On intermediate subfactors of Goodman-de la Harpe-Jones subfactors

FENG XU*
Department of Mathematics
University of California at Riverside
Riverside, CA 92521
E-mail: xufeng@math.ucr.edu

Abstract

In this paper we present a conjecture on intermediate subfactors which is a generalization of Wall’s conjecture from the theory of finite groups. Motivated by this conjecture, we determine all intermediate subfactors of Goodman-Harpe-Jones subfactors, and as a result we verify that Goodman-Harpe-Jones subfactors verify our conjecture. Our result also gives a negative answer to a question motivated by a conjecture of Aschbacher-Guralnick.

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1 Introduction

Let $M$ be a factor represented on a Hilbert space and $N$ a subfactor of $M$ which is irreducible, i.e., $N' \cap M = \mathbb{C}$. Let $K$ be an intermediate von Neumann subalgebra for the inclusion $N \subseteq M$. Note that $K' \cap K \subseteq N' \cap M = \mathbb{C}$, $K$ is automatically a factor. Hence the set of all intermediate subfactors for $N \subseteq M$ forms a lattice under two natural operations $\land$ and $\lor$ defined by:

$$K_1 \land K_2 = K_1 \cap K_2, \quad K_1 \lor K_2 = (K_1 \cup K_2)'' .$$

The commutant map $K \to K'$ maps an intermediate subfactor $N \subseteq K \subseteq M$ to $M' \subseteq K' \subseteq N'$. This map exchanges the two natural operations defined above.

Let $M \subseteq M_1$ be the Jones basic construction of $N \subseteq M$. Then $M \subseteq M_1$ is canonically anti-isomorphic to $M' \subseteq N'$, and the lattice of intermediate subfactors for $N \subseteq M$ is related to the lattice of intermediate subfactors for $M \subseteq M_1$ by the commutant map defined as above.

Let $G_1$ be a group and $G_2$ be a subgroup of $G_1$. An interval sublattice $[G_1/G_2]$ is the lattice formed by all intermediate subgroups $K, G_2 \subseteq K \subseteq G_1$.

By cross product construction and Galois correspondence, every interval sublattice of finite groups can be realized as intermediate subfactor lattice of finite index. Hence the study of intermediate subfactor lattice of finite index is a natural generalization of the study of interval sublattice of finite groups. The study of intermediate subfactors has been very active in recent years (cf. [9], [14], [23], [22], [20], [31], [41] and [38] for only a partial list). There are a number of old problems about interval sublattice of finite groups. It is therefore a natural programme to investigate if these old problems have any generalizations to subfactor setting. The hope is that maybe subfactor theory can provide new perspective on these old problems.

In [44] we consider the problem whether the very simple lattice $M_n$ consisting of a largest, a smallest and $n$ pairwise incomparable elements can be realized as subfactor lattice. We showed in [44] all $M_{2n}$ are realized as the lattice of intermediate subfactors of a pair of hyperfinite type III$_1$ factors with finite depth. Since it is conjectured that infinitely many $M_{2n}$ can not be realized as interval sublattices of finite groups (cf. [2] and [33]), our result shows that if one is looking for obstructions for realizing finite lattice as lattice of intermediate subfactors with finite index, then the obstruction is very different from what one may find in finite group theory.

In 1961 G. E. Wall conjectured that the number of maximal subgroups of a finite group $G$ is less than $|G|$, the order of $G$ (cf. [10]). In the same paper he proved his conjecture when $G$ is solvable. See [27] for more recent result on Wall’s conjecture.

Wall’s conjecture can be naturally generalized to a conjecture about maximal elements in the lattice of intermediate subfactors. More precisely, since $M$ is the maximal element under inclusion, what we mean by maximal elements are those subfactors $K \neq M, N$ with the property that if $K_1$ is an intermediate subfactor and $K \subseteq K_1$, then $K_1 = M$ or $K$. Minimal elements are defined similarly where $N, M$ are not considered as minimal elements. When $M$ is the cross product of $N$ by a
finite group $G$, the maximal elements correspond to maximal subgroups of $G$, and the order of $G$ is the dimension of second higher relative commutant. Hence a natural generalization of Wall’s conjecture is the following:

**Conjecture 1.1.** Let $N \subset M$ be an irreducible subfactor with finite index. Then the number of maximal intermediate subfactors is less than dimension of $N' \cap M_1$ (the dimension of second higher relative commutant of $N \subset M$).

We note that since maximal intermediate subfactors in $N \subset M$ correspond to minimal intermediate subfactors in $M \subset M_1$, and the dimension of second higher relative commutant remains the same, the conjecture is equivalent to a similar conjecture as above with maximal replaced by minimal.

If we take $N$ and $M$ to be cross products of a factor $P$ by $H$ and $G$ with $H$ a subgroup of $G$, then the above conjecture gives a generalization of Wall’s conjecture which we call relative version of Wall’s conjecture. The relative version of Wall’s conjecture states that the number of maximal subgroups of $G$ strictly containing a subgroup $H$ is less than $|G|/|H|$.

In the appendix we give a “subfactor friendly” proof of relative version of Wall’s conjecture when $G$ is solvable. We also discuss a question which is naturally motivated by a conjecture of Aschbacher-Guralnick. This question also partially motivates our work in this paper. A negative answer to this question is presented in §4.5.

When subfactors do not come from groups, with a few exceptions such as [14] and [44], very little is known about their maximal intermediate subfactors. To test conjecture 1.1 it is therefore desirable to determine lattices of intermediate subfactors for more examples of subfactors not coming from groups. As shown in [14], a rich source of such subfactors come from conformal field theories, and the techniques developed in [44] allow one to determine intermediate subfactors in many cases. In [14], as part of effort to classify subfactors with no extra structure, the intermediate subfactor lattice of a Goodman-Harpe-Jones (GHJ) subfactor (cf. [13]) is determined. Since the dual GHJ subfactors are closely related to conformal field theories based on Loop group $LSU(2)$, we can use the method of [44] to determine lattices of intermediate subfactors for GHJ subfactors (The idea is already presented in [44]). In this paper we carry out this idea. Compare to [44], the main difference is that we need to determine structure of a larger ring, but such rings have been determined in [43], [5], [8]. Combing these results we determine intermediate subfactor lattices for all GHJ subfactors. One interesting consequence of our work is that the intermediate subfactors of (dual) GHJ subfactors are again (dual) GHJ subfactors. Also as a result we do not find any counter examples to our conjecture 1.1 and we give a negative answer to question A.12. We also find several surprising intermediate subfactors which are not visible at first sight (cf. figures 22, 18, 41).

Since P. Grossman has proved that all non-commuting quadrilateral of subfactors with bottom two subfactors of type $A$ come form GHJ subfactor of type $D$ (cf. [15]), figures [23] [20] also determined all intermediate subfactor lattices of non-commuting quadrilateral of subfactors with bottom two subfactors of type $A$. 

3
Besides what is already described above, this paper is organized as follows: §2 is a preliminary section on sectors, representations of intermediate subfactors by a pair of sectors, sectors from conformal nets, inductions, Jones-Wassermann subfactors and a description of GHJ subfactors. The simple idea of representations of intermediate subfactors by a pair of sectors will prove to be crucial in later classifications. In §3 we first explain the basic idea in [44] to determine intermediate subfactors by fusion, and we carry out this idea for GHJ subfactors of type $A$, $D$ and $E$ respectively. In §4 we apply the results of §3 to determine the lattice relations of intermediate subfactors, and these lattices are listed.

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2 Preliminaries

For the convenience of the reader we collect here some basic notions that appear in this paper. This is only a guideline and the reader should look at the references such as preliminary sections of [25] for a more complete treatment.

2.1 Sectors

Let $M$ be a properly infinite factor and $\text{End}(M)$ the semigroup of unit preserving endomorphisms of $M$. In this paper $M$ will always be the unique hyperfinite $III_1$ factors. Let $\text{Sect}(M)$ denote the quotient of $\text{End}(M)$ modulo unitary equivalence in $M$. We denote by $[\rho]$ the image of $\rho \in \text{End}(M)$ in $\text{Sect}(M)$.

It follows from [28] and [29] that $\text{Sect}(M)$, with $M$ a properly infinite von Neumann algebra, is endowed with a natural involution $\theta \mapsto \bar{\theta}$; moreover, $\text{Sect}(M)$ is a semiring.

Let $\rho \in \text{End}(M)$ be a normal faithful conditional expectation $\epsilon : M \to \rho(M)$. We define a number $d_\epsilon$ (possibly $\infty$) by:

$$d_\epsilon^{-2} := \max \{ \lambda \in [0, +\infty) | \epsilon(m_+) \geq \lambda m_+, \forall m_+ \in M_+ \}$$

(cf. [PP]).

We define

$$d = \min \{ d_\epsilon | d_\epsilon < \infty \}.$$

$d$ is called the statistical dimension of $\rho$ and $d^2$ is called the Jones index of $\rho$. It is clear from the definition that the statistical dimension of $\rho$ depends only on the unitary equivalence classes of $\rho$. The properties of the statistical dimension can be found in [28], [29], and [30].

Denote by $\text{Sect}_0(M)$ those elements of $\text{Sect}(M)$ with finite statistical dimensions. For $\lambda, \mu \in \text{Sect}_0(M)$, let $\text{Hom}(\lambda, \mu)$ denote the space of intertwiners from $\lambda$ to $\mu$, i.e. $a \in \text{Hom}(\lambda, \mu)$ iff $a \lambda(x) = \mu(x)a$ for any $x \in M$. $\text{Hom}(\lambda, \mu)$ is a finite dimensional vector space and we use $\langle \lambda, \mu \rangle$ to denote the dimension of this space. $\langle \lambda, \mu \rangle$ depends
only on $[\lambda]$ and $[\mu]$. Moreover we have $\langle \nu \lambda, \mu \rangle = \langle \lambda, \nu \mu \rangle$, $\langle \nu \lambda, \mu \rangle = \langle \nu, \mu \lambda \rangle$ which follows from Frobenius duality (See [29]). We will also use the following notation: if $\mu$ is a subsector of $\lambda$, we will write as $\mu \prec \lambda$ or $\lambda \succ \mu$. A sector is said to be irreducible if it has only one subsector.

For any $\rho \in \text{End}(M)$ with finite index, there is a unique standard minimal inverse $\phi_\rho : M \rightarrow M$ which satisfies

$$\phi_\rho(\rho(m)m'\rho(m'')) = m\phi_\rho(m')m'', m, m', m'' \in M.$$  

$\phi_\rho$ is completely positive. If $t \in \text{Hom}(\rho_1, \rho_2)$ then we have

$$d_{\rho_1}\phi_{\rho_1}(mt) = d_{\rho_2}\phi_{\rho_2}(tm), m \in M$$  

(1)

### 2.2 Representation of intermediate subfactors by a pair of sectors

Let $M$ be an AFD type \textit{III}_1 factor and $\rho \in \text{End}(M)$. Let $K$ be a factor such that $\rho(M) \subset K \subset M$. Since $K$ is also AFD, one can choose $\rho_1 \in \text{End}(M)$ with $\rho_1(M) = K$. Then we have $\rho = \rho_1\rho_2$ with $\rho_2 = \rho_1^{-1}\rho \in \text{End}(M)$. Conversely if $\rho = \rho_1\rho_2$ with $\rho_1, \rho_2 \in \text{End}(M)$, then $\rho(M) \subset \rho_1(M) \subset M$. The following lemma follows directly from definitions:

**Lemma 2.1.** Suppose that $\rho_1\rho_2 = \sigma_1\sigma_2, \rho_i, \sigma_i \in \text{End}(M), i = 1, 2$ and $\rho_1(M) \subset \sigma_1(M)$. Then set $\sigma = \sigma_1^{-1}\rho_1 \in \text{End}(M)$ we have $\rho_1 = \sigma_1\sigma_2 = \sigma\rho_2$. Conversely if there is $\sigma \in \text{End}(M)$ such that $\rho_1 = \sigma_1\sigma, \sigma_2 = \sigma\rho_2$, then $\rho_1\rho_2 = \sigma_1\sigma_2$, and $\rho_1(M) \subset \sigma_1(M)$. In addition $\sigma$ is an automorphism if $\rho_1(M) = \sigma_1(M)$.

By Lemma 2.1 we can represent the intermediate subfactor of $\rho$ by pairs $\rho_1, \rho_2$ with $\rho_1\rho_2 = \rho$, and if $\sigma_1, \sigma_2$ represent the same intermediate subfactor iff there is an automorphism $\sigma$ of $M$ such that $\rho_1 = \sigma_1\sigma, \sigma_2 = \sigma\rho_2$. The next lemma shows that we can replace the pair $\rho_1, \rho_2$ by $[\rho_1], [\rho_2]$ when $\rho$ is irreducible:

**Lemma 2.2.** Suppose that $\rho = \rho_1\rho_2 = \sigma_1\sigma_2, \rho_i, \sigma_i \in \text{End}(M), [\rho_i] = [\sigma_i], i = 1, 2$. Then $\rho_1(M) = \sigma_1(M)$.

**Proof** By assumption we have unitaries $U_i \in M$ such that $\rho_i = \text{Ad}_{U_i}\sigma_i, i = 1, 2$. Since $\rho = \rho_1\rho_2 = \sigma_1\sigma_2$, we have $\rho = \text{Ad}_{U_1}\sigma_1(U_2)\rho$. Since $\rho$ is irreducible, $U_1\sigma_1(U_2)$ must be a scalar multiple of identity, and it follows that $\rho_1(M) = \text{Ad}_{U_1}\sigma_1(M) = \text{Ad}\sigma_1(U_2)\sigma_1(M) = \sigma_1(\text{Ad}_{U_2}(M)) = \sigma_1(M)$.

