PAPER

Generation of long- and short-range potentials from atom-molecules to quark-gluon systems by the GPT potential

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Abstract

The general particle transfer (GPT) potential generates not only the Yukawa-type potential but also the $1/r^n$-type potential in the hadron system, where the mass dependence of the transferred (exchanged) particle is clarified. The GPT potential from the atom-molecule system to the quark-gluon system was studied, where pico-meter physics could be highlighted. It is demonstrated that the long-range three-body Efimov potential is connected with the short-range three-body force potential by the GPT potential. Some applications for historical few-body problems in physics are summarized.

1. Introduction

In our previous papers [1–7], it was shown that the general particle transfer (GPT) potential can be applicable to a wide range of physics; for example, it can not only involve the Efimov theory, but also covers a wide hierarchy from atom-molecule to quark-gluon systems [8–20]. The Efimov theory was experimentally confirmed in an ultracold gas of cesium atoms [17] where the two-body scattering length could be infinite for the $1/r^2$ potential [21]. Since the $1/r^2$ potential could not always be generated in the atom-atom interaction, it should be shown why the $1/r^2$ potential enters for the atomic-molecular system.

On the other hand, in hadron systems, the Efimov condition can not usually be satisfied. However, the GPT potential verifies the Efimov properties by a ‘quasi two-nucleon potential’ with the (πN)N systems which gives a Yukawa-type potential for short-range, but the $1/r^n$ potential for long-range where $n$ is related to the mass of the transferred particle [6, 22] as has been discussed in previous papers. However, some extension will be shown in section 2. Other generalizations of the GPT theory will be given in section 3. Applications for some physical systems will be discussed in section 4. Discussion will be given in section 5.

2. Derivation of the GPT potential for the Hadron system

It is well known that the Lippmann-Schwinger equation for the Coulomb potential can not be solved because of the singularity coincidence between that of the potential and of the Green’s function at the on-energy-shell point [23]. In the dispersion theoretical terminology, this fact is represented by the ‘pinching singularity’ at the threshold between the right-hand cut and the left-hand cut [24, 25]. We demonstrated in the previous paper that the kernel of the quasi two-body equation (Q2E) in the three-body problem has a ‘pinching singularity’ at the threshold of Q2E in the same way as in the Coulomb potential case. Therefore, we pointed out that the Q2E potential has a long-range property as similar to that in the case of the Coulomb potential [6, 7].

Let us recall that the ‘quasi two-body potential’ is essential to the ‘particle transfer mechanism’, which is given by the Born term of the Alt-Grassberger- Sandhas (AGS) equation [26–28],

$$Z_{\alpha,\beta}(q_{\alpha}, q_{\beta}; E) = \frac{4\pi \alpha_{\beta}}{E - q_{\alpha}/2\mu_{\beta} + \epsilon_{\beta}},$$

(1)

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where \( g_a \), \( g_b \) are the form factors. Here, \( p_a \) and \( q_a \) are the corresponding two-body momentum and spectator particle momentum. Operator \( \Omega(q_a, q_b; \sigma) \) represents the spin-isospin, angular momentum effects, in the three-body re-coupling coefficient between two-channels, and the channel connection for \( q_a, q_b \), where \( n = 2 \) denotes one particle transfer.

Hereafter we abbreviate the channel notations, i.e.,

\[
\frac{g(p)g(p')}{E - q^2/2 \mu + \epsilon_B}.
\]

Therefore, by omitting the operator, equation (2) becomes

\[
Z(q, q'; E) = \frac{g(p)g(p')}{E - q^2/2 \mu + \epsilon_B} = 2\mu\epsilon_B(q^2 - q'^2).
\]

(3)

\[
\sigma = \sqrt{2\mu|E_0|}; \quad \text{for } E_0 \equiv E + \epsilon_B.
\]

(4)

Since \( g(p), g(p') \) are almost constant near small \( q, q' \) and \( \sigma \), then the Fourier transformation of the operator form could also be represented by

\[
F[Z(q, q'; E)] = \frac{e^{-\sigma|r - r'|}}{|r - r'|} \Omega(r; 2) \frac{e^{-\sigma r}}{r} \rightarrow V_0 \frac{e^{-\sigma r}}{r},
\]

(5)

with \( r = |r - r'| \). Here, \( \sigma \) is an energy dependent parameter given by equation (4), where \( \sigma = 0 \) denotes the Coulomb-like potential. \( V_0 \) is a constant which is defined in a specific region for the eigenvalue of the operator.

