Generalized Springer correspondence for symmetric spaces associated to orthogonal groups

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Abstract. Let $G = GL_N(k)$, where $k$ is an algebraically closed field of characteristic $k \neq 2$, and $\theta$ an involutive automorphism of $G$ such that $H = (G^\theta)^0$ is isomorphic to $SO_N(k)$. Then $G^\theta = \{g \in G | \theta(g) = g^{-1}\}$ is regarded as a symmetric space $G/G^\theta$. Let $G^\theta_{uni}$ be the set of unipotent elements in $G^\theta$. $H$ acts on $G^\theta_{uni}$ by the conjugation. As an analogue of the generalized Springer correspondence in the case of reductive groups, we establish in this paper the generalized Springer correspondence between $H$-orbits in $G^\theta_{uni}$ and irreducible representations of various symmetric groups.

Introduction

In [L1], Lusztig established the theory of generalized Springer correspondence for reductive groups. Let $G$ be a connected reductive group, and $G_{uni}$ the set of unipotent elements in $G$. $G_{uni}$ has finitely many conjugacy classes, and let $\mathcal{N}_G$ be the set of pairs $(C, \mathcal{E})$, where $C$ is a unipotent class, and $\mathcal{E}$ is a $G$-equivariant simple ($\overline{\mathbb{Q}}_l$-) local system on $C$. Lusztig gave a notion of cuspidal pairs for $\mathcal{N}_G$, and showed that $\mathcal{N}_G$ has a natural partition

$$\mathcal{N}_G = \bigsqcup_{c \in \mathcal{J}_G} \mathcal{N}_G^{(c)}.$$

Here $\mathcal{J}_G$ is the set of isomorphism classes of the triples $(L, C_L, \mathcal{E}_L)$, where $L$ is a Levi subgroup of some parabolic subgroup of $G$, and $(C_L, \mathcal{E}_L)$ is a cuspidal pair on $L$. He showed that $\mathcal{W}_L = N_G(L)/L$ is a Coxeter group, and that there exists a natural bijection between $\mathcal{N}_G^{(c)}$ and the set $\mathcal{W}_L^\wedge$ of irreducible representations of $\mathcal{W}_L$ over $\overline{\mathbb{Q}}_l$, namely,

$$(*) \quad \mathcal{N}_G \simeq \bigsqcup_{(L, C_L, \mathcal{E}_L)} \mathcal{W}_L^\wedge,$$

where $(L, C_L, \mathcal{E}_L)$ runs over the classes in $\mathcal{J}_G$. $(*)$ is a generalization of the Springer correspondence, which is a natural injective map $W^\wedge \to \mathcal{N}_G$, where $W$ is the Weyl group of $G$. The correspondence $\mathcal{N}_G^{(c)} \simeq \mathcal{W}_L^\wedge$ is obtained by considering a certain semisimple perverse sheaf on $G$ arising from a finite Galois covering related to $\mathcal{W}_L$, and by restricting each simple component to $G_{uni}$.

In this paper, we consider a similar problem for symmetric spaces associated to orthogonal groups. Let $G = GL_n(k)$, where $k$ is an algebraically closed field of odd characteristic. We consider an involutive automorphism $\theta : G \to G$ such that its subgroup $G^\theta$ of $\theta$-fixed points is isomorphic to $O_N(k)$. Put $H = (G^\theta)^0$. 

1
Let \( \iota : G \to G \) be the anti-automorphism \( g \mapsto g^{-1} \), and consider a closed subset \( G^\theta = \{ g \in G \mid \theta(g) = g^{-1} \} \) of \( G \). As a variety, \( G^\theta \) is isomorphic to the symmetric space \( G/G^\theta \). Let \( G_{\text{uni}}^\theta \) be the set of unipotent elements in \( G^\theta \). \( G^\theta \) and \( G_{\text{uni}}^\theta \) are stable under the conjugation action of \( H \) on \( G \). It is known that \( G_{\text{uni}}^\theta \) has finitely many \( H \)-orbits ([R]).

We consider the set \( \mathcal{N}_G \) of pairs \((\mathcal{O}, \mathcal{E})\), where \( \mathcal{O} \) is an \( H \)-orbit in \( G_{\text{uni}}^\theta \), and \( \mathcal{E} \) is an \( H \)-equivariant simple local system on \( \mathcal{O} \). We can formulate the notion of cuspidal pairs for \( \mathcal{N}_G \), and so obtain a similar set \( \mathcal{N}_G \) as before. However some difficulty occurs for constructing semisimple perverse sheaf on \( G^\theta \) related to a finite Galois covering. Our discussion is based on the choice of a \( \theta \)-stable Borel subgroup \( B \) and a \( \theta \)-stable maximal torus \( T \) contained in \( B \). But in our case, \( T^\theta \) is not maximal \( \theta \)-anisotropic, and the role of semisimple elements becomes restricted in contrast to the case of reductive groups (see 1.4). In order to overcome this difficulty, we introduce a certain subgroup \( D \) of \( U \) such that \( D \) is a subset of \( U_{\text{uni}}^\theta \), where \( U \) is the unipotent radical of \( B \). By making use of \( D \), one can construct a finite Galois covering, and a semisimple perverse sheaf on \( G^\theta \) associated to it. Once this is done, basically similar arguments as in [L1] can be applied for the remaining part. We establish the generalized Springer correspondence. By making use of the restriction theorem analogous to [L1], we give an explicit description of the generalized Springer correspondence, based on the arguments used in Lusztig-Spaltenstein [LS] in the case of classical groups.

Let \( g \) be the Lie algebra of \( G \), and \( \theta : g \to g \) an induced automorphism. We obtain the decomposition \( g = g^+ \oplus g^- \) into eigen-spaces of \( g \), where \( g^\pm = \{ x \in g \mid \theta(x) = \pm x \} \). The set \( g_{\text{nil}}^- \) of nilpotent elements in \( g^- \) is isomorphic to \( G_{\text{uni}}^\theta \), compatible with \( H \)-action. We can consider the set \( \mathcal{N}_g \) of pairs \((\mathcal{O}, \mathcal{E})\), where \( \mathcal{O} \) is an \( H \)-orbit in \( g_{\text{nil}}^- \), and \( \mathcal{E} \) is an \( H \)-equivariant simple local system on \( \mathcal{O} \). In the case where \( k = \mathbb{C} \), Chen-Vilonen-Xue [CVX1] considered a similar problem for \( \mathcal{N}_g \), by making use of the Fourier transform of perverse sheaves on \( g_{\text{nil}}^- \), instead of considering the restriction of perverse sheaves on \( G \) to \( G_{\text{uni}} \). In [CVX2], they proved the generalized Springer correspondence for \( \mathcal{N}_g \), but the explicit description of the correspondence is not yet done. In [LY1], [LY2], [LY3], Lusztig-Yun studied perverse sheaves on \( g_{\text{nil}}^- \) associated to arbitrary symmetric spaces, and more generally, associated to \( \mathbb{Z}/m\mathbb{Z} \)-graded Lie algebras, and established the results closely related to the generalized Springer correspondence. In their case, they also use the Fourier-Deligne transforms, instead of the restriction.

Our result can be viewed as a “global” analogue of the problems considered by them.

Contents

Introduction
1. Preliminaries on symmetric spaces
2. Cuspidal local systems
3. Admissible complexes
4. Sheaves on the variety of semisimple orbits
5. Generalized Springer correspondence
Appendix

1. Preliminaries on symmetric spaces

1.1. Let $G$ be a connected reductive group over an algebraically closed field $k$ of characteristic $\neq 2$, and $\theta : G \to G$ be an involutive automorphism on $G$. Then $G^\theta = \{ g \in G \mid \theta(g) = g \}$ is a reductive subgroup of $G$. We put $H = (G^\theta)^0$. Let $\iota : G \to G$, $g \mapsto g^{-1}$ be the anti-automorphism on $G$. We consider the set $G_i^\theta = \{ g \in G \mid \theta(g) = g^{-1} \}$ of $\theta$-fixed points in $G$. Then $G$ acts on $G_i^\theta$ by $g : x \mapsto gx\theta(g)^{-1}$. This action is called the twisted action of $G$. $G$ acts transitively on each connected component of $G_i^\theta$ with respect to the twisted $G$-action (\cite{R}). Thus there are only finitely many $G$-orbits in $G_i^\theta$, and each $G$-orbit is closed. In particular, the set $\{ g\theta(g)^{-1} \mid g \in G \}$ is a connected component of $G_i^\theta$.

It is known by \cite{St, §7} that there exist a $\theta$-stable Borel subgroup $B$ and a $\theta$-stable maximal torus $T$ of $G$ such that $B \supset T$. We show a lemma.

**Lemma 1.2.** Let $(B, T), (B', T')$ be $\theta$-stable pairs of Borel subgroup and maximal torus of $G$. Then there exists $g \in G^\theta$ such that $gBg^{-1} = B', gTg^{-1} = T'$.

**Proof.** We choose $x \in G$ such that $xBx^{-1} = B'$, $xTx^{-1} = T'$. Then $\theta(x)B\theta(x)^{-1} = xBx^{-1}, \theta(x)T\theta(x)^{-1} = xTx^{-1}$. Hence $x^{-1}\theta(x) \in B \cap N_G(T) = T$. In particular, $x^{-1}\theta(x) = t \in T^\theta$. Since $T^\theta$ is a torus, one can find $t_1 \in T^\theta$ such that $t = (t_1)^2$, i.e., $t = t_1\theta(t_1)^{-1}$. If we put $g = xt_1$, we have $g \in G^\theta$, and $B' = gBg^{-1}, T' = gTg^{-1}$. The lemma is proved. □

**Remark 1.3.** In general, $\theta$-stable pairs $(B, T), (B', T')$ are not necessarily $H$-conjugate, (see 1.9).

1.4. We fix a $\theta$-stable Borel subgroup $B$ and a $\theta$-stable maximal torus $T$ such that $T \subset B$. A torus $S$ is called $\theta$-anisotropic if $\theta(t) = t^{-1}$ for any $t \in S$. A maximal $\theta$-anisotropic torus is a $\theta$-anisotropic torus which is maximal with respect to the inclusion relation. It is known by \cite{V} that a $\theta$-anisotropic torus exists if $\theta \neq id$, and every maximal $\theta$-anisotropic tori are conjugate under $H$. Moreover, any semisimple element in $G_i^\theta$ is contained in some maximal $\theta$-anisotropic torus. Here $T^\theta$ is a $\theta$-anisotropic torus, but in general, it is not maximal $\theta$-anisotropic as the following example shows.

Let $G = GL_2$ and define $\theta : G \to G$ by $\theta(g) = J^{-1}(t^tg^{-1})J$ with $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. We define $x \in G$ by

$$x = \begin{pmatrix} 1 & -\zeta \\ 1/2 & \zeta/2 \end{pmatrix}.$$
where \( \zeta \in k^* \) such that \( \zeta^2 = -1 \). One can check that \( x^{-1}\theta(x) = J \). Let \( T \) be the group of diagonal matrices, and \( B \) the group of upper triangular matrices in \( G \). Then \( (B, T) \) is a \( \theta \)-stable pair of Borel subgroup and maximal torus of \( G \). In this case,

\[
T^\theta = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \mid a \in k^* \right\} = Z(G).
\]

Put \( T_1 = xTx^{-1} \). Since \( x^{-1}\theta(x) = J \in N_G(T) \), \( T_1 \) is also a \( \theta \)-stable maximal torus. One can check that \( T_1^\theta = T_1 \), and so \( T_1 \) is a maximal \( \theta \)-anisotropic torus. Since \( \dim T_1 = 1 \), \( \dim T_1 = 2 \), \( T^\theta \) is not maximal.

1.5. Put \( B_H = B \cap H \), and \( T_H = T \cap H \). Then as remarked in [R, Lemma 5.1], \( B_H \) is a Borel subgroup of \( H \) and \( T_H \) is a maximal torus of \( H \) contained in \( B_H \). Let \( g = \text{Lie} G \) be the Lie algebra of \( G \), and put \( b = \text{Lie} B \), \( t = \text{Lie} T \). The differential \( d\theta \) of \( \theta \) induces an involutive automorphism of \( g \), which we also denote by \( \theta \). Thus, \( b, t \) are \( \theta \)-stable. We have a decomposition \( g = g^+ \oplus g^- \), where \( g^\pm = \{ x \in g \mid \theta(x) = \pm x \} \). Here \( g^+ = \text{Lie} H \), and \( g^- \) coincides with the tangent space of \( G/G^\theta \approx G^\theta \). Let \( \Delta \subset X(T) \) be the root system of \( G \) with respect to \( T \) (here \( X(T) \) is the character group of \( T \)), and \( \Delta_+ \) the set of positive roots in \( \Delta \) with respect to \( (T, B) \). We have the root space decomposition \( g = \text{t} \oplus \bigoplus_{\alpha \in \Delta} g_\alpha \). The map \( \alpha \mapsto \alpha^\theta \) on \( X(T) \) induces a bijection on \( \Delta \), which we denote by \( \sigma \). \( \sigma : \Delta \to \Delta \) is compatible with the root space decomposition, and \( \theta \) gives an isomorphism \( \theta : g_\alpha \to g_{\sigma(\alpha)} \). \( \sigma \) preserves \( \Delta_+ \), and so produces an automorphism of the Dynkin diagram of \( G \). Put

(1.5.1) \[
\Delta_0 = \{ \alpha \in \Delta_+ \mid \sigma(\alpha) = \alpha \}.
\]

\( \sigma \) acts freely on \( \Delta_1 = \Delta_+ - \Delta_0 \), and we denote by \( \Delta_1 \) the set of \( \sigma \)-orbits in \( \Delta_1 \). For each \( \beta = \{ \alpha, \sigma(\alpha) \} \in \Delta_1 \), put \( g_\beta = g_\alpha \oplus g_{\sigma(\alpha)} \). Then \( g_\beta \) is \( \theta \)-stable, and \( g_\beta \) is decomposed as \( g_\beta = g^+_{\beta} \oplus g^-_{\beta} \). If \( \alpha \in \Delta_0 \), then \( \theta \) gives a linear isomorphism \( g_\alpha \to g_\alpha \). Since \( \theta^2 = 1 \), we have \( \theta = \pm 1 \) on \( g_\alpha \), i.e., \( g_\alpha \subset g^+ \) or \( g_\alpha \subset g^- \). We denote by \( \Delta_0^+ \) (resp. \( \Delta_0^- \)) the set of \( \alpha \in \Delta_0 \) such that \( g_\alpha \subset g^+ \) (resp. \( g_\alpha \subset g^- \)). Let \( n \) be the nilpotent radical of \( b \). \( n \) is \( \theta \)-stable, and is decomposed as \( n = n^+ \oplus n^- \). We have a root space decomposition

(1.5.2) \[
n^\pm = \bigoplus_{\beta \in \Delta_1} g^\pm_{\beta} \oplus \bigoplus_{\alpha \in \Delta_0^+} g_\alpha.
\]

1.6. Let \( U \) be the unipotent radical of \( B \), which is \( \theta \)-stable, and put \( U_H = U^\theta = U \cap H \). For each \( \alpha \in \Delta_+ \), we have a one parameter subgroup \( U_\alpha \subset U \) such that \( \text{Lie} U_\alpha = g_\alpha \). We have an isomorphism

\[
U \simeq \prod_{\alpha \in \Delta_+} U_\alpha
\]

for a choice of the total order on \( \Delta_+ \). Here \( \theta : U \to U \) induces an isomorphism \( \theta : U_\alpha \to U_{\sigma(\alpha)} \) for each \( \alpha \), and according to the decomposition in (1.5.2) in the Lie
algebra case, we have

\begin{equation}
U_H \simeq \prod_{(\alpha, \sigma(\alpha)) \in \Delta_1} (U_\alpha U_{\sigma(\alpha)})^\theta \times \prod_{\alpha \in \Delta^+_0} U_\alpha.
\end{equation}

Note that \( U(\Delta_0) = \prod_{\alpha \in \Delta_0} U_\alpha \) is a \( \theta \)-stable closed subgroup of \( U \), and its \( \theta \)-fixed point subgroup coincides with \( \prod_{\alpha \in \Delta^+_0} U_\alpha \).

Concerning \( U^{i\theta} \), we obtain a similar description as in (1.6.1). But since \( i\theta \) is not a group homomorphism, we need a special care. One can find a filtration \( U \supset U_1 \supset U_2 \supset \cdots \) of \( U \) by \( \theta \)-stable normal subgroups, by making use of the commutator relations for \( U \), such that each quotient group \( U_i/U_{i+1} \) is of the form \( U_\alpha U_{\sigma(\alpha)} \) for \( \{\alpha, \sigma(\alpha)\} \in \Delta_1 \) or of the form \( U_\alpha \) for \( \alpha \in \Delta_0^+ \). In that case, \( U^{i\theta} \) has a filtration \( U^{i\theta} \supset U_1^{i\theta} \supset U_2^{i\theta} \supset \cdots \) (by affine subspaces) such that \( U_i^{i\theta}/U_{i+1}^{i\theta} \simeq (U_\alpha U_{\sigma(\alpha)})^{i\theta} \) or \( \simeq U_\alpha^{i\theta} \) accordingly. In particular, \( U_H \) (resp. \( U^{i\theta} \)) is isomorphic to an affine space with \( \dim U_H = \dim \mathfrak{n}^+ \) (resp. \( \dim U^{i\theta} = \dim \mathfrak{n}^- \)).

1.7. In the remaining part of the paper, we concentrate on the symmetric spaces associated to orthogonal groups. Let \( V \) be an \( N \) dimensional vector space over \( \mathbf{k} \), and let \( G = GL_N = GL(V) \) with \( N \geq 2 \). Consider an involutive automorphism \( \theta : G \to G \) defined by \( \theta(g) = J^{-1}(t^g J^{-1})J \), where

\[
J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1_n \\ 0 & 1_n & 0 \end{pmatrix} \quad \text{if } N = 2n + 1,
\]

\[
J = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \quad \text{if } N = 2n,
\]

with \( 1_n \) the identity matrix of degree \( n \). Let \( H = (G^\theta)^0 \). Then \( H \) is a special orthogonal group \( SO_N \) with respect to the symmetric bilinear form \( \langle u, v \rangle = t^{uJv} \) on \( V \) (\( u, v \in V \)), under the identification \( V \simeq \mathbf{k}^N \) via the basis \( \{e_0, e_1, \ldots, e_n, f_1, \ldots, f_n\} \) in the case where \( N = 2n + 1 \), and the basis \( \{e_1, \ldots, e_n, f_1, \ldots, f_n\} \) in the case where \( N = 2n \). It follows that

\[
\langle e_i, f_j \rangle = \langle f_j, e_i \rangle = \delta_{ij} \text{ for } 1 \leq i, j \leq n,
\]

\[
\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0 \text{ for } 1 \leq i, j \leq n,
\]

and \( \langle e_0, e_0 \rangle = 1, \langle e_0, e_i \rangle = \langle e_0, f_i \rangle = 0 \) for \( i \geq 1 \).

Let \( \mathfrak{g} = \text{Lie } G \), and \( \theta : \mathfrak{g} \to \mathfrak{g} \) be the induced automorphism. Then \( \theta(x) = -J^{-1}(t^x)J \) for \( x \in \mathfrak{g} \). Let \( x^* \) be the adjoint of \( x \in \mathfrak{g} \) with respect to the bilinear form \( (, ) \). Then we have \( x^* = J^{-1}(t^x)J \), and so

\[
\mathfrak{g}^\pm = \{ x \in \mathfrak{g} | x^* = \mp x \}.
\]

We have
\[ g^+ = \{ x \in g \mid \langle xv, w \rangle = -\langle v, xw \rangle \}, \]
\[ g^- = \{ x \in g \mid \langle xv, w \rangle = \langle v, xw \rangle \}. \]

In particular, \( g^- \) coincides with the set of self-adjoint matrices in \( g = gl_N \). Correspondingly, \( G^{\theta} \) coincides with the set of non-degenerate self-adjoint matrices in \( gl_N \). In particular, \( G^{\theta} \) is connected, and by 1.1 we have

\[
(1.7.1) \quad G^{\theta} = \{ g\theta(g)^{-1} \mid g \in G \}.
\]

1.8. We fix a \( \theta \)-stable Borel subgroup \( B \) and a \( \theta \)-stable maximal torus \( T \subset B \) as follows. First assume that \( N = 2n + 1 \). Let \( B \) be the subgroup of \( G \) consisting of the matrices of the form

\[
\begin{pmatrix}
 a & 0 & d_1 \\
 t^*d_2 & b_1 & c \\
 0 & 0 & b_2
\end{pmatrix},
\]

where \( a \in k^* \), \( d_1, d_2 \) are (row) vectors in \( k^n \), and \( b_1, b_2, c \) are square matrices of degree \( n \) with \( b_1 \) upper triangular, \( b_2 \) lower triangular. Let \( T \) be the set of all diagonal matrices in \( G \). Then \( B \) is a Borel subgroup of \( G \), and \( T \) is a maximal torus in \( G \) with \( T \subset B \). \( B, T \) are both \( \theta \)-stable. We have

\[
T^{\theta} = \left\{ \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b^{-1} \end{pmatrix} \mid b \in D_n \right\}, \quad T^{\theta} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{pmatrix} \mid a \in k^*, b \in D_n \right\},
\]

where \( D_n \) is the group of diagonal matrices of \( GL_n \). Moreover, we have

\[
B^{\theta} = \left\{ \begin{pmatrix} a & 0 & d_1 \\ t^*d_2 & b & c \\ 0 & 0 & t^*b^{-1} \end{pmatrix} \in B \mid a = \pm 1, d_2 = -a^{-1}t^*d_1b, \quad t^*c = -(t^*d_1d_1 + b^{-1}c)t^*b \right\},
\]
\[
B^{\theta} = \left\{ \begin{pmatrix} a & 0 & d \\ t^*d & b & c \\ 0 & 0 & t^*b \end{pmatrix} \in B \mid t^*c = c \right\}.
\]

Next assume that \( N = 2n \). Let \( B \) be the subgroup of \( G \) consisting of the matrices of the form

\[
\begin{pmatrix} b_1 & c \\ 0 & b_2 \end{pmatrix},
\]

where \( b_1, b_2, c \) are square matrices of degree \( n \) with \( b_1 \) upper triangular, \( b_2 \) lower triangular. Let \( T \) be the set of diagonal matrices in \( G \). Then \( B \) is a Borel subgroup of \( G \) and \( T \) is a maximal torus in \( G \) with \( T \subset B \). \( B, T \) are both \( \theta \)-stable. We have
Moreover, we have
\[
T^\theta = \left\{ \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix} \big| b \in D_n \right\}, \quad T'^\theta = \left\{ \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \big| b \in D_n \right\}.
\]

Moreover, we have
\[
B^\theta = \left\{ \begin{pmatrix} b & c \\ 0 & t_b^{-1} \end{pmatrix} \in B \big| t_c = -b^{-1}c t_b \right\},
\]
\[
B'^\theta = \left\{ \begin{pmatrix} b & c \\ 0 & t_b \end{pmatrix} \in B \big| t_c = c \right\}.
\]

Let \( U \) be the unipotent radical of \( B \). Then \( U \) is \( \theta \)-stable. Put \( B_H = (B^\theta)^0, T_H = (T^\theta)^0 \) and \( U_H = U^\theta \). Thus \( B_H \supset T_H \) is a pair of a Borel subgroup and a maximal torus in \( H \), and \( U_H \) is the unipotent radical of \( B_H \).

**1.9.** Let \( T \subset B \) be as above. For \( i = 1, \ldots, n - 1 \), let \( s_i \) be the permutation of the basis in \( V \) such that \( e_i \leftrightarrow e_{i+1}, f_i \leftrightarrow f_{i+1} \) and that it fixes all other basis. Also for \( i = 1, \ldots, n \), let \( t_i \) be the permutation \( e_i \leftrightarrow f_i \) which fixes all other basis. Assume that \( N \) is odd. Then \( N_H(T) \) is generated by \( s_1, \ldots, s_{n-1}, t_n \) and \( T \). \( N_H(T)/T \) is isomorphic to the Weyl group \( W_n \) of type \( B_n \). On the other hand, \( N_{G^\theta}(T) \) is generated by \( N_H(T) \) and \(-1\), so that \( N_{G^\theta}(T)/T \simeq W_n \times \mathbb{Z}/2\mathbb{Z} \). Hence in view of Lemma 1.2, any \( \theta \)-stable pair \((B', T')\) is \( H \)-conjugate to \((B, T)\).

Next assume that \( N \) is even. In this case, \( N_H(T) \) is generated by \( s_1, \ldots, s_{n-1}, t_{n-1} t_n \) and \( T \). \( N_H(T)/T \) is isomorphic to the Weyl group \( W'_n \) of type \( D_n \). Moreover, \( N_{G^\theta}(T) \) is generated by \( N_H(T) \) and \( t_n \), hence \( N_{G^\theta}(T)/T \simeq W'_n \). One can check that \( B_1 = t_n B t_n^{-1} \) is not contained in the set of \( W'_n \)-conjugates of \( B \). It follows that the \( \theta \)-stable pair \((B_1, T)\) is not \( H \)-conjugate to \((B, T)\), and \((B, T), (B_1, T)\) give representatives of \( H \)-conjugates of \( \theta \)-stable pairs in \( G \).

**1.10.** Let \( \Delta \) and \( \Delta_+ \) be as in 1.5 with respect to the pair \((B, T)\) in 1.8. Let \( t = \text{Lie} T \), and fix the basis of \( t^* \) as \( \{ \epsilon_0, \epsilon_1, \ldots, \epsilon_n, \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_N \} \) in the case where \( N = 2n+1 \), and \( \{ \epsilon_1, \ldots, \epsilon_n, \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_n \} \) in the case where \( N = 2n \). Here \( \epsilon_i \) (resp. \( \tilde{\epsilon}_j, \epsilon_0 \)) is the weight vector corresponding to \( e_i \) (resp. \( f_j, e_0 \)). We write those vectors as \( \bar{\zeta}_1, \ldots, \bar{\zeta}_N \), in the order \( \epsilon_1, \ldots, \epsilon_n, \tilde{\epsilon}_0, \epsilon'_1, \ldots, \epsilon'_n \) if \( N = 2n+1 \), and \( \epsilon_1, \ldots, \epsilon_n, \epsilon'_1, \ldots, \epsilon'_n \) if \( N = 2n \). Thus \( \Delta_+ \) can be written as
\[
\Delta_+ = \{ \bar{\zeta}_i - \bar{\zeta}_j \mid 1 \leq i < j \leq N \}.
\]

It follows from the description of \( B'^\theta \) in 1.8, we see that

\[
\Delta_0^+ = \emptyset, \quad \Delta_0^- = \{ \bar{\zeta}_i - \bar{\zeta}_{N-i+1} \mid 1 \leq i \leq n \}.
\]

Recall that \( U(\Delta_0) = \prod_{\alpha \in \Delta_0} U_\alpha \) is a \( \theta \)-stable subgroup of \( U \). It follows from (1.10.1), we have \( U(\Delta_0) = U(\Delta_0)^{\theta} \), which we denote by \( \mathcal{D} \). By using (1.10.1), \( \mathcal{D} \)
Thus $D$ can be written explicitly as follows.

\[
\mathcal{D} = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1_n & c \\ 0 & 0 & 1_n \end{pmatrix} \in U \ \bigg| \ c : \text{diagonal} \right\} \quad (N = 2n + 1),
\]
\[
\mathcal{D} = \left\{ \begin{pmatrix} 1_n & c \\ 0 & 1_n \end{pmatrix} \in U \ \bigg| \ c : \text{diagonal} \right\} \quad (N = 2n).
\]

Thus $\mathcal{D}$ is $T^\theta$-stable, and $\mathcal{D} \simeq k^n$.

1.11. Let $P = LU_P$ be a $\theta$-stable parabolic subgroup of $G$ containing $B$, where $L$ is the Levi subgroup of $P$ containing $T$ and $U_P$ is the unipotent radical of $P$. Here we consider the special case where $L^\theta \simeq (GL_1)^a \times GL^\theta_{N_0}$ with $N_0 = N - 2a$ for $0 \leq a \leq n$ (we understand that $GL_1^\theta = \{\pm 1\}$, $GL^\theta_0 = \{1\}$). Put $\mathcal{D}_P = \mathcal{D} \cap U_P$. Then $\mathcal{D}_P$ is a closed subgroup of $U_P$, and is contained in $U_P^\theta$. We have $\mathcal{D}_P \simeq k^a$. It is easy to see that $\mathcal{D}_P$ is stable under the conjugation action of $L$ on $U_P$. More precisely, we have

\[
\text{(1.11.1)} \quad \text{Assume that } x \in L^\theta \text{ and } u \in \mathcal{D}_P. \text{ Then we have } xu = ux.
\]

The proof is done by a direct computation by using (1.10.2). Note that in the Lie algebra case, this corresponds to the fact that if $x, y \in g^-$ such that $[x, y] \in g^-$, then $[x, y] = 0$ as $[x, y] \in g^+$. \dots

Remark 1.12. For the comparison, we briefly discuss the case of symplectic groups. Let $V$ be an $N = 2n$-dimensional vector space over $k$, and let $G = GL_N = GL(V)$. The involutive automorphism $\theta : G \to G$ is defined as in 1.7, but by replacing $J$ by $J = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$. Then $H = G^\theta$ is the symplectic group $Sp_N$, and we consider $G^\theta$ with $H$-action. We choose a pair of $\theta$-stable Borel subgroup $B$ and a $\theta$-stable maximal torus $T \subset B$ as in [SS, 1]. Then the root system $\Delta$ of $G$ and the set of positive roots with respect to the pair $(B, T)$ are given similarly to 1.10. In particular, the set $\Delta_0 = \Delta_0^+ \cup \Delta_0^-$ is defined, and $\Delta_0^+$ are determined by using the computation in [SS, 1], namely, we have

\[
\text{(1.12.1)} \quad \Delta_0^+ = \{ \tilde{e}_i - \tilde{e}_{N-i+1} \mid 1 \leq i \leq n \}, \quad \Delta_0^- = \emptyset.
\]

(1.12.1) shows that in the symplectic case, an opposite situation occurs compared to the orthogonal case (1.10.1).

2. Cuspidal local systems

2.1. In this section, we extend the notion of cuspidal local systems in the case of reductive groups given in [L1] to the case of symmetric spaces. Let $G = GL_N$, and $\theta : G \to G$ be as in 1.7. Let $H = (G^\theta)^0$. We denote the twisted action of $G$ on $G^\theta$ given by $g : x \mapsto gx\theta(g)^{-1}$ ($g \in G, x \in G^\theta$) by $g \cdot x$. \dots
Let $P = LU_P$ be a $\theta$-stable parabolic subgroup of $G$ containing $B$, where $L$ is the $\theta$-stable Levi subgroup containing $T$ and $U_P$ is the unipotent radical of $P$. Then $L^\theta \simeq GL_{n_1} \times \cdots \times GL_{n_a} \times GL_{N_0}^\theta$, where $N_0 + \sum_{i=1}^a 2n_i = N$. We have

$$(2.1.1) \quad L^\theta \simeq GL_{n_1} \times \cdots \times GL_{n_a} \times GL_{N_0}^\theta,$$

on which $L^\theta$ acts via the conjugation action. The natural projection $P \to P/U_P \simeq L$ induces a surjective map $\eta_P : P^\theta \to L^\theta$. Note that $L^\theta$ acts on $U_P^\theta$ by conjugation action. Under the isomorphism $L^\theta \simeq L/L^\theta$, $\eta_P$ is regarded as an affine space bundle over $L/L^\theta$, namely,

$$(2.1.2) \quad P^\theta \simeq L \times L^\theta U_P^\theta \to L/L^\theta \simeq L^\theta.$$

We note that

$$(2.1.3) \quad P^\theta_0 \text{ is connected, and we have } P^\theta = \{g\theta(g)^{-1} \mid g \in P\}.$$

In fact, $L^\theta$ is connected by (1.7.1) and by (2.1.1). Since $U_P^\theta$ is connected, $P^\theta_0$ is connected by (2.1.2). On the other hand, $\{g\theta(g)^{-1} \mid g \in P\}$ is a connected subset of $P^\theta$, which has the same dimension as $P^\theta$, hence coincides with $P^\theta$. Thus (2.1.3) holds.

Put $P_H = (P^\theta)_0$ and $L_H = (L^\theta)_0$. Then $P_H$ is the parabolic subgroup of $H$ containing $B_H$. Let $W_H = N_H(T_H)/T_H$ be the Weyl group of $H$. We define a Weyl subgroup $W_L$ of $W_H$ as the Weyl group of $L_H$. Let $\mathcal{O}$ be an $H$-orbit in $G^\theta/\theta$ and $\mathcal{O}_L$ an $L_H$-orbit in $L^\theta$. Let $Z_L^0$ be the connected center of $L$. Then $(Z_L^0)^\theta \simeq (GL_1)^a$.

Consider the varieties

$$Z = \{(x, gP_H, g'P_H) \in G^\theta \times H/P_H \times H/P_H \mid g^{-1}xg \in \eta_P^{-1}((Z_L^0)^\theta \mathcal{O}_L), g'^{-1}xg' \in \eta_P^{-1}((Z_L^0)^\theta \mathcal{O}_L)\},$$

$$Z' = \{(x, gP_H, g'P_H) \in G^\theta \times H/P_H \times H/P_H \mid g^{-1}xg \in \eta_P^{-1}(\mathcal{O}_L), g'^{-1}xg' \in \eta_P^{-1}(\mathcal{O}_L)\}.$$

We consider the partition $H/P_H \times H/P_H = \coprod \mathcal{O}_\omega$ into $H$-orbits, where $\mathcal{O}_\omega$ is an $H$-orbit containing $(P_H, \omega P_H)$ for $\omega \in W_{L_H} \setminus W_H/W_{L_H}$. Let $Z_\omega = p^{-1}(\mathcal{O}_\omega)$, $Z'_\omega = p'^{-1}(\mathcal{O}_\omega)$, where $p : Z \to H/P_H \times H/P_H$, $p' : Z' \to H/P_H \times H/P_H$ are projections onto 2nd and third factors. An orbit $\mathcal{O}_\omega$ is said to be good if $P_H$ and $\omega P_H \omega^{-1}$ have a common Levi subgroup, otherwise $\mathcal{O}_\omega$ is said to be bad. The good orbit corresponds to $\omega$ such that $\omega W_{L_H} = W_{L_H} \omega$.

In order to proceed the induction process smoothly for proving the proposition below, we replace $G$ by groups appearing as a Levi subgroup of some parabolic subgroup of $G$, namely we consider

$$(2.1.4) \quad G = G_0 \times \prod_{i=1}^t (G_i \times G_i),$$
where $G_0 = GL_N, G_i = GL_{N_i}$. We consider an involutive automorphism $\theta : G \to G$ such that $\theta$ acts on $G_i \times G_i$ as a permutation of factors, and $\theta$ acts on $G_0$ so that $G_0^\theta \simeq O_N$. Hence

\begin{equation}
G^\theta \simeq G_0^\theta \times \prod_{i=1}^t G_i.
\end{equation}

Under this setting, $P = LU_P$ and the varieties $Z, Z'$ are defined similarly. In particular, $L^\theta$ can be written as $L^\theta \simeq GL_{n_1} \times \cdots \times GL_{n_a} \times GL_{N_0}^\theta$, where $GL_{N_0}$ is a subgroup of $G_0^\theta$. Put $\nu_H = \dim U^\theta$, and $\overline{\nu} = \nu_{L_H}$. Let $c = \dim \mathcal{O}$ and $\overline{\nu} = \dim \mathcal{O}_L$. Put $r = \dim(Z_L^\theta)^\theta$. We define $\Delta_P$ by

\begin{equation}
\Delta_P = (N - N_0)/2.
\end{equation}

The following result is an extension of [L1, Proposition 1.2]. The proof is done along a similar line under suitable modifications.

**Proposition 2.2.** Under the notation above,

(i) For $\overline{\nu} \in \mathcal{O}_L$, we have $\dim(\mathcal{O} \cap \eta_P^{-1}(\overline{\nu})) \leq (c - \overline{\nu})/2 + \Delta_P/2$.

(ii) For $x \in \mathcal{O}$,

$$\dim\{gP_H \in H/P_H \mid g^{-1}xg \in \eta_P^{-1}(\mathcal{O}_L)\} \leq (\nu_H - c/2) - (\overline{\nu} - \overline{\nu}/2) + \Delta_P/2.$$ 

(iii) Put $d = 2\nu_H - 2\overline{\nu} + \overline{\nu} + r$. Then $\dim Z_\omega \leq d + \Delta_P$ if $\mathcal{O}_\omega$ is good, and $\dim Z_\omega < d + \Delta_P$ if $\mathcal{O}_\omega$ is bad. Hence $\dim Z \leq d + \Delta_P$.

(iv) Put $d' = 2\nu_H - 2\overline{\nu} + \overline{\nu}$. Then $\dim Z_\omega' \leq d' + \Delta_P$ for any $\omega$. Hence $\dim Z' \leq d' + \Delta_P$.

**Proof.** Since the proposition holds in the case where $G = T \times T$ for a torus $T$, we may assume that the proposition holds for a proper Levi subgroup $L$ of $G$.

Consider an orbit $\mathcal{O}_\omega$. Let $w \in W_H$ be a representative of $\omega \in W_{L_H} \setminus W_H/W_{L_H}$, and $\tilde{w} \in N_H(T_H)$ a representative of $w$. In order to show (iii) and (iv), it is enough to see that

\begin{equation}
\dim\{x \in \eta_P^{-1}(Z_L^\theta)^\theta \mid \tilde{w}^{-1}x\tilde{w} \in \eta_P^{-1}(Z_L^\theta)^\theta \}
\leq 2\nu_H - 2\overline{\nu} + \overline{\nu} + r - \dim \mathcal{O}_\omega + \Delta_P,
\end{equation}

\begin{equation}
\dim\{x \in \eta_P^{-1}(\mathcal{O}_L) \mid \tilde{w}^{-1}x\tilde{w} \in \eta_P^{-1}(\mathcal{O}_L)\}
\leq 2\nu_H - 2\overline{\nu} + \overline{\nu} - \dim \mathcal{O}_\omega + \Delta_P,
\end{equation}

and that (2.2.1) is a strict inequality if $\mathcal{O}_\omega$ is bad. Now by (2.1.2), an element $x \in P^\theta \cap wP^\theta$ can be written as $x = \ell \cdot u = \ell' \cdot u'$ with $\ell \in L, \ell' \in wL, u \in U_P^\theta, u' \in wU_P^\theta$. Moreover, there exists a unique element $z \in L \cap wL$ such that $\ell = zy', \ell' = zy$ with $y' \in L \cap wU_P, y \in wL \cap U_P$. Put

$$\tilde{Y} = \{(u, u', y, y', z) \in U_P^\theta \times wU_P^\theta \times (wL \cap U_P) \times (L \cap wL) \times (L \cap wL) \}.$$
\[ y' \cdot u = y \cdot u', \quad z y' (z y')^{-1} \in (Z_L^0)^{\theta} G_L, \quad \hat{w}^{-1} z y \theta(z y)^{-1} \hat{w} \in (Z_L^0)^{\theta} G_L. \]

and let \( Y \) be the quotient of \( \tilde{Y} \) by \( E^\theta \), where \( E = (w L \cap U_P) \times (L \cap w U_P) \times (L \cap w L) \) and \( E^\theta \) acts on \( \tilde{Y} \) by

\[(\ell_1, \ell_2, z_1) : (u, u', y, y', z) \mapsto (\ell_2 u \ell_2^{-1}, \ell_1 u' \ell_1^{-1}, z_1 y \ell_1^{-1}, z_1 y' \ell_2^{-1}, z z_1^{-1}).\]

Then (2.2.1) can be rewritten as

\[(2.2.3) \quad \dim Y \leq 2 \nu_H - 2 \bar{\nu} + \bar{\gamma} + r - \dim \Omega_\omega + \Delta_P,\]

where it is the strict inequality if \( \Omega_\omega \) is bad. Moreover (2.2.2) can be rewritten as the inequality obtained by dropping \((Z_L^0)^{\theta}\) in the definition of \( \tilde{Y} \), and by dropping \( \bar{\gamma} \) from (2.2.3).

We now consider the projection \( Y \to (w L \cap U_P) \times (L \cap w U_P) \times (L \cap w L) \) by \( (u, u', y, y', z) \mapsto (y, y', z) \). For fixed \( y, y', z \), the fibre \( Y_{y, y', z} \) can be written as

\[ Y_{y, y', z} = \{(u, u') \in U_P^{\theta} \times w U_P^{\theta} | \ y' \cdot u = y \cdot u' \}. \]

We note that

\[(2.2.4) \quad \dim Y_{y, y', z} = \dim(U_P \cap w U_P)^{\theta}.\]

In fact, put

\[
\tilde{u} = (y'^{-1} y y')^{-1} \cdot u,
\]

\[
\tilde{u}' = y'^{-1} \cdot u'.
\]

Since \( y \in U_P, y' \in L \), we have \( y'^{-1} y y' \in U_P \). Thus \( \tilde{u} \in U_P^{\theta} \). On the other hand, since \( y' \in w U_P \), we have \( \tilde{u}' \in w U_P^{\theta} \). Thus the variety \( Y_{y, y', z} \) is isomorphic to the variety \( \{ (\tilde{u}, \tilde{u}') \in U_P^{\theta} \times w U_P^{\theta} | \tilde{u} = \tilde{u}' \} \). Hence \( \dim Y_{y, y', z} = \dim(U_P \cap w U_P)^{\theta} \). (2.2.4) holds.

Since the fibres \( Y_{y, y', z} \) have constant dimension, (2.2.3) can be rewritten as follows:

\[(2.2.5) \quad \dim D \leq 2 \nu_H - 2 \bar{\nu} + \bar{\gamma} + r - \dim \Omega_\omega + \Delta_P - \dim(U_P \cap w U_P)^{\theta},\]

where

\[
D = \{(y, y', z) \in (w L \cap U_P) \times (L \cap w U_P) \times (L \cap w L) \mid z y' \theta(z y)^{-1} \in (Z_L^0)^{\theta} G_L, \hat{w}^{-1} z y \theta(z y)^{-1} \hat{w} \in (Z_L^0)^{\theta} G_L, \}/E^\theta;\]

and the action of \( E^\theta \) is defined similarly.