In view of Lemma 2.1 and Lemma 2.2 we introduce the following notation:

**Definition 2.3.** We say that two pairs of sectors $[\rho_1], [\rho_2]$ and $[\sigma_1], [\sigma_2]$ are equivalent if there is an automorphism $\sigma$ of $M$ such that $[\rho_1] = [\sigma_1\sigma], [\sigma_2] = [\sigma_2]$. We denote the equivalence class of such pair $[\rho_1], [\rho_2]$ by $[[\rho_1], [\rho_2]]$. When no confusion arises we will write $[[\rho_1], [\rho_2]]$ simply as $[\rho_1, \rho_2]$. 

5
The following follows from our definition, Lemma 2.1 and Lemma 2.2

**Corollary 2.4.** Let $\rho \in \text{End}(M)$ be irreducible. Then the set of intermediate subfactors between $M$ and $\rho(M)$ can be represented naturally by $[[\rho_1], [\rho_2]]$ such that $\rho_1 \rho_2 = \rho$, and the intermediate subfactor is $\rho_1(M)$. In the following we will denote the intermediate subfactor $\rho_1(M)$ simply by $[[\rho_1], [\rho_2]]$. Then $[[\rho_1], [\rho_2]] \subset [[\sigma_1], [\sigma_2]]$ as intermediate subfactors between $M$ and $\rho(M)$ iff there is $\sigma \in \text{End}(M)$ such that $[\rho_1] = [\sigma_1 \sigma], [\rho_2] = [\sigma \rho_2]$.

### 2.3 Symmetries of a subfactor

**Definition 2.5.** Let $\rho \in \text{End}(M)$. Define $\text{Aut}(\rho) := \{ \sigma \in \text{Aut}(M) | \sigma \rho = \rho \}$.

**Lemma 2.6.** If $\rho$ is irreducible and has finite index, then $\text{Aut}(\rho)$ is a finite group. There is a one to one correspondence between $\sigma \in \text{Aut}(\rho)$ and sector $[\sigma]$ of index one which appears in $[\rho \bar{\rho}]$, and $\sigma$ is constructed from $[\sigma]$ in the following way: if $[\sigma]$ is a sector of index one which appears in $[\rho \bar{\rho}]$, then we can find a unique representative $\bar{\rho} \in \text{Aut}(M)$ in the sector of $[\sigma]$ such that $\sigma \bar{\rho} = \rho$.

**Proof** Let $\sigma \in \text{Aut}(\rho)$. Since $\sigma \rho = \rho$, by Frobenius duality we have $\sigma \in \rho \bar{\rho}$, and $[\sigma]$ is a sector of index one which appears in $[\rho \bar{\rho}]$ with (necessarily) multiplicity one since $\rho$ is irreducible. One the other hand if $[\sigma]$ is a sector of index one which appears in $[\rho \bar{\rho}]$, then we can find a unique representative $\bar{\rho} \in \text{Aut}(M)$ in the sector of $[\sigma]$ such that $\sigma \bar{\rho} = \rho$.

**Definition 2.7.** Let $\bar{\rho} \in \text{End}(M)$. Denote by $U(M)$ (resp. $\rho(M)$) the group of unitaries in $M$ (resp. $\bar{\rho}(M)$). Let $U_1(M)$ be the subgroup of $U(M)$ which consists of unitaries in $M$ whose conjugate action on $M$ preserve $\rho(M)$ as a set. Note that $U(\rho(M))$ is a normal subgroup of $U_1(M)$. Define $N(\bar{\rho}) := \frac{U_1(M)}{U(\rho(M))}$.

**Lemma 2.8.** If $\rho$ is irreducible and has finite index, then $N(\bar{\rho})$ is a finite group isomorphic to $\text{Aut}(\rho)$. In fact there is a one to one correspondence between $U_\sigma \in N(\bar{\rho})$ and sector $[\sigma]$ of index one which appears in $[\rho \bar{\rho}]$, and $U_\sigma$ is constructed from $[\sigma]$ in the following way: if $[\sigma]$ is a sector of index one which appears in $[\rho \bar{\rho}]$, then we can find a unique representative $U_\sigma \in N(\bar{\rho})$ such that $\text{Ad}_{U_\sigma} \bar{\rho} = \rho$.

**Proof** Let $U$ be a unitary representative of an element in $N(\bar{\rho})$. Since $\text{Ad}_{U} \bar{\rho} = \rho \sigma$ for some automorphism $\sigma$ of $M$, by Frobenius duality we have $\sigma \in \rho \bar{\rho}$, and $[\sigma]$ is a sector of index one which appears in $[\rho \bar{\rho}]$ with (necessarily) multiplicity one since $\rho$ is irreducible. One the other hand if $[\sigma]$ is a sector of index one which appears in $[\rho \bar{\rho}]$, then we can find a unique representative $U_\sigma$ such that $\text{Ad}_{U_\sigma} \bar{\rho} = \rho \sigma$. Since $\rho$ is irreducible, the element with representative $U_\sigma$ in the group $N(\bar{\rho})$ must be unique. Since $\rho$ has finite index, there are only finitely many subsectors of $[\rho \bar{\rho}]$, and so $N(\bar{\rho})$ is finite and it is isomorphic to $\text{Aut}(\rho)$ by Lemma 2.6. ■
2.4 Sectors from conformal nets and their representations

We refer the reader to §3 of [25] for definitions of conformal nets and their representations. Suppose a conformal net \( \mathcal{A} \) and a representation \( \lambda \) is given. Fix an open interval \( I \) of the circle and let \( M := \mathcal{A}(I) \) be a fixed type \( III_1 \) factor. Then \( \lambda \) gives rise to an endomorphism still denoted by \( \lambda \) of \( M \). We will recall some of the results of [37] and introduce notations.

Suppose \( \{[\lambda]\} \) is a finite set of all equivalence classes of irreducible, covariant, finite-index representations of an irreducible local conformal net \( \mathcal{A} \). We will use \( \Delta_{\mathcal{A}} \) or simply \( \Delta \) to denote all finite index representations of net \( \mathcal{A} \) and will use the same notation \( \Delta_{\mathcal{A}} \) to denote the corresponding sectors of \( M \).

We will denote the conjugate of \([\lambda]\) by \( \overline{[\lambda]} \) and identity sector (corresponding to the vacuum representation) by \([0]\) if no confusion arises, and let \( N_{\lambda\mu}^\nu = \langle [\lambda][\mu],[\nu] \rangle \). Here \( \langle \mu, \nu \rangle \) denotes the dimension of the space of intertwiners from \( \mu \) to \( \nu \) (denoted by \( \text{Hom}(\mu, \nu) \)). We will denote by \( \{T_e\} \) a basis of isometries in \( \text{Hom}(\nu, \lambda\mu) \).

The univalence of \( \lambda \) and the statistical dimension of (cf. §2 of [16]) will be denoted by \( \omega_\lambda \) and \( d(\lambda) \) (or \( d_\lambda \)) respectively. The unitary braiding operator \( \epsilon(\mu, \lambda) \) (cf. [16]) verifies the following

**Proposition 2.9.** (1) Yang-Baxter-Equation (YBE)

\[
\epsilon(\mu, \gamma)\mu(\epsilon(\lambda, \gamma))\epsilon(\lambda, \mu) = \gamma(\epsilon(\lambda, \mu))\epsilon(\lambda, \gamma)\lambda(\epsilon(\mu, \gamma)).
\]

(2) Braiding-Fusion-Equation (BFE)

For any \( w \in \text{Hom}(\mu\gamma, \delta) \)

\[
\epsilon(\lambda, \delta)\lambda(w) = w\mu(\epsilon(\lambda, \gamma))\epsilon(\lambda, \mu),
\]

\[
\epsilon(\delta, \lambda)w = \lambda(w)\epsilon(\mu, \lambda)\mu(\epsilon(\gamma, \lambda))
\]

\[
\epsilon(\delta, \lambda)^*\lambda(w) = w\mu(\epsilon(\gamma, \lambda)^*)\epsilon(\mu, \lambda)^*
\]

\[
\epsilon(\lambda, \delta)^*\lambda(w) = w\mu(\epsilon(\gamma, \lambda)^*)\epsilon(\lambda, \mu)^*
\]

**Lemma 2.10.** If \( \lambda, \mu \) are irreducible, and \( t_\nu \in \text{Hom}(\nu, \lambda\mu) \) is an isometry, then

\[
t_\nu^*\epsilon(\mu, \lambda)\epsilon(\lambda, \mu)t_\nu = \frac{\omega_\lambda}{\omega_\lambda \omega_\mu}.
\]

By Prop. 2.9 it follows that if \( t_i \in \text{Hom}(\mu_i, \lambda) \) is an isometry, then

\[
\epsilon(\mu, \mu_i)\epsilon(\mu_i, \mu) = t_i^*\epsilon(\mu, \lambda)\epsilon(\lambda, \mu)t_i
\]

We shall always identify the center of \( M \) with \( \mathbb{C} \). Then we have the following

**Lemma 2.11.** If

\[
\epsilon(\mu, \lambda)\epsilon(\lambda, \mu) \in \mathbb{C},
\]

then

\[
\epsilon(\mu, \mu_i)\epsilon(\mu_i, \mu) \in \mathbb{C}, \forall \mu_i \prec \lambda.
\]
Let $\phi_\lambda$ be the unique minimal left inverse of $\lambda$, define:

$$Y_{\lambda\mu} := d(\lambda)d(\mu)\phi_\mu(\epsilon(\mu, \lambda)^*\epsilon(\lambda, \mu)^*),$$

where $\epsilon(\mu, \lambda)$ is the unitary braiding operator (cf. [16]).

We list two properties of $Y_{\lambda\mu}$ (cf. (5.13), (5.14) of [37]):

**Lemma 2.12.**

$$Y_{\lambda\mu} = Y_{\mu\lambda} = Y_{\lambda\mu}^* Y_{\lambda\mu}^{\bar{\mu}}.$$  

$$Y_{\lambda\mu} = \sum_k N_{\lambda\mu}^{\nu} \frac{\omega_\lambda\omega_\mu}{\omega_\nu} d(\nu).$$

We note that one may take the second equation in the above lemma as the definition of $Y_{\lambda\mu}$.

Define $a := \sum_i d_i^2 \omega_i^{-1}$. If the matrix $(Y_{\mu\nu})$ is invertible, by Proposition on P.351 of [37] $a$ satisfies $|a|^2 = \sum_\lambda d(\lambda)^2$.

**Definition 2.13.** Let $a = |a| \exp(-2\pi i c_0/24)$ where $c_0 \in \mathbb{R}$ and $c_0$ is well defined mod $8\mathbb{Z}$.

Define matrices

$$S := |a|^{-1} Y, T := C\text{Diag}(\omega_\lambda)$$

where

$$C := \exp(-2\pi i c_0/24).$$

Then these matrices satisfy (cf. [37]):

**Lemma 2.14.**

$$SS^\dagger = TT^\dagger = \text{id},$$

$$STS = T^{-1}ST^{-1},$$

$$S^2 = \hat{\mathcal{C}},$$

$$T\hat{\mathcal{C}} = \hat{\mathcal{C}}T,$$

where $\hat{\mathcal{C}}_{\lambda\mu} = \delta_{\lambda\bar{\mu}}$ is the conjugation matrix.

Moreover

$$N_{\lambda\mu}^\nu = \sum_\delta \frac{S_{\lambda\delta}S_{\mu\delta}S_{\nu\delta}^*}{S_{1\delta}^*}.$$  

is known as Verlinde formula. The commutative algebra generated by $\lambda$’s with structure constants $N_{\lambda\mu}^\nu$ is called **fusion algebra** of $\mathcal{A}$. If $Y$ is invertible, it follows from Lemma 2.14 (4) that any nontrivial irreducible representation of the fusion algebra is of the form $\lambda \rightarrow \frac{S_{\lambda\mu}}{S_{1\mu}}$ for some $\mu$. 

8
2.5 Induced endomorphisms

Suppose that $\rho \in \text{End}(M)$ has the property that $\gamma = \rho \bar{\rho} \in \Delta_A$. By §2.7 of [32], we can find two isometries $v_1 \in \text{Hom}(\gamma, \gamma^2), w_1 \in \text{Hom}(1, \gamma)$ such that $\bar{\rho}(M)$ and $v_1$ generate $M$ and

$$v_1^* w_1 = v_1^* \gamma(w_1) = d_{\rho}^{-1}$$

$$v_1 v_1 = \gamma(v_1) v_1$$

By Thm. 4.9 of [32], we shall say that $\rho$ is local if

$$v_1^* w_1 = v_1^* \gamma(w_1) = d_{\rho}^{-1}$$

$$v_1 v_1 = \gamma(v_1) v_1$$

Note that if $\rho$ is local, then

$$\omega_{\mu} = 1, \forall \mu < \rho \bar{\rho}$$

For each (not necessarily irreducible) $\lambda \in \Delta_A$, let $\varepsilon(\lambda, \gamma) : \lambda \gamma \rightarrow \gamma \lambda$ (resp. $\check{\varepsilon}(\lambda, \gamma)$), be the positive (resp. negative) braiding operator as defined in Section 1.4 of [43]. Denote by $\lambda_{\varepsilon} \in \text{End}(M)$ which is defined by

$$\lambda_{\varepsilon}(x) := ad(\varepsilon(\lambda, \gamma)) \lambda(x) = \varepsilon(\lambda, \gamma) \lambda(x) \varepsilon(\lambda, \gamma)^*$$

$$\lambda_{\check{\varepsilon}}(x) := ad(\check{\varepsilon}(\lambda, \gamma)) \lambda(x) = \check{\varepsilon}(\lambda, \gamma)^* \lambda(x) \check{\varepsilon}(\lambda, \gamma)^*, \forall x \in M.$$

By (1) of Theorem 3.1 of [43], $\lambda_{\varepsilon} \rho(M) \subset \rho(M), \lambda_{\check{\varepsilon}} \rho(M) \subset \rho(M)$, hence the following definition makes sense.

