Dimension Independent Atomic Decomposition for Dyadic Martingale $\mathbb{H}^1$

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Abstract
We introduce atoms for dyadic atomic $\mathbb{H}^1$ for which the equivalence between the atomic and maximal function definitions is dimension independent. We give sharp, up to $\log(d)$ factor, estimates for the $\mathbb{H}^1 \to L^1$ norm of the special maximal function.

Keywords Function spaces · Martingale Hardy space · Atomic decomposition · Dyadic martingale

Mathematics Subject Classification 42B25 · 60G46

We define a martingale $\mathbb{H}^1$ space on $\mathbb{R}^d$

$$M^* f = \sup_n |E_n f|, \quad \|f\|_{\mathbb{H}^1} = \|M^* f\|_{L^1},$$

where $E_n$ is the conditional expectation operator associated with the dyadic grid of scale $2^n$. There are various equivalent definitions of $\mathbb{H}^1$. In particular, it has been proved in [2] that an equivalent norm can be defined by

$$S^* f = \left( \sum_n |E_n f - E_{n+1} f|^2 \right)^{\frac{1}{2}}, \quad \|f\|_{\mathbb{H}^1} \sim \|S^* f\|_{L^1},$$

with the equivalence constants independent of $d$. As in the Euclidean case, the atomic decompositions of martingale $\mathbb{H}^1$ have been proved based either on the maximal func-
tion or the square function definitions, see [10]. We note that although the atomic norm obtained in [10] (based on the atomic decomposition) is equivalent to the maximal norm for any single $d$, the equivalence constants depend on $d$. The monograph [10] contains a comprehensive list of references related to martingale $H^1$ and atomic decompositions. The aim of this short note is to fine tune the definition of atoms, so that the atomic and maximal function norms are equivalent with constants independent of the dimension $d$. By the results of [2], the same decomposition works for the square function $(2)$ norm.

The motivation for our results is their possible applications. We note that the proposed atomic decomposition can be used to obtain dimension explicit estimates for various classical operators acting on martingale $H^1$. In this note we apply Theorem 6 to estimate the $H^1 \to L^1$ norm of a special radial maximal function modeling the classical Hardy-Littlewood maximal operator. A similar argument works for the heat semigroup, see remark at the end of the paper. We are going to address further questions, in particular dimension explicit estimates for classical SIO acting on $H^1$, in the future.

The study of dimension dependence of classical end-point estimates is not new. See papers [6, 9], where explicit upper estimates have been obtained for weak type $(1,1)$ constants. These results have motivated us to consider the other important end-point case: various Hardy spaces. In this context the issue of dimension explicit statements for atomic decomposition results arise in a natural way. For general background on atomic decompositions and martingale $H^1$ [1, 3–5, 8].

We define atoms.

**Definition 1** A function $a_Q$ on $\mathbb{R}^d$ is an atom associated with a dyadic cube $Q$ if

(a) $\int a_Q = 0$, supp $a_Q \subset Q$,
(b) $\|a_Q\|_{L^1} \leq 1$,
(c) $\|a_Q\|_{L^\infty} \leq \frac{2^{d+1}}{|Q|}$,
(d) we have a decomposition

$$\left\{ x : |a_Q(x)| > \frac{1}{|Q|} \right\} \subset \bigcup_s Q_s,$$

where $Q_s$ are essentially disjoint dyadic cubes, $Q_s \subset Q$, satisfying the following two conditions:

- for $Q^{\#}_s$ being the dyadic parent of $Q_s$ (one scale above)

$$\frac{1}{|Q^{\#}_s|} \int_{Q^{\#}_s} a_Q(x) \, dx \leq \frac{1}{|Q|},$$

- $a_Q$ is constant on each $Q_s$.

**Lemma 2** For an atom $a_Q$ we have

$$\|a_Q\|_{H^1} \leq 2.$$
Proof Observe that, by (a) of Definition 1, we only need to consider averages over cubes contained in $Q$. Pick a dyadic cube $\tilde{Q} \subset Q$. We need to compute the average of $a_Q$ over $\tilde{Q}$. Suppose $\tilde{Q}$ is a cube other than any of the $Q_s$’s. Let $\{Q^#_j\}$ be a family of all maximal $Q^#_s \subset \tilde{Q}$. We denote by $(f)_R$ the average of $f$ over a set $R$. Then $(a_Q)_\tilde{Q} = (a^#_Q)_\tilde{Q}$, where $a^#$ is obtained from $a_Q$ by replacing its value on $Q^#_j$ by the constant $(a_Q)_Q^#_j$. By Definition 1(d) we have $|a^#| \leq C |Q|$ ($C$ independent of the dimension). Now suppose $\tilde{Q}$ is one of the $Q_s$’s. Then $a_Q$ is constant on $\tilde{Q}$ and averaging leaves its value unchanged. Hence $M^*a_Q(x) \leq \frac{C}{|Q|} + |a_Q(x)|$ and the desired $L^1$ estimate follows by Definition 1(b).

\[ \square \]

Remark 3 If we remove condition (d) from Definition 1 and assume the $L^\infty$ estimate of $2^d$ on the entire $Q$, the statement of the above lemma will remain true, but with linear dependence of the implied constant on the dimension $d$. In order to see this, one has to use $\|M^*\|_{L^p \rightarrow L^p} \leq \frac{C}{p-1}$ for $p = 1 + \frac{1}{d}$, an estimate $\|a_Q\|_{L^p} \leq \|a_Q\|_{L^1} \|a_Q\|_{L^\infty}$ and the Hölder inequality. This argument immediately extends to any sublinear operator $T^*$ with explicit control of $\|T^*\|_{L^p \rightarrow L^p}$ and consequently can be used for possible application of the Theorem 6. See Remark 13 for comments on the sharpness of this approach and the role of the condition (d) of Definition 1.

Remark 4 It seems of interest to find the multidimensional, dimension explicit statement of Theorem 2 from [7].

Remark 5 The global bound of $2^d$ imposed on the atoms seems natural. Exactly this bound arises in the proof of the atomic decomposition theorem (below), and atoms $\beta$ appearing in that proof are typical examples. Such bound, together with condition (d) of Definition 1 reappears in various proofs when one attempts to control the dependence on $d$ of constants. We intend to return to these types of questions in future.

