Analysis of baryon properties in large-$N_c$ chiral perturbation theory

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Abstract. We discuss how to compute corrections to one-loop order to the baryon axial-vector current and magnetic moments in heavy baryon chiral perturbation theory in the large-$N_c$ limit, where $N_c$ is the number of colors. Loop graphs with octet and decuplet intermediate states cancel to various orders in $N_c$ as a consequence of the large-$N_c$ spin-flavor symmetry of QCD baryons. These cancellations are explicitly shown for the general case of $N_f$ flavors of light quarks. We also indicate how to compare our resultant expressions with the corresponding ones obtained within conventional heavy baryon chiral perturbation theory at the physical values $N_c = 3, N_f = 3$. We observe an excellent agreement order by order.

1. Introductory remarks
Nowadays it is commonly accepted that QCD is the theory of the strong interactions. However, the analytic calculations of the spectrum and properties of hadrons are not possible within QCD because the theory is strongly coupled at low energies. In this respect, chiral perturbation theory and the $1/N_c$ expansion (where $N_c$ is the number of colors) have shed much light on the subject.

Chiral perturbation theory exploits the symmetry of the QCD Lagrangian under $SU(3)_L \times SU(3)_R \times U(1)_V$ transformations of the three flavors of light quarks in the limit $m_q \to 0$. Chiral symmetry is spontaneously broken by the QCD vacuum to the subgroup $SU(3)_V \times U(1)_V$, giving rise to an octet of Goldstone bosons. Physical observables can be expanded order by order in powers of $p^2/\Lambda^2$ and $m^2(\Pi)/\Lambda^2$, or equivalently, $m_\Pi/\Lambda_\chi$, where $p$ is the meson momentum, $m_\Pi$ is the mass of the Goldstone boson and $\Lambda_\chi$ is the scale of chiral symmetry breaking. When baryons are accounted for in the theory it is convenient to introduce velocity-dependent baryon fields, so that the expansion of the baryon chiral Lagrangian in powers of $m_q$ and $1/M_B$ (where $M_B$ is the baryon mass) is manifest [1, 2]. The resultant theory, referred to as heavy baryon chiral perturbation theory, was first used to compute the chiral logarithmic corrections to the baryon axial vector current for baryon semileptonic decays due to meson loops [1, 2]. While these corrections are large when only octet baryon intermediate states are kept [1], the inclusion of decuplet baryon intermediate states yields sizable cancellations between one-loop corrections [2].

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The $1/N_c$ expansion, on the other hand, has led to a considerable theoretical progress in understanding the spin-flavor structure of QCD baryons [3, 4]. In the large-$N_c$ limit there is an exact contracted $SU(2N_f)$ spin-flavor symmetry (where $N_f$ is the number of light quark flavors) [5], which can be used to classify large-$N_c$ baryon states and matrix elements. Physical quantities can be analyzed in the large-$N_c$ limit, where corrections arise at relative orders $1/N_c$, $1/N_c^2$ and so on; this is the origin of the $1/N_c$ expansion. Various static properties of baryons have been computed within this formalism, namely, masses [3, 4, 6], vector and axial couplings [3, 4, 7, 8] and magnetic moments [7, 9, 10], to name but a few.

In the present paper, we use a combined expansion in $m_q$ and $1/N_c$ to show how to compute one-loop corrections specifically to the baryon axial-vector current and magnetic moment. The groundwork material on this topic has already been settled down in Refs. [11, 12, 13] so we limit ourselves to review some basic material. The $1/N_c$ chiral effective Lagrangian for the lowest-lying baryons was constructed in Ref. [14] and describes the interactions of the spin-$\frac{1}{2}$ baryon octet and the spin-$\frac{3}{2}$ baryon decuplet with the pion nonet. Within this framework it is possible to analyze the renormalization of the baryon axial vector current at the one-loop level [12]. As already pointed out in Refs. [3, 4, 5], there are large-$N_c$ cancellations between individual Feynman diagrams, provided one sums over all baryon states in a complete multiplet of the large-$N_c$ $SU(6)$ spin-flavor symmetry and uses axial coupling ratios given by the large-$N_c$ spin-flavor symmetry. In Ref. [11] the general structure of the various large-$N_c$ cancellations was analyzed. Here we briefly outline the evaluation of the corresponding operator expressions that involve complicated products of commutators and/or anticommutators of $SU(6)$ spin-flavor operators. The reduction of these operator products to a physical operator basis, although tedious, is nevertheless doable. The final expressions explicitly exhibit how these large-$N_c$ cancellations occur.

Also, it is instructive to perform a comparison of the results obtained within the combined formalism with the corresponding ones obtained in conventional heavy baryon chiral perturbation theory (including both octet and decuplet baryons), where no $1/N_c$ expansion is involved. Both approaches agree – the large-$N_c$ cancellations are guaranteed to occur as a consequence of the contracted $SU(6)$ spin-flavor symmetry present in the limit $N_c \to \infty$: No large numerical cancellations between loop diagrams with intermediate octet states and low-energy constants of the next-to-leading order effective Lagrangian, containing the effects of decuplet states, arise.

The present paper is organized as follows. In Sec. 2 we provide a brief review on heavy baryon chiral perturbation theory and introduce our notation and conventions; we also present some expressions for the one-loop corrections to the baryon axial-vector current and magnetic moments within this formalism. In Sec. 3 we give an overview of the $1/N_c$ chiral effective Lagrangian for the lowest-lying baryons. In Secs. 3.1 and 3.2 we discuss, in the combined approach, the one-loop corrections to the baryon-axial vector current and magnetic moments, respectively, and indicate how to compare our expressions with the ones discussed in Sec. 2. We close this paper with some concluding remarks.

