On General boundary value problem for an elliptic higher order equation in the plane with constant real coefficients

Yu A Bogan
Institute of Hydrodynamics, Novosibirsk, Russia
E-mail: bogan@hydro.nsc.ru

Abstract. By means of a new approach, the general boundary value problem for a higher order elliptic equation with two independent variables, and a normal set of boundary conditions and simple complex characteristics is reduced to the Fredholm system of integral equations in a bounded region with a smooth boundary.

1. Introduction
Consider a linear partial differential equation of elliptic type in two variables

\[ L(u) = L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right)u = 0, \]

where

\[ L(\xi, \eta) = \sum_{k=0}^{2m} a_k \xi^k \eta^{2m-k} \]

is a homogeneous polynomial with constant real coefficients of degree \(2m, m \geq 2\) and such that \(L(\xi, \eta) > 0\) (ellipticity) for all real non-vanishing vectors \((\xi, \eta)\). Reduction of boundary value problems to regular integral equations for higher order elliptic equations and systems of equations was discussed in a very general situation by Lopatinskii [1]. S. Agmon in his classical article [2] solved the Dirichlet problem for a higher order elliptic equation in the plane by means of the multiple layer potential, constructed by him. The technique developed by him is very complicated; even in the simplest case of a fourth-order equation it leads on to lengthy computations. Some boundary value problems connected with higher order equations, were discussed later in [3, 4, 5] by the help of the “simple layer” method, advanced by G. Fichera in [3]. It is necessary to recall that S. Agmon considered the very general situation of an elliptic equation with multiple complex characteristics. The considerably simplified situation, when an elliptic equation with real coefficients has simple complex characteristics, is studied in this article. Here necessary computations are nearly trivial. A particular case of a fourth order equation from theory of elasticity was studied by the author in [6]. The case of multiple characteristics can be considered as the result of the limiting transition from the case of the simple ones.
2. Preliminaries

Let $Q$ be a simply-connected bounded plane region with the Lyapunov boundary $\partial Q$; i.e., it is assumed, that its boundary $\partial Q$ has the uniformly Hölder continuous inward normal field $\nu(z)$. Here $C^k(Q)$ and $C^k(\bar{Q})$ are spaces of real $k$-order continuously differentiable functions in $Q$ and $\bar{Q}$, respectively. By $C^{0,\alpha}(Q)$ and $C^{0,\alpha}(\partial Q)$ are meant the spaces of real continuous functions satisfying in $\bar{Q}$ and $\partial Q$ the uniform Hölder condition with the exponent $\alpha$, $0 < \alpha < 1$, respectively. $C^{k,\alpha}(\bar{Q})$ is the subclass of $C^{k}(\bar{Q})$ such that $D^\nu u \in C^{0,\alpha}(\bar{Q}), |\nu| = k$. Here $\nu$ is multi-index $\nu = (k_1, k_2)$, and

$$D^\nu u = \frac{\partial^\nu u}{\partial x_1^{k_1} \partial x_2^{k_2}}, \quad k_1 + k_2 = \nu.$$ 

In a similar way the spaces $C^{l,\alpha}(\partial Q)$ are defined.

The aim of this and next sections is to show that for the general boundary problem

$$L(u) = L(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})u = 0,$$  

(1)

$$B_h(u) = \sum_{i=1}^{n_h} b_{ik} \frac{\partial^{n_h} u}{\partial x_1^{n_h-i} \partial x_2^i} |_{\partial Q} = g_h(s), \quad g_h \in C^{2m-n_h-1,\alpha}(\partial Q), \quad h = 1, 2, \ldots, m.$$  

