A phenomenon of splitting resonant-tunneling one-point interactions

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Abstract

The so-called $\delta'$-interaction as a particular example in Kurasov’s distribution theory developed on the space of discontinuous (at the point of singularity) test functions, is identified with the diagonal transmission matrix, continuously depending on the strength of this interaction. On the other hand, in several recent publications, the $\delta'$-potential has been shown to be transparent at some discrete values of the strength constant and opaque beyond these values. This discrepancy is resolved here on the simple physical example, namely the heterostructure consisting of two extremely thin layers separated by infinitesimal distance. In the three-scale squeezing limit as the thickness of the layers and the distance between them simultaneously tend to zero, a whole variety of single-point interactions is realized. The key point is the generalization of the $\delta'$-interaction to the family for which the resonance sets appear in the form of a countable number of continuous two-dimensional curves. In this way, the connection between Kurasov’s $\delta'$-interaction and the resonant-tunneling point interactions is derived and the splitting of the resonance sets for tunneling plays a crucial role.

Keywords:
Transmission in one-dimensional quantum systems
Resonant tunneling through single-point barriers
Point interactions
Splitting effect
1. Introduction

Starting with the pioneering work by Berezin and Faddeev [1], various exactly solvable models described by the Schrödinger operators with singular zero-range potentials have been studied within the theory of selfadjoint extensions of symmetric operators. These models are specified by the potentials defined on the sets consisting of isolated points and therefore in the literature they are usually referred to as “point interactions” (see monographs [2, 3, 4] for details and references). According to this theory, all the selfadjoint extensions of the kinetic energy operator form a four-parameter family [5, 6], so that there are different ways to define the limit Schrödinger operator being appropriate for a given physical system. A whole body of literature (see, e.g., [7, 8, 9, 10, 11, 12, 13, 14], a few to mention), including the very recent studies [15, 16, 17, 18, 19, 20, 21, 22] with references therein, has been published where the one-dimensional Schrödinger operators were defined via distributions and corresponding two-sided boundary conditions at the points of singularity. The advantage of this “point” approach is the possibility to get the resolvents of these operators in an explicit form, to find their spectra, to compute scattering coefficients, etc.

On the other hand, the distributional part of Schrödinger operators can be treated as the limit of regularized potentials. Within this approach different asymptotic methods are used for realizing limit point interactions. Particularly, in dimension one, the regularized stationary Schrödinger equation

\[-\psi''(x) + V_\varepsilon(x)\psi(x) = E\psi(x),\]  

(1)

where the prime stands for the derivative with respect to the spatial coordinate \(x\) and \(\psi(x)\) is the wavefunction of a particle with energy \(E\), has been used. The potential \(V_\varepsilon(x)\) is supposed to depend on the squeezing parameter \(\varepsilon > 0\), so that in the limit as \(\varepsilon \to 0\), the function \(V_\varepsilon(x)\) is confined to one point. Using the asymptotic approach, most of papers [23, 24, 25, 26, 27, 28, 29, 30] have been devoted to studying the interactions of the point dipole type which are realized in the limit \(V_\varepsilon(x) \to \gamma\delta'(x)\) in the sense of distributions (\(\gamma \in \mathbb{R}\) is a coupling constant). In addition, the Schrödinger operators with \((a\delta' + b\delta)\)-like potentials have been investigated in a series of publications [13, 14, 31, 32, 33].

In the important work [7], Kurasov has developed the distribution theory based on the space of discontinuous at the point of singularity (say, at \(x = 0\))
test functions. Within this theory, it is possible to define rigorously, as a particular example, a point interaction referred in the following to as Kurasov’s $\delta'$-interaction, which is determined by the one-parameter transmission matrix

$$\begin{pmatrix} \psi(+0) \\ \psi'(+0) \end{pmatrix} = \Lambda \begin{pmatrix} \psi(-0) \\ \psi'(-0) \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \theta & 0 \\ 0 & \theta^{-1} \end{pmatrix}, \quad \theta = \frac{2 + \gamma}{2 - \gamma}$$

(2)

where the parameter $\gamma \in \mathbb{R} \setminus \{\pm 2\}$ serves as a coupling constant of this interaction. This transmission matrix has widely been used by many authors (see, e.g., [13, 15, 16, 17, 18, 19, 20]). On the other hand, beginning from the paper [27], for a whole class of conventional approximations of the $\delta'$-potential, Golovaty and coworkers have rigorously established the existence of discrete resonance sets in the $\gamma$-space on which the tunneling through this point barrier appears to be non-zero, whereas beyond these sets the system is fully opaque. Moreover, on the resonance sets they have developed the procedure how to compute the transmission matrix for this tunneling. This type of point interactions may be referred to as resonant-tunneling $\delta'$-potentials. Note that the only common feature of Kurasov’s $\delta'$-interaction and the resonant-tunneling $\delta'$-potential is that the transmission matrices of both these interactions are of the diagonal form. In this regard, it is important to develop an approach within which both these types of interactions could somehow be connected. Therefore the goal of the present paper is to realize both these types of interactions within a unique description starting from the same profile of the potential $V_\varepsilon(x)$ in Eq. (1).

It is fascinating that the connection between Kurasov’s $\delta'$-interaction and the family of resonant-tunneling $\delta'$-potentials can be described on the basis of the most simple physical system. We show that Kurasov’s $\delta'$-potential emerges from the realistic heterostructure consisting of two thin parallel plane layers separated by some distance in the limit as both the thickness of layers and the distance between them simultaneously tend to zero in a certain way. In other squeezing limits, the limit one-point interactions are proved to depend crucially on the relative coming up to zero of the thickness and the distance. As a result, a whole variety of single-point interactions occurs in this limit depending on the way of convergence. Surprisingly, within this approach, it is possible to realize both the Kurasov $\delta'$-interaction and the family of $\gamma\delta'$-potentials with countable sets in the $\gamma$-space at which a non-zero resonant tunneling takes place. The key point is that we have to extend the family of $\delta'$-potentials to wider class of interactions for which the resonance sets become curves instead of points. Another surprising point is that the
δ-potential discovered by Šeba in [23] can also be realized under a certain way of squeezing.

In general, one can consider the structure consisting of arbitrary $N$ separated layers. Then the potential in Eq. (1) can be expressed as a piecewise constant function depending on barrier heights or well depths $h_j \in \mathbb{R}$, widths $l_j$, $j = 1, N$, and the distances between the layers $r_j$, $j = 1, N-1$, such that $|h_j| \to \infty$ and $l_j, r_j \to 0$ as $\varepsilon \to 0$. Using the power-connecting parametrization $h_j = a_j \varepsilon^{-\mu_j}$, $r_j = c_j \varepsilon^{\tau_j}$ ($c_j > 0$) with positive powers $\mu_1, \ldots, \mu_N$ and $\tau_1, \ldots, \tau_{N-1}$, where $a_j \in \mathbb{R}$ may be called characteristic intensities of the layers, the potential $V_\varepsilon(x)$ can be represented in the form of the function $V_\varepsilon(l_1, \ldots, l_N; a_1, \ldots, a_N; \mu_1, \ldots, \mu_N; \tau_1, \ldots, \tau_{N-1}; x)$. The problem to be solved is the finding of the conditions on the parameters $a_1, \ldots, a_N$ and $\mu_1, \ldots, \mu_N; \tau_1, \ldots, \tau_{N-1}$ at which all the possible families of point interactions can be realized in the limit as $\varepsilon \to 0$.

