ABELIAN IDEALS OF A BOREL SUBALGEBRA AND LONG POSITIVE ROOTS

DMITRI I. PANYUSHEV

Independent University of Moscow, Bol’shoi Vlasevskii per. 11
121002 Moscow, Russia
e-mail: panyush@mccme.ru

Let \( b \) be a Borel subalgebra of a simple Lie algebra \( g \). Let \( \mathfrak{Ab} \) denote the set of all Abelian ideals of \( b \). It is easily seen that any \( a \in \mathfrak{Ab} \) is actually contained in the nilpotent radical of \( b \). Therefore \( a \) is determined by the corresponding set of roots. More precisely, let \( t \) be a Cartan subalgebra of \( g \) lying in \( b \) and let \( \Delta \) be the root system of the pair \((g,t)\). Choose \( \Delta^+ \), the system of positive roots, so that the roots of \( b \) are positive. Then \( a = \oplus_{\gamma \in \mathfrak{I}_g^\gamma} g_\gamma \), where \( \mathfrak{I}_g^\gamma \) is a suitable subset of \( \Delta^+ \) and \( g_\gamma \) is the root space for \( \gamma \in \Delta^+ \). It follows that there are finitely many Abelian ideals and that any question concerning Abelian ideals can be stated in terms of combinatorics of the root system.

An amazing result of D. Peterson says that the cardinality of \( \mathfrak{Ab} \) is \( 2^{rk g} \). His approach uses a one-to-one correspondence between the Abelian ideals and the so-called ‘minuscule’ elements of the affine Weyl group \( \hat{W} \). An exposition of Peterson’s results is found in [5]. Peterson’s work appeared to be the point of departure for active recent investigations of Abelian ideals, ad-nilpotent ideals, and related problems of representation theory and combinatorics [1],[2],[3],[4],[5],[6],[7],[8]. We consider \( \mathfrak{Ab} \) as poset with respect to inclusion, the zero ideal being the unique minimal element of \( \mathfrak{Ab} \). Our goal is to study this poset structure. It is easily seen that \( \mathfrak{Ab} \) is a ranked poset; the rank function attaches to an ideal its dimension.

It was shown in [8] that there is a one-to-one correspondence between the maximal Abelian ideals and the long simple roots of \( g \). (For each simple Lie algebra, the maximal Abelian ideals were determined in [10].) This correspondence possesses a number of nice properties, but the very existence of it was demonstrated in a case-by-case fashion. Here we give a conceptual explanation of that empirical observation. More generally, we prove that

- there is a natural mapping \( \tau : \mathfrak{Ab} \to \Delta^+_I \), where \( \mathfrak{Ab} \) is the set of all nontrivial Abelian ideals and \( \Delta^+_I \) is the set of long positive roots, see Proposition 2.4. We say that \( \tau(I) \) is the rootlet of \( I \);
- Each fibre \( \mathfrak{Ab}_\mu := \tau^{-1}(\mu) \) is a poset in its own right, and we prove that \( \mathfrak{Ab}_\mu \) contains a unique maximal and a unique minimal element, see Theorem 3.1.
- If \( I \) is a maximal Abelian ideal, then \( \tau(I) \) is a (long) simple root. Restricting \( \tau \) to \( \mathfrak{Ab}_{max} \), the set of maximal Abelian ideals, yields the above correspondence;

This research was supported in part by the Alexander von Humboldt-Stiftung and RFBI Grant no. 01–01–00756.
The uniqueness of maximal and minimal elements suggests that they can have a nice description. For any $\mu \in \Delta^+_f$, we explicitly describe the minimal ideal in $\mathfrak{Ab}_\mu$ and the corresponding minuscule element of $\widehat{W}$ (Theorem 4.2). Let $I(\mu)_{\text{min}}$ denote the minimal element of $\mathfrak{Ab}_\mu$.

The collection of these ideals has a transparent characterisation: Given $I \in \mathfrak{Ab}$, we have $I = I(\mu)_{\text{min}}$ for some $\mu$ if and only if all roots of $I$ are not orthogonal to $\theta$, the highest root (see Theorem 4.3). We also determine the generators of the ideals $I(\mu)_{\text{min}}$.

In Section 5, the structure of posets $\mathfrak{Ab}_\mu$ is considered. It is shown that $\#(\mathfrak{Ab}_\mu) > 1$ if and only if $(\mu, \theta) = 0$. A criterion is also given for $\#(\mathfrak{Ab}_\mu) > 2$. In fact, I can give a general description of $\mathfrak{Ab}_\mu$ and, in particular, of the maximal element $I(\mu)_{\text{max}} \in \mathfrak{Ab}_\mu$. This description is in accordance with (actually, is inspired by) my computations for all simple Lie algebras, but I cannot give yet a general case-free proof. This description shows that any $\mathfrak{Ab}_\mu$ is isomorphic to the poset of all ideals sitting inside of an Abelian nilpotent radical. More precisely, there are a regular simple subalgebra $\mathfrak{g}(\mu) \subset \mathfrak{g}$ and a maximal parabolic subalgebra $\mathfrak{p}(\mu) \subset \mathfrak{g}(\mu)$ with Abelian nilpotent radical $\mathfrak{p}^{\text{nil}}(\mu)$ such that $\mathfrak{Ab}_\mu$ is isomorphic to the poset of all Abelian $\mathfrak{b}(\mu)$-ideals in $\mathfrak{p}^{\text{nil}}(\mu)$, see Section 3 for details. As is well-known, the latter is isomorphic to the weight poset of a fundamental representation of the Langlands dual Lie algebra $\mathfrak{g}(\mu)^\vee$ [9, 12]. Since this fundamental representation is minuscule, the weight poset of it is isomorphic to the Bruhat poset $W(\mu)/W_\varphi(\mu)$. Here $W(\mu)$ is the Weyl group of $\mathfrak{g}(\mu)$ (or $\mathfrak{g}(\mu)^\vee$) and $W_\varphi(\mu)$ is the stabilizer of the fundamental weight in question. Such posets are also called minuscule. This completely solves the problem of describing the structure of $\mathfrak{Ab}_\mu$.

In Section 6, the general theory developed so far is illustrated with examples related to all simple Lie algebras. We compute $\#(\mathfrak{Ab}_\mu)$ for each $\mu \in \Delta^+$. For $\mathfrak{sl}_n, \mathfrak{sp}_{2n}, G_2,$ and $F_4$, an explicit description of the posets $\mathfrak{Ab}_\mu$ is given. In case of $\mathfrak{sl}_n$, an algorithm is presented for writing out the minuscule element corresponding to an Abelian ideal.

Our proofs are based on the relationship between the Abelian ideals and the minuscule elements in the affine Weyl group. We repeatedly use the procedure of extension of Abelian ideals that follows from this relationship.

1. Preliminaries on Abelian ideals

(1.1) Main notation. $\Delta$ is the root system of $(\mathfrak{g}, \mathfrak{t})$ and $W$ is the usual Weyl group. For $\alpha \in \Delta$, $\mathfrak{g}_\alpha$ is the corresponding root space in $\mathfrak{g}$.

$\Delta^+$ is the set of positive roots and $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$.

$\Pi = \{\alpha_1, \ldots, \alpha_p\}$ is the set of simple roots in $\Delta^+$.

We set $V := t_\mathbb{Q} = \bigoplus_{i=1}^p \mathbb{Q} \alpha_i$ and denote by $(\ , \ )$ a $W$-invariant inner product on $V$. As usual, $\mu^\vee = 2\mu/(\mu, \mu)$ is the coroot for $\mu \in \Delta$. Letting $\widehat{V} = V \oplus \mathbb{Q} \delta \oplus \mathbb{Q} \lambda$, we extend the inner product $(\ , \ )$ on $\widehat{V}$ so that $(\delta, V) = (\lambda, V) = (\delta, \delta) = (\lambda, \lambda) = 0$ and $(\delta, \lambda) = 1$.

$\widehat{\Delta} = \{\Delta + k\delta \mid k \in \mathbb{Z}\}$ is the set of affine real roots and $\widehat{W}$ is the affine Weyl group.

Then $\widehat{\Delta}^+ = \Delta^+ \cup \{\Delta + k\delta \mid k \geq 1\}$ is the set of positive affine roots and $\widehat{\Pi} = \Pi \cup \{\alpha_0\}$ is

---

1this means that the subalgebra is normalized by $\mathfrak{t}$
the corresponding set of affine simple roots. Here \( \alpha_0 = \delta - \theta \), where \( \theta \) is the highest root in \( \Delta^+ \). The inner product \( (\ , \ ) \) on \( \hat{V} \) is \( \hat{W} \)-invariant.

For \( \alpha_i \ (0 \leq i \leq p) \), we let \( s_i \) denote the corresponding simple reflection in \( \hat{W} \). If the index of \( \alpha \in \hat{\Pi} \) is not specified, then we merely write \( s_\alpha \). The length function on \( \hat{W} \) with respect to \( s_0, s_1, \ldots, s_p \) is denoted by \( l \). For any \( w \in \hat{W} \), we set

\[
\hat{N}(w) = \{ \alpha \in \hat{\Delta}^+ | w(\alpha) \in -\hat{\Delta}^+ \}.
\]

If \( w \in W \), then \( \hat{N}(w) \subset \Delta^+ \) and we also write \( N(w) = \hat{N}(w) \) in this case.

**Abelian ideals.** Let \( a \subset b \) be an Abelian ideal. It is easily seen that \( a \subset [b, b] \). Therefore \( a = \bigoplus_{\alpha \in I} g_\alpha \) for a subset \( I \subset \Delta^+ \), which is called the set of roots of \( a \).

As our exposition will be mostly combinatorial, an Abelian ideal will be identified with the respective set of roots. That is, \( I \) is said to be an Abelian ideal, too. Whenever we want to explicitly indicate the context, we say that \( a \) is a geometric Abelian ideal, while \( I \) is a combinatorial Abelian ideal. In the combinatorial context, the definition of an Abelian ideal (subalgebra) can be stated as follows.

\( I \subset \Delta^+ \) is an Abelian ideal, if the following two conditions are satisfied:

(a) for any \( \mu, \nu \in I \), we have \( \mu + \nu \notin \Delta \);

(b) if \( \gamma \in I \), \( \nu \in \Delta^+ \), and \( \gamma + \nu \in \Delta \), then \( \gamma + \nu \in I \).

If \( I \) satisfies only (a), then it is called an Abelian subalgebra.

Following D. Peterson, an element \( w \in \hat{W} \) is said to be minuscule, if \( \hat{N}(w) \) is of the form \( \{ \delta - \gamma \mid \gamma \in I \} \), where \( I \) is a subset of \( \Delta^+ \). It was shown by Peterson that such an \( I \) is a combinatorial Abelian ideal and, conversely, each Abelian ideal occurs in this way, see [2, Prop. 2.8], [5]. Hence one obtains a one-to-one correspondence between the Abelian ideals of \( b \) and the minuscule elements of \( \hat{W} \). If \( w \in \hat{W} \) is minuscule, then \( I_w \) (resp. \( a_w \)) is the corresponding combinatorial (resp. geometric) Abelian ideal. That is,

\[
I_w = \{ \gamma \in \Delta^+ | \delta - \gamma \in \hat{N}(w) \} \quad \text{and} \quad a_w = \bigoplus_{\alpha \in I_w} g_\alpha.
\]

Conversely, given \( I \in \mathfrak{Ab} \), we write \( w(I) \) for the respective minuscule element. Notice that

\[
\dim a_w = \#(I_w) = l(w).
\]

Accordingly, being in combinatorial (resp. geometric) context, we speak about cardinality (resp. dimension) of an ideal. Throughout the paper, \( I \) or \( I_w \) stands for a combinatorial Abelian ideal.

