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A fractional order control model for Diabetes and COVID-19 co-dynamics with Mittag-Leffler function

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Abstract The aim of this paper is to present and analyze the fractional optimal control model for COVID-19 and diabetes co-dynamics, using the Atangana-Baleanu derivative. The positivity and boundedness of the solutions was shown by the method of Laplace transform. The existence and uniqueness of the solutions of the proposed model were established using Banach fixed point Theorem and Leray–Schauder alternative Theorem. The fractional model was also shown to be Hyers-Ulam stable. The model was fitted to the cumulative confirmed daily COVID-19 cases for Indonesia. The simulations of the total number of hospitalized individuals co-infected with COVID-19 and diabetes, at different face-mask compliance levels, when vaccination strategy is maintained reveals that the total number of hospitalized co-infection cases decreases with increase in face-mask compliance levels, while maintaining COVID-19 vaccination. The simulation results show that to curtail COVID-19 and diabetes co-infections, policies and measures to enforce mass COVID-19 vaccination and strict face-mask usage in the public must be put in place. To further cut down the spread of COVID-19 and diabetes co-infection, time dependent controls are added into the fractional model, and the obtained optimal control problem investigated via the Pontryagin’s Maximum Principle.

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1. Introduction

Globally, Diabetes is one of the major causes of morbidity and mortality and is closely linked with various microvascular and macrovascular complications which severely damage the health of the patient [1]. Links between diabetes and infections have been clinically established for decades [2]. Infections such as influenza and pneumonia are more prevalent and severe in elderly persons with type 2 diabetes mellitus (T2DM) [3,4]. It is observed that [5], individuals with comorbidity have an increased risk of developing severe COVID-19 illness than those without comorbidity. Bjorgul et al. [6] mentioned that comorbidities are “diseases or medical conditions unrelated in etiology or causality to the principal diagnosis that coexist with the disease of interest” [7]. People of any age with the following conditions are at increased risk of severe illness from COVID-19: Chronic kidney disease, COPD (chronic obstructive pulmonary disease), immunocompromised state (weakened immune system) from solid organ transplant, Obesity (body mass index [BMI] of 30 or higher), Serious heart conditions, such as heart failure, coronary artery disease, or cardiomyopathies, Sickle cell disease, Type 2 diabetes mellitus, Hypertension or high blood pressure and Neurologic conditions, such as dementia” [7]. Diabetes prevalence is more in patients with severe COVID-19 illness than those with mild symptoms [8]. The authors in [9,10] also observed that patients with diabetes mellitus have higher susceptibility to several viral infections. Based on the survey involving 138 patients, it was investigated that over 43% of them had more than one comorbidities and that “the patients who were admitted to the intensive care unit (ICU) had a higher number of comorbidities (72.2%) than those not admitted to the ICU (37.3%)” [11,12].

Infectious disease modelling has attracted so much attention in recent years [13,14]. These models seek to find solutions to many of the epidemics and plagues that have inflicted man. Several integer-order optimal control co-infection models have been carried out in the literature. For instance, Bonyah et al. [15] considered and analyzed an integer order mathematical model for the co-infection of dengue fever and Zika virus. Zhao et al. [16] investigated and analyzed an integer order co-infection model for Buruli ulcer and Cholera with optimal control, they showed that the infection of buruli ulcer, cholera and its dual infection can be minimized by using both prevention and treatment controls. Just recently, Oname et al. [17] investigated an integer order COVID-19 and Dengue co-infection model with optimal control and cost-effectiveness analysis, using Brazil as a case study. They showed that Dengue-only preventive strategy or COVID-19 only preventive control would be sufficient to combat either of the two diseases in Brazil. A co-infection model (integer order) for HPV and syphilis with optimal control was studied by the authors in [18]. These models have limitations and are unable to capture memory; an essential ingredient in modeling real life problems. Several models have applied fractional optimal control in their study. For instance, Bonyah et al. [19] analyzed a fractional order COVID-19 model with Mittag-Leffler law. They showed that applying fractional optimal control via the Atangana-Baleanu derivative gave the best strategy in managing complex models. Also, the authors in [20] analyzed a fractional optimal control model for COVID-19 with generalized Mittag–Leffler function. They used optimal control analysis to minimize the infection cases and maximize the susceptible people. Sweilam and Al-Mekhlafi [21] studied an optimal control model with time delay for multi-strain tuberculosis via fractional derivative approach. Further, the authors in [22] considered a fractional order optimal control nonlinear mathematical model of tumor under immune suppression. Moreover, fractional optimal control model for tuberculosis infection including the impact of diabetes and resistant strains was studied by [23]. They showed that fractional-order optimal control model can potentially describe more complex dynamics than can the integer model and can easily include the memory effects present in many real world phenomena. The used two numerical schemes: 2LIM and NS2LIM and compared the simulations for the two different schemes. It has proven several times that the Atangana-Baleanu fractional derivative [24], which has non-singular and non-local kernel contains all the properties of fractional derivative and is the most preferred in modelling real life issues dealing with disease outbreaks and controls [25–27]. For detailed and up to date review of models on fractional derivatives, see the works in [28–36,38,37,39]. Some models have also been carried out on COVID-19 and diabetes co-morbidity co-infection in recent times. Kouidere et al. [40] analyzed an integer order control model for COVID-19 highlighting the detrimental effect of quarantine on diabetes. They showed that during epidemic outbreaks such as COVID-19, diabetes must be followed up to avoid complications, so that co-infection with COVID-19 can be minimized. Also, Ssebuliba et al. [41] modeled COVID-19 dynamics in a partially co-morbid population. They reported that infections at the community level increase very rapidly when detection rates for both individuals with or without comorbidities are decreased. Furthermore, Khan et al. [42] developed and studied a discrete time integer order model for diabetics exposed to COVID-19 infection. They showed that to eliminate chaos and period-doubling bifurcation in the system, quarantine was highly necessary in the COVID-19 environment. In addition, the authors in [43] studied an integer order model for COVID-19 and co-morbidity with COVID-19 re-infection. They also examined the effect of COVID-19 re-infection and showed that re-infection could induce backward bifurcation in the model. They pointed that non-pharmaceutical COVID-19 preventive efforts could not only avert new cases of the diseases, but also help reduce co-infection with co-morbidity. Ozkose and Yavuz [44] investigated the co-interactions between COVID-19 and diabetes with hereditary traits, using real data from Turkey. None of the above works considered fractional optimal control or vaccination in their work. As case detection and non-pharmaceutical interventions alone can not curtail COVID-19 spread, amid rising variants of concern (VOC), there is therefore a need to develop a more comprehensive fractional optimal control model for the co-dynamics of COVID-19 and diabetes (via the best and most preferred Atangana-Baleanu fractional derivative) incorporating vaccination in order assess to what extent the current available COVID-19 vaccines could help decrease the spread of COVID-19 and its co-infection with diabetes. The model is fitted to real COVID-19 data from Indonesia.

The rest of the paper is organized as follows: Model formulation and basic properties of the model are presented in Section 2. Stability analysis is given in Section 3. In Section 4, we
present the numerical scheme for the solution of the model. The data fitting and numerical simulations are presented in Section 5, analysis of the fractional optimal control model is given in Section 6 while Section 7 gives the conclusion.

1.1. Preliminaries

This section presents some definitions and theorems applied in the paper.

Definition 1.1 [24]. Suppose a function \( f \in C^1(0, T) \) such that \( T > 0 \) and \( 0 < q \leq 1 \), then Atangana-Baleanu (AB) derivative in Caputo sense is defined by

\[
a^\alpha D^q_{t}f(t) = \frac{G(q)}{1-q} \int_0^t \frac{f'(\tau)}{\Gamma(q)(t-\tau)^{q-1}} d\tau, t > 0.
\]

where \( G(q) = 1 - q + \frac{q}{\Gamma(q)} \) is a normalization function with \( G(0) = G(1) = 0 \) and \( E_\beta \) is a Mittag–Leffler function.

Definition 1.2 [24]. The Atangana-Baleanu fractional integral of order \( q \) with lower limit \( a \) and non-local kernel is defined by

\[
a^\alpha I^q_{t}f(t) = \frac{1-q}{G(q)} f(t) + \frac{q}{G(q) \Gamma(q)} \int_a^t (t-\tau)^{q-1} f(\tau) d\tau.
\]

Definition 1.3 [24]. The Laplace transform of the Atangana-Baleanu (AB) fractional derivative of order \( q \) in Caputo sense is given by

\[
\mathcal{L}\{a^\alpha D^q_{t}f(t)\} = \frac{G(q)}{s^q} \mathcal{L}\{f(s)\} - \frac{1-q}{s^{q-1}} f(0^+),
\]

where \( \mathcal{L} \) is the Laplace transform operator.

