Josephson-based threshold detector for Lévy distributed fluctuations

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We propose a threshold detector for Lévy distributed fluctuations based on a Josephson junction. The Lévy noise current added to a linearly ramped bias current results in clear changes in the distribution of switching currents out of the zero-voltage state of the junction. We observe that the analysis of the cumulative distribution function of the switching currents supplies information on both the characteristics shape parameter \( \alpha \) of the Lévy statistics and the intensity of the fluctuations. Moreover, we discuss a theoretical model which allows from a measured distribution of switching currents to extract characteristic features of the Lévy fluctuations. In view of this results, this system can effectively find an application as a detector for a Lévy signal embedded in a noisy background.

Introduction. – A current-biased Josephson junction (JJ) represents a natural threshold detector for current fluctuations, inasmuch as it is a metastable system operating on an activation mechanism. Actually, the behaviour of a JJ can be depicted as a particle, representing the superconducting phase difference \( \varphi \) across the JJ, in a cosine “washboard” potential with friction [1, 2], see Fig. 1(a). In this picture, the slope of the washboard potential is given by the injected current, and the dynamics of the phase is described by the resistively and capacitively shunted junction (RCSJ) model. The equivalent particle remains near a washboard minimum (correspondingly, the JJ is in the zero-voltage metastable state) until the direct bias current exceeds a critical value, or a fluctuation sets the phase \( \varphi \) in motion along the potential. Indeed, a current fluctuation instantaneously tilts the potential, and a noise-induced escape from a minimum can occur. In correspondence of the escape a voltage develops, as the voltage is related to the velocity of the phase particle. Shortly, if a JJ is set in the fundamental zero-voltage state, noise can cause a passage from this zero-voltage state to the finite voltage “running” state. The statistics of these passages can be exploited to reveal the features of the noise.

After early suggestions [3–5], several proposals for concrete experimental setup of Josephson-based noise detectors have been put forward [6–12]. A scheme to detect the Poissonian character of the charge injection in an underdamped JJ, based on the analysis of the third-order moment of the electrical noise, was proposed in Ref. [3] and a scheme to detect the fourth-order moment of the noise was discussed in Ref. [4]. A threshold detector based on an array of overdamped JJs for the direct measurement of the full counting statistics, through rare over-the-barrier jumps induced by current fluctuations, was suggested in Ref. [5]. Alternatively, in the Coulomb blockade regime, the sensitivity of the JJ conductance to the non-Gaussian character of the applied noise was demonstrated [13, 14]. Most proposals make use of the

\[ U = U_0 [1 - \cos(\varphi) - i_0 \varphi] \]

FIG. 1. (a) The phase particle in a potential minimum of the tilted washboard potential \( U \). The barrier height, \( \Delta U \), and the distance between the minimum and maximum of the potential, \( \Delta x \), are also shown. (b) \( \varphi \) trajectories in the noisy driven case when the Lévy stochastic term dominates. (c) Simplified equivalent circuit diagram for the RCSJ model.

The linearly ramped bias current, \( I_b(t) \), and the noise current, \( I_N(t) \), of the JJ are included in the diagram.
information content of higher moments, beyond the variance, of the electric noise, mainly to discuss the Poissonian character of the current fluctuations. However, experimental measurements of third and fourth moments are actually demanding and error-prone with respect to measurements of dc-transport properties. Indeed, deviations from Gaussian behaviour are typically small and high frequency regimes are usually necessary to retrieving information about fluctuations.

In this Letter we address the issue of Lévy distributed noise characterization through the switching currents distribution (SCD) of a JJ [15, 16]. These stochastic processes could drive the particle, namely, the phase in JJ context, over a very long distance in a single motion event, namely, a flight. Lévy flights well describe transport phenomena in different condensed matter systems [17–32]. Results on Lévy flights were recently reviewed in Refs. [33, 34] and an extensive bibliography on α-stable distributions is maintained online by Nolan [35].

