DEEP NEURAL NETWORKS WITH RELU-SINE-EXPONENTIAL ACTIVATIONS BREAK CURSE OF DIMENSIONALITY IN APPROXIMATION ON HÖLDER CLASS

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Abstract. In this paper, we construct neural networks with ReLU, sine and $2^x$ as activation functions. For general continuous $f$ defined on $[0, 1]^d$ with continuity modulus $\omega_f(\cdot)$, we construct ReLU-sine-2$x$ networks that enjoy an approximation rate $O\left(\omega_f(\sqrt{d}) \cdot 2^{-M} + \omega_f\left(\frac{\sqrt{N}}{2^a}\right)\right)$, where $M, N \in \mathbb{N}^+$ are the hyperparameters related to widths of the networks. As a consequence, we can construct ReLU-sine-2$x$ network with the depth 6 and width $\max\left\{2d\left[\log_2\left(\frac{2d}{1/\alpha}\right)^{1/\alpha}\right], \ 2\left[\log_2\left(\frac{2d}{2^a}\right)^{3/2}\right] + 2\right\}$ that approximates $f \in \mathcal{H}_\mu^\alpha([0, 1]^d)$ within a given tolerance $\epsilon > 0$ measured in $L^p$ norm with $p \in [1, \infty)$, where $\mathcal{H}_\mu^\alpha([0, 1]^d)$ denotes the Hölder continuous function class defined on $[0, 1]^d$ with order $\alpha \in (0, 1]$ and constant $\mu > 0$. Therefore, the ReLU-sine-2$x$ networks overcome the curse of dimensionality in approximation on $\mathcal{H}_\mu^\alpha([0, 1]^d)$. In addition to its super expressive power, functions implemented by ReLU-sine-2$x$ networks are (generalized) differentiable, enabling us to apply SGD to train.

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1. Introduction. In recent years, deep learning has aroused great interest among mathematicians. How to approximate some common function classes with neural network is an important theoretical issue in this field. Some early works can be dated back to the 1980s [8, 14, 13, 27]. These results are mainly focused on sigmoidal networks, i.e., the activation functions are sigmoidal functions. Recently, ReLU networks are attached great interest due to its superior empirical performances in nowadays learning tasks [16]. Comparing to sigmoidal networks, ReLU networks do not suffer from the vanishing gradient problem [10]. Moreover, the ReLU is easy to compute and improves the ability of data representation [3]. In [40], Yarotsky firstly shows how to construct a ReLU network to achieve any approximation accuracy by the idea of Taylor expansion. Suzuki then shows that the ReLU networks can also be built up based on the classical approximation results of B-spline [37]. From a different point of view, Shen et al. construct ReLU networks to achieve any given accuracy by explicitly adjusting the depths and widths [33, 19]. Readers are also referred to some other excellent works related to ReLU networks [31, 12, 18, 20, 11].

Unfortunately, all those results of ReLU networks suffer from the curse of dimensionality in approximation [9], which is a term commonly used to describe of the difficulty of the problem depending on of the dimension exponentially. In the case of network approximation, it is
usually reflected in the fact that the size of the network is exponentially dependent on the approximation error. In fact, Yarotsky already proves that ReLU networks cannot escape the curse of dimensionality in approximation by constructing a lower bound for network size, which is based on the VC dimension of ReLU networks [40].

1.1. Main Contributions. In this paper, we construct neural networks achieving super expressive power with ReLU, sine, and \(2^x\) as activation functions. The constructed ReLU-sine-\(2^x\) networks break the curse of dimensionality in approximation on Hölder continuous function class defined on \([0, 1]^d\) and can be trained by SGD. The main contributions of this paper are summarized as follows. Let \(M, N \in \mathbb{N}^+\) be hyperparameters related to width, we construct deep networks \(\Phi\) with ReLU-sine-\(2^x\) activation functions that enjoy following approximation rate.

- For general continuous function \(f\) defined on \([0, 1]^d\) with continuity modulus \(\omega_f(\cdot)\), we have
  \[
  \|f - \Phi\|_{L^p} \leq O\left(\omega_f(\sqrt{d}) \cdot 2^{-M} + \omega_f\left(\frac{\sqrt{d}}{N}\right)\right),
  \]
  where \(p \in [1, \infty)\), the depth \(L(\Phi) = 6\) and the width \(W(\Phi) = \max\{2d\log_2 N, 2M\}\) as in Theorem 3.4. And
  \[
  \|f - \Phi\|_{L^\infty} \leq O\left(\omega_f(\sqrt{d}) \cdot 2^{-M} + \omega_f\left(\frac{\sqrt{d}}{N}\right)\right),
  \]
  where the depth \(L(\Phi) = 2d + 6\), and width \(W(\Phi) = 3d\left(\max\{2d\log_2 N, 2M\} + 4\right)\) as in Theorem 3.6.
- If \(f \in \mathcal{H}^\alpha_\mu([0, 1]^d),\) the Hölder function class with order \(\alpha \in (0, 1]\) and constant \(\mu > 0\), then we obtain
  \[
  \|f - \Phi\|_{L^p} \leq \epsilon
  \]
  as long as \(p \in [1, \infty)\), the depth \(L(\Phi) = 6\) and width
  \[
  W(\Phi) = \max\left\{2d \left[\log_2 \left(\frac{3\mu}{\varepsilon} \left(\frac{3\mu}{\varepsilon}\right)^{1/\alpha}\right)\right], 2 \left[\log_2 \left(\frac{3\mu\varepsilon^2/2}{2\varepsilon}\right)\right] + 2\right\}.
  \]
  It implies that the constructed \(\Phi\) breaking the curse of dimensionality in approximation on \(\mathcal{H}^\alpha_\mu([0, 1]^d)\), see Corollary 3.8 and Corollary 3.9.
- Functions implemented by \(\Phi\) are (generalized) differentiable [7, 4], thus, they can be trained by first order optimization algorithms such as stochastic gradient descent method.

