Exponents of Primitive Symmetric Companion Matrices

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Abstract

A symmetric companion matrix is a matrix of the form $A + A^T$ where $A$ is a companion matrix all of whose entries are in $\{0, 1\}$ and $A^T$ is the transpose of $A$. In this paper, we find the total number of primitive and the total number of imprimitive symmetric companion matrices. We establish formulas to compute the exponent of every primitive symmetric companion matrix. Hence the exponent set for the class of primitive symmetric companion matrices is completely characterized. We also obtain the number of primitive symmetric companion matrices with a given exponent for certain cases.

Key Words: Symmetric companion matrix, Primitive matrix, Exponent, Exponent set.

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1 Introduction

Let $n$ be a positive integer, and let $A, B \in M_n(\mathbb{R})$, the set of all $n \times n$ real matrices. We denote $ij$-th entry of $A$ by $a_{i,j}$. If all of the entries of $A$ are positive, then $A$ is called a positive matrix, which is denoted by $A > 0$. If all of the entries of $A$ are nonnegative, then $A$ is called a nonnegative matrix, which is denoted by $A \geq 0$. By $A > B$ (resp., $A \geq B$), we mean $A - B > 0$ (resp., $A - B \geq 0$); that is, $a_{i,j} > b_{i,j}$ (resp., $a_{i,j} \geq b_{i,j}$) for all $i$ and $j$. A matrix $A$ is called primitive if it is nonnegative and there exists a positive integer $k$ such that $A^k > 0$. For a primitive matrix $A$, the smallest
such positive integer $k$ is called the \textit{primitive exponent} of $A$, or simply, the \textit{exponent} of $A$, and is denoted by $\text{exp}(A)$. An irreducible matrix that is not primitive is called \textit{imprimitive}. For additional information on irreducible matrices, see H. Minc [13].

Let $\mathbb{B}$ denote the binary Boolean semiring, that is $\mathbb{B}$ is the set $\{0, 1\}$ with arithmetic the same as for the reals, except that $1 + 1 = 1$. The set $\mathcal{M}_n(\mathbb{B})$ is the set of $n \times n$ matrices with entries in $\mathbb{B}$ which forms a semimodule with the usual definitions of addition and multiplication.

Let $A$ be a nonnegative matrix. The \textit{support} of $A$ (sometimes called the \textit{nonzero pattern} of $A$), denoted $\overline{A}$, is the matrix in $\mathcal{M}_n(\mathbb{B})$ such that $\overline{a_{i,j}} = 0$ if $a_{i,j} = 0$, and $\overline{a_{i,j}} = 1$ if $a_{i,j}$ is nonzero. Since the product or sum of two positive numbers is always positive and the magnitude of the nonzero entries in a primitive matrix is not important, we have that a nonnegative matrix $A$ is primitive if and only if $\overline{A}$ is primitive and $\text{exp}(A) = \text{exp}(\overline{A})$.

Let $X$ be a nonempty subset of $\mathcal{M}_n(\mathbb{R})$. The \textit{exponent set} of $X$, denoted $E(X)$, is given by

$$E(X) = \{k \in \mathbb{N} : \text{there exists a primitive } A \in X \text{ with } \text{exp}(A) = k\}.$$ 

Note that $X$ can contain both primitive and imprimitive matrices, and also that $E(X)$ is empty when $X$ contains no primitive matrices.

For positive integers $a$ and $b$ with $a \leq b$, the set of all positive integers $k$ with $a \leq k \leq b$, will be denoted by $\llbracket a, b \rrbracket$. This set partitions into two sets, the subset of all even integers in $\llbracket a, b \rrbracket$, denoted by $\llbracket a, b \rrbracket^e$; and the subset of all odd integers in $\llbracket a, b \rrbracket$, denoted by $\llbracket a, b \rrbracket^o$.

Every positive matrix is a primitive matrix with exponent one, that is, $1 \in E(\mathcal{M}_n(\mathbb{R}))$. In 1950, Helmut Wielandt [21] proved that if $A \in \mathcal{M}_n(\mathbb{R})$ is primitive, then $\text{exp}(A) \leq \omega_n$, where $\omega_n = (n - 1)^2 + 1$. This bound is known as the \textbf{Wielandt bound}. For the proof, see Hans Schneider [18], Holladay and Varga [5], or Perkins [16]. With this remarkable result we have $E(\mathcal{M}_n(\mathbb{R})) \subseteq \llbracket 1, \omega_n \rrbracket$. In 1964, Dulmage and Mendelsohn [4] showed that $E(\mathcal{M}_n(\mathbb{R})) \subseteq \llbracket 1, \omega_n \rrbracket$.

The study of $\text{exp}(A)$ has been focusing on the following problems.
The maximum exponent problem, i.e., to estimate the upper bound of \( E(X) \) for a particular class \( X \) of primitive matrices. Here people generally looks for the ‘best possible’ upper bound, i.e., upper bound \( k \) of \( E(X) \) such that \( k \in E(X) \). Sometimes it is called the exact upper bound.

The set of exponents problem, i.e., to determine \( E(X) \).

The extremal matrix problem, i.e., to determine primitive matrices in \( X \) with the maximum exponent.

The extremal matrix problem can be generalized further as the set of matrices problem, i.e., for any \( k \in E(X) \), determine primitive matrices in \( X \) with the exponent \( k \).

Significant literature is available for these types of problems except for the set of matrices problem. The papers [1], [7], [8], [10], [11], [17], [22] studied exponent sets for different classes of primitive matrices. In 1987, Shao [19] investigated the exponent set of \( S_n \), the class of \((0, 1)\) primitive symmetric matrices. He showed that \( E(S_n) = [1, 2n-2] \setminus [n, 2n-1] \). Later, Liu et al. [11] established the results for the primitive matrices in \( S_n \) with zero trace. Here we consider another class of primitive symmetric matrices of order \( n \). We denote this class by \( PSC_n \), the class of primitive symmetric companion matrices. We show that the exponent set for this class is \( [2, 2n-2] = [1, 2n-2] \setminus [1, 2n-1] \). Theorem 4 of Fuyi et al. [20] and the Wielandt graph in Figure 1 motivate us for such a consideration. Surprisingly, upper bounds as well as even numbers in the exponent sets given in [19] and [11] are attained by matrices that belong to \( PSC_n \). Not only that, in this paper we find the exponent of each \( A \in PSC_n \). Hence we completely solve the set of matrices problem (SMP) for \( PSC_n \). Characterizations of matrices in \( S_n \) with exponent \( x \geq n - 3 \) can be found in literature. For instance, see Lichao and Cai [9]. Furthermore, we obtain the number of matrices which attain the exact upper and lower bound of the exponent set. We refer readers to Theorem 6, 7, and 8 in [4] where the authors found the number of primitive matrices with a given exponent. In 2007, Liu et al. [12] found the number of \( n \times n \) \((0, 1)\) primitive matrices with
exponent \((b - 1)^2 + 1\), where \(b\) is the Boolean rank of the matrix. Research in the area of the set of matrices problem and related counting problem has not seen significant progress in recent years. This problem appears to be harder. The study of this problem is just the beginning, see Kim et al. [6].

Figure 1: The Wielandt graph \(W_n\).

2 Notation

\[
C_n = \begin{cases} 
A \in M_n(\mathbb{R}) : A = \begin{bmatrix} 
0 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 1 \\
a_0 & a_1 & a_2 & a_3 & \ldots & a_{n-1} 
\end{bmatrix}, a_i \in \{0, 1\} 
\end{cases}.
\]

That is \(C_n\) is the set of all companion matrices of polynomials of the form \(x^n - \sum_{i=0}^{n-1} a_i x^i\), where \(a_i \in \{0, 1\}\). Since the first \(n - 1\) rows of every matrix in \(C_n\) are fixed, hence it is sufficient to specify the last row. Thus there is a bijection between \(C_n\) and \(B_n\), where \(B_n\) denotes the set of all binary strings of length \(n\), in particular \(|C_n| = |B_n| = 2^n\), where \(|S|\) denotes the cardinality of the set \(S\). Suppose \(F : M_n(\mathbb{R}) \to M_n(\mathbb{R})\) is given by \(F(A) = A + A^T\), where \(A^T\) is the transpose of \(A\). Then every entry of \(F(A)\), where \(A \in C_n\) is either 0 or 1 except that \(f_{n,n-1} = f_{n-1,n} \in \{1, 2\}\) and that \(f_{n,n} \in \{0, 2\}\), where \((f_{i,j}) = F(A)\).
If \( A \in C_n \), then \( A \) is called a \((0, 1)\) companion matrix and we call \( F(A) \) a symmetric companion matrix. For \( \alpha, e \in \{0, 1\} \), we define

\[
C_{\alpha, e}^n = \{ F(A) : A \in C_n, a_{n1} = \alpha, a_{nn} = e \},
\]

\[
\mathcal{PSC}_{\alpha, e}^n = \{ B \in C_{\alpha, e}^n : B \text{ is primitive} \}.
\]

Elements in \( C_{\alpha, e}^n \) will be denoted by \( A_Y \), where \( Y \in B_{n-3} \) and last row of \( A_Y \) will be the row vector \([\alpha, Y, 1, e]\) or \([\alpha, Y, 2, e]\). Hence corresponding to each \( Y \) there are two elements in \( C_{\alpha, e}^n \) which are both imprimitive or primitive simultaneously and in case of primitivity, both of them have the same exponent.

**Remark 2.1** The number of imprimitive symmetric companion matrices of order \( n \) or the number of matrices in \( \mathcal{PSC}_n \) with a given exponent is always even.

