Property (A) and Oscillation of Third-Order Differential Equations with Mixed Arguments

By

B. Baculíková and J. Džurina
(Technical University of Košice, Slovakia)

Abstract. In this paper we offer criteria for property (A) and the oscillation of the third-order nonlinear functional differential equation with mixed arguments

\[ a(t)(x'(t))^\gamma'' + q(t)f(x[\tau(t)]) + p(t)h(x[\sigma(t)]) = 0, \]

where \( a, q, \tau, p, \sigma \in C([t_0, \infty)), f, h \in C((0, \infty)). \) Throughout the paper, we assume that

- \((H_1)\) \( \gamma \) is the ratio of two positive odd integers,
- \((H_2)\) \( a(t), q(t), p(t) \) are positive,
- \((H_3)\) \( \tau(t) \leq t, \sigma(t) \geq t, \tau(t), \sigma(t) \) are nondecreasing, \( \lim_{t \to \infty} \tau(t) = \infty, \)
- \((H_4)\) \( xf(x) > 0, \ xh(x) > 0, \ f'(x) \geq 0 \) and \( h'(x) \geq 0 \) for \( x \neq 0. \)

Through the paper it is assumed that

\[ R(t) = \int_{t_0}^{t} a^{-1/\gamma}(s)ds \to \infty \quad \text{as} \quad t \to \infty. \]

In some results we shall require the following additional assumptions

- \((H_5)\) \( -f(-xy) \geq f(xy) \geq f(x)f(y) \) for \( xy > 0, \)
- \((H_6)\) \( -h(-xy) \geq h(xy) \geq h(x)h(y) \) for \( xy > 0. \)

By a solution of Eq. \((E)\) we mean a function \( x(t) \in C^1([T_x, \infty)), T_x \geq t_0, \)

which has the property \( a(t)(x'(t))^\gamma \in C^2([T_x, \infty)) \) and satisfies Eq. \((E)\) on \([T_x, \infty). \) We consider only those solutions \( x(t) \) of \((E)\) which satisfy
sup\{|x(t)| : t \geq T\} > 0 \text{ for all } T \geq T_x. \text{ We assume that } (E) \text{ possesses such a solution. A solution of } (E) \text{ is called oscillatory if it has arbitrarily large zeros on } [T_x, \infty) \text{ and otherwise, it is called nonoscillatory. Equation } (E) \text{ is said to be oscillatory if all its solutions are oscillatory.}

\textbf{Remark 1.} All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all } t \text{ large enough.}

We need the following lemma in which the classification of the possible nonoscillatory solutions of } (E) \text{ is presented.}

\textbf{Lemma 1.} \textit{Let } x(t) \text{ be a nonoscillatory solution of } (E). \textit{Then } x(t) \text{ satisfies one of the following conditions}

(I) \quad x(t)x'(t) > 0, \quad x(t)[a(t)x'(t)]'' > 0, \quad x(t)[a(t)x'(t)]''' < 0;

(II) \quad x(t)x'(t) < 0, \quad x(t)[a(t)x'(t)]'' > 0, \quad x(t)[a(t)x'(t)]''' < 0;

ev\textit{eventually.}

\textbf{Proof.} Let } x(t) \text{ be a nonoscillatory solution of } (E), \text{ let us say } x(t) > 0 \text{ for } t \geq t_0. \text{ It follows from } (E) \text{ that } [a(t)x'(t)]'' < 0. \text{ Thus } [a(t)x'(t)]'' \text{ is decreasing and of fixed sign.}

If } [a(t)x'(t)]'' < 0, \text{ then it follows from (1.1) that } a(t)x'(t) < 0, \text{ which implies } x(t) < 0, \text{ a contradiction. So we conclude that } [a(t)x'(t)]'' > 0, \text{ eventually. Consequently, } a(t)x'(t) \text{ is of fixed sign for all } t \text{ large enough. Therefore, either Case (I) or Case (II) holds. The proof is complete.} \qed

Following [7], [13], [14], we say that Eq. (E) has property (A) if every nonoscillatory solution } x(t) \text{ of } (E) \text{ satisfies the Case (II) of Lemma 1.}

For the partial case of } (E), \text{ namely for differential equation}

\[ x''(t) + p(t)x(t) = 0, \]

there always exists a nonoscillatory solution satisfying the Case (II) of Lemma 1 (see e.g. [12]). Therefore, the effort of the authors has been aimed at finding criteria for all nonoscillatory solutions of studied equations to satisfy just the Case (II). Hence various kinds of sufficient conditions for property (A) appeared. In this paper, new criteria for property (A) of } (E) \text{ are established and moreover, we present also oscillation criteria for } (E).

