Numerical mathematical model for calculating ion density in the gas-discharge chamber of a radio-frequency ion thruster

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Abstract. The mathematical model for calculation of the ion density in the gas-discharge chamber of a radio-frequency ion thruster is developed. The model is based on the assumption of higher potential of the central plasma volume comparing to the chamber walls due to higher mobility of electrons. Firstly, the initial approximation for the plasma potential is set and parameters of the ion trajectories in quasistationary electric field are computed. Then, from the assumption of keeping the total ion flow for each trajectory, the ion density is computed as well as the next approximation for plasma potential. The computation is performed iteratively until the change in the ion density is less than the predefined small value. The model was verified by comparing the results of numerical simulation with the analytical solution for spherical volume. The high accuracy of the numerical model is shown. Also, the ion density in the discharge chamber of a radio-frequency ion thruster was calculated with the uniform distribution of neutral atom density and coefficient of ionization rate. It was shown that at such conditions the ion density has a maximum in central part of the discharge chamber and it decreases towards the walls.

1. Introduction

Many of modern spacecraft use electric propulsion for the motion control. For missions requiring significant total pulse, it is advisable to use the thrusters with high specific impulse which will significantly reduce the required supply of propellant on board the spacecraft [1]. Thus, ion thrusters are promising as they have comparatively high specific impulse [2]. At present, ion thrusters are commonly used for deep space exploration [1], [2]. Besides, ion thrusters have comparatively long lifetime. Thus, they can be efficiently used on board the long operating spacecraft. When operating, ion thruster firstly converts the propellant to plasma state. Then ions of the propellant are accelerated by electrostatic force and produce thrust. There are ion thrusters with various mechanisms of propellant ionization: with ionization in a direct current discharge, in a radio-frequency discharge and in microwave one. The DC ion thrusters have the highest efficiency of power consumption, but they have a structurally complicated and thermally loaded cathode unit in a gas discharge chamber. Ion thrusters with ionization of propellant in an alternating electric field have a simpler and potentially more reliable design, so they are also used on board spacecraft. This paper is devoted to mathematical modeling of processes in the discharge chamber of ion thruster with propellant ionization in the radio-frequency discharge. In the discharge chamber of radio-frequency ion thruster (RIT) the electrons are accelerated by vortex electric field which is generated by current in the coil wrapped around the
discharge chamber [3]. When colliding with the accelerated electrons, the atoms of propellant (Xenon) fed to the chamber are ionized. The originated ions are accelerated in the ion-extraction system. To avoid the accumulation of negative charge on the spacecraft elements, electrons from the neutralizer are added to the accelerated ions. A mathematical model of a radio-frequency discharge should describe the behavior of ions, electrons, neutral atoms, and the electromagnetic field in the considered volume. In this paper we consider an approach to numerical simulation of the distribution of ion density in the chamber of a radio-frequency ion thruster.

A significant number of mathematical models of the discharge in the chamber of a radio-frequency ion thruster have been developed by now. The so-called balance models are quite simple and require a small amount of computing resources. They often assume a uniform distribution of plasma parameters over the chamber volume and such models allow one to analyze the integral thruster parameters only [4]. If it is necessary to obtain the distribution of plasma parameters over the chamber volume, models based on magnetohydrodynamic or kinetic plasma modeling methods are usually used. Such models are highly accurate and can calculate the distribution of local plasma parameters, but require significant computational resources [5]. In this paper, we describe an approach to modeling the behavior of ions in a radio-frequency discharge plasma, which allows one to obtain the distribution of local plasma parameters in the considered volume and does not require significant computational resources. The accuracy of the proposed model is less than that of the models based on the magnetohydrodynamic or kinetic approach, but it allows one to significantly decrease the time of calculation and at the same time to analyze the local plasma parameters. Such model may be useful for analyzing the obtained experimental data as well as for developing radio-frequency ion thrusters.

When calculating the ion density, it is required to know the electron velocity distribution, the neutral atom density, as well as the ionization cross section of the atoms of propellant [5]. This paper considers these quantities as the known ones and the behavior of ions in the discharge chamber is considered only.

