Global well-posedness to stochastic reaction-diffusion equations on the real line $\mathbb{R}$ with superlinear drifts driven by multiplicative space-time white noise

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Abstract

Consider the stochastic reaction-diffusion equation with logarithmic nonlinearity driven by space-time white noise:

\[
\begin{aligned}
&du(t, x) = \frac{1}{2} \Delta u(t, x) \, dt + b(u(t, x)) \, dt \\
&\quad + \sigma(u(t, x)) \, W(dt, dx), \quad t > 0, \ x \in I, \\
&u(0, x) = u_0(x), \quad x \in I.
\end{aligned}
\]

When $I$ is a compact interval, say $I = [0, 1]$, the well-posedness of the above equation was established in [DKZ] (Ann. Prob. 47:1,2019). The case where $I = \mathbb{R}$ was left open. The essential obstacle is caused by the explosion of the supremum norm of the solution, $\sup_{x \in \mathbb{R}} |u(t, x)| = \infty$, making the usual truncation procedure invalid. In this paper, we prove that there exists a unique global solution to the stochastic reaction-diffusion equation on the whole real line $\mathbb{R}$ with logarithmic nonlinearity. Because of the nature of the nonlinearity, to get the uniqueness, we are forced to work with the first order moment of the solutions on the space $C_{tem}(\mathbb{R})$ with a specially designed norm

\[
\sup_{t \leq T, x \in \mathbb{R}} \left( |u(t, x)| e^{-\lambda|x|e^{\beta t}} \right),
\]

where, unlike the usual norm in $C_{tem}(\mathbb{R})$, the exponent also depends on time $t$ in a particular way. Our approach depends heavily on the new, precise lower order moment estimates of the stochastic convolution and a new type of Gronwall’s inequalities we obtained, which are of interest on their own right.

Keywords and Phrases: Stochastic reaction-diffusion equations, logarithmic nonlinearity, space-time white noise, stochastic convolution, lower order moment estimates.

AMS Subject Classification: Primary 60H15; Secondary 35R60.

Contents

1 Introduction

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1 Introduction

In this paper, we study the stochastic reaction-diffusion equation on the whole line $\mathbb{R}$ driven by multiplicative space-time white noise given as follows:

$$\begin{align*}
\frac{du(t, x)}{dt} &= \frac{1}{2} \Delta u(t, x) \, dt + b(u(t, x)) \, dt \\
&\quad + \sigma(u(t, x)) \, W(dt, dx), \quad t > 0, x \in \mathbb{R}, \\
 u(0, x) &= u_0(x), \quad x \in \mathbb{R}.
\end{align*}$$

The coefficients $b, \sigma$ are two deterministic measurable functions from $\mathbb{R}$ to $\mathbb{R}$, $W$ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$ defined on some filtrated probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

There exist numerous work in the literature on stochastic reaction-diffusion equations driven by space-time white noise covering a wide range of topics. We refer the reader to [DPZ], [C], [K] and references therein. The majority of the work are focused on stochastic reaction-diffusion equations defined on finite intervals (i.e., the space variable belongs to a fixed finite interval) instead of the whole real line $\mathbb{R}$, partly due to the essential difficulties brought by the non-compactness of the whole space. We like to mention some relevant existing work on the well-posedness of the stochastic reaction-diffusion equations on the real line. In the early paper [S], the author obtained the existence and uniqueness of solutions of stochastic reaction-diffusion equations on the real line under the Lipschitz conditions of the coefficients. Pathwise uniqueness were established in [MP] and [MPS] for stochastic reaction-diffusion equations on the real line with Hölder continuous coefficients.

It is well known that the equation (1.1) admits a unique global solution when the coefficients fulfill the usual Lipschitz condition, in particular, being of linear growth. We are concerned here with the well-posedness of the stochastic reaction-diffusion equation (1.1) with superlinear drift. Several papers in the literature discuss stochastic partial differential equations with locally Lipschitz coefficients that have polynomial growth and/or satisfy certain monotonicity conditions (see [C, DMP, LR], for instance). The typical example of such a coefficient is $b(u) = -u^3$, which has the effect of “pulling the solution back toward the origin.” In the joint paper [DKZ] with Dalang and Khoshnevisan by the second named author, stochastic reaction-diffusion equations(SRDEs) on finite intervals were considered and
it was proved that if the coefficients are locally Lipschitz and of \(|z| \log |z|\)-growth, then the SRDEs is globally well-posed. Unfortunately, the methods in [DKZ] are not valid for SRDEs on the whole line \(\mathbb{R}\) because typically the supremum norm of the solution explodes, i.e., \(|u(t)|_{\infty} = \sup_{x \in \mathbb{R}} |u(t, x)| = \infty\). The global well-posedness on the whole line \(\mathbb{R}\) under the logarithmic non-linearity was left open.

The goal of this article is to fill in this gap. More precisely, we prove that if the drift \(b\) is locally Log-Lipschitz and if \(|b(z)| = O(|z| \log |z|)\), then the stochastic reaction-diffusion equation (1.1) is globally well-posed. The precise statements are given in the next section. Because of the nature of the nonlinearity, we are forced to work with the first order moment of the solutions on the space \(C_{tem}(\mathbb{R})\) with a specially designed norm

\[
\sup_{t \leq T, x \in \mathbb{R}} \left( |u(t, x)| e^{-\lambda|x|e^{\beta t}} \right),
\]

where, unlike the usual norm in \(C_{tem}(\mathbb{R})\), the exponent also depends on time \(t\) in a particular way. We need to establish some new, precise (lower order) moment estimates of stochastic convolution on the real line and hence obtain some a priori estimates of the solution. We like to stress that it is harder to get precise lower order moment estimate than high order for stochastic convolutions. To obtain the pathwise uniqueness, one of the difficulties is that we are not able to apply the usual localization procedure as in the literature because the usual uniform norm of the solution on the real line explodes. To overcome the difficulty, we provide a new type of Gronwall’s inequalities, which is of independent interest.

Now we describe the content of the paper in more details. In Section 2, we present the framework for the stochastic reaction-diffusion equations driven by space-time white noise and state our main results. In Section 3, we will prove two Gronwall-type inequalities and obtain some estimates associated with the heat kernel of the Laplacian operator. In Section 4, we establish new lower order moment estimates of the stochastic convolution with respect to the space-time white noise and obtain a priori estimates of the solutions of the stochastic reaction-diffusion equations. In Section 5, we approximate the coefficients \(b\) and \(\sigma\) by smooth functions and establish the tightness of the laws of the solutions of the corresponding approximating equations. As a consequence, we obtain the existence of weak solution (in the probabilistic sense). Section 6 is devoted to the proof of the pathwise uniqueness of the stochastic reaction-diffusion equation under the local Log-Lipschitz conditions of the coefficients.

2 Statement of main results

Let us recall the following definition.

**Definition 2.1** A random field solution to equation (1.1) is a jointly measurable and adapted space-time process \(u := \{u(t, x) : (t, x) \in \mathbb{R}^+ \times \mathbb{R}\}\) such
that for every \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\),
\[
u(t, x) = P_t u_0(x) + \int_0^t \int \mathbb{R} p_{t-s}(x, y) b(u(s, y)) \, ds \, dy
+ \int_0^t \int \mathbb{R} p_{t-s}(u(s, y)) W(ds, dy), \quad \mathbb{P} - a.s., \tag{2.1}
\]
where \(p_t(x, y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}\), and \(\{P_t\}_{t \geq 0}\) is the corresponding heat semi-group on \(\mathbb{R}\).

**Remark 2.2** The above mild form is equivalent to the weak (in the sense of partial differential equations) formulation of the stochastic reaction-diffusion equations. We refer readers to [WA] for details.

We also recall the so-called \(C_{tem}\) space defined by
\[
C_{tem} := \left\{ f \in C(\mathbb{R}) : \sup_{x \in \mathbb{R}} |f(x)| e^{-\lambda |x|} < \infty \text{ for any } \lambda > 0 \right\},
\]
and endow it with the metric defined by
\[
d(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min \left\{ 1, \sup_{x \in \mathbb{R}} |f(x) - g(x)| e^{-\frac{1}{n} |x|} \right\},
\]
for any \(f, g \in C_{tem}\). Then \(f_n \to f\) in \(C_{tem}\) iff \(\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| e^{-\lambda |x|} \to 0\) as \(n \to \infty\) for any \(\lambda > 0\), and \((C_{tem}, d)\) is a Polish space.

Next we introduce the following conditions of nonlinear term \(b\). Set \(\log_+(u) := \log_+(1 \vee u)\) for any \(u \geq 0\).

(H1) \(b\) is continuous, and there exist two nonnegative constants \(c_1\) and \(c_2\) such that for any \(u \in \mathbb{R}\),
\[
|b(u)| \leq c_1 |u| \log_+ |u| + c_2. \tag{2.2}
\]

(H2) There exist nonnegative constants \(c_3, c_4, c_5\), such that for any \(u, v \in \mathbb{R}\),
\[
|b(u) - b(v)| \leq c_3 |u - v| \log_+ \frac{1}{|u - v|} + c_4 \log_+(|u| \vee |v|) |u - v| + c_5 |u - v|. \tag{2.3}
\]

Note that condition (H2) implies condition (H1). A typical example of function \(b\) that satisfies (H2) is given below.

**Example 2.3** The function \(x \mapsto x \log |x|\) satisfies the local log-Lipschitz condition (H2), more precisely, for any \(x, y \in \mathbb{R}\),
\[
|x \log |x| - y \log |y| | \leq |x - y| \log \frac{1}{|x - y|} + \log_+(|x| \vee |y|) + 1 + \log 2) |x - y|. \tag{2.4}
\]
Proof. Without loss of generality, we may assume $|y| \leq |x|$. We divide the proof into two cases.

Case 1: $|x| \leq 1$. We have

$$
|x \log |x| - y \log |y|| \leq |x - y| \log |x| + |y| \log |x| - \log |y|| \\
\leq |x - y| \log \frac{1}{|x|} + |x - y| \\
\leq |x - y| \left( \log \frac{1}{|x - y|} + \log 2 \right) + |x - y|, \tag{2.5}
$$

since $|x - y| \leq 2|x|$ leads to

$$
\log \frac{1}{|x|} \leq \log \frac{1}{|x - y|} + \log 2. \tag{2.6}
$$

Case 2. $|x| \geq 1$. We have

$$
|x \log |x| - y \log |y|| \leq |x - y| \log |x| + |y| \log |x| - \log |y|| \\
\leq |x - y| \times \log_+ (|x| \vee |y|) + |x - y|, \tag{2.7}
$$

Combining these two cases together yields (2.4). ■

Now we can state the main results of this paper.

**Theorem 2.4** Assume $u_0 \in C_{tem}$ and that (H1) is satisfied. If $\sigma$ is bounded and continuous, then there exists a weak (in the probabilistic sense) solution to the stochastic reaction-diffusion equation (1.1) with sample paths a.s. in $C(\mathbb{R}_+, C_{tem})$.

**Theorem 2.5** Assume $u_0 \in C_{tem}$ and that (H2) is satisfied. If $\sigma$ is bounded and Lipschitz, then the pathwise uniqueness holds for solutions of (1.1) in $C(\mathbb{R}_+, C_{tem})$. Hence there exists a unique strong solution to (1.1) in $C(\mathbb{R}_+, C_{tem})$.

## 3 Preliminaries

In this section, we will provide two Gronwall-type inequalities which play an important role in this paper. Moreover, we also present some estimates associated with the heat kernel of the Laplacian operator which will be used in our analysis later.

Lemma 3.1 is a slight modification of Theorem 3.1 in [W], and is proved in Lemma 7.2 of [SZ2]. We give a short proof here for completeness. Set $\log_+ (r) := \log (r \vee 1)$.

