We show that diffeomorphism invariance of the Maxwell and the Dirac-Hestenes equations implies the equivalence among different universe models such that if one has a linear connection with non-null torsion and/or curvature the others have also. On the other hand local Lorentz invariance implies the surprising equivalence among different universe models that have in general different $G$-connections with different curvature and torsion tensors.

I. INTRODUCTION

In this paper, by using the Clifford and spin-Clifford bundle formalism we present a thoughtful analysis on the concepts concerning diffeomorphism invariance and local Lorentz invariance of Maxwell and Dirac-Hestenes equations. Diffeomorphism invariance implies the equivalence among different universe models such that if one has non-null torsion and curvature, the others also possess similar characters. Local Lorentz invariance implies the astounding equivalence between different universe models that have in general different $G$-connections with different curvature and torsion tensors. This article is organized as follows: after presenting some algebraic preliminaries in Section 2, in Section 3 the invariance of the Maxwell Lagrangian and of Dirac-Hestenes equation, under diffeomorphisms, is investigated from the extensor field formalism viewpoint. Lorentz transformations and the Lienard-Wiechert formulæ are derived in this context. In Section 4 active local Lorentz mappings are introduced, regarding their action on electromagnetic fields. The covariant derivative acting on vector and spinor fields is briefly revisited in the light of the Clifford and spin-Clifford bundle context in Section 5. Next, using that formalism we present in Section 6 the Dirac-Hestenes equation in Riemann-Cartan spacetimes and recall that, in general, in theories of that kind the spin generates torsion. Indeed, it is always emphasized that in a theory where, besides the spinor field, also the tetrad fields and the connection are dynamical variables, the torsion is not zero, because its source is the spin associated with the spinor field. However, in Section 7 we show that to suppose the Dirac-Hestenes Lagrangian is invariant under active rotational gauge transformations implies in an equivalence between torsion free and non-torsion free $G$-connections, and also that we may also have equivalence between spacetimes with null and non-null curvatures.

II. SOME PRELIMINARIES

A Riemann-Cartan spacetime is a pentuple $(M, g, \nabla, \tau_g, \uparrow)$ where $(M, g)$ is an oriented (by $\tau_g \in \sec (\bigwedge^2 T^* M)$) 4-dimensional Lorentzian manifold $M$, equipped with a Lorentz metric $g \in \sec T^2 M$ of signature $(1, 3)$. The operator $\nabla$ denotes the Levi-Civita (metric compatible) connection of $g$ $(\nabla g = 0)$, and in general $T(\nabla) \neq 0$, and $R(\nabla) \neq 0$, where $T$ is the torsion tensor of $\nabla$ and $R$ is the Riemann curvature tensor of $\nabla$. When $\nabla g = 0$ and $T(\nabla) = 0$ the pentuple $(M, g, \nabla, \tau_g, \uparrow)$ is called a Lorentzian manifold. A Lorentzian manifold for which $M \simeq \mathbb{R}^4$ is called a Minkowski spacetime. In this case it is represented by a pentuple $(M, \eta, \nabla, \tau_\eta, \uparrow)$. More details if needed can be found, e.g., in [1].

At each point $e \in M$, we denote respectively by $T_e M$ and $T^*_e M$, the tangent and cotangent spaces. Reference frames are time-like vector fields (pointing to the future) in the world manifold $M$. If $\eta \in \sec T^2 M$ is the metric of Minkowski spacetime, there exists in $M$ a global chart $(M, \varphi)$ with coordinate functions $\{x^\mu\}$ (said to be in the Einstein-Lorentz-Poincaré gauge) such that for the section $\{e_\mu = \frac{\partial}{\partial x^\mu}\}$ of the orthonormal frame bundle $\textbf{PSO}_4(3, \eta)$
we have
\[ \eta(e_\mu, e_\nu) = \text{diag}(1, -1, -1, -1) \]

The pair \((T_M, \eta)\) is called Minkowski vector space. The existence of global coordinates in the Einstein-Lorentz-Poincaré gauge permits to identify all tangent (and cotangent) spaces for all \( \mathfrak{e} \in M \).

Special Relativity (SR) refers to theories that have the Poincaré group as a symmetry group. This theory asserts that there is a class of physically equivalent reference frames, the inertial ones. The Clifford algebra associated with \( \mathbb{R}^{1,3} \) is denoted by \( \mathbb{R}_{1,3} \simeq \mathbb{H}(2) \) and is called the spacetime algebra. The Dirac algebra is \( \mathbb{C} \otimes \mathbb{R}_{4,1} \simeq \mathbb{C}(4) \), the Clifford algebra associated with a 5-dimensional vector space endowed with a scalar product of signature \((4,1)\). Note that given a general Riemann-Cartan spacetime we also have that \( \mathcal{C}(T_\mathfrak{e}M, g_{\mathfrak{e}e}) = \mathbb{R}_{1,3} \). Also, if \( g \in \text{sec} T^0_\mathfrak{e}M \) is the metric associated with the cotangent bundle, we have \( \mathcal{C}(T^*_\mathfrak{e}M, g_{\mathfrak{e}e}) = \mathbb{R}_{1,3} \).

Fields in the Clifford algebra formalism can be taken as sections of the Clifford bundle of multivectors, denoted by \( \mathcal{C}(M, g) = \cup_{\nu} T^\nu \mathcal{C}(T_\mathfrak{e}M, g_{\mathfrak{e}e}) \) or as sections of the Clifford bundle of multiforms, denoted by \( \mathcal{C}(M, g) = \cup_{\nu} T^\nu \mathcal{C}(T^*_\mathfrak{e}M, g_{\mathfrak{e}e}) \), which we shall use in what follows, because it is more convenient for our purposes. By \( \mathcal{C}(M, g) \) we denote the even subalgebra of \( \mathcal{C}(M, g) \). Note that \( \mathcal{C}(T^*_\mathfrak{e}M, g_{\mathfrak{e}e}) \simeq \mathbb{R}_{0,3} \simeq \mathbb{R}_{3,0} \), where \( \mathbb{R}_{3,0} \simeq \mathbb{C}(2) \) is the Pauli algebra. Then, a Clifford field of multiforms will be considered as a section
\[ C \in \text{sec} \bigwedge T^* M \hookrightarrow \text{sec} \mathcal{C}(M, g) \]

where \( \bigwedge T^* M = \bigoplus_{\nu=0}^4 \bigwedge^\nu T^* M \) denotes the exterior algebra of multiforms. The symbol \( \hookrightarrow \) means that \( \bigwedge T^* M \) is embedded in \( \mathcal{C}(M, g) \).