1.2. Related Works. To avoid curse of dimensionality in approximation, one needs more regularity or structures on the target functions. For compositional functions [28], there exists a network with smooth, non-polynomial activation function, constant depth and width \(O(\varepsilon^2)\) to achieve error \(\epsilon\). The functions defined on low dimensional submanifolds are studied in
In [32] it is shown that for functions in $C^2(\Gamma)$, where $\Gamma$ is a smooth $m$-dimensional manifold, there exists a ReLU network with depth 4 and the number of units $O\left(\epsilon^{-m/2}\right)$ to achieve error $\epsilon$. The functions with finite Fourier moment conditions are studied in [24]. In [2] it is shown that there exists a shallow sigmoidal network with depth 2 and width $O\left(\frac{1}{\epsilon}\right)$ to achieve error $\epsilon$. Smooth functions are studied in [19, 42, 22, 39]. In [19] it is shown that for $f \in C^r([0,1]^d)$, to achieve an error $O\left(\|f\|_{C^r([0,1]^d)}N^{-2s/d}L^{-2s/d}\right)$, the depth and width of the ReLU network $\Phi$ are required to be $L(\epsilon) = O\left((L+2)\log(4L)+2d\right)$ and $W(\Phi) = O\left((N+2)\log_{2}(8N)\right)$. Piecewise smooth functions are studied in [18, 26]. In [26] ReLU networks with constant depth and number of weights $O\left(\epsilon^{-2(d-1)/\beta}\right)$, where $\beta$ characterizes smoothness of target functions, are constructed to achieve error $\epsilon$. For analytic functions on $(-1,1)^d$, there exists a ReLU network with depth $L$ and width $d+4$ to achieve accuracy $O\left(\epsilon^{-d\beta(e^{-1}L^{1/2d-1})}\right)$ for any $\delta > 0$ [39]. For band-limited functions, there exists a ReLU network $\Phi$ with depth $L(\Phi) = O\left(\frac{d}{\epsilon}\frac{1}{\log_{2}\left(\frac{1}{\epsilon}\right)}\right)$ and width $W(\Phi) = O\left(\frac{1}{\epsilon}\frac{1}{\log_{2}\left(\frac{1}{\epsilon}\right)}\right)$ to achieve error $\epsilon$ [23]. For functions in Korobov spaces, there exists a ReLU network $\Phi$ with depth $L(\Phi) = O\left(\log_{2}\left(\frac{1}{\epsilon}\right)\right)$ and the number of units $O\left(\frac{1}{\epsilon}\frac{1}{\log_{2}\left(\frac{1}{\epsilon}\right)}\right)^{(d-1)+1}$ to achieve error $\epsilon$ [21]. For measure $\mu$ whose support has a Minkowski dimension $d < D$, where $D$ is the ambient dimension, the approximation error measured in the norm $L^\infty(\mu)$ is roughly $O(W^{-\beta/d})$ where $\beta$ characterizing smoothness of target functions and $W$ is the number of parameters of the ReLU network [24]. For holomorphic mappings, the approximation rate of ReLU network is $O\left(e^{-bW^{1/(d+1)}}\right)$ with $b$ depending on the domain of analyticity and $W$ being the number of weights [25].

Although these works have achieved great achievements, an interesting question we can still ask is that for functions without much additional regularity conditions, can we construct an approximation network which does not suffer from the curse of dimensionality in approximation? For Hölder continuous functions, Shen et al. gives a positive answer by building a ReLU–floor network overcoming curse of dimensionality in approximation [34]. The size of their network can be adjusted by setting different values of depth and width. For example, to approximate a Hölder continuous function on $[0,1]^d$ with Hölder constant $\mu$ and order $\alpha$, there exists a ReLU–floor network with depth $6d+3$ and width max $\left\{d, 5\sqrt{d} \left(\frac{2d}{\epsilon}\right)^{1/\alpha} + 13\right\}$, where $\epsilon$ is the given approximation tolerance. However, it is a pity that the existence of floor activations exhibit using the working horse SGD [29, 17] for training since the gradient vanishes by chain rule. Note that non-piecewise constant and continuous activation functions have also been proposed in [34] in order to use SGD to train.

The rest of the paper is organized as follows. In Section 2, we give some notations and definitions. In Section 3, we present details on the construction of the ReLU-sine-$2^x$ networks with super expressive power. We give a conclusion and a short discussion in Section 4.

2. Notations. The continuity modulus $\omega_f(r)$ of a function $f$ is defined as

$$\omega_f(r) = \sup_{\|x-y\| \leq r} |f(x) - f(y)|.$$
For $\mu > 0$ and $\alpha \in (0, 1]$, the set of Hölder continuous function on $[0, 1]^d$ with constant $\mu$ and order $\alpha$ is defined by

$$\mathcal{H}_\mu^\alpha([0, 1]^d) = \{ f : |f(x) - f(y)| \leq \mu\|x - y\|_2^\alpha, \quad \forall x, y \in [0, 1]^d \}.$$

A function $f : \mathbb{R}^d \to \mathbb{R}^N$ implemented by a neural network is defined by

$$f_0(x) = x,$$

$$f_\ell(x) = \vartheta_\ell(A_\ell f_{\ell-1} + b_\ell) \quad \text{for } \ell = 1, \ldots, L - 1,$$

$$f = f_L(x) := A_L f_{L-1} + b_L,$$

where $A_\ell \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$, $b_\ell \in \mathbb{R}^{N_\ell}$ and the activation function $\vartheta_\ell$ is understood to act component-wise (it is allowed that there are different activation functions in different layers). For simplicity we also use $f$ to present this network. $L$ is called the depth of the network and $\max\{N_\ell, \ell = 0, \ldots, L\}$ is called the width of the network. We will use $\mathcal{L}(f)$ and $\mathcal{W}(f)$ to denote the depth and width of the neural network $f$, respectively. $\sum_{\ell=1}^L N_\ell$ is called number of unites of $f$ and $\{A_\ell, b_\ell\}$ are called the weight parameters.

We now introduce the concept of VC-dimension [38], which plays an important role in the research of neural network approximation. Let $S \subset X$ be a finite subset and $H \subset \{ h : X \to \{0, 1\} \}$. We define by $H_S := \{ h | S : h \in H \}$ the restriction of $H$ to $S$.

**Definition 2.1.** The growth function of $H$ is defined by

$$G_H(m) := \max \{|H_S| : S \subset X, |S| = m\}, \quad \text{for } m \in \mathbb{N}.$$

It is clear that for every set $S$ with $|S| = m$, we have that $|H_S| \leq 2^m$ and hence $G_H(m) \leq 2^m$. We say that a set $S$ with $|S| = m$ for which $|H_S| = 2^m$ is shattered by $H$.

**Definition 2.2.** $\text{VCdim}(H)$ is defined to be the largest integer $m$ such that there exists $S \subset X$ with $|S| = m$ that is shattered by $H$. In other words,

$$\text{VCdim}(H) := \max \{ m \in \mathbb{N} : G_H(m) = 2^m \}.$$

VC-dimension reflects the capacity of a class of functions to perform binary classification of points. The larger VC-dimension is, the stronger the capability to perform binary classification is. For more discussion of VC-dimension, readers are referred to [1].

### 3. Construction of network

In this section, we give detail construction of the ReLU-sine-2$^x$ networks that enjoy super expressive power and can be trained by SGD. We will construct ReLU-sine-2$^x$ networks with depth 6 and depth 2$d + 6$ that approximate functions in $L^p$ norm $p \in [1, \infty)$ and $L^\infty$ norm, respectively.

Inspired by Lemma 7.2 in [1], which shows that sine functions class enjoys an infinite VC-dimension, we give a Lemma 3.1 below, which plays a key role in our network construction.
Lemma 3.1. Define

\[ \mathcal{N} := \{g : g \text{ is a neural network function with depth 3 and width 2 and its activation functions being ReLU and sine} \} \]

Then \( \{2^i\}_{i=1}^{\infty} \) are scattered by \( \mathcal{N} \), i.e., for any given \( n \in \mathbb{N}^+ \), there exist \( g \in \mathcal{N} \) that interpolates \( (2^i, b_i), i = 1, \ldots, n \) with \( b_i \in \{0, 1\} \).