We now compute the difference of \( \dim(U_P \cap w U_P)^{\theta} \) and \( \dim(U_P \cap w U_P)^{\theta} \). According to the decomposition in (2.1.4), \( U_P \cap w U_P \) is the direct product of those subgroups corresponding to \( G_0 \) or \( G_1 \times G_i \) \((1 \leq i \leq t)\) in (2.1.4). For \( G_1 \times G_i \), its
\(\theta\)-fixed part and \(\ell\theta\)-fixed part are isomorphic. Hence we have only to consider the part \(G_0 \cap P\). We assume that \(G_0 \cap L^\theta\) is expressed as in the right hand side of (2.1.1). Put \(n = [N/2], n_0 = [N_0/2]\). The Weyl subgroup \(W_0\) of \(W_H\) corresponding to \(G_0\) is isomorphic to \(S_n \times (\mathbb{Z}/2\mathbb{Z})^{n'}\), where \(n' = n\) (resp \(n' = n - 1\)) if \(N\) is odd (resp. even), which we identify with a subgroup of signed permutations of \(n\) letters \(\{1, \ldots, n\}\). For \(w \in W_H\), let \(w_0\) be the element corresponding to \(w_0\) under the decomposition in (2.1.4). Put

\[
(2.2.6) \quad b_w = \sharp\{i \mid 1 \leq i \leq n - n_0, 1 \leq w_0^{-1}(i) \leq n - n_0\}.
\]

Note that the action of \(\theta\) on \(U \cap wU(P)\) can be described from the formula in (1.5.2), and its group version in 1.6, by using (1.10.1) and (1.10.2). In particular, we have

\[
(2.2.7) \quad \dim(U \cap wU(P))^\theta = \dim(U \cap wU(P))^\theta - b_w.
\]

Since \(\bar{w} \cap wP\) is a parabolic subgroup of \(L\) with a Levi decomposition \(L \cap wP = (L \cap wL)(L \cap wU(P))\), and similarly for \(wL \cap P\), \(wL \cap P = (wL \cap L)(wL \cap U(P))\), we have

\[
\dim(L \cap wL) + \dim(L \cap wU(P)) + \dim(wL \cap U(P)) = \dim L.
\]

Thus

\[
\dim(P \cap wP)^\theta = 2\nu - \dim(T) + \dim(U \cap wU(P))\theta.
\]

It follows, by (2.2.7), that

\[
(2.2.8) \quad \dim(U \cap wU(P))\theta = 2\nu - 2\nu - \dim(O_\omega + b_w).
\]

Hence (2.2.5) is equivalent to the form

\[
(2.2.9) \quad \dim D \leq \bar{c} + r + \Delta_L - b_w.
\]

Thus in order to prove (2.2.1), we have only to show (2.2.9), where the strict inequality holds if \(O_\omega\) is bad.

A similar discussion shows that, in order to prove (2.2.2), we have only to show

\[
(2.2.10) \quad \dim D' \leq \bar{c} + \Delta_L - b_w,
\]

where

\[
D' = \{(y,y',z) \in (wL \cap U(P)) \times (L \cap wU(P)) \times (L \cap wL) \mid zy'\theta(zy')^{-1} \in \mathcal{O}_L, w^{-1}zy\theta(zy)^{-1}w \in \mathcal{O}_L\}/E^\theta.
\]

We consider (2.2.10). Since \(wP\) and \(L\) contain a common maximal torus \(T\), \(Q = L \cap wP\) is a \(\theta\)-stable parabolic subgroup of \(L\) with Levi decomposition \(Q = MU_Q\), where \(M = L \cap wL\) and \(U_Q = L \cap wU(P)\). Hence by replacing \(G, P, L\) by \(L, Q, M\), one can define a map \(\eta_Q : Q^\theta \to M^\theta\) as in the case of \(\eta_P\). Similarly, for a parabolic subgroup \(Q' = wL \cap P = MU_{Q'}\) of \(wL\) with \(U_{Q'} = wL \cap U(P)\), the map \(\eta_{Q'} : Q'^\theta \to M^\theta\)
can be defined. Here \( \eta_Q(zy\theta(zy)^{-1}) = z\theta(z)^{-1} \) for \( z \in M, y' \in U_Q \), and similarly, \( \eta_Q(zy\theta(zy)^{-1}) = z\theta(z) \). We note that (2.2.11) There exist finitely many \((M^\theta)^0\)-orbits \( \hat{\mathcal{O}}_1, \ldots, \hat{\mathcal{O}}_m \) in \( M^\theta \) such that \( z\theta(z)^{-1} \) is contained in \( \bigcup_i \hat{\mathcal{O}}_i \) if \( (y, y', z) \in D' \).

In fact, by definition of \( D' \), \( z\theta(z)^{-1} \) is contained in \( \mathcal{O}_l U_Q \cap M^\theta \), hence its semisimple part \( z_1 \) is contained in \( \mathcal{O}_s \), where \( \mathcal{O}_s \) is a single \( L_H \)-orbit in \( L^\theta \) obtained from a semisimple element, say \( s \in M^\theta \). But \( \mathcal{O}_s \cap M^\theta \) splits into finitely many \((M^\theta)^0\)-orbits. Here note that \( Z_M(s) \) is \( \theta \)-stable, and \( Z_M(s)^{\theta} \) has only finitely many unipotent \( Z_M^0(s)^{\theta} \)-orbits ([R, Proposition 7.4]). (2.2.11) follows from this.

Let \( \pi_3 : D' \to M^\theta \) be the map defined by \( (y, y', z) \mapsto \pi = z\theta(z)^{-1} \). By (2.2.11), \( \pi \in \bigcup_i \hat{\mathcal{O}}_i \), and for each \( \pi \in \hat{\mathcal{O}}_i \), \( \pi_3^{-1}(\pi) \) is isomorphic to the product of the varieties as in (i), namely, \( (\pi_{\mathcal{L}} \cap \eta_{Q}^{-1}(\pi)) \times (\pi_{\mathcal{M}} \cap \eta_{Q}^{-1}(\pi)) \). Hence by induction hypothesis, we have

\[
\dim \pi_3^{-1}(\pi) \leq \frac{1}{2}(\pi - \dim \hat{\mathcal{O}}_i) + \frac{1}{2}(\pi - \dim \hat{\mathcal{O}}_i) + \Delta_Q.
\]

(Note that \( \Delta_Q = \Delta_{Q_1} \)). It follows that \( \dim \pi_3^{-1}(\mathcal{O}_i) \leq \pi + \Delta_Q \). Since this is true for any \( i \), we have \( \dim D' \leq \pi + \Delta_Q \). In order to show (2.2.10), it is enough to see that (2.2.12)

\[
\Delta_Q \leq \Delta_P - b_w.
\]

We can express \( \Delta_Q \) as

\[
\Delta_Q = \#\{i \mid n - n_0 + 1 \leq i \leq n, 1 \leq w_0^{-1}(i) \leq n - n_0\}
\]

By comparing this with (2.2.6), we obtain (2.2.12). Thus (2.2.10) is proved.

Next consider (2.2.9). A similar argument as in the proof of (2.2.10) shows that there exist finitely many orbits \( \hat{\mathcal{O}}_1', \ldots, \hat{\mathcal{O}}_m', \hat{\mathcal{O}}_1'', \ldots, \hat{\mathcal{O}}_m'' \) in \((M^\theta)^0\) such that \( z\theta(z)^{-1} \) is contained in the intersection of \((Z_L^0)^{\theta}(\hat{\mathcal{O}}_1' \cup \cdots \cup \hat{\mathcal{O}}_m')\) and \((Z_m^0)^{\theta}(\hat{\mathcal{O}}_1'' \cup \cdots \cup \hat{\mathcal{O}}_m'')\). Since \( Z_L^0 \) and \( Z_m^0 \) are contained in the center of \( M \), \( z\theta(z)^{-1} \) is contained in \((Z_L^0 \cap Z_m^0)^{\theta}(\hat{\mathcal{O}}_1' \cup \cdots \cup \hat{\mathcal{O}}_m')\) for some \((M^\theta)^0\)-orbits \( \hat{\mathcal{O}}_1', \ldots, \hat{\mathcal{O}}_m' \) in \( M^\theta \). Thus as in the proof of (2.2.10), we have \( \dim D' \leq \pi + \dim(Z_L^0 \cap Z_m^0)^{\theta} + \Delta_Q \). Here \( \dim(Z_L^0 \cap Z_m^0)^{\theta} \leq \pi \), and the strict inequality holds if \( \mathcal{O}_w \) is bad. (2.2.9) follows from this by using (2.2.12). Hence we have proved (iii) and (iv), assuming the induction hypothesis.

Next we show (ii). Put \( Z^\prime_{\theta} = \{ (x, gP_H, g'P_H) \in Z' \mid x \in \theta \} \). If \( Z^\prime_{\theta} \) is empty, then the variety in (ii) is also empty, and the inequality holds. So we assume that \( Z^\prime_{\theta} \) is non-empty. From (iv), we have \( \dim Z^\prime_{\theta} \leq d' + \Delta_P \). Consider the projection \( p : Z^\prime_{\theta} \to \theta \) to the first factor. Then each fibre is isomorphic to the product of two copies of the variety in (ii). Thus we have

\[
\dim\{ gP_H \in H/P_H \mid g^{-1}xg \in \eta_{P}^{-1}(\theta_L) \} = (\dim Z^\prime_{\theta} - \dim \theta)/2
\]

\[
\leq (d' + \Delta_P - c)/2
\]

\[
= \nu_{H} - \nu + (\nu - c + \Delta_P)/2.
\]

Hence (ii) holds.
Finally we show (i). Consider the variety $R = \{(x, gP_H) \in \mathcal{O} \times H/P_H \mid x \in \eta^{-1}_P(\mathcal{O}_L)\}$. By projecting to the first factor, and by using (ii), we see that $\dim R \leq \nu_H - \varpi + (c + \overline{c})/2 + \Delta_p/2$. If we project to the second factor, each fibre is isomorphic to the variety $\mathcal{O} \cap \eta^{-1}_P(\mathcal{O}_L)$. Hence
\[
\dim(\mathcal{O} \cap \eta^{-1}_P(\mathcal{O}_L)) \leq \nu_H - \varpi + (c + \overline{c})/2 + \Delta_p/2 - \dim H/P_H = (c + \overline{c})/2 + \Delta_p/2.
\]

Now we consider the map $\mathcal{O} \cap \eta^{-1}_P(\mathcal{O}_L) \to \mathcal{O}_L$ by $x \mapsto \eta_p(x)$. Then each fibre is isomorphic to the variety considered in (i). Hence the dimension of this variety is $\leq (c + \overline{c})/2 + \Delta_p/2 - \varpi = (c - \overline{c})/2 + \Delta_p/2$. This proves (i). The proposition is proved. \qed

2.3. We keep the setting in 2.1. Let $G_{\text{uni}}^\theta$ be the set of unipotent elements in $G$, and put $G_{\text{uni}}^{\theta} = G^{\theta} \cap G_{\text{uni}}$. By [R, proposition 7.4], $G_{\text{uni}}^{\theta}$ has finitely many $H$-orbits. We define $L_{\text{uni}}^\theta$ for $L$ similarly to $G$. Let $\mathcal{O} \subset G_{\text{uni}}^{\theta}$; $\mathcal{O}_L \subset L_{\text{uni}}^\theta$. Take $u \in \mathcal{O}$, $v \in \mathcal{O}_L$, and fix them. We define varieties
\[
\begin{align*}
Y_{u,v} &= \{gZ^0_{L_H}(v)U^\theta_P \mid g \in H, g^{-1}ug \in \eta^{-1}_P(v)\}, \\
\tilde{Y}_{u,v} &= \{g \in H \mid g^{-1}ug \in \eta^{-1}_P(v)\}.
\end{align*}
\]

Note that $\eta^{-1}_P(v) = (vU_P)^\theta$, hence $Y_{u,v}$ is well-defined. $Z_H(u) \times Z_{L_H}(v)U^\theta_P$ acts on $\tilde{Y}_{u,v}$ by $(x, y) : g \mapsto xgy^{-1}$ $(x \in Z_H(u), y \in Z_{L_H}(v)U^\theta_P, g \in \tilde{Y}_{u,v})$. Let $\phi : \tilde{Y}_{u,v} \to Y_{u,v} \cong \tilde{Y}_{u,v}/Z^0_{L_H}(v)U^\theta_P$, $\varphi : \tilde{Y}_{u,v} \to Z^0_H(u)/\tilde{Y}_{u,v}$ be the quotient maps. We define $\xi : \tilde{\mathcal{O}} = H/Z^0_H(u) \to H/Z_H(u) \simeq \mathcal{O}$ by $gZ^0_H(u) \mapsto gug^{-1}$. Then $\xi$ is a finite Galois covering with Galois group $A_H(u) = Z_H(u)/Z^0_H(u)$. We have the following commutative diagram.

\[
\begin{array}{ccc}
Y_{u,v} & \xleftarrow{\phi} & \tilde{Y}_{u,v} \\
\downarrow{\tau} & & \downarrow{\varphi} \\
\mathcal{O} \cap \eta^{-1}_P(v) & \xleftarrow{\xi^{-1}} & \xi^{-1}(\mathcal{O} \cap \eta^{-1}_P(v)),
\end{array}
\]

where $\tau : g \mapsto g^{-1}ug$, $\varphi : Z^0_H(u)g \mapsto g^{-1}Z^0_H(u)$. Note that $\varphi$ gives an isomorphism $Z^0_H(u)/\tilde{Y}_{u,v} \cong \xi^{-1}(\mathcal{O} \cap \eta^{-1}_P(v))$. Put
\[
\begin{align*}
\delta &= (\dim \mathcal{O} - \dim \mathcal{O}_L)/2 + \Delta_p/2, \\
s &= (\dim Z_H(u) - \dim Z_{L_H}(v))/2 + \Delta_p/2.
\end{align*}
\]

By (2.3.1), we have $\dim(\mathcal{O} \cap \eta^{-1}_P(v)) = \dim \xi^{-1}(\mathcal{O} \cap \eta^{-1}_P(v)) = \dim Z^0_H(u)/\tilde{Y}_{u,v}$. Hence if we put $d_{u,v} = \dim(\mathcal{O} \cap \eta^{-1}_P(v))$, we have
\[
\dim \tilde{Y}_{u,v} = d_{u,v} + \dim Z_H(u),
\]
\[ \dim Y_{u,v} = d_{u,v} + \dim Z_H(u) - \dim Z_{L_H}(v) - \dim U_P^\theta. \]

On the other hand, by Proposition 2.2 (i), we have \( d_{u,v} \leq \delta \). It follows that \( \dim Y_{u,v} \leq s \), and that the equality holds if and only if \( d_{u,v} = \delta \). Let \( I(Y_{u,v}) \) be the set of irreducible components of \( Y_{u,v} \) of dimension \( s \). Similarly, let \( I(\tilde{Y}_{u,v}) \) (resp. \( I_{u,v} \)) be the set of irreducible components of \( \tilde{Y}_{u,v} \) (resp. \( \xi^{-1}(\mathcal{O} \cap \eta_P^{-1}(v)) \)) of dimension \( \delta + \dim Z_H(u) \) (resp. dimension \( \delta \)). By (2.3.1), we have a natural bijection \( I(Y_{u,v}) \cong I(\tilde{Y}_{u,v}) \cong I_{u,v} \).

Put \( A_H(u) = Z_H(u)/Z_H^0(u) \). Since \( Z_H(u) \) acts on \( Y_{u,v} \) from the left, \( A_H(u) \) acts on \( I(Y_{u,v}) \) as permutations of irreducible components. Similarly, \( A_H(u) \) acts on \( I(\tilde{Y}_{u,v}) \), \( I_{u,v} \), and the above bijection turns out to be \( A_H(u) \)-equivariant.

**Definition 2.4.** Let \( \mathcal{O} \) be an \( H \)-orbit in \( G_{\text{uni}}^{\theta} \), and take \( u \in \mathcal{O} \). \( \tau \in A_H(u)^\wedge \) is said to be cuspidal if \( \tau \) does not appear in the permutation representation of \( A_H(u) \) on \( I(Y_{u,v}) \) for any \( \theta \)-stable Levi subgroup \( L \) of any \( \theta \)-stable proper parabolic subgroup \( P \) of \( G \), and for any \( v \in L_{\text{uni}}^{\theta} \). An \( H \)-equivariant simple local system \( \mathcal{E} \) on \( \mathcal{O} \) corresponding to \( \tau \in A_H(u)^\wedge \) is said to be cuspidal if \( \tau \) is cuspidal.

It follows from the definition that if \( \tau \) is cuspidal, then its dual representation \( \tau^* \) is also cuspidal.

**Lemma 2.5.** The local system \( \mathcal{E} \) on \( \mathcal{O} \) is cuspidal if and only if for any proper \( \theta \)-stable parabolic subgroup \( P \) of \( G \), and for any \( v \in L_{\text{uni}}^{\theta} \), we have
\[ H_c^{2\delta}(\mathcal{O} \cap \eta_P^{-1}(v),\mathcal{E}) = 0. \]

**Proof.** Let \( \xi : \tilde{\mathcal{O}} \to \mathcal{O} \) be the finite Galois covering with Galois group \( A_H(u) \). Then \( \xi_*\mathcal{Q}_\ell \) is a semisimple local system on \( \mathcal{O} \) equipped with \( A_H(u) \)-action, and we have \( \mathcal{E} = \text{Hom}_{A_H(u)}(\tau,\xi_*\mathcal{Q}_\ell) \). Hence
\[
H_c^i(\mathcal{O} \cap \eta_P^{-1}(v),\mathcal{E}) \simeq (H_c^i(\mathcal{O} \cap \eta_P^{-1}(v),\xi_*\mathcal{Q}_\ell) \otimes \tau^*)^{A_H(u)} \\
\simeq (H_c^i(\xi^{-1}(\mathcal{O} \cap \eta_P^{-1}(v)),\mathcal{Q}_\ell) \otimes \tau^*)^{A_H(u)},
\]
where \( \tau^* \) is the dual representation of \( \tau \). This implies that the condition \( \tau \) does not appear in the permutation representation of \( I_{u,v} \) is equivalent to the condition \( H_c^{2\delta}(\mathcal{O} \cap \eta_P^{-1}(v),\mathcal{E}) = 0 \). Since \( I_{u,v} \simeq I(Y_{u,v}) \) with \( A_H(u) \)-action, the lemma follows.

**2.6.** More generally, we consider \( G \) and an involution \( \theta : G \to G \) as in (2.1.4). The definition of cuspidal local system \( \mathcal{E} \) on \( \mathcal{O} \) can be generalized to this case, and Lemma 2.5 holds. Let \( \mathcal{E} \) be a local system on \( \mathcal{O} \). Then, under the isomorphism in (2.1.5), \( \mathcal{O} \simeq \mathcal{O}_0 \times \mathcal{O}_1 \times \cdots \times \mathcal{O}_t \), where \( \mathcal{O}_0 \subset G_0^{\theta} \) and \( \mathcal{O}_i \subset (G_i \times G_i)^{\theta} \simeq G_i \) for \( i = 1, \ldots, t \). Thus \( \mathcal{E} \) can be written as \( \mathcal{E} = \mathcal{E}_0 \boxtimes \mathcal{E}_1 \boxtimes \cdots \boxtimes \mathcal{E}_t \), where \( \mathcal{E}_i \) is a local system on \( \mathcal{O}_i \) for each \( i \). By Lemma 2.5, it is easy to see that \( \mathcal{E} \) is cuspidal on \( \mathcal{O} \) if and only if \( \mathcal{E}_i \) is cuspidal on \( \mathcal{O}_i \) for each \( i \). But note that the definition of cuspidality for \( \mathcal{E}_i \) \((i \geq 1)\) is exactly the same as the definition of cuspidality in the
case of reductive groups in [L1, 2.4]. It is well-known that in the case of $GL_n$, there does not exist a cuspidal local system unless $n = 1$. Hence we have the following.

**Lemma 2.7.** Assume that $H = SO_N$. Let $L$ be a $\theta$-stable Levi subgroup of a $\theta$-stable parabolic subgroup of $G$. Let $\mathcal{O}_L$ be an $L_H$-orbit in $L^{\theta}_{uni}$. If there exists a cuspidal local system on $\mathcal{O}_L$, then $L_H \simeq (GL_1)^a \times SO_{N_0}$, where $a = (N - N_0)/2$.

**Remark 2.8.** In the case where $H = Sp_N$, Proposition 2.2 still holds by putting $\Delta_p = 0$.

3. Admissible complexes

3.1. We follow the setting in 1.7, and consider $H = (G^\theta)^0 \simeq SO_N$. Let us fix a $\theta$-stable Borel subgroup $B$ and a $\theta$-stable maximal torus $T$ contained in $B$. By Lemma 1.2, the pair $(B, T)$ is $G^\theta$-conjugate to the specific choice of the Borel subgroup and the maximal torus given in 1.8. Let $U$ be the unipotent radical of $B$. We define a subgroup $\mathcal{D}$ of $U$ by $\mathcal{D} = U(\Delta_0)$. $\mathcal{D}$ is conjugate under $G^\theta$ to the corresponding group defined in 1.10. In particular, we have $\mathcal{D} = U(\Delta_0)^{\theta}$.

Let $P = LU_P$ be a $\theta$-stable parabolic subgroup of $G$ containing $B$ such that $T \subset L$. Here we assume that $L^\theta \simeq (GL_1)^a \times GL_{N_0}$ with $N_0 = N - 2a$ for $0 \leq a \leq n$. Let $\mathcal{O}_L$ be an $L_H$-orbit in $L^{\theta}_{uni}$, and consider $\Sigma = (Z_L^0)^{\theta} \times \mathcal{O}_L \subset L^\theta$. Note that $(Z_L^0)^{\theta} \simeq (GL_1)^a$. Let $(Z_L^0)^{\theta}_{reg}$ be the set of $(t_1, \ldots, t_a) \in (GL_1)^a$ such that $t_i$ are all distinct, under the above isomorphism. Put $\Sigma_{reg} = (Z_L^0)^{\theta}_{reg} \times \mathcal{O}_L$. Then $\Sigma_{reg}$ is open dense in $\Sigma$. Put $\mathcal{D}_P = \mathcal{D} \cap U_P$. Then $\mathcal{D}_P$ is a closed subgroup of $U_P$ contained in $U_P^\theta$ such that $\mathcal{D}_P \simeq k^a$. Moreover by 1.11, $\mathcal{D}_P$ is stable under the conjugation action of $L_H$ on $U_P^\theta$. By (1.11.1), any element in $\Sigma$ commutes with any element in $\mathcal{D}_P$. Hence

\[(3.1.1) \quad \Sigma \mathcal{D}_P \subset P^{\theta}.
\]

Note that $\dim \mathcal{D}_P = a = \Delta_P$ (see (2.1.6)).

Let $\mathcal{D}_P^0$ be the subset of $\mathcal{D}_P \simeq k^a$ consisting of $\xi = (\xi_1, \ldots, \xi_a) \in k^a$ such that $\xi_i \neq 0$ for any $i$. We define varieties

\[\tilde{Y}_{(L, \Sigma)} = \{(x, gL_H) \in G^\theta \times H/L_H \mid g^{-1}xg \in \Sigma_{reg} \mathcal{D}_P^0\} \]

\[Y_{(L, \Sigma)} = \bigcup_{g \in H} g(\Sigma_{reg} \mathcal{D}_P^0)g^{-1}.
\]

We show a lemma.

**Lemma 3.2.** (i) $Y_{(L, \Sigma)}$ is a smooth, irreducible variety with

\[(3.2.1) \quad \dim Y_{(L, \Sigma)} = 2\nu_H - 2\nu_{L_H} + \dim \Sigma + \Delta_P.
\]

(ii) Let $P' = L'U_{P'}$ be another $\theta$-stable parabolic subgroup with $\theta$-stable Levi subgroup $L'$, $(L', \Sigma')$ a pair defined for $L'$, similarly to $(L, \Sigma)$. Assume that $(L', \Sigma')$ is not $H$-conjugate to $(L, \Sigma)$. Then we have $Y_{(L, \Sigma)} \cap Y_{(L', \Sigma')} = \emptyset.$
Proof. For each \( g \in G^0 \), let \( g_s \) be the semisimple part of \( g \) in \( G^0 \), and \( Z_H(g_s) \) be the stabilizer of \( g_s \) in \( H \). Put \( Z = Z^0_{Z_H(g_s)} \), the connected center of \( Z_H(g_s) \). Then \( H(g) = Z_H(Z_g) \) is a Levi subgroup of some parabolic subgroup of \( H \). If \( g \in Y_{(L, \Sigma)} \), \( H(g) \) is a Levi subgroup conjugate to \( L \) under \( H \). By identifying \( H/N_H(LH) \) with the set of Levi subgroups of \( H \) conjugate to \( L \), we define a map \( \zeta : Y_{(L, \Sigma)} \to H/N_H(LH) \) by \( g \mapsto H(g) \). Then one can show that

\[
(3.2.2) \quad \zeta : Y_{(L, \Sigma)} \simeq H \times^{N_H(LH)} \zeta^{-1}(L_H) \to H/N_H(LH)
\]

is a locally trivial fibration with fibre isomorphic to \( \zeta^{-1}(L_H) \). We have

\[
\zeta^{-1}(L_H) = \bigcup_{w \in N_H(LH)/LH} \sigma \left( \mathcal{D}_P^0 \right) w^{-1}.
\]

Here \( N_H(LH)/LH \simeq S_a \ltimes (\mathbb{Z}/2\mathbb{Z})^{a'} \), where \( S_a \) is the symmetric group of degree \( a \), and \( a' = a \) unless \( a = n \) and \( N \) is even, in which case \( a' = n - 1 \). \( w \in N_H(LH)/LH \) leaves \( \Sigma_{\text{reg}} \) stable, and \( \sigma \in S_a \) leaves \( \mathcal{D}_P^0 \) stable. Hence

\[
(3.2.3) \quad \zeta^{-1}(L_H) \simeq \Sigma_{\text{reg}} \times \bigcup_{w \in (\mathbb{Z}/2\mathbb{Z})^{a'}} w \mathcal{D}_P^0 w^{-1}.
\]

Since \( w \mathcal{D}_P^0 w^{-1} \) are mutually disjoint for \( w \in (\mathbb{Z}/2\mathbb{Z})^{a'} \), we see that \( \zeta^{-1}(L_H) \) is a disjoint union of smooth pieces \( \Sigma_{\text{reg}} \times w \mathcal{D}_P^0 w^{-1} \). Hence \( \zeta^{-1}(L_H) \) is smooth, and so \( Y_{(L, \Sigma)} \) is smooth by (3.2.2). Since \( \Sigma_{\text{reg}} \mathcal{D}_P^0 \) is irreducible, \( Y_{(L, \Sigma)} \) is also irreducible. The dimension formula (3.2.1) follows from (3.2.2) and (3.2.3). Thus (i) holds. (ii) is immediate since for \( g \in Y_{(L, \Sigma)} \), \( H(g) \) determines a unique Levi subgroup in \( H \), and once \( L_H = H(g) \) is given, \( g \) determines a unique \( L_H \)-orbit \( \mathcal{O}_L \), hence determines \( \Sigma \) uniquely. The lemma is proved. \( \square \)

Remark 3.3. The definition of \( Y_{(L, \Sigma)} \) given here depends on the special choice of \( P \) as in 3.1. The discussion in 3.1 can not be applied for arbitrary \( L \). So, we cannot discuss the partition of \( G^0 \) in terms of various \( Y_{(L, \Sigma)} \) as given in [L1, 3.1].

3.4. By fixing \( (L, \Sigma) \), put \( Y = Y_{(L, \Sigma)}, \widetilde{Y} = \tilde{Y}_{(L, \Sigma)} \). Recall the map \( \eta_P : P^0 \to L^0 \). Let \( \Sigma \) be the closure of \( \Sigma \) in \( L^0 \). We define varieties \( X, \widetilde{X} \) by

\[
\widetilde{X} = \{(x, gP_H) \in G^0 \times H/P_H \mid g^{-1}xg \in \eta_P^{-1}(\Sigma)\},
\]

\[
X = \bigcup_{g \in H} g(\eta_P^{-1}(\Sigma))g^{-1}
\]

and a map \( \pi : \widetilde{X} \to X \) by \( (x, gP_H) \mapsto x \). Then \( \pi \) is proper and surjective. We have

\[
(3.4.1) \quad \widetilde{X} \simeq H \times^{P_H} \eta_P^{-1}(\Sigma).
\]

We have a lemma.
Lemma 3.5.  
(i) $X$ is a closed irreducible subvariety of $G^{\theta}$ such that the closure $\overline{Y}$ of $Y$ coincides with $X$. We have $\dim \overline{X} = \dim X$.

(ii) The map $(x,gL_H) \mapsto (x,gP_H)$ gives an isomorphism $\gamma : \overline{Y} \cong \pi^{-1}(Y)$.

(iii) $Y$ is an open dense subset of $X$. Hence $Y$ is an irreducible, locally closed smooth subvariety of $G^{\theta}$.

Proof. (i) Since $\eta_P$ is an affine space bundle with fibre isomorphic to $U^\theta_P$ by (2.1.2), we have $\dim \eta_P^{-1}(\Sigma) = \dim \Sigma + \dim U^\theta_P$. Thus (3.4.1) shows that $\overline{X}$ is irreducible with

$$\dim \overline{X} = \dim H - \dim P_H + \pi_P^{-1}(\Sigma) = \dim H - \dim P_H + \dim \Sigma + \dim U^\theta_P = 2\nu_H - 2\nu_{LH} + \dim \Sigma - \dim U^\theta_P + \dim U^\theta_P.$$

By (1.10.1) and (1.10.2), we have $\dim U^\theta_P = \dim U^\theta_P + \Delta_P$. Hence by comparing the last equality with (3.2.1), we have

$$\dim \overline{X} = \dim Y.$$

Since $\overline{X}$ is irreducible and $\pi$ is proper, $X$ is an irreducible closed subset of $G^{\theta}$. Since $Y \subset X$, we have $\overline{Y} \subset X$. We have

$$\dim Y \leq \dim X \leq \dim \overline{X} = \dim Y$$

by (3.5.1). Hence $\dim Y = \dim X = \dim \overline{X}$. Since $X$ and $\overline{Y}$ are irreducible, closed with same dimension, we have $X = \overline{Y}$. This proves (i).

(ii) The map $(x,gL_H) \mapsto (x,gP_H)$ gives a well-defined map $\gamma : \overline{Y} \to \pi^{-1}(Y)$. For $u \in U_P$, $t \in (Z^0_L)^{\theta}_{\text{reg}}$, if $tu = ut$, then $u \in \mathcal{G}_P$. Thus for $1 \neq u \in U^\theta_P$, $t \in (Z^0_L)^{\theta}_{\text{reg}}$, $t^{-1}u^{-1}tu$ produces an element in $U_P - \mathcal{G}_P$. The injectivity of $\gamma$ follows from this. We will show that $\gamma(\overline{Y}) = \pi^{-1}(Y)$. Take an element in $\pi^{-1}(x)$ for $x \in \Sigma_{\text{reg}}^{\theta}_P$, which is of the form $(x,gP_H)$ with $g^{-1}xg \in \eta_P^{-1}(\Sigma)$. Let $x_s$ be the semisimple part of $x$. Then $x_s \in (Z^0_L)^{\theta}_{\text{reg}}$, and $Z^0_H(x_s) = L_H$. On the other hand, since $\Sigma = (Z^0_L)^{\theta}_L$, $g^{-1}xsg \in (Z^0_L)^{\theta}$. In particular, $Z_H(g^{-1}xsg) \supset L_H$. But since $Z_H(g^{-1}xsg) = g^{-1}L_Hg$, we have $g^{-1}L_Hg = L_H$, namely, $g \in N_H(L_H)$. Here note that $N_H(L_H)/L_H \cong S_a \ltimes (\mathbb{Z}/2\mathbb{Z})^{\theta}$. Thus $g \in N_H(L_H)$ leaves $\Sigma_{\text{reg}}$ invariant. By our assumption $g^{-1}xg \in \Sigma U_P$, $g \in N_H(L_H)$ should be contained in the inverse image of $S_a$ under the map $N_H(L_H) \to N_H(L_H)/L_H$. Such a $g$ leaves $\mathcal{G}_P$ invariant also. Hence $(x,gL_H) \in \overline{Y}$, and we have $(x,gP_H) = \gamma((x,gL_H))$. Since any element in $\pi^{-1}(Y)$ is $H$-conjugate to the element discussed above, we see that $\gamma$ is surjective. Hence $\gamma$ is a bijection. The inverse morphism $\gamma^{-1} : \pi^{-1}(Y) \to \overline{Y}$ is constructed from the above discussion. (ii) is proved.

(iii) Since $X$ is irreducible, it is enough to show that $Y$ is an open subset of $X$. Other properties are already shown in Lemma 3.2. Put $(Z^0_L)^{\theta}_1 = (Z^0_L)^{\theta} - (Z^0_L)^{\theta}_{\text{reg}}$ and $\Sigma_1 = (Z^0_L)^{\theta}_1 \overline{\mathcal{G}}_L$. Then $\Sigma_1$ is a closed subset of $\Sigma$, and $X_1 = \bigcup_{g \in H} g(\eta_P^{-1}(\Sigma_1))g^{-1}$ is a
closed subset of $X$. Thus $X - X_1$ is an open subset of $X$, and in fact, it coincides with the subset of $X$ consisting of $x$ such that the semisimple part of $x$ is $H$-conjugate to an element in $(Z_L^0)^\Theta_{reg}$. Let $\Sigma_2 = (Z_L^0)_{\mathcal{O}_L - \mathcal{O}_L}$. Then $\Sigma_2$ is a closed subset of $\Sigma$, and $X_2 = \bigcup_{y \in H} g(\eta_p^{-1}(\Sigma_2))g$ is a closed subset of $X$. Since $Y = X - (X_1 \cup X_2)$, $Y$ is open in $X$. (iii) is proved. \hfill $\square$

3.6. We consider the diagram

$$
\Sigma \xleftarrow{\alpha_0} \tilde{Y} \xrightarrow{\psi_0} \bar{Y} \xrightarrow{\pi_0} Y,
$$

where

$$
\tilde{Y} = \{(x, g) \in G H \mid g^{-1}xg \in \Sigma_{reg} \mathcal{D}_P^0 \},
$$

and maps are defined by $\pi_0(x, g \mathcal{L} H) = x$, $\psi_0(x, g) = (x, g \mathcal{L} H)$, $\alpha_0(x, g) = \eta_p(g^{-1}xg)$. Here $H$ acts on $\bar{Y}$ by $h : (x, g \mathcal{L} H) \mapsto (h x h^{-1}, h g \mathcal{L} H)$, and $\pi_0$ is $H$-equivariant with respect to the conjugation action of $H$ on $Y$. On the other hand, $H \times L_H$ acts on $\tilde{Y}$ by $(h, \ell) : (x, g) \mapsto (h x h^{-1}, h g \ell^{-1})$, and $\psi_0$ is $H$-equivariant. $H \times L_H$ also acts on $\Sigma$ by $(h, \ell) : y \mapsto \ell y \ell^{-1}$, and $\alpha_0$ is $H \times L_H$-equivariant.

Let $\mathcal{E}_1$ be an $L_H$-equivariant simple local system on $\Sigma$. Since $\Sigma = (Z^0_L)^\Theta \times \mathcal{O}_L$, $\mathcal{E}_1$ can be written as $\mathcal{E}_1 = \mathcal{S} \boxtimes \mathcal{E}_1^\dagger$, where $\mathcal{S}$ (resp. $\mathcal{E}_1^\dagger$) is a simple local system on $(Z_L^0)^\Theta$ (resp. $\mathcal{O}_L$). We say that $(\Sigma, \mathcal{E}_1)$ is a cuspidal pair if $\mathcal{E}_1^\dagger$ is a cuspidal local system on $\mathcal{O}_L$. Note that this definition is weaker than the definition of the cuspidal pair in [L1, 2.4]. By abuse of the notation, we also say that $(\mathcal{O}_L, \mathcal{E}_1^\dagger)$ is a cuspidal pair if $\mathcal{E}_1^\dagger$ is cuspidal on $\mathcal{O}_L$. Now consider an arbitrary $\mathcal{E}_1$. Then $\mathcal{E}_1$ is $(H \times L_H)$-equivariant with respect to the trivial action of $H$ on $\Sigma$, and $\alpha_0^* \mathcal{E}_1$ is an $(H \times L_H)$-equivariant local system on $\tilde{Y}$. Since $\psi_0$ is a principal bundle over $\tilde{Y}$ with respect to the free action of $L_H$, there exists a unique local system $\tilde{\mathcal{E}}_1$ on $\tilde{Y}$, up to isomorphism, such that

$$\alpha_0^* \mathcal{E}_1 \simeq \psi_0^* \tilde{\mathcal{E}}_1.$$

Since $\alpha_0$ is smooth with connected fibre $H \times \mathcal{D}_P$, $\tilde{\mathcal{E}}_1$ turns out to be an $H$-equivariant simple local system on $\tilde{Y}$.

Here $\mathcal{W} = N_H(L_H)/L_H \simeq S_a \times (\mathbb{Z}/2\mathbb{Z})^{a'}$ (see the proof of Lemma 3.2). Let $\mathcal{W}_1$ be the stabilizer of $\Sigma_{reg} \mathcal{D}_P^0$ in $\mathcal{W}$. Then $\mathcal{W}_1 \simeq S_a$. We note that $\pi_0 : \tilde{Y} \to Y$ is a finite Galois covering with group $\mathcal{W}_1$. In fact, we have an isomorphism $\tilde{Y} \simeq H \times \mathcal{D}_P \Sigma_{reg} \mathcal{D}_P^0$, and by (3.2.2), the map $\pi_0$ is given by the canonical map

$$\pi_0 : H \times \mathcal{D}_P \Sigma_{reg} \mathcal{D}_P^0 \to H \times N_{H(L_H)} \zeta^{-1}(L_H),$$

where $\zeta^{-1}(L_H) = \bigcup_{w \in (\mathbb{Z}/2\mathbb{Z})^{a'}} w(\Sigma_{reg} \mathcal{D}_P^0) w^{-1}$, see (3.2.3). Thus $\pi_0$ is a finite Galois covering with group $\mathcal{W}_1$.

We consider $(\pi_0)_* \tilde{\mathcal{E}}_1$. Since $\pi_0$ is a finite Galois covering, $(\pi_0)_* \tilde{\mathcal{E}}_1$ is a semisimple local system on $Y$. Put

$$\mathcal{W}_{\mathcal{E}_1} = \{n \in N_H(L_H) \mid n(\Sigma_{reg} \mathcal{D}_P^0)n^{-1} = \Sigma_{reg} \mathcal{D}_P^0, \text{ad}(n)^* \mathcal{E}_1 \simeq \mathcal{E}_1 \}/L_H,$$
where $\text{ad}(n) : \Sigma \to \Sigma, y \mapsto nyn^{-1}$. Then $\mathcal{W}_{\ell_1}$ is a subgroup of $\mathcal{W}_1$. In a similar way as in [L1, Proposition 3.5], one can show that the endomorphism algebra $\mathcal{A}_{\ell_1} = \text{End}((\pi_0)_* \tilde{\mathcal{E}}_1)$ of $(\pi_0)_* \tilde{\mathcal{E}}_1$ is isomorphic to the (twisted) group algebra $\mathbb{Q}_l[\mathcal{W}_{\ell_1}]$. Thus $(\pi_0)_* \tilde{\mathcal{E}}_1$ can be decomposed as

\[(3.6.1) \quad (\pi_0)_* \tilde{\mathcal{E}}_1 \simeq \bigoplus_{\rho \in \mathcal{A}_{\ell_1}} \rho \otimes ((\pi_0)_* \tilde{\mathcal{E}}_1)_\rho,
\]

where $((\pi_0)_* \tilde{\mathcal{E}}_1)_\rho = \text{Hom}_{\mathcal{A}_{\ell_1}}(\rho, (\pi_0)_* \tilde{\mathcal{E}}_1)$ is a simple local system on $Y$ corresponding to $\rho \in \mathcal{A}_{\ell_1}$.

3.7. We consider the following commutative diagram

$$
\begin{array}{ccc}
\Sigma & \xleftarrow{\alpha} & \hat{X} \\
\uparrow & & \uparrow \\
\Sigma & \xleftarrow{\alpha'} & \hat{X}_0
\end{array}
\quad \begin{array}{ccc}
\psi & \rightarrow & \tilde{X} \\
\pi & \rightarrow & X
\end{array}
$$

where

$$
\begin{align*}
\hat{X} &= \{(x, g) \in G^\theta \times H \mid g^{-1}xg \in \eta_F^{-1}(\Sigma)\}, \\
\hat{X}_0 &= \{(x, g) \in G^\theta \times H \mid g^{-1}xg \in \eta_F^{-1}(\Sigma)\}, \\
\tilde{X}_0 &= \{(x, gP_H) \in G^\theta \times H/P_H \mid g^{-1}xg \in \eta_P^{-1}(\Sigma)\}.
\end{align*}
$$

Here $\Sigma, \hat{X}_0, \tilde{X}_0$ are open dense smooth subsets of $\Sigma, \hat{X}, \tilde{X}$, and the horizontal maps are natural inclusions. $\psi$ is defined by $(x, g) \mapsto (x, gP_H)$, $\alpha$ is defined by $(x, g) \mapsto \eta_F(g^{-1}xg)$, and $\psi', \alpha'$ are their restrictions on $\hat{X}_0$. $H$ acts on $\tilde{X}$ by $h : (x, gP_H) \mapsto (hxh^{-1}, hgP_H)$, and $H \times P_H$ acts on $\tilde{X}$ by $(h, p) : (x, g) \mapsto (hxh^{-1}, hgp^{-1})$. With the trivial action of $U_{P_H}$, $P_H$ acts on $\Sigma$, and $\alpha$ is $H \times P_H$-equivariant, $\psi$ is $H$-equivariant. Similar properties hold for $\alpha', \psi'$ as the restriction of $\alpha, \psi$.

