Modified Kerr-Schild Method applied to $f(R)$ Gravity

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We propose an adaptation of the Kerr-Schild method by implementing the correspondence relations (mapping) between Ricci-based Gravity (RBG) and General Relativity (GR). Basically, we generate GR known solutions from a canonical metric with a well-defined form and then obtain the configuration of this solution for the Gravity $f(R)$. The new method allows us to perturb the metric associated with GR with a null geodetic vector, but instead of taking us to a new solution described by Einstein’s gravitational sector, it allows us to configure this solution in an RBG.

**Keywords:** Exact Solutions, Mapping, Modified Gravity, Kerr-Schild, General Relativity

I. INTRODUCTION

Several solutions within the scope of Einstein Gravity can be found in the literature, such as the Schwarzschild solution (solution external to the static black hole without charge), Kerr solution (charged black hole with rotation) and/or Reissner-Nordström solution (black hole charged). And although such solutions present uniqueness, which is a theoretical problem, they provide us with the theoretical archetype that allows us to analyze observational data, in order to verify the existence and understand the distribution and location of compact objects. However, solutions in alternative gravity theories have been increasingly recurrent in the scientific community. We can classify them into two classes: 1) metric theories and non-metric theories. With respect to this last class, we are interested in the Ricci-based gravity (RBG), which is a family of (modified) gravitational theories that present a certain correspondence with the matter sector of General Relativity (GR).

We consider presenting the general properties that make up a RBG, as well as its relationship with the GR. Subsequently, we developed the correspondence relationships for the scalar matter sector between RBG and GR. On these relationships, we obtain alternative equations to the (conventional) mapping method, in order to illustrate the reversibility of the method; it is possible to map solutions from GR to RBG and vice versa.

Roughly speaking, we unified the mapping method with the Kerr-Schild method in order to perturb the generic metric derived from Einstein’s gravity, $g_{\mu\nu}$, to lead us to a descriptive metric of another gravity, the RGB. That is, instead of obtaining a metric that is also an exact vacuum solution for the GR, we obtain a metric that is an exact solution for the RBG, assigning the compliance factor that relates the scalar matter fields between the RBG and the GR.

II. RICCI-BASED GRAVITY

Ricci-based gravity theories configure a modified gravity family, which are described by general action

$$S = \frac{1}{2\kappa^2} \int \sqrt{-\tilde{g}} L_G[g_{\mu\nu}, R_{(\mu\nu)}] + S_m(g_{\mu\nu}, \psi_m), \quad (1)$$

where $L_G[g_{\mu\nu}, R_{(\mu\nu)}]$ is the lagrangean of the gravitational sector $g_{\mu\nu}$ is the metric, $R_{(\mu\nu)}$ is the symmetrized Ricci tensor and $\psi_m$ represents the matter fields associated with the action that characterizes the matter sector, $S_m$. $\kappa^2 = 8\pi G$ in GR.

A. Fundamental properties

RBG facilitates physical and mathematical treatment regarding the analysis of field equations and their solutions. To be considered an RBG, the gravitational theory

- Must provide field equations constructed by the Metric-affine formalism;
- Depends on the symmetrized Ricci tensor (which frees us from projective variance);
- Does not consider that matter is explicitly coupled to the connection, which implies that the matter-connection coupling tensor is null $H_{\alpha}^{\nu\beta} = 0$; and
- Allow the Einstein framework to be set up.

In the literature, with regard to the theories that make up the RBG, we can mention the gravities $f(R)$, $f[R, R_{(\mu\nu)}R_{(\mu\nu)}]$ and EiBI (Eddington-inspired Born-Infeld).

