BSDE, Path-dependent PDE and Nonlinear Feynman-Kac Formula

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Abstract

In this paper, we introduce a type of path-dependent quasilinear (parabolic) partial differential equations in which the (continuous) paths $\omega_t$ on an interval $[0,t]$ becomes the basic variables in the place of classical variables $(t,x) \in [0,T] \times \mathbb{R}^d$. This new type of PDE are formulated through a classical backward stochastic differential equations (BSDEs, for short) in which the terminal values and the generators are allowed to be general function of Brownian paths. In this way we have established a new type of nonlinear Feynman-Kac formula for a general non-Markovian BSDE. Some main properties of regularities for this new PDE was obtained.

Keywords: backward stochastic differential equation, nonlinear Feynman-Kac formula, Itô integral and Itô calculus, Path-dependent PDE.

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1 Introduction

Linear Backward Stochastic Differential Equations (in short BSDE) was introduced by Bismut [2] in 1973. Pardoux and Peng [18, 1990] established the existence and uniqueness theorem for nonlinear BSDEs under a standard Lipschitz condition. Peng [20, 1991] and then Peng and Pardoux [19, 1992] introduced what we called nonlinear Feynman-Kac formula which provides a probabilistic representation for a wide class of (system of) quasi-linear partial differential equations. Since then, especially after the paper of El Karoui-Peng-Quenez [9, 1997] for BSDE, the theory of BSDE have received a wide attention for both theoretical research and applications.

In order to illustrate the above nonlinear Feynman-Kac formula, let us consider the following well-posed BSDE

\[-dY(t) = f(t,Y(t),Z(t))dt - Z(t)dB(t), \quad t \in [0,T],
Y(T) = \varphi(B(T)),\]

(BSDE)

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where $B$ is a given $d$-dimensional Brownian motion, $f = f(t, y, z)$, $\varphi = \varphi(x)$ are two 'good' functions. The unique solution of this BSDE consists of two stochastic process $(Y(t), Z(t))_{0 \leq t \leq T}$ adapting to the natural filtration $(\mathcal{F}_t)_{t \geq 0}$ of Brownian motion $B$ (see Section 2.2 for details), namely, both $Y(t)$ and $Z(t)$ at each time $t \in [0, T]$ are functions of the Brownian path $B(s)_{0 \leq s \leq t}$. For the case $Y(T) = \varphi(B(T))$, there exists a deterministic function $u = u(t, x)$ defined on $[0, T] \times \mathbb{R}^d$ such that $Y(t) = u(t, B(t))$ and $Z(t) = \nabla u(t, B(t))$. We can also prove that the function $u = u(t, x)$, $t \in [0, T]$, $x \in \mathbb{R}^d$, is in fact the unique solution of the following quasi-linear PDE:

$$\partial_t u + \frac{1}{2} \Delta u + f(t, u, \nabla u) = 0, \quad u(T, x) = \varphi(x).$$

(PDE)

This relation permits us to solve the above type of BSDE by PDE. Conversely we can also use the BSDE to solve the PDE. This nonlinear Feynman-Kac formula also provides a nonlinear Monte-Carlo method via the BSDE (3, 11) to solve numerically the PDE.

But in general the terminal condition $Y(T)$ of the BSDE may be a general function of Brownian paths, i.e., $Y(T) = \varphi(B(s)_{0 \leq s \leq T})$, where $\varphi$ is a given functional defined on the space $C([0, T], \mathbb{R}^d)$ of $d$-dimensional continuous paths. In this situation the solution $(Y, Z)$ is regarded as a generalized 'path-dependent' solution of the above BSDE. This problem was raised by the first author and summarized in his ICM2010’s talk [22, Sec.1.3].

With a very different point of view, Dupire [8, 2009] introduced a new type of a functional Itô’s formula which non-trivially generalized the classical one (See Cont and Fournié [4, 5, 6] for a more general and more systematic research). His idea is to introduce a simple but deeply insightful definition of derivatives with respect to path for a given $(\mathcal{F}_t)_{t \geq 0}$-adapted process $Y(t) = u(t, \omega(s)_{0 \leq s \leq t})$, which is a family of functional of Brownian path indexed by time $t$. He then claimed that if a martingale is a $C^{1,2}$-function in his new definition then it is also a solution of a 'functional PDE'. The so-called functional Feynman-Kac formula was also claimed under the same framework.

In this paper we will prove that, under certain smooth assumptions of the given path-dependent functions

$$Y(T) = \Phi(\omega(s)_{0 \leq s \leq T}), \quad f = f(t, \omega(s)_{0 \leq s \leq t}, y, z)$$

with respect to $(y, z)$ and the path $\omega$ (see (H1) and (H2) in Section 3), the solution $(Y(t), Z(t))$ of (BSDE) solves (PDE). More specifically, the path-function $u(t, \omega(s)_{0 \leq s \leq t}) := Y(t, \omega)$ is the unique $C^{1,2}$-solution of (PDE) and $Z(t)$ is the vertical derivative. We then can prove the regularity of the solution of BSDE in the sense of Dupire’s derivatives which gives the existence of $Y(t)$. The results of this paper has non-trivially generalized the ones of Pardoux and Peng [18, 1992] to the path-dependent situation.

The paper is organized as follows. In section 2, we present some existing results on functional Itô’s formula and BSDE that will be used in this paper. In section 3, we establish some estimates and regularity results for the solution of BSDEs with respect to paths. Finally, in section 4, we establish our main results, Theorems 4.1 and 4.3 which provide a one to one correspondence between (BSDE) and the system of path-dependent PDE.

When the coefficients of (BSDE) are only Lipschitz functions, we usually can not obtain the smoothness results given in this paper, thus a new type of viscosity solution is required. We refer to Peng [23, 2011] for the corresponding comparison theorem.
2 Preliminaries

2.1 Functional Itô’s formula

The following notations are mainly from Dupire [8]. Let $T > 0$ be fixed. For each $t \in [0, T]$, we denote by $\Lambda_t$ the set of càdlàg (French abbreviation for “right continuous with left limit”, also often denoted by RCLL) $\mathbb{R}^d$-valued functions on $[0, t]$.

For each $\gamma \in \Lambda_T$ the value of $\gamma$ at time $s \in [0, T]$ is denoted by $\gamma(s)$. Thus $\gamma = \gamma(s)_{0 \leq s \leq T}$ is a càdlàg process on $[0, T]$ and its value at time $s$ is $\gamma(s)$. The path of $\gamma$ up to time $t$ is denoted by $\gamma_t$, i.e., $\gamma_t = \gamma(s)_{0 \leq s \leq t}$. We denote $\Lambda = \bigcup_{t \in [0, T]} \Lambda_t$. We sometimes also specifically write

$$\gamma_t = \gamma(s)_{0 \leq s \leq t} = (\gamma(s)_{0 \leq s < t}, \gamma(t))$$

to indicate the terminal position $\gamma(t)$ of $\gamma_t$ which often plays a special role in this framework. For each $\gamma_t \in \Lambda$ and $x \in \mathbb{R}^d$ we denote $\gamma^*_t = (\gamma(s)_{0 \leq s < t}, \gamma(t) + x)$ which is also an element in $\Lambda_t$.

We are interested in a function $u$ of path, i.e., $u : \Lambda \to \mathbb{R}$. This function $u = u(\gamma_t)$, $\gamma_t \in \Lambda$ can be also regarded as a family of real valued functions:

$$u(\gamma_t) = u(t, \gamma(s)_{0 \leq s \leq t}) = u(t, \gamma(s)_{0 \leq s < t}, \gamma(t)) : \gamma_t \in \Lambda_t, \quad t \in [0, T].$$

We also denote $u(\gamma^*_t) := u(t, \gamma(s)_{0 \leq s < t}, \gamma(t) + x)$, for $\gamma_t \in \Lambda_t$, $x \in \mathbb{R}^d$.

Remark 2.1 It is also very important to understand $u(\gamma^*_t)$ as a function of $t$, $(\gamma(s)_{0 \leq s < t}, \gamma(t))$ and $x$. A typical case is $u(\gamma_t) = u(t, \gamma_{t-}, \gamma(t) + x)$, $t \in [0, T]$, where $\gamma_{t-} = \lim_{s \to t^-} \gamma(s)$.

We now introduce a distance on $\Lambda$. Let $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote the inner product and norm in $\mathbb{R}^d$. For each $0 \leq t \leq \bar{t} \leq T$ and $\gamma_t, \tilde{\gamma}_t \in \Lambda$, we denote

$$\|\gamma_t\| := \sup_{r \in [0, t]} |\gamma(r)|,$$

$$d_\infty(\gamma_t, \tilde{\gamma}_t) := \max(\sup_{r \in [0, t]} \{|\gamma(r) - \tilde{\gamma}(r)|\}, \sup_{r \in [t, \bar{t}]} \{|\gamma(t) - \tilde{\gamma}(r)|\}) + |t - \bar{t}|.$$

It is obvious that $\Lambda_t$ is a Banach space with respect to $\|\cdot\|$. Since $\Lambda$ is not a linear space, $d_\infty$ is not a norm.

