The role of post-selection in generating vortex particles of arbitrary mass, spin, and energy

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There exist many methods to obtain twisted photons with a projection of orbital angular momentum onto a propagation axis. However, despite the big interest in generating massive particles with the phase vortices, the available diffraction techniques are only applicable to the moderately relativistic beams of electron microscopes, to cold neutrons, or non-relativistic atoms. In order to bring the matter waves physics into the high energy domain, one needs to develop alternative methods, applicable for ultrarelativistic energies and for composite particles, including ions, and nuclei. Here, we show that vortex particles of in principle arbitrary mass, spin, and energy, including muons, hadrons, nuclei, and heavy ions, can be generated during photon emission in helical undulators, via Cherenkov radiation, in collisions of charged particles with intense circularly polarized laser beams, in such scattering or annihilation processes as $e\mu \rightarrow e\mu$, $e\mu \rightarrow e\gamma$, $e^- e^+ \rightarrow p\bar{p}$, and so forth. We argue that the key element in obtaining such quantum states is the post-selection protocol due to quantum entanglement between a pair of the final particles and it is largely not the process itself and, in contrast to the classical theory, not the trajectory of the emitting wave-packet’s centroid. The state of a final particle – be it a photon, a proton, or a nucleus – becomes twisted if the azimuthal angle of the other particle’s momentum is weakly measured with a large error or not measured at all, which is often the case when the scattering angle is small. As a result, the requirements to the beam transverse coherence can be greatly relaxed, which enables the generation of vortex particles at accelerators and synchrotron radiation facilities, thus making them a new tool for hadronic and spin studies. This weak-measurement technique can also facilitate the development of sources of hard X-ray and $\gamma$-range twisted photons for the purposes of nuclear and particle physics.

I. INTRODUCTION

Photons with a quantized projection of orbital angular momentum (OAM) onto a propagation axis were predicted in Ref. [1] and found numerous applications in quantum optics and quantum information, optomechanics, biology, astrophysics, and in other fields [2–4]. Along with the conventional diffraction techniques, such twisted states of light can be generated by charged particles during electromagnetic emission in helical undulators [10–15], via non-linear Thomson or Compton scattering [15–19] or – more generally – when the electron’s classical trajectory is helical [20, 21], during Cherenkov radiation and transition radiation in media with different characteristics [22, 23], and so forth. However, despite the proposals to use twisted photons in particle and nuclear studies [6], the highest energy of them achieved so far does not exceed but a few keV [7]. One recent idea to upconvert their frequency is to employ resonant scattering of the optical twisted photons on relativistic ions within the Gamma Factory project at the Large Hadron Collider [6, 23].

The massive counterparts of twisted photons – vortex electrons [25, 26] – were first generated in 2010-2011 with the 300 keV beams of the transmission electron microscopes [27, 29]. They have also attracted much attention outside the microscopy community because of their potential applications in atomic and high-energy physics [25, 26, 30] and, in particular, in hadronic and spin studies, and even in accelerator physics [51, 52] where the classical analogues of the vortex electrons – the so-called angular-momentum-dominated beams – have been known and utilized since the beginning of 2000s (see [53, 55] and references therein).

There are also ongoing discussions of possible experiments with other vortex particles, including elementary fermions heavier than electron, hadrons, ions and atoms, cold neutrons, nuclei, and even spin waves (magnons) [8–11, 14, 19, 51, 52, 56, 57]. Very recently, in 2021, the non-relativistic twisted atoms and molecules have been generated [58]. Clearly, a large class of quantum and classical waves can potentially carry phase vortices, which follows already from the similarities between the classical wave equations and the quantum ones. However, the available diffraction techniques to generate such particles [27, 29, 59, 60] require the beams with high transverse coherence, typical for electron microscopes, and, therefore, they are hardly applicable for relativistic beams of particle accelerators. This circumstance severely limits the development of the matter waves physics.

Indeed, even if the beam current is very low and there is no space-charge effect, the Rayleigh length of a relativistic wave packet – say, of an electron – is usually

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much larger than a distance from a particle source (say, a cathode) to the collision region in a linear accelerator, whereas in a storage ring the rms-width of a charged-particle wave packet does not grow but oscillates with time \[\tau\]. Therefore, a transverse coherence length of a quantum wave packet naturally remains small, and the corresponding quantum effects do not usually play a role in accelerator physics. Thus, in order to boost the physics of vortex waves to the high-energy domain, we need to develop alternative methods to generate vortex states of a large class of quantum objects. Several techniques based on the similarities between the quantum wave packets and the classical beams in accelerators have been recently discussed in \([51,52]\).

In this paper, which is complementary to our Letter \([61]\), we describe in detail a universal and versatile method to generate vortex states of arbitrary mass, spin, and energy, including hard X-ray and \(\gamma\)-range twisted photons, relativistic muons, protons, ions and nuclei. Such particles can be obtained during emission of photons in electromagnetic fields of accelerators and free-electron lasers, via scattering or annihilation processes with freely propagating particles, such as

\[
e\mu \rightarrow e\mu, ep \rightarrow ep, e^{-}e^{+} \rightarrow h\bar{h}, e^{-}e^{+} \rightarrow 2\gamma,
\]

(1)

scattering of light on relativistic ions, which is being discussed within the Gamma Factory project at the LHC \([6,62]\), etc.

Our key observation is that it is largely not the process itself that defines whether a final particle gets a phase vortex or not, but a post-selection protocol due to quantum entanglement between a pair of the final particles. Whereas in the classical theory the radiation field is always twisted if the emitting electron’s path is helical \([20]\), the more general quantum theory developed in this paper predicts that the photons cannot be twisted at all if the emitting particle’s 3-momentum \(\mathbf{p}’\) is measured with the vanishing error, and their OAM mostly depends on the way we post-select the emitting electron and to a much lesser extent on the process kinematics. Our results complement and specify those of the classical theory from Ref.\[20\], but also make them somewhat less optimistic because the conventional post-selection protocol, described, for instance, in the textbooks \([68,69]\), results in no vorticity of the emitted photons whatsoever.

Importantly, we point out a deep analogy between a weak measurement \([63–66]\) of a final particle’s momentum

\[
\mathbf{p}' = \{p'_\perp, \phi', p'_z\},
\]

(2)
in which not all the components are measured with a vanishing error, and a strong (projective) measurement in cylindrical basis \([34]\), in which \(\phi'\) does not belong to the measurable set of quantum numbers. In more detail, if the azimuthal angle of the final particle’s momentum is measured with a large error or not measured at all, this lack of information about the particle’s 3-momentum itself is enough to project the other final particle to the vortex state, thanks to the angle-OAM uncertainty relation \([67]\).

The proposed technique can readily be employed at the existing electron and hadron accelerators, X-ray free-electron lasers and synchrotron radiation facilities with helical undulators such as, for instance, the European XFEL and ESRF, powerful laser facilities such as the Extreme Light Infrastructure, and at the future linear colliders. A system of units with \(\hbar = c = 1\) is used throughout the paper.

II. MEASUREMENT SCENARIOS

A. Evolved wave function of a particle

Let us consider emission of a photon by a charged particle (for definiteness, by an electron), \(e \rightarrow e' + \gamma\), in the lowest order of the perturbation theory in QED. It can be Cherenkov radiation, synchrotron radiation, undulator radiation, transition radiation, and so on. An initial state \(|\text{in}\rangle\) of the electron and an evolved (pre-selected) state of the electron and the photon are connected via an evolution operator \(\hat{S}\) \([68,69]\),

\[
|\psi'\gamma\rangle = \hat{S}|\text{in}\rangle = \sum_{\lambda,\lambda'} \int \frac{d^3p'}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} S_{fi} |\mathbf{k}, \lambda; \mathbf{p}', \lambda'\rangle,
\]

(3)

where we have used the identity for the one-particle states

\[
\sum_{\lambda,\lambda'} \int \frac{d^3p'}{(2\pi)^3} \frac{d^3k}{(2\pi)^3} |k, \lambda; \mathbf{p}', \lambda'\rangle |\mathbf{p}', \lambda'; k, \lambda\rangle = \hat{1}
\]

(4)

and

\[
S_{fi} = \langle k, \lambda; \mathbf{p}', \lambda'| \hat{S}|\text{in}\rangle
\]

(5)

is a matrix element with two final plane-wave states with the momenta \(\mathbf{p}', \mathbf{k}\) and the helicities \(\lambda' = \pm 1/2, \lambda = \pm 1\). Without post-selection, the wave function of the evolved state is generally not factorized into a product of the photon wave function and that of the electron. Indeed, in the momentum representation we have

\[
\langle \mathbf{p}', \mathbf{k}|\psi'\gamma\rangle = \sum_{\lambda,\lambda'} u' e S_{fi},
\]

(6)

where

\[
e \equiv e_{\mathbf{k},\lambda}
\]

(7)

is a photon polarization vector in the Coulomb gauge, \(e \cdot \mathbf{k} = 0\), \(e \cdot e' = 1\), and

\[
u' \equiv u_{\mathbf{p}',\lambda'}
\]

(8)

is a final electron bispinor, normalized as \(\bar{u}'u' = 2m_e\) \([68,69]\). So the final particles are generally entangled.
In order to derive the evolved wave function of the photon alone, irrespectively of the photon detector, we need to disentangle the particles by post-selecting the electron. If it is post-selected to a state $|e'_{\text{det}}\rangle$, the photon’s wave function in momentum space becomes

$$A^{(f)}(k,\omega) = \sum_{\lambda_i=\pm1} e^i S_{fi}, \quad (9)$$

where

$$S_{fi} = (k,\lambda_i; e'_{\text{det}}) S(\text{in}). \quad (10)$$

This simple expression allows one to find in what quantum state the photon ends up as a result of the emission process itself, irrespectively of how this photon is detected, whereas the electron detected state is specified. The Fourier transform

$$A^{(f)}(r,\omega) = \int \frac{d^3k}{(2\pi)^3} E^{(f)}(k,\omega) e^{ikr} \quad (11)$$

defines spatial distribution of the energy at the frequency $\omega$ in the Coulomb gauge, because $E^{(f)}(r,\omega) = i\omega A^{(f)}(r,\omega)$, and so

$$|E^{(f)}(r,\omega)|^2 = \omega^2|A^{(f)}(r,\omega)|^2. \quad (12)$$

We emphasize the difference between this quantum picture and the classical theory: in the latter the electron path is given, the emitting particle experiences no recoil, the coherent properties and the phase of the radiation field are solely defined by the classical trajectory. In particular, when the electron path is circular or spiral the radiation field is always twisted \[20\]. The quantum picture embraces the classical problem as a special case, but it is generally more complex: the evolved state of the emitted photon itself always depends on how the emitting particle (electron) is post-selected. As we show below, if the electron’s 3-momentum is measured with the conventional plane-wave detector, as it is in the overwhelming majority of experiments in particle physics \[69\], the final photon turns out to be not twisted at all, even if the electron wave-packet’s centroid moves along the helical path. Thus, the twisted particles are likely to be not that abundant in Nature as it follows from the classical theory.

Analogously, one can define an evolved wave function of an electron, or of any other fermion, when the photon is post-selected to some state,

$$\psi^{(f)}(p',\epsilon') = \sum_{\lambda_i=\pm1/2} u' S_{fi}. \quad (13)$$

Clearly, the expressions \[9\], \[13\] also hold for scattering and annihilation processes, including reactions with hadrons in the final state. In the latter case, Eq.\[13\] can define an evolved wave function of the hadron.

