Edge sampling is an important topic in network analysis. It provides a natural way to reduce network size while retaining desired features of the original network. Sampling methods that only use local information are common in practice as they do not require access to the entire network and can be parallelized easily. Despite promising empirical performance, most of these methods are derived from heuristic considerations and therefore still lack theoretical justification. To address this issue, we study in this paper a simple edge sampling scheme that uses network local information. We show that when local connectivity is sufficiently strong, the sampled network satisfies a strong spectral property. We quantify the strength of local connectivity by a global parameter and relate it to more common network statistics such as clustering coefficient and Ricci curvature. Based on this result, we also derive a condition under which a hypergraph can be sampled and reduced to a weighted network.

1. Introduction

Network analysis has become an important area in many research domains. It provides a natural way to model and analyze data in the presence of complex interdependence among entities. A network typically consists of a set of nodes representing the entities of interest and a set of edges between nodes encoding the relations between the nodes. For example, in a social network such as Facebook or Twitter, nodes are users and there is an edge between two users if they are friends. Studying the structure of a network provides valuable information about how entities interact and may help predict the formation of different groups [12, 10].

As real-world networks are often very large, it is difficult and often impossible to store or even get access to the entire data set. It is therefore desirable to preprocess the data to reduce the network size before performing any analysis. A natural method that has been studied in network literature is to reduce the number of edges, also known as graph sparsification [6, 41, 40]. For a network $G = (V, E)$ with the set of nodes $V$ and the set of edges $E$, the adjacency matrix $A$ is a symmetric matrix with entries $A_{ij} = 1$ if there is an edge between node $i$ and node $j$, and $A_{ij} = 0$ otherwise. If $\varepsilon$ is sufficiently large so that $\Omega(n \log n)$ edges are retained, then by a concentration inequality [30],

$$\|A_\varepsilon - \varepsilon A\| = O(\varepsilon \|A\| \log n).$$

Here $A_\varepsilon$ is the adjacency matrix of the sparsified network and hereafter $\|\cdot\|$ denotes the operator norm. The advantage of this sampling scheme is that it can be done separately for every edge without additional information from other edges. However, this method
only preserves cuts of large sets of nodes \cite{41}; here, the cut of a set of nodes is the number of edges between that set and its complement in \( V \).

Another sampling scheme that approximately preserves all cuts is proposed by \cite{40}. Instead of using the same probability \( \varepsilon \) as in bond percolation model, each edge is now sampled with a probability proportional to its effective resistance. Under this sampling scheme, the (weighted) sparsified network \( H \) of \( G \), also known as the spectral sparsifier, satisfies

\[
(1 - \varepsilon)x^\top L_G x \leq x^\top L_H x \leq (1 + \varepsilon)x^\top L_G x
\]

for all vectors \( x \). Here \( L_G = D - A \) denotes the Laplacian of \( G \), where \( D \) is the diagonal matrix with degrees \( d_i = \sum_{j \in V} A_{ij} \) on the diagonal; similarly, \( L_H = D_H - W_H \) denotes the Laplacian of \( H \), where \( W_H = (w_{ij}) \) is the weighted adjacency matrix of \( H \) and \( D_H \) is the diagonal matrix with weighted degrees \( \sum_{j=1}^n w_{ij} \) on the diagonal. Despite the strong spectral property, this method requires access to the entire network for computing effective resistances of edges, which may be prohibitive in practice. Also, the computation involves a complicated linear system solver of Spielman and Teng that is not easy to implement in practice. Although some improvements of \cite{40} are now available, they still rely on complicated linear system solvers \cite{19, 18}.