Definition 2.2 (Continuous) A functionals $u : \Lambda \to \mathbb{R}$ is said $\Lambda$-continuous at $\gamma_t \in \Lambda$, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for each $\tilde{\gamma}_t \in \Lambda$ satisfying $d_\infty(\gamma_t, \tilde{\gamma}_t) < \delta$, we have $|u(\gamma_t) - u(\tilde{\gamma}_t)| < \varepsilon$. $u$ is said to be $\Lambda$-continuous if it is $\Lambda$-continuous at each $\gamma_t \in \Lambda$.

Remark 2.3 In our framework we often regard $u(\gamma^*_t)$ as a function of $t$, $\gamma$ and $x$, i.e., $u(\gamma^*_t) = u(t, \gamma(s)_{0 \leq s < t}, \gamma(t) + x)$. Thus, for a fixed $\gamma \in \Lambda_T$, $\gamma(\gamma_t)$ is regarded as a function of $(t, x) \in [0, T] \times \mathbb{R}^d$.

Definition 2.4 Let $u : \Lambda \to \mathbb{R}$ and $\gamma_t \in \Lambda$ be given. If there exists $p \in \mathbb{R}^d$, such that

$$u(\gamma^*_t) = u(\gamma_t) + \langle p, x \rangle + o(|x|), \quad x \in \mathbb{R}^d,$$

Then we say that $u$ is (vertically) differentiable at $\gamma_t$ and denote the gradient of $D_xu(\gamma_t) = p$. $u$ is said to be vertically differentiable in $\Lambda$ if $D_xu(\gamma_t)$ exists for each $\gamma_t \in \Lambda$. We can similarly define the Hessian $D_{xx}u(\gamma_t)$. It is an $\mathbb{S}(d)$-valued function defined on $\Lambda$, where $\mathbb{S}(d)$ is the space of all $d \times d$ symmetric matrices.
For each $\gamma_t \in \Lambda$ we denote
\[\gamma_{t,s}(r) = \gamma(r)\mathbf{1}_{[0,t)}(r) + \gamma(t)\mathbf{1}_{[t,s]}(r), \quad r \in [0,s].\]

It is clear that $\gamma_{t,s} \in \Lambda_s$.

**Definition 2.5** For a given $\gamma_t \in \Lambda$ if we have
\[u(\gamma_{t,s}) = u(\gamma_t) + a(s-t) + o(|s-t|), \quad s \geq t,\]
then we say that $u(\gamma_t)$ is (horizontally) differentiable in $t$ at $\gamma_t$ and denote $D_t u(\gamma_t) = a$. $u$ is said to be horizontally differentiable in $\Lambda$ if $D_t u(\gamma_t)$ exists for each $\gamma_t \in \Lambda$.

**Definition 2.6** Define $C^{1,k}(\Lambda)$ as the set of functionals $u := (u(\gamma_t))_{\gamma_t \in \Lambda}$ defined on $\Lambda$ which are $j$ times horizontally and $k$ times vertically differentiable in $\Lambda$ such that all these derivatives are $\Lambda$-continuous.

**Definition 2.7** $u := (u(\gamma_t))_{\gamma_t \in \Lambda}$ is said to be in $C^{0,2}_{t,j}(\Lambda)$, if $u$ is in $C^{0,2}(\Lambda)$ and, for $\varphi = u, D_x u, D_{xx} u$, we have
\[|\varphi(\gamma_t) - \varphi(\bar{\gamma}_t)| \leq C(1 + ||\gamma_t||^q + ||\bar{\gamma}_t||^q)d_\infty(\gamma_t, \bar{\gamma}_t), \quad \gamma_t, \bar{\gamma}_t \in \Lambda,\]
where $C, q$ are constants depending only on $\varphi$.

**Example 2.8** If $u(\gamma_t) = f(t, \gamma_t(t))$ with $f \in C^{1,1}([0,T]\times\mathbb{R})$, we have
\[D_t F(\gamma_t) = \partial_t f(t, \gamma_t(t)), \quad D_x F(\gamma_t) = \partial_x f(t, \gamma_t(t)),\]
which is the classic derivative. In general, these derivatives also satisfy the classic properties: linearity, product and chain rule.

The following Itô formula was firstly obtained by Dupire [8, 2009] and then by Cont and Fournier [5, 2010] for a more general formulation.

**Theorem 2.9** (Functional Itô’s formula). Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, P)$ be a probability space, if $X$ is a continuous semi-martingale and $u$ is in $C^{1,2}(\Lambda)$, then for any $t \in [0, T]$: \[u(X_t) - u(X_0) = \int_0^t D_s u(X_s) ds + \int_0^t D_x u(X_s) dX(s) + \frac{1}{2} \int_0^t D_{xx} u(X_s) ds, \quad a.s..\]
In particular , if $X$ is a $\mathcal{F}$-Brownian motion,
\[u(X_t) - u(X_0) = \int_0^t D_s u(X_s) ds + \int_0^t D_x u(X_s) dX(s) + \frac{1}{2} \int_0^t D_{xx} u(X_s) ds, \quad a.s..\]

**2.2 Backward Stochastic Differential Equations**

Let $\Omega = C([0, T], \mathbb{R}^d)$ and $P$ the standard Wiener measure defined on $(\Omega, B(\Omega))$. We denote by the canonical process $B(t) = B(t, \omega) = \omega(t), \ t \in [0, T], \ \omega \in \Omega$. Then $B(t)_{0 \leq t \leq T}$ is a $d$-dimensional Brownian motion defined on the probability space $(\Omega, B(\Omega), P)$. Let $\mathcal{N}$ be the collection of all
P-null sets in $\Omega$. For any $0 \leq t \leq r \leq T$, $F_t^\gamma$ denotes the completion of $\sigma(B(s) - B(t); t \leq s \leq r)$, i.e., $F_t^\gamma = \sigma\{B(s) - B(t); t \leq s \leq r, x, y, z \in \mathbb{R}^m\}$. We also write $F_r$ for $F_T^R$ and $F_t$ for $F_T^\gamma$.

For any $0 \leq t \leq T$, we denote by $L^2(F_t)$ the set of all square integrable $F_t$-measurable random variables, $M^2(t,T)$ the space of all $F_t^\gamma$-adapted, $\mathbb{R}^d$-valued processes $(X(s))_{s \in [t,T]}$ with $E[\int_t^T |X(s)|^2 ds] < \infty$ and $S^2(t,T)$ the space of all $F_t^\gamma$-adapted, $\mathbb{R}^d$-valued continuous processes $(X(s))_{s \in [t,T]}$ with $E\left[\sup_{s \in [t,T]} |X(s)|^2\right] < \infty$.

Let us consider a deterministic function $f : \Lambda \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}^m$, which will be in the following the generator of our BSDEs. For the function $f$, we will make the following assumptions:

- There exists constants $C \geq 0$ and $q \geq 0$ such that: for $\gamma_t, \gamma_y \in \Lambda, y, \bar{y} \in \mathbb{R}^m, z, \bar{z} \in \mathbb{R}^{m \times d}$,

$$|f(\gamma_t, y, z) - f(\gamma_t, \bar{y}, \bar{z})| \leq C((1 + \|\gamma_t\|^q + \|\gamma_t\|^q)\|\gamma_t - \gamma_y\| + |y - \bar{y}| + |z - \bar{z}|).$$

The following result on backward stochastic differential equations (BSDEs) is by now well known, for its proof the reader is referred to Pardoux and Peng [15].

**Lemma 2.10** Let $f$ satisfy the above conditions, then for each $\xi \in L^2(F_T)$, the BSDE

$$Y(t) = \xi + \int_t^T f(B_s, Y(s), Z(s))ds - \int_t^T Z(s)dB(s), \quad 0 \leq t \leq T,$$

has a unique adapted solution

$$(Y(t), Z(t))_{0 \leq t \leq T} \in S^2(0, T) \times M^2(0, T).$$

We also shall recall the following basic result on BSDEs, which is the well-known comparison theorem (see El Karoui, Peng, Quenez [9]).

**Lemma 2.11** (Comparison Theorem) We assume $m = 1$. Given two coefficients $f_1$ and $f_2$ satisfying the above assumptions and two terminal values $\xi_1, \xi_2 \in L^2(F_T)$, we denote by $(Y_1, Z_1)$ and $(Y_2, Z_2)$ the solution of BSDE with the data $(\xi_1, f_1)$ and $(\xi_2, f_2)$, respectively. Then we have: If $\xi_1 \geq \xi_2$ and $f_1 \geq f_2$, a.s., then $Y_1(t) \geq Y_2(t)$, a.s., for all $t \in [0, T]$.

## 3 Property of solution of the BSDE

Let us first recall some notations from Pardoux and Peng [20].

$C^k(\mathbb{R}^p, \mathbb{R}^q), C^k_b(\mathbb{R}^p, \mathbb{R}^q), C^k_p(\mathbb{R}^p, \mathbb{R}^q)$ will denote respectively the set of functions of class $C^k$ from $\mathbb{R}^p$ into $\mathbb{R}^q$, the set of functions the set of those functions of class $C^k$ whose partial derivatives of order less than or equal to $k$ are bounded (and hence the function itself grows at most linearly at infinity), and the set of those functions of class $C^k$ which, together with all their partial derivatives of order less than or equal to $k$, grow at most like a polynomial function of the variable $x$ at infinity.