### 2. Generators of Abelian ideals and long positive roots

Given an Abelian ideal \( I \), let us say that \( \gamma \in I \) is a generator of \( I \), if \( \gamma - \alpha \notin I \) for all \( \alpha \in \Delta^+ \). Clearly, this is equivalent to the fact that \( I \setminus \{ \gamma \} \) is still an Abelian ideal. Conversely, if \( \kappa \) is a maximal element of \( \Delta^+ \setminus I \) (i.e., \( (\kappa + \Delta^+) \cap \Delta \subset I \)) and \( (\kappa + I) \cap \Delta = \emptyset \), then \( I \cup \{ \kappa \} \) is an Abelian ideal. These two procedures show that the following is true.
2.1 Proposition. Suppose $I \subset J$ are two Abelian ideals. then there is a chain of Abelian ideals $I = I_0 \subset I_1 \subset \ldots \subset I_m = J$ such that $#(I_{i+1}) = #(I_i) + 1$. In other words, $\mathfrak{A}b$ is a ranked poset, with cardinality (dimension) of an ideal as the rank function.

In the geometric setting, the set of generators has the following description. For an ideal $a = \oplus_{\gamma \in I} \mathfrak{g}_\gamma \subset b$, there is a unique $t$-stable space $\hat{a} \subset a$ such that $a = [b, a] \oplus \hat{a}$. Then $\gamma$ is a generator of $I$ if and only if it is a weight of $\hat{a}$.

However, we need a description of generators of $I$. For example, in the geometric setting, the set of generators has the following description. For an ideal $a$, we can write $a = \oplus_{\gamma \in I} \mathfrak{g}_\gamma \subset b$, where $\mathfrak{g}_\gamma$ is an Abelian ideal. Then $\gamma$ is a generator of $I$ if and only if it is a weight of $\hat{a}$.

2.2 Theorem. Suppose $\gamma \in I_w$. Then $\gamma$ is a generator of $I_w$ if and only if $w(\delta - \gamma) \in -\hat{W}$.

Proof. “$\Rightarrow$”. Suppose $w(\delta - \gamma) = -\alpha_i$. This means that $w^{-1}(\alpha_i) = \gamma - \delta < 0$. Therefore there exists a reduced decomposition of $w$ starting with $s_i$: $w = s_i w'$, where $l(w') = l(w) - 1$. Hence $\hat{N}(w') = \hat{N}(w) \setminus \{\delta - \gamma\}$ and $w'$ is still a minuscule element. Thus, $I_w \setminus \{\gamma\}$ is an Abelian ideal.

“$\Leftarrow$”. Suppose $w(\delta - \gamma) \notin -\hat{W}$, i.e. $w(\delta - \gamma) = -\kappa_1 - \kappa_2$, where $\kappa_i \in \hat{\Delta}^+$. Then $w^{-1}(\kappa_1) + w^{-1}(\kappa_2) = -\delta - \gamma < 0$. Assume for definiteness that $w^{-1}(\kappa_2) < 0$. Since $w^{-1}(\kappa_2) > 0$ and $w(w^{-1}(\kappa_2)) < 0$, we have $w^{-1}(\kappa_2) \in \hat{N}(w)$, i.e. $w^{-1}(\kappa_2) = \delta - \gamma$ for some $\gamma \in I_w \subset \Delta^+$. It follows that $w^{-1}(\kappa_2) = \delta - \gamma$. As $w(\gamma_2 - \gamma) = -\kappa_1 < 0$ and $w$ is minuscule, we must have $\gamma_2 - \gamma < 0$. Thus $\gamma$ is not a generator of $I_w$.

Remark. By a result of Cellini and Papi [3, Theorem 2.6], to any ad-nilpotent ideal of $b$ (not necessarily Abelian), one may attach a unique element of $\hat{W}$. Then one can extend Theorem 2.2 to this setting. However, the proof becomes more involved, since the procedure of shortening does not work for the corresponding elements of $\hat{W}$. I hope to consider related problems in a subsequent publication.

2.3 Theorem. Let $I_w$ be an Abelian ideal and $\gamma \in \Delta^+ \setminus I_w$. Then $I_w \cup \{\gamma\}$ is an Abelian ideal if and only if $w(\delta - \gamma) \in -\hat{W}$.

Proof. “$\Leftarrow$” Suppose $w(\delta - \gamma) = \alpha_i$. Then $l(s_i w) = l(w) + 1$ and $\hat{N}(s_i w) = \hat{N}(w) \cup \{\delta - \gamma\}$. That is, $s_i w$ is again minuscule and hence $I_w \cup \{\gamma\}$ is an Abelian ideal.
“⇒” It is clear that \( \gamma \) is a generator for \( I_w \cup \{ \gamma \} =: I_w. \) By Theorem 1, we then have 
\[ \hat{w}(\delta - \gamma) \in -\hat{\Pi}. \]
Assume that it is \(-\alpha_i. \) Then \( w = s_i \hat{w} \) and \( w(\delta - \gamma) = \alpha_i. \)

Given a non-trivial minuscule \( w \in \hat{W}, \) it was noticed before that \( w(\alpha_i) > 0, i \in \{1, \ldots, p\}, \) and \( w(\alpha_0) < 0. \) Let us study the last element. Let \( \Delta_l^+ \) denote the subset of long roots in \( \Delta^+. \) In the simply-laced case, all roots are proclaimed to be long.

2.4 Proposition. If \( w \) is a non-trivial minuscule element, then \( w(\alpha_0) + \delta \in \Delta_l^+. \)

Proof. Since \( w(\alpha_0) \) is negative, we can write \( w(\alpha_0) = -k \delta - \gamma_0, \) where \( k \in \{0, 1, 2, \ldots\} \) and \( \gamma_0 \in \Delta. \) Recall that \( \alpha_0 = \delta - \theta. \)

a) Assume \( k \geq 2. \) Then \( w(2\delta - \theta) = -(k - 1)\delta - \gamma_0 < 0. \) This contradicts the fact that \( w \) is minuscule.

b) Assume \( k = 0. \) Then \( w(\delta - \theta) = -\gamma_0 \) and \( \gamma_0 \in \Delta_l^+. \) It is clear that \( w \in \hat{W} \setminus W. \)

Write the expression of \( \theta \) through the simple roots: \( \theta = \sum_{i=1}^p n_i \alpha_i \) and set \( \gamma_i = w(\alpha_i). \) Then \( \sum_{i=1}^p n_i \gamma_i = \gamma_0 + \delta. \) Since \( \gamma_i \)'s are positive and \( \gamma_0 \in \Delta, \) there exists a unique \( i_0 \in \{1, \ldots, p\} \) such that \( n_{i_0} = 1, \gamma_{i_0} \in \delta + \Delta \) and \( \gamma_i \in \Delta \) for \( i \neq i_0. \) It follows that the elements \( -\gamma_0, \gamma_j (j \geq 1, j \neq i_0) \) form a basis for \( \Delta. \) Hence there is \( w' \in W \) which takes \( -\gamma_0, \gamma_j (j \neq i_0) \) to \( \alpha_1, \ldots, \alpha_p. \)

Because \( w'(\gamma_{i_0}) \in \delta + \Delta \) and the elements \( w'(\gamma_i) (i = 0, 1, \ldots, p) \) form a basis for \( \hat{\Delta}, \) we see that \( w'(\gamma_{i_0}) = \alpha_0. \) Thus, \( w'w \) takes \( \hat{\Pi} \) to itself and hence \( w'w = 1. \) This is however impossible, since \( w \notin W. \)

Thus, \( k = 1 \) and \( \mu := w(\alpha_0) + \delta = w(2\delta - \theta) \in \Delta. \) Since \( \delta \) is isotropic and \( \theta \) is long, \( \mu \) is long as well. Finally, since \( w \) is minuscule, \( 2\delta - \theta \notin \hat{N}(w). \) Hence \( \mu \) is positive.

Let \( \mathfrak{A} \) denote the set of all non-trivial Abelian ideals. By Proposition 2.4, one obtains the mapping

\[ \tau : \mathfrak{A} \to \Delta_l^+, \]

which is given by

\[ \tau(I_w) = w(\alpha_0) + \delta. \]

The long positive root \( \tau(I_w) \) is said to be the rootlet of the Abelian ideal \( I_w. \) Note that the ideal \( \{ \theta \} \) is the unique minimal element of \( \mathfrak{A} \) and, by Peterson’s result, \( \#(\mathfrak{A}) = 2^{rkg} - 1. \)

2.5 Theorem.

1. The mapping \( \tau \) is onto;

2. If the rootlet of \( I_w \) is not simple, i.e., \( w(\alpha_0) + \delta \in \Delta^+ \setminus \Pi, \) then \( I_w \) is not maximal.

3. If \( \Delta \) is simply-laced and \( \tau(I_w) \) is not simple, then there are at least two maximal Abelian ideals containing \( I_w. \)

Proof. 1. We perform a descending induction on the height of the rootlet of an ideal. The rootlet with maximal height is \( \theta. \) Here one takes \( w = s_0. \) Then \( I_{s_0} = \{ \theta \} \) and \( \tau(I_{s_0}) = \theta. \)

The induction step goes as follows. If \( \mu = \tau(I_w) \) and \( \mu \notin \Pi, \) then there exists an \( \alpha \in \Pi \) such that \( (\alpha, \mu) > 0. \) Then \( \mu' = s_\alpha(\mu) = \mu - n_\alpha \alpha \in \Delta_l^+ \) and \( ht(\mu') = ht(\mu) - n_\alpha. \) Notice that
n_\alpha = 1$ if and only if $\alpha$ is long. Set $\mu'' = \mu - \alpha$. It is again a positive root (not necessarily long).

We have $w(\delta - \theta) = -\delta + \mu'' + \alpha$. Hence $w^{-1}(\mu'') + w^{-1}(\alpha) = 2\delta - \theta$. It follows that

$$
\begin{align*}
&\{ w^{-1}(\mu'') = (k + 2)\delta - \mu_1 \\
&w^{-1}(\alpha) = -k\delta - \mu_2
\end{align*}
$$

for some $k \in \mathbb{Z}$ and $\mu_1, \mu_2 \in \Delta^+$ such that $\mu_1 + \mu_2 = \theta$.