Differential equations with mixed arguments are usually reduced into the corresponding advanced differential inequality

\[ [a(t)x'(t)]''' + q(t)f(x[\tau(t)]) \leq 0, \]

and the delayed differential inequality

\[ [a(t)x'(t)]'' + p(t)h(x[\sigma(t)]) \leq 0, \]
and further only properties of these inequalities are investigated. Consequently, the criteria obtained withhold information either from the delay argument \( t(t) \) and the corresponding functions \( q(t) \) and \( f(u) \), or from the advanced argument \( s(t) \) and the corresponding functions \( p(t) \) and \( h(u) \). In the paper, we offer a technique for obtaining such new criteria for property (A) of \((E)\) that involves both delayed and advanced parts of \((E)\). So, our results are new even for the linear case of \((E)\) and properly complement and extend earlier ones presented in [1]–[16]. Our technique is based on the comparison results in which we compare studied equation with the first order advanced/delayed equations, so that the oscillation of these first order equation yields studied properties of \((E)\).

2. Main results

The structure of our results is the following. At first we present new criteria for property (A) and then we extend these results to cover also the oscillation of \((E)\).

It is convenient to prove our main results by means of a series of lemmas as follows. The first one (see [6]) presents a useful relationship between an existence of positive solutions of the advanced differential inequality and the corresponding advanced differential equation.

**Lemma 2.** Suppose \( p(t), \sigma(t), \) and \( h(u) \) satisfies \((H_2), (H_3), \) and \((H_4), \) respectively. If the first order advanced differential inequality

\[
 z'(t) - p(t)h(z(\sigma(t))) \geq 0
\]

has an eventually positive solution, so does the advanced differential equation

\[
 z'(t) - p(t)h(z(\sigma(t))) = 0.
\]

**Lemma 3.** Assume \( A \geq 0, B \geq 0, \alpha \geq 1. \) Then

\[
 (A + B)^\alpha \geq A^\alpha + B^\alpha.
\]

*Proof.* If \( A = 0 \) or \( B = 0, \) then (2.3) holds. For \( A \neq 0, \) setting \( x = B/A \) condition (2.3) takes the form \((1 + x)^\alpha \geq 1 + x^\alpha, \) which is for \( x > 0 \) evidently true. \( \square \)

**Lemma 4.** Assume \( A \geq 0, B \geq 0, \ 0 < \alpha \leq 1. \) Then

\[
 (A + B)^\alpha \geq \frac{A^\alpha + B^\alpha}{2^{1-\alpha}}.
\]
Proof. We may assume that 0 < A < B. Consider a function \( g(u) = u^x \). Since \( g''(u) < 0 \) for \( u > 0 \), function \( g(u) \) is concave down, that is,

\[
g\left(\frac{A + B}{2}\right) \geq \frac{g(A) + g(B)}{2}
\]

which implies (2.4).

Lemma 5. Let \( \sigma(t) > t \). Assume \( z(t) > 0 \), \( z'(t) > 0 \), \( (a(t)[z'(t)]^\gamma)' > 0 \), eventually. Then for arbitrary \( k \in (0, 1) \)

\[
\frac{z(\sigma(t))}{z(t)} \geq k \frac{R(\sigma(t))}{R(t)},
\]

eventually.

Proof. It follows from the monotonicity of \( w(t) = a(t)[z'(t)]^\gamma \) that

\[
z(\sigma(t)) - z(t) = \int_t^{\sigma(t)} z'(s) ds = \int_t^{\sigma(t)} w^{1/\gamma}(s)a^{-1/\gamma}(s) ds
\]

\[
\geq w^{1/\gamma}(t) \int_t^{\sigma(t)} a^{-1/\gamma}(s) ds = w^{1/\gamma}(t)[R(\sigma(t)) - R(t)].
\]

That is,

\[
(2.6) \quad \frac{z(\sigma(t))}{z(t)} \geq 1 + \frac{w^{1/\gamma}(t)}{z(t)} [R(\sigma(t)) - R(t)].
\]

On the other hand, since \( z(t) \to \infty \) as \( t \to \infty \), then for any \( k \in (0, 1) \) there exists a \( t_1 \) large enough, such that

\[
kz(t) \leq z(t) - z(t_1) = \int_{t_1}^t w^{1/\gamma}(s)a^{-1/\gamma}(s) ds
\]

\[
\leq w^{1/\gamma}(t) \int_{t_1}^t a^{-1/\gamma}(s) ds \leq w^{1/\gamma}(t)R(t)
\]

or equivalently

\[
(2.7) \quad \frac{w^{1/\gamma}(t)}{z(t)} \geq \frac{k}{R(t)}.
\]

Using (2.7) in (2.6), we get

\[
\frac{z(\sigma(t))}{z(t)} \geq 1 + \frac{k}{R(t)} [R(\sigma(t)) - R(t)] \geq k \frac{R(\sigma(t))}{R(t)}.
\]

This completes the proof. \( \square \)
We offer several independent criteria for property (A) of (E). For our further references, let us denote

\[(2.8) \quad Q_1(t) = (t - t_1)^{1/\gamma} a^{-1/\gamma}(t) \left[ \int_{t_1}^{t} q(s) / \tau'(s) \right]^{1/\gamma}, \]

\[(2.9) \quad P_1(t) = (t - t_1)^{1/\gamma} a^{-1/\gamma}(t) \left[ \int_{t}^{\infty} p(s) \right]^{1/\gamma}. \]

In the first two results we shall distinguish whether or not \( \gamma \) is less than one.

