GENERAL CONSTRAINTS ON SPIN OBSERVABLES; APPLICATIONS TO $\bar{p} + p \to \Lambda + \Lambda$
AND TO POLARIZED QUARK DISTRIBUTIONS\(^1\)

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Abstract

We review the density matrix formalism and the positivity conditions for general multiple spin asymmetries, taking as an example the case $\bar{p} + p \to \Lambda + \Lambda$ in which one, two or three spins are analyzed. Some aspects related to quantum information and entangled states are discussed. Some positivity domains for pairs and triplets of spin parameters are displayed, together with the experimental points. The case of inclusive reaction is also treated, taking as an example the spin- and transverse momentum-dependent quark distributions.

1. Introduction

The single- or multiple-spin asymmetries which can be measured using polarized beams, polarized targets or analyzing the final-particle polarizations provide important information about the elementary processes in particle physics. These spin observables may be related by equalities coming from the symmetries of the processes. Besides, they satisfy inequalities expressing the positivity of a Grand Density Matrix, $R$, which describes all possible polarized cross sections. The positivity of $R$ insures that the cross section is positive for any initial and final spin states, including entangled ones. The resulting inequalities provide consistency checks of experimental data, constrain any parametrization of polarized structure functions and, in the future, may be applied to Monte-Carlo event generators with spins.

The positivity conditions are also interesting from the point of view of the quantum information carried by spins. The information about the scattering amplitudes is maximal when all the independent spin observables are measured. There are inequalities that define the allowed domain and are saturated in this case. Conversely, there is a loss of information, in other words an increase of entropy, when some particles are not analyzed.

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or in the case of inclusive reactions. In this case, most of the inequalities may be non-saturated.

There is an abundant literature on positivity conditions. Some key papers of the 60’s are still relevant. See, for instance, Ref. [1] for a survey and some references. The subject has been revisited in recent months due to the results on the reaction

\[ \bar{p} + p \rightarrow \Lambda + \Lambda , \]

measured by the PS185 collaboration at CERN [2]. The polarization of the outgoing hyperon or antihyperon is revealed by its weak decay. Since a polarized target was used in the last runs, observables up to rank 3 can be accessed.

This contribution is a short introduction to the density matrix formalism and derivation of the positivity conditions. In Sections 2-5 we will consider the case of exclusive reactions, illustrating it by reaction (1.1). We will also present in Section 6 some results of an alternative “empirical” method by which the positivity domains of a subset of observables can be very easily discovered.

Finally, we will consider the case of inclusive reactions and obtain the inequalities which must be satisfied by the spin-dependent quark distribution, considered as the probability of the elementary splitting process

\[ \text{nucleon} \rightarrow \text{quark} + X . \]

for a given longitudinal momentum ratio \( x = p_L(\text{quark})/p_L(\text{nucleon}) \).

2. The spin observables

The fully polarized differential cross section of (1.1), more generally \( A + B \rightarrow C + D \), where \( A, B, C \) and \( D \) have spin \( \frac{1}{2} \), can be expressed as

\[ \frac{d\sigma}{d\Omega}(S_A, S_B, S_C, S_D) = \frac{1}{4\text{unpol}} \left( \frac{d\sigma}{d\Omega} \right) \sum_{\lambda\mu\nu\tau} C_{\lambda\mu\nu\tau} S_A^\lambda S_B^\mu S_C^\nu S_D^\tau . \]  

(2.1)

The \( S \)'s are the polarization vectors of pure spin states (\( |S| = 1 \)). In the right-hand side they are promoted to four-vectors with \( S^0 = 1 \). The indices \( \lambda, \mu, \nu, \tau \), run from 0 to 3, whereas latin indices \( i, j, k, l \), take the values 1, 2, 3, or \( x, y, z \). A summation is understood over each repeated index. \( S^x, S^y, S^z \) are measured in a triad of unit vectors \( \{\hat{x}, \hat{y}, \hat{z}\} \) which may differ from one particle to the other. \( C_{\lambda\mu\nu\tau} \) are the correlation parameters. For example, \( C_{0000} \equiv 1 \), \( C_{xy00} \equiv A_{xy} \) is an initial double-spin asymmetry, \( C_{000y} \) is the spontaneous polarization of particle \( D \) along \( \hat{y} \), \( C_{0y0y} \equiv D_{yy} \) is a spin transmission coefficient from \( B \) to \( D \) and \( C_{00xy} \equiv C_{xy} \) is a final spin correlation.