To visualize the effect of Lévy noise, in Fig. 1(b) we show several phase trajectories, in the absence of bias current, characterized by abrupt fluctuations. A Lévy flights distribution exhibits power-law tails and, consequently, second and higher moments diverge. The latter feature poses a relevant complication in relating Lévy flight models to experimental data. The problem is to accurately perform the experimental measurement of a physical quantity that, according to a possibly infinite variance, can suffer limitless fluctuations. A JJ-based threshold detector circumvents this difficulty, since the switching occurs as the phase particle passes a potential barrier [see Fig. 1(a)]

The phase dynamics is obtained by numerically solving the RCSJ model equation [1]

\[
\left( \frac{\Phi_0}{2\pi} \right)^2 i_N \frac{d^2 \varphi}{dt^2} + \frac{\Phi_0}{2\pi} \frac{1}{R} \frac{d \varphi}{dt} + \frac{d}{d \varphi} U = \frac{\Phi_0}{2\pi} I_N, \tag{1}
\]

where \( \Phi_0 = h/(2e) \) is the flux quantum and \( R \) is the normal resistance of the JJ. Moreover, \( U \) is the washboard potential [see Fig. 1(a)]

\[
U = U_0 \left[ 1 - \cos(\varphi) - i_b \varphi \right], \tag{2}
\]

where \( U_0 = (\Phi_0/2\pi) I_c \). The average slope of the potential \( U \) is given by \( i_b(t) = I_b(t)/I_c = v_b t \), where \( v_b = t_{\text{max}}^{-1} \) is the ramp speed. The resulting activation energy barrier \( \Delta U = 2 \left[ \sqrt{1 + t_b^2} - t_b \arcsin(i_b) \right] \) confines the phase in a potential minimum.

Eq. (1) can be recast for convenience in a compact form

\[
m \frac{d^2 \varphi}{dt^2} + \frac{\eta}{m} \frac{d \varphi}{dt} + U_0 \frac{d}{d \varphi} u = U_0 i_N, \tag{3}
\]

where \( m = (\Phi_0/2\pi)^2 C \) is the effective junction mass, the friction is governed by the parameter \( \eta = 1/(RC) \), \( u = U/U_0 \), and \( i_N = I_N/I_c \) is the stochastic term. In these units, \( \omega_p = \sqrt{U_0/m} \). In all simulations we assume the damping \( \eta = 0.1 \omega_p \), the ramp speed \( v_b = 10^{-7} \), and the Lévy noise intensity \( D = 5 \times 10^{-7} \).

To model the Lévy noise sources, we use the algorithm proposed by Weron [37] for the implementation of the Chambers method [38]. The notation \( S_\alpha(\sigma, \beta, \lambda) \) is used for the Lévy distributions [39–41], where \( \alpha \in (0, 2) \) is the stability index, \( \beta \in [-1, 1] \) is called asymmetry parameter, and \( \sigma > 0 \) and \( \lambda \) are a scale and a location parameter, respectively. The stability index produces the asymptotic long-tail power law for the distribution, which for \( \alpha < 2 \) is of the \( |x|^{-(1+\alpha)} \) type, while \( \alpha = 2 \) gives the Gaussian distribution. We consider exclusively symmetric (i.e., with \( \beta = 0 \)) bell-shaped, standard (i.e., with \( \sigma = 1 \) and \( \lambda = 0 \)), stable distributions \( S_\alpha(1, 0, 0) \), with \( \alpha \in (0, 2) \).

Lévy escape. – Escapes over a barrier in the presence of Lévy noise have been thoroughly investigated for the overdamped case [33, 42–44]. If both the distance between neighbour minimum and maximum of a metastable potential and the height of the potential barrier [see Fig. 1(a)] are unitary (\( \Delta x = 1 \) and \( \Delta U = 1 \), respectively), the power-law asymptotic behaviour of the mean escape time \( \tau \) for the Lévy statistics reads [45, 46]

\[
\tau(\alpha, D) = \frac{C_\alpha}{D\mu_\alpha}, \tag{4}
\]

of \( 10^4 \) ramps of maximum duration \( t_{\text{max}} = 10^7 \omega_p^{-1} \) are applied, where \( \omega_p = \sqrt{2eI_c/(\hbar C)} \) and \( C \) are the plasma frequency and the capacitance of the JJ, respectively. Finally, a SCD is obtained.