**Theorem 1.** Let \( 0 < \gamma \leq 1 \). Assume that \((H_6)\) holds and

\[(2.10) \quad \frac{f(u)}{u^{\gamma'}} \geq 1 \quad \text{for} \quad u \neq 0. \]

If the first order advanced differential equation

\[(E_1) \quad z'(t) - P_1(t)e^{-\int_{t_1}^{t} Q_1(s)ds} h^{1/\gamma} \left( e^{\int_{t_1}^{t} Q_1(s)ds} h^{1/\gamma}(z[\sigma(t)]) \right) = 0 \]

is oscillatory, then Eq. (E) has property (A).

**Proof.** Assume the contrary, let \( x(t) \) be a nonoscillatory solution of Eq. (E), satisfying the Case (I) of Lemma 1. We may assume that \( x(t) > 0 \) for \( t \geq t_0 \). Integrating (E) from \( t \) to \( \infty \), one gets

\[(2.11) \quad (a(t)[x'(t)]^{\gamma'})' \geq \int_{t}^{\infty} q(s)f(x[\tau[s]])ds + \int_{t}^{\infty} p(s)h(x[\sigma(s)])ds \]

\[ \geq \int_{t}^{\infty} q(s)x'[\tau(s)]ds + h(x[\sigma(t)]) \int_{t}^{\infty} p(s)ds. \]

On the other hand, the substitution \( \tau(s) = u \) yields

\[ \int_{t}^{\infty} q(s)x'[\tau(s)]ds = \int_{\tau(t)}^{\infty} \frac{q(u)}{\tau'(u)} x'[u]du \]

\[ \geq \int_{\tau(t)}^{\infty} \frac{q(u)}{\tau'(u)} x'[u]du \geq x'(t) \int_{t}^{\infty} \frac{q(u)}{\tau'(u)} du. \]

Using the last inequalities in (2.11), we find

\[(2.12) \quad (a(t)[x'(t)]^{\gamma'})' \geq x'(t) \int_{t}^{\infty} \frac{q(u)}{\tau'(u)} du + h(x[\sigma(t)]) \int_{t}^{\infty} p(s)ds. \]

It follows from the monotonicity of \((a(t)[x'(t)]^{\gamma'})'\) that

\[(2.13) \quad a(t)[x'(t)]^{\gamma} \geq \int_{t_1}^{t} (a(t)[x'(s)]^{\gamma'})'ds \geq (a(t)[x'(t)]^{\gamma})'(t - t_1). \]
Combining (2.13) with (2.12), we are led to

\[(x(t))^\gamma \geq Q_1(t)x(t) + P_1(t)h(x[\sigma(t)]).\]

On the other hand, Lemma 3 implies that

\[x'(t) \geq Q_1(t)x(t) + P_1(t)h^{1/\gamma}(x[\sigma(t)]).\]

Setting

\[x(t) = z(t)e^{\int_{t_1}^{t} Q_1(s)ds},\]

we can easily verify that \(z(t)\) is a positive solution of the advanced differential inequality

\[z'(t) - P_1(t)e^{-\int_{t}^{\sigma(t)} Q_1(s)ds}h^{1/\gamma}(e^{\int_{t}^{\sigma(t)} Q_1(s)ds})h^{1/\gamma}(z[\sigma(t)]) \geq 0.\]

By Lemma 2, we deduce that the corresponding differential equation \((E_1)\) has also a positive solution. A contradiction and the proof is complete. \(\square\)

**Corollary 1.** Let \(0 < \gamma \leq 1\). Assume that \((H_6)\) and (2.10) hold. If

\[\frac{h^{1/\gamma}(u)}{u} \geq 1, \quad |u| \geq 1\]

and

\[\liminf_{t \to \infty} \int_{t}^{\sigma(t)} P_1(s)e^{\int_{s}^{\sigma(t)} Q_1(u)du} ds > \frac{1}{e},\]

then \((E)\) has property \((A)\).