Definition 2.15. If $\lambda \in \Delta_A$ define two elements of $\text{End}(M)$ by

$$a_{\lambda}^\rho(m) := \rho^{-1}(\lambda_{\varepsilon} \rho(m)), \quad \check{a}_{\lambda}^\rho(m) := \rho^{-1}(\lambda_{\check{\varepsilon}} \rho(m)), \forall m \in M.$$

$a_{\lambda}^\rho$ (resp. $\check{a}_{\lambda}^\rho$) will be referred to as positive (resp. negative) induction of $\lambda$ with respect to $\rho$.

Remark 2.16. For simplicity we will use $a_{\lambda}, \check{a}_{\lambda}$ to denote $a_{\lambda}^\rho, \check{a}_{\lambda}^\rho$ when it is clear that inductions are with respect to the same $\rho$.

The endomorphisms $a_{\lambda}$ are called braided endomorphisms in [43] due to its braiding properties (cf. (2) of Corollary 3.4 in [43]), and enjoy an interesting set of properties (cf. Section 3 of [43]). Though [43] focus on the local case which was clearly the most interesting case in terms of producing subfactors, as observed in [31, 4], [5],

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1 We use $v_1, w_1$ instead of $v, w$ here since $v, w$ are used to denote sectors in Section 2.6.

2 We have changed the notations $a_{\lambda}, \check{a}_{\lambda}$ of [43] to $a_{\lambda}, \check{a}_{\lambda}$ of this paper to make some of the formulas such as equation (13) simpler.
that many of the arguments in [43] can be generalized. These properties are also
studied in a slightly different context in [3], [4], [5]. In these papers, the induction
is between $M$ and a subfactor $N$ of $M$, while the induction above is on the same
algebra. A dictionary between our notations here and these papers has been set up
in [45] which simply use an isomorphism between $N$ and $M$. Here one has a choice
to use this isomorphism to translate all endomorphisms of $N$ to endomorphisms of $M$, or
equivalently all endomorphisms of $M$ to endomorphisms of $N$. In [45] the later choice
is made (Hence $M$ in [45] will be our $N$ below). Here we make the first choice which
makes the dictionary slightly simpler. Our dictionary here is equivalent to that of
[45]. Set $N = \bar{\rho}(M)$. In the following the notations from [3] will be given a subscript
BE. The formulas are:

\[ \rho \mid N = i_{BE}, \bar{\rho} \mid N = \bar{i}_{BE} \bar{i}_{BE}, \]
\[ \gamma = \bar{\rho}^{-1}\theta_{BE} \bar{\rho}, \bar{\rho} \gamma = \gamma_{BE}, \]
\[ \lambda = \bar{\rho}^{-1} \lambda_{BE} \bar{\rho}, \varepsilon(\lambda, \mu) = \bar{\rho}(\varepsilon_{BE}(\lambda_{BE}, \mu_{BE})) \]
\[ \varepsilon(\lambda, \mu) = \bar{\rho}(\varepsilon_{BE}(\lambda_{BE}, \mu_{BE})) \]

The dictionary between $a_\lambda \in \text{End}(M)$ in definition 2.15 and $\alpha_\lambda^-$ as in Definition 3.3,
Definition 3.5 of [3] are given by:

\[ a_\lambda = \alpha_{\lambda_{BE}}^+, \bar{a}_\lambda = \alpha_{\lambda_{BE}}^- \]

The above formulas will be referred to as our dictionary between the notations of [43]
and that of [3]. The proof is the same as that of [45]. Using this dictionary one can
easily translate results of [43] into the settings of [3], [4], [5], [6], [7], [8] and vice versa.
First we summarize a few properties from [43] which will be used in this paper: (cf.
Th. 3.1, Co. 3.2 and Th. 3.3 of [43]):

**Proposition 2.17.** (1) The maps $[\lambda] \rightarrow [a_\lambda], [\lambda] \rightarrow [\bar{a}_\lambda]$ are ring homomorphisms;
(2) $a_\lambda \bar{\rho} = \bar{a}_\lambda \bar{\rho} = \bar{\rho} \lambda$;
(3) When $\rho \bar{\rho}$ is local, $la_{\lambda, \mu} = l\bar{a}_{\lambda, \mu} = l\lambda_{BE} \bar{\rho}, a_{\mu} \bar{\rho} = l\bar{a}_{\lambda} \bar{\rho}, \bar{a}_{\mu} \bar{\rho}$;
(4) (3) remains valid if $a_{\lambda}, a_{\mu}$ (resp. $\bar{a}_{\lambda}, \bar{a}_{\mu}$) are replaced by their subsectors.

**Definition 2.18.** $H_{\rho}$ is a finite dimensional vector space over $\mathbb{C}$ with orthonormal
basis consisting of irreducible sectors of $[\lambda_{\rho}], \forall \lambda \in \Delta_A$.

$[\lambda]$ acts linearly on $H_{\rho}$ by $[\lambda][a] = \sum_{b} l\lambda_{a} b [b]$ where $[b]$ are elements in the basis of $H_{\rho}$.
By abuse of notation, we use $[\lambda]$ to denote the corresponding matrix relative to
the basis of $H_{\rho}$. By definition these matrices are normal and commuting, so they can
be simultaneously diagonalized. Recall the irreducible representations of the fusion
algebra of $A$ are given by

\[ \lambda \rightarrow \frac{S_{\lambda \mu}}{S_{1 \mu}}. \]

\[ \sum_{b} \text{denote the sum over the basis } [b] \text{ in } H_{\rho}. \]
Definition 2.19. Assume \( \langle a, b \rangle = \sum_{\mu, i \in \text{Exp}} S_{\mu}^{a} \cdot \phi_{a}^{(\mu, i)}(\mu, i)^{*} \phi_{b}^{(\mu, i)} \) where \( \phi_{a}^{(\mu, i)} \) are normalized orthogonal eigenvectors of \( [\lambda] \) with eigenvalue \( S_{\mu}^{a} \), \( \text{Exp} \) is a set of \( \mu, i \)'s and \( i \) is an index indicating the multiplicity of \( \mu \), and is called the set of exponents of \( H_{\rho} \). Recall if a representation is denoted by \( 0 \), it will always be the vacuum representation.

The following lemma is elementary:

Lemma 2.20. (1):
\[
\sum_{b} d_{b}^{2} = \frac{1}{S_{00}^{2}}
\]
where the sum is over the basis of \( H_{\rho} \). The vacuum appears once in \( \text{Exp} \) and
\[
\phi_{a}^{(1)} = S_{00} d_{a};
\]

(2)
\[
\sum_{i} \frac{\phi_{a}^{(\lambda, i)}(\mu, i)^{*} \phi_{b}^{(\lambda, i)}}{S_{\lambda}^{2}} = \sum_{\nu} l(\nu, b) S_{\nu}^{a} S_{00}^{\lambda}
\]
where if \( \lambda \) does not appear in \( \text{Exp} \) then the right-hand side is zero.

Proof    Ad (1): By definition we have
\[
[a, \bar{\rho}] = \sum_{\lambda} l(a, \bar{\rho}) [\lambda] = \sum_{\lambda} l(a, \lambda \rho) [\lambda]
\]
where in the second = we have used Frobenius reciprocity. Hence
\[
d_{a} d_{\bar{\rho}} = \sum_{\lambda} l(a, \bar{\rho}) d_{\lambda}
\]
and we obtain
\[
\sum_{\lambda} d_{\lambda}^{2} = \sum_{\lambda, a} l(a, \bar{\rho}) d_{\lambda} d_{a} / d_{\rho} = \sum_{a} d_{a}^{2}
\]
(2) follows from definition and orthogonality of \( S \) matrix. ■

In [51] and [43], commutativity among subsectors of \( a_{\lambda}, \tilde{a}_{\mu}, \lambda, \mu \in \Delta \) were studied. We record these results in the following for later use:

Lemma 2.21. (1) Let \( [b] \) (resp. \( [b'] \)) be any subsector of \( a_{\lambda} \) (resp. \( \tilde{a}_{\lambda} \)). Then
\[
[a_{\mu} b] = [b a_{\mu}], [\tilde{a}_{\mu} b] = [b' \tilde{a}_{\mu}] \forall \mu, [b b'] = [b b'];
\]
(2) Let \( [b] \) be a subsector of \( a_{\mu} \tilde{a}_{\lambda} \), then \( [a_{\nu} b] = [b a_{\nu}], [\tilde{a}_{\nu} b] = [b \tilde{a}_{\nu}], \forall \nu \);
(3) If \( \sigma \prec \lambda \rho \), the \( [\mu \sigma] = [\sigma a_{\mu}] = [\sigma \tilde{a}_{\mu}] \).
2.6 Jones-Wassermann subfactors from representation of Loop groups

Let \( G = SU(n) \). We denote \( LG \) the group of smooth maps \( f : S^1 \mapsto G \) under pointwise multiplication. The diffeomorphism group of the circle \( \text{Diff}S^1 \) is naturally a subgroup of \( \text{Aut}(LG) \) with the action given by reparametrization. In particular the group of rotations \( \text{Rot}S^1 \cong U(1) \) acts on \( LG \). We will be interested in the projective unitary representation \( \pi : LG \to U(H) \) that are both irreducible and have positive energy. This means that \( \pi \) should extend to \( LG \rtimes \text{Rot}S^1 \) so that \( H = \bigoplus_{n \geq 0} H(n) \), where the \( H(n) \) are the eigenspace for the action of \( \text{Rot}S^1 \), i.e., \( r_{\theta}^* = \exp(i n \theta) \) for \( \theta \in H(n) \) and \( \dim H(n) < \infty \) with \( H(0) \neq 0 \). It follows from \([36]\) that for fixed level \( k \) which is a positive integer, there are only finite number of such irreducible representations indexed by the finite set

\[
P^k_{++} = \left\{ \lambda \in P \mid \lambda = \sum_{i=1,\ldots,n-1} \lambda_i \Lambda_i, \lambda_i \geq 0, \sum_{i=1,\ldots,n-1} \lambda_i \leq k \right\}
\]

where \( P \) is the weight lattice of \( SU(n) \) and \( \Lambda_i \) are the fundamental weights. We will write \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \), \( \lambda_0 = k - \sum_{1 \leq i \leq n-1} \lambda_i \) and refer to \( \lambda_0, \ldots, \lambda_{n-1} \) as components of \( \lambda \).

We will use \( \Lambda_0 \) or simply 1 to denote the trivial representation of \( SU(n) \). For \( \lambda, \mu, \nu \in P^k_{++} \), define \( N^\nu_{\lambda \mu} = \sum_{\delta \in P^k_{++}} S^{(\delta)}_\lambda S^{(\delta)}_\mu S^{(\delta)}_\nu / S^{(\delta)}_{\Lambda_0} \) where \( S^{(\delta)}_\lambda \) is given by the Kac-Peterson formula:

\[
S^{(\delta)}_\lambda = c \sum_{w \in S_n} \varepsilon_w \exp(i w(\delta) \cdot \lambda 2\pi/n)
\]

where \( \varepsilon_w = \det(w) \) and \( c \) is a normalization constant fixed by the requirement that \( S^{(\delta)}_\mu \) is an orthonormal system. It is shown in \([24]\) P. 288 that \( N^\nu_{\lambda \mu} \) are non-negative integers. Moreover, define \( \text{Gr}(C_k) \) to be the ring whose basis are elements of \( P^k_{++} \) with structure constants \( N^\nu_{\lambda \mu} \). The natural involution \( * \) on \( P^k_{++} \) is defined by \( \lambda \mapsto \lambda^* = \text{the conjugate of} \lambda \text{as representation of} SU(n) \).

We shall also denote \( S^{(\lambda)}_{\Lambda_0} \) by \( S^{(\lambda)}_1 \). Define \( d^{(\lambda)}_\lambda = \frac{S^{(\lambda)}_\lambda}{S^{(\lambda)}_{\Lambda_0}} \). We shall call \( (S^{(\lambda)}_\nu) \) the \( S \)-matrix of \( LSU(n) \) at level \( k \).

The following result is proved in \([42]\) (See Corollary 1 of Chapter V in \([42]\)).

**Theorem 2.22.** Each \( \lambda \in P^k_{++} \) has finite index with index value \( d^{(\lambda)}_\lambda \). The fusion ring generated by all \( \lambda \in P^k_{++} \) is isomorphic to \( \text{Gr}(C_k) \).

**Remark 2.23.** The subfactors in the above theorem are called Jones-Wassermann subfactors after the authors who first studied them (cf. \([21],[42]\)).

