STUDY OF $\sigma(750)$ AND $\rho^0(770)$ PRODUCTION
IN MEASUREMENTS OF $\pi N_\uparrow \rightarrow \pi^+\pi^- N$ ON
A POLARIZED TARGET AT 5.98, 11.85 AND 17.2 GeV/c.

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We present a new and improved model independent amplitude analysis of reactions $\pi^+n_\uparrow \rightarrow \pi^+\pi^-p$ at 5.98 and 11.85 GeV/c and $\pi^-p_\uparrow \rightarrow \pi^-\pi^+n$ at 17.2 GeV/c measured with transversely polarized targets at the CERN Proton Synchrotron. For dipion masses below 1000 MeV the pion production process is described by two $S$-wave and six $P$-wave production amplitudes. Previous analyses suffered from the presence of unphysical solutions for moduli of amplitudes or cosines of their relative phases, causing uncertainties regarding the signal for scalar state $I = 0\ 0^{++}(750)$. To remove the unphysical solutions we use a Monte Carlo approach to amplitude analysis. In each $(m, t)$ bin we randomly varied the input spin density matrix elements 30 000 times within their experimental errors and performed amplitude analysis for each selection. Unphysical solutions were rejected and the physical solutions produced a continuous range of values for moduli, cosines of relative phases and for partial wave intensities. A clear signal for $\sigma(750)$ resonance emerges in all four solutions for $S$-wave intensity $I_S$ at 5.98 and 11.85 GeV/c and in both solutions for $S$-wave amplitude $|S|^2\Sigma$ at 17.2 GeV/c. Its $\pi^+\pi^-$ decay width is estimated to be in the range 200–300 MeV. We find significant suppression of $\rho^0$ production in the amplitudes $|U|^2\Sigma$, $|N|^2\Sigma$ and, at 17.2 GeV/c, in $|L|^2\Sigma$. The mass dependence of amplitudes $|\overline{L}|^2\Sigma$ and $|L|^2\Sigma$ shows unexpected structures within the $\rho^0$ mass region which correlate the mass spectra corresponding to opposite nucleon spins. These features of $P$-wave moduli reveal the essential role of nucleon spin in pion production process and contradict the factorization hypothesis. Our results emphasize the need for a systematic study of pion production on the level of amplitudes.
in a new generation of dedicated experiments with spin at the recently proposed high-intensity hadron facilities.
1. INTRODUCTION

The pion production process $\pi N \rightarrow \pi^+\pi^- N$ has always served to develop our ideas on the dynamics of hadron collisions and hadron production. In 1978, Lutz and Rybicki showed\(^1\) that measurements of pion production in meson-nucleon scattering on transversely polarized target yield in a single experiment enough observables that almost complete and model independent amplitude analysis can be performed. In the kinematic region with dimeson masses below 1000 MeV the dimeson system is produced predominantly in spin states $J = 0$ (S-wave) and $J = 1$ (P-wave). The results enable us to study the pion production and the resonance production on the level of spin-dependent amplitudes rather than spin-averaged cross-sections. In particular, a model independent separation of S-wave and P-wave amplitudes is possible only in measurement on transversely polarized targets.

The high statistics measurement of $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c at CERN-PS on unpolarized target\(^2\) was later repeated with a transversely polarized target\(^3\)-\(^7\) at the same energy. The resulting model independent amplitude analysis\(^3\),\(^4\),\(^5\) provided the first evidence for significant contributions from helicity-nonflip amplitudes with $A_1$ exchange quantum numbers ($I^G = 1^-, J^{PC} = 1^{++}$) which had long been assumed absent. Moreover, the S-wave partial wave intensity $I_S$ showed a clear bump in the 750–800 MeV mass region\(^8\) in one of the two solutions\(^4\),\(^5\),\(^6\) which indicated the possibility of a new scalar state with a mass near $\rho^0$.

Additional information was provided by the first measurement of $\pi^+ n \rightarrow \pi^+\pi^- p$ reaction on polarized deuteron target at 5.98 and 11.85 GeV/c also done at CERN-PS.\(^9\),\(^10\),\(^11\) Amplitude analysis at larger momentum transfer confirmed the
evidence for large $A_1$-exchange contributions and found resonant-like structures in
the $S$-wave partial-wave intensities.\textsuperscript{12} This data also found important $t$-dependent
structures in the moduli of $P$-wave amplitudes within the $\rho^0$ mass region.\textsuperscript{13}

In a recent paper\textsuperscript{14} we focused on the evidence for the scalar state \( I = 0 \)
\( 0^{++}(750) \) coming from all these measurements on polarized targets. We found
that all four solutions for $S$-wave partial wave intensity at 5.98 and 11.85 GeV/c
suggest resonance structure around 750 MeV. Its $\pi^+\pi^-$ decay width depends on
the solution and was estimated to be in the range of 100–250 MeV. The evidence
for this state is strengthened by the fact that the $S$-wave amplitudes $S$ and $\overline{S}$ are
nearly phase degenerate with the dominant resonating $P$-wave amplitudes $L$ and $\overline{L}$. It was suggested\textsuperscript{14} that the \( I = 0 \) \( 0^{++}(750) \) state is best understood as the
lowest-mass scalar gluonium $0^{++}(gg)$. Our results are in agreement with the $S$-
wave partial-wave intensity for $\pi^+\pi^- \rightarrow \pi^0\pi^0$ estimated\textsuperscript{15} from the measurement
of $\pi^+p \rightarrow \pi^0\pi^0\Delta^{++}$ at 8.0 GeV/c.

The data on transversely polarized targets are best analysed in terms of nucleon transversity amplitudes. There are two $S$-wave and six $P$-wave amplitudes. Amplitude analysis expresses analytically\textsuperscript{1,12} the eight moduli and six cosines of relative phases of nucleon transversity amplitudes in terms of measured spin density matrix (SDM) elements. There are two similar solutions. However, in many $(m,t)$ bins the solutions are unphysical: typically a cosine has magnitude larger than one or the two solutions for moduli are complex conjugate with a small imaginary part.

In Ref. 12 and 14 we presented the direct analytical results (taking only real part of complex solutions). The authors of Ref. 3–6 took these analytical solutions as start-
ing values for a $\chi^2$ minimization program which fitted the measured observables to obtain physical values of moduli and cosines.

The occurrence of unphysical solutions is a major difficulty for all amplitude analyses of pion production and the source of uncertainty in the evidence for $I = 0\,0^{++}(750)$ resonance. In this paper we investigate this problem using Monte Carlo methods.\textsuperscript{16,17}

The basic idea of Monte Carlo amplitude analysis is to filter out\textsuperscript{16} the unwanted unphysical solutions and to determine\textsuperscript{17} the range of physical values of moduli and cosines of relative phases. To achieve this we randomly varied the input SDM elements within their errors, performed the amplitude analysis for each new set of the input SDM elements, and retained the resulting moduli and cosines only when all of them had physical values in both solutions. The results presented in this report are based on 30 000 random variations of input SDM elements in each $(m, t)$ bin. The distributions of moduli and cosines define the range of their physical values and their average values\textsuperscript{17} in each $(m, t)$ bin.

The results for the $P$-wave moduli are essentially the same as in the previous analysis\textsuperscript{14} but the changes for the $S$-wave are striking. After filtering out the unwanted unphysical solutions, a clear signal for $I = 0\,0^{++}(750)$ state emerges in all four solutions for $S$-wave partial wave intensity at all 3 energies.

The paper is organized as follows. In Section 2 we review the basic formalism. In Section 3 we describe our Monte Carlo approach to amplitude analysis of $\pi N \rightarrow \pi^+\pi^- N$ reactions on polarized target. The evidence for the $I = 0\,0^{++}(750)$ state is presented in Section 4. In Section 5 we describe the spin dependence of $\rho^0$
production and discuss its unexpected features. In Section 6 we discuss the assumptions involved in the determination of $\pi\pi$ phase shifts from data on $\pi^-p \rightarrow \pi^-\pi^+n$ on unpolarized target and present tests of the key assumption of absence of $A_1$-exchange in measurements on polarized targets. The paper closes with a summary in Section 7.

2. BASIC FORMALISM

For invariant masses below 1000 MeV, the dipion system in reactions $\pi N \rightarrow \pi^+\pi^-N$ is produced predominantly in spin states $J = 0$ ($S$-wave) and $J = 1$ ($P$-wave). The experiments on transversely polarized targets then yield 15 spin-density-matrix (SDM) elements describing the dipion angular distribution. The measured SDM elements are\textsuperscript{10,11}

\begin{equation}
\rho_{ss} + \rho_{00} + 2\rho_{11}, \ \rho_{00} - \rho_{11}, \ \rho_{1-1}
\end{equation}

\begin{equation}
Re\rho_{10}, \ Re\rho_{1s}, \ Re\rho_{0s}
\end{equation}

\begin{equation}
\rho_{ss}^y + \rho_{00}^y + 2\rho_{11}^y, \ \rho_{00}^y - \rho_{11}^y, \ \rho_{1-1}^y
\end{equation}

\begin{equation}
Re\rho_{10}^y, \ Re\rho_{1s}^y, \ Re\rho_{0s}^y
\end{equation}

\begin{equation}
Im\rho_{1-1}^x, \ Im\rho_{10}^x, \ Im\rho_{1s}^x
\end{equation}

The SDM elements (2.1a) are also measured in experiments on unpolarized targets. The observables (2.1b) and (2.1c) are determined by the transverse component of target polarization perpendicular and parallel to the scattering plane $\pi N \rightarrow (\pi^+\pi^-)N$, respectively. The SDM elements (2.1) depend on $s, t, m$ where $s$ is the
c.m. system energy squared, $t$ is the four-momentum transfer squared, and $m$ is the $\pi^+\pi^-$ invariant mass. There are two linear relations among the matrix elements in (2.1):

$$\rho_{ss} + \rho_{00} + 2\rho_{11} = 1$$

$$\rho_{ss}^y + \rho_{00}^y + 2\rho_{11}^y = A$$

where $A$ is the polarized target asymmetry.

The reaction $\pi^+n \rightarrow \pi^+\pi^-p$ is described by pion production amplitudes $H_{\lambda_p,0\lambda_n}(s, t, m, \theta, \phi)$ where $\lambda_p$ and $\lambda_n$ are helicities of the proton and neutron, respectively. The angles $\theta, \phi$ describe the direction of $\pi^+$ in the $\pi^+\pi^-$ rest frame.

The production amplitudes can be expressed in terms of production amplitudes corresponding to definite dipion spin $J$ using an angular expansion

$$H_{\lambda_p,0\lambda_n} = \sum_{J=0}^{\infty} \sum_{\lambda=-J}^{+J} (2J + 1)^{\frac{1}{2}} H_{\lambda\lambda p,0\lambda n}^J(s, t, m)d_J^{\lambda_0}(\theta)e^{i\lambda\phi}$$

where $J$ is the spin and $\lambda$ the helicity of the $\pi^+\pi^-$ dipion system. Our amplitude analysis is carried out in the $t$-channel helicity frame for the $\pi^+\pi^-$ dimeson system. The helicities of the initial and final nucleons are always in the $s$-channel helicity frame.