The most simple situation appears if $\mu_1 = \ldots = \mu_N \equiv \mu$ and $\tau_1 = \ldots = \tau_{N-1} \equiv \tau$. The two cases of a double- and a triple-layer structures ($N = 2, 3$) have been analyzed in detail in the recent work [34]. Here the parameter $\mu$ controls the same rate of shrinking the layers, whereas the parameter $\tau$ describes the rate of decreasing the distance between the layers. Various families of single point interactions have been realized on different two-dimensional $\{\mu, \tau\}$-sets. In the present work we assume the different shrinking of two layers, i.e., $\mu_1 \equiv \mu$ and $\mu_2 \equiv \nu$, so that the power-like connection occurs here between the three parameters: $\mu$, $\nu$ and $\tau$. On the other hand, to keep things simple, we restrict ourselves only to a double-layer structure, but enlarge the number of squeezing parameters from two to three. In this (three-dimensional) space, we will find the open sets where Kurasov’s $\delta'$-interaction as well as Šeba’s $\delta$-potential [23] are defined. Within these sets, under approaching their limiting sets, the splitting of these interactions into countable families of one-point interactions is shown to occur and the description of this phenomenon is the key point of the present paper.

The paper is organized as follows. In Section 2 we present the piecewise constant potential for the double-layer structure and the transmission matrix for this system. The conditions for the resonant tunneling through the double-layer system in the limit as the layer thickness squeezes to one point are derived in Section 3 in a general form. In Section 4 we introduce a two-scale power-connecting parametrization of the layer parameters and describe the splitting of three types of point interactions. The additional parametrization of the distance between the layers is present in Section 5 for
the visualization of the splitting effect and Šeba’s transition. The paper is concluded by Section 6 in which we summarize the results with the discussion of possible extensions.

2. Finite-range potential and its transmission matrix

Consider the system consisting of two separated layers described by the piecewise constant potential

\[
\tilde{V}(h_1, h_2, l_1, l_2, r; x) = \begin{cases}
    h_1 & \text{for } 0 < x < l_1, \\
    h_2 & \text{for } l_1 + r < x < l_1 + r + l_2, \\
    0 & \text{for } -\infty < x < 0, \ l_1 < x < l_1 + r, \\
    l_1 + r + l_2 < x < \infty,
\end{cases}
\]

(3)

where \(h_j \in \mathbb{R}\) (\(h_j > 0\), barrier; \(h_j < 0\), well), \(l_j > 0\) (layer thickness), \(r > 0\) (distance between layers), \(j = 1, 2\). The transmission matrix \(\bar{\Lambda}\) for Eq. (1) with this potential is defined by the relations

\[
\begin{pmatrix}
    \psi(x_2) \\
    \psi'(x_2)
\end{pmatrix} = \bar{\Lambda} \begin{pmatrix}
    \psi(x_1) \\
    \psi'(x_1)
\end{pmatrix}, \quad \bar{\Lambda} = \begin{pmatrix}
    \bar{\lambda}_{11} & \bar{\lambda}_{12} \\
    \bar{\lambda}_{21} & \bar{\lambda}_{22}
\end{pmatrix}.
\]

(4)

It connects the boundary conditions of the wave function \(\psi(x)\) and its derivative \(\psi'(x)\) at \(x = x_1 = 0\) and \(x = x_2 = l_1 + r + l_2\). The notations with the overhead bars have been introduced for the finite-range quantities. Explicitly, the elements of the \(\Lambda\)-matrix that corresponds to the potential (3) are given by

\[
\begin{align*}
\bar{\lambda}_{11} &= [(k_1 l_1) \cos(k_2 l_2) - (k_1/k_2) \sin(k_1 l_1) \sin(k_2 l_2)] \cos(k r) \\
&\quad - [(k_1/k) \sin(k_1 l_1) \cos(k_2 l_2) + (k/k_2) \cos(k_1 l_1) \sin(k_2 l_2)] \sin(k r),
\end{align*}
\]

(5)

\[
\begin{align*}
\bar{\lambda}_{12} &= [(1/k_1) \sin(k_1 l_1) \cos(k_2 l_2) + (1/k_2) \cos(k_1 l_1) \sin(k_2 l_2)] \cos(k r) \\
&\quad + [(1/k) \cos(k_1 l_1) \cos(k_2 l_2) - (k_1/k_2) \sin(k_1 l_1) \sin(k_2 l_2)] \sin(k r),
\end{align*}
\]

(6)

\[
\begin{align*}
\bar{\lambda}_{21} &= - [k_1 \sin(k_1 l_1) \cos(k_2 l_2) + k_2 \cos(k_1 l_1) \sin(k_2 l_2)] \cos(k r) \\
&\quad - [k \cos(k_1 l_1) \cos(k_2 l_2) - (k_1/k_2) \sin(k_1 l_1) \sin(k_2 l_2)] \sin(k r),
\end{align*}
\]

(7)

\[
\begin{align*}
\bar{\lambda}_{22} &= [(k_1 l_1) \cos(k_2 l_2) - (k_2/k_1) \sin(k_1 l_1) \sin(k_2 l_2)] \cos(k r) \\
&\quad - [(k_1/k) \sin(k_1 l_1) \cos(k_2 l_2) + (k_2/k) \cos(k_1 l_1) \sin(k_2 l_2)] \sin(k r),
\end{align*}
\]

(8)

where

\[
k_j := \sqrt{E - h_j}, \quad j = 1, 2, \quad k := \sqrt{E}.
\]

(9)
3. Squeezing limit: Resonance conditions

The squeezing limit of the system given by the potential (3) means that $l_j, r \to 0$ but $|h_j| \to \infty$, $j = 1, 2$. Therefore if the matrix elements (5) - (8) are finite in the squeezing limit, we adopt the following notations:

$$\bar{\Lambda} \to \Lambda = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}, \quad \bar{\lambda}_{ij} \to \lambda_{ij}, \quad i, j = 1, 2,$$

(10)

where the limit elements are denoted without overhead bars. Next, having accomplished the limit procedure, we set $x_1 = -0$ and $\lim_{l_j, r \to 0} x_2 = +0$.

3.1. Two particular cases of point interactions

Consider some trivial cases of the convergence of the $\bar{\lambda}_{ij}$-elements given by Eqs. (5) - (8) in the limit as $l_1, l_2, \sin(kr) \to 0$. The first of these is a $\delta$-like profile of the layers. In this case, we have to assume in the squeezing limit that $h_j l_j = \alpha_j \in \mathbb{R}$, $j = 1, 2$. Then $k_j \to \sqrt{-h_j} \to \sqrt{-\alpha_j/l_j}$ and, as a result, the limit transmission matrix becomes

$$\Lambda = \begin{pmatrix} 1 & 0 \\ \alpha_1 + \alpha_2 & 1 \end{pmatrix},$$

(11)

which describes the $\delta$-interaction with the coupling constant that equals the algebraic sum of the layer intensities.

The second case, which also follows from Eqs. (5) - (8), concerns with a double-well structure ($h_j \leq 0$). Here the transmission across the system occurs perfect ($\Lambda = \pm \mathbb{I}$, $\mathbb{I}$ is the identity matrix) if $\sin(k_j l_j) \to 0$ or if $k_1 = k_2$ and $\cos(k_j l_j) \to 0$. As a result, we obtain the following two types of conditions on the system parameters:

$$\sqrt{-h_1} l_1 = m\pi, \sqrt{-h_2} l_2 = n\pi \quad \text{and} \quad h_1 = h_2, \sqrt{-h_1} l_1 = (n + 1/2)\pi$$

(12)

with $m, n = 0, 1, \ldots$.

3.2. Three types of resonant-tunneling point interactions

The arguments $k_j l_j$ of the trigonometric functions in Eqs. (5) - (8) must be finite under the squeezing of the widths $l_1$ and $l_2$. Therefore since $|h_j| \to \infty$, then $|k_j| \to \infty$, $j = 1, 2$. We assume that $k_j l_j \to A_j$, where $A_j$'s are required to be zero or finite non-zero (either real or imaginary) constants. For the realization of resonant-tunneling (connected) point interactions, the elements
of the limit $\Lambda$-matrix must be finite in the squeezing limit. As can be seen from the explicit representation (5) - (8), the element $\bar{\lambda}_{21}$ appears to be the most singular term in this limit. Hence, we have to assume the following limit:

$$\bar{\lambda}_{21} \to \alpha,$$

where $\alpha \in \mathbb{R}$ is an arbitrary constant. There are two ways of cancellation of divergences in the element $\bar{\lambda}_{21}$ as the layers are squeezed to zero and three types of connected point interactions can be realized as follows.

