Monadic forgetful functors and (non-)presentability for $C^*$- and $W^*$-algebras

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Abstract

We prove that the forgetful functors from the categories of $C^*$- and $W^*$-algebras to Banach $\ast$-algebras, Banach algebras or Banach spaces are all monadic, answering a question of J.Rosický, and that the categories of unital (commutative) $C^*$-algebras are not locally-isometry $\aleph_0$-generated either as plain or as metric-enriched categories, answering a question of I. Di Liberti and Rosický.

We also prove a number of negative presentability results for the category of von Neumann algebras: not only is that category not locally presentable, but in fact its only presentable objects are the two algebras of dimension $\leq 1$. For the same reason, for a locally compact abelian group $G$ the category of $G$-graded von Neumann algebras is not locally presentable.

Key words: $C^*$-algebra; $W^*$-algebra; locally presented; locally generated; monadic; Beck’s theorem; tripleability; enriched

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Introduction

Denote by $C^*_1$ and $C^*_{c,1}$ the categories of unital $C^*$- and unital commutative $C^*$-algebras respectively, and by $\text{Ban}$ the category of Banach spaces with linear maps of norm $\leq 1$ (i.e. contractions) as morphisms. The original impetus for the present paper was provided by [27, Remark 5.2 (3)], asking whether the forgetful functors

$$G : C^*_1 \to \text{Ban} \quad \text{and} \quad G_c : C^*_{c,1} \to \text{Ban}$$

(0-1)

are monadic. Settling for $C^*_1$ to fix ideas, recall (tripleable [19, §VI.3] or [6, §3.3]) that this means that $G$ induces an equivalence between $C^*_1$ and the Eilenberg-Moore category $\text{Ban}^{GF}$ of algebras over the monad $FG$, where $F : \text{Ban} \to C^*_1$ is the left adjoint of $G$.

Per [19, §VI.8] or [3, §3.31] the prototypical monadic functors are the forgetful functors $V \to \text{Set}$ where $V$ is a variety of algebras: (possibly multi-sorted) sets equipped with operations required to satisfy various equations. Examples are the categories of semigroups, monoids, groups, rings, Lie algebras, modules over a fixed ring, etc. etc. The monadicity of the forgetful functors (0-1), then, roughly says that the passage from Banach spaces to (unital) $C^*$-algebras is effected by freely adjoining a number of operations and equations.

We prove that the functors are indeed monadic in Corollary 2.6, along with a number of variations. Aggregating all of the claims of Theorem 2.4 and Corollary 2.6:

Theorem The forgetful functors from the category $C^*_1$ of unital $C^*$-algebras to the categories of unital Banach $\ast$-algebras, unital Banach algebras and Banach spaces are all monadic.

The same holds for commutative ($C^*$- and Banach) algebras.
A number of other problems suggest themselves naturally, all circumscribed under the same general topic of studying operator algebras category-theoretically. For one thing, the obvious modification of the previous result goes through for von Neumann (or $W^*$-)algebras (see Theorem 4.11 and Corollary 4.13, as well as the beginning of Section 4 for a reminder on $W^*$-algebras):

**Theorem** The forgetful functors from the category $W^*_1$ of $W^*$-algebras to the categories of $C^*$-algebras, unital Banach $*$-algebras, unital Banach algebras and Banach spaces are all monadic.

In another direction, the tasks of proving monadicity, constructing (co)limit or adjoint functors, and so on simplify considerably when the categories in question are technically “reasonable”. This might mean, perhaps, that every object can be recovered as a colimit of appropriately “small” objects. To recall, briefly, the relevant concepts from [3, §1.B]:

**Definition 0.1** Let $\kappa$ be a regular cardinal, i.e. one that is not a union of strictly fewer strictly smaller sets.

- A poset $(I, \leq)$ is $\kappa$-directed if every subset of cardinality $< \kappa$ has an upper bound.
- A $\kappa$-directed colimit in a category $C$ is a colimit of a functor $I \to C$ for a $\kappa$-directed poset $(I, \leq)$, where the latter is regarded as a category with exactly one arrow $i \to j$ whenever $i \leq j$.
- An object $c \in C$ is $\kappa$-presentable if $\text{hom}_C(c, -) : C \to \text{Set}$ preserves $\kappa$-directed colimits. An object is presentable if it is $\kappa$-presentable for some $\kappa$.
- A category $C$ is locally $\kappa$-presentable if it is cocomplete and has a set $S$ of $\kappa$-presentable objects so that every object is a $\kappa$-directed colimit of objects from $S$. A category is locally presentable if it is locally $\kappa$-presentable for some $\kappa$.

It has been known for some time that the categories $C^*_1$ and $C^*_c,1$ are locally $\aleph_1$-presentable (this follows, for instance, from [3, Theorem 3.28] and the fact that they are varieties of infinitary algebras [26, Theorem 2.4]). It is natural to ask whether the category of $W^*$-algebras is also locally presentable:

- on the one hand it is certainly cocomplete (Proposition 4.1, [17, Proposition 5.7], etc.);
- while on the other hand the local-presentability claim does seem to occur in some of the literature (see §4.1 for examples).

We will nevertheless prove below a strong negation of local presentability for $W^*_1$ (Theorem 4.2 and Proposition 4.10):

**Theorem** The only presentable objects in the category $W^*_1$ of von Neumann algebras are $\{0\}$ and $\mathbb{C}$.

As a consequence, for any locally compact abelian group $G$, the category $W^*_1,G$ of $G$-graded von Neumann algebras in the sense of Definition 4.9 is not locally presentable.

The notion of presentability (for an object) captures the intuition of being definable by “few generators and relations”: in, say, categories of modules, being $\aleph_0$-presentable in the sense of Definition 0.1 is equivalent to being finitely-presentable in the usual sense (finitely many generators, finitely many relations) [3, §3.10 (2)].

There is, similarly, an abstract formulation for the weaker notion of being definable by few generators (regardless of relations). Having fixed an appropriate class $M$ of monomorphisms in a category $C$ (half of a factorization system [1, Definition 14.1]; those details are not crucial here), recall, say, [2, p.15, Definition] or [9, Definitions 2.1 and 2.5]:

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Definition 0.2 Let $C$ be a cocomplete category and $\kappa$ a regular cardinal.

- An object $c \in C$ is $\kappa$-generated with respect to (wrt) $\mathcal{M}$ if $\text{hom}_C(c, -)$ preserves $\kappa$-directed colimits of morphisms from $\mathcal{M}$.

- $C$ is $\mathcal{M}$-locally $\kappa$-generated if there is a set $S$ of $\kappa$-generated objects wrt $\mathcal{M}$ such that every object is expressible as a $\kappa$-directed colimit of $\mathcal{M}$-morphisms between objects from $S$.

- $C$ is $\mathcal{M}$-locally generated if it is $\mathcal{M}$-locally $\kappa$-generated for some regular cardinal $\kappa$.

When $\mathcal{M}$ is the class of all monomorphisms this recovers [3, Definition 1.67], and as for presentability, the concept embodies the right intuition in familiar cases: an object in a category of modules is $\aleph_0$-generated wrt the class of monomorphisms if and only if it is finitely-generated in the usual sense [3, §3.10 (1)].

[9, Remark 6.10] speculates that the category $C_1^*$ is not isometry-locally $\aleph_0$-generated. The $\aleph_0$-generation notion discussed there is not quite that of Definition 0.2, but rather an enriched version thereof [9, Definitions 4.1 and 4.4]: everything in sight (Banach spaces/algebras, $C^*$-algebras) is regarded as enriched over the category CMet of complete generalized metric spaces, where ‘generalized’ means the distance is allowed values in $[0, \infty]$. One can then reiterate Definition 0.2 by requiring that the colimit-preservation condition

$$\text{hom}(c, \lim\limits_i c_i) \cong \lim\limits_i \text{hom}(c, c_i)$$

take place in the enriching category CMet instead of Set. Either way (enriched or not), we can confirm that speculation: see Proposition 3.1, Corollary 3.2, Proposition 3.3 and Corollary 3.4.

Theorem Let $A$ be a commutative unital $C^*$-algebra and $\mathcal{M}$ the class of isometric $C^*$ morphisms.

(a) $A$ is $\aleph_0$-generated wrt $\mathcal{M}$ in the (plain or enriched) category $C_{c,1}^*$ if and only if it is finite-dimensional.

(b) $A$ is $\aleph_0$-generated wrt $\mathcal{M}$ in the ordinary category $C_1^*$ if and only if it has dimension $\leq 1$.

(c) $A$ is $\aleph_0$-generated wrt $\mathcal{M}$ in the CMet-enriched category $C_1^*$ if and only if it is finite-dimensional.