The “partial-wave” amplitudes $H_{\lambda\lambda_p,0\lambda_n}^J$ can be expressed in terms of nucleon helicity amplitudes with definite $t$-channel exchange naturality. In the case when the $\pi^+\pi^-$ system is produced in the $S$- and $P$-wave states we have

$$0^-\frac{1^+}{2} \rightarrow 0^+\frac{1^+}{2} : H_{0+,0+}^0 = S_0$$

$$H_{0+,0-}^0 = S_1$$
\[0^{-\frac{1+}{2}} \rightarrow 1^{-\frac{1+}{2}} : H^{1}_{0+,0+} = L_0 \quad (2.4b)\]

\[H^{1}_{0+,0-} = L_1\]

\[H^{1}_{\pm1+,0+} = \frac{N_0 \pm U_0}{\sqrt{2}}\]

\[H^{1}_{\pm1+,0-} = \frac{N_1 \pm U_1}{\sqrt{2}}\]

In (2.4), \(0^{-}\) stands for pion, \(\frac{1+}{2}\) for nucleon, \(0^{+}\) for \(J = 0\) dipion state (S-wave), and \(1^{-}\) for \(J = 1\) dipion state (P-wave). At large \(s\), the amplitudes \(N_0\) and \(N_1\) are both dominated by natural \(A_2\) exchange. The amplitudes \(S_n, L_n, U_n, n = 0, 1\) are dominated by unnatural exchanges: \(A_1\) exchange for \(n = 0\) and \(\pi\) exchange for \(n = 1\). The index \(n = |\lambda_n - \lambda_p|\) is nucleon helicity flip.

The data on transversely polarized targets are best analysed in terms of nucleon transversity amplitudes (NTA’s).\(^{1,11,12}\) In our kinematic region we work with two \(S\)-wave and six \(P\)-wave NTA’s of definite naturality defined as follows\(^{1,11,12}\)

\[S = (S_0 + iS_1)/\sqrt{2}, \quad \overline{S} = (S_0 - iS_1)/\sqrt{2}\quad (2.5)\]

\[L = (L_0 + iL_1)/\sqrt{2}, \quad \overline{L} = (L_0 - iL_1)/\sqrt{2}\]

\[U = (U_0 + iU_1)/\sqrt{2}, \quad \overline{U} = (U_0 - iU_1)/\sqrt{2}\]

\[N = (N_0 - iN_1)/\sqrt{2}, \quad \overline{N} = (N_0 + iN_1)/\sqrt{2}\]

The amplitudes \(S, L, U, N\) and \(\overline{S}, \overline{L}, \overline{U}, \overline{N}\) correspond to recoil nucleon transversity “down” and “up”, respectively.\(^{11,12}\) The “up” direction is the direction of normal to the scattering plane defined according to the Basel convention by \(\vec{p}_\pi \times \vec{p}_{\pi\pi}\) where \(\vec{p}_\pi\) and \(\vec{p}_{\pi\pi}\) are the incident pion and dimeson momenta in the target nucleon rest
frame. The $S$-wave amplitudes $S, \overline{S}$ and $P$-wave amplitudes $L, \overline{L}$ have dimeson helicity $\lambda = 0$. The pairs of amplitudes $U, \overline{U}$ and $N, \overline{N}$ are combinations of nucleon helicity amplitudes with dimeson helicities $\lambda = \pm 1$ and have opposite $t$-channel-exchange naturality.

We can now express the observables in terms of amplitudes. In our normalization, the integrated cross section $\Sigma \equiv d^2\sigma/dm dt$ is given by

$$\Sigma = \sum_{n=0,1} |S_n|^2 + |L_n|^2 + |U_n|^2 + |N_n|^2 \quad (2.6)$$

$$= |S|^2 + |\overline{S}|^2 + |L|^2 + |\overline{L}|^2 + |U|^2 + |\overline{U}|^2 + |N|^2 + |\overline{N}|^2$$

The cross section has not been measured in the experiments on polarized targets. Consequently, we will work with normalized amplitudes corresponding to

$$\Sigma = \frac{d^2\sigma}{dm dt} \equiv 1 \quad (2.7)$$

Using (2.6), the relations for SDM elements in terms of normalized helicity amplitudes read as follows.\textsuperscript{1,12}

Unpolarized SDM elements

$$\rho_{ss} + \rho_{00} + 2\rho_{11} = \sum_{n=0} |S_n|^2 + |L_n|^2 + |U_n|^2 + |N_n|^2 \quad (2.8a)$$

$$\rho_{00} - \rho_{11} = \sum_{n=0,1} |L_n|^2 - \frac{1}{2}(|N_n|^2 + |U_n|^2)$$

$$\rho_{1-1} = \sum_{n=0,1} \frac{1}{2}(|N_n|^2 - |U_n|^2)$$

$$\sqrt{2}Re\rho_{10} = \sum_{n=0,1} Re(U_n L_n^*)$$

$$\sqrt{2}Re\rho_{1s} = \sum_{n=0,1} Re(U_n S_n^*)$$

$$Re\rho_{0s} = \sum_{n=0,1} Re(L_n S_n^*)$$
Polarized SDM elements

\[ \rho_{ss}^y + \rho_{00}^y + 2\rho_{11}^y = 2Im(S_0S_1^* + L_0L_1^* + U_0U_1^* + N_0N_1^*) \] (2.8b)

\[ \rho_{00}^y - \rho_{11}^y = Im(2L_0L_1^* - N_0N_1^* - U_0U_1^*) \]

\[ \rho_{1-1}^y = Im(N_0N_1^* - U_0U_1^*) \]

\[ \sqrt{2}Re\rho_{10}^y = Im(U_0L_1^* - U_1L_0^*) \]

\[ \sqrt{2}Re\rho_{1s}^y = Im(U_0S_1^* - U_1S_0^*) \]

\[ Re\rho_{0s}^y = Im(L_0S_1^* - L_1S_0^*) \]

\[ -Im\rho_{1-1}^x = Im(N_0U_1^* + N_1U_0^*) \] (2.8c)

\[ \sqrt{2}Im\rho_{10}^x = Im(N_0L_1^* + N_1L_0^*) \]

\[ \sqrt{2}Im\rho_{1s}^x = Im(N_0S_1^* + N_1S_0^*) \]

Only the polarization dependent SDM elements measure the nucleon helicity flip-nonflip interference. The observables (2.8b) and (2.8c) measure the interference between the amplitudes of the same and opposite naturalities, respectively.

To express the observables in terms of normalized nucleon transversity amplitudes (2.5), we first introduce partial wave cross-sections \( \sigma(A) \) and partial-wave polarizations \( \tau(A) \) defined for amplitudes \( A = S, L, U, N \) as

\[ \sigma(A) = |A_0|^2 + |A_1|^2 = |A|^2 + |\overline{A}|^2 \] (2.9)

\[ \tau(A) = 2\epsilon Im(A_0A_1^*) = |A|^2 - |\overline{A}|^2 \]
where $\epsilon = +1$ for $A = S, L, U$ and $\epsilon = -1$ for $A = N$. In our normalization the reaction cross-section is

$$
\Sigma = \sigma(S) + \sigma(L) + \sigma(U) + \sigma(N) = 1 \quad (2.10)
$$

The relations for SDM elements (2.8a) and (2.8b) in terms of normalized nucleon transversity amplitudes (2.5) and quantities (2.9) read

$$
\rho_{ss} + \rho_{00} + 2\rho_{11} = \sigma(S) + \sigma(L) + \sigma(U) + \sigma(N) \quad (2.11a)
$$

$$
\rho_{00} - \rho_{11} = \sigma(L) - \frac{1}{2}[\sigma(U) + \sigma(N)]
$$

$$
\rho_{1-1} = -\frac{1}{2}[\sigma(U) - \sigma(N)]
$$

$$
\rho_{ss}^{y} + \rho_{00}^{y} + 2\rho_{11}^{y} = \tau(S) + \tau(L) + \tau(U) - \tau(N) \quad (2.11b)
$$

$$
\rho_{00}^{y} - \rho_{11}^{y} = \tau(L) - \frac{1}{2}[\tau(U) - \tau(N)]
$$

$$
\rho_{1-1}^{y} = -\frac{1}{2}[\tau(U) + \tau(N)]
$$

$$
\sqrt{2}Re\rho_{10}^{y} = Re(UL^{*} + \overline{U} \overline{L}^{*}) \quad (2.12a)
$$

$$
\sqrt{2}Re\rho_{1s}^{y} = Re(US^{*} + \overline{U} \overline{S}^{*})
$$

$$
Re\rho_{0s} = Re(LS^{*} + \overline{L} \overline{S}^{*})
$$

$$
\sqrt{2}Re\rho_{10}^{y} = Re(UL^{*} - \overline{U} \overline{L}^{*}) \quad (2.12b)
$$

$$
\sqrt{2}Re\rho_{1s}^{y} = Re(US^{*} - \overline{U} \overline{S}^{*})
$$

$$
Re\rho_{0s}^{y} = Re(LS^{*} - \overline{L} \overline{S}^{*})
$$

The relations (2.11) and (2.12) suggest to introduce new observables which are the sum and difference of the SDM elements (2.8a) and (2.8b). Using the notation of (2.2), the first group of new observables reads

$$
a_{1} = \frac{1}{2}[1 + A] = |S|^2 + |L|^2 + |U|^2 + |\overline{N}|^2 \quad (2.13a)
$$
\[ a_2 = [(\rho_{00} - \rho_{11}) + (\rho_{00}^y - \rho_{11}^y)] = 2|L|^2 - |U|^2 - |N|^2 \]

\[ a_3 = [\rho_{1-1} + \rho_{1-1}^y] = |N|^2 - |U|^2 \]

\[ a_4 = \frac{1}{\sqrt{2}} [\text{Re} \rho_{10} + \text{Re} \rho_{10}^y] = |U||L| \cos(\gamma_{LU}) \] \hspace{1cm} (2.13b)

\[ a_5 = \frac{1}{\sqrt{2}} [\text{Re} \rho_{1s} + \text{Re} \rho_{1s}^y] = |U||S| \cos(\gamma_{SU}) \]

\[ a_6 = \frac{1}{2} [\text{Re} \rho_{0s} + \text{Re} \rho_{0s}^y] = |L||S| \cos(\gamma_{SL}) \]

Similar equations relate the difference of SDM elements to amplitudes of opposite transversity. The second group of observables is defined as

\[ \overline{a}_1 = \frac{1}{2} [1 - A] = |S|^2 + |L|^2 + |U|^2 + |N|^2 \] \hspace{1cm} (2.14a)

\[ \overline{a}_2 = [(\rho_{00} - \rho_{11}) - (\rho_{00}^y - \rho_{11}^y)] = 2|L|^2 - |U|^2 - |N|^2 \]

\[ \overline{a}_3 = [\rho_{1-1} - \rho_{1-1}^y] = |N|^2 - |U|^2 \]

\[ \overline{a}_4 = \frac{1}{\sqrt{2}} [\text{Re} \rho_{10} - \text{Re} \rho_{10}^y] = |U||L| \cos(\gamma_{LU}) \] \hspace{1cm} (2.14b)

\[ \overline{a}_5 = \frac{1}{\sqrt{2}} [\text{Re} \rho_{1s} - \text{Re} \rho_{1s}^y] = |U||S| \cos(\gamma_{SU}) \]

\[ \overline{a}_6 = \frac{1}{2} [\text{Re} \rho_{0s} - \text{Re} \rho_{0s}^y] = |L||S| \cos(\gamma_{SL}) \]

In the Equation (2.13b) and (2.14b) we have introduced explicitly the cosines of relative phases between the nucleon transversity amplitudes.

