Noise resistance of the violation of local causality for pure three-qutrit entangled states

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Abstract
Bell’s theorem started with two qubits (spins 1/2). It is a ‘no-go’ statement on classical (local causal) models of quantum correlations. After 25 years, it turned out that for three qubits the situation is even more astonishing. General statements concerning higher dimensional systems, qutrits, etc, started to appear even later, once the picture with spin (higher than 1/2) was replaced by a broader one, allowing all possible observables. This work is a continuation of the Gdansk effort to take advantage of the fact that Bell’s theorem can be put in the form of a linear programming problem, which in turn can be translated into a computer code. Our results are numerical and classify the strength of the violation of local causality by various families of three-qutrit states, as measured by the resistance to noise. This is previously uncharted territory. The results may be helpful in suggesting which three-qutrit states will be handy for applications in quantum information protocols. One of the surprises is that the W state turns out to reveal a stronger violation of local causality than the GHZ (Greenberger–Horne–Zeilinger) state.

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(Some figures may appear in colour only in the online journal)
1. Introduction

Bell’s theorem [1], celebrated in this special issue, is one of the great steps in our understanding of the physics of the microworld. It signals a much greater departure from the classical picture than the uncertainty relations (note that, e.g., frequency–time uncertainty holds for classical waves). The non-commutative nature of quantum mechanics and the collapse postulate clearly depart from the classical description, but where are the phenomena which directly show the impossibility of a return to a picture suggested by classical mechanics and field theory? Well, this is the Bell theorem for correlations of entangled systems. The theorem tells us that in such a case we cannot have a local causal model obeying all rules of classical (Kolmogorov) probability theory1. Bell’s theorem describes a mathematical property of the quantum mechanical description of nature. However, the universal form of Bell inequalities, in the modern parlance a device independent one, allows one to ask whether nature itself is locally causal. Experiments point to a negative answer, but the matter is still not settled (for a review of photonic experiments, see [2]). Once a single loophole-free experiment is done, the fundamental question will be answered. Still Bell inequalities have a utilitarian aspect. They point to strictly nonclassical phenomena which, as they are impossible to realize with classical means, can lead to new practical applications. Thus apart from their fundamental importance, the continued studies related to Bell’s theorem constitute a search for new nonclassical phenomena, states, and quantum tools for future technologies.

This work will study an important characteristic of entangled states—noise resistance of the violation of local causality. The quantum correlations are fragile, and can be totally blotted out by noise, which is inevitable in any experiment, and thus in all applications. Therefore, one should have estimates of some critical noise parameters to see the usefulness of quantum correlations characteristic for a given state. The states which we shall analyze here go beyond the standard Bell theorem. They are three-particle (subsystem) ones, and they are for systems more complicated than qubits. Such states allow new applications, impossible with the two-qubit entanglement of the pioneering works (see e.g. [3]).

Surprisingly, the first paper showing the extraordinary features of three or more qubits, in a maximally entangled state, appeared 25 years after Bell’s work [4]. For more than a generation, nobody cared to investigate this new realm (with an exception of [5]; however the striking properties of perfect correlations were left undiscovered). Correlations of three or more particles lead to new possibilities in applications, like three-party cryptography, with the third party holding a key to the key of the other two (‘secret sharing’) [6], and the reduction of the communication complexity of some computational tasks [7, 8].

Studies of higher dimensional systems also had a slow start. This was due to the fact that earlier investigations treated such systems as spins higher than 1/2, and employed only observables which are components of the spin operators. Thus, not all unitary transformations between measurement bases were allowed. It is perhaps even more important that the spin components are observables with classical counterparts. Introduction to the discussion of new observables, in the form of multiport interferometers [9], and subsequent generalizations [10, 11], allowed it to be shown that the violation of the Bell inequalities is stronger for higher dimensional systems than for qubits [12] (this work was numerical; an analytic reproduction of the results was given in [13, 14]). This discovery was a signature of possible

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1 There are no tools in quantum mechanics other than quantum states, observables, the principle of unitary evolution and the Born probabilistic rule using all of these. Thus, for example, if a single photon impinges on a 50/50 beam splitter, there is no direct cause in the formalism telling one whether it will be reflected or transmitted. Adding to the description any new parameters outside of quantum mechanics as causes of this or that behavior is in effect the introduction of hidden variables (hidden, as they are not present in the quantum formalism).
practical advantages related to higher dimensional systems (e.g., higher security in quantum cryptography [15]). Noise resistance of the violation of local causality by quantum correlations was introduced in [12] to quantify the strength of violation of local causality by quantum correlations. Previously, in various works, authors were using such parameters as violation factors and numerical differences between a Bell bound and the quantum value. However, such measures do not allow comparison between different Bell inequalities, or versions of the Bell theorem which are not based directly on them, but rather on numerical methods based on linear programming, like in [12]. Notice, as an illustrative example, that any violation of the Clauser–Horne [16] inequality is always by an infinite factor (as its local causal bound is zero).

