Non-relativistic effects on the entanglement entropy of low energy field theories

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We provide the leading contribution to the quantum entanglement in a system described by a general Lagrangian containing first- and second-order time derivatives. This can be seen as an interpolation between a non-relativistic and a relativistic system where only first- or second-order time derivatives appear, respectively. The presence of both terms induces an area law for the entanglement entropy growth in the system together with a finite correlation length. Our analytic predictions for the entanglement entropy and correlation length are successfully compared to previous numerical results in two sample systems: the Mott insulator to superfluid transition and the ground state of ferrimagnetic materials.

In the last decades quantum entanglement has provided a new perspective to well-established concepts in quantum many-body physics [1]. On one hand, quantum entanglement is the key ingredient for the most daring applications of quantum technologies, e.g. see examples in the European quantum technologies roadmap [2]. On the other hand, it is a powerful tool to detect and characterize quantum phase transitions (QPT) [3, 4]. An already well established measure of quantum entanglement is the entanglement entropy. Given a surface that separates the system into two subsystems, it is defined as the von Neumann entropy associated with one of the two subsystems. In Lorentz invariant systems this entropy has been shown to satisfy an area law [5–10], i.e. it scales with the area of the surface up to logarithmic corrections.

In condensed matter physics, non-relativistic quantum field theories arise as low-energy effective approximations for quantum many-body systems. One example are systems near a QPT with a dynamical critical exponent different from one [11], which indicates the different scaling of space and time. A second example are non-Lorentz-invariant systems in a broken phase, i.e. the ground state spontaneously breaks the symmetry of the Hamiltonian. Goldstone’s theorem ensures that the low-energy physics is dictated by the presence of Nambu-Goldstone (NG) bosons [12, 13], but the lack of Lorentz invariance implies a dramatic change in the nature of these bosons [14–17]. Non-relativistic NG bosons have been extensively studied recently [17] and naturally appear in the many-body context [15].

In the present letter we compute the entanglement present in general low-energy non-relativistic field theoretical descriptions. The latter appear both in particle physics at finite chemical potential, e.g. kaon condensation [19], and also in a large number of phenomena in condensed matter physics, see examples in [11]. We obtain simple expressions for the entanglement entropy, which show an area law, and provide the proportionality constant.

Non-relativistic low-energy theory. We consider the general Lagrangian describing the dynamics of a complex field \( \psi(\vec{r}, t) \),

\[
\mathcal{L} = i K_1 \left( \psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) + K_2 \left| \frac{\partial \psi}{\partial t} \right|^2 - K_3 \left| \nabla \psi \right|^2 - c_4 |\psi|^4, \tag{1}
\]

with \( c_4 > 0 \). This can be seen as an interpolation between a purely relativistic Lagrangian and a non-relativistic one. This interpolation is due to the presence of two terms with different order in time derivatives. The \( K_2 \) term which is of second-order resembles a relativistic theory where space and time scale in the same way. On the other hand, the \( K_1 \) term is of first-order and corresponds to a non-relativistic theory where the evolution is dictated by a Schrödinger equation.

The Lagrangian (1) is of interest for the condensed matter community where Lorentz invariance is usually absent [11, 20]. This invariance appears effectively in systems which are particle-hole symmetric, such as superconductors, where the equation of motion has to be symmetric under complex conjugation [21, 22]. This imposes the coefficient of the first-order time derivative to vanish \( K_1 = 0 \). On the other hand, in pure non-relativistic systems, \( K_2 = 0 \), the dynamics is driven by a Schrödinger equation, e.g. superfluid Helium.