We will concentrate on \( n = 2 \) case in this paper. Fix level \( k \geq 1 \), the representations are labeled by half integers \( i \) with \( 0 \leq i \leq k/2 \). For example 0 will label the
vacuum representation, and $1/2$ will label the vector representation. The statistical dimension of $1/2$ is $2\cos(\frac{2\pi}{k+2})$. The fusion rules are given by

$$[i][j] = \sum_{|i-j|\leq l \leq i+j, i+j+l \leq k, i+j+l \in \mathbb{Z}} [l].$$

The $S$ matrix for $n = 2$ is given by $S_{ij} = \sqrt{2}k+2\sin(\frac{\pi(2i+1)(2j+1)}{k+2})$.

### 2.7 Dual Goodman-Harpe-Jones subfactors from Conformal Field Theory

The dual GHJ subfactors are obtained from irreducible sectors of $[i][\rho]$ with $\rho \bar{\rho} \in \Delta_A$ (cf. [13], Appendix of [5]). The induction in the following will be with respect to $\rho$.

**Definition 2.24.** The fusion graph of $[1/2]$ on $H_\rho$ is the graph whose vertices are irreducible sectors in $H_\rho$ and two vertices $a, b$ are connected by an edge if $\langle 1/2a, b \rangle = 1$. We will say that $H_\rho$ is type $A, D, E$ if the fusion graph is $A, D, E$ respectively. By an end point of the fusion graph we mean a vertex on the graph which is connected to only one other point on the graph.

Since the norm of the fusion graph of $[1/2]$ is less than 2, the graph must be $A - D - E$ (cf. [13]). When $\rho = 0$, the GHJ subfactors are Jones subfactors. The fusion graph of $1/2$ is given by:

Figure 1: Fusion graph of $1/2$, type $A_{k+1}$

When $k$ is even, and $[\rho \bar{\rho}] = [0] + [k/2]$. The fusion graph of $[1/2]$ on $H_\rho$ is type $D$ graph. In this case $[i][\rho]$ is irreducible iff $i \neq k/4$. If $4|k$ then $[a_{k/4}] = [b] + [b']$, and if $[\rho \bar{\rho}] = [0] + [g], [g^2] = [0]$, then $[gbg] = [b']$. If $k$ is not divisible by 4, $[k/4][\rho] = [b] + [b']$ where $b, b'$ are irreducible with same index, and $[\rho \bar{\rho}] = [0] + [a_{k/2}]$ and $[k/2][b] = [b']$ (cf. §5.2 of [8]).

**Notation 2.25.** For convenience we shall use $\rho_0$ to denote a fixed endomorphism with $[\rho_0 \bar{\rho}_0] = [0] + [k/2]$.

When the fusion graph of $[1/2]$ is $E_6$, the graph is given by figure 4 (cf. Page 392-393 of [13]).

Note that $k = 10, [b_0] = [a_{3/2}] - [a_{9/2}], [\rho \bar{\rho}] = [0] + [3]$.

When the fusion graph of $[1/2]$ is $E_7$, $k = 16, [\rho \bar{\rho}] = [0] + [4] + [8]$, and the graph is given by figure 5 (cf. Figure 42 of [8]). Note that

$$[b_2] = [5/2\rho] - [3/2\rho], [b_1] = [1/2b_2], [b'_1] = [5/2\rho] - [b_1]$$
The square root of indices of the corresponding subfactors divided by \( d_\rho \) is given by figure 6.

When the fusion graph of \([1/2]\) is \( E_8\), \( k = 28\), \([\rho\bar\rho] = [0] + [5] + [9] + [14]\) (cf. 43), and the graph is given by figure 7. The square root of indices of the corresponding subfactors divided by \( d_\rho \) is given by figure 8. The fusion graph of \( \tilde a_{1/2} \) is given by
Figure 7: Fusion graph of $1/2$, type $E_8$

![Fusion graph of 1/2, type E8](image)

Figure 8: Normalized eigenvectors

![Normalized eigenvectors](image)

Figure 9: Fusion graph of $\tilde{a}_{1/2}$

![Fusion graph of a1/2](image)

$[a_{5/2}] = [b_3'] + [b_3], [b_3] = [a_{1/2}b_4], [\tilde{a}_{5/2}] = [\tilde{b}_3'] + [\tilde{b}_3], [\tilde{b}_3] = [\tilde{a}_{1/2}b_4]$.

3 Classification of intermediate subfactors of dual GHJ subfactors

Let $\sigma$ be a dual GHJ subfactor. We assume that $k \geq 5$ in this section. Recall that a subfactor is maximal if there is no nontrivial intermediate subfactor. To list nontrivial intermediate subfactors of $\sigma$, according to Cor. 2.4 we need to determine all pairs $[\sigma_1, \sigma_2]$ such that $[\sigma_1\sigma_2] = [\sigma]$ with $1 < d_{\sigma_1} < d_{\sigma}$. Since $\sigma_1\sigma_1 < \sigma_2 \in \Delta$, by Definition 2.15 we can apply induction with respect to $\sigma_1$.

Our basic strategy, as explained in the end of §2 of [44], is to consider the fusion graph of $[1/2]$ on $H_{\sigma_1}$. By $A-D-E$ classification, this graph must be one of $A-D-E$ graphs. The following is useful:

**Lemma 3.1.** Let $\sigma$ be a dual GHJ subfactor such that $\sigma = \sigma_1\sigma_2$. Then either $[1/2\sigma_1]$ or both $[a_{1/2}\sigma_2]$ and $[\tilde{a}_{1/2}\sigma_2]$ are both irreducible where the inductions are with respect to $\sigma_1$. In terms of fusion graphs of $[1/2]$ on $\sigma_1$, $[a_{1/2}]$ on $\sigma_2$ and $[\tilde{a}_{1/2}]$ on $\sigma_2$, this means that either $\sigma_1$ is an end point or $\sigma_2$ is an end point on the fusion graphs of $[a_{1/2}]$ on $\sigma_2$ and $[\tilde{a}_{1/2}]$ on $\sigma_2$. 

15
Proof Since $\sigma$ is irreducible, we have $1 = \langle \sigma_1 \sigma_2, \sigma_1 \sigma_2 \rangle = \langle \bar{\sigma}_1 \sigma_1, \sigma_2 \bar{\sigma}_2 \rangle$ On the other hand

$$\langle 1/2 \sigma_1, 1/2 \sigma_1 \rangle = \langle \sigma_1 a_1/2, \sigma_1 a_1/2 \rangle = \langle \bar{\sigma}_1 \sigma_1, a_1^2/2 \rangle = 1 + \langle \bar{\sigma}_1 \sigma_1, a_1 \rangle$$

Similarly

$$\langle a_1/2 \sigma_2, a_1/2 \sigma_2 \rangle = \langle \sigma_2 \sigma_2, a_1^2/2 \rangle = 1 + \langle \sigma_2 \sigma_2, a_1 \rangle$$

$$\langle a_1/2 \sigma_2, a_1/2 \sigma_2 \rangle = \langle \sigma_2 \sigma_2, a_1^2/2 \rangle = 1 + \langle \sigma_2 \sigma_2, a_1 \rangle$$

Since $a_1, \bar{a}_1$ are irreducible by Lemma 2.33 of [44], the lemma follows.

By Lemma 3.1 either $\sigma_1$ or $\sigma_2$ is an end point on a graph of type $A-D-E$, which has two or three end points. This together with the known values of indices greatly reduce possible choices of $\sigma_1, \sigma_2$, and in the next few sections we will determine all such choices.

3.1 Type A

By Example 5.24 of [44], in this case $i$ is maximal if $i \neq k/4$. When $k = 2, 3$ all GHJ subfactors are maximal, and when $k = 4$ a direct computation gives the lattice of intermediate subfactors represented by sector $[1]$ as in figure 13. Assume now $k \geq 5$.

Lemma 3.2. Assume that $[k/4] = [\sigma_1][\sigma_2]$. Then if $0 \leq j \leq k/2$ and $j$ is an integer, then $j \in \text{Exp of } H_{\sigma_1}$ as defined in definition 2.19.

Proof Apply (2) of Lemma 2.20 to $a = \bar{\sigma}_2, b = \sigma_1$, we have

$$\sum_i \phi_a^{(j,i)} \phi_b^{(j,i)*} \frac{S_{ij}}{S_{0j}} = \sum \nu \langle \bar{a}_1 \rho \nu, b \rangle \frac{S_{k/4j}}{S_{0j}}$$

Since $\sum \nu \langle \bar{a}_1 \rho \nu, b \rangle \frac{S_{k/4j}}{S_{0j}}$ up to a nonzero constant is $\sin \left( \frac{(2i+1)\pi}{2} \right) \neq 0$ when $0 \leq j \leq k/2$ and $j$ is an integer, it follows that $j \in \text{Exp of } H_{\sigma_1}$ as defined in definition 2.19.

When $k$ is even and $i = k/4$, there are two cases to consider: when $4|k$, by Lemma 3.2 if $j$ is an integer then $j \in \text{Exp}$, and it follows that $H_{\sigma_1}$ is either type $A$ or $D$. When $H_{\sigma_1}$ is type $A$ then by fusion rule up to right multiplication by an automorphism $[\sigma_1] = [\sigma]$ or $d_{\sigma_1} = 1$; When $H_{\sigma_1}$ is type $D$, by left multiplication by an automorphism on $\sigma_1$ if necessary we may assume that $\rho \in H_{\sigma_1}$, with $[\rho \bar{\rho}] = [0] + [k/2]$. By Lemma 3.1 either $\sigma_1$ or $\sigma_2$ has to be an end. It follows that $[\sigma_1, \sigma_2] = [\rho \bar{\rho}]$ or $[\sigma_1, \sigma_2] = [\rho, \bar{\rho}]$.

When $k$ is not divisible by 4, similar argument shows that $[\sigma_1, \sigma_2] = [b, \bar{\rho}]$ or $[\sigma_1, \sigma_2] = [\rho, b]$.

We summarize the result in the following

Theorem 3.3. Suppose that $k \geq 4$. If $[i] \neq [k/4]$, the corresponding subfactor is maximal; If $k = 4$, the intermediate subfactor of $[1]$ is given by $[\rho \bar{\rho}] [\rho \bar{\rho}] = [0] + [1]$, $\rho b < 2 \rho$; If $4|k, k \geq 8$, the intermediate subfactors of $[k/4]$ is given by $[\rho \bar{\rho}]$ and $[\rho, b \bar{\rho}]$ where $[\rho \bar{\rho}] = [0] + [k/2], pb < k/2 \rho$; If $k$ is even but not divisible by 4, the intermediate subfactors of $[k/4]$ is given by $[b, \bar{\rho}]$ and $[\rho, b]$ where $[\rho \bar{\rho}] = [0] + [k/2], b < k/2 \rho$.  

16
3.2 Type $E_6$

In this section we assume that $\sigma$ is a dual GHJ subfactor appearing in $j\rho$ with $[\rho \bar{\rho}] = [0] + [3]$. Let $\sigma = \sigma_1\sigma_2$.

If $H_{\sigma_1}$ is type $A$, then multiply $\sigma_1$ on the right by an automorphism if necessary, we may assume that $\sigma_1 \in \Delta$, hence $\sigma_1 \in \Delta \rho$. By Lemma 3.1 either $1/2\sigma_1$ is irreducible or $a_{1/2}\sigma_2$ is irreducible. Then from figure 4 it is clear that the possible pairs of $[\sigma_1, \sigma_2]$ are given by

$$[1/2, \rho], [1, \rho], [9/2, \rho], [4, \rho], [1/2, \rho b]$$

If $H_{\sigma_1}$ is type $D$, we may assume that $\sigma_1 \in \Delta \rho_0$, and it follows that $\sigma_2 \in \bar{\rho}_0 \Delta \rho$. We note that $[5\rho b_0] = [\rho b_0], [5\rho a_1] = [\rho a_1]$. By Lemma 2.5 $\rho b_0(M)$ (resp. $\rho a_1(M)$) is contained in a subfactor of index 2 which is the fixed point subalgebra of $\Delta$ under an automorphism determined by $[5]$. Hence there are sectors $x, y$ such that $[\rho_0 x] = [\rho b_0], [\rho_0 y] = [\rho a_1]$. Since $k = 10$, we have $[\bar{\rho}_0 \rho_0] = [0] + [a_{5\rho}^0]$. From $1 = \langle \rho_0 x, \rho_0 x \rangle = 1 + \langle a_{5\rho}^0 x, x \rangle$ we conclude that $[x] \neq [a_{5\rho}^0 x]$. Similarly $[y] \neq [a_{5\rho}^0 y]$. Let us determine all irreducible sectors of $\bar{\rho}_0 \Delta \rho$. Note that $a_{1/2}^{\rho_0}$ acts on $\sigma_2 \in \bar{\rho}_0 \Delta \rho$, and the corresponding fusion graph of $a_{1/2}^{\rho_0}$ is an $A - D - E$ graph. Since $\langle \rho_0 \rho, \bar{\rho}_0 \rho \rangle = \langle \rho_0 \rho, \rho \bar{\rho} \rangle = 1$, $\langle a_{1/2}^{\rho_0} \bar{\rho}_0 \rho, a_{1/2}^{\rho_0} \bar{\rho}_0 \rho \rangle = \langle [0] + [1]\rho_0 \bar{\rho}_0 \rho, \rho \bar{\rho} \rangle = 1$, and $[a_{1/2}^{\rho_0} \bar{\rho}_0 \rho] = \bar{\rho}_0 \rho y = [y] + [a_{5\rho}^0 y]$, it follows that the fusion graph of the action of $a_{1/2}^{\rho_0}$ is given by (The vertices are labeled by all irreducible sectors of $\bar{\rho}_0 \Delta \rho$) figure 10. By Lemma 3.1 and figure 10 it follows that the following is a list of possible intermediate subfactors:

$$[\rho_0, x], [\rho_0, y], [1/2\rho_0, x]$$

Note that from $[\rho_0 x] = [\rho b_0]$ we have

$$[\rho x] < [\rho \bar{\rho}_0 \rho_0] = [\rho]([a_{3/2} - [a_{9/2}]][\bar{\rho} \rho_0] = ([3/2] - [9/2])([0] + [3])[\rho_0] = 2[3/2\rho_0]$$