**Lemma 3.1** Let $X, a, c_1, c_2$ be nonnegative functions on $\mathbb{R}_+$, $M$ an increasing function with $M(0) \geq 1$. Moreover, suppose that $c_1, c_2$ be integrable on finite time intervals. Assume that for any $t \geq 0$,

$$
X(t) + a(t) \leq M(t) + \int_0^t c_1(s)X(s)\,ds + \int_0^t c_2(s)X(s) \log_+ X(s)\,ds, \tag{3.1}
$$
and the above integral is finite. Then for any \( t \geq 0 \),
\[
X(t) + a(t) \leq M(t)\exp(C_2(t)) \exp\left(\exp(C_2(t)) \int_0^t c_1(s) \exp(-C_2(s)) \, ds\right),
\]
where \( C_2(t) := \int_0^t c_2(s) \, ds \).

**Proof.** Fix any \( T > 0 \). Let
\[
Y(t) := M(T) + \int_0^t c_1(s) X(s) \, ds + \int_0^t c_2(s) X(s) \log_+ X(s) \, ds, \quad t \in [0, T].
\]
We see that \( Y \) is almost surely differentiable on \([0, T] \), \( Y(t) \geq 1 \) and
\[
X(t) + a(t) \leq Y(t), \quad \forall \ t \in [0, T].
\]
This leads to
\[
Y'(t) = c_1(t) X(t) + c_2(t) X(t) \log_+ X(t)
\leq c_1(t) Y(t) + c_2(t) Y(t) \log_+ Y(t)
= c_1(t) Y(t) + c_2(t) Y(t) \log Y(t).
\]
Thus,
\[
(\log Y)'(t) \leq c_1(t) + c_2(t) \log Y(t).
\]
Solving this ordinary differential inequality, we get for any \( t \in [0, T] \),
\[
\log Y(t) \leq \exp(C_2(t)) \left[ \log M(T) + \int_0^t c_1(s) \exp(-C_2(s)) \, ds \right].
\]
Therefore, we obtain
\[
X(T) + a(T) \leq Y(T)
\leq M(T)\exp(C_2(T)) \exp\left(\exp(C_2(T)) \int_0^T c_1(s) \exp(-C_2(s)) \, ds\right).
\]
By the arbitrariness of \( T \), (3.2) is deduced. ■

**Lemma 3.2** Let \( Y(t) \) be a nonnegative function on \( \mathbb{R}_+ \). Let \( c_1 \) and \( c_2 \) be non-negative, increasing functions on \( \mathbb{R}_+ \). Let \( \varepsilon \in [0, 1) \) be a constant and \( c_3 : \mathbb{R}_+ \times (\varepsilon, 1) \mapsto \mathbb{R}_+ \) be a function that is increasing with respect to the first variable. Suppose that for any \( \theta \in (\varepsilon, 1) \), the following integral inequality holds
\[
Y(t) \leq c_1(t) \int_0^t Y(s) \, ds + c_2(t) \int_0^t Y(s) \log_+ \frac{1}{Y(s)} \, ds + c_3(t, \theta) \int_0^t Y(s)^{\theta} \, ds, \quad \forall \ t \geq 0.
\]
If for any \( t > 0 \),
\[
\lim_{\theta \to 1^-} (1 - \theta) c_3(t, \theta) < \infty,
\]
then \( Y(t) = 0 \) for any \( t \geq 0 \). In particular, if \( c_3(t, \theta) \leq \frac{c(t)}{1-\theta} \) and \( c \) is an increasing function with respect to \( t \), then (3.8) holds.
Proof. It suffices to prove that for any $T > 0$, $Y(t) = 0$ on $[0, T]$. In order to prove this, let
\[
\delta_T := \limsup_{\theta \to 1^-} (1 - \theta)c_3(T, \theta),
\]
\[
T^* := \min \left\{ T, \frac{1}{3\delta_T}, \frac{e}{3c_2(T)} \right\}. \tag{3.9}
\]
Step 1. We first prove $Y(t) = 0$ for any $t \in [0, T^*]$. Since
\[
\sup_{x \geq 0} \left( x \log_e \frac{1}{x} \right) = \frac{1}{e}, \tag{3.10}
\]
we have
\[
Y(t) \leq c_1(t) \int_0^t Y(s) \, ds + \frac{c_2(t)}{1 - \theta} \int_0^t Y(s)^\theta Y(s)^{1-\theta} \log_e \frac{1}{Y(s)^{1-\theta}} \, ds
\]
\[
+ c_3(t, \theta) \int_0^t Y(s)^\theta \, ds
\]
\[
\leq c_1(t) \int_0^t Y(s) \, ds + \left[ \frac{c_2(t)}{e(1 - \theta)} + c_3(t, \theta) \right] \int_0^t Y(s)^\theta \, ds, \quad \forall t \geq 0. \tag{3.11}
\]
For $t \in [0, T]$, let
\[
\Phi(t) := c_1(T) \int_0^t Y(s) \, ds + \left[ \frac{c_2(T)}{e(1 - \theta)} + c_3(T, \theta) \right] \int_0^t Y(s)^\theta \, ds. \tag{3.12}
\]
Then $Y(t) \leq \Phi(t)$ for any $t \in [0, T]$. Thus,
\[
\frac{d}{dt} \Phi(t) = c_1(T)Y(t) + \left[ \frac{c_2(T)}{e(1 - \theta)} + c_3(T, \theta) \right] Y(t)^\theta
\]
\[
\leq c_1(T)\Phi(t) + \left[ \frac{c_2(T)}{e(1 - \theta)} + c_3(T, \theta) \right] \Phi(t)^\theta. \tag{3.13}
\]
Without loss of generality, we can assume that $\Phi(t) > 0$ for any $t \in (0, T]$, otherwise we can take the zero time to be $\min\{t : \Phi(t) > 0\}$. Multiplying $(1 - \theta)\Phi(t)^{-\theta}$ on both sides of the above inequality yields
\[
\frac{d}{dt} \left( \Phi(t)^{1-\theta} \right) \leq (1 - \theta)c_1(T)\Phi(t)^{1-\theta} + \left[ \frac{c_2(T)}{e} + c_3(T, \theta)(1 - \theta) \right]. \tag{3.14}
\]
Solving the above inequality, we obtain
\[
\Phi(t)^{1-\theta} \leq \left[ \frac{c_2(T)}{e} + c_3(T, \theta)(1 - \theta) \right] \int_0^t e^{(1-\theta)c_1(T)(t-s)} \, ds. \tag{3.15}
\]
Hence
\[
Y(t) \leq \Phi(t) \leq \left\{ \left[ \frac{c_2(T)T^*}{e} + (1 - \theta)c_3(T, \theta)T^* \right] e^{(1-\theta)c_1(T)T^*} \right\}^{1/(1-\theta)}
\]
\[
\leq e^{c_1(T)T^*} \left\{ \left[ \frac{c_2(T)}{e} + (1 - \theta)c_3(T, \theta)T^* \right] \right\}^{1/\theta}. \tag{3.16}
\]
for any $t \in [0, T^*]$. Letting $\theta \to 1$ in the above inequality and in view of the definition of $T^*$, we can see that

$$Y(t) = 0, \quad \forall t \in [0, T^*].$$ \hspace{1cm} (3.17)

Step 2. From (3.7) it follows that

$$Y(t + T^*) \leq c_1(T) \int_0^{t+T^*} Y(s) \, ds + c_2(T) \int_0^{t+T^*} Y(s) \log \frac{1}{Y(s)} \, ds$$

$$+ c_3(T, \theta) \int_0^{t+T^*} Y(s)^\theta \, ds$$

$$\leq c_1(T) \int_T^{T+T^*} Y(s) \, ds + c_2(T) \int_T^{T+T^*} Y(s) \log \frac{1}{Y(s)} \, ds$$

$$+ c_3(T, \theta) \int_T^{T+T^*} Y(s)^\theta \, ds$$

$$\leq c_1(T) \int_0^{t} Y(s + T^*) \, ds + c_2(T) \int_0^{t} Y(s + T^*) \log \frac{1}{Y(s)} \, ds$$

$$+ c_3(T, \theta) \int_0^{t} Y(s + T^*)^\theta \, ds, \quad t \in [0, T - T^*].$$ \hspace{1cm} (3.18)

By Step 1, we see that

$$Y(t + T^*) = 0, \quad t \in [0, T^* \wedge (T - T^*)],$$ \hspace{1cm} (3.19)

that is $Y(t) = 0$ for any $t \in [0, T \wedge 2T^*]$. Repeating this argument, we see that $Y(t) = 0$ for any $t \in [0, T]$. The arbitrariness of $T$ leads $Y(t) = 0$ for any $t > 0$. \blacksquare

Recall that $p_t(x, y) := \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$. Now we state some estimates of the heat kernel $p_t(x, y)$ used in this paper. For any $x \in \mathbb{R}$ and $t > 0$,

$$\int_{\mathbb{R}} p_t(x, y) e^{\eta |y|} \, dy \leq 2e^{\frac{x^2}{4t}} e^{\eta |x|}, \quad \forall \eta \in \mathbb{R},$$ \hspace{1cm} (3.20)

$$\int_{\mathbb{R}} p_t(x, y)^2 e^{\eta |y|} \, dy \leq \frac{1}{\sqrt{\pi t}} e^{\frac{x^2}{4t}} e^{\eta |x|}, \quad \forall \eta \in \mathbb{R},$$ \hspace{1cm} (3.21)

$$\int_{\mathbb{R}} p_t(x, y) e^{\eta |y|} \eta |y| \, dy \leq e^{\frac{x^2}{4t}} e^{\eta |x| |x|} + 2e^{\frac{x^2}{2t}} \left( \eta^2 t + \eta \sqrt{\frac{t}{2\pi}} \right) e^{\eta |x|}, \quad \forall \eta > 0.$$ \hspace{1cm} (3.22)

The above three estimates can be obtained through straightforward calculations. Moreover, the following lemma is needed.

**Lemma 3.3** The following estimates of the heat kernel $p_t(x, y)$ hold.

(i) For any $x, y \in \mathbb{R}$, $\theta \in [0, 1]$, $0 < s < t$,

$$|p_t(x, y) - p_s(x, y)| \leq \left(\frac{2\sqrt{2}}{s^\theta} \right) (p_s(x, y) + p_t(x, y) + p_{2t}(x, y)).$$ \hspace{1cm} (3.23)
(ii) For any \( x, y \in \mathbb{R} \) and \( t > 0 \),
\[
\int_{\mathbb{R}} |p_t(x, z) - p_t(y, z)| \, dz \leq \sqrt{\frac{2}{\pi}} \times \frac{|x - y|}{\sqrt{t}}. \tag{3.24}
\]

(iii) For any \( x, y \in \mathbb{R} \) and \( \eta, t > 0 \),
\[
\int_{\mathbb{R}} |p_t(x, z) - p_t(y, z)| e^{\eta |z|} \, dz \leq 2\sqrt{2} \times \frac{|x - y|}{\sqrt{t}} \times e^{\eta t} \times e^{\eta(|x| + |x - y|)}.
\tag{3.25}
\]

(iv) For any \( x, y \in \mathbb{R} \) and \( \eta, t > 0 \),
\[
\int_{\mathbb{R}} |p_t(x, z) - p_t(y, z)| e^{\eta |z|} |\eta| \, dz \\
\leq \frac{\sqrt{2}|x - y|}{\sqrt{t}} \times \left[ e^{\eta t} \times e^{\eta(|x| + |x - y|)} \eta(|x| + |x - y|) + 2e^{\eta t} \left( 2\eta^2 t + \eta \sqrt{\frac{t}{\pi}} \right) e^{\eta(|x| + |x - y|)} \right]. \tag{3.26}
\]

(v) For any \( x, y \in \mathbb{R} \) and \( 0 < s \leq t \),
\[
\int_{0}^{s} \int_{\mathbb{R}} |p_{t-r}(x, z) - p_{s-r}(y, z)|^2 \, dr \, dz \leq \frac{\sqrt{2} - 1}{\sqrt{\pi}} |t - s|^\frac{3}{2} + \frac{2}{\sqrt{\pi}} |x - y|. \tag{3.27}
\]