The Atangana-Baleanu derivative possesses a non-singular kernel. The choice of non-singular kernel guarantees a better analysis of physical problems unlike other fractional derivatives and other tools in classical calculus.

Lemma 1.1 [47]. Let \( \psi_1 : [0, T] \to \mathbb{R} \) and \( \psi_2 \) a positive function on \( [0, T] \). Suppose that \( \alpha > 0 \) and \( 0 < q < 1 \) are the constants such that the following holds:

\[
\psi_1(t) \leq \psi_2(t) + d \left( (1-q) \psi_2 + \frac{q}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \psi_2(\tau) d\tau \right).
\]

Then there exists a constant \( \gamma_c = \gamma_c(q) \) such that

\[
\psi_1(t) \leq \psi_2(t) + \gamma_c (1-q) \psi_2(t) + \frac{q}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} \psi_2(\tau) d\tau,
\]

\( t \in [0, T] \).

2. Formulation of the model

At any given time, \( t \), the epidemiological states are: unvaccinated susceptible individuals \( \Psi_s(t) \), vaccinated individuals against COVID-19, \( \Psi_h \), diabetic susceptible individuals \( \Psi_d(t) \), COVID-19 infected (non-diabetic) \( \Psi_d(t) \), isolated individuals with COVID-19 (non-diabetic) \( \Psi_i(t) \), COVID-19 recovered individuals (non-diabetic) \( \Psi_{ir}(t) \), individuals with dual infections \( \Psi_{id}(t) \), individuals with dual infections (hospitalized) \( \Psi_{ih}(t) \), COVID-19 recovered individuals (diabetic) \( \Psi_{ir}(t) \). Individuals are recruited into the population at the rate, \( \Theta \). Unvaccinated susceptibles, \( \Psi_s(t) \) acquire COVID-19 at the rate \( \frac{\gamma_s(1-\psi_s)\Psi_s}{N_h} \), where, \( \pi_s \) denotes the COVID-19 contact rate. \( \eta \) is the face-mask compliance rate and \( \vartheta \) denotes the efficacy of the face-mask. Individuals in this epidemiological state acquire diabetes at the rate \( \pi_d \). Furthermore, they die natural death, which is assumed same for individuals in all compartments of the model, at the rate \( \pi_d \). Vaccinated susceptibles, \( \Psi_h(t) \) acquire COVID-19 at a reduced rate \( (1-\chi)\frac{\gamma_s(1-\psi_s)\Psi_s}{N_h} \), where \( \chi \) is the COVID-19 vaccine efficacy. Diabetic susceptibles, \( \Psi_d(t) \) acquire COVID-19 at the rate \( \lambda_d \frac{\gamma_s(1-\psi_s)\Psi_s}{N_h} \), where \( \lambda_d \) is accounting for increased susceptibility rate for diabetics [9,10]. Also, we have assumed that the transmissibility rate for singly infected and co-infected persons is the same, as there is no evidence yet to justify otherwise. The other transitions in the model are given in the model Eqs. 1, with all the model parameters well defined in Table 1.

The total population at time, \( t \) is given by

\[
N_h = \Psi_s + \Psi_h + \Psi_d + \gamma_c + \Psi_{ir} + \Phi_i + \Psi_{ir} + \Phi_{ird} + \Gamma_{ird}.
\]

The model is given thus,

\[
\begin{align*}
\frac{d\Psi_s}{dt} &= F_1(t, \Psi_s(t)) = \Theta - \left( \lambda_d \frac{\gamma_s(1-\psi_s)\Psi_s}{N_h} + \pi_d + \mu_d \right) \Psi_s, \\
\frac{d\Psi_h}{dt} &= F_2(t, \Psi_h(t)) = \Theta - \left( \lambda_d \frac{\gamma_s(1-\psi_s)\Psi_s}{N_h} + \pi_d + \mu_d \right) \Psi_h, \\
\frac{d\Psi_d}{dt} &= F_3(t, \Psi_d(t)) = \Theta - \left( \lambda_d \frac{\gamma_s(1-\psi_s)\Psi_s}{N_h} + \pi_d + \mu_d \right) \Psi_d, \\
\frac{d\Psi_i}{dt} &= F_4(t, \Phi_i(t)) = \Psi_i - (\alpha_i + \epsilon_i + \mu_d) \Phi_i, \\
\frac{d\Phi_i}{dt} &= F_5(t, \Phi_i(t)) = \mu_i + \alpha_i \Phi_i - \gamma_i \Phi_i, \\
\frac{d\Psi_{ir}}{dt} &= F_6(t, \Psi_{ir}(t)) = \Psi_{ir} - (\psi_{ir} + \psi_{ird} + \mu_d) \Psi_{ir}, \\
\frac{d\Psi_{ird}}{dt} &= F_7(t, \Phi_{ird}(t)) = \mu_i + \alpha_i \Phi_{ird} - \gamma_i \Phi_{ird}, \\
\frac{d\Phi_{ird}}{dt} &= F_8(t, \Phi_{ird}(t)) = \mu_i + \alpha_i \Phi_{ird} - \gamma_i \Phi_{ird},
\end{align*}
\]

(1)

with corresponding initial conditions

\[
\begin{align*}
\Psi_s(0) &= \Psi_{s0}, \Psi_h(0) = \Psi_{h0}, \Psi_d(0) = \Psi_{d0}, \gamma_c(0) = \gamma_{c0}, \\
\Phi_i(0) &= \Phi_{i0}, \Phi_{ird}(0) = \Phi_{ird0}, \Gamma_{ird}(0) = \Gamma_{ird0}.
\end{align*}
\]

(2)

2.1. Non-negativity

Here we consider the non-negativity of all the solutions in our system. Application of Laplace transform of the Atangana-Baleanu derivative in Caputo sense plays a crucial in this case. Following similar arguments to those in [26], we suppose that the point \( t \) satisfies

\[
\min_{t \in [0, \tau]} \{ \psi_s(t) - \psi_s(0), \psi_h(t) - \psi_h(0), \psi_d(t) - \psi_d(0), \gamma_c(t) - \gamma_c(0), \psi_{ir}(t) - \psi_{ir}(0), \psi_{ird}(t) - \psi_{ird}(0), \Gamma_{ird}(t) - \Gamma_{ird}(0) \} = 0.
\]

Then let \( \Psi_s(t) = 0 \) and \( \Psi_h(t) < 0 \) for all \( 0 \leq t \leq \tau \), then there exists the quantity \( P \) such that

\[
\frac{d}{dt} \Psi_s(t) - P \Psi_s(t) > 0.
\]

(3)

We now seek for additional force function \( g(t) \) such that

\[
\frac{d}{dt} \Psi_s(t) = P \Psi_s(t) - g(t)
\]

can be ascertained. Thus application of Laplace transform gives
fractional derivative gives a better space for this case which cannot be accommodated in the classical calculus or fractional derivatives with singular kernel.

Similar approach yields the non-negativity of the solutions $\Psi_{\alpha}(t)$, $\Psi_{\beta}(t)$, $Y_t(t)$, $\Phi_t(t)$, $\Gamma_t(t)$, $Y_{\alpha d}(t)$, $\Phi_{\alpha d}(t)$ and $\Gamma_{\alpha d}(t)$ respectively as

\[ \Psi_{\alpha}(t) > \frac{\Psi_{\alpha}(0)}{\delta} E_\delta \left( \frac{q}{\alpha} P t^\alpha \right) > 0, \]

\[ \Psi_{\beta}(t) > \frac{\Psi_{\beta}(0)}{\delta} E_\delta \left( \frac{q}{\alpha} P t^\alpha \right) > 0, \]

\[ \Phi_t(t) > \frac{\Phi_t(0)}{\delta} E_\delta \left( \frac{q}{\alpha} P t^\alpha \right) > 0, \]

\[ \Gamma_t(t) > \frac{\Gamma_t(0)}{\delta} E_\delta \left( \frac{q}{\alpha} P t^\alpha \right) > 0, \]

\[ \Psi_{\alpha d}(t) > \frac{\Psi_{\alpha d}(0)}{\delta} E_\delta \left( \frac{q}{\alpha} P t^\alpha \right) > 0, \]

\[ \Gamma_{\alpha d}(t) > \frac{\Gamma_{\alpha d}(0)}{\delta} E_\delta \left( \frac{q}{\alpha} P t^\alpha \right) > 0. \]