A metric compatible connection \( \nabla \) acting on the tensor bundle defines a covariant derivative acting on Clifford fields, see e.g., [2]. A reference frame \([Z : M \supset U \to TM] \) is a vector field such that \( \eta(Z,Z) = 0 \) in Minkowski spacetime there are infinite global inertial reference frames. These are reference frames for which \( \nabla_\mathfrak{e} \mathfrak{e}, Z = 0 \), for \( \nu = 0,1,2,3 \). For any \( \mathfrak{e} \in M \), \( R(\mathfrak{e}) \mathfrak{e} \) is an inertial reference frame. In the Clifford algebra formalism we can write Eq. [4] as
\[ \gamma_0' = R \gamma_0 R, \]

where \( R \in \text{Spin}^+_3(M) \), i.e., for any \( \mathfrak{e} \in M \), \( R(e) \mathfrak{e} \mathfrak{e} \) is an orthonormal frame and by \( \{\gamma^\alpha\} \in \text{sec} P^+_3(M) \) the respective orthonormal coframe. The Dirac operator acting on sections of \( \mathcal{C}(M, g) \) is the invariant differential operator which maps Clifford fields in Clifford fields, given by
\[ \partial = \gamma^\alpha \nabla_{e_\alpha}, \]
\[ \partial C = \partial \wedge C + \partial \cdot C. \]

When \( \nabla \) is the Levi-Civita connection of \( g \), we have
\[ \partial = \partial \wedge + \partial \cdot = d - \delta, \]

where \( \delta \) is the Hodge codervivative operator. Thus, in this case we can write
\[ \partial C = dC - \delta C. \]

Recall that coordinates functions for \( U \subset M \) are mappings \( x^\mu : M \supset U \to \mathbb{R} \). These mappings can be considered as sections, \( x^\mu \in \text{sec} \bigwedge T^0 U \hookrightarrow \mathcal{C}(U_\mathfrak{e}, g_\mathfrak{e}) \). In the case of a Minkowski spacetime a special set of coordinates naturally adapted to an inertial frame are the ones in the Einstein-Lorentz-Poincaré gauge. They are global coordinate functions such that
\[ \partial x^\mu = \gamma^\mu. \]

In this case the Dirac operator can be written as \( \partial = \gamma^\mu \nabla_{e_\mu} = \gamma^\mu \partial_\mu. \)
III. MAXWELL THEORY AND DIFFEOMORPHISM INVARIANCE

Classical Maxwell theory on a Lorentzian spacetime deals with an electromagnetic field \( F \in \sec \bigwedge^2 T^*M \hookrightarrow \sec \mathcal{C}(M, g) \) generated by a current \( J \in \sec \bigwedge^1 T^*M \hookrightarrow \sec \mathcal{C}(M, g) \), and the motion of probe charges modelled by triples \( (m, q, \sigma) \) in the field \( F \). The field \( F \) satisfies the equations

\[
dF = 0 \quad \delta F = -J. \tag{10}
\]

Eqs. \ref{10} can be written in a general Lorentzian spacetime, taking into account Eq. \ref{8}, as

\[
\partial F = J. \tag{11}
\]

Neglecting radiation reaction, the motion of an arbitrary probe charge of mass \((m, q, \sigma)\) is given by

\[
m \nabla_v v = qv \ast F, \tag{12}
\]

where \( v \equiv g(\sigma, \cdot) \), with \( \sigma \) the tangent vector field to the worldline \( \sigma : \mathbb{R} \to M \) of the charged particle. Eqs. \ref{11} and \ref{12} are intrinsic, i.e., they do not depend on any reference frame and/or coordinates used by observers living on different reference frames. Note that the concept of observer is different from that of a reference frame. An observer is modelled by an integral line of a reference frame \([1, 2]\). Indeed, a reference frame \( \mathbf{e}_0 \) can be viewed as the four-velocity field of a family of test observers whose worldlines are the integral curves of \( \mathbf{e}_0 \), each one can be parametrized by the proper time \( \tau_{\mathbf{e}_0} \) defined up to an additive constant on each curve.

Now, Eqs. \ref{10} can be derived from the following Lagrangian density

\[
\mathcal{L} = \int_U F \wedge \ast F - A \wedge \ast J. \tag{13}
\]

As it is well known, every Lagrangian density written in terms of differential forms is invariant under arbitrary diffeomorphisms \( \mathbf{h} : M \to M \). Under this diffeomorphism the fields, currents and connection transforms under the pullback mapping, i.e.,

\[
\begin{align*}
\eta &\mapsto \mathbf{h}^* \eta, \\
A &\mapsto \mathbf{h}^* A, \\
F &\mapsto \mathbf{h}^* F, \\
J &\mapsto \mathbf{h}^* J, \\
\nabla &\mapsto \mathbf{h}^* \nabla, \\
h^* \nabla_{\mathbf{h}^{-1}} \mathbf{h}^* \mathbf{t} &= \mathbf{h}^* \left( \nabla \mathbf{t} \right)_{\mathbf{h}^{-1}}, \\
\forall \mathbf{t} \in M, \quad \forall \mathbf{v} \in \sec TM, \quad \mathbf{t} \in \sec TM,
\end{align*} \tag{14-16}
\]

where \( TM \) denotes the tensor bundle. The models \((M, g, \nabla, \mathbf{\tau}_g, \uparrow, A, F, J)\) and \((M, g', h^* \nabla, h^* \mathbf{\tau}_g, \uparrow, h^* A, h^* F, h^* J)\) are said to be equivalent in the sense that if Eqs. \ref{10} are satisfied with well defined initial and boundary conditions then \( h^* F \) satisfy the equations

\[
dh^* F = 0, \quad \delta h^* F = -h^* J. \tag{17}
\]

with well defined transformed initial and boundary conditions. However, take into account that the equivalence is realized via the introduction of different universe models that are also declared to be equivalent.