Proof. The proof is based on the bit-extraction technique. For any \( k \in \mathbb{N}^+ \), we demonstrate that there exists a function \( g \) in \( \mathcal{N} \) scattering \( \{2^i\}_{i=1}^{k} \). Let \( b_i = \sum_{j=i+1}^{k} b_j 2^{-j} + \pi \cdot 2^{i-k-1} \), where \( b_i \in \{0, 1\} \), \( i = 1, \ldots, k \). Set \( x_i = 2^i \), then

\[
\sin(2\pi bx_i) = \sin \left( \pi \sum_{j=1}^{k} b_j 2^{-j} + \pi \cdot 2^{i-k-1} \right) = \sin \left( b_i \pi + \pi \cdot \sum_{j=i+1}^{k} b_j 2^{-j} + \pi \cdot 2^{i-k-1} \right),
\]

where the second equality is due to the periodicity of sine function. Since

\[
\left( \frac{1}{2} \right)^k \leq \left( \frac{1}{2} \right)^{k+1-i} \leq \sum_{j=i+1}^{k} b_j 2^{-j} + 2^{i-k-1} \leq 1 - \left( \frac{1}{2} \right)^{k+1-i} \leq 1 - \left( \frac{1}{2} \right)^k,
\]

we have

\[
\sin(2\pi bx_i) \in \begin{cases} [\sin \left( \frac{1}{2^k} \right), 1] , & b_i = 0, \\ [-1, -\sin \left( \frac{1}{2^k} \right)] , & b_i = 1. 
\end{cases}
\]

Define

\[
f(x) = \text{ReLU} \left( \frac{1}{2 \sin(1/2^k)} x + \frac{1}{2} \right) - \text{ReLU} \left( \frac{1}{2 \sin(1/2^k)} x - \frac{1}{2} \right)
\]

\[
= \begin{cases} 1, & x > \sin \left( \frac{1}{2^k} \right), \\ \frac{1}{2 \sin(1/2^k)} x + \frac{1}{2}, & -\sin \left( \frac{1}{2^k} \right) \leq x \leq \sin \left( \frac{1}{2^k} \right), \\ 0, & x < -\sin \left( \frac{1}{2^k} \right), 
\end{cases}
\]

and \( g(x) = f(\sin(2\pi bx)) \), then it is easy to check that

\[
g(x_i) = f(\sin(2\pi bx_i)) = \begin{cases} 1, & b_i = 0, \\ 0, & b_i = 1, 
\end{cases} \quad i = 1, 2, \ldots, k.
\]

The above equation and Definition 2.1 imply that \( \{2^i\}_{i=1}^{k} \) are scattered by \( g(x) \in \mathcal{N} \). \( \square \)

Let \( N \in \mathbb{N}^+, \delta > 0 \), define by a small region\( \Omega(N, \delta, d) = \{ x = [x_1, \ldots, x_i, \ldots, x_d]^T \in \Omega = [0, 1]^d : \text{there exists a coordinate } i \text{ such that } x_i \in \left( \frac{j}{N} - \delta, \frac{j}{N} \right) , \quad j = 1, 2, \ldots, N \} \).
We will prove the approximation to the network outside this region first and go back to this region later.

**Theorem 3.2.** Let $M, N \in \mathbb{N}^+$, $\delta > 0$. For any $f \in C([0, 1]^d)$ with maximum $\tilde{f}$ and minimum $\underline{f}$, there exists a ReLU-sine-$2^x$ network $\Phi$ with $\mathcal{L}(\Phi) = 6$, $\mathcal{W}(\Phi) = \max \{2d/\log_2 N, 2M\}$ such that for all $x \in [0, 1]^d$, $f \leq \Phi(x) \leq \tilde{f}$ and

$$|f(x) - \Phi(x)| \leq \omega_f(\sqrt{d}) \cdot 2^{-M} + \omega_f\left(\frac{\sqrt{d}}{N}\right), \quad \forall x \in [0, 1]^d \setminus \Omega(N, \delta, d).$$

We list the main ideas and steps before the complete proof. The domain $[0, 1]^d \setminus \Omega(N, \delta, d)$ is divided into some uniform small cubes $\{\Omega_\alpha\}_\alpha$ with size parameter $N$. We will construct an approximation network $\Phi$ which is constant on each $\Omega_\alpha$. It is enough to approximate $f$ at grid points of the cubes $\{\Omega_\alpha\}_\alpha$, then approximation on $[0, 1]^d \setminus \Omega(N, \delta, d)$ can be obtained by using continuity modulus and controlling the size of $N$. To see that we first construct two maps $\Phi_1$ and $\Phi_2$, which serve to map each $\Omega_\alpha$ to a specific integer. Then we can approximate $f$ at grid points of the cubes $\{\Omega_\alpha\}_\alpha$ by applying the tool of binary representation. Specifically, we introduce $\phi_{\alpha, j}$ to allocate the integers acquired by $\Phi_1$ and $\Phi_2$ to 0 or 1, depending on the value of the $j$th bit of binary representation of function value at the grid points. Combining them together we have a network $\Phi_3$ which can approximate $f$ at grid points of the cubes $\{\Omega_\alpha\}_\alpha$.

**Proof.** Our construction is similar to [34]. First we divide the region $[0, 1]^d$ into $N^d$ small cubes with the same size. For $\alpha \in \{0, 1, 2, \ldots, N-1\}^d$, define by

$$\Omega_\alpha(N, \delta, d) = \left\{ x \in [0, 1]^d : x_i \in \left[\frac{\alpha_i}{N}, \frac{\alpha_i + 1}{N} - \delta\right], \quad i = 1, 2, \ldots, N \right\}.$$ 

Then

$$[0, 1]^d = \bigcup_{\alpha \in \{0,1,2,\cdots,N-1\}^d} \Omega_\alpha(N, \delta, d) \bigcup \Omega(N, \delta, d).$$

Let $N_1 = \lceil \log_2 N \rceil$. We can build a network approximating the following periodical function

$$h(x) = \begin{cases} 
1, & x \in \left[2k \cdot \frac{2^{N_1}}{N}, (2k+1) \cdot \frac{2^{N_1}}{N}\right) \\
0, & x \in \left[(2k+1) \cdot \frac{2^{N_1}}{N}, (2k+2) \cdot \frac{2^{N_1}}{N}\right) 
\end{cases}, \quad k = 0, 1, 2, \ldots.$$