Equation (2.1) also applies to the case of incomplete initial polarizations, replacing the unit vector \( S_A \) by \( P_A \) with \( |P_A| \leq 1 \) and the same for \( B \). The final polarizations generally depend on the initial ones, e.g.,

\[ P_C \equiv \langle S_C \rangle = \left( \frac{1}{4\text{unpol}} \frac{d\sigma}{d\Omega} \right)^{-1} \nabla_{S_C} \left( \frac{d\sigma}{d\Omega} (P_A, P_B, S_C; S_D = 0) \right) . \]  

(2.2)
3. The density matrix formalism

Let \{\ket{\alpha}\} with \(\alpha = \pm \frac{1}{2}\), be the basic spin states of particle \(A\), \{\ket{\beta}\} those of \(B\), etc. The quantification axis \(\hat{z}\) may differ from one particle to the other. It can be the helicity axis \(p/|p|\) or the transversity axis, \(\hat{n} = p_A \times p_C / |p_A \times p_C|\). We write the spin-dependent amplitude of (1.1) as

\[
(\bra{\bar{\Lambda}}, \gamma \otimes \bra{\Lambda}, \delta) M (\ket{\bar{p}, \alpha} \otimes \ket{p, \beta}) \equiv \bra{\gamma \delta} M \rho_{\alpha \beta} \langle \rangle .
\]

(3.1)

For each spin \(\frac{1}{2}\) particle we have the single-spin density matrix

\[
\rho(P) \equiv \frac{1}{2}(1 + P \cdot \sigma),
\]

(3.2)

For partial polarizations of \(A\) and \(B\), but definite spins of \(C\) and \(D\), the cross-section (2.1) becomes

\[
\frac{d\sigma}{d\Omega} (P_A, P_B, S_C, S_D) = \text{trace}\{M \rho(P_A) \otimes \rho(P_B) M^\dagger \rho(S_C) \otimes \rho(S_D)\}. \tag{3.3}
\]

The final two-spin density matrix is

\[
\rho_{C,D} = \frac{M \rho(P_A) \otimes \rho(P_B) M^\dagger}{\text{trace}\{M M^\dagger\}} . \tag{3.4}
\]

It describes the individual polarizations of \(C\) and \(D\) and their spin correlations. The polarization of the \(C\) is obtained by taking the partial trace over the \(D\) spin variable:

\[
\rho_C = \text{trace}_D \rho_{C,D} , \quad \text{i.e.,} \quad \langle \gamma | \rho_C | \gamma' \rangle = \sum_\delta \langle \gamma | \rho_{C,D} | \gamma' \rangle . \tag{3.5}
\]

The lack of information about a system can be measured by various estimators, among which the entropy \(S = -\text{trace}\{\rho \log \rho\}\) and the rank of \(\rho\). Pure states (maximum information) have zero entropy and unit rank. The entropy (resp. rank) of the initial state is the sum (resp. product) of the single-particle entropies (resp. ranks). The rank of the final density matrix (3.4) is less than or equal to the initial one. Therefore, complete initial polarizations (\(|P(p)| = |P(\bar{p})| = 1\)) lead to a final pure state. It does not imply that the individual polarization of the \(\bar{\Lambda}\), obtained from (3.5), is complete, because the \(\bar{\Lambda} \Lambda\) state may be entangled [3].

Let us now generalize the density matrix in order to describe in an unified way the spin correlations inside the final state and the transmission of polarizations (i.e., of spin information) between the initial and the final particles. For this purpose we consider the fictitious crossed reaction of (1.1),

\[
|\text{vacuum}\rangle \rightarrow p + \bar{p} + \bar{\Lambda} + \Lambda . \tag{3.6}
\]

