Cylindric partitions, $\mathcal{W}_r$ characters and the Andrews–Gordon–Bressoud identities

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Received 30 September 2015
Accepted for publication 26 January 2016
Published 17 March 2016

Dedicated to Prof R J Baxter on his 75th birthday

Abstract

We study the Andrews–Gordon–Bressoud (AGB) generalisations of the Rogers–Ramanujan $q$-series identities in the context of cylindric partitions. We recall the definition of $r$-cylindric partitions, and provide a simple proof of Borodin’s product expression for their generating functions, that can be regarded as a limiting case of an unpublished proof by Krattenthaler. We also recall the relationships between the $r$-cylindric partition generating functions, the principal characters of $\mathfrak{sl}_r$ algebras, the $\mathcal{M}_{r,r+d}$ minimal model characters of $\mathcal{W}_r$ algebras, and the $r$-string abaci generating functions, providing simple proofs for each. We then set $r = 2$, and use two-cylindric partitions to re-derive the AGB identities as follows. Firstly, we use Borodin’s product expression for the generating functions of the two-cylindric partitions with infinitely long parts, to obtain the product sides of the AGB identities, times a factor $(q; q)_{\infty}$, which is the generating function of ordinary partitions. Next, we obtain a bijection from the two-cylindric partitions, via two-string abaci, into decorated versions of Bressoud’s restricted lattice paths. Extending Bressoud’s method of transforming between restricted paths that obey different restrictions, we obtain sum expressions with manifestly non-negative coefficients for the generating functions of the two-cylindric partitions which contains a factor $(q; q)^{-1}_{\infty}$. Equating the product and sum expressions of the same two-cylindric partitions, and canceling a factor of $(q; q)^{-1}_{\infty}$ on each side, we obtain the AGB identities.

Keywords: cylindric partitions, affine and Virasoro characters, Rogers–Ramanujan identities

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1. Introduction

We recall how the Rogers–Ramanujan identities and their Andrews–Gordon extensions appear in statistical mechanical models in terms of counting weighted paths. Next, we motivate our re-derivation of these identities in terms of counting cylindric partitions.

1.1. Baxter’s corner transfer matrices and Rogers–Ramanujan identities

One of Baxter’s many profound contributions to statistical mechanics is the corner transfer matrix method [1–5], which he introduced to compute order parameters in two-dimensional statistical mechanical lattice models with local weights that satisfy Yang–Baxter equations [6]. One of the surprising consequences of the corner transfer matrix method is the appearance of weighted paths whose generating functions coincide with the very expressions that appear on either side of the Rogers–Ramanujan q-series identities, and generalisations thereof, in the context of computing order parameters [7].

1.2. Restricted solid-on-solid models

An important class of statistical mechanical models are the restricted solid-on-solid models based on the Virasoro algebra \( \mathcal{W}_2 \) [8, 9], and their higher-rank extensions based on \( \mathcal{W}_r \), \( r = 3, 4, \ldots \) [10, 11] 5. These models are labelled by two coprime integers \( p \) and \( p' \), such that \( 1 < p < p' \). We denote the models based on \( \mathcal{W}_r \), and labelled by \( p \) and \( p' \), by \( \mathcal{M}^{p,p'}_r \). In this work, we focus on the models \( \mathcal{M}_r^{p,p'} \), that is \( p = r \), then restrict further to \( \mathcal{M}^{2,2k+3}_2 \), \( k = 1, 2, \ldots \).

1.3. Order parameters and Rogers–Ramanujan-type identities

An important set of observables in statistical mechanical models is the set of ‘order parameters’ 3. In the following, we denote the order parameters of the models \( \mathcal{M}^{2,2k+3}_2 \), \( k = 1, 2, \ldots \), by \( \mathcal{O}^{2,2k+3}_s \). The Andrews–Gordon identities follow from deriving two expressions for \( \mathcal{O}^{2,2k+3}_s \), for each \( k \geq 1 \) and \( 0 \leq s \leq k \), one in a sum form with manifestly non-negative coefficients 5, and one in product form, and equating them.

2 There are ‘higher-level’ versions of the restricted solid-on-solid models based on \( \mathcal{W}_2 \), obtained by a fusion procedure [12, 13]. The same applies, at least in principle, to all restricted solid-on-solid models based on higher rank \( \mathcal{W}_r \). The simplest examples of the fused \( \mathcal{W}_2 \) models are related to the \( N = 1 \) supersymmetric Virasoro algebra. These models, or at least the conformal field theories obtained in the critical limits thereof, are relevant to the Bressoud identities discussed in section 9 [14, 15]. In this work, we approach the Bressoud identities in a combinatorial way.

3 The order parameters are also known as ‘local height probabilities’.

4 In all generality, \( \mathcal{W}_2 \) restricted solid-on-solid models are labelled by all possible pairs of coprime integers \( p \) and \( p' \), such that \( p < p' \). Their order parameters, or more generally, the primary fields in the corresponding conformal field theories, are labelled by two integers \( r_1 \) and \( s_1 \), typically chosen such that \( r_1 = 1, 2, \ldots, p - 1 \), and \( s_1 = 1, 2, \ldots, p' - 1 \). When \( p = 2 \), \( r_1 \) is fixed, \( r_1 = 1 \), and only \( s_1 \) varies. The \( \mathcal{W}_r \) restricted solid-on-solid models have a \( \mathbb{Z}_2 \)-symmetry that reduces the number of independent order parameters. In particular, in the models \( \mathcal{M}^{2,2k+3}_2 \), \( k = 1, 2, \ldots, s_1 \) is further restricted to \( s_1 = 1, 2, \ldots, k + 1 \). In this work, we choose to parameterise the order parameters using \( s = s_1 - 1 = 0, 1, \ldots, k \), which is better suited for our purposes, as it represents the height of the starting point of the one-dimensional paths that are used to compute \( \mathcal{O}^{2,2k+3}_s \).

5 In the sequel, we refer to these sum forms as ‘non-negative sum forms’.
1.4. Two-dimensional lattice configurations

To compute $s_{k^{2,2}_{2,2}+3}$, one needs to sum over all possible two-dimensional lattice configurations, such that the state variable in the middle of the lattice is fixed to a specific value that we parameterise in terms of $s$, and the state variables on the boundaries of the lattice, sufficiently far away from the middle of the lattice, are in ground-state positions. There is only one ground state in each $k^{2,2}_{2,2}+3$ model [9]. This is a difficult problem not only because it requires summing over two-dimensional lattice configurations, but also because these configurations are weighted by products of local weights that are typically complicated functions of the rapidity variables.

1.5. One-dimensional paths on restricted one-dimensional lattices

In the corner transfer matrix approach to the order parameters in $k^{2,2}_{2,2}+3$ models, the Yang–Baxter equations are used to reduce the sums over two-dimensional lattice configurations with complicated local weights, to sums over one-dimensional paths, on restricted one-dimensional lattices, that is one-dimensional lattices with finitely many sites. As we will see in later sections, these one-dimensional paths have simple local weights.

In the case of $k^{2,2}_{2,2}+3$, the restricted one-dimensional lattice has $k + 1$ sites labelled by $s = 0, 1, \ldots, k$. The set of paths that are used to compute $O_s^{2,2k+3}$, where $s = 1, 2, \ldots$, and $s = 0, 1, \ldots, k$ is denoted $A^k_s$.

1.6. The paths as walks on a discrete lattice in discrete time

One can think of the paths that are elements of $A^k_s$ as walks on a one-dimensional $(k + 1)$-site lattice, in discrete time $t$, such that (1) each path starts, at $t = 0$, at site $s$, (2) if a path is at site $h \in \{1, 2, \ldots, k - 1\}$ at time $t$, then it must be at site $h' = h + 1$ at time $t + 1$, (3) if a path is at site $h = k$, at time $t$, then it must be at $h' = k - 1$, at time $t + 1$, (4) if a path is at site $h = 0$, at time $t$, then it can be at $h = 0$ or at $h = 1$, at time $t + 1$, that is, a path is allowed to stay at $h = 0$ for as long as it wishes, and finally (5) a path must asymptotically settle at $h = 0$ and stay at $h = 0$ at all $t \geq N$, for large but finite $N$.

1.6.1. Example. The model $M^{2,5}$ has two order parameters $O^2,5$ and $O^1,5$. An example of a path in the set $A^5_0$, whose weighted generating function leads to the order parameter $O^{2,5}_0$, is in figure 1.

1.7. Alternating-sign $q$-series expressions

The generating functions of the one-dimensional paths with simple weights, obtained using the corner transfer matrix method, can be evaluated relatively straightforwardly using inclusion-exclusion, in the form of alternating-sign $q$-series [8, 9].

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Figure 1. A path in $A^5_0$. The vertical axis is the two-site one-dimensional lattice that the paths are walks on. The horizontal axis represents the discrete time $t$ that starts at $t = 0$. The paths starts at $h = 0$ because it is in $A^5_0$. 

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J. Phys. A: Math. Theor. 49 (2016) 164004

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1.7.1. Pochhammer symbol. We need the following standard definition of the Pochhammer symbol $(z; q)_n$ and its limit $(z; q)_\infty$,

$$(z; q)_0 = 1, \quad (z; q)_n = \prod_{i=0}^{n-1} (1 - zq^i), \quad (z; q)_\infty = \prod_{i=0}^{\infty} (1 - zq^i).$$

1.7.2. Example. In $\mathcal{M}^{2,5}_r$, there are two allowed values $s$ for the height of the initial point of the paths, namely $s = 0, 1$, and one obtains

$$O_{r}^{2,5}(q) \sim \frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} (q^{n(10n+1+2s)} - q^{(2n+1)(5n+2-s)}), \quad s = 0, 1,$$

where we have used the proportionality sign $\sim$ to indicate that we have neglected a normalisation constant that need not concern us here.

1.7.3. Bosonic expressions. In all generality, the order parameters in $\mathcal{M}_r^{p,p'}$ can be evaluated in alternating-sign series form. These alternating-sign series coincide with the characters of the minimal $\mathcal{W}_2$ conformal field theories labelled by coprime $p$ and $p'$, when these characters are evaluated in Feigin–Fuchs form [16, 17]. These forms are called bosonic [18–20], since they can be obtained using the oscillators of $(r - 1)$ free boson fields, in a background charge.

1.8. Manifestly non-negative $q$-series expressions

Since the generating functions, computed using the corner transfer matrix method, count weighted paths, it is natural to try to re-write them as $q$-series with non-negative coefficients. This turns out to be less straightforward to do, compared to the derivation of the alternating-sign expression, but can be done in many cases. In the following, we refer to the non-negative-sign expression as sum expressions.

1.8.1. Fermionic expressions. The non-negative sum expressions provide other forms for the corresponding $\mathcal{W}_2$ characters. These forms are called fermionic [18–24].

1.8.2. Example. The alternating-sign expressions on the right-hand side of (2) can be put in non-negative sum form. One way to do this is to compute the generating function of the weighted paths combinatorially [25], to obtain

$$\frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} (q^{n(10n+1+2s)} - q^{(2n+1)(5n+2-s)}) = \sum_{n=0}^{\infty} q^{n^2 + sn}, \quad s = 0, 1.$$

But these sum expressions are nothing but the sum sides of the celebrated Rogers–Ramanujan identities.

1.9. Product $q$-series expressions and Rogers–Ramanujan-type identities

In the case of the order parameters $O_{r}^{2,2k+3}$, $k = 1, 2, \ldots$, one can use $q$-series identities to put the sum expressions in product form\footnote{To be precise, one uses the $q$-series identity to put the alternating-sign sum side in product form, then uses the equality of the alternating-sign form and the non-negative form, proven separately.}. Equating the non-negative sum and the product expressions, one obtains the Rogers–Ramanujan identities and the Andrews–Gordon extensions thereof.
1.9.1. Example. The sum expressions on the right-hand side of (3), can be put in product form as

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+sn}}{(q; q)_n} = \prod_{n=1}^{\infty} \frac{1}{1 - q^{n^2}}, \quad s = 0, 1,
\]

which are the original Rogers–Ramanujan identities.

1.10. Purpose of this work

The Rogers–Ramanujan identities and their Andrews–Gordon extensions are related to the Virasoro algebra \( \mathcal{W}_2 \). The Bressoud identities are related to the \( N = 1 \) supersymmetric extension of the Virasoro algebra. They can be regarded as equalities between sum and product expressions for the generating functions of one-dimensional weighted paths. For models based on \( \mathcal{W}_r \) algebras, \( r = 3, 4, \ldots \), paths are available, but they are walks on restricted \( (r-1) \)-dimensional lattices, and therefore are not as useful, and as easy to work with as in the \( \mathcal{W}_2 \) case.

We re-derive the AGB identities as equalities between sum and product expressions for the generating functions of two-cylindric partitions. Since the latter have useful higher-rank analogues, this approach may lead to a systematic derivation of the higher rank identities\(^7\).

1.11. Outline of contents

In section 2, we recall basic facts related to \( r \)-cylindric partitions, then provide a straightforward proof of Borodin’s product expression for their generating functions. In section 3, we recall basic facts related to the principal characters of \( \hat{\mathfrak{sl}}_r \) algebras, and the \( \mathcal{M}_{r,r+d} \) minimal model characters of \( \mathcal{W}_r \). In section 4, we express the \( r \)-cylindric partitions in terms of \( \mathcal{M}_{r,r+d} \) abaci. In section 5, we state our results for both the Andrews–Gordon and the Bressoud identities. In section 6, we introduce Bressoud’s paths, the related decorated paths, and Bressoud’s transforms between different paths. In section 7, we focus on the \( r = 2 \) case, use the two-string abaci to map the two-cylindric partitions to decorated Bressoud-type paths, then use the Bressoud transforms between paths to obtain sum expressions for the generating functions of the \( r \)-cylindric partitions. Equating the product expressions of section 2 with the sum expressions of this section, we obtain the Andrews–Gordon identities. Section 8 reconsiders the proof in section 7 in terms of abaci. In section 9, we prove the Bressoud identities. Section 10 includes comments and remarks. Three appendixes contain reviews of technical details. Appendix A recalls the \( \hat{\mathfrak{sl}}_r \) Macdonald identity. Appendix B contains basic facts related to \( \hat{\mathfrak{sl}}_r \) weight space and characters. Appendix C is on the Kyoto patterns and their relation to cylindric partitions.

2. Cylindric partitions

We recall basic facts related to cylindric partitions, and provide a simple proof of Borodin’s product expression for their generating functions.

\(^7\) As we will see in section 2, the \( \mathcal{W}_r \) product sides are known from the \( r \)-cylindric partition approach, for all \( r \). The \( \mathcal{W}_r \) sum sides are known from \([26, 27]\). The \( \mathcal{W}_r \) sum sides, for \( r = 4, 5, \ldots \), are yet to be found. The purpose of this work is to pave a possible way to compute them.
2.1. Brief history

Two-row, that is, two-cylindric partitions were introduced, using different terminology, by Burge [28]. They were formally defined in general form and studied by Gessel and Krattenthaler [29]. Special cases were already present in the work of the Kyoto group [30, 31] on the combinatorics of representations of \( \hat{sl}_r \) and its quantum analogue \( U_q(\hat{sl}_r) \) at \( q = 0 \). In another guise, they were used to label the irreducible representations of the Ariki–Koike algebras for certain choices of the parameters of these algebras [32].

