A THEORY OF DORMANT OPERS
ON POINTED STABLE CURVES
— A PROOF OF JOSHI’S CONJECTURE —

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Abstract. In this paper, we develop a general theory of opers over families of pointed stable curves. After proving certain properties concerning the moduli stack classifying pointed stable curves equipped with a dormant oper, we give an explicit formula, which was conjectured by Kirti Joshi, for the generic number of dormant opers of classical type $A_n$.

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1. Twisted logarithmic connections over log schemes

Joshi’s conjecture. (cf. [37], Conjecture 8.1)

\begin{equation}
\deg(\mathcal{O}_{\mathfrak{p}_{\mathfrak{sl}_n}, X/\overline{k}}) = \frac{p^{(n-1)(g-1)-1}}{n!} \cdot \sum_{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n} \frac{(\prod_{i=1}^n \zeta_i)^{(n-1)(g-1)}}{\prod_{i \neq j} (\zeta_i - \zeta_j)^{g-1}}.
\end{equation}

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We shall explain what is displayed just above. Let \( \overline{k} \) be an algebraically closed field of characteristic \( p > 0 \) and \( X \) a connected proper smooth curve of genus \( g > 1 \) over \( \overline{k} \). Suppose that \( p > C \), where \( C \) is a certain explicit constant depending on \( g \) and a fixed positive integer \( n \). It is known (cf. [36], Theorem 5.4.1 and Corollary 6.1.6) that the moduli functor \( \mathcal{O}_p^{\text{Zax}}_{\mathfrak{sl}_n, X} \) classifying dormant \( \mathfrak{sl}_n \)-opers on \( X \) may be represented by a nonempty finite scheme over \( \overline{k} \). Toward a further understanding of this \( \overline{k} \)-scheme \( \mathcal{O}_p^{\text{Zax}}_{\mathfrak{sl}_n, X} \), Kirti Joshi proposed, in [37], the conjectural formula (1) displayed above, computing the degree \( \text{deg}(\mathcal{O}_p^{\text{Zax}}_{\mathfrak{sl}_n, X}/\overline{k}) \) of \( \mathcal{O}_p^{\text{Zax}}_{\mathfrak{sl}_n, X} \) over \( \overline{k} \). (Here, the sum in the right-hand side of the equality (1) is taken over the set of \( n \)-tuples \( (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n \) of \( p \)-th roots of unity in \( \mathbb{C} \) (= the field of complex numbers) satisfying that \( \zeta_i \neq \zeta_j \) if \( i \neq j \)).

0.1. Before preceding, recall that an “oper” (or, a “\( g \)-oper” for a semisimple Lie algebra \( g \)) is, by definition (cf. Definition 2.2.1), a principal homogeneous space (in other words, a torsor) over an algebraic curve equipped with an integrable connection satisfying certain conditions. Opers on an algebraic curve over the field of complex numbers \( \mathbb{C} \) play a central role in integrable systems and representation theory of loop algebras. They were introduced in [5] in the context of the geometric Langlands program, providing a coordinate-free expression for the connections which appeared first in [17] as the phase space of the generalized Korteweg-de Vries hierarchies. Opers on a fixed proper hyperbolic curve \( X_\mathbb{C} \) over \( \mathbb{C} \) forms an affine space, modeled on the base space of Hitchin’s integrable system on the cotangent bundle of the moduli space of bundles.

Also, various equivalent mathematical objects, including certain kinds of differential operators (related to Schwarzian equations) between line bundles, have been studied by many mathematicians. For example, \( \mathfrak{sl}_2 \)-opers were introduced and studied, under the name of indigenius bundles, in the work of R. C. Gunning (cf. [24], §2). One may think of an indigenius bundle as an algebraic object encoding (analytic, i.e., non-algebraic) uniformization data for Riemann surfaces. Moreover, it may be interpreted as a projective structure, i.e., a maximal atlas covered by coordinate charts on \( X_\mathbb{C} \) such that the transition functions are expressed as Möbius transformations. Also, one may find, in the work of C. Teleman (cf. [71]), the objects corresponding to \( \mathfrak{sl}_n \)-opers, under the name of homographic structures.

In the case of characteristic zero, we refer to [7], [8], and [65], etc., for further studies of \( \mathfrak{sl}_n \)-opers, and moreover, [19], [20], and [22], etc., for reviews and expositions concerning \( g \)-opers in \( \mathbb{C} \) (for an arbitrary \( g \)).
0.2. In the present paper, we study $\mathfrak{g}$-opers in arbitrary characteristic, i.e., including the case of positive characteristic. Just as in the case of the theory over $\mathbb{C}$, one may define the notion of a $\mathfrak{g}$-oper and the moduli space classifying $\mathfrak{g}$-opers in characteristic $p > 0$. If $\mathfrak{g} = \mathfrak{sl}_2$, then various properties of such objects (over families of pointed stable curves) were firstly discussed in the context of the $p$-adic Teichmüller theory developed by S. Mochizuki (cf. [52], [53]). (In a different point of view, Y. Ihara developed, in, e.g., [30], [31], a theory of Schwarzian equations in arithmetic context.) Also, for the case where $\mathfrak{g}$ is more general (but the underlying curve is assumed to be proper and smooth over an algebraically closed field), the study of $\mathfrak{g}$-opers in positive characteristic has been executed by K. Joshi, S. Ramanan, E. Z. Xia, J. K. Yu, C. Pauly, T. H. Chen, X. Zhu et al. (cf. [35], [36], [37], [13]).

One of the common key ingredients in the development of these works is the study of the $p$-curvature of the underlying integrable torsor of a $\mathfrak{g}$-oper. Recall that the $p$-curvature of a connection may be thought of as the obstruction to the compatibility of $p$-power structures that appear in certain associated spaces of infinitesimal (i.e., “Lie”) symmetries. We shall say that a $\mathfrak{g}$-oper is dormant (cf. Definition 3.6.1) if its $p$-curvature is identically zero. Dormant $\mathfrak{g}$-opers, which are our principal objects of study in the present paper, contain diverse aspects.

For example, if the underlying curve $X$ is as introduced at the beginning of the Introduction, then the dormant $\mathfrak{sl}_2$-opers on $X$ correspond, in a certain sense, to a certain type of Frobenius-destabilized vector bundles of rank 2 (cf. [58], § 4, Proposition 4.2). This correspondence gives us an approach to understand the Verschibung map between the moduli space of semistable bundles. Also, it follows from work of S. Mochizuki, F. Liu, and B. Osserman (cf. [53], [47], [74]) that there is a relationship between combinatorics concerning rational polytopes (and spin networks) and the geometry of the moduli stack of dormant $\mathfrak{sl}_2$-opers.

For another example, the sheaf of locally exact differentials on $X$ is obtained (cf. [62], Remark 4.1.2) as the sheaf of horizontal sections of a certain dormant $\mathfrak{sl}_{(p-1)}$-oper twisted by an integrable line bundle (or, in other words, a certain $GL_{(p-1)}$-oper (cf. Definition 4.2.1 (i)) with vanishing $p$-curvature). It has remarkable features, as well as profound importance. Indeed, the theta divisor associated to this bundle (cf. [62], Theorem 4.1.1) gives information on the fundamental group of the underlying curve. It leads us to verifying sorts of anabelian phenomena for hyperbolic curves in positive characteristic (cf. [61], [63], [69], [70]).

Thus, dormant $\mathfrak{g}$-opers occur naturally in mathematics, and hence, the following natural question relevant to such objects may arise:

*Can one calculate explicitly the number of dormant $\mathfrak{g}$-opers on a general curve?*
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(Also, it is very worth studying g-opers with nilpotent p-curvature in the context of the p-adic Teichmüller theory (cf. Theorem B in the following). I. I. Bouw and S. Wewers set up (cf. [11]) an equivalence between indigenous bundles (i.e., slg-opers) with nilpotent p-curvature and so-called deformation data. This equivalence allows us to translate the existence problem for deformation data into the existence of polynomial solutions of certain differential equations with additional properties. In [28], Y. Hoshi obtained an advanced understanding of indigenous bundles with nilpotent p-curvature in characteristic three, including a complete list of them (cf. [28], Theorem 6.1) for the case where the underlying curve is of genus two. But, in the present paper, we focus mainly on dormant g-opers rather than g-opers with nilpotent p-curvature.)

We shall review previous results concerning the explicit computation (i.e., the answer to the question displayed above) of the number of dormant g-opers. All the results that have been shown previously are of the case where g = sl2. A theorem (cf. [53], Chap. II, § 2.3, Theorem 2.8) in the p-adic Teichmüller theory due to S. Mochizuki implies that if X is sufficiently general in the moduli stack Mg of proper smooth curves of genus g, then Opzl2,X is finite (as we mentioned above) and étale over $\overline{k}$. It follows that the task of resolving the above question (for the case where g = sl2) may be reduced to the explicit computation of the degree $\deg(\text{Op}_{zl2,X}/\overline{k})$ of $\text{Op}_{zl2,X}$ over $\overline{k}$, that is, to proving Joshi’s conjecture (for the case where g = sl2) displayed at the beginning of the present paper.

In the case of g = 2, S. Mochizuki (cf. [53], Chap. V, § 3.2, Corollary 3.7), H. Lange-C. Pauly (cf. [46], Theorem 2), and B. Osserman (cf. [59], Theorem 1.2) verified (by applying different methods) the equality

(2) $\deg(\text{Op}_{zl2,X}/\overline{k}) = \frac{1}{24} \cdot (p^3 - p)$.

Also, by extending the relevant formulations to the case where X admits marked points and nodal singularities (i.e., X is a pointed stable curve), S. Mochizuki also gave (cf. [53], Introduction, Theorem 1.3) the combinatorial procedure for computing explicitly the value $\deg(\text{Op}_{zl2,X}/\overline{k})$ according to a sort of fusion rules. (These fusion rules will be generalized, in § 6.4 of the present paper, to the case for an arbitrary g). This procedure also includes an explicit description of dormant sl2-opers on each totally degenerate curve (cf. Definition 6.3.1) in terms of the radii of atoms (cf. [53], Chap. V, §0 for the definition of an atom). Also, it leads to the work by F. Liu and B. Osserman. They have shown (cf. [47], Theorem 2.1) that the value $\deg(\text{Op}_{zl2,X}/\overline{k})$ may be expressed as a polynomial with respect to the characteristic p of degree $3g - 3$. This was done by applying Ehrhart’s theory concerning the cardinality of the set of lattice points inside a polytope.

For an arbitrary g and g = sln, Kirti Joshi conjectured, with his amazing insight, an explicit description, as displayed in (1), of the value $\deg(\text{Op}_{zln,X}/\overline{k})$. In [73], the author proved, by grace of the idea and discussion due to K. Joshi
et al. (cf. [36], [37]), the conjecture of Joshi for the case where $n = 2$ and $g$ is an arbitrary. (Note that the statement of [73], Corollary 5.4, it was supposed that $p > 2(g - 1)$. But by combining [73], Corollary 5.4 (applied to infinitely many primes $p$), with [47], Theorem 2.1, one may conclude the asserted equality even if $g$ is an arbitrary. See also [74], Corollary 8.11.) That is, the equality

$$\deg(\mathcal{O}p_{\mathfrak{sl}_2,X/\mathbb{k}}) = \frac{p^{g-1}}{2^{g-1}} \cdot \sum_{\theta=1}^{p-1} \frac{1}{\sin^{2g-2}(\frac{\pi \theta}{p})}$$

holds. After easy calculations, one may verifies that both the equalities (2) and (3) consist with the equality (1). A goal of the present paper is to give (cf. Theorem G asserted below) an affirmative answer to this conjecture of Joshi for the case where $n$ is an arbitrary integer $> 1$. (The conjectural formula (1) was formulated under the condition that $p > C$ where $C = n(n - 1)(n - 2)(g - 1)$. But, (even if $n > 3$) the left-hand side of this conjectural formula now makes sense for all $p > n$ and we give its proof for all $p > n(g - 1)$.)

0.3. The proof of Joshi’s conjecture given in the present paper may be thought of as a simple generalization of the discussion of the case where $n = 2$. As we executed in [73], our discussion for proving Joshi’s conjecture in the present paper follows, to a substantial extent, the ideas discussed in [36], as well as in [37]. Indeed, certain of the results obtained in the present paper are mild generalizations of the results obtained in [36] concerning $\mathfrak{sl}_n$-opers to the case of families of curves over quite general base schemes. (Such relative formulations are necessary in the theory of the present paper, in order to consider deformations of various types of data.)

For example, Proposition 8.3.1 in the present paper corresponds to [36], Theorem 3.1.6 (or [35], §5.3; [68], §2, Lemma 2.1); Proposition 8.3.2 corresponds to [36], Theorem 5.4.1; and Proposition 8.3.3 corresponds to [36], Proposition 5.4.2. Also, the insight concerning the connection with the formula of Y. Holla (cf. Theorem 8.6.1), which is a special case of the Vafa-Intriligator formula, is due to Joshi. Moreover, according to the discussion in [73], we verify the vanishing of obstructions to deformation to characteristic zero of a certain Quot-scheme that is related to $\mathcal{O}p_{\mathfrak{sl}_n,X}$ (cf. Proposition 8.3.3, Proposition 8.4.1, and the discussion in the proof of Theorem 8.6.2). Then we relate the value $\deg(\mathcal{O}p_{\mathfrak{sl}_n,X}/\mathbb{k})$ to the degree of the result of base-changing this Quot-scheme to $\mathbb{C}$ by applying the formula of Y. Holla (cf. Theorem 8.6.1, the proof of Theorem 8.6.2) directly.

An essential fact that allows us to carry out this discussion is the generic (relative to $\mathfrak{M}_g$) étaleness of $\mathcal{O}p_{\mathfrak{sl}_n,X}$ over $\mathbb{k}$. If $n = 2$, then this fact was, as we mentioned above, proved by S. Mochizuki. Unfortunately, however, the remaining case (i.e., $n > 2$) was still unknown. In order to complete the proof
of Joshi’s conjecture for an arbitrary \( n \), we will prove the generic étaleness of \( \mathcal{O}_p^{\mathfrak{sl}_n} \). This is one of the main results of the present paper (cf. Theorem F asserted below). To this end, as worked in \( [53] \) (for the case of \( n = 2 \)), it will be necessary to begin with formulating the notion of an \( \mathfrak{sl}_n \)-oper (or, more generally, a \( \mathfrak{g} \)-oper for an arbitrary semisimple Lie algebra \( \mathfrak{g} \)) on a family of pointed stable curves, as well as developing a theory of such kind of opers. Moreover, for convenience in, e.g., proving Theorem B, as well as, in future research, we shall deal with a sort of generalization (cf. Definition 2.2.1 (i); \( [5] \), § 3.1.14; \( [6] \), § 5.2) of \( \mathfrak{g} \)-opers which we shall refer to as \( (\mathfrak{g}, \hbar) \)-opers for some parameter \( \hbar \).

0.4. The rest of the Introduction is denoted to describe the organization of the present paper.

We begin in § 1 with a general theory of logarithmic connections (in arbitrary characteristic) twisted by a parameter \( \hbar \), which we shall refer to as an \( \hbar \)-log connection (cf. Definition 1.2.1 (i)). We shall describe (cf. Corollary 1.4.2) the behavior of the form of an \( \hbar \)-log connection upon executing a gauge transformation of the underlying torsor. This fact will be used in the proof of the representability of the moduli functor (by an affine space) of opers (cf. Proposition 2.2.5, Proposition 2.7.3, and Proposition 2.7.5). Also, we define, in Definition 1.6.1, the notion of the monodromy of an \( \hbar \)-log integrable torsor (cf. Definition 1.2.1 (iii)) over a family of pointed stable curves.

In § 2, we study a general theory of (the moduli of) \( (\mathfrak{g}, \hbar) \)-opers (for a semisimple Lie algebra \( \mathfrak{g} \) over a fixed perfect field \( k \)) on families of pointed stable curves of arbitrary characteristic. To proceed our discussion, we are forced to work under a certain condition concerning the characteristic \( \text{char}(k) \) of \( k \), i.e., either one of the two conditions \( (\text{Char})_0 \), \( (\text{Char})_p^W \) described in § 2.1. Here, the condition \( (\text{Char})_0 \) means that \( \text{char}(k) = 0 \), and the condition \( (\text{Char})_p^W \) means, roughly speaking, that \( \text{char}(k) \) is a sufficiently large prime relative to the rank of \( \mathfrak{g} \). One of key ingredients in this theory is an invariant called the radius (cf. Definition 2.9.1) associated with each \( (\mathfrak{g}, \hbar) \)-oper and each marked point of the underlying pointed stable curve. This invariant is defined as a certain point of the GIT quotient \( \mathfrak{c} \) (cf. \( [150] \) and the discussion in § 2.8) of the action on \( \mathfrak{g} \) by the adjoint group \( \mathbb{G} \) of \( \mathfrak{g} \). Let \( \rho \) be a \( k \)-rational point of the product \( \mathfrak{c}^\times \) of \( r \) copies of \( \mathfrak{c} \). Then, for each \( \hbar \in k \) and each pair of nonnegative integers \( (g, r) \) satisfying that \( 2g - 2 + r > 0 \), one may define the moduli functor

\[
\mathcal{O}_{p,g,h,r} \quad (\text{resp.}, \quad \mathcal{O}_{p,g,h,\rho,r})
\]

(cf. § 2.3, § 2.9, and § 3.12) classifying pointed stable curves over \( k \) of type \( (g, r) \) together with a \( (\mathfrak{g}, \hbar) \)-oper (resp., a \( (\mathfrak{g}, \hbar) \)-oper of radii \( \rho \)) on it. Here, denote by \( \mathfrak{M}_{g,r} \) (cf. § 1.5) the moduli stack classifying pointed stable curves over \( k \) of type \( (g, r) \). By passing to the morphisms determined by forgetting the data of
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a \((g, h)\)-oper, both \(\mathcal{O}_p_{g,h,g,r}\) and \(\mathcal{O}_p_{g,h,p,g,r}\) may be considered as functors on the category of \(\tilde{\mathcal{M}}_{g,r}\)-schemes. The main result of §2 is Theorem A asserted below, i.e., the representability of these moduli functors, which is a generalization of the results proved by A. Beilinson-V. Drinfeld (cf. [6] §3.4, Theorem; [3], Lemma 3.6) and S. Mochizuki (cf. [52], Chap. I, Proposition 2.11). (Note that the assertion corresponding to Theorem A in the text lies in §3 (cf. Theorem 3.12.1) for convenience of introducing the notations involved.)

**Theorem A (Structures of \(\mathcal{O}_p_{g,h,g,r}\) and \(\mathcal{O}_p_{g,h,p,g,r}\).)**

Let \(h \in k\) and \(\rho \in c^{x_r}(k)\) (where we take \(\rho = \emptyset\) if \(r = 0\)). Then, \(\mathcal{O}_p_{g,h,g,r}\) (resp., \(\mathcal{O}_p_{g,h,p,g,r}\)) may be represented by a relative affine space over \(\tilde{\mathcal{M}}_{g,r}\) of relative dimension \(\mathcal{N}(g)\) (resp., \(\mathcal{N}(g)\)), where

\[
\mathcal{N}(g) = (g - 1) \cdot \dim(g) + \frac{r}{2} \cdot (\dim(g) + \text{rk}(g))
\]

(resp., \(\mathcal{N}(g) = (g - 1) \cdot \dim(g) + \frac{r}{2} \cdot (\dim(g) - \text{rk}(g))\))

(cf. [111]). In particular, \(\mathcal{O}_p_{g,h,g,r}\) (resp., \(\mathcal{O}_p_{g,h,p,g,r}\)) is a nonempty, geometrically connected, and smooth Deligne-Mumford stack over \(k\) of dimension \(3g - 3 + r + \mathcal{N}(g)\) (resp., \(3g - 3 + r + \mathcal{N}(g)\)).

0.5. In §3, we study a theory of \(p\)-curvature associated with an \(h\)-log connection and a theory of (the moduli of) \((g, h)\)-opers in positive characteristic. In addition to the class of dormant \((g, h)\)-opers, there is an important class of \((g, h)\)-opers satisfying a certain condition concerning \(p\)-curvature, which we refer to as \(p\)-nilpotent \((g, h)\)-opers (cf. Definition 3.8.3). The moduli stack classifying \(p\)-nilpotent \((g, h)\)-opers may be obtained as a spacial fiber of a certain \(p\)-curvature analog of the Hitchin fibration (cf. [251]), which was also studied in [30] under the name of the Hitchin-Mochizuki morphism. We shall write

\[
\mathcal{O}_p^{g,h,g,r}_{z_{x_{\ldots}}} \quad \text{ (resp., } \mathcal{O}_p^{g,h,p,g,r}_{z_{x_{\ldots}}}; \text{ resp., } \mathcal{O}_p^{g,h,g,r}_{p\text{-nilp}}; \text{ resp., } \mathcal{O}_p^{g,h,p,g,r}_{p\text{-nilp}})\)
\]

for the moduli stack classifying pointed stable curves over \(k\) of type \((g, r)\) together with a dormant \((g, h)\)-oper (resp., a dormant \((g, h)\)-oper of radii \(\rho\)) resp., a \(p\)-nilpotent \((g, h)\)-oper; resp., a \(p\)-nilpotent \((g, h)\)-oper of radii \(\rho\) on it. The main result of §3 is the following Theorem B (cf. Theorem 3.12.2) and Theorem C (cf. Theorem 3.12.3). Theorem B may be thought of as a generalization of the results proved by S. Mochizuki (cf. [52], Chap I, Theorem 2.3) and T. H. Chen-X. Zhu (cf. [13], Corollary 3.4).

**Theorem B (Structures of \(\mathcal{O}_p^{g,h,g,r}_{p\text{-nilp}}\) and \(\mathcal{O}_p^{g,h,p,g,r}_{p\text{-nilp}}\).)**

Let \(h \in k\) and suppose that if \(r > 0\), then we are given \(\rho \in c^{x_r}(k)\) satisfying that \(\rho^P_h = [0]_k\) (cf. [241]). Then, \(\mathcal{O}_p^{g,h,g,r}_{p\text{-nilp}}\) (resp., \(\mathcal{O}_p^{g,h,p,g,r}_{p\text{-nilp}}\)) may be represented by
a nonempty and proper Deligne-Mumford stack over \( k \) of dimension \(3g - 3 + r\), and the natural projection \( \mathcal{O}_p^{n-nilp} \to \overline{M}_{g,r} \) (resp., \( \mathcal{O}_p^{\rho-nilp} \to \overline{M}_{g,r} \)) is finite and faithfully flat of degree \( p^\mathfrak{n}(\mathfrak{g}) \) (resp., \( p^\mathfrak{n}(\mathfrak{g}) \)). If, moreover, it is satisfied that \( \hbar = 0 \) and \( \rho = [\bar{0}]_k \), then the natural composite

\[
(7) \quad (\mathcal{O}_p^{n-nilp})_{\text{red}} \to (\mathcal{O}_p^{\rho-nilp} \to \overline{M}_{g,r}),
\]

where \( (\mathcal{O}_p^{\rho-nilp})_{\text{red}} \) denotes the reduced stack associated with \( \mathcal{O}_p^{\rho-nilp} \), is an isomorphism of \( k \)-stacks. In particular, \( \mathcal{O}_p^{\rho-nilp} \) is geometrically irreducible.

**Theorem C (Structures of \( \mathcal{O}_p^{\rho-nilp} \) and \( \mathcal{O}_p^{\rho-nilp} \)).**

Let \( \hbar \in k \) and \( \rho \in \mathfrak{c}^\times(r) \) (where we take \( \rho = \emptyset \) if \( r = 0 \)). Then, \( \mathcal{O}_p^{\rho-nilp} \) and \( \mathcal{O}_p^{\rho-nilp} \) may be represented, respectively, by either the empty stack or a proper Deligne-Mumford stack over \( k \), and the natural projection \( \mathcal{O}_p^{\rho-nilp} \to \overline{M}_{g,r} \) is finite. Moreover, the following assertions hold:

(i) If \( \hbar \in k^\times \), then the natural morphism

\[
(8) \quad \prod_{\rho' \in \mathfrak{c}^\times(r)(F_p)} \mathcal{O}_p^{\rho-nilp} \to \mathcal{O}_p^{\rho-nilp}
\]

is an isomorphism of stacks over \( \overline{M}_{g,r} \).

(ii) If \( \hbar = 0 \), then it is necessarily satisfied that \( \rho = [\bar{0}]_k \), and the natural composite

\[
(9) \quad (\mathcal{O}_p^{\rho-nilp})_{\text{red}} \to (\mathcal{O}_p^{\rho-nilp} \to \overline{M}_{g,r})
\]

where \( (\mathcal{O}_p^{\rho-nilp})_{\text{red}} \) denotes the reduced stack associated with \( \mathcal{O}_p^{\rho-nilp} \), is an isomorphism of \( k \)-stacks. In particular, \( \mathcal{O}_p^{\rho-nilp} \) is nonempty and geometrically irreducible of dimension \( 3g - 3 + r \).

**0.6.** In § 4, we consider a translation of \((\mathfrak{sl}_n, \hbar)\)-opers on a fixed pointed stable curve \( X/S = (X/S, \{\sigma_i\}_{i=1}^r) \) (cf. [57]) (in arbitrary characteristic) into various equivalent objects defined on \( X/S \). To this end, we shall introduce the moduli functors, after fixing a certain data \( \mathcal{U} := (\mathfrak{B}, \nabla_0) \) called an \((n, \hbar)\)-determinant.
data (cf. Definition 4.9.1 (i)), as described in the following:

\[ \Omega_{\mathfrak{sl}_n, h, X/S} \] : the moduli functor classifying \((\mathfrak{sl}_n, h)\)-opers on \(X/S\);

\[ \Omega_{\mathrm{GL}_n, h, X/S} \] : the moduli functor (cf. (408)) classifying equivalent classes (cf. Definition 4.2.2) of \((\mathrm{GL}_n, h)\)-opers on \(X/S\) (cf. Definition 4.2.1 (i));

\[ \Omega_{\mathrm{GL}_n, h, U, X/S} \] : the moduli functor (cf. (408)) classifying isomorphism classes of \((\mathrm{GL}_n, h, U)\)-opers (i.e., certain \(h\)-log connections on a fixed filtered vector bundle of rank \(n\)) on \(X/S\) (cf. Definition 4.9.4);

\[ \hbar\text{-Diff}^{\bullet}_{n, U, X/\log S/\log} \] : the moduli functor (cf. (394)) classifying \(h\)-twisted logarithmic crystalline differential operators on the line bundle \(B\) of order \(n\) with unit principal symbol and subprincipal symbol prescribed by \(\nabla_0\).

Here, a \((\mathrm{GL}_n, h)\)-oper on \(X/S\) is, roughly speaking, a rank \(n\) vector bundle on \(X\) equipped with an \(h\)-log connection (over \(S\)) and a complete flag which is not horizontal but obeys a strict form of Griffiths transversality with respect to the connection. The main result of §4 is as follows (cf. Corollary 4.11.3):

**Theorem D (Comparison of the moduli of opers).**

Suppose that there exists an \((n, h)\)-determinant data \(U := (B, \nabla_0)\) for \(X/\log S/\log\). Then, there exists a canonical sequence of isomorphisms

\[
\begin{align*}
\hbar\text{-Diff}^{\bullet}_{n, U, X/\log S/\log} & \xrightarrow{\Gamma_A} \Omega_{\mathrm{GL}_n, h, U, X/S} \xrightarrow{\Gamma_A} \Omega_{\mathrm{GL}_n, h, U, X/S} \xrightarrow{\Gamma_A} \Omega_{\mathfrak{sl}_n, h, X/S}
\end{align*}
\]

of functors.

If the fixed curve \(X/S\) is an unpointed proper smooth curve \(X_C\) over \(\mathbb{C}\) and \(U := (\mathcal{T}^{\otimes \frac{n-1}{2}}_{X_C/\mathbb{C}}, dx_C/\mathbb{C})\) (after choosing a theta characteristic \(\mathcal{T}^{\otimes \frac{1}{2}}_{X_C/\mathbb{C}}\)), where \(dx_C/\mathbb{C}\) denotes the universal derivation \(\mathcal{O}_{X_C} \to \Omega_{X_C/\mathbb{C}}\), then the result similar to Theorem D will be seen in, e.g., [6], [8].

In the case of positive characteristic, one verifies that there is a nice \((n, h)\)-determinant data \(U\) for the tautological curve over \(\mathcal{M}_{g,r}\) (cf. Proposition 4.13.2). By means of such \(U\), one verifies from Theorem D that (cf. Corollary 4.13.3 (i)) there exists a canonical isomorphism of stacks

\[
\begin{align*}
\Omega_{\mathrm{GL}_n, h, U, g,r}^{\otimes} & \xrightarrow{\sim} \Omega_{\mathfrak{sl}_n, h, g,r}^{\otimes}
\end{align*}
\]

where \(\Omega_{\mathrm{GL}_n, h, U, g,r}^{\otimes}\) (cf. (138)) denotes the moduli stack classifying pointed stable curves over \(k\) of type \((g, r)\) together with a dormant \((\mathrm{GL}_n, h, U)\)-oper.
(cf. Definition 4.13.1) on it. In particular, any dormant \( \mathfrak{s}_n \)-oper may be constructed, via projectivization, from a \((\text{GL}_n, h)\)-oper with vanishing \( p \)-curvature. This fact will be used in the proof of the generic étaleness of \( \mathfrak{Op}_{\mathfrak{s}_n, h, \hbar, \rho, g, r} \) asserted in Theorem F.

0.7. In §5, we study the deformation theory of (dormant) \((g, \hbar)\)-opers by means of cohomology. A key property is (cf. Proposition 5.8.1) that the Cartier operator may be identified, in a certain sense, with the differential of the morphism between relevant moduli stacks induced by the assignment from each \( \hbar \)-log connection to its \( p \)-curvature. By applying this property, we describe, toward the study in §7, the behavior of deformations (i.e., the tangent bundle of the moduli stacks) of (dormant) \((g, \hbar)\)-opers in terms of cohomology.

In §6, we analyze the behavior of clutching dormant \((g, \hbar)\)-opers along with the clutching morphism of the underlying stable curves. Under a certain assumption (i.e., the condition (Etale)\( g, h \) described in §6.3), we focus on a factorization property of the generic degree \( \text{deg}(\mathfrak{Op}_{\mathfrak{g}, \hbar, h, \hbar, \rho, g, r, r}/\mathbb{M}_{g, r}) \) of the moduli stack \( \mathfrak{Op}_{\mathfrak{g}, \hbar, h, \hbar, \rho, g, r, r} \) over \( \mathbb{M}_{g, r} \). This value \( \text{deg}(\mathfrak{Op}_{\mathfrak{g}, \hbar, h, \hbar, \rho, g, r, r}/\mathbb{M}_{g, r}) \) is well-defined by virtue of Proposition 6.3.2. Moreover, this generic degree consists with the number of dormant \((g, \hbar)\)-opers on a sufficiently general pointed stable curve of type \((g, r)\).

We see (cf. Theorem 6.4.1) that the function \( N_{g, p, \rho} : \text{Hom}(\mathfrak{g}, \mathbb{C}) \to \mathbb{Z} \) (cf. (635)) encoding these values forms a (nondegenerate) fusion rule. (For reviews and expositions concerning fusion rules, we refer to [4] or [23].) Consequently, the value \( \text{deg}(\mathfrak{Op}_{\mathfrak{g}, \hbar, h, \hbar, \rho, g, r, r}/\mathbb{M}_{g, r}) \) may be calculated combinatorially from the cases where \((g, r) = (0, 3)\) acceding to fusion rules. By means of this function, we introduce a certain commutative ring

\[
\mathfrak{F}_{\mathfrak{g}, p},
\]

which we refer to as the dormant operatic fusion ring of \( \mathfrak{g} \) of level \( p \) (cf. Definition 6.4.2). By applying a general theory of fusion rings, we shall prove some basic properties of the ring \( \mathfrak{F}_{\mathfrak{g}, p} \) as well as the value \( \text{deg}(\mathfrak{Op}_{\mathfrak{g}, \hbar, h, \hbar, \rho, g, r, r}/\mathbb{M}_{g, r}) \) (cf. the discussion in §6.4; Theorem 6.4.3; Remark 6.4.4). One of these results is described as follows (cf. Theorem 6.4.3):

**Theorem E** (Factorization property of \( \text{deg}(\mathfrak{Op}_{\mathfrak{g}, \hbar, h, \rho, g, r, r}/\mathbb{M}_{g, r}) \)).

Let \( g, r \geq 1 \), and \( \rho = (\rho_i)_{i=1}^r \in C^r(\mathbb{F}_p) \). Write \( \text{Hom}(\mathfrak{F}_{\mathfrak{g}, p}, \mathbb{R}) \) for the set of ring homomorphisms \( \mathfrak{F}_{\mathfrak{g}, p} \to \mathbb{R} \) (where \( \mathbb{R} \) denotes the field of real numbers) and \( \text{Cas} := \sum_{\lambda \in \mathbb{C}(\mathbb{F}_p)} \lambda \otimes \lambda \), where \( \otimes \) denotes the multiplication in \( \mathfrak{F}_{\mathfrak{g}, p} \). Then,
we have the equality

\[
\deg\left(\mathcal{O}_p^{\text{Zax}}/\mathcal{M}_{g,r}\right) = \sum_{\chi \in \text{Hom}(\mathcal{F}_{g,r})} \chi(Cas)^{g-1} \prod_{i=1}^{r} \chi(\rho_i).
\]

(13)


0.8. § 7 is devoted to prove the generic étaleness of \(\mathcal{O}_p^{\text{Zax}}\) (for \(n < p\)) over \(\mathcal{M}_{g,r}\). If \(n = 2\), then this fact was already proved, as we give a comment in Remark 7.8.3, by S. Mochizuki (cf. [53], Chap. II, § 2.8, Theorem 2.8). According to the discussions in §§ 5 and 6, it suffices to prove the claim that there is no nontrivial deformation of any dormant \(\mathfrak{sl}_n\)-oper on the pointed stable curve \(\mathcal{P}/k\) of type \((0, 3)\). We shall give a proof of this claim (cf. Corollary 7.7.5) on the basis of the principle that any dormant \(\mathfrak{sl}_n\)-oper on \(\mathcal{P}/k\) may be determined completely by its radii (as observed in [53]). Consequently, we obtain the following theorem (cf. Theorem 7.8.2):

**Theorem F (Generic étaleness of \(\mathcal{O}_p^{\text{Zax}}/\mathcal{M}_{g,r}\)).**

Let \(p\) be a prime, \(n\) a positive integer satisfying that \(n < p\), \(k\) a perfect field of characteristic \(p\), \(h \in k^\times\), and \(\rho \in c_{\mathfrak{sl}_n}(\mathbb{F}_p)\) (where \(\rho := \emptyset\) if \(r = 0\)). Then, the finite (relative) \(\mathcal{M}_{g,r}\)-scheme \(\mathcal{O}_p^{\text{Zax}}\) (cf. Theorem 3.12.3) is, if it is not empty, étale over the points of \(\mathcal{M}_{g,r}\) classifying totally degenerate curves. In particular, \(\mathcal{O}_p^{\text{Zax}}\) is generically étale over \(\mathcal{M}_{g,r}\) (i.e., any irreducible component that dominates \(\mathcal{M}_{g,r}\) admits a dense open subscheme which is étale over \(\mathcal{M}_{g,r}\)).

In § 8, we investigate relations between the moduli stack \(\mathcal{O}_p^{\text{Zax}}\) and \(\text{Quot}\)-schemes involved. As explained in § 0.3 above, we develop, in the present paper, an argument parallel to the argument given in [73], § 4-§ 5. In particular, a canonical isomorphism between such moduli stacks will be constructed (cf. Proposition 8.3.3). Hence, the problem computing the degree of the moduli in consideration reduces to computing the degree of a certain \(\text{Quot}\) scheme in positive characteristic. Moreover, by deforming various objects to objects in characteristic 0, we reduces to computing the degree of a certain \(\text{Quot}\) scheme over \(\mathbb{C}\). On the other hand, there is a formula given by Y. Holla for computing the degree of this \(\text{Quot}\)-scheme; it may be thought of as a spacial case of the Vafa-Intriligator formula which computes explicitly the Gromov-Witten invariants of the Grassmannian. By applying this formula, we thus conclude the following Theorem G (cf. Theorem 8.7.1). This theorem implies that Joshi’s conjecture holds for sufficiently general curves \(X\).
Theorem G (Joshi’s conjecture).
Suppose that \( p > n \cdot (g - 1) \) and \( h \in k^\times \). Then, the generic degree \( \deg(\mathcal{O}_{\mathbf{P}_{\mathfrak{sl}(n,h,g,0)}^\text{zzz}}/\overline{\mathcal{M}}_{g,0}) \)
of \( \mathcal{O}_{\mathbf{P}_{\mathfrak{sl}(n,h,g,0)}^\text{zzz}} \) over \( \overline{\mathcal{M}}_{g,0} \) is given by the following formula:

\[
\deg(\mathcal{O}_{\mathbf{P}_{\mathfrak{sl}(n,h,g,0)}^\text{zzz}}/\overline{\mathcal{M}}_{g,0}) = \frac{p^{n-1}(g-1)-1}{n!} \sum_{(\zeta_1, \ldots, \zeta_n) \in C^\times n, \zeta_i^{p-1}, \zeta_i \neq \zeta_j (i \neq j)} \frac{\left(\prod_{i=1}^{n-1} \zeta_i^{(n-1)(g-1)}\right)}{\prod_{i \neq j} (\zeta_i - \zeta_j)^{g-1}}.
\]

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1. Twisted logarithmic connections over log schemes

We begin by mentioning briefly the notion of an \( h \)-log connection (where \( h \) is a parameter) on a torsor over a log smooth scheme. For the definition and basic properties concerning an \( h \)-connection over a smooth variety over a field, we shall refer to, e.g., [1], [2], [72].

Let us fix a field \( k \), a connected smooth algebraic group \( \mathbb{G} \) over \( k \), where we denote by \( \mathfrak{g} \) its Lie algebra, i.e., the tangent space \( \mathfrak{g} := T_e \mathbb{G} \) (cf. §1.2 for the notation \( T_{(-)}(-) \)) of \( \mathbb{G} \) at the identity \( e \in \mathbb{G}(k) \).

1.1. First, we shall introduce our notation concerning algebraic groups and the associated Lie algebras.
Let \( h : T \to \mathbb{G} \) be a morphism of \( k \)-schemes. We write \( r_h \) (resp., \( l_h \)) for the right-translation (resp., the left-translation) by \( h \), i.e., the composite isomorphism

\[
(15) \quad r_h : T \times_k \mathbb{G} \xrightarrow{(id_T \times m) \circ ((id_T, h) \times id_G)} T \times_k \mathbb{G} \times_k G \xrightarrow{id_T \times m} T \times_k \mathbb{G}
\]

(resp., \( l_h : T \times_k G \xrightarrow{(id_T, h) \times id_G} T \times_k \mathbb{G} \times_k G \xrightarrow{id_T \times m} T \times_k \mathbb{G} \)),

where \( m \) denotes the multiplication in \( \mathbb{G} \) and \( sw \) denotes the isomorphism \( \mathbb{G} \times_k \mathbb{G} \xrightarrow{\sim} \mathbb{G} \times_k \mathbb{G} \) given by switching \((h_1, h_2) \mapsto (h_2, h_1)\). By the right \( \mathbb{G} \)-action \( r_{(-)} \) on \( \mathbb{G} \), we often identify \( g \) with the space of right-invariant vector fields on \( \mathbb{G} \). We shall write

\[
(16) \quad Ad_h := l_h \circ r_{h^{-1}} : T \times_k \mathbb{G} \to T \times_k \mathbb{G}
\]

for any morphism \( h : T \to \mathbb{G} \). Write \( e_T : T \to \mathbb{G} \) for the identity in \( \mathbb{G}(T) \), i.e.,

the composite \( T \to \text{Spec}(k) \xrightarrow{\sim} \mathbb{G} \). The adjoint representation

\[
(17) \quad Ad_G : \mathbb{G} \to \text{GL}(g)
\]

of \( \mathbb{G} \) is, by definition, the morphism of algebraic groups over \( k \) such that \( Ad_G(h) \) coincides with the automorphism \( dAd_h|_{e_T} \) of \( T \times_k g \) (= \( T_{e_T}((T \times_k \mathbb{G}) \), i.e.,

the differential (over \( T \)) of \( Ad_h \) at the identity \( e_T \). By abuse of notation, we often identify \( Ad_G(h) \) with the corresponding \( O_T \)-linear automorphism of \( O_T \otimes_k g \).

Write

\[
(18) \quad \text{ad} := dAd_G|_{e} : g \to \text{End}(g) \quad (= T_{id_g} \text{GL}(g)),
\]

i.e., \( \text{ad}(v)(v') = [v, v'] \) for \( v, v' \in g \). We equip \( g \) (resp., \( \text{End}(g) \)) with the left \( \mathbb{G} \)-action determined by \( Ad_G \) (resp., \( Ad_{GL(g)} \)). In particular, the morphism \( Ad_G \) of algebraic groups carries a left \( \mathbb{G} \)-action on \( \text{End}(g) \), i.e.,

the action given by

\[
(19) \quad (v, a) \mapsto Ad_{GL(g)}(Ad_G(v))(a) \quad (= Ad_G(v) \cdot a \cdot Ad_G(v^{-1}))
\]

for \( v \in \mathbb{G} \) and \( a \in \text{End}(g) \). Then, \( \text{ad} : g \to \text{End}(g) \) is compatible with the respective left \( \mathbb{G} \)-actions on \( g \) and \( \text{End}(g) \).

Recall that the Maurer-Cartan form on \( \mathbb{G} \) is the \( g \)-valued one-form on \( \mathbb{G} \), i.e., an element

\[
(20) \quad \Theta_G \in \Gamma(\mathbb{G}, \Omega_{\mathbb{G}/k} \otimes_k g) \quad (= \Gamma(\mathbb{G}, \Omega_{\mathbb{G}/k}) \otimes_k g)
\]

determined uniquely by the condition that

\[
(21) \quad h^{*}(\Theta_G)(v) = dl_{h^{-1}}|_{(id_T, h)}(v) \in \Gamma(T, \mathcal{O}_T \otimes_k g) \quad (= \Gamma(T, e_T^{*}(\mathcal{T}_{\mathbb{G}/k})))
\]

for any \( h : T \to \mathbb{G} \) and \( v \in \Gamma(T, h^{*}(\mathcal{T}_{\mathbb{G}/k})) \), where \( dl_{h^{-1}}|_{(id_T, h)} \) is the differential

\[
(22) \quad dl_{h^{-1}}|_{(id_T, h)} : (h^{*}(\mathcal{T}_{\mathbb{G}/k}) \xrightarrow{\sim}) (id_T, h)^{*}(\mathcal{T}_{T \times_k \mathbb{G}/T}) \to (id_T, e_T)^{*}(\mathcal{T}_{T \times_k \mathbb{G}/T}) \quad (\xrightarrow{\sim} e_T^{*}(\mathcal{T}_{\mathbb{G}/k}))
\]
of \(l_h\) at the \(T\)-rational point \((\text{id}_T, h) : T \to T \times_k \mathbb{G}\). If \(\pi_G : T \times_k \mathbb{G} \to \mathbb{G}\) denotes the natural projection, then this form \(\Theta_G\) satisfies the following properties:

\[
\begin{align*}
(23) & \quad (d(\pi_G \circ l_h)^\vee \otimes \text{id}_g)(\Theta_G) = d\pi_G^\vee(\Theta_G); \\
& \quad (d(\pi_G \circ r_h)^\vee \otimes \text{id}_g)(\Theta_G) = (\text{id}_{\Omega_{G/k}} \otimes \text{Ad}_G(h^{-1}))(d\pi_G^\vee(\Theta_G)); \\
& \quad d^i\Theta_G + \frac{1}{2} \cdot [\Theta_G \wedge \Theta_G] = 0,
\end{align*}
\]

where \(d^i\) denotes the exterior derivative for one-forms, and \([- \wedge -]\) denotes the \(\mathcal{O}_G\)-bilinear form on \(\Omega_{G/k} \otimes_k \mathfrak{g}\) obtained by tensoring the Lie bracket in \(\mathfrak{g}\) and the wedge product of \(\Omega_{G/k}\). If we equip \(\Gamma(\mathbb{G}, \Omega_{G/k} \otimes_k \mathfrak{g})\) with the left \(\mathbb{G}\)-action obtained by tensoring the left \(\mathbb{G}\)-actions \(d\text{Ad}_G\) on \(\Gamma(\mathbb{G}, \Omega_{G/k})\) and \(\text{Ad}_G\) on \(\mathfrak{g}\), then the first two properties in (23) implies that \(\Theta_G\) lies in the space of \(\mathbb{G}\)-invariants \(\Gamma(\mathbb{G}, \Omega_{G/k} \otimes_k \mathfrak{g})\).

Let \(\mathbb{G}'\) be a connected smooth algebraic group over \(k\) with the Lie algebra \(\mathfrak{g}'\) and \(w : \mathbb{G}' \to \mathbb{G}\) a morphism of algebraic groups over \(k\). (Hence, we have the \(k\)-linear morphism \(dw|_{e'} : \mathfrak{g}' \to \mathfrak{g}\), where \(e'\) denotes the identity in \(\mathbb{G}'\).) Then, one verifies that the Maurer-Cartan form \(\Theta_{G'}\) on \(\mathbb{G}'\) satisfies the equality

\[
(24) \quad (\text{id}_{\Omega_{G'/k}} \otimes dw|_{e'})(\Theta_{G'}) = (dw^\vee \otimes \text{id}_g)(\Theta_G) \in \Gamma(\mathbb{G}', \Omega_{G'/k} \otimes_k \mathfrak{g}).
\]

1.2. Next, we shall introduce our notation and conventions concerning torsors and the definition of an \(h\)-log scheme on a torsor over a \(\log\) scheme. Basic references for the notion of a log scheme (or, more generally, a log stack) are [39], [32], and [38].

For a log stack indicated, say, by \(Y^\log\), we shall write \(Y\) for the underlying stack of \(Y^\log\), and \(\alpha_Y : M_Y \to \mathcal{O}_Y\) for the morphism of sheaves of monoids defining the log structure of \(Y^\log\).

Let \(T^\log := (T, \alpha_T : M_T \to \mathcal{O}_T), Y^\log := (Y, \alpha_Y : M_Y \to \mathcal{O}_Y)\) be fine log stacks (cf. [39], (2.3)) and \(f^\log : Y^\log \to T^\log\) a log smooth (i.e., “smooth” in the sense of [39], (3.3)) morphism. Let us write \(T_{Y^\log/T^\log}\) for the sheaf of logarithmic derivations of \(Y^\log\) over \(T^\log\) (cf. Remark 4.1.1 for the precise definition of \(T_{Y^\log/T^\log}\), and \(\Omega_{Y^\log/T^\log}\) for its dual \(T_{Y^\log/T^\log}'\), i.e., the sheaf of logarithmic differentials of \(Y^\log\) over \(T^\log\). The log smoothness of \(Y^\log\) over \(T^\log\) implies that \(T_{Y^\log/T^\log}\), as well as \(\Omega_{Y^\log/T^\log}\), is locally free of finite rank (cf. [39], Proposition (3.10)).

A logarithmic chart (or log chart) on \(Y^\log\) (over \(T^\log\)) is a collection of data \((U, \{x_j\}_{j=1}^l)\) consisting of an étale \(Y\)-scheme \(U\) (where we equip \(U\) with a log structure obtained by pulling back the log structure of \(Y^\log\), and denote the resulting log stack by \(U^\log\)) and a system of logarithmic coordinates \(\{x_j\}_{j=1}^l \subseteq \Gamma(U, M_U)\) (i.e., \(\Omega_{U^\log/T^\log} \cong \bigoplus_{j=1}^l \mathcal{O}_U \cdot d\log(x_j)\)). We denote by \(\partial_{x_1}, \ldots, \partial_{x_l}\) the dual basis of \(d\log(x_1), \ldots, d\log(x_j)\) (hence \(T_{U^\log/T^\log} \cong \bigoplus_{j=1}^l \mathcal{O}_U \cdot \partial_{x_j}\) ).
For a vector bundle (i.e., a locally free $\mathcal{O}_Y$-module of finite rank) $\mathcal{V}$ on $Y$, we shall denote by $\mathcal{V}(\mathcal{V})$ the relative affine space associated with $\mathcal{V}$, i.e., the spectrum of the symmetric algebra $S_{\mathcal{O}_Y}(\mathcal{V}^\vee)$ on $\mathcal{V}^\vee$ over $\mathcal{O}_Y$.

If $Y$ is assumed to be either a scheme over $k$ or a Deligne-Mumford stack (not necessarily smooth) over $k$, then we write $TY$ for (the total space of) the tangent bundle of $Y$ over $k$, i.e., $TY := \mathcal{V}(T_{Y/k})$. If, moreover, $q$ is a $k$-rational point of $Y$, then we write $T_qY$ for the $k$-vector space defined as the tangent space of $Y$ at $q$.

Let $\pi : \mathcal{E} \to Y$ be a right $G$-torsor over $Y$ in the fppf topology. (Since $G$ is smooth, $\mathcal{E}$ is also locally trivial in the étale topology.) If $\mathfrak{h}$ is a $k$-vector space equipped with a left $G$-action, then we shall write $\mathfrak{h}_\mathcal{E}$ for the vector bundle on $Y$ associated with the relative affine space $\mathcal{E} \times^G \mathfrak{h} := ((\mathcal{E} \times_k \mathfrak{h})/G)$, i.e., $\mathcal{V}(\mathfrak{h}_\mathcal{E}) \cong \mathcal{E} \times^G \mathfrak{h}$. If $\mathfrak{h}'$ is a $k$-vector space equipped with a left $G$-action and $d : \mathfrak{h} \to \mathfrak{h}'$ is a $k$-linear morphism that is compatible with the respective $G$-actions, then we shall write $d_\mathcal{E} : \mathfrak{h}_\mathcal{E} \to \mathfrak{h}'_\mathcal{E}$ for the $\mathcal{O}_Y$-linear morphism arising from $d$ twisted by $\mathcal{E}$.

By pulling-back the log structure of $Y^{\log}$ via $\pi : \mathcal{E} \to Y$, one may obtain a log structure on $\mathcal{E}$; we denote the resulting log stack by $\mathcal{E}^{\log}$. The right $G$-action on $\mathcal{E}$ carries a right $G$-action on the direct image $\pi_* (T_{\mathcal{E}^{\log}}/T^{\log})$ (resp., $\pi_* (T_{\mathcal{E}^{\log}}/Y^{\log})$) of the sheaf of logarithmic derivations $T_{\mathcal{E}^{\log}}/T^{\log}$ (resp., $T_{\mathcal{E}^{\log}}/Y^{\log}$). Denote by
\begin{equation}
(25) \quad \widetilde{T}_{\mathcal{E}^{\log}}/T^{\log}, \quad \widetilde{T}_{\mathcal{E}^{\log}}/Y^{\log}
\end{equation}
the subsheaves of $G$-invariant sections of $\pi_* (T_{\mathcal{E}^{\log}}/T^{\log})$, $\pi_* (T_{\mathcal{E}^{\log}}/Y^{\log})$ respectively. If we identify $\mathfrak{g}$ with the space of right-invariant vector fields on $G$ in a natural way, then $\widetilde{T}_{\mathcal{E}^{\log}}/Y^{\log}$ may be identified with $\mathfrak{g}_\mathcal{E}$, i.e., the adjoint vector bundle associated with $\mathcal{E}$. The natural short exact sequence of tangent bundles (i.e., the dual of the exact sequence in [39], Proposition (3.12)) induces a short exact sequence
\begin{equation}
(26) \quad 0 \longrightarrow \mathfrak{g}_\mathcal{E} \longrightarrow \widetilde{T}_{\mathcal{E}^{\log}}/T^{\log} \overset{a^{\log}_\mathcal{E}}{\longrightarrow} \mathcal{T}_{Y^{\log}}/T^{\log} \longrightarrow 0
\end{equation}
of $\mathcal{O}_Y$-modules. The pair $(\widetilde{T}_{\mathcal{E}^{\log}}/T^{\log}, a^{\log}_\mathcal{E})$ forms a Lie algebroid on $Y$ (in the logarithmic sense), often referred to as the Atiyah algebra of $\mathcal{E}$.

**Definition 1.2.1.**

Let $h \in \Gamma(T, \mathcal{O}_T)$.

(i) A $T$-$h$-logarithmic connection (or $T$-$h$-log connection) on $\mathcal{E}$ is an $\mathcal{O}_Y$-linear morphism $\nabla_\mathcal{E} : \mathcal{T}_{Y^{\log}}/T^{\log} \to \widetilde{T}_{\mathcal{E}^{\log}}/T^{\log}$ such that $a^{\log}_\mathcal{E} \circ \nabla_\mathcal{E} = h \cdot \text{id}_{\mathcal{T}_{Y^{\log}}/T^{\log}}$. (In particular, giving an $S$-$0$-log connection on $\mathcal{E}$ is equivalent to giving an $\mathcal{O}_Y$-linear morphism $\mathcal{T}_{Y^{\log}}/\mathcal{O}_Y \to \mathfrak{g}_\mathcal{E}$.) For simplicity, a $T$-$1$-log connection may be often referred to as a $T$-log connection.
(ii) Let $\nabla_E : T_{Y^\log/T^\log} \to \tilde{T}_{E^\log/T^\log}$ be a $T$-$h$-log connection on $E$. Consider the $O_Y$-linear morphism

$$\psi(E, \nabla_E) : \bigwedge^2 T_{Y^\log/T^\log} \to \mathfrak{g}_E$$

determined by

$$\psi(E, \nabla_E)(a \wedge b) = [\nabla_E(a), \nabla_E(b)] - h \cdot \nabla_E([a, b])$$

for local sections $a, b \in T_{Y^\log/T^\log}$, to which we refer as the curvature of $(E, \nabla_E)$. We shall say that $\nabla_E$ is integrable if $\psi(E, \nabla_E)$ vanishes identically on $Y$. (If $f^\log : Y^\log \to T^\log$ is isomorphic to the underlying morphism of a pointed stable curve over $T$ (cf. §1.5), then any $T$-$h$-log connection is necessarily integrable.)

(iii) An $h$-logarithmic integrable $G$-torsor (or $h$-log integrable $G$-torsor) over $Y^\log/T^\log$ is a pair $(G, \nabla_G)$ consisting of a right $G$-torsor $G$ over $Y$ and an integrable $T$-$h$-log connection $\nabla_G$ on $G$. For simplicity, a 1-log integrable $G$-torsor may be often referred to as a log integrable $G$-torsor.

1.3. Let $T^\log, Y^\log$, and $E$ be as above, and $\nabla_E$ a $T$-$h$-log connection on $E$. Also, let $F$ be a right $G$-torsor over $Y$ and $\eta : F \to E$ an isomorphism of $G$-torsors. By applying the functor $\pi_*(-)$ composed with $(-)^G$ to the differential of $\eta^{-1}$, one obtains an isomorphism $\pi_*(d(\eta^{-1}))^G : \tilde{T}_{E^\log/T^\log} \to \tilde{T}_{F^\log/T^\log}$ of $O_Y$-modules that is compatible with the surjections $a^\log_E$ and $a^\log_F$. Hence, the composite

$$\nabla_{E, \eta} := \pi_*(d(\eta^{-1}))^G \circ \nabla_E : T_{Y^\log/T^\log} \to \tilde{T}_{F^\log/T^\log}$$

forms a $T$-$h$-log connection on $F$.

Let us assume that $F = E$ (i.e., $\eta$ is an automorphism of $E$). Then, the assignment $\eta \mapsto \pi_*(d(\eta^{-1}))^G$ determines a group homomorphism $\text{Aut}_G(E) \to \text{Aut}_{O_Y}(\tilde{T}_{E^\log/T^\log})$, where $\text{Aut}_G(E)$ denotes the group of automorphisms of the right $G$-torsor $E$ and $\text{Aut}_{O_Y}(\tilde{T}_{E^\log/T^\log})$ denotes the group of automorphisms of the $O_Y$-module $\tilde{T}_{E^\log/T^\log}$. Moreover, the assignment $(\eta, \nabla_E) \mapsto \nabla_{E, \eta}$ determines a right action of $\text{Aut}_G(E)$ on the set of $T$-$h$-log connections on $E$. Since $\pi_*(d(\eta^{-1}))^G$ is compatible with the Lie bracket operator in $\tilde{T}_{E^\log/T^\log}$, the integrability of $\nabla_E$ (cf. Definition 1.2.1 (ii)) does not depend on executing the action by any element in $\text{Aut}_G(E)$. In the following, we shall describe the local behavior of the connection $\nabla_E$ upon executing the action by the automorphism $\eta : E \to E$.

Suppose that we are given a trivialization $\tau : E \to Y \times_k G$ of the right $G$-torsor $E$. Write $\pi^\tau_1 : E \to Y$ (resp., $\pi^\tau_2 : E \to G$) for the composite of $\tau$ with the projection $Y \times_k G \to Y$ (resp., $Y \times_k G \to G$) to the first (resp., second) factor.
Then, since $T_{\log/\log}$ is naturally isomorphic to $\pi_Y^* (T_{Y/\log} \oplus T_{G/k})$, we have

$$
(\text{id}_Y, e_Y)^* (T_{\log/\log}) \xrightarrow{\sim} (\text{id}_Y, e_Y)^* (\pi_Y^* (T_{Y/\log} \oplus T_{G/k}))

\xrightarrow{\sim} T_{Y/\log} \oplus (O_Y \otimes_k g).
$$

In particular, we have a canonical isomorphism $\pi_* (d\tau)^G : g \cong \sim O_Y \otimes_k g$. Thus, by means of the identification of $g$ with the space of right-invariant vector fields on $G$, we obtain a canonical isomorphism

$$
\epsilon^\tau_Y : T_{Y/\log} \oplus (O_Y \otimes_k g) \xrightarrow{\sim} T_{\log/\log}
$$

i.e., a split surjection $\epsilon^\tau_Y : \tilde{T}_{\log/\log} \twoheadrightarrow O_Y \otimes_k g$ of the short exact sequence (20). If we write

$$
\nabla_\epsilon := \epsilon^\tau_Y \circ \nabla_\tau, \quad \nabla_{\epsilon, \eta} := \epsilon^\tau_Y \circ \nabla_{\epsilon, \eta},
$$

then the $T\cdot h$-log connections $\nabla_\tau, \nabla_{\epsilon, \eta}$ may be expressed, via the isomorphism (31), as

$$
\nabla_\tau = (h \cdot \text{id}_{T_{Y/\log}}, \nabla_\epsilon), \quad \nabla_{\epsilon, \eta} = (h \cdot \text{id}_{T_{Y/\log}} \circ \nabla^\tau_\epsilon)
$$

respectively.

Observe that if $h_\eta : Y \rightarrow G$ denotes the composite

$$
h_\eta : Y \xrightarrow{(\text{id}_Y, e_Y)} Y \times_k G \xrightarrow{\tau \circ \eta \circ \tau^{-1}} Y \times_k G \xrightarrow{\pi_G} G,
$$

then the automorphism $\tau \circ \eta \circ \tau^{-1}$ of $Y \times_k G$ may be expressed as the left-translation $t_{h_\eta} : Y \times_k G \xrightarrow{\sim} Y \times_k G$ (cf. (15)) by $h_\eta$.

**Proposition 1.3.1.**

The automorphism of $T_{Y/\log} \oplus (O_Y \otimes_k g)$ corresponding, via the isomorphism (31), to $\pi_* (d\eta^{-1})^G$ coincides with

$$
(\text{id}_{T_{Y/\log}}, (dh_\eta \otimes \text{id}_g)(\Theta_G) + \text{Ad}_G (h_\eta)),
$$

where we identify $(dh_\eta \otimes \text{id}_g)(\Theta_G) \in \Gamma(Y, \Omega_{Y/\log} \otimes_k g)$ with the corresponding $O_Y$-linear morphism $T_{Y/\log} \oplus (O_Y \otimes_k g) \rightarrow O_Y \otimes_k g$. In particular, we have the equality

$$
\nabla_\epsilon^\tau_\eta = h \cdot (dh_\eta \otimes \text{id}_g)(\Theta_G) + \text{Ad}_G (h_\eta) \circ \nabla_\epsilon
$$

of morphisms $T_{Y/\log} \rightarrow O_Y \otimes_k g$

**Proof.** The latter assertion follows from the former assertion. We shall prove the former assertion. By the above discussion, we may assume that $E = Y \times_k G$, $\tau = \text{id}_E$, and $\eta = t_{h_\eta}$. Since $\pi_* (dh_\eta)^G : \tilde{T}_{\log/\log} \xrightarrow{\sim} \tilde{T}_{\log/\log}$ is compatible with the surjection $a^\log_E : \tilde{T}_{\log/\log} \twoheadrightarrow T_{Y/\log}$, the automorphism of $T_{Y/\log} \oplus$
\((\mathcal{O}_Y \otimes_k \mathfrak{g})\) corresponding to \(\pi_* (d\eta^G)\) may be expressed as \((\text{id}_{\mathcal{T}_{Y\log/T\log}}, \mathfrak{D}^1_\eta + \mathfrak{D}^2_\eta)\) for some \(\mathcal{O}_Y\)-linear morphisms

\[\mathfrak{D}^1_\eta : \mathcal{T}_{Y\log/T\log} \to \mathcal{O}_Y \otimes_k \mathfrak{g}, \quad \text{and} \quad \mathfrak{D}^2_\eta : \mathcal{O}_Y \otimes_k \mathfrak{g} \to \mathcal{O}_Y \otimes_k \mathfrak{g}.\]

We shall prove that \(\mathfrak{D}^1_\eta = (dh^Y_\eta \otimes \text{id}_\mathfrak{g})(\Theta_G)\) and \(\mathfrak{D}^2_\eta = \text{Ad}_G(h^{-1}_\eta).\)

By the definition of the isomorphism \((31)\), the automorphism of \(\mathcal{T}_{Y\log/T\log} \oplus (\mathcal{O}_Y \otimes_k \mathfrak{g})\) given by \(\pi_* (d\eta^G)\) coincides with \(d\text{Ad}_{h_\eta}|_{e_Y}\). This implies that the restriction \(\mathfrak{D}^2_\eta\) of \(\pi_* (d\eta^G)\) to \(\mathcal{O}_Y \otimes_k \mathfrak{g}\) \((\subseteq \mathcal{T}_{Y\log/T\log} \oplus (\mathcal{O}_Y \otimes_k \mathfrak{g}))\) coincides with \(\text{Ad}_G(h_\eta)\). On the other hand, for any local sections \(\partial \in \mathcal{T}_{Y\log/T\log}\),

\[\mathfrak{D}^1_\eta(\partial) = d\text{Ad}_{h^{-1}_\eta}|_{(\text{id}_Y,e_Y)}((\partial,0)) = dh^{-1}_\eta \circ d\tau_{h_\eta}(e_Y((\partial,0))) = dh^{-1}_\eta(e_Y((\partial,0))) = dh^{-1}_\eta|_{(\text{id}_Y,h_\eta)}(dh_\eta(\partial)) = h^*_\eta(\Theta_G)(dh_\eta(\partial)) = (dh^Y_\eta \otimes \text{id}_\mathfrak{g})(\Theta_G)(\partial).\]

Hence, we conclude that \(\mathfrak{D}^1_\eta = (dh^Y_\eta \otimes \text{id}_\mathfrak{g})(\Theta_G).\) \(\square\)

We shall consider the relationship between the curvature \(\psi^{(\mathcal{E}, \nabla')}\) of \((\mathcal{E}, \nabla')\) (cf. Definition 1.2.1 (ii)) and the curvature \(\psi^{(\mathcal{E}, \nabla, \eta)}\) of \((\mathcal{E}, \nabla, \eta)\). Here, note that the curvature \(\psi^{(\mathcal{E}, \nabla)}\) may be given by assigning

\[\psi^{(\mathcal{E}, \nabla)}(a \wedge b) = [\overline{\nabla}'(a), \overline{\nabla}'(b)] - h \cdot \overline{\nabla}'([a, b]).\]

**Corollary 1.3.2.**

We have the equality

\[\psi^{(\mathcal{E}, \nabla, \eta)} = \text{Ad}_G(h^{-1}_\eta) \circ \psi^{(\mathcal{E}, \nabla')}\]

of morphisms \(\bigwedge^2 \mathcal{T}_{Y\log/T\log} \to \mathfrak{g}_\mathcal{E} \ (= \mathcal{O}_Y \otimes_k \mathfrak{g}).\)
Proof. We shall write \( \mathfrak{D}_h^1 := (dh^\vee \otimes \text{id}_g)(\Theta_G) \) and \( \mathfrak{D}_h^2 := \text{Ad}_G(h^{-1}) \) for simplicity. Then, for local sections \( a, b \in T_{Y_{\log}/T_{\log}} \),

\[
[\mathfrak{D}_h^1(a), \mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}(b)] = [\Theta_G(dh_y(a)), \mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}(b)]
\]

\[
= \mathfrak{D}_h^2([(\text{id} \otimes \text{Ad}_G(h_y))(d\pi^\vee_G(\Theta_G))(d(\text{id}_Y, h_y)(a)), \nabla^p_{\mathfrak{E}}(b))]
\]

\[
= \mathfrak{D}_h^2([(d\pi_G \circ r_{h^{-1}_y})^\vee \otimes \text{id}_g(\Theta_G)(d(\text{id}_Y, h_y)(a)), \nabla^p_{\mathfrak{E}}(b))]
\]

\[
= \mathfrak{D}_h^2((\Theta_G(d\pi_G \circ r_{h^{-1}_y} \circ (\text{id}_Y, h_y)(a)), \nabla^p_{\mathfrak{E}}(b))]
\]

\[
= \mathfrak{D}_h^2((\text{Ad}_G(d\pi_Y)(a)), \nabla^p_{\mathfrak{E}}(b))
\]

\[
= \mathfrak{D}_h^2([0, \nabla^p_{\mathfrak{E}}(b)]) = 0.
\]

Thus, we have

\[
\psi^{(\xi, \nabla_{\mathfrak{E}, \eta})}(a \wedge b)
\]

\[
= \nabla^p_{\mathfrak{E}, \eta}(a), \nabla^p_{\mathfrak{E}, \eta}(b)) - h \cdot \nabla^p_{\mathfrak{E}, \eta}([a, b])
\]

\[
= [h \cdot \mathfrak{D}_h^1(a) + \mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}(a), h \cdot \mathfrak{D}_h^1(b) + \mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}(b)]
\]

\[
- h \cdot (h \cdot \mathfrak{D}_h^1([a, b]) + \mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}([a, b]))
\]

\[
= h^2 \cdot ([\mathfrak{D}_h^1(a), \mathfrak{D}_h^1(b)] - \mathfrak{D}_h^1([a, b]))
\]

\[
+ h \cdot ([\mathfrak{D}_h^1(a), \mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}(b)] + [\mathfrak{D}_h^1(b), \mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}(a)])
\]

\[
+ [\mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}(a), \mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}(b)] + \mathfrak{D}_h^2 \circ \nabla^p_{\mathfrak{E}}([a, b])]
\]

\[
= \mathfrak{D}_h^2([\nabla^p_{\mathfrak{E}}(a), \nabla^p_{\mathfrak{E}}(b)] + \nabla^p_{\mathfrak{E}}([a, b]))
\]

\[
= \text{Ad}_G(h^{-1}) \circ \psi^{(\xi, \nabla_{\mathfrak{E}})}(a \wedge b),
\]

where the fourth equality follows from (41) and the fact that \( \mathfrak{D}_h^1 : T_{Y_{\log}/T_{\log}} \to \mathcal{O}_Y \otimes_k g \) is compatible with the respective Lie bracket operators in the domain and codomain. It completes the proof of Corollary 1.3.2. \( \square \)

1.4. In this subsection, we consider the exponential map, as well as the logarithmic map, between the unipotent elements in \( G \) and nilpotent elements in \( g \) (under a certain condition on the characteristic of \( k ) \). Here, denote by \( p \) the characteristic \( \text{char}(k) \) of \( k \).

Suppose further that \( G \) is semisimple of adjoint type. Fix a Borel subgroup \( B \) of \( G \), and set \( N := [B, B] \). Denote by \( n \) the Lie algebra of \( N \). Let \( G^{\text{unip}} \) (resp., \( g^{\text{nilp}} \)) denotes the Zariski closed subset of all unipotent elements of \( G \) (resp., all nilpotent elements of \( g \)). If \( p = 0 \) (resp., \( p > 0 \), then we shall define

\[
\text{GL}(g)^{p-\text{unip}}
\]
to be the Zariski closed subset of all elements $b$ of $\text{GL}(g)$ satisfying that $(b - \text{id}_g)^m = 0$ for some $m > 0$ (resp., $(b - \text{id}_g)^p = 0$). Also, we shall define

\begin{equation}
\text{End}(g)^p\text{-nilp}
\end{equation}

to be the Zariski closed subset of all elements $a$ of $\text{End}(g)$ satisfying that $a^m = 0$ for some $m > 0$ (resp., $a^p = 0$). We endow the Zariski closed subsets $\mathbb{G}^{\text{unip}}, g^{\text{nilp}}, \text{GL}(g)^p\text{-unip},$ and $\text{End}(g)^p\text{-nilp}$ with the reduced subscheme structures. (The varieties $\mathbb{G}^{\text{unip}}$ and $g^{\text{nilp}}$ are called, respectively, the unipotent variety and the nilpotent variety of $g$ (cf. [43], Chap. VI, p. 256).) In particular, one obtains natural closed immersions $N \to \mathbb{G}^{\text{unip}}$ and $n \to g^{\text{nilp}}$.

Now, we suppose that the characteristic $\text{char}(k)$ of $k$ satisfies either one of the two conditions $(\text{Char})_0, (\text{Char})_p$ described as follows:

$(\text{Char})_0 : \text{char}(k) = 0$;

$(\text{Char})_p : \text{char}(k) > \text{rk}(\mathbb{G})$.

(Here, rk($\mathbb{G}$) denotes the rank of $\mathbb{G}$.) In particular, by [67], §4.2 and §4.6, the adjoint representation $\text{Ad}_G : \mathbb{G} \to \text{GL}(g)$ (cf. [17]) of $\mathbb{G}$ induces an isomorphism $\mathbb{G} \sim \text{Aut}(g)^0$, where $\text{Aut}(g)^0$ denotes the identity component of the group of Lie algebra automorphisms of $g$.

For any $a \in \text{End}(g)^p\text{-nilp}$ and $b \in \text{GL}(g)^p\text{-unip}$, the exponential

\begin{equation}
\exp(a) := \sum_{s=0}^{\infty} \frac{1}{s!} \cdot a^s \quad (= \sum_{s=0}^{p-1} \frac{1}{s!} \cdot a^s, \text{ if } p > 0)
\end{equation}

and the logarithm

\begin{equation}
\log(b) := \sum_{s=1}^{\infty} \frac{(-1)^{s-1}}{s} \cdot (b - \text{id}_g)^s \quad (= \sum_{s=1}^{p-1} \frac{(-1)^{s-1}}{s} \cdot (b - \text{id}_g)^s, \text{ if } p > 0)
\end{equation}

are well-defined. The assignment $a \mapsto \exp(a)$ determines an isomorphism

\begin{equation}
\exp : \text{End}(g)^p\text{-nilp} \sim \text{GL}(g)^p\text{-unip}
\end{equation}
of $k$-schemes and $b \mapsto \log(b)$ determines its inverse.

The action $\text{Ad}_{\text{GL}(g)}$ (resp., $\text{Ad}_{(-)}$) on $\text{End}(g)$ (resp., $\text{GL}(g)$) carries a left action of $\text{GL}(g)$ on $\text{End}(g)^p\text{-nilp}$ (resp., $\text{GL}(g)^p\text{-unip}$), which induces, via $\text{Ad}_G : \mathbb{G} \to \text{GL}(g)$, a left action of $\mathbb{G}$ on it. Then, $\exp$ and $\log$ are compatible with these $\text{GL}(g)$-actions (hence the induced $\mathbb{G}$-actions) on $\text{End}(g)^p\text{-nilp}$ and $\text{GL}(g)^p\text{-unip}$.

By restricting the natural isomorphism $\text{End}(g) \sim T_{\text{id}_g}(\text{GL}(g))$, one may identify $\text{End}(g)^p\text{-nilp}$ with $T_{\text{id}_g}(\text{GL}(g)^p\text{-unip})$. One verifies that the differential of $\exp$ coincides, via this identification, with the identity morphism of $\text{End}(g)^p\text{-nilp}$.

It follows from the condition on $p : \text{char}(k)$ supposed above that the image of $g^{\text{nilp}}$ (resp., $\mathbb{G}^{\text{unip}}$) via $\text{ad} : g \to \text{End}(g)$ (resp., $\text{Ad}_G : \mathbb{G} \to \text{GL}(g)$) lies in
End(\mathfrak{g})^{p\text{-nilp}} (\text{resp.}, \text{GL}(\mathfrak{g})^{p\text{-unip}}); denote by

\begin{equation}
(48) \quad \text{ad}|_{\mathfrak{g}^{\text{nilp}}} : \mathfrak{g}^{\text{nilp}} \to \text{End}(\mathfrak{g})^{p\text{-nilp}} \quad (\text{resp.}, \quad \text{Ad}_G|_{\mathbb{G}^{\text{unip}}} : \mathbb{G}^{\text{unip}} \to \text{GL}(\mathfrak{g})^{p\text{-unip}})
\end{equation}

the resulting morphism obtained by restricting \text{ad} (\text{resp., } \text{Ad}_G). It follows from the definition of \mathfrak{crp} that for each \(a \in \mathfrak{g}^{\text{nilp}}\), its image \(\mathfrak{crp}(\text{ad}(a))\) in \(\text{GL}(\mathfrak{g})^{p\text{-unip}}\) defines a Lie algebra automorphism of \(\mathfrak{g}\). This implies (since \(\mathbb{G} \cong \text{Aut}(\mathfrak{g})^0\)) that the composite \(\mathfrak{crp} \circ \text{ad}|_{\mathfrak{g}^{\text{nilp}}} : \mathfrak{g}^{\text{nilp}} \to \text{GL}(\mathfrak{g})^{p\text{-unip}}\) factors through the injection \(\text{Ad}_G|_{\mathbb{G}^{\text{unip}}}\). Moreover, by the identification \(\text{End}(\mathfrak{g})^{p\text{-nilp}} \cong \mathcal{T}_{\text{id}}(\text{GL}(\mathfrak{g})^{p\text{-unip}})\) obtained above, the resulting morphism \(\mathfrak{g}^{\text{nilp}} \to \mathbb{G}^{\text{unip}}\) defines, by restricting, a morphism

\begin{equation}
(49) \quad \mathfrak{crp}|_n : n \to N
\end{equation}

of \(k\)-schemes. Since \(n\) and \(N\) are irreducible and have the same dimension, the morphism \(\mathfrak{crp}|_n\) turns out to be an isomorphism. Thus, we obtain a commutative square

\begin{equation}
(50) \quad \begin{array}{ccc}
\mathfrak{crp}|_n & \cong & \text{End}(\mathfrak{g})^{p\text{-nilp}} \\
\downarrow \quad \iota & & \downarrow \quad \iota \\
N & \xrightarrow{\text{Ad}_G|_n} & \text{GL}(\mathfrak{g})^{p\text{-unip}},
\end{array}
\end{equation}

where the arrows in this diagram are all compatible with the respective left \(G\)-actions of the domain and codomain.

In the following part of this subsection, we shall relax, when considering the case where \(k\) is of positive characteristic, the condition on char\((k)\) (i.e., \((\text{Char})_p\), supposed above) to a slightly generalized condition \((\text{Char})_p\) described as follows:

\((\text{Char})_p^\Pi : \text{char}(k) = p > 0 \text{ and } \mathbb{G} \text{ is isomorphic to a direct product } \prod_{l=1}^L \mathbb{G}_l, \) \text{ where each } \mathbb{G}_l (l = 1, \cdots, L) \text{ is a connected semisimple algebraic groups over } k \text{ of adjoint type satisfying that } p > \text{rk}(\mathbb{G}_l).

We shall define an isomorphism \(\mathfrak{crp}_N : n \to N\) as follows. If \((\text{Char})_0\) is satisfied, then we define \(\mathfrak{crp}_N := \mathfrak{crp}|_n\). On the other hand, let us suppose that \((\text{Char})_p^\Pi\) is satisfied. Denote by \(\mathfrak{g}_l (l = 1, \cdots, L)\) the Lie algebra of \(\mathbb{G}_l\) mentioned in the statement of \((\text{Char})_p^\Pi\). Then a fixed isomorphism \(\mathbb{G} \cong \prod_{l=1}^L \mathbb{G}_l\) allows us to consider each \(\mathbb{G}_l\) as a subgroup of \(\mathbb{G}\), as well as consider \(\mathfrak{g}_l\) as a sub-Lie algebra of \(\mathfrak{g}\). Let us define an isomorphism

\begin{equation}
(51) \quad \mathfrak{crp}_N : \left( \prod_{l=1}^L n \cap \mathfrak{g}_l \right) \to N \cong \left( \prod_{l=1}^L N \cap \mathbb{G}_l \right)
\end{equation}

to be the direct product \(\prod_{l=1}^L \mathfrak{crp}|_{n \cap \mathfrak{g}_l}\) of the isomorphisms \(\mathfrak{crp}|_{n \cap \mathfrak{g}_l} : n \cap \mathfrak{g}_l \to N \cap \mathbb{G}_l\) (i.e., the morphism \((49)\) of the case where the pair \((\mathbb{G}, \mathcal{B})\) is taken to be
Let $\mathcal{F} = Y \times_k \mathbb{N}$ (resp., $\mathcal{E}_G = Y \times_k G$) be the trivial right $\mathbb{N}$-torsor (resp., $G$-torsor) over $Y$. Let $h : Y \to \mathbb{N}$ be a $Y$-rational points of $\mathbb{N}$, and $\nabla_{\mathcal{E}_G}$ a $T$-connection on $\mathcal{E}_G$. Denotes (by abuse of notation) by $l_h$ the left-translation of $\mathcal{E}_G$ by the composite $Y \to \mathbb{N} \hookrightarrow G$. Then the morphism $\nabla^{id_{Y \times_k G}}_{\mathcal{E}_G,l_h}$ (i.e., the morphism of the case where the triple $(\mathcal{E}, \eta, \tau)$ is taken to be $(\mathcal{E}_G, l_h, \text{id}_{Y \times_k G})$) may be given by

$$\nabla^{id_{Y \times_k G}}_{\mathcal{E}_G,l_h}(\partial) = d\log_{\mathcal{F}}(h)(\partial) + \sum_{s=0}^{\infty} \frac{1}{s!} \cdot \text{ad}(\log_{\mathcal{F}}(h))^s(\nabla^{id_{Y \times_k G}}_{\mathcal{E}_G}(\partial))$$

for any local section $\partial \in \mathcal{T}_{Y}^{\log/T^{\log}}$. 

Thus, by combining Proposition 1.3.1 and Proposition 1.4.1 (and the equality (21)), we have obtained the following Corollary 1.4.2, which will be used in the proof of Proposition 2.2.5 and Proposition 2.7.3.

**Corollary 1.4.2.**

Let $\mathcal{E}_N = Y \times_k \mathbb{N}$ (resp., $\mathcal{E}_G = Y \times_k G$) be the trivial right $\mathbb{N}$-torsor (resp., $G$-torsor) over $Y$. Then the morphism $\nabla^{id_{Y \times_k G}}_{\mathcal{E}_G,l_h}$ (i.e., the morphism of the case where the triple $(\mathcal{E}, \eta, \tau)$ is taken to be $(\mathcal{E}_G, l_h, \text{id}_{Y \times_k G})$) may be given by

$$\nabla^{id_{Y \times_k G}}_{\mathcal{E}_G,l_h}(\partial) = d\log_{\mathcal{F}}(h)(\partial) + \sum_{s=0}^{\infty} \frac{1}{s!} \cdot \text{ad}(\log_{\mathcal{F}}(h))^s(\nabla^{id_{Y \times_k G}}_{\mathcal{E}_G}(\partial))$$

for any local section $\partial \in \mathcal{T}_{Y}^{\log/T^{\log}}$. 

Write

$$\log : N \to n$$

for the inverse of the isomorphism $\mathcal{E}_N$. The commutativity of the diagram (50) implies the following proposition.

**Proposition 1.4.1.**

For each $T$-rational point $h : T \to N$ of $N$, we have the equality

$$\text{Ad}_G(h) = \mathcal{E}_G(\text{ad}(\log_N(h)))$$

of $O_T$-linear automorphisms of $O_T \otimes_k g$.

Since the tangent bundle $\mathcal{T}_g/k$ on the $k$-scheme $g$ is canonically isomorphic to $O_g \otimes_k g$, the differential of $\log : N \to n$ composed with the inclusion $\text{incl} : O_N \otimes_k n \to O_N \otimes_k g$ defines an $O_N$-linear morphism, which we denote (by abuse of notation) by

$$\text{dlog} : \mathcal{T}_{N/k} \to O_N \otimes_k g.$$

This morphism is compatible with both the $G$-actions $\text{Ad}(\cdot)$ on $N$ and $\text{Ad}_G$ on $g$ in an evident sense, and the differential of $\log_N$ at the identity $e \in N$ coincides with the identity morphism of $n$. It follows (cf. (21)) that if we identify $\Theta_N \in \Gamma(N, \Omega_N \otimes_k n)$ with the corresponding $O_N$-linear morphism $\mathcal{T}_{N/k} \to O_N \otimes_k n$, then we have the equality

$$\text{dlog} = \text{incl} \circ \Theta_N.$$
Throughout the present paper, we fix a pair of nonnegative integers $(g, r)$ satisfying that $2g - 2 + r > 0$. Denote by $\overline{\mathcal{M}}_{g, r}$ the moduli stack of $r$-pointed stable curves (cf. [14], Definition 1.1) over $k$ of genus $g$ (i.e., of type $(g, r)$), and by $f_{\text{tau}} : \mathcal{C}_{g, r} \to \overline{\mathcal{M}}_{g, r}$ the tautological curve, with its $r$ marked points $s_1, \ldots, s_r : \overline{\mathcal{M}}_{g, r} \to \mathcal{C}_{g, r}$. Recall (cf. [44], Corollary 2.6 and Theorem 2.7; [15], § 5) that $\overline{\mathcal{M}}_{g, r}$ may be represented by a geometrically connected, proper, and smooth Deligne-Mumford stack over $k$ of dimension $3g - 3 + r$. Also, recall (cf. [38], Theorem 4.5) that $\overline{\mathcal{M}}_{g, r}$ has a natural log structure given by the divisor at infinity, where we shall denote the resulting log stack by $\overline{\mathcal{M}}_{g, r}^{\log}$. Also, by taking the divisor which is the union of the $s_i$'s and the pull-back of the divisor at infinity of $\overline{\mathcal{M}}_{g, r}$, we obtain a log structure on $\mathcal{C}_{g, r}$; we denote the resulting log stack by $\mathcal{C}_{g, r}^{\log}$. $f_{\text{tau}} : \mathcal{C}_{g, r} \to \overline{\mathcal{M}}_{g, r}$ extends naturally to a morphism $f_{\text{log}} : \mathcal{C}_{g, r}^{\log} \to \overline{\mathcal{M}}_{g, r}^{\log}$ of log stacks.

Next, let $S$ be a scheme (or, more generally, a stack) over $k$ and

$$\mathcal{X}/S := (f : X \to S, \{\sigma_i : S \to X\}_{i=1}^r)$$

a pointed stable curve over $S$ of type $(g, r)$, consisting of a (proper) semi-stable curve $f : X \to S$ over $S$ of genus $g$ and $r$ marked points $\sigma_i : S \to X$ ($i = 1, \ldots, r$).

$\mathcal{X}/S$ determines its classifying morphism $s : S \to \overline{\mathcal{M}}_{g, r}$ and an isomorphism $X \simeq S \times_{\overline{\mathcal{M}}_{g, r}} \mathcal{C}_{g, r}$ over $S$. By pulling-back the log structures of $\overline{\mathcal{M}}_{g, r}^{\log}$ and $\mathcal{C}_{g, r}^{\log}$, we obtain log structures on $S$ and $X$ respectively; we denote the resulting log stacks by $S^{\log}$ and $X^{\log}$. The structure morphism $f : X \to S$ extends to a morphism $f^{\log} : X^{\log} \to S^{\log}$ of log schemes, which is log smooth (cf. [39], § 3; [38], Theorem 2.6).

Denote by $D_{\mathcal{X}/S}$ the étale effective relative divisor on $X$ relative to $S$ defined to be the union of the image of the marked points $\sigma_i$ ($i = 1, \ldots, r$). Then, if we denote by $\omega_{\mathcal{X}/S}$ the dualizing sheaf of $X$ over $S$, then $\omega_{\mathcal{X}/S}$ is naturally isomorphic to $\Omega_{X^{\log}/S^{\log}}(-D_{\mathcal{X}/S})$.

In this way, we consider the pointed stable curve $\mathcal{X}/S$ as an object of log geometry.

Next, we introduce the notion of the monodromy of an $h$-log integrable torsor at a marked point of a pointed stable curve.

Let $\mathcal{G}, g$ be as in § 1.1, $S$ a scheme (or, more generally, a stack) over $k$, and $\mathcal{X}/S := (f : X \to S, \{\sigma_i : S \to X\}_{i=1}^r)$ a pointed stable curve over $S$ of type $(g, r)$. Let $u : U \to X$ be an étale morphism. Hence $U$ is a semi-stable curve (in the sense of [45], § 10.3, Definition 3.14) over $S$. Write $\sigma_i^U : U \times_{X, \sigma_i} S \to U$ for the base-change of $\sigma_i$ via $u$, and set the collection of data

$$\mathcal{U}/S := (U/S, \{\sigma_i^U\}_{i=1}^r).$$
We identify the pointed stable curve $X_S$ with $U_S$ of the case where $u : U \to X$ is taken to be $\id_X : X \to X$.) Denote by $D_{U_S}$ the restriction of the (possibly empty) divisor $D_X \subseteq X$ to $U$. Denote by $U_{\sm}$ the smooth locus of $U \setminus \Supp(D_{U_S})$ over $S$, which is a scheme-theoretically dense open subscheme of $U$. By pulling-back the log structure of $X^\log$ via $u$, one may equip $U$ with a log structure; we denote the resulting log scheme by $U^\log$. The restriction of this log structure to the open subscheme $U_{\sm}$ coincides with the pull-back of the log structure of $S^\log$ via $f \circ u$.

For an étale $U$-scheme indicated, say, by the notation "$U_\square$" for a certain index $\square$, we shall use (in response to the notation "$U/S$") the notation "$U_\square/S$" for indicating the base-change of $U/S$ to $U_{\square}/S$. Also, if $t : T \to S$ is an $S$-scheme, then we shall use the notation "$U_T$" for indicating the base-change of $U/S$ via $T \to S$.

Let us fix a right $G$-torsor $\pi : E \to U$ over $U$, which has a structure of log scheme $E^\log$ as we defined in § 1.1. Consider natural morphisms

$$t_U : \mathcal{T}_{U^\log/S^\log} \to \mathcal{T}_{U/S}, \quad t_E : \mathcal{T}_{E^\log/S^\log} \to \mathcal{T}_{E/S},$$

where $\mathcal{T}_{U/S}$ and $\mathcal{T}_{E/S}$ denote the sheaves of (non-logarithmic) derivations of $U/S$ and $E/S$ respectively. Since $\mathcal{T}_{U^\log/S^\log}$ and $\mathcal{T}_{E^\log/S^\log}$ are locally free, and $t_U$, $t_E$ are isomorphisms over the scheme-theoretically dense open subscheme $U_{\sm}$ of $U$, these morphisms are injective. The direct image $\pi_* (\mathcal{T}_{E^\log/S^\log}) \to \pi_* (\mathcal{T}_{E/S})$ of $t_E$ is compatible with the respective natural $G$-actions on $\pi_* (\mathcal{T}_{E^\log/S^\log})$ and $\pi_* (\mathcal{T}_{E/S})$. Hence, $t_E$ yields an injection

$$\tilde{t}_E : \tilde{T}_{E^\log/S^\log} \hookrightarrow \tilde{T}_{E/S} \quad (:= \pi_* (\mathcal{T}_{E/S})^G).$$

Moreover, we obtain a morphism of short exact sequences of $\mathcal{O}_U$-modules:

$$0 \longrightarrow \mathfrak{g}_E \longrightarrow \tilde{T}_{E^\log/S^\log} \longrightarrow \mathcal{T}_{U^\log/S^\log} \longrightarrow 0$$

$$0 \longrightarrow \mathfrak{g}_E \longrightarrow \tilde{T}_{E/S} \longrightarrow \mathcal{T}_{U/S} \longrightarrow 0.$$

Let us combine this diagram and the adjunction morphism $\eta_{(-)} : (-) \to \sigma^U_t (\sigma^U_{- \ast} (-))$ arising from the adjunction relation "$\sigma^U_t (\cdot) + \sigma^U_{\ast} (\cdot)$" (i.e., "the functor $\sigma^U_{\ast} (-)$ is left adjoint to the functor $\sigma^U_t (-)$"). Then, we obtain, for
each $i \in \{1, \ldots, r\}$, a morphism of sequences
\[(63)\]
\[
\begin{array}{ccccccc}
0 & \longrightarrow & \mathfrak{g}_\mathcal{E} & \longrightarrow & \tilde{T}_{\mathcal{E} \log / S_{\log}} & \longrightarrow & \mathcal{T}_{U_{\log} / S_{\log}} \longrightarrow 0 \\
\eta_{\mathfrak{g}_\mathcal{E}} & \downarrow & \eta_{T_{\mathcal{E} \log / S_{\log}}} \circ \tilde{\eta}_\mathcal{E} & \downarrow & \eta_{\mathcal{T}_{U_{\log} / S_{\log}}} \circ \tilde{\eta}_U & \\
0 & \longrightarrow & \sigma_{i*}(\sigma_{i*}(\mathfrak{g}_\mathcal{E})) & \longrightarrow & \sigma_{i*}(\tilde{T}_{\mathcal{E} \log / S_{\log}}) & \longrightarrow & \sigma_{i*}(\mathcal{T}_{U_{\log} / S_{\log}}) \longrightarrow 0.
\end{array}
\]
(The local triviality of the right $\mathbb{G}$-torsor $\mathcal{E}$ in the étale topology implies that the lower horizontal sequence is exact.) Since
\[(64)\]
\[
\sigma_{i*}(\sigma_{i*}(\mathcal{T}_{U/S})) \cong \sigma_{i*}(\sigma_{i*}(\mathcal{T}_{U/S} / t_U(\mathcal{T}_{U_{\log} / S_{\log}}))),
\]
the composite $\eta_{\mathcal{T}_{U_{\log} / S_{\log}}} \circ \tilde{\eta}_U : \mathcal{T}_{U_{\log} / S_{\log}} \rightarrow \sigma_{i*}(\sigma_{i*}(\mathcal{T}_{U/S}))$ (i.e., the right vertical arrow in the diagram (63)) is the zero map.

Now, let us take an $S$-$h$-$\log$ connection $\nabla_\mathcal{E}$ on $\mathcal{E}$. It follows from the above discussion that the composite
\[(65)\]
\[
(\eta_{T_{\mathcal{E} \log / S_{\log}}} \circ \tilde{\eta}_\mathcal{E}) \circ \nabla_\mathcal{E} : \mathcal{T}_{U_{\log} / S_{\log}} \rightarrow \sigma_{i*}(\sigma_{i*}(\tilde{T}_{\mathcal{E} \log / S_{\log}}))
\]
factors through the injection $\sigma_{i*}(\sigma_{i*}(\mathfrak{g}_\mathcal{E})) \hookrightarrow \sigma_{i*}(\sigma_{i*}(\tilde{T}_{\mathcal{E} \log / S_{\log}}))$. The resulting morphism $\mathcal{T}_{U_{\log} / S_{\log}} \rightarrow \sigma_{i*}(\sigma_{i*}(\mathfrak{g}_\mathcal{E}))$ corresponds, via the adjunction relation $\sigma_{i*}(-) \dashv \sigma_{i*}(-)^{\log}$, to a morphism
\[(66)\]
\[
\sigma_{i*}(\mathcal{T}_{U_{\log} / S_{\log}}) \rightarrow \sigma_{i*}(\mathfrak{g}_\mathcal{E}).
\]
Here, observe that there exists a canonical isomorphism
\[(67)\]
\[
\text{triv}_{\sigma_i, U} : \sigma_{i*}(\mathcal{T}_{U_{\log} / S_{\log}}) \sim \mathcal{O}_{U \times X, \sigma_i, S}.
\]
which maps any local section of the form $\sigma_{i*}^{U*}(d\log(x)) \in \sigma_{i*}^{U*}(\mathcal{T}_{U_{\log} / S_{\log}})$ (for a local function $x$ defining the closed subscheme $U \times X, \sigma_i, S$ of $U$) to $1 \in \mathcal{O}_{U \times X, \sigma_i, S}$. Thus, we obtain a global section
\[(68)\]
\[
\mu_{i}(\mathcal{E}, \nabla_\mathcal{E}) \in \Gamma(U \times X, \sigma_i, S, \sigma_{i*}(\mathfrak{g}_\mathcal{E})) \quad (= \Gamma(U \times X, \sigma_i, S, \mathfrak{g}_{\sigma_i}^{U*}(\mathcal{E})))
\]
determined by the image of $1 \in \Gamma(U \times X, \sigma_i, S, \mathcal{O}_{U \times X, \sigma_i, S})$ via the composite
\[(69)\]
\[
\mathcal{O}_{U \times X, \sigma_i, S} \xrightarrow{(\text{triv}_{\sigma_i, U})^{-1}} \sigma_{i*}(\mathcal{T}_{U_{\log} / S_{\log}}) \rightarrow \sigma_{i*}(\mathfrak{g}_\mathcal{E})
\]
where the second morphism is (66).

**Definition 1.6.1.**
We shall refer to $\mu_{i}(\mathcal{E}, \nabla_\mathcal{E})$ as the monodromy of $(\mathcal{E}, \nabla_\mathcal{E})$ at $\sigma_i^U$. 

2. Opers on a family of pointed stable curves

In this section, we discuss the definition and some basic properties of \((\mathfrak{g}, \hbar)\)-opers on a family of pointed stable curves in arbitrary characteristic. If a fixed stable curve has at least one marked point, then one may consider an additional invariant, which is called the \textit{radius} (cf. Definition 2.9.1; Definition 2.9.2), associated, for such a marked point, with the \((\mathfrak{g}, \hbar)\)-oper under consideration.

As one of the main results in this section, we prove (cf. Proposition 2.11.1) that for each pointed stable curve \(X/S = (X/S, \{\sigma_i\}_{i=1}^r)\), the moduli functor (which will be denoted by \(\mathcal{O}p_{\mathfrak{g}, \hbar, \rho, X/S}\)) classifying \((\mathfrak{g}, \hbar)\)-opers on \(X/S\) of a prescribed radii \(\rho\) may be represented by a certain relative affine space over \(S\).

Let \(k, S, X/S, G, g, \) and \(\hbar\) be as in §1. Suppose further (cf. §1.4) that \(G\) is a split connected semisimple algebraic group of adjoint type over \(k\). Let \(\mathbb{G}_m\) denotes the multiplicative group over \(k\).

2.1. To begin with, we shall recall a natural filtration on the Lie algebra \(\mathfrak{g}\).

Let us fix a \textit{pinning} (cf. [54], §1.2) of \(G\), which is, by definition, a collection of data
\[
(70) \quad \mathcal{G} := (G, T, B, \Gamma, \{x_\alpha\}_{\alpha \in \Gamma}),
\]
where \(T\) is a split maximal torus of \(G\), \(B\) is a Borel subgroup containing \(T\), \(\Gamma \subseteq X(T) := \text{Hom}(T, \mathbb{G}_m)\), i.e., the additive group of all characters of \(T\) denotes the set of simple roots in \(B\) with respect to \(T\), and each \(x_\alpha (\alpha \in \Gamma)\) is a generator of \(\mathfrak{g}_\alpha\). Here,
\[
(71) \quad \mathfrak{g}_\beta := \{x \in \mathfrak{g} \mid \text{ad}(t)(x) = \beta(t) \cdot x \text{ for all } t \in T\}
\]
for \(\beta \in X(T)\). Denote by \(\mathfrak{b}, \mathfrak{n}, \) and \(\mathfrak{t}\) the Lie algebras of \(B, N := [B, B]\), and \(T\) respectively (hence \(\mathfrak{t} \subseteq \mathfrak{n} \subseteq \mathfrak{b} \subseteq \mathfrak{g}\)). There exists uniquely a Lie algebra grading
\[
(72) \quad \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j
\]
(i.e., \([\mathfrak{g}_{j_1}, \mathfrak{g}_{j_2}] \subseteq \mathfrak{g}_{j_1 + j_2}\) for \(j_1, j_2 \in \mathbb{Z}\)) satisfying the following conditions:
- \(\mathfrak{g}_j = \mathfrak{g}_{-j} = 0\) for all \(j > \text{rk}(\mathfrak{g})\);
- \(\mathfrak{g}_0 = \mathfrak{t}, \mathfrak{g}_1 = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_\alpha\) and \(\mathfrak{g}_{-1} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha}\).

The associated decreasing filtration
\[
(73) \quad \mathfrak{g}^j := \bigoplus_{l \geq j} \mathfrak{g}_l
\]
\((j \in \mathbb{Z})\) on \(\mathfrak{g}\) is closed under the adjoint action of \(B\) and satisfies the following conditions
- \(\mathfrak{g}_{-j} = \mathfrak{g}\) and \(\mathfrak{g}^j = 0\) for all \(j > \text{rk}(\mathfrak{g})\);
- \(\mathfrak{g}^0 = \mathfrak{b}, \mathfrak{g}^1 = \mathfrak{n}, \) and \([\mathfrak{g}^{j_1}, \mathfrak{g}^{j_2}] \subseteq \mathfrak{g}^{j_1 + j_2}\) for \(j_1, j_2 \in \mathbb{Z}\).
Finally, let us write

\begin{equation}
    p_1 := \sum_{\alpha \in \Gamma} x_{\alpha}, \quad \rho := \sum_{\alpha \in \Gamma} \check{\omega}_{\alpha},
\end{equation}

where \( \check{\omega}_{\alpha} \) denotes the fundamental coweight of \( \alpha \). Then, there exists uniquely a collection \( (y_{\alpha})_{\alpha \in \Gamma} \), where \( y_{\alpha} \) is a generator of \( g^{-\alpha} \), such that if we write

\begin{equation}
    p_{-1} := \sum_{\alpha \in \Gamma} y_{\alpha} \in g_{-1},
\end{equation}

then the set \( \{p_{-1}, 2\rho, p_1\} \) forms an \( sl_2 \)-triple.

Denote by \( W \) the Weyl group of \( (G, T) \). In the rest of the present paper, we suppose either one of the two conditions \( (\text{Char})_0, (\text{Char})_W \) described below:

- \( (\text{Char})_0 : \text{char}(k) = 0; \)
- \( (\text{Char})_W : \text{char}(k) = p > 0 \) and \( p \) does not divide the order of \( W \).

In particular, if \( (\text{Char})_W \) is satisfied, then the condition \( (\text{Char})_P \) described in §1.4 is necessarily satisfied.

2.2. By means of the filtration \( \{g^j\}_{j \in \mathbb{Z}} \) defined above, we construct a filtration on the Lie algebroid associated with a right \( G \)-torsor admitting a structure of \( B \)-reduction. And then, we define the notion of a \( (g, h) \)-oper on a family of pointed stable curves (cf. Definition 2.2.1).

Let \( u : U \to X \) be an \( \acute{e}tale \) morphism of \( k \)-schemes and \( \pi_\mathbb{B} : \mathcal{E}_\mathbb{B} \to U \) a right \( \mathbb{B} \)-torsor over \( U \). Denote by \( \pi_G : (\mathcal{E}_\mathbb{B} \times^B G =: ) \mathcal{E}_G \to U \) the right \( G \)-torsor over \( U \) obtained by executing a change of structure group via the inclusion \( \mathbb{B} \hookrightarrow G \).

The natural morphism \( \mathcal{E}_\mathbb{B} \to \mathcal{E}_G \) yields a canonical isomorphism \( g_{\mathcal{E}_\mathbb{B}} \sim \to g_{\mathcal{E}_G} \).

Moreover, it gives a morphism between short exact sequences obtained by applying (26) to \( \mathcal{E}_\mathbb{B} \) and \( \mathcal{E}_G \) respectively:

\begin{equation}
\begin{array}{cccccc}
0 & \longrightarrow & b_{\mathcal{E}_\mathbb{B}} & \longrightarrow & T_{E_{\log}/S_{\log}}^{\mathcal{E}_\mathbb{B}} & \longrightarrow & T_{U_{\log}/S_{\log}} \longrightarrow & 0 \\
\downarrow \iota_{g/b} & & \downarrow \check{\iota}_{g/b} & & \downarrow \text{id} & & \\
0 & \longrightarrow & g_{\mathcal{E}_G} & \longrightarrow & T_{E_{\log}/S_{\log}}^{\mathcal{E}_G} & \longrightarrow & T_{U_{\log}/S_{\log}} \longrightarrow & 0,
\end{array}
\end{equation}

where both the left-hand and middle vertical arrows \( \iota_{g/b}, \check{\iota}_{g/b} \) are injective. Since \( g^j (\subseteq g) \) is closed under the adjoint action of \( B \) via \( \text{Ad}_{G} \), one may define vector bundles \( g^j_{\mathcal{E}_\mathbb{B}} \) \( (j \in \mathbb{Z}) \) associated with \( \mathcal{E}_\mathbb{B} \times^B g^j \) (cf. §1.1). The collection \( \{g^j_{\mathcal{E}_\mathbb{B}}\}_{j \in \mathbb{Z}} \) forms a decreasing filtration on \( g_{\mathcal{E}_\mathbb{B}} \) \( \sim \to g_{\mathcal{E}_G} \).