B. Conventional plane-wave basis

To begin with, let the initial electron be described as a plane-wave state propagating along the $z$ axis with the momentum $p = \{0,0,|p|\}$ (say, in the problem of Cherenkov radiation below). Following the conventional textbook approach \[68,69\], if it is post-selected to a plane-wave state with the momentum $p' = \{p'_x,p'_y,p'_z\}$ and the helicity $\lambda'$, the matrix element $S^{(pw)}_{fi}$ and the photon’s evolved wave function are both proportional to the momentum conservation delta-function

$$S^{(pw)}_{fi} \propto \delta^{(3)}(p - p' - k) \propto \delta^{(2)}(p'_\perp + k_\perp),$$

$$A^{(f)}(k,\omega) \propto \sum_{\lambda_i=\pm1} e^i \delta(p'_\perp + k_\perp). \quad (14)$$

As soon as the azimuthal angle $\phi'$ of the electron momentum is measured, the photon’s azimuthal angle $\phi_k$ is also set to a definite value

$$\phi_k = \phi' \pm \pi. \quad (15)$$

Thereby, the photon is projected to a plane-wave state without an intrinsic OAM projection on the $z$ axis. One can qualitatively understand this result without the calculations by recalling the uncertainty relation from Ref.\[57\]: the definite azimuthal angle implies a vanishing OAM $z$-projection.

Now let us take a general reaction with the plane-wave states, the matrix element of which is proportional to the energy-momentum conservation delta-function \[68,69\],

$$S^{(pw)}_{fi} \propto \delta^{(4)}(\sum p_{\text{in}} - \sum p_{\text{out}}). \quad (16)$$

If we substitute this to Eq.\[3\] or Eq.\[13\], we immediately see that the evolved wave function of a photon or of a fermion has a well-defined 4-momentum, which automatically means that the $z$ projection of the intrinsic OAM is vanishing.

The post-selection to the plane waves is the most frequently used approach both in calculations and in experiments on particle scattering, annihilation, or photon emission. In contrast to the results of the classical calculations \[10,12,13,20\], in quantum theory it is not the trajectory but the post-selection protocol that defines the vorticity of the final photon. Therefore, when it comes to the photon phase, the direct comparison of the results in the plane-wave basis with the classical theory simply does not have sense. Indeed, the classical electron does not experience recoil, so its angle $\phi'$ is not defined, which is more similar to the scheme (iii) below than to the above plane-wave approach.

C. Cylindrical basis and weak measurements

The post-selection to a plane wave represents a so-called strong (or projective) measurement when all components of the momentum

$$p' = \{p'_x,p'_y,p'_z\} = \{p'_\perp,\phi',p'_z\} \quad (17)$$

are measured with vanishing errors. After a weak measurement, on the contrary, at least some of the components are known with a non-vanishing error (see, e.g.,
[23]. If the error of some component is maximal, we do not effectively measure this component at all. Put simply, a weak measurement is a post-selection to a wave packet, a coherent superposition of plane waves,

\[ |\psi'(\mathbf{p}')\rangle = \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \psi(\mathbf{p}') |\mathbf{p}',\lambda'\rangle. \]  

(18)

The overlap function \( \psi(\mathbf{p}') \) can be of the Gaussian form,

\[
\psi(\mathbf{p}') \propto \prod_i \exp \left\{ - (p'_i - \langle p'_i \rangle)^2 / \sigma_i^2 \right\}.
\]

(19)

where \( i = x, y, z \) or \( i = \rho, \phi, z \). A strong measurement implies that

\[
\sigma_x, \sigma_y, \sigma_z \to 0.
\]

(20)

Alternatively, one can use the cylindrical coordinates \( p'_\perp, \phi', p'_z \) with the corresponding uncertainties \( \sigma_\perp, \sigma_\phi, \sigma_z \). For a weak measurement, at least one of these uncertainties is not vanishing.

To illustrate these ideas, let us distinguish the following three scenarios (see Fig. 1):

(i) We post-select the particles to the plane-wave states and repeat the measurements many times with an ensemble of electrons, each time fixing the detector at a different angle \( \phi' \). The emission rate or a cross section is proportional to

\[
\int_0^{2\pi} \frac{d\phi'}{2\pi} |S_{\phi}^{(pw)}|^2,
\]

which represents an incoherent averaging over the azimuthal angle in the strong-measurement scheme. This is the standard textbook approach [23, 29], in which no phases contribute to the observables.

(ii) Another example of a strong measurement is post-selection to a cylindrical wave (a Bessel state) [34] with the definite \( p'_\perp, p'_z \), the z-projection of the total angular momentum (TAM) \( m' \), and the helicity \( \lambda' \), but undefined \( \phi' \)

\[
|\psi'(\mathbf{p}')\rangle = |p'_\perp, p'_z, m', \lambda'\rangle = \int_0^{2\pi} \frac{d\phi'}{2\pi} e^{i(m' - \lambda')\phi'} |\mathbf{p}', \lambda'\rangle.
\]

(22)

The TAM z-projection operator

\[
\hat{j}_z = \hat{L}_z + \hat{s}_z,
\]

(23)

is a sum of an orbital part \( \hat{L}_z \) and a spin part \( \hat{s}_z \). Both in the coordinate representation and in the momentum one, the former is

\[
\hat{L}_z = [\mathbf{r} \times \mathbf{p}]_z = -i \frac{\partial}{\partial \phi}.
\]

(24)

while the latter depends on the particle spin. Here, \( \phi \) refers to the azimuthal angle either of \( \mathbf{r} \) or of \( \mathbf{p} \), depending on the representation.

The corresponding amplitude

\[
\int_0^{2\pi} \frac{d\phi'}{2\pi} e^{i(m' - \lambda')\phi'} S_{\phi}^{(pw)}
\]

(25)

represents a coherent averaging over the azimuthal angle, whereas the detector is able to measure the TAM with a vanishing error \( \sigma_m \to 0 \). As the azimuthal angle and the z-projection of the angular momentum represent the conjugate variables [67], in this way we also obtain the complete information about the electron, but in the cylindrical basis.

(iii) Consider now an electron emitting a photon in the case we measure the azimuthal angle \( \phi' \) of the final electron momentum with a finite error (see the right panel in Fig. 1)

\[
\sigma_\phi \in (0, 2\pi).
\]

(26)
mentum is incomplete.

**FIG. 2.** Two possible directions of the vectors $\mathbf{k}_\perp$ and $\mathbf{p}_\perp'$ according to Eq. (34).

Generally, in the weak-measurement scheme we post-select to the wave packet (18) with the finite uncertainties $\sigma_\perp, \sigma_\phi, \sigma_z$. The finite uncertainty $\sigma_\phi$ means that we do not know exactly at which azimuthal angle the final electron goes, and the maximal value, $\sigma_\phi \to 2\pi$, corresponds to a special case when the angle is not known at all or, simply, is not measured. The corresponding uncertainty relation [67] says that the larger the error of the angle is, the smaller the corresponding OAM error.

If $\sigma_\phi \to 2\pi$, the information about the electron’s momentum is incomplete, as only two out of three momentum components are known, but the energy is well-defined,

$$\varepsilon' = \sqrt{(p'_\perp)^2 + (p'_{\perp y})^2 + m^2_e}.$$  \hspace{1cm} (27)

The corresponding amplitude is also obtained via the coherent averaging,

$$\frac{2\pi}{0} \int \frac{d\phi'}{2\pi} S^{(pm)}_{ij}(w).$$  \hspace{1cm} (28)

Formally, this expression coincides with (25) at $m' - \lambda' = 0$, but its physical meaning is different. In the strong-measurement scheme (ii), we do measure the TAM projection and the helicity, and we can easily obtain $m' - \lambda' = 0$, but during the weak measurement we do not measure the TAM at all, which implies projection to the state

$$|e_{\text{det}}^{(w)}\rangle = \frac{2\pi}{0} \int \frac{d\phi'}{2\pi} |\mathbf{p}', \lambda'\rangle.$$  \hspace{1cm} (29)

The corresponding function $\psi(\mathbf{p}')$ from Eq. (18)

$$\psi(\mathbf{p}') = (2\pi)^2 \delta(p'_z - \langle p'_z \rangle) \frac{1}{p'_\perp} \delta(p'_\perp - \langle p'_\perp \rangle)$$

does not depend on the angle $\phi'$. The “angle-OAM” uncertainty relation [67] and quantum entanglement play the key role in this weak-measurement technique.

The expansion reverse to Eq. (22) is

$$|\mathbf{p}', \lambda'\rangle = \sum_{m' \lambda' = -\infty}^\infty im' - \lambda' e^{-i(m' - \lambda')\phi'} |p'_\perp, p'_{\perp y}, m', \lambda'\rangle.$$  \hspace{1cm} (30)

Therefore one can look at the weak-measurement scheme (iii) as at an averaging over the phase space during the strong measurement in the cylindrical basis (scheme (ii)),

$$|e_{\text{det}}^{(w)}\rangle = \frac{2\pi}{0} \int \frac{d\phi'}{2\pi} \times \sum_{m' \lambda' = -\infty}^\infty im' - \lambda' e^{-i(m' - \lambda')\phi'} |p'_\perp, p'_{\perp y}, m', \lambda'\rangle,$$  \hspace{1cm} (31)

where the probability to get a definite TAM $m'$ during the strong measurement,

$$|im' - \lambda' e^{-i(m' - \lambda')\phi'}|^2 = 1,$$  \hspace{1cm} (32)

is the same for all $m' - \lambda'$. The measure

$$\int \frac{d\phi'}{2\pi} \sum_{m' \lambda' = -\infty}^\infty$$  \hspace{1cm} (33)

is a total number of the mutually orthogonal states in the particle’s phase space, which is obviously infinite for the delocalized cylindrical waves.

Working in cylindrical coordinates, we use the following representation for the delta-function of the transverse momentum conservation (see Fig. 2):

$$\delta(\mathbf{p}'_\perp + \mathbf{k}_\perp) = \delta(p'_x + k_x) \delta(p'_y + k_y) = \frac{1}{p'_\perp} \delta(p'_\perp - k_\perp) \left( \delta(\phi' - (\phi_k - \pi)) \bigg|_{\phi_k \in [\pi, 2\pi]} + \delta(\phi' - (\phi_k + \pi)) \bigg|_{\phi_k \in [0, \pi]} \right).$$  \hspace{1cm} (34)

Note that only one of these configurations contributes to the evolved state, and there is no interference between them (cf. Ref. [8, 40, 45]).

So, in the simplest weak-measurement scheme with $\sigma_\phi \to 2\pi$ the evolved wave functions of the photon and
of the fermion, Eq. [9] and Eq. [13], are
\[ A^{(f)}_{(w)}(k, \omega) = \sum_{\lambda_{s}=\pm 1} e^{i \int \frac{d\theta'}{2\pi} S_{f_{I}}^{(pw)}} \propto \]
\[ \propto \int \frac{d\theta'}{2\pi} \delta(p_{\perp}' + k_{\perp}) \propto \frac{1}{p_{\perp}} \delta(p_{\perp}' - k_{\perp}), \]
\[ \psi^{(f)}_{(w)}(p', \varepsilon') = \sum_{\lambda'_{s}=\pm 1/2} u' \int \frac{d\theta'}{2\pi} S_{f_{I}}^{(pw)}. \]
(35)
where for the fermion the integration is performed over the azimuthal angle of the weakly-measured particle. The proportionality to \( \delta(p_{\perp}' - k_{\perp}) \) instead of \( \delta(p_{\perp}' + k_{\perp}) \) is a hallmark of the Bessel state with the definite TAM z-projection and its vanishing uncertainty.