2.2. Cylindric partitions as restricted plane partitions

What we refer to, in section 1, as \( r \)-cylindric partitions, are plane partitions that consist of \( r \) rows, with a condition that relates the entries in the 1st and the \( r \)th rows. Because of this condition, one can naturally draw cylindric partitions on a cylinder.

2.3. Cylindric partitions with infinitely long rows

The special cases mentioned in section 2.1 are such that the rows are allowed to be infinitely long. In this work, we will focus on the cylindric partitions with possibly infinitely long rows.

2.4. Borodin’s product expression and relation to affine and \( \mathcal{W}_r \) algebras

A product expression for the generating functions of the \( r \)-cylindric partitions with possibly infinitely long rows, was obtained by Borodin [33]. This product expression shows that the generating functions of these \( r \)-cylindric partitions are closely related to the principally specialised characters of integrable representations of the affine Lie algebra \( \hat{sl}_r \), or equivalently \( A_r^{(1)} \), as well as to characters of specific minimal conformal field theories based on, or equivalently, specific degenerate representations of the \( \mathcal{W}_r \) algebras, the \( r = 2 \) case of which is the Virasoro algebra. If \( \Lambda \) is a level \( d \) dominant integral weight of \( \hat{sl}_r \), then there is a set of cylindric partitions associated with \( \Lambda \), as will be made clear below, whose generating function \( C^\Lambda_r \) satisfies

\[
C^\Lambda_r = \frac{Pr \chi^\Lambda_{\hat{sl}_r}}{(q'; q')_\infty} = \frac{\chi^\Lambda_{\mathcal{W}_r}}{(q; q)_\infty},
\]

where \( Pr \chi^\Lambda_{\hat{sl}_r} \) is the principal character of \( \hat{sl}_r \) labelled by \( \Lambda \), and \( \chi^\Lambda_{\mathcal{W}_r} \) is a normalised character of the \( \mathcal{W}_r \) minimal model \( \mathcal{M}_{c; r+d} \), this character also labelled by \( \Lambda \). Note that the minimal models \( \mathcal{M}_{c; r+d} \), based on the algebras \( \mathcal{W}_r \), are defined only if \( r \) and \( d \) are coprime. Thus in other than such a case, the second equality in (5) should be ignored. However, the first equality continues to hold.

2.5. Ordinary partitions

We define an ordinary partition \( \lambda \) to be a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots) \) for which \( \lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \), with only finitely many non-zero elements. If we write \( \lambda = (\lambda_1, \ldots, \lambda_p) \), then we imply that \( \lambda_i = 0 \) for \( i > p \). The length \( \ell(\lambda) \) of \( \lambda \) is the smallest \( \ell \) such that \( \lambda_{\ell+1} = 0 \), while the weight \( |\lambda| \) of \( \lambda \) is defined by \( |\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)} \).

\[
8 \quad \text{As mentioned above, we will not discuss statistical mechanical models, or conformal field theories, based on supersymmetric extensions of } \mathcal{W}_r, \text{ where the labels } p \text{ and } p' \text{ need not be co-prime, in this work. Instead, we will obtain the corresponding characters, combinatorially, from those related to } \mathcal{W}_r.
\]
2.6. Formal definition of cylindric partitions

Following Gessel and Krattenthaler [29], given two ordinary partitions \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_r) \), and \( d \geq 0 \), a cylindric partition of type \( \lambda/\mu/d \) is an array of non-negative integers \( p_{i,j} \) of the form

\[
\begin{array}{cccc}
\pi_{1,\mu_1+1} & \ldots & \ldots & \pi_{1,\lambda_1} \\
\pi_{2,\mu_2+1} & \ldots & \ldots & \pi_{2,\lambda_2} \\
& \ldots & \ldots & \\
\pi_{r,\mu_r+1} & \ldots & \ldots & \pi_{r,\lambda_r} \\
\end{array}
\]  

such that the entries are weakly decreasing across the rows and down the columns, i.e.

\[
\pi_{i,j} \geq \pi_{i,j+1} \quad (1 \leq i \leq r, \mu_i \leq j < \lambda_i), \tag{7a}
\]

\[
\pi_{i,j} \geq \pi_{i+1,j} \quad (1 \leq i < r, \max \{\mu_i, \mu_{i+1}\} \leq j < \min \{\lambda_i, \lambda_{i+1}\}). \tag{7b}
\]

and also respect the cylindrical condition

\[
\pi_{i,j} \geq \pi_{i,j+d} \quad \max \{\mu_r, \mu_1 - d\} \leq j \leq \min \{\lambda_r, \lambda_1 - d\}. \tag{7c}
\]

Condition (7c) can be regarded as the weakly decreasing column condition, extending to a copy of the first row of the array (6), displaced \( d \) positions to the left and placed below the \( r \)th row. Note that the ordinary partition \( \lambda \) has \( r \) parts, and these parts determine the maximal possible lengths of the \( r \) rows of the \( r \)-cylindric partition of type \( \lambda/\mu/d \). On the other hand, the ordinary partition \( \mu \) also has \( r \) parts, and these parts determine the relative shifts of the \( r \) rows of the \( r \)-cylindric partition of type \( \lambda/\mu/d \).

As mentioned in section 2.3, in this paper, we are primarily interested in cylindric partitions with rows that are allowed to be arbitrarily long. These cylindric partitions are then of type \((\infty')/\mu/d\).

2.6.1. Example. A cylindric partition of type \((\infty')/\mu/d\), such that \( r = 3, \mu = (1, 1, 0) \), and \( d = 4 \), is

\[
\begin{array}{cccc}
5 & 4 & 4 & 2 \ldots \\
3 & 2 & 2 & 1 \ldots \\
4 & 2 & 2 & 1 \ldots \\
\end{array}
\]  

where each dot represents a zero, and these are assumed to extend indefinitely to the right.

2.7. Working on a cylinder

As mentioned above, it is useful to visualise the cylindric condition (7c) by placing a copy of the first row, displaced \( d \) positions to the left, below the \( r \)th row. In fact, it is useful to place copies of the whole cylindric partition above and below the ‘main copy’, so to speak, displaced appropriately.
2.7.1. Example. From the example in paragraph 2.6.1, we obtain

\begin{align*}
5 & 4 4 2 2 1 1 \cdots \\
3 & 2 2 2 1 \cdots \\
4 & 2 2 1 \cdots \\
5 & 4 4 2 2 1 1 \cdots \\
3 & 2 2 2 1 \cdots \\
4 & 2 2 1 \cdots \\
\end{align*}

\begin{equation}
(9)
\end{equation}

2.8. Level-rank duality

By focussing on \(d\) consecutive columns of this array, we see that cylindric partitions are in bijection with those having \(r\) and \(d\) interchanged, and \(\mu\) replaced by its conjugate \(\mu'\). This demonstrates a level-rank duality.

2.9. Labeling cylindric partitions by affine fundamental weights

Let \(\Lambda_0, \Lambda_1, \ldots, \Lambda_{r-1}\) denote the fundamental weights of \(\hat{sl}_r\). The level-\(d\) dominant integral weight lattice of \(\hat{sl}_r\) is then defined by

\[
P^{\Lambda}_{r,d} = \sum_{i=0}^{r-1} m_i \Lambda_i, \quad m_i \in \mathbb{Z}_{\geq 0}, \quad 0 \leq i < r, \sum_{i=0}^{r-1} m_i = d.
\]

A weight \(\sum_{i=0}^{r-1} m_i \Lambda_i\) will be denoted \([m_0, m_1, \ldots, m_{r-1}]\).

Let \(\mathcal{P}_{\Lambda}\) denote the set of all partitions \(\lambda\) for which \(\ell(\lambda) \leq k\). For \(\Lambda = \sum_{i=0}^{r-1} m_i \Lambda_i \in P^{\Lambda}_{r,d}\), we define a corresponding partition \(\hat{\lambda}(\Lambda) = (\mu_1, \mu_2, \ldots)\) by setting \(\mu_j = \sum_{i=j+1}^{r-1} m_i\) for \(j = 1, \ldots, r-1\) and \(\mu_j = 0\) for \(j \geq r\). Thus \(\hat{\lambda}(\Lambda) \in \mathcal{P}_{r-1}\).

2.10. Generating functions

For \(\Lambda \in P^{\Lambda}_{r,d}\) and \(\mu = \hat{\lambda}(\Lambda)\), define \(\mathcal{C}_\Lambda^{\mu}\) to be the set of all cylindric partitions of type \((\infty')/\mu/d\) that have a finite number of non-zero entries, and let \(\mathcal{C}_\Lambda^{\mu}(a)\) be the subset of \(\mathcal{C}_\Lambda^{\mu}\) comprising those \(\pi\) whose entries do not exceed \(a\). In the case of the cylindric partition \(\pi\) of (8) and (9), \(|\pi| = 38\). Define the generating functions

\[
\mathcal{C}_\Lambda^{\mu} = \sum_{\pi \in \mathcal{C}_\Lambda^{\mu}} q^{|\pi|}, \quad \mathcal{C}_\Lambda^{\mu}(a) = \sum_{\pi \in \mathcal{C}_\Lambda^{\mu}(a)} q^{|\pi|}.
\]

2.11. Product formula

There is a remarkably simple product formula for \(\mathcal{C}_\Lambda^{\mu}\), first derived by Borodin [33]. To express this, first construct a length \((r + d)\) binary word \(\hat{\omega}(\Lambda) = \omega_1 \omega_2 \cdots \omega_r \omega_{r+1} \cdots \omega_{r+d}\) from \(\Lambda \in P^{\Lambda}_{r,d}\). This word comprises \(r\) 0’s and \(d\) 1’s, and for \(\Lambda = \sum_{i=0}^{r-1} m_i \Lambda_i\), is defined by putting the \(j\)th 0 at position \(j + \sum_{i=0}^{j-1} m_i\).

Alternatively, consider the partition \(\mu\) as sitting in an \(r \times d\) box. Then on viewing the profile of \(\mu\) as a lattice path passing from the top right of the box to the bottom left, the word \(\hat{\omega}(\Lambda)\) is constructed by interpreting each horizontal step as 1 and each vertical step as 0.
Theorem 1. [33, proposition 5.1] If $\Lambda \in \mathcal{P}_{r,d}^{+}$ and $\omega = \hat{\omega}(\Lambda)$, then

$$C_{\Lambda} = \frac{1}{(q^{r+d}; q_{r+d})_{\infty}} \prod_{1 \leq i < j \leq r+d} \frac{1}{(q^{r+d-j+i}; q_{r+d})_{\infty}} \prod_{1 \leq i < j \leq r+d} \frac{1}{(q^{r+d-i+j}; q_{r+d})_{\infty}}. \quad (12)$$

Below, we provide a straightforward proof of this result, based on the results of [29]. This proof may be viewed as a limiting case of an unpublished proof of Krattenthaler [34]. Before doing so, we illustrate the result, demonstrating that it may be cast in terms of hook-lengths.

2.12. Hook lengths

The hook-length of a node is the number of nodes in the hook comprising the nodes directly above and to the left of the given node.

2.13. The product expression in terms of hook lengths

Consider the case $r = 5$ with $\Lambda = [1, 3, 0, 2, 1]$ so that $d = 7$. From this we obtain $\omega = \hat{\omega}(\Lambda) = 101110011010$. The profile of the corresponding $\mu = (6, 3, 3, 1)$ is then the thick line in figure 2. The exponents $(j - i)$ that appear in the first product on the right side of (12) are inserted in the boxes below the profile in figure 2, and are seen to correspond to hook-lengths. The exponents $(r + d - j + i)$ that appear in the second product are inserted in the boxes above the profile, and are seen to correspond to ‘reverse’ hook-lengths.

Therefore, we obtain

$$C_{5}^{[1,3,0,2,1]} = \frac{1}{(q_{12}; q_{12})_{\infty}(q, q^{3}, q^{9}, q^{11}; q_{12})_{\infty}(q^{2}, q^{5}, q^{6}, q^{7}; q_{12})_{\infty}(q^{4}, q^{8}; q_{12})_{\infty}}.$$ \quad (13)

where $(z_{1}, \ldots, z_{i}; q)_{\infty} = (z_{1}; q)_{\infty} \cdots (z_{i}; q)_{\infty}$. Note that it is also possible to place all $(r \times d)$ exponents below the profile, with all corresponding to genuine hook-lengths, although they will no longer form a rectangle, in general.

2.14. Short proof of theorem 1

In [26], Andrews et al observed that all $\mathcal{M}_{r,d}^{+}$ characters are expressible in product form. This suggests that theorem 1 can be proved by applying the $\hat{\delta}_{l}$ Macdonald identity to the appropriate limit of theorem 2 of [29]. To apply this latter result to the cylindric partitions of...
type $(\infty')/\mu/d$ defined above, we use the case of theorem 2 of [29] for which each $a_j = a$ and each $b_j = 0$, and then take the limit $\lambda \to (\infty')$.

**Theorem 2.** [29, specialisation of theorem 2] If $\Lambda \in \mathcal{P}_{r,d}^+$ and $\mu = \hat{\lambda}(\Lambda)$, then

$$C_{\lambda}^\prime(a) = \sum_{k_1 + \cdots + k_r = 0}^{d \leq k \leq r} \det \left( \frac{q^{(\mu_k-k+r-t+i-t)(r+k+t)(r+d)}}{(q; q)_i} \right).$$

(14)

By taking the $a \to \infty$ limit in (14), we obtain

$$C_{\lambda}^\prime = \lim_{a \to \infty} C_{\lambda}^\prime(a) = \frac{1}{(q; q)_\infty} \sum_{k_1 + \cdots + k_r = 0}^{d \leq k \leq r} \det \left( q^{(\mu_k-k+r-t+i-t)(r+k+t)(r+d)} \right).$$

(15)

The $\hat{\mathfrak{sl}}_r$ Macdonald identity can be expressed in the form$^{10}$,

$$\sum_{k_1 + \cdots + k_r = 0}^{1 \leq i \leq r} \det \left( x_i^{(r+k+t-i)} q^{k(r+k+t)} \right) = (q; q)_\infty^{-1} \prod_{1 \leq i < j \leq r} \left( \frac{x_i}{x_j} q^{-1} \right).$$

(16)

Applying this, with $x_r \to q^{\mu_r-t}$, and $q \to q^{r+d}$, to (15) yields

$$C_{\lambda}^\prime = \frac{(q^{r+d}; q^{r+d})_\infty^{-1}}{(q; q)_\infty} \prod_{1 \leq i < j \leq r} (q^{\mu_i-i+j} q^{r+d-\mu_j+i-j}; q^{r+d})_\infty$$

(17)

theorem 1 follows by considering the terms that remain after cancelling each factor in the final product with a factor from $(q; q)_\infty$, possibly with the help of a diagram as in figure 2.