In this work, sequences of stochastic processes activated by noise fluctuations. However, experimental measurements of third and fourth moments are actually demanding and error-prone with respect to measurements of dc-transport properties. Indeed, deviations from Gaussian behaviour are typically small and high frequency regimes are usually necessary to retrieving information about fluctuations.
The physical interpretation of the previous assumption, Eq. (5), becomes
\[ \tau (\alpha, D) = \left( \frac{\eta^{1-\mu_\alpha} \Delta x^{2-2\mu_\alpha} + \alpha \mu_\alpha}{4^{1-\mu_\alpha} \Delta U^{1-\mu_\alpha} 2^{\alpha \mu_\alpha}} \right) \frac{C_\alpha}{D^{\mu_\alpha}}. \]

The CDF of \( i_{SW} \) as a function of \( i_b \) for a specific initial value of the bias ramp, \( i_0 \), reads
\[ \text{CDF}(i_b|i_0) = 1 - \text{Prob}(i_{SW} > i_b|i_0). \]

Recalling that the distribution of escape times is exponential with rate \( 1/\tau(i_b) \) also for Lévy flight noise [46], the same logic of the seminal paper [48] leads to
\[ P(i_b|i_0) = \mathcal{N} \frac{1}{v_b} \frac{1}{\tau(i_b)} \exp \left[ -\frac{1}{v_b} \int_{i_0}^{i_b} \frac{1}{\tau(i)} \, di \right] \]
for the PDF associated to Eq. (7) as a function of the average escape time \( \tau(i_b) \) (here \( \mathcal{N} \) is an appropriated normalization constant). For the thermal noise, Kramers’ formula entails that escapes across the barrier depend on the barrier height. For Lévy noise, with the same widely employed approximations behind Eq. (6), \( \tau(i_b) \) turns out independent of the barrier height \( \Delta U \), and becomes only a function of \( \Delta x = \pi - 2 \arcsin i_b \), see Eq. (2). The expression of \( \tau(\alpha, D) \), Eq. (6), inserted in Eq. (8) gives for the Lévy statistics (at the first order in \( i_b \))
\[ P(i_b|i_0) \propto \exp \left[ -\left( \frac{2}{\pi} \right)^{\alpha} \frac{i_b D^{\mu_\alpha}}{C_\alpha v_b} \right]. \]

This is a further step forward with respect to results of Ref. [49], concerning the nonsinusoidal potential appropriated for graphene JJs [50–53], inasmuch as the above equation contains the explicit expression for the argument of the exponential.

Notably, the solution of Eq. (8) can be analytically computed and expressed in a compact form by using the function \( \mathcal{F}_\alpha \) defined as
\[ \mathcal{F}_\alpha(i_b) = 2^\alpha \left\{ \frac{2}{2\pi-2\arcsin(i_b)} \left[ E_\alpha \left( \cosh^{-1}(i_b) \right) + E_\alpha \left( -\cosh^{-1}(i_b) \right) \right] + \frac{i_b^{1-\alpha}}{4} \left[ E_\alpha \left( \frac{-i_b}{2} \right) - E_\alpha \left( \frac{i_b}{2} \right) \right] \right\}. \]
where $E_{\alpha}$ is the exponential integral with $\alpha$ argument [54]. Then, the PDF can be written as

$$P(i_b|i_0) = N\frac{dF_\alpha}{d i_b} \exp \left\{ -\frac{\mu_\alpha}{C_\alpha v_b} \left[ F_\alpha(i_b) - F_\alpha(i_0) \right] \right\},$$

where $N$ reads

$$N = \left\{ 1 - \exp \left[ -\frac{\mu_\alpha}{C_\alpha v_b} \left( F_\alpha(1) - F_\alpha(i_0) \right) \right] \right\}^{-1}.$$  

The corresponding CDF is

$$CDF(i_b|i_0) = N \left\{ 1 - \exp \left[ -\frac{\mu_\alpha}{C_\alpha v_b} \left( F_\alpha(i_b) - F_\alpha(i_0) \right) \right] \right\}.$$ (13)

This is the main result of this work, that is to connect the properties of Lévy flights with the accessible quantity of SCDs. It is important to remind the main approximations underlying Eq. (13): it has been assumed that the result obtained for an overdamped system, see Eq. (6), still holds for moderately underdamped systems, and that Eq. (8), which is strictly valid in the adiabatic regime, can be applied to a slowly varying process.

