\]

Observe that by (13) and (18),

\[
-2\beta^2 \int_{\mathbb{R}} \partial_x^3 u \, dx = 2\beta^2 \int_{\mathbb{R}} \partial_x P \partial_x^3 u \, dx = \beta^2 \int_{\mathbb{R}} u \partial_x^4 u \, dx
\
= -2\beta^2 \int_{\mathbb{R}} \partial_x u \partial_x^3 u \, dx + 2\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\
2\alpha \int_{\mathbb{R}} \partial_x^3 u \, dx = 2\beta^2 \int_{\mathbb{R}} u \partial_x^4 u \, dx = 2\alpha \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{19}
\]

Moreover, by (15) and (16),

\[
2\delta^2 \int_{\mathbb{R}} \partial_x^3 \left( u^3 \right) \, dx = -2\delta^2 \int_{\mathbb{R}} \partial_x P \partial_x^3 \left( u^3 \right) \, dx = -2\delta^2 \int_{\mathbb{R}} u \partial_x^4 \left( u^3 \right) \, dx
\
= 2\delta^2 \int_{\mathbb{R}} \partial_x u \partial_x^3 \left( u^3 \right) \, dx = 6\delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{20}
\]

Therefore, multiplying (18) by \(2P\), an integration on \(\mathbb{R}\), (19) and (20) give

\[
\frac{d}{dt} \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 6\delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\
= -2\kappa \int_{\mathbb{R}} Pu^2 \, dx - 2\alpha \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{21}
\]

Due to the Young inequality,

\[
2\kappa \int_{\mathbb{R}} |P|u^2 \, dx \leq \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \kappa^2 \left\| u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4.
\]

It follows from (21) that

\[
\frac{d}{dt} \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 6\delta^2 \left\| u(t, \cdot) \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\
\leq \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \kappa^2 \left\| u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + 2\kappa \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\
\leq \left\| P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| u(t, \cdot) \right\|_{L^4(\mathbb{R})}^4 + C_0 \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]
Therefore, by the Gronwall Lemma and (5), we have that
\[
\| P(t, \cdot) \|_{L^2(R)}^2 + 2\beta^2 e^t \int_0^t e^{-s} \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(R)}^2 ds + 6\beta^2 e^t \int_0^t e^{-s} \| u(s, \cdot) \partial_x u(s, \cdot) \|_{L^2(R)}^2 ds \\
\leq C_0 + C_0 e^t \int_0^t e^{-s} \| u(s, \cdot) \|_{L^4(R)}^4 ds + C_0 e^t \int_0^t e^{-s} \| \partial_x u(s, \cdot) \|_{L^2(R)}^2 ds \\
\leq C(T) + C(T) \int_0^t \| u(s, \cdot) \|_{L^4(R)}^4 ds + C(T) \int_0^t \| \partial_x u(s, \cdot) \|_{L^2(R)}^2 ds,
\]
which gives (14). 

Lemma 3. Let \( T > 0 \). There exists a constant \( C(T) > 0 \), such that
\[
\frac{\beta^2}{6} \| \partial_x u(t, \cdot) \|_{L^2(R)}^2 + \frac{\delta^2}{4} \| u(t, \cdot) \|_{L^4(R)}^4 \leq C(T),
\]
\[
\| P(t, \cdot) \|_{L^2(R)} \leq C(T),
\]
\[
\| u(t, \cdot) \|_{L^2(R)} \leq C(T),
\]
\[
\int_0^t \| \partial_x^2 u(s, \cdot) \|_{L^2(R)}^2 ds \leq C(T),
\]
\[
\int_0^t \| u(s, \cdot) \partial_x u(s, \cdot) \|_{L^2(R)}^2 ds \leq C(T),
\]
\[
\int_0^t \left( \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^3 \left( u^3 \right) \right) ds dx \leq C(T),
\]
\[
\| P \|_{L^\infty((0,T) \times R)} \leq C(T)
\]
\[
\| u \|_{L^\infty((0,T) \times R)} \leq C(T)
\]
\[
\int_0^t \| \partial_x u(s, \cdot) \|_{L^4(R)}^4 ds \leq C(T),
\]
for every \( 0 \leq t \leq T \).

Proof. Let \( 0 \leq t \leq T \). Consider an real constant \( A \), which will specified later. Multiplying (1) by
\[-\beta^2 \partial_x^2 u + \delta^2 u^3 + Au,
\]
we have that
\[
\left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) \partial_t u + 2\kappa \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) u \partial_x u \\
\quad -\beta^2 \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) \partial_x^3 u \\
\quad + \alpha \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) \partial_x^4 u \\
\quad + \delta^2 \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) \partial_x^4 \left( u^3 \right) = 0.
\]
Observe that
\[
\int_R \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) \partial_t u dx \\
= \frac{d}{dt} \left( \frac{\beta^2}{2} \| \partial_x u(t, \cdot) \|_{L^2(R)}^2 + \frac{\delta^2}{4} \| u(t, \cdot) \|_{L^4(R)}^4 + \frac{A}{2} \| u(t, \cdot) \|_{L^2(R)}^2 \right),
\]
\[
2\kappa \int_R \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) u \partial_x u dx = -2\beta^2 \kappa \int_R u \partial_x u \partial_x^2 u dx,
\]
\[
-\beta^2 \int_R \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) \partial_x^3 u dx.
\]
\[
\begin{align*}
&= -\beta^4 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 u dx + \beta^2 \delta^2 \int_{\mathbb{R}} \partial_x \left( u^3 \right) \partial_x^2 u dx + A \beta^2 \int_{\mathbb{R}} \partial_x u \partial_x^3 u dx \\
&= \beta^4 \int_{\mathbb{R}} (\partial_x^4 u)^2 dx - \beta^2 \delta^2 \int_{\mathbb{R}} \partial_x (u^3) \partial_x^4 u dx - A \beta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^2 u dx \\
&= \beta^4 \int_{\mathbb{R}} (\partial_x^4 u)^2 dx - \beta^2 \delta^2 \int_{\mathbb{R}} \partial_x (u^3) \partial_x^4 u dx + A \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
&= \alpha \int_{\mathbb{R}} \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) \partial_x^4 u dx \\
&= \alpha \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \alpha \delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 u \left( u^3 \right) dx - A \alpha \int_{\mathbb{R}} \partial_x u \partial_x^3 u dx \\
&= \alpha \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 u \left( u^3 \right) dx + A \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2, \\
&= \delta^2 \int_{\mathbb{R}} \left( -\beta^2 \partial_x^2 u + \delta^2 u^3 + Au \right) \partial_x^4 u dx \\
&= \delta^2 \beta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 u \left( u^3 \right) dx - \delta^4 \int_{\mathbb{R}} \partial_x u \partial_x^3 \partial_x^3 u \left( u^3 \right) dx - A \delta^2 \int_{\mathbb{R}} \partial_x u \partial_x^3 u \left( u^3 \right) dx \\
&= -\delta^2 \beta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^3 u \left( u^3 \right) dx + \delta^4 \int_{\mathbb{R}} \partial_x^4 \left( u^3 \right) \partial_x^2 u dx + A \delta^2 \int_{\mathbb{R}} \partial_x^2 u \partial_x^3 u \left( u^3 \right) dx.
\end{align*}
\]