The SDM elements (2.8c) form the third group of observables\(^1,12\) which is not used in the present amplitude analysis.

The first group (2.13) involves four moduli \(|S|^2, |L|^2, |U|^2\) and \(|N|^2\), and three cosines of relative phases \(\cos(\gamma_{SL}), \cos(\gamma_{SU})\) and \(\cos(\gamma_{LU})\). The second group (2.14) involves the same amplitudes but with opposite nucleon transversity. In Ref. 12 we derived analytical solution for these amplitudes in terms of observables. For the
first group we obtained a cubic equation for $|L|^2 \equiv x$

$$ax^3 + bx^2 + cx + d = 0 \quad (2.15)$$

with coefficients $a, b, c, d$ expressed in terms of observables $a_i, i = 1, 2, \ldots, 6$ (see Ref. 1, 12). The remaining moduli and the cosines are given by expressions

$$|S|^2 = (a_1 + a_2) - 3|L|^2 \quad (2.16)$$

$$|U|^2 = |L|^2 - \frac{1}{2}(a_2 + a_3)$$

$$|N|^2 = |L|^2 - \frac{1}{2}(a_2 - a_3)$$

$$\cos(\gamma_{LU}) = \frac{a_4}{|L||U|}$$

$$\cos(\gamma_{SU}) = \frac{a_5}{|S||U|}, \quad \cos(\gamma_{SL}) = \frac{a_6}{|S||L|}$$

The solution for the second group (2.14) is similar.

The physical solutions of cubic equation (2.15) for $|L|^2$ must produce physical and normalized moduli and physical values for the cosines

$$0 \leq |A|^2 \leq 1, \ A = L, S, U, \bar{N} \quad (2.17)$$

$$-1 \leq \cos \gamma_k \leq 1, \ k = LU, SU, SL$$

There are similar constraints on the solutions for $|L|^2$ in the second group.

The analytical solution of the cubic equation (2.15) is given in Table 1 of Ref. 12. One solution of (2.15) is always negative and it is rejected. The other two solutions are generally positive and close. However in a number of $(m, t)$ bins we get unphysical values for some cosines and in some cases also negative moduli of amplitudes. In some $(m, t)$ bins the mean values of input SDM elements yield
complex solutions for $|L|^2$ or $|\overline{L}|^2$ or both (with positive real parts). To filter out the unwanted unphysical solutions we now turn to Monte Carlo amplitude analysis.

### 3. MONTE CARLO AMPLITUDE ANALYSIS

The origin of the presented Monte Carlo method lies in the problem of error combination and propagation when the function of input uncertain variables is highly nonlinear. This is our case. The analytical solutions for the amplitude $x = |L|^2$ from the cubic equation (2.15) are highly nonlinear functions of the input observables – see Table 1 of Ref. 12. Even if the input spin density matrix (SDM) elements have Gaussian distributions, the solutions of the cubic equation (2.15) and the amplitudes (2.16) are non-Gaussian distributions as the result of the nonlinearity. The question arises how to estimate the errors on the amplitudes.

In general, when the errors on the input variables are small, one can use a linear approximation to error propagation (Ref. 17). This method yields symmetric errors using $1\sigma$ errors as input. This approximation was used in our previous analyses. Since the errors on polarized SDM elements are not actually small, this approximation is not satisfactory.

In his review paper (Ref. 17) F. James advocates the use of Monte Carlo method as perhaps the only way to calculate the errors in case of nonlinear functions which produce non-Gaussian distributions. The method has the added advantage that it can separate the physical and unphysical solutions, something the linear approximation and the $\chi^2$ method cannot do. The Monte Carlo method was first used to calculate errors of analytical solutions of a cubic equation by H. Palka in an
amplitude analysis of reactions $\pi^- p \rightarrow K^+ K^- n$ and $\pi^- p \rightarrow K^- n$ at 63 GeV/c.

In our Monte Carlo amplitude analysis the input SDM elements were randomly varied within their experimental errors in each $(m, t)$ bin and amplitudes were calculated for each new selection of SDM elements. The input SDM elements were varied independently using a multidimensional random number generator SURAND with an initial seed number 80629.0. Since any sequence of random numbers generated by SURAND is reproducible, our results are also reproducible.

Each selection of SDM elements yields two solutions for amplitudes of group 1 (eqs. (2.13))

$$|S|^2, |L|^2, |U|^2, |N|^2 \hspace{1cm} (3.1)$$

$$\cos(\gamma_{LU}), \cos(\gamma_{SU}), \cos(\gamma_{SL})$$

and two solutions for amplitudes of group 2 (eqs. (2.14))

$$|S'|^2, |L'|^2, |U'|^2, |N'|^2 \hspace{1cm} (3.2)$$

$$\cos(\gamma'_{LU}), \cos(\gamma'_{SU}), \cos(\gamma'_{SL})$$

In each group the solution was classified as physical only when all 4 moduli and all 3 cosines of relative phases had physical values. The selection of SDM elements was classified as pass only when all solutions for amplitudes (3.1) and (3.2) were physical solutions. Otherwise the selection was classified as fail.

The Monte Carlo amplitude analysis program was run with a total number of SDM elements selections $N_{tot} = 10\,000, 20\,000$ and $30\,000$ in each $(m, t)$ bin. Each selection is classified as pass or fail according to the above criteria. The passing
rate, or the ratio $N_{\text{pass}}/N_{\text{tot}}$, was nearly constant in each $(m, t)$ bin as $N_{\text{tot}}$ was increased from 10 000 to 20 000 and 30 000 selections. Consequently no further increases in $N_{\text{tot}}$ were attempted. However, there are considerable variations of $N_{\text{pass}}$ from bin to bin. In Figure 1 we show the $m$-dependence of $N_{\text{pass}}$ for the runs with $N_{\text{tot}} = 30 000$ for reactions $\pi^- p_\uparrow \to \pi^- \pi^+ n$ at 17.2 GeV/c and $\pi^+ n_\downarrow \to \pi^+ \pi^- p$ at 5.98 GeV/c. The results at 11.85 GeV/c are similar. For some reason $N_{\text{pass}}$ is lowest for $m \sim 800$ MeV at all 3 energies. At each energy there is one bin which produced no physical solution (880–900 MeV bin for 17.2 GeV/c, 360–440 MeV bin at 5.98 GeV/c, and 820–870 MeV bin at 11.85 GeV/c). At 17.2 GeV we shall omit also the mass bin 640–660 MeV where we found $N_{\text{pass}} = 3$ only.

The pass and fail SDM elements are retained in each $(m, t)$ bin to determine their distribution, range of values and average value. A priori, the ranges and average values are expected to be different for pass and fail selections and could depend on SDM elements. However, we found that for all unpolarized SDM elements in all $(m, t)$ bins the ranges of values for pass and fail Monte Carlo selections are the same and coincide with the ranges given by the original errors. Also surprisingly, the average values of pass and fail unpolarized SDM elements are equal to the mean values of the input SDM elements. For polarized SDM elements we reach the same conclusion for fail Monte Carlo selections. For pass Monte Carlo selections, the polarized SDM elements have smaller ranges in a few $(m, t)$ bins with lowest $N_{\text{pass}}$ values and their average values differ up to 6% from the mean values of input SDM elements. The situation for $\rho_{ss}^y + \rho^y_{00} + 2\rho_{11}^y$ is typical and is illustrated in the Table 1 for $\pi^- p \to \pi^- \pi^+ n$ at 17.2 GeV/c.
The values of moduli and cosines obtained for each pass Monte Carlo selection were collected to determine their range and average values in each \((m, t)\) bin. To study the distribution of values of moduli and cosines for all \(N_{\text{pass}}\) selections, the moduli were binned in bins of size 0.02 and the cosines in bins of size 0.04. An example of such distributions is shown in Figure 2 where we present solutions 1 and 2 for \(\cos(\gamma_{SL})\) at 5.98 GeV/c for \(m = 520–600\) MeV and \(-t = 0.2–0.4\) (GeV/c).\(^2\) Notice the typical differences between the distributions for Solutions 1 and 2. The distribution for solution 1 yields a range of \(\cos\gamma_{SL}\) from 0.52 to 1.00 with an average value 0.94. The distribution for Solution 2 yields a range of \(\cos\gamma_{SL}\) from 0.31 to 1.00 with an average of 0.64. The differences between the two distributions suggest that the \(\chi^2\) minimization approach used in Ref. 3,4,5 to obtain physical solutions may not be dependable.

The resulting ranges of values and average values for all unnormalized moduli and for cosines are shown in Figure 3 for \(\pi^- p \rightarrow \pi^- \pi^+ n\) at 17.2 GeV/c and in Figure 4 for \(\pi^+ n \rightarrow \pi^\pi^- p\) at 5.98 GeV/c. The figures show the mass dependence of the two solutions for moduli and cosines for \(-t = 0.005–0.2\) (GeV/c)\(^2\) at 17.2 GeV/c and \(-t = 0.2–0.4\) (GeV/c)\(^2\) at 5.98 GeV/c. The unnormalized moduli \(|A|^2\Sigma\) and \(|A|^2\Sigma, A = S, L, U, N\) are calculated using \(\Sigma = d^2\sigma/dmdt\) from Ref. 2 at 17.2 GeV/c and from Ref. 10 at 5.98 and 11.85 GeV/c. A comparison with Figures 1 and 2 of Ref. 14 reveals that the physical solutions for \(S\)-wave moduli and for cosines are much smoother and continuous. The jitter which characterized the unphysical solutions is removed by filtering out the unphysical solutions. We discuss the features of the amplitudes in detail in the next two sections.
For each pass Monte Carlo selection we also calculated the normalized partial-wave cross-sections $\sigma_A$ and partial-wave polarizations $\tau_A$ defined in (2.9) in order to obtain their range of values and average values. Using experimental results for $\Sigma = d^2\sigma/dm^2$, we then calculated partial wave intensities $I_A = \sigma_A \Sigma$ where $A = S, L, U, N$. The solutions for amplitudes with opposite transversities are entirely independent. We can denote the two solutions for normalized nucleon transversity amplitudes as $A(i)$ and $\overline{A}(j)$ with $i = 1, 2$ and $j = 1, 2$. Because the moduli squared in (2.9) are independent, there is a fourfold ambiguity in the partial-wave intensities. Using the indices $i$ and $j$ to label the four solutions, we get

$$I_A(i, j) = [ |A(i)|^2 + |\overline{A}(j)|^2 ] \Sigma$$

(3.3)

where $A = S, L, U, N$. The partial-wave intensities (3.3) are defined in the physical region and depend on the dipion mass $m$ and momentum transfer $t$.