It has been observed that many real networks have very strong non-Euclidian local structure \cite{46, 43}. This reflects the belief that incident nodes exhibit the transitivity property: if \( i \) and \( j \) are connected and \( j \) and \( k \) are connected then it is likely that \( j \) and \( k \) are also connected. One way to quantify the transitivity is via clustering coefficient \cite{46}. For each node \( i \in V \), the local clustering coefficient of \( i \) is defined as the ratio between the number of triangles containing \( i \) and the maximum number of triangles it can form with incident nodes

\[
c_i = \frac{|\{(j,k) \in E : (i,j) \in E, (i,k) \in E\}|}{d_i(d_i - 1)/2}.
\]

The clustering coefficient of a network \( G \) is the average of all local clustering coefficients

\[
c = \frac{1}{|V|} \sum_{i \in V} c_i.
\]

Another measure of network transitivity that has recently attracted much attention is the Ricci curvature \cite{4, 17, 24, 24}. In Riemannian geometry, Ricci curvature is a fundamental quantity that measures the degree to which the local geometry of a manifold deviates from Euclidian geometry. It is well known that a manifold has positive Ricci curvature if and only if the geodesic distance between any two close points is larger than the optimal transportation distance between two small balls around these points \cite{45}. Based on this property, the notion of Ricci curvature has been extended to metric spaces by \cite{31}. In particular, when the metric space is a network equipped with the geodesic distance, Ricci curvature is closely related to the local clustering coefficient \cite{17}.

Network local information has been used by several edge sampling methods that aim at preserving certain features of networks such as number of connected components, diameter, homophily, node centrality measures or community structure \cite{28}. These methods sample each edge of a network according to certain edge scores that depend for instance on Jaccard similarity score \cite{36}, the number of triangles \cite{14} or the number of quadrangles containing the edge \cite{29}; see also \cite{14} for methods based on other local measures. Although these methods have been empirically shown to perform well and can be parallelized easily, to our best knowledge, there is still no theoretical guarantee of their performance. It is
also unclear if other features of networks (besides the targeting features considered) are preserved.

To address this issue, we study in this paper a simple edge sampling scheme similar to methods that use Jaccard similarity or number of triangles [36, 14]. Specifically, we sample each edge \((i, j) \in E\) of a network \(G = (V, E)\) with probability proportional to the number of common neighbors that \(i\) and \(j\) have. For simplicity, we assume that numbers of common neighbors are known for all edges. In practice, these numbers can be computed exactly in parallel fashion [32, 14] or approximated by hashing [36, 37, 38]. We show that when the local connectivity of a network is sufficiently strong, our sampling method satisfies the spectral property (2.3); this provides theoretical evidence supporting edge sampling methods based on local information [36, 29, 14].

Intuitively, as numbers of common neighbors increase, the network transitivity becomes stronger and our sampling scheme becomes more similar to the sampling scheme using effective resistances of edges. In contrast, as numbers of common neighbors decreases, the sampling scheme become more similar to bond percolation. We quantify this intuition using a global measure and relate it to the network transitivity and Ricci curvature. Based on this result, we also derive a condition under which a hypergraph can be sampled and reduced to a weighted graph. While the sparsified network under our sampling scheme approximately preserves all cuts, it is interesting that the approximation accuracy only depends on the average of local features. This result also confirms the usefulness of network local information observed for example in the context of community detection [33, 36, 35, 9, 11].

2. Edge sampling using common neighbors

We are interested in finding a sparsifier of \(G = (V, E)\) that satisfies the spectral notion of similarity introduced by Spielman and Teng [41, 42]. A sparsifier of \(G\) is a weighted sparse network \(H = (V, E_H, W_H)\) such that \(E_H \subseteq E_G\) and (1.1) holds for any vector \(x \in \mathbb{R}^n\). Note that (1.1) is equivalent to \((1-\varepsilon)L_G \preceq L_H \preceq (1+\varepsilon)L_G\) and we write \(X \preceq Y\) if \(Y-X\) is a semi-positive definite matrix.