On the other hand, the diagram (76) induces a composite isomorphism

\begin{equation}
    g_{\mathcal{E}_\mathbb{B}}/g_{\mathcal{E}_\mathbb{B}}^0 \sim \to g_{\mathcal{E}_G}/\iota_{g/b} (b_{\mathcal{E}_\mathbb{B}}) \sim \to \check{T}_{E_{\log}/S_{\log}}^{\mathcal{E}_\mathbb{B}}/\check{\iota}_{g/b} (\check{T}_{E_{\log}/S_{\log}}^{\mathcal{E}_\mathbb{B}}). \tag{77}
\end{equation}
The filtration \( \{ \mathfrak{g}^{j}_{E_{\mathfrak{g}}} \}_{j \leq 0} \) carries, via this composite isomorphism, a decreasing filtration \( \mathcal{T}^{j}_{E_{G}/S_{\log}} \) on \( \mathcal{T}_{E_{G}/S_{\log}} \) satisfying that

\[
\mathcal{T}^{0}_{E_{G}/S_{\log}} = \mathcal{T}_{E_{G}/S_{\log}} / \mathcal{T}_{E_{G}/S_{\log}}^{0}
\]
and

\[
\mathcal{T}^{j}_{E_{G}/S_{\log}} / \mathcal{T}^{j-1}_{E_{G}/S_{\log}} \cong \mathfrak{g}^{j-1}_{E_{\mathfrak{g}}} / \mathfrak{g}^{j}_{E_{\mathfrak{g}}}. \tag{79}
\]

Since each \( \mathfrak{g}^{-\alpha} \) (\( \alpha \in \Gamma \)) is closed under the action of \( \mathbb{B} \), the decomposition \( \mathfrak{g}_{-1} = \mathfrak{g}^{-1} / \mathfrak{g}^{0} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha} \) gives a canonical decomposition

\[
\mathcal{T}^{-1}_{E_{G}/S_{\log}} / \mathcal{T}^{0}_{E_{G}/S_{\log}} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha}_{E_{\mathfrak{g}}}. \tag{80}
\]

**Definition 2.2.1.**

(i) Let

\[
\mathcal{E}^{\bullet} := (\pi_{\mathbb{B}} : \mathcal{E}_{\mathbb{B}} \rightarrow U, \nabla_{\mathcal{E}} : \mathcal{T}_{U_{\log}/S_{\log}} \rightarrow \mathcal{T}^{0}_{E_{G}/S_{\log}})
\]
be a pair consisting of a right \( \mathbb{B} \)-torsor \( \mathcal{E}_{\mathbb{B}} \) over \( U \) and an \( S \)-\( h \)-log connection \( \nabla_{\mathcal{E}} \) (cf. Definition 1.2.1 (i)) on the right \( G \)-torsor \( \pi_{G} : \mathcal{E}_{G} \rightarrow U \) induced by \( \mathcal{E}_{\mathbb{B}} \). We shall say that the pair \( \mathcal{E}^{\bullet} = (\mathcal{E}_{\mathbb{B}}, \nabla_{\mathcal{E}}) \) is a \((\mathfrak{g}, \mathfrak{h})\)-oper on \( U_{/S} \) (cf. §2) if \( \nabla_{\mathcal{E}}(\mathcal{T}_{U_{\log}/S_{\log}}) \subseteq \mathcal{T}^{-1}_{E_{G}/S_{\log}} / \mathcal{T}^{0}_{E_{G}/S_{\log}} \), and for any \( \alpha \in \Gamma \) the composite

\[
\mathcal{T}_{U_{\log}/S_{\log}} \xrightarrow{\nabla_{\mathcal{E}}} \mathcal{T}^{-1}_{E_{G}/S_{\log}} / \mathcal{T}^{0}_{E_{G}/S_{\log}} \rightarrow \mathfrak{g}^{-\alpha}_{E_{\mathfrak{g}}}, \tag{82}
\]

where the third arrow denotes the natural projection with respect to the decomposition \( \mathfrak{g}^{j}_{E_{\mathfrak{g}}} \), is an isomorphism. For simplicity, a \((\mathfrak{g}, 1)\)-oper may be often referred to as a \( \mathfrak{g} \)-oper.

(ii) Let \( \mathcal{E}^{\bullet} := (\mathcal{E}_{\mathbb{B}}, \nabla_{\mathcal{E}}) \), \( \mathcal{F}^{\bullet} := (\mathcal{F}_{\mathbb{B}}, \nabla_{\mathcal{F}}) \) be \((\mathfrak{g}, \mathfrak{h})\)-opers on \( U_{/S} \). An isomorphism from \( \mathcal{E}^{\bullet} \) to \( \mathcal{F}^{\bullet} \) is an isomorphism \( \mathcal{E}_{\mathbb{B}} \cong \mathcal{F}_{\mathbb{B}} \) of right \( \mathbb{B} \)-torsors such that the induced isomorphism \( \mathcal{E}_{G} \cong \mathcal{F}_{G} \) of right \( G \)-torsors is compatible with the respective \( S \)-\( h \)-log connections \( \nabla_{\mathcal{E}}, \nabla_{\mathcal{F}} \).

**Remark 2.2.2.**

Let \( \mathcal{E}^{\bullet} = (\mathcal{E}_{\mathbb{B}}, \nabla_{\mathcal{E}}) \) be a \((\mathfrak{g}, \mathfrak{h})\)-oper on \( U_{/S} \).

(i) Consider the composite isomorphism \( \mathcal{T}_{U_{\log}/S_{\log}} \cong \mathfrak{g}^{-\alpha}_{E_{\mathfrak{g}}} \) for each \( \alpha \in \Gamma \) (cf. (82)) and the definition of the Lie algebra grading \( \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{j} \) (cf. (72)). Then, one verifies that each subquotient \( \mathfrak{g}^{j}_{E_{\mathfrak{g}}} / \mathfrak{g}^{j+1}_{E_{\mathfrak{g}}} \) (\( \mathfrak{g}^{j}_{E_{\mathfrak{g}}} / \mathfrak{g}^{j+1}_{E_{\mathfrak{g}}} \cong \mathfrak{g}^{-\alpha}_{E_{\mathfrak{g}}} \) by (79)) of \( \mathfrak{g}^{j}_{E_{\mathfrak{g}}} \) is isomorphic to a direct sum of finite copies of \( \Omega^{\otimes j}_{U_{\log}/S_{\log}} \). In particular, we have \( \mathfrak{g}^{0}_{E_{\mathfrak{g}}} / \mathfrak{g}^{1}_{E_{\mathfrak{g}}} \cong \Omega^{\otimes \text{rank}(\mathfrak{g})}_{U_{\log}/S_{\log}} \).
(ii) If \( u' : U' \to U \) is an étale morphism, then the restriction \( (u'^* (E_B), u'^* (\nabla_E)) \) of \( \mathcal{E}_\bullet \) to \( U' \) forms a \((\mathfrak{g}, \mathfrak{h})\)-oper on \( \mathfrak{U}'_S \) (cf. § 1.6 for the notation of \( \mathfrak{U}'_S \)).

Also, if \( s' : S' \to S \) is a morphism of schemes, then one may define, in an evident fashion, the base-change of \( \mathcal{E}_\bullet \) via \( s' \) which is a \((\mathfrak{g}, \mathfrak{h})\)-oper on \( \mathfrak{U}'_S \) (cf. \( \mathfrak{U}'_S \)). The notion of restriction and base-change are, respectively, compatible with composition of morphisms.

**Definition 2.2.3.**

Let \((U, x)\) be a log chart (cf. § 1.2) on \( X^{\log} \) over \( S^{\log} \), \( \mathcal{E}_\bullet := (E_B, \nabla_E) \) a \((\mathfrak{g}, \mathfrak{h})\)-oper on \( \mathfrak{U}_S \), and \( \tau : E_B \to U \times_k \mathbb{G} \) a trivialization of the right \( \mathbb{G} \)-torsor \( E_B \). (Hence, it follows from the definition of a \((\mathfrak{g}, \mathfrak{h})\)-oper that \( \nabla_E'(T_{U^{\log}/S^{\log}}) \subseteq \mathcal{O}_U \otimes_k \mathfrak{g}^{-1} \) (cf. \( \mathfrak{U}'_S \)).) We shall say that \( \mathcal{E}_\bullet \) is of precanonical type relative to the triple \((U, x, \tau)\) if the composite

\[
\mathcal{T}_{U^{\log}/S^{\log}} \xrightarrow{\nabla_E} \mathcal{O}_U \otimes_k \mathfrak{g}^{-1} \to \mathcal{O}_U \otimes_k \mathfrak{g}^{-1} = \mathcal{O}_U \otimes_k (\mathfrak{g}^{-1} / \mathfrak{g}^{0})
\]

sends \( \partial_x \) to \( 1 \otimes p_{-1} \) (cf. \( \mathcal{O}_U \otimes_k \mathfrak{g}^{-1} \) for the definition of \( p_{-1} \)).

**Lemma 2.2.4.**

Let \((U, x)\) and \( \mathcal{E}_\bullet \) be as in Definition 2.2.3 and suppose further that \( E_B = U \times_k \mathbb{B} \) (i.e., the trivial right \( \mathbb{B} \)-torsor). Then, there exists uniquely a \( U \)-rational point \( h : U \to \mathbb{T} \) of \( \mathbb{T} \) satisfying the following condition: if \( h_G : U \times_k \mathbb{G} \to U \times_k \mathbb{G} \) is the left-translation (cf. \( \mathfrak{U}_S \)) by the composite \( h_G : U \to \mathbb{G} \) of \( h \) with the inclusion \( \mathbb{T} \hookrightarrow \mathbb{G} \), then \( \mathcal{E}_\bullet \) is of precanonical type relative to the triple \((U, x, h_G)\).

**Proof.** Let \( h : U \to \mathbb{T} \) be an arbitrary \( U \)-rational points of \( \mathbb{T} \), and let \( h_G : U \to \mathbb{G} \) be the composite of \( h \) with \( \mathbb{T} \hookrightarrow \mathbb{G} \). The automorphism \( \pi_*(dh_G)^G \) (cf. § 1.3) of \( \mathcal{T}_{E_B^{\log}/S^{\log}} \) induced by the left-translation \( I_{h_G} \) by \( h_G \) preserves both the filtration \( \{ \mathcal{T}_{E_B^{\log}/S^{\log}} \}_{j \leq 0} \) and the decomposition \( \mathcal{T}_{E_B^{\log}/S^{\log}} / \mathcal{T}_{\mathbb{B}^{\log}_{S^{\log}}} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}_{E_{\mathbb{B}}}^{-\alpha} \) (cf. \( \mathfrak{U}'_S \)). \( I_{h_G} \) acts on each component \( \mathfrak{g}_{E_{\mathbb{B}}}^{-\alpha} \) \((\cong \mathcal{O}_U \otimes_k \mathfrak{g}^{-\alpha})\) as the adjoint action by \( h \), which coincides (by the definition of \( \mathfrak{g}^{-\alpha} \)) the automorphism \( h_\alpha \in \text{GL}(\mathfrak{g}_{E_{\mathbb{B}}}^{-\alpha}) \) given by multiplication by \((\mathfrak{g}^{-\alpha}) \circ h : U \to \mathbb{G}_m \). It follows from the condition on \( \mathbb{G} \) being of adjoint type that the assignment \( h \mapsto (h_\alpha)_{\alpha \in \Gamma} \) determines an isomorphism

\[
U \times_k \mathbb{T} \cong \prod_{\alpha \in \Gamma} \text{GL}(\mathfrak{g}_{E_{\mathbb{B}}}^{-\alpha}).
\]
Thus, since $y_\alpha$ (cf. (75)) is a generator of $\mathfrak{g}^{-\alpha}$, there exists uniquely a $U$-rational point $h : U \to \mathbb{T}$ of $\mathbb{T}$ such that the composite
\begin{equation}
\mathcal{T}_{U^{\log}/S^{\log}} \xrightarrow{\nabla_{\mathcal{E}^0_B}} \tilde{\mathcal{T}}_{\mathcal{E}^{0}_{\log}/S^{\log}}^{-1} \to \tilde{\mathcal{T}}_{\mathcal{E}^{0}_{\log}/S^{\log}}^{-1}/\tilde{\mathcal{T}}_{\mathcal{E}^{0}_{\log}/S^{\log}} \sim \mathfrak{g}^{-\alpha}_{\mathcal{E}^0_B}
\end{equation}
(cf. (82)) sends $\partial_x$ to $1 \otimes y_\alpha$. Thus, by definition of $p_{-1}$, $h$ turns out to be the required $U$-rational point. \hfill \square

The following proposition was proved in [3], § 1.3, Proposition, (or [3], Proposition 2.1,) if $U/S$ is a unpointed smooth curve over $\mathbb{C}$. Similarly, even if $k$ has positive characteristic (and $U$ is an arbitrary étale (relative) scheme over a family of pointed stable curves), our condition (Char) $W$ (which implies (Char)$\Pi$) assumed at the end of § 2.1 enable us to parallel the argument of the proof in loc. cit.

**Proposition 2.2.5.**

A $(\mathfrak{g}, h)$-oper on $\Omega/S$ does not have nontrivial automorphisms.

**Proof.** As the statement is of local nature we may assume that there exists a globally defined log chart $(U, x)$ and the underlying $\mathbb{B}$-torsor $\mathcal{E}_B$ of our $(\mathfrak{g}, h)$-oper $\mathcal{E}^\bullet = (\mathcal{E}_B, \nabla_\mathcal{E})$ is trivial (i.e., $\mathcal{E}_B = U \times_k \mathbb{B}$). Moreover, by Lemma 2.2.4 above, we may assume that $\mathcal{E}^\bullet$ is of precanonical type relative to the triple $(U, x, \text{id}_{U \times_k \mathbb{B}})$.

Let us take an automorphism $\eta$ of $\mathcal{E}^\bullet$, i.e., an automorphism $\eta : U \times_k \mathbb{B} \simto U \times_k \mathbb{B}$ of the right $\mathbb{B}$-torsor $U \times_k \mathbb{B} = \mathcal{E}_B$ satisfying that $\nabla_{\mathcal{E}, \eta} = \nabla_\mathcal{E}$ (cf. (29)). $\eta$ may be given by the left-translation (cf. the discussion in § 1.1) by some $U$-rational point $h_\eta : U \to \mathbb{B}$ of $\mathbb{B}$ (cf. (34)). Let $h_{\eta, \mathbb{T}} : U \to \mathbb{T}$ denotes the composite
\begin{equation}
h_{\eta, \mathbb{T}} : U \xrightarrow{h} \mathbb{B} \to \mathbb{B}/\mathbb{N} \simto \mathbb{T},
\end{equation}
where the third arrow is the inverse of the composite isomorphism $\mathbb{T} \hookrightarrow \mathbb{B} \to \mathbb{B}/\mathbb{N}$. By virtue of the condition that $\nabla_{\mathcal{E}, \eta} = \nabla_\mathcal{E}$, the isomorphism (82) is compatible with the adjoint action on $\mathfrak{g}_{\mathcal{E}^0_B} = (\mathcal{O}_U \otimes_k \mathfrak{g}^{-\alpha})$ by $h_{\eta, \mathbb{T}}$ (cf. the discussion in the proof of Lemma 2.2.4). It follows from the isomorphism (84) that $h_{\eta, \mathbb{T}}$ coincides with the identity in the group $\mathbb{T}(U)$, i.e., $h_\eta$ factors through the inclusion $\mathbb{N} \hookrightarrow \mathbb{B}$; for simplicity, we also write $h_\eta : U \to \mathbb{N}$ for the resulting morphism. Thus, a $U$-rational point $\log(h_\eta) : U \to \mathfrak{n}$ (cf. (32)) may be defined.

By Corollary 1.4.2, we have (since $\nabla_{\mathcal{E}, \eta} = \nabla_\mathcal{E}$) the equality
\begin{equation}
\nabla_\mathcal{E}(\partial) = d\log_{\mathbb{N}}(h_\eta)(\partial_x) + \sum_{s=0}^{\infty} \frac{1}{s!} \cdot \text{ad}(\log_{\mathbb{N}}(h_\eta))^s(\nabla_\mathcal{E}(\partial_x))
\end{equation}
(cf. (32) for the definition of $\nabla_\mathcal{E}$) in $\Gamma(U, \mathcal{O}_U \otimes_k \mathfrak{g})$. 

We shall suppose that $\log_{\mathbb{N}}(h_\eta) \neq 0$. Then,
\begin{equation}
  j_0 := \max\{j \in \mathbb{Z} \mid h_\eta \in \mathfrak{g}^j(U)\}
\end{equation}
is a well-defined positive integer. Let
\begin{equation}
  \log_{\mathbb{N}}(h_\eta)_j : U \to \mathfrak{g}_j \quad \text{and} \quad \nabla_{\mathcal{E}}(\partial_x)_j \in \Gamma(U, \mathcal{O}_U \otimes_k \mathfrak{g}_j)
\end{equation}
be the components of the decompositions of $\log_{\mathbb{N}}(h_\eta) : U \to \mathfrak{g}$ and $\nabla_{\mathcal{E}}(\partial_x) \in \Gamma(U, \mathcal{O}_U \otimes_k \mathfrak{g})$, respectively, with respect to the grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$. By the definition of $j_0$, we see that $\log_{\mathbb{N}}(h_\eta)_j(\partial_x) = 0$ if $j < j_0$, and $d\log_{\mathbb{N}}(h_\eta)(\partial_x) \in \Gamma(U, \mathcal{O}_U \otimes_k \mathfrak{g}^m)$. Since $\nabla_{\mathcal{E}}(\partial_x)_{-1} = 1 \otimes p_{-1}$ and $\nabla_{\mathcal{E}}(\partial_x)_j = 0$ if $j < -1$, the section $\text{ad}(\log_{\mathbb{N}}(h_\eta)_j \circ (\nabla_{\mathcal{E}}(\partial_x)_j))$ lies in $\Gamma(U, \mathcal{O}_U \otimes_k \mathfrak{g}^m)$ if either “$s \geq 2$” or “$s = 1$ and $j + j' \geq j_0$”. Hence, in the component $\Gamma(U, \mathcal{O}_U \otimes_k \mathfrak{g}_{j_0-1})$ of $\Gamma(U, \mathcal{O}_U \otimes_k \mathfrak{g})$, the equality (87) asserts that
\begin{equation}
  \text{ad}(\log_{\mathbb{N}}(h_\eta)_j \circ (\nabla_{\mathcal{E}}(\partial_x)_{-1})) = -\text{ad}(1 \otimes p_{-1})(\log_{\mathbb{N}}(h_\eta)_j) = 0.
\end{equation}
But, since $j_0 > 0$ and $\{p_{-1}, 2\rho, p_1\}$ forms an $\mathfrak{sl}_2$-triple, the morphism $\text{ad}(1 \otimes p_{-1}) : \mathcal{O}_U \otimes_k \mathfrak{g}_{j_0} \to \mathcal{O}_U \otimes_k \mathfrak{g}_{j_0-1}$ is injective. This implies that $\log_{\mathbb{N}}(h_\eta)_j = 0$, which contradicts the definition of $j_0$. Thus, $\log_{\mathbb{N}}(h_\eta)$ must be zero, i.e., $\eta$ is the identity morphism of $\mathcal{E}^\bullet$. This completes the proof of Proposition 2.2.5. \hfill \Box

2.3. Write $\mathcal{E}_{t/U}$ for the small étale site on $U$. By the discussion in Remark 2.2.2 (ii), one may define the stack in groupoids
\begin{equation}
  \mathcal{O}_{\mathfrak{p}_{\mathfrak{g}, h, \mathcal{U}/S}} \to \mathcal{E}_{t/U}
\end{equation}
over $\mathcal{E}_{t/U}$ whose category of sections over an étale morphism $u' : U' \to U$ is the groupoids of $(\mathfrak{g}, h)$-opers on $\mathcal{U}'_S$. One verifies from Proposition 2.2.5 that $\mathcal{O}_{\mathfrak{p}_{\mathfrak{g}, h, \mathcal{U}/S}}$ is fibered in equivalence relations (cf. [18], Definition 3.39).

Also, write $\mathfrak{S}et$ for the category of (small) sets and $\mathfrak{S}ch_{/S}$ for the category of $S$-schemes. By the discussion in Remark 2.2.2 (ii) again, one may define the $\mathfrak{S}et$-valued contravariant functor
\begin{equation}
  \mathfrak{O}_{\mathfrak{p}_{\mathfrak{g}, h, \mathcal{U}/S}} : \mathfrak{S}ch_{/S} \to \mathfrak{S}et
\end{equation}
on $\mathfrak{S}ch_{/S}$ which, to any $S$-scheme $T$, assigns the set of isomorphism classes of $(\mathfrak{g}, h)$-opers on $\mathcal{U}_T$. The natural functor $\mathfrak{O}_{\mathfrak{p}_{\mathfrak{g}, h, \mathcal{U}/S}}(X) \to \mathfrak{O}_{\mathfrak{p}_{\mathfrak{g}, h, \mathcal{U}/S}}(S)$ is an equivalence of categories.

Let $s' : S' \to S$ be a morphism of schemes. The assignment $t' \mapsto s' \circ t'$ (for $S'$-scheme $t' : T' \to S'$) defines a functor
\begin{equation}
  s' : \mathfrak{S}ch_{/S'} \to \mathfrak{S}ch_{/S}.
\end{equation}
Then, one may construct, in an evident fashion, a natural transformation
\begin{equation}
  \beta_{\mathfrak{g}, h, \mathcal{U}/S, S'} : \mathfrak{O}_{\mathfrak{p}_{\mathfrak{g}, h, \mathcal{U}/S}} \circ s' \to \mathfrak{O}_{\mathfrak{p}_{\mathfrak{g}, h, \mathcal{U}/S'}}
\end{equation}of functors $\mathfrak{S}ch_{/S'} \to \mathfrak{S}et$. 
2.4. In §2.4–§2.7, we verify (cf. Proposition 2.7.5) that the functor $\mathcal{O}_{p_{B,h,X/S}}$ may be represented by a certain relative affine space over $S$. First, we consider the case where $g = sl_2$.

Let us write $G^\circ$ for the projective linear group over $k$ of rank 2 (i.e., $G^\circ := PGL_2$), $B^\circ$ for the Borel subgroup of $G^\circ$ defined as the image of upper triangular matrices, and $N^\circ := [B^\circ, B^\circ]$. Also, write $g^\circ$, $b^\circ$, and $n^\circ$ for the Lie algebras of $G^\circ$, $B^\circ$, and $N^\circ$ respectively. In particular,

$$g^\circ = sl_2, \quad g_1^\circ = n^\circ, \quad g_0^\circ = b^\circ/n^\circ, \quad g_{-1}^\circ = g^\circ/b^\circ.$$  

Here, we shall apply an argument similar to the argument given in the proof of [3], §3, Lemma 3.1, or apply, in advance, the results of §4 of the present paper (e.g., if $k$ is of characteristic $p > 2$). According to them, one may easily verify the facts (i)-(v) described below.

(i) One obtains, for each étale morphism $u : U \to X$, a specific right $B^\circ$-torsor

$$\pi_{B^\circ,h,U/S} : E^\dagger_{B^\circ,h,U/S} \to U$$  

(i.e., $\pi_{B,h,U/S} : \mathcal{E}^\dagger_{B,h,U/S} \to U$ defined (422) of the case where $n = 2$) over $U$, and for each log chart $(U,x)$ on $X^{\log}$ over $S^{\log}$, a trivialization

$$\text{triv}_{B^\circ,j,(U,x)} : E^\dagger_{B^\circ,h,U/S} \sim \to U \times_k B$$  

(i.e., $\text{triv}_{B,h,(U,x)} : \mathcal{E}^\dagger_{B,h,U/S} \sim \to U \times_k B$ defined in (427) of the case where $n = 2$) of the $B^\circ$-torsor $E^\dagger_{B^\circ,h,U/S}$. These assignments $u \mapsto E^\dagger_{B^\circ,h,U/S}$ and $(U,x) \mapsto \text{triv}_{B^\circ,j,(U,x)}$ are, respectively, functorial (in an evident sense) with respect to any restriction of $U$, as well as any base-change over $S$.

(ii) The notion of a $(g^\circ, h)$-operator of canonical type (i.e., an $(sl_2, h)$-operator of canonical type II in Definition 4.12.1) are defined. In particular, the underlying $B^\circ$-torsor of any $(g^\circ, h)$-operator on $U/S$ of canonical type is $E^\dagger_{B^\circ,h,U/S}$. Let $u : U \to X$ be an étale morphism of $k$-schemes and $E^\bullet$ a $(g^\circ, h)$-operator on $U/S$. Then, there exists (cf. Remark 4.12.2 (ii)) uniquely a pair

$$(E^\bullet^\circ, \text{can}_{E^\bullet})$$

consisting of a $(g^\circ, h)$-operator $E^\bullet^\circ = (E^\dagger_{B^\circ,h,U/S}, \nabla^\circ_E)$ on $U/S$ of canonical type and an isomorphism $\text{can}_{E^\bullet} : E^\bullet \sim \to E^\bullet^\circ$ of $(g^\circ, h)$-operators. The assignment $E^\bullet \mapsto (E^\bullet^\circ, \text{can}_{E^\bullet})$ is compatible (in an evident sense) with any restriction of $U$, as well as any base-change over $S$.

(iii) For $j = -1, 0, 1$, there exists (cf. §4.5, 434) a canonical isomorphism

$$\Omega^\circ_{U^{\log}/S^{\log}} \sim \to \text{Hom}_U(T_U^{\log}/S^{\log}, (g_j^\circ)E^\dagger_{B^\circ,h,U/S})$$;
of $\mathcal{O}_U$-modules (cf. (95)). In the following, if $R \in \Gamma(U, \Omega_{U^{\log}/S^{\log}}^2)$, then we shall denote by

$$
(100) \quad R^\natural: T_{U^{\log}/S^{\log}} \to n^\natural_{\mathfrak{g}^{\circ}, h, U/S}
$$

the $\mathcal{O}_U$-linear morphism corresponding to $R$ via the isomorphism (99) of the case where $j = 1$.

(iv) Let $E \diamondsuit = (E^{\dagger}_{\mathfrak{g}^{\circ}, h, U/S}, \nabla^{\diamondsuit}_E)$ be a $(\mathfrak{g}^{\circ}, h)$-oper on $U/S$ of canonical type and $R \in \Gamma(U, \Omega_{U^{\log}/S^{\log}}^2)$. Since $R^\natural$ may be thought of as an $\mathcal{O}_U$-linear morphism $T_{U^{\log}/S^{\log}} \to \tilde{T}_{E^{\dagger\log}_{\mathfrak{g}^{\circ}, h, U/S}/S^{\log}}$ via the injection $n^\natural_{\mathfrak{g}^{\circ}, h, U/S} \hookrightarrow \tilde{T}_{E^{\dagger\log}_{\mathfrak{g}^{\circ}, h, U/S}/S^{\log}}$, one obtains an $S$-$h$-log connection

$$
(101) \quad \nabla^{\diamondsuit}_E + R^\natural: T_{U^{\log}/S^{\log}} \to \tilde{T}_{E^{\dagger\log}_{\mathfrak{g}^{\circ}, h, U/S}/S^{\log}}
$$

on $E^{\dagger}_{\mathfrak{g}^{\circ}, h, U/S}$. Moreover, one verifies that the pair

$$
(102) \quad E_{+R} := (E^{\dagger}_{\mathfrak{g}^{\circ}, h, U/S}, \nabla^{\diamondsuit}_E + R^\natural)
$$

forms a $(\mathfrak{g}^{\circ}, h)$-oper on $U/S$ of canonical type. The assignment $(E^{\diamondsuit}, R) \mapsto E_{+R}$ is functorial with respect to any base-change over $S$. In particular, both the fact in (ii) and this assignment (of the case where $U = X$) determine a $\mathbb{V}(f_* (\Omega_{X^{\log}/S^{\log}}^2))$-action

$$
(103) \quad \text{act}_{\mathfrak{g}^{\circ}, h, x/S} : \mathfrak{D}p_{\mathfrak{g}^{\circ}, h, x/S} \times_S \mathbb{V}(f_* (\Omega_{X^{\log}/S^{\log}}^2)) \to \mathfrak{D}p_{\mathfrak{g}^{\circ}, h, x/S}
$$

on $\mathfrak{D}p_{\mathfrak{g}^{\circ}, h, x/S}$ (cf. § 1.2 for the definition of $\mathbb{V}(-)$). Owing to this action, the functor $\mathfrak{D}p_{\mathfrak{g}^{\circ}, h, x/S}$ may be represented by a relative affine space over $S$ modeled on $\mathbb{V}(f_* (\Omega_{X^{\log}/S^{\log}}^2))$ (cf. Remark 4.12.2 (iii)). In particular, the fiber of $\mathfrak{D}p_{\mathfrak{g}^{\circ}, h, x/S}$ over any point of $S$ is nonempty.

(v) Let $s': S' \to S$ be a morphism of schemes and

$$
(104) \quad \beta_{\mathfrak{g}^{\circ}, h, x/S, S'} : \mathfrak{D}p_{\mathfrak{g}^{\circ}, h, x/S} \times_S S' \twoheadrightarrow \mathfrak{D}p_{\mathfrak{g}^{\circ}, s' \circ S'} \to \mathfrak{D}p_{\mathfrak{g}^{\circ}, s' \circ S'}
$$
the morphism \((\Omega^2/X^{\text{log}/S^{\text{log}}})\) in the case where the pair \((g,U)\) is taken to be \((\mathfrak{g}^\circ,X)\). Then, it is an isomorphism and makes the square diagram

\[
\begin{array}{ccc}
\mathcal{D}_g^{\circ\cdot\cdot\cdot,h,x/S} \times S (f^*(\Omega^2_{X^{\text{log}/S^{\text{log}}}})) & \xrightarrow{\text{act}^g_{\circ\cdot\cdot\cdot,h,x/S \times \text{id}_{S'}}} & \mathcal{D}_g^{\circ\cdot\cdot\cdot,h,x/S} \times S' \\
\downarrow & & \downarrow \text{id} \\
(\mathcal{D}_g^{\circ\cdot\cdot\cdot,h,x/S} \times S') \times S^\prime (f^*(\Omega^2_{X \times S^{\text{log}/S^{\text{log}}}})) & \xrightarrow{\text{act}^g_{\circ\cdot\cdot\cdot,h,x/S' \times \text{id}}} & (\mathcal{D}_g^{\circ\cdot\cdot\cdot,h,x/S} \times S') \\
\beta_{g\circ\cdot\cdot\cdot,h,x/S',S'} & & \beta_{g\circ\cdot\cdot\cdot,h,x/S,S'} \\
\end{array}
\]

commute, where the left-hand upper vertical arrow arises from a natural isomorphism \(\mathcal{V}(f^*(\Omega^2_{X^{\text{log}/S^{\text{log}}}})) \times S \xrightarrow{\sim} \mathcal{V}(f^*(\Omega^2_{X \times S^{\text{log}/S^{\text{log}}}}))\) (cf. Proposition 2.6.1, \((119)\)).

2.5. Next, we consider the functor \(\mathcal{D}_g^{\circ\cdot\cdot\cdot,h,x/S}\) of the case where \(g\) is arbitrary. The \(\mathfrak{sl}_2\)-triple \(\{p_1,2\hat{p},p_1\}\) determines an injection

\[
t_g : \mathfrak{g}^\circ \hookrightarrow \mathfrak{g}
\]
satisfying that \(t_g(b^\circ) \subseteq b\). It gives (since both \(G^\circ\) and \(G\) are of adjoint type) the corresponding injection

\[
t_G : G^\circ \hookrightarrow G
\]
satisfying that \(t_G(B^\circ) \subseteq B\). If \(u : U \to X\) is an étale morphism, then the right \(B^\circ\)-torsor \(\mathcal{E}_{B^\circ,h,U/S}^\dagger\) induces, by a change of structure group via \(t_G\), a right \(B\)-torsor

\[
\pi_{B,h,U/S}^\dagger : \mathcal{E}_{B,h,U/S}^\dagger := \mathcal{E}_{B^\circ,h,U/S}^\dagger \times B^\circ, t_G \circ B) \to U
\]
over \(U\), and hence, a right \(G\)-torsor

\[
\pi_{G,h,U/S}^\dagger : \mathcal{E}_{G,h,U/S}^\dagger := \mathcal{E}_{B^\circ,h,U/S}^\dagger \times B(G) \to U.
\]

Also, if \((U,x)\) is a log chart on \(X^{\text{log}}\) over \(S^{\text{log}}\), then the trivialization \(\text{triv}_{B^\circ,h,(U,x)}\) (cf. \((97)\)) induces a trivialization

\[
\text{triv}_{B,h,(U,x)} : \mathcal{E}_{B,h,U/S}^\dagger \xrightarrow{\sim} U \times_k B
\]
of the \(B\)-torsor \(\mathcal{E}_{B,h,U/S}^\dagger\), and hence, a trivialization

\[
\text{triv}_{G,h,(U,x)} : \mathcal{E}_{G,h,U/S}^\dagger \xrightarrow{\sim} U \times_k G
\]
of the \(G\)-torsor \(\mathcal{E}_{G,h,U/S}^\dagger\).
2.6. We introduce and study certain vector bundles, which will be denoted by \( \mathcal{V}_{g,h,\mathcal{U}/S} \) and \( ^c\mathcal{V}_{g,h,\mathcal{U}/S} \).

Consider the space \( \mathfrak{g}^{\text{ad}(p_1)} \) of \( \text{ad}(p_1) \)-invariants, i.e.,
\[
(112) \quad \mathfrak{g}^{\text{ad}(p_1)} := \{ x \in \mathfrak{g} \mid \text{ad}(p_1)(x) = 0 \}.
\]
The \( k \)-vector space \( \mathfrak{g}^{\text{ad}(p_1)} \) has rank equal to \( \text{rk}(\mathfrak{g}) \) (cf. \( [g], \S \ 3.4 \)). The grading
\[
(113) \quad \mathfrak{g}^{\text{ad}(p_1)} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j^{\text{ad}(p_1)}
\]
on \( \mathfrak{g}^{\text{ad}(p_1)} \). The components \( \mathfrak{g}_j^{\text{ad}(p_1)} \) are closed under the adjoint action of \( \mathbb{R}^\circ \) via \( \iota_G \) and satisfies that \( \mathfrak{g}_j^{\text{ad}(p_1)} = 0 \) if \( j \leq 0 \) (cf. \( [g], \S \ 3.4 \)).

Thus, one may construct a vector bundle
\[
(114) \quad \mathcal{V}_{g,h,\mathcal{U}/S} := \Omega_{U^{\log}/S^{\log}} \otimes \mathfrak{g}^{\text{ad}(p_1)}_{\mathfrak{g}^\circ,h,\mathcal{U}/S}
\]
on \( U \) of rank equal to \( \text{rk}(\mathfrak{g}) \), and moreover,
\[
(115) \quad ^c\mathcal{V}_{g,h,\mathcal{U}/S} := \mathcal{V}_{g,h,\mathcal{U}/S}(-D_{\mathcal{U}/S}) \big( \subseteq \mathcal{V}_{g,h,\mathcal{U}/S} \big)
\]
(cf. \( \S \ 1.6 \) for the definition of \( D_{\mathcal{U}/S} \)). Also, the grading \( (113) \) carries a gradings
\[
(116) \quad \mathcal{V}_{g,h,\mathcal{U}/S} \sim \bigoplus_{j \in \mathbb{Z}} \mathcal{V}_{g,h,\mathcal{U}/S,j}, \quad ^c\mathcal{V}_{g,h,\mathcal{U}/S} \sim \bigoplus_{j \in \mathbb{Z}} ^c\mathcal{V}_{g,h,\mathcal{U}/S,j}.
\]
Each component \( \mathcal{V}_{g,h,\mathcal{U}/S,j} \) (resp., \( ^c\mathcal{V}_{g,h,\mathcal{U}/S,j} \)) of \( \mathcal{V}_{g,h,\mathcal{U}/S} \) (resp., \( ^c\mathcal{V}_{g,h,\mathcal{U}/S} \)) is isomorphic to a direct sum of finite copies of \( \Omega_{U^{\log}/S^{\log}}^{(j+1)} \) (resp., \( \Omega_{U^{\log}/S^{\log}}^{(j+1)} \)) (cf. Remark 2.2.2 (i)), and it is satisfied that \( \mathcal{V}_{g,h,\mathcal{U}/S,j} = ^c\mathcal{V}_{g,h,\mathcal{U}/S,j} = 0 \) if \( j \leq 0 \).

Thus, in particular, by applying the isomorphism \( \text{triv}_{\sigma_i,U} : \sigma_i^{U,s}(T_{U^{\log}/S^{\log}}) \sim \mathcal{O}_{U \times X, \sigma_i S} \) (cf. \( (67) \)) for all \( i \), we have an isomorphism
\[
(117) \quad \mathcal{V}_{g,h,\mathcal{U}/S}/^c\mathcal{V}_{g,h,\mathcal{U}/S} \sim \bigoplus_{i=1}^r \sigma_i^{U}\left( \mathcal{O}_{U \times X, \sigma_i S}^{\text{rk}(g)} \right)
\]
of \( \mathcal{O}_U \)-modules. Note that the isomorphism class of the \( \mathcal{O}_U \)-module \( \mathcal{V}_{g,h,\mathcal{U}/S} \) (resp., \( ^c\mathcal{V}_{g,h,\mathcal{U}/S} \)) does not, in fact, depend on the choice of \( h \).

The proof of the following proposition will apply certain basic facts concerning the Lie algebra \( \mathfrak{g} \) reviewed in \( \S \ 2.8 \).

**Proposition 2.6.1.**

The direct image \( f_*(\mathcal{V}_{g,h,X/S}) \) (resp., \( f_*(^c\mathcal{V}_{g,h,X/S}) \)) is a vector bundle on \( S \) of
rank $\mathfrak{N}(\mathfrak{g})$ (resp., $\mathfrak{cN}(\mathfrak{g})$), where

\begin{align}
\mathfrak{N}(\mathfrak{g}) := (g - 1) \cdot \dim(\mathfrak{g}) + \frac{r}{2} \cdot (\dim(\mathfrak{g}) + \text{rk}(\mathfrak{g})) \\
(\text{resp., } \mathfrak{cN}(\mathfrak{g}) := (g - 1) \cdot \dim(\mathfrak{g}) + \frac{r}{2} \cdot (\dim(\mathfrak{g}) - \text{rk}(\mathfrak{g})).
\end{align}

Moreover, if $s : S' \to S$ is a morphism of $k$-schemes, then the natural morphism

\begin{align}
\mathbb{V}(f_*(\mathcal{V}_{\mathfrak{g}, h, \mathcal{X}/S})) \times_S S' \to \mathbb{V}(f_*(\mathcal{V}_{\mathfrak{g}, s^*(h), \mathcal{X}/S}')) \\
(\text{resp., } \mathbb{V}(f_*(\mathcal{cV}_{\mathfrak{g}, h, \mathcal{X}/S})) \times_S S' \to \mathbb{V}(f_*(\mathcal{cV}_{\mathfrak{g}, s^*(h), \mathcal{X}/S}')))
\end{align}

is an isomorphism of $S'$-schemes.

**Proof.** As we discussed after (116), $\mathcal{V}_{\mathfrak{g}, h, \mathcal{X}/S}$ and $\mathcal{cV}_{\mathfrak{g}, h, \mathcal{X}/S}$ are isomorphic to direct sums of $\Omega_{X_{\text{log}}/S_{\text{log}}}^{\otimes l}$'s and $\Omega_{X_{\text{log}}/S_{\text{log}}}^{\otimes l}(-D_{\mathcal{X}/S})$'s $(l \geq 2)$ respectively. But, since

\begin{align}
R^1f_*(-) = R^1f_*(\mathcal{V}_{\mathfrak{g}, h, \mathcal{X}/S}) = R^1f_*(\mathcal{cV}_{\mathfrak{g}, h, \mathcal{X}/S} = 0.
\end{align}

for any $l \geq 2$, one verifies that

\begin{align}
R^1f_*(\mathcal{V}_{\mathfrak{g}, h, \mathcal{X}/S}) = R^1f_*(\mathcal{cV}_{\mathfrak{g}, h, \mathcal{X}/S}) = 0.
\end{align}

Thus, it follows (cf. [25], Chap. III, Theorem 12.11 (b)) that $f_*(\mathcal{V}_{\mathfrak{g}, h, \mathcal{X}/S})$ and $f_*(\mathcal{cV}_{\mathfrak{g}, h, \mathcal{X}/S})$ are vector bundles on $S$ and the morphism (119) is an isomorphism.

We shall compute the rank of $f_*(\mathcal{V}_{\mathfrak{g}, h, \mathcal{X}/S})$ and $f_*(\mathcal{cV}_{\mathfrak{g}, h, \mathcal{X}/S})$. By applying the functor $R^1f_*(-)$ to the short exact sequence

\begin{align}
0 \to \mathcal{V}_{\mathfrak{g}, h, \mathcal{X}/S} \to \mathcal{cV}_{\mathfrak{g}, h, \mathcal{X}/S} \to \bigoplus_{i=1}^r \sigma_{i*}(\mathcal{O}_S^{\oplus \text{rk}(\mathfrak{g})}) \to 0
\end{align}

(cf. (117)), we have (by taking account of the equality (121)) the equality

\begin{align}
\text{rk}(f_*(\mathcal{V}_{\mathfrak{g}, h, \mathcal{X}/S})) = \text{rk}(f_*(\mathcal{cV}_{\mathfrak{g}, h, \mathcal{X}/S})) + r \cdot \text{rk}(\mathfrak{g}).
\end{align}

On the other hand, refer to the fact discussed in § 2.8 that the isomorphism $\mathfrak{iso}_g : \mathfrak{g}^{\text{ad}(p_1)} \cong \mathfrak{c}$ (cf. (152)) is compatible with the $\mathbb{G}_m$-action $\text{Ad}^+$ on $\mathfrak{g}^{\text{ad}(p_1)}$ and the $\mathbb{G}_m$-action on $\mathfrak{c}$ that comes from the homotheties on $\mathfrak{g}$. Hence, if $e_l$ ($l = 1, \cdots, \text{rk}(\mathfrak{g})$) denotes the degree of homogeneous generators $u_l$ in $S_{\text{h}}(\mathfrak{g}^\lor)^G$, 

then $\mathcal{V}_{g,h,x/S}$ is isomorphic to $\bigoplus_{i=1}^{rk(g)} \Omega_{X_{\log}/S_{\log}}^{\otimes e_i}$. It follows that

$$
\text{(124)} \quad \text{rk}(f_*(\mathcal{V}_{g,h,x/S})) = \sum_{i=1}^{rk(g)} \text{rk}(f_*(\Omega_{X_{\log}/S_{\log}}^{\otimes e_i})) \\
= \sum_{i=1}^{rk(g)} ((g - 1) \cdot (2e_i - 1) + r \cdot e_i) \\
= (g - 1) \cdot \dim(g) + \frac{r}{2} \cdot (\dim(g) + \text{rk}(g)) \\
= \aleph(g),
$$

where the third equality follows from the equality $\sum_{i=1}^{rk(g)} (2e_i - 1) = \dim(g)$ (cf. [5], 2.2.4, Remark (iv)). By combining (123) and (124), we obtain the required equalities. \hfill \Box

The injection $g^{\text{ad}(p_1)} \hookrightarrow g$, which is compatible with the respective $B^\otimes$-actions, gives an $O_{U}$-linear injection

$$
\text{(125)} \quad \zeta_{g,h,U/S} : \mathcal{V}_{g,h,U/S} \hookrightarrow \Omega_{U_{\log}/S_{\log}} \otimes g_{g,U/S}^\dagger.
$$

Since $(g^{\otimes})^{\text{ad}(p_1)} = n^\otimes$, the isomorphism (99) of the case where $j = -1$ asserts that $\mathcal{V}_{g^{\otimes},h,U/S}$ is canonically isomorphic to $\Omega_{U_{\log}/S_{\log}}^{\otimes 2}$. Hence, the injection $\iota_g : g^{\otimes} \hookrightarrow g$ induces an injection

$$
\text{(126)} \quad \iota_{g,U/S} : \Omega_{U_{\log}/S_{\log}}^{\otimes 2} (\sim \mathcal{V}_{g^{\otimes},h,U/S}) \hookrightarrow \mathcal{V}_{g,h,U/S}
$$

of vector bundles, in particular, a closed immersion

$$
\text{(127)} \quad \nabla(f_*(\Omega_{X_{\log}/S_{\log}}^{\otimes 2})) \hookrightarrow \nabla(f_*(\mathcal{V}_{g,h,x/S})).
$$

2.7. Now, we prove (cf. Proposition 2.7.5) that $\mathcal{Op}_{g,h,x/S}$ may be represented by a relative affine space over $S$ modeled on $\nabla(f_*(\mathcal{V}_{g,h,x/S}))$.

**Definition 2.7.1.**

A $(g,h)$-oper on $U_{\log}$ of canonical type (or, of canonical type I) is a $(g,h)$-oper on $U_{\log}$ of the form $\mathcal{E}^{\otimes} := (\mathcal{E}_{g,h,U/S}^\dagger, \nabla_{\mathcal{E}}^\otimes)$ such that for any local chart $(U', x')$ on $U_{\log}$ over $S_{\log}$, the following two conditions are satisfied:

- The restriction of $\mathcal{E}^{\otimes}$ to $U'_{\log}$ is of precanonical type relative to the triple $(U', x', \text{triv}_{G,h,(U',x')})$ (cf. Definition 2.2.3 and (111));
- The image of the composite

$$
\nabla_{\mathcal{E}}^{\otimes} : \mathcal{T}_{U_{\log}/S_{\log}} \otimes_{\mathcal{O}_{U'}} \mathcal{O}_{U'} \otimes g^{-1} \rightarrow \mathcal{O}_{U'} \otimes g^0,
$$

(128)
lies in $O_{U'} \otimes_k g^{\text{ad}(p_1)} (\subseteq O_{U'} \otimes g^0)$, where $\tau := \text{triv}_{G,h(U',x')}^*$ and the second arrow denotes the natural projection with respect to the decomposition $O_{U'} \otimes_k g^{-1} \twoheadrightarrow \bigoplus_{j \geq -1} O_{U'} \otimes_k g^j$.

Let $E^\bigcirc := (E^\bigcirc, \nabla_{E^\bigcirc})$ be a $(g^\bigcirc, h)$-oper on $\mathcal{U}/S$. We shall write $t_G* (\nabla_{E^\bigcirc})$ for the $S$-$h$-log connection on $E^\bigcirc \times^{B^\bigcirc, t_G^*} G$ arising from $\nabla_{E^\bigcirc}$ by executing a change of structure group via $t_G : G^\bigcirc \hookrightarrow G$. One verifies that the pair

$$t_G*(E^\bigcirc) := (E^\bigcirc \times^{B^\bigcirc, t_G^*} B, t_G*(\nabla_{E^\bigcirc}))$$

forms a $(g, h)$-oper on $\mathcal{U}/S$. The assignment $E^\bigcirc \mapsto t_G*(E^\bigcirc)$ is functorial with respect to $S$, hence determines a morphism

$$t_D p_{g,h,\mathcal{U}/S} : D p_{g^\bigcirc, h, \mathcal{U}/S} \to D p_{g,h,\mathcal{U}/S}$$

of functors on $\mathcal{S}ch/\mathcal{S}$.

Next, let $E^\bigcirc \downarrow = (E^\bigcirc \downarrow, x/S, \nabla_{E^\bigcirc})$ be a $(g^\bigcirc, h)$-oper on $\mathcal{U}_{x}/S$ of canonical type (cf. §2.4 (ii)) and $R \in \Gamma(U, \mathfrak{V}_{g,h,\mathcal{U}/S})$. Consider the composite injection

$$\mathfrak{V}_{g,h,\mathcal{U}/S} \rightarrow^{\mathfrak{V}_{g,h,\mathcal{U}}/S} \Omega_{U^\text{loc}/S^{\text{log}}} \otimes g^\bigcirc_{\mathfrak{V}_{g,h,\mathcal{U}/S}}$$

$$\rightarrow^{\sim} \text{Hom}_{\mathcal{O}_U} (\mathcal{T}_{U^{\text{loc}/S^{\text{log}}}}, g^\bigcirc_{\mathfrak{V}_{g,h,\mathcal{U}/S}})$$

$$\rightarrow^{\sim} \text{Hom}_{\mathcal{O}_U} (\mathcal{T}_{U_{x}^{\text{loc}/S^{\text{log}}}}, \mathcal{E}_{g,h,\mathcal{U}_{x}/S}^{\text{triv}}).$$

If $R^\mathfrak{V}$ denotes the image of $R$ via this injection, then the sum $t_G*(\nabla_{E^\bigcirc}) + R^\mathfrak{V}$ is an $S$-$h$-log connection on $E^\bigcirc_{g,h,\mathcal{U}/S}$. One verifies that the pair

$$t_G*(E^\bigcirc^\downarrow) + R^\mathfrak{V} := (E^\bigcirc^\downarrow_{g,h,\mathcal{U}_{x}/S}, t_G*(\nabla_{E^\bigcirc}^\downarrow) + R^\mathfrak{V})$$

forms a $(g, h)$-oper on $\mathcal{U}_{x}/S$ of canonical type. Moreover, one verifies easily that following

**Proposition 2.7.2.**

A $(g, h)$-oper on $\mathcal{U}/S$ is of canonical type if and only if it is of the form

$$t_G*(E^\bigcirc^\downarrow) + R^\mathfrak{V} = (E^\bigcirc^\downarrow_{g,h,\mathcal{U}_{x}/S}, t_G*(\nabla_{E^\bigcirc}^\downarrow) + R^\mathfrak{V})$$

for some $(g^\bigcirc, h)$-oper $(E^\bigcirc^\downarrow_{g,h,\mathcal{U}_{x}/S}, \nabla_{E^\bigcirc}^\downarrow)$ of canonical type and $R \in \Gamma(U, \mathfrak{V}_{g,h,\mathcal{U}/S})$.

As is the case of Proposition 2.2.5, the following Proposition 2.7.3 (hence Proposition 2.7.5) was proved in the previous work by A. Beilinson and V. Drinfeld (cf. [6] §3.4, Theorem; [3], Lemma 3.6) if $\mathcal{U}/S$ is a unpointed smooth curve over $\mathbb{C}$. Under our condition that either (Char)$_0$ or (Char)$_W$ (cf. §2.1) is satisfied, one may prove the assertion for more general cases, i.e., the case
where $\Omega_{/S}$ is an arbitrary pointed stable curve. (If $g = s_2$, then the following propositions was proved by S. Mochizuki (cf. [52], Chap. I, Proposition 2.11)).

**Proposition 2.7.3.**
For a $(g,h)$-oper $E^\otimes = (E_B, \nabla_E)$ on $\Omega_{/S}$, there exists uniquely a pair $(E^{\otimes \diamond} , \operatorname{can}_{E^{\otimes \diamond}})$ consisting of a $(g,h)$-oper $E^{\otimes \diamond} = (E^{\otimes \diamond}_{B,h,\Omega_{/S}}, \nabla^{\otimes \diamond}_E)$ on $\Omega_{/S}$ of canonical type and an isomorphism

$$
\operatorname{can}_{E^{\otimes \diamond}} : E^{\otimes \diamond} \cong E^{\otimes \diamond}
$$

of $(g,h)$-opers. Moreover, the assignment $E^{\otimes \diamond} \mapsto (E^{\otimes \diamond} , \operatorname{can}_{E^{\otimes \diamond}})$ is compatible (in an evident sense) with any restriction of $U$, as well as any base-change over $S$.

**Proof.** By Proposition 2.2.5 (and the fact that $O_{p,g,h,\Omega_{/S}}$ is a stack over $\mathfrak{Et}_{/U}$), the statement is easily verified to be of local nature. Thus, we may suppose

$$
\nabla_E \mid_{U \times \mathbb{G}_m} = \operatorname{id} \otimes O_U \otimes_k g^{ad(p_1)}
$$

(cf. (110) for the definition of $\nabla^{(-),0}_E$). Moreover, by Lemma 2.2.4, it suffices to prove the assertion of the case where $E^{\otimes \diamond}$ is of precanonical type relative to the triple $(U, x, \operatorname{id}_{U, x, k})$.

We shall suppose that $\operatorname{Im}(\nabla^{id_{U, x, k},0}_E) \not\subseteq O_U \otimes_k g^{ad(p_1)}$ (cf. (128) for the definition of $\nabla^{(-),0}_E$). Then,

$$
\begin{aligned}
\quad j_0 := \max\{ j \in \mathbb{Z} \mid \operatorname{Im}(\nabla^{id_{U, x, k},0}_E) &\subseteq \bigoplus_{j \geq j_0} O_U \otimes_k g_j^{ad(p_1)} \} \\
\end{aligned}
$$

is a well-defined nonnegative integer. Here, observe that

$$
\begin{aligned}
g_j = \operatorname{ad}(p_{-1})(g_{j+1}) \oplus g_j^{ad(p_1)} , \quad g_j^{ad(p_{-1})} &\cap b = 0 \\
\end{aligned}
$$

$(j \in \mathbb{Z})$. Hence, there exists uniquely a section $h^{log} \in \Gamma(U, O_U \otimes_k g_{j_0+1})$ (equivalently, a $U$-rational point $U \to g_{j_0+1}$ of $g_{j_0+1}$) satisfying the following condition: if $O_U \otimes_k g \twoheadrightarrow \operatorname{ad}(1 \otimes p_{-1})(O_U \otimes_k g_{j_0+1})$ denotes the projection with respect to the composite decomposition

$$
\nabla_E \otimes_k g \sim \bigoplus_{j \in \mathbb{Z}} O_U \otimes_k g_j \sim \bigoplus_{j \in \mathbb{Z}} \operatorname{ad}(1 \otimes p_{-1})(O_U \otimes_k g_j) \otimes O_U \otimes_k g_j^{ad(p_1)}
$$

arising from (136), then the image of $\nabla^{id_{U, x, k},0}_E(\partial_x) \in \Gamma(U, O_U \otimes_k g)$ via this projection coincides with $\operatorname{ad}(1 \otimes p_{-1})(h^{log})$. Since $g_{j_0+1} \subseteq n$, the $U$-rational point $h^{rep} := \operatorname{rep}_N \circ h^{log} : U \to N$ (cf. (50)) may be defined. By Corollary 1.4.2,
we have the equality

\begin{equation}
\nabla_{E,h}\log \left(\partial_x\right) = dh^{\log}(\partial_x) + \sum_{s=0}^{\infty} \frac{1}{s!} \cdot \text{ad}(h^{\log})^s(\nabla_{E,h} \log \left(\partial_x\right))
\end{equation}

in \(\Gamma(U, O_U \otimes_k g)\). By the definitions of \(j_0\) and \(h^{\log}\), the sections \(dh^{\log}(\partial_x)\) and \(\text{ad}(h^{\log})^s(\nabla_{E,h} \log \left(\partial_x\right))\) \((s \geq 2)\) lie in \(\Gamma(U, O_U \otimes_k g^{j_0+1})\). Hence, modulo \(\Gamma(U, O_U \otimes_k g^{j_0+1})\), we obtain from (138) the equality

\begin{equation}
\nabla_{E,h}^{\text{idU} \times k,G,0}(\partial_x) \equiv \nabla_{E,h}^{\text{idU} \times k,G,0}(\partial_x) + \text{ad}(h^{\log})^s(\nabla_{E,h} \log \left(\partial_x\right))
\end{equation}

But, it follows from this equality and the definition of \(h^{\log}\) that \(\nabla_{E,h}^{\text{idU} \times k,G,0}(\partial_x)\) lies in \(\Gamma(U, O_U \otimes_k g^{\text{ad}(p_1)})\) modulo \(\Gamma(U, O_U \otimes_k g^{j_0+1})\). That is,

\begin{equation}
\nabla_{E,h}^{\text{idU} \times k,G,0}(\partial_x) \equiv \nabla_{E,h}^{\text{idU} \times k,G,0}(\partial_x) + \text{ad}(h^{\log})^s(\nabla_{E,h} \log \left(\partial_x\right))
\end{equation}

By continuing inductively the procedure for constructing from \(\nabla_{E,h}^{\text{idU} \times k,G,0}(\partial_x)\) to \(\nabla_{E,h}^{\text{idU} \times k,G,0}(\partial_x)\), one obtains a \((g, h)\)-oper on \(U/S\) of the form \((U \times_k \mathbb{B}, \nabla_{E,h}^\oplus)\) for some \(S\)-log connection \(\nabla_{E,h}^\oplus\) satisfying that \(\text{Im}(\nabla_{E,h}^{\text{idU} \times k,G,0}(\partial_x)) \subseteq \bigoplus_{j \geq j_0} O_U \otimes_k g^{\text{ad}(p_1)}\) and an isomorphism \(\text{can}_{E,h} : (U \times_k \mathbb{B}, \nabla_{E,h}^\oplus) \cong (U \times_k \mathbb{B}, \nabla_{E,h}^\oplus)\) of \((g, h)\)-opers. This proves the existence of the required pair \((U \times_k \mathbb{B}, \nabla_{E,h}^\oplus, \text{can}_{E,h})\). The asserted uniqueness follows from the above discussion. This completes the proof of Proposition 2.7.3.

**Remark 2.7.4.**

In the present paper, for a scheme \(Y\), we shall not distinguish between \(\mathcal{O}_Y\)-modules from the respective associated sheaves (i.e., stacks fibered in sets) on \(\mathcal{E}_Y\).

The proof of Proposition 2.7.3 implies that if we fix a log chart \((U, x)\) on \(X^{\text{log}}\) over \(S^{\text{log}}\), then the assignment

\begin{equation}
\mathcal{E}^\bullet := (\mathcal{E}_B, \nabla_{\mathcal{E}}) \mapsto \nabla^{\text{triv}_{g,h}(U,x)} - (1 \otimes p_{-1})
\end{equation}

determines a canonical isomorphism

\begin{equation}
\text{triv}_{g,h,\mathcal{O}_S}^{\mathcal{O}_p} : \mathcal{O}_{\mathcal{E}^{\text{triv}_{g,h,\mathcal{O}_S}}} \cong \mathcal{O}_U \otimes_k g^{\text{ad}(p_1)}
\end{equation}

of stacks over \(\mathcal{E}_Y\).
The assignment \((E \hat{\otimes}, R) \mapsto \iota_{G^*}(E \hat{\otimes})_{+R}\) discussed above is functorial with respect to \(S\), and hence, determines a morphism
\[
\text{act}_{g,h,x/S} : \mathcal{Op}_{g^\otimes,h,x/S} \times_S V(f_*(V_{g,h,x/S})) \to \mathcal{Op}_{g,h,x/S}
\]
of functors on \(\mathcal{C}\mathcal{h}_f/S\). Here, if \(R_\circ\) is an element in \(\Gamma(S,f_*(\Omega^2_{X_{\log}/S_{\log}}))\) (and \(E \hat{\otimes}\) is as above) there exists a functorial (with respect to \(S\)) isomorphism
\[
\iota_{G^*}(E \hat{\otimes} \otimes_{R_{\circ}}) \sim \iota_{G^*}(E \hat{\otimes})_{+\xi_{g,h,x/S}(R_{\circ})},
\]
(cf. (102)) of \((g,h)\)-opers. Thus, \(\text{act}_{g,h,x/S}\) induces a morphism
\[
\overline{\text{act}}_{g,h,x/S} : \mathcal{Op}_{g^\otimes,h,x/S} \times S V(f_*(V_{g,h,x/S})) \to \mathcal{Op}_{g,h,x/S}
\]
of functors on \(\mathcal{C}\mathcal{h}_f/S\). It follows from Proposition 2.7.3 that \(\xi_{g,h,X/S}\) is an isomorphism, which implies the following proposition (cf. [9] §3.4, Theorem; [3], Lemma 3.6; [52], Chap.1, Proposition 2.11):

**Proposition 2.7.5.**
\(\mathcal{Op}_{g,h,x/S}\) may be represented by a relative affine space over \(S\) modeled on \(V(f_*(V_{g,h,x/S}))\), hence, by a relative affine space of relative dimension \(\mathfrak{g}(g)\) (cf. Proposition 2.6.1). In particular, the morphism \(\overline{\text{act}}_{g,h,x/S} : \mathcal{Op}_{g^\otimes,h,x/S} \to \mathcal{Op}_{g,h,x/S}\) (cf. (130)) is a closed immersion of \(S\)-schemes, and the fiber of \(\mathcal{Op}_{g,h,x/S}\) over any point of \(S\) is nonempty.

**Corollary 2.7.6.**

(i) If \(g\) is simple, then there exists a canonical isomorphism
\[
\mathcal{Op}_{g^\otimes,h,x/S} \times S V(f_*(\bigoplus_{k \geq 1} V_{g,h,x/S,k})) \sim \mathcal{Op}_{g,h,x/S}
\]
of \(S\)-schemes.

(ii) For \(s = 1,2\), suppose that we are given a split connected semisimple algebraic group \(G^s\) of adjoint type over \(k\) and its pinning \(\mathfrak{g}^s := (G^s, T^s, B^s, \Gamma^s, \{x_{a^s}\}_{a^s \in \Gamma^s})\) (cf. (70)). Then, the collection of data
\[
\mathfrak{g}^1 \times \mathfrak{g}^2 := (G^1 \times_k G^2, T^1 \times_k T^2, B^1 \times_k B^2, \Gamma^1 \times \Gamma^2, \{(x_{a^1}, x_{a^2})\}_{(a^1, a^2) \in \Gamma^1 \times \Gamma^2})
\]
forms a pinning of \(G^1 \times_k G^2\). Thus, if \(g^1\) and \(g^2\) are the Lie algebras of \(G^1\) and \(G^2\) respectively, then one may define, by means of the pinning \(\mathfrak{g}^1 \times \mathfrak{g}^2\), the moduli stack \(\mathcal{Op}_{g^1 \times_k g^2,h,x/S}\). Moreover, there exists a canonical isomorphism
\[
\mathcal{Op}_{g^1 \times_k g^2,h,x/S} \sim \mathcal{Op}_{g^1,h,x/S} \times_S \mathcal{Op}_{g^2,h,x/S}
\]
of \(S\)-schemes.

**Proof.** See [9], §3.4, Remark. \(\square\)
Moreover, the definition of the structure of $V(f_*(V_{g,h,x/S}))$-action on $\mathcal{O}_{p,g,h,x/S}$ and the fact explained in §2.4, (iv) imply the following

**Corollary 2.7.7.**

If $s' : S' \to S$ is a morphism of $k$-schemes, the morphism

$$\beta_{g,h,x/S,S'} : \mathcal{O}_{p,g,h,x/S} \times_S S' (\sim \mathcal{O}_{p,g,h,x/S} \circ s') \to \mathcal{O}_{p,s'^*(h),x/S'}$$  

(cf. [44]) is an isomorphism that is compatible (in the sense similar to the compatibility of the diagram (103)) with the respective $V(f_*(V_{g,h,x/S}))$-actions via the isomorphism $V(f_*(V_{g,h,x/S})) \times_S S' \sim V(f_*(V_{g,s'^*(h),x/S'}))$ of (114).

2.8. Let us introduce (cf. Definition 2.9.1) an invariant associated with a marked point of $X/S$ and an $S$-log integrable $G$-torsor, (in particular, a $(g,h)$-oper,) which is called the radius. To begin with, we review some basic facts concerning the quotient of $g$ by the adjoint action of $G$.

We shall identify $g$ and $t$ with $\text{Spec}(S_k(g^\vee))$ and $\text{Spec}(S_k(t^\vee))$ respectively, where for a $k$-vector space $a$ we denote by $S_k(a)$ the symmetric algebra on $a$ over $k$. Consider the GIT quotient $g/\!/G$ (resp., $t/\!/W$) of $g$ (resp., $t$) by the adjoint action of $G$ (resp., the Weyl group $W$ of $(G,T)$ (cf. §2.1)), i.e., the spectrum of the ring of polynomial invariants $S_k(g^\vee)^G$ (resp., $S_k(t^\vee)^W$) on $g$ (resp., $t$). Let us write

$$c := t/\!/W.$$  

(When there is fear of confusion, we write $c_g := t/\!/W$.) A Chevalley’s theorem asserts (cf. [54], Theorem 1.1.1; [43], Chap. VI, Theorem 8.2) that (under the condition that either $(\text{Char})_0$ or $(\text{Char})_p$ is satisfied) the natural morphism $S_k(g^\vee)^G \to S_k(t^\vee)^W$ is an isomorphism, i.e., $c \sim g/\!/G$. Thus, one may define a morphism

$$\chi : g \to c$$  

of $k$-schemes to be the composite of the natural quotient $g \to g/\!/G$ and the inverse of the resulting isomorphism $c \sim g/\!/G$. Let

$$\mathcal{R}os_g : g^{\text{ad}(p_1)} \hookrightarrow g \xrightarrow{\chi} c,$$

be the composite, where the first arrow is given by $s \mapsto s + p_{-1}$ for $s \in g^{\text{ad}(p_1)}$. Then, it is well-known (cf. [54], Lemma 1.2.1) that $\mathcal{R}os_g$ is an isomorphism of $k$-schemes.

If we equip $c$ with the trivial $G$-action, then $\chi$ is compatible with the respective $G$-actions. Hence, $\chi$ factors through the quotient morphism $g \to [g/G]$. (Here, we recall that $[g/G]$ is the quotient stack representing the functor which,
to any $k$-scheme $T$, assigns the groupoid of pairs $(\mathcal{F}, R)$ consisting of a right $G$-torsor $\mathcal{F}$ over $T$ and $R \in \Gamma(T, \mathfrak{g}_T)$. We denote by

$$[\chi] : [\mathfrak{g}/G] \to \mathfrak{c}$$

the resulting morphism, which is compatible with the respective $G$-actions on $[\mathfrak{g}/G]$ and $\mathfrak{c}$.

Consider a $\mathbb{G}_m$-action $\text{Ad}^+$ on $\mathfrak{g}^{\text{ad}(p_1)}$ as follows. (Note that this action on $\mathfrak{g}^{\text{ad}(p_1)}$ has already appeared in the discussion preceding (113).) Identify $B \circ \mathbb{G} \to GL(b)$ via the adjoint action $B \circ \mathbb{G} \sim \to GL(b) = \mathbb{G}_m$. The action of $B \circ \mathbb{G}$ on $b$ via the composite $B \circ \mathbb{G} \to B \circ \mathbb{G} \to GL(b)$ (cf. (17)) induces an action $\text{Ad}$ of $B \circ \mathbb{G}$ on $\mathfrak{g}^{\text{ad}(p_1)}$. Then, we define a new action $\text{Ad}^+$ of $\mathbb{G}_m$ on $\mathfrak{g}^{\text{ad}(p_1)}$ by $\text{Ad}^+(t)(v) := t \cdot \text{Ad}(t)(v)$ for $v \in \mathfrak{g}^{\text{ad}(p_1)}$, $t \in \mathbb{G}_m$. On the other hand, $c(\sim = \text{Spec}(S_k(\mathfrak{g}^{\vee}))$ admits a canonical $\mathbb{G}_m$-action that comes from the homotheties on $\mathfrak{g}$ (i.e., the natural grading on $S_k(\mathfrak{g}^{\vee})$). Then, it is known that the isomorphism $\text{Kos}$ is compatible with the respective $\mathbb{G}_m$-actions of $\mathfrak{g}^{\text{ad}(p_1)}$ and $c$. Moreover, a (non-canonical) choice of homogeneous generator $u_l \in S_k(\mathfrak{g}^{\vee})$ of degree $e_l$ ($l = 1, 2, \ldots$) gives (cf. the discussion following [54], Theorem 1.1.1) an isomorphism

$$k[u_1, \ldots, u_{rk(\mathfrak{g})}] \sim \to S_k(\mathfrak{g}^{\vee})$$

of $k$-algebras. If we denote by

$$(h, \ a) \mapsto h \ast a$$

the $\mathbb{G}_m$-action on $\mathfrak{c}$ just discussed, then, via the isomorphism $c \sim \to \text{Spec}(k[u_1, \ldots, u_{rk(\mathfrak{g})}) (=: A^{rk(\mathfrak{g})})$

induced by (154), this action may be expressed as

$$h \ast (a_1, \ldots, a_{rk(\mathfrak{g}))} = (h^{e_1} \cdot a_1, \ldots, h^{e_{rk(\mathfrak{g})}} \cdot a_{rk(\mathfrak{g})})$$

for $(a_1, \ldots, a_{rk(\mathfrak{g}))} \in A^{rk(\mathfrak{g})}$. The action $\ast$ may be, in a natural fashion, extended uniquely to a morphism

$$\tilde{\ast} : A^1 \times_k \mathfrak{c} \to \mathfrak{c}.$$
Definition 2.9.1.
Let \((\mathcal{G}, \nabla_{\mathcal{G}})\) be an \(h\)-log integrable \(\mathbb{G}\)-torsor over \(U^{\log}/S^{\log}\) and \(i \in \{1, \cdots, r\}\). Write
\[
\rho_i^{(\mathcal{G}, \nabla_{\mathcal{G}})} \in \mathfrak{c}(U \times_{\mathcal{X}, \sigma_i} S)
\]
for the image, via \([\chi] : [\mathfrak{g}/\mathbb{G}] \to \mathfrak{c}\), of the \((U \times_{\mathcal{X}, \sigma_i} S)\)-rational point of \([\mathfrak{g}/\mathbb{G}]\) classifying the pair \((\sigma_i^{U \times_{\mathcal{X}, \sigma_i} S}(\mathcal{G}), \mu_i^{(\mathcal{G}, \nabla_{\mathcal{G}})})\) (cf. Definition 1.6.1 for the definition of \(\mu_i^{(-,-)}\)). We shall refer to \(\rho_i^{(\mathcal{G}, \nabla_{\mathcal{G}})}\) as the radius of \((\mathcal{G}, \nabla_{\mathcal{G}})\) at the marked point \(\sigma_i\).

Definition 2.9.2.
Let \(\mathcal{E}^\bullet := (\mathcal{E}_0, \nabla_{\mathcal{E}})\) be a \((\mathfrak{g}, h)\)-oper on \(\mathfrak{u}/S\) and \(\rho\) a \((U \times_{\mathcal{X}, \sigma_i} S)\)-rational point of \(\mathfrak{c}^{\times r}\), where \(\mathfrak{c}^{\times r}\) denotes the product of \(r\) copies of \(\mathfrak{c}\). (That is, \(\rho\) is an ordered set \(\rho := (\rho_i)_{i=1}^r\) consisting of \(k\)-morphisms \(\rho_i : U \times_{\mathcal{X}, \sigma_i} S \to \mathfrak{c}\).) We shall say that \(\mathcal{E}^\bullet\) is of radii \(\rho\) if \(\rho_i^{(\mathcal{E}_0, \nabla_{\mathcal{E}})} = \rho_i\) for all \(i = 1, \cdots, r\).

The image of the zero vector \(0 \in \mathfrak{t}\) via the quotient \(\mathfrak{t} \to \mathfrak{c}\) determines a \((U \times_{\mathcal{X}, \sigma_i} S)\)-rational point of \(\mathfrak{c}\), for which we shall write \([0]_{U \times_{\mathcal{X}, \sigma_i} S} \in \mathfrak{c}(U \times_{\mathcal{X}, \sigma_i} S)\). Also, we shall write
\[
[0]_{U \times_{\mathcal{X}, \sigma_i} S} := ([0]_{U \times_{\mathcal{X}, \sigma_i} S}; [0]_{U \times_{\mathcal{X}, \sigma_i} S}; \cdots; [0]_{U \times_{\mathcal{X}, \sigma_i} S}) \in \mathfrak{c}^{\times r}(U \times_{\mathcal{X}, \sigma_i} S).
\]
Let \(\rho := (\rho_i)_{i=1}^r \in \mathfrak{c}^{\times r}(S)\). We shall denote by
\[
\mathcal{D}_{\mathfrak{g}, h, \mathfrak{t}, \mathcal{X}/S} : \mathfrak{S}_{\mathcal{X}/S} \to \mathfrak{G}_{\mathcal{X}/S}
\]
the \(\mathfrak{G}_{\mathcal{X}/S}\)-valued contravariant functor on \(\mathfrak{S}_{\mathcal{X}/S}\) which, to any \(S\)-scheme \(t : T \to S\), assigns the set of isomorphism classes of \((\mathfrak{g}, h)\)-opers on \(\mathcal{X}/T\) of radii \(\rho \circ t := (\rho_i \circ t)_{i=1}^r \in \mathfrak{c}^{\times r}(T)\). If \(r = 0\), then by the notation \(\mathcal{D}_{\mathfrak{g}, h, \mathfrak{t}, \mathcal{X}/S} \) we mean the moduli stack \(\mathcal{D}_{\mathfrak{g}, h, \mathcal{X}/S}\).

2.10. Let \(h' \in \Gamma(S, \mathcal{O}_S)\), \(\rho := (\rho_i)_{i=1}^r \in \mathfrak{c}^{\times r}(S)\), and let \(\mathcal{E}^\bullet := (\mathcal{E}_0, \nabla_{\mathcal{E}})\) be a \((\mathfrak{g}, h)\)-oper on \(\mathcal{X}/S\) of radii \(\rho\). The \(\mathcal{O}_X\)-linear morphism
\[
h' \cdot \nabla_{\mathcal{E}} : \mathcal{T}_{X^{\log}/S^{\log}} \to \tilde{\mathcal{T}}_{X^{\log}/S^{\log}}
\]
defined by assigning \(a \mapsto h' \cdot \nabla_{\mathcal{E}}(a)\) is an \(S\)-\((h' \cdot h)\)-log connection on \(\mathcal{E}_{\mathcal{G}}\). One verifies the equality
\[
\rho_i^{(\mathcal{E}_{\mathcal{G}}, h' \cdot \nabla_{\mathcal{E}})} = h' \star \rho_i^{(\mathcal{E}_{\mathcal{G}}, \nabla_{\mathcal{E}})}
\]
(for \(i = 1, \cdots, r\)), where \(\star\) denotes (by abuse of notation) the \(\Gamma(S, \mathcal{O}_S^\times)\)-action
\[
\star : \Gamma(S, \mathcal{O}_S^\times) \times \mathfrak{c}(S) \to \mathfrak{c}(S)
\]
\[
(\star, a) \mapsto h' \cdot a
\]
on the set of $S$-rational points $c(S)$ arising from the $\mathbb{G}_m$-action $\star$ on $c$ (cf. (155)). Moreover, the pair

$$E^\bullet := (E_B, h' \cdot \nabla E)$$

forms a $(g, h' \cdot h)$-oper on $X/S$ of radii $h' \rho := (h' \star_S \rho)^r_{r=1}$. The assignment $E^\bullet \mapsto E^\bullet$ is functorial with respect to $S$, and hence, determines a morphism

$$\text{op}_{x,h'} : \mathcal{O}p_{g,h',\rho,x/S} \to \mathcal{O}p_{g,h',h' \rho,x/S}$$

of functors. If, moreover, $h'' \in \Gamma(S, \mathcal{O}_S^\times)$, then

$$\text{op}_{x,h''} \circ \text{op}_{x,h'} = \text{op}_{x,h'}.$$ 

In particular, we have that

$$\text{op}_{x,h'-1} \circ \text{op}_{x,h'} = \text{op}_{x,h'} \circ \text{op}_{x,h'-1} = \text{id},$$

which implies the following

**Proposition 2.10.1.**

Let $h' \in \Gamma(S, \mathcal{O}_S^\times)$. Then, the morphism $\text{op}_{x,h'}$ is an isomorphism.

2.11. In this subsection, we conclude (cf. Proposition 2.11.1) that for any $\rho := (\rho_i)^r_{i=1} \in c^{xir}(S)$ the functor $\mathcal{O}p_{g,h,\rho,x/S}$ may be represented by a relative affine subspace of $\mathcal{O}p_{g,h,x/S}$ modeled on $V(f^*(\mathcal{O}^g_{h,\rho,x/S}))$. In particular, the fiber of $\mathcal{O}p_{g,h,\rho,x/S}$ over any point of $S$ is nonempty.

To this end, we may suppose (after possibly replacing $S$ with an open subscheme of $S$) that for all $i \in \{1, \cdots, r\}$, there exist an open subscheme $U_i$ of $X$ containing the image of the marked point $\sigma_i : S \to X$ and a trivialization

$$(168) \quad \text{triv}_{\mathbb{B},U_i} : E^\dagger_{\mathbb{B},h,h^\dagger_{U_i}} \cong U_i \times_k \mathbb{B}$$

of the $\mathbb{B}$-torsor $E^\dagger_{\mathbb{B},h,h^\dagger_{U_i}}$.

Let us fix an $i_0 \in \{1, \cdots, r\}$. The trivialization $\text{triv}_{\mathbb{B},U_i}$ determines an isomorphism

$$(169) \quad \text{triv}_{g,U_{i_0}} : (\Omega_{U_{i_0}^{\log}/S^{\log}} \otimes g_{c^\dagger_{\mathbb{B},h,h_{i_0}^{\dagger}}} ) \cong \Omega_{U_{i_0}^{\log}/S^{\log}} \otimes_k g,$$

which gives, by restricting, an isomorphism

$$(170) \quad s_{g,h,x/S}(\mathcal{V}_{g,h,x/S})|_{U_{i_0}} \cong \Omega_{U_{i_0}^{\log}/S^{\log}} \otimes_k g^{\text{ad}(p_1)}.$$
By applying the functor $\sigma^*_\rho(-)$ to the isomorphism (169) and applying the trivialization (67) obtained in §1.6, we obtain a commutative square

$$
\begin{array}{ccc}
\sigma^*_\rho(\mathcal{V}_{g,h,x/S}) & \xrightarrow{\text{triv}^\text{ad}(p_1)} & \mathcal{O}_S \otimes_k \mathfrak{g}^\text{ad}(p_1) \\
\downarrow & & \downarrow \\
\mathfrak{g}_{\sigma^*_\rho(\mathcal{E}^\dagger_{g,h,x/S})} & \xrightarrow{\text{triv}} & \mathcal{O}_S \otimes_k \mathfrak{g}
\end{array}
$$

(171)

of $\mathcal{O}_S$-modules, where both the upper and lower horizontal arrows are isomorphisms and the both sides of vertical arrows are natural injections.

Now, let us fix a $(g, h)$-oper $\mathcal{E}^{\bullet,\diamond} := (\mathcal{E}^\dagger_{g,h,x/S}, \nabla^\diamond, \hat{\nabla})$ on $X/S$ of canonical type (cf. Definition 2.7.1) and of radii $\rho := (\rho_i)_{i=1}^m \in \mathfrak{c}^\times(S)$. Then, the image

$$
\text{triv}^\text{ad}(p_1)(\mathcal{E}^\dagger_{g,h,x/S}, \nabla^\diamond, \hat{\nabla}) \in \Gamma(S, \mathcal{O}_S \otimes_k \mathfrak{g})
$$

(172)

of the monodromy $\mu_{\sigma^*_\rho(\mathcal{E}^\dagger_{g,h,x/S}, \nabla^\diamond, \hat{\nabla})} \in \Gamma(S, \mathfrak{g}_{\sigma^*_\rho(\mathcal{E}^\dagger_{g,h,x/S})})$ may be expressed as $1 \otimes p_{-1} + R^{\mathcal{E}^{\bullet,\diamond}}$ for some $R^{\mathcal{E}^{\bullet,\diamond}} \in \Gamma(S, \mathcal{O}_S \otimes_k \mathfrak{g}^\text{ad}(p_1))$. Consider the composite morphism

$$
\mathfrak{R}^{\mathcal{E}^{\bullet,\diamond}}_{g,h,x_S,\sigma^*_\rho} : \nabla(\sigma^*_\rho(\mathcal{V}_{g,h,x/S})) \to \nabla(\mathfrak{g}_{\sigma^*_\rho(\mathcal{E}^\dagger_{g,h,x/S})}) \to S \times_k \mathfrak{g} \xrightarrow{\text{id}_S \times X} S \times_k \mathfrak{c}
$$

of $S$-schemes, where the first arrow denotes the morphism given by $\mathcal{R} \mapsto \mathcal{R} + (\mathcal{E}^\dagger_{g,h,x/S}, \nabla^\diamond, \hat{\nabla})$ for any local section $\mathcal{R}$ of $\sigma^*_\rho(\mathcal{V}_{g,h,x/S})$. This morphism does not depend on the choice of the trivialization (168). It follows from the isomorphism $\mathfrak{R}^{\mathcal{E}^{\bullet,\diamond}}_g$ (cf. (152)) and the commutativity of the square diagram (171) that $\mathfrak{R}^{\mathcal{E}^{\bullet,\diamond}}_{g,h,x_S,\sigma^*_\rho}$ is an isomorphism.

Let $R$ be an element in $\Gamma(X, \mathcal{V}_{g,h,x/S})$ and write $R^\mathcal{E}$ for the image of $R$ via the injection (131) (as we defined in the discussion preceding Proposition 2.7.2). One verifies that the pair $\mathcal{E}^{\bullet,\diamond}_R := (\mathcal{E}^\dagger_{g,h,x/S}, \nabla^\diamond + R^\mathcal{E})$ forms a $(g, h)$-oper on $X/S$ of canonical type and satisfies the equality

$$
\rho_{\mathcal{E}^{\bullet,\diamond}}_{\mathcal{E}^\dagger_{g,h,x/S}, \nabla^\diamond + R^\mathcal{E}} = \mathfrak{R}^{\mathcal{E}^{\bullet,\diamond}}_{g,h,x_S,\sigma^*_\rho} \circ \mathcal{R} \in \mathfrak{c}(S),
$$

(174)

where $\mathcal{R}$ denotes the $S$-rational point of $\nabla(\sigma^*_\rho(\mathcal{V}_{g,h,x/S}))$ arising from $\sigma^*_\rho(R)$. In particular, (since $\mathfrak{R}^{\mathcal{E}^{\bullet,\diamond}}_{g,h,x_S,\sigma^*_\rho}$ is an isomorphism,) $\rho_{\mathcal{E}^{\bullet,\diamond}}_{\mathcal{E}^\dagger_{g,h,x/S}, \nabla^\diamond + R^\mathcal{E}} = \rho_{\mathcal{E}^{\bullet,\diamond}}$ if and only if $\mathcal{R} : S \to \nabla(\sigma^*_\rho(\mathcal{V}_{g,h,x/S}))$ is the zero section.
We shall apply the above discussion to all $i$. Consider the composite
\[(175)\]
\[
\mathcal{R}os_{g,h,x/S}^{\circ \diamond} : \mathbb{V}(f_*(\mathcal{V}_{g,h,x/S})) \to \prod_{i=1}^r \mathbb{V}(\sigma_i^*(\mathcal{V}_{g,h,x/S})) \to \prod_{i=1}^r S \times_k c,
\]
where the first arrow denotes the morphisms determined by assigning $R \mapsto (\sigma_i^*(R))_{i=1}^r$ for any $R \in \Gamma(X, \mathcal{V}_{g,h,x/S})$. Since the second arrow in (175) is an isomorphism, the scheme-theoretic inverse image via $\mathcal{R}os_{g,h,x/S}^{\circ \diamond}$ of the graph $S \to \prod_{i=1}^r S \times_k c$ corresponding to $[\tilde{0}]_S \in c^{\times r}(S)$ (cf. (160)) is isomorphic to the closed subscheme $\mathbb{V}(f_*(\mathcal{V}_{g,h,x/S}))$ of $\mathbb{V}(f_*(\mathcal{V}_{g,h,x/S}))$. Thus, it follows from the above discussion that for $R \in \Gamma(X, \mathcal{V}_{g,h,x/S})$ the $(g,h)$-oper $\mathcal{E}^{\circ \diamond}_R$ is of radii $\rho$ if and only if $R$ lies in $\Gamma(X, \mathcal{V}_{g,h,x/S})$. That is, the $\mathbb{V}(f_*(\mathcal{V}_{g,h,x/S}))$-action on $\mathfrak{D}p_{g,h,x/S}$ carries a free and transitive $\mathbb{V}(f_*(\mathcal{V}_{g,h,x/S}))$-action on $\mathfrak{D}p_{g,h,\rho,x/S}$ (if $\mathfrak{D}p_{g,h,\rho,x/S}$ is nonempty).

**Proposition 2.11.1.**

Let $\rho = (\rho_i)_{i=1}^r \in c^{\times r}(S)$. Then, the functor $\mathfrak{D}p_{g,h,\rho,x/S}$ may be represented by a relative affine subspace of $\mathfrak{D}p_{g,h,x/S}$ (over $S$) modeled on $\mathbb{V}(f_*(\mathcal{V}_{g,h,x/S}))$. In particular, the fiber of $\mathfrak{D}p_{g,h,\rho,x/S}$ over any point of $S$ is nonempty.

**Proof.** Since we have already defined the free and transitive $\mathbb{V}(f_*(\mathcal{V}_{g,h,x/S}))$-action on $\mathfrak{D}p_{g,h,\rho,x/S}$, it suffices to prove that the fiber of $\mathfrak{D}p_{g,h,\rho,x/S}$ over any point of $S$ is nonempty. To this end, we may assume (by virtue of Corollary 2.7.7) that $S = \text{Spec}(k)$.

Consider the morphism
\[(176)\]
\[
\mathfrak{Rad}_{g,h,x/k} : \mathfrak{D}p_{g,h,x/k} \to c^{\times r}
\]
of $k$-schemes determined by assigning $\mathcal{E}^{\circ \diamond} : (\mathcal{E}_g, \nabla_{\mathcal{E}}) \mapsto (\rho_i^{(\mathcal{E}_g, \nabla_{\mathcal{E}})})_{i=1}^r$ for any $(g,h)$-oper $\mathcal{E}^{\circ \diamond}$. If we fix a $(g,h)$-oper $\mathcal{E}^{\circ \diamond} := (\mathcal{E}^{\circ \diamond}_{B,h}, \nabla_{\mathcal{E}})$ on $\mathcal{X}/k$ of canonical type and of radii $\rho$, then its classifying morphism $[\mathcal{E}^{\circ \diamond}] : \text{Spec}(k) \to \mathfrak{D}p_{g,h,x/k}$ determines a trivialization $\text{triv}_{\mathcal{E}^{\circ \diamond}} : \mathbb{V}(f_*(\mathcal{V}_{g,h,x/S})) \sim \mathfrak{D}p_{g,h,x/k}$ of the affine space $\mathfrak{D}p_{g,h,x/k}$. One verifies that the composite
\[(177)\]
\[
\mathbb{V}(f_*(\mathcal{V}_{g,h,x/k})) \xrightarrow{\text{triv}_{\mathcal{E}^{\circ \diamond}}} \mathfrak{D}p_{g,h,x/k} \xrightarrow{\mathfrak{Rad}_{g,h,x/k}} c^{\times r}
\]
coincides with $\mathfrak{Rad}_{g,h,x/k}^{\circ \diamond}$. In particular, the scheme-theoretic inverse image of $\rho \in c^{\times r}(k)$ via $\mathfrak{Rad}_{g,h,x/k}^{\circ \diamond}$ is isomorphic to $\mathbb{V}(f_*(\mathcal{V}_{g,h,x/k}))$, and hence, of dimension $\mathfrak{N}(g)$ (cf. Proposition 2.6.1).

Here, we shall suppose that $\mathfrak{Rad}_{g,h,x/k}$ is nonjective. Since both $\mathfrak{D}p_{g,h,x/k}$ and $c^{\times r}$ are irreducible, the scheme-theoretic image $\text{Im}(\mathfrak{Rad}_{g,h,x/k})$ of $\mathfrak{Rad}_{g,h,x/k}$
is irreducible and of dimension \(< \dim(\mathfrak{c}^r) = r \cdot \rk(\mathfrak{g})\). Thus, we have
\[
\dim(\mathcal{O}_p_{g,h,x/k}) - \dim(\Im(\text{Rad}_{g,h,x/k}))
= \dim(\mathcal{V}(f_*(\mathcal{V}_{g,h,x/k}))) - \dim(\Im(\text{Rad}_{g,h,x/k}))
> \mathcal{N}(\mathfrak{g}) - r \cdot \rk(\mathfrak{g})
= \mathcal{N}(\mathfrak{g})
\]
(cf. Proposition 2.6.1). It follows from (178) and [25], Chap. II, Exercise 3.22 (b) that every irreducible component of the scheme-theoretic inverse image, via \(\text{Rad}_{g,h,x/k}\), of any point in \(\Im(\text{Rad}_{g,h,x/k})\) is of dimension \(> \mathcal{N}(\mathfrak{g})\). This contradicts the above observation, i.e., that any such inverse image is isomorphic to \(\mathcal{V}(f_*(\mathcal{V}_{g,h,x/k}))\). Hence, \(\text{Rad}_{g,h,x/k}\) is surjective, equivalently, for any \(\rho \in \mathfrak{c}^r(k)\), the scheme-theoretic inverse image of \(\rho\) via \(\text{Rad}_{g,h,x/k}\), which is isomorphic to \(\mathcal{O}_p_{g,h,x/k}\), is nonempty. This completes the proof of Proposition 2.11.1.

**Remark 2.11.2.**
We shall consider canonical isomorphisms between stacks relating to different
types of \(\mathcal{O}_p_{g,h,x/S}\)'s constructed by restricting the isomorphisms obtained in
Corollary 2.7.6 (i), (ii) and Corollary 2.7.7.

(i) Write \(T^\circ\) for the maximal torus of \(G^\circ\) defined as the image of diagonal
matrices, \(t^\circ\) for its Lie algebra, and \(W^\circ\) for the Weyl group of \((G^\circ, T^\circ)\).
Set \(c^\circ := t^\circ / W^\circ\). The injection \(t_\mathfrak{g} : \mathfrak{g}^\circ \hookrightarrow \mathfrak{g}\) (cf. (106)) induces, in a
natural way, a morphism \(t_\mathfrak{c} : c^\circ \rightarrow c\). For \(\rho^\circ := (\rho^\circ)^\iota = (c^\circ)^{\times r}(S)\),
we shall write \(t_\mathfrak{c} \circ \rho^\circ := (t_\mathfrak{c} \circ \rho^\circ)^\iota \in (c^\circ)^{\times r}(S)\). If we suppose that \(\mathfrak{g}\)
is simple, then the isomorphism (146) yields, by restricting to closed
subschemes, an isomorphism
\[
\mathcal{O}_p_{g^\circ, h, \rho^\circ, x/S} \times S \mathcal{V}(f_*(\bigoplus_{k \geq 1} \mathcal{V}_{g,h,x/k})) \sim \mathcal{O}_p_{g^\circ, h, t_\mathfrak{c} \circ \rho^\circ, x/S}
\]
of \(S\)-schemes.

(ii) We shall use the notation in Corollary 2.7.6 (ii). For \(s = 1, 2\), write \(t^s\) for the Lie algebra of \(T^s\), \(W^s\) for the Weyl group of \((G^s, T^s)\), and \(c^s := t^s / W^s\). (Hence, \(W^1 \times W^2\) is the Weyl group of \((G^1 \times_k G^2, T^1 \times_k T^2)\)
and \(c^1 \times_k c^2 = (t^1 \times_k t^2) / (W^1 \times W^2)\).) Also, for \(\rho^s := (\rho^s)^\iota \in (c^s)^{\times r}(S)\)
\((s = 1, 2)\), we shall write
\[
(\rho^1, \rho^2) := ((\rho^s)^\iota)_{i = 1} \in (c^1 \times_k c^2)^{\times r}(S).
\]
Then, the isomorphism (148) yields, by restricting to closed subschemes, an isomorphism
\[
\mathcal{O}_p_{g^1 \times_k g^2, h, (\rho^1, \rho^2), x/S} \sim \mathcal{O}_p_{g^1 \times_k g^2, h, x/S} \times S \mathcal{O}_p_{g^2, h, \rho^2, x/S}
\]
of \(S\)-schemes.
(iii) If $s' : S' \to S$ is a morphism of $k$-schemes, then the isomorphism (149) yields, by restricting to closed subschemes, an isomorphism

$$\beta_{g,h,p,x/S,S'} : \mathcal{O}p_{g,h,p,x/S} \times_S S' \to \mathcal{O}p_{g,s'^*(h),p,s'^*(x)/S'}$$

that is compatible with the respective $\mathbb{R}(\mathcal{V}_{g,s'^*(h),x/s'})$-actions.

3. Operas in positive characteristic

In this section, we focus on $(g,h)$-opers in positive characteristic. In particular, we discuss the notion of $p$-curvature (cf. Definition 3.2.1) attached to each $h$-log integrable torsor, and define (in terms of $p$-curvature) two important classes of $(g,h)$-opers, which are referred to as dormant $(g,h)$-opers (cf. Definition 3.6.1) and $p$-nilpotent $(g,h)$-opers (cf. Definition 3.8.3). Also, we introduce (cf. §3.12) the moduli stack $\mathcal{O}p_{g,h,g,r}$ (resp., $\mathcal{O}p_{g,h,p,g,r}$) classifying pointed stable curves of type $(g,r)$ equipped with a dormant $(g,h)$-oper (resp., of radii $\rho$), as well as the moduli stack $\mathcal{O}p_{g,h,g,r}^{nilp}$ (resp., $\mathcal{O}p_{g,h,p,g,r}^{nilp}$) classifying pointed stable curves of type $(g,r)$ equipped with a $p$-nilpotent $(g,h)$-oper (resp., of radii $\rho$). As consequences of this section, we assert properties concerning the structure of these moduli stacks (cf. Theorem 3.12.1; Theorem 3.12.2; Theorem 3.12.3).

Let us keep the notation $k$, $\mathbb{G}$, $g$, $h$, etc., in §2, and suppose that the condition $(\text{Char})_p^W$ is satisfied.

3.1. Let $T$ be a scheme over $k (\equiv \mathbb{F}_p =: \mathbb{Z}/p\mathbb{Z})$ and $f : Y \to T$ a scheme over $T$. Denote by $F_T : T \to T$ (resp., $F_Y : Y \to Y$) the absolute Frobenius morphism of $T$ (resp., $Y$). The Frobenius twist of $Y$ over $T$ is, by definition, the base-change $Y_T^{(1)} := Y \times_{T,F_T} T$ of $f : Y \to T$ via $F_T : T \to T$. Denote by $f^{(1)} : Y_T^{(1)} \to T$ the structure morphism of the Frobenius twist of $Y$ over $T$. The relative Frobenius morphism of $Y$ over $T$ is the unique morphism $F_{Y/T} : Y \to Y_T^{(1)}$ over $T$ that fits into a commutative diagram of the form

$$\begin{array}{ccc}
Y & \xrightarrow{F_{Y/T}} & Y_T^{(1)} \\
\downarrow f & & \downarrow f^{(1)} \\
T & \xrightarrow{id_T} & T
\end{array}$$

Here, the upper composite in this diagram coincides with $F_Y$ and the right-hand square is, by the definition of $Y_T^{(1)}$, cartesian.
If $\tilde{y} : \tilde{Y} \to Y$ is a morphism of schemes (e.g., $\tilde{Y} = Y \times_T \tilde{T}$ for some $T$-scheme $\tilde{T}$), then (since $F_{Y/T}$ is a finite morphism) the cartesian diagram

$$
\begin{array}{ccc}
\tilde{Y} & \xrightarrow{\tilde{y}} & Y \\
F_{\tilde{Y}/T} \downarrow & & \downarrow F_{Y/T} \\
\tilde{Y}^{(1)} & \xrightarrow{\tilde{y} \times \text{id}_T} & Y^{(1)}
\end{array}
$$

induces a natural isomorphism

$$
(\tilde{y} \times \text{id}_T)^*(F_{Y/T*}(F_{Y/T}(-))) \cong F_{\tilde{Y}/T*}(F_{\tilde{Y}/T*}((\tilde{y} \times \text{id}_T)^*(-)))
$$

between functors from the category of $\mathcal{O}_{Y^{(1)}}$-modules to the category of $\mathcal{O}_{\tilde{Y}^{(1)}}$-modules.

3.2. We recall the definition of the $p$-curvature of a connection (cf., e.g., [73], §3). Let $f : Y^{\log} \to T^{\log}$ be a log smooth morphism of fine log schemes over $k$ and $(\mathcal{E}, \nabla_{\mathcal{E}})$ an $h$-log integrable $\mathbb{G}$-torsor over $Y^{\log}/T^{\log}$ (cf. Definition 1.2.1 (iii)). If $\partial$ is a logarithmic derivation corresponding to a local section of $\mathcal{T}_{Y^{\log}/T^{\log}}$ (resp., $\tilde{\mathcal{T}}_{E^{\log}/T^{\log}} := (\pi_* \mathcal{T}_{E^{\log}/T^{\log}})^G$), then we shall denote by $\partial[p]$ the $p$-th symbolic power of $\partial$ (i.e., “$\partial \mapsto \partial[p]$” as asserted in [55], Proposition 1.2.1), which is also a logarithmic derivation corresponding to a local section of $\mathcal{T}_{Y^{\log}/T^{\log}}$ (resp., $\tilde{\mathcal{T}}_{E^{\log}/T^{\log}}$). The pair $(\mathcal{T}_{Y^{\log}/T^{\log}}, (-)[p])$ (resp., $(\tilde{\mathcal{T}}_{E^{\log}/T^{\log}}, (-)[p])$) forms a restricted Lie algebra over $f^{-1}(\mathcal{O}_T)$. Since $a^{\log}_E \circ \nabla_{\mathcal{E}} = h \cdot \text{id}_{\mathcal{T}_{Y^{\log}/T^{\log}}}$ and $a^{\log}_E(\partial[p]) = (a^{\log}_E(\partial))[p]$ for any local section $\partial$ of $\mathcal{T}_{Y^{\log}/T^{\log}}$, the image of the $p$-linear map from $\mathcal{T}_{Y^{\log}/T^{\log}}$ to $\mathcal{T}_{E^{\log}/T^{\log}}$ defined by assigning $\partial \mapsto (\nabla_{\mathcal{E}}(\partial))[p] - h^{p-1} \cdot \nabla_{\mathcal{E}}(\partial[p])$ is contained in $\mathfrak{g}_{\mathcal{E}} (= \text{Ker}(a^{\log}_E))$. Thus, we obtain an $\mathcal{O}_Y$-linear morphism

$$
F_Y*(\mathcal{T}_{Y^{\log}/T^{\log}}) \to \mathfrak{g}_{\mathcal{E}}
$$

$$
F_Y*(\partial) \mapsto (\nabla_{\mathcal{E}}(\partial))[p] - h^{p-1} \cdot \nabla_{\mathcal{E}}(\partial[p]),
$$

which corresponds, in an evident way, to an element

$$
p^{(E, \nabla)}_{\psi} \in \Gamma(Y, F_Y*(\Omega_{Y^{\log}/T^{\log}}) \otimes \mathfrak{g}_{\mathcal{E}}).
$$

**Definition 3.2.1.**

We shall refer to $p^{(E, \nabla)}_{\psi}$ as the $p$-curvature of $(\mathcal{E}, \nabla_{\mathcal{E}})$.

As is well-known, by identifying $\mathfrak{g}$ with the space of right-invariant vector fields on $G$, the $p$-power operation $\partial \mapsto \partial[p]$ (where $\partial[p]$ coincides with the $p$-th iterate of the derivation $\partial : \mathcal{O}_G \to \mathcal{O}_G$) carries a $p$-Lie algebra structure on $\mathfrak{g}$. This $p$-power operation is compatible with the adjoint action of $G$, and hence,
carries, for each $G$-torsor $E$ over $Y$, a $p$-Lie algebra (over $f^{-1}(O_Y)$) structure $(-)^{[p]}$ on $g_E$; it coincides with the restriction of the $p$-Lie algebra structure on $\mathcal{T}_{\log}/\mathcal{T}^\log$ defined at the beginning of this subsection. Moreover, the assignment $R \mapsto R^{[p]}$ ($R \in \Gamma(Y, g_E)$) defines a $p$-power operation on the quotient stack $[g/G]$ in the way that $(E, R)^{[p]} = ([E, R^{[p]}]) \in [g/G](Y)$. The quotient $g \to [g/G]$ is compatible with the respective $p$-power operations on $g$ and $[g/G]$.

If $(E, \nabla_E)$ is a 0-log integrable $G$-torsor over $Y^{\log}/T^{\log}$, then the $p$-curvature $p\psi^{(E, \nabla_E)}$ may be determined by the condition that

$$\langle p\psi^{(E, \nabla_E)}, F_Y^*(\partial) \rangle = \nabla_E(\partial)^{[p]}$$

for any local section $\partial \in \mathcal{T}_{Y^{\log}/T^{\log}}$, where $\langle -, - \rangle$ denotes the $O_Y$-bilinear pairing $(F_Y^{*}(\Omega_{Y^{\log}/T^{\log}}) \otimes g_E) \times F_Y^{*}(\mathcal{T}_{Y^{\log}/T^{\log}}) \to g_E$ induced by the natural pairing $\Omega_{Y^{\log}/T^{\log}} \times \mathcal{T}_{Y^{\log}/T^{\log}} \to O_Y$.

Let $G'$ be a connected smooth algebraic group over $k$ with the Lie algebra $g'$ and $w : G \to G'$ a morphism of algebraic groups over $k$. If we write $dw : g \to g'$ for the differential of $w$, then it is compatible with the respective $p$-power operations on $g$ and $g'$ (cf. [66], 4.4.9). Hence, we have the following

**Proposition 3.2.2.**

Write $(E_{G'}, \nabla_{E_{G'}})$ for the $\hbar$-log integrable $G'$-torsor over $Y^{\log}/T^{\log}$ obtained from $(E, \nabla_E)$ by executing a change of structure group via $w : G \to G'$. Then, the $p$-curvature $p\psi^{(E_{G'}, \nabla_{E_{G'}})}$ of $(E_{G'}, \nabla_{E_{G'}})$ may be given by

$$p\psi^{(E_{G'}, \nabla_{E_{G'}})} = \text{id}_{F_Y^{*}(\Omega_{Y^{\log}/T^{\log}})} \otimes dw_E(p\psi^{(E, \nabla_E)}),$$

where $dw_E$ denotes the $O_Y$-linear morphism $g_E \to g'_{E_{G'}} (= g'_{E})$ arising from $dw$ twisted by $E$.

