Dissipation-induced squeezing

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We present a method for phase and number squeezing in two-mode Bose systems using dissipation. The effectiveness of this method is demonstrated by considering cold Bose gases trapped in a double-well potential. The extension of our formalism to an optical lattice gives control of the phase boundaries of the steady-state phase diagram, and we discover a new phase characterized by a non-zero condensate fraction and thermal-like particle-number statistics. We also show how to perform amplitude squeezing in a single-mode system using dissipation.

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I. INTRODUCTION

In the usual situations, dissipation, caused by coupling to the environment, is considered to be a serious enemy to quantum-mechanical systems as it leads to a rapid decay of the coherence. Surprisingly, however, an appropriately designed coupling between the system and the reservoir can drive the system into a given pure state \[|\psi\rangle\]. This type of quantum-state engineering, driven by dissipation, recently has attracted considerable interest both theoretically \[9\] and experimentally \[10,11\]. A strong advantage of this approach is that the desired steady state is obtained without active control of the system. This should be contrasted with the standard approach in quantum-state engineering where dynamical control of the system is required (see, e.g., Ref. \[12\]). Another advantage is that the target state can be obtained regardless of the initial state, making the state-engineering protocol insensitive to imperfections in the initial-state preparation.

In this paper, we present a method for phase and number squeezing in two-mode Bose systems using dissipation. Our work is motivated by the importance of squeezed states in matter-wave interferometry \[13,14\]. By using squeezed states, the performance of an interferometer can be increased: Phase-squeezed states improve the accuracy of the readout of the phase difference, and number-squeezed states make longer measurement times possible. The scheme presented here gives a way to perform both phase and number squeezing in cold atomic gases and provides an important building block for dissipation-driven quantum-state engineering. Additionally, by extending our scheme to an optical lattice, we demonstrate that it can produce a new non-equilibrium phase characterized by a non-zero condensate fraction and thermal-like particle number statistics.

Our method can be applied to any two-mode Bose system, such as the one described by the two-site Bose-Hubbard Hamiltonian \[H_{\text{BH}} = -2J\hat{S}_x + U\hat{S}_z^2\]. This Hamiltonian is often used to describe cold Bose gases trapped in a double-well potential. Here, \(J\) is the tunneling matrix element, \(U\) is the on-site interaction, and we have introduced the SU(2) generators defined as \(\hat{S}_x = (\hat{a}_1^\dagger\hat{a}_2 + \hat{a}_2^\dagger\hat{a}_1)/2\), \(\hat{S}_y = -i(\hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2^\dagger\hat{a}_1)/2\), and \(\hat{S}_z = (\hat{a}_1\hat{a}_2 - \hat{a}_2\hat{a}_1)/2\), where \(\hat{a}_i(\hat{a}_i^\dagger)\) annihilates (creates) an atom in mode \(i\).

II. TWO-MODE SQUEEZING

First, we consider a two-state Bose system with a fixed number of particles. We assume that the system is coupled to an environment, leading to dissipative dynamics such that the time evolution of the density operator is governed by the master equation,

\[
\frac{\partial \hat{\rho}}{\partial t} = -i[\hat{H}, \hat{\rho}] + \gamma (2\hat{c}\hat{c}^\dagger - \hat{c}^\dagger\hat{c} - \hat{c}\hat{c}^\dagger\hat{c}^\dagger),
\]

where \(\hat{c}\) is the Lindblad, or jump, operator. We propose the following jump operator for the creation of phase- and number-squeezed states (hereafter called the squeezing jump operator):

\[
\hat{c} \equiv (\hat{a}_1^\dagger + \hat{a}_2^\dagger)(\hat{a}_1 - \hat{a}_2) + \epsilon (\hat{a}_1^\dagger - \hat{a}_2^\dagger)(\hat{a}_1 + \hat{a}_2) = 2(1 + \epsilon)\hat{S}_x - 2i(1 - \epsilon)\hat{S}_y.
\]

Here, \(\epsilon (-1 < \epsilon < 1)\) is a parameter by which we can control the squeezing \[21\]. The jump operator \(\hat{c}\) can be realized in a system of trapped ultracold Bose gas immersed in a background Bose-Einstein condensate of a different species of bosonic atoms \[21,22\].