Similarly

$$[\rho x a_{5\rho}^0] < [\rho \bar{\rho}_0 \rho_0] = [\rho]([a_{3/2} - [a_{9/2}]][\bar{\rho} \rho_0] = ([3/2] - [9/2])([0] + [3])[\rho_0] = 2[3/2\rho_0]$$

By comparing indices we have proved the following

$$[\rho x] = [\rho x a_{5\rho}^0] = [3/2\rho_0]$$

(14)
When $H_{\sigma_1}$ is $E_6$, then there is $\rho_2 \in H_{\sigma_1}$ such that $[\rho_2\bar{\rho_2}] = [0] + [3]$. By the cohomology vanishing result of remark 5.4 in \cite{26}, multiplying $\sigma_1$ on the right by an automorphism if necessary, we can assume that $[\rho_2] = [\rho]$, and so $\sigma_1 \in \Delta\rho$, and $\sigma_2 \in \bar{\rho}\Delta\rho$. The set of irreducible sectors of $\bar{\rho}\Delta\rho$ and fusion graphs of the action of $a_{1/2}, \bar{a}_{1/2}$ are given by \cite{43} and Figure 5 of \cite{34}. By using Lemma 3.1 it is straightforward to determine the following list of possible intermediate subfactors:

$$[\rho, a_{1/2}], [\rho, \bar{a}_{1/2}], [\rho, a_{9/2}], [\rho, \bar{a}_{9/2}], [\rho, a_1], [\rho, \bar{a}_1], [\rho, b], [\rho b, a_{1/2}], [\rho b, \bar{a}_{1/2}], [1/2\rho, b].$$

**Theorem 3.4.** Assume that $\sigma$ is a dual GHJ subfactor appearing in $j\rho$ with $[\rho\bar{\rho}] = [0] + [3]$. Then the following is the list of possible intermediate subfactors that can occur:

$$[1/2, \rho], [1, \rho], [9/2, \rho], [4, \rho], [1/2, \rho b], [\rho_0, x], [\rho_0, y], [1/2\rho_0, x], [\rho, a_{1/2}], [\rho, \bar{a}_{1/2}], [\rho, a_{9/2}]$$

$$[\rho, \bar{a}_{9/2}], [\rho, a_1], [\rho, \bar{a}_1], [\rho, b], [\rho b, a_{1/2}], [\rho b, \bar{a}_{1/2}], [1/2\rho, b].$$

### 3.3 $E_7$ Case

In this section we assume that $\sigma$ is a dual GHJ subfactor appearing in $j\rho$ with $[\rho\bar{\rho}] = [0] + [4] + [8]$. Since $[8\rho] = [\rho]$, by Lemma 2.7 we may assume that $\rho = \rho_0\rho_1$, and similarly

$$[b_1] = [\rho_0\hat{b}_1], [b_1'] = [\rho_0\hat{b}_1'], [b_2] = [\rho_0\hat{b}_2].$$

Let $\sigma = \sigma_1\sigma_2$. If $H_{\sigma_1}$ is type $A$, then multiply $\sigma_1$ on the right by an automorphism if necessary, we may assume that $\sigma_1 \in \Delta$, hence $\sigma_1 \in \Delta\rho$. By Lemma 3.1 either $1/2\sigma_1$ is irreducible or $a_{1/2}\sigma_2$ is irreducible. Then from figure 5 it is clear that the possible pairs of $[\sigma_1, \sigma_2]$ are given by

$$[1/2, \rho], [1, \rho], [3/2, \rho], [1/2, \bar{b}_1'], [1/2, b_2], [1, b_2].$$

If $H_{\sigma_1}$ is type $D$, we may assume that $\sigma_1 \in \Delta\rho_0$, and it follows that $\sigma_2 \in \bar{\rho}_0\Delta\rho$. From $\rho = \rho_0\rho_1$ we get $[\rho_0\rho_1] = [\rho_1] + [g\rho_1]$ where $[\rho_0\rho_0] = [0] + [g], [g^2] = [0]$. It is then easy to determine all irreducible sectors of $\bar{\rho}_0\Delta\rho$. There are two $E_7$ graphs in figure 11 encoding the action of $a_{1/2}^{\rho_0}$.

By Lemma 3.1 and figure 11 it follows that the following is a list of possible intermediate subfactors: $[\rho_0, w]$ where $w$ is one of the vertices in the first graph of figure 11 $[\rho_0 b, \bar{b}_2], [\rho_0 b, gb_2]$, where $\rho_0 b < [4\rho_0], [\rho_0 b\bar{b}_2] = [\rho_0 bg\bar{b}_2] = [3/2\rho], [1/2\rho_0, \rho_1], [1\rho_0, \rho_1], [3/2\rho_0, \rho_1], [1/2\rho_0, \bar{b}_1'], [1/2\rho_0, \bar{b}_2], [1\rho_0, \bar{b}_2]$

Note that from $[5/2\rho] = [3/2\rho] + [b_2]$ we have

$$[5/2\rho\bar{\rho}_1] = [3/2\rho\bar{\rho}_1] + [b_2\bar{\rho}_1], [5/2\rho\bar{\rho}_1 g] = [3/2\rho\bar{\rho}_1 g] + [b_2\bar{\rho}_1 g]$$

On the other hand from $[\rho] = [\rho_0\rho_1]$ and computing index it is easy to derive that

$$[\rho_1\bar{\rho}_1] + [g\rho_1\bar{\rho}_1 g] = 2[0] + [a_4^{\rho_0}]$$

18
Theorem 3.5. Assume that $\sigma$ is a dual GHJ subfactor appearing in $j\rho$ with $[\rho\tilde{\rho}] = [0] + [4] + [8]$. Then the following is the list of all intermediate subfactors that can occur:

\[[1/2, \rho], [1, \rho], [3/2, \rho], [1/2, b_1], [1/2, b_2], [1, b_2]\]
Lemma 3.6. Assume that 

\[ \rho_0 b < [4\rho_0], [\rho_0 b b_2] = [\rho_0 b g b_2] = [3/2\rho], \]

where 
\[ [1/2\rho_0, \rho_1], [1\rho_0, \rho_1], [3/2\rho_0, \rho_1], [1/2\rho_0, \hat{b}_1], [1/2\rho_0, \hat{b}_2], [1\rho_0, \hat{b}_2] \]
\[ [\rho, a_{1/2}, [\rho, a_{3/2}, [\rho, a_{3}], [\rho, a_{1}], [\rho, \tau], [b_1, a_{1/2}, [b_1, \hat{a}_{1/2}, [b_2, a_{1/2}, [b_2, \alpha_{1/2}] \]

where \[ [\tau] = [a_{1/2}](\hat{a}_{5/2} - [\alpha_{3/2}], [\delta] = [a_4] - [\alpha_1] \]

and 
\[ [\rho\tau] = [b_1] \]

\[ 3/2\rho. \]

3.4 Type D

In this section we assume that \( \sigma \) is a dual GHJ subfactor appearing in \( j\rho \) with \( [\rho\delta] = [0] + [k/2] \). Let \( \sigma = \sigma_1\sigma_2 \).

Lemma 3.6. Assume that \( \sigma \) is a dual GHJ subfactor appearing in \( j\rho \) with \( [\rho\delta] = [0] + [k/2] \). Let \( \sigma = \sigma_1\sigma_2 \). If \( 0 \leq i \leq k/2 \) and \( i \) is an integer, and \( \sin \left( \frac{(2i+1)(2j+1)}{k+2} \right) \neq 0 \), then \( i \in \text{Exp of } H_{\sigma_1} \)

as defined in definition 2.17

Proof: When \( j \neq k/4 \), we have \( [\sigma] = [j\rho] \). Apply (2) of Lemma 2.20 to \( a = \rho\sigma_2, b = \sigma_1 \), we have

\[ \sum_{l} \frac{\phi^{(i,l)}_{a} \phi^{(i,l)*}_{b}}{S_{0i}^2} = \sum_{\nu} l\nu a, b) S_{\nu i} S_{0i} = \frac{S_{ji} + S_{(k/2-j)i}}{S_{0i}} \]

When \( j = k/4 \), we have \( [\sigma\delta] = [k/4] \). Apply (2) of Lemma 2.20 to \( a = \rho\sigma_2, b = \sigma_1 \), we have

\[ \sum_{l} \frac{\phi^{(i,l)}_{a} \phi^{(i,l)*}_{b}}{S_{0i}^2} = \sum_{\nu} l\nu a, b) S_{\nu i} S_{0i} = \frac{S_{ji}}{S_{0i}} \]

Note that \( \frac{S_{ji}}{S_{0i}} + \frac{S_{(k/2-j)i}}{S_{0i}} = 0 \) if \( i \) is not an integer. When \( i \) is an integer, up to a nonzero constant \( \frac{S_{ji}}{S_{0i}} + \frac{S_{(k/2-j)i}}{S_{0i}} \) is \( \sin \left( \frac{(2i+1)(2j+1)}{k+2} \right) \), it follows that if \( 0 \leq i \leq k/2 \), \( i \) is an integer, and \( \sin \left( \frac{(2i+1)(2j+1)}{k+2} \right) \neq 0 \), then \( i \in \text{Exp of } H_{\sigma_1} \) as defined in definition 2.17

By Lemma 3.6, if \( i \) is an integer, \( (2i+1)(2j+1) \) is not divisible by \( k+2 \), then \( i \in \text{Exp of } H_{\sigma_1} \). By inspecting exponents on Page 18 of [13], if \( k \neq 10, 16, H_{\sigma_1} \) must be of type A or D. We will see in the following that when \( k = 10, 16 \) \( H_{\sigma_1} \) can be \( E_6, E_7 \) respectively.
3.4.1 Local case: $4|k, k \neq 16$

In this section we assume that $4|k, k \neq 16$ and $\sigma$ is a dual GHJ subfactor appearing in $j \rho$ with $[\rho \bar{\rho}] = [0] + [k/2]$. By our assumption $H_{\sigma_1}$ must be of type $A$ or $D$. If $H_{\sigma_1}$ is type $A$, then multiply $\sigma_1$ on the right by an automorphism if necessary, we may assume that $\sigma_1 \in \Delta$, hence $\sigma_1 \in \Delta \rho$. By Lemma 3.8 either $1/2 \sigma_1$ is irreducible or $a_{1/2} \sigma_2$ is irreducible. It is clear that the possible pairs of $[\sigma_1, \sigma_2]$ are given by

$$[i, \rho], i \neq [k/4], [1/2, \rho b], [1/2, \rho b'],$$

where $[a_{k/4}] = [b] + [b'], [1/2 \rho b] = [1/2 \rho b'] = [\rho a_{k/4 − 1/2}]$.

If $H_{\sigma_1}$ is type $D$, we may assume that $\sigma_1 \in \Delta \rho$, and it follows that $\sigma_2 \in \bar{\rho} \Delta \rho$. It follows that the fusion graph of the action of $a^{\rho_0}_{1/2}$ is given by two $D$ graphs, whose vertices are labeled by irreducible components of $a_i, i \in \Delta$ and $\bar{a}_i, i \in \Delta$ respectively. By Lemma 3.1 it follows that the following is a list of possible intermediate subfactors: $[\rho, w]$ where $w$ is an irreducible component of $a_i, i \in \Delta$,

$$[\rho b, a_{1/2}], [\rho b, \bar{a}_{1/2}], [1/2 \rho, b], [1/2 \rho, b']$$

Theorem 3.7. Assume that $4|k, k \neq 16$ and $\sigma$ is a dual GHJ subfactor appearing in $j \rho$ with $[\rho \bar{\rho}] = [0] + [k/2]$. Then the following is a list of possible intermediate subfactors:

$$[i, \rho], i \neq [k/4], [1/2, \rho b], [1/2, \rho b'],$$

where $[a_{k/4}] = [b] + [b'], [1/2 \rho b] = [1/2 \rho b'] = [\rho a_{k/4 − 1/2}]$ $[\rho, w]$ where $w$ is an irreducible component of $a_i, i \in \Delta$,

$$[\rho b, a_{1/2}], [\rho b, \bar{a}_{1/2}], [1/2 \rho, b], [1/2 \rho, b']$$

3.4.2 $k = 16$

By Lemma 3.6 $H_{\sigma_1}$ may be $E_7$ when $\sigma = 5/2 \rho_0 = \sigma_1 \sigma_2$. Suppose that $H_{\sigma_1}$ is $E_7$. Then we can assume that $\sigma_1 \in \Delta \rho_2$ with $[\rho_2 \bar{\rho}_2] = [0] + [4] + [8]$. It follows that $\sigma_2 \in \bar{\rho}_2 \Delta \rho_0$. But the conjugate of $\bar{\rho}_2 \Delta \rho_0$ has already been determined by Figure 11. By computing index we conclude that either $d_{\sigma_1} = d_{\sigma_2} = d_\rho$, or $d_{\sigma_1} = d_{\rho_2}, d_{\sigma_2} = d_{\bar{\rho}_2}$. By equation 15 we conclude that $[\sigma_1, \sigma_2] = [b_2, \bar{\rho}_1], [\sigma_1, \sigma_2] = [b_2, \bar{\rho}_1 \rho_2], [\sigma_1, \sigma_2] = [\rho_2, \bar{\rho}_2], [\sigma_1, \sigma_2] = [\rho_2, \bar{\rho}_2]$, or $[\sigma_1, \sigma_2] = [\rho_2, \bar{\rho}_2 \rho_2]$

