Weak Pontryagin’s Maximum Principle for Optimal Control Problems Involving a General Analytic Kernel

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Abstract

We prove a duality relation and an integration by parts formula for fractional operators with a general analytical kernel. Based on these basic results, we are able to prove a new Grönwall’s inequality and continuity and differentiability of solutions of control differential equations. This allow us to obtain a weak version of Pontryagin’s maximum principle. Moreover, our approach also allow us to consider mixed problems with both integer and fractional order operators and derive necessary optimality conditions for isoperimetric variational problems and other problems of the calculus of variations.

Key words: fractional operators with general analytical kernels, optimal control problems, Pontryagin extremals, calculus of variations, isoperimetric problems

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1. Introduction

Integration by parts is a powerful tool when two functions are multiplied together, but is also helpful in many other ways. In fact, applications of integration by parts abound, including the laws of Bessel bridges via hypergeometric functions [1], surface measures on levels sets induced by Brownian functionals [2], reductions of Feynman integrals [3], and approximation theory [4].

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Another important tool, allowing to bound a function that is known to satisfy a certain differential or integral inequality by the solution of the corresponding differential or integral equation, is Grönwall’s inequality \[5\]. It provides useful estimates in ordinary and stochastic differential equations \[6\], stability analysis \[5\], and fractional difference \[8\] and differential equations \[9\].

Along the years, conjugation of integration by parts with Grönwall’s inequality has shown a myriad of interesting results in several different areas, e.g., in probability theory and stochastic processes \[10\], systems and control theory \[11\], and fractional optimal control \[12\].

Fractional optimal control and the fractional calculus of variations are concerned with the analysis and derivations of necessary optimality conditions for optimization problems involving fractional operators \[13, 14\]. For smooth and unconstrained data, optimal control problems of Lagrange form can be seen as a generalization of the calculus of variations \[13 \ 16\]. In fact, maximum principles or optimality conditions can be obtained from variational analysis approaches. In particular, the Pontryagin maximum principle \[17\] takes then a special weaker form in which the Hamiltonian maximality condition is reduced to a null derivative of the Hamiltonian function along the extremals \[18\]. Here we prove integration by parts and a Grönwall’s inequality for fractional operators with a general analytical kernel in the sense of Fernandez, Özarslan and Baleanu \[19\], and apply it in the context of the fractional calculus of variations and fractional optimal control.

Recent advancements within the fractional calculus research community has included proposals for general classes of operators, covering many of the numerous diverse definitions of fractional integrals and derivatives under a single generalised operator \[20\]. As already mentioned, here we deal with one such advancement: the Fernandez–Özarslan–Baleanu (FOB) fractional calculus of 2019 \[19\]. Although recent, the ideas of the FOB fractional calculus have already find many interesting applications, for example in the identification of space-dependent source terms in nonlocal problems \[21\], tempered fractional calculus \[22\], and on the determination of a source term for fractional Rayleigh–Stokes equations with random data \[23\]. Here we introduce and develop a FOB fractional optimal control theory.

The manuscript is organized as follows. In Section 2 we briefly review the FOB fractional calculus, recalling necessary notions and results while fixing notation. Our original results begin with Section 3 where we prove an important duality relation (Lemma 9), integration by parts (Lemma 10), and Grönwall’s inequality (Theorem 11). Follows our main applications: in Section 4 we prove continuity and differentiability of solutions to control differential equations (respectively Lemmas 13 and 14), from which we prove a weak version of Pontryagin’s maximum principle for FOB optimal control problems (Theorem 15). As corollaries, we deduce Euler–Lagrange necessary optimality conditions for the fundamental FOB fractional problem of the calculus of variations (Corollary 17) as well as isoperimetric problems (Corollary 18).
2. Preliminaries

We begin by recalling the general analytic kernel fractional calculus of Fernandez, Özarslan and Baleanu, proposed in 2019 [19].

**Definition 1** (See [19]). Let $[a, b]$ be a real interval, $\alpha$ be a real parameter in $[0, 1]$, $\beta$ be a complex parameter with non-negative real part, and $R$ be a positive number satisfying $R > (b - a)^{\text{Re}(\beta)}$. Let $A$ be a complex analytic function on the disc $D(0, R)$ and defined on this disc by the locally uniformly convergent power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$  

The left and right-sided fractional integrals with general analytic kernel of a function $x : [a, b] \to \mathbb{R}$ are defined by

$$A I_{a+}^{\alpha, \beta} x(t) := \int_{a}^{t} (t - s)^{\alpha - 1} A ((t - s)^{\beta}) x(s) ds$$

and

$$A I_{b-}^{\alpha, \beta} x(t) := \int_{t}^{b} (s - t)^{\alpha - 1} A ((s - t)^{\beta}) x(s) ds,$$

respectively.

**Remark 2.** Note that the Riemann–Liouville fractional integral is a special case of Definition 1 given by

$$RL I_{a+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} A_{RL}^{\alpha} x(t),$$

for any arbitrary choice of the analytic function $A$.

**Notation 3.** For any analytic function $A$ as in Definition 1, we denote $A_T$ to be the transformation function

$$A_T(x) := \sum_{n=0}^{\infty} a_n \Gamma(\beta n + \alpha) x^n.$$  

**Lemma 4** (Series formula [19]). For any integrable function $x(t), t \in [a, b]$, the following uniformly convergent series formulas for $A I_{a+}^{\alpha, \beta} x$ and for $A I_{b-}^{\alpha, \beta} x$, as functions on $[a, b]$, hold:

$$A I_{a+}^{\alpha, \beta} x(t) := \sum_{n=0}^{\infty} a_n \Gamma(\beta n + \alpha)^{RL} I_{a+}^{\alpha+n\beta} x(t)$$

and

$$A I_{b-}^{\alpha, \beta} x(t) := \sum_{n=0}^{\infty} a_n \Gamma(\beta n + \alpha)^{RL} I_{b-}^{\alpha+n\beta} x(t),$$

respectively. Here $RL I_{a+}^{\alpha+n\beta}$ and $RL I_{b-}^{\alpha+n\beta}$ are the left and right-sided Riemann–Liouville fractional integrals of order $\alpha + n\beta$, respectively.
Let \(a, b, \alpha, \beta\) and \(A\) be as in Definition \[1\].

