Abstract. We extract from a toric model of the Chekanov-Schlenk exotic torus in $\mathbb{CP}^2$ methods of construction of Lagrangian submanifolds in toric symplectic manifolds. These constructions allow for some control of the monotonicity. We recover this way some known monotone Lagrangians in the toric symplectic manifolds $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ as well as new examples.

1. Introduction

One can reconstruct $\mathbb{RP}^2$, the real part of $\mathbb{CP}^2$ with respect to the standard conjugation map, from its image under the standard moment map of $\mathbb{CP}^2$. The real part projects under the moment map of $\mathbb{CP}^2$ onto the entire moment polytope, each point in the interior of the polytope having four preimages, the points on the interior of the edges on the boundary of the triangle having two preimages, the points on the vertices having one preimage. By taking four copies of the moment polytope and gluing them along the edges according to the prescriptions of the torus action on $\mathbb{CP}^2$ (see [1]), one recovers the real projective plane.

From the Lagrangian submanifolds point of view, $L = \mathbb{RP}^2$ is an important example of monotone Lagrangian submanifold in $\mathbb{CP}^2$. Monotone means that there exists a positive constant $K_L$ such that

$$\forall u \in H_2(\mathbb{CP}^2, L), \int u \omega = K_L \mu_L(u),$$

where $\mu_L : H_2(\mathbb{CP}^2, L) \to \mathbb{Z}$ is the Maslov class of $L$.

The exotic torus of Chekanov and Schlenk (see [7]) is another important example of monotone Lagrangian submanifold of $\mathbb{CP}^2$. The second author proved in [8] that this torus is Hamiltonian isotopic to a torus described by Biran and Cornea in [3]. To do so, she Hamiltonian-isotoped both tori to

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a so-called modified Chekanov torus $\tilde{\Theta}_{\text{Ch}}$. This torus has a nice image by the moment map and can be reconstructed, as the real projective space, out of copies of this image and gluing patterns. The rules for gluing are of two types. The first are coming from the definition of the moment map and are the same as the ones used for the real part. The second are new and we have managed to interpret them as Lagrangian surgeries of two copies of the real part intersecting transversely at one isolated point and cleanly (in the sense of Pozniak [13]) along a circle.

The surgery for two Lagrangian submanifolds intersecting transversely at a point has been developed by Polterovich in [12] and we have modified it to keep a toric description of the result of the surgery. The surgery for two Lagrangian submanifolds intersecting along an isotropic submanifold not reduced to a point is new and we intend to develop it in full generality in a future work. We could show in our case:

**Theorem 1.** The Chekanov and Schlenk torus is Hamiltonian isotopic in $\mathbb{C}P^2$ to a Lagrangian torus obtained from two copies of $\mathbb{R}P^2$ by Lagrangian surgeries at a point and along an isotropic circle.

With our method we can also recover Lagrangian embeddings of some surfaces that were constructed by Givental in [9] and embedded afterwards in $\mathbb{C}P^2$ or $\mathbb{C}P^1 \times \mathbb{C}P^1$. The advantage of this construction is that we have a good control of the monotonicity condition. We knew so far only the monotone Lagrangian embeddings of tori in $\mathbb{C}P^1 \times \mathbb{C}P^1$ and of tori and real projective planes in $\mathbb{C}P^2$ and our method enables us to prove that

**Theorem 2.** There exists a monotone Lagrangian embedding of the connected sum of a surface of genus 2 and a Klein bottle in $\mathbb{C}P^2$ and a monotone Lagrangian embedding of the connected sum of a surface of genus 4 and a Klein bottle in the product $\mathbb{C}P^1 \times \mathbb{C}P^1$.

Note that neither the Klein bottle (see [14]) nor the orientable surface of genus 2 (see [10]) can be embedded as Lagrangian submanifolds of $\mathbb{C}P^2$.

The structure of this article is the following: in Section 2 we study the gluing patterns for the modified Chekanov torus and we describe the surgeries we will use; in Section 3 we give some construction and study the monotonicity of examples we can get with the surgery at a point; finally, in Section 4 we use the surgery for an intersection along a circle to describe two monotone Lagrangian embeddings.

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2. The local models

2.1. A toric model of the exotic torus of Chekanov and Schlenk in $\mathbb{C}P^2$. There is a well-known monotone torus in $\mathbb{C}P^2$ called the Clifford
torus which can be described in homogeneous coordinates as:

\[ T_{\text{Cliff}} = \left\{ \left[ e^{i\alpha} : e^{i\beta} : 1 \right] \mid \alpha, \beta \in [0, 2\pi] \right\}. \]

Its image under the moment map of \( \mathbb{CP}^2 \) (corresponding to the normalization of the symplectic form we use in Section 3)

\[ \mu : \mathbb{CP}^2 \rightarrow \mathbb{R}^2, \quad [z_0 : z_1 : z_2] \mapsto \left( 3 \frac{|z_0|^2}{\sum |z_i|^2}, 3 \frac{|z_1|^2}{\sum |z_i|^2} \right), \]

is the barycenter \((1, 1)\) of the image of \( \mathbb{CP}^2 \), the triangle obtained as the convex hull of the points \((0, 0), (3, 0), (0, 3)\).

In 2004, Chekanov and Schlenk (see [7]) have studied the torus given in homogeneous coordinates of \( \mathbb{CP}^2 \):

\[ \Theta_{\text{CS}} = \left\{ \left[ \frac{1}{\sqrt{2}} \gamma(s) e^{i\theta} : \frac{1}{\sqrt{2}} \gamma(s) e^{-i\theta} : \sqrt{\frac{3}{\pi} - |\gamma(s)|^2} \right] : \theta \in [0, 2\pi], s \in [0, 2\pi] \right\} \]

where \( \gamma : [0, 2\pi] \rightarrow \mathbb{C} \) parametrizes a curve enclosing a domain of area 1 lying in the disk centered in the origin and of area \( 2 + \varepsilon \) of \( \mathbb{C} \), in the half-disk of complex numbers of positive real part (see Figure 1). They have proved (see [7, 5, 6]) that \( \Theta_{\text{CS}} \) is a monotone Lagrangian torus in \( \mathbb{CP}^2 \), nondisplaceable and non-Hamiltonian isotopic to the Clifford torus (therefore called exotic) in \( \mathbb{CP}^2 \).

By its definition, the Chekanov and Schlenk torus projects under the moment map \( \mu \) to a segment lying in the diagonal line of \( \mathbb{R}^2 \). More precisely the image is

\[ \left\{ (x, x) \in \mathbb{R}^2 \mid x \in \left\lbrack \frac{\pi}{2} \rho_{\text{min}}, \frac{\pi}{2} \rho_{\text{max}} \right\rbrack \right\} \]

where \( \rho_{\text{min}} \) is the minimum of \( |\gamma(s)| \) and \( \rho_{\text{max}} \) is the maximum.

There exists a description of this exotic torus more adapted to the toric picture, that is a description that enables to reconstruct the torus from its moment map image as in the case of the real part of \( \mathbb{CP}^2 \).

Such a description can be obtained by considering the modified Chekanov torus of [8]. This torus is a torus Hamiltonian isotopic to the exotic torus of Chekanov and Schlenk and is defined in homogeneous coordinates (with the normalizations of [7]):

\[ \tilde{\Theta}_{\text{Ch}} = \left\{ \left[ \cos(\theta) \gamma(s) : \sin(\theta) \gamma(s) : \sqrt{\frac{3}{\pi} - |\gamma(s)|^2} \right] : \theta, s \in [0, 2\pi] \right\}. \]

The image of the torus under the moment map \( \mu \) can be parametrised by

\[ \mu(\tilde{\Theta}_{\text{Ch}}) = \left\{ \left( \pi \cos^2(\theta) |\gamma(s)|^2, \pi \sin^2(\theta) |\gamma(s)|^2 \right) : \theta, s \in [0, 2\pi] \right\}. \]

It is a trapezoid sitting inside the polytope for \( \mathbb{CP}^2 \) between the two parallel lines \( x + y = \pi \rho_{\text{min}}^2 \) and \( x + y = \pi \rho_{\text{max}}^2 \).

If the curve \( \gamma \) is such that \( \gamma(0) = \rho_{\text{min}}, \gamma(\pi) = \rho_{\text{max}}, \gamma \) is symmetric with respect to the real axis, and each point \( |\gamma(s)| = \rho(s) \) has only one preimage
$s \in (0, \pi)$ (see for example Figure 1], then for $s \neq 0, \pi$ and $\theta \neq 0, \pi$, the point

$$(\pi \cos^2(\theta) |\gamma(s)|^2, \pi \sin^2(\theta) |\gamma(s)|^2)$$

has 8 preimages in the torus.

Recall (see for example [2, 4]) that the image of the moment map for $\mathbb{C}P^2$, or for a general (compact, connected) toric manifold $(M, \omega)$ is a convex polytope $P$ such that the fiber of each point of $P$ is an isotropic torus. Recall also that we have action-angle coordinates on the preimage $\tilde{M}$ of the interior $\tilde{P}$ which is the open dense set in $M$ consisting of all the points where the action of the torus $\mathbb{T}^n$ is free. One can describe this set as

$$\tilde{M} \cong \tilde{P} \times \mathbb{T}^n = \left\{ (x_1, \ldots, x_n, e^{i\theta_1}, \ldots, e^{i\theta_n}) \mid x \in \tilde{P}, \theta \in \mathbb{R}^n / 2\pi\mathbb{Z}^n \right\},$$

where $(x, \theta)$ are the action-angle coordinates for $\omega$:

$$\omega = \sum dx_j \wedge d\theta_j.$$

In the case of $\tilde{\Theta}_{\text{Ch}}$, we parametrise the curve $\gamma$ above the real axis by

$$\gamma(s) = \rho(s)e^{it(s)},$$

such that $t(s) \in [0, t_{\text{max}}], t_{\text{max}} < \frac{\pi}{2}, t(0) = 0, t(\pi) = 0.$

![Figure 1. A suitable curve $\gamma$](image)

Then for a fixed $s$ in $(0, \pi)$ and a fixed $\theta$ in $(0, \frac{\pi}{2})$, the eight preimages of the point

$$(\pi \cos^2(\theta) |\gamma(s)|^2, \pi \sin^2(\theta) |\gamma(s)|^2)$$

...
in $\mu(\tilde{\Theta}_{Ch})$ are given in action-angle coordinates by:

$$A_{\epsilon,k,\ell} = \left( \pi \cos^2(\gamma(s)) \frac{\pi}{2}, \pi \sin^2(\gamma(s)) \frac{\pi}{2}, \epsilon \tau(s) + \frac{\pi}{2} + k\pi, \epsilon \tau(s) + \frac{\pi}{2} + \ell\pi \right)$$

with $\epsilon \in \{-1, 1\}$, $k, \ell \in \{0, 1\}$.

