A REMARK ON THE FARRELL-JONES CONJECTURE

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ABSTRACT. Assuming the classical Farrell-Jones conjecture we produce an explicit (commutative) group ring $R$ and a thick subcategory $C$ of perfect $R$-complexes such that the Waldhausen $K$-theory space $K(C)$ is equivalent to a rational Eilenberg-Maclane space.

1. INTRODUCTION

Our main goal is to prove the following theorem

**Theorem 1.1** (Main result). There exists a commutative ring $R$ and a thick subcategory $C$ of $\text{Perf}(R)$ such that the space $K(C)$ of Waldhausen $K$-theory is equivalent to an Eilenberg-Maclane space.

In our opinion this theorem seems counterintuitive at the first glance. There is very few examples of rings for which the algebraic $K$-theory groups were computed in all degrees (e.g. the $K$-theory of finite fields computed by Quillen). Another source for such computation is the Farrell-Jones conjecture. We will compute explicitly the $K$-groups for some particular (commutative) group ring.

**Conjecture 1.2** (Classical Farrell-Jones [4]). For any regular ring $k$ and any torsionfree group $G$, the assembly map

$$H^n(BG; K(k)) \to K_n(k[G])$$

is an isomorphism for any $n \in \mathbb{Z}$.

We refer to [8] for the definition of the $K$-theory spectrum $K(k)$ of a ring $k$. We recall that $BG$ is the classifying space of the group $G$ and that $k[G]$ is the associated group ring with a natural augmentation $k[G] \to k$. We recall also that $H^n(BG; K(k))$ is the same thing as the $n$-th stable homotopy group of the spectrum $BG_+ \wedge K(k)$. More precisely the assembly map is induced by the following map of spectra

$$BG_+ \wedge K(k) \to K(k[G]).$$

The conjecture admits a positive answer in the case where $k$ is regular ring and $G$ is a torsionfree abelian group, it is a particular case of the main result of [9].

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2. Fibre sequence for Waldhausen K-theory

Notation 2.1. We fix the following notations:

1. Let \( \mathcal{E} \) be any (differential graded) ring. Let \( \text{Mod}_\mathcal{E} \) denotes the (differential graded) model category of \( \mathcal{E} \)-complexes. And \( \text{Perf}(\mathcal{E}) \) denotes the (differential graded) category of perfect (i.e. compact) \( \mathcal{E} \)-complexes.

2. For any (differential graded) ring map \( \mathcal{E} \to A \), \( \text{Perf}(\mathcal{E}, A) \) denotes the thick subcategory of \( \text{Perf}(\mathcal{E}) \) such that \( M \in \text{Perf}(\mathcal{E}, A) \) if and only if \( M \otimes^L_{\mathcal{E}} A \simeq 0 \).

Lemma 2.2. Let \( \mathcal{E} \to A \) be a morphism of (differential graded) rings such that \( A \otimes^L_{\mathcal{E}} A \simeq A \), then
\[
\text{K}(\mathcal{E}, A) \to \text{K}(\mathcal{E}) \to \text{K}(A)
\]
is a fibre sequence of (infinite loop) spaces where \( \text{K}(\mathcal{E}, A) := \text{K}(\text{Perf}(\mathcal{E}, A)) \).

Proof. Let \( w \) be the class of equivalences in \( \text{Mod}_\mathcal{E} \) defined as follows: a map \( P \to P' \) is \( w \)-equivalence if and only if \( A \otimes^L_{\mathcal{E}} P \to A \otimes^L_{\mathcal{E}} P' \) is a quasi-isomorphism (q.i.).

The left Bousfield localization [2] of the model category \( \text{Mod}_\mathcal{E} \) with respect to the class \( w \) exists and it is denoted by \( Lw\text{Mod}_\mathcal{E} \). Since \( A \otimes^L_{\mathcal{E}} A \simeq A \) we obtain a Quillen equivalence
\[
Lw\text{Mod}_\mathcal{E} \xrightarrow{\mathcal{E}} \text{Mod}_A
\]
More precisely, for any \( M \in \text{Mod}_A \) the (derived) counit map \( A \otimes^L_{\mathcal{E}} U(M) \to M \) is a quasi-isomorphism (because it is a quasi-isomorphism for \( A = M \), the functor \( \mathcal{E} \otimes_A - \) commutes with homotopy colimits and \( A \) is a generator for the homotopy category of \( \text{Mod}_A \)). On another hand, the derived unit map \( P \to A \otimes^L_{\mathcal{E}} U(P) \) is an equivalence in \( Lw\text{Mod}_\mathcal{E} \) for any \( P \in \text{Mod}_\mathcal{E} \) by definition. In particular the subcategory of compact objects in \( Lw\text{Mod}_\mathcal{E} \) is equivalent to \( \text{Perf}(A) \). Thus, by [7, theorem 3.3], we have an equivalence of the \( K \)-theory spaces
\[
\text{K}((\text{Perf}(\mathcal{E}), w)) \simeq \text{K}((\text{Perf}(A), \text{q.i.})) := \text{K}(A).
\]
By Waldhausen fundamental theorem [8, Theorem 1.6.4], the sequence of Waldhausen categories
\[
(\text{Perf}(\mathcal{E}), \text{q.i.}) \to (\text{Perf}(\mathcal{E}), \text{q.i.}) \to (\text{Perf}(\mathcal{E}), w)
\]
induces a fibre sequence of \( K \)-theory spaces
\[
\text{K}((\text{Perf}(\mathcal{E}), \text{q.i.})) \to \text{K}(\mathcal{E}) \to \text{K}(A)
\]
where \( \text{Perf}(\mathcal{E})^w \) is the full subcategory of \( \text{Perf}(\mathcal{E}) \) such that \( E \in \text{Perf}(\mathcal{E})^w \) if and only if \( A \otimes^L_{\mathcal{E}} E \simeq 0 \). It is obvious by definition that \( \text{Perf}(\mathcal{E})^w = \text{Perf}(\mathcal{E}, A) \).

