Symbol, Surface operators and $S$-duality

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ABSTRACT: We study rigid surface operators in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theories with gauge groups SO($n$) and Sp($2n$). Using maps $X_S$ and $Y_S$ between these two theories, Wyllard made explicit proposals for how the $S$-duality map should act on certain subclasses of surface operators. We study the maps $X_S$ and $Y_S$ further and simplify the construction of symbol invariant of rigid surface operators by a convenient trick. By consistency checks, we recover and extend the $S$-duality maps proposed by Wyllard. We find new subclasses of rigid surface operators related by $S$-duality. We try to explain the exceptions of $S$-duality maps. We also discuss the extension of the techniques used in the $B_n/C_n$ theories to the $D_n$ theories.
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1 Introduction

Surface operators are supported on a two-dimensional submanifold of spacetime, which are natural generalisations of the Wilson and ’t Hooft operators. In [1], Gukov and Witten initiated a study of surface operators in $\mathcal{N} = 4$ super Yang-Mills theories.

$S$-duality for certain subclass of surface operators is discussed in [2][4]. The $S$-duality conjecture [8] for the $\mathcal{N} = 4$ super Yang-Mills theories in four dimensions asserts that
the theory with gauge group $G$ and complexified coupling constant $\tau$ is equivalent to the theories arising from the transformations $S$ and $T$:

$$S : (G, \tau) \rightarrow (G^L, -1/n_g\tau),$$
$$T : (G, \tau) \rightarrow (G, \tau + 1),$$

where $G^L$ denotes the Langlands dual group of $G$. $n_g$ is 2 for $F_4$, 3 for $G_2$, and 1 for other semisimple classical groups [1]. For example, the Langlands dual groups of $\text{Spin}(2n+1)$ are $\text{Sp}(2n)/\mathbb{Z}_2$. And the langlands dual groups of $\text{SO}(2n)$ are themselves.

In [3], Gukov and Witten extended their earlier analysis [2] of surface operators in the $\mathcal{N} = 4$ super-Yang-Mills theories. They identified a subclass of surface operators called 'rigid' surface operators, preserving half the supersymmetries. Rigid surface operators appear not to suffer from quantum ambiguities, expected to be closed under $S$-duality. There are two types rigid surface operators: unipotent and semisimple. The rigid semisimple surface operators are labelled by pairs of partitions. Unipotent rigid surface operators arise when one of the partitions is empty.

In [4], some proposals for the $S$-duality maps related to rigid surface operators in the $B_n(\text{SO}(2n+1))$ and $C_n(\text{Sp}(2n))$ theories were made. These proposals involved all unipotent rigid surface operators as well as certain subclasses of rigid semisimple operators. A problematic mismatch in the total number of rigid surface operators between the $B_n$ and the $C_n$ theories was pointed out by Wyllard.

In this paper, we attempt to extend the analysis in [4]. With no noncentral rigid conjugacy classes in the $A_n$ theory, we do not discuss surface operators in this case. We omit the discussion of the exceptional groups, which are more complicated. We will focus on theories with gauge groups $\text{SO}(2n)$ and the gauge groups $\text{Sp}(2n)$ whose Langlands dual group is $\text{SO}(2n+1)$. In section 2, we review the construction of rigid surface operators given in [3]. We discuss some mathematical results and definitions as preparation. In section 3, we focus on the invariants of surface operators which are unchanged under the $S$-duality map. We review the symbol invariant proposed in [4]. In particular, we find a simple rule to determine the contribution to symbol of an even row in a partition. In section 4, we introduce two maps $X_S$ and $Y_S$ preserving symbol invariant. Using these maps and the construction of symbols for partitions with only even rows, we simplify the computation of symbols for general rigid surface operators in the $B_n$, $C_n$, and $D_n$ theories.

In section 5, we review the dual pair of rigid surface operators proposed in [4]. By consistency check, we recover the $S$-duality maps proposed by Wyllard and find new subclasses of rigid surface operators related by $S$-duality. As an example, we tabulate all rigid surface operators and their associated invariants in the $\text{SO}(13)$ and $\text{Sp}(12)$ theories. We try to explain exceptions of $S$-duality maps. We find that there are general characteristics in these proposals and make a conjecture.

In section 6, we extend the discussion of the techniques used in the $B_n/C_n$ theories to the $D_n$ theories. We tabulate all rigid surface operators and their associated invariants in the $\text{SO}(12)$ to illustrate our proposals.

In appendix A, we summarizes relevant facts.
2 Surface operators in $\mathcal{N} = 4$ Super-Yang-Mills

In this section, we introduce the relevant backgrounds for our discussion. We closely follow paper [4] to which we refer the reader for more details.

We consider $\mathcal{N} = 4$ super-Yang-Mills theory on $\mathbb{R}^4$ with coordinates $x^0, x^1, x^2, x^3$. The field content: a gauge field as 1-form, $A_\mu$ ($\mu = 0, 1, 2, 3$), four Majorana spinors $\psi^a$ ($a = 1, 2, 3, 4$) and six real scalars, $\phi_I$ ($I = 1, \ldots, 6$). All fields take values in the adjoint representation of the gauge group $G$. Surface operators are defined by prescribing a certain singularity structure of fields near the surface on which the operator is supported. We consider surface operators supported on a $\mathbb{R}^2$ submanifold $D$ which lie at $x_2 = x_3 = 0$.

Since the singularity must be chosen to be compatible with half of supersymmetries, the combinations $A = A_2 \, dx^2 + A_3 \, dx^3$ and $\phi = \phi_2 \, dx^2 + \phi_3 \, dx^3$ must obey Hitchin’s equations [3]

\[
F_A - \phi \wedge \phi = 0, \quad \text{(2.1)}
\]

\[
d_A \phi = 0, \quad d_A \ast A = 0.
\]

A surface operator is defined as a solution to these equations with a prescribed singularity along the surface $\mathbb{R}^2$.

For the superconformal surface operator, setting $x_2 + ix_3 = re^{i\theta}$, the most general possible rotation-invariant Ansatz for $A$ and $\phi$ is

\[
A = a(r) \, d\theta, \quad \phi = -c(r) \, d\theta + b(r) \frac{dr}{r}.
\]  

Substituting this Ansatz into Hitchin’s equations (2.1) and defining $s = -\ln r$, one finds that equations (2.1) reduce to Nahm’s equations

\[
\frac{da}{ds} = [b, c], \quad \frac{db}{ds} = [c, a], \quad \frac{dc}{ds} = [a, b]
\]  

which imply that the constant elements $a$, $b$ and $c$ must commute. Surface operators of this type were treated in [1].

There is another way to obtain conformally invariant surface operator. Nahm’s equations (2.3) are solved by

\[
a = \frac{T_x}{s + 1/f}, \quad b = \frac{T_z}{s + 1/f}, \quad c = \frac{T_y}{s + 1/f},
\]  

where $T_x, T_y$ and $T_z$ are elements of the lie algebra $\mathfrak{g}$, spanning a representation of the $\text{su}(2)$ Lie algebra. These $T_i$’s are in the adjoint representation of the gauge group. The surface operator is actually conformal invariant if the function $f$ allowed to fluctuate.
Besides being defined as the singular solutions of Hitch’s equations, the surface operators can be characterised as the conjugacy class of the monodromy

\[ U = P \exp(\oint A), \tag{2.5} \]

where \( A = A + i\phi \) and the integration is around a circle near \( r = 0 \). Following from (2.1), it is easy to find that \( F = dA + A \wedge A = 0 \), which means that \( U \) is independent of deformations of the integration contour. For the surface operators (2.4), \( U \) becomes

\[ U = P \exp\left(\frac{2\pi}{s + 1/f} T_+\right), \tag{2.6} \]

where \( T_+ \equiv T_x + iT_y \) is nilpotent, corresponding to unipotent surface operator.

There are two types of conjugacy classes in a Lie group: unipotent and semisimple. Semisimple classes can also lead to surface operators. For a semisimple element \( S \) of the gauge group, we can obtain a surface operator with monodromy \( V = SU \), with the following construction. Near the surface \( D \), we require the fields which are solutions to Nahm’s equations satisfy the following restriction

\[ S \Upsilon(r, \theta) S^{-1} = \Upsilon(r, \theta + 2\pi). \tag{2.7} \]

From all the surface operators constructed from conjugacy classes, a subclass of surface operators called rigid surface operator is relatively easy to study. Since they are closed on the \( S \)-duality. The rigid surface operators are expected to be superconformal and not to depend on any parameters. A unipotent conjugacy classes is called rigid\(^1\) if its dimension is strictly smaller than that of any nearby orbit. All rigid orbits have been classified \([3][18]\). A semisimple conjugacy classes \( S \) is called rigid if the centraliser of such class is larger than that of any nearby class. Summary, surface operators are called rigid if they based on monodromies of the form \( V = SU \), with \( U \) is unipotent and rigid and \( S \) is semisimple and rigid.

### 2.1 Some mathematical definitions and results

From the above discussions, we see that a classification of unipotent and semisimple conjugacy classes is needed in order to find the possible surface operators. In this subsection, we will describe classification of rigid surface operators in the \( B_n(\text{SO}(2n+1)), C_n(\text{Sp}(2n)) \) and \( D_n(\text{SO}(2n)) \) theories in detail.