2. Baryons in chiral perturbation theory

The heavy baryon chiral Lagrangian as first introduced in Refs. [1, 2] was constructed in terms of the pion field

$$
\pi = \frac{1}{\sqrt{2}} \begin{pmatrix}
\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta \\
\frac{1}{\sqrt{2}} \pi^- \\
K^-
\end{pmatrix}
= \frac{1}{\sqrt{2}} \begin{pmatrix}
\pi^+ \\
\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta \\
K^0
\end{pmatrix}
= \frac{1}{\sqrt{2}} \begin{pmatrix}
K^+ \\
\frac{1}{\sqrt{2}} \pi^0 + \frac{1}{\sqrt{6}} \eta \\
\bar{K}^0
\end{pmatrix}
- \frac{2}{\sqrt{6}} \eta,
$$

(1)
the baryon octet field

\[ B_v = \begin{pmatrix}
\frac{1}{\sqrt{2}} \Sigma^0_v + \frac{1}{\sqrt{6}} \Lambda_v \\
\Sigma^-_v \\
\Xi^-_v 
\end{pmatrix} \]

(2)

and the baryon decuplet field

\[ T^\mu_{abc} \]

(3)

where \( T^\mu_{abc} \) is a Rarita-Schwinger field which contains both spin-1/2 and spin-3/2 pieces and obeys the constraint \( \gamma^\mu T_\mu = 0 \).

The coupling of the pseudoscalar pion field with the baryon matter fields occurs through the vector and axial vector combinations

\[ V^\mu = \frac{1}{2}(\xi \partial^\mu \xi^\dagger + \xi^\dagger \partial^\mu \xi), \quad A^\mu = \frac{i}{2}(\xi \partial^\mu \xi^\dagger - \xi^\dagger \partial^\mu \xi), \]

(4)

where

\[ \xi = e^{i\pi/f}, \quad \Sigma = \xi^2 = e^{2i\pi/f}, \]

(5)

and \( f \approx 93 \text{ MeV} \) is the pion decay constant. Under \( SU(3)_L \times SU(3)_R \) chiral symmetry, the above fields transform as

\[ \xi \rightarrow L\xi U^\dagger, \quad \Sigma \rightarrow L\Sigma R^\dagger, \]

\[ B \rightarrow UBU^\dagger, \quad T^\mu_{abc} \rightarrow U_a U_b U_c T^\mu_{def}, \]

(6)

where \( U \) is defined implicitly by the transformation of \( \xi \) in Eq. (6).

The most general Lagrangian at lowest order is

\[ \mathcal{L}_{\text{baryon}} = i \text{Tr} \bar{B}_v (v \cdot D) B_v - i \bar{T}^\mu_{v} (v \cdot D) T_{v\mu} + \Delta \bar{T}^\mu_{v} T_{v\mu} + 2\text{DTr} \bar{B}_v S^\mu_{v} \{ A_\mu, B_v \} + 2F \text{Tr} \bar{B}_v S^\mu_{v} \{ A_\mu, B_v \} + C (\bar{T}^\mu_{v} A_\mu B_v + \bar{B}_v A_\mu T^\mu_{v}) + 2\mathcal{H} \bar{T}^\mu_{v} S^\nu_{v} A_v T_{v\mu}, \]

(7)

where \( D, F, C, \) and \( \mathcal{H} \) are the baryon-pion couplings and \( \Delta = m_T - m_B \) is the decuplet-octet mass difference.

On the other hand, to first order in the quark mass matrix, \( \mathcal{M} = \text{diag}(m_u, m_d, m_s) \), the chiral Lagrangian is expressed as

\[ \mathcal{L}^M_{\text{baryon}} = \sigma \text{Tr} [\mathcal{M}(\Sigma + \Sigma^\dagger)] \text{Tr}(\bar{B}_v B_v) - \tilde{\sigma} \text{Tr} [\mathcal{M}(\Sigma + \Sigma^\dagger)] \bar{T}^\mu_{v} T_{v\mu} + b_D \text{Tr} \bar{B}_v \{ \xi^\dagger \mathcal{M}\xi^\dagger + \xi \mathcal{M}\xi, B_v \} + b_F \text{Tr} \bar{B}_v [\xi^\dagger \mathcal{M}\xi^\dagger + \xi \mathcal{M}\xi, B_v] + c\bar{T}^\mu_{v} (\xi^\dagger \mathcal{M}\xi^\dagger + \xi \mathcal{M}\xi) T_{v\mu}. \]

(8)

Here \( \sigma \) and \( \tilde{\sigma} \) are the quark mass parameters of the octet and decuplet, respectively. The flavor octet quark mass splittings are described by the parameters \( b_D \) and \( b_F \) for the baryon octet, and by the parameter \( c \) for the baryon decuplet.
Fig. 1. Feynman diagrams that yield one-loop corrections to the baryon axial-vector current. The dashed lines denote mesons and the solid lines denote baryons. The crossed circle represents the insertion of an axial-vector current.

2.1. Chiral corrections to the baryon axial current

The renormalization of the baryon axial current can now be readily obtained by using the effective Lagrangian (7). The leading nonanalytic correction is a chiral logarithm produced by the one-loop Feynman graphs displayed in Fig. 1. The renormalized baryon axial current can be written as \[ \langle B_i | J^A_{\mu} | B_j \rangle = \left[ \alpha_{ij}^A + \sum_{\Pi} \left( \beta_{ij}^{\Pi, A} - \lambda_{ij}^{\Pi, A} \alpha_{ij}^A \right) \frac{m_{\Pi}^2}{16\pi^2 f^2} \ln \left( \frac{m_{\Pi}^2}{\mu^2} \right) \right] \bar{u}_{B_i} \gamma_{\mu} \gamma_5 u_{B_j}, \]

where \( \alpha_{ij}^A \) is the lowest order result, \( \lambda_{ij}^{\Pi, A} = \lambda_{ij}^{\Pi} + \lambda_{ij}^{\Pi'} \) is the one-loop corrections due to wavefunction renormalization

