LIOUVILLE THEOREMS TO SYSTEM OF ELLIPTIC DIFFERENTIAL INEQUALITIES ON THE HEISENBERG GROUP

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Abstract. In this paper, we establish Liouville theorems for the following system of elliptic differential inequalities
\[
\begin{aligned}
\Delta H u^{m_1} + |\eta|_{H}^{\gamma_1} |u|^{p} &\leq 0, \\
\Delta H v^{m_2} + |\eta|_{H}^{\gamma_2} |v|^{q} &\leq 0,
\end{aligned}
\]
on different unbounded open domains of Heisenberg group \(H\), including the whole space, and half space of \(H\). Here \(p > m_2 > 0\), \(q > m_1 > 0\).

1. Introduction

Let \(H\) be Heisenberg group, which is topologically Euclidean but analytically non-Euclidean. To be precise, \(H = (\mathbb{R}^{2N+1}, \cdot)\) is the space \(\mathbb{R}^{2N+1}\) with the non-commutative law of product
\[
\eta' \circ \eta = (x + x', y + y', \tau + \tau' + 2(x \cdot y' - x' \cdot y)),
\]
fors all \(\eta = (x, y, \tau), \eta' = (x', y', \tau') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}\), where \(\cdot\) denotes the standard scalar product in \(\mathbb{R}^N\). For more information on Heisenberg group, see Section 2.

In this paper, we consider the quasilinear system of elliptic differential inequalities
\[
\begin{aligned}
\Delta H u^{m_1} + |\eta|_{H}^{\gamma_1} |u|^{p} &\leq 0, &\text{in} \ \Omega, \\
\Delta H v^{m_2} + |\eta|_{H}^{\gamma_2} |v|^{q} &\leq 0, &\text{in} \ \Omega,
\end{aligned}
\]
where \(p > m_2 > 0\) and \(q > m_1 > 0\), \(|\eta|_H\) is defined as in (2.1), \(\Delta H\) is the sub-Laplacian on \(H\) (see Section 2), and \(\Omega\) is an unbounded open subset of Heisenberg group \(H\) taking one of the following three forms
\[
\begin{align*}
(1). \ \Omega_1 &:= \mathbb{H}; \\
(2). \ \Omega_2 &:= \{(x, y, \tau) \in \mathbb{H} \mid x_1 > 0\}; \\
(3). \ \Omega_3 &:= \{(x, y, \tau) \in \mathbb{H} \mid \tau > 0\}.
\end{align*}
\]

In the past several decades, more and more attentions has been attracted to the analysis and PDEs on Heisenberg group, see [1, 2, 3, 4, 6, 7, 10, 12, 13, 17, 22, 23]. Recall the celebrated results from Birindelli, Capuzzo Dolcetta and Cutri in [1], they investigated
\[
\Delta H u + |\eta|_{H}^{\gamma} u^p \leq 0, \quad \text{in} \ \Omega,
\]
and proved that

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(a) if \( \gamma > -2 \), and \( 1 < p \leq \frac{Q+\gamma}{Q+1} \), then (1.2) admits no positive solutions in \( \Omega_1 \).

(b) if \( \gamma > -1 \), and \( 1 < p \leq \frac{Q+\gamma}{Q} \), then (1.2) admits no positive solutions in \( \Omega_2 \).

(c) if \( \gamma > 0 \), and \( 1 < p \leq \frac{Q+\gamma}{Q} \), then (1.2) admits no positive solutions in \( \Omega_3 \),

where

\[ Q = 2N + 2, \]

and usually is called the homogeneous dimension of \( \mathbb{H} \).

Later in [17], Pohozaev and Véron removed the assumption of positiveness of solution, and studied

\[ \Delta_H u + |\eta|^{\gamma_1}_H |u|^p \leq 0, \quad \text{in } \mathbb{H}. \]  

(1.3)

They proved that if \( \gamma > -2 \), and \( 1 < p \leq \frac{Q+\gamma}{Q+1} \), then (1.3) admits no locally integrable solution \( u \in L^p_{loc}(\mathbb{H}, |\eta|^{\gamma_1}_H d\eta) \).

In the same paper [17], Pohozaev and Véron also studied system of (1.1) under the special case of \( m_1 = m_2 = 1 \), \( \Omega = \mathbb{H} \), namely

\[
\begin{aligned}
\Delta_H u + |\eta|^{\gamma_1}_H |u|^p &\leq 0, \quad \text{in } \mathbb{H}, \\
\Delta_H v + |\eta|^{\gamma_2}_H |v|^q &\leq 0, \quad \text{in } \mathbb{H}.
\end{aligned}
\]  

(1.4)

They proved that (1.4) admits no solution \((u,v) \in L^p_{loc}(\mathbb{H}, |\eta|^{\gamma_2}_H d\eta) \times L^q_{loc}(\mathbb{H}, |\eta|^{\gamma_1}_H d\eta) \) provided that \( \gamma_1, \gamma_2 > -2 \), and

\[ Q \leq 2 + \min \left\{ \frac{2 + \gamma_1}{p - 1}, \frac{2 + \gamma_2}{q - 1} \right\}, \quad \text{for } p > 1, q > 1. \]

Later, Hamidi and Kirane [4] showed that (1.4) admits no nontrivial solution, if

\[ Q \leq 2 + \max \left\{ \frac{\gamma_1 + 2 + p(\gamma_2 + 2)}{pq - 1}, \frac{\gamma_2 + 2 + q(\gamma_1 + 2)}{pq - 1} \right\}. \]

One can easily check that when \( p > 1, q > 1 \),

\[ \max \left\{ \frac{\gamma_1 + 2 + p(\gamma_2 + 2)}{pq - 1}, \frac{\gamma_2 + 2 + q(\gamma_1 + 2)}{pq - 1} \right\} \geq \min \left\{ \frac{2 + \gamma_1}{p - 1}, \frac{2 + \gamma_2}{q - 1} \right\}. \]

Motivated by the above literature, we would like to generalize the study in two respects: The first is that we aim to remove the positive assumption of \((u,v) \) to problem (1.1) with generalized \( m_1, m_2 > 0 \); the second is that we will study the nonexistence results in three different domains \( \Omega_1, \Omega_2, \Omega_3 \).

