GAUGE THEORY AND FOLIATIONS I;
GERM CORDS VERSUS QUANTUM CORDS

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Abstract. We apply gauge theory to study the space $F_k(M)$ of smooth codimension-$k$ framed foliations on a smooth manifold $M$. The quotient of Maurer-Cartan elements by the action of an infinite dimensional non-abelian gauge groupoid forms a moduli space, which contains $F_k(M)$ as a subspace. The notion of holonomy is naturally extended to this moduli space and the cohomology theory associated with points of this moduli space which correspond to non-singular foliations coincides with Bott cohomology. The quotient of the moduli space under concordance is identified as the space of homotopy classes of maps to the classifying spaces $B\Gamma^g_k$ and $B\Gamma^q_k$. While $B\Gamma^q_k$ is a classic and has been studied since Haefliger, $B\Gamma^q$ (which is a quotient of $B\Gamma^q_k$) carries a simpler topology and offers a rival theory.

1. Introduction

Foliations grew out of Poincaré’s qualitative theory of differential equations and Ehresmann’s connection theory on vector bundles. The central idea in both, is the notion of holonomy or monodromy which dates even farther back to the time of Cauchy and Riemann. Gauge theory and foliations crossed paths several times. Perhaps the first happened in the 40’s as Ehresmann developed the connection theory of vector bundles and generalized Poincaré’s holonomy of a loop lying on a leaf of a foliation. In 70’s, the Godbillon-Vey invariants of foliations were introduced [14] few years prior to the Chern-Simons functional in gauge theory. The similarity between the two intrigued and inspired mathematicians. Most notably, R. Bott introduced the notion of (partial) Bott connection on the normal bundle of a foliation, c.f. [4]. Bott connection is flat on the leaves, generalizing the Reeb’s class (which measures the transverse holonomy expansion [12]) to higher codimensions. Theory of foliations went through rapid developments in 70’s, as Thurston proved his important existence result [29] and Haefliger structures provided a framework for classification of foliations up to concordance [17, 18]. Mather [23, 24] and Thurston [28] proved important theorems about the classifying space of Haefliger structures, which may be compared with classification results for homogenous foliations and flat connections [2, 3]. These classification results linked the study of foliations on spheres to the homotopy theory of the classifying space of Haefliger structures and resulted in several existence and non-existence theorems [16, 27, 28]. Despite these similarities, a path from foliations back to gauge theory has been missing.

Contrary to Ehresmann who saw a foliation in a flat connection, we associate flat connections to framed foliations. Such connections turn out to be gauge equivalent. We examine Frobenius equation from a gauge theoretic point of view and identify nonabelian infinite dimensional gauge groups. If a smooth codimension-one foliation $\mathcal{F}$ on $M$ is given by a 1-form $a_0 \in \Omega^0(M, \mathbb{R})$, Frobenius equation implies that $da_0 = a_1 a_0$ for another 1-form $a_1 \in \Omega^1(M, \mathbb{R})$ which is determined up to scalar multiples of $a_0$. The process may be repeated to obtain the 1-forms $a_2, a_3, a_4, \cdots \in \Omega^1(M, \mathbb{R})$ so that their derivatives satisfy a sequence of equations, starting with $da_1 = a_2 a_0$ and $da_2 = a_3 a_0 + a_2 a_1$. One motivation for the current work is to present the Godbillon-Vey sequences $(a_0, a_1, a_2, \cdots)$ [13, 12] as geometric objects (gauge fields) and observe their local symmetries through gauge action. Gauge theory then suggests a study of the moduli space of foliations, while we are lead to welcome certain singular objects (foliations). Two groupoids are encountered along the journey, as we consider smooth framed foliation of codimension $k$ on a manifold $M$. The first groupoid is $Q_k$ which consists
of formal power series of the form

\[ Y = \sum_{i=1}^{k} \sum_{l} y_{i,l}(t_{1} - x_{1})^{i_{1}} \cdots (t_{k} - x_{k})^{i_{k}} \partial_{l} = \sum_{i,l} y_{i,l}(t - x)^{i} \partial_{l} \]

in the formal variables \( t_{1}, \ldots, t_{k} \) with \( \det(y_{i,j})_{i,j=1}^{k} > 0 \). The index \( I \) runs over the \( k \)-tuples \( (i_{1}, \ldots, i_{k}) \in \mathbb{Z}^{k} \) of non-negative integers and \( \partial_{1}, \ldots, \partial_{l} \) denote the standard unit vectors of \( \mathbb{R}^{k} \).

The power series \( Y \) is realized as an arrow from the source \( x = (x_{1}, \ldots, x_{k}) \in \mathbb{R}^{k} = \text{Obj}(Q_{k}) \) to the target \( y = (y_{1}, \ldots, y_{k}, \theta) \in \mathbb{R}^{k} = \text{Obj}(Q_{k}) \). The groupoid \( Q_{k} \) acts on its Lie algebra \( q_{k} \) of all power series of the form \( A = \sum_{i,l} a_{i,l}(t - x)^{i} \partial_{l} \). Alternatively, a parallel theory is created if we replace the gauge groupoid \( Q_{k} \) with the groupoid \( G_{k} \) of germs of local diffeomorphisms of \( \mathbb{R}^{k} \) and replace \( q_{k} \) with the algebroid \( g_{k} \) of germs of smooth \( \mathbb{R}^{k} \)-valued maps at points of \( \mathbb{R}^{k} \). For every smooth manifold \( M, \Omega^{0}(M, Q_{k}) \) acts on flat \( q_{k} \)-valued 1-forms, which are called quantum cords and are denoted by \( \wedge(M, Q_{k}) \). Similarly, \( \Omega^{0}(M, G_{k}) \) acts on the space \( \wedge(M, G_{k}) \) of germ cords. We write \( \Gamma^{g}_{k} \) and \( \Gamma^{q}_{k} \) for \( G_{k} \) and \( Q_{k} \) if we want to emphasize that the discrete topology is chosen on the space of arrows from \( x \in \mathbb{R}^{k} \) to \( y \in \mathbb{R}^{k} \).

If \( (a_{0}, a_{1}, a_{2}, \ldots) \) is the Godbillon-Vey sequence associated with a transversely oriented codimension one foliation \( \mathcal{F} \) on \( M \), \( A = \sum_{n} a_{n}v^{n} \in \wedge(M, q_{1}) \) would be a quantum cord. Our main theorem may be stated as follows:

**Theorem 1.1.** The space \( \mathcal{F}_{k}(M) \) of smooth framed codimension \( k \) foliations of \( M \) is embedded in the moduli space of germs of cords and quantum cords (upto gauge action) while there are natural classification maps \( c \) and \( c \) from these moduli spaces to the space of all Haefliger \( \Gamma_{k}^{g} \)-structures and \( \Gamma_{k}^{q} \)-structures (respectively), which make the following diagram commutative.

\[
\begin{array}{ccc}
\mathcal{F}_{k}(M) & \xrightarrow{\mathcal{I}_{k}} & \mathcal{M}(M, g_{k}, G_{k}) = \wedge(M, g_{k})/\Omega^{0}(M, G_{k}) \\
\xrightarrow{\text{Id}} & & \xrightarrow{\} \wedge(M, \Gamma_{k}^{g}) \\
\mathcal{F}_{k}(M) & \xleftarrow{\mathcal{I}_{k}} & \mathcal{M}(M, q_{k}, Q_{k}) = \wedge(M, q_{k})/\Omega^{0}(M, Q_{k}) \\
\xrightarrow{\text{T}} & & \xrightarrow{\} \wedge(M, \Gamma_{k}^{q}) \\
\end{array}
\]

The maps \( T \) are obtained by taking Taylor expansions. The classification maps \( c \) and \( c \) induce the maps

\[ c : \mathcal{C}(M, g_{k}) = \wedge(M, g_{k})/\sim_{g} \rightarrow [M, \mathcal{B}^{g}_{k}] \quad \text{and} \quad c : \mathcal{C}(M, q_{k}) = \wedge(M, q_{k})/\sim_{q} \rightarrow [M, \mathcal{B}^{q}_{k}] \]

from the germs concordia (concordance classes of germ cords) and quantum concordia (concordance classes of quantum cords) to the spaces of homotopy classes of maps from \( M \) to the classifying spaces \( \mathcal{B}^{g}_{k} \) and \( \mathcal{B}^{q}_{k} \), respectively.

Let us fix the codimension \( k \) and drop it from the notation for simplicity. Let us denote the set of arrows in \( G = G_{k} \) which start at source \( x \in \mathbb{R}^{k} \) and end at target \( y \in \mathbb{R}^{k} \) by \( G_{x \rightarrow y} \). Similarly, define \( Q_{x \rightarrow y} \) and note that \( G_{0 \rightarrow 0} \) and \( Q_{0 \rightarrow 0} \) are both topological groups, where \( Q_{0 \rightarrow 0} \) has the structure of an infinite dimensional Lie group. Let \( q_{0 \rightarrow 0} \) and \( g_{0 \rightarrow 0} \) denote the Lie algebras of \( Q_{0 \rightarrow 0} \) and \( G_{0 \rightarrow 0} \), respectively. The cords in \( \wedge(M, q_{0 \rightarrow 0}) \) and \( \wedge(M, g_{0 \rightarrow 0}) \) are called the impotent quantum and germ cords respectively. The algebroids \( g_{0 \rightarrow 0} \) and \( q_{0 \rightarrow 0} \) are (respectively) sub-algebroids of the fibers \( g_{0} \) and \( q_{0} \) of \( g \) and \( q \) over \( 0 \in \mathbb{R}^{k} \). An important source of examples of impotent quantum and germ cords is the restriction of a quantum or germ cord to the leaves of a foliation \( \mathcal{F} \in \mathcal{F}_{k}(M) \).

**Theorem 1.2.** For every smooth manifold \( M \), there are natural bijections

\[ \rho_{M}^{g} : \wedge(M, g_{0 \rightarrow 0})/\Omega^{0}(M, G_{0 \rightarrow 0}) \rightarrow \text{Hom}(\pi_{1}(M), G_{0 \rightarrow 0})/G_{0 \rightarrow 0} \quad \text{and} \quad \rho_{M}^{q} : \wedge(M, q_{0 \rightarrow 0})/\Omega^{0}(M, Q_{0 \rightarrow 0}) \rightarrow \text{Hom}(\pi_{1}(M), Q_{0 \rightarrow 0})/Q_{0 \rightarrow 0}. \]

If \( L \) is a leaf of a foliation \( \mathcal{F} \in \mathcal{F}_{k}(M) \) which corresponds to a germ cord \( A \in \wedge(M, g_{0}) \), the conjugacy class of the homomorphism \( \rho_{L}^{g}(A|_{L}) : \pi_{1}(L) \rightarrow G_{0 \rightarrow 0} \) gives the holonomy of the leaf \( L \).
This theorem gives a way to generalize the notion of leaves and their holonomy for singular foliations, i.e. arbitrary gauge equivalence classes

\[ \mathcal{F} \in \bigwedge(M, \mathfrak{g}_0) / \Omega^0(M, \mathcal{G}_{0,\rightarrow}) = \bigwedge(M, \mathfrak{g}) / \Omega^0(M, \mathfrak{g}). \]

A leaf-like map for the singular foliation \( \mathcal{F} \), which corresponds to some \( A \in \bigwedge(M, \mathfrak{g}_0) \), is a diffeomorphism \( f : L \rightarrow M \) from a smooth manifold \( L \) to \( M \) such that \( f^*A \) is impotent. The definition is independent of the choice of the representative \( A \) for the singular foliation \( \mathcal{F} \). Associated with a leaf-like map \( f : L \rightarrow M \), we obtain the conjugacy class of a holonomy map

\[ \rho_{\mathcal{F},L} = \rho^A_L(A|_L) \in \text{Hom}(\pi_1(L), \mathcal{G}_{0,\rightarrow}) / \mathcal{G}_{0,\rightarrow} \cong M(L, \mathcal{G}_{0,\rightarrow}, \mathcal{G}_{0,\rightarrow}). \]

This notion of holonomy generalizes the usual holonomy map for the leaves of non-singular foliations. It is nice to compare this approach with the approaches of [9].

Every flat 1-form may be used to define a twisted differential on differential forms. In particular, each \( A \in \bigwedge(M, \mathfrak{g}_k) \) gives a differential

\[ \nabla_A : \Omega^*_k(M, \mathfrak{g}) \rightarrow \Omega^{*+1}_k(M, \mathfrak{g}) \]

which satisfies \( \nabla_A \circ \nabla_A = 0 \). Here, the subscript \( s_A \) indicates that we only consider the differential forms \( E \in \Omega^*_k(M, \mathfrak{g}) \) which satisfy \( \nabla g = s_A \), i.e. the source maps associated with \( A \) and \( E \) are the same. Correspondingly, we obtain the cohomology groups associated with \( A \) which are denoted by \( H^*_k(M, \mathfrak{g}) \). Similarly, we can define the cohomology groups \( H^*_k(M, \mathfrak{g}, \mathfrak{Q}) \) for every \( A \in \bigwedge(M, \mathfrak{q}_k) \).

**Theorem 1.3.** The cohomology groups \( H^*_k(M, \mathfrak{g}) \) and \( H^*_k(M, \mathfrak{g}, \mathfrak{Q}) \) are independent of the choice of \( A \) and \( \mathfrak{g} \) in their gauge equivalence classes and their isomorphism classes are well-defined for every singular foliation

\[ \mathcal{F} \in \mathcal{M}(M, \mathfrak{g}_k, \mathfrak{G}_k) \quad \text{or} \quad \mathcal{F} \in \mathcal{M}(M, \mathfrak{q}_k, \mathfrak{Q}_k). \]

If \( \mathcal{F} \) is non-singular, the cohomology groups are both isomorphic to the Bott cohomology of \( \mathcal{F} \).

The investigations of this paper indicate that the moduli spaces \( \mathcal{M}(M, \mathfrak{g}_k, \mathcal{G}_k) \) and \( \mathcal{M}(M, \mathfrak{q}_k, \mathfrak{Q}_k) \) share many properties as completions of the space of all smooth framed foliations on \( M \). The notion of concordance for germ cords and quantum cords, gives the quotients \( C(M, \mathfrak{g}_k) \) and \( C(M, \mathfrak{q}_k) \) which may be studied through the classification spaces \( B\Gamma_k^\mathfrak{g} \) and \( B\Gamma_k^\mathfrak{Q} \) of \( \Gamma_k^\mathfrak{g} \) and \( \Gamma_k^\mathfrak{Q} \), respectively. The results of this paper would thus suggest that the space \( [M, B\Gamma_k^\mathfrak{g}] \) of homotopy classes of maps from \( M \) to \( B\Gamma_k^\mathfrak{g} \) may be studied to classify smooth foliations on \( M \), similar to [16], [23] and [28].

We start by the study of the gauge action of \( \text{Diff}^+(\mathbb{R}) \) on \( C^\infty(\mathbb{R}) \)-valued flat 1-forms (cords) in Section 2. In Section 3 we introduce germ cords and quantum cords and their relevance in the study of codimension-one foliations. Section 4 is devoted to the study of holonomy for leaf-like submanifold. In Section 5 we formulate and prove our classification theorems for germ and quantum cords and the corresponding classification of codimension-one foliations. Section 6 is a quick review of the relation between the cohomology theory for germ cords and quantum cords and the Bott cohomology of a foliation. Finally, Section 7 states the main results of the previous sections, which are formulated for foliations of codimension one, to the case of general smooth framed foliations of arbitrary codimension. In [1], we investigate complex cords and residues. The gauge theoretic approach conveyed here takes us to a conjecture which is both an attempt in fixing the Seifert conjecture [26] as well as a complex analogue to the Novikov’s compact leaf theorem [25].

## 2. Flat Connections and Codimension-One Foliations

### 2.1. Diffeomorphisms of \( \mathbb{R} \) as a Gauge Group

Let \( \text{Diff}^+(\mathbb{R}) \) denote the group of orientation preserving diffeomorphisms of \( \mathbb{R} \). The Lie algebra of \( \text{Diff}^+(\mathbb{R}) \) consists of smooth vector fields on \( \mathbb{R} \) with the usual Lie bracket on vector fields, and is thus identified with \( C^\infty(\mathbb{R}) \). Both \( \text{Diff}^+(\mathbb{R}) \) and \( C^\infty(\mathbb{R}) \) are Fréchet spaces. The Lie bracket on \( C^\infty(\mathbb{R}) \) is given by

\[ [\cdot, \cdot] : C^\infty(\mathbb{R}) \times C^\infty(\mathbb{R}) \rightarrow C^\infty(\mathbb{R}), \quad [A, B] := AB' - A'B, \quad \forall A, B \in C^\infty(\mathbb{R}). \]
Let us assume that $M$ is a smooth manifold of dimension $n$. We may consider the smooth $C^\infty(\mathbb{R})$-valued differential forms

$$\Omega^k(M, C^\infty(\mathbb{R})) = \bigoplus_{k=0}^{n} \Omega^k(M, C^\infty(\mathbb{R})) = \bigoplus_{k=0}^{n} \Gamma \left( M, \wedge^k (\mathcal{L}) \otimes_{\mathbb{R}} C^\infty(\mathbb{R}) \right).$$

We denote elements of $\Omega^k(M, C^\infty(\mathbb{R}))$ by capital letters in sans serif font, i.e. $A$, $B$. The differential of $A$ with respect to its $\mathbb{R}$-variable is denoted by $A'$ or $\partial A$. Note that $A'$ is also a smooth differential form with values in $C^\infty(\mathbb{R})$. The Lie bracket of $C^\infty(\mathbb{R})$ induces a Lie bracket on $\Omega^*(M, C^\infty(\mathbb{R}))$, giving it the structure of a differential graded Lie algebra, or a DG Lie algebra for short. Note that for $A \in \Omega^k(M, C^\infty(\mathbb{R}))$ and $B \in \Omega^l(M, C^\infty(\mathbb{R}))$

$$d[A, B] = ((dA)B' - (dA')B) + (-1)^k(A(dB') - A'(dB)) = [dA, B] + (-1)^k[A, dB].$$

A 1-form $A \in \Omega^1(M, C^\infty(\mathbb{R}))$ is called a cord if it satisfies the Maurer-Cartan equation

$$dA + \frac{1}{2}[A, A] = dA - A'A = 0.$$

The space of cords is denoted by $\mathcal{C}(M, C^\infty(\mathbb{R}))$.