Let $\mathcal{E}_1$ be the local system on $\Sigma$ as before. Since $\psi'$ is a principal bundle over $\tilde{X}_0$ with respect to the free action of $P_H$, by a similar argument as in 3.6, there exists a unique local system (up to isomorphism) $\overline{\mathcal{E}}_1$ on $\tilde{X}_0$ such that

$$(\alpha')^* \mathcal{E}_1 \simeq (\psi')^* \overline{\mathcal{E}}_1.$$

Note that $\tilde{Y}$ is an open dense subset of $\tilde{X}_0$, and the restriction of $\overline{\mathcal{E}}_1$ on $\tilde{Y}$ coincides with $\tilde{\mathcal{E}}_1$ defined in 3.6. Put $K_{\overline{\mathcal{E}}_1} = \text{IC}(\tilde{X}, \overline{\mathcal{E}}_1)[\dim \tilde{X}]$. Then $K_{\overline{\mathcal{E}}_1}$ is an $H$-equivariant simple perverse sheaf on $\tilde{X}_0$. Since $\alpha$ is smooth with connected fibre $H \times U_P^\theta$, and $\psi$ is a principal bundle with respect to the free action of $P_H$, the discussion in 3.6 works in the level of perverse sheaves, and we have

$$
\alpha^* \text{IC}(\Sigma, \mathcal{E}_1)[\dim \Sigma] \simeq \psi^* \text{IC}(\tilde{X}, \overline{\mathcal{E}}_1)[\dim \tilde{X}].
$$
We consider the complex $\pi_*K_{\varpi_1}$ on $X = \overline{Y}$. The following result is an analogue of [L1, Proposition 4.5]. Note that here we don’t need to assume that $(\Sigma, E_1)$ is cuspidal.

**Proposition 3.8.** We have $\pi_*K_{\varpi_1} \simeq IC(X, (\pi_0)_*\tilde{E}_1)[\dim X]$.

**Proof.** By Lemma 3.5 and Lemma 3.2, we have $\dim X = 2\nu_H - 2\nu_{L_H} + \dim \Sigma + \Delta_P$. Then the proof of the proposition is done by an entirely similar way as the proof of Proposition 4.5 in [L1], by making use of Proposition 2.2. We omit the details. $\square$

**3.9.** By Proposition 3.8, $\pi_*K_{\varpi_1}$ is a semisimple perverse sheaf. By (3.6.1), it is decomposed as

$$\pi_*K_{\varpi_1} \simeq \bigoplus_{\rho \in \sigma_{E_1}} \rho \otimes IC(X, (\pi_0)_*(\tilde{E}_1)_{\rho})[\dim X].$$

A simple perverse sheaf isomorphic to a direct summand of $\pi_*K_{\varpi_1}$ obtained from the various cuspidal pair $(\Sigma, E_1)$ is called an admissible complex.

**Remark 3.10.** In the case where $P = B$ and $L = T$, we have $\Sigma = T^{\theta}$. Then

$$\tilde{X} = \{(x, gB_H) \in G'^\theta \times H/B_H \mid g^{-1}xg \in B^{t\theta}\},$$

$$X = \bigcup_{g \in H} gB^{t\theta}g^{-1},$$

and $\pi : \tilde{X} \to X$ is the first projection. A similar map can be defined even in the case where $H = Sp_N$. However in that case, we have $\dim \tilde{X} > \dim X$, and Lemma 3.5 does not hold. Nevertheless, in the case where $E_1$ is a constant sheaf $\overline{Q}_l$ on $T^{\theta}$, it was proved in [H] (see also [SS]) that $\pi_*\overline{Q}_l$ is a semisimple complex equipped with $S_m$-action, and some modified formula of (3.9.1) holds. If we consider the exotic symmetric space $G'^\theta \times V$ associated to the symplectic group $Sp(V)$, the map $\pi : \tilde{X} \to X$ is also defined. In that case, an exactly analogous formula of (3.9.1) holds ([SS], [K]).

4. Sheaves on the variety of semisimple orbits

**4.1.** We keep the setting in Section 3. In particular, consider $Y = Y_{(L, \sigma)}$ and $X = \overline{Y}$ with $\Sigma = (Z_L^0)^{t\theta}\sigma_L$. In this section, we assume that the local system $E_1$ on $\Sigma$ is of the form $E_1 = Q_l \boxtimes \sigma_1^{t\theta}$, where $Q_l$ is the constant sheaf on $(Z_L^0)^{t\theta}$ and $E_1^{t\theta}$ is an $L_H$-equivariant simple local system on $\sigma_L$.

Let $\sigma' : G \to T/S_N$ be the Steinberg map with respect to $G = GL_N$. We have a natural embedding $A = T^{t\theta}/S_n \hookrightarrow T/S_N$, where $A$ has a structure of an affine variety. Let $G_A^{t\theta}$ be the subset of $G^{t\theta}$ consisting of $g \in G^{t\theta}$ such that its semisimple part $g_s$ is $H$-conjugate to an element in $T^{t\theta}$. (Note that $T^{t\theta}$ is not a maximal $\theta$-anisotropic torus. See the example in 1.4, which corresponds to our setting with
Then the restriction of $\sigma'$ on $G^\theta_+$ induces a map $\sigma : G^\theta_+ \to A$. Since $\overline{Y} \subset G^\theta_+$, one can consider $\sigma(\overline{Y})$. We have

$$\sigma(\overline{Y}) = \sigma(\eta^{-1}_p(\Sigma)) = \sigma(\Sigma) = \sigma((Z_L^0)^{\theta}) \subset A,$$

which we denote by $A_Y$.

**4.2.** Let $(B', T')$ be another $\theta$-stable pair. By 1.9, the $H$-conjugate class of $(B', T')$ is described as follows. If $N$ is odd, we may choose $(B', T') = (B, T)$. If $N$ is even, then $(B', T') = (B, T)$ or $B' = t_n Bt_n^{-1}, T' = T$, where $t_n \in N_G(T)$ is given as in 1.9. We put $B_1 = t_n Bt_n^{-1}$.

Assume that $N$ is even, and let $P$ be a $\theta$-stable parabolic subgroup of $G$ containing $B$, and $L$ the Levi subgroup of $P$ containing $T$. Then $P' = t_n Pt_n^{-1}$ is the $\theta$-stable parabolic subgroup containing $B_1$, and $L' = t_nLt_n^{-1}$ is the Levi subgroup of $P'$ containing $T$. Note that $L' = L$ if $L \neq T$.

**4.3.** We now consider another $\theta$-stable parabolic subgroup $P'$ containing $B'$ and its $\theta$-stable Levi subgroup $L'$ containing $T$, where $B' = B$ or $B' = B_1$. We consider a set $\Sigma' \subset L'^{\theta}$ similarly to $\Sigma \subset L^{\theta}$. We shall denote by $Y', X'$, etc. various objects associated to $\Sigma', L', P'$ by attaching prime to corresponding objects $Y, X$, etc. associated to $\Sigma, L, P$. Put

$$Z = \{(x, gP_H), (x', g'P'_H)) \in \tilde{X} \times \tilde{X}' \mid x = x'\}$$

Thus $Z$ is isomorphic to the fibre product $\tilde{X} \times_{G^\theta} \tilde{X}'$ of $\tilde{X}$ and $\tilde{X}'$ over $G^\theta$. Here we assume that $X \cap X' \neq \emptyset$, otherwise $Z = \emptyset$. We define a map $\tilde{\sigma} : Z \to A_Y \cap A_Y'$ by the composite $\sigma \circ p_1$ of the first projection $p_1 : Z \to X \cap X'$ and $\sigma$. For $a \in A_Y$, put $Z^a = \tilde{\sigma}(a)^{-1} \subset Z$.

Since $\Sigma = (Z_L^0)^{\theta} \mathcal{O}_L$, $\Sigma$ has a stratification $\Sigma = \bigsqcup_{\beta} \Sigma_\beta$ with smooth strata $\Sigma_\beta = (Z_L^0)^{\theta} \mathcal{O}_\beta$, where we put $\mathcal{O}_L = \bigsqcup_{\beta} \mathcal{O}_\beta$ ($\mathcal{O}_\beta : L_H$-orbit in $L_{uni}^{\theta}$). We put $\Sigma_{\beta_0} = \Sigma = (Z_L^0)^{\theta} \mathcal{O}_L$: the open dense stratum. By defining $\tilde{X}_\beta$ in a similar way as $\tilde{X}_0$ in 3.7, we obtain a stratification $\tilde{X} = \bigsqcup_{\beta} \tilde{X}_\beta$, with $\tilde{X}_0 = \tilde{X}_{\beta_0}$, where $\tilde{X}_\beta$ are locally closed subvarieties of $\tilde{X}$. Given strata $\beta$ of $\Sigma$ and $\beta'$ of $\Sigma'$, we put $Z_{\beta, \beta'}^a = Z^a \cap (\tilde{X}_\beta \times G^\theta \tilde{X}'_{\beta'})$. Then the sets $Z_{\beta, \beta'}^a$ form a partition of $Z^a$ into locally closed pieces, where $Z_{\beta_0, \beta'_0}^a$ is open dense in $Z^a$. Put

$$(4.3.1) \quad d_0 = 2\nu_H - \nu_{L_H} - \nu_{L_H'} + (\dim \mathcal{O}_L + \dim \mathcal{O}_{L'})/2 + (\Delta_P + \Delta_{P'})/2.$$

Take local systems $\mathcal{E}_1$ on $\Sigma$ and $\mathcal{E}_1'$ on $\Sigma'$, and consider the external tensor product $K_{\mathcal{E}_1} \boxtimes K_{\mathcal{E}_1'}$ on $Z$. We define

$$(4.3.2) \quad \widetilde{\mathcal{F}} = \mathcal{H}^{-r'-r}(\mathcal{O}_1(K_{\mathcal{E}_1} \boxtimes K_{\mathcal{E}_1'})),$$

where we put $r = \dim(Z_L^0)^{\theta}, r' = \dim(Z_{L'}^0)^{\theta}$. $\widetilde{\mathcal{F}}$ is a constructible sheaf on $A_Y \cap A_Y'$.
Put $Z_0 = \tilde{X}_0 \times_{G^0} \tilde{X}'_0$. Then $Z_0$ is an open dense subset of $Z$. Let $\tilde{\sigma}_0 : Z_0 \to A_Y \cap A_{Y'}$ be the restriction of $\tilde{\sigma}$ on $Z_0$. Noticing that

$$K_{\sigma_1} \boxtimes K'_{\sigma_1} [-\dim X - \dim X']|_{Z_0} = \overline{\sigma}_1 \boxtimes \overline{\sigma}'_1,$$

we define

$$\mathcal{F} = \mathcal{H}^{-r-r'}((\tilde{\sigma}_0)! (K_{\sigma_1} \boxtimes K'_{\sigma_1})|_{Z_0}) = \mathcal{H}^{2d_0}((\tilde{\sigma}_0)! (\overline{\sigma}_1 \boxtimes \overline{\sigma}'_1)).$$

$\mathcal{F}$ is also a constructible sheaf on $A_Y \cap A_{Y'}$. Since $Z_0$ is open in $Z$, we have a natural map $\mathcal{F} \to \mathcal{F}$. In order to obtain a relationship between $\mathcal{F}$ and $\mathcal{F}$, we prepare a lemma, which is an analogue of [L1, Lemma 5.3]. The proof is done by a similar argument as in the proof of [loc. cit.], by using Proposition 2.2, and we omit the proof.

**Lemma 4.4.**

(i) For any $\beta, \beta'$, we have

$$\dim Z^a_{\beta, \beta'} \leq d_0 - (\dim \sigma_L + \dim \sigma_{L'})/2 + (\dim \sigma + \dim \sigma_{\beta'})/2.$$

In particular, we have $\dim Z^a \leq d_0$.

(ii) The natural map

$$H^k_{c} \oplus \dim X + \dim X' (Z^a_{\beta, \beta'}, \overline{\sigma}_1 \boxtimes \overline{\sigma}'_1) \simeq H^k_c (Z^a_{\beta_0, \beta_0}, K_{\sigma_1} \boxtimes K'_{\sigma_1}) \to H^k_c (Z^a, K_{\sigma_1} \boxtimes K'_{\sigma_1})$$

is an isomorphism for $k > -r - r'$, and is surjective if $k = -r - r'$. It is an isomorphism for $k = -r - r'$ (i.e., for $k + \dim X + \dim X' = 2d_0$) if $a$ is such that $Z^a_{\beta, \beta'}$ is empty whenever exactly one of $\beta, \beta'$ is equal to $\beta_0$ or $\beta'_0$.

**4.5.** Let $\mathcal{F}_a$ (resp. $\mathcal{F}_a$) be the stalk of $\mathcal{F}$ (resp. $\mathcal{F}$) for $a \in A_Y \cap A_{Y'}$. Then we have

$$\mathcal{F}_a \simeq H^c_{-r-r'} (Z^a, K_{\sigma_1} \boxtimes K'_{\sigma_1}),$$

$$\mathcal{F}_a \simeq H^c_{-r-r'} (Z^a_{\beta_0, \beta'_0}, K_{\sigma_1} \boxtimes K'_{\sigma_1}),$$

and the natural map $\mathcal{F}_a \to \mathcal{F}_a$ corresponds to the map in Lemma 4.4 (ii) for $k = -r - r'$. Thus, by Lemma 4.4 (ii), we see that

$$\mathcal{F}_a = \mathcal{H}^{2d_0} (\tilde{\sigma}_0)! (\overline{\sigma}_1 \boxtimes \overline{\sigma}'_1|_{Z_0}).$$

**4.5.1** The natural map of sheaves $\mathcal{F} \to \mathcal{F}$ is surjective.
\[ \mathcal{Z}_0 \] is a constructible sheaf on \( A_Y \cap A_{Y'} \). Recall that \( \mathcal{W} = N(L_H)/L_H \) and \( \mathcal{W}_1 \simeq S_q \) is the subgroup of \( \mathcal{W} \). Let \( N_H(L_H)_1 \) be the inverse image of \( \mathcal{W}_1 \) under the map \( N_H(L_H) \to \mathcal{W} \).

We show a lemma.

**Lemma 4.6.** Assume that \((\Sigma, \delta_1), (\Sigma', \delta'_1)\) are cuspidal pairs. Let \( \mathcal{O} = \mathcal{O}_{\omega} \) be an \( H \)-orbit corresponding to \( \omega \in W_{L_H} \setminus W_H/W_{L'_H} \), and \( w \in W_H \) be a representative of \( \omega \). Let \( \dot{w} \in N_H(T_H) \) be a representative of \( w \).

(i) If \((L, \Sigma)\) and \((L', \Sigma')\) are not conjugate under \( H \), then \( \mathcal{Z}_0 = 0 \).

(ii) Assume that \( P = P', L = L' \) and \( \Sigma = \Sigma' \). If \( \dot{w} \in N_H(T_H) \) is not contained in \( N_H(L_H)_1 \). Then \( \mathcal{Z}_0 = 0 \).

(iii) Assume that \( N \) is even, and \( P = B, P' = B_1, L = L' = T \). Then \( \mathcal{Z}_0 = 0 \).

**Proof.** In order to see \( \mathcal{Z}_0 = 0 \), we have only to show that, for any \( a \in A_Y \cap A_{Y'} \),

\[ H_c^{2d_0}(Z^a \cap Z_0, \overrightarrow{\delta}_1 \otimes \overrightarrow{\delta}_1) = 0. \]

Since \( \dim(Z^a \cap Z_0) \leq d_0 \) (Lemma 4.4 (ii)), by using the fibration \( Z^a \cap Z_0^0 \to \mathcal{O} \), it is enough to see, for any \( a \in A_Y \cap A_{Y'} \) and for any \( g \in H \), \( H_c^{2d_0-2d_0}(V^a, j^*(\overrightarrow{\delta}_1 \otimes \overrightarrow{\delta}_1)) = 0 \), where \( \mathcal{O} \) is the \( H \)-orbit containing \((P_H, \dot{w}P'_H)\) (here \( \dot{w} \in N_H(T_H) \) is a representative of \( w \in W_H \), and

\[ V^a = \{ x \in \sigma^{-1}(a) \mid g^{-1}xg \in \eta_P^{-1}(\Sigma), \dot{w}^{-1}g^{-1}xg\dot{w} \in \eta_P^{-1}(\Sigma') \}, \]

and \( j : V^a \to \Sigma \times \Sigma' \) is defined by \( j(x) = (\eta_P(g^{-1}xg), \eta_P(\dot{w}^{-1}g^{-1}xg\dot{w})) \). We have

\[ g^{-1}xg = (zy') \cdot u = (zy) \cdot u', \]

where \( z \in L \cap wL', y' \in L \cap wU_{P'}, y \in wL' \cap U_P, u \in U_{P'}, u' \in wU_{P'} \). Thus \( V^a \) can be described as follows; put

\[ \tilde{V}^a = \{ (u, u', y, y', z) \in U_{P}^{\theta} \times wU_{P'}^{\theta} \times (wL' \cap U_P) \times (L \cap wU_{P'}) \times (L \cap wL') \]

\[ \mid y' \cdot u = y \cdot u', (zy')^\theta(zy')^{-1} \in \Sigma \cap \sigma^{-1}(a), (zy)^\theta(zy)^{-1} \in \dot{w} \Sigma' \dot{w}^{-1} \cap \sigma^{-1}(a) \}. \]

Then \( V^a \) is isomorphic to the quotient of \( \tilde{V}^a \) by \( E^\theta \), where \( E = (wL' \cap U_P) \times (L \cap wU_{P'}) \times (L \cap wL') \) and the action of \( E^\theta \) is given by the same formula as in the proof of Proposition 2.2. We now consider the map \( \tilde{V}^a/E^\theta \to E/E^\theta \) induced by the projection \( (u, u', y, y', z) \mapsto (y, y', z) \). Its image is given by

\[ \overline{V}^a = \{ (y, y', z) \in E \mid (zy')^\theta(zy')^{-1} \in \Sigma \cap \sigma^{-1}(a), (zy)^\theta(zy)^{-1} \in \dot{w} \Sigma' \dot{w}^{-1} \cap \sigma^{-1}(a) \} / E^\theta. \]

By a similar computation as in (2.2.4), we see that all the fibres of this map are isomorphic to \( (U_{P} \cap wU_{P'})^{\theta} \). Moreover, by a similar discussion as in the proof of (2.2.8), one can show that

\[ \dim(U_P \cap wU_{P'})^{\theta} = 2\nu_H - \nu'_{L_H} - \nu'_{L'_H} - \dim \mathcal{O}_\omega + b', \]
where

\[ b'_w = \#\{ i \mid 1 \leq i \leq n - n_0, 1 \leq w^{-1}(i) \leq n - n'_0 \}. \]

(Here we assume that \( L^\theta \simeq (GL_1)^{(N - N_0)/2} \times GL_{N_0}^0 \) and \( L'^\theta \simeq (GL_1)^{(N - N'_0)/2} \times GL_{N'_0}^0 \), and put \( n_0 = [N_0/2], n'_0 = [N'_0/2] \). If \( B' = B \), a similar argument as in the proof of Proposition 2.2 can be applied. In the case where \( N \) is even, we need to consider \( B' = B_1 \) also. But in this case, if we replace \( W_H = N_H(T_H)/T_H \) by a group \( N_{G^0}(T_H)/T_H \), which is isomorphic to \( S_n \times (\mathbb{Z}/2\mathbb{Z})^n \), a similar argument as in the odd \( N \) case works.)

Thus we are reduced to showing that

\[
H^s_c(\overline{\mathcal{V}}^\omega, \overline{j}^* (\mathcal{E}_1 \boxtimes \mathcal{E}_i')) = 0,
\]

where \( s = \dim \mathfrak{g}_L + \dim \mathfrak{g}_{L'} + \Delta_P + \Delta_{P'} - 2b_w' \), and \( \overline{j} : \overline{\mathcal{V}}^\omega \to \Sigma \times \Sigma' \) is defined by \( (y, y', z) \mapsto ((y \theta(z) y')^{-1}, w^{-1}(y' \theta(z) y')^{-1} \hat{w}) \).

We use the notation \( Q = L \cap wP' \) the \( \theta \)-stable parabolic subgroup of \( L \) with Levi decomposition \( Q = M U_Q \), where \( M = L \cap wL', U_Q = L \cap wU_{P'} \). Similarly, we define \( Q' = wL' \cap P \) the \( \theta \)-stable parabolic subgroup of \( wL' \) with Levi decomposition \( Q' = M U_{Q'} \), where \( U_{Q'} = wL' \cap U_P \). Let \( \pi_3 : \overline{\mathcal{V}}^\omega \to M^o_\theta \) be the map defined by \( (y, y', z) \mapsto z \theta(z)^{-1} \). Since the semisimple part of \( z \theta(z)^{-1} \) is contained in a fixed \( H \)-orbit, \( \pi_3(\overline{\mathcal{V}}^\omega) \) consists of finitely many \( (M^o_\theta)^{\alpha_\omega} \)-orbits \( \hat{\mathcal{E}}_1, \ldots, \hat{\mathcal{E}}_m \) in \( M^o_\theta \). Since \( \dim \overline{\mathcal{V}}^\omega / \Sigma' \leq s/2 \), it is enough to show, for any \( i \), that \( H_c^{s-2d} \pi_3^{-1}(\xi), \overline{j}^* (\mathcal{E}_1 \boxtimes \mathcal{E}_i') \) = 0 for \( \xi \in \hat{\mathcal{E}}_i \). Here \( \pi_3^{-1}(\xi) \) is isomorphic to \( D \times D' \), where

\[
D = \{ q \in Q \cap \Sigma \mid \eta_Q(q) = \xi \},
D' = \{ q' \in Q' \cap \hat{w} \Sigma' \hat{w}^{-1} \mid \eta_{Q'}(q') = \xi \}.
\]

Moreover, the restriction of \( \overline{j}^* (\mathcal{E}_1 \boxtimes \mathcal{E}_i') \) on \( \pi_3^{-1}(\xi) \) corresponds to the tensor product \( \mathcal{E}_{1|b} \boxtimes n^* \mathcal{E}_{i'|s} \) (here \( n : \hat{w} \Sigma' \hat{w}^{-1} \simeq \Sigma' \)). Here by Proposition 2.2 (i), \( 2 \dim D \leq d = \dim \mathfrak{g}_L - \dim \hat{\mathfrak{g}}_i + \Delta_Q, 2 \dim D' \leq d' = \dim \mathfrak{g}_{L'} - \dim \hat{\mathfrak{g}}_i + \Delta_{Q'} \). In a similar way as in (2.2.12), we have

\[
\Delta_Q + b'_w \leq \Delta_P, \quad \Delta_{Q'} + b'_w \leq \Delta_P,
\]

since one can write as

\[
\Delta_Q = \#\{ i \mid n - n_0 + 1 \leq i \leq n, 1 \leq w^{-1}(i) \leq n - n'_0 \},
\Delta_{Q'} = \#\{ i \mid n - n'_0 + 1 \leq i \leq n, 1 \leq w(i) \leq n - n_0 \}.
\]

It follows that

\[
d + d' = s - 2 \dim \hat{\mathfrak{g}}_i + \Delta_Q + \Delta_{Q'} - \Delta_P - \Delta_{P'} + 2b'_w \leq s - 2 \dim \hat{\mathfrak{g}}_i.
\]
Hence by the Künneth formula we are reduced to showing

\[(4.6.2)\quad H^d_c(D, \mathcal{E}_1) \otimes H^d_c(D', n^*\mathcal{E}_1') = 0.\]

Unless \(L = w L', Q = L \cap w P'\) is a proper parabolic subgroup of \(L\), or \(Q' = w L' \cap P'\) is a proper parabolic subgroup of \(w L'\). In that case, since the local systems \(\mathcal{E}_1, \mathcal{E}'_1\) are cuspidal, we have \(H^d_c(D, \mathcal{E}_1) = 0\) or \(H^d_c(D', n^*\mathcal{E}_1') = 0\) by Lemma 2.5. Thus (4.6.2), and so (4.6.1) holds. Now assume that \(B' = B\). Then by the choice of \(P\) and \(P'\), we have \(L = L'\) and \(w \in N_H(L_H)\). If \(\Sigma \cap \Sigma' = \emptyset\), then \(V^0 = \emptyset\), so (4.6.1) holds. This proves (i). Assume that \(w \in N_H(L_H)\), but \(w \notin N_H(L_H)_1\). Then \(Q = L\) and \(\Delta_Q = 0\). Moreover \(\Delta_P > b'_w\). Hence \(d + d' < s - 2 \dim \hat{\mathcal{G}}_i\). This implies that \(H^{s-2 \dim \hat{\mathcal{G}}_i}((\pi^{-1}_3(\xi), \mathcal{J}^s(\mathcal{E}_1 \boxtimes \mathcal{E}'_1))) = 0\), and so (4.6.1) holds. This proves (ii). Next assume that \(N\) is even and \(B' = B_1\). If \(L \neq T\), we have \(L' = L\) (see 4.2), and \(\hat{\mathcal{G}}_i\) is dense in \(V\). This case is essentially the same as the case where \(B' = B\), and \(\hat{\mathcal{G}}_i\) is dense in \(V\). Thus (i), (ii) are proved. So assume that \(L = L' = T\), and \(B' = B_1\). This case is essentially the same as the case where \(\hat{\mathcal{G}}_i\) is dense in \(V\). Hence \(d + d' < s\), and (4.6.2) holds. This proves (iii). The lemma is proved. \(\square\)

\textbf{4.7.} Following [L1, 5.4], we recall the notion of perfect sheaves. A constructible sheaf \(\mathcal{E}\) on an irreducible variety \(V\) is said to be perfect, if

\begin{enumerate}[(i)]
\item \(\mathcal{E} = \operatorname{IC}(V, \mathcal{E}|_{V_0})\), where \(V_0\) is an open dense smooth subset of \(V\) and \(\mathcal{E}|_{V_0}\) is locally constant,
\item the support of any non-zero constructible subsheaf of \(\mathcal{E}\) is dense in \(V\).
\end{enumerate}

In particular, the complex \(\operatorname{IC}(V, \mathcal{E}|_{V_0})\) is reduced to a single sheaf. The following properties hold.

\begin{enumerate}[(4.7.1)]
\item If \(\pi : V' \to V\) is a finite morphism, with \(V'\) smooth, and if \(\mathcal{E}'\) is a locally constant sheaf on \(V'\), then \(\mathcal{E} = \pi_* \mathcal{E}'\) is a perfect sheaf on \(V\).
\item If \(0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_3 \to 0\) is an exact sequence of constructible sheaves on \(V\), with \(\mathcal{E}_1, \mathcal{E}_3\) perfect, then \(\mathcal{E}_2\) is perfect.
\end{enumerate}

\textbf{4.8.} We assume that \((L, \Sigma) = (L', \Sigma')\). We also assume \(\mathbb{O} = \mathbb{O}_\omega\), where a representative of \(\omega\) satisfies the condition \(n = \hat{\omega} \in N_H(L_H)\) with \(\hat{\omega} \in \mathcal{W}'_1\). In this case, we have \(n \Sigma n^{-1} = \Sigma\). The computation in the proof of the lemma shows that

\[(4.8.1)\quad \mathcal{T}_\mathbb{O} \simeq R^s\mathcal{G}(\mathcal{E}_1 \otimes n^*\mathcal{E}_1')\]

where \(\mathcal{G} : \Sigma \to A_Y\) is the restriction of \(\sigma\), \(s = 2 \dim \mathcal{G}_L\) (note that \(\Delta_P = b'_w = b_w\)). Here \(\Sigma = (Z^0_L)^{\hat{\sigma}} \times \mathcal{G}_L\). We denote by \(\pi_1 : \Sigma \to (Z^0_L)^{\hat{\sigma}}\) the projection, and \(\pi_1 : (Z^0_L)^{\hat{\sigma}} \to A_Y\) the restriction of \(\sigma\). Since \(\pi_1\) is a finite morphism, if we put \(\mathcal{G} = R^s(\pi_1)_!(\mathcal{E}_1 \otimes n^*\mathcal{E}_1')\), we have

\[\mathcal{T}_\mathbb{O} = (\pi_1)_!\mathcal{G}.\]

We note that
(4.8.2) \( \mathcal{G} \) is a locally constant sheaf on \((Z^0_L)_{/\theta}\).

In fact, \((Z^0_L)_{/\theta}\) has a structure of a torus. Since \(\mathcal{E}_1 = \tilde{Q} \otimes \mathcal{E}'_1\), \(\mathcal{E}_1\) is \((Z^0_L)_{/\theta}\)-equivariant local system on \(\Sigma\) with respect to the action of \((Z^0_L)_{/\theta}\) on \(\Sigma\) by the left multiplication. Hence \(\mathcal{E}_1 \otimes n^* \mathcal{E}'_1\) is also \((Z^0_L)_{/\theta}\)-equivariant. Since \(\pi_1\) is \((Z^0_L)_{/\theta}\)-equivariant, \(\mathcal{G}\) is \((Z^0_L)_{/\theta}\)-equivariant with respect to the transitive action of \((Z^0_L)_{/\theta}\) on it. Hence \(\mathcal{G}\) is locally constant.

Now \(\Theta_1 : (Z^0_L)_{/\theta} \to A_Y\) is a finite morphism, with \((Z^0_L)_{/\theta}\) smooth, we have, by (4.7.1),

(4.8.3) \(\mathcal{F}_0\) is a perfect sheaf on \(A_Y\).

We now show

**Theorem 4.9.**

(i) Assume that \((L, \Sigma)\) and \((L', \Sigma')\) are not \(H\)-conjugate. Then \(\mathcal{F} = 0\).

(ii) Assume that \(N\) is even, and \(P = B, P' = B_1, L = L' = T\). Then \(\mathcal{F} = 0\).

(iii) Assume that \(P = P', L = L', \Sigma = \Sigma'\). Then \(\mathcal{F}\) is a perfect sheaf on \(A_Y\).

(iv) The natural map of sheaves \(\mathcal{F} \to \tilde{\mathcal{F}}\) is an isomorphism.

**Proof.** Let \(E\) be a locally closed subvariety of \(H/P_H \times H/P'_H\), which is a union of some \(H\)-orbits, and put \(Z^E_0 = q^{-1}(E)\) \((q\) is as in 4.5). One can define a constructible sheaf \(\mathcal{F}_E\) in a similar way as in the definition of \(\mathcal{F}_0\). According to the filtration of \(H/P_H \times H/P'_H\) by various \(E\), we obtain a filtration of \(Z_0\) by various \(Z^E_0\), which yields an exact sequence among various \(\mathcal{F}_E\). In the case of (i), since \(\mathcal{F}_0 = 0\) by Lemma 4.6 (i), we have \(\mathcal{F}_E = 0\) for any \(E\), and so \(\mathcal{F} = 0\). (ii) also follows from Lemma 4.6 (iii). On the other hand, since \(\mathcal{F}_0\) is perfect by (4.8.3), by using the property on exact sequences in (4.7.2), we see that \(\mathcal{F}\) is perfect. This proves (iii).

We show (iv). By (4.5.1) we know that the map \(\mathcal{F} \to \tilde{\mathcal{F}}\) is surjective. Hence it is enough to show that the kernel of \(\mathcal{F} \to \tilde{\mathcal{F}}\) is zero. Put \((A_Y)_{/\theta} = \sigma(\Sigma_{/\theta}) = \Theta_1((Z^0_L)_{/\theta})\). Then \((A_Y)_{/\theta}\) is an open dense subset of \(A_Y\). Since \(\mathcal{F}\) is a perfect sheaf by (i), by the property (ii) of perfect sheaves, it is enough to show that the stalk of this kernel at any point \(a \in (A_Y)_{/\theta}\) is zero. So, by Lemma 4.4 (ii), we have only to show that \(Z^a_{\beta, \beta} = 0\) if \(a \in (A_Y)_{/\theta}\) and if \(\beta \neq \beta_0\) or \(\beta' \neq \beta_0\). Take \((x, gP_H, g'P'_H) \in Z\) such that the semisimple part \(x_s\) of \(x\) is \(H\)-conjugate to an element in \((Z^0_L)_{/\theta}\) and that \(g^{-1}xg \in \eta^{-1}_P(\Sigma), g'^{-1}xg' \in \eta^{-1}_{P'}(\Sigma)\). Then \(Z_H(x_s) = L_H\). We must have \(g^{-1}xg \notin \eta^{-1}_P(\Sigma)\) since if \(g^{-1}xg \notin \eta^{-1}_P(\Sigma)\), then \(\dim Z_H(g^{-1}xg) > \dim L_H\), a contradiction. Similarly, we have \(g'^{-1}xg' \in \eta^{-1}_{P'}(\Sigma)\). This shows that \((x, gP_H, g'P'_H) \in Z^a_{\beta_0, \beta_0}\). (iv) is proved.

**Proposition 4.10.** Assume that \(P = P', L = L', \Sigma = \Sigma'\). Then there exists an isomorphism of sheaves on \(A_Y\);

\[
\mathcal{F} \simeq \bigoplus_{w \in \mathcal{W}_1} \mathcal{F}_{\mathcal{O}(w)},
\]

where \(\mathcal{O}(w)\) is an \(H\)-orbit corresponding to \(w \in \mathcal{W}_1 \subset \mathcal{W} = N_H(L_H)/L_H\). 

\(\Box\)
Proof. Recall that \( \mathcal{F} = R^{2d_0}(\mathcal{O}_0)(\mathcal{E}_1 \otimes \mathcal{E}_1') \). Here \( \mathcal{O}_0 : Z_0 \to A_Y \), where \( Z_0 \simeq X_0 \times_Y X_0 \). By a similar argument as in the proof of [L1, Proposition 5.11], we see that \( \mathcal{O}_0^{-1}((A_Y)_{reg}) \simeq Y \times_Y Y \). Thus the restriction of \( \mathcal{F} \) on \((A_Y)_{reg}\) is the same as \( R^{2d_0}(\mathcal{E}_1)(\mathcal{E}_1 \otimes \mathcal{E}_1) \), where \( \mathcal{E}_1 : Y \times_Y Y \to (A_Y)_{reg} \) is defined similarly to \( \mathcal{O}_0 \). Consider the partition \( Z_0 = \bigcup_{O} Z_0^O \) as before. The pieces \( Z_0^O \) are locally closed subsets in \( Z_0 \). But since \( Y \) is a principal \( \mathcal{U}_1 \)-bundle over \( Y \), the intersection \( Z_0^O \cap (\mathcal{O}_0)^{-1}((A_Y)_{reg}) \) is an open and closed subset. Since \( \mathcal{F}_{O(w)} = 0 \) if \( w \notin \mathcal{U}_1 \) by Lemma 4.6 (ii), we have a natural isomorphism \( \mathcal{F} \simeq \bigoplus_{w \in \mathcal{U}_1} \mathcal{F}_{O(w)} \) over \((A_Y)_{reg}\). Since \( \mathcal{F} \) and \( \bigoplus_{w \in \mathcal{U}_1} \mathcal{F}_{O(w)} \) are both perfect sheaves, this isomorphism can be extended uniquely to the isomorphism over \( A_Y \). The proposition is proved.

4.11. Under the assumption of Proposition 4.10, we consider the sheaf \( \mathcal{F} = R^{2r}(K_{\mathcal{F}_1} \otimes K_{\mathcal{F}'_1}) \). Since \( \mathcal{F} = \sigma \circ p_1 \) with \( p_1 : Z = \tilde{X} \times_X \tilde{X} \to X \) the natural projection, one can write as

\[
\mathcal{F} \simeq R^{2r}(\pi_* K_{\mathcal{F}_1} \otimes \pi_* K_{\mathcal{F}'_1}).
\]

By Proposition 3.8, \( \pi_* K_{\mathcal{F}_1} \otimes \pi_* K_{\mathcal{F}'_1} \) has a natural structure of a module over the algebra \( \mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}'_1} \). Hence \( \mathcal{F} \) inherits a natural action of \( \mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}'_1} \). As in [L1, 3.4], \( \mathcal{A}_{\mathcal{E}_1} \) has a natural decomposition \( \mathcal{A}_{\mathcal{E}_1} = \bigoplus_{w \in \mathcal{E}_1} \mathcal{A}_{\mathcal{E}_1,w} \) such that \( \mathcal{A}_{\mathcal{E}_1,w} \cdot \mathcal{A}_{\mathcal{E}_1,w'} = \mathcal{A}_{\mathcal{E}_1,ww'} \). Then the action of \( \mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}'_1} \) on \( \mathcal{F} \) satisfies the relation (see [L1, 5.12])

\[
(\mathcal{A}_{\mathcal{E}_1,w} \otimes \mathcal{A}_{\mathcal{E}'_1,w'}) \cdot \mathcal{F}_{O(w)} = \mathcal{F}_{O(w1w1')}^{-1}
\]

for \( w \in \mathcal{E}_1, w_1 \in \mathcal{E}_1, w_1' \in \mathcal{E}'_1 \).

We consider the stalk \( \mathcal{F}_1 \) of \( \mathcal{F} \) at \( 1 \in A_Y \). For \( w \in \mathcal{E}_1 \), the stalk \( \mathcal{F}_{O(w),1} \) of \( \mathcal{F}_{O(w)} \) at \( 1 \in A_Y \) is given by (4.8.1),

\[
\mathcal{F}_{O(w),1} \simeq H^2_{c,\dim \mathcal{E}_1}(\mathcal{O}_L, \mathcal{E}_1 \otimes n\mathcal{E}'_1).
\]

Hence \( \mathcal{F}_{O(w),1} \) is a one-dimensional \( \overline{Q}_r \)-vector space if \( w \) satisfies the relation \( n\mathcal{E}'_1 \simeq \mathcal{E}_1^* \), the dual local system of \( \mathcal{E}_1 \), and \( \mathcal{F}_{O(w),1} = 0 \) otherwise. Now assume that \( \mathcal{E}'_1 = \mathcal{E}_1^* \). Then \( n\mathcal{E}'_1 \simeq \mathcal{E}_1^* \) if and only if \( w \in \mathcal{W}_{\mathcal{E}_1} \). Thus we have a decomposition

\[
\mathcal{F}_1 = \mathcal{F}_1 = \bigoplus_{w \in \mathcal{W}_{\mathcal{E}_1}} \mathcal{F}_{O(w),1}
\]

into one-dimensional vector spaces, and (4.11.2) implies that

\[
(\mathcal{A}_{\mathcal{E}_1,w} \otimes \mathcal{A}_{\mathcal{E}'_1,w'}) \cdot \mathcal{F}_{O(w),1} = \mathcal{F}_{O(w1w1')}^{-1}
\]

for \( w, w_1, w_1' \in \mathcal{W}_{\mathcal{E}_1} \). This is nothing but the two-sided regular representation of \( \mathcal{A}_{\mathcal{E}_1} \) if we identify \( \mathcal{A}_{\mathcal{E}_1} \) with \( \mathcal{A}_{\mathcal{E}_1} \) the opposed algebra of \( \mathcal{A}_{\mathcal{E}_1} \). Thus we have proved
Proposition 4.12.  
(i) \( \widetilde{\mathcal{T}} = 0 \) if \( \mathcal{E}_1' \) and \( \mathcal{E}_1^* \) are not conjugate under \( N_H(L_H) \).
(ii) Assume that \( \mathcal{E}_1' = \mathcal{E}_1^* \). Then the stalk \( \mathcal{T} \) of \( \mathcal{T} \) at \( 1 \in A_Y \) is isomorphic to the \( \mathcal{A}_{\mathcal{E}_1} \otimes \mathcal{A}_{\mathcal{E}_1}^0 \)-module \( \mathcal{A}_{\mathcal{E}_1} \) (two-sided regular representation of \( \mathcal{A}_{\mathcal{E}_1} \)).

5. Generalized Springer correspondence

5.1. In this section, we shall establish the generalized Springer correspondence for \( G_{\text{uni}}^\theta \) following the discussion in [L1, §6]. Let \( \mathcal{N}_G \) be the set of pairs \((\mathcal{O}, \mathcal{E})\), where \( \mathcal{O} \) is an \( H \)-orbit in \( G_{\text{uni}}^\theta \), and \( \mathcal{E} \) is an \( H \)-equivariant simple local system on \( \mathcal{O} \). We consider a triple \((L \subset P, \mathcal{O}_L, \xi)\), where \( L \) is a \( \theta \)-stable Levi subgroup of a \( \theta \)-stable parabolic subgroup \( P \) of \( G \), and \( \mathcal{O}_L \) is an \( L \)-orbit in \( L_{\text{uni}}^\theta \), \( \xi \) is an \( L \)-equivariant simple local system on \( \mathcal{O}_L \). Put \( \Sigma = (Z_L^\theta)^\theta \mathcal{O}_L \), and \( \xi = \mathcal{Q}_L \otimes \xi_1^\theta \). Let \( \mathcal{J}_G \) be the set of triples \((L \subset P, \mathcal{O}_L, \xi_1^\theta)\), up to \( H \)-conjugate, such that \( \xi_1 \) is cuspidal on \( \Sigma \). (Note that in the case where \( N \) is odd, \( P \) is determined by \( L \), up to \( H \)-conjugate, hence we need not to write \( P \) in the notation of \( \mathcal{J}_G \) (see [L1, §6]). But in the case where \( N \) is even and \( L = T \), we need to distinguish the two cases \( T \subset B \) and \( T \subset B_1 \).

For each \((L \subset P, \mathcal{O}_L, \xi_1^\theta) \in \mathcal{J}_G \), one can construct a semisimple perverse sheaf \( K = \pi_* \kappa_{\mathcal{P}'_L} \) on \( X = \overline{Y} \) with \( Y = Y_{(L, X)} \). By Proposition 3.8, \( K \) is equipped with an action of \( \mathcal{A}_{\mathcal{E}_1} \), and is decomposed into simple perverse sheaves as follows:

\[
K \simeq \bigoplus_{\rho \in \mathcal{A}_{\mathcal{E}_1}} \rho \otimes K_{\rho},
\]

where \( K_{\rho} = \text{Hom}(\rho, K) \). We consider the restriction \( K|_{G_{\text{uni}}^\theta} \) of \( K \) on \( G_{\text{uni}}^\theta \). Then \( K|_{G_{\text{uni}}^\theta} \) inherits the natural action of \( \mathcal{A}_{\mathcal{E}_1} \). The following result gives the generalized Springer correspondence for \( G_{\text{uni}}^\theta \), which is an analogue of [L1, Theorem 6.5]

Theorem 5.2 (generalized Springer correspondence). Under the notation above,

(i) Let \( K = \pi_* \kappa_{\mathcal{P}'_L} \) for \((L \subset P, \mathcal{O}_L, \xi_1^\theta) \in \mathcal{J}_G \), and put \( r = \dim(Z_L^\theta)^\theta \). Then \( K[-r]|_{X_{\text{uni}}} \) is a semisimple perverse sheaf on \( X_{\text{uni}} = X \cap G_{\text{uni}}^\theta \), and is decomposed into simple perverse sheaves,

\[
K[-r]|_{X_{\text{uni}}} \simeq \bigoplus_{(\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G} V(\mathcal{O}, \mathcal{E}) \otimes \text{IC}(\overline{\mathcal{O}}, \mathcal{E})[\dim \mathcal{O}].
\]

The action of \( \mathcal{A}_{\mathcal{E}_1} \) on \( K[-r]|_{G_{\text{uni}}^\theta} \) induces an action of \( \mathcal{A}_{\mathcal{E}_1} \) on \( V(\mathcal{O}, \mathcal{E}) \), which makes \( V(\mathcal{O}, \mathcal{E}) \) an irreducible \( \mathcal{A}_{\mathcal{E}_1} \) module if it is non-zero. The map \((\mathcal{O}, \mathcal{E}) \mapsto V(\mathcal{O}, \mathcal{E}) \) gives a bijection

\[
\{(\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G \mid V(\mathcal{O}, \mathcal{E}) \neq 0\} \simeq \mathcal{A}_{\mathcal{E}_1}^\text{â€œfinite}.
\]

Moreover, under the above correspondence \((\mathcal{O}, \mathcal{E}) \mapsto \rho \), we have

\[
K_{\rho}[-\dim X]|_{X_{\text{uni}}} \simeq \text{IC}(\overline{\mathcal{O}}, \mathcal{E})[-2d(\mathcal{E})],
\]
where we put \( d_\theta = (\nu_H - \dim \mathcal{O}/2) - (\nu_{L_H} - \dim \mathcal{O}/2) + \Delta_P/2 \).