Consequently, integrating (31) on \( \mathbb{R} \), by (32), we have that

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^4 + A \frac{\beta^2}{2} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
+ \beta^4 \int_{\mathbb{R}} (\partial_x^3 u)^2 dx - 2 \beta^2 \delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 u \left( u^3 \right) dx + \delta^4 \int_{\mathbb{R}} \partial_x^4 \left( u^3 \right) \partial_x^2 u dx \tag{33}
\]

\[
= -\beta^2 (A + \alpha) \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - \delta^2 (A + \alpha) \int_{\mathbb{R}} \partial_x^2 u \partial_x^3 u \left( u^3 \right) dx - A \alpha \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2 \beta^2 \kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx.
\]

Taking

\[ A = -\alpha, \tag{34} \]

by (33), we have that

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^4 - \frac{\alpha}{2} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
+ \beta^4 \int_{\mathbb{R}} (\partial_x^3 u)^2 dx - 2 \beta^2 \delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 u \left( u^3 \right) dx + \delta^4 \int_{\mathbb{R}} \partial_x^4 \left( u^3 \right) \partial_x^2 u dx \tag{35}
\]

\[
= \alpha^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2 \beta^2 \kappa \int_{\mathbb{R}} u \partial_x u \partial_x^2 u dx.
\]

Observe that

\[
\beta^4 \int_{\mathbb{R}} (\partial_x^3 u)^2 dx - 2 \beta^2 \delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_x^3 u \left( u^3 \right) dx + \delta^4 \int_{\mathbb{R}} \partial_x^4 \left( u^3 \right) \partial_x^2 u dx = \int_{\mathbb{R}} \left[ \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 \left( u^3 \right) \right] dx.
\]

Consequently, by (35),

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{\delta^2}{4} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^4 - \frac{\alpha}{2} \left\| u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) \\
+ \int_{\mathbb{R}} \left[ \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^2 \left( u^3 \right) \right] dx \tag{36}
\]
Due to the Young inequality,

\[ 2\beta^2 |\int_R |u \partial_x u| |\partial_x^2 u| dx | \leq \kappa^2 \|u(t, \cdot)\|_{L^2(R)}^2 + \beta^4 \|\partial_x^2 u(t, \cdot)\|_{L^2(R)}^2. \]

Therefore, by (36),

\[
\frac{d}{dt} \left( \frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(R)}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(R)}^4 - \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(R)}^2 \right) \\
+ \int_R \left[ \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^3 \left( u^3 \right) \right] dx \\
\leq \left( \alpha^2 + \beta^4 \right) \|\partial_x^2 u(t, \cdot)\|_{L^2(R)}^2 + \kappa^2 \|u(t, \cdot)\|_{L^2(R)} \|\partial_x u(t, \cdot)\|_{L^2(R)}^2 \\
\leq C_0 \beta^2 \|\partial_x^2 u(t, \cdot)\|_{L^2(R)}^2 + C_0 \delta^4 \|u(t, \cdot)\|_{L^2(R)} \|\partial_x u(t, \cdot)\|_{L^2(R)}^2.
\]

Integrating on \((0, t)\), by (3), we have that

\[
\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(R)}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(R)}^4 - \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(R)}^2 \\
+ \int_0^t \int_R \left[ \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^3 \left( u^3 \right) \right] dsdx \\
\leq C_0 + C_0 \beta^2 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(R)}^2 + C_0 \delta^4 \int_0^t \|u(s, \cdot)\|_{L^2(R)} \|\partial_x u(s, \cdot)\|_{L^2(R)}^2 ds.
\]

Therefore, we obtain that

\[
\frac{\beta^2}{2} \|\partial_x u(t, \cdot)\|_{L^2(R)}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(R)}^4 \\
+ \int_0^t \int_R \left[ \beta^2 (\partial_x^4 u)^2 - \delta^2 \partial_x^3 \left( u^3 \right) \right] dsdx \\
\leq C_0 + C_0 \beta^2 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(R)}^2 + C_0 \delta^4 \int_0^t \|u(s, \cdot)\|_{L^2(R)} \|\partial_x u(s, \cdot)\|_{L^2(R)}^2 ds + \frac{\alpha}{2} \|u(t, \cdot)\|_{L^2(R)}^2 \\
\leq C_0 + C_0 \beta^2 \int_0^t \|\partial_x^2 u(s, \cdot)\|_{L^2(R)}^2 + C_0 \delta^4 \int_0^t \|u(s, \cdot)\|_{L^2(R)} \|\partial_x u(s, \cdot)\|_{L^2(R)}^2 ds + C_0 \|u(t, \cdot)\|_{L^2(R)}^2.
\]

Observe that by (12) and (13),

\[
C_0 \|u(t, \cdot)\|_{L^2(R)}^2 = C_0 \int_R uudx = -C_0 \int_R P \partial_x udx.
\]

Therefore, by the Young inequality,

\[
C_0 \|u(t, \cdot)\|_{L^2(R)}^2 \leq 2 \int_R \left| \frac{\sqrt{3} C_0 P}{2\beta} \right| \left| \frac{\beta \partial_x u}{\sqrt{3}} \right| dx \\
\leq C_0 \|P(t, \cdot)\|_{L^2(R)} + \frac{\beta^2}{3} \|\partial_x u(t, \cdot)\|_{L^2(R)}^2.
\]

If follows from (14) and (37) that

\[
\frac{\beta^2}{6} \|\partial_x u(t, \cdot)\|_{L^2(R)}^2 + \frac{\delta^2}{4} \|u(t, \cdot)\|_{L^4(R)}^4 
\]
Finally, we prove (30). We begin by observing that ([66], Lemma 2.3) says that
\[
\|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq 6 \left( \|u(t, \cdot)\|_{L^2(\mathbb{R})}^2 + \|\partial_x u(t, \cdot)\|_{L^2(\mathbb{R})}^2 \right) \left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]
Therefore, by (22) and (23),
\[
\|\partial_x u(t, \cdot)\|_{L^4(\mathbb{R})}^4 \leq C(T) \left\| \partial_t^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{39}
\]
Integrating (39) on (0, t), by (23), we have (30).
Lemma 4. Let $T > 0$. There exists a constant $C(T) > 0$, such that

\[
\int_0^t \| \partial_x^4 u(s, \cdot) \|^2_{L^2(\mathbb{R})} ds \leq C(T),
\]

(40)

\[
\int_0^t \| \partial_x^3 u(s, \cdot) \|^2_{L^2(\mathbb{R})} ds \leq C(T),
\]

(41)

for every $0 < t \leq T$.