The adjoint action of $\text{Diff}^+(\mathbb{R})$ on $C^\infty(\mathbb{R})$ is described as follows. For each $Y \in \text{Diff}^+(\mathbb{R})$, the function $\text{Adj}_Y : \text{Diff}^+(\mathbb{R}) \to \text{Diff}^+(\mathbb{R})$ defined by

$$\text{Adj}_Y(X) = Y^{-1} \circ X \circ Y \quad \forall \, X \in \text{Diff}^+(\mathbb{R})$$

fixes the identity. Thus, the differential of $\text{Adj}_Y$ gives a smooth map $d\text{Adj}_Y : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ and the adjoint action

$$\star : \text{Diff}^+(\mathbb{R}) \times C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R}), \quad Y \star A := d\text{Adj}_Y(A).$$

**Proposition 2.1.** The adjoint action of $\text{Diff}^+(\mathbb{R})$ on $C^\infty(\mathbb{R})$ is given by

$$Y \star A := \frac{A \circ Y}{Y'} \quad \forall \, Y \in \text{Diff}^+(\mathbb{R}), \, A \in C^\infty(\mathbb{R}).$$

**Proof.** For $A \in C^\infty(\mathbb{R})$, let us assume that $X_s$, for $s \in (-\epsilon, \epsilon)$, is a path in $\text{Diff}^+(\mathbb{R})$ with $X_0 = \text{Id}_{\text{Diff}^+(\mathbb{R})}$ and $\partial_s X_s|_{s=0} = A$. The element $Y \star A \in C^\infty(\mathbb{R})$ is then given by

$$\partial_s \left( (Y^{-1} \circ X_s \circ Y) \right)_{s=0} = \left( ((Y^{-1})' \circ X_s \circ Y) \cdot (\partial_s X_s) \circ Y \right)_{s=0} = ((Y^{-1})' \circ Y) \cdot (A \circ Y) = \frac{A \circ Y}{Y'}.$$

This completes the proof. \qed

Let us assume that $M$ is an oriented smooth manifold. If $Y \in \Omega^0(M, \text{Diff}^+(\mathbb{R}))$ is a function on $M$ with values in $\text{Diff}^+(\mathbb{R})$ and $E \in \Omega^*(M, C^\infty(\mathbb{R}))$ is a $C^\infty(\mathbb{R})$-valued differential form on $M$, we can define $E \circ Y \in \Omega^*(M, C^\infty(\mathbb{R}))$ by

$$(E \circ Y)(t, x) := E(Y(t, x), x) \quad \forall \, t \in \mathbb{R}, \, x \in M.$$

Note that

$$(E \circ Y)' = (E' \circ Y)Y' \quad \text{and} \quad d(E \circ Y) = (dE) \circ Y + (dY)(E' \circ Y).$$

**Proposition 2.2.** The group $\Omega^0(M, \text{Diff}^+(\mathbb{R}))$ acts on $\mathcal{C}(M, C^\infty(\mathbb{R}))$ by

$$\star : \Omega^0(M, \text{Diff}^+(\mathbb{R})) \times \mathcal{C}(M, C^\infty(\mathbb{R})) \to \mathcal{C}(M, C^\infty(\mathbb{R})), \quad Y \star A := \frac{A \circ Y - dY}{Y'}. $$

**Proof.** Note that the action defined above is clearly smooth and that

$$Z \star (Y \star A) = \frac{A \circ Y \circ Z - (dY) \circ Z - (Y' \circ Z)dz}{(Y' \circ Z)'Z'} = \frac{A \circ Y \circ Z - d(Y \circ Z)}{(Y \circ Z)'Z'} = (Y \circ Z) \star A.$$
Let us assume that $A \in \Omega^1(M, C^\infty(\mathbb{R}))$, $B = Y \ast A$ and $F_A = dA + AA'$. We then have
\[
F_B = dB + BB' = \frac{(dA) \circ Y - (A' \circ Y)dY - (dY')B}{Y'} - \frac{((A' \circ Y)Y' - dY' - Y'dY)B}{Y'}.
\]
In particular, if $A \in \Lambda(M, C^\infty(\mathbb{R}))$ then $Y \ast A \in \Lambda(M, C^\infty(\mathbb{R}))$, which completes the proof.

2.2. Classification of cords. One would naturally lean to study the quotient of the space of cords on a smooth manifold $M$ under the gauge action of $\Omega^0(M, \text{Diff}^+(\mathbb{R}))$. It is expected that this quotient is identified with the conjugacy classes of group homomorphisms from $\pi_1(M)$ to $\text{Diff}^+(\mathbb{R})$. Nevertheless, this statement, which comes to mind from working with finite dimensional Lie groups, is in general not true for infinite dimensional Lie groups (and their Lie algebras). The main issue is that the quotient $\Lambda(U, C^\infty(\mathbb{R}))/\Omega^0(U, \text{Diff}^+(\mathbb{R}))$ can be non-zero, and in fact highly non-trivial, for small contractible open subsets $U \subset \mathbb{R}^n$, i.e. the Poincaré Lemma is not satisfied.

We may easily construct $A \in \Lambda(U, C^\infty(\mathbb{R}))$ so that $A$ is not gauge equivalent to zero, i.e. so that $A$ is not of the form $-dY/Y'$, even when $U$ is an open subset of a 1-dimensional manifold (and may thus be identified with $\mathbb{R}$). For instance, let us set
\[
A(t, s) = (1 + t^2)ds, \quad \forall \ (t, s) \in \mathbb{R} \times \mathbb{R}.
\]
If $A = -dY/Y'$, it follows that $(1+t^2)\partial_t Y = \partial_s Y$. If we write $t = \tan(\theta)$ it follows that $\partial_\theta Y + \partial_s Y = 0$. In particular, $Y$ is constant on the images of the curves
\[
\gamma_s : (-\pi/2, \pi/2) \to \mathbb{R}^2 \quad \gamma_s(\theta) = (\tan(\theta), s + \theta).
\]
Nevertheless, this means that $Y(t, s + \pi_2)$ is bounded above by $Y(0, s)$, which is a contradiction.

Definition 2.3. A cord $A \in \Lambda(M, C^\infty(\mathbb{R}))$ is called locally trivial if for every point $x \in M$ there is an open subset $U \subset M$ containing $x$ so that $A|_U$ is gauge equivalent to zero. The space of all locally trivial cords is denoted by $\Lambda^L(M, C^\infty(\mathbb{R}))$.

It is then clear that $\Omega^0(M, \text{Diff}^+(\mathbb{R}))$ takes locally trivial cords to locally trivial cords, and we thus obtain an action of $\Omega^0(M, \text{Diff}^+(\mathbb{R}))$ on $\Lambda^L(M, C^\infty(\mathbb{R}))$. Let us assume that $f : M_1 \to M_2$ is a smooth map between smooth manifolds. If $A$ is locally trivial, it follows that $f^*A$ is also locally trivial. This gives a natural pull-back map
\[
f^* : \Lambda^L(M_2, C^\infty(\mathbb{R})) \to \Lambda^L(M_1, C^\infty(\mathbb{R})).
\]
Let us denote $\Lambda^L(M, C^\infty(\mathbb{R}))/\Omega^0(M, \text{Diff}^+(\mathbb{R}))$ by $\mathcal{M}(M, C^\infty(\mathbb{R}), \text{Diff}^+(\mathbb{R}))$. From the equality $f^*(Y \ast A) = (f^*Y) \ast (f^*A)$, it follows that $f^*$ induces a map
\[
f^* : \mathcal{M}(M_2, C^\infty(\mathbb{R}), \text{Diff}^+(\mathbb{R})) \to \mathcal{M}(M_1, C^\infty(\mathbb{R}), \text{Diff}^+(\mathbb{R})).
\]

Theorem 2.4. For every smooth manifold $M$, there is a natural bijection
\[
\rho_M : \mathcal{M}(M, C^\infty(\mathbb{R}), \text{Diff}^+(\mathbb{R})) \to \text{Hom}(\pi_1(M), \text{Diff}^+(\mathbb{R}))/\text{Diff}^+(\mathbb{R}),
\]
where $\text{Diff}^+(\mathbb{R})$ acts on $\text{Hom}(\pi_1(M), \text{Diff}^+(\mathbb{R}))$ by conjugation. If $f : M_1 \to M_2$ is a smooth map between smooth manifolds, the following diagram is commutative:
\[
\begin{array}{ccc}
\mathcal{M}(M_2, C^\infty(\mathbb{R}), \text{Diff}^+(\mathbb{R})) & \xrightarrow{f^*} & \mathcal{M}(M_1, C^\infty(\mathbb{R}), \text{Diff}^+(\mathbb{R})) \\
\rho_{M_2} \downarrow & & \downarrow \rho_{M_1} \\
\text{Hom}(\pi_1(M_2), \text{Diff}^+(\mathbb{R}))/\text{Diff}^+(\mathbb{R}) & \xrightarrow{f^*} & \text{Hom}(\pi_1(M_1), \text{Diff}^+(\mathbb{R}))/\text{Diff}^+(\mathbb{R})
\end{array}
\]
Thus, a connected manifold has every locally trivial cord is gauge equivalent to zero. Choose \( Y_\alpha \in \Omega^0(U_\alpha, \text{Diff}^+(\mathbb{R})) \) so that \( A|_{U_\alpha} = Y_\alpha \ast 0 \). Over \( U_\alpha \cap U_\beta \) we find

\[
A|_{U_\alpha \cap U_\beta} = Y_\alpha \ast 0 = Y_\beta \ast 0 \Rightarrow (Y_\alpha \circ Y_\beta^{-1}) \ast 0 = 0.
\]

This means that \( c_{\alpha \beta} = Y_\alpha \circ Y_\beta^{-1} : U_\alpha \cap U_\beta \to \text{Diff}^+(\mathbb{R}) \) is locally constant. These maps satisfy the cocycle condition and give a cohomology class in the Čech cohomology \( \check{H}^1(M, \text{Diff}^+(\mathbb{R})) \). The functions \( Y_\alpha \) are well-defined only up to composition with locally constant functions. If \( Z_\alpha = d_\alpha \circ Y_\alpha \), where \( d_\alpha : U_\alpha \to \text{Diff}^+(\mathbb{R}) \) is locally constant, the Čech cocycle \( c_{\alpha \beta}' \) associated with \( \{Z_\alpha\} \) would be given by

\[
c_{\alpha \beta}' = (d_\alpha \circ Y_\alpha) \circ (d_\beta \circ Y_\beta)^{-1} = d_\alpha \circ c_{\alpha \beta} \circ d_\beta^{-1} : U_\alpha \cap U_\beta \to \text{Diff}^+(\mathbb{R}).
\]

This means that the cohomology classes represented by \( \{c_{\alpha \beta}\} \) and \( \{c_{\alpha \beta}'\} \) are the same. Moreover, if we gauge \( A \) by \( Y \in \Omega^0(M, \text{Diff}^+(\mathbb{R})) \), the cocycles associated with \( A \) and \( Y \ast A \) are the same and we obtain a well-defined map

\[
\rho = \rho_M : \mathcal{M}(M, C^\infty(\mathbb{R}), \text{Diff}^+(\mathbb{R})) \to \check{H}^1(M, \text{Diff}^+(\mathbb{R})) \simeq \text{Hom}(\pi_1(M), \text{Diff}^+(\mathbb{R}))/\text{Diff}^+(\mathbb{R}).
\]

If \( \rho(A) = \rho(B) \) it follows that the corresponding cocycles \( \{c^A_{\alpha \beta}\} \) and \( \{c^B_{\alpha \beta}\} \) are related by the locally constant functions \( d_\alpha : U_\alpha \to \text{Diff}^+(\mathbb{R}) \), after we pass to refinement of the coverings associated with \( A \) and \( B \). In particular,

\[
d_\alpha \circ c^A_{\alpha \beta} = c^B_{\alpha \beta} \circ d_\beta \Rightarrow d_\alpha \circ Y_\alpha \circ Y_\beta^{-1} = Z_\alpha \circ Z_\beta^{-1} \circ d_\beta,
\]

where \( Z_\alpha \ast 0 = B|_{U_\alpha} \). It follows from the above equality that \( Z_\beta^{-1} \circ d_\beta \circ Y_\beta = Z_\alpha^{-1} \circ d_\alpha \circ Y_\alpha \) over \( U_\alpha \cap U_\beta \), and we can define \( X \in \Omega^0(M, \text{Diff}^+(\mathbb{R})) \) by \( X|_{U_\alpha} = Z_\alpha^{-1} \circ d_\alpha \circ Y_\alpha \). Over \( U_\alpha \) we can then compute

\[
X \ast B|_{U_\alpha} = (Z_\alpha^{-1} \circ d_\alpha \circ Y_\alpha) \ast (Z_\alpha \ast 0) = Y_\alpha \ast d_\alpha \ast 0 = A|_{U_\alpha}.
\]

Thus, \( A = X \ast B \) and the two cords are gauge equivalent. It follows from here that over simply connected manifolds every locally trivial cord is gauge equivalent to zero.

To finish the proof, we need to show that the map \( \rho = \rho_M \) is surjective. Let us assume that \( \{c_{\alpha \beta} : U_\alpha \cap U_\beta \to \text{Diff}^+(\mathbb{R})\} \) is a cocycle in \( \check{H}^1(M, \text{Diff}^+(\mathbb{R})) \). Choose a smooth partition of unity \( \{\lambda_\alpha : U_\alpha \to \mathbb{R}_{\geq 0}\}_\alpha \) subordinate to the cover \( \{U_\alpha\}_\alpha \) of \( M \) and define

\[
Y_\alpha : U_\alpha \to \text{Diff}^+(\mathbb{R}), \quad Y_\alpha(t, x) = \sum_\gamma \lambda_\gamma(x)c_{\gamma \alpha}(t).
\]

Note that \( \partial_t Y_\alpha > 0 \), and \( Y_\alpha(\cdot, x) \) is thus a diffeomorphism for all \( x \in U_\alpha \). Over the intersections \( U_\alpha \cap U_\beta \) we have \( Y_\alpha \circ c_{\alpha \beta} = Y_\beta \). If we define \( Z_\alpha : U_\alpha \to \text{Diff}^+(\mathbb{R}) \) by \( Z(x, \cdot) = Y(x, \cdot)^{-1} \), we find \( Z_\beta = c_{\alpha \beta} \circ Z_\alpha \) and thus \( Z_\beta \ast 0 = Z_\beta \ast c_{\alpha \beta} \ast 0 = (c_{\alpha \beta} \circ Z_\beta) \ast 0 = Z_\alpha \ast 0 \). In particular, the cords \( Z_\alpha \ast 0 \in \Lambda^2(U_\alpha, C^\infty(\mathbb{R})) \) may be glued together to give \( A \in \Lambda^1(M, C^\infty(\mathbb{R})) \). It is also clear that

\[
\rho(A) = [\{c_{\alpha \beta}\}] \in \check{H}^1(M, \text{Diff}^+(\mathbb{R})).
\]

\[\square\]

3. Lie groupoids, Lie algebroids and gauge theory

3.1. Lie groups and Lie algebroids. A topological groupoid is a small topological category \( \mathcal{H} \) such that all arrows are invertible. This means that the sets \( \text{Obj}(\mathcal{H}) \) and \( \text{Mor}(\mathcal{H}) \) of objects and morphisms of \( \mathcal{H} \) are topological spaces and there are maps

\[
s, t : \text{Mor}(\mathcal{H}) \to \text{Obj}(\mathcal{H}), \quad e : \text{Obj}(\mathcal{H}) \to \text{Mor}(\mathcal{H})
\]

which assign the source \( s = s(\phi) \) and the target \( y = t(\phi) \) to a morphism \( \phi \in \text{Mor}(x, y) \), and the identity map \( e(x) \in \text{Mor}(x, x) \) for \( x, y \in \text{Obj}(\mathcal{H}) \). These maps are required to be continuous and have the appropriate properties, e.g. that \( \phi \circ e(x) = \phi = e(y) \circ \phi \) for every \( \phi \in \text{Mor}(x, y) \). Moreover, we require that the arrows are invertible, meaning that for every \( \phi \in \text{Mor}(x, y) \) there is a unique morphism \( \phi^{-1} \in \text{Mor}(y, x) \) such that \( \phi \circ \phi^{-1} = e(y) \) and \( \phi^{-1} \circ \phi = e(x) \). Let us review some important examples which play important role in this paper.
The covering groupoid. Let \( U = \{ U_\alpha \}_{\alpha \in I} \) be a covering of a smooth manifold \( M \) with open subsets. We can then define a groupoid \( \Gamma_U \) by setting

\[
\text{Obj}(\Gamma_U) = \sqcup_{\alpha \in I} U_\alpha, \quad \text{Mor}(\Gamma_U) = \{ (x, \alpha, \beta) \mid \alpha, \beta \in I, x \in U_\alpha \cap U_\beta \}.
\]

The morphism \((x, \alpha, \beta)\) is usually denoted by \((x \in U_\beta) \xrightarrow{(x, \alpha, \beta)} (x \in U_\alpha)\). The source map sends \((x, \alpha, \beta)\) to \((x \in U_\beta)\) and the target map sends it to \((x \in U_\alpha)\). The composition is defined by setting \((x, \alpha, \beta) \star (x, \beta, \gamma) = (x, \alpha, \gamma)\). This gives a groupoid which is sometimes denoted by \(\Gamma_U\) in the literature.

