Derivation and generation of path-based valid inequalities for transmission expansion planning

J. Kyle Skolfield1 · Laura M. Escobar2 · Adolfo R. Escobedo1

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Abstract
This paper seeks to solve the long-term transmission expansion planning problem in power systems more effectively by reducing the solution search space and the computational effort. The proposed methodology finds and adds cutting planes based on structural insights about bus angle differences along paths. Two lemmas and a theorem are proposed which formally establish the validity of these cutting planes onto the underlying mathematical formulations. These path-based bus angle difference constraints, which tighten the relaxed feasible region, are used in combination with branch-and-bound to find lower bounds on the optimal investment of the transmission expansion planning problem. This work also creates an algorithm that automates the process of finding and applying the most effective valid inequalities, resulting in significantly reduced testing and computational time. The algorithm is implemented in Python, using Gurobi to add constraints and solve the exact DCOPF-based transmission expansion problem. This paper uses two different-sized systems to illustrate the effectiveness of the proposed framework: the GOC 500-bus system and a modified Polish 2383-bus system.

Keywords OR in energy · Mathematical modeling · Mixed-integer linear programming · Transmission expansion planning · Valid inequalities

J. Kyle Skolfield
kyle.skolfield@asu.edu
Laura M. Escobar
lauramonicaesva@gmail.com
Adolfo R. Escobedo
adRes@asu.edu

1 School of Computing and Augmented Intelligence (SCAI), Arizona State University, Tempe, AZ, USA
2 Electrical Engineering Department, São Paulo State University (UNESP), Ilha Solteira, São Paulo, Brazil
Nomenclature

Sets

\( n \in B \) \quad \text{Buses (i.e., nodes)}

\((i, j) \in \Omega\) \quad \text{Corridors (i.e., arcs)}

Parameters

\( c_{ij,k} \) \quad \text{Cost of line } k \text{ in corridor } (i, j)

\( c_n \) \quad \text{Per unit cost of generation at bus } n

\( \omega_{ij}^0 \) \quad \text{Number of established lines in corridor } (i, j)

\( \omega_{ij} \) \quad \text{Maximum number of expansion lines in corridor } (i, j)

\( \bar{s}_n \) \quad \text{Maximum limit of power generation at bus } n

\( d_n \) \quad \text{Active power demand at bus } n

\( \theta_{ij} \) \quad \text{Maximum bus angle difference magnitude}

\( \bar{P}_{ij,k} \) \quad \text{Capacity of candidate line } k \text{ in corridor } (i, j)

\( \bar{P}_{ij,k}^0 \) \quad \text{Capacity of existing line } k \text{ in corridor } (i, j)

\( x_{ij,k} \) \quad \text{Reactance of line } k \text{ in corridor } (i, j)

\( b_{ij,k} \) \quad \text{Susceptance of line } k \text{ in corridor } (i, j)

\( M_{ij} \) \quad \text{Large number (big-} M \text{) used in the disjunctive constraints}

\( \sigma \) \quad \text{Scaling factor to align generation and expansion costs}

Continuous Variables

\( P_{ij,k}^0 \) \quad \text{Active power flow in existing line } k \text{ in corridor } (i, j)

\( P_{ij,k} \) \quad \text{Active power flow in candidate line } k \text{ in corridor } (i, j)

\( g_n \) \quad \text{Active power output of generator at bus } n

\( \theta_n \) \quad \text{Voltage angle at bus } n

Binary Variables

\( y_{ij,k} \) \quad \text{Decision to construct the } k \text{th candidate line in corridor } (i, j)

1 Introduction

1.1 Background

The objective of the Transmission-network Expansion Planning (TEP) problem is to find the least costly investment options in new transmission devices required to ensure proper power system operations into the future (Garver 1970). Optimizing this problem is important because the transmission network belongs to the so-called heavy technologies, which are both expensive and difficult to withdraw or relocate once they are installed (Dominguez 2017). Inadequate long-term planning can lead to low service quality, excessive oversizing, inefficient systems with high operating costs, and delays in the expansion of electricity markets. While new systems are growing in size and the demands imposed on them are increasing, deregulation and other challenges have made meeting those requirements ever more difficult (Lumbreras and Ramos 2016). Hence, it is critical to obtain solutions that maximize cost efficiency to enable the incorporation of more avant-garde technologies into the smart grid. For these reasons, it is necessary to devise new planning methodologies that can effectively deal with the associated combinatorial difficulties of the underlying TEP optimization models.
In its standard form, TEP consists of linear and non-linear functions that include continuous variables (e.g., voltage angles, power flows, etc.) and integer variables (decisions to, e.g., add lines to the network). TEP can be formulated as a non-convex, mixed-integer nonlinear programming problem. It is NP-hard, which makes its solution generally intractable (Latorre et al. 2003). This is exacerbated by the fact that in large-scale systems, the number of network components and associated restrictions can number in the hundreds or thousands. That is, the size and/or topology of the transmission network and the inclusion of discrete variables for representing possible transmission investments lead to a combinatorial explosion of potential solutions. Due to these complications, TEP cannot practically be solved using standard optimization techniques, in general. Different modeling techniques and algorithms have been proposed to expedite solution times (e.g., Haghighat and Zeng 2018; Da Silva et al. 2018; Cabrera et al. 2018; Choi et al. 2006; Wickramarathna and Wickramarachchi 2006). Exact methods require larger computational times when compared to those required by metaheuristic techniques such as Tabu search (Gallego et al. 1998; García-Martínez et al. 2015) and genetic algorithms (Gallego et al. 1998; de Oliveira et al. 2005), among others. However, the latter techniques generally do not provide formal optimality guarantees. In small- and medium-sized systems, the ideal solution can be found using methods such as branch-and-bound or branch-and-cut when a disjunctive integer linear programming model approximation is utilized (Bahiense et al. 2001; Sousa 2011; Di et al. 2013). Such methods provide formal guarantees, but they are demanding computationally. They also include decomposition techniques, such as hierarchical Benders decomposition (e.g. Romero and Monticelli 1994; Binato et al. 2001; Haffner et al. 2001). Additionally, recent work has used Benders decomposition techniques to solve generation and transmission expansion planning together (Jenabi et al. 2015). The valid inequalities presented in this paper can be seen as a complementary technique to these exact methods.