Theorem 3.8. Assume that $k = 16$ and $\sigma$ is a dual GHJ subfactor appearing in $j \rho$ with $[\rho \bar{\rho}] = [0] + [8]$. Then the following is a list of possible intermediate subfactors:

$$[i, \rho], i \neq [k/4], [1/2, \rho b], [1/2, \rho b'],$$

where $[a_{k/4}] = [b] + [b'], [1/2 \rho b] = [1/2 \rho b'] = [\rho a_{k/4 − 1/2}]$ $[\rho, w]$ where $w$ is an irreducible component of $a_i, i \in \Delta$,

$$[\rho b, a_{1/2}], [\rho b, \bar{a}_{1/2}], [1/2 \rho, b], [1/2 \rho, b']$$

21
\[ [b_2, \tilde{\rho}_1], [b_2, \tilde{\rho}_1 g], [\rho_2, \tilde{b}_2], [\rho_2, \tilde{b}_2 g] \]

where
\[ [b_2 \tilde{\rho}_1] = [b_2 \tilde{\rho}_1 g] = [\rho_2 \tilde{b}_2] = [\rho_2 \tilde{b}_2 g] = [5/2\rho]. \]

### 3.4.3 Nonlocal case: \( k \) is even but not divisible by 4, \( k \neq 10 \)

In this section we assume that \( k \) is even but not divisible by 4, \( k \neq 10 \) and \( \sigma \) is a dual GHJ subfactor appearing in \( j\rho \) with \([\rho \tilde{\rho}] = [0] + [k/2]\). By our assumption \( H_{\sigma_1} \) must be of type \( A \) or \( D \). If \( H_{\sigma_1} \) is type \( A \), then multiply \( \sigma_1 \) on the right by an automorphism if necessary, we may assume that \( \sigma_1 \in \Delta \), hence \( \sigma_1 \in \Delta \rho \). By Lemma 3.1 either \( 1/2\sigma_1 \) is irreducible or \( a_{1/2}\sigma_2 \) is irreducible. Then it is clear that the possible pairs of \([\sigma_1, \sigma_2]\) are given by
\[ [i, \rho], i \neq [k/4], [1/2, b], [1/2, b'], \]
where \([\rho a_{k/4} = [b] + b'], [1/2b] = [1/2b'] = [\rho a_{k/4-1/2}]\).

If \( H_{\sigma_1} \) is type \( D \), we may assume that \( \sigma_1 \in \Delta \rho \), and it follows that \( \sigma_2 \in \tilde{\rho}\Delta \rho \). It follows that the fusion graph of the action of \( a_i \sigma_i \) is given by one \( D \) graph, whose vertices are labeled by irreducible components of \( a_i, i \in \Delta \). By Lemma 3.1 it follows that the following is a list of possible intermediate subfactors: \([\rho, w] \) where \( w \) is an irreducible component of \( a_i, i \neq k/4, [b, a_{1/2}], [b, \tilde{a}_{1/2}]\).

**Theorem 3.9.** Assume that \( k \) is even but not divisible by 4, \( k \neq 10 \) and \( \sigma \) is a dual GHJ subfactor appearing in \( j\rho \) with \([\rho \tilde{\rho}] = [0] + [k/2]\). Then the following is a list of possible intermediate subfactors:
\[ [i, \rho], i \neq [k/4], [1/2, b], [1/2, b'], \]
where \([\rho a_{k/4} = [b] + b'], [1/2b] = [1/2b'] = [\rho a_{k/4-1/2}]\), \([\rho, w] \) where \( w \) is an irreducible component of \( a_i, i \in \Delta, [b, a_{1/2}], [b, \tilde{a}_{1/2}]\).

### 3.4.4 \( k = 10 \)

By Lemma 3.6 \( H_{\sigma_1} \) may be \( E_6 \) when \( \sigma = 3/2\rho_0 \). Suppose that \( H_{\sigma_1} \) is \( E_6 \). Then we can assume that \( \sigma_1 \in \Delta \rho_2 \) with \([\rho_2 \tilde{\rho}_2] = [0] + [3]\). It follows that \( \sigma_2 \in \tilde{\rho}_2 \Delta \rho_0 \). But the conjugate of \( \tilde{\rho}_2 \Delta \rho_0 \) has already been determined by Figure 10. By computing index we conclude that \( d_{\sigma_1} = d_{\rho_2} = d_{\sigma_2} \). By equation (14) we conclude that \([\sigma_1, \sigma_2] = [\rho_2, \tilde{x}]\) or \([\sigma_1, \sigma_2] = [\rho_2, \tilde{x}a_5]\).

**Theorem 3.10.** Assume that \( k = 10 \) and \( \sigma \) is a dual GHJ subfactor appearing in \( j\rho \) with \([\rho \tilde{\rho}] = [0] + [5]\). Then the following is a list of possible intermediate subfactors:
\[ [i, \rho], i \neq [k/4], [1/2, \rho b], [1/2, \rho b'], \]
where \([\rho a_{k/4} = [b] + [b'], [1/2\rho b] = [1/2\rho b'] = [\rho a_{k/4-1/2}]\), \([\rho, w] \) where \( w \) is an irreducible component of \( a_i, i \in \Delta, [b, a_{1/2}], [b, \tilde{a}_{1/2}] \) \([\rho_2, \tilde{x}], [\rho_2, \tilde{x}a_5]\) where
\[ [\rho_2 \tilde{x}] = [\rho_2 \tilde{x}a_5] = [3/2\rho]. \]
3.5 $E_8$ case

Assume that $\sigma$ is a dual GHJ subfactor appearing in $j \rho$ with $[\rho \tilde{\rho}] = [0] + [5] + [9] + [14]$. Since $[8 \rho] = [\rho]$, by Lemma 2.3 we may assume that $\rho = \rho_0 \rho_1$. Let $\sigma = \sigma_1 \sigma_2$. If $H_{\sigma_1}$ is type $A$, then multiply $\sigma_1$ on the right by an automorphism if necessary, we may assume that $\sigma_1 \in \Delta$, hence $\sigma_1 \in \Delta \rho$. By Lemma 3.1 either $1/2 \sigma_2$ is irreducible or $a_{1/2} \sigma_2$ is irreducible. Then from Figure 5 in [5] and fusion rules it is clear that the possible pairs of $[\sigma_1, \sigma_2]$ are given by

$$[1/2, \rho], [1, \rho], [3/2, \rho], [2, \rho], [1/2, \rho b_3'], [1/2, b_4], [1, \rho b_4]$$

If $H_{\sigma_1}$ is type $D$, we may assume that $\sigma_1 \in \Delta \rho_0$, and it follows that $\sigma_2 \in \tilde{\rho}_0 \Delta \rho$. From $\rho = \rho_0 \rho_1$ we get $[\tilde{\rho}_0 \rho] = [\rho_1] + [g \rho_1]$ where $[\tilde{\rho}_0 \rho_0] = [1] + [g], [g^2] = [0]$. It is then easy to determine all irreducible sectors of $\tilde{\rho}_0 \Delta \rho$. There are two $E_8$ graphs encoding the action of $a_{1/2}^{\rho_0}$ in Figure 12. By Lemma 3.1 and Figure 12 it follows that the following

$$\begin{array}{c}
\rho_1 b_3' \\
\rho_1 \rho_1 a_2/2 \rho_1 a_1 \rho_1 a_3/2 \rho_1 a_2 \\
\rho_1 b_3 \rho_1 b_4 \\
g \rho_1 a_1 \\
g \rho_1 b_3/3 \\
g \rho_1 a_2 \\
g \rho_1 b_3 \\
g \rho_1 b_4
\end{array}$$

Figure 12: Fusion graph of $a_{1/2}^{\rho_0}$

is a list of possible intermediate subfactors: $[\rho_0, w]$, where $w$ is one of the vertices in the first graph of Figure 12

$$[\rho_0 b, \rho_1], [\rho_0 b, g \rho_1]$$

where $\rho_0 b < [7 \rho_0]$ and $[\rho_0 b \rho_1] = [\rho_0 b g \rho_1] = [2 \rho],

$$[1/2 \rho_0, \rho_1], [1 \rho_0, \rho_1], [3/2 \rho_0, \rho_1], [2 \rho_0, \rho_1], [1/2 \rho_0, \rho_1 b_3'], [1/2 \rho_0, \rho_1 b_4], [1 \rho_0, \rho_1 b_4]$$

When $H_{\sigma}$ is $E_8$, then there is $\rho_2 \in H_{\sigma}$ such that $[\rho_2 \tilde{\rho}_2] = [\rho \tilde{\rho}]$. By the cohomology vanishing result of remark 5.4 in [20], multiplying $\sigma_1$ on the right by an automorphism if necessary, we can assume that $[\rho_2] = [\rho]$, and so $\sigma_1 \in \Delta \rho$, and $\sigma_2 \in \tilde{\rho} \Delta \rho$. The set of irreducible sectors of $\tilde{\rho} \Delta \rho$ and fusion graphs of the action of $a_{1/2}, \tilde{a}_{1/2}$ are given by Figure 5 of [5]. By using Lemma 3.1 and comparing indices it is straightforward to determine the following list of possible intermediate subfactors:

$$[\rho, a_{1/2}], [\rho, \tilde{a}_{1/2}], [\rho, a_{3/2}], [\rho, \tilde{a}_{3/2}], [\rho, a_1], [\rho, \tilde{a}_1], [\rho, a_2], [\rho, \tilde{a}_2], [\rho, a_{1/2} b_3'], [\rho, \tilde{a}_{1/2} b_3'], [\rho b_3', a_{1/2}], [\rho b_3', \tilde{a}_{1/2}], [\rho b_4, a_{1/2}], [\rho b_4, \tilde{a}_{1/2}], [\rho b_4, a_1],$$

23
Given a dual GHJ subfactor $\sigma$

In this section we list the lattices of intermediate subfactors of dual GHJ subfactors.

4 The lattice structure of intermediate dual GHJ subfactors

In this section we list the lattices of intermediate subfactors of dual GHJ subfactors. Given a dual GHJ subfactor $\sigma \sim \Delta \rho$, first we inspect all the pairs $[\sigma_1, \sigma_2]$ listed in Th. 3.3, Th. 3.4, Th. 3.5, Th. 3.7, Th. 3.9, Th. 3.8, Th. 3.10, and Th. 3.11 such that $[\sigma] = [\sigma_1 \sigma_2]$. This gives all intermediate subfactors of $\sigma$. Then we use Cor. 2.4 and known fusion rules to determine the relations between these intermediate subfactors.

The result are listed in the following figures. In each figure indexed by a dual GHJ subfactor $\sigma$ we list all nontrivial intermediate subfactors $[\sigma_1, \sigma_2]$ with $[\sigma_1 \sigma_2] = [\sigma]$. If $[\sigma_1, \sigma_2]$ lies above $[\tau_1, \tau_2]$ and there is a line connecting them, then $[\tau_1, \tau_2] \subset [\sigma_1, \sigma_2]$.

4.1 Type A

When $k$ is odd or $k = 2$, all type A GHJ subfactors are maximal. When $k \geq 4$ is even, all $i \neq k/4$ are maximal.
4.2 Type D

When \( k = 2 \) the GHJ subfactor is maximal. When \( k = 4 \), \( \rho_0 b, \rho_0 b' \) are maximal. When \( k \) is not divisible by 4, \( b, b' \prec k/4 \rho_0 \) are maximal. We note that \([\rho_0 b] = [\rho_0 b' g] \) when \( k \) is divisible by 4, so \( \rho_0 b \) and \( \rho_0 b' \) have identical intermediate subfactor lattice. The lattice in figure 23 for the case when \( k = 6 \) was first obtained in [14] in the setting of type \( II_1 \) factors.
4.3 $E_6$

In this case $\rho, \rho a_5$ are maximal. The lattice of intermediate subfactors of $9/2\rho$ is isomorphic to that of $1/2\rho$ since $[9/2\rho] = [1/2\rho a_5]$ and $a_5$ is an automorphism.
Figure 25: $\rho b_0$, type $E_6$

Figure 26: $1\rho$, type $E_6$

Figure 27: $\rho$, type $E_7$

Figure 28: $1/2\rho$, type $E_7$

Figure 29: $1\rho$, type $E_7$

Figure 30: $\tilde{b}_1$, type $E_7$
The labeling of type $E_7, E_8$ cases are described in Th. 3.5 and Th. 3.11 respectively. Here we use the most complicated case of $2\rho$, type $E_8$ to explain how we obtain the lattice structure. Let us first use Lemma 2.1 to show that $[\rho, a_2] \neq [\rho, \tilde{a}_2]$. If $[\rho, a_2] = [\rho, \tilde{a}_2]$, we can find an automorphism $\sigma$ such that $[\rho \sigma] = [\rho]$, $[\sigma a_2] = [\tilde{a}_2]$. But $\tilde{a}_2 a_2$ is irreducible, and this implies that $[\sigma] = [\tilde{a}_2 a_2]$, contradicting our assumption that $\sigma$ is an automorphism. Alternatively we can argue from $[\rho \sigma] = [\rho]$ that $\sigma \prec \tilde{\rho} \rho$, and from the formula for $\tilde{\rho} \rho$ in [7] we conclude that $[\sigma] = [0]$, and hence $[a_2] = [\tilde{a}_2]$, again a contradiction. A good way to look at the intermediate subfactors of $2\rho$, type $E_8$ is to start with dual GHJ of type $A$: these are pairs with the first components...
Figure 35: $1/2 \rho$, type $E_8$

Figure 36: $1 \rho$, type $E_8$

Figure 37: $3/2 \rho$, type $E_8$

Figure 38: $\rho b_3$, type $E_8$
labeled by a half integer, and there are three of them; for type $D$ the first components are labeled by a half integer multiplied by $\rho_0$, and there are six of them; for type $E$ the first components are labeled the vertices of figure 7, and there are eleven of them.

