Coloured quantum universal enveloping algebras

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Abstract

We define some new algebraic structures, termed coloured Hopf algebras, by combining the coalgebra structures and antipodes of a standard Hopf algebra set \( \mathcal{H} \), corresponding to some parameter set \( Q \), with the transformations of an algebra isomorphism group \( G \), herein called colour group. Such transformations are labelled by some colour parameters, taking values in a colour set \( C \). We show that various classes of Hopf algebras, such as almost cocommutative, coboundary, quasitriangular, and triangular ones, can be extended into corresponding coloured algebraic structures, and that coloured quasitriangular Hopf algebras, in particular, are characterized by the existence of a coloured universal \( R \)-matrix, satisfying the coloured Yang-Baxter equation. The present definitions extend those previously introduced by Ohtsuki, which correspond to some substructures in those cases where the colour group is abelian. We apply the new concepts to construct coloured quantum universal enveloping algebras of both semisimple and nonsemisimple Lie algebras, considering several examples with fixed or varying parameters. As a by-product, some of the matrix representations of coloured universal \( R \)-matrices, derived in the present paper, provide new solutions of the coloured Yang-Baxter equation, which might be of interest in the context of integrable models.

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I INTRODUCTION

Since its introduction, the parametrized (quantum) Yang-Baxter equation (YBE) [1] plays a crucial role in nonlinear integrable systems in physics, such as exactly solvable statistical mechanics models and low-dimensional integrable field theories [2]. Its constant form is also important in knot theory, where it is connected with braid groups [3].

In addition, the YBE has inspired the development of quantum groups and quantum algebras [4], essentially appearing in the literature in two different forms.

In the Faddeev-Reshetikhin-Takhtajan (FRT) formulation [5], to any invertible matrix solution $R$ of the constant YBE, one associates two bialgebras $A(R)$ and $U(R)$ that under certain conditions can lead to two dual Hopf algebras. For both of them, the constant YBE is a sufficient condition for associativity.

In the Drinfeld and Jimbo (DJ) approach [6], one considers one-parameter deformations $U_q(g)$ of the universal enveloping algebras (or quantum universal enveloping algebras (QUEA’s)) of simple Lie algebras $g$. Such quantizations are quasitriangular Hopf algebras, that is there exists a universal $R$-matrix, which is an invertible element of $U_q(g) \otimes U_q(g)$, and which among other properties, satisfies the YBE.

The DJ approach has been completed for nonsemisimple Lie algebras by applying various procedures, such as contractions of QUEA’s of simple Lie algebras [7, 8], and by introducing multiparametric deformations [9, 10].

A link has been established between the FRT and DJ formulations by considering for $R$ the matrix representing the operator $R$ in the fundamental representation of $g$ (see e.g. Ref. [11]). In such a case, the generators of $U(R)$ can indeed be expressed in terms of those of $U_q(g)$. The algebra $A(R)$, associated with $U(R)$, is then related to the quantized functions on the corresponding Lie group $G$, or quantum group, $\text{Fun}_q(G) = G_q$.

In recent years, some integrable models with nonadditive-type solutions $R^{\lambda,\mu} \neq R(\lambda - \mu)$ of the YBE have been discovered [12, 13]. The corresponding YBE

$$R_{12}^{\lambda, \mu} R_{13}^{\lambda, \nu} R_{23}^{\mu, \nu} = R_{23}^{\mu, \nu} R_{13}^{\lambda, \nu} R_{12}^{\lambda, \mu}$$

(1.1)

is referred to in the literature as the ‘coloured’ YBE, the nonadditive (in general multicomponent) spectral parameters $\lambda, \mu, \nu$ being considered as ‘colour’ indices [14].
Constructing solutions of Eq. (1.1) by starting from some quantum algebra has then become a topic of active research. Various approaches have been used for such a purpose [15]–[22]. Among them, one should mention a recent work of Bonatsos et al [22] on a nonlinear deformation $A_q^+(1)$ of $su(2)$, distinct from the DJ one, wherein the colour parameter is related with an involutive automorphism of the algebra, and serves to distinguish between the irreducible representations with the same dimension.

Extending the definitions of quantum groups and quantum algebras by connecting them to coloured $R$-matrices, instead of ordinary ones, is an interesting problem, which so far has not received much attention in the literature.

The generalization of the FRT approach has been discussed by Kundu and Basu-Mallick [19, 20, 23] for some quantizations of $U(gl(2))$ and $Gl(2)$. Such coloured extensions are characterized by generalized algebraic structures, but coalgebra structures identical with those of standard $A(R)$ and $U(R)$ algebras.

In the context of knot theory, Ohtsuki [24] has introduced coloured quasitriangular Hopf algebras, which are characterized by the existence of a coloured universal $R$-matrix, and he has applied his formalism to coloured representations of $U_q(sl(2))$ for $q$ a root of unity. A rather similar, but nevertheless distinct generalization has been independently considered by Bonatsos et al [22] for the above-mentioned $A_q^+(1)$ algebra, which has been endowed with a two-colour quasitriangular Hopf structure.

In the present paper, we extend the DJ formulation of QUEA’s to coloured ones by elaborating on the results of Bonatsos et al [22]. For such a purpose, in the next section we define coloured Hopf algebras in a way that generalizes Ohtsuki’s first attempt. In Secs. III and IV, we demonstrate the present definition generality and usefulness by reviewing various examples of coloured QUEA’s, then summarize and comment on prospects in Sec. V.

II COLOURED HOPF ALGEBRAS

Let $(\mathcal{H}_q, +, m_q, \iota_q, \Delta_q, \epsilon_q, S_q; k)$ (or in short $\mathcal{H}_q$) be a Hopf algebra over some field $k (= \mathbb{C} \text{ or } \mathbb{R})$, depending upon some parameters $q$. Here $m_q : \mathcal{H}_q \otimes \mathcal{H}_q \rightarrow \mathcal{H}_q$, $\iota_q : k \rightarrow \mathcal{H}_q$, $\Delta_q : \mathcal{H}_q \rightarrow \mathcal{H}_q \otimes \mathcal{H}_q$, $\epsilon_q : \mathcal{H}_q \rightarrow k$, and $S_q : \mathcal{H}_q \rightarrow \mathcal{H}_q$ denote the multiplication, unit,
comultiplication, counit, and antipode maps respectively\[25\]. Whenever \(q\) runs over some set \(Q\), called parameter set, we obtain a set of Hopf algebras \(\mathcal{H} = \{\mathcal{H}_q \mid q \in Q\}\). In the examples given in Secs. III and IV, we shall distinguish between two cases, according to whether \(Q\) contains a single element (fixed-parameter case) or more than one element (varying-parameter case).

Let us assume that there exists a set of one-to-one linear maps \(G = \{\sigma_{\nu} : \mathcal{H}_q \rightarrow \mathcal{H}_{q\nu} \mid q, q\nu \in Q, \nu \in C\}\), defined for any \(q \in \mathcal{H}\). They are labelled by some parameters \(\nu\), called colour parameters, taking values in some set \(C\), called colour set. The latter may be finite, countably infinite, or uncountably infinite. Two conditions are imposed on the \(\sigma_{\nu}\)'s:

(i) Every \(\sigma_{\nu}\) is an algebra isomorphism, i.e.,

\[
\sigma_{\nu} \circ m_q = m_{q\nu} \circ (\sigma_{\nu} \otimes \sigma_{\nu}), \quad \sigma_{\nu} \circ \iota_q = \iota_{q\nu};
\]

(ii) \(G\) is a group (called colour group) with respect to the composition of maps, i.e.,

\[
\forall \nu, \nu' \in C, \exists \nu'' \in C : \sigma_{\nu''} = \sigma_{\nu'} \circ \sigma_{\nu} : \mathcal{H}_q \rightarrow \mathcal{H}_{q\nu''} = \mathcal{H}_{q\nu'},
\]

\[
\exists \nu^0 \in C : \sigma_{\nu^0} = \text{id} : \mathcal{H}_q \rightarrow \mathcal{H}_{q^{\nu^0}} = \mathcal{H}_q,
\]

\[
\forall \nu \in C, \exists \nu' \in C : \sigma_{\nu'} = \sigma_{\nu} \equiv (\sigma_{\nu})^{-1} : \mathcal{H}_{q\nu} \rightarrow \mathcal{H}_q.
\]

In Eqs. (2.2) and (2.4), \(\nu''\) and \(\nu'\) will be denoted by \(\nu' \circ \nu\) and \(\nu^i\), respectively.

\(\mathcal{H}, \mathcal{C},\) and \(G\) can be combined into

**Definition II.1** The maps \(\Delta^{\lambda,\mu}_{q,\nu} : \mathcal{H}_{q\nu} \rightarrow \mathcal{H}_q \otimes \mathcal{H}_{q\nu}, \epsilon_{q,\nu} : \mathcal{H}_{q\nu} \rightarrow k,\) and \(S^\mu_{q,\nu} : \mathcal{H}_{q\nu} \rightarrow \mathcal{H}_{q^\nu},\)

defined by

\[
\Delta^{\lambda,\mu}_{q,\nu} \equiv (\sigma^\lambda \otimes \sigma^\mu) \circ \Delta_q \circ \sigma_{\nu}, \quad \epsilon_{q,\nu} \equiv \epsilon_q \circ \sigma_{\nu}, \quad S^\mu_{q,\nu} \equiv \sigma^\mu \circ S_q \circ \sigma_{\nu},
\]

for any \(q \in Q\), and any \(\lambda, \mu, \nu \in C\), are called coloured comultiplication, counit, and antipode respectively.