To see that we consider

$$\phi_{1,1}(x) := \frac{1}{2\sin\delta} \text{ReLU} (x + \sin\delta) - \frac{1}{2\sin\delta} \text{ReLU} (x - \sin\delta)$$

$$= \begin{cases} 
1, & x > \sin\delta, \\
\frac{1}{2\sin\delta}x + \frac{1}{2}, & -\sin\delta \leq x \leq \sin\delta, \\
0, & x < -\sin\delta,
\end{cases}$$
and
\[
\phi_{1,2}(x) := \phi_{1,1} \left( \sin \left( \frac{N\pi}{2N_1} x + \delta \right) \right)
\]
\[
= \begin{cases} 
1, & x \in \left[2k \cdot \frac{2N_1}{N}, (2k + 1) \cdot \frac{2N_1}{N} - 2\delta\right], \\
0, & x \in \left((2k + 1) \cdot \frac{2N_1}{N}, (2k + 2) \cdot \frac{2N_1}{N} - 2\delta\right), \\
\frac{1}{2\sin \theta} \sin \left( \frac{N\pi}{2N_1} x + \delta \right) + \frac{1}{2}, & \text{otherwise},
\end{cases}
\]
for \( k = 0, 1, 2, \cdots \) and \( 0 \leq \delta \leq \frac{1}{2} \). It can be easily verified that \( \phi_{1,2} \) is an approximation of the periodical function \( h(x) \). We next define
\[
\phi_{1,3}^n(x) := \phi_{1,2}(2^n x), \quad \text{for } n = 1, 2, \cdots, N_1,
\]
and let
\[
\Phi_1(x) := (\phi_{1,3}^1(x_1), \cdots, \phi_{1,3}^{N_1}(x_1), \phi_{1,3}^1(x_2), \cdots, \phi_{1,3}^{N_1}(x_2), \cdots, \phi_{1,3}^1(x_d), \cdots, \phi_{1,3}^{N_1}(x_d)). (3.2)
\]

We claim that \( \Phi_1 \) maps each \( \Omega_\alpha(N, \delta, d) \) to a corresponding \( N_1d \)-dimensional vector and for \( \alpha \neq \beta, \Phi_1(\Omega_\alpha(N, \delta, d)) \cap \Phi_1(\Omega_\beta(N, \delta, d)) = \emptyset \), which will be proved in Lemma 3.3.

For any \( \bar{\alpha} \in \mathbb{R}^{N_1d} \), we define
\[
\Phi_2(\bar{\alpha}) = 2^{\sum_{i=1}^{N_1d} a_i 2^{i-1} + 1},
\]
then
\[
\Phi_2 \circ \Phi_1(x) \in \{2, 4, 8, \cdots\}, \quad x \in [0, 1]^d \setminus \Omega(N, \delta, d).
\]

Denote \( \bar{f} \) and \( \underline{f} \) as the maximum and minimum of \( f \) in \([0, 1]^d\), respectively. Define
\[
\tilde{f}(x) = \frac{f(x) - \underline{f}}{\bar{f} - \underline{f}}.
\]
It is clear that \( 0 \leq \tilde{f} \leq 1 \). For any \( \alpha \in \{0, 1, 2, \cdots, N - 1\}^d \), we consider the grid points \( \frac{\alpha}{N} \) in \([0, 1]^d\). Then we express \( \tilde{f} \) in the following form of binary decomposition, that is, for \( \alpha \in \{0, 1, \cdots, N - 1\}^d \), there exists \( \{a_{i_{\alpha,j}}\}_{j=1}^\infty \) with \( a_{i_{\alpha,j}} \in \{0, 1\} \) such that
\[
\tilde{f} \left( \frac{\alpha}{N} \right) = \sum_{j=1}^\infty a_{i_{\alpha,j}} 2^{-j},
\]
where
\[
i_{\alpha} = \sum_{k=1}^{N_1d} \left( \Phi_1 \left( \frac{\alpha}{N} \right) \right)_k 2^{k-1} + 1 \in \{1, 2, \cdots, N_1d\}.\]

For \( j = 1, 2, \cdots, M \), by Lemma 3.1, there exists network \( \phi_{3,j} \) with ReLU and sine activations
such that

$$\phi_{3,j}(2^{i_\alpha}) = a_{i_\alpha,j}, \quad \alpha \in \{0, 1, \cdots, N - 1\}^d.$$  

Define by

$$\Phi_3 = \sum_{j=1}^{M} \phi_{3,j} 2^{-j} \text{ and } \tilde{\Phi} = \Phi_3 \circ \Phi_2 \circ \Phi_1.$$  

Then for \(x \in \Omega_\alpha\), we have

$$|\tilde{\Phi}(x) - \tilde{f}(x)| \leq \sum_{j=M+1}^{+\infty} a_{ij} 2^{-j} + \omega_{\tilde{f}} \left(\frac{\sqrt{d}}{N}\right) = 2^{-M} + \omega_{\tilde{f}} \left(\frac{\sqrt{d}}{N}\right).$$  

Hence for all \(x \in [0, 1]^d \setminus \Omega(N, \delta, d)\),

$$|\tilde{\Phi}(x) - \tilde{f}(x)| \leq 2^{-M} + \omega_{\tilde{f}} \left(\frac{\sqrt{d}}{N}\right).$$  

Denoted by

$$\Phi = (\mathcal{T} - f) \tilde{\Phi} + f,$$

then we can obtain that

$$|\Phi(x) - f(x)| \leq |\mathcal{T} - f| |\tilde{\Phi}(x) - \tilde{f}(x)|$$

$$\leq |\mathcal{T} - f| \cdot 2^{-M} + |\mathcal{T} - f| \cdot \omega_{\tilde{f}} \left(\frac{\sqrt{d}}{N}\right)$$

$$= \omega_{f} \left(\sqrt{d}\right) \cdot 2^{-M} + \omega_{f} \left(\frac{\sqrt{d}}{N}\right).$$  

Since \(0 \leq \phi_{3,j} \leq 1\) for \(1 \leq j \leq M\), \(0 \leq \Phi_3 \leq 1\) and hence \(0 \leq \tilde{\Phi} \leq 1\). Then \(\underline{f} \leq \Phi \leq \overline{f} \).

Last, we calculate the depth and width of \(\Phi\). Obviously, for \(n = 1, 2, \cdots, N_1\), \(\mathcal{L}(\phi_{3,j}^n) = 3, W(\phi_{3,j}^n) = 2\). Then \(\mathcal{L}(\Phi_1) = 3, W(\Phi_1) = 2N_1d\), and \(\mathcal{L}(\Phi_2 \circ \Phi_1) = 4, W(\Phi_2 \circ \Phi_1) = 2N_1d\), and \(\mathcal{L}(\phi_{3,j}) = 3, W(\phi_{3,j}) = 2\). Then, \(\mathcal{L}(\Phi) = 6, W(\Phi) = \max\{2N_1d, 2M\} = \max\{2d|\log_2 N|, 2M\}\).

\[
\text{Remark 3.1. From the proof of Theorem 3.2, we know the activation functions of } \Phi \text{ are the ReLU in the second and fifth layer, the sine in the first and fourth layer and the } 2^x \text{ in the third layer. The same structure also hold for Theorem 3.4, Collorary 3.7 and Collorary 3.8. See Figure 3.1 for the detail on the structure of the constructed ReLU-sine-2^x network } \Phi.\]

The next Lemma states properties of the mapping

$$\Phi(x) := (\phi_1(x_1), \phi^N_{1,1}(x_1), \phi^N_{1,2}(x_2), \cdots, \phi^N_{1,d}(x_d), \cdots)$$  \quad (3.3)

which have been used in the construction of network in Theorem 3.2.