**Example 2.2** For \( n = 3 \), \( B_3 = \{000,001,010,100,011,101,110,111\} \), and

\[
C_3 = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \right\}.
\]

Since the matrices

\[
\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}
\]

are imprimitive, then \( \mathcal{PSC}_3^{0,0} = \emptyset \).

Also, \( \mathcal{PSC}_3^{0,1} = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 2 \end{bmatrix} \right\} \), \( \mathcal{PSC}_3^{1,0} = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \right\} \), and \( \mathcal{PSC}_3^{1,1} = \left\{ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} \right\} \).

Moreover, it is easy to check that \( E(\mathcal{PSC}_3^{0,1}) = \{4\} \) and \( E(\mathcal{PSC}_3^{1,1}) = \{2\} = E(\mathcal{PSC}_3^{1,1}) \).

To the best of our knowledge no one has studied the number of primitive and imprimitive matrices of a given class and also the number of matrices with a given exponent from that class. Furthermore, there is no specific formula for computing the exponent of a given matrix from a given class. We
address here these problems for the class of primitive symmetric companion matrices.

Here is an overview of the content of the paper. Our interest is in finding

1. $|\mathcal{PSC}_n|$, where $\mathcal{PSC}_n = \mathcal{PSC}^{0,0}_n \cup \mathcal{PSC}^{0,1}_n \cup \mathcal{PSC}^{1,0}_n \cup \mathcal{PSC}^{1,1}_n$,

2. $E(F(C_n))$ by finding $E(\mathcal{PSC}^{\alpha,e}_n)$, for $\alpha, e \in \{0, 1\}$ as $E(F(C_n)) = E(\mathcal{PSC}^{0,0}_n) \cup E(\mathcal{PSC}^{0,1}_n) \cup E(\mathcal{PSC}^{1,0}_n) \cup E(\mathcal{PSC}^{1,1}_n)$,

3. $N^{\alpha,e}_n(b)$, where $N^{\alpha,e}_n(b) = |\{A \in \mathcal{PSC}^{\alpha,e}_n : exp(A) = b, b \in E(\mathcal{PSC}^{\alpha,e}_n)\}|$.

In Section 3 we will address Problem 1 that is we will find $|\mathcal{PSC}_n|$. The set $E(\mathcal{PSC}^{0,1}_n \cup \mathcal{PSC}^{1,1}_n)$ and the numbers $N^{0,1}_n(b), N^{1,1}_n(b)$ will be presented in Section 3. In Sections 4 and 5 we will evaluate $E(\mathcal{PSC}^{1,0}_n), N^{1,0}_n(b)$ and $E(\mathcal{PSC}^{0,0}_n), N^{0,0}_n(b)$ respectively for specific values of $b$. At the end of Section 5 we display a table in which $N^{\alpha,e}_n(b), \alpha, e \in \{0, 1\}$ can be found for a small $n$ and for all possible $b$. In the rest of this section we will see a few preliminaries and notations required for rest of the paper.

For each nonnegative matrix $A \in M_n(\mathbb{R})$ we associate a digraph $D(A)$ with vertex set $V(A)$ (or simply $V$) = $[1, n]$ and edge set $E(A) = \{(r, s) \in V \times V : a_{rs} > 0\}$. There are an infinite number of matrices associated with a single digraph. Further if the matrix $A$ is symmetric, then $D(A)$ is an undirected graph or simply a graph. In particular, $D(F(A))$ is a graph for every $A \in C_n$. If $X$ is a digraph, then $X$ is a primitive (imprimitive) if its adjacency matrix is primitive (imprimitive) and $exp(X)$ is defined to be the exponent of its adjacency matrix.

For $A \in C^{\alpha,e}_n$ we define $V_1(A), V_2(A)$ (or simply $V_1, V_2$ if $A$ is clear form the context) by

$V_1 = \{i \in [1, n-1] : a_{ni} = 0\} \text{ and } V_2 = \{i \in [1, n-1] : a_{ni} > 0\}.$

For $U \subseteq [1, n]$ write $U = U_1 \cup U_2 \cup \cdots \cup U_r$, where for each $k \in [1, r]$, $U_k = [i_k, j_k]$ with $1 \leq i_k \leq j_k \leq n$, and for each $k \in [2, r]$, $j_{k-1} + 2 \leq i_k$. We define $m(U) = \max\{|U_1|, |U_2|, \ldots, |U_r|\}$. For $U = \emptyset$, $m(U)$ assume to be zero. For example if $U = \{2, 3, 5, 7, 8, 9, 10, 13, 14\}$, then $U = \{2, 3\} \cup \{5\} \cup \{7, 8, 9, 10, 13, 14\}$.
\(\{7, 8, 9, 10\} \cup \{13, 14\}\) and \(m(U) = |\{7, 8, 9, 10\}| = 4\). From the construction of \(C_{n}^{a, e}\), we have \(n - 1 \in V_2\), and thus \(0 \leq m(V_1) \leq n - 2\).

**Example 2.3** Consider the graph with \(V = [1, 10]\) in Figure 2. For this, we have \(V_1 = \{2\} \cup \{4\} \cup \{6\} \cup \{8\}\), \(m(V_1) = 1\) and \(V_2 = \{1, 3, 5, 7, 9\}\).

![Graphs of imprimitive symmetric companion matrices](image)

**Figure 2**: Graphs of imprimitive symmetric companion matrices

### 3 Number of primitive symmetric companion matrices and \(E(\mathcal{P}SC_{n}^{0,1} \cup \mathcal{P}SC_{n}^{1,1})\)

The following theorem gives the number of primitive symmetric companion matrices. Before continuing recall that a connected graph is a primitive graph if and only if it has a cycle of odd length. From the construction, \(D(F(A))\) is a connected graph for every \(A \in C_n\) as it contains the path graph \(1 - 2 - 3 - \cdots - n\) as a subgraph. Hence in order to find \(|\mathcal{P}SC_{n}^{a, e}|\) it is sufficient to find \(|C_{n}^{a, e} \setminus \mathcal{P}SC_{n}^{a, e}|\). That is we count the number of matrices \(A\) in \(C_n\) such that \(a_{n1} = a, a_{nn} = e\) and every cycle in \(D(F(A))\) is of even length. Further, the two positions of last row of every matrix in \(C_{n}^{a, e}\) are fixed, hence \(|C_{n}^{a, e}| = 2^{n-2}\) or equivalently \(|\mathcal{P}SC_{n}^{a, e}| = 2^{n-2} - |C_{n}^{a, e} \setminus \mathcal{P}SC_{n}^{a, e}|\).
Theorem 3.1 Let \( n \geq 4 \), and \( \alpha \in \{0, 1\} \). Then

1. \( |\mathcal{PSC}_n^{\alpha, 1}| = 2^{n-2} \).

2. \( |\mathcal{PSC}_n^{1, 0}| = \begin{cases} 2^{n-2} & \text{when } n \text{ is odd,} \\ 2^{n-2} - 2^{\frac{n-2}{2}} & \text{when } n \text{ is even.} \end{cases} \)

3. \( |\mathcal{PSC}_n^{0, 0}| = 2^{n-2} - 2^{\frac{n-2}{2}} \).

Proof of Part 1. Let \( B = [b_{i,j}] \in C_n^{\alpha, 1} \). Then \( B \in \mathcal{PSC}_n^{\alpha, 1} \), as there is a loop at the vertex \( n \) in \( D(B) \), which is of odd length. Consequently \( C_n^{\alpha, 1} \setminus \mathcal{PSC}_n^{\alpha, 1} = \emptyset \).

Proof of Part 2.

\( n \) is odd. Similar to proof of Part 1, now there is cycle of length \( n \) in \( D(B) \) for every \( B \in C_n^{1, 0} \).

\( n \) is even. Let \( A \in C_n^{1, 0} \). Then \( 1, n-1 \in V_2(A) \) as \( n-1 \) always belongs to \( V_2(A) \). If \( V_2 \cap [1, n]^{c} \neq \emptyset \) then \( D(A) \) contains an odd cycle hence \( A \in \mathcal{PSC}_n^{1, 0} \). Thus \( A \in C_n^{1, 0} \setminus \mathcal{PSC}_n^{1, 0} \) if and only if \( V_2 \setminus \{1, n-1\} \subseteq \{3, 5, \ldots, n-3\} \). There are \( 2^{\frac{n-4}{2}} \) such cases. Hence the result follows from Remark 2.

Proof of Part 3. In this case, \( A \in C_n^{0, 0} \setminus \mathcal{PSC}_n^{0, 0} \) if and only if \( V_2 \setminus \{n-1\} \subseteq \begin{cases} \{3, 5, 7, \ldots, n-3\} & \text{when } n \text{ is even,} \\ \{2, 4, 6, \ldots, n-3\} & \text{when } n \text{ is odd.} \end{cases} \)

The remainder of the proof is similar to the proof of the even case of Part 2.

Example 3.2 Both the graphs in Figure 2 are imprimitive.

1. Consider the graph with \( V = [1, 10] \) in Figure 3. Then by using Remark 2.1 it is easy to see that the number of imprimitive symmetric companion matrices of order 10 is \( 2^{(10-1)/2} = 16 \). These are the adjacency matrices of subgraphs obtained by removing one or more edges from \( \{10, 3\}, \{10, 5\}, \{10, 7\} \) of that graph.

2. Similarly from the graph with \( V = [1, 11] \) in Figure 2 it is easy to see that the number of imprimitive symmetric companion matrices of order 11 is \( 2^{(11-1)/2} = 32 \).
Henceforth we restrict our investigation to primitive symmetric companion matrices, and focus on finding the exponent set of these matrices. In the paper [19] it is shown that the exponent of a primitive symmetric matrix is at most \(2(n - 1)\). Thus if \(A\) is a primitive symmetric companion of order \(n \geq 2\), then \(\exp(A) \in [2, 2n - 2]\).