**Proof.** It is easy to see that (2.17) implies

\[\int_{t_1}^{\infty} P_1(s)e^{\int_{s}^{\sigma(t)} Q_1(u)du} ds = \infty.\]

Taking into account Theorem 1, it is sufficient to show that \((E_1)\) is oscillatory. Assume the converse, let \((E_1)\) have an eventually positive solution \(z(t)\). Then \(z'(t) > 0\) and so \(z(\sigma(t)) > c > 0\). Integrating \((E_1)\) from \(t_1\) to \(t\), we have, in view of (2.16),

\[z(t) \geq \int_{t_1}^{t} P_1(u)e^{-\int_{t_1}^{u} Q_1(s)ds}h^{1/\gamma}(e^{\int_{u}^{\sigma(t)} Q_1(s)ds})h^{1/\gamma}(z[\sigma(u)])du\]

\[\geq h^{1/\gamma}(c) \int_{t_1}^{t} P_1(u)e^{\int_{u}^{\sigma(t)} Q_1(s)du} du,\]
which together with (2.18) ensures that \( z(t) \to \infty \) as \( t \to \infty \). Therefore, \( z(t) \geq 1 \), eventually. Using (2.16) in \((E_1)\), we see that \( z(t) \) is a positive solution of the differential inequality

\[
(2.19) \quad z'(t) - P_1(t)e^{\int_t^{\sigma(t)} Q_1(u)du}z[\sigma(t)] \geq 0.
\]

On the other hand, by Theorem 2.4.1 in [15], condition (2.17) guarantees that (2.19) has no positive solutions. This is a contradiction and we conclude that \((E)\) has property (A).

**Theorem 2.** Let \( \gamma \geq 1 \). Assume that \((H_6)\) and (2.10) hold. If the first order advanced differential equation

\[
(E_2) \quad z'(t) - \frac{P_1(t)}{2^{(1-\gamma)/\gamma}} e^{-\int_t^{\sigma(t)} Q_1(s)ds} h^{1/\gamma}(e^{2^{(1-\gamma)/\gamma} \int_t^{\sigma(t)} Q_1(s)ds}) h^{1/\gamma}(z[\sigma(t)]) = 0
\]

is oscillatory, then Eq. \((E)\) has property (A).

**Proof.** Suppose that \( x(t) \) is an eventually positive solution of Eq. \((E)\) satisfying the Case (I) of Lemma 1. Then (2.14) holds and Lemma 3 implies that

\[
(2.20) \quad x'(t) \geq 2^{(1-\gamma)/\gamma} [Q_1(t)x(t) + P_1(t)h^{1/\gamma}(x[\sigma(t)])].
\]

Let us denote

\[
x(t) = z(t)e^{\int_t^{\sigma(t)} Q_1(s)ds}.
\]

It is easy to see that \( z(t) \) is a positive solution of the advanced differential inequality

\[
z'(t) - 2^{(1-\gamma)/\gamma} P_1(t)e^{-\int_t^{\sigma(t)} Q_1(s)ds} h^{1/\gamma}(e^{2^{(1-\gamma)/\gamma} \int_t^{\sigma(t)} Q_1(s)ds}) h^{1/\gamma}(z[\sigma(t)]) \geq 0.
\]

By Lemma 2, we deduce that the corresponding differential equation \((E_2)\) has also a positive solution, which is a contradiction and the proof is complete now.

The proof of the following result is similar to that of Corollary 1 and so it can be omitted.

**Corollary 2.** Let \( \gamma \geq 1 \). Assume that \((H_6)\), (2.10), and (2.16) hold. If

\[
(2.21) \quad \liminf_{t \to \infty} \int_t^{\sigma(t)} P_1(u)e^{\int_u^{\sigma(t)} Q_1(s)ds} du > \frac{2^{(\gamma-1)/\gamma}}{e},
\]

then \((E)\) has property (A).
For our next results we need an auxiliary function $\alpha(t) \in C^1([t_0, \infty))$ satisfying

$$a'(t) \geq 0, \quad \alpha(t) < t, \quad \text{and} \quad \psi(t) = \alpha(\sigma(t)) > t.$$  

Denote $\varphi(t) = \alpha(\tau(t))$ and set

$$A(t) = q(t) \left[ \int_{\varphi(t)}^{\tau(t)} a^{-1/\gamma}(s) ds \right]^{\gamma},$$

$$Q_2(t) = \int_t^{\infty} A(\varphi^{-1}(s)) \frac{\varphi'(s)}{\varphi'(\varphi^{-1}(s))} ds,$$

$$P_2(t) = \int_t^{\infty} p(s) dh \left[ \int_{\varphi(t)}^{\tau(t)} a^{-1/\gamma}(s) ds \right].$$

**Theorem 3.** Let $(H_5)$ hold. Assume that there exists a function $\varphi(t) \in C^1([t_0, \infty))$ such that (2.22) holds. If the first order advanced differential equation

$$(E_3) \quad z'(t) - P_2(t)e^{-\int_t^{\infty} Q_2(s) ds} h(e^{\int_t^{\infty} Q_2(s) ds/\gamma}) h(z^{1/\gamma}[\psi(t)]) = 0$$

is oscillatory, then Eq. $E$ has property (A).