For our convenience, throughout the paper, let us denote

\[ \Lambda := 2 + \max \left\{ \frac{\gamma_1 m_1 [m_2(\gamma_1 + 2) + p(\gamma_2 + 2)]}{pq - m_1 m_2}, \frac{\gamma_2 m_2 [m_1(\gamma_2 + 2) + q(\gamma_1 + 2)]}{pq - m_1 m_2} \right\}, \]

and

\[ \alpha := \max \left\{ \frac{p}{p - m_2}, \frac{q}{q - m_1} \right\}. \]

**Theorem 1.1.** When \( \Omega = \Omega_1 \). If \( \gamma_1, \gamma_2 > -2 \), and

\[ Q \leq \Lambda, \]  

then (1.1) admits no nontrivial solution.
Theorem 1.2. When $\Omega = \Omega_2$. If $\gamma_1, \gamma_2 > -1$, and

$$Q \leq \Lambda - \alpha,$$

then (1.1) admits no nontrivial solution.

Theorem 1.3. When $\Omega = \Omega_3$. If $\gamma_1, \gamma_2 > 0$, and

$$Q \leq \Lambda - 2 \alpha,$$

then (1.1) admits no nontrivial solution.

Remark 1.4. Theorem 1.2 and 1.3 hold true respectively for any half-spaces taking the forms of

$$\left\{ \eta \in \mathbb{H} : \sum_{i=1}^{N} a_i x_i + b_i y_i + d > 0, \text{for } (a, b) \in \mathbb{R}^N \times \mathbb{R}^N \setminus \{0\}, d \in \mathbb{R} \right\},$$

and

$$\left\{ \eta \in \mathbb{H} : \sum_{i=1}^{N} a_i x_i + b_i y_i + ct + d > 0, \text{for } a, b \in \mathbb{R}^N, c \in \mathbb{R} \setminus \{0\}, d \in \mathbb{R} \right\}.$$ 

The paper is organized as follows: In Section 2, we prepare some preliminaries. In Section 3, we give the proof of Theorem 1.1 by applying three different test functions. Section 4 is devoted to the proof of Theorems 1.2-1.3.

Throughout the paper, we denote by $c_1, c_2, \ldots, C_1, C_2 \ldots$ some positive constants, which may vary from line to line. And $f \ll h$ means that $f \leq Ch$ for some constant $C > 0$.

2. Preliminaries

The sub-Laplacian $\Delta_{\mathbb{H}}$ on $\mathbb{H}$ is defined, from the vector fields

$$X_i = \partial_{x_i} + 2y_i \partial_\tau, \quad Y_i = \partial_{y_i} - 2x_i \partial_\tau, \quad (i = 1, \ldots, N),$$

by

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} (X_i^2 + Y_i^2) = \sum_{i=1}^{N} \left[ \partial_{x_i}^2 + \partial_{y_i}^2 + 4y_i \partial_{x_i} \partial_{x_\tau} - 4x_i \partial_{y_i} \partial_{y_\tau} + 4(x_i^2 + y_i^2) \partial_{\tau}^2 \right].$$

For a a function $f : \mathbb{H} \to \mathbb{R}$, the horizontal gradient of $f$ is defined as

$$\nabla_{\mathbb{H}} f = (X_1 f, \ldots, X_N f, Y_1 f, \ldots, Y_N f) = \sum_{i=1}^{N} [(X_i f) X_i + (Y_i f) Y_i].$$

Let us define the norm of $\eta \in \mathbb{H}$ by

$$|\eta|_{\mathbb{H}} = \left( |x|^2 + |y|^2 + \tau^2 \right)^{\frac{1}{2}},$$

which is homogeneous of degree 1 with respect to the dilations $\delta_\lambda : (x, y, \tau) \mapsto (\lambda x, \lambda y, \lambda^2 \tau)$ for $\lambda > 0$. And Heisenberg distance between $\eta$ and $h$ on $\mathbb{H}$ is defined by

$$d_{\mathbb{H}}(\eta, h) = |\eta^{-1} \circ h|_{\mathbb{H}},$$

where $\eta^{-1} = -\eta$. 

Let us define the Heisenberg ball of radius $R$ centered at $\eta$ be the set
\[ B_\mathbb{H}(\eta, R) = \{ h \in \mathbb{H} : d_\mathbb{H}(\eta, h) < R \}, \]
it follows that
\[ |B_\mathbb{H}(\eta, R)| = |B_\mathbb{H}(0, R)| = |B_\mathbb{H}(0, 1)| R^Q, \]
where $|B_\mathbb{H}(0, 1)|$ is the volume of the unit Heisenberg ball under Haar measure, which is equivalent to $(2N + 1)$-dimensional Lebesgue measure of $\mathbb{R}^{2N+1}$, and $Q = 2N + 2$ is called the homogeneous dimension of $\mathbb{H}$. For more details concerning the Heisenberg group, one can refer to books as [3, 11], survey papers as [5, 6, 10] and the references therein.

Define
\[ W^{1,p}_{\text{loc}}(\Omega) = \{ u : \Omega \rightarrow \mathbb{R} \mid u, \nabla_\mathbb{H}u \in L^p_{\text{loc}}(\Omega) \}, \]
and let $W^{1,p}(\Omega)$ be the subspace of $W^{1,p}_{\text{loc}}(\Omega)$ of functions with compact support.