2.2. Invariant Region

We investigate the region where the solutions of the system are biologically and mathematically well-posed. The compartments are human population and thus the parameters used in the system are assumed positive. Therefore we consider the total of human population at the time; $t$

\[ N_h = \Psi_h + \Psi_{\alpha h} + \Psi_{\beta} + \Phi_t + \Gamma_t + Y_{\alpha d} + \Phi_{\alpha d} + \Gamma_{\alpha d}. \]

Applying Atangana-Baleanu fractional derivative on both sides and plugging in respective functions gives

\[ \frac{\partial}{\partial t} \Psi_h(t) \leq \Theta_h - \mu_h N_h. \]

Application of Laplace transform and its inverse expressed as a Mittag-Lefler function, we deduce that

\[ \lim_{t \to \infty} N_h(t) \leq \frac{\Theta_h}{\mu_h}. \]

Thus, the biologically feasible domain is

\[ \Lambda = \left\{ (\Psi_h, \Psi_{\alpha h}, \Psi_{\alpha d}, \Psi_{\beta}, \Gamma_t, Y_{\alpha d}, \Phi_{\alpha d}, \Gamma_{\alpha d}) \in \mathbb{R}^8 \mid \Theta_h / \mu_h \right\}. \]

2.3.Existence and Uniqueness of the solution

For any mathematical system, it is always desirable to know that if a solution to the problem exists and if so, is it the only solution? This subsection deals with answering the above posed question with reference to the fractional order model under consideration. Consider

\[ \frac{\partial}{\partial t} \phi(t) = F(t, \phi(t)), \]

\[ \phi(0) = \phi_0, \]

where the vector $\phi = (\Psi_h, \Psi_{\alpha h}, \Psi_{\alpha d}, \Psi_{\beta}, \Gamma_t, Y_{\alpha d}, \Phi_{\alpha d}, \Gamma_{\alpha d})$ represents the compartment variables while $\phi(0) = (\Psi_h(0), \Psi_{\alpha h}(0), \Psi_{\alpha d}(0))$. 

Table 1 Description of parameters in the model equation.

| Parameter | Interpretation | Value (day$^{-1}$) | Reference |
|-----------|----------------|---------------------|-----------|
| $\beta$   | COVID-19 contact rate | 0.5776 | Fitted |
| $\eta$    | Face-mask compliance rate | 0.70 | Variable |
| $\sigma$  | Face-mask efficacy | 0.90 | Variable |
| $\Theta_h$| Recruitment rate | 27352621 | [56] |
| $\mu_{\alpha}$ | Rate of Diabetes development for unvaccinated susceptible humans | 0.3988 | Fitted |
| $\mu_{\alpha d}$ | Rate of Diabetes development for vaccinated susceptible humans | 0.0890 | Fitted |
| $\sigma_r$ | COVID-19 vaccination rate | 0.4 | Fitted |
| $\Gamma_r$ | COVID-19 vaccine efficacy | 0.95 | [57] |
| $\rho$    | Natural death rate | 15 | [43] |
| $v_1$     | Detection rate for individuals in $Y_t$ | 0.541414 | Fitted |
| $v_2$     | Detection rates for individuals in $Y_{\alpha d}$ | 0.5305 | Fitted |
| $p_{1, o1}$ | COVID-19 treatment rates for individuals in $Y_t$ and $\Phi_t$, respectively | 0.13978 | [43] |
| $p_{2, o2}$ | COVID-19 treatment rates for individuals in compartments $Y_{\alpha d}$ and $\Phi_{\alpha d}$, respectively | 0.0015 | [43] |
| $\xi_1$   | COVID-19-related death rate for those in $Y_t$ and $\Phi_t$, respectively | 0.1238 | [43] |
| $\xi_2$   | Diabetes-related death rate for those in $\Phi_{\alpha d}$ | 0.5009 | Fitted |
| $\lambda_0$ | modification parameter | 1.2 | [7] |
To prove the uniqueness of the solution, we show that the operator $\Omega$ has a fixed unique point. Note that the Chebyshev norm $\|\cdot\|_{[0,T]}$ is the supremum norm over the interval $[0,T]$ which is defined by

$$\|\phi\|_{[0,T]} = \sup_{0 \leq t \leq T} |\phi(t)|, \quad \phi \in \mathbb{C}.$$
where \( B_c = \{ \phi \in C^1(\cdot) : ||\phi||_{0,T} \leq \epsilon \} \) As \( F \) is totally continuous, for all \( t \in [0, T] \), we have

\[
|\Omega(\phi)(t)| \leq \|\phi\| + \frac{1}{|\alpha|} \int_0^t |F(\sigma, \phi(\sigma))|d\sigma + \int_0^t \left( \int_0^\sigma |F(\tau, \phi(\tau))|d\tau \right) d\sigma
\]

and

\[
|\Omega(\phi)(t)| \leq \|\phi\| + \frac{1}{|\alpha|} \int_0^t |F(\sigma, \phi(\sigma))|d\sigma + \int_0^t \left( \int_0^\sigma |F(\tau, \phi(\tau))|d\tau \right) d\sigma
\]

Thus \( |\Omega(\phi)(t)| \leq \mathcal{L}_r \) as required.

We also show that the operator \( \Omega \) maps bounded set into equi-continuous sets of \( C^1(\cdot) \). Let \( 0 \leq t_1 \leq t_2 \leq T \) and \( B_r \) be a set of \( C^1(\cdot) \) as considered above. For each \( t \in [0, T] \) we get

\[
|\Omega(\phi)(t) - \Omega(\phi)(t_1)| \leq \frac{1}{|\alpha|} \int_{t_1}^t |F(\sigma, \phi(\sigma))|d\sigma + \int_{t_1}^t \left( \int_0^\sigma |F(\tau, \phi(\tau))|d\tau \right) d\sigma
\]

which proves that \( \Omega \) is equi-continuous as \( t_1 \to t_2 \), the right hand side of the inequality goes to zero. Thus by Arzelà-Ascoli theorem, we conclude that \( \Omega : C^1(\cdot) \to C^1(\cdot) \) is totally continuous.

Finally we show that \( \phi \neq \lambda \Omega(\phi) \) for \( 0 < \lambda < 1 \) and \( \phi \in \rho E \) (boundary of \( E \)), for an open set \( E \subseteq C^1(\cdot) \). Suppose \( \phi = \lambda \Omega(\phi) \) and \( \phi \in C^1(\cdot) \) where \( 0 < \lambda < 1 \), then for every \( t \in [0, T] \), we obtain

\[
\phi(t) = \left( 1 - \frac{\theta}{G(\theta)} \right) F(t, \phi) + \frac{\theta}{G(\theta)G'(\theta)} \int_0^t (t - \sigma)^{\alpha - 1} F(\sigma, \phi(\sigma)) d\sigma
\]

Using \( (B2) \) we get

\[
|\phi(t)| = \frac{1}{|\alpha|} \int_0^t |F(\sigma, \phi(\sigma))|d\sigma + \int_0^t \left( \int_0^\sigma |F(\tau, \phi(\tau))|d\tau \right) d\sigma
\]

where in this case

\[
\|\phi\| = Y_{0,T} \sup \{ ||\phi(\sigma)|| : 0 \leq \sigma \leq T \} \leq \psi_2.
\]

Thus

\[
\psi_2(t) \leq \psi(t) \leq C(1 - \eta \theta) Y_{0,T} \|h\|_\infty\frac{1}{G(\theta)G'(\theta)} + \frac{1}{\alpha} \frac{|\alpha|}{|\alpha|} \int_0^t (t - \sigma)^{\alpha - 1} \psi_2(\sigma) d\sigma,
\]

for every \( t \in [0, T] \).

Using \( \text{Lemma (1.1)} \), there exists \( \gamma_c = Y_c(\cdot) \) such that

\[
\|\psi_2(t)\| \leq W + \gamma_c(\|h\|_\infty) \max\left( \frac{|\alpha|}{|\alpha|}, \frac{|\alpha|}{|\alpha|} \right) \int_0^t (t - \sigma)^{\alpha - 1} W d\sigma.
\]

where

\[
W = \left( 1 - \frac{\theta}{G(\theta)} \right) \int_0^t (t - \sigma)^{\alpha - 1} F(\sigma, \phi(\sigma)) d\sigma.
\]

It follows that

\[
\|\phi(t)\| \leq \left( 1 - \frac{\theta}{G(\theta)} \right) \int_0^t (t - \sigma)^{\alpha - 1} \|h\|_\infty + N\|\phi\|_\infty = N^*.
\]

Thus \( \Omega : E_1 \to C^1(\cdot) \) is continuous. The choice of set \( E \) suggest that \( \phi \notin \rho E \) such that \( \Omega(\phi) = \phi \) for \( 0 < \lambda < 1 \). Thus, an operator \( \Omega \) has a fixed point \( \phi \in E \) by non-linear Leray–Schauder alternative Theorem [54].