An equivalent representation of \( \mathcal{L} \) is,

\[
\mathcal{L} = K_2 \left| \left( \partial_i - i \mu_r \right) \psi \right|^2 - K_3 \left| \nabla \psi \right|^2 - m^2 |\psi|^2 - c_4 |\psi|^4, \tag{2}
\]

where we have introduced the relativistic chemical potential \( \mu_r = K_1 / 2 K_2 \) and the mass \( m^2 / K_2 = c_2 / K_2 + \mu_r^2 \). This second representation was studied in the context of relativistic Bose-Einstein condensates [23, 24]. Fur-
thermore, a more novel application is the study of non-relativistic NG bosons \[^{10,17,25}\] , which naturally appear in systems at finite chemical potential \[^{26–28}\] . It has been shown that the interplay between first- and second-order time derivatives leads to the appearance of massive NG bosons \[^{29}\] .

We consider two distinct sets of systems. The first are physical systems near a QPT, in this case \(\mathcal{L}\) can be seen as a generalization of the Ginzburg-Landau theory \[^{30,31}\] . The second ones are systems which are in a broken phase, here the Lagrangian \[^{1}\] \((c_2 = c_4 = 0)\) describes the low-energy physics of the NG bosons \[^{10,17}\] .

**Low-energy excitations.** The elementary excitations of the model \[^{1}\] have been previously obtained in the Gaussian approximation \[^{21,22}\] . Within this approach one studies quadratic fluctuations above a fixed vacuum, \(\psi = \psi_0 + (\phi_1 + i\phi_2)/2\), which is zero \(\psi_0 = 0\) (non-zero \(\psi_0 = \sqrt{-c_2/(2c_4)}\)) when \(c_2 > 0\) \((c_2 < 0)\). In the following we review the main results, which are later used to compute the entanglement entropy.

When \(c_2 > 0\) the system is in the disordered phase with a vanishing vacuum \(\psi_0 = 0\). In this situation the two excitations \(w_\pm(k)\) are gapped with gap,

\[
\Delta_\pm = \sqrt{\left(\frac{K_1}{2K_2}\right)^2 + \frac{c_2}{K_2}} \pm \frac{K_1}{2K_2}.
\]

At the critical point, \(c_2 = 0\), one of the two excitations becomes gapless and develops a quadratic dispersion relation \(w_- (k) = K_3 k^2/K_1\). Finally, in the ordered phase, \(\psi_0 \neq 0\), one finds a gapless linear dispersion relation and a gapped one,

\[
\begin{align*}
\Delta_H &= \sqrt{2} \left(\frac{K_2}{2K_2} - \frac{c_2}{K_2}\right), \\
\Delta \approx \sqrt{\frac{K_3}{K_2}} - \frac{K_1 K_4}{K_2^2} \frac{1}{\Delta_H} k, \\
\end{align*}
\]

The presence of a gapless mode in the broken phase is expected from Goldstone’s theorem \[^{12,13}\] . The gapped mode is usually called the Higgs mode \[^{21,22}\] and is associated with amplitude excitations of the field.

As we have seen, the nature of the excitations drastically changes in the different phases of the model \[^{1}\] , e.g. gapped and gapless with quadratic or linear dispersion relations. We will show how this affects the quantum entanglement of the system.

**Quantum entanglement.** In the Gaussian approximation, the Schrödinger representation of the ground state \(|0\rangle\), i.e. the vacuum of elementary excitations, \(\Psi [\psi, \psi^*] \equiv \langle \psi, \psi^* |0\rangle\), reads,

\[
\Psi [\psi, \psi^*] = \mathcal{N} \exp \left\{ - \int d\vec{r} d\vec{r}' \psi^*(\vec{r}) \bar{\psi}(\vec{r}', \psi(\vec{r}')) \right\},
\]

where \(\mathcal{N}\) is a normalization factor and \(\bar{\psi}(k) = K_2 (w_+(k) + w_-(k))/2\). Let us emphasize that \(\Psi [\psi, \psi^*]\) is found to depend solely on the sum \(w_+(k) + w_-(k)\). This feature was already found for the partition function at zero temperature \[^{23}\] , and carries over to the entanglement entropy, following the arguments in Refs. \[^{5,7,9}\] .