Theorem 6 (Atomic decomposition) For $f \in H^1$ there exist a sequence of atoms $\{a_{Q_i}\}$ and a sequence of constants $\{\lambda_i\}$ such that

$$f = \sum_i \lambda_i a_{Q_i} \quad \text{in } H^1,$$

and

$$\sum_i |\lambda_i| \leq \|f\|_{H^1}.$$

Proof We start with a series of reductions. By the structure of the dyadic grid we can separate the function $f$ into its components supported on coordinate system “octants”. We can thus only consider $f \in H^1$ supported on the “octant” $[0, \infty)^d$. Clearly $f$ must have mean 0. We now observe, that for any $\epsilon > 0$ we can decompose $f = f_1 + f_2$, with both $f_1, f_2 \in H^1$, $f_1$ supported on some cube $[0, 2^n]^d$ and $\|f_2\|_{H^1} < \epsilon$. We now justify this observation. Let $n \in \mathbb{Z}$ be large enough so that for $Q = [0, 2^n]^d$

$$\left| \int_Q f \right| \leq \epsilon, \quad \text{and} \quad \|M^*f\|_{L^1(Q^c)} \leq \epsilon.$$
We consider
\[ f_1 = f \cdot 1_Q - \frac{1_Q}{|Q|} \int_Q f, \]
\[ f_2 = f - f_1 = f \cdot 1_{Q^c} + \frac{1_Q}{|Q|} \int_Q f. \]

It follows
\[ M^* f_2 = \max \left\{ \sup_{Q_1 \cap Q = \emptyset} 1_{Q_1} |\langle f_2 \rangle_{Q_1}|, \sup_{Q_1 \cap Q \neq \emptyset} 1_{Q_1} |\langle f_2 \rangle_{Q_1}| \right\} \]
\[ = \max \left\{ M^*_1 f_2, M^*_2 f_2 \right\} \]
(recall that we denote by $\langle f \rangle_R$ the average of $f$ over a set $R$). Observe
\[ 1_{Q_1} \sup_{Q_1 \cap Q = \emptyset} |\langle f_2 \rangle_{Q_1}| = 0 \text{ on } Q, \]
and
\[ 1_{Q_1} \sup_{Q_1 \cap Q = \emptyset} |\langle f_2 \rangle_{Q_1}| = 1_{Q_1} \sup_{Q_1 \cap Q = \emptyset} |\langle f \rangle_{Q_1}| \leq M^* f \text{ on } Q^c. \]

We obtain
\[ \|M^*_1 f_2\|_{L^1} \leq \|M^* f\|_{L^1(Q^c)} \leq 2\epsilon. \]

We turn to $M^*_2 f_2$. Suppose $x \in Q$ and $Q_1 \subset Q$. Then
\[ |\langle f_2 \rangle_{Q_1}| = |f_2(x)| = \frac{1}{|Q|} \left| \int_Q f \right|. \]

Suppose now $Q \subset Q_1$
\[ |\langle f_2 \rangle_{Q_1}| \leq \frac{1}{|Q_1|} \left| \int_Q f \right| + \frac{1}{|Q_1|} \int_{Q_1 \setminus Q} |f| \leq \frac{1}{|Q|} \left| \int_Q f \right| + \frac{1}{|Q|} \int_{Q^c} |f| \]
We see, that
\[ \|M^*_2 f_2\|_{L^1(Q)} < 3\epsilon. \]
Consider $x \notin Q$ and $Q \subset Q_1$. We have
\[ |\langle f_2 \rangle_{Q_1}| = |\langle f \rangle_{Q_1}| \leq M^* f(x). \]
Thus
\[ \|M^*_2 f_2\|_{L^1(Q^c)} < \epsilon. \]
Combining all of the above we see that $\|f_2\|_{H^1} < C\epsilon$, where $C$ is absolute. Our observation is therefore proved. Clearly, we can iterate this observation, and further decompose $f_2$. It follows, that to prove the atomic decomposition for an arbitrary

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$f \in \mathcal{H}^1$ it is sufficient to prove it for mean zero functions supported on cubes of the form $[0, 2^n]^d$. Using dyadic homogeneity of $\mathcal{H}^1$ we can further restrict ourselves to the fixed cube $Q = [0, 1]^d$.

We let the family (finite or infinite) $\{Q_{i_1}\}_{i_1}$ consist of the maximal dyadic subcubes of $Q$, for which the average of $f$ is non-zero. By differentiation, a.e. outside of the union of $\{Q_{i_1}\}_{i_1}$ we have $f = 0$. We will define inductively consecutive generations of subcubes. The first generation is the family $\{Q_{i_1}\}_{i_1}$. We now construct the second generation of subcubes $\{Q_{i_1, i_2}\}_{i_2} \subset Q_{i_1}$. Let an integer $R(i_1)$ be defined by

$$2^{R(i_1)} \leq |\langle f \rangle_{Q_{i_1}}| < 2^{R(i_1)+1} , \quad i_1 = 1, 2, \ldots$$

The cubes $Q_{i_1, i_2}$ are the maximal subcubes of $Q_{i_1}$ for which

$$|\langle f \rangle_{Q_{i_1, i_2}}| \geq 2^{R(i_1)+2}.$$ 

In other words $\{Q_{i_1, i_2}\}$ are the moments of the first “break” through the level $2^{R(i_1)+2}$. We define integers $R(i_1, i_2)$ by

$$2^{R(i_1, i_2)} \leq |\langle f \rangle_{Q_{i_1, i_2}}| < 2^{R(i_1, i_2)+1}.$$ 

We iterate the procedure for each of the cubes $Q_{i_1, i_2}$, and we obtain a family of cubes $\{Q_{i_1, i_2, \ldots, i_l}\}_{l=1, 2, \ldots}$