\[ \sqrt{Z_i Z_j} = 1 + \sum_{\Pi} \bar{\lambda}_{ij}^{\Pi} \frac{m_{\Pi}^2}{16\pi^2 f^2} \ln \left( \frac{m_{\Pi}^2}{\mu^2} \right), \]

\[ \bar{\lambda}_{ij}^{\Pi} = \frac{1}{2} \left( \bar{\lambda}_{ij}^{\Pi} + \bar{\lambda}_{ij}^{\Pi'} \right), \]

where \( \bar{\beta}_{ij}^{\Pi, A} = \beta_{ij}^{\Pi, A} + \beta_{ij}^{\Pi'} \) and \( \Pi \) stands for \( \pi, K, \) and \( \eta \) mesons. The unprimed and primed quantities are contributions with intermediate octet and decuplet baryons, respectively. The explicit formulas for the chiral coefficients \( \alpha_{ij}^A, \beta_{ij}^{\Pi, A}, \) and \( \lambda_{ij}^{\Pi} \) can be found in Appendix A of Ref. [12].

The correction to the axial current can be expressed in a more compact form as [12]

\[ \delta \langle B_i | J^A_{\mu} | B_j \rangle = \left( \sum_{\Pi} a_{ij}^{\Pi, A} F(m_{\Pi}) \right) \bar{u}_{B_i} \gamma_{\mu} \gamma_5 u_{B_j}, \]

where

\[ a_{ij}^{\Pi, A} = \beta_{ij}^{\Pi, A} - \lambda_{ij}^{\Pi, A} \alpha_{ij}^A, \]

and

\[ F(m_{\Pi}) = - \frac{m_{\Pi}^2}{16\pi^2 f^2} \ln \left( \frac{m_{\Pi}^2}{\mu^2} \right), \]
where $F(m)$ stands for the integration over the loop and only the non-analytic pieces have been retained. Equation (11) can be split into flavor singlet, flavor octet and flavor-27 pieces in terms of flavor singlet, octet, and 27 linear combinations of $F(\pi)$, $F(K)$, and $F(\eta)$ [14], namely,

$$\delta \langle B_i | J^A_{\mu} | B_j \rangle = \left[ \frac{1}{8} [3F(\pi) + 4F(K) + F(\eta)] b_{ij,1} + \frac{2\sqrt{3}}{5} \left( \frac{3}{2} F(\pi) - F(K) - \frac{1}{2} F(\eta) \right) b_{ij,8} + \left( \frac{1}{3} F(\pi) - \frac{4}{3} F(K) + F(\eta) \right) b_{ij,27} \right] \bar{u}_{B_i} \gamma_\mu \gamma_5 u_{B_j}, \quad (14)$$

where

$$b_{ij,1} = -a_{ij,\pi} + a_{ij,K} + a_{ij,\eta}, \quad (15)$$

$$b_{ij,8} = -\frac{1}{\sqrt{3}} \left( a_{ij,\pi} - \frac{1}{2} a_{ij,K} - a_{ij,\eta} \right), \quad (16)$$

$$b_{ij,27} = -\frac{3}{40} \left( a_{ij,\pi} - 3a_{ij,K} + 9a_{ij,\eta} \right), \quad (17)$$

In Eq. (13) the mass of the meson is denoted by its particle label.

2.2. Chiral corrections to baryon magnetic moments

The chiral Lagrangian with electromagnetism incorporated was derived in Ref. [15]. The one-loop diagrams that contribute to baryon magnetic moments are displayed in Figs. 2 and 3. Corrections arising from these loop graphs are order $O(m_q^{1/2})$ and $O(m_q \ln m_q)$, respectively.

![Feynman diagrams](image)

(a) (b)

**Figure 2.** Feynman diagrams which yield nonanalytic $m_q^{1/2}$ corrections to the magnetic moments of octet baryons. Dashed lines denote mesons and single and double solid lines denote octet and decuplet baryons, respectively. For decuplet baryons and decuplet-octet transitions the diagrams are similar.

For octet baryons, the magnetic moments computed in Ref. [15] can be rewritten as

$$\mu_i = \alpha_i + \sum_{X=\pi,K} \beta_i^{(X)} I(m_X, 0) + \sum_{X=\pi,K} \beta_i^{(X)} I(m_X, \Delta)$$

$$+ \sum_{X=\pi,K,\eta} \frac{1}{32\pi^2 f^2} \left( \gamma_i^{(X)} - 2\gamma_i^{(X)} \alpha_i \right) m_X^2 \ln \frac{m_X^2}{\mu^2}, \quad (18)$$

where $\alpha_i$ corresponds to the tree-level value of baryon $i$, $\beta_i^{(X)}$ and $\beta_i^{(X)}$ are the contributions arising from loop graphs of Fig. 2 and the remaining coefficients come from loop graphs of Fig. 3.

For $\mu_{\Sigma^{-}}$, for instance, the corresponding chiral coefficients listed in Ref. [15] read

$$\beta_{\Sigma^{-}}^{(\pi)} = \frac{2}{3} D^2 + 2F^2, \quad \beta_{\Sigma^{-}}^{(K)} = (D - F)^2, \quad \beta_{\Sigma^{-}}^{(\pi)} = -\frac{1}{18} \epsilon^2, \quad \beta_{\Sigma^{-}}^{(K)} = -\frac{1}{9} \epsilon^2. \quad (19)$$
3. Baryons in large $N_c$ chiral perturbation theory

The $1/N_c$ baryon chiral Lagrangian obtained in Ref. [14] is expressed as

$$
\mathcal{L}_{\text{baryon}} = iD^0 - M_{\text{hyperfine}} + \text{Tr} \left( A^i \lambda^a \right) A^{ia} + \text{Tr} \left( A^i \frac{2I}{\sqrt{6}} \right) A^i + \ldots, \tag{20}
$$

with

$$
D^0 = \partial^0 I + \text{Tr} \left( V^0 \lambda^a \right) T^a + \frac{1}{3} \text{Tr} \left( V^0 I \right) N_c I. \tag{21}
$$

Each term in Eqs. (20) and (21) represents itself a baryon operator which can be expressed in terms of polynomials in the spin-flavor generators $J^i$, $T^a$, and $G^{ia}$ [4]. Here $M_{\text{hyperfine}}$ denotes the spin splittings of the tower of baryon states with spins $\frac{1}{2}, \ldots, \frac{N_c}{2}$ in the flavor representations.