(i): One of the ways of cancellation of divergences in the $\bar{\lambda}_{21}$-element [see Eq. (7)] that provides the limit (13) is the asymptotic equation

$$\tan(kr) = \frac{l_1}{A_1} \cot A_1 + \frac{l_2}{A_2} \cot A_2 .$$

(14)

It follows from this equation that $\sin(kr) \to 0$ as $l_1, l_2 \to 0$ and therefore using this limit in Eq. (13), we find that $\alpha = 0$ in Eq. (13). Using next the resonance condition (14), one can find the asymptotic representation of the diagonal elements of the $\bar{\Lambda}$-matrix. Thus, inserting this condition into the expressions (5) and (8) for $\bar{\lambda}_{11}$ and $\bar{\lambda}_{22}$, we obtain the following three asymptotic representations:

$$\bar{\lambda}_{11} , \bar{\lambda}_{22}^{-1} \to \frac{\cos A_1 - (A_1/kl_1) \sin A_1 \tan(kr)}{\cos A_2} = \frac{\cos A_1}{\cos A_2 - (A_2/kl_2) \sin A_2 \tan(kr)} = - \frac{A_1 l_2 \sin A_1}{A_2 l_1 \sin A_2} .$$

(15)

(ii): The second way of cancellation of divergences in the $\bar{\lambda}_{21}$-element is to assume the equation

$$\frac{l_1}{A_1} \cot A_1 + \frac{l_2}{A_2} \cot A_2 = 0 .$$

(16)

In this case the limit (13) reduces to

$$\frac{A_1 A_2}{l_1 l_2} \sin A_1 \sin A_2 \sin(kr) \frac{k}{\bar{\lambda}_{21}} \to \alpha .$$

(17)

For a given $\alpha \in \mathbb{R}$, from this limit we find the second dependence of $\sin(kr)$ on the widths $l_1, l_2$:

$$\frac{\sin(kr)}{k} = \frac{\alpha l_1 l_2}{A_1 A_2 \sin A_1 \sin A_2} .$$

(18)
Similarly, using the resonance condition (16) as well as the dependence (18), we find the limit diagonal elements of the $\Lambda$-matrix in the second case of cancellation of divergences:

$$\bar{\lambda}_{11}, \bar{\lambda}_{22}^{-1} \to \frac{\cos A_1}{\cos A_2} = -\frac{A_1 l_2 \sin A_1}{A_2 l_1 \sin A_2}. \quad (19)$$

(iii): Finally, the third type of resonant-tunneling point interactions can be realized on the same resonance set defined by Eq. (16), however, for this type we assume that the limit $\sin(kr) \to 0$ proceeds faster (as a function of $l_1, l_2$) than in the asymptotic representation (18). The limit diagonal elements $\lambda_{11}$ and $\lambda_{22}$ are given in this case by the same formulae (19).

4. A power-connecting representation of the layer parameters

One of the ways to study the convergence $\bar{\Lambda} \to \Lambda$ as $l_1, l_2 \to 0$ is the connection of $l_j$ and $h_j$, $j = 1, 2$, through the squeezing parameter $\varepsilon > 0$ using two positive powers $\mu$ and $\nu$ as follows

$$l_1 = \varepsilon, \quad l_2 = \eta \varepsilon^{1-\mu+\nu} \ (0 < \eta < \infty), \quad h_1 = a_1 \varepsilon^{-\mu}, \quad h_2 = a_2 \varepsilon^{-\nu}. \quad (20)$$

Here the coefficients $a_j \in \mathbb{R}, \ j = 1, 2$, are characteristic quantities of the system, so that they may be called the intensities of layers. Inserting the parametrization (20) into the potential $\bar{V}$ [see Eq. (3)] and the matrix $\bar{\Lambda}$, we replace the notations as $\bar{V}(x) \to V_\varepsilon(x)$ and $\bar{\Lambda} \to \Lambda_\varepsilon$. Because of the limit $l_2 \to 0$, the inequality $1 - \mu + \nu > 0$ is necessary. We assume $\mu > 1$ because for $\mu < 1$ the transmission is trivially perfect and the case $\mu = 1$ reduces to the $\delta$-potential with the transmission matrix (11).

4.1. Sets of the existence of the distribution $\delta'(x)$

One can prove that the parametrized function $V_\varepsilon(x)$ converges in the sense of distributions to the derivative delta potential $\gamma \delta'(x)$ with the coupling constant $\gamma$ given below. This convergence takes place on the sets $B_j$, $j = 0, 1, 2$, shown in Fig. 1 in the case if $r = 0$ and $a_1 + a_2 = 0$ with arbitrary positive $\eta$, and defined by

$$B_0 := \{\mu = \nu = 2\}, \quad B_1 := \{1 < \mu < 2, \ \nu = 2(\mu - 1)\}, \quad B_2 := \{\mu = 2, \ 2 < \nu < \infty\}. \quad (21)$$
Figure 1: Schematics of the $\Omega$-set and its limiting (boundary) sets: point $B_0$, lines $B_1$ and $B_2$, defined by Eqs. (21) and (22).
which are the limiting sets of the open set
\[ \Omega := \{ 1 < \mu < 2, 2(\mu - 1) < \nu < \infty \}. \] (22)
The coupling constant \( \gamma \) of the \( \delta' \)-potential is the set function of \( B_j \)'s:
\[ \gamma = \gamma(B_j) = \begin{cases} \frac{a_1}{2} & \text{for } B_0, \\ \frac{\eta}{1} & \text{for } B_1, \\ \frac{1}{1} & \text{for } B_2. \end{cases} \] (23)

In the limit as \( \varepsilon \to 0 \), from Eqs. (9) and (20) we get the asymptotic representation
\[ k_1 \to \sqrt{-a_1 \varepsilon^{-\mu/2}}, \quad k_2 \to \sqrt{-a_2 \varepsilon^{-\nu/2}}, \]
\[ A_1 = \sqrt{-a_1 \varepsilon^{1-\mu/2}}, \quad A_2 = \eta \sqrt{-a_2 \varepsilon^{1-\mu+\nu/2}}. \] (24)

On the sets (21) and (22), the asymptotic formulae (24) provide the finiteness of the arguments of the trigonometric functions in Eqs. (5) - (8), so that all the expressions (14) - (19) can be used in the following in the parametrized form. Thus, the resonance condition (14) can be rewritten as
\[ \tan(kr) = \frac{\varepsilon^{\mu/2}}{\sqrt{-a_1}} \cot\left(\sqrt{-a_1 \varepsilon^{1-\mu/2}}\right) + \frac{\varepsilon^{\nu/2}}{\sqrt{-a_2}} \cot\left(\eta \sqrt{-a_2 \varepsilon^{1-\mu+\nu/2}}\right), \] (25)
well defined on the sets \( \Omega \) and \( B_j, j = 0, 1, 2 \). This representation defines the equation with respect to the intensities \( a_1 \) and \( a_2 \) that depends on the rate of the distance shrinking either to \( r = 0 \) (single-point interactions) or to \( r = n\pi, n \in \mathbb{N} \) (double-point interactions). In the present work, we restrict ourselves to the case of single-point interactions and therefore assume either \( r \equiv 0 \) or \( r \to 0 \).