**Germs of diffeomorphisms.** Let us assume that \( R \) is a smooth oriented manifold. We can define the groupoid of germs of diffeomorphisms of \( R \), denoted by \( \Gamma_R^{q} \), by setting \( \text{Obj}(\Gamma_R^{q}) = R \) and

\[
\text{Mor}(\Gamma_R^{q}) = \bigcup_{x,y \in R} \text{Mor}^q(x,y) \quad \text{where} \quad \text{Mor}^q(x,y) = \{ (f, x, y) \mid f \in \text{Diff}^+(R), f(x) = y \}/_{\sim_q}
\]

and \((f, x, y) \sim_q (g, x, y)\) if there is an open neighborhood \( U \) of \( x \) in \( R \) so that \( f|_U = g|_U \). The equivalence class of \((f, x, y)\) is denoted by \([f, x, y]_q\) or \([f, x]_q\) (note that \( y = f(x) \) is determined by \( f \) and \( x \)). The source map and the target map are then defined by \( s[f, x, y]_q = x \) and \( t[f, x, y]_q = y \), while the map \( e \) is defined by \( e(x) = [\text{Id}_R, x]_q \in \text{Mor}^q(x,x) \). The topology on \( \Gamma_R^{q} \) is defined so that a basis of neighborhoods for \([f, x]_q\) is given by \( \{ [f, y]_q \mid y \in U \} \), where \( U \) is an open set in \( R \) which contains \( x \). Note that the topology induced on \( \text{Mor}^q(x,x) \) is the discrete topology. With this topology \( \text{Mor}(\Gamma_R^{q}) \) has the structure of a manifold which is equipped with the covering maps \( s \) and \( t \) to \( R \).

**Quantum groupoid of diffeomorphisms.** Let us continue to assume that \( R \) is a smooth oriented manifold. We can define the groupoid of quantum diffeomorphisms of \( R \), denoted by \( \Gamma_R^{q} \), by setting \( \text{Obj}(\Gamma_R^{q}) = R \) and

\[
\text{Mor}(\Gamma_R^{q}) = \bigcup_{x,y \in R} \text{Mor}^q(x,y) \quad \text{where} \quad \text{Mor}^q(x,y) = \{ (f, x, y) \mid f \in \text{Diff}^+(R), f(x) = y \}/_{\sim_q}
\]

and \((f, x, y) \sim_q (g, x, y)\) if the Taylor expansions of \( f \) and \( g \) agree at \( x \). The equivalence class of \((f, x, y)\) is denoted by \([f, x, y]_q\) or \([f, x]_q\). The source map and the target map are defined by \( s[f, x, y]_q = x \) and \( t[f, x, y]_q = y \), while the map \( e \) is defined by \( e(x) = [\text{Id}_R, x]_q \in \text{Mor}^q(x,x) \). The topology on \( \Gamma_R^{q} \) is defined so that a basis of neighborhoods for \([f, x]_q\) is given by \( \{ [f, y]_q \mid y \in U \} \), where \( U \) is an open set in \( R \) which contains \( x \). As before, note that the topology induced on \( \text{Mor}^q(x,x) \) is the discrete topology. Unlike \( \Gamma_R^{q} \), the groupoid \( \Gamma_R^{q} \) is not Hausdorff, as different germs may have the same Taylor expansion.

The particular cases where the manifold \( R \) is the real line \( \mathbb{R} \) is of particular interest. In this case, we write \( \Gamma^q = \Gamma^{q}_{\mathbb{R}} \) and \( \Gamma_q = \Gamma^{q}_{\mathbb{R}} \). We write \( \Gamma^k \) for \( \Gamma^{k}_{\mathbb{R}} \) and \( \Gamma_k \) for \( \Gamma^{k}_{\mathbb{R}} \). There is a well-defined **Taylor expansion functor** \( T : \Gamma^{k}_{\mathbb{R}} \to \Gamma^{q}_{\mathbb{R}} \) which is the identity map over the objects and takes \([f, x]_q\) to \([f, x]_q\). This functor is a homomorphism of groupoids.

The topology on \( \Gamma^q \) and \( \Gamma_q \) is not so pleasant if one would like to treat them as Lie groupoids. The arrows of the groupoid \( \Gamma^q \), which are given by the formal power series \( \sum_{m=0}^{\infty} b_m(t-x)^m \) may naturally be identified with the points

\[
(x, y_0, y_1, y_2, \ldots) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^\infty.
\]

This gives a natural candidate for equipping \( \text{Mor}(\Gamma^q) \) with a topology and arriving at a Lie groupoid, i.e. a topological groupoid with the structure of a manifold on the spaces of objects and morphisms. We denote this Lie groupoid by \( \mathcal{Q} \). Note that \( \mathcal{Q} \) and \( \Gamma_q \) are the same as groupoids, but not as topological groupoids. The Lie algebroid associated with \( \mathcal{Q} \) is the vector bundle \( q \) over \( \mathbb{R} \) which is given by

\[
q_s = \left\{ \sum_{m=0}^{\infty} a_m(t-s)^m \mid a_m \in \mathbb{R}, m = 0, 1, 2, \ldots \right\} \quad \forall s \in \mathbb{R}.
\]
A section $A = \sum_{m=0}^{\infty} a_m(s)(t - s)^m$ of the vector bundle $q$ is determined by the smooth functions $a_m : \mathbb{R} \rightarrow \mathbb{R}$ for $m \in \mathbb{Z}^{\geq 0}$.

Equipping $\Gamma^g$ with a more appropriate topology is difficult. In fact, it is a common belief that it is not possible to construct a "good" non-discrete topology on $\text{Mor}^g(x, x)$. For instance, Gromov [15] writes: "there is no useful topology in this space ... of germs of $[C^k]$ sections...". Instead of choosing a topology on $\Gamma^g$, it is enough for to define the notion of a smooth function with values in $\Gamma^g$. A map $A : M \rightarrow \Gamma^g$ from a smooth manifold to $\Gamma^g$ is called smooth if for every $x \in M$ there is an open set $U_x \subset M$ containing $x$, an open interval $I \subset \mathbb{R}$ and a smooth function $\overline{A} : I \times U_x \rightarrow \mathbb{R}$ such that $A(y)$ is represented by $\overline{A}(\cdot, y) : \mathbb{R} \rightarrow \mathbb{R}$ for every $y \in U_x$. We write $\mathcal{G}$ for $\Gamma^g$ if this weak notion of topology is used. It then makes sense to talk about the tangent bundles $T\mathcal{G}$ of the space of arrows. The Lie algebroid $\mathcal{G}$ may thus be defined where for $s \in \mathbb{R}$, the vector space $\mathcal{g}_s$ consists of germs of smooth real-valued functions at $s$. The derivatives of $A$ and $A'$ with respect to the variable $t$ are denoted $A'$ and $A''$, for simplicity.

The homomorphism $T : \Gamma^g \rightarrow \Gamma^g$ may also be regarded as a homomorphism $T : \mathcal{G} \rightarrow \mathcal{Q}$ and induces a homomorphism of vector bundles $T : \mathcal{g} \rightarrow \mathcal{q}$.

3.2. Germ cords, quantum cords and the gauge actions. Let us assume that $M$ is a smooth manifold. A germ $k$-form on $M$ is a smooth $k$-form $A$ on $M$ with values in $\mathcal{g}$. The smooth map $s \circ A$ then induces a well-defined smooth map $s_A : M \rightarrow \mathbb{R}$, which is called the source of $A$. The space of all germ $k$-forms is denoted by $\Omega^k(M, \mathcal{g})$. Similarly, we can define $\Omega^k(M, \mathcal{q})$. The source map defines the source maps

$$s : \Omega^*(M, \mathcal{q}) \rightarrow C^\infty(M) \quad \text{and} \quad s : \Omega^*(M, \mathcal{g}) \rightarrow C^\infty(M),$$

while the target map induces the maps

$$t : \Omega^*(M, \mathcal{q}) \rightarrow \Omega^*(M, \mathbb{R}) \quad \text{and} \quad t : \Omega^*(M, \mathcal{g}) \rightarrow \Omega^*(M, \mathbb{R}).$$

Denoting both source maps by $s$ and both target maps by $t$ is of course an abuse of notation, which will be repeated in many similar situations in this paper. The homomorphisms $T : \Gamma^g \rightarrow \Gamma^g$ induces a Taylor expansions map

$$T : \Omega^*(M, \mathcal{g}) \rightarrow \Omega^*(M, \mathcal{q}).$$

The Lie brackets on the fibers of $\mathcal{g}$ and $\mathcal{q}$ (which is induced by the Lie bracket of these algebroids) induces Lie brackets on $\Omega^*(M, \mathcal{g})$ and $\Omega^*(M, \mathcal{q})$, and are defined only when the source maps match. The Lie bracket is given by

$$[A, B] := AB' - BA' \quad \forall \ A \in \Omega^k(M, \mathcal{q}), B \in \Omega^l(M, \mathcal{g}) \ \text{s.t.} \ s_A = s_B : M \rightarrow \mathbb{R}.$$

A formula for the Lie bracket of $\Omega^*(M, \mathcal{q})$ is given similarly.

**Definition 3.1.** A germ cord is a 1-form $A \in \Omega^1(M, \mathcal{g})$ which satisfies

$$dA + \frac{1}{2}[A, A] = dA + AA' = 0.$$

The space of germ cords on $M$ is denoted by $\wedge(M, \mathcal{g})$.

The gauge action of $\mathcal{G}$ over $\wedge(M, \mathcal{g})$ is defined as follows. We define

$$Y \star A := \frac{A \circ Y - dY}{Y'} \quad \forall \ A \in \wedge(M, \mathcal{g}), \ Y \in \Omega^0(M, \mathcal{G}).$$

As usual, it is understood from the definition that $Y \star A$ makes sense only if $t_Y = s_A$ as smooth functions from $M$ to $\mathbb{R}$. Setting $B = Y \star A$ and $F_A = dA + AA'$, we have $F_B = (F_A \circ Y)/Y'$. In particular, if $A \in \wedge(M, \mathcal{g})$ then $Y \star A \in \wedge(M, \mathcal{g})$. If $Y, Z \in \Omega^0(M, \mathcal{G})$ are gauge maps with $t_Z = s_Y$, we also compute $Z \star (Y \star A) = (Y \circ Z) \star A$. These observations show that $\star$ defines an action of $\Omega^0(M, \mathcal{G})$ on $\Omega^1(M, \mathcal{g})$ which induces action on $\wedge(M, \mathcal{g})$, called the gauge action of $\mathcal{G}$. 

on $\wedge(M, g)$. The gauge action of $Q$ on $\Omega^1(M, q)$ is defined in a similar way, as Equation 3 makes sense for the formal power series. If $A = \sum_{m=0}^{\infty} a_m(t-s)^m$ then

$$F_A = \sum_{m=0}^{\infty} (da_m + (m+1)a_{m+1}ds)(t-s)^m - \sum_{p \neq q} (p-q)a_pa_q(t-s)^{p+q-1}$$

$$\Rightarrow F_A = 0 \iff da_0 + (m+1)a_{m+1}ds = \sum_{p+q=m+1} (p-q)a_pa_q \quad m = 0, 1, \ldots$$

**Definition 3.2.** A quantum cord is a 1-form $A \in \Omega^1(M, g)$ which satisfies $F_A = dA + AA' = 0$ and is locally trivial, meaning that for every point $x \in M$ there is an open set $U_x \subset M$ containing $x$ so that $A|_{U_x}$ is of the form $Y \ast 0$ for some $Y \in \Omega^0(U_x, Q)$. The space of quantum cords on $M$ is denoted by $\wedge(M, g)$.

We will soon see the reason for the extra local triviality condition in the case of quantum cords. The action of $\Omega^0(M, Q)$ on $\Omega^1(M, q)$ induces a gauge action of $Q$ on $\wedge(M, q)$. Note that $T$ induces a well-defined homomorphism $T : \Omega^*(M, g) \rightarrow \Omega^*(M, q)$ which restricts to $T : \wedge(M, g) \rightarrow \wedge(M, q)$. This follows from Lemma 5.1, which will be proved in Section 5.

### 3.3. Foliations of codimension one

Let us denote the fibers of $g$ and $q$ over $0 \in \mathbb{R}$ by $g_0$ and $q_0$. Denote the group of local diffeomorphisms of $(\mathbb{R}, 0)$, which consists of the arrows in $G$ with source and target equal to $0 \in \mathbb{R}$, by $G_{0 \to 0}$. Similarly, let $Q_{0 \to 0}$ denote the group of power series $\sum_{m=1}^{\infty} a_mt^m$ with $a_1 > 0$, which consists of the arrows in $Q$ with source and target equal to $0$. It is then clear that $G_{0 \to 0}$ and $Q_{0 \to 0}$ are both groups. From the gauge action of the groupoids $G$ and $Q$ on $\wedge(M, g)$ and $\wedge(M, q)$ we obtain the gauge actions of $G_{0 \to 0}$ and $Q_{0 \to 0}$ on

$$\wedge(M, g_0) \subset \Omega^1(M, g_0) \quad \text{and} \quad \wedge(M, q_0) \subset \Omega^1(M, q_0),$$

respectively. The target maps

$$t : \Omega^1(M, g) \rightarrow \Omega^1(M, \mathbb{R}) \quad \text{and} \quad t : \Omega^1(M, q) \rightarrow \Omega^1(M, \mathbb{R})$$

may be used to associate a 1-form $a_0$ to every germ cord $A \in \wedge(M, g_0)$, or its quantized image $A = T(A)$. In fact, $A = \sum_{m=0}^{\infty} a_mt^m$, and $a_0$ is the initial term in this Taylor expansion. It follows that $da_0 = a_1 \wedge a_0$. If we further assume that $a_0$ is nowhere zero, it follows that $a_0$ determines a smooth transversely oriented codimension-one foliation on $M$. Let us denote the subspaces of $g_0$ and $q_0$ which consists of the elements which are positive at the origin by $g_0^+$ and $q_0^+$, respectively. Correspondingly, the subsets

$$\wedge(M, g_0^+) \subset \wedge(M, g_0) \quad \text{and} \quad \wedge(M, q_0^+) \subset \wedge(M, q_0)$$

of non-singular germ cords and quantum cords may be defined. The above discussion implies that there are projection maps

$$\text{proj}_g : \wedge(M, g_0^+) \rightarrow F(M) \quad \text{and} \quad \text{proj}_q : \wedge(M, q_0^+) \rightarrow F(M),$$

where $F(M)$ denotes the space of smooth transversely oriented codimension-one foliations on $M$. We abuse the notation and denote $\text{proj}_g$ and $\text{proj}_q$ by $\text{proj}$.

Let us assume that $\mathcal{F} \in F(M)$ is given by a 1-form $a_0 \in \Omega^1(M, \mathbb{R})$ such that $da_0 = a_1 \wedge a_0$, for another 1-form $a_1 \in \Omega^1(M)$. There is a vector field $V$ transverse to $\mathcal{F}$ which satisfies $a_0(V) = -1$. By subtracting a suitable multiple of $a_0$ from $a_1$, we may further assume that $a_1(V) = 0$. The vector field $V$ may be integrated to give the flow $\Phi_t = \Phi_t^V$ on $M$. For every $x \in M$ we can then define $A = A_{a_0, V} \in \Omega^1(M, C^\infty(\mathbb{R}))$ by

$$A(t, x) := \Phi_t^V(a_0)(x) \quad \forall \ x \in M, \ t \in \mathbb{R}.$$  

The 1-form $A$ may be considered as an element in $\Omega^1(M, g_0)$. The Taylor expansion of $A$ is given as follows. Let $L$ denote the Lie derivative corresponding to $V$ and define

$$A = e^{tL}a_0 = \sum_{n=0}^{\infty} \frac{L^n(a_0)t^n}{n!} = \sum_{n=0}^{\infty} a_n t^n.$$
Lemma 3.3. Having fixed the above notation, $A_{a_0,V}$ is a cord and its image in $\Omega^1(M, g_0)$ is a germ cord, while $A_{a_0,V} = T(A)$ is quantum cord.