(ii) For each \((\mathcal{O}, \mathcal{E}) \in \mathcal{X}_G\), there exists a unique \((L \subset P, \mathcal{O}_L, \mathcal{E}_L^\dagger) \in \mathcal{X}_G\) such that \( \text{IC}([\mathcal{O}, \mathcal{E}])\dim \mathcal{O} \) is a direct summand of \( \pi_*K_{\mathcal{O}_L} \). The correspondence \((\mathcal{O}, \mathcal{E}) \mapsto V_{(\mathcal{O}, \mathcal{E})}\) gives a bijection

\[
\mathcal{X}_G \simeq \bigsqcup_{(L \subset P, \mathcal{O}_L, \mathcal{E}_L^\dagger) \in \mathcal{X}_G} \mathcal{A}_{\mathcal{E}_L}^\dagger.
\]

(iii) For \((L \subset P, \mathcal{O}_L, \mathcal{E}_L^\dagger) \in \mathcal{X}_G\), let \( f : \widetilde{X}_0 \to X \) be the restriction of \( \pi : \widetilde{X} \to X \). Under the correspondence in (ii), the condition \((\mathcal{O}, \mathcal{E})\) corresponds to \((L \subset P, \mathcal{O}_L, \mathcal{E}_L^\dagger)\) is that \( \mathcal{O} \subset X \) and that \( \mathcal{E} \) appears as a direct summand in the local system \( R^{2d_\theta} f_*\mathcal{O}_1|_{\mathcal{O}} \). Moreover, in that case, the natural homomorphism, obtained from the embedding \( \widetilde{X}_0 \subset \widetilde{X} \),

\[
R^{2d_\theta} f_*\mathcal{O}_1|_{\mathcal{O}} \to \mathcal{H}^{2d_\theta}(\pi_*K_{\mathcal{O}_L}[−\dim X])|_{\mathcal{E}}
\]

is an isomorphism.

5.3. The remainder of this section is devoted to the proof of the theorem. Let \( P = LU_P \) be a \( \theta \)-stable parabolic subgroup of \( G \), with \( \theta \)-stable Levi subgroup \( L \). Let \( \mathcal{O} \) be an \( H \)-orbit in \( G_{\text{uni}}^\theta \), \( \mathcal{O}_L \) an \( L_H \)-orbit in \( L_{\text{uni}}^\theta \). We consider the following diagram.

\[
\begin{array}{ccc}
V = H \times^{P_H} (\mathcal{O} \cap \eta_P^{-1}(\mathcal{O}_L)) & \to & \mathcal{O} \\
\downarrow f_2 & & \\
V' = H \times^{P_H} \mathcal{O}_L.
\end{array}
\]

(5.3.1)

Here in \( V' \), we consider the action of \( P_H \) on \( \mathcal{O}_L \) such that \( U_{P_H} \) acts trivially. \( f_1 \) is the map induced from the map \((g,x) \mapsto gxg^{-1} \), and \( f_2 \) is the map induced from the map \( \mathcal{O} \cap \eta_P^{-1}(\mathcal{O}_L) \to \mathcal{O}_L \) which is the restriction of the projection \( \eta_P : P^\theta \to L^\theta \). \( H \) acts on \( V, V', \mathcal{O} \) naturally, and \( f_1, f_2 \) are \( H \)-equivariant. Moreover, the action of \( H \) on \( V' \) and on \( \mathcal{O} \) are transitive. Put

\[
d_1 = (\nu_H - \dim \mathcal{O}/2) - (\nu_{L_H} - \dim \mathcal{O}/2) + \Delta_P/2,
\]

\[
d_2 = (\dim \mathcal{O} - \dim \mathcal{O}_L)/2 + \Delta_P/2.
\]

By Proposition 2.2 (ii), (i), all the fibres of \( f_1 \) have dimension \( \leq d_1 \) and all the fibres of \( f_2 \) have dimension \( \leq d_2 \). Moreover, some (or all) fibre of \( f_1 \) has dimension \( d_1 \) if and only if some (or all) fibre of \( f_2 \) has dimension \( d_2 \) since this is equivalent to the condition that \( \dim V = (\nu_H - \nu_{L_H}) + (\dim \mathcal{O} + \dim \mathcal{O}_L)/2 + \Delta_P/2 \). Under this situation, the following result was proved by [L1, (6.1.1)].
Lemma 5.4. Let \( \mathcal{F} \) be an \( H \)-equivariant simple local system on \( \mathcal{O} \), and \( \mathcal{F}' \) be an \( H \)-equivariant simple local system on \( V' \) (note that \( V' \) is a single \( H \)-orbit). Then the multiplicity of \( \mathcal{F} \) in the \( H \)-equivariant local system \( R_{2d_1}(f_1)_!(f_2^* \mathcal{F}') \) on \( \mathcal{O} \) is equal to the multiplicity of \( \mathcal{F}' \) in the \( H \)-equivariant local system \( R_{2d_2}(f_2)_!(f_1^* \mathcal{F}) \) on \( V' \).

5.5. If \((\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G\) is a cuspidal pair, by Lemma 2.5, for any \( \theta \)-stable parabolic subgroup \( P \neq G \), and any \( L_H \)-orbit \( \mathcal{O}_L \) of \( L_{uni}^G \), we have

\[
H_c^{2\delta}(\mathcal{O} \cap \eta_P^{-1}(v), \mathcal{E}) = 0,
\]

where \( \delta = (\dim \mathcal{O} - \dim \mathcal{O}_L)/2 + \Delta_P/2 \) and \( v \in \mathcal{O}_L \). Now let \((\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G\) be an arbitrary element. One can find a \( \theta \)-stable parabolic subgroup \( P \) of \( G \) with the \( \theta \)-stable Levi subgroup \( L \) and an \( L_H \)-orbit \( \mathcal{O}_L \) in \( L_{uni}^G \) satisfying the property

\[
(5.5.1) \quad H_c^{2\delta}(\mathcal{O} \cap \eta_P^{-1}(v), \mathcal{E}) \neq 0
\]

for \( v \in \mathcal{O}_L \). We choose \( P \) a \( \theta \)-stable minimal parabolic subgroup of \( G \) satisfying the property (5.5.1). Note that \( P = G \) satisfies the condition (5.5.1).

Let \( f_2 : \mathcal{O} \cap \eta_T^{-1}(\mathcal{O}_L) \to \mathcal{O}_L \) be the map defined by \( x \mapsto \eta_P(x) \), and let \( \mathcal{E}_1 \) be a \( L_H \)-equivariant simple local system on \( \mathcal{O}_L \) which is a direct summand of \( R_{2\delta}(f_2)_!(\mathcal{E}) \).

Proposition 5.6. The triple \((L \subset P, \mathcal{O}_L, \mathcal{E}_1)\) is uniquely determined by \((\mathcal{O}, \mathcal{E})\). Moreover, \((\mathcal{O}_L, \mathcal{E}_1)\) is a cuspidal pair on \( L_{uni}^G \), namely \((L \subset P, \mathcal{O}_L, \mathcal{E}_1) \in \mathcal{N}_G\).

Proof. Suppose that \((\mathcal{O}_L, \mathcal{E}_1)\) is not cuspidal. Then there exists a \( \theta \)-stable parabolic subgroup \( P' \subset P \) and a Levi subgroup \( L' \) of \( P' \) such that \( L' \subset L \), and an \( L_H \)-orbit \( \mathcal{O}_L' \) in \( L_{uni}^G \) satisfying the property

\[
H_c^{2\delta'}(\mathcal{O}_L \cap \eta^{-1}_{L \cap P'}(v'), \mathcal{E}_1) \neq 0
\]

for \( v' \in \mathcal{O}_L' \), where \( \delta' = (\dim \mathcal{O} - \dim \mathcal{O}_L)/2 + \Delta_{L \cap P'} \). It follows that

\[
H_c^{2\delta'}(\mathcal{O}_L \cap \eta_T^{-1}(\mathcal{O}_L') \cap \mathcal{O}_L \cap \eta_T^{-1}(\mathcal{O}_L')) \neq 0.
\]

The map \( f_2 \) defines a map \( \mathcal{O} \cap \eta_T^{-1}(\mathcal{O}_L') \mapsto \mathcal{O}_L \cap \eta_T^{-1}(\mathcal{O}_L') \) by the restriction, and all the fibres have dimension \( \leq \delta \) by Proposition 2.2. By the Leray spectral sequence, we have

\[
H_c^{2\delta + 2\delta'}(\mathcal{O} \cap \eta_T^{-1}(v'), \mathcal{E}) \neq 0.
\]

Since \( \Delta_P + \Delta_{L \cap P'} = \Delta_{P'} \), we have \( \delta' + \delta = (\dim \mathcal{O} - \dim \mathcal{O}_L)/2 + \Delta_{P'} \). This contradicts the minimality of \( P \). Hence \( \mathcal{E}_1 \) is cuspidal on \( \mathcal{O}_L \).

Let \( \Sigma = (Z^i_\mathcal{O})H \mathcal{O}_L \), and put \( \mathcal{E}_1 = \mathcal{O}_L \boxtimes \mathcal{E}_1 \). Then \((\Sigma, \mathcal{E}_1)\) is a cuspidal pair in \( L \). By Lemma 2.7, \( L_H \) is of the form \( L_H \simeq (GL_1)^a \times SO_{N_0} \). Hence the discussion in Section 3 can be applied. We consider \( Y = Y_{(L, \Sigma)} \) and the map \( \pi : \bar{X} \to X = \bar{Y} \). Let \( \bar{X}_0 \simeq H \times_P \eta_T^{-1}(\Sigma) \) be as in 3.7. The local system \( \bar{\mathcal{E}} \) on \( \bar{X}_0 \) is constructed from \( \mathcal{E}_1 \). Let \( f : \bar{X}_0 \to X \) be the restriction of the map \( \pi : \bar{X} \to X \). Then
the restriction of $f$ on $f^{-1}(\mathcal{O})$ coincides with $f_1$ in 5.3. By the definition of $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^\dagger)$ and by Lemma 5.4, we see that $\mathcal{E}$ is a direct summand of $R^{2d}f_1(\mathcal{E}_1)|_{\mathcal{O}}$, where $d = (\nu_H - \dim \mathcal{O}/2) - (\nu_{L_H} - \dim \mathcal{O}_L/2) + \Delta_P/2$ (note that $d_1 = \delta, d_2 = d$ in the notation in Lemma 5.4).

We show that such a triple $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^\dagger) \in \mathcal{I}_G$ is uniquely determined from $(\mathcal{O}, \mathcal{E})$. Suppose that there exists another $\theta$-stable minimal parabolic subgroup $P'$ satisfying (5.5.1), and $(L' \subset P', \mathcal{O}_{L'}, \mathcal{E}_1^\dagger) \in \mathcal{I}_G$. We can define a map $f' : \tilde{X}_0' \to X'$ and a local system $\mathcal{E}'_1$ on $\tilde{X}_0'$ as before. We see that $\mathcal{E}$ appears as a direct summand in $R^{2d}f_1(\mathcal{E}_1)|_{\mathcal{O}}$. It follows that $R^{2d}f_1(\mathcal{E}_1) \otimes R^{2d}f_1(\mathcal{E}_1^*)|_{\mathcal{O}}$ contains a constant sheaf $\tilde{Q}_l$ on $\mathcal{O}$, where $\mathcal{E}^*$ is the dual local system of $\mathcal{E}_1$. Hence

$$\tag{5.6.1} H^2_c(\mathcal{O}, R^{2d}f_1(\mathcal{E}_1) \otimes R^{2d}f_1(\mathcal{E}_1^*)|_{\mathcal{O}}) \neq 0.$$

Let $p : Z_0 = \tilde{X}_0 \times_{G_0} \tilde{X}_0' \to G_0^\theta$ be the natural projection, and put $Z_{0,\mathcal{O}} = p^{-1}(\mathcal{O})$. Since for each $x \in \mathcal{O}$, $\dim p^{-1}(x) \leq d + d'$, (5.6.1) implies, by the Leray spectral sequence, that

$$\tag{5.6.2} H^2_c(Z_{0,\mathcal{O}}, \mathcal{E}_1 \otimes \mathcal{E}_1^*) \neq 0,$$

where $d_0 = d + d' + \dim \mathcal{O}$ is as in (4.3.1). We consider $Z_0^1 = Z_{0,a,b,c}^1$ for $a = 1 \in A_Y \cap A_Y'$ under the notation in 4.3. Then $Z_{0,\mathcal{O}'}$ form a partition of $Z_0^1$ by locally closed pieces $Z_{0,\mathcal{O}'}$ of dimension $\leq d_0$ (by Lemma 4.4) if $\mathcal{O}'$ varies the $H$-orbits in $G_0^\theta$. It follows from (5.6.2), that

$$H^2_c(Z_{0,\mathcal{O}}, \mathcal{E}_1 \otimes \mathcal{E}_1^*) \neq 0.$$

This means that $\mathcal{F}_1 \neq 0$, where $\mathcal{F}$ is defined with respect to $(L \subset P, \Sigma, \mathcal{E}_1)$ and $(L' \subset P', \Sigma', \mathcal{E}_1^*)$. Then by Theorem 4.9, $P = P', L = L', \Sigma = \Sigma'$. Moreover by Proposition 4.12, $\mathcal{E}_1'$ is $N_H(L_H)$-conjugate to $\mathcal{E}_1$. Thus the triple $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^\dagger)$ is uniquely determined from $(\mathcal{O}, \mathcal{E})$. \hfill \box

### 5.7.

By Proposition 5.6, for each $(\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G$, there exists a unique triple $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^\dagger) \in \mathcal{I}_G$. In this case, we say that $(\mathcal{O}, \mathcal{E})$ belongs to the series $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^\dagger)$. Hence we have a partition

$$\tag{5.7.1} \mathcal{N}_G = \coprod_{\xi \in \mathcal{I}_G} \mathcal{N}_G^{(\xi)},$$

where $\mathcal{N}_G^{(\xi)}$ is the set of all $(\mathcal{O}, \mathcal{E})$ which belong to the series $\xi = (L \subset P, \mathcal{O}_L, \mathcal{E}_1^\dagger)$.

For each $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^\dagger) \in \mathcal{I}_G$, we consider $Y = Y(L, \Sigma)$ and $X = \overline{Y}$. Let $f : \tilde{X}_0 \to X$ be the restriction of $\pi : \tilde{X} \to X$. The following result gives a characterization of the set $\mathcal{N}_G^{(\xi)}$. 


Lemma 5.8. \((\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G \) belong to \(\xi = (L \subset P, \mathcal{O}_L, \mathcal{E}_L)\) if and only if \(\mathcal{O} \subset X\) and \(\mathcal{E}\) is a direct summand of \(R^{2d}f_!(\overline{\mathcal{E}}_1)|_\mathcal{O}\), where \(d = (\nu_H - \dim \mathcal{O}/2) - (\nu_{L_H} - \dim \mathcal{O}_L/2) + \Delta P/2\).

Proof. In the course of the proof of Proposition 5.6, we have already shown that if \((\mathcal{O}, \mathcal{E})\) belongs to \((L \subset P, \mathcal{O}_L, \mathcal{E}_L)\), then \(\mathcal{O} \subset X\) and \(\mathcal{E}\) is a direct summand of \(R^{2d}f_!(\overline{\mathcal{E}}_1)|_\mathcal{O}\). Conversely, assume that \(\mathcal{O} \subset X\) and \(\mathcal{E}\) is a direct summand of \(R^{2df_!(\overline{\mathcal{E}}_1)|_\mathcal{O}\). Then by Lemma 5.4, \(\mathcal{E}_1^v\) is a direct summand of \(R^{2d}(f_2)_!(\mathcal{E})\), where \(\delta\) and \(f_2\) are defined as in 5.5. The discussion in the proof of Proposition 5.6 shows that \((P, L, \mathcal{O}_L)\) satisfies the condition in (5.5.1), and if such a choice of \((P, L, \mathcal{O}_L)\) is not minimal, it contradicts that \((\mathcal{O}_L, \mathcal{E}_L)\) is a cuspidal pair. Hence \((\mathcal{O}, \mathcal{E})\) belongs to \((L \subset P, \mathcal{O}_L, \mathcal{E}_L)\). □

5.9. We fix \((L \subset P, \mathcal{O}_L, \mathcal{E}_L) \in \mathcal{J}_G\). Put \(X_{uni} = X \cap G_{uni}^0\), and \(\tilde{X}_{uni} = \pi^{-1}(X_{uni})\). Thus

\[
\tilde{X}_{uni} = \{(x, gP_H) \in G^{ag} \times H/P_H \mid g^{-1}xg \in \eta_P^{-1}(\mathcal{O}_L)\},
\]

\[
X_{uni} = \bigcup_{g \in H} g(\eta_P^{-1}(\mathcal{O}_L))g.
\]

Let \(\pi_1 : \tilde{X}_{uni} \to X_{uni}\) be the restriction of \(\pi\) on \(\tilde{X}_{uni}\). Since \(\pi_1\) is proper, surjective, \(X_{uni}\) is a closed subset of \(G_{uni}^0\).

Lemma 5.10. \(\tilde{X}_{uni}, X_{uni}\) are irreducible varieties. We have

\[
\dim \tilde{X}_{uni} = \dim X_{uni} = 2\nu_H - 2\nu_{L_H} + \dim \mathcal{O}_L + \Delta P.
\]

Proof. Since \(\tilde{X}_{uni} \simeq H^{\times \nu_H} \eta_P^{-1}(\mathcal{O}_L)\), \(\tilde{X}_{uni}\) is irreducible. Also \(\dim \tilde{X}_{uni} = \dim H/P_H + \dim \eta_P^{-1}(\mathcal{O}_L) = 2\nu_H - 2\nu_{L_H} + \dim \mathcal{O}_L + \Delta P\) (compare with Lemma 3.5). Thus \(\dim \tilde{X}_{uni} = d_0\). Since \(\pi_1\) is surjective, \(\tilde{X}_{uni} \geq X_{uni}\) and \(X_{uni}\) is irreducible. Suppose that \(\dim X_{uni} = \delta < \dim \tilde{X}_{uni}\). Then there exists an open dense subset \(D\) of \(X_{uni}\) such that \(\dim \pi_1^{-1}(x) = \dim \tilde{X}_{uni} - \delta\) for \(x \in D\). We consider the fibre product \(\pi_1^{-1}(D) \times_D \pi_1^{-1}(D)\). It has the dimension \(\dim D + 2(\dim \tilde{X}_{uni} - \delta) = 2\dim \tilde{X}_{uni} - \delta\). On the other hand, \(\pi_1^{-1}(D) \times_D \pi_1^{-1}(D) \subset \tilde{X}_{uni} \times_{X_{uni}} \tilde{X}_{uni} \simeq Z'\), where \(Z'\) is as in 2.1. By Proposition 2.2 (iv), we have \(\dim Z' \leq d_0 = \dim \tilde{X}_{uni}\). It follows that \(\dim \tilde{X}_{uni} \geq 2\dim \tilde{X}_{uni} - \delta\), and so \(\delta \geq \dim \tilde{X}_{uni}\). This is a contradiction, and the lemma follows. □

5.11. We now consider the restriction of \(\pi_*K_{\overline{\mathcal{E}}_1}\) on \(X_{uni}\). Since \(\overline{\Sigma} \simeq \mathcal{Z}^0 \times \mathcal{O}_L\), \(IC(\Sigma, \mathcal{E}_1) \simeq \mathcal{Q}_i \oplus IC(\mathcal{O}_L, \mathcal{E}_L)\). It follows that the restriction of \(K_{\overline{\mathcal{E}}_1} = IC(\tilde{X}, \mathcal{E}_1)|_{\tilde{X}}\) on \(\tilde{X}_{uni}\) coincides with \(K_{\overline{\mathcal{E}}_1}[r]\), where \(K_{\overline{\mathcal{E}}_1} = IC(\tilde{X}_{uni}, \mathcal{E}_1)|_{\tilde{X}_{uni}}\). Here \(\mathcal{E}_1\) is defined from \(\mathcal{E}_1^v\) in a similar way as \(\mathcal{E}_1\) is defined from \(\mathcal{E}_1^v\). (Note that \(\dim \tilde{X} = \dim \tilde{X}_{uni} = r\) by Lemma 5.10 and (3.2.1).) Hence the restriction of \(\pi_*K_{\overline{\mathcal{E}}_1}\) on \(X_{uni}\)
Lemma 5.12. \( \pi_*K_{\pi_1}[-r]|_{X_{uni}} \simeq (\pi_1)_*K_{\pi_1}^1 \) is a perverse sheaf on \( X_{uni} \).

5.13. Since \( \pi_1 \) is proper, by Deligne-Gabber's decomposition theorem, \( (\pi_1)_*K_{\pi_1}^1 \) is a semisimple perverse sheaf on \( X_{uni} \). Since \( \pi_1 \) is \( H \)-equivariant, each simple component in \( (\pi_1)_*K_{\pi_1}^1 \) is an \( H \)-equivariant simple perverse sheaf, hence it is of the form \( K(\mathcal{E}) = IC(\overline{\mathcal{O}}, \mathcal{E})[\dim \mathcal{E}] \) for some \( (\mathcal{O}, \mathcal{E}) \in \mathcal{M}_G \). Here we prepare a lemma.

Lemma 5.14. Take \( (\mathcal{O}, \mathcal{E}), (\mathcal{O}', \mathcal{E}') \in \mathcal{M}_G \), and assume that \( \mathcal{O}, \mathcal{O}' \subset X_{uni} \). Then

\[
\dim H^0_c(X_{uni}, K(\mathcal{O}) \otimes K(\mathcal{E}')) = \begin{cases} 
1 & \text{if } \mathcal{O} = \mathcal{O}' \text{ and } \mathcal{E}' \simeq \mathcal{E}^*, \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. First, by assuming that \( \mathcal{O} \neq \mathcal{O}' \), we show that

\[ (5.14.1) \quad H^0_c(X_{uni}, K(\mathcal{O}) \otimes K(\mathcal{E}')) = 0. \]

In order to show (5.14.1), it is enough to see, by using the hypercohomology spectral sequence, that

\[ (5.14.2) \quad \text{If } H^i_c(X_{uni}, \mathcal{H}^j K(\mathcal{E}) \otimes \mathcal{H}^{j'} K(\mathcal{E}')) \neq 0, \text{ then } i + j + j' < 0. \]

Put \( D_{j,j'} = \text{supp } \mathcal{H}^j K(\mathcal{E}) \cap \text{supp } \mathcal{H}^{j'} K(\mathcal{E}'). \) Since the cohomology in (5.14.2) is not changed if we replace \( X_{uni} \) by \( D_{j,j'} \). Thus we may assume that

\[ H^i_c(D_{j,j'}, \mathcal{H}^j K(\mathcal{E}) \otimes \mathcal{H}^{j'} K(\mathcal{E}')) \neq 0. \]

It follows that \( i \leq 2 \dim D_{j,j'} \). By the condition on the intersection cohomology for \( K(\mathcal{E}) \) and \( K(\mathcal{E}') \), we have

\[
\dim D_{j,j'} \leq \dim \text{supp } \mathcal{H}^j K(\mathcal{E}) \leq -j,
\]

\[
\dim D_{j,j'} \leq \dim \text{supp } \mathcal{H}^{j'} K(\mathcal{E}') \leq -j'.
\]

Hence \( j, j' \leq -\dim D_{j,j'} \). If we assume both are equalities, then we have \( \mathcal{O} = D_{j,j'} = \mathcal{O}' \), a contradiction. Hence we have \( i + j + j' < 0 \), as asserted.

Next assume that \( \mathcal{O} = \mathcal{O}' \). Put \( D = \overline{\mathcal{O}} - \mathcal{O} \). We show the following.

\[ (5.14.3) \quad H^0_c(D, K(\mathcal{O}) \otimes K(\mathcal{E}')) = 0 \quad \text{and} \quad H^{-1}_c(D, K(\mathcal{O}) \otimes K(\mathcal{E}')) = 0. \]

As in the previous argument, we consider the condition that

\[ H^i_c(D_{j,j'}, \mathcal{H}^j K(\mathcal{E}) \otimes \mathcal{H}^{j'} K(\mathcal{E}')) \neq 0, \]

where \( D_{j,j'} = D \cap \text{supp } \mathcal{H}^j K(\mathcal{E}) \cap \text{supp } \mathcal{H}^{j'} K(\mathcal{E}'). \) Again we have \( i \leq 2 \dim D_{j,j'} \). But since \( \dim D < \dim \mathcal{O} \), we have inequalities \( j < -\dim D_{j,j'}, j' < -\dim D_{j,j'} \).
and so \( i + j + j' < -1 \). This proves (5.14.3). By using the cohomology long exact sequence with respect to \( \mathcal{O} \subset \overline{\mathcal{O}} \), (5.14.3) implies that

\[
H^0_{\mathcal{O}}(X_{\text{uni}}, K(\mathcal{O}) \otimes K(\mathcal{O}')) \simeq H^0_{\mathcal{O}}(\mathcal{O}, K(\mathcal{O}) \otimes K(\mathcal{O}')).
\]

Since the cohomology in the right hand side coincides with \( H^2_{\mathcal{O}}(\mathcal{O}, \mathcal{O} \otimes \mathcal{O}') \), we obtain the required formula. The lemma is proved.

5.15. We are now ready to prove (i) of Theorem 5.2. Put \( K = \pi_* K_{\mathcal{O}_1}, K_1 = K[-r]|_{X_{\text{uni}}} = (\pi_1)_* K_{\mathcal{O}_1} \). We have \( \text{End} K \simeq \mathcal{A}_{\mathcal{O}_1} \). Let

\[
\alpha : \text{End} K \to \text{End} K_1
\]

be the natural homomorphism. We show that \( \alpha \) gives rise to an isomorphism. Let us consider the sheaf \( \tilde{\mathcal{F}} \) defined in (4.3.2) with \( \mathcal{O}_1 = \mathcal{O}_1^+ \). Note that \( K_{\mathcal{O}_1} = D(K_{\mathcal{O}_1}) \) and \( \pi_* K_{\mathcal{O}_1} = D(K) \), where \( D \) is the Verdier dual operator. Using the expression (4.11.1), we have

\[
\tilde{\mathcal{F}}_1 \simeq H_{\mathcal{O}}^{-2r}(Z', K_{\mathcal{O}_1} \otimes D(K_{\mathcal{O}_1}))) \\
\simeq H_{\mathcal{O}}^{-2r}(X_{\text{uni}}, K|_{X_{\text{uni}}} \otimes D(K)|_{X_{\text{uni}}}) \\
= H^0_{\mathcal{O}}(X_{\text{uni}}, K_1 \otimes D(K_1)).
\]

Let \( n_{\mathcal{O}} \) be the multiplicity of \( K(\mathcal{O}) \) appearing in the decomposition of the semisimple perverse sheaf \( K_1 \). Since \( n_{\mathcal{O}} \) coincides with the multiplicity of \( K(\mathcal{O}^*) = D(K(\mathcal{O})) \) in the decomposition of \( DK_1 \), by using Lemma 5.14, we have

\[
\dim \tilde{\mathcal{F}}_1 = \dim H^0_{\mathcal{O}}(X_{\text{uni}}, K_1 \otimes D(K_1)) = \sum_{(\mathcal{O}, \mathcal{O}) \in \mathcal{N}} n_{\mathcal{O}}^2.
\]

On the other hand, by Proposition 4.12, \( \dim \tilde{\mathcal{F}}_1 = \dim \mathcal{A}_{\mathcal{O}_1} = \dim \text{End} K \). Since \( \dim \text{End} K_1 = \sum n_{\mathcal{O}}^2 \), we have \( \dim \text{End} K = \dim \text{End} K_1 \). Hence it is enough to show that \( \alpha \) is injective. Now \( K_1 \) can be decomposed as \( K_1 \simeq \bigoplus_{\rho \in \mathcal{A}_{\mathcal{O}_1}} \rho \otimes (K_\rho|_{X_{\text{uni}}}) \), up to shift, and the \( \mathcal{A}_{\mathcal{O}_1} \otimes \mathcal{A}_{\mathcal{O}_1}^0 \)-module structure of \( \tilde{\mathcal{F}}_1 = H^0_{\mathcal{O}}(X_{\text{uni}}, K_1 \otimes D(K_1)) \) is determined from this decomposition. By Proposition 4.12, we know that \( \tilde{\mathcal{F}}_1 \) is the two-sided regular representation of \( \mathcal{A}_{\mathcal{O}_1} \). In particular, \( K_\rho|_{X_{\text{uni}}} \neq 0 \) for any \( \rho \in \mathcal{A}_{\mathcal{O}_1}^\wedge \). Since \( K_\rho|_{X_{\text{uni}}} \) is a direct summand of \( K_1 \), \( K_\rho|_{X_{\text{uni}}} \) is a sum of various \( K(\mathcal{O}) \), at least one summand. Thus \( \alpha \) is injective, and so \( \alpha \) gives an isomorphism. (5.2.1) follows from this. From the above discussion, we have \( K_\rho|_{X_{\text{uni}}} \simeq IC(\overline{\mathcal{O}}, \mathcal{O})[\dim \mathcal{O} + r] \), which is equivalent to (5.2.2). Thus (i) of Theorem 5.2 is proved.

5.16. We prove (ii) and (iii) of Theorem 5.2. Assume that \( \mathcal{O} \subset X \). First we show that the homomorphism in (iii) is surjective. For this, it is enough to see that

\[
H^2_{\mathcal{O}} X(\pi^{-1}(x) - \pi^{-1}(x)\beta_0, K_{\mathcal{O}_1}) = 0 \text{ for any } x \in \mathcal{O},
\]

thus enough to see that
Now we only to show that if \( H^i_c(\pi^{-1}(x)_{\beta}, \mathcal{K}_{X}) \neq 0 \) for any \( \beta \neq \beta_0 \). (Here we put \( \pi^{-1}(x)_{\beta} = \pi^{-1}(x) \cap \tilde{X}_{\beta} \) under the notation in 4.3.) By using the hypercohomology spectral sequence, we have only to show that if \( H^i_c(\pi^{-1}(x)_{\beta}, \mathcal{K}_{X}) \neq 0 \), then \( i + j < 2d_{\beta} - \dim X \). If the cohomology is non-zero, Proposition 2.2 (ii) implies that
\[
i \leq 2 \dim \pi^{-1}(x)_{\beta} \leq 2d_{\beta} - (\dim \mathcal{O}_L - \dim \mathcal{O}_{\beta})
\]
and \( j < - \dim \tilde{X}_{\beta} = - \dim X + (\dim \mathcal{O}_L - \dim \mathcal{O}_{\beta}) \) since \( \beta \neq \beta_0 \). Thus \( i + j < 2d_{\beta} - \dim X \) as asserted. Hence the homomorphism is surjective.

For an \( H \)-equivariant simple local system \( \mathcal{E} \) on \( \mathcal{O} \), we denote by \( m_{\mathcal{E}} \) the multiplicity of \( \mathcal{E} \) in \( R^{2d_{\beta}} f_!(\mathcal{E})_1 |_{\mathcal{O}} \), and by \( \tilde{m}_{\mathcal{E}} \) the multiplicity of \( \mathcal{E} \) in \( \mathcal{H}^{2d_{\beta} - \dim X} K |_{\mathcal{O}} \). Since the homomorphism in (iii) is surjective, we have
\[
m_{\mathcal{E}} \geq \tilde{m}_{\mathcal{E}}.
\]
By (5.2.2), we have \( \mathcal{H}^{2d_{\beta}} (K_{\rho}[\dim X]) |_{\mathcal{O}} \simeq \mathcal{H}^0 (\text{IC}(\mathcal{O}, \mathcal{E})) |_{\mathcal{O}} \simeq \mathcal{E} \) and so \( \tilde{m}_{\mathcal{E}} = n_{\mathcal{E}} \). Hence by the discussion in 5.15, we have
\[
\sum_{(\mathcal{O}, \mathcal{E}) \in \mathcal{K}_G} \tilde{m}_{\mathcal{E}}^2 = \dim \mathcal{A}_{\mathcal{E}_1}.
\]
We shall prove that
\[
\sum_{(\mathcal{O}, \mathcal{E}) \in \mathcal{K}_G} m_{\mathcal{E}}^2 = \dim \mathcal{A}_{\mathcal{E}_1}. \tag{5.16.3}
\]
From the definition of \( m_{\mathcal{E}} \), we have
\[
\dim H^2_{c \dim \mathcal{O}} (\mathcal{O}, R^{2d_{\beta}} f_!(\mathcal{E})_1 \otimes R^{2d_{\beta}} f_!(\mathcal{E})_1) = \sum_{\mathcal{E}} m_{\mathcal{E}}^2,
\]
where in the sum, \( \mathcal{E} \) runs over all the \( H \)-equivariant local systems on a fixed \( \mathcal{O} \). Let \( Z_0 = \tilde{X}_0 \times_X \tilde{X}_0 \) and \( p_1 : Z_0 \to X \) be the first projection. For an \( H \)-orbit \( \mathcal{O} \subset X_{\text{uni}} \), put \( Z_{0, \mathcal{O}} = p_1^{-1}(\mathcal{O}) \). We have a natural map \( Z_{0, \mathcal{O}} \to \mathcal{O} \), and all fibres have dimension \( \leq 2d_{\beta} \). Since \( 2d_{\beta} + \dim \mathcal{O} = d_0 \), where \( d_0 \) is as in (4.3.1), by using the Leray spectral sequence, we have
\[
H^2_{c d_0} (Z_{0, \mathcal{O}}, \mathcal{E}_1 \boxtimes \mathcal{E}_1) \simeq H^2_{c \dim \mathcal{O}} (\mathcal{O}, R^{2d_{\beta}} f_!(\mathcal{E}_1) \otimes R^{2d_{\beta}} f_!(\mathcal{E}_1)).
\]
Now \( Z_{0, \mathcal{O}} = \prod_{\mathcal{E}} Z_{0, \mathcal{O}} \) gives a partition of \( Z_{1, \mathcal{O}_{\beta_0}} = p_1^{-1}(X_{\text{uni}}) \) by locally closed pieces \( Z_{0, \mathcal{O}} \), where \( \dim Z_{0, \mathcal{O}} \leq d_0 \). Hence
\[
\sum_{(\mathcal{O}, \mathcal{E})} m_{\mathcal{E}}^2 = \sum_{\mathcal{E}} \dim H^2_{c d_0} (Z_{0, \mathcal{O}}, \mathcal{E}_1 \boxtimes \mathcal{E}_1) = \dim H^2_{c d_0} (Z_{1, \mathcal{O}_{\beta_0}} \mathcal{E}_1 \boxtimes \mathcal{E}_1) = \dim H^2_{c d_0} (Z_{1, \mathcal{O}_{\beta_0}} \mathcal{E}_1 \boxtimes \mathcal{E}_1)\]
\[ \dim \mathcal{T}_i = \dim \mathcal{A}_{\mathcal{T}_i}. \]

Hence (5.16.3) holds. Comparing (5.16.2) and (5.16.3), we see, by (5.16.1), that \( m_\mathcal{E} = \tilde{m}_\mathcal{E} \). This shows that the natural homomorphism in (iii) is an isomorphism.

By Lemma 5.8, the condition \((\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G\) belongs to \( \xi = (L \subset P, \mathcal{O}_L, \mathcal{E}_1^\mathcal{E}) \in \mathcal{\mathcal{S}}_G\), i.e., \((\mathcal{O}, \mathcal{E}) \in \mathcal{\mathcal{N}}^{(\xi)}_G\), is equivalent to the condition that \( m_\mathcal{E} \neq 0 \). On the other hand, let \( \mathcal{\mathcal{N}}^{(\xi)}_G \) be the set of \((\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G\) such that \( IC(\mathcal{O}, \mathcal{E})[\dim \mathcal{E}] \) appears as a direct summand of \( \pi_\ast K_{\mathcal{T}_1}^{\mathcal{\mathcal{E}}} \) for \( \xi = (L \subset P, \mathcal{O}_L, \mathcal{E}_1^\mathcal{E}) \). Then by Theorem 5.2 (i), the condition \((\mathcal{O}, \mathcal{E}) \in \mathcal{\mathcal{N}}^{(\xi)}_G\) is equivalent to the condition that \( \tilde{m}_\mathcal{E} \neq 0 \). Since \( m_\mathcal{E} = \tilde{m}_\mathcal{E} \), we conclude that \( \mathcal{\mathcal{N}}_G = \mathcal{\mathcal{N}}^{(\xi)}_G \). Now (ii) and (iii) follows from (5.7.1) and Lemma 5.8. This completes the proof of the theorem.

**Remark 5.17.** It is likely that the previous discussion for establishing the generalized Springer correspondence will work for a symmetric space of general type if it satisfies the condition that \( \Delta_0^+ = \emptyset \).

### 6. Restriction Theorem

**6.1.** Let \( P = LU_P \) be as in 3.1. Hence \( L^\theta \simeq (GL_1)^a \times GL_{N_0}^\theta \) with \( N_0 = N - 2a \). We consider a \( \theta \)-stable parabolic subgroup \( Q \) of \( G \) containing \( P \), and the \( \theta \)-stable Levi subgroup \( M \) of \( Q \) containing \( L \). Here we assume that \( M^\theta \simeq (GL_1)^a \times GL_{N_0}^\theta \), where \( N_0 = N - 2a \) with \( a \geq a' \). Let \( \mathcal{O}_L \) be an \( L_H \)-orbit in \( L_{(\text{uni})}^\theta \), and \( \mathcal{E}_1^\mathcal{E} \) a cuspidal local system on \( \mathcal{O}_L \). Put \( \Sigma = (Z_L^\theta)^0 \mathcal{O}_L \), and \( \mathcal{E}_1^\Sigma = Q_L \otimes \mathcal{E}_1^\mathcal{E} \). We consider the complex \( \pi_\ast K_{\mathcal{T}_1}^{\mathcal{\mathcal{E}}} \) on \( X \) obtained from the triple \((L \subset P, \mathcal{O}_L, \mathcal{E}_1^\mathcal{E}) \in \mathcal{\mathcal{S}}_G\). Then the endomorphism algebra \( \text{End}(\pi_\ast K_{\mathcal{T}_1}^{\mathcal{\mathcal{E}}}) \) is isomorphic to \( \mathcal{A}_{\mathcal{E}_1} \), which is a twisted group algebra of \( \mathcal{\mathcal{M}}_{\mathcal{E}_1} \subset \mathcal{\mathcal{M}}_1 \simeq S_a \). By applying a similar discussion for \((L \subset M \cap P, \mathcal{O}_L, \mathcal{E}_1^\mathcal{E}) \in \mathcal{\mathcal{S}}_M\), one can obtain a complex \( \pi_\ast K_{\mathcal{T}_1}^{\mathcal{\mathcal{E}}} \) on \( X' \). Then the endomorphism algebra \( \text{End}(\pi_\ast K_{\mathcal{T}_1}^{\mathcal{\mathcal{E}}}) \simeq \mathcal{A}_{\mathcal{E}_1} \), which is isomorphic to a twisted group algebra of \( \mathcal{\mathcal{M}}_{\mathcal{E}_1} \subset \mathcal{\mathcal{M}}_1 \simeq S_a \). (Here we denote by \( X', \pi' \), etc. by attaching the primes to express the objects corresponding to \( X, \pi, \) etc.). \( \mathcal{A}_{\mathcal{E}_1} \) is canonically identified with the subalgebra of \( \mathcal{A}_{\mathcal{E}_1} \).

We now apply the generalized Springer correspondence for \( H \) and \( M_H = (M^\theta)^0 \). By Theorem 5.2 (ii), if \((\mathcal{O}', \mathcal{E}') \in \mathcal{N}_M\) belongs to the series \((L \subset M \cap P, \mathcal{O}_L, \mathcal{E}_1^\mathcal{E}) \in \mathcal{\mathcal{S}}_M\), then an irreducible representation \( \rho' \in (\mathcal{A}_{\mathcal{E}_1})^\mathcal{\mathcal{E}} \) is determined from \((\mathcal{O}', \mathcal{E}')\). On the other hand, if \((\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G\) belongs to the series \((L \subset P, \mathcal{O}_L, \mathcal{E}_1^\mathcal{E}) \in \mathcal{\mathcal{S}}_G\), then an irreducible representation \( \rho \in \mathcal{A}_{\mathcal{E}_1}^\mathcal{E} \) is determined. We will describe the relationship between \( \rho' \) and \( \rho \).

Take \( \mathcal{O} \) and \( \mathcal{O}' \) as above. Let \( f_{\mathcal{O}, \mathcal{O}'} : \eta_Q^{-1}(\mathcal{O}') \cap \mathcal{O} \to \mathcal{O}' \) be the restriction of the map \( \eta_Q : Q^\theta \to M_{(\mathcal{O}')} \). We define an integer \( m_{\mathcal{O}, \mathcal{O}'} \), as the multiplicity of \( \mathcal{E}' \) in the local system \( H_{2d_{\mathcal{O}, \mathcal{O}'}(f_{\mathcal{O}, \mathcal{O}'}, (\ell))} \), where we put \( d_{\mathcal{O}, \mathcal{O}'} = (\dim \mathcal{O} - \dim \mathcal{O}')/2 + \Delta_Q/2 \). The following result is an analogue of Lusztig’s restriction theorem [L1, Theorem 8.3].