4.2. Resonance conditions and their splitting

For the first type of point interactions we assume that \( r \to 0 \) in such a way that for any \( c > 0 \),
\[ \frac{\sin(kr)}{k} \varepsilon^{1-\mu} \to c. \] (26)

Then on the \( \Omega \)-set and its limiting sets \( B_0, B_1, B_2 \), the resonance condition (14) rewritten in the asymptotic form (25) reduces to
\[ c = \begin{cases} -\frac{1}{a_1} + \frac{1}{\eta a_2} \cot\left(\sqrt{-a_1} / \sqrt{-a_1} + \cot\left(\eta \sqrt{-a_2} / \sqrt{-a_2}\right) / \sqrt{-a_2}\right) & \text{for } \Omega, \\ \cot\left(\sqrt{-a_1} / \sqrt{-a_1} - 1/a_1\right) \cot\left(\eta \sqrt{-a_2} / \sqrt{-a_2} - 1/a_1\right) & \text{for } B_0, \\ \cot\left(\sqrt{-a_1} / \sqrt{-a_1} - 1/\eta a_2\right) & \text{for } B_1, \end{cases} \] (27)
Similarly, for the second type of point interactions we assume that $r \to 0$ in such a way that for any $c_0 > 0$,

$$\frac{\sin(kr)}{k} e^{2(1-\mu)} \to c_0.$$  \hfill (28)

The comparison of the limits (26) and (28) results in $c = 0$ in Eqs. (27) for the second type of interactions. For the third type the $r \to 0$ limit is performed in such a way that $c_0 = 0$ in (28), so that for this type $c = c_0 = 0$. Note also that the linearization of the right-hand side in (27) for $B_0$ with respect to $a_1$ and $a_2$ results in the corresponding right-hand expressions for $B_1$, $B_2$ and $\Omega$.

Thus, the analysis of the resonance sets for all the three types is based on Eqs. (27) with $c \geq 0$. The resonance sets for the $\Omega$-set as solutions to the first equation (27) are illustrated by Fig. 2 for both $c > 0$ (two red curves) and $c = 0$ (green line). The solution with $c > 0$ plotted by the two (red) curves $\sigma_{K,0}$ and $\sigma_{K,1}$ forms the resonance set $\Sigma_K = \sigma_{K,0} \cup \sigma_{K,1}$. The resonance curve $\sigma_{K,0}$ appears to be “pinned” to the origin $a_1 = a_2 = 0$. It can be considered as a background branch of the resonance set and therefore we call it the zeroth resonance curve. In the limit as $c \to 0$, the curve $\sigma_{K,1}$ vanishes “escaping” to infinity, while the zeroth branch straightens to the line (green in Fig. 2)

$$\sigma_L := \{(a_1,a_2) \in \mathbb{R}^2 \mid a_1 + \eta a_2 = 0\},$$  \hfill (29)

called in the following the resonance set for the second and the third types of interactions.

Next, as follows from the set of Eqs. (27) for both the cases with $c > 0$ and $c = 0$, while approaching the limiting sets $B_j$, $j = 0, 1, 2$, within the open set $\Omega$, the splitting or bifurcation of the resonance sets $\Sigma_K$ ($c > 0$) and $\sigma_L$ ($c = 0$) happens and this effect is clearly illustrated by Figs. 3-5. As shown in these figures, each of Eqs. (27) for $B_j$, $j = 0, 1, 2$, admits a countable set of solutions in the form of curves on the $\{a_1,a_2\}$-plane. These resonance curves can be numbered by $n = 0, 1, \ldots$ and we denote them as $\sigma_{c,n}(B_j)$ for $c \geq 0$, which depend on the boundary sets $B_j$, $j = 0, 1, 2$. Hence the total resonance sets become as the set functions:

$$\Sigma_c(B_j) := \bigcup_{n=0}^{\infty} \sigma_{c,n}(B_j), \quad j = 0, 1, 2.$$  \hfill (30)

The set of the curves with $n = 1, 2, \ldots$ may be considered as the detachment from the zeroth curves $\sigma_{c,0}(B_j)$. The comparison of Figs. 3-5 with Fig. 2 clearly illustrates the splitting of the resonance sets $\Sigma_K$ and $\sigma_L$ into $\Sigma_{c>0}(B_j)$
Figure 2: Two disconnected curves ($\sigma_{K,0}$ and $\sigma_{K,1}$, red lines) as a solution of Eq. (27) for the $\Omega$-set with $c = 1/2$ and $\eta = 1$ forming the resonance set $\Sigma_K$. The curve $\sigma_{K,0}$ corresponds to the two barrier-well configurations with $h_1 h_2 < 0$ ($a_1 a_2 < 0$) of the potential (3), while the curve $\sigma_{K,1}$ describes the resonance related to the double-well structure. The point lying on the line $\sigma_{K,1}$ with the coordinates $a_1 = a_2 = -2/c$ (shown with the empty ball) corresponds to the symmetric double-well system with perfect transmission ($\Lambda = -I$). They belong to the family with the conditions (12). The line $\sigma_L$ ($a_1 + \eta a_2 = 0$, green) intersects the zeroth resonance curve $\sigma_{K,0}$ only at the origin $a_1 = a_2 = 0$. The coordinates of the asymptotic (dashed) lines are $a_1 = -1/c$ and $a_2 = -1/\eta c$. When $c \to 0$, the curve $\sigma_{K,0}$ remains pinned to the origin $a_1 = a_2 = 0$ straightening to the line $\sigma_L$ ($\sigma_{K,0} \to \sigma_L$ as $c \to 0$), while the second resonance curve $\sigma_{K,1}$ vanishes escaping to infinity.
Figure 3: The first three (marked with $n = 0, 1, 2$) resonance curves $\sigma_{c>0,n}(B_1)$ (red) and $\sigma_{c=0,n}(B_1)$ (blue) as solutions to Eq. (27) for $B_1$ plotted at $\eta = 1$ with $c = 1/2$ and $c = 0$, respectively. The curve $\sigma_{c>0,2}(B_1)$ (red) is depicted partially. The points $(-1/c, d_{n-1})$ with $n = 1, 2$ are shown with the filled (black) balls. The points shown as the intersection of the first detached resonance curves $\sigma_{c>0,1}(B_1)$ (red) and $\sigma_{c=0,1}(B_1)$ (blue) with the $\sigma_L$-line ($a_1 + \eta a_2 = 0$, green) belong to the resonance sets $\Sigma_{\gamma \delta'}(B_1)$ for the potential $\gamma \delta'(x)$, where $\gamma = \eta a_1/2$ [see Eq. (23) for $B_1$] and $a_1$'s are solutions to the equation $\sqrt{\eta a_1} \cot \sqrt{\eta a_1} = 1 + ca_1$ [see Eq. (27) for $B_1$] with $c = 1/2$ (red) and $c = 0$ (blue).
Figure 4: The first four (marked with \( n = 0, 1, 2, 3 \) resonance curves \( \sigma_{c>0,n}(B_1) \) (red) and \( \sigma_{c=0,n}(B_1) \) (blue) as solutions to Eq. (27) for \( B_2 \) plotted at \( \eta = 2 \) with \( c = 1/2 \) and \( c = 0 \), respectively. The curve \( \sigma_{c>0,3}(B_2) \) (red) is depicted partially. The points \( (d_{n-1}, -1/\eta c) \) with \( n = 1, 2 \) are shown with the filled (black) balls. The points shown as the intersection of the first detached resonance curves \( \sigma_{c>0,1}(B_2) \) (red) and \( \sigma_{c=0,1}(B_2) \) (blue) with the \( \sigma_L \)-line \( (a_1 + \eta a_2 = 0, \text{green}) \) belong to the resonance sets \( \Sigma_{\gamma \delta'}(B_2) \) for the potential \( \gamma \delta'(x) \), where \( \gamma = a_1/2 \) [see Eq. (24) for \( B_2 \)] and \( a_1 \)'s are solutions to the equation \( \sqrt{-a_1 \cot \sqrt{-a_1}} = 1 - ca_1 \) [see Eq. (27) for \( B_2 \)] with \( c = 1/2 \) (red) and \( c = 0 \) (blue).
Figure 5: The first five (marked with $n = 0, 1, 2, 3, 4$) pairs of the resonance curves $\sigma_{c>0,n}(B_0)$ (red) and $\sigma_{c=0,n}(B_0)$ (blue) as solutions to Eqs. (27) for $B_0$ at $\eta = 1$ with $c = 1/2$ and $c = 0$, respectively. The values $a_1 = b_n, d_n, s_n$ (not shown) correspond to $a_2 = b_n, d_n, s_n$ placed vertically. The characteristic points $(\bar{b}_0, d_0), (d_0, b_0)$ for $n = 1$, $(\bar{b}_0, d_1), (d_0, b_1), (\bar{b}_1, d_0)$ and $(d_1, b_0)$ for $n = 2$, $(\bar{b}_1, d_1)$ and $(d_1, b_1)$ for $n = 3$ that belong to the set $\Sigma_{c>0}(B_0)$ can be seen as the intersection of the corresponding vertical and horizontal lines depicted with the dashed lines. The two points $(\bar{d}_0, d_1)$ and $(d_1, d_0)$ shown with the black filled balls and lying on the curve $\sigma_{c=0,2}(B_0)$ are the limits of the pairs $(\bar{b}_0, d_1), (d_0, b_1)$ and $(b_1, d_0), (d_1, b_0)$ as $c \to 0$, respectively.
and $\Sigma_{c=0}(B_j)$, respectively. Here the zeroth curves $\sigma_{c>0,0}(B_j)$ are deformed a bit if compared with the $\sigma_{K,0}$-curve shown by the red line in Fig. 2. The location of the split resonance sets on the $\{a_1,a_2\}$-plane for each set $B_j$ is described below. The characteristic points on this plane are given in terms of the $(n+1)$th root (denoted by $b_n = b_n(\eta)$, $b_n|_{\eta=1} =: \bar{b}_n$; $n = 0, 1, \ldots$) of the equation