In the previous papers, the energy dependence of the potential is smoothed by utilizing the 'Euler integral of the second kind' \([1–7, 29–31]\), let us utilize the operator signs \( F \) and \( \mathcal{E} \) for the Fourier transformation and the so-called Euler transformation,

\[
\mathcal{E}[F[Z(q, q'; E)]] = V_0 \frac{e^{-\sigma r}}{r} \left\{ \frac{e^{-\sigma r}}{r} \right\} = \frac{1}{P} \int_0^{\infty} \sigma n-2 e^{-\sigma} \left\{ \frac{e^{-\sigma r}}{r} \right\} d\sigma \rightarrow \frac{V_0}{r(r + a)^{n+1}} \frac{e^{-\sigma r}}{r}
\]

(7)

with

\[
P = \int_0^{\infty} \sigma n-2 e^{-\sigma} d\sigma = \frac{\Gamma(n - 1)}{a^{n-1}},
\]

(8)

Therefore, the one particle transfer operator \( \Omega(r, 2) \) in equation (5) is generalized to an \( n \) -particle transfer operator \( \Omega(r, n) \) by the weight function \( \sigma n-2 \) in the Euler transformation,

\[
\mathcal{E}[F[Z(q, q'; E)]] \equiv V(n, a; r) = \Omega(r, n) \frac{a^{n-1}}{r(r + a)^{n+1}} \rightarrow V_0(n; r) \frac{a^{n-1}}{r(r + a)^{n-1}},
\]

(9)

(10)

(11)

where \( V_0(n; r) \) is the eigenvalue of the operator. Equation (10) is the so-called GPT potential.

On the other hand, equation (11) becomes

\[
V(n, a; r) \rightarrow V_0(n) \frac{e^{-(n-1)\sigma r/a}}{r},
\]

(12)
$$n = \mu / m_\pi + 1$$

Figure 1. The $n$-$\mu$ relation of equation (15) is illustrated, where $\mu$ is the transferred particle mass in unit of the pion mass $m_\pi$, and $n$ is the index number of the long-range potential. The damping range $a = 1/m_\pi$ in the Euler transformation gives the $1/r^2$ potential for one pion transfer.

Table 1. The GPT potential $\Omega(r; n) a^{n-1}/[r(r + a)^{n-1}]$ is illustrated. The potential properties for the longer and shorter ranges are shown with respect to the parameter $n$. The potential depths $V_0$ and $V_0'$ are given for the short-range and the long-range expansion for the eigenvalue of the operator $\Omega(r; n)$.

| $n$ | $r \ll a$ | GPT- potential | $a \ll r$ |
|-----|------------|----------------|----------|
| 1   | $V_0(1)/r$ | $\Omega(r; 1)/r$ | $V_0'(1)/r$ |
| 2   | $V_0(2) e^{-\mu(r+a)/r}$ | $\Omega(r; 2) a/r(r + a)$ | $V_0'(2) a/r^3$ |
| 3   | $V_0(3) e^{-\mu(r+a)/r}$ | $\Omega(r; 3) a^2/r(r + a)^2$ | $V_0'(3) a^2/r^4$ |
| 4   | $V_0(4) e^{-\mu(r+a)/r}$ | $\Omega(r; 4) a^3/r(r + a)^3$ | $V_0'(4) a^3/r^5$ |
| 5   | $V_0(5) e^{-\mu(r+a)/r}$ | $\Omega(r; 5) a^4/r(r + a)^4$ | $V_0'(5) a^4/r^6$ |
| 6   | $V_0(6) e^{-\mu(r+a)/r}$ | $\Omega(r; 6) a^5/r(r + a)^5$ | $V_0'(6) a^5/r^7$ |
| ... | ...        | ...            | ...      |
| $n$ | $V_0(n) e^{-(n-1)\mu/r}$ | $\Omega(r; n) a^{n-1}/[r(r + a)^{n-1}]$ | $V_0'(n) a^{n-1}/r^n$ |

\[ \equiv V_0(n) \frac{e^{-\mu r}}{r} \quad \text{for} \quad r \ll a \quad (13) \]

\[ \rightarrow V_0'(n) \frac{a^{n-1}}{r^n} \quad \text{for} \quad a \ll r, \quad (14) \]

where $V_0(n)$ and $V_0'(n)$ represent the short-range and the long-range limits of $V_0(n; r)$, respectively. The $n = 2$ case is given by a simple Laplace transformation with equation (7).