Proof. Let $0 < t \leq T$. We begin by observing that

\[
\partial_x^2 \left( u^3 \right) = 6u(\partial_x u)^2 + 3u^2 \partial_x^2 u.
\]

(42)

By (27), we have that

\[
\beta \int_0^t \| \partial_x^4 u(s, \cdot) \|^2_{L^2(\mathbb{R})} ds \leq C(T) - \delta \int_0^t \left[ \partial_x^2 \left( u^3 \right) \right]^2 ds dx
\]

\[
+ 2\beta^2 \delta \int_0^t \partial_x^4 u \partial_x^2 \left( u^3 \right) ds dx.
\]

(43)

Due to the Young inequality,

\[
2\beta^2 \delta \int_0^t \| \partial_x^4 u \| \| \partial_x^2 \left( u^3 \right) \| ds dx \leq \frac{\beta^4}{2} \left[ \int_0^t \| \partial_x^4 u(t, \cdot) \|^2_{L^2(\mathbb{R})} \right]^2 + 2\delta \int_0^t \left[ \partial_x^2 \left( u^3 \right) \right]^2 ds dx.
\]

Therefore, by (43),

\[
\frac{\beta^4}{2} \left[ \int_0^t \| \partial_x^4 u(t, \cdot) \|^2_{L^2(\mathbb{R})} \right] \leq C(T) + \delta \int_0^t \left[ \partial_x^2 \left( u^3 \right) \right]^2 ds dx.
\]

(44)

Observe that (42),

\[
\int_\mathbb{R} \left[ \partial_x^2 \left( u^3 \right) \right]^2 dx = 36 \int_\mathbb{R} u^2(\partial_x u)^4 dx + 9 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 36 \int_\mathbb{R} u^3(\partial_x u)^2 \partial_x^2 u dx
\]

\[
= 36 \int_\mathbb{R} u^2(\partial_x u)^4 dx + 9 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} - 36 \int_\mathbb{R} u^2(\partial_x u)^4 dx
\]

\[
= 9 \left\| u^2(t, \cdot) \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} = 9 \int_\mathbb{R} u^4(\partial_x^2 u)^2 dx.
\]

Therefore, by (23) and (29),

\[
\int_\mathbb{R} \left[ \partial_x^2 \left( u^3 \right) \right]^2 dx \leq 9 \left\| u \right\|^2_{L^\infty((0, T) \times \mathbb{R})} \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}
\]

\[
\leq C(T) \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]

(45)

Integrating (45) on $(0, t)$, by (23), we have that

\[
\int_0^t \left[ \partial_x^2 \left( u^3 \right) \right]^2 ds dx \leq C(T) \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds \leq C(T).
\]

(46)

(40) follows from (44) and (46).

Finally, we prove (41). We begin by observing that

\[
\left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} = \int_\mathbb{R} \partial_x^2 u \partial_x^4 u dx = - \int_\mathbb{R} \partial_x^2 u \partial_x^4 u dx.
\]
Therefore, by the Young inequality,
\[
\left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq \int_{\mathbb{R}} |\partial_x^2 u| |\partial_x^4 u| \, dx \leq \frac{1}{2} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \frac{1}{2} \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2. \tag{47}
\]
(23), (40) and an integration on \((0, t)\) of (47) give (41). \(\square\)

**Lemma 5.** Fix \(T > 0\). There exists a constant \(C(T) > 0\), such that
\[
\left\| \partial_x u \right\|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T), \tag{48}
\]
\[
\left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T), \tag{49}
\]
for every \(0 \leq t \leq T\).

**Proof.** Let \(0 \leq t \leq T\). We begin by observing that by (42),
\[
\partial_x^3 \left( u^3 \right) = 6(\partial_x u)^3 + 18u \partial_x u \partial_x^2 u + 3u^2 \partial_x^3 u. \tag{50}
\]

Multiplying (1) by \(2 \partial_x^4 u\), thanks to (50), an integration on \(\mathbb{R}\) gives
\[
\frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = 2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^4 u \, dx
\]
\[
= -4\kappa \int_{\mathbb{R}} u \partial_x \partial_x^3 u \partial_x^4 u \, dx + 2\beta^2 \int_{\mathbb{R}} \partial_x^4 u \partial_x^6 u \, dx - 2\kappa \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
- 2\beta^2 \int_{\mathbb{R}} \partial_x^6 u \partial_x^4 \left( u^3 \right) \, dx
\]
\[
= -4\kappa \int_{\mathbb{R}} u \partial_x \partial_x^3 u \partial_x^4 u \, dx - 2\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\kappa \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
+ 2\beta^2 \int_{\mathbb{R}} \partial_x^6 u \partial_x^4 \left( u^3 \right) \, dx
\]
\[
= -4\kappa \int_{\mathbb{R}} u \partial_x \partial_x^3 u \partial_x^4 u \, dx - 2\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 - 2\kappa \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
+ 12\beta^2 \int_{\mathbb{R}} (\partial_x u)^3 \partial_x^5 u \, dx + 36\beta^2 \int_{\mathbb{R}} u \partial_x \partial_x^3 u \partial_x^5 u \, dx + 6 \int_{\mathbb{R}} u^2 \partial_x^3 u \partial_x^5 u \, dx.
\]

Therefore, we have that
\[
\frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
= -4\kappa \int_{\mathbb{R}} u \partial_x \partial_x^3 u \partial_x^4 u \, dx - 2\kappa \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 12\beta^2 \int_{\mathbb{R}} (\partial_x u)^3 \partial_x^5 u \, dx
\]
\[
+ 36\beta^2 \int_{\mathbb{R}} u \partial_x \partial_x^3 u \partial_x^5 u \, dx + 6 \int_{\mathbb{R}} u^2 \partial_x^3 u \partial_x^5 u \, dx. \tag{51}
\]