**Definition 2.1.** A pair $(u, v)$ is called a weak solution to system $(1.1)$ if $(u, v) \in W^{1,q}_{\text{loc}}(\Omega) \times W^{1,p}_{\text{loc}}(\Omega)$, and the following inequalities
\[ \int_{\Omega} \left| \eta \right|^{\gamma_1} |v|^p \psi d\eta \leq -\int_{\Omega} \psi \nabla_\mathbb{H} u^{m_1} d\eta, \quad (2.2) \]
and
\[ \int_{\Omega} \left| \eta \right|^{\gamma_2} |u|^q \psi d\eta \leq -\int_{\Omega} \psi \nabla_\mathbb{H} v^{m_2} d\eta, \quad (2.3) \]
are valid for any $0 \leq \psi \in W^{1,q}_{c}(\Omega) \cap W^{1,p}_{c}(\Omega)$.

Let
\[ D_R := B_\mathbb{H}(0, 2kR), \quad \text{for } k \geq 1. \]
Define
\[ D_i := \Omega_i \cap D_R, \quad \text{for } i = 1, 2, 3, \]
and
\[ f_1 := 1, \quad f_2 := x_1^\alpha, \quad f_3 := \tau^\alpha. \]

**Lemma 2.2.** Assume that $(u, v)$ is a solution to $(1.1)$. Let $\varphi \in W^{1,q}_{c}(\Omega) \cap W^{1,p}_{c}(\Omega)$ satisfying that $0 \leq \varphi \leq 1$ and $\text{supp } \varphi \subset D_R$, we have for $i = 1, 2, 3,$ and $j = 1, 2$,
\[ \begin{cases} 
I_i^{1-\frac{m_1 m_2}{pq}} \lesssim (K_{i,1} + L_{i,1})^{\frac{m_1}{q}} (K_{i,2} + L_{i,2})^\frac{m_2}{p}, \\
J_i^{1-\frac{m_1 m_2}{pq}} \lesssim (K_{i,1} + L_{i,1}) (K_{i,2} + L_{i,2})^{\frac{m_2}{p}}.
\end{cases} \quad (2.4) \]
where
\[ I_i := \int_{D_i} |\eta|^{\gamma_i} |v|^p f_i \varphi^b \, d\eta, \quad J_i := \int_{D_i} |\eta|^{\gamma_i} |u|^q f_i \varphi^b \, d\eta, \]
\[ K_{i,j} := \left( \int_{D_i} |\eta|_{\mathbb{H}}^{(1-\lambda_i)\gamma_j} f_i |\Delta_{\mathbb{H}} \varphi|^\lambda \, d\eta \right)^{\frac{1}{\gamma_j}}, \]
\[ L_{i,j} := \left( \int_{D_i} |\eta|_{\mathbb{H}}^{(1-\lambda_i)\gamma_j} f_i |\nabla_{\mathbb{H}} f_i|^\lambda_j |\nabla_{\mathbb{H}} \varphi|^\lambda \, d\eta \right)^{\frac{1}{\gamma_j}}, \]
\[ \lambda_1 := \frac{p}{p-m_2}, \quad \lambda_2 := \frac{q}{q-m_1}. \]