In a radio-frequency inductive discharge the electrons fall onto the walls more intensively, since they have greater mobility compared to ions due to their low mass, and therefore the plasma potential in the central part of the discharge is higher than the potential of the walls. Ions are originated in the chamber volume and are accelerated towards the walls by the resulting potential difference. When the ions reach the walls they are neutralized and return to the chamber volume in the form of neutral atoms.

Thus, the proposed mathematical model is based on the following assumptions:
- A two-dimensional axisymmetric quasistationary problem is considered.
- The density of neutral particles and ionization rate are considered as uniformly distributed and known.
- Only the stationary component of the electric field is taken into account.
- It is assumed that the effect of alternating magnetic fields on ions is negligible.
- It is assumed that when reaching the walls the ions are neutralized, i.e. they become neutral atoms.
- The double charged ions are not taken into account.

2. The calculation of ion density in radio-frequency inductive discharge

Let us specify some potential distribution in the considered volume \( \varphi_0(r, z) \), such that the potential decreases from a certain point inside the volume towards its boundaries. We place one ion at every grid node and simulate their motion towards the chamber walls (figure 1). The electric field at any point in the considered region is expressed by the equation (1):

\[
\vec{E} = -\nabla \varphi
\]  

(1)

The ions are affected by the Coulomb force which is described by equation (2):

\[
\vec{F} = q_e \vec{E}
\]

(2)
Here: $q_e = 1.6 \cdot 10^{-19} \text{Coul}$ is the ion discharge (doubly charged ions are not taken into account in this paper).

It is assumed that the initial ion velocity is equal to the average thermal velocity of the atoms, and the direction of this velocity is considered to be the same as that of the electric field. The acceleration created by electrostatic force is described by the equation (3):

$$a = \frac{qE}{M_i}$$

Here $M_i = 131.3 \cdot 1.6 \cdot 10^{-27} \text{kg}$ is the mass of Xenon atom.

Having defined the time step as $dt$, let’s calculate the velocities and coordinates of the ions in the considered region assuming that the ions move with constant acceleration during the time $dt$. The velocities and coordinates of the ions are described by equations (4-7):

$$v_{ir}(j) = v_{ir}(j-1) + a_{ir}(j-1) \cdot dt$$

$$v_{iz}(j) = v_{iz}(j-1) + a_{iz}(j-1) \cdot dt$$

$$r(j) = r(j-1) + v_{ir}(j-1) \cdot dt + \frac{a_{ir}(j-1)dt^2}{2}$$

$$z(j) = z(j-1) + v_{iz}(j-1) \cdot dt + \frac{a_{iz}(j-1)dt^2}{2}$$

Here: $j$ is the step number, for which trajectory parameters are calculated.

2.1. A balance model for ion trajectories

Now let us mark ring volumes (equation (8)) near the nodes of the grid (figure 2):

$$V_k = 2\pi r \cdot \pi dr^2$$

Here: $k$ is the step number, for which trajectory parameters are calculated.

The amount of the ions which originate in the ring volume during a second is described by equation (9) [6]:

**Figure 1.** Calculation of ion trajectories in the RIT discharge chamber.
Here: $n_i$ is the ion density in the considered point, $n_a$ is the neutral atom density in the considered point, and $\sigma v_i$ is the coefficient of propellant ionization rate.

![Figure 2. Schematic diagram for the ion density computation.](image)

Let us consider the surface $S_k$ (equation (10)), which is a side surface of the truncated cone with a generatrix perpendicular to the considered trajectory. The length of the generatrix is calculated from the assumption that the ions scatter in the direction perpendicular to the trajectory with a thermal velocity:

$$S_{k(j)} = 2\pi r_{(j)} \cdot (dr + 2 \cdot j \cdot V_t dt)$$  \hspace{1cm} (10)

Ion flow through the area element $S_k$ is described by equation (11):

$$N = n_{i(j)} v_{i(j)} S_{k(j)}$$  \hspace{1cm} (11)

By equating (9) to (11), we can obtain an equation (12) for calculating density at the $j^{th}$ point of the trajectory:

$$n_{i(j)} = \frac{n_{i(0)} n_a \sigma v_i V_k}{v_{i(j)} S_{k(j)}}$$  \hspace{1cm} (12)