$$\cot(\eta \sqrt{-b}) = c \sqrt{-b}, \quad -\infty < b < 0,$$

(31)

and the points

$$d_n = d_n(\eta) := -[(n + 1/2)\pi/\eta]^2, \quad d_n|_{\eta=1} =: \tilde{d}_n$$

$$s_n = s_n(\eta) := -(n\pi/\eta)^2, \quad s_n|_{\eta=1} =: \tilde{s}_n$$

(32)

being the solutions of the equations $\cos(\eta \sqrt{-a_j}) = 0$ and $\sin(\eta \sqrt{-a_j}) = 0$ ($j = 1, 2$), respectively. Note that the root $b_n$, $n = 1, 2, \ldots$, is found in the interval $-[(n+1/2)\pi/\eta]^2 < b_n(\eta) < -(n\pi/\eta)^2$, where $b_n \to d_n$ as $c \to 0$. The intersection of the $\sigma_{c}$-line with the resonance sets $\Sigma_{c}(B_j)$ defines the discrete point set for the $\gamma\delta'$-potential, where the coupling constant $\gamma$ is given by Eqs. (23) with $a_1 = -\eta a_2$ and $c \geq 0$ satisfying Eqs. (27). In Figs. 3-6, the red curves belong to $c > 0$ and the blue ones to $c = 0$.

**Description of the resonance sets $\Sigma_{c}(B_1)$ plotted in Fig. 3** For the $B_1$-set, the zeroth resonance curve is located in the region $\{-1/c < a_1 \leq 0, 0 \leq a_2 < \infty\} \cup \{0 \leq a_1 \leq \infty, b_0 < a_2 \leq 0\}$. The asymptotics of the curves with $n = 1, 2, \ldots$ are $(a_1 \to -\infty, a_2 \to b_{n-1})$ and $(a_1 \to +\infty, a_2 \to b_n)$. Each of these curves passes through the points $(-1/c, d_{n-1})$ and $0, s_n$.

**Description of the resonance sets $\Sigma_{c}(B_2)$ plotted in Fig. 4** For the $B_2$-set, the zeroth resonance curve is located in the region $\{\tilde{b}_0 < a_1 \leq 0, 0 \leq a_2 < \infty\} \cup \{0 \leq a_1 < \infty, -1/\eta c < a_2 \leq 0\}$. The asymptotics of the curves with $n = 1, 2, \ldots$ are $(a_1 \to \tilde{b}_n, a_2 \to +\infty)$ and $(a_1 \to \tilde{b}_{n-1}, a_2 \to +\infty)$. Each of these curves passes through the points $(\tilde{d}_{n-1}, -1/\eta c)$ and $(\tilde{s}_n, 0)$.

**Description of the resonance sets $\Sigma_{c}(B_0)$ plotted in Fig. 5** For the $B_0$-set, the zeroth resonance curve is located in the region $\{\tilde{b}_0 < a_1 \leq 0, 0 \leq a_2 < \infty\} \cup \{0 \leq a_1 < \infty, b_0 < a_2 \leq 0\}$. The asymptotics of the $\sigma_{c,n}(B_0)$-curves are $(a_1 \to \tilde{b}_n, a_2 \to +\infty)$ and $(a_1 \to +\infty, a_2 \to b_n)$, $n = 0, 1, \ldots$. Each detached curve passes through the characteristic points $(\tilde{s}_i, s_j)$ with $i + j = n$ and $(\tilde{b}_i, d_j), (\tilde{d}_i, b_j)$ with $i + j = n - 1$. 

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4.3. Splitting of the first type of interactions

Using the parametrization (20), from the asymptotic representation of the diagonal elements of the $\bar{\Lambda}$-matrix given by the limits (15), we obtain the following expressions for $\theta := \lambda_{11} = \lambda_{22}^{-1}$:

$$
\theta = \left\{ \begin{array}{ll}
1 + ca_1 = (1 + \eta ca_2) & \text{for } \Omega, \\
(\cos \sqrt{-a_1} - c\sqrt{-a_1} \sin \sqrt{-a_1}) / \cos(\eta \sqrt{-a_2}) & \\
= \cos \sqrt{-a_1} \left[ \cos(\eta \sqrt{-a_2}) - c\sqrt{-a_2} \sin(\eta \sqrt{-a_2}) \right]^{-1} & \\
= -\sqrt{a_1/a_2} \sin(-a_1/\sin(\eta \sqrt{-a_2})) & \text{for } B_0, \\
(1 + ca_1) / \cos(\eta \sqrt{-a_2}) & \\
= \left[ \cos(\eta \sqrt{-a_2}) - c\sqrt{-a_2} \cos(\eta \sqrt{-a_2}) \right]^{-1} & \\
= a_1/\sqrt{-a_2} \sin(\eta \sqrt{-a_2}) & \text{for } B_1, \\
\cos \sqrt{-a_1} - c\sqrt{-a_1} \sin \sqrt{-a_1} = \cos \sqrt{-a_1}/(1 + \eta ca_2) & \\
= -a_1 \sin(-a_1/\eta a_2) & \text{for } B_2
\end{array} \right.
$$

\[(33)\]

in the limit as $\varepsilon \to 0$ for the first type of interactions ($c > 0$). This single-point interaction realized on the $\Omega$-set under the assumption (26) is referred in the following to as the resonant-tunneling $\delta'_K$-interaction. Setting here

$$
\theta = -\frac{a_1}{\eta a_2} = \frac{2 + \gamma}{2 - \gamma},
$$

we obtain the limit transmission matrix $\Lambda$ in the form of (2). Under the assumption (31), we obtain Kurasov’s $\delta'$-interaction with the intensity $\gamma \in \mathbb{R} \setminus \{\pm 2\}$ defined in the distributional sense on the space of discontinuous at $x = 0$ test functions. For this case one can find the resonance values of $a_1$ and $a_2$ as functions of the strength $\gamma$:

$$
a_1 = \frac{2\gamma}{c_1(2 - \gamma)} \quad \text{and} \quad a_2 = -\frac{2\gamma}{\eta c_1(2 + \gamma)}.
$$

\[(35)\]

The barrier-well structure corresponds to the interval $-2 < \gamma < 2$ ($a_1 > 0$, $a_2 < 0$ for $-2 < \gamma < 0$ and $a_1 < 0$, $a_2 > 0$ for $0 < \gamma < 2$), whereas beyond this interval ($2 < |\gamma| < \infty$), we have the double-well configuration. The boundary conditions beyond the resonance set $\Sigma_K$ are of the Dirichlet type: $\psi(\pm 0) = 0$.