(2)

the Fredholm alternative holds. We assume, that the system of boundary conditions (2) is normal, that is, there are $m$ boundary conditions, $n_i \neq n_j, i \neq j$, the order of any boundary condition is lesser $2m$, and

$$\sum_{i=1}^{n_h} b_{ik} \nu_1^{n_h-i} \nu_2^j \neq 0$$

where $\nu = (\nu_1, \nu_2)$ is the normal vector of the boundary. With the problem (1), (2) is related the next boundary problem for an ordinary differential equation: find the solution $v(t)$ of the equation

$$\sum_{k=0}^{2m} a_k t^k \frac{d^{2m-k} v}{d t^{2m-k}} = 0,$$  

(3)

equal to zero at infinity and satisfying the boundary condition

$$\sum_{i=1}^{n_h} a_i k^i \frac{d^{n_h} v}{d t^{n_h-1}} |_{t=0} = g_h, \quad h = 1, 2, \ldots, m.$$  

(4)

We construct now the complete system of linearly independent solutions of equation (3), choose from them $m$ equal to zero at infinity. Let $\omega(z, t)$ be the matrix, whose columns are chosen $m$ solutions, $\delta(z)$ is the determinant of the matrix, obtained by substitution of $\omega(z, t)$ into the left-hand side of equation (4). Then the Lopatinsky condition means that $\delta(z) \neq 0$ for all $z \in \partial Q$.

It was proved in [7], that the principle assertion pertaining to this circle of problems, consists in the following: the Lopatinsky condition is sufficient for the problem (1)-(2) to be Nöterian, and necessary for this problem to be solvable with the exactness up to finite dimensional sub-spaces. It is known about the problem (1)-(3), that it can be reduced to the system of regular integral equations, if the condition of Lopatinsky is satisfied. We consider below the simplest situation when the equation and boundary conditions have constant coefficients, and the solution belongs to Hölder class of functions. With respect to the problem it is known, that in can be reduced to the system of regular integral equations if the Lopatinsky condition is satisfied.
Our computations in this section are purely formal; their validity will be confirmed later. A general solution of equation (1) can be written as a sum of \( m \) quasi-harmonic functions,

\[
u(x_1, x_2) = \sum_{k=1}^{m} w_k(x_1, x_2).
\]

Since equation (1) is elliptic with real constant coefficients, the characteristic equation

\[
L(1, \lambda) = \sum_{k=0}^{2m} a_k \lambda^{2m-k} = 0
\]

has \( m \) pairs of (distinct) complex conjugate roots \( \lambda_k, \overline{\lambda}_k, Im \lambda_k > 0, k = 1, 2, \ldots, m \). Let \( \lambda_k = \alpha_k + i \beta_k, \beta_k > 0, k = 1 \ldots m \) be roots with a positive imaginary part. A quasi-harmonic function \( w_k(x_1, x_2) \) is defined as a solution of the elliptic second order equation

\[
(\beta_k^2 + \alpha_k^2) \frac{\partial^2 w_k}{\partial x_1^2} - 2\alpha_k\beta_k \frac{\partial^2 w_k}{\partial x_1 \partial x_2} + \frac{\partial^2 w_k}{\partial x_2^2} = 0,
\]

which reduces to Laplacian under the change of independent variables \( y_1 = x_1 + \alpha_k x_2, y_2 = \beta_k x_2 \). In a simply connected region any quasi-harmonic function \( w_k(x_1, x_2), k = 1, \ldots m \) can be represented as a real part of a holomorphic function of the (complex) argument \( z_k = x_1 + \lambda_k x_2, k = 1, \ldots, m \),

\[
w_k(x_1, x_2) = \text{Re} \phi_k(z_k), \quad z_k = x_1 + \lambda_k x_2, \quad k = 1, \ldots, m.
\]

3. Construction of potentials
Put \( z_k = x_1 + \lambda_k x_2, t_k(s) = x_1(s) + \lambda_k x_2(s), k = 1, 2, \ldots, m \). The solution of the problem (1),(2) is a sum of \( m \) analytic functions

\[
u(x_1, x_2) = \sum_{n=1}^{m} \phi_n(x_1 + \lambda_n x_2).
\]