It turns out that \(\bar{\psi}(k)\) can be expressed in the following form \(\bar{\psi}(k) = \sqrt{(ck)^2 + M^2}\) with,

\[
M = \sqrt{\left(\frac{K_1}{2K_2}\right)^2 + \frac{1}{1 - 3\Theta(-c_2)} \frac{c_2}{K_2}}, \quad c = \sqrt{\frac{K_3}{K_2}}.
\]

where we identified \(M\) as an effective mass and \(c\) as the maximum propagation velocity of the excitations. Notice that this calculation of \(\bar{\psi}(k)\) is exact for the unbroken phase \(c_2 \geq 0\), as can be inferred from \[^{9}\] , but is not valid deep inside the broken phase \(c_2 \ll -1\), where the sum of both dispersion relations \[^{1}\] becomes more involved. The effective mass \[^{6}\] has two different contributions. One comes from the quadratic term in the Lagrangian \[^{1}\] and vanishes at the transition, \(c_2 = 0\). The other comes from the presence of the non-relativistic term. Because of that, the correlation length of the system \(\xi = c/M\) remains finite at the transition \(\xi = 2\sqrt{K_3 K_2}/K_1\) and only diverges when the pure relativistic theory is considered, \(K_1 = 0\). We can detect this finite correlation length inspecting the derivative \(\partial M/\partial c_2\) around \(c_2 \approx 0\). A divergence is expected if the correlation length only has a contribution coming from \(c_2\). Instead, if \(K_1 \neq 0\) this derivative remains finite around the transition, see Eq. \[^{9}\] . This is expected since using the parameters \(m\) and \(\mu_r\) the phase transition, \(c_2 = 0\), occurs for a finite value of the mass \(m^2 = K_2 H_r\), which sets the value of the correlation length. Finally, let us mention that for \(c_2 \geq 0\) the effective mass \(M\) corresponds to the original mass \(m/\sqrt{K_2}\) of the Lagrangian \[^{2}\] . This implies that the entanglement entropy is independent of the chemical potential \(\mu_r\) in the unbroken phase, as we will see later. This is a natural extension of the Silver Blaze problem, i.e. at zero temperature thermodynamical observables are independent of the chemical potential up to some critical value \[^{33}\] , to the entanglement entropy.

The Gaussian nature of the ground state wavefunction \[^{5}\] directly implies an area law for the entanglement entropy, \(S_E = aA\) \[^{5,7,9}\] . The proportionality constant, \(a\), can be computed exactly and depends on the correlation length of the system \[^{7,8,34}\] , \(a \sim \xi^{1-D}\) (with \(D\) the number of space dimensions), with logarithmic corrections appearing for odd values of \(D\). Using the explicit expressions \[^{6}\] we find for \(D = 2\) ,

\[
S_E/A = - \frac{1}{6} \sqrt{\frac{K_3}{4K_3 K_3}} + \frac{1}{1 - 3\Theta(-c_2)} \frac{c_2}{K_3},
\]

up to some constant quantity independent of the mass \(M\) \[^{5,7,9}\] . Interestingly, for the ordered phase, \(c_2 \leq 0\), we find a direct connection between the entanglement entropy and the Higgs gap, an idea mentioned in \[^{35}\] .
for the Bose-Hubbard model. From (1), (6), and (7) one gets,

\[ S_E/A = -\frac{\Delta_H}{12c}. \]  

(8)

The entanglement entropy \( S_E/A \) presents a cusp at the transition, \( c_2 = 0 \). The cusp becomes more prominent when \( K_1 \to 0 \), thus providing an indicator of the corresponding divergent correlation length. We can quantify this with the jump of the derivatives of \( S_E/A \) at the transition,

\[ \frac{\partial S_E/A}{\partial c_2} \bigg|_{c_2=0^+} - \frac{\partial S_E/A}{\partial c_2} \bigg|_{c_2=0^-} = -\frac{1}{4K_1} \sqrt{\frac{K_2}{K_3}}. \]  

(9)

Thus, a finite jump would correspond to \( K_1 \neq 0 \) and a divergent one to \( K_1 = 0 \).