Consequently, $C_1^*$ and $C_{c,1}^*$ are not isometry-locally $\aleph_0$-generated, either as plain categories or as CMet-enriched categories.

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1 Preliminaries

The requisite category-theoretic background is amply covered in, say, [19, 6, 3], with more precise references given below, as needed.

For objects $x, y$ of a category $C$, both $\text{hom}_C(x, y)$ and $C(x, y)$ denote the respective set of morphisms, and we depict adjunctions $(F, G)$ with $F$ as the left adjoint as $F \dashv G$.

We also assume some material on operator algebras ($C^*$ or $W^*$), for which the reader can consult any number of excellent sources (some cited below in more detail): [8, 32, 10, 11], etc.

We write
• $\mathcal{C}_1^*$ and $\mathcal{C}_{c,1}^*$ for the categories of unital (commutative) $C^*$-algebras respectively;

• $\text{BanAlg}_1$ and $\text{BanAlg}_{c,1}$ for the categories of unital (commutative) Banach algebras respectively;

• $\text{BanAlg}_{*1}$ and $\text{BanAlg}_{*c,1}$ for the categories of unital (respectively commutative) Banach $*$-algebras;

• and finally, $\text{Ban}$ for the category of complex Banach spaces with contractions (these being the “appropriate” morphisms when handling Banach spaces category-theoretically [3, §1.48]).

2 Monadic categories of $C^*$-algebras

[27, Remark 5.2 (3)] asks whether the forgetful functors

$$G : \mathcal{C}_1^* \to \text{Ban} \quad \text{and} \quad G_c : \mathcal{C}_{c,1}^* \to \text{Ban}$$

are monadic [19, §VI.3] (or tripleable [6, §3.3]), i.e. whether they can be identified with the forgetful functors from the categories of algebras for the monads $GF$ and $G_cF_c$ attached to the adjunctions

$$F \dashv G \quad \text{and} \quad F_c \dashv G_c.$$

Corollary 2.6 gives affirmative answers to those two questions.

We first recall some category-theoretic language and background. Recall ([6, §3.3]):

**Definition 2.1** For a category $\mathcal{C}$:

• A pair of morphisms $\partial_i : A \to B$ $i = 0, 1$ in $\mathcal{C}$ is **reflexive** if the two arrows have a common **section**, i.e. a morphism $t : B \to A$ with

$$\partial_1 t = \text{id}_B = \partial_0 t.$$

• A **reflexive coequalizer** is a coequalizer of a reflexive pair.

• A pair $\partial_i : A \to B$ is **contractible** or **split** if there is an arrow $t : B \to A$ such that

$$\partial_0 t = \text{id}_B, \quad \partial_1 \circ t \circ \partial_1 = \partial_1 \circ t \circ \partial_0.$$  

• A **contractible** or **split coequalizer** in $\mathcal{C}$ consists of a diagram

\[
\begin{array}{ccc}
A & \xrightarrow{t} & B \\
\downarrow{\partial_0} & & \downarrow{e} \\
\downarrow{\partial_1} & & \downarrow{s} \\
\end{array}
\]

such that

$$e \partial_0 = e \partial_1, \quad es = \text{id}, \quad \partial_0 t = \text{id}, \quad \partial_1 t = se.$$  

Just the presence of the arrows and the equations automatically implies that $e$ is a coequalizer for $\partial_i$, $i = 0, 1$ ([6, §3.3, Proposition 2 (a)] or [19, §VI.6, Lemma]), hence the name (‘split coequalizer’). Furthermore, if a contractible pair of parallel arrows has a coequalizer, it is split [6, §3.3, Proposition 2 (c)]; this justifies the terminology coincidence in the last two bullet items.
Given a functor \( G : \mathcal{C} \to \mathcal{D} \), a pair \( \partial_i : A \to B, i = 0, 1 \) in \( \mathcal{C} \) is \( G \)-split or \( G \)-contractible if the pair \( G\partial_i \) is contractible in \( \mathcal{D} \). The notion of being \( G \)-reflexive is defined analogously.

Various criteria ensure that functors are monadic, each useful under appropriate circumstances. Recall two such sets of criteria ([6, §3.3 Theorem 10 and subsequent discussion] as well as [6, §3.5, paragraph preceding Proposition 1]):

**Definition 2.2** A functor \( G : \mathcal{C} \to \mathcal{D} \)

- satisfies the *Crude Tripleability Theorem (CTT)* if
  1. it has a left adjoint;
  2. it reflects isomorphisms;
  3. \( \mathcal{C} \) has coequalizers for those reflexive pairs \( \partial_i, i = 0, 1 \) for which \( G\partial_i \) has a coequalizer, and \( G \) preserves those coequalizers.

- satisfies the *Precise Tripleability Theorem (PTT)* if
  1. it has a left adjoint;
  2. it reflects isomorphisms;
  3. \( \mathcal{C} \) has coequalizers for reflexive \( G \)-split pairs, and \( G \) preserves them.

We also use ‘CTT’ and ‘PTT’ as adjectives (e.g. ‘the functor \( G \) is PTT’).

According to *Beck’s theorem* ([6, §3.3, Theorem 10]), being PTT is equivalent to being monadic, so these terms (along with ‘tripleable’ will be interchangeable). On the other hand, as noted in [6, §3.5, paragraph preceding Proposition 1], being CTT is *sufficient* for monadicity.

The following remark is a consequence of [6, §3.5, Proposition 1 (b)]; we include a proof for completeness.

**Lemma 2.3** Consider functors \( G'' : \mathcal{A} \to \mathcal{B} \) and \( G' : \mathcal{B} \to \mathcal{C} \). If \( G'' \) is CTT and \( G' \) is monadic then the composition \( G' \circ G'' \) is monadic.

**Proof** Since right adjoints and conservative functors are closed under composition, it suffices to show \( \mathcal{A} \) has coequalizers of reflexive \( G'G'' \)-split coequalizer pairs, and \( G'G'' \) preserves them.

Suppose we have a reflexive \( G'G'' \)-split coequalizer pair in \( \mathcal{A} \):

\[
\begin{tikzcd}
A & B \\
& t \arrow[lu, bend right, \partial_0] \arrow[ru, bend left, \partial_1]
\end{tikzcd}
\]  \hspace{1cm} (2-2)

such that \( t \) is the common section for \( \partial_0 \) and \( \partial_1 \), for which there is a split coequalizer diagram in \( \mathcal{C} \):

\[
\begin{tikzcd}
G'G''A & G'G''B \\
G'G''\partial_0 & G'G''\partial_1 \\
& Z_{\mathcal{E}} \arrow[ru, e_{\mathcal{E}}] \arrow[lu, s_{\mathcal{E}}]
\end{tikzcd}
\]  \hspace{1cm} (2-3)
Since $\mathcal{A}$ has reflexive coequalizers, there is a coequalizer $e: B \to Z$ for (2-2). It suffices to show $G'G''$ preserves the coequalizer.

Since $G''$ preserves reflexive coequalizers, $G''e: G''B \to G''Z$ is the coequalizer for the image of (2-2) under $G''$ in $\mathcal{B}$. Since $\mathcal{B}$ has coequalizers of reflexive $G'$-split coequalizer pairs, there is a coequalizer diagram in $\mathcal{B}$:

$$
\begin{array}{ccc}
G''A & \xrightarrow{G''t} & G''B \\
\downarrow{G''\partial_0} & & \downarrow{G''t} \\
G''A & \xrightarrow{G''t} & G''B \\
\downarrow{G''\partial_1} & & \downarrow{e_{\mathcal{B}}} \\
Z_{\mathcal{B}} & & Z_{\mathcal{B}}
\end{array}
$$

and since $G'$ preserves them, $G'e_{\mathcal{B}} \cong e_{\mathcal{B}}$. Now by the uniqueness of coequalizers, we have $G''e \cong e_{\mathcal{B}}$ and so $G'G''e \cong e_{\mathcal{B}}$. ■

**Theorem 2.4** The forgetful functors

$$
G : C^*_1 \to \text{BanAlg}^*_1 \quad \text{and} \quad G_c : C^*_c,1 \to \text{BanAlg}^*_c,1
$$

are both CTT and hence monadic.

**Proof** We focus on $C^*_1$, as the other argument is entirely parallel.

That $G$ is a right adjoint we can see as in, say, [27, §5] (which discusses the forgetful functor to $\text{Ban}$ instead). Isomorphism-reflection, on the other hand, follows from the fact that bijective morphisms of $C^*$-algebras are automatically isometries (and hence invertible) [5, discussion preceding Theorem 1.3.2].