**Definition 5.** The left and right Riemann–Liouville fractional derivatives with general analytic kernels of a function \(x : [a, b] \rightarrow \mathbb{R}\) with sufficient differentiability properties are defined by

\[
A_{RL}D_{a+}^{\alpha, \beta} x(t) = \frac{d}{dt} \left( \tilde{A} I_{1+}^{\alpha, \beta} x(t) \right) \quad \text{and} \quad A_{RL}D_{b-}^{\alpha, \beta} x(t) = -\frac{d}{dt} \left( \tilde{A} I_{1-}^{\alpha, \beta} x(t) \right),
\]

where the function \(\tilde{A}\) used on the right-hand side is an analytic function defined by

\[
\tilde{A}(x) = \sum_{n=0}^{\infty} \tilde{a}_n x^n
\]

and such that \(A \Gamma \cdot \tilde{A} \Gamma = 1\).

**Definition 6.** The left and right Caputo fractional derivatives with general analytic kernels of a function \(x : [a, b] \rightarrow \mathbb{R}\) with sufficient differentiability properties are defined by

\[
A_{C}D_{a+}^{\alpha, \beta} x(t) = \tilde{A} I_{1+}^{\alpha, \beta} x'(t) \quad \text{and} \quad A_{C}D_{b-}^{\alpha, \beta} x(t) = -\tilde{A} I_{1-}^{\alpha, \beta} x'(t),
\]

where the function \(\tilde{A}\) used on the right-hand side is an analytic function defined by

\[
\tilde{A}(x) = \sum_{n=0}^{\infty} \tilde{a}_n x^n
\]

and such that \(A \Gamma \cdot \tilde{A} \Gamma = 1\).

**Remark 7.** Note that the classical integer order derivative is obtained, up to a multiplicative constant, when \(\alpha = 1\) and \(\beta = 0\).

**Theorem 8** (Semi group property \[19\]). Let \(a, b, A\) be as in Definition \[7\] and fix \(\alpha_1, \alpha_2, \beta \in \mathbb{C}\) with non-negative real parts. The semigroup property

\[
A I_{a+}^{\alpha_1, \beta} \circ A I_{a+}^{\alpha_2, \beta} x(t) = A I_{a+}^{\alpha_1 + \alpha_2, \beta} x(t)
\]

is uniformly valid (regardless of \(\alpha_1, \alpha_2, \beta\) and \(f\)) if, and only if, the following condition is satisfied for all non-negative integers \(k\):

\[
\sum_{m+n=k} a_n(\alpha_1, \beta)a_m(\alpha_2, \beta)\Gamma(\alpha_1+n\beta)\Gamma(\alpha_2+n\beta) = a_k(\alpha_1+\alpha_2, \beta)\Gamma(\alpha_1+\alpha_2+k\beta).
\]

3. Fundamental Properties

We begin by proving some technical but important results.

**Lemma 9** (Duality operation). For any functions \(x(t)\) and \(y(t)\), \(t \in [a, b]\), the following duality relation holds:

\[
\int_{a}^{b} x(t) A I_{a+}^{\alpha, \beta} y(t) dt = \int_{a}^{b} y(t) A I_{b-}^{\alpha, \beta} x(t) dt.
\]

**Proof.** By the series formula, we have that

\[
\int_{a}^{b} x(t) A I_{a+}^{\alpha, \beta} y(t) dt = \int_{a}^{b} x(t) \sum_{n=0}^{\infty} a_n \Gamma(\beta n + \alpha) A_{RL} I_{a+}^{\alpha+n\beta} y(t) dt.
\]
Since the series in the right-hand side of (1) is uniformly convergent, it follows that
\[
\int_a^b x(t)^A I_{a+}^{\alpha, \beta} y(t) dt = \sum_{n=0}^\infty a_n \Gamma(\beta n + \alpha) \int_a^b x(t)^{RL I_{a+}^{\alpha+n\beta}} y(t) dt
\]
and, by duality of Riemann–Liouville integral operators, one has
\[
\int_a^b x(t)^{RL I_{a+}^{\alpha+n\beta}} y(t) dt = \int_a^b y(t)^{RL I_{b-}^{\alpha+n\beta}} x(t) dt
\]
for any \( n \in \mathbb{N} \), which leads to
\[
\sum_{n=0}^\infty a_n \Gamma(\beta n + \alpha) \int_a^b x(t)^{RL I_{a+}^{\alpha+n\beta}} y(t) dt = \sum_{n=0}^\infty a_n \Gamma(\beta n + \alpha) \int_a^b y(t)^{RL I_{b-}^{\alpha+n\beta}} x(t) dt.
\]
Therefore, we obtain that
\[
\int_a^b x(t)^A I_{a+}^{\alpha, \beta} y(t) dt = \int_a^b y(t)^{RL I_{b-}^{\alpha, \beta}} x(t) dt.
\]
The proof is complete. \( \square \)

**Lemma 10** (Integration by parts formula). Let \( x \) be a continuous function and \( y \) a continuously differentiable function. Then,
\[
\int_a^b x(t)^{D_{a+}^{\alpha, \beta}} y(t) dt = \left[ y(t)^{D_{a+}^{1-\alpha, \beta}} \right]_a^b + \int_a^b y(t)^{D_{a+}^{\alpha, \beta}} y'(t) dt.
\]