The noise resistance depends on the model used. The simplest one is a ‘white noise’ admixture; see later. Generally, one can define a ‘critical visibility’ [12] as the value of the parameter \( v \), nonnegative, less than or equal to 1, for which, under a fixed set of conditions (like the number of settings, and kinds of observables), a mixed state defined by

\[
\rho = v\rho_{\text{state}} + (1 - v)\rho_{\text{noise}}
\]

loses the uniquely quantum properties of the original state \( \rho_{\text{state}} \) (the symbol \( \rho_{\text{noise}} \) stands for the noise admixed with the state; it is a form of a certain separable density operator, the actual form of which depends on the chosen noise model).

In this work we attempt to obtain noise resistance values for three-qutrit entangled states. This is done using numerical methods, as in [12]. Only some families are analyzed. This is due to the fact that, despite our efforts (for a computer science outline of the current upgraded numerical method, see [17]), the machine time taken to perform the numerical calculations is quite long.

Let us briefly sketch related earlier developments. Acin et al constructed a tight Bell inequality of a three-qutrit system [18]. They showed that the maximal violation is for a maximally entangled GHZ state, with two measurements for each observer defined by three-port beam splitters (tritters) [11]. Moreover, the inequality reproduced a numerical result for the resistance to white noise, with critical visibility \( v_{\text{crit}} = 0.6 \), which was obtained by numerical methods in [19]. In [20], Deng et al studied the quantum violations of three-qutrit GHZ states by using their Bell-type inequality, based on Svetlichny’s ideas [5]. In the case of maximally entangled GHZ states, no violation was reported. In [21] a set of Bell inequalities was derived for three-qutrit systems involving three measurement settings for each subsystem. We have checked that the critical visibility for the inequalities is about 0.77 (all value quotations are for white noise). Grandjean et al suggested a family of Bell inequalities for tripartite and high dimensional systems with two measurement settings [22]. For the two-dimensional case, they can be reduced to the Mermin–Bell [23] inequality, and merge with the CGLMP inequality [14] for the bipartite case. Numerical results in terms of the critical visibility to white noise were shown for several tripartite states, such as a maximally entangled GHZ state, a W state, and the three-qutrit singlet state (also known as the Aharonov state). The critical visibility for the GHZ state was 0.75, while for the three-qutrit singlet state, it was 0.7846 [22].

2. A description of the method

In our numerical analysis (called STEAM ROLLER) we consider some classes of pure states of three qutrits. Three spatially separated observers perform measurements of \( m \) alternative local three-valued observables: \( A_1, A_2, \ldots, A_m \) for Alice, \( B_1, B_2, \ldots, B_m \) for Bob, and \( C_1, C_2, \ldots, C_m \) for Charlie (\( m = 2, 3 \)). We assume that they measure observables which belong to the following families:
(i) U(3) measurements—by this we mean the full family of observables, as defined by orthogonal projectors (the symbol U(3) signifies the fact that all such observables are linked by U(3)-type unitary transformations).

(ii) T—denoting a subset of the above observables which are defined by unitary transformations, starting from the computational basis, which use one unbiased three-port beam splitter (tritter; see [11]) performing a unitary transformation given by the matrix $U_{kl} = e^{2i k l / \sqrt{3}}$, and a set of phase shifters in front of it. This class was singled out, as it played an important role in breaking the limitations linked with spin observables, discussed in the introduction. Also, such observables are rapidly becoming more feasible using the modern methods of integrated optics [24].

By saying that an experiment is locally causal (realistic), we understand that it has a local realistic model, which is equivalent to the existence of a joint probability distribution $p_{lr}(a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_m)$, where $a_i(b_j, c_k) = \{0, 1, 2\}$ denotes the results of the measurement of Alice’s (Bob’s, Charlie’s) $i$th observable. Quantum predictions for the probabilities should be given, if the model exists, by the marginal sums:

$$P(a_i, b_j, c_k|A_i, B_j, C_k) = \sum_{a_i', b_j', c_k'=0} p_{lr}(a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_m),$$

where $P(a_i, b_j, c_k|A_i, B_j, C_k)$ denotes the probability of obtaining the result $a_i$ for Alice, $b_j$ for Bob and $c_k$ for Charlie if they measure observables $A_i$, $B_j$ and $C_k$, respectively, and $i' \neq i$, $j' \neq j$, $k' \neq k$.