Proof. First note that

$$L(a_0) = d(\nu_V(a_0)) + \nu_V(d(a_0)) = \nu_V(a_1 a_0) = a_1(V) a_0 - a_0(V) a_1 = a_1.$$  

We then observe that the derivative $A'$ of $A$ is given by

$$A' = \frac{d}{dt} \Phi^*_t(a_0) = \Phi^*_t(L(a_0)) = \Phi^*_t(a_1).$$

It follows that $dA + AA' = \Phi^*_t(da_0 + a_0 a_1) = 0$. It follows from Lemma 4.2 that $A = T(A)$ is a quantum cord.

Note that $\nu_V A_{a_0,V} = -1$. The cord $A = A_{a_0,V}$ is uniquely specified by the following two conditions:

- $A(0, x) = a_0(x)$ for every $x \in M$.
- We have $\nu_V(A) = -1$.

We call $a_0 = A(0, \cdot)$ the initial term of $A$. More generally, let us assume that $X$ is a $C^\infty(\mathbb{R})$-valued function on $M$ with negative initial term. Consider the equation

$$L(B) + [B, X] + dX = 0.$$  

This is a differential equation for $B$ which defines $B$ (in a neighborhood of the origin in $\mathbb{R}$) once the initial term of $B$ is fixed. In particular, if we set $B(0, x) = X(0, x) A(0, x)$ and solve for $B$, we obtain the 1-form $B = A_{x, V, X} \in \Omega^1(M, g_0)$.

Proposition 3.4. Fix the transversely oriented codimension-1 foliation $\mathcal{F}$ and the transverse vector field $V$. The element $A_{x, V, X} \in \Omega^1(M, g_0)$ is the unique germ cord in $\Lambda(M, g_0)$ which satisfies the following two conditions:

- The initial term $A_{x, V, X}(0, \cdot)$ of $A_{x, V, X}$ defines the foliation $\mathcal{F}$.
- The equation $\nu_V(A) + X = 0$ is satisfied in $\Omega^0(M, g_0)$.

Proof. Let $B = A_{x, V, X}$. We first need to show that $dB = B'B$. Let us assume that

$$T(B) = \sum_{n=0}^\infty b_n t^n \quad \text{and} \quad T(X) = \sum_{n=0}^\infty x_n t^n.$$  

It is then clear that $B'(0, x) = b_1(x)$ is given as $L(b_0) + x_1 b_0 + dx_0$ and we compute

$$b_1 b_0 = (L(b_0) + x_1 b_0 + dx_0) a_0 = (x_0 L(a_0) + dx_0) a_0 = d(x_0 a_0).$$

Thus, $d(b_0) = b_1 b_0$ and the initial term of $E = dB - B'B$ is zero. Moreover, the differential equation of (4) implies $L(B') + dX + BX' - B'' X = 0$ and we can thus compute

$$L(E) + [E, X] = dL(B) - L(B') B - B'L(B) + dBX' - B'BX' - dBX - B'' BX$$

$$= dL(B) - L(B') B - B'L(B) + dBX' - B'(L(B) + dx) - dBX - B(L(B') + dX')$$

$$= dL(B) + dBX' - BdX' - dBX + B'dX$$

$$= d(L(B) + BX' - B'X) = 0.$$  

The equation $L(E) + [E, X] = 0$ and the initial value condition $E(0) = d(b_0) - b_1 b_0 = 0$ uniquely determine $E$ in a neighborhood of the origin. This implies that in a neighborhood of the origin we have $E = 0$, i.e. $B = A_{x, V, X}$ is a germ cord and belongs to $\Lambda(M, g_0)$. Let us denote $\nu_V(B)$ by $C$. It follows that

$$L_V(B) + [B, X] + dX = 0 \quad \Rightarrow \quad (\nu_V \circ d)(C) + [\nu_V(B), X] + (\nu_V \circ d)(X) = 0$$

$$\Rightarrow \quad L(C + X) + [C, C + X] = L(C + X) + [C, X] = 0.$$  

From this last equation, and the uniqueness of solutions for differential equations, it follows that $X = -C$. 

Let $A$ be a germ cord in $\wedge(M, g_0)$ which is compatible with a foliation $\mathcal{F}$. Set $X = -i_V(A)$ and let $B = A_{\mathcal{F}, V, X}$. By definition, $A(0, x) = a_0$ and $B(0, x) = b_0$ differ by multiplication by a non-zero constant. Since $i_V(a_0) = i_V(b_0)$, it follows that the initial terms of $A$ and $B$ agree. Moreover, both $A$ and $B$ satisfy Equation 4, and we thus have $L(B - A) + [B - A, X] = 0$. This differential equation for $B - A$, together with the initial condition that the initial term of $B - A$ vanishes, force $B - A$ to vanish in a neighborhood of the origin in $\mathbb{R}$. □

Proposition 3.4 implies that the germ cords in $\wedge(M, g_0)$ which correspond to a foliation $\mathcal{F}$ are determined by their evaluation over the vector field $V$. This evaluation map takes its values in $\Omega^0(M, g_0)$.

Remark 3.5. The same statement is also true for $q_0$, that the quantum cords in $\wedge(M, q_0)$ which correspond to $\mathcal{F}$ are determined by their evaluation over the vector field $V$. This latter evaluation map takes its values in $\Omega^0(M, q_0)$.

Remark 3.6. Fix $A \in \wedge(M, g)$ and let $A = T(A) = \sum_{m=0}^{\infty} a_m (t - s)^m$. Note that $s = s_A = s_A$ is a smooth function while $a_1 \in \Omega^1(M, \mathbb{R})$. Since $A$ is a germ cord (and $A$ is a quantum cord) it follows that $da_0 = a_1 \wedge (a_0 + ds_A)$. If we further assume that $a = a_0 + ds_A \in \Omega^1(M, \mathbb{R})$ is nowhere zero, it follows that $a$ determines a smooth transversely oriented codimension-one foliation on $M$. If $B = Y \ast A$ with $Y(t, x) = t + s_A(x)$ (and $s_B = 0$) we find $b_0 = a$ while $B \in \wedge(M, g_0)$. This observation implies that every germ cord (respectively, quantum cord) is gauge equivalent to a germ cord (respectively, quantum cord) in $\wedge(M, g_0)$ (respectively, in $\wedge(M, q_0)$). The induced actions of $\Omega^0(M, g_{0-0})$ and $\Omega^0(M, q_{0-0})$ on $\wedge(M, g_0)$ and $\wedge(M, q_0)$ give the moduli spaces

$$\mathcal{M}(M, g_0, g_{0-0}) = \wedge(M, g_0)/\Omega^0(M, g_{0-0}) \quad \text{and} \quad \mathcal{M}(M, q_0, q_{0-0}) = \wedge(M, q_0)/\Omega^0(M, q_{0-0}).$$

The passage from $g_0$ to $q_0$ and $q$ may be viewed as a detour towards classification which is forced by the lack of Lie groups which integrate the Lie algebras $q_0$ and $q_0$. Integrability of Lie algebroids is an interesting question, and the reader is referred to [7] for some nice results/obstructions.

Theorem 3.7. The groups $\Omega^0(M, g_{0-0})$ and $\Omega^0(M, q_{0-0})$ act on $\wedge(M, g_0)$ and $\wedge(M, q_0)$, respectively. Over the space $\mathcal{F}(M)$ of smooth transversely oriented codimension-one foliations of $M$, the actions of $\Omega^0(M, g_{0-0})$ and $\Omega^0(M, q_{0-0})$ preserve the fibers of

$$\text{proj}_g : \wedge(M, g_0) \to \mathcal{F}(M) \quad \text{and} \quad \text{proj}_q : \wedge(M, q_0) \to \mathcal{F}(M).$$

while the actions are transitive and without fixed points on the fibers.

Proof. Let us assume that $Y \in \Omega^0(M, g_{0-0})$ and $T(Y) = \sum_{m=1}^{\infty} y_m t^m$. The foliations given by $A$ and $Y \ast A$ are defined by the 1-forms $A(0) = a_0$ and $A(0)/Y(0) = a_0/y_1$, respectively. Since $y_1$ is a positive valued function on $M$, it follows that the action of $\Omega^0(M, g_{0-0})$ preserves the fibers of $\text{proj}_g$. We then need to show that this latter action is transitive and without fixed points. Fix $\mathcal{F} \in \mathcal{F}(M)$ and the transverse vector field $V$. Let $A = A_{\mathcal{F}, V, 1}$ and given a section $X \in \Omega^0(M, g_0)$ with $X(0) < 0$, solve the equation

$$X = i_V(Y \ast A) = i_V\left(\frac{A \circ Y - dY}{Y'}\right) = -\frac{1}{Y'} L_V(Y)$$

for $Y \in \Omega^0(M, g_{0-0})$. Evaluation at 0 gives $x_0 + 1/y_1 = 0$, which implies $y_1 = -1/x_0 > 0$. Furthermore, the above differential equation uniquely determines $Y$ in a neighborhood of the origin in $\mathbb{R}$ in terms of the given $X \in \Omega^0(M, g_0)$ and $A$. This completes the proof of the theorem for germ cords with the help of Proposition 3.4.

It follows that the gauge action of $Q_{0-0}$ preserves the fibers of $\text{proj}_q$ and that the action is transitive over the fibers. If $Y = \sum_{m=1}^{\infty} y_m t^m \in \Omega^0(M, Q_{0-0})$ preserves a quantum cord

$$A = \sum_{m=0}^{\infty} a_m t^m \in \wedge(M, q_0)$$
with \( a_0 \neq 0 \), we find \( Y' A = A \circ Y - dY \). We may contract this equation using a vector field \( V \) which is transverse to the foliation induced by \( a_0 \) and satisfies \( \tau_V(a_0) = 1 \), to get

\[
Y' \tau_V(A) = \tau_V(A) \circ Y - L_V Y
\]

Let us assume that \( \tau_V(A) = \sum_{m=0}^{\infty} b_m t^m \), where \( b_0 = 1 \). The initial term in the above equation reads as \( y_1 = 1 \). Looking at the coefficient of \( t^n \) gives an equation of the from

\[
y_{n+1} = F_n(y_1, \ldots, y_n, b_1, \ldots, b_n)
\]

which uniquely determines \( y_{n+1} \) by induction. Since \( y_1 = 1 \) and \( y_m = 0 \) for \( m > 1 \) is an obvious solution, it follows that this is the only possibility. This completes the proof of the theorem. \( \square \)

4. LEAVES AND HOLOMONY

4.1. Impotent cords. Let us fix a germ cord \( A \in \wedge (\mathcal{M}, g_0) \). If \( A(0) \) is a nowhere zero 1-form, it defines a transversely oriented codimension one foliation \( \mathcal{F} = \text{proj}(A) \in F(M) \). Associated with every leaf \( L \) of \( \mathcal{F} \) we obtain an immersion \( i_L : L \to M \) which gives the restriction \( A_{|L} = i_L^* A \in \wedge (L, g_0) \). The nature of this germ cord is quite different from the nature of \( A \) in the following sense. Unlike \( A(0) \) which is nowhere zero, \( A_{|L}(0) \) takes its values in \( g_0 \to 0 \subset g_0 \) and \( A_{|L}(0) = 0 \).

**Definition 4.1.** A germ cord \( A \in \wedge (M, g_0) \) is called impotent if \( A \in \Omega^1(M, g_0) \). Similarly, \( A \in \wedge (M, g_0) \) is called impotent if \( A \in \Omega^1(M, g_0) \). The spaces of impotent germ cords and impotent quantum cords are denoted by \( \wedge (M, g_0) \) and \( \wedge (M, q_0) \), respectively.

**Lemma 4.2** (Poincaré Lemma). Every impotent germ cord \( A \in \wedge (M, g_0) \) is locally gauge equivalent to zero, i.e., every \( x \in M \) has an open neighborhood \( U_x \subset M \) so that \( A_{|U_x} \) is gauge equivalent to zero. Similarly, every impotent quantum cord \( A \in \wedge (M, q_0) \) is locally gauge equivalent to zero.

**Proof.** Choose coordinates \( (x_1, \ldots, x_n) \) on a chart \( U \) around \( x \) so that \( x \) corresponds to the origin and \( A \) is given by \( \sum f_i dx_i \) with \( f_i \in \Omega^0(U, g_0) \). After shrinking \( U \), we can assume that for some \( \epsilon > 0 \), the function \( f_i \) is defined for all \( (t, x) \in W = (-\epsilon, \epsilon) \times U \). From \( dA = A^\wedge A \) we get

\[
\partial_i f_j + f_i \partial_j f = \partial_j f_i + f_j \partial_i f \quad \forall \ i, j \in \{1, \ldots, n\}.
\]

The above equation implies that the vector fields \( \xi_i = \partial_i + f_i \partial_t \) commute. We can thus choose new coordinates \( (y_0, y_1, \ldots, y_n) \) on an open neighborhood \( W' \) of \( (0, x) \in W \) so that \( \partial/\partial y_0 = \xi_i, \ y_0 \) agrees with \( t \) over \( x \) and the foliation is given by \( \{y_0 = \text{constant}\} \). Choose \( U_x \subset U \) such that it contains \( x \) and \( (-\delta, \delta) \times U_x \) is a subset of \( W' \). Set \( Y \) equal to \( y_0 \) over \( U_x \). It is then clear that \( Y \in \Omega^0(U_x, g_0) \). Furthermore, we have \( \xi_i Y = 0 \), which means that \( \partial_i Y = f_i Y' = 0 \). This means that \( A_{|U_x} = -dY/Y' = Y \times 0 \) and completes the proof for impotent germ cords. The theorem for impotent quantum cords follows from a similar argument. \( \square \)

The gauge group sends impotent cords to impotent cords. After dividing by the action of the gauge group, we obtain the moduli spaces

\[
\mathcal{M}(M, g_0) = \wedge (M, g_0) / \Omega^0(M, g_0)
\]

and

\[
\mathcal{M}(M, q_0) = \wedge (M, q_0) / \Omega^0(M, q_0),
\]

called the moduli spaces of impotent germ cords and impotent quantum cords, respectively.

**Proposition 4.3.** For every smooth manifold \( M \), there are natural bijections

\[
\rho^B_M : \mathcal{M}(M, g_0) \to \text{Hom}(\pi_1(M), g_0) := \text{Hom}(\pi_1(M), g_0) / g_0
\]

and

\[
\rho^Q_M : \mathcal{M}(M, q_0) \to \text{Hom}(\pi_1(M), Q_0) := \text{Hom}(\pi_1(M), Q_0) / Q_0
\]
If \( f : M_1 \to M_2 \) is a smooth map between smooth manifold, the following diagram is commutative:

\[
\begin{array}{c}
\mathcal{M}(M_2, g_{0 \to 0}, G_{0 \to 0}) \xrightarrow{\rho_{M_2}^g} \mathcal{M}(M_2, q_{0 \to 0}, Q_{0 \to 0}) \\
\xrightarrow{\rho_{M_1}^q} \mathcal{M}(M_1, g_{0 \to 0}, G_{0 \to 0}) \xrightarrow{\rho_{M_1}^g} \mathcal{M}(M_1, q_{0 \to 0}, Q_{0 \to 0}) \\
\xrightarrow{\rho_{M_2}^q} \mathcal{M}(M_2, q_{0 \to 0}, Q_{0 \to 0}) \xrightarrow{\rho_{M_1}^q} \mathcal{M}(M_1, q_{0 \to 0}, Q_{0 \to 0})
\end{array}
\]

**Proof.** The proof is identical with the proof of Theorem 2.4 for the most part, as is sketched below. Pick \( A \in \wedge(M, g_{0 \to 0}) \) and cover \( M \) with open subsets \( U_\alpha \) so that \( A|_{U_\alpha} = Y_\alpha \ast 0 \) for \( Y_\alpha \in \Omega^0(U_\alpha, G_{0 \to 0}) \). Over \( U_\alpha \cap U_\beta \), the transition functions \( c_{\alpha \beta} = Y_\alpha \ast Y_\beta^{-1} : U_\alpha \cap U_\beta \to G_{0 \to 0} \) are then locally constant, since \( c_{\alpha \beta} \ast 0 = 0 \). These maps define a cohomology class in the \( \check{\text{C}} \)ech cohomology \( \check{H}^1(M, G_{0 \to 0}) \). The functions \( Y_\alpha \) are well-defined only upto composition with locally constant functions, but this freedom does not change the cohomology class, as before. Moreover, if we gauge \( A \) by \( Y \in \Omega^0(M, G_{0 \to 0}) \), the cocycles associated with \( A \) and \( Y \ast A \) are the same and we obtain the map

\[
\rho = \rho_M^g : \mathcal{M}(M, g_{0 \to 0}, G_{0 \to 0}) \to \check{H}^1(M, G_{0 \to 0}) \simeq \text{Hom}(\pi_1(M), G_{0 \to 0})/G_{0 \to 0}.
\]

The argument of Theorem 5.3 may be copied to show that \( \rho_M^g \) is injective. This implies that over simply connected domains, every impotent germ cord is gauge equivalent to zero.

If \( \{c_{\alpha \beta} : U_\alpha \cap U_\beta \to G_{0 \to 0}\} \) is a cocycle in \( \check{H}^1(M, G_{0 \to 0}) \), we can choose a smooth partition of unity \( \{\lambda_\alpha : U_\alpha \to \mathbb{R}^\geq 0\}_{\alpha} \) as before and define \( Y_\alpha : U_\alpha \to G_{0 \to 0} \) by \( Y_\alpha(t, x) = \sum_\alpha \lambda_\alpha(x)c_{\alpha \gamma}(t) \). This is well-defined as a germ and we have \( Y_\alpha, Z_\alpha \in H^0(U_\alpha, G_{0 \to 0}) \) where \( Z(\cdot, x) = Y(\cdot, x)^{-1} \). The germs \( Z_\alpha \ast 0 \in \wedge(U_\alpha, g_{0 \to 0}) \) match over the intersections \( U_\alpha \cap U_\beta \). They can thus be glued to give some \( A \in \wedge(M, g_{0 \to 0}) \) with \( \rho_M^g(A) = \{[c_{\alpha \beta}]\} \in \check{H}^1(M, G) \). The proof for impotent quantum cords is completely similar. The commutativity of the cubic diagram is straight-forward from the definitions.