The \( T_+ \) in Eq.(2.6) can be described in block-diagonal basis as follows

\[ T_+ = \begin{pmatrix} T_1^{n_1} & & \\ & \ddots & \\ & & T_l^{n_l} \end{pmatrix}, \tag{2.8} \]

where \( T_1^{n_1} \) is the ‘raising’ generator of the \( n_k \)-dimensional irreducible representation of \( \text{su}(2) \). For the \( B_n, C_n \) and \( D_n \) theories, there are restrictions on the allowed dimensions

\(^1\)The rigid surface operators here correspond to strongly rigid operators in [4].
of the $su(2)$ irreps since $T_+$ should belong to the relevant gauge group. From the block-decomposition (2.8) we see that unipotent (nilpotent) surface operators are classified by the restricted partitions.

A partition $\lambda$ of the positive integer $n$ is a decomposition $\sum_{i=1}^{l} \lambda_i = n$ ($\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$). The integer $l$ is called the length of the partition. There is a one-to-one correspondence between partition and Young tableaux. For instance the partition $3^2 2^3 1$ corresponds to

\[
\begin{array}{cccccc}
 & & & \square & & \\
 & & \square & & \square & \\
 & \square & \square & & & \\
\square & \square & \square & \square & \square & \\
\end{array}
\]

For partitions $\lambda$ and $\kappa$, $\lambda + \kappa$ is the partition with parts $\lambda_i + \kappa_i$. Young diagrams occur in a number of branches of mathematics and physics. They are also useful to construct the eigenstates of Hamiltonian System [5] [6] [7].

Unipotent surface operators in the $B_n$ ($D_n$) theories are in one-to-one correspondence with partitions of $2n+1(2n)$ where all even integers appear an even number of times. Unipotent surface operators in the $C_n$ theories are in one-to-one correspondence with partitions of $2n$ for which all odd integers appear an even number of times. A partition in the $B_n$ or $D_n(C_n)$ theories is called rigid if it has no gaps (i.e. $\lambda_i - \lambda_{i+1} \leq 1$ for all $i$) and no odd (even) integer appears exactly twice. Rigid partition correspond to rigid surface operator.

For the $B_n$, $C_n$ and $D_n$ theories, the rigid semisimple conjugacy classes $S$ correspond to diagonal matrices with elements $+1$ and $-1$ along the diagonal [3]. The matrices $S$ break the gauge group to its centraliser at the Lie algebra level as follows

\[
\begin{align*}
\text{so}(2n+1) & \rightarrow \text{so}(2k+1) \oplus \text{so}(2n-2k), \\
\text{sp}(2n) & \rightarrow \text{sp}(2k) \oplus \text{sp}(2n-2k), \\
\text{so}(2n) & \rightarrow \text{so}(2k) \oplus \text{so}(2n-2k).
\end{align*}
\]

This implies that the rigid semisimple surface operators correspond to pairs of partitions $(\lambda'; \lambda'')$ in the $B_n$, $C_n$, and $D_n$ [3]. In the $B_n$ case, $\lambda'$ is a rigid $B_k$ partition and $\lambda''$ is a rigid $D_{n-k}$ partition. For the $C_n$ theories, $\lambda'$ is a rigid $C_k$ partition and $\lambda''$ is a rigid $C_{n-k}$ partition. For the $D_n$ theories, $\lambda'$ is a rigid $D_k$ partition and $\lambda''$ is a rigid $D_{n-k}$ partition. In the theories under consideration, the rigid unipotent surface operators can be seen as a limiting case $\lambda'' = 0$.

There is a close relationship between the pair of partition $(\lambda'; \lambda'')$ and Weyl group. For Weyl groups in the $B_n$, $C_n$, and $D_n$ theories both conjugacy classes and irreducible unitary representations are in one-to one correspondence with ordered pairs of partitions $[\alpha; \beta]$. $\alpha$ is a partition of $n_\alpha$ and $\beta$ is a partition of $n_\beta$, with $n_\alpha + n_\beta = n$. Though the conjugacy classes and unitary representations are parameterised by ordered pair of partitions there is no canonical isomorphism between the two sets.

The Kazhdan-Lusztig map is a map from the unipotent conjugacy classes of a simple group to the set of conjugacy classes of the Weyl group. This map can be extended to the case of rigid semisimple conjugacy classes [19]. The Springer correspondence is a injective map from the unipotent conjugacy classes of a simple group to the set of unitary representations of the Weyl group. For the classical groups the above two maps can be described.
explicitly in terms of partitions. They correspond to the invariants fingerprint and symbol of partitions in [18], respectively.

We recall the construction of symbol in [4]. For a partition \( \lambda \) in the \( B_n \) theory, we add \( l - k \) to the \( k \)th part of the partition. The following table illustrates this process

\[
\begin{array}{cccc}
\lambda_k & \lambda_1 & \lambda_2 & \cdots & \lambda_l \\
l - k & l - 1 & l - 2 & \cdots & 0 \\
l - k + \lambda_k & l - 1 + \lambda_1 & l - 2 + \lambda_2 & \cdots & \lambda_l
\end{array}
\]

For the terms in the sequence \( l - k + \lambda_k \), we arrange the odd parts in an increasing sequence \( 2f_i + 1 \) from right to left and the even parts in an increasing sequence \( 2g_i \) from right to left as follows

\[
\begin{array}{cccc}
\begin{array}{cccc}
2f_1 + 1 & \cdots & 2f_2 + 1 & 2f_1 + 1 \\
f_1 & \cdots & f_2 & f_1
\end{array} & \begin{array}{cccc}
2g_1 & \cdots & 2g_2 & 2g_1 \\
g_1 & \cdots & g_2 & g_1
\end{array}
\end{array}
\]

Next we calculate the terms \( \alpha_i = f_i - i + 1 \) and \( \beta_i = g_i - i + 1 \). Finally we write the symbol as follows \(^2\)

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 & \cdots \\
\beta_1 & \beta_2 & \cdots
\end{pmatrix}
\]

(2.11)

**Example:** For the \( B_{10} \) partition \( \lambda = 3^3 2^4 1^4 \), the symbol is

\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2
\end{pmatrix}
\]

(2.12)

We can view the two rows of the symbol as two partitions \([1^4, 21^4]\) which is the pair of partitions corresponding to a unitary representation of the Weyl group.

For the \( C_n \) case, we need append an extra 0 as the last part of the partition if the length of the partition is even. \( f_i \) and \( g_i \) are constructed as in the \( B_n \) case. Then we calculate the terms \( \alpha_i = g_i - i + 1 \) and the terms \( \beta_i = f_i - i + 1 \).

**Example:** For the \( C_{10} \) partition \( \lambda = 3^2 2^6 1^2 \), the symbol is

\[
\begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 2
\end{pmatrix}
\]

which is the same as (2.12)\(^3\).

For the \( D_n \) theory, we calculate \( f_i \) and \( g_i \) exactly as in the \( B_n \) case. Then we calculate the terms \( \alpha_i = g_i - i + 1 \) and \( \beta_i = f_i - i + 1 \). The number of \( f_i \) and \( g_i \) are equal in this case. Since the rigid partitions of the \( D_n \) theory which always have at least one part equal to 1, the last entry of the sequence \( \beta_i \) is zero. It is omitted finally.

\(^2\)According to the computation of symbol, the entries in the symbol corresponding to \( \lambda_1, \cdots, \lambda_i \) are independent of that corresponding to \( \lambda_{i+1}, \cdots, \lambda_l \).

\(^3\)In fact, these two partitions are related by the map \( X_S \) introduced in section 4.1.
3 Invariants of surface operators

Invariants of the surface operators do not change under the $S$-duality map [3][4]. The dimension $d$ of associated partition is the most basic invariant of a rigid surface operator. It is calculated as follows [3][18]:

\begin{align}
B_n &: d = 2n^2 + n - \frac{1}{2} \sum_k (s_k')^2 - \frac{1}{2} \sum_k (s_k'')^2 + \frac{1}{2} \sum_{k \text{ odd}} r_k' + \frac{1}{2} \sum_{k \text{ odd}} r_k'', \\
C_n &: d = 2n^2 + n - \frac{1}{2} \sum_k (s_k')^2 - \frac{1}{2} \sum_k (s_k'')^2 - \frac{1}{2} \sum_{k \text{ odd}} r_k' - \frac{1}{2} \sum_{k \text{ odd}} r_k'', \\
D_n &: d = 2n^2 - n - \frac{1}{2} \sum_k (s_k')^2 - \frac{1}{2} \sum_k (s_k'')^2 + \frac{1}{2} \sum_{k \text{ odd}} r_k' + \frac{1}{2} \sum_{k \text{ odd}} r_k''.
\end{align}

(3.1)

where $s_k'$ denotes the number of parts of $\lambda'$s that are larger than or equal to $k$ and $r_k'$ denotes the number of parts of $\lambda'$ that are equal to $k$. Similarly, we define $s_k''$ and $r_k''$ corresponding to $\lambda''$.

The invariant symbol is based on the Springer correspondence which can be extended to rigid semisimple conjugacy classes. One can construct the symbol of this rigid semisimple surface operator by calculating the symbols for both $\lambda'$ and $\lambda''$, then add the entries that are ‘in the same place’ of these two results. An example illustrates the addition rule:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & 2 & 2 & 3
\end{pmatrix}.
\]

(3.2)

There is another invariant fingerprint constructed from $(\lambda'; \lambda'')$ via the Kazhdan-Lusztig map. This invariant is a pair of partitions $[\alpha; \beta]$ associated with the Weyl group conjugacy class. It is checked that the symbol of a rigid surface operator contains the same amount of information as the fingerprint [4]. Compared with the fingerprint invariant, the symbol is much easier to be calculated and more convenient to find the $S$-duality maps of surface operators.

