Centro-Affine Tensor Valuations

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Abstract

We completely classify all measurable SL(n)-covariant symmetric tensor valuations on convex polytopes containing the origin in their interiors. It is shown that essentially the only examples of such valuations are the moment tensor and a tensor derived from $L_p$ surface area measures. This generalizes and unifies earlier results for the scalar, vector and matrix valued case.

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1 Introduction

A map $\mu : S \to \langle A, + \rangle$ defined on a collection of sets $S$ with values in an abelian semi-group $\langle A, + \rangle$ is called a valuation if

$$\mu(P \cup Q) + \mu(P \cap Q) = \mu(P) + \mu(Q)$$

whenever $P, Q, P \cup Q, P \cap Q \in S$.

One of the most influential results from the classical Brunn-Minkowski theory is Hadwiger’s classification of continuous rigid motion invariant valuations $\mu : \mathcal{K}^n \to \mathbb{R}$. Here, $\mathcal{K}^n$ denotes the space of convex bodies, i.e. non-empty compact convex subsets of $\mathbb{R}^n$ equipped with the Hausdorff metric. Hadwiger showed that each such valuation is a linear combination of the intrinsic volumes. The latter are of basic geometric nature and include volume, surface area, mean width and the Euler Characteristic.

Two fundamental quantities that are not covered by Hadwiger’s theorem are Blaschke’s equi-affine and centro-affine surface area. The latter is not translation invariant and in fact, does not even fit in the framework of the Brunn-Minkowski theory. It does, however, belong to the so called $L_p$-Brunn-Minkowski theory, which was shaped by Lutwak [31, 32] in the mid 1990s. It is based on Firey’s $L_p$ addition of convex bodies containing the origin in their interiors. The set of all such convex bodies is denoted by $\mathcal{K}_p^n$. Since then, this theory has become a central part of modern convex geometry (see [40, Chapter 9]). The impact of the $L_p$ theory ultimately led to the discovery of an even more general framework: The Orlicz-Brunn-Minkowski theory (see, e.g., [10, 11, 15, 16, 27, 29, 37, 38, 47, 49, 50]).
A characterization of Blaschke’s equi-affine and centro-affine surface area was finally established in a landmark result by Ludwig and Reitzner in [29], where they classified the natural family of Orlicz affine surface areas. However, one crucial part of the problem remained open since one of their assumptions was a certain behavior of the maps on convex polytopes. The first step to bridge this last gap had already been taken by Ludwig [24], but the complete result was only established very recently by the authors [16]:

1.1 Theorem. A map \( \mu : K_n^o \to \mathbb{R} \) is an upper semicontinuous \( SL(n) \)-invariant valuation if and only if there exist constants \( c_0, c_1, c_2 \in \mathbb{R} \) and a function \( \varphi \in \text{Conc}(\mathbb{R}_+) \) such that

\[
\mu(K) = c_0 \chi(K) + c_1 V(K) + c_2 V(K^*) + \Omega_\varphi(K)
\]

for all \( K \in K_n^o \).

Here, \( \chi \) denotes the Euler Characteristic, \( V \) stands for volume, \( K^* \) is the polar body of \( K \), and the \( \Omega_\varphi \) are Orlicz affine surface areas. The reader is referred to Section 2 and [16] for details.

This centro-affine Hadwiger theorem has a discrete version for valuations defined on \( P_n^o \), i.e. convex polytopes containing the origin in their interiors (see [16, 18]): A map \( \mu : P_n^o \to \mathbb{R} \) is a measurable \( SL(n) \)-invariant valuation if and only if there exist constants \( c_0, c_1, c_2 \in \mathbb{R} \) such that

\[
\mu(P) = c_0 \chi(P) + c_1 V(P) + c_2 V(P^*)
\]

for all \( P \in P_n^o \).

The aim of the present paper is to generalize this result to tensor valued valuations of arbitrary rank. Such a generalization to the vector valued case was already established by Ludwig in [23], where she characterized the moment vector, i.e. the centroid without volume normalization. In a highly influential article, Ludwig [25] was also able to show the corresponding result for matrix valued valuations. In both papers, she assumed compatibility with the whole general linear group. A version for the vector valued case that only assumes compatibility with the special linear group was very recently proved by the authors [18]. The present article is the first one to establish a classification for tensor valuations of arbitrary rank in the context of “centro-affine geometry”.

The study of tensor valuations became the focus of increased attention after Alesker’s breakthrough [1]. The new techniques developed in this paper enabled him to prove a long sought after characterization of the rigid motion compatible Minkowski tensors in [2]. In recent years, tensor valuations were studied intensively (see, e.g., [3, 5, 18, 21, 22, 25, 28, 46]). This is in part due to their applications in morphology and anisotropy analysis of cellular, granular or porous structures (see, e.g., [4, 41–43]).

Let us give two examples of tensor valuations. Write \( \text{Sym}^p(\mathbb{R}^n) \subseteq (\mathbb{R}^n)^{\otimes p} \) for the space of symmetric tensors of rank \( p \in \{0, 1, \ldots \} \). The first example is the moment tensor map \( M^{p, 0} : P_n^o \to \text{Sym}^p(\mathbb{R}^n) \). It is defined as

\[
M^{p, 0}(P) = (n + p) \int_P x^{\otimes p} \, dx,
\]
where integration is with respect to Lebesgue measure and \( \odot \) stands for the symmetric tensor product. Ignoring constant factors, \( M^{1,0}(P) \) is the moment vector and \( M^{2,0}(P) \) corresponds to the Legendre ellipsoid from classical mechanics.

The second example, \( M^{0,p}: \mathcal{P}_o^n \to \text{Sym}^p(\mathbb{R}^n) \), is given by

\[
M^{0,p}(P) = \int_{S^{n-1}} u \odot p \, dS_p(P, u).
\] (1)

Here, \( S^{n-1} \subseteq \mathbb{R}^n \) denotes the Euclidean unit sphere and \( S_p(P, \cdot) \) is the \( L_p \) surface area measure of \( P \) (see Section 2 for details). This tensor vanishes for \( p = 1 \) and corresponds to the Lutwak-Yang-Zhang ellipsoid from [35] for \( p = 2 \).

We will prove that these are essentially the only examples of tensor valuations which are compatible with the \( \text{SL}(n) \). This compatibility is contained in the following definition. The group \( \text{GL}(n) \) acts naturally on \( (\mathbb{R}^n)^\otimes p \) by

\[
\phi \cdot x = \phi \odot^p (x)
\]

for all \( \phi \in \text{GL}(n) \) and \( x \in (\mathbb{R}^n)^\otimes p \). A map \( \mu: \mathcal{P}_o^n \to \text{Sym}^p(\mathbb{R}^n) \) is called \( \text{SL}(n) \)-covariant if

\[
\mu(\phi P) = \phi \cdot \mu(P)
\]

for all \( \phi \in \text{SL}(n) \) and each \( P \in \mathcal{P}_o^n \). Moreover, \( \mu \) is called measurable if it is Borel measurable. We are now in a position to state our main result in dimensions greater or equal than three.

1.2 Theorem. Let \( p \geq 2 \) and \( n \geq 3 \). A map \( \mu: \mathcal{P}_o^n \to \text{Sym}^p(\mathbb{R}^n) \) is a measurable \( \text{SL}(n) \)-covariant valuation if and only if there exist constants \( c_1, c_2 \in \mathbb{R} \) such that

\[
\mu(P) = c_1 M^{p,0}(P) + c_2 M^{0,p}(P^*)
\]

for all \( P \in \mathcal{P}_o^n \).

The \( L_p \) surface area measure appearing in (1) is a central notion of the \( L_p \)-Brunn-Minkowski theory. One of the major problems in the field, the so called \( L_p \) Minkowski problem, asks which measures are \( L_p \) surface area measures (see, e.g., [6,8,31,48]). Moreover, \( L_p \) surface area measures found applications in such diverse fields as affine isoperimetric inequalities (see, e.g., [7,19,34]), Sobolev inequalities (see, e.g., [9,20,30]), valuation theory (see, e.g., [14,26,39,41,45]), and information theory (see, e.g., [28,33,36]).

In [31], Lutwak introduced \( L_p \) surface area measures in connection with Firey’s \( L_p \) addition of convex bodies. In some way or the other, the occurrence of these measures is usually related to this \( L_p \) addition. We want to emphasize that by Theorem 1.2 \( L_p \) surface area measures naturally appear in a completely different context. This clearly underlines the basic character of these measures. For an axiomatic characterization of \( L_p \) surface area measures themselves, we refer to [17].
The operators occurring in Theorem 1.2 are special members of the following family of valuations. For \( r, s \in \{0, 1, \ldots \} \) with \( r + s = p \) set
\[
M^{r,s}(P) = \int_{\partial P} x^{\circ r} \odot u_x^{\circ s} h_P^{1-s}(u_x) \, dH^{n-1}(x).
\]
Here, \( h_P \) denotes the support function of \( P \) and integration is with respect to \((n-1)\)-dimensional Hausdorff measure over the boundary \( \partial P \) of \( P \). The vector \( u_x \) is the outer unit normal vector of \( P \) at the boundary point \( x \). Note that \( H^{n-1} \) almost all boundary points have a unique outer unit normal \( u_x \). To the best of our knowledge, these operators have not yet been studied in general.

Except for the extreme cases \( M_{p,0} \) and \( M_0^{0,p} \circ \ast \), the members of the above family are not \( SL(n) \)-covariant. However, in the plane they can be modified in a simple way so that they possess this \( SL(n) \)-covariance. Indeed, we denote by \( \rho \) the counter-clockwise rotation about an angle of \( \frac{\pi}{2} \) and set
\[
M^{r,s}_\rho(P) = \int_{\partial P} x^{\circ r} \odot (\rho u_x)^{\circ s} h_P^{1-s}(u_x) \, dH^1(x).
\]
The planar version of Theorem 1.2 then reads as follows.

1.3 Theorem. Let \( p \geq 2 \). A map \( \mu : \mathcal{P}_o \rightarrow \text{Sym}^p(\mathbb{R}^2) \) is a measurable \( SL(2) \)-covariant valuation if and only if there exist constants \( c_0, \ldots, c_{p-2}, c_p, c_{p+1} \in \mathbb{R} \) such that
\[
\mu(P) = \sum_{i=0}^{p} c_i M^{i,p-i}_\rho(P) + c_{p+1} \rho \cdot M^0_{p}(P^*)
\]
for all \( P \in \mathcal{P}_o^2 \).

Denote by \( \text{TVal}^p(\mathbb{R}^n) \) the vector space of measurable \( SL(n) \)-covariant valuations \( \mu : \mathcal{P}_o^n \rightarrow \text{Sym}^p(\mathbb{R}^n) \). As explained above, a complete classification of \( \text{TVal}^0(\mathbb{R}^n) \) was established in [16,18], whereas the description of \( \text{TVal}^1(\mathbb{R}^n) \) can be found in [18]. In order to get a complete picture, we finally summarize these results and the main theorems of the present article.

1.4 Theorem. For \( n \geq 3 \) the following holds.

- A basis of \( \text{TVal}^0(\mathbb{R}^n) \) is given by \( \chi, V \) and \( V \circ \ast \).
- A basis of \( \text{TVal}^1(\mathbb{R}^n) \) is given by \( M^{1,0} \).
- For \( p \geq 2 \), a basis of \( \text{TVal}^p(\mathbb{R}^n) \) is given by \( M^{p,0} \) and \( M^{0,p} \circ \ast \).

As was mentioned before, the planar case is different and needs to be treated separately.

1.5 Theorem. For \( n = 2 \) the following holds.
• A basis of $\text{TVal}^0(\mathbb{R}^2)$ is given by $\chi$, $V$ and $V \circ \ast$.

• For $p \geq 1$, a basis of $\text{TVal}^p(\mathbb{R}^2)$ is given by $M^{i,p-1}_\rho$ for $i \in \{0, \ldots, p\} \setminus \{p-1\}$ and $\rho \cdot M^{p,0}_\alpha \circ \ast$.

In this paper, we will actually provide a unified proof of Theorems 1.4 and 1.5. Therefore, we also provide new proofs of the results in [16] and [18] which fit into the general context of tensor valuations with arbitrary rank.

2 Notation and preliminary results

For later use, we collect in this section notation and basic facts. Well known results about convex bodies are stated without references. We refer the reader to the excellent books of Gardner [12], Gruber [13], and Schneider [40] for more information.

Let us begin with two one-dimensional facts. The first one is the solution to Cauchy’s functional equation. As is well known, the only measurable functions $f: \mathbb{R} \to \mathbb{R}$ which satisfy

\[ f(x + y) = f(x) + f(y) \]

for all $x, y \in \mathbb{R}$ are the linear ones. The same holds for functions $f: (0, \infty) \to \mathbb{R}$ and $f: \mathbb{R}^n \to \mathbb{R}$. The second one is a version of Vandermonde’s identity,

\[ \sum_{j=0}^{i} \binom{-\frac{p}{2}}{i-j} \binom{\frac{p}{2}}{j} = 0 \quad (2) \]

for $i \geq 1$. This follows from the equality $(1 + x)^{-\frac{p}{2}}(1 + x)^{\frac{p}{2}} = 1$ by comparing coefficients of the Taylor expansions of the involved functions.

Now, we turn towards higher dimensions. The space $\mathbb{R}^n$, $n \geq 1$, will be equipped with the standard inner product and the norm induced by it. Denote by $e_1, \ldots, e_n \in \mathbb{R}^n$ the canonical basis vectors and write $S^{n-1}$ for the set of all unit vectors with respect to this norm.

Throughout this paper, we fix the standard basis of $(\mathbb{R}^n)^{\otimes p}$ induced by the canonical basis vectors $e_1, \ldots, e_n$. For tensors $x_1, \ldots, x_p \in (\mathbb{R}^n)^{\otimes p}$, their symmetric tensor product is defined as

\[ x_1 \odot \cdots \odot x_p = \frac{1}{p!} \sum_{\sigma \in S_p} x_{\sigma(1)} \odot \cdots \odot x_{\sigma(p)}, \]

where $S_p$ denotes the symmetric group of $\{1, 2, \ldots, p\}$. Note that the normalization is chosen in such a way that $x \odot \cdots \odot x = x \otimes \cdots \otimes x$. Let $K \in (\mathbb{R}^2)^{\otimes p}$ and $\alpha \in \{1, 2\}^p$ be a multiindex. If $\phi \in \text{GL}(2)$, then the action of $\phi$ on $K$ can be written as

\[ \phi \cdot K = \sum_{\alpha} \sum_{\beta} K_{\beta} \phi_{\alpha_1 \beta_1} \cdots \phi_{\alpha_p \beta_p} e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_p}. \quad (3) \]

Here, $K_{\beta}$ denote the coefficients of $K$ with respect to the basis we fixed before. Multiindices will be viewed as being equipped with their standard partial order. Therefore,
we immediately arrive at
\[
\begin{bmatrix}
1 & z \\
0 & 1
\end{bmatrix} \cdot K = \sum_{\beta \geq \alpha} K_{\beta z}[\{i : \beta_i = 2\}] - [\{i : \alpha_i = 2\}].
\] (4)

For the space \(\text{Sym}^p(\mathbb{R}^2)\) of symmetric tensors we fix the basis
\[
e_1 \otimes e_2^i, \quad i = 0, \ldots, p.
\]

Let \(K \in \text{Sym}^p(\mathbb{R}^2)\). The coordinates of \(K\) with respect to this basis are denoted by \(K_i\).

Let \(\phi \in \text{GL}(2)\) we therefore have
\[
\begin{bmatrix}
\phi \cdot e_1 \otimes e_2^i \\
\phi \cdot e_1 \otimes e_2^j
\end{bmatrix}_p = (\phi_{21} \phi_{22})^{i, j}.
\] (5)

Relation (4) and a straightforward computation prove
\[
\begin{bmatrix}
1 & z \\
0 & 1
\end{bmatrix} \cdot K_i = \sum_{j=0}^p (j \cdot 1) K_{j z}^{i-j}.
\] (6)

This in turn yields
\[
\begin{bmatrix}
1 & z \\
0 & 1
\end{bmatrix} \cdot K_i = \sum_{j=0}^p (p-j \cdot 1) K_{j z}^{i-j}.
\] (7)

Let us briefly discuss a tensor version of Cauchy’s functional equation. Let \(F: \mathbb{R}^n \to (\mathbb{R}^n) \otimes p\) be a measurable function with
\[
F(x + y) = F(x) + F(y)
\]
for all \(x, y \in \mathbb{R}^n\). Using the scalar Cauchy equation, it is not hard to show that the component functions of \(F\) are linear. Interpreting tensors as multilinear maps, it therefore follows that there exists an \(\tilde{F} \in (\mathbb{R}^n) \otimes p+1\) such that
\[
F(x)(v_1, \ldots, v_p) = \tilde{F}(v_1, \ldots, v_p, x)
\] (8)
for all \(v_1, \ldots, v_p, x \in \mathbb{R}^n\). In other words, \(F\) can be interpreted as an element of \((\mathbb{R}^n) \otimes p+1\).

Next, we collect some facts about tensor integrals. As usual, integrals over tensors are defined componentwise. Thus, a straightforward calculation in combination with (3) proves for a continuous function \(F: [a, b] \to (\mathbb{R}^n) \otimes p\) that
\[
\int_a^b \phi \cdot F(x) dx = \phi \cdot \int_a^b F(x) dx
\] (9)
for all \(\phi \in \text{GL}(n)\). If \(F: \mathbb{R} \to (\mathbb{R}^n) \otimes p\) is continuous, then one can check the symmetry of its images by looking at certain integrals. Indeed, by the componentwise definition of the integral and differentiation we have
\[
F(x) \in \text{Sym}^p(\mathbb{R}^n) \quad \text{for all} \quad x \in \mathbb{R} \iff \int_0^x F(z) dz \in \text{Sym}^p(\mathbb{R}^n) \quad \text{for all} \quad x \in \mathbb{R}.
\] (10)
We conclude our treatment of integrals with the following injectivity type properties. For $K \in (\mathbb{R}^2)_{\otimes p}$, we infer from (4) that
\[
\int_{0}^{x} \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \cdot K \, dz = 0 \quad \text{for some } x \in \mathbb{R} \setminus \{0\} \quad \iff \quad K = 0.
\] (11)

Hence,
\[
K \mapsto \int_{0}^{1} \begin{pmatrix} 1 & z_0 \\ 0 & 1 \end{pmatrix} \cdot K \, dz
\]
is a linear isomorphism on $(\mathbb{R}^2)_{\otimes p}$. A variant of this implication is
\[
\int_{0}^{x} \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} K \, dz = 0 \quad \text{for some } x \in \mathbb{R} \setminus \{0\} \quad \iff \quad K = 0.
\] (12)

Let a convex polytope $P \in \mathcal{P}_o^n$ be given. In the next paragraph we recall some basic geometric quantities associated with $P$. The first example is the support function $h_P$. This is the function $h_P: \mathbb{R}^n \to \mathbb{R}$ defined by
\[ h_P(x) = \max\{x \cdot y : y \in P\}.
\]
The polar body $P^*$ of $P$ is given by
\[ P^* = \{x \in \mathbb{R}^n : x \cdot y \leq 1 \text{ for all } y \in P\}.
\] Note that for each $\phi \in \text{GL}(n)$ we have
\[ (\phi P)^* = \phi^{-t} P^*, \] (13)
where $\phi^{-t}$ denotes the inverse of the transpose of $\phi$. In particular,
\[ (\lambda P)^* = \lambda^{-1} P^* \] (14)
for all positive $\lambda$. We define the polarity map $^*: \mathcal{P}_o^n \to \mathcal{P}_o^n$ as the function which assigns to each polytope its polar body. It is well known that this map is a homeomorphism involution. The surface area measure $S(P, \cdot)$ is defined for each Borel set $\omega \subseteq S^{n-1}$ as
\[ S(P, \omega) = \mathcal{H}^{n-1}\{x \in \partial P : \exists \text{ an outer unit normal } u_x \text{ at } x \text{ which belongs to } \omega\}.
\]
Surface area measures have their centroid at the origin, i.e.
\[ \int_{S^{n-1}} u \, dS(P, u) = 0 \] (15)
for all $P \in \mathcal{P}_o^n$. The $L_p$ surface area measure $S_p(P, \cdot)$ is given by
\[ S_p(P, \omega) = \int_{\omega} h_P^{1-p}(u) \, dS(P, u).
\]
Next, we will generalize the concept of SL($n$)-covariance a little bit. We write SL$^\pm(n)$ for the set of linear maps having determinant either 1 or $-1$. Let $\varepsilon \in \{0, 1\}$ and $G \subseteq SL^\pm(n)$ be given. A map $\mu: P_o^n \to \text{Sym}^p(\mathbb{R}^n)$ is said to be $G$-$\varepsilon$-covariant or $\varepsilon$-covariant with respect to $G$ if

$$\mu(\phi P) = (\det \phi)^\varepsilon \phi \cdot \mu(P)$$

for every $P \in P_o^n$ and each $\phi \in G$. In order to simplify the notation in the sequel, we write TVal$^\varepsilon_p(\mathbb{R}^n)$ for the vector space of measurable SL$^\pm(n)$-$\varepsilon$-covariant valuations.