We restrict this crossing to \textit{spin} and \textit{flavor} variables (we do not consider the momenta). An initial particle \textit{ket} becomes a final \textit{anti-particle bra} of \textit{opposite spin}, for instance,

\[
|p, \beta\rangle \rightarrow \langle \bar{p}, -\beta| \equiv \langle p, \beta| \text{CPT} , \tag{3.7}
\]
where $C$, $P$ and $T$ are the charge conjugation, parity and time-reversal operators (it does not matter if reaction (3.6) does not conserve energy-momentum). Accordingly, we can rewrite the spin-dependent amplitude (3.1) as

$$
\langle \gamma, \delta | M | \alpha, \beta \rangle = \langle -\alpha, -\beta, \gamma, \delta | M^{\text{crossed}} | \text{vacuum} \rangle \equiv \langle -\alpha, -\beta, \gamma, \delta | \Psi \rangle .
$$

(3.8)

Thus we produce a one-to-one correspondence between the 2-particle transition operator $M$ and a 4-particle state vector $|\Psi\rangle$, which we will call the Grand Wave Function. To shorten the equations, we will introduce the notation $\tilde{\alpha} \equiv -\alpha$, $\tilde{\beta} \equiv -\beta$, etc. For explicit values of $\alpha$, we will use the notations $u$ and $d$ (for “up” and “down”, like for quark isospin states) instead of $+\frac{1}{2}$ and $-\frac{1}{2}$. Therefore (3.7) will be written

$$
|u\rangle \rightarrow \langle \tilde{u}|, \quad |d\rangle \rightarrow \langle \tilde{d} | .
$$

(3.9)

The Grand Density Matrix, $R$, which describes all possible spin correlations in reaction (1.1), is defined by

$$
\langle \tilde{\alpha} \tilde{\beta} \gamma \delta | R | \alpha' \beta' \gamma' \delta' \rangle \equiv \langle \gamma \delta | M | \alpha \beta \rangle \langle \alpha' \beta' | M^{\dagger} | \gamma' \delta' \rangle = \langle \tilde{\alpha} \tilde{\beta} \gamma \delta | \Psi \rangle \langle \Psi | \tilde{\alpha}' \tilde{\beta}' \gamma' \delta' \rangle .
$$

(3.10)

Like ordinary density matrices, it is hermitian and semi-positive. Its trace is given by

$$
\text{trace}(R) = \langle \Psi | \Psi \rangle = \text{trace}(MM^{\dagger}) ,
$$

(3.11)

which is $2^2$ times the unpolarized cross section. Dividing by (3.11), we can re-scale $R$ to unit trace, as a standard density matrix. As can be seen from (3.10), $R$ describes a pure state: $R = |\Psi\rangle \langle \Psi|$, and is therefore of rank one. This is a particular property of exclusive reactions.

The expression (3.4) for the final density matrix can be re-written as

$$
\rho(\tilde{\Lambda}, \Lambda) = \frac{\text{trace}_{\tilde{\alpha}, \tilde{\beta}} \{ R [\rho'(p) \otimes \rho'(\bar{p})] \} \text{trace}(R)}{\text{trace}(R)} ,
$$

(3.12)

where $\rho'(p)$ is the transpose of $\rho(p)$. This transposition is explained in Appendix A.

The Grand Density Matrix can be expressed in terms of the correlation parameters and vice-versa through

$$
R = 2^{-4} \ C_{\lambda \nu \tau} \ \sigma^\tau_\lambda(A) \otimes \sigma^\nu_\mu(B) \otimes \sigma_\nu(C) \otimes \sigma_\tau(D) ,
$$

(3.13)

$$
C_{\lambda \mu \nu \tau} = \text{trace} \{ R \ \left[ \sigma^\tau_\lambda(A) \otimes \sigma^\nu_\mu(B) \otimes \sigma_\nu(C) \otimes \sigma_\tau(D) \right] \} ,
$$

(3.14)

where $\sigma_0$ is the unit $2 \times 2$ matrix.