Then

\[
\frac{\partial^{m_k} u}{\partial x_1^{m_k-p} \partial x_2^p} = \text{Re} \sum_{n=1}^{m} \lambda_n^{p} \phi_n^{(m_k)}(z_k), k = 1, 2, \ldots, m, \tag{5}
\]

and

\[
B_k(u) = \text{Re} \sum_{p=0}^{m_k} \sum_{n=1}^{m} b_{pn} \lambda_n^{p} \phi_n^{(m_k)}(z_n).
\]

Put

\[
\phi_n^{(m_k)}(z_n) = \frac{1}{\pi i} \int_{t_n - z_n} b_n(s) \, dt_n, \quad n = 1, 2, \ldots, m. \tag{6}
\]

As result,

\[
B_k(u) = \text{Re} \sum_{p=0}^{m_k} \sum_{n=1}^{m} a_{pn} \lambda_n^{p} \frac{1}{\pi i} \int_{t_n - z_n} b_n(s) \, dt_n, \quad n = 1, 2, \ldots, m. \tag{7}
\]

Recall, that by Plemelj’s jump formula, when a point \( z \in Q \) tends to a point \( t_0 = x_1(s_0) + ix_2(s_0) \in \partial Q, s_0 \in (0, L) \) inside a region, then
Here \( t_{j0} = x_1(s_0) + \lambda_j x_2(s_0), j = 1, 2, \ldots, m \).

Assume, that densities \( d_n \in C^{0, \alpha}(\partial Q) \) and use Plemelj jump formulas at the boundary for the Cauchy type integral. Choose now a set of \( m \) real functions \( f_k(s), k = 1, 2, \ldots, m \). Determine densities \( d_k(s), k = 1, 2, \ldots, m \) from the linear system of equations

\[
\sum_{p=0}^{m_k} \sum_{n=1}^{m} b_{pk} \lambda_p^d d_n = f_k(s), \quad k = r,
\]

\[
\sum_{p=0}^{m_k} \sum_{n=1}^{m} b_{pk} \lambda_p^d = 0, \quad k \neq r.
\]

Densities \( d_n(s) \) and a potential \( B_k(x_1, x_2) \) are uniquely defined from the system (10).

Denote by \( C = (c_{ij}), i, j = 1, 2, \ldots, m \) the matrix of coefficients before \( d_n, n = 1, 2, \ldots, m \). Let \( f_k(s) \in C^{0, \alpha}(0, L), k = 1, 2, \ldots, m \) – is a vector of \( m \) real valued functions. Choose the densities as a solution of a linear system of equations

\[
\sum_{k=1}^{m} a_{k,s} d_k(s) = f_k(s), \quad k = 1, 2, \ldots, m.
\]

It is obvious, that system has a unique solution, if its determinant \( \delta = det(a_{i,j}) \) is different from zero. Here

\[
a_{i,j} = \sum_{i=0}^{m_k} b_{j,i} \lambda_j^p, i, j = 1, \ldots, m.
\]

Denote by \( \Delta_m \) the Vandermond determinant, built on the numbers \( \lambda_i, i = 1, \ldots, m \). We assert, that the determinant \( \delta \) can be written as

\[
\delta = \Delta_m \delta_1,
\]

where \( \delta_1 \) is a symmetric polynome of roots of the characteristic equation. Indeed, columns of the matrix \( C \) differ only by numbers of the characteristic equation and \( \delta \) becomes equal to zero when for some \( i \neq j \lambda_i = \lambda_j \); hence by the Bezout theorem any difference \( \lambda_i - \lambda_j \) is a multiple of \( det C \). Therefore, \( \delta_1 \) depends only on the coefficients of equation (1).