When $s$ goes to 0 or $\pi$, $t(s)$ goes to 0 and the points in the torus fiber converge (moving along the diagonal direction) towards one of the four points

$$\left( \frac{\pi}{2} + k\pi, \frac{\pi}{2} + \ell\pi \right), k, \ell \in \{0, 1\},$$

see Figure 2.

This describes the gluing of the trapezoid along the segments $x + y = \pi \rho_{\text{min}}^2$ and $x + y = \pi \rho_{\text{max}}^2$. 

**Figure 2.** The points are at the four corners of each of the red dashed squares; when $s$ goes to 0, the corners of the two squares are identified.
2.2. The interpretation of the gluing along the segment $x + y = \pi \rho^2_{\text{min}}$. The gluing along the segment $x + y = \pi \rho^2_{\text{min}}$ can be described as the Lagrangian surgery defined in [12] of two Hamiltonian isotopic copies of the real part $\mathbb{R}P^2$ intersecting transversally at the origin $[0 : 0 : 1]$ (see Section 4.2 for the details).

Let us describe the Lagrangian surgery we will use in the rest of this article which is a slight modification of [12]. Following Polterovich, one does the surgery of two transverse Lagrangians in a local chart around an intersection point. In this chart, one finds an almost-complex structure $j$ such that locally $l_1 = jl_0$ and the local Lagrangian handle is the image of the sphere in $l_0$ times $[-T, T]$, for $T$ large under the map: $(\xi, t) \mapsto e^{-t} \xi + e^t j \xi$. One then connects the handle on its boundaries to the original Lagrangians by some smoothing.

Because we will want to control the monotonicity condition and keep the standard local chart in $\mathbb{C}P^2$, we describe explicitly the Lagrangian surgery we will be using, without the use of an auxiliary almost-complex structure $j$ in $\mathbb{C}^2$.

The handle between two Lagrangian linear subspaces of $\mathbb{C}^2$. Consider the linear $\mathbb{C}^2$ and the two Lagrangian linear subspaces

$$l_0 = \mathbb{R}^2$$

and

$$l_1 = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix} \mathbb{R}^2$$

the image of $l_0$ by the Hamiltonian diffeomorphism defined by the diagonal matrix $\text{diag}(e^{i\alpha}, e^{i\beta})$ for $\alpha$ and $\beta$ not a multiple of $\pi$.

We define a handle $h$ parametrised by:

$$h = \left\{ e^{-t} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + e^t \begin{pmatrix} e^{i\alpha} x_0 \\ e^{i\beta} x_1 \end{pmatrix} \mid t \in \mathbb{R}, x_0^2 + x_1^2 = 1 \right\}.$$ 

It is asymptotic to $l_0$ when $t$ goes to $-\infty$ and to $l_1$ when $t$ goes to $+\infty$. One can check that it is a Lagrangian handle when $\sin(\alpha) = \sin(\beta)$.

Note that for the same reason, the handle:

$$h' = \left\{ e^{-t} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + e^t \begin{pmatrix} e^{i\alpha} (-x_0) \\ e^{i\beta} (-x_1) \end{pmatrix} \mid t \in \mathbb{R}, x_0^2 + x_1^2 = 1 \right\}$$

is also a Lagrangian submanifold asymptotic to $l_0$ and $l_1$ and correspond to the first handle for the couple of angles $(\alpha + \pi, \beta + \pi)$.

Note that when $\sin(\alpha) = -\sin(\beta)$, one can also define two Lagrangian handles parametrized by:

$$h = \left\{ e^{-t} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + e^t \begin{pmatrix} e^{i\alpha} x_0 \\ e^{i\beta} (-x_1) \end{pmatrix} \mid t \in \mathbb{R}, x_0^2 + x_1^2 = 1 \right\}$$

and

$$h' = \left\{ e^{-t} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + e^t \begin{pmatrix} e^{i\alpha} (-x_0) \\ e^{i\beta} x_1 \end{pmatrix} \mid t \in \mathbb{R}, x_0^2 + x_1^2 = 1 \right\}.$$
The smoothing. For some (large) $T$, denote by $h_T$ the image of the handle for $t$ between $-T$ and $T$:

$$h_T = \left\{ e^{-t} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + e^{t} \begin{pmatrix} e^{i\alpha} x_0 \\ e^{i\beta} x_1 \end{pmatrix} \mid t \in [-T, T] \right\}. $$

Fix a parameter $T$ large. We smooth the handle at the ends of $h_T$ as in [12]. Remark that the original surgery of Polterovich corresponds to the case when $\alpha = \beta = \frac{\pi}{2}$ and we can obtain any of our handles from Polterovich’s surgery by applying the linear transformation of $\mathbb{C}^2 = \mathbb{R}^4$ of matrix:

$$\begin{pmatrix} 1 & \cos(\alpha) & 0 & 0 \\ 0 & \sin(\alpha) & 0 & 0 \\ 0 & 0 & 1 & \cos(\beta) \\ 0 & 0 & 0 & \sin(\beta) \end{pmatrix}. $$

Hence we can smooth the handle at the boundary of $h_T$ when $\alpha = \beta = \frac{\pi}{2}$ as in [12] and then take the image of this smoothing by the linear map above to get a smoothing in our case.

Note that this surgery lies inside a big ball of radius $R$, and outside it, the Lagrangian submanifold obtained is the union of $l_0$ and $l_1$. As the linear Lagrangians are homogenous with respect to homotheties centered at the origin of $\mathbb{C}^2$, one can use a conformal transformation to make this Lagrangian surgery happen in a ball $B_0$ of small radius. Equivalently, one can take the handle to be, not the image of the unit sphere in $l_0$, but a smaller one

$$x_0^2 + x_1^2 = \varepsilon_1^2$$

and the smoothing happening outside a ball of radius $\varepsilon_2$ such that the Lagrangian submanifold after surgery identifies with the union of $l_0$ and $l_1$ outside a ball of radius $\varepsilon$ for a parameter $\varepsilon > \varepsilon_2 > \varepsilon_1 > 0$ small.

Controlling the area. Let us study the restriction of this surgery (before conformal transformation) to one factor $\mathbb{C}$ of $\mathbb{C}^2$, for instance the first. The trace of the surgery along this coordinate is given by:

$$h_T^1 = \left\{ e^{-t} + e^{e^{i\alpha}} \mid t \in [-T, T] \right\}$$

for some large $T$, followed by some small smoothing between its ends and the original $l_0$ and $l_1$, together with the symmetric curve about the origin.

Let us evaluate the area between the original $l_0$ and $l_1$ and one of the arcs of the surgery, namely the area in grey in Figure 3.

As before, we can deduce this area from the computation of the area when $\alpha = \frac{\pi}{2}$. In this case, $h_T^1$ is the arc of hyperbola in the plane given by the equation $xy = 1$ so that the area in grey is equal to $2T + 1$. This area is equivalent to $2T$ when $T$ goes to $+\infty$. Moreover, when $T$ is large, the smoothing between the original linear Lagrangians and the arc of hyperbola is very small so that the area is still equivalent to $2T$. 
Figure 3. The intersection of the handle to the first $\mathbb{C}$-factor of $\mathbb{C}^2$

The other cases can be obtained from the $\alpha = \frac{\pi}{4}$ situation by applying the linear transformation of the plane $\mathbb{C} = \mathbb{R}^2$ given by the matrix:

$$
\begin{pmatrix}
1 & \cos(\alpha) \\
0 & \sin(\alpha)
\end{pmatrix},
$$

so that the area considered above is equivalent to $2\sin(\alpha)T$ when $T$ goes to $\infty$.

In particular, when $\sin(\alpha) = \sin(\beta)$, the condition which ensures the surgery to be Lagrangian, the areas between the original Lagrangians and the handle along each coordinate are equivalently the same. For $T$ large enough, we can (and will) do the smoothing so that the areas along each $\mathbb{C}$-factor are equal.

Now, given the conformality property of the surgery, we can make this area as small as we want and equal to some $a(\varepsilon)$ (small) if the surgery is done inside the ball of radius $\varepsilon$.

In a general symplectic manifold. Let $L_0$ and $L_1$ be two Lagrangian submanifolds of a symplectic manifold $W$ intersecting transversally at a point $x_0$. One can take a Darboux chart $U_0$ around $x_0$ symplectomorphic to a ball $B_0$ endowed with the standard symplectic form of $\mathbb{C}^2$ such that under the Darboux map, the two Lagrangians are the intersection of Lagrangian linear subspaces of $\mathbb{C}^2$ with the ball. One is then in the linear situation from above and can perform the surgery in the ball as described provided the sines of angles between the restriction of the linear Lagrangians to each factor of $\mathbb{C}^2$ are the same (up to sign).

2.3. The interpretation of the gluing along the segment $x + y = \pi \rho_{\max}^2$. The gluing along the segment $x + y = \pi \rho_{\max}^2$ can be interpreted as
the Lagrangian surgery along a circle of two Hamiltonian isotopic copies of the real part \( \mathbb{RP}^2 \) intersecting along a circle in the \( \mathbb{CP}^1 \) at infinity

\[
\{ [z_0 : z_1 : z_2] \mid z_2 = 0 \}.
\]

This circle is isotropic and, as in the case of the surgery at a point, the surgery along an isotropic submanifold is a local process that we will now describe.

**The neighbourhood of an isotropic manifold.** Let \( P \) be a symplectic manifold of dimension \( 2n \). Let \( N \) be an isotropic submanifold of dimension \( k \). In [15], Weinstein noticed that the tangent bundle of \( P \) along \( N \) is isomorphic as symplectic vector bundle over \( N \) to:

\[
(TN \oplus TN^*) \oplus SN(N, P),
\]

where \( SN(N, P) = TN^\perp /TN \) is called the symplectic normal bundle of \( N \) in \( P \).

