Exponential sums over finite fields and the large sieve

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ABSTRACT. By using a variant of the large sieve for Frobenius in compatible systems developed in [Kow06a] and [Kow08], we obtain zero-density estimates for arguments of ℓ-adic trace functions over finite fields with values in some algebraic subsets of the cyclotomic integers, when the monodromy groups are known. This applies in particular to hyper-Kloosterman sums and general exponential sums considered by Katz.

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1. INTRODUCTION

1.1. Exponential sums with image in the cyclotomic integers. We consider exponential sums over a finite field $\mathbb{F}_q$ of characteristic $p$ such as:

1. Hyper-Kloosterman sums $K_{n,q}(a)$ of rank $n \geq 2$ given by:

$$
\frac{(-1)^{n-1}}{q^{(n-1)/2}} \sum_{x_1,\ldots,x_n \in \mathbb{F}_q^{*}} e\left(\frac{\text{tr}(x_1 + \cdots + x_n)}{p}\right) (a \in \mathbb{F}_q^*),
$$

for $\text{tr} : \mathbb{F}_q \to \mathbb{F}_p$ the trace map, or more generally hypergeometric sums as introduced in [Kat90, Chapter 8];

2. General exponential sums of the form

$$
\frac{-1}{\sqrt{q}} \sum_{y \in \mathbb{F}_q} e\left(\frac{\text{tr}(xf(y) + h(y))}{p}\right) \chi(g(y)) (x \in \mathbb{F}_q),
$$

for $f, g, h \in \mathbb{Q}(X)$ rational functions and $\chi$ a multiplicative character on $\mathbb{F}_q^*$, such as Birch sums;

3. Functions counting points on families of curves such as

$$
q + 1 - \frac{|X_z(\mathbb{F}_q)|}{q^{1/2}} (z \in \mathbb{Z}_f),
$$

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where $X_z$ is the smooth projective model of the affine hyperelliptic curve $y^2 = f(x)(x-z)$, for $f \in \mathbb{Z}[X]$ fixed squarefree of degree $2g \geq 2$ and $Z_f$ the set of zeros of $f$ in $\mathbb{F}_q$.

In [PG16a], we took the point of view of studying these as (normalized) algebraic integers, in the sense that they all take values in the localization $\mathbb{Z}[\zeta_{4p},\zeta_{2p}]$ (by the evaluation of quadratic Gauss sums).

Using $\ell$-adic methods, we investigated the distribution of their reductions modulo a prime ideal and short sums thereof. Indeed, the examples above are incarnations of trace functions of constructible middle-extension sheaves of $O_\lambda = \mathbb{Z}[\zeta_{4p}]_\lambda$-modules on $\mathbb{P}^1/\mathbb{F}_q$, for $\lambda$ an $\ell$-adic valuation with $\ell \neq p$. This allows to use the works of Deligne and Katz, in particular Deligne’s extension [Del80] of the Riemann hypothesis for varieties over finite fields to weights of étale cohomology groups of sheaves of $\mathbb{Q}_\ell$-modules on varieties over finite fields.

1.2. Zero-density estimates through the large sieve for Frobenius in compatible systems.

1.2.1. Families of curves. The large sieve for Frobenius in compatible systems was developed by Kowalski in [Kow06a] and [Kow08] to obtain results of the type of Chavdarov [Cha97] on zeta functions of families of curves, such as the probability that the denominator has Galois group as large as possible.

In the notations of Example (3) above, Kowalski gets for example (see [Kow08, Section 8.8])

$$P(f(z) \neq 0, |X_z(\mathbb{F}_q)| \in \mathbb{N}^2) := \frac{|z \in \mathbb{F}_q : f(z) \neq 0, |X_z(\mathbb{F}_q)| \in \mathbb{N}^2|}{q} \ll q^{-1} \log q,$$

for $\mathbb{N}^2$ the set of squares of integers.

The large sieve bound ultimately relies on estimates of exponential sums through Deligne’s generalization of the Riemann hypothesis over finite fields.

1.2.2. Exponential sums. In this note, we build on these ideas to obtain zero-density estimates of the form

$$P(t(x) \in A) := \frac{|x \in \mathbb{F}_q : t(x) \in A|}{q} = o(1) \quad (q \to +\infty) \quad (4)$$

where:

- $t : \mathbb{F}_q \to E$ is the trace function associated to a coherent family $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ of sheaves of $O_\lambda$-modules over $\mathbb{F}_p$, for $E$ a number field with rings of integers $O$, and $\Lambda$ a set of valuations on $O$.

- $A \subset E$ is an “algebraic” subset such as the set of $m$-powers ($m \geq 2$), the image of a polynomial, or more generally a set defined by a first-order formula in the language of rings.

This will apply in particular with $E = \mathbb{Q}(\zeta_{4p})$ to Kloosterman sums (1) and exponential sums of the form (2).

For that, we will need:
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– A variant of the large sieve for Frobenius in compatible systems handling sheaves of $O_{\lambda}$-modules instead of sheaves of $\mathbb{Z}_\ell$-modules.
– The construction by Deligne and Katz of examples of the form (1) and (2) as trace functions of $\ell$-adic sheaves.
– Information on monodromy groups:
  – When available, the determination of integral monodromy groups for a density one subset of the valuations, not depending on $p$.
  – Otherwise, results of Larsen and Pink [LP92], [Lar95] to handle sheaves whose monodromy groups are known over $\mathbb{Q}_\ell$ (e.g. by the works of Katz [Kat88], [Kat90]), but not over $F_\lambda$.
  – For sheaves associated with exponential sums of the form (2), conditions and/or normalizations so that arithmetic and geometric monodromy groups coincide.
– Bounds on “Gaussian sums” over:
  – Linear algebraic groups. These follow either from Deligne’s generalization of the Riemann hypothesis over finite fields [Del80] and bounds of Katz on sums of Betti numbers [Kat01], or from explicit computations of D.S. Kim for certain finite groups of Lie type.
  – Subsets of $F_\lambda$ such as powers (Bourgain and others [BC06]) or more generally definable subsets (Kowalski [Kow07], relying on the work of Chatzidakis-van der Dries-Macintyre [CvdDM92]).
  – Uniform estimates in Chebotarev’s density theorem (Maynard [May13]).

1.3. Results for Kloosterman sums. We state some of the results for hyper-Kloosterman sums of rank $n \geq 2$. The bounds are uniform in $p$, thanks to the determination of the monodromy groups over $F_\lambda$ in [PG16c] when $\ell \gg 1$ (with no dependency with respect to $p$).

**Proposition 1.1.** Let $n \geq 2$ be an integer and $\varepsilon > 0$. For $m \geq 2$ coprime to $p$, we have

$$P\left(\text{Kln}_q(x) \in \mathbb{Q}(\zeta_{4p})^m\right) \ll_{m,\varepsilon} p^{\varepsilon \log q \over B_n q^{1/(2B_n)}} \to 0$$

when $q = p^e \to +\infty$ with $e \geq 16B_n$,

$$B_n = \begin{cases} 
\frac{2n^2+n-1}{2} : n \text{ odd} \\
\frac{2n^2+3n+4}{4} : n \text{ even},
\end{cases}$$

and $\mathbb{Q}(\zeta_{4p})^m$ is the set of $m$th powers in $\mathbb{Q}(\zeta_{4p})$. The implicit constant depends only on $m$ and $\varepsilon$.

More generally:
Proposition 1.2. Let $n \geq 2$ be an integer and $\varepsilon > 0$. For almost all $f \in \mathbb{Z}[X]$ of fixed degree, we have
\[
P\left( K_{n,q}(x) \in f(\mathbb{Q}(\zeta_{4p})) \right) \ll_{f,\varepsilon} \frac{p^e \log q}{B_n q^{1/(2B_n)}} \to 0
\]
when $q = p^e \to +\infty$ with $e \geq 16B_n$, for $B_n$ as in (5). The implicit constant depends only on $f$ and $\varepsilon$.

Remarks 1.3.

(1) This can further be extended to definable subsets of $\mathbb{Q}(\zeta_{4p})$ (i.e. defined by a first-order formula in the language of rings), under some technical conditions.

(2) The same bounds hold for unnormalized Kloosterman sums.

(3) Under the general Riemann hypothesis (GRH) for the Dedekind zeta function of $\mathbb{Q}(\zeta_{4p})$, one may take $\varepsilon = 0$ and $e \geq 4B_n + 1$.

(4) By relying on the determination of the monodromy groups over $\overline{\mathbb{Q}}_{\ell}$ by Katz and the results of Larsen-Pink, instead of [PG16c], these results would hold when $p$ is fixed, $e \to +\infty$, with an implicit constant depending on $p$.

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2. The large sieve for Frobenius in compatible systems

We start by recalling the technical setup of trace function over finite fields, before stating a version of the large sieve for Frobenius adapted to our needs. Throughout this section, a number field $E$ with ring of integers $\mathcal{O}$ is fixed, as well as a finite field $\mathbb{F}_q$ of characteristic $p$.

2.1. Trace functions over finite fields.

2.1.1. Definitions. Let $\lambda$ be an $\ell$-adic valuation corresponding to a prime ideal $q$ of $\mathcal{O}$, $E_\lambda$ and $\mathcal{O}_\lambda$ the completions, and $\mathbb{F}_\lambda \simeq \mathcal{O}/q$ the residue field.

Let $A = \overline{\mathbb{Q}}_\ell$, $\mathcal{O}_\lambda$ or $\mathbb{F}_\lambda$. We recall that a constructible middle-extension sheaf of $A$-modules over $\mathbb{P}^1/\mathbb{F}_p$ (or sheaf of $A$-modules over $\mathbb{F}_p$ for simplicity) corresponds to a continuous $\ell$-adic Galois representation
\[
\rho_F : \pi_{1,p} \rightarrow \text{Gal}(\mathbb{F}_p(T)/\mathbb{F}_p) \rightarrow \text{GL}(\mathbb{F}_q) \simeq \text{GL}_n(A),
\]
for $\overline{\mathbb{F}}_q$ the geometric generic point. The associated trace functions are, for $\mathbb{F}_{q,\overline{\mathbb{F}}}$ a finite extension,
\[
t_F = t_{F,q} : \mathbb{F}_q \rightarrow A, \quad x \mapsto \text{tr} \left( \rho_F(\text{Frob}_{x,q}) \ | \ F_{\overline{\mathbb{F}}_q} \right),
\]
where $\text{Frob}_{x,q} \in (D_x/I_x)^\ell \simeq \text{Gal}(\mathbb{F}_q/\mathbb{F}_q)$ is the geometric Frobenius at $x \in \mathbb{F}_q$, for $I_x \leq D_x \leq \pi_{1,p}$ the inertia (resp. decomposition) group at $x$. We will denote by $U_F \subset \mathbb{P}^1$ the maximal open of lissity of $\mathcal{F}$.