Due to (22), (29) and the Young inequality,
\[
4|\kappa| \int_{\mathbb{R}} u \partial_x \partial_x^3 u \partial_x^4 u \, dx \leq 2\kappa^2 \int_{\mathbb{R}} u^2 (\partial_x u)^2 \, dx + 2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
\leq 2\kappa^2 \left\| u \right\|_{L^\infty((0, T) \times \mathbb{R})}^2 \left\| \partial_x u(t, \cdot) \right\|_{L^\infty(\mathbb{R})}^2 + 2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
\leq C(T) + 2 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]
\[
12\beta^2 \int_{\mathbb{R}} |\partial_x u|^3 |\partial_x^5 u| \, dx = 2 \int_{\mathbb{R}} \left\| \beta \sqrt{D_1} \partial_x u \right\| \, dx.
\]
where \( D_1 \) is a positive constant, which will specified later. It follows from (51) that

\[
\frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left( 2 - 3D_1 \right) \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) + C_0 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| \partial_x u \right\|_{L^2((0, T) \times \mathbb{R})}^2 \| \partial_x u(t, \cdot) \|_{L^4(\mathbb{R})}^4 \\
+ \frac{C(T)}{D_1} \left\| \partial_x u \right\|_{L^\infty((0, T) \times \mathbb{R})}^2 \right\| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

Taking \( D_1 = \frac{1}{3} \), we have that

\[
\frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \left\| \partial_x^5 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) + C_0 \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C_0 \left\| \partial_x u \right\|_{L^2((0, T) \times \mathbb{R})}^2 \| \partial_x u(t, \cdot) \|_{L^4(\mathbb{R})}^4 \\
+ C(T) \left\| \partial_x u \right\|_{L^\infty((0, T) \times \mathbb{R})}^2 \right\| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\]

Integrating on \((0, t)\), by (3), (23), (30), (40) and (41), we have that

\[
\left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
\leq C_0 + C(T)t + C_0 \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds + C(T) \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
+ C_0 \left\| \partial_x u \right\|_{L^2((0, T) \times \mathbb{R})}^2 \int_0^t \left\| \partial_x u(s, \cdot) \right\|_{L^4(\mathbb{R})}^4 ds \\
+ C(T) \left\| \partial_x u \right\|_{L^\infty((0, T) \times \mathbb{R})}^2 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 ds \\
\leq C(T) \left( 1 + \left\| \partial_x u \right\|_{L^2((0, T) \times \mathbb{R})}^2 \right).
\]

We prove (48). Thanks to (22), (52) and the Hölder inequality,

\[
(\partial_x u(t, x))^2 = 2 \int_{-\infty}^x \partial_x u \partial_x^2 u dx \leq 2 \int_{\mathbb{R}} |\partial_x u \partial_x^2 u| dx.
\]
\[
\leq 2 \| \partial_x u(t, \cdot) \|_{L^2(\mathbb{R})} \| \partial_x^2 u(t, \cdot) \|_{L^2(\mathbb{R})} \leq C(T) \sqrt{1 + \| \partial_x u \|_{L^2((0, T) \times \mathbb{R})}^2}.
\]

Therefore,
\[
\| \partial_x u \|_{L^\infty((0, T) \times \mathbb{R})}^4 - C(T) \| \partial_x u \|_{L^2((0, T) \times \mathbb{R})}^2 \leq C(T) \leq 0,
\] (53)
which gives (48).

Finally, (49) follows from (48) and (52).  \(\square\)

**Lemma 6.** Fix \(T > 0\). There exists a constant \(C(T) > 0\), such that
\[
\left\| \partial_x^4 u(t, \cdot) \right\|_{L^\infty((0, T) \times \mathbb{R})} \leq C(T),
\] (54)
\[
\left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \beta^2 \int_0^t \left\| \partial_x^2 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T),
\] (55)
for every \(0 \leq t \leq T\).

**Proof.** Let \(0 \leq t \leq T\). We begin by observing that by (42), we have that
\[
\partial_x^4 \left( u^3 \right) = 36 (\partial_x u)^2 \partial_x^2 u + 18 u (\partial_x^2 u)^2 + 6 u \partial_x \partial_x^3 u + 3 u^2 \partial_x^4 u.
\] (56)

Multiplying (1) by \(-2 \partial_x^6 u\), thanks to (56), an integration on \(\mathbb{R}\) gives
\[
\frac{d}{dt} \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -2 \int \partial_x^2 u \partial_t u \, dx
\]
\[
-4\lambda \int \partial_x u \partial_t \partial_x^6 u \, dx - 2\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
+ 2\lambda \int \partial_x^4 u \partial_t \partial_x^2 u \, dx + 2\beta^2 \int \partial_x^2 u \partial_x^6 u \left( u^3 \right) \, dx
\]
\[
= -4\lambda \int (\partial_x u)^2 \partial_x^2 u \, dx - 4\lambda \int \partial_x^2 u \partial_x^3 u \, dx - 2\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
-2\lambda \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 72\beta^2 \int (\partial_x u)^3 \partial_x^2 u \, dx
\]
\[
+ 36\beta^2 \int u (\partial_x^2 u)^2 \partial_x^6 u \, dx + 12\beta^2 \int u \partial_x \partial_x^3 u \partial_x^2 u \, dx
\]
\[
+ 6\beta^2 \int u^2 \partial_x^4 u \partial_x^2 u \, dx.
\]

Therefore, we have
\[
\frac{d}{dt} \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
= -4\lambda \int (\partial_x u)^2 \partial_x^2 u \, dx - 4\lambda \int \partial_x^2 u \partial_x^3 u \, dx - 2\lambda \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2
\]
\[
+ 72\beta^2 \int (\partial_x u)^3 \partial_x^2 u \, dx + 36\beta^2 \int u (\partial_x^2 u)^2 \partial_x^6 u \, dx
\]
\[
+ 12\beta^2 \int u \partial_x \partial_x^3 u \partial_x^2 u \, dx + 6\beta^2 \int u^2 \partial_x^4 u \partial_x^2 u \, dx.
\] (57)

Due to (22), (29), (48), (49) and the Young inequality,
\[
4|\lambda| \int (\partial_x u)^2 \partial_x^2 u \, dx \leq 4|\lambda| \left\| \partial_x u \right\|_{L^\infty((0, T) \times \mathbb{R})} \int \left\| \partial_x u \right\|_{L^2(\mathbb{R})} \, dx
\]
\[
\leq C(T) \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + C(T) \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2,
\]
\[ \leq C(T) + C(T) \left\| \partial_x^3 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}, \]

\[ 4|\kappa| \int_{\mathbb{R}} |u| |\partial_x^3 u| |\partial_x^5 u| \, dx \leq 4|\kappa| \left\| u \right\|_{L^\infty((0, T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^5 u| \, dx \]

\[ \leq C(T) \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + C(T) \left\| \partial_x^5 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]

\[ \leq C(T) + C(T) \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}, \]

\[ 72\delta^2 \int_{\mathbb{R}} |\partial_x u|^3 |\partial_x^3 u| |\partial_x^6 u| \, dx \leq 72\delta^2 \left\| \partial_x u \right\|_{L^\infty((0, T) \times \mathbb{R})}^3 \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^6 u| \, dx \]