**Theorem 6.2.** Assume that \((\mathcal{O}', \mathcal{E}') \in \mathcal{N}_M\) belongs to \((L \subset M \cap P, \mathcal{O}_L, \mathcal{E}_1^\mathcal{E}) \in \mathcal{\mathcal{S}}_M\).
(i) If \( m_{\mathcal{G}, \mathcal{G}'} \neq 0 \), then \( (\mathcal{G}, \mathcal{G}') \in \mathcal{N}_G \) belongs to \( (L \subset P, \mathcal{G}_L, \mathcal{G}_L') \).

(ii) Assume that \( (\mathcal{G}, \mathcal{G}') \in \mathcal{N}_G \) belongs to \( (L \subset P, \mathcal{G}_L, \mathcal{G}_L') \), and let \( \rho \in \mathcal{A}_{\mathcal{G}_L} \) (resp. \( \rho' \in (\mathcal{A}_{\mathcal{G}_L})^\wedge \) ) be the irreducible representation corresponding to \( (\mathcal{G}, \mathcal{G}') \) (resp. \( (\mathcal{G}', \mathcal{G}') \) ). Then we have

\[
\langle \rho|_{\mathcal{A}_{\mathcal{G}_L}} , \rho' \rangle_{\mathcal{A}_{\mathcal{G}_L}} = m_{\mathcal{G}, \mathcal{G}'},
\]

where the left hand side means the multiplicity of \( \rho' \) in the restriction of \( \rho \) on \( \mathcal{A}_{\mathcal{G}_L} \).

### 6.3.

The remainder of this section is devoted to the proof of the theorem. The proof is done by a similar strategy as in [L1]. Although the discussions are parallel to [L1], we give the full proof for the sake of completeness. Take \( (L \subset P, \mathcal{G}_L, \mathcal{G}_L') \in \mathcal{F}_G \), and let

\[
Y = \bigcup_{g \in H^1} g(\Sigma_{\text{reg} \mathcal{D}_P}) g_1^{-1}, \quad \overline{Y} = \bigcup_{g \in H} g \eta^{-1}_P(\Sigma) g_1^{-1}
\]

be as in Section 3. By replacing \( G, P \) by \( M, M \cap P \), we obtain similar varieties,

\[
Y^M = \bigcup_{g_1 \in M_H} g_1(\Sigma_{\text{reg} \mathcal{D}_{M \cap P}}) g_1^{-1}, \quad \overline{Y}^M = X^M = \bigcup_{g_1 \in M_H} g_1 \eta^{-1}_P(\Sigma) g_1^{-1}.
\]

Consider the following varieties,

\[
\tilde{Y} = \{(x, g L_H) \in G^{\alpha} \times H/L_H \mid g^{-1} x g \in \Sigma_{\text{reg} \mathcal{D}_P}\},
\]

\[
\tilde{Y}_M = \{(x, g M_H) \in G^{\alpha} \times H/M_H \mid g^{-1} x g \in Y^M \mathcal{D}_Q\},
\]

\[
\tilde{X} = \{(x, g P_H) \in G^{\alpha} \times H/P_H \mid g^{-1} x g \in \eta^{-1}_P(\Sigma)\},
\]

\[
\tilde{X}_Q = \{(x, g Q_H) \in G^{\alpha} \times H/Q_H \mid g^{-1} x g \in \eta^{-1}_Q(X^M)\}.
\]

Note that \( Y^M \mathcal{D}_Q \subset Q^{\alpha} \) since

\[
g_1(\Sigma_{\text{reg} \mathcal{D}_{M \cap P}}) g_1^{-1} \cdot \mathcal{D}_Q = g_1(\Sigma_{\text{reg} \mathcal{D}_{M \cap P}}) g_1^{-1} = g_1((\Sigma_{\text{reg} \mathcal{D}_P}) g_1^{-1} \subset Q^{\alpha}
\]

for \( g_1 \in M_H \). Let \( \pi_0 : \tilde{Y} \rightarrow Y, \pi : \tilde{X} \rightarrow X \) be the first projections as before. We have the following commutative diagram,

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\pi'} & \tilde{X}_Q & \xrightarrow{\pi''} & X \\
\uparrow j_L & & \uparrow j_M & & \uparrow j \\
\tilde{Y} & \xrightarrow{\pi_0} & \tilde{Y}_M & \xrightarrow{\pi_0'} & Y,
\end{array}
\]

where \( \pi' : (x, g P_H) \mapsto (x, g Q_H), \pi'' : (x, g Q_H) \mapsto x, \pi_0' : (x, g L_H) \mapsto (x, g M_H), \pi_0'' : (x, g M_H) \mapsto x \). (Note that \( \pi_0' \) is well-defined since \( g_1^{-1} x g_1 \in Y^M \mathcal{D}_Q \) for \( x \in \Sigma_{\text{reg} \mathcal{D}_P}, g_1 \in M_H \) by \( \mathcal{D}_P = \mathcal{D}^0_{P \cap M} \times \mathcal{D}^0_Q \).) Moreover, \( j_L : (x, g L_H) \mapsto (x, g P_H), j_M :
(x, gM) ↦→ (x, gQ), j : x ↦→ x are natural inclusions. By definition, π = π" ◦ π, π0 = π"0 ◦ π0. By Lemma 3.5, jL, j give embeddings of Y, Y into X as open dense smooth subsets. Since Y_M = (π")^{-1}(Y), Y_M is open dense in X. Here Y_M ≃ H ×^{M} (Y^M Q_0^M). Since Y^M is smooth by Lemma 3.5, and so Y^M Q_0^M is smooth, thus Y_M is smooth.

Recall that π_0 : Y → Y is a finite Galois covering with Galois group W_1 ≃ S_a. L = (π_0)^* E_1 is a semisimple local system on Y and we have End L ≃ A E_1. Thus L is decomposed as

(6.3.2) L ≃ \bigoplus_{\rho \in A E_1} \rho \otimes L_{\rho},

where L_{\rho} = Hom_{A E_1} (\rho, L). Similarly, π_0' : Y → Y_M is a finite Galois covering with group W_1' ≃ S_{a'} and L' = (π_0')* E_1 is a semisimple local system on Y_M, and we have End L' ≃ A' E_1. We have a decomposition into simple local systems,

(6.3.3) L' ≃ \bigoplus_{\rho' \in (A E_1)^\wedge} \rho' \otimes L'_{\rho'},

where L'_{\rho'} = Hom_{A E_1} (\rho', L'). Since L ≃ (π''_0)* (π_0)* E_1 ≃ (π''_0)* L', by applying (π''_0)* on both sides of (6.3.3), we have

(6.3.4) L ≃ \bigoplus_{\rho' \in (A E_1)^\wedge} \rho' \otimes (π''_0)* L'_{\rho'}.

By comparing (6.3.2) and (6.3.4), we see that (π''_0)* L'_{\rho'} is a semisimple local system whose simple components are of the form L_{\rho} for some \rho \in A E_1^\wedge. Since the action of A E_1^\wedge on L in (6.3.4) is the restriction to A E_1^\wedge of the action of A E_1 on L in (6.3.2), we obtain the following decomposition into simple local systems.

(6.3.5) (π''_0)* L'_{\rho'} ≃ \bigoplus_{\rho \in A E_1^\wedge} L_{\rho} \otimes Q_{|\rho| E_1}^{j^0, j'},

where \langle \rho, \rho' \rangle = \langle \rho \mid A E_1^\wedge, \rho' \rangle_{A E_1^\wedge} is the multiplicity of \rho' in the restriction of \rho on A E_1^\wedge.

6.4. We now consider K_{E_1} = IC(\tilde{X}, \tilde{\mathcal{O}}_1)[\dim \tilde{X}]. By Proposition 3.8, π_0 K_{E_1} is a semisimple perverse sheaf on X, and is decomposed into simple perverse sheaves

(6.4.1) π_0 K_{E_1} ≃ \bigoplus_{\rho \in A E_1^\wedge} \rho \otimes \rho_0.
where \( A_\rho = \text{IC}(X, \mathcal{L}_\rho)[\dim X] \).

Next we shall show that

\[
(6.4.2) \quad \pi'_*K_{\pi_1} \simeq \text{IC}(\tilde{X}_Q, \mathcal{L}')[\dim \tilde{X}_Q].
\]

Under the isomorphisms

\[
(6.4.3) \quad H \times^{P_H} (\eta_{P}^{-1}(\Sigma)) \cong \tilde{X},
\]

\[
H \times^{Q_H} (\eta_Q^{-1}(X^M)) \cong \tilde{X}_Q,
\]

consider the following commutative diagram

\[
\begin{array}{ccc}
H \times^{P_H} (\eta_{P}^{-1}(\Sigma)) & \xleftarrow{\xi} & H \times (Q_H \times^{P_H} (\eta_{P}^{-1}(\Sigma))) \xrightarrow{\tilde{\eta}} \tilde{X}^M \\
\pi' \downarrow & & \downarrow \tilde{\pi} & \downarrow \pi^M \downarrow \\
H \times^{Q_H} (\eta_Q^{-1}(X^M)) & \xleftarrow{\xi} & H \times \eta_Q^{-1}(X^M) \xrightarrow{\eta} X^M,
\end{array}
\]

where \( \xi \) is the quotient map under the free action of \( Q_H \) on \( H \times \eta_Q^{-1}(X^M) \) by

\[
q : (g, y) \mapsto (gq^{-1}, qyq^{-1}),
\]

\( \tilde{\xi} \) is the quotient map under the free action of \( Q_H \) on \( H \times (Q_H \times^{P_H} (\eta_{P}^{-1}(\Sigma))) \) as above, under the natural isomorphism

\[
H \times^{Q_H} (Q_H \times^{P_H} (\eta_{P}^{-1}(\Sigma))) \simeq H \times^{P_H} (\eta_{P}^{-1}(\Sigma)).
\]

\( \eta \) is the composite of the projection \( H \times \eta_Q^{-1}(X^M) \rightarrow \eta_Q^{-1}(X^M) \) and the map \( \eta_Q \).

The map \( \pi^M : \tilde{X}^M \rightarrow X^M \) is the map analogous to \( \pi : \tilde{X} \rightarrow X \) obtained by replacing \( G \) by \( M \). \( \tilde{\eta} \) is the composite of the projection \( H \times (Q_H \times^{P_H} (\eta_{P}^{-1}(\Sigma))) \rightarrow Q_H \times^{P_H} (\eta_{P}^{-1}(\Sigma)) \) and the natural map

\[
Q_H \times^{P_H} (\eta_{P}^{-1}(\Sigma)) \rightarrow M_H \times^{(P_H \cap M_H)} (\eta_{P_H \cap M}^{-1}(\Sigma)) \simeq \tilde{X}^M
\]

obtained from the projection \( Q \rightarrow M \). Under the isomorphisms in \((6.4.3)\), \( \pi' \) coincides with the map induced from the natural inclusion \( H \times \eta_P^{-1}(\Sigma) \hookrightarrow H \times \eta_{Q}^{-1}(X^M) \).

\( \tilde{\pi} = \text{id} \times \tilde{\pi}_1 \), where \( \tilde{\pi}_1 : Q_H \times^{P_H} (\eta_{P}^{-1}(\Sigma)) \rightarrow \eta_Q^{-1}(X^M) \) induced from \( (g, x) \mapsto gxg^{-1} \).

Here \( \eta, \tilde{\eta} \) are \( Q_H \)-equivariant with respect to the natural action of \( M_H \) (and the trivial action of \( U_{Q_H} \)) on \( \tilde{X}^M \) and on \( X^M \). Note that two squares are cartesian squares.

By applying the construction of \( K_{\pi_1} = \text{IC}(\tilde{X}, \mathcal{O}_1)[\dim \tilde{X}] \) to \( M \), one can define the perverse sheaf \( K_{\pi_1} = \text{IC}(\tilde{X}^M, \mathcal{O}_1^M)[\dim \tilde{X}^M] \) on \( \tilde{X}^M \), and we obtain an \( M_H \)-equivariant semisimple perverse sheaf \( (\pi^M)_*K_{\pi_1}M \) on \( X^M \). Here \( \eta \) is a smooth morphism with connected fibre isomorphic to \( H \times U_Q^\theta \), and is \( Q_H \)-equivariant. Then \( \eta^*(\pi^M)_*K_{\pi_1}[\alpha] \) is a \( Q_H \)-equivariant perverse sheaf on \( H \times \eta_Q^{-1}(X^M) \), where
\( \alpha = \dim H + \dim U_Q^\theta \). On the other hand, since \( \xi \) is a \( Q_H \)-principal bundle, there exists a unique perverse sheaf \( K_1 \) on \( H \times Q_H (\eta_Q^{-1}(X^M)) \) such that

\[
(6.4.5) \quad \xi^* K_1[\beta] \simeq \eta^* (\pi_M^*) K_{\overset{\sim}{\sigma}_1^M}[\alpha],
\]

where \( \beta = \dim Q_H \).

A similar construction works also for an \( M_H \)-equivariant perverse sheaf \( K_{\overset{\sim}{\sigma}_1^M} \) on \( \tilde{X}^M \), and one can find a perverse sheaf \( K_2 \) on \( H \times P_H (\eta_P^{-1}(\Sigma)) \), unique up to isomorphism, such that

\[
(6.4.6) \quad \tilde{\xi}^* K_2[\beta] \simeq \tilde{\eta}^* K_{\overset{\sim}{\sigma}_1^M}[\alpha].
\]

Here, if we note that the two squares in the diagram (6.4.4) are cartesian squares, by using the proper base change theorem, (6.4.5) and (6.4.6) imply that

\[
(6.4.7) \quad K_1 \simeq \pi'_* K_2.
\]

Thus, in order to prove (6.4.2), it is enough to see that

\[
(6.4.8) \quad K_2 \simeq \text{IC}(\tilde{X}, \mathcal{E}_1)[\dim \tilde{X}], \quad K_1 \simeq \text{IC}(\tilde{X}_Q, (\pi_0', \mathcal{E}_1))[\dim \tilde{X}_Q].
\]

It follows from the construction that \( K_1 \) (resp. \( K_2 \)) is an intersection cohomology complex on \( \tilde{X}_Q \) (resp. \( \tilde{X} \)) whose support is \( \tilde{X}_Q \) (resp. \( \tilde{X} \)). Hence to prove (6.4.8), we have only to show that

\[
(6.4.9) \quad K_2|_{\tilde{Y}} \simeq \mathcal{E}_1[\dim \tilde{Y}], \quad K_1|_{\tilde{Y}_M} \simeq (\pi_0', \mathcal{E}_1)[\dim \tilde{Y}_M].
\]

Under the isomorphisms

\[
\tilde{Y} \simeq H \times L_H (\Sigma_{\text{reg}} \mathcal{D}_P^0), \quad \tilde{Y}_M \simeq H \times M_H (Y^M \mathcal{D}_Q^0),
\]

we have the following commutative diagram.

\[
(6.4.10) \quad \begin{array}{ccc}
H \times L_H (\Sigma_{\text{reg}} \mathcal{D}_P^0) & \xrightarrow{\xi_0} & H \times (M_H \times L_H (\Sigma_{\text{reg}} \mathcal{D}_P^0)) \\
\pi_0' \downarrow & & \pi_0 \downarrow \\
H \times M_H (Y^M \mathcal{D}_Q^0) & \xrightarrow{\xi_0} & H \times (Y^M \mathcal{D}_Q^0) \\
\end{array} \xrightarrow{\pi_0} \begin{array}{c} \tilde{Y}_M \\
\end{array}
\]

The maps are defined similarly to (6.4.4), and the diagram (6.4.10) enjoys similar properties as (6.4.4). In particular, the squares in the diagram are both cartesian.
We consider the local system $\tilde{E}^M_1$ on $\tilde{Y}^M$. Then one can check that the local system on $H \times^L H (\Sigma_{\text{reg}} \mathcal{G}_P^0)$, obtained from $\tilde{E}^M_1$ by a similar discussion as before, coincides with $\tilde{E}_1$ on $\tilde{Y}$. It follows from (6.4.10) that the local system on $H \times^M H (Y^M \mathcal{G}_Q^0)$, obtained from the local system $(\pi_0^M)^* \tilde{E}^M_1$ on $Y^M$ by a similar discussion as before, coincides with $(\pi_0^M)^* \tilde{E}_1$. (6.4.9) now follows from this by applying the proper base change theorem to (6.4.4) and (6.4.10). Thus (6.4.2) is proved.

6.5. By applying the decomposition of $L'$ in (6.3.3) to the isomorphism in (6.4.2), we obtain the decomposition of $\pi''_* K_{\tilde{E}^M_1}$ into simple perverse sheaves,

\begin{equation}
\pi''_* K_{\tilde{E}^M_1} \simeq \bigoplus_{\rho' \in (\mathcal{A}_{\tilde{E}^M_1})^\wedge} \rho' \otimes B_{\rho'},
\end{equation}

where $B_{\rho'} \simeq \text{IC}(\tilde{X}_Q, \mathcal{L}'_{\rho'})[\dim \tilde{X}_Q]$. By applying $\pi''_*$ on both sides of (6.5.1), we obtain

\begin{equation}
\pi''_* K_{\tilde{E}^M_1} \simeq \bigoplus_{\rho' \in (\mathcal{A}_{\tilde{E}^M_1})^\wedge} \rho' \otimes \pi''_* B_{\rho'}.
\end{equation}

Since $\pi''$ is proper, $\pi''_* B_{\rho'}$ is a semisimple complex by the theorem of Deligne-Gabber. Since $\pi''_* B_{\rho'}$ is a direct summand of a semisimple perverse sheaf $\pi''_* K_{\tilde{E}^M_1}$, $\pi''_* B_{\rho'}$ is a semisimple perverse sheaf, and all of its simple components have support on $X$. Since $\pi''_* B_{\rho'}|_Y \simeq (\pi_0''_* \mathcal{L}'_{\rho'})[\dim X]$, we have

\begin{equation}
\pi''_* B_{\rho'} \simeq \text{IC}(X, (\pi_0''_* \mathcal{L}'_{\rho'}))[\dim X].
\end{equation}

By applying the decomposition of $(\pi_0''_* \mathcal{L}'_{\rho'})$ in (6.3.5), we obtain the following lemma, which is a perverse sheaf version of (6.3.5).

**Lemma 6.6.** For each $\rho' \in (\mathcal{A}_{\tilde{E}^M_1})^\wedge$, $\rho \in \mathcal{A}_{\tilde{E}^M_1}$, put $A_\rho = \text{IC}(X, \mathcal{L}'_{\rho'})[\dim X]$, $B_{\rho'} = \text{IC}(\tilde{X}_Q, \mathcal{L}'_{\rho'})[\dim \tilde{X}_Q]$. Then we have an isomorphism of perverse sheaves on $X$,

\begin{equation}
\pi''_* B_{\rho'} \simeq \bigoplus_{\rho \in \mathcal{A}_{\tilde{E}^M_1}} A_\rho \otimes \mathcal{Q}_{\rho'}^{(p, \rho')},
\end{equation}

6.7. Let $(\mathcal{O}_\rho, \mathcal{E}_\rho) \in \mathcal{N}_G$ be the pair which belongs to $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^L) \in \mathcal{J}_G$ and corresponds to $\rho \in \mathcal{A}_{\tilde{E}^M_1}$. Put $d = \dim X$, and let $d_\rho = d_{\rho\rho}$ be as in Theorem 5.2. By (5.2.2), we have $A_\rho|_{X_\text{uni}} \simeq \text{IC}(\mathcal{O}_\rho, \mathcal{E}_\rho)[d - 2d_\rho]$, hence the following property holds. For any $\tilde{\rho} \in \mathcal{A}_{\tilde{E}^M_1}$, we have

\begin{equation}
\mathcal{H}^{2d_\rho - d} A_{\tilde{\rho}}|_{\mathcal{O}_\rho} \simeq \begin{cases} \mathcal{E}_{\tilde{\rho}}^\wedge & \text{if } \mathcal{O}_{\tilde{\rho}} = \mathcal{O}_\rho, \\ 0 & \text{otherwise}. \end{cases}
\end{equation}

Comparing (6.7.1) with Lemma 6.6, we have the following.
(6.7.2) \( \langle \rho, \rho' \rangle \) coincides with the multiplicity of \( \mathcal{E}_\rho \) in the local system \( \mathcal{H}^{2d_{\rho'}-d}((\pi''_x B_{\rho'})|_{\mathcal{E}_\rho} \).

Put \( (\tilde{X}_Q)_{\text{uni}} = \tilde{X}_Q \cap (G^0_{\text{uni}} \times H/Q_H) \). We will consider the restriction of \( B_{\rho'} \) on \( (\tilde{X}_Q)_{\text{uni}} \). Let \( (\mathcal{O}'_{\rho'}, \mathcal{E}'_{\rho'}) \in \mathcal{M}_M \) be the pair which belongs to \( (L \subset M \cap P, , \mathcal{O}_L, \mathcal{E}_\delta^{\dagger}) \in \mathcal{S}_M \) and which corresponds to \( \rho' \in (\mathcal{A}_1^{\delta})^\wedge \). Put

\[
D = \{(x, gQ_H) \in G^0_{\text{uni}} \times H/Q_H \mid g^{-1}xg \in \eta_Q^{-1}(\mathcal{O}'_{\rho'})\}.
\]

\( D \) is a closed subvariety of \( (\tilde{X}_Q)_{\text{uni}} \). We note that

\[
\text{supp } B_{\rho'}|_{(\tilde{X}_Q)_{\text{uni}}} \subset D.
\]

In fact, as in (6.4.1), we denote by \( A'_{\rho'} \) the simple perverse sheaf on \( X^M \) appearing in the decomposition of \( (\pi^M)_* K_{\mathcal{E}_M} \), which corresponds to \( \rho' \in (\mathcal{A}_1^{\delta})^\wedge \). Then \( B_{\rho'} \) can be constructed from \( A'_{\rho'} \) similarly to \( K_1, K_2 \), by using the diagram (6.4.4). On the other hand, by the generalized Springer correspondence for \( M \), the restriction of \( A'_{\rho'} \) on \( M^0_{\text{uni}} \) coincides with \( \text{IC}(\mathcal{O}'_{\rho'}, \mathcal{E}'_{\rho'}) \), up to shift. (6.7.3) follows from this.

Let \( \pi''_{\rho} \) be the restriction of \( \pi'' \) on \( D \). In view of (6.7.3), (6.7.2) can be rewritten as follows.

(6.7.4) \( \langle \rho, \rho' \rangle \) coincides with the multiplicity of \( \mathcal{E}_\rho \) in the local system \( \mathcal{H}^{2d_{\rho'}-d}((\pi''_x : (B_{\rho'}|_D))|_{\mathcal{E}_\rho} \).

Put \( D_0 = \{(x, gQ_H) \in G^0_{\text{uni}} \times H/Q_H \mid g^{-1}xg \in \eta_Q^{-1}(\mathcal{O}'_{\rho'})\} \). \( D_0 \) is an open subset of \( D \). Let \( \pi''_{D_0} \) be the restriction of \( \pi'' \) on \( D_0 \). We define an integer \( x_{\rho, \rho'} \) as the multiplicity of \( \mathcal{E}_\rho \) in the local system \( \mathcal{H}^{2d_{\rho'}-d}((\pi''_{D_0} : (B_{\rho'}|_{D_0}))|_{\mathcal{E}_\rho} \). We show

**Proposition 6.8.** \( \langle \rho, \rho' \rangle \) coincides with \( x_{\rho, \rho'} \).

**Proof.** First we show that

(6.8.1) \( x_{\rho, \rho'} \geq \langle \rho, \rho' \rangle \).

Consider the long exact sequence of local systems obtained from the open embedding \( D_0 \hookrightarrow D \),

\[
\cdots \rightarrow \mathcal{H}^{2d_{\rho'}-d}((\pi''_{D_0} : (B_{\rho'}|_{D_0}))|_{\mathcal{E}_\rho}) \rightarrow \mathcal{H}^{2d_{\rho'}-d}((\pi''_x : (B_{\rho'}|_D))|_{\mathcal{E}_\rho}) \rightarrow \mathcal{H}^{2d_{\rho'}-d}((\pi''_{D-D_0} : (B_{\rho'}|_{D-D_0}))|_{\mathcal{E}_\rho}) \rightarrow \cdots
\]

If \( \varphi \) is surjective, then (6.8.1) holds. Thus it is enough to see that

\[
\mathcal{H}^{2d_{\rho'}-d}((\pi''_{D-D_0} : (B_{\rho'}|_{D-D_0}))|_{\mathcal{E}_\rho}) = 0.
\]

For this we have only to show that the stalk vanishes at any \( x \in \mathcal{O}_\rho \). Hence we will show that

(6.8.2) \( H^{2d_{\rho'}-d}_c((\pi''^{-1}(x) \cap (D - D_0), B_{\rho'}) = 0 \)
for any $x \in \mathcal{O}_\rho$.

We consider the decomposition of $\mathcal{O}'_{\rho'}$ into $M_H$-orbits, $\mathcal{O}'_{\rho'} = \bigsqcup \mathcal{O}'_{\alpha}$. This implies the decomposition of $D - D_0$ into locally closed subsets, $D = D_0 = \bigsqcup \mathcal{D}_\alpha$, where

$$D_\alpha = \{(x, yQ_H) \in G^\theta \times H/Q_H \mid g^{-1}xg \in \eta_Q^{-1}(\mathcal{O}'_{\alpha})\}.$$ 

In order to show (6.8.2), it is enough to see, for any $x \in \mathcal{O}_\rho$, and for any $D_\alpha$, that

(6.8.3) \[\mathbf{H}^{2d_\rho - d}(\pi''^{-1}(x) \cap D_\alpha, B_{\rho'}) = 0.\]

We remark that the following holds.

(6.8.4) If $H^i_c(\pi''^{-1}(x) \cap D_\alpha, \mathcal{H}^j B_{\rho'}) \neq 0$, then $i + j < 2d_\rho - d$.

In fact, since the cohomology is non-zero, we have

$$i \leq 2\dim(\pi''^{-1}(x) \cap D_\alpha) \leq (2\nu_H - \dim \mathcal{O}_\rho) - (2\nu_{M_H} - \dim \mathcal{O}'_{\alpha}) + \Delta_Q,$$

where the second formula is obtained from Proposition 2.2 (ii), applied to $Q_H \subset H$. On the other hand, the condition $\mathcal{H}^j B_{\rho'}|_{D_\alpha} \neq 0$ is equivalent to the condition that $\mathcal{H}^{j+(d-d')}A'_{\rho'}|_{\mathcal{O}'_{\alpha}} \neq 0$ under the correspondence $B_{\rho'} \leftrightarrow A'_{\rho'}$ through the diagram (6.4.4), where $d' = \dim X^M$. Hence by applying (5.2.2) for $M$, we have

$$j + d \leq (2\nu_{M_H} - \dim \mathcal{O}'_{\rho'}) - (2\nu_{L_H} - \dim \mathcal{O}'_{\alpha}) + \Delta_{P \cap M} \leq (2\nu_{M_H} - \dim \mathcal{O}'_{\alpha}) - (2\nu_{L_H} - \dim \mathcal{O}_{\alpha}) + \Delta_{P \cap M}.$$

Since $\Delta_Q + \Delta_{P \cap M} = \Delta_P$, we obtain (6.8.4).

Now by applying the hypercohomology spectral sequence, we obtain (6.8.3) from (6.8.4). Thus (6.8.2) holds, and so (6.8.1) is proved.

Since $\sum_{\rho' \in (\mathcal{O}'_{\alpha})^\vee} (\dim \rho') x_{\rho, \rho'} = \dim \rho$, in view of (6.8.1), in order to prove the proposition, it is enough to show the following formula.

(6.8.5) \[\sum_{\rho' \in (\mathcal{O}'_{\alpha})^\vee} (\dim \rho') x_{\rho, \rho'} = \dim \rho.\] 

For an $M_H$-orbit $\mathcal{O}'$ in $M_{\mathcal{O}_{\text{uni}}}$, put $x_{\mathcal{O}'} = \sum_{\rho'} (\dim \rho') x_{\rho, \rho'}$, where the sum is taken over all $\rho' \in (\mathcal{O}'_{\alpha})^\vee$ such that $\mathcal{O}'_{\rho'} = \mathcal{O}'$. Then the sum in the left hand side of (6.8.5) coincides with $\sum_{\rho} x_{\mathcal{O}_{\rho'}}$, where the sum is taken over all $M_H$-orbits $\mathcal{O}'$ in $M_{\mathcal{O}_{\text{uni}}}$. We consider $x_{\mathcal{O}'}$ separately. It follows from the decomposition (6.5.1) and from the definition of $x_{\mathcal{O}_{\rho'}}$, we see that $x_{\mathcal{O}_{\rho'}}$ coincides with the multiplicity of $\mathcal{O}_{\rho}$ in
the local system

\[(6.8.6) \quad \mathcal{H}^{2d_\rho - d}(\pi''_{D_{\rho'}}!((\pi'_*K_{\mathcal{F}_1}|_{D_{\rho'}}))|_{\mathcal{O}_\rho}, \]

where \(D_{\rho'}\) is defined similarly to \(D_0\) by replacing \(\theta''_\rho\) by \(\theta'\).

Here we note that the local system given in (6.8.6) coincides with the local system

\[(6.8.7) \quad \mathcal{H}^{2d_\rho - 2d'_{\rho'}}((\pi''_{D_{\rho'}}!((\mathcal{H}^{2d'_{\rho'}}(\pi'_*K_{\mathcal{F}_1}|_{D_{\rho'}}))|_{\mathcal{O}_\rho}, \]

since the discussion in the proof of (6.8.4) shows that if \(\mathcal{H}^i((\pi''_{D_{\rho'}}!\mathcal{F})|_{\mathcal{O}_\rho} \neq 0\) for a sheaf \(\mathcal{F}\), then \(i \leq (2\nu_H - \text{dim } \theta_\rho) - (2\nu_{M_H} - \text{dim } \theta''_\rho) + \Delta_Q = 2d_\rho - 2d'_{\rho'}\) and if \(\mathcal{H}^j(\pi'_*K_{\mathcal{F}_1}|_{D_{\rho'}} \neq 0\), then \(j \leq 2d'_{\rho'} - d)\).

By applying Theorem 5.2 (iii) to \(M\), and by converting it to the sheaves on \(\tilde{X}_0\) by making use of the diagram (6.4.4), we obtain

\[\mathcal{H}^{2d_\rho - d}(\pi'_*K_{\mathcal{F}_1}|_{D_{\rho'}} \simeq \mathcal{H}^{2d'_{\rho'}}(\pi'|_{\tilde{X}_0})\mathcal{O}_{\mathcal{F}_1}|_{D_{\rho'}}),\]

where \(\tilde{X}_0 \simeq H \times_{P_H} \eta_{P}^{-1}(\Sigma)\). Hence (6.8.7) can be rewritten as

\[\mathcal{H}^{2d_\rho - 2d'_{\rho'}}((\pi''_{D_{\rho'}}!((\mathcal{H}^{2d'_{\rho'}}(\pi'|_{\tilde{X}_0})\mathcal{O}_{\mathcal{F}_1}|_{D_{\rho'}}))|_{\mathcal{O}_\rho}).\]

By using the spectral sequence associated to the composite \(\pi = \pi'' \circ \pi'\), we see that the last formula is equal to

\[(6.8.8) \quad \mathcal{H}^{2d_\rho}((\pi'' \circ \pi'|_{\tilde{X}_{0,\rho'}})\mathcal{O}_{\mathcal{F}_1})|_{\mathcal{O}_\rho},\]

where \(\tilde{X}_{0,\rho'} = \tilde{X}_0 \cap \pi'^{-1}(D_{\rho'})\). In particular, \(x_{\rho'}\) coincides with the multiplicity of \(\mathcal{E}_\rho\) in the local system given in (6.8.8).

Here we note that \(\tilde{X}_0 = \bigsqcup_{\theta'_{\rho'}} \tilde{X}_{0,\theta'_{\rho'}}\) gives a partition of \(\tilde{X}_0\) by locally closed pieces \(\tilde{X}_{0,\theta'_{\rho'}}\), where \(\theta'_{\rho'}\) runs over all the \(M_H\)-orbits in \(M_{\text{uni}}^\theta\). By Proposition 2.2 (ii), \(\text{dim}(\tilde{X}_{0,\theta'_{\rho'}} \cap \pi^{-1}(x)) \leq d_\rho\) for any \(x \in \theta'_{\rho}\). Then by applying the cohomology long exact sequence of \(\pi^{-1}(x)\), we see that \(\sum_{x_{\rho'}} \mathcal{E}_\rho\) coincides with the multiplicity of \(\mathcal{E}_\rho\) in the local system

\[(6.8.9) \quad \mathcal{H}^{2d_\rho}((\pi|_{\tilde{X}_0})\mathcal{O}_{\mathcal{F}_1})|_{\mathcal{O}_\rho}.)\]

On the other hand, the stalk at \(x \in \theta'_{\rho}\) of the local system in (6.8.9) coincides with \(H^{2d_\rho}(\mathcal{P}_{\theta_{L,x}}, \mathcal{O}_{\mathcal{F}_1})\), where \(\mathcal{P}_{\theta_{L,x}} = \{gP_H \in H/P_H \mid g^{-1}xg \in \eta^{-1}_P(\theta_L)\}\). By Theorem 5.2, \(H^{2d_\rho}(\mathcal{P}_{\theta_{L,x}}, \mathcal{O}_{\mathcal{F}_1})\) has a natural structure of \(\mathcal{A}_{\theta_{L}} \times A_{H}(x)\)-module, and is decomposed as
(6.8.10) \[ H_c^{2d_H}(\mathcal{P}_{\mathcal{O}_L,x}, \overline{\mathcal{E}}_1) \simeq \bigoplus_{\overline{\rho} \in \mathcal{A}_{\mathcal{O}_L}} V_{\overline{\rho}} \otimes \tau_{\overline{\rho}}, \]

where \( V_{\overline{\rho}} \) is an \( \mathcal{A}_{\mathcal{O}_L} \)-module, which is isomorphic to \( \overline{\rho} \) if it is non-zero, and \( \tau_{\overline{\rho}} \) is an irreducible representation of \( A_H(x) \) corresponding to the local system \( \mathcal{E}_{\overline{\rho}} \) on \( \mathcal{O}_{\overline{\rho}} = \mathcal{O}_{\overline{\rho}} \). In particular, the multiplicity of \( \mathcal{E}_{\overline{\rho}} \) in the local system in (6.8.9) coincides with \( \dim \rho \) if it is non-zero. But by (6.8.1), we know that \( \sum_{\mathcal{O}_\rho} x_{\mathcal{O}_\rho} \geq \dim \rho \). It follows that \( \sum_{\mathcal{O}_\rho} x_{\mathcal{O}_\rho} = \dim \rho \). Thus we have proved (6.8.5), and the proposition follows. \[ \square \]

6.9. We are now ready to prove the theorem. For an \( H \)-orbit \( \mathcal{O} \) in \( G^{\mathcal{O}_L} \) and an \( M_H \)-orbit \( \mathcal{O}' \) in \( M^{\mathcal{O}_L} \), consider the diagram as in (5.3.1), but by replacing \( P \) by \( Q \), namely,

\[
\begin{array}{ccc}
V = H \times^{Q_H} (\mathcal{O} \cap \eta^{-1}_Q(\mathcal{O}')) & \xrightarrow{f_1} & \mathcal{O} \\
\downarrow{f_2} & & \\
V' = H \times^{Q_H} \mathcal{O}'.
\end{array}
\]

By putting
\[
d_1 = (\nu_H - \dim \mathcal{O}/2) - (\nu_{M_H} - \dim \mathcal{O}'/2) + \Delta_{Q}/2,
d_2 = (\dim \mathcal{O} - \dim \mathcal{O}')/2 + \Delta_{Q}/2.
\]

as in 5.3, we can apply Lemma 5.4 for \( Q \).

Since \( \mathcal{O}' \subset X^M \), we have \( V \subset H \times^{Q_H} (\eta^{-1}_Q(X^M)) \). The lower row of the diagram (6.4.4) gives a diagram, where \( \xi_1, \eta_1 \) are restrictions of \( \xi, \eta \) on \( H \times (\mathcal{O} \cap \eta^{-1}_Q(\mathcal{O}')) \).

\[
H \times^{Q_H} (\mathcal{O} \cap \eta^{-1}_Q(\mathcal{O}')) \xrightarrow{\xi_1} H \times (\mathcal{O} \cap \eta^{-1}_Q(\mathcal{O}')) \xrightarrow{\eta_1} \mathcal{O}'.
\]

By using a similar argument as in 6.4, we obtain an \( H \)-equivariant simple local system \( \hat{\mathcal{E}}' \) on \( V \) from an \( H \)-equivariant simple local system \( \mathcal{E}' \) on \( \mathcal{O}' \). On the other hand, \( V' \) is a single \( H \)-orbit, and its stabilizer is isomorphic to \( Z_{M_H}(v)U_{Q_H} \) for \( v \in \mathcal{O}' \). Hence the set of \( H \)-equivariant simple local systems on \( V' \) is in bijection with the set of \( M_H \)-equivariant simple local systems on \( \mathcal{O}' \). We denote by \( \hat{\mathcal{E}}' \) the local system on \( V' \) corresponding to \( \mathcal{E}' \) on \( \mathcal{O}' \). Here we note that

(6.9.1) \[ f_2^* \hat{\mathcal{E}}' \simeq \hat{\mathcal{E}}'. \]

In fact, since both of \( V, V' \) are locally trivial fibration over \( H/Q_H \), we have the following commutative diagram from the embedding of fibres.
\[ \Theta \cap \eta_{Q}^{-1}(\Theta') \xrightarrow{i} H \times_{Q} (\Theta \cap \eta_{Q}^{-1}(\Theta')) \]

(6.9.2)

where \( f_{\Theta, \Theta'} \) is the map defined in 6.1, and \( i, i' \) are inclusions of fibres. Since \( f_{2}^{\ast} \tilde{\Theta}, \tilde{\Theta}' \) are both \( H \)-equivariant local systems, it is enough to show that their restrictions on \( \Theta \cap \eta_{Q}^{-1}(\Theta') \) coincide. Since the restriction of \( \tilde{\Theta} \) on \( \Theta' \) is equal to \( \Theta' \), the restriction of \( f_{2}^{\ast} \tilde{\Theta}' \) coincides with \( f_{\Theta, \Theta'} \tilde{\Theta}' \). On the other hand, it follows from the construction of \( \tilde{\Theta}' \), one can check that the restriction of \( \tilde{\Theta}' \) on \( \Theta \cap \eta_{Q}^{-1}(\Theta') \) coincides with \( \Theta' \).

Thus (6.9.1) holds.

Recall that \( m_{\Theta, \Theta'} \) is the multiplicity of \( \Theta' \) in the local system \( R^{2d_{\Theta, \Theta'}}(f_{\Theta, \Theta'}) \tilde{\Theta} \). Here \( d_{2} = d_{\Theta, \Theta'} \). Since (6.9.2) is the fibre product, we see that the restriction of \( R^{2d_{\Theta, \Theta'}}(f_{2})f_{1}^{\ast}(\Theta) \) on \( \Theta' \) coincides with \( R^{2d_{\Theta, \Theta'}}(f_{\Theta, \Theta'}) \tilde{\Theta}' \). Hence \( m_{\Theta, \Theta'} \) coincides with the multiplicity of \( \tilde{\Theta}' \) in the local system \( R^{2d_{2}}(f_{2})f_{1}^{\ast}(\Theta) \). Then by using (6.9.1) and Lemma 5.4, we obtain

(6.9.3) \( m_{\Theta, \Theta'} \) coincides with the multiplicity of \( \Theta \) in the local system \( R^{2d_{1}}(f_{1}) \tilde{\Theta}' \).

Here the perverse sheaf \( B_{\rho'} \) on \( \tilde{X}_{Q} \) is constructed from \( A'_{\rho'} \) on \( X^{M} \) by using the diagram (6.4.4). The local system \( \tilde{\Theta}' \) on \( V \) is also constructed from \( \Theta' \) on \( \Theta' \) by (6.4.4). If we write \( (\Theta', \Theta') = (\Theta_{\rho'}, \Theta_{\rho'}) \), one can check that \( \mathcal{H}^{2d_{\rho'} - d} B_{\rho'} \mid V \simeq \tilde{\Theta}' \). Hence by (6.9.3), \( m_{\Theta, \Theta'} \) coincides with the multiplicity of \( \Theta \) in the local system \( R^{2d_{1}}(\pi'_{\mid V})^{\ast}(\mathcal{H}^{2d_{\rho'} - d} B_{\rho'} \mid V) \mid \Theta \). If we write \( (\Theta, \Theta) = (\Theta_{\rho}, \Theta_{\rho}) \), we have \( d_{1} = d_{\rho} - d_{\rho}' \). It follows from the definition of \( x_{\rho, \rho'} \) and (6.8.7), that this multiplicity of \( \Theta' \) actually coincides with \( x_{\rho, \rho'} \). Hence by Proposition 6.8, we conclude that \( m_{\Theta, \Theta'} = (\rho, \rho') \). This proves (ii) of the theorem.

We show (i). Take \( (\Theta', \Theta') \in \mathcal{N}_{M} \) which belongs to \( (L \subset M \cap P, E_{L}, E_{L}^{1}) \). Assume that \( m_{\Theta, \Theta'} \neq 0 \). Since \( \mathcal{H}^{2d_{\rho'} - d} \pi_{\ast} K_{\tilde{\Theta}_{1}} \mid V \) contains \( \tilde{\Theta}' \), the above discussion implies that \( \Theta \) appears in the local system \( R^{2d_{1}}(\pi'_{\mid V})^{\ast} R^{2d_{\rho'}}(\pi'_{\mid X_{0, \Theta}}) \mid (\Theta_{1}) \mid V \) on \( \Theta \) with non-zero multiplicity. This is equivalent to saying that \( \Theta \) appears in the local system \( R^{2d_{1} + 2d_{\rho'}}(\pi'_{\mid X_{0, \Theta}}) \mid (\Theta_{1}) \mid V \) with non-zero multiplicity (since \( \mathrm{dim} \pi^{-1}(x) \cap X_{0, \Theta} \leq d_{1} + d_{\rho'} \) for \( x \in \Theta \)). It follows that \( \Theta \) appears in the local system \( R^{2d_{1} + 2d_{\rho'}}(\pi'_{\mid X_{0, \Theta}}) \mid (\Theta_{1}) \mid V \) with non-zero multiplicity, by considering the partition of \( \tilde{X}_{0} \) by locally closed pieces \( \tilde{X}_{0, \Theta'} \). Now (i) follows from Theorem 5.2 (iii) since \( d_{1} + d_{\rho'} = d_{\Theta} \) in the notation there. This complete the proof of Theorem 6.2.