Comparing equations (12) and (13), we have,

\[ n = a\mu + 1 \equiv \mu/m_\pi + 1, \quad (15) \]

where the damping range $a$ in equation (7) should be defined by the ‘inverse of the pion mass’. The result introduces a relation between the exchange particle mass and the index number $n$: i.e., $n = 2$ for $\mu = m_\pi$; $n = 3$ for $\mu = 2m_\pi$; $n = 4$ for $\mu = 3m_\pi$, and so on. The transferred mass relation is shown in figure 1.

Consequently, equations (12) and (14) are summarized in table 1, where the GPT potential for $a \ll r$ becomes the $(n-1)$-pion transfer Yukawa potentials, while for $a \ll r$, $n$-order long range potentials are indicated. These results are also illustrated in figure 2 where $0 \leq n$ represents the branch points on the first Riemann sheet or the physical plane.

Finally, a simple Laplace transformation with $n = 2$ gives one-pion exchange, or a branch point of one-pion production. Therefore, the weight function represents $(n-1)$-$\pi$-meson production. In this context, the operator $\Omega(r; n)$ depends on the index number and the transferred particle mass.
3. Consideration of the GPT potential

3.1. A Nucleon-Nucleon potential

In order to confirm the Efimov effect in the Hadron system, some typical properties are calculated. In the beginning, the deuteron binding energy by the GPT potential of equation (11) is calculated for the case: $n = 2$ of the one pion transfer GPT potential, i.e.,

$$V(2; r) = V_0(2; r) \frac{a}{r(r + a)}$$

where a damping constant $a = 1/m_\pi$ is adopted. For the simplicity, we adjusted the potential depth $V_0$ to reproduce the experimental deuteron binding energy: $E_0 = 2.226$ MeV. In order to obtain an accurate energy level, we used a high precision analysis with more than 100 figures where an energy sequence and the wave functions were obtained. By using these results, the energy ratio $E_n/E_{n+1}$ and the rms radii $\sqrt{\langle r^2 \rangle_1}/\sqrt{\langle r^2 \rangle_n}$ are calculated for the rms radius $\langle r^2 \rangle_n \equiv \langle n|r^2|n \rangle$. The ratios of the energy eigenvalues and the rms radii are monotonous for all the number $n$ in the potential $V_0/r^2$ of equation (17), that is, $E_n/E_{n+1} = 171.0$ and $\sqrt{\langle r^2 \rangle_1}/\sqrt{\langle r^2 \rangle_2} = 13.1$, however $3 \leq n$ is necessary to obtain the constant values for the original GPT potential equation (16) because of the short-range effect. These numerical results are well fitted with the analytical predictions $E_n/E_{n+1} = 170.98$ and $\sqrt{\langle r^2 \rangle_1}/\sqrt{\langle r^2 \rangle_2} = 13.076$ which were discussed in the article [1, 32]. Furthermore, the first excited state of the np triplet state: $E_2 = 13$ keV is obtained, however such an experimental result has not been measured yet [32].

3.2. One pion exchange potential (OPEP)

The one pion exchange nucleon-nucleon potential (OPEP) has the range $r \sim 1/m_\pi$ where the Hamiltonian of the $N-\pi$ interaction: $H'_{NN\pi}$ is given by the pseudo-scalar meson with the meson creation and annihilation operators [33-38]. Therefore, the second order perturbation formula for the N-N interaction is given by using momentum conservation at vertices for the nucleon and meson momenta $q_1$, $q_2$, and $k$,

$$W_2 = \sum_k \left\{ \frac{\langle 0|H_{NN\pi}|m\rangle \langle m|H_{NN\pi}|0 \rangle}{E_0 - [E'(q_1, q_2) + \omega_k]} + (1 \leftrightarrow 2) \right\},$$

where $\omega_k = \sqrt{k^2 + m_\pi^2}$ is the meson energy, and $E_0$ denotes the initial two-nucleon energy.

