Sufficient optimality conditions for a separable product quasiconcave programming *

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Abstract

In mathematical economics, the used functions are, in general, considered to be quasiconcave. Moreover, they are, in many cases, separable of nature. It is known that a local maximum of a quasiconcave function is not, in general, a global maximum. In this paper we will show that this property is true when the quasiconcave function is furthermore separable. Sufficient optimality conditions for a separable quasiconcave programming will be studied in both differentiable and non differentiable cases. Thanks to separability condition, quasiconcave functions have nice properties in optimization problems.

Keywords: Concave, Quasiconcave, Pseudoconcave Function, Separable Product Function, Generalized Superdifferential, modified K.K.T conditions.

1 Introduction

Concavity is a central tool in mathematical economics and optimization theory, but in practice the widely used functions in these areas are considered to be quasiconcave instead of concave. In Arrow-Enthoven (1961) [1], the concave optimization problem was extended to the quasiconcave programming, and sufficient optimality conditions in differentiable case were obtained. Later, several authors have studied the quasiconvex optimality conditions by means of various generalized gradients (see for instance Hassouni [14], Hiriart-Urruty [16], Martinez-Lagaz [19]). In consumer theory, the functions studied in many cases are considered to be separable of nature. Sufficient and necessary condition on separable utility functions to be quasiconcave was given in Yaari [21], Debreu and Koopmans [12], Crouzeix and Lindberg [11], Berdi and Hassouni [5].

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In this paper, the unconstrained optimization problem

$$\text{(USQP)} : \max_{x \in X} f(x)$$

and the constrained optimization problem

$$\text{(CSQP)} : \max_{x \in X} f(x) \text{ subject to } h_j(x) \geq 0 \ (j = 1, \ldots, p)$$

are investigated, where $f$ and $h_j \ (j = 1, \ldots, p)$ are quasiconcave on $X$, and

$$f(x) = \prod_{i=1}^{m} f_i(x_i), \quad x = (x_1, \ldots, x_m) \in X = X_1 \times X_2 \times \ldots \times X_m$$

with $f_i$ is a positive non constant real valued function defined on the nonempty open convex set $X_i$ of $\mathbb{R}^{n_i}$.

The scope of the paper is to obtain sufficient optimality conditions for the unconstrained separable quasiconcave problem (USQP), and first order sufficient optimality conditions of K.K.T type for the constrained separable quasiconcave problem (CSQP) with quasiconcave constraint functions, in both differentiable and non differentiable cases.

The paper is organized as follows: In section 2, we recall some definitions and properties that will be needed, such as some characterizations of quasiconcavity and pseudoconcavity in the differentiable case.

In section 3, we recall the definition and some properties of a multiplicatively separable (MS) function. The basic tool is the multiplicative concavity index introduced by Crouzeix and Kebbour in [10]. Under the separability condition (MS), quasiconcave differentiable functions become more regular (in convex sense), and then nice results of the problems (USQP) and (CSQP) are obtained.

In section 4, we define a generalized superdifferential to extend the notion of pseudoconcavity and some of its fundamental properties without differentiability assumption. Some useful and important generalizations of pseudoconcavity in non differentiable case were introduced and studied under various assumptions by Aussel [2], Hassouni and Jaddar [15], and Komlósi [17] for related works.

Again, under the separability condition (MS), the problems (USQP) and (CSQP) will be studied in the non differentiable case. An appropriate variant of K.K.T conditions of (CSQP) will be provided.

2 Preliminaries and notations.

We recall some definitions and properties that will be needful in the sequel of this paper.

**Definition 2.1.** Let $C$ be a nonempty convex set in $\mathbb{R}^n$ and let $g$ be a real valued function defined on $C$. 
The hypograph of \( g \) is the set
\[
\text{hyp}(g) := \{(x, \mu) \in C \times \mathbb{R} : g(x) \geq \mu\}
\]

For \( \alpha \in \mathbb{R} \), the upper-level set \( U(g, \alpha) \) and the strict upper-level set \( U^*(g, \alpha) \) of \( g \) are defined as follows:
\[
U(g, \alpha) := \{x \in C : g(x) \geq \alpha\}.
\]
\[
U^*(g, \alpha) := \{x \in C : g(x) > \alpha\}.
\]

- \( g \) is said to be **concave** on \( C \) if \( \text{hyp}(g) \) is convex, or equivalently if
\[
g((1 - \lambda)x + \lambda y) \geq (1 - \lambda)g(x) + \lambda g(y)
\]
for all \( x, y \in C \) and for all \( \lambda \in [0; 1] \).