We refer the reader to [Kat88, Chapter 2] for more details and references.
2.1.2. Monodromy groups. If $F$ is a sheaf of $A$-modules over $\mathbb{F}_p$ as above, the arithmetic and geometric monodromy groups of $F$ are the groups

$$G_{\text{geom}}(F) = \rho_F(\pi_{1,p}^{\text{geom}}) \leq G_{\text{arith}}(F) = \rho_F(\pi_{1,p}) \leq \text{GL}_n(A)$$

if $A$ is discrete, and

$$G_{\text{geom}}(F) = \rho_F(\pi_{1,p}^{\text{geom}}) \leq G_{\text{arith}}(F) = \rho_F(\pi_{1,p}) \leq \text{GL}_n(\overline{\mathbb{Q}}_p)$$

if $A = \overline{\mathbb{Q}}_p$, where $\overline{\mathbb{Q}}_p$ denotes Zariski closure, for $\pi_{1,p}^{\text{geom}} := \text{Gal}(\overline{\mathbb{F}}_p(T)_{\text{sep}}/\mathbb{F}_p(T))$.

The works of Katz (see e.g. [Kat88], [Kat90], [KS91]) contain the determination of the monodromy groups over $\mathbb{Q}_p$ of many sheaves of interest, such as Kloosterman sheaves. An important input is the fact that, for pointwise pure of weight 0 sheaves, the connected component of the geometric monodromy group is a semisimple algebraic group by a result of Deligne.

The determination of discrete monodromy groups is usually more difficult, since they have far less structure.

2.2. Coherent families.

**Definition 2.1.** Let $\Lambda$ be a set of valuations on $\mathcal{O}$ and let $U \subset \mathbb{P}^1/\mathbb{F}_p$ be an open affine subset. A family $(F_\lambda)_{\lambda \in \Lambda}$ where $F_\lambda$ is a sheaf of $\mathcal{O}_\lambda$-modules over $\mathbb{F}_p$ with maximal open of lissity $U$ is coherent if:

1. It forms a compatible system: if $\rho_\lambda : \pi_{1,p} \to \text{GL}_n(\mathcal{O}_\lambda)$ is the representation corresponding to $F_\lambda$, then for every $\lambda \in \Lambda$, every finite extension $\mathbb{F}_q/\mathbb{F}_p$ and every $x \in U(F_q)$, the characteristic polynomial

   $$\text{charpol}_{\rho_\lambda(Frob_x,q)} \in \mathcal{O}_\lambda[T]$$

   lies in $E[T]$ and does not depend on $\lambda$.

2. There exists $G \in \{\text{SL}_n, \text{Sp}_{2n}, \text{SO}_{2n+1}, \text{SO}^+_{2n}\}$ such that for every $\lambda$ corresponding to a prime ideal $q \subseteq \mathcal{O}$, the arithmetic and geometric monodromy groups of $\widehat{F}_\lambda := F_\lambda \mod q$ coincide and are conjugate to $G(\mathbb{F}_\lambda)$.

3. The conductor

   $$\text{cond}(\widehat{F}_\lambda) = n + |\text{Sing}(\widehat{F}_\lambda)| + \sum_{x \in \text{Sing}(\widehat{F}_\lambda)} \text{Swan}_x(\widehat{F}_\lambda),$$

   as defined by Fouvry-Kowalski-Michel (see e.g. [FKM15]), is uniformly bounded, independently from $p$.

Note that in particular, the trace function $t = t_{\widehat{F}_\lambda} : \mathbb{F}_q \to \mathcal{O}_\lambda$ (as the opposite of the coefficient of order $n-1$ in the characteristic polynomial) is independent from $\lambda$ and takes values in $E$. More precisely,

$$t(\mathbb{F}_q) \subset \bigcap_{\lambda \in \Lambda} \mathcal{O}_\lambda \cap E = \bigcap_{q \in \Lambda} \mathcal{O}_q = (\text{Spec}(\mathcal{O}) - \Lambda)^{-1} \mathcal{O} \subset E,$$

where $\mathcal{O}_q$ is the localization at the ideal $q$ corresponding to the valuation $\lambda$.

To simplify the notations, we will often drop the index $\lambda$ in the family and leave it implicit.
2.2.1. Fourier transforms. The sheaves we will consider arise by \( \ell \)-adic Fourier transforms, as developed by Deligne, Laumon and others (see [Kat90, Section 7.3], [Kat88, Chapter 5]), corresponding to the discrete Fourier transform on the level of trace functions. This often results in sheaves with large classical monodromy groups, which is part of Condition (2) above.

Concerning Conditions (1) and (3), we recall:

**Lemma 2.2.** Let us assume that \( \mathbb{Q}(\zeta_{4p}) \subseteq E \) and let \( \psi : \mathbb{F}_p \to \mathbb{C} \) be a nontrivial additive character. If \((\mathcal{F}_\lambda)_{\lambda \in \Lambda}\) is a compatible system of Fourier sheaves of \( \mathcal{O}_\lambda \)-modules over \( \mathbb{F}_p \), then the family \((\text{FT}_\psi(\mathcal{F}_\lambda))_{\lambda \in \Lambda}\) is compatible as well and \( \text{cond}(\text{FT}_\psi(\mathcal{F}_\lambda)) \ll (\text{cond}(\widehat{\mathcal{F}}_\lambda))^2 \), where \( \text{FT}_\psi \) denotes the normalized Fourier transform with respect to \( \psi \).

**Proof.** Let \( \mathcal{F} = \mathcal{F}_\lambda \) and \( \mathcal{G} = \text{FT}_\psi(\mathcal{F}) \). By construction, for every finite extension \( \mathbb{F}_q/\mathbb{F}_p \) and every \( a \in U_\mathcal{G}(\mathbb{F}_q) \), the reverse characteristic polynomial \( \det (1 - \text{Frob}_q T | \mathcal{G}) \) is equal to

\[
\prod_{i=0}^{2} \det (1 - \text{Frob}_q T | H^i_c(U_\mathcal{G} \times \overline{\mathbb{F}}_p, \mathcal{F} \otimes \mathcal{L}_{\psi(ax)}))^{(-1)^{i+1}},
\]

where \( \mathcal{L}_{\psi(ax)} \) denotes an Artin-Schreier sheaf and \( H^i_c \) the \( i \)th \( \ell \)-adic cohomology group with compact support. By the Grothendieck-Lefschetz trace formula, this is

\[
\exp \left( \sum_{n \geq 1} S(a, n) \frac{T^n}{n} \right),
\]

where \( S(a, n) = \sum_{x \in U_\mathcal{G}(\mathbb{F}_q^{an})} t_{\mathcal{F}, q^n}(x) \psi(\text{tr}(ax)) \) has image in \( E \) and does not depend on \( \lambda \) by hypothesis, whence the conclusion.

The assertion on the conductors can be found in [FKM15, Proposition 8.2] and [Kat88, Remark 1.10].

2.2.2. Examples. For the examples below, we let \( E = \mathbb{Q}(\zeta_{4p}) \), with ring of integers \( \mathcal{O} = \mathbb{Z}[\zeta_{4p}] \).

**Proposition 2.3** (Kloosterman sheaves). Let \( n \geq 2 \) be a fixed integer. For \( \Lambda_n = \{ q \leq \mathcal{O} \text{ above } \ell \not\equiv n \text{ mod } 4, \ell \equiv 1 \text{ mod } 4, \ell \neq p \} \), there exists a coherent family \((\mathcal{K}_n(\lambda))_{\lambda \in \Lambda_n}\) of sheaves of \( \mathcal{O}_\lambda \)-modules over \( \mathbb{F}_p \) with

\[
G_{\text{geom}}(\mathcal{K}_n(\lambda)) = G_{\text{arith}}(\mathcal{K}_n(\lambda)) = \begin{cases} \text{SL}_n(\mathbb{F}_\lambda) : n \text{ odd} \\ \text{Sp}_n(\mathbb{F}_\lambda) : n \text{ even} \end{cases},
\]

and such that the trace function \( t_{\mathcal{K}_n,q} \) is the Kloosterman sum \( \mathcal{K}_n,q \).

**Proof.** The construction of the Kloosterman sheaves is due to Deligne (see [Kat88] for the construction via recursive Fourier transforms). As already mentioned, the assertion on the integral monodromy groups over \( \mathbb{F}_\lambda \) can be found in [PG16c]. They form a compatible system for \( n \) fixed by Lemma 2.2 applied recursively.

**Remark 2.4.** As an illustration of (6), note that \( \mathcal{K}_n,q : \mathbb{F}_q \to \mathbb{Z}[\zeta_{4p}]_{q^{(n-1)/2}} \).
The following example, when unnormalized (hence replacing $\mathcal{O}_\lambda$ by $\mathbb{Z}_\ell$), was treated in [Kow06a] and [Kow08]:

**Proposition 2.5** (Point counting on families of hyperelliptic curves). Let $f \in \mathbb{Z}[X]$ be a squarefree polynomial of degree $2g \geq 2$, and let $\Lambda$ be the set of $\ell$-adic valuations of $\mathcal{O}$ with $\ell \geq 3$. For $p$ large enough, there exists a coherent family $(\mathcal{F}_f)_{\lambda \in \Lambda}$ of $\ell$-adic sheaves of $\mathcal{O}_\lambda$-modules over $\mathbb{F}_p$, with $G_{\text{geom}}(\mathcal{F}_f) = G_{\text{arith}}(\mathcal{F}_f) \cong \text{Sp}_{2g}(\mathbb{F}_\lambda)$, and such that $t_{\mathcal{F},q}(z)$ is given by (3) for $z \in \mathbb{Z}_f$.