In particular, $p\psi^{(E, \nabla_E)} = 0$ implies that $p\psi^{(E_{G'}, \nabla_{E_{G'}})} = 0$, and vice versa if $dw_E$ is injective (e.g., the case where $G' = \text{GL}(g)$ and $w = \text{Ad}_{G'}$).

**Proof.** Observe that $dw_E$ is compatible with the respective $p$-power operations on $g_E$ and $g'_{E_{G'}}$. Hence, for any local section $\partial \in \mathcal{T}_{Y^{\log}/T^{\log}}$ and a local trivialization $\tau$ of the right $G$-torsor $E$, we have a sequence of equalities:

$$\langle \text{id}_{F_Y^{*}(\Omega_{Y^{\log}/T^{\log}})} \otimes dw_E(p\psi^{(E, \nabla_E)}), F_Y^*(\partial) \rangle = \langle dw_E(p\psi^{(E, \nabla_E)}), F_Y^*(\partial) \rangle$$

$$= dw_E(\langle p\psi^{(E, \nabla_E)}), F_Y^*(\partial) \rangle)$$

$$= dw_E((\nabla_E(\partial))^p - \hbar^{p-1} \cdot \nabla_E(\partial^{[p]}))$$

$$= dw_E((\nabla_E(\partial))^p - \hbar^{p-1} \cdot \nabla_E(\partial^{[p]}))$$

$$= (dw_E \circ \nabla_E(\partial))^p - \hbar^{p-1} \cdot (dw_E \circ \nabla_E)(\partial^{[p]}))$$

$$= \nabla_{E_{G'}}(\partial)^p - \hbar^{p-1} \cdot \nabla_{E_{G'}}(\partial^{[p]}))$$

$$= \langle p\psi^{(E_{G'}, \nabla_{E_{G'}})}, F_Y^*(\partial) \rangle.$$
This completes the proof of Proposition 3.2.2

Finally, we prove the following

**Proposition 3.2.3.**

Let \( \hbar' \in \Gamma(S, \mathcal{O}_S) \) and denote by

\[
\hbar' \cdot \nabla : \mathcal{T}_{Y, \log} \rightarrow \tilde{\mathcal{T}}_{E, \log}
\]

the \( T-(\hbar' \cdot \hbar) \)-log connection on \( E \) (cf. (162)) determined by assigning \( \theta \mapsto \hbar' \cdot \nabla_E(\theta) \) for any local sections \( \theta \in \mathcal{T}_{Y, \log} \). Then, the \( p \)-curvature \( p_{\psi}(E, \hbar' \cdot \nabla_{ad}) \) is given by

\[
(192) \quad p_{\psi}(E, \hbar' \cdot \nabla_{ad}) = \hbar' \cdot p_{\psi}(E, \nabla).
\]

**Proof.** The assertion follows directly from the definition of \( p \)-curvature (cf. \[72], § 2.2). \( \square \)

3.3. We shall construct a canonical \( T-h \)-log connections on the \( G \)-torsor over \( Y \) obtained by pulling-back, via \( F_{\mathcal{Y}/T} \), a right \( G \)-torsor over \( Y^{(1)}_T \).

Let \( \pi : \mathcal{F} \rightarrow Y^{(1)}_T \) be a right \( G \)-torsor over \( Y^{(1)}_T \), and

\[
(193) \quad \pi^{(1)} : F^{*}_{\mathcal{Y}/T}(\mathcal{F}) \rightarrow Y
\]

the \( G \)-torsor over \( Y \) obtained from \( \pi : \mathcal{F} \rightarrow Y^{(1)}_T \) by base-change via the relative Frobenius morphism \( F_{\mathcal{Y}/T} : Y \rightarrow Y^{(1)}_T \) (cf. § 3.1). We write \( F_\bullet \) for the projection

\[
(194) \quad F_\bullet : F^{*}_{\mathcal{Y}/T}(\mathcal{F}) := Y \times_{Y^{(1)}_T, \pi} \mathcal{F} \rightarrow \mathcal{F}
\]

to the second factor (i.e., \( F_\bullet := F_{\mathcal{Y}/T} \times \text{id}_\mathcal{F} \)), which is compatible with the respective \( \mathbb{G} \)-actions of \( F^{*}_{\mathcal{Y}/T}(\mathcal{F}) \) and \( \mathcal{F} \). Consider a morphism of short exact
sequences

\[\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\mathcal{T}_{F_{Y/T}}^{*}(\mathcal{F})_{\log/Y_{\log}} & \xrightarrow{\sim} & F_{\bullet}^{*}(\mathcal{T}_{\mathcal{F}_{\log/Y_{\log}}}^{*}) \\
\downarrow & & \downarrow \\
\mathcal{T}_{F_{Y/T}}^{*}(\mathcal{F})_{\log/Y_{\log}} & \xrightarrow{dF_{\bullet}} & F_{\bullet}^{*}(\mathcal{T}_{\mathcal{F}_{\log/Y_{\log}}}^{*}) \\
\downarrow & & \downarrow \\
\pi^{*}(\mathcal{T}_{Y_{\log}/T_{\log}}) & \xrightarrow{dF_{Y/T}} & \pi^{*}(\mathcal{T}_{Y_{\log}/T_{\log}}^{*}) \\
\downarrow & & \downarrow \\
0 & & 0 \\
\end{array}\]

of \(\mathcal{O}_{F_{Y/T}}^{*}(\mathcal{F})\)-modules, where

(i) the left-hand and right-hand vertical sequences arise from the natural exact sequences of (logarithmic) tangent bundles for \(F_{Y/T}^{*}(\mathcal{F})_{\log/Y_{\log}/T_{\log}}\) and \(\mathcal{F}_{\log/Y_{\log}/T_{\log}}\) respectively and a natural isomorphism

\[(196)\quad \pi^{*}(\mathcal{T}_{Y_{\log}/T_{\log}}^{*}) \xrightarrow{\sim} F_{\bullet}^{*}(\pi^{*}(\mathcal{T}_{Y_{\log}/T_{\log}}^{*}));\]

(ii) the top horizontal arrow denotes the canonical isomorphism obtained by base-change via \(F_{Y/T}\);

(iii) the middle and bottom horizontal arrows are obtained by differentiating (over \(T\)) the morphisms \(F_{\bullet}\) and \(F_{Y/T}\) respectively.

All of the arrows in the diagram (195) are compatible with the respective \(G\)-actions of the domain and codomain. By applying the functor \((\pi^{*}(\_))^{G}\) to
this diagram, we obtain a diagram of $\mathcal{O}_Y$-modules

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
\mathfrak{g}_{X/S}(\mathcal{F}) & \xrightarrow{\sim} & F^*_{Y/T}(\mathfrak{g}_\mathcal{F}) \\
\downarrow & & \downarrow \\
\widetilde{T}^*_{F/Y/T}(\mathcal{F})^{\log/T^\log} & \xrightarrow{(\pi_*^{(1)}(d\mathfrak{F}))^G} & (\pi_*^{(1)}(F^*_{\mathfrak{F}^{\log/T^\log}}))^G \\
\downarrow & & \downarrow \\
\mathcal{T}^{\log/T^\log} & \xrightarrow{(\pi_*^{(1)}(dF_{Y/T}))^G} & F^*_{Y/T}(\mathcal{T}^{(1)}_{F^{\log/T^\log}}) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
\]  

(197)

The two vertical sequences in this diagram are exact. It follows from the definition of $F^*_{Y/T}$ that the bottom horizontal arrow $(\pi_*^{(1)}(dF_{Y/T}))^G$ vanishes identically on $Y$. Hence, the middle horizontal arrow $(\pi_*^{(1)}(d\mathfrak{F}))^G$ factors through the injection $F^*_{\mathfrak{F}^{\log/T^\log}} \hookrightarrow (\pi_*^{(1)}(F^*_{\mathfrak{F}^{\log/T^\log}}))^G$. The resulting morphism $\widetilde{T}^*_{F/Y/T}(\mathcal{F})^{\log/T^\log} \rightarrow F^*_{Y/T}(\mathcal{F})$ determines, via the top horizontal arrow $\mathfrak{g}_{X/S}(\mathcal{F}) \xrightarrow{\sim} F^*_{X/S}(\mathfrak{g}_\mathcal{F})$ in (197), a split surjection of the left-hand vertical sequence in (197). Thus, we obtain a $T$-$1$-log connection

\[
\nabla^\text{can}_\mathcal{F}: \mathcal{T}^{\log/T^\log} \rightarrow \widetilde{T}^*_{F/Y/T}(\mathcal{F})^{\log/T^\log}
\]  

(198)

on $F^*_{Y/T}(\mathcal{F})$ defined to be the corresponding split injection. We shall refer to the $T$-$h$-connection

\[
\nabla^\text{can}_{\mathcal{F},h} := h \cdot \nabla^\text{can}_\mathcal{F}
\]  

(199)

(cf. [11], Theorem (5.1)) as the canonical $T$-$h$-log connection on $F^*_{Y/T}(\mathcal{F})$ (hence $\nabla^\text{can}_{\mathcal{F},1} = \nabla^\text{can}_\mathcal{F}$).

**Proposition 3.3.1.**

$p_{\psi}(F^*_{Y/T}(\mathcal{F}), \nabla^\text{can}_{\mathcal{F},h}) = 0$.

**Proof.** By Proposition 3.2.3, it suffices to prove the assertion of the case where $h = 1$. Observe that the middle horizontal arrow $(\pi_*^{(1)}(d\mathfrak{F}))^G$ in (197) is compatible with the respective $p$-power structures $(-)^{[p]}$ on $\widetilde{T}^*_{F/Y/T}(\mathcal{F})^{\log/T^\log}$ and $(\pi_*^{(1)}(F^*_{\mathfrak{F}^{\log/T^\log}}))^G$. Hence, the kernel $\text{Ker}((\pi_*^{(1)}(d\mathfrak{F}))^G)$ is closed under the $p$-power operator $(-)^{[p]}$ in $\widetilde{T}^*_{F/Y/T}(\mathcal{F})^{\log/T^\log}$. If we equip $\text{Ker}((\pi_*^{(1)}(d\mathfrak{F}))^G)$ with
the induced $p$-power structure, then the natural inclusion \( \ker((\pi^*_x)(dF_x))^G) \hookrightarrow \tilde{\mathcal{F}}_{Y/T}(\mathcal{F})^\log/T_\log \) is evidently a morphism of restricted Lie algebras over \( f^{-1}(\mathcal{O}_T) \), which makes the square diagram

\[
\begin{array}{ccc}
\ker((\pi^*_x)(dF_x))^G & \longrightarrow & \tilde{\mathcal{F}}_{Y/T}(\mathcal{F})^\log/T_\log \\
\downarrow & & \downarrow \text{id} \\
\mathcal{T}_{Y^\log/T_\log} & \longrightarrow & \tilde{\mathcal{F}}_{Y/T}(\mathcal{F})^\log/T_\log
\end{array}
\]

commute. Here, the left-hand vertical arrow of this diagram is, by the above discussion, an isomorphism. In particular, \( \nabla^\text{can} \) is compatible with the respective $p$-power structures on \( \mathcal{T}_{Y^\log/T_\log} \) and \( \tilde{\mathcal{F}}_{Y/T}(\mathcal{F})^\log/T_\log \). This implies, by the definition of $p$-curvature, the validity of Proposition 3.3.1. \( \square \)

3.4. Let \( S \) be a scheme over \( k \), \( \mathcal{X}/S := (f : X \rightarrow S, \{\sigma_i\}_{i=1}^r) \) a pointed stable curve of type \( (g, r) \) over \( S \), and \( u : U \rightarrow X \) an \( \acute{e} \)tale morphism. For \( i = 1, \ldots, r \), we write \( \sigma_i^{(1)} := (\sigma_i \circ F_S, \text{id}_S) : S \rightarrow X_S^{(1)} (= X \times_{S,F_S} S) \). Then, the collection of data \( (X_S^{(1)}/S, \{\sigma_i^{(1)}\}_{i=1}^r) \) forms a pointed stable curve over \( S \) and \( u \times \text{id}_S : U_S^{(1)} \rightarrow X_S^{(1)} \) is an \( \acute{e} \)tale morphism. In response to the notation "\( \mathcal{U}_S \)" for the pair \( (U/S, \{\sigma_i^{(1)}\}_{i=1}^r) \) (cf. [58]), we shall use the notation "\( \mathcal{U}_S^{(1)} \)" for indicating the collection of data \( (U_S^{(1)}/S, \{\sigma_i^{(1)U}\}_{i=1}^r) \).

If the underlying composite of morphisms of schemes \( f \circ u : U \rightarrow S \) is smooth (i.e., the image of \( u \) lies in the smooth locus of \( X \) over \( S \)), then the relative Frobenius morphism \( F_{U/S} : U \rightarrow U_S^{(1)} \) is finite and faithfully flat of degree \( p \). That is, the direct image \( F_{U/S*}(\mathcal{O}_U) \) is a vector bundle on \( U_S^{(1)} \) of rank \( p \). Unfortunately, this is not true in the generality of cases (since \( F_{U/S} : U \rightarrow U_S^{(1)} \) is not flat at a non-smooth point of \( U \)). But, as is shown in Proposition 3.4.1 below, one may verifies that the \( \mathcal{O}_{U_S^{(1)}} \)-module \( F_{U/S*}(\mathcal{O}_U) \) satisfies a certain local property even at a non-smooth point, i.e., that \( F_{U/S*}(\mathcal{O}_U) \) is relatively torsion-free. Here, by relatively torsion-free sheaf of (constant) rank \( n > 0 \) (cf. [34], Definition 1.0.1) on the semistable (in the sense of [48], §10.3, Definition 3.14) curve \( U/S \), we mean a coherent \( \mathcal{O}_U \)-module \( \mathcal{G} \) which is of finite presentation and flat over \( S \) satisfying the following property: on the fiber \( X_s := X \times_{S,S} \text{Spec}(k) \) over each point \( s : \text{Spec}(k) \rightarrow S \) the induced \( \mathcal{O}_{X_s} \)-module \( \mathcal{G}|_{X_s} \) is of rank \( n \) (cf. [29], §1.2, Definition 1.2.2) and has no associated primes of height one. (On the smooth locus of \( X \) over \( S \), any relatively torsion-free sheaf is locally free.) The following Proposition 3.4.1 will be used in the discussion of §5.
Proposition 3.4.1.
For any vector bundle $\mathcal{V}$ on $U$ of rank $n$, the $O_{U_S^{(1)}}$-module $F_{U/S*}(\mathcal{V})$ is relatively torsion-free of rank $n \cdot p$.

Proof. Since $\mathcal{V}$ is locally isomorphic to $O_U^{\oplus m}$ and $F_{U/S*}(\mathcal{V})$ is flat over $S$, we may assume, without loss of generality, that $S = \text{Spec}(k)$ for an algebraically closed field $k$ over $k$ and $\mathcal{V} = O_U$. As explained above, $F_{U/S*}(O_U)$ is locally free of rank $p$ over the smooth locus of $X_k^{(1)}$. Hence, it suffices to prove that $F_{U/S*}(O_U)$ is (relatively) torsion-free of rank $n$ at each nodal point of $U$. For a nodal point $q$ of $U$ with $p := F_{U/k}(q) \in U_k^{(1)}(k)$, we denote by $\hat{\mathcal{O}}_{U,p}$ and $\hat{m}_{U,p}$ the completions of the stalk of $O_{U}^{(1)}$ at $p$ and its maximal ideal respectively. What we have to prove is (cf. [34], the discussion at the beginning of §1.2) that the completion of the stalk of $F_{U/S*}(O_U)$ at $p$ (i.e., the completion $\hat{\mathcal{O}}_{U,q}$ of the stalk of $O_U$ at $q$, viewed as an $\hat{\mathcal{O}}_{U,p}$-module) is isomorphic to $\hat{\mathcal{O}}_{U,q}^{\oplus a} \oplus \hat{m}_{U,p}^{\oplus b}$ for some $a, b \geq 0$. Let us fix an isomorphism $\hat{\mathcal{O}}_{U,q} \cong \mathcal{O}[[x, y]]/(xy)$. By passing to the ring homomorphism $F_{U/k}^* : \hat{\mathcal{O}}_{U,p} \rightarrow \hat{\mathcal{O}}_{U,q}$ (which is verified to be injective) induced by $F_{U/k}$, we shall identify $\hat{\mathcal{O}}_{U,p}$ with the subring $\mathcal{O}[[x, y]]/(xy)$ of $\mathcal{O}[[x, y]]/(xy)$ (hence we have $\hat{m}_{U,p}^{\oplus (p-1)} \cong \mathcal{O}[[x, y]]/(x^p y^p)$). Now, we define a $\mathcal{O}[[x, y]]/(x^p y^p)$-linear morphism

\[(\mathcal{O}[[x, y]]/(x^p y^p)) \oplus (x^p \cdot \mathcal{O}[[x, y]] \oplus y^p \cdot \mathcal{O}[[y, y]])^{\oplus (p-1)} \rightarrow \mathcal{O}[[x, y]]/(xy)\]

given by assigning

\[(A, (x^p \cdot B_i + y^p \cdot C_i)) \mapsto A + \sum_{i=1}^{p-1} (x^i \cdot B_i + y^i \cdot C_i)\]

for $A \in \mathcal{O}[[x, y]]/(x^p y^p)$, and $B_i \in \mathcal{O}[[x, y]]$, $C_i \in \mathcal{O}[[y, y]]$ (i = 1, · · · , p − 1). This morphism is verified to be an isomorphism. That is, $\hat{\mathcal{O}}_{U,q}^{\oplus a}$ is isomorphic to $\hat{\mathcal{O}}_{U,p}^{\oplus (p-1)} \oplus \hat{m}_{U,p}^{\oplus (p-1)}$. This completes the proof of Proposition 3.4.2. □

The following proposition will be used in the proof of Proposition 3.8.1.

Proposition 3.4.2.
Let $\mathcal{V}'$ be a vector bundle on $U_S^{(1)}$ of rank $n$ and denote by

\[w_{\mathcal{V}'} : \mathcal{V}' \rightarrow F_{U/S*}(F_{U/S}^*(\mathcal{V}'))\]

the $O_{U_S^{(1)}}$-linear morphism corresponding, via the adjunction relation “$F_{U/S}^*(-) \dashv F_{U/S*}(-)$” (i.e., “the functor $F_{U/S}^*(-)$ is left adjoint to the functor $F_{U/S*}(-)$”),
to the identity morphism of $F_{U/S}^*(\mathcal{Y}^\nu)$. Then, $w_\mathcal{Y}$ is injective and its cokernel $\text{Coker}(w_\mathcal{Y})$ is relatively torsion-free of rank $n \cdot (p - 1)$.

Proof. As the statement is of local nature, we may assume (since $\mathcal{Y}^\nu$ is locally isomorphic to $\mathcal{O}_{U_S}^{(1)}$) that $\mathcal{Y}^\nu = \mathcal{O}_{U_S}^{(1)} =: \mathcal{O}_U$. The morphism $w_{\mathcal{O}_U}$ is functorial with respect to base-change via any $S$-scheme $\tilde{s} : \tilde{S} \to S$ in the sense that if we write $\tilde{U} := U \times_S \tilde{S}$, then the composite

\[
\begin{align*}
\mathcal{O}_{\tilde{U}}^{(1)} \\
\cong (\text{id}_U \times \tilde{s})^* (\mathcal{O}_{\tilde{U}}^{(1)}) \\
\cong (\text{id}_U \times \tilde{s})^* (F_{U/S}(F_{U/S}^*(\mathcal{O}_U))) \\
\cong F_{\tilde{U}/S}(F_{\tilde{U}/S}^*(\mathcal{O}_{\tilde{U}}))
\end{align*}
\]

coincides with $w_{\mathcal{O}_{\tilde{U}}}$, where the third isomorphism follows from the natural isomorphism \[185\]. The open subscheme $U^\text{sm}$ of $U$ is scheme-theoretically dense and $w_{\mathcal{O}_U}$ is injective over $U^\text{sm}$. It follows that $w_{\mathcal{O}_U}$ is injective (over $U$), which concludes the half part of the assertion.

Moreover, by applying a similar argument to any $\tilde{U}$ as above (by means of the functoriality of $w_{\mathcal{O}_U}$), one concludes that $w_{\mathcal{O}_U}$ is universally injective with respect to base-change over $S$. By \[49\], p. 17, Theorem 1, the cokernel $\text{Coker}(w_{\mathcal{O}_U})$ turns out to be flat over $S$. Thus, it suffices to prove the assertion of the case where $S = \text{Spec}(\overline{k})$ for an algebraically closed field $\overline{k}$ over $k$. We shall only prove the torsion-freeness of $\text{Coker}(w_{\mathcal{O}_U})$ at a nodal point $p$ (since the case of smooth points is technically much simpler). By the discussion in the proof of Proposition 3.4.1, the completion of $w_{\mathcal{O}_U} : \mathcal{O}_{U_{\overline{k}}}^{(1)} \to F_{U/\overline{k}}(F_{U/\overline{k}}^*(\mathcal{O}_{U_{\overline{k}}}^{(1)}))$ at $q$ may be identified with the inclusion $\hat{\mathcal{O}}_{U_{\overline{k}}}^{(1), \hat{q}} \hookrightarrow \hat{\mathcal{O}}_{U_{\overline{k}}}^{(1), \hat{q}} \oplus \hat{m}_{U_{\overline{k}}}^{(p-1)}$ into the first factor. Hence, the completion of $\text{Coker}(w_{\mathcal{O}_U})$ at $p$ is isomorphic to $\hat{m}_{U_{\overline{k}}}^{(p-1)}$, which is torsion-free of rank $p - 1$. This completes the proof of Proposition 3.4.2. \[\square\]

3.5. In the following part of this section, suppose further that $k$ is perfect (i.e., $F_k : \text{Spec}(k) \to \text{Spec}(k)$ is an isomorphism), $G$ may be split over $F_p$, and moreover, the pinning $G_p$ fixed in § 2.1 \[70\], is taken over $F_p$. In particular, the $k$-scheme $c$ (cf. \[150\]) arises from an $F_p$-scheme $c_{F_p}$ (i.e., $c_{F_p} \times_{F_p} k \cong c$), and hence, is isomorphic to $c_k^{(1)}$ via the isomorphism $\text{id}_k \times F_k : c_k^{(1)} \cong c$. We denote, by abuse of notation, the set of $F_p$-rational points of $c_{F_p}$ by $c(F_p)$ (instead of $c_{F_p}(F_p)$).
Proposition 3.5.1.
The square diagram

\[
\begin{array}{ccc}
[g/G] & \xrightarrow{[x]} & c \\
\downarrow (-)[p] & & \downarrow F_{t/k} \\
[g/G] & \xrightarrow{[x]} & c
\end{array}
\]

(205)

is commutative.

Proof. Observe (cf. [66], 4.4.10. Example (2)) that the $p$-power operation on $t$ coincides with the relative Frobenius morphism $F_{t/k} : t \rightarrow t^{(1)}_k (= t)$ and is compatible with the adjoint action of $\mathbb{W}$. Hence, the endomorphism of $c$ induced by the $p$-power operation on $t$ coincides with $F_{t/k}$. But, it also coincides with the endomorphism induced, via the quotient $\chi : g \rightarrow c$, by the $p$-power operation on $g$. Thus, the relative Frobenius morphism $F_{t/k}$ makes the diagram

\[
\begin{array}{ccc}
g & \xrightarrow{} & [g/G] \xrightarrow{[x]} c \\
\downarrow (-)[p] & & \downarrow (-)[p] \downarrow F_{t/k} \\
g & \xrightarrow{} & [g/G] \xrightarrow{[x]} c
\end{array}
\]

(206)

commute, as desired. \qed

Let $(\mathcal{E}, \nabla_\varepsilon)$ be an $h$-log integrable $G$-torsor over $U^{log}/S^{log}$, and suppose that $r > 0$. By passing to the isomorphism

\[
(\text{triv}_{s_i,U})^\psi : (F_U^*(\Omega_{U^{log}/S^{log}}) \cong) \sigma_i^{U*}(\Omega_{U^{log}/S^{log}}) \cong \mathcal{O}_{U \times X,s_i,S}
\]

(207) $(i = 1, \ldots, r)$ arising from the trivialization $\text{triv}_{s_i,U}$ (cf. (G1)), we consider $\sigma_i^{U*}(p_{\psi}(\mathcal{E},\nabla_\varepsilon))$ as an element in $\Gamma(U, \mathcal{G}_{s_i}(\mathcal{E}))$. Under this consideration, one verifies the equality

\[
\sigma_i^{U*}(p_{\psi}(\mathcal{E},\nabla_\varepsilon)) = (\mu_i^{(\mathcal{E},\nabla_\varepsilon)})[p] - h^{p-1} \cdot \mu_i^{(\mathcal{E},\nabla_\varepsilon)}
\]

(208) (cf. Definition 1.6.1 for the definition of $\mu_i^{(\mathcal{E},\nabla_\varepsilon)}$) of elements $\in \Gamma(U, \mathcal{G}_{s_i}(\mathcal{E}))$.

Proposition 3.5.2.
Let $(\mathcal{E}, \nabla_\varepsilon)$ be as above, and suppose further that $p_{\psi}(\mathcal{E},\nabla_\varepsilon) = 0$. Then, we have the equality

\[
F_{t/k} \circ \rho_i^{(\mathcal{E},\nabla_\varepsilon)} = h^{p-1} \sigma_i^{U*}(\mathcal{E},\nabla_\varepsilon) = \rho_i^{(\mathcal{E},h^{p-1}\nabla_\varepsilon)}
\]

(209) (cf. (I58); (I64)). In particular, the following assertions hold:

(i) If, moreover, $h \in \Gamma(S, \mathcal{O}_S^\times)$, then $h^{-1} \simeq \rho_i^{(\mathcal{E},\nabla_\varepsilon)} \in c(\mathbb{F}_p)$;
(ii) If, moreover, $h = 0$ and $S$ is reduced, then $\rho_i^{(\mathcal{E},\nabla_\varepsilon)} = 0 \in c(\mathbb{F}_p)$. 

Proof. First, we prove the asserted equality $\mathbf{210}$. Under the condition that $p_\psi(\mathfrak{c}, \nabla_\mathfrak{c}) = 0$, the equality $\mathbf{208}$ is equivalent to
\begin{equation}
(\mu_1^{(\mathfrak{c}, \nabla_\mathfrak{c})})^p = \hbar^{p-1} \cdot \mu_i^{(\mathfrak{c}, \nabla_\mathfrak{c})}.
\end{equation}
Hence, by Proposition 3.5.1, we have a sequence of equalities
\begin{equation}
F_{i/k} \circ \rho_i^{(\mathfrak{c}, \nabla_\mathfrak{c})} = F_{i/k} \circ [\chi]( (\sigma_i^*(\mathfrak{c}), \mu_i^{(\mathfrak{c}, \nabla_\mathfrak{c})}) )
= [\chi]( (\sigma_i^*(\mathfrak{c}), \mu_i^{(\mathfrak{c}, \nabla_\mathfrak{c})})^p )
= [\chi]( (\sigma_i^*(\mathfrak{c}), (\mu_i^{(\mathfrak{c}, \nabla_\mathfrak{c})})^p ) )
= [\chi]( (\sigma_i^*(\mathfrak{c}), \hbar^{p-1} \cdot \mu_i^{(\mathfrak{c}, \nabla_\mathfrak{c})}) )
= \hbar^{p-1} \cdot \rho_i^{(\mathfrak{c}, \nabla_\mathfrak{c})},
\end{equation}
which gives the asserted equality.

Next, we consider assertions (i) and (ii). Let us fix an isomorphism $\mathfrak{c} \sim \text{Spec}(k[u_1, \ldots, u_{k+1}])$ as in $\mathbf{156}$. By passing to this isomorphism, one may express $\rho_i^{(\mathfrak{c}, \nabla)}$ as
\begin{equation}
\rho_i^{(\mathfrak{c}, \nabla)} := (a_1, \ldots, a_{k+1}) \in A^{\text{rk}(\mathfrak{c})}(U \times X_{\mathfrak{c}}, S)
\end{equation}
for some $a_j \in \Gamma(U \times X_{\mathfrak{c}}, S, \mathcal{O}_{U \times X_{\mathfrak{c}}, S})$ $(j = 1, \ldots, \text{rk}(\mathfrak{c}))$. By this expression and the definition of $\rho$, the equality $\mathbf{209}$ may be described that $a^p = h^{(p-1)e_j} \cdot a_j$ for all $j$.

If $\hbar \in \Gamma(S, \mathcal{O}_S^\times)$ (i.e., the assumption in (i) is satisfied), then
\begin{equation}
a_j = h^{e_j} \cdot a_j \iff (\hbar^{e_j} \cdot a_j)^p = \hbar^{e_j} \cdot a_j \iff \hbar^{e_j} \cdot a_j \in \mathbb{F}_p,
\end{equation}
equivalently, $\rho_i^{(\mathfrak{c}, \nabla)} \in \mathcal{C}(\mathbb{F}_p)$.

On the other hand, if $\hbar = 0$ and $S$ is reduced (i.e., the assumption in (ii) is satisfied), then
\begin{equation}
a_j = h^{e_j} \cdot a_j (= 0) \iff a_j = 0.
\end{equation}
This completes the proof of Proposition 3.5.2. \hfill $\Box$

3.6. Now we proceed with our discussion concerning the moduli stacks $\mathcal{D}p_{g,h,X/S}$ and $\mathcal{D}p_{g,h,\mathfrak{c},X/S}$, for the case of positive characteristic. Denote by
\begin{equation}
\kappa_{g,h,\mathfrak{c},X/S} : \mathcal{O}_{g,h,\mathfrak{c},X/S} \rightarrow \mathcal{O}_{g,h,\mathfrak{c},X/S} \otimes \mathfrak{g}_{g,h,\mathfrak{c},X/S}^\vee
\end{equation}
the morphism of stacks over $Et/U$ which, to any local section of $\mathcal{O}_{g,h,\mathfrak{c},X/S}$ corresponding to a $(\mathfrak{g}, \hbar)$-oper $\mathfrak{E}^\bullet := (\mathfrak{E}_\mathfrak{g}, \nabla_\mathfrak{g})$, assigns (via $\mathcal{O}_{g,h,\mathfrak{c},X/S} \otimes \mathfrak{g}_{g,h,\mathfrak{c},X/S}^\vee \cong F_U^{\bullet}(\mathcal{O}_{g,h,\mathfrak{c},X/S})$) its $p$-curvature $p_\psi(\mathfrak{E}_\mathfrak{g}, \nabla_\mathfrak{g})$. Consider the relative affine space
\begin{equation}
\mathfrak{g}_{g,h,X/S} := \mathcal{V}(f_*(\mathcal{O}_{g,h,\mathfrak{c},X/S} \otimes \mathfrak{g}_{g,h,\mathfrak{c},X/S}^\vee)),
\end{equation}
(cf. [60] for the notation “\(\bigotimes\)” that means a *spindle*) over \(S\), which has the zero section
\[
\varepsilon : S \to \bigotimes_{g,h,x/S}
\]
(cf. [60] for the notation “\(\varepsilon\)” that means a *princess's finger*). By considering the morphism (215) of the case where \(U\) is taken to be \(X\), we obtain a morphism
\[
\kappa_{g,h,x/S} : \mathcal{O}_{g,h,x/S} \to \bigotimes_{g,h,x/S}
\]
(cf. [60] for the notation “\(\kappa\)” that means an *old fairy*) over \(S\). Also, for each \(\rho \in \mathfrak{c} \times \mathfrak{r}(S)\) (where we take \(\rho = \emptyset\) if \(r = 0\)), this morphism gives its restriction
\[
\kappa_{g,h,\rho,x/S} := \kappa_{g,h,x/S}|_{\mathcal{O}_{g,h,\rho,x/S}} : \mathcal{O}_{g,h,\rho,x/S} \to \bigotimes_{g,h,x/S}.
\]
Thus, we obtain uniquely a closed substack
\[
\mathcal{O}_{g,h,\rho,x/S} \ (\text{resp.} \ \mathcal{O}_{g,h,\rho,x/S})
\]
(cf. [60] for the notation “Zzz...” !) of \(\mathcal{O}_{g,h,x/S}\) (resp., \(\mathcal{O}_{g,h,\rho,x/S}\)) for which the square diagram
\[
\begin{array}{ccc}
\mathcal{O}_{g,h,x/S} & \to & \mathcal{O}_{g,h,\rho,x/S} \\
\downarrow & & \downarrow \\
\bigotimes_{g,h,x/S} & \to & \bigotimes_{g,h,\rho,x/S}
\end{array}
\]
is cartesian. If \(r = 0\), then we mean \(\mathcal{O}_{g,h,\emptyset,x/S}\) for the stack \(\mathcal{O}_{g,h,\rho,x/S}\).

**Definition 3.6.1.**
We shall say that a \((g, h)\)-oper \(\mathcal{E} = (\mathcal{E}_\beta, \nabla_\epsilon)\) on \(X/S\) is *dormant* if \(p_{\psi}^{(\mathcal{E}_\beta, \nabla_\epsilon)} = 0\), equivalently, if it is classified by the closed substack \(\mathcal{O}_{g,h,\rho,x/S}\).

**Proposition 3.6.2.**
The closed immersion \(i_{\mathcal{O}_{g,h,x/S}} : \mathcal{O}_{g,h,x/S} \to \mathcal{O}_{g,h,x/S} \ (\text{cf. } (130); \text{Proposition 2.7.4})\) induces, by restricting, a closed immersion
\[
i_{\mathcal{O}_{g,h,x/S}}^{Zzz...} : \mathcal{O}_{g,h,\rho,x/S} \to \mathcal{O}_{g,h,\rho,x/S}
\]
\[
i_{\mathcal{O}_{g,h,x/S}}^{Zzz...} : \mathcal{O}_{g,h,x/S} \to \mathcal{O}_{g,h,x/S}
\]

**Proof.** It follows from Proposition 3.2.2 that if \(\mathcal{E}_\bigotimes\) is a dormant \((g^\bigotimes, h)\)-oper on \(X/S\), then the associated \((g, h)\)-oper \(i_{G^\bigotimes}(\mathcal{E}_\bigotimes)\) (cf. (129)) is dormant. This implies that the composite
\[
\mathcal{O}_{g^\bigotimes,h,x/S} \to \mathcal{O}_{g^\bigotimes,h,x/S} \to \mathcal{O}_{g,h,x/S}
\]
factors through the closed immersion \( \mathcal{O}_{\mathbb{P}^{\text{Zar}}_{\mathfrak{g},h,x/S}} \to \mathcal{O}_{\mathbb{P}^{\text{Zar}}_{\mathfrak{g},h,x/S}} \). The resulting morphism \( t_{\mathbb{P}^{\text{Zar}}_{\mathfrak{g},h,x/S}} : \mathcal{O}_{\mathbb{P}^{\text{Zar}}_{\mathfrak{g},h,x/S}} \to \mathcal{O}_{\mathbb{P}^{\text{Zar}}_{\mathfrak{g},h,x/S}} \) is easily verified to be a closed immersion. \( \square \)

Define an open subscheme \( \mathfrak{o}_c \) of \( c \) to be
\[
(224) \quad \mathfrak{o}_c := c \setminus \text{Im}([0]_k)
\]
(cf. §2.9 for the definition of the closed immersion \([0]_k : \text{Spec}(k) \to c\)). We shall observe the following property concerning the radii of dormant opers:

**Proposition 3.6.3.**

Suppose that \( r > 0 \) and \( h \in \Gamma(S, \mathcal{O}_S^\times) \), and let \( \mathcal{E}^\bullet := (\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E}) \) be a dormant \((\mathfrak{g}, h)\)-oper on \( \mathfrak{X}/S \). Then, the radii \( \rho := (\rho_i^{(\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E})})_{i=1}^r \in \mathfrak{c}^{x\mathfrak{r}}(S) \) lies in the subset \( \mathfrak{o}_c^{x\mathfrak{r}}(S) \) of \( \mathfrak{c}^{x\mathfrak{r}}(S) \). In particular, if \( h \in k^{x\mathfrak{r}} \), then \( h^{-1} \star \rho \in \mathfrak{o}_c^{x\mathfrak{r}}(\mathbb{F}_p) \).

**Proof.** The latter assertion follows from the former assertion and Proposition 3.5.2 (i). We shall consider the former assertion. It suffices to consider the assertion of the case where \( \mathcal{E}^\bullet \) is of canonical type. In particular, the monodromy \( \mu_i^{(\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E})} \) of \( (\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E}) \) at \( \sigma_i \) \((i = 1, \ldots, r)\) may be expressed as \( \mu_i^{(\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E})} := 1 \otimes p - 1 + R_i \), where \( R_i \in \Gamma(S, \sigma_i^*(\mathcal{V}_{\mathfrak{g},h,x/S})) \). Now, we suppose that \( \rho_i^{(\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E})} = [0]_S \) for some \( i \). This implies (since \( \mathfrak{Ros}_{\mathfrak{g}} \) (cf. (152)) is an isomorphism) that \( R_i = 0 \), and hence \( (\mu_i^{(\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E})})_p = 1 \otimes p^{-1} = 0 \). But, by the equality
\[
(208) \quad \sigma_i^*(p_\psi^{(\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E})}) = (\mu_i^{(\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E})})_p - h^{p-1} \cdot \mu_i^{(\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E})} = -h^{p-1} \cdot (1 \otimes p^{-1}) \neq 0.
\]
This contradicts the assumption that \( \mathcal{E}^\bullet \) is dormant. Thus, we see that \( \rho_i^{(\mathcal{E}_\mathcal{B}, \nabla_\mathcal{E})} \neq [0]_S \) for any \( i \), which completes the proof of the former assertion of Proposition 3.6.3. \( \square \)

3.7. For a line bundle \( \mathcal{L} \) on \( X \), we shall write \( \mathcal{L}^\times \) for the \( \mathbb{G}_m \)-torsor associated with \( \mathcal{L} \). We shall denote by
\[
(226) \quad \mathcal{X}^{\nabla}_{t,\mathfrak{X}/S} \quad (\text{resp., } \mathcal{X}^{\nabla}_{t,\mathfrak{X}/S})
\]
the \( S \)-scheme (cf. [18, Theorem 5.23]) representing the \( \mathbf{Set} \)-valued contravariant functor on \( \mathbf{Sh}_{/S} \) (cf. §2.3) which, to any \( S \)-scheme \( t : T \to S \), assigns the set of morphisms \( X \times_T S \to \Omega^{\times}_{X_{\log}/S_{\log}} \times \mathbb{G}_m \mathfrak{c} \) (resp., \( X \times_T S \to (\Omega^{\otimes p}_{X_{\log}/S_{\log}})^{\times} \times \mathbb{G}_m \mathfrak{c} \)) over \( X \). If \( r > 0 \), then one may define a morphism of \( k \)-schemes
\[
(227) \quad \mathfrak{Rad}^{\otimes}_{t,\mathfrak{X}/S} : \mathcal{X}^{\nabla}_{t,\mathfrak{X}/S} \to \mathfrak{c}^{x\mathfrak{r}}
\]
to be the morphism which, to each $T$-rational point of \( \bigotimes_{c}^\nabla_{c,x_S} \) classifying a morphism $w : X \times_S T \to \Omega^{x}_{X^{\log/S^{\log}}} \times_{G^{m}} c$, assigns the collection of data $(w|_{\sigma_i})_{i=1}^{r} \in c^{\times r}(T)$ consisting of the composites

$$w|_{\sigma_i} : T \xrightarrow{\sim} S \times_{\alpha_i} X \times_{T} (\Omega^{x}_{X^{\log/S^{\log}}} \times_{G^{m}} c) \xrightarrow{\sim} S \times_{k} c$$

($i = 1, \ldots, r$), where the last isomorphism arises from the trivialization (67). For each $\rho \in c^{\times r}(S)$, we shall write

$$\bigotimes_{c,\rho,x_S}$$

for the closed subscheme of $\bigotimes_{c,x_S}^\nabla$ defined to be the scheme-theoretic inverse image of $\rho$ via $\mathcal{H}_{c,x_S}$. One verifies that $\bigotimes_{c,x_S}^\nabla$ and $\bigotimes_{c,\rho,x_S}^\nabla$ may be represented by a relative affine space over $S$ of relative dimension $N(\mathfrak{g})$ and $cN(\mathfrak{g})$ (cf. (138)) respectively. Indeed, recall (cf. a discussion in the proof of Proposition 2.6.1) that the morphism $\mathcal{H}_{c,x_S} : \mathfrak{g}^{Ad(P)} \to c$ (cf. (132)) induces an isomorphism

$$\mathbb{V}(f_{c}(\mathcal{V}_{\mathfrak{g}, h,x_S})) \xrightarrow{\sim} \bigotimes_{c}^\nabla_{c,x_S}$$

of $S$-schemes. Moreover, upon passing to this isomorphism, $\bigotimes_{c,\rho,x_S}^\nabla$ may be identified with an affine subspace of $\mathbb{V}(f_{c}(\mathcal{V}_{\mathfrak{g}, h,x_S}))$ modeled on $\mathbb{V}(f_{c}(c\mathcal{V}_{\mathfrak{g}, h,x_S}))$.

Next, write

$$\bigotimes_{c,h,x_S} : \bigotimes_{c,h,x_S} \to \bigotimes_{c,x_S}$$

for the morphism arising naturally from the morphism

$$[((\Omega^{\mathfrak{g}^{p}}_{X^{\log/S^{\log}}} \times_{G^{m}} \mathfrak{g}) / \mathbb{G} \to (\Omega^{\mathfrak{g}^{p}}_{X^{\log/S^{\log}}} \times_{G^{m}} \mathfrak{g})$$

(which is induced by $[\chi] : [\mathfrak{g}/\mathbb{G}] \to c$). If $t : T \to S$ is an $S$-scheme, then there exist natural isomorphisms

$$\bigotimes_{g,t^{*}(h),x_T} \xrightarrow{\sim} \bigotimes_{g,h,x_S} \times_{S,t} T$$

and

$$\bigotimes_{c,t^{*}(h),x_T} \xrightarrow{\sim} \bigotimes_{c,x_S} \times_{S,t} T$$

of $T$-schemes which make the square diagram

$$\begin{array}{ccc}
\bigotimes_{g,t^{*}(h),x_T} & \xrightarrow{\sim} & \bigotimes_{g,h,x_S} \times_{S,t} T \\
\bigotimes_{c,t^{*}(h),x_T} & \xrightarrow{\sim} & \bigotimes_{c,x_S} \times_{S,t} T \\
\bigotimes_{c,x_T} & \xrightarrow{\sim} & \bigotimes_{c,x_S} \times_{S,t} T \\
\xrightarrow{\sim} & \bigotimes_{c,x_T} & \xrightarrow{\sim} \bigotimes_{c,x_S} \times_{S,t} T
\end{array}$$

commute.

Now, we consider relations between the $S$-schemes just defined and those of the case where $\mathfrak{X}_{S}$ is replaced with the Frobenius twist $\mathfrak{X}_{S}^{(1)} = (X^{(1)}_{S} / S, \{\sigma_{i}^{(1)}\}_{i=1}^{r})$ of $\mathfrak{X}_{S}$ (cf. §3.4). Let $\rho := (\rho_{i})_{i=1}^{r} \in c^{\times r}(S)$, and write

$$F_{c/k} \circ \rho := (F_{c/k} \circ \rho_{i})_{i=1}^{r} \in c^{\times r}(S).$$
It follows from the definitions of \( \mathcal{O}_{\mathcal{X}/S} \) (resp., \( \mathcal{O}_{\mathcal{X}/S} \)) (cf. \((233)\)) that there exists a natural isomorphism

\[
(\mathcal{O}_{\mathcal{X}/S}) \sim (\mathcal{O}_{\mathcal{X}/S} \times S_{\mathcal{X}/S}) \quad \text{of } S\text{-schemes. The natural isomorphism}
\]

\[
(\mathcal{O}_{\mathcal{X}/S} \times S_{\mathcal{X}/S}) \rightarrow \mathcal{X}/S \quad \text{gives a morphism}
\]

\[
(\mathcal{O}_{\mathcal{X}/S} \times S_{\mathcal{X}/S}) \rightarrow \mathcal{X}/S \quad \text{of } S\text{-schemes}.
\]

**Proposition 3.7.1.**

The morphism \( \mathfrak{S}_{\mathcal{X}/S} \) (resp., \( \mathfrak{S}_{\mathcal{X}/S} \)) is a closed immersion.

**Proof.** Since \( \mathfrak{S}_{\mathcal{X}/S} \) may be obtained as the composite of \( \mathfrak{S}_{\mathcal{X}/S} \) and the closed immersion \( \mathfrak{S}_{\mathcal{X}/S} \rightarrow \mathfrak{S}_{\mathcal{X}/S} \), it suffices to prove the non-resp’d portion.

Let us fix an isomorphism \( \iota_{\mathcal{X}} : \mathfrak{S} \rightarrow \text{Spec}(k[u_1, \ldots, u_{\text{rk}(g)}) \) as in \((156)\). If \( \mathcal{L} \) is a line bundle on either \( \mathcal{X} \) or \( \mathcal{X}^{(1)} \), then this isomorphism \( \iota_{\mathcal{X}} \) allows us to identify \( \mathcal{L} \times \mathfrak{S} \) with the relative affine space associated with \( \bigoplus_{l=1}^{\text{rk}(g)} \mathcal{L} \otimes e_l \) (i.e., \( \mathcal{L} \times \mathfrak{S} \rightarrow \mathcal{V}(\bigoplus_{l=1}^{\text{rk}(g)} \mathcal{L} \otimes e_l) \)). Hence, there exist isomorphisms of \( S\text{-schemes} 
\]

\[
(39) \quad \mathfrak{S}_{\mathcal{X}/S} \sim \mathfrak{V}(f_*(\bigoplus_{l=1}^{\text{rk}(g)} \mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}}))
\]

and

\[
(40) \quad \mathfrak{S}_{\mathcal{X}/S} \sim \mathfrak{V}(f_*(\bigoplus_{l=1}^{\text{rk}(g)} \mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}})) = \mathfrak{V}(f_*(\bigoplus_{l=1}^{\text{rk}(g)} \mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}})),
\]

where we recall that \( f_1 : \mathcal{X}^{(1)} \rightarrow S \) is, by definition, the structure morphism of the curve \( \mathcal{X}^{(1)} / S \) (cf. \( \S 3.1 \)). For \( l = 1, \ldots, \text{rk}(g), \) we shall write

\[
(41) \quad \omega_l : \mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}} \rightarrow F_{\mathcal{X}/S}(F_{\mathcal{X}/S}(\mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}}))(\cong F_{\mathcal{X}/S}(\mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}})),
\]

for the morphism of \( \mathcal{O}_{\mathcal{X}^{(1)}_{S}} \)-modules corresponding, via the adjunction relation \( "F_{\mathcal{X}/S}(\mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}}) \rightarrow F_{\mathcal{X}/S}(\mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}})" \) (i.e., “the functor \( F_{\mathcal{X}/S}(\mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}}) \) is left adjoint to the functor \( F_{\mathcal{X}/S}(\mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}})"), to the identity morphism of \( F_{\mathcal{X}/S}(\mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}}) \) (\( \cong \mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}} \)). Note that the \( \mathcal{O}_{\mathcal{X}^{(1)}_{S}} \)-module \( F_{\mathcal{X}/S}(\mathcal{O}_{\mathcal{X}^{(1)}_{S},\mathcal{X}^{(1)}_{S},\log,S_{\mathcal{X}/S}}) \) is relatively torsion free of rank \( p \) (cf. Proposition 3.4.1) and \( \omega_l \) is easily verified to be injective over the scheme-theoretically dense open subscheme \( \mathcal{X}^{(1)}_{\text{sm}} \) of \( \mathcal{X}^{(1)} \) (cf. \( \S 3.1.6 \)). Hence, the
morphism $\omega_l$, as well as its direct image $f^{(1)}(\omega_l)$, is injective. On the other hand, since $e_l \geq 2$, we have that $\mathbb{R}^1 f^{(1)}(\Omega^{e_l}_{X_S^{(1)log}/Slog}) = 0$ and

$$(242) \quad \mathbb{R}^1 f^{(1)}_*(F_{X/S*}(\Omega^{p\cdot e_l}_{X^{log}/Slog})) = \mathbb{R}^1 f_*(\Omega^{p\cdot e_l}_{X^{log}/Slog}) = 0.$$ 

By [25], Chap. III, Theorem 12.11 (b), both the domain and codomain of $f^{(1)}_*(\omega_l)$ are locally free, and hence, the construction of the injection $f^{(1)}_*(\omega_l)$ is, in a natural sense, functorial with respect to any base-change over $S$. In particular, by applying a similar argument to the pull-back of $f^{(1)}_*(\omega_l)$ via any base-change over $S$, one concludes that $f^{(1)}_*(\omega_l)$ is universally injective with respect to base-change over $S$. Hence, $\text{Coker}((f^{(1)}_*(\omega_l))$ is flat over $S$ (cf. [10], p. 17, Theorem 1). It follows that the dual of $f^{(1)}_*(\omega_l)$

$$(243) \quad f^{(1)}(\omega_l)^\vee : f^{(1)}_*(F_{X/S*}(\Omega^{p\cdot e_l}_{X^{log}/Slog}))^\vee \rightarrow f^{(1)}_*(\Omega^{p\cdot e_l}_{X_S^{(1)log}/Slog})^\vee$$

is surjective for all $l$.

Since the morphism $\mathcal{O}_{\mathcal{H},X/S}$ may be obtained by (taking the spectrum of) the surjection

$$(244) \quad \mathcal{S}_\mathcal{O}_S(\bigoplus_{l=1}^{\text{rk}(g)} f^{(1)}_*(F_{X/S*}(\Omega^{p\cdot e_l}_{X^{log}/Slog})))^\vee \rightarrow \mathcal{S}_\mathcal{O}_S(\bigoplus_{l=1}^{\text{rk}(g)} f^{(1)}_*(\Omega^{p\cdot e_l}_{X_S^{(1)log}/Slog})))^\vee$$

of $\mathcal{O}_S$-algebras induced by $\bigoplus_{l=1}^{\text{rk}(g)} f^{(1)}_*(\omega_l)$, we conclude that $\mathcal{O}_{\mathcal{H},X/S}$ is a closed immersion.

3.8. We shall write

$$(245) \quad \mathcal{X}^{H-M}_{g,h,x/S} : \mathcal{O}_\mathcal{H}_{g,h,x/S} \rightarrow \mathcal{O}_{\mathcal{H},X/S} \quad (\text{resp., } \mathcal{X}^{H-M}_{g,h,\rho,x/S} : \mathcal{O}_\mathcal{H}_{g,h,\rho,x/S} \rightarrow \mathcal{O}_{\mathcal{H},X/S})$$

for the composite $\mathcal{O}_\mathcal{H}_{g,h,x/S} \circ \mathcal{X}_{g,h,x/S}$ (resp., $\mathcal{O}_\mathcal{H}_{g,h,\rho,x/S} \circ \mathcal{X}_{g,h,\rho,x/S}$).

Here, if $Y$ is a $k$-scheme and $\mathfrak{h} \in \Gamma(Y, \mathcal{O}_Y)$, $\lambda \in \mathfrak{c}(Y)$, then we shall write

$$(246) \quad \lambda^F_{\mathfrak{h}} := \chi \circ ((\mathfrak{g}s_{\mathfrak{g}}^{-1} \circ \lambda)^{[p]} - \mathfrak{h}^{p-1} \cdot (\mathfrak{g}s_{\mathfrak{g}}^{-1} \circ \lambda)) \in \mathfrak{c}(Y),$$

where we regard $\mathfrak{g}s_{\mathfrak{g}}^{-1} \circ \lambda$ as an element of $\mathfrak{g}(Y)$. If $\mathfrak{h} = 0$, then one verifies (cf. Proposition 3.5.1) the equality

$$(247) \quad \lambda^F_{\mathfrak{h}} = F_{\mathfrak{c}/\mathfrak{h}} \circ \lambda$$

(cf. [23]). Also, for each $\mathfrak{r} \in \mathfrak{c}^{x^r}(S)$, we shall write

$$(248) \quad \rho^F_{\mathfrak{h}} := ((\rho_i^F_{\mathfrak{h}})^{r_i}_{i=1} \in \mathfrak{c}^{x^r}(S).$$
Proposition 3.8.1.
For a $k$-scheme $Y$, we shall consider the morphism

$$(249) \quad \Psi_{c,h,Y} : c \times_k Y \to c \times_k Y$$

of $Y$-schemes determined uniquely by the condition that

$$(250) \quad \Psi_{c,h,Y} \circ (\lambda, \tilde{y}) = (\lambda^p, \tilde{y}) : \tilde{Y} \to c \times_k Y$$

for any morphism $\tilde{y} : \tilde{Y} \to Y$ and $\lambda \in c(\tilde{Y})$.

(i) Let us denote by $R_{c,x_kY}$ the $\mathcal{O}_Y$-algebra corresponding to the affine $Y$-scheme $c \times_k Y$ (cf. §3.11). Consider the $\mathbb{G}_m$-action on $R_{c,x_kY}$ opposite to that induced by the $\mathbb{G}_m$-action on $c$ (cf. (153)) and the increasing filtration $\{R_{c,x_kY}^j\}_{j \in \mathbb{Z}}$ associated with this grading, which is, by construction, positive (i.e., $R_{c,x_kY}^j = 0$ if $j < 0$). Then, the $\mathcal{O}_Y$-algebra endomorphism $\Psi^*_{c,h,Y}$ of $R_{c,x_kY}$ corresponding to $\Psi_{c,h,Y}$ is filtered of degree $p$, i.e., $\Psi^*_{c,h,Y}(R_{c,x_kY}^j) \subset R_{c,x_kY}^{j+p}$.

(ii) If $h = 0$, then the morphism $\Psi_{c,0,Y}$ coincides with the relative Frobenius morphism $F_{c,Y}/Y$ of $c \times_k Y (= (c \times_k Y)^{(1)})$ over $Y$.

(iii) If $\text{gr}(R_{c,x_kY})$ denotes the graded $\mathcal{O}_Y$-algebra associated with $R_{c,x_kY}$, then the graded $\mathcal{O}_Y$-algebra endomorphism $\text{gr}(\Psi_{c,h,Y}^*)$ of $\text{gr}(R_{c,x_kY})$ associated with $\Psi_{c,h,Y}^*$ coincides with $\Psi_{c,0,Y}^*$.

(iv) $\Psi_{c,h,S}$ is finite and faithfully flat of degree $p^{rk(g)}$.

Proof. Consider the morphism $\Psi_{c,z,A^1}$ (where $A^1 := \text{Spec}(k[z])$), i.e., the morphism (249) of the case where the pair $(Y, h)$ is taken to be $(A^1, z)$. Note that the original $\Psi_{c,h,Y}$ may be obtained as the fiber of $\Psi_{c,z,A^1}$ over the $S$-rational point $[h] : Y \to A^1$ corresponding to $h \in \Gamma(Y, \mathcal{O}_Y)$. Hence, it suffices to prove the assertion for the case of $\Psi_{c,z,A^1}$.

Let us equip $k[z] \otimes_k S_k(\mathfrak{g})^G$ with the grading induced from the $\mathbb{G}_m$-action $\ast$ on $c \cong \text{Spec}(S_k(\mathfrak{g})^G)$ (cf. (155)). Then the ring homomorphism $k[z] \otimes_k S_k(\mathfrak{g})^G \to k[z] \otimes_k S_k(\mathfrak{g})^G$ corresponding to $\chi \times \text{id}_{A^1}$ preserves the gradings. On the other hand, recall that $\chi \times \text{id}_1 : g \times_k Y \to c \times_k Y$ corresponds to the inclusion $k[z] \otimes_k S_k(\mathfrak{g})^G \hookrightarrow k[z] \otimes_k S_k(\mathfrak{g})^G$. Hence, one verifies from the definition of $\Psi_{c,z,A^1}$ that the corresponding endomorphism $\Psi_{c,z,A^1}^*$ of $k[z] \otimes_k S_k(\mathfrak{g})^G$ is filtered of degree $p$. The proof of assertion (i) is completed.

Assertion (ii) follows from (247), and assertion (iii) follows from the discussion in the proof of assertion (i).

Finally, consider assertion (iv). Let us write $\text{gr}(k[z] \otimes_k S_k(\mathfrak{g})^G)$ for the graded $k[z]$-algebra associated with $k[z] \otimes_k S_k(\mathfrak{g})^G$ and $\text{gr}(\Psi_{c,z,A^1}^*)$ for the graded endomorphism of $\text{gr}(k[z] \otimes_k S_k(\mathfrak{g})^G)$ associated with the filtered morphism $\Psi_{c,z,A^1}^*$. By assertion (ii) and (iii), $\text{gr}(k[z] \otimes_k S_k(\mathfrak{g})^G)$ turns out to be a finite $\text{gr}(\Psi_{c,z,A^1}^*)/(\text{gr}(k[z] \otimes_k S_k(\mathfrak{g})^G))$-module. It follows from a routine argument that $k[z] \otimes_k S_k(\mathfrak{g})^G$ is a finite $\Psi_{c,z,A^1}^*(k[z] \otimes_k S_k(\mathfrak{g})^G)$-module, i.e., $\Psi_{c,z,A^1}$
is finite. Since the domain and codomain of \( \Psi_{c,z,\mathbb{A}^1} \) are irreducible, smooth, and of the same dimension, one verifies immediately that \( \Psi_{c,z,\mathbb{A}^1} \) is also faithfully flat. The computation of its degree may be accomplished by computing the degree of its fiber over the zero section in \( \mathbb{A}^1 \), which coincides with \( \Psi_{c,0,\mathbb{A}^1} (= F_{c,0,\mathbb{A}^1} = F_{c/k} \times \text{id}_{\mathbb{A}^1} \) by assertion (ii)). This completes the proof of assertion (iv). \( \square \)

**Proposition 3.8.2.**

The morphism \( \tilde{\kappa}_{g,h,x/S} \) (resp., \( \hat{\kappa}_{g,h,x/S} \)) factors through the closed immersion \( \mathcal{Z}_{x/S} : \bigotimes_{c} \nabla_{c,x/S}^{(1)} \to \bigotimes_{c} \tilde{\kappa}_{x,S}^{(1)} \) (resp., \( \mathcal{Z}_{c,\tilde{\kappa}x/S} : \bigotimes_{c,\tilde{\kappa}} \nabla_{c,\tilde{\kappa}x/S}^{(1)} \to \bigotimes_{c,\tilde{\kappa}} \tilde{\kappa}_{c,x/S}^{(1)} \)) (cf. (253)), i.e., there exists uniquely a morphism

\[
(251) \quad \bigotimes_{c} \nabla_{c,x/S}^{(1)} \to \bigotimes_{c} \tilde{\kappa}_{c,x/S}^{(1)}
\]

satisfying that \( \mathcal{Z}_{c,x/S} \circ \tilde{\kappa}_{g,h,x/S} = \hat{\kappa}_{g,h,x/S} \) (resp., \( \mathcal{Z}_{c,\tilde{\kappa}x/S} \circ \tilde{\kappa}_{g,h,x/S} = \hat{\kappa}_{g,h,x/S} \)).

**Proof.** First, we consider the non-resp’d portion. Let \( t \) and \( \omega_l \) (\( l = 1, \ldots, \text{rk}(g) \)) be as in the proof of Proposition 3.7.1. In particular, by virtue of them, we obtain an isomorphism

\[
(252) \quad \bigotimes_{c} \nabla_{c,x/S}^{(1)} \cong \bigotimes_{c} \nabla_{c,x/S}^{(1)} \left( F_{X/S}(\Omega_{X/S}^{\log}(X_{S/S})) \right).
\]

and an isomorphism

\[
(253) \quad \bigotimes_{c} \nabla_{c,x/S}^{(1)} \cong \bigotimes_{c} \nabla_{c,x/S}^{(1)} \left( F_{X/S}(\Omega_{X/S}^{\log}(X_{S/S})) \right).
\]

Now, let \( t : T \to S \) be an \( S \)-scheme and \( \mathcal{E}^\bullet = (\mathcal{E}_T, \nabla_\mathcal{E}) \) a \((g, h)\)-oper on \( X/T \). The \( T \)-rational point of \( \mathcal{D}\mathcal{P}_{g,h,x/S} \) classifying \( \mathcal{E}_{\mathcal{O}} \) determines, via \( \hat{\kappa}_{g,h,x/S} \), a \( T \)-rational point of the \( S \)-scheme displayed in the left-hand side of (253). By passing to the isomorphism (253) (and (183)), we obtain the corresponding global section

\[
(254) \quad R_{\mathcal{E}^\bullet} \in \Gamma((X \times S T)^{(1)}_T, \bigoplus_{l=1}^{\text{rk}(g)} F_{X \times S T/S}((X \times S)^\log_T)/T_{\log_T}).
\]

What we want to prove is (by the isomorphism (252)) the assertion that

\[
(255) \quad R_{\mathcal{E}^\bullet} \in \Gamma((X \times S T)^{(1)}_T, \bigoplus_{l=1}^{\text{rk}(g)} \Omega_{(X \times S T)_T}^{\log_T}(X \times S T)^{(1)}_T/T_{\log_T}).
\]

Here, by Proposition 3.4.2, the cokernel of \( \bigoplus \omega_l \) turns out to be relatively torsion-free (of rank \( p - 1 \)). Hence, it suffices to prove the assertion (255).
where \( X \times_{\mathcal{S}} T \) is replaced with the scheme-theoretically dense open subscheme \((X \times_{\mathcal{S}} T)^{\text{sm}}\) (cf. §1.6) of \( X \times_{\mathcal{S}} T \). (In particular, we may assume that \( X \times_{\mathcal{S}} T/T \) is a smooth curve.) But, this assertion follows from an argument similar to the argument discussed in the proof of [13], Theorem 3.1 (or [46], Proposition 3.2). This completes the proof of the non-resp’d portion.

The resp’d portion follows from the non-resp’d portion and the following observation: if \( \mathcal{E}^\bullet = (\mathcal{E}_g, \nabla_\mathcal{E}) \) is a \((g, \hbar)\)-oper on \( \mathfrak{X}/\mathfrak{S} \) of radii \( \rho \) and of canonical type, then

\[
[\chi](\sigma_i^* \psi^!(\mathcal{E}_g, \nabla_\mathcal{E})) = [\chi]((\mu_i^!(\mathcal{E}_g, \nabla_\mathcal{E}))[p] - \hbar^{p-1} \cdot \mu_i^!(\mathcal{E}_g, \nabla_\mathcal{E}))
\]

\[
= [\chi]((\mathfrak{Kos}_g^{-1} \circ \rho)[p] - \hbar^{p-1} \cdot (\mathfrak{Kos}_g^{-1} \circ \rho))
\]

\[
= \rho F^{\hbar}
\]

(cf. (208) for the first equality). □

We shall denote by

\[
\mathcal{O}_p^{\text{p-nilp}}_{g,h,\mathfrak{X}/\mathfrak{S}} \quad \text{(resp., } \mathcal{O}_p^{\text{p-nilp}}_{g,h,\rho,\mathfrak{X}/\mathfrak{S}})\]

the closed substack of \( \mathcal{O}_p_{g,h,\mathfrak{X}/\mathfrak{S}} \) (resp., \( \mathcal{O}_p_{g,h,\rho,\mathfrak{X}/\mathfrak{S}} \)) defined to be the stack-theoretic inverse image, via \( \kappa_\text{H-M}_{g,h,\mathfrak{X}/\mathfrak{S}} \) (resp., \( \kappa_{\text{H-M}}_{g,h,\rho,\mathfrak{X}/\mathfrak{S}} \)), of the composite

\[
\mathfrak{S} \to \bigotimes_{g,h,\mathfrak{X}/\mathfrak{S}} \to \bigotimes_{\mathfrak{t},\mathfrak{X}/\mathfrak{S}}.
\]

In particular, we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_p^{\text{p-nilp}}_{g,h,\mathfrak{X}/\mathfrak{S}} & \longrightarrow & \mathcal{O}_p^{\text{p-nilp}}_{g,h,\rho,\mathfrak{X}/\mathfrak{S}} \\
\downarrow & & \downarrow \\
\mathcal{O}_p_{g,h,\mathfrak{X}/\mathfrak{S}} & \longrightarrow & \mathcal{O}_p_{g,h,\rho,\mathfrak{X}/\mathfrak{S}}
\end{array}
\]

(259)

consisting of closed immersions between \( \mathfrak{S} \)-schemes. The both sides of square diagrams in this diagram are cartesian. If \( r = 0 \), then we mean \( \mathcal{O}_p^{\text{p-nilp}}_{g,h,\emptyset,\mathfrak{X}/\mathfrak{S}} \) for the stack \( \mathcal{O}_p^{\text{p-nilp}}_{g,h,\mathfrak{X}/\mathfrak{S}} \).

**Definition 3.8.3.**

We shall say that a \((g, \hbar)\)-oper \( \mathcal{E}^\bullet \) on \( \mathfrak{X}/\mathfrak{S} \) is \( p \)-nilpotent if it is classified by the closed substack \( \mathcal{O}_p^{\text{p-nilp}}_{g,h,\mathfrak{X}/\mathfrak{S}} \) of \( \mathcal{O}_p_{g,h,\mathfrak{X}/\mathfrak{S}} \).
3.9. Consider the moduli stack $\mathcal{O}_S^{\rho}(\mathcal{E}, \rho)$, where $\mathbb{A}^1 := \text{Spec}(k[z])$ and $\bar{\rho}$ is an $(\mathbb{A}^1 \times_k S)$-rational point of $c^{\times r}$. (We take $\bar{\rho} := 0$ if $r = 0$.) Denote by
\[
\Delta_{\rho, \mathcal{E}/S} : \mathcal{O}_S^{\rho}(\mathcal{E}/S) / \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho]) \to \mathbb{A}^1
\]
the structure morphism of $\mathcal{O}_S^{\rho}(\mathcal{E}/S)$ over $\mathbb{A}^1$. $\mathcal{O}_S^{\rho}(\mathcal{E}/S)$ represents, by definition, the functor which, to any $S$-scheme $T$ assigns the set of pairs $(\mathcal{E}, \mathcal{E}^\bullet)$ consisting of $\mathcal{E}$ the morphism $T \to \mathbb{A}^1$ of $k$-schemes corresponding to $\mathcal{E}$ and the isomorphism class of a $(g, \mathcal{E})$-oper on $\mathcal{E}/T$ of radii $\mathcal{E}^\bullet. \rho$. (cf. (158) for the definition of $\mathcal{E}^\bullet$). One verifies the equality
\[
\mathcal{O}_S^{\rho}(\mathcal{E}/S) / \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho]) \cong \mathcal{O}_S^{\rho}(\mathcal{E}/S) / \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho])
\]
for $S$-schemes, and hence, a closed immersion
\[
\mathcal{O}_S^{\rho}(\mathcal{E}/S) \times \Delta_{\rho, \mathcal{E}/S} : \mathcal{O}_S^{\rho}(\mathcal{E}/S) / \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho]) \to \mathcal{O}_S^{\rho}(\mathcal{E}/S) / \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho])
\]
of $S$-schemes, and hence, a closed immersion
\[
\mathcal{O}_S^{\rho}(\mathcal{E}/S) / \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho]) \cong \mathcal{O}_S^{\rho}(\mathcal{E}/S) / \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho])
\]
Moreover, one may construct a $g_m := \text{Spec}(k[z^{-1}])$-action
\[
g_m \times_k \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho]) \to \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho])
\]
on $\mathcal{O}_S^{\rho}(\mathcal{E}/S)$ determined by assigning $(h', (h, [\mathcal{E}^\bullet])) \mapsto (h' \cdot h, [\mathcal{E}^\bullet])$ (for any $h' \in G_m$). If $\bar{\rho} \in c^{\times r}(G_m \times_k S) (\subseteq c^{\times r}(\mathbb{A}^1 \times S))$, then this action carries a $g_m$-action
\[
g_m \times_k \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho]) \to \mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho])
\]
on $\mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho])$. On the other hand, there exists a natural $g_m$-action
\[
ineseq{\mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho])}{\mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho])}
\]
on $\mathcal{O}_S^{\rho}(\mathcal{E}/S, \mathbb{A}^1, [\rho])$ induced from the homotheties on $g$ (resp., the $g_m$-action $\star : G_m \times_k c \to c$ (cf. (155))).
Let $\mathcal{E} \in \Gamma(S, \mathcal{O}_S^\times) \subseteq \Gamma(S, \mathcal{O}_S)$. For $\rho := (\rho_i^\star)^r_{i=1} \in c^{\times r}(S)$, we shall write
\[
\rho^\star := (\rho_i^\star)^r_{i=1} \in c^{\times r}(\mathbb{A}^1 \times_k S),
\]
where $\rho_i^\star (i = 1, \ldots, r)$ is the composite
\[
\rho_i^\star : \mathbb{A}^1 \times_k S \overset{id_{\mathbb{A}^1} \times \rho_i}{\to} \mathbb{A}^1 \times_k c \overset{\mathcal{E}}{\to} c
\]
(cf. (158) for the definition of $\mathcal{E}$). One verifies the equality
\[
\rho^\star \circ ([h], \text{id}_S) = h \star \rho
\]
(cf. §2.10). Consider the diagram of \((G_m \times_k S)\)-schemes:

\[
\begin{array}{ccc}
G_m \times_k \mathcal{O}_p \times_k S & \longrightarrow & \mathcal{O}_p \times_k \mathcal{O}_p \mathcal{O}_p \times_k S \\
\downarrow & & \downarrow \\
G_m \times_k \mathcal{O}_p \times_k S & \longrightarrow & \mathcal{O}_p \times_k \mathcal{O}_p \times_k S \\
\downarrow & & \downarrow \\
G_m \times_k \mathcal{O}_p \times_k S & \longrightarrow & \mathcal{O}_p \times_k \mathcal{O}_p \times_k S \\
\end{array}
\]

(269)

whose three horizontal arrows are constructed as follows:

- the top horizontal arrow denotes the composite

\[
G_m \times_k \mathcal{O}_p \times_k S \rightarrow G_m \times_k \mathcal{O}_p \times_k S \rightarrow \mathcal{O}_p \times_k \mathcal{O}_p \times_k S
\]

where the first arrow is the product of the identity morphism \(id_{G_m}\) of \(G_m\) and the closed immersion \(\mathcal{O}_p \times_k S \rightarrow \mathcal{O}_p \times_k S\) (cf. (262)), and the second arrow is (264);

- the middle (resp., bottom) horizontal arrow denotes the composite

\[
G_m \times_k \mathcal{O}_p \times_k S \rightarrow G_m \times_k \mathcal{O}_p \times_k S \rightarrow \mathcal{O}_p \times_k \mathcal{O}_p \times_k S
\]

(resp., \(G_m \times_k \mathcal{O}_p \times_k S \rightarrow G_m \times_k \mathcal{O}_p \times_k S \rightarrow \mathcal{O}_p \times_k \mathcal{O}_p \times_k S\)),

where the first arrow is the product of the identity morphism \(id_{G_m}\) of \(G_m\) and the composite of closed immersions

\[
\mathcal{O}_p \times_k S \rightarrow S \times_k S \rightarrow G_m \times_k \mathcal{O}_p \times_k S
\]

(resp., \(\mathcal{O}_p \times_k S \rightarrow S \times_k S \rightarrow G_m \times_k \mathcal{O}_p \times_k S\)),

and the second arrow is (265).

One verifies (cf. Proposition 2.10.1 and Proposition 3.2.3) that this diagram (269) is commutative and compatible, in a natural sense, with any base-change over \(S\). Moreover, all the horizontal arrows in this diagram are isomorphisms. In particular, we have the following

**Proposition 3.9.1.**

The commutative diagram (269) carries a commutative square diagram

\[
\begin{array}{ccc}
G_m \times_k \mathcal{O}_p \times_k S & \longrightarrow & \mathcal{O}_p \times_k \mathcal{O}_p \times_k S \\
\downarrow & & \downarrow \\
G_m \times_k \mathcal{O}_p \times_k S & \longrightarrow & \mathcal{O}_p \times_k \mathcal{O}_p \times_k S \\
\end{array}
\]

(273)

of \((G_m \times_k S)\)-schemes, where
the right-hand vertical arrow denotes the closed immersion $\mathfrak{D}_\mathcal{P}_{\mathcal{G}_m}^{Z_{\mathfrak{g}}\mathcal{X}/\mathcal{O}_S} \rightarrow 
abla_{\mathfrak{g}} \mathcal{P}_{\mathcal{G}_m}^{Z_{\mathfrak{g}}\mathcal{X}/\mathcal{O}_S}$ (cf. (275));

- the left-hand vertical arrow denotes the product of the identity morphism $\text{id}_{\mathcal{G}_m}$ of $\mathcal{G}_m$ and the closed immersion $\mathfrak{D}_\mathcal{P}_{\mathcal{G}_m}^{Z_{\mathfrak{g}}\mathcal{X}/\mathcal{O}_S} \rightarrow \mathfrak{D}_\mathcal{P}_{\mathcal{G}_m}^{Z_{\mathfrak{g}}\mathcal{X}/\mathcal{O}_S}$.

Moreover, both the upper and lower horizontal arrows in this diagram are isomorphisms. In particular for each $\hbar' \in \Gamma(S, \mathcal{O}_S^*)$, we have natural isomorphisms

\[(274) \quad \mathfrak{D}_\mathcal{P}_{\mathcal{G}_m}^{Z_{\mathfrak{g}}\mathcal{X}/\mathcal{O}_S} \cong \mathfrak{D}_\mathcal{P}_{\mathcal{G}_m}^{Z_{\mathfrak{g}}\mathcal{X}/\mathcal{O}_S} \quad \text{and} \quad \mathfrak{D}_\mathcal{P}_{\mathcal{G}_m}^{Z_{\mathfrak{g}}\mathcal{X}/\mathcal{O}_S} \cong \mathfrak{D}_\mathcal{P}_{\mathcal{G}_m}^{Z_{\mathfrak{g}}\mathcal{X}/\mathcal{O}_S}
\]

of $S$-schemes that are compatible with any base-change over $S$.

**Proof.** The last assertion follows from the equality (268). \qed

### 3.10

We shall study the case where $\hbar = 0$. Note that there exists a canonical decomposition

\[(275) \quad \mathfrak{g}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}} \cong \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^j / \mathfrak{g}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^{j+1} = \bigoplus_{j \in \mathbb{Z}} (\mathfrak{g}_j \mathfrak{G}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^j).
\]

Indeed, it follows from the definition of $\mathfrak{E}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^j$ (cf. (109)) and the canonical decomposition of the case where the Lie algebra $\mathfrak{g}$ is taken to be $\mathfrak{g}_{\mathfrak{G},0}$, which will be proved in § 4.14. The natural inclusion $\mathfrak{g}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}} \hookrightarrow \mathfrak{g}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}$ is compatible with the gradings, and hence, determines a composite

\[(276) \quad \mathcal{O}_X \cong \mathfrak{Hom}_{\mathcal{O}_X}(\nabla_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^1, (\mathfrak{g}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^j)^{-1}) \hookrightarrow \mathfrak{Hom}_{\mathcal{O}_X}(\nabla_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^1, (\mathfrak{g}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^j)^{-1}),
\]

where the first arrow denotes the isomorphism asserted in (99) (of the case where $j = -1$). In particular, the image of $1 \in \Gamma(X, \mathcal{O}_X)$ determines an $S$-0-log connection $\nabla_{\text{triv}}$ on $\mathfrak{E}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^1$. The pair $(\mathfrak{E}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^1, \nabla_{\text{triv}})$ forms a $(\mathfrak{g}, 0)$-oper on $\mathcal{X}/\mathcal{S}$ of canonical type and of radii $[0]_S = [(0), (0), \cdots, (0)]_S \in \mathfrak{C}^{*}(S)$ (cf. (160)). The $(\mathfrak{g}, 0)$-oper $(\mathfrak{E}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^1, \nabla_{\text{triv}})$ yields a canonical global section

\[(277) \quad \mathfrak{E}_{\text{triv}} : S \rightarrow \mathfrak{D}_\mathcal{P}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}.
\]

(Note that $\mathfrak{P}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^{\mathfrak{E}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}^1, \nabla_{\text{triv}}} = 0$, and hence, the section $\mathfrak{E}_{\text{triv}}$ factors through the closed immersion $\mathfrak{D}_\mathcal{P}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}} \rightarrow \mathfrak{D}_\mathcal{P}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}$.) This section gives a trivialization

\[(278) \quad \nabla(f_*(\mathfrak{P}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}})) \cong \mathfrak{D}_\mathcal{P}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}
\]

of the relative affine space $\mathfrak{D}_\mathcal{P}_{\mathfrak{G},0,0,\mathcal{X}/\mathcal{S}}$ over $S$. By composing the inverse of (278) and the isomorphism (230) (of the case where $\hbar = 0$), we have a canonical
isomorphism
\( (279) \quad \mathcal{R} : \mathcal{O}_p \to \mathcal{O}_p \)
of \( S \)-schemes. Moreover, for each \( \rho \in \mathcal{O}_p \), we obtain, by restricting (279) to closed subschemes, an isomorphism
\( (280) \quad \mathcal{R} : \mathcal{O}_p \to \mathcal{O}_p \).

**Proposition 3.10.1.**

The composite
\( (281) \quad V_{\mathcal{O}_p} = \mathcal{O}_p \to \mathcal{O}_p \)
\( (282) \quad \mathcal{O}_p \to \mathcal{O}_p \)
of \( \mathcal{O}_p \) (resp., \( \mathcal{O}_p \)) over \( S \). In particular, \( \mathcal{O}_p \) (resp., \( \mathcal{O}_p \)) is finite and faithfully flat of degree \( p^\mathcal{O}_p \) (resp., \( p^\mathcal{O}_p \)).

**Proof.** It suffices to prove the non-resp’d portion (since the proof of the resp’d portion is entirely similar). The composite isomorphism
\( (283) \quad \mathcal{V}(f_*(\mathcal{O}_p)) \to \mathcal{O}_p \)
sends, by construction, any local section \( R \in f_*(\mathcal{O}_p) \) to the local section of \( \mathcal{O}_p \) expressed locally as
\( (284) \quad \mathcal{V}(f_*(\mathcal{O}_p)) \to \mathcal{O}_p \)
for each log chart \( (U, x) \) on \( \mathcal{V} \) over \( S \). But, by Proposition 3.5.1, we have the equality
\( (285) \quad \mathcal{V}(f_*(\mathcal{O}_p)) \to \mathcal{O}_p \)
Hence, the composite (283) may be given by assigning \( R \mapsto F_{\mathcal{O}_p} \circ [\mathcal{V}(f_*(\mathcal{O}_p)) \to \mathcal{O}_p] \) of (283) may be given by assigning \( R \mapsto [\mathcal{V}(f_*(\mathcal{O}_p)) \to \mathcal{O}_p] \). Consequently, the composite \( \mathcal{R} \circ \mathcal{O}_p \), which coincides the composite of
and the inverse of \((230)\), may be given by assigning \(\rho \mapsto F_{t/k} \circ \rho\) for any local section \(\rho\) of the \(S\)-scheme \(\bigotimes_{c,t:S}^\nabla\). This completes the proof of the non-resp’d portion. \(\square\)

In particular, by considering the local description of the inverse image of the composite section \(\mathfrak{Hos}_{g,0,\rho,x/S}^\circ \circ \mathfrak{triv} : S \to \bigotimes_{c,x/S}^\nabla\) (resp., \(\mathfrak{Hos}_{g,0,\rho,x/S}^\circ \circ \mathfrak{triv} : S \to \bigotimes_{c,\rho,x/S}^\nabla\)) via the relative Frobenius morphism \(F : \bigotimes_{c,x/S}^\nabla\) (resp., \(F : \bigotimes_{c,\rho,x/S}^\nabla\)), we have the following

**Corollary 3.10.2.**

*Suppose that we are given \(\rho \in c^r(S)\) satisfying that \(F_{t/k} \circ \rho = [0]_S\). (If \(r = 0\), then we take \(\rho = 0\).) Then, \(\mathcal{O}_{t,0,\rho,x/S}^{p-nilp}\) (resp., \(\mathcal{O}_{t,0,\rho,x/S}^{p-nilp}\)) is finite and faithfully flat over \(S\) of degree \(\mathcal{N}(g)\) (resp., \(\mathcal{N}(g')\)) (cf. \((113)\)). Moreover, \(\mathcal{O}_{t,0,\rho,x/S}^{p-nilp}\) (resp., \(\mathcal{O}_{t,0,\rho,x/S}^{p-nilp}\)) is, Zariski locally on \(S\), isomorphic to the \(S\)-scheme \(S \times_k \text{Spec}(\bigotimes_{l=1}^N k[\varepsilon_l]/\varepsilon_l^m)\) (resp., \(S \times_k \text{Spec}(\bigotimes_{l=1}^N k[\varepsilon_l]/\varepsilon_l^m)\)).*

### 3.11.

For an affine \(S\)-scheme \(t : T \to S\), we shall write \(\mathcal{R}_T\) for the \(\mathcal{O}_S\)-algebra corresponding to \(T\) (i.e., \(t_*(\mathcal{O}_T) \cong \mathcal{R}_T\)). The \(\mathbb{G}_m\)-action \(*\) on \(c\) (cf. \((135)\)) yields \(\mathbb{Z}\)-gradings of \(\mathcal{O}_t^\nabla_{c,x/S}\), \(\mathcal{O}_t^\nabla_{c,\rho,x/S}\), and the \(\mathbb{G}_m\)-action \(Ad^+\) on \(g^{ad(p_1)}\) (cf. \(\S\) 2.8) yields \(\mathbb{Z}\)-gradings of \(\mathcal{R}_V(f_*(\mathcal{V}_{g,h,x/S}))\). These \(\mathcal{O}_S\)-algebras preserve their respective gradings under any base-change over \(S\) and the isomorphism of \(\mathcal{O}_S\)-algebras corresponding to \((230)\) is compatible with the gradings on \(\mathcal{R}_t^\nabla_{c,x/S}\) and \(\mathcal{R}_V(f_*(\mathcal{V}_{g,h,x/S}))\). One verifies that the structures of relative affine space (modeled on \(V(f_*(\mathcal{V}_{g,x/S}))\)) on \(\mathcal{O}_{g,h,x/S}\) carries a well-defined filtration on \(\mathcal{R}_{\mathcal{O}_{g,h,x/S}}\). Indeed, each log chart \((U,x)\) on \(X^{log}\) over \(S^{log}\) associates (by means of the isomorphism \(\Omega_{U^{log}/S^{log}} \cong \mathcal{O}_U \cdot d\log(x)\)) a canonical isomorphism

\[
(286) \quad \text{triv}^\circ_{\mathcal{O}_{g,h,x/S}} : \mathcal{O}_{g,h,U/S} \sim (\mathcal{O}_U \otimes_k g^{ad(p_1)} \sim) \mathcal{V}_{g,h,U/S}
\]

(cf. \((142)\)). If another log chart \((U',x')\) has been chosen, then automorphism \(\text{triv}^\circ_{\mathcal{O}_{g,h,(U',x')}} \circ (\text{triv}^\circ_{\mathcal{O}_{g,h,(U,x)}})^{-1}\) preserves the filtrations. Hence, by passing to \(\text{triv}^\circ_{\mathcal{O}_{g,h,(U,x)}}\), one may construct a well-defined filtration on the affine space \(\mathcal{O}_{g,h,x/S}(S) \cong \mathcal{O}_{g,h,x/S}(X))\), and hence, on \(\mathcal{R}_{\mathcal{O}_{g,h,x/S}}\).

Moreover, it follows from this construction of the filtration that the associated graded \(\mathcal{O}_S\)-algebra \(gr(\mathcal{R}_{\mathcal{O}_{g,h,x/S}})\) is canonically isomorphic to \(\mathcal{R}_V(f_*(\mathcal{V}_{g,h,x/S}))\). Thus, by virtue of the isomorphism \((230)\), there exists a canonical isomorphism

\[
(287) \quad gr(\mathcal{R}_{\mathcal{O}_{g,h,x/S}}) \sim \mathcal{R}_t^\nabla_{c,x/S}.
\]
If $\mathfrak{X}_S$ is a unpointed projective smooth curve over an algebraically closed field of positive characteristic, then the following proposition was proved in \[13\], Lemma 3.3 and Corollary 3.4.

**Proposition 3.11.1.**

(i) The morphism $\kappa_{g,h,X/S}^H \circ \otimes_{c,\mathfrak{X}/S} \nabla \longrightarrow R_{\mathcal{O}_{p,h,x/S}} \mathcal{O}_S$-algebras corresponding to $\kappa_{g,h,X/S}^H$ is filtered of degree $p$, i.e.,

(288) $\kappa_{g,h,X/S}^H (\mathcal{R}_{c,\mathfrak{X}/S}^j (\nabla)) \subseteq \mathcal{R}_{\mathcal{O}_{p,h,x/S}}^{j+p}$

for all $j \in \mathbb{Z}$.

(ii) The morphism of $S$-schemes corresponding to the composite

(289) $\mathcal{R}_{c,\mathfrak{X}/S} \nabla \longrightarrow \mathcal{R}_{\mathcal{O}_{p,h,x/S}} \mathcal{O}_S$

where the second arrow denotes the morphism of graded $\mathcal{O}_S$-algebra associated with $\kappa_{g,h,X/S}^H$, coincides with $\kappa_{g,h,x/S}^H \circ \mathcal{O}_{p,0,x/S} \circ \mathcal{O}_{p,0,x/S}$ (and hence, $F_{10}$ by Proposition 3.9.1).

(iii) $\kappa_{g,h,X/S}^H$ is finite and faithfully flat of degree $p^{N(g)}$.

**Proof.** First, we consider assertion (i). Consider the map

(290) $\kappa_{g,h,X/S}^H (S) : \mathcal{O}_{p,h,X/S}^\bullet (X) \rightarrow (\Omega_{X_S^{(1)log}/Slog}^X \times \mathcal{O}_m \mathcal{c}) (X)$

between the respective sets of global sections $\mathcal{O}_{p,h,X/S}^\bullet (X) (= \mathcal{O}_{p,h,x/S}^\bullet (S))$, $(\Omega_{X_S^{(1)log}/Slog}^X \times \mathcal{O}_m \mathcal{c}) (X) (= \mathcal{R}_{c,\mathfrak{X}/S} \nabla (S))$ (cf. the beginning of §4 for the definition of $\mathcal{O}_{p,h,X/S}^\bullet (X)$) associated with $\kappa_{g,h,X/S}^H$. It may be thought of as the morphism between the sets of global sections associated with the morphism

(291) $\kappa_{c,h,X/S}^H : \mathcal{O}_{p,h,X/S}^\bullet \rightarrow \Omega_{X_S^{(1)log}/Slog}^X \times \mathcal{O}_m \mathcal{c}$

of stacks over $\mathfrak{E}_t/X$, i.e., $\kappa_{g,h,X/S}^H = \kappa_{c,h,X/S}^H (X)$. (Here, we refer to (the proof of) Proposition 3.8.2 for the fact that any section of $(\Omega_{X_S^{(1)log}/Slog}^X \times \mathcal{O}_m \mathcal{c})$ of the form $(\mathcal{E}_G, p,q((\mathcal{E}_G, \nabla)))$ lies in the subsheaf $\Omega_{X_S^{(1)log}/Slog}^X \times \mathcal{O}_m \mathcal{c})$.

Now, let us fix a log chart $(U, x)$ on $X^{log}$ over $S^{log}$. The composition of the isomorphism \[142\] and the isomorphism \[152\] yields a bijection

(292) $\mathcal{O}_{p,h,X/S}^\bullet (U) \sim \mathcal{c}(U)$. 

Also, the section \((\text{id}_U \times F_S)^* (d \log(x)) \in \Gamma(U^{(1)}_S, \Omega^{(1)\log}_{S/\mathcal{G}\log})\) gives a bijection
\[
(\Omega^\times_{X^{(1)\log}/S\log} \otimes \mathbb{G}_m \mathbf{c})(U) \xrightarrow{\sim} \mathbf{c}(U).
\]
This map makes the following diagram commute:
\[
\begin{array}{ccc}
\mathcal{O}_{\mathcal{O}_X/S}(X) & \xrightarrow{\kappa_{t,h,X/S}(X) = \mathcal{K}_{\mathcal{O}_X/S}(S)} & (\Omega^\times_{X^{(1)\log}/S\log} \otimes \mathbb{G}_m \mathbf{c})(X) \\
\downarrow & & \downarrow \\
\mathcal{O}_{\mathcal{O}_X/S}(U) & \xrightarrow{\kappa_{t,h,X/S}(U)} & (\Omega^\times_{X^{(1)\log}/S\log} \otimes \mathbb{G}_m \mathbf{c})(U) \\
\end{array}
\]
where both the right-hand and left-hand upper vertical arrows are restriction maps. Consider the commutativity of the diagram (294) where \((U, x)\) is replaced with various log charts. Then, assertion (i) follows from Proposition 3.8.1 (i) and the definition of the filtrations involved.

Assertion (ii) follows from Proposition 3.8.1 (ii) and (iii).

Finally, we consider assertion (iii). Since \(\text{gr}(\mathcal{R}_{\mathcal{O}_{X/S}})\) is a finite \(\mathcal{R}_{\mathcal{O}_{X/S}}\)-module via the morphism \(\text{gr}(\mathcal{K}_{\mathcal{O}_X/S})\), \(\mathcal{R}_{\mathcal{O}_{X/S}}\) turns out to be a finite \(\mathcal{R}_{\mathcal{O}_{X/S}}\)-module. That is, \(\mathcal{K}_{\mathcal{O}_X/S}\) is finite. Moreover, both \(\mathcal{O}_{\mathcal{O}_X/S}\) and \(\mathcal{O}_{\mathcal{O}_X/S}\) are flat over \(S\) and the respective fibers over any point of \(S\) are irreducible, smooth and of the same dimension. It follows immediately that \(\mathcal{K}_{\mathcal{O}_X/S}\) is also faithfully flat. Now we compute its degree. To this end, we apply the above discussion to the case where the triple \((S, \mathcal{X}_S, h)\) is taken to be \((\mathbb{A}^1 \times_k S, \mathcal{X}_S, h, z)\), where \(\mathbb{A}^1 := \text{Spec}(k[z])\). Hence, \(\mathcal{K}_{\mathcal{O}_{X/S}}\) is finite and faithfully flat. In particular, the degree of \(\mathcal{K}_{\mathcal{O}_{X/S}}\) may be given as the degree of its fiber over the zero section \(S \to \mathbb{A}^1 \times_k S\). That is, the degree of \(\mathcal{K}_{\mathcal{O}_{X/S}}\) coincides with the degree of \(\mathcal{K}_{\mathcal{O}_{X/S}}\). By Proposition 3.10.1, this value equals \(p^{N(h)}\). Moreover, since \(\mathcal{K}_{\mathcal{O}_{X/S}}\) may be obtained as the base-change of \(\mathcal{K}_{\mathcal{O}_{X/S}}\) via the graph \(S \to \mathbb{A}^1 \times_k S\), the degree of \(\mathcal{K}_{\mathcal{O}_{X/S}}\) coincides with the degree of \(\mathcal{K}_{\mathcal{O}_{X/S}}\), i.e., \(p^{N(h)}\). Thus, the proof of assertion (iii) is completed. \(\square\)

Suppose that \(r > 0\), and consider the morphism
\[
\mathcal{R}_{\mathcal{O}_{X/S}} : \mathcal{O}_{\mathcal{O}_{X/S}} \to \mathbf{c}^r,
\]
(cf. (176)) of \(k\)-schemes determined by assigning \(\mathcal{E}^\bullet := (\mathcal{E}_B, \nabla_{\mathcal{E}}) \mapsto (\rho^{(\mathcal{E}_B, \nabla_{\mathcal{E}}), i}_i)\) for any \((g, h)\)-oper \(\mathcal{E}^\bullet\). The square diagram

\[
\begin{array}{ccc}
\mathcal{O}_p g, h, x_j/k & \xrightarrow{\nabla^{H-M\nabla}} & \mathcal{O}_g x_{j/S} \\
\downarrow & & \downarrow \\
\mathcal{R}ad_{g, h, x_j/S} & \xrightarrow{\nabla} & \mathcal{R}ad_{\mathcal{O}_g x_{j/S}} \\
\end{array}
\]

(296)

(cf. (227) for the definition of \(\mathcal{R}ad_{\mathcal{O}_g x_{j/S}}\)) is commutative. Recall that \(\mathcal{O}_p g, h, \rho, x_j/k\) (resp., \(\mathcal{R}ad_{\mathcal{O}_g x_{j/S}}\)) is, by definition, the scheme-theoretic inverse image of \(\rho \in c^{x\rho}(S)\) (resp., \(\rho^F_h \in c^{x\rho}(S)\)) via \(\mathcal{R}ad_{g, h, x_j/S}\) (resp., \(\mathcal{R}ad_{\mathcal{O}_g x_{j/S}}\)). Thus, by virtue of Proposition 3.8.1 (iv) and Proposition 3.11.1 (iii) (and the equality \(c^\mathcal{R}(g) = \mathcal{R}(g) - r \cdot \dim(c)\)), one may conclude, from the square diagram (296), the following proposition.

**Proposition 3.11.2.**

_The morphism_ \(\nabla^{H-M\nabla}_{g, h, x_j/S} : \mathcal{O}_p g, h, \rho, x_j/S \to \mathcal{R}ad_{\mathcal{O}_g x_{j/S}}\) _is finite and faithfully flat of degree_ \(p^{\mathcal{R}(g)}\).

3.12. In the last subsection, we consider the moduli stack of \((g, h)\)-opers on the universally family \(\mathcal{E}_{g, r}/\overline{\mathcal{M}}_{g, r}\) of pointed stable curves (cf. §1.5). For simplicity, we shall write

\[
\begin{array}{ll}
\mathcal{O}_p g, h, g, r, & \mathcal{O}_p g, h, g, r, \\
\mathcal{O}_p g, h, \rho, g, r, & \mathcal{O}_p g, h, \rho, g, r,
\end{array}
\]

for the moduli stacks \(\mathcal{O}_p g, h, e_{g, r}/\overline{\mathcal{M}}_{g, r}\), \(\mathcal{O}_p g, h, e_{g, r}/\overline{\mathcal{M}}_{g, r}\), and \(\mathcal{O}_p g, h, e_{g, r}/\overline{\mathcal{M}}_{g, r}\) respectively. Here, since \(\overline{\mathcal{M}}_{g, r}\) is proper and geometrically connected over \(k\), we have \(\Gamma(\overline{\mathcal{M}}_{g, r}, O_{\overline{\mathcal{M}}_{g, r}}) = k\).

Hence, if \(\mathcal{O}_p g, h, \rho, g, r\) (let alone either \(\mathcal{O}_p g, h, \rho, g, r\) or \(\mathcal{O}_p g, h, \rho, g, r\)) is nonempty, then it is necessarily satisfied that \(h \in k\) and \(\rho \in c^{x\rho}(k)\). By applying the results obtained so far, we obtain the following theorems concerning the structures of these stacks. (Note that Theorem 3.12.3 asserted below is a generalization of the results prove by S. Mochizuki (cf. [52], ChapI, Theorem 2.3) and T. H. Chen-X. Zhu (cf. [13], Corollary 3.4).)

**Theorem 3.12.1 (Structures of** \(\mathcal{O}_p g, h, g, r\) **and** \(\mathcal{O}_p g, h, \rho, g, r\)).

_Let_ \(h \in k\) _and_ \(\rho \in c^{x\rho}(k)\) _where we take_ \(\rho = \emptyset\) _if_ \(r = 0\). _Then_, \(\mathcal{O}_p g, h, g, r\)
(resp., \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\)) may be represented by a relative affine space over \(\overline{\mathfrak{M}}_{g,r}\) of relative dimension \(\mathfrak{N}(g)\) (resp., \('\mathfrak{N}(g)'\)) (cf. (118)). In particular, \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\) (resp., \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\)) is a nonempty, geometrically connected, and smooth Deligne-Mumford stack over \(k\) of dimension \(3g-3+r+\mathfrak{N}(g)\) (resp., \(3g-3+r+\mathfrak{N}(g)\)).

**Proof.** The assertion follows from Proposition 2.7.5 and Proposition 2.11.1. \(\square\)

**Theorem 3.12.2 (Structures of \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\) and \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\))**

Let \(h \in k\) and suppose that if \(r > 0\), then we are given \(\rho \in C^{\infty}(k)\) satisfying that \(\rho_h^F = \{0\}_k\) (cf. (248)). Then, \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\) (resp., \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\)) may be represented by a nonempty and proper Deligne-Mumford stack over \(k\) of dimension \(3g-3+r\), and the natural projection \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}} \to \overline{\mathfrak{M}}_{g,r}\) (resp., \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}} \to \overline{\mathfrak{M}}_{g,r}\)) is finite and faithfully flat of degree \(p^{\mathfrak{N}(g)}\) (resp., \(p^{\mathfrak{N}(g)}\)). If, moreover, it is satisfied that \(h = 0\) and \(\rho = \{0\}_k\), then the natural composite

\[(\mathfrak{O}_{\mathfrak{p}_{g,0,\{0\}_k,g,r}})_{\text{red}} \to (\mathfrak{O}_{\mathfrak{p}_{g,0,\{0\}_k,g,r}}) \to \overline{\mathfrak{M}}_{g,r}\]

where \((\mathfrak{O}_{\mathfrak{p}_{g,0,\{0\}_k,g,r}})_{\text{red}}\) denotes the reduced stack associated with \(\mathfrak{O}_{\mathfrak{p}_{g,0,\{0\}_k,g,r}}\) is an isomorphism of \(k\)-stacks. In particular, \(\mathfrak{O}_{\mathfrak{p}_{g,0,\{0\}_k,g,r}}\) is geometrically irreducible.

**Proof.** The former assertion follows from Proposition 3.11.1 and Proposition 3.11.2. The remaining portion that we have to prove in the former assertion is the nonemptiness of \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\). It is verified as follows; if \(\rho_h^F = \{0\}_k\), then the composite

\[\overline{\mathfrak{M}}_{g,r} \to \bigotimes_{g,h,\mathfrak{c}_{g,r},\overline{\mathfrak{M}}_{g,r}} \bigotimes_{h,\mathfrak{c}_{g,r},\overline{\mathfrak{M}}_{g,r}} \to \bigotimes_{\mathfrak{c}_{g,r},\overline{\mathfrak{M}}_{g,r}}\]

factors through the closed immersion

\[\mathfrak{M}_{g,r} \to \bigotimes_{\mathfrak{c}_{g,r},\overline{\mathfrak{M}}_{g,r}} \bigotimes_{\mathfrak{c}_{g,r},\overline{\mathfrak{M}}_{g,r}} \to \bigotimes_{\mathfrak{c}_{g,r},\overline{\mathfrak{M}}_{g,r}}\]

We shall write \(\mathfrak{c}_{\rho_h^F} : \mathfrak{M}_{g,r} \to \bigotimes_{\mathfrak{c}_{g,r},\overline{\mathfrak{M}}_{g,r}}\) for the resulting closed immersion.

\(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\) may be obtained as the scheme-theoretic inverse image of \(\mathfrak{c}_{\rho_h^F}\) via the faithfully flat morphism \(\mathfrak{c}_{\rho_h^F}^{\text{H-MVF}} : \mathfrak{M}_{g,r} \to \mathfrak{O}_{\mathfrak{p}_{g,0,g,r}}\) (cf. (277)), and hence, is nonempty. The latter assertion follows from Corollary 3.10.2 and the fact that the section \(\mathfrak{c}_{\rho_h^F}^{\text{triv}} : \mathfrak{M}_{g,r} \to \mathfrak{O}_{\mathfrak{p}_{g,0,g,r}}\) factors through the closed immersion \(\mathfrak{O}_{\mathfrak{p}_{g,0,\{0\}_k,g,r}} \to \mathfrak{O}_{\mathfrak{p}_{g,0,g,r}}\). \(\square\)

**Theorem 3.12.3 (Structures of \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\) and \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\))**

Let \(h \in k\) and \(\rho \in C^{\infty}(k)\) (where we take \(\rho = \emptyset\) if \(r = 0\)). Then, \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\) and \(\mathfrak{O}_{\mathfrak{p}_{g,h,\rho,g,r}}\) may be represented, respectively, by either the empty stack or a proper
Deligne-Mumford stack over \( k \), and the natural projection \( \mathcal{O}_p_{g,h,p,g,r}^{zaz} \to \overline{\mathcal{M}}_{g,r} \) is finite. Moreover, the following assertions hold:

(i) If \( h \in k^\times \), then the natural morphism

\[
\bigotimes_{\rho' \in c^{\times r}(\mathbb{F}_p)} \mathcal{O}_p_{g,h,h',p,g,r} \to \mathcal{O}_p_{g,h,g,r}
\]

(cf. (277) for the definition of \( c^r \)) is an isomorphism of stacks over \( \overline{\mathcal{M}}_{g,r} \). If, moreover, \( r = 0 \), then \( \mathcal{O}_p_{g,0,0,g,0}^{zaz} \) is nonempty (cf. Remark 3.12.4 below).

(ii) If \( h = 0 \), then it is necessarily satisfied that \( \rho = [0]_k \), and the natural composite

\[
(\mathcal{O}_p_{g,0,[0]_{k,g},g,r})_{\text{red}} \to (\mathcal{O}_p_{g,0,[0]_{k,g},g,r}) \to \overline{\mathcal{M}}_{g,r}
\]

where \( (\mathcal{O}_p_{g,0,[0]_{k,g},g,r})_{\text{red}} \) denotes the reduced stack associated with \( \mathcal{O}_p_{g,0,[0]_{k,g},g,r} \) is an isomorphism of \( k \)-stacks. In particular, \( \mathcal{O}_p_{g,0,0,g,0}^{zaz} \) is nonempty and geometrically irreducible of dimension \( 3g - 3 + r \).