**Proof.** By definition,
\[
\int_a^b x(t)^{D_{a+}^{\alpha, \beta}} y(t) dt = \int_a^b x(t)^{D_{a+}^{1-\alpha, \beta}} y'(t) dt
\]
and, by the duality formula, it follows that
\[
\int_a^b x(t)^{D_{a+}^{1-\alpha, \beta}} y'(t) dt = \int_a^b y'(t)^{D_{a+}^{\alpha, \beta}} x(t) dt.
\]
Using (standard) integration by parts, we obtain that
\[
\int_a^b y'(t)^{D_{a+}^{\alpha, \beta}} x(t) dt = \left[ y(t)^{D_{a+}^{1-\alpha, \beta}} \right]_a^b - \int_a^b y(t) \frac{d}{dt} \left( \frac{D_{a+}^{1-\alpha, \beta}}{x(t)} \right) dt,
\]
which leads to the desired formula. \( \square \)

**Theorem 11** (Grönwall’s inequality). Let \( \alpha \) be a positive real number and let \( a(\cdot), g(\cdot) \) and \( u(\cdot) \) be non-negative continuous functions on \([0, T]\) with \( g(\cdot) \) monotonic increasing, satisfying \( \max_{t \in [0, T]} g(t) < \frac{1}{T^{\alpha}} \). If
\[
u(t) \leq a(t) + g(t) \left( A I_{a+}^{\alpha, \beta} u \right)(t),
\]
then
\[
u(t) \leq A I_{a+}^{\alpha, \beta} \left( a(t) + g(t) \left( A I_{a+}^{\alpha, \beta} u \right)(t) \right).
\]
then
\[ u(t) \leq a(t) + \sum_{k=1}^{\infty} g^k(t) \left( \mathcal{A} I_{0^+}^{k\alpha,\beta} a \right)(t) \]

for any \( t \in [0,T] \).

**Proof.** Because \( \mathcal{A} I_{0^+}^{\alpha,\beta} \) is a non-decreasing operator, we have
\[
\left( \mathcal{A} I_{0^+}^{\alpha,\beta} u \right)(t) \leq \mathcal{A} I_{0^+}^{\alpha,\beta} \left( a(\cdot) + g(t) \left( \mathcal{A} I_{0^+}^{\alpha,\beta} u \right) \right)(t) = \left( \mathcal{A} I_{0^+}^{\alpha,\beta} a \right)(t) + g(t) \left( \mathcal{A} I_{0^+}^{\alpha,\beta} u \right)(t).
\]

Now, using its semi-group property \(^{(8)}\), we can substitute the previous inequality into \(^{(2)}\), to obtain
\[
u(t) \leq a(t) + g(t) \left( \mathcal{A} I_{0^+}^{\alpha,\beta} a \right)(t) + g^2(t) \left( \mathcal{A} I_{0^+}^{2\alpha,\beta} u \right)(t).
\]

Repeating this procedure up to \( N \) times, we get
\[
u(t) \leq a(t) + \sum_{k=1}^{N-1} g^k(t) \left( \mathcal{A} I_{0^+}^{k\alpha,\beta} a \right)(t) + g^N(t) \left( \mathcal{A} I_{0^+}^{N\alpha,\beta} u \right)(t).
\]

Therefore, when \( N \to \infty \), we have
\[
u(t) \leq a(t) + \sum_{k=1}^{\infty} g^k(t) \left( \mathcal{A} I_{0^+}^{k\alpha,\beta} a \right)(t) + \lim_{N \to \infty} g^N(t) \left( \mathcal{A} I_{0^+}^{N\alpha,\beta} u \right)(t).
\]

It remains to show that the series
\[
\sum_{k=1}^{\infty} g^k(t) \left( \mathcal{A} I_{0^+}^{k\alpha,\beta} a \right)(t)
\]
is convergent and \( \lim_{N \to \infty} g^N(t) \left( \mathcal{A} I_{0^+}^{N\alpha,\beta} u \right)(t) = 0 \), to obtain the desired result.

Using Definition \(^{(1)}\) of the operator \( \mathcal{A} I_{0^+}^{k\alpha,\beta} \), one has
\[
\sum_{k=1}^{\infty} g^k(t) \left( \mathcal{A} I_{0^+}^{k\alpha,\beta} a \right)(t) = \sum_{k=1}^{\infty} g^k(t) \int_0^t (t-s)^{k-1} A((t-s)^{\beta})a(s)ds.
\]

Next,
\[
\left| \sum_{k=1}^{\infty} g^k(t) \left( \mathcal{A} I_{0^+}^{k\alpha,\beta} a \right)(t) \right| \leq \mu \sum_{k=1}^{\infty} |g^k(t)T^{k\alpha-1}| \int_0^t |a(s)|ds
\]
\[
= \mu \sum_{k=1}^{\infty} |g^k(t)T^{k\alpha-1}| \int_0^t |a(s)|ds
\]
\[
\leq \mu \sum_{k=1}^{\infty} |M^k T^{k\alpha-1}| \int_0^t |a(s)|ds,
\]
where $\mu = \sup_{|x|<T^3} A(x)$ and $M = \max_{t \in [0,T]} g(t)$. Hence, we obtain that
\[
\left| \sum_{k=1}^{\infty} g^k(t) \left( A I_0^{\alpha, \beta} a \right)(t) \right| \leq \mu T^{-1} \sum_{k=1}^{\infty} \left| (MT^\alpha)^k \right| \int_0^t |a(s)| ds.
\]