**Proof.** For the construction, see [KS91, Section 10.1], and normalize by a Tate twist. Because of this normalization, [KS91, Theorem 10.1.16] and [KS91, Lemma 10.1.9] show that the arithmetic and geometric monodromy group preserve the same symplectic pairing. Finally, [Hal08, Theorem 2.5] shows that the geometric monodromy group is $\text{Sp}_{2g}$.

\[ \square \]

### 2.3. The large sieve for Frobenius.

**Theorem 2.6.** Let $\Lambda$ be a set of valuations (or equivalently prime ideals) on $\mathcal{O}$. For $L \geq 1$, we write $\Lambda_L = \{ q \in \Lambda : N(q) \leq L \}$. Let $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ be a coherent family, with monodromy group structure $G$, where $\mathcal{F}_\lambda$ corresponds to a representation

$$
\rho_\lambda : \pi_{1,p} \to GL_n(\mathcal{O}_\lambda) \to GL_n(\mathbb{F}_\lambda).
$$

For every $\lambda \in \Lambda$, let $\Omega_\lambda \subset G(\mathbb{F}_\lambda)$ be a conjugacy-invariant subset. Then for all $L \geq 1$,

$$
\frac{|\{ x \in U_{\mathcal{F}_\lambda}(\mathbb{F}_q) : \rho_\lambda(\text{Frob}_{x,q}) \not\in \Omega_\lambda \text{ for all } \lambda \in \Lambda_L \}|}{q} \ll \left( 1 + \frac{L^B}{q^{1/2}} \right) \frac{1}{P(L)},
$$

where the implicit constant is absolute and

$$
P(L) = \sum_{\lambda \in \Lambda_L} \frac{|\Omega_\lambda|}{|G(\mathcal{F}_\lambda)|}, \quad B = \begin{cases} 
2n^2 + n - 1 & : G = \text{SL}_n \\
\frac{2n^2+3n+4}{4} & : G = \text{Sp}_n, \text{ SO}_n^\pm \text{ (n even)} \\
\frac{2n^2-n+3}{4} & : G = \text{SO}_n \text{ (n odd)}.
\end{cases}
$$

**Proof.** This is a variant of [Kow06a, Proposition 3.3] (see also [Kow08, Chapter 8]). For $\lambda, \lambda' \in \Lambda$ distinct, the product map $\pi_{1,p} \to G(\mathbb{F}_\lambda) \times G(\mathbb{F}_{\lambda'})$ is surjective by [Kow06a, Corollary 2.6] (a variant of Goursat’s Lemma), which extends with no modification to the case where $\mathbb{F}_\lambda$ and $\mathbb{F}_{\lambda'}$ do not necessarily have prime order (see [MT11, Part III]). By [MT11, Corollary 24.6], $B = 1 + \dim(G) + \text{rank}(G)/2$.

**Remark 2.7.** Note that in the case $E = \mathbb{Q}((\zeta_d))$ of the examples of Section 1.1, the size of the residue field $\mathbb{F}_\lambda$ corresponding to a prime ideal $q \leq \mathbb{Z}((\zeta_d)$ depends on the multiplicative order modulo $d$ of the prime $\ell$ above which $q$ lies (see [Was97, Theorem 2.13]). In particular, if $d = 4p$, then $|\mathbb{F}_\lambda|$ depends on $p$. This is a new phenomenon compared to the degree 1 case (i.e. $\mathcal{O}_\lambda = \mathbb{Z}_\ell$) studied in [Kow06a] and [Kow08].
3. Traces of random matrices and Gaussian sums

In the next section, we will apply Theorem 2.6 to $\Omega_\lambda = \{ g \in G(\mathbb{F}_\lambda) : \text{tr}(g) \notin A_\lambda \}$, for some $A_\lambda \subset \mathbb{F}_\lambda$. Thus

$$P(L) = \sum_{\lambda \in A_\lambda} P(\text{tr}(g) \notin A_\lambda)$$

(with respect to the counting measure on $G(\mathbb{F}_\lambda)$). In this section, we get estimates on this quantity.

By the orthogonality relations in $\mathbb{F}_\lambda$:

**Proposition 3.1.** Let $G \leq \text{GL}_n(\mathbb{F}_\lambda)$ be a finite group and $A \subset \mathbb{F}_\lambda$. Then

$$P(\text{tr}(g) \in A) = \frac{|A|}{|\mathbb{F}_\lambda|} + O \left( \frac{1}{|G|} \sum_{g \in G} \psi(\text{tr}(g)) \left| \sum_{x \in A} \psi(-x) \right| \right).$$

We expect

$$\frac{1}{|G|} \sum_{g \in G} \psi(\text{tr}(g)) \ll |\mathbb{F}_\lambda|^{-\alpha(G)}$$

for some $\alpha(G) > 0$, and similarly if $A$ is “well-distributed” in $\mathbb{F}_\lambda$,

$$\frac{1}{|A|} \sum_{x \in A} \psi(x) \ll |\mathbb{F}_\lambda|^{-\alpha(A)}$$

for some $\alpha(A) > 0$, in both cases uniformly for all nontrivial $\psi \in \hat{\mathbb{F}}_\lambda$.

Under (8) and (9), Proposition 3.1 becomes

$$P(\text{tr}(g) \in A) = \frac{|A|}{|\mathbb{F}_\lambda|} \left( 1 + O \left( |\mathbb{F}_\lambda|^{-\alpha(G)-\alpha(A)-1} \right) \right).$$

**3.1. Gaussian sums in linear groups (8).**

3.1.1. General result.

**Proposition 3.2.** Let $V = V(\mathbb{F}_\lambda)$ for $V \leq \text{GL}_n$ an algebraic variety over $\mathbb{F}_\lambda$. The bound (8) holds with $\alpha(V) = 1/2$, uniformly for all nontrivial $\psi \in \hat{\mathbb{F}}_\lambda$, unless $\text{tr} : V \to \mathbb{F}_\lambda$ is constant.

**Proof.** Let $\ell' \neq \text{char}(\mathbb{F}_\lambda)$ be an auxiliary prime and let us consider the restriction $\mathcal{L}$ of the Lang torsor $\mathcal{L}_{\psi_{\text{det}}} \otimes \mathbf{A}^2 / \mathbb{F}_\lambda$ to $V$ (see [KR15, Example 7.17]). By the Grothendieck-Lefschetz trace formula,

$$\sum_{g \in V} \psi(\text{tr}(g)) = \sum_{i=0}^{2 \text{dim } V} (-1)^i \text{tr} \left( \text{Frob}_{\mathbb{F}_\lambda} | H^i_c(V \times \overline{\mathbb{F}}_\lambda, \mathcal{L}) \right).$$

By Deligne’s generalization of the Riemann hypothesis over finite fields [Del80],

$$\text{tr} \left( \text{Frob}_{\mathbb{F}_\lambda} | H^i_c(V \times \overline{\mathbb{F}}_\lambda, \mathcal{L}) \right) \ll |\mathbb{F}_\lambda|^{i/2} \text{dim } H^i_c(V \times \overline{\mathbb{F}}_\lambda, \mathcal{L})$$

for $0 \leq i \leq 2 \text{dim } V$, and by the coinvariant formula,

$$\text{tr} \left( \text{Frob}_{\mathbb{F}_\lambda} | H^{2 \text{dim } V}_c(V \times \overline{\mathbb{F}}_\lambda, \mathcal{L}) \right) = 0$$
Gaussian sums over finite groups of Lie type.

Table 1. Cancellation for Gaussian sums over finite groups of Lie type.

| $G$ | $\alpha(G)$ |
|-----|-------------|
| $\text{GL}_n$ | $\frac{n(n-1)}{2}$ |
| $\text{SL}_n$ | $\frac{n^2-1}{2}$ |
| $\text{Sp}_n$, $\text{SO}_n^-$ ($n$ even) | $\frac{n(n+2)}{2}$ |
| $\text{SO}_n$ ($n$ odd) | $\frac{n^g-1}{8}$ |
| $\text{SO}_n^+$ ($n$ even) | $\frac{n(n-2)}{8}$ |

unless $\mathcal{L}$ is geometrically trivial, in which case $\text{tr} : V \to \mathbb{F}_\lambda$ would be constant. Hence

$$\left| \sum_{g \in V} \psi(\text{tr}(g)) \right| \leq |\mathbb{F}_\lambda|^{\dim V - 1/2} \sum_{i=0}^{2 \dim V - 1} \dim H^i_c(V \times \mathbb{F}_\lambda, \mathcal{L}).$$

By [Kat01, Theorem 12], we find that

$$\left| \sum_{g \in V} \psi(\text{tr}(g)) \right| \leq 3|\mathbb{F}_\lambda|^{\dim V - 1/2(2 + d)^{n^2+r}}$$

if $V$ is defined by $r$ polynomials of degree at most $d$. The conclusion follows by [MT11, Corollary 24.6].

3.1.2. Classical finite groups of Lie type. Using the Bruhat decomposition, D.S. Kim actually explicitly evaluated the Gaussian sums (8) for classical finite groups of Lie type. The expressions involve hyper-Kloosterman sums, and applying Deligne’s bound yields the following, which greatly improves Proposition 3.2, in particular as $n$ grows:

**Proposition 3.3.** For $n \geq 1$ and $G = \text{GL}_n(\mathbb{F}_\lambda), \text{SL}_n(\mathbb{F}_\lambda), \text{Sp}_{2n}(\mathbb{F}_\lambda), \text{SO}_{2n}^\pm(\mathbb{F}_\lambda)$ and $\text{SO}_{2n+1}(\mathbb{F}_\lambda)$, the bound (8) holds with $\alpha(G) \geq 1$ given in Table 1.

**Proof.** See [PG16a, Proposition 6.28].

3.2. Gaussian sums in $\mathbb{F}_\lambda$. Let us now consider Bound (9) for various subsets $A \subset \mathbb{F}_\lambda$.

3.2.1. Squares. Let $A = \mathbb{F}_\ell^{\times 2}$ be the subgroup of squares in $\mathbb{F}_\ell^{\times}$ with $\ell > 2$. Using the Legendre symbol and the evaluation of quadratic Gauss sums, we get that (9) holds with $\alpha(A) = 1/2$, uniformly for all nontrivial $\psi \in \hat{\mathbb{F}}_\ell$, corresponding to square-root cancellation since $|A| = (\ell - 1)/2$.

3.2.2. Powers/Multiplicative subgroups. More generally, we have:

**Proposition 3.4.** For $\alpha \in (0, 1/2)$, Bound (9) holds for any $H \leq \mathbb{F}_\lambda^{\times}$ such that $|H| \geq |\mathbb{F}_\lambda|^{1/2+\alpha}$, uniformly for all nontrivial $\psi \in \hat{\mathbb{F}}_\lambda$.

**Proof.** This follows for example from the bound $\sum_{x \in H} \psi(x) \ll |\mathbb{F}_\lambda|^{1/2}$ that is deduced from the Deligne’s extension of the Riemann hypothesis over finite fields (see [PG16a, Proposition 5.7]).
Example 3.5. For $m \geq 2$ fixed and $H = \mathbb{F}_λ^m$ the subgroup of $m$th powers, the condition $|H| \geq |\mathbb{F}_λ|^{1/2+\alpha}$ holds as soon as $|\mathbb{F}_λ|$ is large enough, since $|H| = \frac{|\mathbb{F}_λ|}{(m|\mathbb{F}_λ|-1)}$.

Remark 3.6. When $|H|$ is arbitrarily small (say $|H| \geq |\mathbb{F}_λ|^\delta$ for some $\delta > 0$), the works of Bourgain and others (see e.g. [BC06]) give (9) for some $\alpha = \alpha(\delta) > 0$, up to some necessary restrictions if $\delta \leq 1/2$ and $\mathbb{F}_λ \neq \mathbb{F}_\ell$.