1.2 Aim and contributions

This work considers a DCOPF-based mixed-integer programming version of the static TEP problem which consists of a single investment period occurring at the beginning of the planning horizon and is a subproblem of the dynamic TEP problem. The choice of this model helps illustrate the computational intractability of TEP even for this basic context and is useful for various practical studies. Moreover, it highlights the potential of the fundamental insights introduced herein to be extended to a variety of more complex TEP models with a similar core structure (e.g. Binato et al. 2001; Ploussard et al. 2017; Vinasco et al. 2011, etc.). Explicitly, this work derives and implements a set of theoretical contributions for detecting and including structural information on the underlying network, which is relevant to any DCOPF-based model that incorporates the linear relationship between bus angle differences and power flows (i.e., “$B - \theta$” constraints) into the constraint set. The lemmas proved herein generalize the relationship first formalized in Binato et al. (2001) between the shortest path problem, the longest path problem, and the big-M coefficients used in the standard TEP model. Specifically, they consider arbitrary paths—either alone or in parallel—and their derivation and implementation is independent of the longest path problem. It is important to highlight that the techniques introduced in Binato et al. (2001) only apply to static topologies (i.e., the big-M coefficients do not take advantage of potential topologies that can result from the addition of new lines). On the other hand, the main theorem proved herein (i.e., Theorem 1) is the first to explicitly tighten the big-M constraints mentioned here for dynamic topologies in TEP, which permits improvements beyond considering only the static topology. The insights...
presented in this paper may be applied to a variety of problem classes, since this structural information is common to many power system formulations. Such insights are captured via the concept of valid inequalities, which represent one of the most effective exact solution techniques and are a highly active research area in mathematical programming (Conforti et al. 2014).

Other papers have explored structural insights based on bus angle differences, which serve as the inspiration of this work. In particular, in Escobar and Romero (2017), a subset of the classes of valid inequalities introduced in this paper were applied in an ad hoc manner, in particular, only those from the herein included lemmas, which are proved in the present paper for the first time. It is worthwhile to explain that, while these lemmas provide insights for the major theorems derived in this paper, and their implementation could produce coincidental improvement in Gurobi due to the ordering of constraints, it is analytically impossible for them to reduce the linear relaxation solution space of TEP. This is because the valid inequalities presented in these lemmas are simply linear combinations of the original set of constraints. In short, the cited work lacked the systematic and theoretical depth featured in this work from an operations research perspective. The present paper formally establishes the validity of two classes of valid inequalities used therein via two lemmas, plus one additional class, proved via a theorem, which can in fact reduce the solution space of the linear relaxation.

In addition to these theoretical contributions, this work provides techniques for applying the theory in the form of a heuristic algorithm used to help find promising candidate valid inequalities (also referred to herein as cuts). These techniques are then used to perform computational experiments that show the effectiveness of the proposed valid inequalities in reducing the solution time of two modified benchmark instances. While their effectiveness is shown herein for the static TEP context, the reduction in solution time would be amplified in, for example, stochastic programming approaches to TEP. In these approaches, many scenarios need to be solved with each using the same collection of valid inequalities, since the first-stage decisions usually involve the structure of the network. A similar argument holds for solving the multi-period TEP.

The structure of the paper is as follows: Sect. 2 introduces the disjunctive model used for modeling TEP. Section 3 presents the key insights and intuition for deriving and generating the valid inequalities. Section 4 contains the main contribution of this work, namely the lemmas and theorem which prove the validity of the proposed cuts. Section 5 presents numerical results from testing the application of these theorems on two different test cases, and Sect. 6 summarizes the conclusions drawn from these results.

### 2 Modeling framework

The nonlinear ACOPF model for TEP can be transformed into a mixed-integer linear model with bilinear equations (Zhang 2013). This model is itself transformed into a disjunctive model with binary variables, which requires the incorporation of a large enough disjunctive coefficient (big-M). In the disjunctive model, a binary variable is considered for each candidate line, which converts the original mixed-integer non-linear program into a mixed-integer linear program (MILP). The DCOPF-based model is appropriate for TEP. First, it is widely used in industrial practice, especially for planning purposes (Kocuk et al. 2016). Additionally, this approach is the most common classical optimization approach in the literature (Skolfield 2021; Lumbreras and Ramos 2016). Finally, long-term planning is primarily concerned with active power rather than reactive power, and consequently the assumption in DCOPF that...
active power is much larger than reactive is reasonable. The main concerns that are only captured with an AC model (e.g., stability of the network) can be incorporated in a more short-term, operational perspective (Lumbreras et al. 2014). The full model is as follows.

The objective function (1) is to minimize the joint cost of generation and investments in new lines, with investment considered to be performed at the beginning of the planning horizon:

$$
\min \sum_{(i,j) \in \Omega} \sum_{k=1}^{\tilde{\omega}_{ij}} c_{ij,k} y_{ij,k} + \sum_{n \in B} \sigma c_n g_n.
$$

Here, $c_{ij,k}$ is the cost of each line in corridor $(i, j)$ and binary variable $y_{ij,k}$ represents the decision to add the $k^{th}$ candidate line in corridor $(i, j)$; when $y_{ij,k} = 1$, the $k^{th}$ candidate line is added in corridor $(i, j)$. Additionally, $\tilde{\omega}_{ij}$ is the maximum number of candidate lines considered in corridor $(i, j)$, and $\Omega$ is the set of expansion corridors in the expansion plan. Finally, note that the generation costs in the objective function are weighted by a factor $\sigma$ to make generation costs and planning costs comparable (Mínguez et al. 2018). The model’s constraints are as follows:

$$
\sum_{(n,i) \in \Omega} \left( \sum_{k=1}^{\bar{\omega}_{ni,k}} P_{ni,k}^0 + \sum_{k=1}^{\tilde{\omega}_{ni,k}} P_{ni,k} \right) - \sum_{(i,n) \in \Omega} \left( \sum_{k=1}^{\bar{\omega}_{ij,k}} P_{ij,k}^0 + \sum_{k=1}^{\tilde{\omega}_{ij,k}} P_{ij,k} \right) + g_n = d_n \quad \forall n \in B
$$

(2)

$$
-P_{ij,k}^0 \leq y_{ij,k} - y_{ij,k} \leq P_{ij,k}^0 \quad \forall (i,j) \in \Omega, k \in \{1 \ldots \bar{\omega}_{ij,k}\}
$$

(3)

$$
-P_{ij,k} \leq y_{ij,k} y_{ij,k} \leq P_{ij,k} \quad \forall (i,j) \in \Omega, k \in \{1 \ldots \tilde{\omega}_{ij,k}\}
$$

(4)

$$
-\frac{1}{b_{ij,k}} P_{ij,k}^0 - (\theta_i - \theta_j) = 0 \quad \forall (i,j) \in \Omega, k \in \{1 \ldots \bar{\omega}_{ij,k}\}
$$

(5)

$$
-M_{ij} (1 - y_{ij,k}) \leq -\frac{1}{b_{ij,k}} P_{ij,k} - (\theta_i - \theta_j) \leq M_{ij} (1 - y_{ij,k}) \quad \forall (i,j) \in \Omega, k \in \{1 \ldots \tilde{\omega}_{ij,k}\}
$$

(6)

$$
g_n \leq \bar{g}_n \quad \forall n \in B
$$

(7)

$$
-\bar{\theta} \leq \theta_i - \theta_j \leq \bar{\theta} \quad \forall (i,j) \in \Omega
$$