We now write the decomposition of $f$ into “pre-atoms”

$$f = \sum_{i_1} 1_{Q_{i_1}} (f)_{Q_{i_1}}$$

$$+ \sum_{i_1} \left( - 1_{Q_{i_1}} \langle f \rangle_{Q_{i_1}} + f \cdot 1_{Q_{i_1} \setminus \bigcup_{i_2} Q_{i_1, i_2}} + \sum_{i_2} 1_{Q_{i_1, i_2}} (f)_{Q_{i_1, i_2}} \right)$$

$$+ \sum_{i_1, i_2} \left( - 1_{Q_{i_1, i_2}} \langle f \rangle_{Q_{i_1, i_2}} + f \cdot 1_{Q_{i_1, i_2} \setminus \bigcup_{i_3} Q_{i_1, i_2, i_3}} + \sum_{i_3} 1_{Q_{i_1, i_2, i_3}} (f)_{Q_{i_1, i_2, i_3}} \right)$$

$$\ldots$$

$$+ \sum_{i_1, i_2, \ldots, i_s} \left( - 1_{Q_{i_1, \ldots, i_s}} \langle f \rangle_{Q_{i_1, \ldots, i_s}} + f \cdot 1_{Q_{i_1, \ldots, i_s} \setminus \bigcup_{i_{s+1}} Q_{i_1, \ldots, i_{s+1}}} + \sum_{i_{s+1}} 1_{Q_{i_1, \ldots, i_{s+1}} (f)_{Q_{i_1, \ldots, i_{s+1}}} \right)$$

$$\ldots$$
We call “pre-atoms” associated with dyadic cubes $Q_{i_1, \ldots, i_s}$ the functions $a_{Q_{i_1, \ldots, i_s}}$, which are the normalized elements of the above decomposition

$$a_{Q_{i_1, \ldots, i_s}} = \frac{\omega_{Q_{i_1, \ldots, i_s}}}{\lambda_{Q_{i_1, \ldots, i_s}}} ,$$

where

$$\omega_{Q_{i_1, \ldots, i_s}} = -1_{Q_{i_1, \ldots, i_s}}\langle f \rangle \chi_{Q_{i_1, \ldots, i_s}} + f \cdot 1_{Q_{i_1, \ldots, i_s} \setminus \bigcup_{i_{s+1}} Q_{i_1, \ldots, i_{s+1}}} + \sum_{i_{s+1}} 1_{Q_{i_1, \ldots, i_{s+1}}} \langle f \rangle \chi_{Q_{i_1, \ldots, i_{s+1}}} ,$$

and

$$\lambda_{Q_{i_1, \ldots, i_s}} = 2^{R(i_1, \ldots, i_s)+1} |Q_{i_1, \ldots, i_s}| + 2^{R(i_1, \ldots, i_s)+2} |Q_{i_1, \ldots, i_s}| + \sum_{i_{s+1}} 2^{R(i_1, \ldots, i_{s+1})+1} |Q_{i_1, \ldots, i_{s+1}}| .$$

We include in the above the first “pre-atom” of the decomposition

$$\omega_Q = \sum_{i_1} 1_{Q_{i_1}} \langle f \rangle \chi_{Q_{i_1}} , \quad a_Q = \frac{\omega_Q}{\lambda_Q} ,$$

where

$$\lambda_Q = \sum_{i_1} 2^{R(i_1)} |Q_{i_1}| .$$

We immediately obtain

• $\int a_{Q_{i_1, \ldots, i_s}} = 0$, supp $a_{Q_{i_1, \ldots, i_s}} \subset Q_{i_1, \ldots, i_s}$, $\|a_{Q_{i_1, \ldots, i_s}}\|_{L^1} \leq 1$,

• the decomposition

$$f = \sum_{s=1}^{\infty} \sum_{i_1, \ldots, i_s} \lambda_{Q_{i_1, \ldots, i_s}} \cdot a_{Q_{i_1, \ldots, i_s}} , \quad (3)$$

where the convergence is pointwise. The convergence is actually in $L^1$ which will become clear momentarily, when we estimate the sum of the coefficients. Eventually we will modify the “pre-atoms” on certain cubes to obtain convergence in $\mathbb{H}^1$.

Observe that by the definition of $R(i_1, \ldots, i_s)$ we have

$$2^{R(i_1, \ldots, i_s)} \leq |\langle f \rangle \chi_{Q_{i_1, \ldots, i_s}}| < 2^{R(i_1, \ldots, i_s)+1} ,$$

and similarly for the cubes $Q_{i_1, \ldots, i_{s+1}}$ (with $R(i_1, \ldots, i_s)$ replaced by $R(i_1, \ldots, i_{s+1})$).

Also, by definition,

$$R(i_1, \ldots, i_s) + 2 \leq R(i_1, \ldots, i_{s+1}) .$$

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On the cube $Q_{i_1,...,i_s}$, outside $\bigcup_{i_{s+1}} Q_{i_1,...,i_{s+1}}$ we have

$$|f| \leq 2^{R(i_1,...,i_s)+2}$$

(from the definition of $Q_{i_1,...,i_{s+1}}$ and the differentiation of integrals). Furthermore, for $x \in Q_{i_1,...,i_s} \setminus \bigcup_{i_{s+1}} Q_{i_1,...,i_{s+1}}$

$$|a_{Q_{i_1,...,i_s}}(x)| = \frac{|\langle f \rangle_{Q_{i_1,...,i_s}} - f(x)|}{\lambda_{Q_{i_1,...,i_s}}} \leq \frac{2^{R(i_1,...,i_s)+1} + 2^{R(i_1,...,i_s)+2}}{\lambda_{Q_{i_1,...,i_s}}} \leq \frac{1}{|Q_{i_1,...,i_s}|},$$

while for $x \in Q_{i_1,...,i_{s+1}}$

$$|a_{Q_{i_1,...,i_s}}(x)| = \frac{|-\langle f \rangle_{Q_{i_1,...,i_s}} + \langle f \rangle_{Q_{i_1,...,i_{s+1}}}|}{\lambda_{Q_{i_1,...,i_s}}} \leq \frac{2^{R(i_1,...,i_s)+1} + 2^{R(i_1,...,i_{s+1})+1}}{\lambda_{Q_{i_1,...,i_s}}} \leq \frac{2^{R(i_1,...,i_{s+1})+2}}{\sum_{i_{s+1}} 2^{R(i_1,...,i_{s+1})+1}|Q_{i_1,...,i_{s+1}}|}.$$ Integrating the above two estimates over $Q_{i_1,...,i_s}$ we obtain