Let us now explain why $[\rho b_4, a_1] \subset [\rho, a_2]$ and yet $[\rho b_4, a_1]$ is not a subfactor of $[\rho, \tilde{a}_2]$. Since by Lemma 2.1, $[\rho b_4, a_1] \subset [\rho, b_4 a_1]$, but $b_4 a_1 = [a_2]$, so we have shown that $[\rho b_4, a_1] \subset [\rho, a_2]$. Now if $[\rho b_4, a_1] \subset [\rho, \tilde{a}_2]$, by Lemma 2.1 we can find $\sigma$ such that

$[\rho b_4] = [\rho \sigma], [\sigma a_1] = [\tilde{a}_2]$

It follows that $[\rho, \sigma]$ is an intermediate subfactor of $\rho b_4$. But all such pairs are classified in Th. 3.11, and by inspection we conclude that $[\rho, \sigma] = [\rho, b_4]$. By definition we have an automorphism $\sigma_1$ such that $[\rho \sigma_1] = [\rho], [\sigma_1 a] = [b_4]$. Since by [7] the only subsector of $\bar{\rho} \rho$ which is an automorphism is $[0]$, we have $[\sigma] = [b_4]$, and $[\sigma a_1] = [a_2] = [\tilde{a}_2]$, a contradiction. The rest of relations in Figure 41 are derived in a similar way.

4.5 A negative answer to question [A.12]

From the list of lattices we can easily verify conjecture 1.1 for GHJ subfactors. It is an interesting question to check that whether the stronger conjecture $A.1$ is true for all GHJ subfactors using our list.

We claim that the GHJ subfactor $\bar{\rho} 3/2$ of type $E_7$ gives a negative answer to question [A.12] in the appendix. This subfactor is dual to $3/2 \rho$ whose lattice of intermediate subfactors is given by figure 33. Since $3/2 \rho$ has 12 minimal subfactors, it follows that $\bar{\rho} 3/2$ has 12 maximal subfactors. Note that $[\bar{\rho} 3/2][3/2 \rho] \in \bar{\rho} \Delta \rho$, and it is known by Figure 42 of [8] that the only sector with index 1 which appears in $\bar{\rho} \Delta \rho$ is $[0]$. By Lemma 2.6 we have $\text{Aut}(\bar{\rho} 3/2)$ is trivial. On the other hand it is easy to calculate
[3/2\rho \bar{\rho} 3/2], and we find there are 9 irreducible sectors which can appear in [3/2\rho \bar{\rho} 3/2]. Since 12 > 9, this gives a negative answer to problem A.12.

A A proof of relative version of Wall’s conjecture for solvable groups

As pointed in introduction, Wall proved his conjecture for solvable groups. Another supporting evidence, as observed by V. F. R. Jones, is that the minimal version of conjecture 1.1 holds for subfactors with a commutative $N' \cap M_1$. In this appendix we will prove the relative version of Wall’s conjecture for solvable groups. Our proof is partially inspired by some comments of V. F. R. Jones. Let $G$ acts properly on the hyperfinite type $II_1$ factor $R$, and consider the subfactors $R^G \subset R^H \subset R$ where
$R^G, R^H$ are fixed point subalgebras of $R$ under the action of $G, H$ respectively. Let $K_i, 1 \leq i \leq n$ be the set of maximal subgroups of $G$ which strictly contains $H$. Let $e_i$ be the Jones projections from $R$ onto $R^{K_i}$. Note that $e_i = \frac{1}{|K_i|} \sum_{g \in K_i} g \in \mathbb{C}G$. For any subgroup $K$ of $G$ we will denote by $e_K = \frac{1}{|K|} \sum_{g \in K} g \in \mathbb{C}G$. Note that $\mathbb{C}G$ is a $C^*$ algebra. We denote by $l(G)$ the abelian algebra of complex valued functions on $G$. There are examples when $G$ is a semidirect product of an elementary abelian group $V$ by $G_1$ which acts irreducibly on $V$, and we find that the set $\frac{1}{|G_1|} \sum_{g \in G_1} vgv^{-1}, v \in V$ is not linearly independent (Note that for any $v \in V, vG_1v^{-1}$ is maximal in $G$ by our assumption). However the following modification seems to be interesting:

**Conjecture A.1.** Let $N \subset M$ be an irreducible subfactor with finite Jones index, and let $P_i, 1 \leq i \leq n$ be the set of minimal intermediate subfactors. Denote by $e_i \in N' \cap M_1, 1 \leq i \leq n$ the Jones projections $e_i$ from $M$ onto $P_i$ and $e_N$ the Jones projections $e_N$ from $M$ onto $N$. Then there are vectors $\xi_i, \xi_1 \in N' \cap M_1$ such that $e_i \xi_i, 1 \leq i \leq n, e_N \xi_1 = \xi_1$, and $\xi_i, 1 \leq i \leq n, \xi_1$ are linearly independent.

**Remark A.2.** We note that unlike conjecture 1.1, the conjecture above makes use of the algebra structure of $N' \cap M_1$ and therefore does not immediately imply the dual version or if one replaces minimal by maximal.

By definition conjecture A.1 implies conjecture 1.1 and we shall prove conjecture A.1 for $R^G \subset R, G$ solvable. In fact it is easy to check that for $R^G \subset R$ conjecture A.1 is equivalent to:

**Conjecture A.3.** Let $K_i, 1 \leq i \leq n$ be a set of maximal subgroups of $G$. Then there are vectors $\xi_i \in l(G), 1 \leq i \leq n$ such that $\lambda G \xi_i = 0, \xi_1$ are $K_i$ invariant and linearly independent.

We will prove conjecture A.3 when $G$ is solvable.

**Lemma A.4.** Suppose that $K$ acts irreducibly on an elementary abelian group $V$, and $G$ is the semi-direct product of $V$ by $K$. For each $vKv^{-1}, v \neq 0$, we assign a vector $\xi_v := \delta_v - \frac{1}{|V|} \lambda G 1 \in l(G)$, and for $K$, we assign $\xi_K := \delta_{M_b} - \frac{1}{|V|} \lambda G 1 \in l(G)$ where $b \neq 0$, and we use $\delta_S$ to denote the characteristic function of a set $S \subset G$, and 1 stands for constant function with value 1. Then $\xi_v, v \in V, v \neq 0, \xi_K$ verify conjecture A.3 for $vKv^{-1}, v \in V$.

**Proof** By definition we just have to check that $\xi_v, v \in V, v \neq 0, \xi_K$ are linearly independent. Suppose that $\lambda_v$ are complex numbers such that

$$\sum_{v \neq 0, v \in V} \lambda_v \xi_v + \lambda_0 \xi_K = 0.\]$$

The we have

$$\sum_{v \neq 0, v \in V} \lambda_v \delta_v + \lambda_0 \delta_K - \sum_{v \in V} \lambda_v \frac{1}{|V|} 1 = 0.$$

32
For a fixed $v \in V, v \neq 0$, since $K$ acts irreducibly on $V$, we can find $k \in K$ such that $k(v) := k^{-1}vk \neq b$. It follows that $\delta_{Kb}(vk) = 0$. Evaluate the LHS of the above sum at $vk, v \neq 0$ we have

$$\lambda_v = \frac{1}{|V|} \sum_{v' \in V} \lambda_{v'}.$$ 

Since $kb = k^{-1}(b)k$, evaluating the above sum at $kb$ we have

$$\lambda_0 + \lambda_{k^{-1}(b)} - \frac{1}{|V|} \sum_{v \in V} \lambda_v = 0.$$

Notice that $k^{-1}(b) \neq 0$ since $b \neq 0$, we conclude that $\lambda_0 = 0$, all $\lambda_v, v \neq 0$ are identical to the same value $\lambda$, and $\frac{1}{|V|}(|V| - 1)\lambda = \lambda$. It follows that $\lambda_v = 0, \forall v \in V$. \hfill \blacksquare

**Lemma A.5.** Suppose that $H$ is a normal subgroup of $G$ and $K_i \geq H, 1 \leq i \leq n$ is a set of maximal subgroups of $G$. If conjecture $\text{A.3}$ is true for $K_i/H \leq G/H$, then it is also true for $K_i \leq G, 1 \leq i \leq n$.

**Proof** By assumption we have functions $\xi_i : G/H \to \mathbb{C}$ such that $\xi_i$ is linearly independent and $K_i/H$ invariant, $e_{G/H}\xi_i = 0, 1 \leq i \leq n$. Let $\pi : G \to G/H$ be the projection, then $\xi_i\pi : G \to \mathbb{C}$ is linearly independent and $K_i$ invariant, $e_G\xi_i\pi = 0, 1 \leq i \leq n$.

Recall that for a subgroup $K \leq G$, $\text{Core}_G K := \cap_{g \in G} Kg^{-1}$ is the largest normal subgroup of $G$ that is contained in $K$.

**Lemma A.6.** Let $H_1, H_2$ be subgroups of a finite group $G$. Denote by $H_1H_2$ the set of different elements $g$ which can be written as $h_1h_2$ with $h_1 \in H_1, h_2 \in H_2$. Then $e_{H_1}e_{H_2} = \frac{1}{|H_1H_2|} \sum_{g \in H_1H_2} g$.

**Proof** Let $g = h_1h_2$ with $h_1 \in H_1, h_2 \in H_2$. Then $h_1h_2 = h_1'h_2'$ iff $h_1^{-1}h_1' = h_2h_2'^{-1} \in H_1 \cap H_2$. It follows that for each $g = h_1h_2 \in H_1H_2$, there are $H_1 \cap H_2$ different pairs of $(h_1', h_2') \in H_1 \times H_2$ such that $g = h_1'h_2'$. Hence $|H_1H_2| = \frac{|H_1||H_2|}{|H_1\cap H_2|}$ and the lemma follows from definition. \hfill \blacksquare

**Proposition A.7.** Conjecture $\text{A.3}$ is true for $G$ solvable.

**Proof** The proof goes by induction on $|G|$. Consider $H := \text{Core}_G K_1$. If Core$_G K_i$ does not contain $H$, then $K_iH \neq K_i$, and since $K_iH$ is a subgroup of $G, K_i$ is maximal, we have $K_iH = G$. Suppose that there is at least one $K_i$ such that Core$_G K_i$ does not contain $H$. By induction hypothesis, for the set of $K_i$ with Core$_G K_i$ not containing $H$, we can find vectors $\xi_i$ which verifies conjecture $\text{A.3}$ and for the set of $K_i$ with Core$_G K_i$ containing $H$, we can find vectors $\xi_i$ which verifies conjecture $\text{A.3}$. We claim that such set $\xi_i, 1 \leq i \leq n$ is linearly independent. Suppose that $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$ such that $\sum_{1 \leq i \leq n} \lambda_i \xi_i = 0$. Multiply on the left by $e_H$. We have

$$\sum_{i, \text{Core}_G K_i \geq H} \lambda_i e_H \xi_i + \sum_{i, \text{Core}_G K_i \cap H \neq H} \lambda_i e_H \xi_i = 0.$$
Note that if Core$_G K_i \geq H$, then $e_H \xi_i = e_H e_{K_i} \xi_i = e_{K_i} \xi_i$; if Core$_G K_i$ does not contain $H$, then $e_H \xi_i = e_H e_{K_i} \xi_i = e_G \xi_i = 0$ by Lemma A.6. It follows that

$$\sum_{i, \text{Core}_G K_i \geq H} \lambda_i \xi_i = 0, \quad \sum_{i, \text{Core}_G K_i \cap H \neq H} \lambda_i \xi_i = 0.$$ 

By our assumption $\lambda_i = 0, 1 \leq i \leq n$.

So we are left with the case that Core$_G K_i \geq H, 1 \leq i \leq n$. By replacing $H = \text{Core}_G K_1$ by $H = \text{Core}_G K_j$ for some $1 \leq j \leq n$ we can now assume that Core$_G K_i = H, 1 \leq i \leq n$. If $H$ is nontrivial, by Lemma A.5 we are done.

If $H$ is trivial, by Th. 15.6 of [11], $G$ is the semidirect product of an elementary abelian group $V$ by $K_1$, and the action of $K_1$ on $V$ is irreducible. Moreover by Th.16.1 of [11] all maximal subgroup $K$ of $G$ with Core$_G K = H$ is of the form $vK_1v^{-1}$ for some $v \in V$. By Lemma A.4 we are done. \(\blacksquare\)

**Remark A.8.** The reduction in the proof of the above proposition works for general groups, and conjecture A.3 can be reduced to the case where $G$ is a primitive group, and the set of maximal subgroups have trivial core. Such groups are classified by O’Nan-Scott theorem (cf. §4 of [10]). The first case is when $G$ is the semidirect product of an elementary abelian group $V$ by $K_1$, and the action of $K_1$ on $V$ is irreducible. When $G$ is not solvable, maximal subgroups $K$ of $G$ with trivial core are not conjugates of $K_1$, and our proof above does not work. Such maximal subgroups are related to the first cohomology of $K_1$ with coefficients in $V$, and conjecture A.3 implies that the order of this cohomology is less than $|K_1|$ (cf. Question 12.2 of [19]). Unfortunately even though it is believed that the order of this cohomology is small (cf. [18]), the bound $|K_1|$ has not been achieved yet.