### 3. $\hat{\mathfrak{sl}}_r$ and $\mathcal{W}_r$ characters

We give simple proofs of the relationship between the generating functions of $r$-cylindric partitions and the principally specialised characters of $\hat{\mathfrak{sl}}_r$ and certain characters of $\mathcal{W}_r$.

#### 3.1. $\hat{\mathfrak{sl}}_r$ characters

As detailed in appendix B.2, the Weyl–Kac character formula [38] enables the character of an integrable $\hat{\mathfrak{sl}}_r$-module, with highest weight $\Lambda \in \mathcal{P}_{r,d}^+$, to be written in the form

$$\chi_{\hat{\mathfrak{sl}}_r}^{\Lambda} = e^{\lambda N_{\Lambda}^{\hat{\mathfrak{sl}}_r}} N_{\Lambda}^{\hat{\mathfrak{sl}}_r},$$

(18)

with $N_{\Lambda}^{\hat{\mathfrak{sl}}_r}$ given by

$$N_{\Lambda}^{\hat{\mathfrak{sl}}_r} = \sum_{k_1 + \cdots + k_r = 0}^{d \leq k \leq r} \det \left( x_i^{(-r+d)(-\nu_i+r+t+\nu_{i-1})} q^{(\mu_i-i)(r+k+t)} \right).$$

(19)

where $\mu = \hat{\lambda}(\Lambda)$, $q = e^{-d}$ and $x_1/x_{i+1} = e^{-n}$ for $1 \leq i < r$. Applying the Macdonald identity (74) to the $\Lambda = 0$ case, the denominator of (18) can be written as

---

10 See appendix A.
11 This implies that $x_1, x_2, \ldots, x_r$ are specified only up to an overall factor. However, although not immediately obvious, $N_{\Lambda}^{\hat{\mathfrak{sl}}_r}$, as defined by (19), is independent of such a factor.
The numerator $\mathcal{N}_{0}^{\hat{\lambda}}$ of (18) cannot be written in product form. However, there is a product form for the principal specialisation $\Pr \mathcal{N}_{0}^{\hat{\lambda}}$ of (19) in which $e^{-\alpha_i} \to q$, for each simple root $\alpha_i$. This specialisation is obtained by substituting $q \to q'$ and $x_i \to q'^{-i}$, leading to

$$\Pr \mathcal{N}_{0}^{\hat{\lambda}} = \sum_{k_1+\cdots+k_r=0}^{\infty} \prod_{1 \leq i < j \leq r} \det (q^{(r+d)k_i\delta_{ij}} (\mu_i - \mu_j + z_i + z_j) (\mu_i - \mu_j + z_i + z_j - 1))$$

$$= \sum_{k_1+\cdots+k_r=0}^{\infty} \prod_{1 \leq i < j \leq r} q^{-(\mu_i - \mu_j)} \det (q^{(\mu_i - \mu_j)(k_i - k_j) + (r+d)k_i + (r+d)k_j}).$$

(21)

In view of (20), we also have

$$\Pr \mathcal{N}_{0}^{\hat{\lambda}} = (q'; q')^{\infty} \prod_{1 \leq i < j \leq r} (q'^{-i}, q'^{-j}; q')_{\infty} = \frac{(q'; q')^{\infty}}{(q'; q')_{\infty}}.$$

(22)

Comparing (15) with the ratio of (21) and (22) yields

$$C'_{\xi} = \frac{1}{(q'; q')_{\infty}} \frac{\Pr \mathcal{N}_{0}^{\hat{\lambda}}}{\Pr \mathcal{N}_{0}^{\hat{\lambda}'}}.$$

(23)

The principally specialised character $\Pr \hat{\xi}$ is obtained from $\hat{\xi}$, by setting $e^{-\alpha_i} \to q$, for each simple root $\alpha_i$, and setting $e^{\lambda} = 1$. Combining (18) and (23) proves the first equality in (5).

3.2. $\mathcal{W}_r$ characters

Let $r \leq p < p'$, with $p$ and $p'$ coprime, and let $\xi \in \mathcal{P}_r^{p-p'}$, and $\zeta \in \mathcal{P}_r^{p'-r}$. As detailed in appendix B.3, the expression of [40–42] for the $\mathcal{M}_{r,p}^{p'}$ character of the highest weight $\mathcal{W}_r$ representation labelled by the pair $\xi$ and $\zeta$ can be written in the form

$$\chi_{r,p}^{p'} = \frac{q^{\hat{\lambda}_{\xi,\zeta}}}{(q'; q')_{\infty}} \sum_{k_1+\cdots+k_r=0}^{\infty} q^{p'\hat{\Delta}'_{\xi,\zeta}} \det (q^{(\mu_i - \mu_j)(k_i - k_j) + (r+d)k_i + (r+d)k_j}).$$

(24)

where the partitions $\mu$ and $\nu$ are defined by $\mu = \hat{\lambda}(\xi)$ and $\nu = \hat{\lambda}(\zeta)$, and

$$\hat{\Delta}'_{\xi,\zeta} = \frac{1}{2p'} |p'| (\xi + \rho) - p(\xi + \rho)|^2 - \frac{1}{24} (r - 1).$$

3.3. $\mathcal{W}_r$ characters of type $\chi_{0,0}^{r,r,d}$

In this paper, we are primarily interested in the $\mathcal{M}_{r,r,d}$ characters of $\mathcal{W}_r$ for $d > 0$. In this case $\xi = 0$ and therefore the characters are labelled by a single weight $\zeta \in \mathcal{P}_r^{d}$. On comparing $\chi_{0,0}^{r,r,d}$, given by (24), with (15), we see that the normalised character $\chi_{0,0}^{r,r,d}$ satisfies

$$\chi_{0,0}^{r,r,d} = q^{-\Delta_{0,0}^{r,r,d}} \chi_{0,0}^{r,r,d}.$$

This holds for any affine Lie algebra $g$: see [38, proposition 10.10] and also [39].

12 This holds for any affine Lie algebra $g$: see [38, proposition 10.10] and also [39].
This proves the equality between the first and third expressions in (5).

3.4. \( W \), characters of type \( \chi^{r,p,p'}_{r} \)

In [29], Gessel and Krattenthaler define the notion of \((\alpha, \beta)\)-cylindric partitions of type \( \lambda/\mu/d \), which generalises that defined in section 2.6. This enables an analogue of (25) to be given for the most general minimal model \( W \), characters \( \chi^{r,p,p'}_{r} \).

For \( \zeta = \sum_{i=0}^{r-1} m_i \lambda_i \), \( \xi = \sum_{i=0}^{r-1} n_i \lambda_i \), and \( \mu = \hat{\lambda}(\zeta) \), let \( C^r_{\xi,\zeta} \) be the set of all arrays \( \pi \) of the form (6) with \( \lambda = (\infty') \) that satisfy

\[
\begin{align*}
\pi_{i,j} &\geq \pi_{i,j+1} \quad (1 \leq i \leq r, \, \mu_i \leq j), \\
\pi_{i,j} &> \pi_{i+1,j} - n_i \quad (1 \leq i < r, \, \max \{ \mu_i, \mu_{i+1} \} \leq j), \\
\pi_{r,j} &> \pi_{r,j+d} - n_0 \quad (\max \{ \mu_r, \mu_1 - d \} \leq j)
\end{align*}
\]

instead of (7). In the terminology of [29], \( C^r_{\xi,\zeta} \) is the set of \((0, \beta)\)-cylindric partitions of type \((\infty')/\mu/(p' - r)\), where \( \beta = (-n_1, -n_2, \ldots, -n_{r-1}, -n_0) \). With \( C^r_{\xi,\zeta} \) the generating function of \( C^r_{\xi,\zeta} \), comparing (24) with the appropriate limit of [29, theorem 3] yields

\[
\chi^{r,p,p'}_{r} = (q; q)^{\infty} C^r_{\xi,\zeta}.
\]

3.5. \( r \)-Burge partitions

The cylindric partitions \( C^r_{\xi,\zeta} \) are also \( r \)-Burge partitions, as defined in [43]. To see this, define \( \tilde{\pi}_{i,j} = \pi_{i,\mu_{i+1} + j} \) for \( 1 \leq i \leq r \) and \( j \geq 0 \). Then, for each \( i \), the sequence \( \tilde{\pi}_i = (\tilde{\pi}_{i,1}, \tilde{\pi}_{i,2}, \ldots) \) is a partition. The definition of \( \mu = \hat{\lambda}(\zeta) \) implies that \( \mu_i - \mu_{i+1} = m_i \) for \( 1 \leq i < r \). Therefore, \( \tilde{\pi}_{i,j} = \tilde{\pi}_{i+1,j+m_i} = \pi_{i,j+\mu_i} - \pi_{i+1,j+\mu_i} = -n_i \) for \( 1 \leq i < r \) and \( \tilde{\pi}_{r,j} = \tilde{\pi}_{r,j+\mu_0} = \pi_{r,j} - \pi_{r,j+d} = -n_0 \). After appropriately renaming the parameters and indices, we obtain precisely the conditions [43, (1)].

4. Reckoning with the abacus

We introduce the \( r \)-string abaci which are in bijection with \( r \)-cylindric partitions.

4.1. Introducing the abacus

An \( r \)-string abacus consists of \( r \) infinitely long strings. Each string is a one-dimensional discrete, regular lattice, with infinitely many equally spaced sites. On each string, there is a semi-infinite set of beads. The beads are allowed to occupy the sites on the string, such that only a finite number of beads are to the right of any particular position on the string. The semi-infinite set of beads acts as a Dirac sea from which excitations emerge in the form of beads that move to the right along a string to occupy initially vacant positions\(^{14}\).

\(^{14}\) The \( r \)-string abacus is an \( r \)-fold version of a Maya diagram [44, 45].
4.2. From r-cylindric partitions to r-string abaci

r-cylindric partitions are conveniently represented using r-string abaci. Because of level-rank duality\textsuperscript{15}, two such representations are possible, one with r strings and one with d strings. For the purposes of the current paper, we will use the representation that has r strings.

The r-string abacus that corresponds to an r-cylindric partition is obtained by associating each vacancy on the tth string of the abacus with an entry in the tth row of the cylindric partition. Specifically, scanning the tth row the r-cylindric partition from left to right, the ith entry gives the number of beads that appear on the tth string to the right of the ith vacancy.

4.2.1. Example. The r = 3, d = 4 cylindric partition of (8) corresponds to the 3-string abacus in figure 3. In this example, the beads have been yoked together, that is, attached with finite-length strings that consist of (r - 1) segments each, in such a way that they allow for sets of vacancies. Far to the left, the yoked r-tuples of beads are all of the same shape. This shape corresponds to the partition $$\lambda(\Lambda)$$ in the definition of the r-cylindric partition. These yoked r-tuples are vertical when $$\Lambda = d \Lambda_0$$. Each shape of yokes, and each set of vacancies corresponds to a unique r-cylindric partition.

4.3. The formation of an r-cylindric partition

For $$j > 0$$, let $$\delta_j$$ be the number of entries j in (r consecutive rows of) the cylindrical partition $$\pi \in \mathcal{C}_h\Lambda$$. Then $$\delta_j$$ is the number of vacancies between the jth and (j + 1)-th yoked r-tuples of the corresponding abacus, counting the r-tuples from the right. We immediately have

$$|\pi| = \sum_{j=1}^{\infty} j \delta_j,$$  \hspace{1cm} (28)

\textsuperscript{15} See section 2.8.
We refer to the sequence $\delta(\pi) = (\ldots, \delta_3, \delta_2, \delta_1)$ as the formation of $\pi$. Note that if $\pi \in C'_\lambda(a)$ then $\delta_j = 0$ for $j > a$.

4.4. The cylindric abacus and $\Lambda$ yokes

Corresponding to the augmented viewpoint (9) of the cylindric partition, we can place infinitely many copies of the $r$ strings of the abacus above and below the original abacus, with each copy displaced by $d$ positions further to the right with respect to the copy immediately below it. Equivalently, the $r$ strings may be placed on a cylinder. Doing this in the case of figure 3 leads to a diagram, a portion of which is shown, in figure 4.

In view of the above cylindrical extension, each yoked $r$-tuple of beads should be considered as being of infinite length, consisting of the same repeating sequence of $r$ segments whose horizontal lengths, $m_0, m_1, \ldots, m_{r-1}$, sum to $d$. This way, each $r$-tuple is associated with a dominant affine weight $\Lambda = [m_0, m_1, \ldots, m_{r-1}] \in F^r_{d,d}$. We call this a $\Lambda$-yoke.

4.5. $\Lambda$ yokels

For $\Lambda \in F^r_{d,d}$, we define a $\Lambda$-yoke to be a semi-infinite sequence $\Lambda = (\ldots, \Lambda(3), \Lambda(2), \Lambda(1))$ of elements $\Lambda(j) \in F^r_{d,d}$ for $j \geq 1$, for which there exists $L$ such that $\Lambda(j) = \Lambda$ for all $j > L$. Let $\ell(\Lambda)$ be the smallest such $L$. Note that if $\pi \in C'_\lambda(a)$, then $\ell(\Lambda(\pi)) \leq a$. For $\pi \in C'_\lambda$, we define the $\Lambda$-yokel $\Lambda(\pi) = (\ldots, \Lambda(3), \Lambda(2), \Lambda(1))$, by setting $\Lambda(j)$ to be such that, counting from the right, the $j$th yoke of the abacus corresponding to $\pi$ is a $\Lambda(j)$-yoke.

For each $\pi$, we can thus use the abacus to obtain a pair $(A(\pi), \delta(\pi))$, from which $\pi$ can be uniquely reconstructed. However, not every pair $(A, \delta)$ in which $A$ is a $\Lambda$-yokel and $\delta = (\ldots, \delta_3, \delta_2, \delta_1)$ is an arbitrary sequence of non-negative integers, arises in this way.

4.6. Yokes apart

The process of moving a yoke one position horizontally to the left, without touching another yoke is known as a lift. In the example of figure 4, we see that lifts can be performed on both the first and second yokes. Correspondingly, an inverse lift is a moving of a yoke to the right.