**Proof.** Let
\[ \psi_i = f_i \varphi^b, \quad (i = 1, 2, 3), \]
where \( b > 1 \) is a large enough constant. Note that on \( \partial D_i \),
\[ \psi_i = 0, \]
and
\[ \nabla_{\mathbb{H}} \psi_i = b f_i \varphi^{b-1} \nabla_{\mathbb{H}} \varphi + \varphi^b \nabla_{\mathbb{H}} f_i = 0, \]
with
\[ \nabla_{\mathbb{H}} f_1 = 0, \quad \nabla_{\mathbb{H}} f_2 = \alpha x_1^{\alpha-1} \nabla_{\mathbb{H}} x_1, \quad \nabla_{\mathbb{H}} f_3 = \alpha \tau^{\alpha-1} \nabla_{\mathbb{H}} \tau, \]
and
\[ \nabla_{\mathbb{H}} x_1 = (1, 0, \ldots, 0), \quad \nabla_{\mathbb{H}} \tau = 2(y, x). \]
Substituting \( \psi = \psi_i = f_i \varphi^b \) into (2.2) and (2.3), we obtain
\[ \int_{D_i} |\eta|_{\mathbb{H}}^{\gamma_i} |v|^p f_i \varphi^b \, d\eta \leq - \int_{D_i} u^{m_1} \Delta_{\mathbb{H}} \left( f_i \varphi^b \right) \, d\eta, \quad (2.5) \]
and
\[ \int_{D_i} |\eta|_{\mathbb{H}}^{\gamma_i} |u|^q f_i \varphi^b \, d\eta \leq - \int_{D_i} v^{m_2} \Delta_{\mathbb{H}} \left( f_i \varphi^b \right) \, d\eta. \quad (2.6) \]
Note also that
\[ \Delta_{\mathbb{H}} f_1 = 0, \quad \Delta_{\mathbb{H}} f_2 = \alpha (\alpha-1) x_1^{\alpha-2} \geq 0, \quad \Delta_{\mathbb{H}} f_3 = 4\alpha (\alpha-1) (x^2+y^2)^{\alpha-2} \geq 0, \]
and
\[ \Delta_{\mathbb{H}} \varphi^b = b(b-1) \varphi^{b-2} |\nabla_{\mathbb{H}} \varphi|^2 + b \varphi^{b-1} \Delta_{\mathbb{H}} \varphi \geq b \varphi^{b-1} \Delta_{\mathbb{H}} \varphi. \]
It follows that
\[ \Delta_{\mathbb{H}} \left( f_i \varphi^b \right) = f_i \Delta_{\mathbb{H}} \varphi^b + 2 \nabla_{\mathbb{H}} f_i \cdot \nabla_{\mathbb{H}} \varphi^b + \varphi^b \Delta_{\mathbb{H}} f_i \]
\[ \geq b f_i \varphi^{b-1} \Delta_{\mathbb{H}} \varphi + 2 b \varphi^{b-1} \nabla_{\mathbb{H}} f_i \cdot \nabla_{\mathbb{H}} \varphi. \quad (2.7) \]
Combining (2.5) with (2.7), we obtain
\[ \int_{D_i} |\eta|_{\mathbb{H}}^{\gamma_i} |v|^p f_i \varphi^b \, d\eta \leq b \int_{D_i} |u|^{m_1} f_i \varphi^{b-1} |\Delta_{\mathbb{H}} \varphi| \, d\eta \]
\[ + 2b \int_{D_i} |u|^{m_1} \varphi^{b-1} |\nabla_{\mathbb{H}} f_i| |\nabla_{\mathbb{H}} \varphi| \, d\eta. \]
Applying Hölder’s inequality, we arrive
\[
\int_{D_i} |\eta|_H^{\gamma} |v|^p f_i \varphi^b \, d\eta 
\leq \left( \int_{D_i} |\eta|_H^{\gamma} |u|^q f_i \varphi^b \, d\eta \right)^{\frac{m_1}{q}} \left\{ \left( \int_{D_i} |\eta|_H^{\gamma} |f_i \varphi^{b - \frac{q}{q-m_1}}| \Delta_H \varphi |^{q-m_1} \, d\eta \right)^{\frac{q-m_1}{q}} + \left( \int_{D_i} |\eta|_H^{\gamma} f_i |\nabla_H f_i|^{q-m_1} |\nabla_H \varphi|^{q-m_1} \, d\eta \right)^{\frac{q-m_1}{q}} \right\}. 
\]
Since 0 ≤ φ ≤ 1, we can chose b large enough such that
\[
\int_{D_i} |\eta|_H^{\gamma} |u|^p f_i \varphi^b \, d\eta 
\leq \left( \int_{D_i} |\eta|_H^{\gamma} |u|^q f_i \varphi^b \, d\eta \right)^{\frac{m_1}{q}} \left\{ \left( \int_{D_i} |\eta|_H^{\gamma} |f_i \varphi^{b - \frac{q}{q-m_1}}| \Delta_H \varphi |^{q-m_1} \, d\eta \right)^{\frac{q-m_1}{q}} + \left( \int_{D_i} |\eta|_H^{\gamma} f_i |\nabla_H f_i|^{q-m_1} |\nabla_H \varphi|^{q-m_1} \, d\eta \right)^{\frac{q-m_1}{q}} \right\}. 
\tag{2.8}
\]
Similarly,
\[
\int_{D_i} |\eta|_H^{\gamma} |u|^p f_i \varphi^b \, d\eta 
\leq \left( \int_{D_i} |\eta|_H^{\gamma} |u|^q f_i \varphi^b \, d\eta \right)^{\frac{m_2}{p}} \left\{ \left( \int_{D_i} |\eta|_H^{\gamma} |f_i \varphi^{b - \frac{q}{p-m_2}}| \Delta_H \varphi |^{p-m_2} \, d\eta \right)^{\frac{p-m_2}{p}} + \left( \int_{D_i} |\eta|_H^{\gamma} f_i |\nabla_H f_i|^{p-m_2} |\nabla_H \varphi|^{p-m_2} \, d\eta \right)^{\frac{p-m_2}{p}} \right\}. 
\tag{2.9}
\]
Combining (2.8) with (2.9), we obtain
\[
\begin{cases}
I_1^{1 - \frac{m_1}{pq}} \lesssim (K_{i,1} + L_{i,1})^{\frac{m_1}{q}} (K_{i,2} + L_{i,2}), \\
J_1^{1 - \frac{m_1}{pq}} \lesssim (K_{i,1} + L_{i,1}) (K_{i,2} + L_{i,2})^{\frac{m_1}{p}},
\end{cases}
\]
which completes the proof. \hfill \square

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. When Ω = Ω_1, we have for \(j = 1, 2\),
\[
K_{1,j} = \left( \int_{D_1} |\eta|_H^{1-\lambda_j} |\Delta_H \varphi|^{\lambda_j} \, d\eta \right)^{\frac{1}{\gamma_j}},
\]
and
\[
L_{1,j} = 0.
\]
Then (2.4) becomes
\[
\begin{cases}
I_1^{1 - \frac{m_1}{pq}} \lesssim (K_{1,1})^{\frac{m_1}{q}} K_{1,2}, \\
J_1^{1 - \frac{m_1}{pq}} \lesssim K_{1,1} (K_{1,2})^{\frac{m_1}{p}}.
\end{cases}
\tag{3.1}
\]
In what follows, we would like to use (3.1) with three different types of test functions to prove \((u, v) \equiv (0, 0)\) in \(\mathbb{H}\), respectively.

- One is
  \[
  \varphi(\eta) = \varphi(x, y, \tau) := \phi \left( \frac{|x|^4 + |y|^4 + \tau^2}{R^4} \right),
  \]
  where \(\phi \in C^\infty[0, \infty)\) is a nonnegative function satisfying
  \[
  \phi(t) = 1 \text{ on } [0, 1]; \quad \phi(t) = 0 \text{ on } [2, \infty); \quad |\phi'| \leq C < \infty.
  \]
  Set \(r = \frac{|x|^4 + |y|^4 + \tau^2}{R^4}\). Note that supp \((\nabla_\mathbb{H} \varphi)\) is a subset of \(\Sigma_R = \{ \eta = (x, y, \tau) \in \mathbb{H} \mid R^4 \leq |x|^4 + |y|^4 + \tau^2 \leq 2R^4 \}\).