**Lemma 3.3.** (1) The mapping \(\Phi_1\) defined by (3.2) satisfies

$$\Phi_1 : [0,1]^d \setminus \Omega(N, \delta, d) \to \{0,1\}^{N_1 d}$$

and for each \(\alpha \in \{0,1,2,\cdots,N-1\}^d\), \(\Phi_1(\Omega_\alpha(N, \delta, d))\) is a singleton.

(2) For any \(\alpha,\beta \in \{0,1,2,\cdots,N-1\}^d\), \(\alpha \neq \beta\),

$$\Phi_1(\Omega_\alpha(N, \delta, d)) \cap \Phi_1(\Omega_\beta(N, \delta, d)) = \emptyset.$$  

**Proof.** (1) Let \(\alpha \in \{0,1,2,\cdots,N-1\}^d\). By the definition of \(\Omega_\alpha(N, \delta, d)\) and \(\Phi_1\), it suffices to show that for any \(i = 1,2,\cdots,d\) and \(n = 1,2,\cdots,N_1\) (\(N_1 = \lceil \log_2 N \rceil\)),

$$\left[ \frac{2^n \alpha_i}{N}, \frac{2^n (\alpha_i + 1)}{N} - 2^n \delta \right] \subset \left[ k \cdot \frac{2^{N_1}}{N}, (k + 1) \cdot \frac{2^{N_1}}{N} - 2 \delta \right],$$
for some \( k \in \mathbb{N}_0^+ \). It is equivalent to two inequalities:

\[
\begin{cases}
  k \cdot \frac{2^{N_1}}{N} \\
  (k + 1) \cdot \frac{2^{N_1}}{N} - 2\delta \leq \frac{2^n \alpha}{N}, \\
  (k + 1) \cdot \frac{2^{N_1}}{N} - 2\delta \geq \frac{2^n (\alpha + 1)}{N} - 2^n \delta.
\end{cases}
\]

In the following we show that there exists a \( k \in \mathbb{N}_0^+ \) satisfying

\[
\alpha_i - 2^{N_1 - n} + 1 \leq k \leq \frac{\alpha_i}{2^{N_1 - n}}.
\]

Since \( \alpha_i \in \{0, 1, \ldots, N - 1\} \), there exists \( A_1 \in \mathbb{N}_0^+ \) and \( A_2 \in \{0, 1, \ldots, 2^{N_1 - n} - 1\} \) such that

\[
\alpha_i = A_1 \cdot 2^{N_1 - n} + A_2.
\]

Then

\[
\begin{align*}
A_1 &= \frac{\alpha_i}{2^{N_1 - n}} = -\frac{A_2}{2^{N_1 - n}} \\
A_1 - \frac{\alpha_i - 2^{N_1 - n} + 1}{2^{N_1 - n}} &= 2^{N_1 - n} - 1 - A_2 \geq 0.
\end{align*}
\]

Therefore, we can set \( k = A_1 \) and conclude the result.

(2) By (1) it is sufficient to show that \( \Phi_1 \left( \frac{\alpha}{N} \right) \neq \Phi_1 \left( \frac{\beta}{N} \right) \) for any \( \alpha \neq \beta \). The fact that \( \alpha \neq \beta \) implies there exists an index \( i \), with \( 1 \leq i \leq 2d \) such that \( \alpha_i \neq \beta_i \). We will show there exists an index \( j \), with \( 1 \leq j \leq N_1 \) such that \( f_{\delta,j} \left( \frac{\alpha}{N} \right) \neq f_{\delta,j} \left( \frac{\beta}{N} \right) \). It can be verified by contradiction. Assume that for all \( 1 \leq n \leq N_1 \) there holds \( f_{\delta,j} \left( \frac{\alpha}{N} \right) = f_{\delta,j} \left( \frac{\beta}{N} \right) \), which means

\[
f_2 \left( \frac{2^\alpha}{N} \right) = f_2 \left( \frac{2^\beta}{N} \right)
\]

for all \( 1 \leq n \leq N_1 \) by definition. For \( n = 1 \), \( f_2 \left( \frac{\alpha}{N} \right) = 1 \) when \( \gamma \in \{0, 1, \ldots, 2^{N_1 - 1} - 1\} \) and \( f_2 \left( \frac{\beta}{N} \right) = 0 \) when \( \gamma \in \{2^{N_1 - 1}, 2^{N_1 - 1} + 1, \ldots, N - 1\} \), respectively. Then we have \( \alpha_i, \beta_i \in \{0, 1, \ldots, 2^{N_1 - 1}\} \). Similarly for \( n = 2 \), we can deduce that \( f_2 \left( \frac{2^\alpha}{N} \right) = 1 \) when \( \gamma \in \{0, 1, \ldots, 2^{N_1 - 2} - 1\} \) and \( f_2 \left( \frac{2^\beta}{N} \right) = 0 \) when \( \gamma \in \{2^{N_1 - 2}, 2^{N_1 - 2} + 1, \ldots, 2^{N_1 - 1} - 1\} \), respectively. Thus we obtain that \( \alpha_i, \beta_i \in \{0, 1, \ldots, 2^{N_1 - 2}\} \). The same argument can be applied to \( n = N_1 \), one may find \( \alpha_i, \beta_i \in \{0\} \). This contradicts to the fact that \( \alpha_i \neq \beta_i \).

**Theorem 3.4.** Let \( M, N \in \mathbb{N}^+ \), \( \delta > 0 \), \( p \in [1, +\infty) \). For any \( f \in C \left( [0, 1]^d \right) \), there exists a ReLU-sine-2\(^d\) network \( \Phi \) with \( \mathcal{L}(\Phi) = 6, \mathcal{W}(\Phi) = \max \{2d[\log_2 N], 2M\} \) such that

\[
\| f - \Phi \|_P \leq \omega_f (\sqrt{d}) \cdot 2^{-M} + \omega_f \left( \frac{\sqrt{d}}{N} \right) + \omega_f (\sqrt{d}) \left[ 1 - (1 - N\delta)^d \right]^{1/p}.
\]

**Proof.** By the approximation results in Theorem 3.2, we can compute the error in \( L^p \) norm
as follows:

\[
\int_{[0,1]^d} |f - \Phi|^p \, dx = \int_{[0,1]^d \setminus \Omega(N, \delta, d)} |f - \Phi|^p \, dx + \int_{\Omega(N, \delta, d)} |f - \Phi|^p \, dx
\]
\[
\leq \left[ \omega_f(\sqrt{d}) \cdot 2^{-M} + \omega_f \left( \frac{\sqrt{d}}{N} \right) \right]^p \int_{[0,1]^d \setminus \Omega(N, \delta, d)} \, dx + \omega_f(\sqrt{d}) \int_{\Omega(N, \delta, d)} \, dx
\]
\[
= \left[ \omega_f(\sqrt{d}) \cdot 2^{-M} + \omega_f \left( \frac{\sqrt{d}}{N} \right) \right]^p (1 - N\delta)^d + \omega_f(\sqrt{d}) [1 - (1 - N\delta)^d]^{1/p}
\]
\[
\leq \left\{ \omega_f(\sqrt{d}) \cdot 2^{-M} + \omega_f \left( \frac{\sqrt{d}}{N} \right) + \omega_f(\sqrt{d}) [1 - (1 - N\delta)^d]^{1/p} \right\}^p .
\]

\[\square\]

In [19], an approach of expanding the approximation result from \(x \in [0,1]^d \setminus \Omega(N, \delta, d)\) to the whole region \([0,1]^d\) is developed, which is based on a technique called horizontal shift. The result obtained in [19] is stated as follows.