Proof. Proof of (i). On the one hand, by the mean value theorem, there exists some \( \xi \in [s, t] \) such that
\[
|p_t(x, y) - p_s(x, y)| \\
\leq \frac{1}{\sqrt{2\pi t}} \left| e^{-\frac{(x-y)^2}{2t}} - e^{-\frac{(x-y)^2}{2s}} \right| + \frac{1}{\sqrt{2\pi t}} - \frac{1}{\sqrt{2\pi s}} \left| e^{-\frac{(x-y)^2}{2s}} \right| \\
\leq \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2s}} \times \frac{(x-y)^2}{2s^2} \times |t - s| + \frac{1}{\sqrt{2\pi}} \times \frac{t - s}{2s^3/2} e^{-\frac{(x-y)^2}{2s}} \\
\leq 2\sqrt{2} \times \frac{|t - s|}{s} p_2 t(x, y) + \frac{|t - s|}{2s} p_s(x, y) \\
\leq \frac{2\sqrt{2}|t - s|}{s} (p_s(x, y) + p_t(x, y) + p_2 t(x, y)), \tag{3.28}
\]
where we have used \( z \leq e^z \) for any \( z \geq 0 \). On the other hand,
\[
|p_t(x, y) - p_s(x, y)| \leq p_s(x, y) + p_t(x, y) + p_2 t(x, y). \tag{3.29}
\]
Combining (3.28) and (3.29) together yields (i).
Due to the Fubini theorem, the Fubini theorem and (3.20) leads to

\[
|p_t(x, z) - p_t(y, z)| = \frac{1}{\sqrt{2\pi t}} \int_0^1 \frac{1}{|x - y|} \int_0^1 e^{-\frac{|x - y + \rho(y - x)|^2}{2t}} d\rho \leq \frac{|x - y|}{\sqrt{2\pi t}} \times \int_0^1 e^{-\frac{|x - z + \rho(y - x)|^2}{2t}} \times \frac{|x - z + \rho(y - x)|}{t} d\rho.
\] (3.30)

Proof of (ii). Obviously,

\[
|p_t(x, z) - p_t(y, z)| \leq \frac{2}{t} |x - y| \times \int_0^1 \frac{1}{\sqrt{2\pi t} \times 2t} e^{-\frac{|x - z + \rho(y - x)|^2}{4t}} d\rho.
\] (3.32)

The Fubini theorem and (3.20) leads to

\[
\int_\mathbb{R} |p_t(x, z) - p_t(y, z)| [e^{\eta|z|} |z| d\eta] dz \\
\leq \sqrt{\frac{2}{t}} |x - y| \int_0^1 \left( \frac{1}{\sqrt{2\pi t} \times 2t} e^{-\frac{|x - z + \rho(y - x)|^2}{4t}} e^{\eta|z|} |z| d\rho \right) dz \\
\leq \sqrt{\frac{2}{t}} |x - y| \times \int_0^1 \left( 2e^{\eta^2 t} \times e^{\eta|x + \rho(y - x)|} |d\rho \right) dz \\
\leq 2\sqrt{2} \times \frac{|x - y|}{\sqrt{t}} \times e^{\eta^2 t} \times e^{\eta(|x| + |x - y|)}.
\] (3.33)

Proof of (iv). By (3.32), the Fubini theorem and (3.22), we have

\[
\int_\mathbb{R} |p_t(x, z) - p_t(y, z)| [e^{\eta|z|} |z| d\eta] dz \\
\leq \sqrt{\frac{2}{t}} |x - y| \int_0^1 \left( \frac{1}{\sqrt{2\pi t} \times 2t} e^{-\frac{|x - z + \rho(y - x)|^2}{4t}} e^{\eta|z|} |z| d\rho \right) dz \\
\leq \sqrt{\frac{2}{t}} |x - y| \int_0^1 \left[ e^{\eta^2 t} \times e^{\eta|x + \rho(y - x)|} |x + \rho(y - x)| \\
+ 2e^{\eta^2 t} \left( 2\eta^2 t + \eta \sqrt{\frac{t}{\pi}} \right) e^{\eta|x + \rho(y - x)|} \right] d\rho \\
\leq \sqrt{\frac{2}{t}} |x - y| \times \left[ e^{\eta^2 t} \times e^{\eta(|x| + |x - y|)} \eta(|x| + |x - y|) \\
+ 2e^{\eta^2 t} \left( 2\eta^2 t + \eta \sqrt{\frac{t}{\pi}} \right) e^{\eta(|x| + |x - y|)} \right].
\] (3.34)
Proof of (iv). This inequality can be found in Lemma 6.2 of [S], here we just give the explicit constant by straightforward calculations.
This completes the proof of Lemma 3.3. ■

4 Moment estimates

In this section, we will establish estimates for moments of stochastic convolutions, and obtain some a priori estimates for solutions of equation (1.1).

We begin with the estimates of high order moments of stochastic convolutions. We stress that the precise lower order moment estimates are harder to get.

Lemma 4.1 Let \( h : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be an increasing function. Let \( \{\sigma(s, y) : (s, y) \in \mathbb{R}_+ \times [0, 1]\} \) be a random field such that the following stochastic convolution with respect to space time white noise is well defined. Let \( \tau \) be a stopping time. Then for any \( p > 10 \) and \( T > 0 \), there exists a constant \( C_{p,h(T),T} > 0 \) such that

\[
E \sup_{(t, x) \in [0, T \wedge \tau] \times \mathbb{R}} \left\{ \left| \int_0^t \int_\mathbb{R} p_{t-s}(x, y)\sigma(s, y) W(ds, dy) e^{-h(t)|x|} \right| \right\}^p \leq C_{p,h(T),T} \int_0^{T \wedge \tau} \int_\mathbb{R} |\sigma(t, x)|^p e^{-ph(t)|x|} dx dt.
\]

(4.1)

In particular, if \( \sigma \) is bounded and \( h \) is a positive constant, then the left hand side of (4.1) is finite.

Proof. We employ the factorization method (see e.g. [DPZ]). The proof here is inspired by [SZ1]. Choose \( \frac{3}{2p} < \alpha < \frac{1}{4} - \frac{1}{p} \). This is possible because we assume \( p > 10 \). Let

\[
(J_\alpha \sigma)(s, y) : = \int_0^s \int_\mathbb{R} (s - r)^{-\alpha} p_{s-r}(y, z)\sigma(r, z) W(dr, dz),
\]

(4.2)

\[
(J^{\alpha-1} f)(t, x) : = \frac{\sin \pi \alpha}{\pi} \int_0^t \int_\mathbb{R} (t - s)^{\alpha-1} p_{t-s}(x, y) f(s, y) ds dy.
\]

(4.3)

From the stochastic Fubini theorem (see Theorem 2.6 in [W]), it follows that for any \( (t, x) \in \mathbb{R}_+ \times \mathbb{R} \),

\[
\int_0^t \int_\mathbb{R} p_{t-s}(x, y)\sigma(s, y) W(ds, dy) = J^{\alpha-1}(J_\alpha \sigma)(t, x).
\]

(4.4)

By H"older’s inequality and (3.21), we have

\[
E \sup_{(t, x) \in [0, T \wedge \tau] \times \mathbb{R}} \left\{ \left| \int_0^t \int_\mathbb{R} p_{t-s}(x, y)\sigma(s, y) W(ds, dy) e^{-h(t)|x|} \right| \right\}^p = E \sup_{(t, x) \in [0, T \wedge \tau] \times \mathbb{R}} \left\{ \frac{\sin \pi \alpha}{\pi} \int_0^t \int_\mathbb{R} (t - s)^{\alpha-1} p_{t-s}(x, y) J_\alpha \sigma(s, y) ds dy e^{-h(t)|x|} \right\}^p
\]
\[
\begin{align*}
\leq & \left| \frac{\sin \pi \alpha}{\pi} \right|^p \mathbb{E} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \left\{ \int_0^t (t-s)^{\alpha-1} \times \left( \int_{\mathbb{R}} p_{t-s}(x,y) |J_\alpha \sigma(s,y)| \, dy \right)^p e^{-ph(t)|x|} \right\} \\
\leq & \left| \frac{\sin \pi \alpha}{\pi} \right|^p \mathbb{E} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \left\{ \int_0^t (t-s)^{\alpha-1} \times \left( \int_{\mathbb{R}} p_{t-s}(x,y) e^{\frac{ph(t)}{2}|y| e^{-\frac{ph(t)}{2}|y|} |J_\alpha \sigma(s,y)| \frac{p}{2} \, dy \right)^\frac{2}{p} \, ds \right\}^p e^{-ph(t)|x|} \\
\leq & \left| \frac{\sin \pi \alpha}{\pi} \right|^p \mathbb{E} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \left\{ \int_0^t (t-s)^{\alpha-1} \left( \int_{\mathbb{R}} p_{t-s}(x,y)^2 e^{ph(t)|y|} \, dy \right)^\frac{1}{2} \times \frac{1}{p} \, ds \right\}^p e^{-ph(t)|x|} \\
\leq & \left| \frac{\sin \pi \alpha}{\pi} \right|^p \frac{1}{\sqrt{\pi}} e^{\frac{ph(T)^2}{4}} \times \mathbb{E} \sup_{t \in [0,T]} \left\{ \int_0^t (t-s)^{\alpha-1-\frac{1}{2p}} \times \left( \int_{\mathbb{R}} |J_\alpha \sigma(s,y)| e^{-ph(t)|y|} \, dy \right)^{\frac{1}{2}} \, ds \right\}^p e^{-ph(t)|x|} \\
\leq & \left| \frac{\sin \pi \alpha}{\pi} \right|^p \frac{1}{\sqrt{\pi}} e^{\frac{ph(T)^2}{4}} \times \left( \int_0^T s^{(\alpha-1-\frac{1}{2p}) \frac{p}{p-1}} \, ds \right)^{p-1} \times \left( \int_0^T \int_{\mathbb{R}} \mathbb{E} \left[ |J_\alpha \sigma(s,y)|^p 1_{[0,T]}(s) \right] e^{-ph(s)|y|} \, dy \, ds \right)^{\frac{p-1}{p}} \\
\leq & C'_{p,h(T),T,\alpha} \int_0^T \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^s \int_{\mathbb{R}} (s-r)^{-\alpha} p_{s-r}(y,z) \sigma(r,z) W(dr,dz) \right]^p e^{-ph(s)|y|} \, dy \, ds,
\end{align*}
\]

where we have used the condition \( \alpha > \frac{3}{2p} \) in the last inequality, so that

\[
C'_{p,h(T),T,\alpha} = \left| \frac{\sin \pi \alpha}{\pi} \right|^p \frac{1}{\sqrt{\pi}} e^{\frac{ph(T)^2}{4}} \times \left( \int_0^T s^{(\alpha-1-\frac{1}{2p}) \frac{p}{p-1}} \, ds \right)^{p-1} \times \left( \int_0^T \int_{\mathbb{R}} \mathbb{E} \left[ |J_\alpha \sigma(s,y)|^p 1_{[0,T]}(s) \right] e^{-ph(s)|y|} \, dy \, ds \right)^{\frac{p-1}{p}} T^{\alpha p - \frac{3}{2}}.
\]

For any fixed \( (s,y) \in [0,T] \times \mathbb{R} \), let

\[
M_t := \int_0^t \int_{\mathbb{R}} (s-r)^{-\alpha} p_{s-r}(y,z) \sigma(r,z) 1_{[0,T]}(r) W(dr,dz), \quad t \in [0,s].
\]
Then it is easy to see that \( \{M_t\}_{t \in [0,s]} \) is a martingale. Applying the Bukholder-Davis-Gundy inequality (see Proposition 4.4 in [K] and also [W]), we have for \( t \in [0,s] \),

\[
\mathbb{E}|M_t|^p \leq (4p)^{\frac{p}{2}} \mathbb{E}\langle M \rangle_t^{\frac{p}{2}}
\]

\[
= (4p)^{\frac{p}{2}} \mathbb{E}\left( \int_0^{t \wedge \tau} \int_{\mathbb{R}} (s - r)^{-2\alpha} p_{s-r}(y, z)^2 \sigma(r, z)^2 \, dr \, dz \right)^{\frac{p}{2}}. \tag{4.8}
\]