The Monte Carlo amplitude analysis presented in this paper involves one important simplification, namely the experimental data points are represented by a rectangular distribution of width $2\sigma$. The $2\sigma$ width is motivated by the requirement to study the propagation of $1\sigma$ errors on experimental data through the highly nonlinear expressions for the amplitudes in order to define the errors on amplitudes. The uniformity of the experimental data distributions is due to the use of the uniform random number generator.

The actual experimental data are presumed to have Gaussian distributions. To examine the effect of Gaussian distribution of data on the results of amplitude analysis, we analysed the data at 17.2 GeV/c and 5.98 GeV/c, using an IBM Gaussian
random number generator SNRAND.\textsuperscript{19}

In this calculation the “fail” distributions of SDM elements are approximately Gaussian with a broad range of half-size of $4.\text{–}4.5\sigma$. The “pass” distributions of SDM elements (giving the physical solutions for amplitudes) have a narrower range of approximate half-size $2\sigma$. Most of the points lie within $1\sigma$ half-interval. Between $1\sigma$ and $2\sigma$ the “pass” distributions have only low tails. This means that most physical solutions lie within $1\sigma$ vicinity of central points and none beyond $2\sigma$.

As expected, the distributions for amplitudes (moduli and cosines) are non-Gaussian. Importantly, the average values of distributions are essentially identical to the average values obtained previously. The agreement is such that the black points in the figures do not change. The range of distributions is now broader. The minimum and maximum values change typically by 0–20%. However this increase is due to only low tails of distributions. The broadening of range of values now makes more difficult the estimation of errors. The usual calculation of standard deviation does not make sense in the case of such highly non-Gaussian distribution as is e.g. Solution 1 for $\cos \gamma_{SL}$ (Fig. 2).

Because the average values of amplitudes remain virtually the same and the distributions for amplitudes acquire only low tails, there is no significant change from the results presented in this paper, and the simplification used in our Monte Carlo amplitude analysis is an acceptable approximation.

4. EVIDENCE FOR A SCALAR STATE $I = 0\ 0^{++}$ (750)

After filtering out the unphysical solutions, the evidence for the scalar state $I =$
0 $0^{++}(750)$ becomes much more convincing than were the previous indications.\textsuperscript{14}

The evidence is based on three observations:

(a) Resonance structure of the $S$-wave partial wave intensity $I_S$.

(b) The resonant structure of unnormalized $S$-wave moduli.

(c) The constant relative phase between the $S$-wave amplitudes $S, \overline{S}$ and the dominant $P$-wave amplitudes $L, \overline{L}$.

We first look at the $S$-wave moduli at 17.2 GeV/c at lower momentum transfers which are summarized in Fig. 5. We see that the amplitude $|\overline{S}|^2\Sigma$ has a clear resonant structure in both solutions. This is a change from previous results in Ref. 14 where only the solution 1 was clearly resonating. In both solutions the maximum is at 750–770 MeV with a width at half-height estimated at 175–200 MeV. The moduli $|S|^2\Sigma$ of opposite transversity have smaller magnitudes and do not show a resonant structure. At first sight this might be surprising, but we see the same effect also in $\rho^0$ production amplitudes. For instance, while the amplitudes $|\overline{L}|^2\Sigma$, $|\overline{U}|^2\Sigma$ and $|N|^2\Sigma$ show resonant structures, the amplitudes $|L|^2\Sigma$, $|U|^2\Sigma$ and $|\overline{N}|^2\Sigma$ are smaller and show less resonant structure. We can ascribe this effect to the spin dependent dynamics of resonance production. The difference of magnitudes of $|\overline{S}|^2\Sigma$ and $|S|^2\Sigma$ is due to the presence of $A_1$-exchange i.e. nonvanishing amplitude $S_0$.

The usual signature of a production resonance is a peak or bump in the production cross-section. The $S$-wave partial-wave intensity $I_S = (|S|^2 + |\overline{S}|^2)\Sigma$ at 17.2 GeV/c is shown in Fig. 6. We find a clear resonant signal at or near 750 MeV in two solutions $I_S(1, 1)$ and $I_S(2, 1)$. The other two solutions $I_S(1, 2)$ and $I_S(2, 2)$ do
not show a clear resonant structure. In these solutions $I_S$ is rising up to 770 MeV, then it drops but remains high. The cause of this behaviour is the nonresonating amplitude $|S|^2 \Sigma$ which is large above 800 MeV. We are thus faced with an apparent ambiguity in the $S$-wave intensity observed also in the earlier studies.\textsuperscript{4,5,6} However more fundamental evidence for $\sigma(750)$ production at 17.2 GeV/c at lower $t$ is the fact that both solutions for the amplitude $|\overline{S}|^2 \Sigma$ resonate. We see the ambiguity in $I_S$ only because the amplitudes $|\overline{S}|^2 \Sigma$ and $|S|^2 \Sigma$ happen to have magnitudes which are not very different. We do not see any ambiguity e.g. in the intensity $I_L$ because the amplitude $|\overline{L}|^2 \Sigma$ is much larger than $|L|^2 \Sigma$ which also remains high above 800 MeV.

As seen in Figure 3, the amplitudes $|\overline{L}|^2 \Sigma$ and $|L|^2 \Sigma$ are dominant $\rho^0$ production amplitudes. The relative phase between the $S$-wave amplitudes $\overline{S}$ and $S$ and the $P$-wave amplitudes $\overline{L}$ and $L$ is another important piece of evidence for existence of the $I = 0^{++}(750)$ resonance. In Solution 1 both relative phases $\gamma_{SL}$ and $\gamma_{SL}$ are consistent with zero and we have a phase degeneracy of amplitudes $\overline{S}$ and $\overline{L}$, and $S$ and $L$. In Solution 2 the relative phases are not zero but are small and constant over the considered mass region. Since the amplitude $\overline{L}$ is resonating, the phase degeneracy with amplitude $\overline{S}$ suggests that $\overline{S}$ also resonates with a resonance mass near $\rho_0$.

The $S$-wave intensity at larger momentum transfers $-t = 0.2-0.4$ (GeV/c)$^2$ at 5.98 and 11.85 GeV/c is shown in Fig. 7 and 8, respectively. There is no ambiguity at larger momentum transfers. At both energies we find a clear resonant signal at or near 750 MeV in all solutions. At 5.98 GeV/c, the width at half-height is estimated
to be 270 MeV. We note that at 11.85 GeV/c, the solutions $I_S(2, 1)$ and $I_S(2, 2)$ are consistent with a narrow width of 150 MeV.

The resonant structure of moduli at larger momentum transfers $-t = 0.2–0.4$ (GeV/c)$^2$ at 5.98 GeV/c and 11.85 GeV/c is less clear. At both energies it is the Solution 2 which has a stronger indication for a resonance in both moduli $|S|^2\Sigma$ and $|S|^2\Sigma$. Because of lower statistics, the errors on input SDM elements are larger which results in larger ranges of values for cosines of relative phases. Nevertheless, we also observe that the relative phases $\gamma_{SL}$ and $\gamma_{SL}$ are near zero in Solution 1, and are only slowly varying over the critical $\rho^0$ mass region in Solution 2. The unusually larger variation of $\cos(\gamma_{SL}), \cos(\gamma_{SL}), \cos(\gamma_{SU})$ and $\cos(\gamma_{SU})$ at $m = 980$ MeV could be due to the presence of scalar resonance $f_0(975)$.

We have also fitted the $S$-wave partial-wave intensities $I_S$ to a Breit-Wigner form using the CERN optimization program FUMILI. The parametrization used had the general form

$$I_A(m) = N_A |BW_R|^2$$

(4.1)

where $A = S, L, U, N$ and

$$BW_R = \left(\frac{m}{\sqrt{q}}\right)\sqrt{2J + 1} \frac{m_R\Gamma}{m_R^2 - m^2 - im_R\Gamma}$$

(4.2)

In (4.1) $N_A$ is a normalization constant which includes square of elasticity $x$ and isospin factor. In (4.2) $q$ is the $\pi^-$ momentum in the $\pi^-\pi^+$ rest frame

$$q = |\vec{q}| = \sqrt{0.25m^2 - \mu^2}$$

(4.3)

where $\mu$ is the pion mass. In (4.2), $J$, $m_R$ and $\Gamma$ are spin, mass and width of the
resonance. The mass dependent width is

$$\Gamma = \Gamma_R \left( \frac{q}{q_R} \right)^{2J+1} \frac{D_J(q_R r)}{D_J(q r)}$$

(4.4)

where $q_R = q(m = m_R)$ and $D_J$ are centrifugal barrier functions of Blatt-Weiskopf\textsuperscript{22}

$$D_0(q r) = 1.$$  \hfill (4.5)

$$D_1(q r) = 1 + (q r)^2$$

In (4.5) $r$ is the interaction radius. The factor $m/\sqrt{q}$ in (4.2) arises from the Chew-Low formula.

For $S$-wave $A = S$, $J = 0$, $R = \sigma$ and $\Gamma$ is independent of $r$. The results of the fit to the two resonating solutions for $I_S$ at 17.2 GeV/c are presented in Table 2. We encountered difficulties in fitting the form (4.1) to the solutions $I_S(1, 2)$ and $I_S(2, 2)$. As seen in Fig. 6, the maximum values of the fits (4.1) are systematically well below the maximum values of the experimental points for $I_S$. We conclude that the parametrization (4.1) does not represent the experimental data very well. The likely reason for this is the sudden drop of values of $I_S$ at 790 and 810 MeV which can be traced back to the dip in $|S|^2 \Sigma$ at 790 MeV (see Fig. 5).