In [2], it was pointed that two discrete quantum numbers ‘center’ and ‘topology’ are interchanged under $S$-duality. A surface operator can detect topology then its dual should detect the centre and vice versa. However, there are some puzzles using these discrete quantum numbers to find duality pair [4]. There is another problem that the generating functions for the total number of rigid surface operators show that the number of rigid surface operators in the $B_n$ theory is larger than that in the $C_n$ theory [4], which was first observed in the $B_4/C_4$ theories [3].

In this paper, we ignore these problems for the moment. We focus on the symbol invariant to identify certain subsets of rigid surface operators and make proposals for how the $S$-duality map should acts on surface operators. Hopefully, our constructions will be helpful in making new insight to the surface operator.

3.1 Construction of symbol of partitions with only even rows

In this section, we propose rules to compute symbol of partitions with only even rows. Viewing the contribution to the symbol of each row independently, we find distinct regular patterns of contributions to symbol of even rows.
We derive these calculation rules through examples. First, we calculate the symbol of an old row \(1^{2l+1}\) in the \(B_n\) theory which can be seen as the first row of a partition. According to the definition of symbol, we have

\[
\begin{align*}
\lambda_k &: 1 \ 1 \cdots 1 \\
2l + 1 - k &: 2l \ 2l - 1 \cdots 0 \\
2l + 1 - k + \lambda_k &: 2l + 1 \ 2l \cdots 1
\end{align*}
\]  

(3.3)

which lead to the following sequences increasing from right to left

\[
\begin{align*}
2f_i + 1 &: 2l + 1 \ 2l - 1 \cdots 1 \\
f_i &: l \ l - 1 \cdots 0
\end{align*} \quad \begin{align*}
2g_i &: 2l \ 2l - 2 \cdots 2 \\
g_i &: l \ l - 1 \cdots 1
\end{align*}
\]

Thus we get two sequences \(\alpha_i = f_i - i + 1 : 0, 0, \ldots, 0\) and \(\beta_i = g_i - i + 1 : 1, 1, \ldots, 1\).

So the symbol of \(1^{2l+1}\) is

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & \lambda_{l+1} \\
\lambda_{l+1} & 0 & \cdots & 0
\end{bmatrix}.
\]  

(3.4)

Now we add an even row with \(2m(m < l)\) boxes, which lead to a partition \(2^{2m} \ 1^{2l+1-2m}\) in the \(B_n\) theory. We calculate the symbol as follows

\[
\begin{align*}
\lambda_k &: 2 \ 2 \cdots 2 \ 1 \cdots 1 \\
2l + 1 - k &: 2l \ 2l - 1 \cdots 2l - 2m + 1 \ 2l - 2m \cdots 0 \\
2l + 1 - k + \lambda_k &: 2l + 2 \ 2l + 1 \cdots 2l - 2m + 3 \ 2l - 2m + 1 \cdots 0.
\end{align*}
\]

Compared with (3.3), the first \(2m\) terms in the sequence \(2l + 1 - k\) is added one while the others are unchanged. Then we have

\[
\begin{align*}
2f_i + 1 &: 2l + 1 \ 2l - 1 \cdots 2l - 2m + 3 \ 2l - 2m + 1 \cdots 1 \\
f_i &: l \ l - 1 \cdots l - m + 1 \ l - m \cdots 0.
\end{align*}
\]

Compared with the sequences \(f_i\) corresponding to the first row, nothing is changed. However, each term of the sequences \(g_i\) is added one as follows

\[
\begin{align*}
2g_i &: 2l + 2 \ 2l \cdots 2l - 2m + 4 \ 2l - 2m \cdots 2 \\
g_i &: l + 1 \ l \cdots l - m + 2 \ l - m \cdots 1.
\end{align*}
\]

Then the entries of the top row and bottom row in the symbol are

\[
\begin{align*}
\alpha_i &= f_i - i + 1 : 0, 0, \ldots, 0, 0, \ldots, 0 \\
\beta_i &= g_i - i + 1 : 2, 2, \ldots, 2, 1, \ldots, 1.
\end{align*}
\]

So the symbol of these two rows \(2^{2m} \ 1^{2l+1-2m}\) is

\[
\begin{bmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & \lambda_{l+1} & 0 & \cdots & 0 \\
\lambda_{l+1} & 0 & \cdots & 0 & 1 & \cdots & \lambda_{l+1}
\end{bmatrix}.
\]  

(3.5)
Thus the symbol of the partition which can be formally decomposed into the sum of the contribution of each row
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{pmatrix}^{l+1}_{m}
\]

We get the entries
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{l} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{m}.
\] (3.7)

Formally, the symbol of \(2^{2m}1^{2l+1-2m}\) can be seen as the sum of the contribution of each row
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 2 & 2 & \cdots & 2 \\
\end{pmatrix}^{m}
\]

Compared to formula (3.4), the second row contribute to the symbol as follows
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 2 & 2 & \cdots & 2 \\
\end{pmatrix}^{m} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{l} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}_{m}.
\] (3.6)

Next, if we add another even row of \(2n\) boxes \((n < m)\) to the partition, the new partition is \(3^{2n}2^{2m-2n}1^{2l+1-2m}\). We calculate the symbol of this partition as follows
\[
\begin{pmatrix}
3 & \cdots & 3 & 2 & \cdots & 2 & 1 & \cdots & 1 \\
2l & \cdots & 2l - 2n + 1 & 2l - 2n & \cdots & 2l - 2m + 1 & 2l - 2m & \cdots & 0 \\
2l + 3 & \cdots & 2l - 2n + 4 & 2l - 2n + 2 & \cdots & 2l - 2m + 3 & 2l - 2m + 1 & \cdots & 1 \\
l + 1 & \cdots & l - n + 2 & l - n & \cdots & l - m + 1 & l - m & \cdots & 0 \\
1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
The above three rows correspond to sequences \(\lambda_k, 2l + 1 - k, 2l + 1 - k + \lambda_k\), respectively. And the following three rows correspond to sequences \(2f_i + 1, f_i, \alpha_i\), respectively.

We get the entries \(\beta_i\) of the bottom row of the symbol as follows
\[
\begin{pmatrix}
2g_i & 2l + 2 & \cdots & 2l - 2n + 4 & 2l - 2n + 2 & \cdots & 2l - 2m + 2 & 2l - 2m & \cdots & 2 \\
g_i & l + 1 & \cdots & l - n + 2 & l - n + 1 & \cdots & l - m + 2 & l - m & \cdots & 1 \\
\beta_i & 2 & \cdots & 2 & 2 & \cdots & 2 & 1 & \cdots & 1
\end{pmatrix}
\]

Thus the symbol of the partition \(3^{2n}2^{2m-2n}1^{2l+1-2m}\) is
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{pmatrix}^{n}_{m}
\]

which can be formally decomposed into the sum of the contribution of each row
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\end{pmatrix} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.
\] (3.8)
Compared with (3.7), the contribution to symbol of the second row added to the partition is
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
where the number of '1' is one half of the length of the row.

According to the above discussions, we claim that the contributions to symbol of even rows have formal additivity. Since the first row of a partition in the \(B_n\) theory is odd, we consider partitions with only even rows in the \(C_n\) and \(D_n\) theories.

**Rule 1** \(^4\) For a partition with only even rows in the \(C_n\) and \(D_n\) theories, the row with \(2m\) boxes contribute \(m\) '1' in sequence from right to left in the same row of symbol, while other entries of the symbol are '0'. The contribution to symbol of the adjoining even rows are formed in the same way, except that '1's occupy another row of symbol.

To construct symbol by this rule, the contribution to symbol of the first row is needed as an initial condition. In the appendix, we prove that the longest two rows of a rigid \(C_n\) partition are either even or odd. This pairwise pattern then continues. If the first row has \(2l\) boxes, its contribution to symbol can be calculated as (3.4)
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
l+1 & 0 & \cdots & 0 \\
1 & \cdots & 1
\end{pmatrix}.
\] (3.9)

For the \(D_n\) theory, we can prove that the longest row of a partition always contains even number of boxes. And the following two rows are either both of odd length or both of even length. This pairwise pattern then continues. The contribution to symbol of the first row with \(2l\) boxes is
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
l & 0 & \cdots & 0 \\
l+1 & \cdots & 1
\end{pmatrix}.
\] (3.10)

**Example:** partition \(3^22^11^2\) in the \(D_n\) theory,

\[
\begin{array}{cccc}
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet & \bullet
\end{array}
\] (3.11)

According to **Rule 1**, the symbol is
\[
\sigma_{(3^22^11^2)}^D = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 \end{pmatrix},
\] (3.12)

\(^4\)Addition of an odd row to a partition leads to a partition not in the same theory. We can determine the contribution to symbol of each odd row formally, but there are no simple addition rules for the contribution to symbol\([23][24]\).
where the superscript $D$ indicates it is a partition in the $D_n$ theory.