Let $\mu \in \text{TVal}^p(\mathbb{R}^n)$. Choose a $\theta \in SL^\pm(n) \setminus SL(n)$. For all $P \in P_o^n$ define

$$\mu^0(P) = \frac{1}{2} \left( \mu(P) + \theta \cdot \mu(\theta^{-1} P) \right)$$

and

$$\mu^1(P) = \frac{1}{2} \left( \mu(P) - \theta \cdot \mu(\theta^{-1} P) \right).$$

The SL($n$)-covariance of $\mu$ implies that these definitions do not depend on the choice of $\theta$. Clearly, $\mu^0$ and $\mu^1$ are measurable valuations and $\mu = \mu^0 + \mu^1$. Moreover, it is easy to see that $\mu^0 \in \text{TVal}^0_p(\mathbb{R}^n)$ and $\mu^1 \in \text{TVal}^1_p(\mathbb{R}^n)$. Hence,

$$\text{TVal}^p(\mathbb{R}^n) = \text{TVal}^0_p(\mathbb{R}^n) \oplus \text{TVal}^1_p(\mathbb{R}^n). \quad (16)$$

The convex hull of a set $A \subseteq \mathbb{R}^n$ is written as $[A]$. In the context of double pyramids, the following symbols will usually have a fixed meaning. The letters $a, b, c, d$ will denote positive real numbers with associated line segments $I := [-ae_1, be_1]$ and $J := [-ce_n, de_n]$, respectively. The letters $x, y$ will denote elements of $\mathbb{R}^{n-1}$. In particular, for $n = 2$ we have $J = [-ce_2, de_2]$ and $x, y \in \mathbb{R}$. The letter $B$ will denote an element of $P_o^{n-1}$. For $n = 2$, we say that $a, b, c, d, x, y$ form a double pyramid if

$$\left[I, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix}\right] \cap e_2^\perp = I$$

and for $n \geq 3$, we say that $B, c, d, x, y$ form a double pyramid if

$$\left[B, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix}\right] \cap e_n^\perp = B.$$

If $x = y = 0$, then we call the double pyramid straight. The set of double pyramids will be denoted by $R^n$ and the set of straight double pyramids by $Q^n$. In [24], Ludwig proved that if a real valued valuation $\mu: P_o^n \to \mathbb{R}$ vanishes on all SL($n$)-images of elements in $R^n$, then it vanishes on $P_o^n$. A componentwise application of this fact yields the following result.

**2.1 Theorem.** Let $n \geq 2$. Suppose that $\mu: P_o^n \to (\mathbb{R}^n)^{\otimes p}$ is a valuation which vanishes on all SL($n$)-images of elements in $R^n$. Then $\mu$ vanishes everywhere.
Clearly, for \( n = 2 \), straight double pyramids can be split into two triangles having one side contained in \( e_2^\perp \). The set of such straight triangles

\[
[I, -ce_2] \quad \text{and} \quad [I, de_2]
\]
is denoted by \( T^2 \).

We need an explicit description of \( M_{i,p}^{i,p-i} \) on straight double pyramids. We start with the following calculation. Note that in the next two lemmas we use the common convention that

\[
\binom{i}{j} = 0 \quad \text{if} \quad j < 0 \quad \text{or} \quad j > i. \tag{17}
\]

2.2 Lemma. For \( b, c > 0 \) and \( i \in \{0, \ldots, p\} \), we have

\[
(i + 1)(bc)^{1-p+i} \int_0^1 \left( \frac{bt}{-c(1-t)} \right)^i \left( \frac{b}{c} \right)^{p-i} dt = \sum_{l=0}^p m_{i,l} b^{1+p-l} c^{1-p+i+l} e_1 e_2^\perp \quad \text{where}
\]

\[
m_{i,l} = \binom{p-i-1}{l} + (-1)^j \binom{p-i-1}{l-j-1}
\]

for \( i \neq p \) and

\[
m_{p,l} = (-1)^l.
\]

In particular, for \( i \neq p \), \( m_{i,l} = 0 \) if \( p - i \leq l \leq i \).

Proof. Define \( L \) as the left hand side of (18). Writing the first tensor product in coordinates, using well known results for the Beta function, and writing the second tensor product in coordinates, we calculate

\[
L = (i + 1) \sum_{j=0}^i \binom{i}{j} (-1)^j b^{1+p-2i-j} c^{1-p+i+j} \int_0^1 t^{i-j} (1-t)^j dt e_1 e_2^\perp \left( \frac{b}{c} \right)^{p-i}
\]

\[
= \sum_{j=0}^i (-1)^j b^{1+p+2i-j} c^{1-p+i+j} e_1 e_2^\perp \left( \frac{b}{c} \right)^{p-i}
\]

\[
= \sum_{j=0}^i \sum_{k=0}^{p-i} \binom{p-i-1}{l} (-1)^j b^{1+p-j-k} c^{1-p+i+j+k} e_1 e_2^\perp \left( \frac{b}{c} \right)^{p-j+k}.
\]

Summing first over \( l = j + k \) and keeping convention (17) for the binomial coefficient in mind, this becomes

\[
L = \sum_{l=0}^p \sum_{j=0}^l \binom{p-i-1}{l-j} (-1)^j b^{1+p-l} c^{1-p+i+l} e_1 e_2^\perp \left( \frac{b}{c} \right)^{p-l}.
\]

For \( i = p \), we clearly have

\[
\sum_{j=0}^l \binom{0}{l-j} (-1)^j = (-1)^l.
\]
Assume \( i \neq p \). A well known formula for an alternating sum of binomial coefficients states

\[
\sum_{j=0}^{l} \binom{p-i}{j} (-1)^j = (-1)^l \binom{p-i-1}{l}.
\]

Note that \( p-i \geq 1 \) and that we again use convention (17) for the binomial coefficient from above. Hence,

\[
\sum_{j=0}^{i} \binom{p-i}{l-j} (-1)^j = (-1)^l \sum_{j=l-i}^{l} \binom{p-i}{j} (-1)^j = (-1)^l \binom{p-i-1}{l-i-1}.
\]

With the aid of this result, we can now calculate \( M^{i,p}_{\rho} \) on straight double pyramids.

2.3 Lemma. For \( i \in \{0, \ldots, p\} \), we have

\[
M^{i,p}_{\rho}[I, J] = \frac{1}{i+1} \sum_{l=0}^{p} m_{i,l} \left[ (-1)^{i+l} a^{i+1-i-l} c^{1-p+i+l} + b^{1+i-l} c^{1-p+i+l} \right. \\
\left. + (-1)^l a^{1+i-l} d^{1-p+i+l} + (-1)^{p+i+l} b^{1+i-l} d^{1-p+i+l} \right] e_1^{\rho-l} \circ e_2^{\rho-l}
\]

for all \( a,b,c,d > 0 \), where \( m_{i,l} \) is defined as in Lemma 2.2. In particular, for \( p \geq 1 \), \( M^{p-1,1}_{\rho}[I, J] = 0 \).

Proof. Clearly, the double pyramid \([I, J]\) has four edges. Hence, the defining integral of \( M^{i,p}_{\rho} \) can be split into four integrals along these line segments. Let us consider the edge \([be_1, -ce_2]\). Using the parametrization

\[
\gamma(t) = tbe_1 - (1-t)ce_2, \quad t \in [0,1],
\]

an elementary calculation shows that

\[
\int_{[be_1, -ce_2]} x^{\rho-l} \circ (\rho u_x)^{\rho-p-i} d\mathcal{H}(x)
\]

equals, up to a factor of \( i+1 \), the integral considered in Lemma 2.2. Similar observations are true for the other three edges. Summing the expressions from Lemma 2.2 for all edges yields the desired result. Finally, for \( i \neq p \), note that the terms for \( l = i + 1 \) as well as \( l = p - i - 1 \) cancel out. \( \square \)

Let \( B = [I, J] \). Note that \( B^* = [-a^{-1}, b^{-1}] \times [-c^{-1}, d^{-1}] \). Thus,

\[
[M^{p,0}(B^*)]_p = \frac{1}{p+1} \left( a^{-1} + b^{-1} \right) \left( d^{-p-1} + (-1)^p c^{-p-1} \right).
\]
In combination with the last lemma, we see that
\[ [M_{i,p}^{i,p-i}(B)]_0, \quad i \in \{0, \ldots, p\}\setminus\{p-1\}, \quad \text{and} \quad [\rho \cdot M_{p,0}^{p,0}(B^*)]_0 \]
(19)
do not vanish for all double pyramids \(B\). It follows immediately from their definition, that the \(M_{i,p}^{i,p-i}\) are homogeneous, i.e.
\[ M_{i,p}^{i,p-i}(\lambda P) = \lambda^{2-p+i}M_{i,p}^{i,p-i}(P) \]
(20)
for all \(P \in \mathcal{P}_o^2\) and \(\lambda > 0\). Combining the last two facts and (14), it is easy to see that the family
\[ M_{i,p}^{i,p-i}, i \in \{0, \ldots, p\}\setminus\{p-1\}, \quad \rho \cdot M_{p,0}^{p,0} \circ * \]
is linearly independent.

We also state a few simple facts about \(M_{i,p}^{i,p-i}\) in dimension \(n \geq 2\) for later reference. Clearly, these maps are homogeneous,
\[ M_{i,p}^{i,p-i}(\lambda P) = \lambda^{n-p+i}M_{i,p}^{i,p-i}(P) \]
(21)
for all \(P \in \mathcal{P}_o^n\) and \(\lambda > 0\). The \(e_1^{\otimes p}\)-coordinates of
\[ M_{p,0}^{p,0}(B^*) \quad \text{and} \quad M_{0,p}^{0,p}(B) \]
do not vanish for all crosspolytopes \(B\) except for \(M_{0,1}^{0,1}(B)\). To see this, one can look at crosspolytopes that are sufficiently asymmetric with respect to \(e_1^{\perp}\). Combining the last two facts and (14), it is easy to see that
\[ M_{p,0}^{p,0} \circ * \quad \text{and} \quad M_{0,p}^{0,p} \]
are linearly independent.

Next, we show that \(M_{i,p}^{i,p-i}\), for \(n = 2\), and \(M_{i,p}^{i,p-i}\), for \(n \geq 2\), are valuations. The easiest way to see this is to write them as integrals over the support measure \(\Lambda_{n-1}\) (see [40, Chapter 4]). For the latter, we have
\[ M_{i,p}^{i,p-i}(P) = 2 \int_{\mathbb{R} \times S^{n-1}} x^{\otimes i} \otimes u^{\otimes p-i} (x \cdot u)^{1-p+i} d\Lambda_{n-1}(P, (x, u)) \]
for all \(P \in \mathcal{P}_o^n\). Now, the valuation property, the covariance properties, as well as the continuity of these maps follow from similar properties of the support measure \(\Lambda_{n-1}\). In particular,
\[ M_{i,p}^{i,p-i}(\phi P) = (\det \phi)^{p-i} \phi \cdot M_{i,p}^{i,p-i}(P), \]
(23)
for all \(P \in \mathcal{P}_o^2\) and \(\phi \in \text{SL}^\pm(2)\). Furthermore,
\[ M_{p,0}^{p,0}(\phi P) = \phi \cdot M_{p,0}^{p,0}(P) \quad \text{and} \quad M_{0,p}^{0,p}((\phi P)^*) = \phi \cdot M_{0,p}^{0,p}(P^*) \]
(24)
for all \(P \in \mathcal{P}_o^n\) and \(\phi \in \text{SL}^\pm(n)\).
3 Proof of the Main Results

3.1 The 1-dimensional case

We aim at a description of $TV_{\mathcal{P}}(\mathbb{R}^1)$. Note that $(\mathbb{R}^1)^{\otimes p}$ is always isomorphic to $\mathbb{R}$. Moreover, an SL$(1)$-covariant map is either even or odd. So it suffices to classify even and odd valuations $\mu: \mathcal{P}^1_o \to \mathbb{R}$, respectively. Such classifications were already established in [16, 18] and are stated below. Let us begin with the even case.

3.1 Theorem. Suppose that $\mu: \mathcal{P}^1_o \to \mathbb{R}$ is a measurable valuation. Then $\mu$ is even if and only if there exists a measurable function $F: (0, \infty) \to \mathbb{R}$ such that

$$\mu[-a, b] = F(a) + F(b)$$

for all $a, b > 0$. Moreover, $F(a) = \frac{1}{2}\mu[-a, a]$.

For homogeneous valuations even more can be said. In fact, the function $F$ from the above theorem can be described explicitly.

3.2 Theorem. Suppose that $\mu: \mathcal{P}^1_o \to \mathbb{R}$ is a measurable valuation. Then $\mu$ is even and homogeneous of degree $r \in \mathbb{R}$ if and only if there exists a constant $c \in \mathbb{R}$ such that

$$\mu[-a, b] = c(a^r + b^r)$$

for all $a, b > 0$.

Next, we state the corresponding classifications for odd valuations.

3.3 Theorem. Suppose that $\mu: \mathcal{P}^1_o \to \mathbb{R}$ is a measurable valuation. Then $\mu$ is odd if and only if there exists a measurable function $F: (0, \infty) \to \mathbb{R}$ such that

$$\mu[-a, b] = F(b) - F(a)$$

for all $a, b > 0$. Moreover, $F(a) = \mu[-1, a] + c$ for some constant $c \in \mathbb{R}$.

As before, an immediate consequence of Theorem 3.3 is a classification of odd homogeneous valuations.

3.4 Theorem. Suppose that $\mu: \mathcal{P}^1_o \to \mathbb{R}$ is a measurable valuation. Then $\mu$ is odd and homogeneous of degree $r \in \mathbb{R} \setminus \{0\}$ if and only if there exists a constant $c \in \mathbb{R}$ such that

$$\mu[-a, b] = c(b^r - a^r)$$

for all $a, b > 0$.

The valuation $\mu$ is odd and homogeneous of degree 0 if and only if there exists a constant $c \in \mathbb{R}$ such that

$$\mu[-a, b] = c[\ln(b) - \ln(a)]$$

for all $a, b > 0$.

We remark that every valuation $\mu: \mathcal{P}^1_o \to \mathbb{R}$ can be written as the sum of an even and an odd valuation. Therefore, the above theorems yield a classification of all measurable valuations $\mu: \mathcal{P}^1_o \to \mathbb{R}$. 12
3.2 The 2-dimensional case

3.2.1 Some tensor equations

We begin by solving a sheared version of Cauchy’s functional equation for tensors.

3.5 Lemma. Suppose that $G: \mathbb{R} \to (\mathbb{R}^2)^\otimes p$ is a measurable function. Then $G$ satisfies

$$G(x + y) = G(x) + \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot G(y) \quad (25)$$

for all $x, y \in \mathbb{R}$ if and only if there exists a tensor $K \in (\mathbb{R}^2)^\otimes p$ such that

$$G(x) = \int_0^x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K \, dz. \quad (26)$$

Moreover, if $G$ has symmetric images, then $K \in \text{Sym}^p(\mathbb{R}^2)$. Furthermore, the same results hold if $G$ is only defined on $(0, \infty)$.

Proof. An elementary calculation combined with (9) proves

$$\int_0^{x+y} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K \, dz = \int_0^x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K \, dz + \int_x^{x+y} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K \, dz$$

$$= \int_0^x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K \, dz + \int_0^y \begin{pmatrix} 1 & z + x \\ 0 & 1 \end{pmatrix} \cdot K \, dz$$

$$= \int_0^x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K \, dz + \int_0^y \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \int_0^y \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K \, dz \quad (27)$$

for each $K \in (\mathbb{R}^2)^\otimes p$. So each $G$ defined by (26) satisfies (25).

Now, let $G$ be a solution of (25). By (11) we can find a $K \in (\mathbb{R}^2)^\otimes p$ such that

$$H(x) := G(x) - \int_0^x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K \, dz$$

satisfies $H(1) = 0$. It remains to prove that $H(x) = 0$ for all $x \in \mathbb{R}$. We will show by induction that $H_\alpha = 0$ for all $\alpha \in \{1, 2\}^p$. Assume that $H_\beta = 0$ for all $\beta > \alpha$, which is trivially true for $\alpha = (2, \ldots, 2)$. Equation (25) is clearly a linear functional equation. Hence, by the definition of $H$ and (27), $H$ satisfies (25). So (4) and the induction assumption yield

$$H_\alpha(x + y) = H_\alpha(x) + H_\alpha(y).$$

Thus, $H_\alpha$ satisfies Cauchy’s functional equation. Since $H_\alpha$ is measurable and $H_\alpha(1) = 0$, it follows that $H_\alpha = 0$.

We still have to prove the assertion about symmetric tensors. By assumption,

$$G(x) = \int_0^x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K \, dz \in \text{Sym}^p(\mathbb{R}^2)$$
for all \( x \in \mathbb{R} \). So from (10) we infer that
\[
\begin{pmatrix}
1 & z \\
0 & 1
\end{pmatrix} \cdot K \in \text{Sym}^p(\mathbb{R}^2)
\]
for all \( z \in \mathbb{R} \). Since the above matrix is invertible, also \( K \in \text{Sym}^p(\mathbb{R}^2) \). The proof for \( G: (0, \infty) \to (\mathbb{R}^2)^{\otimes p} \) is exactly the same.

The next result is a slightly different version of Lemma 3.5.

**3.6 Lemma.** Suppose that \( G: \mathbb{R} \to (\mathbb{R}^2)^{\otimes p} \) is a measurable function. Then \( G \) satisfies
\[
G(x + y) = G(x) + \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \cdot G(y) \tag{28}
\]
for all \( x, y \in \mathbb{R} \) if and only if there exists a tensor \( K \in (\mathbb{R}^2)^{\otimes p} \) such that
\[
G(x) = \int_0^x \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} \cdot K \, dz.
\]
Moreover, if \( G \) has symmetric images, then \( K \in \text{Sym}^p(\mathbb{R}^2) \). Furthermore, the same results hold if \( G \) is only defined on \((0, \infty)\).

**Proof.** Define a function \( H: \mathbb{R} \to (\mathbb{R}^2)^{\otimes p} \) by
\[
H(x) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot G(x).
\]
Then \( G \) satisfies (28) if and only if \( H \) satisfies (25). By Lemma 3.5, this happens precisely if there exists a \( J \in (\mathbb{R}^2)^{\otimes p} \) with
\[
H(x) = \int_0^x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot J \, dz.
\]
Rewriting \( H \) in terms of \( G \) and setting
\[
K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot J
\]
closes the proof.

Using Lemma 3.5, we now establish the solution of a more intricate functional equation. In fact, this functional equation will be crucial for the proof of our main theorem.