4. Reduction of the density matrix

It is difficult to have polarized anti-protons. Therefore the practical spin observables in reaction (1.1) concern only $p$, $\Lambda$ and $\bar{\Lambda}$. They are encoded in the sub-density matrix $R(p, \bar{\Lambda}, \Lambda) = \text{trace}_\tilde{\alpha} \{ \rho'(\bar{p}) \ R(\bar{p}, p, \bar{\Lambda}, \Lambda) \}$ with $\rho'(\bar{p}) = \frac{1}{2}I$, more explicitly

$$
\langle \tilde{\beta} \gamma \delta | R(p, \bar{\Lambda}, \Lambda) | \tilde{\beta}' \gamma' \delta' \rangle = \sum_\tilde{\alpha} \langle \tilde{\alpha} \tilde{\beta} \gamma \delta | R(\bar{p}, p, \bar{\Lambda}, \Lambda) | \tilde{\alpha} \tilde{\beta}' \gamma' \delta' \rangle .
$$

(4.1)
This density matrix has dimension $8 \times 8$, which is still rather large to write down the positivity conditions (the original one was $16 \times 16$). It has a non-zero entropy, brought by the $\bar{p}$, and rank 2 because $\bar{\alpha}$ takes two values in (4.1).

An important simplification occurs due to the symmetry of reaction (1.1) under the reflection $\Pi$ about the scattering plane, which reverses the spin components parallel to $\bar{p}$, and rank 2 because $\bar{\alpha}$ and $\bar{\beta}$ move positivities (the original one was 16). The positivity of the Grand Density Matrix comes from the very general, but non-nilpotent, $\bar{\sigma}$ or $\bar{\sigma}_3$ conditions of positivity, or semi-positivity with $N_0$ vanishing eigenvalues, is

$$\Sigma_n > 0 \quad \text{for} \quad n \leq N - N_0 , \quad \Sigma_n = 0 \quad \text{for} \quad N - N_0 < n \leq N .$$

5. The positivity conditions

The positivity of the Grand Density Matrix comes from the very general, but non-nilpotent, requirement that the probability of any process is positive. It is not sufficient to require that the cross section (2.1) is positive for any set of polarizations $\{S_A, S_B, S_C, S_D\}$. Let us suppose, for instance, that (2.1) possesses the factor $(1 + S_C \cdot S_D)$. This factor is positive or null for any $S_B$ and $S_D$. However, it corresponds to a final density matrix of the form $\rho_{C,D} = [1 + \sigma_C^i \otimes \sigma_D^j]/4$ which is non-positive. For example, for the singlet spin state, we have $\sigma_C^i \otimes \sigma_D^j = -3$. The probability that reaction (1.1) produces a $(\bar{\Lambda}, \Lambda)$ system in the singlet state would be negative! Note that the singlet state is entangled.

This shows that positivity has to be tested with non-entangled and entangled states.

Similarly, a factor $(1 - S_A \cdot S_C)$, which leads to the complete spin reversal $S_C = -S_A$ according to (2.2), gives a non-positive $R$ and is therefore forbidden. As an example, let us consider the splitting $\pi \rightarrow q + \bar{q}$ followed by a quark–hadron scattering $q + h \rightarrow q' + h'$ where the $\bar{q}$ is spectator. The intermediate spin correlation is in $(1 - S_q \cdot S_\bar{q})$. If there were a complete spin reversal $S_q = -S_\bar{q}$ in the quark-hadron scattering, it would lead to a final correlation in $(1 + S_q \cdot S_\bar{q})$, which is forbidden as explained before.

The general positivity conditions are as follows: a $N \times N$ hermitian matrix $\rho$ is positive (respectively semi-positive) if all its eigenvalues $r_i$ are positive (resp. positive or null). Let us consider the symmetric functions of the eigenvalues

$$\Sigma_1 = \sum_i r_i , \quad \Sigma_2 = \sum_{i<j} r_i r_j , \quad \Sigma_3 = \sum_{i<j<k} r_i r_j r_k , \quad \cdots \quad \Sigma_N = r_1 r_2 \cdots r_n .$$

$\Sigma_n$ is the sum of the on-diagonal sub-determinants of order $n$ (when a sub-matrix has its diagonal on the diagonal of $\rho$, we call it "on-diagonal"). A necessary and sufficient condition of positivity, or semi-positivity with $N_0$ vanishing eigenvalues, is

$$\Sigma_n > 0 \quad \text{for} \quad n \leq N - N_0 , \quad \Sigma_n = 0 \quad \text{for} \quad N - N_0 < n \leq N .$$
If $\rho$ is (semi-)positive, each of its on-diagonal sub-determinant is (null or) positive. This may provide inequalities simpler than, but redundant with (5.3), in the same manner as $|x^2| < 1$ is redundant with $|x^2| + |y^2| < 1$.