The vector and axial vector combinations of the pion fields are given by

$$
V^0 = \frac{1}{2} \left( \xi \partial^0 \xi^\dagger + \xi^\dagger \partial^0 \xi \right), \quad A^i = \frac{i}{2} \left( \xi \nabla^i \xi^\dagger - \xi^\dagger \nabla^i \xi \right), \tag{22}
$$

which couple to vector and axial vector baryon currents, respectively. The $\ell = 1$ flavor octet axial vector pion combination couples to the flavor octet baryon axial vector current $A^{ia}$. It is a spin-1 object, an octet under $SU(3)$ and its $1/N_c$ expansion is [4]

$$
A^{ia} = a_1 G^{ia} + \sum_{n=2,3}^{N_c} b_n \frac{1}{N_c^{n-1}} D^{ia}_n + \sum_{n=3,5}^{N_c} c_n \frac{1}{N_c^{n-1}} O^{ia}_n, \tag{23}
$$

where $D^{ia}_n$ are diagonal operators with nonzero matrix elements only between states with the same spin, whereas $O^{ia}_n$ are purely off-diagonal operators with nonzero matrix elements only
between states of different spin. The explicit forms for these operators can be found in Ref. [4]. At the physical value \(N_c = 3\), Eq. (23) can be truncated after the fourth term

\[
A^{ia} = a_1 G^{ia} + b_2 \frac{1}{N_c} J^i T^a + b_3 \frac{1}{N_c^2} D_j^{ia} + c_3 \frac{1}{N_c} O_j^{ia},
\]  
(24)

where

\[
D_j^{ia} = \{J^i, \{J^j, G^{ia}\}\},
\]  
(25)

\[
O_j^{ia} = \{J^j, G^{ia}\} - \frac{1}{2} \{J^i, \{J^j, G^{ia}\}\}.
\]  
(26)

On the other hand, in the limit of exact \(SU(3)\) flavor symmetry, the baryon mass operator is defined by

\[
M = \langle B' | \mathcal{H}_{QCD} | B \rangle,
\]  
(27)

where \(\mathcal{H}_{QCD}\) is the QCD Hamiltonian in the chiral limit \(m_i \to 0\). \(M\) transforms as a \((0, 1)\) under \(SU(2) \times SU(3)\) symmetry and its \(1/N_c\) expansion is [14]

\[
M = m_{(0)}^{0,1} N_c I + \sum_{n=2,4} m_{(n)}^{0,1} \frac{1}{N_c^{n-1}} J^a.
\]  
(28)

The first term on the right-hand side of Eq. (28) is the overall spin-independent mass of the baryon multiplet and is removed from the chiral Lagrangian by the heavy baryon field redefinition [1]. The remaining terms are spin-dependent so they define \(M_{\text{hyperfine}}\). For \(N_c = 3\), the hyperfine mass expansion reduces to the operator

\[
M = m_{(2)}^{0,1} \frac{1}{N_c} J^2.
\]  
(29)

### 3.1. Corrections to the baryon axial-vector current in large-\(N_c\) chiral perturbation theory

The leading nonanalytic corrections to the baryon axial current is of the form \(m^2 \ln m^2\) arising from the loop diagrams shown in Fig. 1. The normalized axial current, in the degeneracy limit \(\Delta \to 0\), can be expressed as

\[
A^{kc} + \delta A^{kc},
\]  
(30)

where the correction \(\delta A^{kc}\) reads [11, 12]

\[
\delta A^{kc} = \frac{1}{2} [A^{ia}, [A^{ib}, A^{kc}]] \Pi^{ab} - \frac{1}{2} [T^a, [T^b, A^{kc}]] \Pi^{ab}.
\]  
(31)

The first term on the right-hand side of Eq. (31) comes from three-vertex diagrams, Figs. 1(a-c) whereas the second term is the contribution of the one-vertex diagram, Fig. 1(d). Here \(\Pi^{ab}\) is a symmetric tensor which contains meson loop integrals with the exchange of a single meson. \(\Pi^{ab}\) decomposes into flavor singlet, flavor 8 and flavor 27 representations as [14]

\[
\Pi^{ab} = \frac{1}{8} [3F(\pi) + 4F(K) + F(\eta)] \delta^{ab} + \frac{2\sqrt{3}}{5} \left( \frac{3}{2} F(\pi) - F(K) - \frac{1}{2} F(\eta) \right) d^{88} + \left( \frac{1}{3} F(\pi) - \frac{4}{3} F(K) + F(\eta) \right) (\delta^{88} d^{88} - \frac{1}{8} \delta^{ab} - \frac{3}{5} d^{88} d^{88}),
\]  
(32)
where \( F(m) \) is a function of the meson mass \( m \) and is defined in this formalism by the integral
\[
\delta^{ij} F(m) = \frac{1}{f^2} \int \frac{d^4k}{(2\pi)^4} \frac{i^3(k^i)(-k^j)}{k^2 - m^2(k \cdot v)^2},
\]
for the first term in Eq. (31), and by the integral
\[
F(m) = \frac{1}{f^2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2},
\]
for the second term in Eq. (31).