The general weak-measurement scheme with the finite uncertainty \( 0 < \sigma_{\phi} < 2\pi \) employs a function \( g(\phi, \sigma_{\phi}) \), and so
\[ A^{(f)}_{(w)}(k, \omega) = \sum_{\lambda_{s}=\pm 1} e^{i \int \frac{d\theta'}{2\pi} g(\phi', \sigma_{\phi}) S_{f_{I}}^{(pw)}}, \]
\[ \psi^{(f)}_{(w)}(p', \varepsilon') = \sum_{\lambda'_{s}=\pm 1/2} u' \int \frac{d\theta'}{2\pi} g(\phi', \sigma_{\phi}) S_{f_{I}}^{(pw)} \].
(36)
where \( g(\phi', 0) = \delta(\phi' - \phi) \) and \( g(\phi', 2\pi) = 1 \). In particular, when \( \sigma_{\phi} - 2\pi < 2\pi, \) the function \( g \) is smooth. Clearly, the latter case describes a Bessel-like wave packet with a mean TAM z-projection and a finite TAM uncertainty, \( \Delta_{j_{z}} \ll \langle j_{z} \rangle \). In what follows, we discuss the scheme with \( \sigma_{\phi} \to 2\pi \) for simplicity.

Finally, note that this weak-measurement scheme can somewhat surprisingly be closely related to the quasi-classical regime of electromagnetic emission when the emitting particle (an electron for simplicity) experiences very small recoil and the scattering angle is vanishing, \( \theta' \ll 1 \). For instance, in a number of scattering and emission processes, including Cherenkov radiation, synchrotron radiation, and the Compton effect, this angle can be of the order of
\[ \theta' \sim 10^{-6} - 10^{-5} \]
(37)
for realistic parameters. Therefore, precise measurements of the azimuthal angle \( \phi' \) can technically be very challenging at these small polar angles. This is the reason why the predictions of this genuinely quantum weak-measurement scheme are somewhat similar to those of the classical theory.

III. WORKING IN THE TAM BASIS

In the conventional plane-wave approach (the scheme (i)), the observable emission rates or the cross sections do not depend on the wave functions’ phases. Here, we are interested in the phases of the evolved wave functions, which is why an analysis of the general phases of the electron bispinors is needed before coming to examples.

A. When phases matter

A two-component spinor \( w^{(\lambda)} \), an eigenstate of the helicity operator \( \hat{s} \cdot \hat{p}/|\hat{p}| \), can be expanded into a series over the eigenstates of the \( \hat{j}_{z} \) operator,
\[ w^{(\lambda)} = \sum_{\sigma=\pm 1/2} w^{(\sigma)} d^{(1/2)}_{\sigma\lambda}(\theta) e^{-i(\sigma-\lambda)\phi}, \]
\[ w^{(1/2)} = (1, 0)^{T}, w^{(-1/2)} = (0, 1)^{T}, \]
\[ \hat{s}_{z} w^{(\sigma)} = \sigma w^{(\sigma)}, \hat{j}_{z} w^{(\lambda)} = \lambda w^{(\lambda)}, \]
(38)
and \( \theta \) is the momentum polar angle, \( p_{\perp} = |\hat{p}| \sin \theta, p_{z} = |\hat{p}| \cos \theta \). The functions
\[ d^{(1/2)}_{\sigma\lambda}(\theta) = \delta_{\sigma\lambda} \cos(\theta/2) - 2\sigma \delta_{\sigma,-\lambda} \sin(\theta/2) \]
(39)
are sometimes called the small Wigner functions and described, for instance, in Ref.[77].

The choice of the overall phase of the spinor \( w^{(\lambda)} \) defines the eigenvalue of the \( \hat{j}_{z} \) operator. The above phase \( e^{i\lambda\phi} \) corresponds to the eigenvalue \( \lambda \) of the TAM operator and without this phase the eigenvalue would be vanishing, \( \hat{j}_{z} w^{(\lambda)} = 0 \). The latter choice is made in the textbook [68], where the Dirac electron bispinor \( \tilde{u}_{\mu,\lambda}, \) also a helicity state, looks as follows:
\[ \tilde{u}_{\mu,\lambda} \equiv \sum_{\sigma=\pm 1/2} u_{\sigma\lambda}(\sigma) d^{(1/2)}_{\sigma\lambda}(\theta) e^{-i\sigma\phi}, \]
\[ u_{\sigma\lambda}(\sigma) = (\sqrt{\varepsilon + m_{e}w^{(\sigma)}}, 2\lambda(\varepsilon - m_{e}w^{(\sigma)}))^{T}, \]
\[ \hat{s}_{z} u_{\sigma\lambda}(\sigma) = \sigma u_{\sigma\lambda}(\sigma), \hat{j}_{z} \tilde{u}_{\mu,\lambda} = 0, \]
(40)
Clearly, the operator \( \hat{j}_{z} \) commutes with the Dirac Hamiltonian, so its eigenvalue is a conserved quantum number.

The phase \( e^{i\lambda\phi} \) in Eq.[38] yields the same general phase of the bispinor \( u_{\mu,\lambda}, \) which changes the orbital part and shifts the TAM to a non-vanishing value,
\[ u_{\mu,\lambda} \equiv \sum_{\sigma=\pm 1/2} u_{\sigma\lambda}(\sigma) d^{(1/2)}_{\sigma\lambda}(\theta) e^{-i(\sigma-\lambda)\phi}, \]
\[ \hat{j}_{z} u_{\mu,\lambda} = \lambda u_{\mu,\lambda}. \]
(41)
Such a choice of the phase is made in the textbook [69].

The difference between the two choices is clearly seen when the \( z \) axis is chosen along the momentum \( \hat{p}, \theta \to 0 \). The state [40] with \( \langle j_{z} \rangle = 0 \) still depends on the azimuthal angle \( \phi \)
\[ w^{(\lambda)}(\theta \to 0) \to w^{(\sigma)} e^{-i\sigma\phi}, \]
(42)
which is somewhat unnatural because it leads to a non-vanishing OAM. The state with \( \langle j_{z} \rangle = \lambda \) does not have this phase factor. In what follows, we denote as \( \tilde{u}_{\mu,\lambda} \) the bispinors with a vanishing TAM projection (the choice of [68]), while we keep \( u_{\mu,\lambda} \) for the bispinor with the TAM
projection $\langle \hat{J}_z \rangle = \lambda$ (the choice of $[68]$). The corresponding short-hand notations are

$$\tilde{u}_{p\lambda} \equiv \tilde{u} \quad \text{and} \quad u_{p\lambda} \equiv u,$$  \hspace{1cm} (43)

respectively. So, whereas the overall phases are not relevant in the strong-measurement of momentum, in which the probability depends on $|S_{fi}^{(pw)}|^2$, they become important in the weak-measurement scheme when finding the evolved states of particles.

### B. Transition current

When calculating the matrix elements in the head-on geometry, we deal with the transition currents in momentum space. Let us calculate the current between two free fermions with the mass $m$, the energies $\varepsilon$ and $\varepsilon'$, and with the definite TAM z-projections $\lambda$ and $\lambda'$. Let the initial fermion propagate along the $z$ axis.

$$\mathbf{p} = \{0, 0, |\mathbf{p}|\};$$ \hspace{1cm} (44)

the angles of the final fermion are $\theta', \phi'$. By using the above formulas we get

$$\bar{u}_{p'\lambda'} \gamma^\mu u_{p\lambda} = J^\mu e^{i(\lambda-\lambda')\phi'},$$

$$J^0 = \delta^{(1/2)}_{\lambda\lambda'}(\theta') \left( \sqrt{\varepsilon + m_e\varepsilon' + m_e} + 2\lambda 2\lambda' \sqrt{\varepsilon - m_e\varepsilon' - m_e} \right),$$

$$J = \left( \sqrt{\varepsilon' + m_e\varepsilon' - m_e} + 2\lambda 2\lambda' \sqrt{\varepsilon' - m_e\varepsilon' - m_e} \right) \chi_0 - \sqrt{2} \delta^{(1/2)}_{\lambda\lambda'}(\theta') \chi_{2\lambda} e^{-i\lambda\phi'\varepsilon},$$ \hspace{1cm} (45)

where $\gamma^\mu$ are the 4 × 4 Dirac matrices in the standard representation $[68]$ and

$$\chi_0 = (0, 0, 1)^T, \quad \chi_{\pm 1} = \mp \frac{1}{\sqrt{2}} (1, \pm i, 0)^T,$$

$$\chi^\dagger_{\lambda} \cdot \chi_{\lambda'} = \delta_{\lambda\lambda'},$$

$$\hat{s}_z \chi_{\lambda} = i \chi_0 \times \chi_{\lambda} = \chi_{\lambda}, \quad \Lambda = 0, \pm 1.$$ \hspace{1cm} (46)

Here, $\hat{s}_z$ is a photon spin operator. We have also used the equality

$$\left( w(\sigma') \right)^{\dagger} \left\{ 1, \sigma \right\} w(\sigma) = \left\{ \delta_{\sigma\sigma'}, 2\sigma \left( \chi_0 \delta_{\sigma\sigma'} - \sqrt{2} \delta_{\sigma, 0} \chi_{2\sigma} \right) \right\},$$ \hspace{1cm} (47)

where $\sigma$ are the Pauli matrices. Note that $J^0$ does not depend on $\phi'$.

Likewise, we can calculate the current when the initial particle propagates opposite to the $z$ axis.

$$\mathbf{p} = \{0, 0, -|\mathbf{p}|\};$$ \hspace{1cm} (48)

The corresponding spinors of the incoming particle we denote as $u_{p\lambda}(-\mathbf{p})$ and $\omega(\lambda)(-\mathbf{p})$. We arrive at

$$\bar{u}_{p'\lambda'} \gamma^\mu u_{p\lambda}(-\mathbf{p}) = J^\mu(-\mathbf{p}) e^{-i(\lambda + \lambda')\phi'},$$

$$J^0(-\mathbf{p}) = d^{(1/2)}_{\lambda\lambda'}(\theta') \left( \sqrt{\varepsilon + m_e\varepsilon' + m_e} + 2\lambda 2\lambda' \sqrt{\varepsilon - m_e\varepsilon' - m_e} \right),$$

$$J(-\mathbf{p}) = \left( \sqrt{\varepsilon' + m_e\varepsilon' - m_e} + 2\lambda 2\lambda' \sqrt{\varepsilon' - m_e\varepsilon' - m_e} \right) \chi_0 + \sqrt{2} d^{(1/2)}_{\lambda\lambda'}(\theta') \chi_{-2\lambda} e^{i\lambda\phi'\varepsilon}.$$ \hspace{1cm} (49)

Importantly, $u_{p\lambda}(-\mathbf{p}) \neq u_{p, -\lambda}$, but $\omega(\lambda)(-\mathbf{p}) = \omega(-\lambda)$, and the latter transition current cannot be obtained from Eq. (45) by simply swapping $\lambda \to -\lambda$. Now we are ready to give some examples.