The following directional derivatives will be used frequently in the sequel.

**Definition 3.1** If $\Phi$ is an $\mathbb{R}^m$-valued function on $\Lambda_T$. $\Phi$ is said to be in $C^2(\Lambda_T)$, if for each $x \in \Lambda_T$ and $t \in [0, T]$, there exist $p_1 \in \mathbb{R}^d$ and $p_2 \in \mathbb{R}^{d \times d}$ such that $p_2$ is symmetric and

$$\Phi(\gamma_t^x) - \Phi(\gamma) = (p_1, x) + \frac{1}{2} (p_2, x \otimes x) + o(|x|^2), \quad x \in \mathbb{R}^d,$$
we denote
\[ \Phi'_{\gamma}(\gamma) := p_1, \quad \Phi''_{\gamma}(\gamma) := p_2. \]

\( \Phi \) is said to be in \( C^2_{l,lip}(\Lambda_T) \) if \( \Phi'_{\gamma}(\gamma) \) and \( \Phi''_{\gamma}(\gamma) \) exist for all \( \gamma \in \Lambda_T, t \in [0, T] \), and for some constants \( C, k \geq 0 \) depending only on \( \Phi \),
\[
|\Phi(\gamma) - \Phi(\bar{\gamma})| \leq C(1 + ||\gamma||^k + ||\bar{\gamma}||^k)||\gamma - \bar{\gamma}||, \quad \gamma, \bar{\gamma} \in \Lambda_T,
\]
\[
|\Psi_{\gamma}(\gamma) - \Psi_{\gamma}(\bar{\gamma})| \leq C(1 + ||\gamma||^k + ||\bar{\gamma}||^k)(|t - s| + ||\gamma - \bar{\gamma}||), \quad \gamma, \bar{\gamma} \in \Lambda_T, t, s \in [0, T],
\]
with \( \Psi_{\gamma}(\gamma) = \Phi'_{\gamma}(\gamma), \Phi''_{\gamma}(\gamma) \). Analogous, we can define \( C^2(\Lambda_T), C^2_{l,lip}(\Lambda_t), C^1_{l,lip}(\Lambda_t) \).

In this rest of this paper we shall make use of the following assumptions on the generator \( f \) and the terminal \( \Phi \) of our BSDE:

(H1): Suppose \( \Phi \) is an \( \mathbb{R}^m \)-valued function on \( \Lambda_T \). Moreover, \( \Phi \) is in \( C^2_{l,lip}(\Lambda_T) \).

(H2): Suppose \( f(\gamma_t, y, z) \) is a given continuous function on \( \Lambda \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \). For any \( \gamma_t \in \Lambda \) and \( s \in [0, t], (x, y, z) \rightarrow f(\gamma_t, \gamma_s, y, z) \) is of class \( C^3_p(\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d}, \mathbb{R}^m) \) and the first order partial derivatives as well as their derivatives of up to order two with respect to \( (y, z) \) are uniformly bounded; for any \( (y, z), \gamma_t \mapsto f(\gamma_t, y, z) \) is of class \( C^1_{l,lip}(\Lambda_t) \), \( \gamma_t \mapsto f_y(\gamma_t, y, z), f_z(\gamma_t, y, z) \) is of class \( C^1_{l,lip}(\Lambda_t) \).

(H3): Suppose \( f(\gamma_t, y, z) = \bar{f}(t, \gamma_t(t), y, z) \), where \( \bar{f} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^m \) is such that \( (t, x, y, z) \rightarrow \bar{f}(t, x, y, z) \) is of class \( C^0,3_p([0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d}, \mathbb{R}^m) \) and the first order partial derivatives as well as their derivatives of up to order two with respect to \( (y, z) \) are uniformly bounded.

It is obvious that assumption (H3) implies assumption (H2).

Assume (H1) and (H2) hold. For any \( \gamma_t = (\gamma(s))_{0 \leq s \leq t} \in \Lambda \), let \( (Y_{\gamma_t}(s), Z_{\gamma_t}(s))_{t \leq s \leq T} \) denote the unique element of \( S^2[t, T] \) which solves the following BSDE:

\[ Y_{\gamma_t}(s) = \Phi(B^n) + \int_{s}^{T} f(B^n, Y_{\gamma_t}(r), Z_{\gamma_t}(r))dr - \int_{s}^{T} Z_{\gamma_t}(r)dB(r), \quad (2) \]

where,
\[ B^n(u) := \gamma(t)I_{[0, t]}(u) + (\gamma(t) + B(u) - B(t))I_{[t, T]}(u). \]

It is clear, for \( \gamma_t, \bar{\gamma_t} \in \Lambda \) with \( \bar{t} \geq t \), we have
\[ B^n(u) - B^{\bar{\gamma}_t}(u) = I_{[0, t]}(u)(\gamma_t(u) - \bar{\gamma}_t(u)) + I_{[t, \bar{t}]}(u)(\gamma_t(u) - \bar{\gamma}_t(u)) + B(u) - B(t) \]
\[ + I_{[t, T]}(u)(\gamma_t(t) + B(t) - B(t) - \bar{\gamma}_t(t)) \]

It follows easily from the existence result in [19]:

Corollary 3.2 For each \( \gamma_t \in \Lambda \), the BSDE (2) has a unique solution \( (Y_{\gamma_t}(s), Z_{\gamma_t}(s))_{t \leq s \leq T} \) and \( Y_{\gamma_t}(t) \) defines a deterministic mapping from \( \Lambda \) to \( \mathbb{R}^m \).

3.1 Regularity of solution of the BSDE

We next establish higher order moment estimates for the solution of BSDE (2). For each \( z \in \mathbb{R}^{m \times d} \), we denote \( ||z|| = \sqrt{tr(z^*z)} \).
Lemma 3.3 For any $p \geq 2$, there exist $C_p$ and $q$ depending on $C, T, k, p$, such that:

$$E[\sup_{s \in [t,T]} |Y_{\gamma_t}(s)|^p] \leq C_p (1 + \|\gamma_t\|^q),$$  \hspace{1cm} (3)

$$E[\int_t^T \|Z_{\gamma_t}(s)\|^2 ds]^2 \leq C_p (1 + \|\gamma_t\|^q).$$  \hspace{1cm} (4)

Proof. Applying Itô’s formula to $\varphi(x) = |x|^p$ yields that,

$$|Y_{\gamma_t}(s)|^p + \frac{p}{2} \int_s^T |Y_{\gamma_t}(r)|^{p-2} |Z_{\gamma_t}(r)|^2 dr + \frac{p}{2} (p-2) \int_s^T |Y_{\gamma_t}(r)|^{p-4} \langle Z_{\gamma_t}, Z_{\gamma_t}^* Y_{\gamma_t}, Y_{\gamma_t} \rangle(r) dr$$

$$= |\Phi(B^{\gamma_t})|^p + p \int_s^T |Y_{\gamma_t}(r)|^{p-2} \langle Y_{\gamma_t}(r), f(B_t^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r)) \rangle dr - p \int_s^T |Y_{\gamma_t}(r)|^{p-2} \langle Y_{\gamma_t}(r), Z_{\gamma_t}(r) dB(r) \rangle.$$  \hspace{1cm} (5)

After localization, we get that

$$E[|Y_{\gamma_t}(s)|^p + \frac{p}{2} (p-1) \int_s^T |Y_{\gamma_t}(r)|^{p-2} |Z_{\gamma_t}(r)|^2 dr$$

$$\leq E[|\Phi(B^{\gamma_t})|^p + p \int_s^T |Y_{\gamma_t}(r)|^{p-2} \langle Y_{\gamma_t}(r), f(B_t^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r)) \rangle dr].$$

From Hölder’s inequality and Young’s inequality, we have for any $s \leq T$ and any $\delta > 0$ that,

$$\int_s^T |Y_{\gamma_t}(r)|^{p-2} \langle Y_{\gamma_t}(r), f(B_t^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r)) \rangle dr$$

$$\leq \int_s^T |Y_{\gamma_t}(r)|^{p-1} |f(B_t^{\gamma_t}, 0, 0)| dr + C \int_s^T |Y_{\gamma_t}(r)|^{p-2} |Z_{\gamma_t}(r)| dr$$

$$\leq \left(\frac{p-1}{p} + \frac{C}{\delta} \right) \int_s^T |Y_{\gamma_t}(r)|^p dr + \frac{1}{p} \int_s^T |f(B_t^{\gamma_t}(0), 0, 0)|^p dr + \frac{C_p^2}{\delta} \int_s^T |Y_{\gamma_t}(r)|^p dr.$$  \hspace{1cm} (6)

Hence,

$$E[|Y_{\gamma_t}(s)|^p + \frac{p}{2} (p-1) - \delta Cp \int_s^T |Y_{\gamma_t}(r)|^p |Z_{\gamma_t}(r)|^2 dr$$

$$\leq E[|\Phi(B^{\gamma_t})|^p + \frac{1}{p} \int_s^T |f(B_t^{\gamma_t}(0), 0, 0)|^p dr + (p-1 + \frac{C_p^2}{\delta}) \int_s^T |Y_{\gamma_t}(r)|^p dr].$$