(8)

$$
y_{ij,k} \in \{0, 1\} \quad \forall (i,j) \in \Omega, k \in \{1 \ldots \tilde{\omega}_{ij,k}\}
$$

(9)

$$
g_n \geq 0, \theta_{unr.} \quad \forall n \in B
$$

(10)

$$
P_{ij,k}^0, P_{ij,k,unr.} \quad \forall (i,j) \in \Omega, k \in \{1 \ldots \tilde{\omega}_{ij,k}\}
$$

(11)

Constraint (2) interrelates the active power flows that arrive at and leave bus $n$ through both existing and candidate lines and the demand and supply of active power at bus $n$. Constraint (3) represents the limit of active power flow through the current network in corridor $(i,j)$, where $P_{ij,k}^0$ is the power flow in the $k^{th}$ existing line. Constraint (4) represents the limit of active power flow through the candidate lines in corridor $(i,j)$. Constraints (5) and (6) together represent Kirchhoff’s second law, either for each existing line or each candidate line to be added to the transmission system, respectively; stated otherwise, these constraints reflect the link between incident buses $i$ and $j$. Constraint (6) becomes active when the decision variable $y_{ij,k}$ takes the value of 1, i.e. when that candidate line is built; otherwise, a sufficiently large big-$M$ parameter $M_{ij}$ ensures that the constraint is extraneous for the model. Finding the best value for $M_{ij}$ requires solving a shortest path problem in connected networks but a
longest path problem in disconnected networks, which is itself an NP-hard problem (Binato et al. 2001). Because power networks are generally connected including in the instances tested herein, except for perhaps a handful of considered buses, these problems are solved while pre-processing the networks, to use the best possible big-$M$ parameter for each pair of buses. Constraint (7) presents the limits of the active power supply for generators, where a bus $n$ with no generator is assumed to have $\bar{g}_n = 0$. Constraint (8) enforces the maximum bus angle difference for adjacent bus-pairs $(i, j) \in \Omega$, i.e. those bus-pairs connected by a corridor. Finally, (9), (10) and (11) give the variable domains.

3 Motivating the path-based valid inequalities

Due to the combinatorial explosion of TEP, it is not possible to find an optimal solution for large-scale systems using standard, off-the-shelf algorithms. The computational difficulty of the problem is related directly to the size of the system to be analyzed. However, other factors increase computational difficulty, including the connectivity of the buses or how well the system is enmeshed. This is complicated by the “Braess Paradox,” according to which adding more transmission lines can create a more inefficient system (O’Neill et al. 2005).

To solve NP-hard problems, it is often useful to investigate the structural characteristics of a particular instance. This knowledge can be highly valuable when it comes to designing effective exact solution methods (Wolsey and Nemhauser 2014; Conforti et al. 2014). One key application of this knowledge is to derive valid inequalities (VIs): additional problem constraints that preserve the original solution space $P$ but may otherwise reduce an associated relaxed solution space $P^R \subseteq \mathbb{R}^n$, where $P \subset P^R$. Formally, for the set $P \subset \mathbb{R}^n$, the coefficient vector $\pi = (\pi_1, \ldots, \pi_n) \in \mathbb{R}^n$, and the constant $\pi_0 \in \mathbb{R}$, the inequality $\pi y \leq \pi_0$ is called a valid inequality for $P$ if it is satisfied by all points $y \in P$ (i.e., $P$ is the TEP solution space). Because the solution of MILP typically proceeds by solving a sequence of linear relaxations, adding structurally useful VIs as cutting planes can reduce the number of such linear problems solved in a branch-and-bound framework, thus decreasing the computational time necessary to solve the overall problem (Wolsey and Nemhauser 2014). The proposed method seeks to provide mechanisms that reduce the size of the solution space by incorporating structural information of TEP that can eliminate unpromising settings of the decision variables.

The structural insights derived in this work stem from the relationships between the bus angle and power flow decision variables that characterize DCOPF-based transmission system models. Specifically, if there is an existing line with index $k$ in corridor $(i, j) \in \Omega$, with $i < j$, an angular VI relating the difference between $\theta_i$ and $\theta_j$ can be obtained through $P_{ij,k}$ (the flow along the line), as follows:

$$\theta_i - \theta_j = -\frac{1}{b_{ij,k}} P_{ij,k} = x_{ij,k} \bar{P}_{ij,k} \leq x_{ij,k} \bar{P}_{ij,k}, \quad (12)$$

where $b_{ij}$, $x_{ij}$, and $\bar{P}_{ij,k}$ are the line susceptance, line reactance, and flow capacity, respectively. The right-hand side of this inequality is referred to henceforth as a capacity-reactance product and is used to improve angular VIs presented herein. Note that (12) is a direct result of (4)–(6). The main insight of this work is to leverage such adjacent-bus VIs to derive formal restrictions on non-adjacent buses and on buses connected via multiple parallel paths in the network. That is, the TEP model (and the DCOPF model, generally) provides only simple angular constraints for the buses that are directly connected via a transmission line.
However, by forming a single path connecting adjacent buses in the transmission network, these VIs can be combined into potentially tighter path-based constraints relating the initial bus angle and the terminating bus angle of said path and the sequence of flow restrictions of each corridor along the path. Even stronger restrictions may be obtained from the combination of VIs along parallel paths—two otherwise disjoint paths which share initial and terminating buses—by taking the tighter of the separate bus angle difference restrictions or, equivalently, flow restrictions. An example application of these insights is illustrated in Fig. 1 via a stylized bus-line diagram consisting of bus set $B = \{i_0, i_1, i_2\}$, corridor set $\Omega = \{(i_0, i_1), (i_0, i_2), (i_1, i_2)\}$, and single lines between each pair of buses with reactances $x_{i_0,i_1} = x_{i_1,i_2} = x$, $x_{i_0,i_2} = 3x$ and capacities $\bar{P}_{i_0,i_1} = \bar{P}_{i_1,i_2} = \bar{P}_{i_0,i_2} = \bar{P}$. For this simple example, and for all future numerical examples, we assume that there can be at most one existing line and at most one candidate line per corridor. This allows us to increase visual clarity by dropping the third index of each variable.

In Fig. 1, three path-based VIs (adjacent to each transmission line) are obtained by considering the capacity-reactance products of every pair of buses in the network [see (12)]. Moreover, by combining two of these VIs, a tighter VI for bus angles $\theta_{i_0}$ and $\theta_{i_2}$ is obtained (see the boxed expression). It is important to remark that this constraint would be valid even in the absence of a direct transmission line between $\theta_{i_0}$ and $\theta_{i_2}$, i.e. if it were an expansion corridor. In larger networks, many such VIs can be constructed, which may or may not tighten the model’s simple bus angle difference constraints. In highly enmeshed electric systems, the number of parallel paths can increase exponentially, depending on the specific network properties (Kavitha et al. 2009). Consequently, it may be prohibitive to identify and verify the strength of each possible VI for large-size systems. Instead, this work will identify the most effective of these constraints and provide data-driven insights through the use of relaxation models that are easier to solve.