We now will give a general procedure to find exponent of a primitive matrix.

Let \(A\) be a primitive matrix and let \(\exp(A : i, j)\) be the smallest positive integer \(k\) such that there exists a walk of length \(\ell\) from vertex \(i\) to vertex \(j\) in \(D(A)\) for all \(\ell \geq k\). Equivalently, for \(B = A^\ell\), \(b_{i,j} > 0\) for all \(\ell \geq k\). Let \(\exp(A : i)\) is the smallest positive integer \(p\) such that there exists a walk of length \(p\) from vertex \(i\) to any vertex \(j\) in \(D(A)\). Equivalently, every entry in the \(i^{th}\) row of \(A^p\) is positive. As a consequence, every entry in the \(i^{th}\) row of \(A^{p+1}\) is also positive.

The following result can be found in Brualdi and Ryser [2].

**Lemma 3.3** Let \(A \in M_n(\mathbb{R})\) be a primitive matrix. Then \(\exp(A) = \max_{1 \leq i, j \leq n} \exp(A : i, j) = \max_{1 \leq i \leq n} \exp(A : i)\).

**Example 3.4** Consider the Fan graph, \(F_8\), in Figure 3, whose adjacency matrix is a symmetric companion matrix. It is easy to check that \(\exp(F_8 : 8, 1) = 1, \exp(F_8 : 8, 8) = 2\). Also note that \(\exp(F_8 : i) = 2\) for \(1 \leq i \leq n\). Hence \(\exp(F_8) = 2\).

Hence our objective is to find \(\exp(A : i, j)\) for all \(i, j \in [1, n]\). In this paper, a cycle at a vertex \(i \in [1, n]\), we mean a walk from \(i\) to \(i\), where repetition of internal vertices is allowed. Where as an elementary cycle or circuit we mean a cycle and repetition of vertices is allowed only at initial and terminal vertices. If there is a loop at a vertex \(i\), then \(\exp(A : i, i) = 1\). For the remaining cases, see Proposition 4 in Liu et al. [11]. It shows that \(\exp(A : i, j) = d(i, j) + 2k\) for some nonnegative integer \(k\). We include the proof for the sake of completeness.

**Proposition 3.5** (11) Let \(A \in M_n(\mathbb{R})\) be a symmetric primitive matrix
Figure 3: The Fan graph $F_8$

and $i, j \in [1, n]$. Then $\exp(A : i, j) = \ell - 1$, where $\ell$ is the length of the shortest walk from $i$ to $j$ in $D(A)$ such that $2 \nmid d(i, j) + \ell$, and $d(i, j)$ is the distance between $i$ and $j$ in the graph $D(A)$. In particular, if $a_{ii} = 0$, then $\exp(A : i, i) = \ell - 1$, where $\ell$ is the length of shortest odd cycle from $i$ to $i$.

If $A \in M_n(\mathbb{R})$ is a symmetric primitive matrix, then as discussed earlier $D(A)$ is a connected graph. As a consequence $ij^{th}$ entry of $A^{d(i, j)+2k}$ is positive for every nonnegative integer $k$. Suppose $\ell$ is the length of the shortest walk from $i$ to $j$ with parity different from that of $d(i, j)$, then $ij^{th}$ entry of $A^{(\ell-1)+t}$ is positive for every nonnegative integer $t$. Hence the result follows.

Corollary 3.6 ([11]) Let $G$ be a primitive simple graph, and let $i$ and $j$ be any pair of vertices in $V(G)$. If there are two walks $P_1, P_2$ from $i$ to $j$ with lengths $k_1$ and $k_2$ respectively, where $2 \nmid k_1 + k_2$, then $\exp(G : i, j) \leq \max\{k_1, k_2\} - 1$.

The following theorem provides a necessary condition for the exponent of primitive symmetric matrix to be odd.

Theorem 3.7 ([20]) Let $X$ be a primitive graph and suppose that $\exp(X)$
is odd. Then $X$ contains two vertex disjoint odd cycles.

From the construction of sets $PSC_n^{\alpha,e}$, if $A \in PSC_n^{\alpha,e}$, then every cycle in $D(A)$ contains the vertex $n$. Hence the following result.

**Corollary 3.8** If $A \in PSC_n^{\alpha,e}$, then $\exp(A)$ is even.

Thus, if $d(i,j)$ is odd, then $\exp(A : i,j) < \exp(A)$. Before proceeding to next result, we need the following notation.

**Definition 3.1** Let $B_{n}^{q,k} \subseteq B_{n}$ denote the set of all binary strings of length $n$ with $q$ zeros and having at least one longest subword of zeros of length $k$. For example, $B_4^{2,6} = \{100100, 010100, 010010, 001100, 001010, 001001\}$. Consequently, a necessary condition for $B_{n}^{q,k}$ to be nonempty is $n \geq q \geq k \geq 0$. An immediate observation is that $B_{n} = \bigcup_{q=0}^{n} \bigcup_{k=0}^{q} B_{n}^{q,k}$, thus $2^n = \sum_{q=0}^{n} \sum_{k=0}^{q} F_n(q,k)$, where $F_n(q,k) = \# B_n^{q,k}$. The value of $F_n(q,k)$ is defined to be zero whenever $n < 0$. For more results on $F_n(q,k)$ see Monimala Nej, A. Satyanarayana Reddy [14]. M.A. Nyblom in [15] denoted $S_r(n)$ for the set of all binary strings of length $n$ without $r$-runs of ones, where $n \in \mathbb{N}$ and $r \geq 2$, and $T_r(n) = |S_r(n)|$. For example, if $n = 3, r = 2$, then $S_2(3) = \{000, 011, 010, 100, 101\}, T_2(3) = 5$.

**Theorem 3.9**

1. Let $A \in PSC_n^{1,1},$ where $n \geq 4$. Then $\exp(A) = 2(t+1)$, where $t = \lfloor \frac{m(V_1)+1}{2} \rfloor$. And $N_{n}^{1,1}(2(t+1)) = 2 \sum_{q=n-m(V_1)}^{n-3} F_{n-3}(q, m(V_1))$.

2. Let $A \in PSC_n^{0,1},$ where $n \geq 4$. Suppose $\min(V_2) = h$ and $t = \max \left\{ h-1, \lfloor \frac{m(V_1^h)}{2} \rfloor \right\}$, where $V_1^h = V_1 \setminus [1, h-1]$. Then $\exp(A) = 2(t+1)$. And

$$N_{n}^{0,1}(2(t+1)) = 2 \left[ \sum_{i=0}^{t-2} \sum_{k=2t-1}^{2t} \sum_{q=k}^{n-i-4} F_{n-i-4}(q,k) + T_{2t+1}(n-t-3) \right].$$
Proof of Part 1. Let $A \in \mathcal{PSC}_{n}^{1,1}$, then from Corollary 3.6, $\exp(A : i, j) \leq 2$ for all $i, j \in V_2$ and for $j \in V_1$, $i \in V_1 \exp(A : i, j) \leq (i - i') + 2$ where $i' = \text{max}([1, i] \cap V_2)$. Now it is sufficient to find $\exp(A : i, j)$ for all $i, j \in V_1$. Suppose $m(V_1) \neq 0$. Then by definition of $m(V_1)$ there exists $U \subseteq V_1$ such that $|U| = m(V_1)$. Suppose $U = \{i_1, i_2, \ldots, i_{m(V_1)}\}$, where $i_a = i_{a - 1} + 1, a \in [2, m(V_1)]$. The result follows by observing the fact that $\exp(A : i, j) \leq \exp(A : i_t, i_t) = 2t + 2$, for all $i, j \in V_1$.

Before finding $N_{n}^{1,1}(2(t + 1))$ one can observe that the exponent of a matrix depends on $m(V_1)$. Since $a_{n,1}a_{n,n-1}a_{n,n} > 0$, there is a bijection between sets $\mathcal{PSC}_{n}^{1,1}$ and $B_{n-3}$. In particular there is a bijection between $B_{n-3}^{m(V_1)}$ and $\{A \in \mathcal{PSC}_{n}^{1,1} : \exp(A) = 2[\frac{m(V_1) + 1}{2}] + 2\}$. To conclude one has to invoke Remark 2.3.

Proof of Part 2. Suppose $r = \lfloor \frac{m(V_1)^h + 1}{2} \rfloor$. From the proof of first part we have $\exp(A : i, j) \leq 2(r + 1)$ for $i, j \in V_1^h$. Further there exists $i \in V_1^h$ with $\exp(A : i, i) = 2r + 2$. Thus the proof follows by observing that $\exp(A : j, j') \leq \exp(A : 1, 1) = 2h$ for $j, j' \in \lfloor 1, h - 1 \rfloor$ and $\exp(A : j, j') \leq h + r + 1$, where $j \in \lfloor 1, h - 1 \rfloor$ and $j' \in V_1^h$.

Finally the number of primitive matrices with a given exponent follows easily from the definitions of $F_n(q, k)$ and $T_r(n)$.

**Example 3.10** Consider the graphs in the Figure 4. Both of them have exponent six.

For $n = 7$, number of matrices with exponent six is given by $N_{7}^{0,1}(6) = 2 \left[ \sum_{k=3}^{4} \sum_{q=k}^{3} F_3(q, k) + T_3(2) \right] = 10$. Thus there are 10 matrices in $\mathcal{PSC}_{7}^{0,1}$ with exponent 6.

**Corollary 3.11** Let $n \geq 4$. Then $E(\mathcal{PSC}_{n}^{1,1}) = \{2(t + 1) : t \in [0, \lfloor \frac{n+2}{2} \rfloor]\}$ and $E(\mathcal{PSC}_{n}^{0,1}) = \{2t : t \in [2, n - 1]\}$.

