LINDELÖF HYPOTHESIS AND THE ORDER OF THE MEAN-VALUE OF $|\zeta(s)|^{2k-1}$ IN THE CRITICAL STRIP

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Abstract. The main subject of this paper is the mean-value of the function $|\zeta(s)|^{2k-1}$ in the critical strip. On Lindelöf hypothesis we give a solution to this question for some class of disconnected sets. This paper is English version of our paper [5].

1. Introduction
1.1. E.C. Titchmarsh had began with the study of the mean-value of the function $|\zeta(\sigma+it)|^{\omega}$, $1/2 < \sigma \leq 1$, $0 < \omega$,
where $\omega$ is non-integer number. [6] (comp. [2], p. 278). Next, Ingham and Davenport have obtained the following result (see [1], [2], comp. [7], pp. 132, 133)

$$\frac{1}{T} \int_{1}^{T} |\zeta(\sigma+it)|^{2\omega} dt = \sum_{n=1}^{\infty} \frac{d_{\omega}(n)}{n^{\sigma}} + O(1), \omega \in (0,2], T \to \infty.$$  (1.1)

Let us remind that:
(a) for $\omega \in \mathbb{N}$ the symbol $d_{\omega}(n)$ denotes the number of decompositions of $n$ into $\omega$-factors ,
(b) in the case $\omega$ is not an integer, we define $d_{\omega}(n)$ as the coefficient of $n^{-\omega}$ in the Dirichlet series for the function $\zeta^{\omega}(s)$ converging for all $\sigma > 1$.

1.2. Next, for

$$\omega = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$$

it follows from (1.1) that the orders of mean-values

$$\frac{1}{T} \int_{1}^{T} |\zeta(\sigma+it)| dt, \frac{1}{T} \int_{1}^{T} |\zeta(\sigma+it)|^{3} dt$$

are determined. But a question about the order of mean-value of

$$|\zeta(\sigma+it)|^{2l+1}, l = 2, 3, \ldots, \frac{1}{2} < \sigma < 1$$

remains open.

In this paper we give a solution to this open question on the assumption of truth of the Lindelöf hypothesis for some infinite class of disconnected sets. In a

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particular case we obtain the following result: on Lindelöf hypothesis we have

\[
1 - |o(1)| < \frac{1}{H} \int_T^{T+H} |\zeta(\sigma + it)|^{2k-1} dt < \\
< \sqrt{F(\sigma, 2k-1) + |o(1)|}, \quad H = T^\epsilon, \quad k = 1, 2, \ldots, 0 < \epsilon,
\]

where

\[
F(\sigma, \omega) = \sum_{n=1}^{\infty} \frac{d^2(n)}{n^{2\sigma}},
\]

and \(\epsilon\) is an arbitrarily small number.

The proof of our main result is based on our method (see [4]) for the proof of new mean-value theorem for the Riemann zeta-function

\[
Z(t) = e^{i\theta(t)} \zeta \left( \frac{1}{2} + it \right)
\]

with respect of two infinite classes of disconnected sets.

2. Main formulas

We use the following formula: on Lindelöf hypothesis

\[
\zeta^k(s) = \sum_{n \leq t^\delta} \frac{d_k(n)}{n^s} + \mathcal{O}(t^{-\lambda}), \quad \lambda = \lambda(k, \delta, \sigma) > 0,
\]

\[
s = \sigma + it, \quad \frac{1}{2} < \sigma < 1, \quad t > 0
\]

(see [7], p. 277) for every natural number \(k\), where \(\delta\) is any given positive number less than 1. Let us remind that

\[
d_k(n) = \mathcal{O}(n^\eta),
\]

where \(0 < \eta\) is an arbitrarily small number. Of course, (see (2.1), (2.2))

\[
\zeta^k(s) = \mathcal{O} \left( \sum_{n \leq t^\delta} d_k(n)n^{-\sigma} \right) = \mathcal{O} \left( t^{\delta \eta + \delta(1-\sigma)} \right) = \\
= \mathcal{O} \left( t^{(n+1/2)\delta} \right).
\]

Let

\[
t \in [T, T + H], \quad H = T^\epsilon, \quad 2\delta n + 2\delta < \epsilon.
\]