Choosing $\delta$ small enough such that $\frac{p}{2} (p-1) - \delta Cp > 0$ and applying Gronwall’s inequality implies that

$$\sup_{s \in [t,T]} E[|Y_{\gamma_t}(s)|^p + \int_s^T |Y_{\gamma_t}(r)|^{p-2} |Z_{\gamma_t}(r)|^2 dr] \leq C_p (1 + \|\gamma_t\|^q).$$

On the other hand, still using the first equality of this proof and choosing $\delta$ appropriately, we deduce the existence of a constant $C_p^2$ such that

$$|Y_{\gamma_t}(s)|^p \leq |\Phi(B^{\gamma_t})|^p + C_p^2 \int_s^T (|Y_{\gamma_t}(r)|^p + |f(B_t^{\gamma_t}(0, 0, 0))|^p) dr$$

$$- p \int_s^T |Y_{\gamma_t}(r)|^{p-2} \langle Y_{\gamma_t}(r), Z_{\gamma_t}(r) dB(r) \rangle.$$
Hence, from Burkholder-Davis-Gundy’s inequality,
\[
E \left[ \sup_{s \in [t,T]} |Y_{\gamma_t}(s)|^p \right] \leq E[|\Phi(B^{\gamma_t})|^p + C_p^2 \int_t^T |Y_{\gamma_t}(r)|^p + |f(B^{\gamma_t}, 0, 0)|^p dr] \\
+ C_p^4 \sqrt{E \int_t^T |Y_{\gamma_t}(r)|^{2p-2} \|Z_{\gamma_t}(r)\|^2 dr}.
\]

Then we can get
\[
E \left[ \sup_{s \in [t,T]} |Y_{\gamma_t}(s)|^p \right] \leq C_p(1 + \|\gamma_t\|^q).
\]

Now according to Itô’s formula, we have
\[
\int_t^T \|Z_{\gamma_t}(r)\|^2 dr = |\Phi(B^{\gamma_t})|^2 - |Y_{\gamma_t}(t)|^2 + 2 \int_t^T \langle Y_{\gamma_t}(r), f(B^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r)) \rangle dr \\
- 2 \int_t^T \langle Y_{\gamma_t}(r), Z_{\gamma_t}(r) dB(r) \rangle,
\]

thus
\[
E \left[ \int_t^T \|Z_{\gamma_t}(r)\|^2 dr \right] \leq C_p^2 E[|\Phi(B^{\gamma_t})|^p + |Y_{\gamma_t}(t)|^p + \int_t^T \langle Y_{\gamma_t}(r), f(B^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r)) \rangle dr] \\
+ \int_t^T \langle Y_{\gamma_t}(r), Z_{\gamma_t}(r) dB(r) \rangle.
\]

From Hölder’s inequality and Young’s inequality
\[
E \left[ \int_t^T \langle Y_{\gamma_t}(r), f(B^{\gamma_t}, Y_{\gamma_t}(r), Z_{\gamma_t}(r)) \rangle dr \right] \\
\leq C_p^4 \left[ \int_t^T |f(B^{\gamma_t}, 0, 0)|^p dr + \sup_{s \in [t,T]} |Y_{\gamma_t}(s)|^p + \int_t^T |Y_{\gamma_t}(s)| \|Z_{\gamma_t}(s)\| ds \right].
\]

For any \( \delta > 0 \),
\[
E \left[ \int_t^T \|Z_{\gamma_t}(r)\|^2 dr \right] \leq C_p^4 E \left[ \int_t^T |f(B^{\gamma_t}, 0, 0)|^p dr + (1 + \frac{1}{2\delta}) \sup_{s \in [t,T]} |Y_{\gamma_t}(s)|^p \\
+ \delta \int_t^T \|Z_{\gamma_t}(s)\|^2 ds \right] + |\Phi(B^{\gamma_t})|^p + |Y_{\gamma_t}(t)|^p + \int_t^T |Y_{\gamma_t}(s)|^2 \|Z_{\gamma_t}(s)\|^2 ds \right] \\
\leq C_p^4 \left[ (1 + \|\gamma_t\|^q) + \delta (1 + \delta) \right] E \left[ \int_t^T \|Z_{\gamma_t}(r)\|^2 dr \right].
\]

Choosing \( \delta \) small enough to ensure \( (1 + \delta)\delta C_p^4 < 1 \) thus
\[
E \left[ \int_t^T \|Z_{\gamma_t}(r)\|^2 dr \right] \leq C_p(1 + \|\gamma_t\|^q),
\]
which completes the proof. ■

We immediately have
Corollary 3.4 Assuming \( h \) is in \( M^2(t, T) \), let \((Y_{\gamma_t}(s), Z_{\gamma_t}(s))_{t \leq s \leq T}\) be the adapted solution to the BSDE:

\[
Y_{\gamma_t}(s) = \Phi(B^h) + \int_s^T [f(B^{\gamma_r}_u, Y_{\gamma_r}(r), Z_{\gamma_r}(r)) + h(r)]dr - \int_s^T Z_{\gamma_r}(r)dB(r).
\]

Then for any \( p \geq 2 \), there exist \( C_p \) and \( q \) depending on \( C, T, k, p \), such that:

\[
E[\sup_{s \in [t, T]} |Y_{\gamma_t}(s)|^p] \leq C_p(1 + \|\gamma_t\|^q + \int_t^T |h(s)|^pds), \tag{5}
\]

\[
E[\int_t^T \|Z_{\gamma_t}(s)\|^2ds] \leq C_p(1 + \|\gamma_t\|^q + \int_t^T |h(s)|^pds). \tag{6}
\]

We need the regularity properties of the solution of BSDE with respect to the “parameter” \( \gamma_t \).

For convenience, we define \( Y_{\gamma_t}(s), Z_{\gamma_t}(s) \) for any \( t, s \in [0, T] \), \( \gamma_t \in \Lambda \) by \( Y_{\gamma_t}(s) = Y_{\gamma_t}(s \vee t) \), while \( Z_{\gamma_t}(s) = 0 \) for \( s < t \).

Proposition 3.5 For any \( p \geq 2 \), there exist \( C_p \) and \( q \) depending on \( C, T, k, p \), such that for any \( t, \bar{t} \in [0, T] \), and \( \gamma_t, \bar{\gamma}_t \in \Lambda \), \( h, \tilde{h} \in \mathbb{R} \setminus \{0\} \),

\[
(i) \quad E[\sup_{u \in [0, T]} |Y_{\gamma_t}(u) - Y_{\bar{\gamma}_t}(u)|^p] \leq C_p(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q)(d_{\infty}(\gamma_t, \bar{\gamma}_t))^p + |t - \bar{t}|^{\bar{q}_t},
\]

\[
(ii) \quad E[\int_0^T \|Z_{\gamma_t}(u) - Z_{\bar{\gamma}_t}(u)\|^2du] \leq C_p(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q)(d_{\infty}(\gamma_t, \bar{\gamma}_t))^p + |t - \bar{t}|^{\bar{q}_t},
\]

\[
(iii) \quad E[\sup_{u \in [0, T]} |\Delta^1_{h}Y_{\gamma_t}(u) - \Delta^1_{h}Y_{\bar{\gamma}_t}(u)|^p]
\]

\[
\leq C_p(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q + |h|^{\bar{q}})(d_{\infty}(\gamma_t, \bar{\gamma}_t))^p + |t - \bar{t}|^{\bar{q}_t},
\]

\[
(iv) \quad E[\int_0^T \|\Delta^1_{h}Z_{\gamma_t}(u) - \Delta^1_{h}Z_{\bar{\gamma}_t}(u)\|^2du]
\]

\[
\leq C_p(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q + |h|^{\bar{q}})(d_{\infty}(\gamma_t, \bar{\gamma}_t))^p + |t - \bar{t}|^{\bar{q}_t},
\]

where \( \Delta^1_{h}Y_{\gamma_t}(s) = \frac{1}{h}(Y_{\gamma_t^h, 1}(s) - Y_{\gamma_t}(s)), \Delta^1_{h}Z_{\gamma_t}(s) = \frac{1}{h}(Z_{\gamma_t^h, 1}(s) - Z_{\gamma_t}(s)) \) and \( (e_1, \cdots, e_m) \) is an orthonormal basis of \( \mathbb{R}^m \).