The results of the fits to the $S$-wave intensity $I_S$ at 5.98 GeV/c are presented in Table 3 and shown in Figure 7. Again, the maximum values of the Breit-Wigner fits (4.1) are systematically below the maximum experimental points of $I_S$. Because of large errors on $I_S$ at this energy, the resonance parameters in Table 3 have also larger errors.
5. SPIN DEPENDENCE OF $\rho^0$ PRODUCTION

Resolution of the $\rho^0$ peak seen in the reaction cross-section $d^2\sigma/dmdt$ into its individual spin components (8 moduli of nucleon transversity amplitudes) is shown in Figures 3 and 4 at 17.2 and 5.98 GeV/c, respectively. We stress that this model independent resolution is possible only in measurements on polarized targets. We discussed the $S$-wave contributions in the preceding Section. In this Section we focus on the $P$-wave contributions. We observe several important features in the spin dependence of $\rho^0$ production on the level of amplitudes.

(a) The mass spectrum on the level of amplitudes depends on meson helicity $\lambda$ and on the spin of the recoil nucleon. At low momentum transfer (Fig. 3) the unnatural exchange amplitudes $|L|^2\Sigma$ and $|U|^2\Sigma$ with recoil nucleon spin up are larger than the amplitudes $|L|^2\Sigma$ and $|U|^2\Sigma$ with recoil nucleon spin down. The opposite is true for natural exchange amplitudes $|N|^2\Sigma$ and $|N|^2\Sigma$. These findings are similar to observations reported in Fig. 10 of Ref. 4. At larger momentum transfers (Fig. 4) we also find $|L|^2\Sigma$ and $|N|^2\Sigma$ larger than $|L|^2\Sigma$ and $|N|^2\Sigma$, respectively, but $|U|^2\Sigma \approx |U|^2\Sigma$. These large differences between the unnatural amplitudes with recoil nucleon spin up and down are due to the contributions from large and nontrivial $A_1$-exchange amplitudes $L_0$ and $U_0$. The same interpretation was given in Ref. 4 and 5. We note that the $t$-dependence of amplitudes in the $\rho^0$ mass region also shows evidence for large $A_1$ exchange amplitudes (see Fig. 7 and 15 of Ref. 4, and Fig. 1 and 3 of Ref. 12).

(b) We find unexpected suppression of $\rho^0$ production in several amplitudes. At low momentum transfer (Fig. 3) the moduli $|N|^2\Sigma$ and $|U|^2\Sigma$ are small and flat and do
not show the expected $\rho^0$ peak. Also, while the amplitude $|L|^2\Sigma$ is still large, it does not show a clear $\rho^0$ peak but rather a broad structure. At the larger momentum transfers (Fig. 4) we find $\rho^0$ suppression in the amplitudes $|\mathcal{U}|^2\Sigma$ and $|U|^2\Sigma$ which are small relative to the other $P$-wave amplitudes.\(^{(23)}\)

(c) The mass spectra of the dominant amplitudes $|\mathcal{L}|^2\Sigma$ and $|L|^2\Sigma$ show unexpected narrow structures within the $\rho^0$ mass region which are not seen in the spin-averaged cross-section $I_L$. The $P$-wave intensities $I_L$, $I_N$ and $I_U$ are shown in Fig. 9 and 10 for 5.98 and 17.2 GeV/c, respectively. We notice the narrow range of values in these partial-wave intensities compared to the range of the moduli. This indicates that the values of moduli with recoil nucleon spin up and down are highly correlated for each $P$-wave amplitude.

First we look at the amplitudes at 5.98 GeV/c (Fig. 9). The mass spectrum of the recoil nucleon spin up amplitude $|\mathcal{L}|^2\Sigma$ shows a clear dip at 757 MeV and a peak at 807 MeV. The opposite behaviour is seen in the nucleon spin down amplitude $|L|^2\Sigma$ which peaks at 757 MeV and has a dip at 807 MeV. These spin correlated structures within $\rho_0$ mass region do not appear in the partial-wave intensity $I_L$ which shows a structureless $\rho^0$ line shape.

Next we note a similar situation in the natural exchange amplitudes but at a different mass. The spectrum of $|\mathcal{N}|^2\Sigma$ shows a maximum at 782 MeV which is associated with a pronounced dip in $|\mathcal{N}|^2\Sigma$ at the same mass. The partial wave intensity $I_N$ again shows a structureless line shape expected from a $\rho^0$.

One should note however that the apparently most significant ($2 \sim 3\sigma$) narrow structure at 807 MeV is due to a single deviating date point in the $\rho_{1-1}^y - \rho_n^y$ (see

\(^{(26)}\)
Fig. 8 of Ref. 10). This deviation is not observed in the $\rho_{ss}^{yy} + \rho_{00}^{yy} + 2\rho_{11}^{yy}$, also involving the difference between $|L|^2$ and $|\overline{L}|^2$. This shows that the structures at 5.98 GeV/c can be treated as indications only.

Partial-wave intensity $I_L$ at 17.2 GeV/c at lower momentum transfer (Fig. 10) shows a structureless line shape which peaks at 790 MeV. The expectation that the same structureless line shape is reproduced on the level of amplitudes is not met again. Instead we find spin correlated structures in the amplitude $|\overline{L}|^2$ and $|\overline{L}|^2$. The amplitude $|\overline{L}|^2$ peaks at 790 MeV and has a lower value at 770 MeV. The opposite is seen in the amplitude $|L|^2$ which has a lower value at 790 MeV and peaks at 770 MeV.

The partial-wave intensities presented in Fig. 9 and 10 are solution $I_A(1,1), A = L, U, N$. The other solutions are similar.

At this point one may ask how significant is the evidence for the structures in the $\rho$ line shape on the level of amplitudes. We note that the structures occur in $(m, t)$ bins with relatively high statistics which provides a measure of confidence in the results. Also, the $t$-evolution of mass dependence of moduli of normalized amplitudes shows $t$-dependent structures within $\rho^0$ mass region (see Fig. 1 of Ref. 13). Another possibility is to fit Breit-Wigner form (4.1) to the spin-averaged partial wave intensity $I_L$ and superimpose this Breit-Wigner fit (suitably normalized) over the mass spectra in moduli $|\overline{L}|^2$ and $|L|^2$. The numerical results of the fits to $I_L$ are given in Tables 4 and 5. In Fig. 11 we show the results for reaction $\pi^+n \to \pi^+\pi^-p$ at 5.98 GeV/c and $-t = 0.2 - 0.4$ (GeV/c)$^2$. The Breit-Wigner fit to $I_L$ peaks at 780 MeV. Superimposed over the moduli $|\overline{L}|^2$ and $|L|^2$, the
Breit-Wigner fit shows that the narrow structures in their mass spectra are statistically significant. At 757 MeV, the structures are 1σ effect (dip in $|L|^{2\Sigma}$ and peak in $|L|^{2\Sigma}$). At 807 MeV, the structures are 2–3σ effect (peak in $|L|^{2\Sigma}$ and dip in $|L|^{2\Sigma}$). In Fig. 10 we show the results for $\pi^- p \to \pi^- \pi^+ p$ at 17.2 GeV/c and $-t = 0.005 - 0.2$ (GeV/c). While the Breit-Wigner form fits the intensity $I_L$ very well we see that the mass spectrum of amplitude $|L|^{2\Sigma}$ is narrower than the Breit-Wigner fit and the mass spectrum of amplitude $|L|^{2\Sigma}$ is broader than the Breit-Wigner fit to $I_L$. At present, the evidence for structures in the $\rho$-line shape of amplitudes at larger $t$ is still preliminary and insufficient for theoretical analyses. New experiments on polarized targets with significantly higher statistics than the 60,000 events (at 5.98 GeV/c) in the Saclay measurement\textsuperscript{10} are required to confirm the existence of such structures and to investigate the $t$-dependence of $\rho$ line shape on the level of amplitudes.\textsuperscript{13}

We now briefly discuss the physical significance of the observed spin effects in $\rho^0$ production. The presence of $A_1$-exchange (feature (a)) will be discussed in the next Section. Here we will focus on the features (b) and (c).

Following the discovery of resonances in hadron interactions, it has always been believed that the production and decay of resonances were separate events. For instance, the reaction $\pi^+ n \to \pi^+ \pi^- p$ was thought of as a two step process in the $\rho^0$ mass region: $\rho^0$ production $\pi^+ n \to \rho^0 p$ followed by $\rho^0$ decay $\rho^0 \to \pi^+ \pi^-$. In this picture the $S$-matrix elements factorize into production and decay matrix elements

$$T(\pi^+ n \to \pi^+ \pi^- p) = T(\pi^+ n \to \rho^0 p)\phi(\rho^0)T(\rho^0 \to \pi^+ \pi^-)$$ \hspace{1cm} (5.1)
where $\phi(\rho^0)$ is a $\rho^0$ propagator leading to the Breit-Wigner description of dipion mass dependence of modulus of each production amplitude $|T(\pi^+n \rightarrow \pi^+\pi^-p)|^2$ as well as the production cross-section $\Sigma = d^2\sigma/dm dt$.

It is expected from (5.1) that the same $\rho^0$ resonance line shape seen in the spin-averaged cross section $d^2\sigma/dm dt$ will appear also in the modulus of every $P$-wave spin-dependent production amplitude. The suppression of $\rho^0$ production observed in several spin amplitudes and described above in part (b) seems to invalidate the factorization hypothesis and the underlying simple picture of resonance production. However, there is no theoretical explanation for the suppression of $\rho^0$ production in these spin amplitudes (Fig. 3 and 4). Evidently, the spin amplitudes showing $\rho_0$ suppression cannot be fitted with Breit-Wigner form.

The factorization hypothesis (5.1) also implies that the line shape of the mass spectrum of the decaying resonance does not depend on the nucleon spin and on the momentum transfer $t$. However, the line shapes of the dominant amplitudes $|T|^2\Sigma$ and $|L|^2\Sigma$ at larger $t$ show unexpected structures within the $\rho^0$ mass region which correlate the mass spectra corresponding to opposite nucleon spins. Comparison of the structures in Fig. 9 and 10 indicates a change as we go from low to larger momentum transfer. This suggests that the narrow dips and peaks observed in these moduli within the $\rho^0$ mass region evolve with $t$ (see also Ref. 13). This $t$-dependence of the resonance line shape in the amplitudes is also at variance with the factorization hypothesis.

The suppression of $\rho^0$ production and the narrow structures observed in dominant amplitudes in $\rho^0$ mass region represent a new information on pion production
and raise questions about the nature of hadron resonances. A resonance which is a pole in the amplitude would lead to simple resonance peaks in all moduli without structures within their widths. We also note that the usual models of meson resonances as $q\bar{q}$ bound states do not predict their hadronic widths and line shape.