3.2.3. Definable subsets. For $R$ a ring and $\varphi(x)$ a first-order formula in one variable in the language of rings, we define $\varphi(R) = \{a \in R : \varphi(a) \text{ holds}\}$.

Example 3.7. For $\varphi(x) = (\exists y : x = y^2)$, the set $\varphi(R)$ is the subset of squares, as in the previous section. More generally, we can take $\varphi(x) = (\exists y : x = g(y))$ for any polynomial $g \in \mathbb{Z}[Y]$.

We recall:

**Theorem 3.8** (Chatzidakis-van den Dries-Macintyre [CvdDM92]). For every formula $\varphi(x)$ in one variable in the language of rings, there exists a finite set $C(\varphi) \subset (0, 1) \cap \mathbb{Q}$ such that for every finite field $\mathbb{F}_λ$, $|\varphi(\mathbb{F}_λ)| = C(\lambda, \varphi)|\mathbb{F}_λ| + O(\mathbb{F}_λ^{1/2})$ (11) with $C(\lambda, \varphi) \in C(\varphi)$, or $|\varphi(\mathbb{F}_λ)| \ll \mathbb{F}_λ^{-1/2}$.

The implicit constants depend only on $\varphi$.

When $\varphi(x) = (\exists y : f(y) = x)$ for some polynomial $f \in \mathbb{Z}[X]$, Theorem 3.8 also appears in [BSD59] (using the Weil conjectures for curves) as:

**Proposition 3.9.** If $f \in \mathbb{F}_λ[X]$ of degree $d$ has full Galois group $S_d$, then (11) holds with

$$C(\varphi) = \left\{ \sum_{n=1}^{d} (-1)^{n+1} \frac{1}{n!} \right\} \subset (0, 1).$$

If $f \in \mathbb{Z}[X]$ of degree $d$ has full Galois group $S_d$ (a generic condition by [vdW34]), then the same holds true for the reductions in almost all characteristics.

This is extended to $f \in \mathbb{Q}(X)$ in [Coh70].

The following combined with Theorem 3.8 shows that Gaussian sums over definable subsets exhibit square-root cancellation:

**Theorem 3.10** ([Kow07, Theorem 1, Corollary 12, Remark 19]). Let $\varphi(x)$ be a formula in one variable in the language of rings such that $|\varphi(\mathbb{F}_λ)|$ is not bounded as $|\mathbb{F}_λ| \to +\infty$. Then, if $\psi \in \hat{\mathbb{F}}_λ$ is nontrivial, the bound (9) for $A = \varphi(\mathbb{F}_λ)$ holds with $\alpha(A) = 1/2$, with an implicit constant depending only on $\varphi$. 

We continue to fix a number field $E$ with ring of integers $\mathcal{O}$.

4.1. General result.

**Proposition 4.1.** Let $t : \mathbb{F}_q \to E$ be the trace function over $\mathbb{F}_q$ associated to a coherent family $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ of sheaves of $\mathcal{O}_\lambda$-modules over $\mathbb{F}_p$ with monodromy group structure $G$. For $A \subset E$ and $\lambda \in \Lambda$, we denote by $A_\lambda \subset \mathcal{F}_\lambda$ the reduction of $A \cap \mathcal{O}_\lambda$ modulo $q$. Assume that

$$\sup_{\lambda \in \Lambda} \frac{|A_\lambda|}{|\mathcal{F}_\lambda|} < 1.$$  \hfill (12)

Then

$$P(t(x) \in A) \ll \frac{1}{|A_L|}$$ with $L = \left\lfloor q^{1/2}\right\rfloor$, \hfill (13)

where $B > 0$ is as in Theorem 2.6, with an absolute implicit constant.

**Proof.** For every $\lambda \in \Lambda$, we let $\Omega_\lambda = \{g \in G_\lambda : \text{tr } g \notin A_\lambda\}$, which is clearly conjugacy-invariant. We may reduce $t : \mathbb{F}_q \to \mathcal{O}_\lambda$ to $t : \mathbb{F}_q \to \hat{A}_\lambda$. By Theorem 2.6,

$$P(t(x) \in A) \ll \frac{1}{q} \frac{|A_\lambda|}{\mathcal{F}_\lambda} \left(1 + \frac{L}{q^{1/2}}\right) \frac{1}{P(L)},$$

where $P(L) = \sum_{\lambda \in \Lambda} P(\text{tr } g \notin A_\lambda)$. By (10) (Proposition 3.1),

$$P(\text{tr } g \in A_\lambda) = \frac{|A_\lambda|}{|\mathcal{F}_\lambda|} \left(1 + O\left(\frac{1}{|\mathcal{F}_\lambda| \cdot \alpha(G) + \alpha(A_\lambda) - 1}\right)\right) \ll \frac{|A_\lambda|}{|\mathcal{F}_\lambda|},$$

since $\alpha(G) \geq 1$ by Proposition 3.3. Therefore, we get that for any $L \geq 1$,

$$P(t(x) \in A) \ll \left(1 + \frac{L}{q^{1/2}}\right)^{-1} |A_L|^{-1} \left(1 - \max_{\lambda \in \Lambda} \frac{|A_\lambda|}{|\mathcal{F}_\lambda|}\right)^{-1}.$$ \hfill \Box

**Remark 4.2.** If we assume more generally that the monodromy group of $\mathcal{F}_\lambda$ is $G(\mathbb{F}_q)$ for $G \leq \text{GL}_n$ any linear group over $\mathbb{F}_q$, the results holds if $\alpha(A_\lambda) \geq 1/2$ for all $\lambda \in \Lambda$, by Proposition 3.2. Interestingly, in the case of $\text{SL}_n$, $\text{Sp}_{2n}$ and $\text{SO}_{n}^\pm$, Proposition 3.3 gives much more cancellation, so that we do not even need information about the $\alpha(A_\lambda)$.

To apply Proposition 4.1, we need the local densities assumption (12) and lower bounds on $|A_L|$. We treat these aspects in the next two sections.

4.2. Lower bounds on $|A_L|$. For our applications, we will mainly consider $\Lambda$ to be either:

**Examples 4.3.**

1. The full set $\Lambda_{0,p}$ of prime ideals of $\mathcal{O}$ which do not lie above $p$.
2. For $m \geq 2$ and $C \subset (\mathbb{Z}/m)^\times$, the set of prime ideals $q$ of $\mathcal{O}$ not lying above $p$ such that $|\mathcal{F}_q| \in C$.
3. The restriction of these to ideals having degree 1 over $\mathbb{Q}$.
More generally, let $F/E$ be a fixed finite Galois extension of number fields with Galois group $H$, $C \subset H$ be a conjugacy-stable subset, and

\[
\Lambda(C) = \{ \mathfrak{q} \leq E \text{ prime, not ramified in } F : \text{Frob}_q \in C \} \quad (14)
\]

\[
\Lambda_1(C) = \{ \mathfrak{q} \in \Lambda(C) \text{ of degree 1 over } \mathbb{Q} \}.
\]

Example 4.3 (1) then corresponds to $E = F$, while (2) corresponds to $E = E(\zeta_n)$ with $H \cong (\mathbb{Z}/m)^\times$.

By Chebotarev’s density theorem, if $E$ and $F$ are fixed,

\[
|\Lambda(C)| \gg |\Lambda_1(C)| \gg \frac{|C|}{|H| \log L} \quad (L \to +\infty)
\]

with an absolute implicit constant. Hence, if $F$ and $E$ do not depend on $p$, (13) is

\[
P(t(x) \in A) \ll_C H \quad \frac{\log q}{Bq^{1/2B}} \to 0 \quad (q = p^e \to +\infty). \quad (16)
\]

If $E$ and/or $F$ depend on $p$ (e.g., for Kloosterman sums, where $E = \mathbb{Q}(\zeta_{lp})$), we must either fix $p$ or deal with uniformity with respect to $E$ and $F$. We discuss this situation in the following paragraphs.

4.2.1. Uniformity in the prime ideal theorem. By [Fri80] (extending Chebychev’s method to number fields), if $E/\mathbb{Q}$ is normal\footnote{In Friedlander’s paper, it is only assumed that $E$ is in a tower of normal extensions. If $E/\mathbb{Q}$ is itself normal, we can improve the result by using more a precise version of Stark’s estimates [Sta74] on the residue at 1 of the Dedekind zeta function of $E$.}, then

\[
\pi_E(L) = |\{ \mathfrak{q} \leq E \text{ prime : } N(\mathfrak{q}) \leq L \}| \gg \frac{L}{\log(2L)^{1+\varepsilon}\Delta_E^{1/2+\varepsilon}}
\]

for $\Delta_E = |\text{disc} \mathbb{Q}(E)|$, and any $\varepsilon > 0$ if $n_E = [E : \mathbb{Q}] \gg \varepsilon 1$. This nontrivial only when $L \gg \Delta_E^{1/2+\varepsilon}$ for some $\varepsilon' > 0$.

4.2.2. Uniformity in Chebotarev’s density theorem. The best unconditional result is due to Lagarias-Odlyzko and Serre (see [Ser81, Section 2.2]), showing that (15) holds with an absolute implicit constant under the restriction $\log L \gg n_E(\log \Delta_E)^2$.

Assuming the generalized Riemann hypothesis (GRH) for the Dedekind zeta function of $E$, this range can be improved to $L \gg (\log \Delta_E)^{2+\varepsilon}$ for an arbitrary $\varepsilon > 0$ (see [Ser81, Section 2.4]).

4.2.3. Cyclotomic fields. If $E = \mathbb{Q}(\zeta_d)$, $F = E(\zeta_m)$ are cyclotomic fields, it is possible to improve the unconditional uniform range in Chebotarev’s density theorem by relying on estimates for primes in arithmetic progressions.

Proposition 4.4. For $d, m \geq 1$ coprime integers, let $E = \mathbb{Q}(\zeta_d)$ and $F = E(\zeta_m)$. For $C \subset \text{Gal}(F/E) \cong (\mathbb{Z}/m)^\times$, we have

\[
|\Lambda(C)| \gg |\Lambda_1(C)| \gg \frac{|C|L}{(dm)^\varepsilon \varphi(m) \log L}
\]

when either:

1. $\varepsilon > 0$ and $L \geq (dm)^8$, or
2. under GRH, $\varepsilon = 0$ and $L \geq (dm)^{2+\varepsilon'}$ for some $\varepsilon' > 0$. 

\[
\text{l}
\]
Proof. Since every unramified rational prime of ramification index $f_\ell$ (equal to the order of $\ell$ in $(\mathbb{Z}/d)^\times$) gives rise to $\varphi(d)/f_\ell$ primes ideals with norm $\ell^{f_\ell}$, 

$$|\Lambda(C)_L| \geq \varphi(d) \sum_{f | \varphi(d)} |\{\ell \leq L^{1/f} \text{ prime : } \ell \nmid \Delta_E, f_\ell = f, \ell^f \in C\}|$$

$$\geq \varphi(d) |\{\ell \leq L \text{ prime : } \ell \equiv 1 \pmod{d}, \ell \in C\}|.$$ 

If $(d, m) = 1$, then by the Chinese remainder theorem 

$$|\Lambda(C)_L| \geq \varphi(d) \left[ \sum_{c \in C} \pi(c, dm, L) - \omega(d) \right],$$

where $\pi(a, d, L) = |\{\ell \leq L \text{ prime : } \ell \equiv a \pmod{d}\}|$ for $a \in (\mathbb{Z}/d)^\times$. Uniformly, the explicit formula gives 

$$\pi(a, d, L) \gg \frac{L}{\varphi(d)d^e \log L} \quad (17)$$

under (1) (the best current unconditional range, by [May13, Theorem 3.3], using Linnik-type arguments) or (2) assuming GRH. \hfill \Box

Remark 4.5. Similarly, this shows that for a Galois extension $E/\mathbb{Q}$, the set of primes ideals with inertia degree 1 has natural density 1, so we cannot hope to substantially improve the lower bound by taking into account the $f > 1$.

