Discriminating between models for the dark energy

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Recent measurements suggest our universe has a substantial dark energy component, which is usually interpreted in terms of a cosmological constant. Here we examine how much the form of this dark energy can be modified while still retaining an acceptable fit to the high redshift supernova data. We first consider changes in the dark energy equation of state and then explore a model in which the dark energy is interpreted as a fluid with a bulk viscosity.

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1. INTRODUCTION

Publication of the analysis of type Ia supernovae red shift data by the High-z Supernova Search Team [1] and the Supernova Cosmology Project [2] provided the first indication that the universe is accelerating. The data favor an interpretation in which most of the energy in the universe is some form of dark energy capable of providing negative pressure. The usual candidate for the dark energy is a cosmological constant $\Lambda$, which, as the name suggests, gives a time independent dark energy density. Both groups analyze their data assuming contributions from ordinary matter and a cosmological constant. They present their results as $\chi^2$ contours the plane of the parameters $\Omega_M$ and $\Omega_\Lambda$, defined below, which are a measure of the relative contributions of matter and dark energy [1, 2]. From these plots (e.g., Fig. 1 and Table 1), it is clear that values of $\Omega_M$ and $\Omega_\Lambda$ are most probably those of an accelerating cosmology.

While this picture is attractive because of its simplicity, one can ask to what extent the data can distinguish between alternative hypotheses for the behavior of the dark energy. Investigations into this question are generally framed in terms of determining the equation of state of the dark component ($X$), that is, determining $w = p_X/\rho_X$, where $p_X$ is the pressure of the dark component and $\rho_X$ is its energy density, as a function of the scale parameter $R(t)$ [3, 4]. It is possible to quantify the effect of any particular $w$ by fitting the supernova data and comparing the resulting value of $\chi^2$ with the one for $w = -1$, which corresponds to the case of a cosmological constant.

In the next Section, we review briefly the strategy for extracting the cosmological parameters $\Omega_M$ and $\Omega_\Lambda$ from the effective magnitude data for type Ia supernovae, and use a straightforward $\chi^2$ minimization to reproduce the results of Refs. [1] and [2]. In Section 3 we use the values of $\chi^2$ for $w = -1$ to explore several modifications to the equation of state including a search for the minimum $\chi^2$ for the case $w = \text{constant}$. Changes in the pressure-density relation of this type or similar modifications, including models using a Chaplygin equation of state [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] or generalized Cardassian expansion [18, 19, 20], have been considered before. Our discussion is included here as a prologue to Section 4 where we present a model in which the dark component is a fluid with a bulk viscosity. It is shown that this model provides an equally good fit to the supernova data, predicts an accelerating universe and has mass density fluctuations which grow. All of this occurs at the expense of some entropy production. Finally, we conclude with some comments on the current supernova data’s capacity to discriminate between models of the dark energy.

2. FITTING THE SUPERNOVAE DATA WITH A COSMOLOGICAL CONSTANT

If $R(t)$ is the scale parameter in the Robertson-Walker metric and $k$ its curvature constant, the Friedmann equations with a cosmological constant are [21],

$$ \frac{\dot{R}^2}{R^2} + \frac{k}{R^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3}, $$  

(1)
R = −4πG3(ρ + 3p) + Λ3,

where $G$ is Newton’s constant, $\rho$ is the density of matter, $p$ its pressure and a dot denotes time differentiation.

Conservation of energy and momentum gives the additional relation

\[ d(\rho R^3) = -pd(R^3). \]

Letting $H^2 = \dot{R}^2/R^2$, and denoting the present time by the subscript 0, Eq. (1) can be divided by $H_0^2$, the square of the Hubble constant, to give

\[ 1 = \frac{1}{H_0^2} \left( \frac{8\pi G}{3} \rho_0 + \frac{\Lambda}{3} \frac{k}{R_0^2} \right) \equiv (\Omega_M + \Omega_\Lambda + \Omega_k), \]

at present and

\[ H^2 = H_0^2 \left( \Omega_M \rho_0 + \Omega_\Lambda + \Omega_k \frac{R_0^2}{R^2} \right). \]

in general. Assuming the pressure of ordinary matter is negligible, Eq. (3) gives

\[ \rho \rho_0 = R_0^3 R^3, \]

and, in terms of the scaled variables $y = R/R_0$ and $\tau = H_0(t - t_0)$, Eq. (5) becomes

\[ \left( \frac{dy}{d\tau} \right)^2 = \frac{1}{y} \left( \Omega_M - (\Omega_M + \Omega_\Lambda - 1)y + \Omega_\Lambda y^3 \right). \]

From Eq. (4), the time history of the universe can be obtained as

\[ H_0(t - t_0) = \int_1^y dy' \frac{\sqrt{y'}}{\sqrt{\Omega_M - (\Omega_M + \Omega_\Lambda - 1)y' + \Omega_\Lambda y'^3}}. \]

Owing to the cubic behavior of the denominator of Eq. (8), the $\Omega_M - \Omega_\Lambda$ plane is divided into different categories of histories depending on whether a zero of the cubic occurs for $y < 0$, $0 < y < 1$, or $y > 1$. These are, respectively, those universes which continue to increase in size from $y = 0$, those which never achieve $y = 0$ and those which start at $y = 0$, increase to a maximum size and then collapse. The boundaries of these regions can be seen, for example, in Fig. 1 along with the line corresponding to flat cosmologies, $\Omega_\Lambda = 1 - \Omega_M$. Also, at the present time, Eq. (2) can be written

\[ \frac{\dot{R}_0}{R_0} = H_0^2 \left( -\frac{\Omega_M}{2} + \Omega_\Lambda \right), \]

and hence the universe is accelerating if $\Omega_\Lambda > \Omega_M/2$.