It is easy to prove the following proposition:
Proposition II.2 The coloured comultiplication, counit, and antipode maps, defined in Eq. (2.5), transform under the colour group $G$ as

$$
\left( \sigma_\alpha^{\lambda} \otimes \sigma_\beta^{\mu} \right) \circ \Delta^{\alpha,\beta}_{q,\nu} = \Delta^{\lambda,\mu}_{q,\nu} \circ \sigma_\gamma^{\gamma},
$$

$$
\epsilon_{q,\nu} \circ \sigma_\alpha^{\alpha} = \epsilon_{q,\nu},
$$

$$
\sigma_\mu^{\mu} \circ S^{\alpha}_{q,\nu} = S^{\mu}_{q,\nu} = S^{\mu}_{q,\beta} \circ \sigma_\nu^{\beta}, \tag{2.6}
$$

and satisfy generalized coassociativity, counit, and antipode axioms

$$
\left( \Delta^{\alpha,\beta}_{q,\lambda} \otimes \sigma_\mu^{\gamma} \right) \circ \Delta^{\lambda,\mu}_{q,\nu} = \left( \sigma_\lambda^{\alpha} \otimes \Delta^{\beta,\gamma}_{q,\mu'} \right) \circ \Delta^{\lambda',\mu'}_{q,\nu},
$$

$$
\left( \epsilon_{q,\nu} \otimes \sigma_\mu^{\alpha} \right) \circ \Delta^{\lambda,\mu}_{q,\nu} = \left( \sigma_\lambda^{\alpha} \otimes \epsilon_{q,\mu'} \right) \circ \Delta^{\lambda',\mu'}_{q,\nu} = \sigma_\nu^{\alpha},
$$

$$
m_{q^{\alpha}} \circ \left( S^{\alpha}_{q,\lambda} \otimes \sigma_\mu^{\alpha} \right) \circ \Delta^{\lambda,\mu}_{q,\nu} = m_{q^{\alpha}} \circ \left( S^{\alpha}_{q,\lambda} \otimes \sigma_\mu^{\alpha} \right) \circ \Delta^{\lambda',\mu'}_{q,\nu} = \iota_{q^{\alpha}} \circ \epsilon_{q,\nu}, \tag{2.7}
$$

as well as generalized bialgebra axioms

$$
\Delta^{\lambda,\mu}_{q,\nu} \circ m_{q^{\alpha}} = \left( \epsilon_{q,\nu} \otimes m_{q^{\alpha}} \right) \circ \left( \epsilon_{q,\nu} \otimes \tau \otimes \id \right) \circ \left( \Delta^{\lambda,\mu}_{q,\nu} \otimes \Delta^{\lambda,\mu}_{q,\nu} \right),
$$

$$
\Delta^{\lambda,\mu}_{q,\nu} \circ \iota_{q^{\alpha}} = \iota_{q^{\alpha}} \otimes \iota_{q^{\alpha}},
$$

$$
\epsilon_{q,\nu} \circ m_{q^{\alpha}} = \epsilon_{q,\nu} \otimes \epsilon_{q,\nu},
$$

$$
\epsilon_{q,\nu} \circ \iota_{q^{\alpha}} = 1_k. \tag{2.8}
$$

Here $\sigma_\mu^{\lambda}$ is the element of $G$ defined by

$$
\sigma_\mu^{\lambda} \equiv \sigma_\lambda \circ \sigma_\mu, \tag{2.9}
$$

$\tau$ is the twist map, i.e., $\tau(a \otimes b) = b \otimes a$, $1_k$ denotes the unit of $k$, and no summation is implied over repeated indices.

Proof. The various results are obtained by combining standard Hopf algebra axioms [25] with Definition II.1 and Eqs. (2.4)–(2.4). Consider for instance the first equation in (2.7). The map on the left-hand side, $$
\left( \Delta^{\alpha,\beta}_{q,\lambda} \otimes \sigma_\mu^{\gamma} \right) \circ \Delta^{\lambda,\mu}_{q,\nu} : \mathcal{H}_{q^{\alpha}} \to \mathcal{H}_{q^{\lambda}} \otimes \mathcal{H}_{q^{\mu}} \to \mathcal{H}_{q^{\gamma}} \otimes \mathcal{H}_{q^{\beta}} \otimes \mathcal{H}_{q^{\gamma}},
$$
can be proved to be identical with that on the right-hand one, $$
\left( \sigma_\lambda^{\alpha} \otimes \Delta^{\beta,\gamma}_{q,\mu'} \right) \circ \Delta^{\lambda',\mu'}_{q,\nu} : \mathcal{H}_{q^{\gamma}} \to
$$
\( \mathcal{H}_{q^\lambda} \otimes \mathcal{H}_{q^\mu} \rightarrow \mathcal{H}_{q^\alpha} \otimes \mathcal{H}_{q^\beta} \otimes \mathcal{H}_{q^\gamma} \), as follows:

\[
\begin{align*}
(\Delta^\alpha_\lambda \otimes \sigma^\gamma_\mu) \circ \Delta^\lambda_\mu &= \left( (\sigma^\alpha \otimes \sigma^\beta) \circ \Delta_\lambda \circ \sigma_\lambda \right) \otimes \left( (\sigma^\gamma \otimes \sigma^\mu) \circ \Delta_\gamma \circ \sigma_\mu \right) \\
&= \left( (\sigma^\alpha \otimes \sigma^\beta) \circ \Delta_\lambda \otimes \sigma_\lambda \right) \circ \left( \Delta_\gamma \circ \sigma_\gamma \right) \\
&= \left( \sigma^\alpha \otimes (\sigma^\beta \circ \sigma^\gamma) \right) \circ (\Delta_\lambda \otimes \sigma_\lambda) \circ \left( \sigma^\gamma \otimes (\sigma^\beta \circ \sigma^\gamma) \right) \circ \Delta_\gamma \circ \sigma_\gamma.
\end{align*}
\]

From Proposition II.2, it is straightforward to obtain

**Corollary II.3** If Eqs. (2.1)–(2.4) are satisfied, then for any \( q \in \mathcal{Q} \), any \( \nu \in \mathcal{C} \), and \( q, \nu \equiv q^\nu \), \( (\mathcal{H}_{q^+}, m_q, \iota_q, \Delta^\nu_\mu, \epsilon_{q^\nu}, S_{q^\nu}^\nu; k) \) is a Hopf algebra over \( k \) with comultiplication \( \Delta^\nu_\mu \), counit \( \epsilon_{q^\nu} \), and antipode \( S_{q^\nu}^\nu \), defined by particularizing Eq. (2.5).

**Remark.** In particular, for \( \nu = \nu^0 \), we get back the original Hopf structure of \( \mathcal{H}_q \).

Generalizing the result contained in Corollary II.3, we are led to introduce

**Definition II.4** A set of Hopf algebras \( \mathcal{H} \), endowed with coloured comultiplication, counit, and antipode maps \( \Delta^\lambda_\mu, \epsilon_{q^\nu}, S_{q^\nu}^\nu \), as defined in (2.5), is called coloured Hopf algebra, and denoted by any one of the symbols \( (\mathcal{H}, +, m_q, \iota_q, \Delta^\nu_\mu, \epsilon_{q^\nu}, S_{q^\nu}^\nu; k, \mathcal{Q}, \mathcal{C}, \mathcal{G}) \), \( (\mathcal{H}, \mathcal{C}, \mathcal{G}) \), or \( \mathcal{H}^c \).

As in standard Hopf algebras, the coloured antipode \( S_{q^\nu}^\nu \) satisfies some additional properties.

**Proposition II.5** The coloured antipode \( S_{q^\nu}^\nu \) of a coloured Hopf algebra \( \mathcal{H}^c \) fulfils the relations

\[
\begin{align*}
S_{q^\nu}^\nu \circ m_{q^\nu} &= m_{q^\nu} \circ \tau \circ \left( S_{q^\nu}^\nu \otimes S_{q^\nu}^\nu \right), & S_{q^\nu}^\nu \circ \iota_{q^\nu} &= \iota_{q^\nu}, \\
(S_{q^\nu}^\nu \otimes S_{q^\nu}^\nu) \circ \Delta^\lambda_\mu &= \tau \circ \Delta^\nu_\gamma \circ S_{q^\nu}^\nu, & \epsilon_{q^\nu} \circ S_{q^\nu}^\nu &= \epsilon_{q^\nu}. \tag{2.10}
\end{align*}
\]
Proof. Eq. (2.10) (resp. (2.11)) is obtained by combining Eqs. (2.1)–(2.5) with the first (resp. second) line in the following equation

\[ S_q \circ \sigma_q = \sigma_q \circ \tau \circ (S_q \otimes S_q), \quad S_q \circ \epsilon_q = \epsilon_q, \]

expressing the fact that \( S_q \) is an algebra (resp. coalgebra) antiautomorphism.

Let us now assume that the members of the Hopf algebra set \( \mathcal{H} \) are almost cocommutative Hopf algebras [25], i.e., for any \( q \in \mathcal{Q} \) there exists an invertible element \( R_q \in \mathcal{H}_q \otimes \mathcal{H}_q \) (completed tensor product), such that

\[
\tau \circ \Delta_q(a) = R_q \Delta_q(a) R_q^{-1} \quad \text{(2.12)}
\]

for any \( a \in \mathcal{H}_q \).

We may then introduce

**Definition II.6** Let \( \mathcal{R}^c \) denote the set of elements \( \mathcal{R}_q^\lambda,\mu \in \mathcal{H}_q^\lambda \otimes \mathcal{H}_q^\mu \), defined by

\[
\mathcal{R}_q^\lambda,\mu \equiv (\sigma^\lambda \otimes \sigma^\mu) (\mathcal{R}_q), \quad \text{(2.13)}
\]

where \( q \) runs over \( \mathcal{Q} \), and \( \lambda, \mu \) over \( \mathcal{C} \).

The following result can be easily obtained:

**Proposition II.7** If the Hopf algebras \( \mathcal{H}_q \) of \( \mathcal{H} \) are almost cocommutative, then \( \mathcal{R}_q^\lambda,\mu \), as defined in (2.13), is invertible with \( (\mathcal{R}_q^\lambda,\mu)^{-1} \) given by

\[
(\mathcal{R}_q^\lambda,\mu)^{-1} = (\sigma^\lambda \otimes \sigma^\mu) (\mathcal{R}_q^{-1}), \quad \text{(2.14)}
\]

and

\[
\tau \circ \Delta_{q,\lambda}(a) = \mathcal{R}_q^\lambda,\mu \Delta_{q,\lambda}(a) \left( \mathcal{R}_q^\lambda,\mu \right)^{-1} \quad \text{(2.15)}
\]

for any \( a \in \mathcal{H}_q^\nu \). If in addition, the almost cocommutative Hopf algebras \( (\mathcal{H}_q, \mathcal{R}_q) \) are (i) coboundary, (ii) quasitriangular, or (iii) triangular, then \( \mathcal{R}_q^\lambda,\mu \) also satisfies the relations...
\( (i) \)
\[
\mathcal{R}_{q,12}^{\alpha,\beta} \left( \Delta_{q,\lambda}^{\alpha,\beta} \otimes \sigma_{\mu}^{\gamma} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \mathcal{R}_{q,23}^{\beta,\gamma} \left( \sigma_{\lambda}^{\alpha} \otimes \Delta_{q,\mu'}^{\gamma,\lambda} \right) \left( \mathcal{R}_{q}^{\lambda',\mu'} \right),
\]
\[
\mathcal{R}_{q,21}^{\lambda,\mu} = \tau \left( \mathcal{R}_{q}^{\mu,\lambda} \right) = \left( \mathcal{R}_{q}^{\mu,\lambda} \right)^{-1},
\]
\[
\left( \epsilon_{q,\lambda} \otimes \epsilon_{q,\mu} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = 1_{k},
\]
\tag{2.16}

\( (ii) \)
\[
\left( \Delta_{q,\lambda}^{\alpha,\beta} \otimes \sigma_{\mu}^{\gamma} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \mathcal{R}_{q,13}^{\alpha,\gamma} \mathcal{R}_{q,23}^{\beta,\gamma},
\]
\[
\left( \sigma_{\lambda}^{\alpha} \otimes \Delta_{q,\mu}^{\beta,\gamma} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \mathcal{R}_{q,13}^{\alpha,\gamma} \mathcal{R}_{q,12}^{\alpha,\beta},
\]
\tag{2.17}

\( (iii) \)
\[
\left( \Delta_{q,\lambda}^{\alpha,\beta} \otimes \sigma_{\mu}^{\gamma} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \mathcal{R}_{q,13}^{\alpha,\gamma} \mathcal{R}_{q,23}^{\beta,\gamma},
\]
\[
\left( \sigma_{\lambda}^{\alpha} \otimes \Delta_{q,\mu}^{\beta,\gamma} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \mathcal{R}_{q,13}^{\alpha,\gamma} \mathcal{R}_{q,12}^{\alpha,\beta},
\]
\[
\mathcal{R}_{q,21}^{\lambda,\mu} = \left( \mathcal{R}_{q}^{\lambda,\mu} \right)^{-1},
\]
\tag{2.18}

respectively.