**3.7 Lemma.** Let \( \varepsilon \in \{0, 1\} \) and \( F: (0, \infty) \to (\mathbb{R}^2)^{\otimes p} \) be a measurable function. The function \( F \) satisfies
\[
F(t) = \begin{pmatrix} 1 & 0 \\ \frac{1}{st} & 1 \end{pmatrix} \cdot F\left(\frac{st}{s+1}\right) + (-1)^{\varepsilon} \begin{pmatrix} s & t \\ \frac{t}{s+1} & 0 \end{pmatrix} \cdot F\left(\frac{t}{s+1}\right) \tag{29}
\]
and

\[ (-1)^{c} \begin{pmatrix} 0 & s \\ \frac{1}{s} & 0 \end{pmatrix} \cdot F(s) = F(s) \] (30)

for all \( s, t > 0 \) if and only if there exists a tensor \( K \in (\mathbb{R}^2)^{\otimes p} \) with

\[ (-1)^{c} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot K = K \] (31)

such that

\[ F(x) = \int_{0}^{x} \left( \begin{array}{cc} 1 & z \\ -\frac{1}{x} & 1 - \frac{x}{z} \end{array} \right) \cdot K \, dz, \quad x > 0. \] (32)

**Proof.** Define a function \( G: (0, \infty) \to (\mathbb{R}^2)^{\otimes p} \) by

\[ G(x) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot F(x). \]

We will first show that \( F \) satisfies (29) and (30) for all \( s, t > 0 \) if and only if \( G \) satisfies

\[ G(x + y) = G(x) + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot G(y) \] (33)

and

\[ (-1)^{c} \begin{pmatrix} -1 & x \\ 0 & 1 \end{pmatrix} \cdot G(x) = G(x) \] (34)

for all \( x, y > 0 \). In order to do so, we consider the coordinate transformation \( s = \frac{x}{y} \) and \( t = x + y \). Multiplying (29) by

\[ \begin{pmatrix} 1 \\ -\frac{1}{t} \\ 1 \end{pmatrix} \]

and rewriting the resulting equation in terms of \( x \) and \( y \) shows that (29) is equivalent to

\[ \begin{pmatrix} 1 \\ -\frac{1}{x+y} \\ 1 \end{pmatrix} \cdot F(x + y) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \cdot F(x) + (-1)^{c} \begin{pmatrix} \frac{y}{x} \\ x + y \\ 1 \end{pmatrix} \cdot F(y). \]

By the definition of \( G \), the last equation holds precisely if

\[ G(x + y) = G(x) + (-1)^{c} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot G(y). \] (35)

Clearly, \( F \) satisfies (30) if and only if \( G \) satisfies (34). In combination with (35) this proves the desired equivalence

\( F \) satisfies (29) and (30) for all \( s, t > 0 \) if and only if \( G \) satisfies (33) and (34) for all \( x, y > 0 \).
From Lemma 3.5 we infer that $G$ solves (33) if and only if
\[ G(x) = \int_0^x \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \cdot K \, dz \]
for some tensor $K \in (\mathbb{R}^2)^{\otimes p}$. By (9) and a substitution we obtain
\[ \left( -1 \begin{array}{c} x \\ 0 \\ 1 \end{array} \right) \cdot \int_0^x \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \cdot K \, dz = \int_0^x \left( -1 \begin{array}{c} x \cdot z \\ 0 \\ 1 \end{array} \right) \cdot K \, dz. \]
Using (11), we see that $G$ satisfies (34) if and only if
\[ (-1)^{\varepsilon} \left( -1 \begin{array}{c} 0 \\ 0 \\ 1 \end{array} \right) \cdot K = K. \]
Rewriting $G$ in terms of $F$ concludes the proof.

### 3.2.2 Splitting over pyramids

Let $\mu \in \text{TVal}_p^p(\mathbb{R}^2)$. We say that $\mu$ splits over pyramids if the following three conditions hold. First, there is a measurable map $\tilde{\mu} : \mathbb{T}^2 \to \text{Sym}^p(\mathbb{R}^2)$ with
\[ \mu[I,J] = \tilde{\mu}[I,-ce_2] + \tilde{\mu}[I,de_2] \]
for all $a, b, c, d > 0$. Recall that by our notation convention we set $I = [-ae_1,be_1]$ and $J = [-ce_2,de_2]$. Second, for all $c, d > 0$ the maps
\[ I \mapsto \tilde{\mu}[I,-ce_2] \quad \text{and} \quad I \mapsto \tilde{\mu}[I,de_2] \]
are valuations on $P^1_0$. Third, $\tilde{\mu}$ is $\varepsilon$-covariant with respect to the transformations
\[ \left( \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right) \quad \text{and} \quad \left( \begin{array}{c} 1 \\ 0 \\ -1 \end{array} \right). \]

In Subsection 3.2.3 we will construct splittings explicitly. However, for now we assume that such a splitting exists.

Clearly, a double pyramid can be divided into two separately tilted triangles. The idea of the next lemma is to compare the value of $\mu$ on a double pyramid with the values of a splitting on these triangles. As it turns out, the error term in this comparison has suprisingly nice properties.

In the sequel, we will repeatedly use the following obvious fact. If $a, b > 0$ and $x, y \in \mathbb{R}$ are given, then for sufficiently small $c, d > 0$ the numbers $a, b, c, d, x, y$ form a double pyramid.

#### 3.8 Lemma. Let $\mu \in \text{TVal}_p^p(\mathbb{R}^2)$. If $\mu$ splits over pyramids, then there exists a family of functions $F^I : \mathbb{R}^2 \to \text{Sym}^p(\mathbb{R}^2)$ such that
\[ \mu\left[ I, -c \left( \begin{array}{c} x \\ 1 \end{array} \right), d \left( \begin{array}{c} y \\ 1 \end{array} \right) \right] = \left( \begin{array}{c} 1 \\ x \\ 0 \end{array} \right) \cdot \tilde{\mu}[I,-ce_2] + \left( \begin{array}{c} 1 \\ y \\ 0 \end{array} \right) \cdot \tilde{\mu}[I,de_2] + F^I(x,y) \quad (36) \]
for all \( a, b, c, d > 0 \) and \( x, y \in \mathbb{R} \) which form a double pyramid. Furthermore, each \( F^I \) satisfies

\[
F^I(x, y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot F^I(0, y - x)
\]

(37)

and

\[
F^I(0, x + y) = F^I(0, x) + \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot F^I(0, y)
\]

(38)

for all \( x, y \in \mathbb{R} \).

Proof. Let \( a, b > 0 \) and \( x, y \in \mathbb{R} \) be given. Choose \( c, d > 0 \) such that \( a, b, c, d, x, y \) form a double pyramid. For sufficiently small \( r > 0 \) the valuation property implies

\[
\mu \left[ I, \left( -c \begin{pmatrix} x \\ 1 \end{pmatrix}, \left( y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \right] + \mu \left[ I, \left( -r \begin{pmatrix} y \\ 1 \end{pmatrix}, \left( y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \right] = \\
\mu \left[ I, \left( -c \begin{pmatrix} x \\ 1 \end{pmatrix}, \left( y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \right] + \mu \left[ I, \left( -r \begin{pmatrix} y \\ 1 \end{pmatrix}, \left( y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \right].
\]

Since \( \mu \) is SL\(^{\pm}(2)\)-\( \varepsilon \)-covariant and splits over pyramids, we have

\[
\mu \left[ I, \left( -c \begin{pmatrix} x \\ 1 \end{pmatrix}, \left( y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \right] - \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cdot \bar{\mu}[I, de_2] = \\
\mu \left[ I, \left( -c \begin{pmatrix} x \\ 1 \end{pmatrix}, \left( y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \right] - \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cdot \bar{\mu}[I, re_2].
\]

In other words, the expression on the left hand side is independent of \( d \). Similarly,

\[
\mu \left[ I, \left( -c \begin{pmatrix} x \\ 1 \end{pmatrix}, \left( y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \right] - \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \bar{\mu}[I, -ce_2]
\]

is independent of \( c \). Consequently, the term

\[
\mu \left[ I, \left( -c \begin{pmatrix} x \\ 1 \end{pmatrix}, \left( y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \right) \right] - \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \bar{\mu}[I, -ce_2] - \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cdot \bar{\mu}[I, de_2]
\]

is independent of \( c \) and \( d \). This proves the existence of functions \( F^I \) which satisfy (36). Next, we establish relation (37). By the SL\(^{\pm}(2)\)-\( \varepsilon \)-covariance of \( \mu \) and equation (36)
we have
\begin{align*}
\mu \left[ I, -c \left( \frac{x}{1} \right), d \left( \frac{y}{1} \right) \right] \\
= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \mu \left[ I, -ce_2, d \left( \frac{y-x}{1} \right) \right] \\
= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \left( \tilde{\mu}[I, -ce_2] + \begin{pmatrix} 1 & y-x \\ 0 & 1 \end{pmatrix} \cdot \tilde{\mu}[I, de_2] + F^I(0, y-x) \right) \\
= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \tilde{\mu}[I, -ce_2] + \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cdot \tilde{\mu}[I, de_2] + \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot F^I(0, y-x).
\end{align*}

A glance at (36) quickly yields (37).
It remains to show (38). For sufficiently small \( r > 0 \) the valuation property implies
\begin{align*}
\mu \left[ I, -c \left( \frac{x}{1} \right), d \left( \frac{y}{1} \right) \right] + \mu \left[ I, -re_2, re_2 \right] = \mu \left[ I, -c \left( \frac{x}{1} \right), re_2 \right] + \mu \left[ I, -re_2, d \left( \frac{y}{1} \right) \right].
\end{align*}

Using (36) and the fact that \( \mu \) splits over pyramids give
\begin{align*}
F^I(x, y) = F^I(x, 0) + F^I(0, y).
\end{align*}

With the aid of (37) we finally arrive at
\begin{align*}
\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot F^I(0, y-x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot F^I(0, -x) + F^I(0, y).
\end{align*}

Replacing \( x \) by \( -x \) immediately yields (38).

Now, we are going to use the solution of the sheared Cauchy equation (25) to get a more explicit representation for the \( F^I \).

3.9 Lemma. Let \( \mu \in \text{Val}^p_\varepsilon(\mathbb{R}^2) \). If \( \mu \) splits over pyramids, then there exists a family of tensors \( K^I \in \text{Sym}^p(\mathbb{R}^2) \) such that
\begin{align*}
\mu \left[ I, -c \left( \frac{x}{1} \right), d \left( \frac{y}{1} \right) \right] = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \tilde{\mu}[I, -ce_2] + \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \cdot \tilde{\mu}[I, de_2] + \int_x^y \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K^I \, dz \tag{39}
\end{align*}
for all \( a, b, c, d > 0 \) and \( x, y \in \mathbb{R} \) which form a double pyramid.

Proof. Fix an interval \( I \) and let \( F^I \) be the function from Lemma 3.8. By (38) and Lemma 3.5 there exists a \( K^I \in \text{Sym}^p(\mathbb{R}^2) \) with
\begin{align*}
F^I(0, x) = \int_0^x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K^I \, dz \tag{40}
\end{align*}
for all $x \in \mathbb{R}$. From relations (37), (40), a substitution, and (9) we obtain
\[
F^{I}(x, y) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot F^{I}(0, y - x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \int_{0}^{y-x} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K^{I} \, dz
\]
\[
= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot \int_{x}^{y} \begin{pmatrix} 1 & z - x \\ 0 & 1 \end{pmatrix} \cdot K^{I} \, dz
\]
\[
= \int_{x}^{y} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \cdot K^{I} \, dz.
\]

Thus, equation (36) immediately implies (39). \qed

3.2.3 The main results

After these preparations we will now prove our description of TValp($\mathbb{R}^2$). We start by showing that every $\mu \in TValp(\mathbb{R}^2)$ splits over pyramids. Recall that we fixed a basis $e_i^{\oplus p-i} \circ e_2^i$, $i = 0, \ldots, p$
and that the $i$-th coordinate of $\mu$ with respect to this basis is denoted by $\mu_i$. 

3.10 Lemma. Each $\mu \in TValp(\mathbb{R}^2)$ splits over pyramids. Furthermore, there exists a splitting with the following two properties: For $i \in \{0, \ldots, p\}$ and $a, b, d > 0$,
\[
\tilde{\mu}_i[I, de_2] = d^{2i-p} \tilde{\mu}_i[I, e_2]
\]
if $i + \varepsilon$ is even and
\[
\tilde{\mu}_i[I, e_2] = 0
\]
if $i + \varepsilon$ is odd.

Proof. We begin with the simple observation that $J \mapsto \mu_i[I, J]$ is a measurable valuation. By the SL±(2)-$\varepsilon$-covariance of $\mu$ we therefore obtain
\[
\mu_i[I, -J] = (-1)^{i+\varepsilon} \mu_i[I, J].
\]

Let $c, d > 0$. Define a map $\tilde{\mu} : T^2 \to Sym^p(\mathbb{R}^2)$ componentwise by
\[
\tilde{\mu}_i[I, -ce_2] = \frac{1}{2} \mu_i[I, -ce_2, ce_2], \quad \tilde{\mu}_i[I, de_2] = \frac{1}{2} \mu_i[I, -de_2, de_2]
\]
for even $i + \varepsilon$ and
\[
\tilde{\mu}_i[I, -ce_2] = -\mu_i[I, -e_2, ce_2], \quad \tilde{\mu}_i[I, de_2] = \mu_i[I, -e_2, de_2]
\]
for odd $i + \varepsilon$. 19
Next, we will show that relations (41) and (42) hold. If \( i + \varepsilon \) is even, then the definition of \( \tilde{\mu} \) and the SL\(^+\)(2)-\(\varepsilon\)-covariance of \( \mu \) yield
\[
\tilde{\mu}_i[I,de_2] = \frac{1}{2} \mu_i[I,-de_2,de_2]
= \frac{1}{2} \left[ \begin{pmatrix} \frac{1}{d} & 0 \\ 0 & d \end{pmatrix} \mu[I,-e_2,e_2] \right],
= \frac{1}{2} d^{2i-\mu_i}[dI,-e_2,e_2],
= d^{2i-\mu_i}[dI,e_2].
\]
Hence, relation (41) holds. If \( i + \varepsilon \) is odd, then the definition of \( \tilde{\mu} \) and (43) give
\[
\tilde{\mu}_i[I,e_2] = \mu_i[I,-e_2,e_2] = 0.
\]
So \( \tilde{\mu} \) satisfies (42).

It remains to show that \( \tilde{\mu} \) is actually a splitting. First, suppose that \( i + \varepsilon \) is even. From (43) and Theorem 3.1 we infer
\[
\mu_i[I,J] = \frac{1}{2} \mu_i[I,-ce_2,ce_2] + \frac{1}{2} \mu_i[I,-de_2,de_2]
= \tilde{\mu}_i[I,-ce_2] + \tilde{\mu}_i[I,de_2].
\]
Second, let \( i + \varepsilon \) be odd. By relation (43) and Theorem 3.3 we obtain
\[
\mu_i[I,J] = \mu_i[I,-e_2,de_2] - \mu_i[I,-e_2,ce_2]
= \tilde{\mu}_i[I,-ce_2] + \tilde{\mu}_i[I,de_2].
\]
So \( \tilde{\mu} \) has the additivity property required for a splitting. From the definition of \( \tilde{\mu} \) and the respective properties of \( \mu \) it follows easily that \( \tilde{\mu} \) possesses the desired valuation and covariance property.

Recall from Lemma 3.9 that a splitting can be used to describe \( \mu \) on double pyramids. Our next result reveals that the above splitting can be modified in such a way that it is determined by a function \( F: (0, \infty) \to \text{Sym}^p(\mathbb{R}^2) \). Moreover, the error term in Lemma 3.9 can be calculated explicitely.

3.11 Lemma. Let \( \mu \in \text{TVal}^p(\mathbb{R}^2) \). There exist a measurable function \( F: (0, \infty) \to \text{Sym}^p(\mathbb{R}^2) \) and a constant \( k \in \mathbb{R} \) such that
\[
\mu \left[ I, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right] = \left( \begin{pmatrix} \frac{1}{c} & 0 \\ 0 & c \end{pmatrix} \cdot (-1)^p F(ac) + (-1)^\varepsilon \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot F(bc) \right)
+ \left( \begin{pmatrix} \frac{1}{d} & dy \\ 0 & d \end{pmatrix} \cdot (-1)^\varepsilon \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot F(ad) + F(bd) \right)
+ k \left( (-1)^{p+1} a^{p-2} + b^{p-2} \right) \int_x^y \begin{pmatrix} 1 & 2z \\ 0 & 1 \end{pmatrix} \cdot \varepsilon^p dz \quad (44)
\]
for all \( a, b, c, d > 0 \) and \( x, y \in \mathbb{R} \) which form a double pyramid. Furthermore, \( k = 0 \) if \( p + \varepsilon \) is odd.
Proof. Let $\tilde{\mu}$ be the splitting from Lemma 3.10 and suppose that $a, b, c, d > 0$ and $x, y \in \mathbb{R}$ form a double pyramid. By Lemma 3.9 equation (6), a basic fact about binomial coefficients, and an index shift we obtain

$$\mu_i \left[ I, -c \left( \begin{array}{c} x \\ 1 \\ 1 \end{array} \right), d \left( \begin{array}{c} y \\ 1 \end{array} \right) \right]$$

$$= \left[ \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right] \cdot \tilde{\mu}[I, -ce_2] + \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \cdot \tilde{\mu}[I, de_2] + \int_x^y \left( \begin{array}{c} 1 \\ 0 \\ 1 \end{array} \right) \cdot K^I(dz),$$

$$= \sum_{j=1}^p \binom{j}{i} x^{j-i} \tilde{\mu}_j[I, -ce_2] + y^{j-i} \tilde{\mu}_j[I, de_2] + \sum_{j=1}^p \binom{j}{i} y^{j-i} - x^{j-i+1} \frac{K^I_j}{j - i + 1}$$

$$= \sum_{j=1}^p \binom{j}{i} x^{j-i} \tilde{\mu}_j[I, -ce_2] + y^{j-i} \tilde{\mu}_j[I, de_2] + \sum_{j=1}^p \binom{j}{i} y^{j-i} - x^{j-i+1} \frac{K^I_j}{j - i + 1}$$

$$= \sum_{j=1}^p \binom{j}{i} x^{j-i} \tilde{\mu}_j[I, -ce_2] + y^{j-i} \tilde{\mu}_j[I, de_2] + \sum_{j=i+1}^{p+1} \binom{j}{i} y^{j-i} - x^{j-i} \frac{K^I_{j-1}}{j - i + 1}$$

for all $i \in \{0, \ldots, p\}$. As usual, we write $J = [-ce_2, de_2]$. Rearranging sums in the last formula gives

$$\mu_i \left[ I, -c \left( \begin{array}{c} x \\ 1 \\ 1 \end{array} \right), d \left( \begin{array}{c} y \\ 1 \end{array} \right) \right] = \mu_i[I, J] + \sum_{j=i+1}^p \binom{j}{i} x^{j-i} \left( \tilde{\mu}_j[I, -ce_2] - \frac{K^I_{j-1}}{j} \right)$$

$$+ \sum_{j=i+1}^p \binom{j}{i} y^{j-i} \left( \tilde{\mu}_j[I, de_2] + \frac{K^I_{j-1}}{j} \right) + \binom{p+1}{i} y^{p+1-i} - x^{p+1-i} \frac{K^I_p}{p+1}.$$  \hspace{1cm} (45)

Assume that we know the following. First, there exists a constant $k \in \mathbb{R}$ with

$$K^I_p = k((-1)^{p+1}a^{-p-2} + b^{-p-2}).$$  \hspace{1cm} (46)

Second, there exist measurable functions $F_j : (0, \infty) \to \mathbb{R}$, $j \in \{0, \ldots, p\}$, such that

$$\tilde{\mu}_j[I, -ce_2] - \frac{K^I_{j-1}}{j} = e^{2j-p}((-1)^p F_j(ac) + (-1)^{j+\varepsilon} F_j(bc))$$  \hspace{1cm} (47)

and

$$\tilde{\mu}_j[I, de_2] + \frac{K^I_{j-1}}{j} = d^{2j-p}((-1)^{p+j+\varepsilon} F_j(ad) + F_j(bd)).$$  \hspace{1cm} (48)

for $j \neq 0$. Third, for these functions also the equality

$$\mu_i[I, J] = e^{2i-p}((-1)^p F_i(ac) + (-1)^{i+\varepsilon} F_i(bc)) + d^{2i-p}((-1)^{p+i+\varepsilon} F_i(ad) + F_i(bd))$$  \hspace{1cm} (49)
holds for all \( i \in \{0, \ldots, p\} \). Under these assumptions, we can plug (46), (47), (48), and (49) into (45). This results in

\[
\mu_i \left[ I, -c \left( \begin{array}{c} x \\ 1 \end{array} \right), d \left( \begin{array}{c} y \\ 1 \end{array} \right) \right] = \sum_{j=i}^{p} \left( \begin{array}{c} j \\ i \end{array} \right) x^{j-i} c^{2j-p} \left( (-1)^{p} F_j(ac) + (-1)^{j+i} F_j(bc) \right) \\
+ \sum_{j=i}^{p} \left( \begin{array}{c} j \\ i \end{array} \right) y^{j-i} d^{2j-p} \left( (-1)^{p+j+i} F_j(ad) + F_j(bd) \right) \\
+ k \left( (-1)^{p+1} a^{-p-2} + b^{-p-2} \right) \left( \frac{p + 1}{i} \right) y^{p+1-i} - x^{p+1-i}.
\]

But by (6) this is, in coordinates, precisely what we want to show.