In order to understand the action of our squeezing jump operator, we first consider the ideal case where there is no Hamiltonian and the dynamics is driven by the dissipative terms alone. In the double-well setting, for example, this can be achieved by making the potential barrier between the wells high enough so that \(J = 0\). The interaction \(U\) can be reduced using Feshbach resonances. We start by calculating the amount of squeezing in the steady state. It can be characterized by the normalized...
U/γ gives the analytical value of \( P[\rho] \). The solid lines in Fig. 1 show the time evolution of the squeezing and purity for \( N = 100, J = 0 \), and (a) \( \epsilon = -0.50 \), (b) \( \epsilon = 0.74 \). The solid (dashed) line shows the numerically obtained exact solution for \( U/\gamma = 0 (U/\gamma = 0.5) \), and the dashed-dotted line gives the analytical value of \( \xi_{P,N} \) evaluated using Eq. (3). The initial state is the coherent state \( \phi_{0n}\).}

\[
\xi_{P} \equiv \frac{2\langle \Delta S_{y,z}^{2} \rangle}{|\langle S_{y,z} \rangle|}, \quad \xi_{N}^{2} \equiv \frac{2\langle \Delta S_{y,z}^{2} \rangle}{|\langle S_{y,z} \rangle|},
\]

respectively, where \( \langle \Delta S_{y,z}^{2} \rangle = \langle S_{y,z}^{2} \rangle - \langle S_{y,z} \rangle^{2} \). In this work, the average spin is always parallel to the \( x \) axis, \( \langle S \rangle = \langle \hat{S}_{x} \rangle \hat{\phi}_{0} \). Using the coherent-state approximation \( \langle \hat{S}_{y,z}^{2} \rangle \approx \langle \hat{S}_{y,z} \rangle^{2} \) with \( \langle \hat{S}_{y,z} \rangle \approx N/2 + O(N^{0}) \), which holds for any value of \( \epsilon \), the number of particles \( N \gg 1 \), and the truncation scheme based on the Bogoliubov backreaction formalism \( \langle \hat{S}_{y,z} \rangle \langle \hat{S}_{y,z} \rangle + \langle \hat{S}_{y,z} \rangle \langle \hat{S}_{y,z} \rangle + \langle \hat{S}_{y,z} \rangle \langle \hat{S}_{y,z} \rangle \approx 2\langle \hat{S}_{y,z} \rangle \langle \hat{S}_{y,z} \rangle \), the equations of motion for \( \langle \hat{S}_{y,z}^{2} \rangle \) and \( \langle \hat{S}_{y,z}^{2} \rangle \) become

\[
\frac{d}{dt}\langle \hat{S}_{y,z}^{2} \rangle \approx -4N\gamma (1 - \epsilon^{2})\langle \hat{S}_{y,z}^{2} \rangle + N^{2}\gamma (1 + \epsilon)^{2}.
\]

In the second term on the right-hand side, the upper sign corresponds to \( y \), and the lower one corresponds to \( z \). The time constant \( \tau \) of \( \langle \hat{S}_{y}^{2} \rangle \) and \( \langle \hat{S}_{z}^{2} \rangle \), thus, is \( \tau^{-1} \approx 4N\gamma (1 - \epsilon^{2}) \). Since \( \langle \hat{S}_{y} \rangle = \langle \hat{S}_{z} \rangle = 0 \) throughout the time evolution, in the steady state, we get

\[
\langle \Delta S_{y,z}^{2} \rangle = \langle \hat{S}_{y,z}^{2} \rangle \approx \frac{N}{4} \left( 1 + \epsilon^{2} \right).
\]

where, again, the upper sign corresponds to \( y \) and the lower sign corresponds to \( z \). We see that a phase-squeezed state characterized by \( \xi_{P} < 1 \) is obtained for \( \epsilon < 0 \), while \( \epsilon > 0 \) yields a number-squeezed state for which \( \xi_{N} < 1 \).

In general, the steady state of the time evolution is a mixed state. The amount of mixedness can be quantified using purity \( P[\rho] = \{ (N + 1) \text{Tr}[\rho^{2}] - 1 \} / N \). For a pure state, \( P[\rho] = 1 \), while the completely mixed state gives \( P[\rho] = 0 \). The solid lines in Fig. 1 show the time evolution of the squeezing and purity for \( N = 100, J = 0 \) obtained by numerically solving the master equation (1) with the squeezing jump operator (2). We have set \( \epsilon = -0.5 \) (upper panels) and 0.74 (lower panels). The value \( \epsilon = 0.74 \) is chosen to demonstrate that our method can produce the same (and even larger) amount of squeezing than a different method used in an experiment with cold atomic gases [17]. This figure clearly demonstrates that, using the jump operator (2), we can obtain almost pure phase- and number-squeezed states.