Proof. \((Y_{\gamma_t} - Y_{\bar{\gamma}_t}, Z_{\gamma_t} - Z_{\bar{\gamma}_t})\) can be formed as a linearized BSDE:

\[
Y_{\gamma_t}(r) - Y_{\bar{\gamma}_t}(r) = \Phi(B^h) - \Phi(B^\bar{h}) + \int_r^T [f(B^{\gamma_r}_u, Y_{\gamma_r}(u), Z_{\gamma_r}(u)) - f(B^{\bar{\gamma}_r}_u, Y_{\bar{\gamma}_r}(u), Z_{\bar{\gamma}_r}(u))]du
\]

\[
- \int_u^T (Z_{\gamma_r}(u) - Z_{\bar{\gamma}_r}(u))dB(u),
\]

\[
= \Phi(B^h) - \Phi(B^\bar{h}) + \int_r^T [\alpha_{\gamma_t, \gamma_t}(u)(Y_{\gamma_t}(u) - Y_{\bar{\gamma}_t}(u)) + \beta_{\gamma_t, \gamma_t}(u)(Z_{\gamma_t}(u) - Z_{\bar{\gamma}_t}(u)) + \tilde{f}_{\gamma_t, \gamma_t}(u)]du
\]

\[
- \int_r^T (Z_{\gamma_r}(u) - Z_{\bar{\gamma}_r}(u))dB(u), \quad r \in [t \vee \bar{t}, T],
\]

\[
E[\sup_{u \in [t \vee \bar{t}, T]} |Y_{\gamma_t}(u) - Y_{\bar{\gamma}_t}(u)|^p] \leq C_p(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q)(d_{\infty}(\gamma_t, \bar{\gamma}_t))^p + |t - \bar{t}|^{\bar{q}_t},
\]

\[
E[\int_{t \vee \bar{t}}^T \|Z_{\gamma_t}(u) - Z_{\bar{\gamma}_t}(u)\|^2du] \leq C_p(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q)(d_{\infty}(\gamma_t, \bar{\gamma}_t))^p + |t - \bar{t}|^{\bar{q}_t},
\]

\[
E[\sup_{u \in [t \vee \bar{t}, T]} |\Delta^1_{h}Y_{\gamma_t}(u) - \Delta^1_{h}Y_{\bar{\gamma}_t}(u)|^p]
\]

\[
\leq C_p(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q + |h|^{\bar{q}})(d_{\infty}(\gamma_t, \bar{\gamma}_t))^p + |t - \bar{t}|^{\bar{q}_t},
\]

\[
E[\int_{t \vee \bar{t}}^T \|\Delta^1_{h}Z_{\gamma_t}(u) - \Delta^1_{h}Z_{\bar{\gamma}_t}(u)\|^2du]
\]

\[
\leq C_p(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q + |h|^{\bar{q}})(d_{\infty}(\gamma_t, \bar{\gamma}_t))^p + |t - \bar{t}|^{\bar{q}_t},
\]

where \( \Delta^1_{h}Y_{\gamma_t}(s) = \frac{1}{h}(Y_{\gamma_t^h, 1}(s) - Y_{\gamma_t}(s)), \Delta^1_{h}Z_{\gamma_t}(s) = \frac{1}{h}(Z_{\gamma_t^h, 1}(s) - Z_{\gamma_t}(s)) \) and \( (e_1, \cdots, e_m) \) is an orthonormal basis of \( \mathbb{R}^m \).
where
\[
\alpha_{\gamma_t, \tilde{\gamma}_t}(u) = \int_0^1 \frac{\partial f}{\partial y}(B^\gamma_u, Y_u + \theta(Y_u - Y_{\gamma_t}(u)), Z_{\gamma_t}(u) + \theta(Z_{\gamma_t}(u) - Z_{\tilde{\gamma}_t}(u)))d\theta,
\]
\[
\beta_{\gamma_t, \tilde{\gamma}_t}(u) = \int_0^1 \frac{\partial f}{\partial z}(B^\gamma_u, Y_u + \theta(Y_u - Y_{\gamma_t}(u)), Z_{\gamma_t}(u) + \theta(Z_{\gamma_t}(u) - Z_{\tilde{\gamma}_t}(u)))d\theta,
\]
\[
\hat{f}_{\gamma_t, \tilde{\gamma}_t}(u) = f(B^\gamma_u, Y_{\gamma_t}(u), Z_{\gamma_t}(u)) - f(B^\gamma_u, Y_{\gamma_t}(u), Z_{\tilde{\gamma}_t}(u)).
\]
By assumptions (H1) and (H2),
\[
|\hat{f}_{\gamma_t, \tilde{\gamma}_t}(u)| + |\Phi(B^\gamma_t) - \Phi(B^\gamma_{\tilde{\gamma}_t})| \leq 2C \left( 1 + \|B^\gamma_t\|^q + \|B^\gamma_{\tilde{\gamma}_t}\|^q \right) \sup_u |B^\gamma_t(u) - B^\gamma_{\tilde{\gamma}_t}(u)|,
\]
we know that the first two inequalities hold true after applying Corollary 3.4 to the above linearized BSDE.

For the last two inequalities, we also can write \((-\Delta^1_Y Y_{\gamma_t}, \Delta^1_Y Z_{\gamma_t})\) as the solution of the following linearized BSDE:
\[
\Delta^1_Y Y_{\gamma_t}(r) = \frac{1}{\underline{h}}(\Phi(B^{\gamma_{\tilde{\gamma}_t}}) - \Phi(B^{\gamma_t})) + \int_0^T \left( \alpha_{\gamma_{\tilde{\gamma}_t}, \gamma_t}(u) \Delta^1_Y Y_{\gamma_t}(u) + \beta_{\gamma_{\tilde{\gamma}_t}, \gamma_t}(u) \Delta^1_Y Z_{\gamma_t}(u) + \frac{1}{\underline{h}} \hat{f}_{\gamma_t, \tilde{\gamma}_t}(u) \right) du - \int_0^T \Delta^1_Y Z_{\gamma_t}(u) dB_u.
\]
Then the same calculus as above implies that:
\[
E\left[ \sup_{s \in [0,T]} |\Delta^1_Y Y_{\gamma_t}(s)|^p + |\int_0^T \|\Delta^1_Y Z_{\gamma_t}(s)\|^2 ds|^{\frac{p}{2}} \right] \leq C_p \left( 1 + \|\gamma_t\|^q + |\gamma_t|^q \right).
\]
Finally, we consider
\[
\Delta^1_Y Y_{\gamma_t}(r) - \Delta^1_Y Y_{\tilde{\gamma}_t}(r) = \frac{1}{\underline{h}}(\Phi(B^{\gamma_{\tilde{\gamma}_t}}) - \Phi(B^{\gamma_t})) - \frac{1}{\underline{h}}(\Phi(B^{\gamma_{\tilde{\gamma}_t}}) - \Phi(B^{\gamma_{\tilde{\gamma}_t}}))
\]
\[
+ \int_0^T \left( \alpha_{\gamma_{\tilde{\gamma}_t}, \gamma_t}(u) \Delta^1_Y Y_{\gamma_t}(u) - \alpha_{\gamma_{\tilde{\gamma}_t}, \gamma_t}(u) \Delta^1_Y Y_{\tilde{\gamma}_t}(u) \right) du
\]
\[
+ \int_0^T \left( \beta_{\gamma_{\tilde{\gamma}_t}, \gamma_t}(u) \Delta^1_Y Z_{\gamma_t}(u) - \beta_{\gamma_{\tilde{\gamma}_t}, \gamma_t}(u) \Delta^1_Y Z_{\tilde{\gamma}_t}(u) \right) du
\]
\[
- \int_0^T \left( \frac{1}{\underline{h}} \hat{f}_{\gamma_{\tilde{\gamma}_t}, \gamma_t}(u) - \frac{1}{\underline{h}} \hat{f}_{\gamma_{\tilde{\gamma}_t}, \tilde{\gamma}_t}(u) \right) du
\]
\[
- \int_0^T \left( \Delta^1_Y Z_{\gamma_t}(u) - \Delta^1_Y Z_{\tilde{\gamma}_t}(u) \right) dB_u, \quad r \in [t \lor \tilde{t}, T]
\]
Thus \((\hat{Y}(r), \hat{Z}(r)) := \left( \Delta^1_Y Y_{\gamma_t}(r) - \Delta^1_Y Y_{\tilde{\gamma}_t}(r), \Delta^1_Y Z_{\gamma_t}(r) - \Delta^1_Y Z_{\tilde{\gamma}_t}(r) \right)\) solves the BSDE
\[
\hat{Y}(r) = \frac{1}{\underline{h}}(\Phi(B^{\gamma_{\tilde{\gamma}_t}}) - \Phi(B^{\gamma_t})) - \frac{1}{\underline{h}}(\Phi(B^{\gamma_{\tilde{\gamma}_t}}) - \Phi(B^{\gamma_{\tilde{\gamma}_t}}))
\]
\[
+ \int_t^r \left[ \alpha_{\gamma_{\tilde{\gamma}_t}, \gamma_t}(u) \hat{Y}(r) + \beta_{\gamma_{\tilde{\gamma}_t}, \gamma_t}(u) \hat{Z}(u) + \hat{f}(u) \right] du - \int_r^T \hat{Z}(u) dB_u,
\]
with
\[
\tilde{f}(u) := [\alpha_h^{(h)}(u) - \alpha_h^{(k)}(u)]\Delta_h\gamma_t(u) + [\beta_h^{(h)}(u) - \beta_h^{(k)}(u)]\Delta_h\gamma_t(u)
+ 1 \frac{1}{h}\tilde{f}_{\gamma_t}^{(h)}(u) - \frac{1}{h}\tilde{f}_{\gamma_t}^{(k)}(u)
\]

By assumptions (H1) and (H2), there exist some \(\delta, \varepsilon_u \in [0, 1]\),
\[
\Phi(B_t^{(h)}) - \Phi(B_t^{(k)}) = \langle \Phi'(B_t^{(h)}), e_t \rangle h,
\]
\[
\tilde{f}_{\gamma_t}^{(h)}(u) = \langle f'_{\gamma_t}(B_t^{(h)}), Y_{\gamma_t}(u), Z_{\gamma_t}(u), e_t \rangle h,
\]
again by the Corollary 3.4 we know that the last two inequalities holds true.