Hence we have

**Definition II.8** A coloured, almost cocommutative Hopf algebra is a pair \((\mathcal{H}^{c}, \mathcal{R}^{c})\), where \(\mathcal{H}^{c}\) is a coloured Hopf algebra, \(\mathcal{R}^{c} = \{ \mathcal{R}_{q}^{\lambda,\mu} | q \in \mathcal{Q}, \lambda, \mu \in \mathcal{C} \}\), and \(\mathcal{R}_{q}^{\lambda,\mu}\), defined in \((2.13)\), satisfies Eqs. \((2.14)\) and \((2.15)\). A coloured, almost cocommutative Hopf algebra \((\mathcal{H}^{c}, \mathcal{R}^{c})\) is said to be coboundary, quasitriangular, or triangular if \(\mathcal{R}_{q}^{\lambda,\mu}\) satisfies Eq. \((2.16)\), \((2.17)\), or \((2.18)\), respectively. In the case of a coloured quasitriangular Hopf algebra, the set \(\mathcal{R}^{c}\) is called the coloured universal \(\mathcal{R}\)-matrix of \((\mathcal{H}^{c}, \mathcal{R}^{c})\).

The terminology used for \(\mathcal{R}^{c}\) in Definition \(II.8\) is justified by the following proposition:

**Proposition II.9** Let \((\mathcal{H}^{c}, \mathcal{R}^{c})\) be a coloured quasitriangular Hopf algebra. Then
\[
\mathcal{R}_{q,12}^{\lambda,\mu} \mathcal{R}_{q,13}^{\lambda,\nu} \mathcal{R}_{q,23}^{\mu,\nu} = \mathcal{R}_{q,23}^{\mu,\nu} \mathcal{R}_{q,13}^{\lambda,\nu} \mathcal{R}_{q,12}^{\lambda,\mu},
\]
\[
\left( \epsilon_{q,\lambda} \otimes \sigma_{\mu}^{\alpha} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \left( \sigma_{\lambda}^{\alpha} \otimes \epsilon_{q,\mu'} \right) \left( \mathcal{R}_{q}^{\lambda',\mu'} \right) = 1_{k},
\]
\[
\left( \sigma_{q,\lambda}^{\alpha} \otimes \sigma_{\mu}^{\beta} \right) \left( \mathcal{R}_{q}^{\lambda,\mu} \right) = \left( \sigma_{\lambda}^{\alpha} \otimes \left( \mathcal{S}_{q,\beta}^{\mu'} \right)^{-1} \right) \left( \mathcal{R}_{q}^{\lambda',\mu'} \right) = \left( \mathcal{R}_{q}^{\alpha,\beta} \right)^{-1},
\]
\tag{2.19}
where \(1_{q^\nu}\) denotes the unit element of \(H_{q^\nu}\), and \((S_{q,\nu}^\mu)^{-1} : H_{q^\nu} \to H_{q^\nu}\) is given by \((S_{q,\nu}^\mu)^{-1} = \sigma^\nu \circ S_q^{-1} \circ \sigma_\mu\).

Remarks. (1) The first equation in (2.19) shows that the elements of the coloured universal \(\mathcal{R}\)-matrix satisfy the coloured YBE, as given in (1.1). (2) Following common use for standard Hopf algebras, we shall also call \(\mathcal{R}^c\) coloured universal \(\mathcal{R}\)-matrix if its elements satisfy Eq. (2.13) and the coloured YBE.

In those cases where the colour group \(G\) is abelian, one can always transform the colour parameters so as to make them additive. Let therefore \(\nu(p)\) be such that \(\nu'(p') \circ \nu(p) = (\nu' \circ \nu)(p + p')\), \(\nu(0) = \nu^0\), \(\nu'(0) = \nu'(0)\), and let denote \(H_{q^\nu(p)}\) by \(A_p\). The coloured comultiplication, counit, antipode, and universal \(\mathcal{R}\)-matrix, introduced in Definitions II.1 and II.6, can then be written as \(\Delta_{q^\nu(p)}: A_p \to A_p \otimes A_p\), \(\epsilon: A_p \to \mathbb{k}\), \(\Delta_{q^\nu(p)}: A_p \to A_{-p}\), and the invertible elements \(\mathcal{R}_{p_1p_2} \equiv R_{q^\nu(p_1),\nu(p_2)}\) of \(A_0 \otimes A_0\) satisfy the defining relations of an Ohtsuki’s coloured quasitriangular Hopf algebra [24], i.e.,

\[
\begin{align*}
(\Delta_{p_1p_2} \otimes \text{id}) \circ \Delta_{p_1+p_2,p_3} &= (\text{id} \otimes \Delta_{p_2p_3}) \circ \Delta_{p_1,p_2+p_3}, \\
(\epsilon \otimes \text{id}) \circ \Delta_{0,p} &= (\text{id} \otimes \epsilon) \circ \Delta_{p,0} = \text{id}, \\
m_p \circ (S_{-p} \otimes \text{id}) \circ \Delta_{-p,p} &= m_p \circ (\text{id} \otimes S_{-p}) \circ \Delta_{p,-p} = \iota_p \circ \epsilon, \\
\tau \circ \Delta_{p_2p_1}(a) &= R_{p_1p_2}^{-1} \Delta_{p_1p_2}(a) R_{p_1p_2}^{-1}, \\
(\Delta_{p_1p_2} \otimes \text{id})(R_{p_1+p_2,p_1}p) &= R_{p_1p_3} R_{p_2p_3,23}, \\
(\text{id} \otimes \Delta_{p_2p_3})(R_{p_1+p_2,p_3}) &= R_{p_1p_3} R_{p_1p_2,12},
\end{align*}
\]

(2.20)

with \(m_p \equiv m_{q^\nu(p)}\), \(\iota_p \equiv \iota_{q^\nu(p)}\), and \(a \in A_{p_1+p_2}\). Hence the latter is a substructure of \((\mathcal{H}^c, R^c)\).

Remark. As opposed to Ohtsuki’s coloured Hopf algebras, those considered in the present paper are also valid for nonabelian colour groups. Such a generalization is significant as it will be shown in the next two sections that coloured Hopf algebras with such colour groups can indeed be constructed.
III EXAMPLES OF COLOURED QUEA’S WITH FIXED PARAMETERS

In the present section, we construct various examples of coloured quasitriangular Hopf algebras, for which the underlying Hopf algebra set $\mathcal{H}$ reduces to a single element $\mathcal{H}_q$ (hence $Q = \{ q \}$ and $q^\nu = q$), which is some QUEA $U_q(g)$.

A The standard quantum algebra $U_q(sl(2))$

We begin by considering the simplest case of QUEA, namely the standard DJ deformation of $U(sl(2))$ [6], i.e., $U_q(sl(2))$ where $k = \mathbb{C}$ and $q = \exp(\eta) \in \mathbb{C} \setminus \{ 0 \}$, whose universal $R$-matrix was obtained in Ref. [26]. Although this example might look over-simple, it nevertheless serves three important purposes: to illustrate the fact that any QUEA can be easily transformed into a coloured one, to show that this can be achieved in various ways, and to demonstrate that some of them may involve a nonabelian colour group.

1 The colour group $G = S_2$

The quantum algebra $U_q(sl(2))$ is generated by three operators $J_3, J_\pm$, satisfying the commutation relations

$$[J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_3 \frac{\sinh(2\eta J_3)}{\sinh(\eta)},$$

(3.1)

where $[x]_q \equiv (q^x - q^{-x})/(q - q^{-1})$.

Such relations are left invariant under the transformation $\sigma(J_3) = -J_3, \quad \sigma(J_+) = J_-, \quad \sigma(J_-) = J_+$. Hence, defining $C = \{ +1, -1 \}$, we get a finite, abelian colour group $G = \{ \sigma^+ = \text{id}, \sigma^- = \sigma \}$, isomorphic to the symmetric group $S_2$. The action of $\sigma^\nu, \nu = \pm 1$, on the generators can be written in compact form as

$$\sigma^\nu (J_3) = \nu J_3, \quad \sigma^\nu (J_\pm) = J_\pm \nu.$$  \hspace{1cm} (3.2)

By using the results of Refs. [6, 26], and Definitions II.1 and II.6 of the present paper, we obtain the following coloured comultiplication, counit, antipode, and universal $R$-matrix:

$$\Delta^{\lambda,\mu}_{q,\nu} (J_3) = (\lambda \nu) J_3 \otimes 1 + (\mu \nu) 1 \otimes J_3.$$
\[ \Delta_{\lambda,\mu}^{\nu} (J_{\pm}) = J_{\pm \lambda \nu} \otimes q^\mu J_3 + q^{-\lambda} J_3 \otimes J_{\pm \mu \nu}, \]
\[ \epsilon_{\nu, \nu}(X) = 0, \quad X \in \{ J_3, J_\pm \}, \]
\[ S_{\nu}^{\mu} (J_3) = -\mu \nu J_3, \quad S_{\nu}^{\mu} (J_\pm) = -q^{\pm \nu} J_{\pm \mu \nu}, \]
\[ R_{q}^{\lambda,\mu} = q^{2 \lambda \mu J_3 \otimes J_3} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q !} q^{n(n-1)/2} (q^\lambda J_3 J_\lambda)^n \otimes (q^{-\mu} J_3 J_{-\mu})^n, \]

where \([n]_q ! \equiv [n]_q [n-1]_q \cdots [1]_q\) for \(n \in \mathbb{N}^+\), and \([0]_q ! \equiv 1\).

The matrix representation of the coloured universal \(R\)-matrix in any finite-dimensional representation of \(U_q(sl(2))\) provides us with a matrix solution \(R_{q}^{\lambda,\mu}\) of the coloured YBE, corresponding to discrete colour parameters \(\lambda, \mu = \pm 1\). For instance, in the two-dimensional representation of \(U_q(sl(2))\),
\[ D(J_3) = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D(J_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D(J_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

we get a (renormalized) \(4 \times 4\) coloured \(R\)-matrix \(R_{q}^{\lambda,\mu} \equiv q^{1/2} (D \otimes D) (R_{q}^{\lambda,\mu})\), whose components are given by
\[ R_{q}^{+,+} = (R_{q}^{-,-})^t = R_q = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \]
\[ R_{q}^{+-} = (R_{q}^{-,-})^t = \begin{pmatrix} 1 & 0 & 0 & q-1 \\ 0 & q & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

where \(t\) stands for matrix transposition.