If we neglect the nucleon-recoil effect by the meson creation and annihilation, or adopt a static approximation [35] $0 = E_0 \approx E'(q_1, q_2)$ assuming $\Delta = m_\pi/M_N = 0.14703$ is small, Therefore, the static approximation of equation (18) becomes

$$W_2 \approx \sum_k \left\{ \frac{\langle 0|H_{NN\pi}|m\rangle \langle m|H_{NN\pi}|0 \rangle}{-\omega_k} + (1 \leftrightarrow 2) \right\},$$
which gives the Yukawa potential or the OPEP,

\[ V^\Psi(r) = \Omega^\Psi(r; 2) \frac{e^{-m_\pi r}}{r}, \]

\[ \Omega^\Psi(r; 2) = \frac{f_2^2}{3} \left\{ \sigma_1 \cdot \sigma_2 + \left(1 + \frac{3}{m_\pi r} + \frac{3}{m_\pi^2 r^2}\right) S_{12}\right\}, \]

\[ S_{12} = \frac{3}{r^2} (\sigma_1 \cdot r)(\sigma_2 \cdot r) - \sigma_1 \cdot \sigma_2, \]

However the exact ‘quasi two-body potential’ for \((\pi\mathrm{N})-\mathrm{N}\) should be obtained by equation (1) or equation (18) for \(E_0 \simeq E'(q_1, q_2)\) [35].

### 3.3. The three-body force potentials

In the quasi two-body one pion exchange \(N_1-(N_2 \pi_3)\) GPT potential, the two-body form factors for the \((N_1 \pi_3)\) \((N_2 \pi_3)\), and \((N_1 N_2)\) systems are represented by the GPT potentials for the systems \(\pi_3-(N_1 \pi_4)\), \(\pi_3-(N_2 \pi_4)\) and \(N_1-(N_2 \pi_4)\). The process is repeated by the same way. In this context, a nucleon exchange \(\pi(N_1 N_2)\) GPT potential is given by the index number with \(2 \ll n\) which is negligible due to the long range property in equation (15) [6].

Therefore, the unique long range GPT potential:

\[ V(2, a; r_\delta) = V_\delta(2; r_\delta) \equiv \Omega_\delta(r_\delta; 2)a_{ij} / r_\delta^2 \rightarrow \frac{V_\delta(2)}{r_\delta^2} \]

could survive for any channels in the long range region.

In the hadron system, since the GPT potential is generated by another GPT potential with a pion exchange, and again and again like a 'fractal structure', therefore such a potential could be given by a 'nonlinear three-body potential'.

1. Let us propose a nonlinear three-body potentials, by using equation (22),

\[ V_{\text{GPT}}(r_1, r_2, r_3) \rightarrow V_{\text{long}}(r_1, r_2, r_3) = \left[ \sum_{i=j=1}^{3} \{ V_{\Omega}(2) / r_\delta^2 \}^{-1} \right]^{-1} \]

\[ = \frac{1}{r_{12}^2 / V_{120} + r_{23}^2 / V_{230} + r_{31}^2 / V_{310}} \approx \frac{1}{\alpha_{12} r_{12}^2 + \alpha_{23} r_{23}^2 + \alpha_{31} r_{31}^2} \]

\[ \rightarrow \left. \frac{V_0}{r_{12}^2 + r_{23}^2 + r_{31}^2} \equiv \frac{1}{\rho^2}, \right. \]

where equation (25) is called the three-body Efimov potential. Therefore, the nonlinear potential equation (24) satisfies the three-body long range 'Efimov potential' of equation (25) for the case \(\alpha_{12} = \alpha_{23} = \alpha_{31} = 1 / V_0\) where \(\alpha_{ij} = 1 / V_{\Omega}(2) \rightarrow 1 / V_0\) is required [7].