It is easy to check that the exact upper bound for the class of symmetric primitive matrices in [19] belongs to $E(\mathcal{PSC}_{n}^{0,1})$. And the graph which achieve this bound is in Figure 5.
4 Exponents of matrices in $\mathcal{PSC}_n^{1,0}$

If $A \in \mathcal{PSC}_n^{1,0}$, then we know that $a_{n,n-1}$ is either 1 or 2. Thus the last row of $A$ is of the form $1Ya_{n,n-1}0$, where $Y \in B_{n-3}$. Recall that we denote an element in $\mathcal{PSC}_n^{1,0}$ as $A_Y$. From Corollary 3.8 we know that if $A \in \mathcal{PSC}_n^{1,0}$, then $\exp(A) = 2d$ for some $d \in \mathbb{N}$. In particular, we have the following result.

**Theorem 4.1** Let $n \in \mathbb{N}, n \geq 4$. Then $E(\mathcal{PSC}_n^{1,0}) = \{2t : t \in \llbracket 1, \left\lfloor \frac{n-1}{2} \right\rfloor \rrbracket \}$.

If $Y \in B_{n-3}^{k,k}$ and $k$ is even, then we claim that $\exp(A_Y)$ is $k + 2$ or equivalently $m(V_1) + 2$. For example, if $Y = \underbrace{11 \cdots 11}_{n-3} \in B_{n-3}^{0,0}$, then $A_Y =$
\[
\begin{bmatrix}
0 & 1 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 1 & 0 & \cdots & 0 & 1 \\
0 & 1 & 0 & 1 & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 2 \\
1 & 1 & 1 & 1 & \cdots & 2 & 0 \\
\end{bmatrix}
\]
and clearly \(A^n_2 > 0\). Hence \(2 \in E(\mathcal{P}\mathcal{S}C_{n}^{1,0})\). We now suppose that \(k \in [2, n-3]^r\). In this case \(V_1 = [a + 1, a + k]\) for some \(1 \leq a \leq n - k - 2\) and \(V_2 = [1, a] \cup [a + k + 1, n - 1]\). As a consequence if \(i, j \in V_2\) then \(\text{exp}(A_Y : i, j) \leq 2\). Furthermore, it is easy to see that for every \(i, j \in [1, n]\), \(\text{exp}(A_Y : i, j) \leq \text{exp}(A_Y : p, p) = k + 2\) where \(p = a + \frac{k}{2} + 1\) or \(a + \frac{k}{2}\). Finally the upper bound is achieved from Corollary 1 in Delorme and Sole [3]. Hence the result follows.

Next result is the main result of this section, which expresses the exponent of \(A \in \mathcal{P}\mathcal{S}C_{n}^{1,0}\) in terms of certain parameters related to \(V_1\). It can be used as an algorithm to determine the exponent of a primitive matrix of this class. To define these parameters, recall that we expressed \(V_1\) as \(V_1 = U_1 \cup U_2 \cup \cdots \cup U_r\), where \(U_i\) is a set of consecutive numbers in \(V_1\) and for \(i > j \in [1, r]\), \(|\text{min}(U_i) - \text{max}(U_j)| \geq 2\). Then the maximum odd number among \(|U_1|, |U_2|, \ldots, |U_r|\) is denoted \(\text{mo}(V_1)\). Thus \(\text{mo}(V_1) = m(V_1)\), if \(m(V_1)\) is odd. Because of the primitivity of \(A\), \(|U_i|\) is even for some \(i\).

Also, let \(q_{V_1}\) be the smallest even number in \{|U_1|, |U_2|, \ldots, |U_r|\}, and let \(\text{se}(V_1) = \begin{cases} 0 & \text{if } i, i + 1 \in V_2 \text{ for some } i \in [1, n - 2], \\
q_{V_1} & \text{otherwise.} \end{cases}\)
That is \(\text{se}(V_1) + 3\) is the smallest odd cycle length in \(D(A)\). Now corresponding to each \(U_i\), \(1 \leq i \leq r\) there is an elementary cycle denoted \(c_{U_i}\) (or simply \(c_i\)), with its vertex set \(V(c_{U_i}) = U_i \cup \{p_i, q_i\}\), where \(p_i, q_i \in V_2\). If \(p_i < q_i\), then we call \(p_i\) and \(q_i\) as the first vertex and the last vertex of \(c_{U_i}\), respectively.

It is clear that \(\ell(c_{U_i})\), the length of the cycle \(c_{U_i}\) is equal to \(|U_i| + 3\). Let \(C(A) = \{c_{U_i} : 1 \leq t \leq r\} \cup \{\text{cycles of length } 3\}\). Since there exists \(i \in [1, r]\) such that \(|U_i|\) is even, hence there exists \(c \in C(A)\) such that \(\ell(c)\) is odd. From this observation it is easy to see, for each cycle \(d_1 \in C(A)\) with an even length there exists a cycle \(d_2 \in C(A)\) with \(\ell(d_2) = \text{se}(V_1) + 3\) such
that $\ell(d_1 + d_2)$ is $\ell(d_1) + se(V_1) + 1$ or $\ell(d_1) + se(V_1) + 3$, where $d_1 + d_2$ is a cycle in $D(A)$ with $V(d_1 + d_2) = V(d_1) \cup V(d_2)$. Now $V(d_1 + d_2)$ contains at least three vertices from $V_2$. In particular, $|V(d_1 + d_2) \cap V_2| = 3$ or 4. Hence we let the first vertex and last vertex of $d_1 + d_2$ be same as the first vertex and last vertex of $d_1$ respectively. A cycle $d_2$ with this property is called an associate of $d_1$. Let $C'(A) = \{c \in C(A) : \ell(c) \text{ is odd} \} \cup \{d_1 + d_2 : d_1, d_2 \in C(A), \ell(d_1) \text{ is even}, d_2 \text{ is an associate of } d_1\}$. 

**Example 4.2** Consider the graph in Figure 6. 

![Figure 6: An example illustrate the parameters](image)

Here $V_1 = \{2, 3, 4, 5\} \cup \{7\} \cup \{9, 10\}$ and $C(A) = \{c_1, c_2, c_3\}$ with $V(c_1) = \{1, 2, 3, 4, 5, 6, 12\}$, $V(c_2) = \{6, 7, 8, 12\}$, $V(c_3) = \{8, 9, 10, 11, 12\}$. Hence $\ell(c_1) = 7$ and 1, 6 are the first vertex, last vertex of $c_1$ respectively. Also $m(V_1) = 4$, $mo(V_1) = 1$, $se(V_1) = 2$. Hence $C'(A) = \{c_1, c_2 + c_3, c_3\}$.

**Theorem 4.3** Let $n \geq 4$ be an integer and $A \in \mathcal{PS}_{n}^{1,0}$. 

1. If $m(V_1)$ is odd, then  

$$exp(A) = \begin{cases} 
\ m(V_1) + se(V_1) + 5 & \text{if there exists } c \in C'(A) \text{ with } c = d_1 + d_2, \ell(d_1) = m(V_1) + 3 \\
\ m(V_1) + se(V_1) + 3 & \text{and } |V(d_1 + d_2) \cap V_2| = 4, \\
\ m(V_1) + se(V_1) + 3 & \text{otherwise.} 
\end{cases}$$
2. If \( m(V_1) \) is even, and \( mo(V_1) \geq m(V_1) - se(V_1) - 1 \), then

\[
\exp(A) = \begin{cases} 
    mo(V_1) + se(V_1) + 5 & \text{if there exists } c \in C'(A) \text{ with } c = d_1 + d_2, \ell(d_1) = mo(V_1) + 3 \text{ and } |V(d_1 + d_2) \cap V_2| = 4, \\
    mo(V_1) + se(V_1) + 3 & \text{otherwise.}
\end{cases}
\]

3. In the remaining cases \( \exp(A) = m(V_1) + 2 \).

Recall that if \( A \) is a symmetric primitive matrix and the \( ij \)-th entry of \( A^r \) is positive, then the \( ij \)-th entry of \( A^{r+2} \) is positive. Further if \( i, j \in V(c) \), where \( c \) is a cycle of odd length, then there are two paths from \( i \) to \( j \) in \( c \) such that sum of their lengths is \( \ell(c) \). Hence from Corollary 3.6 the smallest odd cycle containing \( i, j \) plays an important role in finding \( \exp(A : i, j) \).

We suppose that \( c \in C'(A) \) with \( p \) as its first vertex. Then for all \( i, j \in V(c) \), we have

\[
\exp(A : i, j) \leq \begin{cases} 
    \ell(c) - 1 & \text{when } c \in C(A), \\
    \ell(c) + se(V_1) & \text{when } c = d_1 + d_2, \text{ and } \ell(c) = \ell(d_1) + se(V_1) + 1, \\
    \ell(c) + se(V_1) + 2 & \text{otherwise.}
\end{cases}
\]

And from Proposition 3.5, equality holds when \( i = j = p + \lfloor \ell(c) - 1 \rfloor \). Now we attempt to find a similar bound for \( \exp(A : i, j) \) when \( i, j \) belongs to different cycles in \( C'(A) \).

To prove the result it is sufficient to show that if \( \gamma, \delta \in C'(A) \) with \( \ell(\gamma) \geq \ell(\delta) \), then \( \exp(A : i, j) \leq \ell(\gamma) - 1 \), for \( i \in V(\gamma) \) and \( j \in V(\delta) \).