Conversely, one can embed any manifold which is the base of a symplectic vector bundle as an isotropic submanifold of a symplectic manifold such that the tangent bundle looks like this:

**Theorem 2.1 (The existence theorem, Weinstein [16]).** Let \( N \) be a manifold of dimension \( k \) and \( E \to N \) a symplectic vector bundle with fibre dimension \( 2(n - k), k \leq n \). Then \( N \) can be embedded as an isotropic submanifold of a symplectic manifold \( P(E) \) of dimension \( 2n \) such that the tangent bundle of \( P(E) \) along \( N \) is isomorphic as symplectic vector bundle to the sum \( (TN \oplus TN^*) \oplus E \).

This space \( P(E) \) is the Whitney sum \( P(E) = T^*N \oplus E \) as Weinstein explains in [15]. The symplectic structure on \( P(E) \) is not canonical and is described in [16].

And we have a uniqueness result:

**Theorem 2.2 (Weinstein [15]).** *The isotropic manifold theorem:* Let \( N \) be a manifold of dimension \( k \). Then the extensions of \( N \) to a 2\( n \)-dimensional symplectic manifold in which \( N \) is isotropic are classified, up to local symplectomorphism about \( N \), by the isomorphism classes of 2\( (n - k) \)-dimensional symplectic vector bundle over \( N \).

This means that if \( N \) is an isotropic submanifold of a symplectic manifold \( P \), then a neighbourhood of \( N \) in \( P \) is symplectomorphic to a neighbourhood of the embedding of \( N \) in \( P(E) \) for \( E = SN(N, P) \).

We will extend the surgery at an intersection point of two Lagrangian submanifolds to the surgery of two Lagrangian submanifolds intersecting cleanly (in the sense of Pozniak [13]) along an isotropic submanifold. In his thesis, Pozniak proves
Theorem 2.3 (Pozniak [13]). If two Lagrangian submanifolds $L_0$ and $L_1$ of a symplectic manifold $P$ intersect cleanly along $N$, that is if $N = L_0 \cap L_1$ and for each $x \in N$, $T_x N = T_x L_0 \cap T_x L_1$, then there exists a vector bundle $L \to N$ such that a neighbourhood of $N$ in $P$ is symplectomorphic to a neighbourhood of $N$ in $T^*L$, $L_0$ being mapped to the zero section of $T^*L$ and $L_1$ to the conormal of $N$ in $T^*L$.

In this setting, identifying $L_0$ and $L_1$ with their image in $T^*L$, one can see that $E = SN(N, T^*L)$ is isomorphic to the Whitney sum of the vector bundles $L \to N$ and $L^* \to N$ and that in the Whitney sum $P(E) = T^*N \oplus E$, the Lagrangian $L_0$ is mapped to the zero section in the $T^*N$-summand and to $L \oplus \{0\}$ in the $E$-summand and the Lagrangian $L_1$ is mapped to the zero section in the $T^*N$-summand and to $\{0\} \oplus L^*$ in the $E$-summand, so that the intersection of $L_0$ and $L_1$ is the sum of the zero-section of $T^*N$ and the transverse intersection in each fibre of $E$ of $L_x \oplus \{0\}$ with $\{0\} \oplus L^*_x$.

The surgery we will construct in this neighbourhood will also fiber over $N$, be equal to the zero-section in the $T^*N$-summand and will resolve the intersection in each fiber of $E$, so that it is enough to define it in the symplectic normal bundle $E$.

The bundle surgery. In this paper, the constructions are done only in real dimension 4 with the clean intersection of two Lagrangians along an isotropic circle. The symplectic normal bundle $E = SN(N, T^*L)$ is then a rank 2 symplectic vector bundle. There exists only one rank 2 symplectic vector bundle over the circle, the trivial bundle $S^1 \times \mathbb{C}$. However, for the rank 1 Lagrangian subbundle $L \to N$, it can be the trivial line bundle or the non-orientable line bundle over the circle. So one will be in real dimension 4 in one of the following case:

- either $E$ is $S^1 \times \mathbb{C}$ and $L \to N$ is $S^1 \times \mathbb{R}$, the trivial real subbundle;
- or $E$ can be described as $[0, 1] \times \mathbb{C}$ identifying the fiber at 0 and the fiber at 1 with the opposite of the identity, i.e. multiplication by $-1$ and $L \to N$ is the associated subbundle $[0, 1] \times \mathbb{R}$ with the same identification.

Now if the restriction of the Lagrangian $L_1$ to $E$ is the associated subbundle with fiber $e^{\alpha} \mathbb{R}$ as it will be the case in our examples, one can perform a surgery in dimension 1 parametrized in each fiber as

$$\{ e^{-t}x + e^{i\alpha}x \mid t \in [-T, T], x \in \mathbb{R}, x^2 = \varepsilon_1^2 \}$$

or

$$\{ e^{-t}x + e^{i\alpha}(-x) \mid t \in [-T, T], x \in \mathbb{R}, x^2 = \varepsilon_1^2 \},$$

followed by a smoothing at the end.

Now note that in this situation, the restrictions of $L_0$ and $L_1$ are invariant by multiplication by $-1$, so the handles and the smoothing can be made invariant as well. Therefore, in each case, the trivial or the non-trivial symplectic bundle over $S^1$, the change of trivialisation preserves the construction so that the handle can be defined globally and fibers over $S^1$. 
3. Construction of new monotone Lagrangian submanifolds using the surgery at a point

3.1. A non-orientable monotone Lagrangian in $\mathbb{C}P^2$. In the following sections, we explain how to get via a Lagrangian surgery on two copies of $\mathbb{R}P^2$ a monotone Lagrangian connected sum of a Klein bottle and an orientable surface of genus two in $\mathbb{C}P^2$.

3.1.1. The construction. We will take two copies of $\mathbb{R}P^2$ in $\mathbb{C}P^2$ that intersect in three points exactly, the three points of $\mathbb{C}P^2$ projecting on the three corners of the image of the moment map. Let us consider:

$$L_0 = \{ [x_0 : x_1 : x_2] \in \mathbb{C}P^2 | x_0, x_1, x_2 \in \mathbb{R} \}$$

and

$$L_1 = \{ [e^{i \pi \frac{2}{3}} x_0 : e^{-i \pi \frac{2}{3}} x_1 : x_2] \in \mathbb{C}P^2 | x_0, x_1, x_2 \in \mathbb{R} \}.$$

These are two copies of $\mathbb{R}P^2$ in $\mathbb{C}P^2$, $L_1$ being obtained from $L_0$ by a Hamiltonian isotopy. Indeed, $L_1$ is the image of $L_0$ through the map given by the action of the following diagonal matrix on the two first coordinates of the homogeneous coordinates:

$$A = \begin{pmatrix} e^{i \pi \frac{2}{3}} & 0 \\ 0 & e^{-i \pi \frac{2}{3}} \end{pmatrix}.$$ 

It is the time one of the transformation given by

$$A_t = \begin{pmatrix} e^{it \pi \frac{2}{3}} & 0 \\ 0 & e^{-it \pi \frac{2}{3}} \end{pmatrix}$$

$A_t \in SU(2)$ and it defines a Hamiltonian diffeomorphism $\Phi_t$ of $\mathbb{C}P^2$.

One can check that these two copies of $\mathbb{R}P^2$ intersect in the three points of homogeneous coordinates: $[0 : 0 : 1]$, $[0 : 1 : 0]$, $[1 : 0 : 0]$.

We want to perform a Lagrangian surgery at each of the intersection points as described in Section 2.2. Let us give the choices of handles we make for the construction.

At $[0 : 0 : 1]$, the local chart is

$$[z_0 : z_1 : z_2] \mapsto \left( \frac{z_0}{z_2}, \frac{z_1}{z_2} \right),$$

so that locally, $L_0$ is the real plane

$$l_0 = \{(x_0, x_1) | x_0, x_1 \in \mathbb{R} \}$$

and $L_1$ is

$$l_1 = \{(e^{i \pi \frac{2}{3}} x_0, e^{-i \pi \frac{2}{3}} x_1) | x_0, x_1 \in \mathbb{R} \}.$$ We are in the case when $\sin(\pi) = -\sin(-\frac{\pi}{3})$, so that we need the modified version of the handle to do the Lagrangian surgery. We will use the one defined by the smoothing of:

$$\left\{ e^{-t} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + e^t \begin{pmatrix} e^{i \pi \frac{2}{3}} x_0 \\ e^{-i \pi \frac{2}{3}} (-x_1) \end{pmatrix} | x_0^2 + x_1^2 = e^t \right\}.$$
At $[0 : 1 : 0]$, the local chart is
\[ [z_0 : z_1 : z_2] \mapsto \left( \frac{z_0}{z_1}, \frac{z_2}{z_1} \right), \]
so that locally, $L_0$ is the real plane
\[ l_0 = \{(x_0, x_2) \mid x_0, x_1 \in \mathbb{R} \} \]
and $L_1$ is
\[ l_1 = \{(e^{i\frac{2\pi}{3}}x_0, e^{i\frac{\pi}{3}}x_2) \mid x_0, x_2 \in \mathbb{R} \}. \]
As $\sin(2\pi/3) = \sin(\pi/3)$, we can use the first description of the handle to define the Lagrangian surgery:
\[ \left\{ e^{-t} \begin{pmatrix} x_0 \\ x_2 \end{pmatrix} + e^{t} \begin{pmatrix} e^{i\frac{2\pi}{3}}x_0 \\ e^{i\frac{\pi}{3}}x_2 \end{pmatrix} \left| t \in \mathbb{R} \right. \right\}. \]

At $[1 : 0 : 0]$, the local chart is
\[ [z_0 : z_1 : z_2] \mapsto \left( \frac{z_1}{z_0}, \frac{z_2}{z_0} \right), \]
so that locally, $L_0$ is the real plane
\[ l_0 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R} \} \]
and $L_1$ is
\[ l_1 = \{(e^{-i\frac{2\pi}{3}}x_1, e^{-i\frac{\pi}{3}}x_2) \mid x_1, x_2 \in \mathbb{R} \}. \]
As $\sin(-2\pi/3) = \sin(-\pi/3)$, we can also use the first description of the handle to do the Lagrangian surgery. But for the monotonicity condition to be satisfied in Section 3.1.2, we will use instead the smoothing of the $h'$-handle:
\[ \left\{ e^{-t} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + e^{t} \begin{pmatrix} e^{-i\frac{2\pi}{3}}(-x_1) \\ e^{-i\frac{\pi}{3}}(-x_2) \end{pmatrix} \left| t \in \mathbb{R} \right. \right\}. \]

After these surgeries, the projection of the Lagrangian $L$ we constructed will be contained in the polytope obtained from the polytope of $\mathbb{C}\mathbb{P}^2$ by cutting the three vertices as in Figure 4.