6.10. By making use of the discussion in 2.3, we reformulate the restriction theorem in a more convenient form for the application. We keep the notation in 6.1. Let \( (\Theta, \Theta) \in \mathcal{N}_{L} \) and \( (\Theta', \Theta') \in \mathcal{N}_{M} \). We fix \( u \in \Theta, v \in \Theta' \). Let \( \tau \in A_{H}(u) \) be the irreducible representation corresponding to \( \Theta \), and \( \tau' \in A_{M_{u}}(v) \) the irreducible
representation corresponding to $\mathcal{E}'$. As in 2.3, but by replacing $L$ by $M$, we define varieties

$$Y_{u,v} = \{ gZ_{M_H}^0(v)U_Q^0 \mid g \in H, g^{-1}ug \in \eta_Q^{-1}(v) \},$$

$$\tilde{Y}_{u,v} = \{ g \in H \mid g^{-1}ug \in \eta_Q^{-1}(v) \}.$$ 

$Z_H(u) \times Z_{M_H}(v)$ acts on $Y_{u,v}$ by $(g_1, g_2) : gZ_{M_H}^0(v)U_Q^0 \mapsto g_1gg_2^{-1}Z_{M_H}^0U_Q^0$. By the discussion in 2.3, we have

$$\dim Y_{u,v} \leq s = (\dim Z_H(u) - \dim Z_{M_H}(v))/2 + \Delta_Q/2.$$ 

Let $I(Y_{u,v})$ be the set of irreducible components in $Y_{u,v}$ of dimension $s$. Then $A_H(u) \times A_{M_H}(v)$ acts on $I(Y_{u,v})$ as the permutations. We denote by $\varepsilon_{u,v}$ the corresponding permutation representation of $A_H(u) \times A_{M_H}(v)$. As a corollary to the restriction theorem, we have the following result (cf [LS, 0.4]).

**Corollary 6.11.** Assume that $(\mathcal{E}, \mathcal{E}) \in \mathcal{N}_G$ corresponds to $\rho \in \mathcal{M}_m^\lambda$ and $(\mathcal{E}', \mathcal{E}') \in \mathcal{N}_m$ corresponds to $\rho' \in (\mathcal{M}_m^\rho)^\land$. Then

$$(6.11.1) \quad \langle \rho, \rho' \rangle_{\mathcal{M}_m^\rho} = \langle \tau \otimes \tau'^*, \varepsilon_{u,v} \rangle_{A_H(u) \times A_{M_H}(v)}.$$ 

**Proof.** Assume that $I(Y_{u,v}) \neq \emptyset$. Then it follows from the discussion in 2.3 that $\dim(\mathcal{E} \cap \eta_Q^{-1}(v)) = d_{\mathcal{E}, \mathcal{E}'} = (\dim \mathcal{E} - \dim \mathcal{E}')/2 + \Delta_Q/2$. As in 6.1, let $m_{\mathcal{E}, \mathcal{E}'}$ be the multiplicity of $\mathcal{E}'$ in the local system $R^{2d_{\mathcal{E}, \mathcal{E}'}(f_{\mathcal{E}, \mathcal{E}'})}$. Then $m_{\mathcal{E}, \mathcal{E}'}$ coincides with the multiplicity of $\tau'^*$ in the $A_{M_H}(v)$-module $H_c^{2d_{\mathcal{E}, \mathcal{E}'}(f_{\mathcal{E}, \mathcal{E}'})}$. We consider a similar diagram as in (2.3.1). We use the same notation there. $Z_H(u) \times Z_{M_H}(v)$ acts on $\tilde{Y}_{u,v}, Z_H^0(u)\tilde{Y}_{u,v}$, and $\phi, \psi$ are $Z_H(u) \times Z_{M_H}(v)$-equivariant. On the other hand, let $\xi : \mathcal{E} = H/\tilde{Z}_H^0(u) \rightarrow \mathcal{E} = H/Z_H(u)$ be the natural map. Then $Z_{M_H}(v)$ acts on $\xi^{-1}(\mathcal{E} \cap \eta_Q^{-1}(v))$ under the isomorphism $\xi^{-1}(\mathcal{E} \cap \eta_Q^{-1}(v)) \simeq Z_H(u)\tilde{Y}_{u,v}$, and the map $\xi^{-1}(\mathcal{E} \cap \eta_Q^{-1}(v)) \rightarrow (\mathcal{E} \cap \eta_Q^{-1}(v))$ becomes $Z_{M_H}$-equivariant. Under this situation, we can identify $H_c^{2d_{\mathcal{E}, \mathcal{E}'}(\mathcal{E} \cap \eta_Q^{-1}(v))}$ with the multiplicity space of $\tau$ in the $A_H(u)$-module $H_c^{2d_{\mathcal{E}, \mathcal{E}'}(\mathcal{E} \cap \eta_Q^{-1}(v))}$. Moreover, this identification is compatible with the action of $A_{M_H}(v)$ on both cohomologies. It follows that $m_{\mathcal{E}, \mathcal{E}'}$ coincides with the multiplicity of $\tau \otimes \tau'^*$ in the $A_H(u) \times A_{M_H}(v)$-module $H_c^{2d_{\mathcal{E}, \mathcal{E}'}(\mathcal{E} \cap \eta_Q^{-1}(v))}$. From the discussion in 2.3, this $A_H(u) \times A_{M_H}(v)$-module is isomorphic to the permutation representation $\varepsilon_{u,v}$ on the set $I(Y_{u,v})$. Thus the corollary follows from Theorem 6.2.

## 7. Unipotent Orbits

**7.1.** We follow the notation in 1.7. Since $g^-$ is the set of self-adjoint matrices in $g = gl_N$, and $G^\theta$ is the set of non-degenerate self-adjoint matrices in $g$, the map
$x \mapsto x - 1$ gives an isomorphism between $G_{\text{uni}}^\theta$ and $\mathfrak{g}_{\text{nil}}$. This map is clearly $H$-equivariant. In this way, we have a natural bijection between the set of $H$-orbits in $G_{\text{uni}}^\theta$ and the set of $H$-orbits in $\mathfrak{g}_{\text{nil}}$.

For an integer $m \geq 0$, let $\mathcal{P}_m$ be the set of partitions $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_r \geq 0$ such that $\sum_i \lambda_i = m$. $m = |\lambda|$ is called the size of $\lambda$, and the maximal number $k = \ell(\lambda)$ such that $\lambda_k \neq 0$ is called the length of $\lambda$. For $\lambda = (\lambda_1, \ldots, \lambda_r)$, put

$$n(\lambda) = \sum_{i=1}^r (i - 1)\lambda_i.$$  

(7.1.1)

It is known that the set of unipotent conjugacy classes in $G = GL_N$ is in bijection with $\mathcal{P}_N$, via Jordan normal form. We denote by $\mathcal{O}_\lambda$ the unipotent class in $G$ corresponding to $\lambda \in \mathcal{P}_N$. Similarly, the set of nilpotent orbits in $\mathfrak{g}_{\text{nil}}$ is in bijection with $\mathcal{P}_N$. We also denote by $\mathcal{O}_\lambda$ the corresponding nilpotent orbit in $\mathfrak{g}_{\text{nil}}$, if there is no fear of confusion. For $x \in \mathcal{O}_\lambda \subset \mathfrak{g}_{\text{nil}}$, we define a Jordan basis $\{v_{ij} \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$ of $V$ with respect to $x$ by $xv_{ij} = v_{i,j-1}$ if $j \geq 2$, and $xv_{i1} = 0$. The following result gives a parametrization of $G^\theta$-orbits in $G_{\text{uni}}^\theta$.

**Lemma 7.2.** For any $\lambda \in \mathcal{P}_N$, $\mathcal{O}_\lambda \cap G_{\text{uni}}^\theta \neq \emptyset$. Any two elements in $\mathcal{O}_\lambda \cap G_{\text{uni}}^\theta$ is conjugate under $G^\theta$. Thus the set of $G^\theta$-orbits in $G_{\text{uni}}^\theta$ is in bijection with $\mathcal{P}_N$.

**Proof.** It is enough to prove the corresponding fact for $\mathfrak{g}_{\text{nil}}$. The argument below is in part due to [CVX1, Lemma 2.2]. First assume that $\mathcal{O}_\lambda \cap \mathfrak{g}_{\text{nil}} \neq \emptyset$. Take $x \in \mathcal{O}_\lambda \cap \mathfrak{g}_{\text{nil}}$, with $\lambda = (\lambda_1, \ldots, \lambda_r)$. We show that there exists a Jordan basis $\{v_{ij}\}$ of $V$ with respect to $x$ satisfying the following property:

$$\langle v_{i,j}, v_{i',j'} \rangle = \begin{cases} 1 & \text{if } i = i' \text{ and } j + j' = \lambda_i - 1, \\ 0 & \text{otherwise.} \end{cases}$$  

(7.2.1)

Since $\langle \ker x^{\lambda_1 - 1}, \im x^{\lambda_1 - 1} \rangle = 0$, there exists $v_1 \in V$ such that $\langle v_1, x^{\lambda_1 - 1}v_1 \rangle \neq 0$. We may assume that $\langle v_1, x^{\lambda_1 - 1}v_1 \rangle = 1$. Then we have $\langle x^iv_1, x^jv_1 \rangle = 0$ unless $i + j = \lambda_1 - 1$, in which case it is equal to 1. Let $W_1$ be the subspace of $V$ spanned by $v_1, xv_1, \ldots, x^{\lambda_1 - 1}v_1$. Then dim $W_1 = \lambda_1$, and the restriction of the bilinear form $\langle \cdot, \cdot \rangle$ on $W_1$ is non-degenerate. Hence we can write as $V = W_1 \oplus W_1^\perp$, and $W_1^\perp$ is stable by $x$. The Jordan type of $x|_{W_1}$ is $\lambda' = (\lambda_2, \ldots, \lambda_r)$. Thus by induction on $N = \dim V$, one can find a Jordan basis $\{v_{ij}\}$ satisfying the property (7.2.1).

If we take another $x' \in \mathcal{O}_\lambda \cap \mathfrak{g}_{\text{nil}}$, we can find a Jordan basis $\{v'_{ij}\}$ satisfying (7.2.1). The map $g : v_{ij} \mapsto v'_{ij}$ determines $g \in G^\theta$, and we have $gx = x'$. Thus $\mathcal{O}_\lambda \cap \mathfrak{g}_{\text{nil}}$ consists of a single $G^\theta$-orbit.

Next we show that $\mathcal{O}_\lambda \cap \mathfrak{g}_{\text{nil}} \neq \emptyset$. Take $x \in \mathcal{O}_\lambda \subset \mathfrak{g}_{\text{nil}}$, and choose a Jordan basis $\{v_{ij}\}$ of $V$ with respect to $x$. We define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $V$ by a similar condition as in (7.2.1). Then $x$ is self-adjoint with respect to this bilinear form, namely, we have $\langle xv, w \rangle = \langle v, xw \rangle$ for any $v, w \in V$. By 1.7, this implies that some $G$-conjugate of $x$ is contained in $\mathfrak{g}_{\text{nil}}$. Hence $\mathcal{O}_\lambda \cap \mathfrak{g}_{\text{nil}} \neq \emptyset$. The lemma is proved. □
7.3. We denote by $O_\lambda$ the $G^\theta$-orbit in $G^\theta_{\text{uni}}$ corresponding to $\lambda \in \mathcal{P}_N$. Also by the same symbol $O_\lambda$, we denote the $G^\theta$-orbit in $g^\theta_{\text{nil}}$. For $x \in O_\lambda \subset g^\theta_{\text{nil}}$, a Jordan basis $\{v_{i,j}\}$ of $V$ with respect to $x$ satisfying the property in (7.2.1) is called a normal basis of $x$.

We shall compute $\dim Z_{G^\theta}(x)$ for $x \in O_\lambda$. First we recall the well-known result in the case of $Z_G(x)$ for $x \in O_\lambda$. Take $x \in O_\lambda \subset g_{\text{nil}}$, and put $E_x = \{y \in \text{End}(V) \mid xy = yx\}$. Note that $\dim Z_G(x) = \dim E_x$. Let $\{v_{i,j}\}$ be a Jordan basis of $V$ with respect to $x$. Following [AH, Proposition 2.8], we define

$y_{i_1,i_2,s}(v_{i,j}) = \begin{cases} v_{i_2,j-s} & \text{if } i = i_1, s + 1 \leq j \leq \lambda_i, \\ 0 & \text{otherwise.} \end{cases}$

Then

$\{y_{i_1,i_2,s} \mid i_1, i_2 \leq \ell(\lambda), \max\{0, \lambda_{i_1} - \lambda_{i_2}\} \leq s \leq \lambda_{i_1} - 1\}$

gives rise to a basis of the $k$-vector space $E_x$. Hence we have

(7.3.1) $\dim E_x = \sum_{i_1 \geq i_2} \lambda_{i_2} + \sum_{i_1 < i_2} \lambda_{i_1} = N + 2n(\lambda)$.

We show a lemma.

**Lemma 7.4.** For $x \in O_\lambda \subset G^\theta_{\text{uni}}$, we have $\dim Z_{G^\theta}(x) = n(\lambda)$.

**Proof.** It is enough to show the corresponding fact for $x \in O_\lambda \subset g^\theta_{\text{nil}}$. Thus we have only to show that $\dim Z_{g^\theta}(x) = n(\lambda)$. Here $E_x = Z_g(x)$ is stable under $\theta$, and we can decompose $E_x = (E_x)^+ \oplus (E_x)^-$, where $(E_x)^+ = Z_{g^+}(x)$. We choose a normal basis of $x$, and define $y_{i_1,i_2,s}$ by using this basis. Then by 1.7,

$\{y_{i_1,i_2,s} + y_{i_2,i_1,s+(\lambda_{i_2} - \lambda_{i_1})} \mid i_1 \leq i_2, 0 \leq s \leq \lambda_{i_1} - 1\}$

gives a basis of $(E_x)^-$ and

$\{y_{i_1,i_2,s} - y_{i_2,i_1,s+(\lambda_{i_2} - \lambda_{i_1})} \mid i_1 < i_2, 0 \leq s \leq \lambda_{i_1} - 1\}$

gives a basis of $(E_x)^+$. It follows that

$\dim(E_x)^+ = n(\lambda), \quad \dim(E_x)^- = n(\lambda) + N.$

The lemma follows from this. \qed

**Remark 7.5.** In the case where $k = \mathbb{C}$, the lemma follows from the general result of Kostant-Rallis [KR]. In fact, by [KR, Proposition 5], we have

(7.5.1) $\dim[\mathfrak{g}^+, x] = \dim[\mathfrak{g}, x]/2$
for $x \in \mathfrak{g}^\circ$. Now assume that $x \in \mathcal{O}_\lambda$. By (7.3.1), \( \dim[\mathfrak{g}, x] = \dim \mathbb{O}_\lambda = N^2 - (N + 2n(\lambda)) \). If we notice that \( \dim G^\theta = (N^2 - N)/2 \), (7.5.1) implies that \( \dim \mathcal{O}_\lambda = \dim G^\theta - n(\lambda) \), and the lemma follows.

7.6. Take \( x \in \mathcal{O}_\lambda \subset \mathfrak{g}_{\text{nil}} \). We write \( \lambda \in \mathcal{P}_N \) as \( \lambda = (a_1^{m_1}, a_2^{m_2}, \ldots, a_h^{m_h}) \) with \( a_1 > a_2 > \cdots > a_h > 0 \). Then by using a normal basis of \( x \), we have a decomposition of \( V \) into \( x \)-stable subspaces

\[
(7.6.1) \quad V = \bigoplus_{1 \leq i \leq h} \tilde{J}_i,
\]

where \( \tilde{J}_i \)'s are mutually orthogonal, and the restriction of \( x \) on \( \tilde{J}_i \) has the Jordan type \( (a_i^{m_i}) \). By this decomposition, we have an embedding of groups

\[
(7.6.2) \quad \prod_{1 \leq i \leq h} G_i^\theta \subset G^\theta,
\]

where \( G_i = GL(\tilde{J}_i) \) is a \( \theta \)-stable subgroup of \( G \). One can find a subspace \( J_i \) of \( \tilde{J}_i \) such that \( \dim J_i = m_i \) and that \( \tilde{J}_i = J_i \oplus x_1 J_i \oplus \cdots \oplus x^{a_i-1} J_i \). We define a quadratic form \( Q_i \) on \( J_i \) by

\[
(7.6.3) \quad Q_i(v) = \langle v, x^{a_i-1}v \rangle, \quad (v \in J_i).
\]

Then \( Q_i \) is non-degenerate on \( J_i \). Let \( O_i = O(J_i) \) be the orthogonal group on \( J_i \) with respect to the quadratic form \( Q_i \). For each \( g \in O_i \), we define a map \( \tilde{g} : \tilde{J}_i \to \tilde{J}_i \) by \( \tilde{g}(k^x v) = x^k (gv) \) for \( 0 \leq k \leq a_i - 1, v \in J_i \). Then \( \tilde{g} \) commutes with the action of \( x \) on \( \tilde{J}_i \), and \( \tilde{g} \) preserves the bilinear form \( \langle \cdot, \cdot \rangle \) on \( \tilde{J}_i \). Hence \( \tilde{g} \in Z_{G^\theta}(x) \). It is clear that \( g \mapsto \tilde{g} \) gives a homomorphism \( O_i \to Z_{G^\theta}(x) \) whose image is isomorphic to \( O_i \). Thus we can construct a closed subgroup of \( Z_{G^\theta}(x) \) which is isomorphic to \( \prod_{1 \leq i \leq h} O_i \). Actually, this subgroup is the reductive part of \( Z_{G^\theta}(x) \), and, as in the classical case, one can show that

\[
(7.6.4) \quad Z_{G^\theta}(x) \simeq \left( \prod_{1 \leq i \leq h} O_i \right) \rtimes U_1,
\]

where \( U_1 \) is a connected unipotent normal subgroup of \( Z_{G^\theta}(x) \).

7.7 Put \( A_{G^\theta}(x) = Z_{G^\theta}(x)/Z_{G^\theta}^0(x) \) for \( x \in \mathcal{O}_\lambda \). Then by (7.6.4), \( A_{G^\theta}(x) \simeq (\mathbb{Z}/2\mathbb{Z})^h \). We write an element \( \alpha \in A_{G^\theta}(x) \) as \( \alpha = (\alpha_1, \ldots, \alpha_h) \) with \( \alpha_i = \pm 1 \), where \( \alpha_i = -1 \) is the generator of the \( i \)-th component \( \mathbb{Z}/2\mathbb{Z} \). Let \( A_H(x) = Z_H(x)/Z_H^0(x) \) as before. Since \( Z_{G^\theta}^0(x) = Z_H^0(x) \), \( A_H(x) \) is a subgroup of \( A_{G^\theta}(x) \). It follows from the discussion in 7.6, we see easily that...
A partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ is called an even partition if $\lambda_i$ is even for each $i$. Then (7.7.1) can be written as

\begin{equation}
A_H(x) = \{ \alpha = (\alpha_1, \ldots, \alpha_h) \in A_{G^\theta}(x) \mid \prod_{a_i \text{ odd}} \alpha_i = 1 \}.
\end{equation}

Proposition 7.8. 
(i) Assume that $N$ is odd. Then each $G^\theta$-orbit $O_\lambda$ is a single $H$-orbit.

(ii) Assume that $N$ is even. If $\lambda$ is not an even partition, $O_\lambda$ is a single $H$-orbit. If $\lambda$ is an even partition, $O_\lambda$ splits into two $H$-orbits.

Proof. Since $G^\theta = H \cup H\zeta$ for some $\zeta \in G^\theta - H$, we have $G^\theta x = Hx \cup H(\zeta x)$. Thus $G^\theta x = Hx$ if $\zeta \in Z_{G^\theta}(x)$, namely, if $Z_{G^\theta}(x) \supseteq Z_H(x)$. If $N$ is odd, $-1 \in Z_{G^\theta}(x) - Z_H(x)$ satisfies this condition. If $N$ is even and $\lambda$ is not an even partition, then $Z_{G^\theta}(x) \supseteq Z_H(x)$ by (7.7.2). Hence $O_\lambda$ is a single $H$-orbit. Now assume that $N$ is even and $\lambda$ is an even partition. In this case, $Z_{G^\theta}(x) = Z_H(x)$. We have

\[ G^\theta/Z_{G^\theta}(x) \simeq (H \prod \zeta H)/Z_H(x) \simeq H/Z_H(x) \prod \zeta H/Z_H(x). \]

Hence $O_\lambda$ splits into two $H$-orbits.

\[ \square \]

7.9. Let $O_\lambda$ be the regular unipotent class in $G_{uni}$, where $\lambda = (N) \in \mathcal{P}_N$. Then $O_\lambda$ is open dense in $G_{uni}$. Since $O_\lambda \cap G^\theta_{uni} \neq \emptyset$, $O_\lambda \cap G^\theta_{uni}$ is an open dense subset of $G^\theta_{uni}$. In the case where $N$ is odd, $O_\lambda \cap G^\theta_{uni} = O_\lambda$ is a single $H$-orbit. Thus $O_\lambda$ is the unique $H$-orbit in $G^\theta_{uni}$ such that $G^\theta_{uni} = O_\lambda$. In particular, $G^\theta_{uni}$ is an irreducible variety. In the case where $N$ is even, $O_\lambda = O_\lambda \cap G^\theta_{uni}$ splits into two $H$-orbits since $(N)$ is an even partition. Thus if we decompose $O_\lambda$ into two $H$-orbits, $O_\lambda = O'_\lambda \cup O''_\lambda$, $G^\theta_{uni} = \overline{O'_\lambda} \cup \overline{O''_\lambda}$ gives a decomposition of $G^\theta_{uni}$ into irreducible components. Summing up the above arguments, we have

Lemma 7.10. Let $\lambda = (N) \in \mathcal{P}_N$.

(i) Assume that $N$ is odd. Then $G^\theta_{uni}$ is irreducible, which is the closure of the $H$-orbit $O_\lambda$.

(ii) Assume that $N$ is even. Then $G^\theta_{uni}$ consists of two irreducible components, which are the closures of $H$-orbits $O'_\lambda$ and $O''_\lambda$. We have $\overline{O'_\lambda} \cap \overline{O''_\lambda} = \overline{O'_\lambda}$, where $O'_\lambda$ is the $H$-orbit corresponding to $\lambda' = (N - 1, 1) \in \mathcal{P}_N$, and $\overline{O'_\lambda}$ is irreducible. In particular, we have

\[ \overline{O'_\lambda} - \overline{O'_{\lambda'}} = O'_\lambda, \quad \overline{O''_\lambda} - \overline{O'_{\lambda'}} = O''_\lambda. \]

Proof. It is enough to show the latter statement of (ii). It is well-known that $O_{\lambda'}$ is the subregular unipotent class in $G_{uni}$ and that $G_{uni} - O_{\lambda'} = \overline{O_{\lambda'}}$. Since $\overline{O_{\lambda'}} =
\[ \bigcup_{\mu \leq \lambda} \mathcal{O}_\mu \] (here \( \mu \leq \lambda \) is the dominance order on \( \mathcal{P}_N \)), we have \( G_{\text{uni}}^\theta - \mathcal{O}_\lambda = \overline{\mathcal{O}}_\lambda \). (ii) follows from this. Note that \( \overline{\mathcal{O}}_\lambda \) is a single \( H \)-orbit since \( \lambda \) is not an even partition, so \( \overline{\mathcal{O}}_\lambda \) is irreducible. \( \square \)

**7.11.** Let \( \lambda = (N) \in \mathcal{P}_N \). We discuss the relationship between the regular unipotent orbit \( \mathcal{O}_\lambda \) and the varieties \( X_{\text{uni}} \) appeared in 5.9. We consider the case where \( P = B, L = T, \) and \( \Sigma = T \). Let \( Y = Y_{(L, \Sigma)} \) and \( X = \overline{Y} \). Then \( X_{\text{uni}} \) is given by

\[
(7.11.1) \quad X_{\text{uni}} = \bigcup_{g \in H} g U^\sigma g^{-1},
\]

where \( U \) is the unipotent radical of \( B \). Under the notation in 1.7, we define \( x - 1 \in g_{\text{nil}} \) by

\[
(x - 1) : f_1 \mapsto f_2 \mapsto \cdots \mapsto f_n \mapsto e_0 \mapsto e_n \mapsto \cdots \mapsto e_2 \mapsto e_1 \mapsto 0
\]
in the case where \( N \) is odd, and by

\[
(x - 1) : f_1 \mapsto f_2 \mapsto \cdots \mapsto f_n \mapsto e_n \mapsto \cdots \mapsto e_2 \mapsto e_1 \mapsto 0
\]
in the case where \( N \) is even. Then \( x \in U^\sigma \cap \mathcal{O}_\lambda \). In particular, \( X_{\text{uni}} \) contains an \( H \)-orbit containing \( x \).

Assume that \( N \) is odd. In this case \( \mathcal{O}_\lambda \) is a single \( H \)-orbit, and is contained in \( X_{\text{uni}} \). Since \( X_{\text{uni}} \) is irreducible, \( X_{\text{uni}} \) is a closed subset of \( G_{\text{uni}}^\theta \), and \( \overline{\mathcal{O}}_\lambda = G_{\text{uni}}^\theta \), we conclude that

\[
(7.11.2) \quad X_{\text{uni}} = G_{\text{uni}}^\theta.
\]

Assume that \( N \) is even. In this case, we need to consider the pairs \((B, T)\) and \((B_1, T)\), where \( B_1 = t_n B t_n^{-1} \) as in 1.9. Put \( U_1 = t_n U t_n^{-1} \). Then \( X_{\text{uni}} \) associated to \((B, T)\) is defined as in (7.11.1), which we denote by \( X_{\text{uni}}^+ \). A similar variety is defined by replacing \( U^\sigma \) by \( U_1^\sigma \) in (7.11.1), which we denote by \( X_{\text{uni}}^- \). Note that \( X_{\text{uni}}^- = \zeta X_{\text{uni}}^+ \zeta^{-1} \) for \( \zeta \in G^\theta \). Let \( \mathcal{O}_\lambda^+ \) be the \( H \)-orbit containing \( x \). Then another \( H \)-orbit contained in \( \mathcal{O}_\lambda \) is given by \( \mathcal{O}_\lambda^- = \zeta \mathcal{O}_\lambda^+ \zeta^{-1} \). We have \( \mathcal{O}_\lambda^+ \subset X_{\text{uni}}^+ \) and \( \mathcal{O}_\lambda^- \subset X_{\text{uni}}^- \). Note that \( X_{\text{uni}}^\pm \) are irreducible, closed subsets of \( G_{\text{uni}}^\theta \). By Lemma 7.10 (ii), \( \overline{\mathcal{O}}_\lambda^+ \cup \overline{\mathcal{O}}_\lambda^- \) gives a decomposition of \( G_{\text{uni}}^\theta \) into irreducible components. This implies that

\[
(7.11.3) \quad X_{\text{uni}}^+ = \overline{\mathcal{O}}_\lambda^+, \quad X_{\text{uni}}^- = \overline{\mathcal{O}}_\lambda^-,
\]

and \( G_{\text{uni}}^\theta = X_{\text{uni}}^+ \cup X_{\text{uni}}^- \) gives a decomposition into irreducible components.
8. STRUCTURE OF THE ALGEBRA $\mathcal{A}_{\mathfrak{g}_1}$

8.1. We follow the notion in 6.1. Let $Q$ be as in 6.1. Here we assume that $M_H \simeq GL_1 \times SO_{N-2}$. Hence $Q_H$ is a maximal parabolic subgroup of $H$, and $H/Q_H$ can be identified with the set of isotropic lines in $P(V)$. Let $\mathcal{O}$ be an $H$-orbit in $G_{\mathfrak{un}_1}^\theta$ and $\mathcal{O}'$ an $M_H$-orbit in $M_{\mathfrak{un}_1}^\theta$. We fix $u \in \mathcal{O}, v \in \mathcal{O}'$, and consider the varieties, $Y_{u,v}, \tilde{Y}_{u,v}$ as in 6.10. Let $\mathcal{O}'$ be the $M^\theta$-orbit in $M_{\mathfrak{un}_1}^\theta$ containing $\mathcal{O}'$. Put

\begin{equation}
\tilde{Y}_{u,\mathcal{O}'} = \{ g \in H \mid g^{-1}ug \in \eta_Q^{-1}(\mathcal{O}') \}.
\end{equation}

Then $Q_H$ acts on $\tilde{Y}_{u,\mathcal{O}'}$ by $x : g \mapsto gx^{-1}$, and $\tilde{Y}_{u,\mathcal{O}'}/Q_H$ is a locally closed subset of $H/Q_H$. Let $\tau : \tilde{Y}_{u,\mathcal{O}'} \to \mathcal{O}'$ be the map defined by $g \mapsto \eta_Q(g^{-1}ug)$. Then $\tau$ is $M_H$-equivariant, and for any $v' \in \mathcal{O}'$, $\tau^{-1}(v') \simeq \tilde{Y}_{u,v}$. Hence $\dim \tilde{Y}_{u,\mathcal{O}'} = \dim \tilde{Y}_{u,v} + \dim \mathcal{O}'$, and we have

\begin{equation}
\dim \tilde{Y}_{u,\mathcal{O}'}/Q_H = \dim \tilde{Y}_{u,v} + \dim \mathcal{O}' - \dim Q_H.
\end{equation}

8.2. Let $\mathfrak{m} = \text{Lie } M$, and consider the subvariety $\mathfrak{m}_{\mathfrak{nil}}^-$ of $\mathfrak{m}$ on which $M_H$ acts. We denote by the same symbol $\mathcal{O}$ (resp. $\mathcal{O}'$) the $H$-orbit in $\mathfrak{g}_{\mathfrak{nil}}^-$ (resp. $M_H$-orbit in $\mathfrak{m}_{\mathfrak{nil}}^-$) corresponding to $\mathcal{O}$ (resp. $\mathcal{O}'$) as in Section 7. In the Lie algebra setting, the map $\eta_Q : \mathfrak{q}^{-\theta} \to \mathfrak{m}^{-\theta}$ is defined similarly, where $\mathfrak{q} = \text{Lie } Q$, and $\tilde{Y}_{x,\mathcal{O}'}$ is defined, for $x \in \mathcal{O} \subset \mathfrak{g}_{\mathfrak{nil}}^-$, similarly to (8.1.1). We have $\tilde{Y}_{u,\mathcal{O}'} = \tilde{Y}_{x,\mathcal{O}'}$ for $x = u-1$.

By Lemma 7.2, the set of $G^\theta$-orbits in $G_{\mathfrak{un}_1}^\theta$ is parametrized by $\mathcal{P}_N$, $\mathcal{O}$ corresponds to $\lambda \in \mathcal{P}_N$ if $u \in \mathcal{O}$ has Jordan type $\lambda$ as an element in $G_{\mathfrak{un}_1}^\theta$. Similarly, the set of $M^\theta$-orbits in $M_{\mathfrak{un}_1}^\theta$ is parametrized by $\mathcal{P}_{N-2}$.

Now assume that $\mathcal{O}$ corresponds to $\lambda \in \mathcal{P}_N$, and $\mathcal{O}'$ corresponds to $\lambda' \in \mathcal{P}_{N-2}$. Take $x \in \mathcal{O} \subset \mathfrak{g}_{\mathfrak{nil}}^-$. Put $W = \text{Ker } x$, and let $P(W)^0$ be the set of isotropic lines in $P(W)$, namely, $P(W)^0 = \{ (v) \in P(W) \mid \langle v, v \rangle = 0 \}$. Note that the form $\langle , \rangle$ induces a non-degenerate symmetric bilinear form on $\overline{V} = V_1^+ / V_1$ for $V_1 \in P(W)^0$, and $x|_{\overline{V}}$ is self-adjoint with respect to this form. Here $gl(\overline{V})$ is $H$-conjugate to $\mathfrak{m}$, and $x|_{\overline{V}}$ gives an element in $\mathfrak{m}_{\mathfrak{nil}}^-$ under this isomorphism. It is easy to see that

\begin{equation}
\tilde{Y}_{x,\mathcal{O}'}/Q_H \simeq \{ V_1 \in P(W)^0 \mid x|_{V_1^+ / V_1} : \text{Jordan type } \lambda' \},
\end{equation}

8.3. We will see the variety on the right hand side on (8.2.1) more precisely. Write $\lambda \in \mathcal{P}_N$ as $\lambda = (a_1^{m_1}, a_2^{m_2}, \ldots, a_h^{m_h})$ as in 7.6, and put $W^i = \text{Ker } x \cap \text{Im } x^{a_i - 1}$. We have a filtration of $W$ by subspaces

\begin{equation}
W = W^h \supseteq W^{h-1} \supseteq \cdots \supseteq W^1 \supseteq W^0 = \{ 0 \},
\end{equation}

where $\dim W^i / W^{i-1} = m_i$ for $i = 1, \ldots, h$. For $v, v' \in W^i$, put $\langle v, v' \rangle = \langle v, v' \rangle$, where $v' \in V$ is such that $x^{a_i - 1}v_1 = v'$. By using a normal basis of $x \in \mathfrak{g}_{\mathfrak{nil}}^-$, one can check that this gives a well-defined symmetric bilinear form on $W^i$, which is identically zero on $W^{i-1}$. The induced form $\langle , \rangle$ on $W^i / W^{i-1}$ gives a non-degenerate symmetric bilinear form on $W^i / W^{i-1}$. For each non-zero $v \in W^i$, let $V_1 = \langle v \rangle$ be the
line spanned by \( v \). Since \( v \in \ker x \), \( x \) induces a linear map \( \overline{x} = x|_{\overline{V}} \) on \( \overline{V} = V^i_1/V_1 \). We have \( \overline{x} \in \mathfrak{gl}(\overline{V})_{\text{nil}} \cong m_{\text{nil}} \). Define a variety \( S_i \) of \( V \) such that \( W^{i-1} \subset S_i \subset W^i \) by \[
abla \triangleq \{ v \in W^i \mid (v,v) = 0 \}.
\]

We consider the following conditions on \( \lambda \in \mathcal{P}_N \) and \( \lambda' \in \mathcal{P}_{N-2} \).

(A) The Young diagram of \( \lambda' \) is obtained from that of \( \lambda \) by replacing one row of length \( a_i \) by a row of length \( a_i - 2 \).

(B) The Young diagram of \( \lambda' \) is obtained from that of \( \lambda \) by replacing two rows of length \( a_i \) by two rows of length \( a_i - 1 \) (in this case, we assume that \( m_i \geq 2 \)).

Moreover, we divide (A) into two cases (A\(_i\)) and (A\(_i\)\(_\prime\)). Here (A\(_i\)) is the case where \( a_{i+1} \leq a_i - 2 \), and (A\(_i\)\(_\prime\)) is the case where \( a_{i+1} = a_i - 1 \).

The following lemma can be checked easily, by using a normal basis of \( x \). (It is reduced to the case where \( \lambda = (a^m) \) with \( m = 1 \) or 2).

Lemma 8.4. Under the notation above, let \( \lambda' \) be the Jordan type of \( \overline{x} \in \mathfrak{gl}(\overline{V})_{\text{nil}} \). Then we have

(i) If \( v \notin S_i \), \( \lambda' \) is obtained from \( \lambda \) by the procedure (A).

(ii) If \( v \in S_i \) (in this case \( m_i \geq 2 \)), \( \lambda' \) is obtained from \( \lambda \) by the procedure (B).

The following corollary is immediate from Lemma 8.4.

Corollary 8.5. \( \overline{Y}_{x,\theta'} = \emptyset \) unless \( \lambda' \) is obtained from \( \lambda \) by (A\(_i\)), (B\(_i\)) for some \( i \). If \( \overline{Y}_{x,\theta'} \neq \emptyset \), then

\[
\dim \overline{Y}_{x,\theta'}/Q_H = \begin{cases} 
    m_1 + \cdots + m_i - 1, & \text{case (A)}_i, \\
    m_1 + \cdots + m_i - 2, & \text{case (B)}_i.
\end{cases}
\]

8.6. Recall that \( \dim Y_{u,v} \leq s \), where \( s = (\dim Z_H(u) - \dim Z_{M_H}(v))/2 + \Delta_\theta/2 \) by 6.10. Since we know by Lemma 7.4 that \( \dim Z_H(u) = n(\lambda) \) if \( u \in G_{\text{uni}}^\theta \) has Jordan type \( \lambda \), the number \( s \) can be computed explicitly. We have the following.

Proposition 8.7. Assume that \( u \in \mathcal{O} \) has type \( \lambda \), \( v \in \mathcal{O}' \) has type \( \lambda' \). Then \( Y_{u,v} \) is non-empty if and only if \( \lambda' \) is obtained from \( \lambda \) by the procedure (A\(_i\)), or (B\(_i\)). We have \( \dim Y_{u,v} = s \) if and only if \( \lambda' \) is obtained from \( \lambda \) by (A\(_i\)).

Proof. Since \( Y_{u,v} \neq \emptyset \) if and only if \( \overline{Y}_{x,\theta'} \neq \emptyset \), the first assertion follows from Corollary 8.5. Now assume \( Y_{u,v} \neq \emptyset \). Since \( \dim Y_{u,v} = \dim \overline{Y}_{x,\theta'}/Q_H \), \( \dim Y_{u,v} \) can be computed from the formula in Corollary 8.5. Since \( M_H \cong GL_1 \times SO_{N-2} \), \( \dim Z_{M_H}(v) = n(\lambda') + 1 \) by Lemma 7.4. Then we have

\[
(8.7.1) \quad \dim Z_H(u) - \dim Z_{M_H}(v) = \begin{cases} 
    2(m_1 + \cdots + m_i - 1) - 1, & \text{case (A)}_i, \\
    2(m_1 + \cdots + m_i - 1), & \text{case (A)}_i', \\
    2(m_1 + \cdots + m_i - 2), & \text{case (B)}_i.
\end{cases}
\]
Since $\Delta_Q = 1$, we have

$$s = \begin{cases} m_1 + \cdots + m_i - 1 & \text{case } (A'_i), \\ (m_1 + \cdots + m_i - 1) + 1/2 & \text{case } (A''_i), \\ (m_1 + \cdots + m_i - 2) + 1/2 & \text{case } (B_i). \end{cases}$$

The proposition follows. \qed

8.8. For $x \in g_{\text{nil}}$, let $\mathcal{B}_x = \{gB_H \in H/B_H \mid g^{-1}x \in \text{Lie } B_H\}$ be the Springer fibre of $x$. We also define $\mathcal{P}_x = \{gQ_H \in H/Q_H \mid g^{-1}x \in \text{Lie } Q_H\}$ and a natural map $\pi_Q : \mathcal{B}_x \to \mathcal{P}_x$, $gB_H \mapsto gQ_H$. $\mathcal{P}_x$ can be identified with $P(W)^0$. Thus $\mathcal{P}_x$ is partitioned into locally closed pieces $\mathcal{P}_{x,\lambda} \simeq \bar{Y}_{x,\theta'/Q_H}$, where $\theta'$ is the $M^\theta$-orbit corresponding to $\lambda' \in \mathcal{P}_{N-2}$, and we have a partition of $\mathcal{B}_x$ into locally closed pieces

(8.8.1) \[ \mathcal{B}_x = \coprod_{\lambda' \in \mathcal{P}_{N-2}} \pi_Q^{-1}(\mathcal{P}_{x,\lambda'}). \]

Assume that $x \in \mathcal{O}$ has type $\lambda$. By Lemma 8.4, if $\pi_Q^{-1}(\mathcal{P}_{x,\lambda'}) \neq \emptyset$, then $\lambda'$ is obtained from $\lambda$ by the procedure given in (i) or (ii) in Lemma 8.4. If $v \in W$ is given, $\bar{x} \in g(V)^{-\theta}$ is defined as above. We denote by $\mathcal{B}_{\bar{x}}$ the corresponding Springer fibre for $SO(V)$. By applying Proposition 2.2 (ii) for the case $P = B, L = T$, we have

(8.8.2) \[ \dim \mathcal{B}_x \leq \nu_H - \dim \mathcal{O} + \Delta_B/2 \]

\[ = \dim Z_H(x)/2, \]

where $\Delta_B = [N/2] = \dim T_H$.

The following result was proved by [CVX1, Proposition 3.1] in the case where $N$ is odd, and $k = \mathbb{C}$. The discussion below seems to be simpler. (But this result will not be used in later discussions.)

**Proposition 8.9.** Assume that $x \in \mathcal{O}$ has type $\lambda$. Then $\dim \mathcal{B}_x = \dim Z_H(x)/2$ if and only if $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathcal{P}_N$ satisfies the condition that $\lambda_i$ is even for $i \geq 2$ (hence $\lambda_1$ is odd (resp. even) if $N$ is odd (resp. even)).