For a detailed description of the construction of the action described in the equation (1), as well as understanding the fundamental properties of this family of theories, see [1].
B. Field equations

By varying the action (1) via Metric-affine formalism, we obtain two general and independent field equations:

$$\frac{\partial L_G}{\partial g^{\mu\nu}} - \frac{L_G}{2} = \kappa^2 T^{\mu\nu}$$  \hspace{1cm} (2)

$$\nabla_\mu \left( \sqrt{-q} \frac{\partial L_G}{\partial R_{\mu\nu}} \right) = 0$$  \hspace{1cm} (3)

The (2) equation is the metric equation and is second order. The tensor energy momentum is described as being $T_{\mu\nu} = -\frac{2}{\sqrt{-q}} \frac{\delta S}{\delta g^{\mu\nu}}$. The second equation, (3), is about connection. Note that nullity is justified by our imposed restriction (non-coupling of matter to connection); the variation of the matter action $S_m$ with respect to the connection is null.

III. FUNDAMENTALS FOR RBG-GR MAPPING

Introducing an auxiliary metric $q_{\mu\nu}$ that relates to the equation (3), with the proviso that

$$\nabla_\mu \left( \sqrt{-q} q_{\mu\nu} \right) = \nabla_\mu \left( \sqrt{-q} \frac{\partial L_G}{\partial R_{\mu\nu}} \right),$$  \hspace{1cm} (4)

that is, it is written by the Levi-Civita connection, $\nabla_\mu q_{\mu\nu} = 0$, which implies that the components of the connection are described by Christoffel’s symbols – as in Einstein:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} q^{\lambda\alpha} \left( \partial_\mu q_{\alpha\nu} + \partial_\nu q_{\mu\alpha} - \partial_\alpha q_{\mu\nu} \right),$$  \hspace{1cm} (5)

we can “reproduce” the dimensional form of Einstein’s field equation.

On the metrics $g_{\mu\nu}$ and $q_{\mu\nu}$, they are related through a deformation matrix $\Omega_{\mu\nu}$:

$$q_{\mu\nu} = g_{\mu\nu} \Omega_{\mu\nu}. \hspace{1cm} (6)$$

This deformation matrix depends only on the matter fields (scalar, electromagnetic and/or fluid) and its components depend exclusively on the gravitational model used. Because, it is the gravitational model considered that will provide us with the compliance factor that relates the mapping of the RBG (non-metric theory) with the GR (metric theory). In addition, nonlinearly, as well as the shape of the gravitational sector, are fully mapped to the matter sector.

Due to the RBG-GR [2] correspondence, the RBG enables us to develop the space-time geometry with curvature described by the matter sector of the metric $q_{\mu\nu}$, with the same shape as the field equation of General Relativity, which we call Einstein-frame:

$$q^{\mu\alpha} G_{\alpha\nu} \equiv \tilde{G}_{\mu\nu}(q) = q^{\mu\alpha} \left( R_{\mu\nu}(q) - \frac{1}{2} q_{\mu\nu} R(q) \right) = \kappa^2 \tilde{T}_{\mu\nu}.$$  \hspace{1cm} (7)

And in this way, the modified Einstein tensor, $\tilde{G}_{\mu\nu}(q)$ is related to the matter sector of the RBG through the expression

$$\tilde{G}_{\nu\mu}(q) = \frac{\kappa^2}{|\Omega|^{1/2}} \left[ T^\mu_{\nu} - \left( L_G + \frac{T}{2} \right) q_{\nu\mu} \right]. \hspace{1cm} (8)$$

$|\Omega|^{1/2}$ is the determinant of the deformation matrix. According to our option, we will establish the mapping relationships for the matter sector, as we will generate a scalar matter solution.