**Proof.** The former assertion follows from the fact that both \( \mathcal{O}_p_{g,h,g,r} \) and \( \mathcal{O}_p_{g,h,p,g,r} \) are closed substacks of \( \mathcal{O}_p_{g,h,p,g,r} \). Assertion (i) follows from Proposition 3.5.2 (i) and Proposition 3.6.3. Assertion (ii) follows from Proposition 3.5.2 (ii), Proposition 3.12.2, and the fact that the section \( \mathcal{O}_p_{g,h,p,g,r} \) factors through the closed immersion \( \mathcal{O}_p_{g,0,0,g,0} \to \mathcal{O}_p_{g,0,0,g,r} \).

**Remark 3.12.4.**

(i) Let \( h \in k \) and \( \rho \in c^{\times r}(\mathbb{F}_p) \) (where we take \( \rho = 0 \) if \( r = 0 \)). The closed immersion \( \mathcal{O}_p_{g,0,0,h,g,0} \to \mathcal{O}_p_{g,0,0,h,p,g,r} \) arising from the discussion in Remark 2.11.2 (i) induces, by restricting to closed substacks, closed immersions

\[
\mathcal{O}_p_{g,0,0,h,p,g,r}^{zaz} \to \mathcal{O}_p_{g,0,0,h,p,g,r}^{zaz}
\]

and \( \mathcal{O}_p_{g,0,0,h,p,g,r}^{p-nilp} \to \mathcal{O}_p_{g,0,0,h,p,g,r}^{p-nilp} \)

of (relative) \( \overline{\mathcal{M}}_{g,r} \)-schemes (cf. Proposition 3.6.2).

If \( r = 0 \) and \( h = 1 \), then \( \mathcal{O}_p_{g,0,0,1,0} \) is known to be nonempty (cf. [35], Theorem 5.4.1; [27], Corollary 5.4). It follows that \( \mathcal{O}_p_{g,0,0,1,0}^{zaz} \) (as well as \( \mathcal{O}_p_{g,0,0,h,0}^{zaz} \) for any \( h \in k^\times \) by virtue of Proposition 3.9.1) is nonempty.

(ii) Also, the isomorphism \( \mathcal{O}_p_{g,0,0,1,0} \) obtained in Remark 2.11.2 (ii) induces isomorphisms

\[
\mathcal{O}_p_{g,0,0,1,0,0} \cong \mathcal{O}_p_{g,0,0,1,0,0} \times \overline{\mathcal{M}}_{g,r} \overline{\mathcal{M}}_{g,r}
\]

and

\[
\mathcal{O}_p_{g,0,0,1,0,0}^{p-nilp} \cong \mathcal{O}_p_{g,0,0,1,0,0}^{p-nilp} \times \overline{\mathcal{M}}_{g,r} \overline{\mathcal{M}}_{g,r}
\]
of (relative) $\overline{M}_{g,r}$-schemes.

4. Vector bundles associated with opers

This section is devoted to study $(\mathfrak{g}, \hbar)$-opers of the case where $\mathfrak{g} = \mathfrak{sl}_n$. We shall consider translation of them into various equivalent mathematical objects, including certain kinds of (logarithmic) differential operators on a line bundle (cf. § 4.5). Also, we introduce the moduli functors classifying such objects respectively. One goal of this section is to construct a canonical sequence of isomorphisms

$$
\hbar \cdot \text{Diff}_{n, U, X/S}^{\log} \rightarrow \mathcal{O}p_{\mathfrak{gl}_n, \hbar, U, X/S} \rightarrow \mathcal{O}p_{\mathfrak{gl}_n, \hbar, U, X/S} \rightarrow \mathcal{O}p_{\mathfrak{sl}_n, \hbar, X/S}
$$

between these functors (cf. Corollary 4.11.3), where we write

$$
\mathcal{O}p_{\mathfrak{sl}_n, \hbar, X/S} := \mathcal{O}p_{\mathfrak{sl}_n, \hbar, X/S}
$$

for convenience of notation. In particular, we shall prove the representability of these moduli functors by a certain relative affine space.

For $\mathfrak{g}, U/S$ as before, we shall write

$$
\mathcal{O}p_{\mathfrak{g}, \hbar, U/S} : \mathcal{E}t/U \rightarrow \mathcal{S}et
$$

for the $\mathcal{S}et$-valued functor on $\mathcal{E}t/U$ which, to any étale $U$-scheme $U'$, assigns the set of isomorphism classes of $(\mathfrak{g}, \hbar)$-opers on $U'/S$. In particular, the natural functor $\mathcal{O}p_{\mathfrak{g}, \hbar, U/S}(U') \rightarrow \mathcal{O}p_{\mathfrak{g}, \hbar, U/S}(U'')$ is an equivalence of categories (cf. Proposition 2.2.5), and $\mathcal{O}p_{\mathfrak{g}, \hbar, U/S}$ is, in fact, a $\mathcal{S}et$-valued sheaf on $\mathcal{E}t/U$.

In this section, we fix a positive integer $n$ and $\hbar \in \Gamma(S, \mathcal{O}_S)$.

4.1. Let $T^{\log} := (T, \alpha_T : M_T \rightarrow \mathcal{O}_T), Y^{\log} := (Y, \alpha_Y : M_Y \rightarrow \mathcal{O}_Y)$ be fine log schemes (cf. § 1.2) and $f^{\log} : Y^{\log} \rightarrow T^{\log}$ a log smooth morphism. Also, let $\nabla$ be a rank $n$ ($> 0$) vector bundle on $Y$ (or, more generally, an $\mathcal{O}_Y$-module). By a $T$-$\hbar$-log connection on $\nabla$, we mean (cf. [22], § 2.2) an $f^{-1}(\mathcal{O}_T)$-linear morphism

$$
\nabla : \nabla \rightarrow \Omega_{Y^{\log}/T^{\log}} \otimes \nabla
$$

satisfying the condition that

$$
\nabla(a \cdot m) = \hbar \cdot d_{Y^{\log}/T^{\log}}(a) \otimes m + a \cdot \nabla(m)
$$
for local sections \(a \in \mathcal{O}_Y, m \in \mathcal{V}\), where \(d_{Y^\log/T^\log}\) denotes the universal logarithmic derivation \(\mathcal{O}_Y \to \Omega_{Y^\log/T^\log}\). As we shall prove in Remark 4.1.1 below, one may identify (cf. [40], Lemma 2.2.3, for the case where \(Y\) is a smooth scheme over an algebraically closed field of characteristic 0) a \(T\)-\(h\)-log connection on \(\mathcal{V}\) with a \(T\)-\(h\)-log connection on the \(\GL_n\)-torsor associated with \(\mathcal{V}\) (in the sense of Definition 1.2.1 (i)).

An \(h\)-log integrable vector bundle on \(Y^\log/T^\log\) (of rank \(n\)) is a pair \((\mathcal{F}, \nabla_{\mathcal{F}})\) consisting of a vector bundle on \(Y\) (of rank \(n\)) and a \(T\)-\(h\)-log connection \(\nabla_{\mathcal{F}}\) on \(\mathcal{F}\). If, moreover, \(\mathcal{F}\) is of rank 1, then we shall say that \((\mathcal{F}, \nabla_{\mathcal{F}})\) is an \(h\)-log integrable line bundle on \(Y^\log/T^\log\). For simplicity, a 1-log integrable vector bundle may be often referred to as a log integrable vector bundle.

Let \((\mathcal{F}, \nabla_{\mathcal{F}})\) be an \(h\)-log integrable vector bundle on \(Y^\log/T^\log\). We shall write \(\text{det}(\nabla_{\mathcal{F}})\) (resp., \(\nabla_{\mathcal{F}}^\vee\)) for the \(T\)-\(h\)-log connection on the determinant \(\text{det}(\mathcal{F})\) (resp., the dual \(\mathcal{F}^\vee\)) of \(\mathcal{F}\) induced naturally by \(\nabla_{\mathcal{F}}\). If, moreover, \((\mathcal{G}, \nabla_{\mathcal{G}})\) is an \(h\)-log integrable vector bundle on \(Y^\log/T^\log\), then we shall write \(\nabla_{\mathcal{F}} \otimes \nabla_{\mathcal{G}}\) for the \(T\)-\(h\)-log connection on the tensor product \(\mathcal{F} \otimes \mathcal{G}\) induced by \(\nabla_{\mathcal{F}}\) and \(\nabla_{\mathcal{G}}\). Also, for \(n \geq 1\), we shall write \(\nabla_{\mathcal{F}}^\otimes n\) for the associated \(T\)-\(h\)-log connection on the \(n\)-fold tensor product \(\mathcal{F}^\otimes n\) of \(\mathcal{F}\).

**Remark 4.1.1.**

In this remark, we shall construct a bijective correspondence between the set of \(T\)-\(h\)-log connections (in the sense described above) on a rank \(n\) vector bundle \(\mathcal{V}\) on \(Y\) and the set of \(T\)-\(h\)-log connections on the \(\GL_n\)-torsor associated with \(\mathcal{V}\) (in the sense of Definition 1.2.1 (i)).

Recall (cf. [55], Definition 1.1.2) that a logarithmic derivation \(\partial\) of \(Y^\log\) over \(T^\log\) (i.e., a section of \(\mathcal{T}_{Y^\log/T^\log}\)) is, by definition, given as a pair

\[(311) \quad (D : \mathcal{O}_Y \to \mathcal{O}_Y, \delta : M_Y \to \mathcal{O}_Y),\]

where \(D\) is a (non-logarithmic) derivation on \(\mathcal{O}_Y\) and \(\delta\) is a monoid homomorphism from \(M_Y\) to the underlying additive monoid of \(\mathcal{O}_Y\) satisfying the following two conditions:

(i) \(D(\alpha_Y(m)) = \alpha_Y(m) \cdot \delta(m)\) for any local section \(m \in M_Y\).

(ii) \(D(f^*(t)) = \delta(f^*(n)) = 0\) for any local sections \(t \in \mathcal{O}_T\) and \(n \in M_T\).

Consider the sheaf \(\mathcal{E}nd_{f^{-1}(\mathcal{O}_T)}(\mathcal{V})\) of \(f^{-1}(\mathcal{O}_T)\)-linear endomorphisms of \(\mathcal{V}\), which admits a natural Lie bracket operator \([-,-]\). We have an \(f^{-1}(\mathcal{O}_T)\)-linear morphism

\[(312) \quad i : \mathcal{O}_Y \hookrightarrow \mathcal{E}nd_{f^{-1}(\mathcal{O}_Y)}(\mathcal{V})\]

which, to any local section \(a \in \mathcal{O}_Y\), assigns the locally defined endomorphism \(i(a)\) of \(\mathcal{V}\) given by multiplication by \(a\). This morphism is (since \(\mathcal{V}\) is locally free) injective, and defines a structure of \(\mathcal{O}_Y\)-bimodule on \(\mathcal{E}nd_{f^{-1}(\mathcal{O}_Y)}(\mathcal{V})\) by

\[(313) \quad (a \cdot D)(v) := (i(a) \circ \partial D)(v) \quad \text{and} \quad (\partial D \cdot a)(v) := (\partial D \circ i(a))(v),\]
where \(a, v,\) and \(\tilde{D}\) are local sections of \(\mathcal{O}_Y, V,\) and \(E_{nd}^{-1}(\mathcal{O}_T)(\mathcal{V})\) respectively. \(E_{nd}^{\mathcal{O}_Y}(\mathcal{V})\) may be thought of as an \(\mathcal{O}_Y\)-sub(bi)module of \(E_{nd}^{-1}(\mathcal{O}_T)(\mathcal{V})\), more precisely, coincides with the sub-Lie algebra consisting of local sections that commute with \(f^{-1}(\mathcal{O}_T)\).

Let us define an \(\mathcal{O}_Y\)-submodule

\[
\mathcal{D}er_{\mathcal{Y}/\mathcal{Y}/\mathcal{T}}
\]

of \(E_{nd}^{-1}(\mathcal{O}_T)(\mathcal{V})\) to be the sheaf consisting of local sections \(\tilde{D} \in E_{nd}^{-1}(\mathcal{O}_T)(\mathcal{V})\) satisfying that

\[
[D, E_{nd}^{\mathcal{O}_Y}(\mathcal{V})] \subseteq E_{nd}^{\mathcal{O}_Y}(\mathcal{V}), \text{ and } [\tilde{D}, i(\mathcal{O}_Y)] \subseteq i(\mathcal{O}_Y).
\]

Write

\[
\mathcal{D}er_{\mathcal{Y}/\mathcal{Y}/\mathcal{T}}^{\log}
\]

for the Zariski sheaf on \(Y\) which, to any open subscheme \(U\) of \(Y\), assigns the set of pairs

\[
(\tilde{D}, \tilde{\delta}),
\]

where \(\tilde{D}\) is an element of \(\Gamma(U, \mathcal{D}er_{\mathcal{Y}/\mathcal{Y}/\mathcal{T}})\) and \(\tilde{\delta}\) is a monoid homomorphism from \(M_Y|_U\) to the underlying additive monoid of \(E_{nd}^{-1}(\mathcal{O}_T)(\mathcal{V})|_U\), such that \(\tilde{\delta}(f^*(M_T)|_U) = 0\) and

\[
\tilde{D}(\alpha_Y(m) \cdot v) = \alpha_Y(m) \cdot \tilde{\delta}(m)(v)
\]

for any local sections \(m \in M_Y|_U, v \in V|_U\).

Let \(\tilde{\delta} := (\tilde{D}, \tilde{\delta})\) be a section of the sheaf \(\mathcal{D}er_{\mathcal{Y}/\mathcal{Y}/\mathcal{T}}^{\log}\). By virtue of the condition \(\mathcal{D}er_{\mathcal{Y}/\mathcal{Y}/\mathcal{T}}^{\log}\), one may obtain a well-defined \(f^{-1}(\mathcal{O}_T)\)-linear endomorphism

\[
D_{\tilde{\delta}} := i^{-1}([\tilde{D}, i(-)]) : \mathcal{O}_Y \to \mathcal{O}_Y
\]

of \(\mathcal{O}_Y,\) which forms a derivation on \(\mathcal{O}_Y\). Observe the following sequence of equalities:

\[
\alpha_Y(m) \cdot (-\tilde{D} + \tilde{\delta}(m))(v) = -\alpha_Y(m) \cdot \tilde{D}(v) + \alpha_Y(m) \cdot \tilde{\delta}(m)(v)
\]

\[
= -\alpha_Y(m) \cdot \tilde{D}(v) + \tilde{D}(\alpha_Y(m) \cdot v)
\]

\[
= D_{\tilde{\delta}}(\alpha_Y(m)) \cdot v,
\]

where \(m\) and \(v\) are local sections of \(M_Y\) and \(V\) respectively. Hence, we have

\[
i(\alpha_Y(m)) \circ (-\tilde{D} + \tilde{\delta}(m)) \in i(\mathcal{O}_Y).
\]

But, since \(f^{\log}\) is log smooth, one verifies, by means of \([39, \text{Theorem 3.5}]), that \(-\tilde{D} + \tilde{\delta}(m)\) lies in \(i(\mathcal{O}_Y)\). The resulting morphism

\[
\delta_{\tilde{\delta}} := i^{-1}(-\tilde{D} + \tilde{\delta}(-)) : M_Y \to \mathcal{O}_Y
\]
is verified to be a monoid homomorphism. The sequence of equalities \((3.20)\)
also implies that the pair
\[
(323)\quad \alpha^\log_V(\tilde{\delta}) := (D_{\tilde{\partial}}, \delta_{\tilde{\partial}})
\]
satisfies the conditions (i) (as well as the condition (ii)) described above. Thus, \(\alpha^\log_V(\tilde{\delta})\) forms a logarithmic derivation of \(Y^\log\) over \(T^\log\), equivalently, determines a section of \(\mathcal{T}^\log_{Y^\log / T^\log}\).

The assignment \(\tilde{\delta} \mapsto \alpha^\log_V(\tilde{\delta})\) determines an \(O_Y\)-linear morphism
\[
(324)\quad \alpha^\log_V : \mathcal{D}er^\log_{V/Y/T} \to \mathcal{T}^\log_{Y^\log / T^\log},
\]
where we equip \(\mathcal{D}er_{V/Y/T}\) with a structure of \(O_Y\)-module given by the multiplication \((a, \tilde{D}) \mapsto a \cdot \tilde{D}\) (for \(a \in O_Y, \tilde{D} \in End_{f^{-1}(O_T)(V)}\)) described in \((3.13)\).

The morphism
\[
(325)\quad \mathcal{E}nd_{O_Y}(V) \to \text{Ker}(\alpha^\log_V)
\]
\[
\tilde{D} \mapsto (\tilde{D}, \text{const}_{\tilde{D}}),
\]
where \(\text{const}_{\tilde{D}}\) denotes the constant map \(M_Y \to \mathcal{E}nd_{f^{-1}(O_T)(V)}\) with constant value \(\tilde{D}\), is an isomorphism. Here, we shall write \(\pi : E_V \to Y\) for the right \(GL_n\)-torsor over \(Y\) associated with \(V\). Then, \(\mathcal{E}nd_{O_Y}(V)\) is canonically isomorphic to \(\mathfrak{gl}_{E_V}\). Moreover, by noting that \(\mathcal{D}er_{V/Y/T}\) is canonically isomorphic to \(\mathfrak{T}_{E_V/T} (:= \pi_* (\mathcal{T}_{E_V/T})^{GL_n})\) (cf. [40], Lemma 2.2.3), one verifies that there exists a canonical \(O_Y\)-linear isomorphism
\[
(326)\quad \mathcal{U} : \mathfrak{T}_{E_V^\log / T^\log} \cong \mathcal{D}er^\log_{V/Y/T}
\]
which fits into an isomorphism of short exact sequences
\[
(327)\quad \begin{array}{cccccc}
0 & \longrightarrow & \mathfrak{gl}_{E_V} & \longrightarrow & \mathfrak{T}_{E_V^\log / T^\log} & \longrightarrow & \mathcal{T}_{Y^\log / T^\log} \\
& & \downarrow & & \downarrow & & \downarrow \text{id} \\
& & \mathcal{U} & & \text{id} & & \\
0 & \longrightarrow & \mathcal{E}nd_{O_Y}(V) & \longrightarrow & \mathcal{D}er_{V/Y/T} & \longrightarrow & \mathcal{T}_{Y^\log / T^\log} & \longrightarrow & 0,
\end{array}
\]
where the upper horizontal sequence is \((26)\) for the case where the pair \((G, \mathcal{E})\) is taken to be \((GL_n, E_V)\).

Now, suppose that we are given a \(T\)-\(h\)-log connection \(\nabla : \mathcal{T}_{Y^\log / T^\log} \to \mathfrak{T}_{E_V^\log / T^\log}\) on \(E_V\) (in the sense of Definition 1.2.1 (i)). Then, there exists uniquely a \(T\)-\(h\)-log connection \(\nabla_V : V \to \Omega_{Y^\log / T^\log} \otimes V\) on \(V\) (in the sense described above) such that
\[
(328)\quad (\nabla_V(v), \partial) = \tilde{D}_{\text{Un}(\partial)}(v)
\]
for any local sections \(\partial \in \mathcal{T}_{Y^\log / T^\log}\) and \(v \in V\), where
\( \langle - , - \rangle \) denotes the pairing \((\Omega^\log_{Y/T} \otimes V) \times T^\log_{Y/T} \to V\) induced by the natural pairing \(\Omega^\log_{Y/T} \times T^\log_{Y/T} \to \mathcal{O}_Y\);

\( \tilde{D}_Y \circ \nabla(\partial) \) denotes the first factor of \(\mathcal{U} \circ \nabla(\partial) \in \mathcal{D}el^\log_{V/Y/T}\) under the expression of \(\mathcal{U} \circ \nabla(\partial)\) as in [317].

This assignment \(\nabla \mapsto \nabla\) gives the bijective correspondence between the set of \(T\text{-}\hbar\text{-log}\) connections on \(E\) and the set of \(T\text{-}\hbar\text{-log}\) connections on \(V\), as desired.

4.2. Now, we define the notion of a \((\text{GL}_n, \hbar)\)-oper and a certain equivalence relation on the set of \((\text{GL}_n, \hbar)\)-opers. As asserted in Proposition 4.3.2, the quotient set of \((\text{GL}_n, \hbar)\)-opers by this equivalence relation corresponds bijectively to the isomorphic classes of \((\mathfrak{sl}_n, \hbar)\)-opers.

Let \(k\) be a field of arbitrary characteristic, \(S\) a scheme over \(k\), \(X/S := (f : X \to S, \{\sigma_i\}_{i=1}^r)\) a pointed stable curve over \(S\) of type \((g, r)\), and \(u : U \to X\) an étale morphism of \(k\)-schemes.

**Definition 4.2.1.**

(i) A \((\text{GL}_n, \hbar)\)-oper on \(U/S\) is a collection of data

\[
\mathcal{F}^\circ := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}_j\}_{j=0}^n),
\]

where

- \(\mathcal{F}\) is a vector bundle on \(U\) of rank \(n\);
- \(\nabla_{\mathcal{F}}\) is an \(S\text{-}\hbar\text{-log}\) connection \(\mathcal{F} \to \Omega^\log_{U/S} \otimes \mathcal{F}\) on \(\mathcal{F}\);
- \(\{\mathcal{F}_j\}_{j=0}^n\) is a decreasing filtration

\[
0 = \mathcal{F}^n \subseteq \mathcal{F}^{n-1} \subseteq \cdots \subseteq \mathcal{F}^0 = \mathcal{F}
\]

on \(\mathcal{F}\) by vector bundles on \(U\)

satisfying the following conditions:

1. The subquotients \(\mathcal{F}^j/\mathcal{F}^{j+1}\) \((0 \leq j \leq n-1)\) are line bundles;
2. \(\nabla_{\mathcal{F}}(\mathcal{F}^j) \subseteq \Omega^\log_{U/S} \otimes \mathcal{F}^{j-1} \) \((1 \leq j \leq n-1)\);
3. The \(\mathcal{O}_U\text{-linear}\) morphism

\[
\kappa_{\mathcal{F}}^j : \mathcal{F}^j/\mathcal{F}^{j+1} \to \Omega^\log_{U/S} \otimes (\mathcal{F}^{j-1}/\mathcal{F}^j)
\]

defined by assigning \(\pi \mapsto \overline{\nabla_{\mathcal{F}}(a)}\) for any local section \(a \in \mathcal{F}^j\)

(where \(\overline{(-)}\)'s denote the images in the respective quotients), which is well-defined by virtue of the condition (2), is an isomorphism.

For simplicity, a \((\text{GL}_n, 1)\)-oper may be often referred to as a \(\text{GL}_n\text{-oper}\).

(ii) Let \(\mathcal{F}^\circ := (\mathcal{F}, \nabla_{\mathcal{F}}, \{\mathcal{F}_j\}_{j=0}^n)\), \(\mathcal{G}^\circ := (\mathcal{G}, \nabla_{\mathcal{G}}, \{\mathcal{G}_j\}_{j=0}^n)\) be \((\text{GL}_n, \hbar)\)-opers on \(U/S\). An isomorphism from \(\mathcal{F}^\circ\) to \(\mathcal{G}^\circ\) is an isomorphism \(\mathcal{F}, \nabla_{\mathcal{F}} \simeq (\mathcal{G}, \nabla_{\mathcal{G}})\) of \(\hbar\text{-log}\) integrable vector bundles that is compatible with the respective filtrations \(\{\mathcal{F}_j\}_{j=0}^n\) and \(\{\mathcal{G}_j\}_{j=0}^n\).
Let \( \mathcal{F}^\triangledown := (\mathcal{F}, \nabla_\mathcal{F}, \{\mathcal{F}^j\}_{j=0}^n) \) be a \((\text{GL}_n, \hbar)\)-oper on \( \mathcal{U}/S \), and \((\mathcal{L}, \nabla_\mathcal{L})\) an \( \hbar \)-log integrable line bundle on \( \mathcal{U}^{\log}/S^{\log} \). We regard \( \{\mathcal{F}^j \otimes \mathcal{L}\}_{j=0}^n \) as a decreasing filtration on \( \mathcal{F} \otimes \mathcal{L} \). One verifies easily that the collection of data
\[
(332) \quad \mathcal{F}^\triangledown \otimes \mathcal{L} := (\mathcal{F} \otimes \mathcal{L}, \nabla_\mathcal{F} \otimes \nabla_\mathcal{L}, \{\mathcal{F}^j \otimes \mathcal{L}\}_{j=0}^n)
\]
(cf. § 4.1 for the definition of \( \nabla_\mathcal{F} \otimes \nabla_\mathcal{L} \)) forms a \((\text{GL}_n, \hbar)\)-oper on \( \mathcal{U}/S \).

**Definition 4.2.2.**

Let \( \mathcal{F}^\triangledown := (\mathcal{F}, \nabla_\mathcal{F}, \{\mathcal{F}^j\}_{j=0}^n) \), \( \mathcal{G}^\triangledown := (\mathcal{G}, \nabla_\mathcal{G}, \{\mathcal{G}^j\}_{j=0}^n) \) be \((\text{GL}_n, \hbar)\)-opers on \( \mathcal{U}/S \).

We shall say that \( \mathcal{F}^\triangledown \) is equivalent to \( \mathcal{G}^\triangledown \) if there exists an \( \hbar \)-log integrable line bundle \((\mathcal{L}, \nabla_\mathcal{L})\) on \( \mathcal{U}^{\log}/S^{\log} \) such that the \((\text{GL}_n, \hbar)\)-oper \( \mathcal{F}^\triangledown \otimes \mathcal{L} \) is isomorphic (cf. Definition 4.2.1 (ii)) to \( \mathcal{G}^\triangledown \).

If \( u' : U' \to U \) is an étale morphism and \( \mathcal{F}^\triangledown := (\mathcal{F}, \nabla_\mathcal{F}, \{\mathcal{F}^j\}_{j=0}^n) \) is a \((\text{GL}_n, \hbar)\)-oper on \( \mathcal{U}/S \), then its restriction
\[
(333) \quad u'^* (\mathcal{F}^\triangledown) := (u'^* (\mathcal{F}), u'^* (\nabla_\mathcal{F}), \{u'^* (\mathcal{F}^j)\}_{j=0}^n)
\]
to \( U' \) forms a \((\text{GL}_n, \hbar)\)-oper on \( \mathcal{U}' \). The notion of restriction is compatible with both composition of étale morphisms and the equivalence relation defined above. Thus, one obtains the \( \text{Set} \)-valued sheaf
\[
(334) \quad \mathcal{O}_{\text{GL}_n, \mathcal{U}/S} : \mathcal{E}_U \to \text{Set}
\]
on \( \mathcal{E}_U \) associated with the presheaf which, to any étale \( U \)-scheme \( U' \), assigns the set of equivalence classes of \((\text{GL}_n, \hbar)\)-opers on \( \mathcal{U}' \).

**4.3.** For the rest of this section, we assume that

the characteristic \( \text{char}(k) \) of the base field \( k \) is either zero or a prime \( p \) satisfying that \( n < p \).

This assumption is equivalent to the condition that either \((\text{Char})_0\) or \((\text{Char})^W_p\) for the case where \( \mathbb{G} \) is taken to be \( \text{PGL}_n \) (i.e., the projective linear algebraic group) is satisfied. In particular, under this assumption, the Lie algebra of \( \text{PGL}_n \), which is of adjoint type, is isomorphic to \( \mathfrak{sl}_n \) (via the composite isomorphism \( \mathfrak{sl}_n \hookrightarrow \mathfrak{gl}_n \to \mathfrak{p} \mathfrak{g} \mathfrak{l}_n \)). The following lemma will be elemental, but shows clearly how to apply this assumption to the discussions in the rest of this section.

**Lemma 4.3.1.**

(i) Let \( \mathcal{L} \) be a line bundle on \( U \). Suppose that we are given an \( S \)-\( \hbar \)-log connection \( \nabla_{\mathcal{L}^{\otimes n}} \) on the \( n \)-fold tensor product \( \mathcal{L}^{\otimes n} \) of \( \mathcal{L} \). Then, there exists uniquely an \( S \)-\( \hbar \)-log connection \( \nabla \) on \( \mathcal{L} \) satisfying that \( \nabla^{\otimes n} = \nabla_{\mathcal{L}^{\otimes n}} \) (cf. § 4.1).
(ii) Let $E_{GL_n}$ be a GL$_n$-torsor over $U$. Suppose that we are given a pair $(\nabla_{PGL_n}, \nabla_{G_m})$, where

- $\nabla_{PGL_n}$ denotes an $S$-$\h$-log connection on the PGL$_n$-torsor $E_{PGL_n} := E \times^{GL_n} PGL_n$ induced by $E_{GL_n}$ via the natural quotient $\mu_{PGL_n} : GL_n \rightarrow PGL_n$;
- $\nabla_{G_m}$ denotes an $S$-$\h$-log connection on the $G_m$-torsor $E_{G_m} := E \times^{GL_n} G_m$ induced by $E_{GL_n}$ via the determinant map $\mu_{G_m} : GL_n \rightarrow G_m$.

Then, there exists uniquely an $S$-$\h$-log connection $\nabla_{GL_n}$ on $E$ which induces $\nabla_{PGL_n}$, $\nabla_{G_m}$ via $\mu_{PGL_n}$, $\mu_{G_m}$, respectively.

Proof. Consider the morphism

$$ (335) \quad \tilde{\tau}_{(L^x)^{\log}/S^{\log}} \rightarrow \tilde{\tau}_{(L^{\otimes n})^{\log}/S^{\log}} $$

(cf. §3.7 for the definition of $(-)^x$) induced by the $n$-th power morphism $L^x \rightarrow (L^{\otimes n})^x$ (resp., the morphism $(\mu_{PGL_n}, \mu_{G_m}) : GL_n \rightarrow PGL_n \times_k G_m$). Since this $n$-th power morphism (resp., the morphism $(\mu_{PGL_n}, \mu_{G_m})$) is étale by the assumption on char($k$), the morphism (335) is verified to be an isomorphism. This implies assertion (i) (resp., assertion (ii)).

We shall construct a morphism of sheaves $\Lambda_{n,h,U/S} : \mathcal{O}P_{GL_n,h,U/S} \rightarrow \mathcal{O}P_{\mathfrak{sl}_n,h,U/S}$.

In the following, we shall write

$$ (336) \quad B (\subseteq PGL_n) $$

for the Borel subgroup of PGL$_n$ defined to be the image of upper triangular matrices in GL$_n$. Let $U' : U' \rightarrow U$ be an étale morphism of $k$-schemes, $\mathcal{F}^\circ = (F, \nabla_F, \{\mathcal{F}^j\}_{j=0}^n)$ a (GL$_n$, $\h$)-oper on $U'/S$. Denote by $(\mathcal{F}_{PGL_n}, \nabla_{PGL_n})$ the $\h$-log integrable PGL$_n$-torsor on $U'^{\log}/S^{\log}$ arising from $(\mathcal{F}, \nabla_F)$ via the change of structure group GL$_n \rightarrow PGL_n$. The data of the filtration $\{\mathcal{F}^j\}_{j=0}^n$ may be interpreted as giving a $B$-reduction $\mathcal{F}_B$ of $\mathcal{F}_{PGL_n}$. It follows from the definition of a (GL$_n$, $\h$)-oper that the pair $\mathcal{F}^\bullet := (\mathcal{F}_B, \nabla_{PGL_n})$ forms an (sl$_n$, $\h$)-oper on $U'/S$. The isomorphism class $[\mathcal{F}^\bullet]$ of the (sl$_n$, $\h$)-oper $\mathcal{F}^\bullet$ does not depend on the choice of a representative in the equivalence class defining $\mathcal{F}^\circ$. Moreover, the assignment $\mathcal{F}^\circ \mapsto [\mathcal{F}^\bullet]$ is compatible with restriction to any étale $U'$-schemes, and hence, determines a morphism

$$ (337) \quad \Lambda_{n,h,U/S} : \mathcal{O}P_{GL_n,h,U/S} \rightarrow \mathcal{O}P_{\mathfrak{sl}_n,h,U/S} $$

of sheaves.

**Proposition 4.3.2.**

$\Lambda_{n,h,U/S}$ is an isomorphism.
Proof. We only consider the injectivity of $\Lambda^\triangleright \rightarrow \bullet$ (since the surjectivity is technically much simpler). It suffices to prove that if $\mathcal{F}^\triangleright := (\mathcal{F}, \nabla_\mathcal{F}, \{\mathcal{F}^j\}_{j=0}^n)$ and $\mathcal{G}^\triangleright := (\mathcal{G}, \nabla_\mathcal{G}, \{\mathcal{G}^j\}_{j=0}^n)$ are (GL$_n$, $\hbar$)-opers on $U/S$ whose associated $(\mathfrak{sl}_n, \hbar)$-opers $\mathcal{F}^\bullet := (\mathcal{F}_B, \nabla_{\mathcal{F}_{PGL_n}})$ and $\mathcal{G}^\bullet := (\mathcal{G}_B, \nabla_{\mathcal{G}_{PGL_n}})$ are isomorphic, then $\mathcal{F}^\triangleright$ is equivalent to $\mathcal{G}^\triangleright$. Let us suppose this hypothesis, i.e., that there exists an isomorphism

$$h_{PGL_n} : \mathcal{F}_{PGL_n} \rightarrow \mathcal{G}_{PGL_n}$$

that is compatible with the respective $B$-reductions $\mathcal{F}_B$, $\mathcal{G}_B$ and $S$-$\hbar$-connections $\nabla_{\mathcal{F}_{PGL_n}}$, $\nabla_{\mathcal{G}_{PGL_n}}$. Then, there exist a line bundle $\mathcal{L}$ on $U$ and an isomorphism $h : \mathcal{F} \rightarrow \mathcal{G} \otimes \mathcal{L}$. Let us denote by

$$h_{det} : \det(\mathcal{F}) \rightarrow \det(\mathcal{G}) \otimes \mathcal{L}^\otimes n$$

the isomorphism induced by $h$, and define an $S$-$\hbar$-log connection $\nabla_{\mathcal{L}^\otimes n}$ on $\mathcal{L}$ to be $\nabla_{\mathcal{L}^\otimes n} := \det(\nabla_\mathcal{G})^\vee \otimes \det(\nabla_\mathcal{F})$ via the isomorphism $id_{\det(\mathcal{G})^\vee} \otimes h_{det} : \det(\mathcal{G})^\vee \otimes \det(\mathcal{F}) \rightarrow \mathcal{L}^\otimes n$. By Lemma 4.3.1 (i), there exists uniquely an $S$-$\hbar$-log connection $\nabla_\mathcal{L}$ on $\mathcal{L}$ satisfying that $\nabla_{\mathcal{L}^\otimes n} = \nabla_{\mathcal{L}^\otimes n}$. Thus, we have obtained an $\hbar$-log integrable line bundle $(\mathcal{L}, \nabla_\mathcal{L})$ for which both $h_{PGL_n}$ and $h_{det}$ are compatible with the respective $S$-$\hbar$-connections on the domain and codomain. By the asserted uniqueness of Lemma 4.3.1 (ii), $h$ is compatible with the respective $S$-$\hbar$-log connections $\nabla_\mathcal{F}$ and $\nabla_\mathcal{G}$. Finally, if $B_{GL_n}$ denotes the Borel subgroup of GL$_n$ consisting of upper triangular matrices, then $B_{GL_n} \rightarrow B \times_{PGL_n} GL_n$. Hence, since the isomorphism $h_{PGL_n}$ is compatible with the respective $B$-reduction structures, $h$ is compatible with the respective filtrations $\{\mathcal{F}^j\}_{j=0}^n$ and $\{\mathcal{G}^j \otimes \mathcal{L}\}_{j=0}^n$. Consequently, $\mathcal{F}^\triangleright$ is equivalent to $\mathcal{G}^\triangleright$, and this completes the proof of the injectivity of $\Lambda^\triangleright \rightarrow \bullet$. \hfill $\square$

4.4. We define the sheaf of $\hbar$-twisted logarithmic crystalline differential operators (or $\hbar$-tlcdo’s for short)

$$\mathcal{D}_{h,U^{log}/S^{log}}^{\leq \infty}$$

on $U^{log}$ over $S^{log}$ to be the Zariski sheaf on $U$ generated, as a sheaf of rings, by $\mathcal{O}_U$ and $\mathcal{T}_{U^{log}/S^{log}}$ subject to the following relations:

- $f_1 \ast f_2 = f_1 \cdot f_2$;
- $f_1 \ast \xi_1 = f_1 \cdot \xi_1$;
- $\xi_1 \ast \xi_2 - \xi_2 \ast \xi_1 = \hbar \cdot [\xi_1, \xi_2]$;
- $f_1 \ast \xi_1 - \xi_1 \ast f_1 = \hbar \cdot \xi_1(f_1)$,

for local sections $f_1$, $f_2 \in \mathcal{O}_U$ and $\xi_1$, $\xi_2 \in \mathcal{T}_{U^{log}/S^{log}}$, where $\ast$ denotes the multiplication in $\mathcal{D}_{h,U^{log}/S^{log}}^{\leq \infty}$. 
In a usual sense, the order \((\geq 0)\) of a given \(h\)-\(t\) \(\log\) \(O\) is well-defined. Hence, \(D_{h,U,log/\log}^{\leq j}\) admits, for each \(j \geq 0\), the subsheaf
\[
D_{h,U,log/\log}^{\leq j} (\subseteq D_{h,U,log/\log}^{\leq \infty})
\]
consisting of \(h\)-\(t\) \(\log\)’s of order \(< j\). \(D_{h,U,log/\log}^{\leq j} (j = 0, 1, 2, \ldots, \infty)\) admits two different structures of \(O_U\)-module — one as given by left multiplication (where we denote this \(O_U\)-module by \(D_{h,U,log/\log}^{\leq j}\)), and the other given by right multiplication (where we denote this \(O_U\)-module by \(D_{h,U,log/\log}^{r\leq j}\)) —. In particular, we have \(D_{h,U,log/\log}^{\leq 0} = 0\) and \(D_{h,U,log/\log}^{r\leq 1} = 0 = O_U\). We shall not distinguish the \(O_U\)-module \(D_{h,U,log/\log}^{t\leq j}\) (resp., \(D_{h,U,log/\log}^{r\leq j}\)) from its associated sheaf on \(\mathcal{E}_t\) in a natural fashion (cf. the comment in Remark 2.7.4).

The set \(\{D_{h,U,log/\log}^{\leq j}\}_{j \geq 0}\) forms an increasing filtration on \(D_{h,U,log/\log}^{\leq \infty}\) satisfying that
\[
\bigcup_{j \geq 0} D_{h,U,log/\log}^{\leq j} = D_{h,U,log/\log}^{\leq \infty}, \quad \text{and} \quad D_{h,U,log/\log}^{\leq j+1}/D_{h,U,log/\log}^{\leq j} \xrightarrow{\sim} T_{U,log/\log}^{\leq j}
\]
for any \(j \geq 0\). In particular, if \(\text{gr}(D_{h,U,log/\log}^{\leq \infty})\) denotes the graded \(O_U\)-algebra associated with this filtration, then there exists a canonical isomorphism
\[
\text{gr}(D_{h,U,log/\log}^{\leq \infty}) \xrightarrow{\sim} S_{O_U}(T_{U,log/\log})
\]
of graded \(O_U\)-algebras (where the two structures of \(O_U\)-module on \(\text{gr}(D_{h,U,log/\log}^{\leq \infty})\) induced by \(D_{h,U,log/\log}^{t,\leq \infty}\) and \(D_{h,U,log/\log}^{r,\leq \infty}\) are identical).

Let \(\mathcal{V}\) be a vector bundle on \(U\) (or more generally, an \(O_U\)-module), and consider the tensor product \(D_{h,U,log/\log}^{\leq j} \otimes \mathcal{V}\) (resp., \(\mathcal{V} \otimes D_{h,U,log/\log}^{\leq j}\)) of \(\mathcal{V}\) with the \(O_U\)-module \(D_{h,U,log/\log}^{\leq j}\) (resp., \(D_{h,U,log/\log}^{\leq j}\)). In the following, we shall regard the \(D_{h,U,log/\log}^{\leq j} \otimes \mathcal{V}\) (resp., \(\mathcal{V} \otimes D_{h,U,log/\log}^{\leq j}\)) as being equipped with a structure of \(O_U\)-module arising from the structure of \(O_U\)-module \(D_{h,U,log/\log}^{\leq j}\) (resp., \(D_{h,U,log/\log}^{\leq j}\)) of \(D_{h,U,log/\log}^{\leq j}\).

Next, let \(\nabla : \mathcal{V} \rightarrow \Omega_{U,log/\log} \otimes \mathcal{V}\) be an \(S\)-\(h\)-\(log\) connection on \(\mathcal{V}\). One may associate \(\nabla\) with a structure of left \(D_{h,U,log/\log}^{\leq \infty}\)-module
\[
\nabla : D_{h,U,log/\log}^{\leq \infty} \otimes \mathcal{V} \rightarrow \mathcal{V},
\]
(which is \(O_U\)-linear) determined uniquely by the condition that \(\nabla(\partial \otimes v) = \langle \nabla(v), \partial \rangle\) for any local sections \(v \in \mathcal{V}\) and \(\partial \in T_{U,log/\log}\), where \(\langle -, - \rangle\) denotes the pairing \((\Omega_{U,log/\log} \otimes \mathcal{V}) \times T_{U,log/\log} \rightarrow \mathcal{V}\) induced by the natural paring \(T_{U,log/\log} \otimes \Omega_{U,log/\log} \rightarrow O_U\). One verifies that this assignment \(\nabla \mapsto \nabla\) determines a bijective correspondence between the set of \(S\)-\(h\)-\(log\) connections on \(\mathcal{V}\) and the set of structures of left \(D_{h,U,log/\log}^{\leq \infty}\)-module \(D_{h,U,log/\log}^{\leq \infty} \otimes \mathcal{V} \rightarrow \mathcal{V}\) on \(\mathcal{V}\). Such a bijective correspondence holds even if the condition that \(n < p\) (cf.
the beginning of § 4.3) is not necessarily satisfied (cf. the discussion following Proposition 8.3.1).

4.5. Let us fix a line bundle $\mathcal{B}$ on $U$. For $j = 0, \ldots, n$, we shall define $\mathfrak{B}^j_{U^\log/S^\log}$ (resp., $\mathfrak{B}^j_{U^{\log}/S^{\log}}$) to be the $\mathcal{O}_U$-module

\begin{equation}
\mathfrak{B}^j_{U^\log/S^\log} := \mathcal{H}om_{\mathcal{O}_U}(\mathcal{B}, (\Omega^\otimes_{U^\log/S^\log} \otimes \mathcal{B}) \otimes \mathcal{D}^j_{h,U^\log/S^\log})
\end{equation}

(resp., $\mathfrak{B}^j_{U^{\log}/S^{\log}} := \mathcal{H}om_{\mathcal{O}_U}(\mathcal{T}^\otimes_{U^{\log}/S^{\log}} \otimes \mathcal{B}^j, \mathcal{D}^j_{h,U^\log/S^\log} \otimes \mathcal{B}^j)$).

(Here, we shall recall (cf. § 4.4) that the structure of $\mathcal{O}_U$-module on $(\Omega^\otimes_{U^\log/S^\log} \otimes \mathcal{B}) \otimes \mathcal{D}^j_{h,U^\log/S^\log}$ and $(\mathfrak{B}^j_{U^\log/S^\log} \otimes \mathcal{B}^j)$ mentioned in § 4.4.) We shall refer to a global section of $\mathfrak{B}^j_{U^\log/S^\log}$ as an $h$-twisted logarithmic crystalline differential operator on $\mathcal{B}$ (or $h$-tlcd on $\mathcal{B}$ for short) of order $< j$.

The set $\{\mathfrak{B}^j_{U^\log/S^\log}\}_{j=0}^n$ (resp., $\{\mathfrak{B}^j_{U^{\log}/S^{\log}}\}_{j=0}^n$) forms an increasing filtration on $\mathfrak{B}^n_{U^\log/S^\log}$ (resp., $\mathfrak{B}^n_{U^{\log}/S^{\log}} = 0$ (resp., $\mathfrak{B}^0_{U^{\log}/S^{\log}}$). The isomorphism $\mathcal{D}^j_{U^\log/S^\log} / \mathcal{D}^j_{h,U^\log/S^\log} \sim \mathcal{T}^\otimes_{U^\log/S^\log}$ of (342) yields an isomorphism

\begin{equation}
\mathfrak{B}^{j+1}_{U^\log/S^\log} / \mathfrak{B}^j_{U^\log/S^\log} \sim \Omega^{\otimes(n-j)}_{U^\log/S^\log} (\text{resp., } \mathfrak{B}^{j+1}_{U^{\log}/S^{\log}} / \mathfrak{B}^j_{U^{\log}/S^{\log}} \sim \Omega^{\otimes(n-j)}_{U^{\log}/S^{\log}})
\end{equation}

for each $j = 0, \ldots, n$. Write

\begin{equation}
\mathfrak{Sym}^{j+1} : \mathfrak{B}^{j+1}_{U^\log/S^\log} \rightarrow \Omega^{\otimes(n-j)}_{U^\log/S^\log} (\text{resp., } \mathfrak{Sym}^j : \mathfrak{B}^j_{U^{\log}/S^{\log}} \rightarrow \Omega^{\otimes(n-j)}_{U^{\log}/S^{\log}})
\end{equation}

\begin{align*}
\text{for the composite of the quotients } &\mathfrak{B}^{j+1}_{U^\log/S^\log} \twoheadrightarrow \mathfrak{B}^{j+1}_{U^{\log}/S^{\log}} / \mathfrak{B}^j_{U^{\log}/S^{\log}} \text{ (resp., } \\
&\mathfrak{B}^{j+1}_{U^\log/S^\log} \twoheadrightarrow \mathfrak{B}^{j+1}_{U^{\log}/S^{\log}} / \mathfrak{B}^j_{U^{\log}/S^{\log}})
\end{align*}

and the isomorphism (346).

Consider the composite isomorphism

\begin{equation}
\circ \beta^j : \mathfrak{B}^j_{U^\log/S^\log} \sim \Omega^{\otimes n}_{U^\log/S^\log} \otimes \mathcal{B} \otimes \mathcal{D}^j_{h,U^\log/S^\log} \otimes \mathcal{B}^j \sim \mathfrak{B}^j_{U^\log/S^\log}
\end{equation}

of $\mathcal{O}_U$-bimodules, where the first and second isomorphisms follow from the definition of $\mathfrak{B}^j_{U^\log/S^\log}$ and $\mathfrak{B}^j_{U^{\log}/S^{\log}}$ respectively. Then, one verifies that

\begin{equation}
\mathfrak{Sym}^{j'} \circ \circ \beta^j = \mathfrak{Sym}^j \text{ and } \circ \beta^j \mid_{\mathfrak{B}^j_{U^{\log}/S^{\log}}} = \circ \beta^{j'}
\end{equation}

for each $j' \leq j$.

We shall define the sheaf $h-Diff_{n,B,U^\log/S^\log}$ on $Et/U$ to be

\begin{equation}
h-Diff_{n,B,U^\log/S^\log} := (\mathfrak{Sym}^{n+1})^{-1}(1) (\subset \mathfrak{B}^{n+1}_{U^\log/S^\log}).
\end{equation}

i.e., the sheaf consisting of $h$-tlced $\mathcal{D}$ on $\mathcal{B}$ of order $n$ with $\mathfrak{Sym}^{n+1}(\mathcal{D}) = 1 \in \mathcal{O}_U$. Note that $h-Diff_{n,B,U^\log/S^\log}$ has sections locally on $U$ and admits a natural $(\text{Ker}(\mathfrak{Sym}^{n+1})) = \mathfrak{B}^n_{U^\log/S^\log}$-torsor structure over $U$. 
4.6. Next, we shall define (cf. Definition 4.6.1 below) a certain type of 
(GL_n, h)-opers on Σ/S with a prescribed underlying filtered vector bundle.

Here, observe that \( \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}' \) is a rank \( n \) vector bundle on \( U \), which admits a decreasing filtration \( \{ \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}' \}_{j=0}^n \).

**Definition 4.6.1.**

(i) A (GL_n, h, B)-oper on \( \Sigma/S \) is an \( S \)-h-log connection \( \nabla^\bigcirc \) on \( \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}' \) such that the collection of data

\[
\nabla^\bigcirc := \{ \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}' \otimes \{ \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}' \}_{j=0}^n \}
\]

forms a (GL_n, h)-oper on \( \Sigma/S \).

(ii) Let \( \nabla_1^\bigcirc, \nabla_2^\bigcirc \) be (GL_n, h, B)-opers on \( \Sigma/S \). We shall say that \( \nabla_1^\bigcirc \) is isomorphic to \( \nabla_2^\bigcirc \) if the associated (GL_n, h)-opers \( \nabla_1^\bigcirc \) and \( \nabla_2^\bigcirc \) are isomorphic (cf. Definition 4.2.1 (ii)).

If \( u': U' \to U \) is an étale morphism and \( \nabla^\bigcirc \) is a (GL_n, h, B)-oper on \( \Sigma/S \), then the restriction \( u^*(\nabla^\bigcirc) \) forms a (GL_n, h, u^*(B))-oper on \( \Sigma'/S' \). Moreover, the notion of restriction is compatible with composition of étale morphisms. Thus, one may define an \( \text{Set} \)-valued presheaf

\[
O_{p_{GL_n,h,B,\Sigma/S}}^\bigcirc : \mathcal{E}t_U \to \text{Set}
\]

on \( \mathcal{E}t_U \) which, to any étale \( U \)-scheme \( u': U' \to U \), assigns the set of isomorphism classes of (GL_n, h, u^*(B))-opers on \( \Sigma'/S' \).

4.7. In this subsection, we discuss an interaction between \( h^{-}\text{Diff}_{n,B,U^\log/S^{\log}} \)'s as well as \( O_{p_{GL_n,h,B,\Sigma/S}}^\bigcirc \)'s for different \( B \)'s.

Let \( \mathcal{L} := (\mathcal{L}, \nabla_{\mathcal{L}}) \) be an \( h \)-log integrable line bundle on \( U^\log/S^{\log} \) and \( \nabla^\bigcirc \) a (GL_n, h, B)-oper on \( \Sigma/S \). The \( S \)-h-log connection \( \nabla_{\mathcal{L}} \otimes \nabla^\bigcirc \) (cf. §4.1) on the tensor product \( \mathcal{L} \otimes \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}' \) corresponds (cf. §4.4) to a structure of left \( \mathcal{D}_{h,U^\log/S^{\log}}^< \)-module

\[
(\nabla_{\mathcal{L}} \otimes \nabla^\bigcirc)^\mathcal{L} : \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes (\mathcal{L} \otimes \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}') \to \mathcal{L} \otimes \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}' .
\]

Consider the left \( O_{\Sigma_U} \)-linear composite

\[
\text{can}_{\nabla^\bigcirc, \mathcal{L}} : \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes (\mathcal{B}' \otimes \mathcal{L}) \to \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes (\mathcal{L} \otimes \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}') \]

\[
\to \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes (\mathcal{L} \otimes \mathcal{D}_{h,U^\log/S^{\log}}^\leq \otimes \mathcal{B}') .
\]
where the first arrow arises from the identification \( {}^t \mathcal{D}^1_{h, U^\log/S^\log} = {}^{r^c} \mathcal{D}^1_{h, U^\log/S^\log} = \mathcal{O}_U \) and the second arrow arises from the inclusions \( \mathcal{D}^{<n}_{h, U^\log/S^\log} \hookrightarrow \mathcal{D}^{<\infty}_{h, U^\log/S^\log} \). It follows from the definition of a \((\text{GL}_n, \hbar)\)-oper that the composite \( \text{can}_{\nabla_{\otimes \mathbb{L}}} \) is an isomorphism. By passing to this isomorphism, \( \nabla_L \otimes \nabla_{\mathbb{L}} \) carries an \( S\)-\( h \)-log connection on \( \mathcal{D}^{<n}_{h, U^\log/S^\log} \otimes (\mathcal{B}^\vee \otimes \mathbb{L}) \), which we denote by

\[
(355) \quad \nabla_{\otimes \mathbb{L}}^\vee.
\]

One verifies that \( \nabla_{\otimes \mathbb{L}}^\vee \) forms a \((\text{GL}_n, \hbar, \mathcal{B} \otimes \mathcal{L}^\vee)\)-oper on \( \mathcal{U}/S \) and the assignment \( \nabla^\vee \mapsto \nabla_{\otimes \mathbb{L}}^\vee \) is compatible with restriction to any étale \( U \)-scheme \( U' \). Thus, we obtain a morphism

\[
(356) \quad \nabla_{B \to \mathcal{B} \otimes \mathcal{L}^\vee} : \mathcal{O}p_{\text{GL}_n, h, \mathcal{B}, \mathcal{U}/S} \to \mathcal{O}p_{\text{GL}_n, h, \mathcal{B} \otimes \mathcal{L}^\vee, \mathcal{U}/S}
\]

of sheaves on \( \mathcal{E}_t/S \).

Next, let \( \mathcal{D}^\bullet \in \Gamma(U, h\text{-Diff}_{n, \mathcal{B} \otimes \mathcal{L}^\vee}^{\otimes \mathbb{L}}) \). The composite

\[
(357) \quad \mathcal{T}^{\otimes n}_{U^\log/S^\log} \otimes (\mathcal{B} \otimes \mathcal{L}^\vee)^\vee \to \mathcal{L} \otimes \mathcal{T}^{\otimes n}_{U^\log/S^\log} \otimes \mathcal{B}^\vee
\]

\[
\text{id}_\mathcal{L} \otimes \circ \beta^{n+1}(\mathcal{D}^\bullet) : \mathcal{L} \otimes \mathcal{D}^{<n+1}_{h, U^\log/S^\log} \otimes \mathcal{B}^\vee
\]

\[
\text{can}_{\nabla_{\otimes \mathbb{L}}^\vee} : \mathcal{D}^{<n+1}_{h, U^\log/S^\log} \otimes (\mathcal{B}^\vee \otimes \mathbb{L})
\]

corresponds, via \( \circ \beta^{n+1} \) (cf. (348)), to an element \( \mathcal{D}^\bullet \in \Gamma(U, h\text{-Diff}_{n, \mathcal{B} \otimes \mathcal{L}^\vee}^{\otimes \mathbb{L}}) \), i.e., an \( h \)-tlcdo \( \mathcal{D}^\bullet \) on \( \mathcal{B} \otimes \mathcal{L}^\vee \) of order \( n \) with \( \text{Stab}^n(\mathcal{D}^\bullet) = 1 \). The assignment \( \mathcal{D}^\bullet \mapsto \mathcal{D}^\bullet_{\otimes \mathbb{L}} \) determines a morphism

\[
(358) \quad \mathcal{D}_{B \to \mathcal{B} \otimes \mathcal{L}^\vee} : h\text{-Diff}_{n, \mathcal{B} \otimes \mathcal{L}^\vee, U^\log/S^\log} \to h\text{-Diff}_{n, \mathcal{B} \otimes \mathcal{L}^\vee, U^\log/S^\log}
\]

of sheaves on \( \mathcal{E}_t/U \).

**Proposition 4.7.1.**

*Both \( \mathcal{D}_{B \to \mathcal{B} \otimes \mathcal{L}^\vee} \) and \( \mathcal{D}_{B \to \mathcal{B} \otimes \mathcal{L}^\vee} \) are isomorphisms.*

**Proof.** The assertion follows directly from the equalities

\[
(359) \quad \mathcal{D}_{B \to \mathcal{B} \otimes \mathcal{L}^\vee} \circ \mathcal{D}_{(B \otimes \mathcal{L}) \to (B \otimes \mathcal{L}^\vee) \otimes \mathbb{L}} = \text{id}, \quad \mathcal{D}_{(B \otimes \mathcal{L}^\vee) \to (B \otimes \mathcal{L}^\vee) \otimes \mathbb{L}} \circ \mathcal{D}_{B \to \mathcal{B} \otimes \mathcal{L}^\vee} = \text{id},
\]

and similarly,

\[
(360) \quad \mathcal{D}_{B \to \mathcal{B} \otimes \mathcal{L}^\vee} \circ \mathcal{D}_{(B \otimes \mathcal{L}) \to (B \otimes \mathcal{L}^\vee) \otimes \mathbb{L}} = \text{id}, \quad \mathcal{D}_{(B \otimes \mathcal{L}^\vee) \to (B \otimes \mathcal{L}^\vee) \otimes \mathbb{L}} \circ \mathcal{D}_{B \to \mathcal{B} \otimes \mathcal{L}^\vee} = \text{id}.
\]

\( \square \)
4.8. We shall construct a morphism \( \Lambda_{n, h, B, \mathcal{U}/S} \) of sheaves from \( h\text{-Diff}_{n, B, \mathcal{U}/S} \) to \( \mathcal{O}_{\mathcal{P}^\dagger_{GL_n, h, B, \mathcal{U}/S}} \).

Let \( \mathcal{D}^{\dagger} : \mathcal{B} \to \Omega_{U_{\log}/S_{\log}} \otimes \mathcal{B} \otimes \mathcal{D}_{h, U_{\log}/S_{\log}}^{<n+1} \) be a global section of \( h\text{-Diff}_{n, B, \mathcal{U}/S} \).

If we consider \( \mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} \otimes \mathcal{B}^\vee \) as being equipped with a structure of left \( \mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} \)-module in the natural fashion, then one may construct its quotient

\[
\mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} / \left( \text{Im}(\circ \beta^{n+1}(\mathcal{D}^{\dagger})) \right)
\]

by the \( \mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} \)-submodule generated by the image \( \text{Im}(\circ \beta^{n+1}(\mathcal{D}^{\dagger})) \) of the morphism

\[
\circ \beta^{n+1}(\mathcal{D}^{\dagger}) : \mathcal{T}_{U_{\log}/S_{\log}}^{<n} \otimes \mathcal{B}^\vee \to \mathcal{D}_{h, U_{\log}/S_{\log}}^{<n+1} \otimes \mathcal{B}^\vee
\]

The condition that \( \mathcal{Symb}_n(\mathcal{D}^{\dagger}) = 1 \) implies that the composite

\[
\mathcal{D}_{h, U_{\log}/S_{\log}}^{<n} \otimes \mathcal{B}^\vee \hookrightarrow \mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} \otimes \mathcal{B}^\vee \to \mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} / \left( \text{Im}(\circ \beta^{n+1}(\mathcal{D}^{\dagger})) \right)
\]

is an isomorphism of \( \mathcal{O}_X \)-modules. Consider the composite

\[
\mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} \otimes (\mathcal{D}_{h, U_{\log}/S_{\log}}^{<n} \otimes \mathcal{B}^\vee) \to \mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} \otimes \mathcal{B}^\vee
\]

\[
\to \mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} / \left( \text{Im}(\circ \beta^{n+1}(\mathcal{D}^{\dagger})) \right)
\]

where the first arrow arises from the multiplication \( * \) (cf. §4.4) in \( \mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} \) and the third arrow denotes the inverse of \( \mathcal{D}_{h, U_{\log}/S_{\log}}^{<\infty} \). It determines (cf. §4.4) an \( S \)-\( h \)-log connection

\[
\mathcal{D}^\dagger : \mathcal{D}_{h, U_{\log}/S_{\log}}^{<n} \otimes \mathcal{B}^\vee \to \Omega_{U_{\log}/S_{\log}} \otimes (\mathcal{D}_{h, U_{\log}/S_{\log}}^{<n} \otimes \mathcal{B}^\vee)
\]

on \( \mathcal{D}_{h, U_{\log}/S_{\log}}^{<n-1} \otimes \mathcal{B}^\vee \). One verifies easily that the collection of data

\[
\mathcal{D}^\dagger := (\mathcal{D}_{h, U_{\log}/S_{\log}}^{<n} \otimes \mathcal{B}^\vee, \mathcal{D}^\dagger, \{\mathcal{D}_{h, U_{\log}/S_{\log}}^{<n-j} \otimes \mathcal{B}^\vee\}_{j=0}^{n})
\]

forms a \( (\text{GL}_n, h, B) \)-oper on \( \mathcal{U}/S \). The assignment \( \mathcal{D}^{\dagger} \mapsto \mathcal{D}^\dagger \) is functorial with respect to restriction to open subsets of \( U \), and hence, determines a morphism

\[
\Lambda_{n, h, B, \mathcal{U}/S} : h\text{-Diff}_{n, B, \mathcal{U}/S} \to \mathcal{O}_{\mathcal{P}^\dagger_{GL_n, h, B, \mathcal{U}/S}}
\]

of presheaves on \( \mathcal{E}t/U \).

**Proposition 4.8.1.**

\( \Lambda_{n, h, B, \mathcal{U}/S} \) is an isomorphism of presheaves. In particular, \( \mathcal{O}_{\mathcal{P}^\dagger_{GL_n, h, B, \mathcal{U}/S}} \) is a sheaf on \( \mathcal{E}t/U \).
Proof. We shall construct an inverse to $\Lambda_{n,h,\mathcal{B},U/S}^\bullet$. Let

\begin{equation}
\nabla^\diamond : \mathcal{D}_{h,U^{\log}/S^{\log}}^{<n} \otimes \mathcal{B}^\vee \to \mathcal{O}_{U^{\log}/S^{\log}} \otimes (\mathcal{D}_{h,U^{\log}/S^{\log}}^{<n} \otimes \mathcal{B}^\vee)
\end{equation}

be a $(\text{GL}_n, h, \mathcal{B})$-oper on $\mathcal{U}/S$. It corresponds, according to the discussion in §4.4, a structure of left $\mathcal{D}_{h,U^{\log}/S^{\log}}$-linear morphism

\begin{equation}
\nabla^\diamond : \mathcal{D}_{h,U^{\log}/S^{\log}}^\infty \otimes (\mathcal{D}_{h,U^{\log}/S^{\log}}^{<n} \otimes \mathcal{B}^\vee) \to \mathcal{D}_{h,U^{\log}/S^{\log}}^{<n} \otimes \mathcal{B}^\vee.
\end{equation}

For $j \geq 0$, consider the restriction

\begin{equation}
\nabla^\diamond_{\mathcal{D}}^{<j} := \nabla^\diamond|_{\mathcal{D}_{h,U^{\log}/S^{\log}}^{<j} \otimes \mathcal{B}^\vee} : \mathcal{D}_{h,U^{\log}/S^{\log}}^{<j} \otimes \mathcal{B}^\vee \to \mathcal{D}_{h,U^{\log}/S^{\log}}^{<n} \otimes \mathcal{B}^\vee
\end{equation}

of $\nabla^\diamond$ to the $\mathcal{O}_U$-submodule

\begin{equation}
\mathcal{D}_{h,U^{\log}/S^{\log}}^{<j} \otimes \mathcal{B}^\vee = \mathcal{D}_{h,U^{\log}/S^{\log}}^{<j} \otimes (\mathcal{D}_{h,U^{\log}/S^{\log}}^{<1} \otimes \mathcal{B}^\vee) \subseteq \mathcal{D}_{h,U^{\log}/S^{\log}}^\infty \otimes (\mathcal{D}_{h,U^{\log}/S^{\log}}^{<n} \otimes \mathcal{B}^\vee).
\end{equation}

It follows from the definition of a $(\text{GL}_n, h, \mathcal{B})$-oper that $\nabla^\diamond_{\mathcal{D}}^{<n}$ is an isomorphism and $\nabla^\diamond_{\mathcal{D}}^{<n+1}$ is surjective. Also, we consider the short exact sequence

\begin{equation}
0 \to \mathcal{D}_{h,U^{\log}/S^{\log}}^{<n} \otimes \mathcal{B}^\vee \to \mathcal{D}_{h,U^{\log}/S^{\log}}^{<n+1} \otimes \mathcal{B}^\vee \to \mathcal{T}_{U^{\log}/S^{\log}}^\otimes \mathcal{B}^\vee \to 0,
\end{equation}

of $\mathcal{O}_U$-modules arising (by tensoring with $\mathcal{B}^\vee$) from the natural sequence

\begin{equation}
0 \to \mathcal{D}_{h,U^{\log}/S^{\log}}^{<n} \to \mathcal{D}_{h,U^{\log}/S^{\log}}^{<n+1} \to \mathcal{T}_{U^{\log}/S^{\log}}^\otimes \to 0
\end{equation}

(cf. (342)). Then, the composite $\nabla^\diamond_{\mathcal{D}}^{<n-1} \circ \nabla^\diamond_{\mathcal{D}}^{<n+1}$ determines a split surjection of this exact sequence (372), equivalently, a split injection

\begin{equation}
\nabla^\bullet : \mathcal{T}_{U^{\log}/S^{\log}}^\otimes \otimes \mathcal{B}^\vee \hookrightarrow \mathcal{D}_{h,U^{\log}/S^{\log}}^{<n+1} \otimes \mathcal{B}^\vee,
\end{equation}

which is a global section of $\mathbb{E}_{h,U^{\log}/S^{\log}}^{n+1}$ with $^c\mathbb{Symb}_{n+1}^\bullet (\nabla^\bullet) = 1$. Since the isomorphism $^c\beta^{n+1}$ is compatible with $^c\mathbb{Symb}_{n+1}$ and $^c\mathbb{Symb}_{n+1}$ (cf. (349)), $(^c\beta^{n+1})^{-1}(\nabla^\bullet)$ lies in the set of global sections of $h-\mathcal{D}_{def}^\bullet$. One verifies easily that the assignment $\nabla^\diamond \mapsto (^c\beta^{n+1})^{-1}(\nabla^\bullet)$ is functorial with respect to restriction to any étale $U$-schemes, and the resulting morphism

$\mathcal{D}_{U^{\log},h,\mathcal{B},U/S} \to h-\mathcal{D}_{def}^\bullet$ determines an inverse to $\Lambda_{n,h,\mathcal{B},U/S}^\bullet$. This completes the proof of Proposition 4.8.1. \qed

**Proposition 4.8.2.**
Let $\mathcal{L} = (\mathcal{L}, \nabla_{\mathcal{L}})$ be an $h$-log integrable line bundle on $U^{\log}/S^{\log}$. The square
diagram

\[ \begin{array}{ccc}
\text{\textit{\textbf{h-Diff}}}_{n,B, U^\log/S^\log} & \xrightarrow{\Lambda_{n,h,B, \Omega/S}} & \mathcal{O}_{GL_n,h,B, \Omega/S} \\
\downarrow & & \downarrow \\
\text{\textit{\textbf{h-Diff}}}_{n,B \otimes L^\vee, U^\log/S^\log} & \xrightarrow{\Lambda_{n,h,B \otimes L^\vee, \Omega/S}} & \mathcal{O}_{GL_n,h,B \otimes L^\vee, \Omega/S}
\end{array} \]

(375)

is commutative.

**Proof.** The assertion follows from the definitions of the various morphisms in (375). \[\square\]

**Proposition 4.8.3.**

Let \( U' \) be an \'{e}tale \( U \)-scheme and \( \nabla^\otimes \in \Gamma(U', \mathcal{O}_{GL_n,h,B, \Omega/S}) \), \( \nabla_{+R} \in \Gamma(U', \mathcal{B}^n_{U^\log/S^\log}) \).

(i) Write \( \nabla_{+R}^\otimes \) for the \( S \)-log connection on \( \mathcal{D}^{<n}_{h,U^\log/S^\log} \otimes \mathcal{B}^n|_{U'} \) defined to be

\[ \nabla_{+R}^\otimes := \nabla^\otimes - (\nabla^\otimes \mathcal{D}^{<n})^{-1} \circ \gamma((\mathcal{D}^{<n}_R)) \circ \nabla^\otimes \mathcal{D}^{<n}, \]

where

- \( \nabla^\otimes \mathcal{D}^{<n} \) is the isomorphism (of the case where the \( U \) is taken to be \( U' \)) given in the proof of Proposition 4.8.1 (cf. (370));
- \( \gamma \) denotes the injection

\[ \mathcal{B}^n_{U^\log/S^\log} \hookrightarrow \Omega_{U^\log/S^\log} \otimes \text{End}_U(\mathcal{D}^{<n}_{h,U^\log/S^\log} \otimes \mathcal{B}^n), \]

given by composition with the quotient

\[ \mathcal{T}_{U^\log/S^\log} \otimes (\mathcal{D}^{<n}_{h,U^\log/S^\log} \otimes \mathcal{B}^n) \to \mathcal{T}_{U^\log/S^\log} \otimes (\mathcal{T}_{U^\log/S^\log}^{\otimes(n-1)} \otimes \mathcal{B}^n) = \mathcal{T}_{U^\log/S^\log}^{\otimes n} \otimes \mathcal{B}^n. \]

Then, \( \nabla_{+R}^\otimes \) forms a \((GL_n,h,B|_{U'})\)-oper on \( \Omega'/S \) i.e., determines an element of \( \Gamma(U', \mathcal{O}_{GL_n,h,B, \Omega/S}) \).

(ii) Consider the structure of \( \mathcal{B}^n_{U^\log/S^\log}\)-torsor on \( \mathcal{O}_{GL_n,h,B, \Omega/S} \) induced, via the isomorphism \( \Lambda_{n,h,B, \Omega/S} \), by the structure of \( \mathcal{B}^n_{U^\log/S^\log}\)-torsor on \( \text{\textit{\textbf{h-Diff}}}_{n,B, U^\log/S^\log} \) (cf. § 4.5). Then, it is given by

\[ \mathcal{O}_{GL_n,h,B, \Omega/S} \times \mathcal{B}^n_{U^\log/S^\log} \to \mathcal{O}_{GL_n,h,B, \Omega/S} \]

\[ \nabla_{+R}^\otimes \mapsto \nabla_{+R} \]

**Proof.** The assertion follows from the definition of the isomorphism \( \Lambda_{n,h,B, \Omega/S} \). \[\square\]
4.9. Next, we consider restricting the isomorphism $\Lambda_{n, h, B, U/S}$ to an isomorphism between a subsheaf of $h\cdot \mathcal{D}_{n, h, B, U/S, log}$ classifying $h$-tlcds with prescribed subprincipal symbol and a subsheaf of $\mathcal{O}_{P_{\mathrm{GL}_n, h, B}}$ classifying $(\mathrm{GL}_n, h, B)$-opers with prescribed determinant.

**Definition 4.9.1.**

(i) An $(n, h)$-determinant data for $U^\log$ over $S^\log$ is a pair $U := (B, \nabla_0)$ consisting of a line bundle $B$ on $U$ and an $S$-$h$-log connection $\nabla_0$ on the determinant $\det(D_{n, h, U^\log / S^\log}^\log \otimes B')$. (In particular, since $\det(D_{n, h, U^\log / S^\log}^\log \otimes B')$ is a line bundle, giving a 0-determinant data for $U^\log$ over $S^\log$ is equivalent to giving a line bundle $B$ together with a global section of $\Omega_{U^\log / S^\log}$.)

(ii) Let $U := (B, \nabla_0)$ and $U' := (B', \nabla'_0)$ be $(n, h)$-determinant data for $U^\log$ over $S^\log$. An isomorphism from $U$ to $U'$ is an isomorphism $B \sim B'$ of $\mathcal{O}_U$-modules such that the induced isomorphism $\det(D_{n, h, U^\log / S^\log}^\log \otimes B') \sim \det(D_{n, h, U'^\log / S'^\log}^\log \otimes B'^\prime)$ is compatible with the respective $S$-$h$-connections $\nabla_0$ and $\nabla'_0$.

**Remark 4.9.2.**

(i) By means of the isomorphism in (342), one may express the line bundle $\det(D_{n, h, U^\log / S^\log}^\log \otimes B')$ appeared in Definition 4.9.1 as follows:

\[
\det(D_{n, h, U^\log / S^\log}^\log \otimes B') \sim \bigotimes_{j=0}^{n-1} \left( \left( \bigotimes_{j'=0}^{n-j-1} \mathcal{T}_{U^\log / S^\log}^{\otimes j'} \right) \otimes (B')^{\otimes n} \right)
\]

(ii) Let $U := (B, \nabla_0)$ be an $(n, h)$-determinant data for $U^\log$ over $S^\log$. If $u' : U' \to U$ is an étale morphism, then the restriction

\[
u'^* (U) := (u'^* (B), u'^* (\nabla_0))
\]

to $U'$ forms an $(n, h)$-determinant data for $U'^\log$ over $S'^\log$.

Also, if $s' : S' \to S$ is a morphism of $k$-schemes, then the base-change

\[
(id_U \times s')^* (U) := ((id_U \times s')^* (B), (id_U \times s')^* (\nabla_0))
\]

via $s'$ forms an $(n, h)$-determinant data for $(U \times_S S')^\log$ over $S'^\log$.

Both the notion of restriction to étale $U$-schemes and base-change over $S$ are compatible with composition of morphisms.
(iii) Suppose that $X/S$ is an unpointed smooth curve of genus $g > 1$. Then, $X/S$ necessarily admits at least étale locally on $S$, an $(n, \hbar)$-determinant data $\mathbb{U} = (\mathcal{B}, \nabla_0)$.

Indeed, let us denote by $\mathcal{Pic}^d_{X/S}$ the relative Picard scheme of $X/S$ classifying the set of (equivalence classes, relative to the equivalence relation determined by tensoring with a line bundle pulled back from the base $S$) of $d$ invertible sheaves on $X$. Then, the morphism

$$\mathcal{Pic}^{(-g+1)(n-1)}_{X/S} \to \mathcal{Pic}^{(-g+1)n(n-1)}_{X/S} : [\mathcal{L}] \mapsto [\mathcal{L}^{\otimes n}]$$

given by multiplication by $n$ is finite and étale (since $n < p = \text{char}(k)$).

Here, observe that

$$\text{deg}(\mathfrak{T}_{X/S}^{\otimes \frac{n(n-1)}{2}}) = (-2g + 2) \cdot \frac{n(n-1)}{2} = (-g + 1)n(n - 1).$$

Hence, the equivalence class $[\mathfrak{T}_{X/S}^{\otimes \frac{n(n-1)}{2}}]$ represented by the line bundle $\mathfrak{T}_{X/S}^{\otimes \frac{n(n-1)}{2}}$ determines an $S$-rational point of $\mathcal{Pic}^{(-g+1)n(n-1)}_{X/S}$, which lifts, étale locally on $S$, to a point of $\mathcal{Pic}^{(-g+1)(n-1)}_{X/S}$. Equivalently, after possibly replacing $S$ with an étale covering of $S$, there exists a line bundle $\mathcal{N}$ on $X$ satisfying that $\mathcal{N}^{\otimes n} \simeq \mathfrak{T}_{X/S}^{\otimes \frac{n(n-1)}{2}}$. Then, we have

$$\det(D_{h}^{n}_{X/S} \otimes \mathcal{N}^{\vee}) \simeq \mathfrak{T}_{X/S}^{\otimes \frac{n(n-1)}{2}} \otimes (\mathcal{N}^{\vee})^{\otimes n} \simeq \mathcal{O}_X$$

(cf. the sequence of isomorphisms discussed in (i)). The universal derivation $h \cdot d_{X/S} : \mathcal{O}_X \to \Omega_{X/S}$ multiplied by $h$ defines, via this composite isomorphism, an $S$-$\hbar$-(log )connection on $\det(D_{h}^{n}_{X/S} \otimes \mathcal{N}^{\vee})$. Thus, we have an $(n, \hbar)$-determinant data $(\mathcal{N}, h \cdot d_{X/S})$, as desired.

But, unlike the classical case (i.e., the case where $X$ is a proper smooth curve over an algebraically closed field) or the above case, a nodal curve $X/k$ may not admit any such $(n, \hbar)$-determinant data $\mathbb{U} = (\mathcal{B}, \nabla_0)$ satisfying that $\det(D_{h}^{n}_{X/S} \otimes \mathcal{B}^{\vee}) \simeq \mathcal{O}_X$ (cf. the discussion in [14], 1.2.4 Example).

In the following, suppose that we are given an $(n, \hbar)$-determinant data $\mathbb{U} = (\mathcal{B}, \nabla_0)$ for $U^{\log}$ over $S^{\log}$.

**Lemma 4.9.3.**

(i) For an $\hbar$-log integrable line bundle $\mathcal{L} = (\mathcal{L}, \nabla_{\mathcal{L}})$, we shall consider the pair

$$\mathbb{U} \otimes \mathcal{L}^{\vee} := (\mathcal{B} \otimes \mathcal{L}^{\vee}, \nabla_0 \otimes \mathcal{L}^{\otimes n}_{\mathcal{L}^{\vee}}),$$

where we regard $\nabla_0 \otimes \nabla_L^{\otimes n}$ as an $S$-h-log connection on $\det(D_{h,U}^{<n}/\log \otimes (\mathcal{B}^\vee \otimes \mathcal{L}))$ ($\cong \det(D_{h,U}^{<n}/\log \otimes \mathcal{B}^\vee) \otimes \mathcal{L}^{\otimes n}$). Then, $U \otimes \mathcal{L}^\vee$ forms an $(n,h)$-determinant data for $U^{\log}/\log$.

(ii) Conversely, if $U' = (\mathcal{B}', \nabla'_0)$ is an $(n,h)$-determinant data for $U^{\log}/\log$, then there exists an $h$-log integrable line bundle $\mathcal{L}' = (\mathcal{L}', \nabla_{\mathcal{L}'})$, which is uniquely determined up to isomorphism, such that $U' \otimes \mathcal{L}'$ is isomorphic to $U$ (cf. Definition 4.9.1 (ii)).

\textit{Proof}. Assertion (i) is clear from the definition of an $(n,h)$-determinant data. Assertion (ii) follows from Lemma 4.3.1 (i). \qed

**Definition 4.9.4.**

A $(\text{GL}_n, h, \mathbb{U})$-oper on $\mathfrak{U}/S$ is a $(\text{GL}_n, h, \mathcal{B})$-oper $\nabla^\circ$ on $\mathfrak{U}/S$ satisfying that $\det(\nabla^\circ) = \nabla_0$.

We shall write

\begin{equation}
\mathcal{O}_{p_{\text{GL}_n, h, \mathbb{U}, \mathfrak{U}/S}}
\end{equation}

for the subsheaf of $\mathcal{O}_{p_{\text{GL}_n, h, \mathcal{B}, \mathfrak{U}/S}}$ classifying $(\text{GL}_n, h, \mathbb{U})$-opers. Also, we shall write

\begin{equation}
h\text{-diff}_{n, \mathbb{U}, U^{\log}/\log} : (\Lambda_{n, h, \mathbb{U}, \mathfrak{U}/S}^\circ)^{-1}(\mathcal{O}_{p_{\text{GL}_n, h, \mathbb{U}, \mathfrak{U}/S}}).
\end{equation}

Evidently, the isomorphism $\Lambda_{n, h, \mathbb{U}, \mathfrak{U}/S}^\circ : h\text{-diff}_{n, \mathbb{U}, U^{\log}/\log} \cong \mathcal{O}_{p_{\text{GL}_n, h, \mathbb{U}, \mathfrak{U}/S}}$ of sheaves on $\mathcal{E}_U$.

4.10. Next, we shall verify the fact that $\mathcal{O}_{p_{\text{GL}_n, h, \mathbb{U}, \mathfrak{U}/S}}$ has a natural $\mathbb{B}^{n-1}_{U^{\log}/\log}$-torsor structure.

Let $\nabla^\circ$ be a $(\text{GL}_n, h, \mathbb{U})$-oper on $\mathfrak{U}/S$ and $\mathbb{R} \in \Gamma(U, \mathbb{B}^{n}_{U^{\log}/\log})$. Recall (cf. Proposition 4.8.3) the $\mathbb{B}^{n}_{U^{\log}/\log}$-torsor structure on $\mathcal{O}_{p_{\text{GL}_n, h, \mathcal{B}, \mathfrak{U}/S}}$, which sends $\nabla^\circ$, via the action by $\mathbb{R}$, to the $(\text{GL}_n, h, \mathbb{U})$-oper

\begin{equation}
\nabla^\circ \mapsto R := \nabla^\circ - (\nabla^\circ D^{<n})^{-1} \circ \gamma(\langle \right)_{\beta^n (\mathbb{R})} \circ \nabla^\circ D^{<n}.
\end{equation}

Thus, $\det(\nabla^\circ_{\mapsto R}) = \nabla_0$ if and only if the trace of

\begin{equation}
(\nabla^\circ D^{<n})^{-1} \circ \gamma(\langle \right)_{\beta^{n+1} (\mathbb{R})} \circ \nabla^\circ D^{<n} \in \Gamma(U, \Omega_{U^{\log}/\log})
\end{equation}
vanishes identically on $U$. But,
\begin{equation}
(392) \quad \text{trace}(\nabla^{\otimes \mathcal{D}^<n})^{-1} \circ \gamma(\otimes \beta^n(\mathcal{D})) \circ \nabla^{\otimes \mathcal{D}^<n} = \text{trace}(\gamma(\otimes \beta^n(\mathcal{D})))
\end{equation}
\begin{align*}
&= \mathcal{S}ym^n(\otimes \beta^n(\mathcal{D})) \\
&= \mathcal{S}ym^n(\mathcal{D}). \\
\end{align*}

Hence, the section of $\mathcal{O}^\nu_{\mathcal{G}_{\mathcal{L}}/\mathcal{B}_{\mathcal{M}/\mathcal{S}}}$ classifying $\nabla^{\otimes \mathcal{D}^<R}$ lies in $\mathcal{O}^\nu_{\mathcal{G}_{\mathcal{L}}/\mathcal{B}_{\mathcal{M}/\mathcal{S}}}$ if and only if $\mathcal{O}^\nu_{\mathcal{G}_{\mathcal{L}}/\mathcal{B}_{\mathcal{M}/\mathcal{S}}}$ has sections locally on $U$. Consequently, the structure of $\mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}$-torsor on $\mathcal{O}^\nu_{\mathcal{G}_{\mathcal{L}}/\mathcal{B}_{\mathcal{M}/\mathcal{S}}}$ carries a structure of $\mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}$-subtorsor on $\mathcal{O}^\nu_{\mathcal{G}_{\mathcal{L}}/\mathcal{B}_{\mathcal{M}/\mathcal{S}}}$, as well as on $\mathcal{H}_{\mathcal{U}^\log/\mathcal{S}^\log}$.

If $t : T \to S$ is a morphism of $k$-schemes, then the pull-back of sections gives a morphism
\begin{equation}
(393) \quad \Gamma(X, \mathcal{H}_{\mathcal{U}^\log/\mathcal{S}^\log}) \to \Gamma(X \times_S T, \mathcal{H}_{\mathcal{U}^\log/\mathcal{S}^\log})
\end{equation}
(cf. Remark 4.9.2 (ii) for the definition of $(id_X \times t)^\ast(\mathcal{U}))$. This notion of pull-back is compatible, in an evident sense, with composition of morphisms. Thus, one may define the $\mathcal{S}et$-valued contravariant functor
\begin{equation}
(394) \quad \mathcal{H}_{\mathcal{U}^\log/\mathcal{S}^\log}^\bullet : \mathcal{S}et / S \to \mathcal{S}et
\end{equation}
on any $\mathcal{S}et$-scheme $t : T \to S$, assigns the set of global sections $\Gamma(X \times_S T, \mathcal{H}_{\mathcal{U}^\log/\mathcal{S}^\log})$ (cf. Remark 4.9.2 (ii) for the definition of $(id_X \times t)^\ast(\mathcal{U}))$.

**Proposition 4.10.1.**

(i) $f_\ast(\mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1})$ is a vector bundle on $S$ of rank
\begin{equation}
(395) \quad (g - 1)(n^2 - 1) + \frac{r}{2}(n + 2)(n - 1) \quad (= \mathfrak{N}(\mathfrak{L}_n))
\end{equation}
(cf. (113)).

(ii) $\mathcal{H}_{\mathcal{U}^\log/\mathcal{S}^\log}^\bullet$ may be represented by a relative affine space over $S$ modeled on $\mathbb{V}(f_\ast(\mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1}))$. In particular, the fiber of $\mathcal{H}_{\mathcal{U}^\log/\mathcal{S}^\log}^\bullet$ over any point of $S$ is nonempty.

**Proof.** First, we consider assertion (i). Since $\mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1} / \mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1} \cong \mathcal{S}ymb_{\mathcal{U}^\log/\mathcal{S}^\log}^{(n-1)}$ (cf. (343)), it follows from Grothendieck-Serre duality that, for $j = 0, \ldots, n-2$,
\begin{equation}
\begin{align*}
\mathbb{R}_1 f_\ast(\mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1} / \mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1}) &\cong \mathbb{R}_1 f_\ast(\mathcal{O}_{\mathcal{U}^\log/\mathcal{S}^\log}^{(n-j)}) \\
&\cong f_\ast(\mathcal{T}_{\mathcal{U}^\log/\mathcal{S}^\log}^{(n-1-j)}(-D_{X/S}))^\vee = 0.
\end{align*}
\end{equation}

It follows that $\mathbb{R}_1 f_\ast(\mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1} / \mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1}) = 0$, and hence, $f_\ast(\mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1})$ is a vector bundle on $S$ (cf. [25], Chap. III, Theorem 12.11 (b)); its rank $\text{rk}(f_\ast(\mathcal{B}_{\mathcal{U}^\log/\mathcal{S}^\log}^{n-1}))$
may be calculated by

\[ \text{rk}(f_*(\mathcal{B}_{X^{\log}/S^{\log}}^{n-1})) = \sum_{j=0}^{n-2} \text{rk}(\Omega_X^{(n-j)}) \]
\[ = \sum_{j=0}^{n-2} (g - 1)(2n - 2j - 1) + r(n - j) \]
\[ = (g - 1)(n^2 - 1) + \frac{r}{2} \cdot (n + 2)(n - 1) \]
\[ = \aleph(\mathfrak{g}(\mathfrak{s}\mathfrak{l}_n)), \]

where the third equality arises from the Riemann-Roch theorem. This completes the proof of the former assertion.

Next, we consider assertion (ii). Apply the isomorphism (389) and the discussion in §4.10. Then, \( f_*(h_{\text{Diff}}^{\bullet}_{n, U, X^{\log}/S^{\log}}) \) admits a free and transitive action of \( f_*(\mathcal{B}_{X^{\log}/S^{\log}}^{n-1}) \). Since \( \mathbb{R} f_*(\mathcal{B}_{X^{\log}/S^{\log}}^{n-1}) = 0 \), the sheaf \( f_*(h_{\text{Diff}}^{\bullet}_{n, U, X^{\log}/S^{\log}}) \) admits sections locally on \( S \). Moreover, this equality implies that the direct image \( f_*(h_{\text{Diff}}^{\bullet}_{n, U, X^{\log}/S^{\log}}) \) is compatible, in an evident sense, with pull-back via any morphism \( S' \to S \). Thus, \( f_*(h_{\text{Diff}}^{\bullet}_{n, U, X^{\log}/S^{\log}}) \) may be represented by a relative affine space over \( S \) modeled on the vector bundle \( f_*(\mathcal{B}_{X^{\log}/S^{\log}}^{n-1}) \), and is isomorphic to \( h_{\text{Diff}}^{\bullet}_{n, U, X^{\log}/S^{\log}} \). This completes the proof of the latter assertion. \( \square \)

4.11. We shall compare \( \mathcal{O}_p^{\bigodot}_{GL_n, h, U, S} \) with \( \overline{\mathcal{O}}_p^{\bigodot}_{GL_n, h, U, S} \).

**Proposition 4.11.1.**

The natural morphism

\[ \Lambda_{n, h, U, S}^{\bigodot} : \mathcal{O}_p^{\bigodot}_{GL_n, h, U} \to \overline{\mathcal{O}}_p^{\bigodot}_{GL_n, h, U, S} \]

determined by assigning, to each \( (GL_n, h, U) \)-oper \( \nabla^{\bigodot} \), the equivalence class represented by the \( (GL_n, h) \)-oper \( \nabla^{\bigodot} \) (cf. Definition 4.6.1 (i)), is an isomorphism.

**Proof.** First, we consider the injectivity of \( \Lambda_{n, h, U, S}^{\bigodot} \). It suffices to prove that if \( \nabla_1^{\bigodot} \) and \( \nabla_2^{\bigodot} \) are \( (GL_n, h, U) \)-opers on \( U_{/S} \) that are equivalent, then they are, in fact, isomorphic locally on \( U \). After possibly replacing \( U \) with an open covering...
of \(\mathcal{U}\), we may assume that there exists an \(h\)-log integrable line bundle \((\mathcal{L}, \nabla_{\mathcal{L}})\) on \(U^{\log}/S^{\log}\) and an isomorphism

\[
(399) \quad \alpha : (\mathcal{L} \otimes \mathcal{D}^n_{h,U^{\log}/S^{\log}} \otimes B^\vee, \nabla_{\mathcal{L}} \otimes \nabla_1^\vee) \simto (\mathcal{D}^n_{h,U^{\log}/S^{\log}} \otimes B^\vee, \nabla_2^\vee)
\]

of \((\text{GL}_n, h)\)-opers, i.e., that is compatible with the respective filtrations. In particular, by restricting \(\alpha\) to the \((n - 1)\)-st filtrations, we obtain an isomorphism

\[
(400) \quad \mathcal{L} \otimes \mathcal{D}^{\leq 1}_{h,U^{\log}/S^{\log}} \otimes B^\vee \simto \mathcal{D}^{\leq 1}_{h,U^{\log}/S^{\log}} \otimes B^\vee,
\]

which implies that \(\mathcal{L} \cong \mathcal{O}_U\). In the following we shall identify \(\mathcal{L}\) with \(\mathcal{O}_U\) via a specific isomorphism \(\nabla_{\mathcal{L}} \simto \mathcal{O}_U\). Denote by \(\alpha_{\det}\) the automorphism of \(\text{det}(\mathcal{D}^n_{h,U^{\log}/S^{\log}} \otimes B^\vee)\) induced by \(\alpha\). We have a sequence of equalities

\[
(401) \quad \nabla_0 = \alpha_{\det}^*(\nabla_0) = \alpha_{\det}^*(\text{det}(\nabla_2^\vee)) = \text{det}(\nabla_{\mathcal{L}} \otimes \nabla_1^\vee) = \nabla_2^{\otimes n} \otimes \nabla_0.
\]

It follows that \((\mathcal{L}^{\otimes n}, \nabla_{\mathcal{L}}^{\otimes n}) \cong (\mathcal{O}_X, d)\), and hence, \((\mathcal{L}, \nabla_{\mathcal{L}}) \cong (\mathcal{O}_X, d)\) by Lemma 4.3.1 (i). That is, \(\nabla_1^{\otimes} = 0\), by definition, isomorphic to \(\nabla_2^{\otimes}\), and this completes the injectivity of \(\Lambda^n_{h,U^{\log}/S}\).

Next, consider the subjectivity of \(\Lambda^n_{h,U^{\log}/S}\). We shall prove that, for each \((\text{GL}_n, h)\)-oper \(F^{\otimes} := (\mathcal{F}, \nabla_{\mathcal{F}}, \{F^j\}_{j=0}^{n})\) on \(\mathcal{U}/S\), there exists a \((\text{GL}_n, h, \mathcal{U})\)-oper \(\nabla_{\mathcal{F}}^{\otimes}\) on \(\mathcal{U}/S\) whose associated \((\text{GL}, h)\)-oper \(\nabla_{\mathcal{F}}^{\otimes}\) (cf. Definition 4.6.1 (i)) is equivalent to \(F^{\otimes}\). By means of the isomorphisms \(I_{F^{\otimes}} = (j + 1 \leq l \leq n)\) (cf. \(331\)), we obtain a canonical composite isomorphism

\[
(402) \quad F^j/F^{j+1} \simto (F^{j+1}/F^{j+2}) \otimes T^{\otimes j}_{U^{\log}/S^{\log}} \simto \cdots \simto F^{n-1} \otimes T^{\otimes n-1-j}_{X^{\log}/S^{\log}}
\]

\((j = 0, \ldots, n - 1)\). By applying these isomorphisms for all \(j\), we obtain a composite isomorphism

\[
(403) \quad \text{det}(\mathcal{F}) \simto \bigotimes_{j=0}^{n-1} (F^j/F^{j+1}) \simto \bigotimes_{j=0}^{n-1} T^{\otimes (n-1-j)}_{U^{\log}/S^{\log}} \otimes (F^{n-1} \otimes S^{\otimes n}).
\]

If we write \(B_{\mathcal{F}} := (B \otimes F^{n-1})^{\vee}\), then the composite isomorphism \((403)\) and the composite isomorphism in Remark 4.9.2 give an isomorphism

\[
(404) \quad (\text{det}(\mathcal{F} \otimes B_{\mathcal{F}}) \simto) \text{det}(\mathcal{F}) \otimes B_{\mathcal{F}}^{\otimes n} \simto \text{det}(\mathcal{D}^n_{h,U^{\log}/S^{\log}} \otimes B^\vee)
\]

By Lemma 4.3.1 (i), there exists uniquely an \(S\)-log connection \(\nabla_{\mathcal{F}}\) on \(B_{\mathcal{F}}\) such that \((\text{det}(\nabla_{\mathcal{F}} \otimes \nabla) = \text{det}(\nabla_{\mathcal{F}}) \otimes \nabla^{\otimes n})\) coincides, via the isomorphism \((404)\), with \(\nabla_0\). It follows from the definition of a \((\text{GL}_n, h)\)-oper that the morphism

\[
(405) \quad (\mathcal{D}^n_{h,U^{\log}/S^{\log}} \otimes B^\vee) \simto \mathcal{D}^n_{h,U^{\log}/S^{\log}} \otimes (F^{n-1} \otimes B_{\mathcal{F}}) \rightarrow \mathcal{F} \otimes B_{\mathcal{F}}
\]

obtained by restricting \((\nabla_{\mathcal{F}} \otimes \nabla)^{D}\) to the \(\mathcal{O}_U\)-submodule \(\mathcal{D}^n_{h,U^{\log}/S^{\log}} \otimes (F^{n-1} \otimes B_{\mathcal{F}})\) of \(\mathcal{D}^n_{h,U^{\log}/S^{\log}} \otimes (\mathcal{F} \otimes B_{\mathcal{F}})\) is an isomorphism. Consequently, the collection
of data
\[(406) \quad \mathcal{F}_\otimes^\otimes := (\mathcal{F} \otimes \mathcal{B}_F, \nabla_F \otimes \nabla, \{\mathcal{F}^i \otimes \mathcal{B}_F\}_{i=0}^n)\]
forms, via the isomorphism \[(405)\], a \((\text{GL}_n, h, \mathcal{U})\)-oper on \(\mathcal{U}/S\) whose associated \((\text{GL}_n, h)\)-oper is equivalent to \(\mathcal{F}_\otimes^\otimes\). This completes the proof of the asserted subjectivity. \(\square\)

By combining Proposition 4.3.2, the discussion following Definition 4.9.4, and Proposition 4.11.1, we have obtained the following theorem.

**Theorem 4.11.2 (Comparison of the moduli of opers I).**
Suppose that there exists an \((n, h)\)-determinant data \(\mathcal{U} := (\mathcal{B}, \nabla_0)\) for \(U^\log\) over \(S^\log\). Then, there exists a canonical sequence of isomorphisms
\[(407) \quad \mathcal{h}\text{-Diff}_{n, \mathcal{U}, U^\log/S^\log}^\bullet \longrightarrow \mathcal{Op}_{\text{GL}_n, h, \mathcal{U}/S}^\otimes \longrightarrow \overline{\mathcal{Op}}_{\text{GL}_n, h, \mathcal{U}/S}^\otimes \longrightarrow \mathcal{Op}_{\text{sl}_n, h, \mathcal{U}/S}^\bullet\]
of sheaves on \(\mathcal{E}_t/U\).

Let us write
\[(408) \quad \mathcal{Op}_{\text{GL}_n, h, x/S}^\otimes (\text{resp., } \overline{\mathcal{Op}}_{\text{GL}_n, h, x/S}^\otimes) : \text{Sch}/S \rightarrow \text{Set}\]
for the \(\text{Set}\)-valued contravariant functor on \(\text{Sch}/S\) which, to any \(S\)-scheme \(t : T \rightarrow S\), assigns the set \(\Gamma(X \times_S T, \mathcal{Op}_{\text{GL}_n, h, (id_X \times t)^* (\mathcal{U}), x/T}^\otimes)\) (resp., \(\Gamma(X \times_S T, \overline{\mathcal{Op}}_{\text{GL}_n, h, x/T}^\otimes)\). By applying Proposition 4.10.1 (i), (ii) and the above theorem, we have the following corollary.

**Corollary 4.11.3 (Comparison of the moduli of opers II).**
Suppose that there exists an \((n, h)\)-determinant data \(\mathcal{U} := (\mathcal{B}, \nabla_0)\) for \(X^\log\) over \(S^\log\). Then, the isomorphism \[(407)\] induces a canonical sequence of isomorphisms
\[(409) \quad \mathcal{h}\text{-Diff}_{n, \mathcal{U}, X^\log/S^\log}^\bullet \longrightarrow \mathcal{Op}_{\text{GL}_n, h, \mathcal{U}, x/S}^\otimes \longrightarrow \overline{\mathcal{Op}}_{\text{GL}_n, h, x/S}^\otimes \longrightarrow \mathcal{Op}_{\text{sl}_n, h, \mathcal{U}, x/S}^\bullet\]
of functors, where recall that \(\mathcal{Op}_{\text{sl}_n, h, x/S}^\bullet := \mathcal{Op}_{\text{sl}_n, h, x/S}\). In particular, the functor \(\mathcal{Op}_{\text{sl}_n, h, x/S}\) may be represented by a relative affine space over \(S\) of relative dimension
\[(410) \quad (g - 1)(n^2 - 1) + \frac{r}{2} \cdot (n + 2)(n - 1) \left(= \mathbb{A}(\text{sl}_n)\right)\]
(cf. \[(118)\]).
Corollary 4.11.4.
Let $U$ be as in Corollary 4.11.3 and $\mathcal{L} = (\mathcal{L}, \nabla_{\mathcal{L}})$ an $\hbar$-log integrable line bundle on $X^{\log}/S^{\log}$. Then, the isomorphisms $\Gamma \hat{\otimes} B^{\hbar} \rightarrow \mathcal{D}_{B^{\hbar}}$ (cf. (358)) and $\Gamma \hat{\otimes} B^{\hbar} \rightarrow \mathcal{D}_{B^{\hbar}}$ (cf. (359)) carry naturally isomorphisms

\[ (411) \quad \Gamma \hat{\otimes} B^{\hbar} \rightarrow \mathcal{D}_{B^{\hbar}}, \quad \mathcal{O} \rightarrow \mathcal{D}_{\mathcal{O}} \]

and

\[ (412) \quad \Gamma \hat{\otimes} B^{\hbar} \rightarrow \mathcal{D}_{\mathcal{O}}, \quad \mathcal{O} \rightarrow \mathcal{D}_{\mathcal{O}} \]

respectively. Moreover, the diagram

\[ (413) \quad \begin{array}{ccc}
\Gamma \hat{\otimes} B^{\hbar} & \rightarrow & \mathcal{O} \\
\downarrow \Gamma \hat{\otimes} B^{\hbar} & & \downarrow \Gamma \hat{\otimes} B^{\hbar} \\
\mathcal{D}_{\mathcal{O}} & \rightarrow & \mathcal{O}
\end{array} \]

consisting of isomorphisms of $S$-schemes is commutative (cf. Proposition 4.8.2).

Proof. The assertion follows from the definition of the morphisms involved and the equivalence relation defined in Definition 4.2.2. $\square$

4.12. Let us fix an $(n, \hbar)$-determinant data $\mathbb{U} := (\mathbb{B}, \nabla_0)$. Denote by

\[ (414) \quad \pi_{\mathcal{O}_{\mathbb{U}}, \mathbb{U}/S} : \mathcal{E}_{\mathbb{U}, \mathbb{U}/S} \rightarrow \mathcal{E}_{\mathbb{U}, \mathbb{U}/S} \]

the right $\mathcal{O}_{\mathbb{U}}$-torsor over $\mathbb{U}$ associated with $\mathcal{D}^{\leq n}_{\mathcal{O}_{\mathbb{U}}, \mathbb{U}/S} \otimes \mathcal{B}^{\nu}$, i.e., the $\mathbb{U}$-scheme representing the functor

\[ (415) \quad \mathcal{I}_{\mathcal{O}_{\mathbb{U}}}(\mathcal{O}_{\mathbb{U}}, \mathcal{D}_{\mathcal{O}_{\mathbb{U}}, \mathbb{U}/S}^{\leq n} \otimes \mathcal{B}^{\nu}) : \mathcal{E}_{\mathbb{U}} \rightarrow \mathcal{S} \mathcal{E} \mathcal{T} \mathcal{G} \mathcal{T} \]

Then, its adjoint vector bundle $(\mathfrak{g}_{\mathbb{U}})_{\mathcal{E}_{\mathbb{U}, \mathbb{U}/S}^{\nu}}$ is canonically isomorphic to the vector bundle $\mathcal{E}_{\mathbb{U}}(\mathcal{D}_{\mathcal{O}_{\mathbb{U}}, \mathbb{U}/S}^{\leq n} \otimes \mathcal{B}^{\nu})$ on $\mathbb{U}$.

For each $j \in \mathbb{Z}$, we shall denote by

\[ (416) \quad (\mathfrak{g}_{\mathbb{U}})_{\mathcal{E}_{\mathbb{U}, \mathbb{U}/S}^{\nu}}^{j} \]

the $\mathcal{O}_{\mathbb{U}}$-submodule of the vector bundle $\mathcal{E}_{\mathbb{U}}(\mathcal{D}_{\mathcal{O}_{\mathbb{U}}, \mathbb{U}/S}^{\leq n} \otimes \mathcal{B}^{\nu})$ consisting of local sections $w$ satisfying that $w(\mathcal{D}_{\mathcal{O}_{\mathbb{U}}, \mathbb{U}/S}^{\leq n} \otimes \mathcal{B}^{\nu}) \subseteq \mathcal{D}_{\mathcal{O}_{\mathbb{U}}, \mathbb{U}/S}^{\leq n-j} \otimes \mathcal{B}^{\nu}$. 