- \( g \) is said to be **quasiconcave** on \( C \) if the upper-level set \( U(g, \alpha) \) is convex for all \( \alpha \in \mathbb{R} \), or equivalently if
\[
g((1 - \lambda)x + \lambda y) \geq \min \{g(x); g(y)\}
\]
for all \( x, y \in C \) and for all \( \lambda \in [0; 1] \).

- \( g \) is said to be **semi-strictly quasiconcave** on \( C \) if
\[
\forall x_1, x_2 \in C, x_1 \neq x_2 : \ g(x_2) > g(x_1) \Rightarrow g((1 - \lambda)x_1 + \lambda x_2) > g(x_1)
\]
for all \( \lambda \) in \((0, 1)\).

For a full description of concavity and quasiconvexity we refer to [6, 8, 13, 20].

- \( g \) is said to be convex (quasiconvex, semi-strictly quasiconvex) if \( -g \) is concave (quasiconcave, semi-strictly quasiconcave).

- If \( g \) is positive, it is said to be **logarithmically concave** (log-concave for short) if \( \ln \circ g \) is concave.

The following properties are rather direct consequences of the definitions.

- \( g \) concave \( \implies \) \( g \) log-concave \( \implies \) \( g \) quasiconcave.
  
  Let \( \varphi : g(C) \rightarrow \mathbb{R} \);

- If \( g \) is quasiconcave on \( C \) and \( \varphi \) is nondecreasing then \( \varphi \circ g \) is quasiconcave.

- If \( g \) is convex (concave) and \( \varphi \) is nondecreasing and convex (concave) then \( \varphi \circ g \) is convex (concave).

- If \( g \) is convex (concave) and \( \varphi \) is nonincreasing and concave (convex) then \( \varphi \circ g \) is concave (convex).

  In particular, if \( g \) is positive and concave on \( C \), then \( \frac{1}{g} \) is convex on \( C \).
Given $x \in C$ and $d \in \mathbb{R}^n$, let us define

$$I_{x,d} := \{ t \in \mathbb{R} : x + td \in C \}$$

$$g_{x,d}(t) = g(x + td), \quad t \in I_{x,d}$$

Then, $g$ is concave (quasiconcave, log-concave) if and only if for every $x \in C$ and $d \in \mathbb{R}^n$, the function $g_{x,d}$ is concave (quasiconcave, log-concave) on the interval $I_{x,d}$.

Let’s recall the well known characterization of quasiconcavity under differentiability assumption.

**Proposition 2.1.** (Arrow-Enthoven [1]) Let $g$ be a differentiable function defined on a nonempty open convex set $X$ of $\mathbb{R}^n$. Then, $g$ is quasiconcave on $X$ if and only if for all $x_1, x_2 \in X$:

$$\langle \nabla g(x_1), x_2 - x_1 \rangle < 0 \Rightarrow g(x_2) < g(x_1)$$

Under differentiability assumption, the following proposition gives a necessary condition of quasiconcavity by means of the strict upper level-set.

**Proposition 2.2.** Let $g$ be a differentiable quasiconcave function defined on a convex set $X$ of $\mathbb{R}^n$, and $x, \bar{x} \in X$. Then,

$$x \in clU^*(g, g(\bar{x})) \Rightarrow \langle \nabla g(\bar{x}), x - \bar{x} \rangle \geq 0$$

By extending the inequality in Proposition 2.1, the notion of pseudoconcavity was introduced as a generalized concavity which plays an important role in applied mathematics such as, optimization theory and mathematical economics. Let’s recall the definition and some properties of pseudoconcavity.

**Definition 2.2.** Let $g$ be a differentiable function defined on an open nonempty convex set $X$ of $\mathbb{R}^n$.

$g$ is said to be pseudoconcave on $X$ if for all $x_1, x_2 \in X$

$$\langle \nabla g(x_1), x_2 - x_1 \rangle \leq 0 \Rightarrow g(x_2) \leq g(x_1)$$

$g$ is said to be pseudoconvex on $X$ if $(-g)$ is pseudoconcave.

**Proposition 2.3.** (Mangasarian [18]) Let $g$ be differentiable on $X$. Then, if $g$ is pseudoconcave on $X$, then it is semi-strictly quasiconcave on $X$.

**Proposition 2.4.** Let $g$ be a differentiable function on an open convex set $X$ of $\mathbb{R}^n$. Then, $g$ is pseudoconcave on $X$ if and only if the restriction of $g$ to any line segment in $X$ is pseudoconcave. (see [13]).