The existence of coequalizers (reflexive or not) is not an issue: not only is $C^*_1$ cocomplete (i.e. has arbitrary colimits), but as noted in [9, Remark 6.10], it is locally $\aleph_1$-presentable in the sense of [3, Definition 1.17], because it can be realized [26, Theorem 2.4] as a variety of algebras equipped with $\aleph_0$-ary operations [3, Theorem 3.28].

It thus remains to argue that $G$ preserves reflexive coequalizers. Here too, much more is true: it creates arbitrary coequalizers (Lemma 2.5).

**Lemma 2.5** The forgetful functor

$$
G : C^*_1 \to \text{BanAlg}^*_1
$$

from unital $C^*$-algebras to unital Banach $*$-algebras creates arbitrary coequalizers in the sense of [6, §1.7, discussion following Proposition 3].

The analogous statement holds for the respective categories of commutative $C^*$- and Banach $*$-algebras.

**Proof** We once more focus on the non-commutative version to fix ideas, but the argument goes through virtually verbatim in general.

Consider a parallel pair $f, g : A \to B$ of unital $C^*$-morphisms. The coequalizer in $\text{BanAlg}^*_1$ is obtained by annihilating the closed ideal

$$
I := B \cdot \{f(a) - g(a) \mid a \in A\} \cdot B
$$

and then equipping the quotient $B/I$ with the largest Banach-space norm making $\pi : B \to B/I$ contractive: for $x \in B/I$,

$$
\|x\| := \inf\{\|b\| \mid b \in B, \pi(b) = x\}.
$$
But because $I \trianglelefteq B$ is a closed *-ideal, this is already a $C^*$-norm on $B/I$ [5, §1.3, Corollary 2]. In conclusion, for two parallel unital $C^*$-morphisms the coequalizer constructions in $C^*$ and $\text{BANALG}_1^*$ coincide.

Theorem 2.4 in turn implies

**Corollary 2.6** The following forgetful functors are all monadic:

(a) From $\mathcal{C}_1^*$ to unital Banach *-algebras, unital Banach algebras, or Banach spaces.

(b) From $\mathcal{C}_{c,1}^*$ to unital commutative Banach *-algebras, unital commutative Banach algebras, or Banach spaces.

**Proof** In each case the functor in question decomposes as one of the CTT (Theorem 2.4) forgetful functors

$$
\mathcal{C}_1^* \rightarrow \text{BANALG}_1^* \quad \text{or} \quad \mathcal{C}_{c,1}^* \rightarrow \text{BANALG}_{c,1}^*,
$$

followed by forgetful functors

$$
\text{BANALG}_1^* \rightarrow \text{BANALG}_1, \quad \text{BANALG}_1^* \rightarrow \text{BAN},
$$

or

$$
\text{BANALG}_{c,1}^* \rightarrow \text{BANALG}_{c,1}, \quad \text{BANALG}_{c,1}^* \rightarrow \text{BAN}.
$$

That these last four functors are monadic follows either directly from [27, Theorem 5.1] or from very slight alterations to its proof, so the conclusion is a consequence of Lemma 2.3. ■

3 \ $\aleph_0$-generation

As noted in [9, Remark 6.10] (and recalled above in the course of the proof of Corollary 2.6) the categories $\mathcal{C}_1^*$ and $\mathcal{C}_{c,1}^*$ are both locally $\aleph_1$-presentable.

By contrast, the same [9, Remark 6.10] speculates that $\mathcal{C}_1^*$ is unlikely to be $\mathcal{M}$-locally $\aleph_0$-generated, where $\mathcal{M}$ is the class of unital $C^*$-embeddings (which are, in particular, automatically isometric). That this is indeed the case will follow from the identification of those commutative $C^*$-algebras that are $\aleph_0$-generated with respect to $\mathcal{M}$. As much of [9, Remark 6.10] makes sense for $\mathcal{C}_1^*$ and $\mathcal{C}_{c,1}^*$ either as plain categories or as categories enriched over metric spaces, we treat discuss the settings separately.

3.1 Ordinary categories of $C^*$-algebras

**Proposition 3.1** A commutative $C^*$-algebra is $\aleph_0$-generated w.r.t. the class $\mathcal{M}$ of $C^*$ embeddings

(a) in $\mathcal{C}_{c,1}^*$ if and only if it is finite-dimensional;

(b) and in $\mathcal{C}_1^*$ if and only if it has dimension $\leq 1$.

**Proof** Some claims can be treated simultaneously for both (a) and (b).

(\(=\)): **algebras of dimension $\leq 1$.** That is, either the zero algebra of the scalars. That these are indeed $\mathcal{M}$-generated (in either category) is immediate.

(\(\Rightarrow\)): **ruling out infinite-dimensional algebras in both (a) and (b).** Consider an $\mathcal{M}$-$\aleph_0$-generated commutative $C^*$-algebra $A \in \mathcal{C}_{c,1}^*$. We have $A \cong C(X)$ for some compact $X$ by
Gelfand-Naimark [8, Theorem III.2.2.4]. Denoting by \( X_d \) the underlying set \( X \) with the discrete topology, we now have a surjection
\[
\beta X_d \to X
\]
from the Stone-Čech compactification [21, §38, Definition preceding Exercises] of \( X_d \). Because \( X_d \) is discrete its Stone-Čech compactification is totally disconnected [21, §38, Exercise 7 (c)], and hence profinite [31, Tag 08ZY]: it is a filtered limit of finite (discrete) spaces, say
\[
\beta X_d = \lim_{\leftarrow i} X_i, \quad |X_i| < \infty.
\]
Dualizing back to commutative \( C^* \)-algebras, this gives us a one-to-one morphism
\[
C(X) \to \lim_{\rightarrow i} C(X_i)
\]
into a directed colimit of embeddings. The \( \aleph_0 \)-generation hypothesis then ensures that said embedding factors through some \( C(X) \to C(X_i) \), and since \( C(X_i) \) is finite-dimensional \( A \cong C(X) \) must be too.

(b) \( \Rightarrow \): ruling out algebras of dimension \( \geq 2 \) in \( C^*_1 \). First, note that the isometry-\( \aleph_0 \)-generation property is inherited by quotients, so it suffices to focus on the two-dimensional \( C^* \)-algebra \( A \cong \mathbb{C}^2 \).

A unital morphism \( A \to B \) simply picks out a projection in \( B \). All the statement claims, then, is that there are directed colimits
\[
B = \lim_{\rightarrow i} B_i
\]
of unital \( C^* \)-algebras that contain projections which belong to none of the individual \( B_i \). This is well known; [18, Example 1.3], for instance, explains how to construct such a projection (denoted on [18, p.711] by \( x \)) in the “Fermion algebra” of [25, discussion preceding Proposition 6.4.3]: the directed limit
\[
B = \lim_{\rightarrow n} M_{2^n}
\]
of the inclusions
\[
M_{2^n} \ni a \mapsto a \otimes 1 \in M_{2^n} \otimes M_2 \cong M_{2^{n+1}}.
\]

(a) \( \Leftarrow \): arbitrary finite-dimensional algebras in \( C^*_c,1 \). The finite-dimensional commutative \( C^* \)-algebras are those of continuous functions on finite discrete spaces, so upon dualizing by Gelfand-Naimark the claim is as follows: for every filtered limit
\[
X = \lim_{\leftarrow i} X_i
\]
of (surjections of) compact Hausdorff spaces, a continuous map \( \pi : X \to F \) to a finite discrete set must factor through one of the surjections \( \pi_i : X \to X_i \).

For a map \( f \) defined on \( X \) (such as \( \pi \) or the \( \pi_i \)) set
\[
\text{DIFF}(f) := \{(x, y) \in X^2 \mid f(x) \neq f(y)\}.
\]
These sets are always open for continuous \( f \) into Hausdorff spaces.

The realization of \( X \) as a limit ensures that every \( (x, y) \in \text{DIFF}(\pi) \) has some open neighborhood of the form
\[
U_{x,y,i} \subseteq \text{DIFF}(\pi_i)
\]
for some \( i \). Because \( \pi : X \to F \) has finite discrete codomain \( \text{DIFF}(\pi) \) is also closed in \( X^2 \), and hence compact covered by only finitely many \( U_{x,y,i} \). The map \( \pi \) will then factor through \( \pi_j \) for \( j \) dominating all \( i \) appearing among these finitely many \( U_{x,y,i} \), finishing the proof. \( \blacksquare \)
As announced above, Proposition 3.1 implies

**Corollary 3.2** The categories $C^*_1$ and $C^*_{c,1}$ are not isometry-locally $\aleph_0$-generated.

**Proof** Indeed, according to Proposition 3.1 the only commutative unital $C^*$-algebras that can be recovered as directed colimits of embeddings of isometry-$\aleph_0$-generated subalgebras are the AF algebras of [7, Definition 7.1.1]. Dualizing via Gelfand-Naimark [8, Theorem III.2.2.4], these are the algebras of the form $C(X)$ for profinite $X$. For that reason, commutative $C^*$-algebras cannot all be such colimits. ■

### 3.2 Metric-space-enriched categories

[16, §1.2] defines the notion of a $V$-category (often also called a $V$-enriched category) for a monoidal [16, §1.1] category $V$. That source quickly specializes in [16, §1.6] to symmetric monoidal closed [16, §§1.4, 1.5] categories $V$.