### 2.4 Basic reproduction number

The model possesses a disease free equilibrium \( \Omega_0 \) which we obtained by setting the right hand side of the system (1) to zero which yields

\[
\Omega_0 = \left( \begin{array}{c} \Psi^*_{\Psi}, \Psi^*_{\Psi}, \Psi^*_{\Psi}, \Psi^*_{\Psi}, \Psi^*_{\Psi}, \Psi^*_{\Psi}, \Psi^*_{\Psi}, \Psi^*_{\Psi} \\ 0, 0, 0, 0, 0, 0, 0, 0 \end{array} \right).
\]

The linear stability of the disease free equilibrium, \( \Omega_0 \), can be established using the next generation operator method [49] on the system (1).

The next generation matrices are, respectively, given by

\[
F = \left( \begin{array}{ccccccc} \pi_1(1 - \eta \theta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \pi_1(1 - \eta \theta) & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda_1 \pi_c(1 - \eta \theta) & 0 & \lambda_1 \pi_c(1 - \eta \theta) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 \pi_c & 0 & 0 & 0 & 0 & 0 \end{array} \right)
\]

where

\[
\mathcal{M}_1 = (v_1 + \rho_1 + \mu_1), \quad \mathcal{M}_2 = (\omega_2 + \xi_2 + \mu_2),
\]

\[
\mathcal{M}_3 = (\nu_2 + \rho_2 + \mu_2), \quad \mathcal{M}_4 = (\xi_2 + \mu_2 + \omega_2 + \mu_2).
\]

Using the approach shown in [49], the basic reproduction number of the model (1) is given by (where \( \rho \) is the spectral radius)

\[
\mathcal{R}_{0} = \rho(F^{-1})
\]

The result below follows from Theorem 2 in [49].

**Lemma 2.1.** The DFE \( \Omega_0 \) of the model is locally asymptotically stable (LAS) if \( \mathcal{R}_0 < 1 \), and unstable if \( \mathcal{R}_0 > 1 \).

### 2.5 Local asymptotic stability of the disease free equilibrium (DFE) of the co-infection model

**Theorem 2.3.** The DFE, \( \Omega_0 \), of the model (1) is locally asymptotically stable (LAS) if \( \mathcal{R}_0 < 1 \), and unstable if \( \mathcal{R}_0 > 1 \).

**Proof.** The local stability of the model (1) is analyzed by the Jacobian matrix of the system (1) evaluated at the disease-free equilibrium, \( \Omega_0 \), given by:

\[
\begin{pmatrix}
\pi_1(1 - \eta \theta) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \pi_1(1 - \eta \theta) & 0 & 0 & 0 & 0 & 0 & 0 \\
\lambda_1 \pi_c(1 - \eta \theta) & 0 & \lambda_1 \pi_c(1 - \eta \theta) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \lambda_2 \pi_c & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

with

\[
\mathcal{M}_1 = (v_1 + \rho_1 + \mu_1), \quad \mathcal{M}_2 = (\omega_2 + \xi_2 + \mu_2),
\]

\[
\mathcal{M}_3 = (\nu_2 + \rho_2 + \mu_2), \quad \mathcal{M}_4 = (\xi_2 + \mu_2 + \omega_2 + \mu_2).
\]

Using the approach shown in [49], the basic reproduction number of the model (1) is given by (where \( \rho \) is the spectral radius)

\[
\mathcal{R}_0 = \rho(F^{-1})
\]

The result below follows from Theorem 2 in [49].

**Lemma 2.1.** The DFE \( \Omega_0 \) of the model is locally asymptotically stable (LAS) if \( \mathcal{R}_0 < 1 \), and unstable if \( \mathcal{R}_0 > 1 \).
$\begin{vmatrix} 0 & 0 & - 1 & 0 & 0 & 0 & - 1 & 0 & 0 \\ - 1 & 0 & 0 & 0 & 0 & 0 & - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ - 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}$

where,

$\tilde{\pi} = \pi_1(1 - \eta \tilde{\theta})$,  $\mathcal{H}_1 = v_1 + \rho_1 + \mu_b$,
$\mathcal{H}_2 = \omega_1 + \xi_1 + \mu_b$,  $\mathcal{H}_3 = v_2 + \rho_2 + \mu_b$,
$\mathcal{H}_4 = \xi_1 + \xi_2 + \omega_2 + \mu_b$.

The characteristic equation of the above matrix is given by

$\left[ (\mathbf{\bar{\epsilon}} + \mathbf{\bar{\mu}}(\mathbf{\bar{\mu}} + \mathbf{\bar{\mu}} + \mathbf{\bar{\mu}}) + \mathbf{\bar{\mathcal{H}}}) (\mathbf{\bar{\mathcal{H}}} + \mathbf{\bar{\mu}} (1 - \mathbf{\bar{\mathcal{R}}})) \right] - \mathbf{\bar{\mu}}^2 = 0$.

Following the Routh-Hurwitz criterion, the equation above will have roots with negative real parts if and only if $\mathbf{\bar{\mathcal{R}}} < 1$. Thence, the DFE, $\mathcal{F}_0$ is locally asymptotically stable if $\mathbf{\bar{\mathcal{R}}} < 1$.

2.6. Global stability of the DFE of the model

We employ the approach in [50] to investigate the global asymptotic stability (GAS) of the DFE of the model. The system (1) is re-written as:

$\frac{d^\alpha}{dt^\alpha} D_t^\alpha \mathcal{F} = P(\mathcal{F}, \mathcal{Y})$,  $\mathcal{F} = (\mathcal{Y}, \mathcal{Y}_a, \Psi_b, \mathcal{G}) \in \mathbb{R}^5$

where $\mathcal{F} = (\Psi_b, \Psi_{ab}, \Psi_b, \mathcal{G}, \mathcal{G}) \in \mathbb{R}^5$ represents the number of non-infectious compartments and $\mathcal{Y} = (\mathcal{Y}, \mathcal{Y}_a, \Psi_b, \Psi_{ab}) \in \mathbb{R}^4$ represents the number of infectious compartments. $\mathcal{F}_0 = (\mathcal{F}, \mathcal{Y})$ denotes the DFE of the system. For local asymptotic stability, the following conditions must be met:

$\Delta_1$: For $\frac{d^\alpha}{dt^\alpha} D_t^\alpha \mathcal{F} = P(\mathcal{F}, \mathcal{Y})$, $\mathcal{F}$ is globally asymptotically stable (GAS).

$\Delta_2$: $\Psi(\mathcal{F}, \mathcal{Y}) = \Psi(\mathcal{F}, \mathcal{Y}) = 0$, $Q(\mathcal{F}, \mathcal{Y}) \geq 0$ for $F(\mathcal{F}, \mathcal{Y}) \in \Omega$, where $\Psi = D^\alpha P(\mathcal{F}, \mathcal{Y})$, $\Omega$ is the invariant region. If (1) fulfills the above conditions, then the following result holds:

**Theorem 2.4.** The fixed point $\mathcal{F}_0 = (\mathcal{F}, 0)$ is a globally asymptotic stable (GAS) equilibrium of (1) if $\mathbf{\bar{\mathcal{R}}} < 1$ (LAS) and that conditions $\Delta_1$ and $\Delta_2$ are met.

Proof\:

$\frac{d^\alpha}{dt^\alpha} D_t^\alpha \mathcal{F} = P(\mathcal{F}, \mathcal{Y}) = \begin{cases} \Theta_1, & \text{if } (\mathcal{F}_0 + \mathcal{Y}_a) - (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_b + (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_{ab} \\ 0, & \text{if } (\mathcal{F}_0 + \mathcal{Y}_a) - (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_b + (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_{ab} \end{cases}$

$\mathcal{F}_0 = (\mathcal{F}, \mathcal{Y}) = \begin{cases} 0, & \text{if } (\mathcal{F}_0 + \mathcal{Y}_a) - (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_b + (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_{ab} \\ 0, & \text{if } (\mathcal{F}_0 + \mathcal{Y}_a) - (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_b + (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_{ab} \end{cases}$

where $\mathcal{F}$ represents the number of non-infectious compartments and $\mathcal{Y}$ represents the number of infectious compartments.