Remarks 4.6. (1) The range (2) in (17) holds unconditionally for all $a$ for almost all $d$ by the Bombieri-Vinogradov theorem.

(2) By a conjecture of Montgomery, one may be able to take $\varepsilon = 0$ and $L \gg (dm)^{1+\varepsilon}$ for any $\varepsilon > 0$. By Barban-Davenport-Halberstam, Montgomery and Hooley, this holds true in (17) for almost all $d$ and almost all $a$.

4.2.4. Consequence for Proposition 4.1.

Proposition 4.7. Under the hypotheses of Proposition 4.1 and (12), with $E/\mathbb{Q}$ normal, $F/E$ a finite Galois extension with Galois group $H$, $C \subset H$ a conjugacy-invariant subset and $\Lambda = \Lambda_C$ or $\Lambda_1(C)$ as in (14), we have that for any $\varepsilon > 0$:

(1) If $F = E$ is normal, 

$$P(t(x) \in A) \ll \frac{\Delta_F^{1/2+\varepsilon} (\log q)^{1+\varepsilon}}{B^{1+\varepsilon}q^{1/(2B)}},$$

which is nontrivial when $\Delta_F^{B+\varepsilon'} = o(q)$ for some $\varepsilon' > 0$.

(2) Under GRH, if $q \gg (\log \Delta_E)^{2B+\varepsilon}$, 

$$P(t(x) \in A) \ll \frac{m|C| \log q}{B q^{1/(2B)}} \ll_{m,C} \frac{\log q}{B q^{1/(2B)}}.$$
4.7 Assume that \( E = \mathbb{Q}(\zeta_d) \) and \( F = \mathbb{Q}(\zeta_m) \) with \( (d,m) = 1 \). If \( q \geq (dm)^{16B} \), then
\[
P(t(x) \in A) \ll \frac{m|C|(dm)^{e} \log q}{Bq^{1/(2B)}} \ll_m C \frac{d^e \log q}{Bq^{1/(2B)}}.
\]

4.2.5. The case \( E = \mathbb{Q}(\zeta_{4p}) \). For exponential sums, we are interested in the case \( E = \mathbb{Q}(\zeta_{4p}) \), with \( n_E = 2(p-1) \) and \( \Delta_E = 4^{2p-3}p^{2(p-2)} \). The restrictions \( q \gg g(E) \) (for some \( g(E) = g(n_E, \Delta_E) \geq 1 \)) of Proposition 4.7 impose limitations on the range of \( e,p \) when \( q = p^e \to +\infty \):

**Corollary 4.8.** Under the hypotheses of Proposition 4.7 for \( E = \mathbb{Q}(\zeta_{4p}) \) and \( F = \mathbb{Q}(\zeta_m) \) with \( (m, 4p) = 1 \), we have
\[
P(t(x) \in A) \ll \frac{m|C|(pm)^{e} \log q}{Bq^{1/(2B)}} \ll_m C \frac{p^e \log q}{Bq^{1/(2B)}} \to 0 \quad (q = p^e \to +\infty)
\]
when either
1. \( \varepsilon > 0 \) and \( e \geq 16B \), or
2. under GRH, \( \varepsilon = 0 \) and \( e > 4B \).

**Remarks 4.9.**
1. Had we not taken advantage of the fact that \( E \) is a cyclotomic field, the best unconditional results would have forced to take \( q = p^e \to +\infty \) with \( e \gg p \).
2. Under Montgomery’s conjecture, we may take \( \varepsilon = 0 \) and \( e > 2B \). Without an improvement in the error term of the large sieve bound (13), \( e = 2B + 1 \geq 10 \) is the minimal value the method could handle.

4.3. Local densities. In this section, we finally give examples of sets \( A \subset E \) for which the local densities assumption (12) holds.

4.3.1. Powers/finite index subgroups.

**Proposition 4.10.** Let \( E, \mathcal{O} \) be as in Proposition 4.1 and for \( m \geq 2 \), let
\[
\Lambda = \{ q \leq \mathcal{O} : |q| \equiv 1 \pmod{m} \}
\]
be the set of Example 4.3 (2). Then (12) holds for \( A = E^m \subset E \).

**Proof.** We have \( A_{\lambda} = \mathbb{F}_\lambda^m \), and for \( |\mathbb{F}_\lambda| \geq 3 \),
\[
\frac{|A_{\lambda}|}{|\mathbb{F}_\lambda|} = \left( 1 - \frac{1}{|\mathbb{F}_\lambda|} \right) \frac{1}{(|\mathbb{F}_\lambda|, m)} + \frac{1}{m} \left( \frac{1}{|\mathbb{F}_\lambda|} - 1 \right) < 1.
\]

4.3.2. Definable subsets.

**Proposition 4.11.** Let \( E, \mathcal{O} \) and \( \Lambda \) be as in Proposition 4.1 and let \( \varphi(x) \) be a first order formula in one variable in the language of rings such that:
1. Neither \( |\varphi(\mathbb{F}_\lambda)| \) nor \( -\varphi(\mathbb{F}_\lambda) | \) are bounded as \( |\mathbb{F}_\lambda| \to +\infty \).
2. For every \( \lambda \in \Lambda \) corresponding to an ideal \( q \), \( \varphi(E) \cap \mathcal{O}_q \pmod{q} \) is contained in \( \varphi(\mathbb{F}_\lambda) \).

Then (12) holds with \( A = \varphi(E) \subset E \).
Proof. Condition (2) implies that $A_\lambda \subset \varphi(F_\lambda)$ for all $\lambda \in \Lambda$, so that by Theorem 3.8, $\limsup_{|F_\lambda| \to +\infty} |A_\lambda| / |F_\lambda| \leq \max \varphi < 1$. □

Remark 4.12. Condition (2) of Proposition 4.11 holds if both

(a) $\varphi(E) \cap O_q \subset \varphi(O_q)$, and
(b) $\varphi(O_q) \mod q \subset \varphi(F_\lambda)$

hold. Note that:

– Condition (a) holds when $\text{char} F_\lambda \gg 1$ if $\varphi(x) = (\exists y : f(x) = y)$ for some $f \in \mathbb{Z}[X]$. Indeed, for $x \in E$, we have $\lambda(\varphi(x)) = \deg(\varphi)\lambda(x)$ if no coefficient of $\varphi$ is divisible by $\text{char}(F_\lambda)$.

– Condition (b) holds if $\varphi$ contains no negations or implications. On the other hand, for $\varphi(x) = (\exists y : x = y^2)$, the reduction of a nonsquare in $O$ may be a square in $F_\lambda$.

Example 4.13. Consider the case $\varphi(x) = (\exists y : f(x) = y)$ for $f \in \mathbb{Z}[X]$. Then Proposition 4.11 applies for almost all $f$ of fixed degree up to restricting to a cofinite subset of $\Lambda$. Indeed:

– By Proposition 3.9, Condition (1) of Proposition 4.11 holds for almost all $f$ of fixed degree.

– Condition (2) holds if $\text{char}(F_\lambda) \gg 1$ by Remark 4.12.

5. Examples

5.1. Kloosterman sums. By Proposition 2.3 and the local densities estimates from Propositions 4.10 and 4.11, Corollary 4.8 yields respectively Proposition 1.1 and:

Proposition 5.1. Let $\varphi(x)$ be a first-order formula in the language of rings as in Proposition 4.11. Then, for $n \geq 2$ and $\epsilon > 0$,

$$P\left( \text{Kl}_{n,q}(x) \in \varphi(\mathbb{Q}^e) \right) \ll_{\varphi,\epsilon} \frac{p^{\epsilon\log q}}{B_nq^{1/2B_n}} \to 0$$

(18) when $q = p^\epsilon \to +\infty$ with $\epsilon \geq 16B_n$, for $B_n$ as in (5). The implicit constant depends only on $\varphi$ and $\epsilon$.

Proposition 1.2 is a particular case of the latter, using Example 4.13.

5.1.1. Results for unnormalized sums. Replacing $A$ by $q^{(n-1)/2}A$ in Proposition 4.1 and using uniformity shows that the above results also hold for unnormalized Kloosterman sums.

5.1.2. Galois actions. When considering densities of the form (18), it is interesting to take into account the following Galois actions:

1. For all $\sigma \in \text{Gal}(F_q/F_p) \cong \mathbb{Z}/e$ and $x \in F_q^e$,

$$\text{Kl}_{n,q}(x) = \text{Kl}_{n,q}(\sigma(x))$$

The orbit of $x$ has size $\deg(x) \in \{1, \ldots, e\}$. Fisher [Fis92, Corollary 4.25] has actually shown that if $p > (2n^{2e} + 1)^2$, the Kloosterman sums are distinct up to this action.
(2) For $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong \mathbb{F}_p^\times$, corresponding to $c \in \mathbb{F}_p^\times$ and $x \in \mathbb{F}_q^\times$, we have

$$\sigma(Kn,q(x)) = Kn,q(cx).$$

Moreover, orbits have size $p^{\frac{p-1}{(p-1,n)}} \in \{(p-1)/n, \ldots, p-1\}$.