In the end, we would like to point out that the contribution to symbol of the odd rows and even rows of a partition are independent of each other. We illustrate this fact through an example. We add three equal rows $3^{2m}$ to a row with $2l + 1$ ($l > m$) boxes in the $B_n$ theory, with a decomposition $3^{2m} = 2^{2m} + 1^{2m}$. The decomposition means adding the two rows $2^{2m}$ firstly then adding the row $1^{2m}$. From the calculation of symbol, the rows $2^{2m}$ will not alter the parity of the sequences $(2l + 1) - k + \lambda_k$, but do turn $f_i$ to $f_i + 1$ and $g_i$ to $g_i + 1$. Thus its contribution to symbol is

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & \underbrace{1 \cdots 1}_m \\
0 & 0 & \cdots & 0 & \underbrace{1 \cdots 1}_m
\end{pmatrix}
\]

According to the Rule 1, the third row $1^{2m}$ contribute to symbol as follows

\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \underbrace{1 \cdots 1}_m
\end{pmatrix}
\]

which is equal to the contribution to symbol calculated by the formula (4.4) when it is added to the row $1^{2l+1}$ firstly. The contribution to symbol will not be changed when one replace rows $2^{2m}$ by another two rows with the same parity, since the entries in the symbol corresponding to $\lambda_1, \cdots, \lambda_l$ are independent of that corresponding to $\lambda_{l+1}, \cdots, \lambda_l$. This property and the fact that even or old rows occur pair following the first row in the $B_n$ theory are crucial in searching the $S$-duality pairs.

4 Construction of symbol of rigid partitions

In the previous section, we found an addition rule to calculate symbol of a partition with only even rows by summing the contributions of each row independently. In this section, we introduce two maps $X_S$ and $Y_S$ preserving symbol, which translate partition with only odd rows to partition with only even rows. Combining these maps and employing the addition rule, we can calculate the symbol of a partition in a simple manner.

4.1 Map $X_S$

First, we introduce some concepts. The transpose partition $\lambda^t$ is obtained by interchanging the roles of the rows and columns of the Young tableau corresponding to $\lambda$. For instance

\[
\begin{pmatrix}
\begin{array}{ccc}
\otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes
\end{array}
\end{pmatrix}^t = \begin{pmatrix}
\begin{array}{ccc}
\otimes & \otimes & \otimes \\
\otimes & \otimes & \otimes
\end{array}
\end{pmatrix}
\]

A partition $\lambda$ is called special if its transpose partitions satisfy following condition

\[
\begin{array}{ll}
B_n: & \lambda^t \text{ is orthogonal}, \\
C_n: & \lambda^t \text{ is symplectic}, \\
D_n: & \lambda^t \text{ is symplectic}.
\end{array}
\]

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It is easy to find that all rows of a rigid special partition are odd in the $B_n$ theory. On the contrary, all rows in the Young tableau corresponding to a rigid special partition are even in the $C_n$ and $D_n$ cases.

In [3], it is pointed out that the special rigid unipotent surface operators in the $B_n$ and $C_n$ theories are related by $S$-duality. The proposed $S$-duality map $X_S$ performs in the following way

$$X_S : \begin{array}{c} m^{2n_m+1}(m - 1)^{2n_m-1}(m - 2)^{2n_m-2} \cdots 2^{n_2} 1^{2n_1} \\
\end{array} \mapsto \begin{array}{c} m^{2n_m}(m - 1)^{2n_m-1+2}(m - 2)^{2n_m-2-2} \cdots 2^{n_2+2} 1^{2n_1-2}. \end{array}$$

Here $m$ has to be odd such that the partition on the left is in the $B_n$ theory. It is clear that the map is a bijection so that $X_S^{-1}$ is well defined.

The map $X_S$ can be described as removing one box from the end of the $(2k+1)th$ row in the Young tableau and taking this box to the end of the $(2k)th$ row as shown in Fig.1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{The black boxes of the left Young tableau are removed. And then they are putted at the end of the rows below, denoted as gray boxes. Under the map $X_s$, old rows become even rows.}
\end{figure}

Obviously, this map preserves the rigidity conditions. Note that the black boxes at the end of the first row disappear. So the number of boxes in the left Young tableau is one more than that of boxes in the right Young tableau, which is consistent with the fact that the right partition is in the $C_n$ theory.

The map $X_S$ preserves symbol. The symbol invariant can be calculated by the definition

$$\begin{pmatrix}
0 \cdots 0 & 1 \cdots 1 & 1 \cdots 1 \\
n_1 & 1 \cdots 1 & 2 \cdots 2 \\
n_3 & n_2 & n_3 \\
\end{pmatrix}$$

which is consistent with the result computed by Rule 1 in the previous section. When $m = 1$, this map is defined as

$$X_S : 1^{2n+1} \mapsto 1^{2n}.$$ 

Acting on an empty set, the inverse map $X_S^{-1}$ give the partition ’1’

$$X_S^{-1} : \emptyset \mapsto 1.$$ 

In [4], Wyllard proposed an algorithm to construct dual of unipotent operators in the $B_n$ theory: Firstly, split the Young tableau $\rho$ into tableau $\rho_{\text{even}}$ constructed from even rows only and one $\rho_{\text{odd}}$ constructed from the old rows only. Then take the map $X_S$ turns $\rho_{\text{odd}}$ to a partition with only even rows. While $\rho_{\text{even}}$ is left unchanged. Finally, the duality
partition corresponding to \( \rho \) in \( C_n \) theory is \((X_S \rho_{\text{odd}}, \rho_{\text{even}})\). This duality map is denoted by \( WB \) preserving the symbol
\[
WB : (\lambda, \emptyset)_B \rightarrow (\lambda_{\text{old}} + \lambda_{\text{even}}, \emptyset) \rightarrow (X_S \lambda_{\text{old}}, \lambda_{\text{even}})_{C_n}. \tag{4.5}
\]
An example illustrates this process.

**Example**: For the \( B_{16} \) partition \( 5 4^2 3^2 1 \), the Young tableau is
\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{example_tableau.png}
\end{array}
\tag{4.6}
\]
Let us split it into partitions \( \rho_{\text{odd}} \) and \( \rho_{\text{even}} \)
\[
\begin{array}{c}
\includegraphics[width=0.2\textwidth]{split_tableau.png}
\end{array} \rightarrow \left( \begin{array}{c}
\includegraphics[width=0.2\textwidth]{left_split.png} ; \includegraphics[width=0.2\textwidth]{right_split.png}
\end{array} \right). \tag{4.7}
\]
The map \( X_S \) take the special \( B_n \) partition \( \rho_{\text{odd}} \) to a special \( C_n \) partition
\[
X_S : \begin{array}{c}
\includegraphics[width=0.2\textwidth]{left_partition.png}
\end{array} \rightarrow \begin{array}{c}
\includegraphics[width=0.2\textwidth]{right_partition.png}
\end{array}. \tag{4.8}
\]
After leaving the partition \( \rho_{\text{even}} \) on the right side of (4.7) untouched, we arrive at a rigid semisimple surface operator \((2^4 1^4 ; 2^4 1^4)\) in the \( C_n \) theory
\[
WB : \begin{array}{c}
\includegraphics[width=0.2\textwidth]{wb_tableau.png}
\end{array} \rightarrow \left( \begin{array}{c}
\includegraphics[width=0.2\textwidth]{left_wb_split.png} ; \includegraphics[width=0.2\textwidth]{right_wb_split.png}
\end{array} \right).
\]
Using **Rule 1**, we have
\[
\sigma_C^{(2^41^4, 2^41^4)} = \sigma_C^{(2^41^4)} + \sigma_C^{(2^41^4)} = \\
\begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix} + 
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix} + 
\begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix} + 
\begin{pmatrix}
0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
which is consistent with the symbol of the \( B_{16} \) partition \( 5 4^2 3^2 1 \).

Summary, under the map \( X_S \), we have
\[
\sigma_B^{(\lambda)} = \sigma_C^{(X_S \lambda_{\text{odd}}, \lambda_{\text{even}})} = \sigma_C^{(X_S \lambda_{\text{odd}})} + \sigma_C^{(\lambda_{\text{even})}}. \tag{4.9}
\]
The symbol of the \( B_n \) partition \( \rho_{\text{odd}} \) is equal to that of the image of the map \( X_S \). The contribution to symbol of \( \lambda_{\text{even}} \) as even rows in the \( B_n \) theory is equal to that of \( \lambda_{\text{even}} \) in the \( C_n \) theory.
4.2 Map YS

Now we introduce another map \( YS \) which maps the odd-row tableaux \( \rho_{\text{odd}} \) in the \( C_n \) side to a rigid \( D_{n-k} \) partition \( \rho_{\text{even}} \)

\[
YS : m^{2n_m+1} (m-1)^{2n_{m-1}} (m-2)^{2n_{m-2}} \ldots 2^{n_2} 1^{2n_1} \mapsto m^{2n_m} (m-1)^{2n_{m-1}+2} (m-2)^{2n_{m-2}-2} \ldots 2^{n_2-2} 1^{2n_1+2}
\]

where \( m \) has to be even in order for the first element to be a \( C_k \) partition. This map is a bijection which takes a special \( C_k \) partition to a special \( D_k \) partition. It is clear that \( Ys \) preserves the number of boxes and symbol. A simple example illustrates this rule.

In [4], Wyllard proposed an algorithm to construct dual of the unipotent operators on the \( C_n \) side: Firstly, split the Young tableau \( \rho \) into even-row tableau \( \rho_{\text{even}} \) and odd-row \( \rho_{\text{odd}} \) tableaux. Then take the map \( X^{-1}S \) turns the even-row tableau \( \rho_{\text{even}} \) to a \( B_k \) tableau and the map \( YS \) turn the old-row tableau \( \rho \) to a \( D_k \) tableau. This duality map is denoted by \( WC \),

\[
WC : (\lambda, \emptyset)_C \rightarrow (\lambda_{\text{odd}} + \lambda_{\text{even}}, \emptyset) \rightarrow (X^{-1}_S \lambda_{\text{even}}, YS \lambda_{\text{odd}})_B.
\]

which has been proved preserve the symbol invariant in [4].