$\Psi(\mathcal{F}, \mathcal{Y}) = \begin{cases} \Theta_1, & \text{if } (\mathcal{F}_0 + \mathcal{Y}_a) - (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_b + (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_{ab} \end{cases}$

$\mathcal{F}_0 = (\mathcal{F}, \mathcal{Y}) = \begin{cases} 0, & \text{if } (\mathcal{F}_0 + \mathcal{Y}_a) - (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_b + (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_{ab} \end{cases}$

so that

$\Psi(\mathcal{F}, \mathcal{Y}) - \Psi(\mathcal{F}, \mathcal{Y}) = \begin{cases} \Theta_1, & \text{if } (\mathcal{F}_0 + \mathcal{Y}_a) - (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_b + (\mathcal{Y}_a + \mathcal{Y}_a) \Psi_{ab} \end{cases}$

It is seen that, $\Psi(\mathcal{F}, \mathcal{Y}) \geq 0$. Thus, the DFE is GAS.

3. Hyers-Ulam's Stability analysis of the fractional system

In this subsection, we employ similar approach in [46] where Hyers-Ulam's stability approach will be used in the fractional system (1) and (2) given by the initial value problem (4) and (5).

**Definition 3.1** [48]. The initial value problem (4) and (5) is called Hyers-Ulam stable if for every $\phi \in C^4([0, T], \mathbb{R})$ satisfying

$\phi(t) - \frac{q}{G(q)} F(t, \phi(t)) - \frac{q}{G(q) F(t, \phi(t))} \int_0^t (t - \phi)^{-1} F(t, \phi(t)) d\phi \leq q, \quad q > 0$,

there exists a solution $\phi(t) \in C^4([0, T], \mathbb{R})$ of (4) and (5) such that $|\phi(t) - \phi(t)| \leq \phi$, where $q > 0$ for all $t \in [0, T]$. Note that $C^4([0, T], \mathbb{R})$ is the Banach space of the total continuous function defined in $[0, T]$. 
Theorem 3.1. Suppose that $\phi = \hat{\phi}$ is satisfied, then the initial value problem (4) and (5) which denotes the fractional model (1) and (2) is Hyers-Ulam stable.

Proof. Recall that Theorem (2.1) guarantees that there exists a unique solution to the fractional model. Let $\hat{\phi}(t)$ be another unique solution to the fractional model. Note that every solution of (7) is also a solution of the IVP and vice versa. Also,

$$\|\hat{\phi}(t) - \hat{\phi}(\tilde{t})\| \leq \frac{(1 - \phi)}{G(t)} \|F(\phi, \phi)\| + \frac{\phi}{G(t) T(t)} \int_0^1 (t - \phi)^{q-1} |F(\phi, \phi)| d\phi,$$

where $\mathcal{L}_r > 0$ is the Lipschitz constant. Let $\mathcal{L}_r = p$ and $q = \left(\frac{1}{\mathcal{L}_r} + \frac{4}{\mathcal{L}_r^2 + \mathcal{L}_r + 1}\right)$, so that

$$\|\hat{\phi}(t) - \hat{\phi}(\tilde{t})\| \leq p q, \quad t \in [0, T], \quad p, q > 0. \quad (15)$$

Adopting the inequality (15) into the fractional system (1) and (2) yields the following

$$\|\Psi(t) - \Psi(\tilde{t})\| \leq p q t_1, \quad \|\Psi_a(t) - \Psi_a(\tilde{t})\| \leq p q t_2, \quad \|\Psi_d(t) - \Psi_d(\tilde{t})\| \leq p q t_3,$$

$$\|\Psi(t) - \Psi(\tilde{t})\| \leq p q t_4, \quad \|\Psi_a(t) - \Psi_a(\tilde{t})\| \leq p q t_5,$$

$$\|\Psi_d(t) - \Psi_d(\tilde{t})\| \leq p q t_6, \quad \|\Psi(t) - \Psi(\tilde{t})\| \leq p q t_7,$$

$$\|\Psi_a(t) - \Psi_a(\tilde{t})\| \leq p q t_8, \quad \|\Psi_d(t) - \Psi_d(\tilde{t})\| \leq p q t_9.$$

Hence the solution of the fractional model (1) and (2) of order $q$ is Hyers-Ulam stable.

4. Numerical scheme for the Atangana-Baleenu model

The numerical scheme given in [51] shall now be applied to approximate the Atangana-Baleenu fractional polynomial. The numerical method used combines the two-step Lagrange fractional polynomial and the fundamental theorem of fractional calculus. This method has proven to be highly accurate and efficient, user-friendly and converges quickly to the exact solution even with a large step of discretization [51].

Applying Atangana-Baleenu fractional integral on both sides of the initial value problem (4) and (5) gives

$$\phi(t) = \phi_0 + \frac{1 - \phi}{G(t)} F(t, \phi(t)) + \frac{\phi}{G(t) T(t)} \int_0^t (t - \phi)^{q-1} F(\phi, \phi) d\phi.$$ 

At $t = t + (\varepsilon + 1)h$, where $h = t + 1 - t$ is the time space, the above equation discretizes to

$$\phi(t + 1) = \phi_0 + \frac{1 - \phi}{G(t)} F(t, \phi(t)) + \frac{\phi}{G(t) T(t)} \sum_{n=0}^{\epsilon} (t - \phi)^{q-1} F(\phi, \phi) d\phi. \quad (16)$$

When we adopt Lagrange two-points interpolation into (16), we have

$$\phi(t+1) - \phi(\theta) = \frac{(1 - \phi)}{G(t)} F(t, \phi(t)) + \frac{\phi}{G(t) T(t)} \sum_{n=0}^{\epsilon} (t - \phi)^{q-1} F(\phi, \phi) d\phi.$$ 

If we apply (17), then (1) and (2) becomes,

$$\Psi(t+1) = \Psi(t_1) + \frac{1 - \phi}{G(t)} F(t, \Psi(t_1)) + \frac{\phi}{G(t) T(t)} \sum_{n=0}^{\epsilon} (t - \phi)^{q-1} F(\phi, \phi) d\phi.$$ 

5. Numerical simulations

5.1. Estimation of initial conditions and parameters

Since the total population of Indonesia is 273,523,621 [56], we set $\Psi(t) = 260,000,000$. Also, we set $\Psi(t) = 345,605$, following from the estimate in [56]. The total number of recovered individuals is set as $\Gamma(t) = 993,117$ [56]. The paper [58] gave some diabetes report. Hence, the diabetic susceptible population is set at $\Psi(t) = 16,000,000$. For the other initial conditions, we assumed: $\Phi(t) = 1,090,000$. The $fmincon$ optimization toolbox [59] is used for the model fitting. The model (1) is fitted to the cumulative confirmed daily COVID-19 cases for Indonesia from February 11, 2021 to August 26, 2021. As shown in Fig. 1, the fractional order model (1) fits well to the COVID-19 data sets for Indonesia. The estimated reproduction number is $R_0 = 0.9148$. 

$$\phi(t+1) - \phi(\theta) = \frac{(1 - \phi)}{G(t)} F(t, \phi(t)) + \frac{\phi}{G(t) T(t)} \sum_{n=0}^{\epsilon} (t - \phi)^{q-1} F(\phi, \phi) d\phi.$$ 

$$\phi(t+1) - \phi(\theta) = \frac{(1 - \phi)}{G(t)} F(t, \phi(t)) + \frac{\phi}{G(t) T(t)} \sum_{n=0}^{\epsilon} (t - \phi)^{q-1} F(\phi, \phi) d\phi.$$ 

$$\phi(t+1) - \phi(\theta) = \frac{(1 - \phi)}{G(t)} F(t, \phi(t)) + \frac{\phi}{G(t) T(t)} \sum_{n=0}^{\epsilon} (t - \phi)^{q-1} F(\phi, \phi) d\phi.$$ 

$$\phi(t+1) - \phi(\theta) = \frac{(1 - \phi)}{G(t)} F(t, \phi(t)) + \frac{\phi}{G(t) T(t)} \sum_{n=0}^{\epsilon} (t - \phi)^{q-1} F(\phi, \phi) d\phi.$$ 

$$\phi(t+1) - \phi(\theta) = \frac{(1 - \phi)}{G(t)} F(t, \phi(t)) + \frac{\phi}{G(t) T(t)} \sum_{n=0}^{\epsilon} (t - \phi)^{q-1} F(\phi, \phi) d\phi.$$ 

$$\phi(t+1) - \phi(\theta) = \frac{(1 - \phi)}{G(t)} F(t, \phi(t)) + \frac{\phi}{G(t) T(t)} \sum_{n=0}^{\epsilon} (t - \phi)^{q-1} F(\phi, \phi) d\phi.$$
5.2. Simulation of the Fractional model without optimal control