The choice of these copies of $\mathbb{R}\mathbb{P}^2$ and these surgeries is motivated by the monotonicity condition we will want to prove for this construction in the next section. Let us describe the restriction of the surgery along the "coordinate"-$\mathbb{C}\mathbb{P}^1$s, that is the projective lines which are the preimages of the edges on the boundary of the moment polytope and can be defined by the vanishing of one of the homogeneous coordinates. We will describe the case of
\[ \{(z_0 : z_1 : z_2) \mid z_0 = 0 \}, \]
the other coordinate-$\mathbb{C}\mathbb{P}^1$s being similar.

Along the sphere $z_0 = 0$, $L_0$ and $L_1$ are two circles intersecting transversally at the north and the south pole. We have locally in the chart $\mathbb{C}$ at $[0 : 0 : 1]$, $L_0$ on the real axis and $L_1$ on the axis $e^{-i\frac{\pi}{3}}\mathbb{R}$. In this chart, the
intersection of the surgery with \( \{ z_0 = 0 \} \) we choose consists in two curves (see Figure 5-left): inside a ball centered at the origin and of area \( \varepsilon \) they lie in two opposite "quadrants" defined by these two axis, that is in the quadrants making an angle of \( 2\pi/3 \).

Locally in \([0 : 1 : 0]\) we have a similar picture but with curves in the quadrants making an angle of \( \pi/3 \) (see Figure 5-right).

Away from the small neighbourhoods where we do the surgery, namely on the part where we glue the two charts, the restriction of \( L \) is the restriction of \( L_0 \) and \( L_1 \) to this complex projective line. One sees then that the restriction of \( L \) to this \( \mathbb{CP}^1 \) is one circle joining \( L_0 \) and \( L_1 \) through the two handles constructed at each intersection point and looking like the seam of a tennis ball (see Figure 4).

If we would have chosen the first handle we described in Section 2.2 at \([0 : 0 : 1]\), the restriction of \( L \) to \( z_0 = 0 \) would have been the union of two circles.
Figure 6. The intersection of the surgery with \( \{ z_0 = 0 \} \)

Actually, the choice of handles we made are such that in each of the other coordinate-\(\mathbb{CP}^1\)s (namely \(z_1 = 0\) and \(z_2 = 0\)), the restriction of \(L\) is one circle joining \(L_0\) and \(L_1\) through the two handles constructed at each intersection point at the poles of \(\mathbb{CP}^1\).

3.1.2. Monotonicity. Let us normalize the symplectic form on \(\mathbb{CP}^2\) such that the area of a projective line is 3:

\[
\int_{\mathbb{CP}^1} \omega = 3.
\]

With this normalization, \(\mathbb{CP}^2\) is monotone with monotonicity constant 1:

\[
\forall v \in H_2(\mathbb{CP}^2), \int_v \omega = c_1(T\mathbb{CP}^2)(v).
\]

and any monotone Lagrangian submanifold \(L\) will have a monotonicity constant equal to half the monotonicity constant of \(\mathbb{CP}^2\) (see [11]), namely \(\frac{1}{2}\):

\[
\forall u \in H_2(\mathbb{CP}^2, L), \int_u \omega = \frac{1}{2} \mu_L(u),
\]

where \(\mu_L\) is the Maslov class of \(L\).

**Theorem 3.1.** The construction of Section 3.1.1 produces a monotone Lagrangian embedding of the connected sum of a Klein bottle and a compact orientable surface of genus 2 in \(\mathbb{CP}^2\).

**Proof.** Topologically, the surgery at a point between two copies of the real projective plane gives a connected sum of these two spaces, namely a Klein bottle. Then attaching a 2-dimensional handle corresponds to a connected sum with a torus, so that the Lagrangian submanifold constructed in 3.1.1 is diffeomorphic to the connected sum

\[
L \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# T^2 \# T^2 \cong K \# \Sigma_2
\]
where $K$ is a Klein bottle and $\Sigma_2$ is a compact orientable surface of genus 2.

We know that $H_2(\mathbb{C}P^2, L) \cong H_2(\mathbb{C}P^2) \oplus H_1(L)$ as $H_2(L) = 0$. Let us examinate the monotonicity condition on each factor of this direct sum.

On the factor $H_2(\mathbb{C}P^2)$, we already have the monotonicity condition from the one on $\mathbb{C}P^2$ so that we only need to verify the monotonicity condition on the disks representing generators of $H_1(L)$. This means also that for a given generator of $H_1(L)$, it is enough to satisfy the monotonicity condition for one choice of disk with boundary this generator, as the symplectic invariants for another disc with the same boundary will differ by the invariants coming from a sphere (the sphere obtained by gluing the two disks along their boundary) where the condition is satisfied.

Now $H_1(L) = \mathbb{Z}^5 \oplus \mathbb{Z}/2$ can be generated by the two loops generating the first homology groups of each copy of $\mathbb{R}P^2$, the three loops inside each handle generating the homology of the handle and three loops "between" the handles (see Figure 7).

![Three loops between the handles](image)

Figure 7. Three loops between the handles

Any loop sating in one of the original copies of $\mathbb{R}P^2$, $L_0$ or $L_1$, satisfies the monotonicity condition because $L_0$ and $L_1$ are monotone.

One can take a representative in homology of a loop inside one handle to be the image of the circle $\{(x_0, x_1), x_0^2 + x_1^2 = \varepsilon^2\}$, for $\varepsilon > \varepsilon_1$ small, in a chart on one of the copies of the Lagrangian $\mathbb{R}P^2$, say for example $L_0$. We take a disc in $\mathbb{C}P^2$ with boundary this circle in $L$ and compute the symplectic invariants of that disc. One can for example choose the disc in $L_0$ which was
cut out of $L_0$ to built $L$. But as the disc is Lagrangian, the two invariants, area and Maslov class, vanish on this disc.

One is left to check the monotonicity condition on the circles between handles. One could prove that the monotonicity condition is satisfied on the three circles drawing the tennis ball seam on the coordinate-$\mathbb{CP}^1$s we described at the end of the previous section and the discs they bound on these $\mathbb{CP}^1$. Unfortunately these circles are not in our set of generators for $H_1(L)$ (they go twice around the handle). But we can use for the generators loops which partially follow these seams. Let us describe a loop $\gamma$ we can choose between the handles created at $[0 : 0 : 1]$ and $[0 : 1 : 0]$ and a disk it bounds. Two other loops between handles can be constructed in a similar way.

The loop $\gamma$ is almost entirely lying in the coordinate $\mathbb{CP}^1$ of homogeneous equation $z_0 = 0$. See Figure 8.

![Figure 8. The loop $\gamma$ in green](image)

It is based at a point where $L$ coincides with $L_0$, for example the point $a$ of local coordinates $(z_0, z_1) = (0, \varepsilon)$ in the local chart at $[0 : 0 : 1]$. From this point, follow the handle at $[0 : 0 : 1]$ along the path parametrized by

$$\{(0, e^{-t}\varepsilon_1 + e^t e^{-i\frac{\pi}{3}}(-\varepsilon_1))\}$$

(we include here the smoothing by considering we can locally take the parametrization of the handle for $t$ varying from $-\infty$ to $+\infty$) till the point $b$ of local coordinates $(z_0, z_2) = (0, e^{-i\frac{\pi}{3}}\varepsilon)$. Then follow $L_1 \cap \{z_0 = 0\}$ "up" towards $[0 : 1 : 0]$ till the point $c$ of local coordinates $(z_0, z_2) = (0, e^{i\frac{\pi}{3}}(-\varepsilon))$ in the local chart at $[0 : 1 : 0]$. Next, $\gamma$ goes back to $L_0$ through the handle at $[0 : 1 : 0]$, following "backwards" the path parametrized by

$$(0, e^{-t}(-\varepsilon_1) + e^t e^{i\frac{\pi}{3}}(-\varepsilon_1))$$
in local coordinates in the chart at \([0 : 1 : 0]\) till it reaches the point \(d\) of local coordinates \((0, -\varepsilon)\) in the chart at \([0 : 0 : 1]\). The path then follows \(L_0 \cap \{z_0 = 0\}\) "down" to \([0 : 0 : 1]\) till the point \(e\) of local coordinates \((0, -\varepsilon)\) in the chart at \([0 : 0 : 1]\). Now we close the loop \(\gamma\) with a path contained in \(L_0 \cap L\) but leaving the coordinate-\(\mathbb{C}P^1\) \(\{z_0 = 0\}\) by following the half circle parametrized in the chart at \([0 : 0 : 1]\) by

\[
\{(-\varepsilon \sin(t), -\varepsilon \cos(t)) | t \in [0; \pi]\}.
\]

This loop encloses a disk \(u\) in \(\mathbb{C}P^2\) which can be described as the union of the portion of sphere \(\{z_0 = 0\}\) lying between \(L_0\) and \(L_1\) in the sector making a \(\frac{\pi}{3}\)-angle and delimited by \(\gamma\) in the "north", the portion of the same sphere in the \(\frac{2\pi}{3}\)-sector between \(\gamma\) and the segment in \(L_0\) (but not \(L\)) of coordinates in the chart at \([0 : 0 : 1]\)

\[
[e; a] = \{(0, z_2) | z_2 \in [-\varepsilon, \varepsilon]\},
\]

and the half disk in \(L_0\) enclosed between this segment \([e; a]\) and \(\gamma\) (the part of the disk \(u\) in \(\{z_0 = 0\}\) and the half disk are glued along the segment \([e; a]\)).

This disk \(u\) and the similar ones we can built between the other handles together with the disks considered before generate \(H_2(\mathbb{C}P^2, L)\), so that the monotonicity of \(L\) will follow from the next two lemmas which compute the area and the Maslov class of \(u\).

\[\square\]

**Lemma 3.2.** The disk \(u\) has area \(\frac{1}{2}\).

**Proof.** We will compute the area of \(u\) by adding the area of the three portions we described above.

As we noticed in Section 2.2 we can make the Lagrangian surgery such that the areas between the restrictions to each \(C\)-factor of \(\mathbb{C}^2\) of the original Lagrangians and the handles are small and equal. We do the surgeries as in Section 2.2 so that these areas are equal to \(a(\varepsilon)\) small at each of the intersection points.