For each edge \((i, j) \in E\), let \(T_{ij}\) be the number of common neighbors of \(i\) and \(j\). To form a sparsifier \(H = (V, E_H, W_H)\), we sample \(m\) edges of \(G\) independently according to a multinomial distribution with probabilities

\[
p_{ij} = \frac{2}{\sum_{(i,j) \in E_G} T_{ij} + 2}.
\]

(2.1)

If an edge \((i, j) \in E_G\) of the original graph \(G\) is selected, we add an edge to \(H\) with the weight \((mp_{ij})^{-1}\). Weights of edges of \(H\) are summed if edges are selected more than once.

**Theorem 2.1** (Edge sampling using common neighbors). Consider an undirected connected graph \(G = (V, E)\). For each edge \((i, j) \in E\) denote by \(T_{ij}\) the number of common neighbors of \(i\) and \(j\). Let \(\varepsilon \in (0, 1)\) and denote

\[
\alpha = \frac{1}{n} \sum_{(i,j) \in E_G} \frac{2}{T_{ij} + 2}.
\]

(2.2)

Form a weighted graph \(H\) by sampling \(\alpha n \log n / \varepsilon^2\) edges of \(G\) as described above. Then with probability at least \(1 - 1/n\) the following holds:

\[
(1 - \varepsilon)L_G \preceq L_H \preceq (1 + \varepsilon)L_G.
\]

(2.3)
Parameter $\alpha$ in (2.2) measures the average strength of network local connectivity. When the local connectivity is strong, i.e. $\alpha = O(1)$, Theorem 2.1 shows that we can preserve network topology if we locally sample and retain $O(n \log n)$ edges. In contrast, if the local connectivity is weak (for example when $T_{ij} = O(1)$) then $p_{ij}$ are similar, therefore the sampling scheme is similar to bond percolation. Table 1 shows the value of $\alpha$ and clustering coefficient $c$ for some well-known real-world networks. Note that while these networks are relatively sparse, their values of $\alpha$ are quite small, which suggests that real-world networks have strong local connectivity. The proof of Theorem 2.1 is given in Appendix A.

| Data                          | $n$  | Average degree | Clustering coefficient | $\alpha$ |
|-------------------------------|------|----------------|------------------------|-----------|
| Karate Club [47]              | 34   | 4.59           | 0.59                   | 1.40      |
| Dolphins [25]                 | 62   | 5.13           | 0.30                   | 1.58      |
| Political Blogs [1]           | 1222 | 27.36          | 0.36                   | 3.04      |
| Ego-Facebook [26]             | 4039 | 43.69          | 0.62                   | 1.96      |
| Astro Physics Collaboration [22] | 18772 | 21.10          | 0.68                   | 1.94      |
| Enron Email [21]              | 36692 | 10.02          | 0.72                   | 1.56      |

Table 1. Statistics of some real-world networks.

3. LOWER BOUND ON PARAMETER $\alpha$

The following lemma provides a lower bound for parameter $\alpha$ in term of clustering coefficient $c$ and degrees $d_i$.

**Lemma 3.1** (Lower bound on parameter $\alpha$). For any undirected connected graph the following holds:

$$\alpha \geq \frac{1}{4c + \frac{n}{2} \sum_{i \in V} \frac{1}{d_i}}.$$ (3.1)

**Proof.** For each node $i$, denote by $N_i$ and $t_i$ the set of neighbors of $i$ and the number of triangles that contain $i$, respectively. Using Lemma A.2, we have

$$\sum_{j \in N_i} \frac{2}{T_{ij} + 2} \geq \sum_{j \in N_i} \frac{2|N_i|^2}{(T_{ij} + 2)} = \frac{d_i^2}{t_i + d_i} \geq \frac{1}{2c_i + \frac{1}{d_i}}.$$

Summing over all nodes $i$ and applying Lemma A.2 again, we obtain

$$\sum_{(i,j) \in E} \frac{4}{T_{ij} + 2} \geq \sum_{i \in V} \frac{1}{2c_i + \frac{1}{d_i}} \geq \frac{|V|^2}{\sum_{i \in V} \left(2c_i + \frac{1}{d_i}\right)} = \frac{n}{2c + \frac{1}{n} \sum_{i \in V} \frac{1}{d_i}}.$$