**4.2. Monodromy for impotent cords.** Let us assume that \( A \in \wedge(M, g_{0 \to 0}) \) is an impotent germ cord. Every element \( Y \in G_{0 \to 0} \) defines the map

\[
DY : g_{0 \to 0} = T_{id}G_{0 \to 0} \to T_Y G_{0 \to 0}.
\]

We can use this map to define a connection \( H^A \subset TM \times TG_{0 \to 0} \) by

\[
H^A_{\xi, Y} = \{(\zeta, DY(A(\zeta))) \mid \zeta \in T_{id} M\}.
\]

Since \( A \) satisfies \( dA + [A, A]/2 = 0 \), it follows that that \( H^A \) gives a foliation \( \mathcal{F}^A \) of \( M \times G_{0 \to 0} \) and a foliation \( \tilde{\mathcal{F}}^A \) of \( \tilde{M} \times G_{0 \to 0} \), where \( \tilde{M} \) denotes the universal cover of \( M \). For constructing this foliation, the weak notions of smoothness on \( g_0 \) and \( G_{0 \to 0} \) suffice.

Fix a point \( x \in M \) and a corresponding pre-image \( \tilde{x} \in \tilde{M} \) of \( x \) under the covering map. Every \( \theta \in \pi_1(M, x) \) may be lifted to a path \( \theta \) on the leaf of \( \tilde{\mathcal{F}}^A \) which passes through \( (\tilde{x}, Id_{G_{0 \to 0}}) \in \tilde{M} \times G_{0 \to 0} \).
The monodromy map

\[ \phi = \phi_A : \pi_1(M, x) \to \mathcal{G}_{0 \to 0}, \quad \phi(\theta) := \pi_{\mathcal{G}_{0 \to 0}}(\tilde{\theta}(1)) \]

is defined by projecting \( \tilde{\theta}(1) \in \tilde{M} \times \mathcal{G}_{0 \to 0} \) onto its second factor. It follows that \( \tilde{\theta}(1) = (\theta \tilde{x}, \phi_M(\theta)) \), where \( \theta \tilde{x} \) denotes the image of \( \tilde{x} \) under the deck transformation corresponding to \( \theta \). Moreover, since \((\tilde{x}, Id_{\mathcal{G}_{0 \to 0}})\) and \((\theta \tilde{x}, \phi(\theta))\) are on the same leaf of \( \tilde{\mathcal{F}}^A \), for every \( Y \in \mathcal{G}_{0 \to 0} \) the points \((\tilde{x}, Y)\) and \((\theta \tilde{x}, \phi(\theta) \circ Y)\) are also on the same leaf of \( \tilde{\mathcal{F}}^A \).

Every other pre-image of \( x \) under the covering map is of the form \( \tilde{y} = \gamma \tilde{x} \). If we use \( \tilde{y} \) instead of \( \tilde{x} \) in defining \( \phi \), we obtain another map \( \phi' \), with the property that the points \((\tilde{y}, Id_{\mathcal{G}_{0 \to 0}})\) and \((\theta \tilde{y}, \phi'(\theta))\) are on the same leaf. On the other hand \((\tilde{y}, Id_{\mathcal{G}_{0 \to 0}})\) is on the same leaf as \((\tilde{x}, \phi(\gamma)^{-1})\). If follows that

\[ \phi'(\theta) = \phi(\gamma)^{-1} \phi(\theta) \phi(\gamma), \quad \forall \theta \in \pi_1(M, x). \]

In particular, the conjugacy class of the representation \( \phi_A : \pi_1(M, x) \to \mathcal{G}_{0 \to 0} \) does not depend on the choice of the pre-image \( \tilde{x} \) of \( x \). On the other hand, if we gauge the germ cord \( A \) by a section \( Y \in \Omega^0(M, \mathcal{G}_{0 \to 0}) \), one can easily show that the monodromy map \( \phi : \pi_1(M) \to \mathcal{G}_{0 \to 0} \) changes by conjugation by \( Y(x) \in \mathcal{G}_{0 \to 0} \).

The above discussion gives a second construction which constructs the map

\[ \rho^\delta_M : \mathcal{M}(M, \mathcal{G}_{0 \to 0}, \mathcal{G}_{0 \to 0}) \to \overline{\Hom}(\pi_1(M), \mathcal{G}_{0 \to 0}) = \Hom(\pi_1(M), \mathcal{G}_{0 \to 0})/\mathcal{G}_{0 \to 0} \]

in an explicit way, by assigning the monodromy homomorphism \( \phi_A \in \overline{\Hom}(\pi_1(M), \mathcal{G}_{0 \to 0}) \) to every \( A \in \mathcal{M}(M, \mathcal{G}_{0 \to 0}, \mathcal{G}_{0 \to 0}) \). A similar discussion gives an explicit description of the correspondence

\[ \rho^\delta_M : \mathcal{M}(M, \mathcal{G}_{0 \to 0}, \mathcal{Q}_{0 \to 0}) \to \overline{\Hom}(\pi_1(M), \mathcal{Q}_{0 \to 0}) = \Hom(\pi_1(M), \mathcal{Q}_{0 \to 0})/\mathcal{Q}_{0 \to 0} \]

by assigning the quantum monodromy map \( \phi_A \) to \( A \in \mathcal{M}(M, \mathcal{Q}_{0 \to 0}, \mathcal{Q}_{0 \to 0}) \).

There is a third (geometric) way to understand the monodromy map as follows. Let us assume that \( A \in \Lambda(M, \mathcal{G}_{0 \to 0}) \) is represented by a smooth differential form on \((-\epsilon, \epsilon) \times M \) such that \( A(0, x) = 0 \) for all \( x \in M \). As discussed in the proof of Lemma 4.2, \( A \) defines a foliation on \((-\epsilon, \epsilon) \times M \). In fact, the 1-form \( B = A - dt \in \Omega^1((-\epsilon, \epsilon) \times M, \mathbb{R}) \) satisfies \( dB = dA = A' dt = A'B \), which implies the Frobenius condition \( BdB = 0 \). It thus gives a foliation \( \mathcal{F}_A \) on \((-\epsilon, \epsilon) \times M \). Since \( A(0, x) = 0 \), \( \{0\} \times M \) is one of the leaves of \( \mathcal{F}_A \). Let us fix \( x \in M \) and \( \gamma : [0,1] \to M \) so that \( \gamma(0) = \gamma(1) = x \). The positive number \( \delta > 0 \) may be chosen so that \( \gamma \) may be lifted (in a unique way) to a curve \( \gamma_1 : [0,1] \to (-\epsilon, \epsilon) \times M \) with image on the leaf passing through \( (t, x) \) so that \( \gamma_1(0) = (t, x) \) and \( \gamma_1(s) = \gamma(s) \) for all \( s \in [0,1] \). Here \( \pi_M : (-\epsilon, \epsilon) \times M \to M \) denotes the projection map over the second factor, while the projection map over the first factor is denoted by \( \pi_R \). It is easy to show that the value \( \pi_R(\gamma_1(1)) \in \mathbb{R} \) is independent of the choice of \( \gamma \) in its homotopy class \( [\gamma] \in \pi_1(M, x) \). Let us denote this value by \( \phi_{[\gamma]}(t) \). Since \( \{0\} \times M \) is a leaf, \( \phi_{[\gamma]}(0) = 0 \). It follows that \( \phi_{[\gamma]} \) is smooth and that the map \( \phi : \pi_1(M, x) \to \mathcal{G}_{0 \to 0} \) which sends \( [\gamma] \) to the germ of \( \phi_{[\gamma]} \) is a homomorphism. Moreover, the conjugacy class of this homomorphism remains invariant under gauge, and is equal to \( \phi_A \). This point of view brings us very close to the notion of holonomy for the leaves of a foliation on \( M \).

4.3. Holonomy of leaves. Let us assume that \( \mathcal{F} \in \mathcal{F}(M) \) is a transversely oriented codimension one foliation on \( M \) and that \( L \) is a leaf of \( \mathcal{F} \). \( \mathcal{F} \) corresponds to the gauge equivalence class of a germ cord \( A \in \Lambda(M, \mathcal{G}_{0 \to 0}) \). The restriction \( A|_L \) of \( A \) to \( L \) is impotent and we thus obtain a homomorphism \( \phi_{\mathcal{F}|_L} \in \overline{\Hom}(\pi_1(L), \mathcal{G}_{0 \to 0}) \). If \( x \in L \) is a fixed point, using a transverse arc we can also define a holonomy homomorphism \( \rho_L : \pi_1(L, x) \to \mathcal{G}_{0 \to 0} \), and the conjugacy class of this homomorphism is independent of the choice of \( x \) and the transverse arc.

**Proposition 4.4.** For every leaf \( L \) of a smooth transversely oriented codimension-one foliation \( \mathcal{F} \) of a smooth manifold \( M \), the conjugacy classes of the holonomy homomorphism \( \rho_L : \pi_1(L, x) \to \mathcal{G}_{0 \to 0} \) and the monodromy representation \( \phi_{\mathcal{F}|_L} : \pi_1(L, x) \to \mathcal{G}_{0 \to 0} \) in \( \overline{\Hom}(\pi_1(L), \mathcal{G}_{0 \to 0}) \) are the same.
Proof. Let us assume that \( \mathcal{F} \in \mathcal{F}(M) \) is a transversely oriented codimension one foliation on \( M \) given by a 1-form \( a \in \Omega^1(M, \mathbb{R}) \), \( V \) is a transverse vector field with \( \iota_V(a) = -1 \) and \( A = A_{a,V} \) is the corresponding cord. Denote the flow of \( V \) by \( \Phi_t \) (thus, \( A = \Phi^*_t(a) \)). Associated with \( A \) we obtain a 1-form \( B = A - dt = \Phi^*_t(a) - dt \in \Omega^1(\mathbb{R} \times M, \mathbb{R}) \) and a foliation \( \mathcal{F}_A \) on \( \mathbb{R} \times M \) as before. If we define \( F : \mathbb{R} \times M \to \mathbb{R} \times M \) by \( F(t, x) = (t, \Phi_t(x)) \), it follows that \( B = F^*(a) \). The foliation \( \mathcal{F}_A \) is thus given as the image of the product foliation \( \mathbb{R} \times \mathcal{F} \) on \( \mathbb{R} \times M \) under the map \( F \).

Suppose that \( L \) is a leaf of \( \mathcal{F} \) and fix \( x \in L \). Our third description of the monodromy map

\[
\rho^L : \mathcal{M}(L, g_0, G_{0-0}) \to \text{Hom}(\pi_1(L), G_{0-0})
\]

may be used to describe the homomorphism \( \phi = \phi_{A|L} \) as follows. The foliation associated with \( A|_L \) is the restriction of \( \mathcal{F}_A \) to \( (-\epsilon, \epsilon) \times L \subset \mathbb{R} \times M \). For every small value of \( t \), the curve \( \gamma_t \) is mapped to a curve \( \theta_t = F \circ \gamma_t \) by \( F \). Note, however, that \( \pi_\mathbb{R} \circ \theta_t = \pi_\mathbb{R} \circ \gamma_t \). In particular, \( \phi = \phi_{\gamma_t} \in G_{0-0} \) may be computed as the return map of the curves \( \{ \theta_t \} \) for small values of \( t \). We then observe that \( \theta_t(0) = F(t, x) = (t, \Phi_t(x)) \). Moreover, \( \theta_t(1) = F(\phi(t), x) = (\phi(t), \Phi_{\phi(t)}(x)) \). We can parametrize the transverse arc \( \{ \Phi_t(x) \mid t \in (-\epsilon, \epsilon) \} \) to the foliation \( \mathcal{F} \) in \( M \) by \( t \in (-\epsilon, \epsilon) \) and then the above considerations imply that \( \rho_{\mathcal{L}}(\gamma)(t) = \phi(t) \), completing the proof of the proposition.

Suppose that a foliation \( \mathcal{F} \in \mathcal{F}(M) \) is compatible with a germ cord \( A \in \Lambda(M, g_0) \) and that \( \Gamma(A) = \sum_{m} a_m t^m \). Proposition 4.4 and Proposition 4.3 imply that in the Taylor expansion of the holonomy map along a closed curve \( \gamma \), the initial term is obtained by integrating \( a_1 \) along \( \gamma \). This observation generalizes a proposition of Ghys in [12], which identifies the first derivative of the holonomy map along a foliation \( \mathcal{F} \) given by a 1-form \( a_0 \in \Omega^1(M, \mathbb{R}) \) with the integral of \( a_1 = L_V(a_0) \) along the closed curves representing the elements of \( \pi_1(L, x) \).

The above observations suggest the following extension of the concept of leaves and their holonomy to singular foliations.

Definition 4.5. Let \( \mathcal{F} \in \mathcal{M}(M, g_0, G_{0-0}) \) denote the gauge equivalence class of \( A \in \Lambda(M, g_0) \). A leaf-like map for the singular foliation \( \mathcal{F} \) is a diffeomorphism \( f : L \to M \) from a smooth manifold \( L \) to \( M \) such that \( f^*A \) is impotent.

The definition is clearly independent of the choice of the representative \( A \) for the singular foliation \( \mathcal{F} \). Associated with a leaf-like map \( f : L \to M \), we obtain the conjugacy class of a holonomy map

\[
\rho_{\mathcal{F}, L} = \rho^L(A|_L) \in \text{Hom}(\pi_1(L), G_{0-0}) \simeq \mathcal{M}(L, g_0, G_{0-0}).
\]

This notion of holonomy generalizes the usual holonomy map for the leaves of non-singular foliations, by Proposition 4.4. This generalization may be compared with other generalizations of the notion of holonomy for singular foliations, and in particular [9].

5. Classification of germ cords and quantum cords

5.1. A Poincaré lemma for cords. The purpose of this section is to state and prove a classification theorem for germ cords and quantum cords up to gauge equivalence. A survey of approaches to classification of foliations may be found in [20]. The basis of any such theorem is a local classification lemma, which shows that up to gauge equivalence, every cord is locally trivial. We refer to such statement as a Poincaré Lemma.

Lemma 5.1 (Poincaré Lemma). Every germ cord \( A \in \Lambda(M, g) \) is locally gauge equivalent to the trivial cord, i.e. for every \( x \in M \) there is an open neighborhood \( U_x \subset M \) of \( x \) such that \( \mathcal{A}|_{U_x} = \mathcal{Y}_x \star 0 \) for some \( \mathcal{Y}_x \in \Omega^0(U_x, G) \).

Proof. Let us assume that \( A \in \Lambda(M, g) \) is a germ cord and that \( s_A : M \to \mathbb{R} \) is the corresponding source map. One can then represent \( A \) as the germ of a differential form \( A \in \Omega^1(U, \mathbb{R}) \), where \( U \) is an open neighborhood of \( \Delta_A = \{(s_A(x), x) \in \mathbb{R} \times M \mid x \in M \} \).

By making \( U \) smaller, if necessary, we can assume that \( A \) satisfies \( d_M A = A' A \), which is equivalent to \( d_U(A - dt) = A'(A - dt) \). In particular, \( B = A - dt \in \Omega^1(U, \mathbb{R}) \) defines a codimension one foliation \( \mathcal{B} \) on \( \mathbb{R} \times U \). The local triviality of \( \mathcal{B} \) then follows from the Poincaré Lemma.
\( \mathcal{F}_A \) on \( U \) which is transverse to vertical lines \( \ell_y = U \cap (\mathbb{R} \times \{y\}) \) for all \( y \in M \). For every \( u \in U \) let us denote the leaf of \( \mathcal{F}_A \) through \( u \) by \( L_u \). Given \( x \in M \) we can choose an open neighborhood \( U_x \subset M \) of \( x \) and an open subset \( \ell_x' \subset \ell_x \subset \mathbb{R} \times \{x\} \) which contains \( (s_A(x), x) \in \Delta_A \), such that the union of leaves of the foliation \( \mathcal{F}_A \) which cut \( \ell_x' \), intersect \( U \cap (\mathbb{R} \times U_x) \) in a box \( W \) around \( x \). By this, we mean that associated with every \( y \in U_x \) and every \( r \in \ell_x' \) there is a unique point \( w = w(r, y) \in \ell_y \cap W \) such that the connected component of \( L_w \cap W \) which contains \( w \) also contains \( r \). Moreover, every \( w \in W \) is of the form \( w(r, y) \) for some \( y \in U_x \) and some \( r \in \ell_x' \). Over the box \( W \), we can define the real-valued function \( Y_x \) so that the restriction of \( Y_x \) to every plaque is constant. The map \( Y_x \) defines a smooth function \( Y_x: U_x \to \mathcal{G} \) such that \( A|_{U_x} = Y_x \ast 0 = -dY_x/Y_x' \). In particular, every germ cord \( A \in \wedge(M, g_0) \) is locally gauge equivalent to zero. □

**Remark 5.2.** Our earlier assumption that every quantum cord is the locally trivial is made to replace the above lemma, which is only available for germ cords. It follows that the image of every germ cord under \( \Gamma \) is automatically locally trivial, and is thus a quantum cord.