**Proof.** In the notation of 8.8, the restriction of the map $\pi_Q : \mathcal{B}_x \to \mathcal{P}_x$ on $\pi_Q^{-1}(\mathcal{P}_{x,\lambda'})$ is surjective, and each fibre is isomorphic to $\mathcal{B}_{\bar{x}}$. Hence

$$\dim \pi_Q^{-1}(\mathcal{P}_{x,\lambda'}) = \dim \mathcal{P}_{x,\lambda'} + \dim \mathcal{B}_{\bar{x}},$$

and $\dim \mathcal{B}_x = \max\{\dim \pi_Q^{-1}(\mathcal{P}_{x,\lambda'}) \mid \lambda' \in \mathcal{P}_{N-2}\}$. $\dim \mathcal{P}_{x,\lambda'}$ is given in Corollary 8.5. If we denote $\mathcal{H} = SO(V)$, then $\dim \mathcal{B}_{\bar{x}} \leq \dim Z_{\mathcal{H}}(\bar{x})/2$, and $\dim Z_{\mathcal{H}}(\bar{x}) = n(\lambda')$. Thus by using a similar argument as in the proof of Proposition 8.7, we see that $\dim \mathcal{B}_x = \dim Z_H(x)/2$ if and only if there exists $i$ such that $\lambda' \in \mathcal{P}_{N-2}$ is obtained...
from $\lambda$ by $(A'_i)$ and that $\dim \overline{\mathcal{B}_\tau} = \dim Z_{\overline{\mathcal{F}}(\tau)}/2$. The proposition follows from this by induction on $N$. \hfill \Box

8.10. $Z_H(x)$ acts on $\tilde{Y}_{x,\sigma'}/Q_H$. Put $C_i = W^i - S_i$, and let $p(C_i), p(S_i^*)$ be the image of $C_i, S_i^* = S_i - \{0\}$ under the map $p : W^i - \{0\} \to P(W)$. Then by Lemma 8.4, the variety $\tilde{Y}_{x,\sigma'}/Q_H$ is isomorphic to $p(C_i)$ or $p(S_i^*)$ if $\sigma'$ corresponds to $\lambda'$, which is obtained from $\lambda$ as in Lemma 8.4 for some $i$. The corresponding action of $Z_H(x)$ on $p(C_i)$ or $p(S_i^*)$ is given as follows; The natural action of $Z_H(x)$ on $V$ leaves $W^i$ invariant. Moreover, the bilinear form $(\ , )$ on $W^i$ is $Z_H(x)$-invariant. Hence $S_i$ and $C_i$ are stable by the action of $Z_H(x)$. The induced action on $p(S_i^*)$ or $p(C_i)$ coincides with the action of $Z_H(x)$ on $\tilde{Y}_{x,\sigma'}/Q_H$. We show the following lemma.

Lemma 8.11. (i) The action of $Z_H(u)$ on $\tilde{Y}_{u,\sigma'}/Q_H$ is transitive.

(ii) The action of $Z_H(u) \times Z_{M_H}(v)$ on $Y_{u,v}$ is transitive.

Proof. (i) It is enough to show that $Z_H(x)$ acts transitively on $p(C_i)$ or $p(S_i^*)$. Here we only show the case of $p(C_i)$, since this is the case needed in later discussions. The case of $p(S_i^*)$ is proved similarly. Put $a = \dim W^i, b = m_i = \dim W^i/W^{i-1}$ and $r = [b/2]$. Take $v \in C_i$. Then there exists a basis $\{v_1, \ldots, v_a\}$ of $W^i$ such that $\{v_{b+1}, \ldots, v_a\}$ gives a basis of $W^{i-1}$, satisfying the condition that $v = v_1 + v_b$ and that, for any $w = \sum_j x_j v_j \in W^i$,

$$(w, w) = x_1 x_b + x_2 x_{b-1} + \cdots + x_r x_{b-r} + \delta x_{r+1}^2,$$

where $\delta = 1$ if $b$ is odd and $\delta = 0$ if $b$ is even. Let $C_i^0$ be the set of $w = \sum_j x_j v_j \in C_i$ such that $x_j \neq 0$. Thus $C_i^0$ is an open dense subset of $C_i$ containing $v$. We show (8.11.1) For any $w \in C_i^0$, there exists $\phi_w \in Z_H(x)$ such that $\phi_w(v) = \gamma^{-1}w$, where we put $(w, w) = \gamma^2 \in k^*$. In fact, for a given $w \in C_i^0$, we choose $\gamma$ such that $(w, w) = \gamma^2$, and define a map $\phi'_w : W^i \to W^i$ by

$$\left\{\begin{align*}
v &\mapsto \gamma^{-1}w, \\
v_b &\mapsto x_1^{-1}v_b, \\
v_j &\mapsto v_j - x_{b-j+1}v_b &\text{if } 1 < j < b, \\
v_j &\mapsto v_j &\text{if } j > b.
\end{align*}\right.$$

It is easy to check that $\phi'_w$ preserves the symmetric bilinear form $(\ , )$ on $W^i$, and the restriction of $\phi'_w$ on $W^{i+1}$ is identity. Moreover $\det \phi'_w = 1$. For $\lambda = (\lambda_1, \ldots, \lambda_r)$, let $\{w_{i,j}\}$ be a normal basis of $x$. For any $i$, we define $V^i$ as the subspace of $V$ spanned by $\{w_{k,j} \ | \ 1 \leq k \leq i\}$. Then $V^i$ is an $x$-stable subspace of $V$ such that $V^i \cap \text{Ker } x = W^{i+1}$. The linear map $\phi'_w$ on $W^i$ can be extended in a canonical way to a linear automorphism $\phi'_w$ on $V^i$ which commutes with $x$, and preserves the form $(\ , )$. We extend $\phi'_w$ to a linear map $\phi_w$ on $V$ by defining $w_{k,j} \mapsto w_{k,j}$ for any $k > a$. Then $\phi_w \in Z_H(x)$, and satisfies the condition in (8.11.1). Thus (8.11.1) was proved.
Now (8.11.1) shows that the action of $Z_H(x)$ on $p(C_i)$ is transitive on $p(C_0)$. For any $v \in C_i$, we can find such $C_0^i$ containing $v$. Hence such $p(C_0^i)$ covers whole $p(C_i)$. Since $p(C_0^i)$ is open dense in $p(C_i)$, we conclude that $Z_H(x)$ acts transitively on $p(C_i)$. Thus (i) is proved.

(ii) It follows from (i) that the action of $Z_H(u)$ on $Y_{u,v}$ is transitive modulo $Q_H$, namely, for any $gZ_{M_H}(v)\theta^Q \in Y_{u,v}$, there exists $z \in Z_H(u)$ and $q \in Q_H$ such that $gqZ_{M_H}(v)\theta^Q = Z_{M_H}(v)\theta^Q$. Here we may replace $q$ by $m \in M_H$. Then the definition of $Y_{u,v}$ implies that $m^{-1}(g^{-1}ug) \in \eta^{-1}_Q(v)$. But this implies that $m \in M_H(v)$. Hence $Z_H(u) \times Z_{M_H}(v)$ acts transitively on $Y_{u,v}$. (ii) is proved. □

8.12. Take $x \in \theta \subset g_{\text{nil}}$, where the Jordan type of $x$ is $\lambda$. The structure of the group $A_H(x)$ is described in 7.7. Now assume that $x \in g_{\text{nil}}$ and let $\overline{x} \in m_{\text{nil}}$ be the image of $x$ under the map $\eta_Q : q \to m$. Assume that the Jordan type of $\overline{x}$ is $\lambda'$. $A_{M_H}(\overline{x})$ is described similarly. Put $Z_{Q_H}(x) = Z_H(x) \cap Q_H$, and let $A_{Q_H}(x)$ be the image of $Z_{Q_H}(x)$ under the natural map $Z_H(x) \to A_H(x)$. The projection $Q_H \to M_H$ induces a map $Z_{Q_H}(x) \to Z_{M_H}(\overline{x})$, which gives a natural homomorphism $\varphi : A_{Q_H}(x) \to A_{M_H}(\overline{x})$.

Now assume that $\lambda'$ is obtained from $\lambda$ by $(A'_i)$. In this case, one can check that $A_{Q_H}(x) = A_H(x)$. Hence $\varphi$ gives rise to a map $\varphi : A_H(x) \to A_{M_H}(\overline{x})$. The image $\text{Im} \varphi \subset A_{M_H}(\overline{x})$ is described as follows; we consider two cases, according to the case where $a_{i+1} = a_i - 2$ or $a_{i+1} < a_i - 2$ (note that $a_{i+1} \leq a_i - 2$ by the assumption $(A'_i)$).

Case I. $a_{i+1} = a_i - 2$. In this case, $\lambda' = (a_1^{m_1}, \ldots, a_i^{m_i-1}, a_{i+1}^1, \ldots, a_h^{m_h})$. Then $A_{M_H}(\overline{x}) = \{\alpha' = (\alpha'_1, \ldots, \alpha'_i, \alpha'_i, \ldots, \alpha'_h)\}$ (here we ignore $\alpha'_i$ if $m_i = 1$). $\varphi$ is given by $\alpha_j \mapsto \alpha'_j$ for $j \neq i$, and $\alpha_i \mapsto \alpha'_i$. We have $\text{Im} \varphi = \{\alpha' \in A_{M_H}(\overline{x}) | \alpha'_i = 1\}$, which is an index two subgroup of $A_{M_H}(\overline{x})$ if $m_i \geq 2$, and $\varphi$ is surjective if $m_i = 1$.

Case II. $a_{i+1} < a_i - 2$. In this case, $\lambda' = (a_1^{m_1}, \ldots, a_i^{m_i-1}, a_i - 2, a_{i+1}^{m_{i+1}}, \ldots, a_h^{m_h})$. Then $A_{M_H}(\overline{x}) = \{\alpha' = (\alpha'_1, \ldots, \alpha'_i, \alpha'_i, \ldots, \alpha'_h)\}$. The map $\varphi$ is given by $\alpha_j \mapsto \alpha'_j$ if $j \neq i$, and $\alpha_i \mapsto \alpha'_i$. We have $\text{Im} \varphi = \{\alpha' \in A_{M_H}(\overline{x}) | \alpha'_i = 1\}$, which is an index 2 subgroup of $A_{M_H}(\overline{x})$ if $m_i \geq 2$, and $\varphi$ is an isomorphism if $m_i = 1$.

8.13. We now consider the variety $Y_{u,v} = Y_{x,\overline{x}}$ under the assumption that $\lambda$ is obtained from $\lambda$ by $(A'_i)$. Recall that $I(Y_{x,\overline{x}})$ is the set of irreducible components of $Y_{x,\overline{x}}$ of dimension $s$. By Proposition 8.7, we have $\dim Y_{x,\overline{x}} = s$, hence $I(Y_{x,\overline{x}}) \neq \emptyset$. By Lemma 8.11, $Z_H(x) \times Z_{M_H}(\overline{x})$ acts transitively on $Y_{x,\overline{x}}$, hence $A_H(x) \times A_{M_H}(\overline{x})$ acts transitively on $I(Y_{x,\overline{x}})$. In order to determine the permutation representation $\varepsilon_{x,\overline{x}}$ of $A_H(x) \times A_{M_H}(\overline{x})$ on $I(Y_{x,\overline{x}})$, it is enough to determine an isotropy subgroup of $A_H(x) \times A_{M_H}(\overline{x})$. By the definition of $Y_{x,\overline{x}}$, we can write as

$$Y_{x,\overline{x}} = \bigcup_{\alpha \in A_H(x), \alpha' \in A_{M_H}(\overline{x})} \alpha Z^0_H(x)U^0_Q Z^0_{M_H}(\overline{x}) \alpha'.$$

Hence $Z^0_H(x)U^0_Q Z^0_{M_H}(\overline{x})$ is an irreducible component in $Y_{x,\overline{x}}$. We denote by $E$ the stabilizer of the component $Z^0_H(x)U^0_Q Z^0_{M_H}(\overline{x})$ in $A_H(x) \times A_{M_H}(\overline{x})$. We have the following lemma.
Lemma 8.14. Under the notation as above, we have

\[ E = \begin{cases} 
\{ (\alpha, \alpha') \in A_H(x) \times A_{M_H}(\overline{x}) \mid \alpha_j = \alpha'_j \ (j \neq i), \ \alpha_i = \alpha'_{i+1} \}, & \text{case (I)}, \\
\{ (\alpha, \alpha') \in A_H(x) \times A_{M_H}(\overline{x}) \mid \alpha_j = \alpha'_j \ (j \neq i), \ \alpha_i = \alpha'_i \}, & \text{case (II)}. 
\end{cases} \]

Proof. Since \( A_H(x) = A_{Q_H}(x) \), we have, for \( (\alpha, \alpha') \in A_H(x) \times A_{M_H}(\overline{x}) \),

\[ \alpha A^0_H(x) U^\theta Q A^0_{M_H}(\overline{x}) \alpha' = A^0_H(x) U^\theta Q A^0_{M_H}(\overline{x}) \varphi(\alpha) \alpha'. \]

Hence the lemma follows from the discussion in 8.12. \( \square \)

8.15. We denote an element \( \tau \) of \( A_{G^0}(x)^\wedge \simeq (\mathbb{Z}/2\mathbb{Z})^h \) as \( \tau = (\tau_1, \ldots, \tau_h) \in (\mathbb{Z}/2\mathbb{Z})^h \), with \( \tau_i = \pm 1 \). Let \( \tau_i = (-1)_i \) be the character of \( A_{G^0}(x) \) corresponding to the generator of the \( i \)-th component \( \mathbb{Z}/2\mathbb{Z} \). Then \( A_H(x)^\wedge \) is given as

\[ A_H(x)^\wedge = A_{G^0}(x)^\wedge / \prod_{a_i : \text{odd}} (-1)_i, \]

hence, we may identify \( A_H(x)^\wedge \) with the following subset of \( A_{G^0}(x)^\wedge \),

\[ (8.15.1) \quad A_H(x)^\wedge = \{ \tau = (\tau_1, \ldots, \tau_h) \in (\mathbb{Z}/2\mathbb{Z})^h \mid \tau_{i_0} = 1 \}, \]

where \( i_0 \) is the index such that \( a_{i_0} \) is the largest odd number among \( a_1, \ldots, a_h \).

Let \( \lambda, \lambda' \) be as in 8.12, and we return to the setting that \( u \in \mathcal{O} \subset G^0_{\text{univ}}, \ v \in \mathcal{O}' \subset M^0_{\text{univ}} \). Then we can write \( \tau \in A_H(u)^\wedge = (\tau_1, \ldots, \tau_h) \). As in 8.12, according to the case (I) or (II), we can write \( \tau' \in A_{M_H}(v)^\wedge \) as \( \tau' = (\tau'_1, \ldots, \tau'_i, \tau'_{i+1}, \ldots, \tau'_h) \) or \( \tau' = (\tau'_1, \ldots, \tau'_i, \tau'_{i+1}, \ldots, \tau'_h) \). Note that this notation is compatible with the identification in (8.15.1). We define a subset \( D \) of \( A_H(u)^\wedge \times A_{M_H}(v)^\wedge \) by

\[ D = \begin{cases} 
\{ (\tau, \tau') \in A_H(u)^\wedge \times A_{M_H}(v)^\wedge \mid \tau_j = \tau'_j \ (j \neq i), \ \tau_i = \tau'_{i+1} \}, & \text{case (I)}, \\
\{ (\tau, \tau') \in A_H(u)^\wedge \times A_{M_H}(v)^\wedge \mid \tau_j = \tau'_j \ (j \neq i), \ \tau_i = \tau'_i \}, & \text{case (II)}. 
\end{cases} \]

Let \( \varepsilon_{u,v} \) be the permutation representation of \( A_H(x) \times A_{M_H}(y) \) on \( I(Y_{u,v}) \) as before. It follows from the previous discussions, we have

Proposition 8.16. \( A_H(u) \times A_{M_H}(v) \)-module \( \varepsilon_{u,v} \) can be decomposed into irreducible modules as

\[ \varepsilon_{u,v} \simeq \bigoplus_{(\tau, \tau') \in D} \tau \otimes \tau'. \]

8.17. We generalize the notation of \( \tau \in A_H(u)^\wedge \) as follows; let \( u \in \mathcal{O} \) with type \( \lambda \). Write \( \lambda \in \mathcal{P}_N \) as \( \lambda = (\lambda_1, \ldots, \lambda_N) \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0 \). Correspondingly, we consider the symbol \( \tau = (\tau_1, \ldots, \tau_N) \) satisfying the following properties,
(i) $\tau_i = \pm 1$ for any $i$, and $\tau_i = 1$ if $\lambda_i = 0$,
(ii) $\tau_i = \tau_j$ if $\lambda_i = \lambda_j$,
(iii) $\tau_i = 1$ for $i$ such that $\lambda_i$ is the largest odd number among $\lambda_1, \ldots, \lambda_N$.

The set of such symbols $\tau$ is in bijection with $A_H(u)\wedge$. If $\mathcal{E}$ is a local system on $\mathcal{O}$ corresponding to $\tau \in A_H(u)^\wedge$, we denote it by $\mathcal{E} = \mathcal{E}_\tau$.

8.18. Take $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^{\dagger}) \in \mathcal{M}_G$. We assume that $L_H \simeq (GL_1)^a \times SO_{N_0}$ with $N_0 + 2a = N$. Assume that the Jordan type of $\mathcal{O}_L$ is $\nu = (\nu_1, \ldots, \nu_{N_0})$, and $\mathcal{E}_1^{\dagger} = \mathcal{E}_\sigma^\dagger$ with $\sigma = (\sigma_1, \ldots, \sigma_{N_0})$. For $i = 0, 1, \ldots, a$, let $P^{(i)}$ be the $\theta$-stable parabolic subgroup of $G$ containing $P$, and $L^{(i)}$ the $\theta$-stable Levi subgroup of $P^{(i)}$ containing $L$ such that $L^{(i)}_H \simeq (GL_1)^{a-i} \times SO_{N_0+2i}$. Hence $P^{(a-1)}_H = Q_H$ is the maximal parabolic subgroup of $H$, and $P^{(0)}_H = P_H$. We consider $\pi : \tilde{X} \to X$ with respect to $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^{\dagger})$ and consider the semisimple perverse sheaf $K = \pi_* K_{\tilde{E}_1}$ on $X$. We can define a similar complex $K^{(i)}$ on $X^{(i)} \subset (L^{(i)})^\theta$, by replacing $G$ by $L^{(i)}$. The following lemma is a generalization of Proposition 8.9.

Lemma 8.19. Let $(\mathcal{O}, \mathcal{E}) \in \mathcal{M}_G$, and assume that $(\mathcal{O}, \mathcal{E})$ belongs to $(L \subset P, \mathcal{O}_L, \mathcal{E}_1^{\dagger})$. Then the Jordan type $\lambda = (\lambda_1, \ldots, \lambda_N)$ of $\mathcal{O}$ satisfies the condition

\[(8.19.1) \quad \lambda_i - \nu_i \in 2\mathbb{Z}_{\geq 0} \quad \text{for each } i.\]

(Here we write $\nu = (\nu_1, \ldots, \nu_N)$ by putting $\nu_i = 0$ for $i > N_0$.)

Proof. We can formulate a similar property as in the lemma by replacing $G$ by $L^{(i)}$. We shall prove the claim of the lemma by induction on $i$. So we may assume that the claim holds for $i = a - 1$, i.e., in the case where $P^{(i)}_H$ is the maximal parabolic subgroup $P^{(a-1)}_H = Q_H$. Let $\rho \in \mathcal{A}_\mathcal{E}_{\tilde{E}_1}$ be the irreducible character corresponding to $(\mathcal{O}, \mathcal{E})$ under the generalized Springer correspondence. Let $\mathcal{A}_{\tilde{E}_1}$ be the subalgebra of $\mathcal{A}_{\tilde{E}_1}$ associated to $Q_H$, and take $\rho' \in (\mathcal{A}_{\tilde{E}_1}^{\dagger})^\wedge$ such that $(\rho, \rho') \neq 0$. Let $(\mathcal{O}', \mathcal{E}') \in \mathcal{M}_M$ be the pair corresponding to $\rho'$ under the generalized Springer correspondence. Take $u \in \mathcal{O}, v \in \mathcal{O}'$, and write $\mathcal{E} = \mathcal{E}_\tau, \mathcal{E}' = \mathcal{E}_{\tau'}$ for $\tau \in A_H(u)\wedge, \tau' \in A_{M_H}(v)\wedge$. Then by Corollary 6.11, $\tau \otimes \tau'$ appears in the decomposition of $\varepsilon_{u,v}$. In particular, $I(Y_{u,v})$ is non-empty. Hence by Proposition 8.7, the Jordan type $\lambda'$ of $\mathcal{O}'$ is obtained from $\lambda$ by the procedure $(A_i')$ for some $i$. By induction hypothesis, $(\mathcal{O}', \mathcal{E}')$ satisfies the claim of the lemma. Hence $\lambda$ also satisfies the claim. The lemma is proved. \hfill \Box

Proposition 8.20. Let $\mathcal{O}_0$ be a unique $H$-orbit in $X_{\text{uni}}$ such that $\mathcal{O}_0 \cap \eta_P^{-1}(\mathcal{O}_L)$ is open dense in $\eta_P^{-1}(\mathcal{O}_L)$.

(i) $\mathcal{O}_0$ is the unique open dense orbit contained in $X_{\text{uni}}$.

(ii) The Jordan type of $\mathcal{O}_0$ is given by $\lambda = (2a + \nu_1, \nu_2, \ldots, \nu_N)$.

Proof. (i) Put $\mathcal{O} = \mathcal{O}_0$. By the assumption, $\eta_P^{-1}(\mathcal{O}_L) \subset \mathcal{O}$. Since $X_{\text{uni}}$ is a union of $H$-conjugates of $\eta_P^{-1}(\mathcal{O}_L), X_{\text{uni}} \subset \mathcal{O}$. As $\mathcal{O} \subset X_{\text{uni}}$, we have $\overline{\mathcal{O}} = X_{\text{uni}}$. Since $X_{\text{uni}}$ is irreducible, $\mathcal{O}$ is uniquely determined.

(ii) By Lemma 5.10, $\dim X_{\text{uni}} = 2\nu_H - 2\nu_{L_H} + \dim \mathcal{O}_L + a$ (here $a = \Delta_P$). Hence $\dim \mathcal{O} = \dim H - \dim L_H + \dim \mathcal{O}_L + a$. But $\dim L_H - \dim \mathcal{O}_L - a = \dim Z_{T_H}(v)$ for
\( v \in \mathcal{O}_L \), where \( \overline{H} = SO_{N_0} \). Take \( u \in \mathcal{O} \), and put \( \lambda = (2a + \nu_1, \nu_2, \ldots, \nu_N) \). If we note that \( n(\lambda) = n(\nu) \), by using Lemma 7.4 we have

\[
\dim Z_H(u) = \dim Z_{\overline{\mathcal{O}}} (v) = n(\nu) = n(\lambda).
\]

On the other hand, let \( d_{\mathcal{O}} \) be as in (5.2.2). The previous computation shows that \( d_{\mathcal{O}} = 0 \). Thus \( R^{d_{\mathcal{O}}} f_{\overline{\mathcal{O}}} = f_{\overline{\mathcal{O}}} \). Since \( \eta \rho^{-1}(\mathcal{O}_L) \cap \mathcal{O} \) is open dense in \( \eta^{-1}(\mathcal{O}_L) \), we see that \( f_{\overline{\mathcal{O}}} \rho \not\equiv 0 \). Hence by Theorem 5.2 (iii), the pair \((\mathcal{O}, \mathcal{E})\) belongs to \((L \subset P, \mathcal{O}_L, \mathcal{E}_1)\) for some local system \( \mathcal{E} \) on \( \mathcal{O} \). Then by Lemma 8.19, the Jordan type of \( \mathcal{O} \) satisfies the condition (8.19.1). One can check that if \( \lambda' \neq \lambda \) satisfies the condition (8.19.1), then \( n(\lambda') > n(\lambda) \). This implies, by (8.20.1), that the Jordan type of \( \mathcal{O} \) is equal to \( \lambda \). The proposition is proved.

We can now prove the following theorem, which is a counter-part of [L1, Theorem 9.2] in the symmetric space case.

**Theorem 8.21.** Let the notations be as in 8.18.

(i) The algebra \( \mathcal{A}_{\mathcal{E}_i} \) is isomorphic to the group algebra \( \mathcal{Q}_i[S_a] \).

(ii) Let \( \mathcal{E}_0 = \mathcal{E}_\tau \) be the local system on \( \mathcal{O}_0 \) defined by \( \tau = (\tau_1, \ldots, \tau_N) \) with \( \tau_i = \sigma_i \) for \( i = 1, \ldots, N_0 \), and \( \tau_i = 1 \) for \( i > N_0 \). Then \( (\mathcal{O}_0, \mathcal{E}_0) \) belongs to \((L \subset P, \mathcal{O}_L, \mathcal{E}_1)\). \( \mathcal{E}_0 \) is the unique local system on \( \mathcal{O}_0 \) such that \((\mathcal{O}_0, \mathcal{E}_0)\) belongs to \((L \subset P, \mathcal{O}_L, \mathcal{E}_1)\).

(iii) Under the isomorphism \( \mathcal{A}_{\mathcal{E}_i} \simeq \mathcal{Q}_i[S_a] \) in (i), \((\mathcal{O}_0, \mathcal{E}_0)\) corresponds to the unit representation of \( S_a \).

**Proof.** The statement of the theorem can be formulated by replacing \( G \) by \( L^{(i)} \). In the case where \( i = 0 \) the claim of the theorem is trivial. By induction on \( i \), we may assume that the claim holds for \( i = a - 1 \), i.e., for \( L_H^{(a - 1)} = M_H \). Let \((\mathcal{O}', \mathcal{E}_0') \) be the pair in \( \mathcal{M}_M \) defined similarly to \((\mathcal{O}_0, \mathcal{E}_0)\) for \( H \). Let \( \lambda' \) be the Jordan type of \( \mathcal{O}' \). From the proof of Proposition 8.20, we know that there exists a pair \((\mathcal{O}_0, \mathcal{E}')\) which belongs to \((L \subset P, \mathcal{O}_L, \mathcal{E}_1)\) for some local system \( \mathcal{E}' \) on \( \mathcal{O}_0 \). Let \( \rho \in \mathcal{A}_{\mathcal{E}_i}^{\wedge} \) be the irreducible character corresponding to \((\mathcal{O}_0, \mathcal{E}')\). Also define \( \rho'' \in (\mathcal{A}_{\mathcal{E}_i}^{\wedge})^w \) as the character corresponding to \((\mathcal{O}_0', \mathcal{E}_0')\). Let \( \rho'' \in (\mathcal{A}_{\mathcal{E}_i}^{\wedge})^w \) be a character appearing in the restriction of \( \rho \) on \( \mathcal{A}_{\mathcal{E}_i}^{\wedge} \), and let \((\mathcal{O}'', \mathcal{E}'')\) be the corresponding pair in \( \mathcal{M}_M \). By Corollary 6.11 and Proposition 8.7, the Jordan type \( \lambda'' \) of \( \mathcal{O}'' \) is obtained from \( \lambda' \) by the procedure \((A_i')\) for some \( i \). On the other hand, by Lemma 8.19, \( \lambda'' \) satisfies the condition (8.19.1). It follows that the multiplicity \( \langle \rho, \rho'' \rangle = 0 \) unless \( \lambda'' = \lambda' \). Now assume that \( \lambda'' = \lambda' \), i.e., \( \mathcal{O}' = \mathcal{O}_0' \). In this case, by our assumption, we have \((\mathcal{O}'', \mathcal{E}'') = (\mathcal{O}_0', \mathcal{E}_0')\), hence \( \rho'' = \rho' \). We write \( \mathcal{E} = \mathcal{E}_\xi \) for \( \xi \in A_H(u)^\wedge \), and \( \mathcal{E}_0' = \mathcal{E}_{\tau'} \) for \( \tau' \in A_M(v) \) with \( v \in \mathcal{O}_0' \). By Corollary 6.11, \( \langle \rho, \rho' \rangle \) coincides with the multiplicity of \( \xi \otimes \tau' \) in \( \varepsilon_{a,v} \). Since \( \langle \rho, \rho' \rangle \neq 0 \), by Proposition 8.16, we see that \( \xi = \tau \) and that \( \langle \rho, \rho' \rangle = 1 \). This shows that \( \mathcal{E} = \mathcal{E}_0 \), and (ii) holds.

The above discussion shows that the restriction of \( \rho \) on \( \mathcal{A}_{\mathcal{E}_i}^{\wedge} \) coincides with \( \rho' \). Since \( \rho' \) is a one-dimensional representation by our assumption, we see that \( \rho \) is one-dimensional. Recall that \( \mathcal{A}_{\mathcal{E}_i} \) is isomorphic to a twisted group algebra \( \mathcal{Q}_i[S_{\mathcal{E}_i}] \) (see 3.6). By making use of the one-dimensional representation \( \rho \), in a similar way as in the proof of Theorem 9.2 in [L1], we can construct an algebra isomorphism
$A_{\xi_{1}} \cong \mathbb{Q}[\mathcal{W}_{\xi_{1}}]$, where $\rho$ corresponds to the unit representation on $\mathcal{W}_{\xi_{1}}$. But in the definition of $\mathcal{W}_{\xi_{1}}$ in 3.6, $n \in N_{K}(L_{H})$ induces a trivial automorphism $\text{ad}(n)$ on $L_{H}$, hence acts trivially on $\mathcal{O}_{L}$ if the image of $n$ in $\mathcal{W}$ is contained in $\mathcal{W}_{1}$. It follows that $\mathcal{W}_{\xi_{1}} \cong S_{a}$. Thus (i) and (iii) holds. The theorem is proved. \hfill \Box

**Remark 8.22.** In [L1, Proposition 9.5], the unipotent class corresponding to the sign representation of $N_{G}(L)/L$ was determined. In the symmetric space case, however, the behavior of the $H$-orbit corresponding to the sign representation of $S_{a}$ is more complicated. For the description of this $H$-orbit, one has to wait for the determination of the whole generalized Springer correspondence.

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### 9. Determination of the generalized Springer correspondence

**9.1.** In order to obtain an exact parametrization of $H$-orbits in $G_{\text{uni}}^{\theta}$ in the case where $N$ is even, we need some preliminaries. Assume that $N = 2n$. Let $P^{+}$ be a $\theta$-stable parabolic subgroup of $G$ containing $B$ and $L^{+}$ the $\theta$-stable Levi subgroup of $G$ containing $T$ such that $L^{+}_{H} \cong GL_{n}$. Then we can write as

$$L^{+}_{H} = \left\{ \begin{pmatrix} a & 0 \\ 0 & t^{-1}a \end{pmatrix} \mid a \in GL_{n} \right\}, \quad L^{+\theta} = \left\{ \begin{pmatrix} a & 0 \\ 0 & t_{a} \end{pmatrix} \mid a \in GL_{n} \right\}.$$ 

Thus $L^{+\theta}$ is in natural bijection with $GL_{n}$, and the conjugation action of $L^{+}_{H}$ on $L^{+\theta}$ coincides with the conjugation action of $GL_{n}$ on $GL_{n}$. Let $\mathcal{O}_{H}^{\lambda}$ be the $L^{+}_{H}$-orbit in $L^{+\theta}$ corresponding to the unipotent class of $GL_{n}$ corresponding to $\lambda \in \mathcal{P}_{n}$. Let $\mathcal{O}$ be the unique $H$-orbit in $G^{\theta}$ such that $\eta_{F^{+}}^{-1}(\mathcal{O}_{H}^{\lambda}) \cap \mathcal{O}$ is open dense in $\eta_{F^{+}}^{-1}(\mathcal{O}_{H}^{\lambda})$, where $\eta_{F^{+}} : P^{+\theta} \to L^{+\theta}$ is defined similarly as before. Similarly to the map $\pi : \tilde{X} \to X$ in 3.4, we define

$$\tilde{X}_{\lambda}^{+} = \{(x, gP^{+}_{H}) \in G_{\text{uni}}^{\theta} \times H/P^{+}_{H}, \ g^{-1}xg \in \eta_{F^{+}}^{-1}(\mathcal{O}_{H}^{\lambda})\},$$

$$X_{\lambda}^{+} = \bigcup_{g \in H} g\eta_{F^{+}}^{-1}(\mathcal{O}_{H}^{\lambda})g^{-1},$$

and let $\pi_{\lambda} : \tilde{X}_{\lambda}^{+} \to X_{\lambda}^{+}$ be the first projection. Then $\pi_{\lambda}$ is proper surjective, and $X_{\lambda}^{+}$ is an irreducible closed subset of $G^{\theta}$. The following result is an analogue of Proposition 8.20.

**Lemma 9.2.** For $\lambda = (\lambda_{1}, \ldots, \lambda_{k}) \in \mathcal{P}_{n}$, put $2\lambda = (2\lambda_{1}, \ldots, 2\lambda_{k}) \in \mathcal{P}_{N}$.

(i) $\mathcal{O}$ is the unique open dense orbit in $X_{\lambda}^{+}$.
(ii) The Jordan type of $\mathcal{O}$ is equal to $2\lambda$.
(iii) $\dim \tilde{X}_{\lambda}^{+} = \dim X_{\lambda}^{+}$.
Proof. The proof of (i) is similar to the proof of Proposition 8.20 (i). We show (ii) and (iii). First compute the dimension of $\widetilde{X}_\lambda^+$. Since $\widetilde{X}_\lambda^+ \cong H \times P^+_\eta^{-1}(\mathcal{O}_\lambda^1)$, 
\[
\dim \widetilde{X}_\lambda^+ = \dim U^\theta_{P^+} + \dim \mathcal{O}_{P^+}^1 + \dim U^\theta_{P^+} \\
= \dim U_{P^+} + \dim \mathcal{O}_\lambda^1 \\
= 2n^2 - n - 2n(\lambda)
\]
since $\dim U_{P^+} = n^2$, $\dim \mathcal{O}_{P^+}^1 = n^2 - n - 2n(\lambda)$ (known result for $GL_n$, see (7.3.1)). Since $\dim H = 2n^2 - n$, and $\pi_\lambda$ is surjective, we have 
\[
\dim X_\lambda^+ \leq \dim H - 2n(\lambda).
\]
If $\mathcal{O}'$ is an $H$-orbit of type $2\lambda$, then $\dim \mathcal{O}' = \dim H - 2n(\lambda)$ by Lemma 7.4. Hence in order to prove (ii) and (iii), it is enough to show that $X_\lambda^+$ contains an element of Jordan type $2\lambda$. Choose a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ of $V$ as in 1.7. Consider the following map $x : V \to V$; 
\[
(9.2.1) \quad f_1 \mapsto f_2 \mapsto \cdots \mapsto f_{\lambda_1} \mapsto e_{\lambda_1} \mapsto e_{\lambda_1-1} \mapsto \cdots \mapsto e_1 \mapsto 0, \\
f_{\lambda_1+1} \mapsto f_{\lambda_1+2} \mapsto \cdots \mapsto f_{\lambda_1+\lambda_2} \mapsto e_{\lambda_1+\lambda_2} \mapsto e_{\lambda_1+\lambda_2-1} \mapsto \cdots \mapsto e_{\lambda_1+1} \mapsto 0, \\
\cdots \cdots \\
f_{\lambda_1+\ldots+\lambda_k-1+1} \mapsto \cdots \mapsto f_n \mapsto e_n \mapsto \cdots \mapsto e_{\lambda_1+\ldots+\lambda_k-1+1} \mapsto 0.
\]
Then $u = x + 1 \in \eta^{-1}_{P^+}(\mathcal{O}_\lambda^1)$ and the Jordan type of $u$ is $2\lambda$. Thus (ii) and (iii) hold. The lemma is proved. \hfill \Box

9.3. For an even partition $2\lambda$ of $N$, we denote by $\mathcal{O}_{2\lambda}^+$ the unique open dense orbit in $X_{\lambda}^+$ given in Lemma 9.2. We denote by $\mathcal{O}_{2\lambda}^-$ another $H$-orbit contained in $\mathcal{O}_{2\lambda}$. Note that (9.2.1) gives an explicit representative of the $H$-orbit $\mathcal{O}_{2\lambda}^+$. In particular, all the $\mathcal{O}_{2\lambda}^+$ are contained in $X_{uni} = \bigcup_{g \in H} gU^\theta g^{-1}$. Hence $\mathcal{O}_{(2n)}^+$ coincides with the unique open dense orbit $\mathcal{O}_0$ in $X_{uni}$ given in Proposition 8.20.

Let $t_n \in N_{G^\theta}(T) - N_H(T)$ be as in 1.9. Then $t_n \mathcal{O}_{2\lambda}^+ t_n^{-1} = \mathcal{O}_{2\lambda}^-$. Put $P^+ = t_n P^+ t_n^{-1}$ and $L^- = t_n L^+ t_n^{-1}$. Replacing $P^+, L^+$ by $P^-, L^-$, we can define a variety $X_{\lambda}^-$ similarly to $X_{\lambda}^+$. Then $\mathcal{O}_{2\lambda}^-$ is characterized as the unique open dense orbit contained in $X_{\lambda}^-$. The representatives of $\mathcal{O}_{2\lambda}$ are obtained by applying $\text{ad}(t_n)$ on $x$ in (9.2.1).

We have a refinement of Proposition 8.7

Lemma 9.4. Assume that $\lambda, \lambda'$ are even partitions. Let $Y_{u,v}$ be as in Proposition 8.7, where $u \in \mathcal{O}_{\lambda}^\varepsilon$, $v \in \mathcal{O}_{\lambda'}^\varepsilon$ with $\varepsilon, \varepsilon' \in \{1, -1\}$. If $I(Y_{u,v}) \neq \emptyset$, then we have $\varepsilon = \varepsilon'$.

Proof. Assume that $u \in \mathcal{O}_{\lambda}^+$. We choose a representative $u$ such that $x = u - 1$ is given as in (9.2.1). Then $\mathfrak{f} \in \mathcal{O}_{\lambda'}^\pm$ is obtained from $x$, for example, by replacing 
\[
f_1 \mapsto f_2 \mapsto \cdots \mapsto f_{\lambda_1} \mapsto e_{\lambda_1} \mapsto \cdots \mapsto e_2 \mapsto e_1 \mapsto 0
\]
by
\[ f_2 \mapsto \cdots \mapsto f_{\lambda_1} \mapsto e_{\lambda_1} \mapsto \cdots \mapsto e_2 \mapsto 0. \]
Thus \( v = \bar{v} + 1 \) is contained in a similar variety \( X^+_\lambda \) as \( X^+_\lambda \) defined for \( \lambda' \). Hence \( v \in \mathcal{O}_X^+ \). If \( u \in \mathcal{O}_X^- \), we can apply the same argument by replacing \( X^+_\lambda \) by \( X^-_\lambda \), and obtain that \( v \in \mathcal{O}_X^- \). The lemma is proved. \( \square \)

9.5. Assume that \( N \geq 1 \) is an odd integer. A pair \((\lambda, \tau)\) is called a signed partition if \( \lambda = (\lambda_1, \ldots, \lambda_N) \) is a partition of \( N \), and if \( \tau = (\tau_1, \ldots, \tau_N) \) satisfies the condition (cf. 8.17) that

(i) \( \tau_i = \pm 1 \), and \( \tau_i = 1 \) if \( \lambda_i = 0 \),
(ii) \( \tau_i = \tau_j \) if \( \lambda_i = \lambda_j \),
(iii) \( \tau_{i_0} = 1 \) where \( i_0 \) is the index such that \( \lambda_{i_0} \) is the largest odd number among \( \lambda_1, \ldots, \lambda_N \).

We denote by \( \Psi_N \) the set of signed partitions of \( N \). Note that, by Proposition 7.8 (i), the set of \( H \)-orbits in \( G^\theta_{\mathfrak{u}} \) is parametrized by \( \mathcal{P}_N \). We denote by \( \mathcal{O}_\lambda \) the \( H \)-orbit corresponding to \( \lambda \in \mathcal{P}_N \). Then as in 8.17, \( H \)-equivariant simple local system on \( \mathcal{O}_\lambda \) can be expressed as \( \mathcal{E}_r \) for \( (\lambda, \tau) \in \Psi_N \), and the map \( (\lambda, \tau) \mapsto (\mathcal{O}_\lambda, \mathcal{E}_r) \) gives a bijection \( \Psi_N \curvearrowright \mathcal{N}_G \) if \( N \geq 3 \).

Next assume that \( N \geq 0 \) is an even integer. In this case, for each partition \( \lambda \in \mathcal{P}_N \), we prepare two copies \( \lambda^\pm \), and assume that \( \lambda^+ = \lambda^- \) if \( \lambda \) is not an even partition. A signed partition \((\lambda^\pm, \tau)\) is defined similarly as above, for each \( \lambda^+ \) and \( \lambda^- \). We denote by \( \Psi_N \) the set of signed partitions of \( N \). Note that if \( N = 0 \), we regard the empty partition \( \emptyset \) as an even partition, so we consider \((\emptyset^+, 1)\) and \((\emptyset^-, 1)\).

By Proposition 7.8 (ii), the set of \( G^\theta \)-orbits in \( G^\theta_{\mathfrak{u}} \) is parametrized by \( \mathcal{P}_N \). \( G^\theta \)-orbit \( \mathcal{O}_\lambda \) is a single \( H \)-orbit unless \( \lambda \) is an even partition, in which case, \( \mathcal{O}_\lambda \) splits into two \( H \)-orbits. By 9.3, we denote those two \( H \)-orbits by \( \mathcal{O}_\lambda^+ \) and \( \mathcal{O}_\lambda^- \). By abuse of the notation, we denote \( \mathcal{O}_\lambda \) by \( \mathcal{O}_\lambda^+ = \mathcal{O}_\lambda^- \) if \( \lambda \) is not an even partition. Thus we have a bijective map \( \Psi_N \curvearrowright \mathcal{N}_G \) by \((\lambda^\pm, \tau) \mapsto (\mathcal{O}_\lambda^\pm, \mathcal{E}_r)\), if \( N \geq 2 \).

9.6. Assume that \( N \) is odd. Take integers \( N_0 \geq 1, a \geq 0 \) such that \( N = 2a + N_0 \).

For a fixed \( \xi = (\nu, \sigma) \in \Psi_{N_0} \), we define a map \( \Gamma_\xi : \mathcal{P}_a \to \Psi_N \) as follows; write \( \nu = (\nu_1, \ldots, \nu_N), \sigma = (\sigma_1, \ldots, \sigma_N) \) by putting \( \nu_i = 0, \sigma_i = 1 \) for \( i > N_0 \). For each \( \mu = (\mu_1, \ldots, \mu_N) \in \mathcal{P}_a \), define integers \( \lambda_1, \ldots, \lambda_N \) by

\[
(9.6.1) \quad \lambda_i = \nu_i + 2\mu_i \quad \text{for } i = 1, \ldots, N.
\]

Then \( \lambda = (\lambda_1, \ldots, \lambda_N) \in \mathcal{P}_N \), and the pair \((\lambda, \sigma)\) gives a well-defined element in \( \Psi_N \). By definition, \( \Gamma_\xi(\mu) = (\lambda, \sigma) \). In the case where \( N \) is even, the map \( \Gamma_\xi : \mathcal{P}_a \to \Psi_N \) is defined similarly, but we consider \( \xi^\pm = (\nu^\pm, \sigma) \), and put \( \Gamma_\xi(\mu) = (\lambda^\pm, \sigma) \).