As $w$ is minuscule, neither of the elements in the RHS is negative (for instance, if $w^{-1}(\alpha)$ were negative, i.e., $k \geq 0$, then $w(\mu_2) = -k\delta - \alpha_2 < 0$, which contradicts the fact that $w$ is minuscule). It follows that $k + 2 > 0$ and $-k > 0$, hence $k = -1$. In particular, we have $w(\delta - \mu_2) = \alpha \in \Pi$. It then follows from Theorem 2.3 that $w' = s_\alpha w$ is again a minuscule element and $I_{w'} = I_w \cup \{\mu_2\}$. The previous formulae show that $\tau(I_{w'}) = s_\alpha(\mu) = \mu'$. Obviously, any positive long root can be obtained from $\theta$ through a suitable sequence of simple reflections. Hence the assertion.

2. The previous argument also shows that if $\tau(I_w) \notin \Pi$, then $I_w$ is contained in a larger Abelian ideal.

3. As above, $\mu = \tau(I_w)$. Making use of the induction argument from part 1, we may reduce the problem to the case, where $ht(\mu) = 2$. Then $\mu = \alpha_1 + \alpha_2$ – the sum of two simple roots. Again the argument from part 1 (with $\alpha_1$ and $\alpha_2$ in place of $\mu''$ and $\alpha$) shows that there are two different Abelian extensions of $I_w$; namely, $I_{w_1} = I_w \cup \{\mu_1\}$ and $I_{w_2} = I_w \cup \{\mu_2\}$, where $w^{-1}(\alpha_1) = \delta - \mu_1$ and $w^{-1}(\alpha_2) = \delta - \mu_2$. But $I_w \cup \{\mu_1, \mu_2\}$ is not Abelian, since $\mu_1 + \mu_2 = \theta$.

\[ \square \]

Remark. In the doubly-laced case, it may happen that the rootlet of an Abelian ideal is not simple, but the ideal lies in a unique maximal one. For instance, let $\mathfrak{g}$ be the simple Lie algebra of type $F_4$. We use Vinberg–Onishchik’s numbering of simple roots [13]. If $\mu = 2\alpha_2 + \alpha_3$, then $\tau^{-1}(\mu)$ consists of two ideals (of dimension 7 and 8). In the notation of Table 1 in Section 3, $\tau^{-1}(\mu) = \{I_7', I_8\}$. The only maximal ideal containing these two is $I_9$. Denoting by $\Pi_l$ the set of long simple roots in $\Pi$, we record an important consequence of the theorem.

2.6 Corollary. If $I_w$ is a maximal Abelian ideal, then $w(\alpha_0) + \delta \in \Pi_l$.

Thus, denoting by $\text{Ab}_{\text{max}}$ the set of all maximal Abelian ideals, we obtain the mapping

$$
\bar{\tau} : \text{Ab}_{\text{max}} \to \Pi_l ,
$$

which is the restriction of $\tau$ to $\text{Ab}_{\text{max}}$. By Theorem 2.5, $\bar{\tau}$ is onto. We shall prove below that $\bar{\tau}$ is actually one-to-one. It turns out that the correspondence obtained between the maximal Abelian ideals and the long simple roots is precisely the one described in [8]. So that our present results provide an a priori proof for some empirical observations in that paper.
3. Basic properties of posets $\mathbf{Ab}_\mu$

Given $\mu \in \Delta_\ell^+$, let $\mathbf{Ab}_\mu$ denote the fibre of $\mu$ for $\tau : \check{\mathbf{Ab}} \to \Delta_\ell^+$. The following useful equality is a consequence of Peterson’s result:

$$\sum_{\mu \in \Delta_\ell^+} \#(\mathbf{Ab}_\mu) = 2^{\text{rk } \vartheta} - 1 .$$

Each $\mathbf{Ab}_\mu$ is a poset in its own right, and it appears that cutting $\check{\mathbf{Ab}}$ into pieces parametrized by $\Delta_\ell^+$ has a number of good properties.

3.1 Theorem. For any $\mu \in \Delta_\ell^+$, we have

(i) the poset $\mathbf{Ab}_\mu$ contains a unique maximal and a unique minimal element;

(ii) The dimension of the minimal Abelian ideal in $\mathbf{Ab}_\mu$ is equal to $1 + (\rho, \theta^\vee - \mu^\vee)$.

(iii) If $I, J \in \mathbf{Ab}_\mu$ and $I \subset J$, then any intermediate ideal also belong to $\mathbf{Ab}_\mu$. In particular, $\mathbf{Ab}_\mu$ is a ranked poset.

The proof of this result consists of several parts. The uniqueness of the minimal (resp. maximal) element will be proved in Proposition 3.6 (resp. Proposition 3.7), and the dimension formula for the minimal ideal is proved in Theorem 4.2. The latter is a by-product of an explicit description of the minimal ideal in $\mathbf{Ab}_\mu$ obtained in Section 4. Part (iii) is proved in Corollary 3.3.

To prove the theorem, we look at the procedure of extension of Abelian ideals in more details. If $I, J \in \mathbf{Ab}$, $\text{dim } J = \text{dim } I + 1$, and $I \subset J$, then we say that $J$ is an (Abelian) elementary extension of $I$. Given $I = I_w$, it follows from Theorem 2.3 that an elementary extension of $I_w$ is possible if and only if $w(\delta - \gamma) = \alpha_i \in \check{\Pi}$ for some $\gamma \in \Delta^+$. Then one can replace $w$ with $w' = s_iw$ and $I_w$ with $I_{w'} = I_w \cup \{\gamma\}$. The passage $w \rightarrow s_iw$ is also said to be an elementary extension (via the reflection $s_i$). Let us realize what happens with the rootlet under this procedure. Recall that $\Delta$ (or, more generally, the root lattice) has a standard partial order; one writes $\mu \preceq \nu$, if $\nu - \mu$ is a sum of positive roots.

3.2 Proposition. Suppose $I_{w'}$ is an elementary extension of $I_w$, as above. Then $\tau(I_{w'}) = s_i(\tau(I_w)) \preceq \tau(I_w)$. Moreover, if $w' = s_0w$ (i.e., $i = 0$), then $\tau(I_w) = \tau(I_{w'})$.

Proof. Set $\nu := w(\alpha_0) + \delta$, the rootlet of $I_w$. Then the rootlet of $I_{w'}$ is $s_iw(\alpha_0) + \delta = s_i(\nu - \delta) + \delta = s_i(\nu)$. We have two equalities: $w(\delta - \gamma) = \alpha_i$ and $w(\delta - \theta) = \nu - \delta$. Consider two possibilities for $i$.

(a) $i \neq 0$. Here we have

$$(\alpha_i, \nu) = (\alpha_i, \nu - \delta) = (\delta - \gamma, \delta - \theta) = (\gamma, \theta) \geq 0 ,$$

as $\delta$ is isotropic. It follows that $s_i(\nu) = \nu - (\nu, \alpha_i^\vee)\alpha_i \preceq \nu$.

(b) $i = 0$. As $\alpha_0 = \delta - \theta$, we obtain

$$0 \leq (\gamma, \theta) = (\nu - \delta, \delta - \theta) = -(\nu, \theta) \leq 0 .$$
Hence \((\gamma, \theta) = (\nu, \theta) = 0\) and \(s_0^w(\alpha_0) + \delta = s_0(\nu) = \nu\).

\[\square\]

### 3.3 Corollary.

If \(I, J \in \mathfrak{Ab}\) and \(I \subset J\), then \(\tau(J) \leq \tau(I)\). In particular, if \(I, J \in \mathfrak{Ab}_\mu\), then any intermediate ideal also belong to \(\mathfrak{Ab}_\mu\).

**Proof.** Obviously, for any pair \(I \subset J\) of Abelian ideals there is a sequence of elementary extensions that makes \(J\) from \(I\).

The following result will be our main tool in induction arguments.

### 3.4 Proposition.

Let \(I = I_w\) be an Abelian ideal. Suppose \(I\) has two different elementary extensions \(I_1 = I \cup \{\gamma_1\}\) and \(I_2 = I \cup \{\gamma_2\}\). Write \(s_i w\) for the minuscule element corresponding to \(I_i\), \(i = 1, 2\).

1. If \(\tilde{I} := I_1 \cup I_2\) is not Abelian, then \(\tau(I_1) = \alpha_2\), \(\tau(I_2) = \alpha_1\), and \(\tau(I) = \alpha_1 + \alpha_2\). Moreover, \(\alpha_1, \alpha_2 \in \Pi_i\).

2. If \(\tilde{I}\) is Abelian, then \(s_1 s_2 = s_2 s_1\) and \(w(\tilde{I}) = s_1 s_2 w\);

3. If \(\tau(\tilde{I}) = \tau(I_1)\), then \(\tilde{I}\) is Abelian as well and \(\tau(I_2) = \tau(\tilde{I})\).

**Proof.** The equalities \(s_i w = w(I_i)\) and \(I_i = I \cup \{\gamma_i\}\) mean together that

\[
\begin{align*}
w(\delta - \gamma_i) &= \alpha_i \in \hat{\Pi}, \\
i &= 1, 2.
\end{align*}
\]

1. Assume that \(I_1 \cup I_2\) is not Abelian. Since both \(I_1\) and \(I_2\) are Abelian, the only possibility for this is that \(\gamma_1 + \gamma_2 \in \Delta^+\).

If \(\gamma_1 + \gamma_2 \neq \theta\), then there is an \(\alpha \in \Pi\) such that \(\gamma_1 + \gamma_2 + \alpha\) is a (positive) root. Then \(\gamma_1 + \alpha \in \Delta\) or \(\gamma_2 + \alpha \in \Delta\) (Exercise!). If, for instance, the second condition is satisfied, then \(\gamma_2 + \alpha \in I\) and \(\gamma_1 \in I_1\), which contradicts the fact that \(I_1\) is Abelian.

Hence \(\gamma_1 + \gamma_2 = \theta\).

Now, taking the sum of Equations \(3.3\) yields

\[
\alpha_1 + \alpha_2 = w(2\delta - \gamma_1 - \gamma_2) = w(\delta - \theta) + \delta = \tau(I).
\]

Since \(\tau(I) \in \Delta_1^+\), we have \(\alpha_1, \alpha_2 \in \Pi_i\). It follows that \(\tau(I_1) = s_1(\alpha_1 + \alpha_2) = \alpha_2\) and \(\tau(I_2) = s_2(\alpha_1 + \alpha_2) = \alpha_1\).

2. The presence of the elementary extension \(I_1 \mapsto I_1 \cup \{\gamma_2\} = \tilde{I}\) shows that \(w(\tilde{I}) = s_2 w(I_1) = s_2 s_1 w\) and \(s_2 w(\delta - \gamma_1) \in \hat{\Pi}\). The latter means that \(s_2(\alpha_1)\) is a simple root. It follows that \(s_2(\alpha_1) = \alpha_1\) and hence \(s_2 s_1 = s_1 s_2\).

3. Under the assumption \(\tau(I) = \tau(I_1)\), the first case cannot occur. Hence \(\tilde{I}\) is Abelian. Since \(s_1, s_2\) commute, we have \(s_2 s_1 w(\alpha_0) + \delta = s_2(\nu) = s_2 w(\alpha_0) + \delta\), i.e., \(\tau(\tilde{I}) = \tau(I_2)\).

\[\square\]

### 3.6 Proposition.

For any \(\mu \in \Delta_i^+\), the poset \(\mathfrak{Ab}_\mu\) has a unique minimal element.