Hence by the local property of the stochastic integral (see Lemma A.1 in Appendix of [SZ1]), we get

\[
\mathbb{E}\left[ \int_0^s \int_{\mathbb{R}} (s - r)^{-\alpha} p_{s-r}(y, z) \sigma(r, z) W(dr, dz) \right]_{1[0,\tau](s)}^p = \mathbb{E}\left[ \int_0^s \int_{\mathbb{R}} (s - r)^{-\alpha} p_{s-r}(y, z) \sigma(r, z) W(dr, dz) \right]_{1[0,\tau](s)}^p \leq \mathbb{E}\left[ \int_0^s \int_{\mathbb{R}} (s - r)^{-2\alpha} p_{s-r}(y, z)^2 \sigma(r, z)^2 \, dr \, dz \right]^{\frac{p}{2}}. \tag{4.9}
\]

Note that \( p_t(x, y) \leq (2\pi t)^{-\frac{1}{2}} \) for any \( t > 0 \) and \( x, y \in \mathbb{R} \). Using (4.9), Hölder’s inequality and the Fubini theorem, we get

\[
\int_0^T \int_{\mathbb{R}} \mathbb{E}\left[ \int_0^s \int_{\mathbb{R}} (s - r)^{-2\alpha} p_{s-r}(y, z)^2 \sigma(r, z)^2 \, dr \, dz \right]^{\frac{p}{2}} e^{-ph(s)|y|} dy ds
\]

\[
\leq (4p)^{\frac{p}{2}} \int_0^T \int_{\mathbb{R}} \mathbb{E}\left\{ \int_0^s \int_{\mathbb{R}} (s - r)^{-2\alpha} p_{s-r}^2(y, z) \sigma^2(r, z) \, dr \, dz \right\}^{\frac{p}{2}} e^{-ph(s)|y|} dy ds
\]

\[
\leq \left( \frac{4p}{\sqrt{2\pi}} \right)^{\frac{p}{2}} \int_0^T \int_{\mathbb{R}} \mathbb{E}\left[ \left\{ \int_0^T \left[ \int_{\mathbb{R}} (s - r)^{-2\alpha} \frac{1}{2} p_{s-r}^2(y, z) \, dr \right]^{\frac{p}{2}} e^{-ph(s)|y|} dy \right\} \right]^{\frac{p}{2}} ds
\]

\[
\leq C_{p,h(T),T,\alpha}^{''''} \int_0^T \int_{\mathbb{R}} \left| \sigma(r, z) \right|^p e^{-ph(r)|z|} \, dz dr, \tag{4.10}
\]

where (3.20) and the condition \( \alpha < \frac{1}{4} - \frac{1}{p} \) are used to see that

\[
C_{p,h(T),T,\alpha}^{''''} = \left( \frac{4p}{\sqrt{2\pi}} \right)^{\frac{p}{2}} \times T \times \left( \int_0^T r^{-\left(2\alpha + \frac{1}{2}\right)\frac{p}{2}} dr \right)^{\frac{p-2}{2}} \times 2e^{\frac{2h(T)^2}{p}}
\]

\[
= \left( \frac{4p}{\sqrt{2\pi}} \right)^{\frac{p}{2}} \times \left( \frac{p-2}{2-2-2\alpha p} \right)^{\frac{p-2}{2}} T^{\frac{p}{2}-\alpha p} \times 2e^{\frac{2h(T)^2}{p}}. \tag{4.11}
\]
Combining (4.5) with (4.10), we obtain
\[
\mathbb{E} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y) \sigma(s,y) W(ds,dy) e^{-h(t)|x|} \right|^p \right\} \\
\leq C_{p,h(T),T} \mathbb{E} \int_0^{T \wedge \tau} \left| \sigma(r,z) \right|^p e^{-ph(r)|z|} \, dz \, dr,
\]
where
\[
C_{p,h(T),T} = \min_{\frac{3}{p} < \alpha < \frac{3}{p} + 1} C'_{p,h(T),T,\alpha} \times C''_{p,h(T),T,\alpha}.
\]

In view of (4.6) and (4.11), a straightforward calculation leads to
\[
C_{p,h(T),T} < 2\sqrt{2} p^2 \left( \frac{2}{\pi} \right)^{\frac{2}{p}} \left( \frac{1}{\sqrt{2\pi}} \right)^{\frac{2}{p}+1} \left( \frac{6p-8}{p-10} \right)^{\frac{2p-2}{2}} T^{\frac{p}{2}} e^{\frac{3}{4}h(T)^2}.
\]

**Proposition 4.2** Let \( h : \mathbb{R} \rightarrow \mathbb{R} \) be an increasing function. Let \( \{ \sigma(s,y) : (s,y) \in \mathbb{R} \times [0,1] \} \) be a random field such that the following stochastic convolution with respect to the space time white noise is well defined. Let \( \tau \) be a stopping time. Then for any \( \epsilon, T > 0 \), and \( 0 < p \leq 10 \), there exists a constant \( C_{\epsilon,p,h(T),T} \) such that
\[
\mathbb{E} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} \right| p_{t-s}(x,y) \sigma(s,y) W(ds,dy) e^{-h(t)|x|} \right\}^p \\
\leq \epsilon \mathbb{E} \sup_{(t,x) \in [0,T] \times \mathbb{R}} \left( \left| \sigma(t,x) \right| e^{-h(t)|x|} \right)^p \\
+ C_{\epsilon,p,h(T),T} \mathbb{E} \int_0^{T \wedge \tau} \left| \sigma(t,x) \right|^p e^{-ph(t)|x|} \, dx \, dt.
\]

**Remark 4.3** The constant \( C_{\epsilon,p,h(T),T} \) is increasing with respect to \( T \) and \( C_{\epsilon,p,h(0),0} = 0 \).

**Proof.** The following proof is inspired by [SZ1]. The proof is divided into two steps.

**Step 1.** We first show that for any \( \rho, T > 0 \) and \( q > 10 \),
\[
\mathbb{P} \left( \sup_{(t,x) \in [0,T] \times \mathbb{R}} \left[ \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y) \sigma(s,y) W(ds,dy) e^{-h(t)|x|} \right] > \rho \right) \\
\leq \mathbb{P} \left( \int_0^{T \wedge \tau} \left| \sigma(s,y) \right|^q e^{-qh(s)|y|} \, dy ds > \rho^q \right) \\
+ \frac{C_{q,h(T),T}}{\rho^q} \mathbb{E} \min \left\{ \rho^q, \int_0^{T \wedge \tau} \left| \sigma(s,y) \right|^q e^{-qh(t)|y|} \, dy ds \right\}.
\]

Here the constant \( C_{q,h(T),T} \) is the constant \( C_{p,h(T),T} \) in (4.1) with \( p \) replaced by \( q \).
To prove (4.16), we set
\[ \Omega_\rho := \left\{ \omega \in \Omega : \int_0^{T \wedge \tau} \int_{\mathbb{R}} |\sigma(s,y)|^q e^{-qh(s)|y|} \, dy \, ds \leq \rho^q \right\}. \quad (4.17) \]

By Chebyshev’s inequality, we have
\[
\begin{align*}
&\mathbb{P} \left( \sup_{(t,x) \in [0,T \wedge \tau] \times \mathbb{R}} \left[ \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\sigma(s,y) \, W(ds,dy) \right] e^{-h(t)|x|} > \rho \right) \\
\leq &\mathbb{P} \left( \Omega \setminus \Omega_\rho \right) + \mathbb{P} \left( \sup_{(t,x) \in [0,T \wedge \tau] \times \mathbb{R}} \left[ \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\sigma(s,y) \, W(ds,dy) \right] \mathbb{1}_{\Omega_\rho} e^{-h(t)|x|} > \rho \right) \\
\leq &\mathbb{P} \left( \int_0^{T \wedge \tau} \int_{\mathbb{R}} |\sigma(s,y)|^q e^{-qh(s)|y|} \, dy \, ds > \rho^q \right) \\
&+ \frac{1}{\rho^q} \mathbb{E} \sup_{(t,x) \in [0,T \wedge \tau] \times \mathbb{R}} \left\{ \mathbb{1}_{\Omega_\rho} \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\sigma(s,y) \, W(ds,dy) \right\} e^{-h(t)|x|}. \\
& \quad (4.18)
\end{align*}
\]

Now, we introduce the random field
\[ \tilde{\sigma}(s,y) := \sigma(s,y)\mathbb{1}_{\{\omega \in \Omega : \int_0^{T \wedge \tau} \int_{\mathbb{R}} |\sigma(r,y)|^q e^{-qh(r)|y|} \, dy \, dr \leq \rho^q \}}. \quad (4.19) \]

Note that the stochastic integral of \( \tilde{\sigma}(\cdot, \cdot) \) with respect to the space time white noise \( W \) is well defined. By the local property of the stochastic integral (see Lemma A.1 in Appendix of [SZ1]),
\[
\mathbb{1}_{\Omega_\rho} \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\tilde{\sigma}(s,y) \, W(ds,dy) \\
= \mathbb{1}_{\Omega_\rho} \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\sigma(s,y) \, W(ds,dy), \quad \mathbb{P} - a.s.. \quad (4.20)
\]

Hence using the bound (4.1), we get
\[
\begin{align*}
&\mathbb{E} \sup_{(t,x) \in [0,T \wedge \tau] \times \mathbb{R}} \left\{ \mathbb{1}_{\Omega_\rho} \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\sigma(s,y) \, W(ds,dy) \right\}^q e^{-qh(t)|x|} \\
\leq &\mathbb{E} \sup_{(t,x) \in [0,T \wedge \tau] \times \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\tilde{\sigma}(s,y) \, W(ds,dy) \right\}^q e^{-qh(t)|x|} \\
\leq &C_{q,h(T),T} \mathbb{E} \int_0^{T \wedge \tau} |\tilde{\sigma}(s,y)|^q e^{-qh(s)|y|} \, ds \\
\leq &C_{q,h(T),T} \mathbb{E} \min \left\{ \rho^q, \int_0^{T \wedge \tau} |\sigma(s,y)|^q e^{-qh(s)|y|} \, dy \, ds \right\}. \quad (4.21)
\end{align*}
\]

Combining (4.18) with (4.21), we obtain (4.16).