2 The colour group \(G = Gl(1, \mathbb{C})\)

The commutation relations are also left invariant under the transformations
\[ \sigma^{\nu} (J_3) = J_3, \quad \sigma^{\nu} (J_\pm) = \nu^{\pm 1} J_\pm, \]

where \(\nu \in \mathbb{C} = \mathbb{C} \setminus \{0\}\). Since \(\nu' \circ \nu = \nu' \nu, \nu^0 = 1, \nu^1 = \nu^{-1}\), the colour group \(G\) is now isomorphic to the abelian Lie group \(Gl(1, \mathbb{C})\).
The corresponding coloured quasitriangular Hopf algebra is defined by (3.1) and
\[
\Delta_{q,\nu}^{\lambda,\mu} (J_3) = J_3 \otimes 1 + 1 \otimes J_3, \quad \Delta_{q,\nu}^{\lambda,\mu} (J_\pm) = \left( \frac{\lambda}{\nu} \right)^{\pm 1} J_\pm \otimes q^{J_3} + \left( \frac{\mu}{\nu} \right)^{\pm 1} q^{-J_3} \otimes J_\pm,
\]
\[
\epsilon_{q,\nu} (X) = 0, \quad X \in \{ J_3, J_\pm \},
\]
\[
S_{q,\nu}^\mu (J_3) = - J_3, \quad S_{q,\nu}^\mu (J_\pm) = - \left( \frac{\mu q}{\nu} \right)^{\pm 1} J_\pm,
\]
\[
R_{q}^{\lambda,\mu} = q^{2J_3 \otimes J_3} \sum_{n=0}^{\infty} \left( \frac{1 - q^{-2}}{[n]_q} \right)^n q^{n(n-1)/2} \left( \lambda q^{J_3} J_+ \right)^n \otimes \left( \mu^{-1} q^{-J_3} J_- \right)^n.
\]

(3.7)

In the two-dimensional representation (3.4) of \( U_q(sl(2)) \), we obtain a 4\( \times \)4 matrix solution of the coloured YBE,
\[
R_{q}^{\lambda,\mu} = q^{1/2} (D \otimes D) (R_{q}^{\lambda,\mu}) = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & \lambda \mu^{-1} (q^{-1} - q) & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix},
\]
(3.8)
depending upon continuous colour parameters \( \lambda, \mu \in \mathbb{C} \setminus \{ 0 \} \). Similar results can be derived for other finite-dimensional representations.

3 The colour group \( \mathcal{G} = Gl(1, \mathbb{C}) \otimes S_2 \)

The automorphisms, considered in Subsubsecs. I and II, can be combined by defining
\[
\sigma^\nu \equiv \sigma^{(\nu_1, \nu_2)} \equiv \sigma^{(\nu_1, +)} \circ \sigma^{(1, \nu_2)}, \quad \text{where } \sigma^{(\nu_1, +)} \text{ and } \sigma^{(1, \nu_2)} \text{ are given by Eqs. (3.6) and (3.2)},
\]
respectively. Hence the colour set is the cartesian product \( \mathcal{C} = (\mathbb{C} \setminus \{ 0 \}) \times \{ +1, -1 \} \),
\[
\sigma^\nu (J_3) = \nu_2 J_3, \quad \sigma^\nu (J_\pm) = \nu_1^{\pm \nu_2} J_{\mp \nu_2},
\]
(3.9)
and \( \nu' \circ \nu \equiv (\nu'_1, \nu'_2) \circ (\nu_1, \nu_2) = (\nu'_1 \nu_1^{\nu'_2}, \nu'_2 \nu_2) \), \( \nu^0 = (1, +) \), \( \nu^i = (\nu_1^{-\nu_2}, \nu_2) \). The colour group \( \mathcal{G} \) is nonabelian, and is a semidirect product group, \( Gl(1, \mathbb{C}) \otimes S_2 \). The subgroup \( Gl(1, \mathbb{C}) \) is indeed invariant, whereas \( S_2 \) is not, since \( \nu' \circ (\nu_1, +) \circ \nu'^i = (\nu_1^2, +) \), but \( \nu' \circ (1, \nu_2) \circ \nu'^i = ((\nu_1^2)^{1-\nu_2}, \nu_2) \).

Eqs. (3.7) and (3.8) are now replaced by
\[
\Delta_{q,\nu}^{\lambda,\mu} (J_3) = (\lambda_2 \nu_2) J_3 \otimes 1 + (\mu_2 \nu_2) 1 \otimes J_3,
\]
\[
\Delta_{q,\nu}^{\lambda,\mu} (J_\pm) = \lambda_1^{\pm \nu_2} \nu_1^{\mp 1} J_{\pm \nu_2} \otimes q^{\mu_2 J_3} + \mu_1^{\pm \nu_2} \nu_1^{\mp 1} q^{-\lambda_2 J_3} \otimes J_{\pm \nu_2},
\]

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\[ \epsilon_{q,\nu}(X) = 0, \quad X \in \{ J_3, J_\pm \}; \]
\[ S^{\mu}_{q,\nu}(J_3) = -\mu_2 \nu_2 J_3, \quad S^{\mu}_{q,\nu}(J_\pm) = -\mu_1^{\pm \mu_2 \nu_2} \nu_1^{\mp 1} q^{\pm \nu_2} J_{\pm \mu_2 \nu_2}, \]
\[ R_{q}^{\lambda,\mu} = q^{2 \lambda_2 \mu_2 J_3 \otimes J_3} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q} q^{n(n-1)/2} \left( \lambda_1^2 q^{\lambda_2 J_3 J\lambda_2} \right)^n \otimes \left( \mu_1^{-\mu_2} q^{-\mu_2 J_3 J_{-\mu_2}} \right)^n, \quad (3.10) \]

and

\[ R_{q}^{(\lambda_1,+),(\mu_1,+)} = \left( R_{q}^{(\lambda_1,-),(\mu_1,-)} \right)^t = \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & \lambda_1^{-1} (q - q^{-1}) & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}, \]
\[ R_{q}^{(\lambda_1,+),(\mu_1,-)} = \left( R_{q}^{(\lambda_1,-),(\mu_1,+)} \right)^t = \begin{pmatrix}
1 & 0 & 0 & \lambda_1 \mu_1 (q - q^{-1}) \\
0 & q & 0 & 0 \\
0 & 0 & q & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}, \quad (3.11) \]

respectively. The \(4 \times 4\) matrices defined in Eqs. (3.3), (3.8), and (3.11) are known five-vertex solutions of Eq. (1.1) [13].

B The two-parameter quantum algebra \(U_{q,s}(gl(2))\)

The next example deals with the two-parameter deformation of \(U(gl(2))\) [9], whose universal \(\mathcal{R}\)-matrix was given in Ref. [27]. Such an example is quite significant since \(U_{q,s}(gl(2))\) and its corresponding quantum group have played an important role both in generating some matrix solutions of the coloured YBE [17, 19], and in constructing a coloured extension of the FRT formalism [19, 20, 23].

The quantum algebra \(U_{q,s}(gl(2))\), for which \(k = \mathbb{C}\) and \(q, s \in \mathbb{C}\setminus\{0\}\), is generated by four operators \(J_3, J_\pm, Z\), with commutation relations

\[ [J_3, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = [2J_3]_q, \quad [Z, J_3] = [Z, J_\pm] = 0, \quad (3.12) \]

and coalgebra and antipode depending upon both parameters \(q\) and \(s\) (hence \(\mathcal{Q} = \{(q, s)\}\)).

Eq. (3.12) is left invariant under the transformations

\[ \sigma^\nu (J_3) = J_3, \quad \sigma^\nu (J_\pm) = J_\pm, \quad \sigma^\nu (Z) = \nu Z, \quad (3.13) \]
where \( \nu \in \mathcal{C} = \mathbb{C} \setminus \{0\} \). As in Subsubsec. A.2, the colour group is therefore \( \mathcal{G} = \text{Gl}(1, \mathbb{C}) \).

The coloured maps and universal \( \mathcal{R} \)-matrix are easily obtained as

\[
\Delta_{q,s,\nu}^{\lambda,\mu} (J_3) = J_3 \otimes 1 + 1 \otimes J_3, \quad \Delta_{q,s,\nu}^{\lambda,\mu} (Z) = \frac{\lambda}{\nu} Z \otimes 1 + \frac{\mu}{\nu} 1 \otimes Z,
\]

\[
\Delta_{q,s,\nu}^{\lambda,\mu} (J_{\pm}) = J_{\pm} \otimes q^{J_{\mu}} \left( \frac{s}{q} \right)^{\pm \mu Z} + q^{-J_3(qs)^{\pm \lambda Z}} \otimes J_{\pm},
\]

\[
\epsilon_{q,s,\nu} (X) = 0, \quad X \in \{ J_3, J_{\pm}, Z \},
\]

\[
S_{q,s,\nu}^{\mu} (J_3) = -J_3, \quad S_{q,s,\nu}^{\mu} (Z) = -\frac{\mu}{\nu} Z, \quad S_{q,s,\nu}^{\mu} (J_{\pm}) = -q^{\pm 1} s^{\mp 2 \mu Z} J_{\pm},
\]

\[
\mathcal{R}_{q,s}^{\lambda,\mu} = q^{2(J_3 \otimes J_3 - \lambda Z \otimes J_3 + \mu J_3 \otimes Z)} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{n(n-1)/2} \left( q^{J_3(qs) - \lambda Z} J_{\pm} \right)^n \otimes \left( q^{-J_3} \left( \frac{s}{q} \right)^{\mu Z} J_{\pm} \right)^n.
\]

By considering for instance the matrix elements of \( \mathcal{R}_{q,s}^{\lambda,\mu} \) in the \( 2 \times 2 \) defining representation of \( U_{q,s}(\text{gl}(2)) \), given by Eq. (3.4) and

\[
D(Z) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\]

we obtain the following \( 4 \times 4 \) matrix solution of the coloured YBE,

\[
R_{q,s}^{\lambda,\mu} \equiv q^{1/2} (D \otimes D) \left( \mathcal{R}_{q,s}^{\lambda,\mu} \right) = \begin{pmatrix} q^{1-\lambda+\mu} & 0 & 0 & 0 \\ 0 & q^{\lambda+\mu} (q - q^{-1}) s^{-\lambda+\mu} & 0 & 0 \\ 0 & 0 & q^{-\lambda-\mu} & 0 \\ 0 & 0 & 0 & q^{1+\lambda-\mu} \end{pmatrix}.
\]

The latter coincides with the coloured \( R \)-matrix previously derived by Burdík and Hellinger [17] by considering \( 2 \times 2 \) representations of \( U_{q,s}(\text{gl}(2)) \) characterized by different eigenvalues \( \lambda, \mu \) of \( Z \).