The proof is complete by dividing both sides of this inequality by $2n$. \qed

**Remark 3.2** (Order of parameter $\alpha$ in Theorem 2.1). Lemma 3.1 provides a lower bound for $\alpha$ in (2.2). If $c \geq 1/n \sum_{i \in V} 1/d_i$ then the constant $\alpha$ in Theorem 2.1 satisfies $\alpha \geq 1/c$. The proof of Lemma 3.1 suggests that the upper bound $\alpha \lesssim 1/c$ also holds if $T_{ij}$ are similar for most of edges of $G$. Table 1 provides a numerical evidence supporting this heuristic argument.
Clearly, the bound $\alpha \lesssim 1/c$ does not hold for all networks. Below is an example of a network for which two sides of (3.1) are of different orders. Let $G = R_n \cup K_n$ be the union of a random $d$-regular graph $R_n$ ($c = d/n$) of size $n$ and a complete graph $K_n$ ($c = 1$), also of size $n$ (we can connect $R_n$ and $K_n$ by an arbitrary edge to make $G$ a connected graph). If $d = o(n)$ then an easy calculation shows that the left hand-side of (3.1) is of order $n^2/d$ while the right hand-side is of order $n$.

4. Upper bound on parameter $\alpha$

In this section we recall the definition of Ricci curvature for graphs and show that if the Ricci curvature of a graph is bounded from below by some constant $\kappa_0 > 0$ then $\alpha \leq 1/\kappa_0$.

For a graph $G$, denote by $d(i,j)$ the length of a shortest path connecting nodes $i$ and $j$. We attach to each node $i$ of $G$ a uniform probability measure $m_i$ with support being the set of neighbors of $i$:

$$m_i(k) = \begin{cases} \frac{1}{d_i} & \text{if } k \in N_i \\ 0 & \text{otherwise.} \end{cases}$$

The optimal transportation distance between $m_i$ and $m_j$ is defined as follows:

$$W_1(m_i,m_j) = \inf_{\xi \in \Pi(m_i,m_j)} \sum_{(k,k') \in V \times V} d(k,k')\xi(k,k'),$$

where $\Pi(m_i,m_j)$ is the set of all probability measures on $V \times V$ with marginals $m_i$ and $m_j$. Intuitively, $\xi(k,k')$ represents the mass transported from $k$ to $k'$, and $W_1(m_i,m_j)$ is the optimal cost for moving a unit mass distributed evenly among neighbors of $i$ to neighbors of $j$. With this notion of distance between probability measures on $G$, the Ricci curvature of two nodes $i$ and $j$ is defined by

$$\kappa(i,j) = 1 - \frac{W_1(m_i,m_j)}{d(i,j)}.$$ 

Figure 1 shows Zachary’s karate club network [47] together with information of Ricci curvatures of incident nodes (edges). Edges with negative curvatures are in blue, positive curvatures – in red and zero curvatures – in black; widths of edges are proportional to magnitudes of curvatures.

A lower bound $\kappa \geq \kappa_0 > 0$ on the curvature implies that $W_1(m_i,m_j) \leq (1-\kappa_0)d(i,j)$ for each pair of nodes $i$ and $j$. In particular, if $i$ and $j$ are neighbors then $W_1(m_i,m_j) \leq 1-\kappa_0$. Note that if $G$ is a connected graph then the inverse is also true: If $W_1(m_i,m_j) \leq 1-\kappa_0$ holds for all pairs of neighbors $i$ and $j$ then $W_1(m_i,m_j) \leq (1-\kappa_0)d(i,j)$ holds for all $(i,j) \in V \times V$ by a triangle inequality.

For social networks, positive Ricci curvature reflects the idea that people are better off with the help of friends. Imagine that person $i$ needs to transfer money to person $j$ who is not his friend. Without knowing the best route to reach $j$, $i$ divides the money and asks his friends to help him transfer the money to $j$. Similarly, $j$ asks his friends to accept the transferred money on his behalf. Positive Ricci curvature ensures that the cost of transferring money in this way is smaller than the cost of sending money directly along a shortest path.