A. Mapping the scalar matter sector

Unlike what is done in [2, 3], we chose to obtain the mapping relations considering the case of a real scalar field described by an arbitrary lagrangian $K(Z, \phi)$, where the kinetic term $Z = q^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$. There may be a potential for the scalar field $V(\phi)$, in such a way that the action that describes the matter sector has the form

$$\tilde{S}_m(Z, \phi) = -\frac{1}{2} \int d^4 x \sqrt{-q} K(Z, \phi). \hspace{1cm} (9)$$

This choice allows us to consider the form of the gravitational action of the Ricci-based and at the same time consider the description of the matter sector, as well as the curvature of the GR. Note that besides this last equation does not depend on the connection, its metric variation will lead to an energy-momentum tensor as in (6). That said, by varying the action (9), we have to

$$\delta \tilde{S}_m(Z, \phi) = -\frac{1}{2} \int d^4 x \sqrt{-q} \frac{\delta q^{\mu\nu}}{q^{\mu\nu}} \left( K_2 Z^{\mu\nu} - \frac{K(Z, \phi)}{2} \right),$$

that takes us to the energy-momentum tensor for the matter fields

$$\tilde{T}_{\mu\nu} = K_2 Z^{\mu\nu} - \frac{2}{\kappa^2} K(Z, \phi) \Delta \mu \nu. \hspace{1cm} (11)$$

Where $K_2 \equiv dK/dZ$ and $Z^{\mu\nu} \equiv q^{\mu\alpha} \partial_\alpha \phi \partial_\nu \phi$. Given the dependency of the matrix $\Omega_{\mu\nu}$, and how it can be written as a (nonlinear) function of the tensor $T_{\mu\nu}$ which makes it possible to match the Ricci-based with Einstein. Thus, we assume that the aforementioned matrix can be written as a power series of the energy-momentum tensor described in (11):

$$\Omega_{\mu\nu} = \tilde{f}_0(Z, \phi) \delta_{\mu\nu} + \tilde{f}_1(Z, \phi) \tilde{T}_{\mu\nu} + \tilde{f}_2(Z, \phi) \tilde{T}_{\mu\nu} \tilde{T}^\alpha_{\nu} + \ldots \hspace{1cm} (12)$$

where $\tilde{f}_0, 1, \ldots (Z, \phi)$ are functions of matter fields and explicitly depend on the gravitational model used to describe the dynamics of the sector. Namely, the linearity of the curvature scalar described in the usual General Relativity, $\tilde{R} = \kappa^2 \tilde{T}$, is reproduced for our Einstein framework, $\tilde{R}(\phi) = \kappa^2 \tilde{T}$, when we only work with the term $f_0(Z, \phi) \delta_{\mu\nu}$. This occurs in the Starobinsky model of gravity $f(R)$, and this implies that the Starobinsky model $f(R) = R + \alpha R^2$, is linear with the deformation matrix:

$$\Omega_{\mu\nu} = \tilde{f}_0(Z, \phi) \delta + f_R \delta_{\mu\nu}. \hspace{1cm} (13)$$
Although it seems counterintuitive, the (13) equation allows us to deeply analyze the philosophy behind the RBG-GR correspondence. Beforehand, see that in [2], for example, the deformation matrix is described in terms of matter coupled to the metric $g_{\mu\nu}$, and this implies that the general field equations of Ricci-based are described by the $g_{\mu\nu}$ metric. Through our construction, notice that our RBG’s field equations are still described in terms of the metric $g_{\mu\nu}$, however, we chose to build the deformation matrix in terms of the metric $g_{\mu\nu}$ (which describes Einstein’s framework) to show that just as the mapping can be reversible (from RBG to GR or from GR to RBG) it is also possible to obtain the correspondence relationships in terms of the auxiliary metric. For this to be possible, we need to be careful with the description of the curvature scalar; whether it depends on $g_{\mu\nu}$ or $q_{\mu\nu}$.