6. TESTS OF ASSUMPTIONS USED IN DETERMINATIONS OF $\pi\pi$ PHASE SHIFTS.

We have presented a model independent and solution independent evidence for the scalar state $I = 0 \, 0^{++}(750)$ in measurements of $\pi N_{\uparrow} \to \pi^+\pi^- N$ on polarized target. The question arises how to understand the absence of such a state in the accepted solution for $S$-wave phase shift $\delta_0$ in $\pi\pi$ scattering.\(^{2,26−29}\)

Of course, there are no actual measurements of pion-pion scattering and there is no partial-wave analysis of $\pi\pi \to \pi\pi$ in the usual sense. The $\pi\pi$ phase shifts are determined indirectly from measurements of $\pi^- p \to \pi^- \pi^+ n$ on unpolarized target using extrapolations into unphysical region of momentum transfer $t$ and several necessary enabling assumptions. Some of these crucial assumptions lead to predictions for polarized spin-density-matrix (SDM) elements and are thus directly testable in measurements on polarized targets. As we shall see in detail below, these assumptions are invalidated in a major way by the data on polarized targets. We must use the results of measurements on polarized targets to judge the validity of $\pi\pi$ phase shift determinations, and not vice versa. We are thus led to the conclusion, that the indirect and model dependent determinations of $\pi\pi$ phase shifts cannot be correct. This explains the absence of $I = 0 \, 0^{++}(750)$ resonance in $\delta_0$ phase shift.
from these analyses.

To unveil the assumptions used in determinations of $\pi\pi$ phase shifts we will trace step by step the procedure used by Estabrooks and Martin.$^{27,28}$ Their approach has the advantage of being the most transparent and of using the least number of assumptions. Moreover, their assumptions are common to all other determinations of $\pi\pi$ phase shifts from unpolarized $\pi N \to \pi^+\pi^- N$ data.$^{2,26,29}$ We shall restrict our discussion to dipion masses below 1000 MeV where $S$- and $P$-wave dominate.

The starting point are dimeson helicity $\lambda = 0$ pion exchange amplitudes $S_1$ and $L_1$ in the $t$-channel. It is assumed that the $t$- and $m$-dependence in these amplitudes factorizes$^{27,28}$

$$S_1(m, t) = \frac{\sqrt{-t}}{t - \mu^2} F_0(t) \frac{m}{\sqrt{q}} f_0(m)$$

$$L_1(m, t) = \frac{\sqrt{-t}}{t - \mu^2} F_1(t) \frac{m}{\sqrt{q}} \sqrt{3} f_1(m)$$

where $t$ is the momentum transfer at the nucleon vertex, $m$ and $q$ are dipion mass and $\pi^-$ momentum in the $\pi^-\pi^+$ c.m. frame. The form factors $F_J(t)$ describe the $t$-dependence and the functions $f_J(m), J = 0, 1$ describe the mass dependence. Further, the functions $f_J(m)$ are assumed to be partial wave amplitudes in $\pi^-\pi^+ \to \pi^-\pi^+$ reaction at c.m. energy $m$

$$f_0 = \frac{2}{3} f_0^{I=0} + \frac{1}{3} f_0^{I=2}$$

$$f_1 = f_1^{I=1}$$

The partial wave amplitudes $f_J^I$ with definite isospin $I$ are defined so that in the
\(\pi\pi\) elastic region

\[ f^I_J = \sin \delta^I_J e^{i\delta^I_J} \]  \hspace{1cm} (6.3)

The \(\pi\pi\) phase shifts \(\delta^I_J\) can be obtained by extrapolating the production amplitudes \(S_1\) and \(L_1\) from the physical region \(t < 0\) to \(t = \mu^2\). One cannot determine both \(I = 0\) and \(I = 2\) S-wave phase shifts and so values for \(f^2_0\) obtained in analyses of \(\pi^+p \rightarrow \pi^+\pi^+n\) data were used.\(^{27,28}\)

There is no theoretical proof of factorization (6.1) and identification (6.2) of functions \(f_J\) with \(\pi\pi\) partial wave amplitudes. Strictly speaking, the relations (6.1)–(6.3) are definitions of \(\pi\pi\) phase shifts. It is by no means obvious that these phase shifts would coincide with \(\pi\pi\) phase shifts determined directly from real pion-pion scattering. Only such comparison would test the assumption (6.2) which is not testable in measurements of \(\pi N \rightarrow \pi^+\pi^- N\).

Essential element in the determination of \(\pi\pi\) phase shifts is the knowledge of the production amplitudes \(S_1\) and \(L_1\). But an inspection of equations (2.8a) reveals that the amplitudes \(S_1\) and \(L_1\) cannot be calculated from the data on unpolarized target without additional assumptions. There are simply more amplitudes than data. To proceed further all determinations of \(\pi\pi\) phase shifts must assume that all \(A_1\)-exchange amplitudes vanish, i.e.

\[ S_0 = L_0 = U_0 \equiv 0 \]  \hspace{1cm} (6.4)

With the assumptions (6.4) and notation (2.9) the equations (2.8a) now read

\[ c_1 = \rho_{ss} + \rho_{00} + 2\rho_{11} = |S_1|^2 + |L_1|^2 + |U_1|^2 + \sigma_N \equiv 1 \]  \hspace{1cm} (6.5)
\[ c_2 = \rho_{00} - \rho_{11} = |L_1|^2 - \frac{1}{2}|U_1|^2 - \frac{1}{2}\sigma_N \]
\[ c_3 = \rho_{11} = \frac{1}{2}\sigma_N - \frac{1}{2}|U_1|^2 \]
\[ c_4 = \sqrt{2}Re\rho_{10} = Re(U_1 L_1^*) = |U_1||L_1|\cos(\chi_{LU}) \]
\[ c_5 = \sqrt{2}Re\rho_{1s} = Re(U_1 S_1^*) = |U_1||S_1|\cos(\chi_{SU}) \]
\[ c_6 = Re\rho_{0s} = Re(L_1 S_1^*) = |L_1||S_1|\cos(\chi_{SL}) \]

where the relative phases satisfy identity

\[ \chi_{LU} - \chi_{SU} + \chi_{SL} = (\phi_{L_1} - \phi_{U_1}) - (\phi_{S_1} - \phi_{U_1}) + (\phi_{S_1} - \phi_{L_1}) = 0 \quad (6.6) \]

The equations (6.5) are formally similar to the set (2.13) and can be similarly solved.

We obtain

\[ |S_1|^2 = 1 + 2c_2 - 3|L_1|^2 \quad (6.7) \]
\[ |U_1|^2 = |L_1|^2 - (c_2 + c_3) \]
\[ \sigma_N = |L_1|^2 - (c_2 - c_3) \]
\[ \cos \chi_{LU} = \frac{c_4}{|L_1||U_1|} \]
\[ \cos \chi_{SU} = \frac{c_5}{|S_1||U_1|}, \quad \cos \chi_{SL} = \frac{c_6}{|S_1||L_1|} \]

For \(|L_1|^2 \equiv x\) we have a cubic equation

\[ ax^3 + bx^2 + cx + d = 0 \quad (6.8) \]

where

\[ a = 3 \]
\[ b = (1 + 5c_2 + 3c_3) \]
\[ c = (1 + 2c_2)(c_2 + c_3) - 3c_4^2 + c_5^2 + c_6^2 \]
\[ d = (1 + 2c_2)c_4^2 + (c_2 + c_3)c_6^2 - 2c_4 c_5 c_6 \]
Using the data on unpolarized target at 17.2 GeV/c [Ref. 2], Estabrooks and Martin found in all \((m, t)\) bins 3 real solutions. One of the solutions is always negative and it is rejected. The other two solutions for \(|L_1|^2\) yield two solutions for \(|S_1|^2\). Thus there are two solutions for \(S^-\) and \(P^-\) wave phase shifts below 1000 MeV. Solution 1 for \(\delta_0^0\) is nonresonating while the solution 2 resonates at the mass around 770 MeV with a width about 150 MeV. The resonating solution was rejected because it disagreed with a \(\pi^0\pi^0\) mass spectrum from a low statistics experiment on \(\pi^-p \rightarrow \pi^0\pi^0n\) at 8 GeV/c.

The absence of \(A_1\) exchange amplitudes is absolutely crucial for the determination of \(\pi\pi\) phase shifts from unpolarized target data on \(\pi^-p \rightarrow \pi^-\pi^+n\). Fortunately, it is an assumption that can be tested directly in experiments on polarized targets. Using the definitions (2.4) of nucleon transversity amplitudes, the assumptions (6.4) imply that the moduli of unnatural exchange amplitudes with recoil nucleon spin up and down must be equal, i.e.

\[
|S|^2 = |S|^2, \quad |\mathcal{L}|^2 = |L|^2, \quad |\mathcal{U}|^2 = |U|^2
\]  

These predictions are in sharp disagreement with the results of model independent amplitude analysis of the data on polarized target at 17.2 GeV/c (Fig. 3) which finds

\[
|S|^2 > |S|^2, \quad |\mathcal{L}|^2 > |L|^2, \quad |\mathcal{U}|^2 > |U|^2
\]  

Particularly important is the large difference between \(|\mathcal{L}|^2\) and \(|L|^2\) (see also Fig. 10) which can be accounted for only by a large and nontrivial \(A_1\)-exchange contribution from the amplitude \(L_0\). The assumptions (6.4) are thus badly violated. This implies
that the determination of amplitudes $|L_1|^2$ and $|S_1|^2$ through equations (6.7) and (6.8) cannot be correct. This in turn means through (6.1) that the $\pi\pi$ phase shifts cannot be correctly determined from the data on unpolarized target.

Using the assumptions (6.4) in equations (2.8b) leads to the following predictions for polarized SDM elements:

$$\rho_{ss}^y + \rho_{00}^y + 2\rho_{11}^y = -2(\rho_{00}^y - \rho_{11}^y) = +2\rho_{1-1}^y \quad (6.11)$$

$$\text{Re}\rho_{10}^y = \text{Re}\rho_{1s}^y = \text{Re}\rho_{0s}^y \equiv 0 \quad (6.12)$$

The data for polarized SDM elements clearly rule out these predictions as is shown in Fig. 12 and 13 for $\pi^-p \to \pi^-\pi^+n$ reaction at 17.2 GeV/c. We find that $\rho_{ss}^y + \rho_{00}^y + 2\rho_{11}^y$ and $-2(\rho_{00}^y - \rho_{11}^y)$ have large magnitudes but opposite signs while $2\rho_{1-1}^y$ has a small magnitude. The interference terms $\text{Re}\rho_{10}^y$, $\text{Re}\rho_{1s}^y$, and $\text{Re}\rho_{0s}^y$ are all dissimilar and have large non zero values. On the basis of this evidence we again must conclude that the determinations of $\pi\pi$ phase shifts from unpolarized data on $\pi N \to \pi^+\pi^-N$ are questionable.

If the pion exchange amplitudes $S_1$ and $L_1$ cannot be reliably determined from the data on unpolarized targets, the question arises whether these amplitudes can be determined from the data on polarized targets. Unfortunately the data on polarized target do not allow the separation of $\pi$- and $A_1$-exchange amplitudes. In fact it has been shown recently\textsuperscript{31} that the pion exchange amplitudes $S_1$ and $L_1$ can be expressed in terms of data on polarized target plus the $A_1$-exchange amplitudes $S_0$ and $L_0$. These relations show explicitly that the determination of $\pi\pi$ phase shifts from data on polarized target depends on the model used for $A_1$-exchange
amplitudes $S_0$ and $L_0$. We conclude that in the absence of a reliable model for $A_1$-exchange or real pion-pion scattering data the $\pi\pi$ phase shifts remain undetermined.