$$\|a_{Q_{i_1,...,i_s}}\|_{L^1} \leq 3.$$ Observe

$$\sum_{s=1}^{\infty} \sum_{i_1,...,i_s} |\lambda_{Q_{i_1,...,i_s}}| = \sum_{s=1}^{\infty} \sum_{i_1,...,i_s} (2^{R(i_1,...,i_s)+1} + 2^{R(i_1,...,i_s)+2}) |Q_{i_1,...,i_s}| +$$

$$+ \sum_{s=1}^{\infty} \sum_{i_1,...,i_{s+1}} 2^{R(i_1,...,i_{s+1})+1} |Q_{i_1,...,i_{s+1}}| \leq 4 \sum_{s=1}^{\infty} \sum_{i_1,...,i_s} 2^{R(i_1,...,i_s)+1} |Q_{i_1,...,i_s}| =$$

$$= 8 \sum_{k=1}^{\infty} 2^k \sum_{0 pts = 1,2,...,i_1,...,i_s, R(i_1,...,i_s) = k} |Q_{i_1,...,i_s}| = (**).$$
We make the following 2 observations:

(a) for a fixed \( R(i_1, \ldots, i_s) = R(j_1, \ldots, j_t) \) the cubes \( Q_{i_1, \ldots, i_s} \) and \( Q_{j_1, \ldots, j_t} \) are essentially disjoint. This follows, since if they weren’t essentially disjoint, one would have to contain the other, which is impossible (unless, of course, they are the same cube).

(b) we have

\[
Q_{i_1, \ldots, i_s} \subset \{ x : M^* f(x) > 2^{R(i_1, \ldots, i_s) - 1} \}
\]

so

\[
(\ast\ast) \leq C \sum_{k=1}^{\infty} 2^k \left\{ x : M^* f(x) > 2^{k-1} \right\} \leq C \| M^* f \|_{L^1},
\]

with the constant \( C \) absolute. Clearly, the above also gives

\[
|\lambda_{Q}| = \sum_{i_1} 2^{R(i_1)} |Q_{i_1}| \leq C \| M^* f \|_{L^1},
\]

with the same absolute constant. We will further decompose the \( a_{Q_{i_1, \ldots, i_s}} \).

Let \( Q_{i_1, \ldots, i_s+1} \) be the dyadic, immediate, parent of \( Q_{i_1, \ldots, i_s+1} \). The cubes \( Q_{i_1, \ldots, i_s+1} \) need not to be disjoint, so we initially only consider the family of maximal ones. We call this family of maximal cubes \( \{ Q_{i} \}_{i \in I} \). We fix one such maximal cube, and call it \( Q_{i_0} \). By the construction it is contained in \( Q_{i_1, \ldots, i_s} \). We denote by \( Q_1, \ldots, Q_n \) those cubes among the immediate dyadic descendants of \( Q_{i_0} \) which belong to the set \( \{ Q_{i_1, \ldots, i_s+1} \}_{s+1} \) (there is at least one such), while we denote by \( Q_{n+1}, \ldots, Q_{2^d} \) the remaining descendants \( (0 < n \leq 2^d) \). Since the cube \( Q_{i_0} \) is not one of the chosen cubes, thus

\[
|\langle f \rangle_{Q_{i_0}}| < 2^{R(i_1, \ldots, i_s) + 2}.
\]

Similarly,

\[
|\langle f \rangle_{Q_k}| < 2^{R(i_1, \ldots, i_s) + 2} \quad k = n + 1, \ldots, 2^d,
\]

\[
|\langle f \rangle_{Q_k}| \geq 2^{R(i_1, \ldots, i_s) + 2} \quad k = 1, \ldots, n.
\]

For \( k = 1, \ldots, n \) let us denote by \( \alpha_k \) the constant value of \( a_{Q_{i_1, \ldots, i_s}} \) on \( Q_k \), that is

\[
\alpha_k = \frac{\langle f \rangle_{Q_{i_1, \ldots, i_s}} - \langle f \rangle_{Q_k}}{\lambda_{Q_{i_1, \ldots, i_s}}}.
\]
We observe the following

\[
\lambda_{Q_1,\ldots,i_s} \sum_{k=1}^n |Q_k| \alpha_k = -|Q_k| n \langle f \rangle_{Q_1,\ldots,i_s} + \sum_{k=1}^n |Q_k| \langle f \rangle_{Q_k} \\
= -|Q_k| n \langle f \rangle_{Q_1,\ldots,i_s} + |Q_{i_0}^\#| \langle f \rangle_{Q_{i_0}^\#} - \\
- \sum_{k=n+1}^{2^d} |Q_k| \langle f \rangle_{Q_k}.
\]

Dividing by \( |Q_k| \) (all of these are equal) we get

\[
\lambda_{Q_1,\ldots,i_s} \sum_{k=1}^n \alpha_k = 2^d \langle f \rangle_{Q_{i_0}^\#} - n \langle f \rangle_{Q_1,\ldots,i_s} - \sum_{k=n+1}^{2^d} \langle f \rangle_{Q_k}.
\]

We thus obtain

\[
\left| \sum_{k=1}^n \alpha_k \right| \leq \frac{2^d \cdot (2^{R(i_1,\ldots,i_s)+1} + 2 \cdot 2^{R(i_1,\ldots,i_s)+2})}{\lambda_{Q_1,\ldots,i_s}} < \frac{2 \cdot 2^d}{|Q_1,\ldots,i_s|}.
\]

Let

\[
\tilde{\alpha} = \frac{1}{n} \sum_{k=1}^n \alpha_k,
\]

and let us adjust the value of the pre-atom \( a_{Q_1,\ldots,i_s} \) on cubes \( Q_k \) from \( \alpha_k \) to \( \tilde{\alpha} \) (\( k = 1, \ldots, n \)). We proceed with the above adjustment procedure for all of the maximal cubes from the family \( \{Q_i^\#\}_{i \in I} \). Call the adjusted pre-atom \( a_{Q_1,\ldots,i_s,1} \). Observe that these new adjusted functions have the same support, the same mean, and the \( L^1 \) norm is still \( \leq 1 \). Additionally, the new functions satisfy:

- **outside** \( \bigcup_{i_s+1} Q_{i_1,\ldots,i_s+1} \) we have

  \[
  |a_{Q_1,\ldots,i_s,1}(x)| \leq \frac{1}{|Q_{i_1,\ldots,i_s}|},
  \]

- **on** \( \bigcup_{i_s+1 \in \Lambda} Q_{i_1,\ldots,i_s+1} \) we have

  \[
  |a_{Q_1,\ldots,i_s,1}(x)| \leq \frac{2 \cdot 2^d}{|Q_1,\ldots,i_s|},
  \]

  where the summation extends over those cubes \( Q_{i_1,\ldots,i_s+1} \) whose immediate parent belongs to \( \{Q_i^{\#}\}_{i \in I} \).

- \( \{ x : |a_{Q_1,\ldots,i_s,1}(x)| \geq \frac{1}{|Q_{i_1,\ldots,i_s}|} \} \subset \bigcup_{i_s+1} Q_{i_1,\ldots,i_s+1} \).
and the respective parent $Q^\#_{i_0}$ of each cube $Q_{i_1, \ldots, i_x+1}$ with $i_x+1 \in \Lambda$ satisfies
\[
\frac{1}{|Q^\#_{i_0}|} \int_{Q^\#_{i_0}} a_{Q_{i_1, \ldots, i_x}, 1}(x) \, dx = \frac{1}{|Q^\#_{i_0}|} \int_{Q^\#_{i_0}} a_{Q_{i_1, \ldots, i_x}, 1}(x) \, dx + \\
+ \sum_{k=n+1}^{2^d} \int_{Q_k} a_{Q_{i_1, \ldots, i_x}, 1}(x) \, dx
\]
\[
= \frac{1}{|Q^\#_{i_0}|} \sum_{k=1}^{n} \hat{\alpha} \cdot |Q_k| + \sum_{k=n+1}^{2^d} \int_{Q_k} a_{Q_{i_1, \ldots, i_x}, 1}(x) \, dx
\]
\[
= \frac{1}{|Q^\#_{i_0}|} \int_{Q^\#_{i_0}} a_{Q_{i_1, \ldots, i_x}}(x) \, dx.
\]

To keep notation simple we drop the dependence on $Q_{i_1, \ldots, i_x+1}$ when referring to $Q_k$, $n$, $\alpha_k$ and $\hat{\alpha}$, and hope this does not lead to confusion. We let $b_1 = a_{Q_{i_1, \ldots, i_x}} - a_{Q_{i_1, \ldots, i_x}, 1}$. Observe that $b_1 \neq 0$ only on
\[
S = \bigcup_{i_{x+1} \in \Lambda} \bigcup_{k=1}^{n} Q_k
\]
and has mean 0. By the construction we have
\[
\|b_1\|_{L^1(S)} \leq 2\|a_{Q_{i_1, \ldots, i_x}}\|_{L^1(S)}.
\]

Fix $i \in \Lambda$ and let $\beta_i = b_1 \cdot \mathbf{1}_{Q^\#_{i_0}}$, where $Q^\#_{i_0}$, $i_0 \in I$ is the parent of $Q_i$. Since $\beta_i$ is constant on each $Q_k$, $1 \leq k \leq n$, it satisfies
\[
|Q^\#_i| \|\beta_i\|_{L^\infty} \leq 2^d \sum_{k=1}^{n} |Q_k| \cdot |\alpha_k - \hat{\alpha}| = 2^d \|\beta_i\|_{L^1},
\]
and hence $\frac{\beta_i}{\|\beta_i\|_{L^1}}$ is an atom in the sense of Definition 1. That the condition (d) of the definition is satisfied is clear—the family of parent cubes $Q^\#$ reduces to the single cube $Q^\#_{i_0}$. Thus we have the atomic decomposition
\[
b_1 = \sum_{i \in \Lambda} \|\beta_i\|_{L^1} \beta_i.
\]

We now proceed in consecutive generations of adjustments, to adjust the value of $a_{Q_{i_1, \ldots, i_x}, 1}$ on remaining cubes $Q_{i_1, \ldots, i_x+1}$, $i_x+1 \notin \Lambda$. For each $Q^\#_i$ we consider, in turn, the cubes $Q_{n+1}, \ldots, Q_{2^d}$, starting with, say $Q = Q_{n+1}$. Let again $\{Q^\#_i\}_{i \in I'}$ be the family of maximal cubes like the family we considered before, but now within $Q_{n+1}$. We repeat the above procedure and obtain new adjusted “pre-atom” $a_{Q_{i_1, \ldots, i_x}, 2}$
obtained by modification of $a_{Q_1^i}^1, \ldots, a_{Q_1^i}^2$ on appropriate subcubes of the cubes $Q_i^#$, $i \in I'$. We observe that the supports of $b_1$ and $b_2$ are disjoint. We continue recurrently packing up the cubes $Q_i^#$. Clearly, we may have infinitely many iterations. As we finish, we have the atom $a_{Q_1^i}^1, \ldots, a_{Q_1^i}^\infty$ satisfying conditions of Definition 1 (verification of that is immediate, once we observe that the “pre-atom” $a_{Q_1^i}^1, \ldots, a_{Q_1^i}^1$ has only been modified on cubes $Q$ for which we use Definition 1(d)), and a sequence of correction atoms $\beta_j$’s of disjoint supports. We thus have

$$\sum_j \|\beta_j\|_{L^1} \leq 2\|a_{Q_1^i}^1, \ldots, a_{Q_1^i}^\infty\|_{L^1}.$$ 

The theorem follows.