The matrix $\rho$ depends on $N^2$ real parameters. They can be $\text{Re}(\rho_{ii'})$ for $i \leq i'$ and $\text{Im}(\rho_{ii'})$ for $i < i'$, or the correlation parameters, which are linear combinations of them. In the $N^2$-dimensional parameter space, the domain of positivity of $\rho$ is a convex half-cone. Its intersection with the hyperplane $\Sigma_1 \equiv \text{trace}(\rho) = 1$ is a finite convex domain $D$. The boundary of $D$ is a sheet of the hypersurface $\Sigma_N \equiv \det(\rho) = 0$. It is a $(N^2-2)$-dimensional manifold of degree $N$. On this boundary, $\rho$ is only semi-positive. The other conditions, $\Sigma_n \geq 0$ for $n = 2, \cdots N - 1$ define domains which include $D$. These “auxiliary” conditions serve to eliminate the other sheets of the hypersurface $\Sigma_N = 0$. The hypersurface where $\Sigma_n$, or any on-diagonal sub-determinant, vanishes is externally tangent to $D$.

As we have seen, for an exclusive reaction the Grand Density Matrix $R$ is of rank one. Therefore all the $\Sigma_n$’s are vanishing for $n \geq 2$ and all the positivity constraints are saturated. It can be shown that $R$ is on a ”corner” of $D$. On the contrary, when much information is lost through non-detected or non-analyzed particles, $R$ is ”deep inside” $D$.

6. Empirical approach

The search for inequalities is straightforward using the density matrix method, but does not reveal at once the shape of the allowed domains. Also, when one writes the conditions on the density matrix, one gets in general a combination of several spin observables, and thus one has to make appropriate combinations of inequalities to obtain constraints on two or three given observables of interest.

To circumvent this difficulty, the following method was used in Ref. [5]. The real and imaginary parts of the amplitudes were chosen randomly, and the spin observables were computed using their explicit expression in terms of amplitudes. This detects which pairs or triplets of observables fulfill inequalities, and then these inequalities can be derived by straightforward calculus. The case of pairs of observables is extensively discussed in Ref. [5], and preliminary results on triplets presented at the LEAP2003 conference [6]. A sample of the results is displayed in Figs. 1 and 2.

Notice that only a few types of inequalities are encountered. For pairs of observables, say $X$ and $Y$, each being typically restricted to $[-1, +1]$, one gets the following possibilities

- nothing: $X$ and $Y$ might reach any point of the square $[-1, +1]^2$,
- the disk $X^2 + Y^2 \leq 1$,
- a triangle $4Y^2 \leq (1 + X)^2$.

For triplets of observables, say $X$, $Y$ and $Z$, the following situations are obtained

- nothing, any point of cube $[-1, +1]^3$ is allowed,
- a sphere $X^2 + Y^2 + Z^2 \leq 1$,
- a cone $(1 + Z)^2 \geq 4X^2 + 4Y^2$,
- a cubic of the type $X^2 + Y^2 + Z^2 \pm XYZ \leq 1$. 

Figure 1: Pair of observables restricted to the unit disk (left, here polarisation and $C_{ll}$ are shown) or to a triangle (right, here $C_{nn}$ and $C_{lm}$ are shown). The small grey dots correspond to hypothetical, randomly generated, amplitudes, and the larger dots, to actual data.

Figure 2: Triplet of observables restricted to the inner volume a cone or of a cubic. The small dots correspond to randomly generated amplitudes, the larger ones (partly hidden) to actual data.

The latter case is the most interesting, since the domain is restricted in space of three observables, but each projection cover the whole square, i.e., there is no restriction for any pair observables within $(X,Y,Z)$. The border has the shape of a twisted cushion.
7. Inclusive case: the spin-dependent parton densities

As an example of inclusive reaction, let us consider now the elementary process (1.2), which we rewrite for fixed momenta and spin vectors as

\[ N(p, S_N) \rightarrow q(k, S_q) + X, \quad (7.1) \]

with \( k = xp + k_T \). The probability of (7.1) is the spin- and \( k_T \)-dependent quark density in the nucleon. All what we will say below also applies to the quark fragmentation \( q \rightarrow \text{baryon} + X \), only commuting \( q \) and \( N \), or to any inclusive reaction of the type

\[ A \uparrow + B \rightarrow C \uparrow + X, \quad (7.2) \]

already treated by Doncel and Méndez \[4\]. Thus the problem was solved long ago. One can also relate the inequalities in (7.2) to those of the crossed reaction \[7\]

\[ A_1 \uparrow + A_2 \uparrow \rightarrow A_3 \uparrow + X, \quad (7.3) \]

by the correspondence \( S(A_2) \leftrightarrow -S(C) \).