The series converges providing that $|M| < \frac{1}{T^\alpha}$. Moreover, according to the necessary condition of convergence of an infinite series, one has
\[
\lim_{k \to \infty} g^k(t) A I_0^{N\alpha, \beta} = 0
\]
for all $t \in [0, T]$, which leads to
\[
\lim_{N \to \infty} g^N(t) \left( A I_0^{N\alpha, \beta} u \right)(t) = 0.
\]

4. Main Results

Here we consider a basic optimal control problem, which consists to find a piecewise continuous control function $u(\cdot) \in PC ([a, b]; \mathbb{R}^m)$ and its corresponding state trajectory $x(\cdot) \in PC^1 ([a, b]; \mathbb{R}^n)$, solution to problem

\[
J[x(\cdot), u(\cdot)] = \int_a^b w(t)L(t, x(t), u(t)) dt.
\]

4.1. Continuity of solutions of control differential equations

Now, we consider the following control differential equation:
\[
\dot{A} D_{a_+}^{\alpha, \beta} x(t) = f(t, x(t), u(t)), \quad t \in [a, b],
\]
where $x(\cdot) \in PC^1 ([a, b]; \mathbb{R}^n)$ represents the state trajectory of (4), $u(\cdot) \in PC ([a, b]; \mathbb{R}^m)$ is the control input, and function $f$ is assumed to be Lipschitz-continuous with respect to both $x$ and $u$. 

Notation 12. We set $w(\cdot) := \frac{(b - \cdot)^{a-1} A \left( (b - \cdot)^{\beta} \right)}{\Gamma(\alpha) A(1)}$, so that
\[
J[x(\cdot), u(\cdot)] = \int_a^b w(t)L(t, x(t), u(t)) dt.
\]
Lemma 13. Let us denote by \( u^* \) a precise control input to (1), and \( x^* \) its associated state trajectory. Suppose that \( u^\varepsilon \) is a control perturbation around the control input \( u^* \), that is, for all \( t \in [a,b] \), \( u^\varepsilon(t) = u^*(t) + \varepsilon h(t) \), where \( h(\cdot) \in PC([a,b];\mathbb{R}^m) \) is a variation function and \( \varepsilon \in \mathbb{R} \). Denote by \( x^\varepsilon \) its corresponding state trajectory, solution of

\[
\bar{A} D^\alpha_{a^+} x^\varepsilon(t) = f(t, x^\varepsilon(t), u^\varepsilon(t)), \quad x^\varepsilon(a) = x_a.
\]

Then, we have that \( x^\varepsilon \) converges to \( x^* \) when \( \varepsilon \) tends to zero.

Proof. By definition of the differential operator, we have

\[
\bar{A} D^\alpha_{a^+} x^\varepsilon(t) - \bar{A} D^\alpha_{a^+} x^*(t) = f(t, x^\varepsilon(t), u^\varepsilon(t)) - f(t, x^*(t), u^*(t)).
\]

By linearity of the operator and applying its inverse operation,

\[
x^\varepsilon - x^* = \bar{A} I^\alpha_{a^+} \left( f(t, x^\varepsilon(t), u^\varepsilon(t)) - f(t, x^*(t), u^*(t)) \right),
\]

where \( \bar{A}_\Gamma \cdot A_\Gamma = 1 \). Next,

\[
\|x^\varepsilon - x^*\| \leq \bar{A} I^\alpha_{a^+} \left( \|f(t, x^\varepsilon(t), u^\varepsilon(t)) - f(t, x^*(t), u^*(t))\| \right)
\]

and, by the Lipschitz-property of \( f \) and the non-decreasing property of \( \bar{A} I^\alpha_{a^+} \),

\[
\|x^\varepsilon - x^*\| \leq \bar{A} I^\alpha_{a^+} \left( L_1 \|x^\varepsilon - x^*\| + \bar{A} I^\alpha_{a^+} \left( L_2 |\varepsilon h(t)| \right) \right) = L_2 |\varepsilon| \bar{A} I^\alpha_{a^+} \left( \|h(t)\| \right) + L_1 \bar{A} I^\alpha_{a^+} \left( \|x^\varepsilon - x^*\| \right).
\]

Now, applying Grönwall’s inequality (Theorem 2), it follows that

\[
\|x^\varepsilon - x^*\| \leq L_2 |\varepsilon| \bar{A} I^\alpha_{a^+} \left( \|h(t)\| \right) + \sum_{k=1}^{\infty} L_1 \left[ \bar{A} I^\alpha_{a^+} \left( \|h(t)\| \right) \right]^k
\]

\[
= |\varepsilon| L_2 \left[ \bar{A} I^\alpha_{a^+} \left( \|h(t)\| \right) \right] + \sum_{k=1}^{\infty} L_1 \left( \bar{A} I^\alpha_{a^+} \left( \|h(t)\| \right) \right)^k.
\]

Finally, due to Theorem 2 if \( L_1 < \frac{1}{T^\alpha} \), then the series

\[
\sum_{k=1}^{\infty} L_1 \left( \bar{A} I^\alpha_{a^+} \left( \|h(t)\| \right) \right)^k
\]

converges and, by taking the limit when \( \varepsilon \to 0 \), we obtain the desired result, that is, \( x^\varepsilon(t) \to x^*(t) \) for all \( t \in [a,b] \).
4.2. Differentiability of solutions

The following result is useful for the proof of our necessary optimality condition in Section 4.3.