### IV. EXAMPLE 1: CHERENKOV RADIATION

Consider Cherenkov radiation,

$$e(p) \to e'(p') + \gamma(k),$$ \hspace{1cm} (50)

in a transparent medium with weak frequency dispersion and a refractive index $n(\omega)$; see Fig.\textsuperscript{3}. The incoming electron is described as a plane-wave state with $\mathbf{p} = \{0, 0, |\mathbf{p}|\}$ and the plane-wave matrix element is

$$S_{fi}^{(pw)} = -ieN(2\pi)^4 \delta^{(4)}(p - p' - k) \bar{u}'^\gamma u e^{\mu},$$ \hspace{1cm} (51)

where $N$ is a normalization constant and

$$p = \{\varepsilon, 0, 0, |\mathbf{p}|\}, \quad p' = \{\varepsilon', p'_\perp \cos \phi', p'_\perp \sin \phi', p'_z\},$$

$$p'_\perp = |\mathbf{p}'| \sin \theta', \quad k = \{\omega, k_\perp \cos \phi_k, k_\perp \sin \phi_k, k_z\},$$

$$|\mathbf{k}| = \sqrt{k^2_\perp + k_z^2} = \omega n(\omega), \quad \theta_k = |\mathbf{k}| \sin \theta_k.$$ \hspace{1cm} (52)

The phases are chosen so that the electron bispinors are eigenfunctions of the TAM z-projection operator with the eigenvalues $\lambda = \pm 1/2, \lambda' = \pm 1/2$.

If the electron is detected in the above weak-measurement scheme, we do not know the photon momentum’s azimuthal angle. Then, the photon evolved state is (see Eq.\textsuperscript{43})

$$A(f)(\mathbf{f}, \omega) = \frac{2\pi}{0} \frac{d\phi'}{2\pi} \sum_{\lambda_\gamma = \pm 1} e S_{fi}^{(pw)} = -ieN \times$$

$$\times \mathbf{n} \times \left[ \frac{2\pi}{0} \int \frac{d\phi'}{2\pi} (2\pi)^4 \delta^{(4)}(p - p' - k) \bar{u}' \gamma u \right],$$ \hspace{1cm} (53)

where $\mathbf{n} = \mathbf{k}/|\mathbf{k}|$, we employ the Coulomb gauge with $e^\mu = \{0, e\}$, and the summation over the photon helicities is done with the following formula

$$\sum_{\lambda_\gamma = \pm 1} e_i e_j^* = \delta_{ij} - n_i n_j.$$ \hspace{1cm} (54)

Substituting Eq.\textsuperscript{54} to Eq.\textsuperscript{53}, we arrive at
Thus, the two terms with \( \chi_0 e^{i(\lambda-\lambda')\phi_k} \) and \( \chi_{2\lambda} e^{-i(\lambda+\lambda')\phi_k} \) in \( F \) are eigenvectors of \( \hat{s}_z \) operator with the eigenvalues 0 and \( 2\lambda \), respectively, and the eigenfunctions of the OAM projection operator

\[
\hat{L}_z = -i \frac{\partial}{\partial \phi_k},
\]

with the eigenvalues \( \lambda - \lambda' \) and \( -\lambda - \lambda' \), respectively. Therefore, the vector \( F \) itself is an eigenvector of the photon TAM projection operator \( \hat{\gamma}^{(\gamma)}_z = \hat{s}_z + \hat{L}_z \) with the eigenvalue \( \lambda - \lambda' \).

Similarly, using the representation

\[
n = x_0 \cos \theta_k - \frac{1}{\sqrt{2}} (\chi_{1+} e^{-i\phi_k} - \chi_{1-} e^{i\phi_k}) \sin \theta_k
\]

it is not difficult to prove that the vector \( n(nF) \) and, therefore, the photon state \( A^{(f)} \) itself are eigenvectors of \( \hat{\gamma}^{(\gamma)}_z \) operator with a definite value of the photon TAM projection:

\[
\hat{\gamma}^{(\gamma)}_z A^{(f)} = (\lambda - \lambda') A^{(f)},
\]

where

\[
\lambda - \lambda' = \begin{cases} 0, & \text{if } \lambda' = \lambda \text{ (no electron TAM flip)}, \\ 2\lambda = \pm 1, & \text{if } \lambda' = -\lambda \text{ (an electron TAM flip)} \end{cases}.
\]

The transverse momentum of this Bessel beam \( k_\perp \) equals to the transverse momentum of the final electron and, therefore, it can be written via the electron scattering angle \( \theta' \),

\[
k_\perp = p_\perp = |p'\sin \theta'|.
\]

As the electron scattering angle is usually negligibly small for Cherenkov radiation, \( \theta' \ll 1 \), it is technically very challenging to precisely measure the azimuthal angle \( \phi' \). In this sense, Cherenkov radiation with no post-selection of \( \phi' \) can be viewed as a natural source of twisted photons, coherently emitted to all the azimuthal angles on the Cherenkov cone. We emphasize that even the state with the vanishing eigenvalue of \( \hat{\gamma}^{(\gamma)}_z \) in Eq. (55) is not a plane wave but a cylindrical one with the relativistic spin-orbit interaction (SOI).

Clearly, Eq. (60) is in full agreement with the conservation law of the TAM z-projection \( \hat{\gamma}_z = \hat{\gamma}^{(\gamma)}_z + \hat{\gamma}^{(\gamma)}_j \). Now recall that the conservation law of the TAM takes place not only for the pure states with \( \langle j_z \rangle = \pm 1/2 \), but also for the
mixed states with $\langle \hat{j}_z \rangle \in [-1/2,1/2], \langle \hat{j}'_z \rangle \in [-1/2,1/2]$. In the latter case, instead of Eq. (60) we have

$$\langle \hat{j}_z \rangle^{(\gamma)} = \langle \hat{j}_z \rangle - \langle \hat{j}'_z \rangle,$$

(62)

where for the mixed electron states the photon’s TAM lies in the interval $\langle \hat{j}_z \rangle^{(\gamma)} \in [-1,1]$. In particular, when the initial electron is unpolarized, $\langle \hat{j}_z \rangle = 0$, and the final electron’s TAM is not measured, $\langle \hat{j}'_z \rangle = 0$, the photon’s TAM is also vanishing, although the photon is still twisted.

Finally, when the initial electron is in a pure twisted state [20] with the TAM $z$-projection

$$m = \pm 1/2, \pm 3/2, ..., \quad (63)$$

its bispinor $u \equiv u_{p\lambda}$ with

$$p = \{p_\perp \cos \phi, p_\perp \sin \phi, p_z\} \quad (64)$$

transforms as (see Eq. (22))

$$u_{p\lambda} \rightarrow u_{p\perp p_z, m\lambda} = i^{-(m-\lambda)} \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(m-\lambda)\phi} u_{p\lambda}. \quad (65)$$

The further integration over $\phi$ and $\phi'$ can be performed analytically. As a result, the photon TAM transforms as $\lambda - \lambda' \rightarrow m - \lambda'$:

$$\langle \hat{j}^{(\gamma)} \rangle A^{(f)} = (m - \lambda') A^{(f)}. \quad (66)$$

So, vorticity of the incoming electron is transferred to the emitted photon.

Clearly, the same post-selection protocol can also be applied for other emission processes, including synchrotron radiation, transition radiation, diffraction radiation, Smith-Purcell radiation, and so on.

V. EXAMPLE 2: NON-LINEAR COMPTON SCATTERING AND UNDULATOR RADIATION

A. Post-selection to a plane wave

In a circularly polarized laser wave with the following potential [68] [70]

$$A^\mu = a_1^\mu \cos(kx) + a_2^\mu \sin(kx),$$

$$A^2 = \frac{1}{2} \begin{pmatrix} -a_1^2 & -a_2^2 & 0 & 0 \\ a_1^2 & a_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$a_1 a_2 = (ka_1) = (ka_2) = 0, \quad kx = \omega t - \mathbf{k} \cdot \mathbf{r} \quad (67)$$

an electron is conventionally described with a Volkov state [68] [70] [71]

$$\psi_{p\lambda}(\mathbf{r}, t) = N_e \left( 1 + \frac{e}{2(pk)} \langle \gamma k \rangle \langle \gamma A \rangle \right)$$

$$\times u \exp \left\{ -ipx - \frac{ie}{(pk)} \int d\varphi \left( (pA) - \frac{e}{2} A^2 \right) \right\}, \quad (68)$$

which is exact solution of the Dirac equation and where $\bar{u} u = 2m_e, N_e$ is a normalization constant, and the second term in the pre-exponential factor is due to the spin,

$$\langle \gamma k \rangle \langle \gamma A \rangle = \frac{1}{2} \int \frac{d\varphi}{\pi} F^{\mu\nu} \sigma_{\mu\nu},$$

$$\sigma_{\mu\nu} = \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu), \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \quad (69)$$

Note that the Volkov state (68) transforms into a plane wave when the laser field is off, $A \rightarrow 0$. The case when the incoming electron is twisted and is in a Bessel-Volkov state [69] is studied hereafter. The final photon’s wave function is also a plane wave,

$$A^\mu = N_e e^{i\omega t + ik \cdot \mathbf{r}}, \quad k^\mu e^{\mu} = 0. \quad (70)$$

The corresponding matrix element is

$$S^{(pw)}_{fi} = -ie \int d^4x \bar{\psi}_{p'\lambda'}(\gamma) \psi_{p\lambda} (A^\mu)^* \quad (71)$$

where the final electron is also in the Volkov state,

$$\bar{\psi}_{p'\lambda'}(\mathbf{r}, t) = N_{e'} \bar{u}' \left( 1 + \frac{e}{2(p'k)} \langle \gamma k \rangle \langle \gamma A \rangle \right)$$

$$\times \exp \left\{ ip'x + \frac{ie}{(p'k)} \int d\varphi \left( (p'A) - \frac{e}{2} A^2 \right) \right\}, \quad (72)$$

where $\bar{u}'u' = 2m_{e'}$.

Because this problem can be solved exactly [68] [70] [71], its results can be applied for an approximate description of similar problems where an electron also moves along a helical path, the simplest example being emission in
a helical undulator \([72, 74]\). As shown, for instance, in Ref.\([73]\), the spectral distributions of the radiated energy of both the processes are quantitatively very similar for ultrarelativistic electrons with \(\varepsilon/\mu \gg 1\) and the small recoil \(\omega'/\varepsilon \ll 1\). We emphasize that although the electron is usually post-selected to the Volkov (plane-wave) state in calculations of the non-linear QED effects \([68, 70, 71]\), this implies a \textit{strong measurement} of the electron’s quasi-momentum \(q'\) with the vanishing uncertainties of all its components (the scheme (i) from Sec.\([II\,C]\)).

Importantly, this may \textit{not} necessarily happen in practice in the quasi-classical emission regime if we do not have the complete information about the final electron’s state. Indeed, when the recoil is small and the electron scattering angle is vanishing, \(\theta' \ll 1\), the electron stays in the laser field or inside an undulator, its energy is thereby implicitly measured, but the momentum’s azimuthal angle may not be known at all, simply because it is practically challenging to precisely measure this angle at the polar angles of \(\theta' \sim 10^{-6} - 10^{-5}\) rad. In this regime, the electron quasi-momentum is weakly measured within the scheme (iii) from Sec.\([II\,C]\) and the photon itself is automatically projected to the twisted state (see Fig.\([4]\)). We will return to this weak-measurement scheme in the next sub-section.