**Proposition 3.5** (Theorem 2.1, [19]). Given any \(\epsilon > 0\), \(N, L, K \in \mathbb{N}^+\), and \(\delta \in (0, \frac{1}{3K}]\), assume \(f \in C([0,1]^d)\) and \(\tilde{\phi}\) is a network with width \(W(\tilde{\phi}) = N\) and depth \(L(\tilde{\phi}) = L\). If

\[|f(x) - \tilde{\phi}(x)| \leq \epsilon, \quad x \in [0,1]^d \setminus \Omega(K, \delta, d),\]

then there exists a new network \(\phi\) with width \(W(\phi) = 3^d(N + 4)\) and depth \(L(\phi) = L + 2d\) such that

\[|f(x) - \phi(x)| \leq \epsilon + d \cdot \omega_f(\delta), \quad x \in [0,1]^d.\]

Moreover, the activation functions of \(\phi\) are the activation functions of \(\tilde{\phi}\) and ReLU.

**Remark 3.2.** Note that Theorem 2.1 in [19] is applied for ReLU networks. However, its argument can be extended to network with any activation functions easily.

**Theorem 3.6.** Let \(M, N \in \mathbb{N}^+\), \(\delta > 0\). For any \(f \in C([0,1]^d)\), there exists a ReLU-sine-2s network \(\Phi\) with \(L(\Phi) = 2d + 6\), \(W(\Phi) = 3^d(\max \{2d\log_2 N, 2M\} + 4)\) such that

\[|f(x) - \Phi(x)| \leq \omega_f(\sqrt{d}) \cdot 2^{-M} + \omega_f \left( \frac{\sqrt{d}}{N} \right) + d \cdot \omega_f(\delta), \quad x \in [0,1]^d.\]

**Proof.** This Theorem follows directly from Theorem 3.2 and Proposition 3.5. \(\square\)

**Remark 3.3.** In Theorem 3.6 (also Corollary 3.9 below), activation functions of \(\Phi\) in the first and fourth layers are the sine and the third layer the 2s, while in other layers are all the ReLU.

For \(f \in \mathcal{H}_p^\alpha([0,1]^d)\), the continuity modulus \(\omega_f(r)\) can be bounded by Hölder constant \(\mu\), i.e., \(\omega_f(r) \leq \mu r^\alpha\). Hence we are able to obtain a series of more explicit approximation results that breaking the curse of dimensionality in approximation, i.e., to achieve an approximation error of \(\epsilon\), the depth and the width depend on \(\epsilon\) only polynomially rather than exponentially.

**Corollary 3.7.** Let \(\delta > 0\). For any \(f \in \mathcal{H}_p^\alpha([0,1]^d)\) and \(\epsilon > 0\), there exists a ReLU-sine-
2x network Φ with \( \mathcal{L}(\Phi) = 6, W(\Phi) = \max \left\{ 2d \left[ \log_2 (\sqrt{d}) \right]^{1/\alpha} , 2 \left[ \log_2 \left( \frac{\mu d^{\alpha/2}}{\epsilon} \right) \right] + 2 \right\} \) such that

\[
|f(x) - \Phi(x)| \leq \epsilon, \quad x \in [0, 1]^d \setminus \Omega \left( \left[ \sqrt{d} \right]^{1/\alpha} \right).
\]

**Proof.** From Theorem 3.2, there exists a ReLU − sin − 2x network Φ with \( \mathcal{L}(\Phi) = 6, W(\Phi) = \max \left\{ 2d \left[ \log_2 N \right], 2M \right\} \) such that for \( x \in [0, 1]^d \setminus \Omega(N, \delta, d) \).

Set \( \mu d^{\alpha/2} \cdot 2^{-M} = \mu \left( \sqrt{d} \right)^{\alpha} = \frac{\epsilon}{3} \). Then \( M = \left[ \log_2 \left( \frac{\mu d^{\alpha/2}}{\epsilon} \right) \right] + 1, N = \left[ \sqrt{d} \left( \frac{\epsilon}{\mu} \right)^{1/\alpha} \right] \). \( \square \)

**Corollary 3.8.** Let \( p \in [1, +\infty) \). For any \( f \in \mathcal{H}_p^\alpha ([0, 1]^d) \) and \( \epsilon > 0 \), there exists a ReLU-sine-2x network Φ with \( \mathcal{L}(\Phi) = 6, W(\Phi) = \max \left\{ 2d \left[ \log_2 (\sqrt{d}) \right]^{1/\alpha} , 2 \left[ \log_2 \left( \frac{\mu d^{\alpha/2}}{\epsilon} \right) \right] + 2 \right\} \) such that

\[
\|f - \Phi\|_{L^p} \leq \epsilon.
\]

**Proof.** Let the parameters

\[
\mu d^{\alpha/2} \cdot 2^{-M} = \mu \left( \sqrt{d} \right)^{\alpha} = \mu d^{\alpha/2} \left[ 1 - (1 - N\delta)^d \right]^{1/p} = \frac{\epsilon}{3}
\]

in Theorem 3.4, we obtain the desired result. \( \square \)

**Corollary 3.9.** For any \( f \in \mathcal{H}_p^\alpha ([0, 1]^d) \) and \( \epsilon > 0 \), there exists a ReLU-sine-2x network Φ with \( \mathcal{L}(\Phi) = 2d + 6, W(\Phi) = 3^d \left( \max \left\{ 2d \left[ \log_2 \left( \sqrt{d} \right)^{1/\alpha} \right] , 2 \left[ \log_2 \frac{3\mu d^{\alpha/2}}{2\epsilon} \right] + 2 \right\} + 4 \right) \) such that

\[
|f(x) - \Phi(x)| \leq \epsilon, \quad x \in [0, 1]^d.
\]

**Proof.** Applying Theorem 3.6 and setting

\[
\mu d^{\alpha/2} \cdot 2^{-M} = \mu \left( \sqrt{d} \right)^{\alpha} = \mu d^{\alpha/2} \left( \frac{\epsilon}{3} \right),
\]

yields the result. \( \square \)

The results in Corollary 3.8 and 3.9 show that our proposed ReLU-sine-2x networks overcome the curse of dimensionality in approximation on Hölder Class. We should mention some related works on constructing networks that break curse of dimensionality in approximation. In [42], to achieve accuracy \( \epsilon \), a network with ReLU and any Lipschitz periodic activations with the total number of weights \( O \left( \log^2 \frac{1}{\epsilon} \right) \) is built. In [35], the authors constructed a three hidden layer network that achieves the same approximation power as the ReLU-sine-2x network constructed...
here. They use floor, $2^x$ and step functions as activation functions. In the consideration of applying SGD for training, they propose using “continuous version” activation functions, i.e., utilizing piecewise linear functions to approximate the floor and step activation functions. The resulting “continuous version” network still enjoy the super expressive power. However, the directional derivative of the piecewise linear functions may blow up since it depends on $1/\epsilon$, see Table 3.1. In recent work of Yarotsky [41], network with \{sin, arcsin\} activation is constructed to approximate continuous functions $f \in C([0, 1]^d)$ with precision $\epsilon$. The main feature of the sin-arcsin network is that the size is $O(d^2)$ and independent on $\epsilon$. Hence such a network overcomes curse of dimensionality in approximation. We summarize the related works in Table 3.1.