Summary, we have

\[
\sigma_{(\lambda)}^C = \sigma_{(X^{-1}_S \lambda_{\text{even}}, YS \lambda_{\text{odd}})}^B = \sigma_{(X^{-1}_S \lambda_{\text{even}})}^B + \sigma_{(YS \lambda_{\text{odd}})}^D.
\]

Since the maps \( X_S \) preserve symbol, the above equation reduces to

\[
\sigma_{(\lambda)}^C = \sigma_{(\lambda_{\text{even}})}^C + \sigma_{(YS \lambda_{\text{odd}})}^D.
\]

For the unipotent surface operators in the \( D_n \) theory, Wyllard made the following proposal

\[
WD : (\lambda, \emptyset)_D \rightarrow (\lambda_{\text{odd}} + \lambda_{\text{even}}, \emptyset) \rightarrow (\lambda_{\text{even}}, YS \lambda_{\text{odd}})_D.
\]

Start by splitting the corresponding tableau into even-and old-row tableaux, leaving the even-row tableau unchanged and applying the map \( YS \) to the odd-row tableau. The map \( YS \) turn an old-row tableau into an even-row tableau, which has been proved preserve the symbol invariant in [4]. Summary, we have

\[
\sigma_{(\lambda, \emptyset)}^D = \sigma_{(\lambda_{\text{even}})}^D + \sigma_{(YS \lambda_{\text{odd}})}^D.
\]
4.3 Calculating symbol in a simple way

From the above discussions, using the formulas (4.9), (4.12), and (4.14), a natural construction of symbol of a rigid semisimple surface operator \((\lambda, \rho)\) in the \(B_n\), \(C_n\), and \(D_n\) theories now presents itself:

- For a rigid semisimple surface operator \((\lambda, \rho)\) in the \(B_n\) theory, by using the formulas (4.9) and (4.14), the symbol is
  \[
  \sigma^{B}_{(\lambda,\rho)} = \sigma^{B}_{(\lambda)} + \sigma^{D}_{(\rho)} = \sigma^{C}_{(\lambda_{\text{even}})} + \sigma^{Y}_{(X_{S}\lambda_{\text{old}})} + \sigma^{D}_{(\rho_{\text{even}})} + \sigma^{D}_{(Y_{S}\rho_{\text{old}})}.
  \]

  The four terms on the right can be calculated by Rule 1, since the partitions \(\lambda_{\text{even}}, X_{S}\lambda_{\text{old}}, \rho_{\text{even}},\) and \(Y_{S}\rho_{\text{old}}\) only have even rows.

- For a rigid semisimple surface operator \((\lambda, \rho)\) in the \(C_n\) theory, by using the formulas (4.12), the symbol is
  \[
  \sigma^{C}_{(\lambda,\rho)} = \sigma^{C}_{(\lambda)} + \sigma^{C}_{(\rho)} = \sigma^{D}_{(\lambda_{\text{even}})} + \sigma^{D}_{(Y_{S}\lambda_{\text{old}})} + \sigma^{D}_{(\rho_{\text{even}})} + \sigma^{D}_{(Y_{S}\rho_{\text{old}})}.
  \]

- For a rigid semisimple surface operator \((\lambda, \rho)\) in the \(D_n\) theory, by using the formula (4.14), the symbol is
  \[
  \sigma^{D}_{(\lambda,\rho)} = \sigma^{D}_{(\lambda)} + \sigma^{D}_{(\rho)} = \sigma^{D}_{(\lambda_{\text{even}})} + \sigma^{D}_{(Y_{S}\lambda_{\text{old}})} + \sigma^{D}_{(\rho_{\text{even}})} + \sigma^{D}_{(Y_{S}\rho_{\text{old}})}.
  \]

Hence, for a rigid surface operator in the \(B_n\), \(C_n\), and \(D_n\) theories, the symbol can be constructed by using the formulas (4.15), (4.16), and (4.17).

5 Rigid surface operators in the \(B_n/C_n\) theories

In [4], Wyllard made explicit proposals for how the \(S\)-duality map should act on unipotent surface operators and certain subclasses of semisimple surface operators. These proposals, with common characteristics, are given in the first subsection. In the second and third subsections, we will make new proposals for certain subclasses of semisimple surface operators, with evidences provided.

5.1 Proposals for \(S\)-duality maps for surface operators

The \(S\)-duality maps proposed in [4] passed all consistency checks.

For rigid unipotent operators \((\lambda, \emptyset)\) of the \(B_n\) theory

The \(S\)-duality map is

\[
WB : (\lambda, \emptyset)_B \rightarrow (\lambda_{\text{odd}} + \lambda_{\text{even}}, \emptyset) \rightarrow (X_{S}\lambda_{\text{old}}, \lambda_{\text{even}})_C
\]

which is the map (4.5).

Example: For the \(B_{16}\) partition, \(\lambda = 5 \times 4^2 \times 3^3 \times 2^4 \times 1^3\), applying the map \(WB\), we find
which leads to the semisimple $C_{16}$ surface operator $(2^4 1^8, 2^6 1^4)$.

For surface operators $(1; \delta)$ of the $B_n$ theory

The $S$-duality map is

$$WB1 : (1; \delta)_B \rightarrow (1; \delta_{\text{even}} + \delta_{\text{odd}}) \rightarrow (X_S 1 + Y_S^{-1} \delta_{\text{even}} ; \delta_{\text{odd}})_C. \quad (5.3)$$

Split the partition $\delta$ into even and odd rows. Apply $Y_S^{-1}$ to the even-row tableau and leave the odd-row tableau unchanged.

We can prove that the map $WB1$ preserve the symbol. For the surface operator $(1; \delta)_B$, using the formula (4.15), the symbol is

$$\sigma^B(1; \delta) = \sigma^B_1 + \sigma^D(\delta) = \sigma^C(X_S 1) + \sigma^D(\delta_{\text{even}}) + \sigma^D(Y_S \delta_{\text{old}}) = \sigma^D(\delta_{\text{even}}) + \sigma^D(Y_S \delta_{\text{old}}). \quad (5.4)$$

For the surface operator $(X_S 1 + Y_S^{-1} \delta_{\text{even}} ; \delta_{\text{odd}})_C$, using the formula (4.16), the symbol is

$$\sigma^C(X_S 1 + Y_S^{-1} \delta_{\text{even}} ; \delta_{\text{odd}}) = \sigma^C(X_S 1 + Y_S^{-1} \delta_{\text{even}}) + \sigma^C(\delta_{\text{even}}) = \sigma^D(Y_S \delta_{\text{even}}) + \sigma^D(Y_S \delta_{\text{old}})$$

which is equal to the formula (5.4).

For rigid unipotent operators $(\lambda, \emptyset)$ of the $C_n$ theory

The $S$-duality map is

$$WC : (\lambda, \emptyset)_C \rightarrow (\lambda_{\text{odd}} + \lambda_{\text{even}}, \emptyset) \rightarrow (X_S^{-1} \lambda_{\text{even}}, Y_S \lambda_{\text{odd}})_B, \quad (5.5)$$

which is the map (4.11).

For semisimple surface operators $(\rho; \rho)$ of the $C_n$ theory

The $S$-duality map is

$$WCC : (\rho; \rho)_C \rightarrow (\rho_{\text{even}} + \rho_{\text{odd}}; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (\rho_{\text{even}} + X_S^{-1} \rho_{\text{even}} ; \rho_{\text{odd}} + Y_S \rho_{\text{odd}})_B. \quad (5.6)$$

The first step is to split two equal tableaux into even-row tableaux $\rho_{\text{even}}$ and odd-row tableaux $\rho_{\text{odd}}$. The second step is to apply the map $X_S$ to one of the odd-row tableaux and apply the map $Y_S^{-1}$ to the even-row tableau in the other semisimple factor. Next add the altered and unaltered even-row tableaux to form one of the two partitions in a semisimple $B_n$ operator. Finally, do the same to the odd-row tableaux. For the resulting partition, the first partition is a $B_k$ partition and the second factor is a $D_{n-k}$ partition.

We can prove that the map $WCC$ preserve the symbol. For the surface operator $(\rho; \rho)_C$, using the formula (4.16), the symbol is

$$\sigma^C(\rho; \rho) = \sigma^C_0 + \sigma^C(\rho_{\text{even}}) + \sigma^D(Y_S \rho_{\text{odd}}) + \sigma^C(Y_S \rho_{\text{odd}}) + \sigma^D(Y_S \rho_{\text{odd}}). \quad (5.7)$$
For the surface operator $(\rho_{\text{even}} + X_{S}^{-1}\rho_{\text{even}} : \rho_{\text{odd}} + Y_{S}\rho_{\text{odd}})_{B}$, using the formula (4.15), the symbol is

$$
\sigma^{B}_{(\rho_{\text{even}} + X_{S}^{-1}\rho_{\text{even}} : \rho_{\text{odd}} + Y_{S}\rho_{\text{odd}})_{B}} = \sigma^{B}_{(\rho_{\text{even}} + X_{S}^{-1}\rho_{\text{even}})} + \sigma^{D}_{(\rho_{\text{odd}} + Y_{S}\rho_{\text{odd}})}
$$

$$
= \sigma^{C}_{(\rho_{\text{even}})} + \sigma^{C}_{(X_{S}(X_{S}^{-1}\rho_{\text{even}}))} + \sigma^{D}_{(\rho_{\text{odd}})} + \sigma^{D}_{(Y_{S}\rho_{\text{odd}})}
$$

which is equal to the formula (5.7).