We utilize the above ideas to generate a set of structurally useful VIs based on the flows in the solution to TEP that follow single and parallel paths. To this end, we make use of three relaxed models. By solving a subset of these models, each of which takes significantly less time to solve than the full MILP, we can generate a set of structural backbones. These are flow patterns that suggest single paths and parallel paths that are more likely to occur than others in the solution to the original problem. In particular, for any single path or parallel paths which share a common flow direction in the solution of each of a combination of relaxation models, we consider adding a VI. The technique of using these relaxation models in this
way will be denoted the low-effort heuristic, first implemented in a non-algorithmic way in Escobar and Romero (2017). Three models are used: the linear model, where the restriction on the binary variables \( y_{ij,k} \) is relaxed, allowing them to be continuous within the interval \([0, 1]\); the transportation model, where the restriction that flows on all lines obey (5) and (6) is relaxed; and the hybrid model, which is similar to the transportation model, but in which only (6) is relaxed.

### 4 Path-based angular valid inequalities derivation and theorems

This section will introduce the main theorem which is the fundamental contribution of this work. For this purpose, a graph with candidate lines (dotted edges) and existing lines (solid edges) is presented in Fig. 2. An example of these lines can be seen between buses \( i_1 \) and \( i_2 \), where there is one candidate line and one existing line. This graph will be used to illustrate an application of each lemma and theorem.

We say then that \((i, j)\) is an established corridor of \( G \) if \( \omega_{i,j}^0 > 0 \) (i.e., there is an existing line along the corridor); otherwise we say that \((i, j)\) is an expansion corridor. To better clarify instances when we must distinguish among individual lines within each corridor along a path, we introduce the vector \( \hat{k}_\rho = \left( k_{i_0,i_1}, \ldots, k_{|\rho|-1,i_{|\rho|-1}} \right) \subseteq \left\{ 1, \ldots, \omega_{i_0,i_1}^0, \ldots, 1, \ldots, \omega_{i_{|\rho|-1},i_{|\rho|-1}}^0 \right\} \) to denote any vector of valid line-indices \( k_{ij} \) within each established corridor \((i, j)\) along a path \( \rho \). Then, for ease of presentation, we refer to \( x_{ij,k} \), where \( k \) encapsulates a valid setting of element \( ij \) of vector \( \hat{k} \), i.e. \( k \in \{1, \ldots, \omega_{i,j}^0\} \). For example, if the expansion line \((i_1, i_2)\) in Fig. 2 is built, then both \( 1, 1 \) and \( 1, 2 \) are valid settings of the elements in path \((i_0, i_1), (i_1, i_2)\). Thus, in each upcoming lemma and proof, whenever \( k \) is used as a line index, it is shorthand for \( k_{ij} \) when there is no ambiguity. Additionally, because these problems traditionally specify corridors from a lower-index bus to a higher-index bus, we define \( \tilde{P}_{ij,k} = \text{sgn}(j - i) \cdot P_{ij,k} \), where \( \text{sgn}(i - j) = 1 \) if \( i > j \) and \( \text{sgn}(i - j) = -1 \) if \( i < j \). Define \( \tilde{P}_{ij,k}^0 \) analogously for \( P_{ij,k}^0 \).

#### 4.1 Single path over established corridors

**Lemma 1** Let \( \rho = (i_0, i_1), \ldots, (i_{|\rho|-1}, i_{|\rho|}) \) represent a directed path over established corridors in \( G \). For \((i, j)\) \( \in \rho \), set coefficient vector \( \pi = (\pi_0, \pi_1, \ldots, \pi_{|\rho|}) \in \mathbb{R}^{|\rho|+1} \) as,

\[
\pi_j = \begin{cases} 
\sum_{(i,m) \in \rho} x_{im,k} \cdot \tilde{P}_{im,k}^0, & \text{if } j = 0 \\
\text{sgn}(i - j)x_{ij,k}, & \text{otherwise},
\end{cases}
\tag{13}
\]
where \( k \in \{1, \ldots, \omega_{ij}^0\} \) is fixed for each corridor \((i, j)\), but may vary between corridors. Then the following two-sided inequality is valid for TEP for any \( k_{\rho} \):

\[
-\pi_0 \leq \sum_{(i,j) \in \rho} \pi_j \tilde{P}_{ij,k}^0 \leq \pi_0. \tag{14}
\]

**Proof** According to (5), the flow along any fixed, existing line \( k \) of corridor \((i, j)\) is given by

\[
\tilde{P}_{ij,k}^0 = sgn(i - j) b_{ij,k} (\theta_i - \theta_j), \tag{15}
\]
or equivalently,

\[
(\theta_i - \theta_j) = \pi_j \tilde{P}_{ij,k}^0, \tag{16}
\]

where \((i, j) \in \rho\). Hence, the bus angle difference for consecutive bus-pairs \((i_0, i_1), (i_1, i_2), (i_2, i_3), \ldots, (i_{|\rho|-1}, i_{|\rho|})\) in \( \rho \) can be written as:

\[
\begin{align*}
\theta_{i_1} - \theta_{i_0} &= sgn(i_0 - i_1) x_{i_0 i_1,k} \tilde{P}_{i_0 i_1,k}^0 = \pi_1 \tilde{P}_{i_0 i_1,k}^0, \\
\theta_{i_2} - \theta_{i_1} &= sgn(i_1 - i_2) x_{i_1 i_2,k} \tilde{P}_{i_1 i_2,k}^0 = \pi_2 \tilde{P}_{i_1 i_2,k}^0, \\
& \vdots \\
\theta_{i_{|\rho|-1}} - \theta_{i_{|\rho|}} &= sgn(i_{|\rho|-1} - i_{|\rho|}) x_{i_{|\rho|-1} i_{|\rho|},k} \tilde{P}_{i_{|\rho|-1} i_{|\rho|},k}^0 = \pi_{|\rho|} \tilde{P}_{i_{|\rho|-1} i_{|\rho|},k}^0.
\end{align*}
\]

When these equations are summed, this creates a telescoping effect on the left-hand side, which yields the following bus angle difference equation for the starting and ending buses in \( \rho \):

\[
\begin{align*}
\theta_{i_{|\rho|}} - \theta_{i_0} &= \sum_{(i,j) \in \rho} \pi_j \tilde{P}_{ij,k}^0 \\
& \leq \sum_{(i,j) \in \rho} \left| \pi_j \tilde{P}_{ij,k}^0 \right| \\
& \leq \sum_{(i,j) \in \rho} x_{ij,k} \tilde{P}_{ij,k}^0 = \pi_0, \tag{17}
\end{align*}
\]

where the latter inequality is obtained by adding the rightmost inequalities from (3). By a similar argument we have that,

\[
\begin{align*}
\sum_{(i,j) \in \rho} \pi_j \tilde{P}_{ij,k}^0 & \geq \sum_{(i,j) \in \rho} \left| \pi_j \tilde{P}_{ij,k}^0 \right| \\
& \geq \sum_{(i,j) \in \rho} x_{ij,k} \tilde{P}_{ij,k}^0 \\
& = -\pi_0. \tag{20}
\end{align*}
\]

Since every corridor considered has at least one existing line to select and fix as \( k \), and (15) holds for any line in corridor \((i, j)\), this establishes the validity of (14).