The first formulæ in Eq. \ref{17} is clearly diffeomorphically invariant since it is a well known result that \( dh^* = h^* d. \) The second equation is also diffeomorphically invariant because the pullback mapping can be represented by an invertible dislocated extensor field \( \mathbf{h}^{-1} : \bigwedge T^* M \to \bigwedge T^* M \) such that its exterior power extension satisfies \( h^{-1} X = \mathbf{h}^* X \), for any \( X \in \sec \bigwedge T^* M \hookrightarrow \sec \mathcal{C}(M, g) \) and moreover we can easily show that \( \mathbf{h}^{-1} \)

\[
\ast h^{-1} = \ast g^{-1} \ast h. \tag{18}
\]

where \( \ast \) and \( \ast h \) denote the Hodge star operators associated with \( g \) and \( g' \). In this way the equation \( \delta h^* F = -h^* J \)

implies the equation \( \delta F = J \). Indeed, we have

\[
h^* F = \ast^{-1} d \ast h^* F = h^{-1} \ast d \ast F = h^{-1} \delta F = h^{-1} J. \tag{19}
\]

The active formulation of the Principle of Relativity implies that if the set of geometrical objects \((J, F, (m, e, \sigma))\) living on Minkowski spacetime \((M, \eta, \nabla, \tau_\eta, \uparrow)\) satisfies Eqs. \ref{11} and \ref{12}, with physically realizable initial and boundary conditions, then any other set \((J, \tilde{F}, (m, e, \tilde{\sigma}))\), with

\[
\tilde{F} = l^* F, \quad \tilde{J} = l^* J, \quad \tilde{v} = l^* v, \tag{20}
\]

where
where $l$ a Lorentz mapping, $e \mapsto le$, and $l^*$ denotes the pullback mapping, will satisfy

$$\partial F = \bar{J}$$

and

$$m \nabla_{\bar{\nu}} \bar{v} = e \bar{v}_{,\bar{\nu}} F,$$

with also physically realizable initial and boundary conditions. It is trivial to see, e.g., the validity of Eq. \ref{21}, for indeed since in this case, $g = l^* \eta = \eta$ we have that $\star l^* = \eta$. Note moreover that $l$ is conveniently defined in terms of coordinate transformations by

$$x'^\mu (le) = x^\mu (e), \quad x^\mu (le) = (L^{-1})^\mu_{\nu} x^\nu (e),$$

where $(L^\mu_\nu) \in SO^1_{1,3}$ is a Lorentz transformation. Observe that the coordinate functions $x'^\mu$ satisfy

$$\partial x'^\mu = \gamma'^\mu = dx'^\mu.$$

These coordinate functions are, of course, naturally adapted coordinates in the Einstein-Lorentz-Poincaré gauge to the reference frame $\gamma'_0$.

Now consider a velocity boost in the $\gamma_1$-direction. We write \ref{28} (with $\gamma = (1 - v^2)^{-\frac{1}{2}}$)

$$L = \begin{pmatrix} \gamma & -v \gamma & 0 & 0 \\ -v \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L^{-1} = \begin{pmatrix} \gamma & v \gamma & 0 & 0 \\ -v \gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{27}$$

Consider, moreover, the frames $\gamma_0$, $\gamma'_0$ and $\gamma''_0$, and the orthonormal sets $\{\gamma_\mu\}$, $\{\gamma'_\mu\}$ and $\{\gamma''_\mu\}$, with \ref{29}

$$\gamma'_\mu = R_{\gamma_\mu} R^{-1} = (L^{-1})^\alpha_{\mu} \gamma_\alpha, \quad \gamma''_\mu = R^{-1} \gamma_\mu R = L^\alpha_{\mu} \gamma_\alpha, \quad x''^\mu = (L^{-1})^\mu_{\nu} x'^\nu. \tag{29}$$

We have in details,

$$\gamma_0 = \frac{1}{\sqrt{1 - v^2}}(\gamma_0 + v \gamma_1), \quad \gamma'_1 = \frac{1}{\sqrt{1 - v^2}}(v \gamma_0 + \gamma_1), \quad \gamma'_2 = \gamma_2, \quad \gamma'_3 = \gamma_3, \tag{30}$$

$$x'^0 = \frac{1}{\sqrt{1 - v^2}}(x^0 - vx^0), \quad x'^0 = \frac{1}{\sqrt{1 - v^2}}(x^0 - vx^0), \quad x'^2 = x^2, \quad x'^3 = x^3, \tag{31}$$

$$\gamma''_0 = \frac{1}{\sqrt{1 - v^2}}(\gamma_0 - v \gamma_1), \quad \gamma''_1 = \frac{1}{\sqrt{1 - v^2}}(v \gamma_0 - \gamma_1), \quad \gamma''_2 = \gamma_2, \quad \gamma''_3 = \gamma_3, \tag{32}$$

$$x''^0 = \frac{1}{\sqrt{1 - v^2}}(x^0 + vx^0), \quad x''^0 = \frac{1}{\sqrt{1 - v^2}}(x^0 + vx^0), \quad x''^2 = x^2, \quad x''^3 = x^3. \tag{33}$$

Now, consider a charge at rest at the origin of the $\gamma_0$ frame. Its field is

$$F|_e = \frac{1}{2} F_{\mu \nu}(x(e)) \gamma'^\mu \wedge \gamma'^\nu,$$

with

$$F_{\mu \nu}(x(e)) = \frac{q x^i(e)}{|x(e)|^3}, \quad F_{ij}(x(e)) = 0,$$

$$|x(e)| = \sqrt{(x^1(e))^2 + (x^2(e))^2 + (x^3(e))^2}. \tag{35}$$
By definition, for any \( u, w \in \sec TM \),

\[
F|_\epsilon (u|_\epsilon, w|_\epsilon) = F|_\epsilon (L_\epsilon u|_\epsilon, L_\epsilon w|_\epsilon),
\]

from where we get

\[
\bar{F}_{\mu\nu}(x(\epsilon)) = (L^{-1})^\alpha_\mu (L^{-1})^\beta_\nu F_{\mu\nu}(x(\epsilon)).
\]