For $\Lambda', \Lambda'' \in F^r_{d,d}$, consider a $\Lambda'$-yoke on an abacus to the left of a $\Lambda''$-yoke, with $\delta$ the number of vacancies between them. We see that performing a lift on the $\Lambda'$-yoke, or an inverse lift on the $\Lambda'$-yoke, results in the two yokes being separated by $\delta' = \delta - r$ vacancies. By repeating the process, we see that there is a minimal possible number of vacancies between a $\Lambda'$-yoke and a $\Lambda''$-yoke, which we denote $\Delta(\Lambda', \Lambda'')$, and that

$$\delta \in r\mathbb{Z}_{\geq 0} + \Delta(\Lambda', \Lambda'').$$

4.6.1. Examples. We find that

$$\Delta([4, 0, 0], [3, 1, 0]) = 1, \quad \Delta([4, 0, 0], [0, 3, 1]) = 5,$$

$$\Delta([4, 0, 0], [1, 0, 3]) = 6.$$  

Note that it is not necessarily the case that $\Delta(\Lambda', \Lambda'') < r$. In the $r = 2$ case, we obtain the explicit expression

Though the term ‘lift’ makes sense in the context of the ‘patterns’ of [31], it seems a misnomer in the context of our formulation. We explain the correspondence between these patterns and cylindric partitions in appendix C. The notions of ‘tightening’ and ‘tight’ in [46] are equivalent to those of ‘lift’ and ‘highest-lift’ here.
\[ \Delta([d - x, x], [d - y, y]) = |x - y|, \]  

which results from direct consideration of suitable yokes on the abacus.

4.7. Highest-lift abaci

Let \( \pi \in C_\lambda \) and let \( \Delta(\pi) = (\ldots, \lambda(3), \lambda(2), \lambda(1)) \) and \( \delta(\pi) = (\ldots, \delta_3, \delta_2, \delta_1) \). Let \( \pi' \) be obtained from \( \pi \) by applying an inverse lift to the \( j \)th yoke. If \( \delta'(\pi') = (\ldots, \delta'_3, \delta'_2, \delta'_1) \) then \( \delta'_j = \delta_j + r \), \( \delta'_{j-1} = \delta_{j-1} - r \) (if \( j > 1 \)) and \( \delta'_1 = \delta_1 \) otherwise. Then (28) gives \( |\pi'| = |\pi| + r \). It follows that the generating function of all abaci obtained by applying inverse lifts to \( \pi \) is \( q^{[k]}(q^r; q^r)_\infty^{-1} \).

A cylindric partition \( \pi \) (or abacus) for which no lifts are possible is known as a highest-lift configuration. Let \( H' \subset C_\lambda \) be the subset of highest-lift configurations. Note that \( \pi \in H' \) is completely characterised by \( L' \). Moreover, every \( \pi' \in C_\lambda \) for which \( \Lambda(\pi') = \Lambda(\pi) \) is obtained from \( \pi \) by a sequence of inverse lifts. It follows that if \( H'_\lambda \) denotes the generating function for \( H' \), then

\[ C'_\lambda = \frac{1}{(q'; q')_\infty} H'_\lambda. \]  

Comparing this result with the first equality in (5), which was proved in section 3.1, then shows that

\[ H'_\lambda = \Pr \chi_{\hat{\lambda}}. \]  

4.8. Note on crystal graphs

The previous result is also a consequence of the crystal graph theory of representations of \( \hat{sl}_r \) described in [31]. There, so-called ‘patterns’ are used to label the nodes of the crystal graph. As we show in appendix C, these patterns encode cylindric partitions. The subset of the full set of patterns which, in [31], are known as ‘highest-lift’ are in bijection with the nodes of the crystal graph of an irreducible representation of \( \hat{sl}_r \). These highest-lift patterns correspond to those cylindric partitions that are designated highest-lift in this section. Note, however, that the crystal graph theory provides a combinatorial model for the full characters \( \chi_{\hat{\lambda}} \) of \( \hat{sl}_r \).

5. The Andrews–Gordon–Bressoud identities. Outline of results

Our main results are theorem 3 and corollary 4.

5.1. Main results

In this and the following sections, we concentrate on the \( r = 2 \) case. Here \( \Lambda \in P^+_{2,d} \) takes the form \([d - i, i] \) for \( 0 \leq i \leq d \), and \( \hat{\lambda}(\lambda) = (i, 0) \). We set \( p = d + 2 \). Then, expanding the determinant of (14) yields

\[ C^2_\lambda(a) = \sum_{k \in \mathbb{Z}} \frac{q^{k(2k-1)p+2k(i+1)}}{(q'; q)_h+2k(q'; q)_h-2k} - \sum_{k \in \mathbb{Z}} \frac{q^{k(2k-1)p-(2k-1)(i+1)}}{(q'; q)_h+2k(q'; q)_h-2k+1} \]

\[ = \sum_{j \in \mathbb{Z}} (-1)^j \frac{q^{j(j+1)p-j(i+1)}}{(q'; q)_{h-j}(q'; q)_{h+j}}, \]  

(34)
having combined the two summations, using \( j = -2k_i \) in the first and \( j = 2k_i - 1 \) in the second. Taking the \( a \to \infty \) limit then gives

\[
C_\lambda^2 = \frac{1}{(q; q)_\infty} \sum_{j \in \mathbb{Z}} (-1)^j q^{j(j+1)p-j(i+1)}.
\]  

(35)

From (17), we also have

\[
C_\lambda^2 = \frac{(q^{p+1}, q^{p-i-1}, q^p; q)_\infty}{(q; q)_\infty}. \tag{36}
\]

When \( p \) is odd, as indicated in footnote 12, the Virasoro character \( \chi_{1, i+1}^{2p} = (q; q)_\infty C_\lambda^2 \).

### 5.2. Bailey chains

Expression (34) is at the heart of Bailey chain methods to derive generalisations of the Rogers–Ramanujan identities [47–51] 17. These methods lead to non-negative sum expressions, which when, in the case of (34), are equated with (36), yield generalisations of the Rogers–Ramanujan identities that are originally due to Andrews [53], Gordon [54] and Bressoud [55]. Here, we make a modest addition to the understanding of these identities, by showing how they arise from simple manipulations of cylindric partitions. In fact, the manipulations will be made on what we refer to as decorated \( \mathcal{B} \)-paths, which are modified versions of Bressoud’s lattice paths [25]. They come in two versions, appropriate to the cases of odd and even \( d \), and are defined in sections 6.2 and 9.1 below. The bijections between such paths and the cylindric partitions are described in sections 7.3 and 9.5.

For convenience, we will state the sum-type expressions for \( C_{a \lambda}^2 \) in the form of a theorem and a corollary. Note that there are different forms for the cases of odd and even \( d \). These correspond to the coprime and non-coprime cases discussed above. The theorem makes use of generalised \( q \)-multinomials \([a^{(f)}_{\lambda \mu}]_q\) expressed in terms of partitions. For \( a, f \geq 0 \) and \( \lambda \in \mathcal{P}_k \), define

\[
[a^{(f)}_{\lambda \mu}]_q = \begin{cases} \frac{(q^f; q^f)_\lambda}{(q; q)_{\lambda_1-\lambda_f} \cdots (q; q)_{\lambda_{i-1}-\lambda_f} \cdots (q; q)_{\lambda_{i-1}-\lambda_f} \cdots (q; q)_{\lambda_{i+1}-\lambda_f} \cdots (q; q)}_\lambda \lambda_f & \text{if } |\lambda| \leq a, \\ 0 & \text{if } |\lambda| > a. \end{cases} \tag{37}
\]

If \( f = 1 \), then the superscript ‘(1)’ can be omitted. Note that this generalises the definition of the \( q \)-binomial because for the one part partition \( \lambda = (n) \) we have \( [a^{(1)}_{n \mu}]_q = [a \mu n]_q \).

**Theorem 3.** Let \( d \geq 2 \) and \( 0 \leq i \leq k \) with \( k = \lfloor d/2 \rfloor \). Then, for \( \lambda = [d - i, i] \in \mathcal{P}_{2a}^+ \) and \( a \geq 0 \),

\[
C_{a \lambda}^2(a) = \begin{cases} \frac{1}{(q; q)_a} \sum_{\lambda \in \mathcal{P}_k} q^{\lambda_1^2 + \cdots + \lambda_f^2 + \lambda_{i+1}^2 + \cdots + \lambda_k^2} \left[ a \right]_{\lambda \mu}^{(1)} & \text{if } d \text{ is odd}, \\ \frac{1}{(q^2; q^2)_a} \sum_{\lambda \in \mathcal{P}_k} q^{\lambda_1^2 + \cdots + \lambda_f^2 + \lambda_{i+1}^2 + \cdots + \lambda_k^2} \left[ a \right]_{\lambda \mu}^{(2)} & \text{if } d \text{ is even}. \end{cases} \tag{38}
\]

Taking the limit \( a \to \infty \) of (38) leads to the following corollary.

17 For a review and further reference to Bailey chain methods, see [52].
Corollary 4. Let $d \geq 2$ and $0 \leq i \leq k$ with $k = \lfloor d/2 \rfloor$. Then, for $\Lambda = [d - i, i] \in P_{2,d}^+$,

$$C_\Lambda^2 = \frac{1}{(q; q)_\infty} \sum_{\lambda \in P_{2,d}} \frac{q^{\lambda_1 + \lambda_2 + \cdots + \lambda_k + \lambda_{i+1} + \lambda_{i+2} + \cdots + \lambda_k}}{(q; q)_{\lambda_1 - \lambda_2}(q; q)_{\lambda_2 - \lambda_3}(q; q)_{\lambda_3 - \lambda_4}(q; q)_{\lambda_{k-1} - \lambda_k}(q; q)^2_{\lambda k}}$$

if $d$ is odd,

$$C_\Lambda^2 = \frac{1}{(q; q)_\infty} \sum_{\lambda \in P_{2,d}} \frac{q^{\lambda_1 + \lambda_2 + \cdots + \lambda_k + \lambda_{i+1} + \lambda_{i+2} + \cdots + \lambda_k}}{(q; q)_{\lambda_1 - \lambda_2}(q; q)_{\lambda_2 - \lambda_3}(q; q)_{\lambda_3 - \lambda_4}(q; q)_{\lambda_{k-1} - \lambda_k}(q^2; q^2)^{\lambda k}}$$

if $d$ is even. \hspace{1em}(39)

Equating these expressions (39) with (36) then yields the aforementioned Andrews–Gordon–Bressoud Rogers–Ramanujan type identities.

5.3. Plan of proof

To prove theorem 3, we first show that the rhs of 3 gives the generating functions for the corresponding decorated $B$-paths. This is done in theorems 10 and 16 for the cases of odd and even $d$ respectively. The proof is then completed by showing that there are weight-preserving bijections between the cylindric partitions and the corresponding decorated $B$-paths. Lemmas 11 and 17 state the implied identities between their generating functions.

6. $B$-paths, decorated paths and transforms

We recall Bressoud’s paths, define a decorated version of these paths, then recall Bressoud’s transforms that interpolate paths that satisfy different conditions.

6.1. $B$-paths

We define a $B$-path $h$ to be a semi-infinite sequence $h = (h_0, h_1, h_2, \cdots)$ satisfying $h_i \in \mathbb{Z}_{\geq 0}$ and $h_{i+1} - h_i \in \{0, \pm 1\}$ for $i \geq 0$, with $h_{i+1} = h_i$ only if $h_i = 0$. For $0 \leq b \leq k$ define $A_b^k$ to be the set of all $B$-paths $h = (h_0, h_1, h_2, \cdots)$ for which $h_0 = b$ and $0 \leq h_i \leq k$ for $i > 0$, and for which there exists $L \geq 0$ with $h_i = 0$ for all $i \geq L$. Such paths were introduced in [25].

18 The $B$ in $B$-paths, etc is for Bressoud who introduced these very paths, as well as the transforms that relate paths that satisfy different conditions in [25]. Since the work in [25] was motivated by Burge’s work [28], which in turn was motivated by Bailey’s work [36, 37], $B$ may equally stand for Bressoud, Burge and Bailey. Another version of these paths was used by Warnaar in [58, 59] to obtain non-negative sum forms of characters in minimal Virasoro models labelled by $p$ and $p' = p + 1$. ‘Half-lattice paths’ were defined in [60, 61], and used to derive non-negative $q$-series expressions for Virasoro characters related to non-thermal perturbations of minimal models. ‘Fused paths’ were defined in [62], and used to study alternating-sign $q$-series expressions for Virasoro characters in higher-level non-unitary minimal models.

19 In [63, 64], an equivalent version of Bressoud’s $B$-paths was introduced. In the latter version, the paths must always change heights.

Equating these expressions (39) with (36) then yields the aforementioned Andrews–Gordon–Bressoud Rogers–Ramanujan type identities.
6.1.1. Example. A picture of \( h \in \mathbf{A}_b^k \) is obtained by linking the points \((0, h_0), (1, h_1), (2, h_2), (3, h_3)\) on the plane. Figure 5 shows the picture of a typical element \( h \in \mathbf{A}_b^3 \).

6.1.2. Peaks and local weights. A path \( h \) has a peak at \( i > 0 \), if both \( h_{i-1} < h_i \) and \( h_{i+1} < h_i \). Each peak is assigned a weight equal to its position \( i \). These weights are the local weights referred to in section 1.5.

6.1.3. The weight of a path. The total weight, or simply the weight \( w_t(h) \) of a path \( h \in \mathbf{A}_b^k \) is the sum of its local weights, that is the sum of the positions of its peaks:

\[
wt(h) = \sum_{\substack{i > 0 \\ h_{i-1} < h_i < h_{i+1}}} i .
\]  

The generating function of the weighted paths \( A_b^k \) is

\[
A_b^k = \sum_{h \in \mathbf{A}_b^k} q^{wt(h)} .
\]  

6.1.4. The number of peaks of a path. An important attribute of a \( \mathbf{B} \)-path \( h \) is the number of peaks \( np(h) \) of the path\(^{20}\). We define \( \mathbf{A}_b^k(a) = \{ h \in \mathbf{A}_b^k | np(h) = a \} \) to be the set of paths with the same number of peaks \( a \). The corresponding generating function is

\[
A_b^k(a) = \sum_{h \in \mathbf{A}_b^k(a)} q^{wt(h)} .
\]

Lemma 5. For \( k > 0 \),

\[
A_b^k(a) = q^a A_b^{k-1}(a) .
\]

Proof. Each \( h \in \mathbf{A}_b^k(a) \) is necessarily such that \( h_0 = k \) and \( h_i = k - 1 \). Therefore \( i = 1 \) not a peak. Removing the first segment of \( h \) therefore results in an element \( h' \in \mathbf{A}_b^{k-1}(a) \). Moreover, every element \( h' \in \mathbf{A}_b^{k-1}(a) \) arises from a unique \( h \in \mathbf{A}_b^k(a) \) in this way. Because each peak of \( h' \) is one less than the corresponding peak in \( h \), and there are \( a \) peaks, (43) immediately follows.

6.2. Decorated \( \mathbf{B} \)-paths

A decorated \( \mathbf{B} \)-path \( \tilde{h} \) is a pair \((h, h')\) where \( h \) is a \( \mathbf{B} \)-path and \( h' \) is any sequence \( h' = (h_0', h_1', h_2', \ldots) \) of non-negative integers, for which there exists \( M \geq 0 \) such that \( h_i' = 0 \) for \( i > M \).