Following the standard procedure \([68, 70, 71]\), we expand the matrix element into series over the harmonic numbers \(s \geq 1\), regroup the terms in the pre-exponential factor with the same indices of the Bessel functions \(J_s\) and \(J_{s\pm 1}\), and represent the result as follows:

\[
S_{ji}^{(pw)} = \sum_{s=1}^{\infty} S_{ji}^{(s)} = -i e N \sum_{s=1}^{\infty} (2\pi)^4 \delta^{(4)}(q + sk - q' - k') \times \sum_{\sigma=0,\pm 1} (e'_\sigma)^* \hat{u}^\dagger \Gamma^\sigma_\mu u J_{s+\sigma}(\Sigma) e^{i(s+\sigma)\xi},
\]

(73)

Here, \(N = N_x N_y N_z\) is a normalization constant, the electron quasi-momenta are

\[
q^\mu = p^\mu + c^2 k^\mu \frac{a^2}{2(pk)}, (q')^\mu = (p')^\mu + c^2 k'^\mu \frac{a^2}{2(p'k)}
\]

and we have also denoted

\[
\Sigma = e \sqrt{\left(\frac{(pa_1)}{(pk)} - (p'a_1)\right)^2 + \left(\frac{(pa_2)}{(pk)} - (p'a_2)\right)^2}, \quad \xi = \arctan\left[\frac{(pa_2)/(pk) - (p'a_2)/(p'k)}{(pa_1)/(pk) - (p'a_1)/(p'k)}\right].
\]

(75)

The “dressed” vertex \(\Gamma^\mu_\sigma\) is

\[
\Gamma^\mu_\sigma = \{\Gamma^\mu_\sigma, \Gamma^\mu_{\sigma+1}, \Gamma^\mu_{\sigma-1}\} = \Gamma^\mu_{\sigma+1} - a^\mu \frac{e}{2(pk)} (\gamma k), \quad \Gamma^\mu_{\sigma-1} = \Gamma^\mu_{\sigma+1} - a^\mu \frac{e}{2(pk)} (\gamma k),
\]

\[
= \left\{ \gamma^\mu - k^\mu \frac{e^2 A^2}{2(pk)(p'k)} (\gamma k), \frac{1}{2}(\gamma a_-) \left( \frac{\gamma (\gamma k)^2}{(p'k)} - \frac{1}{(pk)} \right) + k^\mu \frac{e}{(pk)} \right\} - a^\mu \frac{e}{2(pk)} (\gamma k),
\]

\[
\frac{1}{2}(\gamma a_+) \left( \frac{\gamma (\gamma k)^2}{(p'k)} - \frac{1}{(pk)} \right) + k^\mu \frac{e}{(pk)} - a^\mu \frac{e}{2(pk)} (\gamma k),
\]

\[a^\mu = a^\mu_0 = 0, (ka_\pm) = 0. \]

(76)

We emphasize that these expressions are still the standard ones for the non-linear Compton scattering \([68, 70, 71]\), just written differently.

In what follows, we only study the head-on collision, for which

\[
p = \{\varepsilon, 0, 0, |p|\}, \quad k = \{\omega, 0, 0, -\omega\}, \quad k' = \{\omega', k'_\perp \cos \phi_{k'}, k'_\perp \sin \phi_{k'}, k'_\parallel\}
\]

\[
P = \{\varepsilon', p_\perp' \cos \phi', p_\perp' \sin \phi', p_\parallel'\}, \quad p_\parallel' = |p| \sin \theta',
\]

\[
a^\mu_0 = a(0, 0, 0, 0), a^\mu_0 = a(0, 0, 0, 0), a^\mu_0 = a(0, 0, 0, 0),
\]

\[
a^\mu_\pm = a(0, \varepsilon, 0, 0), \quad a^\mu_\pm = a(0, 0, \varepsilon, 0), \quad a^\mu_\pm = a(0, 0, 0, \varepsilon),
\]

\[
\delta^{(4)}(q + sk - q' - k') = \delta(q^0 + s\omega - (q')^0 - \omega') \delta(q_z + s\omega - (q')_z - \omega') \delta^{(2)}(p_\perp + k_\perp'),
\]

(77)

and the transverse delta-function can again be represented in cylindrical coordinates, according to Eq.\([74]\).

In this case,

\[
(pa_1) = (pa_2) = 0,
\]

(78)

the vertex \(\Gamma^\mu_\sigma\) does not depend on the azimuthal angles of the final particles, and so

\[
\tan \xi = \frac{p_y'}{p_x'} = \tan \phi' \Rightarrow \xi = \phi' + \pi g, \quad g = 0, 1, 2, \ldots
\]

(79)

For \(g = 0\), we have

\[
\xi = \phi' = \phi_{k'} \pm \pi
\]

(80)
due to the transverse delta-function. The argument of the Bessel functions becomes
\[ \Sigma = \frac{ea}{(p\kappa)} \rho'_\perp = \frac{ea}{(p\kappa)} \rho'_\parallel \equiv \rho'_\parallel, \]  
(81)
where
\[ \rho'_\parallel = \frac{ea}{(p\kappa)} = \frac{e\sqrt{-A^2}}{mc} = \frac{ea}{mc} \]
(82)
is a classical radius of the final electron’s path (see the problem No.2 to Sec.47 in Ref.75 and Fig.4) and
\[ \eta = \frac{e\sqrt{-A^2}}{mc} = \frac{ea}{mc} \]
(83)
is a classical field strength parameter [68, 70, 71]. Thus, the radius of the classical helical trajectory inside the laser wave naturally arises in the quantum theory. For an optical photon and a relativistic electron, we have roughly
\[ \rho'_\parallel \sim \eta \lambda_c \frac{mc}{\omega} \sim \frac{\eta}{\gamma} \times 0.1 [\mu m], \]
(84)
which reaches the atomic scale,
\[ \rho'_e \sim 0.1 \text{ nm} - 1 \text{ nm}, \]
(85)
for \( \gamma' \sim 10 - 100, \eta \sim 0.1 - 1. \)

Thus, the evolved state of the emitted photon looks as follows:
\[ A^{(f,s)}(k', \omega') = \sum_{s=1}^{\infty} A^{(f,s)}(k', \omega'), \]
(86)
where the sum over \( \sigma \) has appeared because of the spin terms [69]. Summation over the photon helicities \( \lambda \) is done as in Eq.54. Then, choosing the final electron bispinor as an eigenstate of \( \gamma' \) operator with the eigenvalue \( \lambda' = \pm 1/2 \), as in Sec.III.A the evolved state becomes
\[ |e'_\text{det}\rangle^{(w)} = \int_0^{2\pi} \frac{d\phi'}{2\pi} |q', \lambda '\rangle, \]
(87)
sure the electron’s angle \( \phi' \) alone, as often takes place in the quasiclassical emission regime with the small scattering angle, \( \theta' \ll 1 \), then the photon itself appears in the twisted state, irrespectively of how we detect it. The electron’s detected state in the weak-measurement scheme is the azimuthal angle \( \phi_k \).}

B. Post-selection with a weak measurement

1. General analysis

In the strong-measurement scheme, we detect photons emitted in a laser wave or in a helical undulator with a detector placed at an angle \( \phi'_k \), which automatically projects the electron to the plane-wave (Volkov) state \( |q', \lambda'\rangle \) with \( \phi' = \phi'_k \pm \pi. \) However, if we do not measure the photon wave function by integrating Eq.87
\[ A^{(f,s)}(k', \omega') = \int_0^{2\pi} \frac{d\phi'}{2\pi} A^{(f,s)}(k', \omega') = \]
(86)
\[ = -ieN(2\pi)^3(q^0 + s\omega - (q')^0 - \omega')\delta(q_z + s k_z - q'_z - k'_z) \]
\[ \times n' \times \left[ n' \times \sum_{\sigma' = \pm 1/2} \sum_{\sigma = \pm 1} d^{(1/2)}_{\sigma' \lambda'}(\theta') \bar{u}^{(\sigma)'}_{\epsilon' \lambda'} \bar{u}^{(\sigma)}_{\epsilon' \lambda'} \Gamma_{\sigma} u J_{s+\sigma}(\rho'_e k'_\perp) \right] \]
(89)
Even without the calculation of \( \bar{u}^{(\sigma)'}_{\epsilon' \lambda'} \Gamma_{\sigma} u \), it is now clear that the photon represents a cylindrical wave with the
Next, we use the representation

$$\ddot{u}^{(σ')}_c = \Gamma \sigma \Gamma \dot{u} \equiv G^{(↑↑)} \delta_{λ,σ'} + G^{(↑↓)} \delta_{λ,-σ'},$$  \hspace{1cm} (90)$$

where the vectors $G^{(↑↑)} = G^{(↑↑)}_{σλ'λ}, G^{(↑↓)} = G^{(↑↓)}_{σλ'λ}$ are found as

$$G^{(↑↑)}_{0σλ'λ} = x_0 \left( f^{(2)}_{σλ'λ} - \frac{mm_eω^2}{2(pk)(p'k')} \right) \left( f^{(2)}_{σλ'λ} + f^{(2)}_{σλ'λ} \right),$$

$$G^{(↑↓)}_{0σλ'λ} = -\sqrt{2} \sum_σ \left( \frac{mm_eω}{2} \right) \left( \delta_{σ,σ'} \left( \frac{1}{(p'k')} - \frac{1}{(pk)} \right) \right),$$

$$G^{(↑↑)}_{±1σλ'λ} = \pm \sqrt{2} \sum_σ \frac{mm_eω}{2} \left( \frac{1}{(p'k')} - \frac{1}{(pk)} \right) \left( f^{(2)}_{σλ'λ} - f^{(2)}_{σ′λ'λ} \right),$$

$$f^{(1)}_{σλ'λ} = 2 \sqrt{2} \sqrt{ε + m_e}\sqrt{ε' + m_e} + 2λ2'\sqrt{ε - m_e}\sqrt{ε' - m_e},$$

$$f^{(2)}_{σλ'λ} = 2 \sqrt{2} \sqrt{ε - m_e}\sqrt{ε' + m_e} + 2λ2'\sqrt{ε + m_e}\sqrt{ε' - m_e}. $$  \hspace{1cm} (91)

Note that they do not explicitly depend on the 4-momentum $k'$ of the final photon. The vectors $x_0, x_{±1}$ are given in Eq. (16).