By this method, we could define a 'generalized nonlinear three-body GPT (3GPT)' potential by an 'entangled channel' form with equation (10),

\[ V_{\text{GPT}}(r_1, r_2, r_3) \equiv \left[ \sum_{i=j=1}^{3} \{ V_{\Omega}(n; r_\delta) \}^{-1} \right]^{-1} \]

\[ \equiv \left[ \sum_{i=j=1}^{3} \left( \frac{\Omega_\delta(r_{ij}; n)a_{ij}^{n-1}}{r_{ij} + a_{ij}^{n-1}} \right) \right]^{-1}, \]

which could be generalized to \(N\)-body form with

\[ \rightarrow \left[ \sum_{i=j=1}^{N} \left( \frac{\Omega_\delta(r_{ij}; n)a_{ij}^{n-1}}{r_{ij} + a_{ij}^{n-1}} \right) \right]^{-1}. \]

(2) While the short-range part equation (13) of equation (27) gives a nonlocal three-body potential for the three-nucleon case,

\[ V_{\text{GPT}}(r_1, r_2, r_3) \rightarrow V_{\text{short}}(r_1, r_2, r_3) = \left[ \sum_{i=j=1}^{3} \{ V_{\Omega}'(2)e^{-r_{ij} / r_\delta} \}^{-1} \right]^{-1}, \]

where it is usually known as an attractive potential: \(V_{\Omega}' < 0\). This is the so-called three-body force (3BF) potential in the three-nucleon system. Therefore, the triton binding energy is compensated by an attractive three-body (nucleon) potential; however a nuclear ground state energy by a three-cluster calculation is usually
corrected by a repulsive three-cluster force. The reason why the sign difference occurs will be understood in the next paragraph.

(3) For the three-cluster system, equation (29) could be replaced by the Gaussian-type form,

\[ V_{\text{GPT}}(r_1, r_2, r_3) \rightarrow V_{\text{short}}(r_1, r_2, r_3) = \left[ \sum_{i,j=1}^{3} \left( W_{ij0} e^{-\frac{1}{r_{ij}}} \right) \right]^{-1}, \]

where \( W_{ij0} \) expansion was adopted for a very short-range region with \( 0 < W_{ij0} \) which is closely related to the anti-symmetrization between clusters by the Pauli principle [39, 40].

Finally, it should be stressed that the ‘short range three-body force (3BF) potential’ could be smoothly continued to the ‘long range three-body force potential’ equation (25) through the mediation of the GPT potential with equation (27).

Therefore, the total three-body Hamiltonian of the hadron system is given by

\[ H = K + V_{\text{linear}} + V_{\text{short}} + V_{\text{long}}, \]

\[ V_{\text{linear}} = V_{12} + V_{23} + V_{31}, \]

\[ V_0 = V_{\text{had}} + V_{\text{C}}, \]

where \( K \) is the three-body kinetic energy and the linear potential \( V_{\text{linear}} \) consists of the two-body hadron and the Coulomb potential for the three-body system which is calculated by the well-known Faddeev equation [27]. \( V_{\text{short}} \) and \( V_{\text{long}} \) are nonlinear short-range and the long-range three-body hadron potentials defined by equation (27).

In the molecular system, for example, the CsH\(_2\)Pd\(_{12}\) cub-octahedron cluster takes on a three-body CsH\(_2\) quasi-molecule in the Pd\(_{12}\) cage [36, 37]. Therefore, the entire Hamiltonian could be represented by using the hadron potentials and the Coulomb potentials with the electron effects,

\[ H = K + V_{\text{linear}} + V_{\text{short}} + V_{\text{long}} + V_{\text{Coul}}, \]

\[ V_{\text{Coul}} = V_{\text{ions}} + V_{\text{electrons}}, \]

where \( V_{\text{Coul}} \) indicates the Coulomb effects of the other ions and electrons. In such a molecular system, the calculation range should be considered from the 10\(^{-2}\)fm region to several \( 10^7 \)fm region in one stretch with 100 figures accuracy. The system has been considered as a nano scale nuclear fusion reactor Pd\(_{12}\) [36, 37].

### 4. Application of the GPT potential

#### 4.1. Some hierarchies from atom-molecule to quark-gluon systems

The GPT potential is introduced for a general particle exchange system such as the photon, the electron, the meson, the nucleon, the hyperon, the nucleus, even for the quark and gluon etc.. Therefore, it could be applicable for many different hierarchical systems.

(1) The Atom-Molecule Systems:

The electric-bond for the molecular system has been understood as given by the photon exchange or the electron exchange. The GPT potential with \( n=1 \) indicates only the 1/r potential for all the region which corresponds to a massless particle exchange making an ‘ionic bond’ by a photon exchange or the Coulomb potential (see table 1). While, for the \( n=2 \) GPT potential, the short-range potential has a Yukawa-type, a 1/r\(^2\)-type for the long-range where a nonzero mass particle like an electron could get involved in making a molecule, therefore (n-1)-electron exchanges generate a molecule of the ‘covalent bonds’ in chemistry [41, 42].