**Example:** For the $C_{14}$ partition, $(4 \ 3 \ 2 \ 1 \ 2 \ 1 \ 2)$, we have

$$
\begin{align*}
&\begin{pmatrix}
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}
\end{pmatrix}
\end{align*}
$$

$$
\mapsto
\begin{pmatrix}
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}
\end{pmatrix} + \begin{pmatrix}
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\begin{array}{cccc}
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\ast & \ast & \ast & \ast \\
\end{array}
\end{pmatrix}
$$

which leads to a semisimple operator in the $B_{n}$ theory.

### 5.2 S-duality maps for rigid semisimple operators in the $C_{n}$ theory

In this subsection, we discuss the possible forms of $S$-duality maps of rigid surface operators $(\lambda ; \rho)$ in the $C_{n}$ theory and extend maps proposed in previous section, with the same manipulation rule at the level of Young tableaux. Start by splitting the partition $\lambda$ into even-row and odd-row tableaux and do the same for the partition $\rho$. The map $WC$ (5.5) fix the possible form of the $S$-duality map of rigid surface operator as follows

$$
CB : (\lambda ; \rho)_{C} \rightarrow (\lambda_{\text{even}} + \lambda_{\text{odd}} ; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (X_{S}^{-1}\lambda_{\text{even}} + A ; Y_{S}\lambda_{\text{odd}} + B)_{B},
$$

where $A$ and $B$ denote the uncertain parts related to the partition $\rho$. Setting $\lambda = \rho$, comparing with the map $WCC$ (5.6), we find that $A = \rho_{\text{even}}$ and $B = \rho_{\text{odd}}$. So the $S$-duality map of a general rigid semisimple surface operator $(\lambda ; \rho)$ can be fixed as follows

$$
CB : (\lambda ; \rho)_{C} \rightarrow (\lambda_{\text{even}} + \lambda_{\text{odd}} ; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (X_{S}^{-1}\lambda_{\text{even}} + \rho_{\text{even}} ; Y_{S}\lambda_{\text{odd}} + \rho_{\text{odd}})_{B}. \quad (5.9)
$$

The inverse map $CB^{-1}$ for a rigid semisimple surface operator in the $B_{n}$ theory is

$$
CB^{-1} : (\lambda ; \rho)_{B} \rightarrow (X_{S}\rho_{\text{odd}} + Y_{S}^{-1}\rho_{\text{even}} ; \lambda_{\text{even}} + \rho_{\text{old}})_{C}. \quad (5.10)
$$

To be well defined, the images of the maps $CB$ and $CB^{-1}$ should be rigid semisimple surface operators.
5.3 S-duality maps for rigid semisimple operators in the \( B_n \) theory

The map \( WB \) (5.1) fix the possible forms of the \( S \)-duality map of rigid surface operators \((\lambda;\rho)\) in the \( B_n \) theory as follows

\[
BC : (\lambda;\rho)_B \rightarrow (\lambda_{\text{even}} + \lambda_{\text{odd}}; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (X_S \lambda_{\text{old}} + A; \lambda_{\text{even}} + B)_C
\]

where \( A \) and \( B \) denote the uncertain parts related to the partition \( \rho \). Setting \( \lambda = 1 \) and comparing with the map \( WB1 \) (5.3), we find that there are two choices for the parts \( A \) and \( B \). The first one is \( A = Y_{S}^{-1} \rho_{\text{even}} \) and \( B = \rho_{\text{old}} \) and the second one is \( A = \rho_{\text{old}} \) and \( B = Y_{S}^{-1} \rho_{\text{even}} \).

For the first choice, the \( S \)-duality map can be fixed as follows

\[
BC1 : (\lambda;\rho)_B \rightarrow (\lambda_{\text{even}} + \lambda_{\text{odd}}; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (X_S \lambda_{\text{old}} + Y_{S}^{-1} \rho_{\text{even}}; \lambda_{\text{even}} + \rho_{\text{old}})_C \quad (5.11)
\]

which is equal to the map \( CB^{-1} \) (5.10). To be well defined, the images of the maps \( CB \) and \( CB^{-1} \) should be rigid semisimple surface operators.

For the second choice, the \( S \)-duality map can be fixed as follows

\[
BC2 : (\lambda;\rho)_B \rightarrow (\lambda_{\text{even}} + \lambda_{\text{odd}}; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (X_S \lambda_{\text{old}} + \rho_{\text{old}}; \lambda_{\text{even}} + Y_{S}^{-1} \rho_{\text{even}})_C. \quad (5.12)
\]

However, it is not consistent with the map \( CB \) (5.9) for all rigid partition pair \((\lambda;\rho)_B\).

Applying the map \( CB \) to the rigid surface operator \((\lambda;\rho) = (\lambda,\emptyset)\) in the \( C_n \) theory, we have

\[
CB : (\lambda,\emptyset)_C \rightarrow (\lambda_{\text{old}} + \lambda_{\text{even}},\emptyset) \rightarrow (X_S^{-1} \lambda_{\text{even}},Y_{S} \lambda_{\text{old}})_B.
\]

Then applying the inverse map of \( CB \) to \((\lambda';\rho')_B = (X_S^{-1} \lambda_{\text{even}},Y_{S} \lambda_{\text{old}})_B\), we have

\[
BC2 : (\lambda;\rho)_B \rightarrow (X_S (X_S^{-1} \lambda_{\text{even}})_{\text{old}}; Y_{S}^{-1} (Y_{S} \lambda_{\text{old}})_{\text{even}}) = (\lambda_{\text{even}}; \lambda_{\text{odd}})_C
\]

which is not equal to \((\lambda,\emptyset)_C\). For certain subclasses rigid surface operators, we find a map \( BC2 \) which is consistent with the map \( CB \) in section 5.5.

To be well defined, the images of all the \( S \)-duality maps should be rigid semisimple surface operators.

5.4 S-duality maps for rigid surface operators in the \text{SO}(13) and \text{Sp}(12) theories

In the following table, we list all rigid surface operators in the \( \text{Sp}(12) \) and \( \text{SO}(13) \) theories to illustrate the proposed \( S \)-duality maps. The first column is the type of the duality maps. The second and third columns list pairs of partitions corresponding to the surface operators in the \( B_n \) and \( C_n \) theories. The other columns are the dimension, symbol invariant, and fingerprint invariant of the surface operator, respectively. Combining the discussions in the following subsections, all the candidates \( S \)-duality pairs of rigid surface operators can fit into the proposal duality relationships. Even the mismatch in the total number of rigid surface operators in the \( B_n \) and \( C_n \) theories can be explained.
| Num | Type | Sp(12) | SO(13) | Dim | Symbol | Fingerprint |
|-----|------|--------|--------|-----|--------|-------------|
| 1   | CB   | $\{12; 0\}$ | $\{13; 0\}$ | 0   | $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ | $[1^6; 0]$ |
| 2   | CB   | $\{21; 0\}$ | $\{1; 12\}$ | 12  | $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ | $[1^5; 1]$ |
| 3   | CB   | $\{10; 1^2\}$ | $\{2^2; 1^3; 0\}$ | 20  | $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$ | $[21^4; 0]$ |
| 4   | CB   | $\{2^3; 1^6; 0\}$ | $\{1; 2^2; 1^8\}$ | 30  | $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $[1^3; 1^3]$ |
| 5   | CB eo| $\{21; 1^2\}$ | $\{1; 3; 1^10\}$ | 30  | $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ | $[1^3; 1^3]$ |
| 6   | CB   | $\{1^8; 1^4\}$ | $\{2^4; 1^5; 0\}$ | 32  | $\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$ | $[2^2; 1^2; 0]$ |
| 7   | CB   | $\{2^4; 1^4; 0\}$ | $\{3^2; 1^6; 0\}$ | 36  | $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ | $[1^2; 1^4]$ |
| 8   | CB eo| $\{1^8; 21^2\}$ | $\{1^9; 1^4\}$ | 36  | $\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ | $[1^2; 1^4]$ |
| 9   | CB   | $\{1^6; 1^6\}$ | $\{2^6; 1; 0\}$ | 36  | $\begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix}$ | $[2^3; 0]$ |
| 10  | CB   | $\{2^5; 1^2; 0\}$ | $\{1; 2^4; 1^4\}$ | 40  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ | $[1; 1^5]$ |
| 11  | CB eo| $\{21; 1^4\}$ | $\{1^5; 1^8\}$ | 40  | $\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ | $[1; 1^5]$ |
| 12  | CB eo| $\{1^6; 21^4\}$ | $\{1^7; 1^6\}$ | 42  | $\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ | $[0; 1^6]$ |
| 13  | CB   | $\{3^2; 21^4; 0\}$ | $\{1^3; 2^2; 1^6\}$ | 44  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ | $[31^2; 1]$ |
| 14  | N1   | $\{2^3; 1^4; 1^2\}$ | $\{2^2; 1; 1^8\}$ | 44  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$ | $[31^2; 1]$ |
| 15  | CB   | $\{21^6; 21^2\}$ | $\{1; 3^2; 2^1; 1^5\}$ | 44  | $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ | $[21^2; 2]$ |
| 16  | N2   | $\{2^4; 1^2; 1^2\}$ | $\{2^2; 1^5; 1^4\}$ | 48  | $\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ | $[31; 1^2]$ |
| 17  | CB   | $\{21^4; 21^4\}$ | $\{1; 3^2; 2^4; 1\}$ | 48  | $\begin{pmatrix} 2 & 2 & 2 \end{pmatrix}$ | $[2^2; 2]$ |
| 18  | CB eo| $\{2^3; 1^2; 1^4\}$ | $\{1^5; 2^2; 1^4\}$ | 50  | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 \end{pmatrix}$ | $[3; 1^3]$ |
| 19  | –    | –      | $\{2^2; 13; 1^6\}$ | 50  | $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 \end{pmatrix}$ | $[3; 1^3]$ |
| 20  | –    | –      | $\{2^4; 1; 1^4\}$ | 52  | $\begin{pmatrix} 0 & 1 & 1 \\ 2 & 2 \end{pmatrix}$ | $[3; 1^3]$ |
| Num | Type | Sp(12) | SO(13) | Dim | Symbol | Fingerprint |
|-----|------|--------|--------|-----|--------|-------------|
| 21  | N3   | (2^3 \cdot 1^2; 21^2) | (1^3; 3 \cdot 2^2 \cdot 1^3) | 54  | \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \end{pmatrix} | [3; 1; 2] |
| 22* | CB   | (3^2 \cdot 1^2; 1^2) | (2^2 \cdot 1; 2^2 \cdot 1^4) | 54  | \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 0 \end{pmatrix} | [41; 1] |
| 23  | -    | (5^2 \cdot 1^2) | (1^5; 3 \cdot 2^2 \cdot 1) | 56  | \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix} | [3; 2; 1] |
| 24  | -    | (2^2 \cdot 1; 3 \cdot 2^2 \cdot 1) | (2^2 \cdot 1; 3 \cdot 2^2 \cdot 1) | 60  | \begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix} | [\emptyset; 2^3] |