It is of interest to compare our results for $S$-wave intensity with previous $\pi\pi$ phase shift analyses to assess the effect of $A_1$ exchange and the contribution of isospin $I = 2$ $S$-wave amplitudes.

In Fig. 14 and 15 we compare $S$-wave cross-sections normalized to one at maximum value. The data points correspond to $I_S(1, 1)$ and $I_S(2, 2)$ at 17.2 GeV/c in the physical region of $t$. The curves in Fig. 14 (taken from Ref. 15) are predictions of various determinations of $\pi\pi$ phase shifts. The curves A and D are the two solutions of Estabrooks and Martin (Ref. 27). The curves B and C are predictions from Ref. 32 and 33, respectively. The predictions A, B and C show broad structures in $I_S$ around 750 MeV where the data on polarized target require a narrower structure. The broad structure of predictions A, B and C does not agree well with the current algebra prediction above threshold (curve E, Ref. 34).

The solid curves in Fig. 15 are prediction from a recent fit to $\delta_0^0$ phase shift data by Zou and Bugg.\(^3\)\(^5\) Results from another recent fit by Au, Morgan and Pennington\(^3\)\(^6\) are very similar. The predictions are broad and peak at 860 MeV while our data are much narrower and peak at 750–770 MeV. The predictions from phase shift analyses\(^3\)\(^5\),\(^3\)\(^6\) were calculated using $I_S = |f_0|^2$ where $f_0 = \frac{2}{3} f_0^{I=0}$ with $f_0^{I=0}$ given by (6.3).

All curves in Fig. 14 and the solid curves in Fig. 15 correspond to $I = 0$ $S$-wave intensity in $\pi^+\pi^- \rightarrow \pi^+\pi^-$ reaction. The data points in Fig. 14 and 15 correspond to $S$-wave intensity in $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c in the physical region of $t$ and
include contribution from isospin $I = 2$ S-wave amplitudes. The question arises to what extent the $I = 2$ S-wave amplitudes affect the resonance shape of $\delta(750)$ state seen in the S-wave intensity data $I_S(1,1)$. We cannot answer this question using the data on polarized target since these data do not allow isolation of the two $I = 2$ S-wave amplitudes. However, we can use (6.2) and the data for $I = 2$ phase shift $\delta_0^2$ to calculate the S-wave intensity $I_S = |f_0|^2$ in $\pi^+\pi^-$ scattering with the $I = 2$ contribution.

The experimental information about $I = 2$ S-wave phase shifts comes from the analysis of $\pi^+p \rightarrow \pi^+\pi^+n$ reaction. We used the results from the 1977 analysis of Hoogland et al. in Ref. 37. This analysis presents two very similar results for $\delta_0^2$ based on two different methods (method A and method B). The dashed line in Fig. 15 shows the S-wave intensity of Zou and Bugg with the $I = 2$ correction using $\delta_0^2$ from method A in Ref. 37. The method B yields essentially identical curve. We see that the $I = 2$ correction maintains the overall shape of S-wave intensity in $\pi^+\pi^- \rightarrow \pi^+\pi^-$ and does not secure agreement with the data from $\pi^-p \rightarrow \pi^-\pi^+n$ analysis. We can conclude that the principal cause of the difference between the $\pi\pi$ phase shift analyses and our amplitude analysis of $\pi^- p \rightarrow \pi^-\pi^+n$ data on polarized target is the existence of large $A_1$ exchange amplitudes in this reaction.
7. SUMMARY

The measurements of reactions $\pi^- p^\uparrow \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c and $\pi^+ n^\uparrow \rightarrow \pi^+ \pi^- p$ at 5.98 and 11.85 GeV/c on polarized targets enable a model-independent amplitude analysis for dipion masses below 1000 MeV where $S$-wave and $P$-wave contributions dominate the pion production process. The amplitude analysis yields two similar solutions for 8 moduli and 6 cosines of relative phases. In most $(m,t)$ bins the analytical solution produces unphysical values of some cosines or moduli, violating the conditions $2.17$. These conditions are equivalent to inequalities constraints on SDM elements.\textsuperscript{12,38} To avoid unphysical solutions the constraints should be imposed during the optimization of maximum likelihood function in data tapes analysis in future experiments. This will require the use of methods for constrained optimization.\textsuperscript{16,39,40,41}

The previous amplitude analyses\textsuperscript{14} found evidence for a new scalar state $I = 0^0_{0^+} (750)$ which can be interpreted as the lowest mass scalar gluonium $0^{++} (gg)$.\textsuperscript{14} Because of the significance of this finding it is important to see what happens to the scalar mass spectra when the contamination by unphysical solutions is removed. To filter out the unphysical solutions we used a Monte Carlo approach. The input SDM elements were randomly varied within errors 30 000 times in each $(m,t)$ bin and amplitude analysis was performed for each selection. Only when the 8 moduli and 6 cosines attain physical values in both solutions are their values retained and partial-wave intensities and polarizations are calculated. All such physical solutions are collected and these collections then define the distribution, range and average values for each modulus, cosine of relative phase, partial wave intensity and
polarization.

After filtering out the unwanted unphysical solutions, a clear signal for \( I = 0^{++}(750) \) resonance emerges in all solutions for partial wave intensity \( I_S \) at larger momentum transfers at 5.98 and 11.85 GeV/c, and in two solutions at lower momentum transfers at 17.2 GeV/c. The ambiguity at 17.2 GeV/c is caused by the non-resonating amplitude \( |S|^2 \Sigma \) which masks the resonating behaviour of \( |\overline{S}|^2 \Sigma \). The mass spectra in \( I_S \) peak at or near 750 MeV. The width at half-height is about 230 MeV and 270 MeV at 17.2 and 5.98 GeV/c, respectively. At 11.85 GeV/c, the solutions \( I_S(2,1) \) and \( I_S(2,2) \) are consistent with a narrow width of 150 MeV. At 17.2 GeV/c, the amplitude \( |\overline{S}|^2 \Sigma \) clearly resonates in both solutions at 750–770 MeV with a width at half-height estimated to be 175 MeV. The production of \( I = 0^{++}(750) \) appears suppressed in the amplitude \( |S|^2 \Sigma \). There is a clear phase degeneracy between amplitudes \( \overline{S} \) and \( \overline{L} \), and \( S \) and \( L \), also indicating resonant structure of the \( S \)-wave near \( \rho^0 \) mass.

The results for \( \rho^0 \) production generally confirm the findings of previous analyses.\(^4\,12,\,13,\,14\) We find significant suppression of \( \rho^0 \) production in amplitudes \( |\overline{N}|^2 \Sigma \) and \( |U|^2 \Sigma \) at all energies. There is an additional \( \rho^0 \) suppression in \( |L|^2 \Sigma \) at 17.2 GeV/c and in \( |\overline{U}|^2 \Sigma \) at 5.98 GeV/c. Moreover, the line shapes of \( |\overline{L}|^2 \Sigma \) and \( |L|^2 \Sigma \) show unexpected structures within \( \rho^0 \) mass region which correlate the mass spectra corresponding to opposite nucleon spins. These narrow structures are not observed in the spin averaged partial wave intensities \( L_L \) which show the expected structureless \( \rho^0 \) peak. There is no theoretical explanation for these important features of \( \rho^0 \) production which contradict the factorization hypothesis.
All determinations of $\pi\pi$ phase shifts from unpolarized data on $\pi^-p \to \pi^-\pi^+n$ rely on the absence of $A_1$-exchange amplitudes. This essential assumption leads to predictions for polarized SDM elements which are clearly ruled out by the data on polarized target. In our Monte Carlo amplitude analysis we find a clear evidence for large and nontrivial $A_1$-exchange contributions from $S$-wave and $P$-wave amplitudes, in particular at 17.2 GeV/c. On the basis of this evidence we come to the conclusion that the usual determinations of $\pi\pi$ phase shifts cannot be correct. This also explains why the accepted solution for $\delta_0^0$ phase shift shows no evidence for the $\sigma(750)$ resonance.

Production of scalar state $I = 0^++(750)$, the suppression of $\rho^0$ production in certain production amplitudes and the possible narrow structures in moduli in $\rho^0$ mass region are unexpected and important findings of measurements of $\pi N \to \pi^+\pi^-N$ reactions on polarized targets. These new phenomena should be further investigated experimentally in a new generation of dedicated high statistics experiments with polarized targets.$^{42,43}$

Acknowledgments

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\[
\frac{1}{4}(a_1 + a_2) \leq 1
\]

\[
\frac{1}{2}|a_3| \leq \frac{4}{3} - \left[\frac{1}{3}(a_1 + a_2) - \frac{1}{2}a_2\right]
\]

\[
|a_4| \leq \sqrt{1 - \frac{1}{2}(a_2 + a_3)}
\]

Also, in eqs. (2.16a) of ref. 12 \(\rho_{00} + \rho_{11}\) should read \(\rho_{00} - \rho_{11}\).

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| INPUT | PASS | FAIL |
|-------|------|------|
| MASS  | MEAN | ERROR | MIN  | MAX  | AVRG | MIN  | MAX  | AVRG |
| 0.610 | -0.702 | 0.096 | -0.796 | -0.606 | -0.653 | -0.798 | -0.606 | -0.703 |
| 0.630 | -0.620 | 0.085 | -0.704 | -0.535 | -0.585 | -0.705 | -0.535 | -0.624 |
| 0.650 | -0.762 | 0.078 | -0.823 | -0.808 | -0.816 | -0.840 | -0.684 | -0.761 |
| 0.670 | -0.596 | 0.067 | -0.663 | -0.529 | -0.600 | -0.663 | -0.528 | -0.594 |
| 0.690 | -0.535 | 0.057 | -0.584 | -0.479 | -0.509 | -0.592 | -0.479 | -0.534 |
| 0.710 | -0.592 | 0.050 | -0.642 | -0.542 | -0.585 | -0.642 | -0.542 | -0.591 |
| 0.730 | -0.610 | 0.046 | -0.656 | -0.564 | -0.595 | -0.656 | -0.564 | -0.610 |
| 0.750 | -0.588 | 0.043 | -0.631 | -0.547 | -0.591 | -0.631 | -0.546 | -0.587 |
| 0.770 | -0.489 | 0.043 | -0.532 | -0.447 | -0.486 | -0.532 | -0.447 | -0.488 |
| 0.790 | -0.557 | 0.043 | -0.598 | -0.514 | -0.543 | -0.599 | -0.514 | -0.555 |
| 0.810 | -0.411 | 0.046 | -0.410 | -0.366 | -0.380 | -0.457 | -0.365 | -0.410 |
| 0.830 | -0.436 | 0.053 | -0.489 | -0.383 | -0.420 | -0.489 | -0.383 | -0.436 |
| 0.850 | -0.461 | 0.057 | -0.488 | -0.404 | -0.426 | -0.518 | -0.404 | -0.460 |
| 0.870 | -0.340 | 0.060 | -0.399 | -0.280 | -0.315 | -0.401 | -0.280 | -0.342 |
| 0.890 | -0.372 | 0.064 | – | – | – | -0.436 | -0.308 | -0.371 |

