Exact coherent matter-wave solitons induced and controlled by laser field

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We find a set of exact solutions of coherent bright solitons in the quasi-one-dimensional (1D) Bose-Einstein condensate (BEC) trapped in a harmonic potential, by using a Gaussian laser well (barrier) with oscillating position to balance the repulsive (attractive) interatomic interaction. The bright solitons do not deform in propagation and are controlled accurately by the laser driving which resonates with the trapping potential. The solitonic motion is more stable for the repulsive BEC than that of the attractive BEC. The results reveal a different kind of soliton trains compared to that reported recently in Phys. Rev. Lett. 100, 164102 (2008) and suggest an experimental scheme for generating and controlling the coherent matter-wave solitons.

PACS numbers: 03.75.Lm, 41.75.Jv, 03.75.Kk, 32.80.Lg

Soliton in a Bose-Einstein condensate (BEC) is a kind of important nonlinear phenomena, which has been investigated experimentally and theoretically [1]-[11]. The formation and propagation of BEC solitons were observed by magnetically tuning the atom-atom interaction from repulsive to attractive [1,2]. The theoretical works demonstrated that the matter-wave bright solitons can be created in a BEC, through the modulational instability [3,6] and quantum phase fluctuations [7]. Propagation feature of the solitons is the breathing oscillation which is mostly controlled by the harmonic trap [12,13]. In the harmonic trapping case, the Gaussian ansatz was used to fit the profiles of solitons [3,7,14,15]. The Gaussian-shaped optical potentials have been applied for investigating the BEC solitons [16,17,18] and quantum tunneling [19,20,21,22]. It is worth noting that the balance between nonlinearity and dispersion was found in the seminal study of soliton [23]. Recently, the new balances between the atom-atom interaction and the Gaussian and/or periodical potentials are demonstrated [17,24]. For some special forms of external potential and interaction intensity, the exact soliton solutions in BECs have also been reported [16,22,26,27].

The quantum states governed by the linear Schrödinger equation with inseparable space-time variables are very important but had to find. The coherent state of a harmonic oscillator is a nice example of such states, which has been widely applied to physics and optics [28,29] and is also extended to the case of wavepacket trains [13,30]. Can the coherent wavepacket trains exist in a harmonically trapped BEC system governed by the Gross-Pitaevskii equation (GPE)? This is usually impossible, because of the nonlinearity in GPE. However, when we employ the laser field to balance the nonlinear term, seeking exact coherent states of the GPE could become possible. Demonstrating the exact coherent state of GPE and its experimental feasibility is our main motivation in this paper.

By using the balance technique and applying the oscillating Gaussian lasers, we find n exact solutions of coherent soliton trains in the quasi-1D BEC. When n = 0 is considered, for the attractive or repulsive BEC the interatomic interaction is balanced by the Gaussian barrier or well respectively, and in the both cases the soliton solutions possesses the same form of coherent state. The coherent bright soliton oscillates like a classical harmonic oscillator with the trapping frequency, which agrees with Strecker’s experimental results [3]. However, compared to the deformed soliton trains observed in experiment, our bright solitons have different properties, namely their shapes are kept in propagation, their behaviors are controlled accurately by the laser field and their motions possess more stability for the repulsive BEC rather than the attractive one. Based on the capacity of current experiments, such bright solitons can be observed in a BEC.

We consider a BEC consisting of N identical Bose atoms and being transferred into a cigar-shaped harmonic trap. The potential that takes into account the combination of the magnetic trap with the laser sheet reads $V(x,t) = \frac{1}{2}m\omega_r^2 x^2 + V_L(x,t)$ with $V_L(x,t)$ containing the Gaussian-shaped laser potential [22,21,24]. Let the transverse frequency $\omega_r$ be much greater than the axial frequencies $\omega_x$, the dynamics of the system is governed by the quasi-1D GPE [3,15]

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + [V(x,t) + g_{1d}|\psi|^2]\psi, \quad (1)$$

where we have assumed the transverse wave function being in ground state of a harmonic oscillator such that the quasi-1D interaction intensity related to the s-wave scattering length $a_s$, atomic mass $m$ and number of condensed atoms $N$ reads $g_{1d} = Nm\omega_r g_0/(2\pi \hbar) = 2N\hbar\omega_r a_s$ for the normalized wave-function $\psi$. The norm $|\psi|^2$ is the probability density and $N|\psi|^2$ the density of atomic number. Setting $l_r = \sqrt{\hbar/(m\omega_r)}$, $l_x = \sqrt{\hbar/(m\omega_x)}$, we normalize the time, space, wave function and laser potential by $\omega_r^{-1}$, $l_x$, $\sqrt{g_{1d}}$ and $\hbar\omega_r$ respectively, then the interaction intensity becomes $g_{1d} =$

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The required laser frequency of the solitonic center are controlled by the oscillating amplitude and frequency 

\[ V_L(x, t) + g_{1d}|\psi|^2 = \mu \]