Let \( p_\gamma \) and \( p_\delta \) be the first vertex of \( \gamma \) and \( \delta \) respectively. Suppose \( q, q' \) are nonnegative integers such that \( \ell(\gamma) - q - 2 > q \) and \( \ell(\delta) - q' - 2 > q' \). For \( q \leq q' \), from vertex \( p_\gamma + q \) to \( p_\delta + q' \) there are paths of length \( q + q' + 2 \) and \( \ell(\gamma) + q' - q \) which are in opposite parity because \( \ell(\gamma) - 2 \) is odd. Hence 

\[
\exp(A : p_\gamma + q, p_\delta + q') \leq \max\{q + q' + 2, \ell(\gamma) + q' - q\} - 1 \leq \ell(\gamma) - 1.
\]

For \( q > q' \) there are paths of length \( q + q' + 2 \) and \( \ell(\delta) + q - q' \) from vertex \( p_\gamma + q \) to \( p_\delta + q' \) and 

\[
\exp(A : p_\gamma + q, p_\delta + q') \leq \max\{q + q' + 2, \ell(\delta) + q - q'\} - 1 \leq \ell(\gamma) - 1.
\]
Thus $\exp(A : i, j) \leq \ell(\gamma) - 1$, for $i \in V(\gamma)$ and $j \in V(\delta)$. Hence the result follows from Lemma 3.3.

It is easy to see that Theorem 4.1 also follows as a corollary of the above theorem. In the remaining part of this section

1. we will evaluate $N_{n}^{1,0}(b)$ when $b = 2$ and $b = \max[2, n - 1]^c$.

2. And for the remaining values of $b$, we will provide bounds for $N_{n}^{1,0}(b)$.

If $Y \in B_{n-3}^{k,k}$, then $\exp(A_Y)$ is either $k + 2$ or $k + 3$ depending on the parity of $k$. From [14] it is known that for $k \geq 1$ $F_{n}^{k,k} = (n - k) + 1$. Hence we have $4(n - b) + 2 \leq N_{n}^{1,0}(b)$. And $N_{n}^{1,0}(b) \leq 2 \left\lfloor \sum_{k=\lfloor b/2 \rfloor}^{n-3} F_{n-3}^{q,k} \right\rfloor$ follows from Theorem 4.3. In order to improve the upper bound of $N_{n}^{1,0}(b)$ we partition the set $\mathcal{PSC}_{n,k}^{1,0}$ as $\mathcal{PSC}_{n,k}^{1,0} = \bigcup_{k=\lfloor b/2 \rfloor}^{n-3} \mathcal{PSC}_{n,k}^{1,0}$, where $\mathcal{PSC}_{n,k}^{1,0} = \{A_Y \in \mathcal{PSC}_{n}^{1,0} : Y \in B_{n-3}^{k,k}\}$.

Note that if $A \in \mathcal{PSC}_{n,k}^{1,0}$, then $m(V_1) = k$ and $se(V_1) \in [0,k]^c$.

If $n \geq 4$, then it is easy to verify that $2 \notin E(\mathcal{PSC}_{n,k}^{1,0})$ if and only if $k \geq 1$ and $E(\mathcal{PSC}_{n,0}^{1,0}) = \{2\}$. As a consequence we get $|\mathcal{PSC}_{n,k}^{1,0}| = 2$ and $N_{n}^{1,0}(2) = 2$. The following result evaluates $E(\mathcal{PSC}_{n,k}^{1,0})$ for various values of $n$ and $k$ through which the upper bound of $N_{n}^{1,0}(b), b \geq 4$ can be improved.

**Theorem 4.4** Let $n \geq 5$ and $k$ be positive integers.

1. If $n$ is odd and $n \geq 2(k + 2) + 1$, or if $n$ is even and $n \geq 3(k + 2) - 1$, then $E(\mathcal{PSC}_{n,k}^{1,0}) = \{k + 2, 2(k + 2)\}^c$.

2. If $k$ is odd and $n \in [k + 3, 2(k + 2) - 1]^c$, then $E(\mathcal{PSC}_{n,k}^{1,0}) = \{k + 2, n - 1\}^c$.

3. If $k$ is even and $n \in [k + 3, 2(k + 2) - 1]^c \setminus \{k + 5, k + 7\}$, then $E(\mathcal{PSC}_{n,k}^{1,0}) = \{k + 2, n - 1\}^c \setminus \{k + 4\}$. And for $n \in \{k + 5, k + 7\}$, $E(\mathcal{PSC}_{n,k}^{1,0}) = \{k + 2, n - 1\}^c$. 

17
4. If \( k \) is odd and \( n \in \left[ k+5, 3(k+1) \right] \), then \( E(\mathcal{PSC}_{n,k}^{1,0}) = \left[ k+3, l \right] \), where \( l = \left\{ \begin{array}{ll}
+ 2k \left\lfloor \frac{n-k-3}{4} \right\rfloor + 3 & \text{if } n-k-4 = 4 \left\lfloor \frac{n-k-4}{4} \right\rfloor + 1, \\
+ 2k \left\lfloor \frac{n-k-4}{4} \right\rfloor + 5 & \text{if } n-k-4 = 4 \left\lfloor \frac{n-k-4}{4} \right\rfloor + 3.
\end{array} \right. \)

5. If \( k \) is even and \( n \in \left[ k+4, 3k+2 \right] \), then \( E(\mathcal{PSC}_{n,k}^{1,0}) = \left[ k+2, n-k \right] \) but if \( n-k \leq k+2 \), then \( E(\mathcal{PSC}_{n,k}^{1,0}) = \left\{ k+2 \right\} \).

For \( k \) is even and \( n = 3(k+1) + 1 \), \( E(\mathcal{PSC}_{n,k}^{1,0}) = \left[ k+2, 2(k+1) \right] \).

First observe that the lower bound of \( E(\mathcal{PSC}_{n,k}^{1,0}) \) will be attained by \( A_Y \) where \( Y \in B_{n,k}^{k,k-3} \).

Proof of part \( 1 \). For any element in \( \mathcal{PSC}_{n,k}^{1,0} \) the maximum value of \( \text{se}(V_1) \) is either \( k \) or \( k-1 \) depending on whether \( k \) is even or odd respectively. Then from Theorem \( 4.3 \) the upper bound \( 2(k+2) \) belongs to \( E(\mathcal{PSC}_{n,k}^{1,0}) \) whenever \( n \geq 2(k+2) + 1 \) when \( n \) is odd or \( n \geq 3(k+2) - 1 \) when \( n \) is even.

- Suppose \( n \geq 2(k+2) + 1 \) is odd and

\[
Y = \left(0, \ldots, 0, 1, 0, 0, 0, \ldots, 0, 1, 0, \ldots, 1, 0\right),
\]

where \( t \in \left[0, k-1\right] \) and \( 2 \nmid t+k \). Then the exponent of \( A_Y \in \mathcal{PSC}_{n,k}^{1,0} \) is \( k + t + 5 \).

- Suppose \( n \geq 3(k+2) - 1 \) is even and \( k \) is odd. If we choose

\[
Y = \left(0, \ldots, 0, 1, 0, 1, 0, 0, 0, 1, 0, \ldots, 0, 1, \ldots, 0, 1, 0\right),
\]

where \( t \in \left[0, k-1\right] \), then the exponent \( A_Y \) is \( k + t + 5 \). Similarly for \( k \) is even, the exponent \( A_Y \) is \( k + t + 5 \), whenever

\[
Y = \left(0, \ldots, 0, 1, 0, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 1, 0\right)
\]

and \( t \in \left[1, k-1\right] \).

Finally, \( A_Y \in \mathcal{PSC}_{n,k}^{1,0} \) has exponent \( k + 4 \), if

\[
Y = \left(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 1, 0\right).
\]

18
Proof of part 2 Suppose $n \neq k + 4$ and
\[ Y = (0, \ldots, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0), \]
where $t \in \llbracket 0, n - k - 6 \rrbracket^c$. Then $\exp(A_Y)$ is $k + t + 5$ and $E(\mathcal{P}SC_{n,k}^{1,0}) = \llbracket k + 2, n - 1 \rrbracket^c$.

Proof of part 3 For $n \in \llbracket k + 9, 2(k + 2) - 1 \rrbracket^c$, there are no elements in $\mathcal{P}SC_{n,k}^{1,0}$ with exponent $k + 4$. Because then $\text{mo}(V_1) + \text{se}(V_1) = k + 1$ or $\text{mo}(V_1) + \text{se}(V_1) = k - 1$. And $\text{mo}(V_1) + \text{se}(V_1) = k + 1$ implies $\text{se}(V_1) = k$ and $\text{mo}(V_1) = 1$ and the exponent of the corresponding elements is $k + 6$. Since $n - k - 4 \leq k - 1$, $\text{mo}(V_1) + \text{se}(V_1) = k - 1$ is not possible. For $n = k + 5$ and $n = k + 7$, $A_Y$ has exponent $k + 4$, where $Y = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) \in B_{k+2}$ and $Y = (0, 1, 0, \ldots, 0, 1, 0) \in B_{k+4}$ respectively. And for $n \in \llbracket k + 7, 2(k + 2) - 1 \rrbracket^c$,
\[ \exp(A_Y) = k + t + 5, \]
where $Y = (0, \ldots, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ and $t \in \llbracket 0, n - k - 6 \rrbracket^c$. Hence the result follows in this case.

Proof of part 4 Suppose $d = n - k - 4$ and $d' = \lfloor \frac{d}{2} \rfloor$. Then for this part either $d = 4d' + 1$ or $d = 4d' + 3$ and $\text{se}(V_1) \in \llbracket 0, 2d' \rrbracket^c$.

For $d = 4d' + 1$, consider $Y = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, \ldots, 0)$, where $t \in \llbracket 0, 2d' \rrbracket^c$. Then $\exp(A_Y) = k + t + 3$. If $d = 4d' + 3$, then the exponent of $A_Y$ is $k + t + 5$, where
\[ Y = (0, \ldots, 0, 1, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, 1, 0, \ldots, 0) \]
and $t \in \llbracket 0, 2d' \rrbracket^c$. Hence the result follows in this case.