Let \( N \geq 0 \) be an integer. An element \((\lambda, \tau) \in \Psi_N \) (or \((\lambda^\pm, \tau) \in \Psi_N \)) is called cuspidal if

(i) \( \lambda_i - \lambda_{i+1} \leq 2 \) for \( i = 1, \ldots, N \) (here we put \( \lambda_{N+1} = 0 \)),
(ii) If \( \lambda_i - \lambda_{i+1} = 2 \), then \( \tau_i \neq \tau_{i+1} \).
We denote by $\mathcal{C}_N$ the set of triples $(N_0, \nu, \sigma)$ such that $N - N_0 \in 2\mathbb{Z}_{\geq 0}$ and $\xi = (\nu, \sigma) \in \Psi_{N_0}$ is a cuspidal element. For $c \in \mathcal{C}_N$, we denote by $\Psi_N^{(c)}$ the image of $\Gamma_c : \mathcal{P}_a \to \Psi_N$, where $a = (N - N_0)/2$. Clearly $\Gamma_c$ gives a bijection $\mathcal{P}_a \cong \Psi_N^{(c)}$. We also denote by $\Psi_N^{(0)}$ the set of cuspidal elements in $\Psi_N$.

**Proposition 9.7.** There exists a partition

\[(9.7.1) \quad \Psi_N = \coprod_{c \in \mathcal{C}_N} \Psi_N^{(c)}.\]

**Proof.** For simplicity we assume that $N$ is odd. The case where $N$ is even is dealt similarly. Assume that $(\lambda, \tau) \in \Psi_N$ is not cuspidal. Then there exists some $i$ such that $\lambda_i > \lambda_{i+1} + 2$ or that $\lambda_i = \lambda_{i+1} + 2$ with $\tau_i = \tau_{i+1}$. Put $\lambda' = (\lambda'_1, \ldots, \lambda'_N) \in \mathcal{P}_{N-2}$, where $\lambda'_i = \lambda_i - 2, \lambda'_j = \lambda_j$ for $j \neq i$. If we put $\tau' = \tau$, we have $(\lambda', \tau') \in \Psi_{N-2}$ by our assumption. By induction on $N$, we may assume (9.7.1) holds for $\Psi_{N-2}$. Hence there exists $c = (N_0, \nu, \sigma) \in \mathcal{C}_{N-2}$ such that $(\lambda', \tau') \in \Psi_{N-2}^{(c)}$. In particular, $(\lambda', \tau') = \Gamma_c(\mu')$ for some $\mu' \in \mathcal{P}_a$, where $\xi = (\nu, \sigma)$ and $a = (N - 2 - N_0)/2$. Here $\mu' = (\mu'_1, \ldots, \mu'_N)$ satisfies the condition that $\mu'_{i-1} \geq \mu'_i + 1$, and if we define $\mu = (\mu_1, \ldots, \mu_N) \in \mathcal{P}_{a+1}$ by $\mu_i = \mu'_i + 1, \mu_j = \mu'_j$ for $j \neq i, (\lambda, \tau) = \Gamma_c(\mu)$. Hence $(\lambda, \tau) \in \Psi_N^{(c)}$. $c$ is determined by $(\lambda, \tau)$ uniquely. In fact, suppose that $(\lambda, \tau) \in \Psi_N^{(c)}$ for another $c'$. The above argument shows, since $(\nu, \sigma)$ is cuspidal, that $(\lambda', \tau') \in \Psi_{N-2}^{(c')}$.

Hence by induction on $N$, we have $c = c'$.

**9.8.** We have a natural parametrization of $S_a^\land$ by $\mathcal{P}_a$. We denote by $\rho_\mu$ the irreducible representation of $S_a$ corresponding to $\mu \in \mathcal{P}_a$. (Here the partition $(a)$ corresponds to the unit representation, $(1^a)$ corresponds to the sign representation.)

Under the identification $\Psi_N \cong \mathcal{N}_G$ in 9.5, we denote by $\mathcal{N}_G^{(0)}$ the image of $\Psi_N^{(0)}$. We also denote by $\mathcal{N}_G^{(c)}$ the subset of $\mathcal{N}_G$ which is the image of $\Psi_N^{(c)}$. For each $c = (N_0, \nu, \sigma) \in \mathcal{C}_N$ with $\xi = (\nu, \sigma)$, the map $\Gamma_c$ induces a bijection $\tilde{\Gamma}_c : S_a^\land \cong \mathcal{N}_G^{(c)}$, combined with $\mathcal{P}_a \simeq S_a^\land$. Thus, by Proposition 9.7, we have a bijective map

\[(9.8.1) \quad \tilde{\Gamma} : \coprod_{c \in \mathcal{C}_N} S_a^\land \simeq \coprod_{c \in \mathcal{C}_N} \mathcal{N}_G^{(c)} = \mathcal{N}_G\]

where $a = (N - N_0)/2$ for $c = (N_0, \nu, \sigma) \in \mathcal{C}_N$.

Recall the definition of $\mathcal{N}_G^{(\xi)}$ in 5.7 for $\xi = (L \subset P, \mathcal{O}_L, \mathcal{E}_1^\xi) \in \mathcal{I}_G$. The following result gives a combinatorial description of the generalized Springer correspondence for the symmetric space associated to orthogonal groups.

**Theorem 9.9.** Let the notations be as above.

(i) $\mathcal{N}_G^{(0)}$ coincides with the set of cuspidal pairs in $\mathcal{N}_G$.

(ii) $c = (N_0, \nu, \sigma) \mapsto \xi = (L \subset P, \mathcal{O}_L, \mathcal{E}_1^\xi)$ gives a bijection $\mathcal{C}_N \cong \mathcal{I}_G$ such that $\mathcal{N}_G^{(c)} = \mathcal{N}_G^{(\xi)}$, where $L$ is such that $L_H \simeq (GL_1)^a \times SO_{N_0}$, and $(\mathcal{O}_L, \mathcal{E}_1^\xi) = (\mathcal{O}_{\nu}, \mathcal{E}_1^\xi)$. 

(iii) The nap $\tilde{\Gamma}$ in (9.8.1) gives a bijection in Theorem 5.2 (ii).

Proof. We prove the theorem by induction on $N$. In the case where $N = 2, 3$, the claim (iii) is verified directly, see Appendix. First consider the case where $N$ is odd. Assume that the theorem holds for $L \neq G$. We show (iii). Choose $(L \subset P, \mathcal{O}_L, \mathcal{O}_L^\dagger) \in \mathcal{S}_G$ with $L \neq G$. By (i) for $L$, we can attach $(N_0, \nu, \sigma) \in \mathcal{C}_N$ to $(L \subset P, \mathcal{O}_L, \mathcal{O}_L^\dagger)$. Take $\rho = \rho_\mu \in S_{\chi}^\wedge$, and let $(\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G$ be the pair which belongs to $(L \subset P, \mathcal{O}_L, \mathcal{O}_L^\dagger)$ corresponding to $\rho$. Put $(\lambda, \tau) = \Gamma_\xi(\mu)$ for $\xi = (\nu, \sigma)$. In order to prove (iii), it is enough to see that $\mathcal{O} = \mathcal{O}_\lambda, \mathcal{E} = \mathcal{E}_\tau$. Let $Q_H$ be the maximal parabolic subgroup of $H$ containing $P$ and $M_H$ its Levi subgroup containing $L$ as in Section 8. Let $\rho' = \rho_\mu'$ be an irreducible character of $S_{a-1}$, and $(\mathcal{O}', \mathcal{E}') \in \mathcal{N}_M$ the pair which belongs to $(L \subset M \cap P, \mathcal{O}_L, \mathcal{O}_L^\dagger)$ corresponding to $\rho'$. By induction hypothesis, one can write as $(\mathcal{O}', \mathcal{E}') = (\mathcal{O}_\lambda', \mathcal{E}_\tau')$ with $(\lambda', \tau') = \Gamma_\xi(\mu')$. By Corollary 6.11 and Proposition 8.7, if $\langle \rho, \rho' \rangle \neq 0$, then the Jordan type of $\mathcal{O}$ is obtained from $\lambda'$ by the inverse operation of $(A_i')$ for some $i$. This is true for any $\rho'$ such that $\langle \rho, \rho' \rangle \neq 0$. In the case where $a \geq 3$, this condition determines the Jordan type of $\mathcal{O}$ uniquely, and we have $\mathcal{O} = \mathcal{O}_\lambda$. Moreover, in this case, by Proposition 8.16, $\mathcal{E}$ is determined uniquely from $\mathcal{E}_\tau$, namely we have $\mathcal{E} = \mathcal{E}_\tau$. Thus we have proved (iii) for $a \geq 3$.

Now assume that $a = 1$. In this case, $S_1 = \{1\}$, and $\pi_* K_{G|_T}^L |_{x_{\text{uni}}} = 1C(\tilde{\mathcal{O}}_\lambda, \mathcal{E}_\tau)$ up to shift, where $\lambda' = (\nu_1 + 2, \nu_2, \ldots, \nu_N), \tau' = (\sigma_1, \ldots, \sigma_N)$ by Theorem 8.21. Hence the claim holds. Next assume that $a = 2$. In this case, $S_2^\wedge$ has two representations, the unit representation $\rho(2)$ and the sign representation $\rho(12)$. By Lemma 8.19, the possibility for $(\mathcal{O}, \mathcal{E})$ belonging to $(L \subset P, \mathcal{O}_L, \mathcal{O}_L^\dagger)$ is only $\mathcal{O} = \mathcal{O}_\lambda$ with $\lambda = (\nu_1 + 4, \nu_2, \ldots, \nu_N)$ or $\lambda = (\nu_1 + 2, \nu_1 + 2, \nu_3, \ldots, \nu_N)$. By Theorem 8.21, $\rho(2)$ corresponds to $(\mathcal{O}_0, \mathcal{E}_0) = (\mathcal{O}_\lambda, \mathcal{E}_\tau), \nu, \sigma)$, where $\lambda = (\nu_1 + 4, \nu_2, \ldots, \nu_N)$ and $\tau = (\sigma_1, \ldots, \sigma_N)$. Hence $\rho(12)$ corresponds to $(\mathcal{O}_\lambda, \mathcal{E}_\tau)$ with $\lambda = (\nu_1 + 2, \nu_2 + 2, \nu_3, \ldots, \nu_N)$. The restriction of $\rho(12)$ on $S_1$ is $\rho(1)$ which corresponds to $(\mathcal{O}_\lambda, \mathcal{E}_\tau)$. Thus, by a similar argument as above, $\tau$ is determined from $\tau'$, namely we see that $(\lambda, \tau) = \Gamma_\xi(1^{(2)})$. Thus the claim holds for $a = 2$. Hence (iii) is proved for odd $N$.

Next consider the case where $N$ is even. We choose $(L \subset P, \mathcal{O}_L, \mathcal{O}_L^\dagger) \in \mathcal{S}_G$. Assume that the Jordan type of $\mathcal{O}_L$ is not an even partition. Then a similar argument as before works. (Note that in that case the Jordan type of $\mathcal{O}$ is not even for any $(\mathcal{O}, \mathcal{E})$ belonging to $(L \subset P, \mathcal{O}_L, \mathcal{O}_L^\dagger)$ by Lemma 8.19.) Thus we assume that the Jordan type of $\mathcal{O}_L$ is even. Then by the same reason, the Jordan type of $\mathcal{O}$ is even for such $(\mathcal{O}, \mathcal{E})$. The previous argument shows that if $(\mathcal{O}, \mathcal{E})$ corresponds to $\rho = \rho_\mu \in S_{\chi}^\wedge$, then the Jordan type of $\mathcal{O}$ is $\lambda$, and $\mathcal{E} = \mathcal{E}_\tau$ for $\Gamma_\xi(\mu) = (\lambda, \tau)$. But we have to show that $\mathcal{O} = \mathcal{O}_\lambda^\pm$ if $\Gamma_\xi(\mu) = (\lambda^\pm, \tau)$. This is done by using Lemma 9.4. Thus (iii) is proved for even $N$ similarly as above.

We show (i) and (ii). Assume that $N$ is even. If $N_0 = 0$, then $a = n$, and there exists two triples $c^+, c^- \in \mathcal{C}_N$, where $c^\pm = (0, \emptyset^\pm, 1)$. We have $\Psi(c^\pm) \simeq \mathcal{P}_n$. On the other hand, $N_0 = 0$ corresponds to the case where $L = T, \mathcal{O}_L \subset \{1\}$ and $\mathcal{O}_L^\dagger = \mathcal{Q}_T^1$ : constant sheaf. Hence there exist two triples $(T \subset B, \{1\}, \mathcal{Q}_T^1), (T \subset B_1, \{1\}, \mathcal{Q}_T^1) \in \mathcal{S}_G$, where $B_1 = t_nBt_n^{-1}$. By (iii), the Springer correspondence gives a bijection $\Psi(c^\pm) \simeq \mathcal{N}_G(c^\pm).$ For other cases (for any $N$), $c = (N_0, \nu, \sigma) \in \mathcal{C}_N$ determines a
unique triple \((L \subset P, O_L, E^t_1) \in \mathcal{G}\). By (iii), and by Proposition 9.7, this proves (i). (ii) follows from (i). The theorem is proved. \(\square\)

9.10. Following the discussion in [L1, 10.6], we shall give a formula which describes the cardinality of \(\Psi^{(0)}_N\), namely, the number of the cuspidal pairs \((O, E)\) in \(\mathcal{G}\). Note that the computation below is much simpler compared to the case of generalized Springer correspondence for orthogonal groups.

Let \(t\) be an indeterminate. Recall that the partition function \(p(n) = |\mathcal{P}_n|\) is defined by the formula \(\prod_{i=1}^{\infty} (1 - t^i)^{-1} = \sum_{n \geq 0} p(n)t^n\). We define functions \(q_1(n), q_2(n)\) for \(n \in \mathbb{Z}_{\geq 0}\) as follows:

\[
\prod_{i=1}^{\infty} (1 + t^i)^2 = \sum_{n \geq 0} q_1(n)t^n,
\]

\[
\prod_{i=1}^{\infty} (1 + t^{2i}) = \sum_{n \geq 0} q_2(n)t^n.
\]

Proposition 9.11. The cardinality \(|\Psi^{(0)}_N| = |\mathcal{G}^{(0)}_N|\) of cuspidal pairs is given by the formula,

\[
|\Psi^{(0)}_N| = \begin{cases} 
\frac{1}{2} q_1(N) & \text{if } N \geq 3 : \text{odd}, \\
\frac{1}{2} q_1(N) + \frac{3}{2} q_2(N) & \text{if } N \geq 2 : \text{even}.
\end{cases}
\]

Proof. Let \(x_N\) be the number of elements in \(\Psi_N\), and \(x'_N\) (resp. \(x''_N\)) the number of elements \((\lambda, \tau) \in \Psi_N\) such that \(\lambda\) is not an even partition (resp. even partition). Hence \(x_N = x'_N + x''_N\). Then as in [L1, 10.6],

\[
1 + \sum_{N \geq 1} \left(2x'_N + x''_N/2\right)t^N = \sum_{i_1, i_2, \ldots \geq 0} 2^{\{a_i > 0\}} t^{i_1+2i_2+3i_3+\cdots} = (1 + \sum_{i \geq 1} 2t^i) \left(1 + \sum_{i_2 \geq 1} 2t^{i_2}\right) \cdots = \frac{1 + t}{1 - t}\frac{1 + t^2}{1 - t^2} \cdots = \prod_{i=1}^{\infty} (1 - t^{2i})^{-1} \prod_{i=1}^{\infty} (1 + t^i)^2.
\]

It follows that

\[
(9.11.1) \quad 2x'_N + x''_N/2 = \sum_{0 \leq a \leq N/2} p(a)q_1(N - 2a).
\]
By a similar computation shows that

\[ 1 + \sum_{N \geq 2; \text{even}} \left( \frac{x''}{2} \right) t^N = \sum_{i_1, i_2, \ldots \geq 0} 2^{i(i + 1)} t^{2(i_1 + 2i_2 + \ldots)} \]

\[ = (1 + \sum_{i_1 \geq 1} 2t^{2i_1})(1 + \sum_{i_2 \geq 1} 2t^{4i_2}) \cdots \]

\[ = \frac{1 + t^2}{1 - t^2} \frac{1 + t^4}{1 - t^4} \cdots \]

\[ = \prod_{i=1}^{\infty} (1 - t^{2i})^{-1} \prod_{i=1}^{\infty} (1 + t^{2i}). \]

Hence we have

\[(9.11.2) \quad \frac{x''}{2} = \sum_{0 \leq a \leq N/2} p(a) q_2(N - 2a). \]

Now assume that \( N \) is even. Then by (9.11.1) and (9.11.2), we have

\[ x_N = x'_N + x''_N = \sum_{0 \leq a \leq N/2} p(a) \left\{ \frac{1}{2} q_1(N - 2a) + \frac{3}{2} q_2(N - 2a) \right\}. \]

On the other hand, by the generalized Springer correspondence (Theorem 9.9), we have

\[ |\Psi_N| = \sum_{0 \leq a \leq N/2} |S^a| |\Psi_N^{(0)}| \]

Then by induction on \( N \), we obtain the formula for \( |\Psi_N^{(0)}| \). The case where \( N \) is odd is similar (in this case, \( x''_N = 0 \)). \qed

10. Induction

10.1. We consider the group \( G \) and \( \theta : G \to G \) as in (2.1.4). We fix a \( \theta \)-stable pair \((B, T)\) of a Borel subgroup \( B \) and a maximal torus of \( G \). Let \( P \) be a \( \theta \)-stable parabolic subgroup of \( G \) containing \( B \) and \( L \) the \( \theta \)-stable Levi subgroup of \( P \) containing \( T \). As before, let \( \eta_P : P^{i^\theta} \to L^{i^\theta} \) be the natural projection. Consider the following diagram

\[(10.1.1) \quad L^{i^\theta} \overset{\psi}{\leftarrow} \widehat{X}^P \overset{\varphi'}{\to} \overline{X}^P \overset{\varphi''}{\to} G^{i^\theta}, \]

where
\[ \tilde{X}^P = \{(x, g) \in G^\theta \times H \mid g^{-1}xg \in P^\theta \}, \]
\[ \tilde{X} = \{(x, gP_H) \in G^\theta \times H/P_H \mid g^{-1}xg \in P^\theta \}, \]
and \( \varphi' : (x, g) \mapsto (x, gP_H), \varphi'' : (x, gP_H) \mapsto x, \) and \( \psi : (x, g) \mapsto \eta_P(g^{-1}xg). \) Then \( \varphi', \psi \) are smooth with connected fibres. Moreover, \( H \times P_H \) acts on \( \tilde{X}^P \) by \( (h, p) : (x, g) \mapsto (hxh^{-1}, hgp^{-1}) \), and \( H \) acts on \( \tilde{X} \) by \( h : (x, gP_H) \mapsto (hxh^{-1}, hgpH) \). \( \varphi', \varphi'' \) are \( H \)-equivariant, and \( \psi \) is \( H \times P_H \)-equivariant with respect to the trivial action of \( H \) and the action of \( P_H \) on \( L^\theta \) induced from the map \( P_H \to L_H \).

Let \( K \) be an \( L_H \)-equivariant perverse sheaf on \( L^\theta \), which is regarded as an \( H \times P_H \)-equivariant perverse sheaf. Since \( \psi \) is smooth with connected fibre, there exists a perverse sheaf \( \psi^*K[\alpha] \) on \( \tilde{X}^P \), where \( \alpha \) is the dimension of the fibre. \( \psi^*K[\alpha] \) is \( H \times P_H \)-equivariant, and since \( \varphi' \) is a locally trivial principal \( P_H \)-bundle, there exists a perverse sheaf \( K_1 \) on \( \tilde{X}^P \) such that \( \psi^*K[\alpha] \simeq \varphi'^*K_1[\beta] \), where \( \beta = \dim P_H \).

We define \( \text{ind} K = \varphi''K_1 \). Since \( \varphi'' \) is proper, \( \text{ind} K \) is a semisimple complex on \( G^\theta \). Since \( K_1 \) is \( H \)-equivariant, \( \text{ind} K \) is \( H \)-equivariant. We also write \( \text{ind} K = \text{ind}^G_P K \). \( \text{ind} K \) is an analogue of the induction functor of the character sheaves ([L2]).

Let \( Q \) be a \( \theta \)-stable parabolic subgroup of \( G \) containing \( P \), and \( M \) the \( \theta \)-stable Levi subgroup of \( Q \) containing \( L \). Then \( M \cap P \) is a \( \theta \)-stable parabolic subgroup of \( M \) with Levi subgroup \( L \). Thus one can define functors \( \text{ind}^M_{M \cap P} \) and \( \text{ind}^G_Q \). The following transitivity property can be proved in a similar way as in [L2, Proposition 4.2].

**Proposition 10.2.** Let \( K \) be an \( L_H \)-equivariant perverse sheaf on \( L^\theta \). Assume that \( \text{ind}^M_{M \cap P} K \) is an \( M \)-equivariant perverse sheaf on \( M^\theta \). Then we have

\[ \text{ind}^G_Q(\text{ind}^M_{M \cap P} K) \simeq \text{ind}^G_P K. \]

### 10.3.
Returning to the original setting, we consider \( G = GL_N \), and let \( P \) be the \( \theta \)-stable parabolic subgroup such that \( L_H \simeq (GL_1)^a \times SO_{N_0} \), where \( N = N_0 + 2a \). Let \( Q \) be the \( \theta \)-stable parabolic subgroup of \( G \) containing \( P \) such that \( M_H \simeq GL_a \times SO_{N_0} \). Thus \( (M \cap P)_H \simeq B_a \times SO_{N_0} \), where \( B_a \) is a Borel subgroup of \( GL_a \) such that \( B_H \cap (GL_a \times GL_a)^\theta \simeq B_a \). Let \( \xi = (\xi_L, \xi^\xi_L) \) be a cuspidal pair on \( L^\theta_\text{uni} \), and consider the \( L_H \)-equivariant perverse sheaf \( K_\xi = \text{IC}(\mathcal{O}_{\xi_L}, \xi^\xi_L)[\dim \mathcal{O}_{\xi_L}] \) on \( L^\theta \). Since \( \text{ind}^M_{M \cap P} K_\xi \) is isomorphic to \( (\text{ind}^{GL_a}_{B_a} K_0) \boxtimes K_\xi \), where \( K_0 \) is the constant sheaf \( \delta_q \) on \( \{1\} \subset T_a \) (\( T_a \) is the maximal torus of \( B_a \)), by a well-known result of Borho-MacPherson ([BM]) for \( GL_a \), \( \text{ind}^M_{M \cap P} K_\xi \) is a semisimple perverse sheaf on \( M^\theta_\text{uni} \), equipped with \( S_a \)-action, and is decomposed as

\[
\text{ind}^M_{M \cap P} K_\xi \simeq \bigoplus_{\mu \in \mathcal{P}_a} \rho_\mu \otimes (K_\mu^a \boxtimes K_\xi),
\]
where $K^a_\mu = \mathrm{IC}(\bar{\mathcal{O}}^\mu_* \mathcal{Q}_\mu)[\dim \Theta^a_\mu]$ (here $\Theta^a_\mu$ is the unipotent class in $GL_a$ with Jordan type $\mu$). Thus $\text{ind}^M_{\mathcal{P}^a} K_\xi$ is a $M_H$-equivariant perverse sheaf on $M^\theta$, and one can apply the functor $\text{ind}_Q^G$ on it. By the transitivity of induction (Proposition 10.2), we have

\begin{equation}
\text{ind}_P^G K_\xi \simeq \bigoplus_{\mu \in \mathcal{P}_a} \rho_\mu \otimes \text{ind}_Q^G (K^a_\mu \boxtimes K_\xi). \tag{10.3.2}
\end{equation}

On the other hand, by comparing the construction of $\text{ind}_P^G K_\xi$ with the complex $\pi_* K_{\widetilde{\mathcal{F}}_1}$ constructed in 3.7, we see that $\text{ind}_P^G K_\xi \simeq \pi_* K_{\widetilde{\mathcal{F}}_1}[-r]|_{X_{\text{uni}}}$. Hence by Theorem 5.2 together with Theorem 9.9, we have

\begin{equation}
\text{ind}_P^G K_\xi \simeq \bigoplus_{\mu \in \mathcal{P}_a} \rho_\mu \otimes K_{\Gamma_\xi(\mu)}, \tag{10.3.3}
\end{equation}

where $K_{\Gamma_\xi(\mu)} = \mathrm{IC}(\bar{\mathcal{O}}, \mathcal{E})[\dim \mathcal{O}]$ if $(\Theta^a_\mu, \mathcal{E}) \in \mathcal{N}_G$ corresponds to $\Gamma_\xi(\mu) \in \Psi_N$. $\text{ind}_Q^G (K^a_\mu \boxtimes K_\xi)$ is an $H$-equivariant semisimple complex on $G^\theta$. Since it is a direct summand of the semisimple perverse sheaf on $G^\theta$, it is a semisimple perverse sheaf. By counting the multiplicities $\sum \dim \rho_\mu$, we see that $\text{ind}_Q^G (K^a_\mu \boxtimes K_\xi)$ is a simple perverse sheaf on $G^\theta_{\text{uni}}$. Hence we have

\begin{equation}
\text{ind}_Q^G (K^a_\mu \boxtimes K_\xi) \text{ coincides with one of the simple component appearing in the right hand side of (10.3.3).} \tag{10.3.4}
\end{equation}

**Remark 10.4.** If the isomorphisms in (10.3.2) and (10.3.3) are compatible with $S_a$-action, we will immediately get the isomorphism $\text{ind}_Q^G (K^a_\mu \boxtimes K_\xi) \simeq K_{\Gamma_\xi(\mu)}$. Note that the diagram in 3.7 with respect to $\pi : \tilde{X} \to X$ corresponds to the diagram (10.1.1) with respect to $\text{ind}_P^G$. However, in the case of $\text{ind}_Q^G$, we cannot construct an analogous diagram of 3.6. So, it is not certain whether $\text{ind}_Q^G$ commutes with the action of $S_a$. Nevertheless we will show in the discussion below that the above isomorphism actually holds.

**10.5.** Let $\mathcal{O}'$ be an $M_H$-orbit in $M^\theta_{\text{uni}}$. Then there exists a unique $H$-orbit $\mathcal{O}$ in $G^\theta_{\text{uni}}$ such that $\mathcal{O} \cap \eta_{\text{uni}}^{-1}(\mathcal{O}')$ is open dense in $\eta_{\text{uni}}^{-1}(\mathcal{O}')$. $\mathcal{O}$ is called the $H$-orbit induced from $\mathcal{O}'$, and is denoted by $\mathcal{O} = \text{Ind}_Q^G \mathcal{O}'$. Put

\begin{align*}
\tilde{X}_{\mathcal{O}'} &= \{(x, gQ_H) \in G^\theta \times H/Q_H \mid g^{-1} x g \in \eta_{\text{uni}}^{-1}(\mathcal{O}')\}, \\
X_{\mathcal{O}'} &= \bigcup_{g \in H} g \eta_{\text{uni}}^{-1}(\mathcal{O}') g,
\end{align*}

and let $\pi_{\mathcal{O}'} : \tilde{X}_{\mathcal{O}'} \to X_{\mathcal{O}'}$ be the first projection. Then $\pi_{\mathcal{O}'}$ is proper, surjective. By a similar argument as in the proof of Proposition 8.20, we see that $\mathcal{O}$ is the unique open dense orbit contained in $X_{\mathcal{O}'}$. Let $\mathcal{O}_L$ be the $L_H$-orbit in $L^\theta_{\text{uni}}$ as before,
and consider $\mathcal{O}' = \mathcal{O}_\mu^a \times \mathcal{O}_L$ for each $\mu \in \mathcal{P}_a$. We write $\mathcal{O}' = \mathcal{O}_\mu'$, and express $\pi_{\mathcal{O}'} : \tilde{X}_{\mathcal{O}'} \to X_{\mathcal{O}'}$ as $\pi_\mu : \tilde{X}_\mu \to X_\mu$. The following result shows that the induction of $H$-orbits can be realized in the level of perverse sheaves.

**Proposition 10.6.** Let $\mu \in \mathcal{P}_a$. Under the notation above,

(i) $\text{Ind}_Q^G(K_\mu \boxtimes K_\xi) \simeq K_{\Gamma_\xi(\mu)}$.

(ii) Let $(\mathcal{O}, \mathcal{E}) \in \mathcal{N}_G$ be the pair corresponding to $\Gamma_\xi(\mu) \in \Psi_N$. Then $\mathcal{O} = \text{Ind}_Q^G(\mathcal{O}_\mu^a \times \mathcal{O}_L)$. In particular, the Jordan type of $\mathcal{O}$ is equal to $\nu + 2\mu$.

(iii) $\dim \tilde{X}_\mu = \dim X_\mu$.

**Proof.** First we note the following property, which is obtained from the explicit description of the generalized Springer correspondence in Theorem 9.9 (iii).

(10.6.1) Let $(\mathcal{O}, \mathcal{E}), (\mathcal{O}', \mathcal{E}') \in \mathcal{N}_G$. Assume that $(\mathcal{O}, \mathcal{E})$ corresponds to $\Gamma_\xi(\mu)$, and $(\mathcal{O}', \mathcal{E}')$ corresponds to $\Gamma_\xi(\mu')$. If the Jordan type of $\mathcal{O}$ is the same as that of $\mathcal{O}'$, then $\mu = \mu'$.

We prove (i) by the backward induction on $n(\mu)$. Assume that the statement holds for $\mu'$ such that $n(\mu') > n(\mu)$. Put $\text{Ind}_Q^G(K_\mu \boxtimes K_\xi) = IC(\mathcal{O}', \mathcal{E}')[\dim \mathcal{O}']$. It follows from the definition of $\text{Ind}_Q^G$, $\mathcal{O}'$ is open dense in $X_\mu$. Let $\mathcal{O}_{\mu,\mu',\nu}$ be the unipotent class in $M$ such that its intersection with $M^{\mu,\theta}$ coincides with $\mathcal{O}_\mu = \mathcal{O}_\mu^a \times \mathcal{O}_L$. Then $\eta_Q^{-1}(\mathcal{O}') \subset \mathcal{O}_{\mu,\mu',\nu} U_Q$. It is known from the theory of Hall polynomials (see [M]) that the unipotent class in $GL_N$ which has open dense intersection with $\mathcal{O}_{\mu,\mu',\nu} U_Q$ has Jordan type $2\mu + \nu$, and all other unipotent classes with non-zero intersection with $\mathcal{O}_{\mu,\mu',\nu} U_Q$ have Jordan type $< 2\mu + \nu$. Hence the Jordan type of $\mathcal{O}'$ is of the form $2\mu + \nu \leq 2\mu + \nu$. By induction hypothesis, for any $\mu''$ such that $n(\mu'') > n(\mu)$, the pair $(\mathcal{O}_1, \mathcal{E}_1)$ such that $\mathcal{O}_1$ has Jordan type $2\mu'' + \nu$ is already assigned to $\text{Ind}_Q^G(K_{\mu''} \boxtimes K_\xi)$. Thus $n(\mu') = n(\mu)$, and so the Jordan type of $\mathcal{O}'$ must coincide with $2\mu + \nu$. Hence $(\mathcal{O}', \mathcal{E}')$ corresponds to $\Gamma_\xi(\mu)$ by (10.6.1). This proves (i).

We show (ii) and (iii). It follows from the definition that $\mathcal{O}'$ coincides with $\text{Ind}_Q^G \mathcal{O}_\mu'$. Thus (ii) follows from (i). We have

$$\dim X_\mu = \dim \mathcal{O}' = \dim H - n(\nu + 2\mu).$$

On the other hand, since $\tilde{X}_\mu \simeq H \times^{Q_\mu} \eta_Q^{-1}(\mathcal{O}')$, we have

$$\dim \tilde{X}_\mu = \dim H - \dim Q_H + \dim \mathcal{O}_\mu' + \dim U_Q^{\theta}.$$ 

Since $\dim \mathcal{O}_\mu' = (a^2 - a - 2n(\mu)) + (\dim SO_{N_0} - n(\nu))$, $\dim U_Q^{\theta} = \dim U_Q^a + a$, we conclude that $\dim \tilde{X}_\mu = \dim X_\mu$. Thus (iii) holds. The proposition is proved. \hfill \Box

**Remark 10.7.** Proposition 10.6 is a generalization of Proposition 8.20 and of Lemma 9.2. But in contrast to the previous cases, in this discussion, we don’t need to show the existence of an element of Jordan type $\nu + 2\mu$ inside of $X_\mu$. 
Appendix

We give some examples of the generalized Springer correspondence. Here we use a simplified notation to denote the signed partition \((\lambda, \tau)\). If \(\lambda = (a_1^{m_1}, a_2^{m_2}, \ldots)\), we only denote the signature \(\pm 1\) corresponding to the block \(a_i^{m_i}\) as \(\pm\). For example, \((1^3; +1, +1, +1)\) is written as \((1^3; +)\) and \((2^21^2; -1, -1, +1, +1)\) is written as \((2^21^2; -+)\). If \(c = (N_0, \nu, \sigma) \in \mathcal{C}_N\), the set \(\Psi_N^{(c)}\) is the set of signed partitions corresponding to \(\rho_\mu \in S_a^N\) with \(a = (N - N_0)/2\), which we denote by \(S_a; (\nu, \sigma)\). Let \(\Psi_N^{(0)}\) be the set of cuspidal elements in \(\Psi_N\). In the first column, we only list up the elements which are not contained in \(\Psi_N^{(0)}\).

- \(N = 3\).

| \((\lambda, \tau)\) | \(S_1; (1; +)\) | \(S_1; (3; +)\) |
|---------------------|----------------|----------------|
| \((1^3; +)\)      | \((1, 1^3; +)\) | \((1)\)       |

\(\Psi_N^{(0)} = \{(1^3; +), (21; ++), (21; +)\}\).

- \(N = 5\).

| \((\lambda, \tau)\) | \(S_2; (1; +)\) | \(S_1; (21; ++)\) | \(S_1; (21; -+)\) | \(S_1; (1^3; +)\) |
|---------------------|----------------|----------------|----------------|----------------|
| \((5; +)\)      | \((2)\)     | \((1)\)     | \((1)\)     | \((1)\)     |
| \((41; ++)\)    | \((1^2)\)   | \((1)\)     | \((1)\)     | \((1)\)     |
| \((41; -+)\)    | \((1^2)\)   | \((1)\)     | \((1)\)     | \((1)\)     |
| \((32; ++)\)    | \((1^2)\)   | \((1)\)     | \((1)\)     | \((1)\)     |
| \((31^2; ++)\)  | \((1^2)\)   | \((1)\)     | \((1)\)     | \((1)\)     |

\(\Psi_N^{(0)} = \{(32; -), (31^2; --), (2^21; ++), (2^21; -+), (21^3; ++), (21^3; -+), (1^5; +)\}\).
\[ N = 7 \]

\[
\begin{array}{|c|c|c|c|c|}
\hline
(\lambda, \tau) & S_3; (1; +) & S_2; (21; ++), & S_2; (21; +) & S_2; (1^3; +) \\
\hline
(7; +) & (3) & (2) & & \\
(61; ++) & & (21) & (2) & \\
(61; ++) & & & & \\
(52; ++) & & & & \\
(52; ++) & & & & \\
(51^2; ++) & & & (2) & \\
(51^2; ++) & & & & \\
(43; ++) & & & (1^2) & \\
(43; ++) & & & (1^2) & \\
(421; ++++) & & & & \\
(421; --++) & & & & \\
(41^3; ++) & & & & \\
(41^3; --) & & & & \\
(3^21; ++) & & & & \\
(32^2; ++) & & & (1^3) & \\
(31^4; ++) & & & (1^2) & \\
\hline
\end{array}
\]

The correspondence in the case of \( S_1 \) is as follows;
\[
\begin{align*}
S_1; (32; --) & \leftrightarrow (52; --), & S_1; (31^2; --) & \leftrightarrow (51^2; --), \\
S_1; (2^21; ++) & \leftrightarrow (421; ++), & S_1; (2^21; +--) & \leftrightarrow (421; --), \\
S_1; (2^13; ++) & \leftrightarrow (41^3; ++), & S_1; (2^13; +--) & \leftrightarrow (41^3; --), \\
S_1; (1^5; +) & \leftrightarrow (31^4; ++). \\
\end{align*}
\]

Here
\[
\Psi_N^{(0)} = \{(421; --++), (421; ++--), (3^21; +--), (32^2; --), \\
(321^2; +++), (321^2; +++), (321^2; +--), (321^2; --+), \\
(31^4; +--), (2^31; +++), (2^31; +--), (2^21^3; ++), (2^21^3; --), \\
(21^5; +++), (21^5; --), (1^7; +)\}.
\]

\[ N = 2 \]

\[
\begin{array}{|c|c|c|}
\hline
(\lambda, \tau) & S_1; (\emptyset^+; +) & S_1; (\emptyset^-; +) \\
\hline
((2)^+; +) & (1) & \\
((2)^-; +) & & (1) \\
\hline
\end{array}
\]

\[
\Psi_N^{(0)} = \{(2)^+; --), (2)^-(--; --), (1^2; +)\}.
\]
\[ N = 4. \]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
(\lambda, \tau) & S_2: (\theta^+; +) & S_2: (\theta^-; +) & S_1: (\theta^+; -) & S_1: (\theta^-; -) & S_1: (1^2; +) \\
\hline
((4)^+; +) & (2) & (2) & (1) & & \\
((4)^+; -) & & & & & \\
((4)^-; +) & & & & & \\
((4)^-; -) & & & & & \\
(31; ++) & (1^2) & (1^2) & & & \\
((2^2)^+; +) & (1^2) & (1^2) & & & \\
((2^2)^-; +) & & & & & \\
(\lambda, \tau) & S_3: (\theta^+; +) & S_3: (\theta^-; +) & S_2: (\theta^+; -) & S_2: (\theta^-; -) & S_2: (1^2; +) \\
\hline
((6)^+; +) & (3) & (3) & (2) & (2) & \\
((6)^+; -) & & & & & \\
((6)^-; +) & & & & & \\
((6)^-; -) & & & & & \\
(51; ++) & & & & & \\
(51; -) & & & & & \\
((42)^+; ++) & (21) & (1^2) & (1^2) & & \\
((42)^+; +) & & & & & \\
((42)^+; -) & & & & & \\
((42)^-; ++) & & & & & \\
((42)^-; +) & & & & & \\
((42)^-; -) & & & & & \\
(41^2; ++) & & & & & \\
(41^2; +) & & & & & \\
(3^2; +) & & & & & \\
(31^3; ++) & & & & & \\
((2^3)^+; +) & (1^3) & (1^3) & & & \\
((2^3)^-; +) & & & & & \\
\hline
\end{array}
\]

\[ \Psi_N^{(0)} = \{ (31; -), ((2^2)^+; -), ((2^2)^-; -), (21^2; ++), (21^2; -), (1^4; +) \}. \]

\[ N = 6. \]

The correspondence in the case of \( S_1 \) is as follows:

\[
\begin{align*}
(31; -) & \leftrightarrow (51; -), \\
((2^2)^+; -) & \leftrightarrow ((42)^+; -), \\
((2^2)^-; -) & \leftrightarrow ((42)^-; -), \\
(21^2; ++) & \leftrightarrow (41^2; ++), \\
(21^2; -) & \leftrightarrow (41^2; -), \\
(1^4; +) & \leftrightarrow (31^3; ++).
\end{align*}
\]
Here
\[ \Psi_N^{(0)} = \{(42)^+, (+-), (42)^-, (+-), (321; ++), (321; ++-),
(321; +-+), (321; ++-), (31^3; +-), ((2^3)^+; -), ((2^3)^-; -),
(2^21^2; ++), (2^21^2; -+), (21^4; ++), (21^4; -+), (1^6; +)\}.

REFERENCES

[BM] W. Borho and R. MacPherson; Représentations des groupes des Weyl et homologie
d’intersection pour les variétés nilpotentes, C.R. Acad. Sci. Paris t. 292 (1981), série A,
707-710.

[CVX1] T.H. Chen, K. Vilonen and T. Xue; Springer correspondence for symmetric spaces,
preprint, arXiv: 1510.05986v2.

[CVX2] T.H. Cen, K. Vilonen and T. Xue; Springer correspondence for the split symmetric pair
in type A, preprint, arXiv:1608.06034v1.

[H] A. Henderson; Fourier transform, parabolic induction, and nilpotent orbits, Transformation
Groups 6 (2001), 353 - 370.

[K] S. Kato; An exotic Deligne-Langlands correspondence for symplectic groups, Duke Math.
J. 148 (2009), 306 - 371.

[KR] B. Kostant and S. Rallis; Orbits and representations associated with symmetric spaces,
Amer. J. Math. 93 (1971), 753 - 809.

[L1] G. Lusztig; Intersection cohomology complexes on a reductive group, Invent. Math.75
(1984), 205-272.

[L2] G. Lusztig; Character sheaves, I, Adv. in Math. 56 (1985), 193 - 237.

[LS] G. Lusztig and N. Spaltenstein; On the generalized Springer correspondence for classical
groups, in Algebraic Groups and Related Topics, Adv. Studies in Pure Math. 6, (1985),
pp.289 - 315.

[LY1] G. Lusztig and Z. Yun; \( \mathbb{Z}/m \)-graded Lie algebras and perverse sheaves, I, preprint. arXiv:
1602.05524v2.

[LY2] G. Lusztig and Z Yun; \( \mathbb{Z}/m \)-graded Lie algebras and perverse sheaves, II, preprint. arXiv:
1604.00659v1.

[LY3] G. Lusztig and Z.Yun; \( \mathbb{Z}/m \)-graded Lie algebras and perverse sheaves, III, Graded double
affine Hecke algebra, preprint, arXiv: 1607.07916v2.

[M] I.G. Macdonald; “Symmetric functions and Hall polynomials”, Clarendon Press, Oxford,
1995.

[R] R.W. Richardson; Orbits, invariants, and representations associated to involutions of
reductive groups, Inv. Math. 66 (1982), 287-312.

[St] R. Steinberg; Endomorphisms of linear algebraic groups, Mem. Amer. Math. Soc. 80,
(1968).

[SS] T. Shoji and K. Sorlin; Exotic symmetric space over a finite field, I, Transformation
Groups, 18 (2013), 877 - 929.

[V] T. Vust; Opération de groupes réductifs dans un type de cônes presque homogenes, Bull.
Soc. Mat. France 102 (1974), 317-334.
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