C  The three-parameter quantum algebra \( U_{q,s_1,s_2}(\text{sl}(3) \oplus u(1) \oplus u(1)) \)

Similar considerations to those in Subsec. B can be carried through for some other multi-parametric QUEA’s of nonsemisimple Lie algebras. As an example, let us consider the three-parameter deformation of \( U(\text{sl}(3) \oplus u(1) \oplus u(1)) \), constructed by Burdík and Hellinger [17].
It is generated by eight operators $H_i$, $X_i^\pm$, $Z_i$, $i = 1, 2$, with relations
\[
[H_i, X_j^\pm] = \pm 2X_i^\pm, \quad [H_i, X_j^\mp] = \mp X_j^\pm \quad (i \neq j),
\]
\[
[X_i^+, X_j^-] = \delta_{ij} [H_i]_q, \quad [X_1^\pm, X_2^\pm] = 0,
\]
\[
q^{-1/2}X_1^\pm X_3^\pm - q^{1/2}X_3^\pm X_1^\pm = q^{1/2}X_2^\pm X_3^\pm - q^{-1/2}X_3^\pm X_2^\pm = 0,
\]
\[
[Z_i, H_j] = [Z_i, X_j^\pm] = 0,
\]
where $X_3^\pm \equiv q^{1/2}X_1^\pm X_2^\pm - q^{-1/2}X_2^\pm X_1^\pm$. Here $k = \mathbb{C}$ and $Q = \{ (q, s_1, s_2) \}$, where $q$, $s_1$, $s_2 \in \mathbb{C} \setminus \{0\}$, and $s_1$, $s_2$ make their appearance only in the coalgebra structure and the antipode.

Eq. (3.17) is left invariant under the transformations
\[
\sigma^\nu (H_i) = H_i, \quad \sigma^\nu (X_i^\pm) = X_i^\pm, \quad \sigma^\nu (Z_i) = \nu_i Z_i,
\]
where $\nu \equiv (\nu_1, \nu_2) \in \mathcal{C} = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$. Hence the colour group $\mathcal{G}$ is a direct product group, $Gl(1, \mathbb{C}) \otimes Gl(1, \mathbb{C})$.

The coloured maps and universal $\mathcal{R}$-matrix are given by
\[
\Delta_{q,s_1,s_2,\nu}^{\lambda,\mu} (H_i) = H_i \otimes 1 + 1 \otimes H_i, \quad \Delta_{q,s_1,s_2,\nu}^{\lambda,\mu} (Z_i) = \frac{\lambda_i}{\nu_i} Z_i \otimes 1 + \frac{\mu_i}{\nu_i} 1 \otimes Z_i,
\]
\[
\Delta_{q,s_1,s_2,\nu}^{\lambda,\mu} (X_i^\pm) = X_i^\pm \otimes q^{H_i/2} (q_{s_1})^{\pm \mu_i Z_i/2} + q^{-H_i/2} \left( \frac{s_i}{q} \right)^{\pm \lambda_i Z_i/2} \otimes X_i^\pm,
\]
\[
\epsilon_{q,s_1,s_2,\nu} (X) = 0, \quad X \in \{ H_i, X_i^\pm, Z_i \},
\]
\[
S_{q,s_1,s_2,\nu}^{\mu} (H_i) = -H_i, \quad S_{q,s_1,s_2,\nu}^{\mu} (Z_i) = -\frac{\mu_i}{\nu_i} Z_i, \quad S_{q,s_1,s_2,\nu}^{\mu} (X_i^\pm) = -q^{\pm 2} s_i^{\pm \mu Z_i X_i^\pm},
\]
\[
\mathcal{R}_{q,s_1,s_2}^{\lambda,\mu} = q^{\sum_{i,j} s_i^{\dagger} \delta_{ij} H_i - \mu_j H_i - \mu_i Z_i + H_i \otimes H_j} E_{q^{-2}} \left( \alpha e^\lambda \otimes f^\mu_1 \right)
\times E_{q^{-2}} \left( -\alpha e^\lambda \otimes f^\mu_3 \right) E_{q^{-2}} \left( \alpha e^\lambda \otimes f^\mu_2 \right),
\]
where $\alpha \equiv 1 - q^{-2}$, $E_x (A)$ is the $q$-exponential
\[
E_x (A) = \sum_{n=0}^{\infty} \frac{x^{-n(n-1)/4}}{[n]_{x^{1/2}}!} A^n,
\]
a is the Cartan matrix of $sl(3)$, $a^{-1}$ its inverse,
\[
a = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix}, \quad a^{-1} = \frac{1}{3} \begin{pmatrix}
2 & 1 & 0 \\
1 & 2 & 0 \\
0 & 0 & 3
\end{pmatrix},
\]

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and

\[ e^\lambda_i = q^{H_i/2} \left( \frac{s_i}{q} \right)^{-\lambda_i Z_i/2} X_i^+, \quad f^\mu_i = q^{-H_i/2} (qs_i)^{\mu_i Z_i/2} X_i^- \]
\[ e^\lambda_3 = e^\lambda_1 e^\lambda_2 q^{-1} e^\lambda_1, \quad f^\mu_3 = f^\mu_1 f^\mu_2 q^{-1} f^\mu_2 f^\mu_1. \]  

(3.22)

In the \( 3 \times 3 \) defining representation of \( U_{q,s_1,s_2}(sl(3) \oplus u(1) \oplus u(1)) \), the coloured universal \( R \)-matrix, given in Eq. (3.19), gives rise to a \( 9 \times 9 \) matrix solution of the coloured YBE. As it coincides with the matrix given in Eqs. (28) and (29) of Ref. [17], we shall not reproduce it here.

IV  EXAMPLES OF COLOURED QUEA’S WITH VARYING PARAMETERS

The examples considered in the present section differ from those constructed in Sec. I by the fact that the parameter set \( Q \) now contains more than one element and the transformations of the colour group \( \mathcal{G} \) in general change the parameters (hence \( q^\nu \neq q \)). Although we simultaneously deal with elements \( a \) belonging to different Hopf algebras \( \mathcal{H}_{q^\nu} \) of the set \( \mathcal{H} \), we denote them in the same way in order not to overload notation by adding an extra index referring to the corresponding algebra, or colour parameter. This should cause no confusion since, from the context, it is always clear to which Hopf algebra \( a \) belongs.

A  The nonstandard quantum algebra \( U_h(sl(2)) \)

We begin by considering the nonstandard (Jordanian) deformation of \( U(sl(2)) \) [28], known as \( U_h(sl(2)) \), \( h \in \mathbb{C}\{0\} \), which can be obtained by contracting the standard DJ deformation \( U_q(sl(2)) \) [29]. Its universal \( R \)-matrix was derived in Refs. [30, 31].

\( U_h(sl(2)) \) is generated by three operators \( H, J_\pm \), satisfying the commutation relations [28]

\[ [H, J_+] = 2 \frac{\sinh(hJ_+)}{h}, \quad [J_+, J_-] = H, \]
\[ [H, J_-] = -J_- \cosh(hJ_+) - \cosh(hJ_+) J_- \].

(4.1)
Its universal $\mathcal{R}$-matrix assumes a very simple form \[31\] provided one chooses another basis, whose generators $A, A_\pm$ are defined in terms of the old ones by

$$A = e^{hJ_3}, \quad A_+ = J_+, \quad A_- = e^{hJ_+} \left( J_- - \frac{h}{4} \sinh(hJ_+) \right). \quad (4.2)$$

Eq. (4.1) is then transformed into

$$[A, A_+] = e^{2hA_+} - 1 \frac{1}{h}, \quad [A, A_-] = -2A_+ + hA^2, \quad [A_+, A_-] = A. \quad (4.3)$$

Such relations are left invariant under the transformations

$$\sigma^\nu(A) = A, \quad \sigma^\nu(A_+) = \nu A_+, \quad \sigma^\nu(A_-) = \nu^{-1} A_-,$$

where $\nu \in \mathbb{C} = \mathbb{C}\{0\}$, with the proviso that $h$ becomes $h^\nu = \nu h$ (hence $\mathcal{Q} = \mathbb{C}\{0\}$). The $\sigma^\nu$'s are therefore isomorphic mappings from $U_h(sl(2))$ to $U_{h^\nu}(sl(2))$, and define a colour group $\mathcal{G} = Gl(1, \mathbb{C})$. The corresponding coloured maps and universal $\mathcal{R}$-matrix are given by the relations

$$\Delta^{\lambda,\mu}_{h,\nu}(A_+) = \frac{\lambda}{\nu} A_+ \otimes 1 + \frac{\mu}{\nu} 1 \otimes A_+, \quad \Delta^{\lambda,\mu}_{h,\nu}(A) = A \otimes e^{2\mu hA_+} + 1 \otimes A,$$

$$\Delta^{\lambda,\mu}_{h,\nu}(A_-) = \frac{\nu}{\lambda} A_- \otimes e^{2\mu hA_+} + \frac{\nu}{\mu} 1 \otimes A_-,$$

$$\epsilon_{h,\nu}(X) = 0, \quad x \in \{A, A_\pm\},$$

$$S^{\mu}_{h,\nu}(A_+) = -\frac{\mu}{\nu} A_+, \quad S^{\mu}_{h,\nu}(A) = -Ae^{-2\mu hA_+}, \quad S^{\mu}_{h,\nu}(A_-) = -\frac{\nu}{\mu} A_- e^{-2\mu hA_+},$$

$$\mathcal{R}^{\lambda,\mu}_h = \exp \{-\lambda h A_+ \otimes A\} \exp \{\mu h A \otimes A_+\}. \quad (4.5)$$

The two-dimensional defining representation of $U_h(sl(2))$ is still given by Eq. (3.4), where $D(H) = 2D(J_3)$, hence [32]

$$D(A) = \begin{pmatrix} 1 & -h \\ 0 & -1 \end{pmatrix}, \quad D(A_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad D(A_-) = \begin{pmatrix} h & -h^2/4 \\ 1 & 0 \end{pmatrix}. \quad (4.6)$$

From Eq. (4.3), it is straightforward to obtain

$$R^{\lambda,\mu}_h \equiv (D \otimes D) \left( \mathcal{R}^{\lambda,\mu}_h \right) = \begin{pmatrix} 1 & \mu h & -\lambda h & (\lambda - \mu + \lambda \mu)h^2 \\ 0 & 1 & 0 & \lambda h \\ 0 & 0 & 1 & -\mu h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (4.7)$$

To the best of our knowledge, this $4 \times 4$ matrix solution of the coloured YBE is new. Higher-dimensional solutions of the latter could be obtained in a similar way by considering other finite-dimensional irreducible representations of the quantum algebra $U_h(sl(2))$ [31, 33].
The standard quantum oscillator algebra \( U_z^{(s)}(h(4)) \)

The next example is the standard deformation \( U_z^{(s)}(h(4)) \) of the oscillator algebra \( U(h(4)) \), which was first derived by contracting \( U_q(gl(2)) \) \cite{ref}, then recently obtained in a more convenient basis \cite{ref} by using the Lyakhovsky and Mudrov formalism \cite{ref}. This quantum algebra has been used to construct a solution of the coloured YBE connected with some link invariants \cite{ref}.