Ricci curvature is also closely related to a simple random walk on a graph. If $\kappa \geq \kappa_0 > 0$ then [31] shows that the spectral gap between the two largest eigenvalues of the transition probability matrix $D^{-1}A$ is bounded from below by $\kappa_0$ (see also [4] for an improvement
Figure 1. Zachary’s karate club network [47]. Edges with negative Ricci curvatures are in blue, positive Ricci curvatures – in red and zero Ricci curvatures – in black; widths of edges are proportional to magnitudes of Ricci curvatures.

of the bound). Thus, Ricci curvature of a graph controls how fast a simple random walk on that graph mixes.

**Lemma 4.1** (Upper bound on parameter \( \alpha \)). If the Ricci curvature \( \kappa \) of a graph \( G \) satisfies \( \kappa \geq \kappa_0 \) for some constant \( \kappa_0 > 0 \) then \( \alpha \leq 1/\kappa_0 \).

**Proof.** Consider an edge \((i, j)\) of \( G \). The masses of \( m_i \) and \( m_j \) are evenly distributed among neighbors of \( i \) and \( j \), respectively. To transport \( m_i \) to \( m_j \), except those masses at common neighbors of \( i \) and \( j \) that may not have to be moved, we need to move the rest along routes of distances at least one. Since the masses that do not require transportation is at most \( c_{ij} \cdot \min\{1/d_i, 1/d_j\} \) and \( d(i, j) = 1 \), we have

\[
1 - \kappa_0 \geq W_1(m_i, m_j) \geq 1 - c_{ij} \cdot \min\{1/d_i, 1/d_j\}.
\]

This implies \( c_{ij} \geq \kappa_0 / \min\{1/d_i, 1/d_j\} \). Summing over all edges \((i, j)\) of \( G \), we obtain

\[
\sum_{(i, j) \in E} \frac{2}{2 + c_{ij}} \leq \frac{2}{\kappa_0} \sum_{(i, j) \in E} \min\{1/d_i, 1/d_j\} = \frac{1}{\kappa_0} \sum_{i \in V} \sum_{j \in N_i} \min\{1/d_i, 1/d_j\} \leq \frac{n}{\kappa_0}.
\]

For the last inequality we use the fact that \( \sum_{j \in N_i} \min\{1/d_i, 1/d_j\} \leq 1 \). The proof is complete. \( \square \)

**Remark 4.2.** Lemma 4.1 requires that Ricci curvatures of all edges of graph \( G \) are bounded from below by \( \kappa_0 \). This assumption can be relaxed by a weaker assumption that the number of edges with Ricci curvatures less than \( \kappa_0 \) is of order \( O(n) \). With this assumption, the upper bound of \( \alpha \) becomes \( \alpha \leq 1/\kappa_0 + O(1) \).

5. Sparsifying Hypergraphs

Strong local conectivity of a network is often caused by the fact that each node belongs to several tightly connected small groups [13]. To simplify the analysis, we assume in this section that within each small group, all nodes are connected. Under this assumption, a network can be modeled by a hypergraph \( \mathcal{G} = (V, \mathcal{E}) \) which consists of a set of nodes
V and a set of hyperedges $E$ where each hyperedge is a subset of $V$. We derive in this section a condition under which a hypergraph can be sampled and reduced to a weighted network. This provides an example when our sampling scheme works well and may be useful in practice as a computational acceleration technique.

The Laplacian previously defined for networks can be naturally extended to hypergraphs through clique expansion [34, 2]. For a hypergraph $\mathcal{G} = (V, E)$, the evaluation of the Laplacian $L_G$ at a vector $x$ is defined by

$$L_G(x) = \sum_{e \in E} \sum_{i, j \in e} (x_i - x_j)^2.$$ 

If we view $x$ as a function on $V$ then $L_G(x)$ measures the smoothness of $x$ and it occurs naturally in many problems of estimating smooth functions [39, 5, 16, 20, 23].