It remains to prove (46), (47), (48), and (49). In order to do so, fix \( a, b > 0 \) and \( x, y \in \mathbb{R} \). The SL\( ^\pm(2) \)-\( \varepsilon \)-covariance of \( \mu \) with respect to the reflection at \( e_1^\perp \) and the origin yields

\[
\mu \left[ -I, -c \left( \begin{array}{c} x \\ 1 \end{array} \right), d \left( \begin{array}{c} y \\ 1 \end{array} \right) \right] = (-1)^{\varepsilon} \left( \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right) \cdot \mu \left[ I, -c \left( \begin{array}{c} x \\ 1 \end{array} \right), d \left( \begin{array}{c} y \\ 1 \end{array} \right) \right]
\]

and

\[
\mu \left[ -I, -c \left( \begin{array}{c} x \\ 1 \end{array} \right), d \left( \begin{array}{c} y \\ 1 \end{array} \right) \right] = (-1)^{\varepsilon} \mu \left[ I, -d \left( \begin{array}{c} y \\ 1 \end{array} \right), c \left( \begin{array}{c} x \\ 1 \end{array} \right) \right].
\]

For sufficiently small \( c, d > 0 \) all arguments of \( \mu \) in (50) and (51) are double pyramids. Hence, we can apply representation (39) to (50) and (51). Thus,

\[
\left( \begin{array}{c} 1 \\ x \\ 0 \\ 1 \end{array} \right) \cdot \tilde{\mu}[-I, -ce_2] + \left( \begin{array}{c} 1 \\ y \\ 0 \\ 1 \end{array} \right) \cdot \tilde{\mu}[I, -ce_2] + \int_{x}^{y} \left( \begin{array}{c} 1 \\ z \\ 0 \\ 1 \end{array} \right) \cdot K^{-I} \ dz =
\]

\[
(-1)^{\varepsilon} \left( \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right) \cdot \tilde{\mu}[I, -ce_2] + (-1)^{\varepsilon} \cdot \tilde{\mu}[I, -ce_2] + (-1)^{\varepsilon} \int_{-z}^{y} \left( \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right) \cdot K^{-I} \ dz
\]

and

\[
\left( \begin{array}{c} 1 \\ x \\ 0 \\ 1 \end{array} \right) \cdot \tilde{\mu}[-I, -ce_2] + \left( \begin{array}{c} 1 \\ y \\ 0 \\ 1 \end{array} \right) \cdot \tilde{\mu}[-I, -ce_2] + \int_{x}^{y} \left( \begin{array}{c} 1 \\ z \\ 0 \\ 1 \end{array} \right) \cdot K^{-I} \ dz =
\]

\[
(-1)^{p} \left( \begin{array}{c} 1 \\ y \\ 0 \\ 1 \end{array} \right) \cdot \tilde{\mu}[I, -d e_2] + (-1)^{p} \left( \begin{array}{c} 1 \\ x \\ 0 \\ 1 \end{array} \right) \cdot \tilde{\mu}[I, d e_2] + (-1)^{p} \int_{y}^{x} \left( \begin{array}{c} 1 \\ z \\ 0 \\ 1 \end{array} \right) \cdot K^{-I} \ dz.
\]

The terms involving \( \tilde{\mu} \) in the above equations cancel due to the \( \varepsilon \)-covariance of splittings. Applying elementary transformations to the remaining integrals therefore proves

\[
\int_{x}^{y} \left( \begin{array}{c} 1 \\ z \\ 0 \\ 1 \end{array} \right) \cdot K^{-I} \ dz = (-1)^{\varepsilon+1} \int_{x}^{y} \left( \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \right) \cdot K^{I} \ dz.
\]
and
\[ \int_x^y \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K^{-I} \, dz = (-1)^{p+1} \int_x^y \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \cdot K^I \, dz. \]
Thus, the injectivity property (11) implies
\[ K^{-I} = (-1)^{p+1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot K^I \]
and
\[ K^{-I} = (-1)^{p+1} K^I. \]

Let \( j \in \{0, \ldots, p\} \). Writing the last equation componentwise shows
\[ K^{-I}_j = (-1)^{p+1} K^I_j. \]
This relation implies that \( I \mapsto K^I_j \) is either even or odd. Bearing (11) and (39) in mind, it is easy to see that \( I \mapsto K^I_j \) is also a measurable valuation. If we combine (52) and (53), then we have in addition
\[ K^I_j = 0 \quad \text{for } j + \varepsilon \text{ odd.} \]

Next, fix \( a, b, d > 0 \). From the \( \text{SL}^+(2) \)-covariance of \( \mu \) we deduce
\[ \mu \left[ dI, - \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right] = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \cdot \mu \left[ I, -d \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right]. \]
In particular, the 0-th component of \( \mu \) satisfies
\[ \mu_0 \left[ dI, - \begin{pmatrix} x \\ 1 \end{pmatrix}, \begin{pmatrix} y \\ 1 \end{pmatrix} \right] = d^p \mu_0 \left[ I, -d \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right]. \]
Note that both arguments of \( \mu \) in the last equation are double pyramids for sufficiently small \( x \) and \( y \). So we can apply (15) to (56). Therefore, both sides of (56) are polynomials in \( x \) and \( y \) on a small rectangle around the origin. Comparing the coefficients of \( y^{p+1} \) yields
\[ \frac{K^d_I}{p+1} = d^{-p-2} \frac{K^p_I}{p+1}. \]
In other words, \( I \mapsto K^I_p \) is \((-p-2)\)-homogeneous. Recall from (54) that \( I \mapsto K^I_p \) is a measurable valuation which is either even or odd. So Theorems 3.2 and 3.4 prove the existence of a constant \( k \in \mathbb{R} \) such that (16) holds. From (55) we also know that \( K^I_p \) vanishes if \( p+\varepsilon \) is odd. Thus, \( k = 0 \) if \( p+\varepsilon \) is odd.

Let \( j \in \{1, \ldots, p\} \) and suppose that \( j + \varepsilon \) is odd. In order to get information on \( K^I_{j-1} \) we proceed similar as before. Indeed, comparing the coefficients of \( y^j \) in (55) yields
\[ \mu_j[dI, e_2] + \frac{K^d_{j-1}}{j} = d^{p-2j} \left( \mu_j[I, de_2] + \frac{K^I_{j-1}}{j} \right). \]
Since $j + \varepsilon$ is odd, we can use (42) to get

$$\hat{\mu}_j[I, de_2] + \frac{K_{j-1}^I}{j} = d^{2j-p}K_{j-1}^{dI}. \quad (57)$$

Again, recall that $I \mapsto K_{j-1}^I$ is a measurable valuation which is either even or odd. So Theorems 3.1 and 3.3 prove the existence of a measurable function $F_j : (0, \infty) \rightarrow \mathbb{R}$ such that

$$\frac{K_{j-1}^I}{j} = (-1)^{p+1}F_j(a) + F_j(b).$$

Plugging this into the right hand side of (57) gives

$$\hat{\mu}_j[I, de_2] + \frac{K_{j-1}^I}{j} = d^{2j-p}\left((-1)^{p+1}F_j(ad) + F_j(bd)\right).$$

A corresponding formula also exists for triangles contained in the lower half plane. Indeed, combining the last equality with the $\varepsilon$-covariance of splittings yields

$$\hat{\mu}_j[I, -ce_2] - \frac{K_{j-1}^I}{j} = c^{2j-p}\left((-1)^{p}F_j(ac) - F_j(bc)\right).$$

Therefore, we established (47) and (48) for odd $j + \varepsilon$. Next, let $j + \varepsilon$ be even. The $\varepsilon$-covariance of splittings shows

$$\hat{\mu}_j[-I, -e_2] = (-1)^p\hat{\mu}_j[I, -e_2].$$

Consequently, the map $I \mapsto \hat{\mu}_j[I, -e_2]$ is either even or odd. Moreover, it is a measurable valuation by definition. Now Theorems 3.1 and 3.3 show that there exists a measurable function $F_j : (0, \infty) \rightarrow \mathbb{R}$ such that

$$\hat{\mu}_j[I, -e_2] = (-1)^{p}F_j(a) + F_j(b).$$

From the $\varepsilon$-covariance of splittings and (41) we infer

$$\hat{\mu}_j[I, -ce_2] = c^{2j-p}\hat{\mu}_j[cI, -e_2],$$

which results in

$$\hat{\mu}_j[I, -ce_2] = c^{2j-p}\left((-1)^{p}F_j(ac) + F_j(bc)\right).$$

Using again the $\varepsilon$-covariance of splittings, we also have the following representation

$$\hat{\mu}_j[I, de_2] = d^{2j-p}\left((-1)^{p}F_j(ad) + F_j(bd)\right)$$

for triangles contained in the upper half plane. Recall from (55) that $K_{j-1}^I = 0$. Therefore, the last two equations prove (47) and (48) for even $j + \varepsilon$. 

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Finally, we have to show the validity of (49). For \( i \geq 1 \), this is a simple consequence of adding (47) and (48). The case \( i = 0 \) remains. For \( p \neq 0 \), set \( F_0(x) = (-1)^{x}x^{p}F_p(x) \) and note that the SL\(^x(2)\)-covariance of \( \mu \) implies

\[
\mu_0[I, J] = (-1)^{\xi}\mu_p[\tilde{J}, \tilde{I}],
\]

where \( \tilde{I} := [-ae_2, be_2] \) and \( \tilde{J} := [-ce_1, de_1] \). Since we already have (49) for \( i = p \), this and the definition of \( F_0 \) prove (49) for \( i = 0 \). This last step does not work for \( p = 0 \). For \( p = 0 \) and \( \varepsilon = 0 \) one can easily deduce (49) as in [16, Lemma 3.5]. For \( p = 0 \) and \( \varepsilon = 1 \) the situation is a little different. We refer to [16, Lemma 3.6] and [18, Theorem 2.3] for a proof that (49) holds with \( F_0 = 0 \) in this case.

Let \( \mu \in \text{TVal}^p(\mathbb{R}^2) \). The last lemma shows that, up to an additive term, \( \mu \) is determined by a function \( F \). This motivates the following definition. We say that a measurable function \( F : (0, \infty) \rightarrow \text{Sym}^p(\mathbb{R}^2) \) describes \( \mu \) if

\[
\mu_i[I, -c(x_1), d(y_1)] = (-1)^{p}x^i \cdot F(ac) + (-1)^{\xi}x^{-c} \cdot F(bc)
+ (-1)^{\xi}d^i \cdot F(ad) + (-1)^{\xi}d^{-c} \cdot F(bd) \tag{58}
\]

for all \( a, b, c, d > 0 \) and \( x, y \in \mathbb{R} \) which form a double pyramid. In coordinates (58) reads as

\[
\mu_i[I, -c(x_1), d(y_1)] = \sum_{j=i}^{p} x^j \cdot e_1^j \cdot F_j(ac) + (-1)^{\xi} \sum_{j=i}^{p} y^j \cdot e_2^j \cdot F_j(bd) \tag{59}
\]

which can be easily seen using (6).

In general, there is some freedom in the choice of the describing function \( F \). So it makes sense to single out a particular \( F \) with useful additional properties. This will be done in the next lemma.

3.12 Lemma. Let \( \mu \in \text{TVal}^p(\mathbb{R}^2) \). If \( \mu \) can be described by some measurable function, then there exists a measurable \( \tilde{F} : (0, \infty) \rightarrow \text{Sym}^p(\mathbb{R}^2) \) and a constant \( k \in \mathbb{R} \) such that

\[
\tilde{F} + k \ln e_1^0 \otimes e_2^0 \tag{60}
\]

also describes \( \mu \) and

\[
(-1)^{\xi} \begin{pmatrix} 0 & a \\ \frac{1}{a} & 0 \end{pmatrix} \cdot \tilde{F}(a) = \tilde{F}(a) \tag{61}
\]

holds for all \( a > 0 \).

If \( p \) as well as \( p \) are even and \( \xi = 1 \), then \( \tilde{F}_\frac{p}{2} = 0 \). In all other cases we have \( k = 0 \).
There is a slight abuse of notation in (60), since we write
\[ k \ln e_1 \circ e_2 \circ e_1 \circ e_2 \]
even in cases where the above tensor might not be defined, i.e. for odd \( p \). However, as stated in Lemma 3.12 in all such cases \( k = 0 \) holds. This actually means that the corresponding term does not show up at all and hence \( \tilde{F} \) itself describes \( \mu \). This notation has the advantage that we can avoid distinctions of cases in the sequel.

**Proof.** Denote by \( F \) the measurable function which describes \( \mu \). Let \( j \in \{0, \ldots, p\} \) and assume that \( p \) has a different parity than \( j + \varepsilon \). Then it follows from (59) that a constant can be added to \( F_j \) without changing (58). Therefore, without loss of generality, we assume that \( F_j(1) = 0 \) for all such \( j \).

By the SL\( ^\pm(2) \)-covariance of \( \mu \) we have
\[ \mu[-ae_1, be_1, -ce_2, de_2] = (-1)^\varepsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mu[-ce_1, de_1, -ae_2, be_2] \]
for all \( a, b, c, d > 0 \). If we plug representation (58) into the above terms we obtain
\[
(\varepsilon + 1) \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \cdot F(ac) + \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \cdot F(bc) + (-1)^\varepsilon \begin{pmatrix} 1 & d \\ 0 & -d \end{pmatrix} \cdot F(ad) + \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix} \cdot F(bd)
\]

\[
= (-1)^p \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} \cdot F(ac) + \begin{pmatrix} 1 & d \\ 0 & -d \end{pmatrix} \cdot F(ad) + \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \cdot F(bc) + (-1)^\varepsilon \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix} \cdot F(bd).
\]

(62)

We start with the case where \( p \) is even. For \( j \in \{0, \ldots, p\} \) such that \( j + \varepsilon \) is even we choose \( b = a \) and \( c = d = 1 \) in (62). Thus,
\[ 4F_j(a) = (-1)^\varepsilon 4a^{p-2j}F_{p-j}(a). \]  

(63)

For \( j \in \{0, \ldots, p\} \) such that \( j + \varepsilon \) is odd we set \( b = d = 1 \) in (62). Together with the assumption \( F_j(1) = 0 \) for such \( j \) we obtain
\[ c^{2j-p}F_j(ac) - c^{2j-p}F_j(c) - F_j(a) = (-1)^\varepsilon \left( a^{p-2j}F_{p-j}(ac) - a^{p-2j}F_{p-j}(a) - F_{p-j}(c) \right). \]

(64)

Define measurable functions \( G_j : (0, \infty) \to \mathbb{R} \) by
\[ G_j(a) = a^{2j-p}F_j(a) - (-1)^\varepsilon F_{p-j}(a). \]

An immediate consequence of this definition is the fact that
\[ G_{p-j}(a) = (-1)^{\varepsilon+1} a^{p-2j} G_j(a). \]

(65)
Using the definition of $G_j$, equation (64) can be written as

$$G_j(ac) = G_j(a) + a^{2j-p}G_j(c).$$  \hspace{1cm} (66)

First, suppose that $j \neq \frac{p}{2}$. The left hand side of this equation is symmetric in $a$ and $c$. So interchanging the roles of $a$ and $c$ yields

$$G_j(a) + a^{2j-p}G_j(c) = G_j(c) + c^{2j-p}G_j(a).$$

Therefore, we arrive at

$$G_j(a) = \frac{G_j(c)}{1 - c^{2j-p}} \left(1 - a^{2j-p}\right)$$

for all $a, c > 0$. If we choose $c = 2$ and define constants $g_j \in \mathbb{R}$ by

$$g_j = \frac{G_j(2)}{1 - 2^{2j-p}},$$

then we have

$$G_j(a) = g_j \left(1 - a^{2j-p}\right).$$

Plugging the definition of $G_j$ into this relation shows

$$a^{2j-p}F_j(a) - (-1)^{\varepsilon}F_{p-j}(a) = g_j \left(1 - a^{2j-p}\right).$$

From (65) we infer that $g_{p-j} = (-1)^{\varepsilon}g_j$. Hence, rearranging terms gives

$$F_j(a) + g_j = (-1)^{\varepsilon}a^{p-2j}(F_{p-j}(a) + g_{p-j}).$$  \hspace{1cm} (67)

Next, assume that $j = \frac{p}{2}$ and $\varepsilon = 1$. In this case, equation (65) is of the form

$$G_{\frac{p}{2}}(ac) = G_{\frac{p}{2}}(a) + G_{\frac{p}{2}}(c).$$

This is one of Cauchy’s classical functional equations. Its solution is well known to be

$$G_{\frac{p}{2}}(a) = 2g_{\frac{p}{2}} \ln(a)$$

for some constant $g_{\frac{p}{2}} \in \mathbb{R}$. In terms of $F_{\frac{p}{2}}$ this reads as

$$F_{\frac{p}{2}}(a) = g_{\frac{p}{2}} \ln(a).$$  \hspace{1cm} (68)

Now, we are in a position to define our desired function $\tilde{F}$ by

$$\tilde{F}_j = \begin{cases} 
F_j & \text{for } j \in \{0, \ldots, p\} \text{ such that } j + \varepsilon \text{ is even} \\
F_j + g_j & \text{for } j \in \{0, \ldots, p\} \setminus \{\frac{p}{2}\} \text{ such that } j + \varepsilon \text{ is odd} \\
0 & \text{for } j = \frac{p}{2} \text{ even and } \varepsilon = 1 \\
F_j & \text{for } j = \frac{p}{2} \text{ odd and } \varepsilon = 0.
\end{cases}$$

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For \( \frac{p}{2} \) even and \( \varepsilon = 1 \) set \( k = g_{\frac{p}{2}} \). In all other cases set \( k = 0 \). A glance at (59) reveals that an addition of constants for even \( p \) and odd \( j + \varepsilon \) does not change (59). This and (68) imply that
\[
\tilde{F} + k \ln e_1 ^{\otimes 2} \otimes e_2 ^{\otimes 2}
\]
describes \( \mu \). We still need to show that \( \tilde{F} \) satisfies (61). In coordinates, we have to prove
\[
\tilde{F}_j(a) = (-1)^\varepsilon a^{p-2j} \tilde{F}_{p-j}(a).
\]
Let \( j = \frac{p}{2} \). If \( \varepsilon = 0 \), then this is trivially true. For \( \varepsilon = 1 \) we have to distinguish two cases. First, assume \( \frac{p}{2} \) is even. Then \( \tilde{F}_{\frac{p}{2}} = 0 \) and thus (69) obviously holds. Second, let \( \frac{p}{2} \) be odd. Then (68) implies (69). For \( j \neq \frac{p}{2} \), the desired equality follows directly from (63) and (67).