The density calculated by (12) is summed up with the density in the nodes located near the $j^{th}$ point of the trajectory calculated using the same relation at $S_k = S_{k(0)}$. For this, at each value of $j$ the values of the coordinates $r_{v(j)}$, $z_{v(j)}$, $r_{n(j)}$ and $z_{n(j)}$ of the boundaries of the conical area $S_{k(j)}$ are computed. Then the condition is checked if there grid nodes are inside the quadrangle formed by points $(r_{v(j)}, z_{v(j)})$, $(r_{n(j)}, z_{n(j)})$, $(r_{v(j-1)}, z_{v(j-1)})$, $(r_{n(j-1)}, z_{n(j-1)})$ (figure 3). If the grid node is inside the quadrangle, the value of ion density from the $j^{th}$ point of the trajectory is being added to the value of ion density in the grid node.

Besides, the ion fluxes in the grid nodes are calculated at each iteration. The calculation of ion density and the fluxes in the grid node with coordinates $(f, k)$ is described by the equations (13), (14) and (15).

$$\left(n_i\right)^{(f,k)}_{\Sigma} = \sum_{l} \sum_{m} \left(n_i\right)^{(l,m)}_j$$  \hspace{1cm} (13)

$$\left(n_i v_i\right)^{(f,k)}_{\Sigma} = \sum_{l} \sum_{m} \left(n_i\right)^{(l,m)}_j \left(v_{i}\right)^{(l,m)}_j$$  \hspace{1cm} (14)

$$\left(n_i v_i\right)^{(f,k)}_{\Sigma} = \sum_{l} \sum_{m} \left(n_i\right)^{(l,m)}_j \left(v_{i}\right)^{(l,m)}_j$$  \hspace{1cm} (15)
Here: \((l, m)\) is the trajectory number (trajectory numbers coincide with the coordinates of the grid nodes, from which they start), \(j\) is the number of step of the trajectory at which the grid node with coordinates \((f, k)\) is near the trajectory \((l, m)\), \(\langle n_i \rangle_{j}^{(l,m)}\) is the ion density calculated at the \(j\)th step of the trajectory \((l, m)\), \((v_{ir})_{j}^{(l,m)}\) and \((v_{iz})_{j}^{(l,m)}\) are respectively the radial and axial components of the ion velocity calculated at the \(j\)th step of the trajectory \((l, m)\), \(\langle n_i v_i \rangle_{r}^{(f,k)}\) and \(\langle n_i v_i \rangle_{z}^{(f,k)}\) are radial and axial components of the total ion flux at the grid node with coordinates \((f, k)\), respectively.

When calculating the total ion density and ion flux at the grid nodes in accordance with Equations 13-15, the trajectory summation is performed only when the grid node \((f, k)\) is near the \(j\)th point of the trajectory \((l, m)\) (figure 3).

![Figure 3. Defining the grid nodes which are located near the \(j\)th point of the trajectory.](image)

Then, the radial and axial components of the average ion velocity at the grid nodes are calculated (equations (16) and (17)):

\[
\langle v_{ir} \rangle^{(f,k)} = \frac{\langle n_i v_i \rangle_{r}^{(f,k)}}{\langle n_i \rangle_{r}^{(f,k)}}
\]  

(16)

\[
\langle v_{iz} \rangle^{(f,k)} = \frac{\langle n_i v_i \rangle_{z}^{(f,k)}}{\langle n_i \rangle_{z}^{(f,k)}}
\]  

(17)

The obtained values of the average ion velocity at the grid nodes are used as the initial ion velocities for the calculation of trajectories at the subsequent iteration.

Having calculated density at the grid nodes, we calculate the plasma potential at each point of the considered area. It is known that the density of particles in a potential field is described by the equation (18):

\[
n_i = n_{i0} \cdot \exp\left(-\frac{q \varphi}{kT_e}\right)
\]  

(18)

Here: \(n_{i0}\) is the maximal ion density in the considered region, \(k = 1.38 \cdot 10^{-23}\) is the Boltzmann constant, \(T_e\) is the electron temperature. Therefore we obtain the equation (19):

\[
\varphi = -\ln\left(\frac{n_i}{n_{i0}}\right) \cdot \frac{kT_e}{q_e}
\]  

(19)

Then the obtained potential distribution is used to recalculate the ion density in the way described above. The calculation is performed iteratively until the difference between the densities calculated at the considered iteration and at the previous one becomes less than a certain predetermined small value.
3. Results and discussion

The described model was used to develop the program code. Using this code, the further calculations were carried out.