Note that the exact form of the steady state and show that our jump operator (2) drives the system into a squeezed state. The jump operator can be written as \( \hat{c} = 4\sqrt{\epsilon} e^{\Delta S_{y}^{2}} e^{-\Delta S_{z}^{2}} \), where \( \chi = \arctanh[(1 - \epsilon)/(1 + \epsilon)] \).

If \( N \) is even, one of the eigenvalues is equal to zero, the corresponding normalized eigenstate is \( \phi_{0n} \propto e^{\Delta S_{y}^{2}}N/2 \).

Here, we use the notation \( |j \rangle \) for a state that has \( j \) particles in mode 1. State \( \phi_{0n} \) is a stationary state of the dynamics and can be written as \( \phi_{0n} = \sum s_{n=1/2} \alpha_{n} |N/2 - n \rangle \), where

\[
\alpha_{n} = A \left( N \right)^{-1/2} \times \frac{N/2}{N/2} \left( \frac{N/2}{N/2} \right) \left( \frac{1 - \sqrt{\epsilon}}{1 + \sqrt{\epsilon}} \right)^{2s-|n|},
\]

and \( A \) is a normalization factor. State \( \phi_{0n} \) is a phase-squeezed (\( \epsilon < 0 \)), a number-squeezed (\( \epsilon > 0 \)), or a coherent (\( \epsilon = 0 \)) state. When \( N \) is large, we can approximate

\[
\alpha_{n} \approx (2\pi \langle \Delta S_{y}^{2} \rangle)^{-1/4} e^{-n^{2}/(4\langle \Delta S_{y}^{2} \rangle)}.
\]

In deriving these results, \( N \) was assumed to be even. If \( N \) is odd, the steady state can be a mixed state. However, the non-zero elements of the vector \( \phi_{s} \phi_{n} \) scale as \( \epsilon^{N+1/2} \), and \( \phi_{0n} \) becomes an approximate dark state of the jump operator for large \( N \). Thus, in the large \( N \) limit, \( \phi_{0n} \) is a steady state, regardless of whether \( N \) is even or odd.

Now, we introduce the two-site Bose-Hubbard Hamiltonian in addition to the dissipative terms. For simplicity, we assume \( J = 0 \). We have checked numerically that the effect of non-zero \( J \) is much less important than that of non-zero \( U \) and does not change the results qualitatively.

The interaction term has the Fock states \( |j \rangle \) as the steady states, while the dissipative part drives the system toward state \( \phi_{0n} \), which is close to the coherent state \( \phi_{0n}|\epsilon=0 \). The steady state resulting from the competition between these terms is a mixed state. The \( N \) dependence of the purity and squeezing of the steady state decreases with increasing \( N \) and, at large \( N \), these quantities are determined only by \( \epsilon \) and \( U/\gamma \). The disappearance of the
in the steady state. The equations of motion for normalized correlation functions $\langle \hat{a}^n \hat{a}^m \rangle$ do not contain higher-order correlation functions and form a closed set of equations. Thus, the average particle numbers $\langle \hat{a}^\dagger \hat{a} \rangle$, the number fluctuation $\langle \Delta \hat{n}^2 \rangle \equiv \langle (\hat{a}^\dagger \hat{a})^2 \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2$, and the second-order coherence function $g^{(2)} \equiv \langle \hat{a}^\dagger (\hat{a})^2 \rangle / \langle \hat{a}^\dagger \hat{a} \rangle^2$ for the steady state can be calculated exactly,

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{\lambda^2}{1 - \epsilon^2} + \frac{\epsilon^2}{1 - \epsilon},$$  

(11)

$$\langle \Delta \hat{n}^2 \rangle = \lambda^2 \frac{1 + \epsilon}{(1 - \epsilon)^3} + \frac{2\epsilon^2}{(1 - \epsilon^2)^2},$$  

(12)

and

$$g^{(2)} = 1 + \frac{\epsilon}{(\hat{a}^\dagger \hat{a})^2} \left[ \frac{2\lambda^2}{(1 - \epsilon)^3} + \frac{\epsilon(1 + \epsilon^2)}{(1 - \epsilon^2)^2} \right].$$  

(13)