Now, let \( p \) be odd and \( j \in \{0, \ldots, p\} \). We suppose further that \( j + \varepsilon \) is even. Then choosing \( b = c = d = 1 \) in (62) together with the assumption that \( F_j(1) = 0 \) yields
\[
F_j(a) + (-1)^\varepsilon F_{p-j}(1) = (-1)^\varepsilon a^{p-2j} F_{p-j}(a).
\]
(70)
We are already in a position to define \( \tilde{F} \) and \( k \) by
\[
\tilde{F}_j = \begin{cases} F_j + (-1)^\varepsilon F_{p-j}(1) & \text{for } j \in \{0, \ldots, p\} \text{ such that } j + \varepsilon \text{ is even} \\ F_j & \text{for } j \in \{0, \ldots, p\} \text{ such that } j + \varepsilon \text{ is odd} \end{cases}
\]
and \( k = 0 \), respectively. Similar to before we see that \( \tilde{F} \) describes \( \mu \). Moreover, from (70) follows (69), which in turn yields (61).

Next, we are going to deduce a crucial linear equation.

**3.13 Lemma.** Let \( \mu \in \mathrm{TVal}_p(\mathbb{R}^2) \). If \( \mu \) can be described by some measurable function, then it can also be described by a measurable \( F: (0, \infty) \to \mathrm{Sym}^p(\mathbb{R}^2) \) with the following properties:

- There exists a tensor \( C \in \mathrm{Sym}^p(\mathbb{R}^2) \) such that
  \[
  F(t) = \begin{pmatrix} 1 & 0 \\ \frac{1}{s} & 1 \end{pmatrix} \cdot F\left(\frac{st}{s+1}\right) + (-1)^\varepsilon \begin{pmatrix} s & t \\ \frac{t}{s+1} & 0 \end{pmatrix} \cdot F\left(\frac{\sqrt{st}}{0 \ \sqrt{st}}\right) \cdot C
  \]
  holds for all \( s, t > 0 \).

- For all \( s > 0 \) we have
  \[
  (-1)^\varepsilon \begin{pmatrix} 0 & s \\ \frac{1}{s} & 0 \end{pmatrix} \cdot F(s) = F(s).
  \]
(72)

- The tensor \( C \) satisfies
  \[
  (-1)^\varepsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot C = C \quad \text{and} \quad (-1)^\varepsilon \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \cdot C = C.
  \]
(73)
Proof. By Lemma 3.12 there exists a function $\tilde{F}$ and a constant $k$ such that

$$F = \tilde{F} + k \ln e_1^{\frac{\xi}{2}} \odot e_2^{\frac{\xi}{2}}$$

describes $\mu$. Let us remark that, without further mentioning, we will use that $k$ vanishes in most cases. In fact, we know from Lemma 3.12 that $k = 0$ except for $p$ even, $\frac{\eta}{2}$ even, and $\varepsilon = 1$. Thus, relations such as $(-1)^{\xi} k = -k$ and $(-1)^{p} k = k$ will be implicitly used.

It easily follows from (61) that

$$(-1)^{\xi} \begin{pmatrix} 0 & -u \\ -\frac{1}{u} & 0 \end{pmatrix} \cdot \tilde{F}(uv) = (-1)^{p} \begin{pmatrix} \frac{1}{v} & 0 \\ 0 & v \end{pmatrix} \cdot \tilde{F}(uv)$$  \hspace{1cm} (74)

for all $u, v > 0$ and

$$(-1)^{\xi} \begin{pmatrix} 0 & t \\ 1 & 0 \end{pmatrix} \cdot \tilde{F}(t) = \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \cdot \tilde{F}(t)$$  \hspace{1cm} (75)

for all $t > 0$.

Let $s, t, u, v$ be positive real numbers and consider the triangle $T$ with corners $se_1 + te_2$, $-ue_1$ and $-ve_2$. We can write $T$ in two different ways involving double pyramids. In fact, a simple calculation shows that on the one hand

$$T = \left[-ue_1, \frac{sv}{t+v}e_1, -ve_2, t \begin{pmatrix} \frac{1}{v} \\ 1 \end{pmatrix} \right],$$

and on the other hand

$$T = \left[0, -1 \right] \left[-ve_1, \frac{tu}{s+u}e_1, -s \left(\frac{1}{s} \right), ue_2 \right].$$

Since $F$ describes $\mu$, equation (58) holds. Applying this to the first representation of $T$ gives

$$\mu(T) = (-1)^{p} \begin{pmatrix} \frac{1}{v} & 0 \\ 0 & v \end{pmatrix} \cdot F(uv) + (-1)^{\xi} \begin{pmatrix} \frac{1}{v} & 0 \\ 0 & -v \end{pmatrix} \cdot F\left(\frac{sv^2}{t+v}\right)$$

$$+ (-1)^{\xi} \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix} \cdot F(tu) + \begin{pmatrix} 1 & s \\ 0 & t \end{pmatrix} \cdot F\left(\frac{stv}{t+v}\right),$$

whereas the second representation and the SL(2)-covariance of $\mu$ yield

$$\mu(T) = (-1)^{p} \begin{pmatrix} 0 & -s \\ s & -t \end{pmatrix} \cdot F(sv) + (-1)^{\xi} \begin{pmatrix} 0 & s \\ \frac{1}{s} & t \end{pmatrix} \cdot F\left(\frac{stu}{s+u}\right)$$

$$+ (-1)^{\xi} \begin{pmatrix} 0 & -u \\ -\frac{1}{u} & 0 \end{pmatrix} \cdot F(uv) + \begin{pmatrix} 0 & -u \\ \frac{1}{u} & 0 \end{pmatrix} \cdot F\left(\frac{tu^2}{s+u}\right).$$
So the right hand sides of the last two equations must be equal. In the resulting equation
we can plug in the definition of $F$ and use (74). By doing so, it turns out that the terms
containing $u$ and those containing $v$ can be separated. We therefore arrive at

$$L(s, t, u) = R(s, t, v),$$

(76)

where

$$L(s, t, u) := (-1)^{\varepsilon} \left( \begin{array}{c} -\frac{1}{t} \\ 0 \\ s \\ t \end{array} \right) \cdot F(tu) - (-1)^{\varepsilon} \left( \begin{array}{c} 0 \\ \frac{1}{s} \\ s \\ t \end{array} \right) \cdot F\left( \frac{stu}{s + u} \right)$$

and

$$R(s, t, v) := \left( \begin{array}{c} 0 \\ \frac{1}{s} \\ s \\ t \end{array} \right) \cdot F(sv) - \left( \begin{array}{c} \frac{1}{t} \\ 0 \\ s \\ t \end{array} \right) \cdot F\left( \frac{stv}{t + v} \right)$$

$$- (-1)^{\varepsilon} \left( \begin{array}{c} \frac{1}{v} \\ 0 \\ 1 \\ -v \end{array} \right) \cdot F\left( \frac{sv^2}{t + v} \right) - 2k \ln(v) e_1^\circ \otimes e_2^\circ.$$ 

A straightforward calculation proves

$$\left( \begin{array}{c} \frac{1}{t} \\ 0 \\ s \end{array} \right) \cdot L(s, t, s) = L(1, st, 1) + 2k \ln(s) e_1^\circ \otimes e_2^\circ.$$ 

(77)

Similarly, we have

$$(-1)^{\varepsilon} \left( \begin{array}{c} 0 \\ \frac{1}{t} \\ 1 \\ 0 \end{array} \right) \cdot R(s, t, t) = L(1, st, 1) + 2k \ln(t) e_1^\circ \otimes e_2^\circ.$$ 

(78)

Choose $s = t = 1$ in (78). A glance at (76) then shows

$$L(1, 1, 1) = (-1)^{\varepsilon} \left( \begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \end{array} \right) \cdot R(1, 1, 1) = (-1)^{\varepsilon} \left( \begin{array}{c} 0 \\ 1 \\ 1 \\ 0 \end{array} \right) \cdot L(1, 1, 1).$$

(79)

From (76) we infer that $L(s, t, u)$ is independent of $u$. In particular, $L(s, t, u) = L(s, t, s)$.

By (77) we therefore have

$$L(s, t, u) = \left( \begin{array}{c} s \\ 0 \\ \frac{1}{s} \end{array} \right) \cdot L(1, st, 1) + 2k \ln(s) e_1^\circ \otimes e_2^\circ.$$ 

Consequently,

$$\left( \begin{array}{c} -t \\ s \\ \frac{1}{t} \end{array} \right) \cdot L(s, t, u) = \left( \begin{array}{c} -st \\ 1 \\ \frac{1}{st} \end{array} \right) \cdot L(1, st, 1) + 2k \ln(s) \left( \begin{array}{c} -1 \\ st \\ 1 \end{array} \right) \cdot e_1^\circ \otimes e_2^\circ.$$ 

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Set \( u = 1 \) in this equation and plug in the definition of \( L(s, t, 1) \) afterwards. Then we obtain
\[
F(t) - \left( \frac{1}{st} \begin{array}{cc} 0 \\ 1 \\ s \\ t \\ s+1 \end{array} \right) \cdot F\left( \frac{st}{s+1} \right) - (-1)^{\varepsilon} \left( \begin{array}{cc} s \\ t \\ s \\ t \\ s+1 \end{array} \right) \cdot F\left( \frac{t}{s+1} \right) = \\
(-1)^{\varepsilon} \left( \begin{array}{cc} -st \\ 1 \\ st \\ 1 \\ st \end{array} \right) \cdot L(1, st, 1) - 2k \ln(s) \left( \begin{array}{cc} -1 \\ st \\ s \\ t \\ s+1 \end{array} \right) \cdot e_1 \otimes e_2 \otimes e_2.
\]

Define a function \( \tilde{H} : (0, \infty) \rightarrow \text{Sym}^p(\mathbb{R}^2) \) by
\[
\tilde{H}(s) = \left( \begin{array}{cc} -s \\ 0 \\ 1 \\ s \end{array} \right) \cdot L(1, s, 1).
\]
Using this definition, the last equation reads as
\[
F(t) - \left( \frac{1}{st} \begin{array}{cc} 0 \\ 1 \\ s \\ t \\ s+1 \end{array} \right) \cdot F\left( \frac{st}{s+1} \right) - (-1)^{\varepsilon} \left( \begin{array}{cc} s \\ t \\ s \\ t \\ s+1 \end{array} \right) \cdot F\left( \frac{t}{s+1} \right) = \\
(-1)^{\varepsilon} \tilde{H}(st) - 2k \ln(s) \left( \begin{array}{cc} -1 \\ st \\ 1 \\ 0 \\ t \\ s \\ t \\ s+1 \end{array} \right) \cdot e_1 \otimes e_2 \otimes e_2. \quad (80)
\]
This is a functional equation for \( F \) whose homogeneous part coincides with the one of the desired equation (71). However, we still need to simplify the inhomogeneous part. A multiplication of (80) by
\[
\left( \begin{array}{cc} 1 \\ 0 \\ t \end{array} \right)
\]
proves
\[
\left( \begin{array}{cc} 1 \\ 0 \\ t \end{array} \right) \cdot F(t) - \left( \frac{1}{st} \begin{array}{cc} 0 \\ 1 \\ s \\ t \\ s+1 \end{array} \right) \cdot F\left( \frac{st}{s+1} \right) - (-1)^{\varepsilon} \left( \begin{array}{cc} s \\ t \\ s \\ t \\ s+1 \end{array} \right) \cdot F\left( \frac{t}{s+1} \right) = \\
(-1)^{\varepsilon} \left( \begin{array}{cc} 1 \\ 0 \\ t \\ 1 \\ 0 \end{array} \right) \cdot \tilde{H}(st) - 2k \ln(s) \left( \begin{array}{cc} -1 \\ st \\ 1 \\ 0 \\ t \\ s \\ t \\ s+1 \end{array} \right) \cdot e_1 \otimes e_2 \otimes e_2. \quad (81)
\]
Next, we replace \( s \) by \( \frac{1}{s} \) in (81) and multiply the equation by
\[
(-1)^{\varepsilon} \left( \begin{array}{cc} 0 \\ 1 \\ 1 \end{array} \right).
\]
This yields the following:
\[
(-1)^{\varepsilon} \left( \begin{array}{cc} 0 \\ 1 \\ t \\ 0 \end{array} \right) \cdot F(t) - (-1)^{\varepsilon} \left( \begin{array}{cc} 0 \\ t \\ 1 \\ 0 \end{array} \right) \cdot F\left( \frac{t}{s+1} \right) = \\
\left( \begin{array}{cc} 0 \\ 1 \\ t \\ 0 \end{array} \right) \cdot \tilde{H}\left( \frac{t}{s} \right) - 2k \ln(s) \left( \begin{array}{cc} 0 \\ t \\ 0 \\ 1 \\ t \\ s \\ t \\ s+1 \end{array} \right) \cdot e_1 \otimes e_2 \otimes e_2. \quad (82)
\]
We subtract (81) from (82) and see that on the resulting left hand side only terms involving \( F(t) \) remain. For these terms we plug in the definition of \( F \) and apply (75). We then arrive at

\[
(-1)^\varepsilon \begin{pmatrix}
1 & 0 \\
0 & t
\end{pmatrix} \cdot \tilde{H}(st) - 2k \ln(s) \begin{pmatrix}
-1 & st \\
0 & t
\end{pmatrix} \cdot e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}} - k \ln(t) \begin{pmatrix}
1 & 0 \\
0 & t
\end{pmatrix} \cdot e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}} = \\
\begin{pmatrix}
0 & t \\
1 & 0
\end{pmatrix} \cdot \tilde{H}(\frac{t}{s}) - 2k \ln(s) \begin{pmatrix}
0 & t \\
-1 & \frac{t}{s}
\end{pmatrix} \cdot e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}} + k \ln(t) \begin{pmatrix}
0 & t \\
1 & 0
\end{pmatrix} \cdot e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}}.
\]

Setting \( s = t \) and rearranging terms yields

\[
\tilde{H}(t^2) = (-1)^\varepsilon \begin{pmatrix}
0 & t \\
1 & 0
\end{pmatrix} \cdot \tilde{H}(1) + 2k \ln(t) \begin{pmatrix}
0 & 1 \\
-1 & \frac{t^2}{1^2}
\end{pmatrix} \cdot e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}} \\
- 2k \ln(t)e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}} - 2k \ln(t) \begin{pmatrix}
0 & 1 \\
-1 & \frac{t^2}{1^2}
\end{pmatrix} \cdot e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}}. \tag{83}
\]

Now, choose \( t = 1 \). Then we obtain

\[
\tilde{H}(1) = (-1)^\varepsilon \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \cdot \tilde{H}(1). \tag{84}
\]

If we plug this back into (83), then we obviously get

\[
\tilde{H}(t^2) = \begin{pmatrix}
t & 0 \\
0 & \frac{t^2}{1^2}
\end{pmatrix} \cdot \tilde{H}(1) + 2k \ln(t) \begin{pmatrix}
0 & 1 \\
-1 & \frac{t^2}{1^2}
\end{pmatrix} \cdot e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}} \\
- 2k \ln(t)e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}} - 2k \ln(t) \begin{pmatrix}
0 & 1 \\
-1 & \frac{t^2}{1^2}
\end{pmatrix} \cdot e_1^{\oplus \bar{t}} \circ e_2^{\oplus \bar{t}}. \tag{85}
\]

Moreover, by the definition of \( \tilde{H} \), relation (79), and the definition of \( \tilde{H} \) again we have

\[
\tilde{H}(1) = \begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix} \cdot L(1,1,1) \\
= (-1)^\varepsilon \begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix} \cdot L(1,1,1) \\
= (-1)^\varepsilon \begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
-1 & 1 \\
0 & 1
\end{pmatrix}^{-1} \cdot \tilde{H}(1) \\
= (-1)^\varepsilon \begin{pmatrix}
-1 & 0 \\
1 & 1
\end{pmatrix} \cdot \tilde{H}(1).
\]
Define $C = \hat{H}(1)$. Then (84) and the last lines prove (73). Furthermore, set

$$H(s, t) = (-1)^{\varepsilon} \left( \frac{\sqrt{st}}{0} \frac{0}{\sqrt{st}} \right) \cdot C
- k \ln(st) \left( \frac{0}{-1} \frac{1}{\sqrt{st}} \right) \cdot e_1 \otimes e_2 + k \ln(st) e_1 \otimes e_2
+ k \ln(st) \left( \frac{-1}{0} \frac{st}{1} \right) \cdot e_1 \otimes e_2 - 2k \ln(s) \left( \frac{-1}{0} \frac{st}{1} \right) \cdot e_1 \otimes e_2.
$$

Now, replace $t$ by $\sqrt{st}$ in (85). This yields a formula for $\hat{H}(st)$. Plugging this formula into (80) shows that $F$ satisfies

$$F(t) = \left( \frac{1}{s t} \frac{0}{1} \right) \cdot F\left( \frac{st}{s + 1} \right) + (-1)^{\varepsilon} \left( \frac{s}{t} \frac{0}{1} \right) \cdot F\left( \frac{t}{s + 1} \right) + H(s, t). \quad (86)$$

If we can show that $k = 0$, then we are done. Indeed, if $k$ vanishes, then the definition of $H(s, t)$ and the last equation prove the desired functional equation (71) for $F$. Moreover, the definition of $F$ and (61) would imply (72).

So let us turn to the proof that $k = 0$. Let $p$ be even such that $\frac{p}{2}$ is also even and suppose that $\varepsilon = 1$. Recall that this is the only combination of $p$ and $\varepsilon$ for which we have to prove something since in all other cases we already know from Lemma 3.12 that $k = 0$. First, assume $p = 0$. If we plug $F = k \ln$ into (58), then for all $k$ the right hand side of this equation is always equal to 0. Hence, we can just set $k = 0$.