The results presented above are very general and affect a large variety of models whose low energy dynamics can be captured by the Lagrangian (1). In the following we provide two prominent examples where our results can be directly applied.

**The Bose-Hubbard model.** Our formulae apply readily to one of the best explored QPTs, the Mott-insulator (MI) to superfluid (SF) transition in the Bose-Hubbard (BH) model. This model describes a Bose gas with contact-like interactions confined in a \( D \)-dimensional hypercubic optical lattice at zero temperature \( [36] \). The second quantized Hamiltonian reads,

\[ H = -J \sum_{\langle i,j \rangle} (b_i^\dagger b_j + h.c.) + \frac{U}{2} \sum_i (b_i^\dagger)^2 b_i^2 - \mu \sum_i b_i^\dagger b_i, \]

(10)

where \( b_i (b_i^\dagger) \) are bosonic annihilation (creation) operators on site \( i = 1, ..., L \), \( \mu \) is the chemical potential, \( J \) is the hopping strength and \( U \) is proportional to the two-boson interaction strength.

The QPT has been observed in ultracold atomic gases experiments \([37]\). Near the critical transition point an effective low-energy description of the system is applicable \([11, 36, 38]\). This effective description coincides with the Lagrangian (1), where the order parameter is proportional to the vacuum expectation value of the bosonic annihilation operator \( \psi(\vec{r},t) \propto (b_1(t)) \). The coefficients of the Lagrangian (1) can be expressed in terms of the BH parameters \([11, 39]\), \( c_2 = \mu(U - \mu)/(U + \mu) - zJ, K_1 = -\partial_\mu c_2, K_2 = -(1/2)\partial_\mu^2 c_2 \) and \( K_3 = J \). Where we introduced the coordination number \( z = 2D \), and we write the expressions focusing on the first MI lobe with mean density \( n = 1 \).

The MI-SF transition occurs at \( c_2 = 0 \) and can be crossed along two different possible paths changing the hopping \( J \), indicated in Fig. 1 as (a) and (b). In (a) there is particle-hole symmetry, or a relativistic low-energy description, identified with the condition \( K_1 = 0 \) which corresponds to crossing the transition at the tip of the lobe \( \mu/U = \sqrt{2} - 1 \). In (b) this symmetry is absent,
$K_1 \neq 0$. It corresponds to crossing the transition at any other point away from the tip of the lobe. The comparison between our analytical formulas for the entanglement entropy along these two paths ([1]), and the numerical results reported in [25] is shown in Fig. 2 (panels 1 and 2). The agreement is in all cases extremely good with only small discrepancies deep inside the SF phase, where deviations are expected due to the Gaussian approximation.

To confirm that the correlation length remains finite at the transition in the non-relativistic case, $K_1 \neq 0$ we computed the derivative of the entropy along the trajectory (b). Clearly, a finite jump is observed which coincides with the value predicted by Eq. [9]. This finite correlation length found at the MI-SF transition for $K_1 \neq 0$ seems to indicate the change of nature of the transition with respect to the case with $K_1 = 0$ which has an infinite correlation length.

From [7] we predict that the entanglement entropy does not depend on $\mu$, in the unbroken phase. This result, applied to the BH model, implies that inside the MI phase the entanglement entropy does not depend on the value of the chemical potential $\mu$. This is actually simply the abovementioned Silver Blaze problem [33]. Therefore, the behavior of entanglement entropy along trajectories (a) and (c) in Fig. 1 is identical, although in (c) we fix $c_2 = 0$ and vary $K_1$ and in (a) we fix $K_1 = 0$ and vary $c_2$. On the SF side (close to the critical point) we predict that the entanglement entropy should decrease linearly with the chemical potential, trajectory (d). This is in agreement with the numerical calculation close to the critical point, see Fig. 2 (panel 3).