Considering the full action of the gauge groupoids on spaces of cords gives the moduli spaces
\[
\mathcal{M}(M, g, \mathcal{G}) = \wedge(M, g)/\Omega^0(M, \mathcal{G}) \quad \text{and} \quad \mathcal{M}(M, q, \mathcal{Q}) = \wedge(M, q)/\Omega^0(M, \mathcal{Q}),
\]
which are called the moduli space of germ cords and the moduli space of quantum cords, respectively. Theorem 3.7 shows that the space \( \mathcal{F}(M) \) of smooth transversely oriented codimension one foliations on \( M \) may be identified with a subset of both of the moduli spaces. In fact, Remark 3.6 implies that there are bijections
\[
\mathcal{F}(M) \cong \mathcal{M}(M, g, \mathcal{G}) \quad \text{and} \quad \mathcal{F}(M) \cong \mathcal{M}(M, q, \mathcal{Q}),
\]
which sit in a commutative diagram
\[
\begin{array}{ccc}
\mathcal{F}(M) & \cong & \mathcal{M}(M, g, \mathcal{G}) \\
Id & \quad \uparrow \quad & \uparrow \\
\mathcal{F}(M) & \cong & \mathcal{M}(M, q, \mathcal{Q})
\end{array}
\]

### 5.2. The Čech cohomology

Recall that a map \( Y: M \to \mathcal{G} \), which assigns an arrow \( Y(x) \) in \( \mathcal{G} \) to each point \( x \in M \), is smooth if for every \( x \in M \) one can find an open set \( U \) containing \( x \) so that \( Y_U \) can be represented by a smooth real-valued map on \( \mathbb{R} \times U \) which is still denoted by \( Y \), so that \( Y(x) \) is given by \( Y(\cdot, x): \mathbb{R} \to \mathcal{G} \) at each point \( x \in U \). The smooth section \( Y: M \to \mathcal{G} \) is locally constant if for every point \( x \in M \) there is a smooth local diffeomorphism \( f \) from a neighborhood of \( s_Y(x) \) to a neighborhood of \( t_Y(x) \) and a neighborhood \( U \) of \( x \) so that \( Y(f) \) is given as the germ of \( f \) at \( s_Y(y) \) for every point \( y \in U \). Similarly, a map \( Y: M \to \mathcal{Q} \) from \( M \) to the arrows of the groupoid \( \mathcal{Q} \) which is given by \( Y = \sum_{m=0}^{\infty} y_m(x)(t-s(x))^m \) is smooth if the functions \( y_m: M \to \mathbb{R} \) and \( s = s_Y: M \to \mathbb{R} \) are smooth, and is called locally constant if \( dy_m = (m+1)y_{m+1}ds \) for all \( m \in \mathbb{Z}^{\geq 0} \). This condition may be described as
\[
dY = \sum_{m=0}^{\infty} (dy_m - (m+1)y_{m+1}ds)(t-s)^m = 0.
\]
Locally constant maps to \( \mathcal{G} \) and \( \mathcal{Q} \) are in fact smooth maps to \( \Gamma^g \) and \( \Gamma^q \), respectively.

The spaces of locally constant functions with values in \( \Gamma^g \) and \( \Gamma^q \) over a manifold \( M \) is denoted by \( \Omega^0(M, \Gamma^g) \) and \( \Omega^0(M, \Gamma^q) \), respectively. Correspondingly, we can define the Čech cohomology groups \( \check{H}^1(M, \Gamma^g) \) and \( \check{H}^1(M, \Gamma^q) \). For this purpose, associated with each open cover \( \mathcal{U} = \{U_\alpha\}_\alpha \) of \( M \), we can construct the spaces of cocycles \( \check{C}^1(\mathcal{U}, \Gamma^g) \) and \( \check{C}^1(\mathcal{U}, \Gamma^q) \), as well as the spaces of coboundaries \( \check{B}^1(\mathcal{U}, \Gamma^g) \) and \( \check{B}^1(\mathcal{U}, \Gamma^q) \). An element of \( \check{C}^1(\mathcal{U}, \mathcal{G}) \) consists of a union of locally constant maps \( c_{\alpha \beta}: U_\alpha \cap U_\beta \to \Gamma^g \) from \( U_\alpha \cap U_\beta \) to the arrows of \( \Gamma^g \) which satisfy the cocycle condition \( c_{\alpha \beta} \circ c_{\beta \gamma} = c_{\alpha \gamma} \). In other words, a cocycle in \( \check{C}^1(\mathcal{U}, \Gamma^g) \) is a continuous groupoid homomorphism
from $\Gamma^d$ to $\Gamma^g$. The space $C^1(\mathcal{U}, \Gamma^g)$ is defined similarly using locally constant maps with values in $Q$ and a cocycle in $C^1(\mathcal{U}, \Gamma^g)$ is a continuous groupoid homomorphism from $\Gamma^d$ to $\Gamma^g$. The space of coboundaries $B^1(\mathcal{U}, \Gamma^g)$ consists of a union of locally constant maps
\[ c_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \Gamma^g \]
which come from locally constant maps $\{b_\alpha : U_\alpha \rightarrow \Gamma^g\}_\alpha$ in the sense that $c_{\alpha\beta} = b_\alpha \circ b_\beta^{-1}$ over the intersections $U_\alpha \cap U_\beta$. The coboundaries are the groupoid homomorphisms from $\Gamma^d$ to $\Gamma^g$ which are conjugate to the trivial homomorphism. Again, we can define $B^1(\mathcal{U}, \Gamma^\partial)$ in a similar way. We then set
\[ \hat{H}^1(\mathcal{U}, \Gamma^g) := C^1(\mathcal{U}, \Gamma^g) / B^1(\mathcal{U}, \Gamma^g) \quad \text{and} \quad \hat{H}^1(\mathcal{U}, \Gamma^\partial) := C^1(\mathcal{U}, \Gamma^\partial) / B^1(\mathcal{U}, \Gamma^\partial). \]
Considering the refinements of the coverings, we can define the limits of $\hat{H}^1(\mathcal{U}, \Gamma^g)$ and $\hat{H}^1(\mathcal{U}, \Gamma^\partial)$, which are $\hat{H}^1(M, \Gamma^g)$ and $\hat{H}^1(M, \Gamma^\partial)$, respectively. The quantization functor $\mathcal{T} : \Gamma^g \rightarrow \Gamma^\partial$ induces the maps
\[ \mathcal{T} : C^1(\mathcal{U}, \Gamma^g) \rightarrow C^1(\mathcal{U}, \Gamma^\partial), \quad \mathcal{T} : B^1(\mathcal{U}, \Gamma^g) \rightarrow B^1(\mathcal{U}, \Gamma^\partial) \quad \text{and} \quad \mathcal{T} : \hat{H}^1(M, \Gamma^g) \rightarrow \hat{H}^1(M, \Gamma^\partial). \]
It can be shown that $\hat{H}^1(M, \Gamma^g) \simeq \hat{H}^1(\mathcal{U}, \Gamma^g)$ and $\hat{H}^1(M, \Gamma^\partial) \simeq \hat{H}^1(\mathcal{U}, \Gamma^\partial)$ if the cover $\mathcal{U} = \{U_\alpha\}_\alpha$ consists only of contractible open subsets of $M$.

**Theorem 5.3.** There are natural one to one correspondences
\[ c : \mathcal{M}(M, g, \mathcal{G}) \rightarrow \hat{H}^1(M, \Gamma^g) \quad \text{and} \quad c : \mathcal{M}(M, q, \mathcal{Q}) \rightarrow \hat{H}^1(M, \Gamma^\partial). \]
from the moduli space of germ cords and yje moduli space of quantum cords to the Čech cohomology spaces with coefficients in $\Gamma^g$ and $\Gamma^\partial$, respectively.

**Proof.** Given $A \in \Lambda(M, g)$, we can cover $M$ with finitely many open sets $\{U_\alpha\}_\alpha$ so that $A|_{U_\alpha}$ is gauge equivalent to zero. One can then pick the sections $Y_\alpha \in \Omega^0(U_\alpha, \mathcal{G})$ such that $A|_{U_\alpha} = Y_\alpha \ast 0$. Over the intersections $U_\alpha \cap U_\beta$ we obtain $Y_\alpha \ast 0 = Y_\beta \ast 0$, which implies that $d(Y_\alpha \circ Y_\beta^{-1}) = 0$, or that
\[ c_{\alpha\beta} = Y_\alpha \circ Y_\beta^{-1} : U_\alpha \cap U_\beta \rightarrow \mathcal{G} \]
is locally constant. Note that over $U_\alpha \cap U_\beta$, we have $s_Y_\alpha = t_{Y_\beta^{-1}} = s_{Y_\beta} = s_A$ and the compositions $c_{\alpha\beta} = Y_\alpha \circ Y_\beta^{-1}$ are thus well-defined. The above argument gives the smooth locally constant maps
\[ c_{\alpha\beta} = Y_\alpha \circ Y_\beta^{-1} : U_\alpha \cap U_\beta \rightarrow \Gamma^g \quad \sim \quad c(A) = \{c_{\alpha\beta}\}_{\alpha,\beta} \in \hat{H}^1(M, \Gamma^g). \]
It is not hard to see that this class is well-defined. The role of the covering is not important as we can always pass to a common refinement for two different given covers. If $A|_{U_\alpha} = X_\alpha \ast 0 = Y_\alpha \ast 0$, then $d_a = X_\alpha \circ Y_\alpha^{-1} : U_\alpha \rightarrow \Gamma^g$ is locally constant and
\[ d_a \circ (Y_\alpha \circ Y_\beta^{-1}) = X_\alpha \circ Y_\beta^{-1} = (X_\alpha \circ Y_\beta^{-1}) \circ d_\beta \quad \forall \alpha, \beta \]
which implies that the definition of $c(A)$ is independent of the choice of $\{Y_\alpha\}_\alpha$. The map
\[ c : \Lambda(M, g) \rightarrow \hat{H}^1(M, \Gamma^g) \]
is thus well-defined.

We then explore the equality $c(A) = c(B)$ for $A, B \in \Lambda(M, g)$. Let us choose an open cover $\mathcal{U} = \{U_\alpha\}_\alpha$ for $M$ so that $A|_{U_\alpha} = X_\alpha \ast 0$ and $B|_{U_\alpha} = Y_\alpha \ast 0$ for $X_\alpha, Y_\alpha \in \Omega^0(U_\alpha, \mathcal{G})$. It follows that, after passing to a refinement of the cover $\mathcal{U}$, we can assume that over $U_\alpha \cap U_\beta$
\[ d_a \circ (Y_\alpha \circ Y_\beta^{-1}) = (X_\alpha \circ X_\beta^{-1}) \circ d_\beta \quad \text{for locally constant} \quad d_a \in \Omega^0(U_\alpha, \Gamma^g). \]
If we set $Z_\alpha = X_\alpha^{-1} \circ d_a \circ Y_\alpha \in \Omega^0(U_\alpha, \mathcal{G})$, it follows that $Z_\alpha = Z_\beta$ over $U_\alpha \cap U_\beta$. In particular, $Z_\alpha$ is the restriction of a global section $Z \in \Omega^0(M, \mathcal{G})$ to $U_\alpha$. Note that $s_Z|_{U_\alpha} = s_{Y_\alpha} = s_A|_{U_\alpha}$. From $X_\alpha \circ Z|_{U_\alpha} = d_a \circ Y_\alpha$ it follows that
\[ (Z \ast A)|_{U_\alpha} = Y_\alpha \ast d_a \ast 0 = Y_\alpha \ast 0 = B|_{U_\alpha} \quad \Rightarrow \quad Z \ast A = B. \]
If $Z \star A = B$ for some $Z \in \Omega^0(M, \mathcal{G})$, it is also implied from the above argument that $c(A) = c(B)$. We thus obtain a well-defined injective map

$$c : \mathcal{M}(M, \mathfrak{g}, \mathcal{G}) = \land(M, \mathfrak{g})/\Omega^0(M, \mathcal{G}) \to \tilde{H}^1(M, \Gamma^g).$$

To complete the proof for germ cords, we then need to show that the map $c$ is surjective. Let us assume that a cocycle $c_{\alpha\beta} : U_\alpha \cap U_\beta \to \Gamma^g$ represents an element of $\tilde{H}^1(M, \Gamma^g)$. We may further assume that $U_\alpha$ are all contractible and that $U_\alpha = \cup_{\gamma \neq \alpha} (U_\alpha \cap U_\gamma)$. It is implied that there are source maps $s_\alpha : U_\alpha \to \mathbb{R}$ such that

$$s_{\alpha\beta} = s_\beta|_{U_\alpha \cap U_\beta} \quad \text{and} \quad t_{\alpha\beta} = s_\alpha|_{U_\alpha \cap U_\beta} \quad \forall \alpha, \beta.$$

Choose a smooth partition of unity $\{\lambda_\alpha : U_\alpha \to \mathbb{R}^\geq 0\}$ subordinate to the cover $\mathcal{U} = \{U_\alpha\}_\alpha$ of $M$ and define $Y_\alpha \in \Omega^0(U_\alpha, \mathcal{G})$ by

$$Y_\alpha^{-1}(t, x) = \sum_{\gamma \neq \alpha} \lambda_\gamma(x)c_{\gamma\alpha}(t, x) \quad \forall (t, x) \in W_\alpha,$$

where $W_\alpha$ denotes an open neighborhood of

$$\Delta_\alpha = \{(s_\alpha(x), x) \in \mathbb{R} \times U_\alpha \mid x \in U_\alpha\}.$$ 

Note that the source map for the right-hand-side of the above equation stays equal to $s_\alpha$, and the expression on the right-hand-side is thus well-defined. Moreover, the derivative of $Y_\alpha^{-1}$ with respect to $t$ is positive and $Y_\alpha^{-1}(x) \in \mathcal{G}$ for all $x$. In particular, we can define the inverse of this arrow, which would be $Y_\alpha : U_\alpha \to \mathcal{G}$. The target map for $Y_\alpha$ is $t_{Y_\alpha} = s_\alpha$, while its source map is

$$s_{Y_\alpha}(x) = t_{Y_\alpha}^{-1}(x) = \sum_\gamma \lambda_\gamma(x)c_{\gamma\alpha}(s_\alpha(x), x) = \sum_\gamma \lambda_\gamma(x)s_\gamma(x).$$

This means that the source maps of $Y_\alpha$ define a well-defined map $s : M \to \mathbb{R}$ and that $s_{Y_\alpha} = s|_{U_\alpha}$ for all $\alpha$. Let us set $A_\alpha = Y_\alpha \star 0 \in \land(U_\alpha, \mathcal{G})$. We then compute

$$Y_\beta^{-1} = \sum_\gamma \lambda_\gamma c_{\gamma\beta} = (\sum_\gamma \lambda_\gamma c_{\gamma\alpha}) \circ c_{\beta\alpha} = Y_\alpha^{-1} \circ c_{\gamma\beta}$$

$$\Rightarrow A_\alpha|_{U_\alpha \cap U_\beta} = Y_\alpha \star 0 = (c_{\alpha\beta} \circ Y_\alpha) \star 0 = Y_\alpha \star (c_{\alpha\beta} \star 0) = A_\beta|_{U_\alpha \cap U_\beta}.$$ 

In particular, $A_\alpha \in \land(U_\alpha, \mathcal{G})$ match over the intersections to give a global germ cord $A \in \land(M, \mathcal{G})$. It is clear from the construction that $c(A)$ is the cocycle we started with. This completes the proof for germ cords.

The proof for quantum cords is completely similar, as discussed below. If follows from the proof for the germ cords that one can associate a well-defined Čech cohomology class $c(A) \in \tilde{H}^1(M, \Gamma^q)$ to every $A \in \land(M, \mathcal{Q})$. If $A = T(A)$ then $c(A) = T(c(A))$. This gives a map

$$c : \mathcal{M}(M, \mathfrak{q}, \mathcal{Q}) = \land(M, \mathfrak{q})/\Omega^0(M, \mathcal{Q}) \to \tilde{H}^1(M, \Gamma^q).$$

We then need to show that $c$ is surjective. The key point is that given a cocycle

$$c = \{c_{\alpha\beta} : U_\alpha \cap U_\beta \to \Gamma^q\}_{\alpha, \beta} \in \tilde{H}^1(M, \Gamma^q)$$

we can construct the sections $Y_\alpha : U_\alpha \to \mathcal{Q}$ using a partition of unity so that $\{Y_\alpha \star 0\}_\alpha$ match over the inclusions, and give a global quantum cord $A \in \Omega^1(M, \mathcal{Q})$. The equalities $A|_{U_\alpha} = Y_\alpha \star 0$ imply that $A$ is locally trivial and hence an element of $\land(M, \mathcal{Q})$.