The electric and magnetic parts of the pullback fields in the \( \gamma_0 \) frame are

\[
\bar{\mathbf{E}}(x(\epsilon)) = q \left\{ \frac{x^1(\epsilon)}{|x|}, \frac{\gamma x^2(\epsilon)}{|x|^2}, \frac{\gamma x^3(\epsilon)}{|x|^2} \right\},
\]

\[
\bar{\mathbf{B}}(x(\epsilon)) = q \left\{ 0, \gamma v \frac{x^3(\epsilon)}{|x|^3}, -\gamma v \frac{x^2(\epsilon)}{|x|^3} \right\},
\]

and using Eq.(34) we finally have

\[
\bar{\mathbf{E}}(x(\epsilon)) = q \gamma \left\{ \frac{x^1(\epsilon) + vz^0(\epsilon)}{|R(\epsilon)|^3}, \frac{x^2(\epsilon)}{|R(\epsilon)|^3}, \frac{x^3(\epsilon)}{|R(\epsilon)|^3} \right\},
\]

\[
\bar{\mathbf{B}}(x(\epsilon)) = v \times \bar{\mathbf{E}}(x(\epsilon)),
\]

where \( v = (-v, 0, 0) \) and

\[
R(\epsilon) = \sqrt{\gamma^2 (x^1(\epsilon) + vz^0(\epsilon))^2 + (x^2(\epsilon))^2 + (x^3(\epsilon))^2}. \tag{40}
\]

Eqs.(39) give the field of a charge \( q \) moving in the negative \( x^1 \)-direction, as can be calculated directly from the Lienard-Wiechert potential formula.

We can also write for the field \( F \),

\[
F|_\epsilon = \frac{1}{2} F_{\mu\nu}(x(\epsilon))\gamma^\mu \wedge \gamma^\nu = \frac{1}{2} \bar{F}'_{\mu\nu}(x'(\epsilon))\gamma'^\mu \wedge \gamma'^\nu, \tag{41}
\]

where

\[
\bar{F}'_{\mu\nu}(x'(\epsilon)) = (L^{-1})^\alpha_\mu (L^{-1})^\beta_\nu F_{\mu\nu}(x(\epsilon)). \tag{42}
\]

and we have for the electric and magnetic fields in the \( \gamma_0' \) frame,

\[
\mathbf{E}'(x'(\epsilon)) = q \gamma \left\{ \frac{x^1(\epsilon) + vz^0(\epsilon)}{|R'(\epsilon)|^3}, \frac{x^2(\epsilon)}{|R'(\epsilon)|^3}, \frac{x^3(\epsilon)}{|R'(\epsilon)|^3} \right\},
\]

\[
\mathbf{B}'(x'(\epsilon)) = v \times \mathbf{E}'(x'(\epsilon)), \tag{43}
\]

with

\[
R'(\epsilon) = \sqrt{\gamma^2 (x^1(\epsilon) + vz^0(\epsilon))^2 + (x^2(\epsilon))^2 + (x^3(\epsilon))^2}. \tag{44}
\]

We see the \( \gamma_0' \) observers perceive (of course, through measurements) the field \( F \) as the field of a charged particle moving with constant velocity in the negative \( x^1 \)-direction, which is intuitively obvious. Note that \( \gamma_0 \) observers perceive the field \( \bar{F} \) in the same way that their colleagues at \( \gamma_0' \) realize \( F \). Finally the observers (at rest) in the frame \( \gamma_0'' \) realize the field \( \bar{F} \) as the field of a particle at rest in that frame.

All these results are classical[31], although not explained in general with rigor.

In definitive, the observers at rest in \( \gamma_0 \) can write

\[
F = \mathbf{E} + \gamma_5 \mathbf{B},
\]

\[
\bar{F} = \bar{\mathbf{E}} + \gamma_5 \bar{\mathbf{B}},
\]

with \( \mathbf{E} = F^i(x) \sigma_i, \mathbf{B} = \frac{1}{2} \varepsilon^{ijk} F_{jk} \sigma_i, \bar{\mathbf{E}} = \bar{F}^i(x) \sigma_i, \bar{\mathbf{B}} = \frac{1}{2} \varepsilon^{ijk} \bar{F}_{jk} \sigma_i, \sigma_i = \gamma_i \gamma_0 \) and the observers at \( \gamma_0' \) can write

\[
F = \mathbf{E}' + \gamma_5' \mathbf{B}'. \tag{47}
\]

The relations of all these fields are well-defined and have precise physical meaning.
IV. ACTIVE LOCAL LORENTZ ROTATIONS OF THE ELECTROMAGNETIC FIELD

Action (13) is also invariant under local (i.e., spacetime point dependent) Lorentz transformations. This statement is trivial once we use the Clifford bundle formalism. Indeed, taking into account that
\[ F \wedge \star g F = (F \cdot F) \tau_g, \]  
we see that if we perform an \textit{active} Lorentz transformation
\[ F \mapsto R F = R F R^{-1}, \]  
where \( R \in \text{sec Spin}^e_{1,3}(M) \hookrightarrow \text{sec} \mathcal{Cl}^0(M, g) \), since \( \tau_g = \gamma^5 \) which commutes with even sections of the Clifford bundle, we have
\[ F \wedge \star g F = R F \wedge \star g R F. \]  

What is the meaning of the field \( R F \)? A trivial calculation, as shown originally by Hestenes [11], reveals that in the case where \( R \) is a constant Lorentz transformation in Minkowski spacetime, the components of \( R F \) in the \( \gamma_0 \) inertial frame field are the components of \( F \) as seen in the \( \gamma_0' \) inertial frame. But the important question, that is the source of much confusion in the literature arises: is \( R F \) a solution of Maxwell equations with a transformed source term \( R J R^{-1} \)? The answer in the \textit{Clifford bundle} \( \mathcal{Cl}(M, \eta) \) formalism is in general negative. Indeed, if
\[ dF = 0, \quad \delta F = -J, \]  
in general
\[ d(R F R^{-1}) \neq 0, \quad \delta(R F R^{-1}) \neq R J R^{-1}. \]  
This can be easily seen in the Clifford bundle formalism, since in general,
\[ \theta(R F R^{-1}) \neq R(\theta F) R^{-1}, \]  
because, of course, in general, \( \gamma^\mu R \neq R \gamma^\mu \). After recalling the concept of \textit{generalized} gauge covariant derivatives (\( G \)-connections) in the context of Dirac theory we shall investigate if it is possible in some sense to generalize Maxwell equation in order to have local Lorentz invariance.