The area of the first portion is then the area of the \(\frac{\pi}{3}\)-sector, namely one sixth of the total area of the sphere, minus the area lost at the handle, namely \(a(\varepsilon)\). The area in the other sector of the coordinate projective line is the area gained through the handle, that is \(a(\varepsilon)\). The contribution of the portions of the disk in the coordinate sphere is thus \(\frac{1}{2}\). The contribution of the half-disk in \(L_0\) is zero as this half-disk lies totally in a Lagrangian submanifold. \[\square\]

**Lemma 3.3.** The disk \(u\) has Maslov class 1.

**Proof.** The first crucial remark is that the disk \(u\) is lying entirely in the chart at \([0 : 0 : 1]\), so that the tangent bundle of \(\mathbb{C}P^2\) is already trivialized along the disk when we work in this chart. Now, to compute the Maslov class of this disk, we will write the loop in the Lagrangian Grassmannian we have along the boundary \(\gamma\) as the action of a loop \(A(t)\) of matrices of \(U(2)\) on the reference linear Lagrangian space \(\mathbb{R}^2\) of \(\mathbb{C}^2\). The Maslov class \(\mu(u)\) is
then the degree of the square of the determinant of $A$ seen as a map from $S^1$ to $S^1$.

To describe this action, we will decompose the action along the different portions of the loop we considered above. The loop $\gamma$ is the concatenation of the paths $\gamma_1$ from $a$ to $b$, $\gamma_2$ from $b$ to $c$, ... and, $\gamma_5$ from $e$ to $a$. On each of these paths, we will decompose the action of $U(2)$ so that the Maslov class of $u$ can be written as the product of these different paths of matrices.

At the point $a$, we are in $L_0 \cap L$, with $L_0$ a linear Lagrangian in the chart, so that the submanifold identifies with its tangent space, namely $\mathbb{R}^2$, our reference linear Lagrangian subspace.

Between the points $b$ and $c$, we stay on $L_1$, the tangent space is identically equal to the linear Lagrangian subspace $l_1 = \{(e^{i\pi/2}x_0, e^{-i\pi/2}x_1) | x_0, x_1 \in \mathbb{R}\}$ so that $A(t)$ is the identity along $\gamma_2$ and this portion has no contribution to the degree.

Similarly, along $\gamma_4$ and $\gamma_5$, we stay on the same linear Lagrangian (either $l_1$ or $l_0$) so that the matrix $A(t)$ is again the identity along these portions of $\gamma$.

We are left to compute the contributions of the handles to the Maslov class.

Computing the contribution of the handle at $[0 : 0 : 1]$ along $\gamma_1$ is straightforward because we have the parametrization of the handle explicitly written in this chart, namely:

\[
\left\{ e^{-t} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + e^{t} \begin{pmatrix} e^{i\pi} x_0 \\ e^{-i\pi} (-x_1) \end{pmatrix} \right| t \in \mathbb{R}, x_0^2 + x_1^2 = \varepsilon_1^2 \right\}.
\]

Then the tangent spaces to that handle along the points in $\{z_0 = 0\}$ can be parametrized by:

\[
\left\{ e^{-t} \begin{pmatrix} X_0 \\ -\varepsilon_1 T \end{pmatrix} + e^{t} \begin{pmatrix} e^{i\pi} X_0 \\ e^{-i\pi} (-\varepsilon_1) T \end{pmatrix} \right| t, X_0 \in \mathbb{R} \right\}.
\]

For $t$ going to $-\infty$, the tangent space is asymptotic to $\mathbb{R}^2$ for which we can take the canonical basis $\{(1,0), (0,1)\}$. Through the handle, the vectors $(X_0, -\varepsilon_1 T) = (1,0)$ and $(X_0, -\varepsilon_1 T) = (0,1)$ are mapped to $(X_0, -\varepsilon_1 T) = (e^{i\pi}, 0) and (X_0, -\varepsilon_1 T) = (0, e^{-i\pi})$, a basis of $l_1$ through a path of matrices homotopic to

\[
A_1(s) = \begin{pmatrix} e^{-is\pi/2} & 0 \\ 0 & e^{is\pi/2} \end{pmatrix}
\]

for $s$ going from $s = 0$ to $s = 1$. The determinant of $A_1$ being identically equal to 1, this part of $\gamma$ will not contribute to the degree.

The contribution of the handle at $[0 : 1 : 0]$ can also be computed thanks to the explicit parametrization of the handle, but we have first to write it in the chart at $[0 : 0 : 1]$ for our computation. In that chart, the handle is now parametrized by

\[
\left\{ \begin{pmatrix} e^{-t} x_0 + e^{t} e^{2i\pi} x_0 \\ e^{-t} x_2 + e^{t} e^{2i\pi} x_2 \end{pmatrix} \right| \frac{1}{(e^{-t} x_2 + e^{t} e^{2i\pi} x_2)^2} t \in \mathbb{R}, x_0^2 + x_2^2 = \varepsilon_1^2 \}
\].

so that the tangent spaces are described by
\[
\left\{ \left( e^{-t}X_0 + e^{t}e^{i\frac{2\pi}{3}}x_0, -e^{-t}x_2T + e^{t}e^{i\frac{2\pi}{3}}x_2T \right) \middle| \begin{array}{l}
  t, T, X_0 \in \mathbb{R} \\
  x_2 = -\varepsilon_1
\end{array} \right\}.
\]

When \( t \) tends to \(-\infty \), the handle is indeed asymptotic to \( l_0 \) and when \( t \) tends to \(+\infty \), the handle is asymptotic to \( l_1 = \{(e^{i\frac{2\pi}{3}}x_0, e^{-i\frac{2\pi}{3}}x_1) | x_0, x_1 \in \mathbb{R} \} \). The canonical basis \( \{(1, 0), (0, 1)\} \) is mapped through the handle to \( \{(e^{i\frac{2\pi}{3}}, 0), (0, -e^{-i\frac{2\pi}{3}})\} \)

The action on the first coordinate can be homotopic to the path in \( U(1) \)
\[ s \in [0, 1] \mapsto e^{i\frac{2\pi s}{3}} \]
or to \( s \in [0, 1] \mapsto e^{-i\frac{2\pi s}{3}} \). But for \( t = 0 \) at the middle of the handle, one can check via the formula that the image of the vector \( (1, 0) \) is positively proportional to the vector \( (e^{i\frac{2\pi}{3}}, 0) \) so that the path is
\[ s \in [0, 1] \mapsto e^{i\frac{2\pi s}{3}}. \]

For the second coordinate, in a similar manner one can act either by a path homotopic to \( s \in [0, 1] \mapsto e^{i\frac{2\pi s}{3}} \) or to \( s \in [0, 1] \mapsto e^{-i\frac{2\pi s}{3}} \). For \( t = 0 \) at the middle of the handle, one can check via the formula that the image of the vector \( (0, 1) \) is positively proportional to the vector \( (0, -e^{i\frac{2\pi}{3}}) \) so that the path is
\[ s \in [0, 1] \mapsto e^{-i\frac{4\pi s}{3}}. \]

The contribution of the handle along \( \{z_0 = 0\} \) from \( l_0 \) to \( l_1 \) is thus homotopic to
\[ s \in [0, 1] \mapsto \begin{pmatrix} e^{i\frac{2\pi s}{3}} & 0 \\ 0 & e^{-i\frac{4\pi s}{3}} \end{pmatrix}. \]

But along \( \gamma_3 \) we move from \( L_1 \) to \( L_0 \) so that the matrix of the action of \( U(2) \) along this portion of the boundary is homotopic to
\[ s \in [0, 1] \mapsto \begin{pmatrix} e^{-i\frac{\pi s}{3}} & e^{4i\frac{\pi s}{3}} \\ 0 & e^{+4i\frac{\pi s}{3}} \end{pmatrix}, \]

whose determinant squared is equal to
\[ [0, 1] \mapsto S^1 \quad s \mapsto e^{2i\pi} \]

which is of degree 1. In conclusion, the Maslov class of \( u \) being the sum of all the contributions of the portions of the loop \( \gamma \) is equal to 1. \( \square \)

### 3.2. A monotone \( K \# \Sigma_4 \) in \( \mathbb{CP}^1 \times \mathbb{CP}^1 \)

In this section, we explain the construction of a monotone Lagrangian embedding of the connected sum of a Klein bottle and a surface of genus 4 in the product \( \mathbb{CP}^1 \times \mathbb{CP}^1 \).

Let us normalize the symplectic form on \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) such that the area of a projective line is 2:
\[ \int_{\mathbb{CP}^1} \omega = 2. \]
With this normalisation, $\mathbb{CP}^1 \times \mathbb{CP}^1$ is monotone with monotonicity constant 1:
\[
\forall v \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1), \int_v \omega = c_1(T(\mathbb{CP}^1 \times \mathbb{CP}^1))(v),
\]
and any monotone Lagrangian submanifold $L$ will have a monotonicity constant equal to $\frac{1}{2}$:
\[
\forall u \in H_2(\mathbb{CP}^1 \times \mathbb{CP}^1), L, \int_u \omega = \frac{1}{2} \mu_L(u).
\]

To construct a monotone $K\#\Sigma_4$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$, we take two Hamiltonian isotopic copies of the real part of $\mathbb{CP}^1 \times \mathbb{CP}^1$, i.e. two Lagrangian tori, that intersect in four points and perform a suitable Lagrangian surgery at this four points.

The image of the resulting Lagrangian $L$ under the moment map will be contained in the original moment polytope of the ambient symplectic manifold chopped at its four corners (see Figure 9).

![Figure 9](image-url)

**Figure 9.** The image of $L$ under the moment map is contained and smoothly approximates the shaded polytope.

In view of the monotonicity condition, we have here two possible choices for the copies of the real part and corresponding surgeries.