Note, that it is the place, where we get a distinction from the standard approach. We see, that there is no necessity to use the fundamental solution of the equation, to substitute it into the corresponding Green’s formula, reducing the boundary problem to the system of singular integral equations. In the standard approach densities \( d_k k = 1, 2, \ldots, m \) are chosen to be real functions, and there is no necessity to solve the previous system of equations. By Cramer’s formula we obtain, that

\[
d_k = \delta^{-1} \sum_{j=1}^{m} f_j A_{j,k}, \quad k = 1, 2, \ldots, m.
\]

Here \( A_{j,k} \) is the cofactor of \( a_{j,k} \). Therefore, the function \( B_k(u) \) can be written as

\[
B_k(u) = Re \sum_{n=1}^{m} a_{k,n} \frac{1}{\pi i \delta} \int \sum_{j=1}^{m} g_j A_{j,n} \frac{dt_n}{t_n - z_n}.
\]
Rewrite the previous equation as

\[ B_k u = Re \sum_{n=1}^{m} a_{k,n} \frac{1}{\pi i \delta} \int_{\partial Q} \sum_{j=1}^{m} g_j A_{j,n} \left( \frac{d t_n}{t_n - z_n} - \frac{d t_k}{t_k - z_k} \right) + \]

\[ Re \sum_{n=1}^{m} a_{k,n} \frac{1}{\pi i \delta} \int_{\partial Q} \sum_{j=1}^{m} g_j A_{j,n} \frac{d t_k}{t_k - z_k}. \]  \hspace{1cm} (12)

By known properties of determinants we have the relations

\[ \sum_{n=1}^{m} a_{k,n} A_{k,n} = \delta, \; k = 1, 2, \ldots m, \sum_{n=1}^{m} a_{k,n} A_{l,n} = 0, \; k \neq l. \]  \hspace{1cm} (13)

Then for any \( k = 1, 2, \ldots m \), we can write \( B_k(u) \) as

\[ B_k(u) = Re \frac{1}{\pi i} \int_{\partial Q} f_k \frac{d t_k}{t_k - z_k} + Re \frac{1}{\pi i \delta} \sum_{n=1}^{m} a_{k,n} \int_{\partial Q} \sum_{j=1}^{m} f_j A_{j,n} \left( \frac{d t_n}{t_n - t_n0} - \frac{d t_k}{t_k - t_k0} \right). \]  \hspace{1cm} (14)

As result, we obtain at the boundary the system of equations:

\[ f_k(s_0) + Re \frac{1}{\pi i} \int_{\partial Q} g_k \frac{d t_k}{t_k - t_k0} + Re \frac{1}{\pi i \delta} \sum_{n=1}^{m} a_{k,n} \int_{\partial Q} \sum_{j=1}^{m} f_j A_{j,n} \left( \frac{d t_n}{t_n - t_n0} - \frac{d t_k}{t_k - t_k0} \right) = g_k(s_0), \; k = 1, 2, \ldots, m. \]  \hspace{1cm} (15)

It is obvious, that expression (14), specialized at the straight boundary, for example, half-plane \( z = 0 \), gives an explicit rapidly decaying solution of problem (1)-(2), in a half plane. As equation (1) has real coefficients and is strongly elliptic, all summands in (14) are compact operators in \( C^{0,\alpha}(\partial Q) \) and the boundary value problem (1)-(2) is Fredholm with zero index.

Summing up, we get

Theorem. Assume, that a bounded region \( Q \) has a Lyapunov boundary \( \partial Q \), boundary data \( g_k(s) \in C^{2m-m_k-1}, k = 1, 2, \ldots, m \). Then the boundary value problem (1)-(2) is Fredholm in the Hölder class of functions, if the determinant \( \delta \neq 0 \); It means, that

- a) The homogeneous boundary problem (1)-(2) has only finite number of linearly independent solutions;
- b) if the homogeneous boundary problem (1)-(2) does not have nonzero solutions, then the corresponding inhomogeneous problem always has a unique solution and
- c) if the number of linearly independent solutions of the homogeneous problem is equal to \( s \), then for solvability of the inhomogeneous problem (1)-(2) it is necessary and sufficient, that the vector \( g = (g_1(s), g_2(s), \ldots, g_m(s)) \) of boundary data has to satisfy \( s \) conditions of orthogonality

\[ \int_{\partial Q} f v^j d s = 0, \]

where \( v^j \) are linearly independent solutions of the homogeneous adjoint problem.
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