**Proof.** Assuming the converse, we admit that $x(t)$ is a positive solution of Eq. $E$, satisfying the Case (I) of Lemma 1. Furthermore, the monotonicity of $y(t) = a(t)[x'(t)]^{1/\gamma} > 0$, implies

$$x(t) \geq \int_{\sigma(t)}^{t} (a(s)(x'(s))^{1/\gamma})^{1/\gamma} a^{-1/\gamma}(s) ds$$

$$\geq y^{1/\gamma}(\alpha(t)) \int_{\sigma(t)}^{t} a^{-1/\gamma}(s) ds,$$

eventually. Combining (2.26) together with (2.11), we find

$$y'(t) \geq \int_t^{\infty} A(s)y(\varphi(s)) ds + P_2(t) h[y^{1/\gamma}(\psi(t))].$$

Arguing as in the proof of Theorem 1, we see that the transformation $\varphi(s) = u$ leads to

$$\int_t^{\infty} A(s)y[\varphi(s)] ds \geq y(t) \int_t^{\infty} A(\varphi^{-1}(u)) \frac{\varphi'(\varphi^{-1}(u))}{\varphi'(\varphi^{-1}(u))} du = Q_2(t)y(t),$$

246 B. Baculíková and J. Džurina
which, in view of (2.27), guarantees that \( y(t) \) is a positive solution of the differential inequality

\[
y'(t) - Q_2(t)y(t) - P_2(t)h[y^{1/\gamma}(\psi(t))] \geq 0.
\]

Putting

\[
y(t) = z(t)e^{\int_t^T Q_2(s)ds},
\]

it can easily be checked that \( z(t) \) is a positive solution of the advanced differential inequality

\[
z'(t) - P_2(t)e^{-\int_t^T Q_2(s)ds} h(\phi(t)Q_2(t)d\gamma)h(z^{1/\gamma}(\psi(t))) \geq 0.
\]

It follows from Lemma 2 that the corresponding differential equation \((E_3)\) also has a positive solution. This contradicts our assumption and we conclude that Eq. \((E)\) has property \((A)\). \(\square\)

Using the similar arguments as in the proof of Corollary 1, it can be easily verified that the following result holds true.

**Corollary 3.** Assume that \((H_6)\) and (2.10) hold. If

\[
\frac{h(u^{1/\gamma})}{u} \geq 1, \quad |u| \geq 1
\]

and

\[
\liminf_{t \to -\infty} \int_t^{\psi(t)} P_2(s)e^{\int_s^T Q_2(u)d\gamma} ds > \frac{1}{e},
\]

then \((E)\) has property \((A)\).

We offer another criterion for property \((A)\) of \((E)\) based on the properties of the first order delayed equation. Let us denote

\[
P_3(t) = kp(t) \frac{R^\gamma(\sigma(t))}{R^\gamma(t)} \left[ \int_{t_1}^{\psi(t)} a^{-1/\gamma}(s) (s - t_1)^{1/\gamma} ds \right]^{\gamma},
\]

\[
Q_3(t) = q(t) \left[ \int_{t_1}^{\psi(t)} a^{-1/\gamma}(s) (s - t_1)^{1/\gamma} ds \right],
\]

where \( k \in (0,1) \) is chosen arbitrarily.

**Theorem 4.** Assume that \((H_5)\) and \((H_6)\) hold. Let

\[
\frac{h(u^{1/\gamma})}{u} \geq 1, \quad |u| \neq 0.
\]
If for some $k \in (0, 1)$ the first order delayed differential equation
\[(E_4) \quad w'(t) + Q_3(t)e^{\int_{t_1}^{t} P_3(s)ds} f\left(e^{-\int_{t_1}^{t_0} P_3(s)ds/\gamma}\right)f(w^{1/\gamma}[\tau(t)]) = 0\]
is oscillatory, then Eq. \((E)\) has property \((A)\).