It has been known that a differentiable pseudoconcave function $g$ is quasiconcave and has a maximum at $x$ whenever $\nabla g(x) = 0$. In [8], Crouzeix and Ferland have shown that this property is a necessary and sufficient condition for pseudoconcavity.
Proposition 2.5. (Theorem 2.2. \[8\]) Let $g$ be a differentiable and quasiconcave function on an open convex set $X \subset \mathbb{R}^n$. Then $g$ is pseudoconcave on $X$ if and only if $g$ has a local maximum at $x \in X$ whenever $\nabla g(x) = 0$.

Definition 2.3. Let $f$ be a real function defined on a convex set $C \subset \mathbb{R}^n$. For $r \in \mathbb{R} \setminus \{0\}$, let $f_r$ be defined by: $f_r(x) = e^{rf(x)}$.

$f$ is said to be $r$-concave if $f_r$ is concave whenever $r > 0$ and convex whenever $r < 0$.

$f$ is 0-concave if it is concave. (For more details on $r$-concavity/convexity see Avriel \[4\]).

Proposition 2.6. (Theorem 6.1. \[4\]) Let $r$ be any real number and let $f$ be a differentiable $r$-concave function on a convex set $C \subset \mathbb{R}^n$. Then $f$ is pseudoconcave on $C$.

3 Optimality conditions under Separability: The differentiable case.

In this section and the next one, we will study sufficient optimality conditions for separable quasiconcave programming when the objective function is multiplicatively decomposed. First, we recall the definition and some properties of a separable product function.

Definition 3.1. Let $X$ be a subset of $\mathbb{R}^n$. A function $f: X \to \mathbb{R}$ is said to be multiplicatively separable if it satisfies the following condition:

\[(MS) : \text{there exist subsets } X_i \text{ of } \mathbb{R}^{n_i}, \text{ and functions } f_i : X_i \to \mathbb{R} \text{ (} i = 1, ..., m \text{) such that}\]

$$f(x) = \prod_{i=1}^{m} f_i(x_i)$$

where $x = (x_1, ..., x_m) \in X = X_1 \times \cdots \times X_m$ and $\sum_{i=1}^{m} n_i = n$.

First, we recall a necessary condition for the function $f$ to be quasiconcave. See \[5\]

Proposition 3.1. (Lemma 3.4. \[5\]) Let $f$ be a real function defined on a convex set $X \subset \mathbb{R}^n$ verifying the condition (MS). If $f$ is quasiconcave on $X$, then all $f_i$ (i = 1, ..., m) are quasiconcave on $X_i$.

Now we recall the definition and some properties of the multiplicative concavity index of a function introduced by Crouzeix and Kebbour in \[10\]. Such an index was the basic tool to study the (generalized)concavity of a function.

Definition 3.2. (Crouzeix-Kebbour \[10\]) The multiplicative concavity index $i_{cv}(f)$ of a function $f: X \to (0, \infty)$ is defined as follows:
If there exists \( \lambda < 0 \) such that \( f^\lambda \) is not convex then
\[
i_{cv}(f) = \sup\{\mu < 0 : f^\mu \text{ is convex}\}
\]
If \( f^\lambda \) is convex for every \( \lambda < 0 \), then
\[
i_{cv}(f) = \sup\{\mu > 0 : f^\mu \text{ is concave}\}
\]
The following proposition is an immediate consequence of Definition 3.2. (See [10, 11]).

**Proposition 3.2.** Let \( X \) and \( f \) as in Proposition 3.1. Then,

(a) If \( i_{cv}(f) > -\infty \), then \( f \) is quasiconcave;
(b) \( f \) is log-concave if and only if \( i_{cv}(f) \geq 0 \);
(c) \( f \) is concave if and only if \( i_{cv}(f) \geq 1 \);
(d) \( f \) is constant if and only if \( i_{cv}(f) = +\infty \) and \( X \) is open;
(e) Let \( \alpha > 0 \), then \( i_{cv}(f^\alpha) = \frac{1}{\alpha} i_{cv}(f) \);
(f) \( i_{cv}(f) = \inf\{i_{cv}(f_{x,d}) : x \in X, d \in \mathbb{R}^n \setminus \{0\}\} \).

The following proposition, which reveals an interesting property of separable quasiconcave functions, will be used frequently in the sequel of this paper (see [5]).