Step 2. Let now \( 0 < p \leq 10 \). From (4.16) and Lemma A.2 in Appendix of [SZ1], it follows that
\[
\mathbb{E} \sup_{(t,x) \in [0,T \wedge \tau] \times \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_{t-s}(x,y)\sigma(s,y) \, W(ds,dy) \right\}^p e^{-h(t)|x|}. \]
where \( \lambda, \kappa > 0 \) defined above. We have used the following Young inequality

\[
\int_0^\infty pp^{p-1}\mathbb{P} \left( \sup_{(t,x)\in[0,T\wedge \tau] \times \mathbb{R}} \left[ \int_0^t \int_\mathbb{R} |\sigma(s,y)|^q e^{-q \rho h(s)} |y| \, dy \, ds + \rho^q \right] > \rho \right) \, d\rho
\]

\[
\leq \int_0^\infty pp^{p-1}\mathbb{P} \left( \int_0^{T\wedge \tau} \int_\mathbb{R} |\sigma(s,y)|^q e^{-q \rho h(s)} |y| \, dy \, ds > \rho^q \right) \, d\rho
\]

\[
+ C_{q,h(T),T} \int_0^\infty pp^{p-1-q} \mathbb{E} \left\{ \rho^q \int_0^{T\wedge \tau} \int_\mathbb{R} |\sigma(s,y)|^q e^{-q \rho h(s)} |y| \, dy \, ds \right\} \, d\rho
\]

\[
= C_{p,q,h(T),T} \mathbb{E} \left( \sup_{(s,y)\in[0,T\wedge \tau] \times \mathbb{R}} \left( |\sigma(s,y)| e^{-h(s)} |y| \right)^{\frac{q-p}{q}} \times \left( \int_0^{T\wedge \tau} \int_\mathbb{R} |\sigma(s,y)|^p e^{-p h(s)} |y| \, dy \, ds \right)^{\frac{\rho}{q}} \right)
\]

\[
\leq \epsilon \mathbb{E} \sup_{(s,y)\in[0,T\wedge \tau] \times \mathbb{R}} \left( |\sigma(s,y)| e^{-h(s)} |y| \right)^{p}
\]

\[
+ C_{p,q,h(T),T} \times C_{e,p,q,h(T),T} \mathbb{E} \int_0^{T\wedge \tau} \int_\mathbb{R} |\sigma(s,y)|^p e^{-p h(s)} |y| \, dy \, ds,
\] (4.22)

where

\[
C_{p,q,h(T),T} := 1 + \frac{q}{q-p} C_{q,h(T),T},
\] (4.23)

and we have used the following Young inequality

\[
ab \leq \epsilon \mathbb{E} C_{p,q,h(T),T} a^{\frac{q}{q-p}} + C_{e,p,q,h(T),T} b^{\frac{q}{q}}, \quad \forall a, b > 0,
\] (4.24)

\[
C_{e,p,q,h(T),T} := p \left( \frac{q-p}{\epsilon / C_{p,q,h(T),T}} \right)^{\frac{q-p}{q}}.
\] (4.25)

Set

\[
C_{e,p,h(T),T} := \inf_{q>10} C_{p,q,h(T),T} \times C_{e,p,q,h(T),T}.
\] (4.26)

Combining (4.23) and (4.24) gives

\[
C_{e,p,h(T),T} = \inf_{q>10} \left\{ \frac{p}{q-p} \epsilon^{1-\frac{a}{q}} \left( q - p + q C_{q,h(T),T} \right)^{\frac{q}{p}} \right\}.
\] (4.26)

where the constant \( C_{q,h(T),T} \) is bounded by the right hand side of (4.14) with \( p \) replaced by \( q \). Now, (4.15) follows from (4.22) with the constant \( C_{e,p,h(T),T} \) defined above.

Next, we will establish an a priori estimate of solutions to (1.1). Throughout this paper, we will use the following notations. For \( \lambda, \kappa > 0 \), set

\[
\beta(\lambda, \kappa) := \max \left\{ \frac{\lambda^2}{2}, 4\kappa \right\},
\] (4.27)

\[
T^*(\lambda, \kappa) := \frac{1}{2\beta(\lambda, \kappa)} \left[ 1 + \log \left( \frac{4\beta(\lambda, \kappa)}{\lambda^2 \log \frac{\beta(\lambda, \kappa)}{2\kappa}} \right) \right].
\] (4.28)

It is easy to see that for any \( \kappa > 0 \), \( T^*(\lambda, \kappa) \to \infty \) as \( \lambda \to 0 \).
Lemma 4.4 Assume that (H1) is satisfied and \( \sigma \) is bounded. Let \( u \) be a solution of (1.1). Set also
\[
V(t, x) := \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) \sigma(u(s, y)) W(ds, dy). \tag{4.29}
\]
Then for any \( \lambda > 0 \) and \( T \leq T^*(\lambda, c_1) \), there exists a constant \( C_{\lambda, c_1, T} \) such that the following a priori estimate holds for \( \mathbb{P} \)-a.s.,
\[
\sup_{t \leq T, x \in \mathbb{R}} \left( |u(t, x)| e^{-\lambda|x|\beta t} \right) 
\leq C_{\lambda, c_1, T} \times \left\{ 1 + 2e^2 \sup_{x \in \mathbb{R}} |u_0(x)| e^{-\lambda|x|} \right\} 
+ 2 \sup_{(t, x) \in [0, T] \times \mathbb{R}} \left( |V(t, x)| e^{-\lambda|x|} \right) 
+ e^{4e_1 T} e^{2\beta T - 1}, \tag{4.30}
\]
where we write \( \beta \) instead of \( \beta(\lambda, c_1) \) for simplicity, and the constant \( c_1 \) is same as that in condition (H1).

Remark 4.5 Lemma 4.4 actually implies that the solutions of (1.1) don’t blow up in the space \( C_{\text{tem}} \), since we can take sufficiently small \( \lambda > 0 \) such that \( T^*(\lambda, c_1) \) can be larger than any given number.

Proof. Set
\[
U(T) := \sup_{(t, x) \in [0, T] \times \mathbb{R}} \left( |u(t, x)| e^{-\lambda|x|\beta t} \right).
\]
From (2.1), we have
\[
U(T) \leq \sup_{t \leq T, x \in \mathbb{R}} \left( |P_t u_0(x)| e^{-\lambda|x|\beta t} \right) + \sup_{t \leq T, x \in \mathbb{R}} \left( |V(t, x)| e^{-\lambda|x|} \right) 
+ \sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_0^T \int_{\mathbb{R}} p_{t-s}(x, y)b(u(s, y)) \, ds \, dy \, e^{-\lambda|x|\beta t} \right\}. \tag{4.31}
\]
Now we estimate the three terms on the right hand side of the above inequality.
\[
\sup_{t \leq T, x \in \mathbb{R}} \left( |P_t u_0(x)| e^{-\lambda|x|\beta t} \right) 
\leq \left\{ \int_{\mathbb{R}} p_t(x, y) u_0(y) \, dy \right\} e^{-\lambda|x|},
\]
\[
\leq \sup_{y \in \mathbb{R}} \left( |u_0(y)| e^{-\lambda|y|} \right) \times \sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_{\mathbb{R}} p_t(x, y) e^{\lambda|y|} \, dy \times e^{-\lambda|x|} \right\},
\]
\[
\leq 2e^{2\beta T} \sup_{y \in \mathbb{R}} \left( |u_0(y)| e^{-\lambda|y|} \right), \tag{4.32}
\]
where we have used (3.20). Applying Lemma 4.1 and using the boundedness of \( \sigma \), we get that for any \( p, q, T > 0 \),
\[
\mathbb{E} \sup_{t \leq T, x \in \mathbb{R}} \left( |V(t, x)|^p e^{-q|x|} \right) < \infty. \tag{4.33}
\]
In particular,
\[
\sup_{t \leq T, x \in \mathbb{R}} \left( |V(t, x)| e^{-\lambda|x|} \right) < +\infty, \quad \mathbb{P} - a.s.. \tag{4.34}
\]

The nonlinear term can be estimated as follows. By (2.2) and \( \log_+ (ab) \leq \log_+ a + \log_+ b \) for any \( a, b > 0 \), we have

\[
\begin{aligned}
&\sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_t-s(x, y) b(u(s, y)) \, ds dy \right\} e^{-\lambda|x| e^\beta t} \\
\leq & \sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_t-s(x, y) \left( c_1 |u(s, y)| \log_+ |u(s, y)| + c_2 \right) \, ds dy \times e^{-\lambda|x| e^\beta t} \right\} \\
\leq & c_2 T + c_1 \sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_t-s(x, y) e^{\lambda|y| e^\beta s} \times \left( |u(s, y)| e^{-\lambda|y| e^\beta s} \right) \right. \\
& \left. \times \log_+ \left( \left( |u(s, y)| e^{-\lambda|y| e^\beta s} \right) \times e^{\lambda|y| e^\beta s} \right) \, ds dy \times e^{-\lambda|x| e^\beta t} \right\} \\
\leq & c_2 T + c_1 \sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_t-s(x, y) e^{\lambda|y| e^\beta s} \, dy ds \times e^{-\lambda|x| e^\beta t} \right\} \\
& + c_1 \sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_0^t \sup_{y \in \mathbb{R}} \left( |u(s, y)| e^{-\lambda|y| e^\beta s} \right) \\
& \times \int_{\mathbb{R}} p_t-s(x, y) e^{\lambda|y| e^\beta s} \times |y| e^\beta s \, dy ds \times e^{-\lambda|x| e^\beta t} \right\} \\
= : & c_2 T + I + II. \tag{4.35}
\end{aligned}
\]

Note that the function \( x \mapsto x \log_+ x \) is increasing on \([0, \infty)\), so we have

\[
I \leq c_1 \sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_0^t \sup_{y \in \mathbb{R}, r \leq s} \left[ \left( |u(r, y)| e^{-\lambda|y| e^\beta r} \right) \times \log_+ \left( |u(r, y)| e^{-\lambda|y| e^\beta r} \right) \right] \\
\times 2 e^{\frac{\lambda^2 c_2 (t-s) e^{2\beta s}}{2}} e^{\lambda|x| e^\beta d} \, ds \times e^{-\lambda|x| e^\beta t} \right\} \\
\leq 2 c_1 \sup_{t \leq T} \left\{ \sup_{s \leq t} \left( e^{\frac{\lambda^2 c_2 (t-s) e^{2\beta s}}{2}} \right) \int_0^t U(s) \log_+ U(s) \, ds \right\} \\
\leq 2 c_1 e^{\frac{\lambda^2 c_2 T - 1}{2}} \int_0^T U(s) \log_+ U(s) \, ds, \tag{4.36}
\]

where we have used (3.20) and

\[
\max_{s \in [0, t]} e^{\frac{\lambda^2 c_2 (t-s) e^{2\beta s}}{2}} = e^{\frac{\lambda^2 c_2 T - 1}{2}}. \tag{4.37}
\]

For the term \( II \), we estimate as follows

\[
II \leq c_1 \sup_{t \leq T, x \in \mathbb{R}} \left\{ \int_0^t \sup_{y \in \mathbb{R}} \left( |u(s, y)| e^{-\lambda|y| e^\beta s} \right) \\
\times \int_{\mathbb{R}} p_t-s(x, y) e^{\lambda|y| e^\beta s} \, dy ds \times e^{-\lambda|x| e^\beta t} \right\}
\]

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\[
\begin{align*}
\lambda^{2}(|x|^{2} & + |y|^{2} + C_{\lambda,\beta,t}e^{\lambda|x|e^{\beta s}}) ds \times e^{-\lambda|y|^{2}}
\leq c_{1} \sup_{t \leq T, x \in \mathbb{R}} \left\{ \sup_{s \leq t, y \in \mathbb{R}} \left( |u(s, y)| e^{-\lambda|y|e^{\beta s}} \right) \times \frac{1}{\beta} \sup_{s \leq t} \left( e^{\lambda^{2}(|x|^{2}+|y|^{2})} \right) \right. \\
& \left. \times \int_{0}^{t} \frac{d}{ds} e^{\lambda|x|e^{\beta s}} ds \times e^{-\lambda|y|e^{\beta t}} \right\} \\
& + c_{1} \sup_{t \leq T} \left\{ C_{\lambda,\beta,t} \int_{0}^{t} \sup_{s \leq r, y \in \mathbb{R}} \left( |u(r, y)| e^{-\lambda|y|e^{\beta r}} \right) ds \right\} \\
& \leq \frac{c_{1}}{\beta} e^{\frac{\lambda^{2}}{\beta^{2}} \beta^{2} T - 1} U(T) + C_{\lambda,c_{1},T} \int_{0}^{T} U(s) ds, \quad (4.38)
\end{align*}
\]

where we have used (3.22), (4.37), and that the constant \( C_{\lambda,\beta,t} \) is increasing with respect to \( t > 0 \). Note that

\[
\frac{c_{1}}{\beta} e^{\frac{\lambda^{2}}{\beta^{2}} \beta^{2} T - 1} \leq \frac{1}{2} \iff T \leq T^{*}(\lambda, c_{1}) = \frac{1}{2\beta} \left[ 1 + \log \left( \frac{4\beta}{\lambda^{2}} \log \frac{\beta}{2c_{1}} \right) \right].
\]

Hence for \( T \leq T^{*}(\lambda, c_{1}) \),

\[
II \leq \frac{1}{2} U(T) + c_{1} C_{\lambda,\beta,T} \int_{0}^{T} U(s) ds. \quad (4.40)
\]

Combining (4.31), (4.32), (4.34), (4.35), (4.36) and (4.40) together, we obtain that for \( T \leq T^{*}(\lambda, c_{1}) \),

\[
U(T) \leq 2e^{\frac{\lambda^{2}}{2} T} \sup_{y \in \mathbb{R}} \left( |u_{0}(y)| e^{-\lambda|y|} \right) + \sup_{t \leq T, x \in \mathbb{R}} \left( |V(t, x)| e^{-\lambda|x|} \right) \\
+ c_{2} T + 2c_{1} e^{\frac{\lambda^{2}}{\beta^{2}} \beta^{2} T - 1} \int_{0}^{T} U(s) \log_{+} U(s) ds \\
+ \frac{1}{2} U(T) + C_{\lambda,c_{1},T} \int_{0}^{T} U(s) ds. \quad (4.41)
\]

Subtracting \( \frac{1}{2} U(T) \) on both sides of the above inequality, and then applying the log Gronwall inequality (see Lemma 3.1), (4.30) is deduced.