If we admix with the three-qutrit state the ‘white’ noise, in the form of a completely random process, the quantum probabilities $P_v(a_i, b_j, c_k|A_i, B_j, C_k)$ are given by

$$P_v(a_i, b_j, c_k|A_i, B_j, C_k) = v P(a_i, b_j, c_k|A_i, B_j, C_k) + \frac{1 - v}{27},$$

where $P(a_i, b_j, c_k|A_i, B_j, C_k)$ is the quantum prediction for the state without the noise admixture. For such noisy states there always exists the critical parameter $v_{\text{crit}}$, such that for $v < v_{\text{crit}}$ there exists a local realistic joint probability distribution $p_{lr}$ which satisfies the set of constraints (2). Our task, for a given $\rho$, is to find the critical parameter $v_{\text{crit}}$. This can be done by means of linear programming.

The numerical procedure was first used in [25], and further developed in [12], to show that violation of local realism is stronger (more white noise resistant) for two qubits than for two qutrits and that this increases with $N$. Later the method was also used for the analysis of violations of local realism by two qubits in all possible pure entangled states [26] and different kinds of qubit quantum states, which are often discussed in the context of quantum information [27]. The method is described in details in [17, 27].

3. Results

We applied the numerical method to different kinds of three-qutrit quantum states, namely, the generalized GHZ state, a family of the Dicke states and the singlet state. The critical visibility for two and three settings per side was determined. Selected results are presented below.

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2 In our analysis we also consider a product noise admixture described in section 3.1.

3 The optimal measurement settings for the most important examples are shown in [28].
3.1. Generalized GHZ states

We considered the following class of pure states of three qutrits (generalized GHZ states, see figure 1):

$$|\psi(\alpha)\rangle = \cos \alpha |000\rangle + \frac{1}{\sqrt{2}} \sin \alpha (|111\rangle + |222\rangle).$$

(4)

We have found a range of the $\alpha$ parameter of (4), for which the critical visibility is lower than the lowest previously known critical visibility for white noise (0.43) for three-qutrit states, given in [19]. The lowest critical visibility resulting from our analysis is now 0.4256, and corresponds to the rank-2 state $|\psi(90^\circ)\rangle$. A similar effect was observed for two qutrits and reported in [26, 29]. This phenomenon is visible if one uses the most general local three-dimensional observables [U(3)], whereas for the observables based on transformations T it is unobservable.

The lowest critical visibility for transformations T only and two measurement settings per side (0.5931) is achieved for an asymmetric state $|\psi_{\text{asym}}\rangle = |\psi(\sim 50^\circ)\rangle$. For the symmetric state $|\psi_{\text{sym}}\rangle = |\psi(54.74^\circ)\rangle = \frac{1}{\sqrt{3}} (|000\rangle + |111\rangle + |222\rangle)$, the critical visibility is equal to 0.6000 as in [18, 19]. Surprisingly, for three measurement settings per side the plot of the critical visibility as a function of $\alpha$ becomes more symmetric. The minimal critical visibility (0.5000) is achieved by the symmetric proper three-qutrit GHZ state $|\psi_{\text{sym}}\rangle$. This is exactly the same as the well known value for the case of a three-qubit GHZ state, and two settings per party.

There are some ranges of the parameter $\alpha$ for which the observables generated by U(3) give a critical visibility for three measurement settings per side better than the one obtained for two measurement settings ($46^\circ \leq \alpha \leq 58^\circ$). An example is provided by the state $|\psi_{\text{sym}}\rangle$, for which the critical visibilities for two and three measurement settings are equal to 0.5264 and 0.5000, respectively.

In the case of the two-qutrit GHZ state, upon increasing the number of settings to 3 one cannot observe reduction of the critical visibility [26].

![Figure 1](image-url)
Figure 2. The critical visibilities \( v_{\text{crit}} \) for the states \( |\psi(\alpha)\rangle \) for the case in which all observers use any observables (U(3) case) and two measurement settings. The dotted line (blue) corresponds to the product noise admixture and it is compared with the results for white noise (dashed, black). Note the ‘singular’ point for 90° with visibility 1/2, for the product noise. This is for the rank-2 state (effectively, a three-qubit GHZ state). Also note that in the case of a perfect three-qutrit GHZ state, the product noise and the white noise are described by identical operators. This is reflected in the figure by the mid-crossing point of the product and white noise curves. The other crossing points seem accidental.

We have also studied a product noise admixture of the form \( v\rho + (1 - v)\rho_A \otimes \rho_B \otimes \rho_C \), where \( \rho_i \) is the reduced density matrix of system \( i \). The results, compared with the white noise ones, are shown in figure 2. Note that the state \( |\psi(0)\rangle \) (product state) does not violate local realism, whereas for \( \alpha \) close to 0 the state (4) violates local realism for visibility approaching 0. A situation of a different kind is that for \( \alpha = 90^\circ \). The critical visibility for the rank-2 state \( |\psi(90^\circ)\rangle \) is equal to 0.5 and immediately drops to almost 0 in the neighborhood of 90°.