On the other hand, the Efimov theory requires that the two-body system should have an infinite scattering length, which is given by the 1/r\(^2\)-type potential. The Caesium (Cesium) atomic system was investigated to confirm the Efimov theory [17]; however it was not very clear where the 1/r\(^2\) potential comes from. The GPT theory has a solution that one electron exchange leads the 1/r\(^2\)-type potential for the criterion equation (15): \( n = \mu/m_e + 1 \) with the infinite scattering length. In this context, let us call the electron the ‘Efimov particle’ in the molecular system.

Analogously, the pion should be the ‘Efimov particle’ in the hadron systems. As a matter of fact, since the other atomic system does not necessarily generate the 1/r\(^2\) potential, then the Efimov effect could not occur. For example, 1/r\(^4\) and 1/r\(^2\) potentials are the Van der Waals potentials [43, 44], where the five- and six-electron exchanges take part in the potentials which occur in one of the ‘covalent bonds’ in chemistry [41, 42]. Although, the Van der Waals potential is one of the long range potentials, the potential gives neither an infinite scattering length nor the Efimov effect.
(2) The nuclear systems:
We mentioned above that the GPT potential mediates the short-range and the long-range hadron three-body force potentials. This fact is confirmed in a neutron rich nucleus such as $^{6}\text{He}$, where the ground state is adjusted by the long-range three-body force potential in equation (27) and in table 1 for $n = 3$,

$$V_{\text{GPT}}(r_1, r_2, r_3) = \sum_{i<j=1}^{3} \left( \frac{\Omega_0(r_{ij}^{-n} a_{ij}^{n-1})}{r_{ij}^{n} (r_{ij} + a_{ij})^{n-1}} \right)^{-1} - 1,$$

$$= \sum_{i<j=1}^{3} \left( \frac{\Omega_0(r_{ij}^{-3} a_{ij}^{2})}{r_{ij} (r_{ij} + a_{ij})^{2}} \right)^{-1} - 1,$$

$$= \sum_{i<j=1}^{3} \left( \frac{V_{\text{GPT}}(3)}{r_{ij}^{3}} \right)^{-1} - 1,$$

$$= \frac{1}{\beta_{22} r_{12}^{3} + \beta_{23} r_{23}^{3} + \beta_{31} r_{31}^{3}}. \quad (36)$$

Therefore, if we adopt $\beta_{22} = \beta_{23} = \beta_{31} = 1/V_0$, and $\beta_0 = 1/V_{00} \equiv 1/V_0(<0)$, then we obtain the long-range three-body force potential,

$$V_{\text{long}}(r_{12}, r_{23}, r_{34}) = \frac{V_0}{r_{12}^{3} + r_{23}^{3} + r_{34}^{3} + 1} = \frac{V_0}{\rho^3 + 1}. \quad (37)$$

where the factor ‘1’ in the denominator is adopted to truncate the short-range part of the potential which is seen in equation (34) of the article by Thompson et al [45, 46].

Furthermore, equation (37) suggests that the two-pion exchange entangled long-range three-body force is essential to reproduce the ground state binding energy of $^{6}\text{He}$. It should be emphasized that the ground state energy has been adjusted by the ‘repulsive’ three-body short-range force for the usual nuclei; however, the ground state energy for the neutron rich nuclei with a nuclear halo could be reproduced by ‘attractive’ three-body long range forces.

(3) Quark-gluon systems:
If we put $n \to -|n|$ in equation (10), we obtain,

$$V(n, a; r) = \Omega(r; n) \frac{a^{-1}}{r(r + a)^{n-1}}$$

$$= \Omega(r; |n|) \frac{(r + a)^{n+1}}{a^{n+1} r}$$

$$\rightarrow \frac{V_0(|n|)}{r} \quad \text{for} \quad r \ll a, \quad (38)$$

$$\rightarrow \frac{V_0'(|n|)}{a^{n+1}} \quad \text{for} \quad a \ll r. \quad (39)$$