The final expression for the wave function is

$$A^{(j,s)}_{w}(k', ω') = -ie(-1)^s + lam - χ' N(2π)^3 δ(q^0 + sw_0 - (q')^0 - ω') δ(q_z + sk_z - q'_z - k'_z) \frac{1}{k'_⊥} \delta(p'_⊥ - k'_⊥)$$

$$× n' \times \sum_{σ=0,±1} J_{σ+σ}(p'_⊥) e^{i(s+σ-λ')ϕ_{σ'}} \left[ d^{(1/2)}_{σλ'λ}(θ') G^{(↑↑)}_{σλ'λ} e^{iλϕ_{σ'}} + d^{(1/2)}_{σλ'λ}(θ') G^{(↑↓)}_{σλ'λ} e^{-iλϕ_{σ'}} \right], $$  \hspace{1cm} (92)

which is obviously a Bessel beam with the SOI, revealed by the sum over $σ$, and the following TAM projection:

$$\dot{j}^{(j)}_z A^{(j,s)}_{w}(k', ω') = (s + λ - χ') A^{(j,s)}_{w}(k', ω'), $$  \hspace{1cm} (93)

whereas the TAM uncertainty is vanishing. Here, in contrast to Eq. (60) for Cherenkov radiation above, the harmonic number $s$ appears in the r.h.s. Analogously to Cherenkov radiation, the transverse momentum of this Bessel photon is defined by the electron scattering angle $θ'$ as

$$k'_⊥ = |p'| \sin θ'. $$  \hspace{1cm} (94)

Importantly, the twisted photons can also be generated in the genuinely quantum non-linear regime with $η ~ θ' ~ 1$, but only within the weak-measurement scheme.

If needed, one can also obtain the Bessel beam in the coordinate representation according to Eq. (11). To this end, it is convenient to divide the vector potential (92) into the following parts:

$$A^{(j,s)}_{w}(k', ω') \equiv A^{(j,s)}_G + A^{(j,s)}_{n'};$$

$$A^{(j,s)}_G \propto G, A^{(j,s)}_{n'} \propto n'(n' \cdot G), $$  \hspace{1cm} (95)

where $G$ is either $G^{(↑↑)}_{σλ'λ}$ or $G^{(↑↓)}_{σλ'λ}$. For the former part, we obtain

$$A^{(j,s)}_G(r, ω') = ie(-1)^s + lam - χ' N(2π)^3 δ(q^0 + sw - (q')^0 - ω') e^{iz(q_z + sk_z - q'_z)}$$

$$× \sum_{σ=0,±1} J_{σ+σ}(p'_⊥) e^{i(s+σ-λ')ϕ_{σ'}} \left[ d^{(1/2)}_{σλ'λ}(θ') G^{(↑↑)}_{σλ'λ} e^{iλϕ_{σ'}} J_{σ+σ-λ'}(ρp'_⊥) + \right]$$

$$+ i^{-λ} d^{(1/2)}_{σ-λ'λ}(θ') G^{(↑↓)}_{σλ'λ} e^{-iλϕ_{σ'}} J_{σ+σ-λ'}(ρp'_⊥) \right], $$  \hspace{1cm} (96)

where $ϕ_φ$ is the azimuthal angle of the vector

$$r = (ρ \cos φ_{φ}, ρ \sin φ_{φ}, z), $$  \hspace{1cm} (97)

and the following identity has been used

$$\int_0^{2π} \frac{dφ}{2π} e^{ix \cos ϕ + iℓϕ} = i^ℓ J_ℓ(x), $$  \hspace{1cm} (98)
Clearly, the part \( A^{(f,s)}_{\ell}(r, \omega') \) is much more cumbersome, whereas the momentum representation above is more compact.

The spatial distributions of the energy of this Bessel beam
\[
|E^{(f,s)}_{\omega'}(r, \omega')|^2 \propto \frac{1}{T} \int d\omega'' |E^{(f,s)}_{\omega}(r, \omega')|^2
\]
represent a typical doughnut, as shown in Fig.\[\text{3}\] Here \( T \) is a very large “observation time” obtained when squaring the delta-function in Eq.\[\text{(96)}\] (see, e.g., \[\text{68}\]). Importantly, whereas the radius of the emitting electron’s classical path \[\text{(82)}\] does not change much during the emission of soft photons, \( \rho'_e \approx \rho_e \), the radius of the first Bessel ring is of the order of \( 1/p'_\perp \approx 1/e'\theta' \), which does not necessarily coincide with \( \rho_e \). Indeed, the electron scattering angle is usually small and limited from above by \[\text{above} \[\text{116}\],
\[
\theta' \leq \frac{2s}{\sqrt{1 + \eta^2}} \frac{\omega}{m_e},
\]
so the photon transverse momentum is
\[
k'_\perp \leq e' \frac{2s}{\sqrt{1 + \eta^2}} \frac{\omega}{m_e},
\]
and it can reach the keV scale for GeV electrons and optical laser photons. That is why in the relativistic case we have
\[
\frac{(p'_e)^{-1}}{\rho'_e} \approx \frac{2\omega}{\eta m_e} \theta' \geq \frac{\sqrt{1 + \eta^2}}{s \eta}.
\]
Very roughly, one can think of \((k'_\perp)^{-1} = (p'_e)^{-1}\) as of the transverse coherence length of the twisted photon, although this quantity cannot be quantitatively defined

\[
A^{(f,s)}(k'_\perp, \omega') \propto \sum_{\sigma=0,1,2} J_\sigma(p'_e k'_\perp) e^{i(s-\lambda')\phi_k'} \left( d^{(1/2)}_{\sigma\lambda}(\theta') G_{\sigma\lambda\lambda}^{(1)} e^{i\lambda\phi_k'} + d^{(1/2)}_{\sigma\lambda}(\theta') G_{\sigma\lambda\lambda}^{(1)} e^{-i\lambda\phi_k'} \right) \approx \left( \delta_{\lambda\lambda} G_{-1\lambda\lambda}^{(1)} + \delta_{-\lambda\lambda} G_{-1\lambda\lambda}^{(1)} \right)
\]

where we have only taken the first term in expansion of the Bessel function in series,
\[
J_{s-1}(p'_e k'_\perp) \approx \frac{(p'_e \omega \theta_k')^{s-1}}{2^{s-1}(s-1)!}.
\]

In other words, the SOI vanishes in this approximation.

Indeed, the fact that only the term with \( \sigma = -1 \) survives for relativistic energies is the reason why we have a minimum at \( \rho = 0 \) for \( j^{(1)}_2 \) = 0 in the left panel of Fig.\[\text{3}\] whereas there is a maximum in the central panel at \( \rho = 0 \) for \( j^{(2)}_1 = 1 \). The Bessel function in Eq.\[\text{96}\] that defines the shape of these distributions is \( J_{s-1}(p'_e k'_\perp) \rightarrow J_{s-1}(\rho p'_\perp) \), which yields \( J_1 \) in the former for an unnormalized Bessel beam. Thus, in the linear regime with \( \eta \ll 1 \) the radius of the first Bessel ring is much larger than the radius of the electron’s classical path, while in the non-linear regime with \( \eta \gg 1 \) this ratio is of the order of unity, as shown in Fig.\[\text{3}\]. The opposite case with \( \eta \gg 1 \) corresponds to a constant crossed field instead of the plane wave \[\text{68, 70}\], and the above ratio gets larger than \( 1/s \).

2. Quasi-classical regime

Let us study in more detail the emission of twisted photons, \( \omega' \ll \varepsilon \), by a relativistic electron, \( \varepsilon \gg m_e \), which stays relativistic after the emission, \( \varepsilon' \gg m_e \). The radiated energy is concentrated at the small angles,
\[
\theta_{k'} \ll 1.
\]
We call this the quasi-classical regime. Due to the delta-function \( \delta(p'_\perp - k'_\perp) \) in Eq.\[\text{90}\],
\[
\sin \theta' / \sin \theta_{k'} = \frac{\omega'}{p'} \approx \frac{\omega'}{\varepsilon'} \ll 1,
\]
which is why the electron scattering angle \( \theta' \) is yet smaller than the photon emission angle,
\[
\theta' \ll \theta_{k'} \ll 1.
\]
In this case
\[
d^{(1/2)}_{\sigma\lambda}(\theta') = \delta_{\sigma\lambda} + O(\theta'),
\]
and only the term with \( \sigma = -1 \) survives in the sum in Eq.\[\text{92}\] in the leading approximation.

As follows from the r.h.s. of Eq.\[\text{107}\], the term with \( G^{(\uparrow\uparrow)} \) corresponds to the emission without an electron spin flip, whereas the term with \( G^{(\uparrow\downarrow)} \) implies that the electron spin flips when the photon is emitted. Now let us recall that the vectors \( G^{(\uparrow\uparrow)} \), \( G^{(\uparrow\downarrow)} \) only depend on the electron energies but not on the emission angles \( \theta_{k'}, \phi_{k'} \). As can be easily seen from Eq.\[\text{91}\], the following estimates hold in the relativistic case,
\[
\left| f^{(1)}_{\lambda\lambda} + f^{(2)}_{\lambda\lambda} \right| \approx 4\sqrt{\varepsilon'} m',
\]
\[
\left| f^{(1)}_{\lambda\lambda} - f^{(2)}_{\lambda\lambda} \right| \approx 2\sqrt{\varepsilon' \varepsilon} \frac{m}{\varepsilon},
\]
\[
f^{(2)}_{\lambda\lambda} \approx \sqrt{\varepsilon'} \frac{m}{\varepsilon}(\frac{m}{\varepsilon} - \frac{m}{\varepsilon}).
\]
As a result
\[ |G_{-1-\lambda\lambda}^{(\uparrow\uparrow)}|/|G_{-1\lambda\lambda}^{\uparrow\uparrow}| = O(m/\varepsilon), \tag{110} \]

because \( \varepsilon' \ll \varepsilon \). In other words, the amplitude with an electron spin flip \( \delta_{-\lambda\lambda'} G_{-1-\lambda\lambda}^{(\uparrow\downarrow)} \) is roughly \( \varepsilon/m \gg 1 \) times

\[
A(f,s)(k',\omega') \propto \sum_{\sigma=0,\pm 1} \approx \frac{(\rho'_e \omega' \theta_{k'})^{s-1}}{2^{s-1}(s-1)!} e^{i(s-1)\phi_{k'}} \delta_{\lambda\lambda'} G_{-1-\lambda\lambda}^{(\uparrow\downarrow)},
\]

\[
G_{-1\lambda\lambda}^{(\uparrow\uparrow)} \approx 2\sqrt{2} \eta m_e \omega \sqrt{\varepsilon \varepsilon'} \chi_{1+1} \left( \frac{1}{(p'k)} - \frac{1}{(pk)} \right) + \frac{1}{(pk)} \right). \tag{111} \]

When the electron recoil is completely neglected (the classical limit or the Thomson scattering), we have

\[ \varepsilon' \to \varepsilon, \omega' \to s \omega, \tag{112} \]

and so

\[
G_{-1\lambda\lambda}^{(\uparrow\uparrow)} \approx 2\sqrt{2} \rho_e \omega \varepsilon \chi_{1+1},
\]

\[
\approx 2\sqrt{2} \varepsilon (\rho_e \omega)^s \frac{(s \theta_{k'})^{s-1}}{2^{s-1}(s-1)!} e^{i(s-1)\phi_{k'}} \delta_{\lambda\lambda'} \chi_{1+1} \tag{113} \]

where \( \rho_e = ea/(pk) = \rho' e = ea/(p'k) \) is the classical radius of the electron's helical path in the plane wave from Eq. (82); see Ref. [73]. In Fig. 6 we present comparison of the general angular dependence of \( |A(f,s)(k',\omega')|^2 \), smaller than the amplitude \( \delta_{\lambda\lambda'} G_{-1-\lambda\lambda}^{(\uparrow\downarrow)} \) without the spin flip, which looks natural for the quasi-classical regime. As a result, only \( G_{-1\lambda\lambda}^{(\uparrow\uparrow)} \) survives and we finally get

\[
\gamma \rho A(f,s)(w) = s A(f,s). \tag{114} \]

The angular distribution and the phase in Eq. (113) exactly coincide with those of the far-field undulator radiation (see the Appendix A), whereas the TAM (114) is in accord with the classical calculations for the undulator radiation and for the non-linear Thomson scattering [10, 12, 18, 20].