### 5.5 Other S-duality maps

To be a well defined S-duality map, the image of BC1 should be rigid semisimple surface operators, with rigid partitions $X_{S\lambda_{\text{old}}}^\lambda$ and $Y_{S^{-1}\rho_{\text{even}}}^{-1}$ in the $C_n$ theory. Applying the map $BC1$ to the rigid surface operator $(1^{2m+1}; 1^{2n})$, we have the surface operator $(X_{S}1^{2m+1} + Y_{S}^{-1}1^{2n}; \emptyset)$. For $n \geq m$, the resulting partition $32^{2m-1}1^{2(n-m)}$ is not a rigid partition in the $C_n$ theory. For $n \leq m$, we draw the same conclusion.

However, there is an $S$-duality pair in the table of previous subsection:

$$S : (1^{2m+1}; 1^{2n})_B \rightarrow (1^{2m}; 21^{2(n-1)})_C.$$  \hspace{1cm} (5.13)

We can prove that the map $S$ preserve the symbol invariant as well as dimension

$$d_{(1^{2m+1}; 1^{2n})_B} = d_{(1^{2m}; 21^{2(n-1)})_C}.$$  

It suggest us to propose the following $S$-duality map

$$BC_{oe} : (\lambda_{\text{old}}; \rho_{\text{even}})_B \rightarrow (X_{S}\lambda_{\text{old}}; Y_{S}^{-1}\rho_{\text{even}})_C.$$ \hspace{1cm} (5.14)

By using the map $X_{S}$, the partition $1^{2m+1}$ in the $B_n$ theory become a partition $1^{2m}$ in the $C_n$ theory. By using the map $Y_{S}^{-1}$, the partition $1^{2n}$ in the $D_n$ theory become the partition $21^{2(n-1)}$ in the $C_n$ theory.

The map $BC_{oe}$ is a special case of the map $BC2$ (5.12), whose inverse map can be proposed as follows

$$CB_{eo} : (\lambda_{\text{even}}; \rho_{\text{old}})_C \rightarrow (X_{S}^{-1}\lambda_{\text{even}}; Y_{S}\rho_{\text{old}})_B.$$  

The following example of the map $BC2$ in the preceding table is not included in the map $S$ (5.13).

**Example**: The 18th duality pair

$$CB_{eo} : (2^1 \cdot 1^2; 1^4)_C \rightarrow (1^5; 2^2 \cdot 1^4)_B.$$
5.6 Exceptions

There are several $S$-duality pairs in the table of section 5.4 not belonging to any proposed $S$-duality maps. We discuss $S$-duality maps of the 14th, 16th, 19th, and 21th pairs independently.

The 14th duality pair: we have the map $N_1 : (2^2 1^3)_B \to (2^3 1^4 ; 1^2)_C$.

\[
\begin{aligned}
&\left( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} : \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned} \right) \mapsto \left( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} ; \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned}
\]

\[
\begin{aligned}
&\rightarrow (Y_S^{-1}(X_S + X_S) ; 1)
\end{aligned}
\]

\[
\begin{aligned}
&= \left( \begin{array}{c}
\begin{array}{c}
\end{array} \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} : \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned}
\right)
\]

(5.15)

The 16th duality pair: we have the map $N_2 : (2^2 1^5 ; 1^4)_B \to (2^4 1^2 ; 1^2)_C$.

\[
\begin{aligned}
&\left( \begin{array}{c}
\begin{array}{c}
\end{array} : \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \right) \mapsto \left( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} ; \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned} \right)
\]

\[
\begin{aligned}
&\rightarrow (X_S + X_S ; 1)
\end{aligned}
\]

\[
\begin{aligned}
&= \left( \begin{array}{c}
\begin{array}{c}
\end{array} \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} : \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned}
\right)
\]

(5.16)

The 21th duality pair: we have the map $N_3 : (1^3 ; 3^2 1^3)_B \to (2^3 1^2 ; 2^1 2)_C$.

\[
\begin{aligned}
&\left( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} : \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \right) \mapsto \left( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} ; \begin{array}{c}
\begin{array}{c}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned} \right)
\]

\[
\begin{aligned}
&\rightarrow (Y_S^{-1}(X_S + X_S + X_S) ; 1)
\end{aligned}
\]

\[
\begin{aligned}
&= \left( \begin{array}{c}
\begin{array}{c}
\end{array} \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} : \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned}
\right)
\]

(5.17)

For the 19th duality pair, the $B_n$ surface operator $(1^3 ; 3^2 1^3)_B$ do not have candidate duals. Since there is only one $C_n$ rigid surface operator $(2^3 1^2 ; 2^1 2)_C$ have the same invariants with it, we propose the following $S$-duality map artificially.

The 19th duality pair: we have the map $N : (2^2 1^3 ; 1^6)_B \to (2^3 1^2 ; 2^1 2)_C$.

\[
\begin{aligned}
&\left( \begin{array}{c}
\begin{array}{c}
\end{array} : \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} \right) \mapsto \left( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} : \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned}
\]

\[
\begin{aligned}
&\rightarrow (Y_S^{-1}(X_S + X_S) ; 1)
\end{aligned}
\]

\[
\begin{aligned}
&= \left( \begin{array}{c}
\begin{array}{c}
\end{array} \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array} : \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{aligned}
\right)
\]

(5.18)

For the 20th, 23th, and 24th $S$-duality pairs, the $B_n$ surface operators do not have candidate duals. In fact, for these operators, we could not find the $S$-duality surface operators in the $C_n$ theory by splitting the factors of the rigid semisimple surface operator in the $B_n$ theory to an even-row tableau and an old-row tableau, and then applying maps $X_S$ and $Y_S^{-1}$ to them.

The above examples imply the following conjecture.

\footnote{It imply that there should be a mechanism for mapping several rigid surface operators with the same invariants in the $B_n$ theory to a rigid surface operator in the $C_n$ theory.}

The above examples imply the following conjecture.
Conjecture: All the rigid surface operators in the $C_n$ theories can be obtained through the manipulations of splitting the semisimple factors of a rigid surface operator in the $B_n$ theories into even tableaux and old tableaux, the actions of $X_S$ on part of old tableaux and $Y_S^{-1}$ on part of the even tableaux, and combinations of these building blocks if necessary.

6 Rigid surface operators in the $D_n$ theory

Since the Langlands dual groups of $D_n(SO(2n))$ are themselves, applying the $S$-duality map to a rigid semisimple surface operator, we would obtain a surface operator in the same theory. For unipotent surface operators, Wyllard made the following $S$-duality map in [4]

$$WD: (\lambda, \emptyset)_D \rightarrow (\lambda_{\text{old}} + \lambda_{\text{even}}, \emptyset) \rightarrow (\lambda_{\text{even}}, Y_S \lambda_{\text{old}})_D.$$ (6.1)

which is the map (4.13).

For the semisimple rigid $D_n$ surface operators of the form $(\rho; \rho)$, Wyllard made the following proposal in [4]

$$WDD: (\rho; \rho)_D \rightarrow (\rho_{\text{even}} + \rho_{\text{odd}}; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (\rho_{\text{even}} + Y_S^{-1} \rho_{\text{even}}; \rho_{\text{odd}} + Y_S \rho_{\text{odd}})_D.$$ (6.2)

Split the two equal tableaux into even-row and odd-row tableaux. Next apply $Y_S^{-1}$ to one of the even-row tableau and $Y_S$ to one of the odd-row tableau. Then add the unchanged even-row tableau and the transformed even-row tableau. Do the same for the odd-row tableau.