If $\varphi$ is a first-order formula in the language of rings, let $A_p = \varphi(\mathbb{Q}(\zeta_p))$. Since $\sigma(A_p) = A_p$ for all $\sigma \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$, we can define an equivalence relation $\sim$ on $\{x \in \mathbb{F}_q^\times : Kn,q(x) \in A_p\}$ generated by $x \sim cx$ for all $c \in \mathbb{F}_p^\times$, $x \in \mathbb{F}_q^\times$, and we have

$$\left|\{x \in \mathbb{F}_q^\times : Kn,q(x) \in A_p\}/ \sim \right| = \frac{\left|\{x \in \mathbb{F}_q^\times : Kn,q(x) \in A_p\}\right|(p-1,n)}{p-1} \ll_n \frac{\left|\{x \in \mathbb{F}_q^\times : Kn,q(x) \in A_p\}\right|}{p-1}.$$

If Corollary 5.1 holds, this yields

$$\left|\{x \in \mathbb{F}_q^\times : Kn,q(x) \in A_p\}/ \sim \right| \ll_n \varepsilon q^{1-1/(2B_n)} \log q.$$

Remark 5.2. The right-hand side can tend to 0 with $p \to +\infty$ only when $e < \frac{2B_n}{2B_n-1}$. Since $\frac{2B_n}{2B_n-1} \in (1, 2)$, this is the case only for $e = 1$. Unfortunately, our estimate on the number of primes ideals of bounded norm in $\mathbb{Q}(\zeta_p)$ requires to take $e \gg 1$. If it could be extended to $e = 1$ (but see Remarks 4.9 (2)), the above would show that for $p$ large enough, there is no $x \in \mathbb{F}_q^\times$ such that $Kn,p(x) \in \varphi(\mathbb{Q}(\zeta_p))$.

5.2. Exploiting monodromy over $C$. As we mentioned in the previous section, determining integral monodromy groups (as required by Definition 2.1 (2)), say for almost all valuations $\lambda$, is usually difficult.

By using some deep results of Larsen and Pink (relying in particular on the classification of finite simple groups in [Lar95]), the following result allows to obtain coherent families from the knowledge of the monodromy groups over $\overline{\mathbb{Q}}_\ell$, up to passing to a subfamily of density 1 depending on $p$.

Theorem 5.3. Let $E \subset C$ be a Galois number field with ring of integers $O$ and let $\Lambda$ be a set of valuations on $O$ of natural density 1. Let $(\mathcal{F}_\lambda)_{\lambda \in \Lambda}$ be a compatible system with $\mathcal{F}_\lambda$ a sheaf of $O_\lambda$-modules over $\overline{\mathbb{Q}}_\ell$. We assume that Condition (3) from Definition 2.1 holds and that:

$(2')$ There exists $G \in \{\text{SL}_n, \text{Sp}_{2n}\}$ such that for every $\lambda \in \Lambda$, the arithmetic monodromy group of $\mathcal{F}_\lambda \otimes \overline{\mathbb{Q}}_\ell$ is conjugate to $G(\overline{\mathbb{Q}}_\ell)$.

Then there exists a subset $\Lambda_p \subset \Lambda$ of natural density 1, depending on $p$ and on the family, such that $(\mathcal{F}_\lambda)_{\lambda \in \Lambda_p}$ is coherent.

After using Theorem 5.3, we may apply Proposition 4.1 with the coherent subfamily $(\mathcal{F}_\lambda)_{\lambda \in \Lambda_p}$ to get

$$P(t(x) \in A) \ll \frac{1}{|\Lambda_p|} \ll_p \frac{1}{|\Lambda|},$$

(19)
when \( L = \lfloor q^{1/(2B)} \rfloor \to +\infty \), with the implicit constant depending on \( p \) and on the original family.

### 5.2.1. Proof of Theorem 5.3.

The idea of the argument, based on [LP92] and [Lar95], is due to Katz and appears partly in [Kow06a, p. 29], [Kow06b, p. 7], [Kow08, pp. 188–189] (however see Remark 5.5 below), and [Kat12, Section 7].

To reduce as much as possible to the situation of [LP92] and [Lar95], we consider the subset \( \Lambda_1 \subset \Lambda \) corresponding to ideals of degree 1 over \( \mathbb{Q} \), so that \( E_\lambda = \mathbb{Q}_\ell, \mathcal{O}_\lambda = \mathbb{Z}_\ell \) and \( F_\lambda = \mathbb{F}_\ell \) if \( \lambda \in \Lambda_1 \) is an \( \ell \)-adic valuation. By [Jan05, 4.7.1], for any \( S \subset \text{Spec}(\mathcal{O}) \), the Dirichlet density of \( S \) is equal to the Dirichlet density of the elements of \( S \) having degree 1 over \( \mathbb{Q} \). In particular, \( \Lambda_1 \) has Dirichlet density 1, and actually natural density 1 by the proof of Proposition 4.4.

In the notations of [Lar95, Section 3], we have the compact \( F \)-group \( \pi_{1,p} \) with compatible system of representations

\[
(\rho_\lambda = \rho_{F,\lambda} : \pi_{1,p} \to \text{GL}_n(\mathcal{O}_\lambda) = \text{GL}_n(\mathbb{Z}_\ell))_{\lambda \in \Lambda_1}
\]

and Frobenius \( \text{Frob}_\alpha \) for \( \alpha \in \mathcal{A} = \{(x,p^n) : n \geq 1, x \in \mathbb{F}_p^n\} \). Note that \( G \) is a simply connected reductive group scheme over \( \mathbb{Z} \), and by hypothesis \( \rho_\lambda \) is semisimple.

For every \( \lambda \in \Lambda \), we let \( G_\lambda = G_{E_\lambda}, \Gamma_\lambda = \rho_\lambda(\pi_{1,p}) \leq G(\mathcal{O}_\lambda) \) the integral monodromy group, \( \hat{\Gamma}_\lambda := \Gamma_\lambda \mod \lambda \) its reduction, and

\[
B = \{ \lambda \in \Lambda : \hat{\Gamma}_\lambda \leq G(\mathbb{F}_\ell) \} \subset \Lambda
\]

the set of valuations where the monodromy group is smaller than expected. We let moreover:

- For every \( \alpha \in \mathcal{A} \),

\[
\xi(\alpha) \in \mathcal{O}_\lambda[T] \subseteq E[T]
\]

the characteristic polynomial of \( \rho_\lambda(\text{Frob}_\alpha) \) (which does not depend on \( \lambda \in \Lambda \) by hypothesis).

- \( K \subset \xi(G_\mathbb{Q}) \) the \( \mathbb{Q} \)-rational closed subvariety given by [Lar95, (3.8)]. Since it is Zariski-closed, there exists a constant \( C_\alpha \geq 0 \) such that \( \xi(\alpha) \mod \ell \notin K \mod \ell \) if \( \ell > C_\alpha \).

- \( \mathcal{A}' \subset \mathcal{A} \) the set of the \( \alpha \in \mathcal{A} \) such that:

1. \( \rho_\lambda(\text{Frob}_\alpha) \) is regular with respect to \( \text{GL}_n \) (see [Lar95, (3.4)], [LP92, (4.5)]) for every \( \lambda \in \Lambda \).

2. \( \xi(\alpha) \notin K \).

By [Lar95, (3.11)], \{\text{Frob}_\alpha : \alpha \in \mathcal{A}'\} \subset \pi_{1,p} \) is still dense and by [LP92, (4.7)]:

1. \( \rho_\lambda(\text{Frob}_\alpha) \) lies in a unique maximal torus of \( T_{\lambda,\alpha} \) of \( G_{E_\lambda} \).

2. \( \xi(\alpha) \) is associated to a torus \( T_\alpha \) in \( \text{GL}_n(E) \), unique up to \( \text{GL}_n(E) \)-conjugacy, such that \( T_\alpha \times_E E_\lambda \) is conjugate to \( T_{\lambda,\alpha} \).

3. The splitting field of these tori is equal to the splitting field \( L_\alpha \) of \( \xi(\alpha) \) over \( E \) [LP92, (4.4)].

- \( C_\alpha \geq 1 \) such that \( L_\alpha/\mathbb{Q} \) is unramified at any \( \ell > C_\alpha \).
- $L$ the intersection of the $L_\alpha$ for $\alpha \in \mathcal{A}'$, so that $\mathbb{Q} \subset E \subset L \subset L_\alpha$. We decompose

$$B = (\Lambda \setminus \Lambda_1) \cup \bigcup_{x \in \text{Gal}(L/E)^\beta} B_x, \quad B_x = \{ \lambda \in \Lambda_1 \cap B : [\lambda, L/E] = x \}.$$  

The upper natural density of $B$ is

$$\overline{\delta}(B) = \limsup_{S \to +\infty} \frac{|\lambda \in B : N(\lambda) \leq S|}{|\lambda \in \Lambda : N(\lambda) \leq S|} \leq \overline{\delta}(\Lambda \setminus \Lambda_1) + \sum_{x \in \text{Gal}(L/E)^\beta} \overline{\delta}(B_x) = \sum_{x \in \text{Gal}(L/E)^\beta} \overline{\delta}(B_x).$$

Let us fix a class $x \in \text{Gal}(L/E)^\beta$ and an $\ell'$-adic valuation $\lambda' \in \Lambda_1$ with Frobenius $[\lambda', L/E] = x$. If $\lambda \in B_x$, then $\Gamma_{\lambda}$ is a proper subgroup of $G(F_\lambda) = G(F_\ell)$. By [Lar95, (1.1), (1.19)], when $\ell \gg 1$, every maximal subgroup of $G(F_\ell)$ is of the form $H(F_\ell)$, for $H \subset G_{Z_\ell}$ a smooth $Z_\ell$-subgroup scheme. By [Lar95, (3.1)] (see also [Lar95, (3.8)]), it follows that there exists a maximal proper reductive $\mathbb{Q}_\ell$-subgroup $N$ of $G_\lambda$ (containing a Levi component of $H_{\mathbb{Q}_\ell}$) such that

$$\text{FM}(\lambda, \alpha) \in \text{FM}_{N_0} \subsetneq \text{FM}_{G_\lambda}$$

for every $\alpha \in \mathcal{A}'$ such that $\ell > D_\alpha = \max(C_{\alpha}, C_{\alpha}')$, where:

- $\text{FM}(\lambda, \alpha)$ is the isomorphism class of the Frobenius module (i.e. free $\mathbb{Z}$-module of finite rank with an endomorphism of finite order) arising from the character group of the maximal torus $T_{\lambda, R} \leq G_\lambda$ containing $\rho_\lambda(F_\alpha)$, with the action of $\text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell) = \text{Gal}(\overline{\mathbb{Q}}_\ell/F_\ell)$. By [Lar95, (3.14)], this depends only on $[\ell, L_\alpha/\mathbb{Q}] = [\lambda, L_\alpha/E]$ up to isomorphism.

- $\text{FM}_{G_\lambda}$ and $\text{FM}_{N_0}$ are the set of isomorphism classes of Frobenius modules arising from unramified tori of $G_\lambda$, resp. $N_0$.