**Proposition 3.3.** (Theorem 3.12.[6]) For \( i = 1, \ldots, m \), let \( X_i \) be a non-empty open convex subset of \( \mathbb{R}^m \), and let \( f_i \) be a positive non-constant function defined on \( X_i \), and \( f \) be the function defined on the product space \( X = X_1 \times X_2 \times \ldots \times X_m \) by
\[
f(x_1, \ldots, x_m) = \prod_{i=1}^{m} f_i(x_i)
\]
i) The function \( f \) is quasiconcave if and only if one of the following holds:
   a) all functions \( f_i \) are log-concave.
   b) all functions \( f_i \) except one are log-concave and
   \[
   \sum_{i=1}^{n} \frac{1}{i_{cv}(f_i)} \leq 0
   \]  
   (1)
   ii) If \( f \) is quasiconcave then
   \[
   \frac{1}{i_{cv}(f)} = \sum_{i=1}^{n} \frac{1}{i_{cv}(f_i)}
   \]  
   (2)
   with the convention \( \frac{1}{0} = \infty \).

**Remark 3.1.** Notice that if all \( f_i \) (\( i = 1, \ldots, m \)) are differentiable log-concave, then so is the function \( f \), and then it is pseudoconcave. If not, there exists \( i_0 \in \{1, \ldots, m\} \) such that \( f_{i_0} \) is not log-concave. It is clear that \( f_{i_0} \) is \( i_{cv}(f_{i_0}) \)-concave with \( i_{cv}(f_{i_0}) < 0 \), then, from Proposition 2.6, \( f_{i_0} \) is pseudoconcave on \( X \), hence, \( f \) is also pseudoconcave on \( X \). (For more details see the proof of Theorem 5.3. in Crouzeix-Hassouni [9]).
Unconstrained problem.

Consider the unconstrained problem

\[(USQP): \quad \max_{x \in X} f(x)\]

where \(f\) satisfies the condition (MS) in Definition 3.1.

**Theorem 3.1.** Assume that all \(f_i\) are differentiable, and \(f\) is quasiconcave on \(X\). Then,

i) If \(\bar{x}\) is a critical point of \(f\) then \(\bar{x}\) is a global maximum of \(USQP\).

ii) If \(\bar{x}\) is a local maximum of \(USQP\) then it is a global maximum.

**Proof.** i) Suppose that \(\bar{x}\) is a critical point, that is \(\nabla f(\bar{x}) = 0\), since \(\langle \nabla f(\bar{x}), x - \bar{x} \rangle = 0 \forall x \in X\) then, by the pseudoconcavity of \(f\), one has \(f(x) \leq f(\bar{x})\) for all \(x \in X\).

ii) Let \(\bar{x}\) be a local maximum of \(f\), then there is a neighbourhood \(\mathcal{N}(\bar{x})\) of \(\bar{x}\), such that for all \(x \in \mathcal{N}(\bar{x}) \cap X\) we have \(f(x) \leq f(\bar{x})\).

Let \(x \in X\) such that \(x \not\in \mathcal{N}(\bar{x})\). There exists \(\lambda \in (0,1)\) such that \(\bar{x} = (1-\lambda)\bar{x} + \lambda x \in \mathcal{N}(\bar{x}) \cap X\). By Proposition 2.3, the restriction of \(f\) to the line segment \([\bar{x}, x]\) is pseudoconcave. If \(f(x) > f(\bar{x})\) then by the semi-strict quasiconcavity of \(f\) (see Proposition 2.3), one has \(f(\bar{x}) > f(\bar{x})\) which is a contradiction. Thus \(\bar{x}\) is a global maximum.

Constrained problem.

Now consider the constrained problem

\[(CSQP): \quad \max_{x \in X} f(x) \quad \text{subject to} \quad h_j(x) \geq 0 \quad j = 1, \cdots, p\]

where \(f\) satisfies the condition (MS) in Definition 3.1.

Define the feasible set \(F = \{x \in X : h_j(x) \geq 0, \quad j = 1, \cdots, p\}\).

**Theorem 3.2.** Assume that \(f\) and all \(h_j\) \((j = 1, \cdots, p)\) are differentiable and quasiconcave on \(X\). Let \(\bar{x}\) be a feasible point such that \(\nabla h_j(\bar{x}) \neq 0\) for all \(j\) and \(h_j(\bar{x}) > 0\) for some \(j\).