We now give a brief rundown of previous results on COVID-19 and diabetes/co-morbidity co-infection: Koudere et al. [40], in their work on integer model for COVID-19 and diabetes, showed that during epidemic outbreak such as COVID-19, diabetics must be followed up to avoid complications. Ssebuliba et al. [41] in a paper on COVID-19 dynamics in a partially co-morbid population, reported that infections at the community level increase very rapidly when detection rates for both individuals with or without co-morbidities are decreased. Also, Khan et al. [42] in a work on integer order model for diabetics exposed to COVID-19 infection, showed that in order to eliminate chaos and period-doubling bifurcations, quarantine was a necessity in the COVID-19 environment. Furthermore, the authors in [43] in paper on COVID-19 and co-morbidity with COVID-19 re-infection, showed that re-infection could induce backward bifurcation in the model. Moreover, Ozkose and Yavuz [44], in a recent work on COVID-19 and diabetes with hereditary traits, using real data from Turkey, showed that the Caputo fractional derivative greatly influenced the dynamics of the two diseases in the population. As treatment and non-pharmaceutical interventions alone can not curtail COVID-19 spread, amid rising variants of concern, our current work considers a more comprehensive fractional optimal control model for the co-dynamics of COVID-19 and diabetes (via the best and most preferred Atangana-Baleanu derivative) incorporating vaccination in order assess to what extent the current available COVID-19 vaccines could decrease the spread of COVID-19 and its co-infection with diabetes. This was not considered by previous studies in this area.

Simulation of the total number of individuals co-infected with COVID-19 and diabetes, at different face-mask compliance levels, when vaccination strategy is maintained, is depicted by Fig. 3. Here, the fractional order value, \( \alpha = 0.80 \) and the COVID-19 contact rate \( \sigma_c = 1.0991 \). The vaccine efficacy is assumed \( \chi = 0.95 \) and vaccination rate \( \eta = 0.4 \). The face-mask is assumed to be 95% effective in preventing new infections. The face-mask compliance level \( \eta \) is varied from 0.15 to 0.60. It is observed that the total number of hospitalized co-infection cases decreases with increase in face-mask compliance levels. The simulation results show that to curtail COVID-19 and diabetes co-infections, policies and measures to enforce mass COVID-19 vaccination and strict face-mask usage in the public must be enforced.

Simulation of the total number of individuals infected with COVID-19, when COVID-19 vaccination is not administered and when it is implemented, is depicted by Fig. 4. Here, the fractional order value, \( \alpha = 0.80 \), the COVID-19 contact rate is set at \( \sigma_c = 1.0991 \), so that the basic reproduction number is \( R_{0\infty} = 2.4390 > 1 \). When only face-mask usage is strictly enforced, but with no COVID-19 vaccination, the infection cases increase steadily. However, it is observed that incorporating vaccination strategy with vaccine efficacy at \( \chi = 0.95 \) and vaccination rate at \( \eta = 0.4 \) per day, will avert about 1,648,000 new COVID-19 cases after 200 days. Also, Simulation of the total number of individuals infected with COVID-19, when COVID-19 vaccination is not administered and when it is implemented, is depicted by Fig. 5. Here, also, the fractional order value, \( \alpha = 0.80 \), the COVID-19 contact rate is set at \( \sigma_c = 1.0991 \), so that the basic reproduction number is \( R_{0\infty} = 2.4390 > 1 \). The plot reveals that incorporating vaccination strategy with vaccine efficacy at \( \chi = 0.95 \) and vaccination rate at \( \eta = 0.4 \) per day, will avert about 6,991,000 new active COVID-19 cases at the end of 200 days.

Simulations of the total number of individuals infected with COVID-19, at different face-mask compliance levels, when vaccination strategy is maintained, is depicted by Fig. 6. Here, the fractional order value, \( \alpha = 0.80 \). Also, the COVID-19 contact rate \( \sigma_c = 1.0991 \) while the vaccine efficacy is set at \( \chi = 0.95 \) and vaccination rate, \( \eta = 0.4 \). The face-mask is assumed to be 95% effective in preventing new infections. The face-mask compliance level \( \eta \) is varied from 0.15 to 0.60. It is observed that the combined strategy leads to significant decrease in the co-infection new cases over time. The simulations of the total number of hospitalized individuals co-infected with COVID-19 and diabetes, at different

![Fig. 1](image.jpg) Fitting the model to cumulative number of confirmed COVID-19 cases for Indonesia.
Fig. 2  Simulations of the total number of individuals co-infected with COVID-19 and diabetes, at different face-mask compliance levels. Here, the fractional order value is $\varphi = 0.80$. Also, $\pi_c = 1.0991, \chi = 0.95, \sigma_v = 0.4, \vartheta = 0.95$, while $\eta$ is varied from 0.16 to 0.60. All other parameters are as in Table 1.

Fig. 3  Simulations of the total number of isolated and hospitalized individuals co-infected with COVID-19 and diabetes, at different face-mask compliance levels. Here, the fractional order value is taken as $\varphi = 0.80$. Also, $\pi_c = 1.0991, \chi = 0.95, \sigma_v = 0.4, \vartheta = 0.95$, while $\eta$ is varies from 0.16 to 0.60. All other parameters are as in Table 1.

Fig. 4  Simulations of the total number of individuals infected with COVID-19, when COVID-19 vaccination is not administered and when it is implemented. Here, the fractional order value, $\varphi = 0.80$. Also, $\pi_c = 1.0991, \chi = 0.95, \sigma_v = 0.01$. All other parameters are as in Table 1.
COVID-19, at different face-mask compliance levels, when vaccination strategy is maintained, is depicted by Fig. 7. Here, the fractional order value, $\varphi = 0.80$. Also, the COVID-19 contact rate $\pi_c = 1.0991$ while the vaccine efficacy is set at $\chi = 0.95$ and vaccination rate, $\pi_v = 0.4$. The face-mask is assumed to be 95% effective in preventing new infections. The face-mask compliance level $\eta$ is varied from 0.15 to 0.60. It is seen from the figure that the higher the compliance level in face-mask usage, the lower the number of hospitalized patients with COVID-19.

### 6. Analysis of the optimal control model

In this section, we incorporate time dependent controls into the fractional order model. We define the controls as follows:

- The control $u_1(t)$ represents COVID-19 preventive efforts
- The control $u_2(t)$ is the treatment efforts for those infected with COVID-19

The controls $u_1$ and $u_2$ satisfies $0 \leq u_1, u_2 \leq 0.9$. Thus,

$$
\begin{align*}
\dot{\Psi}_c(t) &= F_1(\Psi_c(t)) - \Theta_c - \left(\frac{\gamma_1}{\Psi_c(t) + \zeta_1} + \gamma_2\right)\Phi_c, \\
\dot{\Psi}_v(t) &= F_2(\Psi_v(t)) - \Theta_v - \left(\frac{\gamma_3}{\Psi_v(t) + \zeta_3} + \gamma_4\right)\Phi_v, \\
\dot{\Psi}_d(t) &= F_3(\Psi_d(t)) - \Theta_d - \left(\frac{\gamma_5}{\Psi_d(t) + \zeta_5} + \gamma_6\right)\Phi_d, \\
\dot{\Phi}_c(t) &= F_4(\Phi_c(t)) - \gamma_{11}\Phi_c - \left(\frac{\gamma_7}{\Phi_c(t) + \zeta_7} + \gamma_8\right)\Phi_c, \\
\dot{\Phi}_v(t) &= F_5(\Phi_v(t)) - \gamma_{12}\Phi_v - \left(\frac{\gamma_9}{\Phi_v(t) + \zeta_9} + \gamma_{10}\right)\Phi_v, \\
\dot{\Gamma}_c(t) &= F_6(\Gamma_c(t)) - \gamma_{13}\Gamma_c - \left(\frac{\gamma_{14}}{\Gamma_c(t) + \zeta_{14}} + \gamma_{15}\right)\Gamma_c,
\end{align*}
$$

with corresponding initial conditions

$$
\begin{align*}
\Psi_c(0) &= \Psi_{c0}, \Psi_v(0) = \Psi_{v0}, \Psi_d(0) = \Psi_{d0}, \Phi_c(0) = \Phi_{c0}, \Phi_v(0) = \Phi_{v0}, \Gamma_c(0) = \Gamma_{c0}, \\
Y_c(0) &= Y_{c0}, Y_v(t) = Y_{v0}, Y_d(t) = Y_{d0}, \Phi_c(t) = \Phi_{c0}, \Phi_v(t) = \Phi_{v0}, \Gamma_c(t) = \Gamma_{c0}.
\end{align*}
$$