V. COVARIANT DERIVATIVE IN THE CLIFFORD BUNDLE

Let \( \{ e_a \}, \{ e'_a \} \in \text{sec} P_{SO_{1,3}}(M) \) two orthonormal frames and \( \{ \theta^a \}, \{ \theta'^a \} \in \text{sec} P_{SO_{1,3}}(M) \) the respective dual bases satisfying
\[ \theta^a(e_b) = \delta^a_b, \quad \theta^a \cdot \theta^b = \eta^{ab}, \quad \theta'^a(e'_b) = \delta_b^a, \quad \theta'^a \cdot \theta'^b = \eta'^{ab}. \]  

Let be \( R \in \text{Spin}_{1,3}(M) \hookrightarrow \text{sec} \mathcal{Cl}^0(M, g) \), i.e., \( R \hat{R} = 1 \) such that
\[ \theta'^a = R \theta^a R^{-1}. \]  
It is well-known that the covariant derivative \( \nabla_X \) of a Clifford multiform \( A \in \text{sec} \bigwedge T^* M \hookrightarrow \text{sec} \mathcal{Cl}(M, g) \) in the direction of the vector field \( X \in \text{sec} TM \) in the \textit{gauge} determined \( \{ e_a \} \in \text{P}_{SO_{1,3}}(M) \) is given by
\[ \nabla_X A = \partial_X(A) + \frac{1}{2} [\omega_X, A], \]  
where \( \partial_X \) is the Pfaff derivative of form fields, defined by
\[ \partial_X A := \frac{1}{p!} X(A_{\mu_1 \ldots \mu_p}) \theta^{\mu_1} \wedge \cdots \wedge \theta^{\mu_p} \]  

and where \( \omega_X \in \text{sec} \bigwedge^2 T^* M \hookrightarrow \text{sec} \mathcal{C}(M, g) \) is a \( \bigwedge^2 T^* M \)-valued connection calculated at \( X \) in the given gauge.

We define
\[
\nabla_X \theta^a = \frac{1}{2} [\omega_X, \theta^a], \quad \nabla_X \theta'^a = \frac{1}{2} [\omega'_X, \theta'^a],
\]
from where we find that:
\[
\omega'_X = R \omega_X R^{-1} + (\nabla_X R) R^{-1}
\]
From the fact that
\[
\nabla_{e_a} \theta^b = -\omega_{ac} \theta^c = \frac{1}{2} [\omega, \theta^c], \quad \nabla_{e'_a} \theta'^b = -\omega'_{ac} \theta'^c = \frac{1}{2} [\omega'_a, \theta'^c]
\]

it follows the expressions
\[
\omega_a = \frac{1}{2} \omega_{bc} \theta_b \wedge \theta_c \in \text{sec} \bigwedge^2 T^* M \hookrightarrow \text{sec} \mathcal{C}(M, g), \quad \omega_{a}^{bc} = -\omega_{a}^{cb}
\]
\[
\omega'_a = \frac{1}{2} \omega'^{bc} \theta'_b \wedge \theta'_c \in \text{sec} \bigwedge^2 T^* M \hookrightarrow, \quad \omega'^{a}_{bc} = -\omega'^{a}_{cb}
\]

### A. Covariant Derivative of Spinor Fields

The covariant derivative of the representative of a Dirac-Hestenes spinor field is a kind of gauge covariant derivative. Let us explain what we mean by this wording.

Let \( \nabla_{e_a} \) be the spinor covariant derivative that acting on sections of the left spin-Clifford bundle, i.e., on \( \chi \in \text{sec} \mathcal{C}(\text{Spin}_{1,3}(M, g)) \). The representative \( \nabla_{e_a} \) acts on the gauge representatives of \( \chi \) in the Clifford bundle. Consider two spin coframes \( \Xi, \Xi' \in \text{PSpin}_{1,3}(M) \), such that \( s(\Xi) = \{ \theta^a \} \) and \( s(\Xi') = \{ \theta'^a \} \) where \( s : \text{PSpin}_{1,3}(M) \to \text{PSpin}_{1,3}(M) \) is the fundamental map connecting those bundles. Suppose that two different spinor fields \( \chi \in \text{sec} \mathcal{C}(\text{Spin}_{1,3}(M, g)) \) and \( \Phi \in \text{sec} \mathcal{C}(\text{Spin}_{1,3}(M, g)) \) have in the spin frames \( \Xi \) and \( \Xi' \) the same representative \( \chi \in \text{sec} \mathcal{C}(M, g) \). Then in each spin frame the representative of the spin covariant derivative is given by
\[
\nabla_{e_a}^{(s)} \chi = \partial_X \chi + \frac{1}{2} \omega_{a}^{bc} \theta_b \wedge \theta_c \chi, \\
\nabla_{e'_a}^{(s)} \chi = \partial_X \chi + \frac{1}{2} \omega'^{a}_{bc} \theta'_b \wedge \theta'_c \chi,
\]
where \( \omega'_a \) and \( \omega_a \) are related as in Eq. (59) and \( X \in \text{sec} T M \).

Now, let \( \psi_{\Xi} \in \text{sec} \mathcal{C}(M, g) \) and \( \psi_{\Xi'} = \psi_{\Xi} R^{-1} \in \text{sec} \mathcal{C}(M, g) \) be the representatives of \( \Phi \in \text{sec} \mathcal{C}(\text{Spin}_{1,3}(M, g)) \) in two different spin frames \( \Xi \) and \( \Xi' \). We have, as it is easy to verify:
\[
\nabla_{e_a}^{(s)} \psi_{\Xi'} = (\nabla_{e_a}^{(s)} \psi_{\Xi}) R^{-1},
\]
which shows that the representative of the spinor covariant derivative in the Clifford bundle is a kind of gauge covariant derivative.

**Remark 1** From now on we call \( \nabla_{e_a}^{(s)} \) simply the spinor derivative. In each gauge, if \( A \in \text{sec} \mathcal{C}(M, g) \) and \( \psi_{\Xi} \in \text{sec} \mathcal{C}(M, g) \) is the representative of \( \Phi \in \text{sec} \mathcal{C}(\text{Spin}_{1,3}(M, g)) \) we have
\[
\nabla_{e_a}^{(s)} (A \psi_{\Xi}) = \nabla_{e_a}^{(s)} (A \psi_{\Xi}) = A (\nabla_{e_a}^{(s)} \psi_{\Xi})
\]