[9, §6] places $C^*_1$ and $C^*_{c,1}$ (and categories of Banach spaces, etc.) in this enriched context, with $CMet$ [4, Examples 2.3 (2)] as the enriching category $V$. This is the category of complete generalized metric spaces with non-expansive maps (also: ‘contractions’) as morphisms:

$$f : (X,d_X) \to (Y,d_Y), \quad d_Y(fx,fx') \leq d_X(x,x'), \forall x,x' \in X,$$

and ‘generalized’ means that the distance is allowed infinite values. We have an enriched counterpart to Proposition 3.1.

**Proposition 3.3** A commutative $C^*$-algebra is isometry-locally $\aleph_0$-generated in either of the $CMet$-enriched categories $C^*_1$ or $C^*_{c,1}$ if and only if it is finite-dimensional.

**Proof** We prove the two implications separately.

1. **finite-dimensional commutative $\Rightarrow \aleph_0$-generated.** Since finite-dimensional commutative $C^*$-algebras are of the form $C^n$, we have to examine morphisms

$$\phi : C^n \to A = \lim_{i} A_i$$

for a colimit of embeddings. The projection $e_1 := (1,0,\cdots,0) \in C^n$ is arbitrarily approximable by a projection $p \in A_i$ for some $i$ [35, Proposition L.2.2], so our original morphism $\phi : C^n \to A$ is arbitrarily approximable by morphisms mapping $e_1$ to such a projection $p \in A_i$.

Choosing such an approximation, assume now that $\phi(e_1) = p \in A_i$. We can work only with the indices $j \geq i$ (so that $p$ can be regarded as a projection in all $A_j$), and consider the morphism

$$\phi|_{(1-e_1)C^n} : (1-e_1)C^n \to pAp = \lim_{i} pA_ip.$$

A repetition of the previous procedure will now allow us to detach a further minimal projection $e_2$ from the $(n-1)$-dimensional $(1-e_1)C^n \cong C^{n-1}$ and assume, upon approximating, that $\phi(e_2) = q \in A_j$ for some $j$, etc.

In short: the original morphism $\phi : C^n \to A$ is arbitrarily approximable, uniformly on the unit ball of $C^n$, by $C^*$-morphisms that factor through some $A_i$.

2. **$\aleph_0$-generated commutative $\Rightarrow$ finite-dimensional.** We can repurpose the relevant portion of the proof of Proposition 3.1: write, once more, $A = C(X)$ for an infinite compact Hausdorff $X$, and consider the embedding

$$\iota : C(X) \to C(\beta X_d)$$
for the Stone-Čech compactification \( \beta X_d \) of the discrete space underlying \( X \).
\( \beta X_d \) is the inverse limit of its surjections \( \beta X_d \to X_F \) onto the finite discrete space associated to a finite partition \( F \) of \( X_d \), so that
\[
C(\beta X_d) \cong \lim_{\longrightarrow} C(X_F).
\]
If \( C(X) \) were isometry-locally \( \aleph_0 \)-generated in the enriched sense, we could at least approximate \( \iota \) arbitrarily well by morphisms \( C(\beta X_d) \to C(X_F) \) for various finite partitions \( F \) of \( X \). This means that for every \( \varepsilon > 0 \) there is some finite partition \( F = F_\varepsilon \) such that
\[
\forall f \in C(X) \ \exists f' \in C(X_F) : \| f - f' \| < \varepsilon \| f \|.
\]
In other words: every continuous function \( f \in C(X) \) is \( \varepsilon \| f \| \)-close to some function \( f' \) constant along every part of the finite partition
\[
F = (F_0, \cdots, F_n), \quad X = \bigsqcup_i F_i
\]
But if \( X \) is infinite, some \( F_i \) contains two distinct points \( x \neq y \in X \). We can then find a continuous
\[
f \in C(X), \quad \| f \| = 1, \quad f(x) = 0, \quad f(y) = 1
\]
and such a function is at least 1 apart in the uniform norm from any \( f' \in C(X_F) \).
\[\blacksquare\]
This in turn entails the enriched version of Corollary 3.2.

**Corollary 3.4** The CMet-enriched categories \( C^*_1 \) and \( C^*_{c,1} \) are not isometry-locally \( \aleph_0 \)-generated.

**Proof** As in the proof of Corollary 3.2, using Proposition 3.3 in place of Proposition 3.1. \[\blacksquare\]

With slightly more effort, we can transport part of Proposition 3.3 over to non-commutative \( C^* \)-algebras. The key point is

**Proposition 3.5** For any finite-dimensional \( C^* \)-algebra \( B \), a unital \( C^* \)-morphism
\[
\phi : B \to A := \lim_{\longrightarrow} A_i
\]
into a directed colimit of embeddings is arbitrarily approximable by morphisms \( B \to A_i \).

**Proof** Finite-dimensional \( C^* \)-algebras are finite products
\[
B \cong M_{n_1} \times \cdots \times M_{n_k}.
\]
Separating out the individual central factors \( M_{n_i} \) by means of minimal central projections as in the proof of Proposition 3.3, we can focus on the individual matrix algebras \( M_{n_i} \). In short, this allows us to assume \( B = M_n \) for the rest of the proof.

The matrix algebra \( M_n \) can be realized as the universal \( C^* \)-algebra generated by two unitaries \( S \) and \( U \), with
\[
S^n = U^n = 1, \quad USU^{-1} = \zeta S \quad (3-1)
\]
for some primitive \( n^{th} \) root of unity \( \zeta \).

We will henceforth identify \( M_n \) with its image through (the automatically-injective) \( \phi : M_n \to A \). The argument consists of a number of steps. Throughout, we write ‘\( \simeq \)’ to mean ‘is close to’. This will avoid having to keep careful track of \( \varepsilon \) estimates,
(1) First, $U$ can be approximated arbitrarily well with some order-$n$ unitary $U' \in A_i$, since the $C^*$-algebra it generates is commutative and finite-dimensional, and the commutative case has already been handled (in Proposition 3.3). Working only with indices dominating $i$, we can henceforth assume that all $A_j$ contain $U'$.

(2) Next, $S$ was a $\zeta$-eigenvector for $U$ and $U'$ is close to $U$, so the $U'$-$\zeta$-eigenvector

$$S' := \frac{1}{n} \sum_{s=0}^{n-1} \zeta^{-s} \cdot (U')^s S (U')^{-s}$$

is close to $S$, because

$$S = \frac{1}{n} \sum_{s=0}^{n-1} \zeta^{-s} \cdot U^s S U^{-s}.$$  

(3) Since conjugation by $U'$ leaves all $A_i$ invariant, we can further assume that $S' \simeq S$ belongs to one (and hence all) of the $A_i$.

(4) Being close to $S$ the, element $S'$ is invertible [35, Lemma 4.2.1]. Its polar decomposition [35, §1.5]

$$S' = S'' P, \ S'', P \in A_i$$

then provides a unitary $U'$-$\zeta$-eigenvector $S''$. We still have $S'' \simeq S$, because $S$ itself was unitary to begin with.

(5) The continuity of the spectrum on normal operators [14, Solution 105] together with $S'' \simeq S$ and

$$U' S'' (U')^{-1} = \zeta S''$$

show that the spectrum $\sigma(S'')$ of $S''$ is a subset of the unit circle invariant under multiplication by $\zeta$ and concentrated around $\{\zeta^s \mid 0 \leq s < n\}$. This latter condition means that $\sigma(S'')$ is contained in the union of $n$ mutually-disjoint closed arcs

$$I_s \ni \zeta^s, \ 0 \leq s < n.$$  

We can then find a continuous self-map $f : S^1 \to S^1$, equivariant under multiplication by $\zeta$, that compresses each $I_s$ onto $\zeta^s$. The operator $f(S'')$ obtained by functional calculus [8, §II.2.3] now has all of the desired properties:

- it is unitary of order $n$;
- satisfies

$$U' f(S'') (U')^{-1} = \zeta f(S'');$$

- and is contained in $A_i$ and close to $S'' \simeq S$.

We have thus achieved the desired goal of approximating $U$ and $S$ by unitaries in $A_i$ satisfying the same defining relations (3-1) (with $U'$ and $f(S'')$ in place of $U$ and $S$ respectively).