Proof of part 5 Let $n \in \llbracket k + 4, 2(k + 1) \rrbracket^c$. Then for the elements in $\mathcal{P}SC_{n,k}^{1,0}$ either $\text{mo}(V_1)$ does not exists or $\text{mo}(V_1) + \text{se}(V_1) \leq \text{mo}(V_1) - 3$, which gives $E(\mathcal{P}SC_{n,k}^{1,0}) = \{k + 2\}$.

For $n \in \llbracket 2(k + 2), 3k + 4 \rrbracket^c$, maximum value of $\text{mo}(V_1) + \text{se}(V_1)$ is $n - k - 5$. Then the upper bound for the set $E(\mathcal{P}SC_{n,k}^{1,0})$ is $n - k$ or $n - k - 2$ according as $n \in \llbracket 2(k + 2), 3k + 2 \rrbracket^c$ or $n = 3k + 4$ respectively. And this upper bound
will be attained by $A_Y$, where $Y = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$. Also for $2(k + 2) \leq n \leq 3k + 4$, exponent of $A_Y$ is $k + t + 5$, where $Y = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ with $t \in [1, n - 2k - 7]$. And the exponent of $A_Y$ is $k + 4$, where $Y = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 1)$. Hence the proof is complete.

Recall that $N_n^{1,0}(b) \leq 2 \left( \sum_{k=\frac{b}{2}}^{b} \sum_{q=k}^{n} F_{q,k}^{n-3} \right)$. Using the above theorem, by restricting some of the values of $k$ in the first summation, we will improve this upper bound of $N_n^{1,0}(b)$ for certain values of $b$.

Let $S_n^{1,0}(b)$ denote the set of all possible values of $k$ in the first summation. Then $S_n^{1,0}(b) \subseteq \left[ \frac{b-4}{2}, b-2 \right]$. The following result provides all possible values in $S_n^{1,0}(b)$.

**Corollary 4.5** Let $b \in E({\mathcal{PSC}}_n^{1,0}) \setminus \{2\}$.

1. Then $b - 2 \in S_n^{1,0}(b)$.

2. If $n \in [b + 5, 2b - 5]$, then $S_n^{1,0}(b) = \left[ \frac{b-4}{2}, b - 2 \right] \setminus \{b - 4\}$.

3. Let $n$ be even. Then by Part 4 of the above theorem $\left[ \frac{b-4}{2}, \left\lfloor \frac{n+1}{3} \right\rfloor - 2 \right] \subseteq S_n^{1,0}(b)$. For the remaining cases that is when $k \geq \left\lfloor \frac{n+1}{3} \right\rfloor - 1$, we have the following.

   (a) If $k$ is odd, then $k \in S_n^{1,0}(b)$, provided $l \geq b$, where $l$ is as defined in Part 4 of the above theorem.

   (b) If $k$ is even, $k \leq n - b$, and $n \in [2k+4, 3k+2]$, then $k \in S_n^{1,0}(b)$. If $n = 3k + 4$ and $b \leq \frac{2(n-1)}{3}$, then $k \in S_n^{1,0}(b)$.

For example, $S_{20}^{1,0}(12) = \left[ \frac{b-4}{2}, b - 2 \right] \setminus \{b - 4\}$. Hence there is no improvement, where as $S_{20}^{1,0}(16) = \{9, 11, 13, 14\}$, a significant improvement.
Theorem 4.6 Let $n \in \mathbb{N}$ and $b = \max\{[2, n - 1]\}$. Then

$$N^1_0(n)(b) = \begin{cases} 
2(2n - 7) & \text{if } n \geq 5, n \text{ is odd}, \\
18 & \text{if } n \geq 8, n \text{ is even}.
\end{cases}$$

We divide the proof into the even and odd cases.

$n$ is odd. From Part 2 of Corollary 4.5 a necessary condition for a matrix $A_Y \in \mathcal{PSC}^{1,0}_{n,k}$ having exponent $n - 1$ is $k \in \left[\frac{n-5}{2}, n-3\right]$ where $Y \in B^{q,k}_{n,k-3}$. Here $q \leq 2(k+1)$ follows from the fact that $k \leq q \leq n-3$ and $n-k-4 \leq k+1$.

Since exponent of $A_Y$ is $n - 1$, hence from Theorem 4.3 we have

- if $k \in \left[\frac{n-5}{2}, n-6\right]$, then $q \in \{n-4, n-5\}$.
- if $k = \frac{n-5}{2}$, then $q = n-5$.
- if $k \in \{n-5, n-4\}$, then $q = n-4$ and when $k = n-3$, $q$ is also $n-3$.

Thus $A_Y$ will have exponent $n - 1$ only if $Y \in S$, where

$$S = \bigcup_{k=\frac{n-5}{2}}^{n-6} B^{n-5,k}_{n-3} \bigcup_{k=\frac{n-3}{2}}^{n-4} B^{n-4,k}_{n-3} \bigcup B^{n-3,n-3}_{n-3}.$$  

When $q = k = n-3$ there are only two matrices with exponent $n - 1$, in the remaining cases there are four matrices with exponent $n - 1$. Hence the result follows in this case.

$n$ is even. Suppose $n \geq 10$. From Part 3 of Corollary 4.5 a necessary condition for a matrix $A \in \mathcal{PSC}^{1,0}_{n,k}$ having exponent $n - 2$ is $k \in \{n-7, n-5, n-4\}$.

- When $k = n-4$, $n-5$ it is easy to see that $q = n-4$ and $q = n-5$ respectively and exponent of $A_Y$ is $n - 2$ for all $Y \in B^{k,k}_{n-3}$.
- When $k = n-7$, we have $n-7 \leq q \leq n-4$. But $\exp(A_Y) \neq n-2$ if $q = n-7$ or $q = n-4$. Further when $q = n-6$ or $q = n-5$, there are elements in $\mathcal{PSC}^{1,0}_{n,n-7}$ whose exponent is $n-2$ provided $se(V_i) = 0$.  

21
Thus $A_Y \in \mathcal{PSC}_{n}^{1,0}$ will have exponent $n - 2$ only if $Y \in S$, where

$$S = \bigcup_{q=n-6}^{n-5} B_{n-3}^{q,n-7} \bigcup B_{n-3}^{n-5,n-5} \bigcup B_{n-3}^{n-4,n-4}.$$ 

For $n = 8$ possible values of $q$ and $k$ will be same as above except $k = 1$ whenever $q = 3$. And modified value will be $k = 2$ and $q = 3$. Now for $n \geq 8$ it is easy to check that when $k = n - 7$, then for each $q$ there are four matrices with exponent $n - 2$. Hence the result follows in this case.

\[\square\]

When $n = 6$, the number of primitive matrices with exponent 4 is 10.

5 \hspace{1cm} \textbf{Exponents of matrices in } \mathcal{PSC}_{n}^{0,0} \hspace{1cm} \textbf{Suppose that } n \geq 4 \text{ and } A \in \mathcal{PSC}_{n}^{0,0}. \text{ Then } \exp(A) \geq 4 \text{ as } \exp(A : 1, 1) \geq 4. \text{ Liu et al.} \ [11] \text{ showed that the exponent of a primitive symmetric matrix with trace zero is at most } 2(n - 2). \text{ Hence } E(\mathcal{PSC}_{n}^{0,0}) \subseteq [4, 2(n - 2)]^c. \text{ The following result shows the equality of these sets.} \text{ Proof of the result is available in} \ [11] \text{. We include the proof for the sake of completeness.}

\textbf{Theorem 5.1} \hspace{0.5cm} \text{Suppose } n \geq 4. \text{ Then } E(\mathcal{PSC}_{n}^{0,0}) = \{2t : t \in [2, n - 2]\}.

If $A \in \mathcal{PSC}_{n}^{0,0}$, then $a_{n,1} = a_{n,n} = 0$ and $a_{n,n-1} = 1$ or 2. If the last row of $A$ is $[0 \ldots 0 1 1 \ldots 1 2 0]$, where $t \in [1, n - 3]$, then it is easy to see that $\exp(A) = 2(t + 1)$.

Now it is easy to see that the upper bound for the class of symmetric primitive $(0,1)$-matrices with zero trace is $2(n - 2)$. And this upper bound will be attained by the graph in Figure 7.

Our next goal is to find $\exp(A)$ for $A \in \mathcal{PSC}_{n}^{0,0}, n \geq 4$. We will continue with the notation used in previous sections. Recall that $\min(V_2) = h$ and
Figure 7: A lollipop graph

$V_1^h = V_1 \setminus [1, h-1]$. It is clear that $2 \leq h \leq n - 2$. We denote the cycle in $C(A)$ whose first vertex is $h$ as $c^h$. The following observations are easy to verify.

1. For all $i, j \in [1, h-1]$, $\exp(A : i, j) \leq 2(h-1) + \exp(A : h, h)$. The equality holds for $i = j = 1$. Also $\exp(A : h, h) = \begin{cases} se(V_1^h) + 2 & \text{if } \ell(c^h) = se(V_1^h) + 3, \\ se(V_1^h) + 4 & \text{otherwise.} \end{cases}$

2. For all $i, j \in V \setminus [1, h-1]$, $\exp(A : i, j)$ can be found from the previous section.

3. For $i \in [1, h-1]$ and $j \in V \setminus [1, h-1]$, $\exp(A : i, j) \leq (h-1) + \exp(h, j)$. The equality holds for $i = 1$.