**Proof.** Assume \(\tilde{I}_1, \tilde{I}_2\) are two different minimal elements of \(\mathfrak{Ab}_\mu\). Clearly \(I := \tilde{I}_1 \cap \tilde{I}_2\) is again an Abelian ideal, but \(\tau(I)\) is strictly less than \(\mu\).

The ideal \(\tilde{I}_1\) can be obtained from \(I\) via a chain of elementary extensions, say

\[
I \rightarrow I \cup \{\varepsilon_1\} \rightarrow \ldots \rightarrow I \cup \{\varepsilon_1, \ldots, \varepsilon_n\} = \tilde{I}_1.
\]
Similarly, let \( I \to I \cup \{ \eta \} \) be the first step in the chain of extensions leading from \( I \) to \( \tilde{I}_2 \). Set \( I(k, 0) = I \cup \{ \zeta_1, \ldots, \zeta_k \} \) and \( I(k, 1) = I \cup \{ \zeta_1, \ldots, \zeta_k, \eta \} \), \( 0 \leq k \leq n \). By construction, \( I(0, 1) \) and \( I(k, 0) \) are Abelian ideals. Consider the sequence of statements depending on \( k \):

\[(C_k) \quad I(k, 0) \neq \tilde{I}_1, \ I(k, 1) \text{ is Abelian, and } \mu = \tau(\tilde{I}_1) \leq \tau(I(k, 1)).\]

Claim. For any \( k \geq 0 \), \((C_k)\) implies \((C_{k+1})\).

Note that \((C_0)\) is true. (The last inequality follows from the equality \( \tau(\tilde{I}_1) = \tau(\tilde{I}_2) \) and Corollary 3.3.) Therefore, granting the claim, we conclude that \((C_n)\) is also true. But this is nonsense, since \( I(n, 0) = \tilde{I}_1 \). This contradiction shows that \( \mathfrak{Ab}_\mu \) cannot have two minimal elements. Thus, it remains to prove the Claim.

Proof of the Claim. By assumption, we have two elementary extensions:

\[I(k, 0) \to I(k + 1, 0) \text{ and } I(k, 0) \to I(k, 1).\]

If \( w := w(I(k, 0)) \), then \( w(I(k, 1)) = s'w \) and \( w(I(k+1, 0)) = s''w \) for some simple reflections \( s', s'' \).

1. Assume that \( I(k+1, 1) \) is not Abelian. Applying Proposition 3.4(1) to the above triplet of ideals, we obtain \( \tau(I(k, 0)) = \alpha' + \alpha'', \tau(I(k + 1, 0)) = \alpha', \) and \( \tau(I(k+1, 1)) = \alpha'' \), where \( \alpha', \alpha'' \in \Pi_l \). Since \( I(k + 1, 0) \subset \tilde{I}_1 \), we have \( \tau(\tilde{I}_1) = \alpha' \). On the other hand, our assumptions give \( \tau(\tilde{I}_1) \leq \tau(I(k, 1)) = \alpha'' \). Whence \( \alpha' \leq \alpha'' \). This contradiction shows that \( I(k + 1, 1) \) is Abelian.

2. Since \( I(k+1, 1) \) is Abelian, Proposition 3.4(2) says that \( s' s'' = s'' s' \) and \( w(I(k+1, 1)) = s' s'' w \). It follows that

\[
\tau(I(k, 1)) = s' \tau(I(k, 0)) = \tau(I(k, 0)) - n' \alpha',
\]

and, since \( I(k+1, 0) \subset \tilde{I}_1 \),

\[
\tau(\tilde{I}_1) \leq \tau(I(k, 1)) = \tau(I(k, 0)) - n' \alpha'.
\]

Hence \( \tau(\tilde{I}_1) \leq \tau(I(k, 0)) - n' s' - n'' s'' = \tau(\tilde{I}_1) \).

3. If \( I(k+1, 0) = \tilde{I}_1 \), then the inequalities in the previous part of the proof imply that

\[
\tau(I(k, 0)) - n'' \alpha'' \leq \tau(I(k, 0)) - n' \alpha'.
\]

Hence \( n' = n'' = 0 \). Then \( \mu = \tau(\tilde{I}_1) = \tau(I(k, 0)) \). Thus, \( I(k, 0) \) is smaller than \( \tilde{I}_1 \) and has the same rootlet, which contradicts the minimality of \( \tilde{I}_1 \). Hence \( I(k + 1, 0) \neq \tilde{I}_1 \), and the claim is proved. \( \square \)

This completes the proof of Proposition 3.6. \( \square \)

In what follows, \( I(\mu)_{\text{min}} \) stands for the minimal element of \( \mathfrak{Ab}_\mu \).

3.7 Proposition. For any \( \mu \in \Delta^+_l \), the poset \( \mathfrak{Ab}_\mu \) has a unique maximal element.
Proof. By Proposition 3.6, any ideal \( I \subset \mathfrak{A}_{\mu} \) can be obtained from \( I(\mu)_{min} \) via a chain of elementary extensions. Moreover, it follows from Corollary 3.3 that each ideal in this chain belong to \( \mathfrak{A}_{\mu} \). Another consequence is that if \( I, J \in \mathfrak{A}_{\mu} \), then \( I \cap J \in \mathfrak{A}_{\mu} \) as well.

Suppose \( I_1, I_2 \in \mathfrak{A}_{\mu} \). Let us prove that \( I_1 \cup I_2 \in \mathfrak{A}_{\mu} \). Consider the set \( I_2 \setminus I_1 \) and pick there a maximal element with respect to \( \preceq \), say \( \gamma_2 \). Arguing by induction, it suffices to prove that \( I_1 \cup \{\gamma_2\} \) lies in \( \mathfrak{A}_{\mu} \). Similarly, take a maximal element \( \nu_1 \in I_1 \setminus I_2 \). Applying Proposition 3.4(3) to the ideal \( I = I_1 \cap I_2 \in \mathfrak{A}_{\mu} \) and the roots \( \nu_1, \gamma_2 \), we conclude that \( I \cup \{\nu_1, \gamma_2\} \) is in \( \mathfrak{A}_{\mu} \). If \( I' := I \cup \{\nu_1\} \neq I_1 \), then take a maximal element \( \nu_2 \in I_1 \setminus I' \). Then one applies Proposition 3.4(3) to \( I' \) and \( \nu_2, \gamma_2 \). We eventually obtain \( I_1 \cup \{\gamma_2\} \in \mathfrak{A}_{\mu} \).

Since \( I_1 \cup I_2 \in \mathfrak{A}_{\mu} \) for any pair \( I_1, I_2 \in \mathfrak{A}_{\mu} \), we see that \( \mathfrak{A}_{\mu} \) has a unique maximal element. \( \square \)

3.8 Corollary. The map \( \bar{\tau} : \mathfrak{A}_{\mu_{max}} \to \Pi_{l} \) is bijective.

Proof. It follows from Corollary 2.6 and Proposition 3.7 that the maximal Abelian ideals are precisely the maximal elements of the posets \( \mathfrak{A}_{\alpha}, \alpha \in \Pi_{l} \). \( \square \)

In what follows, \( I(\mu)_{max} \) stands for the maximal element of \( \mathfrak{A}_{\mu} \). We also say that \( I(\mu)_{min} \) is the \( \mu \)-minimal and \( I(\mu)_{max} \) is the \( \mu \)-maximal ideal.

4. \( \mu \)-MINIMAL IDEALS AND THEIR PROPERTIES

In this section, an explicit description of \( I(\mu)_{min} \) is given for any \( \mu \in \Delta_{l}^{+} \). We also characterise the set of all \( \mu \)-minimal ideals and find the generators of \( I(\mu)_{min} \).

4.1 Theorem. Let \( w \in W \) be an element of minimal length such that \( w(\theta) = \mu \). Then

1. \( l(w) = (\rho, \theta^{\vee} - \mu^{\vee}) \);
2. \( N(w^{-1}) = \{\gamma \in \Delta^+ \mid (\gamma, \mu^{\vee}) = -1\} \).

In particular, the set \( \{u \in W \mid u(\theta) = \mu\} \) contains a unique element of minimal length.

Proof. 1. Recall that \( (\rho, \alpha^{\vee}) = 1 \) for all \( \alpha \in \Pi \). A straightforward calculation shows that, for any \( \nu \in \Delta \) and \( \alpha \in \Pi \),

\[
(\rho, s_{\alpha}(\nu)^{\vee}) = (\rho, \nu^{\vee}) - (\alpha, \nu^{\vee}).
\]

If \( \nu \) is long, then \( |(\alpha, \nu^{\vee})| \leq 1 \). It follows that, for any \( w' \in W \) with the property \( w'(\theta) = \mu \), we have \( l(w') \geq (\rho, \theta^{\vee} - \mu^{\vee}) \). On the other hand, if \( \mu \in \Delta_{l}^{+} \) and \( \mu \neq \theta \), then one can always find an \( \alpha \in \Pi \) such that \( (\alpha, \mu^{\vee}) = 1 \). This means that starting with \( \mu \) and moving \( \theta \), one can reach \( \theta \) after applying exactly \( (\rho, \theta^{\vee} - \mu^{\vee}) \) simple reflections.

2. Set \( \Delta_{l}^{+}(i) = \{\gamma \in \Delta^+ \mid (\gamma, \mu^{\vee}) = i\} \). We are to show that \( \Delta_{l}^{+}(i)(-1) = N(w^{-1}) \). Let us compare the cardinalities of these two sets. By the first part of the proof, \( \#N(w^{-1}) = (\rho, \theta^{\vee} - \mu^{\vee}) \). On the other hand, one has the system of two equations

\[
\begin{cases}
(\rho, \mu^{\vee}) = 1 + \frac{1}{2}(\#\Delta_{l}^{+}(1) - \#\Delta_{l}^{+}(-1)) \\
2(\rho, \theta^{\vee}) - 2 = \#\Delta_{\theta}(1) = \#\Delta_{\mu}(1) = \#\Delta_{l}^{+}(1) + \#\Delta_{l}^{+}(-1).
\end{cases}
\]

The first equality stems from the very definition of \( \rho \), whereas in the second equation we use
the fact that $\theta$ is dominant and that $\mu$ and $\theta$ are $W$-conjugate. From the above system we deduce that 
$\#\Delta_\mu^+(1) = (\rho, \theta^\vee - \mu^\vee) = \#N(w^{-1})$. 
On the other hand, if $\gamma \in \Delta_\mu^+(1)$, then $(w^{-1}(\gamma), \theta) = -1$. Hence $w^{-1}(\gamma)$ is negative and $N(w^{-1}) \supset \Delta_\mu^+(1)$.

Notice that we also proved that if $u \in W$ is any element taking $\theta$ to $\mu$, then $N(u^{-1}) \supset \Delta_\mu^+(1)$. In what follows, we write $w_\mu$ for the unique element of minimal length in $W$ that takes $\theta$ to $\mu$.