$B^\vee$ for all $l = 0, \ldots, n$. These $\mathcal{O}_U$-submodules $\{(\mathfrak{gl}_n)^j_{\mathfrak{g}_{n,h,\mathcal{M}/S}}\}_{j \in \mathbb{Z}}$ form a decreasing filtration on $\mathcal{E}nd_{\mathcal{O}_U}(\mathcal{D}^{<n}_{h,\mathcal{M}/S} \otimes B^\vee)$:

\begin{equation}
0 = (\mathfrak{gl}_n)^n_{\mathfrak{g}_{n,h,\mathcal{M}/S}} \subset (\mathfrak{gl}_n)^{n-1}_{\mathfrak{g}_{n,h,\mathcal{M}/S}} \subset \cdots \subset (\mathfrak{gl}_n)^1_{\mathfrak{g}_{n,h,\mathcal{M}/S}} = \mathcal{E}nd_{\mathcal{O}_U}(\mathcal{D}^{<n}_{h,\mathcal{M}/S} \otimes B^\vee).
\end{equation}

Next, consider the natural inclusion

\begin{equation}
\text{diag} : \mathcal{O}_U \to \mathcal{E}nd_{\mathcal{O}_U}(\mathcal{D}^{<n}_{h,\mathcal{M}/S} \otimes B^\vee)
\end{equation}

which, to any local section $a \in \mathcal{O}_U$, assigns the locally defined endomorphism $\text{diag}(a)$ of $\mathcal{D}^{<n}_{h,\mathcal{M}/S} \otimes B^\vee$ given by multiplication by $a$. Denote by

\begin{equation}
\pi_{\text{PGL}_n,\mathcal{M}/S} : \mathcal{E}^\dagger_{n,h,\mathcal{M}/S} \to U
\end{equation}

the right $\text{PGL}_n$-torsor over $U$ induced, via a change of structure group $\text{GL}_n \to \text{PGL}_n$, by $\mathcal{E}^\dagger_{n,h,\mathcal{M}/S}$. Then, its associated adjoint vector bundle $(\mathfrak{pgl}_n)^\dagger_{\mathfrak{g}_{n,h,\mathcal{M}/S}}$ is canonically isomorphic to the quotient

\begin{equation}
\mathcal{E}nd_{\mathcal{O}_U}(\mathcal{D}^{<n}_{h,\mathcal{M}/S} \otimes B^\vee) := \mathcal{E}nd_{\mathcal{O}_U}(\mathcal{D}^{<n}_{h,\mathcal{M}/S} \otimes B^\vee)/\text{Im}(\text{diag}).
\end{equation}

Write

\begin{equation}
(\mathfrak{pgl}_n)^j_{\mathfrak{g}_{n,h,\mathcal{M}/S}} := (\mathfrak{gl}_n)^j_{\mathfrak{g}_{n,h,\mathcal{M}/S}} / ((\mathfrak{gl}_n)^j_{\mathfrak{g}_{n,h,\mathcal{M}/S}} \cap \text{Im}(\text{diag})).
\end{equation}

The filtration $\{\mathcal{D}^{<n-j}_{h,\mathcal{M}/S} \otimes B^\vee\}_{j=0}^n$ carries a structure of $B$-reduction

\begin{equation}
\pi_{B,h,\mathcal{M}/S} : \mathcal{E}^\dagger_{B,h,\mathcal{M}/S} \to U
\end{equation}

on $\mathcal{E}^\dagger_{n,h,\mathcal{M}/S}$ (cf. the discussion following Lemma 4.3.1 for the definition of the filtration $B$ of $\text{PGL}_n$). The adjoint vector bundle associated with $\mathcal{E}^\dagger_{B,h,\mathcal{M}/S}$ is canonically isomorphic to $(\mathfrak{pgl}_n)^0_{\mathfrak{g}_{n,h,\mathcal{M}/S}}$. For each $j = -n, \ldots, n$, the $\mathcal{O}_U$-module $(\mathfrak{pgl}_n)^j_{\mathfrak{g}_{n,h,\mathcal{M}/S}}$ is closed under the adjoint action of $B$ (arising from the $B$-action on $\mathcal{E}^\dagger_{B,h,\mathcal{M}/S}$), and

\begin{equation}
(\mathfrak{pgl}_n)^j_{\mathfrak{g}_{n,h,\mathcal{M}/S}} / (\mathfrak{pgl}_n)^{j+1}_{\mathfrak{g}_{n,h,\mathcal{M}/S}} \simeq (\mathfrak{g}_{\mathcal{M}/S} \otimes B^\vee)^{m_j},
\end{equation}

where $m_j = n - |j|$ (resp., $m_j = n - 1$) if $j \neq 0$ (resp., $j = 0$).

**Definition 4.12.1.**

An $(\mathfrak{sl}_n, h)$-oper on $\mathcal{M}/S$ of canonical type II is an $(\mathfrak{sl}_n, h)$-oper obtained, via a change of structure group $\text{GL}_n \to \text{PGL}_n$, from $\mathcal{D}^\natural$ (cf. (366)) for some $\mathcal{D}^\natural \in \Gamma(U, h\text{-Diff}_{\mathfrak{sl}_n,\mathcal{M}/S})$. (In particular, the underlying $B$-torsor of any $(\mathfrak{sl}_n, h)$-oper of canonical type II coincides with $\mathcal{E}^\dagger_{B,h,\mathcal{M}/S}$.)
Note that by Corollary 4.11.4, whether an \((\mathfrak{sl}_n, \hbar)\)-oper is of canonical type II or not does not depend on the choice of an \((n, \hbar)\)-determinant data \(U\) in Definition 4.12.1. Also, an \((\mathfrak{sl}_n, \hbar)\)-oper on \(\mathcal{U}/S\) of canonical type II arises from a unique \(\mathcal{D}^\bullet \in \Gamma(U, h\mathcal{D}iff_{n, U, \mathcal{U}/S}^\bullet)\).

**Remark 4.12.2.**

In consideration of the discussion in §2.4, we mention some properties concerning the right PGL\(_n\)-torsor \(\mathcal{E}^\dagger_{n, h, \mathcal{U}/S}\).

(i) Let us fix a log chart \((U, x)\) on \(X_{\log}\) over \(S_{\log}\) (cf. §1.2). By means of the dual base \(\partial_x \in \Gamma(U, T_{U_{\log}/S_{\log}})\) of \(d\log(x)\), we obtain canonically a composite isomorphism
\[
\mathcal{D}^n_{h, U_{\log}/S_{\log}} \otimes \mathcal{B}^\vee \xrightarrow{\sim} \bigoplus_{j=0}^{n-1} \mathcal{O}_U \cdot \partial_x^j \otimes \mathcal{B}^\vee \xrightarrow{\sim} (\mathcal{B}^\vee)_{(n-j)}
\]
that is compatible with the filtration \(\{\mathcal{D}^n_{h, U_{\log}/S_{\log}} \otimes \mathcal{B}^\vee\}_{j=0}^n\) on \(\mathcal{D}^n_{h, U_{\log}/S_{\log}} \otimes \mathcal{B}^\vee\) and the filtration \(\{(\mathcal{B}^\vee)_{(n-j)}\}_{j=0}^n\) on \((\mathcal{B}^\vee)_{(n-j)}\). It yields an isomorphism
\[
\mathcal{E}^\dagger_{n, h, \mathcal{U}/S} \xrightarrow{\sim} \mathcal{I}som_{\mathcal{O}_U}(\mathcal{O}_{\lceil U_{\log}}, \mathcal{D}^n_{h, U_{\log}/S_{\log}} \otimes \mathcal{B}^\vee) \xrightarrow{\sim} (\mathcal{B}^\vee)^x \times_{\mathbb{G}_m} (U \times \kappa \text{GL}_n)
\]
(cf. §3.7 for the definition of \((-)^x\)) of right GL\(_n\)-torsors, which induces a trivialization
\[
\mathcal{E}^\dagger_{n, h, \mathcal{U}/S} \xrightarrow{\sim} U \times \kappa \text{PGL}_n
\]
of the right PGL\(_n\)-torsor \(\mathcal{E}^\dagger_{n, h, \mathcal{U}/S}\). Moreover, one obtains a canonical trivialization
\[
\text{triv}_{B, h(U, x)} : \mathcal{E}^\dagger_{B, h, \mathcal{U}/S} \xrightarrow{\sim} U \times \kappa B
\]
of the right \(B\)-torsor \(\mathcal{E}^\dagger_{B, h, \mathcal{U}/S}\).

(ii) It follows from Proposition 2.2.5 and Corollary 4.11.3 that for each \((\mathfrak{sl}_n, \hbar)\)-oper \(\mathcal{E}^\bullet\) on \(\mathcal{U}/S\) there exists uniquely a pair
\[
(\mathcal{E}^{\circ \bullet}, \text{can}_{\mathcal{E}^\bullet})
\]
consisting of an \((\mathfrak{sl}_n, \hbar)\)-oper on \(\mathcal{U}/S\) of canonical type II and an isomorphism \(\text{can}_{\mathcal{E}^\bullet} : \mathcal{E}^\bullet \xrightarrow{\sim} \mathcal{E}^{\circ \bullet}\) of \((\mathfrak{sl}_n, \hbar)\)-opers. In particular, the underlying \(B\)-torsors of \((\mathfrak{sl}_n, \hbar)\)-opers are all isomorphic to \(\mathcal{E}^{\circ \bullet}_{B, h, \mathcal{U}/S}\), which does not depends on the choice of \(B\). By construction, the assignment \(\mathcal{E}^\bullet \mapsto (\mathcal{E}^{\circ \bullet}, \text{can}_{\mathcal{E}^\bullet})\) is compatible with any restriction of \(U\), as well as any base-change over \(S\).

(iii) It follows from Proposition 4.10.1 (ii), Corollary 4.11.4, and the isomorphism \(\circ \beta^n\) (cf. [348]) that \(\mathcal{D}\mathfrak{p}_{\mathfrak{sl}_n, h, \mathcal{U}/S}\) may be regarded as a relative affine
space over $S$ modeled on $\mathbb{V}(f_*(\mathcal{B}_{X^{log}/S^{log}}))$. We review this structure of relative affine space as follows. First, we observe that the composite
\[ \mathcal{B}_{X^{log}/S^{log}} \hookrightarrow \mathcal{B}_{X^{log}/S^{log}}^{n-1} \]
\[ \xrightarrow{\gamma} \Omega_{X^{log}/S^{log}} \otimes \mathcal{E}nd_{\mathcal{O}_U}(\mathcal{D}_{h,X^{log}/S^{log}}^{\leq n} \otimes \mathcal{B}^\vee) \]
\[ \xrightarrow{} \Omega_{X^{log}/S^{log}} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{D}_{h,X^{log}/S^{log}}^{\leq n} \otimes \mathcal{B}^\vee) \]
\[ \sim \hom_{\mathcal{O}_X}(\mathcal{T}_{X^{log}/S^{log}}, (\mathfrak{pgl}_n)_{\mathcal{E}nd_{\mathcal{O}_U}(D_{h,U^{log}/S^{log}}^{\leq n})}) \]
(cf. (377) for the definition of $\gamma$) is injective. Let $\mathcal{E}^{\bullet, \diamond} = (\mathcal{E}_{B,h,x/S}^{\bigotimes}, \nabla^\bigotimes)$ be an $(\mathfrak{sl}_n, \hbar)$-oper on $X/S$ of canonical type II and $R \in \Gamma(U, \mathcal{B}_{X^{log}/S^{log}}^{n-1})$. Write $R$ for the image of $R$ via the composite injection (429). Then, one verifies that the pair
\[ \mathcal{E}_{+R}^{\bullet, \diamond} := (\mathcal{E}_{B,h,x/S}^{\bigotimes}, \nabla^\bigotimes + R^\bigotimes) \]
forms an $(\mathfrak{sl}_n, \hbar)$-oper on $X/S$ of canonical type II. Moreover, the assignment $(\mathcal{E}^{\bullet, \diamond}, R) \mapsto \mathcal{E}_{+R}^{\bullet, \diamond}$ defines the action on $\mathfrak{D}_{n,h,x/S}^{\mathfrak{pgl}_n}$ of canonical type II, which gives the structure of relative affine space mentioned above (cf. Proposition 4.8.3).

Remark 4.12.3.
We shall study the adjoint vector bundle $(\mathfrak{pgl}_n)_{\mathcal{E}nd_{\mathcal{O}_U}(D_{h,U^{log}/S^{log}}^{\leq n})}$ of the case where $\hbar = 0$. Consider the symmetric algebra
\[ S_{\mathcal{O}_X}(\mathcal{T}_{U^{log}/S^{log}}) := \bigoplus_{j \geq 0} S^j_{\mathcal{O}_U}(\mathcal{T}_{U^{log}/S^{log}}) \]
on $\mathcal{T}_{U^{log}/S^{log}}$ over $\mathcal{O}_U$, where $S^j_{\mathcal{O}_U}(\mathcal{T}_{U^{log}/S^{log}})$ denotes the $j$-th symmetric power of $\mathcal{T}_{U^{log}/S^{log}}$ (hence $S^j_{\mathcal{O}_U}(\mathcal{T}_{U^{log}/S^{log}}) = \mathcal{T}_{U^{log}/S^{log}}^{\otimes j}$). It follows from the definition of $\mathcal{D}_U^{\leq \infty}$ (cf. § 4.4) that there exists a canonical isomorphism
\[ \mathcal{D}_{0,U^{log}/S^{log}}^{\leq \infty} \sim S_{\mathcal{O}_X}(\mathcal{T}_{U^{log}/S^{log}}) \]
that is compatible with the respective increasing filtrations $\{\mathcal{D}_{0,U^{log}/S^{log}}^{\leq j}\}_{j \geq 0}$ and $\{\bigoplus_{j \geq j'} S^j_{\mathcal{O}_U}(\mathcal{T}_{U^{log}/S^{log}})\}_{j \geq 0}$. In particular, we have an isomorphism
\[ \mathcal{D}_{0,U^{log}/S^{log}}^{\leq n} \otimes \mathcal{B}^\vee \sim \bigoplus_{j=0}^{n-1} \mathcal{T}_{U^{log}/S^{log}}^{\otimes j} \otimes \mathcal{B}^\vee \]
of \( \mathcal{O}_U \)-modules, and this isomorphism gives (by the definition of \( \mathcal{E}_{n,h,U/S}^+ \)) a canonical decomposition

\[
(\text{434}) \quad (\mathfrak{pgl}_n)_{\mathcal{E}_{n,h,U/S}}^j \xrightarrow{\mathcal{E}_{n,h,U/S}} \bigoplus_{l=j}^{n-1} (\mathfrak{pgl}_n)_{\mathcal{E}_{n,h,U/S}}^{l} / (\mathfrak{pgl}_n)_{\mathcal{E}_{n,h,U/S}}^{l+1}
\]

for \(-n \leq j \leq n\), that is compatible with the respective \( B \)-actions.

4.13. Finally, we consider the case of positive characteristic. That is, suppose (cf. § 4.3) that the characteristic \( \text{char}(k) \) of \( k \) is a prime \( p \) satisfying that \( n < p \).

As a remarkable point of dealing with opers in positive characteristic is that there always exists (even if \( X/S \) is not necessarily a unpointed smooth curve (cf. Remark 4.9.2 (iii))) an \((n, \hbar)\)-determinant data. In particular, any \((\mathfrak{sl}_n, \hbar)\)-oper may be necessarily constructed from a globally defined \((\text{GL}_n, \hbar, \mathbb{U})\)-oper (as well as a certain \( \hbar \)-tlcdo on \( B \)) for some \((n, \hbar)\)-determinant data \( \mathbb{U} = (B, \nabla_0) \).

Note that the classical \( p \)-curvature map (cf., e.g., [41], § 5, p. 190 for the case where \( \hbar = 1 \)) of \( \hbar \)-log integrable vector bundle is compatible, in the evident sense, with the \( p \)-curvature map of the associated \( \hbar \)-log integrable \( \text{GL}_n \)-torsor. In the following, we shall not distinguish between these definitions of the \( p \)-curvature map.

**Definition 4.13.1.**

Let \( \mathbb{U} := (B, \nabla_0) \) be an \((n, \hbar)\)-determinant data for \( U^\log \) over \( S^\log \). We shall say that a \((\text{GL}_n, \hbar, \mathbb{U})\)-oper \( \nabla^\circ \) on \( \mathbb{U}/S \) is **dormant** if \( p^\psi(D^{<n}_{h,U^\log/S^\log} \otimes B^\vee, \nabla^\circ) = 0 \).

Here, apply Proposition 3.2.2 to the case where the collection of data \((G, G', w : G \to G')\) is taken to be \ "{(GL}_n, G_m, \mu_{G_m} : \text{GL}_n \to G_m}\), where \( \mu_{G_m} \) denotes the determinant map. Then, for an \((n, \hbar)\)-determinant data \( \mathbb{U} \) satisfying that \( p^\psi(D^{<n}_{h,U^\log/S^\log} \otimes B^\vee, \nabla_0) \neq 0 \), it turns out that there are no dormant \((\text{GL}_n, \hbar, \mathbb{U})\)-opers. Hence, to proceed with our discussion, we should consider whether there exists at least an \((n, \hbar)\)-determinant data \( \mathbb{U} := (B, \nabla_0) \) with \( p^\psi(D^{<n}_{h,U^\log/S^\log} \otimes B^\vee, \nabla_0) = 0 \) or not. But, one may prove the following proposition.

**Proposition 4.13.2.**

(i) There exists an \((n, \hbar)\)-determinant data \( \mathbb{U} := (B, \nabla_0) \) for \( U^\log \) over \( S^\log \) with \( p^\psi(D^{<n}_{h,U^\log/S^\log} \otimes B^\vee, \nabla_0) = 0 \).

(ii) Let \( \mathbb{U} \) be as asserted in (i) and \( \nabla^\circ \) be a \((\text{GL}_n, \hbar, \mathbb{U})\)-oper on \( \mathbb{U}/S \). Write \( \mathcal{E}^\circ = (\mathcal{E}_S, \nabla_\mathcal{E}) \) for the \((\mathfrak{sl}_n, \hbar)\)-oper on \( \mathbb{U}/S \) associated, via the composite
Proposition 3.3.1, its composite isomorphism, an S resp’d portion of (335), one verifies that \( p \) is dormant if and only if \( \mathcal{E} \) is dormant (cf. Definition 3.6.1).

Proof. First, we shall consider assertion (i). Since \( p \) does not divide \( n \), one may choose a pair of nonnegative integers \((k, l)\) satisfying that \( p \cdot k = n \cdot l + \frac{n(q-1)}{2} \).

Let us take \( B = \Omega_{X}^{\log} \). Then,

\[
\det(D_{h,U}^{<n}/S_{\log} \otimes \mathcal{B}^{v}) \sim \mathcal{T}_{U}^{\otimes n} \log_{S} \otimes \mathcal{T}_{U}^{\otimes (l-n)} \log_{S} \text{can} \otimes p \sim F_{U/S}^{n}((\text{id}_{U} \times F_{S})^{*}(\mathcal{T}_{U}^{\otimes k} \log_{S} \otimes \mathcal{T}_{U}^{\otimes (l-n)} \log_{S} \text{can} ))
\]

(cf. Remark 4.9.2 (i) for the first isomorphism). The canonical \( S \)-\( h \)-connection \( \nabla_{\text{can}}^{\log}(\text{id}_{U} \times F_{S})^{*}(\mathcal{T}_{U}^{\otimes k} \log_{S} \otimes \mathcal{T}_{U}^{\otimes (l-n)} \log_{S} \text{can} )_{h} \) (cf. §3.3) on \( F_{U/S}^{n}((\text{id}_{U} \times F_{S})^{*}(\mathcal{T}_{U}^{\otimes k} \log_{S} \otimes \mathcal{T}_{U}^{\otimes (l-n)} \log_{S} \text{can} )) \) carries, via this composite isomorphism, an \( S \)-\( h \)-log connection \( \nabla_{0} \) on \( \det(D_{h,U}^{<n}/S_{\log} \otimes \mathcal{B}^{v}) \). By Proposition 3.3.1, its \( p \)-curvature is zero, and hence, we have obtained the pair \((B, \nabla_{0})\) which forms a required \((n, h)\)-determinant data.

Next, we shall consider assertion (ii). By applying the isomorphism in the resp’d portion of (335), one verifies that \( p_{U}((D_{h,U}^{<n}/S_{\log} \otimes \mathcal{B}^{v}), \nabla^{v}) = 0 \) if and only if the \( p \)-curvature of both its associated \( h \)-log integrable PGL\(_n\)-torsor (i.e., \((\mathcal{E}_{\text{PGL}n}, \nabla_{\mathcal{E}})\)) and its associated \( h \)-log integrable \( \mathbb{G}_{m} \)-torsor (i.e., \((\det(D_{h,U}^{<n}/S_{\log} \otimes \mathcal{B}^{v}), \nabla_{0})\)) are zero. Hence, the assertion follows from the hypothesis that the \( p \)-curvature of \((\det(D_{h,U}^{<n}/S_{\log} \otimes \mathcal{B}^{v}), \nabla_{0})\) is zero.

If \( U \) is as asserted in Proposition 4.13.2 (i), then we shall denote by

\[
\mathcal{D}_{\mathbf{p}}^{\nabla^{v}Z_{aa}...}
\]

the subfunctor of \( \mathcal{D}_{\mathbf{p}}^{\nabla^{v}Z_{aa}...} \) classifying dormant \((\text{GL}_{n}, h, U)\)-opers. By restricting the isomorphism \( \Gamma \Lambda_{n,h,U,X/S}^{\nabla^{v}Z_{aa}...} \) in Corollary 4.11.3, we obtain an isomorphism between \( \mathcal{D}_{\mathbf{p}}^{\nabla^{v}Z_{aa}...} \) and \( \mathcal{D}_{\mathbf{p}}^{\nabla^{v}Z_{aa}...} \) as asserted in the following.

Corollary 4.13.3.

(i) There exists a canonical isomorphism

\[
\Gamma \Lambda_{n,h,U,X/S}^{\nabla^{v}Z_{aa}...} : \mathcal{D}_{\mathbf{p}}^{\nabla^{v}Z_{aa}...} \otimes \mathcal{D}_{\mathbf{p}}^{\nabla^{v}Z_{aa}...} \rightarrow \mathcal{D}_{\mathbf{p}}^{\nabla^{v}Z_{aa}...} \otimes \mathcal{D}_{\mathbf{p}}^{\nabla^{v}Z_{aa}...}
\]

of schemes over \( S \).

(ii) If, moreover, \( \mathcal{L} = (\mathcal{L}, \nabla_{\mathcal{L}}) \) is an \( h \)-log integrable line bundle on \( X^{\log} \) with vanishing \( p \)-curvature. (Hence, the \((n, h)\)-determinant data \( U \otimes \mathcal{L}^{v} := (B \otimes \mathcal{L}^{v}, \nabla_{0} \otimes \mathcal{L}^{v}) \) satisfies the condition required in Proposition 4.13.2 (i), i.e., that \( p_{U}((\det(D_{h,U}^{<n}/S_{\log} \otimes (B \otimes \mathcal{L}^{v}), \nabla_{0} \otimes \mathcal{L}^{v}))) = 0. \) Then, by
restricting the isomorphism $\Gamma_{U \to U \otimes \mathcal{L}^\vee}$ asserted in Corollary 4.11.4, we obtain an isomorphism

$$\Gamma_{U \to U \otimes \mathcal{L}^\vee} : \mathcal{O}_{GL_n,h,U,X/S} \rightarrow \mathcal{O}_{GL_n,h,U \otimes \mathcal{L}^\vee,X/S}$$

of schemes over $S$ which makes the square diagram

$$(439) \Gamma_{\Delta_{n,h,U,X/S}} : \mathcal{O}_{GL_n,h,U,X/S} \rightarrow \mathcal{O}_{GL_n,h,U \otimes \mathcal{L}^\vee,X/S}$$

commute.

Proof. The assertion follows from Proposition 4.11.2 (ii) and Corollary 4.11.4.

5. Deformation theory of opers

In this section, we study de Rham cohomology of various log integrable vector bundles arising from a $(\mathfrak{g}, \hbar)$-oper, and describe the deformation space (in some senses) of the $(\mathfrak{g}, \hbar)$-oper (i.e., the tangent bundle of various moduli stacks classifying opers) in terms of cohomology. After proving some properties concerning computations of cohomology, we conclude (cf. Corollary 5.12.2) that if the moduli stack $\mathcal{O}_{p_{\mathfrak{g},\hbar,p,g,r}}$ of dormant opers is unramified over $\mathbb{M}_{g,r}$ at a point, then it is also flat (hence étale) at this point. This assertion will be one of the important parts in proving Joshi’s conjecture.

We shall review notation as follows; let $k$ be a perfect field (cf. §3.5), $G$ a split connected semisimple algebraic group of adjoint type over $k$ (with the Lie algebra $\mathfrak{g}$) which admits a pinning $\mathcal{G}$ (cf. (7.0)) and satisfies either one of the two conditions $(\text{Char})_0, (\text{Char})^W_p$ described in §2.1. Also, let $S$ be a scheme over $k$ and $X/S := (f : X \to S, \{\sigma_i\}_{i=1}^r)$ a pointed stable curve over $S$ of type $(g,r)$.

5.1. First, we recall two spectral sequences associated with a morphism of sheaves on $X$. 
Let $\nabla : \mathcal{K}^0 \to \mathcal{K}^1$ be a morphism of sheaves of abelian groups on $X$. It may be thought of as a complex concentrated at degree 0 and 1; we denote this complex by
\begin{equation}
\mathcal{K}^* \{\nabla\}
\end{equation}
(\text{where } \mathcal{K}^i \{\nabla\} := \mathcal{K}^i \text{ for } i = 0, 1). For $i = 0, 1, \ldots$, one may define the sheaf
\begin{equation}
\mathcal{R}^i f_* (\mathcal{K}^* \{\nabla\})
\end{equation}
on $S$, where $\mathcal{R}^i f_* (-)$ is the $i$-th hyper-derived functor of $\mathcal{R}^0 f_* (-)$ (cf. \cite{[1]}, (2.0)). In particular, $\mathcal{R}^0 f_* (\mathcal{K}^* \{\nabla\}) = f_* (\text{Ker}(\nabla))$.

Consider the spectral sequence
\begin{equation}
\check{E}^{p,q}_2 := \mathcal{R}^q f_* (\mathcal{K}^p \{\nabla\}) \Rightarrow \mathcal{R}^{p+q} f_* (\mathcal{K}^* \{\nabla\}),
\end{equation}
which we call the \textit{Hodge to de Rham spectral sequence} of the complex $\mathcal{K}^* \{\nabla\}$. This spectral sequence \((442)\) yields a short exact sequence
\begin{equation}
0 \to \text{Coker}(\mathcal{R}^0 f_* (\nabla)) \xrightarrow{\check{e}_2^{0}} \mathcal{R}^1 f_* (\mathcal{K}^* \{\nabla\}) \xrightarrow{\check{e}_2^1} \text{Ker}(\mathcal{R}^1 f_* (\nabla)) \to 0
\end{equation}
of $\mathcal{O}_S$-modules, where $\mathcal{R}^i f_* (\nabla) \ (i = 0, 1)$ denotes the morphism $\mathcal{R}^i f_* (\mathcal{K}^0) \to \mathcal{R}^i f_* (\mathcal{K}^1)$ obtained by applying the functor $\mathcal{R}^i f_* (-)$ to the morphism $\nabla$.

Also, we consider the spectral sequence
\begin{equation}
\check{E}^{p,q}_2 := \mathcal{R}^q (\mathcal{H}^q (\mathcal{K}^* \{\nabla\})) \Rightarrow \mathcal{R}^{p+q} f_* (\mathcal{K}^* \{\nabla\}),
\end{equation}
where $\mathcal{H}^q (\mathcal{K}^* \{\nabla\})$ denotes the $q$-th cohomology sheaf of the complex $\mathcal{K}^* \{\nabla\}$, and call it the \textit{conjugate spectral sequence} of $\mathcal{K}^* \{\nabla\}$. This spectral sequence \((444)\) induces a short exact sequence
\begin{equation}
0 \to \mathcal{R}^1 f_* (\text{Ker}(\nabla)) \xrightarrow{\check{e}_2^1} \mathcal{R}^1 f_* (\mathcal{K}^* \{\nabla\}) \xrightarrow{\check{e}_2^0} f_* (\text{Coker}(\nabla)) \to 0
\end{equation}
of $\mathcal{O}_S$-modules.

5.2. Let $\rho = (\rho_i)_{i=1}^r \in \mathfrak{c}^{(r)}(S)$, and $\mathcal{E}^\bullet = (\mathcal{E}_{B,h,X/S}^\dagger, \nabla_{\mathcal{E}})$ be a $(\mathfrak{g}, h)$-oper on $\mathcal{X}/S$ of canonical type and of radii $\rho$. Denote by
\begin{equation}
\nabla^\text{ad} : \mathfrak{g}_{\mathcal{E}_{B,h,X/S}}^\dagger \to \Omega^{\text{X}^{\text{log}} / S^{\text{log}}} \otimes \mathfrak{g}_{\mathcal{E}_{B,h,X/S}}^\dagger
\end{equation}
the $S$-$h$-log connection on the adjoint vector bundle $\mathfrak{g}_{\mathcal{E}_{B,h,X/S}}^\dagger$ (induced by $\nabla_{\mathcal{E}}$ via the adjoint representation $\text{Ad}_{\mathcal{G}} : \mathcal{G} \to \text{GL}(\mathfrak{g})$). It follows from the definition of a $(\mathfrak{g}, h)$-oper (and the fact that $[p, \mathfrak{g}] \subseteq \mathfrak{g}^{j-1}$) that
\begin{equation}
\nabla^\text{ad}_{i,j} (\mathfrak{g}_{\mathcal{E}_{B,h,X/S}}^\dagger) \subseteq \Omega^{\text{X}^{\text{log}} / S^{\text{log}}} \otimes \mathfrak{g}_{\mathcal{E}_{B,h,X/S}}^{j-1}
\end{equation}
for any $j \in \mathbb{Z}$. We shall write
\begin{equation}
\nabla^\text{ad}_{i,j} : \mathfrak{g}_{\mathcal{E}_{B,h,X/S}}^j \to \Omega^{\text{X}^{\text{log}} / S^{\text{log}}} \otimes \mathfrak{g}_{\mathcal{E}_{B,h,X/S}}^{j-1}
\end{equation}
for the resulting morphism. Moreover, \( \nabla_{\mathcal{E}}^{\text{ad}(j)} \) induces, by passing to the quotients, a well-defined \( \mathcal{O}_X \)-linear morphism

\[
\nabla_{\mathcal{E}}^{\text{ad}(j/j+1)} : \mathfrak{g}_{\mathfrak{b},h,X/S}^{j+1} / \mathfrak{g}_{\mathfrak{b},h,X/S}^{j+1} \rightarrow \Omega_X^{\text{log}} / \Omega_X^{\text{log}} \otimes (\mathfrak{g}_{\mathfrak{b},h,X/S}^{j+1} / \mathfrak{g}_{\mathfrak{b},h,X/S}^{j+1}).
\]

In the following, we shall study the morphism \( \nabla_{\mathcal{E}}^{\text{ad}(j/j+1)} \) in some detail. For \( \Box = -1 \) or \( 1 \), we shall write \( \mathfrak{g}_{\text{ad}(p_0)} \) for the space of \( \text{ad}(p_0) \)-coinvariants (i.e., \( \mathfrak{g}_{\text{ad}(p_0)} := \mathfrak{g}/\text{Im}(\text{ad}(p_0)) \)). \( \mathfrak{g}_{\text{ad}(p_1)} \) admits a grading \( \mathfrak{g}_{\text{ad}(p_1)} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{\text{ad}(p_1),j} \) induced, via the quotient \( \mathfrak{g} \rightarrow \mathfrak{g}_{\text{ad}(p_1)} \), by the grading \( \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j \) (cf. \( \mathfrak{g}_{\text{ad}(2\rho)} \) in \( \S 2.1 \)). One verifies from the fact that \( \{p_{-1}, 2\rho, p_1\} \) forms an \( \mathfrak{sl}_2 \)-triple (cf. \( \S 2.5 \)) that the two composites

\[
\begin{align*}
\mathfrak{g}_{\text{ad}(p_1)} & \twoheadrightarrow \mathfrak{g} \rightarrow \mathfrak{g}_{\text{ad}(p_{-1})}, & \mathfrak{g}_{\text{ad}(p_{-1})} & \twoheadrightarrow \mathfrak{g} \rightarrow \mathfrak{g}_{\text{ad}(p_1)}
\end{align*}
\]

are isomorphisms. The second composite \( \mathfrak{g}_{\text{ad}(p_{-1})} \rightarrow \mathfrak{g}_{\text{ad}(p_1)} \) in \( \mathfrak{g}_{\text{ad}(p_1)} \) is compatible with the respective gradings, i.e., induces an isomorphism

\[
\mathfrak{g}_{\text{ad}(p_{-1}),j} \sim \mathfrak{g}_{\text{ad}(p_1),j}
\]

(cf. \( \mathfrak{g}_{\text{ad}(1)} \)) for any \( j \in \mathbb{Z} \). On the other hand, since we have assumed that either \( (\text{Char})_0 \) or \( (\text{Char})_{\Pi} \) is satisfied, there exists (cf. \( \mathfrak{g}_{\text{ad}(1)} \), Chap. VI, Theorem 5.1) a \( \mathcal{G} \)-invariant nondegenerate symmetric \( k \)-bilinear form

\[
\mathfrak{B} \mathfrak{I} \mathfrak{f} : \mathfrak{g} \times \mathfrak{g} \rightarrow k
\]

on \( \mathfrak{g} \). Moreover, it follows from \( \mathfrak{g}_{\text{ad}(1)} \), Chap. VI, Lemma 5.2, that the isomorphism \( \mathfrak{g} \sim \mathfrak{g}^\vee \) determined by \( \mathfrak{B} \mathfrak{I} \mathfrak{f} \) gives, via passing to the quotients \( \mathfrak{g} \rightarrow \mathfrak{g}_{\text{ad}(p_1)} \) and \( \mathfrak{g}^\vee \rightarrow (\mathfrak{g}_{\text{ad}(p_1)})^\vee \) (i.e., the dual of the inclusion \( \mathfrak{g}_{\text{ad}(p_1)} \rightarrow \mathfrak{g} \)), an isomorphism

\[
\mathfrak{g}_{\text{ad}(p_1)} \sim (\mathfrak{g}_{\text{ad}(p_1)})^\vee.
\]

This isomorphism gives, via taking the gradings, an isomorphism

\[
\mathfrak{g}_{\text{ad}(p_1),j} \sim (\mathfrak{g}_{j-1})^\vee
\]

for \( j \in \mathbb{Z} \). The isomorphisms \( \mathfrak{g}_{\text{ad}(p_1)} \) and \( \mathfrak{g}_{\text{ad}(p_1),j} \) yields an exact sequence

\[
0 \rightarrow (\mathfrak{g}_{\text{ad}(p_1)})^\vee \rightarrow \mathfrak{g}_{\text{ad}(p_{-1})} \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}_{\text{ad}(p_1)} \rightarrow 0.
\]

By taking the gradings of the vector spaces in \( \mathfrak{g}_{\text{ad}(p_1)} \) and applying the isomorphisms \( \mathfrak{g}_{\text{ad}(p_{-1})} \) and \( \mathfrak{g}_{\text{ad}(p_1),j} \), we obtain an exact sequence

\[
0 \rightarrow (\mathfrak{g}_{j-1})^\vee \rightarrow \mathfrak{g}^j / \mathfrak{g}^{j+1} \rightarrow \mathfrak{g}^j / \mathfrak{g}^{j-1} \rightarrow \mathfrak{g}^{j-1} / \mathfrak{g}^j \rightarrow \mathfrak{g}_{j-1} \rightarrow 0
\]

\((j \in \mathbb{Z})\). If \( j \leq 1 \) (resp., \( j \geq 0 \)), then \( \mathfrak{g}_{j-1} = 0 \) (resp., \( (\mathfrak{g}_{j-1})^\vee = 0 \)), and moreover, the resulting short exact sequence defined by \( \mathfrak{g}_{\text{ad}(p_{-1})} \) is split. We note that the morphisms in \( \mathfrak{g}_{\text{ad}(p_{-1})} \) are all compatible with the respective \( \mathfrak{B} \)-actions on the domain and codomain.
There exists, by the definition of $\mathcal{V}_{\Sigma, h, x/S}$, an isomorphism

$$\mathcal{g}^{\text{ad}(p_1)} \overset{\nabla}{\to} \Omega_{X, \log}^{\otimes} \otimes \mathcal{V}_{\Sigma, h, x/S}$$

that is compatible with the respective gradings. The composite $\mathcal{g} \to \mathcal{g}^{\text{ad}(p_1)} \overset{\nabla}{\to} (\mathcal{g}^{\text{ad}(p_1)})^{\vee}$ (cf. (453)) induces a morphism

$$\mathcal{g}^{\vee \Omega}_{\Sigma, h, x/S} : \mathcal{g}^{\vee \Omega}_{\Sigma, h, x/S} \to \Omega_{X, \log}^{\otimes} \otimes \mathcal{V}^{\vee}_{\Sigma, h, x/S}$$

which satisfies (by (454)) that

$$(\mathcal{g}^{\text{ad}(p_1)})^{\vee} \otimes \mathcal{V}^{\vee}_{\Sigma, h, x/S} \to \mathcal{g}^{\vee \Omega}_{\Sigma, h, x/S}$$

for any $j \in \mathbb{Z}$.

Now, let us choose an integer $j$ satisfying that $j \leq 1$. Then, $\mathcal{g}_{j-1}^{\text{ad}(p_1)} = 0$, and (456) becomes the split short exact sequence

$$0 \to (\mathcal{g}_{j-1}^{\text{ad}(p_1)})^{\vee} \to \mathcal{g}^{j}/\mathcal{g}^{j+1} \to \mathcal{g}^{j-1}/\mathcal{g}^{j} \to 0.$$

The second arrow $(\mathcal{g}_{j-1}^{\text{ad}(p_1)})^{\vee} \to \mathcal{g}^{j}/\mathcal{g}^{j+1}$ in (460) induces the second arrow in the sequence

$$0 \to \Omega_{X, \log}^{\otimes} \otimes \mathcal{V}^{\vee}_{\Sigma, h, x/S} \otimes (-j, -j-1) \to \mathcal{g}^{j}/\mathcal{g}^{j+1} \to \mathcal{g}^{j-1}/\mathcal{g}^{j} \to 0.$$

It follows from the definition of $\nabla^{\text{ad}(j+1)}$ (cf. (449)) and the split exact sequence (460) that the sequence (461) forms a split short exact sequence of $\mathcal{O}_{X}$-modules.

Next, let us choose an integer $j$ satisfying that $j \geq 0$. Then, $(\mathcal{g}_{j-1}^{\text{ad}(p_1)})^{\vee} = 0$, and (456) becomes the split short exact sequence

$$0 \to \mathcal{g}^{j}/\mathcal{g}^{j+1} \to \mathcal{g}^{j-1}/\mathcal{g}^{j} \to \mathcal{g}^{j-1}/\mathcal{g}^{j} \to 0.$$

The third arrow $\mathcal{g}^{j-1}/\mathcal{g}^{j} \to \mathcal{g}^{j-1}/\mathcal{g}^{j}$ in (462) induces the third arrow in the sequence

$$0 \to \mathcal{g}^{j}/\mathcal{g}^{j+1} \to \mathcal{g}^{j-1}/\mathcal{g}^{j} \to \mathcal{g}^{j-1}/\mathcal{g}^{j} \to 0.$$

It follows from the definition of $\nabla^{\text{ad}(j+1)}$ and (450) that the sequence (463) forms a split short exact sequence of $\mathcal{O}_{X}$-modules.

The following Lemma 5.2.1 will be used in the proof of Proposition 5.2.2.
Lemma 5.2.1.
Let us fix $j \in \mathbb{Z}$.

(i) The composite

\[
\begin{align*}
&f_*(\mathcal{V}^j_{g,h,X/S}) \to f_*(\Omega^\log_{X/S} \otimes g^j_{g,h,X/S}) \to \text{Coker}(f_*(\nabla^\text{ad}(j+1))),
\end{align*}
\]

where $\varsigma^j_{g,h,X/S}$ denotes the morphism obtained by taking the $j$-th grading of $\varsigma_{g,h,X/S}$ (cf. (125)), is an isomorphism of $\mathcal{O}_S$-modules. In particular, we have an isomorphism

\[
\begin{align*}
f_*(\mathcal{V}^j_{g,h,X/S}) \sim \text{Coker}(f_*(\nabla^\text{ad})).
\end{align*}
\]

(ii) The composite

\[
\begin{align*}
\text{Ker}(\mathbb{R}^1 f_*(\mathcal{V}^\text{ad}(j))) &\to \mathbb{R}^1 f_*(g^j_{g,h,X/S}) \\
\mathbb{R}^1 f_*(\mathcal{V}^\text{ad}(j)) &\to \mathbb{R}^1 f_*(\Omega^\log_{X/S} \otimes (\mathcal{V}^{\text{ad}}_{g,h,X/S}/\mathcal{V}^{\text{ad}}_{g,h,X/S}))
\end{align*}
\]

where $\varsigma^\text{ad}_{g,h,X/S}$ denotes the morphism obtained by taking the $j$-th grading of $\varsigma_{g,h,X/S}$ (cf. (458)), is an isomorphism of $\mathcal{O}_S$-modules. In particular, we have an isomorphism

\[
\begin{align*}
\text{Ker}(\mathbb{R}^1 f_*(\mathcal{V}^\text{ad})) &\sim \mathbb{R}^1 f_*(\Omega^\log_{X/S} \otimes \mathcal{V}^\text{ad}_{g,h,X/S}).
\end{align*}
\]

Proof. We shall only consider assertion (ii) since assertion (i) follows from a similar argument.
If \( j \geq 0 \), then we have \( \text{Ker}(\mathbb{R}^1 f_*(\nabla_{E}^{\text{ad}(j/j+1)}) = 0 \) (since (463) is split) and \( V_{g,h,X/S}/V_{g,h,X/S} = 0 \). Hence, it suffices to prove the case where \( j < -1 \). Consider the morphism of sequences
\[
\begin{array}{ccc}
0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \\
\mathbb{R}^1 f_*(g^j_{e_{b,h,x/S}}) & \longrightarrow & \mathbb{R}^1 f_*(\nabla_{E}^{\text{ad}(j+1)}) \\
\downarrow & & \downarrow \\
\mathbb{R}^1 f_*(g^j_{e_{b,h,x/S}}) & \longrightarrow & \mathbb{R}^1 f_*(\Omega_{X/S}^{\log} \otimes g^{j-1}_{e_{b,h,x/S}}) \\
\downarrow & & \downarrow \\
\mathbb{R}^1 f_*(g^j_{e_{b,h,x/S}}/g^{j+1}_{e_{b,h,x/S}}) & \longrightarrow & \mathbb{R}^1 f_*(\Omega_{X/S}^{\log} \otimes (g^{j-1}_{e_{b,h,x/S}}/g^{j+1}_{e_{b,h,x/S}})) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
\]

Since \( g^j_{e_{b,h,x/S}}/g^{j+1}_{e_{b,h,x/S}} \) and \( \Omega_{X/S}^{\log} \otimes (g^{j-1}_{e_{b,h,x/S}}/g^{j+1}_{e_{b,h,x/S}}) \) are direct sums of finite copies of \( \Omega_{X/S}^{\log} \) (cf. Remark 2.2.2 (i)), we have that
\[
\begin{equation}
(469) \quad f_*(g^{j+1}_{e_{b,h,x/S}}) = f_*(\Omega_{X/S}^{\log} \otimes (g^{j-1}_{e_{b,h,x/S}}/g^{j+1}_{e_{b,h,x/S}})).
\end{equation}
\]

By (469) and the fact that \( X/S \) is of relative dimension 1 (hence \( \mathbb{R}^2 f_*(-) = 0 \)), the both sides of vertical sequences in (468) are exact. On the other hand, by the split short exact sequence (461), the bottom horizontal arrow in (468) is surjective. By induction on \( j \), one verifies that both the top and middle horizontal arrows in (468) (of the case for any \( j < -1 \)) are surjective. Hence, by applying the “snake lemma” to (468), we obtain a short exact sequence
\[
(470) \quad 0 \rightarrow \text{Ker}(\mathbb{R}^1 f_*(\nabla_{E}^{\text{ad}(j+1)}) \rightarrow \text{Ker}(\mathbb{R}^1 f_*(\nabla_{E}^{\text{ad}(j)}) \rightarrow \text{Ker}(\mathbb{R}^1 f_*(\nabla_{E}^{\text{ad}(j/j+1)}) \rightarrow 0.
\]
Also, we obtain a morphism of short exact sequences

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{R}^1 f_*(\nabla^{\text{ad}(j+1)}) \\
\downarrow & & \downarrow \\
\text{Ker}(\mathbb{R}^1 f_*(\nabla^{\text{ad}(j+1)})) & \longrightarrow & \mathbb{R}^1 f_*(\Omega_{X^{\log}/S^{\log}} \otimes (\mathcal{V}_{g,h,x/S}^j)^\vee) \\
\downarrow & & \downarrow \\
\text{Ker}(\mathbb{R}^1 f_*(\nabla^{\text{ad}(j+1)})) & \longrightarrow & \mathbb{R}^1 f_*(\Omega_{X^{\log}/S^{\log}} \otimes (\mathcal{V}_{g,h,x/S}^j)^\vee) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & 0
\end{array}
\]

(471)

It follows from the split short exact sequence (461) that the bottom horizontal arrow in (471) is an isomorphism. Hence, by induction on \(j\), we conclude that the middle horizontal arrow in (471) is an isomorphism. This completes the proof of Lemma 5.2.1 (ii).

For the assertion of Proposition 5.2.2, described below, of the case where \(g = \mathfrak{sl}_2\), we refer to [52], Chap. I, Theorem 2.8.

**Proposition 5.2.2.**

(i) \(\mathbb{R}^0 f_*(\mathcal{K}^\bullet[\nabla^{\text{ad}}]) \sim f_*(\text{Ker}(\nabla^{\text{ad}})) = 0\).

(ii) There exists a natural short exact sequence

\[
\begin{array}{ccc}
0 & \longrightarrow & f_*(\mathcal{V}_{g,h,x/S}) \\
\longrightarrow & \longrightarrow & \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^{\text{ad}}]) \\
\longrightarrow & \longrightarrow & \mathbb{R}^1 f_*(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{V}_{g,h,x/S}^j) \\
\longrightarrow & \longrightarrow & 0
\end{array}
\]

(472)

of \(\mathcal{O}_S\)-modules. In particular, \(\mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^{\text{ad}}])\) is a vector bundle on \(S\) of rank \((2g - 2 + r) \cdot \dim(g)\).

(iii) \(\mathbb{R}^2 f_*(\mathcal{K}^\bullet[\nabla^{\text{ad}}]) \sim \mathbb{R}^1 f_*(\text{Coker}(\nabla^{\text{ad}})) = 0\).
Proof. First, we shall consider assertion (i). Consider the morphism of short exact sequences

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
\mathcal{g}^{j+1}_{\mathfrak{E},b,h,x/S} & \Omega^{\log/S_{\log}} \otimes \mathcal{g}^{j}_{\mathfrak{E},b,h,x/S} \\
\downarrow & \downarrow \\
\mathcal{g}^{j}_{\mathfrak{E},b,h,x/S} / \mathcal{g}^{j+1}_{\mathfrak{E},b,h,x/S} & \Omega^{\log/S_{\log}} \otimes (\mathcal{g}^{j-1}_{\mathfrak{E},b,h,x/S} / \mathcal{g}^{j}_{\mathfrak{E},b,h,x/S}) \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

(\(j \in \mathbb{Z}\)). It follows from the exact sequence (473) that the bottom horizontal arrow in (473) is injective. By descending induction on \(j\), we conclude (upon applying the functor \(f_{\ast}(-)\)) that

\[
f_{\ast}(\nabla^{\text{ad}(j)}) : f_{\ast}(\mathcal{g}^{0}_{\mathfrak{E},b,h,x/S}) \to f_{\ast}(\Omega^{\log/S_{\log}} \otimes \mathcal{g}^{-1}_{\mathfrak{E},b,h,x/S})
\]

is injective. On the other hand, since both \(\mathcal{g}^{j}_{\mathfrak{E},b,h,x/S} / \mathcal{g}^{j+1}_{\mathfrak{E},b,h,x/S}\) and \(\Omega^{\log/S_{\log}} \otimes (\mathcal{g}^{j-1}_{\mathfrak{E},b,h,x/S} / \mathcal{g}^{j}_{\mathfrak{E},b,h,x/S})\) are direct sums of finite copies of \(\Omega^{\log/S_{\log}} \otimes (\mathcal{g}^{(j-1)}_{\mathfrak{E},b,h,x/S} / \mathcal{g}^{j}_{\mathfrak{E},b,h,x/S})\), we have

\[
f_{\ast}(\mathcal{g}^{0}_{\mathfrak{E},b,h,x/S}) = f_{\ast}(\mathcal{g}^{-1}_{\mathfrak{E},b,h,x/S}) = \cdots = f_{\ast}(\mathcal{g}^{j}_{\mathfrak{E},b,h,x/S})
\]

and

\[
f_{\ast}(\Omega^{\log/S_{\log}} \otimes \mathcal{g}^{-1}_{\mathfrak{E},b,h,x/S}) = f_{\ast}(\Omega^{\log/S_{\log}} \otimes \mathcal{g}^{-2}_{\mathfrak{E},b,h,x/S}) = \cdots = f_{\ast}(\Omega^{\log/S_{\log}} \otimes \mathcal{g}^{j}_{\mathfrak{E},b,h,x/S}).
\]

Hence, by (474), (475), and (476), we conclude that \(f_{\ast}(\nabla^{\text{ad}})\) is injective, i.e., Ker\((f_{\ast}(\nabla^{\text{ad}})) = \mathbb{R}^{0}f_{\ast}(\mathcal{K}^{\ast}[\nabla^{\text{ad}}]) = 0\). This completes the proof of assertion (i).

Assertion (ii) follows from the short exact sequence (443), Lemma 5.2.1 (i), (ii). The computation of the rank of \(\mathbb{R}^{1}f_{\ast}(\mathcal{K}^{\ast}[\nabla^{\text{ad}}])\) follows from Proposition 2.6.1 and the isomorphism

\[
\mathbb{R}^{1}f_{\ast}(\Omega^{\log/S_{\log}} \otimes \mathcal{V}_{\mathfrak{E},b,h,x/S}) \cong f_{\ast}(\mathcal{V}_{\mathfrak{E},b,h,x/S})
\]
arising from Grothendieck-Serre duality.

Finally, assertion (iii) follows from an argument similar to the argument of the proof of (i). \qed

5.3. Next, we shall describe the deformation sheaf of a \((g, h)\)-oper by means of cohomology associated with a certain complex.

Consider the \(f^{-1}(\mathcal{O}_S)\)-linear morphism

\begin{equation}
\nabla^{\text{ad}\otimes}_g : \tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S} \to \Omega_X^{\log/\log S} \otimes \tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S}
\end{equation}

determined uniquely by the condition that

\begin{equation}
(\nabla^{\text{ad}\otimes}_g(s), \partial) = [s, \nabla_\mathcal{E}(\partial)] - \nabla_\mathcal{E}([a^{\log}_\mathcal{E}_G(s), \partial])
\end{equation}

(cf. (26) for the definition of \(a^{\log}_\mathcal{E}_G\)), where

- \(s\) and \(\partial\) denote any local sections of \(\tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S}\) and \(\mathcal{T}_X^{\log/\log S}\) respectively;
- \(\langle -, - \rangle\) denotes the \(\mathcal{O}_X\)-bilinear pairing

\begin{equation}
(\Omega_X^{\log/\log S} \otimes \tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S}) \times \mathcal{T}_X^{\log/\log S} \to \tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S}
\end{equation}

induced by the natural paring \(\Omega_X^{\log/\log S} \times \mathcal{T}_X^{\log/\log S} \to \mathcal{O}_X\).

The restriction of \(\nabla^{\text{ad}\otimes}_g\) to \(g^{\log}_{\mathcal{E}_G} \subseteq \tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S}\) coincides with \(\nabla^{\text{ad}}_\mathcal{E}\).

**Lemma 5.3.1.**
The image of \(\nabla^{\text{ad}\otimes}_g\) is contained in \(\Omega_X^{\log/\log S} \otimes g^{\log}_{\mathcal{E}_G} \subseteq \tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S}\). Moreover, the image of the composite

\begin{equation}
\nabla^{\text{ad}\otimes}_g \circ \tilde{\iota}_{\mathcal{E}_G} : \tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S} \to \Omega_X^{\log/\log S} \otimes \tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S}
\end{equation}

(cf. (76) for the definition of \(\tilde{\iota}_{\mathcal{E}_G}\)) is contained in \(\Omega_X^{\log/\log S} \otimes \mathcal{G}^{-1}_{\mathcal{E}_G} \subseteq \tilde{\mathcal{T}}_{\mathcal{E}_G}^{\log}_{g, h, \log S}\).
Proof. Since $a^\log_{E_{\mathfrak{g},h,x/S}}$ is compatible with the Lie bracket structures on $\tilde{T}_{E_{\mathfrak{g},h,x/S}}^{\log}/S^{\log}$ and $T_{X^{\log}/S^{\log}}$, we have, for local sections $s \in \tilde{T}_{E_{\mathfrak{g},h,x/S}}^{\log}/S^{\log}$ and $\partial \in T_{X^{\log}/S^{\log}}$,

\begin{equation}
\begin{aligned}
& a^\log_{E_{\mathfrak{g},h,x/S}}([s, \nabla_{E}(\partial)] - \nabla_{E}(a^\log_{E_{\mathfrak{g},h,x/S}}(s), \partial)) \\
= & [a^\log_{E_{\mathfrak{g},h,x/S}}(s), a^\log_{E_{\mathfrak{g},h,x/S}} \circ \nabla_{E}(\partial)] - a^\log_{E_{\mathfrak{g},h,x/S}} \circ \nabla_{E}(a^\log_{E_{\mathfrak{g},h,x/S}}(s), \partial)) \\
= & [a^\log_{E_{\mathfrak{g},h,x/S}}(s), h \cdot \partial] - h \cdot [a^\log_{E_{\mathfrak{g},h,x/S}}(s), \partial] \\
= & 0.
\end{aligned}
\end{equation}

That is, the image of $\nabla^{\text{ad}^\oplus}$ is contained in $\Omega_{X^{\log}/S^{\log}} \otimes \text{Ker}(a^\log_{E_{\mathfrak{g},h,x/S}}) = \Omega_{X^{\log}/S^{\log}} \otimes g^\gg_{E_{\mathfrak{g},h,x/S}}$.

Moreover, it follows from the definition of a $(g, h)$-oper that the image of the composite $\nabla^{\text{ad}^\oplus} \circ \tilde{g}/b$ is contained in $\tilde{T}_{E_{\mathfrak{g},h,x/S}}^{\log}/S^{\log}$ and hence, in

\begin{equation}
\Omega_{X^{\log}/S^{\log}} \otimes (g^\gg_{E_{\mathfrak{g},h,x/S}} \cap \tilde{T}_{E_{\mathfrak{g},h,x/S}}^{\log}/S^{\log}) = \Omega_{X^{\log}/S^{\log}} \otimes g^\gg_{E_{\mathfrak{g},h,x/S}} \cap
\end{equation}

By Lemma 5.3.1, we obtain an $f^{-1}(O_S)$-linear morphism

\begin{equation}
\nabla^{\text{ad}^\oplus}_{E_{\mathfrak{g}}}: \tilde{T}_{E_{\mathfrak{g},h,x/S}}^{\log}/S^{\log} \to \Omega_{X^{\log}/S^{\log}} \otimes g^\gg_{E_{\mathfrak{g},h,x/S}} \cap
\end{equation}

given by the condition that

\begin{equation}
(\nabla^{\text{ad}^\oplus}_{E_{\mathfrak{g}}}(s), \partial) = [s, \nabla_{E}(\partial)] - \nabla_{E}(a^\log_{E_{\mathfrak{g},h,x/S}}(s), \partial])
\end{equation}

$(s \in \tilde{T}_{E_{\mathfrak{g},h,x/S}}^{\log}/S^{\log}, \partial \in T_{X^{\log}/S^{\log}})$. The following lemma will be used in the discussions in § 5.4 and § 5.12.

**Lemma 5.3.2.**

$R^2 f_*(\mathcal{K}^\bullet[\nabla_{E}^{\text{ad}(0)})]) = R^2 f_*(\mathcal{K}^\bullet[\nabla_{E_{\mathfrak{g}}}^{\text{ad}^\oplus}]) = 0$.

**Proof.** By applying the Hodge to de Rham spectral sequence (142), $R^2 f_*(\mathcal{K}^\bullet[\nabla_{E}^{\text{ad}(0)})])$ and $R^2 f_*(\mathcal{K}^\bullet[\nabla_{E_{\mathfrak{g}}}^{\text{ad}^\oplus}])$ may be considered as the cokernel of the morphisms

\begin{equation}
R^1 f_* (\nabla_{E}^{(0)}) : R^1 f_* (g^0_{E_{\mathfrak{g},h,x/S}}) \to R^1 f_* (\Omega_{X^{\log}/S^{\log}} \otimes g^\gg_{E_{\mathfrak{g},h,x/S}} \cap
\end{equation}

and

\begin{equation}
R^1 f_* (\nabla_{E_{\mathfrak{g}}}^{\text{ad}^\oplus}) : R^1 f_* (\tilde{T}_{E_{\mathfrak{g},h,x/S}}^{\log}) \to R^1 f_* (\Omega_{X^{\log}/S^{\log}} \otimes g^\gg_{E_{\mathfrak{g},h,x/S}} \cap
\end{equation}
of the case where \( T \) \( R \to j \) and the composite

\[
\text{of (488) is exact.}
\]

\( \nabla \) of (443) applied to respectively. (Here, we recall the fact that the restriction of \( \nabla^{\text{ad} \circledast} \) to \( g^{0}_{\mathcal{E}_{b}} \)

\( \subseteq \tilde{T}_{\log}^{1, \text{log}}_{E_{b}, h, X/S} \) coincides with \( \nabla^{\text{ad}(0)} \).) Thus, it suffices to prove that the

morphism \( R^{1} f_{*}(\nabla^{\text{ad}(0)}) \) is surjective. But, since the short exact sequence (461) of the case where \( j = 0 \) or 1 is split, we obtain

\[
Coker(R^{1} f_{*}(\nabla^{\text{ad}(0/1)})) = Coker(R^{1} f_{*}(\nabla^{\text{ad}(1/2)})) = 0.
\]

Thus, (by considering the diagram (468)) the assertion reduces to proving the claim that \( R^{1} f_{*}(\nabla^{\text{ad}(2)}) \) is surjective. But, this claim follows from the fact that

\( R^{1} f_{*}(\Omega^{1}_{X/\mathcal{S}} \log \otimes \mathcal{G}^{1}_{c, X')_{S} / S_{\log}}) = 0 \) (cf. Remark 2.2.2 (i)).

\[ \square \]

5.4. Any abelian sheaf \( \mathcal{F} \) on \( X \) be thought of as a complex concentrated at degree 0. For \( n \in \mathbb{Z} \), we define the complex \( \mathcal{F}[n] \) to be (the complex) \( \mathcal{F} \)

shifted down by \( n \), so that \( \mathcal{F}[n]^{-n} = \mathcal{F} \) and \( \mathcal{F}[n]^{i} = 0 \) \( (i \neq -n) \).

Consider the short exact sequence

\[
0 \to \mathcal{K}^{\bullet}[\nabla^{\text{ad}(0)}] \to \mathcal{K}^{\bullet}[\nabla^{\text{ad} \circledast}] \to \mathcal{T}_{X/\mathcal{S}} \log \to 0
\]

of complexes, where the second arrow is obtained by restricting \( \nabla^{\text{ad} \circledast} \) to \( g^{0}_{\mathcal{E}_{b}} \)

and the third arrow is obtained by the surjection \( \mathcal{G}^{1}_{c, X')_{S} / S_{\log} \to \mathcal{T}_{X/\mathcal{S}} \log \) (cf. (26)). By applying the functor \( R^{1} f_{*}(\circledast) \), we obtain a sequence

\[
0 \to R^{1} f_{*}(\mathcal{K}^{\bullet}[\nabla^{\text{ad}(0)}]) \to R^{1} f_{*}(\mathcal{K}^{\bullet}[\nabla^{\text{ad} \circledast}]) \to R^{1} f_{*}(\mathcal{T}_{X/\mathcal{S}} \log) \to 0
\]

of \( \mathcal{O}_{S} \)-modules. It follows from Lemma 5.3.2 and the equality \( f_{*}(\mathcal{T}_{X/\mathcal{S}} \log) = 0 \)

that the sequence (490) is exact.

On the other hand, consider the short exact sequence (491)

\[
0 \to \text{Coker}(f_{*}(\nabla^{\text{ad}(0)})) \to R^{1} f_{*}(\mathcal{K}^{\bullet}[\nabla^{\text{ad}(0)}]) \to \text{Ker}(R^{1} f_{*}(\nabla^{\text{ad}(0)})) \to 0
\]

of (443) applied to \( \nabla = \nabla^{\text{ad}(0)} \). It follows from Lemma 5.2.1 that \( \text{Ker}(R^{1} f_{*}(\nabla^{\text{ad}(0)})) = 0 \)

and the composite

\[
\mathcal{F}(\mathcal{G}_{b, h, X/S}) \to \text{Coker}(f_{*}(\nabla^{\text{ad}(0)})) \to R^{1} f_{*}(\mathcal{K}^{\bullet}[\nabla^{\text{ad}(0)}])
\]

is an isomorphism. Thus, by combining (490) and the composite isomorphism (492), we obtain a short exact sequence

\[
0 \to f_{*}(\mathcal{V}_{b, h, X/S}) \to R^{1} f_{*}(\mathcal{K}^{\bullet}[\nabla^{\text{ad} \circledast}]) \to R^{1} f_{*}(\mathcal{T}_{X/\mathcal{S}} \log) \to 0
\]

of \( \mathcal{O}_{S} \)-modules.
Proposition 5.4.1.
Let \( s : S \to \mathcal{M}_{g,r} \) and \( s^\bullet : S \to \mathcal{O}_{g,h,g,r} \) (cf. [297] for the definition of \( \mathcal{O}_{g,h,g,r} \)) be the \( S \)-rational points classifying \( \mathcal{X}/S \) and the pair \( (\mathcal{X}/S, \mathcal{E}^\bullet) \) respectively. Let
\[
\Xi_{\mathcal{X}/S} : R^1 f_* (\mathcal{T}_{\mathcal{X}^{\text{log}}/S^{\text{log}}}) \sim s^*(\mathcal{T}_{\mathcal{M}_{g,r}/k})
\]
be the canonical isomorphism obtained by well-known generalities concerning deformation theory (cf. [39], Proposition (3.14)). Also, let
\[
\Xi_{\mathcal{E}^\bullet/\mathcal{X}/S} : f_*(\mathcal{V}_{g,h,\mathcal{X}/S}) \sim s^*(\mathcal{T}_{\mathcal{O}_{g,h,g,r}/\mathcal{M}_{g,r}})
\]
be the canonical isomorphism arising from the structure of relative affine space on \( \mathcal{O}_{g,h,\mathcal{X}/S} \) modeled on \( \mathcal{V}(f_*(\mathcal{V}_{g,h,\mathcal{X}/S})) \) (cf. Proposition 2.7.5). Then, there exists a canonical isomorphism
\[
\Xi_{\mathcal{E}^\bullet} : R^1 f_* (\mathcal{K}^* [\nabla_{\mathcal{E}^\bullet}]) \sim s^*(\mathcal{T}_{\mathcal{O}_{g,h,g,r}/k})
\]
which fits into the following diagram of \( \mathcal{O}_S \)-modules:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\downarrow & & \downarrow \\
f_*(\mathcal{V}_{g,h,\mathcal{X}/S}) & \xrightarrow{\Xi_{\mathcal{E}^\bullet/\mathcal{X}/S}} & s^*(\mathcal{T}_{\mathcal{O}_{g,h,g,r}/\mathcal{M}_{g,r}}) \\
\downarrow & & \downarrow \\
R^1 f_* (\mathcal{K}^* [\nabla_{\mathcal{E}^\bullet}]) & \xrightarrow{\Xi_{\mathcal{E}^\bullet}} & s^*(\mathcal{T}_{\mathcal{O}_{g,h,g,r}/k}) \\
\downarrow & & \downarrow \\
R^1 f_* (\mathcal{T}_{\mathcal{X}^{\text{log}}/S^{\text{log}}}) & \xrightarrow{\Xi_{\mathcal{X}/S}} & s^*(\mathcal{T}_{\mathcal{M}_{g,r}/k}) \\
\downarrow & & \downarrow \\
0 & \to & 0 \\
\end{array}
\]

where the left vertical sequence is (493) and the right vertical sequence is the natural exact sequence of tangent bundles. In particular, the sequence (493) is exact.

Proof. The assertion follows from the structure of relative affine space on \( \mathcal{O}_{g,h,\mathcal{X}/S} \) and the assumption that \( \mathcal{E}^\bullet \) is of canonical type. Indeed, one may apply an argument (in the case where \( \mathcal{X}/S \) is a pointed stable curve over an arbitrary scheme \( S \)) similar to the argument (in the case where \( \mathcal{X}/S \) is an unpointed smooth curve over \( \mathbb{C} \)) given in [12]. By the explicit description of the hypercohomology sheaf \( R^1 f_* (\mathcal{K}^* [\nabla_{\mathcal{E}^\bullet}]) \) in terms of the Čech double complex associated with \( \mathcal{K}^* [\nabla_{\mathcal{E}^\bullet}] \), one verifies that \( R^1 f_* (\mathcal{K}^* [\nabla_{\mathcal{E}^\bullet}]) \) may be naturally
identified with the deformation space (relative to $S$) of $(\mathcal{X}_S, \mathcal{E}_\bullet)$, i.e., the $\mathcal{O}_S$-module $s_\bullet^\ast(\mathcal{T}_{\mathcal{O}_p, b, p, g, r/k})$. In particular, for completing the proof of Proposition 5.4.1, we refer to [12], Proposition 4.1.3 and Proposition 4.3.1.

5.5. Next, we shall construct from $\nabla_{\mathcal{E}_S}^\text{ad}$ a certain morphism $\nabla_{\mathcal{E}_S}^\text{ad}$ and prove (cf. Proposition 5.5.2) that the cohomology of the complex corresponding to this morphism may be identified with the tangent space of $\mathcal{O}_p g, b, p, g, r$ at the point classifying $\mathcal{E}_\bullet$.

Write $Tg, Tc$ (cf. §1.2) for the total space of the tangent bundles of $g, c$ over $k$ respectively, and $d\chi : Tg \to Tc$ for the differential of the morphism $\chi : g \to c$ (cf. [151]). Suppose that $r > 0$ and that for each $i \in \{1, \cdots, r\}$, there exists a trivialization

\[ \text{triv}_{B^\circ, \sigma_i} : \sigma_i^\ast(\mathcal{E}_{B^\circ, b, h, X/S}) \cong S \times_k B^\circ \]

of the right $B^\circ$-torsor $\sigma_i^\ast(\mathcal{E}_{B^\circ, b, h, X/S})$. Note that such a trivialization necessarily exists after possibly replacing $S$ with an open subscheme of $S$. A change of structure group via $\iota_G|_{B^\circ} : B^\circ \hookrightarrow G$ induces an isomorphism

\[ \text{triv}_{g, \sigma_i} : \sigma_i^\ast(g_{\mathcal{E}_{B^\circ, b, h, X/S}}) \cong \mathcal{O}_S \otimes_k g. \]

By passing to this isomorphism (499), the monodromy $\mu_i^{(\mathcal{E}_0, \nabla_\mathcal{E})} \in \Gamma(S, \sigma_i^\ast(g_{\mathcal{E}_{B^\circ, b, h, X/S}}))$ (cf. [68]) corresponds to a global section of $\mathcal{O}_S \otimes_k g$, equivalently, an $S$-rational point $S \to g$ of $S$; for simplicity, we shall write $\mu_i$ for this $S$-rational point. Consider the differential of $\chi$ at $\mu_i$, i.e., the morphism

\[ d\mu_i \chi : (\nabla(\mathcal{O}_S \otimes_k g) \xrightarrow{\sim} Tg \times_{g, \mu_i} S \to Tc \times_{c, \chi \circ \mu_i} S \]

of affine spaces over $S$ obtained as the pull-back of $d\chi$ via the composite $\chi \circ \mu_i : S \to c$. We shall define a sheaf

\[ \mathcal{K}^{-1}[c_{\nabla_{\mathcal{E}_b}^\text{ad}}] \]

on $X$ to be the $\mathcal{O}_X$-submodule of $\Omega_{X^{\text{log}}/S^{\text{log}} \otimes g_{\mathcal{E}_{B, b, h, X/S}}^{-1}}$ consisting of local sections $s$ such that the local section $\sigma_i^\ast(s)$ of $\nabla(\sigma_i^\ast(\Omega_{X^{\text{log}}/S^{\text{log}} \otimes g_{\mathcal{E}_{B, b, h, X/S}}^{-1}}))$ is mapped to
the zero section via the composite

\[ \mathcal{V}(\sigma^*_\varepsilon(\Omega_{X^{\text{log}}/S^{\text{log}}} \otimes \mathcal{G}_{e,h,x/S}^{-1})) \rightarrow \mathcal{V}(\sigma^*_\varepsilon(\Omega_{X^{\text{log}}/S^{\text{log}}} \otimes \mathcal{G}_{e,h,x/S}^{\tau}))) \]

\[ \mathcal{V}(\text{triv}_{\varepsilon,x}^{\tau} \otimes \text{id}) \mathcal{V}(\sigma^*_\varepsilon(\mathcal{G}_{e,h,x/S}^{\tau})) \]

\[ \mathcal{V}(\text{triv}_{B} \otimes \sigma_i) \mathcal{V}(\mathcal{O}_S \otimes \mathfrak{g}) \]

\[ d_{\mu_i} \mathcal{T}_c \times c_{\chi_{\mu_i}} S. \]

Note that this definition of the subsheaf \( \mathcal{K}^1[\nabla_{\varepsilon_{\varepsilon}}^\oplus] \) of \( \Omega_{X^{\text{log}}/S^{\text{log}}} \otimes \mathcal{G}_{e,h,x/S}^{-1} \) does not depend on the choice of a trivialization \( \text{triv}_{B} \otimes \sigma_i \) in \[198\]. Hence, even if there does not exist globally any trivialization \[198\], the sheaf \( \mathcal{K}^1[\nabla_{\varepsilon_{\varepsilon}}^\oplus] \) on \( X \) may be constructed.

**Lemma 5.5.1.**

The image of \( \nabla_{\varepsilon_{\varepsilon}}^\oplus \) (cf. \[184\]) is contained in \( \mathcal{K}^1[\nabla_{\varepsilon_{\varepsilon}}^\oplus] \) \( (\subseteq \Omega_{X^{\text{log}}/S^{\text{log}}} \otimes \mathcal{G}_{e,h,x/S}^{-1}) \).

**Proof.** For any scheme \( Y \) over \( k \), we shall write \( Y_\varepsilon := Y \times \text{Spec}(k) \text{Spec}(k[\varepsilon]/(\varepsilon^2)) \), and write \( \text{pr}_Y : Y_\varepsilon \rightarrow Y \) for the natural projection.

Let us fix \( i \in \{1, \ldots, r\} \) and an affine open subscheme \( U \) of \( X \) with \( U \cap \text{Im}(\sigma_i) \neq \emptyset \). For a logarithmic derivation \( \partial \in \Gamma(U, \mathcal{T}_{e,h,x/S}^{\text{log}} \otimes \mathcal{G}_{e,h,x/S}^{-1}) \), we shall prove that \( \nabla_{\varepsilon_{\varepsilon}}^\oplus(\partial) \) lies in \( \Gamma(U, \mathcal{K}^1[\nabla_{\varepsilon_{\varepsilon}}^\oplus]) \).

Consider the natural isomorphism

\[ \Gamma((\mathcal{E}_{e,h,x/S}^{\tau} \mid U), \mathcal{O}_{e,h,x/S}^{\tau} \mid U)) \Rightarrow \Gamma((\mathcal{E}_{e,h,x/S}^{\tau} \mid U), \mathcal{O}_{e,h,x/S}^{\tau} \mid U)) \oplus \varepsilon \cdot \Gamma((\mathcal{E}_{e,h,x/S}^{\tau} \mid U), \mathcal{O}_{e,h,x/S}^{\tau} \mid U)). \]

The automorphism of \( \Gamma((\mathcal{E}_{e,h,x/S}^{\tau} \mid U), \mathcal{O}_{e,h,x/S}^{\tau} \mid U)) \) given, via \[503\], by \((a, \varepsilon \cdot b) \mapsto (a, \varepsilon \cdot (\partial(a) + b)) \) determines (since \( (\mathcal{E}_{e,h,x/S}^{\tau} \mid U)) \) is affine) an automorphism of \( (\mathcal{E}_{e,h,x/S}^{\tau} \mid U) \) over \( S_\varepsilon \); we denote by \( \eta_\theta \) this automorphism of the right B-torsor \( (\mathcal{E}_{e,h,x/S}^{\tau} \mid U) \). It follows from the construction of the isomorphism \( \Xi_{e \bullet} : \mathbb{R}_1 f_* (\mathcal{K}^1[\nabla_{\varepsilon_{\varepsilon}}^\oplus]) \Rightarrow \mathcal{S}_{h}^{\bullet}(\mathcal{T}_{e,h,g,r}^{\text{log}}/k) \) (cf. the proof of Proposition 5.4.1 or the discussion given in \[12\]) asserted in Proposition 5.4.1 that we have the equality

\[ \eta_\theta^\sharp(\text{pr}_U^* (\nabla_{\varepsilon} \mid U)) = \text{pr}_U^* (\nabla_{\varepsilon} \mid U) + \varepsilon \cdot \nabla_{\varepsilon_{\varepsilon}}^\tau(\partial)^\sharp, \]
where $\nabla_{\mathfrak{c}}^{ad\oplus}(\partial)^2$ denotes the morphism $\mathcal{T}_{U^{log}/S^{log}} \to \hat{\mathcal{T}}_{(c, h, X/S)|u}^{log}/S^{log}$ corresponding to $\nabla_{\mathfrak{c}}^{ad\oplus}(\partial)$ via the composite
\begin{equation}
\Gamma(U, \Omega_{X^{log}/S^{log}} \otimes \mathfrak{g}_{(c, h, X/S)}^{-1}) \to \Gamma(U, \Omega_{X^{log}/S^{log}} \otimes \hat{\mathcal{T}}_{(c, h, X/S)^{log}/S^{log}})
\end{equation}
\[\to \text{Hom}_{\mathcal{O}_U} (\mathcal{T}_{U^{log}/S^{log}}, \hat{\mathcal{T}}_{(c, h, X/S)|u}^{log}/S^{log}).\]

But, since $G$ acts trivially on $\mathfrak{c}$, the radii of the $h$-log integrable $G$-torsor $((\mathcal{E}_{G, h, X/S}|u), \text{pr}_U((\nabla_{\mathfrak{c}}|U)))$ and the radii of its pull-back $((\mathcal{E}_{G, h, X/S}|u), \eta_\partial(\text{pr}_U((\nabla_{\mathfrak{c}}|U))))$ via the automorphism $\eta_\partial$ coincide. This implies that the $(U \times_{X, \sigma} S)_{\text{rational}}$-point of $\mathfrak{c}$ corresponding to the radius of $((\mathcal{E}_{G, h, X/S}|u), \eta_\partial(\text{pr}_U((\nabla_{\mathfrak{c}}|U))))$ through the $(U \times_{X, \sigma} S)$-rational point $\mu_\partial|_{U \times_{X, \sigma} S} : U \times_{X, \sigma} S \to \mathfrak{c}$. That is, the $(U \times_{X, \sigma} S)$-rational point of $\mathcal{T}_{\mathfrak{g}} \times_{\mathfrak{g}, \mu_\partial} S$ corresponding to the monodromy of $((\mathcal{E}_{G, h, X/S}|u), \eta_\partial(\text{pr}_U((\nabla_{\mathfrak{c}}|U))))$ at the marked point $\text{pr}_U((\nabla_{\mathfrak{c}}|U))$ is, the $(U \times_{X, \sigma} S)$-rational point $\mu_\partial|_{U \times_{X, \sigma} S} : U \times_{X, \sigma} S \to \mathfrak{c}$ is sent, via $d_{\mu_\partial}$, to the zero section of $\mathcal{T}_{\mathfrak{c}} \times_{\mathfrak{c}, \mu_\partial} S$ over $U \times_{X, \sigma} S$. It follows (by recalling the equality (504)) that $\nabla_{\mathfrak{c}}^{ad\oplus}(\partial)$ lies in $\Gamma(U, \mathcal{K}^{\log}[\nabla_{\mathfrak{c}}^{ad\oplus}])$, and hence, completes the proof of Lemma 5.5.1.

By Lemma 5.5.1, $\nabla_{\mathfrak{c}}^{ad\oplus}$ yields an $f^{-1}(\mathcal{O}_S)$-linear morphism
\begin{equation}
\mathfrak{c}\nabla_{\mathfrak{c}}^{ad\oplus} : \hat{\mathcal{T}}_{c, h, X/S}^{log}/S^{log} \to \mathcal{K}^{\log}[\mathfrak{c}\nabla_{\mathfrak{c}}^{ad\oplus}].
\end{equation}

One verifies easily that $\mathcal{V}_{\mathfrak{g}, h, X/S} \subseteq \mathcal{K}^{\log}[\mathfrak{c}\nabla_{\mathfrak{c}}^{ad\oplus}]$. The resulting inclusion yields the second arrow of the sequence
\begin{equation}
0 \to \mathcal{V}_{\mathfrak{g}, h, X/S}[-1] \to \mathcal{K}^{\log}[\mathfrak{c}\nabla_{\mathfrak{c}}^{ad\oplus}] \to \mathcal{T}_{X^{log}/S^{log}} \to 0
\end{equation}
where the third arrow arises from the surjection $\mathfrak{a}_{\mathfrak{c}}^{log}$. By applying the functor $f_*(-)$ to this sequence, we obtain a sequence
\begin{equation}
0 \to f_*(\mathcal{V}_{\mathfrak{g}, h, X/S}) \to f_*(\mathcal{K}^{\log}[\mathfrak{c}\nabla_{\mathfrak{c}}^{ad\oplus}]) \to f_*(\mathcal{T}_{X^{log}/S^{log}}) \to 0
\end{equation}
of $\mathcal{O}_S$-modules.

**Proposition 5.5.2.**

Let $s$ and $\Xi_{X/S}$ be as in Proposition 5.4.1, and $\mathfrak{c}^S : S \to \mathfrak{O}_{\mathfrak{G}_{h, r}}$ the $S$-rational point classifying the pair $(\mathfrak{X}/S, \mathfrak{E}^\bullet)$. Let
\begin{equation}
\mathfrak{c}^S \Xi_{\mathfrak{c}^S, X/S} : f_*((\mathcal{V}_{\mathfrak{g}, h, X/S}) \to f_*((\mathcal{K}^{\log}[\mathfrak{c}\nabla_{\mathfrak{c}}^{ad\oplus}]) \to f_*((\mathcal{T}_{\mathfrak{g}, h, r}/S^{log}))
\end{equation}
be the canonical isomorphism arising from the structure of relative affine space on $\mathfrak{O}_{\mathfrak{G}_{h, r}, X/S}$ modeled on $\mathcal{V}(f_*(\mathcal{V}_{\mathfrak{g}, h, X/S}))$. Then, there exists a canonical isomorphism
\begin{equation}
\mathfrak{c}^S \Xi_{\mathfrak{c}^S} : \mathfrak{R}^1 f_*((\mathcal{K}^{\log}[\mathfrak{c}\nabla_{\mathfrak{c}}^{ad\oplus}]) \to \mathfrak{R}^1 f_*((\mathcal{T}_{\mathfrak{g}, h, r}/S^{log}))
\end{equation}
which fits into the following diagram of $\mathcal{O}_S$-modules:

\[
\begin{array}{ccc}
0 & \to & 0 \\
\downarrow & & \downarrow \\
f^*_s(\mathcal{V}_{g,h,x/S}) & \xrightarrow{c^*_{\mathcal{V}_{g,h,x/S}}} & c^*_s(\mathcal{T}_{\mathcal{O}_S g,h,p,g,r/\mathcal{M}_{g,r}}) \\
\downarrow & & \downarrow \\
\mathbb{R}^1 f_*(\mathcal{K}^{*}[c^{\text{ad}}_\varepsilon]) & \xrightarrow{c^*_{\mathcal{V}_{g,h,x/S}}} & c^*_s(\mathcal{T}_{\mathcal{O}_S g,h,p,g,r/\mathcal{M}_{g,r}}) \\
\downarrow & & \downarrow \\
\mathbb{R}^1 f_*(\mathcal{T}_{\mathcal{X}_{\log}/S_{\log}}) & \xrightarrow{c^*_{\mathcal{V}_{g,h,x/S}}} & c^*_s(\mathcal{T}_{\mathcal{M}_{g,r}/k}) \\
\downarrow & & \downarrow \\
0 & & 0,
\end{array}
\]

where the left vertical sequence is (508) and the right vertical sequence is the natural exact sequence of tangent bundles. In particular, the sequence (508) is exact. Finally, this diagram is compatible, in an evident sense, with the diagram (497).

**Proof.** The assertion follows from the definition of $\mathcal{K}^{1}[c^{\text{ad}}_\varepsilon]$ and an argument similar to the argument given in the proof of Proposition 5.4.1. □

### 5.6

Let us consider the adjoint operator

\[
\text{ad}(\mu_i^{(\xi^1_{G,h,x/S}, \nabla_\xi)}) : \sigma^*_i(\mathfrak{g}_{\xi^1_{G,h,x/S}}) \to \sigma^*_i(\mathfrak{g}_{\xi^1_{G,h,x/S}})
\]

\[(i \in \{1, \cdots, r\})\] determined by $\mu_i^{(\xi^1_{G,h,x/S}, \nabla_\xi)}$. If we apply the isomorphism $\sigma^*_i(\mathfrak{g}_{\xi^1_{G,h,x/S}}) \cong \sigma^*_i(\Omega_{\mathcal{X}_{\log}/S_{\log}} \otimes \mathfrak{g}_{\xi^1_{G,h,x/S}})$ induced by the trivialization (67), then the endomorphism

\[
\text{ad}(\mu_i^{(\xi^1_{G,h,x/S}, \nabla_\xi)}) \in \text{End}_{\mathcal{O}_S}(\sigma^*_i(\mathfrak{g}_{\xi^1_{G,h,x/S}})) = \Gamma(S, \sigma^*_i(\mathcal{E}_{\text{End}_{\mathcal{O}_S}}(\mathfrak{g}_{\xi^1_{G,h,x/S}})))
\]

may be thought of as the monodromy at the marked point $\sigma_i$ of the $\hbar$-log integrable vector bundle $(\mathfrak{g}_{\xi^1_{G,h,x/S}}, \nabla_\xi^{\text{ad}})$ (viewed as an $\hbar$-log integrable $\text{GL}(\mathfrak{g})$-torsor).

The following proposition will be used in the proof of Proposition 5.10.1.
Proposition 5.6.1.

Both \( \text{Ker}(\text{ad}(\mu_i^{\langle \mathcal{E}_{g,h,X/S,\nabla \varepsilon}^l \rangle})) \) and \( \text{Coker}(\text{ad}(\mu_i^{\langle \mathcal{E}_{g,h,X/S,\nabla \varepsilon}^l \rangle})) \) are vector bundles on \( S \) of rank \( \text{rk}(g) \).

Proof. Observe that the \((g, h)\)-oper \( (\mathcal{E}_{g,h,X/S,\nabla \varepsilon}) \) may be obtained as the pull-back of the tautological \((g, h)\)-oper on \( \mathcal{C}_{g,r} \times_{\mathfrak{m}_{g,r}} \mathfrak{D} p_{g,h,g,r} \) via its classifying morphism \( s^\bullet : S \to \mathfrak{D} p_{g,h,g,r} \). Hence, without loss of generality, we may assume that \( S \) is reduced (e.g., \( S = \mathfrak{D} p_{g,h,g,r} \)). Under this assumption, any coherent \( \mathcal{O}_S \)-module \( \mathcal{H} \) is locally free of rank \( l \in \mathbb{Z}_{\geq 0} \) if and only if for any closed point \( s \in S \), the fiber \( \mathcal{H} \otimes_{\mathcal{O}_s} \mathbb{k}(s) \) of \( \mathcal{H} \) (where \( \mathbb{k}(s) \) denotes the residue field of \( s \)) is a \( k(s) \)-vector space of rank \( l \). Hence, it suffices to prove the assertion of the case where \( S = \text{Spec}(k) \). Then, (since \( \mathcal{E}^\bullet \) is of canonical type) the monodromy \( \mu_i^{\langle \mathcal{E}_{g,h,X/S,\nabla \varepsilon}^l \rangle} \) may be expressed as \( p_{-1} + R \in g \) for some \( R \in g^{\text{ad}(p_1)} \). But, such an element \( p_{-1} + R \) is known to be regular (cf. [54], Lemma 1.2.1), and hence, both the kernel and cokernel of the adjoint operator \( \text{ad}(p_{-1} + R) \) are of rank \( \text{rk}(g) \). This completes the proof of Proposition 5.6.1. \( \square \)

5.7. In this subsection, we shall recall the Cartier operator associated with a \((1)\)-log integrable vector bundle. For the rest of this section, we suppose that \( k \) is of characteristic \( p > 0 \) (i.e., the condition \((\text{Char})^\mathbb{W}_p \) is satisfied).

Let \((\mathcal{F}, \nabla_{\mathcal{F}})\) be a \((1)\)-log integrable vector bundle on \( X^{\log}/S^{\log} \) with vanishing \( p \)-curvature. Although \( \nabla_{\mathcal{F}} \) is not \( \mathcal{O}_X \)-linear, but it may be thought, via the underlying homeomorphism of \( F_{X/S} \) (cf. §3.1), of as an \( \mathcal{O}_{X_S^{(1)}} \)-linear morphism

\[
F_{X/S}^*(\nabla_{\mathcal{F}}) : F_{X/S}^*(\mathcal{F}) \to F_{X/S}^*(\Omega_X^{\log}/S^{\log} \otimes \mathcal{F}).
\]

In particular, both \( F_{X/S}^*(\text{Ker}(\nabla_{\mathcal{F}})) \) (= \( \text{Ker}(F_{X/S}^*(\nabla_{\mathcal{F}})) \)) and \( F_{X/S}^*(\text{Coker}(\nabla_{\mathcal{F}})) \) (= \( \text{Coker}(F_{X/S}^*(\nabla_{\mathcal{F}})) \)) may be thought of as \( \mathcal{O}_{X_S^{(1)}} \)-modules.

The Cartier operator (cf. [54], Proposition 1.2.4) associated with \((\mathcal{F}, \nabla_{\mathcal{F}})\) is, by definition, a unique \( \mathcal{O}_{X_S^{(1)}} \)-linear morphism

\[
C^{(\mathcal{F}, \nabla_{\mathcal{F}})} : F_{X/S}^*(\Omega_X^{\log}/S^{\log} \otimes \mathcal{F}) \to \Omega_{X_S^{(1)}}^{\log}/S^{\log} \otimes F_{X/S}^*(\mathcal{F})
\]
satisfying the following condition: for any locally defined logarithmic derivation \( \partial \in \mathcal{T}_{X^{\log}/S^{\log}} \) and any local section \( a \in \Omega_X^{\log}/S^{\log} \otimes \mathcal{F}, \)

\[
\langle C^{(\mathcal{F}, \nabla_{\mathcal{F}})}(\partial)(a), (\text{id}_X \times F_S)^*(\partial) \rangle = \langle a, \partial(p) \rangle - \nabla_{\mathcal{F}}(\partial)^{\circ(p-1)}(\langle a, \partial \rangle),
\]

where \( \nabla_{\mathcal{F}}(\partial)^{\circ(p-1)} \) denotes the \((p-1)\)-st iterate of the endomorphism \( \nabla_{\mathcal{F}}(\partial) \) of \( \mathcal{F} \), and \( \langle - , - \rangle \) in the both sides of this equality denote the pairings induced by the natural pairing \( \Omega_X^{\log}/S^{\log} \times \mathcal{T}_{X^{\log}/S^{\log}} \to \mathcal{O}_X \). Moreover, since \((\mathcal{F}, \nabla_{\mathcal{F}})\) has
vanishing $p$-curvature, there exists (cf. [55], the discussion following Proposition 1.2.4) a commutative square

$$
\begin{array}{ccc}
F_{X/S*}(\Omega_{X^\log/S^\log} \otimes \mathcal{F}) & \xrightarrow{C^{(F, \nabla_F)}} & \Omega_{X^\log/S^\log} \otimes F_{X/S*}(\mathcal{F}) \\
\downarrow & & \uparrow \\
F_{X/S*}(\text{Coker}(\nabla_F)) & \xrightarrow{C^{(F, \nabla_F)}} & \Omega_{X^\log/S^\log} \otimes F_{X/S*}(\text{Ker}(\nabla_F)),
\end{array}
$$

(517)

where the left-hand vertical arrow and the right-hand vertical arrow denote the $\mathcal{O}_{X^\log}^\log$-linear morphisms induced by the surjection $\Omega_{X^\log/S^\log} \otimes \mathcal{F} \twoheadrightarrow \text{Coker}(\nabla_F)$ and the injection $\text{Ker}(\nabla_F) \hookrightarrow \mathcal{F}$ respectively. Finally, it follows from [50], Theorem 3.1.1, that the morphism $\overline{C}^{(F, \nabla_F)}$ (i.e., the lower horizontal arrow of the diagram (517)) is an isomorphism. (Here, we note that since $X^\log/S^\log$ is of Cartier type, the notion of the exact relative Frobenius map in the statement of loc. cit. coincides with $F_{X/S}$).

The projection formula gives an isomorphism

$$
\Omega_{X^\log/S^\log} \otimes F_{X/S*}(\mathcal{F}) \sim F_{X/S*}(F_{X/S}(\Omega_{X^\log/S^\log}) \otimes \mathcal{F})
$$

(518)

$$
\sim F_{X/S*}(\Omega_{X^\log/S^\log} \otimes \mathcal{F}).
$$

Composition of $C^{(F, \nabla_F)}$ with this isomorphism yields a morphism

$$
\Gamma(U, C^{(F, \nabla_F)}): \Gamma(U, \Omega_{X^\log/S^\log} \otimes \mathcal{F}) \rightarrow \Gamma(U, \Omega_{X^\log/S^\log} \otimes \mathcal{F})
$$

(519)

for each open subscheme $U$ of $X$.

5.8. Let $k_\epsilon := k[\epsilon]/(\epsilon^2)$, and for any scheme $Y$, we write $Y_\epsilon := Y \times_{\text{Spec}(k)} \text{Spec}(k_\epsilon)$. Also, write $pr_Y : Y_\epsilon \rightarrow Y$ for the natural projection and $\text{in}_Y : Y \hookrightarrow Y_\epsilon$ for the natural closed immersion. Let $U$ be an open subscheme of $X$. Then, we have the following proposition. (For the case where $G = \mathbb{G}_m$, i.e., the case of log integrable line bundles, we refer to [42], (7.2.2), Proposition.)

**Proposition 5.8.1.**

Let $(\mathcal{E}, \nabla_\epsilon)$ be a log integrable $\mathbb{G}$-torsor over $U^\log/S^\log$ with vanishing $p$-curvature, and $R \in \Gamma(U, \Omega_{U^\log/S^\log} \otimes \mathfrak{g}_\epsilon)$. Denote by $R^\epsilon$ the $\mathcal{O}_U$-linear morphism $T_{U^\log/S^\log} \rightarrow \mathfrak{g}_\epsilon (\subseteq \mathfrak{T}_{U^\log/S^\log})$ corresponding to $R$. Then, the $p$-curvature

$$
p_{U^\epsilon}(pr_U^*(\mathcal{E}), pr_U^*(\nabla_\epsilon) + \epsilon \cdot R^\epsilon) \in \Gamma(U_\epsilon, \Omega_{U_\epsilon^\log/S_\epsilon^\log} \otimes \mathfrak{g}_{pr_U^*(\mathcal{E})})
$$

(520)

of the log integrable $\mathbb{G}$-torsor $(pr_U^*(\mathcal{E}), pr_U^*(\nabla_\epsilon) + \epsilon \cdot R^\epsilon)$ over $U_\epsilon^\log/S_\epsilon^\log$ satisfies the equality

$$
p_{U^\epsilon}(pr_U^*(\mathcal{E}), pr_U^*(\nabla_\epsilon) + R^\epsilon) = -\epsilon \cdot \Gamma(U, C^{(\mathcal{E}, \nabla_\epsilon)})(R),
$$

(521)
where $\nabla^{ad}_E$ denotes, as in (440), the $S$-log connection on $g_E$ induced by $\nabla_E$. In particular, if the $p$-curvature of $(pr_U^*(\mathcal{E}), pr_U^*(\nabla_E) + \epsilon \cdot R^2)$ vanishes, then $R$ may be expressed, Zariski locally on $U$, as $R = \nabla^{ad}_E(R')$ for some local section $R'$ of $g_E$.

**Proof.** After possibly replacing $U$ with an open covering of $U$, we may suppose that there exists a globally defined log chart $(U, x)$ on $U^{\log}$ over $S^{\log}$. The dual base $\partial_x$ corresponding to $d\log(x)$ satisfies the equality $\partial_x^{(p)} = \partial_x$ (cf. [55], Remark 1.2.2). On the one hand, we may obtain a sequence of equalities between local sections in the enveloping algebra of the Lie algebroid $(\tilde{T}_{E^{\log}/S^{\log}}, a_E^{\log})$ (cf. [19], Appendix, A.3):

\begin{equation}
\langle -\epsilon \cdot \Gamma(U, C^{(g_E, \nabla_E^{ad})}(R), pr_U^*(\partial_x)^\otimes p) = -\epsilon \cdot \langle \Gamma(U, C^{(g_E, \nabla_E^{ad})}(R), (id_X \times F_S)^* (\partial_x) \rangle \\
= -\epsilon \cdot \langle R, \partial_x^{(p)} \rangle + \epsilon \cdot (\nabla_E^{ad}(\partial_x))^{(p-1)}(\langle R, \partial_x \rangle) \\
= -\epsilon \cdot \langle R, \partial_x \rangle + \epsilon \cdot \langle R, \partial_x \rangle + \epsilon \cdot \text{ad}(\nabla_E(\partial_x))^{(p-1)}(\langle R, \partial_x \rangle) \\
= -\epsilon \cdot \langle R, \partial_x \rangle + \epsilon \cdot \sum_{i=0}^{p-1} (-1)^i \cdot \left( \binom{p-1}{i} \cdot \nabla_E(\partial_x)^{p-1-i} \cdot \langle R, \partial_x \rangle \cdot \nabla_E(\partial_x)^i \right) \\
= -\epsilon \cdot \langle R, \partial_x \rangle + \epsilon \cdot \sum_{i=0}^{p-1} \nabla_E(\partial_x)^{p-1-i} \cdot \langle R, \partial_x \rangle \cdot \nabla_E(\partial_x)^i,
\end{equation}

where the first equality follows from (516). On the other hand, we obtain a sequence of equalities:

\begin{equation}
\langle \psi(\pr_U^*(\mathcal{E}), \pr_U^*(\nabla_E) + \epsilon \cdot R^2), \pr_U^*(\partial_x)^\otimes p) = (\pr_U^*(\nabla_E) + \epsilon \cdot R^2)(\pr_U^*(\partial_x)^{(p)} - (\pr_U^*(\nabla_E) + \epsilon \cdot R^2)(\pr_U^*(\partial_x)^{(p)}) \\
= (\pr_U^*(\nabla_E(\partial_x)) + \epsilon \cdot \langle R, \partial_x \rangle)^p - \pr_U^*(\nabla_E(\partial_x^{(p)})) + \epsilon \cdot \langle R, \partial_x \rangle \\
= \epsilon \cdot \sum_{i=0}^{p-1} \nabla_E(\partial_x)^{p-1-i} \cdot \langle R, \partial_x \rangle \cdot \nabla_E(\partial_x)^i \\
+ \pr_U^*(\nabla_E(\partial_x)^p + \nabla_E(\partial_x^{(p)})) - \epsilon \cdot \langle R, \partial_x \rangle \\
= -\epsilon \cdot \langle R, \partial_x \rangle + \epsilon \cdot \sum_{i=0}^{p-1} \nabla_E(\partial_x)^{p-1-i} \cdot \langle R, \partial_x \rangle \cdot \nabla_E(\partial_x)^i.
\end{equation}

Thus, by combining (522) and (523), we obtain the asserted equality (521).

The latter assertion follows from this equality and the commutativity of the square diagram (517). \qed
Corollary 5.8.2.
Let $(E, \nabla_E)$ be a log integrable $G$-torsor over $U^{\log} / S^{\log}$ such that $\nabla_{E, \nabla_E} = 0$ and the underlying right $G$-torsor $E$ is isomorphic to the pull-back $(\text{pr}_U \circ \text{in}_U)^*(E)$ of $E$. Then, $(E, \nabla_E)$ is Zariski locally on $U$, isomorphic to the log integrable $G$-torsor $((\text{pr}_U \circ \text{in}_U)^*(E), (\text{pr}_U \circ \text{in}_U)^*(\nabla_E))$.

Proof. Write $E_0 := \text{in}_U^*(E)$ and $\nabla_{E_0} := \text{in}_U^*(\nabla_E)$. Then, the log integrable $G$-torsor $(E_0, \nabla_{E_0})$ over $U^{\log} / S^{\log}$ has vanishing $p$-curvature. We may suppose that $E = \text{pr}_U^*(E_0)$ and $U$ is affine. In the following discussion, we shall consider $\mathcal{T}_{U^{\log}/S^{\log}}$ (resp., $\tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}}$) as a subsheaf of $\mathcal{T}_{U^{\log}/S^{\log}}$ (resp., $\tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}}$) via the composite inclusion

\begin{equation}
\mathcal{T}_{U^{\log}/S^{\log}} \hookrightarrow \mathcal{T}_{U^{\log}/S^{\log}} \oplus \epsilon \cdot \mathcal{T}_{U^{\log}/S^{\log}} \sim \mathcal{T}_{U^{\log}/S^{\log}}
\end{equation}

\begin{equation}
\text{(resp., } \tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}} \hookrightarrow \tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}} \oplus \epsilon \cdot \tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}} \sim \tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}}).
\end{equation}

Now, let

\begin{equation}
R \in \Gamma(U, \Omega_{U^{\log}/S^{\log}} \otimes \mathfrak{g}_E) \quad (\subseteq \Gamma(U, \Omega_{U^{\log}/S^{\log}} \otimes \tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}}))
\end{equation}

be the element corresponding to the $\mathcal{O}_U$-linear morphism

\begin{equation}
R^s := \frac{1}{\epsilon} \cdot (\nabla_E - \text{pr}_U^*(\nabla_{E_0}))|_{\mathcal{T}_{U^{\log}/S^{\log}}} : \mathcal{T}_{U^{\log}/S^{\log}} \rightarrow \tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}}.
\end{equation}

It follows from the latter assertion of Proposition 5.8.1 that there exists, Zariski locally on $U$, an element $R'$ in $\Gamma(U, \mathfrak{g}_{E_0})$ (\subseteq $\Gamma(U, \tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}})$) with $\nabla_{E_0}^{ad}(R') = R$. Consider the automorphism $\eta_R$ of $E$ corresponding to the automorphism of $\Gamma(E, \mathcal{O}_E)$ defined as $\text{id}_{\Gamma(E, \mathcal{O}_E)} + \epsilon \cdot R'$, where we regard $R'$ as a logarithmic derivation on $\Gamma(E, \mathcal{O}_E)$ via $\tilde{\mathcal{T}}_{E_0^{\log}/S^{\log}} \hookrightarrow \tilde{T}_{E_0^{\log}/S^{\log}}$. One verifies easily that the $\mathcal{O}_U$-linear automorphism $d_\eta_R$ of $\tilde{T}_{E_0^{\log}/S^{\log}}$ obtained by differentiating $\eta_R$ may be expressed as $d_\eta_R + \epsilon \cdot [R', -]$. Hence, for any local section $\partial$ of $\mathcal{T}_{U^{\log}/S^{\log}}$, we have

\begin{equation}
d\eta_R \circ \text{pr}_U^*(\nabla_{E_0})(\partial) = (\text{id}_{\tilde{T}_{E_0^{\log}/S^{\log}}} + \epsilon \cdot [R', -]) \circ \text{pr}_U^*(\nabla_{E_0})(\partial)
\end{equation}

\begin{equation}
= \text{pr}_U^*(\nabla_{E_0})(\partial) + \epsilon \cdot [R', \text{pr}_U^*(\nabla_{E_0})(\partial)]
\end{equation}

\begin{equation}
= \text{pr}_U^*(\nabla_{E_0})(\partial) + \epsilon \cdot (\nabla_{E_0}^{ad}(R'), \partial)
\end{equation}

\begin{equation}
= \text{pr}_U^*(\nabla_{E_0})(\partial) + \epsilon \cdot (R, \partial)
\end{equation}

\begin{equation}
= \text{pr}_U^*(\nabla_{E_0})(\partial) + \epsilon \cdot (\frac{1}{\epsilon} \cdot (\nabla_E - \text{pr}_U^*(\nabla_{E_0}))(\partial))
\end{equation}

\begin{equation}
= \nabla_E(\partial),
\end{equation}

where $(-, -)$ denotes the pairing

\begin{equation}
(\Omega_{U^{\log}/S^{\log}} \otimes \mathfrak{g}_E) \times \mathcal{T}_{U^{\log}/S^{\log}} \subseteq (\Omega_{U^{\log}/S^{\log}} \otimes \mathfrak{g}_E) \times \mathcal{T}_{U^{\log}/S^{\log}} \rightarrow \mathfrak{g}_E.
\end{equation}
induced by the natural pairing $\Omega_{U_\epsilon^\log/S_\epsilon^\log} \times \mathcal{T}_{U_\epsilon^\log/S_\epsilon^\log} \rightarrow \mathcal{O}_U$. Thus, the automorphism $\eta_R$ is compatible with the $S$-log connections $\nabla_{\xi}$ and $\text{pr}_U^*(\nabla_{\xi_0})$. That is, $(\mathcal{E}_\xi, \nabla_{\xi_0})$ is locally isomorphic to $(\text{pr}_U^*(\mathcal{E}_0), \text{pr}_U^*(\nabla_{\xi_0}))$, as desired. \qed

**Proposition 5.8.3.**

Let $(\mathcal{F}, \nabla_{\mathcal{F}})$ be a log integrable vector bundle on $U^\log/S^\log$ with vanishing $p$-curvature. Then, both $F_{U/S_*}(\text{Ker}(\nabla_{\mathcal{F}}))$ and $F_{U/S_*}(\text{Coker}(\nabla_{\mathcal{F}}))$ are relatively torsion-free sheaves on $U_S^{(1)}$ (cf. § 3.4) of rank $\text{rk}(\mathcal{F})$.