\( U_z^{(s)}(h(4)) \) is generated by four operators \( N, M, A_{\pm} \), satisfying the commutation relations

\[
\begin{align*}
[N, A_{\pm}] &= \pm A_{\pm}, \\
[A_{-}, A_{+}] &= \frac{\sinh(zM)}{z}, \\
[M, N] &= [M, A_{\pm}] = 0.
\end{align*}
\]

(4.8)

Here we assume \( k = \mathbb{C} \) and \( z \in \mathbb{Q} = \mathbb{C} \setminus \{0\} \). The algebra defining relations (4.8) are left invariant under the transformations

\[
\begin{align*}
\sigma^{\nu}(N) &= N, \\
\sigma^{\nu}(M) &= \nu_+ \nu_- M, \\
\sigma^{\nu}(A_{+}) &= \nu_+ A_{+}, \\
\sigma^{\nu}(A_{-}) &= \nu_- A_{-},
\end{align*}
\]

(4.9)

where \( \nu \equiv (\nu_+, \nu_-) \), provided \( z \) is changed into \( z^{\nu} = \nu_+ \nu_- z \). Hence the colour set is the cartesian product \( \mathcal{C} = (\mathbb{C}\setminus\{0\}) \times (\mathbb{C}\setminus\{0\}) \), and the colour group is the direct product group \( \mathcal{G} = Gl(1, \mathbb{C}) \otimes Gl(1, \mathbb{C}) \).

The corresponding coloured maps and universal \( \mathcal{R} \)-matrix are given by

\[
\begin{align*}
\Delta^{\lambda, \mu}_{z, \nu}(N) &= N \otimes 1 + 1 \otimes N, \\
\Delta^{\lambda, \mu}_{z, \nu}(M) &= \frac{\lambda_+ \lambda_-}{\nu_+ \nu_-} M \otimes 1 + \frac{\mu_+ \mu_-}{\nu_+ \nu_-} 1 \otimes M, \\
\Delta^{\lambda, \mu}_{z, \nu}(A_{+}) &= \frac{\lambda_-}{\nu_-} A_{+} \otimes 1 + \frac{\mu_+}{\nu_+} e^{-\lambda_+ \lambda_- zM} \otimes A_{+}, \\
\Delta^{\lambda, \mu}_{z, \nu}(A_{-}) &= \frac{\lambda_+}{\nu_+} A_{-} \otimes e^{\mu_+ \mu_- zM} + \frac{\mu_-}{\nu_-} 1 \otimes A_{-}, \\
\epsilon_{z, \nu}(X) &= 0, \quad X \in \{N, M, A_{\pm}\}, \\
S^{\mu}_{z, \nu}(N) &= -N, \quad S^{\mu}_{z, \nu}(M) = -\frac{\mu_+ \mu_-}{\nu_+ \nu_-} M, \quad S^{\mu}_{z, \nu}(A_{\pm}) = -\frac{\mu_\pm}{\nu_\pm} A_{\pm} e^{\pm \mu_+ \mu_- zM}, \\
\mathcal{R}^{\lambda, \mu}_{z} &= \exp\{-\lambda_+ \lambda_- zM \otimes N\} \exp\{-\mu_+ \mu_- zN \otimes M\} \\
&\quad \times \exp\{2\lambda_- \mu_+ zA_- \otimes A_{+}\}.
\end{align*}
\]

(4.10)

In the 3 × 3 matrix representation of \( U_z^{(s)}(h(4)) \) defined by

\[
\begin{align*}
D(N) &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
D(M) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\end{align*}
\]
\[ D(A_+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad D(A_-) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

the coloured universal \( R \)-matrix is represented by the \( 9 \times 9 \) matrix

\[
R^{\lambda,\mu}_z \equiv (D \otimes D) \left( R^{\lambda,\mu}_z \right) = \begin{pmatrix} 1_3 & 2\lambda_+ - \mu_+ zD(A_+) & -\lambda_+ \lambda_- zD(N) \\ 0_3 & 1_3 - \mu_+ \mu_- zD(M) & 0_3 \\ 0_3 & 0_3 & 1_3 \end{pmatrix},
\]

where \( 1_3 \) and \( 0_3 \) denote the \( 3 \times 3 \) unit and null matrices respectively.

**Remarks.**

1. Since in physical applications, \( A_+ \) (resp. \( A_- \)) is interpreted as a creation (resp. annihilation) operator, and \( N \) as a number operator, one actually deals there with a real form of \( U^{(s)}_z(h(4)) \), corresponding to the star operation (Hermitian conjugation)

\[
N^\dagger = N, \quad M^\dagger = M, \quad A_\pm^\dagger = A_\mp,
\]

and to real or imaginary values of \( z \). Eq. (4.13) restricts \( \nu_\pm \) in Eq. (4.9) to values satisfying the condition \( \nu_- = \overline{\nu_+} \), where the bar denotes complex conjugation. Hence, in such a case, \( \mathcal{C} = \mathbb{C} \setminus \{0\} \), and \( \mathcal{G} = \text{Gl}(1, \mathbb{C}) \).

2. It is easy to endow \( U^{(s)}_z(h(4)) \) with a coloured Hopf structure corresponding to a nonabelian colour group by combining transformations (4.9) with the elements of \( S_2 \), defined by \( \sigma^+ = \text{id} \), and \( \sigma^-(N) = -N, \sigma^-(M) = -M, \sigma^-(A_\pm) = A_\mp \) (implying \( z^\pm = z \)). This can be done along the same lines as in the example discussed in Subsubsec. III.A.3.

### C The one-parameter nonstandard quantum oscillator algebra \( U^{(n)}_z(h(4)) \)

Instead of the standard deformation of the oscillator algebra, dealt with in Subsec. B, we consider here the one-parameter nonstandard type I\(_+\) deformation of the same, constructed in Ref. [34], and denoted there by \( U^{(n)}_z(h(4)) \). For such an algebra, Eq. (4.8) is replaced by

\[
[N, A_+] = \frac{e^{zA_+} - 1}{z}, \quad [N, A_-] = -A_-, \quad [A_-, A_+] = Me^{zA_+}, \\
[M, N] = [M, A_\pm] = 0,
\]

where we choose \( z \in \mathcal{Q} = \mathbb{C} \setminus \{0\} \).
The defining relations (4.14) are left invariant under the transformations

$$\sigma^\nu(N) = N, \quad \sigma^\nu(M) = \nu M, \quad \sigma^\nu(A_+) = \nu A_+, \quad \sigma^\nu(A_-) = A_-,$$

provided $z$ is changed into $z^\nu = \nu z$. Hence $C = C \setminus \{0\}$, and $G = Gl(1, \mathbb{C})$.

The counterparts of Eqs. (4.10) and (4.12) are now

$$\Delta^\lambda_{z,\nu} (N) = N \otimes e^{\mu z A_+} + 1 \otimes N, \quad \Delta^\lambda_{z,\nu} (M) = \frac{\lambda}{\nu} M \otimes 1 + \frac{\mu}{\nu} 1 \otimes M,$$

$$\Delta^\lambda_{z,\nu} (A_+) = \frac{\lambda}{\nu} A_+ \otimes 1 + \frac{\mu}{\nu} 1 \otimes A_+, \quad \Delta^\lambda_{z,\nu} (A_-) = A_- \otimes e^{\mu z A_+} + 1 \otimes A_- + \mu z N \otimes M e^{\mu z A_+},$$

$$\epsilon_{z,\nu}(X) = 0, \quad X \in \{N, M, A_\pm\},$$

$$S^\mu_{z,\nu} (N) = -N e^{-\mu z A_+}, \quad S^\mu_{z,\nu} (M) = -\frac{\mu}{\nu} M, \quad S^\mu_{z,\nu} (A_+) = -\frac{\mu}{\nu} A_+,$$

$$S^\mu_{z,\nu} (A_-) = -A_- e^{-\mu z A_+} + \mu z N e^{-\mu z A_+},$$

$$\mathcal{R}^\lambda_{z,\nu} = \exp\{-\lambda z A_+ \otimes N\} \exp\{\mu z N \otimes A_+\},$$

and

$$R^\lambda_{z,\nu} \equiv (D \otimes D) \left( \mathcal{R}^\lambda_{z,\nu} \right) = \begin{pmatrix} 1_3 & 0_3 & 0_3 \\ 0_3 & 1_3 + \mu z D(A_+) & -\lambda z D(N) \\ 0_3 & 0_3 & 1_3 \end{pmatrix},$$

respectively, where $D$ is again defined by Eq. (4.11).

Remarks. (1) Eq. (4.14) is not compatible with the star operation (4.13). (2) In Eq. (4.13), we could multiply both $M$ and $A_-$ by some extra parameter, but this would not modify the coloured universal $\mathcal{R}$-matrix, as given in Eq. (4.16).

D The three-parameter nonstandard quantum oscillator algebra $U_{\vartheta, \beta, \delta}^{(II)} (h(4))$

Similar results can be obtained for more complicated deformations of the oscillator algebra. In the present subsection, we consider the three-parameter nonstandard deformation constructed in Ref. [34], where it is denoted by $U_{\vartheta, \beta, \delta}^{(II)} (h(4))$. The algebra defining relations are

$$[N, A_+] = A_+ - \beta_- V(-\vartheta), \quad [N, A_-] = -A_- - \beta_+ V(\vartheta),$$

$$[A_-, A_+] = M, \quad [M, N] = [M, A_\pm] = 0,$$
provide new matrix solutions of the coloured YBE. respectively, where

\[ \nu \equiv (\nu_+, \nu_-), \]

we assume \((\vartheta, \beta_+, \beta_-) \in \mathcal{Q} = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})\).

The transformations

\[ \sigma^\nu(N) = N, \quad \sigma^\nu(M) = \nu_+ \nu_- M, \quad \sigma^\nu(A_+) = \nu_+ A_+, \quad \sigma^\nu(A_-) = \nu_- A_-, \]

where \(\nu \equiv (\nu_+, \nu_-)\), leave Eq. (4.18) invariant, provided \(\vartheta, \beta_+, \beta_-\) are changed into \(\vartheta^\nu = \nu_+ \vartheta, \beta^\nu_+ = \nu_+^2 \beta_+, \beta^\nu_- = \nu_+ \nu_-^2 \beta_-\), respectively. Hence \(\mathcal{C} = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})\), and \(\mathcal{G} = \text{Gl}(1, \mathbb{C}) \otimes \text{Gl}(1, \mathbb{C})\).