Proof. Assume that Eq. \((E)\) has not property \((A)\). Then there exists a nonoscilatory solution $x(t)$ of Eq. \((E)\), satisfying the Case (I) of Lemma 1. We may assume that $x(t) > 0$. Put $k_1 = k^{1/\gamma}$. Applying Lemma 5, we see that $x(t)$ satisfies
\[(2.34) \quad x(\sigma(t)) \geq k_1 \frac{R(\sigma(t))}{R(t)} x(t),\]
eventually, let us say for $t \geq t_1$. Let us denote $z(t) = (a(t)[x'(t)]^{1/\gamma})'$. Then it is easy to verify that
\[(2.35) \quad z'(t) + q(t)f[x(\tau(t))] + p(t)h \left[k_1 \frac{R(\sigma(t))}{R(t)} x(t)\right] \leq 0,\]
Obviously, \((2.13)\) can be written in the form
\[a(t)[x'(t)]^{1/\gamma} \geq z(t)(t - t_1).\]
Integrating from $t_1$ to $t$, we are lead to
\[x(t) \geq \int_{t_1}^{t} z^{1/\gamma}(s)a^{-1/\gamma}(s)(s - t_1)^{1/\gamma} ds \geq z^{1/\gamma}(t) \int_{t_1}^{t} a^{-1/\gamma}(s)(s - t_1)^{1/\gamma} ds.\]
Combining with \((2.35)\), we have
\[z'(t) + q(t)f \left[z^{1/\gamma}(\tau(t)) \int_{t_1}^{\tau(t)} a^{-1/\gamma}(s)(s - t_1)^{1/\gamma} ds\right]
+ p(t)h \left[k_1 \frac{R(\sigma(t))}{R(t)} z^{1/\gamma}(t) \int_{t_1}^{t} a^{-1/\gamma}(s)(s - t_1)^{1/\gamma} ds\right] \leq 0\]
By \((2.33)\), we get
\[z'(t) + P_3(t)z(t) + Q_3(t)f(z^{1/\gamma}[\tau(t)]) \leq 0.\]
We set
\[z(t) = w(t)e^{\int_{t_1}^{t} P_3(s)ds}.\]
It is evidently that $z(t)$ is a positive solution of the delayed differential inequality
\[w'(t) + Q_3(t)e^{\int_{t_1}^{t} P_3(s)ds} f(e^{-\int_{t_1}^{t_0} P_3(s)ds/\gamma})f(w^{1/\gamma}[\tau(t)]) \leq 0.\]
It follows from Theorem 1 in [17] that the corresponding differential equation (E₄) has also a positive solution. A contradiction.

**Corollary 4.** Assume that (H₅), (H₆), and (2.33) hold. If

\[
(2.36) \quad \frac{f(u^{1/2})}{u} \geq 1, \quad 0 < |u| \leq 1
\]

and for some \( k \in (0, 1) \)

\[
(2.37) \quad \lim_{t \to \infty} \inf \int_{t(t)}^{t} \! Q_3(s)e^{\int_{s}^{t}P_3(u)\,du} \, ds > \frac{1}{e},
\]

then (E) has property (A).

**Proof.** It is obviously that (2.37) implies

\[
(2.38) \quad \int_{t_0}^{t_1} \! Q_3(s)e^{\int_{t_1}^{t}P_3(u)\,du} \, ds = \infty.
\]

By Theorem 4, it is sufficient to show that (E₄) is oscillatory. Assume the converse, let (E₄) have an eventually positive solution \( w(t) \). Then \( w'(t) < 0 \). We claim that \( \lim_{t \to \infty} w(t) = 0 \). If not, then there exists some \( \ell > 0 \) such that \( w(\tau(t)) > \ell \). Integrating (E₄) from \( t_1 \) to \( t \), we have, in view of (2.36),

\[
\begin{align*}
  w(t_1) &= w(t) + \int_{t_1}^{t} \! Q_3(s)e^{\int_{s}^{t}P_3(u)\,du} \, ds \left( f(e^{-\int_{t_1}^{t}P_3(u)\,du/2})f(w^{1/2}[\tau(s)]) \right) \, ds \\
  &\geq f(\ell^{1/2}) \int_{t_1}^{t} \! Q_3(s)e^{\int_{s}^{t}P_3(u)\,du} \, ds.
\end{align*}
\]

Letting \( t \to \infty \), we get a contradiction and we conclude that \( \lim_{t \to \infty} z(t) = 0 \). Thus, \( 0 \leq z(t) \leq 1 \). Now, using (2.36) in (E₄), one can verify that \( z(t) \) is a positive solution of the differential inequality

\[
(2.39) \quad w'(t) + Q_3(t)e^{\int_{t_1}^{t}P_3(s)\,ds}w[\tau(t)] \leq 0.
\]

But, by Theorem 2.4.1 in [15], condition (2.37) ensures that (2.39) has no positive solutions. This is a contradiction and we conclude that (E) has property (A).

Now, we eliminate the nonoscillatory solutions of (E) satisfying the Case (II) of Lemma 1.

**Theorem 5.** Assume that there exists a function \( \xi(t) \in C^1([t_0, \infty)) \) such that

\[
(2.40) \quad \xi'(t) \geq 0, \quad \xi(t) > t, \quad \text{and} \quad \eta(t) = \tau(\xi(\xi(t))) < t.
\]
If the first order delay equation
\[ (E_5) \quad z'(t) + a^{-1/\gamma}(t) \left[ \int_t^{\xi(t)} \int_u^{\xi(u)} q(s) ds du \right]^{1/\gamma} f^{1/\gamma}(z[\tau(t)]) = 0 \]
is oscillatory, then Eq. (E) has no nonoscillatory solution satisfying the Case (II) of Lemma 1.