**Table 1.** The mass dependence of $\rho_{ss}^y + \rho_{00}^y + 2\rho_{11}^y$ in $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c for $-t = 0.005 - 0.2$ (GeV/c). The input mean values and errors are compared with the ranges and average values of $\rho_{ss}^y + \rho_{00}^y + 2\rho_{11}^y$ for pass and fail Monte Carlo selections for $N_{tot} = 30 000$. 
| Solution | $I_S$  | $m_\sigma$ | $\Gamma_\sigma$ | $N_S$   | $\chi^2/d.o.f.$ |
|----------|--------|------------|----------------|--------|----------------|
| (1,1)    | 764±6  | 273±24     | 1.97±0.07      | 0.360  |
| (2,1)    | 761±12 | 290±54     | 2.16±0.16      | 0.129  |

Table 2. Results of the fits to two resonating solutions of $S$-wave intensity $I_S$ in $\pi^- p \to \pi^- \pi^+ n$ at 17.2 GeV/c using the parametrization (4.1).
| Solution | $I_S$ (MeV) | $m_\sigma$ (MeV) | $\Gamma_\sigma$ (MeV) | $N_S$ | $\chi^2/d.o.f.$ |
|----------|-------------|------------------|-----------------------|-----|-------------|
| (1,1)    | 732 ± 17    | 244 ± 36         | 0.60 ± 0.12           | 0.60 ± 0.12 | 0.913       |
| (1,2)    | 700 ± 33    | 300 ± 66         | 1.21 ± 0.22           | 1.21 ± 0.22 | 0.126       |
| (2,1)    | 740 ± 32    | 296 ± 116        | 1.02 ± 0.29           | 1.02 ± 0.29 | 0.204       |
| (2,2)    | 711 ± 20    | 300 ± 60         | 1.70 ± 0.21           | 1.70 ± 0.21 | 0.238       |

**Table 3.** Results of the fits to the four solutions of $S$-wave intensity $I_S$ in $\pi^+n \rightarrow \pi^+\pi^-p$ at 5.98 GeV/c using the parametrization (4.1).
| Solution | $I_L$  | $m_\rho$ (MeV) | $\Gamma_\rho$ (MeV) | $r$ (GeV$^{-1}$) | $N_L$ | $\chi^2$/d.o.f. |
|----------|--------|---------------|---------------------|-----------------|------|----------------|
| (1,1)    | 777 ± 1 | 160 ± 2       | 4.9 ± 0.5           | 4.52 ± 0.03     | 2.918|
| (1,2)    | 777 ± 1 | 157 ± 3       | 4.5 ± 0.7           | 4.50 ± 0.05     | 1.106|
| (2,1)    | 777 ± 1 | 160 ± 3       | 4.8 ± 0.7           | 4.50 ± 0.04     | 1.551|
| (2,2)    | 777 ± 1 | 157 ± 3       | 4.3 ± 0.6           | 4.48 ± 0.04     | 1.596|

**Table 4.** Results of the fits to the four solutions of $P$-wave intensity $I_L$ in $\pi^-p \rightarrow \pi^-\pi^n$ at 17.2 GeV/c using the parametrization (4.1).
Table 5. Results of the fits to the four solutions of $P$-wave intensity $I_L$ in $\pi^+n \rightarrow \pi^+\pi^-p$ at 5.98 GeV/c using the parametrization (4.1).

| Solution | $I_L$    | $m_\rho$  | $\Gamma_\rho$ | $r$    | $N_L$    | $\chi^2$/d.o.f. |
|----------|----------|------------|----------------|--------|----------|-----------------|
| (1,1)    | 780±3    | 195 ± 8    | 5.9 ± 1.1      | 1.64 ± 0.05 | 0.670   |
| (1,2)    | 779±5    | 189 ± 12   | 4.3 ± 1.3      | 1.59 ± 0.07 | 0.214   |
| (2,1)    | 778±5    | 192 ± 11   | 4.7 ± 1.3      | 1.59 ± 0.07 | 0.206   |
| (2,2)    | 779±5    | 182 ± 12   | 3.6 ± 1.0      | 1.55 ± 0.07 | 0.212   |
Figure Captions

Fig. 1. Number of physical solutions out of 30 000 Monte Carlo selections of SDM elements as a function of dipion mass for $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c and
$-t = 0.005 - 0.2$ (GeV/c),\(^2\) and for $\pi^+ n \rightarrow \pi^+ \pi^- p$ at 5.98 GeV/c and $-t = 0.2 - 0.4$ (GeV/c).\(^2\)

Fig. 2 Distributions of physical values of $\cos(\gamma_{SL})$ for $\pi^+ n \rightarrow \pi^+ \pi^- p$ at 5.98 GeV/c
and $-t = 0.2 - 0.4$ (GeV/c)\(^2\) in the dipion mass bin $520 \leq m \leq 600$ MeV. The
distribution for Solution 1 yields a range of $\cos(\gamma_{SL})$ from 0.52 to 1.00 with an
average value of 0.94. The distribution for Solution 2 yields a range of $\cos(\gamma_{SL})$
from 0.31 to 1.00 with an average of 0.64.

Fig. 3. The mass dependence of physical solutions for moduli squared of $S$-wave and $P$-
wave nucleon transversity amplitudes and cosines of their relative phases in the
reaction $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.29 GeV/c and momentum transfers $-t = 0.005 -$
0.2 (GeV/c).\(^2\) The results are in the $t$-channel dipion helicity frame.

Fig. 4. The mass dependence of physical solutions for moduli squared of $S$-wave and
$P$-wave nucleon transversity amplitudes and cosines of their relative phases in
the reaction $\pi^+ n \rightarrow \pi^+ \pi^- p$ at 5.98 GeV/c and momentum transfers $-t = 0.2 -$
0.4 (GeV/c).\(^2\) The results are in the $t$-channel dipion helicity frame.

Fig. 5. The mass dependence of physical solutions for moduli squared of $S$-wave
transversity amplitudes $|\Sigma S|$ and $S$ in $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c and mo-
mentum transfer $-t = 0.005 - 0.2$ (GeV/c).\(^2\) Both solutions for the amplitude
$|\Sigma S|^2$ resonate at 750–770 MeV while the amplitude $|S|^2$ is nonresonating in
both solutions.
Fig. 6. Four solutions for $S$-wave partial wave intensity $I_S$ in the reaction $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c and $-t = 0.005$–0.2 (GeV/c). The curves are fits to the Breit-Wigner form (4.1). The fitted parameters are given in Table 2.

Fig. 7. Four solutions for $S$-wave partial wave intensity $I_S$ in the reaction $\pi^+ n \rightarrow \pi^+ \pi^- p$ at 5.98 GeV/c and $-t = 0.2$–0.4 (GeV/c). The curves are fits to the Breit-Wigner form (4.1). The fitted parameters are given in Table 3.

Fig. 8. Four solutions for $S$-wave partial wave intensity $I_S$ in the reaction $\pi^+ n \rightarrow \pi^+ \pi^- p$ at 11.85 GeV/c and $-t = 0.2$–0.4 (GeV/c).

Fig. 9. Comparison of mass dependence of unnormalized moduli of $P$-wave production amplitudes and associated partial wave intensities in reaction $\pi^+ n_\uparrow \rightarrow \pi^+ \pi^- p$ at 5.98 GeV/c and $-t = 0.2$–0.4 (GeV/c). Shown are solutions 1. The other combinations of solutions are similar. The lines are to guide the eye.

Fig. 10. Comparison of mass dependence of unnormalized moduli of $P$-wave production amplitudes and associated partial wave intensities in reaction $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c and $-t = 0.005$–0.2 (GeV/c). Shown are solutions 1. The other combinations of solutions are similar. Also shown is the Breit-Wigner fit to spin-averaged partial wave intensity $I_L(1,1)$. The Breit-Wigner fit is scaled and compared with the moduli of spin amplitudes $|\tilde{L}|^2 \Sigma$ and $|L|^2 \Sigma$.

Fig. 11. Breit-Wigner fit to spin-averaged partial wave intensity $I_L(1,1)$ in $\pi^+ n \rightarrow \pi^- \pi^+ p$ at 5.98 GeV/c for $-t = 0.2$–0.4 (GeV/c). The Breit-Wigner fit is scaled and compared with the moduli of spin amplitudes $|\tilde{L}|^2 \Sigma$ and $|L|^2 \Sigma$.

Fig. 12. Test of predictions $\rho_{ss}^y + \rho_{00}^y + 2\rho_{11}^y = -2(\rho_{00}^y - \rho_{11}^y) = +2\rho_{1}^y$ due to vanishing $A_1$-exchange in the reaction $\pi^- p \rightarrow \pi^- \pi^+ n$ at 17.2 GeV/c and $-t = 0.005$–0.2.
Fig. 13. Test of predictions $Re\rho_{10}^y = Re\rho_{1s}^y = Re\rho_{0s}^y = 0$ due to vanishing $A_1$-exchange in the reaction $\pi^- p \to \pi^- \pi^+ n$ at 17.2 GeV/c and $-t = 0.005–0.2$ (GeV/c).\(^2\)

Fig. 14. $S$-wave intensity normalized to one at maximum value. The data correspond to solutions $I_S(1, 1)$ and $I_S(2, 2)$ at 17.2 GeV/c. The smooth curves are predictions based on $\pi^+\pi^- \to \pi^+\pi^-$ analyses (A and D from Ref. 27, B from Ref. 32 and C from Ref. 33) and on current algebra and PCAC (E from Ref. 34). The curves are taken from Ref. 15. Notice that the curves correspond to $I = 0$ $S$-wave intensity in $\pi^+\pi^- \to \pi^+\pi^-$ reaction while the data correspond to $S$-wave intensity in $\pi^- p \to \pi^- \pi^+ n$ in the physical region of $t$ and include contribution from $I = 2$ $S$-waves.

Fig. 15. The $I = 2$ contribution to $S$-wave intensity in $\pi^+\pi^-$ scattering. The solid curve is $I = 0$ $S$-wave intensity in $\pi^+\pi^- \to \pi^+\pi^-$ from Ref. 35. The dashed curve shows this $S$-wave intensity with $I = 2$ contribution included. The $I = 2$ phase shifts were taken from Ref. 37. The data correspond to solution $I_S(1, 1)$ and $I_S(2, 2)$ at 17.2 GeV/c.