### Table 3.1: Previous works and our result ($\epsilon$ denotes the approximation accuracy)

| Paper | Function class | Activation(s) | Depth | Width |
|-------|----------------|---------------|-------|-------|
| [40]  | $C^s([0, 1]^d)$ | ReLU | $O\left(\log d \log \frac{1}{\epsilon}\right)$ | $O\left(d^{s+1} \log d \left(\frac{1}{\epsilon}\right)^{d/s} \log \frac{1}{\epsilon}\right)$ |
| [42]  | $C^s([0, 1]^d) (s \geq 1)$ | ReLU, sine | $O\left(d \log \frac{1}{\epsilon}\right)$ | $O\left(d^{2s} \log \frac{1}{\epsilon}\right)$ |
| [34]  | $\mathcal{H}_\mu^s([0, 1]^d)$ | ReLU, floor | $256d + 3$ | $O\left(d \alpha^s \log \frac{1}{\epsilon}\right)$ |
| [35]  | $\mathcal{H}_\mu^s([0, 1]^d)$ | $\rho_1, \rho_2, \rho_1^3$ | 4 | $O\left(d \log d \log \frac{1}{\epsilon}\right)$ |
| [41]  | $C([0, 1]^d)$ | \{sin, arcsin\} | $O(d^2)$ | $O(d^2 \log d \log \frac{1}{\epsilon})$ |

This paper $\mathcal{H}_\mu^s([0, 1]^d)$ | ReLU, sine and $2^x$ | 6 | $O\left(d \log d \log \frac{1}{\epsilon}\right)$ |

| 1 | $\varphi_1(x) = \begin{cases} n - 1, & x \in [n - 1, n - \delta], \\ (x - n + \delta) / \delta, & x \in [n - \delta, n], \end{cases}$ for any $n \in \mathbb{Z}$, $\rho_2 = 3^x$, $\varphi_2(x) = \tilde{T}(\cos(2\pi x))$, |
| 2 | $\tilde{T}(x) := \begin{cases} 1 - x / \cos \left(\frac{2\pi}{d}\right), & x \in [0, \cos \left(\frac{2\pi}{d}\right)], \\ 1, & x \in (-\infty, 0) \cup (\cos \left(\frac{2\pi}{d}\right), \infty). \end{cases}$ |

**Remark 3.4.** Comparing with Corollary 3.8, there is an additional constant factor which is exponentially depending on dimension $d$ in the width of network in Corollary 3.9. The factor $3^d$ is introduced by Proposition 3.5 since we want to expand Corollary 3.7 to the whole region. Even so, a factor such as $\epsilon^{-d}$ does not appear in the depth and width of our network, which appears and leads to curse of dimensionality in approximation in many previous results of ReLU networks [40, 37, 33].

Furthermore, if we don’t pursue pointwise accuracy and only interest in approximation in $L^p$ norm with $p \in [1, \infty)$, Corollary 3.7 and 3.8 provide powerful and practical results, where the depth are 6 and the constant factors in width only depend on dimension $d$ at most in terms of $d \log d$.

An important issue in practical learning tasks such as classification and regressions is to determine the parameters in network $\Phi$ with data. Since $\Phi$ are (generalized) differentiable [7, 4], we can utilize the workhorse SGD for training.

**Remark 3.5.** We now compare our ReLU-sine-$2^x$ network with the ReLU-sine network appearing in [42]. The depth and width of the former are 6 and $O\left(\log_2 \frac{1}{\epsilon}\right)$, respectively (Corollary 3.8) while the depth and width of the latter are both $O\left(\log_2 \frac{1}{\epsilon}\right)$. Despite the width of two networks are of the same order, the depth of our network, a constant being independent of approximation error and dimension, is much less than the ReLU-sine network in [42].
4. Numerical Experiment. In this section we will give two simple examples to show the approximation ability of the proposed deep neural network. Let

\[ f_1 = \prod_1^d \sin(\pi x_i), \quad f_2 = \prod_1^d x_i^2, \]  

with space dimension \( d = 3 \). The loss function is chosen as the least square:

\[ L_i(\Phi) = \mathbb{E}_{X \sim U([0,1]^d)} \left[ (\Phi(X)^2 - 2\Phi(X)f_i(X))^2 \right] \quad i = 1, 2, \]  

where \( U([0,1]^d) \) stands for the uniform distribution on \([0,1]^d\). Then we use the stochastic gradient decent (SGD) type algorithm to minimize the loss by taking samples \( X_i \sim U([0,1]^d) \).

In our experiments, we use the Adam [15] optimizer with \( 1e5 \) epochs and \( 1e5 \) batch size. The learning rate is initially set to be \( 3e^{-3} \) and reduced by 0.99 in every 5000 epochs.

The construction of the network is as Figure 3.1. The width of the first layer is \( 4d \). The second layer and the first layer are fully connected in each dimension. Before the activation of the third layer, a truncation is applied to avoid the exponential blow-up. The width of the fourth layer is 8. There are \( 12d + 25 \) neurons in total.

The result is shown in Figure 4.1. The first row plots the landscape of \( \Phi \) on the diagonal line of \([0,1]^d\) and its reference. The second row is the \( L^2 \) error of the approximation: \( ||\Phi - f_i||_{L^2} \). One may find that the SGD algorithm successfully minimizes the loss in the proposed neural network architecture.

![Fig. 4.1: The numerical result of SGD optimization. The first row demonstrates the landscape of \( \Phi \) on the diagonal line of \([0,1]^d\). The second row is the approximation error in \( L^2 \) norm.](image-url)
5. Conclusion. In this paper, we construct neural networks with ReLU, sine and $2^x$ as activation functions that overcome the curse of dimensionality in approximation on the Hölder continuous function class defined on $[0,1]^d$. The proposed ReLU-sine-$2^x$ network functions are (generalized) differentiable, enabling us to apply SGD to train in practical learning tasks.

There are several avenues for further study. First, due to the theoretical advantages established here, the practical performances of the ReLU-sine-$2^x$ networks in real world applications deserves careful evaluations. Second, whether or not the generalization errors of ReLU-sine-$2^x$ networks in supervised learning can break the curse of dimensionality in approximation on number of samples is also of immense current interest.