### VI. DIRAC-HESTENES EQUATION ON RIEMANN-CARTAN SPACETIMES

Let \( (M, g, \nabla, \tau_g, \uparrow) \) be a Riemann-Cartan spacetime. The Dirac-Hestenes Lagrangian written for a representative \( \psi \in \text{sec} \mathcal{C}(M, g) \) in a given gauge of a Dirac-Hestenes spinor field \( \Psi \in \text{sec} \mathcal{C}(\text{Spin}_{1,3}(M, g)) \) reads...
\[ \mathcal{L}(x, \psi, \partial^{(s)}\psi) = \mathcal{L}(x, \psi, \partial^{(s)}\psi) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \]

\[ = \left( \partial^{(s)}\psi \theta^0 \theta^2 \theta^1 \cdot \psi - m \psi \cdot \psi \right) \sqrt{\det g} \right) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \] (67)

where \( \{x^\mu\} \) are the coordinate function of a local chart \((U, \varphi)\) of the maximal atlas of \(M\) and \(\partial^{(s)}\) the representative of the spin-Dirac operator in the Clifford bundle is given by:

\[ \partial^{(s)}\psi = \theta^a \nabla^{(s)}_{e_a} \psi = \theta^a \left( \partial_{e_a} \psi + \frac{1}{2} \omega_{e_a} \psi \right), \] (68)

with \( \theta^a = \hbar^a dx^a \). The variational principle used with the Lagrangian density (Eq. (67)) gives after some algebra [12]

\[ \partial^{(s)}\psi \theta^2 \theta^1 + \frac{1}{2} T \psi \theta^0 \theta^2 \theta^1 - m \psi \theta^0 = 0, \] (69)

where

\[ T = T_{ab} \theta^a. \] (70)

is called the torsion covector. Note that in a Lorentzian manifold \(T = 0\) and we obtain the Dirac-Hestenes equation on a Lorentzian manifold. We observe moreover that the matrix representation of Eq. (69) coincides with an equation first proposed by Hehl and Datta [13]. Eq. (69) is manifestly covariant under a passive gauge transformation as it is trivial to verify taking into account Eq. (64). We also recall that spinors transforms as scalars under diffeomorphism and thus it is easy to verify that the Dirac-Hestenes equation is invariant under diffeomorphisms.

We observe yet that, if we tried to get the equation of motion related to a Dirac-Hestenes spinor field on a Riemann-Cartan spacetime, directly from the equation on Minkowski spacetime by using the principle of minimal coupling, we would miss the term \( \frac{1}{2} T \psi \theta^2 \theta^1 \) appearing in Eq. (69). Is this a bad result? According to [13] the answer is yes, because there, a supposed complete theory, where the \( \{\theta^a\} \) and the \( \{\omega_{e_a}\} \) are dynamical fields, the spinor field generates torsion. To put more spice on this issue, let us next analyze what active Lorentz invariance would imply.

**VII. MEANING OF ACTIVE LORENTZ INVARIANCE OF THE DIRAC-HESTENES LAGRANGIAN**

In the proposed gauge theories of the gravitational field, it is said that the Lagrangians and the corresponding equations of motion of physical fields must be invariant under arbitrary active local Lorentz rotations. In this section we briefly investigate how to mathematically implement such an hypothesis and what is its meaning for the case of a Dirac-Hestenes spinor field on a Riemann-Cartan spacetime. The Lagrangian we shall investigate is the one given by Eq. (67), which we now write with all indices indicating the representative gauge (i.e., spin coframe)

\[ \mathcal{L}(x, \psi_\Xi, \partial^{(s)}\psi_\Xi) = \left( \theta^a \nabla^{(s)}_{e_a} \psi_\Xi \theta^0 \theta^2 \theta^1 \cdot \psi_\Xi - m \psi_\Xi \cdot \psi_\Xi \right) \sqrt{\det g}. \] (Dirac-Hestenes)

Observe that the Dirac-Hestenes Lagrangian has been written in a fixed (passive gauge) individualized by a spin coframe \( \Xi \) and we already know that it is invariant under passive gauge transformations \( \psi_\Xi \mapsto \psi_\Xi = \psi_\Xi R^{-1} (R \hat{R} = 1) \), \( R \in \text{sec Spin}^+_3(M) \hookrightarrow \text{sec Cl}(M, g) \) once the ‘connection’ 2-form \( \omega_V \) transforms as given in Eq. (65), i.e.,

\[ \frac{1}{2} \omega_V \mapsto R \frac{1}{2} \omega_V R^{-1} + (\nabla_V R) R^{-1}. \] (71)

Under an active rotation (gauge) transformation the fields transform in new fields given by

\[ \psi_\Xi \mapsto \psi'_\Xi = R \psi_\Xi, \quad \psi'_\Xi = R \psi_\Xi R^{-1} \]
\[ \theta^m \mapsto \theta'^m = R \theta^m R^{-1} = \Lambda^m_n \theta^n, \]
\[ e_m \mapsto e'^m = (\Lambda^{-1})^m_n e_n. \] (72)
Now, according to the mathematical ideas behind gauge theories, we must search for a new connection $\nabla'$ such that the Lagrangian results invariant. This will be the case if connections $\nabla$ and $\nabla'$ are representatives of a $G$-connection as introduced in [32].

\[ \nabla_{e_m}^{(s)}(R\psi_{\Sigma}) = R\nabla_{e_m}^{(s)}\psi_{\Sigma}, \quad (73) \]

or

\[ \nabla_{e_m}^{(s)}(R\psi_{\Sigma}) = \Lambda_m^n R\nabla_{e_m}^{(s)}\psi_{\Sigma}, \quad (74) \]

Also, taking into account the structure of a representative of a spinor covariant derivative in the Clifford bundle we may verify that in order for Eq. (74) to be satisfied we need that the Pfaff derivative transforms as

\[ \partial_{e_n} \rightarrow \partial_{e_n}' = \Lambda_m^n \partial_{e_m}, \quad (75) \]

and that the connection transforms as

\[ \omega_{e_n}' = \Lambda_m^n (R\omega_{e_m} R^{-1} - 2\partial_{e_m}(R) R^{-1}), \]

or

\[ \omega_{e_n}' = R\omega_{e_m} R^{-1} - 2\partial_{e_m}(R) R^{-1}. \quad (76) \]

Under these conditions we have:

\[ [(\theta^a \nabla_{e_a}^{(s)} \psi_{\Sigma}), \theta^b \theta^c \theta^d \theta^e \cdot \psi_{\Sigma} - m \psi_{\Sigma} \cdot \psi_{\Sigma}'] \sqrt{\det g}, \]

\[ = [(\theta^a \nabla_{e_a}^{(s)} \psi_{\Sigma}), \theta^b \theta^c \theta^d \theta^e \cdot \psi_{\Sigma} - m \psi_{\Sigma} \cdot \psi_{\Sigma}'] \sqrt{\det g}, \quad (77) \]

and we get

\[ \mathcal{L}(x, \psi_{\Sigma}, \mathcal{D}(s) \psi_{\Sigma}) = \mathcal{L}(x, \psi_{\Sigma}, \mathcal{D}(s) \psi_{\Sigma}). \quad (78) \]