Let $M \in \text{FM}_{G_\lambda} \setminus \text{FM}_{N_0}$. As in [Lar95, (3.15)], and [LP92, (8.2)], we will show that

\begin{equation}
(*) \text{ For every } R \geq 1, \text{ there exist } \alpha_1, \ldots, \alpha_R \in \mathcal{A}' \text{ such that } M = \text{FM}(\lambda', \alpha_1) \text{ with } L_{\alpha_1} \text{ linearly disjoint}. \end{equation}

Assuming this, it follows that for $\ell > \max_{1 \leq i \leq R} D_{\alpha_i}$, then for $1 \leq i \leq R$,

$$[\lambda, L_{\alpha_i}/E] = [\ell, L_{\alpha_i}/\mathbb{Q}] \neq [\ell', L_{\alpha_i}/\mathbb{Q}] = [\lambda', L_{\alpha_i}/E]$$

in $\text{Gal}(L_{\alpha_i}/\mathbb{Q}) \supseteq \text{Gal}(L_{\alpha_i}/E)$, since $M \neq \text{FM}(\lambda, \alpha_i)$. Therefore, by Chebotarev’s theorem,

$$\overline{\delta}(B_x) \leq \overline{\delta}(\{ \lambda \in \Lambda : [\lambda, L_{\alpha_i}/E] \neq [\lambda', L_{\alpha_i}/E] \text{ for } 1 \leq i \leq R \})$$

$$= \left( 1 - \frac{1}{n!} \right)^R \frac{|x|}{|\text{Gal}(L/E)|}$$

since $[L_{\alpha_i} : E] \leq n!$ and by linear disjointness. Hence $\overline{\delta}(B) \leq (1 - 1/n!)^R$ for every $R \geq 1$, so that $B$ has natural density $0$ by taking $R \to +\infty$.

\footnote{Here, this means that for any $2 \leq i \leq R$, $L_{\alpha_1}, \ldots, L_{\alpha_{i-1}}, L_{\alpha_i}$ are linearly disjoint over $L$, i.e. their intersection is equal to $L$.}
We now prove \((\ast)\). It suffices to show that for any finite Galois extension \(F/L\), there exists \(\alpha \in \mathcal{A}'\) such that \([M] = FM(\lambda', \alpha)\) with \(L_{\alpha}\) and \(F\) linearly disjoint over \(L\). We proceed as in \([LP92, (8.2)]\) (where \(E = \mathbb{Q}\)).

For \(K_1, \ldots, K_m\) the intermediate fields of \(F/L\) normal over \(L\) and minimal with respect to inclusion with this property, we have that \(L_{\alpha}\) is linearly disjoint with \(F\) over \(L\) if and only if \(K_i \nsubseteq L_{\alpha}\) for all \(1 \leq i \leq m\). This holds in particular if for every \(i\) there exists \(\lambda_i \in \Lambda_1\) corresponding to a prime that splits in \(L_{\alpha}\), but not in \(K_i\).

For every \(1 \leq i \leq m\), let \(\beta_i \in \mathcal{A}'\) be such that \(K_i \nsubseteq E_{\beta_i}\). By minimality of \(K_i\), we have \(E_{\beta_i} \cap K_i = L\), so that \(\text{Gal}(L_{\beta_i}/L) \times \text{Gal}(K_i/L)\) is contained in \(\{(\sigma_1, \sigma_2) \in \text{Gal}(L_{\beta_i}/E) \times \text{Gal}(K_i/E) : \sigma_1 |_{L} = \sigma_2 |_{L}\} \cong \text{Gal}(L_{\beta_i}K_i/E)\).

By Chebotarev’s theorem, the set of \(\lambda \in \text{Spec}(O)\) that split in \(L_{\beta_i}\) but does not split in \(K_i\), has positive Dirichlet density, so the same holds for the \(\lambda \in \Lambda_1\) with this property, since \(\Lambda_1\) has Dirichlet density 1. Hence, there exists \(\lambda_i \in \Lambda_1 \setminus \{\lambda'\}\) that splits in \(L_{\alpha}\) but not in \(K_i\), and we may suppose all the \(\lambda_i\) distinct.

By \([LP92, (7.5.3)]\), there exists \(\alpha \in \mathcal{A}'\) such that:

1. \(T_{\lambda', \alpha}\) is conjugate in \(GL_n(E_{\lambda'})\) to the unramified maximal torus of \(G_{\lambda'}\) corresponding to \(M\), so \([M] = FM(\lambda', \alpha)\).
2. \(T_{\lambda, \alpha}\) is conjugate in \(GL_n(E_{\lambda})\) to \(T_{\lambda_i, \beta_i}\). Since \(\lambda_i\) splits in \(L_{\beta_i}\), this torus is split, so that \(\lambda_i\) also splits in \(L_{\alpha}\).

This concludes the argument.

Finally, concerning the geometric integral monodromy group \(\Gamma^{\text{geom}}_\lambda = \rho_\lambda(\pi^{\text{geom}}_{1,p}) \leq \Gamma_\lambda\), note that:

1. \(\hat{\Gamma}_\lambda / \hat{\Gamma}^{\text{geom}}_\lambda\) is a finite quotient of \(\pi_{1,p}/\pi^{\text{geom}}_{1,p}\) \(\cong \hat{\mathbb{Z}}\), hence a finite cyclic group.
2. If \(|F_\lambda| \gg 1\), the group \(G'(F_\lambda) := G(F_\lambda)/Z(G(F_\lambda))\) is simple non-abelian (see e.g. \([MT11, \text{Theorem 24.17}]\)).

Hence, by (2), if \(\hat{\Gamma}^{\text{geom}}_\lambda \leq G(F_\lambda)\), then it is contained in \(Z(G(F_\lambda))\), so that:

\[
G'(F_\lambda) \cong \frac{\hat{\Gamma}_\lambda / \hat{\Gamma}^{\text{geom}}_\lambda}{Z(G(F_\lambda)) / \hat{\Gamma}^{\text{geom}}_\lambda}
\]

would be cyclic by (1), a contradiction. \(\square\)
Remarks 5.4. (1) If $F_{\lambda}$ is pointwise pure of weight 0, we see as in [KS91, 9.2.4] that if $C$ is a maximal compact subgroup of $G(\mathbb{Q}_\ell)$, then for every $\alpha \in \mathcal{A}$, the semisimple part $\rho(\text{Frob}_\alpha)^{ss}$ gives a well-defined conjugacy class in $C$ with minimal polynomial $\xi(\alpha)$. If $K$ is defined by the polynomials $f_1, \ldots, f_m \in \mathbb{Q}[X]$, then we can take uniformly

$$C_{\alpha} = \max_{\alpha \in \mathcal{A}} \max_{1 \leq i \leq m} |f_i(\xi(\alpha))| < \infty.$$  

(2) We consider compatible systems of representations $\rho_{\lambda} : \pi \to \text{GL}_n(\mathcal{O}_{\lambda})$, where $\lambda$ is a valuation on the ring of integers $\mathcal{O}$ of a number field $E/\mathbb{Q}$, while the results in [LP92, Part II] and [Lar95] are stated for the case $E = \mathbb{Q}$. One needs to be cautious before stating the natural generalizations of the results of Larsen and Pink. For example, under the notations of the theorem, the maximal subgroups of $G(\mathbb{F}_\lambda)$ are not all of the form $H(\mathbb{F}_\lambda)$ for $H \subset G_{\mathcal{O}_{\lambda}}$ a smooth $\mathcal{O}_{\lambda}$-subgroup scheme, unless $F_{\lambda} = F_{\ell}$ as in [Lar95, (1.1), (1.19)]: for instance, one has subfield subgroups.

(3) This does not apply to $\text{SO}_n$, since it is not simply connected, and this assumption is required for [Lar95, (1.19)]. In even dimension, note that one would need additional input to determine the type (+ or −) of the monodromy groups over $\mathbb{F}_\lambda$.

5.2.2. Arithmetic and geometric monodromy groups. Often, only the geometric monodromy group is determined, while Theorem 5.3 and Definition 2.1 require knowledge of the arithmetic monodromy. By twisting a sheaf $F_{\lambda}$ by a constant or a Tate twist, it is often possible to get a sheaf $F'_{\lambda}$ with

$$G_{\text{geom}}(F_{\lambda}) = G_{\text{geom}}(F'_{\lambda}) \leq G_{\text{arith}}(F'_{\lambda}) \leq G_{\text{geom}}(F'_{\lambda}),$$

so that $G_{\text{geom}}(F'_{\lambda}) = G_{\text{arith}}(F'_{\lambda}) = G_{\text{geom}}(F'_{\lambda})$. Examples will be given in the next sections.

Remark 5.5. In [Kow06a], [Kow06b] and [Kow08], the results of Larsen-Pink are applied to deduce the geometric monodromy group over $\mathbb{F}_\ell$ from the geometric group over $\mathbb{Q}_\ell$. However, this is incorrect since the geometric group does not contain a dense subset of the Frobenius. Moreover, note that the arithmetic monodromy group is not contained in $\text{Sp}_{2g}(\mathbb{Q}_\ell)$ (but in $G\text{Sp}_{2g}(\mathbb{Q}_\ell)$).

For the unnormalized family of first cohomology groups of hyperelliptic curves, this is not an issue because the results of J.-K. Yu and C. Hall also apply to give the geometric monodromy groups. Alternatively, one may normalize by a Tate twist as in Proposition 2.5 and apply Theorem 5.3 to the normalized sheaf (see above).

For the characteristic 2 example of [Kow06b, Proposition 3.3], the result of Hall can also be applied because the local monodromy at 0 is a unipotent pseudoreflection. Again, one could also apply Theorem 5.3 after normalizing.

On the other hand, the statement [Kow06a, Theorem 6.1] must be modified to assume for example that the arithmetic monodromy group is $\text{Sp}$, or that the geometric monodromy groups over $\mathbb{F}_\ell$ are known for all $\ell > 1$.

5.3. General exponential sums.
5.3.1. Construction of coherent families.

**Proposition 5.6** (Exponential sums (2), \( h = 0, \chi = 1 \)). Let \( f \in \mathbb{Q}(X) \) and let \( Z_f \) be the set of zeros of \( f' \) in \( \mathbb{C} \), with cardinality \( k_f \). We assume that either:

- \((H_1)\): \( k_f = |Z_f'| \) is even, \( \beta = \sum_{z \in Z_f'} f'(z) = 0 \), and if \( s_1 - s_2 = s_3 - s_4 \) with \( s_i \in f(Z_f') \), then \( s_1 = s_3, s_2 = s_4 \) or \( s_1 = s_2, s_3 = s_4 \).
- \((H_2)\): \( f \) is odd, and if \( s_1 - s_2 = s_3 - s_4 \) with \( s_i \in f(Z_f') \), then \( s_1 = s_3, s_2 = s_4 \) or \( s_1 = s_2, s_3 = s_4 \) or \( s_1 = -s_4, s_2 = -s_3 \).