a. Analyzing: If we consider in the equation (13) that $f(R) = R$, note that $f_R = 1$ and the deformation matrix is really linear with the identity matrix. However, if we chose to use a modified gravity descriptive function, the Starobinsky model, for example, the curvature scalar should be described using the correspondence relations between a lagrangean of matter described by the metric $g_{\mu\nu}$ and the matter Lagrangian described by $q_{\mu\nu}$. We already know this last matter lagrangesan, which is the one described by the action $S_m(Z, \phi)$. Thus, action $S_m(X, \phi)$ has the same form as the equation (9), except for the matter that is coupled to the metric $g_{\mu\nu}$ of Ricci-based. Therefore, the action

$$S_m(X, \phi) = -\frac{1}{2} \int d^4x \sqrt{-g} P(X, \phi),$$

(14)

takes us to the energy-momentum tensor of RBG, in the form

$$T^\nu_\mu = P_X X^\nu_\mu - \frac{1}{2} P(X, \phi) \delta^\mu_\nu,$$

(15)

for $P_X = dP/dX$ and $X^\nu_\mu = g^{\mu\alpha} \partial_\alpha \phi \partial_\nu \phi$.

b. Correspondence relations Because the kinetic term $Z$ is idempotent, any power of the energy-momentum tensor at (11) can be written as a linear combination of $\delta$ with the kinetic term $\hat{Z}$. In general, we are left with

$$\hat{\Omega}^\nu_\mu = f_0(Z, \phi) \delta^\nu_\mu + \hat{f}_1(Z, \phi) Z^\nu_\mu + \hat{f}_2(Z, \phi) Z^\nu_\mu Z^\alpha_\nu + \cdots,$$

(16)

and as in this case, the relation of the metrics with the scalar fields is given by

$$g^{\mu\alpha} \partial_\alpha \phi = g^{\mu\alpha} (\Omega^{-1})^\lambda_\alpha \partial_\lambda \phi \rightarrow (\Omega^{-1})^\lambda_\alpha \partial_\lambda \phi = (\hat{C} + \hat{D}Z) \partial_\alpha \phi,$$

we concluded that

$$Z^\nu_\mu = (\hat{C} + \hat{D}Z + \hat{E}Z^2 + \cdots) X^\nu_\mu,$$

(17)

Where $\hat{C} - \hat{D}...$ are descriptive functions of the gravitational model and depend on it. The simplest case can be studied by the relation

$$Z^\nu_\mu = (\hat{C} + \hat{D}Z) X^\nu_\mu.$$

(18)

With respect to the Starobinsky model, we identified that $C(Z, \phi) = [\Omega]^{-\frac{2}{3}} f_R$ and $D(Z, \phi) = 0$. Thus, the relationship between the objects $X$ and $Z$ is such that

$$X = \frac{f_R}{|\Omega|^{-1/2}} Z = f_R^{-1} Z,$$

(19)

where $|\Omega|^{-1/2} = f_R$, due to matrix construction. Since the curvature scalar is associated with $X$, de

$$R|\Omega|^{-1/2} = \kappa^2(2P - XP_X),$$

(20)

we write

$$R = |\hat{\Omega}|^{1/2} \kappa^2 \left[f_R|\hat{\Omega}|^{1/2} Z - 4V(\phi)\right],$$

(21)

and with this, the correspondence relations for the Lagrangean densities previously established in [2], have the form

$$XP_X = ZK_Z|\hat{\Omega}|^{-1/2}$$

(22)

$$\rightarrow P(X, \phi) = K(Z, \phi)|\hat{\Omega}|^{-1/2}.$$  

(23)

Note that the sign of the exponent of the determinant follows the construction of the matrix described in (12) and, consequently, follows the relationship between the objects $Z-X$, (18).