\[ \leq 2C(T) \int_{\mathbb{R}} \left\| \partial_x^3 u \right\|^2_{L^1(\mathbb{R})} \left\| \beta \sqrt{D_2} \partial_x^6 u \right\| \, dx \]

\[ \leq \frac{C(T)}{D_2} \left\| \partial_x^3 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \beta^2 D_2 \left\| \partial_x^6 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]

\[ \leq \frac{C(T)}{D_2} + \beta^2 D_2 \left\| \partial_x^6 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}, \]

\[ 36\delta^2 \int_{\mathbb{R}} |u| |(\partial_x^2 u)^2| |\partial_x^6 u| \, dx \leq 36\delta^2 \left\| u \right\|_{L^\infty((0, T) \times \mathbb{R})} \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^6 u| \, dx \]

\[ \leq 2C(T) \int_{\mathbb{R}} (\partial_x^2 u)^2 |\partial_x^6 u| \, dx = 2 \int_{\mathbb{R}} \left( \frac{C(T)(\partial_x^2 u)^2}{\beta \sqrt{D_2}} \right) \left\| \beta \sqrt{D_2} \partial_x^6 u \right\| \, dx \]

\[ \leq \frac{C(T)}{D_2} \left\| \partial_x^3 u \right\|^2_{L^1(\mathbb{R})} \left\| \partial_x^3 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \beta^2 D_2 \left\| \partial_x^6 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]

\[ 12\delta^2 \int_{\mathbb{R}} |u| |\partial_x u| |\partial_x^2 u| |\partial_x^6 u| \, dx \leq 12\delta^2 \left\| u \right\|_{L^\infty((0, T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_x^2 u| |\partial_x^6 u| \, dx \]

\[ \leq 2C(T) \int_{\mathbb{R}} |\partial_x u| |\partial_x^2 u| |\partial_x^6 u| \, dx \leq 2C(T) \left\| \partial_x u \right\|_{L^\infty((0, T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^2 u| |\partial_x^6 u| \, dx \]

\[ \leq 2C(T) \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^6 u| \, dx = 2 \int_{\mathbb{R}} \left( \frac{C(T)(\partial_x^3 u)^2}{\beta \sqrt{D_2}} \right) \left\| \beta \sqrt{D_2} \partial_x^6 u \right\| \, dx \]

\[ \leq \frac{C(T)}{D_2} \left\| \partial_x^3 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \beta^2 D_2 \left\| \partial_x^6 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}, \]

\[ 6\delta^2 \int_{\mathbb{R}} u^2 |\partial_x^3 u| |\partial_x^6 u| \, dx \leq 6\delta^2 \left\| u \right\|_{L^\infty((0, T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^6 u| \, dx \]

\[ \leq 2C(T) \int_{\mathbb{R}} |\partial_x^3 u| |\partial_x^6 u| \, dx \leq 2 \int_{\mathbb{R}} \left( \frac{C(T)(\partial_x^3 u)^2}{\beta \sqrt{D_2}} \right) \left\| \beta \sqrt{D_2} \partial_x^6 u \right\| \, dx \]

\[ \leq \frac{C(T)}{D_2} \left\| \partial_x^3 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \beta^2 D_2 \left\| \partial_x^6 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}, \]

where $D_2$ is a positive constant which will specified later. It follows from (57) that

\[ \frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + 2\beta^2 (1 - 2D_2) \left\| \partial_x^6 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]

\[ \leq C(T) \left( 1 + \frac{1}{D_2} \right) + C(T) \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]

\[ + C(T) \left\| \partial_x^2 u \right\|^2_{L^\infty((0, T) \times \mathbb{R})} \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \frac{C(T)}{D_2} \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \]

\[ + \frac{C(T)}{D_2} \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}. \]
Taking $D_2 = \frac{1}{2}$, we have that
\[
\frac{d}{dt} \left\| \partial_x^3 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \beta^2 \left\| \partial_x^6 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
\leq C(T) + C(T) \left\| \partial_x^5 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + C(T) \left\| \partial_x^5 u \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \\
+ C(T) \left\| \partial_x^4 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + C(T) \left\| \partial_x^4 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})}.
\]
(3), (23), (40), (41), (49) and an integration on $(0, t)$ give
\[
\frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \beta^2 \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds \\
\leq C_0 + C(T) t + C(T) \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds \\
+ C(T) \int_0^t \left\| \partial_x^5 u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds + C(T) \int_0^t \left\| \partial_x^4 u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds \\
+ C(T) \left\| \partial_x^2 u \right\|^2_{L^\infty((0,T) \times \mathbb{R})} + C(T) \left\| \partial_x^2 u \right\|^2_{L^2(\mathbb{R})} ds \\
\leq C(T) \left( 1 + \left\| \partial_x^2 u \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \right).
\]
We prove (54). Thanks to (49), (58) the Hölder inequality,
\[
(\partial_x^2 u(t, x))^2 = 2 \int_{-\infty}^\infty \partial_x^2 u \partial_x^4 u dy \leq 2 \int_{\mathbb{R}} \left\| \partial_x^2 u \right\| \left\| \partial_x^4 u \right\| dx \\
\leq 2 \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})} \left\| \partial_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})} \leq C(T) \sqrt{\left( 1 + \left\| \partial_x^2 u \right\|^2_{L^\infty((0,T) \times \mathbb{R})} \right)}.
\]
Therefore,
\[
\left\| \partial_x^2 u \right\|^4_{L^\infty((0,T) \times \mathbb{R})} - C(T) \left\| \partial_x^2 u \right\|^2_{L^\infty((0,T) \times \mathbb{R})} - C(T) \leq 0,
\]
which gives (54).

Finally, (55) follows from (54) and (58).

**Lemma 7.** Fix $T > 0$. There exists a constant $C(T) > 0$, such that
\[
\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\| \partial_t u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds \leq C(T),
\]
for every $0 \leq t \leq T$.