The correction to the axial current can be organized as follows
\[
\delta A^{kc} = \frac{1}{2} \left( [A^{ia}, [A^{ib}, A^{kc}]] - [T^a, [T^b, A^{kc}]] \right) \left( \Pi_1^{ab} + \Pi_8^{ab} + \Pi_2^{ab} \right).
\]

For instance, for the singlet contribution we get
\[
[A^{ia}, [A^{ib}, A^{kc}]] \Pi_1^{ab} = \frac{1}{8} [3F(\pi) + 4F(K) + F(\eta)] [A^{ia}, [A^{ia}, A^{kc}]],
\]
and to relative order in \( 1/N_c^2 \) in the \( 1/N_c \) expansion one has [12]
\[
[A^{ia}, [A^{ia}, A^{kc}]] = \\
\left[ \left( \frac{3}{4} N_F - \frac{1}{N_F} \right) a_1^3 - 2 \left( \frac{1}{N_F} + \frac{1}{N_c} \right) a_1^2 b_2 + \left( \frac{1}{2} - \frac{1}{N_F} + \frac{N_F - 2}{N_c} N_F - 3 \frac{N_F}{N_c^2} \right) a_1 b_2^2 \right] \\
- \left( 1 + 2 \frac{N_F}{N_c} - 2 \frac{N_F - 4}{N_c^2} \right) a_1^2 b_3 - \left( 1 + 2 \frac{N_F}{N_c} - \frac{N_F}{N_c^2} \right) a_1^2 c_3 \right] G^{kc} \\
+ \frac{1}{N_c} \left[ \left( \frac{9}{4} N_F - \frac{1}{N_F} + 2 \right) a_1^2 b_2 + 2 \left( 1 + \frac{N_F}{N_c} \right) \left( 1 - \frac{1}{N_F} \right) a_1 b_2^2 \right] \\
- 3 \left( 1 + \frac{N_F}{N_c} \right) a_1^2 b_3 - \frac{1}{2} \left( 1 + \frac{N_F}{N_c} \right) a_1^2 c_3 \right] j^k T^c \\
+ \frac{1}{N_c^2} \left[ \left( \frac{13}{4} N_F - \frac{3}{N_F} + 4 \right) a_1^2 b_3 + \frac{1}{2} (1 + N_F) a_1^2 c_3 \right] D_3^{kc} \\
+ \frac{1}{N_c^2} \left[ \left( N_F - \frac{8}{3} + 2 \right) a_1^2 b_3 + \left( \frac{15}{4} N_F - \frac{1}{N_F} + 3 \right) a_1^2 c_3 \right] \mathcal{O}_3^{kc} + \mathcal{O} \left( \frac{1}{N_c^3} \right). \tag{37}
\]

Similar expressions can be obtained for the octet and 27 contributions; they can be found in Ref. [12] and will not be repeated here.

Let us remark that there is a nontrivial \( N_c \) dependence of the matrix elements of the generators \( J^i, T^a \) and \( G^{ia} \) in the weight diagrams for the \( SU(3) \) flavor representations of the spin-\( \frac{1}{2} \) and spin-\( \frac{3}{2} \) baryons [4]. In other words, factors of \( T^a/N_c \) and \( G^{ia}/N_c \) are of order 1 somewhere in the weight diagram, whereas factors of \( J^i/N_c \) are of order \( 1/N_c \) everywhere. If we restrict ourselves to baryons with spins of order unity, the \( N_c \) counting rules can be summarized as [11]
\[
T^a \sim N_c, \quad G^{ia} \sim N_c, \quad J^i \sim 1. \tag{38}
\]

Under these assumptions, one naively could assert that the left-hand-side of Eq. (37) is order \( N_c^3 \), a factor of \( N_c \) for each axial-vector current operator. However, after explicit computation, we notice that the right-hand-side of Eq. (37) is at most of order \( N_c \), according to counting rules (38), i.e., there are large-\( N_c \) cancellations in the evaluation of the double commutator.
provided one sums over all baryon states in a complete multiplet of the large-$N_c$ $SU(6)$ spin-flavor symmetry, i.e., over both the octet and decuplet, and uses axial coupling ratios given by the large-$N_c$ spin-flavor symmetry.

Similarly, for the contribution of the Feynman diagram Fig. 1(d) one finds [12]

\[
\delta^{ab}[T^a, [T^b, A^{kc}]] = N_f A^{kc}, \tag{39}
\]

\[
d^{abc}[T^a, [T^b, A^{kc}]] = \frac{N_f}{2} d^{cde} A^{ke}, \tag{40}
\]

\[
\delta^{a8} \delta^{b8}[T^a, [T^b, A^{kc}]] = f^{cde} f^{ced} A^{ke}. \tag{41}
\]

In the double commutator involved in Eqs. (39)-(41) we can observe that there are again large-$N_c$ cancellations in the structure.

3.1.1. Comparison with heavy baryon chiral perturbation theory. Equation (35) can be straightforwardly compared with the analogous expression obtained in conventional chiral perturbation theory, Eq. (15)-(17), by making use of the identifications [14]

\[
D = \frac{1}{2} a_1 + \frac{1}{6} b_3, \quad F = \frac{1}{3} a_1 + \frac{1}{6} b_2 + \frac{1}{9} b_3, \quad C = -a_1 - \frac{1}{2} c_3, \quad H = -\frac{3}{2} a_1 - \frac{3}{2} b_2 - \frac{5}{2} b_3. \tag{42}
\]

Both expressions agree order by order [12].