4.2 Theorem. Set $\tilde{w}_\mu = w_\mu s_0 \in \tilde{W}$. Then

1. $\tilde{w}_\mu(\alpha_0) + \delta = \mu$;
2. $\tilde{w}_\mu$ is minuscule;
3. the ideal $I_{\tilde{w}_\mu}$ is contained in \{ $\gamma \in \Delta^+ | (\gamma, \theta) > 0$ \};
4. $I_{\tilde{w}_\mu} = I(\mu)_{\min}$, the minimal element of $\mathfrak{Ab}_\mu$, and $\#(I_{\tilde{w}_\mu}) = (\rho, \theta^\vee - \mu^\vee) + 1$.

Proof. 1. Obvious.

2. Suppose $(\rho, \theta^\vee - \mu^\vee) = k \geq 1$ and let $w_\mu = s_{i_k} \ldots s_{i_1}$ be a reduced decomposition. We argue by induction on $k$. Set $u := s_{i_k} \ldots s_{i_1} \in W$ and $\nu := u(\theta)$. Then $l(u) = k - 1$ and $s_{i_k}(\nu) = \mu$. Using Theorem 1.1, we obtain

\[
k - 1 \geq (\rho, \theta^\vee - \nu^\vee) = (\rho, \theta^\vee - \mu^\vee) - (\alpha_{i_k}, \nu^\vee) = k - (\alpha_{i_k}, \nu^\vee).
\]

Since $\nu$ is long, $(\alpha_{i_k}, \nu^\vee) = 1$. It follows that $(\rho, \theta^\vee - \nu^\vee) = k - 1$ and hence $u = w_\nu$. Set $\tilde{w}_\nu = w_\nu s_0$. By the induction assumption, $\tilde{w}_\nu$ is minuscule. To prove that $\tilde{w}_\nu = s_{i_k} \tilde{w}_\nu$ is minuscule, one has to verify that $\tilde{w}_\nu(\delta - \gamma_{i_k}) = \alpha_{i_k}$ for some $\gamma_{i_k} \in \Delta^+$ (see Theorem 2.3). In other words, it should be proved that $\delta - \tilde{w}_\nu^{-1}(\alpha_{i_k}) \in \Delta^+$. As we shall see, this is a direct consequence of previous formulae. Indeed, $\tilde{w}_\nu^{-1}(\alpha_{i_k}) = s_0 w_\nu^{-1}(\alpha_{i_k})$ and

\[
(\theta, w_\nu^{-1}(\alpha_{i_k})) = (w_\nu(\theta), \alpha_{i_k}) = (\nu, \alpha_{i_k}) > 0.
\]

The latter shows that $w_\nu^{-1}(\alpha_{i_k}) \in \Delta^+$ and $(\alpha_0, w_\nu^{-1}(\alpha_{i_k})) < 0$. Therefore $s_0 w_\nu^{-1}(\alpha_{i_k}) = w_\nu^{-1}(\alpha_{i_k}) - \theta + \delta$. Thus, $\delta - \tilde{w}_\nu^{-1}(\alpha_{i_k}) = \theta - w_\nu^{-1}(\alpha_{i_k}) \in \Delta^+$, and we are done.

3. Again, we argue by induction on $l(w_\mu)$. Using the notation of the previous part of the proof, it suffices to observe that $I_{\tilde{w}_\mu} = I_{\tilde{w}_\nu} \cup \{ \theta - w_\nu^{-1}(\alpha_{i_k}) \}$ and $(w_\nu^{-1}(\alpha_{i_k}), \theta^\vee) = 1$.

4. If $I_{\tilde{w}_\mu}$ were not minimal in $\mathfrak{Ab}_\mu$, then one could shorten $\tilde{w}_\mu$, so that to obtain a minuscule element giving the ideal with the same rootlet. But this is impossible for length reason, as $w_\mu$ has minimal possible length among the elements taking $\theta$ to $\mu$. The dimension of this ideal is already computed in Theorem 3.2. Finally, $\#(I_{\tilde{w}_\mu}) = l(\tilde{w}_\mu) = l(w_\mu) + 1$.

Thus, we have completed the proof of Theorem 3.2.

Set $\mathcal{H} = \{ \gamma \in \Delta^+ | (\theta, \gamma) > 0 \}$. It is the set of the roots for the standard Heisenberg subalgebra of $\mathfrak{g}$. That is, $\mathfrak{h} = \bigoplus_{\gamma \in \mathcal{H}} \mathfrak{g}_\gamma$ is a Heisenberg subalgebra of $\mathfrak{g}$. Clearly, $\mathfrak{h}$ is a non-Abelian ideal of $\mathfrak{b}$.
The previous exposition shows that one has a distinguished collection of Abelian ideals \( \{I(\mu)_{\text{min}} \mid \mu \in \Delta_i^+\} \) and the corresponding subset of minuscule elements of \( \tilde{W} \). These sets admit the following characterizations:

4.3 Theorem. The following conditions are equivalent for \( I_w \in \mathfrak{Ab} \):

(i) \( I_w = I(\mu)_{\text{min}} \) for some \( \mu \in \Delta_i^+ \);
(ii) \( I_w \subset H \);
(iii) \( w = w's_0 \), where \( w' \in W \).

Proof. (i) \( \Rightarrow \) (ii). This is proved in Theorem 4.2.

(ii) \( \Rightarrow \) (iii). Assume that a reduced decomposition of \( w' \) contains \( s_0 \), say \( w' = w_2s_0w_1 \). Since \( s_0w_1s_0 \) is also minuscule (see Section 2), we may assume without loss that \( w_2 = 1 \), i.e., a reduced decomposition of \( w' \) begins with \( s_0 \). Hence, there is the elementary extension \( w_1s_0 \rightarrow s_0w_1s_0 \). It was already shown that in this case one adds to the ideal \( I_{w_1s_0} \) a root which is orthogonal to \( \theta \), see Proposition 3.2(b).

(iii) \( \Rightarrow \) (i). We argue by induction on \( l(w') \). Suppose a reduced decomposition of \( w' \) starts with \( s_i \), i.e., \( w = s_iw''s_0 \) and \( w''s_0 \) is also minuscule. By the induction hypothesis, \( I_{w''s_0} = I(\nu)_{\text{min}} \), where \( \nu = w''(\theta) \). Then \( w'' = w_\nu \) and \( l(w'') = (\nu, \theta^\vee - \nu^\vee) \). Set \( \mu = s_i(\nu) = w''(\theta) \). Then \( \mu = \tau(I_w) \) and our goal is to prove that \( s_iw'' = w_\mu \). Since \( w''s_0 \rightarrow s_iw''s_0 \) is an elementary extension, we have \( w''s_0(\delta - \gamma) = \alpha_i \in \Pi \) for some \( \gamma \in \Delta^+ \). It follows that \( s_0(\delta - \gamma) \neq \delta - \gamma \). This yields \( (\theta^\vee; \gamma) = 1 \) and \( s_0(\delta - \gamma) = \theta - \gamma \). Hence \( w''(\theta - \gamma) = \alpha_i \).

Therefore

\[
(\alpha_i, \nu^\vee) = (w''(\theta - \gamma), w''(\theta^\vee)) = (\theta - \gamma, \theta^\vee) = 1.
\]

This equality implies that \( (\nu, \theta^\vee - \nu^\vee) = (\nu, \theta^\vee - \nu^\vee) + 1 = 1 + l(w'') = l(w') \). By Theorem 3.1, this means that \( w' = s_iw'' \in W \) is the shortest element taking \( \theta \) to \( \mu \), and we are done. \( \square \)

4.4 Corollary. There is a natural one-to-one correspondence between the Abelian b-ideals in the Heisenberg subalgebra and the long positive roots.

The next result describes the order relation on the set of \( \mu \)-minimal ideals.

4.5 Theorem. For any \( \mu, \nu \in \Delta_i^+ \), we have \( I(\mu)_{\text{min}} \subset I(\nu)_{\text{min}} \iff \nu \leq \mu \). That is, the poset of \( \mu \)-minimal elements is anti-isomorphic to the poset \( (\Delta_i^+, \leq) \).

Proof. “\( \Rightarrow \)” This is contained in Corollary 3.3.

“\( \Leftarrow \)” Let us show that \( w_\nu = w'w_\mu \), where \( l(w') = (\rho, \mu^\vee - \nu^\vee) \). Indeed,

(a) The inequality \( l(w') \geq l(w_\nu) - l(w_\mu) = (\rho, \mu^\vee - \nu^\vee) \) is clear.
(b) The opposite inequality can be proved by induction. Set \( \mu - \nu = \sum_{\alpha} k_\alpha \alpha \), where \( k_\alpha \geq 0 \). Since \( |\mu| = |\nu| \), we obtain \( (\nu, \sum k_\alpha \alpha) < 0 \). Hence there exist an \( \alpha \in \Pi \) such that \( k_\alpha > 0 \) and \( (\alpha, \nu) < 0 \). Then \( \nu \leq s_\alpha(\nu) = \nu + (|\mu|^2/|\alpha|^2) \alpha \leq \mu \). (One should use here the fact that, since \( \mu \) and \( \nu \) are long, \( k_\alpha \) is divisible by \( |\mu|^2/|\alpha|^2 \).)

Thus, the minuscule element \( \tilde{w}_\nu \) is obtained from \( \tilde{w}_\nu \) via a sequence of elementary extensions and hence \( I(\mu)_{\text{min}} \subset I(\nu)_{\text{min}} \). \( \square \)
Finally, we give a description of the generators for $\mu$-minimal ideals. If $w = s_0$, then $I_{s_0} = \{\theta\}$ and everything is clear. So that we may assume that $\mu \neq \theta$, i.e., $\tilde{w}_\mu = w_\mu s_0$ and $w_\mu \neq 1$.

### 5.1 Theorem.
In this case the rootlet of an ideal has to be orthogonal to $\theta$.

**Proof.** By Theorem 2.2, $\mu > 0$. By Theorem 4.2(3), $(\gamma, \theta) > 0$. Therefore the LHS is equal to $w_\mu(\theta - \gamma) = \mu - w_\mu(\gamma)$ and $\mu + \alpha = w_\mu(\gamma) \in \Delta$. Hence $\alpha \in \Pi$ and $\mu + \alpha$ is a root.

This argument can be reversed. Given $\alpha \in \Pi$ such that $(\alpha, \mu^\vee) = -1$, we set $\gamma = w_\mu^{-1}(\alpha + \mu)$. As $(\alpha + \mu, \mu^\vee) \neq -1$, it follows from Theorem 4.1(2) that $\gamma > 0$. The rest is clear. $\square$

### 5.6 Proposition.
For $\mu \neq \theta$, there is a bijection between the generators of $I(\mu)_{\text{min}}$ and the roots $\alpha \in \Pi$ such that $\alpha + \mu \in \Delta$ (i.e., $(\alpha, \mu^\vee) = -1$). The generator corresponding to such an $\alpha$ is $w_\mu^{-1}(\alpha + \mu)$.

**Proof.** By Theorem 2.2, $\gamma \in \Delta^+$ is a generator if and only if $w_\mu s_0(\delta - \gamma) = -\alpha \in \hat{\Pi}$. By Theorem 4.2(3), $(\gamma, \theta) > 0$. Therefore the LHS is equal to $w_\mu(\theta - \gamma) = \mu - w_\mu(\gamma)$ and $\mu + \alpha = w_\mu(\gamma) \in \Delta$. Hence $\alpha \in \Pi$ and $\mu + \alpha$ is a root.