**Proof.** Let $0 \leq t \leq T$. Multiplying (1) by $2\partial_t u$, we have
\[
2(\partial_t u) + 4\kappa u \partial_x u \partial_t u - 2\beta^2 \partial_x^6 u \partial_t u + 2\alpha \partial_x^4 u + 2\delta \partial_x^2 u \partial_x^4 \left( u^3 \right).
\]
Since
\[
-2\beta^2 \int_{\mathbb{R}} \partial_x^6 u \partial_t u dx = \beta^2 \frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})},
\]
\[
2\alpha \int_{\mathbb{R}} \partial_x^4 u \partial_t u dx = \alpha \frac{d}{dt} \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})},
\]

\[
\beta^2 \left\| \partial_x^2 u(t, \cdot) \right\|^2_{L^2(\mathbb{R})} + \int_0^t \left\| \partial_t u(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds \leq C(T).
\]
Due to (22), (29), (48), (49), (54), (55) and the Young inequality,

\[
\begin{align*}
\frac{d}{dt} \left( \beta^2 \left\| \partial_x^3 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \partial_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + 2 \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) &= -4K \int_{\mathbb{R}} u \partial_x u \partial_t u dx - 2\delta^2 \int_{\mathbb{R}} \partial_t u \partial_x^3 u^3 dx \\
&= -4K \int_{\mathbb{R}} u \partial_x u \partial_t u dx - 2\delta^2 \int_{\mathbb{R}} (\partial_x u)^2 \partial_t u dx - 36\delta^2 \int_{\mathbb{R}} u (\partial_x^2 u)^2 \partial_t u dx \\
&\quad - 12\delta^2 \int_{\mathbb{R}} u \partial_x u \partial_x^3 u \partial_t u dx.
\end{align*}
\]

Due to (22), (29), (48), (49), (54), (55) and the Young inequality,

\[
\begin{align*}
4|\mathbb{T}| \int_{\mathbb{R}} |u| |\partial_x u| |\partial_t u| dx &= 4|\mathbb{T}| \left\| u \right\|_{L^\infty((0,T) \times \mathbb{R})} \int_{\mathbb{R}} |\partial_x u| |\partial_t u| dx \\
&\leq 2C(T) \int_{\mathbb{R}} |\partial_x u| |\partial_t u| dx \\
&\leq \frac{C(T)}{D_3} \left\| \partial_x u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + D_3 \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
&\leq \frac{C(T)}{D_3} + D_3 \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
\end{align*}
\]
where $D_3$ is a positive constant, which will be specified later. Therefore, by (61),

$$
\frac{d}{dt} \left( \beta^2 \left\| \dot{\partial}_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \dot{\partial}_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + (2 - 5D_3) \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq \frac{C(T)}{D_3} + \frac{C(T)}{D_3} \left\| \dot{\partial}_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
$$

Taking $D_3 = \frac{1}{5}$, we have that

$$
\frac{d}{dt} \left( \beta^2 \left\| \dot{\partial}_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \dot{\partial}_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \right) + \left\| \partial_t u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) + C(T) \left\| \dot{\partial}_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2.
$$

It follows from (3), (41) and an integration on $(0, t)$ that

$$
\beta^2 \left\| \dot{\partial}_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \alpha \left\| \dot{\partial}_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \\
\leq C_0 + C(T) t + C(T) \int_0^t \left\| \dot{\partial}_x^4 u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T).
$$

Therefore, by (49),

$$
\beta^2 \left\| \dot{\partial}_x^4 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 + \int_0^t \left\| \partial_t u(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \\
\leq C(T) - \alpha \left\| \dot{\partial}_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \\
\leq C(T) + |\alpha| \left\| \dot{\partial}_x^2 u(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 \leq C(T),
$$

which gives (59). $\Box$

Lemma 8. Fix $T > 0$. There exists a constant $C(T) > 0$, such that

$$
\int_0^t \left\| \partial_t P(s, \cdot) \right\|_{L^2(\mathbb{R})}^2 \, ds \leq C(T), \quad (62)
$$

for every $0 \leq t \leq T$.

Proof. Let $0 \leq t \leq T$. Multiplying (18) by $2\partial_t P$, thanks to (42), an integration on $\mathbb{R}$ gives

$$
2 \left\| \partial_t P(t, \cdot) \right\|_{L^2(\mathbb{R})}^2 = -4k \int_{\mathbb{R}} u^2 \partial_t P \, dx + 2\beta^2 \int_{\mathbb{R}} \dot{\partial}_x^4 u \partial_t P \, dx - 2\kappa \int_{\mathbb{R}} \dot{\partial}_x^2 u \partial_t P \, dx \\
- 12\delta^2 \int_{\mathbb{R}} \partial_x^3 u \partial_t P \, dx - 36\delta^2 \int_{\mathbb{R}} u \partial_x^2 \partial_t^2 u \partial_t P \, dx \\
- 6\delta^2 \int_{\mathbb{R}} u^2 \partial_x^2 u \partial_t P \, dx. \quad (63)
$$
Due to (22), (29), (48), (49), (55) and the Young inequality,

\[ 4|\kappa| \int_R u^2|\partial_x P|dx = 2 \int_R \left| \frac{2\kappa u^2}{D_4} \right| \sqrt{D_4} \partial_t P \cdot dx \]

\[ \leq \frac{4\kappa^2}{D_4} \| u(t, \cdot) \|^4_{L^4(\mathbb{R})} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})} \]

\[ \leq \frac{C(T)}{D_4} \| u(t, \cdot) \|^4_{L^4(\mathbb{R})} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})} \]

\[ \leq \frac{C(T)}{D_4} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})}, \]

\[ + 2\beta^2 \int_R |\partial^5_x u| |\partial_t P|dx = 2 \int_R \left| \frac{\beta^2 \partial^5_x u}{\sqrt{D_4}} \right| \sqrt{D_4} \partial_t P \cdot dx \]

\[ \leq \frac{\beta^4}{D_4} \| \partial^5_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})}, \]

\[ 2|\alpha| \int_R |\partial^3_x u| |\partial_t P|dx = 2 \int_R \left| \frac{\alpha \partial^3_x u}{\sqrt{D_4}} \right| \sqrt{D_4} \partial_t P \cdot dx \]

\[ \leq \frac{\alpha^2}{D_4} \| \partial^3_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})} \]

\[ \leq \frac{C(T)}{D_4} \| \partial^3_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})} \]

\[ \leq \frac{C(T)}{D_4} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})}, \]

\[ 12\beta^2 \int_R |\partial_x u|^3 |\partial_t P|dx = 12\beta^2 \| \partial_x u \|^2_{L^\infty((0,T) \times \mathbb{R})} \int_R |\partial_x u| |\partial_t P| \cdot dx \]

\[ \leq 2C(T) \int_R |\partial_x u| |\partial_t P|dx = 2 \int_R \left| \frac{C(T)\partial_x u}{\sqrt{D_4}} \right| \sqrt{D_4} \partial_t P \cdot dx \]

\[ \leq \frac{C(T)}{D_4} \| \partial_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})} \]

\[ \leq \frac{C(T)}{D_4} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})}, \]

\[ 36\kappa^2 \int_R |u| |\partial_x u| |\partial^2_x u| |\partial_t P|dx \leq 36\kappa^2 \| u \|_{L^\infty((0,T) \times \mathbb{R})} \int_R |\partial_x u| |\partial^2_x u| |\partial_t P| \cdot dx \]