**An application** Let $\varphi$ be a radial kernel, with support in a ball of radius $1 + \frac{1}{d}$, with its radial profile constant on a ball of radius $1$, linear for $1 \leq |x| \leq 1 + \frac{1}{d}$, such that \( \int \varphi = 1 \). Let $\varphi_t$ be the $L^1$ normalized dilation, $\varphi_t(x) = \frac{1}{t^d} \varphi(\frac{x}{t})$. We will prove the following

**Theorem 7** For an atom $a$ satisfying the axioms of Definition 1, supported on the cube $[0, 1]^d$, we have

$$\|\sup_{t \leq \frac{1}{d}} \varphi_t * a\|_{L^1} \leq Cd \log(d) \|a\|_{L^1}$$  \hspace{1cm} (4)

with $C$ an absolute constant. As a consequence, the operator norm of the maximal function

$$Mf(x) = |\sup_{t > 0} \varphi_t * f(x)|, \hspace{1cm} (5)$$

acting from $\mathbb{H}^1 \rightarrow L^1$, is at most $Cd \log(d)$.

We also prove the lower estimate $C\frac{d}{\log(d)}$ for the $\mathbb{H}^1 \rightarrow L^1$ norm of the maximal function (5). See comments following the proof of Theorem 7.

**Proof** Let us fix an atom $a$ supported on $Q = [0, 1]^d$. We begin with a sequence of lemmas.

**Lemma 8** Let $Q$ be a cube with sidelength $l(Q)$, $y \in Q$, $t \geq \frac{d^2 l(Q)}{2}$ and

$$1_t = 1_B, \text{ where } B = B(0, t(1 + \frac{1}{d})), \hspace{1cm} (ball \ of \ center \ 0 \ and \ radius \ t(1 + \frac{1}{d})).$$

We then have

$$|\varphi_t(x - y_c) - \varphi_t(x - y)| \leq C \cdot \frac{d}{t} \cdot 1_t(x - y) \cdot \frac{1}{|B|} \cdot |y - y_c|, \hspace{1cm} (6)$$

where $y_c$ is the center of $Q$. Consequently, for a mean-zero function $a$ supported on a cube $Q$, satisfying $\|a\|_{L^\infty} \leq 1$, with $Q$ and $t$ as above, we have

$$|\varphi_t * a(0)| \leq C \cdot \frac{d^3 l(Q)}{t} \cdot \|1_t\|_{L^1(Q)} \cdot \frac{1}{|B|}.$$  \hspace{1cm} (7)
Proof The first assertion (6) follows from the mean-value theorem. Observe, that

\[ \|\nabla \varphi\|_{\infty} = \frac{d}{c_d}, \] where \( c_d \sim v_d \), the volume of the unit ball.

Moreover, if \( x - y \) is outside \( B \),

\[ |x - y| \geq t(1 + \frac{2}{d}) \geq t \left( 1 + \frac{1}{d} \right) \Rightarrow \varphi_t(x - y) = 0. \]

Also,

\[ |x - y_c| \geq |x - y| - |y - y_c| \geq t \left( 1 + \frac{2}{d} \right) - |y - y_c| \geq t \left( 1 + \frac{1}{d} \right), \]

since

\[ |y - y_c| \leq \sqrt{d} t |Q| \leq \frac{t}{d}. \]

Thus \( \varphi_t(x - y_c) = 0 \) and this justifies \( \mathbb{1}_t \) on the right hand side of (6). The second assertion (7) follows immediately.

We have the following corollary to Lemma 8, whose proof we leave to the reader:

**Corollary 9** Suppose \( a_1, a_2, \ldots \) are mean-zero functions supported on disjoint cubes of sidelengths \( \rho \), all contained in some cube \( Q \). Assume \( \|a_i\|_{L^\infty} \leq 1 \) and \( t \geq d^{3/2} \rho/2 \). Then

\[ \left| \varphi_t * \sum_i a_i(0) \right| \leq C \cdot \frac{d^{3/2} \rho}{t |B|} \cdot \|\mathbb{1}_t\|_{L^1(Q)}. \] (8)

We recall that according to Definition 1(d) for each atom \( a \), supported on a cube \( Q \), we have distinguished cubes \( Q_s \) such that \( a \) is constant (with value no greater than \( 2d^{1/2} + 1/|Q| \)) on each of the \( Q_s \)’s. We will call these distinguished cubes “black”. For a black cube \( Q \) the value of the atom \( a \) on \( Q \) will be denoted \( a_Q \).

**Lemma 10** Let us fix an integer \( s \) with \( 2^{-s} \approx td^{1/2} \). Let \( \mathcal{M} \) be the family of the maximal \( Q^# \) with sidelengths \( \leq 2^{-s} \) (cubes \( Q^# \) are the parent cubes of black cubes given by Definition 1(d) for the fixed atom \( a \)). The atom \( a \) decomposes as a sum

\[ a = a_1^s + a_2^s + a_3^s, \] (9)

where

\[
\begin{align*}
a_1^s(x) &= \begin{cases}
1_{Q^#}(x)/|Q^#| \int_{Q^#} a & : \text{on any black cube with sidelength } \geq 2^{-s}, \\
a(x) & : \text{otherwise},
\end{cases} \\
a_2^s(x) &= \sum_i 1_{Q_i}(x) a(x),
\end{align*}
\]
where $Q_i$’s are all the black cubes with sidelengths $\geq 2^{-s}$, and

$$a_3^i(x) = \begin{cases} a(x) - \frac{1}{|Q^\#|} \int_{Q^\#} a : x \in Q^\#, Q^\# \in M, \\ : \text{otherwise.} \end{cases}$$

We have

$$|\varphi_t * a_3^i| \leq 1,$$  \hspace{1cm} (10)

and, moreover, for $t \leq \frac{1}{d}$

$$\text{supp}(\varphi_t * a_3^i) \subset (1 + \frac{4}{d})Q.$$  \hspace{1cm} (11)

For a cube $Q$ and a positive number $s$, $sQ$ means cube with the same center as $Q$, and sidelength equal to $s$ times the sidelength of $Q$.