This argument can be reversed. Given $\alpha \in \Pi$ such that $(\alpha, \mu^\vee) = -1$, we set $\gamma = w_\mu^{-1}(\alpha + \mu)$. As $(\alpha + \mu, \mu^\vee) \neq -1$, it follows from Theorem 4.1(2) that $\gamma > 0$. The rest is clear.

### 5. More on the structure of $\mathfrak{Ab}_\mu$

We already know that each $\mathfrak{Ab}_\mu$ contains a unique maximal and a unique minimal element. In this section, we first answer the question: when is the cardinality of $\mathfrak{Ab}_\mu$ equal to 1? An important observation concerning cardinality stems from Proposition 3.2. It was proved there that the elementary extension via the reflection $s_0$ does not affect the rootlet; and in this case the rootlet of an ideal has to be orthogonal to $\theta$. What we prove now is that this gives a necessary and sufficient condition for $\#(\mathfrak{Ab}_\mu) > 1$.

### 5.1 Theorem.
(i) $\#(\mathfrak{Ab}_\mu) > 1$ if and only if $(\mu, \theta) = 0$ (i.e., $\mu \notin \mathcal{H}$).

(ii) If $(\mu, \theta) = 0$, then the non-empty poset $\mathfrak{Ab}_\mu \setminus \{I(\mu)_{\text{min}}\}$ has a unique minimal element, say $I'$. Here $I' = I(\mu)_{\text{min}} \cup \{\gamma\}$, where $\gamma = w_\mu^{-1}(\theta)$. The corresponding minuscule element is $s_0\tilde{w}_\mu = s_0w_\mu s_0$.

**Proof.**
(i) In view of Theorem 3.1, it is clear that $\#(\mathfrak{Ab}_\mu) > 1$ if and only if $I(\mu)_{\text{min}}$ has an elementary extension that does not change the rootlet. So, we stick to considering possible elementary extensions of $I(\mu)_{\text{min}}$. This is based on the explicit description in Theorem 4.2.

1. Since $\mu$ is the rootlet, we have

   $\tilde{w}_\mu(\delta - \theta) = \mu - \delta$.  

   (5.2)

   Suppose there is an elementary extension of $I(\mu)_{\text{min}}$, i.e., we have a $\gamma \in \Delta^+$ such that

   $\tilde{w}_\mu(\delta - \gamma) = \alpha \in \hat{\Pi}$.  

   (5.3)

   There are two possibilities for $\alpha$.

   (a) $\alpha = \alpha_i \in \Pi$. Rewriting Eq. (5.3) as $s_0(\delta - \gamma) = w_\mu^{-1}\alpha_i$, we see that $s_0(\delta - \gamma) \in \Delta$. This can only happen if $(\alpha_0, \delta - \gamma) > 0$, i.e., $(\theta, \gamma) > 0$ (and then $s_0(\delta - \gamma) = \theta - \gamma$). Combining Equations (5.2) and (5.3), we obtain $(\mu, \alpha_i) > 0$ and hence $s_i(\mu) \neq \mu$. Thus, any elementary extension via a simple reflection from $W$ changes the rootlet of $I(\mu)_{\text{min}}$. 


(b) $\alpha = \alpha_0$. Here we get the chain following inequalities:

$$0 \leq (\theta, \gamma) = (\delta - \theta, \delta - \gamma) = (\mu - \delta, \delta - \theta) = -(\mu, \theta) \leq 0.$$ 

Thus, we have the conclusion: if $I(\mu)_{min}$ has an extension that does not change the rootlet, then this extension uses the reflection $s_0$ and the condition $(\mu, \theta) = 0$ should be satisfied. This proves the “only if” part.

2. Suppose $(\theta, \mu) = 0$. We wish to find an elementary extension of $I(\mu)_{min}$ that does not change the rootlet $\mu$. Recall that $\bar{w}_\mu = w_\mu s_0$. Take $\gamma = w_\mu^{-1}(\theta)$. From the description of $w_\mu^{-1}$ (see Theorem 4.2(2)), it follows that $\theta \not\in N(w_\mu^{-1})$, i.e., $\gamma \in \Delta^+$. Furthermore, $(\gamma, \theta) = (w_\mu(\gamma), w_\mu(\theta)) = (\theta, \mu) = 0$.

Hence $\bar{w}_\mu(\delta - \gamma) = \delta - \theta = \alpha_0$ and $s_0(\mu) = \mu$. Thus, $I(\mu)_{min} \cup \{\gamma\}$ is an Abelian ideal lying in $\mathfrak{Ab}_\mu$.

(ii) This is essentially proved in the previous part of proof, since $s_0$ is the only possible reflection that can be used for constructing an elementary extension of $I(\mu)_{min}$ with the rootlet $\mu$. $\blacksquare$

Remark. We have proved that, for $I(\mu)_{min}$, there is at most one elementary extension which lies inside $\mathfrak{Ab}_\mu$, and, if exists, this extension always exploits the reflection $s_0$. But if $I \in \mathfrak{Ab}_\mu$ is not minimal, then there can exist an elementary extension via $s_i$ ($i \neq 0$) that does not change the rootlet.

Now, we accomplish the following step in describing cardinality of $\mathfrak{Ab}_\mu$. That is, a criterion will be given for $\#(\mathfrak{Ab}_\mu) > 2$. We already know that the condition $(\mu, \theta) = 0$ is necessary.

5.4 Proposition. Suppose $\mu \in \Delta^+_i$ and $(\mu, \theta) = 0$. Then

$$\#(\mathfrak{Ab}_\mu) > 2 \iff \exists \alpha_i \in \Pi \text{ such that } (\alpha_i, \theta) > 0 \text{ and } (\alpha_i, \mu) = 0.$$ 

If these conditions are satisfied, then next element of $\mathfrak{Ab}_\mu$ is

$$I'' = I(\mu)_{min} \cup \{w_\mu^{-1}(\theta), w_\mu^{-1}(\theta - \alpha_i)\}.$$ 

Proof. In view of Theorem 5.1(ii), it is clear that $\#(\mathfrak{Ab}_\mu) > 2$ if and only if $I' = I_{s_0 w_\mu s_0}$ has an elementary extension with the same rootlet. So, we stick to considering possible extensions of $I_{s_0 w_\mu s_0}$.

“$\Leftarrow$” We show that $s_i s_0 w_\mu s_0$ is again minuscule and the corresponding rootlet is again $\mu$. The second condition is satisfied, since $(\alpha_i, \mu) = 0$ and hence $s_i(\mu) = \mu$. The condition that $s_i s_0 w_\mu s_0$ is minuscule is equivalent, in view of Theorem 2.3, to $s_0 w_\mu s_0(\delta - \gamma) = \alpha_i$ for some $\gamma \in \Delta^+$, i.e., $\delta - s_0 w_\mu^{-1} s_0(\alpha_i) \in \Delta^+$. Using the definition of $w_\mu$ and the assumptions, the last expression is equal to $w_\mu^{-1}(\theta - \alpha_i)$. Since $(\mu, \theta - \alpha_i) = 0$, we deduce from Theorem 1.2(2) that $\theta - \alpha_i \not\in N(w_\mu^{-1})$, that is, $w_\mu^{-1}(\theta - \alpha_i)$ is positive.

“$\Rightarrow$” Suppose there is an elementary extension of $I_{s_0 w_\mu s_0}$ that does not affect $\mu$, i.e., there is a $\gamma \in \Delta^+$ such that

$$(5.5) \quad s_0 w_\mu s_0(\delta - \gamma) = \alpha_i$$
and \( s_i(\mu) = \mu \). Clearly, \( i \neq 0 \), i.e., \( \alpha_i \in \Pi \). Since \( s_i(\mu) = \mu \), we have \((\alpha_i, \mu) = 0\). Thus, it remains to prove that \((\alpha_i, \theta) > 0\). If not, then \((\alpha_i, \theta) = 0\) and hence \( s_0(\alpha_i) = \alpha_i \). Then Eq. (5.3) can be written as \( \delta - \gamma = s_0 w^{-1}_\mu(\alpha_i) \). As \((\theta, w^{-1}_\mu(\alpha_i)) = (\mu, \alpha_i) = 0\), the right-hand side is equal to \( w^{-1}_\mu(\alpha_i) \in \Delta \). This contradiction proves that \((\alpha_i, \theta) > 0\).

\[ \square \]

Remark. If \( \mathfrak{g} \neq \mathfrak{sl}_n \), then there is only one simple root that is not orthogonal to \( \theta \). In any case, this condition is easy to verify in practice.

Actually, I can give a description of \( I(\mu)_{\text{max}} \) and \( \mathfrak{Ab}_\mu \), which is consistent with both the previous results and my computations in Section 5.3, but I cannot find a general case-free proof yet. In order to provide a stronger motivation and more evidences in favour of the following description, let us look again at previous results of this section. We have proved that

- if \((\mu, \theta) = 0\), then \( \mathfrak{Ab}_\mu = \{I(\mu)_{\text{min}}\} \);
- if \((\mu, \theta) > 0\) and there is no simple roots \( \alpha \in \Pi \) such that \((\theta, \alpha) > 0\) and \((\alpha, \mu) = 0\), then \( \mathfrak{Ab}_\mu = \{I(\mu)_{\text{min}}, I^1\} \), where \( I^1 = I(\mu)_{\text{min}} \cup \{\gamma\} \) and \( \gamma = w^{-1}_\mu(\theta) \);
- if \((\mu, \theta) > 0\) and \( \alpha \in \Pi \) satisfies the conditions \((\theta, \alpha) > 0\) and \((\alpha, \mu) = 0\), then one can further extend \( I^1 \) as follows: \( I'' = I^1 \cup \{\gamma'\} \), where \( \gamma' = w^{-1}_\mu(\theta - \alpha) \).

These first steps of constructing extensions show that each time one adds to \( I(\mu)_{\text{min}} \) some roots that are orthogonal to \( \nu \). Moreover, the following is true.

5.6 Proposition. Suppose \( \alpha_1, \ldots, \alpha_t \) is a chain of simple roots such that \((\theta, \alpha_1) > 0\), \((\alpha_i, \alpha_{i+1}) > 0\) \((i = 1, \ldots, t - 1)\), and \((\theta, \mu) = (\alpha_1, \mu) = \ldots = (\alpha_t, \mu) = 0\). Then \( \#(\mathfrak{Ab}_\mu) \geq t + 1 \). More precisely,

\[
\{I^{(0)}, I^{(1)}, \ldots, I^{(t)}\} \subset \mathfrak{Ab}_\mu,
\]

where \( I^{(0)} = I(\mu)_{\text{min}} \) and \( I^{(i+1)} = I^{(i)} \cup \{w^{-1}_\mu(\theta - \alpha_1 - \ldots - \alpha_i)\} \).