We seek to minimize the number of COVID-19-infected and the co-infection cases as well as the cost of implementing the controls $u_1(t)$ and $u_2(t)$, subject to the state system \((19)\). The following quadratic objective functional is defined. This quadratic form was also used in \([43]\).

$$
J[u_1, u_2] = \int_0^T \left[ \Psi_c(t) + \Psi_v(t) + \frac{\alpha_1}{2} u_1^2(t) + \frac{\alpha_2}{2} u_2^2(t) \right] dt \quad (20)
$$

$T$ is the final time. An optimal control, $u_1^*, u_2^*$, is to be found, such that

$$
J(u_1^*, u_2^*) = \min \{ J(u_1, u_2^*) \} \quad (21)
$$

where $U = \{ (u_1^*, u_2^*) \}$ such that $u_1^*, u_2^*$ are measurable with $0 \leq u_1^* \leq 0.9, 0 \leq u_2^* < 0.9$, for $t \in [0, T]$ is the control set.

The Pontryagin’s Maximum Principle \([52]\) transforms \((0)\) \((19)-(21)\) into a problem of minimizing a Hamiltonian, $J$, with respect to the control functions, $u_1, \text{and} u_2$:

$$
\begin{align*}
J &= \Psi_c(t) + \Psi_v(t) + \frac{\alpha_1}{2} u_1^2(t) + \frac{\alpha_2}{2} u_2^2(t) \\
&+ \lambda_{\Psi_c} \left[ \Theta_c - \left(\frac{\gamma_1}{\Psi_c(t) + \zeta_1} + \gamma_2\right)\Phi_c \right] \\
&+ \lambda_{\Psi_v} \left[ \Theta_v - \left(\frac{\gamma_3}{\Psi_v(t) + \zeta_3} + \gamma_4\right)\Phi_v \right] \\
&+ \lambda_{\Psi_d} \left[ \Theta_d - \left(\frac{\gamma_5}{\Psi_d(t) + \zeta_5} + \gamma_6\right)\Phi_d \right] \\
&+ \lambda_{\Phi_c} \left[ \gamma_{11}\Phi_c - \left(\frac{\gamma_7}{\Phi_c(t) + \zeta_7} + \gamma_8\right)\Phi_c \right] \\
&+ \lambda_{\Phi_v} \left[ \gamma_{12}\Phi_v - \left(\frac{\gamma_9}{\Phi_v(t) + \zeta_9} + \gamma_{10}\right)\Phi_v \right] \\
&+ \lambda_{\Gamma_c} \left[ \gamma_{13}\Gamma_c - \left(\frac{\gamma_{14}}{\Gamma_c(t) + \zeta_{14}} + \gamma_{15}\right)\Gamma_c \right] \\
&+ \lambda_{\Psi_c} \left[ \frac{\gamma_1}{\Psi_c(t) + \zeta_1} + \gamma_2\right] \Phi_c \\
&+ \lambda_{\Psi_v} \left[ \frac{\gamma_3}{\Psi_v(t) + \zeta_3} + \gamma_4\right] \Phi_v \\
&+ \lambda_{\Psi_d} \left[ \frac{\gamma_5}{\Psi_d(t) + \zeta_5} + \gamma_6\right] \Phi_d \\
&+ \lambda_{\Phi_c} \left[ \frac{\gamma_7}{\Phi_c(t) + \zeta_7} + \gamma_8\right] \Phi_c \\
&+ \lambda_{\Phi_v} \left[ \frac{\gamma_9}{\Phi_v(t) + \zeta_9} + \gamma_{10}\right] \Phi_v \\
&+ \lambda_{\Gamma_c} \left[ \frac{\gamma_{14}}{\Gamma_c(t) + \zeta_{14}} + \gamma_{15}\right] \Gamma_c \\
&+ \lambda_{\Psi_c} \left[ \frac{\gamma_1}{\Psi_c(t) + \zeta_1} + \gamma_2\right] \Phi_c \\
&+ \lambda_{\Psi_v} \left[ \frac{\gamma_3}{\Psi_v(t) + \zeta_3} + \gamma_4\right] \Phi_v \\
&+ \lambda_{\Psi_d} \left[ \frac{\gamma_5}{\Psi_d(t) + \zeta_5} + \gamma_6\right] \Phi_d \\
&+ \lambda_{\Phi_c} \left[ \frac{\gamma_7}{\Phi_c(t) + \zeta_7} + \gamma_8\right] \Phi_c \\
&+ \lambda_{\Phi_v} \left[ \frac{\gamma_9}{\Phi_v(t) + \zeta_9} + \gamma_{10}\right] \Phi_v \\
&+ \lambda_{\Gamma_c} \left[ \frac{\gamma_{14}}{\Gamma_c(t) + \zeta_{14}} + \gamma_{15}\right] \Gamma_c \\
&+ \lambda_{\Psi_c} \left[ \frac{\gamma_1}{\Psi_c(t) + \zeta_1} + \gamma_2\right] \Phi_c \\
&+ \lambda_{\Psi_v} \left[ \frac{\gamma_3}{\Psi_v(t) + \zeta_3} + \gamma_4\right] \Phi_v \\
&+ \lambda_{\Psi_d} \left[ \frac{\gamma_5}{\Psi_d(t) + \zeta_5} + \gamma_6\right] \Phi_d \\
&+ \lambda_{\Phi_c} \left[ \frac{\gamma_7}{\Phi_c(t) + \zeta_7} + \gamma_8\right] \Phi_c \\
&+ \lambda_{\Phi_v} \left[ \frac{\gamma_9}{\Phi_v(t) + \zeta_9} + \gamma_{10}\right] \Phi_v \\
&+ \lambda_{\Gamma_c} \left[ \frac{\gamma_{14}}{\Gamma_c(t) + \zeta_{14}} + \gamma_{15}\right] \Gamma_c
\end{align*}
$$

Theorem 6.1. For an optimal control set $u_1, u_2$ that minimizes $J$ over $U$, there are adjoint variables, $\lambda_1, \lambda_2, \ldots, \lambda_9$ satisfying

$$
\begin{align*}
\bigg(d_t \lambda_i\bigg) &= -\frac{\partial J}{\partial \dot{u_i}}, \\
\text{with,} \\
\lambda_i(t_0) &= 0, \quad \text{where,}
\end{align*}
$$

$$
\begin{align*}
\lambda_1 &= \Psi_{c0}, \lambda_2 = \Psi_{v0}, \lambda_3 = \Psi_{d0}, \lambda_4 = \Phi_{c0}, \lambda_5 = \Phi_{v0}, \lambda_6 = \Phi_{d0}, \\
\lambda_7 &= \Gamma_{c0}, \lambda_8 = \Gamma_{v0}, \lambda_9 = \Gamma_{d0}.
\end{align*}
$$
Following from [52], there exist adjoint variables satisfying:

Further, let $U^* = (u_1^*, u_2^*)$ be an optimal control pair and $\Psi_1^*, \Psi_2^*, \Psi_3^*, \Psi_4^*, \Phi_1^*, \Phi_2^*, \Phi_3^*, \Phi_4^*$ be the corresponding state solutions. Following from [52], there exist adjoint variables satisfying:

Differentiating the Hamiltonian, $J$ with respect to the controls $(u_1, u_2)$ at $t$, we have

Therefore, following from [53], the following is obtained

Fig. 6 Simulations of the total number of individuals infected with COVID-19, at different face-mask compliance levels. Here, the fractional order value is, $\varphi = 0.80$. Also, $\pi_c = 1.0991$, $\chi = 0.95$, $\sigma = 0.4$, $\vartheta = 0.95$, while $\eta$ varies from 0.15 to 0.60. All other parameters are as in Table 1.

Fig. 7 Simulations of the total number of isolated and hospitalized individuals infected with COVID-19, at different face-mask compliance levels. Here, the fractional order value, $\varphi = 0.80$. Also, $\pi_c = 1.0991$, $\chi = 0.95$, $\sigma = 0.4$, $\vartheta = 0.95$, while $\eta$ is varied from 0.16 to 0.60. All other parameters are as in Table 1.
Fig. 8  Simulation of the Fractional optimal control model for individuals co-infected with COVID-19 and diabetes, when only COVID-19 preventive strategy ($u_1$) is implemented. Here, the fractional order value, $\vartheta = 0.90$. Also, $\pi_c = 0.5776$. All other parameters are as in Table 1.