Direct calculation yields that
  \[
  \Delta_\mathbb{H} \varphi = \sum_{i=1}^{N} \left[ \partial^2_{x_i} \varphi + \partial^2_{y_i} \varphi + 4y_i \partial^2_{x_i, \tau} \varphi - 4x_i \partial^2_{y_i, \tau} \varphi + 4(x_i^2 + y_i^2) \partial^2_{\tau} \varphi \right]
  = 16R^{-8} \phi''(r) \left[ |x|^6 + |y|^6 + \tau^2(|x|^2 + |y|^2) + 2\tau x \cdot y(|x|^2 - |y|^2) \right]
  + 4(4 + N)R^{-4} \phi'(r)(|x|^2 + |y|^2).
  \]

Thus,
  \[
  |\Delta_\mathbb{H} \varphi| \lesssim R^{-2}.
  \] (3.3)

It follows that
  \[
  K_{1,j} = \left( \int_{\Sigma_R} |\eta|^{(1-\lambda_j)\gamma_j} |\Delta_\mathbb{H} \varphi|^{\lambda_j} d\eta \right)^{\frac{1}{\lambda_j}}
  \lesssim R^{(1-\lambda_j)\gamma_j} R^{-2\lambda_j} \int_{\Sigma_R} d\eta
  \lesssim R^{\frac{(1-\lambda_j)\gamma_j + Q}{\lambda_j} - 2}.
  \] (3.4)

Inserting (3.4) into (3.1), we compute
  \[
  \begin{aligned}
  I_1 &= R^{\sigma_{I_1}} , \\
  J_1 &= R^{\sigma_{J_1}} ,
  \end{aligned}
  \]
  where
  \[
  \sigma_{I_1} := \frac{Q(pq - m_1m_2)}{pq} - \frac{m_1(p\gamma_2 + m_2\gamma_1)}{pq} - \frac{2(m_1 + q)}{q} ,
  \]
  \[
  \sigma_{J_1} := \frac{Q(pq - m_1m_2)}{pq} - \frac{m_2(q\gamma_1 + m_1\gamma_2)}{pq} - \frac{2(m_2 + p)}{p}.
  \] (3.6) (3.7)

Note that \(\sigma_{I_1} \leq 0\) or \(\sigma_{J_1} \leq 0\) if and only if (1.5) holds. In the case \(\sigma_{I_1} \leq 0\), the integral \(I_1\), increasing in \(R\), is bounded uniformly with respect to \(R\).

Applying the monotone convergence theorem, we conclude that \(|\eta|^{\gamma_1} |v|^p\) is in \(L^1(\mathbb{H})\). Note that instead of the first inequality of (3.5) we have, more precisely,
  \[
  I_1 \lesssim \left( \int_{\Sigma_R} |\eta|^{\gamma_1} |v|^p \varphi d\eta \right)^{\frac{m_1m_2}{pq}} R^{\sigma_{I_1}} \lesssim \left( \int_{\Sigma_R} |\eta|^{\gamma_1} |v|^p \varphi d\eta \right)^{\frac{m_1m_2}{pq}}.
  \]
Finally, using the dominated convergence theorem, we obtain
\[
\lim_{R \to +\infty} \int_{\Sigma_R} |\eta| H^2 |v|^p \varphi \, d\eta = 0.
\]

Therefore,
\[
\lim_{R \to +\infty} I_1 = 0,
\]
which implies that \(v \equiv 0\) in \(H\) and thus \(u \equiv 0\) in \(U\) via (2.9). The proof in the case \(\sigma_{\Omega_1} \leq 0\) is analogous.

- Another is
  \[
  \varphi(\eta) := \omega(\eta) \xi_k(\eta), \quad \text{for fixed } k \in \mathbb{N},
  \]
  where
  \[
  \omega(\eta) = \begin{cases} 
  1, & \rho < R, \\
  \left(\frac{R}{\rho}\right)^{-\delta}, & \rho \geq R,
  \end{cases}
  \]
  and
  \[
  \xi_k(\eta) = \begin{cases} 
  1, & 0 \leq \rho \leq kR, \\
  2 - \frac{\rho}{kR}, & kR \leq \rho \leq 2kR, \\
  0, & \rho \geq 2kR,
  \end{cases}
  \]
  with \(\rho := |\eta|_H\), and
  \[
  \delta > \frac{\Lambda + (1 - \lambda_j) \gamma_j}{\lambda_j} - 2, \quad \text{for } j = 1, 2. \tag{3.8}
  \]

Note that \(\text{supp}(\nabla_H \varphi)\) is a subset of
\[
\Sigma'_R = \{ \eta = (x, y, \tau) \in H \mid \eta \in B_H(0, 2kR) \setminus B_H(0, R) \}.
\]

According to [1], we have
\[
\Delta_H \rho = \frac{Q - 1}{\rho} \Psi(\eta),
\]
where the function \(\Psi\) is defined by
\[
\Psi(\eta) = \frac{|x|^2 + |y|^2}{\rho^2} = |\nabla_H \rho|^2, \quad \text{for } \eta \neq 0.
\]

Note that \(0 \leq \Psi \leq 1\), it is not difficult to check that
\[
|\nabla_H \omega| = \delta R^\delta \rho^{-(\delta + 1)} |\nabla_H \rho| \lesssim R^\delta \rho^{-(\delta + 1)}, \tag{3.9}
\]
\[
|\Delta_H \omega| = \delta R^\delta \rho^{-(\delta + 2)} \left| \rho \Delta_H \rho - (\delta + 1) |\nabla_H \rho|^2 \right| \lesssim R^\delta \rho^{-(\delta + 2)}, \tag{3.10}
\]
and
\[
|\nabla_H \xi_k| = (kR)^{-1} |\nabla_H \rho| \lesssim (kR)^{-1}, \tag{3.11}
\]
\[
|\Delta_H \xi_k| = (kR)^{-1} |\Delta_H \rho| \lesssim (kR \rho)^{-1}. \tag{3.12}
\]