If \( p \) is large enough, there exists a family \( (\mathcal{G}_f)_{\lambda \in \Lambda_{f,p}} \) of \( \ell \)-adic sheaves of \( \mathcal{O}_\lambda \)-modules over \( \mathbb{F}_p \), with trace function

\[
x \mapsto -\frac{1}{\sqrt{q}} \sum_{y \in \mathbb{F}_q} e\left(\frac{\text{tr}(xf(y))}{p}\right) \quad (x \in \mathbb{F}_q).
\]

Moreover, there exists \( \alpha_p \in \mathbb{Q} \) and a set of valuations \( \Lambda' = \Lambda'_{f,p} \) of density 1 on \( E' = E(\alpha_p) \), depending only on \( f \) and \( p \), such that

\[
(\mathcal{G}_f \otimes \alpha_p \mathcal{O}_\lambda')_{\lambda \in \Lambda'}
\]

is a coherent family of sheaves of \( \mathcal{O}_\lambda' \)-modules over \( \mathbb{F}_p \), for \( \mathcal{O}' \) the ring of integers of \( E' \), with monodromy group structure

- \( G = \text{SL}_{k_f} \) if \((H_1)\) holds.
- \( G = \text{Sp}_{k_f} \) if \((H_2)\) holds, and one may take \( \alpha_p = 1 \).

**Proof.** See [Kat90, Theorem 7.9.4, Lemmas 7.10.2.1, 7.10.2.3] for the construction and [Kat90, 7.9.7, 7.10] for the determination of \( G_{\text{geom}}(\mathcal{G}_f) \) over \( \mathbb{C} \). The family forms a compatible system by Lemma 2.2. The definition over \( \mathcal{O}_\lambda \) comes from the definition of the \( \ell \)-adic Fourier transform on the level of sheaves of \( \mathcal{O}_\lambda \)-modules (see [Kat88, Chapter 5]). Under our hypotheses, \( G_{\text{geom}}(\mathcal{G}_f) \) contains \( \text{SL}_{k_f}(\mathbb{Q}_\ell) \), resp. \( \text{Sp}_{k_f}(\mathbb{Q}_\ell) \). Moreover:

- In the \((H_2)\) case, \( G_{\text{arith}}(\mathcal{G}_f) \leq \text{Sp}_{k_f}(\mathbb{F}_\lambda) \) by [Kat90, 7.10.4 (3)], and we can apply Theorem 5.3.
- In the \((H_1)\) case, since \( \pi_{1,p}/\pi_{1,\text{geom}} \simeq \mathbb{Z} \) is abelian, there exists by Clifford theory an element \( \beta_p \in \mathcal{O}_\lambda' \cap \mathbb{Q} \) not depending on \( \lambda \) (since we have a compatible system) such that the arithmetic determinant is isomorphic to \( \beta_p \otimes \mathcal{O}_\lambda \).

As in 5.2.2, we obtain that with \( \alpha_p = \beta_p^{-1/k_f} \in \mathcal{O}_\lambda' \) for any valuation \( \lambda' \) extending \( \lambda \), the arithmetic and geometric monodromy groups of \( \mathcal{G}_f \otimes \alpha_p \mathcal{O}_\lambda' \), coincide and are conjugate to \( \text{SL}_{k_f}(\mathbb{Q}_\ell) \), so that we can apply Theorem 5.3.

Example 5.7. The hypotheses hold for the polynomial \( f = aX^{r+1} + bX \), where \( a, b \in \mathbb{Z} \), \( r \in \mathbb{Z} - \{1\} \), \( rab \neq 0 \), with \((H_1)\) if \( r \) is odd and \((H_2)\) otherwise, or for the polynomial \( f = X^u - naX \), where \( a \in \mathbb{Z} \) nonzero and \( n \geq 3 \), with \((H_2)\).

The following include for example Birch sums (with \( h = X^3 \)):
Proposition 5.8 (Exponential sums (2), $f = X$, $\chi = 1$, $h$ polynomial). Let $h = \sum_{i=1}^{n} a_i X^i \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 3$ with $n \neq 7, 9$ and $a_{n-1} = 0$. If $p$ is large enough, there exists a family $(G_h)_{\lambda \in \Lambda_0, p}$ of $\ell$-adic sheaves of $\mathcal{O}_\lambda$-modules over $\mathbb{F}_p$ with trace function

$$x \mapsto \frac{-1}{\sqrt{q}} \sum_{y \in \mathbb{F}_q} e \left( \frac{\text{tr}(xy + h(y))}{p} \right) (x \in \mathbb{F}_q).$$

Moreover, there exists $\alpha_p \in \overline{\mathbb{Q}}$ and a set of valuations $\Lambda' = \Lambda'_{h, p}$ of density 1 on $E' = E(\alpha_p)$, depending only on $h$ and $p$, such that

$$(G_h \otimes \alpha_p \mathcal{O}_\lambda')_{\lambda \in \Lambda'}$$

is a coherent family of sheaves of $\mathcal{O}_\lambda'$-modules over $\mathbb{F}_p$, for $\mathcal{O}'$ the ring of integers of $E'$, with monodromy group structure:

1. $G = \text{Sp}_{n-1}$ if $n$ is odd and $h$ has no monomial of even positive degree, and one may take $\alpha_p = 1$.
2. $G = \text{SL}_{n-1}$ otherwise.

Proof. This is similar to the proof of Proposition 5.6. See [Kat90, 7.12] for the construction of the sheaves and the determination of the monodromy groups over $\mathbb{C}$. In the symplectic case, ibidem shows that the arithmetic monodromy group is itself contained in $\text{Sp}_{n-1}$. \hfill \square

Proposition 5.9 (Exponential sums (2), $f$ polynomial, $\chi \neq 1$). Let

- $h \in \mathbb{Q}(X)$ with a pole of order $n \geq 1$ at $\infty$.
- $f \in \mathbb{Z}[X]$ odd nonzero of degree $d$ with $(d, n) = 1$.
- $\chi$ a character of $\mathbb{F}_p^*$ of order $r \geq 2$.
- $g \in \mathbb{Q}(X)$ odd nonzero, with the order of any zero or pole not divisible by $r$.

For $p$ large enough, there exists a family $(G_{h, f, X, g})_{\lambda \in \Lambda_0, p}$ of $\ell$-adic sheaves of $\mathcal{O}_\lambda$-modules over $\mathbb{F}_p$ with trace function (2).

Moreover, if we assume that there exists $L \in \mathbb{Q}(X)$ odd with $L(x)^r = g(x)g(-x)$ and either $N = \text{rank}(G) \neq 7, 8$ or $|n - d| \neq 6$, then there exists a set of valuations $\Lambda_p \subset \Lambda_{0, p}$ of density 1, depending only on $h, f, g, \chi$ and $p$, such that

$$(G_{h, f, X, g})_{\lambda \in \Lambda_p}$$

is a coherent family, with monodromy group structure $G = \text{Sp}_N$.

Proof. This is again similar to the proof of Proposition 5.6. See [Kat90, 7.7, 7.13 (Sp-example(2))] for the construction of the sheaves and the determination of the monodromy groups over $\mathbb{C}$; [Kat90, 7.13] shows that the arithmetic monodromy group is itself contained in $\text{Sp}_N$. \hfill \square

Remark 5.10. If $L$ is even, there exists $\alpha_p \in \{\pm 1\}$ such that the arithmetic and geometric monodromy groups over $\mathbb{C}$ of $\alpha_p \otimes G_{h, f, X, g}$ coincide and are conjugate to $\text{SO}_X(\mathbb{C})$ (see [Kat90, 7.14 (O-example(2))]). However, Theorem 5.3 does not apply in that case (see Remarks 5.4 (3)).
5.3.2. Zero-density estimates. Hence, for the three families above, we get by Corollary 4.8 with Propositions 4.10 and 4.11:

**Proposition 5.11.** We fix a prime \( p \) and we set \( q = p^r \). Let \( t : \mathbb{F}_q \to \mathbb{Q}(\zeta_{4p}) \) be the trace function associated with one of the families from Propositions 5.6, 5.8 or 5.9, and let \( B \) as in (7).

For \( \varphi(x) \) a first-order formula in the language of rings as in Proposition 4.11,

\[
P(t(x) \in \varphi(\mathbb{Q}(\zeta_{4p}))) \leq_{p, \varphi} \frac{\log q}{B q^{2/3}} \to 0 \quad (e \to +\infty).
\]

In particular, for almost all \( f \in \mathbb{Z}[X] \) of fixed degree (such as \( f(X) = X^m \) for \( m \geq 2 \) coprime to \( p \)),

\[
P(t(x) \in f(\mathbb{Q}(\zeta_{4p}))) \leq_{p, f} \frac{\log q}{B q^{2/3}} \to 0 \quad (e \to +\infty).
\]

**Proof.** In the symplectic case, this is immediate. In the special linear cases, we get the result for the twisted trace function \( t' : \mathbb{F}_q \to \mathcal{O}_{\lambda}', t'(x) = \alpha_p^e t(x) \). The result for the unnormalized function is obtained as in Section 5.1.1, replacing \( A \) by \( \alpha_p^{-e} A \) in Proposition 4.1, using uniformity. \( \square \)

**Remark 5.12.** In the special linear case, the implicit constant depends on \( p \) both because of the use of Theorem 5.3, and because of the twisting factor \( \alpha_p \).

5.3.3. Galois actions. Note that for

\[
\frac{1}{\sqrt{q}} \sum_{y \in \mathbb{F}_q} e\left( \frac{\text{tr}(xf(y) + h(y))}{p} \right) \chi(g(y))
\]

with \( h(Y) = Y^m \) and \( f(Y) = Y^n \) (\( m, n \in \mathbb{Z} \)), we have \( \sigma_m(t(x)) = t(e^{m-n}x) \) for all \( x \in \mathbb{F}_q^x \), where \( \sigma_m \in \text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \cong \mathbb{F}_p^x \) corresponds to \( e^m \in \mathbb{F}_p^x \) for some \( e \in \mathbb{F}_p^x \). Hence, it again makes sense to study the integer

\[
\frac{|\{ x \in \mathbb{F}_q^x : t(x) \in \varphi(\mathbb{Q}(\zeta_{4p})) \}|(p-1, m-n)}{p-1} \ll_{m,n} \frac{|\{ x \in \mathbb{F}_q^x : t(x) \in \varphi(\mathbb{Q}(\zeta_{4p})) \}|}{p-1}
\]

when \( \varphi(x) \) is a first-order formula in the language of rings. However, doing so requires an estimate of the form (4) uniform in \( p \), for example through a more precise knowledge of the integral monodromy instead of relying on Theorem 5.3.

5.4. Hypergeometric sums. The same methods also apply to the hypergeometric sheaves defined by Katz [Kat90, Chapter 8], generalizing Kloosterman sums: under some conditions, the arithmetic and geometric monodromy groups over \( \mathbb{Q}_\ell \) coincide and are conjugate to \( \text{SL}_n \), without needing to twist (see [PG16b, Proposition 7.7]).

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