If there exist \(\lambda_j \in \mathbb{R}, j = 1, \cdots, p\), such that

\[
\nabla f(\bar{x}) + \sum_{j=1}^{p} \lambda_j \nabla h_j(\bar{x}) = 0, \quad (3)
\]

\[
\lambda_j h_j(\bar{x}) = 0, \quad j = 1, \cdots, p, \quad (4)
\]

\[
\lambda_j \geq 0, \quad j = 1, \cdots, p, \quad (5)
\]

then \(\bar{x}\) is a global solution of \(CSQP\).
Proof. Assume, by contradiction, that there exists \( x_0 \in F \) such that \( f(x_0) > f(\bar{x}) \). By the pseudoconcavity of \( f \) we have \( \langle \nabla f(\bar{x}), x_0 - \bar{x} \rangle > 0 \).

Since \( h_j(x_0) \geq 0 = h_j(\bar{x}), \ j \in J(\bar{x}) = \{ j/h_j(\bar{x}) = 0 \} \), the quasiconcavity of \( h_j \) implies
\[
\langle \nabla h_j(\bar{x}), x_0 - \bar{x} \rangle \geq 0, \ j \in J(\bar{x})
\]
From the complementarity condition (4), we have
\[
\langle \nabla f(\bar{x}), x_0 - \bar{x} \rangle + \sum_{j=1}^{p} \langle \lambda_j h_j(\bar{x}), (x_0 - \bar{x}) \rangle > 0,
\]
which contradicts (3).

Example 3.1. Consider the Cobb-Douglas utility function defined by:
\[
u(x_1, x_2, ..., x_n) = \prod_{i=1}^{n} u_i(x_i)
\]
where \( u_i(x_i) = x_i^{\alpha_i} \) with \( x_i > 0 \) and \( \alpha_i > 0 \); \( i = 1, ..., n \)
Consider the problem of maximization of \( u \) with constraint budget:
\[
(P): \quad \max u(x) \quad \text{subject to} \quad G(x) \leq B
\]
where \( G(x) = p_1 x_1 + p_2 x_2 + ... + p_n x_n \).

From Proposition 3.3., \( u \) is quasiconcave, and since it is differentiable with \( \nabla u(x) \neq 0 \) for all \( x \), then, by Proposition 2.3., \( u \) is pseudoconcave.

Let \( H(x) = B - p_1 x_1 - p_2 x_2 - ... - p_n x_n \). It is clear that \( H \) is quasiconcave and differentiable.

Let \( \bar{x} = (\bar{x}_1, ..., \bar{x}_n) \) a feasible point, that is \( H(\bar{x}) \geq 0 \), and let \( \lambda \in \mathbb{R} \) such that \( (\bar{x}, \lambda) \) satisfies the KKT conditions (3), (4) and (5) in Theorem 3.2.

By (5) one has: \( \alpha_i \bar{x}_1^{\alpha_i} \bar{x}_2^{\alpha_2} \cdots \bar{x}_i^{\alpha_i-1} \cdots \bar{x}_n^{\alpha_n} = \lambda p_i \) for all \( i = 1, ..., n \).

Since \( \bar{x}_i > 0, \alpha_i > 0, p_i > 0, (i = 1, ..., n) \) then \( \lambda \neq 0 \), and \( \frac{\prod_{j=1}^{n} (\bar{x}_j)^{\alpha_j}}{\lambda} = \frac{p_i}{\alpha_i} \bar{x}_i \), thus \( \bar{x}_i = \frac{\alpha_i p_i}{\alpha_1 p_1} \bar{x}_1 \)
for all \( i = 1, ..., n \).

By (4) and since \( \lambda \neq 0 \) one has \( H(\bar{x}_1, ..., \bar{x}_n) = 0 \), i.e. \( p_1 \bar{x}_1 + ... + p_n \bar{x}_n = B \), then;
\[
\bar{x}_1 = \frac{\alpha_1 B}{(\alpha_1 + ... + \alpha_n) p_1}, \quad \text{thus} \quad \bar{x} = \left( \frac{\alpha_1 B}{(\alpha_1 + ... + \alpha_n) p_1}, ..., \frac{\alpha_n B}{(\alpha_1 + ... + \alpha_n) p_n} \right)
\]
is a solution of (P).

4 Optimality conditions under Separability: The non differentiable case.

In this section we will study the problems (USQP) and (CSQP) studied in the previous section when the objective function and the constrained functions are not necessarily differentiable.

First, notice that in Proposition 3.3., any assumption of differentiability of \( f \) is required.