Second, suppose that $p \neq 0$. Multiplying (86) by

$$\left( \frac{1}{x + y} \frac{0}{1} \right)
$$

and then setting $s = \frac{x}{y}$ and $t = x + y$ yields

$$\left( \frac{1}{x+y} \frac{0}{1} \right) \cdot F(x+y) = \left( \frac{1}{x+y} \frac{0}{1} \right) \cdot F(x) - \left( \frac{0}{y} \frac{x+y}{1} \right) \cdot F(y) + \left( \frac{1}{x+y} \frac{0}{1} \right) \cdot H\left( \frac{x}{y}, x+y \right)$$

for all $x, y > 0$. Define a function $G: (0, \infty) \to \text{Sym}^p(\mathbb{R}^2)$ by

$$G(x) = \left( \frac{1}{x} \frac{0}{1} \right) \cdot F(x).$$

Then the previous equation becomes

$$G(x+y) = G(x) - \left( \frac{-1}{0} \frac{x+y}{1} \right) \cdot G(y) + \left( \frac{1}{x+y} \frac{0}{1} \right) \cdot H\left( \frac{x}{y}, x+y \right).$$

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From this it follows easily that the $p$-th component of $G$ satisfies
\[ G_p(x + y) - G_p(x) + G_p(y) = \left[ \begin{pmatrix} 1 & 0 \\ \frac{1}{x+y} & 1 \end{pmatrix} \right] \cdot H \left( \frac{x}{y}, x + y \right) \] . \tag{87}

In particular, by setting $x = y$, we derive that
\[ G_p(x) = \left[ \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{pmatrix} \right] \cdot H(1, x) . \tag{88} \]

Next, we determine the inhomogeneity of (87) explicitly. By the definition of $H$ we have
\[
\begin{align*}
\left( \begin{pmatrix} 1 & 0 \\ \frac{1}{x+y} & 1 \end{pmatrix} \right) \cdot H \left( \frac{x}{y}, x + y \right) &= (-1)^{\varepsilon} \left( \begin{pmatrix} 1 & 0 \\ \frac{1}{x+y} & 1 \end{pmatrix} \right) \left( \begin{pmatrix} \sqrt{\frac{x(x+y)}{y}} & 0 \\ 0 & \sqrt{\frac{y}{x(x+y)}} \end{pmatrix} \right) \cdot C \\
&\quad - k \ln \left( \frac{x(x+y)}{y} \right) \begin{pmatrix} 0 & 1 \\ -1 & \frac{1}{x} \end{pmatrix} \cdot e_1 ^{\otimes \frac{\varepsilon}{2}} \otimes e_2 ^{\otimes \frac{\varepsilon}{2}} \\
&\quad + k \ln \left( \frac{x(x+y)}{y} \right) \begin{pmatrix} 1 & 0 \\ \frac{1}{x+y} & 1 \end{pmatrix} \cdot e_1 ^{\otimes \frac{\varepsilon}{2}} \otimes e_2 ^{\otimes \frac{\varepsilon}{2}} \\
&\quad + k \ln \left( \frac{x(x+y)}{y} \right) \begin{pmatrix} -1 & \frac{x(x+y)}{x+y} \\ \frac{1}{x-y} & \frac{y}{x+y} \end{pmatrix} \cdot e_1 ^{\otimes \frac{\varepsilon}{2}} \otimes e_2 ^{\otimes \frac{\varepsilon}{2}} \\
&\quad - 2k \ln \left( \frac{x}{y} \right) \begin{pmatrix} -1 & \frac{x(x+y)}{x+y} \\ \frac{1}{x-y} & \frac{y}{x+y} \end{pmatrix} \cdot e_1 ^{\otimes \frac{\varepsilon}{2}} \otimes e_2 ^{\otimes \frac{\varepsilon}{2}} .
\end{align*}
\]

Recall that $\frac{\varepsilon}{2}$ is even and $\varepsilon = 1$. Thus, applying (71) to the first term and (53) to the other ones gives
\[
\begin{align*}
\left[ \begin{pmatrix} 1 & 0 \\ \frac{1}{x+y} & 1 \end{pmatrix} \right] H \left( \frac{x}{y}, x + y \right) \bigg|_p &= -(x+y)^{-\frac{\varepsilon}{2}} \sum_{j=0}^{p} C_j \left( \frac{x}{y} \right)^{\frac{\varepsilon-j}{2}} \\
&\quad + k \left( (x+y)^{-\frac{\varepsilon}{2}} - x^{-\frac{\varepsilon}{2}} + y^{-\frac{\varepsilon}{2}} \right) \ln \left( \frac{x(x+y)}{y} \right) - 2ky^{-\frac{\varepsilon}{2}} \ln \left( \frac{x}{y} \right) .
\end{align*}
\]

If we set $x = y$ in this equation and recall relation (88), then we arrive at
\[ G_p(x) = x^{-\frac{\varepsilon}{2}} \left( k \ln(x) - \sum_{j=0}^{p} C_j \right) . \]

Plugging the last two expressions into (87) yields
\[
\begin{align*}
(x+y)^{-\frac{\varepsilon}{2}} &\left( k \ln(x+y) - \sum_{j=0}^{p} C_j \right) - x^{-\frac{\varepsilon}{2}} \left( k \ln(x) - \sum_{j=0}^{p} C_j \right) + y^{-\frac{\varepsilon}{2}} \left( k \ln(y) - \sum_{j=0}^{p} C_j \right) = \\
- (x+y)^{-\frac{\varepsilon}{2}} \sum_{j=0}^{p} C_j \left( \frac{x}{y} \right)^{\frac{\varepsilon-j}{2}} + k \left( (x+y)^{-\frac{\varepsilon}{2}} - x^{-\frac{\varepsilon}{2}} + y^{-\frac{\varepsilon}{2}} \right) \ln \left( \frac{x(x+y)}{y} \right) - 2ky^{-\frac{\varepsilon}{2}} \ln \left( \frac{x}{y} \right) ,
\end{align*}
\]

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which we rewrite to
\[
\begin{align*}
k \ln(x) &= \left((x + y)^{-\frac{x}{y}} - y^{-\frac{x}{y}} - (x + y)^{-\frac{y}{x}}\right)^{-1} \left(\sum_{j=0}^{p} C_j \frac{x}{y}^j \right) \\
&\quad + (x + y)^{-\frac{y}{x}} \sum_{j=0}^{p} C_j \frac{x}{y}^j + k \left((x + y)^{-\frac{y}{x}} - x^{-\frac{y}{x}}\right) \ln(y) + k \left(x^{-\frac{x}{y}} - y^{-\frac{y}{x}}\right) \ln(x + y)
\end{align*}
\]

Note that in this step we used the fact that \( p \neq 0 \). Fix a \( y > 0 \). Using the standard branch of the logarithm, the left hand side can be extended to a holomorphic function on \( \mathbb{C} \setminus (-\infty, 0] \) whereas the right hand side can be extended to a meromorphic function on \( \mathbb{C} \setminus (-\infty, -y] \). If \( k \neq 0 \), then the identity theorem for holomorphic functions would imply that the left hand side could be further extended continuously at some point on the negative real axis, which is impossible. Thus, \( k \) has to be zero.

The last lemma shows that there exists a describing function \( F \) which satisfies a linear functional equation. By solving this equation, we will now describe such functions completely.

**3.14 Lemma.** Let \( \mu \in \text{TVal}_p^p(\mathbb{R}^2) \) be described by some measurable function. For \( p = 0 \) and \( \varepsilon = 0 \) there exist constants \( k_1, k_2 \in \mathbb{R} \) such that
\[
F(x) = k_1 x + k_2, \quad x > 0,
\]
describes \( \mu \). For \( p = 0 \) and \( \varepsilon = 1 \) the function \( F = 0 \) describes \( \mu \).

Let \( p \geq 1 \). Unless \( p \) is odd and \( \varepsilon = 1 \), there exists a tensor \( K \in \text{Sym}_p(\mathbb{R}^2) \) with
\[
K_{p-1} = 0 \quad \text{and} \quad (-1)^{p}(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}) \cdot K = K
\]
such that
\[
F(x) = \int_0^x \begin{pmatrix} 1 & 0 \\ -\frac{1}{x} & 1 - \frac{z}{x} \end{pmatrix} \cdot K \, dz, \quad x > 0,
\]
describes \( \mu \).

**Proof.** We can assume that \( \mu \) is described by a function \( F: (0, \infty) \to \text{Sym}_p(\mathbb{R}^2) \) that satisfies the conclusions of Lemma 3.13. As in the proof of Lemma 3.7, multiplying (71) by
\[
\begin{pmatrix} 1 & 0 \\ \frac{1}{x+y} & 1 \end{pmatrix}
\]
and then setting \( s = \frac{x}{y} \) and \( t = x + y \) yields
\[
\begin{pmatrix} 1 & 0 \\ \frac{1}{x+y} & 1 \end{pmatrix} \cdot F(x + y) = \begin{pmatrix} 1 & 0 \\ \frac{1}{x+y} & 1 \end{pmatrix} \cdot F(x) + (-1)^{p}(\begin{smallmatrix} \frac{x}{y} & x+y \\ y & 1 \end{smallmatrix}) \cdot F(y)
\]
\[
+ (-1)^{p}(\begin{pmatrix} \frac{x+y}{y} \\ y & x+y \end{pmatrix} \cdot C.
\]

for all $x, y > 0$. Define a function $G : (0, \infty) \rightarrow \text{Sym}^p(\mathbb{R}^2)$ by

$$G(x) = \begin{pmatrix} 1 & 0 \\ \frac{1}{x} & 1 \end{pmatrix} \cdot F(x).$$

By the symmetry relation (72) we have

$$(-1)^\varepsilon \begin{pmatrix} -1 & x \\ 0 & 1 \end{pmatrix} \cdot G(x) = G(x).$$

Thus, an elementary calculation yields

$$G(x + y) = G(x) + \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \cdot G(y) + (-1)^\varepsilon \left( \frac{x}{\sqrt{x+y}} \sqrt{\frac{y}{x+y}} \right)^{-\frac{p}{2}} \sum_{j=0}^{p} C_j \left( \frac{x}{y} \right)^{\frac{p}{2}-j} \cdot C.$$  (90)

First, let $p = 0$. For $\varepsilon = 0$, equation (90) simplifies to

$$G_0(x + y) = G_0(x) + G_0(y) + C_0.$$  (90)

This is an inhomogeneous version of Cauchy’s functional equation. Therefore, the solution is given by $G_0(x) = k_1 x + k_2$ for some constant $k_1$ and $k_2 = -C_0$. Since $F = G_0$ for $p = 0$, the case $\varepsilon = 0$ is settled. If $\varepsilon = 1$, then equation (72) directly implies $F = 0$. This concludes the proof of the scalar case $p = 0$.

Second, let $p \geq 1$. For $x, y > 0$ define

$$h(x, y) = (-1)^\varepsilon (x+y)^{\frac{p}{2}} \sum_{j=0}^{p} C_j \left( \frac{x}{y} \right)^{\frac{p}{2}-j}.\quad (91)$$

With this definition, equation (90) simplifies to

$$G_p(x + y) = G_p(x) + G_p(y) + h(x, y).$$

This clearly implies that $h$ is a symmetric function. By the last relation, we can calculate $G_p(x + y + 1)$ in two different ways. On the one hand,

$$G_p(x + y + 1) = G_p(x) + G_p(y + 1) + h(x, y + 1)$$

$$= G_p(x) + G_p(y) + G_p(1) + h(x, y + 1) + h(1, y)$$

and on the other hand

$$G_p(x + y + 1) = G_p(x + 1) + G_p(y) + h(x + 1, y)$$

$$= G_p(x) + G_p(y) + G_p(1) + h(x + 1, y) + h(x, 1).$$
Consequently, for all \( x, y > 0 \) we have
\[
h(x, y + 1) + h(1, y) = h(x + 1, y) + h(x, 1). \tag{92}
\]

Definition \([91]\) clearly extends to all \( x \in \mathbb{C} \) with \( 0 < |x| < |y| \). Let \( y > 2 \) and write \( B_1^\times(0) \) for the punctured unit disc \( \{ x \in \mathbb{C} : 0 < |x| < 1 \} \). The identity theorem for holomorphic functions and \([92]\) imply that
\[
h(x^2, y + 1) + h(1, y) = h(x^2 + 1, y) + h(x^2, 1) \tag{93}
\]
holds for all \( x \in B_1^\times(0) \). Therefore, we can use this equation to compare coefficients of the respective Laurent series. We consider the Laurent expansions at zero and write, for example, \( [x^j] h(x^2, y) \) for the coefficient of \( x^j \) in the Laurent expansion of \( x \mapsto h(x^2, y) \). The functions \( x \mapsto h(1, y) \) and \( x \mapsto h(x^2 + 1, y) \) are holomorphic on \( B_1(0) \). Hence,
\[
[x^j] h(1, y) = [x^j] h(x^2 + 1, y) = 0
\]
for \( j < 0 \). Moreover, we obviously have
\[
[x^0] h(1, y) = [x^0] h(x^2 + 1, y).
\]
So by \([93]\) we deduce for all \( j \leq 0 \) that
\[
[x^j] h(x^2, y + 1) = [x^j] h(x^2, 1). \tag{94}
\]
We need a series expansion of \( x \mapsto h(x, y) \). Since \( h \) is symmetric, we can also look at \( x \mapsto h(y, x) \). For this map, using the Taylor expansion of \( x \mapsto (x + y)^{-\frac{p}{2}} \) at zero and the first relation of \([73]\), we obtain by a rearrangement of the involved sums that
\[
h(x, y) = \sum_{i=0}^{\infty} \sum_{j=0}^{i \wedge p} \binom{\frac{-p}{2}}{i-j} C_{p-j} x^i y^{-i}
\]
for all \( x \in \mathbb{R} \) with \( 0 < x < y \). Here, \( i \wedge p \) denotes the minimum of \( i \) and \( p \). This relation directly yields the Laurent expansion of \( h(x^2, y) \) at zero. Thus, \([94]\) for the coefficients of \( x^{2i-p} \) gives
\[
\sum_{j=0}^{i} \binom{\frac{-p}{2}}{i-j} C_{p-j} (y + 1)^{-i} = \sum_{j=0}^{i} \binom{\frac{-p}{2}}{i-j} C_{p-j}
\]
for \( i \in \{0, \ldots, \lfloor \frac{p}{2} \rfloor \} \). Clearly, this can only hold if
\[
\sum_{j=0}^{i} \binom{\frac{-p}{2}}{i-j} C_{p-j} = 0 \tag{95}
\]
for \( i \in \{1, \ldots, \lfloor \frac{p}{2} \rfloor \} \).
We will now prove that \( C_p \) determines all other components of \( C \). This is clear once we have shown that
\[
(-1)^i C_i = C_{p-i} = \binom{p^2 - i}{i} C_p
\]
for \( i \in \{0, \ldots, \lfloor \frac{p}{2} \rfloor \} \). The first relation of (73) in coordinates is precisely the first equation. The second is shown by induction on \( i \). Clearly, the desired equality holds for \( i = 0 \). So let \( i \geq 1 \) and assume that it holds for all \( j < i \). Then the induction assumption and the Vandermonde identity (2) yield
\[
\sum_{j=0}^{i} \binom{-\frac{p}{2}}{i - j} C_{p-j} = C_p \sum_{j=0}^{i-1} \binom{-\frac{p}{2}}{i - j} \binom{\frac{p}{2}}{j} + C_{p-i}
\]
\[
 = - \binom{\frac{p}{2}}{i} C_p + C_{p-i}.
\]
Now (95) implies (96).

First, suppose that \( p + \epsilon \) is odd. Looking at the 0-th coordinate of the second part of (73), we see that \( C_0 = (-1)^p C_0 \).

So \( C_0 = 0 \) and, by (96), also \( C = 0 \). From (71), (72), and Lemma 3.7 we therefore conclude that \( F \) has the desired form.

Second, let \( p \) as well as \( \frac{p}{2} \) be even and suppose that \( \epsilon = 0 \). From the second part of (73), equation (7), an index change, relation (96), and the binomial theorem we get
\[
C_p = \left[ \begin{array}{cc} -1 & 0 \\ -1 & 1 \end{array} \right] C_p = \sum_{j=0}^{p} (-1)^{p-j} C_j = \sum_{j=0}^{\frac{p}{2}} (-1)^{j} C_j + \sum_{j=0}^{\frac{p}{2}-1} (-1)^{j} C_{p-j}
\]
\[
= 2C_p \sum_{j=0}^{\frac{p}{2}} \binom{\frac{p}{2}}{j} (-1)^{j} - (-1)^{\frac{p}{2}} C_p = -C_p.
\]
Thus, \( C_p \) is equal to zero. As before, this implies \( C = 0 \) and that \( F \) has the desired form.

Third, assume that \( p \) is even, \( \frac{p}{2} \) is odd and \( \epsilon = 0 \). Using (7) and (96), it is not hard to see that the constant function
\[
-C_p e_1 \odot \frac{\epsilon}{2} \odot e_2 \odot \frac{\epsilon}{2}
\]
\[\text{is a solution for (71) and (72). Thus, the sum}\]
\[\tilde{F}(x) = F(x) + C_p e_1 \odot \frac{\epsilon}{2} \odot e_2 \odot \frac{\epsilon}{2}\]
\[\text{satisfies (29) and (39). Lemma 3.7 again shows that } \tilde{F} \text{ has the desired form. Moreover, a glance at (59) reveals that, for this combination of } p, \frac{p}{2}, \text{ and } \epsilon, \text{ we can add a constant}\]
term to the $\frac{p}{2}$-th coordinate without changing the fact that $F$ describes $\mu$. Thus, $\tilde{F}$ also describes $\mu$.

Finally, we have to show that we can choose $K_{p-1} = 0$. For $\varepsilon = 0$ we already know $K_{p-1} = 0$ from Lemma 3.7. So let $p$ be even and $\varepsilon = 1$ and set $K = e_1 \odot e_2^{p-1}$. If we plug this into (89), then we get

$$F(x) = \left(\begin{array}{c} x \\ 0 \\ 1 \end{array}\right) \cdot \int_0^x \left(\begin{array}{c} \frac{1}{x} \\ -1 \\ 1 - \frac{x}{2} \end{array}\right) \odot e_1 \odot e_2^{p-1} \, dz.$$ 

The substitution $z/x = u$ and the definition of the action $\cdot$ show

$$F(x) = \left(\begin{array}{c} x \\ 0 \\ 1 \end{array}\right) \cdot \int_0^1 \left(\begin{array}{c} 1 \\ -1 \end{array}\right) \odot \left(\begin{array}{c} u \\ 1-u \end{array}\right)^{p-1} \, du.$$ 

An application of Lemma 2.2 with $b = 1$, $c = -1$ and $i = p - 1$ proves

$$F(x) = \frac{x^p}{p} e_1^{p-1} - \frac{1}{p} e_2^{p}.$$ 

Plugging this into (58), we see that the right hand side is always equal to 0. This completes the proof. $\square$

Finally, our main result in the planar case can be deduced by a counting argument.

**Proof of Theorem 1.5.** Suppose that $p \geq 1$. We denote by $\text{Val}_{\varepsilon,F}(\mathbb{R}^2)$ the vector space of valuations from $\text{Val}_{\varepsilon}(\mathbb{R}^2)$ which can be described by some function $F$. Furthermore, define two subspaces of $\text{Sym}^p(\mathbb{R}^2)$ by

$$V_\varepsilon = \left\{ K \in \text{Sym}^p(\mathbb{R}^2) : K_{p-1} = 0 \right\}.$$ 

Unless $p$ is odd and $\varepsilon = 1$, Lemma 3.14, the $\text{SL}(2)$-covariance of $\mu$, and Theorem 2.1 show the existence of an injective linear map from $\text{Val}_{\varepsilon,F}(\mathbb{R}^2)$ to $V_\varepsilon$. An immediate consequence is that the inequality

$$\dim \text{Val}_{\varepsilon,F}(\mathbb{R}^2) \leq \dim V_\varepsilon$$ 

holds unless $p$ is odd and $\varepsilon = 1$.

First, let $p$ be even. By Lemma 3.11, the $\text{SL}(2)$-covariance of $\mu$, and Theorem 2.1 we have

$$\dim \text{Val}_0^p(\mathbb{R}^2) \leq \dim \text{Val}_{0,F}^p(\mathbb{R}^2) + 1$$

and

$$\dim \text{Val}_1^p(\mathbb{R}^2) = \dim \text{Val}_{1,F}^p(\mathbb{R}^2).$$

Using (16), the two relations above, and (97), we conclude

$$\dim \text{Val}_0^p(\mathbb{R}^2) \leq \dim V_0 + \dim V_1 + 1.$$ 

(98)
In coordinates, the second condition on $K$ in the definition of $V_\varepsilon$ reads as $(-1)^{\varepsilon+p+i}K_i = K_i$, $i \in \{0, \ldots, p\}$. Therefore, the dimensions of $V_\varepsilon$ satisfy

$$\dim V_0 = \frac{p}{2} + 1 \quad \text{and} \quad \dim V_1 = \frac{p}{2} - 1.$$ 

Now, (98) implies that $\text{TVal}^p(\mathbb{R}^2)$ is at most $(p+1)$-dimensional.