\[ \leq 2C(T) \int_R |\partial_x u| |\partial^2_x u| |\partial_t P|dx \leq 2C(T) \| \partial_x u \|_{L^\infty((0,T) \times \mathbb{R})} \int_R |\partial^2_x u| |\partial_t P| \cdot dx \]

\[ \leq 2C(T) \int_R |\partial^2_x u| |\partial_t P|dx = 2 \int_R \left| \frac{C(T)\partial^2_x u}{\sqrt{D_4}} \right| \sqrt{D_4} \partial_t P \cdot dx \]

\[ \leq \frac{C(T)}{D_4} \| \partial^2_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})} \]

\[ \leq \frac{C(T)}{D_4} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})}, \]

\[ 6\alpha^2 \int_R |u|^2 |\partial^3_x u| |\partial_t P|dx \leq 6\alpha^2 \| u \|^2_{L^\infty((0,T) \times \mathbb{R})} \int_R |\partial^3_x u| |\partial_t P| \cdot dx \]

\[ \leq 2C(T) \int_R |\partial^3_x u| |\partial_t P|dx = 2 \int_R \left| \frac{C(T)\partial^3_x u}{D_4} \right| D_4 \partial_t P \cdot dx \]

\[ \leq \frac{C(T)}{D_4} \| \partial^3_x u(t, \cdot) \|^2_{L^2(\mathbb{R})} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})} \]

\[ \leq \frac{C(T)}{D_4} + D_4 \| \partial_t P(t, \cdot) \|^2_{L_2(\mathbb{R})}. \]
It follows from (63) that
\[ 2 \left(1 - 3D_6\right) \| \partial_t P(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)}{D_4} + \frac{\beta^2}{D_4} \| \partial_t^3 P(t, \cdot) \|_{L^2(\mathbb{R})}^2. \]

Taking \( D_4 = \frac{1}{\beta} \), we have that
\[ \| \partial_t P(t, \cdot) \|_{L^2(\mathbb{R})} \leq C(T) + 6\beta^4 \| \partial_t^3 P(t, \cdot) \|_{L^2(\mathbb{R})}^2. \] (64)

Integrating on \((0, t)\), by (49), we obtain
\[ \int_0^t \| \partial_t P(s, \cdot) \|_{L^2(\mathbb{R})} ds \leq C(T) t + 6\beta^4 \int_0^t \| \partial_t^3 P(s, \cdot) \|_{L^2(\mathbb{R})}^2 ds \leq C(T), \]
which gives (62). \( \square \)

3. Proof of Theorem 1

This section devoted to the proof of Theorem 1.

**Proof of Theorem 1.** Fix \( T > 0 \). Thanks to Lemmas 3, 4, 5, 6, and the Cauchy-Kovalevskaya Theorem [67], we have that \( u \) is solution of (1) and (8) holds. In particular, by Lemma 1, we get (9). Moreover, by Lemmas 2, 3, 4, 5, 6, and Remark 1, we have that
\[ P \in H^1((0, T) \times \mathbb{R}) \cap L^\infty(0, T; H^4(\mathbb{R})). \]

We prove (10). Let \( u_1 \) and \( u_2 \) be two solutions of (1), which verify (8), which is

\[
\begin{align*}
\partial_t u_1 + \kappa \partial_x u_1^2 - \beta^2 \partial_x^6 u_1 + \alpha \partial_x^4 u_1 + \beta^2 \partial_x^4 \left( u_1^3 \right) &= 0, \quad t > 0, \ x \in \mathbb{R}, \\
u_1(0, x) &= u_{1,0}(x), \quad x \in \mathbb{R}, \\
\partial_t u_2 + \kappa \partial_x u_2^2 - \beta^2 \partial_x^6 u_2 + \alpha \partial_x^4 u_2 + \beta^2 \partial_x^4 \left( u_2^3 \right) &= 0, \quad t > 0, \ x \in \mathbb{R}, \\
u_2(0, x) &= u_{2,0}(x), \quad x \in \mathbb{R},
\end{align*}
\]

Then, the function
\[ \omega = u_1 - u_2 \] (65)

is the solution of the following Cauchy problem:
\[
\begin{align*}
\partial_t \omega + \kappa \partial_x \left( u_1^2 - u_2^2 \right) - \beta^2 \partial_x^6 \omega + \alpha \partial_x^4 \omega + \beta^2 \partial_x^4 \left( u_1^3 - u_2^3 \right) &= 0, \quad t > 0, \ x \in \mathbb{R}, \\
\omega(0, x) &= u_{1,0}(x) - u_{2,0}(x), \quad x \in \mathbb{R}.
\end{align*}
\] (66)

Since
\[
\begin{align*}
2 \int_\mathbb{R} \omega \partial_t \omega \, dx &= \frac{d}{dt} \| \omega(t, \cdot) \|^2_{L^2(\mathbb{R})}, \\
2k \int_\mathbb{R} \partial_x \left( u_1^2 - u_2^2 \right) \omega \, dx &= -2k \int_\mathbb{R} \left( u_1^2 - u_2^2 \right) \partial_x \omega \, dx, \\
-2\beta^2 \int_\mathbb{R} \omega \partial_x^6 \omega \, dx &= 2\beta^2 \| \partial_x^3 \omega(t, \cdot) \|^2_{L^2(\mathbb{R})}, \\
2\alpha \int_\mathbb{R} \omega \partial_x^4 \omega \, dx &= -2\alpha \int_\mathbb{R} \partial_x^4 \omega \partial_x \omega \, dx, \\
2\beta^2 \int_\mathbb{R} \omega \partial_x^4 \left( u_1^3 - u_2^3 \right) \, dx &= -2\beta^2 \int_\mathbb{R} \partial_x^4 \omega \partial_x \left( u_1^3 - u_2^3 \right) \, dx,
\end{align*}
\]
multiplying (66) by $2\omega$, an integration on $\mathbb{R}$ gives

$$
\frac{d}{dt} \|\omega(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \|\partial_x^3 \omega(t, \cdot)\|^2_{L^2(\mathbb{R})} = 2\kappa \int_{\mathbb{R}} (u_1^2 - u_2^2) \partial_x \omega dx - 2\alpha \int_{\mathbb{R}} \partial_x \omega \partial_x^2 \omega dx + 2\delta^2 \int_{\mathbb{R}} \partial_x^3 \omega \partial_x (u_1^3 - u_2^3) \, dx. 
$$

(67)

Observe that thanks to (65),

$$
u_1^2 - u_2^2 = (u_1 + u_2)\omega, \quad u_1^3 - u_2^3 = (u_1^2 + u_2 + u_1 u_2)\omega.
$$

(68)