**Corollary A.9.** The relative version of Wall’s conjecture is true for solvable groups.

**Proof** Let $K_i, 1 \leq i \leq n$ be a set of maximal subgroups of $G$ strictly containing $H$. By Prop. A.7 we can find vectors $\xi_i \in \ell(G), 1 \leq i \leq n$ such that $e_G \xi_i = 0$, $\xi_i$ are $K_i$ invariant and linearly independent. Since $K_i \geq H$, we have $e_H \xi_i = e_H e_{K_i} \xi_i = e_{K_i} \xi_i = \xi_i, 1 \leq i \leq n$. It follows that $\xi_i$ is $H$ invariant, and can be thought as functions on $\ell(G/H)$. Since $e_G \xi_i = 0$, we conclude that $1, \xi_i, 1 \leq i \leq n$ are linearly independent functions on $\ell(G/H)$ and the corollary follows. \(\blacksquare\)

At the end of this appendix we discuss a question which is motivated by the following conjecture of Aschbacher-Guralnick in [1]:

**Conjecture A.10.** Let $G$ be a finite group. Then the number of conjugacy classes of maximal subgroups is less or equal to the number of conjugacy classes of $G$.

Conjecture A.10 was proved in [1] for solvable $G$. Here we give a slightly proof (with strict inequality) in the spirit of proof of Prop. A.7.

**Proposition A.11.** If $G$ is a finite solvable group, then the number of conjugacy classes of maximal subgroups is less than the number of conjugacy classes of $G$. 

34
Proof Let $K_i, 1 \leq i \leq n$ be a set of representatives of conjugacy classes of maximal subgroups of $G$. Let $C_1, \ldots, C_k$ be the conjugacy classes of $G$. Define $f_i := \sum_{g \in G} g e_{K_i} g^{-1} - |G| e_G, 1 \leq i \leq n$, $h_j := \sum_{g \in C_j} g, 1 \leq j \leq k$. Note that $e_G f_i = 0, e_G = |G|^{-1} \sum_{1 \leq j \leq k} h_j, h_j, 1 \leq j \leq k$ is linearly independent, and $f_i$ is in the space spanned by $h_j, 1 \leq j \leq k$. We claim that $f_i, 1 \leq i \leq n$ are linearly independent. This will prove the proposition since $f_i$ is in the space spanned by $h_j, 1 \leq j \leq k$, and $\sum_{1 \leq j \leq k} h_j f_i = 0$.

Suppose that $\lambda_i \in \mathbb{C}, 1 \leq i \leq n$ such that $\sum_{1 \leq i \leq n} \lambda_i f_i = 0$. Fix $g$ and $1 \leq i \leq n$. If $j \neq i$, then $g K_i g^{-1}$ is not conjugate to $h K_j h^{-1}, h \in G$ and by Th. 16.2 of [11] we have $g K_i g^{-1} h K_j h^{-1} = G$, and by Lemma [A.6] $g e_{K_i} g^{-1} h e_{K_j} h^{-1} = e_G$. Multiply $\sum_{1 \leq i \leq n} \lambda_i f_i = 0$ on the left by $g e_{K_i} g^{-1}$ we have

$$
\lambda_i g e_{K_i} g^{-1} f_i = g e_{K_i} g^{-1} \sum_{h \in G} h e_{K_i} h^{-1} - |G| e_G = 0, \forall g \in G.
$$

It follows that $|\lambda_i|^2 f_i f_i^* = 0$, and since $\mathbb{C} G$ is a $C^*$ algebra, $\lambda_i f_i = 0$. By looking at the coefficient of identity element of $G$ in $f_i$ we conclude that $\lambda_i(|G|/|K_i| - 1) = 0$. since $|G| > |K_i|$ we conclude that $\lambda_i = 0$. \hfill \blacksquare

The following question is motivated by conjecture A.10

Question A.12. Let $N \subset M$ be an irreducible subfactor with finite index. Let $\text{Aut}(M|N) := \{ \alpha \in \text{Aut}(M) | \alpha(n) = n, \forall n \in N \}$. We say that two intermediate subfactors $P_1, P_2$ are conjugate if there is an $\alpha \in \text{Aut}(M|N)$ such that $\alpha(P_1) = (P_2)$. Is the number of conjugacy classes of maximal or minimal subfactors less or equal to the number of irreducible representations of $N' \cap M_1$?

Remark A.13. There is a similar formulation of the above question using $N(M|N)$ and Lemma 2.8.

Take $N = M^G \subset M$. Then $\text{Aut}(M|M^G) = G, N' \cap M_1 = \mathbb{C} G$. The conjugacy classes of maximal subfactors is the same as the conjugacy classes of minimal subgroups of $G$, and it is easy to see that the number conjugacy classes of minimal subgroups of $G$ is less than the number of conjugacy class of $G$, which is the same as the number of irreducible representations of $N' \cap M_1 = \mathbb{C} G$. On the other hand the conjugacy classes of minimal subfactors is the same as the conjugacy classes of maximal subgroups of $G$, and question A.12 is equivalent to conjecture A.10.

Let $M$ be the cross product of $N$ by $G$. Then $\text{Aut}(M|N)$ is isomorphic to the set of one dimensional representations of $G$, and $N' \cap M_1 = l(G)$. In this case $\text{Aut}(M|N)$ preserves every intermediate subfactor.

The conjugacy classes of minimal subfactors can be identified as the set of minimal subgroups of $G$, and it is easy to see that the number of minimal subgroups of $G$ is less than $|G|$, which is the same as the number of irreducible representations of $N' \cap M_1 = l(G)$. On the other hand the conjugacy classes of maximal subfactors can be identified as the set of maximal subgroups of $G$, and question A.12 is equivalent to Wall’s conjecture (with $\leq$ instead of $<$).
However, as shown in §4.5 question A.12 has a negative answer for general sub-factors. Is there any natural modification of the statement in question A.12 so that it has a better chance of being true while still generalizing conjecture A.10?

References

[1] M. Aschbacher and R. Guralnick, Some applications of the first cohomology group, J. Algebra 90 (1984), no. 2, 446–460.

[2] R. Baddeley and A. Lucchini, On representing finite lattices as intervals in subgroup lattices of finite groups, J. Algebra 196 (1997), 1-100.

[3] J. Böckenhauer, D. E. Evans, Modular invariants, graphs and α-induction for nets of subfactors. I., Comm.Math.Phys., 197, 361-386, 1998.

[4] J. Böckenhauer, D. E. Evans, Modular invariants, graphs and α-induction for nets of subfactors. II., Comm.Math.Phys., 200, 57-103, 1999.

[5] J. Böckenhauer, D. E. Evans, Modular invariants, graphs and α-induction for nets of subfactors. III., Comm.Math.Phys., 205, 183-228, 1999.

[6] J. Böckenhauer, D. E. Evans, Modular invariants from subfactors: Type I coupling matrices and intermediate subfactors, Comm. Math. Phys. 213 (2000), no. 2, 267–289.

[7] J. Böckenhauer, D. E. Evans, Y. Kawahigashi, On α-induction, chiral generators and modular invariants for subfactors, Comm.Math.Phys., 208, 429-487, 1999. Also see math.OA/9904109.

[8] J. Böckenhauer, D. E. Evans, Y. Kawahigashi, Chiral structure of modular invariants for subfactors, Comm.Math.Phys., 210, 733-784, 2000.

[9] D. Bisch and V. F. R. Jones Algebras associated to intermediate subfactors, Invent. Math. 128 (1997), no. 1, 89–157.

[10] J. D. Dixon and B. Mortimer, Permutation groups, Graduate Texts in Mathematics, 163. Springer-Verlag, New York, 1996. xii+346 pp.

[11] Klaus Doerk and Trevor Hawkes, Finite soluble groups, 1939- Berlin ; New York : W. de Gruyter, 1992.

[12] W. Feit, An interval in the subgroup lattice of a finite group which is isomorphic to M7, Algebra Universalis 17 (1983), no. 2, 220–221.

[13] F. Goodman, P. de la Harpe and Vaughan F. R. Jones Coxeter graphs and towers of algebras. Mathematical Sciences Research Institute Publications, 14. Springer-Verlag, New York, 1989. x+288 pp.

[14] P. Grossman and Vaughan F. R. Jones, Intermediate subfactors with no extra structure, J. Amer. Math. Soc. 20 (2007), no. 1, 219–265.

[15] P. Grossman, Forked Temperley-Lieb algebras and intermediate subfactors, J. Funct. Anal. 247 (2007), no. 2, 477–491.

[16] D. Guido & R. Longo, Relativistic invariance and charge conjugation in quantum field theory, Commun. Math. Phys. 148 (1992) 521–551.

[17] D. Guido & R. Longo, The conformal spin and statistics theorem, Commun. Math. Phys. 181 (1996) 11–35.

[18] R. M. Guralnick and C. Hoffman, The first cohomology group and generation of simple groups, Groups and geometries (Siena, 1996), 81–89, Trends Math., Birkhäuser, Basel, 1998.
[19] R. M. Guralnick, W. Kantor, M. Kassabov and A. Lubotzky, Presentations of finite simple groups: profinite and cohomological approaches, Groups Geom. Dyn. 1 (2007), no. 4, 469–523.

[20] M. Izumi, R. Longo & S. Popa, A Galois correspondence for compact groups of automorphisms of von Neumann Algebras with a generalization to Kac algebras, J. Funct. Analysis, 155, 25-63 (1998).

[21] V. F. R. Jones, Fusion en algèbres de von Neumann et groupes de lacets (d’après A. Wassermann). (French) [Fusion in von Neumann algebras and loop groups (after A. Wassermann)], Séminaire Bourbaki, Vol. 1994/95. Astérisque No. 237 (1996), Exp. No. 800, 5, 251–273.

[22] V. F. R. Jones, Two subfactors and the algebraic decompositions of bimodules over II_1 factors, Preprint available from http://math.berkeley.edu/~vfr/.

[23] Vaughan F. R. Jones and F. Xu, Intersections of finite families of finite index subfactors, Internat. J. Math. 15 (2004), no. 7, 717–733.

[24] V. G. Kac, “Infinite Dimensional Lie Algebras”, 3rd Edition, Cambridge University Press, 1990.

[25] V. G. Kac, R. Longo and F. Xu, Solitons in affine and permutation orbifolds, Comm. Math. Phys. 253 (2005), no. 3, 723–764.

[26] Y. Kawahigashi and R. Longo, Classification of two-dimensional local conformal nets with c < 1 and 2-cohomology vanishing for tensor categories, Comm. Math. Phys. 244 (2004), no. 1, 63–97.

[27] M. W. Liebeck, L. Pyber and A. Shalev, On a conjecture of G. E. Wall, J. Algebra 317 (2007), no. 1, 184–197.

[28] R. Longo, Index of subfactors and statistics of quantum fields. I, Commun. Math. Phys. 126 (1989) 217–247.

[29] R. Longo, Index of subfactors and statistics of quantum fields. II, Commun. Math. Phys. 130 (1990) 285–309.

[30] R. Longo, Minimal index and braided subfactors, J. Funct. Anal. 109 (1992), 98–112.

[31] R. Longo, Conformal subnets and intermediate subfactors, Commun. Math. Phys. 237 n. 1-2 (2003), 7–30.

[32] R. Longo & K.-H. Rehren, Nets of subfactors, Rev. Math. Phys. 7 (1995) 567–597.

[33] Péter P. Pálfy, Groups and lattices, 429-454, Groups St Andrews 2001 in Oxford, London Mathematical Society Lecture Note Series 305. Edited by C. M. Campbell, E. F. Robertson and G. C. Smith.

[34] M. Pimsner, & S. Popa, Entropy and index for subfactors, Ann. Scient. Ec. Norm. Sup. 19 (1986), 57–106.

[35] S. Popa, Correspondence, INCREST manuscript, 1986.

[36] A. Pressley and G. Segal, “Loop Groups” Oxford University Press 1986.

[37] K.-H. Rehren, Braid group statistics and their superselection rules, in “The Algebraic Theory of Superselection Sectors”, D. Kastler ed., World Scientific 1990.

[38] T. Teruya and Y. Watatani, Lattices of intermediate subfactors for type III factors, Arch. Math. (Basel) 68 (1997), no. 6, 454–463.

[39] V. G. Turaev, Quantum invariants of knots and 3-manifolds, Walter de Gruyter, Berlin, New York 1994.

[40] G. E. Wall, Some applications of the Eulerian functions of a finite group, J. Austral. Math. Soc. 2 1961/1962 35–59.
[41] Y. Watatani, *Lattices of intermediate subfactors*, J. Funct. Anal. 140 (1996), no. 2, 312–334.

[42] A. Wassermann, Operator algebras and Conformal field theories III, *Invent. Math.* 133 (1998), 467-538.

[43] F. Xu, *New braided endomorphisms from conformal inclusions*, Commun. Math. Phys. 192 (1998) 347–403.

[44] F. Xu, *On representing some lattices as lattices of intermediate subfactors of finite index*, arXiv:math/0703248, to appear in Advance in Mathematics.

[45] F. Xu, *3-manifold invariants from cosets*, Journal of Knot theory and its ramifications, Vol. 14, no. 1 (2005), 21-90.