Since

\[
\sum_{T^\delta \leq n \leq (T+H)^\delta} 1 = \mathcal{O}(T^{\delta+\epsilon-1})
\]

then

\[
\sum_{T^\delta \leq n \leq t^\delta} \frac{d_k(n)}{n^s} = \mathcal{O} \left( T^{\delta \eta + \delta \sigma} \cdot \sum_{T^\delta \leq n \leq (T+H)^\delta} 1 \right) = \\
= \mathcal{O}(T^{\delta \lambda_1 - \delta \sigma + \delta + \epsilon - 1}) = \mathcal{O}(T^{-\lambda_1}),
\]

where

\[
\lambda_1 = 1 - \delta - \epsilon + \delta \sigma - \delta \eta > 0,
\]
(of course, for sufficiently small $\epsilon$ the inequality (2.6) holds true). Next, for
\begin{equation}
\lambda_2 = \lambda_2(k, \delta, \sigma, \epsilon, \eta) = \min\{\lambda, \lambda_1\} > 0
\end{equation}
the following formula (see (2.1), (2.5) – (2.7))
\begin{equation}
\zeta^k(s) = \sum_{n<T^\delta} \frac{d_k(n)}{n^s} + \mathcal{O}(T^{-\lambda_2}), \quad t \in [T, T + H]
\end{equation}
holds true. Since
\begin{equation}
\zeta^k(s) = U_k(\sigma, t) + iV_k(\sigma, t)
\end{equation}
then - on Lindelöf hypothesis - we obtain from (2.8) the following main formula
\begin{equation}
U_k(\sigma, t) = 1 + \sum_{2 \leq n < T^\delta} \frac{d_k(n)}{n^\sigma} \cos(t \ln n) + \mathcal{O}(T^{-\lambda_2}),
\end{equation}
\begin{equation}
V_k(\sigma, t) = -\sum_{2 \leq n < T^\delta} \frac{d_k(n)}{n^\sigma} \sin(t \ln n) + \mathcal{O}(T^{-\lambda_2}),
\end{equation}
$t \in [T, T + H]$.

3. The first class of lemmas

Let us denote by
\[\{t_\nu(\tau)\}\]
an infinite set of sequences that we defined (see [4], (1)) by the condition
\begin{equation}
\vartheta[t_\nu(\tau)] = \pi \nu + \tau, \quad \nu = 1, \ldots, \tau \in [-\pi, \pi],
\end{equation}
of course,
\[t_\nu(0) = t_\nu,
\]
where (see [7], pp. 220, 329)
\begin{equation}
\vartheta(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma \left(\frac{1}{2} + \frac{t}{2}\right),
\end{equation}
\begin{equation}
\vartheta'(t) = \frac{1}{2} \ln t + \mathcal{O}\left(\frac{1}{t}\right),
\end{equation}
\begin{equation}
\vartheta''(t) \sim \frac{1}{2t}.
\end{equation}

3.1. The following lemma holds true.

**Lemma 1.** If
\[2 \leq m, n < T^\delta
\]
then
\begin{equation}
\sum_{T \leq t_\nu \leq T + H} \cos\{t_\nu(\tau) \ln n\} = \mathcal{O}\left(\frac{\ln T}{\ln n}\right),
\end{equation}
\begin{equation}
\sum_{T \leq t_\nu \leq T + H} \cos\{t_\nu(\tau) \ln(mn)\} = \mathcal{O}\left(\frac{\ln T}{\ln(mn)}\right),
\end{equation}
\begin{equation}
\sum_{T \leq t_\nu \leq T + H} \cos\left\{t_\nu(\tau) \ln \frac{m}{n}\right\} = \mathcal{O}\left(\frac{\ln T}{\ln \frac{m}{n}}\right), \quad m > n,
\end{equation}
and
where the \( \mathcal{O} \)-estimates are valid uniformly for \( \tau \in [-\pi, \pi] \).