The announced consequence is now simply a rephrasing of Proposition 3.5.

**Corollary 3.6** Finite-dimensional $C^*$-algebras are isometry-locally $\aleph_0$-generated in the CMET-enriched category $C^*_1$.  

$\blacksquare$
The adjunctions between

- $\mathcal{C}_1^*$ and $\text{Ban}$;
- and similarly, $\mathcal{C}_{c,1}^*$ and $\text{Ban}$

can be put to much good use, but we caution the reader that they are not enriched over $\text{CMet}$. This is already visible when working with the simplest examples, as the following discussion prompted by [28] illustrates (focusing on $\mathcal{C}_{c,1}^*$, i.e. commutative $C^*$-algebras).

**Example 3.7** As noted on [30, p.173], the left adjoint $F$ to the forgetful functor $\mathcal{C}_{c,1}^* \to \text{Ban}$ (the so-called *Banach-Mazur functor*) can be described explicitly as

$$\text{Ban} \ni X \mapsto C(X_1^*) \in \mathcal{C}_{c,1}^*,$$

where $X_1^*$ is the unit ball of the Banach-space dual $X^*$, equipped with the weak* topology (wherein $X_1^*$ is compact, by the Banach-Alaoglu theorem [29, Theorem 3.15]).

Consider the one-dimensional Banach space $\mathbb{C}$. We then have $F(\mathbb{C}) \cong C(D)$, where $D \subset \mathbb{C}$ is the unit disk. The adjunction $F \dashv \text{FORGET}$ gives a bijection

$$\text{hom}_{\mathcal{C}_{c,1}^*}(C(D), \mathbb{C}) \cong \text{hom}_{\text{Ban}}(\mathbb{C}, \mathbb{C}),$$

which however is not an isometry in $\text{CMet}$ (or indeed, even a homeomorphism). To see this, note that

- The metric space $\text{hom}_{\text{Ban}}(\mathbb{C}, \mathbb{C})$ is $D$ with its usual metric, identifying a contraction

  $$T : \mathbb{C} \to \mathbb{C}$$

  with $T(1) \in D$.

- On the other hand, $\text{hom}_{\mathcal{C}_{c,1}^*}(C(D), \mathbb{C})$ is the *spectrum* [8, §II.2.1.4] $D$ of $C(D)$, metrized with the uniform distance on the unit ball of $C(D)$. Since for every two distinct points $x_{\pm1} \in D$ we can find a continuous function

  $$f : D \to [-1,1], \quad f(x_{\pm}) = \pm1.$$

  (e.g. by the *Tietze extension theorem*) [21, Theorem 35.1], the distance between any two distinct points of $D$ is 2. The topology acquired by $D$ as the metric space $\text{hom}_{\mathcal{C}_{c,1}^*}(C(D), \mathbb{C})$ is thus discrete. ♦

**Example 3.7** in fact illustrates a broader principle: as a $\text{CMet}$-enriched category, $\mathcal{C}_{c,1}^*$ is no more interesting than the plain category $\mathcal{C}_{c,1}^*$:

**Proposition 3.8** For any two commutative $C^*$-algebras $A, B \in \mathcal{C}_{c,1}^*$, the metric space

$$\text{hom}_{\mathcal{C}_{c,1}^*}(A, B) \in \text{CMet}$$

is discrete, with distance 2 between any two distinct elements.
Proof The proof is essentially contained in Example 3.7. We have
\[ A \cong C(X), \quad B \cong C(Y) \]
for compact Hausdorff \( X \) and \( Y \), and distinct morphisms \( \varphi \neq \psi : A \to B \) dualize to distinct continuous maps
\[ \varphi^* \neq \psi^* : Y \to X. \]
This means that
\[ x_+ := \varphi^*(y) \neq \psi^*(y) =: x_-, \quad \text{for some} \ y \in Y, \]
and a continuous norm-1 function
\[ f : X \to [-1, 1], \quad \varphi(x_\pm) = \pm 1 \]
will be mapped by \( \varphi \) and \( \psi \) to norm-1 functions on \( Y \) that are at least 2 apart. This shows that \( d(\varphi, \psi) \geq 2 \). On the other hand, because \( \varphi \) and \( \psi \) are contractions, we have
\[ d(\varphi, \psi) = \sup_{\|f\| \leq 1} \|\varphi(f) - \psi(f)\| \leq 2. \]
This concludes the proof that the distance between two distinct morphisms between two commutative \( C^* \)-algebras is precisely 2. \( \blacksquare \)

Remark 3.9 Nothing like Proposition 3.8 holds for the category \( C^*_1 \) of arbitrary (unital but not necessarily commutative) \( C^* \)-algebras: the identity automorphism of a non-commutative \( C^* \)-algebra \( A \) will generally be approximable arbitrarily well by inner automorphisms
\[ A \ni a \mapsto uau^* \in A \]
for unitaries \( u \in A \) close to 1. \( \blacklozenge \)

Proposition 3.8 has some consequences for the structure of \( C^*_{c,1} \) as a CMet-enriched category. First, recall the following notion from \([16, \S3.7]\).

Definition 3.10 A \( \mathcal{V} \)-enriched category \( C \) over a symmetric monoidal closed category \( \mathcal{V} \) is \( \mathcal{V} \)-tensored (or tensored over \( \mathcal{V} \)) if for objects \( x \in \mathcal{V} \) and \( c \in C \) the \( \mathcal{V} \)-functor
\[ \mathcal{V}(x, C(c, -)) : C \to \mathcal{V} \]
is representable. In that case we denote the representing object by \( x \otimes c \). \( \blacklozenge \)

Proposition 3.11 The CMet-enriched category \( C^*_{c,1} \) is tensored over CMet.

Proof It will be enough to prove that for \( X \in \text{CMet} \) and commutative \( C^* \)-algebras \( A, B \in C^*_{c,1} \) we have natural identifications
\[ C^*_{c,1}(X \otimes A, B) \cong \text{CMet}(X, C^*(A, B)) \] (3-2)
of sets (for some appropriate \( X \otimes A \in C^*_{c,1} \)) \( \blacklozenge \) Proposition 3.8 will then provide the identification as metric spaces, since both sides are discrete metric spaces with distance 2 between any two distinct points.
Appealing again to Proposition 3.8 for the discrete metric structure of $C^*_c(A, B)$, the right-hand side of (3-2) (regarded as a set) is nothing but

$$\text{Set}(X/ \sim, C^*(A, B)),$$

where ‘$\sim$’ is the smallest equivalence relation on $X$ that identifies two points less than 2 apart. But then we can simply define the tensor product $X \otimes A$ as

$$X \otimes A := (X/ \sim) \otimes A = A^{\otimes (X/ \sim)},$$

where

- the second ‘$\otimes$’ denotes the tensored structure of $C^*_c$ over Set (rather than CMet);
- and the last equality identifies that tensor product with the corresponding copower [16, §3.7] of $A$, i.e. coproduct in $C^*_c$ of copies of $A$.

As explained, this concludes the argument. ■

4 $W^*$-algebras

We make a few remarks on the category $W^*_1$ of von Neumann (or $W^*$-)algebras ([8, §III], [32, Definitions II.3.2 and III.3.1], etc.). These are the (unital) $C^*$-algebras that satisfy any of a number of mutually equivalent conditions:

- they are realizable as $C^*$-subalgebras $A \subseteq B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ so that $A = A''$, the double commutant (i.e. the set of operators commuting with everything that centralizes $A$);
- or $A \subseteq B(\mathcal{H})$ is closed in any of the six “weak” operator topologies of [32, §II.2] (weak, $\sigma$-weak, strong, $\sigma$-strong, strong*, $\sigma$-strong*) [11, Part I, §3.4, Theorem 2];
- or $A$ is a $C^*$-algebra that is also a dual Banach space (whose predual is then automatically unique) [32, Theorem III.3.5 and Corollary III.3.9].

The morphisms in $W^*_1$ are the unital normal [8, Proposition III.2.2.2] $C^*$-morphisms, i.e. those that are continuous for the $\sigma$-weak topologies induced by the realizations of the von Neumann algebras as duals.

The resulting category $W^*_1$ is almost that studied in [12], the difference being that the latter source considers possibly non-unital morphisms: this unitality is what the ‘1’ subscript on ‘$W^*_1$’ is meant to remind the reader of. In fact, it will be convenient to also, on occasion, refer to the category of $W^*_1$: von Neumann algebras with normal, not-necessarily-unital morphisms (if only to contrast several results and phenomena). The symbol for this latter category will be $W^*$.