**Proof.** Assume the converse. Let \( x(t) \) be a positive solution of Eq. (E), satisfying the Case (II) of Lemma 1. It follows from (E) that
\[ (2.41) \quad [a(t)[x'(t)]^{\gamma}]'' + q(t)f(x[\tau(t)]) \leq 0. \]
An integration of (2.41) from \( t \) to \( \xi(t) \) yields
\[ (a(t)(x'(t))^{\gamma})' \geq \int_t^{\xi(t)} q(s_1)f(x[\tau(s_1)]) ds_1 \geq f(x[\tau(\xi(t))]) \int_t^{\xi(t)} q(s_1) ds_1. \]
Integrating from \( t \) to \( \xi(t) \) once more, we get
\[ -a(t)(x'(t))^{\gamma} \geq \int_t^{\xi(t)} f(x[\tau(s_1)]) ds_1 \int_s^{\xi(s_1)} q(s_1) ds_1 ds_2 \]
\[ \geq f(x[\eta(t)]) \int_t^{\xi(t)} \int_s^{\xi(s_1)} q(s_1) ds_1 ds_2. \]
Consequently, \( x(t) \) is a positive solution of the delayed differential inequality
\[ x'(t) + a^{-1/\gamma}(t) \left[ \int_t^{\xi(t)} \int_u^{\xi(u)} q(s) ds du \right]^{1/\gamma} f^{1/\gamma}(x[\tau(t)]) \leq 0. \]
But, by Theorem 1 in [17], we see that the corresponding differential equation \( (E_5) \) has also a positive solution. This contradiction finishes the proof. \( \square \)

**Corollary 5.** Assume that (2.36) holds and there exists a function \( \xi(t) \in C^1([t_0, \infty)) \) such that (2.40) is satisfied. If
\[ (2.42) \quad \liminf_{t \to \infty} \int_{\xi(t)}^{\xi(u)} a^{-1/\gamma}(v) \left[ \int_v^{\xi(u)} q(s) ds du \right]^{1/\gamma} dv > \frac{1}{e}, \]
then Eq. (E) has no nonoscillatory solution satisfying the Case (II) of Lemma 1.

**Proof.** Since (2.42) guarantees the oscillation of \( (E_5) \), the proof immediately follows from Theorem 5. \( \square \)

Picking up our previous theorems, we get the following oscillation criterion for (E).
Theorem 6. Assume that \((E_5)\) is oscillatory and at the same time at least one of the equation \((E_1)-(E_4)\), is oscillatory, then Eq. \((E)\) is oscillatory.

We illustrate all our results in the following example.

Example 1. Consider the third order nonlinear differential equation with mixed arguments

\begin{equation}
(t^{-1}(x'(t))^3)'' + \frac{q}{t^6} x^3(\beta t) + \frac{p}{t^6} x^3(\delta t) = 0,
\end{equation}

with \(q > 0\), \(p > 0\), \(0 < \beta < 1\), \(\delta > 1\). Conditions \((2.21)\), \((2.30)\), \((2.37)\) reduce to

\begin{equation}
p^{1/3} \delta^{\beta^{3/4}q^{1/3}20^{1/3}} \ln \delta > \frac{20^{1/3}}{e},
\end{equation}

\begin{equation}
p\delta^4(1 - \beta^{4/3})^3 \beta \delta^{3/4} q^{1 - \beta^{1/3}} \ln(\beta \delta) > \left(\frac{4}{3}\right)^3 \frac{5}{3}.
\end{equation}

\begin{equation}
q \beta^5 - 3/5 \rho \delta^4 \ln \left(\frac{1}{\beta}\right) > \left(\frac{5}{3}\right)^3 \frac{1}{e},
\end{equation}

respectively, where we have set \(x(t) = \beta t\). Then by Corollaries 2–4, Eq. \((2.43)\) enjoys property \((A)\), provided that one of the conditions \((2.44)-(2.46)\) holds. It is easy to verify that for

\[ k > 0 \quad \text{such that} \quad k^3(3k + 4)(3k + 5) = q\beta^{-3k} + p\delta^{-3k}, \]

one such admissible solution is \(x(t) = t^{-k}\).

On the other hand, we set \(\xi(t) = \omega t\), where \(\omega = (\sqrt{\beta} + 1)/(2\sqrt{\beta})\). Then condition \((2.42)\) converts to

\begin{equation}
q \left(1 - \frac{1}{\omega^4}\right) \left(1 - \frac{1}{\omega^2}\right) \ln^3 \left(\frac{1}{\beta \omega^2}\right) > \frac{20}{e^3}.
\end{equation}

Then by Corollary 5, equation \((2.43)\) is oscillatory, provided that \((2.47)\) holds and at the same time one of \((2.44)-(2.46)\) is satisfied.