Thus, not only the emitted energy distributions \( |E_{(f,s)}^{(\downarrow\uparrow)}|^2 \) are nearly the same for the Compton scattering and for the emission in a helical undulator in the quasi-classical regime [73], but the phases of the fields do
also coincide in this approximation. Importantly, however, the phase $e^{i(s-1)\phi}$ itself does not guarantee that the field carries angular momentum because the overall factor with the delta-function also plays a role. For instance, if the latter factor contains $\delta(\phi_k - (\phi' \pm \pi))$ within the conventional plane-wave approach, the TAM $z$-projection is vanishing, regardless of the phase. It is only the weak-measurement scheme that provides vorticity of the final photons because the latter turn out to be cylindrical rather than plane waves.

3. Bessel-Volkov electron

Let us now turn to the case when the incoming electron is twisted with the TAM $m = \pm 1/2, 3/2, ..., m_e$ and it is in the Bessel-Volkov state [53]. Clearly, this time we obtain

$$\langle j_z(\gamma) | A^{(f,s)}_{(w)} | s + m - \lambda' \rangle = \langle j_z(\gamma) \rangle = s + \langle j_z \rangle - \langle j_z \rangle,$$

whereas the most general formula for the mixed states is

$$\langle j_z(\gamma) \rangle = s + \langle j_z \rangle - \langle j_z \rangle.$$

For the “unpolarized” electrons with $\langle j_z \rangle = \langle j_z \rangle = 0$, the photon’s TAM is simply $s$. This latter result is in agreement with the predictions of the classical theory [10, 12, 18, 20].

Thus, the highly twisted photons with $\langle j_z(\gamma) \rangle > 1$ can be obtained via the non-linear Compton effect.

- Or, alternatively, by taking a moderately powerful laser, for which only the linear scattering with $s = 1$ takes place, but with an incoming vortex (Bessel-Volkov) electron with $m \gg 1$.

As the highly twisted electrons with $m \sim 100 - 1000$ have already been obtained [20], the latter scheme seems to be technically easier. Besides, if the vortex electron is ultrarelativistic, the energy of the scattered highly twisted photons can easily reach MeV range [3]. The means for accelerating the vortex electrons to ultrarelativistic energies have been recently discussed by one of us in Ref. [52].

We emphasize that in the quantum regime the twisted photons are not naturally generated in a circularly-polarized laser wave or in a helical undulator, as it seems from the classical theory [10, 12, 18, 20]. In quantum approach, the photon turns out to be twisted only if the measurement error of the azimuthal angle of the final electron momentum is finite. When the recoil angle $\theta'$ is very small, the azimuthal angle $\phi'$ simply loses its sense, which is why this weak-measurement scheme yields the results similar to those of the classical theory.

VI. EXAMPLE 3: HEAVIER PARTICLES

A. Leptons

An advantage of the weak-measurement technique is that it can in principle be applied to particles of any mass, spin, or energy. We start with the QED scattering with a lepton heavier than electron,

$$e^+\mu^- \rightarrow e^+\mu^-\text{ or } e^+\tau^- \rightarrow e^+\tau^-.$$  (117)

For definiteness we take the process

$$e^-(p_1) + \mu^-(p_2) \rightarrow e^-(p_3) + \mu^-(p_4)$$  (118)

as an example, where the electron and the muon collide head-on and have the momenta

$$p_1 = \{\varepsilon_1, 0, 0, |p_1|\}, \quad \varepsilon_1 = \sqrt{m_e^2 + |p_1|^2},$$

and

$$p_2 = \{\varepsilon_2, 0, 0, -|p_2|\}, \quad \varepsilon_2 = \sqrt{m^2 + |p_2|^2},$$

respectively, in the laboratory frame of reference. When the final electron is detected in the weak-measurement scheme, the evolved wave function of the muon, according to Eq. (115), is

$$\psi^{(f)}(\mu) = \frac{1}{2\pi} \int d\phi \sum_{\lambda = \pm 1/2} u_4 S^{(pw)}_{fi} G(\mu, \lambda),$$

where $u_4 \equiv u_{\mu \lambda}$ and the matrix element reads

$$S^{(pw)}_{fi} = (2\pi)^4 N \delta(\lambda') (p_1 + p_2 - p_3 - p_4) \times \frac{4\pi^2 e^2}{q^2} (\bar{u}_3 \gamma^\alpha u_1) (\bar{u}_4 \gamma^\alpha u_2), \quad q = p_1 - p_3.$$  (121)
where the photon propagator is taken in the Feynman gauge. In contrast to the previous section, the 4-vector \( q \) is a momentum transfer.

The muon current can be presented as
\[
\bar{u}_4 \gamma_\alpha u_2 = J_\alpha^{(\mu)} e^{-i(\lambda_2 + \lambda_3) \phi_4},
\]
(122)
where \( J_\alpha^{(\mu)} \) is given in Eq. (49) with \( m_e \rightarrow m_\mu \). After the integration, the electron current becomes
\[
J_\alpha^{(e)}(\phi_3 = \phi_4 \pm \pi) e^{i(\lambda_1 - \lambda_3)(\phi_4 \pm \pi)}.
\]
(123)
Taking into account that
\[
\chi_{-2\lambda_3} \cdot \chi_{2\lambda_1} = -\delta_{\lambda_2\lambda_1},
\]
(124)
it is easy to check that the scalar product
\[
\mathcal{J} = J_\alpha^{(e)}(\phi_3 = \phi_4 \pm \pi)(J^{(\mu)})^\alpha
\]
(125)
does not depend on \( \phi_4 \). Therefore,
\[
\psi^{(f)}_\mu \propto \sum_{\lambda_4 = \pm 1/2} \bar{u}_4 J e^{i(\lambda_1 - \lambda_2 - \lambda_3) \phi_4},
\]
(126)
where \( \bar{u}_4 = u_4 e^{-i\lambda_4 \phi_4} \) has a vanishing TAM, see Eq. (40). As a result,
\[
\hat{\psi}_{4,\tau}^{(f)\tau}(\lambda_1 - \lambda_2 - \lambda_3) \psi^{(f)}_\mu,
\]
(127)
i.e. the muon is twisted with the TAM \( z \)-projection \( \lambda_1 - \lambda_2 - \lambda_3 \).

Similarly, for the twisted incoming electron with \( m = \pm 1/2, 3/2, \ldots \), the TAM of the final muon becomes
\[
m - \lambda_2 - \lambda_3.
\]
(128)
Dealing with the unpolarized initial muon and the final electron, the incoming electron’s TAM \( m \) is simply transferred to the muon,
\[
\langle \hat{\psi}_z^{(\mu)} \rangle = m.
\]
(129)

Clearly, the same TAM-transfer can be realized in other processes like \( e^- e^+ \leftrightarrow 2\gamma, e^- e^+ \rightarrow \tau^- \tau^+ \), etc. It has been recently shown in Ref. [49] that a hallmark of the muon’s twisted state is the modification of the electron spectra as an incoming muon decays into an electron and two neutrinos. Such a modification can be observed in experiment, which would prove the vortex state of the muon.

**B. Hadrons**

Finally, we examine scattering off a hadron taking as an example elastic scattering of an electron by a proton with a mass \( m_p \),
\[
e(p_1) + p(p_2) \rightarrow e(p_3) + p(p_4),
\]
(130)
in the same head-on geometry (see Fig. 7) with \( p_3 = \{ \varepsilon_3, p_{3,\perp} \cos \phi_3, p_{3,\perp} \sin \phi_3, p_{3,z} \} \) and similarly for \( p_4 \). Generalization to other hadrons, ions, nuclei, or to the inelastic processes due to the non-electromagnetic forces (say, to deep-inelastic ep scattering) is straightforward.

The plane-wave matrix element is
\[
S_{fi}^{(pw)} = i(2\pi)^4 N \delta^{(4)}(p_1 + p_2 - p_3 - p_4)
\]
\[
\times \frac{4\pi e^2}{q^2} (\bar{u}_3 \gamma^{\mu} u_1) (\bar{u}_4 \Gamma_\mu u_2),
\]
(131)
where
\[
\Gamma_\mu = F_1 \gamma_\mu + F_2 \sigma_{\mu\nu} q^\nu
\]
(132)
is a hadronic vertex, which is parametrized with two form-factors \[68, 69\]

\[F_1 = F_1(q^2, P^2), F_2 = F_2(q^2, P^2),\]  
(133)

where

\[q^2 = (p_1 - p_3)^2, P^2 = (p_4 + p_2)^2 / 4\]  
(134)

do not depend on the angle \(\phi\) of the final proton. So the form-factors do not depend on it either. Here also

\[\sigma^{\mu\nu} = \frac{1}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) = (\alpha, i\Sigma),\]  
(135)

where \(\alpha, \Sigma\) are the \(4 \times 4\) Dirac matrices in the standard representation \[68\].

Then, one can prove that

\[\bar{u}_3(\phi_3 = \phi_4 \pm \pi)\gamma^\mu u_4 \gamma^\nu_2 u_4 \Gamma_{\mu\nu} \phi_3 \phi_4.\]  
(136)

Dealing with the 4-vector

\[\sigma^{\mu\nu} q_\nu = -\{\alpha \cdot q, \alpha q_0 + i\Sigma \times q\},\]  
(137)

at the form-factor \(F_2\), we need to calculate the following averages (cf. with Eq.(45) and Eq.(49)):

\[
\begin{align*}
\bar{u}_{\mu ' \lambda} \alpha u_{\mu \lambda} &= (2\lambda \sqrt{\varepsilon - m_p} \sqrt{\varepsilon' + m_p} - 2\lambda' \sqrt{\varepsilon + m_p} \sqrt{\varepsilon' - m_p})  \\
&\times 2\lambda \left(d_{\lambda \lambda}^{(1/2)}(\theta') \chi_{0} - \sqrt{2} d_{\lambda \lambda}^{(1/2)}(\theta') \chi_{2\lambda} e^{-2i\lambda \theta'}\right) e^{i(\lambda' - \lambda) \phi'}, \\
\bar{u}_{\mu ' \lambda} \Sigma u_{\mu \lambda} &= (\sqrt{\varepsilon + m_p} \sqrt{\varepsilon' + m_p} - 2\lambda \sqrt{\varepsilon - m_p} \sqrt{\varepsilon' - m_p})  \\
&\times 2\lambda \left(d_{\lambda \lambda}^{(1/2)}(\theta') \chi_{0} - \sqrt{2} d_{\lambda \lambda}^{(1/2)}(\theta') \chi_{2\lambda} e^{-2i\lambda \theta'}\right) e^{i(\lambda' - \lambda) \phi'}, \\
\bar{u}_{\mu ' \lambda} \alpha u_{\mu \lambda}( -p) &= (2\lambda \sqrt{\varepsilon - m_p} \sqrt{\varepsilon' + m_p} - 2\lambda' \sqrt{\varepsilon + m_p} \sqrt{\varepsilon' - m_p})  \\
&\times 2\lambda \left(-d_{\lambda \lambda}^{(1/2)}(\theta') \chi_{0} + \sqrt{2} d_{\lambda \lambda}^{(1/2)}(\theta') \chi_{2\lambda} e^{i\lambda \phi'}\right) e^{-i(\lambda' + \lambda) \phi'}, \\
\bar{u}_{\mu ' \lambda} \Sigma u_{\mu \lambda}( -p) &= (\sqrt{\varepsilon + m_p} \sqrt{\varepsilon' + m_p} - 2\lambda \sqrt{\varepsilon - m_p} \sqrt{\varepsilon' - m_p})  \\
&\times 2\lambda \left(-d_{\lambda \lambda}^{(1/2)}(\theta') \chi_{0} + \sqrt{2} d_{\lambda \lambda}^{(1/2)}(\theta') \chi_{2\lambda} e^{i\lambda \phi'}\right) e^{-i(\lambda' + \lambda) \phi'}. \\
\end{align*}
\]
(138)

Here, two former averages correspond to the proton moving along the \(z\) axis, whereas two latter ones are calculated for the proton moving opposite to this axis. We have also denoted for simplicity \(\phi_4 \equiv \phi', \lambda_4 \equiv \lambda', \lambda \equiv \lambda_2, \theta_4 \equiv \theta'\).