Thus the effect of splitting the $\delta'_K$-interaction occurs while approaching the limiting $B_j$-sets from the $\Omega$-set. On these sets, the limit transmission matrix $\Lambda$ is of the form (2) where the element $\theta$ is determined by Eqs. (33) defined
on the resonance sets $\Sigma_{c>0}(B_j)$, $j = 0, 1, 2$. We denote these point interactions as $\delta'_{c>0}(B_j)$ in despite of they are no more the point dipoles (double-well configurations are present together with barrier-well ones). Schematically, we denote this type of splitting as the mapping $\delta'_K \rightarrow \delta'_{c>0}(B_j)$, $j = 0, 1, 2$. Similarly, outside the resonance sets $\Sigma_{c>0}(B_j)$, the limit point interactions satisfy the Dirichlet boundary conditions $\psi(\pm 0) = 0$.

4.4. Splitting of the second and the third types of interactions

Similarly, using the parametrization (20) in Eqs. (18) and (19), we get the representation of the diagonal elements of the limit $\Lambda$-matrix for the second type of interactions ($\theta := \lambda_{11} = \lambda_{22}^{-1}$):

$$\theta = \begin{cases} 
1 & \text{for } \Omega, \\
\cos \sqrt{-a_1}/\sin(\eta \sqrt{-a_2}) & \text{for } B_0, \\
\cos(\eta \sqrt{-a_2}) = a_1/\sqrt{-a_2} \sin \eta \sqrt{-a_2} & \text{for } B_1, \\
\cos \sqrt{-a_1} = \sqrt{-a_1} \sin \sqrt{-a_1}/\eta a_2 & \text{for } B_2 
\end{cases}$$

(36)

and

$$\lambda_{21} = \alpha = c_0 \sqrt{a_1 a_2} \begin{cases} 
\eta \sqrt{a_1 a_2} & \text{for } \Omega, \\
\sin \sqrt{-a_1} \sin(\eta \sqrt{-a_2}) & \text{for } B_0, \\
\sqrt{-a_1} \sin(\eta \sqrt{-a_2}) & \text{for } B_1, \\
\eta \sqrt{-a_2} \sin \sqrt{-a_1} & \text{for } B_2, 
\end{cases}$$

(37)

where $a_1$ and $a_2$ satisfy the resonance conditions (27) with $c = 0$ being defined on the resonance sets $\sigma_L$ (for $\Omega$) and $\Sigma_{c=0}(B_j)$, $j = 0, 1, 2$.

The transmission matrix for the point interaction realized on the $\Omega$-set corresponds to the potential $\alpha \delta(x)$. In the following we denote this point interaction as the $\delta_S$-interaction, which is defined on the line $\sigma_L$. While approaching the limit sets $B_j$, the splitting of the $\delta_S$-interaction occurs resulting in the point interactions with the transmission matrix of the form

$$\Lambda = \begin{pmatrix} \theta & 0 \\ \alpha & \theta^{-1} \end{pmatrix}$$

where the elements $\theta$ and $\alpha$ are given by Eqs. (36) and (37), respectively. We denote these split interactions as $(\delta - \delta')_{c=0}(B_j)$ and thus one can use the mapping notation: $\delta_S \rightarrow (\delta - \delta')_{c=0}(B_j)$, $j = 0, 1, 2$.

For the third type of interactions $c_0 = 0$ and therefore $\alpha = 0$. According to Eq. (36) for $\Omega$, the transmission matrix on the $\Omega$-set is identically the unit ($\Lambda = I$) and therefore the realized interactions on this set are reflectionless. In the following we denote them as $I_R$, while the point interactions realized
on the limit $B_j$-sets can be denoted by $\delta'_{c=0}(B_j)$. Thus, one can use the mapping notation: $I_R \to \delta'_{c=0}(B_j), j = 0, 1, 2$.

As follows from Eqs. (33) and (36), for the $n$th curve passing through the points $(0, s_n)$ in Fig. 3, $(s_i, 0)$ in Fig. 4, $(s_i, s_j), i + j = n$, and $(d_{2j+1}, d_{2j+1})$, $2j + 1 = n$, $i, j = 0, 1, \ldots$, in Fig. 5 we have $\theta = (-1)^n$ resulting in the perfect transmission for the first and the third types of point interactions. These points, which are indicated in the figures with the empty balls, satisfy the conditions (12).

5. Geometric representation of the splitting effect

The power-connecting parametrization (20) can be extended by adding a power parameter for the distance $r$. To this end, we introduce the additional (third) power $\tau$ that describes the rate of shrinking the distance $r$ to one point, setting

$$r = c\varepsilon^\tau, \quad c \geq 0, \quad \tau > 0. \quad (38)$$

Then, adding the third dimension $\tau$ to the $\{\mu, \nu\}$-plane, the dihedral angle formed by the sets $B_j \times \{0 < \tau < \infty\}, j = 0, 1, 2$, can be considered. The cut off this angle with the plane $\tau = \mu - 1$ forms the trihedral angle with the vertex at the point $P_1$ as shown in Fig. 6. In the limit as $\varepsilon \to 0$, the trihedral angle surface appears to be the region where $\delta'(x)$ can be defined in the sense of distributions. Therefore we denote this surface by $S_{\delta'}$ and the notations of its elements indicated in Figs. 6 and 7 will be explained below.

5.1. Point interactions in the interior of the trihedral angle: Šeba’s transition

Consider first the volume interior of the trihedral angle that consists of the volume regions $V_1 := \Omega \times \{\mu - 1 < \tau < 2(\mu - 1)\}$ and $V_2 := \Omega \times \{2(\mu - 1) < \tau < \infty\}$, and the plane $Q_S := \Omega \times \{\tau = 2(\mu - 1)\}$ separating these regions (see Fig. 7). The point interactions realizing on these sets appear to be quite different. In the volume set $V_1$, the point interactions are separated with the boundary conditions of the Dirichlet type $\psi(\pm 0) = 0$ and therefore the transmission in this region is zero. Contrary, the point interactions in the volume region $V_2$ are reflectionless (they are denoted by $I_R$) with the resonance set $\sigma_L$. Note that beyond the $\sigma_L$-set, the point interactions are fully non-transparent. Thus, in the interior of the $S_{\delta'}$-surface, the $Q_S$-plane with partial transmission serves as a transition region from the set $V_1$ of opaque behavior to the volume $V_2$ of perfect transmission.
Figure 6: Schematics of the trihedral angle surface $S_3'$ formed by vertex $P_1$, edges $K_1, L_1, N_1 \cup P_2 \cup N_2$ (line $N_1$ and point $P_2$ shown in Fig. 7), and planes $Q_1 \cup K_2 \cup Q_2$ (line $K_2$ shown in Fig. 7) and $O_1 \cup L_2 \cup O_2$. The interior of the angle is volume set $V_1 \cup Q_S \cup V_2$ (plane $Q_S$ shown in Fig. 7).
Figure 7: Schematics of plane $Q_K$ with its boundary sets (point $P_1$, lines $K_1$ and $L_1$) and plane $Q_S$ with its boundary sets (point $P_2$, lines $K_2$ and $L_2$). The edges $K_1$, $L_1$ and $N_1 \cup P_2 \cup N_2$ form the trihedral angle surface $S_{\delta'}$ with vertex at point $P_1$. 
In physical terms, the point interactions realized in the volume region $V$ exhibit the transition of transmission that occurs on the resonance $\sigma_L$-set while varying the rate of increasing distance $r$ between the layers in the potential (3). For sufficiently slow squeezing this distance $[\mu - 1 < r < 2(\mu - 1)]$, the limit point interaction is opaque, for intermediate shrinking $[\tau = 2(\mu - 1)]$ the interaction becomes partially transparent ($\delta$-well) and for fast shrinking $[2(\mu - 1) < \tau < \infty]$ the transmission is perfect. In other words, the plane $Q_S$ separates the region $V_1$ of full reflection and the region $V_2$ of perfect transmission. Therefore the point interaction realized on the plane $Q_S$ may be called the resonant-tunneling $\delta_S$-interaction and that in the region $V_2$ the resonant-tunneling reflectionless $I_R$-interaction.