Fig. 9  Simulation of the Fractional optimal control model for isolated and hospitalized individuals co-infected with COVID-19 and diabetes, when only COVID-19 preventive strategy ($u_1$) is implemented. Here, the fractional order value, $\vartheta = 0.90$. Also, $\pi_c = 0.5776$. All other parameters are as in Table 1.

Fig. 10  Control profile when only COVID-19 preventive strategy ($u_1$) is implemented. Here, the fractional order value, $\vartheta = 0.90$. Also, $\pi_c = 0.5776$. All other parameters are as in Table 1.
Fig. 11 Simulations of the Fractional optimal control model for individuals co-infected with COVID-19 and diabetes, when only COVID-19 treatment control is implemented. Here, the fractional order value, $\varrho = 0.90$. Also, $\pi_c = 0.5776$. All other parameters are as in Table 1.

Fig. 12 Simulations of the Fractional optimal control model for isolated and hospitalized individuals co-infected with COVID-19 and diabetes, when only COVID-19 treatment control is implemented. Here, the fractional order value, $\varrho = 0.90$. Also, $\pi_c = 0.5776$. All other parameters are as in Table 1.

Fig. 13 Control profile when only COVID-19 treatment control ($u_2$) is implemented. Here, the fractional order value, $\varrho = 0.90$. Also, $\pi_c = 0.5776$. All other parameters are as in Table 1.
6.1. Numerical Scheme for the Fractional Optimal Control

Consider a general initial value problem [19]

\[
\frac{d\psi}{dt}(t) = f(t, \psi(t)),
\]

\[
\psi(0) = \psi_0.
\]

Applying the fundamental theorem of fractional calculus to Eq. 30 above, we have that

\[
\begin{aligned}
\psi(t) &= \psi_0 + \frac{1}{\Gamma(q)} \int_0^t (t - \zeta)^{q-1} f(\zeta, \psi(\zeta)) d\zeta,
\end{aligned}
\]

with \( q > 0 \) and \( \psi_0 \) given.\( \tag{30} \)

\[\text{Eq. 30 above, we have that} \]

\[\begin{aligned}
\psi(t) &= \psi_0 + \frac{1}{\Gamma(q)} \int_0^t (t - \zeta)^{q-1} f(\zeta, \psi(\zeta)) d\zeta,
\end{aligned}\]

outside the fractional order value, \( \psi_0 \) and \( \psi_0 \) given.

With the help of the normalization function, \( G(q) = 1 - q + \frac{\psi_0}{\psi_0} \) at \( t = 0 \), after discretization, we have

\[
\begin{aligned}
\psi(t_n) &= \psi_0 + \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1} (t_n - \xi_i)^{q-1} f(\xi_i, \psi(\xi_i)) + \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1} \int_{\xi_i}^{t_n} (t_n - \xi_i)^{q-1} f(\xi, \psi(\xi)) d\xi,
\end{aligned}
\]

\[\text{Approximating} \]

\[
\begin{aligned}
\psi(t_n) &= \psi_0 + \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1} (t_n - \xi_i)^{q-1} f(\xi_i, \psi(\xi_i)) + \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1} \int_{\xi_i}^{t_n} (t_n - \xi_i)^{q-1} f(\xi, \psi(\xi)) d\xi,
\end{aligned}
\]

\[\text{We then obtain} \]

\[
\begin{aligned}
\psi(t_n) &= \psi_0 + \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1} (t_n - \xi_i)^{q-1} f(\xi_i, \psi(\xi_i)) + \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1} \int_{\xi_i}^{t_n} (t_n - \xi_i)^{q-1} f(\xi, \psi(\xi)) d\xi,
\end{aligned}
\]

\[\text{High stability is obtained by replacing} \]

\[
\begin{aligned}
\psi(t_n) &= \psi_0 + \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1} (t_n - \xi_i)^{q-1} f(\xi_i, \psi(\xi_i)) + \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1} \int_{\xi_i}^{t_n} (t_n - \xi_i)^{q-1} f(\xi, \psi(\xi)) d\xi,
\end{aligned}
\]

\[\text{The scheme is thus used in Eq. 34} \]

6.2. Simulation of the Fractional optimal control model

Simulation of the Fractional optimal control model for individuals co-infected with COVID-19 and diabetes, when only COVID-19 preventive strategy (\( u_1 \)) is implemented, is depicted in Fig. 8. Here, the fractional order value, \( q = 0.90 \), while the COVID-19 contact rate, \( r_1 = 0.5776 \). It is observed from this figure that a total of 67,906,600 co-infected cases were averted after 200 days. Also, this strategy averts about 434,700 hospitalized co-infected cases, after about 100 days, as shown in Fig. 9. The control profile for this strategy is shown in Fig. 10. This shows that COVID-19 preventive strategy involving vaccination and face-mask usage will not only reduce new COVID-19 cases, but will equally cut down COVID-19 and diabetes co-infection cases. Public health agencies and policy makers must therefore drive the campaign for mass vaccination of susceptible individuals and usage of face-masks and sanitizers in public places. This effort will have very high positive impact on diabetics, so as to reduce complications with severe COVID-19 illness.

Simulations of the Fractional optimal control model for individuals co-infected with COVID-19 and diabetes, when only COVID-19 treatment strategy (\( u_2 \)) is implemented, is depicted in Fig. 11. Here, the fractional order value, \( q = 0.90 \), while the COVID-19 contact rate, \( r_2 = 0.5776 \). It is observed from this figure that a total of 67,906,600 co-infected cases were averted after 200 days. Also, this strategy averts about 177,900 hospitalized co-infected cases, after about 100 days, as shown in Fig. 12. The control profile for this strategy is presented in Fig. 13. This shows that effective treatment for COVID-19 treatment will not just reduce new covid-19 cases but will also help bring down the total hospitalized co-infected cases in the population.

7. Conclusion

In this article, we have analyzed a fractional order model for COVID-19 and diabetes co-dynamics, using the Atangana-Baleanu derivative. The positivity and boundedness of the solutions were established by the method of Laplace transform. The existence and uniqueness of the model solution were established with the aid of Banach fixed point theorem and Leray–Schauder alternative Theorem. The disease free equilibrium of the model was shown to be locally and globally asymptotically stable when the associated reproduction number is below one. Furthermore, the fractional model was shown to be Hyers-Ulam stable. Using the finitenon function in the Optimization Toolbox of MATLAB, the model was fitted to the cumulative confirmed daily COVID-19 cases for Indonesia from February 11, 2021 to August 26, 2021. Some parameters related to COVID-19 and its co-infection with diabetes were estimated from the data fitting. The simulations of the total number of isolated and hospitalized individuals co-infected with COVID-19 and diabetes, at different face-mask compliance levels, when vaccination strategy is maintained reveals that the total number of isolated and hospitalized co-infection cases decreases with increase in face-mask compliance levels, while maintaining COVID-19 vaccination. The simulation results show that to curlat COVID-19 and diabetes co-infections, policies and measures to enforce mass COVID-19 vaccination and strict face-mask usage in the public must be put in place. To further mitigate the spread of COVID-19 and diabetes co-infection, time dependent controls are added into the fractional model, and the obtained optimal control problem investigated via the Pontryagin’s Maximum Principle. As different COVID-19 variants continues to spread across the world, efforts must be enhanced to avert co-infection with other co-morbidities, especially among high risk population such as those with diabetes. This model has shown that fractional derivative is the most ideal in dealing with complex epidemiic models such as COVID-19 and diabetes co-infection.

Our model focused basically on COVID-19 and diabetes co-infection. The emergence of highly transmissible and deadlier variants of COVID-19 calls for more research in this area, as co-infection between COVID-19 and different variants of COVID-19 will be an interesting area for further research. Further studies can consider more than one fractional derivative for the solution of the model. We may also consider using semi-analytic methods such as the Laplace Adomian decompo-
sition method or Homotopy analysis method for the solution of the model, taking different sub-populations.

Authors’ contributions Statement: Andrew Omame: performed the calculations and wrote in part the original draft and edited the final manuscript. Ugochukwu Nwajeri: performed the calculations and wrote in part the original draft. Mujahid Abbas: designed the concept of the paper, supervised the work and edited the final manuscript. Chibueze Onyenyeche: Participated in the investigation, and wrote in part the original draft.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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