**Proof** The only assertions requiring proof are (10) and (11). Estimate (10) follows from the fact that, by the definition of an atom, $a_3^i$ is bounded by 1. Assertion (11) follows from the support considerations: $\text{supp}^\prime t \subset B(0, (1 + \frac{1}{d}) \cdot t) \subset B(0, (1 + \frac{1}{d}) \cdot \frac{1}{d})$, $\text{supp} a \subset Q$ while $(1 + \frac{4}{d})Q = [-\frac{2}{d}, 1 + \frac{2}{d}]^d$.

Applying directly the decomposition (9), (10) and (11) we obtain the following corollary.

**Corollary 11** We have

$$|\varphi_t * a(x)| \leq C \cdot \mathbb{1}_{(1 + \frac{4}{d})Q}(x) + \left( \sum_{0 < t Q_i \text{ black}, l(Q_i) \geq d2^{-s}} |\alpha_{Q_i}| \cdot \mathbb{1}_{(1 + 4/d)Q_i}(x) \right) +$$

$$+ \left( \sum_{0 < t Q_i \text{ black}, 2^{-s} \leq l(Q_i) < d2^{-s}} |\alpha_{Q_i}| \cdot \varphi_t * \mathbb{1}_{Q_i}(x) \right) + |\varphi_t * a_3^i(x)|,$$

$$= I + II + III + IV$$

where, as before, we denote by $l(Q)$ the sidelength of $Q$.

The first two summands give rise to the $L^1$ control of the maximal function with constants independent of the dimension. For the third summand we have the uniform in $t$ estimate

$$III \leq \sum_{Q_i \text{ black,}} |\alpha_{Q_i}| \sup_{d^{-1}l(Q_i) \leq t \leq l(Q_i)d^{\frac{3}{2}}} \varphi_t * \mathbb{1}_{Q_i}(x)$$

$$\leq \sum_{Q_i \text{ black,}} |\alpha_{Q_i}| \left( \varphi_{t_0} + \int_{d^{-1}l(Q_i) \leq t \leq l(Q_i)d^{\frac{3}{2}}} |\partial_t \varphi_t| dt \right) * \mathbb{1}_{Q_i}(x),$$
where \( t_0 = d^{-1} l(Q_i) \). Observe the following estimate (\( r = |x|, \phi(|x|) = \varphi(x) \))

\[
|\partial_t \varphi_t(x)| = \left| \partial_t \left( \phi \left( \frac{r}{t} \right) \frac{1}{rt} \right) \right| \\
\leq \left| \frac{1}{t} \right| + \left| \phi' \left( \frac{r}{t} \right) \frac{1}{r^2t^2} \right| + \left| \frac{d}{r^2t^3} \phi \left( \frac{r}{t} \right) \right| \\
\leq \frac{1}{c_d} \frac{\mathbb{1}_{B(1+\frac{1}{d})} \left( \frac{r}{t} \right)}{t^{d+2}} \cdot r \cdot d + \frac{1}{c_d} \frac{d}{t^{d+1}} \frac{\mathbb{1}_{B(1+\frac{1}{d})} \left( \frac{r}{t} \right)}{r} \\
\leq \frac{3}{c_d} \frac{d}{t^{d+1}} \frac{\mathbb{1}_{B(1+\frac{1}{d})} \left( \frac{r}{t} \right)}{r},
\]

where \( c_d \) is the constant from the proof of Lemma 8, and we have used the fact that \( \frac{1}{t} \leq 1 + \frac{1}{d} \). Consequently

\[
\| \partial_t \varphi_t \|_{L^1} \leq C \frac{d}{t},
\]

and we obtain

\[
\int_{d^{-1} l(Q_i) \leq t \leq l(Q_i)} \| \partial_t \varphi_t \|_{L^1} \leq C d \log(d). \tag{12}
\]

The estimate \( \| \varphi_{t_0} \|_{L^1} \leq C d \) is immediate. As a consequence, we have

\[
\| III \|_{L^1} \leq C d \log(d) \| a \|_{L^1}.
\]

The last summand IV will be estimated using Corollary 9. We have the following obvious observation.

**Lemma 12** We have

\[
a_3^s(x) = \sum_{n \geq s} \left( E_{-n} a(x) - E_{-n-1} a(x) \right)
\]

\[
= \sum_{n \geq s} \sum_{Q \in D_n} \left( \frac{|Q|}{|Q|} \int_Q a - \sum_{0 < |Q'|} \left( \frac{|Q'|}{|Q|} \right) \int_{Q'} a \right)
\]

\[
= \sum_{n \geq s} \sum_{0 < |Q'|} \text{remainder} + \text{all } Q \in C(Q), \text{ not black}
\]

\[
= I + II,
\]

where \( D_n \) denotes the family of dyadic cubes of sidelengths \( 2^{-n} \), and \( C(Q) \) denotes the family of immediate dyadic descendants of a cube \( Q \).

Observe, that for a fixed \( n \geq s, Q \in D_n, Q \) of type contained in \( I \), the \( a_Q = a_3^s \cdot \mathbb{1}_Q \)

\[
a_Q(x) = \frac{\mathbb{1}_Q(x)}{|Q|} \int_Q a - \sum_{0 < |Q'|} \left( \frac{|Q'|}{|Q|} \right) \int_{Q'} a
\]
satisfy the assumptions of Corollary 9 with \( \rho = 2^{-n} = 2^{-s-2^{-l}} \), and we can sum up with respect to \( l \). As a result, we obtain a dimension free \( L^\infty \) bound. Let \( Q \in D_n \), \( n \geq s \) be of the type appearing in \( II \), that is such that at least one \( Q' \in C(Q) \) is black. We will then say that \( Q \) has type 2. We decompose \( a_Q \) further into average 0 functions

\[
a_Q(x) = b_Q(x) + e_Q(x)
\]

where

\[
b_Q(x) = \frac{1}{|Q|} \int_{Q': Q' \text{ black}} a - \sum_{Q' \in C(Q) \text{ black}} \frac{1}{|Q'|} \int_{Q'} a
\]

and

\[
e_Q(x) = \frac{1}{|Q|} \int_{Q': Q' \text{ not black}} a - \sum_{Q' \in C(Q) \text{ not black}} \frac{1}{|Q'|} \int_{Q'} a
\]

Observe, that the family \( e_Q(x) \) satisfies again the condition of the Corollary 9, so by the preceding case argument we get

\[
\left| \sum_{Q \in D_n, Q \text{ of type 2}} \varphi_t * e_Q \right| \leq C \cdot 2^{-l}
\]

and we again sum up to obtain a dimension free \( L^\infty \) bound.