Despite the difference between $W^*$ (central to [12]) and $W^*_1$ (studied here), the arguments of loc. cit. do go through with only minor modifications to provide arbitrary coproducts in $W^*_1$ by assembling together finite coproducts [12, §5] and filtered colimits [12, §7]. Coequalizers are also easily constructed [12, §2, p.43]: for parallel $W^*_1$-morphisms $f, g : A \to B$

simply quotient $B$ by the $\sigma$-weakly-closed ideal generated by

$$\{f(a) − g(a) \mid a \in A\}.$$

All of this is to show that, per [19, §V.2, Theorem 1], we have ([17, Proposition 5.7])

**Proposition 4.1** The category $W^*_1$ is cocomplete. ■
4.1 (Non-)presentability

One occasionally finds claims or suggestions to the effect that \( \mathcal{W}_1^* \) is locally presentable. This is stated outright in [22], for instance, and claimed for \( i\mathbb{R}\)-graded von Neumann algebras in [24, paragraph following Definition 3.4] (the grading will not make much of a difference; see §4.2). We prove here the following strong negation of local presentability.

**Theorem 4.2** The only presentable objects in the category \( \mathcal{W}_1^* \) of von Neumann algebras with normal unital morphisms are the zero algebra and \( \mathbb{C} \).

Before turning to the proof, note the obvious consequence:

**Corollary 4.3** The category \( \mathcal{W}_1^* \) of von Neumann algebras is not locally presentable.

**Proof of Theorem 4.2** We will argue that a von Neumann algebra \( A \) of dimension \( \geq 2 \) admits, for arbitrarily large regular cardinals \( \kappa \), a morphism

\[
A \to \left( \kappa\text{-directed } \lim_{i} A \right) \text{ not factoring through any } A_i. \tag{4-1}
\]

We will make a series of simplifications to this end.

**Step 1: reducing to Cartesian factors.** If \( A \) admits a morphism (4-1) so does \( B \times A \), by simply applying the functor \( B \times - \) (Cartesian product with \( B \)) to (4-1): indeed, that functor is easily seen to preserve directed colimits, so we obtain a morphism

\[
B \times A \to B \times \lim_{i} A_i \cong \lim_{i}(B \times A_i)
\]

That it cannot factor through any of the individual \( B \otimes A_i \) will follow, in the particular cases we consider:

- the connecting morphisms in the directed diagram producing \( \lim_{i} A_i \) will injective morphisms, as will (4-1);
- \( A \) will be flat (‘plat’ in [12, §5]), in the sense that the canonical surjections

\[
B \otimes A \to B \otimes A_i, B \in \mathcal{W}_1^*
\]

onto the spatial tensor product (the ‘\( \hat{\otimes} \)’ of [12, discussion preceding Lemme 8.1], ‘\( \otimes \)’ of [8, §III.1.5.4] or [32, Definition IV.5.1], etc.) are all isomorphisms;

•

- \( A \) will be flat (‘plat’ in [12, §5]), in the sense that the canonical surjections

\[
B \otimes A \to B \otimes A_i, B \in \mathcal{W}_1^*
\]

onto the spatial tensor product (the ‘\( \hat{\otimes} \)’ of [12, discussion preceding Lemme 8.1], ‘\( \otimes \)’ of [8, §III.1.5.4] or [32, Definition IV.5.1], etc.) are all isomorphisms;
• whence applying the $\hat{\otimes}$-to-$\otimes$ surjection to (4-2) we obtain
\[
\hat{B} \hat{\otimes} A \cong B \otimes A \subseteq B \otimes (\lim_i A_i),
\]
using the fact that $\otimes$ produces injections when applied to injections [12, Proposition 8.1].

• We can then argue that $B \otimes A$ is not contained in any $B \otimes A_i$ by observing that [32, Corollary IV.5.10]
\[
(B \otimes A) \cap (B \otimes A_i) = B \otimes (A \cap A_i) \subset B \otimes A;
\]
and the properness of the latter inclusion follows from that of $A \cap A_i \subset A$ by applying some functional ([32, equation (1) following Definition IV.5.1]) of the form $\varphi_B \otimes \varphi_A$ where $\varphi_B \in B_*$ and $\varphi_A \in A_*$ vanishes on $A \cap A_i$ but not on $A$.

**Step 3: reducing the problem to abelian von Neumann algebras and matrix algebras.**
For an arbitrary von Neumann algebra $A$ we have its canonical direct-product decomposition [32, Theorem V.1.19]
\[
A \cong A_I \times A_{II_1} \times A_{II_}\infty \times A_{III}
\]
into factors of types $I$, $II_1$, etc. Step 1 above then allows us to focus on the individual single-type Cartesian factors.

Those of types $II$ or $III$ always decompose as $A \cong A' \otimes M_2$ [32, Proposition V.1.35] and $M_2$ is flat [12, Proposition 8.6] (the term ‘discret’ used there is synonymous to ‘type-I’), so Step 2 reduces the entire discussion to $M_2$.

On the other hand, a type-I von Neumann algebra decomposes [32, Theorem V.1.27] as a product of the form
\[
\prod_{\alpha} (C_{\alpha} \otimes B(\mathcal{H}_{\alpha}))
\]
for various cardinals $\alpha$, where $C_{\alpha}$ is abelian and $\mathcal{H}_{\alpha}$ is an $\alpha$-dimensional Hilbert space.

If we have any non-vanishing factors for $\alpha \geq 2$ we can again peel off a matrix-algebra tensorand from $B(\mathcal{H}_{\alpha})$ and proceed as before. On the other hand, the $\alpha = 1$ factor reduces to an abelian von Neumann algebra.

Having thus reduced the discussion to von Neumann algebras that are either

• abelian of dimension $\geq 2$;
• or matrix algebras $M_n$, $n \geq 2$,
we relegate those two cases to two separate results: Proposition 4.7 and Proposition 4.8 respectively. ■

**Remark 4.4** A point of clarification is perhaps in order regarding the maximal tensor product denoted above by $\hat{\otimes}$. As noted, it coincides with $\mu \otimes$ of [12, §8]; this might appear surprising at first, given that

• the morphisms in $W^*_\mu$ and $W^*$ are different;
• while the tensor products $\hat{\otimes}$ and $\otimes$ supposedly have the same universality property, each phrased in the respective category: $A \bullet B$ (for either choice of ‘$\bullet$’) is supposed to be the universal recipient of two mutually commuting morphisms from $A$ and $B$.  

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The problem seems to be that the universality property of $\otimes$ is (or at least appears to us to be) misstated in [12, Proposition 8.2].

Specifically, that universality property is stated as above, via couples of commuting-image morphisms, but by construction (reverting to the notation in [12, §8]) the two morphisms $A_i \mu A_1 \otimes A_2$, $i = 1, 2$ send the respective units $e_{A_i}$ to the same element $g(e_{A_1} \otimes e_{A_2})$. It follows that the $A_1 \otimes A_2$ construction only classifies those pairs of morphisms out of $A_i$, $i = 1, 2$ which

- have commuting images, as discussed previously;
- and additionally, send the units of $A_i$, $i = 1, 2$ onto the same projection of the codomain.

With this caveat, it is no longer surprising that the two tensor products coincide: the categories differ, but so does the (phrasing of the) universality property.

We need the following simple observation on realizing $\ell^\infty$ von Neumann algebras as sufficiently-directed colimits.

**Lemma 4.5** Let $\kappa$ be a regular cardinal and denote by $\kappa_*$ the set $\kappa \cup \{\ast\}$ (i.e. $\kappa$ with an additional distinguished point). For each subset $S \subset \kappa$ of cardinality $< \kappa$, denote by

$$C_S \subset \ell^\infty(\kappa_*)$$

the unital von Neumann subalgebra generated by the minimal projections associated to the elements of $S$. The canonical morphism

$$\lim_{S} C_S \to \ell^\infty(\kappa_*), \quad \text{sets } S \text{ ordered by inclusion} \quad (4-3)$$

is an isomorphism, thus realizing $\ell^\infty(\kappa_*)$ as a $\kappa$-directed colimit in $\mathcal{W}_1^\ast$.

**Proof** That the colimit is $\kappa$-directed follows from the regularity of $\kappa$: a union of fewer than $\kappa$ sets of cardinality $< \kappa$ again has cardinality $< \kappa$.