**Proof.** We use the van der Corput’s method. Let (see (5.3))

\[
\varphi_1(\nu) = \frac{1}{2\pi} t_\nu(\tau) \ln n.
\]

Next, (see (2.4), (3.1), (3.2))

\[
\varphi_1'(\nu) = \ln n 2\varphi_1''(\nu) = -\frac{\pi \ln n}{2\{\varphi'[t_\nu(\tau)]\}^2} \varphi''\{t_\nu(\tau)\} < 0,
\]

\[
0 < A \ln n \leq \varphi_1'(\nu) = \frac{\ln n}{\ln \frac{T}{2\pi} + \mathcal{O}(\frac{1}{T})} = \frac{\ln n}{\ln \frac{T}{2\pi} + \mathcal{O}(\frac{H}{T})} < \frac{\ln n}{\ln \frac{T}{2\pi} + \mathcal{O}(\frac{H}{T})} < \frac{1}{4},
\]

\((A > 0, \text{ since } \delta \text{ may be sufficiently small}). \text{ Hence, (see [7], p. 65 and p. 61, Lemma 4.2)}

\[
\sum_{T \leq t_\nu \leq T + H} \cos\{t_\nu(\tau) \ln n\} = 
\int_{T \leq t_\nu \leq T + H} \cos\{2\pi \varphi_1(x)\} dx + \mathcal{O}(1) = \mathcal{O} \left( \frac{\ln T}{\ln n} \right),
\]

i.e. the estimate (3.3) holds true. The estimates (3.4) and (3.5) follow by the similar way. \(\Box\)

### 3.2. The following lemma holds true.

**Lemma 2.** On Lindelöf hypothesis we have

\[
(3.6) \quad \sum_{T \leq t_\nu \leq T + H} U_k[\sigma, t_\nu(\tau)] = \frac{1}{2\pi} H \ln \frac{T}{2\pi} + \mathcal{O}(H).
\]

**Proof.** Let us remind that

\[
(3.7) \quad \sum_{T \leq t_\nu \leq T + H} 1 = \frac{1}{2\pi} H \ln \frac{T}{2\pi} + \mathcal{O}(1),
\]

(see [3], (23)). Next, (see (2.10), (3.4))

\[
\sum_{T \leq t_\nu \leq T + H} U_k[\sigma, t_\nu(\tau)] = \frac{1}{2\pi} H \ln \frac{T}{2\pi} + \mathcal{O}(1) + \mathcal{O}(T^{-\lambda_2} H \ln T) + \sum_{2 \leq n < T^\delta} \frac{d_k(n)}{n^\sigma} \sum_{T \leq t_\nu \leq T + H} \cos\{t_\nu(\tau) \ln n\} = 
\]

\[
= \frac{1}{2\pi} H \ln T + \mathcal{O}(1) + \mathcal{O}(T^{-\lambda_2} H \ln T) + w_1.
\]
Since (see (2.2), (2.4), (3.3))

\[ w_1 = \mathcal{O}\left( T^{\delta \eta} \ln T \sum_{2 \leq n \leq T^\delta} \frac{1}{\sqrt{n \ln n}} \right) = \]

\[ = \mathcal{O}\left( T^{\delta \eta} \ln T \left( \sum_{2 \leq n < T^{\delta/2}} + \sum_{T^{\delta/2} \leq n < T^\delta} \right) \frac{1}{\sqrt{n \ln n}} \right) = \]

\[ = \mathcal{O}(T^{\delta \eta + \delta/2}) = \mathcal{O}(H), \]

then from (3.8) the formula (3.6) follows. □

4. Theorem 1

4.1. Next, we define the following class of disconnected sets (comp. [4], (3)):

(4.1) \[ G(x) = \bigcup_{T \leq t \leq T + H} \{ t : t\nu(-x) < t < t\nu(x) \}, \quad 0 < x \leq \frac{\pi}{2}. \]

Let us remind that (see [4], (7))

\[ t\nu(x) - t\nu(-x) = \frac{4x}{\ln \frac{2}{\pi}} + \mathcal{O}\left( \frac{xH}{T \ln^2 T} \right), \]

\[ t\nu(-x), t\nu(x) \in [T, T + H]. \]

Of course,

(4.3) \[ m\{G(x)\} = \frac{2x}{\pi} H + \mathcal{O}(x), \]

(see (3.7), (4.2)), where \( m\{G(x)\} \) stands for the measure of \( G(x) \).