6.2.1. Deaks. We say that \( \tilde{h} \) has \( h_i' \) deaks, for ‘degenerate peaks’, at \( i \), these being in addition to a normal peak if \( h_{i-1} < h_i < h_{i+1} \). For \( 0 < b \leq k \) we define \( \mathbf{A}_b^k \) to be the set of all decorated \( \mathbf{B} \)-paths \( \tilde{h} = (h, h') \) for which \( h \in \mathbf{A}_b^k \). The picture of \((h, h') \in \mathbf{A}_b^k\) is obtained

\[^{20}\text{The number of peaks of a \( \mathbf{B} \)-path is called the ‘number of particles’ in physics-motivated literature, including [18–23], and related works.}\]
from that of \( h \) by, for each \( i \) with \( h_i^0 > 0 \), placing the value of \( h_i^0 \) in a small circle at the point \((i, h_i)\).

6.2.2. Example. An example is given in figure 6.

The number of peaks \( n_p(\tilde{h}) \) and weight \( w_t(\tilde{h}) \) of \( \tilde{h} = (h, h^*) \in \mathcal{A}^k_b \) are defined by

\[
\begin{align*}
    n_p(\tilde{h}) &= n_p(h) + \ell(h^*), \quad (44a) \\
    w_t(\tilde{h}) &= w_t(h) + |h^*|, \quad (44b)
\end{align*}
\]

where \( \ell(h^*) = \sum_{i=0}^{\infty} h_i^0 \) and \( |h^*| = \sum_{i=1}^{\infty} i h_i^0 \). Note that \( h^0 \) is, in effect, a partition, with \( h_i^0 \) giving the multiplicity of the part \( i \) for \( i > 0 \), and \( |h^*| \) giving its weight. However, \( \ell(h^*) \) is a variant on the standard definition of partition length because, here, zero parts are counted. We then define the set \( \mathcal{A}^k_b \{ a \} = \{ \tilde{h} \in \mathcal{A}^k_b \mid n_p(\tilde{h}) = a \} \) and the generating functions

\[
\begin{align*}
    \tilde{A}^k_b &= \sum_{h \in \mathcal{A}_b} q^{w_t(h)}, \quad \tilde{A}^k_b(a) = \sum_{\tilde{h} \in \mathcal{A}^k_b \{ a \}} q^{w_t(\tilde{h})}. \quad (45)
\end{align*}
\]

Lemma 6. For \( 0 \leq b \leq k \),

\[
\tilde{A}^k_b(a) = \sum_{n=0}^{a} \frac{1}{(q; q)_{a-n}} A^k_b(n). \quad (46)
\]

Proof. The required expression is the generating function for decorated \( \mathcal{B} \)-paths \( \tilde{h} = (h, h^*) \) with fixed number of peaks \( n_p(h) = a \). By (44a), \( 0 \leq n_p(h) \leq a \) with \( \ell(h^*) = a - n_p(h) \). Therefore,

\[
\tilde{A}^k_b(a) = \sum_{n=0}^{a} A^k_b(n) \sum_{h^* \mid \ell(h^*) = a-n} q^{h^*}. \quad (47)
\]

The second sum here is given by \((q; q)_{a-n}^{-1} \), this being the generating function for partitions with at most \((a-n)\) parts. Then, (46) immediately follows. \( \square \)

6.3. The \( \mathcal{B} \)-transform

Here, we describe a way to transform a decorated \( \mathcal{B} \)-path \( \tilde{h} \in \mathcal{A}^k_b \) to obtain a (non-decorated) \( \mathcal{B} \)-path \( h' \in \mathcal{A}^{k+1}_b \). This action extends the notion of ‘volcanic uplift’ described in [25], and we refer to it as a \( \mathcal{B} \)-transform.

Let \( \tilde{h} = (h, h^*) \), and define \( h^+ = (h_0^+, h_1^+, h_2^+, \ldots) \) by setting \( h_i^+ = h_i^0 + 1 \) if \( i \) is a peak, and \( h_i^+ = h_i^0 \) if \( i \) is not a peak. Note that \( \sum_{i=0}^{\infty} h_i^+ = n_p(\tilde{h}) \). The \( \mathcal{B} \)-path \( h' \) is obtained from \( h \) simply by, for \( i = \ldots, 2, 1, 0 \), inserting \( h_i^+ \) NE-SE pairs at position \((i, h_i)\). In effect,
each peak of $h$ gives rise to a peak of $h'$ of height one greater. By regarding $h^\circ > 0$ deaks as a sequence of $h^\circ$ degenerate peaks at $(i, h_i)$, this statement applies to them as well.

To illustrate the action of the $B$-transform, it maps the decorated $B$-path $\hat{h}$ of figure 7 to the $B$-path $h'$ of figure 8. Note that in this example $np(h') = np(\hat{h}) = 10$ and $wt(h') = wt(\hat{h}) = 100$.

Lemma 7. For $0 \leq b \leq k$, let $h' \in \mathcal{A}_b^{k+1}$ be obtained from the action of a $B$-transform on $\hat{h} \in \mathcal{A}_b^k(a)$. Then

\begin{align*}
np(h') &= a, \\
wt(h') &= wt(\hat{h}) + a^2.
\end{align*}

(48a) \hspace{1cm} (48b)

Proof. Altogether, $\hat{h}$ has $a$ peaks and deaks. The first expression holds because each of these gives rise to a (genuine) peak of $h'$, with all peaks of $h'$ obtained thus. Let $p_1, p_2, \ldots, p_a$ be, in non-decreasing order, the peaks and deaks of $\hat{h}$. The peaks of $h'$ are then at positions $p_1 + 1, p_2 + 3, \ldots, p_a + 2a - 1$. From (40), it follows that

$$wt(h') = \sum_{i=1}^{a} p_i + 1 + 3 + 5 + \cdots + (2a - 1),$$

(49)

which immediately gives (48b). \hfill \Box

Lemma 8. For $0 \leq b \leq k$,

$$A_b^{k+1}(a) = q^a A_b^k(a).$$

(50)

Proof. This immediately follows from lemma 7 once it is shown that the $B$-transform provides a bijection between $\mathcal{A}_b^k(a)$ and $\mathcal{A}_b^{k+1}(a)$. This is so because if $h \in \mathcal{A}_b^{k+1}(a)$, the unique $\hat{h} \in \mathcal{A}_b^k(a)$ from which it arose is obtained by removing the two edges next to each peak, and taking account of the multiplicities of the downgraded peaks to give peaks and deaks. \hfill \Box
Lemma 9. If $0 \leq b < k$ then,

\[ \tilde{A}_b^k (a) = \sum_{n=0}^{a} \frac{q^n}{(q; q)_{a-n}} \tilde{A}^{k-1}_b (n). \] (51a)

If $k > 0$ then

\[ \tilde{A}_b^k (a) = \sum_{n=0}^{a} \frac{q^{n(n+1)}}{(q; q)_{a-n}} \tilde{A}^{k-1}_{b-1} (n). \] (51b)

Proof. The first expression here results from substituting the $k \to k-1$ case of (50) into (46). The second expression results from first substituting (43) into the $b = k$ case of (46), and then applying the $k \to k-1$ and $b \to k-1$ case of (50).

7. Proof of the Andrews–Gordon identities

We prove the Andrews–Gordon identities by using abaci to map cylindric partitions to $B$-paths, then use $B$-transforms.

7.1. The sum side of the Andrews–Gordon identities

We now obtain the desired fermionic-type expression for $\tilde{A}_b^k (a)$ by concatenating together a sequence of $B$-transforms.

Theorem 10. If $k \geq 1$ and $0 \leq b < k$ then

\[ \tilde{A}_b^k (a) = \sum_{a \geq n_0 \geq n_1 \geq \cdots \geq n_k \geq 0} \frac{q^{n_0^2 + n_0^2 + \cdots + n^2 + n_1 + n_2 + \cdots + n_k}}{(q; q)_{a-n_1} (q; q)_{n_1-n_2} \cdots (q; q)_{n_{k-1}-n_k} (q; q)_{n_k} (q; q)_{n_k}}. \] (52)

Proof. Each $(h, h^0) \in \tilde{A}_b^k (a)$ has $h = (0, 0, 0, \ldots)$. Because $h^0$ here is a partition unconstrained apart from having no more than $a$ parts, $\tilde{A}_b^0 (a) = (q; q)_{a-1}$. The two cases of (51) then imply that

\[ \tilde{A}_0^1 (a) = \sum_{n=0}^{a} \frac{q^n}{(q; q)_{a-n} (q; q)_{n}}, \] (53a)

\[ \tilde{A}_1^1 (a) = \sum_{n=0}^{a} \frac{q^{n^2 + n}}{(q; q)_{a-n} (q; q)_{n}}. \] (53b)

These are the $k = 1$ cases of (52). We now proceed to prove (52) by induction on $k$, by assuming that (52) holds for $k \to k-1$ and $0 \leq b < k-1$. We consider the case $0 \leq b \leq k-1$ and the case $b = k$ separately.
7.1.1. The case $b \leq k - 1$. Using (51a), and the induction hypothesis yields

$$
\tilde{\Lambda}_b(a) = \sum_{n_i=0}^{a} \frac{q^{n_i^2 + n_i}}{q^{|n_i|-n_i}} \tilde{\Lambda}_{b-1}(n_i)
= \sum_{n_i=0}^{a} \frac{q^{n_i^2 + n_i}}{q^{|n_i|-n_i}} \sum_{n_{i+1}, \ldots, n_i \geq 0} \frac{q^{n_i^2 + n_i + \cdots + n_i^2 + n_i + n_i + \cdots + n_i}}{(q; q)_{n_i-n_i-1} (q; q)_{n_i-n_i-2} \cdots (q; q)_{n_i-n_i}},
$$

(54)

which is the $b \leq k - 1$ case of (52), as desired.

7.1.2. The case $b = k$. Using (51b), and the induction hypothesis yields

$$
\tilde{\Lambda}_k(a) = \sum_{n_i=0}^{a} \frac{q^{n_i^2 + n_i}}{q^{|n_i|-n_i}} \tilde{\Lambda}_{k-1}(n_i)
= \sum_{n_i=0}^{a} \frac{q^{n_i^2 + n_i}}{q^{|n_i|-n_i}} \sum_{n_{i+1}, \ldots, n_i \geq 0} \frac{q^{n_i^2 + n_i + \cdots + n_i^2 + n_i + n_i + \cdots + n_i}}{(q; q)_{n_i-n_i-1} (q; q)_{n_i-n_i-2} \cdots (q; q)_{n_i-n_i}},
$$

(55)

which is the $b = k$ case of (52), as desired.

7.2. Characterising decorated $B$-paths

For $\tilde{h} \in (h, h') \in \hat{\Lambda}_k(a)$, let $\{i_j\}_{j=1}^{a}$ be the set of peaks and deaks of $\tilde{h}$ with $0 \leq i_a \leq i_{a-1} \leq \cdots \leq i_1$ (for each value that is repeated, all but one of them corresponds to a deak), and let $b_j = h_i$ for $1 \leq j \leq a$. In addition, for convenience, set $i_{a+1} = 0$, $b_{a+1} = b$, and $\delta_j = i_j - i_{j+1}$ for $1 \leq j \leq a$. Then, (44b) gives

$$
\text{wt}(\tilde{h}) = \sum_{j=1}^{a} i_j = \sum_{j=1}^{a} \delta_j.
$$

(56)

We claim that $\tilde{h}$ is completely determined by the points $\{ (i_j, b_j) \}_{j=1}^{a}$. This follows because, for $1 \leq j \leq a$, there is no peak strictly between $i_{j+1}$ and $i_j$, and therefore the sequence of $\delta_j$ edges of the path $h$ can only take one form, namely, a sequence of SE edges followed by a sequence of E edges at height 0 followed by a sequence of NE edges, with any of these sequences of zero length. Thus $\tilde{h}$ is determined uniquely. Moreover, depending on whether there are non-zero or zero E edges, either

$$
\delta_j > b_j + b_{j+1} \text{ or } |b_j - b_{j+1}| \leq \delta_j \leq b_j + b_{j+1} \text{ with } \delta_j - |b_j - b_{j+1}| \in 2\mathbb{Z}.
$$

(57a)

These conditions are sufficient to determine whether $\{ (i_j, b_j) \}_{j=1}^{a}$ corresponds to a $\tilde{h} \in \hat{\Lambda}_k(a)$.

7.3. Bijection from cylindric partitions to decorated $B$-paths

For $d$ odd, set $d = 2k + 1$. Then, for $A = [d - x, x] \in \mathbb{P}_x^+$, let $b = \max \{ k - x, x - k - 1 \}$ so that $0 \leq b \leq k$. In what follows, for $\pi \in \mathcal{C}^2_A(a)$, we use $A(\pi) = (\ldots, A^{(3)}, A^{(2)}, A^{(1)})$ and $\delta(\pi) = (\ldots, \delta_3, \delta_2, \delta_1)$ to construct an element $\tilde{h} \in \hat{\Lambda}_k(a)$ by specifying the set of points $\{ (i_j, b_j) \}_{j=1}^{a}$ of its $a$ peaks and deaks. We then show that this defines a weight-preserving bijection between $\mathcal{C}^2_A(a)$ and $\hat{\Lambda}_k(a)$. 

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First set \( i_j = \sum_{m=j}^{a} \delta_m \) for \( 1 \leq j \leq a \). Then, for \( j > 0 \), let \( x_j \) be such that \( N^{(j)} = [d - x_j, x_j] \), and obtain \( b_j \) by
\[
b_j = \begin{cases} 
k - x_j & \text{if } x_j \leq k, \\
x_j - k - 1 & \text{if } x_j > k, 
\end{cases}
\]
i.e. \( b_j = \max\{k - x_j, x_j - k - 1\} \). Note that \( 0 \leq b_j \leq k \) with \( b_{a+1} = b \). Making use of (31), we then find that if both \( x_{j+1} \leq k \) and \( x_j \leq k \), or both \( x_{j+1} > k \) and \( x_j > k \), then
\[
\delta_j \in 2 \mathbb{Z}_{\geq 0} + |b_{j+1} - b_j|.
\]
Otherwise
\[
\delta_j \in 2 \mathbb{Z}_{\geq 0} + b_{j+1} + b_j + 1.
\]
After noting that one of \( |b_{j+1} - b_j| \) and \( (b_{j+1} + b_j + 1) \) is even and one odd, we see that \( \{i_j, b_j\}_{j=1}^{a} \) satisfies the conditions (57), thereby determining a unique element \( \tilde{h} \in \mathcal{A}_k(a) \).