Thus to find $\exp(A)$, it is sufficient to find $\exp(A : h, j)$, where $j \in V \setminus [1, h]$. It is now clear that we will use the results of the previous section to find $\exp(A : h, j)$ but now they are applied to $V_1^h$, instead of $V_1$.

**Example 5.2** For an example, consider the graph in Figure 8. There we have $V_1 = \{1, 3, 4, 6, 8, 9\}$, $V_2 = \{2, 5, 7, 10\}$, $\min(V_2) = h = 2$. Hence $V_1^h = \{1, 3, 4, 6, 8, 9\} \setminus \{1\} = \{3, 4, 6, 8, 9\} = \{3, 4\} \cup \{6\} \cup \{8, 9\}$. Let $c^2$ be the vertex set of elementary cycle $c^2$ and $\exp(A : 2, 2) = 4$, $\exp(A :
1, 1) = 2 + exp(A : 2, 2) = 6. Thus to find exp(A), it is now sufficient to find exp(A : 2, j), where j ∈ [3, 11].

Figure 8: An Example

The following result provides an upper bound on exp(A : h, i) whenever i belongs to the vertex set of an elementary cycle.
Lemma 5.3 Let $A \in \mathcal{P}SC_n^{0,0}$ and $c \in C(A)$, $i \in V(c)$.

1. If $\ell(c)$ is even, then
   \[\exp(A : h, i) \leq \begin{cases} 
   \frac{\ell(c)}{2} + se(V_1^h) + 1 & \text{if } \ell(c) = se(V_1^h) + 3, \\
   \frac{\ell(c)}{2} + se(V_1^h) + 3 & \text{otherwise.} 
   \end{cases}\]

2. If $\ell(c)$ is odd and $\ell(c) > se(V_1^h) + 3$, then
   \[\exp(A : h, i) \leq \begin{cases} 
   \frac{\ell(c) + se(V_1^h) + 1}{2} & \text{if } \ell(c) = se(V_1^h) + 3, \\
   \frac{\ell(c) + se(V_1^h) + 3}{2} & \text{otherwise.} 
   \end{cases}\]

3. If $\ell(c) = se(V_1^h) + 3$, then $\exp(A : h, i) \leq se(V_1^h) + 2$.

Proof of Part 1 Let $p$ be the first vertex of $c$ and $i = p + \frac{\ell(c) - 2}{2}$. Let $h - n - p - (p + 1) - (p + 2) - \cdots - (i - 1) - i$ and $h - n - (p + \ell(c) - 2) - (p + \ell(c) - 3) - \cdots - (i + 1) - i$ be two paths from $h$ to $i$ with parity different from that of $\frac{\ell(c) - 2}{2} + 2$ and containing all the vertices of $c$. If $\ell(c) = se(V_1^h) + 3$ then there exist a path of length $\frac{\ell(c) - 2}{2} + se(V_1^h) + 3$, otherwise there is a path of length $\frac{\ell(c) - 2}{2} + se(V_1^h) + 5$ satisfying the required conditions. Hence in this case the result follows by applying Corollary 3.6.

Proof of Part 2 Suppose $\ell(c) = se(V_1^h) + 3$ and $p$ be the first vertex of $c$. Let $l_1 = \frac{\ell(c) - se(V_1^h) - 3}{2}$ and $l_2 = \frac{\ell(c) + se(V_1^h) - 1}{2}$. Consider vertices $i = p + l_1$ and $i' = p + l_2$. Then by a proof similar to the proof of part 1, there are two paths from $h$ to $i$ of lengths $l_1 + 2$ and $l_1 + se(V_1^h) + 3$ such that first path contains the vertex set $\{n, p, p + 1, p + 2, \ldots, i\}$ and second path contains the vertex set $\{n, p + \ell(c) - 2, p + \ell(c) - 3, \ldots, i'\}$.

For vertex $j$, $i < j < i'$, there are paths from $h$ to $j$ of lengths $r$ and $s$ such that $r + s = \ell(c) + 2$ and $\max\{r, s\} < l_1 + se(V_1^h) + 3$.

For $\ell(c) \neq se(V_1^h) + 3$, choose $i = p + l_1 - 1$ and $i' = p + l_2 + 1$. And proceed in similar manner by replacing paths $l_1 + 2$, $l_1 + se(V_1^h) + 3$ with $l_1 + 1$, $l_1 + se(V_1^h) + 4$ respectively. Hence from Corollary 3.6 the result follows for this case.
Proof of Part 3. Let $i \in V(c)$. The from Corollary 3.6 there exists two paths from $h$ to $i$ of lengths $r$ and $s$ such that $r + s = \ell(c) + 2$ and $\max\{r, s\} \leq \ell(c)$. Hence $\exp(A : h, i) \leq se(V_1^h) + 2$ for all $i \in V(c)$.

Figure 9 illustrates the proof of part 2 of the above result. This graph is a subgraph of a graph with $se(V_1) = 2$ and $\ell(c^h) = 5$. Choose $i = p + 2$ and $i' = p + 5$, then there is a path of length $se(V_1) + 1$ between $i$ and $i'$.

Figure 9: A subgraph of a graph with $n$ vertices, $se(V_1) = 2$ and $\ell(c^h) = 5$

The following result is the main result of this section which can be used as an algorithm to compute $\exp(A)$, where $A \in \mathcal{PSC}_n^{0,0}$. 

26
Theorem 5.4

1. Let $A \in \mathcal{PSC}_{n}^{0,0}$ and $m(V_{1}^{h})$ is odd. Then

$$
\exp(A) = \begin{cases} 
2h' + \text{se}(V_{1}^{h}) & \text{if } h' \geq \frac{m(V_{1}^{h}) + 5}{2}, \\
m(V_{1}^{h}) + \text{se}(V_{1}^{h}) + 5 & \text{if } 2 \leq h' < \frac{m(V_{1}^{h}) + 3}{2} \text{ and there exists } c \in C'(A) \\
m(V_{1}^{h}) + \text{se}(V_{1}^{h}) + 3 & \text{otherwise},
\end{cases}
$$

where $h' = \begin{cases} h & \text{if } \ell(c^{h}) = \text{se}(V_{1}^{h}) + 3, \\
h + 1 & \text{if } \ell(c^{h}) \neq \text{se}(V_{1}^{h}) + 3.
\end{cases}$

2. Let $A \in \mathcal{PSC}_{n}^{0,0}$ and $m(V_{1}^{h})$ be even. Suppose $\text{mo}(V_{1}^{h})$ does not exists or $\text{mo}(V_{1}^{h}) \leq m(V_{1}^{h}) - \text{se}(V_{1}^{h}) - 3$. Then

$$
\exp(A) = \begin{cases} 
m(V_{1}^{h}) + 2 & \text{if } 2 \leq h \leq \frac{m(V_{1}^{h}) - \text{se}(V_{1}^{h})}{2}, \\
2h + \text{se}(V_{1}^{h}) & \text{if } h \geq \frac{m(V_{1}^{h}) - \text{se}(V_{1}^{h}) + 2}{2} \text{ and } \ell(c^{h}) = \text{se}(V_{1}^{h}) + 3, \\
2h + \text{se}(V_{1}^{h}) + 2 & \text{if } h \geq \frac{m(V_{1}^{h}) - \text{se}(V_{1}^{h}) + 2}{2} \text{ and } \ell(c^{h}) \neq \text{se}(V_{1}^{h}) + 3.
\end{cases}
$$

Proof of Part $\square$ Suppose $\ell(c^{h}) = \text{se}(V_{1}^{h}) + 3$. Then $\exp(A : h, h) = \text{se}(V_{1}^{h}) + 2$ and

1. for all $i, j \in [1, h - 1]$, $\exp(A : i, j) \leq 2h + \text{se}(V_{1}^{h})$.

2. for all $i, j \in V \setminus [1, h - 1]$ $\exp(A : i, j) \leq$

$$
\begin{cases} 
m(V_{1}^{h}) + \text{se}(V_{1}^{h}) + 5 & \text{if there exists } c \in C'(A) \\
& \text{with } c = d_{1} + d_{2}, \ell(d_{1}) = m(V_{1}^{h}) + 3 \\
m(V_{1}^{h}) + \text{se}(V_{1}^{h}) + 3 & \text{otherwise.}
\end{cases}
$$

3. $i \in [1, h - 1]$ and $j \in V \setminus [1, h - 1]$, $\exp(A : i, j) \leq h + \frac{m(V_{1}^{h})}{2} + \text{se}(V_{1}^{h})$.
Now we have $2h + \text{se} (V^h_1) \leq m(V^h_1) + \text{se}(V^h_1) + 3$ and $h + \frac{m(V^h_1) + 3}{2} + \text{se}(V^h_1) \leq m(V^h_1) + \text{se}(V^h_1) + 3$ whenever $h \leq \frac{m(V^h_1) + 3}{2}$. For $h \geq \frac{m(V^h_1) + 5}{2}$, $m(V^h_1) + \text{se}(V^h_1) + 5 \leq 2h + \text{se}(V^h_1)$ and $h + \frac{m(V^h_1) + 3 + \text{se}(V^h_1)}{2} \leq 2h + \text{se}(V^h_1)$. Hence the result follows from Lemma 3.3. For $\ell(c^h) \neq \text{se}(V^h_1) + 3$, $\exp(A : h, h) = \text{se}(V^h_1) + 4$ and we imitate the proof of the case $\ell(c^h) = \text{se}(V^h_1) + 3$ with $\exp(A : h, h) = \text{se}(V^h_1) + 4$.