For a given spectator state \( X \), the matrix element in spin Hilbert space can be written as \( \langle \beta | M_X | \alpha \rangle \). Like in (3.6 - 3.7), we consider the fictitious crossed process

\[ \bar{X} \rightarrow \bar{N}(p, -S_N) + q(k, S_q), \quad (7.4) \]

where \( |\bar{X}\rangle \equiv \text{CPT} \ |X\rangle \). Note that we have also moved the spectator system \( X \) to the initial state. The Grand Wave Function and Grand Density Matrix are then defined by

\[ \langle \bar{a} \bar{\beta} | \Psi_X \rangle = \langle \beta | M_X | \alpha \rangle, \quad (7.5) \]

\[ R = \sum_X |\Psi_X \rangle \langle \Psi_X|. \quad (7.6) \]

Thus \( R \) corresponds to a statistical mixture. Its rank \( r \) is the dimension of the sub-space spanned by the vectors \( |\Psi_X\rangle \) in the \( (\bar{N}q) \) spin Hilbert space. It cannot exceed the number of possible quantum states of the spectator system. In general \( r > 1 \), which means that some information is lost, taken away by the spectator partons.

In our case \( R \) has dimension 4 \( \times \) 4 and depends on \textit{a priori} 16 correlation parameters \( C_{\mu\nu} \) through the analog of (3.13). However, like in the 2 \( \rightarrow \) 2 reaction (1.1), the plane defined by \( p \) and \( k_T \) is a symmetry plane. It is therefore convenient to use the transversity basis with \( \hat{z} \) normal to this plane, instead of the helicity basis (unless one integrates over \( k_T \)). In this basis \( R \) is even under \( \sigma_x \rightarrow -\sigma_x, \sigma_y \rightarrow -\sigma_y \) and the only non-vanishing coefficients are \( C_{00} \equiv 1, C_{0x}, C_{0y}, C_{zz}, C_{zx}, C_{xy}, C_{yx}, C_{yy} \) and \( C_{yx} \). If we take \( \hat{x} \) along \( p \) these parameters are respectively proportional to \( f_1, -h_{1T}^+, f_{1T}^+, h_1 - h_{1T}, g_1, h_{1L}, g_{1T} \) and \( h_1 + h_{1T} \) of Ref. \[8\]; all kinematical factors in \( p_T/M \) or \( p_T^2/(2M^2) \) removed. However the following inequalities are independent of the choice of the \( x \) and \( y \) axes in the production plane. Let us introduce

\[ \frac{1 \pm C_{zz}}{2} \equiv D_{nn}^\pm, \quad \frac{C_{0z} \pm C_{zz}}{2} \equiv A_n^\pm, \quad \frac{C_{xx} \pm C_{yy}}{2} \equiv U^\pm, \quad \frac{C_{xy} \pm C_{yx}}{2} \equiv V^\pm. \quad (7.7) \]
Putting the $|\bar{N}q\rangle$ basic spin states in the order $\{|\bar{u}u\rangle, |\bar{ud}\rangle, |\bar{du}\rangle, |\bar{dd}\rangle\}$, one has

$$R = \frac{1}{2} \begin{pmatrix} D_{nn}^+ + A_n^+ & 0 & 0 & U^- - iV^+ \\ 0 & D_{nn}^- - A_n^- & U^- + iV^+ & 0 \\ 0 & 0 & U^+ + iV^- & 0 \\ U^+ + iV^- & 0 & 0 & D_{nn}^+ - A_n^+ \end{pmatrix}.$$ \hfill (7.8)

As expected from the symmetry about the $(x, y)$ plane, $R$ is block-diagonal in two rank-2 sub-matrices, which obey the separate positivity conditions:

$$(D_{nn}^\pm)^2 \geq (A_n^\pm)^2 + (U^\pm)^2 + (V^\pm)^2,$$ \hfill (7.9)

and

$$D_{nn}^\pm \geq 0, \quad \text{i.e.,} \quad |C_{zz}| \equiv |D_{nn}| \leq 1,$$ \hfill (7.10)

which agrees with the results of Bacchetta et al \cite{8} and of Ref. \cite{4}.