**Proof.** By the isomorphism \( \mathcal{O}^{(\mathcal{F}, \nabla_{\mathcal{F}})} \) (cf. (517)), it suffices to consider only the case of $F_{X/S_*}(\text{Ker}(\nabla_{\mathcal{F}}))$. If $t : T \rightarrow S$ is an $S$-scheme, then we denote by $(\mathcal{F}_T, \nabla_{\mathcal{F}_T})$ the log integrable vector bundle on $(U \times_T S)^{\log}$ defined to be the pull-back of $(\mathcal{F}, \nabla_{\mathcal{F}})$ via $(\text{id}_U \times t) : U \times_T S \rightarrow U$. Consider the natural isomorphism of functors

\[
(id_{U_S^{(1)}} \times t)^*(F_{U/S_*})(-) \sim F_{U \times_T S/T_*}(t^*U)(-) \quad (530)
\]

It follows from the definition of the Cartier operator (cf. (516)) that $C^{(\mathcal{F}_T, \nabla_{\mathcal{F}_T})}$ may be identified, via the isomorphism (530), with the pull-back of $C^{(\mathcal{F}, \nabla_{\mathcal{F}})}$ via $t : T \rightarrow S$. On the other hand,

\[
F_{X \times_S T/T_*}(\text{Coker}(\nabla_{\mathcal{F}_T})) \sim (id_{U_S^{(1)}} \times t)^*(\text{Coker}(\nabla_{\mathcal{F}})) \quad (531)
\]

Hence, the right-hand vertical arrow in the square diagram (517) is compatible, in an evident sense, with the base-change via $t : T \rightarrow S$. That is, the natural morphism

\[
(id_{U_S^{(1)}} \times t)^*(F_{U/S_*}(\text{Ker}(\nabla_{\mathcal{F}}))) \rightarrow F_{U \times_T S/T_*}(\text{Ker}(\nabla_{\mathcal{F}_T})) \quad (532)
\]

is an isomorphism, and hence, the natural morphism

\[
(id_{U_S^{(1)}} \times t)^*(F_{U/S_*}(\text{Ker}(\nabla_{\mathcal{F}}))) \rightarrow (id_{U_S^{(1)}} \times t)^*(F_{U/S_*}(\mathcal{F})) \quad (533)
\]

is injective. Moreover, by applying this argument to various $S$-schemes $t : T \rightarrow S$, we conclude that $F_{U/S_*}(\text{Ker}(\nabla_{\mathcal{F}})) \rightarrow F_{U/S_*}(\mathcal{F})$ is universally injective with respect to base-change over $S$. This implies that $F_{X/S_*}(\mathcal{F})/F_{X/S_*}(\text{Ker}(\nabla_{\mathcal{F}}))$ is flat over $S$ (cf. [49], p. 17, Theorem 1). By [31], Proposition 1.1.1 (1), $F_{U/S_*}(\text{Ker}(\nabla_{\mathcal{F}}))$ turns out to be relatively torsion-free. (The computation of its rank $\text{rk}(F_{U/S_*}(\text{Ker}(\nabla_{\mathcal{F}})))$ follows, e.g., from the equivalence of categories (648)). This completes the proof of Proposition 5.8.3. \qed

By Corollary 5.8.2 and Proposition 5.8.3, we conclude the following

**Corollary 5.8.4.**

Let $(\mathcal{F}_\xi, \nabla_{\mathcal{F}_\xi})$ be a log integrable vector bundle of rank $n \geq 0$ on $U_\epsilon^\log/S_\epsilon^\log$ with
vanishing $p$-curvature. Then, the $\mathcal{O}_{U_{(1)}^{(i)}}$-module $F_{U_1/S_1}(\text{Ker}(\nabla_{\mathcal{F}_{s}}))$ is relatively torsion-free sheaf of rank $n$ on $U_{(1)}^{(i)}$, and there exists a canonical isomorphism
\begin{equation}
F_{U_1/S_1}(\text{Ker}(\nabla_{\mathcal{F}_{s}}))|_{U_{(1)}^{(i)}} \sim F_{U_1/S_1}(\text{Ker}(\nabla_{\mathcal{F}_s}|_U)).
\end{equation}
of $\mathcal{O}_{U_{(1)}^{(i)}}$-modules.

5.9. In this subsection, we study duality between the cohomology sheaves of the complexes associated with a log integrable vector bundle with vanishing $p$-curvature.

If we denote by $\omega_{X/S}$ and $\omega_{X_1/S}$ the dualizing sheaves (cf. §1.5) of $X/S$ and $X_1/S$ respectively, then
\begin{equation}
\omega_{X/S} \sim \Omega_{X/S}^{\log}(-D_{X/S}), \quad F_{X/S}(\omega_{X_1/S}) \sim \omega_{X/S}.
\end{equation}
Hence, for a vector bundle $\mathcal{G}$ on $X$, we obtain a composite isomorphism
\begin{equation}
\Psi^\mathcal{G} : \mathcal{H}om_{\mathcal{O}_{X_1/S}}(F_{X/S}(\mathcal{G}), \omega_{X_1/S}) \sim \mathcal{H}om_{\mathcal{O}_{X}}(\mathcal{G}, \omega_{X/S})
\end{equation}
\begin{equation}
\sim F_{X/S}(\Omega_{X/S}^{\log} \otimes \mathcal{G}^{\vee}(-D_{X/S})),
\end{equation}
where first isomorphism follows from [25], Chap III, Exercise 6.10 (b). In the following, we shall write
\begin{equation}
F_{X/S}(\mathcal{G})^{\mathcal{W}} := \mathcal{H}om_{\mathcal{O}_{X_1/S}}(F_{X/S}(\mathcal{G}), \omega_{X_1/S})
\end{equation}
for simplicity.

Now, let $(\mathcal{F}, \nabla_{\mathcal{F}})$ be a log integrable vector bundle on $X^{\log}/S^{\log}$ of rank $n$ with vanishing $p$-curvature. By applying the contravariant functor $(-)^{\mathcal{W}}$ ($:= \mathcal{H}om_{\mathcal{O}_{X_1/S}}(-, \omega_{X_1/S})$) to the morphism (514) for the case of the fixed $(\mathcal{F}, \nabla_{\mathcal{F}})$, one obtains a morphism
\begin{equation}
F_{X/S}(\nabla_{\mathcal{F}})^{\mathcal{W}} : F_{X/S}(\Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F})^{\mathcal{W}} \to F_{X/S}(\mathcal{F})^{\mathcal{W}}
\end{equation}
of $\mathcal{O}_{X_1^{(i)}}$-modules. On the other hand, consider the $S$-log connection $\nabla_{\mathcal{F}}^\vee : \mathcal{F}^{\vee} \to \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}^{\vee}$ on the dual $\mathcal{F}^{\vee}$ arising from $\nabla_{\mathcal{F}}$ (cf. §4.1). Moreover, we shall write
\begin{equation}
\nabla_{\mathcal{F}}^\vee(-D_{X/S}) : \mathcal{F}^{\vee}(-D_{X/S}) \to \Omega_{X^{\log}/S^{\log}} \otimes \mathcal{F}^{\vee}(-D_{X/S})
\end{equation}
for the $S$-log connection on $\mathcal{F}^{\vee}(-D_{X/S})$ obtained as the restriction of $\nabla_{\mathcal{F}}^\vee$ to $\mathcal{F}^{\vee}(-D_{X/S}) \subseteq \mathcal{F}^{\vee}$. As the case of $\nabla_{\mathcal{F}}$, $F_{X/S}(\nabla_{\mathcal{F}}^\vee(-D_{X/S}))$ may be thought of as a morphism of $\mathcal{O}_{X_1^{(i)}}$-modules. One verifies easily that both $\nabla_{\mathcal{F}}$ and $\nabla_{\mathcal{F}}^\vee(-D_{X/S})$ have vanishing $p$-curvature.
Proposition 5.9.1.

(i) The square diagram

\[
\begin{array}{ccc}
F_{X/S*}(\Omega^{\log}_{X/S} \otimes F)^{\mathbb{W}} & \xrightarrow{\sim} & F_{X/S*}(\mathcal{F}^\vee(-D_{X/S})) \\
F_{X/S*}(\nabla F)^{\mathbb{W}} & \downarrow & F_{X/S*}(\nabla F)^{\mathbb{W}} \\
F_{X/S*}(\mathcal{F})^{\mathbb{W}} & \xrightarrow{\sim} & F_{X/S*}(\Omega^{\log}_{X/S} \otimes \mathcal{F}^\vee(-D_{X/S}))
\end{array}
\]

(540)

is anti-commutative, i.e.,

\[
F_{X/S*}(\nabla F^\vee(-D_{X/S})) \circ \Psi^{\Omega^{\log}_{X/S} \otimes F} = -\Psi^F \circ F_{X/S*}(\nabla F)^{\mathbb{W}}.
\]

(ii) The diagram (540) induces canonical isomorphisms

\[
F_{X/S*}(\text{Coker}(\nabla F))^{\mathbb{W}} \xrightarrow{\sim} F_{X/S*}(\text{Ker}(\nabla F^\vee(-D_{X/S})))
\]

and

\[
F_{X/S*}(\text{Ker}(\nabla F))^{\mathbb{W}} \xrightarrow{\sim} F_{X/S*}(\text{Coker}(\nabla F^\vee(-D_{X/S}))).
\]

Proof. We first consider assertion (i). By Proposition 3.4.1 and [34], Proposition 1.1.3, the \(O_{X_S^{\text{triv}}}^{(1)}\)-module at the left-hand upper corner of (540) is relatively torsion-free (of rank \(n \cdot p\)). Hence, it suffices to verify the commutativity of the diagram (540) over the scheme-theoretically dense open subscheme \(X^{\text{sm}}\) of \(X\) (cf. § 1.6). But, since we have supposed that \((\mathcal{F}, \nabla F)\) has vanishing \(p\)-curvature, \((\mathcal{F}, \nabla F)\) is, Zariski locally on \(X^{\text{sm}}\), isomorphic to the direct sum \((\mathcal{O}^{\text{sm}}_X, d_{X/S}^{\text{sm}})\) of the trivial (log) integrable line bundle (cf. (648)), where \(d_{X/S}\) denotes the universal derivation \(\mathcal{O}_X \to \Omega_{X/S}\). Hence, it suffices to consider the case where \(X^{\text{sm}} = X\) (hence \(D_{X/S} = \emptyset\) and \(\omega_{X/S} = \Omega^{\log}_{X/S} = \Omega_{X/S}\) and \((\mathcal{F}, \nabla F) = (\mathcal{O}_X, d_{X/S})\).

We shall prove this assertion. It is well-known that the trace morphism \((F_{X/S*}(F_{X/S*}(\Omega_{X_S^{(1)}}) = \text{Coker}(\text{Ker}(\nabla F)))) \to \Omega_{X_{S}^{(1)}}\) coincides (after composing with the inclusion \(\Omega_{X_{S}^{(1)}}^{(1)} \hookrightarrow \Omega_{X_{S}^{(1)}}^{(1)} \otimes F_{X/S*}(\mathcal{O}_X)\)) with the Cartier operator \(C^{(\mathcal{O}_{X,S}^{(1)}, d_{X/S})}\) associated with \((\mathcal{O}_X, d_{X/S})\). Hence, if \(\mathcal{G}\) is as in (549) and \(\alpha, \beta\) are local sections of \(F_{X/S*}(\Omega_{X/S} \otimes \mathcal{G}^\vee)\), \(F_{X/S*}(\mathcal{G})\) respectively, then we have the equality

\[
(\Psi\mathcal{G})^{-1}(\alpha)(\beta) = C^{(\mathcal{O}_X, d_{X/S})}(\langle \alpha, \beta \rangle),
\]

where \(\langle -, - \rangle\) denotes the pairing

\[
F_{X/S*}(\Omega_{X/S} \otimes \mathcal{G}^\vee) \times F_{X/S*}(\mathcal{G}) \to F_{X/S*}(\Omega_{X/S})
\]

induced by the natural pairing \(\mathcal{G}^\vee \times \mathcal{G} \to \mathcal{O}_X\). Accordingly, for any local sections \(a, b\) of \(F_{X/S*}(\mathcal{O}_X)\) (which we regard also as local sections of \(\mathcal{O}_X\), we
have a sequence of equalities as follows:

\[(546)\quad (F_{X/S}(d_{X/S}))^\mathcal{W} \circ (\Psi_{\Omega_{Xlog/Slog}})^{-1}(a))(b) = (\Psi_{\Omega_{Xlog/Slog}})^{-1}(a)(d_{X/S}(b)) = C(O_X,d_{X/S})(a \cdot d_{X/S}(b)) = C(O_X,d_{X/S})(d_{X/S}(a \cdot b)) - C(O_X,d_{X/S})(d_{X/S}(a) \cdot b) = -(\Psi_{O_X})^{-1}(d_{X/S}(a))(b) = (-((\Psi_{O_X})^{-1} \circ F_{X/S*}(d_{X/S})(a))(b),\]

where the third equality follows from the commutativity of the diagram \[(517)\].

This implies the required equality \[(541)\], and completes the proof of assertion (i).

Next, we consider assertion (ii). By Proposition 5.8.3, \(F_{X/S*}(\text{Coker}(\nabla_F))\) is relatively torsion-free of rank \(n\). On the other hand, \(F_{X/S*}(\Omega_{Xlog/Slog} \otimes \mathcal{F})\) (by Proposition 3.4.1) relatively torsion-free of rank \(n \cdot p\). Hence, it follows from \[33\], Proposition 1.1.1 (i), that \(F_{X/S*}(\text{Im}(\nabla_F))\) is relatively torsion-free of rank \(n \cdot (p - 1)\). Moreover, by Proposition 1.1.2 in \textit{loc. cit.}, the sequence

\[(547)\quad 0 \to F_{X/S*}(\text{Coker}(\nabla_F))^\mathcal{W} \to F_{X/S*}(\Omega_{Xlog/Slog} \otimes \mathcal{F})^\mathcal{W} \to F_{X/S*}(\text{Ker}(\nabla_F))^\mathcal{W} \to 0\]

arising naturally from \(F_{X/S*}(\nabla_F)\) is exact, and hence, induces the asserted isomorphisms. \(\square\)

Next, consider the tensor product

\[(548)\quad (\mathcal{F}^\vee(-D_{X/S}) \otimes \mathcal{F}, \nabla_F^\vee(-D_{X/S}) \otimes \nabla_F)\]

(cf. §4.1) of the log integrable vector bundles \((\mathcal{F}^\vee(-D_{X/S}), \nabla_F^\vee(-D_{X/S}))\) and \((\mathcal{F}, \nabla_F)\). The natural pairing

\[(549)\quad \mathcal{F}^\vee(-D_{X/S}) \otimes \mathcal{F} \to \mathcal{O}_X(-D_{X/S})\]

is compatible with the respective \(S\)-log connections \(\nabla_F^\vee(-D_{X/S}) \otimes \nabla_F\) and \(d_{Xlog/Slog}(-D_{X/S})\), where \(d_{Xlog/Slog}(-D_{X/S})\) denotes the restriction \(\mathcal{O}_X(-D_{X/S}) \to \Omega_{Xlog/Slog}(-D_{X/S})\) of the universal logarithmic derivation \(d_{Xlog/Slog}\). By composing this pairing and the cup product in the de Rham cohomology, we obtain
a composite $\mathcal{O}_S$-bilinear pairing

\begin{equation}
\mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^\vee(-D_{X/S})]) \times \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla\nabla]) \rightarrow \mathbb{R}^2 f_*(\mathcal{K}^\bullet[\nabla^\vee(-D_{X/S}) \otimes \nabla\nabla])
\end{equation}

\begin{equation}
\rightarrow \mathbb{R}^2 f_*(\mathcal{K}^\bullet[d(-D_{X/S})])
\end{equation}

\begin{equation}
\cong \mathbb{R}^1 f_*(\omega_{X/S})
\end{equation}

\begin{equation}
\cong \mathcal{O}_S,
\end{equation}

which we denote by $\oint$. In particular, this morphism yields a morphism

\begin{equation}
\oint' : \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^\vee(-D_{X/S})]) \rightarrow \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla\nabla])
\end{equation}

of $\mathcal{O}_S$-modules. By the definition of $\oint$ (and by, e.g., the explicit description of the cup product in the de Rham cohomology in terms of the Čech double complex), the restriction of $\oint$ to the image of $\partial_2[\nabla^\vee(-D_{X/S})] \times \partial_2[\nabla\nabla]$ (cf. (445)) vanishes identically. Thus, the morphism $\oint'$ fits into the following diagram:

\begin{equation}
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
\mathbb{R}^1 f_*(\text{Ker}(\nabla^\vee(-D_{X/S}))) & \rightarrow & f_*(\text{Coker}(\nabla\nabla))
\end{array}
\end{equation}

\begin{equation}
\begin{array}{ccc}
\rightarrow & \rightarrow & \\
\downarrow & \downarrow \\
\mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^\vee(-D_{X/S})]) & \rightarrow & \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla\nabla])
\end{array}
\end{equation}

\begin{equation}
\begin{array}{ccc}
& \rightarrow & \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\end{equation}

where we recall that the left-hand vertical sequence is exact (cf. (445)).

**Corollary 5.9.2.**

The morphisms $\oint'$, $\oint''$, and $\oint'''$ are all isomorphisms. In particular, the right-hand vertical sequence of (552) is exact, and the $\mathcal{O}_S$-bilinear paring $\oint$ is non-degenerate.
Proof. Consider the composite isomorphism
\[ (553) \quad R^1 f_*(\text{Ker}(\nabla^{\text{ad}}_F(-D_{X/S}))) \xrightarrow{\sim} R^1 f_*(\text{Ker}(\nabla^{\text{ad}}_F(-D_{X/S})))) \]
\[ \xrightarrow{\sim} f_*^{(1)}(F_{X/S*}(\text{Ker}(\nabla^{\text{ad}}_F(-D_{X/S}))))^{\vee} \]
\[ \xrightarrow{\sim} f_*^{(1)}(F_{X/S*}(\text{Coker}(\nabla_\mathcal{F})))^{\vee} \]
\[ \xrightarrow{\sim} f_*(\text{Coker}(\nabla_\mathcal{F}))^{\vee}, \]
where the second isomorphism arises from Grothendieck-Serre duality and the third isomorphism follows from the isomorphism (542). One verifies from the definition of the $O_S$-bilinear morphism $\mathcal{F}$ that this composite coincides with $\mathcal{F}''$. Thus, $\mathcal{F}''$ is an isomorphism. Also, a similar argument (together with the isomorphism (543)) shows that $\mathcal{F}'''$ is an isomorphism, and hence, the right-hand vertical sequence of (552) is exact. Moreover, by applying the "five lemma" to the diagram (552), we conclude that $\mathcal{F}'$ is an isomorphism (i.e., $\mathcal{F}$ is nondegenerate). \[ \square \]

5.10. Recall the assumption that $\mathcal{E} = (\mathcal{E}_{X/h,S}, \nabla_\mathcal{E})$ is a $(\mathfrak{g}, h)$-oper on $X/S$ of canonical type. Suppose further that $h$ lies in $\Gamma(S, O_S)$ and $\mathcal{E}$ is dormant. If we consider the $(\mathfrak{g}, 1)$-oper $\mathcal{E}_{X/h,-1} := (\mathcal{E}_{X/h,S}, \nabla_\mathcal{E}_{X/h,-1} := h^{-1} \cdot \nabla_\mathcal{E})$, then the $S$-log connection $h^{-1} \cdot \nabla^{\text{ad}}_{\mathcal{E}}$ on $\mathcal{E}_{X/h,S}$ may be identified with $\nabla^{\text{ad}}_{\mathcal{E}_{X/h,-1}}$, i.e., the $S$-log connection on $\mathcal{E}_{X/h,S}$ arising from $\nabla^{\text{ad}}_{\mathcal{E}_{X/h,-1}}$ via $\text{Ad}_G : G \rightarrow \text{GL}(\mathfrak{g})$.

One verifies equalities
\[ (554) \quad \text{Ker}(\nabla^{\text{ad}}_{\mathcal{E}_{X/h,-1}}) = \text{Ker}(\nabla^{\text{ad}}_{\mathcal{E}}), \quad \text{Im}(\nabla^{\text{ad}}_{\mathcal{E}_{X/h,-1}}) = \text{Im}(\nabla^{\text{ad}}_{\mathcal{E}}) \]
of subsheaves of $\mathcal{E}_{X/h,S}$ and $\Omega_{X/S} \otimes \mathcal{E}_{X/h,S}$ respectively; hence, we also have a natural isomorphism
\[ (555) \quad \text{Coker}(\nabla^{\text{ad}}_{\mathcal{E}_{X/h,-1}}) \xrightarrow{\sim} \text{Coker}(\nabla^{\text{ad}}_{\mathcal{E}}). \]

Proposition 5.10.1. The direct image $f_*(\text{Coker}(\nabla^{\text{ad}}_{\mathcal{E}}))$ (resp., $R^1 f_*(\text{Ker}(\nabla^{\text{ad}}_{\mathcal{E}}))$) is a vector bundle on $S$ of rank $\mathfrak{n}(\mathfrak{g})$ (resp., $\mathfrak{n}(\mathfrak{g})$) (cf. (118)).

Proof. By (554) and (555), we may assume that $h = 1$. Observe that since $X^{(1)}_S$ is flat over $S$ and both $F_{X/S*}(\text{Ker}(\nabla^{\text{ad}}_{\mathcal{E}}))$ and $F_{X/S*}(\text{Coker}(\nabla^{\text{ad}}_{\mathcal{E}}))$ are flat $O_{X^{(1)}_S}$-modules (cf. Proposition 5.8.3), both $f_*(\text{Coker}(\nabla^{\text{ad}}_{\mathcal{E}}))$ and $R^1 f_*(\text{Ker}(\nabla^{\text{ad}}_{\mathcal{E}}))$ are flat over $S$. Also, observe that $R^2 f_*(\text{Ker}(\nabla^{\text{ad}}_{\mathcal{E}})) = 0$ (since $X/S$ is of relative dimension 1) and
\[ (556) \quad R^1 f_*(\text{Coker}(\nabla^{\text{ad}}_{\mathcal{E}})) \xrightarrow{\sim} R^2 f_*(K^\bullet(\nabla^{\text{ad}}_{\mathcal{E}})) \xrightarrow{\sim} 0. \]
(by Proposition 5.2.2). Hence, it follows from \([25]\), Chap. III, Theorem 12.11 (b), that both \(f_*(\text{Coker}(\nabla^{\text{ad}}_E))\) and \(\mathbb{R}^1 f_*(\text{Ker}(\nabla^{\text{ad}}_E))\) are vector bundles on \(S\).

In the following, we shall compute the rank of these vector bundles. First, by Corollary 5.9.2 for the case where the log integrable vector bundle \((\mathcal{F}, \nabla_{\mathcal{F}})\) is taken to be \((g, \nabla^{\text{ad}}_E)\), there exists an isomorphism

\[
\int'' : \mathbb{R}^1 f_*(\text{Ker}(\nabla^{\text{ad}}_E)(-D_{X/S})) \xrightarrow{\sim} f_*(\text{Coker}(\nabla^{\text{ad}}_E))^{\vee}.
\]

(In particular, \(\mathbb{R}^1 f_*(\text{Ker}(\nabla^{\text{ad}}_E)(-D_{X/S}))\) is a vector bundle on \(S\) of rank equal to the rank of \(f_*(\text{Coker}(\nabla^{\text{ad}}_E))\).) On the other hand, the \(G\)-invariant nondegenerate symmetric \(k\)-bilinear form \(\mathfrak{B}(\varepsilon) : g \times g \to k\) (cf. \([452]\)) induces an isomorphism \(g_{c,t} \xrightarrow{\sim} g_{c,t}^{\vee}\) that is compatible with the respective \(S\)-log connections \(\nabla^{\text{ad}}_E\) and \(\nabla^{\text{ad}}_{\varepsilon}\). Also, this isomorphism induces, by restricting, an isomorphism

\[
g_{c,1,x/S}(-D_{X/S}) \xrightarrow{\sim} g_{c,1,x/S}^{\vee}(-D_{X/S})
\]

that is compatible with \(\nabla^{\text{ad}}_E(-D_{X/S})\) and \(\nabla^{\text{ad}}_{\varepsilon}(-D_{X/S})\). It follows that

\[
\mathbb{R}^1 f_*(\text{Ker}(\nabla^{\text{ad}}_E)(-D_{X/S})) \xrightarrow{\sim} \mathbb{R}^1 f_*(\text{Ker}(\nabla^{\text{ad}}_{\varepsilon})(-D_{X/S})).
\]

By composing (557) with (559), we obtain an isomorphism

\[
\mathbb{R}^1 f_*(\text{Ker}(\nabla^{\text{ad}}_E)(-D_{X/S})) \xrightarrow{\sim} f_*(\text{Coker}(\nabla^{\text{ad}}_E))^{\vee}
\]

and, in particular, the equality

\[
\text{rk}(\mathbb{R}^1 f_*(\text{Ker}(\nabla^{\text{ad}}_E)(-D_{X/S}))) = \text{rk}(f_*(\text{Coker}(\nabla^{\text{ad}}_E))).
\]

Next, consider the natural injection of short exact sequences

\[
\begin{array}{ccccccc}
0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
g_{c,1,x/S}(-D_{X/S}) & \xrightarrow{\nabla^{\text{ad}}_{\varepsilon}(-D_{X/S})} & \Omega^{\text{log}/S \log} \otimes g_{c,1,x/S}(-D_{X/S}) & \downarrow & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
g_{c,1,x/S} & \xrightarrow{\nabla^{\text{ad}}_{\varepsilon}} & \Omega^{\text{log}/S \log} \otimes g_{c,1,x/S} & \downarrow & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bigoplus_{i=1}^r \sigma^*_i(g_{c,1,x/S}) & \longrightarrow & \bigoplus_{i=1}^r \sigma^*_i(\Omega^{\text{log}/S \log} \otimes g_{c,1,x/S}) & \downarrow & \downarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0,
\end{array}
\]
where the bottom horizontal arrow coincides with $\bigoplus_{i=1}^r \sigma_{i*}(\text{ad}(\mu_i^{(E^\dagger_{G,1,x/S},\nabla_E)}))$ (cf. §5.6). By applying the “snake lemma” to this diagram, we obtain an exact sequence

\[(563)\quad 0 \to \text{Ker}(\nabla_{E^\dagger}^{\text{ad}}(-D_{X/S})) \to \text{Ker}(\nabla_{E^\dagger}^{\text{ad}}) \to \bigoplus_{i=1}^r \sigma_{i*}(\text{Ker}(\text{ad}(\mu_i^{(E^\dagger_{G,1,x/S},\nabla_E)}))) \to \text{Coker}(\nabla_{E^\dagger}^{\text{ad}}(-D_{X/S})) \to \text{Coker}(\nabla_{E^\dagger}^{\text{ad}}).
\]

Here, we consider the square diagram

\[(564)\quad \begin{array}{c}
F_{X/S*}(\text{Coker}(\nabla_{E^\dagger}^{\text{ad}}(-D_{X/S}))) \quad \longrightarrow \quad F_{X/S*}(\text{Coker}(\nabla_{E^\dagger}^{\text{ad}})) \\
\Omega_{X(1)}^{1,\log} \otimes \text{Ker}(\nabla_{E^\dagger}^{\text{ad}}(-D_{X/S})) \quad \longrightarrow \quad \Omega_{X(1)}^{1,\log} \otimes \text{Ker}(\nabla_{E^\dagger}^{\text{ad}}),
\end{array}
\]

where both the upper and lower horizontal arrow are natural morphisms arising from the inclusion $\mathfrak{g}_{E^\dagger_{G,1,x/S}}(-D_{X/S}) \hookrightarrow \mathfrak{g}_{E^\dagger_{G,1,x/S}}$, and the left-hand and right-hand vertical arrows denote the isomorphisms $\mathcal{O}_{(E^\dagger_{G,1,x/S},\nabla_{E^\dagger})}$ and $\mathcal{O}_{(E^\dagger_{G,1,x/S},\nabla_{E^\dagger})}$ (cf. (517)) respectively. It follows from the construction of Cartier operators that this diagram is commutative. Since the lower horizontal arrow in (564) is injective, the upper horizontal arrow in (564), which may be considered as the last arrow in the sequence (563), turns out to be injective. Thus, (563) yields a short exact sequence

\[(565)\quad 0 \to \text{Ker}(\nabla_{E^\dagger}^{\text{ad}}(-D_{X/S})) \to \text{Ker}(\nabla_{E^\dagger}^{\text{ad}}) \to \bigoplus_{i=1}^r \sigma_{i*}(\text{Ker}(\text{ad}(\mu_i^{(E^\dagger_{G,1,x/S},\nabla_E)}))) \to 0.
\]

But, one may observe the composite isomorphisms

\[(566)\quad \mathbb{R}^1 f_*\left(\bigoplus_{i=1}^r \sigma_{i*}(\text{Ker}(\text{ad}(\mu_i^{(E^\dagger_{G,1,x/S},\nabla_E)})))\right) \xrightarrow{\sim} \bigoplus_{i=1}^r \mathbb{R}^1 \text{id}_{S*}(\text{Ker}(\text{ad}(\mu_i^{(E^\dagger_{G,1,x/S},\nabla_E)}))) \xrightarrow{\sim} 0
\]

and

\[(567)\quad f_*(\text{Ker}(\nabla_{E^\dagger}^{\text{ad}})) \xrightarrow{\sim} \mathbb{R}^0 f_*(\mathcal{K}^*(\nabla_{E^\dagger}^{\text{ad}})) \xrightarrow{\sim} 0
\]
(cf. Proposition 5.2.2 (i)). Hence, by applying the functor $R^1f_*(-)$ to (565), we obtain a short exact sequence
\[(568)\quad 0 \to \bigoplus_{i=1}^r \text{Ker}(\text{ad}(\mu_i^{(\mathcal{E}_{0,1,x/S},\nabla)})) \to R^1f_*(\text{Ker}(\nabla^\text{ad}(-D_{X/S}))) \to R^1f_*(\text{Ker}(\nabla^\text{ad})) \to 0.\]

Since $R^1f_*(\text{Ker}(\nabla^\text{ad}))$ was verified, by the above discussion, to be a vector bundle, the exact sequence (568) shows that $R^1f_*(\text{Ker}(\nabla^\text{ad}(-D_{X/S})))$ is a vector bundle on $S$. Moreover, its rank satisfies the equality
\[(569)\quad \text{rk}(R^1f_*(\text{Ker}(\nabla^\text{ad}))) = \text{rk}(R^1f_*(\text{Ker}(\nabla^\text{ad}(-D_{X/S})))) + \text{rk}(\bigoplus_{i=1}^r \text{Ker}(\text{ad}(\mu_i^{(\mathcal{E}_{0,1,x/S},\nabla)})))
= \text{rk}(R^1f_*(\text{Ker}(\nabla^\text{ad}))) + r \cdot \text{rk}(\mathfrak{g}),\]
where the second equality follows from Proposition 5.6.1.

Finally, by Proposition 5.2.2 (ii) and the short exact sequence (445) applied to the case where $\nabla = \nabla^\text{ad}$, one obtains the equality
\[(570)\quad \text{rk}(f_*(\text{Coker}(\nabla^\text{ad}))) + \text{rk}(R^1f_*(\text{Ker}(\nabla^\text{ad}))) = \text{rk}(R^1f_*(\mathcal{K}^\bullet[\nabla^\text{ad}])))
= (2g - 2 + r) \cdot \text{dim}(\mathfrak{g}).\]

Thus, the equations (561), (569), and (570) deduce the equalities
\[(571)\quad \text{rk}(f_*(\text{Coker}(\nabla^\text{ad}))) = \text{\text{N}}(\mathfrak{g}), \quad \text{rk}(R^1f_*(\text{Ker}(\nabla^\text{ad}))) = \text{\text{N}}(\mathfrak{g}).\]

This completes the proof of Proposition 5.10.1. \(\square\)

5.11. Write
\[(572)\quad \hat{\psi}_{g,h,g,r} : \mathcal{O}_p_{g,h,g,r} \to \mathfrak{g}_{g,h,g,r}\]
for the morphism (218) in the case where the pointed stable curve $X/S$ is taken to be the tautological curve $\mathcal{E}_{g,r}$ over $\mathcal{M}_{g,r}$. In this section, we suppose further that $h = 1$ and then, describe, by means of Cartier operator, the differential of the morphism $\hat{\psi}_{g,1,g,r}$ (i.e., the case where $h = 1$) at the classifying morphism of the dormant $\mathfrak{g}$-oper $\mathcal{E}^\bullet$ under consideration.

**Lemma 5.11.1.**
The subsheaves $\text{Im}(\nabla_{\mathcal{E}G}^\text{ad}(\mathcal{E}^\bullet)), \text{Im}(\nabla_{\mathcal{E}}^\text{ad})$ of $\Omega_{X/S} \otimes \mathfrak{g}_{g,1,x/S} \otimes \mathcal{E}^\bullet (cf. Lemma 5.3.1)$ are coincides.
Proof. Consider the $\mathcal{O}_X$-linear endomorphism $\nabla_\alpha$ of $\mathcal{F}_{\mathcal{E}_G,1,X/S}^{log}$ determined by assigning $s \mapsto s - \nabla_\mathcal{E} \circ \alpha^\log_{\mathcal{E}_G,1,X/S}(s)$ for any local section $s \in \mathcal{F}_{\mathcal{E}_G,1,X/S}^{log}$.

Then,

\begin{equation}
\langle \nabla^\ad_{\mathcal{E}_G}(\nabla_\alpha(s)), \partial \rangle
= \langle \nabla^\ad_{\mathcal{E}_G}(s - \nabla_\mathcal{E} \circ \alpha^\log_{\mathcal{E}_G,1,X/S}(s)), \partial \rangle
\end{equation}

\begin{equation}
= [s - \nabla_\mathcal{E} \circ \alpha^\log_{\mathcal{E}_G,1,X/S}(s), \nabla_\mathcal{E}(\partial)] - \nabla_\mathcal{E}([\alpha^\log_{\mathcal{E}_G,1,X/S}(s), \partial])
\end{equation}

\begin{equation}
= [s, \nabla_\mathcal{E}(\partial)] - \nabla_\mathcal{E}([\alpha^\log_{\mathcal{E}_G,1,X/S}(s), \partial])
\end{equation}

\begin{equation}
- \nabla_\mathcal{E}([\alpha^\log_{\mathcal{E}_G,1,X/S}(s), \partial]) - ([\alpha^\log_{\mathcal{E}_G,1,X/S}(s), \partial])
\end{equation}

\begin{equation}
= [s, \nabla_\mathcal{E}(\partial)] - \nabla_\mathcal{E}([\alpha^\log_{\mathcal{E}_G,1,X/S}(s), \partial])
\end{equation}

\begin{equation}
= \langle \nabla^\ad_{\mathcal{E}_G}(s), \partial \rangle,
\end{equation}

where $\langle -, - \rangle$ denotes the $\mathcal{O}_X$-bilinear pairing

\begin{equation}
(\Omega_{X^{log}/S^{log}} \otimes \mathcal{F}_{\mathcal{E}_G,1,X/S}^{log}) \times \mathcal{T}_{X^{log}/S^{log}} \rightarrow \mathcal{F}_{\mathcal{E}_G,1,X/S}^{log}
\end{equation}

induced by the natural pairing $\Omega_{X^{log}/S^{log}} \times \mathcal{T}_{X^{log}/S^{log}} \rightarrow \mathcal{O}_X$. This implies that $\text{Im}(\nabla^\ad_{\mathcal{E}_G}) = \text{Im}(\nabla^\ad_{\mathcal{E}_G} \circ \nabla_\alpha)$. But,

\begin{equation}
\alpha^\log_{\mathcal{E}_G,1,X/S} \circ \alpha^\log_{\mathcal{E}_G,1,X/S}(s)
\end{equation}

\begin{equation}
= \alpha^\log_{\mathcal{E}_G,1,X/S}(s) - (\alpha^\log_{\mathcal{E}_G,1,X/S} \circ \nabla_\mathcal{E}) \circ \alpha^\log_{\mathcal{E}_G,1,X/S}(s)
\end{equation}

\begin{equation}
= \alpha^\log_{\mathcal{E}_G,1,X/S}(s) - \text{id} \circ \alpha^\log_{\mathcal{E}_G,1,X/S}(s)
\end{equation}

\begin{equation}
= 0,
\end{equation}

so the image of $\nabla_\alpha$ is contained in $\mathfrak{g}_{\mathcal{E}_G}$ (denoted $\text{Ker}(\alpha^\log_{\mathcal{E}_G,1,X/S})$). Hence, since $\nabla^\ad_{\mathcal{E}_G} \big|_{\mathfrak{g}_{\mathcal{E}_G}} = \nabla^\ad_{\mathcal{E}_G}$, we have equalities

\begin{equation}
\text{Im}(\nabla^\ad_{\mathcal{E}_G}) = \nabla^\ad_{\mathcal{E}_G}(\text{Im}(\nabla_\alpha)) = \nabla^\ad_{\mathcal{E}_G}(\mathfrak{g}_{\mathcal{E}_G}) = \text{Im}(\nabla^\ad_{\mathcal{E}_G}).
\end{equation}

This completes the proof of Lemma 5.11.1. \qed
By Lemma 5.11.1 above, the square diagram
\[
\begin{array}{ccc}
\tilde{\mathcal{E}}^{\text{log}}_{\log, \log} & \longrightarrow & 0 \\
\n & \downarrow \alpha_{\log} & \\
\Omega_{X_{\log}/S_{\log}} \otimes \mathfrak{g}_{c,1,1}/S_{\log} & \longrightarrow & \text{Coker}(\nabla_{\log}^{\text{ad}}), \\
\end{array}
\]
(577)

where the lower horizontal arrow denotes the natural surjection, is commutative. Thus, this diagram defines a morphism $\mathcal{K}^* [\nabla_{\log}^{\text{ad}}] \to \text{Coker}(\nabla_{\log}^{\text{ad}})[-1]$ of complexes. By applying the functor $\mathbb{R}^1 f_*(-)$ to the composite
\[
\mathcal{K}^* [\nabla_{\log}^{\text{ad}}] \to \mathcal{K}^* [\nabla_{\log}^{\text{ad}}] \to \text{Coker}(\nabla_{\log}^{\text{ad}})[-1],
\]
(578)

we obtain a morphism
\[
e^{\oplus}_{S} : \mathbb{R}^1 f_*(\mathcal{K}^* [\nabla_{\log}^{\text{ad}}]) \to f_*(\text{Coker}(\nabla_{\log}^{\text{ad}}))
\]
(579)

of $\mathcal{O}_S$-modules.

Next, the composite
\[
\Omega_{X_{S}}^{\text{log}}(\log) \otimes F_{X/S_*(\text{Ker}(\nabla_{\log}^{\text{ad}}))) \hookrightarrow F_{X/S_*(\Omega_{X_{S}}^{\text{log}}(\log) \otimes \mathfrak{g}_{c,1}^{\log}/S_{\log}))
\]
\[
\sim F_{X/S_*(\Omega^{\text{log}}_{X_{S}}(\log) \otimes \mathfrak{g}_{c,1}^{\log}/S_{\log}))
\]
(580)

induces, by applying the functor $f_*(-)$, an injection
\[
f_{S}^{(1)}(\Omega_{X_{S}}^{\text{log}}(\log) \otimes F_{X/S_*(\text{Ker}(\nabla_{\log}^{\text{ad}})))) \hookrightarrow f_{S}^{(1)}(F_{X/S_*(\Omega^{\text{log}}_{X_{S}}(\log) \otimes \mathfrak{g}_{c,1}^{\log}/S_{\log}))
\]
\[
\sim f_{S}(\Omega^{\text{log}}_{X_{S}}(\log) \otimes \mathfrak{g}_{c,1}^{\log}/S_{\log}))
\]
(581)

Now, let $s : S \to \mathfrak{m}_{g,r}$ and $s^*: S \to \mathfrak{D}_{g,1,g,r}$ be as in Proposition 5.4.1 for the dormant $\mathfrak{g}$-oper $\mathcal{E}^*$ under consideration. Then, the composite $\hat{\nu}_{g,1,g,r} \circ s^*$ coincides with the zero section $\nu \circ s : S \to \mathfrak{m}_{g,r} \to \mathfrak{D}_{g,1,g,r}$ over $S$. We write
\[
\nu_{s} := \nu \circ s \quad (= \hat{\nu} \circ s^*) : S \to \mathfrak{D}_{g,1,g,r}
\]
(582)

The differential of $\hat{\nu}_{g,1,g,r}$ (over $k$) at $s^*$ determines a morphism
\[
d_{s}^* : \mathfrak{m}_{g,1,g,r} : s^*(\mathcal{T}_{\mathfrak{D}_{g,1,g,r}/k}) \to \mathfrak{m}_{s}^*(\mathcal{T}_{\mathfrak{D}_{g,1,g,r}/k})
\]
(583)

of $\mathcal{O}_S$-modules.

Also, the morphism $\nu_{s}$ determines naturally a decomposition
\[
\mathfrak{m}_{s}^*(\mathcal{T}_{\mathfrak{D}_{g,1,g,r}/k}) \sim \mathfrak{m}_{s}^*(\mathcal{T}_{\mathfrak{D}_{g,1,g,r}/k}) \oplus \mathfrak{m}_{s}^*(\mathcal{T}_{\mathfrak{D}_{g,1,g,r}/\mathfrak{m}_{g,r}})
\]
(584)
Here, the second factor $\mathfrak{e}_s^s(\mathcal{T}_{\mathcal{O}_{g,1,g,r}/\mathbb{P}_{g,r}})$ of the left-hand side of (584) has, by definition, an isomorphism

\[
f_*(\Omega^p_{X_{\log}/S_{\log}} \otimes g_{\mathcal{E}_{1,x/S}}) \sim \mathfrak{e}_s^s(\mathcal{T}_{\mathcal{O}_{g,1,g,r}/\mathbb{P}_{g,r}}).
\]

Finally, recall that the conjugate spectral sequence of $\mathcal{K}^*[\nabla^\text{ad}]$ (cf. (444)) induces a short exact sequence

\[
0 \to \mathbb{R}^1 f_*(\text{Ker}(\nabla^\text{ad})) \xrightarrow{e_s^\text{ad}} \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^\text{ad}]) \xrightarrow{e_s^\text{ad}} f_*(\text{Coker}(\nabla^\text{ad})) \to 0
\]

(cf. (445)).

**Proposition 5.11.2.**

The following square diagram is anti-commutative:

\[
\begin{array}{ccc}
\mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^\text{ad}]) & \xrightarrow{\Xi_{\mathcal{K}\mathcal{S}}} & \mathfrak{s}^s(\mathcal{T}_{\mathcal{O}_{g,1,g,r}/k}) \\
\downarrow & & \downarrow \\
f^1_*(F_{X/S_1}*(\text{Coker}(\nabla^\text{ad}))) & \xrightarrow{\mathfrak{e}_s^s(\mathcal{T}_{\mathcal{O}_{g,1,g,r}/k})} & f^1_*(\Omega^p_{X_{\log}/S_{\log}} \otimes F_{X/S_1}*(\text{Ker}(\nabla^\text{ad})))) \\
\downarrow & & \downarrow \Xi_{\mathcal{K}\mathcal{S}} \\
f^1_*(\Omega^p_{X_{\log}/S_{\log}} \otimes g_{\mathcal{E}_{1,x/S}}) & \xrightarrow{585} & \mathfrak{e}_s^s(\mathcal{T}_{\mathcal{O}_{g,1,g,r}/\mathbb{P}_{g,r}}),
\end{array}
\]

where the left-hand second vertical arrow denotes the isomorphism obtained by applying the functor $f^1_*(-)$ to the Cartier isomorphism $\mathcal{C}^\text{ad}(\mathfrak{e}_{1,x/S}, \nabla^\text{ad})$ (cf. (517)) of $(\mathfrak{e}_{1,x/S}, \nabla^\text{ad})$, and the right-hand third vertical arrow denotes the projection to the second factor.

**Proof.** Write $\Xi_1$ for the composite of the three left-hand vertical arrows in (587). Also, write $\Xi_2$ for the composite of $\Xi_{\mathcal{K}\mathcal{S}}$ and the three right-hand vertical arrows in (581) and moreover, the inverse of the isomorphism (585). In the following, we shall show the equality $\Xi_1 = \Xi_2$.

We may suppose, without loss of generality, that $S$ is affine. Choose a finite covering of $X$ by affine open sets $\{U_\alpha\}_{\alpha \in I}$, indexed by a set $I$, such that, on each $U_\alpha$, there exists a globally defined logarithmic coordinate $x_\alpha$ (i.e., $\Omega_{U_\alpha_{\log}/S_{\log}} \cong \mathcal{O}_{U_\alpha} \cdot d\log(x_\alpha)$). Then, $\Gamma(S, \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^\text{ad}]))$ may be calculated as the total cohomology of the Čech double complex

\[
C^p,q[\nabla^\text{ad}]: = \check{C}^p((U_\alpha)_{\alpha}, \mathcal{K}^q[\nabla^\text{ad}])
\]
associated with $K^\bullet[\nabla_{\text{ad}}]$ (cf. [41], (3.4)). That is,

$$\Gamma(S, \mathbb{R}^1 f_* (K^\bullet[\nabla_{\text{ad}}])) \cong \mathbb{H}^i (\text{Tot}^\bullet (C^\bullet \cdot [\nabla_{\text{ad}}])).$$

Now let $t$ be an element in $\Gamma(S, \mathbb{R}^1 f_* (K^\bullet[\nabla_{\text{ad}}]))$. By applying the isomorphism (589), $t$ may be represented by a 1-cocycle $t$ of $\text{Tot}^\bullet (C^\bullet \cdot [\nabla_{\text{ad}}])$. This 1-cocycle $t$ may be given by a pair

$$t = (a, (b_\alpha)_{\alpha \in I})$$

consisting of a Čech 1-cocycle

$$a \in \check{C}^1 (\{ U_\alpha \}_\alpha, \check{T}_{\log}^{\text{ad}})$$

and a Čech 0-cochain

$$(b_\alpha)_{\alpha \in I} \in \check{C}^0 (\{ U_\alpha \}_\alpha, \Omega_{X/S} \otimes \mathfrak{g}^{-1}_{\text{ad}, 1, x/S})$$

which agree under $\nabla_{\text{ad}}$ and the Čech coboundary map.

First, we consider the image of $t$ via $\Xi_1$. The element $\xi^p_\alpha (t) \in \Gamma(X, \text{Coker}(\nabla_{\text{ad}}))$ ($\subseteq \check{C}^0 (\{ U_\alpha \}_\alpha, \text{Coker}(\nabla_{\text{ad}}))$) corresponds to the image of $b$ via the composite

$$\check{C}^0 (\{ U_\alpha \}_\alpha, \Omega_{X/S} \otimes \mathfrak{g}^{-1}_{\text{ad}, 1, x/S}) \to \check{C}^0 (\{ U_\alpha \}_\alpha, \Omega_{X/S} \otimes \mathfrak{g}^{-1}_{\text{ad}, 1, x/S})$$

One verifies that the global section $\Xi_1 (t)$ of $\Omega_{X/S}^{\otimes \phi} \otimes \mathfrak{g}^{\otimes \phi}_{\text{ad}, 1, x/S}$ may be expressed, on each $U_\alpha$ ($\alpha \in I$), as $C^{\phi \cdot \text{ad}}_{\text{ad}, 1, x/S} (b_\alpha)$.

Next, we consider the image of $t$ via $\Xi_2$. The image of $a$ via the composite

$$\check{C}^1 (\{ U_\alpha \}_\alpha, \check{T}_{\log}^{\text{ad}}) \to \check{C}^1 (\{ U_\alpha \}_\alpha, \check{T}_{X/S \otimes \mathfrak{g}}) \to \Gamma(S, \mathbb{R}^1 f_* (T_{X/S \otimes \mathfrak{g}}))$$

induced by $a^{\log}$ corresponds to a 1-st order thickening $\check{X}_{/S} := (X_{/S}, \{ \otimes_i \})$ of $X_{/S}$ to a pointed stable curve over $S_{\epsilon} := S \times_k \text{Spec}(k[\epsilon]/(\epsilon^2))$. Moreover, the element $a$ represents (in $\Gamma(S, \mathbb{R}^1 f_* (\check{T}_{\log}^{\text{ad}}))$) a 1-st order thickening of $\mathfrak{g}_{\text{ad}, 1, x/S}$ to a right $\mathbb{B}$-torsor over $\check{X}$, which is (by the isomorphism $\Xi_{\text{ad}}$ and Proposition 2.7.3) isomorphic to $\mathfrak{g}_{\text{ad}, 1, \check{X}_{/S}}$. Let us write $\check{U}_{a, \epsilon} := U_a \times_{\check{X}} \check{X}$, and write $\pi_a : \check{U}_{a, \epsilon} \to U_a$ for the natural projection. Then, the log integrable $G$-torsor on $\check{X}_{/S}$ corresponding to $\Xi_{\text{ad}} (a)$ is, on each $\check{U}_{a, \epsilon}$, isomorphic to

$$(\pi_a^* (\mathfrak{g}_{\text{ad}, 1, \check{X}_{/S}} | U_a), \pi_a^* (\nabla \mathfrak{g} | U_a) + \epsilon : b_0^a),$$
where \( b^\alpha_\delta \) denotes the \( \mathcal{O}_{U_\alpha} \)-linear morphism \( T_{U_\alpha}^{\log / S^{\log}} \to g_{c_\delta,1,1,x/S}^t \) corresponding to \( b_\alpha \). Then, one verifies that the global section \( \Xi_2(t) \) of \( \Omega_{X_\delta}^{\log / S^{\log}} \otimes g_{c_\delta,1,x/S}^t \) may be expressed, on \( U_\alpha \), as

\[
\Xi_2(t)|_{U_\alpha} = \frac{1}{\alpha} \cdot \psi \left( (\alpha_{c_\delta,1,x/S}|_{U_\alpha}), g^{(\log / S^{\log})}_{c_\delta,1,x/S} \right)
\]

\[
( = \frac{1}{\alpha} \cdot \left( \psi (\alpha_{c_\delta,1,x/S}|_{U_\alpha}), g^{(\log / S^{\log})}_{c_\delta,1,x/S} \right) - \pi^* (\psi (c_{c_\delta,1,x/S}^{1}|_{U_\alpha} \log u_\alpha)) \).
\]

Hence, it follows from Proposition 5.8.1 that \( \Xi_1(\vec{t}) = -\Xi_2(\vec{t}) \) (for any \( \vec{t} \in \Gamma(S, \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^{ad}_{\mathcal{E}_\delta}])) \)). This completes the proof of Proposition 5.11.2. \( \square \)

5.12. Consider the natural composite

\[
\varpi : \Omega_{X_\delta}^{\log / S^{\log}} \otimes g_{c_\delta,1,x/S}^{-1} \to \Omega_{X_\delta}^{\log / S^{\log}} \otimes g_{c_\delta,1,x/S}^t \to \text{Coker}(\nabla_{\mathcal{E}_\delta}^{ad})
\]

By Lemma 5.11.1, the image of \( \nabla^{ad}_{\mathcal{E}_\delta} \) is contained in \( \text{Ker}(\varpi) \). Hence, one may restrict the codomain of \( \nabla^{ad}_{\mathcal{E}_\delta} \) to \( \text{Ker}(\varpi) \), and obtain an \( f^{-1}(\mathcal{O}_S) \)-linear morphism

\[
\nabla^{Z_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta} : \tilde{T}_{c_\delta}^{\log / S^{\log}} \to \text{Ker}(\varpi).
\]

Also, by restricting the domain of \( \nabla^{Z_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta} \) to \( g_{c_\delta,1,x/S}^{-1} \) (\( \subseteq \tilde{T}_{c_\delta}^{\log / S^{\log}} \)), one obtains an \( f^{-1}(\mathcal{O}_S) \)-linear morphism

\[
\nabla^{Z_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta} : g_{c_\delta,1,x/S}^t \to \text{Ker}(\varpi).
\]

The natural short exact sequence

\[
0 \to \mathcal{K}^*[\nabla^{Z_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}] \to \mathcal{K}^*[\nabla^{G_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}] \to T_{X_\delta}^{\log / S^{\log}}[0] \to 0
\]

yields (since \( f_*(T_{X_\delta}^{\log / S^{\log}}) = 0 \)) an exact sequence

\[
0 \to \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^{G_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}]) \to \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^{G_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}]) \to \mathbb{R}^1 f_*(T_{X_\delta}^{\log / S^{\log}})
\]

of \( \mathcal{O}_S \)-modules.

On the other hand, \( \nabla^{Z_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta} \) also gives naturally an exact sequence

\[
0 \to \mathcal{K}^*[\nabla^{G_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}] \to \mathcal{K}^*[\nabla^{ad_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}] \to \text{Im}(\varpi)[-1] \to 0.
\]

Thus, by taking hypercohomology sheaves, we obtain an exact sequence

\[
0 \to \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^{G_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}]) \to \mathbb{R}^1 f_*(\mathcal{K}^*[\nabla^{ad_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}]) \to f_*(\text{Im}(\varpi))
\]

\[
\to \mathbb{R}^2 f_*(\mathcal{K}^*[\nabla^{ad_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}]),
\]

\[
\to \mathbb{R}^2 f_*(\mathcal{K}^*[\nabla^{ad_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}]),
\]

\[
\to \mathbb{R}^2 f_*(\mathcal{K}^*[\nabla^{ad_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}]),
\]

\[
\to \mathbb{R}^2 f_*(\mathcal{K}^*[\nabla^{ad_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}]),
\]

\[
\to \mathbb{R}^2 f_*(\mathcal{K}^*[\nabla^{ad_{\mathcal{E}_\delta}}_{\mathcal{E}_\delta}]),
\]
where $\mathbb{R}^2 f_*(\mathcal{K}^\bullet[\nabla^{\text{ad}_g}_3]) = 0$ by Lemma 5.3.2. This exact sequence implies an isomorphism

\[(604)\quad \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^{\text{ad}_g}_3]) \xrightarrow{\sim} \text{Ker}(e^\circ_{\circ})\]

of $\mathcal{O}_S$-modules (cf. (579) for the definition $e^\circ_{\circ}$).

**Corollary 5.12.1.**

Let $\hat{s}^\bullet : S \to \mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}$ be the classifying morphisms of the pair $(X/S, \mathcal{E}^\bullet)$.

(i) If, moreover, $\hat{c}^\circ_{\circ : S \to \mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}}$ denotes the $S$-rational point of $\mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}$ classifying $(X/S, \mathcal{E}^\bullet)$ (where we recall that $\mathcal{E}^\bullet$ is of radii $\rho$), then, by restricting the isomorphism $\Xi_{\mathcal{E}^\bullet}$, we obtain canonical isomorphisms

\[(605)\quad \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^{\text{zzz}...}_{\mathcal{E}_3}]) \xrightarrow{\sim} \hat{s}^\circ_{\circ}
(\mathcal{T}_{\mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}/k) \xrightarrow{\sim} \hat{s}^\circ_{\circ}(\mathcal{T}_{\mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}/k)\]

and

\[(606)\quad \mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^{\text{zzz}...}_{\mathcal{E}_3}]) \xrightarrow{\sim} \hat{c}^\circ_{\circ
(\mathcal{T}_{\mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}/k) \xrightarrow{\sim} \hat{s}^\circ_{\circ}(\mathcal{T}_{\mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}/k).\]

(ii) The composite isomorphisms (605) and (606) fit into the isomorphism of exact sequences

\[(607)\quad \begin{array}{ccc}
\mathbb{R}^1 f_*(\mathcal{K}^\bullet[\nabla^{\text{zzz}...}_{\mathcal{E}_3}]) & \xrightarrow{\sim} & \hat{s}^\circ_{\circ}(\mathcal{T}_{\mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}/k) \\
\downarrow & & \downarrow \\
\mathbb{R}^1 f_*(\mathcal{T}_{X^\log/S^\log}) & \xrightarrow{\sim} & \hat{s}^\circ_{\circ}(\mathcal{T}_{\mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}/k),
\end{array}\]

where the left-hand vertical sequence is (605), the right-hand vertical sequence is the natural exact sequence of tangent bundles, and both $s$ and $\Xi_{X/S}$ are as in Proposition 5.4.1.

**Proof.** By the five lemma applied to the diagram (607), it suffices to prove only that the two morphisms in (605) are isomorphisms.

It follows from Proposition 5.11.2 that $\hat{s}^\circ_{\circ}(\mathcal{T}_{\mathcal{O}_{\mathbb{P}^{\text{zzz}...}_{g,1,g},r}/k)$ is naturally isomorphic to $\text{Ker}(\Xi_{1})$ (cf. the proof of Proposition 5.11.2 for the definition of $\Xi_{1}$). But, the two lower arrows in the left-hand vertical sequence of the diagram (587) are injective. It follows that $\text{Ker}(e^\circ_{\circ}) \xrightarrow{\sim} \text{Ker}(\Xi_{1})$, and hence, that the first morphism of (605) is an isomorphism.
Moreover, it follows from Proposition 3.12.3 (i) that $\mathcal{D}_p^{\text{zaz...}}$ is a disjoint union of $\mathcal{D}_p^{\text{zaz...}}$ s for various $\rho \in c^x(F)$. This implies that the second morphism of (603) is an isomorphism, which completes the proof of Corollary 5.12.1.

Corollary 5.12.2.

Let $h \in k^\times$ and $\rho \in c^x(k)$ (where $\rho := 0$ if $r = 0$). Also, let $k(v)$ be a field over $k$ and $v : \text{Spec}(k(v)) \to \mathcal{D}_p^{\text{zaz...}}$ a $k(v)$-rational point of $\mathcal{D}_p^{\text{zaz...}}$. Then, the $k(v)$-vector space $v^*(\mathcal{D}_p^{\text{zaz...}}/(\mathcal{D}_p^{\text{zaz...}}/k))$ is of rank $\geq 3g - 3 + r$. If, moreover, the natural projection $\mathcal{D}_p^{\text{zaz...}} \to \mathfrak{M}_{g,r}$ is unramified at $v$, then this projection is also flat (hence, étale) at $v$.

Proof. In the following, if $F$ is a sheaf on a scheme $Y$, we shall write, for simplicity, $\Gamma(F)$ for the group of its global sections (i.e., $\Gamma(F) := \Gamma(Y, F)$). By Proposition 3.9.1, we may assume, without loss of generality, that $h = 1$. The $k(v)$-rational point $v$ classifies a pointed stable curve $X/k(v)$ (where we denote by $f : X \to \text{Spec}(k(v))$ the structure morphism of its underlying semistable curve) and a dormant $g$-oper $\mathcal{E}^\bullet := (\mathcal{E}_B, \nabla_\mathcal{E})$ of canonical type on it. It follows from the isomorphism (604) and Corollary 5.12.1 that $\Gamma(v^*(\mathcal{D}_p^{\text{zaz...}}/k))$ is isomorphic to $\Gamma(\text{Ker}(e^\oplus))$, i.e., the kernel of the morphism

\[(608) \quad \Gamma(e^\oplus) : \Gamma(\mathbb{R}^1 f_* (K^\bullet[\nabla_\mathcal{E}^{\text{ad}}])) \to \Gamma(f_* \text{Coker}(\nabla_\mathcal{E}^{\text{ad}}))).\]

But, by Theorem 3.12.1, Proposition 5.4.1 (cf. [153], and Proposition 5.10.1, we have

\[(609) \quad \text{rk}(\Gamma(\text{Ker}(e^\oplus))) \geq \text{rk}(\Gamma(\mathbb{R}^1 f_* (K^\bullet[\nabla_\mathcal{E}^{\text{ad}}]))) - \text{rk}(\Gamma(f_* \text{Coker}(\nabla_\mathcal{E}^{\text{ad}})))) = (\mathbb{N}(g) + 3g - 3 + r) - \mathbb{N}(g) = 3g - 3 + r.\]

This completes the proof of the former assertion.

The latter assertion follows from the former assertion. Indeed, by Corollary 5.12.1, the condition that $\mathcal{D}_p^{\text{zaz...}} \to \mathfrak{M}_{g,r}$ is unramified at $v$ is equivalent to the condition that the composite

\[(610) \quad \Gamma(\text{Ker}(e^\oplus)) \to \Gamma(\mathbb{R}^1 f_* (K^\bullet[\nabla_\mathcal{E}^{\text{ad}}])) \to \Gamma(\mathbb{R}^1 f_* (\mathcal{T}_{X^{\text{log}}/k(v)}^{\text{log}}))\]

is injective. But, if it is satisfied, then, since

\[(611) \quad \text{rk}(\Gamma(\text{Ker}(e^\oplus))) \geq 3g - 3 + r = \text{rk}(\Gamma(\mathcal{T}_{X^{\text{log}}/k(v)}^{\text{log}})) ;\]

this composite is also an isomorphism. Hence, $\text{rk}(\Gamma(\text{Ker}(e^\oplus))) = 3g - 3 + r$ (i.e., the inequality (609) is, in fact, an equality) and the morphism (608) must be surjective. In particular, the third arrow $H^1(K^\bullet[\nabla_\mathcal{E}^{\text{ad}}]) \to H^0(\text{Im}(\varpi))$ in
(603) is surjective, and hence, $H^2(K^\bullet [\nabla_{E_B}]) = 0$. This means that any 1-st order deformation of $(X_{/k(v)}, E^\bullet)$ is unobstructed. Thus, we conclude that $\mathcal{D}_{g,h,p,g,r} \to \overline{\mathcal{M}}_{g,r}$ is smooth (hence, flat) at $v$. □

6. DORMANT OPERATIC FUSION RING

In this section, we analyze behavior of clutching dormant $(g, h)$-opers along with the clutching morphism of the underlying pointed stable curves. Under a certain assumption (i.e., the condition (Etale)$_{g,h}$ in §6.3), it makes sense to speak of the generic degree $\deg(\mathcal{D}_{g,h,p,g,r}^{\text{zzz}}/\overline{\mathcal{M}}_{g,r})$ over $\overline{\mathcal{M}}_{g,r}$ of the moduli stack $\mathcal{D}_{g,h,p,g,r}$ classifying dormant opers, which counts the number of dormant $(g, h)$-opers on a sufficiently general pointed stable curve. We see (cf. Theorem 6.4.1) that the function encoding these numbers satisfies a factorization rule, which is well-known as _fusion rule_. By means of this function, we construct a certain commutative ring referred to as the _dormant operatic fusion ring_ $\mathcal{F}_{g,p}$ (cf. Definition 6.4.2), and then, gives a computation of the value $\deg(\mathcal{D}_{g,h,p,g,r}^{\text{zzz}}/\overline{\mathcal{M}}_{g,r})$ (cf. Theorem 6.4.3). In both this section and the next section, we suppose that $k$ is a perfect field satisfying the condition (Char)$_p^W$ (cf. §2.1).

6.1. Let $n$ be a positive integer, and suppose that we are given a collection of data:

\begin{equation}
(612) \quad \mathcal{D} := (\Gamma, \{ (g_j, r_j) \}_{j=1}^n, \{ \lambda_j \}_{j=1}^n),
\end{equation}

where

- $\Gamma$ is a finite graph of $n$ vertices, numbered 1 through $n$;
- $(g_j, r_j)$ ($j = 1, \cdots, n$) is a pair of nonnegative integers such that $2g_j - 2 + r_j > 0$;
- $\lambda_j$ ($j = 1, \cdots, n$) is an injection $\lambda_j : E_j \hookrightarrow \{1, \cdots, r_j\}$ of sets, where $E_j$ denotes the set of ends of edges emanating from the $j$-th vertex of the graph $\Gamma$. (Here, if an edge runs from the $j$-th vertex to itself, then it defines two elements of $E_j$; if an edge runs from the $j$-th vertex to a different vertex, then it defines one element of $E_j$.)

Let $\{X_{j/S}\}_{j=1}^n$ be an ordered set of pointed stable curves over $S = \text{Spec}(k)$, where the $j$-th curve $X_{j/S} := (X_j/S, \{ \sigma_i : S \to X_i \}_{i})$ is of type $(g_j, r_j)$ and the underlying semistable curve $X_j$ is _irreducible_. Once a collection of data $\mathcal{D}$
described above have been specified, we may use it to glue together \( \{X_{j/S}\}_{j=1}^{n} \) to obtain a new pointed stable curve \( \mathfrak{X}/S := (X/S, \{\sigma_i : S \to X\}_{i=1}^{r}) \) in such a way that

- the dual graph of \( \mathfrak{X}/S \) is given by \( \Gamma \);
- the \( j \)-th vertex corresponds, as an irreducible component of \( X \), to \( X_j \);
- if \( \epsilon \) is an edge with ends \( \epsilon_1 \) (attached to the \( j_1 \)-st vertex) and \( \epsilon_2 \) (attached to the \( j_2 \)-nd vertex) such that \( \lambda_{j_1}(\epsilon_1) = a \) and \( \lambda_{j_2}(\epsilon_2) = b \), then \( \epsilon \) corresponds to a node on \( X \) obtained by gluing together \( X_{j_1} \) at the \( a \)-th marked point to \( X_{j_2} \) at the \( b \)-th marked point;
- we order the marked points \( \{\sigma_i\}_{i=1}^{r} \) of \( \mathfrak{X}/S \) lexicographically, i.e., a marked point lying on \( X_j \) comes before a marked point lying on \( X_{j'} \) (if \( j < j' \)), and among marked points lying on \( X_j \), we take the ordering induced by the original ordering of marked points on \( X_j \).

The genus “\( g \)”, and the number of marked points “\( r \)”, of \( \mathfrak{X}/S \) may be computed combinatorially from the collection of data \( D \). One may extend, in the evident way, this construction to the case where each \( \mathfrak{X}_{i/S} \) is any (possibly reducible) pointed stable curve over an arbitrary scheme \( S \) over \( k \). Moreover, by carrying out the above construction in the universal case, we obtain a morphism of moduli stacks:

\[
\text{Clut}_D : \prod_{j=1}^{n} \overline{M}_{g_j, r_j} \to \overline{M}_{g, r}.
\]

**Definition 6.1.1.**

We shall refer (cf. [53], Chap. I, Definition 3.11) to such a collection of data \( D \) as a clutching data for a pointed stable curve of type \( (g, r) \), and to \( \text{Clut}_D \) as the clutching morphism associated with \( D \).

6.2. Let \( S \) be an arbitrary scheme over \( k \), and \( D \) as above.

**Definition 6.2.1.**

A set of radii for \( D \) over \( S \) is an ordered set

\[
\rho_D := \{\rho^j\}_{j=1}^{n},
\]

where each \( \rho^j \) is an element \( \rho^j := (\rho^j_{i})_{i=1}^{r_j} \in \mathbb{C}^{\times r_j}(S) \) satisfying that for every edge \( \epsilon \) of \( \Gamma \) with ends \( \epsilon_1 \) (attached to the \( j_1 \)-st vertex of \( \Gamma \)) and \( \epsilon_2 \) (attached to the \( j_2 \)-nd vertex of \( \Gamma \)) such that \( \lambda_{j_1}(\epsilon_1) = a \) and \( \lambda_{j_2}(\epsilon) = b \), we have \( \rho^j_{a} = \rho_{b}^{j_2} \).

Let us fix a set of radii \( \rho_D := \{\rho^j\}_{j=1}^{n} \) for \( D \) over \( k \) and, for each \( j = 1, \cdots, n \), a pointed stable curve \( \mathfrak{X}_{j/S} := (X_j/S, \{\sigma_{j,i}\}_{i=1}^{r_j}) \) over \( S \) as in the above notations.
Suppose that, on each $X_{j/S}$, we are given a $(\mathfrak{g}, \mathfrak{h})$-oper $\mathcal{E}^{\mathfrak{g}, \mathfrak{h}} := (\mathcal{E}_{B, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}}, \nabla_{\mathfrak{g}, \mathfrak{h}})$ of canonical type and of radii $\rho^j$. Then, one may prove the claim that

these $(\mathfrak{g}, \mathfrak{h})$-opers $\mathcal{E}^{\mathfrak{g}, \mathfrak{h}}$ glue together uniquely to form a $(\mathfrak{g}, \mathfrak{h})$-oper $\mathcal{E}^{\mathfrak{g}, \mathfrak{h}} := (\mathcal{E}_{B, h, X/S}^{\mathfrak{g}, \mathfrak{h}}, \nabla_{\mathfrak{g}, \mathfrak{h}})$ of canonical type on $X/S$.

We shall prove this claim as follows. Let us denote by $X_{j/S} := (X/S, \{\sigma_i\}_{i=1}^n)$ the pointed stable curve obtained by gluing together $\{X_{j/S}\}_{j=1}^n$ by means of $D$. The clutching morphism $\text{Clut}_D$ associated with $D$ gives a natural morphism $\text{Clut}_j : X_j \rightarrow X$ ($j = 1, \cdots, n$) of $S$-schemes. Here, recall (cf. §1.5) that for each $j$, both $X_j$ and $S$ admit log structures pulled back from $\mathcal{E}^{\log}_{B, h, j/S}$ and $\mathcal{M}^{\log}_{g_j, r_j}$, respectively, via the classifying morphism of $X_{j/S}$. Denote by $X^{\log}_j$ and $S^{\log}$ the resulting log schemes (hence $X^{\log}_j = X^{\log}$ as the usual notation). On the other hand, we shall write, as usual, $S^{\log}$ (resp., $X^{\log}$) for the log scheme obtained by equipping $S$ (resp., $X$) with the log structure pulled back from $\mathcal{M}^{\log}_{g, r}$ (resp., $\mathcal{E}^{\log}_{B, h}$) via the classifying morphism of $X/S$. Also, write $X^{\log}_{j/S}$ for the log scheme obtained by equipping $X_j$ with the log structure pulled back from the log structure of $X^{\log}$ via $\text{Clut}_j$. The structure morphism $X_j \rightarrow S$ of the semistable curve $X_j/S$ extends to a morphism $X^{\log}_{j/S} \rightarrow S^{\log}$ of log schemes. Moreover, the natural morphism $X^{\log}_{j/S} \rightarrow X_j$ of log schemes (where we consider $X_j$ as being equipped with the trivial log structure) yields a log étale morphism $\mathfrak{e}_j : X^{\log}_{j/S} \rightarrow X^{\log}_j \times_{S^{\log}} S^{\log}$ over $S^{\log}$. (Note that the underlying morphism between $S$-schemes of $\mathfrak{e}_j$ coincides with the identity morphism of $X_j$.) This morphism $\mathfrak{e}_j$ induces an isomorphism $\Omega_{X^{\log}_{j/S}/S^{\log}} \cong \text{Clut}_j^\dagger(\Omega_{X^{\log}/S^{\log}})$ of $\mathcal{O}_X$-modules. Hence, by the constructions of $\mathcal{E}_{B, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}}$ and $\mathcal{E}_{B, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}}$ (cf. (108) and (422)), there exists a canonical isomorphism

$$\varphi_{B, j} : \mathcal{E}_{B, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}} \times_{X, \text{Clut}_j} X_j \cong \mathcal{E}_{B, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}}$$

of right $B$-torsors. By applying a change of structure group, we obtain a canonical isomorphism

$$\varphi_{G, j} : \mathcal{E}_{G, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}} := \mathcal{E}_{G, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}} \times_{X, \text{Clut}_j} X_j \cong \mathcal{E}_{G, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}}$$

of right $G$-torsors. Thus, by passing to the log étale morphism $\mathfrak{e}_j$ and the isomorphism $\varphi_{G, j}$, one may obtain, from the $h$-log integrable $G$-torsor $(\mathcal{E}_{G, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}}, \nabla_{\mathfrak{g}, \mathfrak{h}})$, an $h$-log integrable $G$-torsor

$$\mathcal{E}_{G}^{\mathfrak{g}, \mathfrak{h}} := (\mathcal{E}_{G, h, X_{j/S}}^{\mathfrak{g}, \mathfrak{h}}, \nabla_{\mathfrak{g}, \mathfrak{h}}, \nabla_{\mathfrak{g}, \mathfrak{h}})$$

over $X_{j/S}^{\log}$.

Now, let us choose an edge $e$ of $\Gamma$, which has ends $e_1$ (attached to the $j_1$-st vertex of $\Gamma$) and $e_2$ (attached to the $j_2$-nd vertex of $\Gamma$). Also, let us choose a
log chart \((U, x)\) on \(X^{\log}\) over \(S^{\log}\) satisfying that
\[ U \cap \text{Im}(\text{Clut}_{j_1} \circ \sigma_{\lambda_{j_1}(\epsilon_1)}) (= U \cap \text{Im}(\text{Clut}_{j_2} \circ \sigma_{\lambda_{j_2}(\epsilon)})) \neq \emptyset. \]
For \(\square = 1\) or \(2\), the pair \(\tau_{j_0} := (U_{j_0} := U \times_{X_{j_0}} X, \text{Clut}_{j_0}^*(x))\) forms a log chart on \(X^{\log, \tau_{j_0}}\) over \(S^{\log}\). We shall write
\[ S^U := U_{j_1} \times_{X_{j_1}} \sigma_{\lambda_{j_1}(\epsilon)} S (= U_{j_2} \times_{X_{j_2}} \sigma_{\lambda_{j_2}(\epsilon)} S), \]
which is, by the definition of \(U\), nonempty. Define \(\text{triv}_{G, h, \tau_{j_0}}^{\log, \lambda} X\) to be the composite isomorphism
\[ \text{triv}_{G, h, \tau_{j_0}}^{\log, \lambda} X : E_{G, h, X_{j_0}/S}^{\log, \lambda} \times_X \tau_{j_0} S^U \rightarrow \text{Clut}_{G, h, \tau_{j_0}}^\circ \tau_{j_0} \]
\[ \rightarrow (U_{j_0} \times_k G) \times_{X_{j_0}} \sigma_{\lambda_{j_0}(\epsilon)} S^U \]
\[ \rightarrow G \times_k S^U, \]
where the first arrow denotes the isomorphism \(\nu_{G, j_1} \times \text{id}_{S^U}\) and the second arrow denotes the isomorphism \(\text{triv}_{G, h, \tau_{j_0}}^{\log, \lambda} X, \sigma_{\lambda_{j_0}(\epsilon)}(\cdot)\) (cf. (111)). The composite isomorphism
\[ t_{j_1 \rightarrow j_2} : (\text{triv}_{G, h, \tau_{j_0}}^{\log, \lambda} X)^{-1} \circ \text{triv}_{G, h, \tau_{j_1}}^{\log, \lambda} X \]
of \(G\)-torsors over \(S^U\) induces an isomorphism
\[ d t_{j_1 \rightarrow j_2} : \Gamma(S^U, \sigma_{\lambda_{j_1}(\epsilon)}) |_{S^U} \rightarrow \Gamma(S^U, \sigma_{\lambda_{j_2}(\epsilon)}) |_{S^U} \]
Then, one may verify the equality
\[ d t_{j_1 \rightarrow j_2} (\mu_{\lambda_{j_1}(\epsilon)}) = \mu_{\lambda_{j_2}(\epsilon)} \]
(cf. Definition 1.6.1 for the definition of \(\mu^\circ (\cdot)\)). (Indeed, \(E^{\log, \lambda, \circ}_G\) is assumed to be of canonical type and the morphism \(\text{Clut}_{G, h, X_{j_0}/S}^\circ \sigma_{\lambda_{j_0}(\epsilon)}(\cdot)\) (cf. (173)) is an isomorphism. Hence the assumption
\[ (\rho_{\lambda_{j_1}(\epsilon)} \circ \rho_{\lambda_{j_1}(\epsilon)} \circ \log_X)(\cdot) = \rho_{\lambda_{j_2}(\epsilon)} \circ \rho_{\lambda_{j_2}(\epsilon)} (\cdot) \in \mathfrak{c}(S^U) \]
implies the asserted equality.)

By means of the isomorphisms \(\text{Clut}_j\)'s and \(t_{j_1 \rightarrow j_2}\)'s, one may glue \(\{E^{\log, \lambda, \circ}_G\}_{j=1}^n\) together to an \(h\)-log integrable \(G\)-torsor \((E^{\log, \lambda, \circ}_G, \nabla_\epsilon)\) over \(X^{\log}/S^{\log}\). By this construction, the pair \(E^{\log, \lambda, \circ}_G := (E^{\log, \lambda, \circ}_G, \nabla_\epsilon)\) forms a \((g, h)\)-oper on \(X/S\) of canonical type. Moreover, one verifies the uniqueness of such a \((g, h)\)-oper, i.e., that if a \((g, h)\)-oper \(E^{\log, \lambda, \circ}_G := (E^{\log, \lambda, \circ}_G, \nabla_\epsilon)\) on \(X/S\) of canonical type satisfies that \(\text{Clut}_{j_1}^* (E^{\log, \lambda, \circ}_G, \nabla_\epsilon) \simeq E^{\log, \lambda, \circ}_G (j = 1, \cdots, n)\), then \(E^{\log, \lambda, \circ}_G \simeq E^{\log, \lambda, \circ}_G\). This completes the proof of the claim. (Note that if all \(E^{\log, \lambda, \circ}_G\)'s are dormant, then the resulting \(E^{\log, \lambda, \circ}_G\) may be verified to be dormant.)
The radii of $E^\circ$ are constructed from $\rho^j$ ($j = 1, \ldots, n$) by means of the clutching data $\mathcal{D}$ in the evident way; we denote the resulting radii of $E^\circ$ by $\tilde{\rho}_D \in \mathcal{C}^{\times r}(S)$.

By carrying out the above construction in the universal case, we obtain a commutative square diagram of moduli stacks:

\[
\begin{array}{ccc}
\prod_{j=1}^n \mathcal{O}_{p_{g,h,\rho^j,g_j,r_j}} & \xrightarrow{\text{Clut}_D} & \mathcal{O}_{\mathcal{P}_{g,h,\tilde{\rho}_D,g,r}} \\
\downarrow & & \downarrow \\
\prod_{j=1}^n \mathcal{M}_{g_j,r_j} & \xrightarrow{\text{Clut}_D} & \mathcal{M}_{g,r}.
\end{array}
\]

Moreover, by restricting this diagram, we obtain a commutative square diagram:

\[
\begin{array}{ccc}
\prod_{j=1}^n \mathcal{O}_{p_{g,h,\rho^j,g_j,r_j}} & \xrightarrow{\text{Clut}_D} & \mathcal{O}_{\mathcal{P}_{g,h,\tilde{\rho}_D,g,r}} \\
\downarrow & & \downarrow \\
\prod_{j=1}^n \mathcal{M}_{g_j,r_j} & \xrightarrow{\text{Clut}_D} & \mathcal{M}_{g,r}.
\end{array}
\]

Here, recall (cf. Theorem 3.12.3 (i)) that if $\hbar \in k^{\times}$, then $\mathcal{O}_{p_{g,h,\tilde{\rho}_D,g,r}}$ is empty unless the radii $\tilde{\rho}_D$ is of the form $\tilde{\rho}_D = \hbar \cdot \rho$ for some $\rho \in \mathcal{C}^{\times r}(\mathbb{F}_p)$. In the case where $\tilde{\rho}_D = \hbar \cdot \rho$ for some $\rho \in \mathcal{C}^{\times r}(\mathbb{F}_p)$, this diagram (627) satisfies the following property.

**Theorem 6.2.2.**

Let $\mathcal{D} := (\Gamma, \{(g_j, r_j)\}_{j=1}^n, \{\lambda_j\}_{j=1}^n)$ be a clutching data for a pointed stable curve of type $(g, r)$, and $\hbar \in k^{\times}$, $\rho \in \mathcal{C}^{\times r}(\mathbb{F}_p)$. Then, the following commutative square diagram is cartesian:

\[
\begin{array}{ccc}
\prod_{\rho_D := \{\rho^j\}_{j=1}^n} \mathcal{O}_{p_{g,h,\rho^j,g_j,r_j}} & \xrightarrow{\text{Clut}_{D,\rho_D}} & \mathcal{O}_{p_{g,h,\hbar \cdot \rho,g,r}} \\
\downarrow & & \downarrow \\
\prod_{j=1}^n \mathcal{M}_{g_j,r_j} & \xrightarrow{\text{Clut}_D} & \mathcal{M}_{g,r}.
\end{array}
\]

where the disjoint union at the upper-left corner is taken over sets of radii $\rho_D$ for $\mathcal{D}$ over $\mathbb{F}_p$ with $\tilde{\rho}_D = \rho$.

**Proof.** Consider the morphism

\[
\prod_{\rho_D := \{\rho^j\}_{j=1}^n} \mathcal{O}_{p_{g,h,\rho^j,g_j,r_j}} \rightarrow \prod_{j=1}^n \mathcal{M}_{g_j,r_j} \times \mathcal{M}_{g,r} \mathcal{O}_{p_{g,h,\hbar \cdot \rho,g,r}}.
\]

In the following, we shall prove that for any $k$-scheme $S$, the map between the respective sets of $S$-rational points induced by this morphism is bijective.
Since the injectivity is easily verified, it suffices to prove the surjectivity. Let \( \eta \) be an \( S \)-rational point of \( \prod_{j=1}^{n} \overline{\mathbb{M}}_{g_j, r_j} \times \mathbb{M}_{g, r} \mathcal{O}_p_{g, h, h\ast p, g, r} \) classifying the collection of data

\[
(630) \quad \{ \mathfrak{x}_{j/S} := (X_j/S, \{ \sigma_{j_{i_{1_{j}}}r_{j_{i_{j}}}} \}) \}_{j=1}^{r}, \quad \mathfrak{x}_{/S} := (X/S, \{ \sigma_{i_{j}} \}_{i=1}^{r}), \quad \mathcal{E}_{\bullet}^{\bullet} := (\mathcal{E}_{g, h, X_{j/S}}^{\dagger}, \nabla_{\mathfrak{e}_{j}}),
\]

where

- each \( \mathfrak{x}_{j/S} (j = 1, \ldots, n) \) is a pointed stable curve over \( S \) of type \( (g_j, r_j) \);
- \( \mathfrak{x}_{/S} \) denotes the pointed stable curve over \( S \) of type \( (g, r) \) obtained by clutching pointed stable curves \( \mathfrak{x}_{j/S} \) by means of the clutching data \( \mathcal{D} \);
- \( \mathcal{E}_{\bullet}^{\bullet} \) is a dormant \( (g, h) \)-oper on \( \mathfrak{x}_{/S} \) of canonical type and of radii \( h \ast \rho \).

By reversing the steps, discussed above, in the construction of \( \mathcal{E}_{\bullet}^{\bullet} \) by means of \( \{ \mathcal{E}_{\bullet}^{\bullet} \}^{\tau}_{j=1} \), we obtain, for each \( j \), a \( (g, h) \)-oper \( \mathcal{E}_{\bullet}^{\bullet} := (\mathcal{E}_{g, h, X_{j/S}}^{\dagger}, \nabla_{\mathfrak{e}_{j}}) \) on \( \mathfrak{x}_{j/S} \) of canonical type. Note that all \( \mathcal{E}_{\bullet}^{\bullet} \) are verified to be dormant. If \( \rho_{\tau}^{j} := (\rho_{\tau}^{j})^{\tau}_{i=1} \) (\( \in \mathfrak{c}_{g, h, x_{j/S}}^{\mathfrak{r}}(S) \)) denotes the radii of \((\mathcal{E}_{g, h, x_{j/S}}, \nabla_{\mathfrak{e}_{j}})\), then the ordered set \( \rho_{\tau, \mathcal{D}} := \{ \rho_{\tau}^{j} \}_{j=1}^{n} \) forms a set of radii for \( \mathcal{D} \) over \( S \) (cf. Definition 6.2.1). It follows from Proposition 3.5.2 (i) that \( h^{-1} \ast \rho_{\tau}^{i} \in \mathfrak{c}(\mathbb{F}_{p}) \) \( (j = 1, \ldots, r \text{ and } i = 1, \ldots, r_{j}) \). This implies that the \( S \)-rational point \( \eta : S \to \prod_{j=1}^{n} \overline{\mathbb{M}}_{g_j, r_j} \times \mathbb{M}_{g, r} \mathcal{O}_p_{g, h, h\ast p, g, r} \) factors through the morphism \( \mathfrak{C}(\mathcal{U}_{D, \rho_{\tau}, \mathcal{D}}) : \prod_{j=1}^{n} \mathcal{O}_p_{g, h, h\ast p, g, r} \to \mathcal{O}_p_{g, h, h\ast p, g, r} \).

This completes the proof of the asserted surjectivity. \( \square \)

6.3. We shall recall a certain kind of pointed stable curves, which we refer to as totally degenerate (cf. Definition 6.3.1).

Suppose that the field \( k \) under consideration is algebraically closed. If \( \mathfrak{x}_{/k} := (X/k, \{ \sigma_{i} \}_{r=1}^{r}) \) is a pointed stable curve over \( k \) of type \( (g, r) \), then we shall write \( \{ \tau_{X_{j/k}}^{j} \}_{j=1}^{r} \) for the (possibly empty) set of nodes in \( X \) and \( \nu_{\mathfrak{x}_{/k}} : \prod_{l=1}^{L_{X_{j/k}}} \to X \) for the normalization of \( X \), where each \( X_{l} (l = 1, \ldots, L_{X_{j/k}}) \) is a proper smooth connected curve over \( k \).

Denote by \( [0], [1], \text{ and } [\infty] \) the \( k \)-rational points of the projective line \( \mathbb{P}^{1} \) over \( k \) determined by the values 0, 1, \text{ and } \infty \text{ respectively}\). The collection of data

\[
(631) \quad \mathfrak{p}_{/k} := (\mathbb{P}^{1}/k, \{ [0], [1], [\infty] \})
\]

(after ordering suitably the points \( [0], [1], [\infty] \)) forms a unique (up to isomorphism) pointed stable curve of type \( (0, 3) \).
Definition 6.3.1.
We shall say that a pointed stable curve $\mathcal{X}_k := (X/k, \{\sigma_i\}_{i=1}^r)$ is \textit{totally degenerate} if, for any $l = 1, \ldots, L_{\mathcal{X}_k}$, the pointed stable curve
\begin{equation}
\mathcal{X}_{l/k} := (X_{l/k}, \nu_{\mathcal{X}_{l/k}}^{-1}(\{\sigma_i\}_{i=1}^r \cup \{\tau_{i,j}^{l}\}_{j=1}^s) \cap X_{l}(k))
\end{equation}
(cf. the notations defined above) is isomorphic to $\mathcal{P}/k$.

There exists a bijective (in a natural sense) correspondence between the set of isomorphism classes of totally degenerate curves over $k$ of type $(g, r)$ and the clutching data $\mathcal{D} = (\Gamma, \{(g_j, r_j)\}_{j=1}^n, \{\lambda_j\}_{j=1}^n)$ for a pointed stable curve of type $(g, r)$ satisfying that $(g_j, r_j) = (0, 3)$ for any $j = 1, \ldots, n$. Indeed, the assignment from such a clutching data $\mathcal{D}$ to the totally degenerate curve classified by the clutching morphism
\begin{equation}
\text{Clut}_{\mathcal{D}} : (\text{Spec}(k) \sim \to) \prod_{j=1}^n \overline{M}_{0,3} \to \overline{M}_{g,r}
\end{equation}
associated with $\mathcal{D}$ determines this bijective correspondence.

Now, we consider the following condition concerning $g$ and $\hbar$:

\text{(Etale)}_{g, \hbar} : for any $\rho \in c^{x \times r}(\mathbb{F}_p)$, the finite $k$-scheme $\mathfrak{Q}_{g, \hbar, \hbar \star \rho, 0, 3}$ is unramified (hence étale) over $k$.

Since we have supposed that $k$ is algebraically closed, the condition \text{(Etale)}_{g, \hbar} is equivalent to the condition that each $\mathfrak{Q}_{g, \hbar, \hbar \star \rho, 0, 3}$ is isomorphic to a disjoint union of finite copies of $\text{Spec}(k)$. By Proposition 3.9.1, it does not depend on the choice of $\hbar \in k^\times$ whether the condition \text{(Etale)}_{g, \hbar} is satisfied or not.

Proposition 6.3.2.
Let $\hbar \in k^\times$ and suppose that the condition \text{(Etale)}_{g, \hbar} is satisfied with respect to the pair $(g, \hbar)$ under consideration. Then, for any pair of nonnegative integers $(g, r)$ satisfying that $2g - 2 + r > 0$ and $\rho \in c^{x \times r}(\mathbb{F}_p)$ (where $\rho := \emptyset$ if $r = 0$), $\mathfrak{Q}_{g, \hbar, \hbar \star \rho, g, r}$ is (finite, by virtue of Theorem 3.12.3, and) étale over the points of $\overline{M}_{g,r}$ classifying totally degenerate curves. In particular, $\mathfrak{Q}_{g, \hbar, \hbar \star \rho, g, r}$ is generically étale over $\overline{M}_{g,r}$ (i.e., any irreducible component that dominates $\overline{M}_{g,r}$ admits a dense open subscheme which is étale over $\overline{M}_{g,r}$).

\textbf{Proof.} Let $\mathcal{D} = (\Gamma, \{(g_j, r_j)\}_{j=1}^n, \{\lambda_j\}_{j=1}^n)$ be a clutching data corresponding, via the bijective correspondence mentioned above, to a totally degenerate curve $\mathcal{X}_k$ (hence $(g_j, r_j) = (0, 3)$ for all $j$). We shall apply Theorem 6.2.2 to this $\mathcal{D}$ and $\rho \in c^{x \times r}(\mathbb{F}_p)$ given in the assertion of Proposition 6.3.2. Then, by the condition \text{(Etale)}_{g, \hbar}, one verifies that the left-hand vertical arrow in the diagram (628) is unramified. This implies that the projection $\mathfrak{Q}_{g, \hbar, \hbar \star \rho, g, r} \to \overline{M}_{g,r}$ (i.e., the right-hand vertical arrow in (628) is unramified over the point of $\overline{M}_{g,r}$).
classifying $\mathcal{X}/k$. But, it follows from Corollary 5.12.2 that $\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...}$ is also étale over this point, which completes the former assertion.

The latter assertion, i.e., the generic étaleness of $\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...}$, follows from the former assertion. Indeed, since $\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...}$ is irreducible, any irreducible component that dominates $\overline{\mathbb{M}}_{g,r}$ surjects onto $\overline{\mathbb{M}}_{g,r}$. In particular, the fiber of $\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...}$ over the point classifying a totally degenerate curve is nonempty. Thus, by the former assertion and the open nature of flatness, the étale locus in this component of $\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...}$ (over $\overline{\mathbb{M}}_{g,r}$) forms a nonempty (hence, dense) open subscheme. \hfill $\square$

Thus, if the condition $\text{(Etale)}_{g,h}$ is satisfied, then it makes sense to speak of the generic degree

\begin{equation}
\deg(\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...}) \quad (\in \mathbb{Z}_{\geq 0})
\end{equation}

of $\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...}$, which is, by definition, the degree of the fiber of the morphism $\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...} \to \overline{\mathbb{M}}_{g,r}$ over the generic point of $\overline{\mathbb{M}}_{g,r}$. By Proposition 3.9.1, the value $\deg(\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...})$ does not depend on the choice of $h \in k^\times$. The generic étaleness of $\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...}$ over $\overline{\mathbb{M}}_{g,r}$ implies that if $\mathcal{X}/k$ is a sufficiently general pointed stable curve of type $(g,r)$ over an algebraically closed field $k$ of characteristic $p$, then the number of dormant $(g,h)$-opers on $\mathcal{X}/k$ is exactly $\deg(\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...})$. We shall prove later (cf, Proposition 7.8.1 or Theorem 7.8.2), the condition $\text{(Etale)}_{g,h}$ is satisfied if $g = \mathfrak{sl}_n$ (for $n < p$), and hence, $\deg(\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...})$ is well-defined. One of our main interests in the present paper is, as we explained in Introduction, the explicit computation of the value $\deg(\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...})$ (i.e., the case where $g = \mathfrak{sl}_n$ and $r = 0$).

6.4. In this section, we shall introduce a certain commutative ring encoding the factorization rules for dormant opers, which we refer to as the dormant operatic fusion ring $\mathfrak{F}_{g,p}^{zaz...}$ (cf. Definition 6.4.2). After that, by applying a general theory of fusion rings discussed in [1], we perform (cf. Theorem 6.4.3) a computation of the value $\deg(\mathcal{O}_{\overline{\mathbb{M}}_{g,r}}^{zaz...})$ by means of (the set of characters of) $\mathfrak{F}_{g,p}^{zaz...}$.

We supposed that $\text{(Etale)}_{g,h}$ is satisfied. Recall from [4], §5, the definition of a fusion rule and the fusion ring associated with it. Let $I$ be a finite set, with an involution $\lambda \mapsto \lambda^*$ ($\lambda \in I$), i.e., $(-)^* = \text{id}_I$. We shall denote by $\mathbb{N}^I$ the free commutative monoid generated by $I$, i.e., the set of sums $\sum_{\alpha \in I} n_\alpha \alpha$ with $n_\alpha \in \mathbb{N}$; we shall always identify $I$ with a subset of $\mathbb{N}^I$ in the natural fashion. The involution of $I$ extends by linearity to an involution $x \mapsto x^*$ of
A fusion rule on $I$ is a map $N : \mathbb{N}^{(I)} \to \mathbb{Z}$ satisfying the following three conditions:

(i) $N(0) = 1$, and $N(\alpha) > 0$ for some $\alpha \in I$;

(ii) $N(x^*) = N(x)$ for every $x \in \mathbb{N}^{(I)}$;

(iii) For $x, y \in \mathbb{N}^{(I)}$, we have $N(x + y) = \sum_{\lambda \in I} N(x + \lambda) \cdot N(y + \lambda^*)$.

Also, let us recall that the kernel of a fusion rule $N$ is the set of elements $\alpha$ in $I$ such that $N(\alpha + x) = 0$ for all $x \in \mathbb{N}^{(I)}$; one says that $N$ is nondegenerate if its kernel is empty.

Now, in our situation, let us take $I = \mathcal{C}(\mathbb{F}_p)$, which contains a specific element $[0]_{\mathbb{F}_p}$ (cf. § 2.9), together with the identity map id : $\mathcal{C}(\mathbb{F}_p) \to \mathcal{C}(\mathbb{F}_p)$ as the involution. Here, we recall that $\mathcal{C}^{x_r}(\mathbb{F}_p) = (\mathcal{C}(\mathbb{F}_p) \setminus \{[0]_{\mathbb{F}_p}\})^{x_r}$ (cf. (224)) and, by Theorem 3.12.3, that the element $\rho \in \mathcal{C}^{x_r}(\mathbb{F}_p)$ lies in $\mathcal{C}^{x_r}(\mathbb{F}_p)$ if $\mathcal{O}_{\mathbb{F}_{g,1,\rho,0,r}^{zaa\ldots}}$ is nonempty. For $\rho := (\rho_i)_{i=1}^r \in \mathcal{C}^{x_r}(\mathbb{F}_p)$ $(r \geq 1)$, we shall write $\overline{\rho}$ for the element $\overline{\rho} := \sum_{i=1}^r \rho_i \in \mathbb{N}(\mathcal{C}(\mathbb{F}_p))$. Then, there exists uniquely a well-defined map

\begin{equation}
(635) \quad N_{g,p,0}^{zaa\ldots} : \mathbb{N}(\mathcal{C}(\mathbb{F}_p)) \to \mathbb{Z}
\end{equation}

determined by the following four conditions (i)-(iv):

(i) $N_{g,p,0}^{zaa\ldots}(x + [0]_{\mathbb{F}_p}) = N_{g,p,0}^{zaa\ldots}(x)$ for any $x \in \mathbb{N}(\mathcal{C}(\mathbb{F}_p))$;

(ii) $N_{g,p,0}^{zaa\ldots}(0) = 1$ and $N_{g,p,0}^{zaa\ldots}(\alpha) = 0$ for all $\alpha \in \mathcal{C}^{x_r}(\mathbb{F}_p)$;

(iii) For $\alpha, \beta \in \mathcal{C}(\mathbb{F}_p)$, we have $N_{g,p,0}^{zaa\ldots}(\alpha + \beta) = 0$ if $\alpha \neq \beta$, and $N_{g,p,0}^{zaa\ldots}(\alpha + \beta) = 1$ if $\alpha = \beta$;

(iv) For $r \geq 3$ and $\rho \in \mathcal{C}^{x_r}(\mathbb{F}_p)$, we have $N_{g,p,0}^{zaa\ldots}(\overline{\rho}) = \deg(\mathcal{O}_{\mathbb{F}_{g,1,\rho,0,r}^{zaa\ldots}})$.

By applying Theorem 6.2.2 and Proposition 6.3.2, one may verify easily the following theorem.

**Theorem 6.4.1.**

The map $N_{g,p,0}^{zaa\ldots} : \mathbb{N}(\mathcal{C}(\mathbb{F}_p)) \to \mathbb{Z}$ is a nondegenerate fusion rule on $\mathcal{C}(\mathbb{F}_p)$.

By means of the map $N_{g,p,0}^{zaa\ldots}$, one may define a multiplication law $\odot : \mathbb{Z}(\mathbb{F}_p) \times \mathbb{Z}(\mathbb{F}_p) \to \mathbb{Z}(\mathbb{F}_p)$ on $\mathbb{Z}(\mathbb{F}_p)$ by putting

\begin{equation}
(636) \quad \alpha \odot \beta := \sum_{\lambda \in \mathcal{C}(\mathbb{F}_p)} N_{g,p,0}^{zaa\ldots}(\alpha + \beta + \lambda) \cdot \lambda
\end{equation}

($\alpha, \beta \in \mathcal{C}(\mathbb{F}_p)$) and extending by bilinearity. As shown in [4], the proof of Proposition 5.3, this multiplication law is commutative, associative and unital.

**Definition 6.4.2.**

We shall refer to the ring $\mathbb{Z}(\mathbb{F}_p)$ with the multiplication $\odot$ as the dormant
operatic fusion ring of $\mathfrak{g}$ of level $p$, and denote it by
\[(637) \ \widetilde{\mathfrak{s}}_{\mathfrak{g},p}^{\text{zzz}}.\]
(Note that the unit element of the ring $\widetilde{\mathfrak{s}}_{\mathfrak{g},p}^{\text{zzz}}$ is $[0]_{\mathbb{F}_p}$.)

In addition to $N_{\mathfrak{g},\hbar,0}^{\text{zzz}}$, there exists a collection of well-defined maps
\[(638) \ N_{\mathfrak{g},p,g}^{\text{zzz}} : \mathbb{N}(\mathfrak{c}(\mathbb{F}_p)) \to \mathbb{Z}\]
$(g = 1, 2, \cdots)$ determined by the following conditions:

(i) $N_{\mathfrak{g},p,1}^{\text{zzz}}(0) = 1$;

(ii) For $g \geq 2$, we have $N_{\mathfrak{g},p,g}^{\text{zzz}}(0) = \deg(\mathfrak{Op}_{\mathfrak{g},1,g,0}/M_{g,0})$;

(iii) For $\rho \in \mathfrak{c}^{x_\mathfrak{g}}(\mathbb{F}_p) \ (r \geq 1)$, we have $N_{\mathfrak{g},p,g}^{\text{zzz}}(\rho) = \deg(\mathfrak{Op}_{\mathfrak{g},1,p,g,r}/M_{g,r})$.

By applying Theorem 6.2.2 to a clutching data whose underlying graph has exactly one vertex and one edge, one verifies the equality
\[(639) \ N_{\mathfrak{g},p,g}^{\text{zzz}}(x) = \sum_{\lambda \in \mathfrak{c}(\mathbb{F}_p)} N_{\mathfrak{g},p,g-1}^{\text{zzz}}(x + 2\lambda)\]
for $x \in \mathbb{N}(\mathfrak{c}(\mathbb{F}_p))$ and $g \geq 1$. Moreover, the collection of data $\{N_{\mathfrak{g},p,g}^{\text{zzz}}\}_{g \geq 0}$ satisfies the following rule:
\[(640) \ N_{\mathfrak{g},p,g_1+g_2}^{\text{zzz}}(x + y) = \sum_{\lambda \in \mathfrak{c}(\mathbb{F}_p)} N_{\mathfrak{g},p,g_1}^{\text{zzz}}(x + \lambda) \cdot N_{\mathfrak{g},p,g_2}^{\text{zzz}}(y + \lambda)\]
for $x, y \in \mathbb{N}(\mathfrak{c}(\mathbb{F}_p))$ and $g_1, g_2 \geq 0$. Thus, for any $\hbar \in k^\times$ and $\rho \in \mathfrak{c}^{x_\mathfrak{g}}(\mathbb{F}_p)$, the generic degree of $\mathfrak{Op}_{\mathfrak{g},\hbar,\rho,p,g,r}$ over $M_{g,r}$ may be calculated combinatorially from the values $\deg(\mathfrak{Op}_{\mathfrak{g},1,p,0,3}/M_{g,0})$ (for various $\rho \in \mathfrak{c}^{x_3}(\mathbb{F}_p)$) according to these rules (and the fusion rule $N_{\mathfrak{g},p,0}^{\text{zzz}}$).