Write now,

\[ \omega_{e_n}' = \frac{1}{2} \omega_{m}^{kl} \theta_k \wedge \theta_l = \frac{1}{2} \omega_{m}^{kl} \theta_k \wedge \theta_l \in \sec \mathfrak{C} \ell(M, g), \]

\[ \omega_{e_n} = \frac{1}{2} \omega_{m}^{kl} \theta_k \wedge \theta_l = \frac{1}{2} \omega_{m}^{kl} \theta_k \wedge \theta_l \in \sec \mathfrak{C} \ell(M, g), \]

\[ U = e^F, \quad F = \frac{1}{2} \lbrack \mathfrak{F}_{rs} \theta_{rs} \in \sec \mathfrak{C} \ell(M, g) \].

Recall that

\[ \omega_{n}^{rs} = \eta^{rs} \omega_{anb} \eta^{ab} = \omega_{nb}^{r} \eta^{ab}, \]

\[ \omega_{nk} = \omega_{n}^{rs} \eta_{nk}. \quad (80) \]

Then, from Eqs. (16), (29) and (80) we get

\[ \omega_{nk}^{r} = \Lambda_{q}^{b} \omega_{mb}^{r} \Lambda_{p}^{c} \Lambda_{m}^{k} - \eta_{nk} \Lambda_{n}^{m} \partial_{e_m} (F^{rs}). \quad (81) \]

Now, we recall that the components of the torsion tensors $T$ and $T'$ related to the (tensorial) connections $\nabla$ and $\nabla'$ in the orthonormal basis $\{e_r \otimes \theta^m \wedge \theta^k\}$ are given by

\[ T_{nk}^{r} = \omega_{nk}^{r} - \omega_{kn}^{r} - \epsilon_{nk}^{r}, \]

\[ T_{nk}'^{r} = \omega_{nk}'^{r} - \omega_{kn}'^{r} - \epsilon_{nk}'^{r}. \quad (82) \]

where $[e_n, e_k] = \epsilon_{nk}^{r} e_r$.

Let us suppose that we start with a torsion free connection $\nabla$. This means that $\epsilon_{nk}^{r} = \omega_{nk}^{r} - \omega_{kn}^{r}$. Then

\[ T_{nk}^{r} = \Lambda_{p}^{b} \Lambda_{q}^{a} \Lambda_{m}^{r} \Lambda_{mb}^{p} - \epsilon_{nk}^{r} - \partial_{e_m} (F_{rs}) \left[ \eta_{nk} \Lambda_{n}^{m} - \eta_{sn} \Lambda_{n}^{m} \right], \quad (83) \]

and we see that $T' = 0$ only for very particular gauge transformations.

We then conclude that to suppose the Dirac-Hestenes Lagrangian is invariant under active rotational gauge transformations implies in an equivalence between torsion free and non-torsion free connections. Note also that we may have equivalence between spacetimes with null and non-null curvatures, as it is easily to verify. It is always emphasized that in a theory where besides $\psi$, also the the tetrad fields $\theta^a$ and the connection $\omega$ are dynamical variables, the torsion is not zero, because its source is the spin of the $\psi$ field. Well, this is true in particular gauges, because as showed above it seems that it is always possible to find gauges where the torsion is null.
A. The Case of the Local Lorentz Invariance of the Electromagnetic Field Equations

If we are prepared to accept as equivalent spacetimes with different curvatures and torsion tensors then we can modify Maxwell equations in such a way that they are formally invariant under local Lorentz transformations. We start with Maxwell theory on a general Riemann-Cartan spacetime \((M, g, \nabla, \tau_g, \uparrow)\), where we propose that Maxwell equation is given by

\[ \partial F = J, \]

where \(F \in \text{sec} \Lambda^2 T^* M \hookrightarrow \text{sec} \mathcal{C}(M, g)\) and \(J \in \text{sec} \Lambda^1 T^* M \hookrightarrow \text{sec} \mathcal{C}(M, g)\) and \(\partial = \partial^a \nabla_a = d - \delta\) is the Dirac operator in a particular \((\text{fiducial} \) gauge) where the spacetime metric is a Lorentzian one.

Next we propose that \(F\) and all \(\text{RFR}^{-1}\) are gauge equivalent (in different but equivalent spacetime models \((M, g, \nabla, \tau_g, \uparrow)\) and \((M, g, \nabla', \tau_{g'}, \uparrow)'\)) and that the Dirac operator in \((M, g, \nabla', \tau_{g'}, \uparrow)'\) \(R \) is \(\tilde{\partial} = \partial^a \nabla_a'\), \(\partial^a = R\partial^a R^{-1}\), \(\partial^a(e^b_0) = \delta^a_b\) and \(R \in \text{Spin}_{1,3}(M) \subset \text{sec} \mathcal{C}(M, g)\). As we can easily verify with the formulas of the last section we have

\[ R \tilde{\partial} F = \tilde{J}, \]  \hspace{1cm} (84)

and we may say that distinct electromagnetic fields are also classified as distinct equivalence classes, where \(F\) and \(\tilde{F}\) represent the same field in different gauges.

Note finally that formally we may say that under a change of gauge model the Dirac operator transforms as

\[ \partial \rightarrow \tilde{\partial} = R \partial R^{-1}. \]  \hspace{1cm} (85)

Such an equation has been used by other authors in the past, but there, its clear mathematical meaning is lacking. In Appendix we present a context in which an equation like Eq. (85) makes its appearance.