(3)
to transfer Eq. (2) to the linear Schrödinger equation

\[ i \frac{\partial \psi}{\partial t} = - \frac{1}{2} \frac{\partial^2 \psi}{\partial x^2} + \left[ \frac{1}{2} \frac{x^2}{\mu} + \psi \right], \]

(4)
where \( \mu \) denotes an undetermined constant determined by the laser profile and is in units of \( \hbar \omega_x \). Obviously, by the balance condition we mean that the external optical potential and the internal interaction reach into an indifferent equilibrium experimentally, namely their sum equates a constant \( \mu \). The exact solutions of Eq. (2) must obey the balance condition (3) and linear equation (4) simultaneously. Therefore, only the properties common to the nonlinear Eq. (3) and linear Eq. (4) can be kept in the balance solution. For the Gaussian potential \( V_L(x, t) \) the soliton solution of the nonlinear Eq. (3) and coherent-state solution of the linear Eq. (4) can be in the same form and could coexist thereby.

The exact solutions of extended coherent states of Eq. (4) read

\[ \psi_n = R_n(x, t)e^{i\theta_n(x, t)}, \quad n = 0, 1, 2, \ldots ; \]
\[ R_n = (\frac{1}{\sqrt{2\pi n!}})^{1/2} H_n(x-x_0 \cos t)e^{-\frac{1}{4}(x-x_0 \cos t)^2}, \]
\[ \theta_n = - \left[ (\frac{1}{2} + \mu + n) t + x_0 x \sin t - \frac{1}{4} x_0^2 \sin 2t \right], \]

(5)
which can be proved by inserting it directly into Eq. (4). Here \( H_n \) denotes the Hermite polynomial with \( x_0 \) being the amplitude of the center position of Gaussian packet, which can be adjusted by the oscillating amplitude of laser position. Applying Eq. (5) to Eq. (3) shows the profile of required laser field

\[ V_L = \mu - \frac{g_{1d}}{\sqrt{2\pi n!}} H_n^2(x-x_0 \cos t)e^{-\frac{1}{4}(x-x_0 \cos t)^2}. \]

(6)
Clearly, for constant interaction \( g_{1d} \) and \( n = 0 \) \( (H_0 = 1) \) case Eq. (6) describes the oscillating Gaussian potential which agrees with that used in \[ 16, 32 \]. At \( x_0 = 0 \) it becomes the well known time-independent form \[ 12, 13, 21 \] which leads Eq. (5) to the exact stationary state of Eq. (2). For any \( n \) Eq. (6) denotes a multi-well (-barrier) which includes the well-known double-well with \( n = 1 \). The absolute value \( |g_{1d}| \) determines the required laser intensity. Obviously, Eq. (5) describes the oscillating single soliton for \( n = 0 \) and the soliton trains for \( n > 0 \) \[ 1, 13 \]. The oscillating amplitude \( x_0 \) and frequency \( \omega = 1(\omega_x) \) of the solitonic center are controlled by the oscillating laser strictly. The required laser frequency \( \omega_L \) is equal to the trapping frequency \( \omega_x \), that means resonance between the trapping and driving potentials. The same frequency fixes width and height of the Gaussian laser and solitons, through the axial length \( l_x = \sqrt{\hbar/(m \omega_x)} \) of harmonic oscillator. On the other hand, rewriting Eq. (5) as \( \psi_n = \phi_n(x, t)e^{-iE_n \gamma \hbar}, \) we obtain the Floquet quasienergy \( E_n = (\frac{1}{2} + \mu + n) \) adjusted by the laser parameter \( \mu \). The corresponding Floquet state obeys \( \phi_n(x, t + 2\pi) = \phi_n(x, t) \).

It is interesting noting that the soliton solutions of Eq. (5) possess the same form for the attractive and repulsive interatomic interactions. However, for different interactions the solitons are associated with different shapes of the required Gaussian lasers. Taking \( n = 0 \) as an example, Eq. (6) exhibits that the laser barriers correspond to the attractive interaction with \( g_{1d} < 0 \) and the laser wells are associated with the repulsive one with \( g_{1d} > 0 \).

When we consider \( N = 10^4 \) \( ^7 \)Li atoms with mass \( m \) being 7 times of the proton mass \( m_p \) and take the experimental parameters \[ 1 \] \( a_s = \pm 1.5 \text{nm}, \omega_x = 20 \text{Hz}, \omega_r = 800 \text{Hz}, \) the harmonic oscillator lengths and the interaction intensity become \( l_x = \sqrt{\frac{4}{10}} = 21.22 \text{µm} \) and \( g_{1d} = 2N\omega_r a_s/(\omega_x l_x) = \pm 56.55 \).