3.2. Baryon magnetic moments in large-$N_c$ chiral perturbation theory
In the large-$N_c$ limit, the baryon magnetic moments possess the same kinematical properties as the baryon axial-vector couplings; as a result, the operators used to describe these quantities are practically identical [7]. The baryon magnetic moment operator is also a spin-1 object and an octet under $SU(3)$. For definiteness, in analogy with Eq. (24), the $1/N_c$ expansion of the operator which yields the baryon magnetic moment is [13]

\[
M^{kc} = m_1 G^{kc} + m_2 \frac{1}{N_c} D^{kc}_2 + m_3 \frac{1}{N_c^2} D^{kc}_3 + m_4 \frac{1}{N_c^2} O^{kc}_3, \tag{43}
\]

where we have truncated the series at the physical value $N_c = 3$. If we assume $SU(3)$ symmetry, the unknown coefficients $m_i$ are independent of $k$ so they are unrelated to the ones of expansion (24) in this limit. The magnetic moments are proportional to the quark charge matrix $Q = \text{diag}(2/3, -1/3, -1/3)$, so they can be separated into isovector and isoscalar components, $M^{k3}$ and $M^{k8}$, respectively. Accordingly, from Eq. (43), we define the baryon magnetic moment operator as

\[
M^k = M^{kQ} = M^{k3} + \frac{1}{\sqrt{3}} M^{k8}. \tag{44}
\]

Hereafter the spin index $k$ of $M^k$ will be set to 3 whereas the flavor index $Q$ will stand for $Q = 3 + (1/\sqrt{3})8$ so any operator of the form $X^Q$ should be understood as $X^3 + (1/\sqrt{3})X^8$.

Now, one-loop corrections to baryon magnetic moments emerge from the Feynman graphs depicted in Fig. 2 and 3, which contribute to orders $\mathcal{O}(m_q^{1/2})$ and $m_q \ln m_q$, respectively.
Graphs from Fig. 2 involve $\pi$ and $K$ emission and reabsorption only (the $\eta$ meson does not contribute). For degenerate heavy baryons interacting with mesons, these diagrams depend on a function $I(m_{\Pi})$ of the meson mass $m_{\Pi}$, which is obtained by performing the Feynman loop integration. Thus, in the $\Delta \to 0$ limit, we have

$$M_{\text{fig}2}^k = \epsilon^{ijk} A^{ia} A^{jb} \Gamma^{ab}.$$  \hspace{1cm} (45)

Here $\Gamma^{ab}$ is an antisymmetric tensor which contains the integrals over the loops and is given in Ref. [7]. $i\Gamma^{ab}$ can be represented by a Hermitian matrix which is diagonal in a basis corresponding to particles of definite quantum numbers. This matrix has eight eigenvalues: four of them are zero and correspond to the four neutral mesons, two of them are equal and opposite eigenvalues $\pm I(m_K)$ corresponding to $K^{\pm}$, respectively, and the remaining two are equal and opposite eigenvalues $\pm I(m_{\pi})$ corresponding to $\pi^{\pm}$, respectively. Thus [7]

$$\Gamma^{ab} = A_0 \Gamma_0^{ab} + A_1 \Gamma_1^{ab} + A_2 \Gamma_2^{ab},$$  \hspace{1cm} (46)

where the coefficients $A_i$ are linear combinations of the functions $I(m_{\pi})$ and $I(m_K)$ and read

$$A_0 = \frac{1}{3} [I(m_{\pi}) + 2I(m_K)],$$  \hspace{1cm} (47)

$$A_1 = \frac{1}{3} [I(m_{\pi}) - I(m_K)],$$  \hspace{1cm} (48)

$$A_2 = \frac{1}{\sqrt{3}} [I(m_{\pi}) - I(m_K)],$$  \hspace{1cm} (49)

and the tensors $\Gamma_i^{ab}$ are written as

$$\Gamma_0^{ab} = f^{ab}Q,$$  \hspace{1cm} (50)

$$\Gamma_1^{ab} = f^{ab}Q,$$  \hspace{1cm} (51)

$$\Gamma_2^{ab} = f^{acQ}d^e_8 - f^{bcQ}d^a_8 - f^{abe}d^c_8.$$  \hspace{1cm} (52)

Although $\Gamma_0^{ab}$ and $\Gamma_1^{ab}$ are both SU(3) octets, they have quite different physical interpretations. The former transforms as the electric charge whereas the latter also transforms as the electric charge but rotated by $\pi$ in isospin space [7]. In what follows any operator of the form $X^Q$ should be understood as $X^3 - (1/\sqrt{3})X^8$. Notice also that $X^Q$ and $X^\overline{Q}$ fall into different octet representations. On the other hand, $\Gamma_2^{ab}$ breaks SU(3) as $10 + \overline{10}$ [7].

In the degeneracy limit $\Delta \to 0$ and retaining only the nonanalytic pieces in $m_{\eta}$, the integral over the loop, which comprises the proper factors to give the correct dimensions, can be expressed as [15]

$$I(m_{\Pi}) = \frac{1}{8\pi^2 f^2} M_N m_{\Pi},$$  \hspace{1cm} (53)

where $f$ is the pion decay constant and $M_N$ and $m_{\Pi}$ denote the nucleon and the meson masses, respectively.