As for (4-3) being an isomorphism, we can check this by examining the functors $\mathcal{W}_1^\ast \to \text{Set}$ represented by the two objects. First, note that

$$\text{hom}(C_S, -) : \mathcal{W}_1^\ast \to \text{Set}$$

is the functor that picks out a projection (the image of the characteristic function $\chi_S \in C_S$) and a partition of that projection into an $S$-indexed set of mutually-orthogonal projections. Passing to the limit,

$$\lim_{S} \text{hom}(C_S, -) \cong \text{hom} \left( \lim_{S} C_S, - \right)$$

is the functor that picks out a projection $p$ (in whatever von Neumann algebra the functor is being applied to) and decomposes it as a $\kappa$-indexed partition of mutually-orthogonal projections. But this is also the functor represented by $\ell^\infty(\kappa_*)$: the unit of the non-unital von Neumann subalgebra

$$\ell^\infty(\kappa) \subset \ell^\infty(\kappa_*)$$

gets mapped to $p$, and the characteristic function $\chi_{\{\ast\}}$ of the leftover singleton $\{\ast\} = \kappa_* \setminus \kappa$ gets mapped to $1 - p$. $\blacksquare$
Remark 4.6 In particular, Lemma 4.5 gives an example (or rather a family of examples) of arbitrarily-directed families

\[ A_S \subset A \]

of von Neumann subalgebras for which \( \lim_{S} A_S \to A \) is not one-to-one: simply take \( A = \ell^\infty(\kappa) \) (not \( \kappa^* \)!) and \( A_S \) to be the unital von Neumann subalgebra generated by the minimal projections attached to elements of \( S \subset \kappa \) for \( |S| < \kappa \). Lemma 4.5 implies that

\[
\lim_{S} A_S \to A \hspace{1cm} (4-4)
\]

can be identified with the first-factor projection

\[
\ell^\infty(\kappa^*) \cong A \times \mathbb{C} \to A \cong \ell^\infty(\kappa).
\]

For Lemma 4.5 to go through, it is crucial that we work with the category of \( W^* \)-algebras with unital morphisms. By contrast, computing the colimit in the category studied in [12], of \( W^* \)-algebras with normal, possibly non-unital morphisms, the more “reasonable” morphism

\[
\lim_{S} C_S \to \ell^\infty(\kappa)
\]

is an isomorphism: both objects, in that case, represent the functor

\[
(W^*\text{-algebra } A) \mapsto (\kappa\text{-partitioned projections in } A) \in \text{Set}.
\]

For the non-unital \( W^* \) category [12, examples following Remarque 7.2] similarly illustrate the phenomenon of morphisms (4-4) out of directed colimits failing to be one-to-one despite \( A_S \subset A \) being so.

Proposition 4.7 An abelian von Neumann algebra of dimension \( \geq 2 \) is not \( \kappa \)-presentable in \( W_1^* \) for any regular cardinal \( \kappa \).

Proof Isolating out the minimal projections, an abelian von Neumann algebra \( A_1 \) decomposes as \( A_1 \times A_2 \) with \( A_1 \cong \ell^\infty(S) \) for some set \( S \) and \( A_2 \) non-atomic, in the sense that it has no minimal non-zero projections (the term is measure-theoretic [13, §40]; diffuse is a synonym: [8, §III.4.8.8], [25, §7.11.16], [23, §1], etc.).

The diffuse factor \( A_2 \) is of the form \( L^\infty(X, \mu) \) for a non-atomic [13, §40] positive measure \( \mu \) on a locally compact Hausdorff space \( X \) [11, Part I, §7.3, Theorem 1]. By general measure-space structure theory ([20, Theorems 1 and 2] or [36, Theorems 2 and 3]) we can decompose \( A_2 \) as a product of copies of \( L^\infty(\mathbb{R}, \mu_{\text{Lebesgue}}) \), and in turn the latter decomposes as

\[
L^\infty(\mathbb{R}, \mu) \cong L^\infty(\mathbb{R}, \mu)^2 \cong L^\infty(\mathbb{R}, \mu) \mathbb{C}^2.
\]

By the reduction steps in the proof of Theorem 4.2, the problem then boils down to \( \mathbb{C}^2 \).

All of this is available provided the diffuse factor \( A_2 \) doesn’t vanish. If it does, \( A_1 \) is \( \ell^\infty(S) \) for \( |S| \geq 2 \), it surjects onto \( \mathbb{C}^2 \), and again the problem reduces to the latter provided we map it into a directed colimit of embeddings.

All in all, we can now focus on \( \mathbb{C}^2 \). To that end, simply note the embedding \( \mathbb{C}^2 \subset \ell^\infty(\kappa^*) \) (notation as in Lemma 4.5) that sends one of the two minimal projections, say \( p \in \mathbb{C}^2 \), to the characteristic function

\[
\chi_\kappa \in \ell^\infty(\kappa^*) = \ell^\infty(\kappa \sqcup \{\ast\}).
\]

This is an embedding into a \( \kappa \)-directed colimit by Lemma 4.5, but cannot factor through any of the \( C_S \subset \ell^\infty(\kappa^*) \), \( |S| < \kappa \) because none \( C^* \)-subalgebras of those contain \( \chi_\kappa \).

\[ \blacksquare \]
Proposition 4.8 A matrix algebra $M_n = M_n(\mathbb{C})$, $n \geq 2$ is not $\kappa$-presentable in $W_1^*$ for any regular cardinal $\kappa$.

Proof We specialize to $n = 2$, in order to fix ideas and ease the notational load of the proof; the general argument does not pose substantial additional difficulties.

Consider a $\kappa$-dimensional Hilbert space $\mathcal{H}$, with a fixed orthonormal basis $\{e_{s,i} \in \mathcal{H}, s \in \kappa, i = 0, 1\}$.

The extra 0, 1 indices are there to aid in working with our choice of $n = 2$; in general the index $i$ would run from 0 to $n-1$. Fix, for each $s \in \kappa$, a partial isometry $u_s$ that implements an equivalence between the projections $p_{s,i}$ onto $\mathbb{C}e_{s,i}$, $i = 0, 1$:

$$u_s^*u_s = p_{s,0}, \quad u_su_s^* = p_{s,1}.$$ 

For a subset $S \subset \kappa$ of strictly smaller cardinality write

- $\mathcal{H}_S$ for the closed span of $\{e_{s,i} \mid s \in S, i = 0, 1\}$;
- $M_{2,S^c}$ (the ‘c’ superscripts stands for ‘complement’) for the copy of $M_2$ spanned by $\sum_{s \notin S} p_{s,i}, i = 0, 1, \sum_{s \notin S} u_s$ and $\sum_{s \notin S} u_s^*$

(this is a non-unital $W^*$-subalgebra of $B(\mathcal{H})$);
- and $B_S$ for the direct sum $B_S := B(\mathcal{H}_S) \oplus M_{2,S^c}$.

A number of observations are immediate:

1. The map $S \mapsto B_S$ is monotone for inclusion, and makes $B_S \subset B(\mathcal{H})$ into a $\kappa$-directed family of von Neumann subalgebras.

2. Every $B_S$ is 2-homogeneous (a slight extension of the language used in [25, §5.5.6], for instance): there is a projection $p \in B_S$ equivalent to $1 - p$ in the sense that $u^*u = p, \quad uu^* = 1 - p$ for some $u \in B_S$ or, equivalently, there is a unital $W^*$-morphism $M_2 \to B_S$.

3. The $B_S$ jointly generate $B(\mathcal{H})$ as a von Neumann algebra, i.e.

$$B := \lim_{\longrightarrow} B_S \to B(\mathcal{H})$$

is onto.

Consider, now, a morphism $M_2 \to B(\mathcal{H})$ that sends the projection

$$p := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
to, say, \( p_{T,0} := \sum_{s \in T} p_s,0 \) for some subset \( T \subseteq \kappa \), \(|T| = \kappa = |T^c|\).

The surjection (4-5) is, like all \( W^*\)-algebra surjections [11, Part I, §3.4, Corollary 3], identifiable with

\[
B \ni b \mapsto bz \in Bz
\]

for some central projection \( z \in B \). The 2-homogeneity of the \( B_S \) implies that of \( B(1 - z) \), so the morphism \( M_2 \rightarrow B(\mathcal{H}) \) lifts along (4-5) to

\[
M_2 \rightarrow B \cong Bz \times B(1 - z) \rightarrow Bz \cong B(\mathcal{H})
\]  (4-6)

by preserving the \( Bz \) component of the morphism we set out with and supplementing it with an arbitrary \( M_2 \rightarrow B(1 - z) \).

Because \( p_{T,0} \) is not contained in any of the \( B_S \subseteq B(\mathcal{H}) \), (4-6) cannot factor through any \( B_S \subseteq \lim_{\rightarrow} S B_S = B \). We thus have a morphism into a \( \kappa \)-directed colimit which doesn’t factor through any of the colimit constituents: precisely what we sought. ■

4.2 Additional structure: graded \( W^*\)-algebras

To return to the motivating discussion, we recall briefly the framework relevant to [24]. That paper works extensively with what it calls \( I \)-graded von Neumann algebras, where

- \( I \) is the imaginary line \((i\mathbb{R}, +)\) regarded as a locally compact abelian group;
- and being \( I \)-graded is a specialization of the following notion ([24, Definition 3.1]).