The counterparts of Eqs. (4.10) and (4.12) are now

\[
\Delta^\lambda_{\vartheta, \beta^+, \beta^-}^\mu \nu(N) = \text{exp} \left\{ \lambda_+ \lambda_- \mathcal{M} \otimes (\vartheta N + \mu_+ \beta_+ A_+ + \mu_- \beta_- A_-) \right\} \text{exp} \left\{ \mu_+ \mu_- (\vartheta N + \mu_+ \beta_+ A_+ + \mu_- \beta_- A_-) \otimes \mathcal{M} \right\},
\]

and

\[
R^\lambda_{\vartheta, \beta^+, \beta^-}^\mu \equiv (D \otimes D) \left( R^\lambda_{\vartheta, \beta^+, \beta^-}^\mu \right)
\]

\[
= \begin{pmatrix}
1_3 & \lambda_- \mu_- \beta_- D(M) & -\lambda_+ \lambda_- (\vartheta D(N) + \mu_+ \beta_+ D(A_+)) + \mu_- \beta_- D(A_-) \\
0_3 & 1_3 + \mu_+ \mu_- \vartheta D(M) & \lambda_+ \mu_- \beta_- D(M) \\
0_3 & 0_3 & 1_3
\end{pmatrix},
\]

respectively, where \(D\) is again defined by Eq. (1.11). Eq. (1.22), as well as Eq. (1.17), provide new matrix solutions of the coloured YBE.
Remark. The real form of $U_{\vartheta, \beta_+, \beta_-}^{(1)}(h(4))$, corresponding to the star operation \((4.13)\), is obtained for $\vartheta = -\overline{\vartheta}$, and $\beta_- = -\overline{\beta_+}$. The colour parameters are then restricted by the condition $\nu_- = \overline{\nu_+}$, so that we are left with $C = \mathbb{C}\setminus\{0\}$, and $G = Gl(1, \mathbb{C})$.

E The standard three-dimensional quantum Euclidean algebra $U_w(e(3))$

In the present subsection, we consider the three-dimensional quantum Euclidean algebra $U_w(e(3))$, which was obtained by contracting the standard DJ deformation of $so(4)$ \[8\].

A basis of $U_w(e(3))$ is made of six operators $J_3$, $J_\pm$, $P_3$, $P_\pm$, generating rotations and translations in the $w \to 0$ limit respectively, and satisfying the commutation relations

\[
\begin{align*}
[J_3, J_\pm] &= \pm J_\pm, & [J_+, J_-] &= 2J_3 \cosh(2wP_3), \\
[J_3, P_\pm] &= [P_3, J_\pm] = \pm P_\pm, & [J_\pm, P_\pm] &= \pm \frac{\sinh(2wP_3)}{w}, \\
\end{align*}
\]

\begin{equation}
(4.23)
\end{equation}

Here we assume $k = \mathbb{R}$, and $w \in \mathbb{Q} = \mathbb{R}\setminus\{0\}$, which is compatible with the star operation usually imposed on $U(e(3))$, namely

\[
\begin{align*}
J_3^\dagger &= J_3, & J_\pm^\dagger &= J_\mp, & P_3^\dagger &= P_3, & P_\pm^\dagger &= P_\mp. \\
\end{align*}
\]

\begin{equation}
(4.24)
\end{equation}

The algebra defining relations \((4.23)\) are left invariant under the transformations

\[
\sigma^\nu(J_3) = J_3, \quad \sigma^\nu(J_\pm) = J_\mp, \quad \sigma^\nu(P_3) = \nu P_3, \quad \sigma^\nu(P_\pm) = \nu P_\mp,
\]

\begin{equation}
(4.25)
\end{equation}

where $\nu \in \mathbb{R}\setminus\{0\}$, provided $w$ is changed into $w^\nu = \nu w$. Hence the colour set and the colour group are $C = \mathbb{R}\setminus\{0\}$, and $G = Gl(1, \mathbb{R})$, respectively.

The coloured comultiplication, counit, antipode, and universal $\mathcal{R}$-matrix are easily found to be given by

\[
\begin{align*}
\Delta_{w,\nu}^{\lambda,\mu}(J_3) &= J_3 \otimes 1 + 1 \otimes J_3, & \Delta_{w,\nu}^{\lambda,\mu}(P_3) &= \frac{\lambda}{\nu} P_3 \otimes 1 + \frac{\mu}{\nu} 1 \otimes P_3, \\
\Delta_{w,\nu}^{\lambda,\mu}(J_\pm) &= J_\pm \otimes e^{\pm wP_3} + e^{-\lambda wP_3} \otimes J_\pm + w \left( \lambda P_\pm \otimes e^{wP_3} J_3 - \mu e^{-\lambda wP_3} J_3 \otimes P_\pm \right), \\
\Delta_{w,\nu}^{\lambda,\mu}(P_\pm) &= \frac{\lambda}{\nu} P_\pm \otimes e^{wP_3} + \frac{\mu}{\nu} e^{-\lambda wP_3} \otimes P_\pm,
\end{align*}
\]

\begin{equation}
(4.26)
\end{equation}
\[ \epsilon_{w,\nu}(X) = 0, \quad X \in \{J_3, J_\pm, P_3, P_\pm\}, \]
\[ S^\mu_{w,\nu}(J_3) = -J_3, \quad S^\mu_{w,\nu}(P_3) = -\frac{\mu}{\nu}P_3, \]
\[ S^\mu_{w,\nu}(J_\pm) = -(J_\pm \pm 2\mu w P_\pm), \quad S^\mu_{w,\nu}(P_\pm) = -\frac{\mu}{\nu}P_\pm, \]
\[ R^\lambda\mu_w = \exp \left\{ 2w(\lambda P_3 \otimes J_3 + \mu J_3 \otimes P_3) \right\} \exp \left\{ B^\lambda\mu_w \arcsinh \left( 2w A^\lambda\mu_w \right) / (w A^\lambda\mu_w) \right\} \]
\[ \times \left( 1 + 4w^2 \left( A^\lambda\mu_w \right)^2 \right)^{-1/2}, \]
\[ (4.26) \]

where

\[ A^\lambda\mu \equiv w Q^\lambda_+ \otimes Q^\mu_-, \]
\[ B^\lambda\mu \equiv w \left( L^\lambda_+ \otimes Q^\mu_- + Q^\lambda_+ \otimes L^\mu_- \right) - w^2 \left( 2Q^\lambda_+ \otimes Q^\mu_- + Q^\lambda_+ \otimes J_3 Q^\mu_- - J_3 Q^\lambda_+ \otimes Q^\mu_- \right), \]
\[ L^\lambda_\pm \equiv e^{\pm \lambda w P_3} J_\pm, \quad Q^\lambda_\pm \equiv \lambda e^{\pm \lambda w P_3} P_\pm. \]
\[ (4.27) \]

The quantum Euclidean algebra \( U_w(e(3)) \) admits the 4 \( \times \) 4 matrix representation

\[ D(J_3) = -ie_{12} + ie_{21}, \quad D(J_\pm) = \pm e_{13} - ie_{23} \pm e_{31} + ie_{32}, \]
\[ D(P_3) = e_{34}, \quad D(P_\pm) = e_{14} \pm ie_{24}, \]
\[ (4.28) \]

where \( e_{ij} \) denotes the matrix with entry 1 in row \( i \) and column \( j \), and zeros everywhere else. In such a representation, the coloured universal \( R \)-matrix is represented by the 16 \( \times \) 16 matrix

\[ R^\lambda\mu_w \equiv (D \otimes D) \left( R^\lambda\mu_w \right) \]
\[ = \begin{pmatrix}
1_4 & -2i\mu w D(P_3) & -2\mu w D(P_-) & 2\lambda w D(J_-) \\
2i\mu w D(P_3) & 1_4 & -2i\mu w D(P_-) & 2i\lambda w D(J_-) \\
2\mu w D(P_-) & 2i\mu w D(P_-) & 1_4 & 2\lambda w D(J_3) \\
0_4 & 0_4 & 0_4 & 1_4
\end{pmatrix}, \]
\[ (4.29) \]

which is a new solution of the coloured YBE.

Remark. Similar results can be obtained for the two-dimensional quantum Euclidean algebra \( U_w(e(2)) \), but in such a case no coloured universal \( R \)-matrix is known.
As a last example, we consider the null-plane deformations $U_z(\text{iso}(D - 1, 1))$ of the Poincaré algebras in $D = 2$ [30], $D = 3$ [37], and $D = 4$ dimensions [38]. Since the results look quite similar for different $D$ values, we only list here those for $D = 4$.

The quantum algebra $U_z(\text{iso}(3, 1))$ is generated by ten operators $K_3, J_3, P_+, P_-, P_1, P_2, E_1, E_2, F_1, F_2$, which, in the $z \to 0$ limit, go over into the following combinations of Poincaré algebra generators in the usual physical basis, $J_i$ (rotations), $K_i$ (boosts), and $P_{\mu}$ (translations), where $i = 1, 2, 3$, and $\mu = 0, 1, 2, 3$: $P_{\pm} = (P_0 \pm P_3)/2$, $E_1 = (K_1 + J_2)/2$, $E_2 = (K_2 - J_1)/2$, $F_1 = (K_1 - J_2)/2$, $F_2 = (K_2 + J_1)/2$. Their nonvanishing commutators are given by

$$
[K_3, P_+] = \frac{e^{2zP_+} - 1}{2z}, \quad [K_3, P_-] = -P_- - zP_1^2 - zP_2^2,
$$
$$
[K_3, E_i] = E_i e^{2zP_+}, \quad [K_3, F_i] = -F_i - 2zK_3P_i,
$$
$$
[J_3, P_i] = -\epsilon_{ij3}P_j, \quad [J_3, E_i] = -\epsilon_{ij3}E_j, \quad [J_3, F_i] = -\epsilon_{ij3}F_j,
$$
$$
[E_i, P_j] = \delta_{ij} \frac{e^{2zP_+} - 1}{2z}, \quad [F_i, P_j] = \delta_{ij} \left( P_- + zP_1^2 + zP_2^2 \right),
$$
$$
[E_i, F_j] = \delta_{ij} K_3 + \epsilon_{ij3}J_3 e^{2zP_+}, \quad [P_+, F_i] = -P_i, \quad [P_-, F_i] = -P_i,
$$

where $i, j$ run over 1, 2. Here we assume $k = \mathbb{R}$, and $z \in \mathcal{Q} = \mathbb{R} \setminus \{0\}$.