As an example using Fig. 2 the path \( \rho^2 := (i_0, i_4), (i_4, i_5) \) creates the example two-sided VI:
\[- \bar{P}_{l_0,i_4} x_{l_0,i_4} - \bar{P}_{i_4,i_5} x_{i_4,i_5} \leq P_{l_0,i_1} x_{l_0,i_1} + P_{i_1,i_2} x_{i_1,i_2} \leq \bar{P}_{l_0,i_4} x_{l_0,i_4} + \bar{P}_{i_4,i_5} x_{i_4,i_5} \]

On the same note, in Fig. 2, the path $\rho^1 := (i_0, i_1, i_2, i_3, i_4)$ is an established path, which creates the example two-sided VI:

\[- \bar{P}_{l_0,i_4} x_{l_0,i_4} - \bar{P}_{i_4,i_5} x_{i_4,i_5} \leq P_{l_0,i_1} x_{l_0,i_1} + P_{i_1,i_2} x_{i_1,i_2} \leq \bar{P}_{l_0,i_4} x_{l_0,i_4} + \bar{P}_{i_4,i_5} x_{i_4,i_5} \]

4.2 Parallel paths over established corridors

**Lemma 2** Let $\rho^1, \ldots, \rho^m$ represent $m > 1$ alternative directed paths over established corridors in $G$ with the same starting/ending buses but with non-overlapping intermediate buses; that is, $i'_0 = i''_0$, $i'_r | \rho^r | = i''_r | \rho^{r'} |$, and $|i'_r | \rho^r | - 1 \cap |i''_r | \rho^{r'} | - 1 = \emptyset$ for $1 \leq r, r' \leq m$ with $r \neq r'$.

Setting coefficient vectors $\pi^r = (\pi_0^r, \pi_1^r, \ldots, \pi_{\rho^r - 1}^r) \in \mathbb{R}^{\rho^r + 1}$ according to (13) for each path $\rho^r$, the following two-sided inequalities are valid for TEP for any $k^r$.

\[- \min(\pi_0^0)^m_{n=1} \leq \sum_{(i,j) \in \rho^r} \pi_j^r \bar{P}_{ij,k} \leq \min(\pi_0^0)^m_{n=1} \quad \text{for} \quad r = 1, \ldots, m. \tag{23} \]

**Proof** Since paths $\rho^r$ and $\rho^{r'}$ share the same starting/ending buses, this gives that $\theta_{i'_0 | \rho^r |} - \theta_{i'_0 | \rho^{r'} |} = \theta_{i''_0 | \rho^{r'} |} - \theta_{i''_0 | \rho^r |}$ or equivalently, with $k$ defined as in Lemma 1,

\[
\sum_{(i,j) \in \rho^r} \pi_j^r \bar{P}_{ij,k} \sum_{(i,j) \in \rho^{r'}} \pi_j^{r'} \bar{P}_{ij,k} = \sum_{(i,j) \in \rho^r} \pi_j^r \bar{P}_{ij,k} \quad \text{for} \quad 1 \leq r, r' \leq m \quad \text{with} \quad r \neq r'
\]

according to the respective telescoped bus angle difference equations of the starting and ending buses associated with each path [e.g., see (17)]. Thus, the proof is completed by joining together the two-sided inequalities,

\[-\pi_0^r \leq \sum_{(i,j) \in \rho^r} \pi_j^r \bar{P}_{ij,k} \leq \pi_0^r \quad \text{for} \quad r = 1, \ldots, m, \]

each of which is valid due to Lemma 1. \(\square\)

Continuing the example from Sect. 4.1, in Fig. 2, $\rho^1$ creates an established parallel path with $\rho^2$. Assuming that path $\rho^2$ is the path with lower capacity-reactance product creates the example two-sided VI:

\[- \bar{P}_{l_0,i_4} x_{l_0,i_4} - \bar{P}_{i_4,i_5} x_{i_4,i_5} \leq P_{l_0,i_1} x_{l_0,i_1} + P_{i_1,i_2} x_{i_1,i_2} + P_{i_2,i_3} x_{i_2,i_3} \leq \bar{P}_{l_0,i_4} x_{l_0,i_4} + \bar{P}_{i_4,i_5} x_{i_4,i_5} \]

4.3 Parallel paths over established and expansion corridors

Consider two buses, $\theta_n$ and $\theta_m$, in a network. Let $C$ denote the set of all paths starting at $\theta_n$ and ending at $\theta_m$. For any path $\rho_r \in C$, let $CR(\rho_r) = \sum_{(i,j) \in \rho_r} x_{ij} \bar{P}_{ij}$ denote the cumulative capacity-reactance product of one line from each corridor along that path. Let $\bar{\rho}$ denote a path...
from this set such that \( CR(\rho) = \max \{ CR(\rho_r) \} \), and let \( \rho \) similarly denote a path from this set such that \( CR(\rho) = \min \{ CR(\rho_r) \} \). Further, let \( N_e(\rho_r) \) denote the number of expansion corridors in the path \( \rho_r \). Note that the theorem below is stated and proved in the context of a network that meets the assumptions of all tested instances for simplicity of presentation: namely that all candidate lines for a given corridor, \((i, j)\) have identical properties (e.g., susceptibility, capacity, etc.), so that additionally we can order the candidate lines. In other words, \( y_{ij,k+1} \leq y_{ij,k} \). However, the result can be easily generalized by considering line paths, where the path is along individual lines rather than corridors. The details of this generalized theorem are provided in the appendix.

**Theorem 1** The following are valid inequalities for TEP, for all paths \( \rho_r \in C \):

\[
|\theta_n - \theta_m| \leq CR(\rho_r) + \left( CR(\rho) - CR(\rho_r) \right) \left( N_e(\rho_r) - \sum_{(i,j) \in \rho_r} I_{ij} y_{ij,1} \right), \tag{24}
\]

where \( I_{ij} \) is used as shorthand for the indicator function \( I \left( \omega_{ij,1} - 1, \omega_{ij} = 0 \right) \) (i.e. to identify expansion corridors).