**Lemma 14** (Differentiability of perturbed trajectories). There exists a function \( \eta \) defined on \([a, b]\) such that

\[ x^*(t) = x^*(t) + \alpha \eta(t) + o(\epsilon). \]

**Proof.** Since \( f \in C^1 \), we have for any fixed index \( j \) that

\[
f_j(t, x^*, u^*) = f_j(t, x^*, u^*) + \sum_{i=1}^{n} (x_i^* - x_i^*) \frac{\partial f_j(t, x^*, u^*)}{\partial x} + \sum_{i=1}^{m} (u_i^* - u_i^*) \frac{\partial f_j(t, x^*, u^*)}{\partial u} + o(|x_i^* - x_i^*|, |u_i^* - u_i^*|).
\]

Observe that \( u_i^* - u_i^* = \epsilon h_i(t) \) and \( u_i^* \to u_i^* \) when \( \epsilon \to 0 \) and, by Theorem 13, we have \( x_i^* \to x_i^* \) when \( \epsilon \to 0 \). Thus, the residue term can be expressed in terms of \( \epsilon \) only, that is, the residue is \( o(\epsilon) \). Therefore, for all indexes \( j \in \{1, \ldots, n\} \), we have the vector expression

\[
f(t, x^*, u^*) = f(t, x^*, u^*) + (x^* - x^*) J_x(t, x^*, u^*) + \epsilon h(t) J_u(t, x^*, u^*) + o(\epsilon),
\]

where \( J_x(t, x^*, u^*) \) and \( J_u(t, x^*, u^*) \) are the Jacobian matrices of \( f \) with respect to \( x \) and \( u \), respectively, and evaluated at \((t, x^*, u^*)\), that is,

\[
J_x(t, x^*, u^*) = \left( \frac{\partial f}{\partial x_{ij}} \right)_{1 \leq i, j \leq n} \quad \text{and} \quad J_u(t, x^*, u^*) = \left( \frac{\partial f}{\partial u_{ij}} \right)_{1 \leq i \leq m, 1 \leq j \leq n}.
\]

Next, we have

\[
\frac{\partial f}{\partial x_{ij}} = \frac{\partial}{\partial x_j} J_x(t, x^*, u^*) + \epsilon h(t) J_u(t, x^*, u^*) + o(\epsilon),
\]

and this leads to

\[
\lim_{\epsilon \to 0} \left[ \frac{\partial f}{\partial x_{ij}} (x^* - x^*) - \frac{(x^* - x^*)}{\epsilon} \cdot J_x(t, x^*, u^*) - h(t) \cdot J_u(t, x^*, u^*) \right] = 0,
\]

that is,

\[
\frac{\partial f}{\partial x_{ij}} \left( \lim_{\epsilon \to 0} \frac{x^* - x^*}{\epsilon} \right) = \lim_{\epsilon \to 0} \frac{(x^* - x^*)}{\epsilon} \cdot J_x(t, x^*, u^*) + h(t) \cdot J_u(t, x^*, u^*).
\]

It remains to prove the existence of the limit \( \lim_{\epsilon \to 0} \frac{x^* - x^*}{\epsilon} \). For this purpose, we set \( \eta := \lim_{\epsilon \to 0} \frac{x^* - x^*}{\epsilon} \). It is easy to see that the limit exists as solution of the following system of fractional differential equations:

\[
\begin{cases}
\frac{\partial}{\partial \epsilon} D_{a+}^{\alpha, \beta} \eta(t) = \eta(t) \cdot J_x(t, x^*, u^*) + h(t) \cdot J_u(t, x^*, u^*), \\
\eta(a) = 0.
\end{cases}
\]

This ends the proof. \( \square \)
4.3. Pontryagin’s maximum principle

The following result is a necessary optimality condition of Pontryagin type for problem (3).

**Theorem 15** (Pontryagin Maximum Principle for (3)). If \((x^*(\cdot), u^*(\cdot))\) is an optimal pair for \((3)\), then there exists \(\lambda_0 \in \{0, 1\}\) and \(\lambda(\cdot) \in PC^1([a, b]; \mathbb{R}^n)\), called the adjoint variables, such that the following conditions hold for all \(t\) in the interval \([a, b]\):

- **the nontriviality condition**
  \((\lambda_0, \lambda) \neq (0, 0); (6)\)

- **the optimality condition**
  \(\nabla_a H ((t, x^*(t), u^*(t), \lambda_0, \lambda(t)) = 0; (7)\)

- **the adjoint system**
  \(\hat{A}_{\cdot}R_{\cdot}D_{\cdot}^a\lambda(t) = \nabla_x H ((t, x^*(t), u^*(t), \lambda_0, \lambda(t)); (8)\)

- **the transversality condition**
  \(\hat{A}_{\cdot}L_{\cdot}I_{\cdot}^b\lambda(b) = 0; (9)\)

where function \(H, \) defined by
\[
H(t, x, u, \lambda_0, \lambda) = \lambda_0 w(t)L(t, x, u) + \sum_{j=1}^n \lambda_j f_j(t, x, u),
\]
is called the Hamiltonian.

**Proof.** Let \((x^*(\cdot), u^*(\cdot))\) be solution of the problem, \(h(\cdot) \in PC([a, b]; \mathbb{R}^n)\) be a variation, and \(\epsilon\) a real constant. Define \(u^\epsilon(t) = u^*(t) + \epsilon h(t)\), so that \(u^\epsilon \in PC([a, b]; \mathbb{R}^m)\). Let \(x^\epsilon\) be the corresponding trajectory to the control \(u^\epsilon\), meaning it is the state solution to the following system:
\[
\hat{A}_{\cdot}D_{\cdot}^{a-b}x^\epsilon(t) = f(t, x^\epsilon(t), u^\epsilon(t)), \quad x^\epsilon(a) = x_a. (10)
\]

Note that \(u^\epsilon(t) \to u^*(t)\) for all \(t \in [a, b]\) whenever \(\epsilon \to 0\). Further, the first derivative of \(u^\epsilon\) with respect to \(\epsilon\) at \(\epsilon = 0\) can be obtained as
\[
\left.\frac{\partial u^\epsilon(t)}{\partial \epsilon}\right|_{\epsilon=0} = h(t). (11)
\]