The algebra defining relations (4.30) are left invariant under the transformations

$$
\sigma^\nu(K_3) = K_3, \quad \sigma^\nu(J_3) = J_3, \quad \sigma^\nu(P_+) = \nu_1 \nu_2 P_+, \quad \sigma^\nu(P_-) = \nu_1 \nu_2^{-1} P_-,
$$
$$
\sigma^\nu(P_i) = \nu_1 P_i, \quad \sigma^\nu(E_i) = \nu_2 E_i, \quad \sigma^\nu(F_i) = \nu_2^{-1} F_i,
$$

where $\nu \equiv (\nu_1, \nu_2) \in \mathcal{C} = (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\})$, provided $z$ is changed into $z^\nu = \nu_1 \nu_2 z$. The corresponding colour group is $\mathcal{G} = \text{Gl}(1, \mathbb{R}) \otimes \text{Gl}(1, \mathbb{R})$.

The coloured maps and universal $\mathcal{R}$-matrix are found to be given by

$$
\Delta_{z,\nu}^\lambda(J_3) = J_3 \otimes 1 + 1 \otimes J_3, \quad \Delta_{z,\nu}^\lambda(P_+) = \frac{\lambda_1 \lambda_2}{\nu_1 \nu_2} P_+ \otimes 1 + \frac{\mu_1 \mu_2}{\nu_1 \nu_2} 1 \otimes P_+,
$$
\[ \Delta_{\lambda,\mu}^\nu(P_\pm) = \frac{\lambda_1 \nu_2}{\lambda_2 \nu_1} P_\mp \otimes e^{2 \mu_1 \mu_2 z} P_\pm + \frac{\mu_1 \nu_2}{\mu_2 \nu_1} 1 \otimes P_\pm, \]

\[ \Delta_{\lambda,\mu}^\nu(P_1) = \frac{\lambda_1}{\nu_1} P_1 \otimes e^{2 \mu_1 \mu_2 z} P_\pm + \frac{\mu_1}{\nu_1} 1 \otimes P_i, \quad \Delta_{\lambda,\mu}^\nu(E_i) = \frac{\lambda_2}{\nu_2} E_i \otimes 1 + \frac{\mu_2}{\nu_2} 1 \otimes E_i, \]

\[ \Delta_{\lambda,\mu}^\nu(F_1) = \frac{\nu_2}{\lambda_2} F_1 \otimes e^{2 \mu_1 \mu_2 z} P_\pm + \frac{\nu_2}{\mu_2} 1 \otimes F_1 - \frac{2 \lambda_1 \mu_2 \nu_2}{\lambda_2} z P_\mp \otimes E_1 e^{2 \mu_1 \mu_2 z} P_\pm, \]

\[ -2 \lambda_1 \nu_2 z P_2 \otimes J_3 e^{2 \mu_1 \mu_2 z} P_\pm, \]

\[ \Delta_{\lambda,\mu}^\nu(F_2) = \frac{\nu_2}{\lambda_2} F_2 \otimes e^{2 \mu_1 \mu_2 z} P_\pm + \frac{\nu_2}{\mu_2} 1 \otimes F_2 - \frac{2 \lambda_1 \mu_2 \nu_2}{\lambda_2} z P_\mp \otimes E_2 e^{2 \mu_1 \mu_2 z} P_\pm, \]

\[ +2 \lambda_1 \nu_2 z P_1 \otimes J_3 e^{2 \mu_1 \mu_2 z} P_\pm, \]

\[ \Delta_{\lambda,\mu}^\nu(K_3) = K_3 \otimes e^{2 \mu_1 \mu_2 z} P_\pm + 1 \otimes K_3 - 2 \lambda_1 \mu_2 z P_1 \otimes E_1 e^{2 \mu_1 \mu_2 z} P_\pm \]

\[ -2 \lambda_1 \mu_2 z P_2 \otimes E_2 e^{2 \mu_1 \mu_2 z} P_\pm, \]

\[ \epsilon_{\nu,\nu}(X) = 0, \quad X \in \{ K_3, J_3, P_\mp, P_i, E_i, F_i \}, \]

\[ S_{\lambda,\mu}^\nu(J_3) = -J_3, \quad S_{\lambda,\mu}^\nu(P_\pm) = -\frac{\mu_1 \mu_2}{\nu_1 \nu_2} P_\pm, \quad S_{\lambda,\mu}^\nu(P_\pm) = -\frac{\mu_1 \nu_2}{\mu_2 \nu_1} P_\mp e^{-2 \mu_1 \mu_2 z} P_\pm, \]

\[ S_{\lambda,\mu}^\nu(P_1) = -\frac{\mu_1}{\nu_1} P_1 e^{-2 \mu_1 \mu_2 z} P_\pm, \quad S_{\lambda,\mu}^\nu(E_i) = -\frac{\mu_2}{\nu_2} E_i, \]

\[ S_{\lambda,\mu}^\nu(F_1) = -\frac{\nu_2}{\mu_2} (F_1 + 2 \mu_1 \mu_2 z P_\mp E_1 + 2 \mu_1 \mu_2 z P_2 J_3) e^{-2 \mu_1 \mu_2 z} P_\pm, \]

\[ S_{\lambda,\mu}^\nu(F_2) = -\frac{\nu_2}{\mu_2} (F_2 + 2 \mu_1 \mu_2 z P_\mp E_2 - 2 \mu_1 \mu_2 z P_1 J_3) e^{-2 \mu_1 \mu_2 z} P_\pm, \]

\[ S_{\lambda,\mu}^\nu(K_3) = -(K_3 + 2 \mu_1 \mu_2 z P_1 E_1 + 2 \mu_1 \mu_2 z P_2 E_2) e^{-2 \mu_1 \mu_2 z} P_\pm, \]

\[ \mathcal{R}_{\lambda,\mu}^\nu = \exp \{ 2 \lambda_2 \mu_1 z E_2 \otimes P_2 \} \exp \{ 2 \lambda_3 \mu_1 z E_1 \otimes P_1 \} \exp \{ -2 \lambda_1 \lambda_2 z P_\pm \otimes K_3 \} \]

\[ \times \exp \{ 2 \mu_1 \mu_2 z K_3 \otimes P_\pm \} \exp \{ -2 \lambda_1 \mu_2 z P_1 \otimes E_1 \} \]

\[ \times \exp \{ -2 \lambda_1 \mu_2 z P_2 \otimes E_2 \}. \] (4.32)

The quantum Poincaré algebra \( U_z(\text{iso}(3,1)) \) admits the \( 5 \times 5 \) matrix representation

\[ D(K_3) = e_{14} + e_{41}, \quad D(J_3) = e_{23} - e_{32}, \quad D(P_\pm) = \frac{1}{2} (e_{10} + e_{40}), \]

\[ D(P_-) = e_{10} - e_{40}, \quad D(P_1) = e_{20}, \quad D(P_2) = e_{30}, \]

\[ D(E_1) = \frac{1}{2} (e_{12} + e_{21} - e_{24} + e_{42}), \quad D(E_2) = \frac{1}{2} (e_{13} + e_{31} - e_{34} + e_{43}), \]

\[ D(F_1) = e_{12} + e_{21} + e_{24} - e_{42}, \quad D(F_2) = e_{13} + e_{31} + e_{34} - e_{43}. \] (4.33)

where rows and columns are labelled by 0, 1, 2, 3, 4, and \( e_{ij} \) has the same meaning as in Subsec. In such a representation, the coloured universal \( \mathcal{R} \)-matrix gives rise to the
following new $25 \times 25$ matrix solution of the coloured YBE:

$$R_{\lambda,\mu}^z \equiv (D \otimes D) \left( R_{\lambda,\mu}^z \right)$$

$$= \begin{pmatrix}
1_5 & 0_5 & 0_5 & 0_5 & 0_5 \\
-\lambda_1 \lambda_2 z D(K_3) & 1_5 & \lambda_2 \mu_1 z D(P_1) & \lambda_2 \mu_1 z D(P_2) & 2 \mu_1 \mu_2 z D(P_2) \\
-2 \lambda_1 \mu_2 z D(E_1) & \lambda_2 \mu_1 z D(P_1) & 1_5 & 0_5 & -\lambda_2 \mu_1 z D(P_1) \\
-2 \lambda_1 \mu_2 z D(E_2) & \lambda_2 \mu_1 z D(P_2) & 0_5 & 1_5 & -\lambda_2 \mu_1 z D(P_2) \\
-\lambda_1 \lambda_2 z D(K_3) & 2 \mu_1 \mu_2 z D(P_1) & \lambda_2 \mu_1 z D(P_1) & \lambda_2 \mu_1 z D(P_2) & 1_5
\end{pmatrix}. \quad (4.34)$$

Remark. The $\kappa$-deformations $U_\kappa(iso(D - 1,1))$ of the $D$-dimensional Poincaré algebras can be transformed into coloured Hopf algebras along the same lines as $U_z(iso(D - 1,1))$, but in such a case no coloured universal $\mathcal{R}$-matrix is known.

V CONCLUSION

In the present paper, we did introduce some new algebraic structures, termed coloured Hopf algebras, by combining the coalgebra structures and antipodes of a standard Hopf algebra set with the transformations of an algebra isomorphism group, called colour group. We did show that various classes of Hopf algebras, such as almost cocommutative, coboundary, quasitriangular, and triangular ones, can be extended into corresponding coloured structures, and that coloured quasitriangular Hopf algebras, in particular, are characterized by the existence of a coloured universal $\mathcal{R}$-matrix, satisfying the coloured YBE.

Finally, we did apply the new concepts to QUEA’s of both semisimple and nonsemisimple Lie algebras, and did prove by means of examples that the colour group may be chosen as a finite or infinite, abelian or nonabelian group. Through such constructions, we did demonstrate that the coloured Hopf algebras defined here significantly generalize those previously introduced by Ohtsuki [24], because the latter are restricted to abelian colour groups, in which case they reduce to substructures of the former.

It is worth noting that some of the matrix representations of coloured universal $\mathcal{R}$-matrices constructed in the present paper, as well as those that would be obtained in higher-dimensional representations, provide new solutions of the coloured YBE, which might be of interest in the context of integrable models.
It is also important to stress that the applicability of the coloured Hopf algebra new concept is not confined to QUEA’s of Lie algebras. As we plan to show elsewhere, QUEA’s of Lie superalgebras may also provide a suitable starting point for constructing coloured Hopf algebras.

Other types of Hopf algebras might be used as well, such as those arising in the FRT formalism. The resulting coloured algebraic structures would significantly differ from those previously constructed by Kundu and Basu-Mallick [19, 20, 23], since the latter have the same coalgebra structure as the original Hopf algebras, whereas for the former it is the algebra structure that would be left unchanged. Further investigation of possible relationships between both types of coloured algebraic structures would be highly desirable.

In the examples considered in the present paper, no effort has been made to determine the maximal colour group — hence the maximal coloured Hopf structure — compatible with a given Hopf algebra set. Similarly, the restrictions on the colour parameters imposed by considering a given real form of a complex Hopf algebra have not been systematically investigated. Solving such problems might be interesting topics for future study.

Generalizing to coloured algebraic structures the duality relationship between pairs of Hopf algebras $U_q(g)$ and $G_q$, as highlighted in the universal $\mathcal{T}$-matrix formalism [40], might also be a promising direction for future investigation.

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