Furthermore, let \( C^0 \subseteq C \) denote the set of paths comprised solely of established corridors, with \( \rho^0_r \) denoting an element of this set. Additionally, let \( CR(\rho^0) = \min \{ CR(\rho^0_r) \} \). If \( C^0 \) is nonempty, then the above inequalities can be strengthened as follows:

\[
|\theta_n - \theta_m| \leq CR(\rho_r) + \left( CR(\rho^0) - CR(\rho_r) \right) \left( N_e(\rho_r) - \sum_{(i,j) \in \rho_r} I_{ij} y_{ij,1} \right) \tag{25}
\]

Before proving this theorem, it is worth noting specifically the ways in which it generalizes the basic result in Binato et al. (2001) about the big-M coefficient in the standard TEP model. The referenced paper gives the big-M parameter as \( M_{kl} \geq x_{kl}^1 \min C_{kl} \), where \( C_{kl}^\min \) is the solution to the shortest path or longest path problem between bus \( k \) and bus \( l \) when the buses are connected or disconnected, respectively. For simple networks, Theorem 1 reduces to these cases as is shown in the proof. However, the formulation in Binato et al. (2001) does not permit multiple lines within the same corridor, and it considers only the static topology of the network. Theorem 1 improves the calculation of big-M coefficients in the TEP formulation by considering multiple lines in each corridor and many topologies that can be realized through the addition of new lines. By doing so, it is capable of producing tighter bounds by considering arbitrary paths. Note that in connected networks with one line per corridor, Lemma 1 reduces to the shortest path problem, while the longest path problem is only relevant when adding new buses or on networks with islands. On the other hand, Theorem 1 can be applied to improve on the big-M coefficient obtained from the shortest path problem through the selection of any pair of paths which share the same initial and terminating buses. The proof of Theorem 1 follows.

**Proof** The telescopied bus angle difference Eq. (17) can be written if and only if corridors \( (i_0, i_1), \ldots, (i_{|\rho| - 1}, i_{|\rho|}) \) are each serviced by transmission lines (i.e., all consecutive bus-pairs must be connected). We then consider two cases: either there are no expansion corridors in \( \rho_r \) (or all expansion corridors in \( \rho_r \) have at least one candidate line built) or there is at least one expansion corridor in \( \rho_r \) with no candidate line built.

**Case 1** There are no expansion corridors in \( \rho_r \), or all expansion corridors in \( \rho_r \) have at least one candidate line built.
Since $\mathbb{I}_{ij,1} = 0$ indicates that there are existing lines servicing corridor $(i, j)$, the equation \( N_e (\rho_r) - \sum_{(i,j) \in \rho_r} \mathbb{I}_{ij} y_{ij,1} \) holds if and only the path $\rho_r$ consists entirely of serviced corridors, that is, for any expansion corridor in the path $\rho_r$, at least one candidate line has been built. In this case, the arguments from Lemma 1 hold for the path $\rho_r$, and we have that $|\theta_h - \theta_m| \leq \pi_0$. However, note that $\pi_0 = \sum_{(i,j) \in \rho_r} x_{ij} \bar{P}_{ij} = CR(\rho_r)$, so in fact we have $|\theta_h - \theta_m| \leq CR(\rho)$, for all $r$.

Case 2 There is at least one expansion corridor in $\rho_r$ with no candidate line built.

In this case, we have $\left( N_e (\rho_r) - \sum_{(i,j) \in \rho_r} \mathbb{I}_{ij} y_{ij,1} \right) \geq 1$. Then in all cases, the inequality $|\theta_h - \theta_m| \leq CR(\rho)$ holds. That is, the bus angle difference between $\theta_h$ and $\theta_m$ is bounded by the largest possible cumulative capacity-reactance product along any path between those buses. Additionally, if $\theta_h$ and $\theta_m$ are connected by a path $\rho_r$ that consists solely of established corridors, then by Lemma 1 we again have $|\theta_h - \theta_m| \leq \pi_0 = CR(\rho_r)$. In fact, by Lemma 2, given any collection of alternative directed paths, $\{\rho^1, \ldots, \rho^r\}$, we have $|\theta_h - \theta_m| \leq \min \{\pi_k^0\}_{k=1}^r$. Similarly to case 1, note that by selecting $\{\rho^1, \ldots, \rho^r\}$ to be all paths solely along established corridors from $\theta_h$ to $\theta_m$, $\min \{\pi_k^0\}_{k=1}^r = CR(\rho^0)$, thus the inequality $|\theta_h - \theta_m| \leq CR(\rho^0)$ holds. Thus the inequalities (24) and (25) are valid in all cases. \( \square \)

As an example of the new valid inequalities described in this theorem, consider again Fig. 2. In this figure, we consider the single paths $\rho^3 := (i_0, i_1), (i_1, i_2), (i_2, i_5), (i_5, i_6), (i_6, i_3)$ and $\rho^4 := (i_0, i_4), (i_4, i_5), (i_5, i_6), (i_6, i_3)$. Additionally, a new single path, $\rho^5$ is created when line $(i_2, i_3)$ is added where $\rho^5 := (i_0, i_1), (i_1, i_2), (i_2, i_3)$ which creates the following VI:

$$|\theta_0 - \theta_3| \leq \sum_{(i,j) \in \rho^5} \bar{P}_{ij} x_{ij} + \left( \min \left\{ \sum_{(i,j) \in \rho^3} \bar{P}_{ij} x_{ij}, \sum_{(i,j) \in \rho^4} \bar{P}_{ij} x_{ij} \right\} - \sum_{(i,j) \in \rho^5} \bar{P}_{ij} x_{ij} \right) \times (1 - y_{i_2,i_3})$$

One important result to note about this theorem is how the coefficients on the right hand side relate to the $M_{ij}$ values in (6). Case 1 can be seen as simply summing those constraints in (6) for each $(i, j) \in \rho_k$, using the best calculated values of big-$M$ as described in Sect. 2 (that is, either by a shortest path problem if bus $i$ is connected to bus $j$ or a longest path problem otherwise). Case 2 allows the conditional use in this summation of the tighter big-$M$ calculated by a shortest path problem, if enough candidate lines have been built to connect bus $i$ and bus $j$. This is what permits these VIs to provide a strictly smaller solution space than their linear relaxation.