Hence
\[
\text{K}(\mathcal{E}, A) \to \text{K}(\mathcal{E}) \to \text{K}(A)
\]
is a homotopy fibre sequence of spaces.

\( \Box \)

A similar result can be found in [5, Theorem 0.5] and in [1, Lemma 5.1].
3. Farrell-Jones conjecture

**Notation 3.1.** We fix the following notations:

1. $k = \mathbb{F}_2$ is the finite field with two elements.
2. $R$ is the group algebra $k[Q]$, where $Q$ is the additive abelian group of rational numbers.

**Proposition 3.2.** If $V$ is a rational vector space and $A$ is a finite abelian group then

$$H_*(BV; Z) = \begin{cases} 
Z & \text{if } n = 0 \\
V & \text{if } n = 1 \\
0 & \text{else}
\end{cases}$$

and

$$H_*(BV; A) = \begin{cases} 
A & \text{if } n = 0 \\
0 & \text{else}
\end{cases}$$

**Lemma 3.3.**

$$\pi_n K(R) := K_n(R) = \begin{cases} 
K_n(k) & \text{if } n \neq 1 \\
Q & \text{if } n = 1
\end{cases}$$

*Proof.* By Quillen theorem [6], the algebraic $K$-theory of the finite field $k$ is given by

$$K_n(k) = \begin{cases} 
Z & \text{if } n = 0 \\
0 & \text{if } n \text{ even} > 0 \\
Z/(2^j - 1) & \text{if } n = 2j - 1 \text{ and } j > 0
\end{cases}$$

Since $Q$ is a rational vector space and $K_n(k)$ are finite abelian groups (for $n > 0$) then by proposition 3.2, we have that

$$H_p(BQ; K_q(k)) = \begin{cases} 
Q & \text{if } p = 1 \text{ and } q = 0 \\
K_q(k) & \text{if } p = 0 \text{ and } q \geq 0 \\
0 & \text{else}
\end{cases}$$

The second page $E_{p,q}^2 = H_p(BQ; K_q(k))$ of the converging Atiyah-Hirzebruch spectral sequence [4]

$$H_p(BQ; K_q(k)) \Rightarrow H_{p+q}(BQ; K(k))$$

has graphically the following shape:
where the differentials $d^2 : E^2_{p,q} \to E^2_{p-2, q+1}$ are obviously identical to 0. It means that the spectral sequence collapses, hence in our particular case it implies that

$$H_p(BQ; K_q(k)) = H_{p+q}(BQ; K(k)).$$

Since the Farrell-Jones conjecture is true in the case of torsion-free abelian groups [9], we obtain that

$$K_n(R) \cong H_n(BQ; K(k)) = \begin{cases} K_n(k) & \text{if } n \neq 1 \\ \mathbb{Q} & \text{if } n = 1 \end{cases}$$

\[\square\]

**Lemma 3.4.** There is a fibre sequence of Waldhausen $K$-theory spaces given by

$$K(R, k) \to K(R) \to K(k)$$

**Proof.** Since $k$ is a finite field (in particular a finite abelian group) and $\mathbb{Q}$ is a rational vector space, it follows by [12] that

$$H_n(BQ; k) = \text{Tor}_n^R(k, k) = \begin{cases} k & \text{if } n = 0 \\ 0 & \text{else} \end{cases}$$
therefore $k \otimes^L_R k \simeq k$. The conclusion follows from lemma 2.2 when $k = \mathcal{A}$ and $R = \mathcal{E}$.

**Theorem 3.5.** With the same notation, the $K$-theory space of the thick subcategory $\text{Perf}(R, k)$ is equivalent to the Eilenberg-MacLane space $BQ$.

**Proof.** Since the Farrell-Jones conjecture is true for $G = \mathbb{Q}$. Combining lemma 3.4 and lemma 3.3, we have by Serre’s long exact sequence that the homotopy groups of the homotopy fibre $K(R, k)$ of $K(R) \to K(k)$ are given by

$$K_n(R, k) = \begin{cases} \mathbb{Q} & \text{if } n = 1 \\ 0 & \text{else} \end{cases}$$

and by definition $K(R, k) := K(\text{Perf}(R, k))$, hence we have proved the main theorem 1.1. □

**References**

1. Hongxing Chen and Changchang Xi. Recollements of derived categories II: Algebraic K-theory. arXiv:1212.1879, 2012.
2. Philip S. Hirschhorn. *Model categories and their localizations*. Number 99. American Mathematical Soc., 2009.
3. Mark Hovey. *Model categories*. Number 63. American Mathematical Soc., 2007.
4. Wolfgang Lück. K-and L-theory of group rings. arXiv:1003.5002, 2010.
5. Amnon Neeman and Andrew Ranicki. Noncommutative localisation in algebraic K-theory I. *Geometry & Topology*, 8(3):1385–1425, 2004.
6. Daniel Quillen. On the cohomology and K-theory of the general linear groups over a finite field. *Annals of Mathematics*, pages 552–586, 1972.
7. Steffen Sagave. On the algebraic K-theory of model categories. *Journal of Pure and Applied Algebra*, 190(1):329–340, 2004.
8. Friedhelm Waldhausen. Algebraic K-theory of spaces. In *Algebraic and Geometric Topology*, pages 318–419. Springer, 1985.
9. Christian Wegner. The Farrell–Jones conjecture for virtually solvable groups. *Journal of Topology*, 8(4):975–1016, 2015.

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