Let $G = (V, E, W)$ be a weighted graph such that $(i, j) \in E$ if and only if both $i$ and $j$ belong to at least one hyperedge of $\mathcal{G}$; the weight $w_{ij}$ of edge $(i, j)$ is the number of hyperedges that both $i$ and $j$ belong to. It is easy to see that $L_G(x) = x^T L_G x$ for every $x$, where $L_G$ is the Laplacian of the weighted graph $G$ defined by

$$x^T L_G x = \sum_{(i,j) \in E} w_{ij} (x_i - x_j)^2.$$ 

Thus, if we are only interested in smoothness induced by $G$ of functions on $V$ then we can replace $\mathcal{G}$ with $G$. We call $G$ the weighted graph induced by $\mathcal{G}$.

To form a sparsifier $H = (V, E_H, W_H)$ of $G$, we sample with replacement $m$ edges of $G$ with probability

$$P_{ij} = \frac{C_{ij}^{-1}}{\sum_{(i,j) \in E_G} C_{ij}^{-1}}, \quad \text{where } C_{ij} = \sum_{e \in E : \{i,j\} \subseteq e} |e|. \quad (5.1)$$

If an edge $(i, j) \in E_G$ is selected, we add $(i, j)$ to $E_H$ with weight $(mP_{ij})^{-1}$. Again, weights are summed up if edges are sampled more than once. Next lemma shows that a condition similar to (2.2) holds if each node of $\mathcal{G}$ belongs to a finite number of hyperedges.

**Lemma 5.1** (Upper bound on parameter $\alpha$ for hypergraphs). Let $\mathcal{G} = (V, E)$ be a hypergraph and $G = (V, E, W)$ be the weighted graph induced by $\mathcal{G}$. If each node of $\mathcal{G}$ belongs to at most $d$ hyperedges of $\mathcal{G}$ then

$$\sum_{(i,j) \in E_G} C_{ij}^{-1} \leq dn/2. \quad (5.2)$$

**Proof.** By definition of $E_G$ we have

$$\sum_{(i,j) \in E_G} C_{ij}^{-1} \leq \sum_{e \in E_G} \sum_{\{i,j\} \subseteq e} C_{ij}^{-1}.$$ 

Since $C_{ij} \geq |e|$ for each $e \in E_G$ that contains $\{i,j\}$ and there are $|e|(|e| - 1)/2$ pairs $\{i,j\} \in e$, it follows from above inequality that

$$\sum_{(i,j) \in E_G} C_{ij}^{-1} \leq \sum_{e \in E_G} \frac{|e| - 1}{2} \leq \frac{1}{2} \sum_{e \in E_G} |e| \leq \frac{dn}{2}. \quad \square$$

For the last inequality we use the assumption that each node belongs to at most $d$ hyperedges.
Remark 5.2 (Tightness of the bound (5.2)). Without further assumptions on \( G \), the dependence of the right-hand-side of (5.2) on \( d \) is optimal. To see this, consider the following example. Let \( k > 0 \) be an integer, \( n = k^2 \) and \( V_1, ..., V_k \) be a partition of \([n] := \{1, ..., n\}\) such that each \( V_i \) contains exactly \( k \) elements \( V_{i1}, ..., V_{ik} \). For each \( i \in [k] \), let \( \sigma_i \) be a permutation of \([k]\) given by \( \sigma_i(j) = i + j \) (mode \( k \)). Define the set of hyperedges of \( G \) as a collection of subsets of the form

\[
\left\{ V_{ij}, V_{2\sigma_i(j)}, ..., V_{k\sigma_i^{-1}(j)} \right\}, \quad 1 \leq i, j \leq k.
\]

It is easy to see that every node of \( G \) is contained in exactly \( d = k \) hyperedges and every pair of nodes of \( G \) is contained in at most one hyperedge. A simple calculation shows that 

\[
\sum_{(i,j) \in E_G} 1/C_{ij} = (d-1)n/2.
\]