**Proposition 3.6** For each \(\gamma_t \in \Lambda\), \(\{Y_{\gamma_t}^{(t)}(s), s \in [0, T], x \in \mathbb{R}^m\}\) has a version which is a.e. of class \(C^{0,2}([0, T] \times \mathbb{R}^m)\).

**Proof.** To simplify presentation, we shall prove only the case when \(m = d = 1\), as the higher dimensional case can be treated in the same way without substantial difficulty. Thus, for each \(h, \bar{h} \in \mathbb{R}\{0\}\) and \(k, \bar{k} \in \mathbb{R}\),
\[
E[\sup_{s \in [0, T]}|Y_{\gamma_t}^{(t)}(s) - Y_{\gamma_t}^{(k)}(s)|^p] \leq C_p(1 + ||\gamma_t||^q)(|k - \bar{k}|^p),
\]
\[
E[\int_0^T|Z_{\gamma_t}^{(t)}(s) - Z_{\gamma_t}^{(k)}|^2 ds] \leq C_p(1 + ||\gamma_t||^q)(|k - \bar{k}|^p),
\]
\[
E[\sup_{s \in [0, T]}|\Delta_h Y_{\gamma_t}^{(t)}(s) - \Delta_h Y_{\gamma_t}^{(k)}(s)|^p] \leq C_p(1 + ||\gamma_t||^q + |h|^p + \bar{h}|^p)(|k - \bar{k}|^p + |h - \bar{h}|^p),
\]
\[
E[\int_0^T|\Delta_h Z_{\gamma_t}^{(t)}(s) - \Delta_h Z_{\gamma_t}^{(k)}|^2 ds] \leq C_p(1 + ||\gamma_t||^q + |h|^p + \bar{h}|^p)(|k - \bar{k}|^p + |h - \bar{h}|^p).
\]
Therefore, using the Kolmogorov’s criterion, the existence of a continuous derivative of \(Y_{\gamma_t}^{(t)}(s)\) with respect to \(x\) follows easily from the above estimate, as well as the existence of a mean-square derivative of \(Z_{\gamma_t}^{(t)}(s)\) with respect to \(x\), which is mean square continuous in \(x\). We denote them by \((D_x Y_{\gamma_t}^{(t)}, D_x Z_{\gamma_t}^{(t)})\).

We now prove the existence of the continuous second derivative of \(Y_{\gamma_t}^{(t)}(s)\) with respect to \(x\).

By Proposition 3.5 \((D_x Y_{\gamma_t}^{(t)}, D_x Z_{\gamma_t}^{(t)})\) is the solution of the following linearized BSDE:
\[
D_x Y_{\gamma_t}^{(t)} = \Phi'_{\gamma_t}(B_t^{(k)}) + \int_s^T[f_y(B_t^{(k)}), Y_{\gamma_t}(r), Z_{\gamma_t}(r)]D_x Y_{\gamma_t}(r) + f_z(B_t^{(k)}, Y_{\gamma_t}(r), Z_{\gamma_t}(r))D_x Z_{\gamma_t}(r)dr
\]
\[ + \int_s^Tf_y'(B_t^{(k)}, Y_{\gamma_t}(r), Z_{\gamma_t}(r))dY_{\gamma_t}(r) - \int_s^Tf_y'(B_t^{(k)}, Y_{\gamma_t}(r), Z_{\gamma_t}(r))dB(r).
\]
Then, applying Proposition 3.5 we have: for each \(h, \bar{h} \in \mathbb{R}\{0\}\) and \(k, \bar{k} \in \mathbb{R}\) we have
\[
E[\sup_{s \in [0, T]}|\Delta_h D_x Y_{\gamma_t}^{(t)}(s) - \Delta_h D_x Y_{\gamma_t}^{(k)}(s)|^p] \leq C_p(1 + ||\gamma_t||^q)(|k - \bar{k}|^p + |h - \bar{h}|^p),
\]
\[
E[\int_0^T|\Delta_h D_x Z_{\gamma_t}^{(t)}(s) - \Delta_h D_x Z_{\gamma_t}^{(k)}|^2 ds] \leq C_p(1 + ||\gamma_t||^q)(|k - \bar{k}|^p + |h - \bar{h}|^p),
\]
which completes the proof. ■

Now we define:

\[ u(\gamma_t) := Y_{\gamma_t}(t), \quad \text{for } \gamma_t \in \Lambda. \] (7)

By the definition of vertical derivative and Proposition 3.5, we have the following corollary

**Corollary 3.7** \( u(\gamma_t) \) is \( \Lambda \)-continuous and \( D_xu(\gamma_t) \), \( D_{xx}u(\gamma_t) \) exist, moreover they are both \( \Lambda \)-continuous. Furthermore, \( u \in C_{t,\text{lip}}^{0,2}(\Lambda) \).

### 3.2 Path regularity of process Z.

In Pardoux and Peng [18], when the terminal of BSDE is the state-dependent case when \( f = \bar{f}(t, \gamma(t), y, z) \) and \( \Phi = \varphi(\gamma(T)) \), it is shown that \( Z \) and \( Y \) are connected in the following sense under appropriate assumptions:

\[ Z_{\gamma_t}(s) = \partial_xu(s, \gamma_t(t) + B(s) - B(t)). \]

In this section, we extend this result to the path-dependent case. Indeed, we have below a formula relating \( Z \) with \( Y \).

**Proposition 3.8** Under assumptions \((H1)-(H2)\), for each fixed \( \gamma_t \in \Lambda \), the process \( (Z_{\gamma_t}(s))_{s \in [t,T]} \) has a continuous version with the form,

\[ D_xu(B^\gamma_{\gamma_t}(s)) = Z_{\gamma_t}(s), \quad \text{for each } s \in [t,T], \quad \text{a.s.}. \]

A direct consequence of Proposition 3.8 is the following corollary.

**Corollary 3.9** Assume that the same conditions of Proposition 3.8. Then, for the above continuous version \( Z_{\gamma_t} \), for each \( p \geq 2 \), there exists a constant \( C_p > 0 \), depending on \( C, T, k, p \), such that

\[ Z_{\gamma_t}(s) \leq C_p(1 + \|B^\gamma_{\gamma_t}(s)\|^q), \quad \forall s \in [t,T], \quad P - \text{a.s.}, \]

and

\[ E[\sup_{s \in [t,T]} \|Z_{\gamma_t}(s)\|^p] \leq C_p(1 + \|\gamma_t\|^q). \]

Before proceeding to the proof, we need the following lemma essentially from Pardoux and Peng [18] (the Lemma 2.5).

**Lemma 3.10** Let \( \gamma_t \) be given. For some \( \bar{t} \in [t,T] \), suppose \( \Phi(\gamma) = \varphi(\gamma(\bar{t}), \gamma(T) - \gamma(\bar{t})) \), where \( \varphi \) is in \( C^3_p(\mathbb{R}^{2d}, \mathbb{R}^m) \). Let \( f(\gamma_s, y, z) = f_1(s, \gamma_s(s), y, z)I_{[0,\bar{t}]>(s) + f_2(s, \gamma_s(s) - \gamma_s(\bar{t}), y, z)I_{[\bar{t},T]}(s) \), where \( f_1 : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^m \), \( f_2 : [0, T] \times \mathbb{R}^2 \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \rightarrow \mathbb{R}^m \) satisfy the assumption \((H3)\), then for each \( s \in [t,T] \),

\[ D_xu(B^\gamma_{\gamma_t}(s)) = Z_{\gamma_t}(s), \quad \text{a.s.}. \]
In this case BSDE (2) is rewritten, for $s \in [t, \bar{t}]$, BSDE (2) is

$$Y_{\gamma_s}(u) = \varphi(\gamma_s(s)) + B(\bar{t}) - B(s), B(T) - B(t)) - \int_u^T Z_{\gamma_s}(r)dB(r)$$

$$+ \int_u^T f_2(r, \gamma_s(s)) + B(r) - B(s), B(T) - B(t), Y_{\gamma_s}(r), Z_{\gamma_s}(r))dr, \quad u \in [t, T],$$

$$Y_{\gamma_s}(u) = Y_{\gamma_s}(\bar{t}) - \int_u^\bar{t} Z_{\gamma_s}(r)dB(r)$$

$$+ \int_s^\bar{t} f_1(r, \gamma(s) + B(r) - B(s), Y_{\gamma_s}(r), Z_{\gamma_s}(r))dr, \quad u \in [s, \bar{t}],$$

for $s \in [\bar{t}, T]$, BSDE (2) is

$$Y_{\gamma_s}(u) = \varphi(\gamma_s(\bar{t}), \gamma_s(s) - \gamma_s(\bar{t}) + B(T) - B(s)) - \int_u^T Z_{\gamma_s}(r)dB(r)$$

$$+ \int_u^T f_2(r, \gamma_s(\bar{t}), \gamma_s(s) - \gamma_s(\bar{t}) + B(s) - B(s), Y_{\gamma_s}(r), Z_{\gamma_s}(r))dr, \quad u \in [s, T].$$