To illustrate the potential of these VIs, consider Fig. 1 again but with the line connecting bus $i_1$ to bus $i_2$ as a candidate line instead of an existing line. Then one VI provided by this theorem is

$$|\theta_{i_2} - \theta_{i_0}| \leq 2x \bar{P} + (3x \bar{P} - 2x \bar{P})(1 - y_{i_1i_2}) \Rightarrow |\theta_{i_2} - \theta_{i_0}| \leq 2x \bar{P} + x \bar{P}(1 - y_{i_1i_2})$$

(26)

By comparison, the best constraints (including linear combinations of constraints) relating these two buses in the original TEP model are

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\begin{align}
|\theta_{i2} - \theta_{i0}| & \leq 3x\bar{P} 
\tag{27}
|\theta_{i2} - \theta_{i1}| & \leq 4x\bar{P}(1 - y_{i1i2}) 
\tag{28}
|\theta_{i1} - \theta_{i0}| & \leq x\bar{P} 
\tag{29}
|\theta_{i2} - \theta_{i0}| & \leq x\bar{P} + 4x\bar{P}(1 - y_{i1i2}), 
\tag{30}
\end{align}

where (27)–(29) are directly from (5) and (6), and (30) is the sum of (28) and (29). If \( P_{LR} \) is the polytope of the linear relaxation of the original TEP model for this simple network, and \( P'_{LR} \) is the polytope of the linear relaxation of the original TEP model together with (26), then obviously \( P'_{LR} \subseteq P_{LR} \). In fact, it can be demonstrated that \( P'_{LR} \subset P_{LR} \).

To find a solution in \( P'_{LR} \), but not in \( P_{LR} \), assume without loss of generality that \( \theta_{i1} \geq \theta_{i2} \), then the following system of inequalities relating bus angles \( \theta_{i0} \) and \( \theta_{i2} \) (which represents a point satisfying (27) and (30) but violating (26)) must be satisfied:

\begin{align}
\theta_{i2} - \theta_{i0} & > 2x\bar{P} + x\bar{P}(1 - y_{i1i2}) \quad \tag{31}
\theta_{i2} - \theta_{i0} & \leq 3x\bar{P} \quad \tag{32}
\theta_{i2} - \theta_{i0} & \leq x\bar{P} + 4x\bar{P}(1 - y_{i1i2}) \quad \tag{33}
\end{align}

By joining the right hand sides of (31) with (33) and (32) and (33), this system must then satisfy

\begin{align}
2x\bar{P} + x\bar{P}(1 - y_{i1i2}) & < 3x\bar{P} \quad \tag{34}
2x\bar{P} + x\bar{P}(1 - y_{i1i2}) & < x\bar{P} + 4x\bar{P}(1 - y_{i1i2}) \quad \tag{35}
\end{align}

It is easy to see that (34) is true when \( y_{i1i2} > 0 \) and (35) is true when

\begin{align}
x\bar{P} & < 3x\bar{P}(1 - y_{i1i2}) \\
\Rightarrow & \quad 1 < 3(1 - y_{i1i2}) \\
\Rightarrow & \quad y_{i1i2} < 2/3.
\end{align}

That is, (34) and (35) are both satisfied when \( 0 < y_{i1i2} < 2/3 \). For example, the point \( y_{i1i2} = 0.5, \theta_{i2} = 2.75x\bar{P}, \theta_{i0} = 0 \) satisfies inequalities (31)–(33), i.e., it is in \( P_{LR} \) but not \( P'_{LR} \).

### 5 Computational tests and results

The experiments consider two TEP instances: the GOC 500-bus instance and the Polish 2383-bus instance. These are first solved without adding any valid inequalities. They are then solved with valid inequalities according to the low-effort heuristic described in Sect. 3. The details of the tested instances and the results associated with each are contained in Sects. 5.1 and 5.2, respectively. The algorithmic steps of the experiment are as follows:

1. The low-effort heuristic method, explained in Sect. 3, is applied. This includes solving a selection of the three relaxations, i.e., a given subset of the linear relaxation, the transportation relaxation, and the hybrid relaxation.
2. The solution flows from the chosen relaxations are overlaid onto the same graph. Simple single paths of same-direction flows are found connecting each pair of buses in the network using a breadth-first search. For larger instances, the number of such paths for any fixed pair of buses is capped at 200 to prevent memory issues.
Table 1  Low demand GOC 500-bus results

| Relaxation models | Average computational times (s) |
|-------------------|--------------------------------|
| TR                | 5.99                           |
| HR                | 4.34                           |
| LR                | 5.50                           |
| TR ⊕ HR           | 5.61                           |
| TR ⊕ LR           | 5.51                           |
| HR ⊕ LR           | 5.49                           |
| TR ⊕ HR ⊕ LR      | 5.45                           |
| N/A               | 48.16                          |

3. All single, simple paths with the same initial and final bus are combined to form pairs of parallel paths.
4. The original problem is passed to Gurobi for optimization. Each time a new incumbent solution is found, a random pair of parallel paths is used to generate two VIs, which are added as lazy constraints. If all paths have been used, the optimization simply continues.

It should be noted that in each of the tested instances, all candidate lines for a given corridor, \((i, j)\), have identical properties (e.g., susceptance, capacity, cost, etc.). Because of this property, we enforce the additional set of symmetry-breaking constraints \(y_{ij,k+1} \leq y_{ij,k}, \forall k \in \{1 \ldots \bar{\omega}_{ij} - 1\}\), since each line is interchangeable. First, testing is performed on the GOC 500-bus instance from Birchfield et al. (2016) to showcase the potential for the effectiveness of the proposed path-based VIs in a relatively small size instance. This instance is used to test the algorithm in both high- and low-demand scenarios. Then, testing is performed on the Polish 2383-bus system in order to show their effectiveness in a more realistically sized and designed instance. The algorithm is implemented in Python, and the disjunctive model is solved using Gurobi version 9.0.2. All tests are run on the ASU High Performance Computing Agave Cluster, which has compute nodes with two Intel Xeon E5-2680 v4 CPUs running at 2.40 GHz.

5.1 GOC 500-bus system

This is a synthetic system designed for the ARPA-e Grid Optimization Competition (GOC), developed in Birchfield et al. (2016). The instance is composed of 500 buses, 224 generators, and 732 transmission corridors. Of those, 193 are designated as candidates for expansion (i.e., these corridors have no existing lines). We tested this system with two different demand and congestion profiles, representing a scenario of peak demand and one of off-peak demand. Congestion scaling was done as in Zhang (2013) to account for the effect of high temperatures on transmission line capacity, coinciding with the peak demand scenario. In both cases, the algorithm specified at the beginning of this section was used to solve the system 50 times for all eight possible combinations of the relaxation models.

Table 1 summarizes the results from the low demand, low congestion scenario. For this table and for future tables, “N/A” refers to the time spent solving the model with no VIs added (i.e., with none of the relaxation models solved). Furthermore, “TR,” “HR,” and “LR” correspond to the transportation, hybrid, and linear relaxations, respectively. The table entries report the average runtimes in seconds. For each combination of relaxation models, at least two and no more than forty valid inequalities were added to the model, which results in
Table 2: High demand GOC 500-bus results

| Relaxation models | Average computational times (s) |
|-------------------|--------------------------------|
| TR                | 95.46                          |
| HR                | 116.59                         |
| LR                | 84.33                          |
| TR ⊕ HR           | 96.71                          |
| TR ⊕ LR           | 92.98                          |
| HR ⊕ LR           | 104.87                         |
| TR ⊕ HR ⊕ LR      | 115.59                         |
| N/A               | 1916.01                        |

A minimal increase in computational effort. Solving solely the hybrid relaxation model to generate the VIs produced the fastest average computation time of 4.34 s, but any combination of relaxation models resulted in at least an $8 \times$ improvement in solution time over the model with no VIs added, which took 48.16 s.