Similarly, by Lemma \([13]\) it follows that \(x^\epsilon(t) \to x^*(t)\), for each fixed \(t\), as \(\epsilon \to 0\). Also, from Lemma \([14]\) the first derivative of \(x^\epsilon\) with respect to \(\epsilon\) at \(\epsilon = 0\),
\[
\left.\frac{\partial x^\epsilon(t)}{\partial \epsilon}\right|_{\epsilon=0}. (12)
\]
exists for each $t$. The objective functional at $(x^*, u^*)$ is

$$\mathcal{J}(x^*, u^*) = \int_a^b Lw(t) \left( t, x^*(t), u^*(t) \right) dt.$$  

This functional can be extended to handle abnormal multipliers in the following way:

$$\mathcal{J}_{\lambda_0}(x^*, u^*) = \lambda_0 \mathcal{J}(x^*, u^*) = \int_a^b \lambda_0 w(t) L \left( t, x^*(t), u^*(t) \right) dt,$$

where $\lambda_0 \in \{0, 1\}$ and the case $\lambda_0 = 0$ is known as the abnormal case. Next, we introduce the adjoint vector function $\lambda$. Let $\lambda(\cdot)$ be in $PC^1([a, b]; \mathbb{R}^n)$, to be determined. By the integration by parts formula, we have, for any fixed index $j$, that

$$\int_a^b \lambda_j(t) \frac{\partial}{\partial x_j} x_j(t) dt = \left[ x_j(t) \lambda_j(t) \right]_a^b - \int_a^b x_j(t) \lambda_j'(t) dt. $$

and, summing up for $j = 1, \ldots, n$, we get the following expression in inner product form:

$$\int_a^b \lambda(t) \cdot \frac{\partial}{\partial x} x'(t) dt = \int_a^b x'(t) \cdot \lambda(t) dt - x'(b) \cdot \lambda(b) + x'(a) \cdot \lambda(a).$$

Adding this zero to the expression $\mathcal{J}_{\lambda_0}(x^*, u^*)$ gives

$$\phi(\epsilon) = \mathcal{J}_{\lambda_0}(x^*, u^*) = \int_a^b \left[ \lambda_0 w(t) L \left( t, x^*(t), u^*(t) \right) + \lambda(t) \cdot \frac{\partial}{\partial x} x(t) \right. + \left. -x'(t) \cdot \frac{\partial}{\partial x} x'(t) \lambda(t) \right] dt - x'(b) \cdot \lambda(b) + x'(a) \cdot \lambda(a),$$

which by (10) is equivalent to

$$\phi(\epsilon) = \mathcal{J}_{\lambda_0}(x^*, u^*) = \int_a^b \left[ \lambda_0 w(t) L \left( t, x^*(t), u^*(t) \right) + \lambda(t) \cdot f(t, x^*(t), u^*(t)) \right. + \left. -x'(t) \cdot \lambda(t) \right] dt - x'(b) \cdot \lambda(b) + x'(a) \cdot \lambda(a).$$

The overall optimization problem is now reduced to the study of function $\phi$ and, for this purpose, we must have $\phi$ non identically zero ($\phi \neq 0$). To ensure this, it is sufficient to consider that $(\lambda_0, \lambda(t)) \neq (0, 0)$, meaning that both multipliers can not vanish simultaneously. The maximum of $\mathcal{J}_{\lambda_0}$ occurs at $(x^*, u^*) = (x^0, u^0)$, so the derivative of $\phi(\epsilon)$ with respect to $\epsilon$ at $\epsilon = 0$ must vanish, that is,

$$0 = \phi'(0) = \frac{d}{d\epsilon} J_{\lambda_0}(x^*, u^*) |_{\epsilon=0}$$

$$= \int_a^b \left[ \lambda_0 w(t) \left( \sum_{i=1}^n \frac{\partial L}{\partial x_i} \frac{\partial x_i}{\partial \epsilon} |_{\epsilon=0} + \sum_{i=1}^m \frac{\partial L}{\partial u_i} \frac{\partial u_i}{\partial \epsilon} |_{\epsilon=0} \right) \right].$$
where using (11) and (12), and rearranging the terms, we obtain that

\[
H = \left( \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} \frac{\partial x_i^*(t)}{\partial \epsilon} \bigg|_{\epsilon=0} + \sum_{i=1}^{m} \frac{\partial f_i}{\partial u_i} \frac{\partial u_i^*(t)}{\partial \epsilon} \bigg|_{\epsilon=0} \right) + \cdots + \lambda_n(t) \sum_{i=1}^{n} \frac{\partial f_n}{\partial x_i} \frac{\partial x_i^*(t)}{\partial \epsilon} \bigg|_{\epsilon=0} + \lambda_n(t) \sum_{i=1}^{m} \frac{\partial f_n}{\partial u_i} \frac{\partial u_i^*(t)}{\partial \epsilon} \bigg|_{\epsilon=0} - \cdots - A_{RL} D_{b-}^{1-\alpha,\beta} \lambda_n(t) \frac{\partial x_n^*(t)}{\partial \epsilon} \bigg|_{\epsilon=0} \]

where the partial derivatives of \( L \) and \( f = (f_1, \cdots, f_n) \) with respect to \( x \) and \( u \), \( x = (x_1, \cdots, x_n) \) and \( u = (u_1, \cdots, u_m) \), are evaluated at \( (t, x^*(t), u^*(t)) \). Thus, using (11) and (12), and rearranging the terms, we obtain that

\[
\int_a^b \left[ \left( \lambda_0 w(t) \nabla_x L + \lambda(t) J_x - A_{RL} D_{b-}^{1-\alpha,\beta} \lambda(t) \right) \frac{\partial x^*(t)}{\partial \epsilon} \bigg|_{\epsilon=0} \right. \\
+ \left. \lambda_0 w(t) \nabla_u L + \lambda(t) J_u \right] \frac{\partial u^*(t)}{\partial \epsilon} \bigg|_{\epsilon=0} \right] dt - \tilde{\lambda} I_{b-}^{1-\alpha,\beta} \lambda(b) \frac{\partial x^*(b)}{\partial \epsilon} \bigg|_{\epsilon=0} = 0,
\]

where \( J_x \) and \( J_u \) are the Jacobian matrices of \( f \), respectively with respect to \( x \) and \( u \) evaluated at \( (t, x^*(t), u^*(t)) \), that is,

\[
J_x = \left( \frac{\partial f_i}{\partial x_i} \right)_{1 \leq i, j \leq n} \quad \text{and} \quad J_u = \left( \frac{\partial f_i}{\partial u_i} \right)_{1 \leq i \leq m, 1 \leq j \leq n}.
\]