Finally, we concluded that

$$P(X, \phi) = \frac{1}{|\hat{\Omega}|^{1/2}} \left(\frac{f(R)}{\kappa^2} + ZK_Z - K\right)$$

(24)

and

$$P_X = \frac{\hat{C} + \hat{D}Z}{|\hat{\Omega}|^{1/2}} K_Z.$$  

(25)

In what follows, we are interested in obtaining a solution for the matter density $K = K(Z, \phi)$,

$$K(Z, \phi) = \frac{f(R)}{\kappa^2} + |\hat{\Omega}|^{1/2}(XP_X - P),$$

(26)

since it is the matter sector of the Einstein-frame. In order to explain the conformal factor of the Starobinsky model, we have to consider the equations (13) and (19). Note that to modify Einstein’s sector of matter, we need to spell out $f_R$ in terms of the kinetic term $Z$. In this sense, due to the construction of our deformation matrix, we need to change the inverse relationship between the kinetic terms $Z$ and $X$. In our case, $Zf_R^{-1} = X$, and in conventional mapping (RBG as the original severity) the terms are related by $Zf_R = X$. Thus, the factors $f_R^{-1}(q)$ clearly tell us which metric to adopt in relation to the construction of relations. And this allows us to establish, when convenient, which matter is considered for the construction of mapping relations. If we use the factor $f_R^{-1}$ our correspondence relations will be written in terms of the curvature $R(q)$. If we adopt $f_R$, we describe the relations for $R(g)$. The transition can
be easily demonstrated. For reasons of practicality and identification, let us consider
\[ Z_{q\rightarrow\text{map}} f_R^{-1} = X_{q\rightarrow\text{map}} \rightarrow f_R = X_{\text{map}}/Z_{\text{map}} \] (27)
then,
\[ \frac{X_{q\rightarrow\text{map}}}{Z_{q\rightarrow\text{map}}} = \frac{Z_{\text{map}}}{X_{\text{map}}} \rightarrow X_{\text{map}} = Z_{\text{map}} f_R. \] (28)
In other words, to obtain \( f_R \) in terms of the Lagrangian of matter \( P(X, \phi) \) introducing the kinetic term of the GR, \( Z \), as seen in (28), we just need to change \( Z \) for \( X \) and \( X \) for \( Z \). This “trick” is quite efficient, especially when the deformation matrix \( \Omega^{\alpha}_{\nu} \) is not linear the identity matrix, as in the case of quadratic gravity \( f(R, R^{\mu\nu} R_{\mu\nu}) \). Therefore, as we are interested in clarifying \( f_R \), we identified that
\[ f_R = 1 + 2\alpha R = (1 - 2\alpha\kappa^2 Z)^{-1}. \] (30)

IV. MODIFIED KERR-SCHILD (MK-S)

The Kerr-Schild method consists of generating solutions for black holes in General Relativity using canonical geometry, the Minkowski metric. As it is possible to map the solutions of Einstein’s gravity to RBGs [2], we propose to apply the Kerr-Schild method to the gravity \( f(R) \) in order to find solutions related to it and obtain the mapped solution of the GR.

The original form of the method is
\[ g_{\mu\nu} = \eta_{\mu\nu} + \epsilon H V_{\mu} V_{\nu}, \] (31)
where \( g_{\mu\nu} \) is the new metric, \( \eta_{\mu\nu} \) is the Minkowski metric, \( \epsilon \) the linearity parameter, \( H \) a scalar function, and \( V_{\mu} \) a null geodetic vector that must satisfy the following conditions:
\[ V_{\mu} V_{\nu} = 0, \] (32)
\[ (\nabla_{\nu} V_{\mu}) V_{\nu} = 0. \] (33)
Although the factor \( \epsilon H V_{\mu} V_{\nu} \) is a perturbation in spacetime, the new metric remains linear with the field equations.