Proof. Argue by induction on \( t \). The induction step is the same as the proof of Proposition 5.3. \[ \square \]

After this preparations, I can state a general description of \( I(\mu)_{\text{max}} \) and \( \mathfrak{Ab}_\mu \). Let \( \tilde{\Gamma} \) be the extended Dynkin diagram of \( \mathfrak{g} \). It has the “usual” nodes that correspond to the roots in \( \Pi \) and the “extra” node corresponding to \(-\theta\). Let us delete from \( \tilde{\Gamma} \) all nodes such that the corresponding roots are not orthogonal to \( \mu \). The remaining graph can be disconnected. Let \( \Gamma_\mu \) denote the connected component of it that contains the node corresponding to \(-\theta\). For instance, if \((\mu, \theta) > 0\), then \( \Gamma_\mu = \emptyset \). Clearly, \( \Gamma_\mu \) is the Dynkin diagram of a regular simple Lie subalgebra of \( \mathfrak{g} \). Call this subalgebra \( \mathfrak{g}(\mu) \). If \( \alpha_1, \ldots, \alpha_k \) are all simple roots of \( \mathfrak{g} \) that correspond to the usual nodes of \( \Gamma_\mu \), then \( \{\theta, -\alpha_1, \ldots, -\alpha_k\} \) can be taken as a set of simple roots for \( \mathfrak{g}(\mu) \), and one can consider the respective set of positive roots. Let \( \mathfrak{b}(\mu) \) be the Borel subalgebra corresponding to the chosen set of positive roots, and let \( \mathfrak{b}_{\text{nil}}(\mu) \) be the opposite Borel subalgebra. With this convention, let \( \mathfrak{p}(\mu) \supset \mathfrak{b}(\mu) \) be the maximal parabolic subalgebra of \( \mathfrak{g}(\mu) \) determined by \( \theta \) (i.e. \( \theta \) is the only simple root of \( \mathfrak{g}(\mu) \) that is not a root of the Levi subalgebra of \( \mathfrak{p}(\mu) \)). Let \( M_\mu \) be the set of roots of \( \mathfrak{p}^{\text{nil}}(\mu) \), the nilpotent radical of \( \mathfrak{p}(\mu) \). It is
obvious that the nilpotent radical constructed in this way is Abelian, i.e., for any \( \gamma \in M_\mu \) the coefficient of \( \theta \) can be only 1. Thus,

\[
M_\mu = \{ \theta - \sum_{i=1}^{k} c_i \alpha_i \mid c_i \geq 0 \} \cap \Delta .
\]

Notice that \( \{ \alpha_1, \ldots, \alpha_k \} \) is a proper subset of \( \Pi \), since \( \mu \neq 0 \). Therefore \( M_\mu \subset \Delta^+ \). Then the promised description of \( I(\mu)_{\max} \) is

\[
I(\mu)_{\max} = I(\mu)_{\min} \cup w_{\mu}^{-1}(M_\mu) .
\]

Furthermore, to get an arbitrary (combinatorial) Abelian ideal in \( \mathfrak{A} \mathfrak{b}_\mu \), one should take any subset \( A \subset M_\mu \) such that the corresponding geometric subspace \( \oplus_{\gamma \in \mathfrak{A} \mathfrak{b}_\gamma} \subset p_{\mu}^{nil} \) be \( \mathfrak{b}_{\mu}^{-} \)-stable. It is easily seen that the last condition is satisfied if and only if \( \oplus_{\gamma \in \mathfrak{A} \mathfrak{b}_\gamma} \subset p_{\mu}^{nil} \) is \( \mathfrak{b}_{\mu}^{-} \)-stable, where \( \mathfrak{A} = M_\mu \setminus A \). In other words, \( A \subset M_\mu \) gives rise to an element of \( \mathfrak{A} \mathfrak{b}_\mu \) if and only if \( \mathfrak{A} \) is a combinatorial \( \mathfrak{b}_{\mu}^{-} \)-ideal. It follows that \( \mathfrak{A} \mathfrak{b}_\mu \) is anti-isomorphic to the poset of \( \mathfrak{b}_{\mu}^{-} \)-ideals in \( p_{\mu}^{nil} \). Since the latter is symmetric, the prefix “anti” can be removed. The posets of ideals in an Abelian nilpotent radical are known as minuscule posets, see e.g. [8, 12]. The minuscule posets have a number of interesting properties; they are rank-symmetric, rank-unimodal, Gaussian, Sperner, etc., see [1].

What can we prove in general in this situation? First, since each root in \( M_\mu \) is orthogonal to \( \mu \), we have, by Theorem 4.1(2), that \( w_{\mu}^{-1}(M_\mu) \subset \Delta^+ \). Second, using the definition of \( I(\mu)_{\min} \), it is not hard to prove that any subset \( I(\mu)_{\min} \cup w_{\mu}^{-1}(A) \) is an Abelian subalgebra of \( \Delta^+ \). But it is not theoretically clear why all these subsets are ideals in \( \Delta^+ \) and why these lie in \( \mathfrak{A} \mathfrak{b}_\mu \). However, a direct verification shows that this construction gives the correct description in all cases.

6. Examples

Here we present our computations for all simple Lie algebras.

(6.1) \( \mathfrak{g} = \mathfrak{sl}_n \). We assume that \( \mathfrak{b} \) is the space of upper-triangular matrices. Then the positive roots are identified with the pairs \((i, j)\), where \(1 \leq i < j \leq n\). Here \( \alpha_i = (i, i + 1) \) and \( \theta = (1, n) \). An Abelian \( \mathfrak{b} \)-ideal is represented by a right-aligned Ferrers diagram such that the number of rows plus the number of columns is at most \( n \). The unique north-east corner of the diagram corresponds to \( \theta \) and the south-west corners give the generators of the corresponding ideal (see also [3, 3.3]). In this case, it is easy to explicitly describe the posets \( \mathfrak{A} \mathfrak{b}_\mu \). If \( \mu = (i, j) \), then

\[
I(i, j)_{\max} = \{(p, q) \mid j \leq q \& p \leq i\} \text{ and } I(i, j)_{\min} = \{(1, q) \mid j \leq q\} \cup \{(p, n) \mid 2 \leq p \leq i\} .
\]

In other words, \( I(i, j)_{\max} \) is the rectangle with the low-left corner at \((i, j)\) and \( I(i, j)_{\min} \) is the “north-east” hook contained in this rectangle, see also Figure [1]. Here \( \#I(i, j)_{\max} = i(n + 1 - j) \) and \( \#I(i, j)_{\min} = n + i - j \). It follows that \( \#I(i, j)_{\max} = \#I(i, j)_{\min} \) if and only
if \( i = 1 \) or \( j = n \), i.e., precisely for the roots that are not orthogonal to \( \theta \). It is not hard to compute that
\[
\#Ab_{(i,j)} = \binom{n + i - j - 1}{i - 1}.
\]
This shows again that \( \#Ab_{(i,j)} = 1 \) if and only if \( i = 1 \) or \( j = n \). This equality is also in accordance with Proposition 5.4. It is curious to observe that the assignment \((i, j) \mapsto \#Ab_{(i,j)}\) gives exactly the Pascal triangle (rotated through the angle 45°).

There is an explicit algorithm for writing out the minuscule element for any \( I \in Ab \). Namely, the minuscule element corresponding to \( I(i,j)_{\min} \) is equal to \((s_{i-1} \ldots s_2 s_1)(s_j \ldots s_{n-2} s_{n-1})s_0\). This can be interpreted as a filling of the respective hook, see Figure 1.

\[\begin{array}{c|c|c|c}
  s_j & \cdots & s_{n-2} s_{n-1} & s_0 \\
  \vdots & & s_1 & \\
  \vdots & & s_2 & \\
  \vdots & & \cdots & \\
  (i,j) & \cdots & s_{i-1} & \\
\end{array}\]

**Figure 1.** The filling of a hook

Note that the products in parentheses, which correspond to the leg and the arm of the hook, commute, so that their order is irrelevant. For an arbitrary Abelian ideal, one should decompose the corresponding Ferrers diagram as the union of ‘north-east’ hooks, and then fill in each hook according to the above rule. The resulting minuscule element is the product of the corresponding hook elements; the first factor corresponds to the smallest hook, etc. The best way for understanding all this is to look at the concrete example.

Consider the Abelian ideal \( I \) in \( \mathfrak{sl}_{10} \) with generators \((1, 5), (2, 7), (3, 8), (4, 9)\). Here the Ferrers diagram is decomposed as the union of three hooks and the corresponding filling is depicted in Figure 2. The minuscule element \( w(I) \) is \( s_0(s_2 s_1)(s_8 s_9)s_0(s_3 s_2 s_1)(s_5 s_6 s_7 s_8 s_9)s_0 \).

\[\begin{array}{cccccccc}
  s_5 & s_6 & s_7 & s_8 & s_9 & s_0 \\
  s_8 & s_9 & s_0 & s_1 & \\
  s_0 & s_1 & s_2 & \\
  & s_2 & s_3 & \\
\end{array}\]

**Figure 2.** The decomposition and filling of the Ferrers diagram for an Abelian ideal in \( \mathfrak{sl}_{10} \)
(6.2) \( g = \mathfrak{so}_{2n+1} \) or \( \mathfrak{so}_{2n} \). In the standard notation, the set of long positive roots is \( \Delta^+_I = \{ \varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n \} \).

Here \( \theta = \varepsilon_1 + \varepsilon_2 \) and \( \mathcal{H} \cap \Delta^+_I = \{ \varepsilon_i \pm \varepsilon_j \mid i = 1, 2 \& j \geq 3 \} \cup \{ \theta \} \). By Theorem \( \text{[5.1]} \), \( \#(\mathfrak{Ab}_\mu) = 1 \) for any \( \mu \in \mathcal{H} \cap \Delta^+_I \). By Proposition \( \text{[5.4]} \), we obtain \( \#(\mathfrak{Ab}_{\varepsilon_1, \varepsilon_2}) = 2 \) and \( \#(\mathfrak{Ab}_{\varepsilon_3, \varepsilon_4}) = 2 \) \((j \geq 4)\). Straightforward computations for the other roots show that \( \#(\mathfrak{Ab}_{\varepsilon_i, \varepsilon_j}) = 2^{i-2} \), if \( i \geq 3 \). Let us demonstrate how all this is related to the description of \( \mathfrak{Ab}_\mu \) in Section \( \text{[3]} \).

Take, for instance, \( \mu = \alpha_{n-2} = \varepsilon_{n-2} - \varepsilon_{n-1} \) for \( \mathfrak{so}_{2n} \). Then \( g(\mu) = \) \( \begin{cases} 0, & \text{if } n = 4 \\ \mathfrak{sl}_2, & \text{if } n = 5 \\ \mathfrak{so}_{2n-6}, & \text{if } n \geq 6 \end{cases} \). For \( n \geq 6 \), the Abelian nilpotent radical in \( g(\mu) \) corresponding to \( \theta \) has dimension \((n-3)(n-4)/2\).

This number is just the difference \( \dim I(\alpha_{n-2})_{\max} - \dim I(\alpha_{n-2})_{\min} \). Hence \( \dim I(\alpha_{n-2})_{\max} = \frac{(n-3)(n-4)}{2} + 2n - 3 = (n^2 - 3n + 1)/2 \), cf. \( \text{[3, Figure 3]} \). In this case, \( g(\mu) \simeq g(\nu) \) and \( \#(\mathfrak{Ab}_\mu) \) is the dimension of the half-spinor representation of \( \mathfrak{so}_{2n-6} \), i.e., \( 2^{n-4} \).