Secondary, we recall that the notion of pseudoconcavity can be extended to non differentiable case by means of a generalized superdifferential instead of the classical gradient.
Let’s define an abstract superdifferential in the same sense as the abstract subdifferential defined by Aussel et al in [3].

**Definition 4.1.** We call superdifferential, denoted by \( \partial S \), any operator which associates a subset \( \partial S f(x) \) of \( \mathbb{R}^n \) to any upper semi-continuous function \( f : X \to \mathbb{R} \cup \{-\infty\} \) and any \( x \in X \) such that \( f(x) \) is finite, and satisfies the following properties:

- **(P1)** \( \partial S f(x) = \{x^* \in \mathbb{R}^n : \langle x^*, y-x \rangle + f(x) \geq f(y), \forall y \in X\} \) whenever \( f \) is concave;
- **(P2)** \( 0 \in \partial S f(x) \) whenever \( x \) is a local maximum of \( f \);
- **(P3)** \( \partial S (f + g)(x) \subseteq \partial S f(x) + \partial S g(x) \) whenever \( g \) is a real-valued concave continuous function which is \( \partial S \)-differentiable at \( x \).

where \( g \) is \( \partial S \)-differentiable at \( x \) means that both \( \partial S g(x) \) and \( \partial S (-g)(x) \) are non-empty.

**Examples 4.1.** Let’s recall the Clarcke-Rockafellar subdifferential \( \partial CR \) and the upper-Dini subdifferential \( \partial D^+ \) for a lower-semicontinuous function \( f : X \to \mathbb{R} \cup \{+\infty\} \):

\[
\partial CR f(x) := \{x^* \in X^* : \langle x^*, v \rangle \leq f^+(x; v), \forall v \in X\}
\]

with \( f^+(x; v) = \sup_{\epsilon > 0} \lim_{x' \to x} \inf_{v' \in B_\epsilon(v)} \frac{f(x' + tv') - f(x')}{t} \),

and

\[
\partial D^+ f(x) := \{x^* \in X^* : \langle x^*, v \rangle \leq f^D^+(x, v), \forall v \in X\}
\]

with \( f^D^+(x, v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t} \)

where the notation \( x' \to_f x \) means that \( x' \to x \) and \( f(x') \to f(x) \).

Let’s define the Clarcke-Rockafellar superdifferential \( \partial CR \) and the upper-Dini superdifferential \( \partial D^+ \) for an upper-semicontinuous function \( f : X \to \mathbb{R} \cup \{-\infty\} \):

\[
\partial CR f(x) := -\partial CR(-f)(x)
\]

\[
\partial D^+ f(x) := -\partial D^+(-f)(x)
\]

These two superdifferentials check, among others, the above abstract superdifferential’s properties in Definition 4.1.

We recall that \( \partial CR \) and \( \partial D^+ \) contain the best known subdifferentials such as the lower Hadamard subdifferential \( \partial H^- \), the Fréchet subdifferential \( \partial F \) and the lipschitz subdifferential \( \partial LS \).

In the sequel, we will use the symbol \( \partial S \) to mean either \( \partial CR \) or \( \partial D^+ \).

The following proposition, which extend Proposition 2.1. in [2], is an immediate consequence of Definition 4.1 and Theorem 2.1. in [2].

**Proposition 4.1.** Let \( X \) be a nonempty convex subset of \( \mathbb{R}^n \) and let \( f : X \to \mathbb{R} \) be an upper-semicontinuous function. Then, the following assertions are equivalent:
i) $f$ is quasiconcave;

ii) $\left( \exists x^* \in \partial f(x) : \langle x^*, y - x \rangle < 0 \right) \Rightarrow f(y) > f(z) \quad \forall z \in [x; y])$

From Definition 4.1 and Proposition 4.1, the Proposition 2.2 can be extended as follows:

**Proposition 4.2.** Let $f$ be an upper semi-continuous and quasiconcave real valued function on a convex set $X$ of $\mathbb{R}^n$, and let $x, \bar{x} \in X$. Then,

$$x \in clU^f(f, f(\bar{x})) \Rightarrow \langle \bar{x}^*, x - \bar{x} \rangle \geq 0 \quad \forall \bar{x}^* \in \partial f(\bar{x})$$

Now, we extend the definition and some properties of pseudocoercivity to the non differentiable case.