In Figs. 1a and 1b we show the spatiotemporal evolutions of the atomic densities \( R_0^2(x, t) \) and \( R_0^2(x, t) \) respectively for the laser parameter \( x_0 = 10(l_x) = 212.2 \text{µm} \). The former is a single Gaussian wave and the latter is a double Gaussian at any time for both \( g_{1d} > 0 \) and \( g_{1d} < 0 \) cases. These bright solitons oscillate their centers with amplitude \( x_0 \). Hereafter, all the parameters in any figure are dimensionless. The laser potential \( V_L(x, t) \) of Eq. (6) with parameters \( \mu = 10, x_0 = 10 \) is plotted as in Fig. 2 for four sets of parameters respectively. In Figs. 2a and 2c with positive \( g_{1d} \), we observe the single and double wells respectively for any time. The wells oscillate their center positions as the increase of time. From Figs. 2b and 2d with negative \( g_{1d} \), we find that at any time the laser potentials describe the single and double barriers with oscillating centers.

To see the details of the soliton motions and to analyze the stability of the system, in Fig. 3 we show spatial profiles of the soliton and total potential functions \( V(x, t) = \frac{1}{2}x^2 + V_L(x, t) \) at several different times for the same parameters with Fig. 2a. It is revealed that in the
time as a quasi-particle the soliton falls into a potential well. According to the well-known criterion of dynamical stability, such soliton motion is stable. In the transportation process of BEC soliton, the laser potential $V_L(x, t)$ plays a role of optical tweezer [32, 33].

When the interaction intensity is changed from $g_{1d} = 56.55$ to $g_{1d} = -56.55$, Fig. 3 is correspondingly changed to Fig. 4. In the latter figure, the solitonic shape and evolution have no change. However, the potential deforms with different pattern compared to Fig. 3 such that the soliton is no longer located on the center of a potential well. Particularly, Figs. 4b and 4d exhibit that sometimes as a quasi-particle the soliton lies at the top of potential barrier. The corresponding soliton motion may be dynamically unstable thereby.

Similarly, in the double-soliton case with $n = 1$, by comparing Fig. 1b with Figs. 2c and 2d, we find that the solitons fall on the optical wells or barriers for the repulsive or attractive BEC. Therefore, the exact soliton pair is dynamically stable for the repulsive BEC or unstable for the attractive one.

In conclusion, we have investigated the repulsive and attractive quasi-1D BECs held in the combination potential of the magnetic trap and the Gaussian laser sheet with oscillating position. It is demonstrated that when the laser potential balances the interatomic interaction, the exact bright soliton trains can be generated. The corresponding $n$ soliton solutions agree with the extended coherent states of harmonic oscillator. The soliton trains fit the periodical motions of laser centers and keep their shapes of Gaussian waves, that agree with Strecker’s experiment partly [1]. The solitonic width, height, oscillating amplitude and frequency are controlled by the laser field accurately. The required optical potentials contain the Gaussian wells and barriers for the repulsive and attractive BECs respectively, which resonate with the trapping potential. For $n = 0$ case the optical well is similar to the quantum dot generated by a focused beam of red-detuned laser light [33]. Although the solitonic profile of

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**FIG. 2:** Spatiotemporal evolutions of the potential functions plotted from Eq. (6) for (a) $n = 0$, $g_{1d} = 56.55$, (b) $n = 0$, $g_{1d} = -56.55$, (c) $n = 1$, $g_{1d} = 56.55$, (d) $n = 1$, $g_{1d} = -56.55$. The potential wells and barriers are found respectively for the different cases.

**FIG. 3:** Spatial profiles of the soliton (solid curves) and potential functions (dashed curves) for the parameters of Fig. 2a and at (a) $t = 0$, (b) $t = \pi/2$, (c) $t = \pi$ and (d) $t = 5\pi/4$. It is observed that the soliton keeps its shape and oscillates its position. The soliton center is located on the center of a potential well for any time.

**FIG. 4:** Spatial profiles of the soliton (solid curves) and potential functions (dashed curves) for the same parameters with Fig. 2b and at (a) $t = 0$, (b) $t = \pi/2$, (c) $t = \pi$ and (d) $t = 5\pi/4$. At $t = 0$ and $t = \pi$ the soliton is not located on the position of minimal potential. For $t = \pi/2$ and $t = 5\pi/4$ the soliton is located on the position of potential barrier.
fixed $n$ is same for the both interaction cases, the soliton of repulsive BEC is more stable than that of the attractive BEC. It is worth noting that in some time intervals the BEC solitons are transported toward fixed direction and the laser field behaves like a quantum tweezer realized in previous works [33, 34]. Therefore, the exact bright solitons could be observed and controlled experimentally by oscillating the laser position and adjusting the system parameters.

It is also noted that the exact soliton trains are similar to the results of Ref. [16]. However, the required external potentials and interatomic interactions are different for the both cases. The spatiotemporal-dependent interaction intensity used in [16] is not required in our systems. This could bring convenience to the experimental observation of the soliton trains.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant Nos. 10575034 and 10875039.

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