In this section, we will extend above maps to a general rigid semisimple surface operator $(\lambda; \rho)$, with the same manipulation rule at the level of Young tableaux. Start by splitting the partition $\lambda$ into even-row and odd-row tableaux and do the same for the partition $\rho$. The map $WD$ fix the possible form of the $S$-duality map as follows

$$DD: (\lambda; \rho)_D \rightarrow (\lambda_{\text{even}} + \lambda_{\text{odd}}; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (\lambda_{\text{even}} + A; Y_S \lambda_{\text{odd}} + B)_D$$

where $A$ and $B$ denote the uncertain parts related to the partition $\rho$. Setting $\lambda = \rho$ and comparing with the map $WDD$, we find that $A = Y_S^{-1} \rho_{\text{even}}$ and $B = \rho_{\text{odd}}$. So the $S$-duality map of a general rigid semisimple surface operator $(\lambda; \rho)$ is fixed as follows

$$DD: (\lambda; \rho)_D \rightarrow (\lambda_{\text{even}} + \lambda_{\text{odd}}; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (\lambda_{\text{even}} + Y_S^{-1} \rho_{\text{even}}; \rho_{\text{odd}} + Y_S \lambda_{\text{odd}})_D.$$ (6.3)

The inverse map of $DD$ is

$$DD^{-1}: (\lambda; \rho)_D \rightarrow (\lambda_{\text{even}} + \lambda_{\text{odd}}; \rho_{\text{odd}} + \rho_{\text{even}}) \rightarrow (\lambda_{\text{even}} + Y_S^{-1} \rho_{\text{even}}; \rho_{\text{odd}} + Y_S \lambda_{\text{odd}})_D.$$ which is equal to the map $DD$ since the $D_n$ theories are self-dual.

---

A rigid surface operator in the $B_n$ theories do not have candidate duals if it can not leads to a rigid $C_n$ surface operator through these manipulations rules. This conjecture implies the mismatch on the total number of rigid surface operators in the $B_n$ and $C_n$ theories.
To illustrate the results in this section, we list all the $S$-duality pairs of rigid surface operators in the $SO(12)$ theories.

| Num | Type  | $SO(12)$                      | $SO(12)$                      | Dim | Symbol | Fingerprint |
|-----|-------|--------------------------------|--------------------------------|-----|---------|-------------|
| 1   | $WD$  | $(1^{12}; \emptyset)$          | $(1^{12}; \emptyset)$          | 0   | $\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ | $[1^6; \emptyset]$ |
| 2   | $WD$  | $(2^2 1^8; \emptyset)$         | $(2^2 1^8; \emptyset)$         | 16  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ | $[2^1 4^4; \emptyset]$ |
| 3   | $WD$  | $(2^4 1^4; \emptyset)$         | $(2^4 1^4; \emptyset)$         | 24  | $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ | $[2^2 1^2; \emptyset]$ |
| 4   | $WD$  | $(1^8, 1^4)$                   | $(3^2 2^1 1^5; \emptyset)$     | 32  | $\begin{pmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ | $[1^2; 1^4]$ |
| 5   | $WDD$ | $(3^2 4^1; \emptyset)$         | $(1^6; 1^6)$                   | 36  | $\begin{pmatrix} 2 & 2 & 2 \\ 0 & 0 \end{pmatrix}$ | $[0; 1^6]$ |
| 6   | $D_{even}$ | $(2^2 1^4; 1^4)$ | $(2^2 1^4; 1^4)$            | 40  | $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 \end{pmatrix}$ | $[3^1 1^2]$ |
| 7   | $DD$  | $(3^2 2^1; 1^4)$               | $(3^2 2^1; 1^4)$               | 48  | $\begin{pmatrix} 3 & 3 \\ 0 \end{pmatrix}$ | $[3; 2 1]$ |

The first three rigid surface operators pairs are self-duality under the map $WD$ (6.1). The 4th $S$-duality pair of rigid surface operators also belong to the map $WD$. The 5th pair of rigid surface operators belong to the map $WDD$ (6.2). While the 7th pair belongs to the extended map $DD$ (6.3)

$$DD : (3^2 2^1; 1^4) \rightarrow (3^2 2^1; 1^4)$$

which is not proposed by Wyllard [4].

For the 6th rigid surface operators pairs, we can not obtain a rigid semisimple $D_n$ surface operator by applying the map $DD$ or other manipulations to the rigid surface operator $(2^2 1^4; 1^4)$. With only one rigid surface operator of dimension 40, it must be self-duality. So we propose the following $S$-duality map for this rigid surface operator

$$D_{even} : (\lambda_{even}; \rho_{even}) \rightarrow (\lambda_{even}; \rho_{even}).$$

### 7 Summary and open problems

We have found simple rules to construct the symbol of a partition with only even rows. Using these calculation rules and the maps $X_S$, $Y_S$, we have simplified the computation of $symbols$ for the partitions in the $B_n$, $C_n$, and $D_n$ theories. These calculation rules of symbol are very convenient to search $S$-duality maps. By consistency checks, we have recovered and extended the $S$-duality maps proposed by Wyllard [4]. We have also found a new subclasses of rigid surface operators related by $S$-duality. We tried to explain the exceptions of $S$-duality maps and pointed out some common characteristics for the surface
operators related by $S$-duality. The above techniques used in the $B_n/C_n$ theories can be extended to the $D_n$ theories.

In addition to the symbol invariant, we should continue the analysis by computing other invariants of surface operators, such as the fingerprint invariant, center and topology of these proposals made in [4] for complement. The computation of the fingerprint invariant is complicated, which we hope to find rules to simplify. Simple and even explicit formulas of the fingerprint and the symbol are needed.

As shown in the table of the surface operators in the $SO(13)$ and $Sp(12)$ theories, there are still several candidates duality pairs of rigid surface operators can not fit into the proposed dual maps in this paper. Even more serious, there is a mismatch on the total number of rigid surface operators in the $B_n$ and $C_n$ theories. The physical reason for the mismatch is still unknown. Maybe we should also take account of the weakly rigid surface operators discussed in [3] or the quantum effect to resolve the discrepancy issue. Clearly more work is required since above reasons.

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A Summary relevant facts

A.1 Rigid Partitions in the $B_n$, $C_n$, and $D_n$ theories

Firstly, we prove that the longest row of a partition always contains an odd number of boxes in the $B_n$ theory. For a rigid partition, even integers appear an even number of times, so the sum of all old integers is old which means the number of old integers is old. It is imply that the length of the partition, which is the sum of the number of old integers and even integers, is old. So the longest row contains an odd number of boxes.

Next, We prove that the following two rows of the first row are either both of odd length or both of even length, this pairwise pattern then continues. Assuming these facts are true for the first $2i + 1$ rows of a partition, we prove that the next two rows are either both odd length or both even length. Assuming that the $(2i + 2)$-th row is old with length $2m + 1$, if the $(2i + 3)$-th row is even with length $2n$ then there are $(2m + 1) - (2n)$ integers $2i + 2$ in the partition. It is a contradictory to the fact that even integers appear an even number of times in a rigid $B_n$ partition. So the $(2i + 3)$-th row is old. Using the same method, we can prove that the $(2i + 3)$-th row is even if the $(2i + 2)$-th row is even.

Similarly, we can prove that the longest two rows in a rigid $C_n$ partition both contain either an even or an odd number number of boxes. This pairwise pattern then continues.

We can also prove that the longest two rows in a rigid $D_n$ partition always contains an even number of boxes. And the following two rows are either both of even length or both of old length. This pairwise pattern then continue.
A.2 Symbol of the first row of a partition in the \( C_n \) and \( D_n \) theories

According to the Rule 1 in section 3, the contribution to symbol of the first even row of a partition in the \( C_n \), \( D_n \) theories is needed as an initial conditions. For the first row with \( 2l \) boxes of a partition in the \( C_n \) theory, we have

\[
\lambda_k \ 1 \cdots 1 \ 0 \\
2l + 1 - k : \ 2l \cdots 1 \ 0 \\
2l + 1 - k + \lambda_l - k + 1 : \ 2l + 1 \cdots 2 \ 0
\]

then

\[
2f_i + 1 : 2l + 1 \cdots 3 \quad 2g_i : 2l \cdots 2 \ 0 \\
f_i : \ l \cdots 1 \quad g_i : \ l \cdots 1 \ 0
\]

Finally, we get \( \alpha_i = g_i - i + 1 : 0, \cdots, 0, 0 \) and \( \beta_i = f_i - i + 1 : 1, \cdots, 1, 0 \). So the contribution to symbol of the first row is

\[
\begin{pmatrix}
0 \\
0 \cdots 0 \\
1 \cdots 1 \\
l
\end{pmatrix}
\]  \hspace{1cm} (A.1)

Next, we check the Rule 1 for the \( C_n \) theory. We add a row with \( 2m \) boxes to the first row, then the partition is \( 2^{2m} \ 1^{2l-2m} \). The contribution to symbol of the second row is

\[
\begin{pmatrix}
0 \\
0 \cdots 0 \\
1 \cdots 1 \\
m
\end{pmatrix}
\]  \hspace{1cm} (A.2)

If two rows with lengths of \( 2m \) and \( 2n \) \((m > n)\) boxes are added to the first row, the contribution to symbol is

\[
\begin{pmatrix}
0 \\
0 \cdots 0 \\
1 \cdots 1 \\
1 \cdots 1 \\
m \\
m \cdots \cdots n
\end{pmatrix}
\]  \hspace{1cm} (A.3)

which is consistent with the Rule 1.

Compared with the \( B_n \) case, the terms \( g_i \) and \( f_i \) reverse roles for the calculation of symbol in the \( D_n \) theory. So the first row with \( 2l \) boxes contribute \( l \ '1' \) in another row of the symbol

\[
\begin{pmatrix}
1 \cdots 1 \\
0 \cdots 0 \\
l \cdots 1
\end{pmatrix}
\]  \hspace{1cm} (A.4)

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