Table 2 summarizes the results from the high demand, high congestion scenario. Note that all solution times for this scenario are larger than those for the low demand, low congestion scenario. In this case, solving the hybrid relaxation alone or with both the linear and transport relaxations resulted in the longest average computation time (116.59 and 115.59 s, respectively). However, as the original model with no VIs added took 1916.01 s to solve to optimality, even these slower solves are a $16 \times$ improvement in solution speed. Solving only the linear relaxation produced the greatest improvement in computation time. This took only 84.33 s on average, for a roughly $22 \times$ improvement in solution speed.

### 5.2 Polish 2383-bus system

We use the Polish 2383-bus system adapted for TEP in Mínguez et al. (2018), which consists of 2383 buses, 327 generators, and 2896 total corridors. The system has been modified as follows: while the original 2383-bus system has candidate lines along established corridors, these options were removed and 120 of the existing lines have been removed and replaced with one candidate line each, while the remaining 2776 corridors do not allow for any expansion. This modified instance is available upon request from the corresponding author. Due to the size of this instance, solution times were dramatically increased when compared to the GOC 500-bus instance. Table 3 thus reports computation times in minutes instead of seconds to help clarify the difference in performance for this model.

The box and whisker plot in Fig. 3 helps demonstrate most strongly the scale of the improvements afforded by the VIs. The three best improvements in computation time are provided when solving solely the linear relaxation, solely the hybrid relaxation, or the two relaxations in conjunction to generate VIs. Each of these cases solves in approximately 263 min (4.4 h), which is a $30 \times$ improvement in speed over solving the 2383-bus model without any VIs (144 h). Although using the transportation model improves solution times, even in combination with other models, the overall improvement is much lower than in the cases which do not solve the transportation model. This suggests that both the hybrid and linear relaxations provide better VIs than the transportation model.
### Table 3  Polish 2383-bus results

| Relaxation models      | Average computational times (mins) |
|------------------------|-----------------------------------|
| TR                     | 495.36                            |
| HR                     | 262.94                            |
| LR                     | 263.71                            |
| TR ⊕ HR                | 455.27                            |
| TR ⊕ LR                | 450.53                            |
| HR ⊕ LR                | 262.60                            |
| TR ⊕ HR ⊕ LR           | 312.45                            |
| N/A                    | 8055.27                           |

### Fig. 3  Box and whisker plot of 2383-bus improvements

### 6 Conclusions and future work

This work presents a new mathematical framework and an algorithm that uses a mixed-integer linear programming model, valid inequalities, and a low-effort heuristic method for solving TEP. The objective is to reduce the total computational effort of planning with exact methods. This work is a significant improvement of the preliminary studies carried out in Escobar and Romero (2017), in which the solutions were found after manual analysis of the test system, creation of cuts using two classes of the valid inequalities introduced in this paper (specifically from Lemmas 1 and 2), which at that time had been implemented without proof, and tests made with different cut combinations. However, this work automates each step of the process and formally establishes the validity of three types of valid inequalities.

Computational tests show the effectiveness of Theorem 1. Most remarkably, valid inequalities generated from this theorem reduce the solution time of TEP by up to a factor of 30×. The results also suggest their potential use in the solution of larger-scale problems. Additionally, the results serve as a guide for selecting which heuristics to apply and combine to obtain...
the most computationally beneficial VIs. Distinct heuristics offer trade-offs in efficiency for instances of varying sizes and expected computational effort.

In future work, we will perform a polyhedral study on the strength of the proposed VIs and we will conduct further studies to determine the most effective use of the presented theorems on instances with particular network structures. In particular, as the size of a system increases, the number of possible paths, and thus the number of possible valid inequalities, increases at an exponential rate. Finding and adding all these inequalities takes significant computational time, and the sheer number added does not necessarily improve the performance of the optimization engine. Thus, additional testing is planned to determine how to select an ideal subset of single path and parallel path inequalities to help decrease total solution time, particularly in large systems. Fine tuning of the implementation, including in regards to the optimal use of user cuts and lazy constraints, such as this will allow us to solve more complex problems, such as the L-1 reliability on TEP (Escobar et al. 2018) and planning with uncertainty due to renewables as well as incorporating new technology such as FACTS devices (Sahraei-Ardakani and Hedman 2015) and reconductoring (Skolfield et al. 2021).

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Appendix

In order to generalize Theorem 1, we introduce new definitions. Given a path ρk, a line path ℓk is a sequence of exactly one line per corridor (i, j) ∈ ρk. The kth line in corridor (i, j) will be denoted (i, j, k) for the purposes of a line path. For example, in a network with 3 lines per corridor, the simple path ρ = (1, 2), (2, 3) might have line paths ℓ1 = (1, 2, 1), (2, 3, 3), ℓ2 = (1, 2, 3), (2, 3, 3), or ℓ3 = (1, 2, 2), (2, 3, 2). That is, ℓ1 is comprised of the first line from corridor (1, 2) and the third line from corridor (2, 3). In this basic case, there are 9 possible such line paths corresponding to the path ρ. Additionally, an established line path is a line path composed entirely of existing lines, hence it corresponds to a path composed of only established paths. Let Cℓ be the set of all line paths. Let Ne(ℓk) denote the number of candidate lines in the line path ℓk, when Ne is applied as a function to a line path instead of a path. Let Iijr k represent the indicator function for candidate lines (i.e., Iijr k = 1 means that line (i, j, k) is a candidate line). Given the above definitions and notations, Theorem 2 follows immediately from Theorem 1.

Theorem 2 The following are valid inequalities for TEP, for all line paths ℓk ∈ Cℓ:

$$|\theta_n - \theta_m| \leq CR(\ell_k) + \left(CR(\ell) - CR(\ell_k)\right) \left(N_e(\ell_k) - \sum_{(i,j,r)\in\ell_k} I_{ijr}y_{ijr}\right).$$  \hspace{1cm} (36)

Furthermore, let C0 ⊆ C denote the set of paths comprised solely of established corridors, with ℓk 0 denoting an element of this set. Additionally, let CR(ℓk 0) = min{CR(ℓk)}. If C0 is nonempty, then the above inequalities can be strengthened as follows:

$$|\theta_n - \theta_m| \leq CR(\ell_k) + \left(CR(\ell^0) - CR(\ell_k)\right) \left(N_e(\ell_k) - \sum_{(i,j,r)\in\ell} I_{ijr}y_{ijr}\right).$$  \hspace{1cm} (37)
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