5.3. The classifying spaces. Theorem 5.3 implies that the gauge equivalence classes of germ cords on $M$ are in correspondence with equivalence classes of Haefliger structures, with values in $\Gamma^g$. On the other hand, the commutative diagram of Equation 5 suggests that the space of equivalence classes of Haefliger $\Gamma^q$-structures also has all rights to be studied as the generalization of space of foliations. In particular, the concordance classes of $\Gamma^g$ and $\Gamma^q$ structures on $M$ are in correspondence with the homotopy classes of maps from $M$ to the classifying spaces $BT^g$ and $BT^q$ associated with the groupoids $\Gamma^g$ and $\Gamma^q$, respectively.
Definition 5.4. The germ cords $A_0, A_1 \in \wedge(M, g)$ are called concordant if there is a germ cord $A \in \wedge(M \times [0, 1], g)$ with $A|_{M \times \{i\}} = A_i$ for $i = 0, 1$. Similarly, $A_0, A_1 \in \wedge(M, q)$ are concordant if there is a quantum cord $A \in \wedge(M \times [0, 1], q)$ with $A|_{M \times \{i\}} = A_i$ for $i = 0, 1$. We write $A_0 \sim_g A_1$ if $A_0, A_1 \in \wedge(M, g)$ are concordant and write $A_0 \sim_q A_1$ if $A_0, A_1 \in \wedge(M, q)$ are concordant.

If $A_0$ and $A_1$ are concordant, we can choose the (germ) cord $A$ connecting them so that $i^*_s A = A_0$ for $s \in [0, \varepsilon)$ and $i^*_s A = A_1$ for $s \in (1 - \varepsilon, 1]$. This is needed when we glue the cords to show that concordance is an equivalence relation.

Our first observation addresses the compatibility of the concept of concordance with the action of the gauge group on germ cords and quantum cords.

Proposition 5.5. If $A_0, A_1 \in \wedge(M, g)$ (or in $\wedge(M, q)$) are gauge equivalent, then they are concordant.

Proof. Let us assume that $A_0, A_1 \in \wedge(M, g)$ and $A_1 = Y \ast A_0$. For every point $x \in M$ the gauge function $Y$ is given by $Y(t, y) \in \mathbb{R}$ for $y \in U_x \subset M$ and $t \in \mathbb{R}$ for a sufficiently small neighborhood $U_x$ of $x \in M$. We can then define

$$Z(t, y, s) = t + e^{-s} \cdot (Y(t, y) - t), \quad \forall y \in U_x \subset M, \quad s \in [0, 1], \quad t \in \mathbb{R}.$$ 

For every $(y, s) \in U_x \times [0, 1]$, the above definition gives a function $Z(y, s) = Z(\cdot, y, s)$ from $\mathbb{R}$ to $\mathbb{R}$, and it is not hard to show that $Z(y, s)$ is a diffeomorphism if $Y(y) = Y(\cdot, y)$ is a diffeomorphism, e.g. since its $t$-derivative is positive. From this construction, we obtain a gauge element $Z \in \Omega^0(M \times [0, 1], G)$. The restriction of $Z$ to $M \times \{0\}$ is the identity map, while the restriction of $Z$ to $M \times \{1\}$ is $Y$. Let us abuse the notation and denote the pull-back of $A_0$ on $M \times [0, 1]$ (using the projection map over the first factor) by $A_0$. Since $A_0 \in \wedge(M \times [0, 1], g)$ it follows that

$$A := Z \ast A_0 \in \wedge(M \times [0, 1], g), \quad i^*_s A = A_0 \quad \text{and} \quad i^*_0 A = Y \ast A_0 = A_1.$$ 

This completes the proof of the proposition for germ cords. The proof for quantum cords is completely similar.

The concordance classes of germ cords and quantum cords form

$$C(M, g) = \wedge(M, g)/\sim_g \quad \text{and} \quad C(M, q) = \wedge(M, q)/\sim_q,$$

which are called the germ concordia and the quantum concordia of $M$ respectively. If $f : M_1 \to M_2$ is a smooth map, we obtain the induced pull-back maps

$$f^* : C(M_2, g) \to C(M_1, g) \quad \text{and} \quad f^* : C(M_2, q) \to C(M_1, q)$$

It follows from Proposition 5.5 that the germ and quantum concordia are quotients of the moduli spaces $\mathcal{M}(M, g, G)$ and $\mathcal{M}(M, q, Q)$. Correspondingly, there are quotient maps

$$\pi_g : \mathcal{M}(M, g, G) \to C(M, g) \quad \text{and} \quad \pi_q : \mathcal{M}(M, q, Q) \to C(M, q).$$

The converse of Proposition 5.5 is true in some cases. The concept of a cord may be defined using the flat 1-forms with values in $C^\infty(S^1)$, which is the Lie algebra associated with $\text{Diff}^+(S^1)$. The corresponding cords are called the circle cords. The space of circle cords is denoted by $\wedge(M, C^\infty(S^1))$. We take the following proposition as a justification for the relation between $\mathcal{M}(M, g, G)$ and $C(M, g)$.

Proposition 5.6. If $A_0, A_1 \in \wedge(M, C^\infty(S^1))$ are concordant then they are gauge equivalent. In particular, there is an injective map

$$\rho_{S^1, M} : \mathcal{M}(M, C^\infty(S^1), \text{Diff}^+(S^1)) \to \text{Hom}(\pi_1(M), \text{Diff}^+(S^1))/\text{Diff}^+(S^1),$$

whose image is identified with the kernel of the Euler class obstructions map

$$e : \text{Hom}(\pi_1(M), \text{Diff}^+(S^1))/\text{Diff}^+(S^1) \to H^2(\pi_1(M), \mathbb{Z}).$$

Proof. Let $A_0, A_1 \in \wedge(M, C^\infty(S^1))$ be concordant and let $A \in \wedge(M \times [0, 1], C^\infty(\mathbb{R}))$ be a cord connecting $A_0$ to $A_1$. The cord $A$ is then given as

$$A(t, x, s) = A_s(t, x) + B_s(t, x) ds, \quad A_s \in \wedge(M, C^\infty(S^1)), \quad B_s \in \Omega^0(M, C^\infty(S^1)).$$
The reader is referred to [19] and [12, 11] for more on the construction of the Godbillon-Vey invariant. This result is in complete contrast with the topology of this intersection is studied in [21]. Among more recent results, one can mention the work [20] which correspond to the same point of [6, 11] and foliations admitting a projective transversal structure [22].

Correspondingly, there is a composition map \( \Gamma^g \rightarrow \Gamma^g \) for circle cords, the last part of the proposition follows from the proof of Theorem 5.3 and an standard observation that the kernel of the obstruction map \( e \) is identified with horizontal foliations of \( M \times S^1 \).

Since \( B\Gamma^g \) naturally (and classically) arises from the study of Haefliger structures on a manifold \( M \) up to concordance, some very interesting results are already available in the literature about the topology of \( B\Gamma^g \). Mather and Thurston proved that \( B\Gamma^g \) is 2-connected [23], [28]. Moreover, Thurston showed [27] that \( H_i(B\Gamma^g; \mathbb{Z}) = 0 \) for \( i = 0, 1, 2 \) while there is a surjection

\[
g_0 : H_3(B\Gamma^g; \mathbb{Z}) \longrightarrow \mathbb{R}
\]

given by the Godbillon-Vey invariant, c.f. [14] and [5]. This result is in complete contrast with the case of homogenous foliations [6] and foliations admitting a projective transversal structure [22]. The reader is referred to [19] and [12, 11] for more on the construction of the Godbillon-Vey invariants.

It is also interesting to study the fiber of the projection map from \( M(M, g, G) \) over a point of \( [M, B\Gamma^g] \). The structure of the intersection of this fiber with \( F(M) \), especially in dimension 3, is studied in a number of interesting papers. Near a taut foliation (of a closed 3-manifold), the topology of this intersection is studied in [21]. Among more recent results, one can mention the work of Eynard-Bontemps [10], where she shows that any two non-singular foliations on a 3-manifold \( M \) which correspond to the same point of \( [M, B\Gamma^g] \) are in the same connected component of \( F(M) \), meaning that the corresponding integrable plane fields are in the same connected component among all integrable plane fields on \( M \). It is still open whether the corresponding connected component is path connected or not.

Nevertheless, the relation between \( \Gamma^g \) and \( F(M) \) is not studied in the literature, and the topology of \( B\Gamma^g \) is not known. The homomorphism \( T : \Gamma^g \rightarrow \Gamma^g \) gives a continuous map \( T : B\Gamma^g \rightarrow B\Gamma^g \). Correspondingly, there is a composition map

\[
T : [M, B\Gamma^g] \rightarrow [M, B\Gamma^g], \quad T[f] = [T \circ f] \quad \forall [f] \in [M, B\Gamma^g].
\]

Let \( \bar{H}^1(M, \Gamma^g) / \sim_q \) denote the space of concordance classes of \( \Gamma^g \)-Haefliger structures on a manifold \( M \). Similarly, let \( \bar{H}^1(M, \Gamma^g) / \sim_q \) denote the space of concordance classes of \( \Gamma^g \)-Haefliger structures on \( M \). Under the identification of \( \bar{H}^1(M, \Gamma^g) / \sim_q \) with \([M, B\Gamma^g] \) and identification of \( \bar{H}^1(M, \Gamma^g) / \sim_q \)
with \([M, B\Gamma^8]\), the map \(T\) defined above is identified with the map induced by the map \(T\) which appears in the last column of the diagram in Equation 5.

6. The cohomology theory of cords

6.1. The cohomology groups. The algebroids \(q\) and \(q\) correspond to differential graded Lie algebras and attached cohomology theories which control the deformations of these algebroids, c.f. [8]. Given a cord \(A \in \wedge(M, g)\), let \(\Omega^i_{sA}(M, g)\) denote the subspace of \(\Omega^i(M, g)\) which consists of section \(E\) with \(s_E = s_A\). For \(A \in \wedge(M, q)\) we may define \(\Omega^i_{sA}(M, q)\) in a similar way. Define the twisted differential

\[
\nabla_A : \Omega^i_{sA}(M, g) \rightarrow \Omega^{i+1}(M, g) \quad \nabla_A(B) := dB + [A, B].
\]

The new differential satisfies \(\nabla_A \circ \nabla_A = 0\) and may be used to define the cohomology groups \(H^i_q(M, A)\). Similarly, we can define the cohomology groups \(H^i_q(M, A)\) for \(A \in \wedge(M, q)\). We study the basic properties of these cohomology groups in this section.

The group of diffeomorphisms of \(M\) acts on all objects considered above in a compatible way. Given an element \(\phi : M \rightarrow M\) in \(\text{Diff}^+(M)\) and \(A, B \in \Omega^\bullet(M, g)\), we have \(\phi^*A, \phi^*B \in \Omega^\bullet(M, g)\) and \([\phi^*A, \phi^*B] = \phi^*[A, B]\). Thus \(\phi^* \wedge(M, g) = \wedge(M, g)\), i.e. the pull-back of a germ cord (respectively, a quantum cord) is another germ cord (respectively, another quantum cord). Moreover, it follows that \(\nabla_{\phi^*} A(\phi^* B) = \phi^*(\nabla_A(B))\) and we thus obtain the natural isomorphisms

\[
\phi^*_q : H^i_q(M, A) \rightarrow H^i_q(M, \phi^* A) \quad \text{and} \quad \phi^*_q : H^i_q(M, A) \rightarrow H^i_q(M, \phi^* A).
\]

Let us fix a pair of germ cords \(A\) and \(B\) in \(\wedge(M, g)\) which are gauge equivalent. There is a gauge element \(Y \in \Omega^\bullet(M, g)\) such that \(Y \ast A = B\). We then define the homomorphism

\[
\Phi_{A \rightarrow B} : \Omega^\bullet_{sA}(M, g) \rightarrow \Omega^\bullet_{sB}(M, g), \quad \Phi_{A \rightarrow B}(W) := \frac{W \circ Y}{Y'}.
\]

Note that \(\Phi_{A \rightarrow B}\) is an isomorphism.

**Proposition 6.1.** For every two gauge equivalent germ cords \(A, B \in \wedge(M, g)\) the following diagram is commutative.

\[
\begin{array}{ccc}
\Omega^\bullet_{sA}(M, g) & \xrightarrow{\nabla_A} & \Omega^\bullet_{sB}(M, g) \\
\Phi_{A \rightarrow B} \downarrow & & \Phi_{A \rightarrow B} \downarrow \\
\Omega^\bullet_{sA}(M, g) & \xrightarrow{\nabla_B} & \Omega^\bullet_{sB}(M, g)
\end{array}
\]

In particular, \(\Phi_{A \rightarrow B}\) defines an isomorphism \(\Phi_{A \rightarrow B} : H^\bullet_q(M, A) \rightarrow H^\bullet_q(M, B)\).

**Proof.** This is a straightforward computation:

\[
(\nabla_B \circ \Phi_{A \rightarrow B})(W) = d\left(\frac{W \circ Y}{Y'}\right) + A \circ Y - dY \left(\frac{W' \circ Y - (W \circ Y)Y''}{(Y')^2}\right)
- \left(A' \circ Y - \frac{dY'}{Y'} - \frac{A \circ Y - dYY'}{(Y')^2}\right) \frac{W \circ Y}{Y'}
= \left(\frac{dW \circ Y + dY(W \circ Y)}{Y'} - \frac{dY'(W \circ Y)}{(Y')^2}\right)
+ \left(\frac{A \circ Y - dY(W' \circ Y) - (A' \circ Y)(W \circ Y)}{Y'} + \frac{(dY')(W \circ Y)}{(Y')^2}\right)
= \frac{(dW + AW' - A(W') \circ Y}{Y'}
= (\Phi_{A \rightarrow B} \circ \nabla_A)(W)
\]

\[\square\]
Proposition 6.1 implies that $H^*_g(M,A)$, for $A \in \wedge (M,g)$ corresponding to a fixed foliation $\mathcal{F}$, form a system of cohomology groups together with the isomorphisms $\Phi_{A\to B}$, and it makes sense to talk about the natural cohomology group

$$H^*_g(M,\mathcal{F}) = \bigsqcup_A H^*(M,A)$$

where the union is over all germ cords $A$ corresponding to the foliation $\mathcal{F}$ and $x \sim y$ if $x \in H^*_g(M,A)$ for some $A$ and $y = \Phi_{A\to B}(x) \in H^*_g(M,B)$ for some other germ cord $B$ corresponding to $\mathcal{F}$.

The cohomology groups are closed under Lie bracket. In fact, for given $Z \in \Omega^k(M,g)$ and $W \in \Omega^l(M,g)$ we have

$$\nabla_A[Z,W] = [dZ,W] + (-1)^k [Z,dW] + AZW'' - AZ''W - A'ZW' + A'Z'W$$

$$+ (1)^k [Z,dW] + (AZW'' - A'ZW' - AZ''W + A''Z'W)$$

$$= [\nabla_A(Z),W] + (-1)^k [Z,\nabla_A(W)].$$

We thus obtain a well-defined bracket

$$[\cdot,\cdot] : H^*_g(M,A) \otimes H^*_g(M,A) \to H^*_{g+1}(M,A).$$

Given a gauge function $Y \in \Omega^0(M,g)$, let $B = Y \ast A$, we thus have

$$\Phi_{A\to B}[Z,W] = [\Phi_{A\to B}(C), \Phi_{A\to B}(D)].$$

In particular, we obtain well-defined graded Lie bracket maps

$$[\cdot,\cdot] : H^*_g(M,\mathcal{F}) \otimes H^*_g(M,\mathcal{F}) \to H^*_{g+1}(M,\mathcal{F}).$$

The whole discussion may be repeated when we consider differential forms with values in $q$ and the corresponding cohomology groups $H^*_q(M,A)$. The outcome is the differential graded Lie algebra $H^*_q(M,\mathcal{F}) = \oplus_k H^*_q(M,\mathcal{F}).$

6.2. Cohomology groups of non-singular foliations. Let us assume that $A = A_{a,V} \in \Omega^1(M,g)$ denote the germ cord with $s_A = 0$ associated with a foliation $\mathcal{F}$ which is constructed from a 1-form $a$ and a transverse vector field $V$ so that $a(V) = -1$. Furthermore, let

$$A = T(a) = e^{tL}a \in \Omega^1(M,q_0)$$

denote the corresponding quantum cord, where $L = L_V$ denotes differentiation in the direction of $V$. Let us denote the subset of $\Omega^*(M,\mathbb{R})$ consisting of the forms $w$ with $\iota_V(w) = 0$ by $\Omega^*_a,V(M,\mathbb{R})$.