**Non-relativistic Nambu-Goldstone bosons.** The Lagrangian [1] (with $c_2 = c_4 = 0$) has been proposed as the low-energy description of NG bosons without assuming Lorentz invariance [16, 17]. Specifically, the Lagrangian corresponds to the case where the original system has a rotational symmetry $SO(3)$ which is broken down to $SO(2)$, i.e. the ground state chooses a particular direction. Therefore, the original complex scalar field can be identified with two NG fields $\psi(\vec{r}, t) = \pi_1(\vec{r}, t) + i\pi_2(\vec{r}, t)$. One corresponds to a type-B NG boson and has a quadratic dispersion relation [17]. The other one is the so-called gapped partner [26, 29]. This coincides with the general classification given in [10, 32].

From our previous discussion one sees that these systems with type-B NG bosons and a gapped partner exhibit a finite correlation length given by $\xi = 2\sqrt{\mu K_2/K_1}$ although they present quadratic gapless excitations. Therefore, they should obey an area law corresponding to this correlation [7] $s_E/A \approx \xi^{1-D}$. This has to be compared with the relativistic case with NG bosons with linear dispersion relation (type-A), where an infinite correlation length (of the order of the system size) is expected.

Here we focus in the 1D case where we do not expect the entanglement entropy to scale with the system size (up to logarithmic corrections) due to the finite correlation length. It should instead saturate to the value $s_E \sim \log \xi$ [7, 8, 34]. To see the applicability of our results we consider a particular type of systems, ferrimagnets.

Ferrimagnets are spin systems living on a two-sublattice $A \cup B$. The spin operators $\vec{S}_i$ in sublattice $A$ are of magnitude $s_1$ and the spin operators $\vec{r}_j$ in sublattice $B$ are of magnitude $s_2$. Typically, the ground state of the system exhibits ferrimagnetic order, i.e. an anti-alignment of the spins living on different sublattices. In this case, the different magnitude of the spins induces a total magnetization density in the system $m \sim |s_1 - s_2|$. This is an indicator of the non-zero expectation value of the commutator of two broken charges $\langle [\vec{S}^r, \vec{S}^s] \rangle \neq 0$, which breaks Lorentz invariance [17].

The low-energy effective description of ferrimagnets is known to be the Lagrangian [1, 46]. Thus, we expect a quadratic Type-B NG boson $\rho = pk^2$ and a gapped partner $w = \Delta$, where $\rho$ and $\Delta$ are the spin stiffness and energy gap, respectively. In terms of the coefficients of [1] (for $c_2 = 0$), we identify $\rho = K_3/K_1$ and $\Delta = K_1/K_2$. Which allows us to write the general expression of the correlation length (in units of the lattice spacing)\footnote{In order to see the applicability of Eq. (11) we consider a microscopic Hamiltonian which has been extensively studied with different techniques [33, 45, 47], $H = J \sum_{<i,j>} \vec{S}_i \cdot \vec{r}_j$. The numerical works found a very short correlation length eventough they also found gapless excitations [33, 45, 47], a feature which is well explained in our model. Indeed, Eq. (11) allows us to predict the} \footnote{\begin{equation}
\xi = 2\sqrt{\frac{\rho}{\Delta}}.
\end{equation}}

$\xi = 2\sqrt{\frac{\rho}{\Delta}}$. (11)
value of the correlation length for several ferrimagnets using previously obtained numerical results for the $\rho$ and $\Delta$. The results are summarized in Fig. 3. In all cases, our $\xi$s are smaller than those predicted by spin-wave theory, $\xi^{-1} = \log(s_1/s_2)$ [43], and closer to the only accurate value obtained using matrix product states [35, 45] for $(s_1 = 1, s_2 = 1/2)$. The discrepancy could be attributed to the difference between the lattice version of the model and our continuum approach.

Outlook We have presented analytic formulas for the leading contribution to the entanglement entropy in non-relativistic low energy descriptions. They have been successfully confronted with two prominent examples: the Mott insulator to superfluid transition and non-relativistic Nambu-Goldstone bosons. Our results also apply to particle physics models with a fixed chemical potential. Moreover, the recent experimental breakthrough in measuring entanglement entropy in ultracold atomic gases [48] provides a tool to experimentally test our findings.

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