Consider the following system of quasilinear parabolic differential equations, defined on $(s, x, y) \in [\bar{t}, T] \times \mathbb{R}^2$ and parameterized by $y \in \mathbb{R}$,

$$\begin{cases}
\partial_s v_2(s, x, y) + \frac{1}{2}\partial_{yy} v_2(s, x, y) + f_2(s, x, y, v_2(s, x, y), \partial_y v_2(s, x, y)) = 0, \\
v_2(T, x, y) = \varphi(x, y),
\end{cases}$$

and then, another one defined on $(s, x) \in [t, \bar{t}] \times \mathbb{R}$,

$$\begin{cases}
\partial_s v_1(s, x) + \frac{1}{2}\partial_{xx} v_1(s, x) + f_1(s, x, v_1(s, x), \partial_x v_1(s, x)) = 0, \\
v_1(t, x) = v_2(\bar{t}, x, 0).
\end{cases}$$

Following the Theorem 3.1 and 3.2 of Pardoux and Peng [13] and the definition of $u$, we have $v_2$ is of class $C^{1,2}([\bar{t}, T] \times \mathbb{R}^2, \mathbb{R})$, $v_1$ is of class $C^{1,2}([t, \bar{t}] \times \mathbb{R}, \mathbb{R})$, and

$$u(\gamma_s) = v_1(s, \gamma_s(s))I_{[t, \bar{t}]}(s) + v_2(s, \gamma_s(\bar{t}), \gamma_s(s) - \gamma_s(\bar{t}))I_{[\bar{t}, T]}(s).$$

Furthermore, we have, a.s.,

$$Y_{\gamma_s}(s) = v_1(s, \gamma(t) + B(s) - B(t)), \quad t \leq s \leq \bar{t},$$

$$Y_{\gamma_s}(s) = v_2(s, \gamma(t) + B(\bar{t}) - B(t), B(s) - B(\bar{t})), \quad \bar{t} \leq s \leq T,$$

$$Z_{\gamma_s}(s) = \partial_x v_1(s, \gamma(t) + B(s) - B(t)), \quad t \leq s \leq \bar{t},$$

$$Z_{\gamma_s}(s) = \partial_y v_2(s, \gamma(t) + B(\bar{t}) - B(t), B(s) - B(\bar{t})), \quad \bar{t} \leq s \leq T.$$
Thus $Z_{\gamma_t}(s)$ have a continuous version. In particular,

$$Z_{\gamma_t}(t) = D_x u(\gamma_t), \quad \gamma_t \in \Lambda,$$

which completes the proof. ■

We now give the proof of Proposition 3.8.

**Proof of Proposition 3.8** For each fixed $t \in [0, T)$ and positive integer $n$, we introduce a mapping $\gamma^n(\bar{s}_t) : \Lambda_t \to \Lambda_t$:

$$\gamma^n(\bar{s}_t)(r) = \bar{s}_n I_{[0,t]}(r) + \sum_{k=0}^{n-1} \gamma_n(t^k + 1 \wedge s) I_{[t^k \wedge s, t^k + 1 \wedge s]}(r) + \gamma_n(s) I_{[s]}(r),$$

where $t^k = t + \frac{k(T-t)}{n}$, $k = 0, 1, \ldots, n$, and set

$$\Phi^n(\bar{s}) := \Phi(\gamma^n(\bar{s})), \quad f^n(\bar{s}, y, z) := f(\gamma^n(\bar{s}), y, z).$$

For each $n$, there exist some $\varphi_n$ defined on $\Lambda_t \times \mathbb{R}^{n \times d}$ and $\psi_n$ defined on $[t, T] \times \Lambda_t \times \mathbb{R}^{n \times d} \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$ such that,

$$\Phi^n(\bar{s}) = \varphi_n(\bar{s}, \gamma_n(t^0), \cdots, \gamma_n(t^n)), \quad f^n(\bar{s}, y, z) = \psi_n(s, \gamma_n(t^1), \cdots, \gamma_n(t^n), y, z).$$

Indeed, set

$$\varphi_n(\bar{s}_t, x_1, \cdots, x_n) := \Phi(\gamma(t) I_{[0,t]}(s) + \sum_{k=1}^{n} x_k I_{[t^k \wedge s, t^k + 1 \wedge s]}(s) + x_n I_{[T]}(s)),
\varphi_n(\bar{s}_t, x_1, \cdots, x_n) := \varphi_n(\bar{s}_t, \gamma(t) + x_1, \gamma(t) + x_2, \cdots, \gamma(t) + x_n).$$

By the assumptions (H1) and (H2), $\varphi_n(\bar{s}_t, x_1, \cdots, x_n)$ is a $C^3_p$-function of $x_0, \cdots, x_n$ for each fixed $\bar{s}_t$, in particular, $\partial_x \varphi_n(x_1, \cdots, x_n) = \Phi_n(\gamma(t) I_{[0,t]}(s) + \sum_{k=1}^{n} x_k I_{[t^k \wedge s, t^k + 1 \wedge s]}(s) + x_n I_{[T]}(s))$.

Furthermore, we can check that $\psi_n(s, \gamma_n(t^1), \cdots, x_n, y, z)$ satisfy assumption (H3) for each fixed $\bar{s}_t$.

Consider the following BSDE, for any $\bar{t} \geq t$, $\bar{s}_t \in \Lambda_t$,

$$Y^{(n)}_{\bar{s}_t}(s) = \Phi^{(n)}(B^{\bar{s}_t}) + \int_s^T f^n(B^{\bar{s}_t}, Y^{(n)}_{\bar{s}_t}(r), Z^{(n)}_{\bar{s}_t}(r)) dr - \int_s^T Z^{(n)}_{\bar{s}_t}(r) dB(r), \quad s \in [\bar{t}, T],$$

and set

$$u^{(n)}(\bar{s}_t) = Y^{(n)}_{\bar{s}_t}(\bar{t}), \quad \bar{s}_t \in \Lambda.$$

Iterating the argument as the above Lemma 3.10 we can get for each $s \in [t, T]$,

$$D_x u^{(n)}(B^{\gamma})(s) = Z^{(n)}_{\gamma}(s), \quad a.s..$$
By the Corollary 3.4,

\[ |u^{(n)}(\gamma_t) - u(\gamma_t)| \]

\[ \leq C E[(\Phi^{(n)}(B^\gamma_t) - \Phi(B^\gamma_t))^2 + \int_0^T |f(B^\gamma_t, Y^{(n)}_{\gamma_t}(r), Z^{(n)}_{\gamma_t}(r)) - f^n(B^\gamma_t, Y^{(n)}_{\gamma_t}(r), Z^{(n)}_{\gamma_t}(r))|^2 dr] \frac{1}{n^4} \]

\[ \leq C_1 E[(1 + \|\gamma\|^k) + \sup_s |B(s) - B(t)|^k] \|\gamma^n(B^\gamma_t) - B^\gamma_t\|^2 ] \]

\[ \leq C_1((1 + \|\gamma\|^k)(E[\sup_s |B(s) - \sum_{k=1}^{n-1} B(t_{k+1}^n) J_{[t_{k+1}^n, t_{k+1}^n]}(s)|^4 + \|\gamma^n(\gamma_t) - \gamma_t\|]) \]

\[ \leq C_2(1 + \|\gamma\|^k)(\frac{1}{n^4} + \|\gamma^n(\gamma_t) - \gamma_t\|). \]

Moreover, we have

\[ |D_x u^{(n)}(\gamma_t) - D_x u(\gamma_t)| \leq C_2(1 + \|\gamma\|^k)(\frac{1}{n^4} + \|\gamma^n(\gamma_t) - \gamma_t\|), \]

\[ |D_{xx} u^{(n)}(\gamma_t) - D_{xx} u(\gamma_t)| \leq C_2((1 + \|\gamma\|^k)(\frac{1}{n^4} + \|\gamma^n(\gamma_t) - \gamma_t\|). \]

It follows from, for each \( p \geq 2, \)

\[ \lim E[\sup_{s \in [t,T]} |D_x u^{(n)}(B_s^\gamma) - D_x u(B_s^\gamma)|^p] \]

\[ \leq C_2 \lim E[\sup_{s \in [t,T]} |(1 + \|B_s^\gamma\|^k)(\frac{1}{n^4} + \|\gamma^n(B_s^\gamma) - B_s^\gamma\|)|^p] \]

\[ = 0, \]

and \( \lim E[f^T |Z_{\gamma}(u) - Z_{\gamma}(u)|^2 du|] = 0 \)

that

\[ D_x u(B_s^\gamma) = Z_{\gamma}(s), \ dP \times ds - a.e. \text{ on } [t,T], \]

which completes the proof.

4 Path-dependent PDE

We now relate our BSDE to the following system of path-dependent version of the Kolmogorov backward equation:

\[
\begin{align*}
D_t u(\gamma_t) + \frac{1}{2} D_{xx} u(\gamma_t) + f(\gamma_t, u(\gamma_t), D_x u(\gamma_t)) & = 0, \quad \gamma_t \in \Lambda, \ t \in [0,T), \\
\sigma(\gamma) & = \Phi(\gamma), \quad \gamma \in \Lambda_T.
\end{align*}
\]

(8)

where \( u : \Lambda \to \mathbb{R}^m \) is a function on \( \Lambda \). We immediately obtain.