Thus, the one-loop correction arising from Fig. 2 can be decomposed into the pieces emerging from the 8 and $10 + \overline{10}$ representations as follows,

$$M_{\text{fig}2}^k = A_0 M_{8,\text{fig}2}^{kQ} + A_1 M_{8,\text{fig}2}^{\overline{Q}} + A_2 M_{10 + \overline{10},\text{fig}2}^{kQ},$$  \hspace{1cm} (54)

where the different contributions read

$$M_{8,\text{fig}2}^{kc} = \epsilon^{ijk} f^{abc} A^{ia} A^{jb},$$  \hspace{1cm} (55)
and
\[ M^{kc}_{10+10,\text{fig}2} = \epsilon^{ijk} (f^{aecn}d^{bce8} - f^{becn}a^{ec8} - f^{abe}d^{ec8})A^i A^j. \] (56)

The reduction of the operator products contained in \( M^{kc}_8 \) and \( M^{kc}_{10+10} \) to the order considered here are given in Ref. [13]. Gathering together partial results, we have for example, for the octet contribution,
\[
M^{kc}_{8,\text{fig}2} = \left[ -\frac{N_c + 3}{2} a_1 b_2 - \frac{3(N_c + 3)}{N_c^2} a_1 b_3 \right] G^{kc} + \frac{1}{2} \left[ a_1^2 - \frac{1}{N_c^2} (2a_1 b_3 - 9a_1 c_3 + 3b_3^2) \right] D^{kc}_2 \\
- \frac{1}{N_c^2} \left[ \frac{N_c + 3}{2} a_1 c_3 + \frac{3}{N_c} b_2 b_3 \right] D^{kc}_4 - \frac{1}{N_c^2} \left[ N_c a_1 b_2 + (N_c + 3)a_1 b_3 + \frac{3}{2} a_1 c_3 + 3 b_2 c_3 \right] \\
\times O^{kc}_3 + \frac{1}{N_c^2} a_1 c_3 D^{kc}_4 - \frac{1}{N_c^2} b_2 c_3 O^{kc}_5 + \ldots 
\] (57)

For the other flavor representation the corresponding expression can be found in Ref. [13]. Equation (57) has been rearranged to exhibit explicitly leading and subleading terms in \( 1/N_c \). It is now evident that this expression yields matrix elements at most of order \( O(N_c^2) \), according to the \( N_c \)-counting rules discussed above. In addition, \( f \) and \( M_N \) are \( O(\sqrt{N_c}) \) and \( O(N_c) \), respectively, so the one-loop contribution \( M^{kc}_{\text{fig}2} \), Eq. (54), is order \( O(N_c) \). In the limit of small \( m_s \), the symmetry breaking part of \( M^{kc}_{\text{fig}2} \) is \( O(m_s^{1/2}) \) so the overall contribution of Eq. (54) to baryon magnetic moments is \( O(m_s^{1/2} N_c) \).

As for nonanalytic corrections of order \( m_s \ln m_q \), we can separate the different contribution as those coming from Figs. 3(a-d) and that from Fig. 3(e). For the first kind we have
\[
M^{k}_{\text{fig}3a-d} = \left[ T^a, \left[ T^b, M^K \right] \right] \Pi^{ab}. \] (58)

Here, \( \Pi^{ab} \) is again a symmetric tensor which contains meson loop integrals and has the same structure as (32) except that this time
\[
F(m_{11}, \mu) = \frac{m_{11}^2}{32\pi^2 f^2} \ln \frac{m_{11}^2}{\mu^2}, \] (59)
where \( \mu \) is the scale of dimensional regularization and only nonanalytic terms in \( m_q \) have been kept.

Now, in the operator reduction of the structure (58) some subtleties arise. The appearance of the new parameters \( m_i \) makes unfeasible the direct application of Eqs. (37), for instance, to obtain the corresponding loop contribution \( M^{k}_{\text{fig}3ad} \). Indeed, new terms need be calculated. We remark that, because the operator basis is complete, the reduction is doable. In Ref. [13] the full calculation is presented and will not be repeated here. What is important to emphasize is the large-\( N_c \) cancellations observed is the structure of this contribution, as expected.

On the other hand, the contribution to the magnetic moments of the Feynman diagram displayed in Fig. 3(e) possesses the structure
\[
M^{k}_{\text{fig}3e} = \left[ T^a, \left[ T^b, M^K \right] \right] \Pi^{ab}, \] (60)
where \( \Pi^{ab} \) is the symmetric tensor introduced in Eq. (32). Equation (60) can also be separated into flavor singlet, flavor octet, and flavor 27 pieces as
\[
M^{k}_{\text{fig}3e} = F_1 M^{k}_{1,\text{fig}3e} + F_8 M^{kQ}_{8,\text{fig}3e} + F_{27} M^{kQ}_{27,\text{fig}3e}, \] (61)
where this time the group structures of the double commutator read

**Flavor singlet contribution**

\[ M^{kc}_{1,\text{fig3e}} = [T^a, [T^a, M^{kc}]] = N_f M^{kc}. \] (62)

**Flavor octet contribution**

\[ M^{kc}_{8,\text{fig3e}} = d^{abc} [T^a, [T^b, M^{kc}]] = \frac{N_f}{2} d^{ce8} M^{ke}. \] (63)

**Flavor 27 contribution**

\[ M^{kc}_{27,\text{fig3e}} = [T^8, [T^8, M^{kc}]] = f^{ce8} f^{8eg} M^{kg}. \] (64)

Let us notice that, in order for \( M^{kc}_{27,\text{fig3e}} \) to be a truly 27 contribution, singlet and octet pieces must be subtracted off. In the above equations, the free flavor index \( c \) will be set to \( Q = 3 + (1/\sqrt{3})8 \). By doing this, expression (64) as it stands, will vanish.

The total correction arising from Fig. 3 is then given by

\[ M^k_{\text{fig3}} = M^k_{\text{fig3a}} - d + M^k_{\text{fig3e}}. \] (65)

Corrections to the baryon magnetic moments are then obtained by computing the matrix elements of operator \( M^k_{\text{fig3}} \) between \( SU(6) \) baryon states, namely,

\[ \mu^k_B = \langle B | M^k_{\text{fig3}} | B \rangle, \] (66)

where \( B \) stands for either an octet or a decuplet baryon. For decuplet to octet transition magnetic moments, we also have

\[ \mu^k_{TB} = \langle T | M^k_{\text{fig3}} | B \rangle. \] (67)

The singlet piece of \( M^k_{\text{fig3}} \) yields magnetic moments that satisfy the Coleman-Glashow relations whereas violations to them are due to the 8 and 27 pieces.