**Definition 4.9** For a locally compact abelian group \( \mathcal{G} \) (always Hausdorff, for us), a \( \mathcal{G} \)-graded von Neumann algebra is a von Neumann algebra \( M \) equipped with an action by the Pontryagin dual

\[
\hat{\mathcal{G}} := \{ \text{continuous group morphisms } \mathcal{G} \rightarrow \mathbb{S}^1 \}
\]

by \( W^*\)-algebra automorphisms, such that for every \( m \in M \) the map

\[
\mathcal{G} \ni g \mapsto gm \in M
\]

is continuous from \( \mathcal{G} \) with its locally-compact topology to \( M \) equipped with its weak* topology.

We write \( \mathcal{W}_{I,G}^* \) for the category of \( \mathcal{G} \)-graded von Neumann algebras, with morphisms being those in \( \mathcal{W}_I^* \) which intertwine the \( \hat{\mathcal{G}} \)-actions.

We remind the reader that

- \( \mathcal{G} \mapsto \hat{\mathcal{G}} \) is a duality on the category of locally compact (Hausdorff) abelian groups, i.e. an involutive contravariant equivalence (as follows, for instance, from [15, Theorems 7.7 and 7.63]).

- The continuity requirement for the \( \hat{\mathcal{G}} \)-action on \( M \) imposed in Definition 4.9 is the standard one in the theory of locally-compact-group actions on von Neumann algebras: see e.g. [33, §X.1, Definition 1.1 and Proposition 1.2].

We can now directly address the claim made in passing in [24, paragraph following Definition 3.4] that the category of \( I \)-graded von Neumann algebras is locally presentable. The following result, generalizing Corollary 4.3, shows that this cannot be the case.
Proposition 4.10 The category $\mathcal{W}_{1,G}^*$ of $G$-graded von Neumann algebras is not locally presentable for any locally compact abelian group $G$.

Proof We can regard any von Neumann algebra as carrying the trivial $G$-grading, i.e. the trivial $\hat{G}$-action; this gives a (fully faithful) functor $F : \mathcal{W}_1^* \to \mathcal{W}_{1,G}^*$.

On the other hand, for every $\gamma \in G$ and $M \in \mathcal{W}_{1,G}^*$ equipped with a $\hat{G}$-action

$$\triangleright : \hat{G} \times M \to M,$$

we can consider the degree-$\gamma$ component of $M$, defined by

$$M_\gamma := \{m \in M \mid p \triangleright m = \gamma(p)m, \forall p \in \hat{G}\}.$$

In particular, selecting the $\hat{G}$-invariants

$$M^{\hat{G}} := M_1 = \{m \in M \mid p \triangleright m = m, \forall p \in \hat{G}\} \subseteq M$$

gives a functor

$$\mathcal{W}_{1,G}^* \ni M \xrightarrow{G} M^{\hat{G}} \in M$$

that is right adjoint to $F$.

Since $F$ is a left adjoint, it is cocontinuous. This means that any morphism

$$M \to \lim_i M_i$$

in $\mathcal{W}_1^*$ witnessing that $M$ is not $\kappa$-presentable (for some regular cardinal $\kappa$) can be regarded as such a diagram in $\mathcal{W}_{1,G}^*$ instead.

It follows from Theorem 4.2 that the only objects in $\mathcal{W}_{1,G}^*$ carrying a trivial $G$-grading that are presentable (for any cardinal) are $\{0\}$ and $\mathbb{C}$. This finishes the proof, since in a locally presentable category every object is presentable for some cardinal [3, Remark 1.30 (1)].

4.3 Back to monadic forgetful functors

It is natural to consider, at this stage, analogues of Theorem 2.4 and Corollary 2.6 in the $W^*$ setup.

Theorem 4.11 The forgetful functor $G : \mathcal{W}_1^* \to \mathcal{C}_1^*$ satisfies the Crude Tripleability Theorem, so is in particular monadic.

Proof We verify the conditions listed in Definition 2.2.

(1) The embedding $\mathcal{W}_1^* \to \mathcal{C}_1^*$ does indeed have a left adjoint, namely the double-dual functor $A \mapsto A^{**}$ [32, Theorem III.2.4].

(2) Isomorphism-reflection is well known, and in fact much more can be said: a purely algebraic isomorphism of $*$-algebras between two $W^*$-algebras is norm-continuous and normal [34, Theorem 3].

(3) $\mathcal{W}_1^*$ being cocomplete (Proposition 4.1), all coequalizers exist (not just reflexive ones).
It thus remains to check that $G$ preserves reflexive coequalizers. Consider, then, a reflexive pair $\partial_i : A \to B$, $i = 0, 1$ in $W^*_1$, with a common right inverse $t : B \to A$:

$$\partial_1 t = \text{id}_B = \partial_0 t.$$  \hfill (4-7)

The idempotent endomorphisms

$$e_i := t\partial_i, \; i = 0, 1$$

of $A$ satisfy

$$e_0e_1 = e_1, \quad e_1e_0 = e_0$$  \hfill (4-8)

as a consequence of (4-7). Because $t$ is one-to-one we have $\ker e_i = \ker \partial_i$, and furthermore these kernels are the ideals generated by central projections $z_i \in A$ [11, Part I, §3.3, Corollary 3]. (4-8) says that each $e_i$ is faithful on $(1 - z_j)A$, $j \neq i$, and hence each of the projections $z_i$, $i = 0, 1$ dominates the other. But this means that

$$z_1 = z_0 =: z, \quad \text{so that } \ker \partial_1 = \ker \partial_0 = zA.$$

The conclusion, then, is that

- the maps $\partial_i$, $i = 0, 1$ both annihilate $zA$;
- and they agree on the complementary subspace $(1 - z)A$: for each $i = 0, 1$ and $a \in (1 - z)A$, the elements

$$a \quad \text{and} \quad (1 - z)(t\partial_i)(a)$$

are both mapped to the same element $\partial_i(a) \in B$. Since

- $\partial_i$ is injective on $(1 - z)A$;
- and $(1 - z)t(\cdot) : B \to (1 - z)A$ is injective (being a right inverse to the restrictions $\partial_i|_{(1 - z)A}$),

we have $\partial_1(a) = \partial_1(a)$. This means that the coequalizer of the pair $(\partial_i)_{i=0,1}$ is precisely $\text{id}_B : B \to B$, clearly preserved by the forgetful functor.  \hfill \blacksquare

**Remark 4.12** In the latter part of the proof of Theorem 4.11, it was crucial to restrict the discussion to reflexive-coequalizer preservation: the forgetful functor $G : W^*_1 \to C^*_1$ does not preserve arbitrary coequalizers.

Consider, for instance, the pair $\partial_i : B(\mathcal{H}) \to B(\mathcal{H})$, $i = 0, 1$ for a countably-infinite-dimensional Hilbert space $\mathcal{H}$, where $\partial_0 = \text{id}$ and $\partial_1$ is conjugation by the unitary

$$u := 1 - 2p, \quad p \in B(\mathcal{H}) \text{ a projection of rank 1}.$$  

Because $u$ differs from the identity by a compact [8, §I.8] operator, every difference

$$\partial_1(a) - \partial_0(a), \; a \in A$$

belongs to the ideal [8, I.8.1.2] $K(\mathcal{H}) \subset B(\mathcal{H})$ of compact operators.

Now, on the one hand, because $\dim \mathcal{H} = \aleph_0$ (as a Hilbert space), $K(\mathcal{H})$ is the unique proper non-zero ideal of $B(\mathcal{H})$ [8, Proposition III.1.7.11] so the $C^*$ cokernel of $(\partial_i)_{i=0,1}$ is the surjection

$$B(\mathcal{H}) \to B(\mathcal{H})/K(\mathcal{H})$$
onto the Calkin algebra [8, I.8.2] of $\mathcal{H}$.

On the other hand, $B(\mathcal{H})$ is a factor [8, §I.9.1.5] (its center is $\mathbb{C}$), so it has no proper, non-zero weak$^*$-closed ideals [11, Part I, §3.3, Corollary 3]. This means that the cokernel of $(\partial_i)_{i=0,1}$ in $W_1^*$ is the zero algebra.

**Corollary 4.13** The forgetful functors from $W_1^*$ to any of the categories $\text{BANALG}_1^*$, $\text{BANALG}_1$ or $\text{BAN}$ are all monadic.

**Proof** As in the proof of Corollary 2.6: said functors decompose as

- $W_1^* \to C_1^*$, which is CTT Theorem 4.11;

- followed by forgetful functors from $C_1^*$ to the respective categories, all of which are monadic Corollary 2.6.

Lemma 2.3 thus applies to conclude.

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