4.2. The following theorem holds true.

**Theorem 1.** On Lindel"of hypothesis

(4.4) \[ \int_{G(x)} U_k(\sigma, t) dt = \frac{2x}{\pi} H + o\left( \frac{xH}{\ln T} \right). \]

First of all, we obtain from (4.4) by (4.3) the following

**Corollary 1.** On Lindel"of hypothesis

(4.5) \[ \frac{1}{m(G(x))} \int_{G(x)} U_k(\sigma, t) dt = 1 + o\left( \frac{1}{\ln T} \right). \]

Next, we obtain from (4.4) the following

**Corollary 2.** On Lindel"of hypothesis

(4.6) \[ \int_{G(x)} |U_k(\sigma, t)| dt \geq \frac{2xH}{\pi} \{1 - o(1)\}. \]

Since (see (2.3))

\[ |\zeta(s)|^{2k-1} = \sqrt{U_{2k-1}^2 + V_{2k-1}^2} \geq |U_{2k-1}| \]

then we obtain from (4.6), \( k \to 2k - 1 \), the following
Corollary 3. On Lindelöf hypothesis

\begin{equation}
\int_{G(x)} |\zeta(\sigma + it)|^{2k-1} dt \geq \frac{2xH}{\pi}(1 - |\sigma(1)|).
\end{equation}

4.3. In this part we shall give the proof of the Theorem 1. Since (see (3.1), (3.2))

\begin{equation}
\int_{-\pi}^{\pi} U_k[\sigma, t \nu(\tau)]d\tau = \int_{-\pi}^{\pi} U_k[\sigma, t \nu(\tau)] \left( \frac{dt \nu(\tau)}{d\tau} \right)^{-1} \frac{dt \nu(\tau)}{d\tau} d\tau =
\end{equation}

\begin{equation}
= \ln P_0 \int_{-\pi}^{\pi} U_k[\sigma, t \nu(\tau)] \left( \frac{dt \nu(\tau)}{d\tau} \right) d\tau +
\end{equation}

\begin{equation}
+ O \left( x \max \{|\zeta_k|\} \frac{H}{T} \max \left\{ \left( \frac{dt \nu(\tau)}{d\tau} \right) \right\} \right) =
\end{equation}

\begin{equation}
= \ln P_0 \int_{t \nu(-x)}^{t \nu(x)} U_k[\sigma, t]dt + O \left( x \frac{T^{\delta \gamma + \delta/2 + 2\epsilon - 1}}{\ln T} \right),
\end{equation}

where the max is taken with respect to the segment \([T, T + H]\). Consequently, (see (2.3), (3.7), (4.1) and (4.2))

\begin{equation}
\sum_{T \leq t \nu \leq T + H} \int_{-\pi}^{\pi} U_k[\sigma, t \nu(\tau)]d\tau =
\end{equation}

\begin{equation}
= \ln P_0 \int_{G(x)} U_k(\sigma, t)dt + O(xT^{\delta \gamma + \delta/2 + 2\epsilon - 1} +
\end{equation}

\begin{equation}
+ O \left( xT^{(\eta + 1/2)\delta} \right). \ln T
\end{equation}

Now, the integration (3.0) by \(\tau \in [-\pi, \pi]\)

gives the formula

\begin{equation}
\ln P_0 \int_{G(x)} U_k(\sigma, t)dt + O(xT^{\delta \gamma + \delta/2 + 2\epsilon - 1}) =
\end{equation}

\begin{equation}
= \frac{x}{\pi} H \ln \frac{T}{2\pi} + O(xT^{\delta \gamma + \delta/2})
\end{equation}

and from this by (2.3) the formula (4.4) follows immediately (here \(\epsilon\) is arbitrarily small number). 
\(\square\)
5. The second class of lemmas

5.1. Let

\[ S_1(t) = \sum_{2 \leq n \leq T^t} \frac{d_k(n)}{n^\sigma} \cos(t \ln n), \]

(5.1)

\[ w_2(t) = \{S_1(t)\}^2. \]

The following lemma holds true.

**Lemma 3.**

\[ \sum_{T \leq t \leq T + H} w_2[t_\nu(\tau)] = \]

(5.2)

\[ = \{F(\sigma, k) - 1\} \cdot \frac{1}{4\pi} H \ln \frac{T}{2\pi} + o(H), \]

(on \(F(\sigma, k)\) see (1.3)).