3.2. Dicke states

We also tested a family of the three-qutrit Dicke states:

\[
|D_{13}^0\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle),
\]

\[
|D_{13}^2\rangle = \frac{1}{\sqrt{15}} (|002\rangle + |020\rangle + |200\rangle + 2(|011\rangle + |101\rangle + |110\rangle)),
\]

\[
|D_{13}^3\rangle = \frac{1}{\sqrt{10}} (|012\rangle + |021\rangle + |102\rangle + |120\rangle + |201\rangle + |210\rangle + 2|111\rangle).
\]

The results for U(3) measurements are presented in table 1. The most robust quantum properties are observed for the state \( |D_{13}^0\rangle \), which is simply the qubit-like W state. The state is more robust against white noise than the symmetric GHZ state. Moreover, the state violates local realism for any admixture of the product noise. Also, the qutrit Dicke states are more noise resistant than the three-qubit W state, for which the two-setting value is 0.6442, while the three-setting one is 0.6330. But the most surprising result is that the quantum nature of the correlations of the state \( |D_{13}^0\rangle \) is totally resistant to the product noise (critical visibility zero), whereas for a three-qubit W state one has 0.662 (two settings per observer) and 0.660 (three settings).
Table 1. Critical visibilities for the three-qutrit Dicke states, two and three measurement settings, and white and product noise admixtures.

| State | White noise | Product noise |
|-------|-------------|---------------|
|       | Two settings | Three settings | Two settings | Three settings |
| $|D_1^3\rangle$ | 0.4993 | 0.4866 | 0.0 | 0.0 |
| $|D_2^3\rangle$ | 0.5363 | 0.5285 | 0.5925 | 0.5894 |
| $|D_3^3\rangle$ | 0.5338 | 0.5108 | 0.5409 | 0.5409 |

3.3. The singlet state

Finally, we analyzed the three-qutrit Aharonov singlet state [30]:

$$|\text{singlet}\rangle = \frac{1}{\sqrt{6}} (|012\rangle - |021\rangle - |102\rangle + |120\rangle + |201\rangle - |210\rangle).$$

Surprisingly, the critical visibilities obtained for all U(3)-type observables for white and product noise are the same and read 0.6456 for two measurement settings and 0.6219 for three settings.

4. Conclusions

The numerical method presented here once more proved to be a useful tool for seeking strictly quantum correlations in situations in which the full set of tight Bell inequalities is either unknown, or is too vast to be useful. The values of the critical visibility obtained are either the best so far, or apply to previously unstudied cases. Note that the numerical code which we used is equivalent to the full set of tight Bell inequalities for the given problem. The Bell inequalities that are violated by the states are not specified. They can be obtained by using the method of [33].

We observe that in many cases, allowing three settings implies a lower threshold visibility. Some results are very surprising, like the full noise resistance of the Dicke W state, in the case of product noise. Also this state outperforms the three-qutrit GHZ state as regards resistance with respect to white noise. However this should not be upsetting news for the authors of the GHZ paper, as they were the first ones to discuss W states [31] in the context of quantum correlations. Note that an earlier result showed that three-qubit W states outperform the original GHZ ones as regards white noise resistance if the number of subsystems is greater than 10 [32].

All of these results suggest that in the case of three-qutrit states it might be the case that possible applications in quantum information protocols could be most exciting in the case of the W state.

The fact that the rank-2 GHZ state ($\alpha = 90^\circ$) has the highest resistance to white noise needs a closer look. Note that the state is effectively a three-qubit one. If one uses observables $T$ which have eigenvectors which are unbiased with respect to the local computational bases, all of that disappears; such states have classical correlations. The high white noise resistance stems from the fact that part of the noise affects only a ‘blind’ third detector, which is in the case of zero noise not registering any counts, because the observables, for which we get this low noise value are such that their non-trivial eigenstates are superpositions of $|1\rangle$ and $|2\rangle$, while the third one is always $|0\rangle$ (we have effectively a two-qubit coincidence interference).
That is, the effect seems to be due to the fact that white noise is a good model of ‘dark counts’, rather than noise due to decoherence or distortion in the channels leading to the detectors.

Note also the peculiar behavior of the resistance to product noise of generalized GHZ states, which are very close to the ‘three-qubit’ one; see figure 2 and its caption. If $\alpha$ is less than $90^\circ$ but very close to $90^\circ$ (up to our numerical accuracy), we have an almost perfect resistance to the product noise. Suddenly, for $\alpha = 90^\circ$ one has the critical value of $0.5000$ (almost as in the case of the three-qutrit GHZ state $|\psi_{\text{sym}}\rangle$, $\nu_{\text{crit}} = 0.5263$). This means that different tight Bell inequalities are optimal for the ‘three-qubit’ GHZ state and for the irreducibly three-qutrit states very close to it. This phenomenon, as the product noise could in a way model a distortion in channels, might be useful in transmitting the GHZ correlations.

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