(6.3) \( g = \mathfrak{sp}_{2n} \). In this case, there is only a few long roots:

\( \Delta^+_I = \{ 2\varepsilon_i \mid 1 \leq i \leq n \} , \)

and \( \theta = 2\varepsilon_1 \). We have \( I(2\varepsilon_i)_{\min} = \{ \varepsilon_1 + \varepsilon_i, \ldots, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1 \} \) and \( I(2\varepsilon_i)_{\max} = \{ \varepsilon_k + \varepsilon_j \mid k \leq j \leq i \} \). The sole generator of \( I(2\varepsilon_i)_{\min} \) (resp. \( I(2\varepsilon_i)_{\max} \)) is \( \varepsilon_1 + \varepsilon_i \) (resp. \( 2\varepsilon_i \)). The minuscule element \( w(I(2\varepsilon_i)_{\min}) = s_i s_{i-1} \ldots s_2 s_1 s_0 \). Using the matrix presentation of \( \mathfrak{sp}_{2n} \) (see e.g. \( \text{[3, 3.3]} \)), it is easily seen that there is a one-to-one correspondence between the ideals in \( \mathfrak{Ab}_{2\varepsilon_i} \) and the Abelian ideals of \( \mathfrak{sp}_{2l-2} \). Therefore \( \#(\mathfrak{Ab}_{2\varepsilon_i}) = 2^{l-1} \). It is also possible to give an algorithm for writing out the minuscule element corresponding to an Abelian ideal in terms of filling of a shifted Ferrers diagram.

(6.4) \( g = \mathfrak{F}_4 \). Here we have 12 long positive roots and 15 non-trivial Abelian ideals. The set \( \mathcal{H} \cap \Delta^+_I \) consists of 9 roots. Hence the fibre \( \mathfrak{Ab}_\mu \) contains a unique ideal for these 9 roots and consists of two ideals for the other 3 roots. The computations of rootlets and minuscule elements are presented in Table 1. We follow the numbering of simple roots from \( \text{[13, Tables]} \), and the root \( \sum_{i=1}^{4} c_i \alpha_i \) is denoted by \( (c_1 c_2 c_3 c_4) \). For instance, \( \theta = (2432) \).

The notation \( I_n \) means that the ideal has cardinality \( n \). To distinguish different ideals with the same cardinality, we use ‘prime’. The third, fourth, and fifth columns represent the ideal, the corresponding minuscule element, and the rootlet, respectively.

The maximal Abelian ideals are \( I^0_8 \) and \( I_9 \).

(6.5) \( g = \mathfrak{G}_2 \). Here \( \#(\mathfrak{Ab}) = \#(\Delta^+_I) = 3 \), so that everything is easy. Let \( \alpha \) (resp. \( \beta \)) be the short (resp. long) simple root. Then

\[ I_1 = \{ 3\alpha + 2\beta \}, \quad w(I_1) = s_0, \quad \tau(I_1) = 3\alpha + 2\beta; \]
\[ I_2 = \{ 3\alpha + 2\beta, 3\alpha + \beta \}, \quad w(I_2) = s_\beta s_0, \quad \tau(I_2) = 3\alpha + \beta; \]
\[ I_3 = \{ 3\alpha + 2\beta, 3\alpha + \beta, \beta \}, \quad w(I_3) = s_\alpha s_\beta s_0, \quad \tau(I_3) = \beta; \]
Table 1. The Abelian b-ideals in \( F_4 \)

| No. | \#I | \( I \) | \( w(I) \) | \( \tau(I) \) |
|-----|-----|--------|--------|--------|
| 1   | 1   | \{θ\} | \( s_0 \) | \( θ \) |
| 2   | 2   | \{θ, 2431\} | \( s_4s_0 \) | 2431 |
| 3   | 3   | \{θ, 2431, 2421\} | \( s_3s_4s_0 \) | 2421 |
| 4   | 4   | \{θ, 2431, 2421, 2321\} | \( s_2s_3s_4s_0 \) | 2221 |
| 5   | 5   | \( I_5^0 = I_4 \cup \{2221\} \) | \( s_3s_2s_3s_4s_0 \) | 2211 |
| 6   | 5   | \( I_5^5 = I_4 \cup \{1321\} \) | \( s_1s_2s_3s_4s_0 \) | 0211 |
| 7   | 6   | \( I_6^0 = I_5^0 \cup \{2211\} \) | \( w_0^6 = s_4s_3s_2s_3s_4s_0 \) | 2210 |
| 8   | 6   | \( I_6^6 = I_5^0 \cup \{1321\} \) | \( w_0^6 = s_1s_2s_3s_4s_0 \) | 0211 |
| 9   | 7   | \( I_7^0 = I_6^0 \cup \{2211\} \) | \( w_0^7 = s_0w_0^6 \) | 2210 |
| 10  | 7   | \( I_7^6 = I_6^0 \cup \{1321\} \) | \( w_0^6 = s_1w_0^6 = s_4w_0^6 \) | 0210 |
| 11  | 7   | \( I_7^{10} = I_6^0 \cup \{1221\} \) | \( w_0^{10} = s_2w_0^6 \) | 0011 |
| 12  | 8   | \( I_8^1 = I_7^0 \cup \{1321\} \) | \( w_1^8 = s_1w_1^7 = s_0w_1^7 \) | 0210 |
| 13  | 8   | \( I_8^8 = I_7^0 \cup \{1221\} \) | \( w_8^8 = s_3w_8^7 = s_4w_8^7 \) | 0010 |
| 14  | 8   | \( I_8^{14} = I_7^0 \cup \{0221\} \) | \( w_8^{14} = s_3w_8^7 \) | 0001 |
| 15  | 9   | \( I_9^2 = I_8^0 \cup \{1221\} \) | \( w_9 = s_2w_8^6 = s_0w_8^6 \) | 0010 |

(6.6) \( g = E_n, n = 6, 7, 8 \). Set \( \Delta^+(i) = \{ \mu \in \Delta^+ \mid \#(\mathfrak{A}_\mu) = i \} \) and \( m_i = \#\Delta^+(i) \). Note that \( \Delta^+(i) = \mathcal{H} \). The output of our calculations of numbers \( m_i \) is given in Table 2, where we include only the columns containing nonzero entries. The rightmost column is the control one.

Table 2.

| \( E_6 \) | \( m_1 \) | \( m_2 \) | \( m_3 \) | \( m_4 \) | \( m_5 \) | \( m_6 \) | \( m_8 \) | \( m_{12} \) | \( \sum i m_i \) |
|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 21     | 9      | 4      | -      | -      | 2      | -      | -      | 26 - 1 |
| 33     | 15     | 8      | 4      | -      | 2      | -      | 1      | 27 - 1 |
| 57     | 27     | 16     | 10     | 6      | 3      | 1      | -      | 28 - 1 |

An explicit description of the subsets \( \Delta^+(i) \)'s is also obtained. Again, we follow the numbering of simple roots from Table 3 and denote the root \( \sum_{i=1}^n c_\iota \alpha_i \) by \( \langle c_1 \, c_2 \ldots c_n \rangle \). For instance, the highest root of \( E_6 \) (resp. \( E_7 \)) is \( (1 \, 2 \, 3 \, 2 \, 1 \, 2) \) (resp. \( (1 \, 2 \, 3 \, 4 \, 3 \, 22) \)).

References

1. G.E. Andrews, C. Krattenthaler, L. Orsina and P. Papi. ad-nilpotent b-ideals in \( \mathfrak{sl}(n) \) having a fixed class of nilpotence: combinatorics and enumeration, Trans. Amer. Math. Soc. 354 (2002), 3835–3853.
2. P. Cellini and P. Papi. ad-nilpotent ideals of a Borel subalgebra, J. Algebra 225 (2000), 130–141.
3. P. Cellini and P. Papi. ad-nilpotent ideals of a Borel subalgebra II, J. Algebra, to appear (= math.RT/0106057).
Table 3.

|   | $\Delta^+_1$ | $\Delta^+_2$ | $\Delta^+_3$ | $\Delta^+_4$ | $\Delta^+_5$ | $\Delta^+_6$ | $\Delta^+_8$ | $\Delta^+_{12}$ |
|---|---|---|---|---|---|---|---|---|
| $E_6$ | $c_6 > 0$ | $c_6 = 0$ | $\{\alpha_1 + \alpha_2, \alpha_2, \alpha_4 + \alpha_5, \alpha_4\}$ | $c_3 > 0$ | $c_3 = 0$ | $\{\alpha_1, \alpha_5\}$ | $-$ | $-$ |
| $E_7$ | $c_6 > 0$ | $c_6 = 0$ | $c_6 = c_5 = c_4 = 0$ | $c_5 > 0$ | $c_4 > 0$ | $c_4 > 0$ or $c_3 > 0$ | $-$ | $\{\alpha_2, \alpha_1 + \alpha_2\}$ | $-$ | $\{\alpha_1\}$ |
| $E_8$ | $c_1 > 0$ | $c_1 = 0$ | $c_1 = c_2 = 0$ | $c_1 = c_2 = c_3 = 0$ | $c_4 = 0$ | $c_3 = c_4 = 0$ | $c_5 = 0$ | $\{\alpha_8, \alpha_6, \alpha_6 + \alpha_7\}$ | $\{\alpha_7\}$ | $-$ |

[4] P. Cellini, P. M. Frairia and P. Papi. Compatible Discrete Series, Preprint arXiv: math.RT/0207275.
[5] B. Kostant. The set of abelian ideals of a Borel subalgebra, Cartan decompositions, and discrete series representations, Intern. Math. Res. Notices, no. 5, (1998), 225–252.
[6] C. Krattenthaler, L. Orsina and P. Papi. Enumeration of ad-nilpotent $\mathfrak{b}$-ideals for simple Lie algebras, Adv. in Appl. Math. 28(2002), 478–522.
[7] L. Orsina and P. Papi. Enumeration of ad-nilpotent ideals of a Borel subalgebra in type A by class of nilpotence, C. R. Acad. Sci. Paris, Sér. Math. 330(2000), no. 8, 651–655.
[8] D. Panyushev and G. Röhrle. Spherical orbits and Abelian ideals, Adv. Math. 159(2001), 229–246.
[9] R. Proctor. Bruhat lattices, plane partition generating functions, and minuscule representations, European J. Combin. 5(1984), 331–350.
[10] G. Röhrle. On Normal Abelian Subgroups of Parabolic groups, Annales de l’Institut Fourier, 48(5) (1998), 1455–1482.
[11] R.P. Stanley. “Enumerative Combinatorics”, vol. 1,2. Cambridge Univ. Press, 1997, 1999.
[12] J.R. Stembridge. On minuscule representations, plane partitions and involutions in complex Lie groups, Duke Math. J. 73(1994), 469–490.
[13] Э.Б. Винберг, А.Л. Онищен. Семинар по группам Ли и алгебраическим группам. Москва: “Наука” 1988 (Russian). English translation: A.L. ONISHCHIK and E.B. VINBERG. “Lie groups and algebraic groups”, Berlin: Springer, 1990.