**Definition 4.2.** Let $X$ be a nonempty convex subset of $\mathbb{R}^n$ and let $f : X \rightarrow \mathbb{R}$ be an upper-semicontinuous function. $f$ is said to be pseudocoercive with respect to $\partial f$ (in short $\partial$-pseudocoercive) if, for any $x, y \in X$, one has

$$\left( \exists x^* \in \partial f(x) : \langle x^*, y - x \rangle \leq 0 \right) \Rightarrow f(y) \leq f(x).$$

As in the differentiable case, every $\partial$-pseudocoercive function satisfies the following fundamental properties:

**Proposition 4.3.** Let $f$ be an upper-semicontinuous function on $X$. Then, $f$ is $\partial$-pseudocoercive on $X$ if and only if the restriction of $f$ to any line segment in $X$ is $\partial$-pseudocoercive.

The proof follows from the radial property of the generalized derivatives $f^+$ and $f^D+$.

**Proposition 4.4.** Let $f : X \rightarrow \mathbb{R}$ be an upper-semicontinuous and radially continuous function. Then, the following assertions are equivalent:

(i) $f$ is $\partial$-pseudocoercive;

(ii) $f$ is quasiconcave and $(0 \in \partial f(x) \Rightarrow f$ has a global maximum at $x)$.

The proof is a direct consequence of Definition 4.2 and Proposition 4.1.

**Unconstrained problem**

Let $f$ be a real valued function defined on a non-empty open convex set $X$ of $\mathbb{R}^n$ satisfying the separability condition $(MS)$ in Definition 3.1. Consider the unconstrained problem:

$$(USQP) : \max_{x \in X} f(x)$$

**Theorem 4.1.** Assume that $f$ is upper-semicontinuous, quasiconcave and radially continuous on $X$. Then, if $\bar{x}$ solves $(USQP)$ locally, then it is a global solution.
Proof. By the separability condition \((MS)\) and from Proposition \(3.3\) and Remark \(3.1\), the function \(f\) is either log-concave or \(r\)-concave, and then it is \(\widehat{\partial}\)-pseudoconcave. If \(\bar{x}\) is a local maximum of \(f\), then by \((P2)\) in Definition \(3.1\), \(0 \in \widehat{\partial} f(\bar{x})\), thus from Proposition \(4.4\), \(\bar{x}\) is a global maximum of \(f\).

\(\square\)

**Corollary 4.1.** Let \(f\) as in Theorem \(4.1\). Assume that \(\bar{x} \notin clU^* (f, f(\bar{x}))\). Then \(\bar{x}\) is a global solution of \((USQP)\).

Proof. If \(\bar{x} \notin clU^* (f, f(\bar{x}))\), then there exists a neighbourhood \(\mathcal{N}(\bar{x}) \subset X\) of \(\bar{x}\) such that \(\mathcal{N}(\bar{x}) \cap clU^* (f, f(\bar{x})) = \emptyset\), thus \(\bar{x}\) is a local maximum of \(f\), hence by the \(\widehat{\partial}\)-pseudoconcavity of \(f\) and Theorem \(4.1\), \(\bar{x}\) is a global solution of \((USQP)\). \(\square\)

**Constrained problem.**

Let \(f\) be an upper-semicontinuous and quasiconcave real valued function defined on \(X\) satisfying the separability condition \((MS)\) in Definition \(3.1\). Consider the constrained problem:

\[
(CSQP): \quad \max_{x \in X} f(x) \quad \text{subject to} \quad h_j(x) \geq 0 \quad j = 1, \ldots, p
\]

where \(h_j\) \((j = 1, \ldots, p)\) are quasiconcave and upper-semicontinuous on \(X\).

Let’s define the feasible set \(\mathcal{F} = \{x \in X : h_j(x) \geq 0, \quad j = 1, \ldots, p\}\).

For \(x \in \mathcal{F}\), denote \(J(x) = \{j : h_j(x) = 0\}\).

**Proposition 4.5.** Let \(\bar{x}\) be a feasible point of \((CSQP)\). For \(j \in J(\bar{x})\), assume that \(0 \notin \widehat{\partial} h_j(x)\) whenever \(x \in \mathcal{F}\) and \(h_j(x) = 0\). Then, \(\langle \bar{x}^* , x - \bar{x} \rangle \geq 0 \forall \bar{x}^* \in \widehat{\partial} h_j(\bar{x})\).