A. APPENDIX: REPRESENTATION \((M, V, \bullet)\) OF MINKOWSKI SPACETIME AND MAXWELL EQUATIONS

In the affine structure \((M, V, \bullet)\) if an arbitrary event \(e_0\) in \(M\) is fixed, any other event \(e \in M\) can be represented by a vector \(x(e) \in V\). We write

\[ x(e) = e - e_0. \]  \hspace{1cm} (A.86)

Recall that \((V, \bullet) \simeq \mathbb{R}^{1,3}\). With the identification given by Eq. (A.86), Clifford fields of multivectors are now represented by mappings

\[ C : \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}^{1,3}. \]  \hspace{1cm} (A.87)

The constant vector fields \(e_0 = (1, 0, 0, 0), e_1 = (0, 1, 0, 0), e_2 = (0, 0, 1, 0), e_3 = (0, 0, 0, 1)\) define coordinates \(\{x^\mu\}\) in the Einstein-Lorentz gauge for a point event \(e\), i.e.,

\[ x(e) = x^\mu(e) e_\mu. \]  \hspace{1cm} (A.88)

Under an active Lorentz transformation, generated by \(R \in \text{Spin}_{1,3} \subset \mathbb{R}^{0,4}_{1,3}\) the position vector is mapped as \(x \rightarrow x'\). The correct interpretation is that \(x'\) is the position vector of a point event \(e' \neq e\). We have

\[ x(e') = e' - e_0 = Rx(e)\tilde{R}. \]  \hspace{1cm} (A.89)

Thus an active rotation is really a diffeomorphism (with a \textit{fixed point}, namely \(e_0\)) in \(M\). The action of \(R\) on a Clifford field, say, an electromagnetic field \(F(x(e)) \in \text{sec} \Lambda^2 \mathbb{R}^{1,3} \hookrightarrow \mathbb{R}^{1,3}\) must be interpreted as a mapping

\[ F(x(e)) \mapsto F'(x'(e')) = RF(x(e))\tilde{R}. \]  \hspace{1cm} (A.90)

From now on we omit the event labels \(e, e'\) when no confusion results. Consider a Lorentz boost

\[ R = \exp \left( \frac{\hbar}{2} e_1 e_0 \right) \]  \hspace{1cm} (A.91)
with \( \tanh \chi = ||\vec{v}|| \), \( \vec{v} = (v, 0, 0) \). We write
\[
\vec{x}' = R\vec{x}\tilde{R} = x^\mu \Re_{\mu} \tilde{R} = x^\mu e'_\mu = x^\mu e_\mu.
\] (A.92)

From Eq. (A.92) we see that the coordinates of the event \( e' \) with respect to \( \{e'_\mu\} \) are the same as the event \( e \) with respect to \( \{e_\mu\} \). Then the coordinates \( \{x^\mu\} \) and \( \{x'\mu\} \) are related by Eq. (A.91). Under these conditions, if \( F(x) \) is the field generated at \( \vec{x} \) by a charge at rest in the \( e_0 \) frame at position \( \vec{x} \), then \( F'(x') = RF(x)\tilde{R} \) is the field generated by a charge at rest in the \( e'_0 \) frame at the point \( \vec{x}' \). A trivial calculation shows that \( e_0 \) frame observers perceive (of course through measurements) the field \( F'(x') \) as the field generated by a charge moving in the positive \( x \)-direction with velocity \( \vec{v}' = (v, 0, 0) \).

In the formalism used in this section the Dirac operator is represented by the vector derivative \( \partial_\vec{x} \), such that
\[
\partial_\vec{x} x^\mu = e^\mu, \quad e^\mu \bullet e_\nu = \delta^\mu_\nu.
\] (A.93)

Hestenes [11] claims that any Lorentz transformation \( R \) sends \( \vec{v} \) into \( \vec{v}' = (v, 0, 0) \). Then, the coordinates \( \{x'\mu\} \) sent by \( R \) are related by Eq. (A.94). Under these conditions, if \( F(x) \) is the field generated at \( \vec{x} \) by a charge at rest in the \( e_0 \) frame at position \( \vec{x} \), then \( F'(x') = RF(x)\tilde{R} \) is the field generated by a charge at rest in the \( e'_0 \) frame at the point \( \vec{x}' \). A trivial calculation shows that \( e_0 \) frame observers perceive (of course through measurements) the field \( F'(x') \) as the field generated by a charge moving in the positive \( x \)-direction with velocity \( \vec{v}' = (v, 0, 0) \).

Then, if \( \partial_\vec{x} F(x) = J(x) \), we have
\[
\partial_\vec{x} F'(x') = J'(x').
\] (A.96)

We now know which is the mathematical meaning of the operator \( \partial_\vec{x}' \) satisfying Eq. (A.94). It has been given by our theory.

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[15] S. Weinberg, Gravitation and Cosmology, J. Wiley & Sons, Inc., New York 1972.
[16] Another popular representation of Minkowski spacetime is the structure \( (M, V, \bullet) \) where \( M = (M, V) \) is an affine space and \( V \simeq \mathbb{R}^4 \) is a vector space, endowed with a Lorentzian scalar product \( \bullet \) of signature \((1, 3)\) and which is oriented and time-oriented. This will be seen in the Appendix.
[17] This will be used in the Appendix.
[18] Details on the meaning of this statement can be found, e.g., in [2].
[19] The definition of reference frame in general, and inertial reference frame in particular will be given below.
For details, see, e.g., [3, 4].

See, e.g., [5], for a rigorous definition of Clifford and spinor fields.

Of course, the Clifford algebra of multiforms associated with a Minkowski spacetime will be denoted by $\text{Cl}(M, \eta)$, where $\eta$ is the metric of the cotangent bundle.

Please do not confuse this concept with the concept of a frame, that is a section of the frame bundle.

Inertial reference frames did not exist in general in an arbitrary Lorentzian spacetime, see, e.g., [1, 6].

See [2] for details of the concept of coordinates naturally adapted to a given general frame $Z$.

The $m_i \in \mathbb{R}^+$ are the masses and the $q_i \in \mathbb{R} - \{0\}$ are the charges of the particles.

Observe that for that boosts, $(L^{-1})^\dagger = L^{-1}$, where $\dagger$ means transpose.

Note that we also have $\gamma'^\mu = R^\mu_\nu \gamma^\nu, \gamma'^\mu = dx'^\mu = \partial x'^\mu, \gamma^\mu = dx^\mu = \partial x^\mu$, etc.

Note that $\gamma'^5 = \gamma^5$.

Note that $\nabla^s$ and $\nabla'^s$ are connection in the spin-Clifford bundle of DHSF whereas $\nabla^{(s)}$ and $\nabla'^{(s)}$ the effective covariant derivative operators acting on the representatives of DHSF in the Clifford bundle.

We note that in [14], authors choose a different approach to the transformations of $\partial_x, F$, etc. Analysis of the meaning of the transformed fields in that case is more difficult, and will be not discussed here.