Either one can choose to take one copy of the real part to be
\[
L_0 = \{(x_0 : x_1, u_0 : u_1) \in \mathbb{CP}^1 \times \mathbb{CP}^1 | x_0, x_1, u_0, u_1 \in \mathbb{R}\}
\]
and the second one its "rotation" by $i$:
\[
L_1 = \{(e^{i\pi}x_0 : x_1, e^{i\pi}u_0 : u_1) \in \mathbb{CP}^1 \times \mathbb{CP}^1 | x_0, x_1, u_0, u_1 \in \mathbb{R}\}.
\]

They intersect each another in the four points with homogeneous coordinates:
\[
([0 : 1], [0 : 1]), ([1 : 0], [0 : 1]), ([0 : 1], [1 : 0]), ([1 : 0], [1 : 0])
\]
the preimage of the four corners under the standard moment map. Then choose the following handles at each of the intersection points

- at \((0 : 1), [0 : 1]\), \(l_0 = \{(x_0, u_0) | x_0, u_0 \in \mathbb{R}\}, l_1 = \{(ix_0, iu_0) | x_0, u_0 \in \mathbb{R}\}, \) and we put the handle:

\[
\left\{ e^{-t} \left( \begin{array}{c} x_0 \\ u_0 \end{array} \right) + e^{t} \left( \begin{array}{c} ix_0 \\ iu_0 \end{array} \right) \left| t \in \mathbb{R} \right. \right. \left. \left. x_0^2 + u_0^2 = \varepsilon_1 \right\},
\]

- at \((1 : 0), [0 : 1]\), \(l_0 = \{(x_1, u_0) | x_1, u_0 \in \mathbb{R}\}, l_1 = \{(-ix_1, iu_0) | x_1, u_0 \in \mathbb{R}\}, \) and we put the handle:

\[
\left\{ e^{-t} \left( \begin{array}{c} x_1 \\ u_0 \end{array} \right) + e^{t} \left( \begin{array}{c} -i(-x_1) \\ iu_0 \end{array} \right) \left| t \in \mathbb{R} \right. \right. \left. \left. x_1^2 + u_0^2 = \varepsilon_1 \right\},
\]

- at \((0 : 1), [1 : 0]\), \(l_0 = \{(x_0, u_1) | x_0, u_1 \in \mathbb{R}\}, l_1 = \{(ix_0, -iu_1) | x_0, u_1 \in \mathbb{R}\}, \) and we put the handle:

\[
\left\{ e^{-t} \left( \begin{array}{c} x_0 \\ u_1 \end{array} \right) + e^{t} \left( \begin{array}{c} ix_0 \\ -i(-u_1) \end{array} \right) \left| t \in \mathbb{R} \right. \right. \left. \left. x_0^2 + u_1^2 = \varepsilon_1 \right\},
\]

- at \((1 : 0), [1 : 0]\), \(l_0 = \{(x_1, u_1) | x_1, u_1 \in \mathbb{R}\}, l_1 = \{(-ix_1, -iu_1) | x_1, u_1 \in \mathbb{R}\}, \) and we put the handle:

\[
\left\{ e^{-t} \left( \begin{array}{c} x_1 \\ u_1 \end{array} \right) + e^{t} \left( \begin{array}{c} -i(-x_1) \\ -i(-u_1) \end{array} \right) \left| t \in \mathbb{R} \right. \right. \left. \left. x_1^2 + u_1^2 = \varepsilon_1 \right\},
\]

so that the intersection of the Lagrangian obtained by surgery with any of the \(\mathbb{C}P^1\) which are preimages of the edges on the boundary of the moment polytope is one circle of the shape of the tennis ball seam as before.

As in the previous section, one cannot take this circle to check the monotonicity of the surface as it goes around the handle twice, but we can construct similar loops going only once around the handle and check that they satisfy the monotonicity condition.

Alternatively, one can choose the surgeries such that the intersection of the Lagrangian after surgery with each coordinate-\(\mathbb{C}P^1\) is a union of two circles. To ensure monotonicity in this case, we need each circle produced to enclose an area slightly bigger than the one we get with the choice of \(L_1\) above.

We will take the same Lagrangian \(L_0\) and for \(L_1\) its following Hamiltonian isotopic copy:

\[
L_1 = \left\{ \left( x_0^{(\frac{n}{2}+\delta)} x_1 | x_1 \right), \left( x_0^{(\frac{n}{2}+\delta)} u_0 : u_1 \right) \right\} \in \mathbb{C}P^1 \times \mathbb{C}P^1 \left| x_0, x_1, u_0, u_1 \in \mathbb{R} \right,
\]

for a small positive parameter \(\delta\) that will be fixed later. They intersect each another again in the four corners of the moment polytope of \(\mathbb{C}P^1 \times \mathbb{C}P^1\).

In each chart around the intersection points, we are making the following choices:
• at \([0 : 1], [0 : 1]\), \(L_0\) and \(L_1\) are the linear subspaces \(l_0 = \{(x_0, u_0)|x_0, u_0 \in \mathbb{R}\}\), \(l_1 = \{(e^{i(\frac{\pi}{2} + \delta)}x_0, e^{i(\frac{\pi}{2} + \delta)}u_0)|x_0, u_0 \in \mathbb{R}\}\), and we put the handle:
\[
\begin{aligned}
&\left\{ e^{-t} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} + e^t \begin{pmatrix} e^{i(\frac{\pi}{2} + \delta)}x_0 \\ e^{i(\frac{\pi}{2} + \delta)}u_0 \end{pmatrix} \right| t \in \mathbb{R} \\
&\quad \quad \quad \quad \quad \quad x_0^2 + u_0^2 = \varepsilon_1 \end{aligned}
\]

• at \([1 : 0], [0 : 1]\), \(L_0\) and \(L_1\) are the linear subspaces \(l_0 = \{(x_1, u_0)|x_1, u_0 \in \mathbb{R}\}\), \(l_1 = \{(e^{-i(\frac{\pi}{2} + \delta)}x_1, e^{i(\frac{\pi}{2} + \delta)}u_0)|x_1, u_0 \in \mathbb{R}\}\), and we put the handle:
\[
\begin{aligned}
&\left\{ e^{-t} \begin{pmatrix} x_1 \\ u_0 \end{pmatrix} + e^t \begin{pmatrix} e^{-i(\frac{\pi}{2} + \delta)}x_1 \\ e^{i(\frac{\pi}{2} + \delta)}(-u_0) \end{pmatrix} \right| t \in \mathbb{R} \\
&\quad \quad \quad \quad \quad \quad x_1^2 + u_0^2 = \varepsilon_1 \end{aligned}
\]

• at \([0 : 1], [1 : 0]\), \(L_0\) and \(L_1\) are the linear subspaces \(l_0 = \{(x_0, u_1)|x_0, u_1 \in \mathbb{R}\}\), \(l_1 = \{(e^{i(\frac{\pi}{2} + \delta)}x_0, e^{-i(\frac{\pi}{2} + \delta)}u_1)|x_0, u_1 \in \mathbb{R}\}\), and we put the handle:
\[
\begin{aligned}
&\left\{ e^{-t} \begin{pmatrix} x_0 \\ u_1 \end{pmatrix} + e^t \begin{pmatrix} e^{i(\frac{\pi}{2} + \delta)}(-x_0) \\ e^{-i(\frac{\pi}{2} + \delta)}u_1 \end{pmatrix} \right| t \in \mathbb{R} \\
&\quad \quad \quad \quad \quad \quad x_0^2 + u_1^2 = \varepsilon_1 \end{aligned}
\]

• at \([1 : 0], [1 : 0]\), \(L_0\) and \(L_1\) are the linear subspaces \(l_0 = \{(x_1, u_1)|x_1, u_1 \in \mathbb{R}\}\), \(l_1 = \{(e^{-i(\frac{\pi}{2} + \delta)}x_1, e^{-i(\frac{\pi}{2} + \delta)}u_1)|x_1, u_1 \in \mathbb{R}\}\), and we put the handle:
\[
\begin{aligned}
&\left\{ e^{-t} \begin{pmatrix} x_1 \\ u_1 \end{pmatrix} + e^t \begin{pmatrix} e^{-i(\frac{\pi}{2} + \delta)}(-x_1) \\ e^{-i(\frac{\pi}{2} + \delta)}(-u_1) \end{pmatrix} \right| t \in \mathbb{R} \\
&\quad \quad \quad \quad \quad \quad x_1^2 + u_1^2 = \varepsilon_1 \end{aligned}
\]

**Theorem 3.4.** The construction produces a monotone Lagrangian embedding of a compact surface \(K \# \Sigma_4\) in \(\mathbb{C}P^1 \times \mathbb{C}P^1\) for an appropriate choice of \(\delta\).

**Proof.** Note that even though the two Lagrangian submanifolds \(L_0\) and \(L_1\) are oriented (they are tori), one can check that given an orientation of the two tori, two of these handles do not preserve the orientation (this cannot be avoided, it is related to the fact that the signs of the intersection points cancel in pairs for any choice of orientation). Therefore, we get through these four surgeries a non-orientable Lagrangian which is the connected sum of the two initial tori with one torus and two Klein bottles. It is diffeomorphic to

\[ L \cong K \# \Sigma_4. \]

Following the remarks from Section 2.2, we can do the surgery in each corner of the moment map such that the areas between the handle and the initial Lagrangians are small and equal along each \(\mathbb{C}P^1\) preimage of the boundary of the moment polytope. With the choice of handles we made above, the intersection of \(L\) with each coordinate-\(\mathbb{C}P^1\) is composed of two circles lying in the sectors of the coordinate sphere making an angle \(\frac{\pi}{2} + \delta\).
We will pick one of these circles and the disk \( u \) it encloses in one of the \( \frac{\pi}{2} + \delta \)-sectors. The area of this disk is equal to the difference of the area of one sector and \( 2a(\varepsilon) \):

\[
\frac{1}{2} + \frac{\delta}{\pi} - 2a(\varepsilon).
\]

We can now fix \( \delta \) (i.e. take \( \delta = 2\pi a(\varepsilon) \)) such that the area of \( u \) is equal to \( \frac{1}{2} \).

We can compute that the Maslov class of \( u \) is equal to \( 1 \). Let us take \( u \) in the projective line of homogeneous equation \( z_0 = 0 \) in the homogeneous coordinates \(([z_0 : z_1], [u_0 : u_1])\) of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \). In a similar way as in Section 3.1.2 we trivialize the tangent bundle of \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) over \( u \) by considering \( u \) in the chart of \(([0 : 1], [0 : 1])\). As before, one can decompose the loop along its boundary in four paths: two paths lying on the restrictions of the initial Lagrangians to this chart and two paths inside the handles. For the first type of paths, as the restrictions of \( L_0 \) and \( L_1 \) are linear in the chart, they will not contribute to the Maslov class. The handle at \(([0 : 1], [0 : 1])\) is contributing with a path from \( L_0 \) to \( L_1 \) homotopic to

\[
s \in [0, 1] \mapsto \begin{pmatrix} e^{i s (\frac{\pi}{2} + \delta)} & 0 \\ 0 & e^{-i s (\frac{\pi}{2} - \delta)} \end{pmatrix},
\]

so that the Maslov class of \( u \) is the degree of

\[
s \in [0, 1] \mapsto e^{-i s 4 \delta} e^{i s (2\pi + 4\delta)}
\]

that is equal to 1.