**Theorem 4.1** Assume that assumptions (H1) and (H2) hold and let \( u \in C^{1,2}(\Lambda) \) be a solution of the equation (8). Then we have \( u(\gamma_t) = Y_{\gamma_t}(t), \) for each \( \gamma_t \in \Lambda, \) where \( (Y_{\gamma_t}(s), Z_{\gamma_t}(s))_{t \leq s \leq T} \) is the unique solution of the BSDE (3). Consequently, the path-dependent PDE (8) has at most one solution.
Proof. Applying the functional Itô formula \([2.9]\) to \(u(B^\gamma_t)\) on \(s \in [t, T)\), we have
\[
du(B^\gamma_s) = (D_x u(B^\gamma_s) + \frac{1}{2} D_{xx} u(B^\gamma_s))ds + D_x u(B^\gamma_s)dB(s).
\]

Since \(u\) solves PDE \([3]\), thus
\[
-du(B^\gamma_s) = f(B^\gamma_t, u(B^\gamma_s)), D_x u(B^\gamma_s))ds - D_x u(B^\gamma_s)dB(s),
\]
which, with \(u(B^\gamma_s) = \Phi(B^\gamma_s)\) and \(u \in C^{1,2}(\Lambda)\), implies that \((Y_\gamma(s), Z_\gamma(s)) = (u(B^\gamma(s)), D_x u(B^\gamma(s)))\) is the unique solution of BSDE \([2]\). In particular \(u(\gamma_t) = Y_\gamma(t)\), which completes the proof. ■

By using this theorem and the classical comparison theorem of BSDE (Lemma \([2.11]\)), we have the comparison theorem of Path-dependent PDE:

**Corollary 4.2** We assume \(m = 1\) and that \(f = f_i, \Phi = \Phi_i, i = 1, 2\) satisfy the same assumptions as in Theorem \([7, 8]\) as well as:
- \(f_1(\gamma_t, y, z) \leq f_2(\gamma_t, y, z)\), for each \((\gamma_t, y, z) \in (\Lambda \times \mathbb{R} \times \mathbb{R}^d)\);
- \(\Phi_1(\gamma_T) \leq \Phi_2(\gamma_T)\), for each \(\gamma_T \in \Lambda_T\).

Let \(u_i \in \mathbb{C}^{1,2}(\Lambda)\) be the solution of equation \([3]\) associated with \((f, \Phi) = (f_i, \Phi_i), i = 1, 2\). Then we also have \(u_1(\gamma_t) \leq u_2(\gamma_t)\), for each \(\gamma_t \in \Lambda\).

We are now in a position to prove the converse to the above result:

**Theorem 4.3** We make assumptions \((H1)-(H2)\). Then the function \(u\) defined in \([7]\) is the unique \(\mathbb{C}^{1,2}(\Lambda)\)-solution of the path-dependent PDE \([3]\).

Proof. From Corollary \([3.7]\), \(u \in \mathbb{C}^{0,2}(\Lambda)\). Let \(\delta \geq 0\) be such that \(t + \delta \leq T\). From the definition of \(u\),
\[
u(B^\gamma_{t+\delta}) = Y_\gamma(t + \delta).
\]

Hence
\[
u(\gamma_{t+\delta}) - \nu(\gamma_t) = u(\gamma_{t+\delta}) - u(\gamma_{t+\delta}) + u(B^\gamma_{t+\delta}) - u(\gamma_t).
\]

By the proof of Proposition \([3.8]\) we get
\[
u(\gamma_{t+\delta}) - \nu(\gamma_t) = \lim_{n \to \infty} \left[ u^{(n)}_{\gamma_{t+\delta}}(\gamma_{t+\delta}) - u^{(n)}_{\gamma_{t+\delta}}(B^\gamma_{t+\delta}) \right]
- \int_t^{t+\delta} f(B^\gamma_s, Y_\gamma(s), Z_\gamma(s))ds + \int_t^{t+\delta} Z_\gamma(s)dB(s).
\]

From Lemma \([3.10]\) and the Proposition 3.2 of Pardoux and Peng \([13]\), we have
\[
u^{(n)}(\gamma_{t+\delta}) - u^{(n)}(B^\gamma_{t+\delta})
= \int_t^{t+\delta} D_x u^{(n)}_{\gamma_{t+\delta}}ds - \int_t^{t+\delta} D_x u^{(n)}(B^\gamma_t)ds
- \int_t^{t+\delta} D_x u^{(n)}_{\gamma_{t+\delta}}dB(s) - \frac{1}{2} \int_t^{t+\delta} D_{xx} u^{(n)}_{\gamma_{t+\delta}}dB(s).
\]
Thus
\[
\begin{align*}
&u(\gamma_{t,t+\delta}) - u(\gamma_t)
\quad = -\int_t^{t+\delta} D_x u(B^\gamma_s) dB(s) - \frac{1}{2} \int_t^{t+\delta} D_{xx} u(B^\gamma_s) ds \\
&\quad - \int_t^{t+\delta} f(B^\gamma_s, Y_{\gamma_t}(s), Z_{\gamma_t}(s)) ds + \int_t^{t+\delta} Z_{\gamma_t}(s) dB(s) + \lim_{n \to \infty} C^n,
\end{align*}
\]
where
\[
C^n = \int_t^{t+\delta} D_x u^n(\gamma_{t,s}) ds - \int_t^{t+\delta} D_x u^n(B^\gamma_s) ds.
\]

It is easy to check that:
\[
|C^n| \leq C \delta \sup_{s \in [t, t+\delta]} |B(s) - B(t)|.
\]
Taking expectation on both sides and following the Proposition 3.8,
\[
\lim_{\delta \to 0} \frac{u(\gamma_{t,t+\delta}) - u(\gamma_t)}{\delta} = -\frac{1}{2} D_{xx} u(\gamma_t) - f(\gamma_t, u(\gamma_t), D_x u(\gamma_t)),
\]

hence, \( u \in C^{1,2}(\Lambda) \) and it satisfies the equation (8).

**Corollary 4.4** We make assumptions (H1)-(H2). Then \((u(B_t), D_x u(B_t))\) is the unique solution of the BSDE (1).

**Remark 4.5** In the case where \( \Phi(\gamma) = \varphi(\gamma(T)) \), for some \( \varphi \in C^3_p(\mathbb{R}^m) \) and \( f \) satisfies assumption (H3), the above results is the nonlinear Feynman-Kac formula, which given by Peng [20] and Pardoux-Peng [19].

**Remark 4.6** Suppose \( k = 1 \), and let
\[
f(t, y, z) = c(t)y,
\]
where \( f \) satisfy assumption (H3). In that case the BSDE (2) has the explicit solution:
\[
Y_{\gamma_t}(s) = \Phi(B^\gamma_s) e^{\int_t^s c(r)dr} - \int_s^T e^{\int_r^s c(r)dr} Z_{\gamma_t}(u) dB(u),
\]
and
\[
Y_{\gamma_t}(t) = E[\Phi(B^\gamma_t) e^{\int_t^T c(r)dr}].
\]

**Example 4.7** Suppose \( \Phi : \Lambda_T \to \mathbb{R} \) :
\[
\Phi(\gamma) = \int_0^T \varphi(\gamma(s)) ds,
\]
for some \( \varphi \in C^3_0(\mathbb{R}) \). It is obvious that \( \Phi \) satisfies assumption (H1).

From the above remark, for each \( t \in [0, T] \), \( \gamma_t \in \Lambda_t \)
\[
u(\gamma_t) = \int_0^t \varphi(\gamma_t(s)) ds e^{\int_t^s c(r)dr} + \int_t^T e^{\int_r^T c(r)dr} E[\varphi(\gamma_t(t) + B(s) - B(t))] ds.
\]
By the classic Feynman-Kac formula, \( \forall s \in [0, T] \) and \( x \in \mathbb{R} \),
\[
u^*(t, x) = E[\phi(x + B(s) - B(t))], \quad t \leq s
\]
is the solution of the following parabolic differential equation :
\[
\begin{cases}
\frac{\partial \nu^*}{\partial t} + \frac{1}{2} \frac{\partial^2 \nu^*}{\partial x^2} = 0, & t \in [0, s) \\
\nu^*(s, x) = \phi(x).
\end{cases}
\]
then
\[
u(\gamma_t) = \int_0^t \phi(\gamma_t(s))dse_T^T c(r)dr + \int_t^T e^{s^T c(r)dr} \nu^*(t, \gamma_t(t))ds.
\]
By the definitions of horizontal derivative and vertical derivative, thus
\[
D_t \nu(\gamma_t) = -c(t)\nu(\gamma_t) + e^{s^T c(r)dr} \int_t^T \partial_t \nu^*(t, \gamma_t(t))ds,
\]
\[
D_x \nu(\gamma_t) = e^{s^T c(r)dr} \int_t^T \partial_x \nu^*(t, \gamma_t(t))ds,
\]
\[
D_{xx} \nu(\gamma_t) = e^{s^T c(r)dr} \int_t^T \partial_{xx}^2 \nu^*(t, \gamma_t(t))ds.
\]
It is obvious that
\[
D_t \nu(\gamma_t) + \frac{1}{2} D_{xx} \nu(\gamma_t) = -c(t)\nu(\gamma_t),
\]
which satisfies the equation (8).

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