Second, let $p$ be odd. Assume that $\varepsilon = 0$. In this case we have

$$\dim V_0 = \frac{p-1}{2} + 1.$$

From (97) we deduce that

$$\dim \text{TVal}_{0,F}^p(\mathbb{R}^2) \leq \frac{p-1}{2} + 1.$$

Note that for this combination of $p$ and $\varepsilon$, the constant $k$ in Lemma 3.14 vanishes. Hence, $\text{TVal}_{0}^p(\mathbb{R}^2) = \text{TVal}_{0,F}^p(\mathbb{R}^2)$, which in turn gives

$$\dim \text{TVal}_{0}^p(\mathbb{R}^2) \leq \frac{p-1}{2} + 1. \quad (99)$$

Consider the map $R: \text{TVal}_{0}^p(\mathbb{R}^2) \to \text{TVal}_{1}^p(\mathbb{R}^2)$ defined by

$$R(\mu)(P) = \rho \cdot \mu(P^*), \quad P \in P_0^2,$$

where as before, $\rho$ denotes the counter-clockwise rotation about an angle of $\frac{\pi}{2}$. Since $R \circ R = -\text{Id}$, the map $R$ is an isomorphism. Consequently, the spaces $\text{TVal}_{0}^p(\mathbb{R}^2)$ and $\text{TVal}_1^p(\mathbb{R}^2)$ have the same dimension. From (99) and (16) we infer that also in this case $\text{TVal}_0^p(\mathbb{R}^2)$ is at most $(p+1)$-dimensional.

Since the $p+1$ valuations from the statement of the theorem are linearly independent and have the desired properties, the proof is completed.

For $p = 0$ we can argue analogously. We get

$$\dim \text{TVal}_{0}^0(\mathbb{R}^2) = 3 \quad \text{and} \quad \dim \text{TVal}_{1}^0(\mathbb{R}^2) = 0.$$ \hfill $\Box$

We conclude this section with the dual result of Theorem 1.5. A map $\mu: P_0^n \to \text{Sym}^p(\mathbb{R}^n)$ is said to be $\text{SL}(n)$-contravariant if

$$\mu(\phi P) = \phi^{-t} \cdot \mu(P)$$

for all $P \in P_0^n$ and each $\phi \in \text{SL}(n)$. The vector space of all measurable $\text{SL}(n)$-contravariant valuations will be denoted by $\text{TVal}_p(\mathbb{R}^n)$.

3.15 Theorem. For $n = 2$ the following holds.

- A basis of $\text{TVal}_0(\mathbb{R}^2)$ is given by $\chi$, $V$ and $V \circ \cdot$. 

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• For \( p \geq 1 \), a basis of \( \text{TV}_{\rho}(\mathbb{R}^2) \) is given by \( \rho \cdot M_{i,p}^{i,p} \) for \( i \in \{0, \ldots, p\} \setminus \{p-1\} \) and \( M^{p,0} \circ \ast \).

Proof. The map \( S: \text{TV}_{\rho}(\mathbb{R}^2) \to \text{TV}_{\rho}(\mathbb{R}^2) \) defined by

\[
S\mu = \rho \cdot \mu
\]

is an isomorphism. The result now follows directly from Theorem 1.5. ☐

3.3 The \( n \)-dimensional case

In this section we will prove Theorem 1.4 by induction over the dimension. During the induction step, we will encounter a tensor valuation that might not be symmetric. This makes it necessary to establish some results for non-symmetric tensor valuations first.

Note that the definition of \( \text{SL}(n) \)-contravariance given at the end of the previous section extends in an obvious way to maps \( \mu: \mathcal{P}_o \to (\mathbb{R}^n)^{\otimes p} \). Also recall that by our notation convention we set \( J = [-cn, de_n] \).

3.16 Lemma. Let \( n \geq 2 \) and \( \mu: \mathcal{P}_o \to (\mathbb{R}^n)^{\otimes p} \) be a measurable \( \text{SL}(n) \)-contravariant valuation. If \( \mu \) satisfies

\[
\mu[B, J] = 0 \quad (100)
\]

for all \( B \in \mathcal{P}_o^{n-1} \) and \( c,d > 0 \), then there exists a family of measurable functions \( F^B: (\mathbb{R}^{n-1})^2 \to (\mathbb{R}^n)^{\otimes p} \) with

\[
F^B(x, y) = \mu \left[ B, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right]
\]

for all \( B, c, d \) and \( x, y \in \mathbb{R}^{n-1} \) which form a double pyramid. Furthermore, each \( F^B \) satisfies

\[
F^B(x, y) = \begin{pmatrix} \text{Id} & 0 \\ -x^t & 1 \end{pmatrix} \cdot F^B(0, y - x) \quad (101)
\]

and

\[
F^B(0, x + y) = F^B(0, x) + \begin{pmatrix} \text{Id} & 0 \\ -x^t & 1 \end{pmatrix} \cdot F^B(0, y) \quad (102)
\]

for all \( x, y \in \mathbb{R}^{n-1} \).

Proof. The arguments which will be used are similar to the ones in the proof of Lemma 3.8. Let \( B \in \mathcal{P}_o^{n-1} \) and \( x, y \in \mathbb{R}^{n-1} \) be given. Choose \( c,d > 0 \) such that \( B, c, d, x, y \) form a double pyramid. For sufficiently small \( r > 0 \) the valuation property implies

\[
\mu \left[ B, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right] + \mu \left[ B, -r \begin{pmatrix} y \\ 1 \end{pmatrix}, r \begin{pmatrix} y \\ 1 \end{pmatrix} \right] =
\]

\[
\mu \left[ B, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, r \begin{pmatrix} y \\ 1 \end{pmatrix} \right] + \mu \left[ B, -r \begin{pmatrix} y \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right].
\]
By assumption (100) and the SL(2)-contravariance of \( \mu \) this simplifies to

\[
\mu \left[ B, -c \left( x \begin{array}{c} 1 \\ 1 \end{array} \right), d \left( y \begin{array}{c} 1 \\ 1 \end{array} \right) \right] = \mu \left[ B, -c \left( x \begin{array}{c} 1 \\ 1 \end{array} \right), r \left( y \begin{array}{c} 1 \\ 1 \end{array} \right) \right].
\]

Hence, the expression

\[
\mu \left[ B, -c \left( x \begin{array}{c} 1 \\ 1 \end{array} \right), d \left( y \begin{array}{c} 1 \\ 1 \end{array} \right) \right]
\]

is independent of \( d \). By analogous arguments we see that it is also independent of \( c \). Therefore, the family \( F^B \) is well defined.

The SL(\( n \))-contravariance of \( \mu \) implies (101). So it remains to prove (102). The valuation property of \( \mu \) implies for sufficiently small \( r > 0 \) that

\[
\mu \left[ B, -c \left( x \begin{array}{c} 1 \\ 1 \end{array} \right), d \left( y \begin{array}{c} 1 \\ 1 \end{array} \right) \right] + \mu \left[ B, -re_n, re_n \right] = \mu \left[ B, -c \left( x \begin{array}{c} 1 \\ 1 \end{array} \right), re_n \right] + \mu \left[ B, -re_n, d \left( y \begin{array}{c} 1 \\ 1 \end{array} \right) \right].
\]

By (100) and the definition of \( F^B \) we therefore obtain

\[
F^B(x, y) = F^B(x, 0) + F^B(0, y).
\]

Combining this with (101) gives

\[
\left( \begin{array}{cc} \text{Id} & 0 \\ -x^t & 1 \end{array} \right) \cdot F^B(0, y - x) = \left( \begin{array}{cc} \text{Id} & 0 \\ -x^t & 1 \end{array} \right) \cdot F^B(0, -x) + F^B(0, y).
\]

Replace \( x \) by \(-x\) in this equation. Then a matrix multiplication proves (102). \( \square \)

Our next result deals with valuations which are not only compatible with the special linear group, but with \( \text{GL}^+(n) \), i.e. linear maps with positive determinant. We say that a map \( \mu: \mathcal{P}^n \to (\mathbb{R}^n)^{\otimes p} \) is \( \text{GL}^+(n) \)-contravariant if there exists a \( q \in \mathbb{R} \) such that

\[
\mu(\phi P) = (\det \phi)^q \phi^{-t} \cdot \mu(P)
\]

for all \( P \in \mathcal{P}^n \) and each \( \phi \in \text{GL}^+(n) \). Clearly, every \( \text{GL}^+(n) \)-contravariant map is also \( \text{SL}(n) \)-contravariant.

3.17 Theorem. Let \( \mu: \mathcal{P}^2 \to (\mathbb{R}^2)^{\otimes p} \) be a measurable \( \text{GL}^+(2) \)-contravariant valuation. If

\[
\mu[I, J] = 0
\]

for all \( a, b, c, d > 0 \), then \( \mu \) vanishes everywhere.

Proof. By Theorem 2.4 and the SL(\( n \))-contravariance of \( \mu \) it is enough to show that \( \mu \) vanishes on double pyramids. So let \( a, b, c, d > 0 \) and \( x, y \in \mathbb{R} \) form a double pyramid. With the aid of (9), we obtain from Lemmas 3.16 and 3.6 that

\[
\mu \left[ I, -c \left( x \begin{array}{c} 1 \\ 1 \end{array} \right), d \left( y \begin{array}{c} 1 \\ 1 \end{array} \right) \right] = \int_x^y \left( \begin{array}{cc} 1 & 0 \\ -z & 1 \end{array} \right) \cdot K^t \, dz \tag{103}
\]
for some tensor $K^I \in (\mathbb{R}^2)^{\otimes p}$. We have to show that $K^I = 0$.

Since $\mu$ is GL$^+(2)$-contravariant, there exists a $q \in \mathbb{R}$ with

$$\mu \left[ rI, -c \left( \frac{x}{1} \right), 1 \right] = r^{2q-p} \mu \left[ I, -\frac{c}{r} \left( \frac{x}{1} \right), -\frac{d}{r} \left( \frac{y}{1} \right) \right] = r^{2q-p} \int_x^y \left( 1, 0 \right)^t \cdot K^I \cdot dz$$

for all $r > 0$. Consequently,

$$\int_x^y \left( 1, 0 \right)^t \cdot K^I \cdot dz = r^{2q-p} \int_x^y \left( 1, 0 \right)^t \cdot K^I \cdot dz.$$  

From (12) we deduce that $I \mapsto K^I_\alpha$ is $(2q-p)$-homogeneous for all $\alpha \in \{1, 2\}^p$. Similarly,

$$\mu \left[ -I, -c \left( \frac{x}{1} \right), d \left( \frac{y}{1} \right) \right] = (-1)^p \mu \left[ I, -d \left( \frac{y}{1} \right), c \left( \frac{x}{1} \right) \right] = (-1)^{p+1} \int_x^y \left( 1, 0 \right)^t \cdot K^I \cdot dz.$$  

As before, we conclude that $I \mapsto K^I_\alpha$ is either even or odd. Since $\mu$ is a valuation, so is $I \mapsto K^I_\alpha$. In fact, this follows from (103) and (12). By Theorems 3.2 and 3.4 we therefore have

$$K^I_\alpha = \begin{cases} k_\alpha [a^{2q-p} + (-1)^{p+1} b^{2q-p}] & \text{for } 2q-p \neq 0, \\ k_\alpha \ln(a) - \ln(b) & \text{for } 2q-p = 0 \text{ and } p \text{ even}, \\ k_\alpha & \text{for } 2q-p = 0 \text{ and } p \text{ odd}, \end{cases}$$

where $k_\alpha \in \mathbb{R}$ is some constant. Define a tensor $K \in (\mathbb{R}^2)^{\otimes p}$ componentwise by $K_\alpha = k_\alpha$.

It remains to prove that $K$ vanishes.

Let $s, u > 0$. Consider the triangle $T$ with corners at $se_1 + e_2$, $-ue_1$ and $-e_2$. A simple calculation shows that $T$ can be written in two different ways, namely

$$T = \left[ -ue_1, \frac{s}{2}e_1, -e_2, \left( \frac{s}{1} \right) \right]$$

and

$$T = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \left[ -e_1, \frac{u}{s+u}e_1, -s \left( \frac{1}{s} \right), ue_2 \right].$$

By the SL(2)-contravariance of $\mu$ and (108) we get

$$\int_0^s \left( 1, 0 \right)^t \cdot K[-ue_1, ze_1] \cdot dz = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \int_0^1 \left( 1, 0 \right)^t \cdot K[-e_1, ze_1] \cdot dz.$$  

(104)

For the tensor $K$ from above, define two tensor polynomials by

$$P(s) = \int_0^s \left( 1, 0 \right)^t \cdot K \cdot dz \quad \text{and} \quad Q(s) = \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \int_0^0 \left( 1, 0 \right)^t \cdot K \cdot dz.$$  

Recall that we have to prove $K = 0$. So by (12) it is enough to show that either $P(s)$ or $Q(s)$ vanishes.
Let $\alpha \in \{0, 1\}^p$. It suffices to show $P_\alpha(s) = 0$ or $Q_\alpha(1)$ = 0 for all positive $s$. First, assume $2q - p \neq 0$. By the representation of $K^I$, the $\alpha$-component of equation (104) simplifies to

$$
\left( u^{2q-p} + (-1)^{p+1} \left( \frac{s}{2} \right)^{2q-p} \right) P_\alpha(s) = \left( 1 + (-1)^{p+1} \left( \frac{u}{s + u} \right)^{2q-p} \right) Q_\alpha(\frac{1}{s}).
$$

If $2q - p > 0$, let $u$ tend to infinity. Note that the limit of the right hand side exists and is finite. Thus, $P_\alpha(s) = 0$. Next, suppose that $2q - p < 0$. The last equation is clearly equivalent to

$$
u^{2q-p}\left( P_\alpha(s) + (-1)^p(s + u)^{p-2q}Q_\alpha(\frac{1}{s}) \right) = (-1)^p \left( \frac{s}{2} \right)^{2q-p} P_\alpha(s) + Q_\alpha(\frac{1}{s}). \tag{105}
$$

Let $u$ tend to 0. Since the right hand side is constant in $u$, we must have

$$Q_\alpha(\frac{1}{s}) = (-1)^{p+1}s^{2q-p}P_\alpha(s).
$$

If we set $u = st$ in (105) and use the last relation we obtain

$$P_\alpha(s) \left( t^{2q-p} - (t + 1)^{p-2q} + (-1)^p(1 - 2^{p-2q}) \right) = 0
$$

for all $t > 0$. Obviously, this implies $P_\alpha(s) = 0$.

Second, let $2q - p = 0$ and $p$ be even. In this case, the $\alpha$-component of (104) is

$$\left( \ln(u) - \ln(\frac{s}{2}) \right) P_\alpha(s) = -\ln \left( \frac{u}{s + u} \right) Q_\alpha(\frac{1}{s}).
$$

The limit $u \to \infty$ implies that $P_\alpha(s) = 0$.

Finally, let $2q - p = 0$ and $p$ be odd. Then we have

$$P_\alpha(s) = Q_\alpha(\frac{1}{s}).
$$

The left hand side is a polynomial in $s$ without constant term and the right hand side is a polynomial in $\frac{1}{s}$ without constant term. Clearly, both polynomials have to be zero. 

Inductively, we will now extend this result to arbitrary dimensions. Let us collect some notation before stating the next theorem. In our context, an $n$-dimensional cross polytope is the convex hull of $n$ line segments

$$[-a_ie_i, b_ie_i], \quad i = 1, \ldots, n,$$

where, for all $i$, the numbers $a_i$ and $b_i$ are positive.

Let $\alpha \in \{0, 1\}^p$. We define a subspace $U_\alpha$ of $(\mathbb{R}^n)^\otimes p$ by

$$U_\alpha = \operatorname{span}\left\{ x_1 \otimes \cdots \otimes x_p : x_i = e_n \text{ if } \alpha_i = 1, \text{ and } x_i \in \mathbb{R}^{n-1} \times \{0\} \text{ if } \alpha_i = 0 \right\}.$$
In other words, the multiindex $\alpha$ indicates the positions of $e_n$ in a tensor product. Note that

$$(\mathbb{R}^n)^{\otimes p} = \bigoplus_{\alpha \in \{0,1\}^p} U_{\alpha}.$$ 

For $C \in (\mathbb{R}^n)^{\otimes p}$ we denote by $C_{\alpha}$ the projection of $C$ onto $U_{\alpha}$. If $i$ is the total number of indices in $\alpha$ which are equal to one, then $C_{\alpha}$ will be viewed as an element of $(\mathbb{R}^{n-1})^{\otimes p-i}$.

**3.18 Theorem.** Let $n \geq 2$ and $\mu : \mathcal{P}_o^n \to (\mathbb{R}^n)^{\otimes p}$ be a measurable $\text{GL}^+(n)$-contravariant valuation. If $\mu$ vanishes on crosspolytopes, then it vanishes everywhere.

**Proof.** We will prove the theorem by induction. The case $n = 2$ is just a reformulation of Theorem 3.17. So let $n \geq 3$ and assume that the theorem holds in dimension $n - 1$. Fix numbers $c, d > 0$. For $\alpha \in \{0,1\}^p$ set

$$\nu(B) = \mu_{\alpha}[B, J], \quad B \in \mathcal{P}_o^{n-1}.$$ 

Since $B \mapsto \nu(B)$ satisfies the induction assumption, it vanishes everywhere. Hence,

$$\mu[B, J] = 0$$

for all $B \in \mathcal{P}^{n-1}$ and $c, d > 0$. From Lemma 3.16 we therefore obtain a family of functions $F^B$ such that

$$F^B(x, y) = \mu \left[ B, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right]$$

for all $x, y \in \mathbb{R}^{n-1}$ whenever $B, c, d, x, y$ form a double pyramid. Set $G^B(x) = F^B(0, x)$. Next, we deduce two properties of $G^B$. First, equation (102) becomes

$$G^B(x + y) = G^B(x) + \begin{pmatrix} \text{Id} & 0 \\ -x^t & 1 \end{pmatrix} \cdot G^B(y).$$

By the $\text{GL}^+(n)$-contravariance of $\mu$, there exists a $q \in \mathbb{R}$ such that

$$\mu \left[ \phi B, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right] = (\det \phi)^q \left( \phi^{-t} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \mu \left[ B, -c \begin{pmatrix} \phi^{-1} x \\ 1 \end{pmatrix}, d \begin{pmatrix} \phi^{-1} y \\ 1 \end{pmatrix} \right] \right)$$

for all $\phi \in \text{GL}^+(n-1)$. Therefore,

$$G^{\phi B}(x) = (\det \phi)^q \left( \phi^{-t} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot G^B(\phi^{-1} x) \right).$$

Projecting onto the subspace $U_{\alpha}$, $\alpha \in \{0,1\}$, immediately proves

$$G^B_{\alpha}(x) = (\det \phi)^q \phi^{-t} \cdot G^{\phi B}_{\alpha}(\phi^{-1} x).$$

(108)

Note that by (101) and Theorem 2.1 it is enough to prove $G^B = 0$ for all $B \in \mathcal{P}_o^{n-1}$.
We will show by induction that $G^B_\alpha = 0$. Assume that $G^B_\beta = 0$ for all $\beta \in \{0,1\}^p$ with $\beta < \alpha$, which is trivially true for $\alpha = (0,\ldots,0)$. Equation (107) and the induction assumption imply

$$G^B_\alpha(x+y) = G^B_\alpha(x) + G^B_\alpha(y).$$

Clearly, the map $x \mapsto G^B_\alpha(x)$ is measurable. If $i$ denotes the number of ones in $\alpha$, then Equation (108) implies that $G^B_\alpha(x)$ can be viewed as an element of $(\mathbb{R}^{n-1})^{\otimes p-i+1}$, say $\tilde{G}^B$. Equation (108) implies that $\tilde{G}^B\phi = (\det \phi)^q \tilde{G}^B$. Hence, $B \mapsto \tilde{G}^B$ is $GL(n-1)$-contravariant. It is also a measurable valuation because $\mu$ has these properties. If we can show that this map vanishes on crosspolytopes, then we can apply our initial induction assumption and the proof is completed.

So let $B \in P_{n-1}^o$ be a crosspolytope and fix some $j \in \{1,\ldots,n-1\}$. Since $n \geq 3$, we can choose a coordinate $k \in \{1,\ldots,n-1\} \setminus \{j\}$. Let $\phi \in SL(n)$ be the map with $e_k \mapsto e_n$ and $e_n \mapsto -e_k$ such that all other canonical basis vectors stay fixed. Note that since $B$ is a crosspolytope, there exists a $\tilde{B} \in P_{n-1}^o$ and a line segment $\tilde{J}$ in the span of $e_n$ with $\phi[B,-ce_n,d(e_j+e_n)] = [\tilde{B},\tilde{J}]$.