This is enough in order to check the monotonicity of this Lagrangian \( K \# \Sigma_4 \) in \( \mathbb{CP}^1 \times \mathbb{CP}^1 \) as the relative homology group \( H_2(\mathbb{CP}^1 \times \mathbb{CP}^1, K \# \Sigma_4) \) is generated by disks with boundary either on \( L_0 \) or \( L_1 \) (which satisfy the monotonicity condition as \( L_0 \) and \( L_1 \) are monotone) and the disks considered above in the \( \mathbb{CP}^1 \)'s.

\[\square\]

4. Construction of monotone Lagrangian submanifolds using the local model along an isotropic circle

4.1. Case of two copies of a torus intersecting along two circles in \( \mathbb{CP}^1 \times \mathbb{CP}^1 \). Let \( P = \mathbb{CP}^1 \times \mathbb{CP}^1 \) with the product symplectic form with the same normalisation as before.

Let us consider the two following Hamiltonian isotopic copies of the real part of \( P \):

\[
L_0 = \{ ([x_0 : x_1], [u_0 : u_1]) \in \mathbb{CP}^1 \times \mathbb{CP}^1 | x_0, x_1, u_0, u_1 \in \mathbb{R} \}
\]
Lagrangian. In the symplectic normal bundle of \( N \) and point \( p \) the trivial complex line bundle over \( N \) is \( \mathbb{C} \). Let us describe now the choice of surgeries we make to produce a monotone Lagrangian. The two Lagrangians \( L_0 \) and \( L_1 \) intersect exactly along two isotropic circles:

\[
L_0 \cap L_1 = \{[0 : 1]\} \times \mathbb{R} \mathbb{P}^1 \cup \{[1 : 0]\} \times \mathbb{R} \mathbb{P}^1.
\]

As we saw in Section 2.3, this means that the fibrewise surgery with fixed parameter \( \varepsilon \) globalise in a bundle surgery over \( N \) and \( N \times e^{i \frac{\pi}{2}} \mathbb{R} \) respectively. The same construction can be done along the other intersection circle \( N' = \{[1 : 0]\} \times \mathbb{R} \mathbb{P}^1 \).

Let us study the neighbourhood of \( N = \{[0 : 1]\} \times \mathbb{R} \mathbb{P}^1 \) in \( P \). The circle \( N \) can be covered by the following charts of \( \mathbb{C} \mathbb{P}^1 \times \mathbb{C} \mathbb{P}^1 \):

\[
\phi_0 : U_0 = \{[z_0 : z_1] | z_1 \neq 0\} \times \{(w_0 : w_1) | w_0 \neq 0\} \rightarrow \mathbb{C} \oplus \mathbb{C} \quad \phi_1 : U_1 = \{[z_0 : z_1] | z_1 \neq 0\} \times \{(w_0 : w_1) | w_1 \neq 0\} \rightarrow \mathbb{C} \oplus \mathbb{C}
\]

so that \( T(\mathbb{C} \mathbb{P}^1 \times \mathbb{C} \mathbb{P}^1) \mid N \) can be trivialised along \( N_0 = N \cap U_0 \) and \( N_1 = N \cap U_1 \) as

\[
T(\mathbb{C} \mathbb{P}^1 \times \mathbb{C} \mathbb{P}^1) \mid N_j \cong N_j \times (\mathbb{C} \oplus \mathbb{C}),
\]

where the first summand is just \( \mathbb{C} = T_{[1,0]} \mathbb{C} \mathbb{P}^1 \). In these trivialisations,

\[
T N_j \cong N_j \times \{0\} \oplus \mathbb{R}
\]

and we have

\[
T N_j \cong N_j \times (\mathbb{C} \oplus \mathbb{R})
\]

so that

\[
SN(N, \mathbb{C} \mathbb{P}^2) \mid N_j = TN_j / T N_j \cong N_j \times \mathbb{C},
\]

and the change of trivialisation from \( N_0 \times \mathbb{C} \) to \( N_1 \times \mathbb{C} \) in a fibre of a point \( ([1 : 0], [c : d]) \) of \( N_0 \cap N_1 \) is the identity as the fibre at each point is \( \mathbb{C} = T_{[1,0]} \mathbb{C} \mathbb{P}^1 \). We are in the case where the symplectic normal bundle is the trivial complex line bundle over \( N \).

Now, in the trivialisations, the restriction of the initial Lagrangians to \( SN(N, \mathbb{C} \mathbb{P}^1 \times \mathbb{C} \mathbb{P}^1) \mid N_j \) are:

- for \( L_0 \): \( N_j \times \mathbb{R} \),
- for \( L_1 \): \( N_j \times e^{i \frac{\pi}{2}} \mathbb{R} \),

with the identity as change of trivialisation so that the restrictions of \( L_0 \) and \( L_1 \) are globally products \( N \times \mathbb{R} \) and \( N \times e^{i \frac{\pi}{2}} \mathbb{R} \) respectively.

As we saw in Section 2.3, this means that the fibrewise surgery with fixed parameter \( \varepsilon \) globalise in a bundle surgery over \( N \).

The same construction can be done along the other intersection circle \( N' = \{[1 : 0]\} \times \mathbb{R} \mathbb{P}^1 \).

Let us describe now the choice of surgeries we make to produce a monotone Lagrangian. In the symplectic normal bundle of \( N \), for any fibre \( C \) of a point \( p \in N \) we choose the surgery

\[
\left\{ e^{-t} x_0 + e^{t} e^{i \frac{\pi}{2}} x_0 \big| x_0 \in \mathbb{R}, x_0^2 = \varepsilon_1^2 \right\}.
\]
For the other intersection circle $N'$, the symplectic normal bundle is again trivial and the restriction of $L_0$ and $L_1$ are $N' \times \mathbb{R}$ and $N' \times e^{-i\pi/2} \mathbb{R}$ respectively, and we choose the following handle:

$$\left\{ e^{-t}x_1 + e^{t}e^{-i\pi/2} (-x_1) \mid x_1 \in \mathbb{R}, x_1^2 = \varepsilon^2 \right\}.$$

One can remark that the Lagrangian constructed via these surgeries fibers over $N$ (and $N'$ respectively) as both handles and the Lagrangians $L_0$ and $L_1$ do, the projection being the restriction of the projection of the product onto its second factor:

$$pr_2 : \mathbb{CP}^1 \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1.$$

It actually lies in the $\mathbb{CP}^1$-bundle over $N$, restriction of $pr_2$ to $\mathbb{CP}^1 \times \mathbb{RP}^1$. In a fiber $\mathbb{CP}^1$ of this fibration, with our choice of surgery, the restriction of $L$ is one circle in the tennis ball seam shape cutting the fiber into two disks of equal area 1. As through this fibration we see that the Lagrangian is just a product of $N$ and the tennis ball seam, the Maslov class of any of these disks of area 1 is the Maslov class of the tennis ball seam in the fiber $\mathbb{CP}^1$, that is 2.

This shows that the Lagrangian we constructed is a torus and that it is monotone.

Unfortunately, this torus is not new, it is Hamiltonian isotopic to the real part of $\mathbb{CP}^1 \times \mathbb{CP}^1$. For the isotopy we could just take the extension of the exact Lagrangian isotopy that isotopes in each fiber of $pr_2$ the tennis ball seam on the real line $\mathbb{RP}^1$. We have proved:

**Theorem 4.1.** The Lagrangian bundle surgery construction above produces a monotone Lagrangian embedding of a torus $L$ in $\mathbb{CP}^1 \times \mathbb{CP}^1$ which is Hamiltonian isotopic to the real part of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and projects through the moment map to a smooth interior approximation of the shaded polytope in Figure 10.

4.2. Recovering the Chekanov and Schlenk torus: Case of two copies of $\mathbb{RP}^2$ intersecting in a point and along a circle in $\mathbb{CP}^2$. We detail here how our method produces a torus Hamiltonian isotopic to the model torus we started with, i.e. the exotic torus of Chekanov and Schlenk.

**Theorem 4.2.** With a Lagrangian surgery at a point and a Lagrangian surgery along a circle of two copies of $\mathbb{RP}^2$ one can construct a monotone Lagrangian embedding of the torus in $\mathbb{CP}^2$ that projects through the moment map to a smooth interior approximation of the shaded polytope in Figure 11.

**Proof.** Let us consider the following two Hamiltonian isotopic copies of $\mathbb{RP}^2$ in $\mathbb{CP}^2$:

$$L_0 = \{ [x_0 : x_1 : x_2] \in \mathbb{CP}^2 \mid x_0, x_1, x_2 \in \mathbb{R} \}$$

and

$$L_1 = \{ [e^{i\alpha}x_0 : e^{i\alpha}x_1 : x_2] \in \mathbb{CP}^2 \mid x_0, x_1, x_2 \in \mathbb{R} \},$$
Figure 10. The image of $L$ under the moment map is contained and smoothly approximates the shaded polytope.

Figure 11. The image of the torus under the moment map is contained and smoothly approximates the shaded polytope.

for some $\alpha \in (0, \pi)$. The two Lagrangians $L_0$ and $L_1$ intersect exactly at the point $[0 : 0 : 1]$ and along the isotropic circle

$$N = \{ [x_0 : x_1 : 0] \mid (x_0, x_1) \in \mathbb{R}^2 \setminus \{(0, 0)\} \}.$$  

We will proceed to a Lagrangian bundle surgery along the isotropic circle and a simple Lagrangian surgery at the point $[0 : 0 : 1]$.