The potential becomes 1/r-type with any $|n|$ for $r \ll a$, but $r^{n+1}$-type for $a \ll r$. It seems that the $|n|$ case represents ‘zero’ mass particle transfer for $r \ll a$, while for $a \ll r$, $r^{n+1}$-order potential could exist. Therefore, it seems that $n < 0$ could correspond to the quark-quark interaction, because of the confinement property [47] (see table 2), where the negative $n$ represents the branch points of the second Riemann sheet in figure 2. If it indicates the quark-quark system with gluon exchanges, then $\Omega(r; |n|)$ is given by the spin-isospin, color and flavor operators, and gives different potential depths $V_{0}(|n|)$ and $V_{0}'(|n|)$ for the short- and the long-range regions. If $0 < V_{0}'(|n|)$ is given, then equation (40) is the confinement potential. The quark-quark scattering observables on the second Riemann sheet could not be observed [47].

4.2. Analyticity of AGS kernel and a new calculation method by the GPT concept
The kernel of the AGS equation is given by the Born term: equation (1) and the propagator $\tau(E; q)$,

$$\tau(E; q) \sim \frac{1}{E - q^2/2\mu + \epsilon_B}. \quad (41)$$

The AGS Born term corresponds to a potential of the quasi two-body equation in the three-body system where the branch point of the left-hand (potential) cut is the Q2T: $q = \sqrt{2\mu(E + \epsilon_B)}$. On the other hand, the scattering cut by equation (41) is also defined in the region from $q = \sqrt{2\mu(E + \epsilon_B)}$ to $+\infty$. Therefore, the analytic structure is illustrated by the complex $q$-plane where the pinching singularity between the left and right-hand cuts occurs at the Q2T. However, in such a case, the equation could not be solved because of the pinching singularity. Therefore, if we adopt an approximation (Aprr-1) that the left-hand cut is given on the real axis from
is the origin with nucleon exchange potentials. While if the right-hand cut could be started from the Q2T till atoms, mesons, nucleons, nuclei, hyperons, and perhaps quark-gluons where such a particle is denoted by the point $A$ and $Q2T$ (Appr. I) where the large round curve is neglected by taking an infinite radius $R \to \infty$.

Figure 3. The complex energy plane with the real and the imaginary axes. The solid thick line along the left real axis running from a point $A$ ($r=2$) to $-\infty$ is the potential cut of the quasi two-body equation. Several small round points on the cut indicate the one-pion ($n=2$), two-pion ($n=3$), and also including nucleon transfer points, so on. The solid thick line along the real axis is the quasi two-body scattering cut from the Q2T ($r=1$) to $+\infty$, and including the three-body scattering cut from 'three-body break up threshold (3BT: $n=0$) to $+\infty$, respectively. The integral path is illustrated by red curve on the ‘physical’ (1st Riemann) sheet: $0 \leq q < \infty$ along the right hand cut which turns at A and Q2T (Appr. I) where the large round curve is neglected by taking an infinite radius $R \to \infty$.

Table 2. The GPT potential for the unphysical Riemann energy plane. By putting $n \to -|n|$ in the original GPT potential: $\Omega(r; n) a^{-n+1}/[r(a+n+1)] \to \Omega(r; |n|) (r + a)^{-n+1}/[a(a+n+1)]$ is illustrated, which corresponds to the dotted path for the second Riemann sheet in figure 2. The potential properties for the long- and short-ranges are shown with respect to the parameter $a$ and $n$. $V_0(|n|)$ and $V_0(|n|)$ are constants.

| $|n|$ | $r \ll a$ | GPT potential | $a \ll r$ |
|------|----------|---------------|----------|
| 0    | $V_0(0)/r$ | $\Omega(r; 0)(r + a)/ar$ | $V_0(0)/a$ |
| 1    | $V_0(1)/r$ | $\Omega(r; 1)(r + a)^2/2a^2$ | $V_0(1)r/2a^2$ |
| 2    | $V_0(2)/r$ | $\Omega(r; 2)(r + a)^3/3a^3$ | $V_0(2)r^2/3a^3$ |
| 3    | $V_0(3)/r$ | $\Omega(r; 3)(r + a)^4/4a^4$ | $V_0(3)r^3/4a^4$ |
| 4    | $V_0(4)/r$ | $\Omega(r; 4)(r + a)^5/5a^5$ | $V_0(4)r^4/5a^5$ |
| 5    | $V_0(5)/r$ | $\Omega(r; 5)(r + a)^6/6a^6$ | $V_0(5)r^5/6a^6$ |
| 6    | $V_0(6)/r$ | $\Omega(r; 6)(r + a)^7/7a^7$ | $V_0(6)r^6/7a^7$ |
| 7    | $V_0(7)/r$ | $\Omega(r; 7)(r + a)^8/8a^8$ | $V_0(7)r^7/8a^8$ |
| ...  | ...      | ...           | ...      |
| $|n|$ | $V_0(|n|)/r$ | $\Omega(r; |n|)[(r+a)^{n+1}/a^{n+1}]$ | $V_0(|n|)r^n/a^{n+1}$ |