For $0 > \epsilon > \epsilon^*$, where $\epsilon^* < 0$ is a value such that $\langle \Delta \hat{n}^2 \rangle = \lambda^2$, we obtain a number-squeezed state $\langle \Delta \hat{n}^2 \rangle < \lambda^2$, which has a non-classical nature, characterized by $g^{(2)} < 1$. If $\epsilon > 0$, we obtain a number-anti-squeezed state $\langle \Delta \hat{n}^2 \rangle > \lambda^2$.

\[\text{IV. LATTICE SYSTEM}\]

Finally, we consider the application of our squeezing jump operator $\hat{c}$ to cold Bose gases in an optical lattice described by the Bose-Hubbard Hamiltonian. A natural extension of $\hat{c}$ for the lattice system is the following jump operator acting on sites $i$ and $j$:

$$\hat{c}_{ij} = (\hat{a}_i^\dagger + \hat{a}_j^\dagger)(\hat{a}_i - \hat{a}_j) + \epsilon(\hat{a}_i^\dagger - \hat{a}_j^\dagger)(\hat{a}_i + \hat{a}_j).$$  

(14)

Here, $\hat{a}_i$ is the annihilation operator at site $i$. The time evolution is given by the master equation,

$$\frac{\partial \hat{\rho}}{\partial t} = -i[H, \hat{\rho}] + (\gamma/2) \sum_{\langle i,j \rangle} (2\hat{c}_{ij} \hat{\rho} \hat{c}_{ij}^\dagger - \hat{c}_{ij}^\dagger \hat{c}_{ij} \hat{\rho} - \hat{\rho} \hat{c}_{ij}^\dagger \hat{c}_{ij}).$$  

(15)

To study the qualitative effect of the squeezing jump operator $\hat{c}$, we employ the generalized mean-field Gutzwiller approach. It consists of a product ansatz for the density operator $\hat{\rho} = \bigotimes \hat{\rho}_i$, where $\hat{\rho}_i \equiv \text{Tr}_\mathcal{E}[\hat{\rho}]$ are the reduced local density operators for site $i$ and the site-decoupling approximation $H = \sum_i \hat{h}_i$, with

$$\hat{h}_i = -J \sum_{\langle i', j \rangle} \langle \hat{a}_{i'} \rangle \hat{a}_{i'}^\dagger \langle \hat{a}_j \rangle + \langle \hat{a}_{i'}^\dagger \rangle \hat{a}_j - \mu \hat{n}_i + (U/2)\hat{n}_i(\hat{n}_i - 1)$$  

(16)

and $\hat{n}_i \equiv \hat{a}_i^\dagger \hat{a}_i$.

We are interested in the region of the higher filling factor $n \equiv \langle \hat{n} \rangle \gtrsim 3$ where the filling factor dependence becomes small and we can obtain a universal result. Besides, it has been shown that distinct commensurability effects are absent even at low $n$. In the following calculations, we consider a homogeneous system (hereafter,
we omit the site index $i$ with $\bar{n} = 4$ as an example. Because of the large $\bar{n}$, we expect that the generalized mean-field Gutzwiller approach is quantitatively reliable for higher dimensions. We choose a pure coherent state as the initial state of the local density operator and study the resulting steady state of the time evolution.

Figure 4(a) shows the local number fluctuation $\langle \Delta \hat{n}_i^2 \rangle^{1/2} \equiv (\langle \hat{n}_i^2 \rangle - \bar{n}_i^2)^{1/2}$ as a function of $U/\gamma z$ ($z$ is the coordination number) for $J/\gamma = 1$ and $\epsilon = -0.2, 0, 0.2$. As in the two-mode case, positive (negative) $\epsilon$ yields smaller (larger) $\langle \Delta \hat{n}_i^2 \rangle^{1/2}$ corresponding to number squeezing (anti-squeezing). In Fig. 4(b), we show the condensate fraction $|\psi|^2/\bar{n}$ of the steady state. Here, $\psi \equiv \langle \hat{\rho} \rangle$ is the order parameter. Similar to the two-mode system, the interaction term favors states with a definite number of particles and, thus, suppresses the off-diagonal order. Hence, $|\psi|^2/\bar{n}$ decreases monotonically with increasing $U$ and finally vanishes. The boundary between the phase with zero and non-zero $|\psi|^2/\bar{n}$ is shown by the dashed lines in Fig. 4(c) for various values of $J$. We find that the region of the condensed phase $|\psi|^2/\bar{n} > 0$ grows (shrinks) for positive (negative) $\epsilon$. This allows us to control the phase boundary of the non-equilibrium phase diagram Fig. 4(c) by changing the value of $\epsilon$.