Consequently, by (67),

$$
\frac{d}{dt} \|\omega(t, \cdot)\|^2_{L^2(\mathbb{R})} + 2\beta^2 \|\partial_x^3 \omega(t, \cdot)\|^2_{L^2(\mathbb{R})} 
= 2\kappa \int_{\mathbb{R}} (u_1 + u_2)\omega \partial_x \omega dx - 2\alpha \int_{\mathbb{R}} \partial_x \omega \partial_x^2 \omega dx 
+ 2\alpha \int_{\mathbb{R}} \partial_x^3 \omega \partial_x (u_1^2 + u_2 + u_1 u_2) \omega dx 
+ 2\delta^2 \int_{\mathbb{R}} \partial_x^3 \omega \partial_x (u_1^2 + u_2 + u_1 u_2) \omega dx.
$$

(69)

Observe that since $u_1, u_2 \in H^3(\mathbb{R})$, for every $0 \leq t \leq T$, we have that

$$
\|u_1\|_{L^\infty([0,T] \times \mathbb{R})}, \|\partial_x u_1\|_{L^\infty([0,T] \times \mathbb{R})} \leq C(T),
\|u_2\|_{L^\infty([0,T] \times \mathbb{R})}, \|\partial_x u_2\|_{L^\infty([0,T] \times \mathbb{R})} \leq C(T).
$$

(70)

Consequently, thanks to (70), we obtain that

$$
|\partial_x u_1 + \partial_x u_2| \leq C(T),
|2u_1 \partial_x u_1 + 2u_2 \partial_x u_2 + u_2 \partial_x u_1 + u_1 \partial_x u_2| \leq C(T),
|u_1^2 + u_2 + u_1 u_2| \leq C(T).
$$

(71)

Due to (71) and the Young inequality,

$$
|\kappa| \int_{\mathbb{R}} |\partial_x u_1 + \partial_x u_2| \omega^2 \, dx \leq C(T) \|\omega(t, \cdot)\|^2_{L^2(\mathbb{R})},
2|\alpha| \int_{\mathbb{R}} |\partial_x \omega| |\partial_x^3 \omega| \, dx \leq \int_{\mathbb{R}} \left| \frac{2\alpha \partial_x \omega}{\beta} \right| \left| \beta \partial_x^2 \omega \right| \, dx \leq \frac{2\alpha^2}{\beta^2} \|\partial_x \omega(t, \cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \|\partial_x^3 \omega(t, \cdot)\|^2_{L^2(\mathbb{R})},
2\delta^2 \int_{\mathbb{R}} |u_1^2 + u_2 + u_1 u_2| |\omega| |\partial_x^3 \omega| \, dx \leq C(T) \int_{\mathbb{R}} |\omega| |\partial_x^3 \omega| \, dx = \int_{\mathbb{R}} \left| \frac{C(T) \omega}{\beta} \right| \left| \beta \partial_x^2 \omega \right| \, dx \leq C(T) \|\omega(t, \cdot)\|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{2} \|\partial_x^3 \omega(t, \cdot)\|^2_{L^2(\mathbb{R})},
2\delta^2 \int_{\mathbb{R}} |u_1^2 + u_2 + u_1 u_2| \|\partial_x \omega\| \|\partial_x^3 \omega\| \, dx
$$
Taking

Therefore, by the Young inequality,

\[ C(T) \| \partial_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C(T) \int_{\mathbb{R}} \| \partial_x \omega \|_{L^2(\mathbb{R})}^2 dx = -C(T) \int_{\mathbb{R}} \omega \partial^2_x \omega dx. \]

Therefore, by the Young inequality,

\[ C(T) \| \partial_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq C(T) \int_{\mathbb{R}} \frac{\omega}{\sqrt{D_5}} \left| \sqrt{D_5} \partial^2_x \omega \right| dx \]

\[ \leq \frac{C(T)D_5}{2} \| \partial^2_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{C(T)D_5}{2} \| \partial^3_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2, \]

where \( D_5 \) is a positive constant, which will be specified later. Observe again that

\[ \frac{C(T)D_5}{2} \| \partial^2_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 = C(T)D_5 \int_{\mathbb{R}} \partial^2_x \omega \partial^2_x \omega dx = -\frac{C(T)D_5}{2} \int_{\mathbb{R}} \partial_x \omega \partial^3_x \omega dx. \]

Consequently, by the Young inequality,

\[ \frac{C(T)D_5}{2} \| \partial^2_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)D_5}{4D_6} \int_{\mathbb{R}} \left| \frac{\partial_x \omega}{\sqrt{D_6}} \right| \sqrt{D_6} \partial^3_x \omega dx \]

\[ \leq \frac{C(T)D_5}{4D_6} \| \partial^2_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{C(T)D_6D_6}{4} \| \partial^3_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2, \]

where \( D_6 \) is a positive constant, which will be specified later. It follows from (73) and (74) that

\[ C(T) \left( 1 - \frac{D_5}{4D_6} \right) \| \partial_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)D_5}{2D_5} \| \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{C(T)D_5D_5}{4D_6} \| \partial^3_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2. \]

Taking

\[ D_5 = 2D_6, \]

We have

\[ \frac{C(T)}{2} \| \partial_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)}{4D_6} \| \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \frac{C(T)D_6^2}{2} \| \partial^3_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2. \]

Therefore,

\[ C(T) \| \partial_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq \frac{C(T)}{D_6} \| \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 + C(T)D_6^2 \| \partial^3_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2. \]  

It follows from (72) and (75) that

\[ \frac{d}{dt} \| \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 + \left( \frac{\beta^2}{2} - C(T)D_6^2 \right) \| \partial^2_x \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2 \leq \frac{1}{D_6} \| \omega(t, \cdot) \|_{L^2(\mathbb{R})}^2. \]
Choosing \( D_6 = \frac{|\beta|}{\sqrt{3C(T)}} \), we have that
\[
\frac{d}{dt} \| \omega(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\beta^2}{6} \left\| \partial_x^3 \omega(t, \cdot) \right\|^2_{L^2(\mathbb{R})} \leq C(T) \| \omega(t, \cdot) \|^2_{L^2(\mathbb{R})}.
\]

The Gronwall Lemma and (66) gives
\[
\| \omega(t, \cdot) \|^2_{L^2(\mathbb{R})} + \frac{\beta^2 e^{C(T)t}}{6} \int_0^t e^{-C(T)s} \left\| \partial_x^3 \omega(s, \cdot) \right\|^2_{L^2(\mathbb{R})} ds \leq e^{C(T)t} \| \omega_0 \|^2_{L^2(\mathbb{R})}.
\]

(10) follows from (65) and (76).

4. Conclusions

We considered the high order convective Cahn-Hilliard type equations that describe the faceting of a growing surface, or the dynamics of phase transitions in ternary oil-water-surfactant systems. We proved the well-posedness of the We considered the Cauchy problem and proved its well-posedness when the initial condition has zero mean and belongs to \( H^3 \).

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