Write $\text{Hom}(\widetilde{\mathfrak{s}}_{\mathfrak{g},p}^{\text{zzz}}, \mathbb{R})$ (where $\mathbb{R}$ denotes the field of real numbers) for the set of ring homomorphisms $\widetilde{\mathfrak{s}}_{\mathfrak{g},p}^{\text{zzz}} \to \mathbb{R}$ and
\[(641) \ \mathcal{C}as := \sum_{\lambda \in \mathfrak{c}(\mathbb{F}_p)} \lambda \otimes \lambda.\]
Since the involution on $I = \mathfrak{c}(\mathbb{F}_p)$ is assumed to be the identity morphism, it follows from [4], Proposition 6.1, that the $\mathbb{Q}$-algebra $\widetilde{\mathfrak{s}}_{\mathfrak{g},p}^{\text{zzz}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to a product $\prod K_l$ of totally real finite extensions $K_l$ of $\mathbb{Q}$. In particular, the $\mathbb{R}$-algebra homomorphism
\[(642) \ \widetilde{\mathfrak{s}}_{\mathfrak{g},p}^{\text{zzz}} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}^{\text{Hom}(\widetilde{\mathfrak{s}}_{\mathfrak{g},p}^{\text{zzz}}, \mathbb{R})}\]
given by assigning $x \otimes 1 \mapsto (\chi(x))_{\chi \in \text{Hom}(\widetilde{\mathfrak{s}}_{\mathfrak{g},p}^{\text{zzz}}, \mathbb{R})}$ is an isomorphism (cf. [4], Corollary 6.2 (a)). Thus, an explicit knowledge of the isomorphism (642) (i.e.,
of the ring homomorphisms \( \chi : \mathfrak{F}_{g,p}^{2ax} \to \mathbb{R} \) will allow us to perform any computation that we need in the ring \( \mathfrak{F}_{g,p}^{2ax} \). For example, by applying [4], Proposition 6.3, one has the following

**Theorem 6.4.3** (Factorization property of \( \deg(\mathfrak{Op}_{g,h,h*p,g,r}^{2ax}/\mathfrak{m}_{g,r}) \)).

Let \( g, r \geq 1 \), and \( \rho = (\rho_i)_{i=1}^r \in c^{\times r}(\mathbb{F}_p) \). Then, we have the equality

\[
(643) \quad \deg(\mathfrak{Op}_{g,h,h*p,g,r}^{2ax} / \mathfrak{m}_{g,r}) = N_{g,p,\rho}^{2ax} \left( \sum_{i=1}^r \rho_i \right)
= \sum_{\chi \in \text{Hom}(\mathfrak{F}_{g,p}^{2ax}, \mathbb{R})} \chi(Cas)^{g-1} \cdot \prod_{i=1}^r \chi(\rho_i).
\]

**Remark 6.4.4.**

According to the discussion in [4, §6.4], we shall review a relation among the structure constants \( N_{\alpha,\beta,\gamma} := N_{g,p,\rho}^{2ax} (\alpha + \beta + \gamma) \) (cf. (636)) in \( \mathfrak{F}_{g,p}^{2ax} \) with respect to the basis \( c(\mathbb{F}_p) \).

For \( \alpha \in c(\mathbb{F}_p) \), consider the endomorphism \( m_\alpha \) of the underlying \( \mathbb{R} \)-vector space of \( \mathfrak{F}_{g,p}^{2ax} \otimes \mathbb{R} \) defined as the multiplication by \( \alpha \). The matrix \( N_\alpha \) corresponding to \( m_\alpha \) with respect to the basis \( c(\mathbb{F}_p) \) is given by \( N_\alpha := (N_{\alpha,\beta,\gamma})_{\beta,\gamma \in c(\mathbb{F}_p)} \). On the other hand, the matrix corresponding to \( m_\alpha \) with respect to the standard basis of \( \mathbb{R}^{\text{Hom}(\mathfrak{F}_{g,p}^{2ax}, \mathbb{R})} \) coincides with \( D_\alpha \), where for each \( x \in \mathfrak{F}_{g,p}^{2ax} \), \( D_x \) denotes the diagonal matrix whose diagonal entries are \( \{\chi(x)\}_{\chi \in \text{Hom}(\mathfrak{F}_{g,p}^{2ax}, \mathbb{R})} \). Thus, by means of the base-change matrix

\[
(644) \quad \Sigma := \left( \frac{1}{\sqrt{\chi(Cas)}} \cdot \chi(\lambda) \right)_{\lambda \in \text{c}(\mathbb{F}_p), \chi \in \text{Hom}(\mathfrak{F}_{g,p}^{2ax}, \mathbb{R})},
\]

(where we note that \( \chi(Cas) = \sum_{\alpha \in \text{c}(\mathbb{F}_p)} \chi(\alpha) \) is a positive integer, and \( \Sigma \) is orthogonal), we have the equality

\[
(645) \quad N_\alpha = \Sigma^{-1} \cdot D_\alpha \cdot \Sigma.
\]

In other words, the matrix \( \Sigma \) simultaneously diagonalize all matrices \( N_\alpha \) (\( \alpha \in c(\mathbb{F}_p) \)).
7. Generic étaleness of the moduli of \((\mathfrak{sl}_n, \hbar)\)-opers

We shall focus on the case where the Lie algebra \(g\) is taken to be of classical type \(A_{n-1}\), i.e., \(\mathfrak{sl}_n\), under the condition that \(n < p\). The goal of this section is to prove Theorem 7.8.2 which asserts that the moduli stack \(\mathcal{O}p_{\mathfrak{sl}_n, \hbar, \hbar^* \rho, g, r}^{\text{Zar}}\) (where \(\hbar \in k^\times\), \(\rho \in e_{\mathfrak{sl}_n}(F_p)\)) is generically étale over \(\mathcal{M}_{g, r}\), i.e., any irreducible component that dominates \(\mathcal{M}_{g, r}\) admits a dense open subscheme which is étale over \(\mathcal{M}_{g, r}\). Here, we wish to emphasize the importance of the open density of the étale locus in \(\mathcal{O}p_{\mathfrak{sl}_n, \hbar, \hbar^* \rho, g, r}^{\text{Zar}}\). As we shall see in Theorem 8.6.2 and its proof, the properties stated in Theorem 7.8.2 enable us to relate a numerical calculation in characteristic zero to the degree of certain moduli spaces of interest in positive characteristic.

7.1. Let \(S\) be a scheme over \(k\), \(X/S := (X/S, \{\sigma_i\}_{i=1}^r)\) a pointed stable curve over \(S\) of type \((g, r)\), \(U\) a nonempty open subscheme of \(X\), and \((\mathcal{F}, \nabla_\mathcal{F})\) a log integrable vector bundle on \(U_{\log}/S_{\log}\). We shall write

\[
(646) \quad \mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})} := F^*_U(F_{U/S}(\text{Ker}(\nabla_\mathcal{F}))).
\]

The natural inclusion \(F_{U/S}(\text{Ker}(\nabla_\mathcal{F})) \hookrightarrow F_{U/S}(\mathcal{F})\) corresponds, via the adjunction relation \(\nabla^*_U(-) \dashv F^*_U(-)\), an \(\mathcal{O}_U\)-linear morphism

\[
(647) \quad \nu^{(\mathcal{F}, \nabla_\mathcal{F})}_U : \mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})} \to \mathcal{F}.
\]

If we consider the canonical \(S\)-log connection \(\nabla_{\text{can}}^{\mathcal{F}_{U/S}(\text{Ker}(\nabla_\mathcal{F}))} / \mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})}\) on \(\mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})}\) (cf. §3.3), then the morphism \(\nu^{(\mathcal{F}, \nabla_\mathcal{F})}_U\) is compatible with the respective connections \(\nabla_{\text{can}}^{\mathcal{F}_{U/S}(\text{Ker}(\nabla_\mathcal{F}))} / \mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})}\) and \(\nabla_\mathcal{F}\).

If the underlying scheme \(U\) of \(U_{\log}\) coincides with \(U_{\text{sm}}\) (cf. §1.6) (i.e., the natural morphism \(\Omega_{U/S} \to \Omega_{U_{\log}/S_{\log}}\) is an isomorphism), then it is known (cf. [11], §5, p. 190, Theorem 5.1) that \(p^\psi_{(\mathcal{F}, \nabla_\mathcal{F})} = 0\) if and only if \(\nu_U^{(\mathcal{F}, \nabla_\mathcal{F})}\) is an isomorphism. In particular, the assignments \(\mathcal{V} \mapsto (F^*_{U/S}(\mathcal{V}), \nabla_{\text{can}}^{\mathcal{V}})\) and \((\mathcal{F}, \nabla_\mathcal{F}) \mapsto \text{Ker}(\nabla_\mathcal{F})\) determine an equivalence of categories

\[
(648) \quad \left(\text{the category of vector bundles on } U^{(1)}_{U/S}\right) \sim \left(\text{the category of log integrable vector bundles on } U_{\log}/S_{\log} \text{ with vanishing } p\text{-curvature}\right)
\]

that is compatible with the formation of tensor products (hence also symmetric and exterior products).
7.2. Until the beginning of the last subsection, we suppose that the field \( k \) is algebraically closed, \( S = \text{Spec}(k) \), and the underlying semistable curve \( X \) of \( \mathcal{X}_k \) is smooth over \( k \). (Hence, \( X^{\text{sm}} = X \setminus \text{Supp}(\mathcal{P}_X/k) \) and \( F_{X/k} \) is finite and faithfully flat of degree \( p \).) Write \( k_\epsilon := k[\epsilon]/(\epsilon^2) \) and \( X_\epsilon := X \times_{\text{Spec}(k)} \text{Spec}(k_\epsilon) \). Also, write \( \text{pr}_X : X_\epsilon \to X \) for the natural projection and \( \text{in}_X : X \to X_\epsilon \) for the natural closed immersion. In the following, the symbol \((-\)) denotes either the presence or absence of \( \epsilon \).

Let \((\mathcal{F}_{(-)}, \nabla_{\mathcal{F}_{(-)}})\) be a log integrable vector bundle on \( X^{\log}/k^{\log} \) of rank \( n \) satisfying that \( p_{\psi}((\mathcal{F}_{(-)}, \nabla_{\mathcal{F}_{(-)}})) = 0 \). We shall prove the claim that \( \nu_z((\mathcal{F}_{(-)}, \nabla_{\mathcal{F}_{(-)}})) \) is (in spite of the presence or absence of \( \epsilon \)) universally injective with respect to base-change over \( k_\epsilon \). First, suppose that the symbol \((-\)) is the absence of \( \epsilon \). Then, the claim follows from the fact that \( \mathcal{A}_{\text{Ker}(\nabla)} \) is locally free (over the smooth curve \( X \) over the field \( k \)) and \( \nu_z((\mathcal{F}_{(-)}, \nabla_{\mathcal{F}_{(-)}})) \) is an isomorphism over \( X^{\text{sm}} \) (by the equivalence of categories \([41, 42]\)). Next, suppose that the symbol \((-\)) is the presence of \( \epsilon \). By Corollary 5.8.2, the log integrable vector bundle \((\mathcal{F}_z, \nabla_{\mathcal{F}_z})\) on \( X^{\log}/k^{\log} \) is, Zariski locally on \( X_\epsilon \), isotrivial, i.e., isomorphic to the pull-back \(( (\text{pr}_X \circ \text{in}_X)^* (\mathcal{F}_z), (\text{pr}_X \circ \text{in}_X)^* (\nabla_{\mathcal{F}_z}) ) \). Hence, \( \nu_z((\mathcal{F}_z, \nabla_{\mathcal{F}_z})) \) (which may be, Zariski locally on \( X_\epsilon \), identified with the pull-back \(( (\text{pr}_X \circ \text{in}_X)^* (\nu_z((\mathcal{F}_{(-)}, \nabla_{\mathcal{F}_{(-)}}))) \)) is universally injective with respect to base-change over \( k_\epsilon \). This completes the proof of the claim.

The morphism \( \nu_z((\mathcal{F}_{(-)}, \nabla_{\mathcal{F}_{(-)}})) \) fits into a short exact sequence

\[
0 \to \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}_{(-)}})} \to \mathcal{F}_{(-)} \to \bigoplus_{i=1}^{r} \Lambda_{(-)} i \to 0
\]

of \( \mathcal{O}_X \)-modules, where for each \( i \in \{1, \cdots, r\} \), \( \Lambda_{(-)} i \) denotes an \( \mathcal{O}_{X_{(-)}} \)-module supported on \( \text{Im}(\sigma_i) \) (\( \subseteq X_{(-)} \)). Then, the claim just discussed shows that each \( \Lambda_{(-)} i \) is flat over \( k_{(-)} \) (cf. \([19]\), p. 17, Theorem 1). \( \Lambda_{(-)} i \) admits a unique \( k_{(-)} \)-log connection

\[
\nabla_{\mathcal{F}_{(-)} i} : \Lambda_{(-)} i \to \Omega_{X_{(-)}/k_{(-)}} \otimes \Lambda_{(-)} i
\]

such that the natural surjection \( \mathcal{F}_{(-)} \to \Lambda_{(-)} i \) is compatible with the respective \( k_{(-)} \)-log connections \( \nabla_{\mathcal{F}_{(-)}} \) and \( \nabla_{\mathcal{F}_{(-)} i} \).

Next, suppose that we are given a decreasing filtration \( \{\mathcal{F}^{j}_i\}^n_{j=0} \) on \( \mathcal{F}_{(-)} \) for which the triple \( \mathcal{F}^{0}_i := (\mathcal{F}_{(-)}, \nabla_{\mathcal{F}_{(-)}}, \{\mathcal{F}^{j}_i\}^n_{j=0}) \) forms a \( \text{GL}_n \)-oper on \( X_{(-)}/k_{(-)} \) (cf. Definition 4.2.1 (i)). One may construct a decreasing filtration \( \{\mathcal{A}^{j}_{\text{Ker}(\nabla_{\mathcal{F}_{(-)}})}\}^n_{j=0} \) on \( \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}_{(-)}})} \) by putting as

\[
\mathcal{A}^{j}_{\text{Ker}(\nabla_{\mathcal{F}_{(-)}})} := (\nu_z((\mathcal{F}_{(-)}, \nabla_{\mathcal{F}_{(-)}}))^{-1}(\mathcal{F}^{j}_i)).
\]
Also, one may construct a decreasing filtration \( \{ \Lambda^j_{(-i)} \}_{j=0}^n \) on each \( \Lambda_{(-i)} \) by putting as

\[
\Lambda^j_{(-i)} := \mathsf{pr}_{(-i)} \circ \nu_i^{(F_{(-)} \otimes \nabla_{F_{(-)})}}(F^j_{(-)}),
\]

where \( \mathsf{pr}_{(-i)} \) denotes the projection \( \bigoplus_{i=1}^r \Lambda_{(-i)} \to \Lambda_{(-i)} \) to the \( i \)-th factor. The collection of data \( \{ \bigoplus_{i=1}^r \Lambda^j_{(-i)} \}_{j=0}^n \) defines a filtration on \( \bigoplus_{i=1}^r \Lambda_{(-i)} \). The subquotients \( \Lambda^j_{(-i)}/\Lambda^{j+1}_{(-i)} \) are verified to be flat over \( k_{(-)} \). Indeed, if \( (-) \) denotes the presence of “\( \varepsilon \)”, then each \( \Lambda^j_{(-i)} \) is Zariski locally on \( X_\varepsilon \), isotrivial with fiber \( \text{in}^*_X(\Lambda^j_{(-i)}) \). Both the morphisms \( \nu_{\varepsilon}^{(F_{(-)} \otimes \nabla_{F_{(-)})}}(F^j_{(-)}) \) and \( \nu_{\varepsilon}^{(F_{(-)} \otimes \nabla_{F_{(-)})}}(F^j_{(-)}) \) in (649) are compatible with the filtrations just obtained. If we write

\[
\text{gr}^j_{F_{(-)}} := \mathcal{A}^j_{\text{Ker}(\nabla_{F_{(-)})}}/\mathcal{A}^{j+1}_{\text{Ker}(\nabla_{F_{(-)})}};
\]

then, by taking the subquotients, the sequence (649) yields a short exact sequences

\[
0 \to \text{gr}^j_{F_{(-)}} \xrightarrow{\text{gr}^j(\nu_{\varepsilon}^{(F_{(-)} \otimes \nabla_{F_{(-)})}}(F^j_{(-)}) \to F^j_{(-)}/F^j_{(-)} \to \bigoplus_{i=1}^r \Lambda^j_{(-i)}/\Lambda^{j+1}_{(-i)} \to 0
\]

\((j = 0, \ldots, n - 1)\) of \( \mathcal{O}_{X_{(-)}} \)-modules.

Recall that \( \nu_{\varepsilon}^{(F_{(-)} \otimes \nabla_{F_{(-)})}} \) is compatible with the respective \( k_{(-)} \)-log connections \( \nabla^\text{can}_{F_{X/\mathcal{E}(\text{Ker}(\nabla_{F_{(-)})}} \) and \( \nabla_{F_{(-)}} \) (cf. §7.1), and is (by the definition of \( \mathcal{A}^j_{\text{Ker}(\nabla_{F_{(-)})}} \)) compatible with the respective filtrations \( \{ \mathcal{A}^j_{\text{Ker}(\nabla_{F_{(-)})}} \}_{j=0}^n \) and \( \{ F^j_{(-)} \}_{j=0}^n \). The isomorphism

\[
\mathfrak{Fs}^j_{F_{(-)}} : F^j_{(-)}/F^{j+1}_{(-)} \cong \Omega^\log_{X_{(-)/k_{(-)}}} \otimes F^{j-1}_{(-)}/F^j_{(-)}
\]
(cf. (331)) induces a morphism of short exact sequences of $\mathcal{O}_{X(-)}$-modules

$$
\begin{array}{cccc}
0 & 0 & \downarrow & \\
\downarrow & \downarrow & \downarrow & \\
\text{gr}^j\mathcal{F}(-) & \text{ts}^j_{\mathcal{F}(-)} & \Omega_{X \log/k \log}^{(-)} & \text{gr}^{j-1}\mathcal{F}(-) \\
\text{gr}^j(\nu_\mathcal{F}(-),\nabla\mathcal{F}(-)) & \Omega_{X \log/k \log}^{(-)} \otimes \text{gr}^{j-1}\mathcal{F}(-) & \text{id} \otimes \text{gr}^j(\nu_\mathcal{F}(-),\nabla\mathcal{F}(-)) & \\
\end{array}
$$

(656)

$$
\begin{array}{cccc}
\mathcal{F}^j(-)/\mathcal{F}^{j+1}(-) & \mathcal{F}^{j-1}(-)/\mathcal{F}^j(-) & \downarrow & \\
\text{gr}^j(\nu_\mathcal{F}(-),\nabla\mathcal{F}(-)) & \mathcal{F}^{j-1}(-)/\mathcal{F}^j(-) & \text{id} \otimes \text{gr}^j(\nu_\mathcal{F}(-),\nabla\mathcal{F}(-)) & \\
\bigoplus_{i=1}^r \Lambda_i^j(-)/\Lambda_i^{j+1}(-) & \bigoplus_{i=1}^r \Omega_{X \log/k \log}^{(-)} \otimes (\Lambda_i^{j-1}(-)/\Lambda_i^j(-)) & \\
\downarrow & \downarrow & \\
0 & 0 & \\
\end{array}
$$

where the left-hand vertical sequence is (654) and the right-hand vertical sequence is (654) (of the case where the index $j$ is replaced with $j - 1$) tensored with $\Omega_{X \log/k \log}^{(-)}$. One verifies that the top horizontal arrow $\text{ts}^j_{\mathcal{F}(-)}$ is injective, and the components

$$
\text{ts}^j_{\mathcal{F}(-)} : \Lambda_i^j(-)/\Lambda_i^{j+1}(-) \rightarrow \Omega_{X \log/k \log}^{(-)} \otimes (\Lambda_i^{j-1}(-)/\Lambda_i^j(-))
$$

(657)

$(i = 1, \cdots, r)$ in the bottom horizontal arrow are surjective. One obtains from this diagram the equality

$$
(658) \text{deg(gr}_{\mathcal{F}}) = 2g - 2 + r + \text{deg(gr}^{-1}_{\mathcal{F}}) + \sum_{i=1}^r (\text{deg}(\Lambda_i^{j-1}/\Lambda_i^j) - \text{deg}(\Lambda_i^j/\Lambda_i^{j-1})).
$$

This equality will be used in the proof of Corollary 7.3.3 and then, Corollary 7.4.2.

7.3. In this section, we study the local description of a given GL$_n$-oper at the marked points of $\mathcal{X}/k$.

Denote by $[\sigma_i]$ $(i = 1, \cdots, r)$ the reduced effective divisor on $X$ defined by the image of $\sigma_i$. For $l \geq 0$, there exists uniquely a $k$-log connection

$$
(659) \nabla_{l,[\sigma_i]} : \mathcal{O}_X(l \cdot [\sigma_i]) \rightarrow \Omega_{X \log/k \log} \otimes \mathcal{O}_X(l \cdot [\sigma_i])
$$
on the line bundle $\mathcal{O}_X(l \cdot [\sigma_i])$ whose restriction to $\mathcal{O}_X (\subseteq \mathcal{O}_X(l \cdot [\sigma_i]))$ coincides with the universal logarithmic derivation $d_{X \log/k \log} : \mathcal{O}_X \rightarrow \Omega_{X \log/k \log}$. The
monodromy of \((\mathcal{O}_X(l \cdot [\sigma_i]), \nabla_{l \cdot [\sigma_i]})\) at \(\sigma_i\), considered as an element of \(k\), satisfies the equality

\[
\mu_i^{(\mathcal{O}_X(l \cdot [\sigma_i]), \nabla_{l \cdot [\sigma_i]})} = -\tilde{I}
\]

where \(\overline{(-)}\) denotes the image via the quotient \(\mathbb{Z} \to \mathbb{F}_p \subseteq k\). Let \(\tilde{l}\) be the unique integer satisfying that \(0 \leq \tilde{l} \leq p - 1\) and \(l = \tilde{l} + p \mod p\). One verifies the equality

\[
\mathcal{A}_{\ker(\nabla_{l \cdot [\sigma_i]})} = \mathcal{O}_X((l - \tilde{l}) \cdot [\sigma_i])
\]

of \(\mathcal{O}_X\)-submodules of \(\mathcal{O}_X(l \cdot [\sigma_i])\) (via the injection \(\nu_2^{(\mathcal{O}_X(l \cdot [\sigma_i]), \nabla_{l \cdot [\sigma_i]}): \mathcal{A}_{\ker(\nabla_{l \cdot [\sigma_i]})} \to \mathcal{O}_X(l \cdot [\sigma_i])]\).

Suppose that \(0 \leq l \leq p - 1\), and \(\tilde{\Lambda}_i\) denotes the quotient \(\mathcal{O}_X(l \cdot [\sigma_i]) / \mathcal{O}_X\). Then, the \(k\)-log connection \(\nabla_{l \cdot [\sigma_i]}\) carries a \(k\)-log connection

\[
\nabla_{l \cdot [\sigma_i]}: \tilde{\Lambda}_i \to \Omega_{X}^{\log} \otimes \tilde{\Lambda}_i
\]

on \(\tilde{\Lambda}_i\) in the manner that the natural surjection \(\mathcal{O}_X(l \cdot [\sigma_i]) \to \tilde{\Lambda}_i\) is compatible with the respective \(k\)-log connections \(\nabla_{l \cdot [\sigma_i]}\) and \(\nabla_{l \cdot [\sigma_i]}\). If we fix a local function \(t \in \mathcal{O}_X\) defining \(\sigma_i\), then \(\tilde{\Lambda}_i\) may be naturally expressed as \(\tilde{\Lambda}_i = \bigoplus_{m=-1}^{l-1} k \cdot t^m\)
and, via this expression, \(\nabla_{l \cdot [\sigma_i]}\) may be given by assigning \(t^m \mapsto m \cdot t^m\). In particular, this observation implies the following Lemma [7.3.1] which will be used in the proof of Lemma [7.6.1].

**Lemma 7.3.1.**

Let \(\mathcal{V}\) be a vector bundle on \(X_k^{(1)}\) and \(h : F^*_{X/k}(\mathcal{V}) \to \tilde{\Lambda}_i\) an \(\mathcal{O}_X\)-linear morphism that is compatible with the respective \(k\)-log connections \(\nabla_{\mathcal{V}}^{\text{can}}\) and \(\nabla_{l \cdot [\sigma_i]}\). Then, \(h\) is the zero map.

Also, one verifies easily the following Lemma [7.3.2] which will be used in the proof of Proposition [7.7.4].

**Lemma 7.3.2.**

For \(l_1, l_2 \geq 0\), we have the equality

\[
\nabla_{l_1 \cdot [\sigma_i]} \otimes \nabla_{l_2 \cdot [\sigma_i]} = \nabla_{(l_1 + l_2) \cdot [\sigma_i]}
\]

upon passing to the natural identification

\[
\mathcal{O}_X(l_1 \cdot [\sigma_i]) \otimes \mathcal{O}_X(l_2 \cdot [\sigma_i]) \cong \mathcal{O}_X((l_1 + l_2) \cdot [\sigma_i]).
\]

If, moreover, \(l_1 + l_2 < p\), then

\[
\ker(\nabla_{l_1 \cdot [\sigma_i]} \otimes \nabla_{l_2 \cdot [\sigma_i]}) = \mathcal{O}_{X_k^{(1)}}.
\]
Also, if, moreover, \(2p > l_1 + l_2 \geq p\), then
\[
\ker(\nabla_{l_1, \sigma_i} \otimes \nabla_{l_2, \sigma_i}) = \mathcal{O}_{X_{\kappa}}^{(1)}(\sigma_i^{(1)})
\]
(cf. § 3.4 for the definition of \(\sigma_i^{(1)}\)).

Let \(t\) be as above. Since \(\nabla_{l_1, \sigma_i}(t^p \cdot a) = t^p \cdot \nabla_{l_1, \sigma_i}(a)\) for any local section \(a \in \mathcal{O}_X(l \cdot [\sigma_i])\), it makes sense to speak of the \(t^p\)-adic completion of the log integrable line bundle \((\mathcal{O}_X(l \cdot [\sigma_i]), \nabla_{l, \sigma_i})\); it is isomorphic to \(\widehat{\mathcal{O}} := \mathcal{O}_{\text{Spf}(k[[t]])}\) together with the regular singular connection
\[
\widehat{\nabla}_l := dx_{\log/k\log} - l \cdot \frac{dt}{t},
\]
i.e.,
\[
(\mathcal{O}_X(l \cdot [\sigma_i]), \nabla_{l, \sigma_i})^\wedge \sim (\widehat{\mathcal{O}}, \widehat{\nabla}_l).
\]

Noe, let \((\mathcal{F}, \nabla_{\mathcal{F}})\) be a log integrable vector bundle on \(X^{\log/k\log}\) of rank \(n\) with vanishing \(p\)-curvature. It follows from [59], Corollary 2.10, that the \(t^p\)-adic completion \((\mathcal{F}, \nabla_{\mathcal{F}})^\wedge\) of \((\mathcal{F}, \nabla_{\mathcal{F}})\) is isomorphic to a direct sum of various \((\widehat{\mathcal{O}}, \widehat{\nabla}_l)\)'s. For each such \((\mathcal{F}, \nabla_{\mathcal{F}})\), we shall fix an isomorphism
\[
\vartheta_i^{(\mathcal{F}, \nabla_{\mathcal{F}})} : (\mathcal{F}, \nabla_{\mathcal{F}})^\wedge \sim \bigoplus_{l=1}^n (\widehat{\mathcal{O}}, \widehat{\nabla}_{m_{i,l}}),
\]
where \(0 \leq m_{i,l_1} \leq m_{i,l_2} < p\) if \(l_1 < l_2\). In particular, if \(\vartheta_i^{(\mathcal{F}, \nabla_{\mathcal{F}})}\) and \(\vartheta_i^{(\mathcal{O}_X(l \cdot [\sigma_i]), \nabla_{l, \sigma_i})}\) denotes the \(t^p\)-adic completions of \(\nu_i^{(\mathcal{F}, \nabla_{\mathcal{F}})}\) and \(\nu_i^{(\mathcal{O}_X(l \cdot [\sigma_i]), \nabla_{l, \sigma_i})}\) respectively, then \(\vartheta_i^{(\mathcal{F}, \nabla_{\mathcal{F}})}\) and the isomorphism (668) induce a composite isomorphism
\[
(670) \quad \Lambda_i \sim \text{Coker}(\vartheta_i^{(\mathcal{F}, \nabla_{\mathcal{F}})}) \sim \text{Coker}(\bigoplus_{l=1}^n (\widehat{\nabla}_{m_{i,l}})) \sim \bigoplus_{l=1}^n \Lambda_{m_{i,l}}
\]
(cf. (649) for the definition of \(\Lambda_i\)) that is compatible with the respective \(k\)-log connections \(\nabla_{\mathcal{F},i}\) (cf. (650)) and \(\bigoplus_{l=1}^n \nabla_{m_{i,l}}\).

Also, we shall define an ordered set
\[
(671) \quad \mathfrak{K}^{(\mathcal{F}, \nabla_{\mathcal{F}})} := (-m_{i,l})_{l=1}^n
\]
of elements in \(\mathbb{Z}\). The ordered set \(\mathfrak{K}^{(\mathcal{F}, \nabla_{\mathcal{F}})}\) depends neither on the choice of the function \(t\) nor the choice of the isomorphism (669). Let us choose a Borel subgroup (resp., a maximal torus) of \(\text{PGL}_n\) as the image \(B\) (cf. (330)) of upper triangular matrices (resp., the image \(T\) of diagonal matrices) in \(\text{GL}_n\). The Lie algebra of \(T\) may be naturally identified with the cokernel \(\text{Coker}(\Delta)\) of the diagonal inclusion \(\Delta : k \hookrightarrow k^{\otimes n}\). The symmetric group \(\mathfrak{S}_n\) of \(n\) letters acts on this \(k\)-vector space by permutation. The set of \(k\)-rational points \(\mathfrak{c}_{\text{st}_n}(k)\) of the \(k\)-scheme \(\mathfrak{c}_{\text{st}_n}\) (cf. § 2.8), which is naturally isomorphic to \(\mathfrak{c}_{\text{pGL}_n}\) since \(n < p\),
may be identified with the quotient set \( \text{Coker}(\Delta)/\mathfrak{S}_n \). Denote by \( \pi_n : \mathbb{Z}^{\oplus n} \to \text{Coker}(\Delta)/\mathfrak{S}_n \) the natural surjection. If we regard \( \mathfrak{R}_{\sigma_i}^{(F,\nabla_F)} \) as an element of \( \mathbb{Z}^{\oplus n} \), then the radii of \( (F,\nabla_F) \) coincides with the \( r \)-tuple \( (\pi_n(\mathfrak{R}_{\sigma_i}^{(F,\nabla_F)}))^r \) upon passing to the identification \( (\text{Coker}(\Delta)/\mathfrak{S}_n)^\times r = \mathfrak{c}_{\mathbb{A}_n}^{x}\mathbb{F}_p \), i.e.,

\[
(672) \quad \rho_i^{(F,\nabla_F)} = \pi_n(\mathfrak{R}_{\sigma_i}^{(F,\nabla_F)})
\]

for any \( i = 1, \cdots, r \).

**Proposition 7.3.3.**

Suppose that there exists a decreasing filtration \( \{F_j\}_{j=0}^n \) on \( F \) by which the triple \( (F,\nabla_F,\{F_j\}_{j=0}^n) \) forms a \( \text{GL}_n \)-oper on \( \mathfrak{X}/k \). Let

\[
(673) \quad \hat{\vartheta}_i^{(F,\nabla_F)} : (F,\nabla_F)^\wedge \to \bigoplus_{l=1}^n (\hat{\mathcal{O}}, \hat{\nabla}_{m_{i,l}})
\]

be as in \( (669) \).

(i) Let \( v \) be a global section of the \( t^p \)-adic completion \( \hat{F} \) of \( F \) which generates (formally) the \( \hat{\mathcal{O}} \)-submodule \( \hat{F}^{n-1} \), and write

\[
(674) \quad \hat{\vartheta}_i^{(F,\nabla_F)}(v) := (u_l(t))_{l=1}^n \in k[[t]]^{\oplus n} (= \Gamma(S\mathfrak{p}(k[[t]]), \hat{\mathcal{O}})).
\]

Then, each \( u_l(t) \) lies in \( k[[t]]^\times \), equivalently, \( u_l(0) \neq 0 \).

(ii) The elements \( m_{i,1}, \cdots, m_{i,n} \) of \( \mathfrak{R}_{\sigma_i}^{(F,\nabla_F)} \) are mutually distinctive.

**Proof.** First, we consider assertion (i). By passing to the assignment \( w \mapsto w \cdot \frac{dt}{t} \), we shall identify \( \hat{F} \) with \( \hat{F} = (\Omega_{\text{log}}/k_{\text{log}} \otimes \mathcal{F})^{\wedge} \). Then, the \( t^p \)-adic completion \( \hat{\nabla}_F \) of \( \nabla_F \) may be thought of as an endomorphism of \( \hat{F} \). We shall write \( \hat{\nabla}_F^\circ j \) for the \( j \)-th iterate of this endomorphism \( \hat{\nabla}_F \), where \( \hat{\nabla}_F^\circ 0 := \text{id}_{\hat{F}} \). It follows from the definitions of a \( \text{GL}_n \)-oper and the section \( v \) that the set \( \{\hat{\nabla}_F^\circ j(v)\}_{j=0}^{n-1} \) forms a basis of \( \hat{F} \). On the other hand, since \( \hat{\vartheta}_i^{(F,\nabla_F)} \) is compatible with the \( k \)-log connections, we have the equality

\[
(675) \quad \hat{\vartheta}_i^{(F,\nabla_F)}(\hat{\nabla}_F^\circ j(v)) = (\hat{\nabla}_{m_{i,l}}^\circ j(u_l(t)))_{l=1}^n
\]

(\( j = 0, \cdots, n-1 \)). Hence, for each \( l \), the set \( \{\hat{\nabla}_{m_{i,l}}^\circ j(u_l(t))\}_{j=0}^{n-1} \) generates the trivial \( \hat{\mathcal{O}} \)-module \( \hat{\mathcal{O}} \). Here, let us write

\[
(676) \quad u_l(t) := \sum_{s=0}^{\infty} u_{l,s} \cdot t^s
\]

(\( u_{l,s} \in k \)). Then, \( \hat{\nabla}_{m_{i,l}}(u_l(t)) \) may be expressed as

\[
(677) \quad \hat{\nabla}_{m_{i,l}}^\circ j(u_l(t)) = \sum_{s=0}^{\infty} (\bar{\sigma} - \bar{m}_{i,l})^j \cdot u_{l,s} \cdot t^s.
\]
Thus, one verifies that \( \{ \hat{\nabla}_{m_i,j}^n(u_l(t)) \}_{j=0}^{n-1} \) generates \( \mathcal{O} \) if and only if \( \mathbf{m}_{i,j}^l \cdot u_{l,s} \neq 0 \) for some \( j \geq 0 \), equivalently, \( u_{t,0} (= u_t(0)) \neq 0 \). This completes the proof of assertion (i).

Next, we consider assertion (ii). Since \( \{ \hat{\nabla}_{m_i,j}^n(u_l(t)) \}_{j=0}^{n-1} \) forms a basis of \( \mathcal{O} \), the \( n \times n \) matrix

\[
(678) \quad \left( \hat{\nabla}_{m_i,a}^{(b-1)}(u_a(t)) \right|_{t=0})_{a,b=1}^{n}
\]

whose \((a,b)\)-entry equals \( \hat{\nabla}_{m_i,a}^{(b-1)}(u_a(t))|_{t=0} \) is regular. But, by means of the expressions (677), we have the equality

\[
(679) \quad \det((\hat{\nabla}_{m_i,a}^{(b-1)}(u_a(t))|_{t=0})_{a,b=1}^{n}) = \det((\mathbf{m}_{i,a}^{b-1} \cdot u_{a,0})_{a,b=1}^{n}) \cdot \prod_{a=1}^{n} (u_{a,0})^{n} \cdot \prod_{1 \leq a_2 < a_1 \leq n} (\mathbf{m}_{a_2,a_1} - \mathbf{m}_{i,a_2}),
\]

where the factor \( \prod_{a=1}^{n} (u_{a,0})^{n} \) is not zero by virtue of assertion (i). This implies that the elements \( \mathbf{m}_{i,1}, \cdots, \mathbf{m}_{i,n} \) (hence the elements \( m_{i,1}, \cdots, m_{i,n} \)) are mutually distinctive, and implies the validity of assertion (ii).

**Corollary 7.3.4.**

Let \( \{ F_j \}_{j=0}^{n} \) be as in Proposition 7.3.3 and both \( \Lambda_j \) and \( \{ \Lambda^j \}_{j=0}^{n} \) as defined in the discussion in § 7.2 (cf. (649) and (652)) applied to \( (F, \nabla_F) \) under consideration. Then, for any \( j = 0, \cdots, n-1 \), we have the equality

\[
(680) \quad \deg(\Lambda^j / \Lambda^{j+1}) = m_{i,j+1}.
\]

In particular,

\[
(681) \quad \deg(\text{gr}_F^j) - \deg(\text{gr}_F^0) = j \cdot (2g - 2 + r) - \sum_{i=1}^{r} (m_{i,j+1} - m_{i,1})
\]

(cf. (653) for the definition of \( \text{gr}_F^j \)).

**Proof.** Since the latter assertion follows from the former assertion and the equality (658), we only prove the former assertion.

Let \( \psi_{j}^{(F, \nabla_F)} \), \( v \), and \( u_t(t) = \sum_{s=0}^{\infty} u_{t,s} \cdot t^s \) (\( l = 1, \cdots, n \)) be as in (the proof of) Proposition 7.3.3. In a similar way to observing the equality (679), the \( (n-j-1) \times (n-j-1) \) matrix

\[
(682) \quad \left( \hat{\nabla}_{m_{i,n+1-a}^{(b-1)}(u_{n+1-a}, t))|_{t=0} \right)_{a,b=1}^{n-j-1}
\]
over $k$ turns out to be regular. Hence, since $k[[t]]^\times = k^\times \oplus t \cdot k[[t]]$, the $(n - j - 1) \times (n - j - 1)$ matrix
\[ (\hat{\otimes}_{m_{i,n+1-a}}^n (u_{n+1-a}(t)))_{a,b=1}^{n-j-1} \]
over $k[[t]]$ is regular. By passing to the isomorphism $\vartheta_i^{(F,\nabla_F)}$, we see (since $\hat{\mathcal{F}}/\hat{\mathcal{F}}^{j+1}$ is generated formally by sections $\{\hat{\otimes}_F^j(v)\}_{0 \leq j' \leq n-j-2}$) that any element of $\hat{\mathcal{F}}/\hat{\mathcal{F}}^{j+1}$ admits a representative $w$ in $\hat{\mathcal{F}}$ such that for each $a = 1, \ldots, n-j-1$, the $(n+1-a)$-th factor of $\vartheta_i^{(F,\nabla_F)}(w)$ equals zero. This implies that the section $t^{m_{i,j+1}} \cdot \bar{w}$ of $\hat{\mathcal{F}}/\hat{\mathcal{F}}^{j+1}$ lies in the image of $\vartheta_i^{(F,\nabla_F)}$. (Indeed, the image of $\bigoplus_{i=1}^n \hat{\mathcal{V}}_i^{(\mathcal{O}_X(m_i,\nu_i),\nabla_{m_i}^{\nu_i})}$ coincides with the $\hat{\mathcal{O}}$-submodule $\bigoplus_{i=1}^n t^{m_{i,i}} \cdot \hat{\mathcal{O}}$ of $\bigoplus_{i=1}^n \hat{\mathcal{O}}_i$.) Here, note that $\Lambda_i/\Lambda_i^{j+1}$ is isomorphic to the cokernel of the $t^p$-adic completion of $gr^j((\nu_i^{(F,\nabla_F)}) : gr^j_F \to \mathcal{F}/\mathcal{F}^{j+1}$, and $\mathcal{F}/\mathcal{F}^{j+1}$ is locally free of rank one. Hence, $\Lambda_i/\Lambda_i^{j+1}$ is generated by a single section and annihilated by $t^{m_{i,j+1}}$. It follows that
\[ \deg(\Lambda_i/\Lambda_i^{j+1}) \leq m_{i,j+1}. \]

On the other hand, by the composite isomorphism (670),
\[ \sum_{j=0}^{n-1} \deg(\Lambda_i/\Lambda_i^{j+1}) = \deg(\Lambda) = \deg(\bigoplus_{l=1}^n \bar{\Lambda}_i,m_{i,l}) = \sum_{l=1}^n m_{i,l}. \]

Thus, by (684) and (685), we conclude that $\deg(\Lambda_i/\Lambda_i^{j+1}) = m_{i,j+1}$, as desired. \qed

7.4. Let us fix a line bundle $\mathcal{B}$ on $X$ and write $\mathcal{F} := \mathcal{D}_{1,n}^{\leq n} \otimes \mathcal{B}^\vee$ and $\mathcal{F}^{j} := \mathcal{D}_{1,n}^{\leq n-j} \otimes \mathcal{B}^\vee$ for $j = 0, \ldots, n$. Let $\nabla_{\mathcal{F}}$ be a $(\text{GL}_n, 1, \mathcal{B})$-oper on $X/k$ (cf. Definition 4.6.1). The sheaf $\text{End}_{\mathcal{O}_X}(\mathcal{F})$ of $\mathcal{O}_X$-linear endomorphisms of $\mathcal{F}$ may be naturally identified with the adjoint vector bundle associated with the right $\text{GL}_n$-torsor corresponding to $\mathcal{F}$. Hence, the vector bundle $\text{End}_{\mathcal{O}_X}(\mathcal{F})$ may be thought of as being equipped with a $k$-log connection $\nabla_{\mathcal{F}}^\text{ind}$ induced by $\nabla_{\mathcal{F}}$ via the adjoint representation $\text{Ad}_{\text{GL}_n} : \text{GL}_n \to \text{GL(}gl_n)(\text{cf. (17)}).$ One may identify this $k$-log connection $\nabla_{\mathcal{F}}^\text{ind}$ with $\nabla_{\mathcal{F}}^\vee \otimes \nabla_{\mathcal{F}}$ upon passing to the isomorphism $\mathcal{F}^\vee \otimes \mathcal{F} \cong \text{End}_{\mathcal{O}_X}(\mathcal{F})$. Denote by $\text{diag} : \mathcal{O}_X \hookrightarrow \text{End}_{\mathcal{O}_X}(\mathcal{F})$ the inclusion which, to any local section $a \in \mathcal{O}_X$, assigns the locally defined endomorphism $\text{diag}(a)$ of $\mathcal{F}$ given by multiplication by $a$. If $\mathcal{F}_{\text{PGL}_n}$ denotes the right $\text{PGL}_n$-torsor induced by $\mathcal{F}$ via a change of structure group $\text{GL}_n \to \text{PGL}_n$, then the adjoint vector bundle associated with $\mathcal{F}_{\text{PGL}_n}$ is canonically isomorphic to the cokernel
\[ \text{End}_{\mathcal{O}_X}(\mathcal{F}) := \text{Coker(}\text{diag)} \]
of $\text{diag}$. In particular, $\mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{F})$ admits a $k$-log connection

$$\nabla^{\text{ad}}_{\mathcal{F}^{\text{PGL}_n}} : \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{F}) \to \Omega_{X^{\log}/k^{\log}} \otimes \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{F})$$

induced by $\nabla^{\text{ad}}_{\mathcal{F}}$.

Consider the morphism

$$\mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{F}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}, \mathcal{F})$$

given by assigning $h \mapsto h \circ \nu^{(\nabla_{\mathcal{F}}, \nabla_{\mathcal{F}})}_i$ for any local section $h \in \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{F})$, and the morphism

$$\mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}) \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}, \mathcal{F})$$

given by assigning $h \mapsto \nu^{(\nabla_{\mathcal{F}}, \nabla_{\mathcal{F}})}_i \circ h$. These morphisms are injective, and we shall regard both $\mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{F})$ and $\mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})})$ as $\mathcal{O}_X$-submodules of $
abla_{\mathcal{F}}$.

Next, let $\mathcal{A}^j_{\text{Ker}(\nabla_{\mathcal{F}})}$ and $\log^j_{\mathcal{F}}$ $(j = 0, 1, \cdots)$ be as defined in (651) and (653) respectively, and write

$$\mathcal{B}^n_{\mathcal{X}^{\log}/k^{\log}} \subset \Omega_{X/k} \otimes (\log^0_{\mathcal{F}})^{\vee} \otimes \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}.$$ 

We regard $\mathcal{B}^n_{\mathcal{X}^{\log}/k^{\log}}$ as an $\mathcal{O}_X$-submodule of $\Omega_{X^{\log}/k^{\log}} \otimes \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})})$ (and hence, of $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}, \mathcal{F})$) via the composite injection

$$\Omega_{X/k} \otimes (\log^0_{\mathcal{F}})^{\vee} \otimes \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})} \hookrightarrow \Omega_{X/k} \otimes \mathcal{A}^0_{\text{Ker}(\nabla_{\mathcal{F}})} \otimes \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}$$

$$\simeq \Omega_{X/k} \otimes \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})})$$

$$\simeq \Omega_{X^{\log}/k^{\log}} \otimes \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})})$$

where the first arrow arises from the natural surjection $\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})} \to \log^0_{\mathcal{F}}$ and the third arrow denotes the injection induced by the inclusion $\Omega_{X/k} \hookrightarrow \Omega_{X^{\log}/k^{\log}}$.

**Proposition 7.4.1.**

Let

$$\mathcal{B}^{n-1}_{\mathcal{X}^{\log}/k^{\log}} \hookrightarrow \Omega_{X^{\log}/k^{\log}} \otimes \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{F}) \quad (= \Omega_{X^{\log}/k^{\log}} \otimes \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{D}^{0}_{1,X^{\log}/k^{\log}} \otimes \mathcal{B}^{0}_{\mathcal{F}}))$$

be as discussed in (429). (By applying this injection and (688), we regard $\mathcal{B}^{n-1}_{\mathcal{X}^{\log}/k^{\log}}$ as an $\mathcal{O}_X$-submodule of $\Omega_{X^{\log}/k^{\log}} \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})}, \mathcal{F})).$ Then, we have the equality

$$\mathcal{B}^{n-1}_{\mathcal{X}^{\log}/k^{\log}} = \mathcal{B}^{n-1}_{\mathcal{X}^{\log}/k^{\log}} \cap (\Omega_{X/k} \otimes \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})})).$$

**Proof.** It is easily verified that

$$\mathcal{B}^{n-1}_{\mathcal{X}^{\log}/k^{\log}} \subseteq \mathcal{B}^{n-1}_{\mathcal{X}^{\log}/k^{\log}} \cap (\Omega_{X/k} \otimes \mathcal{E}n\!d_{\mathcal{O}_X}(\mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})})).$$
In the following, we shall prove that the inverse inclusion holds. Each local section of the sheaf at the right-hand side of (695) may be considered as a locally defined morphism \( h : \mathcal{A}_{\Ker(\nabla)} \to \Omega_{X/\mathbb{K}^\log} \otimes F \) satisfying the following conditions:

1. \( \text{Im}(h) \subseteq \Omega_{X/\mathbb{K}^\log} \otimes \mathcal{A}_{\Ker(\nabla)} \);
2. \( h \) extends to a morphism \( \tilde{h} : F \to \Omega_{X/\mathbb{K}^\log} \otimes F \);
3. \( \tilde{h}(F^1) = 0 \).

Here, observe that \( F^1 \) is locally free and \( \mathcal{A}^1_{\Ker(\nabla)} \hookrightarrow F^1 \) is an isomorphism over the dense open subscheme of \( X \). Thus, by the condition (iii), we have \( h(\mathcal{A}^1_{\Ker(\nabla)}) (= \tilde{h}(\mathcal{A}^1_{\Ker(\nabla)})) = 0 \). By virtue of the condition (i), \( h \) turns out to come from a locally defined morphism \( h' : \mathcal{A}_{\Ker(\nabla)} \to \Omega_{X/\mathbb{K}} \otimes \mathcal{A}_{\Ker(\nabla)} \) with \( h'(\mathcal{A}^1_{\Ker(\nabla)}) = 0 \), i.e., a local section of \( \mathcal{B}^{n-1,1}_{X/\mathbb{K}^\log} \). This completes the proof of the asserted inclusion relation.

**Corollary 7.4.2.**

Suppose further that \( g = 0 \). Then, we have the equality

\[
\Gamma(X, \mathcal{B}^{n-1,1}_{X/\mathbb{K}^\log}) \cap \Gamma(X, \Omega_{X/\mathbb{K}} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}_{\Ker(\nabla)})) = 0.
\]

**Proof.** By Proposition 7.4.1 we have the equality

\[
\Gamma(X, \mathcal{B}^{n-1,1}_{X/\mathbb{K}^\log}) = \Gamma(X, \Omega_{X/\mathbb{K}} \otimes \mathcal{E}nd_{\mathcal{O}_X}(\mathcal{A}_{\Ker(\nabla)})) = 0.
\]

Hence, it suffices to prove that \( \Gamma(X, \mathcal{B}^{n-1,1}_{X/\mathbb{K}^\log}) = 0 \).

\( \mathcal{B}^{n-1,1}_{X/\mathbb{K}^\log} \) admits a filtration \( \Omega_{X/\mathbb{K}} \otimes (\mathcal{B}^{0}_{\nabla}) \otimes \mathcal{A}^1_{\Ker(\nabla)} \) with the associated grading \( \Omega_{X/\mathbb{K}} \otimes (\mathcal{B}^{0}_{\nabla}) \otimes \mathcal{B}^{j}_{\nabla} \) \( (j = 0, \cdots, n) \). It follows from the latter assertion of Corollary 7.3.4 and the condition \( g = 0 \) (hence \( \deg(\Omega_{X/\mathbb{K}}) = -2 \)) that

\[
\deg(\Omega_{X/\mathbb{K}} \otimes (\mathcal{B}^{0}_{\nabla}) \otimes \mathcal{B}^{j}_{\nabla}) = -2 + j \cdot (-2 + r) - \sum_{i=1}^{r}(m_{i,j+1} - m_{i,1}).
\]

On the other hand, since the integers \( m_{i,1}, \cdots, m_{i,n} \) are mutually distinctive (by Proposition 7.3.3), we have

\[
\sum_{i=1}^{r}(m_{i,j+1} - m_{i,1}) = \sum_{i=1}^{r}\sum_{l=1}^{j}(m_{i,l+1} - m_{i,l}) \geq \sum_{i=1}^{r}\sum_{l=1}^{j}1 = j \cdot r.
\]

Thus, it follows from (698) and (699) that

\[
\deg(\Omega_{X/\mathbb{K}} \otimes (\mathcal{B}^{0}_{\nabla}) \otimes \mathcal{B}^{j}_{\nabla}) \leq -2 + j \cdot (-2 + r) - j \cdot r = -2 - 2 \cdot j < 0.
\]
Hence, we have \( \Gamma(X, \Omega_{X/k} \otimes (\text{gr}_F^0)^{\vee} \otimes \text{gr}_F^j) = 0 \). By descending induction on \( j \),
one verifies that \( \Gamma(X, \Omega_{X/k} \otimes (\text{gr}_F^0)^{\vee} \otimes \mathcal{A}_{\text{Ker}(\nabla_F)}) = 0 \) for all \( j \geq 0 \), in particular,

\[
\Gamma(X, \mathcal{E}_{X|k}^{n-1,A})(= \Gamma(X, \Omega_{X/k} \otimes (\text{gr}_F^0)^{\vee} \otimes \mathcal{A}_{\text{Ker}(\nabla_F)})) = 0.
\] (701)

This completes the proof of Corollary 7.4.2.

7.5. It follows from well-known generalities of deformation theory (cf. [57],
Proposition 3.6) that the set of isomorphism classes \( \text{Def}^{(\mathcal{F}, \nabla_F)}_X \) of deformations
over \( X_\varepsilon/k_\varepsilon \) of \( (\mathcal{F}, \nabla_F) \) (i.e., the set of log integrable vector bundles \( (\mathcal{F}_\varepsilon, \nabla_{\mathcal{F}_\varepsilon}) \)
on \( X_\varepsilon/k_\varepsilon \) together with an isomorphism \( \text{in}_X^\varepsilon(\mathcal{F}_\varepsilon, \nabla_{\mathcal{F}_\varepsilon}) \sim (\mathcal{F}, \nabla_F) \) corresponds bijectively to (the underlying set of) the 1-st hypercohomology group
\( \mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_\mathcal{F}^\text{ad}]) \); denote by

\[
\text{Def}^{(\mathcal{F}, \nabla_F)}_X \sim \mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_\mathcal{F}^\text{ad}])
\] (702)

this bijective correspondence. Also, the set of isomorphism classes of deformations over \( X_\varepsilon \) of \( \mathcal{F} \)
corresponds bijectively to \( H^1(X, \text{End}_{\mathcal{O}_X}(\mathcal{F})) \). The assignment \( (\mathcal{F}_\varepsilon, \nabla_{\mathcal{F}_\varepsilon}) \mapsto \mathcal{F}_\varepsilon \) determines, via these correspondences, a morphism
\( \mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_\mathcal{F}^\text{ad}]) \rightarrow H^1(X, \text{End}_{\mathcal{O}_X}(\mathcal{F})) \) of \( k \)-vector spaces. One verifies that this morphism coincides with \( 'e_\varepsilon[\nabla_\mathcal{F}^\text{ad}] \) (cf. [143] for the definition of \( 'e_\varepsilon[-] \)). Hence, the subspace

\[
\text{Ker}'(e_\varepsilon[\nabla_\mathcal{F}^\text{ad}]) \subseteq \mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_\mathcal{F}^\text{ad}])
\] (703)

corresponds, via the correspondence (702), to the deformations \( (\mathcal{F}_\varepsilon, \nabla_{\mathcal{F}_\varepsilon}) \) of
\( (\mathcal{F}, \nabla_F) \) satisfying that \( \mathcal{F}_\varepsilon \cong \text{pr}_X^* \mathcal{F} \).

Next, let

\[
\nabla_{\mathcal{F}}^{\text{zzz}} : \text{End}_{\mathcal{O}_X}(\mathcal{F}) \rightarrow \text{Im}(\nabla_\mathcal{F}^\text{ad})
\] (704)

be the morphism obtained by restricting the codomain of \( \nabla_\mathcal{F}^\text{ad} \). Since the complex \( \mathcal{K}^\bullet[\nabla_\mathcal{F}^{\text{zzz}}] \) is quasi-isomorphic to \( \text{Ker}(\nabla_\mathcal{F}^\text{ad})[0] \) via the natural morphism
\( \text{Ker}(\nabla_\mathcal{F}^\text{ad})[0] \rightarrow \mathcal{K}^\bullet[\nabla_\mathcal{F}^{\text{zzz}}] \), we have an isomorphism

\[
H^1(X, \text{Ker}(\nabla_\mathcal{F}^\text{ad})) \sim \mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_\mathcal{F}^{\text{zzz}}]).
\] (705)

Since \( \mathcal{K}^0[\nabla_\mathcal{F}^{\text{zzz}}] = \mathcal{K}^0[\nabla_\mathcal{F}^\text{ad}] \), the morphism

\[
\mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_\mathcal{F}^{\text{zzz}}]) \rightarrow \mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_\mathcal{F}^\text{ad}])
\] (706)

induced by the natural morphism \( \mathcal{K}^\bullet[\nabla_\mathcal{F}^{\text{zzz}}] \rightarrow \mathcal{K}^\bullet[\nabla_\mathcal{F}^\text{ad}] \) is injective. It follows from Proposition 5.8.1 that the image of (706) corresponds, via the correspondence (702), to the deformations \( (\mathcal{F}_\varepsilon, \nabla_{\mathcal{F}_\varepsilon}) \) of \( (\mathcal{F}, \nabla_F) \) with vanishing p-curvature. In the following, we shall identify, for simplicity, \( \mathbb{H}^1(X, \mathcal{K}^\bullet[\nabla_\mathcal{F}^{\text{zzz}}]) \) with its image via the injection (706).
Now, let us fix an element
\[ \varrho \in \ker(\nu|_{\mathcal{V}_{\mathcal{F}}}) \cap \mathbb{H}^1(X, \mathcal{K}^\bullet[\mathcal{V}_{\mathcal{F}}]) \] (707)
By the above discussion, \( \varrho \) corresponds, via (702), to a log integrable vector bundle \((\mathcal{F}_e, \nabla_{\mathcal{F}_e})\) on \( X_{\epsilon}^{\log}/k_{\epsilon}^{\log} \) (together with an isomorphism \( \iota_X^{\epsilon}(\mathcal{F}_e, \nabla_{\mathcal{F}_e}) \sim (\mathcal{F}, \nabla_{\mathcal{F}}) \)) such that \( \mathcal{F}_e \cong \text{pr}_X^*(\mathcal{F}) \) and \( p_{\gamma}^{\epsilon}(\mathcal{F}_e, \nabla_{\mathcal{F}_e}) = 0 \). Since \( \text{Coker}(\nu|_{\mathcal{V}_{\mathcal{F}}}) \) (= \( \bigoplus_{i=1}^r \Lambda_i \)) (cf. (649)) is flat over \( k_\epsilon \) (cf. the discussion in § 7.2), the injection
\[ \nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}(\mathcal{F}_e, \nabla_{\mathcal{F}_e}) : \mathcal{A}_{\ker(\nabla_{\mathcal{F}})} \to \mathcal{F}_e \] (708)
determines an element \([\nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}(\mathcal{F}_e, \nabla_{\mathcal{F}_e})]\) of the tangent space
\[ \mathcal{T}_{[\nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}(\mathcal{F}_e, \nabla_{\mathcal{F}_e})]}^{n, \deg(\mathcal{A}_{\ker(\nabla_{\mathcal{F}})})} \] (709)
to the Quot-scheme \( \text{Quot}_{\mathcal{F}/X/k}^{n, \deg(\mathcal{A}_{\ker(\nabla_{\mathcal{F}})})} \) (cf. § 8.1) at the point \([\nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}(\mathcal{F}, \nabla_{\mathcal{F}})]\) corresponding to the injection \( \nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}(\mathcal{F}, \nabla_{\mathcal{F}}) : \mathcal{A}_{\ker(\nabla_{\mathcal{F}})} \to \mathcal{F} \). In particular, we obtain an element
\[ \text{def}_{\nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}(\mathcal{F}, \nabla_{\mathcal{F}})}([\nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}(\mathcal{F}, \nabla_{\mathcal{F}})]) \in \text{Hom}_{\mathcal{O}_X}(\mathcal{A}_{\ker(\nabla_{\mathcal{F}})}, \bigoplus_{i=1}^r \Lambda_i) \] (710)
(cf. (744)).

Next, let us take an element
\[ \varrho' \in \Gamma(X, \Omega_{X_{\epsilon}^{\log}/k_{\epsilon}^{\log}} \otimes \text{End}_{\mathcal{O}_X}(\mathcal{F})) \] (711)
whose image via the composite surjection
\[ \Gamma(X, \Omega_{X_{\epsilon}^{\log}/k_{\epsilon}^{\log}} \otimes \text{End}_{\mathcal{O}_X}(\mathcal{F})) \to \text{Coker}(\Gamma(X, \nabla_{\mathcal{F}})) \xrightarrow{\nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}[\nabla_{\mathcal{F}}]} \ker(\nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}[\nabla_{\mathcal{F}}]) \] (712)
coincides with \( \varrho \). Since \( \varrho \in \mathbb{H}^1(X, \mathcal{K}^\bullet[\mathcal{V}_{\mathcal{F}}]) \), \( \varrho' \) may be, Zariski locally on \( X \), expressed as
\[ \varrho' = \nabla_{\mathcal{F}}(\varrho'') \] (713)
for some local section \( \varrho'' \) of \( \text{End}_{\mathcal{O}_X}(\mathcal{F}) \). Here, consider the morphism
\[ \text{End}_{\mathcal{O}_X}(\mathcal{F}) \to \text{Hom}_{\mathcal{O}_X}(\mathcal{A}_{\ker(\nabla_{\mathcal{F}})}, \bigoplus_{i=1}^r \Lambda_i) \] (714)
given by assigning \( h \mapsto \nu|_{\mathcal{V}_{\mathcal{F}}}^{\gamma}(\mathcal{F}, \nabla_{\mathcal{F}}) \circ h \circ \nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}(\mathcal{F}, \nabla_{\mathcal{F}}) \) (cf. (649) for the definition of \( \nu|_{\mathcal{V}_{\mathcal{F}}}^{\epsilon}(\mathcal{F}, \nabla_{\mathcal{F}}) \)). Then, the image of the local sections \( \varrho'' \) via this morphism glue together to a well-defined global section
\[ \varrho'' \in \text{Hom}_{\mathcal{O}_X}(\mathcal{A}_{\ker(\nabla_{\mathcal{F}})}, \bigoplus_{i=1}^r \Lambda_i) \] (715).
By the construction of the bijection (714) (cf. [64], the discussion in the proof of Proposition 4.4.4), one obtains the following proposition.

**Proposition 7.5.1.**
We have the equality
\[ \text{def} \nu'_2(F, \nabla_F) (\nu'_2(F, \nabla_F)) = g'' \tag{716} \]
In particular, \( \nu'_2(F, \nabla_F) \) is isomorphic to the pull-back of \( \nu'_2(F, \nabla_F) \) via \( \text{pr}_X \) (in the sense of Lemma 7.6.1 below) if and only if any local section \( g'' \) of \( \text{End}_{O_X}(F) \) constructed above lies in the \( O_X \)-submodule
\[ \mathcal{E}nd_{O_X}(A_{Ker(\nabla_F)}) \cap \mathcal{E}nd_{O_X}(F) \tag{717} \]
of \( \text{Hom}_{O_X}(A_{Ker(\nabla_F)}, F) \) (cf. (688) and (689)).

**Corollary 7.5.2.**
Let \( (F, \nabla_F) \) be as above. Suppose further that the vector bundle \( F_{X/k} / \text{pr}_X (\text{Ker}(\nabla_F)) \) on \( X_{k, \ell} (\cong X_{k, \ell}^{(1)}) \) (cf. Corollary 5.8.2) is isomorphic to \( \text{pr}^*_{X_{k, \ell}} (F_{X/k} / \text{pr}_X (\text{Ker}(\nabla_F))) \) and \( \nu'_2(F, \nabla_F) \) is isomorphic to the pull-back of \( \nu'_2(F, \nabla_F) \) via \( \text{pr}_X \). Then the element \( g' \) of \( \Gamma(X, \Omega_{X/k}^{\log} / k \otimes \text{End}_{O_X}(F)) \) constructed in (711) lies in the subspace
\[ \Gamma(X, \Omega_{X/k}^{\log} / k \otimes \text{End}_{O_X}(A_{Ker(\nabla_F)})) \cap \Gamma(X, \Omega_{X/k}^{\log} / k \otimes \text{End}_{O_X}(F)). \tag{718} \]

**Proof.** In the following, we shall write \( \nabla := F_{X/k} (\text{Ker}(\nabla_F)) \) for simplicity. Consider the \( k \)-log connections \( \nabla^\text{can} \otimes \nabla^\text{can}, \nabla^\text{can} \otimes \nabla_F \) on \( \text{End}_{O_X}(A_{Ker(\nabla_F)}), \text{Hom}_{O_X}(A_{Ker(\nabla_F)}, F) \) respectively. The inclusion
\[ \text{End}_{O_X}(F) \hookrightarrow \text{Hom}_{O_X}(A_{Ker(\nabla_F)}, F) \tag{719} \]
(resp., \( \text{End}_{O_X}(A_{Ker(\nabla_F)}) \hookrightarrow \text{Hom}_{O_X}(A_{Ker(\nabla_F)}, F) \))
defined in (688) (resp., (689)) is compatible with the respective \( k \)-log connections \( \nabla^\text{ad} \) (resp., \( \nabla^\text{can} \otimes \nabla^\text{can} \)) and \( \nabla^\text{can} \otimes \nabla_F \). But, it follows from Proposition 7.5.1 that the local sections \( g'' \) of \( \text{End}_{O_X}(F) \) lies in \( \text{End}_{O_X}(A_{Ker(\nabla_F)}) \cap \text{End}_{O_X}(F) \). Hence, the element \( g' \) (\( = \nabla^\text{ad}(g'') \)) of
\[ \Gamma(X, \Omega_{X/k}^{\log} / k \otimes \text{End}_{O_X}(F)) \subset \Gamma(X, \Omega_{X/k}^{\log} / k \otimes \text{Hom}_{O_X}(A_{Ker(\nabla_F)}, F)) \tag{720} \]
lies in \( \Gamma(X, \text{Im}(\nabla^\text{can} \otimes \nabla^\text{can})) \).

Next, consider the exact sequence
\[ \Gamma(X, \text{End}_{O_X}(A_{Ker(\nabla_F)})) \to \Gamma(X, \text{Im}(\nabla^\text{can} \otimes \nabla^\text{can})) \to H^1(X, \text{Ker}(\nabla^\text{can} \otimes \nabla^\text{can})) \tag{721} \]
induced by the short exact sequence
\[ 0 \to \text{Ker}(\nabla^\text{can} \otimes \nabla^\text{can}) \to \text{End}_{O_X}(A_{Ker(\nabla_F)}) \to \text{Im}(\nabla^\text{can} \otimes \nabla^\text{can}) \to 0. \tag{722} \]
Here, note that $\nabla^{\text{can}}_{\mathcal{V}} \otimes \nabla^{\text{can}}_{\mathcal{V}}$ may be identified with the canonical $k$-log connection

\begin{equation}
\nabla^{\text{can}}_{\text{End}(\mathcal{V})} : F^*_X(\text{End}(\mathcal{V})) \to \Omega_X^{\text{log}/k} \otimes F^*_X(\text{End}(\mathcal{V}))
\end{equation}

on $F^*_X(\text{End}(\mathcal{V}))$, where $\text{End}(\mathcal{V}) := \text{End}_{\mathcal{O}_X}(\mathcal{V})$, via the natural isomorphism $F^*_X(\text{End}(\mathcal{V})) \simeq \text{End}_{\mathcal{O}_X}(A_{Ker(\nabla_{\mathcal{V}})})$. In particular,

\begin{equation}
F^*_X(\text{Ker}(\nabla^{\text{can}}_{\mathcal{V}} \otimes \nabla^{\text{can}}_{\mathcal{V}})) \simeq F^*_X(\text{Ker}(\nabla_{\text{End}(\mathcal{V})}^{\text{can}})) \sim \mathcal{V},
\end{equation}

and hence,

\begin{equation}
H^1(X, \text{Ker}(\nabla^{\text{can}}_{\mathcal{V}} \otimes \nabla^{\text{can}}_{\mathcal{V}})) \sim H^1(X^{(1)}, \text{End}(\mathcal{V})).
\end{equation}

It follows from well-known generalities concerning deformation theory that there exists a canonical bijective correspondence between $H^1(X^{(1)}, \mathcal{V})$ and the set $\text{Def}^F_{\mathcal{X}/k_*}(\text{Ker}(\nabla_{\mathcal{V}}))$ of isomorphism classes of deformations over $X^{(1)}$ of the vector bundle $F^*_X(\text{Ker}(\nabla_{\mathcal{V}}))$ on $X^{(1)}$. By passing to (725), we obtain a bijection

\begin{equation}
\text{Def}^F_{\mathcal{X}/k_*}(\text{Ker}(\nabla_{\mathcal{V}})) \sim H^1(X, \text{Ker}(\nabla^{\text{can}}_{\mathcal{V}} \otimes \nabla^{\text{can}}_{\mathcal{V}})).
\end{equation}

The image of $\varphi' \in \Gamma(X, \text{End}_{\mathcal{O}_X}(A_{Ker(\nabla_{\mathcal{V}})}))$ via the second arrow in (721) corresponds, via this bijection (726), to the deformation $F^*_X(\text{Ker}(\nabla_{\mathcal{V}}))$ of $F^*_X(\text{Ker}(\nabla_{\mathcal{V}}))$. But, since $F^*_X(\text{Ker}(\nabla_{\mathcal{V}}))$ was assumed to be the trivial deformation, this image of $\varphi'$ equals the zero element. Both this fact and the exactness of (721) imply that there exists an element

\begin{equation}
\varphi' \in \Gamma(X, \text{End}_{\mathcal{O}_X}(A_{Ker(\nabla_{\mathcal{V}})}))
\end{equation}

which is sent to $\varphi'$ by the first arrow in (721). By the definition of the canonical $k$-log connection (cf. §3.3), the morphism $\nabla^{\text{can}}_{\mathcal{V}} \otimes \nabla^{\text{can}}_{\mathcal{V}} := \nabla^{\text{can}}_{\text{End}(\mathcal{V})}$ factors through the inclusion

\begin{equation}
\Omega_{X/k} \otimes \text{End}_{\mathcal{O}_X}(A_{Ker(\nabla_{\mathcal{V}})}) \hookrightarrow \Omega_{X^{\text{log}}/k} \otimes \text{End}_{\mathcal{O}_X}(A_{Ker(\nabla_{\mathcal{V}})}).
\end{equation}

Thus, the element $\varphi'$, which is the image of $\varphi'$ via $\nabla^{\text{can}}_{\mathcal{V}} \otimes \nabla^{\text{can}}_{\mathcal{V}}$, lies in $\Gamma(X, \Omega_{X/k} \otimes \text{End}_{\mathcal{O}_X}(A_{Ker(\nabla_{\mathcal{V}})}))$. This completes the proof of Corollary 7.5.2.

7.6. In this section, we prove (cf. Corollary 7.6.2) that any deformation of a $(\text{GL}_n, 1, \mathbb{U})$-oper (cf. Definition 4.9.4) is, in a certain sense, uniquely determined by the sheaf of its horizontal sections.

Let $\mathcal{B}$ and $\mathcal{F}$ be as above, and $\nabla_0$ a $k$-log connection on $\text{det}(\mathcal{F})$ for which the pair $\mathbb{U} := (\mathcal{B}, \nabla_0)$ forms an $(n, 1)$-determinant data for $X^{\text{log}}$ over $k^{\text{log}}$ (cf. Definition 4.9.1) and satisfying that $p^{\mathbb{U}}_{\text{det}(\mathcal{F})}(\nabla_0) = 0$. (By Proposition 4.13.2, there exists at least one such $(n, 1)$-determinant data.) Write $\mathcal{F}_\epsilon := \mathcal{D}^{<n}_{1, X^{\text{log}}/k^{\text{log}}} \otimes \text{pr}_X^*(\mathcal{B}')$, which is canonically isomorphic to $\text{pr}_X^*(\mathcal{F})$. 

Now, let $\nabla_{\mathcal{F}}$ be a dormant $(\GL_n, 1, \mathbb{U})$-oper on $X_{e/k}$ and denote by $\nabla_{\mathcal{F}}$ the $(\GL_n, 1, \mathbb{U})$-oper on $X_{/k}$ obtained as the pull-back of $\nabla_{\mathcal{F}}$ via $\in_X : X \to X_e$.

**Lemma 7.6.1.**

The morphism $\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})} : \mathcal{A}_{\ker(\nabla_{\mathcal{F}})} \to \mathcal{F}_e$ is isomorphic to the pull-back of $\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})} : \mathcal{A}_{\ker(\nabla_{\mathcal{F}})} \to \mathcal{F}$ via $\pr_X$. More precisely, there exists an isomorphism $\alpha : \mathcal{A}_{\ker(\nabla_{\mathcal{F}})} \cong \pr_X^*(\mathcal{A}_{\ker(\nabla_{\mathcal{F}})})$ of $\mathcal{O}_X$-modules which makes the following square diagram commute:

$$
\begin{array}{ccc}
\mathcal{A}_{\ker(\nabla_{\mathcal{F}})} & \xrightarrow[\alpha]{\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}} & \mathcal{F}_e \\
\downarrow & & \downarrow \\
\pr_X^*(\mathcal{A}_{\ker(\nabla_{\mathcal{F}})}) & \xrightarrow{\pr_X^*(\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})})} & \pr_X^*(\mathcal{F}),
\end{array}
$$

(729)

where the right-hand vertical arrow denotes the isomorphism arising from the natural isomorphism $\mathcal{D}_n^{(\log)} \cong \pr_X^*(\mathcal{D}_n^{(\log)}).$

**Proof.** Since the cokernel $\bigoplus_{i=1}^r \Lambda_i$ of $\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}$ is flat over $k_e$ (cf. §7.2), the injection $\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}$ determines a tangent vector

$$
[\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}] \in \mathcal{T}_{\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}} \Quot(\mathcal{F}/\mathcal{F}/T)^{n, \deg(\mathcal{A}_{\ker(\nabla_{\mathcal{F}})})}
$$

(730)

to the Quot scheme $\Quot$ at the point $[\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}] \in \Quot$ corresponding to $\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}$ (cf. §8.1). By means of the isomorphism (774), the tangent vector $[\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}]$ corresponds to an injection

$$
\text{def}_{\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}}([\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}]) : \mathcal{A}_{\ker(\nabla_{\mathcal{F}})} \to \bigoplus_{i=1}^r \Lambda_i
$$

(731)

of $\mathcal{O}_X$-modules. One verifies from the construction of the isomorphism (774) (cf. [61], Proposition 4.4.4) that the morphism $\text{def}_{\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}}([\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}])$ is compatible with the respective $k$-log connections $\nabla_{\ker(\nabla_{\mathcal{F}})}$ and $\bigoplus_{i=1}^r \nabla_{i,k}$ (cf. [650]). But, by Lemma 7.3.1 and the composite isomorphism (670), this morphism turns out to be the zero map. This implies that $\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}$ is the trivial deformation of $\nu_{\mathcal{F}}^{(\mathcal{F}, \nabla_{\mathcal{F}})}$.

\[\square\]

**Corollary 7.6.2.**

Suppose further that $g = 0$ and the vector bundle $F_{X_{e/k}}(\ker(\nabla_{\mathcal{F}}))$ on $X^{(1)}_{e,k}$

$$
(\cong X^{(1)}_{e,k})
$$

is isomorphic to $\pr_X^{(1)}(F_{X_{e/k}}(\ker(\nabla_{\mathcal{F}})))$. Then the $(\GL_n, 1, \mathbb{U})$-oper
\( \nabla_{\mathcal{F}} \) is isotrivial, i.e., isomorphic to the \((\text{GL}_n, 1, \mathbb{U})\)-oper obtained as the pullback \( \text{pr}_X^* (\nabla_{\mathcal{F}}) \) of \( \nabla_{\mathcal{F}} \) via \( \text{pr}_X \).

**Proof.** By the composite isomorphism asserted in Corollary 4.11.3, we may assume, without loss of generality, that the \( \mathfrak{sl}_n \)-oper \( \mathcal{E}_\epsilon := (\mathcal{E}_{\epsilon, 1, \mathcal{X}/k, \nabla_{\mathcal{E}}}) \) associated with \( \nabla_{\mathcal{F}} \) is of canonical type II (cf. Definition 4.12.1). In particular, if \( \mathcal{E}_\epsilon := (\mathcal{E}_{\epsilon, 1, \mathcal{X}/k, \nabla_{\mathcal{E}}}) \) denotes the \( \mathfrak{sl}_n \)-oper on \( \mathcal{X}/k \) obtained as the pull-back of \( \mathcal{E}_\epsilon \) via \( \text{pr}_X \), then both \( \mathcal{E}_\epsilon \) and the pull-back \( \text{pr}_X^* (\mathcal{E}_\epsilon) \) of \( \mathcal{E}_\epsilon \) are of canonical type II.

Write \( \text{Op-Def}_{\mathcal{F}}^{\nabla_{\mathcal{F}}} \) for the set of isomorphism classes of deformations over \( \mathcal{X}/k \) of the \((\text{GL}_n, 1, \mathbb{U})\)-oper \( \nabla_{\mathcal{F}} \). According to Corollary 4.11.3, Proposition 5.4.1, and the discussion in §5.4, there exists a canonical bijection

\[
(732) \quad \text{Op-Def}_{\mathcal{F}}^{\nabla_{\mathcal{F}}} \sim \mathbb{H}^1(X, \mathcal{K}^* [\nabla_{\mathcal{E}}^{\text{ad}(0)}]).
\]

Also, the morphism \( \mathbb{H}^1(X, \mathcal{K}^* [\nabla_{\mathcal{E}}^{\text{ad}(0)}]) \to \mathbb{H}^1(X, \mathcal{K}^* [\nabla_{\mathcal{E}}^{\text{ad}}]) \) induced by the natural morphism \( \mathcal{K}^* [\nabla_{\mathcal{E}}^{\text{ad}(0)}] \to \mathcal{K}^* [\nabla_{\mathcal{E}}^{\text{ad}}] \) is injective (cf. the discussion in the proof of Proposition 5.2.2 (i)). Thus, \( \text{Op-Def}_{\mathcal{F}}^{\nabla_{\mathcal{F}}} \) may be thought of as a subset of \( \mathbb{H}^1(X, \mathcal{K}^*[\nabla_{\mathcal{E}}^{\text{ad}}]) \). Moreover, by Remark 4.12.2 (iii), \( \text{Op-Def}_{\mathcal{F}}^{\nabla_{\mathcal{F}}} \) corresponds (since \( \mathcal{E}_\epsilon \) is of canonical type II) bijectively to \( \Gamma(X, \mathcal{B}_{X, \log}^{n-1}) \). The resulting composite injection

\[
(733) \quad \Gamma(X, \mathcal{B}_{X, \log}^{n-1}) \hookrightarrow \text{Op-Def}_{\mathcal{F}}^{\nabla_{\mathcal{F}}} \hookrightarrow \mathbb{H}^1(X, \mathcal{K}^*[\nabla_{\mathcal{E}}^{\text{ad}}])
\]

coincides (by the discussion in Remark 4.12.2 (iii)) with the composite injection

\[
(734) \quad \Gamma(X, \mathcal{B}_{X, \log}^{n-1}) \hookrightarrow \Gamma(X, \Omega_{X, \log} \otimes \mathcal{E} \text{nd}_{\mathcal{O}_X}(\mathcal{F}))
\]

\[
\to \mathbb{H}^1(X, \mathcal{K}^*[\nabla_{\mathcal{E}}^{\text{ad}}])
\]

\[
\sim \mathbb{H}^1(X, \mathcal{K}^*[\nabla_{\mathcal{E}}^{\text{ad}}])
\]

(cf. (129) for the definition of the first arrow), where the second arrow arises from the natural morphism \( \Omega_{X, \log} \otimes \mathcal{E} \text{nd}_{\mathcal{O}_X}(\mathcal{F})[-1] \to \mathcal{K}^* [\nabla_{\mathcal{E}}^{\text{ad}}] \) and the third arrow arises from the natural surjection \( \mathcal{E} \text{nd}_{\mathcal{O}_X}(\mathcal{F}) \twoheadrightarrow \mathcal{E} \text{nd}_{\mathcal{O}_X}(\mathcal{F}) \).

Now, let \( \mathcal{g} \) be the element of \( \mathbb{H}^1(X, \mathcal{K}^*[\nabla_{\mathcal{E}}^{\text{ad}}]) \) corresponding, via the bijection (702), to the deformation \( (\mathcal{F}, \nabla_{\mathcal{F}}) \) of \( (\mathcal{F}, \nabla_{\mathcal{F}}) \). Also, let \( \mathcal{g}' \) be an element of \( \Gamma(X, \Omega_{X, \log} \otimes \mathcal{E} \text{nd}_{\mathcal{O}_X}(\mathcal{F})) \) constructed in the manner of §7.5 for the case of this \( \mathcal{g} \). Then, by the construction of \( \text{Op-Def}_{\mathcal{F}}^{\nabla_{\mathcal{F}}} \), one may choose \( \mathcal{g}' \) as satisfying that

\[
(735) \quad \mathcal{g}' \in \Gamma(X, \mathcal{B}_{X, \log}^{n-1}).
\]

Next, since we have assumed that \( \nabla_{\mathcal{F}} \) is dormant and \( \mathcal{F}_\epsilon \sim \text{pr}_X^* (\mathcal{F}) \), the element \( \mathcal{g} \) lies in the subspace \( \text{Ker}(e_\mathcal{g} [\nabla_{\mathcal{F}}]) \cap \mathbb{H}^1(X, \mathcal{K}^*[\nabla_{\mathcal{F}}^{\text{ad}}]) \) (cf. §7.5). If \( \mathcal{g}' \)
is as in (711) for the case of \((\mathcal{F}, \nabla_{\mathcal{F}})\) under consideration, then, by Corollary 7.5.2 and Lemma 7.6.1, we see that
\[
\varrho' \in \Gamma(X, \Omega_{X/k} \otimes \text{End}_{O_X}(A_{\text{Ker}(\nabla_{\mathcal{F}})})) \cap \Gamma(X, \Omega_{X,k}^{\log} \otimes \text{End}_{O_X}(A_{\text{Ker}(\nabla_{\mathcal{F}})})).
\]

By combining (735) and (736), we see that
\[
\varrho' \in \Gamma(X, \bigotimes_{1}^{\star} B_{n-1} \Omega_{X,k}^{\log}) \cap \Gamma(X, \Omega_{X/k} \otimes \text{End}_{O_X}(A_{\text{Ker}(\nabla_{\mathcal{F}})})).
\]

But, since \(g = 0\), it follows from Corollary 7.4.2 that \(\varrho' = 0\). This implies that \(\nabla_{\mathcal{F}}\) is the trivial deformation of \(\nabla_{\mathcal{F}}\), i.e., isomorphic to \(\text{pr}_X^*(\nabla_{\mathcal{F}})\), as desired. \(\square\)

7.7. Now, we study the case of a pointed stable curve of type \((0,3)\). Let \(\mathbb{P}^1_k := (\mathbb{P}^1/k, \{[0], [1], [\infty]\})\) be as in (631). (In particular, the Frobenius twist \(\mathbb{P}^1_k^{(1)}\) of the underlying curve \(\mathbb{P}^1/k\), i.e., the projective line over \(k\), is isomorphic to \(\mathbb{P}^1\).) Recall the Birkhoff-Grothendieck’s theorem which asserts that for any vector bundle \(\mathcal{V}\) on \(\mathbb{P}^1\) of rank \(n \geq 1\) is isomorphic to a direct sum of \(n\) line bundles:
\[
\mathcal{V} \cong \bigoplus_{j=1}^{n} \mathcal{O}_{\mathbb{P}^1}(w_j),
\]
where \(w_{j_1} \leq w_{j_2}\) if \(j_1 < j_2\). The ordered set \((w_j)_{j=1}^{n}\) of integers depends only on the isomorphism class of the vector bundle \(\mathcal{V}\). In this situation, we shall say that \(\mathcal{V}\) is of type \((w_j)_{j=1}^{n}\).

By induction on \(n\), one may verify easily the following

**Lemma 7.7.1.**

Suppose that we are given, for \(\square = 1, 2\), a rank \(n\) vector bundle \(\mathcal{V}_\square\) on \(\mathbb{P}^1\) of type \((w_{\square,j})_{j=1}^{n}\) and an injection \(\mathcal{V}_1 \hookrightarrow \mathcal{V}_2\) of \(\mathcal{O}_{\mathbb{P}^1}\)-modules. Then, for any \(j \in \{1, \ldots, n\}\), it is satisfied that \(w_{1,j} \leq w_{2,j}\).

The above lemma deduces the following lemma, which will be used in the proof of Proposition 7.7.4

**Lemma 7.7.2.**

Let \(s\) be an integer and \(\{\mathcal{V}_i\}_{i \in \mathbb{Z}_{\geq 0}}\) a set of rank \(n\) vector bundles on \(\mathbb{P}^1\) such that each \(\mathcal{V}_i\) is of degree \(s + l\) and type \((w_{i,j})_{j=1}^{n}\) (hence \(\sum_{j=1}^{n} w_{i,j} = s + l\)) satisfying that \(w_{i,n} - w_{i,1} \leq 2\). Also, suppose that we are given a sequence of \(\mathcal{O}_{\mathbb{P}^1}\)-linear injections
\[
\mathcal{V}_0 \hookrightarrow \mathcal{V}_1 \hookrightarrow \mathcal{V}_2 \hookrightarrow \cdots.
\]
Then, there exists \(l_0 \in \mathbb{Z}_{\geq 0}\) such that \(\mathcal{V}_{l_0}\) is of type \((w_{l_0,j})_{j=1}^{n}\) satisfying that \(w_{l_0,n} - w_{l_0,1} \leq 1\).
Proof. It suffices to consider the case where $w_{0,n} - w_{0,1} = 2$. Let $l_1 := \min \{l \in \mathbb{Z}_{\geq 0} \mid w_{l,n} \neq w_{0,n} \}$. (In particular, we have $w_{l-1,n} = w_{0,n}$.) Lemma \ref{7.7.1} and the condition that $w_{0,n} - w_{0,1} = 2$ and $w_{l,n} - w_{l,1} \leq 2$ implies that $w_{l,1} = w_{0,1} + 1$. Hence,

\begin{equation}
(740) \quad w_{l-1,n} - w_{l-1,1} = w_{0,n} - (w_{0,1} + 1) = 1.
\end{equation}

Consequently, the integer $l_0 := l_1 - 1$ is as desired. \hfill \square

In the following, let $\mathbb{U} := (\mathcal{B}, \nabla_0)$ be an $(n, 1)$-determinant data for $\mathbb{P}^{1, \log}$ over $k^{\log}$ satisfying that $p_{\psi}/(\det(\mathcal{D}^{<n}_{1, \mathbb{P}^{1, \log}/k^{\log}} \otimes \mathcal{B}^\vee), \nabla_0) = 0$. Write

\begin{equation}
(741) \quad \mathcal{F} := \mathcal{D}^{<n}_{1, \mathbb{P}^{1, \log}/k^{\log}} \otimes \mathcal{B}^\vee, \quad \text{and} \quad \mathcal{F}_e := \mathcal{D}^{<n}_{1, \mathbb{P}^{1, \log}/k^{\log}} \otimes \mathfrak{p}_{\mathbb{P}^{1}}^*(\mathcal{B}^\vee).
\end{equation}

**Proposition 7.7.3.**

Let $\nabla_\mathcal{F}$ be a dormant $(\text{GL}_n, 1, \mathbb{U})$-oper on $\mathbb{P}/k$. Suppose that the rank $n$ vector bundle $F_{\mathbb{P}^{1}/k^*}(\text{Ker}(\nabla_\mathcal{F}))$ on $\mathbb{P}^{1,(1)}_k$ is of type $(w_i)_{i=1}^n$, i.e., decomposes into a direct sum of $n$ line bundles:

\begin{equation}
(742) \quad F_{\mathbb{P}^{1}/k^*}(\text{Ker}(\nabla_\mathcal{F})) \cong \bigoplus_{i=1}^n \mathcal{O}_{\mathbb{P}^{1}(i)}(w_i)
\end{equation}

(satisfying that $w_{j_1} \leq w_{j_2}$ if $j_1 < j_2$). Then, we have the inequality

\begin{equation}
(743) \quad w_n - w_1 \leq 2.
\end{equation}

Proof. By means of the decomposition (742), the rank $n$ vector bundle $\mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})} := F_{\mathbb{P}^{1}/k^*}(F_{\mathbb{P}^{1}/k^*}(\text{Ker}(\nabla_\mathcal{F})))$ on $\mathbb{P}^1$ decomposes into the direct sum

\begin{equation}
(744) \quad \mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})} \cong \bigoplus_{l=1}^n \mathcal{O}_{\mathbb{P}^{1}}(p \cdot w_l).
\end{equation}

On the other hand, recall that $\mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})}$ admits a decreasing filtration $\{\mathcal{A}^j_{\text{Ker}(\nabla_\mathcal{F})}\}_{j=0}^n$ with $\text{gr}^j_\mathcal{F} := \mathcal{A}^j_{\text{Ker}(\nabla_\mathcal{F})}/\mathcal{A}^{j+1}_{\text{Ker}(\nabla_\mathcal{F})}$ ($j = 0, \ldots, n-1$). It follows from Proposition \ref{7.3.3} and the latter assertion of Corollary \ref{7.3.4} (and the condition that $(g, r) = (0, 3)$) that

\begin{equation}
(745) \quad \deg(\text{gr}^j_\mathcal{F}) < \deg(\text{gr}^{j+1}_\mathcal{F}) < \cdots < \deg(\text{gr}^0_\mathcal{F}).
\end{equation}

Write

\begin{equation}
(746) \quad \xi' : \mathcal{O}_{\mathbb{P}^{1}}(p \cdot w_n) \hookrightarrow \mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})} \quad \text{(resp.,} \quad \xi'' : \mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^{1}}(p \cdot w_1))
\end{equation}

for the inclusion into the $n$-th factor (resp., the projection to the 1-st factor) upon identifying $\mathcal{A}_{\text{Ker}(\nabla_\mathcal{F})}$ with $\bigoplus_{l=1}^n \mathcal{O}_{\mathbb{P}^{1}}(p \cdot w_l)$ by means of the decomposition
Also, write
\[ n' := \max\{ j \mid \text{Im}(\xi') \subseteq A'_{Ker(\nabla_F)} \} \]
(resp., \( n'' := \max\{ j \mid \xi'(A'_{Ker(\nabla_F)}) \neq 0 \} \)).

Then, \( \xi' \) (resp., \( \xi'' \)) induces a nonzero morphism
\[
\xi' : O_{\mathbb{P}^1}(p \cdot w_n) \to A'_n \text{Ker}(\nabla_F) / A'_n+1 \text{Ker}(\nabla_F) \quad (=: \text{gr}^{n'}_F)
\]
(resp., \( \xi'' : (\text{gr}^{n''}_F :=) A''_{Ker(\nabla_F)} / A''_{Ker(\nabla_F)} \to O_{\mathbb{P}^1}(p \cdot w_1) \))
between line bundles (on the smooth curve \( \mathbb{P}^1 \) over \( k \)). In particular \( \xi' \) and \( \xi'' \) are injective, and hence, we have
\[
p \cdot w_n = \deg(O_{\mathbb{P}^1}(p \cdot w_n)) \leq \deg(\text{gr}^{n'}_F) \leq \deg(\text{gr}^0_F)
\]
and
\[
p \cdot w_1 = \deg(O_{\mathbb{P}^1}(p \cdot w_1)) \geq \deg(\text{gr}^{n''}_F) \geq \deg(\text{gr}^{n-1}_F),
\]
where the last inequalities in (749) and (750) respectively follow from (745).
But, it follows from the latter assertion of Corollary 7.3.4 that
\[
\deg(\text{gr}^0_F) - \deg(\text{gr}^{n-1}_F) = -(n - 1) \cdot (2 \cdot 0 - 2 + 3) + \sum_{i=1}^{r}(w_{i,n} - w_{i,1})
\]
\[
< -n + 1 + 3 \cdot p.
\]
By combining (749), (750), and (751), we obtain
\[
w_n - w_1 \leq \frac{1}{p} \cdot (\deg(\text{gr}^0_F) - \deg(\text{gr}^{n-1}_F)) < 3 - \frac{n - 1}{p}.
\]
This implies (since both \( m_n \) and \( m_1 \) are integers) that \( m_n - m_1 \leq 2 \), as desired.

Let \( \mathcal{L}' := (\mathcal{L}', \nabla_{\mathcal{L}'} ) \) be a log integrable line bundle on \( \mathbb{P}^{1\log} / k^{\log} \) satisfying that \( p_\psi'(\mathcal{L}', \nabla_{\mathcal{L}'}) = 0 \). Then, the \( (n, 1) \)-determinant data \( \mathcal{L}^{\vee} := (\mathcal{B} \otimes \mathcal{L}^{\vee}, \nabla_0 \otimes \nabla_{\mathcal{L}'\vee}^{\otimes n} ) \)
(cf. Lemma 4.9.3) satisfies the equality \( p_\psi'(\det(F \otimes \mathcal{L}'), \nabla_0 \otimes \nabla_{\mathcal{L}'\vee}^{\otimes n} ) = 0 \). Also, recall that \( \nabla_{F \otimes \mathcal{L}'} := \nabla_F \otimes \nabla_{\mathcal{L}'} \) forms a dormant \((\text{GL}_n, 1 \otimes \mathcal{L}^{\vee})\)-oper on \( \mathbb{P} / k \). As we prove in the following proposition, the vector bundle \( F_{\mathbb{P}^{1}/k^*(\text{Ker}(\nabla_{F \otimes \mathcal{L}'}))} \) satisfies a certain nice property for a suitable choice of \( \mathcal{L}' \).

**Proposition 7.7.4.**

Let \( \nabla_F \) be a dormant \((\text{GL}_n, 1 \otimes \mathcal{L}')\)-oper on \( \mathbb{P} / k \). Then, there exists a log integrable line bundle \( \mathcal{L} := (\mathcal{L}, \nabla_{\mathcal{L}}) \) on \( \mathbb{P}^{1\log} / k^{\log} \) such that the rank \( n \) vector bundle \( F_{\mathbb{P}^{1}/k^*(\text{Ker}(\nabla_{F \otimes \mathcal{L}}))} \) on \( \mathbb{P}^{1(1)}_k \) is of type \( (w_i)_{i=1}^n \) satisfying that \( w_n - w_1 \leq 1 \).
Proof. For convenience of the following discussion, we write $U_0 := U$, $\nabla_{F,0} := \nabla_F$. In the following, we shall construct, by means of the pair $(U_0, \nabla_{F,0})$, a new $(n,1)$-determinant data $U_1$, and a dormant $(\text{GL}_n, 1, U_1)$-oper $\nabla_{F,1}$ on $\mathbb{P}/k$.

Let $\mathfrak{g}_{[0]}^{(F,\nabla_F,0)} := (-m_l)_{l=1}^n$ (where $0 \leq m_l < p$ if $l_1 < l_2$), and consider the log integrable line bundle

\begin{equation}
\mathfrak{g}_1 := (O_{p^1}((p - m_n) \cdot [0]), \nabla_{(p-m)_n} \cdot [0])
\end{equation}

on $\mathbb{P}^{1\log}/k^{\log}$, which has vanishing $p$-curvature. As we observed preceding Proposition 7.7.4 the $(n,1)$-determinant data

\begin{equation}
U_1 := (B \otimes O_{p^1}((p - m_n) \cdot [0])^v, \nabla_0 \otimes \nabla_{(p-m)_n} \cdot [0])
\end{equation}

satisfies the equality

\begin{equation}
p_{\psi}((\det(F((p-m)_n) \cdot [0]), \nabla_0 \otimes \nabla_{(p-m)_n} \cdot [0]) = 0
\end{equation}

(\text{where } F((p-m)_n) := F \otimes O_{p^1}((p-m)_n) \cdot [0]). Also,

\begin{equation}
\nabla_{F,1} := \nabla_{F,0} \otimes \nabla_{(p-m)_n} \cdot [0]
\end{equation}

forms a dormant $(\text{GL}_n, 1, U_1)$-oper on $\mathbb{P}/k$. Moreover, the inclusion $F \hookrightarrow F((p-m)_n) \cdot [0]$ yields an $O_{p^1}$-linear injection

\begin{equation}
f_1 : F_{p^1/k^s}(\text{Ker}(\nabla_{F,0})) \hookrightarrow F_{p^1/k^s}(\text{Ker}(\nabla_{F,1}))
\end{equation}

between rank $n$ vector bundles. Thus, we have obtained (from $(U_0, \nabla_{F,0})$) a collection of data

\begin{equation}
U_1, \nabla_{F,1}, f_1.
\end{equation}

Here, we shall compare the degree of the vector bundle $F_{p^1/k^s}(\text{Ker}(\nabla_{F,0}))$ with that of the vector bundle $F_{p^1/k^s}(\text{Ker}(\nabla_{F,1}))$. Since the restriction of $\mathfrak{g}_1$ to the open subscheme $\mathbb{A}^1 := \mathbb{P}^1 \setminus \text{Im}([0])$ of $\mathbb{P}^1$ is isomorphic to the trivial log integrable line bundle $(O_{\mathbb{A}^1}, d_{\mathbb{A}^1}/k)$, the natural inclusion

\begin{equation}
(\text{Ker}(\nabla_{F,0}) \hookrightarrow \text{Ker}(\nabla_{F,0}) \otimes \text{Ker}(\nabla_{(p-m)_n} \cdot [0]) \hookrightarrow \text{Ker}(\nabla_{F,1})
\end{equation}

induces an isomorphism

\begin{equation}
\text{Ker}(\nabla_{F,0})|_{\mathbb{A}^1} \cong \text{Ker}(\nabla_{F,1})|_{\mathbb{A}^1}
\end{equation}

of $O_{\mathbb{A}^1}$-modules.

Now, let us fix a local function $t \in O_{p^1}$ defining $[0]$, and

\begin{equation}
\vartheta_{(F,\nabla_{F,0})} : (F, \nabla_{F,0})^\wedge \cong \bigoplus_{l=1}^n (\partial, \nabla_{m_l})
\end{equation}

be the isomorphism where the marked point $\sigma_i$ is taken to be $[0]$ and the $(F, \nabla_{F,0})$ is taken to be the log integrable vector bundle $(F, \nabla_{F,0})$ under
consideration. By taking the respective sheaves of horizontal sections, one obtains from (761) an isomorphism

\[ \vartheta_{[0]}^{(F,\nabla_{\mathcal{F},0})} : \text{Ker}(\nabla_{\mathcal{F},0})^\wedge \xrightarrow{\sim} \bigoplus_{l=1}^{n} t^{m_l} \cdot \mathcal{O}_{\text{Spf}(k[[t^p]])} \]

of \( \mathcal{O}_{\text{Spf}(k[[t^p]])} \)-modules. Also, the isomorphism (761) tensored with \( \mathcal{O} \) gives an isomorphism

\[ \vartheta_{[0]}^{(F(p-m_n):[0]),\nabla_{\mathcal{F},1})} : (F((p-m_n) \cdot [0]), \nabla_{\mathcal{F},1})^\wedge \xrightarrow{\sim} \bigoplus_{l=1}^{n} O \cdot \hat{\nabla}_{p-m_n+m_l} \]

Here, note that \( \text{Ker}(\hat{\nabla}_p) = \mathcal{O}_{\text{Spf}(k[[t^p]])} \) (by Lemma 7.3.2). Thus, the isomorphism (763) induces an isomorphism between the respective sheaves of horizontal sections

\[ \vartheta_{[0]}^{(F(p-m_n):[0]),\nabla_{\mathcal{F},1})} : (F((p-m_n) \cdot [0]), \nabla_{\mathcal{F},1})^\wedge \xrightarrow{\sim} \bigoplus_{l=1}^{n} t^{p-m_n+m_l} \cdot \mathcal{O}_{\text{Spf}(k[[t^p]])} \]

where \( m'_l = m_l \) if \( l = 1, \ldots, n-1 \) and \( m'_n = m_l - p \) if \( l = n \). The \( t^p \)-adic completion of the inclusion (759) may be identified, via the isomorphisms (762) and (764) (and the isomorphism \( \text{Ker}(\nabla_{(p-m_n):[0]})^\wedge \xrightarrow{\sim} t^{p-m_n} \cdot \mathcal{O}_{\text{Spf}(k[[t^p]])} \)), with the morphism

\[ \bigoplus_{l=1}^{n} t^{m_l} \cdot \mathcal{O}_{\text{Spf}(k[[t^p]])} \otimes t^{p-m_n} \cdot \mathcal{O}_{\text{Spf}(k[[t^p]])} \xrightarrow{\oplus} \bigoplus_{l=1}^{n} t^{p-m_n+m'_l} \cdot \mathcal{O}_{\text{Spf}(k[[t^p]])} \]

\[ \sum_{s=1}^{S} (a_{s,l})^{n}_{l=1} \otimes b_s \quad \mapsto \quad \sum_{s=1}^{S} (a_{s,l} \otimes b_s)^{n}_{l=1} \]

The cokernel of this morphism is an \( \mathcal{O}_{\text{Spf}(k[[t^p]])} \)-module of degree one, which is obtained as the quotient of the \( n \)-th component in the direct sum \( \bigoplus_{l=1}^{n} t^{p-m_n+m'_l} \cdot \mathcal{O}_{\text{Spf}(k[[t^p]])} \). Thus, by combining this fact and the isomorphism (760), we conclude that

\[ \text{deg}(F_{p^1/k^*}(\text{Ker}(\nabla_{\mathcal{F},1}))) = \text{deg}(F_{p^1/k^*}(\text{Ker}(\nabla_{\mathcal{F},0}))) + 1. \]

By iterating the above procedure for constructing \((\mathbb{U}_1, \nabla_{\mathcal{F},1})\) from \((\mathbb{U}_0, \nabla_{\mathcal{F},0})\), we obtain a set of the pairs \( \{(\mathbb{U}_l, \nabla_{\mathcal{F},l})\}_{l \geq 0} \), where each \( \mathbb{U}_l \) is a \((n, 1)\)-determinant data for \( \mathbb{P}^{1\log} \) over \( k^{1\log} \) and \( \nabla_{\mathcal{F},l} \) is a dormant \((\text{GL}_m, 1, U_l)\)-oper on \( \mathbb{P}/k \). Moreover, by applying, to each \((\mathbb{U}_l, \nabla_{\mathcal{F},l})\), the same procedure as the above procedure for constructing \( f_1 \) from \((\mathbb{U}_0, \nabla_{\mathcal{F},0})\), we obtain a sequence of \( \mathcal{O}_{\mathbb{P}^{1(1)}/k} \)-linear injections

\[ F_{p^1/k^*}(\text{Ker}(\nabla_{\mathcal{F},0})) \xrightarrow{f_1} F_{p^1/k^*}(\text{Ker}(\nabla_{\mathcal{F},1})) \xrightarrow{f_2} F_{p^1/k^*}(\text{Ker}(\nabla_{\mathcal{F},2})) \xrightarrow{f_3} \cdots. \]

Hence, it follows from Lemma 7.7.2 and Proposition 7.7.3 that there exists \( l_0 \geq 0 \) such that \( F_{p^1/k^*}(\text{Ker}(\nabla_{\mathcal{F},l_0})) \) is of type \( (w_l)_{l=1}^{n} \) (for some \( w_l \in \mathbb{Z} \)) satisfying
that $w_n - w_1 \leq 1$. That is, the tensor product $\mathcal{L} := \bigotimes_{l=1}^n \mathcal{L}_l$ becomes the required log integrable line bundle. This completes the proof of Proposition 7.7.4. \hfill \Box

**Corollary 7.7.5.**

Let $\nabla_F$ be a dormant $(\text{GL}_n, 1, \text{pr}^\ast_{p^1}(\mathbb{U}))$-oper on $\mathcal{P}_{/k}$, where $\text{pr}^\ast_{p^1}(\mathbb{U})$ denotes the $(n, 1)$-determinant data for $\mathbb{P}^{\text{log}}_{\mathbf{U}}$ over $k^{\text{log}}$ obtained by pulling back the data $\mathbb{U}$ via $\text{pr}_{p^1}$. Write $\nabla_F$ for the $(\text{GL}_n, 1, \mathbb{U})$-oper on $\mathcal{P}_{/k}$ obtained as the pull-back of $\nabla_F$ via $\text{in}_{p^1}$. Then, $\nabla_F$ is the trivial deformation of $\nabla_F$, i.e., isomorphic to $\text{pr}_{p^1}^\ast(\nabla_F)$.