## 5 Existence of weak solutions

In this section, we assume that (H1) is satisfied and that \( \sigma \) is bounded, continuous. We will approximate the coefficients \( b \) and \( \sigma \) by Lipschitz continuous functions and establish the existence of weak solutions of the stochastic reaction-diffusion equation.

Let \( \varphi \) be a nonnegative smooth function on \( \mathbb{R} \) such that the support of \( \varphi \) is contained in \((-1, 1)\) and \( \int_{\mathbb{R}} \varphi(x) \, dx = 1 \). Let \( \{\eta_{n}\}_{n \geq 1} \) be a sequence of symmetric smooth functions such that for any \( n \geq 1, 0 \leq \eta_{n} \leq 1, \eta_{n}(x) = 1 \)
if $|x| \leq n$, and $\eta_n(x) = 0$ if $|x| \geq n + 2$. Define

$$b_n(x) := n \int_{\mathbb{R}} b(y) \varphi(n(x - y)) \, dy \times \eta_n(x), \quad (5.1)$$

$$\sigma_n(x) := n \int_{\mathbb{R}} \sigma(y) \varphi(n(x - y)) \, dy \times \eta_n(x). \quad (5.2)$$

Assume that $\sigma$ is bounded by a constant $K_\sigma$, that is

$$|\sigma(z)| \leq K_\sigma, \quad \forall z \in \mathbb{R}. \quad (5.3)$$

Then it is easy to check that there exist constants $L_n, L_b$ and $K_n$ such that for any $x, y \in \mathbb{R},$

$$|b_n(x) - b_n(y)| \leq L_n |x - y|, \quad (5.4)$$

$$|b_n(x)| \leq c_1 |x| \log_+ |x| + L_b (|x| + 1), \quad (5.5)$$

$$|\sigma_n(x) - \sigma_n(y)| \leq K_n |x - y|, \quad (5.6)$$

$$|\sigma_n(x)| \leq K_\sigma, \quad (5.7)$$

where the constant $c_1$ is same as that in condition (H1), and the constant $L_b$ is independent of $n$. Moreover, if $x_n \to x$ in $\mathbb{R}$, then

$$b_n(x_n) \to b(x), \quad (5.8)$$

$$\sigma_n(x_n) \to \sigma(x). \quad (5.9)$$

For $n \geq 1$, consider the following stochastic equation on the real-line $\mathbb{R}$,

$$u_n(t, x) = P_t u_0(x) + \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) b_n(u_n(s, y)) \, ds \, dy$$

$$+ \int_0^t \int_{\mathbb{R}} p_{t-s} \sigma_n(u_n(s, y)) \, W(ds, dy). \quad (5.10)$$

It is known (see [S, MPS, MP]) that for each $n \geq 1$, there exists a unique solution $u_n$ to the above equation. Moreover, the sample paths of $u_n$ are a.s. in $C(\mathbb{R}_+, C_{tem})$. The following result is a uniform bound for the solutions $u_n$.

**Lemma 5.1** Assume $u_0 \in C_{tem}$ and (H1). Suppose that $\sigma$ is bounded and continuous. Then for any $p \geq 1$ and $\lambda, T > 0$, we have

$$\sup_{n \geq 1} \mathbb{E} \left[ \sup_{t \leq T, x \in \mathbb{R}} \left( |u_n(t, x)| e^{-\lambda |x|} \right)^p \right] < \infty. \quad (5.11)$$

**Proof.** It suffices to prove this lemma for sufficiently large $p$ and sufficiently small $\lambda$. Fix $T > 0$. As $\lim_{\lambda \to 0} T^*(\lambda, c_1) = \infty$, there exists a positive constant $\lambda_T$ such that $T \leq T^*(\lambda, c_1)$ for all $\lambda \leq \lambda_T$. For any fixed $\lambda \leq \lambda_T$, we choose $\lambda_0 > 0$ so that $2\lambda_0 e^{\beta(\lambda_0, c_1)T} = \lambda$. In the following we write $\beta$ for $\beta(\lambda_0, c_1)$ to simplify the notation. Define

$$U_n(r) := \sup_{t \leq T, x \in \mathbb{R}} \left( |u_n(t, x)| e^{-\lambda_0 |x| e^{\beta r}} \right), \quad r \in [0, T]. \quad (5.12)$$
It remains to prove
\[
\sup_{n \geq 1} \mathbb{E}[U_n(T)]^p < \infty. \quad (5.13)
\]

From (5.10) we have
\[
U_n(r) \leq \sup_{t \leq r, x \in \mathbb{R}} \left( |P_t u_0(x)| e^{-\lambda_0 |x| e^{\beta t}} \right)
\]
\[+
\sup_{t \leq r, x \in \mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) b_n(u_n(s, y)) \, ds \, dy \right| e^{-\lambda_0 |x| e^{\beta t}} \right\}
\]
\[+
\sup_{t \leq r, x \in \mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) \sigma_n(u_n(s, y)) W(ds, dy) \right| e^{-\lambda_0 |x| e^{\beta t}} \right\}.
\]
\[(5.14)
\]

In the above inequality, the first term can be estimated the same as (4.32). Let
\[
V_n(t, x) = \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) \sigma_n(u_n(s, y)) W(ds, dy).
\]
\[(5.15)
\]
Then by (5.7) and Lemma 4.1, we have for any \( p > 0 \),
\[
\sup_{n \geq 1} \mathbb{E} \sup_{t \leq T, x \in \mathbb{R}} \left\{ |V_n(t, x)| e^{-\lambda_0 |x| e^{\beta t}} \right\}^p
\]
\[\leq \sup_{n \geq 1} \mathbb{E} \sup_{t \leq T, x \in \mathbb{R}} \left\{ |V_n(t, x)| e^{-\lambda_0 |x|} \right\}^p
\]
\[\leq C_{\lambda_0, K, T, p} < \infty.
\]
\[(5.16)
\]

On the other hand,
\[
\sup_{t \leq r, x \in \mathbb{R}} \left\{ \left| \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) b_n(u_n(s, y)) \, ds \, dy \right| e^{-\lambda_0 |x| e^{\beta t}} \right\}
\]
\[\leq \sup_{t \leq r, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) |c_1| u_n(s, y) | \log_+ |u_n(s, y)|
\]
\[+ L_b (|u_n(s, y)| + 1) \, ds \, dy \times e^{-\lambda_0 |x| e^{\beta t}} \right\}
\]
\[\leq c_1 \sup_{t \leq r, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) |u_n(s, y) | \log_+ |u_n(s, y)| \, ds \, dy \times e^{-\lambda_0 |x| e^{\beta t}} \right\}
\]
\[+ L_b \sup_{t \leq r, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} p_{t-s}(x, y) |u_n(s, y)| \, ds \, dy \times e^{-\lambda_0 |x| e^{\beta t}} \right\} + L_b r.
\]
\[(5.17)
\]
Now using (5.17) and following a similar proof as that of Lemma 4.4 we obtain
\[
U_n(T) \leq C_{\lambda_0, c_1, L_b, T} \times \left\{ 1 + 2L_b T + 4 e^{\frac{\lambda_0 x}{T}} \sup_{x \in \mathbb{R}} \left| u_0(x) \right| e^{-\lambda_0 |x|}
\]
\[+ 2 \sup_{t \leq T, x \in \mathbb{R}} \left( |V_n(t, x)| e^{-\lambda_0 |x|} \right) \right\}^{e^{c_1 T} e^{2 \lambda_0 T - 1}}.
\]
\[(5.18)
\]
Hence it follows from (5.16) that

\[
\mathbb{E}[U_n(T)^p] \leq C_{\lambda_0, c_1, L_0, T, p} \left[ 1 + \sup_{x \in \mathbb{R}} \left( |u_0(x)| e^{-\lambda_0|x|} \right) p \epsilon^p T e^{\frac{3}{4} \beta^2 T^{-1}} \right]
\]

\[
+ C_{\lambda_0, c_1, L_0, T, p} \mathbb{E} \left[ \sup_{t \leq T, x \in \mathbb{R}} \left( |V_n(t, x)| e^{-\lambda_0|x|} \right) p \epsilon^p T e^{\frac{3}{4} \beta^2 T^{-1}} \right]
\]

\[
\leq C_{\lambda_0, c_1, L_0, K_\sigma, T, p, ||u_0||_{\lambda_0, \infty}}
\]

(5.19)

where \( ||u_0||_{\lambda_0, \infty} := \sup_{x \in \mathbb{R}} (|u_0(x)| e^{-\lambda_0|x|}) \). Note that the last constant in
(5.19) is independent of \( n \). Hence (5.13) is proved, completing the proof of the lemma.

We will apply a Kolmogorov type tightness criterion (see Lemma 6.3 of [S]) to establish the tightness of the law of \( \{u_n\} \) in \( C(\mathbb{R}_+, C_{tem}) \). This is given in the following lemma.

Define

\[
X_n(t, x) := \int_0^t \int_{\mathbb{R}} p_{t-r}(x, z)b_n(u_n(r, z)) \, dr \, dz, \quad n \geq 1.
\]

(5.20)

**Lemma 5.2** Let \( u_0 \in C_{tem} \). Assume that (H1) holds and that \( \sigma \) is continuous with \( K_\sigma := \sup_{z \in \mathbb{R}} |\sigma(z)| < \infty \). Then for any \( \lambda, T > 0, p \geq 1 \) and \( \theta \in (0, 1) \), there exist constants \( C_{\lambda, c_1, L_0, K_\sigma, T, p, \theta, u_0} \) and \( C_{K_\sigma, T, p} \) independent of \( n \) such that

\[
\mathbb{E} \left( |X_n(t, x) - X_n(s, y)|^p e^{-\lambda|x|} \right) \leq C_{\lambda, c_1, L_0, K_\sigma, T, p, \theta, u_0} \left( |t - s|^\theta p + |x - y|^{p} \right),
\]

(5.21)

\[
\mathbb{E} \left( |V_n(t, x) - V_n(s, y)|^p e^{-\lambda|x|} \right) \leq C_{K_\sigma, T, p} \left( |t - s|^{\frac{p}{4}} + |x - y|^{\frac{p}{2}} \right),
\]

(5.22)

for any \( s, t \in [0, T] \) and \( x, y \in \mathbb{R} \) with \( |x - y| \leq 1 \). In particular, the family \( \{u_n\} \) is tight in \( C(\mathbb{R}_+, C_{tem}) \).