If we integrate over $k_T$, the only surviving parameters are $C_{00} \equiv 1$, $C_{xx} \equiv \Delta q(x)/q(x)$ and $C_{yy} = C_{zz} \equiv \delta q(x)/q(x)$, where $q(x)$, $\Delta q(x)$ and $\delta q(x)$ are the quark number, quark helicity and quark transversity \cite{9,10} distributions. One obtains the Soffer inequality \cite{11}:

$$2\delta q(x) \leq q(x) + \Delta q(x).$$ \hfill (7.11)

In a simple model of quark distribution where $X$ just consists in a scalar di-quark, all the inequalities (7.9 - 7.10) of the $k_T$-dependent case are saturated. Indeed, such an object has no spin to carry quantum information away; a fully polarized nucleon delivers a fully polarized quark \cite{10}. This is no more the case if we integrate over the degree of freedom $k_T$. Nevertheless, the Soffer bound keeps saturated.

8. Conclusions and outlook

The formalism developed in the 60’s remains extremely powerful to analyze the consistency of spin observables. However, it needs some freshening and new methods are needed to quickly derive the inequalities within a subset of accessible observables. We hope to have worked in this direction.

One of the basic tool, already used in \cite{4}, is a fictitious crossing which gathers all the particles on the same side. It is expressed as a partial matrix transposition. Usual crossing also links different physical reactions, just reversing the polarization vectors.

Particle spin physics also touches the more general theory of quantum information, in particular with the concept of entanglement. The fact that the particles considered here have definite momenta is not essential. The inequalities obtained in particle physics could also apply to “gates” between other kinds of quantum information channels like optical fibers.

In a forthcoming article, we shall provide more details about the positivity conditions and their physical interpretation. In particular, we will show explicitly that the method of the Grand Density Matrix gives the same constraints on observables that the empirical approach based on randomly generated amplitudes.

Appendix A. Effect of crossing on the operators acting on an initial particle

Together with (3.7), we have $\langle \alpha' | \rightarrow | \bar{\alpha}' \rangle$,

$$|\alpha\rangle \langle \alpha' | \rightarrow | \bar{\alpha}' \rangle \langle \bar{\alpha} |,$$ \hfill (A.1)

9
and for a linear combination of such elementary operators,
\[ \sum_{\alpha, \alpha'} |\alpha\rangle A_{\alpha\alpha'} \langle \alpha'| \rightarrow \sum_{\alpha, \alpha'} |\bar{\alpha}'\rangle A_{\alpha\alpha'} \langle \bar{\alpha} | , \]

(A.2)

Here we have assumed that crossing acts linearly on operators. Indeed (3.7) is the product of two anti-linear operations: (i) applying CPT (ii) changing a ket into a bra. Equation (A.2) amounts to the matrix transposition \( A \rightarrow A^t \), provided we choose the same ordering for the crossed basis vectors \( \{ |\bar{u}\rangle, |\bar{d}\rangle \} \) as in the initial basis vectors \( \{ |u\rangle, |d\rangle \} \) (the ordering in magnetic number \( s_z \) is reversed: it becomes \( \{ | -\frac{1}{2}\rangle, | +\frac{1}{2}\rangle \} \)). For a single-spin matrix density, the transformation is
\[ \rho = \frac{1}{2}(1 + \mathbf{P} \cdot \sigma) \rightarrow \rho^t = \rho^\dagger = \frac{1}{2}(1 + \mathbf{P} \cdot \sigma) , \]

(A.3)

where \( \mathbf{P} = -\mathbf{P} \) due to spin reversal, and \( \sigma_i = -\sigma_i^t \) are the Pauli matrices for the \( \{ \bar{2} \} \) representation of SU(2) (which is not often used, due to the equivalence \( \{ 2 \} \leftrightarrow \{ \bar{2} \} \)).

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