Proof of Part 2 Suppose $\ell(c^h) = \text{se}(V^h_1) + 3$. Then $\exp(A : h, h) = \text{se}(V^h_1) + 2$. If $\text{mo}(V^h_1)$ does not exist or $\text{mo}(V^h_1) \leq m(V^h_1) - \text{se}(V^h_1) - 3$, then

1. for all $i, j \in [[1, h - 1]]$, $\exp(A : i, j) \leq 2h + \text{se}(V^h_1)$.
2. for all $i, j \in V \setminus [[1, h - 1]]$, $\exp(A : i, j) \leq m(V^h_1) + 2$.
3. for $i \in [[1, h - 1]]$ and $j \in V \setminus [[1, h - 1]]$, $\exp(A : i, j) \leq h + \frac{m(V^h_1) + \text{se}(V^h_1)}{2} + 1$.

Now we have $h + \frac{m(V^h_1) + \text{se}(V^h_1)}{2} + 1 \leq m(V^h_1) + 2$ and $2h + \text{se}(V^h_1) \leq m(V^h_1) + 2$ whenever $h \leq \frac{m(V^h_1) - \text{se}(V^h_1)}{2}$. For $h \geq \frac{m(V^h_1) - \text{se}(V^h_1) + 2}{2}$, $h + \frac{m(V^h_1) + \text{se}(V^h_1)}{2} + 1 \leq 2h + \text{se}(V^h_1)$ and $m(V^h_1) + 2 \leq 2h + \text{se}(V^h_1)$. Hence the result follows from Lemma 3.3. For $\ell(c^h) \neq \text{se}(V^h_1) + 3$, $\exp(A : h, h) = \text{se}(V^h_1) + 4$ and the rest of the proof is similar to the previous case.

**Note 5.5** If $\text{mo}(V^h_1)$ exist and $\text{mo}(V^h_1) > m(V^h_1) - \text{se}(V^h_1) - 3$, then $\exp(A)$ will be determined by Part 2 of Theorem 5.4 with $m(V^h_1) = \text{mo}(V^h_1)$.

The following theorem is the last result of this section. This will evaluate $N^{0,0}_n(4)$ and $N^{0,0}_n(2n - 4)$. Part 1 of the Theorem 5.6 can be verified with Theorem 2 in [11].

**Theorem 5.6**

1. If $n \geq 4$, then $N^{0,0}_n(2n - 4) = 2$.
2. If \( n \geq 5 \), then

\[
2 \left[ \sum_{q=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left( \begin{array}{c} n-2q-4 \\ q \end{array} \right) \right] \leq N^{0,0}_n(4) \leq 2 \left[ \sum_{q=1}^{\left\lfloor \frac{2n-7}{3} \right\rfloor} F_{n-5}(q, 1) + \sum_{q=2}^{\left\lfloor \frac{2n+2}{3} \right\rfloor} F_{n-5}(q, 2) + 1 \right].
\]

Proof of Part 1. From Theorem 5.4 it is clear that only exponent involving \( h \) may be \( 2(n-2) \), because the maximum value that \( m(V^h_1) + se(V^h_1) \) can be is \( n-3 \). Also we have \( 2 \leq h \leq n-2 \). Suppose \( h = n-k \) for some \( k \geq 2 \).

For \( k > 2 \) it can easily be checked that no suitable values for \( se(V^1) \) exist which will provide matrices with exponent \( 2(n-2) \). For \( k = 2 \), \( se(V^1) = 0 \) and the corresponding matrices have exponent \( 2(n-2) \).

Proof of Part 2. Suppose \( A \in \mathcal{PSC}^{0,0}_n \) and \( \exp(A) = 4 \). Then \( A \) is in one of the following classes:

1. from Part 1 of Theorem 5.4: \( h = 2 \), \( m(V^h_1) = 1 \), \( se(V^h_1) = 0 \), \( \ell(c^h) = 3 \) and \( |V(d_1 + d_2) \cap V_2| = 3 \) for each cycle \( c \in C'(A) \) with \( c = d_1 + d_2 \), \( \ell(d_1) = 4 \). That is the last row of \( A \) is of form 011Y10, where \( Y \in \bigcup_{q=1}^{\left\lfloor \frac{2n-7}{3} \right\rfloor} B_{n-5}^{1,0} \) such that each zero in \( Y \) has two consecutive ones to its left or right. Here the upper bound follows because for each \( q \) there will be at least \( \left\lfloor \frac{3x-2}{q} \right\rfloor \) ones.

2. from Part 2 of Theorem 5.4: \( h = 2 \), \( m(V^h_1) = 0 \) or \( m(V^h_1) = 2 \), \( mo(V^h_1) \) does not exists, \( se(V^h_1) = 0 \), \( \ell(c^h) = 3 \). That is the last row of \( A \) is of form 011Y10, where \( Y \in B_{n-5}^{0,0} \) or \( Y \in \bigcup_{q=1}^{\left\lfloor \frac{n+2}{3} \right\rfloor} B_{n-5}^{2q,2} \). More precisely there are \( \sum_{q=0}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \left( \begin{array}{c} n-2q-4 \\ q \end{array} \right) \) matrices.

3. from Note 5.5: \( h = 2 \), \( m(V^h_1) = 2 \), \( mo(V^h_1) = 1 \), \( se(V^h_1) = 0 \), \( \ell(c^h) = 3 \) and \( |V(d_1 + d_2) \cap V_2| = 3 \) for each cycle \( c \in C'(A) \) with \( c = d_1 + d_2 \), \( \ell(d_1) = 4 \). That is the last row of \( A \) is of form 011Y10, where \( Y \in \bigcup_{q=3}^{\left\lfloor \frac{2n+2}{3} \right\rfloor} B_{n-5}^{q,2} \) and \( Y \) must contains a subword of zeros of length 1 and each such subword has two consecutive ones to its left or right.
Hence the result follows. ■

**Conclusion:** For a given $n$, it is quite difficult to find out the exact value of $N_n^\alpha,0(b)$, $\alpha \in \{0, 1\}$, $b \geq 4$ and $b \neq \max \left( E(\mathcal{P}SC_n^\alpha,0) \right)$. 

Table 1: Number of Matrices $N_{n}^{\alpha,e}(b)$ for small $n$

| $N_{n}^{\alpha,e}(b)$ | $\alpha, e$ | 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 | 18 |
|-----------------------|------------|----|----|----|----|----|----|----|----|----|
| $N_{3}^{\alpha,e}(b)$ | 1, 1       | 2  | 2  |    |    |    |    |    |    |    |
|                       | 0, 1       |    |    |    |    |    |    |    |    |    |
|                       | 1, 0       |    |    |    |    |    |    |    |    |    |
|                       | 0, 0       |    |    |    |    |    |    |    |    |    |
| $N_{4}^{\alpha,e}(b)$ | 1, 1       | 2  | 2  | 2  |    |    |    |    |    |    |
|                       | 0, 1       |    |    |    |    |    |    |    |    |    |
|                       | 1, 0       |    |    |    |    |    |    |    |    |    |
|                       | 0, 0       |    |    |    |    |    |    |    |    |    |
| $N_{5}^{\alpha,e}(b)$ | 1, 1       | 2  | 6  | 4  | 2  | 2  |    |    |    |    |
|                       | 0, 1       |    | 2  |    |    |    |    |    |    |    |
|                       | 1, 0       |    |    |    |    |    |    |    |    |    |
|                       | 0, 0       |    |    |    |    |    |    |    |    |    |
| $N_{6}^{\alpha,e}(b)$ | 1, 1       | 2  | 12 | 8  | 4  | 2  | 2  |    |    |    |
|                       | 0, 1       |    | 10 |    | 6  |    |    |    |    |    |
|                       | 1, 0       |    | 10 |    | 4  |    |    |    |    |    |
|                       | 0, 0       |    | 12 |    | 6  |    |    |    |    |    |
| $N_{7}^{\alpha,e}(b)$ | 1, 1       | 2  | 24 | 14 | 10 | 4  | 2  | 2  |    |    |
|                       | 0, 1       |    | 14 |    | 6  |    |    |    |    |    |
|                       | 1, 0       |    | 16 |    | 4  |    |    |    |    |    |
|                       | 0, 0       |    | 12 |    | 2  |    |    |    |    |    |
| $N_{8}^{\alpha,e}(b)$ | 1, 1       | 2  | 46 | 22 | 14 | 4  | 2  | 2  |    |    |
|                       | 0, 1       |    | 22 |    | 8  |    |    |    |    |    |
|                       | 1, 0       |    | 36 |    | 10 |    |    |    |    |    |
|                       | 0, 0       |    | 12 |    | 2  |    |    |    |    |    |
| $N_{9}^{\alpha,e}(b)$ | 1, 1       | 2  | 86 | 48 | 34 | 6  | 4  | 2  | 2  |    |
|                       | 0, 1       |    | 48 |    | 10 |    |    |    |    |    |
|                       | 1, 0       |    | 66 |    | 22 |    |    |    |    |    |
|                       | 0, 0       |    | 22 |    | 8  |    |    |    |    |    |
| $N_{10}^{\alpha,e}(b)$| 1, 1       | 2  | 160| 88 | 78 | 14 | 2  | 2  | 2  | 2  |
|                       | 0, 1       |    | 88 |    | 16 |    |    |    |    |    |
|                       | 1, 0       |    | 110|    | 18 |    |    |    |    |    |
|                       | 0, 0       |    | 42 |    | 12 |    |    |    |    |    |

Range of $N_{n}^{0,0}(4)$ from Thm 5.6:

- $2 \leq N_{5}^{0,0}(4) \leq 2$
- $2 \leq N_{6}^{0,0}(4) \leq 4$
- $4 \leq N_{7}^{0,0}(4) \leq 8$
- $6 \leq N_{8}^{0,0}(4) \leq 12$
- $8 \leq N_{9}^{0,0}(4) \leq 26$
- $12 \leq N_{10}^{0,0}(4) \leq 46$
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