Note also that \(\varphi(\eta) \uparrow \omega(\eta)\) as \(k \to \infty\), and for every \(\lambda \geq 1\),
\[
|\Delta_H \varphi|^\lambda \lesssim \omega^\lambda |\Delta_H \xi_k|^\lambda + \xi_k^\lambda |\Delta_H \omega|^\lambda + |\nabla_H \omega|^\lambda |\nabla_H \xi_k|^\lambda.
\]
Then

\[(K_{1,j})^{\lambda_j} = \int_{\mathbb{H}} |\eta|^{(1-\lambda_j)\gamma_j} |\Delta_{\mathbb{H}} \varphi|^{\lambda_j} d\eta\]

\[\lesssim \int_{B_{3R}(0,2kR) \setminus B_{3R}(0,kR)} |\eta|^{(1-\lambda_j)\gamma_j} \omega^{\lambda_j} |\Delta_{\mathbb{H}} \xi_k|^{\lambda_j} d\eta\]

\[+ \int_{B_{3R}(0,2kR) \setminus B_{3R}(0,kR)} |\eta|^{(1-\lambda_j)\gamma_j} \xi_k^{\lambda_j} |\Delta_{\mathbb{H}} \omega|^{\lambda_j} d\eta\]

\[+ \int_{B_{3R}(0,2kR) \setminus B_{3R}(0,kR)} |\eta|^{(1-\lambda_j)\gamma_j} |\nabla_{\mathbb{H}} \omega|^{\lambda_j} |\nabla_{\mathbb{H}} \xi_k|^{\lambda_j} d\eta\]

\[= M_1 + M_2 + M_3. \quad (3.13)\]

By (3.12), we have

\[M_1 \lesssim (kR)^{-\lambda_j} \int_{B_{3R}(0,2kR) \setminus B_{3R}(0,kR)} \omega^{\lambda_j} \rho^{(1-\lambda_j)\gamma_j} \rho^{-\lambda_j} d\eta\]

\[\lesssim (kR)^{-\lambda_j} \left( \sup_{B_{3R}(0,2kR) \setminus B_{3R}(0,kR)} \omega^{\lambda_j} \right) \int_{B_{3R}(0,2kR) \setminus B_{3R}(0,kR)} \rho^{(1-\lambda_j)\gamma_j - \lambda_j} d\eta\]

\[\lesssim (kR)^{-\lambda_j} \left( \frac{kR}{R} \right)^{-\delta\lambda_j} \int_{kR}^{2kR} \rho^{(1-\lambda_j)\gamma_j - \lambda_j + Q - 1} d\rho\]

\[\lesssim k^{(1-\lambda_j)\gamma_j - (\delta+2)\lambda_j + Q} R^{(1-\lambda_j)\gamma_j - 2\lambda_j + Q}. \quad (3.14)\]

By (3.10), we have

\[M_2 \lesssim R^{\delta\lambda_j} \int_{B_{3R}(0,2kR) \setminus B_{3R}(0,kR)} \rho^{(1-\lambda_j)\gamma_j} \rho^{-(\delta+2)\lambda_j} d\eta\]

\[\lesssim R^{\delta\lambda_j} \int_{kR}^{2kR} \rho^{(1-\lambda_j)\gamma_j - (\delta+2)\lambda_j + Q - 1} d\rho\]

\[\lesssim \frac{(2k)^{(1-\lambda_j)\gamma_j - (\delta+2)\lambda_j + Q - 1}}{(1 - \lambda_j)\gamma_j - (\delta + 2)\lambda_j + Q} R^{(1-\lambda_j)\gamma_j - 2\lambda_j + Q}. \quad (3.15)\]

By (3.9) and (3.11), we have

\[M_3 \lesssim (kR)^{-\lambda_j} R^{\delta\lambda_j} \int_{B_{3R}(0,2kR) \setminus B_{3R}(0,kR)} \rho^{(1-\lambda_j)\gamma_j} \rho^{-(\delta+1)\lambda_j} d\eta\]

\[\lesssim (kR)^{-\lambda_j} R^{\delta\lambda_j} \int_{kR}^{2kR} \rho^{(1-\lambda_j)\gamma_j - (\delta+1)\lambda_j + Q - 1} d\rho\]

\[\lesssim k^{(1-\lambda_j)\gamma_j - (\delta+2)\lambda_j + Q} R^{(1-\lambda_j)\gamma_j - 2\lambda_j + Q}. \quad (3.16)\]

A combination of (3.13), (3.14), (3.15) and (3.16) yields that

\[K_{1,j} \lesssim \left( k^{(1-\lambda_j)\gamma_j - (\delta+2)\lambda_j + Q} + \frac{(2k)^{(1-\lambda_j)\gamma_j - (\delta+2)\lambda_j + Q - 1}}{(1 - \lambda_j)\gamma_j - (\delta + 2)\lambda_j + Q} \right) \frac{1}{\lambda_j} R^{(1-\lambda_j)\gamma_j + Q - 2}. \]

It follows from (1.5) and (3.8) that \((1 - \lambda_j)\gamma_j - (\delta + 2)\lambda_j + Q < 0\). Thus, upon taking \(k \to \infty\), we obtain

\[K_{1,j} \lesssim R^{(1-\lambda_j)\gamma_j + Q - 2}. \quad (3.17)\]
Inserting (3.17) into (3.1), we derive (3.5) once again. By substituting $\Sigma'_R$ for $\Sigma_R$, and arguing as we did as above, we can get the desired result.

- The third is

$$\varphi(\eta) := \frac{1}{n} \sum_{k=n+1}^{2n} \varphi_k(\eta), \quad \text{for fixed } n \in \mathbb{N},$$

where $\{\varphi_k\}_{k \in \mathbb{N}}$ is a sequence satisfying that each $\varphi_k$ is a Lipschitz function such that $\text{supp}(\varphi_k) \subset B_H(0, 2^k)$, $\varphi_k = 1$ in a neighborhood of $B_H(0, 2^{k-1})$, and

$$|\Delta_H \varphi_k| \begin{cases} \lesssim \frac{1}{2^{2k-1}}, & \eta \in B_H(0, 2^k) \setminus B_H(0, 2^{k-1}), \\ = 0, & \text{otherwise}. \end{cases}$$

Note that $\varphi = 1$ on $B_H(0, 2^n)$, $\varphi = 0$ outside $B_H(0, 2^{2n})$, and $0 \leq \varphi \leq 1$ on $H$. Note also that for distinct $k$, $\text{supp}(\Delta_H \varphi_k)$ are disjoint with each other.