the point ‘$A$’ to $-\infty$ in figure 3, then the left-hand cut includes the full effects of $n$-pion transfers and also several nucleon exchange potentials. While if the right-hand cut could be started from the Q2T till $+\infty$, then the Q2T is the origin with $E_{min} = E + \epsilon_0 = 0 = q$, and the integral variable becomes $0 \leq q < \infty$ (see figure 3).

On the other hand, if the origin of the scattering cut is given by taking the origin at the 3BT: $E = 0 = q$ (see figure 4), and the potential cut is the same as the Appr. I, then the method should be another approximation (Appr.-II) which corresponds to the usual three-body Faddeev calculation [27, 28]. In the Appr.-II, the Q2T is a moving pole with the three-body free energy $E$, where the singularity is given by a $1/r$-type GPT potential with a massless particle transfer. In the Faddeev calculation, such a singular pole and the logarithmic potential cut have been avoided by using a contour deformation.

The numerical difference between Appr.-I, and Appr.-II has been carefully investigated by Hiratsuka et al for the triplet $n$-p scattering length, $\pi$-D scattering length, the $nd$ total cross section, the differential cross sections, the $nd$ doublet scattering length and the triton binding energy. They claimed that those results by the Appr.-I method are a better fit to the experimental results than the case of Appr.-II [32, 48–50].

5. Discussion

It has been understood that a potential is generated by a particle transfer which is given by photons, electrons, atoms, mesons, nucleons, nuclei, hyperons, and perhaps quark-gluons where such a particle is denoted by the
word ‘general particle’ in this method. Therefore, an action of particle transfer is essential; however, it was not very clear why the long-range potential enters in the case of a non zero mass particle before the GPT theory. Efimov started from a long-range potential with an infinite scattering length; therefore, the so-called Efimov physics papers do not discuss about these problems [8, 9, 51].

It is known that the $1/r^2$-type potential gives an infinite scattering length [9, 21]; the scattering length is usually finite in nuclear physics which is defined by the scattering amplitude at zero energy. However the calculated scattering amplitude diverges at zero energy because of the threshold singularity. The scattering length (or amplitude) at zero energy has been obtained by an extrapolation from the values at several very small energies, or a gradient of the wave function for a short range potential [38]. Finally, it should be recalled that the phase shift analysis is based on the short range nuclear potential theory, which could depend on a proper truncation.

Since the pion is the lightest exchange particle in the hadron system, all other exchange particles are heavier when the index number becomes $2 < n$. Even for the cluster-cluster interaction, it would be written by one pion exchange in the long-range limit. The solution (or integration) ‘started from the 3BT’ such as Appr.II could be only available for the unbound ‘Efimov- or the Borromean-system’ [52–55].

The GPT potential represents a supplemental property in quantum dynamics which could be compared with the case of the ‘Lorentz transformation’ in special relativity where it satisfies $\beta = v/c = 0$. The GPT potential does not change any traditional nuclear physics except for the behavior at the threshold. The GPT potential tells a beginning of the ‘pico-meter science’, which could exist between atomic-molecular science and nuclear science, such as ‘low energy nuclear fusion’ problems [36, 37, 56].

Finally, it should be emphasized that the GPT potential does not suggest the ‘fifth force’ but should be considered in a category of the ‘unified theory’. However, as mentioned above, it is interesting that some unknown particles such as $n=1$, and $n=0$ were predicted which should be clarified. The paper proposes a ‘fundamental correction’ to the potential form which has been overlooked for very many years.

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Data availability statement

No new data was created or analyzed in this study.
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