**Proof.** If $\mathcal{L}$ is a log integrable line bundle on $\mathbb{P}^{\text{log}}_{/k}$, then we have obtained (cf. Corollary 4.11.4) an isomorphism

$$
\Gamma \hat{\otimes}_{\mathbb{U} \otimes \mathcal{L}'_{\mathcal{P}_{/k}}} : \mathcal{D} \hat{\otimes}_{\text{GL}_n, 1, \mathcal{U}_{/k}} \otimes_{\text{GL}_n, 1, \mathbb{U} \otimes \mathcal{L}'_{\mathcal{P}_{/k}}} \mathcal{D}
$$

of $k$-schemes. The trivial deformation of the $(\text{GL}_n, 1, \mathbb{U})$-oper classified by a $k$-rational point $q \in \mathcal{D} \hat{\otimes}_{\text{GL}_n, 1, \mathcal{U}_{/k}}(k)$ corresponds, via this isomorphism, to the trivial deformation of the $(\text{GL}_n, 1, \mathbb{U})$-oper classified by $\Gamma \hat{\otimes}_{\mathbb{U} \otimes \mathcal{L}'_{\mathcal{P}_{/k}}} \circ q \in \mathcal{D} \hat{\otimes}_{\text{GL}_n, 1, \mathbb{U} \otimes \mathcal{L}'_{\mathcal{P}_{/k}}}(k)$. Hence, by Proposition 7.7.4, we may assume (after possibly replacing $\mathbb{U}$ with $\mathbb{U} \otimes \mathcal{L}'_{\mathcal{P}_{/k}}$ for some log integrable line bundle $\mathcal{L}$ on $\mathbb{P}^{\text{log}}_{/k}$) that the rank $n$ vector bundle $F_{p^1/k^s}(\text{Ker}(\nabla_F))$ on $\mathbb{P}^{(1)}_{1}$ is of type $(w_i)_{i=1}^n$ satisfying that $w_n - w_1 \leq 1$.

Let us recall the canonical bijective correspondence between the underlying set of $H^1(\mathbb{P}^{(1)}_{1}, \mathcal{E}_{\text{nd}}_{\mathcal{O}_{\mathbb{P}^{(1)}_{1}}(\mathbb{P}^{(1)}_{1/k^s}(\text{Ker}(\nabla_F))))$ and the set of isomorphism classes of deformations over $\mathbb{P}^{(1)}_{1,k}$ of the vector bundle $F_{p^1/k^s}(\text{Ker}(\nabla_F))$. If we fix an isomorphism $F_{p^1/k^s}(\text{Ker}(\nabla_F)) \cong \bigoplus_{l=1}^n \mathcal{O}_{\mathbb{P}^{(1)}_{1}}(w_l)$, then we have

$$
H^1(\mathbb{P}^{(1)}_{1}, \mathcal{E}_{\text{nd}}_{\mathcal{O}_{\mathbb{P}^{(1)}_{1}}(\mathbb{P}^{(1)}_{1/k^s}(\text{Ker}(\nabla_F))))
\cong H^1(\mathbb{P}^{(1)}_{1}, \mathcal{E}_{\text{nd}}_{\mathcal{O}_{\mathbb{P}^{(1)}_{1}}(\bigoplus_{l=1}^n \mathcal{O}_{\mathbb{P}^{(1)}_{1}}(w_l)))
\cong H^1(\mathbb{P}^{(1)}_{1}, \bigoplus_{l', l=1}^n \mathcal{O}_{\mathbb{P}^{(1)}_{1}}(w_l - w_{l'}))
\cong \bigoplus_{l, l'=1}^n H^0(\mathbb{P}^{(1)}_{1}, \mathcal{O}_{\mathbb{P}^{(1)}_{1}}(-w_l + w_{l'} - 2))
\cong 0,
$$

(770)

where the third isomorphism arises from Serre duality and $\Omega_{\mathbb{P}^{(1)}_{1/k}} \cong \mathcal{O}_{\mathbb{P}^{(1)}_{1}}(-2)$, and the fourth isomorphism follows from the assumption that $(\max\{w_{l'} - \)
This implies that $F_{p^1/k,*}(\text{Ker}(\nabla_F))$ is the trivial deformation of $F_{p^1/k,*}(\text{Ker}(\nabla_F))$. Thus, the assertion follows from Corollary 7.6.2.

7.8. Consequently, one may obtain the following Proposition 7.8.1 and then, Theorem 7.8.2.

**Proposition 7.8.1.**

Let $n$ be a positive integer satisfying that $n < p$, $k$ an algebraically closed field of characteristic $p > 0$, $X/k := (X/k, \{\sigma_i\}_{i=1}^r)$ a totally degenerate curve (cf. Definition 6.3.1) over $k$, and moreover, $\hbar \in k^\times$, $\rho \in \mathfrak{c}^r_{stn}(\mathbb{F}_p)$ (where we take $\rho = \emptyset$ if $r = 0$). Then, any dormant $\text{(sl}_n, \hbar)\text{-oper}$ on $X/k$ over $k^e := k[e]/e^2$ of radii $\rho$ is the trivial deformation of the dormant $\text{(sl}_n, \hbar)\text{-oper}$ on $X/k$ obtained as the pull-back of $E^{\bullet}$ via $\iota_{X/k}$. Equivalently, the finite $k$-scheme $\mathcal{O}_{p_{\text{sl}_n, h, \rho, X/k}}^{\text{Zax...}}$ (cf. Theorem 3.12.3) is either empty or a disjoint union of finite copies of $\text{Spec}(k)$.

**Proof.** We shall apply Theorem 6.2.2 to the case where the clutching data $D = (\Gamma, \{(g_j, r_j)\}_{j=1}^m, \{\lambda_j\}_{j=1}^m)$ corresponds, in the manner of the discussion following Definition 6.3.1, to the totally degenerate curve $X/k$. In particular, it is satisfied that $(g_j, r_j) = (0, 3)$ for any $j$. Hence, the assertion reduces to the case where the totally degenerate curve $X/k$ is taken to be $\mathbb{P}^1/k$. Moreover, by Proposition 3.9.1, it suffices to consider the case where $\hbar = 1$. Thus, the assertion follows from Corollary 4.11.3 and Corollary 7.7.5.

**Theorem 7.8.2 (Generic étaleness of $\mathcal{O}_{p_{\text{sl}_n, h, h*, \rho, g, r}}^{\text{Zax...}}/\mathbb{M}_{g,r}$).**

Let $p$ be a prime, $n$ a positive integer satisfying that $n < p$, $k$ a perfect field of characteristic $p$, $\hbar \in k^\times$, and $\rho \in \mathfrak{c}^r_{stn}(\mathbb{F}_p)$ (where $\rho := \emptyset$ if $r = 0$). Then, the finite (relative) $\mathbb{M}_{g,r}$-scheme $\mathcal{O}_{p_{\text{sl}_n, h, h*, \rho, g, r}}^{\text{Zax...}}$ (cf. Theorem 3.12.3) is, if it is not empty, étale over the points of $\mathbb{M}_{g,r}$ classifying totally degenerate curves. In particular, $\mathcal{O}_{p_{\text{sl}_n, h, h*, \rho, g, r}}^{\text{Zax...}}$ is (since $\mathbb{M}_{g,r}$ is irreducible) generically étale over $\mathbb{M}_{g,r}$ (i.e., any irreducible component that dominates $\mathbb{M}_{g,r}$ admits a dense open subscheme which is étale over $\mathbb{M}_{g,r}$).

**Proof.** The assertion follows from Proposition 6.3.2 and Proposition 7.8.1.

**Remark 7.8.3.**

If $g = \text{sl}_2$ (hence $p$ is an odd prime), then a stronger result than Theorem 7.8.2 asserted above was proved by S. Mochizuki in the work of $p$-adic Teichmüller theory (cf. [53]). In fact, S. Mochizuki have shown (cf. [53], Chap. II, §2.8, Theorem 2.8 and its proof) that $\mathcal{O}_{p_{\text{sl}_2, h, h*, \rho, g, r}}^{\text{Zax...}}$ is, if it is not empty, represented...
by a geometrically connected, smooth, and proper Deligne-Mumford stack over $k$ of dimension $3 - 3 + r$, and the natural projection $\mathcal{O}_p^{\text{zzz}}_{\mathfrak{sl}_n,h,h^*p,g,r} \to \overline{M}_{g,r}$ is (in addition to the fact asserted in Theorem 7.8.2) faithfully flat.

In particular, it makes sense to speak of the generic degree

$$\deg(\mathcal{O}_p^{\text{zzz}}_{\mathfrak{sl}_n,h,h^*p,g,r}/\overline{M}_{g,r}) \in \mathbb{Z}_{\geq 0}$$

of $\mathcal{O}_p^{\text{zzz}}_{\mathfrak{sl}_n,h,h^*p,g,r}$ over $\overline{M}_{g,r}$, as well as the dormant operatic fusion ring $\mathfrak{F}_p^{\text{zzz}}_{\mathfrak{sl}_n,p}$ of $\mathfrak{sl}_n$ of level $p$. In light of the above Theorem 7.8.2, it will be natural to ask whether the following conjecture holds or not.

**Conjecture 7.8.4.**

Let $k$ be a perfect field of of characteristic $p > 0$, $\mathfrak{g}$ a semisimple Lie algebra of adjoint type over $k$ satisfying the condition $(\text{Char})_p$ (cf. § 2.1), and $h \in k^*$, $\rho \in \mathfrak{c}_\mathfrak{g}^{\times r}(\mathbb{F}_p)$ (where we take $\rho = \emptyset$ if $r = 0$). Then, the finite (relative) $\overline{M}_{g,r}$-scheme $\mathcal{O}_p^{\text{zzz}}_{\mathfrak{g},h^*p,g,r}$ is, if it is not empty, equidimensional of dimension $3g - 3 + r$ (hence any irreducible component dominates $\overline{M}_{g,r}$), and generically étale over $\overline{M}_{g,r}$.

**8. Relations with Quot-schemes**

In this last section, we shall give an affirmative answer to Joshi’s conjecture exhibited in Introduction. As we explained in the end of § 6.3, it suffices to calculate explicitly the value of the generic degree $\deg(\mathcal{O}_p^{\text{zzz}}_{\mathfrak{sl}_n,1,g,0}/\overline{M}_{g,0})$ of the moduli stack $\mathcal{O}_p^{\text{zzz}}_{\mathfrak{sl}_n,1,g,0}$ over $\overline{M}_{g,0}$ (since $\mathcal{O}_p^{\text{zzz}}_{\mathfrak{sl}_n,1,g,0}/\overline{M}_{g,0}$ is finite and generically étale by Theorem 7.8.2). To this end, it will be necessary to relate $\mathcal{O}_p^{\text{zzz}}_{\mathfrak{sl}_n,1,g,0}$ to certain Quot-schemes (cf. Proposition 8.3.3). Consequently, by applying a formula given by Y. Holla for computing the degree of these Quot-schemes, we give, in the last subsection (cf. Theorem 8.7.1), an explicit computation of $\deg(\mathcal{O}_p^{\text{zzz}}_{\mathfrak{sl}_n,1,g,0}/\overline{M}_{g,0})$.

**8.1.** Here, to prepare for the discussion in the following, we introduce notions for Quot-schemes in arbitrary characteristic.

Let $T$ be a noetherian scheme, $Y$ a geometrically connected, proper, and smooth curve of genus $g > 1$ over $T$ (hence $\deg(\Omega_{Y/T}) = \deg(\Omega_{Y/T}) = 0$).
$$2g - 2$$ and $$\mathcal{E}$$ a vector bundle on $$Y$$. Denote by

$$(772) \quad \text{Quot}_{\mathcal{E}/Y/T}^{n,d} : \mathcal{S}ch_{/T} \to \text{Set}$$

the $$\text{Set}$$-valued functor on $$\mathcal{S}ch_{/T}$$ which to any morphism $$t' : T' \to T$$ associates the set of isomorphism classes of injective morphisms of coherent $$\mathcal{O}_{Y \times_{T} T'}$$-modules

$$(773) \quad \eta : \mathcal{F} \to (\text{id}_{Y} \times t')^{*}(\mathcal{E})$$

such that the cokernel $$\text{Coker}(\eta)$$ of $$\eta$$ is flat over $$T'$$ (which, since $$Y/T$$ is smooth of relative dimension 1, implies that $$\mathcal{F}$$ is locally free), and $$\mathcal{F}$$ is of rank $$n$$ and degree $$d$$. It is known (cf. [18], Chap. 5, §5.5, Theorem 5.14) that $$\text{Quot}_{\mathcal{E}/Y/T}^{n,d}$$ may be represented by a proper scheme over $$T$$.

Suppose further that $$T = \text{Spec}(k)$$ for some field $$k$$ and we are given an $$\mathcal{O}_{Y}$$-linear morphism $$\eta_{0} : \mathcal{F} \to \mathcal{E}$$ such that $$\mathcal{F}$$ is of rank $$n$$ and degree $$d$$. Then, the tangent space $$\mathcal{T}_{[\eta_{0}]} \text{Quot}_{\mathcal{E}/Y/k}^{n,d}$$ at the point $$[\eta_{0}] \in \text{Quot}_{\mathcal{E}/Y/k}^{n,d}(k)$$ corresponding to $$\eta_{0}$$ may be naturally identified with the $$k$$-vector space $$\text{Hom}_{\mathcal{O}_{Y}}(\mathcal{F}, \text{Coker}(\eta_{0}))$$ (cf. [64], Proposition 4.4.4). Denote by

$$(774) \quad \text{def}_{\eta_{0}} : \mathcal{T}_{[\eta_{0}]} \text{Quot}_{\mathcal{E}/Y/T}^{n,d} \to \text{Hom}_{\mathcal{O}_{Y}}(\mathcal{F}, \text{Coker}(\eta_{0}))$$

this canonical isomorphism of $$k$$-vector spaces.

8.2. Let $$S$$ be a scheme of characteristic $$p > 0$$ and $$n$$ a positive integer satisfying that $$n < p$$. Fix a smooth pointed stable curve $$\mathcal{X}/S := (f : X \to S, \emptyset)$$ of type $$(g, 0)$$, i.e., a geometrically connected, proper, and smooth curve $$X/S$$ of genus $$g > 1$$. (Hence, both the log structures of $$X_{\text{log}}$$ and $$S_{\text{log}}$$ obtained in the manner of §1.5 are trivial.) Suppose that we are given a line bundle $$\mathcal{B}$$ on $$X$$ together with an isomorphism

$$(775) \quad \text{triv}_{\mathcal{B}} : (\text{det}(\mathcal{D}_{1, X/S}^{\leq n} \otimes \mathcal{B}^{\vee}) \cong) \quad \mathcal{T}_{X/S}^{\otimes n(n - 1)} \otimes (\mathcal{B}^{\vee})^{\otimes n} \to \mathcal{O}_{X}.$$  

(By the discussion in Remark 4.9.2 (iii), $$\mathcal{X}/S$$ necessarily admits, at least étale locally on $$S$$, such a line bundle.) Denote by

$$(776) \quad d^{\mathcal{B}} : \text{det}(\mathcal{D}_{1, X/S}^{\leq n} \otimes \mathcal{B}^{\vee}) \to \Omega_{X/S} \otimes \text{det}(\mathcal{D}_{1, X/S}^{\leq n} \otimes \mathcal{B}^{\vee})$$

the $$S$$-(log )connection on $$\text{det}(\mathcal{D}_{1, X/S}^{\leq n} \otimes \mathcal{B}^{\vee})$$ corresponding, via the isomorphism (775), to the universal derivation $$d_{X/S} : \mathcal{O}_{X} \to \Omega_{X/S}$$. Then, the pair

$$(777) \quad \mathcal{U} := (\mathcal{B}, d^{\mathcal{B}})$$

forms an $$(n, 1)$$-determinant data for $$X$$ over $$S$$ (cf. Definition 4.9.1 (i)). Here, we shall write

$$(778) \quad \mathcal{B}^{\vee} := \mathcal{T}_{X/S}^{\otimes (n - 1)} \otimes \mathcal{B}^{\vee}$$
for simplicity. Note that the isomorphism \( \text{triv}_B \) (cf. (775)) yields an isomorphism
\[
\mathcal{B}^2 \otimes n \cong \mathcal{T}_{X/S}^{n(n-1)/2}.
\]

Now, let us consider the Quot-scheme
\[
\text{Quot}^{n,0}_{F_{X/S}}(B^\vee)/X_S^{(1)}/S
\]
discussed above in the case where the data \((T, Y, E, n, d)\) is taken to be \((S, X_S^{(1)}, F_{X/S}(B^\vee), n, 0)\). Denote by
\[
\tilde{\eta} : \tilde{F} \to (\text{id}_{X_S^{(1)}} \times \tau)^*(F_{X/S}(B^\vee))
\]
the tautological injection of sheaves on \(X_S^{(1)} \times_S \text{Quot}^{n,0}_{F_{X/S}}(B^\vee)/X_S^{(1)}/S\), where \(\tau\) denotes the structure morphism \(\text{Quot}^{n,0}_{F_{X/S}}(B^\vee)/X_S^{(1)}/S \to S\) of the \(S\)-scheme \(\text{Quot}^{n,0}_{F_{X/S}}(B^\vee)/X_S^{(1)}/S\). The determinant bundle \(\det(\tilde{F})\) of \(\tilde{F}\) determines a classifying morphism
\[
\det : \text{Quot}^{n,0}_{F_{X/S}}(B^\vee)/X_S^{(1)}/S \to \text{Pic}^0_{X_S^{(1)}/S}
\]
to the relative Picard scheme \(\text{Pic}^0_{X_S^{(1)}/S}\). (Here, recall (cf. Remark 4.9.2 (iii)) that \(\text{Pic}^0_{X_S^{(1)}/S}\) classifies the set of equivalence classes of degree 0 line bundles on \(X_S^{(1)}/S\)). Thus, one obtains the closed subscheme
\[
\text{Quot}^{n,0,\text{O}}_{F_{X/S}}(B^\vee)/X_S^{(1)}/S
\]
of \(\text{Quot}^{n,0}_{F_{X/S}}(B^\vee)/X_S^{(1)}/S\) defined to be the scheme-theoretic inverse image, via \(\text{det}\), of the identity section of \(\text{Pic}^0_{X_S^{(1)}/S}\).

8.3. Next, we discuss a certain relationship between \(\text{Quot}^{n,0,\text{O}}_{F_{X/S}}(B^\vee)/X_S^{(1)}/S\) and \(\mathfrak{D}_p^{\text{GL}_n,1,\mathbb{U},X_S}/S\) (cf. (436)), where \(\mathbb{U}\) is as above.

Let us consider the rank \(p\) vector bundle
\[
\mathcal{A}_{B^\vee} := F^*_X(F_{X/S}(B^\vee))
\]
on \(X\) (cf. (616)), which admits the canonical \(S\)-(log)connection \(\nabla_{\text{can}}^{F_{X/S}}(B^\vee) : \mathcal{A}_{B^\vee} \to \Omega_{X/S} \otimes \mathcal{A}_{B^\vee}\) (cf. §3.3). By using this \(S\)-connection, one may define a
Let $\nabla_{X/S, (B^\gamma)}^\can : \mathcal{D}_{1, X/S}^{<\infty} \otimes \mathcal{A}_{B^\gamma} \to \mathcal{A}_{B^\gamma}$ be the structure of left $\mathcal{D}_{1, X/S}^{<\infty}$-module on $\mathcal{A}_{B^\gamma}$ corresponding, via the bijective correspondence asserted in §4.4, to $\nabla_{X/S, (B^\gamma)}^\can$. Consider the composite
\begin{equation}
(789) \quad \gamma : \mathcal{D}_{1, X/S}^{p} \otimes \mathcal{A}_{B^\gamma}^{p-1} \hookrightarrow \mathcal{D}_{1, X/S}^{<\infty} \otimes \mathcal{A}_{B^\gamma} \xrightarrow{\nabla_{X/S, (B^\gamma)}^\can} \mathcal{A}_{B^\gamma},
\end{equation}
where the first arrow arises from the inclusions $\mathcal{D}_{1, X/S}^{p} \hookrightarrow \mathcal{D}_{1, X/S}^{<\infty}$ and $\mathcal{A}_{B^\gamma}^{p-1} \hookrightarrow \mathcal{A}_{B^\gamma}$. It follows from Proposition 8.3.1 that this composite $\gamma$ is an isomorphism of $\mathcal{O}_X$-modules that is compatible with the respective filtrations $\{\mathcal{D}_{1, X/S}^{<\infty} \otimes \mathcal{A}_{B^\gamma}^{p-1}\}_{j=0}^{p}$ and $\{\mathcal{A}_{B^\gamma}^{p-1}\}_{j=0}^{p}$. Hence, by taking quotients, we obtain an isomorphism
\begin{equation}
(790) \quad (\mathcal{D}_{1, X/S}^{p} / \mathcal{D}_{1, X/S}^{<\infty}) \otimes \mathcal{A}_{B^\gamma}^{p-1} \sim \mathcal{A}_{B^\gamma}^{p} / \mathcal{A}_{B^\gamma}^{n}.
\end{equation}
On the other hand, one verifies that there exists a canonical isomorphism
\begin{equation}
(791) \quad (\mathcal{D}_{1, X/S}^{p} / \mathcal{D}_{1, X/S}^{<\infty}) \otimes \mathcal{A}_{B^\gamma}^{p-1} \sim \mathcal{D}_{1, X/S}^{<\infty} \otimes (\mathcal{T}_{X/S}^{\otimes (p-n)} \otimes \mathcal{A}_{B^\gamma}^{p-1}).
\end{equation}
Moreover, consider the isomorphism $\mathcal{T}_{X/S}^{\otimes (p-n)} \otimes \mathcal{A}_{B^x}^{p-1} \rightarrow \mathcal{B}^x$ arising from the isomorphism (788). Thus, by composing this isomorphism, the isomorphism (790), and the isomorphism (791), we obtain an isomorphism

$$\Phi : \mathcal{D}_{1, X/S}^{<n} \otimes \mathcal{B}^x \rightarrow \mathcal{A}_{B^x} / \mathcal{A}_{B^x}^n$$

of $\mathcal{O}_X$-modules. In the following, we shall write

$$\mathcal{F} := \mathcal{D}_{1, X/S}^{<n} \otimes \mathcal{B}^x$$

and

$$\mathcal{F}^j := \mathcal{D}_{1, X/S}^{<n-j} \otimes \mathcal{B}^x \quad (j = 0, \ldots, n)$$

for simplicity.

**Proposition 8.3.2.**

Let $\eta : \mathcal{V} \rightarrow F_{X/S*}(\mathcal{B}^x)$ be the injection classified by an $S$-rational point of $\text{Quot}_{F_{X/S*}(\mathcal{B}^x), X}^{n, 0}(\mathcal{B}^x) / S$.

(i) The composite

$$\tilde{\eta} : F_{X/S*}(\mathcal{V}) \rightarrow F_{X/S*}(\mathcal{V}) \rightarrow \mathcal{A}_{B^x} \rightarrow \mathcal{A}_{B^x} / \mathcal{A}_{B^x}^n$$

of the pull-back

$$F_{X/S*}(\eta) : F_{X/S*}(\mathcal{V}) \rightarrow F_{X/S*}(F_{X/S*}(\mathcal{B}^x)) (=: \mathcal{A}_{B^x})$$

of $\eta$ and the natural surjection $\mathcal{A}_{B^x} \rightarrow \mathcal{A}_{B^x} / \mathcal{A}_{B^x}^n$ is an isomorphism of $\mathcal{O}_X$-modules.

(ii) Suppose further that $\eta$ corresponds to an $S$-rational point of the closed subscheme $\text{Quot}_{F_{X/S*}(\mathcal{B}^x), X}^{n, 0, \mathcal{O}}(\mathcal{B}^x) / S$ (cf. (783)). Denote by

$$\nabla_{\mathcal{F}}^{\text{can}, F} : \mathcal{F} \rightarrow \Omega_{X/S} \otimes \mathcal{F}$$

the $S$-(log )connection on $\mathcal{F}$ corresponding, via the composite isomorphism $\tilde{\eta}^{-1} \circ \Phi : \mathcal{F} \rightarrow F_{X/S*}(\mathcal{V})$ (cf. (792)), to the canonical $S$-(log )connection $\nabla_{\mathcal{V}}^{\text{can}, F} : F_{X/S*}(\mathcal{V}) \rightarrow \Omega_{X/S} \otimes F_{X/S*}(\mathcal{V})$ on $F_{X/S*}(\mathcal{V})$. Then, $\nabla_{\mathcal{V}}^{\text{can}, F}$ forms a dormant $(\text{GL}_n, 1, U)$-oper on $\mathcal{X}_S$ (cf. Definition 4.9.4).

**Proof.** First, we consider assertion (i). Since $F_{X/S*}(\mathcal{V})$ and $\mathcal{A}_{B^x} / \mathcal{A}_{B^x}^n$ are flat over $S$, it suffices, by considering the various fibers over $S$, to verify the case where $S = \text{Spec}(k)$ for a field $k$. If we write

$$\text{gr}^j_{F_{X/S*}(\mathcal{V})} := F_{X/S*}(\eta)^{-1}(\mathcal{A}_{B^x}^j) / F_{X/S*}(\eta)^{-1}(\mathcal{A}_{B^x}^{j+1})$$

($j = 0, \ldots, p - 1$), then it follows immediately from the definitions that the coherent $\mathcal{O}_X$-module $\text{gr}^j_{F_{X/S*}(\mathcal{V})}$ admits a natural embedding

$$\text{gr}^j(F_{X/S*}(\eta)) : \text{gr}^j_{F_{X/S*}(\mathcal{V})} \hookrightarrow \mathcal{A}_{B^x}^j / \mathcal{A}_{B^x}^{j+1}$$
into the subquotient \( \mathcal{A}_{B^\vee}^j / \mathcal{A}_{B^\vee}^{j+1} \) of \( \mathcal{A}_{B^\vee} \). Since this subquotient is a line bundle (over the smooth curve \( X \) over \( k \)), one verifies that \( \text{gr}_j^j F_{X/S}(\mathcal{V}) \) is either trivial or a line bundle. In particular, since \( F_{X/S}^*(\mathcal{V}) \) is of rank \( n \), the cardinality of the set \( I := \{ j \mid \text{gr}_j^j F_{X/S}(\mathcal{V}) \neq 0 \} \) is exactly \( n \). Next, let us observe that the pull-back \( F_{X/S}^*(\eta) \) of \( \eta \) via \( F_{X/S} \) is compatible with the respective connections \( \nabla_{\mathcal{V}}^\text{can} \) and \( \nabla_{F_{X/S}^*(\mathcal{V})}^\text{can} \). Thus, it follows from Proposition 8.3.1 that \( \text{gr}_j^{j+1} F_{X/S}^*(\mathcal{V}) \neq 0 \) implies \( \text{gr}_j^j F_{X/S}(\mathcal{V}) \neq 0 \). But this implies that \( I = \{ 0, 1, \ldots, n-1 \} \), and hence that the composite

\[
\tilde{\eta} : F_{X/S}^*(\mathcal{V}) \xrightarrow{F_{X/S}^*(\eta)} \mathcal{A}_{B^\vee} \to \mathcal{A}_{B^\vee} / \mathcal{A}_{B^\vee}^n
\]

is an isomorphism at the generic point of \( X \). On the other hand, observe that

\[
\deg(F_{X/S}^*(\mathcal{V})) = p \cdot \deg(\mathcal{V}) = p \cdot 0 = 0
\]

and

\[
\deg(\mathcal{A}_{B^\vee} / \mathcal{A}_{B^\vee}^n) = \deg(D_{X/S}^\times \otimes \mathcal{B}^\vee) = \deg(\mathcal{O}_X) = 0,
\]

where the first equality of (801) follows from the isomorphism \( \tilde{\eta} \) and the second equality follows from the definition of \( \mathcal{B} \). Thus, by comparing the respective degrees of \( F_{X/S}^*(\mathcal{V}) \) and \( \mathcal{A}_{B^\vee} / \mathcal{A}_{B^\vee}^n \), we conclude that the composite \( \tilde{\eta} \) is an isomorphism of \( \mathcal{O}_X \)-modules. This completes the proof of assertion (i).

Next, consider assertion (ii). Let us write

\[
F_{X/S}^*(\mathcal{V})^\circ := (F_{X/S}^*(\mathcal{V}), \nabla_{\mathcal{V}}^\text{can}, \{ F_{X/S}^*(\eta)^{-1}(\mathcal{A}_{B^\vee}^j) \}_{j=0}^n).
\]

By the above discussion, the morphism \( \text{gr}_j^j F_{X/S}^*(\mathcal{V})(j = 0, \ldots, n-1) \) turns out to be an isomorphism. Moreover, since \( F_{X/S}^*(\eta) \) is compatible with the respective \( S \)-connections \( \nabla_{\mathcal{V}}^\text{can} \) and \( \nabla_{F_{X/S}^*(\mathcal{V})}^\text{can} \), \( \text{gr}_j^j F_{X/S}^*(\mathcal{V}) \) fits into the following commutative square

\[
\begin{array}{ccc}
\text{gr}_j^j F_{X/S}^*(\mathcal{V}) & \xrightarrow{\text{gr}_j^j F_{X/S}^*(\eta)} & \mathcal{A}_{B^\vee}^j / \mathcal{A}_{B^\vee}^{j+1} \\
\downarrow_{\text{ts}^j_{F_{X/S}^*(\mathcal{V})}} & & \downarrow_{\text{ts}^j_{\mathcal{A}_{B^\vee}}}
\end{array}
\]

\[
\Omega_{X/S} \otimes \text{gr}_j^{-1} F_{X/S}^*(\mathcal{V}) \xrightarrow{\text{id}_{\Omega_{X/S}} \otimes \text{gr}_j^{-1} F_{X/S}^*(\eta)} \Omega_{X/S} \otimes (\mathcal{A}_{B^\vee}^{j-1} / \mathcal{A}_{B^\vee}^j).
\]

This implies (by Proposition 8.3.1) that \( F_{X/S}^*(\mathcal{V})^\circ \) forms a \( \text{GL}_n \)-oper on \( \mathfrak{X}_{/S} \), and hence \( \nabla_{\mathcal{V}}^\text{can, F} \) turns out, via the isomorphism \( \tilde{\eta}^{-1} \circ \tilde{\tau} \), to form a \( (\text{GL}_n, 1, \mathcal{B}) \)-oper on \( \mathfrak{X}_{/S} \).

Finally, by the assumption on \( \eta \), one may choose an isomorphism \( \nu : \text{det}(\mathcal{V}) \xrightarrow{\sim} \mathcal{O}_{X^{(1)}} \). The automorphism \( F_{X/S}^*(\nu) \circ \text{det}(\tilde{\eta}^{-1} \circ \tilde{\tau}) \circ \text{triv}_{\mathcal{B}^\vee}^{-1} \) of \( \mathcal{O}_X \) (cf. (775)) for
the definition of \( \text{triv}_R \) coincides with the automorphism \( m_R \) given by multiplication by an element \( R \) of \( \Gamma(X, \mathcal{O}_X)^\times \). But, \( X/S \) is geometrically connected, proper, and smooth, so \( R \) lies in \( \Gamma(S, \mathcal{O}_S) \) (hence \( d_{X/S}(R) = 0 \)). It follows that \( m_R \) is compatible with \( d_{X/S} \). Hence, by the definition of \( d_B \) (cf. (776)), the \( S \)-connection \( d_{X/S} \) on \( \mathcal{O}_X \) corresponds to the \( S \)-connection \( d_B \) on \( \det(F) \) via the composite isomorphism

\[
(m_R \circ \text{triv}_S =) \quad F_{X/S}^*(\nu) \circ \det(\eta^{-1} \circ \overline{\tau}) : \det(F) \sim \mathcal{O}_X.
\]

On the other hand, the pull-back \( F_{X/S}^*(\nu) : F_{X/S}^*(\det(V)) \sim \mathcal{O}_X \) of \( \nu \) is compatible with the respective \( S \)-connections \( \nabla_{\det(V)}^{\text{can}} (= \det(\nabla_{\nu}^{\text{can}})) \) and \( d_{X/S} \). This implies that \( \det(\nabla_{\nu}^{\text{can},F}) \) coincides with \( d_B \).

Consequently, (since it is easily verified that \( p_\psi(F, \nabla_{\nu}^{\text{can},F}) = 0 \)) \( \nabla_{\nu}^{\text{can},F} \) is a dormant \( (\text{GL}_n, 1, \mathbb{U}) \)-oper, and this completes the proof of assertion (ii). □

By applying the above proposition, we may conclude that the moduli space \( \mathcal{O}_\mathcal{p}_{\text{GL}_n, 1, \mathbb{U}, X/S} \) is isomorphic to the Quot-scheme \( \text{Quot}^n_{\mathcal{O}_{F_{X/S}^*(\mathcal{B}^\vee)/X_S^{(1)}/S}} \) as asserted below.

**Proposition 8.3.3.**

Let \( \mathcal{X}/S \) and \( \mathcal{U} \) be as above. Then there exists an isomorphism of \( S \)-schemes

\[
\text{Quot}^n_{\mathcal{O}_{F_{X/S}^*(\mathcal{B}^\vee)/X_S^{(1)}/S}} \sim \mathcal{O}_\mathcal{p}_{\text{GL}_n, 1, \mathbb{U}, X/S}.
\]

**Proof.** The assignment

\[
[\eta : \mathcal{V} \to F_{X/S}^*(\mathcal{B}^\vee)] \mapsto \nabla_{\nu}^{\text{can},F},
\]

discussed in Proposition 8.3.2 (ii), determines a map

\[
\alpha_S : \text{Quot}^n_{\mathcal{O}_{F_{X/S}^*(\mathcal{B}^\vee)/X_S^{(1)}/S}}(S) \to \mathcal{O}_\mathcal{p}_{\text{GL}_n, 1, \mathbb{U}, X/S}(S)
\]

between the respective sets of \( S \)-rational points. By the functoriality of the construction of \( \alpha_S \) with respect to \( S \), it suffices to prove the bijectivity of \( \alpha_S \).

The injectivity of \( \alpha_S \) follows from the observation that any injection \( \eta : \mathcal{V} \to F_{X/S}^*(\mathcal{B}^\vee) \) classified by \( \text{Quot}^n_{\mathcal{O}_{F_{X/S}^*(\mathcal{B}^\vee)/X_S^{(1)}/S}}(S) \) may be identified, via applying the functor \( F_{X/S}(-) \), with the composite of the inclusion \( \text{Ker}(\nabla_{\nu}^{\text{can},F}) \hookrightarrow \mathcal{F} \) and the natural surjection \( \mathcal{F} \to \mathcal{F}/\mathcal{F}^1 (= \mathcal{B}^\vee) \).

Next, we consider the surjectivity of \( \alpha_S \). Let \( \nabla_{\mathcal{F}} \) be a dormant \( (\text{GL}_n, 1, \mathbb{U}) \)-oper on \( \mathcal{X}/S \). Consider the composite

\[
(F_{X/S}^*(\text{Ker}(\nabla_{\mathcal{F}})) =:) \quad \mathcal{A}_{\text{Ker}(\nabla_{\mathcal{F}})} \sim \mathcal{F} \to \mathcal{B}^\vee
\]
of the isomorphism $\nu_{\mathbf{t}}^{(F, \nabla_F)} : A_{Ker(\nabla)} \xrightarrow{\sim} F$ (cf. § 7.1) with the natural surjection $F \twoheadrightarrow F/F^1 (= B^\vee)$. This composite determines a morphism

\[(809) \quad \eta_{\nabla_F} : A_{Ker(\nabla_F)} \rightarrow F_{X/S}^* (F_{X/S*}(B^\vee)) (= A_{B^\vee})\]

via the adjunction relation "$F_{X/S}^* (-) \dashv F_{X/S*} (-)$" and pull-back by $F_{X/S}$.

Here, we claim that $\eta_{\nabla_F}$ is injective. Indeed, the composite

\[(810) \quad \eta_{\nabla_F} \circ (\nu_{\mathbf{t}}^{(F, \nabla_F)})^{-1} : F \rightarrow A_{B^\vee}\]

is (tautologically, by construction!) compatible with the respective natural surjections $F \rightarrow B^\vee$, $A_{B^\vee} \rightarrow B^\vee$ to $B^\vee$. Note that since $\nabla_F$ is assumed to be a $(GL_n, 1, \mathbb{U})$-oper, we have the equality

\[(811) \quad F^{j+1} = \text{Ker}(F^{j+1} \nabla_F |_{F^j} \rightarrow \Omega_{X/S} \otimes F \rightarrow \Omega_{X/S} \otimes (F/F^j))\]

of $\mathcal{O}_X$-submodules of $F$. Hence, the compatibility of the $S$-connections $\nabla_F$ and $\nabla_{F_{X/S}(B^\vee)}$ via $\eta_{\nabla_F} \circ (\nu_{\mathbf{t}}^{(F, \nabla_F)})^{-1}$ implies that $\eta_{\nabla_F} \circ (\nu_{\mathbf{t}}^{(F, \nabla_F)})^{-1}$ is also compatible with the respective subquotients on $F$ and $A_{B^\vee}$. The resulting morphisms $\text{gr}^j(\eta_{\nabla_F} \circ (\nu_{\mathbf{t}}^{(F, \nabla_F)})^{-1})(j = 0, \ldots, p - 1)$ between the respective subquotients fit into the commutative square

\[\begin{array}{ccc}
F^j / F^{j+1} & \xrightarrow{\text{gr}^j(\eta_{\nabla_F} \circ (\nu_{\mathbf{t}}^{(F, \nabla_F)})^{-1})} & A_{B^\vee}^j / A_{B^\vee}^{j+1} \\
\Omega_{X/S} \otimes (F^{j-1} / F^j) & \downarrow \text{id} \otimes \text{gr}^j(\eta_{\nabla_F} \circ (\nu_{\mathbf{t}}^{(F, \nabla_F)})^{-1}) & \Omega_{X/S} \otimes (A_{B^\vee}^{j-1} / A_{B^\vee}^j) \\
\end{array}\]

Since $\text{gr}^0(\eta_{\nabla_F} \circ (\nu_{\mathbf{t}}^{(F, \nabla_F)})^{-1}) = \text{id}_{B^\vee}$ as we proved above, it follows from induction on $j$ that the morphisms $\text{gr}^j(\eta_{\nabla_F} \circ (\nu_{\mathbf{t}}^{(F, \nabla_F)})^{-1})$ are injective, and hence, $\eta_{\nabla_F} \circ (\nu_{\mathbf{t}}^{(F, \nabla_F)})^{-1}$ (as well as $\eta_{\nabla_F}$) is injective. This completes the proof of the claim.

Moreover, by applying a similar argument to the pull-back of $\eta_{\nabla_F}$ via any base-change over $S$, one concludes that $\eta_{\nabla_F}$ is universally injective with respect to base-change over $S$. This implies that $\text{Coker}(\eta_{\nabla_F})$ is flat over $S$ (cf. [19], p. 17, Theorem 1).

Now, denote by

\[(812) \quad (\eta_{\nabla_F})^\vee : F_{X/S*}(\text{Ker}(\nabla_F)) \rightarrow F_{X/S*}(B^\vee)\]

the morphism obtained by restricting $\eta_{\nabla_F}$ to the respective subsheaves of horizontal sections in $F$ and $A_{B^\vee}$. The pull-back of $(\eta_{\nabla_F})^\vee$ via $F_{X/S}$ may be identified with $\eta_{\nabla_F}$, and $F_{X/S*}(\text{Coker}((\eta_{\nabla_F})^\vee))$ is naturally isomorphic to $\text{Coker}(\eta_{\nabla_F})$. Thus, it follows from the faithful flatness of $F_{X/S}$ that $(\eta_{\nabla_F})^\vee$ is injective, and $\text{Coker}((\eta_{\nabla_F})^\vee)$ is flat over $S$. On the other hand, since the determinant
of \((\mathcal{F}, \nabla_{\mathcal{F}})\) is trivial, \(\det(F_{X/S}((\text{Ker}(\nabla_{\mathcal{F}}))))\) is isomorphic to the trivial \(O_{X}^{(1)}\)-module \(O_{X}^{(1)}\) (by the equivalence of categories \([648]\)). Consequently, \((\eta_{\nabla_{\mathcal{F}}})^{\nabla_{\mathcal{F}}}\) determines an \(S\)-rational point of \(\text{Quot}^{n,0}_{F_{X/S}}(X^{(1)}/S)^{\otimes_{\mathcal{B}}}/X^{(1)}/S\) that is mapped by \(\alpha_{S}\) to the \(S\)-rational point of \(\text{Op}_{\text{Gr}_{n,1,U,X}/S}\) corresponding to \(\nabla_{\mathcal{F}}\). This implies that \(\alpha_{S}\) is surjective, and hence, completes the proof of Proposition 8.3.3. \(\square\)

8.4. Next, we shall relate \(\text{Quot}^{n,0}_{F_{X/S}}(X^{(1)}/S)^{\otimes_{\mathcal{B}}}/X^{(1)}/S\) to \(\text{Quot}^{n,0}_{F_{X/S}}(X^{(1)}/S)^{\otimes_{\mathcal{B}}}/X^{(1)}/S\).

By pulling back line bundles on \(X^{(1)}\) via the relative Frobenius \(F_{X/S}: X \to X^{(1)}\), we obtain a morphism

\[
\text{Pic}_{X^{(1)}/S}[n] \xrightarrow{\eta} \text{Pic}_{X/S} \quad [\mathcal{N}] \mapsto [F_{X/S}^{*}(\mathcal{N})]
\]

of \(S\)-schemes. We shall denote by

\[
\text{Pic}_{X^{(1)}/S}[n]
\]

the \(S\)-scheme defined to be the scheme-theoretic inverse image, via this morphism, of the identity section of \(\text{Pic}_{X/S}^{0}\). It is well-known (cf. \([16]\), EXPOSE VII, §4.3; \([51]\), §8, Proposition 8.1 and Theorem 8.2; \([50]\), APPENDIX, Lemma (1.0)) that \(\text{Pic}_{X^{(1)}/S}[n] \) is finite and faithfully flat over \(S\) of degree \(p^{g}\) and, moreover, \(\text{étale}\) over the points \(s\) of \(S\) such that the fiber of \(X/S\) at \(s\) is ordinary. (Recall that the locus of \(\mathfrak{M}_{g,r}\) classifying ordinary smooth curves is open and dense.) Then we have the following

**Proposition 8.4.1.**

There exists an isomorphism of \(S\)-schemes

\[
\text{Quot}_{F_{X/S}}^{n,0}(X^{(1)}/S)^{\otimes_{\mathcal{B}}}/X^{(1)}/S \times_{S} \text{Pic}_{X^{(1)}/S}[n] \xrightarrow{\sim} \text{Quot}_{F_{X/S}}^{n,0}(X^{(1)}/S)^{\otimes_{\mathcal{B}}}/X^{(1)}/S^{'}.\]

**Proof.** It suffices to prove that there is a bijection between the respective sets of \(S\)-rational points that is functorial with respect to \(S\).

Let \((\eta: \mathcal{V} \to F_{X/S}((\mathcal{L}'), \mathcal{N}))\) be a pair of data corresponding to an \(S\)-rational point of the \(S\)-scheme exhibited on the left-hand side of \((816)\). It follows from the projection formula that the composite

\[
\eta_{\mathcal{N}}: \mathcal{V} \otimes \mathcal{N} \xrightarrow{\eta \otimes \text{id}_{\mathcal{N}}} F_{X/S}^{*}((\mathcal{B}')) \otimes \mathcal{N} \xrightarrow{\sim} F_{X/S}^{*}((\mathcal{B'} \otimes F_{X/S}^{*}(\mathcal{N}))) \xrightarrow{\sim} F_{X/S}^{*}((\mathcal{B'} \otimes \mathcal{O}_{X})) \xrightarrow{\sim} F_{X/S}^{*}((\mathcal{B'}))
\]
defines an element of \( \text{Quot}_{F_{X/S}(B^e)/X^0_S/S}^{n,0} (S) \). Thus, we obtain a map
\[
\gamma_S : (\text{Quot}_{F_{X/S}(B^e)/X^0_S/S}^{n,0} \times s \text{Pic}^0_{X^0_S/S}[n])(S) \to \text{Quot}_{F_{X/S}(B^e)/X^0_S/S}^{n,0} (S),
\]
which is verified to be functorial with respect to \( S \).

Conversely, let \( \eta : \mathcal{V} \to F_{X/S}(B^e) \) be an injection classified by an element of \( \text{Quot}_{F_{X/S}(B^e)/X^0_S/S}^{n,0} (S) \). Fix a pair of positive integers \( (a, b) \) satisfying that \( 1 + n \cdot a = p \cdot b \). Let us consider the injection
\[
\eta_{\det(\mathcal{V})^{\otimes a}} : \mathcal{V} \otimes \det(\mathcal{V})^{\otimes a} \to F_{X/S}(B^e),
\]
i.e., the morphism defined in the same fashion as the construction of \( \eta_N \) (cf. (817)), where "\( N \)" is taken to be \( \det(\mathcal{V})^{\otimes a} \). Here, we observe that
\[
\det(\mathcal{V} \otimes \det(\mathcal{V})^{\otimes a}) \xrightarrow{\sim} \det(\mathcal{V}) \otimes \det(\mathcal{V})^{\otimes n \cdot a}
\]
\[
\xrightarrow{\sim} \det(\mathcal{V})^{\otimes p \cdot b}
\]
\[
\xrightarrow{\sim} (\text{id}_X \times F_S)^*(F_X^*(\det(\mathcal{V}))^{\otimes b}).
\]
Also, observe that
\[
F_{X/S}(\det(\mathcal{V})) \xrightarrow{\sim} \det(F_X^*(\mathcal{V})) \xrightarrow{\mathcal{T}^{-1}} \det(D_{1,X/S}^{\leq n} \otimes B^e) \xrightarrow{\sim} \mathcal{O}_X,
\]
where the second arrow is an isomorphism by Proposition 8.3.2 (i), and the third isomorphism is (775). (In particular, we have \( F_X^*(\det(\mathcal{V}))^{\otimes (-a)} \cong \mathcal{O}_X \).) Hence, the determinant of \( \mathcal{V} \otimes \det(\mathcal{V})^{\otimes a} \) is trivial. Consequently, the pair of data
\[
(\eta_{\det(\mathcal{V})^{\otimes a}}, \det(\mathcal{V})^{\otimes (-a)})
\]
defines an \( S \)-rational point of \( \text{Quot}_{F_{X/S}(B^e)/X^0_S/S}^{n,0} \times s \text{Pic}^0_{X^0_S/S}[n]. \) One verifies easily that this assignment \( \eta \mapsto (\eta_{\det(\mathcal{V})^{\otimes a}}, \det(\mathcal{V})^{\otimes (-a)}) \) determines an inverse to \( \gamma_S \). This completes the proof of Proposition 8.4.1.

8.5. Recall that \( \mathcal{Dp}_{\text{stab}, h, g, 0} \) is nonempty (cf. Theorem 3.12.3 (i)) and the natural projection \( \mathcal{Dp}_{\text{stab}, h, g, 0} \to \overline{\mathcal{M}}_{g, 0} \) is finite and generically étale (cf. Theorem 7.8.2). In the following, we shall verify the claim that
\[
\text{there exists a dense open substack } \mathcal{W} \text{ of } \overline{\mathcal{M}}_{g, 0} \text{ over which the projection } \mathcal{Dp}_{\text{stab}, h, g, 0}/\overline{\mathcal{M}}_{g, 0} \text{ is finite and étale.}
\]
(One may prove, of course, the same claim in more general situations.)

Let \( \{ \mathcal{I}_l \}_{l=1} \) be the set of irreducible components of \( \mathcal{Dp}_{\text{stab}, h, g, 0} \) that dominates \( \overline{\mathcal{M}}_{g, 0} \). For each \( l = 1, \cdots, L \), there exists an open substack \( \mathcal{I}_l \) of \( \mathcal{Dp}_{\text{stab}, h, g, 0} \) that lies in \( \mathcal{I}_l \setminus \bigcup_{l' \neq l} \mathcal{I}_{l'} \), contains the generic point of \( \mathcal{I}_l \), and is étale over
\(\mathcal{M}_{g,0}\). In particular, the open substacks \(\mathcal{I}_1, \ldots, \mathcal{I}_L\) are mutually disjoint. Denote by \(\mathcal{W}'\) the open substack of \(\mathcal{M}_{g,0}\) defined to be the complement of the image of \(\mathcal{O}_{p_{s_{\mathcal{I}_1}}}^{2ax...}) \setminus \bigcup_{l=1}^L \mathcal{I}_l\). Evidently, the open substack \(\mathcal{W}'\) is nonempty and moreover dense. If \(\pi : \mathcal{O}_{p_{s_{\mathcal{I}_1}}}^{2ax...}) \to \mathcal{M}_{g,0}\) denotes the natural projection, then \(\pi^{-1}(\mathcal{W}')\) decomposes into a disjoint union \(\bigcup_{l=1}^L \mathcal{I}_l \cap \pi^{-1}(\mathcal{W}')\). Each restriction \(\pi|_{\mathcal{I}_l \cap \pi^{-1}(\mathcal{W}')} : \mathcal{I}_l \cap \pi^{-1}(\mathcal{W}') \to \mathcal{W}'\) of \(\pi\) is a dominant and generically finite morphism of finite type between integral stacks. One verifies (cf. [25], Chap II, Exercise 3.7) that there exists a dense open substack \(\mathcal{W}_i\) of \(\mathcal{W}'\) (hence, of \(\mathcal{M}_{g,0}\)) such that \(\pi|_{\mathcal{I}_i \cap \pi^{-1}(\mathcal{W}')} : \mathcal{I}_i \cap \pi^{-1}(\mathcal{W}') \to \mathcal{W}_i\) is finite and étale over \(\mathcal{W}_i\). Then, by the above discussion, the restriction \(\pi^{-1}(\mathcal{W}')(= \bigcup_{l=1}^L \mathcal{I}_l \cap \pi^{-1}(\mathcal{W}'))\) is finite and étale over \(\mathcal{W}_i\) and hence, this \(\mathcal{W}_i\) satisfies the required conditions. This completes the proof of the claim.

Now, suppose that the curve \(X_S\) under consideration is ordinary (cf. the discussion preceding Proposition 8.4.1) and the classifying morphism \(S \to \mathcal{M}_{g,0}\) of \(X_S\) factors through the open immersion \(\mathcal{W} \to \mathcal{M}_{g,0}\). By Corollary 4.13.3 and Proposition 8.3.3 (and Theorem 3.12, 3 (i)), \(\text{Quot}^{n,\mathcal{O}}_{F_{X/S}(\mathcal{B}^c)/X^0_S/S}\) is isomorphic to \(\mathcal{O}_{p_{s_{\mathcal{I}_1}}}^{2ax...})_{X_S} \) (which is nonempty). Hence, by Proposition 8.4.1, both \(\text{Quot}^{n,\mathcal{O}}_{F_{X/S}(\mathcal{B}^c)/X^0_S/S}\) and \(\text{Quot}^{n,\mathcal{O}}_{F_{X/S}(\mathcal{B}^c)/X^0_S/S}\) are nonempty and finite and étale over \(S\). In particular, it makes sense to speak of the degree

\[
(823) \quad \deg(\text{Quot}^{n,\mathcal{O}}_{F_{X/S}(\mathcal{B}^c)/X^0_S/S}), \quad \deg(\text{Quot}^{n,\mathcal{O}}_{F_{X/S}(\mathcal{B}^c)/X^0_S/S})
\]

over \(S\) of \(\text{Quot}^{n,\mathcal{O}}_{F_{X/S}(\mathcal{B}^c)/X^0_S/S}\) and \(\text{Quot}^{n,\mathcal{O}}_{F_{X/S}(\mathcal{B}^c)/X^0_S/S}\) respectively. Moreover, by Proposition 3.9.1 and the discussion preceding Proposition 8.4.1, one verifies (for \(h \in k^\times\)) the following equalities:

\[
(824) \quad \deg(\mathcal{O}_{p_{s_{\mathcal{I}_1}}}^{2ax...})_{\mathcal{M}_{g,r}} = \deg(\mathcal{O}_{p_{s_{\mathcal{I}_1}}}^{2ax...})_{\mathcal{M}_{g,r}} = \deg(\mathcal{O}_{p_{s_{\mathcal{I}_1}}}^{2ax...})_{\mathcal{M}_{g,r}} = \deg(\text{Quot}^{n,\mathcal{O}}_{F_{X/S}(\mathcal{B}^c)/X^0_S/S})
\]

Therefore, to determine the value of \(\deg(\mathcal{O}_{p_{s_{\mathcal{I}_1}}}^{2ax...})_{\mathcal{M}_{g,r}}\), it suffices to calculate the value \(\deg(\text{Quot}^{n,\mathcal{O}}_{F_{X/S}(\mathcal{B}^c)/X^0_S/S})\).

8.6. In the following, we review a numerical formula concerning the degree of a certain Quot-scheme over the field of complex numbers \(\mathbb{C}\) and relate it to the degree of the Quot-scheme in positive characteristic.
Let $C$ be a smooth proper curve over $\mathbb{C}$ of genus $g > 1$. If $n$ is an integer, and $\mathcal{E}$ is a vector bundle on $C$ of rank $m$ and degree $d$ with $1 \leq n \leq m$, then we define invariants

\begin{equation}
\begin{aligned}
e_{\text{max}}(\mathcal{E}, n) &:= \max\{\deg(\mathcal{F}) \in \mathbb{Z} \mid \mathcal{F} \text{ is a subbundle of } \mathcal{E} \text{ of rank } n\}, \\
s_n(\mathcal{E}) &:= d \cdot n - m \cdot e_{\text{max}}(\mathcal{E}, n).
\end{aligned}
\end{equation}

(Here, we recall that one verifies immediately, for instance, by considering an embedding of $\mathcal{E}$ into a direct sum of $n$ line bundles, that $e_{\text{max}}(\mathcal{E}, n)$ is well-defined.)

We review some facts concerning these invariants (cf. [26; 45; 27]). Denote by $\mathcal{U}^{m,d}_{C}$ the moduli space of stable bundles on $C$ of rank $m$ and degree $d$ (cf. [45], § 1). It is known that $\mathcal{U}^{m,d}_{C}$ is irreducible (cf. the discussion at the beginning of [15], § 2). Thus, it makes sense to speak of a “sufficiently general” stable bundle in $\mathcal{U}^{m,d}_{C}$, i.e., a stable bundle that corresponds to a point of the scheme $\mathcal{U}^{m,d}_{C}$ that lies outside some fixed closed subscheme. If $\mathcal{E}$ is a sufficiently general stable bundle in $\mathcal{U}^{m,d}_{C}$, then it holds (cf. [45], § 1) that $s_n(\mathcal{E}) = n(m-n)(g-1)+\epsilon$, where $\epsilon$ is the unique integer such that $0 \leq \epsilon < m$ and $s_n(\mathcal{E}) = n \cdot d \mod m$. Also, the number $\epsilon$ coincides (cf. [27], § 1) with the dimension of every irreducible component of the Quot-scheme $\text{Quot}^{n,e_{\text{max}}(\mathcal{E}, n)}_{\mathcal{E}/C/\mathbb{C}}$. If, moreover, the equality $s_n(\mathcal{E}) = n(m-n)(g-1)$ holds (i.e., dim($\text{Quot}^{n,e_{\text{max}}(\mathcal{E}, n)}_{\mathcal{E}/C/\mathbb{C}}$) = 0), then $\text{Quot}^{n,e_{\text{max}}(\mathcal{E}, n)}_{\mathcal{E}/C/\mathbb{C}}$ is étale over Spec($\mathbb{C}$) (cf. [27], § 1). Finally, under this particular assumption, a formula for the degree of this Quot-scheme was given by Y. Holla as follows.

**Theorem 8.6.1.**

Let $C$ be a connected, proper, and smooth curve over $\mathbb{C}$ of genus $g > 1$, $\mathcal{E}$ a sufficiently general stable bundle in $\mathcal{U}^{m,d}_{C}$. Write $(a, b)$ for the unique pair of integers such that $d = a \cdot m - b$ with $0 \leq b < m$. Also, we suppose that the equality

\begin{equation}
s_n(\mathcal{E}) = n(m-n)(g-1),
\end{equation}

or equivalently, the equality

\begin{equation}
e_{\text{max}}(\mathcal{E}, n) = (d \cdot n - n(m-n)(g-1))/m
\end{equation}

holds (cf, the above discussion). Then, the degree deg($\text{Quot}^{n,e_{\text{max}}(\mathcal{E}, n)}_{\mathcal{E}/C/\mathbb{C}}/\mathbb{C}$) of $\text{Quot}^{n,e_{\text{max}}(\mathcal{E}, n)}_{\mathcal{E}/C/\mathbb{C}}$ over Spec($\mathbb{C}$) is given by the following formula:

\begin{equation}
\text{deg}(\text{Quot}^{n,e_{\text{max}}(\mathcal{E}, n)}_{\mathcal{E}/C/\mathbb{C}}/\mathbb{C}) = \frac{(-1)^{(n-1)(b-n-(g-1)n^2)/m} \cdot m^{n(g-1)}}{n!} \cdot \sum_{\zeta_1, \ldots, \zeta_n} \prod_{i=1}^{n} \prod_{i \neq j} (\zeta_i - \zeta_j)^{b-g+1}.
\end{equation}
where \(\zeta_i^m = 1\), for \(1 \leq i \leq n\) and the sum is over tuples \((\zeta_1, \cdots, \zeta_n)\) with \(\zeta_i \neq \zeta_j\).

**Proof.** The assertion follows from [27], §4, Theorem 4.2, where “\(k\)” (respectively, “\(r\)” ) corresponds to our \(n\) (respectively, \(m\)). \(\square\)

By applying this formula, we conclude the same kind of formula for certain vector bundles in positive characteristic, as follows.

**Theorem 8.6.2.**

Let \(n\) be a positive integer with \(n < p\), \(k\) an algebraically closed field of characteristic \(p\), \(X\) a connected, proper, and smooth curve over \(k\) of genus \(g > 1\), and \(\mathcal{B}^g\) a line bundle on \(X\) satisfying that \(\mathcal{B}^g \otimes m \cong \mathcal{T}_{X/k}^{n(n-1)}\) (cf. (779)). Suppose that \(X/k\) is sufficiently general in \(\overline{\mathcal{M}}_{g,0}\). (Here, we recall that \(\overline{\mathcal{M}}_{g,0}\) is irreducible (cf. [15], §5); thus, it makes sense to speak of a “sufficiently general” \(X/k\), i.e., an \(X/k\) that determines a point of \(\overline{\mathcal{M}}_{g,0}\) that lies outside some fixed closed substack.) Then, \(\text{Quot}^{n,0}_{F_{X/k}(\mathcal{B}^g)/X_k^{(1)}/k}\) is (nonempty and) finite and étale over \(\text{Spec}(k)\). If, moreover, we suppose that \(p > n \cdot (g - 1)\), then the degree \(\text{deg}(\text{Quot}^{n,0}_{F_{X/k}(\mathcal{B}^g)/X_k^{(1)}/k}/k)\) of \(\text{Quot}^{n,0}_{F_{X/k}(\mathcal{B}^g)/X_k^{(1)}/k}\) over \(\text{Spec}(k)\) is given by the following formula:

\[
\text{deg}(\text{Quot}^{n,0}_{F_{X/k}(\mathcal{B}^g)/X_k^{(1)}/k}/k) = \frac{p^{n(g-1)}}{n!} \cdot \sum_{\{\zeta_1, \cdots, \zeta_n\} \in \mathbb{C} \times n} \frac{(\prod_{i=1}^{n} \zeta_i)^{(n-1)(g-1)}}{\prod_{i \neq j} (\zeta_i - \zeta_j)^{g-1}}. 
\]

**Proof.** Suppose that the curve \(X\) is ordinary and classified by a \(k\)-rational point in the dense open substack \(\mathcal{W}\) (obtained in the discussion at the beginning of §8.5) of \(\overline{\mathcal{M}}_{g,0}\). By the definitions of \(\mathcal{W}\) and ordinarity of curves (cf, the discussion preceding (824)), \(\text{Quot}^{n,0}_{F_{X/k}(\mathcal{B}^g)/X_k^{(1)}/k}\) is (nonempty and) finite and étale over \(\text{Spec}(k)\).

In the following, we determine the value of \(\text{deg}(\text{Quot}^{n,0}_{F_{X/k}(\mathcal{B}^g)/X_k^{(1)}/k})\). Denote by \(W\) the ring of Witt vectors with coefficients in \(k\) and \(K\) the fraction field of \(W\). Since \(\dim(X_k^{(1)}) = 1\), which implies that \(H^2(X_k^{(1)}, \mathcal{T}_{X_k^{(1)}/k} = 0\), it follows from well-known generalities concerning deformation theory that \(X_k^{(1)}\) may be lifted to a smooth proper curve \(X_W^{(1)}\) over \(W\) of genus \(g\). In a similar vein, the fact that \(H^2(X_k^{(1)} \cdot \text{End}_{X_k^{(1)}}(F_{X/k}(\mathcal{B}^g))) = 0\) implies that \(F_{X/k}(\mathcal{B}^g)\) may be lifted to a vector bundle \(\mathcal{E}_W\) on \(X_W^{(1)}\).

Now let \([\eta]\) be the \(k\)-rational point of \(\text{Quot}^{n,0}_{F_{X/k}(\mathcal{B}^g)/X_k^{(1)}/k}\) classifying an injective morphism \(\eta : \mathcal{F} \to F_{X/k}(\mathcal{B}^g)\). The tangent space \(\mathcal{T}_{[\eta]}\text{Quot}^{n,0}_{F_{X/k}(\mathcal{B}^g)/X_k^{(1)}/k}\)
to $\text{Quot}_{F_{X/k^*}(B^2)/X_{k^*}^{(1)}/k}$ at $[\eta]$ may be naturally identified, as we referred in §8.1, with the $k$-vector space $\text{Hom}_{X_{k^*}^{(1)}}(\mathcal{F}, \text{Coker}(\eta))$, and the obstruction to lifting $\eta$ to any first order thickening of Spec$(k)$ is given by an element of $\text{Ext}_{X_{k^*}^{(1)}}^1(\mathcal{F}, \text{Coker}(\eta))$. On the other hand, since $\text{Quot}^{n,0}_{F_{X/k^*}(B^2)/X_{k^*}^{(1)}/k}$ is, as was verified above, étale over Spec$(k)$, it holds that $\text{Hom}_{X_{k^*}^{(1)}}(\mathcal{F}, \text{Coker}(\eta)) = 0$, and hence, $\text{Ext}_{X_{k^*}^{(1)}}^1(\mathcal{F}, \text{Coker}(\eta)) = 0$ by Lemma 8.6.3 below. This implies that $\eta$ may be lifted to a $W$-rational point of $\text{Quot}^{n,0}_{E_{W/X_{W}^{(1)}/W}}$, and hence that $\text{Quot}^{n,0}_{E_{W/X_{W}^{(1)}/W}}$ is finite and étale over $W$ by Lemma 8.5.4 and the vanishing of $\text{Hom}_{X_{k^*}^{(1)}}(\mathcal{F}, \text{Coker}(\eta))$.

Now it follows from a routine argument that $K$ may be supposed to be a subfield of $\mathbb{C}$. Denote by $X_{C}^{(1)}$ the base-change of $X_{W}^{(1)}$ via the morphism Spec$(\mathbb{C}) \to \text{Spec}(W)$ induced by the composite embedding $W \hookrightarrow K \hookrightarrow \mathbb{C}$, and by $\mathcal{E}_{C}$ the pull-back of $\mathcal{E}_{W}$ via the natural morphism $X_{C}^{(1)} \to X_{W}^{(1)}$. Thus, we obtain equalities

\begin{equation}
\deg(\text{Quot}^{n,0}_{F_{X/k^*}(B^2)/X_{k^*}^{(1)}/k}) = \deg(\text{Quot}^{n,0}_{E_{W/X_{W}^{(1)}/W}}) = \deg(\text{Quot}^{n,0}_{\mathcal{E}_{C}/X_{C}^{(1)}/C}/\mathbb{C}).
\end{equation}

To prove the required formula, we calculate the degree $\deg(\text{Quot}^{n,0}_{\mathcal{E}_{C}/X_{C}^{(1)}/C}/\mathbb{C})$ by applying Theorem 8.6.1.

By [68], §2, Theorem 2.2, $F_{X/k^*}(B^2)$ is stable. Since the degree of $\mathcal{E}_{C}$ coincides with the degree of $F_{X/k^*}(B^2)$, $\mathcal{E}_{C}$ is a vector bundle of degree $\deg(\mathcal{E}_{C}) = (p-n)(g-1)$ (cf. the proof of Lemma 8.6.3). On the other hand, one verifies easily from the definition of stability and the properness of Quot schemes (cf. [18], §5.5, Theorem 5.14) that $\mathcal{E}_{C}$ is a stable vector bundle. Next, let us observe that $\text{Quot}^{n,0}_{\mathcal{E}_{C}/X_{C}^{(1)}/C}$ is zero-dimensional (cf. the discussion above), which, by the discussion preceding Theorem 8.6.1, implies that $s_n(\mathcal{E}_{C}) = n(p-n)(g-1)$. Thus, by choosing the deformation $\mathcal{E}_{W}$ of $F_{X/k^*}(B^2)$ appropriately, we may assume, without loss of generality, that $\mathcal{E}_{C}$ is sufficiently general in $\mathcal{U}_{X_{C}^{(1)}}^{p(p-n)(g-1)}$ that Theorem 8.6.1 holds. Now we compute (cf. the discussion preceding Theorem 8.6.1):

\begin{equation}
\epsilon_{\max}(\mathcal{E}_{C}, n) = \frac{1}{p} \cdot (\deg(\mathcal{E}_{C}) \cdot n - s_n(\mathcal{E}_{C}))
= \frac{1}{p} \cdot ((p-n)(g-1) \cdot n - n \cdot (p-n)(g-1))
= 0.
\end{equation}
If, moreover, we write \((a, b)\) for the unique pair of integers such that \(\deg_C(\mathcal{E}_C) = p \cdot a - b\) with \(0 \leq b < p\), then it follows from the hypothesis \(p > n \cdot (g - 1)\) that \(a = g - 1\) and \(b = n \cdot (g - 1)\). Thus, by applying Theorem 8.6.1 in the case where the data

\[\left( C, \mathcal{E}, p, (g - 1)(p - n), n, g - 1, n \cdot (g - 1), 0 \right)\]

we obtain that

\[\deg(\text{Quot}_{n, 0}^* F_{X/k}(\mathcal{B})^g) = \left((-1)^{(n-1)(n(g-1)-1)/(g-1)} \cdot \frac{p^n}{n!} \sum_{\zeta_1, \ldots, \zeta_n} \frac{\left(\prod_{i=1}^{n} \zeta_i\right)^{(n-1)(g-1)} \cdot \left(\prod_{i \neq j} (\zeta_i - \zeta_j)^{g-1}\right)}{\prod_{i \neq j} (\zeta_i - \zeta_j)}\right)\]

This completes the proof of the required equality. □

The following lemma was used in the proof of Theorem 8.6.2.

**Lemma 8.6.3.**

Let \(k, X/k, \) and \(\mathcal{B}^g\) be as in Theorem 8.6.2, and \(\eta : F \to F_{X/k}(\mathcal{B}^g)\) an injection classified by a \(k\)-rational point of \(\text{Quot}^{n, 0}_{F_{X/k}(\mathcal{B}^g)}/X_k^{(1)}/k\). Then \(\text{Coker}(\eta)\) is a vector bundle on \(X_k^{(1)}\), and it holds that

\[\dim_k(\text{Hom}_{\mathcal{O}_{X_k^{(1)}}}(F, \text{Coker}(\eta))) = \dim_k(\text{Ext}^{1}_{\mathcal{O}_{X_k^{(1)}}}(F, \text{Coker}(\eta))).\]

**Proof.** First, we verify that \(\text{Coker}(\eta)\) is a vector bundle. Since \(F_{X/k} : X \to X_k^{(1)}\) is faithfully flat, it suffices to verify that the pull-back \(F_{X/k}(\text{Coker}(\eta))\) is a vector bundle on \(X\). Recall (cf. Proposition 8.3.2) that the composite

\[F_{X/k}(\mathcal{F}) \overset{F_{X/k}(\eta)}{\to} \mathcal{A}_{\mathcal{B}^g} = F_{X/k}(F_{X/k}(\mathcal{B})) \to \mathcal{A}_{\mathcal{B}^g}/\mathcal{A}_{\mathcal{B}^g}^n\]

of the pull-back \(F_{X/k}(\eta)\) of \(\eta\) with the natural surjection \(\mathcal{A}_{\mathcal{B}^g} \to \mathcal{A}_{\mathcal{B}^g}/\mathcal{A}_{\mathcal{B}^g}^n\) is an isomorphism. This implies that the natural composite \(\mathcal{A}_{\mathcal{B}^g}^n \overset{\mathcal{A}_{\mathcal{B}^g}^n}{\to} \mathcal{A}_{\mathcal{B}^g} \to F_{X/k}(\text{Coker}(\eta))\) is an isomorphism, and hence that \(F_{X/k}(\text{Coker}(\eta))\) is a vector bundle, as desired.

Next we consider the asserted equality. Since the morphism \(F_{X/k} : X \to X_k^{(1)}\) is finite, it follows from well-known generalities concerning cohomology that
we have an equality of Euler characteristics $\chi(F_X/k \ast (B \bowtie)) = \chi(B \bowtie)$. Thus, it follows from the Riemann-Roch theorem that

\begin{equation}
\deg(F_X/k \ast (B \bowtie)) = \chi(F_X/k \ast (B \bowtie)) - \text{rk}(F_X/k \ast (B \bowtie)) \cdot (1 - g) \\
= \chi(B \bowtie) - p(1 - g) \\
= (p - n)(g - 1),
\end{equation}

and, since $\text{rk}(\text{Hom}_{\mathcal{O}_{X_k}}(\mathcal{F}, \text{Coker}(\eta))) = n(p - n)$, that

\begin{equation}
\deg(\text{Hom}_{\mathcal{O}_{X_k}}(\mathcal{F}, \text{Coker}(\eta))) = n \cdot \deg(\text{Coker}(\eta)) - (p - n) \cdot \deg(\mathcal{F}) \\
= n \cdot \deg(F_X/k \ast (B \bowtie)) - 0 \\
= n(p - n)(g - 1).
\end{equation}

Finally, by applying the Riemann-Roch theorem again, we obtain equalities

\begin{equation}
\dim_k(\text{Hom}_{\mathcal{O}_{X_k}}(\mathcal{F}, \text{Coker}(\eta))) - \dim_k(\text{Ext}^1_{\mathcal{O}_{X_k}}(\mathcal{F}, \text{Coker}(\eta))) \\
= \deg(\text{Hom}_{\mathcal{O}_{X_k}}(\mathcal{F}, \text{Coker}(\eta))) + \text{rk}(\text{Hom}_{\mathcal{O}_{X_k}}(\mathcal{F}, \text{Coker}(\eta))) \cdot (1 - g) \\
= n(p - n)(g - 1) + n(p - n)(1 - g) \\
= 0.
\end{equation}

\begin{equation}
\deg(\text{Hom}_{\mathcal{O}_{X_k}}(\mathcal{F}, \text{Coker}(\eta))) - \deg(\text{Coker}(\eta)) - (p - n) \cdot \deg(\mathcal{F}) \\
= n(p - n)(g - 1).
\end{equation}

8.7. Finally, we shall conclude the paper with following corollary, which yields an affirmative answer to Joshi’s conjecture (for sufficiently general curves).

**Theorem 8.7.1 (Joshi’s conjecture).**

Suppose that $p > n \cdot (g - 1)$ and $h \in k^\times$. Then, the generic degree $\deg(\mathcal{O}_{p_{sl_n,b,g,0}/\overline{M}_{g,0}})$ of $\mathcal{O}_{p_{sl_n,b,g,0}}$ over $\overline{M}_{g,0}$ is given by the following formula:

\begin{equation}
\deg(\mathcal{O}_{p_{sl_n,b,g,0}/\overline{M}_{g,0}}) = \frac{p^{(n-1)(g-1)-1}}{n!} \cdot \sum_{\sum_{i=1}^{n} (\zeta_i) = 1, \zeta_i \neq \zeta_j (i \neq j)} \prod_{i=1}^{n} (\prod_{i \neq j} (\zeta_i - \zeta_j)^{g-1}).
\end{equation}

**Proof.** Let us fix a proper smooth curve $X/k$ and line bundle $B \bowtie$ on $X$ for which Theorem 8.5.3 holds. Then it follows from Theorem 8.6.2 and the discussion
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in § 8.5 that

\[
\deg(\mathcal{O}_{\text{sl}}^{\text{reg}, \ldots, \text{reg}}_{\mathcal{M}_{g,0}/\overline{\mathcal{M}_{g,0}}})
= \frac{1}{p^g} \cdot \deg(\text{Quot}^{n,0}_{X/k}(B^+X^{(1)}/k)/k)
= \frac{p^{(n-1)(g-1)-1}}{n!} \cdot \sum_{(\zeta_1, \ldots, \zeta_n) \in \mathbb{C} \times \mathbb{n}} \frac{(\prod_{i=1}^{n} \zeta_i)^{(n-1)(g-1)}}{\prod_{i \neq j} (\zeta_i - \zeta_j)^{g-1}}.
\]

\[\square\]

\text{Guten Abend, gut' Nacht,}
\text{Mit Rosen bedacht,}
\text{Mit Näglein besteckt,}
\text{Schlupf’ unter die Deck’;}
\text{Morgen früh, wenn Gott will,}
\text{Wirst du wieder geweckt.}

@\text{Guten Abend, gut’ Nacht,}
\text{Von Englein bewacht,}
\text{Die zeigen im Traum}
\text{Dir Christkindleins Baum;}
\text{Schlaf’ nun selig und süß,}
\text{Schau’ im Traum’s Paradies.}

Johannes Brahms, \textit{Wiegenlied}, “Op.” \textbf{49-4}, (1868).

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INDEX OF NOTATION

\( a^\log_a \) anchor map in Atiyah algebra of \( \mathcal{E} \), § 1.2

\( \mathcal{A}_\mathcal{L} := F^*_{\mathcal{L}/\mathcal{S}}(\mathcal{F}_{\mathcal{U}/\mathcal{S}}(\mathcal{L})) \), § 7.1

\( A^1 \) affine line over \( k \), (§ 3.8), § 3.9

\( \text{ad}(a) \) adjoint operator determined by \( a \), § 1.1

\( \text{Ad}_G \) adjoint representation of \( G \), § 1.1

\( \text{Ad}_h \) inner automorphism induced by \( h \), § 1.1

\( \text{ad} \) Lie algebra of \( \mathbb{B} \), § 2.1

\( \mathfrak{b} \) Lie algebra of \( \mathfrak{B} \), § 2.4

\( B \) upper triangular Borel subgroup of \( \text{PGL}_n \), § 4.3

\( \mathcal{B} \) line bundle on \( U \), § 4.5

\( B^\mathbb{Z} := T^{\otimes(n-1)} \otimes \mathcal{B} \), § 8.2

\( \mathfrak{b}^\mathfrak{B}_{U^\log/S^\log} := \mathcal{H}om_{\mathcal{O}_U}(\mathcal{B},(\mathcal{O}_{U^\log/S^\log} \otimes \mathcal{B}) \otimes \mathcal{D}_{U^\log/S^\log}) \), § 4.5

\( \mathfrak{b}^\mathfrak{g}_{U^\log/S^\log} := \mathcal{H}om_{\mathcal{O}_U}(\mathcal{T}_{U^\log/S^\log} \otimes \mathcal{B}^\prime \otimes \mathcal{D}_{U^\log/S^\log}) \), § 4.5

\( \mathfrak{B} \) Borel subgroup of \( G \), § 1.4

\( \mathfrak{B}^\mathbb{B} \) upper triangular Borel subgroup of \( G^\mathbb{B} \), § 2.4

\( c \) GIT quotient \( t/\mathbb{W} \) of \( t \) by \( \mathbb{W} \)-action, § 2.8

\( \sigma^c := c \setminus \text{Im}([0], \mathcal{E}) \), § 3.6

\( C(F, \nabla_F) \) Cartier operator associated with \( (F, \nabla_F) \), § 5.7

\( \mathcal{C}_{g,r} \) tautological curve over \( \mathcal{M}_{g,r} \), § 1.5

\( \mathcal{C}_a \) := \( \sum_{\lambda \in \Lambda_{\mathfrak{a}}} \lambda \), λ \( \Lambda \), § 6.4

\( \mathcal{C}_{\text{ult}} \) \( \mathcal{D} \)-morphed morphism associated with \( \mathcal{D} \), § 6.1

\( d_\mathcal{E} \) linear morphism \( d \) twisted by \( \mathcal{E} \), § 1.2

\( d_{Y^\log/T^\log} \) universal logarithmic derivation on \( Y^\log \) over \( T^\log \), § 4.1

\( D_{X/S} \) étale effective relative divisor determined by the marked points of \( X/S \), § 1.5

\( D \) clutching data for pointed stable curve of type \( (g, r) \), § 6.1

\( D^\mathbb{B}_{h,U^\log/S^\log} \) sheaf of \( h \)-twisted logarithmic crystalline differential operators (i.e., \( h \)-tledno's) of order \( < \mathbb{B} \) \( (\mathbb{B} = 0, 1, \cdots, \infty) \) on \( U^\log \) over \( S^\log \), § 4.4

\( \text{det}(\nabla_F) \) \( T \)-\( h \)-log connection on \( \text{det}(\mathcal{F}) \) induced by \( \nabla_F \), § 4.1

\( e_{\mathfrak{ao}}[\nabla] (\mathfrak{a} = \mathfrak{a} \text{ or } \mathfrak{b}) \) morphism arising from Hodge to de Rham spectral sequence of \( \mathcal{K}^\mathfrak{a}[\nabla] \), § 5.1

\( e_{\mathfrak{ao}}[\nabla] (\mathfrak{a} = \mathfrak{a} \text{ or } \mathfrak{b}) \) morphism arising from conjugate spectral sequence of \( \mathcal{K}^\mathfrak{a}[\nabla] \), § 5.1

\( \mathcal{E}^\mathfrak{a} \) \( (\mathfrak{a}, h) \)-oper, § 2.2

\( \mathcal{E}^\mathfrak{a}_{\mathfrak{a}+}(\mathfrak{g}, h)-\text{oper of canonical type}, \mathfrak{a} \), § 2.4

\( \mathcal{E}_{\mathfrak{b}, \mathfrak{h}, U/S} \) underlying \( \square \)-torsor \( (\square = \mathfrak{B} \text{ or } \mathfrak{G}) \) of \( (\mathfrak{g}, h) \)-oper of canonical type, § 2.4, § 2.5

\( \mathcal{E}_{\mathfrak{b}, n, \mathfrak{h}, U/S} \) \( \mathfrak{gl}_{\mathfrak{n}}\)-torsor associated with \( \mathcal{D}_{\mathfrak{n}, \mathfrak{h}, U/S}^\mathfrak{b} \otimes \mathcal{B} \), § 4.12

\( \mathcal{E}_{\mathfrak{b}, n, \mathfrak{h}, U/S}^\mathfrak{b} \) \( \mathfrak{gl}_{\mathfrak{n}}\)-torsor induced by \( \mathcal{E}_{\mathfrak{b}, n, \mathfrak{h}, U/S} \), § 4.12

\( \mathcal{E}^\mathfrak{a}_{\mathfrak{b}, \mathfrak{h}, U/S} \) \( B \)-reduction of \( \mathcal{E}_{\mathfrak{b}, n, \mathfrak{h}, U/S} \), § 4.12

\( \mathcal{E}_U \) small étale site on \( U \), § 2.3

\( \mathfrak{e}_P \) exponential map from \( n \) to \( \mathbb{N} \), § 1.4

\( F_Y \) absolute Frobenius morphism of \( Y \), § 3.1

\( F_{Y/T} \) relative Frobenius morphism of \( Y \) over \( T \), § 3.1

\( F_{\mathfrak{a}} \) \( (\mathfrak{gl}_{\mathfrak{n}}, h) \)-oper, § 4.2

\( F_{\mathfrak{a}} \) \( (\mathfrak{gl}_{\mathfrak{n}}, h) \)-oper \( F_{\mathfrak{a}} \) twisted by \( (\mathcal{L}, \nabla_L) \), § 4.2

\( \mathcal{F}[n] \) complex \( \mathcal{F} \) shifted down by \( n \), § 5.4

\( \mathfrak{f}_{\mathfrak{g}, p} \) dormant operatic fusion ring of \( \mathfrak{g} \) of level \( p \), § 6.4

\( \mathfrak{g} \) Lie algebra of \( G \), § 1.1

\( \mathfrak{g}^\mathbb{B} \) Lie algebra of \( G^\mathbb{B} \), § 2.4

\( (g, r) \) pair of nonnegative integers satisfying that \( 2g - 2 + r > 0 \), § 1.5

\( \mathfrak{g} \) connected smooth (or semisimple) algebraic group (of adjoint type) over \( k \), § 1.1 (§ 1.4)

\( \mathfrak{g}_m \) multiplicative group over \( k \), § 2.1

\( \mathfrak{g}^\mathbb{B} \) pinning of \( G \), § 2.1

\( \mathfrak{G} \) projective linear group \( \text{PGL}_2 \) of rank 2, § 2.4

\( \mathfrak{h}_{\mathfrak{g}} \) vector bundle associated with \( \mathfrak{g} \), § 1.2

\( \mathfrak{h} \) parameter, § 1.2

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h-Diff \textsuperscript{\textbullet} \text{sheaf of h-tlcdo's on } B \text{ of order } n \text{ with prescribed symbol depending on } \Box (\Box = B \text{ or } U), \S 4.5, \S 4.9

h-Diff \textsuperscript{\textbullet} \text{moduli stack classifying h-tlcdo's on } B \text{ of order } n \text{ with prescribed symbol depending on } U, \S 4.10

\mathfrak{S}_{\epsilon(p),X/S} \text{closed immersion} \nabla \epsilon_p \xrightarrow{[\epsilon_p]} \nabla X/S, \S 3.7

k \text{ (perfect) field, } \S 1.1 (\S 5.1)

\mathcal{K}^* [\nabla] \text{complex associated with } \nabla, \S 5.1

\text{isos} \text{ morphism given by Kostant section, } \S 2.8

\text{iso} \text{ morphism} \nabla \epsilon_p \xrightarrow{\text{triv.}} \S 3.10

\text{ts}_F \text{ } j\text{-th Kodaira-Spencer map of } F, \S 4.2

l_h \text{left-translation by } h, \S 1.1

\text{log} \text{logarithmic map from } N \text{ to } K, \S 1.4

\mathfrak{M}_{g,r} \text{moduli stack classifying } r\text{-pointed stable curves of type } (g,r), \S 1.5

n \text{Lie algebra of } N, \S 1.4

n^\circ \text{Lie algebra of } N^\circ, \S 2.4

\mathbb{N} \text{unipotent radical of } B, \S 1.4

\mathbb{N}^\circ \text{unipotent radical of } B^\circ, \S 2.4

\mathcal{N}^{\text{azt}}_{g,p,g} \text{function on } \mathbb{N}^\circ(\mathfrak{g}) \text{ assigning}
\text{deg}(\mathcal{O}_{p,g,p,g}^{\text{azt}}(U/U)), \S 6.4

\mathcal{O}_{x,h} \text{morphism from } \mathcal{O}_{p,g,h,p,g,x} \text{given by multiplication by } h', \S 2.10

\mathcal{O}_{p,g,h,U/S} \text{stack in groupoids over } \mathcal{E}_{U/S} \text{ classifying } (g,h)-\text{opers on } U/S, \S 2.3

\mathcal{O}_{p,\bullet} \text{étale sheaf of isomorphism classes of } (g,h)-\text{opers on } U/S, \S 4.1

\mathcal{O}_{p,\mathcal{O}_{\text{GL}}^\circ} \text{étale sheaf of equivalence classes of } (\text{GL}_n,h)-\text{opers on } U/S, \S 4.2

\mathcal{O}_{p,\mathcal{O}_{\text{GL}}^\circ} \text{étale sheaf of isomorphism classes of } (\text{GL}_n,h,\mathbb{D})\text{-opers } (\mathbb{D} = B \text{ or } U) \text{on } U/S, \S 4.6, \S 4.9

\mathcal{O}_{p,g,h,p,g,x} \text{moduli stack classifying } (g,h)-\text{opers on } X/S (\text{of radii } \rho), \S 2.9

\mathcal{O}_{p,g,h,p,g,r} \text{moduli stack classifying pointed stable curves over } k \text{ of type } (g,r) \text{equipped with a } (g,h)-\text{oper } (\text{of radii } \rho), \S 3.12
\mathcal{O}_{p,g,h,p,g,r}^{\text{bor}} \text{moduli stack classifying dormant } (g,h)-\text{opers on } X/S (\text{of radii } \rho), \S 3.6

\mathcal{O}_{p,g,h,p,g,r}^{\text{bor}} \text{moduli stack classifying pointed stable curves over } k \text{ of type } (g,r) \text{equipped with a dormant } (g,h)-\text{oper } (\text{of radii } \rho), \S 3.12

\mathcal{O}_{p,g,h,p,g,r}^{\text{n-nilp}} \text{moduli stack classifying } p\text{-nilpotent } (g,h)-\text{opers on } X/S (\text{of radii } \rho), \S 3.8

\mathcal{O}_{p,g,h,p,g,r}^{\text{n-nilp}} \text{moduli stack classifying pointed stable curves over } k \text{ of type } (g,r) \text{equipped with a } p\text{-nilpotent } (g,h)-\text{oper } (\text{of radii } \rho), \S 3.12

\mathcal{O}_{\text{GL}_{n,h,U/S}} \text{moduli stack classifying } (\text{GL}_n,h,U)-\text{opers on } X/S, \S 4.11

\mathcal{O}_{\text{GL}_{n,h,U/S}}^{\text{bor}} \text{moduli stack classifying equivalence classes of } (\text{GL}_n,h)-\text{opers on } X/S, \S 4.11

p_1 \text{sum of fixed generators of } g_1, \S 2.1

p_1 \text{sum of fixed generators of } g_1, \S 2.1

\mathfrak{P}_{/k} \text{pointed stable curve of type } (0,3), \S 6.3

\text{Pic}^{\text{od}}(X/S) \text{relative Picard scheme of } X/S \text{classifying degree } d \text{ line bundles,} \S 4.9, \S 8.2

\text{Quot}^{d}_{\mathcal{E}Y/T} \text{Quot-scheme classifying } \mathcal{E}_{Y}\text{-submodules of } \mathcal{E} \text{of rank } n \text{ and degree } d, \S 8.1

\text{Quot}^{d}_{\mathcal{E}Y/T} \text{Quot-scheme classifying } \mathcal{E}_{Y}\text{-submodules of } \mathcal{E} \text{of rank } n \text{ with trivial determinant, } \S 8.2

\mathfrak{r}_T \text{right-translation by } h, \S 1.1

\mathcal{R}_T \mathcal{O}_S\text{-algebra corresponding to } T, (\S 3.8), \S 3.11

\mathcal{S}_{\mathcal{O}_Y}(V) \text{symmetric algebra on } V \text{over } \mathcal{O}_Y, \S 1.2, \S 4.12

\mathcal{S}_{\mathfrak{k}}(\mathfrak{a}) \text{symmetric algebra on } \mathfrak{a} \text{ over } k, \S 2.8

\text{Sch}/S \text{category of } S\text{-schemes, } \S 2.3

\mathcal{S} \text{category of sets, } \S 2.3

\ast \mathcal{S}_{\mathfrak{M}} \text{natural surjection}
\mathcal{B}_{U/S}^{\text{mod}} \text{surjection } \ast \mathcal{O}_{U/S}^{\text{mod}} (\mathbb{D} = \mathbb{N} \text{ or } \mathbb{N}), \S 4.5
\[\mathcal{T}_Y^{\log/T_0^{\log}}\] sheaf of logarithmic
derivatives of \(Y^{\log}\) over \(T_0^{\log}\), § 1.2
\[\mathcal{T}_Y^{\log/T_0^{\log}}\] subsheaf of \(G\)-invariant
sections of \(\pi_*(\mathcal{T}_Y^{\log/T_0^{\log}})\), § 1.2
\(T\) maximal torus of \(G\), § 2.1
\(\text{triv}_{\gamma, U}\) trivialization of \(\sigma_1^* (\mathcal{T}_U^{\log/G^{\log}})\),
§ 1.6
\(\text{triv}_{\alpha, \partial, U, (x, \nu)}\) trivialization of \(\mathcal{E}_{\partial, \alpha, U, S}\)
\((\square = \mathbb{B} \text{ or } G)\) w.r.t. log chart \((U, x)\),
§ 2.4, § 2.5
\(\mathcal{T}Y\) (total space of) tangent bundle of
\(Y\), § 1.2
\(\mathcal{T}Y^{\text{tang}}\) smooth locus of \(U \setminus \text{Supp}(D_{U, S})\),
§ 1.6
\(U\) \((n, h)\)-determinant data, § 4.9
\(\mathcal{U}_{\mathcal{X}/S}\) restriction of \(\mathcal{X}/S\) to étale scheme
\(U\) over \(X\), § 1.6
\(\nu^\vee\) dual vector bundle of \(\nu\), § 1.2
\(\nu_{g, h, U, S}^\perp := \Omega_{U^{\log}/S^{\log}} \otimes \text{ad}(p_1)\), § 2.6
\(\mathcal{C}^1_{g, h, U, S} := \mathcal{C}^1_{g, h, U, S}(D_{U, S})\), § 2.6
\(\mathcal{V}_{\alpha, \partial, U, S} := \text{Hom}_{\mathcal{C}_{g, h, U, S}}(\mathcal{V}, \mathcal{W}_{S/\mathcal{X}_{g, h, U, S}})\), § 5.9
\(\mathcal{V}(\mathcal{Y})\) relative affine space associated
with \(\mathcal{V}\), § 1.2
\(\mathcal{W}\) Weyl group of \((G, T)\), § 2.1
\(\mathcal{X}_{\mathcal{S}}\) pointed stable curve of type \((g, r)\),
§ 1.5
\(\mathcal{Y}_{T}^{(1)}\) Frobenius twist of \(Y\) over \(T\), § 3.1
\(\bigotimes^j\mathcal{B}_{\mathcal{Y}^{T_0^{\log}}} \otimes \mathcal{B}_{\mathcal{Y}^{T_0^{\log}}}\), § 4.5
\(\Theta_{\mathcal{G}}\) Maurer-Cartan form on \(\mathcal{G}\), § 1.1
\(\iota_g\) injection determined by \(\mathfrak{g}_0\)-triple in
\(g\), § 2.5
\(\iota_G\) injection into \(G\) corresponding to \(\iota_g\),
§ 2.5
\(\xi_{g, h, \mathcal{X}/S}\) morphism
\(\mathcal{D}_{g, h, \mathcal{X}/S} \to \mathcal{D}_{g, h, \mathcal{X}/S}\) given by
\(p\)-curvature, § 3.6
\(\xi_{h, \mathcal{M}^{\mathcal{V}}(\mathcal{N})}\) Hitchin-Mochizuki
morphism from \(\mathcal{D}_{g, h, (\mathcal{P}, \mathcal{X}/S)}\), § 3.8
\(\lambda_{\mathcal{C}}^\mathcal{F}\) := \(\mathcal{F}_{c/k} \circ \lambda\), § 3.8
\(\lambda_{\mathcal{C}}^\mathcal{D}\) isomorphism of sheaves
\(\mathcal{D}_{g, h, \mathcal{X}/S} \to \mathcal{D}_{g, h, \mathcal{X}/S}\), § 4.3
\(\Gamma_{\mathcal{C}}^\mathcal{F}\) isomorphism of stacks
\(\mathcal{D}_{g, h, \mathcal{X}/S} \to \mathcal{D}_{g, h, \mathcal{X}/S}\), § 4.11
\(\Lambda_{\mathcal{C}}^\mathcal{D}\) isomorphism of sheaves
\(h^{} \mathcal{D}_{g, h, \mathcal{X}/S} \to \mathcal{D}_{g, h, \mathcal{X}/S}\), § 4.11
\(\Gamma_{\mathcal{C}}^\mathcal{D}\) isomorphism of stacks
\(h^{} \mathcal{D}_{g, h, \mathcal{X}/S} \to \mathcal{D}_{g, h, \mathcal{X}/S}\), § 4.11
\(\Lambda_{\mathcal{C}}^\mathcal{D}\) isomorphism of sheaves
\(\mathcal{D}_{g, h, \mathcal{X}/S} \to \mathcal{D}_{g, h, \mathcal{X}/S}\), § 4.11
\(\Gamma_{\mathcal{C}}^\mathcal{D}\) isomorphism of stacks
\(\mathcal{D}_{g, h, \mathcal{X}/S} \to \mathcal{D}_{g, h, \mathcal{X}/S}\), § 4.11
\(\mu_{\mathcal{C}}^{(c, \nabla_{\mathcal{E}})}\) monodromy of \((c, \nabla_{\mathcal{E}})\) at \(\sigma_1^0\),
§ 1.6
\(\nu_q^{(F, \nabla_{\mathcal{E}})}\) morphism \(A_{\text{Ker}(\nabla_{\mathcal{E}})} \to \mathcal{F}\)
arising from the adjunction relation
\(F_{U/S}(-) \to \mathcal{D}_{U/S}(-)\), § 3.1
\(\rho_D\) set of radii for \(D\) over \(S\), § 6.2
\(\rho_{\mathcal{G}}(\mathcal{G}, \nabla_{\mathcal{G}})\) radius of \((\mathcal{G}, \nabla_{\mathcal{G}})\) at \(\sigma_1\), § 2.9
\(\psi_{\mathcal{E}}(\mathcal{E}_{\mathcal{E}})\) curvature of \((\mathcal{E}, \nabla_{\mathcal{E}})\), § 1.2
\(\rho_{\mathcal{E}}(\mathcal{E}_{\mathcal{E}})\) \(p\)-curvature of \((\mathcal{E}, \nabla_{\mathcal{E}})\), § 3.2
\(\Psi_{\mathcal{E}, \gamma, Y}\) endomorphism of \(\mathcal{E} \times_k Y\) given
by \(\lambda \mapsto \lambda^F\), § 3.8
\(\chi\) Chevalley map, i.e., adjoint quotient from \(\mathcal{G}\) to \(\mathcal{C}\), § 2.8
\(\chi(\mathcal{X})\) adjoint quotient from \([\mathcal{G}/\mathcal{G}]\) to \(\mathcal{C}\),
§ 2.8
\(\omega_{\mathcal{X}/S}\) dualizing sheaf of \(X\) over \(S\), § 1.5
\(\Omega_{\mathcal{Y}^{T_0^{\log}}}\) sheaf of logarithmic
differentials of \(Y^{T_0^{\log}}\), § 1.2
\(\mathfrak{N}(\mathcal{G}) := (g - 1) \cdot \dim(\mathcal{G}) + \frac{1}{2} \cdot (\dim(\mathcal{G})
+ \text{rk}(\mathcal{G}))\), § 2.6
\(\mathfrak{N}(\mathcal{G}) := (g - 1) \cdot \dim(\mathcal{G}) + \frac{1}{2} \cdot (\dim(\mathcal{G})
- \text{rk}(\mathcal{G}))\), § 2.6
\([0]_S\) zero element in \(\mathfrak{c}(S)\), § 2.9
\([0]_S\) \(r\)-tuple of \([0]_S\), § 2.9
\(\partial_{\mathcal{O}}\) base dual of \(d\mathfrak{g}(x)\), § 1.2
\(\partial[p]\) \(p\)-th symbolic power of \(\partial\), § 3.2
\(\nabla_{\mathcal{E}, \eta}\) log connection \(\nabla_{\mathcal{E}}\) transformed by
\(\eta\), § 1.3
\(\nabla_{\mathcal{E}, \eta}\) difference between \(\nabla_{\mathcal{E}}\) and trivial connection w.r.t. trivialization \(\tau\),
§ 1.3
\[\nabla_{F}^{\text{can}}\text{ (or }\nabla_{F,h}^{\text{can}}\text{) \ canonical \ } T-1\text{-log}
\] 
\[\text{connection (or } T-h\text{-log connection) on } F_{Y/T}(F)\text{, }\S 3.3\]

\[\nabla_{F}^{T}\text{ \ } T-h\text{-log connection on } F^{\vee}\text{ induced by }\nabla_{F}\text{, }\S 4.1\]

\[\nabla_{F} \otimes \nabla_{G}\text{ \ } T-h\text{-log connection on } F \otimes G\text{ induced by }\nabla_{F}\text{ and }\nabla_{G}\text{, }\S 4.1\]

\[\nabla_{F}^{\otimes n}\text{ \ } T-h\text{-log connection on } F^{\otimes n}\text{ induced by }\nabla_{F}\text{, }\S 4.1\]

\[\nabla^{\bigoplus} (GL_{n}, h, B)\text{-oper, }\S 4.6\]

\[\nabla_{\otimes E}\text{ \ } (GL_{n}, h, B \otimes L)\text{-oper }\nabla^{\bigoplus}\text{ twisted by }\mathcal{L}, \S 4.7\]

\[\nabla^{\text{ad}}_{E}\text{ \ } S-h\text{-log connection on } g^{\cdot}_{E,B,h,x/S}\text{ induced by }\nabla_{E}, \S 5.2\]

\[\nabla^{\text{ad}}_{E}^{(j)}\text{ \ } \text{restriction of }\nabla^{\text{ad}}_{E}\text{ to } g^{j}_{E,B,h,x/S}, \S 5.2\]

\[\nabla^{\text{ad}}_{E}^{(j)+1}\text{ morphism from } g^{j}_{E,B,h,x/S}/g^{j+1}_{E,B,h,x/S}\text{ induced by }\nabla^{\text{ad}}_{E}, \S 5.2\]

\[\nabla^{\text{ad}}_{E}^{(j)}\text{ extension of }\nabla^{\text{ad}}_{E}^{(j)}\text{ to } T^{\cdot \log}_{E,B,h,x/S}/\mathcal{G}^{\log}, \S 5.3\text{ (}\S 5.5\text{)}\]

\[\nabla_{l,[\sigma_{i}]}\text{ \ } k\text{-log connection on } \mathcal{O}_{X}(l \cdot [\sigma_{i}])\text{ inducing } d^{\log}_{X}l, \S 8.3\]

\[\star\text{ \ } \mathcal{G}_{m}\text{-action on }\mathfrak{c}, \S 2.8\]

\[\star\text{ \ } \text{extension } \mathbb{A}^{1} \times_{k} \mathfrak{c} \to \mathfrak{c}\text{ of }\star, \S 2.8\]

\[\mathbb{A}_{B,h,x/S}^{\mathfrak{c}((\rho))}\text{ \ } \text{relative affine space over } S\text{ associated with }\Omega^{x}_{X^{\log}} \times^{\mathcal{G}^{m}} \mathfrak{c}, \S 3.7\]

\[\mathbb{V}_{E}(\mathfrak{c}(\rho))^{x/S}\text{ \ } \text{relative affine space over } S\text{ associated with }\Omega^{\mathfrak{c}}_{X^{\log}} \times^{\mathcal{G}^{m}} \mathfrak{c}, \S 3.7\]

\[\Omega^{x}_{X,h,x/S}\text{ \ } \text{morphism}\]

\[\mathbb{A}_{B,h,x/S} \to \mathbb{V}_{E}^{x/S}\text{ given by }\chi, \S 3.7\]

\[\text{zero section of }\mathbb{A}_{B,h,x/S}, \S 3.6\]

\[\text{triv}\text{ \ } \text{section of }\mathcal{D} p_{0,0,x/S}\text{ determined by trivial } (\mathfrak{g}, 0)\text{-oper, }\S 3.10\]

\[\text{isomorphism from } \mathcal{O}^{\otimes}_{E,B,h,x/S}\text{ given by twisting by }\mathcal{L}^{\vee}, \S 4.7\]

\[\text{isomorphism from } \mathcal{D} p_{E}^{\otimes}_{E,B,h,x/S}\text{ given by twisting by }\mathcal{L}^{\vee}, \S 4.11\]