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REFERENCES

[1] Martin Anthony and Peter L Bartlett, Neural network learning: Theoretical foundations, cambridge university press, 2009.
[2] Andrew R Barron, Universal approximation bounds for superpositions of a sigmoidal function, IEEE Transactions on Information theory, 39 (1993), pp. 930–945.
[3] Yoshua Bengio, Aaron Courville, and Pascal Vincent, Representation learning: A review and new perspectives, IEEE transactions on pattern analysis and machine intelligence, 35 (2013), pp. 1798–1828.
[4] Julius Berner, Dennis Elbrächter, Philipp Grohs, and Arnulf Jentzen, Towards a regularity theory for relu networks–chain rule and global error estimates, in 2019 13th International conference on Sampling Theory and Applications (SampTA), IEEE, 2019, pp. 1–5.
[5] Minshuo Chen, Haoming Jiang, and Tuo Zhao, Efficient approximation of deep relu networks for functions on low dimensional manifolds, Advances in Neural Information Processing Systems, (2019).
[6] Charles K Chui and Hrushikesh N Mhaskar, Deep nets for local manifold learning, Frontiers in Applied Mathematics and Statistics, 4 (2018), p. 12.
[7] Frank H Clarke, Optimization and nonsmooth analysis, SIAM, 1990.
[8] George Cybenko, Approximation by superpositions of a sigmoidal function, Mathematics of control, signals and systems, 2 (1989), pp. 303–314.
[9] David L Donoho et al., High-dimensional data analysis: The curses and blessings of dimensionality, AMS math challenges lecture, 1 (2000), p. 32.
[10] Xavier Glorot, Antoine Bordes, and Yoshua Bengio, Deep sparse rectifier neural networks, in Proceedings of the fourteenth international conference on artificial intelligence and statistics, JMLR Workshop and Conference Proceedings, 2011, pp. 315–323.
[11] Rémi Gribonval, Gitta Kutyniok, Morten Nielsen, and Felix Voigtlaender, Approximation spaces of deep neural networks, arXiv preprint arXiv:1905.01208, (2019).
[12] Ingo Gühring, Gitta Kutyniok, and Philipp Petersen, Error bounds for approximations with deep relu neural networks in $w_s$, $p$ norms, Analysis and Applications, 18 (2020), pp. 803–859.
[13] Kurt Hornik, Approximation capabilities of multilayer feedforward networks, Neural networks, 4 (1991), pp. 251–257.
[14] Kurt Hornik, Maxwell Stinchcombe, and Halbert White, Multilayer feedforward networks are universal approximators, Neural networks, 2 (1989), pp. 359–366.
[15] D. Kingma and J. Ba, Adam: A method for stochastic optimization, Computer Science, (2014).
Alex Krizhevsky, Ilya Sutskever, and Geoffrey E Hinton, *Imagenet classification with deep convolutional neural networks*, Advances in neural information processing systems, 25 (2012), pp. 1097–1105.

Yann A LeCun, Léon Bottou, Genevieve B Orr, and Klaus-Robert Müller, *Efficient backprop*, in Neural Networks: Tricks of the trade, Springer, 2012, pp. 9–48.

Shiyu Liang and Rayadurgam Srikant, *Why deep neural networks for function approximation?*, arXiv preprint arXiv:1610.04161, (2016).

Yann A LeCun, Léon Bottou, Genevieve B Orr, and Klaus-Robert Müller, *Efficient backprop*, in Neural networks: Tricks of the trade, Springer, 2012, pp. 9–48.

Shiyu Liang and Rayadurgam Srikant, *Why deep neural networks for function approximation?*, arXiv preprint arXiv:1610.04161, (2016).

Jianfeng Lu, Zuowei Shen, Haizhao Yang, and Shijun Zhang, *Deep network approximation for smooth functions*, arXiv preprint arXiv:2001.03040, (2020).

Zhou Lu, Hongming Pu, Feicheng Wang, Zhiqiang Hu, and Liwei Wang, *The expressive power of neural networks: A view from the width*, arXiv preprint arXiv:1709.02540, (2017).

Hadrien Montanelli and Qiang Du, *New error bounds for deep relu networks using sparse grids*, SIAM Journal on Mathematics of Data Science, 1 (2019), pp. 78–92.

Hadrien Montanelli and Qiang Du, *Error bounds for deep relu networks using the kolmogorov-arnold superposition theorem*, Neural Networks, 129 (2020), pp. 1–6.

Hadrien Montanelli, Haizhao Yang, and Qiang Du, *Deep relu networks overcome the curse of dimensionality for bandlimited functions*, arXiv preprint arXiv:1903.00735, (2019).

Ryumei Nakada and Masaaki Imaizumi, *Adaptive approximation and estimation of deep neural network with intrinsic dimensionality*, arXiv preprint arXiv:1907.02177, (2019).

Johannes Schmidt-Hieber, *Deep relu network approximation characterized by number of neurons*, arXiv preprint arXiv:1906.05497, (2019).

Zuowei Shen, Haizhao Yang, and Shijun Zhang, *Deep network approximation characterized by number of neurons*, arXiv preprint arXiv:1906.05497, (2019).

Allan Pinkus, *Approximation theory of the mlp model*, Acta Numerica 1999: Volume 8, 8 (1999), pp. 143–195.

Tomaso Poggio, Huishikesh Mhaskar, Lorenzo Rosasco, Brando Miranda, and Qianli Liao, *Why and when can deep-but not shallow-networks avoid the curse of dimensionality: a review*, International Journal of Automation and Computing, 14 (2017), pp. 503–519.

Herbert Robbins and Sutton Monro, *A stochastic approximation method*, The annals of mathematical statistics, (1951), pp. 400–407.

Johannes Schmidt-Hieber, *Deep relu network approximation of functions on a manifold*, arXiv preprint arXiv:1908.00695, (2019).

Johannes Schmidt-Hieber et al., *Nonparametric regression using deep neural networks with relu activation function*, Annals of Statistics, 48 (2020), pp. 1875–1897.

Uri Shaham, Alexander Cloninger, and Ronald R Coifman, *Provable approximation properties for deep neural networks*, Applied and Computational Harmonic Analysis, 44 (2018), pp. 537–557.

Zuowei Shen, Haizhao Yang, and Shijun Zhang, *Deep network approximation characterized by number of neurons*, arXiv preprint arXiv:1906.05497, (2019).

Dmitry Yarotsky, *Error bounds for approximations with deep relu networks*, Neural Networks, 94 (2017), pp. 103–114.

Dmitry Yarotsky and Anton Zhevnerchuk, *The phase diagram of approximation rates for deep neural networks with discrepancy being reciprocal of width to power of depth*, arXiv preprint arXiv:2102.10911, (2021).

Dmitry Yarotsky, *Elementary supereXpressive activations*, arXiv preprint arXiv:2102.10911, (2021).
networks, arXiv preprint arXiv:1906.09477, (2019).