Then we have for any $\lambda > 0$,

$$|\Delta_H \varphi|^\lambda = n^{-\lambda} \sum_{k=n+1}^{2n} |\Delta_H \varphi_k|^\lambda.$$

It follows that

$$(K_{1,j})^{\lambda_j} = n^{-\lambda_j} \int_H |\eta|^{(1-\lambda_j)\gamma_j} \sum_{k=n+1}^{2n} |\Delta_H \varphi_k|^{\lambda_j} d\eta$$

$$= n^{-\lambda_j} \sum_{k=n+1}^{2n} \int_{B_H(0, 2^k) \setminus B_H(0, 2^{k-1})} |\eta|^{(1-\lambda_j)\gamma_j} |\Delta_H \varphi_k|^{\lambda_j} d\eta$$

$$\lesssim n^{-\lambda_j} \sum_{k=n+1}^{2n} 2^{k(1-\lambda_j)\gamma_j} 2^{-2k\lambda_j} 2^{kQ}$$

$$\lesssim n^{-\lambda_j+1} 2^{(1-\lambda_j)\gamma_j-2\lambda_j+Q} n.$$  

Thus,

$$K_{1,j} \lesssim n^{-1+\frac{Q}{2} \left( \left( \frac{1-\lambda_j+Q}{\lambda_j} \right)^2 - 1 \right)}.$$  

Inserting (3.18) into (3.1), we compute

$$\left\{ \begin{array}{l} \frac{1}{I_1} \lesssim n^{-\frac{m_{1,m_2}}{p_1}} 2^{n\sigma_{f_1}}, \\ \frac{1}{I_1} \lesssim n^{-\frac{m_1+m_2}{p_1}} 2^{n\sigma_{f_1}}, \end{array} \right.$$  

where $\sigma_{f_1}$ and $\sigma_{f_2}$ are defined as in (3.6) and (3.7).

Note that $\sigma_{f_1} \leq 0$ or $\sigma_{f_2} \leq 0$ if and only if (1.5) holds. In the case $\sigma_{f_1} \leq 0$, by taking $n \to \infty$, we conclude from the first inequality of (3.19) that

$$\lim_{n \to +\infty} I_1 = 0,$$

which implies that $v \equiv 0$ in $\mathbb{H}$ and thus $u \equiv 0$ in $\mathbb{H}$ via (2.9). The proof in the case $\sigma_{f_1} \leq 0$ is analogous.  \[\square\]
4. Proof of Theorems 1.2-1.3

Proof of Theorem 1.2. When $\Omega = \Omega_2$, we have for $j = 1, 2,$

$$K_{2,j} = \left( \int_{D_2} |\eta|^{(1-\lambda_j)\gamma_j} x_1^{\alpha - \lambda_j} |\Delta_{\mathbb{H}} \varphi|^{\lambda_j} \, d\eta \right)^{\frac{1}{\lambda_j}},$$

and

$$L_{2,j} = \alpha \left( \int_{D_2} |\eta|^{(1-\lambda_j)\gamma_j} x_1^{\alpha - \lambda_j} |\nabla_{\mathbb{H}} x_1|^{\lambda_j} |\nabla_{\mathbb{H}} \varphi|^{\lambda_j} \, d\eta \right)^{\frac{1}{\lambda_j}}.$$ 

Then (2.4) reads as

$$\begin{cases}
I_1^{\frac{1}{\lambda_j}} \lesssim (K_{2,1} + L_{2,1})^{\frac{m_1}{p}} \left( K_{2,2} + L_{2,2} \right), \\
J_2^{\frac{1}{\lambda_j}} \lesssim (K_{2,1} + L_{2,1}) \left( K_{2,2} + L_{2,2} \right)^{\frac{m_2}{p}}.
\end{cases} \quad (4.1)$$

Let us use the first test function $\varphi(\eta)$ defined by (3.2). In view of (3.3), we obtain

$$K_{2,j} \lesssim \left( R^{(1-\lambda_j)\gamma_j} R^{\alpha - \lambda_j} \int_{\Sigma_R} d\eta \right)^{\frac{1}{\lambda_j}} \lesssim R^{\frac{(1-\lambda_j)\gamma_j + \alpha + Q}{\lambda_j} - 2}. \quad (4.2)$$

Also, we calculate

$$|\nabla_{\mathbb{H}} \varphi| = \left( \sum_{i=1}^N |X_i \varphi|^2 + |Y_i \varphi|^2 \right)^{\frac{1}{2}}$$

$$= 4R^{-4} |\phi'(r)| \left[ |x|^6 + |y|^6 + \tau^2 (|x|^2 + |y|^2) + 2\tau x \cdot y (|x|^2 - |y|^2) \right]^{\frac{1}{2}}$$

$$\lesssim R^{-1}. \quad (4.3)$$

Thus,

$$L_{2,j} \lesssim \left( R^{(1-\lambda_j)\gamma_j} R^{\alpha - \lambda_j} \int_{\Sigma_R} d\eta \right)^{\frac{1}{\lambda_j}} \lesssim R^{\frac{(1-\lambda_j)\gamma_j + \alpha + Q}{\lambda_j} - 2}. \quad (4.4)$$

Inserting (4.2) and (4.4) into (4.1), we compute

$$\begin{cases}
I_1^{\frac{1}{\lambda_j}} \lesssim R^{\sigma_{I_2}}, \\
J_2^{\frac{1}{\lambda_j}} \lesssim R^{\sigma_{J_2}},
\end{cases} \quad (4.5)$$

where

$$\sigma_{I_2} := \frac{(\alpha + Q) (pq - m_1 m_2)}{pq} - \frac{m_1 (p\gamma_2 + m_2 \gamma_1)}{pq} - \frac{2(m_1 + q)}{q},$$

$$\sigma_{J_2} := \frac{(\alpha + Q) (pq - m_1 m_2)}{pq} - \frac{m_2 (q\gamma_1 + m_1 \gamma_2)}{pq} - \frac{2(m_2 + p)}{p}.$$ 