It is then clear that $d_{a,V}$ restricts to a Bott differential

$$d_{a,V} : \Omega^*_a,V(M,\mathbb{R}) \to \Omega^{*+1}_a,V(M,\mathbb{R}).$$

To see this, it is enough to note that if $\iota_V(w) = 0$ then

$$\iota_V(d_{a,V}(w)) = \iota_V(dw + aL(w) - bw) = L(w) + \iota_V(a)L(w) - a\iota_V(L(w)) = \iota_V(b)w = 0.$$

Here $b = L(a)$ is the derivative of $a$ in the direction of $V$. We can then define $H^*_a,V(M,\mathbb{R})$ as the cohomology of the chain complex $(\Omega^*_a,V(M,\mathbb{R}),d_{a,V})$:

$$H^*_a,V(M) := \frac{\{ w \in \Omega^*(M,\mathbb{R}) \mid \iota_V(w) = 0 \text{ and } d_{a,V}(w) = 0 \}}{\{ d_{a,V}(z) \mid z \in \Omega^{*+1}(M,\mathbb{R}) \text{ and } \iota_V(z) = 0 \}}.$$

Theorem 6.2. With the above notation fixed, we have

$$H^*_q(M,\mathcal{F}) \simeq H^*_q(M,\mathcal{F}) \simeq H^*_a,V(M).$$

Proof. We first prove the isomorphism $H^*_q(M,\mathcal{F}) \simeq H^*_a,V(M)$. Given $X = \sum_n x_nt^n$ in $\Omega^*(M,q_0)$ we can inductively define

$$Y = \sum_n y_nt^n \in \Omega^{*+1}(M,q_0), \quad y_n = \begin{cases} 0 & \text{if } n = 0, \\ \frac{1}{n}\iota_V(x_{n-1} + dy_{n-1}) & \text{if } n \geq 1. \end{cases}$$

The outcome is the differential graded Lie algebra $H^*_q(M,\mathcal{F}) = \oplus_k H^*_q(M,\mathcal{F})$. The whole discussion may be repeated when we consider differential forms with values in $q$ and the corresponding cohomology groups $H^*_q(M,A)$. The outcome is the differential graded Lie algebra $H^*_q(M,\mathcal{F}) = \oplus_k H^*_q(M,\mathcal{F})$. The whole discussion may be repeated when we consider differential forms with values in $q$ and the corresponding cohomology groups $H^*_q(M,A)$.
From this definition, it follows that \( \nu(Y) = \nu(Y') = 0 \) and \( Y' = \nu(X + dY) \). This implies
\[
\nu(X + \nabla_A(Y)) = \nu(X + dY + AY' - A'Y) = Y' + \nu(A)Y' - \nu(A')Y = 0.
\]
In particular, if \( \nabla_A(X) = 0 \) and \( X \) represents a cohomology class in \( H^*_q(M, A) \), after replacing \( X \) with \( X + \nabla_A(Y) \) which represents the same cohomology class as \( X \), we can assume \( \nu(X) = 0 \) (and thus \( \nu(X') = 0 \)). In particular, \( \nu(d(X)) = L(X) \). Applying \( \nu \) to the two sides of \( \nabla_A(X) = 0 \) we find \( L(X) - X' = 0 \), which is equivalent to \( X = e^{L_t}x_0 \). From \( \nabla_A(X) = 0 \) we obtain
\[
0 = d(e^{L_t}x_0) + (e^{L_t} A)(e^{L_t} L(x_0)) - (e^{L_t} b)(e^{L_t} x_0) = e^{L_t} (d(x_0) + aL(x_0) - bx_0) \]
\[
\Leftrightarrow 0 = d(x_0) + aL(x_0) - bx_0.
\]
In particular, \( x_0 \in \Omega^*_{a,V}(M, \mathbb{R}) \) is in the kernel of \( d_{a,V} \) and uniquely determines \( X \), such that \( \nabla_A(X) = 0 \) and \( \nu(X) = 0 \).

Let us now assume that \( X \) is of the form \( \nabla_A(Y) \) and satisfies \( \nu(X) = 0 \). We can then assume that \( \nu(Y) = 0 \) as well, possibly after replacing \( Y \) by some \( Y + \nabla_A(Z) \). The above considerations imply that \( X = e^{L_t}x_0 \). If we look at the initial terms in the equation \( X = \nabla_A(Y) \) we conclude
\[
x_0 = d(y_0) + ay_1 - by_0 \Rightarrow 0 = \nu(x_0) = L(y_0) + \nu(a) y_1 - \nu(b) y_0 \Rightarrow y_1 = L(y_0).
\]
Thus, \( x_0 = d_{a,V}(y_0) \), which completes the proof of the isomorphism \( H^*_q(M, \mathcal{F}) \simeq H^*_{a,V}(M) \).

The isomorphism \( H^*_q(M, \mathcal{F}) \simeq H^*_{a,V}(M) \) is proved in a completely similar manner, as sketched below. Suppose that \( A = A_{a,V} \in \Lambda(M, q_0) \) correspond to the foliation \( \mathcal{F} \). If \( X \in \Omega^*(M, q_0) \) represents a cohomology class in \( H^*_q(M, A) \), we can construct \( Y \) so that \( Y(0) = 0 \) and \( Y' = \nu(X + dY) \) is satisfied. After replacing \( X \) with \( X + \nabla_A(Y) \) we can assume that \( \nu(X) = \nu(X') = 0 \). Let \( x_0 \) denote the initial term of \( X \). We then have \( d(x_0) + aL(x_0) - L(a)x_0 = 0 \). In particular, \( x_0 \in \text{Ker}(d_{a,V}) \) and it uniquely determines \( X \) in a neighborhood of \( 0 \in \mathbb{R} \) through the differential equation \( X' = L(X) \), with the initial condition \( X(0) = x_0 \). Furthermore, for this solution we find that \( E = \nabla_A(X) \) satisfies \( E(0) = 0 \) and \( E' = L(E) \). This differential equation implies that \( E \) vanishes in a neighborhood of \( 0 \in \mathbb{R} \). Finally, if \( X = \nabla_A(Y) \), we can assume that \( \nu(Y) = 0 \). After replacing \( Y \) with \( Y + \nabla_A(Z) \) if necessary, we can further assume that \( \nu(Y) = 0 \). Looking at the initial terms on the two sides of \( X = \nabla_A(Y) \) we find \( y_1 = L(y_0) \) and \( x_0 = d(y_0) + ay_1 - by_0 \), which means that \( x_0 = d_{a,V}(y_0) \). This observation completes the proof of the theorem.

In the remainder of this subsection, we will focus on the computation of \( H^*_q(M, \mathcal{F}) \) in a number of special cases. The above theorem implies that the corresponding results remain valid for the groups \( H^*_q(M, \mathcal{F}) \).

**Corollary 6.3.** For every transversely oriented codimension-one foliation \( \mathcal{F} \) on the \( n \)-dimensional manifold \( M \), \( \text{H}^*_q(M, \mathcal{F}) = 0 \).

**Proof.** This is an immediate corollary of Theorem 6.2.

Let us now assume that \( a \in \Omega^1(M, \mathbb{R}) \) is a closed nowhere zero one-from. Then \( a \) defines a foliation \( \mathcal{F} = \mathcal{F}^a \) on \( M \) and the corresponding quantum cord is \( A = a \). The foliation \( \mathcal{F} \) may be lifted to the universal cover \( \tilde{M} \) of \( M \) using the covering map \( \pi : \tilde{M} \to M \) to give the foliation \( \tilde{\mathcal{F}} \). This foliation corresponds to the quantum cord \( \tilde{A} = \pi^*A \). Let us denote the leaf space of the foliation \( \tilde{\mathcal{F}} \) by \( \tilde{L} = L_{\tilde{\mathcal{F}}} \). We can also define \( L = L_{\mathcal{F}} \) to be the quotient of \( \tilde{L} \) under the covering map \( \pi \). We call a function on \( L \) **smooth** if it lifts to a smooth function on \( \tilde{L} \). In particular, the restriction of any such function to the closure of any leaf \( \ell \) of \( \mathcal{F} \) is constant. With the above notation fixed, the group \( H^*_q(M, \mathcal{F}) \) may then be computed in a relatively easy way, using Theorem 6.2.

**Corollary 6.4.** If \( a \in \Omega^1(M, \mathbb{R}) \) is a closed nowhere zero one-from which gives the foliation \( \mathcal{F} \),
\[
\text{H}^*_q(M, \mathcal{F}) \simeq C^\infty(L_{\mathcal{F}}, \mathbb{R}).
\]
Proof. Theorem 6.2 identifies $H^0_a(M, F)$ with the kernel of $d_{a,V}$ (for a corresponding transverse vector field $V$). A function $f_0 \in C^\infty(M, \mathbb{R})$ is in the kernel of $d_{a,V}$ if and only if its restriction to every leaf of $\mathcal{F}$ is constant. Such a function gives a section in $C^\infty(L_x, \mathbb{R})$. Conversely, any function in $C^\infty(L_x, \mathbb{R})$ gives a smooth function from $M$ to $\mathbb{R}$ which remains constant on the leaves of $\mathcal{F}$, which is in the kernel of $d_{a,V}$.

In fact, most of the above argument may be repeated for arbitrary foliations to compute their zero cohomology group. The equation $df_0 + aL(f_0) - b_{f_0} = 0$ is satisfied in the transverse direction, i.e. the image of the left-hand-side under $\nu V$ is automatically zero. The equation is thus equivalent to the equalities $df_0 - b_{f_0} = 0$ on all leaves of $\mathcal{F}$. Note that the restriction of $b = L(a)$ to the leaves of $\mathcal{F}$ is closed, since $dL(a) = L(da) = L^2(a)a$. The 1-form $b$ would then define the cohomology groups $H^1_0(\ell, \mathbb{R})$ for every leaf $\ell$ of $\mathcal{F}$. For this purpose, we use the twisted differential

$$d_b : \Omega^*(\ell, \mathbb{R}) \to \Omega^{*+1}(\ell, \mathbb{R}), \quad d_b(X) := d(X) - L(b)X.$$ 

The above argument shows that for every leaf $\ell$ of $\mathcal{F}$, $f_0$ is a section of $H^0_0(\ell, \mathbb{R})$, which is zero unless $b$ is exact on $\ell$. If $b = dg$ on $\ell$, it follows that $f_0 = cg e^{gt}$ for some constant $c_\ell \in \mathbb{R}$. In particular, if $g_0$ is not bounded above, the bounded function $f_0$ is forced to be zero.

The 1-form $b = L(a)$ satisfies $da = ba$, and changing the vector field $V$ would correspond to choosing other $b$ with this property. If $db' = b'a$ for another 1-form $b'$, it follows that $b' = b + ha$ for some function $h$. In particular, the restriction of $b$ to the leaves of $\mathcal{F}$ only depends on $a$. If $a$ is changed to $e^h a$, where $h$ is forced to be bounded above, the restriction of $b$ to the leaves of $\mathcal{F}$ is changed to $b + dh$. The set of points $D = D_x \subset L_x$ where the restriction of $b$ is not of the form $dg$ for some real valued function which is bounded above, is thus independent of the choice of $a$ and $b$, and only depends on the foliation $\mathcal{F}$ and functions in $H^0_0(M)$. Following this approach, every cohomology class $X \in H^1_0(M, \mathcal{F})$ may be studied using its restrictions to the leaves.

7. Foliations of higher codimension

7.1. The groupoids and the corresponding algebras. In this section, we generalize our constructions in the previous sections to the case of foliations of higher codimension. The first step would be generalizing the Lie groupoids $G$ and $Q$ and the corresponding algebroids $g$ and $q$. Most computations remain completely similar to the case of codimension one foliations.

Let us denote the groupoid of germs of local diffeomorphisms of $\mathbb{R}^k$ by $G_k$. The objects of $G_k$ are the points in $\mathbb{R}^k$ and the arrows from $x \in \mathbb{R}^k$ to $y \in \mathbb{R}^k$ are the germs of orientation preserving diffeomorphisms $f : \mathbb{R}^k \to \mathbb{R}^k$ which send $x$ to $y$. The equivalence class of $f$ is denoted by $[(f, x, y)]_g$ or $[f, x]_g$. Note that $[f, x]_g = [g, x]_g$ if $f = g$ in an open neighborhood of $x \in \mathbb{R}^k$. Similarly, we can define $Q_k$ by requiring that $[f, x]_q = [g, x]_q$ if the Taylor expansions of $f$ and $g$ match at $x$. The equivalence class $[f, x]_q$ may then be represented by a formal power series

$$[f, x]_q = Y = \sum_{i=1}^{k} \sum_{t_i(x_i - x_k) \in \mathbb{Z}^k} y_{i,I}(t_1 - x_1)^{i_1} \cdots (t_k - x_k)^{i_k} \partial_i = \sum_{i,I} y_{i,I}(t - x)^I \partial_i.$$ 

Here, $\partial_1, \ldots, \partial_k$ denote the unit vector of $\mathbb{R}^k$ and have a formal nature in the above expression. Moreover, we have

$$y_{i,I} = \frac{\partial^{|I|} f_i}{(\partial t_1)^{i_1} \cdots (\partial t_k)^{i_k}}(x_1, \ldots, x_k), \quad \text{where } f = (f_1, \ldots, f_k) \text{ and } I = (i_1, \ldots, i_k).$$

Being a local diffeomorphism means that the determinant of the matrix $\det(Y) = \det(y_{i,j})_{i,j=1}^k$ is positive. Correspondingly, we can define the algebroids $g_k$ and $q_k$ which are fibered over $\mathbb{R}^k$. The fiber of $q_k$ over $x \in \mathbb{R}^k$ consists of the formal power series $A = \sum_{i,I} a_{i,j}(t - x)^I \partial_i$. One can choose the discrete topology on the space of arrows from $x$ to $y$ in $G_k$ and $Q_k$ to arrive at the groupoids $\Gamma^g_k$.
and $\Gamma^q_k$. These two groupoids are the same as $G_k$ and $Q_k$ (respectively) except that their topologies are different. There are source maps and target maps
\[ s, t : G_k \to \mathbb{R}^k, \quad s, t : Q_k \to \mathbb{R}^n, \quad s, t : g_k \to \mathbb{R}^k \quad \text{and} \quad s, t : q_k \to \mathbb{R}^n. \]

The groupoids $G_k$ and $Q_k$ act on $g_k$ and $q_k$, respectively, and the action is given by
\[ Y \ast A := (A \circ Y)(Y)^{-1} \quad \forall Y \in G_k, \ A \in g_k \quad \text{such that} \ t_Y = s_A. \]

Here, $Y'$ is the $k \times k$ matrix whose entries consist of different first order derivatives of $Y = (Y_1, \ldots, Y_k)$ with respect to the variables $t_1, \ldots, t_k$. Since $\det(Y)$ is positive, it follows that $Y'$ is invertible (both in the germ case and the quantum case).

### 7.2. Germ cords and quantum cords.

There is a Lie bracket on $\Omega^*(M, g_k)$ (and an induced Lie bracket on $\Omega^*(M, q_k)$). For this purpose, we define
\[ [\sum_{i=1}^k A_i \partial_i, \sum_{j=1}^k B_j \partial_j] := \sum_{i,j} (A_i(\partial_i B_j) - B_i(\partial_i A_j)) \partial_j. \]

The germ cords and quantum cords may then be defined as before. A germ cord is a smooth section $A \in \Omega^1(M, g_k)$ which satisfied $dA + \frac{1}{2}[A, A] = 0$. This means that $A = (A_1, \ldots, A_k)$ and that $dA_i + \sum_j A_j(\partial_i A_i) = 0$ for $i = 1, \ldots, k$. The space of germ cords and quantum cords are denoted by $\Lambda(M, g_k)$ and $\Lambda(M, q_k)$, respectively. As before, $\Omega^0(M, g_k)$ and $\Omega^0(M, q_k)$ act on $\Lambda(M, g_k)$ and $\Lambda(M, q_k)$, respectively. Note that a quantum cord is always assumed to be the image of a germ cord. Alternatively, we always restrict our attention to locally trivial flat 1-forms with values in $q_k$.

The quotients give the moduli spaces of germ cords and quantum cords
\[ \mathcal{M}(M, g_k, G_k) = \Lambda(M, g_k)/\Omega^0(M, G_k) \quad \text{and} \quad \mathcal{M}(M, q_k, Q_k) = \Lambda(M, q_k)/\Omega^0(M, Q_k). \]

As in the commutative diagram of Equation 5, one can restrict attention to the cords with values in the fiber of $g_k$ over $0 \in \mathbb{R}^k$ (or the fiber of $q_k$ over $0 \in \mathbb{R}^k$). If $A = (A_1, \ldots, A_k)$ is such a germ cord, $A(0)$ is a $k \times k$ matrix with real values. If the determinant of this matrix is everywhere positive on $M$, $A$ corresponds to a smooth framed foliation $\mathcal{F}$ on $M$. Let us denote the space of all such framed foliations of codimension $k$ by $\mathcal{F}_k(M)$. The action of the gauge group $G_k$ preserves the foliation $\mathcal{F}$ associated with $A$ and we thus obtain the following commutative diagram.

\[ \mathcal{F}_k(M) \xrightarrow{\mathcal{F}_k(G_k)} \mathcal{M}(M, g_k, G_k) \]

\[ \begin{array}{c}
\downarrow \quad \text{Id} \\
\mathcal{F}_k(M) \xrightarrow{\mathcal{F}_k(Q_k)} \mathcal{M}(M, q_k, Q_k)
\end{array} \]

The proof of Theorem 5.3 may then be repeated to prove the following more general form of it.

**Theorem 7.1.** There are natural classification maps
\[ c : \mathcal{M}(M, g_k, G_k) \to \tilde{H}^1(M, \Gamma^g_k) \quad \text{and} \quad c : \mathcal{M}(M, q_k, Q_k) \to \tilde{H}^1(M, \Gamma^q_k) \]

from the moduli spaces of germ cords and quantum cords to the space of $\Gamma^g_k$ and $\Gamma^q_k$ structures on a manifold $M$. This classification map induces the maps
\[ c : \mathcal{C}(M, g_k) = \Lambda(M, g_k)/\sim_g \to [M, BT^g_k] \quad \text{and} \quad c : \mathcal{C}(M, q_k) = \Lambda(M, q_k)/\sim_q \to [M, BT^q_k] \]

from the germs concordia (concordance classes of germ cords) and quantum concordia (concordance classes of quantum cords) to the spaces of homotopy classes of maps from $M$ to the classifying spaces $BT^g_k$ and $BT^q_k$, respectively.
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