That the map so defined is a bijection follows because, given values \( \{i_j, b_j\}_{j=1}^{a} \) that satisfy the conditions in section 7.2, values \( x_a, x_{a-1}, \ldots, x_1 \) with \( 0 \leq x_j \leq 2k + 1 \) are uniquely determined by reversing the above construction. This is done by noting that with \( i_{a+1} = 0 \), \( b_{a+1} = b \), and \( \delta_j = i_j - i_{j+1} \) for \( 1 \leq j \leq a \), (59) implies that
\[
(x_{j+1} \leq k \iff x_j \leq k) \iff \delta_j \equiv b_{j+1} + b_j \pmod{2},
\]
for \( 1 \leq j \leq a \). Then, with \( x_{a+1} \) such that \( \Lambda = [d - x_{a+1}, x_{a+1}] \), each value \( x_j \) is determined from \( b_j, b_{j+1}, \delta_j \) and \( x_{j+1} \) using first (60) to determine whether \( x_j \leq k \) or \( x_j > k \), and then making use of the appropriate case of (58). On setting \( N^{(j)} = [d - x_j, x_j] \) for \( 1 \leq j \leq a \), and \( N^{(a)} = \Lambda \) and \( \delta_a = 0 \) for \( j > a \), \( \pi \in \mathcal{C}_k(a) \) is uniquely determined such that \( \Lambda(\pi) = (\ldots, N^{(3)}, N^{(2)}, N^{(1)}) \) and \( \delta(\pi) = (\ldots, \delta_3, \delta_2, \delta_1) \), thereby demonstrating the map to be bijective.

From (56) and (28), \( \text{wt}(\tilde{h}) = \sum_{j=1}^{a} |\delta_j| = |\pi| \), thereby showing the bijection to be weight-preserving.

7.4. The product side of the Andrews–Gordon identities

The bijection of the previous subsection implies the following result

**Lemma 11.** For \( k \geq 0 \), let \( \Lambda \in \mathcal{P}^{2k+1}_{2k+1} \) be of the form \( \Lambda = [k + 1 + b, k - b] \) or \( \Lambda = [k - b, k + 1 + b] \) for \( 0 \leq b \leq k \). Then, for each \( a \geq 0 \),
\[
\mathcal{C}_k(a) = \mathcal{A}_a^{k}(a).
\]

Using this result, and setting \( b = k - i \) and \( n_j = \lambda_{k+1-j} \), the odd \( d \) case of theorem 3 follows from theorem 10.

8. Translating the proof in terms of abaci

We consider how the proof of theorem 10 translates to the abaci through the bijection of section 7.3.
Expression (52) results from the application of \((k + 1)\) \(B\)-transforms At the \(i\)th stage \((0 \leq i \leq k)\), the factor \((q; q)_{n_i-1}^{-1}\) in (52) corresponds to the placement of degenerate peaks on a \(B\)-path \(h\). When translated to abaci, these correspond to the placement of yokes of varying shapes.

8.2. Interpretation of proof in terms of yoke moves

The generating function for \(A_0^k(a)\) given by (52) is constructed by altering a sequence of \(B\)-transforms (encoded in (50)) with multiplication by \((q; q)_{n_i-1}^{-1}\) for fixed \(n_{i+1}\) and summed over \(n_i\) with \(0 \leq n_i \leq n_{i+1}\) (encoded in (46)). Here \(1 \leq i \leq k\) and we take \(n_{k+1} = a\). It is instructive to interpret these transformations between generating functions for decorated \(B\)-paths in terms of the abaci of the cylindric partitions, through the above bijection.

In the first place, (50) with \(k \rightarrow i - 1\) and \(a \rightarrow n_i\) is interpreted in terms of each yoke changing shape: for the rightmost \(n_i\) yokes, a \([m_0, m_1]\)-yoke changes to either a \([m_0, m_1 + 2]\)-yoke or a \([m_0, m_1 - 2]\)-yoke depending on whether \(m_0 < m_1\) or \(m_0 \geq m_1\) respectively; other than those rightmost \(n_i\), the Dirac sea of \([i + 1 + b, i - 1 - b]\)-yokes become \([i + 1 + b, i - b]\)-yokes. In addition, the distance between the \(j + 1\)th and \(j\)th yokes increases by 2 for \(1 \leq j < n_i\), increases by 1 for \(j = n_i\), and remains 0 for \(j > n_i\). This process is demonstrated by the transition between the abacus in figure 9 and that in figure 10.

In these abaci, we have labelled the beads in the \(j\)th yoke with \(j\), where \(1 \leq j \leq n_i = 10\) (substituting the label ‘10’ by ‘0’). Note that the abaci in figures 9 and 10 correspond respectively to the decorated path in figure 7 and the undecorated path in figure 8.

To interpret (46), consider an element \(h \in A_k^k(n_i)\) and its corresponding abacus \(\pi\). All but the rightmost \(n_i\) yokes of \(\pi\) belong to the Dirac sea and have weight \([i + 1 + b, i - b]\). The rightmost \(n_i\) yokes are unconstrained, apart from each being of a weight of level \(2i + 1\), and being separated from one-another by at least two vacancies.

As described in section 6.2, multiplication by \((q; q)_{n_i-1}^{-1}\) enumerates all sets of positions of \(n_{i+1} - n_i\) deaks along the path \(h\), the positions being encoded in a sequence \(h^\circ = (h_0^\circ, h_1^\circ, h_2^\circ, \ldots)\) for which \(\ell(h^\circ) = n_{i+1} - n_i\). In terms of abaci, this corresponds to enumerating all possible abaci \(\pi^\circ\) obtained from \(\pi\) by inserting \(n_{i+1} - n_i\) yokes between the rightmost \(n_i + 1\) yokes of \(\pi\). For a particular \(h^\circ = (h_0^\circ, h_1^\circ, h_2^\circ, \ldots)\), for each \(j \geq 0\) for which \(h_j^\circ > 0\), the \(h_j^\circ\) yokes are inserted such that there are exactly \(j\) vacancies to their left. Their weight depends on the value of \(h_j^\circ\); either \([i + 1 + h_j, i - h_j]\) or \([i - h_j, i + 1 + h_j]\), with the choice between these two determined by (29).
Alternatively, all the abaci $\pi'$ may be generated by starting with $\pi$ and performing a sequence of ‘moves’ on the yokes numbered $n_i + 1$ to $n_{i+1}$. Note that $\pi$ itself is the abacus corresponding to $h^0 = (n_{i+1} - n_i, 0, 0, 0, \ldots)$. Each move corresponds to changing the deak positions from one $h^0 = (h_{i0}^0, \ldots, h_{i+1}^0, \ldots)$ to $(h_{i0}^0, \ldots, h_{i+1}^0 - 1, h_{i+1}^0 + 1, \ldots)$ for some $j$. Such a move on the abacus is simply the shifting of one bead from the yoke corresponding to the deak at $j$ one position to the right. Which of the two beads moved may be determined from $\pi$. In brief, if only one of the two beads can move then the one chosen is such that the resulting yoke weight $[m_0', m_1']$ has $|m_0' - m_1'|$ a minimum. If neither bead can move, then the yoke is necessarily immediately adjacent to another of the same weight. The move (and subsequent moves) are then performed on the beads of this yoke instead.

9. Proof of the Bressoud identities

We prove Bressoud’s identities using a direct extension of the proof of the Andrews–Gordon identities used in section 7.

9.1. Even $B$-paths

For $0 \leq b \leq k$ define $B^k_b$ to be the subset of $A^k_b$ comprising all $h = (h_0, h_1, h_2, \ldots)$ for which if $h_i = k$ then $i \equiv k - b \pmod{2}$. These paths were also originally considered in [25]. It is easy to see that this condition implies that there are an even number of horizontal segments to the left of each peak $i$ for which $h_i = k$. Note that the path $h \in A^k_b$ of figure 5 is not an element of $B^k_0$. We then define the set $B^k_b(a) = \{ h \in B^k_b \}$ for $h_0 = a$ and the generating functions

$$B^k_b = \sum_{h \in B^k_b} q^{wt(h)}, \quad B^k_b(a) = \sum_{h \in B^k_b(a)} q^{wt(h)}. \quad (62)$$

We have the following analogue of lemma 5

**Lemma 12.** For $k > 0$,

$$B^k_b(a) = q^a B^k_{b-1}(a). \quad (63)$$

**Proof.** Each $h \in B^k_b(a)$ is necessarily such that $h_0 = k$ and $h_1 = k - 1$. Therefore $i = 1$ not a peak. Removing the first segment of $h$ therefore results in an element $h' \in A^{k-1}_b(a)$ in which each peak is one less than the corresponding peak in $h$. Thus $np(h') = np(h) = a$ and $wt(h') = wt(h) - a$. Moreover, because the parity of the startpoint has changed, $h' \in B^k_{b-1}(a)$. Then, because every element $h' \in B^k_{b-1}(a)$ arises from a unique $h \in B^k_b(a)$ in this way, (63) immediately follows.

The paths $h \in B^k_0$ are now decorated much as before. For $k > 0$, we define $\tilde{B}^k_0$ to be the subset of $\tilde{A}^k_0$ comprising all decorated $B$-paths $\tilde{h} = (h, h^0)$ for which $h \in B^k_b$ and $h^0$ is any sequence $h^0 = (h_{i0}^0, h_{i1}^0, h_{i2}^0, \ldots)$ of non-negative integers, for which there exists $M \geq 0$ such

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that $h_i^o = 0$ for $i > M$. For the $k = 0$ case (and only this case), we restrict the $h^o$ to sequences $h^o = (h_0^o, h_1^o, h_2^o, \ldots)$ for which $h_i^o = 0$ for $i \text{ odd}$.

We then define the set $\tilde{B}_b^k(a) = \{ \tilde{h} \in \tilde{B}_b^k | n\tilde{p}(\tilde{h}) = a \}$ and the generating functions

$$
\tilde{B}_b^k(a) = \sum_{\tilde{h} \in \tilde{B}_b^k} q^{n(\tilde{h})}, \quad \tilde{B}_b^k(a) = \sum_{\tilde{h} \in \tilde{B}_b^k(a)} q^{n(\tilde{h})}.
$$

Lemma 13. For $k > 0$ and $0 \leq b \leq k$,

$$
\tilde{B}_b^k(a) = \sum_{n=0}^{a} \frac{1}{(q^b; q^a-b-n)} B_b^k(a).
$$

Proof. This is proved in exactly the same way as lemma 6.

9.2. $B$-transform for even $B$-paths

For $\tilde{h} \in \tilde{A}_b$, the action of the $B$-transform of section 6.3 yields $h' \in A_b^{k+1}$ each of whose peaks is of the opposite parity to that of the corresponding peak of $\tilde{h}$. In particular, the $B$-transform defines a map from $\tilde{h} \in \tilde{B}_b^k$ to $h' \in B_b^{k+1}$.

Lemma 14. For $0 \leq b \leq k$,

$$
B_b^{k+1}(a) = q^{a^2} \tilde{B}_b^k(a).
$$

Proof. The proof of lemma 8 shows that the $B$-transform provides a bijection between $\tilde{A}_b(a)$ and $A_b^{k+1}(a)$. The same is thus true for $\tilde{B}_b^k(a)$ and $B_b^{k+1}(a)$. The required result then immediately follows from lemma 7.

Lemma 15. If $0 \leq b < k$ then

$$
\tilde{B}_b^k(a) = \sum_{n=0}^{a} \frac{q^{n^2}}{(q^b; q^a-b-n)} \tilde{B}_b^{k-1}(a).
$$

If $k > 0$ then

$$
\tilde{B}_b^k(a) = \sum_{n=0}^{a} \frac{q^{n(n+1)}}{(q^b; q^a-b-n)} \tilde{B}_b^{k-1}(a).
$$

Proof. The first expression here results from substituting the $k \rightarrow k-1$ case of (66) into (65). The second expression results from first substituting (63) into the $b = k$ case of (65), and then applying the $k \rightarrow k-1$ and $b \rightarrow k-1$ case of (66).

21 This means that for all $k \geq 0$, $\tilde{B}_b^k$ comprises all decorated paths $\tilde{h}$ whose peaks and deaks $i$ for which $h_i = k$ satisfy $i \equiv k-b \pmod{2}$. 

26
9.3. The sum side of the Bressoud identities

Concatenating the above $B$-transforms together yields

**Theorem 16.** If $k \geq 1$ and $0 \leq b \leq k$ then

$$
\tilde{B}_b^k(a) = \sum_{\alpha \geq n_{k-1} \geq \cdots \geq n_1 \geq 0} a^{n_k} \left( \prod_{i=1}^{k} b_i \right)^{(n_{k-1})} \left( \prod_{i=1}^{k} (q; q)_{n_i} \right) \left( \prod_{i=1}^{k} (q; q)_{n_{i-1} - n_i} \right),
$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$.

(68)

**Proof.** Each $(h, h') \in \tilde{B}_b^k(a)$ has $h = (0, 0, 0, \ldots)$. Because $h'$ here is a partition with at most $a$ parts, all of which are even, $\tilde{B}_b^k(a) = (q^2; q^2)^a$. The two cases of (67a) then imply that

$$
\tilde{B}_b^k(a) = \sum_{n=0}^{a} \left( \prod_{i=1}^{k} (q; q)_{n_i} \right) \left( \prod_{i=1}^{k} (q; q)_{n_{i-1} - n_i} \right),
$$

(69a)

$$
\tilde{B}_b^k(a) = \sum_{n=0}^{a} \left( \prod_{i=1}^{k} (q; q)_{n_i} \right) \left( \prod_{i=1}^{k} (q; q)_{n_{i-1} - n_i} \right),
$$

(69b)

which are the $k = 1$ cases of (68). The proof of the general case then proceeds exactly as the proof of theorem 10.

Note that the rhs of (68) is identical with the $i = k - b$ case of (39), as is seen on setting $n_j \rightarrow \lambda_{k+1-j}$.

9.4. Characterising decorated even $B$-paths

Because $\tilde{B}_b^k(a)$ is a subset of $\tilde{A}_b^k(a)$, the conditions (57) on the points $\{(i_j, b_j)\}_{j=1}^a$ of $\tilde{h} \in \tilde{B}_b^k(a)$ continue to apply. The extra condition that applies to $\tilde{B}_b^k(a)$ is that, for $1 \leq j \leq a$,

$$
b_j = k \Rightarrow i_j \equiv k - b \pmod{2}.
$$

(70)

Each set of points $\{(i_j, b_j)\}_{j=1}^a$ that satisfies and (57) and (70) determines a unique element $\tilde{h} \in \tilde{B}_b^k(a)$.

9.5. Bijection from cylindrical partitions to decorated even $B$-paths

For $d$ even, set $d = 2k$. Let $\Lambda = [d - x, x] \in \mathcal{P}_{2k}^d$. For now, we only consider $x \leq k$. Set $b = k - x$, whereupon $0 \leq b \leq k$. In what follows, for $\pi \in \mathcal{C}_b^a(a)$, we use $\Pi(\pi) = (\ldots, N^{(3)}, N^{(2)}, N^{(1)})$ and $\delta(\pi) = (\ldots, \delta_3, \delta_2, \delta_1)$ to construct an element $\tilde{h} \in \tilde{B}_b^k(a)$ by specifying the set of points $\{(i_j, b_j)\}_{j=1}^a$ of its $a$ peaks and deaks. We then show that this defines a weight-preserving bijection between $\mathcal{C}_b^a(a)$ and $\tilde{B}_b^k(a)$.