Consider now the situation when the thickness of both the layers in the potential (3) squeezes first to zero forming the $\delta$-profiles located at $x = 0$ and $x = r$, and then the $r \to 0$ limit is carried out. As a result, within the parametrizations (20) and (38) at $\eta = 1$ we get the following asymptotic representation for the potential (3):

$$V_\varepsilon(x) \to \varepsilon^{1-\mu} [a_1 \delta(x) + a_2 \delta(x - r)] = (c/r)^\vartheta [a_1 \delta(x) + a_2 \delta(x - r)]$$

with $\vartheta := (\mu - 1)/\tau$ in the limit as $r \to 0$. This potential has the same form used in [23] (see Theorem 3 therein). The transmission matrix of this interaction can be computed and, as a result, we find

$$\Lambda_r = \begin{pmatrix} 1 + c^\vartheta r^{1-\vartheta}a_1 & r \\ (c/r)^\vartheta (a_1 + a_2 + c^\vartheta r^{1-\vartheta}a_1a_2) & 1 + c^\vartheta r^{1-\vartheta}a_2 \end{pmatrix}. \quad (40)$$

It follows from this matrix that on the line $a_1 + a_2 = 0$ at $\vartheta = 1/2$ [on the plane $\tau = 2(\mu - 1)$] we have in the limit as $r \to 0$ the resonant $\delta$-interaction with the limit transmission matrix $\Lambda_{r \to 0} = \begin{pmatrix} 1 \\ -ca_1^2 \\ 0 \\ 1 \end{pmatrix}$, i.e., the result established by Šeba [23], which agrees with Eqs. (36) and (37) on the $\Omega$-set for $c_0 = c$.

In its turn, at $\vartheta = 1$ (on the plane $\tau = \mu - 1$) the $r \to 0$ limit of the matrix (40) reduces to the limit $\Lambda$-matrix with the diagonal elements for $\Omega$ corresponding to Kurasov’s $\delta_K$-interaction. Here the cancellation procedure of divergences in the off-diagonal term results in the resonance condition (27) on $\Omega$ with $\eta = 1$. At this condition for all $\vartheta \in (0, 1/2)$ the limit $\Lambda$-matrix is the unit, while for $\vartheta \in (1/2, 1)$ the limit point interactions are separated satisfying the Dirichlet conditions $\psi(\pm 0) = 0$. In physical
terms, the value $\vartheta = 1/2$ may be called a “transition” point (at which the transmission is partial) separating the opaque interaction from that with perfect transmission. Thus, all the results obtained above for the potential \((39)\) appear to be in agreement with those obtained for both the planes $Q_K$ and $Q_S$: at $\vartheta = 1$ we have the resonance set defined by Eq. (27) for $\Omega (\eta = 1)$ on the plane $Q_K$, while at $\vartheta = 1/2$, i.e., on the plane $Q_S$, the strength of the $\delta_S$-interaction is $\alpha = -c a_1^2$ describing the bound state with $\kappa := \sqrt{-E} = -\alpha/2$ ($E < 0$). Note that the point interactions with full reflection also occur on the boundaries of the volume set $V_1$: line $N_1 := B_0 \times \{1 < \tau < 2\}$ and planes $Q_1 := B_1 \times \{\mu - 1 < \tau < 2(\mu - 1)\}$ and $O_1 := B_2 \times \{1 < \tau < 2\}$.

5.2. Splitting of the interactions of the first type

Using the parametrization \((38)\) in Eq. (26), we find that the point interactions of the first type are realized on the plane $\tau = \mu - 1$. More precisely, the $\delta'_K$-interaction is realized on the plane set $Q_K := \Omega \times \{\tau = \mu - 1\}$ and its splitting occurs at the vertex $P_1 := B_0 \times \{\tau = 1\}$ and on the edges $K_1 := B_1 \times \{\tau = \mu - 1\}$ and $L_1 := B_2 \times \{\tau = 1\}$. Therefore for the first type one can write the following transitions:

$$Q_K \rightarrow W, \quad \Sigma_K \rightarrow \Sigma_{c>0}(W), \quad \delta'_K \rightarrow \delta'_{c>0}(W), \quad W = P_1(B_0), K_1(B_1), L_1(B_2). \quad (41)$$

5.3. Splitting of the interactions of the second type

Using the parametrization \((38)\) in Eq. (28), we find that the point interactions of the second type are realized on the plane $\tau = \mu - 1$ if $c_0 = c$. Here the $\delta_S$-interaction is realized on the plane set $Q_S$ and its splitting occurs at the vertex $P_2 := B_0 \times \{\tau = 2\}$ and on the edges $K_2 := B_1 \times \{\tau = 2(\mu - 1)\}$ and $L_2 := B_2 \times \{\tau = 2\}$. Therefore for the first type one can write the following transitions:

$$Q_S \rightarrow W, \quad \sigma_L \rightarrow \Sigma_{c=0}(W), \quad \delta_S \rightarrow (\delta-\delta')_{c=0}(W), \quad W = P_2(B_0), K_2(B_1), L_2(B_2). \quad (42)$$

5.4. Splitting of the interactions of the third type

For the third type of interactions $c_0 = 0$ in Eq. (28). In this case, the parametrization \((38)\) in Eq. (28) leads to the existence of point interactions in the volume set $V_2$, which is found above the $Q_S$-plane. The resonance set for these interactions is the same as for the $\delta_S$-interaction, i.e., $\sigma_L$, but now
\(\alpha \equiv 0\). Hence, due to Eq. (36) for \(\Omega\), this family of resonant tunneling point interactions appears to be reflectionless and we denote it by \(I_R\). The splitting of these interactions occurs on the limit sets of \(V_2\): edge \(N_2 := B_0 \times \{2 < \tau < \infty\}\) and planes \(Q_2 := B_1 \times \{2(\mu - 1) < \tau < \infty\}\) and \(O_2 := B_2 \times \{2 < \tau < \infty\}\). The diagonal elements of the limit \(\Lambda\)-matrix are defined by Eqs. (36) for \(B_j\), \(j = 0, 1, 2\). Thus, one can write the mappings:

\[V_2 \to \mathcal{W}, \sigma_L \to \Sigma_{c=0}(\mathcal{W}), I_R \to \delta'_{c=0}(\mathcal{W}), \mathcal{W} = N_2(B_0), Q_2(B_1), O_2(B_2).\]

(43)

6. Concluding remarks

We have studied the pointwise convergence of the transmission matrices for the double-layer system in the squeezing limit as both the thickness of the layers and the distance between them tend to zero simultaneously. Using the \(\{\mu, \nu, \tau\}\)-parametrization defined by Eqs. (20) and (38) that determines the three-scale squeezing of the system, the three types of single-point interactions with resonant-tunneling behavior have been realized. The corresponding resonance sets and the transmission \(\Lambda\)-matrices have been derived, treating thus the reflection-transmission properties of the double-layer system. In particular, on the plane \(Q_K\) we have defined Kurasov’s \(\delta'_K\)-interaction \([7]\) for which the diagonal element \(\theta\) in the transmission matrix (2) is given by Eq. (34). Under approaching the limiting sets of this plane, the countable splitting of the \(\delta'_K\)-interaction occurs that describes the resonant tunneling through the system. Unexpectedly, it has been found that Šeba’s \(\delta_S\)-interaction introduced in the work \([23]\) can also be included into the scheme developed in the present paper.

For convenience of the presentation, we have used the three-dimensional diagram for these powers illustrated by Figs. 6 and 7, where the whole variety of the sets corresponds to the family of single-point interactions realized on these sets. These sets determine how rapidly the squeezing of the distance between the layers proceeds in comparison with shrinking the thickness of the layers. The results can be summarized as follows.

- The realization of (both connected and separated) point interactions occurs in the trihedral angle \(V \cup S_{\delta'}\), where \(S_{\delta'}\) is the surface on which the \(\delta'\)-potential is well defined in the sense of distributions.