**Proof.** It suffices to prove this lemma for sufficiently large \( p \) and sufficiently small \( \lambda \). Fix \( T > 0 \). As \( \lim_{\lambda \to 0} T^*(\lambda, c_1) = \infty \), there exists a positive constant \( \lambda_T \) such that \( T \leq T^*(\lambda, c_1) \) for all \( \lambda \leq \lambda_T \). For any fixed \( \lambda \leq \lambda_T \), we choose \( \lambda_0 > 0 \) so that

\[
2\lambda_0 e^{\beta(\lambda_0, c_1)T} = \lambda.
\]

(5.23)

In the following we write \( \beta \) for \( \beta(\lambda_0, c_1) \) to simplify the notation. Without loss of generality, we assume \( t \geq s \).

\[
|X_n(t, x) - X_n(s, y)| e^{-\lambda|x|}
\]

\[
\leq |X_n(t, x) - X_n(s, x)| e^{-\lambda|x|} + |X_n(s, x) - X_n(s, y)| e^{-\lambda|x|}
\]
By (i) of Lemma 3.3, (3.22) and (3.20), we have

\[
\begin{align*}
    &\leq \int_0^s \int_{\mathbb{R}} \left[p_{t-r}(x, z) - p_{s-r}(x, z)\right] b_n(u_n(r, z)) \, dr \, dz \left| e^{-\lambda|x|} \right| \\
    &+ \int_s^t \int_{\mathbb{R}} p_{t-r}(x, z) b_n(u_n(r, z)) \, dr \, dz \left| e^{-\lambda|x|} \right| \\
    &+ \int_0^s \int_{\mathbb{R}} \left[p_{s-r}(x, z) - p_{s-r}(y, z)\right] b_n(u_n(r, z)) \, dr \, dz \left| e^{-\lambda|x|} \right| \\
    &=: J_1 + J_2 + J_3. \quad (5.24)
\end{align*}
\]

By the choice of $\lambda_0$, we have

\[
\begin{align*}
    e^{\lambda_0|z|} e^{\lambda_0 x} \leq 1, \quad \forall r \in [0, T]. \quad (5.26)
\end{align*}
\]

Therefore

\[
\begin{align*}
    J_1 &\leq (2\sqrt{2})^\theta |t - s|^\theta \times \int_0^s \frac{dr}{(s - r)^\theta} \, dr \times C_{\lambda_0, c_1, Lb, T} \\
    &\times \left\{ \sup_{r \leq T, z \in \mathbb{R}} \left| u_n(r, z)\right| e^{-\lambda_0|z|} \right| e^{\lambda_0 x} + \sup_{r \leq T, z \in \mathbb{R}} \left| u_n(r, z)\right| e^{-\lambda_0|z|} \right| e^{\lambda_0 x} \\
    &\times \log_+ \left[ \sup_{r \leq T, z \in \mathbb{R}} \left| u_n(r, z)\right| e^{-\lambda_0|z|} \right] + 1 \right\}. \quad (5.27)
\end{align*}
\]

By Lemma 5.1, (5.19) and the fact that $\theta \in (0, 1)$, we deduce from (5.27) that

\[
E J_1^\theta \leq C_{\lambda_0, c_1, Lb, K, T, p, \theta, ||u_0||_{\lambda_0, \infty}} |t - s|^{\theta p}. \quad (5.28)
\]
Similarly, we can show that

\[ EJ_2^p \leq C_{\lambda_0,c_1,L_b,K_\sigma,T,p,\|u_0\|_{\lambda_0,\infty}} |t - s|^p. \]  

(5.29)

Now we estimate \( J_3 \). By (ii)-(iv) of Lemma 3.3, we have

\[
J_3 \leq \int_0^s \int_\mathbb{R} |p_{s-r}(x,z) - p_{s-r}(y,z)| \times \\
[c_1 |u_n(r,z)| \log_+ |u_n(r,z)| + L_b(\|u_n(r,z)\| + 1)] \, dr \, dz \times e^{-\lambda|x|} \\
\leq \int_0^s \int_\mathbb{R} |p_{s-r}(x,z) - p_{s-r}(y,z)| \\
\times \left\{ c_1 e^{\lambda_0 |y-z| e^\beta r} \lambda_0 |z| e^{\beta r} \times \sup_{z \in \mathbb{R}} \left( |u_n(r,z)| e^{-\lambda_0 |z| e^{\beta r}} \right) \\
+ c_1 e^{\lambda_0 |y-z| e^\beta r} \times \sup_{z \in \mathbb{R}} \left[ |u_n(r,z)| e^{-\lambda_0 |z| e^{\beta r}} \log_+ \left( |u_n(r,z)| e^{-\lambda_0 |z| e^{\beta r}} \right) \right] \\
+ L_b e^{\lambda_0 |y-z| e^\beta r} \times \sup_{z \in \mathbb{R}} \left( |u_n(r,z)| e^{-\lambda_0 |z| e^{\beta r}} + L_b \right) \right\} \, dz \, dr \times e^{-\lambda|x|} \\
\leq |x - y| \times \int_0^s \frac{1}{\sqrt{s - r}} \times \left\{ \sup_{r \leq T, z \in \mathbb{R}} \left( |u_n(r,z)| e^{-\lambda_0 |z| e^{\beta r}} \right) \\
\times C_{\lambda_0,c_1,L_b,T} \left[ e^{\lambda_0 (|x| + |x-y|) e^\beta r} \lambda_0 (|x| + |x-y|) e^{\beta r} + e^{\lambda_0 (|x| + |x-y|) e^\beta r} \right] \\
+ \sup_{r \leq T, z \in \mathbb{R}} \left( |u_n(r,z)| e^{-\lambda_0 |z| e^{\beta r}} \right) \times \log_+ \left( \sup_{r \leq T, z \in \mathbb{R}} \left( |u_n(r,z)| e^{-\lambda_0 |z| e^{\beta r}} \right) \right) \right\} \, dr \times e^{-\lambda|x|}. \\
(5.30)
\]

Due to the fact that \( |x - y| \leq 1 \) and the choice of \( \lambda_0 \), we have

\[
e^{\lambda_0 (|x| + |x-y|) e^\beta r} \lambda_0 (|x| + |x-y|) e^{\beta r} \times e^{-\lambda|x|} \\
\leq e^{\lambda_0 (|x| + 1) e^\beta r} \lambda_0 (|x| + 1) e^{\beta r} \times e^{-\lambda(|x| + 1)} e^\lambda \\
\leq e^\lambda, \quad \forall r \in [0,T]. \]  

(5.31)

Hence in view of (5.19), we see that

\[
EJ_3^p \leq C_{\lambda_0,c_1,L_b,K_\sigma,T,p,\|u_0\|_{\lambda_0,\infty}} |x - y|^p. \]  

(5.32)

Combining (5.24), (5.28), (5.29) and (5.32) together yields

\[
\mathbf{E} \left( |X_n(t,x) - X_n(s,y)| e^{-\lambda|x|} \right)^p \leq C_{\lambda_0,c_1,L_b,K_\sigma,T,p,\|u_0\|_{\lambda_0,\infty}} \left( |t - s|^{\beta p} + |x - y|^p \right), \\
(5.33)
\]

where the constant \( \lambda_0 \) is determined by \( \lambda, T, c_1 \) according to (5.23). Thus, (5.21) is proved.
Now we prove (5.22). Observe that
\[
|V_n(t, x) - V_n(s, y)|
\leq \left| \int_0^s \int_{\mathbb{R}} [p_{t-r}(x, z) - p_{s-r}(x, z)] \sigma_n(u_n(r, z)) W(dr, dz) \right|
+ \left| \int_s^t \int_{\mathbb{R}} p_{t-r}(x, z) \sigma_n(u_n(r, z)) W(dr, dz) \right|
+ \left| \int_0^t \int_{\mathbb{R}} [p_{s-r}(x, z) - p_{s-r}(y, z)] \sigma_n(u_n(r, z)) W(dr, dz) \right|
=: I_1 + I_2 + I_3.
\]
(5.34)

So
\[
|V_n(t, x) - V_n(s, y)|^p \leq 3^{p-1}(I_1^p + I_2^p + I_3^p).
\]
(5.35)

By the BDG inequality, (5.7), and (v) of Lemma 3.3, we get
\[
\mathbb{E}I_1^p \leq \mathbb{E} \left[ \int_0^s \int_{\mathbb{R}} [p_{t-r}(x, z) - p_{s-r}(x, z)]^2 \sigma_n(u_n(r, z))^2 dzd\rho \right]^\frac{p}{2}
\leq K^p_\sigma \times \left[ \int_0^s \int_{\mathbb{R}} [p_{t-r}(x, z) - p_{s-r}(x, z)]^2 dzd\rho \right]^\frac{p}{2}
\leq C_{K_\sigma, T, p}|t - s|^{\frac{p}{4}}.
\]
(5.36)

Similarly, we have
\[
\mathbb{E}I_3^p \leq C_{K_\sigma, T, p}|x - y|^{\frac{p}{2}}.
\]
(5.37)

For the term \(I_2\), the uniform boundedness of \(\sigma_n\) gives
\[
\mathbb{E}I_2^p \leq \mathbb{E} \left[ \int_s^t \int_{\mathbb{R}} p_{t-r}(x, z)^2 \sigma_n(u_n(r, z))^2 dzd\rho \right]^\frac{p}{2}
\leq K^p_\sigma \times \left[ \int_s^t \int_{\mathbb{R}} [p_{t-r}(x, z)]^2 dzd\rho \right]^\frac{p}{2}
\leq K^p_\sigma \times \left[ \int_s^t \frac{1}{2\sqrt{\pi(t - r)}} dr \right]^\frac{p}{2}
\leq C_{K_\sigma, T, p}|t - s|^{\frac{p}{4}}.
\]
(5.38)

Combining (5.35), (5.36), (5.37) and (5.38) together, we obtain (5.22). This completes the proof of Lemma 5.2. ■

**Proof of Theorem 2.4.** We have established in Lemma 5.2 that \(\{u_n\}\) is tight in \(C(\mathbb{R}_+, C_{tem})\). By Prokhorov’s theorem and the modified version of Skorokhod’s representation theorem whose proof can be found in Appendix C of [BHR], we may assume that \(d(u_n, u) \to 0\) (not relabelled) a.s. in \(C(\mathbb{R}_+, C_{tem})\) for some process \(u\) on some probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), in other words, for any \(\lambda > 0, T \geq 0,\)
\[
\sup_{t \leq T, x \in \mathbb{R}} \left( u_n(t, x) - u(t, x) |e^{-\lambda|x|} \right) \to 0, \quad \tilde{\mathbb{P}} - a.s.. \quad (5.39)
\]
By the dominated convergence theorem, and using (5.39), (5.8), (5.9), (5.5) and (5.7), one can deduce that for any \((t, x) \in \mathbb{R}_+ \times \mathbb{R}\),
\[
\int_0^t \int_\mathbb{R} p_{t-r}(x, z)b_n(u_n(r, z)) \, dr \, dz \to \int_0^t \int_\mathbb{R} p_{t-r}(x, z)b(u(r, z)) \, dr \, dz, \tag{5.40}
\]
\(\bar{P}\)-a.s. as \(n \to \infty\), and
\[
\int_0^t \int_\mathbb{R} p_{t-r}(x, z)\sigma_n(u_n(r, z)) W(dr, dz) \to \int_0^t \int_\mathbb{R} p_{t-r}(x, z)\sigma(u(r, z)) W(dr, dz), \tag{5.41}
\]
in the sense of \(L^p(\bar{\Omega})\) for any \(p \geq 1\) as \(n \to \infty\). Therefore, we see that \(u\) is a weak solution of (1.1). \(\blacksquare\)

6 Pathwise uniqueness

In this section, we prove the pathwise uniqueness of solutions to (1.1) and hence obtain the strong solution.

**Proof of Theorem 2.5.** Since condition (H2) implies condition (H1), there exists a weak solution to (1.1) according to Theorem 2.4. We only show the pathwise uniqueness for solutions of (1.1). The existence of strong solutions then follows from the Yamada-Watanabe theorem.