As in section 7.3, set $i_j = \sum_{m=j}^a \delta_m$ for $1 \leq j \leq a$, set $i_{a+1} = 0$, and, for $j > 0$, let $x_j$ be such that $N^{(j)} = [d - x_j, x_j]$, and obtain $b_j$ using (58). In particular, $b_j = b$ for $j > a$. Because the procedure of section 7.3 has already been shown to lead to an element $\tilde{h} \in \tilde{A}_b^k(a)$, it is only necessary to check, additionally, that (70) holds. First, we claim that
for $1 \leq j \leq a + 1$. That this holds for $j = a + 1$ is immediate. Because $i_{j+1} = i_j - \delta_j$, the $j \rightarrow j + 1$ case of (71) implies that $x_{j+1} \leq k \iff i_j \equiv \delta_j + b_{j+1} - b \pmod 2$. Thereupon, (71) itself is obtained from (60), and the claim is proved by induction. Now, if $b_j = k$ then (58) implies that $x_j = 0$, whereupon the above claim gives $i_j \equiv k - b \pmod 2$, as required.

To show that the map from $C^2_{k} \rightarrow B^k_{k}(a)$ so described is a bijection, first note that given values $((i_j, b_j))_{j=1}^a$, the argument of section 7.3 uniquely determines values $x_{a}, x_{a-1}, \ldots, x_{1}$, for which $0 \leq x_j \leq 2k + 1$ for $1 \leq j \leq k$. However, here we require each $x_j \leq 2k$. We must show that $x_j = 2k + 1$ is not possible. Indeed, $x_j = 2k + 1$ can only arise from (58) if $b_j = k$. However, for such a case, (70) combined with (71) would give $x_j \leq k$, and then from (58) that $x_j = 0$. This establishes the bijection, and, as in section 7.3, it is weight-preserving.

9.6. The product side of the Bressoud identities

From the above bijection, we obtain

**Lemma 17.** For $k \geq 0$, let $\Lambda \in \mathcal{P}^{2k}$ be of the form $\Lambda = [k + b, k - b]$ for $0 \leq b \leq k$. Then, for each $a \geq 0$,

$$C^2_{k}(a) = B^k_{b}(a).$$

(72)

Using this result, the even $d$ case of theorem 3 then follows from theorem 16, after setting $b = k - i$ and $n_j = \lambda_{k+1-j}$.

Note that the above analysis omits the cases for which $\Lambda = [k - b, k + b]$ for $0 < b \leq k$. From the point of view of the cylindric partitions, it is easy to see that $C^2_{[k-b,k+b]}(a) = C^2_{[k+b,k-b]}(a)$, and thus (72) leads to sum-type expressions for these cases. On the other hand, the above bijection into decorated even $\mathcal{B}$-paths extends naturally so that this case maps into an analogous set of decorated paths $\tilde{h} = (h_j, h^e)$ for which the constraint on $h = (h_0, h_1, h_2, \ldots)$ that if $h_i = k$ then $i \equiv k - b \pmod 2$, is replaced by the constraint that if $h_i = k$ then $i \neq k - b \pmod 2$. However, applying the methods of the previous sections to these paths (starting with the $k = 1$ case instead of $k = 0$), leads to sum-type expressions for their generating functions that are exactly those obtained from (72) and the equality $C^2_{[k-b,k+b]}(a) = C^2_{[k+b,k-b]}(a)$.

9.7. The interpretation of the proof in terms of abaci

In the even-level case, an interpretation for the expression (68) for the generating function $B^k_{k}(a)$ in terms of moves of the yokes of the corresponding abaci can be given that is very similar to that described in section 8.2 for the odd-level cases. One difference to the description given there is that if there is a peak or deak at $j$ then the weight of the corresponding yoke depends on $h_j$; it is either $[i + h_j, i - h_j]$ or $[i - 1 - h_j, i + 1 + h_j]$, with the choice between these two determined by (29) (the weight of each yoke in the Dirac sea is $[i + b, i - b]$).
10. Discussion

10.1. Summary of results

In this paper, we have brought together known results on product expressions for the characters of \( \mathfrak{sl}_r \), \( \mathcal{W}_r \), and the generating functions of cylindric partitions, giving simple proofs of each, and showing how they are related. In addition, we have applied combinatorial methods to the cylindric partitions in the \( r = 2 \) case to obtain non-negative sum expressions for their generating functions. Equating the sum and the product expressions, and cancelling a common factor, we retrieved the known Andrews–Gordon–Bressoud extensions of the Rogers–Ramanujan identities.

10.2. Higher-rank identities

The methods of the current paper can be extended to investigate Rogers–Ramanujan-type identities that are related to higher-rank affine Lie algebras. More precisely, what we have in mind are identities based on affine \( A_n^{(1)} \) Lie algebras, \( n = 2, 3, \ldots \) As we will show elsewhere, simple proofs of the \( \mathcal{W}_n \) Rogers–Ramanujan-type identities of [26] \(^ {22} \) can be obtained using the methods of the current paper. On the other hand, identities related to higher-rank algebras other than the affine \( A_n^{(1)} \) Lie algebras, \( n = 2, 3, \ldots \), such as those obtained in [65], which are related to the twisted affine \( A_n^{(2)} \) Lie algebras, \( n = 1, 2, \ldots \), and whose characters are not, to the best of our knowledge, related to the enumeration of cylindrical plane partitions, presumably lie outside the limited scope of our methods.

10.3. An intermediate set of ordinary partitions

In the proof of the sum side of the Andrews–Gordon identities, a term \( (q; q)_{\lambda}^{-1} \) appears in the summand (see (38)), in addition to and on equal footing with the terms that must appear in any proof of these identities. A similar additional term appears in the proof of the product side of the Andrews–Gordon identities. In the limit \( a \to \infty \), these extra terms become \( (q; q)_{\lambda}^{-1} \), the generating function of ordinary partitions, and cancel in the final result, leading to the known identities.

Though they eventually cancel, the presence of these terms is crucial in a proof based on cylindric partitions. In a sense, this is the main lesson learnt of Burge’s work on Rogers–Ramanujan-type identities [28]. Extending the set of combinatorial objects that we work with, by introducing a set of ordinary partitions into the mix, greatly simplifies the proofs. The same factor appeared in [26]. A similar remark applies to the proof of the Bressoud identities.

10.4. Cylindric partitions in other contexts

The cylindric partitions that appear in this work have also appeared in Postnikov’s work on quantum Schubert calculus [66] \(^ {23} \). More recently, cylindric partitions have appeared in AGT-type computations of conformal blocks in minimal conformal field theories based on \( \mathcal{W}_r \) times a \( U(1) \) factor [43, 67–70]. This relation was discussed briefly in section 3.5. In this context, the \( (q; q)_{\lambda}^{-1} \) factor that appears in our derivations of the Andrews–Gordon–Bressoud identities, in sections 7 and 9, and that cancels so that we end up with the known forms of these identities, is the result of the action of a Heisenberg algebra whose presence is crucial to the derivation of amenable expressions for the conformal blocks.

\(^ {22} \) See also [27].
\(^ {23} \) We thank Krattenthaler for bringing Postnikov’s work to our attention.
Acknowledgments

We thank Christian Krattenthaler for discussions of W H Burge’s work, that took place almost twenty years ago. We obtained the bijection that we used in this work to derive the sum-side of the Andrews–Gordon–Bressoud identities while attempting to reconstruct Christian’s explanations. OF wishes to thank S Corteel for discussions on the topic of this work, part of which was carried out during a visit to the Laboratoire d’Informatique Algorithmique: Fondements et Applications [LIAFA], Université Paris Diderot 7. He also wishes to thank S Corteel and LIAFA for kind hospitality and the Fondation Sciences Mathematiques de Paris for financial support during his visit. Both authors were supported by the Australian Research Council.

Appendix A. The sl̂r Macdonald identity

The sl̂r Macdonald identity may be expressed in the form [72, theorem 1.61]

\[
\sum_{\sigma \in S_r} (-1)^{\ell(\sigma)} \prod_{k_1+\ldots+k_r=0} x_{\sigma(i)} q^{k_1+i-\sigma(i)} q^{\sum_k k_1+i} = (q^r; q)_\infty^{-1} \prod_{1 \leq i < j \leq r} \left( \frac{x_i}{x_j} q^{\frac{x_i}{x_j}} q \right),
\]

(73)

where \(S_r\) is the symmetric group and \(\ell(\sigma)\) is the length of \(\sigma \in S_r\). Expressing the signed sum over \(S_r\) as a determinant, leads to the following form of the identity:

\[
\sum_{k_1+\ldots+k_r=0}^{\leq s} \det (x_i^{k_1+i} q^{\sum_k k_1+i} = (q^r; q)_\infty^{-1} \prod_{1 \leq i < j \leq r} \left( \frac{x_i}{x_j} q^{\frac{x_i}{x_j}} q \right),
\]

(74)

We can also express this identity in a slightly different form. Renaming each \(k_{\sigma(i)}\) by \(k_i\) in the second sum of (73), then exchanging the order of the two sums, and finally expressing the signed sum over \(S_r\) as a determinant, leads to the following equation

\[
\sum_{k_1+\ldots+k_r=0}^{\leq s} \det (x_i^{k_1+i} q^{\sum_k k_1+i} = (q^r; q)_\infty^{-1} \prod_{1 \leq i < j \leq r} \left( \frac{x_i}{x_j} q^{\frac{x_i}{x_j}} q \right).
\]

(75)

Appendix B. sl̂r weight space and characters

B.1. Cartan data

Let \(\{\alpha_i\}_{i=0}^{r-1}\) be the simple roots of \(sl̂r\), and \(\{\Lambda_i\}_{i=0}^{r-1}\) the corresponding fundamental weights. The dual \(h^*\) of the Cartan subalgebra \(h\) of \(sl̂r\) has basis \(\{\Lambda_0, \alpha_0, \alpha_1, \ldots, \alpha_{r-1}\}\). An alternative basis is \(\{\delta, \Lambda_0, \Lambda_1, \ldots, \Lambda_{r-1}\}\), where the null root is defined by \(\delta = \sum_{i=0}^{r-1} \alpha_i\). The Weyl vector \(\rho\) is defined by \(\rho = \sum_{i=0}^{r-1} \Lambda_i\). The above bases are related by introducing orthonormal vectors \(e_0, e_2, \ldots, e_r\) and setting

24 Although much of this material is similar to that in section 2 of [73], our \(\epsilon_i\) corresponds to their \(e_{i-1}\). Note that in view of (77a), our system is such that the difference between the coefficients of \(\epsilon_i\) and \(\epsilon_{i-1}\) is equal to the length of the \(i\)th row of the corresponding partition \(\lambda(\Lambda_i)\), defined below.
\[ \Lambda_i = \epsilon_1 + \cdots + \epsilon_i - \epsilon + \Lambda_0 \quad (0 < i < r), \]  
(76a)

\[ \alpha_0 = \epsilon_r - \epsilon_1 + \delta, \quad \alpha_i = \epsilon_i - \epsilon_{i+1} \quad (0 < i < r), \]  
(76b)

with \( \epsilon = \frac{1}{r} \sum_{j=1}^{r} \epsilon_j \). To each \( \Lambda = \sum_{i=0}^{r-1} m_i \lambda_i \in \mathcal{P}_{r,d} \), there is a corresponding partition\(^{25}\) \( \tilde{\lambda}(\Lambda) = (\mu_1, \ldots, \mu_{r-1}) \in \mathcal{P}_{r-1} \) defined by \( \mu_j = \sum_{i=j}^{r-1} m_i \) for \( j = 1, \ldots, r-1 \). Note that \( \mu_1 \leq d \) (and \( \mu_r = 0 \)). Then (76a) gives

\[ \Lambda = d \Lambda_0 + \sum_{i=1}^{r} \mu_i \epsilon_i - \epsilon \sum_{i=1}^{r} \mu_i, \]  
(77a)

\[ \rho = r \Lambda_0 + \sum_{i=1}^{r} (r - i) \epsilon_i - \epsilon \sum_{i=1}^{r} (r - i) = r \Lambda_0 - \sum_{i=1}^{r} \epsilon_i + \epsilon \sum_{i=1}^{r} i, \]  
(77b)

\[ \Lambda + \rho = (r + d) \Lambda_0 + \sum_{i=1}^{r} (\mu_i - i) \epsilon_i - \epsilon \sum_{i=1}^{r} (\mu_i - i). \]  
(77c)

An inner product \( \langle \cdot | \cdot \rangle \) on \( \mathfrak{h}^* \) is defined by, in addition to \( \langle \epsilon | \epsilon \rangle = 1 \), setting \( \langle \delta | \Lambda_0 \rangle = \langle \Lambda_0 | \epsilon \rangle = \langle \Lambda_0 | \epsilon \rangle = 0 \).  
(78)

This implies that

\[ \langle \Lambda_i | \Lambda_j \rangle = \min \{ i, j \} \frac{\alpha_i | \alpha_j \rangle}{\langle \alpha_i | \alpha_j \rangle} \delta_{ij}, \quad \langle \alpha_i | \alpha_j \rangle = \delta_{ij}, \quad \langle \alpha_i | \alpha_j \rangle = 2 \delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}, \]  
(79)

for \( 0 \leq i, j < r \), where \( \delta_{ij} = 1 \) if \( i = j \), and \( \delta_{ij} = 0 \) otherwise. Additionally, we obtain

\[ \langle \epsilon | \epsilon \rangle = \langle \epsilon | \epsilon \rangle = \frac{1}{r}, \quad \langle \alpha_i | \epsilon \rangle = 0. \]  
(80)

For any affine Lie algebra \( \mathfrak{g} \) the Weyl group \( W \) of \( \mathfrak{g} \) is the group generated by Weyl reflections \( \{ s_{\alpha_i} \}_{\alpha_i \in \Pi} \), where \( \Pi \) is the set of simple roots, which act on \( \Lambda \in \mathfrak{h}^* \) according to

\[ s_{\alpha_i} \Lambda = - \frac{2}{\langle \Lambda | \alpha_i \rangle} \langle \Lambda | \alpha_i \rangle \]  
(81)

For the simple root \( \alpha_i \in \Pi \), set \( s_i = s_{\alpha_i} \). If \( w = s_{i_0} s_{i_1} \cdots s_{i_r} \), and \( w \) cannot be written as a shorter product of the generators, then we say that \( \ell \) is the length of \( w \) and write \( \ell(w) = \ell \).