In [7], an analytical solution is given for the spatial distribution of the plasma potential for the case when ion production is proportional to the ion density for the plane, cylindrical, and spherical geometry of the plasma region. Such dependence was used to verify the obtained mathematical model. For this, from the distribution of potential using equation (18) the density distribution along the radius of a spherical volume was calculated.

A spherical computational region with a radius $r_c = 50$ mm, which is shown in figure 4, was simulated. Z axis is the axis of symmetry. Then, uniform rectangular grid was built. The grid is shown in figure 5. Before computation, the grid nodes which are inside the computational area were found. The nodes appeared to be outside the considered area were excluded from the consideration. The initial plasma potential as well as the initial ion density distribution were set, and the parameters of ion trajectories in the electric field were computed.

![Figure 4. The spherical calculation area.](image1)

![Figure 5. The uniform rectangular mesh used for calculation.](image2)
Then using the described numerical model the distribution of relative ion density along the sphere radius which is shown in figure 6 was calculated. The calculation was performed with a uniform distribution of the neutral atom density and the ionization rate coefficient. The results of comparing the numerical calculation with the analytical solution are shown in figure 7.

![Figure 6](image6.png)  
**Figure 6.** The calculated ion density distribution in spherical volume, 1/m³.

![Figure 7](image7.png)  
**Figure 7.** Comparison of the numerical calculation of the ion density distribution along the radius of the spherical region with the analytical solution.

As we can see from figure 7, the described model allows one to obtain an ion density distribution with rather good accuracy (the maximum discrepancies between the results of numerical calculation and the analytical solution do not exceed 10%). The mathematical model can be used to calculate the ion density when modeling the discharge in radio-frequency ion thrusters.

After the validation of mathematical model, a gas discharge in the chamber of a radio-frequency ion thruster with a constant and uniform distribution of the neutral atom density and with a uniform distribution of the ionization rate coefficient was simulated. A hemispherical geometry of the discharge chamber with a diameter of 100 mm was chosen for simulation (figure 8).
Figure 8. The initial geometry for calculating the ion density in the discharge chamber of a radio-frequency ion thruster.

Figure 9 shows the uniform rectangular mesh used for the calculation.

We can define the value of neutral atom density and of ionization rate from the balance model of a radio-frequency ion thruster [4]. For the calculation, the following values were chosen: 

\[ n_a = 6.3 \cdot 10^{17} \, \text{m}^{-3}, \ \sigma v_i = 5 \cdot 10^{14} \, \text{m}^2 \, \text{s}^{-1} \]

Figure 10 shows the results of calculation of the ion density in the discharge chamber of radio-frequency ion thruster.
As we can see from figure 10, when a neutral atom density and an ionization rate coefficient are uniformly distributed the ion density has a maximum in the central part of the considered region and it decreases towards the wall. This is caused by a significant acceleration of ions by a radial electric field arising due to the intensive escape of electrons to the walls. While keeping the total ion flow, the ion density decreases with their velocity growth.

4. Conclusion

A numerical mathematical model based on the balance ratios is developed which allows one to obtain the distribution of ion density in the gas-discharge chamber of a radio-frequency ion thruster. The calculation method is based on iterative recalculation of ion density from the electric potential distribution and then of the value of electric potential for the next iteration. Ion density distribution is calculated from numerically computed ion trajectories in predefined electric field. The developed model can be used to simulate radio-frequency inductive gas discharges being sustained at a low gas pressure.

The model validation was carried out using the spherical initial geometry for which there is an analytical solution. A comparison of the ion density distribution obtained by numerical calculation with the analytical solution showed that the disagreement is less than 10% which indicates the applicability of the described model for calculating the ion density in a radio-frequency discharge.

The ion density in the discharge chamber of a radio-frequency ion thruster was calculated for uniform distribution of neutral atom density and ionization rate. It was shown that at such conditions the ion density has a maximum in central part of the discharge chamber and it decreases towards the walls.

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