Suppose that \(u, v\) are two solutions of equation (1.1) that belong to the space \(C(\mathbb{R}_+, C_{lem})\). We are going to show that \(u = v\). Fix \(T > 0\), and take \(\lambda > 0\) sufficiently small so that \(T \leq T^*(\lambda, c_4)\). In this section, we write \(\beta\) for \(\beta(\lambda, c_4)\) for simplicity. Let \(M > 0\) and \(0 < \delta < e^{-1}\). Define stopping times
\[
\tau_M := \inf \left\{ t > 0 : \sup_{x \in \mathbb{R}} \left( |u(t, x)| e^{-\lambda|x|e^{\beta t}} \right) \geq M \right\},
\]
\[
\tau^{\delta} := \inf \left\{ t > 0 : \sup_{x \in \mathbb{R}} \left( |v(t, x)| e^{-\lambda|x|e^{\beta t}} \right) \geq \delta \right\},
\]
\[
\tau_{M}^{\delta} := \tau_M \land \tau^{\delta} \land T,
\]
with the convention that \(\inf \emptyset = +\infty\). Define also
\[
Z(r) := \mathbb{E} \sup_{t \leq r \land \tau_{M}^{\delta}, x \in \mathbb{R}} \left( |u(t, x) - v(t, x)| e^{-\lambda|x|e^{\beta t}} \right). \tag{6.42}
\]

Obviously
\[
Z(r)
\]
\[
\leq \mathbb{E} \sup_{t \leq r \land \tau_{M}^{\delta}, x \in \mathbb{R}} \left\{ \int_0^t \int_\mathbb{R} p_{t-s}(x, y)|b(u(s, y)) - b(v(s, y))| \, ds \, dy \times e^{-\lambda|x|e^{\beta t}} \right\}
\]
\[
+ \mathbb{E} \sup_{t \leq r \land \tau_{M}^{\delta}, x \in \mathbb{R}} \left\{ \left| \int_0^t \int_\mathbb{R} p_{t-s}(x, y)|\sigma(u(s, y)) - \sigma(v(s, y))| W(ds, dy) \right| e^{-\lambda|x|e^{\beta t}} \right\}
\]
\[
=: I + J. \tag{6.43}
\]
Now we estimate the term $I, J$ separately. By condition (H2), we have

$$
I \leq \mathbb{E} \sup_{t \leq r \wedge \tau^u_{M}, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} |p_{t-s}(x, y)| c_3 |u(s, y) - v(s, y)| \times \log_+ \frac{1}{|u(s, y) - v(s, y)|} \frac{1}{d s d y} \times e^{-\lambda|x|e^{\beta t}} \right\}
$$

$$
= \mathbb{E} \sup_{t \leq r \wedge \tau^u_{M}, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} |p_{t-s}(x, y)| c_4 \log_+ \left( |u(s, y)| \vee |v(s, y)| \right) \times |u(s, y) - v(s, y)| \frac{1}{d s d y} \times e^{-\lambda|x|e^{\beta t}} \right\}
$$

$$
= \mathbb{E} \sup_{t \leq r \wedge \tau^u_{M}, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} |p_{t-s}(x, y)| c_5 |u(s, y) - v(s, y)| \frac{1}{d s d y} \times e^{-\lambda|x|e^{\beta t}} \right\}
$$

$$
=: I_1 + I_2 + I_3. \quad (6.44)
$$

First, we estimate the term $I_1$. By the fact that the function $x \mapsto x \log \frac{1}{x}$ is increasing and concave on $(0, e^{-1})$, (3.20) and (4.37), we get

$$
I_1 \leq c_3 \mathbb{E} \sup_{t \leq r \wedge \tau^u_{M}, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} |p_{t-s}(x, y)| e^{\lambda|y|e^{\beta s}} |u(s, y) - v(s, y)| e^{-\lambda|y|e^{\beta s}} \times \log_+ \frac{1}{|u(s, y) - v(s, y)| e^{-\lambda|y|e^{\beta s}}} \frac{1}{d y d s} \times e^{-\lambda|x|e^{\beta t}} \right\}
$$

$$
\leq c_3 \mathbb{E} \sup_{t \leq r \wedge \tau^u_{M}, x \in \mathbb{R}} \left\{ \int_0^t \sup_{y \in \mathbb{R}} \left[ |u(s, y) - v(s, y)| e^{-\lambda|y|e^{\beta s}} \right] \times \log_+ \frac{1}{|u(s, y) - v(s, y)| e^{-\lambda|y|e^{\beta s}}} \frac{1}{d y d s} \times e^{-\lambda|x|e^{\beta t}} \right\}
$$

$$
\leq 2c_3 e^{2\beta^2 r^{1-1}} \int_0^t \mathbb{E} \sup_{\rho \leq s \wedge \tau^u_{M}, y \in \mathbb{R}} \left( |u(s, y) - v(s, y)| e^{-\lambda|y|e^{\beta \rho}} \right) \times \log_+ \frac{1}{\sup_{\rho \leq s \wedge \tau^u_{M}, y \in \mathbb{R}} \left( |u(s, y) - v(s, y)| e^{-\lambda|y|e^{\beta \rho}} \right)} \frac{1}{d s}
$$

$$
\leq 2c_3 e^{2\beta^2 r^{1-1}} \int_0^r Z(s) \log_+ \frac{1}{Z(s)} d s, \quad (6.45)
$$

where (3.20) was used. Note that

$$
\log_+(ab) \leq \log_+ a + \log_+ b.
$$

By the definition of $\tau^u_{M}$, we have

$$
I_2 \leq c_4 \mathbb{E} \sup_{t \leq r \wedge \tau^u_{M}, x \in \mathbb{R}} \left\{ \int_0^t \int_{\mathbb{R}} |p_{t-s}(x, y)| \log_+ \left( e^{\lambda|y|e^{\beta s}} \right) \right\}
$$

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\[ + \log_+ \left( |u(s, y)|e^{-\lambda|y|e^{\beta s}} \right) \vee \left( |v(s, y)|e^{-\lambda|y|e^{\beta s}} \right) \]
\[ \times \left( |u(s, y) - v(s, y)|e^{-\lambda|y|e^{\beta s}} \right)e^{\lambda|y|e^{\beta s}} \, dy \times e^{-\lambda|x|e^{\beta t}} \}
\[ \leq c_4 \mathbb{E} \sup_{t \leq r \wedge \tau_M} \left\{ \int_0^t \sup_{y \in \mathbb{R}} \left( |u(s, y) - v(s, y)|e^{-\lambda|y|e^{\beta s}} \right) \right. \]
\[ \times \int_{\mathbb{R}} p_{t-s}(x, y)e^{\lambda|y|e^{\beta s}} \lambda|y|e^{\beta s} \, dy \times e^{-\lambda|x|e^{\beta t}} \}
\[ + c_4 \log_+(M) \mathbb{E} \sup_{t \leq r \wedge \tau_M} \left\{ \int_0^t \sup_{y \in \mathbb{R}} \left( |u(s, y) - v(s, y)|e^{-\lambda|y|e^{\beta s}} \right) \right. \]
\[ \times \int_{\mathbb{R}} p_{t-s}(x, y)e^{\lambda|y|e^{\beta s}} \, dy \times e^{-\lambda|x|e^{\beta t}} \}
\[ \leq \frac{1}{2} Z(r) + c_4 C_{\lambda, \beta, M, r} \int_0^r Z(s) \, ds, \quad (6.46) \]

where the last inequality holds for the same reason as the derivation of (4.38)-(4.40) with constant \( c_1 \) replaced by constant \( c_4 \). Similarly, \( I_3 \leq 2c_3 e^{\frac{\lambda^2}{2} \sigma^2 t^{3r-1}} \int_0^r Z(s) \, ds. \quad (6.47) \)

For the term \( J \), we use the estimate established in Proposition 4.2 to obtain
\[ J \leq \epsilon \mathbb{E} \sup_{s \leq r \wedge \tau_M, y \in \mathbb{R}} \left( |\sigma(u(s, y)) - \sigma(v(s, y))|e^{-\lambda|y|e^{\beta s}} \right) \]
\[ + C_{\epsilon, \lambda, \beta, r} \mathbb{E} \int_0^{r \wedge \tau_M} \int_{\mathbb{R}} |\sigma(u(s, y)) - \sigma(v(s, y))|e^{-\lambda|y|e^{\beta s}} \, dy \, ds, \]

where the constant \( C_{\epsilon, \lambda, \beta, r} \) is the constant \( C_{\epsilon, p, h(T), r} \) appeared in (4.15) by taking \( p = 1 \), \( T = r \) and \( h(T) = \lambda e^{\beta r} \). Since \( \sigma \) is bounded and Lipschitz, there exists two nonnegative constants \( K_\sigma \) and \( L_\sigma \) such that
\[ |\sigma(x)| \leq K_\sigma, \quad \forall x \in \mathbb{R}, \]
\[ |\sigma(x) - \sigma(y)| \leq L_\sigma |x - y|, \quad \forall x, y \in \mathbb{R}. \]

Hence for any \( 0 < \theta < 1 \), we have
\[ J \leq \epsilon L_\sigma Z(r) + C_{\epsilon, \lambda, \beta, r} \mathbb{E} \int_0^{r \wedge \tau_M} \sup_{y \in \mathbb{R}} \left\{ \left( |\sigma(u(s, y)) - \sigma(v(s, y))|e^{-\lambda|y|e^{\beta s}} \right)^\theta \right. \]
\[ \times \int_{\mathbb{R}} |\sigma(u(s, y)) - \sigma(v(s, y))|^{1-\theta} e^{-(1-\theta)\lambda|y|e^{\beta s}} \, dy \right\} \, ds \]
\[ \leq \epsilon L_\sigma Z(r) + \frac{(2K_\sigma)^{1-\theta} L_\sigma^\theta C_{\epsilon, \lambda, \beta, r}}{(1-\theta)\lambda} \mathbb{E} \int_0^{r \wedge \tau_M} \sup_{y \in \mathbb{R}} \left( |u(s, y) - v(s, y)|e^{-\lambda|y|e^{\beta s}} \right)^\theta \, ds \]
\[ \leq \epsilon L_\sigma Z(r) + \frac{(2K_\sigma)^{1-\theta} L_\sigma^\theta C_{\epsilon, \lambda, \beta, r}}{(1-\theta)\lambda} \int_0^r Z(s)^\theta \, ds. \quad (6.48) \]
Combining (6.43)-(6.48) together, we obtain that
\[
Z(r) \leq \left( \frac{1}{2} + \epsilon L_\sigma \right) Z(r) + C_{\lambda,M,c_4,c_5,r} \int_0^r Z(s) \, ds \\
+ 2 c_3 e^{\frac{\beta r-1}{2}} \int_0^r Z(s) \log_+ \frac{1}{Z(s)} \, ds + \frac{(2 K_\sigma)^{1 - \theta} L_\sigma^\theta C_{\epsilon,\lambda,\beta,r}}{(1 - \theta) \lambda} \int_0^r Z(s)^\theta \, ds. 
\]
(6.49)

Taking for example \( \epsilon = \frac{1}{4} L_\sigma \), subtracting \( \left( \frac{1}{2} + \epsilon L_\sigma \right) Z(r) \) from both sides of the above inequality, and then applying the special Gronwall-type inequality established in Lemma 3.2, we obtain
\[
Z(r) \equiv 0, \quad \forall r \geq 0. 
\]
(6.50)

Since the solutions of (1.1) don’t blowup, let \( M \to \infty \) to obtain \( \mathbb{P}\)-a.s.,
\[
u(t, x) = v(t, x), \quad \forall x \in \mathbb{R}, \quad \forall t \in [0, \tau_\delta \wedge T]. 
\]
(6.51)

This implies that \( \tau_\delta \geq T \), \( \mathbb{P}\)-a.s., otherwise it contradicts the definition of \( \tau_\delta \). By the arbitrariness of \( T \), we obtain that for \( \mathbb{P}\)-a.s.,
\[
\nu(t, x) = v(t, x), \quad \forall (t, x) \in \mathbb{R}_+ \times \mathbb{R}. 
\]
(6.52)

This completes the proof the pathwise uniqueness. ■

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