We are left with the estimate for

\[
J = \sum_{n \geq s} \left| \sum_{Q \in D_n, Q \text{ of type 2}} \varphi_t * b_Q(x) \right|
\]

where we have an additional relation \( 2^{-s} d^{3/2} \simeq t \). We have

\[
J \leq \sum_{Q \text{ of type 2}} \sup_{t \geq d^{3/2} l(Q)} \left| \varphi_t * b_Q(x) \right|
\]

and the right hand side does not depend on \( t \). Observe that by a standard cancellation argument, using (6), we get

\[
\left| \varphi_t * b_Q(x) \right| \leq C \left\| b_Q \right\|_{L^1} \frac{d^{3/2} l(Q)}{t |B|} \int_1 \left| I_t(x-y)b_Q(y) \right| dy,
\]

where, as before, \( y_c \) is the center of the cube \( Q \), \( B = B(0, t(1 + \frac{2}{d})) \), \( I_t = 1_B \) and \( t \geq d^{3} l(Q) \). Consequently,

\[
\left| \varphi_t * b_Q(x) \right| \leq C \left\| b_Q \right\|_{L^1} \frac{d^{3/2} l(Q) 1_{B(Y_c, t(1+2/d))}(x)}{t |B|}.
\]
We have, for $|x - y_c| \geq (1 + 1/d) d^{3/2} l(Q)$

$$\sup_{t \geq d^{3/2} l(Q)} \frac{d^{3/2} l(Q) \cdot \mathbb{1}_{B(x, t(1+2/d))}(x)}{t|B|} = \frac{d^{3} l(Q)(1 + 2/d)^{d+1}}{|B(0, 1 + 2/d)| |x - y_c|^{d+1}}.$$ 

and, integrating in polar coordinates, the expression (13) has $L^1$ norm bounded by $C\|b\|_{L^1} d$ ($b$ is the required sum of $b_Q$’s). Since the case $|x - y_c| \leq (1 + 1/d) d^{3/2} l(Q)$ is immediate, the main estimate (4) follows.

The estimates of the maximal function over the intervals $\frac{1}{d} \leq t \leq d^{3/2}$ and $t \geq d^{3/2}$ follow similarly to (12), (13). We leave the details for the reader. Theorem 7 follows.

We now briefly sketch the argument leading to the maximal function estimates from below. We recall that $B$, $|B|$ denote the unit ball in $\mathbb{R}^d$ and its Lebesgue measure.

First observe, that for $A = 2^{[\log(d)]} \approx d^2$, the function

$$h(x) = 2^{-d} \mathbb{1}_{[-1,1]^d}(x) - (2A)^{-d} \mathbb{1}_{[-A,A]^d}(x) = h_1(x) - h_2(x)$$

(14)

defined on $\mathbb{R}^d$ has $\mathbb{H}^1$ norm of order $\log(d)$. This can be easily checked using the formula

$$h(x) = \sum_{s=0}^{2[\log(d)]-1} 2^{-ds} \mathbb{1}_{[-2^s,2^s]^d}(x) - 2^{-d(s+1)} \mathbb{1}_{[-2^{s+1},2^{s+1}]^d}(x) = \sum_{s=0}^{2[\log(d)]-1} h^s(x)$$

(15)

It can be easily checked that the expectation of each $h^s$ over the grid of the dyadic cubes of sidelength $2^l$, $l \geq s + 1$ vanish, and that the expectation over the grid of dyadic cubes of sidelength $2^l$, $l \leq s$ leaves $h^s$ unchanged. Consequently $h^s$ has its $\mathbb{H}^1$ norm equal to 2.

We then consider the linearized maximal operator $Th(x) = \varphi_t(x) \ast h(x)$, where we will assume $t(x) = |x| + 4 \leq 3d$ for $|x| \leq 2d$. Observe that $Th_2(x) = (2A)^{-d}$ for $|x| \leq Cd$ and this function restricted to the ball of radius $2d$ has the $L^1$ norm of order $O(1)$ (and even smaller).

Now observe, that the $L^1$ norm of $Th_1(x)$ restricted to the ring $d \leq |x| \leq 2d$ is at least $cd$, where $c > 0$ is dimension free. The crucial observation is that if $t(x) = |x| + 4$, the ball $B(x, t(x))$ covers all points in the support of $h_1$ lying below (that is in the direction of $x$) the hyperplane passing through 0 and perpendicular to $x$. Since $\varphi(x) \geq \frac{c}{|B|} 1_B$ ($B$—the unit ball), where $c_0$ is a dimension free constant, as a result we have $Th_1(x) \geq \frac{c}{2|B|} (|x| + 4)^{-d}$. and the statement follows by integration in polar coordinates.

Remark 13 Let $p_t$ denote the classical heat semigroup kernel, and $P^*$ its associated maximal operator. Applying the above argument together with the estimate

$$\|\partial_t p_t\|_{L^1} \leq \frac{C}{t} d^{1/2}$$
one can obtain estimates

\[ C \frac{d^{\frac{1}{2}}}{\log(d)} \leq \| P^* \|_{H^1 \to L^1} \leq C d^{\frac{1}{2}} \log(d) \]

We note, that approach based on the “near \( L^1 \)” method sketched in Remark 3 and on Rota’s theorem seems to give the upper estimate no better than \( Cd \).

**Remark 14** The following, easy to prove, inequality is very useful in obtaining the estimates from below

\[ M R(a)(x) \leq M a(x) \tag{16} \]

where \( M \) denotes the maximal function with respect to a radial kernel, and \( R(a) \) is the radialisation of a function \( a \):

\[ R(a)(r) = \int_{S^{d-1}} a(r\sigma) \, d\sigma, \]

where \( d\sigma \) is the normalized Lebesgue measure on the unit sphere. Using (16) one can obtain lower bound \( Cd \) for an example considered in Theorem 7. We do not present any further details.

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