Proof. For \(j \in J(\bar{x})\), and \(x \in \mathcal{F}\) such that \(h_j(x) = 0\), if \(0 \notin \widehat{\partial} h_j(x)\), then, by \((P2)\) in Definition \(4.1\), \(x\) is not a local maximum of \(h_j\). Suppose that there exists \(\bar{x}^* \in \widehat{\partial} h_j(\bar{x})\), such that \(\langle \bar{x}^* , x - \bar{x} \rangle < 0\). Since \(h_j(x) = h_j(\bar{x}) = 0\), Proposition \(4.2\) yields \(x \notin clU^* (h_j, h_j(x))\). Thus \(x\) is a local maximum of \(h_j\), which is a contradiction. \(\square\)

Let’s define a variant of the well known K.K.T. conditions that we show as a sufficient optimality conditions for the constrained problem \((CSQP)\).

**Definition 4.3.** We say that a pair \((\bar{x}, \bar{\lambda})\) \(\in X \times \mathbb{R}^p\) satisfies the modified Karush-Kuhn-Tucker conditions \((m-K.K.T. \text{ conditions})\) if it satisfies the super-gradient condition:

\[
0 \in \widehat{\partial} f(\bar{x}) + \sum_{j=1}^{p} \lambda_j \widehat{\partial} h_j(\bar{x}) - N_X(\bar{x}) \quad (6)
\]

where \(N_X(\bar{x})\) is the normal cone of \(X\) at \(\bar{x}\), and also the usual complementary slackness conditions:

\[
\lambda_j h_j(\bar{x}) = 0, \quad j = 1, \ldots, p \quad (7)
\]

\[
h_j(\bar{x}) \geq 0, \quad j = 1, \ldots, p \quad (8)
\]

\[
\lambda_j \geq 0, \quad j = 1, \ldots, p \quad (9)
\]
Let’s recall the Slater constraint qualification: there exists \( \tilde{x} \) in \( X \), called a Slater point for (CSQP), such that \( h_j(\tilde{x}) > 0 \) for some \( j \in \{1, \ldots, p\} \).

**Theorem 4.2.** Let \( \tilde{x} \) be a feasible point of (CSQP). Assume that (CSQP) has a Slater point, and \( 0 \not\in \partial h_j(x) \) whenever \( x \in F \) and \( h_j(x) = 0 \). If there exists \( \tilde{\lambda} \in \mathbb{R}^p \) such that \((\tilde{x}, \tilde{\lambda})\) satisfies the m-K.K.T. conditions, then \( \tilde{x} \) is a solution of (CSQP).

**Proof.** Assume, by contradiction, that there exists a feasible point \( x_0 \) such that \( f(x_0) > f(\tilde{x}) \). By Proposition \[3.3\] because of separability condition, \( f \) is actually \( \partial \)-pseudoconcave and then, for all \( \bar{x}^* \in \partial f(\tilde{x}) \) one has \( \langle \bar{x}^*, x_0 - \tilde{x} \rangle > 0 \).

Since \( N_X(\tilde{x}) \) coincides with the normal cone of convex analysis when \( X \) is convex (see \[7\]), then for all \( v \in N_X(\tilde{x}) \) one has \( \langle v, x_0 - \tilde{x} \rangle \leq 0 \). Thus, for all \( \bar{x}^* \in \partial f(\tilde{x}) \), \( \bar{x}_{\tilde{j}}^* \in \partial h_{\tilde{j}}(\tilde{x}) \) and \( v \in N_X(\tilde{x}) \) one has:

\[
\langle \bar{x}^* - v, x_0 - \tilde{x} \rangle > 0 \tag{10}
\]

If \( j \in J(\tilde{x}) \), by Proposition \[4.5\], one has for all \( \bar{x}_{\tilde{j}}^* \in \partial h_{\tilde{j}}(\tilde{x}) \):

\[
\langle \bar{x}_{\tilde{j}}^*, x_0 - \tilde{x} \rangle \geq 0 \tag{11}
\]

From the condition \( 7 \), \( \lambda_j = 0 \) for all \( j \not\in J(\tilde{x}) \). Adding \( 11 \) for \( j = 1, \ldots, p \), and combining with \( 10 \) we get:

\[
\langle \bar{x}^*, x_0 - \tilde{x} \rangle + \sum_{j=1}^{p} \lambda_j \langle \bar{x}_{\tilde{j}}^*, x_0 - \tilde{x} \rangle - \langle v, x_0 - \tilde{x} \rangle > 0
\]

for all \( \bar{x}^* \in \partial f(\tilde{x}) \), \( \bar{x}_{\tilde{j}}^* \in \partial h_{\tilde{j}}(\tilde{x}) \), \( (j = 1, \ldots, p) \) and \( v \in N_X(\tilde{x}) \), which contradicts \( 6 \). \( \square \)

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