In the mapping method [2–4], one of the main relevant relationships is the geometric relationship between the metrics, (6), where the auxiliary metric is \( q_{\mu\nu} \) describes the geometry of Einstein’s gravitational spacetime, the metric \( g_{\mu\nu} \) describes the spacetime associated with an RBG and \( \Omega^{\alpha}_{\nu} \) is a matrix deformation that relates the metrics associated with their respective gravitational theories. In this sense, given the generic form of the method
\[ \bar{g}_{\mu\nu} = g_{\mu\nu} + \epsilon H V_{\mu} V_{\nu}, \] (34)
we will consider that the new metric of (31) (associated with Einstein’s gravity), coincides with the auxiliary metric \( q_{\mu\nu} \) of the mapping method. We chose this choice due to the fact that the RBG-GR correspondence relations are arranged in the perspective that RBG describes the gravitational dynamics in the real context (besides the theoretical one). So from (6), we have to
\[ \eta_{\mu\nu} + \epsilon HV_{\mu} V_{\nu} = \eta_{\mu\nu} + \epsilon 2HV_{\mu} V_{\nu} = \eta_{\mu\nu} + \epsilon HV_{\mu} V_{\nu}. \] (36)
With the implementation of the Kerr-Schild method, the perturbation in the Einstein-frame changes the dynamics of the field equations; the relation (36) tells us that if \( \epsilon \neq 0 \), the perturbative term takes us to an alternative gravity. And if \( \epsilon = 0 \), we return the mapping relationship.

A. Properties and motivations

As the matrix \( \Omega^{\alpha}_{\nu} \) depends solely on matter and is directly related to the energy-momentum tensor, we guarantee that the perturbation in GR leads us to RBG. The main motivation for implementing the method is that, in this way, we will be able to expand the archetype of applications of the mapping method in relation to Einstein’s gravity solutions, since the Kerr-Schild method allows us to generate exact (known) solutions from back background metric. An important point to be mentioned is that RBG’s are described via affine metric formalism and Einstein’s Relativity by metric formalism. As Einstein’s gravity is invariant under the metric-affine formalism (due to the connection coincide with Levi-Civita, with coordinates described by the Christoffel symbols), we assume that the linearity of Einstein’s field equations can be mapped to the RBG equations. That is, although the scalar of curvature GR is linear to the energy-momentum tensor and not to RBG, we assume that the non-linearity in RBG is given by the perturbative factor.

V. GENERATING EXACT SOLUTIONS

We will analyze the application of K-SM in the form of canonical Schwarzschild coordinates of General Relativity. Later, we will analyze this configuration (or new solution) from the perspective of \( f(R) \) gravity.

Remember that when we impose the condition that the parameter \( \epsilon \neq 0 \), we are guaranteeing that the perturbation will lead to another gravitational dynamic. And
that, to analyze a solution in the context of RBG’s, it is enough to make explicit the metric $g_{\mu\nu}$ of the equation (36). However, as the composition of the matrix follows the gravitational model used, we do not need, initially, to explain the geometry of the RBG.

As can be seen, the Kerr-Schild method was applied to generate BTZ (Bañados-Teitelboim-Zanelli) and Reissner-Nordström [7] solutions. For the present work, we will use the Starobinsky model, but we will not generate a solution minimally coupled to Maxwell’s electromagnetism, since the solution coincides with Einstein’s gravity [8]; we will explore the extension of the method starting from the line element described by any metric of type $g_{\mu\nu}$, as described in the (36) equation. Using Schwarzchild’s canonical line element [9–11],

$$ds^2_{GR} = -e^\nu dt^2 + e^\lambda dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

(37)

where $\nu$ and $\lambda$ are metric functions of the radial coordinate, the equations (32) and (33) lead us to the following null (and geodetic) vector:

$$V_\mu = (F, F, 0, 0).$$

(38)

$F$ is any constant. The new metric associated with the disturbance is now described as

$$\bar{g}_{\mu\nu} = \begin{pmatrix} -e^\nu + eHF^2 & -eHF^2 & 0 & 0 \\ -eHF^2 & e^\lambda + eHF^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}. $$

(39)

And according to [9], because of the special way that the tensor energy moment can be written

$$r^2 \bar{T}_{\mu\nu} = \xi \left( \partial_\mu U \partial_\nu U - \frac{1}{2} \bar{g}_{\mu\nu} \partial^\alpha \partial_\alpha U \partial_\beta U \right),$$