For any affine Lie algebra \( \mathfrak{g} \), \( W \) can be written as a semi-direct product

\[ W = T \ltimes \mathcal{W}_0, \]  
(82)

where \( T \) is an infinite abelian group and \( \mathcal{W}_0 \) is the Weyl group of the corresponding simple Lie algebra \( \mathfrak{g}_0 \) [38, section 3]. With \( \mathfrak{h}_0^0 \subset \mathfrak{h}^* \) the weight space of \( \mathfrak{g}_0 \), the group \( T \) is generated by certain elements \( t_{\omega} \), indexed by \( \omega \in \mathfrak{h}_0^0 \), that act on \( \Lambda \in \mathfrak{h}^* \) according to

\[ t_{\omega} \Lambda = \Lambda + \langle \Lambda | \delta \rangle \omega - \left( \langle \Lambda | \alpha \rangle + \frac{1}{2} \langle \alpha | \alpha \rangle \langle \Lambda | \delta \rangle \right) \delta. \]  
(83)

It can be shown that each \( \ell(t_{\omega}) \in 2\mathbb{Z}_{\geq 0} \).

In the case of \( \mathfrak{sl}_r \), \( T = \{ t_{\omega} \}_{\omega \in \mathfrak{h}_0} \), where \( \mathfrak{h}_0 = \{ \eta_1, \cdots, \eta_r \} \) the root lattice of \( \mathfrak{sl}_r \), given explicitly by \( Q_r = \{ n_1 \alpha_1 + n_2 \alpha_2 + \cdots + n_{r-1} \alpha_{r-1} \ | \ n_i \in \mathbb{Z} \} \). Alternatively, because each \( \alpha_i = \epsilon_i - \epsilon_{i+1} \),

\[ Q_r = \{ k_1 \epsilon_1 + k_2 \epsilon_2 + \cdots + k_r \epsilon_r \ | \ k_i \in \mathbb{Z}, k_1 + k_2 + \cdots + k_r = 0 \}. \]  
(84)

\(^{25}\) This partition is the tableau signature \( (f_1, \ldots, f_r) \) described in section 2 of [73], when \( f_r = 0 \). There, adding or removing a column of length \( r \) does not change the corresponding element of \( \mathcal{P}_{r,d} \).
Also in the case $g = \hat{sl}_r$, $W_0 \cong \mathfrak{S}_r$ and this acts naturally on the indices of the $e_i$ in that $\sigma(e_i) = e_{\sigma(i)}$ for all $\sigma \in \mathfrak{S}_r$. After defining $\sigma(\delta) = \delta$ and $\sigma(\Lambda_0) = \Lambda_0$, the action of $\mathfrak{S}_r$ is extended linearly to the whole of $\mathfrak{h}^\ast$.

B.2. Affine characters

For an affine Lie algebra $g$, the character $\chi_\Lambda^g$ of an integrable highest weight $g$-module of highest weight $\Lambda$ is given by the following theorem

**Theorem 18.** [38, section 10.4]

$$
\chi_\Lambda^g = e^\Lambda \frac{\mathcal{N}_\Lambda}{\mathcal{N}_0},
$$

where

$$
\mathcal{N}_\Lambda = \sum_{w \in W} (-1)^{f(w)} e^{w(\Lambda + \rho) - (\Lambda + \rho)},
$$

in which $W$ is the Weyl group of $g$.

In the $\hat{sl}_r$ case, noting (82), each element $w \in W$ is of the form $w = t_{\sigma}r$ for $\alpha = \sum_{i=1}^r k_i \epsilon_i$ with $\sum_{i=1}^r k_i = 0$, and $\sigma \in \mathfrak{S}_r$. For such $w$, use of (77e) and (83) yields

$w(\Lambda + \rho) - (\Lambda + \rho) = (t_{\sigma}(\sigma(\Lambda + \rho)) - \sigma(\Lambda + \rho)) + \sigma(\Lambda + \rho) - (\Lambda - \rho)$

$$
= (r + d)\sum_i k_i \epsilon_i - \frac{1}{2} (r + d) \sum_i k_i^2 \delta + \sum_i \epsilon_i - \mu_{\sigma(i)} + \sigma(i) \epsilon_{\sigma(i)},
$$

where each summation is over $1 \leq i \leq r$. On setting $x_i = e^{-\epsilon_i}$ and $q = e^{-\delta}$, we then obtain

$$
e^{w(\Lambda + \rho) - (\Lambda + \rho)} = \prod_{i=1}^r x_i^{-(r + d)k_i + \mu_{\sigma(i)} - \mu_i + \epsilon_i - \sigma_i} q^{\delta_k + \frac{1}{2}(r + d)k_i^2}.
$$

Note that $(-1)^{f(w)} = (-1)^{f(r)}$. Substituting (88) into (86) and writing the sum as a sum over $k_1, \ldots, k_r$ with $k_1 + \cdots + k_r = 0$, and a signed sum over $\sigma \in \mathfrak{S}_r$, and then expressing the latter sum as a determinant, we obtain

$$
\mathcal{N}_{\hat{sl}_r} = \sum_{k_1, \ldots, k_r} \det_{1 \leq i, j \leq r} (x_i^{-(r + d)k_j - \mu_i + \epsilon_j - \sigma_j} q^{\delta_k + \frac{1}{2}(r + d)k_i^2}).
$$

In the $\Lambda = 0$ case, we find that

$$
\mathcal{N}_{\hat{sl}_r} = \sum_{k_1, \ldots, k_r} \det_{1 \leq i, j \leq r} (x_i^{-r k_j + \epsilon_j - \sigma_j} q^{\delta_k + \frac{1}{2}(r + d)k_i^2}) = (q; q)_r^{-1} \prod_{1 \leq i < j \leq r} \left( \frac{x_i}{x_j} \frac{q_{ij}}{q_{ji}} \right),
$$

the final equality resulting from the Macdonald identity (74) after noting that the sign of each $k_i$ in the sum can be changed. This identity (90) is the denominator identity for $\hat{sl}_r$ [38, equation (10.4.4)].

In section 3.1 of the main text, we evaluate the principal specialisations of $\chi^g_\Lambda$ and $\mathcal{N}_{\hat{sl}_r}$. These are the specialisations for which $e^\Lambda \to 1$ and $e^{-\epsilon_i} \to q$ for each $\epsilon_i \in \Pi$. Because
\[ \delta = \sum_{i=0}^{\ell} \alpha_i, \text{ we also have } e^{-\delta} \rightarrow q^\delta. \]  
Note that, because \( \alpha_i = \epsilon_i - \epsilon_{i+1} \) and \( x_i = e^{-\epsilon_i} \), the specialisation \( e^{-\alpha_i} \rightarrow q^{-\alpha_i} \) is effected by \( x_i = q^{-\epsilon_i} \).

**B.3. \( \mathcal{W} \) characters**

For \( \rho \leq p < p' \) with \( p \) and \( p' \) coprime, the \( \mathcal{M}_{\rho,p}' \) minimal model character of the highest weight \( \mathcal{W}_\ell \) representation labelled by the pair \( \zeta \in \mathcal{P}_{\rho,p} \) and \( \zeta' \in \mathcal{P}_{\rho',p'} \) is given by \([40-42],\)

\[ \chi_{\zeta,\zeta'}^{\rho,p'} = \frac{1}{\eta(q)^{p'-1}} \sum_{\alpha \in \mathcal{Q}, \xi' \in \mathcal{E}_\xi} (-1)^{\ell(\alpha)} q^{\rho p' \ell(\alpha) + (\zeta + \rho \xi' - \zeta' \alpha + \sigma(\zeta') \alpha / p')} \ell! \ell'. \]

Here, \( \eta(q) = q^{1/24} \) is the Dedekind \( \eta \)-function, and \( \mathcal{Q} \) is given by \([84],\)

We now express \([91],\) in terms of \( \mu \) and \( \nu \), defined by \( \mu = (\mu_1, \mu_2, \ldots, \mu_r) = \hat{\lambda}(\zeta) \) and \( \nu = (\nu_1, \nu_2, \ldots, \nu_r) = \hat{\lambda}(\zeta') \). For the exponent of \([91],\),

\[ \left| \alpha - \frac{\xi + \rho}{p'} + \frac{\sigma(\zeta + \rho)}{p'} \right|^2 = |\alpha|^2 - \frac{2}{p} (\alpha(\xi + \rho) + \frac{\sigma^{-1}(\alpha)\zeta + \rho)}{p'} + \frac{\zeta + \rho}{p'} \right|^2. \]

From \([84],\), each \( \alpha \in \mathcal{Q} \) is of the form \( \alpha = \sum_{i=1}^{r} k_i \epsilon_i \) with \( \sum_{i=1}^{r} k_i = 0 \). Then \( \sigma^{-1}(\alpha) = \sum_{i=1}^{r} k_i \epsilon_i \). After writing

\[ \langle \sigma^{-1}(\xi + \rho) \zeta + \rho \rangle = \langle \sigma^{-1}(\xi + \rho) - (\xi + \rho) \zeta + \rho \rangle + \langle \xi + \rho \zeta + \rho \rangle, \]

and using the expression \([77c],\) for \( \xi + \rho \) and \( \zeta + \rho \), we obtain

\[ \left| \alpha - \frac{\xi + \rho}{p'} + \frac{\sigma(\zeta + \rho)}{p'} \right|^2 = p' \sum_{i=1}^{r} \left( \frac{\xi + \rho}{p} - \zeta + \rho \right)^2 \]

where we define

\[ \hat{\Delta}_{\xi,\zeta}^r = \frac{1}{2p'} \left| \xi + \rho \right|^2 - \frac{\zeta + \rho}{p'} \]

Substituting this into \([91],\), writing the signed sum over \( \mathcal{E}_\xi \) as a determinant, and using \( \eta(q) = q^{1/24} \) as a determinant, and using \( \eta(q) = q^{1/24} \), yields

\[ \chi_{\zeta,\zeta'}^{\rho,p'} = q^{r \ell(\xi - \zeta) + \ell(\xi - \zeta') / 24} \sum_{k_{i,j} \geq 0} q^{\rho' \ell(\xi' - \zeta')} \det_{1 \leq i,j \leq r} (q^{\rho' \ell(\xi' - \zeta')} - q^{\rho' \ell(\xi' + \nu - \nu')}) \]

**Appendix C. Kyoto patterns and cylindric partitions**

In \([31, \text{section } 3],\) the authors define a pattern to be a two-index array of integers \( t_{jk} \) for \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}_{\geq 0} \), which is subject to:

1. For each \( j \), there exists \( \gamma_j \) such that the sequence \( \{ \gamma_j - t_{jk} \}_{k \geq 0} \) is a partition;

Note that only ratios \( x_i/y_i \) are required.
(2) $t_{jk} \leq t_{j+1,k}$ for all $j$ and $k$;
(3) $t_{j+d,k} = t_{jk} + r$ for all $j$ and $k$.

Such a pattern is said to be normalised if $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_d < r$.

We will now show that such a normalised sequence encodes a cylindric partition of type $(\infty')/\mu/d$ where the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_r)$ is defined so that its conjugate $\mu'$ is given by $\mu'_j = (\gamma_d - \gamma_1, \gamma_d - \gamma_2, \ldots, \gamma_d - \gamma_{d-1}, 0)$. The $j$th column of the corresponding plane partition $\pi$ is then obtained from the sequence $\{t_{jk}\}_{k \geq 0}$ by, for $k > 0$, placing $t_{jk} - t_{j,k-1}$ entries $k$ in that column. In other words, the values in the $j$th column are the parts of the partition conjugate to $\{\gamma_j - t_{jk} \}_{k \geq 0}$. It is easily checked that the cylindric partition $\pi$ so defined satisfies the conditions (7).

For $1 \leq j \leq d$, let $\lambda^{(j)}$ denote the partition $\{\gamma_j - t_{jk} \}_{k \geq 0}$. The array $\{t_{jk}\}$ is then also encoded in the multipartition $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(d)})$. Because these multipartitions inherit a cylindrical embedding property, they are termed cylindrical multipartitions (in [31, 74] the boxes of $\lambda$ are coloured in a certain periodic manner). Such multipartitions were first considered in [30].

We thus see that translating between a normalised pattern $\{t_{jk}\}$, a cylindric partition and a cylindrical multipartition is straightforward. For example, consider the cylindrical multipartition of example 2.12 of [74] for which $r = 4$, $d = 3$ and $\Lambda = (0, 0, 1, 2)$. Here $\lambda = ((10, 10, 8, 4, 4), (9, 9, 1, 1), (10, 7, 1))$. The description above then leads to the plane partition

\[
\begin{array}{ccccccc}
10 & 4 & 1 & \cdot & \cdot & \cdot & \\
10 & 9 & 7 & 4 & \cdot & \cdot & \\
10 & 9 & 1 & \cdot & \cdot & \cdot & \\
8 & 1 & \cdot & \cdot & \cdot & \cdot & \\
10 & 4 & 1 & \cdot & \cdot & \cdot & \\
10 & 9 & 7 & 4 & \cdot & \cdot & \\
10 & 9 & 1 & \cdot & \cdot & \cdot & \\
8 & 1 & \cdot & \cdot & \cdot & \cdot & \\
\end{array}
\]

Through the correspondence between patterns and plane partitions described above, this condition translates to state that for each $k > 0$, there exists at least one row of $\pi$ in which no entry $k$ appears. This is also equivalent to the condition on multipartitions described in [74 proposition 2.11]²⁸. Multipartitions that respect this condition index the irreducible representations of the Ariki–Koike algebras $U_q(\tilde{\mathfrak{sl}}_r)$, where each parameter is a certain root of unity [32]. The connection between this topic and the crystal basis theory of $U_q(\tilde{\mathfrak{sl}}_r)$ was elucidated and exploited in [74].

The patterns $\{t_{jk}\}$ that pertain to irreducible characters of $\tilde{\mathfrak{sl}}_r$ require a further constraint. Proposition 3.4 of [31] designates a normalised pattern $\{t_{jk}\}$ as highest-lift if, for each $k > 0$, there exists $j$ such that $t_{j+1,k-1} > t_{jk}$. Through the correspondence between patterns and plane partitions described above, this condition translates to state that for each $k > 0$, there exists at least one row of $\pi$ in which no entry $k$ appears. This is also equivalent to the condition on multipartitions described in [74 proposition 2.11]²⁸. Multipartitions that respect this condition index the irreducible representations of the Ariki–Koike algebras $\mathcal{H}(v; u_1, \ldots, u_d)$, in the cases where each parameter is a certain root of unity [32]. The connection between this topic and the crystal basis theory of $U_q(\tilde{\mathfrak{sl}}_r)$ was elucidated and exploited in [74].

²⁷ In [74], this is labelled by $[2, 1, 0, 0] \in P_{L_+}$. However, to agree with the conventions of the current paper, it should be labelled by $[0, 0, 1, 2]$, or any cyclic permutation of the indices therein.

²⁸ If $\mu = \tilde{\mu}(\Delta)$, the colouring of the boxes enables the full character $\chi^\Delta_{\mu}$ to be obtained by enumerating the highest-lift multipartitions.
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