Setting \( H = \lambda_0 w(t) L + \lambda \cdot f \), it follows that

\[
\int_a^b \left[ \left( \nabla_x H - A_{RL} D_{b-}^{1-\alpha,\beta} \lambda(t) \right) \frac{\partial x^*(t)}{\partial \epsilon} \bigg|_{\epsilon=0} + \nabla_u H \cdot h(t) \right] dt \\
- \tilde{\lambda} I_{b-}^{1-\alpha,\beta} \lambda(b) \frac{\partial x^*(b)}{\partial \epsilon} \bigg|_{\epsilon=0} = 0,
\]

where the partial derivatives of \( H \) are evaluated at \( (t, x^*(t), u^*(t), \lambda_0, \lambda(t)) \). Now, choosing

\[
A_{RL} D_{b-}^{1-\alpha,\beta} \lambda(t) = \nabla_x H, \quad \text{with} \quad \tilde{\lambda} I_{b-}^{1-\alpha,\beta} \lambda(b) = 0,
\]

that is, given the adjoint equation (8) and the transversality condition (9), one obtains

\[
\int_a^b \nabla_u H \cdot h(t) = 0
\]

and, by the fundamental lemma of the calculus of variations [24], we have the optimality condition (7):

\[
\nabla_u H \cdot (t, x^*(t), u^*(t), \lambda_0, \lambda(t)) = 0.
\]

This concludes the proof.
**Example 16.** Let us consider the following problem:

\[
\int_0^2 w(t) \left[ \| x(t) - (t^2, e^{1-t}) \|^2 + \| u(t) - (t^2 e^{-t}, -t^6) \|^2 \right] dt \rightarrow \max,
\]

\[
\begin{align*}
\frac{\mathcal{D}D_{a+}^{\frac{1}{2}}}{\mathcal{D}C_{a+}} x_1(t) &= u_1(t), \\
\frac{\mathcal{D}D_{a+}^{\frac{1}{2}}}{\mathcal{D}C_{a+}} x_2(t) &= u_2^2(t) + 2t^6 u_2(t), \\
x_1(0) &= 0, \\
x_2(0) &= e,
\end{align*}
\]

where \( w(t) = \frac{(2-t)^{\frac{1}{2}}}{\pi} A ((2-t)^{\pi}) \). In order to apply Theorem 15, let us define the normal Hamiltonian:

\[
H(t, x, u, \lambda) = -w(t) \left[ \| x(t) - (t^2, e^{1-t}) \|^2 + \| u(t) - (t^2 e^{-t}, -t^6) \|^2 \right] + \lambda_1 u_1 + \lambda_2 u_2^2 + 2 \lambda_2 t^6 u_2.
\]

We have:

- **by the optimality condition,**
  \[
  \begin{align*}
  \lambda_1(t) &= -2 \left( u_1(t) - t^2 e^{-t} \right), \\
  2 \left( u_2(t) + t^6 \right) (1 + \lambda_2(t)) &= 0;
  \end{align*}
  \]

- **by the adjoint system,**
  \[
  \begin{align*}
  \frac{\mathcal{D}D_{a+}^{\frac{1}{2}}}{\mathcal{D}C_{a+}} x_1(t) &= 2 \left( x_1(t) - t^2 \right), \\
  \frac{\mathcal{D}D_{a+}^{\frac{1}{2}}}{\mathcal{D}C_{a+}} x_2(t) &= 2 \left( x_2(t) - e^{1-t} \right);
  \end{align*}
  \]

- **by the transversality condition,**
  \[
  \begin{align*}
  \frac{\mathcal{D}D_{a+}^{\frac{1}{2}}}{\mathcal{D}C_{a+}} \lambda_1(2) &= 0, \\
  \frac{\mathcal{D}D_{a+}^{\frac{1}{2}}}{\mathcal{D}C_{a+}} \lambda_2(2) &= 0.
  \end{align*}
  \]

Therefore, we observe that equations (15), (16) and (17) are satisfied by the following pair of functions: \( x(t) = (t^2, e^{1-t}) \) and \( u(t) = (t^2 e^{-t}, -t^6) \). These are the Pontryagin extremals of the problem.

**4.4. Calculus of variations**

The problem of the calculus of variations involving fractional operators with a general analytic kernel consists to find a piecewise continuously differentiable curve \( x \) solution of

\[
\mathcal{J}[x(\cdot)] = \frac{1}{\Gamma(\alpha)A(1)} \frac{\mathcal{D}D_{a+}^{\alpha,\beta}}{\mathcal{D}C_{a+}} \left[ L \left( \cdot, x(\cdot), \frac{\mathcal{D}D_{a+}^{\alpha,\beta}}{\mathcal{D}C_{a+}} x(\cdot) \right) \right] (b) \rightarrow \max,
\]

subject to

\[
x(a) = x_a, \quad x(b) = x_b,
\]

(18)
where function $L$ is piecewise continuously differentiable, that is,

$$L \in PC^1([a, b] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}).$$

The aforementioned variational problem (18) is a special case of our optimal control problem (3).