From the definition of $G^B$, the $SL(n)$-contravariance of $\mu$, and (109), it follows that $G^B_\alpha(e_j) = 0$. In particular, also $G^B_\alpha(x_j) = 0$ and since $x \mapsto G^B_\alpha(x)$ is linear, we conclude that $\tilde{G}^B = 0$.

Let us now come back to the symmetric setting. For $i \in \{0,\ldots,p\}$ define subspaces $U_i$ of $\operatorname{Sym}^p(\mathbb{R}^n)$ by

$$U_i = \operatorname{span}\{x_1 \odot \cdots \odot x_{p-i} \odot e_n^{\otimes i} : x_1,\ldots,x_{p-i} \in \mathbb{R}^{n-1} \times \{0\}\}.$$  

As before, $\operatorname{Sym}^p(\mathbb{R}^n)$ is the direct sum of these subspaces, i.e.

$$\operatorname{Sym}^p(\mathbb{R}^n) = \bigoplus_{i=0}^p U_i.$$  

For $C \in \operatorname{Sym}^p(\mathbb{R}^n)$ we denote by $C_i$ the projection of $C$ onto $U_i$, and $C_i$ will be viewed as an element of $\operatorname{Sym}^{p-i}(\mathbb{R}^{n-1})$. We remark that for the planar case $\operatorname{Sym}^p(\mathbb{R}^2)$, this notation coincides with the one for tensor components used before.

**3.19 Lemma.** Let $n \geq 2$ and $\mu \in \operatorname{TVal}_p(\mathbb{R}^n)$. If $\mu$ vanishes on all crosspolytopes, then it vanishes everywhere.

**Proof.** Let $n = 2$. By Theorem 3.15 we can write $\mu$ as a linear combination

$$\mu(P) = \sum_{i=0}^p c_i \phi \cdot M_i^{p-i}(P) + c_{p+1} M_{p+0}(P^*), \quad P \in P_{n-1}^o.$$
The assumption that \( \mu \) vanishes on crosspolytopes yields
\[
\sum_{i=0}^{p} c_i \left[ \rho \cdot M_{i,p}^{p-1}(B) \right]_p + c_{p+1} \left[ M_{p,0}(B^*) \right]_p = 0
\]
for all crosspolytopes \( B \). By (14) and (20) all these operators have different degrees of homogeneity. Hence, we can compare coefficients. Since by (19),
\[
\left[ \rho \cdot M_{i,p}^{p-1}(B) \right]_p \quad \text{and} \quad \left[ M_{p,0}(B^*) \right]_p
\]
do not vanish for all crosspolytopes \( B \), all \( c_i \) have to be zero, which in turn settles the case \( n = 2 \).

Let \( n \geq 3 \) and assume that the theorem holds in dimension \( n - 1 \). Exactly as in the beginning of the proof of Theorem 3.18 we obtain
\[
\mu[B, J] = 0
\]
for all \( B \in \mathcal{P}_o^{n-1} \) and \( c, d > 0 \).

Again, we can apply Lemma 3.16 to obtain a family of functions \( F^B \) such that
\[
F^B(x, y) = \mu \left[ B, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right]
\]
for all \( x, y \in \mathbb{R}^{n-1} \) whenever \( B, c, d, x, y \) form a double pyramid. Set \( G^B(x) = F^B(0, x) \). As before, we now deduce some properties of \( G^B \). First, equation (102) becomes
\[
G^B(x + y) = G^B(x) + \left( \begin{array}{cc} \text{Id} & 0 \\ -x^t & 1 \end{array} \right) \cdot G^B(y).
\]
By the SL(\( n \))-contravariance of \( \mu \) we have
\[
\mu \left[ \phi B, -c \begin{pmatrix} x \\ 1 \end{pmatrix}, d \begin{pmatrix} y \\ 1 \end{pmatrix} \right] = \left( \phi^{-t} \begin{pmatrix} 0 & 0 \\ 0 & \det \phi \end{pmatrix} \right) \cdot \mu \left[ B, -c \det \phi \begin{pmatrix} 1 \\ \det \phi^{-1} x \end{pmatrix}, d \det \phi \begin{pmatrix} 1 \\ \det \phi^{-1} y \end{pmatrix} \right]
\]
for all \( \phi \in \text{GL}^+(n - 1) \). By the definition of \( G^B \) we therefore get
\[
G^{\phi B}(x) = \left( \phi^{-t} \begin{pmatrix} 0 & 0 \\ 0 & \det \phi \end{pmatrix} \right) \cdot G^B \left( \begin{pmatrix} 1 \\ \det \phi^{-1} x \end{pmatrix} \right).
\]
In particular,
\[
G^{\phi B}_{\phi^{-1} B}(x) = (\det \phi) \phi^{-t} \cdot G^{\phi B}_{\phi^{-1} B} \left( \begin{pmatrix} 1 \\ \det \phi^{-1} x \end{pmatrix} \right).
\]
By (101) and Theorem 2.1 it suffices again to prove \( G^B = 0 \) for all \( B \in \mathcal{P}_o^{n-1} \).
We will show by induction that \( G_i^B = 0 \), \( i \in \{0, \ldots, p\} \). Assume that \( G_j^B = 0 \) for all \( j \in \{0, \ldots, p\} \) with \( j < i \), which is trivially true for \( i = 0 \). Equation (109) together with the induction assumption proves

\[
G_i^B(x + y) = G_i^B(x) + G_i^B(y).
\]

Since \( x \mapsto G_i^B(x) \) is also measurable, it can be interpreted as an element of \((\mathbb{R}^{n-1})^\otimes (p-i+1)\), say \( \tilde{G}^B \). Note that \( \tilde{G}^B \) need no longer be symmetric. Equation (110) implies

\[
\tilde{G}^B_B = (\det \phi)^{i-1} \phi^{-1} \cdot \tilde{G}^B
\]

for each \( \phi \in \text{GL}^+(n-1) \). Thus, \( B \mapsto \tilde{G}^B \) is a measurable \( \text{GL}^+(n-1) \)-contravariant valuation. With precisely the same argument as at the end of the proof of Theorem 3.18 it follows that \( \tilde{G}^B \) vanishes for crosspolytopes. Our initial induction assumption then implies that \( \tilde{G}^B = 0 \), which in turn yields \( G_i^B = 0 \).

Before we continue, let us collect some notation. For \( a_1, b_1, \ldots, a_{n-1}, b_{n-1} > 0 \) and \( c, d > 0 \) define \( n \) line segments by

\[
I_1 = [a_1e_1, b_1e_1], \ldots, I_{n-1} = [-a_{n-1}e_{n-1}, b_{n-1}e_{n-1}] \quad \text{and} \quad J = [-ce_n, de_n].
\]

Furthermore, set

\[
\tilde{I}_{n-1} = [a_{n-1}e_n, b_{n-1}e_n] \quad \text{and} \quad \tilde{J} = [-ce_{n-1}, de_{n-1}].
\]

Finally, define \( B = [I_1, \ldots, I_{n-2}, I_{n-1}] \) and \( \tilde{B} = [I_1, \ldots, I_{n-2}, -\tilde{J}] \).

**3.20 Theorem.** For \( n \geq 3 \) the following holds.

- A basis of \( \text{TVal}_0(\mathbb{R}^n) \) is given by \( \chi, V \) and \( V \circ \ast \).
- A basis of \( \text{TVal}_1(\mathbb{R}^n) \) is given by \( M^p, 0 \circ \ast \).
- For \( p \geq 2 \), a basis of \( \text{TVal}_p(\mathbb{R}^n) \) is given by \( M^p, 0 \circ \ast \) and \( M^0, p \).

**Proof.** We already know three, one, and two linearly independent elements of \( \text{TVal}_0(\mathbb{R}^n) \), \( \text{TVal}_1(\mathbb{R}^n) \), and \( \text{TVal}_p(\mathbb{R}^n) \) for \( p \geq 2 \), respectively. By Lemma 3.19 and Theorem 2.1 it is enough to prove that if we restrict the maps in \( \text{TVal}_p(\mathbb{R}^n) \), \( p \geq 0 \), to crosspolytopes, the resulting spaces are at most three-, one-, and two-dimensional, respectively.

We will prove this by induction over the dimension. Before we do so, let us collect some prerequisites. For \( i \in \{0, \ldots, p\} \) and fixed positive numbers \( c \) and \( d \) consider the map

\[
B \mapsto \mu_i[B, J], \quad B \in \mathcal{P}_o^{n-1},
\]

where \( J := [-ce_n, de_n] \). By the \( \text{SL}(n) \)-contravariance of \( \mu \) we obtain

\[
\mu[B, rJ] = \begin{pmatrix} r^{-1} & \mathbf{I} & 0 \\ 0 & 1 \\ r \end{pmatrix} \cdot \mu[r^{-1}B, J]
\]
for all $r > 0$ and 
\[
\mu[B, -J] = \begin{pmatrix} \vartheta^t & 0 \\ 0 & -1 \end{pmatrix} \cdot \mu[\vartheta B, J]
\]
for every $\vartheta \in \text{SL}^\pm(n)$ with $\det \vartheta = -1$. By projecting onto the subspace $U_i$ we get 
\[
\mu_i[B, r J] = r^{\frac{p-i}{p}} \mu_i[r^{\frac{1}{p}} B, J]
\]  
and 
\[
\mu_i[B, -J] = (-1)^i \vartheta^t \cdot \mu_i[\vartheta B, J].
\]
Define intervals as in (111). By the SL($n$)-contravariance of $\mu$ again we have 
\[
\mu[I_1, \ldots, I_{n-1}, I_n, \lambda J] = \begin{pmatrix} \text{Id} & 0 \\ 0 & \lambda \end{pmatrix} \cdot \mu[I_1, \ldots, I_{n-1}, -\tilde{J}, \lambda \tilde{I}_{n-1}]
\]
for all $\lambda > 0$. With the above definitions of $B$ and $\tilde{B}$ we therefore get 
\[
(\mu_0)_i[B, \lambda J] = \lambda^i (\mu_0)_i[\tilde{B}, \tilde{I}_{n-1}].
\]
Here, the second indices denote projections in Sym$^p(\mathbb{R}^{n-1})$ and Sym$^{p-i}(\mathbb{R}^{n-1})$, respectively. In other words, on the left hand side we have the component with 0 times $e_n$ and $i$ times $e_{n-1}$, and on the right hand side the component with $i$ times $e_n$ and 0 times $e_{n-1}$.

Now, we can start with our induction. Let $n = 3$. From the SL(3)-contravariance of $\mu$, it follows that the map (112) is SL(2)-contravariant. By Theorem 3.15 and the convention that $\mu_i$ will be viewed as an element of $(\mathbb{R}^2)^{\otimes p - i}$, we therefore have 
\[
\mu_i[B, J] = t_i^j \mu^{p-i,0}(B^*) + \sum_{j=0}^{p-i} k_{i, j}^J \rho \cdot M_{p}^{j, p-i-j}(B)
\]
for $i \neq p$ and 
\[
\mu_p[B, J] = l_i^j V(B^*) + k_{p, 0}^J V(B) + m_p^J,
\]
for $i = p$, where $k_{i, j}^J, l_i^J, m_p^J \in \mathbb{R}$ and $k_{i, p-i-1}^J = 0$. Note that the operators on the right hand side are now operators in dimension 2.

Let $i \neq p$. If we plug (116) into (113) and use the appropriate degrees of homogeneity from (20) and (114), we obtain 
\[
l_i^J \mu^{p-i,0}(B^*) + \sum_{j=0}^{p-i} k_{i, j}^J \rho \cdot M_{p}^{j, p-i-j}(B) = \\
r^{-1-i} l_i^J \mu^{p-i,0}(B^*) + \sum_{j=0}^{p-i} r^{1-i+j} k_{i, j}^J \rho \cdot M_{p}^{j, p-i-j}(B).
\]
As in the proof of Lemma \([3, 19]\) we can compare coefficients in this equation. This shows that \(J \mapsto l_i^J\) and \(J \mapsto k_{i,j}^J\) are homogeneous. For \(i = p\) we can argue similarly. We conclude that \(J \mapsto l_i^J\), \(J \mapsto k_{i,j}^J\) and \(J \mapsto m_p^J\) have degrees of homogeneity \(-1 - i\), \(1 - i + j\), and \(-p\), respectively.

With the same procedure, we obtain by \((111)\) and \((23)\) that

\[
l_i^{-J} = (-1)^i l_i^J, \quad k_{i,j}^{-J} = (-1)^{p+j} k_{i,j}^J \quad \text{and} \quad m_p^{-J} = (-1)^i m_p^J. \tag{118}\]

In particular, the maps \(J \mapsto l_i^J\), \(J \mapsto k_{i,j}^J\) and \(J \mapsto m_p^J\) are even or odd. Comparing coefficients again, also proves that these maps \(l_i^J\), \(k_{i,j}^J\) and \(m_p^J\) are measurable valuations with respect to \(J\). From Theorems \([22]\) and \([23]\) we deduce that \(l_i^J\), \(k_{i,j}^J\) and \(m_p^J\) are determined by constants \(l_i \in \mathbb{R}\), \(k_{i,j} \in \mathbb{R}\) and \(m_p \in \mathbb{R}\), respectively.

For \(p = 0\), we are already done by \((117)\). So assume \(p \geq 1\). If we plug representation \((116)\) and \((117)\) into \((115)\) for \(i = p\) and use the homogeneity of \(k_{i,j}^J\), \(l_i^J\) and \(m_p^J\) with respect to \(J\), we obtain

\[
\lambda^{-1} l_0^J \left[ M_{\rho}^{p,0}(B^*) \right]_p + \sum_{j=0}^{p} \lambda^{1+j} k_{0,j}^J \left[ \rho \cdot M_{\rho}^{p-j}(B) \right]_p = \lambda^{-1} l_0^J V(\tilde{B}^*) + \lambda k_{0,0}^J V(\tilde{B}) + m_p^J.
\]

Therefore, \(m_p = 0\) and \(k_{0,j} = 0\), \(j \neq 0\). Furthermore, \(k_{p,0}\) is a multiple of \(k_{0,0}\) and \(l_p\) is a multiple of \(l_0\).

In the case \(p = 1\) we know \(k_{0,0} = 0\) from \((116)\) and we are done. Assume \(p \geq 2\). We already know that

\[
\mu_0[B, J] = l_i^J M_{\rho}^{p,0}(B^*) + k_{i,j}^J M_{\rho}^{p,0}(B).
\]

If we plug this and representation \((116)\) into \((115)\) for \(i \neq p\) and use the homogeneity of \(k_{i,j}^J\), \(l_i^J\) and \(m_p^J\) with respect to \(J\), we obtain

\[
\lambda^{-1} l_0^J \left[ M_{\rho}^{p,0}(B^*) \right]_i + \lambda k_{0,0}^J \left[ M_{\rho}^{p,0}(B) \right]_i =
\lambda^{-1} l_0^J \left[ M_{\rho}^{p-1,0}(B^*) \right]_0 + \sum_{j=0}^{p-i} \lambda^{1+j} k_{j,i}^J \left[ \rho \cdot M_{\rho}^{p-1-j}(\tilde{B}) \right]_0.
\]

Therefore, \(k_{i,j} = 0\), \(j \neq 0\). Furthermore, \(k_{i,0}\) is a multiple of \(k_{0,0}\) and \(l_i\) is a multiple of \(l_0\). For the preceding argument, note that \(\left[ M_{\rho}^{p-1-j}(\tilde{B}) \right]_0\), \(j \in \{0, \ldots, p-i\} \setminus \{p-i-1\}\), and \(\left[ M_{\rho}^{p-1,0}(B^*) \right]_0\) do not vanish for all choices of intervals by \((19)\) and the \(\text{SL}(2)\)-contravariance of these operators. This completes the proof for \(n = 3\).

Next, assume \(n > 3\) and that the theorem already holds in dimension \(n - 1\). From the \(\text{SL}(n)\)-contravariance of \(\mu\), it follows that the map \((112)\) is \(\text{SL}(n-1)\)-contravariant. By the induction assumption we have

\[
\mu_i[B, J] = l_i^J M_{\rho}^{p-i,0}(B^*) + k_{i,j}^J M_{\rho}^{p-i,0}(B) \tag{119}\]

for \(i \neq p\) and

\[
\mu_p[B, J] = l_p^J V(B^*) + k_j^J V(B) + m_p^J, \tag{120}\]

for \(i = p\).
for $i = p$, where $l_i^J, k_i^J, m_p^J \in \mathbb{R}$ and $k_{p-1}^J = 0$. Note that the operators on the right hand side are now operators in dimension $n - 1$.

Let $i \neq p$. If we plug (119) into (115) and use the appropriate degrees of homogeneity from (21) and (14), we obtain

$$l_i^J M^{p-i,0}(B^*) + k_i^J M^{0,i-0}(B) = r^{-1} l_i^J M^{p-i,0}(B^*) + r^{-1} k_i^J M^{0,i-0}(B).$$

As before, using (21) and (14), we can compare coefficients. This shows that $J \mapsto l_i^J$ and $J \mapsto k_i^J$ are homogeneous. For $i = p$ we can argue similarly. We conclude that $J \mapsto l_i^J$, $J \mapsto k_i^J$ and $J \mapsto m_p^J$ have degrees of homogeneity $-1 - i$, $1 - i$, and $-p$, respectively.

With the same procedure, we obtain by (114) and (24) that

$$l_i^{-J} = (-1)^i l_i^J, \quad k_i^{-J} = (-1)^i k_i^J \quad \text{and} \quad m_p^{-J} = (-1)^i m_p^J. \quad (121)$$

In particular, the maps $J \mapsto l_i^J$, $J \mapsto k_i^J$ and $J \mapsto m_p^J$ are even or odd. As before we can argue that these maps are measurable valuations. From Theorems 3.2 and 3.4 we deduce that $l_i^J$, $k_i^J$ and $m_p^J$ are determined by constants $l_i \in \mathbb{R}$, $k_i \in \mathbb{R}$ and $m_p \in \mathbb{R}$, respectively.

For $p = 0$, we are already done by (120). So assume $p \geq 1$. If we plug representation (119) and (120) into (115) for $i = p$ and use the homogeneity of $l_i^J$, $k_i^J$ and $m_p^J$ with respect to $J$, we obtain

$$\lambda^{-1} l_0^J \left[ M^{p,0}(B^*) \right]_0 + \lambda k_0^J \left[ M^{0,p}(B) \right]_0 = \lambda^{-1} l_0^J \left[ M^{p,0}(B^*) \right]_0 + \lambda k_0^J \left[ M^{0,p}(B) \right]_0.$$ 

Therefore, $m_p = 0$. Furthermore, $k_0$ is a multiple of $k_0$ and $l_0$ is a multiple of $l_0$.

In the case $p = 1$ we know $k_0 = 0$ from (119) and we are done. Assume $p \geq 2$. If we plug representation (119) into (115) for $i \neq p$ and use the homogeneity of $l_i^J$ and $k_i^J$ with respect to $J$, we obtain

$$\lambda^{-1} l_0^J \left[ M^{p,0}(B^*) \right]_0 + \lambda k_0^J \left[ M^{0,p}(B) \right]_0 = \lambda^{-1} l_0^J \left[ M^{p-i,0}(B^*) \right]_0 + \lambda k_0^J \left[ M^{0,p-i}(B) \right]_0.$$ 

As before, using (22), we conclude that $k_1$ is a multiple of $k_0$ and $l_i$ is a multiple of $l_0$. \qed

Finally, we prove our classification for TVal$^p(\mathbb{R}^n)$.

**Proof of Theorem 1.4.** The map $S$: TVal$^p(\mathbb{R}^n) \to TVal_p(\mathbb{R}^n)$ defined by

$$S \mu = \mu \circ *$$

is an isomorphism. The result now follows directly from Theorem 3.20. \qed

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