Let us first understand the neighbourhood of $N$ in $\mathbb{CP}^2$. The circle $N$ can be covered by the following charts of $\mathbb{CP}^2$:

$$\phi_0 : U_0 = \{ [z_0 : z_1 : z_2] \mid z_0 \neq 0 \} \longrightarrow \mathbb{C} \oplus \mathbb{C}$$

$$[z_0 : z_1 : z_2] \quad \mapsto \quad \left( \frac{z_1}{z_0}, \frac{z_2}{z_0} \right)$$
and
\[
\phi_1 : \quad U_1 = \{[z_0 : z_1 : z_2] \mid z_1 \neq 0\} \quad \longrightarrow \quad \mathbb{C} \oplus \mathbb{C} \\
\quad [z_0 : z_1 : z_2] \quad \longmapsto \quad \left(\frac{z_0}{z_1}, \frac{z_2}{z_1}\right)
\]
so that \(T\mathbb{CP}^2|_N\) can be trivialised along \(N_0 = N \cap U_0\) and \(N_1 = N \cap U_1\) as
\[
T\mathbb{CP}^2|_{N_j} \cong N_j \times (\mathbb{C} \oplus \mathbb{C}).
\]
In these trivialisations,
\[
TN|_{N_j} \cong N_j \times (\mathbb{R} \oplus \{0\})
\]
and we have
\[
TN|_{N_j} \cong N_j \times (\mathbb{R} \oplus \mathbb{C})
\]
so that
\[
SN(N, \mathbb{CP}^2)|_{N_j} = TN|_{N_j} / TN|_{N_j} \cong N_j \times \mathbb{C},
\]
and the change of trivialisation from \(N_0 \times \mathbb{C}\) to \(N_1 \times \mathbb{C}\) in a fibre over a point \([a : b : 0]\) of \(N_0 \cap N_1\) is \([(a : b : 0), Z] \mapsto ([a : b : 0], \frac{a}{b}Z)\). As the intersection \(N_0 \cap N_1\) retracts on \(\{(1 : 1 : 0), [-1 : 1 : 0]\}\), we have only two changes of trivialisation to consider: in \([1 : 1 : 0]\) where it is the identity; another in \([-1 : 1 : 0]\) where it is minus the identity.

Now, in the trivialisations, the trace of the initial Lagrangians in \(SN(N, \mathbb{CP}^2)|_{N_j}\) are:

- for \(L_0\): \(N_j \times \mathbb{R}\),
- for \(L_1\): \(N_j \times e^{-i\alpha}\mathbb{R}\),

with the same change of trivialization as before.

One can then make a Lagrangian surgery in each fibre \(\mathbb{C}\), and it globalizes to a Lagrangian subbundle over \(N\) in the symplectic normal bundle.

We can also do a surgery at the transverse intersection point \([0 : 0 : 1]\), so that we get an embedded Lagrangian submanifold out of the surgeries on \(L_0\) and \(L_1\).

We can show that for some choices of handles and of \(\alpha\), this Lagrangian submanifold is monotone. Take \(\alpha = \frac{4\pi}{3} + \delta\), for \(\delta > 0\) a small parameter to be determined later. We proceed to the bundle surgery such that in each fiber \(\mathbb{C}\) over a point of \(N\) it is parametrized
\[
\left\{ e^{-t}x_2 + e^t e^{-i\alpha}x_2 \mid t \in \mathbb{R}, x_2 \in \mathbb{R}, x_2^2 = \varepsilon^2_1 \right\},
\]
followed by a symmetric smoothing.

In the chart at the transverse intersection point \([0 : 0 : 1]\), the two Lagrangians are the linear Lagrangian subspaces \(l_0 = \mathbb{R}^2\) and \(l_1 = \{(e^{i\alpha}x_0, e^{i\alpha}x_1) \mid x_0, x_1 \in \mathbb{R}\}\) and we choose the handle
\[
\left\{ e^{-t} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + e^t \begin{pmatrix} e^{i\alpha}x_0 \\ e^{i\alpha}x_1 \end{pmatrix} \right\} \quad \left( x_0^2 + x_1^2 = \varepsilon^2_1 \right).
\]
After surgery, the restriction to the coordinate-$\mathbb{CP}^1$s of homogeneous equations $z_0 = 0$ (resp. $z_1 = 0$) of the Lagrangian we constructed is the union of two circles enclosing disks of area

$$\frac{1 + \delta}{2\pi} - 2\alpha(\varepsilon)$$

and we can check with the same method as before that the Maslov class of these disks is equal to 2. Then we can adjust $\delta$ such that the area of each of these disks is exactly 1. This is enough to check the monotonicity as one can remark that this Lagrangian is a circle subbundle over $N$ in the normal bundle of $\{z_2 = 0\}$, so that it is a torus. Let us denote it by $\Theta_{\text{surg}}$. We just have checked the monotonicity condition on one generator of the relative second homology group, the monotonicity on a second generator is immediate as it can be represented by a disk with boundary on $L_0$ or $L_1$ for which this condition is satisfied.

Theorem 4.3. The Lagrangian torus $\Theta_{\text{surg}}$ is Hamiltonian isotopic to the modified Chekanov torus in $\mathbb{CP}^2$ and consequently also to the Chekanov and Schlenk exotic torus.

Proof. We use the strategy of [8] and prove that the torus obtained by surgery is invariant under the same Hamiltonian action (called $\rho_{\text{Ch}}$ in [8]) as the modified Chekanov torus.

The modified Chekanov torus is invariant under the Hamiltonian circle action $\rho_{\text{Ch}}$ defined on $\mathbb{CP}^2$ by applying the following matrix to the homogeneous coordinates:

$$\begin{pmatrix}
\cos(\theta) & -\sin(\theta) & 0 \\
\sin(\theta) & \cos(\theta) & 0 \\
0 & 0 & 1
\end{pmatrix},$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

One can note that this circle action preserves $L_0, L_1, N$ and $[0 : 0 : 1]$.

In the chart around $[0 : 0 : 1]$, it restricts to a ball of $\mathbb{C}^2$ to the action defined by the rotation matrix

$$\begin{pmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{pmatrix}.$$

In particular,

- it preserves the handle of the local surgery at the point $[0 : 0 : 1]$,
- one can ask the smoothing to be invariant under the action, so that the entire surgery is preserved,
- the action commutes with the homotheties defining the conformal transformation.

Moreover, $L_0$ is the orbit under this action of its restriction to the projective line of homogeneous equation $z_0 = 0$ and even of $\{[0 : x_1 : x_2] \in \mathbb{CP}^2 \mid x_1, x_2 \in \mathbb{R}, x_1 \geq 0\}$. Similarly, $L_1$ is the orbit under this action of $\{[0 : e^{i\alpha} x_1 : x_2] \in \mathbb{CP}^2 \mid x_1, x_2 \in \mathbb{R}, x_1 \geq 0\}$, and the handle is the orbit of one of
its branches in its restriction to $z_0 = 0$, for instance of $\{0 : e^{-t}x_1 + e^t e^{i\alpha}x_1 : x_2\} \in \mathbb{CP}^2 | x_1, x_2 \in \mathbb{R}, x_1 = +\varepsilon_1\}$.

One can describe the bundle handle in homogeneous coordinates as

$$\{(x_0 : x_1 : e^{-t}x_2 + e^t e^{-i\alpha}x_2) \in \mathbb{CP}^2 | x_0, x_1 \in \mathbb{R}, x_2 = \varepsilon_1\}$$

and see that it is preserved by the circle action $\rho_{Ch}$. In fact, it is the orbit of one of its branches in $z_0 = 0$, for instance:

$$\{0 : x_1 : e^{-t}x_2 + e^t e^{-i\alpha}x_2\} \in \mathbb{CP}^2 | x_0, x_1 \in \mathbb{R}, x_2 = +\varepsilon_1\}.$$

We can also take a smoothing that is preserved by the circle action and the orbit under the circle action of the smoothing in $z_0 = 0$.

This shows that the torus $\Theta_{surg}$ is the orbit under the circle action $\rho_{Ch}$ of one of the two circles that constitute its restriction to $z_0 = 0$. Actually, as the third homogeneous coordinate in the bundle handle never vanishes, the torus $\Theta_{surg}$ lies in the complement of $z_2 = 0$, that is the ball of capacity 3 around $[0 : 0 : 1]$, and is also in this ball the orbit under the circle action of the circle lying in its restriction to the half plane of equations $z_0 = 0$ and $\Re(z_0) \geq 0$.

As this circle is by construction of area 1, one can isotope it inside this half-plane to the curve $\gamma$ used to define $\Theta_{Ch}$. This isotopy composed with the Hamiltonian circle action gives an exact Lagrangian isotopy between the two tori that can be extended to a Hamiltonian isotopy of $\mathbb{CP}^2$ as in [8].

**REFERENCES**

[1] Miguel Abreu and Leonardo Macarini. Remarks on Lagrangian intersections in toric manifolds. *Trans. Amer. Math. Soc.*, 365(7):3851–3875, 2013.

[2] Michèle Audin. *Torus actions on symplectic manifolds*, volume 93 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, revised edition, 2004.

[3] Paul Biran and Octav Cornea. Rigidity and uniruling for Lagrangian submanifolds. *Geom. Topol.*, 13(5):2881–2989, 2009.

[4] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 1764 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[5] Yuri Chekanov and Felix Schlenk. Twist tori I: Construction and classification. *in preparation*.

[6] Yuri Chekanov and Felix Schlenk. Twist tori II: Non-displaceability. *in preparation*.

[7] Yuri Chekanov and Felix Schlenk. Notes on monotone Lagrangian twist tori. *Electron. Res. Announc. Math. Sci.*, 17:104–121, 2010.

[8] Agnès Gadbled. On exotic monotone Lagrangian tori in $\mathbb{CP}^2$ and $S^2 \times S^2$. *J. Symplectic Geom.*, 11(3):343–361, 2013.

[9] A. B. Givental. Lagrangian imbeddings of surfaces and the open Whitney umbrella. *Funktsional. Anal. i Prilozhen.*, 20(3):35–41, 96, 1986.

[10] Hông-Vân Lê. A minimizing deformation of Legendrian submanifolds in the standard sphere. *Differential Geom. Appl.*, 21(3):297–316, 2004.

[11] Yong-Geun Oh. Floer cohomology of Lagrangian intersections and pseudoholomorphic disks. I. *Comm. Pure Appl. Math.*, 46(7):949–993, 1993.

[12] Leonid Polterovich. The surgery of Lagrange submanifolds. *Geom. Funct. Anal.*, 1(2):198–210, 1991.
[13] Marcin Poźniak. Floer homology, Novikov rings and clean intersections. In *Northern California Symplectic Geometry Seminar*, volume 196 of *Amer. Math. Soc. Transl. Ser. 2*, pages 119–181. Amer. Math. Soc., Providence, RI, 1999.

[14] V. V. Shevchishin. Lagrangian embeddings of the Klein bottle and the combinatorial properties of mapping class groups. *Izv. Ross. Akad. Nauk Ser. Mat.*, 73(4):153–224, 2009.

[15] Alan Weinstein. *Lectures on symplectic manifolds*. American Mathematical Society, Providence, R.I., 1977. Expository lectures from the CBMS Regional Conference held at the University of North Carolina, March 8–12, 1976, Regional Conference Series in Mathematics, No. 29.

[16] Alan Weinstein. Neighborhood classification of isotropic embeddings. *J. Differential Geom.*, 16(1):125–128, 1981.

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