**Corollary 17.** If $x^*$ is solution to problem (18), then it satisfies the following Euler–Lagrange equation:

$$A_{RL}D_{b^+}^{\alpha, \beta} \left[ w(t) \nabla_u L \left( t, x^*(t), A_{CD}^{\alpha, \beta} x^*(t) \right) \right] + w(t) \nabla_x L \left( t, x^*(t), A_{CD}^{\alpha, \beta} x^*(t) \right) = 0.$$  

(19)

**Proof.** It is easy to see that the optimal control problem (3) coincides, in the particular case when $A_{CD}^{\alpha, \beta} x(t) = u(t)$, with the problem of the calculus of variations defined in (18). Next, we define the Hamiltonian function

$$H(t, x, u, \lambda_0, \lambda) = \lambda_0 w(t) L(t, x, u) + \lambda \cdot u$$

and, by application of Theorem 15, we have:

- from optimality condition (7),

$$\lambda(t) = -\lambda_0 w(t) \nabla_u L((t, x^*(t), u^*(t)).$$

(20)

It follows that $\lambda_0 = 0$ implies $\lambda(t) \equiv 0$, which is not a possibility by the non-triviality condition (8). Therefore, $\lambda_0 = 1$.

Following the adjoint system (8), we have

$$A_{RL}D_{b^+}^{\alpha, \beta} \lambda(t) = w(t) \nabla_x L((t, x^*(t), u^*(t)).$$

(21)

Combining (20) and (21), we obtain the Euler–Lagrange equation (19). \qed

**Isoperimetric problems**

An important class of variational problems are the fractional isoperimetric problems [25, 26]. The isoperimetric problem involving fractional operators with a general analytic kernel consists to find a piecewise continuously differentiable curve $x$ solution of

$$\mathcal{J}[x(\cdot)] = \frac{1}{\Gamma(\alpha) A(1)} A_{a+}^{\alpha, \beta} \left[ L \left( \cdot, x(\cdot)^{\alpha, \beta} x(\cdot) \right) \right] (b) \longrightarrow \max,$$

subject to

$$A_{a+}^{\alpha, \beta} y \left( t, x(t), A_{CD}^{\alpha, \beta} x(t) \right) = l,$$

$$x(a) = x_a, \quad x(b) = x_b.$$  

(22)

The isoperimetric problem (22) is also a particular case of our optimal control problem (3).
Corollary 18. If \( x^* \) is solution to problem (22), then it satisfies the following fractional differential equation:

\[
\frac{A_{RL}D_{b^-}^{\alpha, \beta}}{\lambda} \left[ \nabla_u \left( w(t) \hat{L} + \lambda \hat{y} \right) \right] + \nabla_x \left( w(t) \hat{L} + \lambda \hat{y} \right) = 0,
\]

where \( \hat{L} = L \left( t, x^*(t), \frac{CD_{a^+}^{\alpha, \beta}}{\lambda} x^*(t) \right) \), \( \hat{y} = y \left( t, x^*(t), \frac{CD_{a^+}^{\alpha, \beta}}{\lambda} x^*(t) \right) \) and \( \lambda \) is a nonzero real constant.

Proof. We rewrite the isoperimetric problem (22) in the following way:

\[
J[x(\cdot), u(\cdot)] = \frac{1}{\Gamma(\alpha)A(1)} \frac{\hat{L}}{\lambda} [L (\cdot, x(\cdot), u(\cdot))] (b) \rightarrow \max,
\]

subject to

\[
\begin{align*}
\frac{CD_{a^+}^{\alpha, \beta}}{\lambda} x(t) &= u(t), \\
\frac{CD_{a^+}^{\alpha, \beta}}{\lambda} z(t) &= y \left( t, x(t), u(t) \right); \\
x(a) &= x_a, \quad x(b) = x_b, \\
z(a) &= 0, \quad z(b) = l.
\end{align*}
\]

Hence, it is easy to see that the isoperimetric problem (22) is a particular case of problem (3). The Hamiltonian function is here given by

\[
H(t, x, u, \lambda_0, \lambda) = \lambda_0 w(t)L(t, x, u) + \mu \cdot u + \lambda y
\]

and, by application of Theorem 15, we have:

- from the optimality condition (7), that
  \[
  \mu = -\lambda_0 w(t)\nabla_L L(t, x^*(t), u^*(t)) - \lambda \nabla_y y(t, x^*(t), u^*(t));
  \]

from the adjoint system (8), that

\[
\frac{A_{RL}D_{b^-}^{\alpha, \beta}}{\lambda} \left( \frac{\mu}{\lambda} \right) = \begin{pmatrix} \lambda_0 w(t)\nabla_x L(t, x^*(t), u^*(t)) + \lambda \nabla_y y(t, x^*(t), u^*(t)) \\ 0 \end{pmatrix}, \tag{24}
\]

Now, following the non-triviality condition (9), if \( \lambda_0 = 0 \), then

\[
\mu = -\lambda \nabla_y y(t, x^*(t), u^*(t))
\]

and

\[
\frac{A_{RL}D_{b^-}^{\alpha, \beta}}{\lambda} = 0.
\]

As a consequence, \( \lambda \) must be a non-zero constant in order to have

\[
(\lambda_0, \mu, \lambda) \neq (0, 0, 0).
\]

Therefore, combining (23) and (24), we obtain that

\[
\frac{A_{RL}D_{b^-}^{\alpha, \beta}}{\lambda} \left[ \nabla_u \left( w(t) \hat{L} + \lambda \hat{y} \right) \right] + \nabla_x \left( w(t) \hat{L} + \lambda \hat{y} \right) = 0.
\]

The proof is complete.
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