- The \(Q_K\)-interaction realized on the plane \(Q_K\) is identified by the transmission matrix of the type (2) where the element \(\theta\) is given by Eq. (34).
The resonance set $\Sigma_K$ for this interaction, being a solution to Eq. (27) for $\Omega$, consists of two curves $\sigma_{K,0}$ and $\sigma_{K,1}$ on the $\{a_1, a_2\}$-plane as illustrated by Fig. 2.

- The plane $Q_S$ splits the volume region $V$ into the set $V_1$ of separated (opaque) interactions satisfying the Dirichlet conditions $\psi(\pm 0) = 0$ and the set $V_2$ of the reflectionless interactions denoted by $I_R$. The $\delta_S$-interaction realized on the set $Q_S$ is defined by the transmission matrix with the elements (36) and (37) for $\Omega$. The resonance sets for both the $\delta_S$- and $I_R$-interactions are determined by the line $\sigma_L$.

- The splitting phenomenon occurs as the $(\mu, \nu, \tau)$-points on the open sets $Q_K$, $Q_S$ and $V_2$ are approaching their limiting sets. These limits can schematically be presented as the mappings

$$Q_K \to K_1, L_1, P_1; \quad Q_S \to K_2, L_2, P_2; \quad V_2 \to Q_2, O_2, N_2.$$  

The zeroth resonance sets $\sigma_{K,0} \subset \Sigma_K$ and $\sigma_L$ as single curves passing through the origin $a_1 = a_2 = 0$ split into countable sets. The splitting of these sets are schematically described as mappings by Eqs. (41) - (43). The comparison of Fig. 2 with Figs. 3 - 5 graphically illustrates the splitting effect. Similarly to the limit $\sigma_{K,0} \to \sigma_L$, the continuous transformation $\Sigma_{c>0} \to \Sigma_{c=0}$ takes place as $c \to 0$, despite the sets $K_1, L_1$ and $P_1$ are disconnected from $K_2 \cup Q_2, L_2 \cup O_2$ and $P_2 \cup N_2$, respectively.

- In the case when the potential (3) parametrized by Eqs. (20) and (38) converges to the distribution $\gamma\delta'(x)$ defined on the $S_{\gamma'}$-surface, the resonance sets $\Sigma_{c}(B_j)$ are restricted to the countable point sets $\Sigma_{\gamma\delta'}(B_j) = \Sigma_{c}(B_j) \cap \sigma_L$.

The splitting phenomenon described in the present paper seems to occur for any multi-layer system. Thus, in the case of $N$ layers separated equidistantly and determined by intensities $a_1, \ldots, a_N$ (as described in Introduction), the $N$-dimensional $S_{\gamma'}$-hypersurface for the existence of the distribution $\delta'(x)$ could be defined. In the $(N + 1)$-dimensional open set surrounded by this surface, the $\delta_K$, $\delta_S$- and $I_R$-interactions should be realized and their countable splitting on some limiting sets located on the $S_{\gamma'}$-hypersurface seems to take place. Therefore the approach developed here can be a starting point for further studies on the realization of point interactions in one dimension using a more general analysis.
Acknowledgments

The author acknowledges the financial support from the Department of Physics and Astronomy of the National Academy of Sciences of Ukraine under Project No. 0117U000240. He would like to express gratitude to Yaroslav Zolotaryuk for stimulating discussions and valuable suggestions.

References

References

[1] F.A. Berezin, L.D. Faddeev, Sov. Math. Dokl. 2 (1961) 372; Math. USSR Dokl. 137 (1961) 1011 (Engl. transl.).

[2] Yu.N. Demkov, V.N. Ostrovskii, Zero-Range Potentials and Their Applications in Atomic Physics, Plenum Press, New York, 1988 (Leningrad University Press, Leningrad, 1975).

[3] S. Albeverio, F. Gesztesy, R. Høegh-Krohn, H. Holden, Solvable Models in Quantum Mechanics, second ed. with appendix by P. Exner, AMS Chelsea, RI, 2005.

[4] S. Albeverio, P. Kurasov, Singular Perturbations of Differential Operators: Solvable Schrödinger-Type Operators, Cambridge University Press, Cambridge, 1999.

[5] S. Albeverio, L. Dąbrowski, P. Kurasov, Lett. Math. Phys. 45 (1998) 33-47.

[6] S. Albeverio, S.-M. Fei, P. Kurasov, Lett. Math. Phys. 59 (1998) 33-47.

[7] P. Kurasov, J. Math. Anal. Appl. 201 (1996) 297-323.

[8] F.A.B. Coutinho, Y. Nogami, J.F. Perez, J. Phys. A: Math. Gen. 30 (1997) 3937-3945.

[9] F.A.B. Coutinho, Y. Nogami, L. Tomio, J. Phys. A: Math. Gen. 32 (1999) 4931-4942.

[10] T. Cheon, T. Fülöp, I. Tsutsui, Ann. Phys. (NY) 294 (2001) 1-23.
[11] I. Tsutsui, T. Fülöp, T. Cheon, J. Math. Phys. 42 (2001) 5687-5697.
[12] S. Albeverio, L. Nizhnik, Lett. Math. Phys. 65 (2003) 27-35.
[13] M. Gadella, J. Negro, L.M. Nieto, Phys. Lett. A 373 (2009) 1310-1313.
[14] M. Gadella, M.L. Glasser, L.M. Nieto, Int. J. Theor. Phys. 50 (2011) 2144-2152.
[15] R.-J. Lange J. High Energy Phys. JHEP11 (2012) 1-32.
[16] J.F. Brasche, L.P. Nizhnik, Methods Funct. Anal. Topol. 19 (2013) 4-15 (arXiv:1112.2545v1 [math.FA]).
[17] M. Gadella, M.A. García-Ferrero, S. González-Martín, F.H. Maldonado-Villamizar, Int. J. Theor. Phys. 53 (2014) 1614-1627.
[18] R.-J. Lange, J. Math. Phys. 56 (2015) 122105.
[19] V.L. Kulinskii, D.Y. Panchenko, Physica B 472 (2015) 78-83.
[20] M. Gadella, J. Mateos-Guilarte, J.M. Muñoz-Castañeda, L.M. Nieto, J. Phys. A: Math. Theor. 49 (2016) 015204.
[21] K. Konno, T. Nagasawa, R. Takahashi, Ann. Phys. (NY) 375 (2016) 91-104.
[22] K. Konno, T. Nagasawa, R. Takahashi, Ann. Phys. (NY) 385 (2017) 729-743.
[23] P. Šeba, Rep. Math. Phys. 24 (1986) 111-120.
[24] P.L. Christiansen, N.C. Arnbak, A.V. Zolotaryuk, V.N. Ermakov, Y.B. Gaididei, J. Phys. A: Math. Gen. 36 (2003) 7589-7600.
[25] A.V. Zolotaryuk, P.L. Christiansen, S.V. Iermakova, J. Phys. A: Math. Gen. 39 (2006) 9329-9338.
[26] F.M. Toyama, Y. Nogami, J. Phys. A: Math. Theor. 40 (2007) F685-F690.
[27] Y.D. Golovaty, S.S. Man’ko, Ukrainian Math. Bull. 6 (2009) 169-203. (arXiv:0909.1034v1 [math.SP]).
[28] A.V. Zolotaryuk, Phys. Lett. A 374 (2010) 1636-1641.

[29] Y.D. Golovaty, R.O. Hryniv, J. Phys. A: Math. Theor. 43 (2010) 155204; 44 (2011) 049802.

[30] Y.D. Golovaty, R.O. Hryniv, Proc. R. Soc. Edinb. A 143 (2013) 791-816.

[31] A.V. Zolotaryuk, Y. Zolotaryuk, J. Phys. A: Math. Theor. 44 (2011) 375305; 45 (2012) 119501.

[32] Y. Golovaty, Methods Funct. Anal. Topol. 18 (2012) 243-255.

[33] Y. Golovaty, Integr. Equ. Oper. Theor. 75 (2013) 341-362.

[34] A.V. Zolotaryuk, J. Phys. A: Math. Theor. 50 (2017) 225303.