We note that the behavior of $|\psi|^2/\bar{n}$ depends strongly on whether $\epsilon < 0$ or $\epsilon \geq 0$ [see Fig. 4(b)]. In the latter case, $|\psi|^2/\bar{n}$ decreases suddenly to zero (solid and dashed lines) while in the case of negative $\epsilon$, $|\psi|^2/\bar{n}$ first drops to a non-zero value and then gradually decreases to zero with further increasing $U/\gamma z$ (dashed-dotted line).

To better understand the properties of the state with a small but non-zero $|\psi|^2/\bar{n}$, we show the matrix elements $\rho_{nn}$ (in the Fock-state basis) of the steady-state density operator $\hat{\rho}$ at $U/\gamma z = 3.6$ for $\epsilon = -0.2$ and $J/\gamma = 1$ in Fig. 4(c). For comparison, we also show $\rho_{nn}$ for the condensed state with large $|\psi|^2/\bar{n}$ at $U/\gamma z = 3$ and a state with $|\psi|^2/\bar{n} = 0$ at $U/\gamma z = 5$. The state with $|\psi|^2/\bar{n} = 0$ is well described by the thermal state $\rho_{nn} = \bar{n}^n/(\bar{n} + 1)^{n+1}$, which does not have off-diagonal elements [see Figs. 4(a) and 4(d)]. Figure 4(a) clearly shows that the diagonal elements $\rho_{nn}$ at $U/\gamma z = 3.6$ are very close to $\rho_{nn}$ of the thermal state. Simultaneously, the steady state at $U/\gamma z = 3.6$ has non-zero off-diagonal elements [Fig. 4(c)]. We call such a state the thermal-condensed state. It is characterized by almost thermal particle-number statistics and a non-zero condensate fraction. We also note that the thermal-condensed state has slightly larger $\langle \Delta \hat{n}_i^2 \rangle^{1/2}$ than the thermal state [see Fig. 4(a)]. The thermal-condensed state appears as a new phase in the non-equilibrium steady-state phase diagram for $\epsilon < 0$. It is represented by the yellow region in Fig. 4(c). This phase is separated from the ordinary condensed phase by a jump in the order parameter [see dashed-dotted line in Fig. 4(b)] and it differs from the thermal phase by having a non-zero order parameter.
and a new phase characterized by a non-zero condensate
the steady-state phase diagram. It also allows us to real-
ize dissipation. When applied to an optical lattice, our
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FIG. 5: (Color online) The ratio of the exact values of \( \langle \hat{S}_x \rangle \)
and \( \sqrt{\langle \hat{S}_x^2 \rangle} \) to their approximate value, \( N/2 \). The parameters
are the same as for the solid lines in Fig. 6. \( N = 100 \), \( J = 0 \),
and \( U/\gamma = 0 \). The upper panels correspond to \( \epsilon = -0.50 \),
and the bottom ones correspond to \( \epsilon = 0.74 \).

V. CONCLUSION

We have proposed a way to produce number- and phase-squeezed states in a two-mode Bose system using dissipation. When applied to an optical lattice, our scheme can be used to control the phase boundaries of the steady-state phase diagram. It also allows us to realize a new phase characterized by a non-zero condensate fraction and thermal-like particle-number statistics.

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Appendix A: Expectation values of \( \hat{S}_x \) and \( \hat{S}_x^2 \) and Bogoliubov backreaction formalism

In the derivation of Eq. (5), we have used the following approximation:

\[
\langle \hat{S}_x \rangle, \sqrt{\langle \hat{S}_x^2 \rangle} \approx \frac{N}{2} \tag{A1}
\]

In order to illustrate the validity of this approximation, we plot the ratio of the exact and approximate expectation values in Fig. 5. The values of the parameters
are the same as in Fig. 6. Note that the approximation (A1) holds very well, and the relative error is less than 1% for \( \epsilon = -0.5 \) (upper panels) and less than 2.5% for \( \epsilon = 0.74 \) (lower panels) throughout the time evolution.

We have observed that the error of the approximation (A1) for the final steady states scales roughly as \( \sim 1/N \) and, thus, becomes even smaller for larger numbers of particles.