Lemma 5.3 (Sparsification of hypergraphs). Let \( G = (V,E) \) be a hypergraph and \( G = (V, E, W) \) be the weighted graph induced by \( G \). Let \( \varepsilon \in (0,1) \) and assume that each node of \( G \) belongs to at most \( d \) hyperedges of \( G \). Form a weighted graph \( H \) by sampling \( 4dn \log n/\varepsilon^2 \) edges of \( G \) with probability \( P_{ij} \). Then with probability at least \( 1 - 1/n \) the following holds:

\[
(1 - \varepsilon)L_G \leq L_H \leq (1 + \varepsilon)L_G.
\]

Proof. The proof of this lemma is similar to the proof of Theorem 2.1 with one exception that we use the bound (5.2) in Lemma 5.3 instead of condition (2.2). \( \square \)

6. Discussion

We study in this paper an edge sampling algorithm that uses only the numbers of common neighbors of incident nodes. These simple statistics provide an easy way to measure the strength of network local connectivity through parameter \( \alpha \). However, in practice we often have access to not only the numbers of common neighbors but also local networks around edges. In that case, we should use the information of these local networks, provided that it is available or easily computed. Measuring the strength of local connectivity through local networks is more challenging and we leave it for future work.

Appendix A. Proof of Theorem 2.1

To prove Theorem 2.1, we will use the following result about concentration of the sum of random matrices [44].

Theorem A.1 (Concentration of sum of matrices). Let \( Y_k \) be independent \( n \times n \) random matrices such that \( Y_k \geq 0 \) and \( \|Y_k\| \leq M \) for all \( 1 \leq k \leq m \). Let \( S_m = \sum_{k=1}^m Y_k \) and \( E = \sum_{k=1}^m \|EY_k\| \). Then for every \( \varepsilon \in (0,1) \) we have

\[
\mathbb{P}\left\{ \|S_m - ES_m\| > \varepsilon E \right\} \leq n \cdot \exp\left( -\frac{\varepsilon^2 E}{4M} \right).
\]

Proof of Theorem 2.1. Let \( X \) be a random matrix that takes one of the \( |E_G| \) matrix values:

\[
X = \frac{1}{p_{ij}} (e_i - e_j)(e_i - e_j)^T \quad \text{with probability } \ p_{ij},
\]

where \( (i,j) \in E_G \) and \( p_{ij} \) is defined by (2.1). Then

\[
\mathbb{E}X = \sum_{(i,j) \in E_G} p_{ij} \times \frac{1}{p_{ij}} (e_i - e_j)(e_i - e_j)^T = L_G.
\]

(A.1)
Let $X_k$ be $m$ independent copies of $X$. By the sampling scheme we have

$$L_H = \frac{1}{m} \sum_{k=1}^{m} X_k, \quad \mathbb{E} L_H = L_G.$$  

Denote by $L_G^{-1}$ the Moore-Penrose pseudoinverse of $L_G$ and by $L_G^{-1/2}$ the squared root of $L_G^{-1}$. Note that the kernel of the map $L_G$ is an one-dimensional vector space spanned by the all-one vector $\mathbf{1}$ and it is contained in the kernel of $L_H$. Therefore inequality (2.3) is equivalent to

$$(1 - \varepsilon)I_1 \preceq \frac{1}{m} \sum_{k=1}^{m} L_G^{-1/2} X_k L_G^{-1/2} \preceq (1 + \varepsilon)I_1, \quad (A.2)$$

where $I_1 = I - (1/n)\mathbf{1}\mathbf{1}^T$ is the identity map on the $(n-1)$-dimensional subspace orthogonal to the all-one vector $\mathbf{1}$.