We have performed extensive numerical simulations to check the validity of results given by Eqs. (6) and (13). In Fig. 3 we show the marginal CDF, i.e., restricted to the maximum bias $i_b = 0.6$, for $\alpha \in [0.1 - 1.1]$ and $\mu_\alpha = 1$. The choice of these values for $\alpha$ and $i_b$ arises from practical considerations, since Eqs. (6) and (13) are more accurate for low bias currents and low $\alpha$ values, respectively. For these values the Lévy flight jump features dominate, while in the opposite limits, $i_b \simeq 1$ and $\alpha \simeq 2$, the Gaussian characteristics set in. Accordingly, in the considered range of values the effects of the Gaussian noise contribute can be safely ignored. The main panel of Fig. 3 shows also the numerical curves obtained by fitting of Eq. (13). The agreement between computational results and the theoretical analysis, see Eq. (13), is quite accurate for $\alpha < 1$. For $\alpha \gtrsim 1$ the statistics of switches becomes undistinguishable from the uniform distribution (the bisector in Fig. 3). Thus, the model we proposed can be used to determine $\alpha$ from switching currents measurements (as the other parameters are known), but it proves to be especially valuable for $\alpha < 1$.

In the inset of Fig. 3 we show with red circles the estimate of the coefficient $C_\alpha$ obtained by numerical fitting of Eq. (13) of the marginal CDFs shown in the main panel. The estimates of the values of $C_\alpha \gtrsim 1$ significantly deviate from both the numerical estimates given in Ref. [46] and the analytical estimate obtained in Ref. [45]. However, these differences can be ascribed to: i) an overdamped rather than underdamped dynamics; ii) a fixed rather than a slowly varying potential barrier; iii) a cubic rather than a cosine potential.

Conclusions. – We have investigated the switching currents distributions (SCDs) in Josephson junctions in the presence of a Lévy noise source. Lévy distributed fluctuations are characterized by scale-free jumps or Lévy flights. Consequently, we expect the SCDs to exhibit a peculiar behaviour markedly different from the Gaussian noise case. The aim is to detect the characteristics of the Lévy noise from SCDs. Specifically, depending on the value of the stability distribution index, $\alpha$, we have numerically found that: i) for $0 < \alpha < 1$, the SCDs are peaked at zero bias current; ii) for $\alpha \simeq 1$, the SCD is roughly flat; iii) finally, for $1 < \alpha < 2$, the SCDs are peaked at high bias currents (alike the usual Gaussian noise induced peak) and slowly decrease at low bias currents. A peculiar behaviour can be observed also in the cumulative distribution function (CDF) curves, that at a given value of $i_b$ decrease with increasing $\alpha$. Moreover, CDFs are convex for $\alpha < 1$, and concave for $\alpha > 1$ (the case $\alpha = 1$ corresponds to a linear CDF).

A theoretical good estimate of the SCDs can be retrieved on the basis of the Fulton adiabatic approach [48] and assuming that the average escape time for the Lévy guided overdamped case can be extended to moderately damped systems. These theoretical findings are confirmed by the abovementioned numerical observations. Moreover, the theoretical approach recovers a previous result [49], where a phenomenological linear approximation has been applied [see Eq.(9)]. Finally, we achieve, from the SCDs through the theoretical model.
[see Eqs. (11) and (13)] the estimate of the universal (i.e., barrier height independent) noise coefficient $C_\alpha$ and then, if the other parameters are known, the value of the stability index $\alpha$.

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