Note that $\sigma_{I_2} \leq 0$ or $\sigma_{J_2} \leq 0$ if and only if (1.6) holds. In the case $\sigma_{I_2} \leq 0$, the integral $I_2$, increasing in $R$, is bounded uniformly with respect to $R$. Applying the monotone convergence theorem, we conclude that $|\eta|_{H^2}^3 |v|^p x_1^\alpha$ is in $L^3 (\Omega_2)$. Note that instead of the first inequality of (4.5) we have, more precisely,

$$I_2 \lesssim \left( \int_{\Omega_2 \cap \Sigma_R} |\eta|_{H^2}^3 |v|^p x_1^\alpha \varphi^b \, d\eta \right)^{\frac{m_1 m_2}{pq}} R^{\sigma_{I_2}} \lesssim \left( \int_{\Omega_2 \cap \Sigma_R} |\eta|_{H^2}^3 |v|^p x_1^\alpha \varphi^b \, d\eta \right)^{\frac{m_1 m_2}{pq}}.$$
Finally, using the dominated convergence theorem, we obtain
\[ \lim_{R \to +\infty} \int_{\Omega_2 \cap \Sigma_R} |\eta|_{H^1}^2 |v|^p x_1^{\alpha} \varphi^b d\eta = 0. \]
Therefore,
\[ \lim_{R \to +\infty} I_2 = 0, \]
which implies that \( v \equiv 0 \) in \( \Omega_2 \) and thus \( u \equiv 0 \) in \( \Omega_2 \) via (2.9). The proof in the case \( \sigma_{J_3} \leq 0 \) is analogous.

**Proof of Theorem 1.3.** When \( \Omega = \Omega_3 \), we have for \( j = 1, 2, \)
\[ K_{3,j} = \left( \int_{D_3} |\eta|_{H^1}^2 (1 - \lambda_j) |\gamma_j x_1^{\alpha} \Delta_{\eta} |\lambda_j \varphi | d\eta \right)^{\frac{1}{\gamma_j}}, \]
and
\[ L_{3,j} = \alpha \left( \int_{D_3} |\eta|_{H^1}^2 (1 - \lambda_j) |\gamma_j x_1^{\alpha - \lambda_j} |\nabla_{\eta} \gamma_j |\Delta_{\eta} |\lambda_j \varphi | d\eta \right)^{\frac{1}{\gamma_j}}. \]
Then (2.4) reads as
\[ \begin{cases} I_3^{1 - \frac{m_1 m_2}{pq}} \lesssim (K_{3,1} + L_{3,1}) \frac{m_1}{\tau} (K_{3,2} + L_{3,2}), \\ J_3^{1 - \frac{m_1 m_2}{pq}} \lesssim (K_{3,1} + L_{3,1}) (K_{3,2} + L_{3,2}) \frac{m_2}{\tau^2}. \end{cases} \] (4.6)
We also use the first test function \( \varphi(\eta) \) defined by (3.2). By (3.3), we obtain
\[ K_{3,j} \lesssim \left( R^{(1 - \lambda_j) \gamma_j} R^{2\alpha - 2\lambda_j} \int_{\Sigma_R} d\eta \right)^{\frac{1}{\gamma_j}} \lesssim R^{(1 - \lambda_j) \gamma_j + 2\alpha + Q}. \] (4.7)
By (4.3), we obtain
\[ L_{3,j} \lesssim \left( R^{(1 - \lambda_j) \gamma_j} R^{2\alpha - \lambda_j} R^{\lambda_j - \lambda_j} \int_{\Sigma_R} d\eta \right)^{\frac{1}{\gamma_j}} \lesssim R^{(1 - \lambda_j) \gamma_j + 2\alpha + Q}. \] (4.8)
Inserting (4.7) and (4.8) into (4.6), we compute
\[ \begin{cases} I_3^{1 - \frac{m_1 m_2}{pq}} \lesssim R^{\sigma_{I_3}}, \\ J_3^{1 - \frac{m_1 m_2}{pq}} \lesssim R^{\sigma_{J_3}}, \end{cases} \] (4.9)
where
\[ \sigma_{I_3} := \frac{(2\alpha + Q)(pq - m_1 m_2)}{pq} - \frac{m_1(p\gamma_2 + m_2\gamma_1)}{pq} - \frac{2(m_1 + q)}{q}, \]
\[ \sigma_{J_3} := \frac{(2\alpha + Q)(pq - m_1 m_2)}{pq} - \frac{m_2(q\gamma_1 + m_1\gamma_2)}{pq} - \frac{2(m_2 + p)}{p}. \]
Note that \( \sigma_{I_3} \leq 0 \) or \( \sigma_{J_3} \leq 0 \) if and only if (1.7) holds. In the case \( \sigma_{I_3} \leq 0 \), the integral \( I_3 \), increasing in \( R \), is bounded uniformly with respect to \( R \). Applying the monotone convergence theorem, we conclude that \( \eta_{H^1}^2 |v|^p x_1^{\alpha} \varphi^b \) is in \( L^1 (\Omega_3) \). Note that instead of the first inequality of (4.9) we have, more precisely,
\[ I_3 \lesssim \left( \int_{\Omega_3 \cap \Sigma_R} |\eta|_{H^1}^2 |v|^p x_1^{\alpha} \varphi^b d\eta \right)^{\frac{m_1 m_2}{pq}} \lesssim \left( \int_{\Omega_3 \cap \Sigma_R} |\eta|_{H^1}^2 |v|^p x_1^{\alpha} \varphi^b d\eta \right)^{\frac{m_1 m_2}{pq}}. \]
Finally, using the dominated convergence theorem, we obtain
\[
\lim_{R \to +\infty} \int_{\Omega \cap \Sigma_R} |\eta|^{2 \alpha} |v|^p r^\alpha \varphi^b d\eta = 0.
\]
Therefore,
\[
\lim_{R \to +\infty} I_3 = 0,
\]
which implies that \( v \equiv 0 \) in \( \Omega_3 \) and thus \( u \equiv 0 \) in \( \Omega_3 \) via (2.9). The proof in the case \( \sigma_{J_3} \leq 0 \) is analogous. \( \square \)

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