Another approximation used in our article was based on the so-called Bogoliubov backreaction formalism [22], which approximates the expectation value of a product of three operators as

\[
\langle \hat{S}_i \hat{S}_j \hat{S}_k \rangle \approx \langle \hat{S}_i \hat{S}_j \hat{S}_k \rangle + \langle \hat{S}_i \hat{S}_j \rangle \langle \hat{S}_k \rangle + \langle \hat{S}_i \hat{S}_k \rangle \langle \hat{S}_j \rangle + \langle \hat{S}_j \hat{S}_k \rangle \langle \hat{S}_i \rangle - 2 \langle \hat{S}_i \rangle \langle \hat{S}_j \rangle \langle \hat{S}_k \rangle \tag{A2}
\]

In the derivation of \( \langle \Delta S_x^2 \rangle \) in Eq. (5), we used

\[
\langle \hat{S}_x \hat{S}_y \rangle + \langle \hat{S}_y \hat{S}_x \rangle \approx \langle \hat{S}_x \hat{S}_y \rangle + \langle \hat{S}_x \rangle \langle \hat{S}_y \rangle + \langle \hat{S}_y \rangle \langle \hat{S}_x \rangle - 4 \langle \hat{S}_x \rangle \langle \hat{S}_y \rangle^2 \tag{A3}
\]

and, in the derivation of \( \langle \Delta S_x^2 \rangle \), we used

\[
\langle \hat{S}_x \hat{S}_x \rangle + \langle \hat{S}_x \rangle \approx \langle \hat{S}_x \rangle + \langle \hat{S}_x \rangle - 4 \langle \hat{S}_x \rangle \langle \hat{S}_x \rangle^2 \tag{A4}
\]
In Fig. 6 we plot the exact values of $\langle \hat{S}_x \hat{S}_y \rangle + \langle \hat{S}_y \hat{S}_x \rangle$ and $\langle \hat{S}_x \hat{S}_z \rangle + \langle \hat{S}_z \hat{S}_x \rangle$ divided by their approximate values given by Eqs. (A3) and (A4), respectively. The values of the parameters are the same as those in Fig. 1. We see that the relative error is less than 3% for $\epsilon = -0.50$ (upper panels) and is less than 7% for $\epsilon = 0.74$ (lower panels). As for $\langle \hat{S}_y \rangle$ and $\langle \hat{S}_z \rangle$, the error of the approximations (A3) and (A4) for the final steady states scales roughly as $\sim 1/N$, which justifies the use of the approximation for large $N$ as in the case of Fig. 1.

Finally, we note that the approximations are valid for the final states of the time evolution regardless of the initial state. In the calculations of Fig. 1 we chose the coherent state as the initial state. The system is, however, driven to the squeezed states no matter how the initial state is chosen [the proof has been given in the paragraph of Eq. (6) in Sec. II]. The squeezed states satisfy the approximate equations (A1) and (A3)-(A4) to the accuracy shown in Figs. 5 and 6. 

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[22] There we consider two narrow wells embedded in a wide harmonic potential. Each narrow well holds either state $\phi_1$ or state $\phi_2$, corresponding to $a_1$ and $a_2$. States $\phi_1$ and $\phi_2$ are Raman coupled to an even-parity state $\phi_e$ (with the Rabi frequencies $\Omega_1$ and $-\Omega_1$, respectively) and an odd-parity one $\phi_o$ (with equal Rabi frequency $\Omega_2$) in the wide harmonic potential. Atoms excited to $\phi_e$ and $\phi_o$ decay into $\phi_1$ and $\phi_2$ by emitting Bogoliubov excitations in the background Bose-Einstein condensate.
[23] M. Kitagawa and M. Ueda, Phys. Rev. A 47, 5138 (1993).
[24] See the Appendix A for a more detailed discussion on the validity of the approximation.
[25] A. Vardi and J. R. Anglin, Phys. Rev. Lett. 86, 568 (2001).
[26] For the steady states of our setup, the relative error of this approximation used in the derivation of Eq. (5) scales roughly as $1/N$.
[27] G. S. Agarwal, J. Opt. Soc. Am. B 5, 1940 (1988).
[28] At first sight, this is somewhat counterintuitive because $\epsilon < 0$ suppresses local phase fluctuations. This result can be understood by noting that, in the steady state, $|\rho_{n+i,n-i}| < \rho_{nn}$ for any integer $i$ such that $-n \leq i \leq n$ (see Fig. 1). In the number-squeezed state, the distribution of the diagonal elements $\rho_{nn}$ becomes more peaked. This makes having larger off-diagonal elements possible and enhances the condensate fraction. Numerical anti-squeezing results in the opposite behavior.