**Proof.** First of all we have

\[ w_2(t) = \sum_m \sum_n \frac{d_k(m)d_k(n)}{(mn)^\sigma} \cos(t \ln m) \cos(t \ln n) = \]

(5.3)

\[ = \frac{1}{2} \sum_m \sum_n \frac{d_k(m)d_k(n)}{(mn)^\sigma} \cos\{t \ln(mn)\} + \]

\[ + \sum_{n < m} \frac{d_k(m)d_k(n)}{(mn)^\sigma} \cos\left(t \ln \frac{m}{n}\right) + \frac{1}{2} \sum_n \frac{d_k^2(n)}{n^{2\sigma}} = \]

\[ = w_{21}(t) + w_{22}(t) + w_{23}(t). \]

Now we have:

by (2.2), (2.4) and (3.4)

(5.4)

\[ \sum_{T \leq t \leq T + H} w_{21}[t_\nu(\tau)] = O\left( T^{\delta \eta} \ln T \cdot \sum_{2 \leq m, n < T^t} \frac{1}{\sqrt{mn \ln (mn)}} \right) = \]

\[ = O(T^{2\delta \eta + \delta} \ln T) = o(H); \]

by (2.2), (2.4) and (5.5) and by \[7\], p. 116, Lemma, \((T \rightarrow T^\delta), \)

(5.5)

\[ \sum_{T \leq t \leq T + H} w_{22}[t_\nu(\tau)] = O\left( T^{\delta \eta} \ln T \cdot \sum_{2 \leq n < m \leq T^t} \frac{1}{\sqrt{mn \ln \frac{m}{n}}} \right) = \]

\[ = O(T^{2\delta \eta + \delta} \ln^2 T) = o(H); \]

by (1.3), \(\omega \rightarrow k\), and by (2.2)

\[ 2_{23} = \frac{1}{2} \sum_{n=2}^{\infty} \frac{d_k^2(n)}{n^{2\sigma}} - \frac{1}{2} \sum_{n \geq T^t} \frac{d_k^2(n)}{n^{2\sigma}} = \]

(5.6)

\[ = \frac{1}{2} \{F(\sigma, k) - 1\} + O\left( \int_{T^t}^{\infty} x^{\eta - 2\sigma} dx \right) = \]

\[ = \frac{1}{2} \{F(\sigma, k) - 1\} + O(T^{-\delta(2\sigma - 1 - \eta)}); \]
(of course, $2\sigma - 1 - \eta > 0$ since $\eta$ is arbitrarily small). Finally, by (2.4), (3.7) and (5.6) we obtain

\begin{equation}
\sum_{T \leq t \leq T+H} w_{23} = \{F(\sigma, k) - 1\} \frac{1}{4\pi} H \ln \frac{T}{2\pi} + o(H).
\end{equation}

Hence, from (5.3) by (5.4) – (5.7) the formula (5.2) follows. □

Next, the following lemma holds true.

**Lemma 4.** On Lindelöf hypothesis

\begin{equation}
\sum_{T \leq t \leq T+H} U_k^2[\sigma, t_\nu(\tau)] = \{F(\sigma, k) - 1\} \frac{1}{2\pi} G \ln \frac{T}{2\pi} + o(H).
\end{equation}

**Proof.** Since (see (2.10), (5.1))

\begin{equation}
U_k^2(\sigma, t) = 1 + S_1 + O(T^{-\lambda_2}),
\end{equation}

then

\begin{equation}
U_k^2(\sigma, t) = 1 + w_2 + 2S_1 + O(|S_1|T^{-\lambda_2}) + O(T^{-2\lambda_2}).
\end{equation}

Now we have:

by (2.4)

\begin{equation}
\sum_{T \leq t \leq T+H} S_1[t_\nu(\tau)] = O(T^{\delta_\eta+\delta/2}) = o(H);
\end{equation}

by (2.4), (3.7) and (5.2)

\begin{equation}
\sum_{T \leq t \leq T+H} |S_1|T^{-\lambda_2} =
\end{equation}

\begin{equation}
= O \left\{ T^{-\lambda_2} \sqrt{H \ln T} \left( \sum_{T \leq t \leq T+H} w_2[t_\nu(\tau)] \right)^{1/2} \right\} =
\end{equation}

\begin{equation}
= O(T^{-\lambda_2} H \ln T) = o(H).
\end{equation}

Consequently, from (5.9) by (3.7) the formula (5.8) follows. □

5.2. Let

\begin{equation}
S_2(t) = \sum_{2 \leq n < T^s} \frac{d_k(n)}{n^{\sigma}} \sin(t \ln n),
\end{equation}

\begin{equation}
w_3(t) = \{S_2(t)\}^2.
\end{equation}

The following lemma holds true.