Coefficients of \((2.43)\) are in very general form, consequently the criteria obtained \((2.44)-(2.47)\) are also very general and cover wide scale of differential equations. For particular case of \((2.43)\) with \(\beta = 3/4\) and \(\delta = 2\), namely for

\begin{equation}
(t^{-1}(x'(t))^3)'' + \frac{q}{t^6} x^3\left(\frac{3}{4} t\right) + \frac{p}{t^6} x^3(2t) = 0,
\end{equation}

conditions \((2.44)\) and \((2.46)\) for property \((A)\) reduces to

\[ p2^{0.68424q^{1/3}} > 2.9899, \quad \text{and} \quad q \left(\frac{4}{3}\right)^{0.82012p^{1/3}} > 5.9209, \]
respectively, while oscillation criterion (2.47) takes the easily readable form

\[ q > 4658.73. \]

3. **Summary**

In this paper, we have presented new comparison theorems for deducing the oscillation/property (A) of \((E)\) from the oscillation of a set of the suitable first order delay/advanced differential equations. Our technique makes use of both delayed and advanced parts of \((E)\). The presented method essentially simplifies the examination of the third order equations and what is more, it supports backward the research on the first order delay/advanced differential equations. Our results here extend and complement latest ones of Agarwal et al. [1]–[3], Grace et al. [11], Cecchi et al. [7], Parhi and Padhi [16] and the presents authors [4]–[6], [9]. The suitable illustrative example is also provided.

**Acknowledgement**

This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0008-10.

**References**

[1] Agarwal, R. P., Shien, S. L. and Yeh, C. C., Oscillation criteria for second order retarded differential equations, Math. Comput. Modelling, 26 (1997), 1–11.

[2] Agarwal, R. P., Grace, S. R. and O’Regan, D., On the oscillation of certain functional differential equations via comparison methods, J. Math. Anal. Appl., 286 (2003), 577–600.

[3] Agarwal, R. P., Grace, S. R. and Smith, T., Oscillation of certain third order functional differential equations, Adv. Math. Sci. Appl., 16 (2006), 69–94.

[4] Baculíková, B. and Džurina, J., Oscillation of Third-Order Neutral Differential Equations, Math. Comput. Modelling, 52 (2010), 215–226.

[5] Baculíková, B. and Džurina, J., Oscillation of third-order nonlinear differential equations, Appl. Math. Lett., 24 (2011), 466–470.

[6] Baculíková, B., Properties of third order nonlinear functional differential equations with mixed arguments, Abstr. Appl. Anal., 2011 (2011), Art. ID 857860, 15 pp.

[7] Cecchi, M., Došlý, Z. and Marini, M., On third order differential equations with property A and B, J. Math. Anal. Appl., 231 (1999), 509–525.

[8] Džurina, J., Asymptotic properties of third order delay differential equations, Czechoslovak Math. J., 45(120) (1995), 443–448.

[9] Džurina, J., Comparison theorems for functional differential equations with advanced argument, Boll. Un. Mat. Ital. A (7), 7 (1993), 461–470.

[10] Erbe, L. H., Kong, Q. and Zhang, B. G., *Oscillation Theory for Functional Differential Equations*, Monographs and Textbooks in Pure and Applied Mathematics, 190, Marcel Dekker, Inc., New York, 1995.
Third-Order Differential Equations with Mixed Arguments

[11] Grace, S. R., Agarwal, R. P., Pavani, R. and Thandapani, E., On the oscillation of certain third order nonlinear functional differential equations, Appl. Math. Comput., 202 (2008), 102–112.

[12] Hartman, P. and Wintner, A., Linear differential and difference equations with monotone solutions, Amer. J. Math., 75 (1953), 731–743.

[13] Kusano, T. and Naito, M., Comparison theorems for functional differential equations with deviating arguments, J. Math. Soc. Japan, 3 (1981), 509–533.

[14] Kusano, T., Naito, M. and Tanaka, K., Oscillatory and asymptotic behaviour of solutions of a class of linear ordinary differential equations, Proc. Roy. Soc. Edinburgh Sect. A, 90 (1981), 25–40.

[15] Ladde, G. S., Lakshmikantham, V. and Zhang, B. G., Oscillation Theory of Differential Equations with Deviating Arguments, Monographs and Textbooks in Pure and Applied Mathematics, 110, Marcel Dekker, Inc., New York, 1987.

[16] Parhi, N. and Padhi, S., On oscillation and asymptotic property of a class of third-order differential equations, Czechoslovak Math. J., 49 (1999), 21–33.

[17] Philos, Ch. G., On the existence of nonoscillatory solutions tending to zero at \( \infty \) for differential equations with positive delay, Arch. Math. (Basel), 36 (1981), 168–178.

nuna adreso:

B. Baculíková
Department of Mathematics
Faculty of Electrical Engineering and Informatics
Technical University of Košice
Letná 9, 042 00 Košice
Slovakia
E-mail: blanka.baculikova@tuke.sk

J. Džurina
Department of Mathematics
Faculty of Electrical Engineering and Informatics
Technical University of Košice
Letná 9, 042 00 Košice
Slovakia
E-mail: jozef.dzurina@tuke.sk

(Ricevita la 21-an de aprilo, 2011)
(Reviziita la 12-an de novembro, 2011)