(40)

the components of the Einstein tensor are related to the equation (40) by the following relation

$$R_{\mu\nu}(q) = -\xi \partial_\mu U \partial_\nu U.$$  

(41)

$U$ is a scalar that depends on $r$ and $t$ and $\xi$ is a constant parameter. The equation (41) implies that $g^{\alpha\beta} U_{,\alpha\beta} = U_{\mu\nu} = 0$ and we identify that the scalar function $H$ has the form

$$H = \frac{1}{2} \left( \nu'^2 + \frac{\nu'^2}{2} - \frac{\nu' \nu''}{2} \right) = \xi U'^2 e^{\lambda - \nu} - \frac{\nu'}{r}$$

(42)

and that $F \equiv U$. Where $U'$ denotes the differentiation in $r$ and $\bar{U}$ in $t$. From the field equations and according to the relations of (42), the scalar $U$ is associated with the metric functions that follow the exponential and that $U' \bar{U} = 0$. With respect to the new line element, we have identified that $\nu = \alpha F$ and $e^\lambda = r^4 F^2 e^{\alpha F}/\omega^2$. Choosing the same coordinate transformations as in [9]

$$r = r(\bar{r}) = r(U) \rightarrow U_{\mu} = (1, 0, 0, 0),$$

(43)

the constant $\omega$ disappears due to the linear relationship of $(r)$ and $t$, in accordance with the transformation (43), such that after the perturbation the line element described in (37) takes the following form:

$$ds^2_{GR} = -e^{\alpha r} dt^2 + e^{\nu r} dr^2 + \frac{d\theta^2 + \sin^2 \theta d\phi^2}{W^2}.$$  

(44)

The metric function $W$, in the asymptotically flat regime, is equivalent to

$$W = e^{\alpha r/2} \sinh (\gamma r),$$

(45)

where the constant $\alpha$ is the netonian mass of the solution at the asymptotic limit, $\alpha = -2M$, and $\gamma \equiv \sqrt{\alpha^2 + 2k^2}/2$.

### A. Modifying the line element

The line element described by the equation (44) is a spherically symmetric static solution for scalar matter fields, Wyman’s solution. Recently, this solution was mapped and new compact objects were found (new solutions)[1, 12]. The mapped solution can be generated taking into account the modified equation of the Kerr-Schild method, equation (36). Thus, from the mapping relations [2], it is easy to see that $\Omega^\alpha = f_R \delta^\alpha_{\nu}$. By Krocke’s property, making $\alpha = \nu$, we can make explicit the metric associated with RBG

$$g_{\mu\nu} = f_R^{-1} (q_{\mu\nu} + eHV_\mu V_\nu),$$

(46)

and which, consequently, implies that

$$ds^2_{f(R)} = f_R^{-1} ds^2_{GR},$$

(47)

which is the line element mapped to the $f(R)$ gravity.

### VI. CONCLUSION

The Kerr-Schild tensor perturbation method and the RBG-GR Mapping method are algebraic alternatives to generate exact solutions. Because RBG is a class of modified gravity theories, RBG’s retrieve Einstein Gravity in vacuum, $T_{\mu\nu} = 0$; RBG’s solutions fall to GR, in a vacuum. And that was the main motivation to consider the hybrid application of both methods.

We made a brief analysis of the RBG’s family and then explained the physical foundations behind the mapping method and the correspondence between RBG and GR. Later, we develop the modified Kerr-Schild method and apply it to the simple form of the Schwarzschild line element and obtain the Wyman solution. Finally, we show how it is possible to apply the Kerr-Schild method to a modified gravity, in particular an RBG, since we can “generate” the same shape as the line element that was mapped.
Therefore, the scope of this work concerns an alternative way, in addition to the Kerr-Schild and Mapping method, when separated, to generate exact solutions from metrics that describe different gravitational scenarios.

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