The evolved wave function of the final proton is defined by Eq.(35),

\[\psi^{(f)}_p = \sum_{\lambda_4 = \pm 1/2} \int_0^{2\pi} d\phi_4 \frac{2\pi}{2\pi} u_4 S_{j_4}^{(p\mu)} \]  
(139)

Clearly, within the same weak-measurement scheme (see Fig.7) the proton is twisted,

\[\hat{J}_{4, z} \psi^{(f)}_p = (\lambda_1 - \lambda_2 - \lambda_3) \psi^{(f)}_p.\]  
(140)

When the incoming electron is twisted with the TAM \(z\)-projection \(m\), we have

\[\lambda_1 - \lambda_2 - \lambda_3 \to m - \lambda_2 - \lambda_3\]  
(141)

If the initial hadron is unpolarized and the electron polarization is not measured, the electron’s TAM is transferred to the hadron,

\[\langle \hat{J}_{z}^{(p)} \rangle = m.\]  
(142)

Clearly, this transfer can also be realized for other particles, including neutrons, pions, atoms, ions, nuclei, and so forth.

\[\text{VII. CONCLUSION}\]

We have theoretically demonstrated that the vortex particles of arbitrary mass, spin, and energy can be generated in a number of customary processes of photon emission, lepton and hadron scattering, or annihilation. In contrast to the classical theory, the key condition that determines vorticity is not the incoming particle’s trajectory, but the post-selection protocol with the weak measurement of the azimuthal angle. It is this scheme that most adequately describes the emission of twisted photons when the recoil is very small, and in that case we have the complete agreement with the classical theory.

In the quantum regime, however, the use of the conventional plane-wave protocol results in no vorticity of the emitted photons whatsoever, even if the emitting wave packet’s centroid moves along a helical path.

We have given explicit examples for the simplest case when the measurement error of the angle is maximized, \(\sigma_{\phi} \to 2\pi\), and the resultant vortex states turn out to be the Bessel beams of photons, electrons, muons, hadrons, nuclei, etc. Clearly, when this error is smaller, the resultant quantum states would be the Bessel-like wave packets with a finite quantum uncertainty of the total angular momentum but with the same central value, which is defined by the TAM conservation law.

The transverse momentum of the resultant twisted particle is defined by this momentum of the other particle whose angle is weakly measured. Thus, selecting the
weakly measured particles scattered at larger polar angles – say, in relativistic weakly measured particles scattered at larger polar angles – one can obtain the vortex beams with the desired transverse momentum. For instance, for the hard X-ray or γ-range twisted photons obtained in collisions of intense laser pulses with GeV electrons or ions (say, at the Gamma Factory [61, 62]), the transverse momenta can reach the keV scale.

Importantly, the proposed technique does not alter the number of particles produced in a specific reaction, compared to the conventional plane-wave scenario. This method can readily be implemented for the production of vortex particles at synchrotron radiation facilities and free-electron lasers with the helical undulators, such as the SASE3 undulator beamline of the European XFEL, at the powerful laser facilities aimed at studying the non-linear QED phenomena, such as, for instance, the Extreme Light Infrastructure, and at the existing and future lepton and hadron colliders.

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Appendix A: Undulator radiation by a single electron in the quasi-classical regime

In classical electrodynamics within the paraxial approximation, the homogeneous wave equation for the slowly-varying amplitude of the electric field in the frequency domain $\tilde{E}_\perp$ can be written as follows (the Gaussian units with $c = 1$ are used):

$$ D \left[ \tilde{E}_\perp(z, r_\perp, \omega) \right] = g(\gamma) \frac{\partial \tilde{E}_\perp}{\partial z}, $$ (A1)

where the differential operator $D$ is defined by

$$ D = \left( \nabla^2 + 2i\omega \frac{\partial}{\partial z} \right), $$ (A2)

$\nabla^2$ being the Laplacian operator over transverse Cartesian coordinates. The source-term vector can be written in terms of the Fourier transform of the transverse current density, $\tilde{j}_\perp(z, r_\perp, \omega)$, and of the charge density, $\tilde{\rho}(z, r_\perp, \omega)$ (both being macroscopic quantities treated as given), as

$$ g(\gamma) = -4\pi \exp[-i\omega z] \left( i\omega \tilde{j}_\perp - \nabla \tilde{\rho} \right). $$ (A3)

For a helical undulator, we set a constrained electron motion

$$ r_{o\perp}(z) = r_{ox} e_x + r_{oy} e_y $$ (A4)

with

$$ r_{ox}(z) = \frac{K}{\gamma o k_w} \cos(k_w z), \quad r_{oy}(z) = \frac{K}{\gamma o k_w} \sin(k_w z) $$ (A5)

with the constant longitudinal speed along the $z$ axis. Here

$$ k_w = 2\pi \frac{\lambda_w}{\lambda}, \quad K = \frac{e H_w}{k_w m_e} $$ (A6)

$\lambda_w$ is an undulator period, and $H_w$ is the maximal modulus of the undulator magnetic field on-axis. The classical parameter $K$ is analogous to $\eta$ from Eq. (83).

Solution of the wave equation is found to be:

$$\tilde{E}_\perp(z, r_\perp, \omega) = \int^{z'}_{-\infty} dz' \frac{1}{z' - z} \int dr'_\perp \exp \left\{ i\omega \left[ \frac{r_\perp - r'_\perp}{2(z - z')} \right]^2 + i \left[ \int_{0}^{z'} dz'' \frac{\omega}{2\gamma^2(z')} \right] \right\} \times \left[ (i\omega \rho_{o\perp}(z') - \nabla \tilde{\rho}) \hat{\rho}(z', r'_\perp - r_{o\perp}(z'), \omega) \right],$$ (A7)

Here $\gamma(z) = 1/\sqrt{1 - v_{ox}(z)^2}$, and $\nabla \tilde{\rho}$ represents the gradient operator with respect to the source point, while $(z, r_\perp)$ indicates the observation point. The further calculations are very similar to those for a planar undulator presented in detail in Ref. [78].

Integration by parts of the gradient term, and introduction of a new integration variable $l = r'_\perp - r_{o\perp}(z')$ gives the following expression in the far-zone limit for relativistic energies, $\gamma \gg 1$, and for the small emission angles $\theta_x = x/z \ll 1, \theta_y = y/z \ll 1$:
\[
\tilde{E}_\perp (z, \mathbf{r}_\perp, \omega) = \frac{i\omega}{z} \int d\ell \int_{-\infty}^{\infty} dz' \tilde{\rho}(z', l, \omega) \exp \left[ i\Phi_T(z', l, \omega) \right] \times \left[ \left( \frac{K}{\gamma} \sin (k_w z') + \theta_x \right) e_x + \left( -\frac{K}{\gamma} \cos (k_w z') + \theta_y \right) e_y \right] \tag{A8}
\]

where

\[
\Phi_T = \omega \left\{ \frac{z'}{2\gamma^2} \left[ 1 + K^2 + \gamma^2 (\theta_x^2 + \theta_y^2) \right] \right\} - \frac{K\theta_x \cos (k_w z') - K\theta_y \sin (k_w z')}{\gamma k_w} + \Phi_0 \tag{A9}
\]

with \(\Phi_0 = \omega \left\{ -(\theta_x l_x + \theta_y l_y) + \frac{z}{2} (\theta_x^2 + \theta_y^2) \right\} \). Further use of the resonant approximation leads to

\[
\tilde{E}_{\perp,s} = \frac{K \omega_0}{2\gamma z} \int_{-\infty}^{\infty} dl \int_{-\infty}^{\infty} dl' \int_{-\infty}^{\infty} dz' \tilde{\rho}(z', l, \omega) \exp \left\{ i\Phi_0 \right\} \exp \left\{ is \frac{\Delta\omega_s}{\omega_s} k_w z' \right\} \times \frac{1}{(s-1)!} \left( s \frac{K\gamma}{1 + K^2} \right)^{s-1} (\theta_y - i\theta_x)^{s-1} (e_x + ie_y) . \tag{A10}
\]

where \(s\) is the harmonic number (cf. with Sec.V), \(\omega = \omega_s + \Delta\omega_s\). A model for the single-electron emission is obtained by setting

\[
\tilde{\rho}(z, l, \omega) = \tilde{g}_0 (l) f(\omega), \tag{A11}
\]

for \(z\) in the range \((-L_w/2, L_w/2)\) and zero otherwise, \(L_w\) is the undulator length, and \(\tilde{g}_0 = e\delta(l)\) and \(f(t) = \delta(t - t_e) \rightarrow f(\omega) = \exp(i\omega t_e)\). The direct substitution explicitly gives

\[
\tilde{E}_{\perp,s} = -e \frac{K \omega}{2\gamma z} \exp \left[ i\frac{\omega_0}{2}(\theta_x^2 + \theta_y^2) \right] L_w \sin \left[ Cs_L_w/z + (\theta_x^2 + \theta_y^2) \omega_0 L_w/(4) \right] \times \frac{1}{(s-1)!} \left( s \frac{K\gamma}{1 + K^2} \right)^{s-1} (\theta_y - i\theta_x)^{s-1} (e_x + ie_y) . \tag{A12}
\]

where we have ignored the unimportant phase \(\omega t_e\). \(C_s\) is the so-called detuning parameter,

\[
C_s = \frac{\omega}{2\gamma z} - sk_w = s \frac{\omega - \omega_0}{\omega_0} k_w, \\
\omega_0 = 2sk_w \gamma_z^2, \gamma_z^2 = \frac{\gamma^2}{1 + K^2} . \tag{A13}
\]

Finally, we note that \(e_x + ie_y = -\sqrt{2} \chi_{+1} \) with \(\chi_{+1}\) from Eq. (A16). Going to cylindrical coordinates \(\{\theta_x, \theta_y\} = \theta \{\cos \phi, \sin \phi\}\) and an infinitely long undulator, \(L_w \to \infty\), we obtain in Eq. (A12) the same angular dependence and the same phase as for the non-linear Compton (or rather Thomson) scattering in the quasi-classical regime with the circularly polarized laser wave; see Eq. (13) in Sec.VB2 where \(\theta_k, \phi_k\) are analogous to \(\theta, \phi\), respectively.

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