### 3.2.1. Total one-loop corrections to baryon magnetic moments

At this point, we can summarize our findings and provide analytic results. Thus, the final expression of the magnetic moment of baryon \( B \) up to one-loop order can be organized as

\[ \mu_B = \mu^{(0)}_B + \mu^{(\text{fig2})}_B + \mu^{(\text{fig3})}_B. \] (68)

Applications of this expression are given in Ref. [13] and will not be repeated here.
3.3. Comparison with conventional heavy baryon chiral perturbation theory.

It is instructive to compare our computation of baryon magnetic moments at the physical value $N_c = 3$ with the one obtained in the framework of conventional heavy baryon chiral perturbation theory, i.e., the effective field theory with no $1/N_c$ expansion. In Ref. [14] it has been shown that there is a one-to-one correspondence between the parameters of the $1/N_c$ baryon chiral Lagrangian at $N_c = 3$ and the octet and decuplet chiral Lagrangian. The baryon-pion couplings are related to the coefficients of the $1/N_c$ expansion of $A^{\mu \nu}$, Eq. (24), at $N_c = 3$ by Eq. (42). On the other hand, the magnetic moments in conventional heavy baryon chiral perturbation theory are parametrized by four $SU(3)$ invariants $\mu_D, \mu_F, \mu_C$ and $\mu_T$ [15] while in the present analysis they are parametrized in terms of $m_i$, with $i = 1, \ldots, 4$, introduced in Eq. (43). The relations between these parameters read

$$
\begin{align*}
\mu_D &= \frac{1}{2} m_1 + \frac{1}{6} m_3, \\
\mu_F &= \frac{1}{3} m_1 + \frac{1}{6} m_2 + \frac{1}{9} m_3, \\
\mu_C &= \frac{1}{2} m_1 + \frac{1}{2} m_2 + \frac{5}{6} m_3, \\
\mu_T &= -2 m_1 - m_4.
\end{align*}
$$

(69)

In the literature, there are some analyses of baryon magnetic moments within heavy baryon chiral perturbation theory which allow us to carry out a comparison of our respective results in the limit $\Delta \to 0$, where $\Delta$ is the decuplet-octet mass difference. The work by Jenkins et. al. [15] about octet baryons allows a full comparison between one-loop corrections whereas the papers by Geng et. al. [17] for decuplet baryons, and Arndt and Tiburzi [18] for decuplet-octet transitions only allow partial comparisons of contributions emerging from loop graphs of Fig. 2.

For octet baryons, we obtain a remarkable agreement with the theoretical expressions displayed in Eq. (16) of Ref. [15], except for the global factor $-5/2$ missing in the loop contributions of Fig. 2(c) of this reference [which corresponds to Fig. 3(b) in the present paper]. When fixing this omission, the agreement is achieved for all nine observables.

As for decuplet baryons, starting from Eq. (17) of Ref. [17] and working in the limit $\Delta \to 0$, we find a very good agreement in both calculations.

Finally, Arndt and Tiburzi [18] present the calculation of baryon decuplet to octet electromagnetic transition form factors in quenched and partially quenched chiral perturbation theory, and provide the corresponding $SU(3)$ coefficients emerging from these schemes. They also present the counterparts of such coefficients that appear in chiral perturbation theory. These are precisely the coefficients we need to compare with. We find that, except for for transitions $\Sigma^* + \Sigma^*$ and $\Xi^0 \Xi^0$, the theoretical expressions differ by a global sign.

We would like to close this section by stating that, to the order of approximation implemented here, both approaches lead to the same results. This fact causes no surprise. Previous works have shown this matching for baryon masses [14] and axial-vector couplings [19] in a systematic way.

4. Summary and conclusions

In this paper we show how to compute corrections to some baryon properties to one-loop order within heavy baryon chiral perturbation theory in the large-$N_c$ limit. We specialize our analysis to the computation of the baryon axial-vector current and magnetic moments and contrast them with their corresponding counterparts obtained in the frame of conventional heavy baryon chiral perturbation theory. Thus, in Sec. 2 we introduce some basic material on heavy baryon chiral perturbation theory and then we proceed with a review of some basic material on large-$N_c$ chiral perturbation theory in Sec. 3.
As a starting point, we use the fact that in the large-$N_c$ limit both the baryon axial-vector couplings and the baryon magnetic moments share the same kinematical properties so they can be analyzed in terms of the same operators. Hence, in Sec. 3.1, we construct the $1/N_c$ expansion of baryon axial-vector current operator and compute the corrections which emerge from Feynman diagrams at one-loop order. In Sec. 3.2 we thus give the baryon magnetic moment operator based on the expansion deduced for the axial-vector current operator. We also analyze one-loop corrections in the degeneracy limit $\Delta \equiv M_T - M_B \to 0$, where $M_T$ and $M_B$ are the $SU(3)$ invariant masses of the decuplet and octet baryon multiplets, respectively.

The final analytic expressions can be crosschecked with other calculations and with experiment. Existing analytic results in conventional heavy baryon chiral perturbation theory allow a detailed comparison at one-loop order. Barring a few exceptions (global multiplicative factors and/or opposite signs), the comparison with existing analytic results is a successful one. The advantage of our approach is that one only needs to construct a universal operator $A^{kc} + \delta A^{kc}$, evaluate the matrix elements of this operator between $SU(6)$ baryon states and compute the integrals over the loops. Here we perform the analysis to relative order $O(1/N_c^2)$. It is now clear that if we had involved ourselves in evaluating higher order contributions, we would have faced a much more complicated computation, perhaps not yet needed.

Returning to the main discussion about the comparison of this approach with conventional heavy baryon chiral perturbation theory, it should be emphasized that these two formulations yield to identical results. Nonetheless, in a given context, one or the other might be more inviting for computational ease.

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