**Lemma 5.**

\begin{equation}
\sum_{T \leq t \leq T+H} w_3[t_\nu(\tau)] = \{F(\sigma, k) - 1\} \frac{1}{4\pi} H \ln \frac{T}{2\pi} + o(H).
\end{equation}
Proof. Since (comp. (5.3))
\[
\begin{align*}
    w_3(t) &= \sum_{m,n} d_k(m) d_k(n) \sin(t \ln m) \sin(t \ln n) = \\
    &= -\frac{1}{2} \sum_{m,n} d_k(m) d_k(n) \cos(t \ln(mn)) + \\
    &+ \sum_{n < m} d_k(m) d_k(n) \cos(t \ln m) + \\
    &+ \frac{1}{2} \sum_n d_k^2(n) = w_{31}(t) + w_{32}(t) + w_{33}(t),
\end{align*}
\]
then we obtain by the way (5.3) – (5.7) our formula (5.11).
\[
\square
\]

Next, the following lemma holds true

**Lemma 6.** On Lindelöf hypothesis
\[
\sum_{T \leq t \leq T + H} V_k^2[\sigma, t, \nu(\tau)] = \\
= \{F(\sigma, k) - 1\} \frac{1}{4\pi} H \ln \frac{T}{2\pi} + o(H).
\]

**Proof.** Since (see (5.11))
\[
V_k(\sigma, t) = -S_2 + O(T^{-\lambda_2}),
\]
then
\[
V_k^2(\sigma, t) = w_3 + O(T^{-\lambda_2} |S_2|) + O(T^{-2\lambda_2}).
\]
Consequently, the proof may be finished in the same way as it was done in the case of our Lemma 4 (comp. (5.12), (5.13)).
\[
\square
\]

Since (see (2.10))
\[
|\zeta(s)|^{2k} = U_k^2 + V_k^2,
\]
then by (5.8), (5.11) we obtain the following.

**Lemma 7.** On Lindelöf hypothesis
\[
\sum_{T \leq t \leq T + H} |\zeta[\nu(\tau)]|^{2k} = \frac{1}{2\pi} F(\sigma, k) H \ln \frac{T}{2\pi} + o(H).
\]

### 6. Theorem 2 and main Theorem

Now we obtain from (5.15) by the way very similar to than one used in the proof of the Theorem 1, the following.

**Theorem 2.** On Lindelöf hypothesis
\[
\int_{G(x)} |\zeta(\sigma + it)|^{2k} dt = \frac{2\pi}{\pi} F(\sigma, t) H + o \left( \frac{xH}{\ln T} \right).
\]
Further, from (6.1) we obtain
Corollary 4. On Lindelöf hypothesis

\[(6.2) \int_{G(x)} |\zeta(\sigma + it)|^{2k-1} dt < \frac{2xH}{\pi} \sqrt{F(\sigma, 2k-1)} \cdot \{1 + |o(1)|}\.
\]

Indeed, by (4.3), (6.1) we have

\[
\int_{G(x)} |\zeta(\sigma + it)|^{2k} dt < \\
< \sqrt{m\{G(x)\}} \left( \int_{G(x)} |\zeta(\sigma + it)|^{4k-2} dt \right)^{1/2} < \\
< \frac{2xH}{\pi} \sqrt{F(\sigma, 2k-1)} \cdot \{1 + |o(1)|}\.
\]

Finally, from (4.7), (6.2) we obtain our main result:

Theorem. On Lindelöf hypothesis

\[(6.3) \quad 1 - |o(1)| < \frac{1}{m\{G(x)\}} \int_{G(x)} |\zeta(\sigma + it)|^{2k-1} dt < \\
< \sqrt{F(\sigma, 2k-1) + |o(1)|}\.
\]

Remark 1. The question about the order of the mean-value of the function

\[|\zeta(\sigma + it)|^{2k-1}, \quad k = 1, 2, \ldots\]

defined on infinite class of disconnected sets \{G(x)\} is answered by the inequalities (6.3).

Remark 2. Inequalities (6.2) follows from (6.3) as a special case for \(x = \pi/2\), (see (2.3), (2.4), (4.3)).

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