To prove (A.2), we apply Theorem A.1 to $Y_k := L_G^{-1/2} X_k L_G^{-1/2}$. Since $X_k \geq 0$ and $\mathbb{E} X_k = L_G$ by (A.1), it follows that $Y_k \geq 0$ and $\|\mathbb{E} Y_k\| = \|I_1\| = 1$. To bound $\|Y_k\|$, note that $Y_k$ takes one of the following matrix values

$$\frac{1}{p_{ij}} \begin{pmatrix} L_G^{-1/2}(e_i - e_j) \end{pmatrix} \begin{pmatrix} L_G^{-1/2}(e_i - e_j) \end{pmatrix}^T, \quad (i, j) \in E_G.$$ 

By (2.1) and assumption (2.2) we have $1/p_{ij} \leq n\alpha(c_{ij} + 2)/2$. Therefore

$$\|Y_k\| \leq \max_{(i, j) \in E_G} \frac{n\alpha(c_{ij} + 2)}{2} \cdot (e_i - e_j)^T L_G^{-1}(e_i - e_j). \quad (A.3)$$

For each $(i, j) \in E_G$, let $N_{ij}$ be the set of common neighbors of $i$ and $j$. Denote by $G_{ij} = (V_{ij}, E_{ij})$ the subgraph of $G$ such that $V_{ij} = \{i, j\} \cup N_{ij}, E_{ij} = \{(i, j), (i, k), (j, k) : k \in N_{ij}\}$.

Thus, $G_{ij}$ contains $c_{ij} + 2$ vertices and $2c_{ij} + 1$ edges. Since $G_{ij}$ is a subgraph of $G$, it follows that $L_{G_{ij}} \preceq L_G$. On the $(c_{ij} + 1)$-dimensional subspace spanned by $\{e_k : k \in V_{ij}\}$ and orthogonal to all-one vector $\mathbf{1}$, both $L_G$ and $L_{G_{ij}}$ are nonsingular, therefore $L_{G_{ij}}^{-1} \succeq L_G^{-1}$ (see e.g. [15, Corollary 7.7.4]). In particular,

$$(e_i - e_j)^T L_G^{-1}(e_i - e_j) \leq (e_i - e_j)^T L_{G_{ij}}^{-1}(e_i - e_j). \quad (A.4)$$

We claim that the right hand side of (A.4) is equal to $2/(c_{ij} + 2)$. Let $x = L_{G_{ij}}^{-1}(e_i - e_j)$. Then $L_{G_{ij}} x = e_i - e_j$ and by comparing the $i$-th and $j$-th components of $L_{G_{ij}} x$ and $e_i - e_j$, we have

$$(c_{ij} + 1)x_i - x_j - \sum_{k \in N_{ij}} x_k = 1, \quad x_i - (c_{ij} + 1)x_j + \sum_{k \in N_{ij}} x_k = 1.$$ 

Adding these equalities, we obtain $x_i - x_j = 2/(c_{ij} + 2)$. Since the right hand side of (A.4) is $(e_i - e_j)^T x = x_i - x_j$, the claim is proved. Together with (A.3) and (A.4) this implies $\|Y_k\| \leq n\alpha$. Therefore by Theorem A.1 we have

$$\mathbb{P} \left\{ \frac{1}{m} \sum_{k=1}^{m} Y_k - I_1 \right\} \geq \varepsilon \leq n \cdot \exp \left( \frac{-\varepsilon^2 m}{4\alpha n} \right).$$

Inequality (A.2) then follows by choosing $m = 8\alpha n \log n/\varepsilon^2$. \qed
Lemma A.2. For positive numbers $x_1, x_2, \ldots, x_k$ the following inequality holds
\[(x_1 + x_2 + \cdots + x_k) \left( \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} \right) \geq k^2.\]
The two sides are equal if and only if $x_1 = x_2 = \cdots = x_k$.

Proof. Using the inequality of arithmetic and geometric means, we have
\[x_1 + x_2 + \cdots + x_k \geq k(x_1x_2\cdots x_k)^{1/k}, \quad \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_k} \geq k(x_1x_2\cdots x_k)^{-1/k}.\]
Lemma A.2 follows directly from these inequalities. \qed

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