The extraction of the cosmological parameters $\Omega_M$ and $\Omega_\Lambda$ is achieved by relating the apparent magnitudes of $m(z)$ of the type Ia supernovae and their absolute magnitudes $M$ to their luminosity distance $d_L(z)$ as

\[ m(z) - M = 5 \log_{10} (d_L(z)) + 25, \]

where $d_L$ is measured in megaparsecs. The luminosity distance of a supernova at red shift $z$ can be expressed as

\[ d_L(z) = \frac{c(1 + z)}{H_0 \sqrt{|\Omega_k|}} \left\{ \frac{\sin z}{1 + \sin z} \right\} \left[ \sqrt{|\Omega_k|} \int_0^z dz' \frac{1}{\sqrt{(1 + z')^2(1 + z'\Omega_M) - z'(2 + z')\Omega_\Lambda}} \right]. \]
with \( c \) denoting the velocity of light. In Eq. (11), \( \sin \) is used for \( \Omega_k > 0 \), \( \sinh \) is used for \( \Omega_k < 0 \), and the unmodified square bracket is used for \( \Omega_k = 0 \).

To obtain a fit to the data, we write the apparent magnitude as

\[
m(z, \Omega_M, \Omega_A, \mathcal{M}) = 5 \log_{10}(D_L) + \mathcal{M},
\]

where \( D_L = c^{-1}H_0d_L \) and \( \mathcal{M} = M + 5 \log_{10}(cH_0^{-1}) + 25 \) is an additive constant. The parameters \( \Omega_M, \Omega_A \) and \( \mathcal{M} \) are then determined by minimizing

\[
\chi^2 = \sum_i \frac{(m_{\exp}(z_i) - m(z_i, \Omega_M, \Omega_A, \mathcal{M}))^2}{\sigma_i^2},
\]

with \( \sigma_i \) denoting the error in \( m_{\exp}(z_i) \).

Our fit to the data presented in Ref. [2] is shown in Fig. 1. To avoid repetition, we restrict our fits to this data. We have obtained similar results for the data in Ref. [1]. When computing points on the \( \Delta \chi^2 = \chi^2 - \chi^2_{\text{min}} \) contours, we choose the minimum value of \( \mathcal{M} \) at each point \( (\Omega_M, \Omega_A) \). This is easily done because the minimization condition for \( \mathcal{M} \) can be solved explicitly in terms of \( \Omega_M \) and \( \Omega_A \). The values of \( \Delta \chi^2 \) plotted \((2.30, 4.61, 6.18)\) would correspond to 67.3%, 90.0% and 95.5% confidence contours if the errors were Gaussian.

The contours are not precisely elliptical indicating non-Gaussian behavior, and this is addressed in the detailed analysis of Refs. [1] and [2]. Fortunately, our more naive analysis yields very similar \( \Delta \chi^2 \) contours, which we can use to assess the quality of the fits resulting from different assumptions about the dark component.

For the most part, the fit shown in Fig. 1 lies above the dashed line separating accelerating from non-accelerating cosmologies. While the best fit is not a flat cosmology, the \( \chi^2 \) values in Table I show that requiring \( \Omega_M + \Omega_A = 1 \) does not dramatically alter the quality of the fit.

### 3. VARYING THE EQUATION OF STATE

To generalize beyond the case of a constant dark energy, we can write Eq. (11) as

\[
\frac{R^2}{R'^2} = H_0^2 \left( \Omega_M \frac{\rho_M}{\rho_M0} + \Omega_X \frac{\rho_X}{\rho_X0} + \Omega_k \frac{R_0^2}{R^2} \right),
\]

and determine the dark energy density \( \rho_X \) using the energy conservation equation

\[
d\rho_X = -3(\rho_X + p_X) \frac{dR}{R},
\]

together with the equation of state

\[
p_X = w(R) \rho_X.
\]

For models of this type, the generalization of Eq. (4) is

\[
\left( \frac{dy}{d\tau} \right)^2 = \frac{1}{y} \left( \Omega_M - (\Omega_M + \Omega_X - 1)y + \Omega_X e^{-3 \int y \, w(y')} dy' \right),
\]

where \( \Omega_X \) is

\[
\Omega_X = \frac{8\pi G}{3H_0^2} \rho_{X0}.
\]

When \( w(y) \) is a constant, say \( w = -a, a > 0 \), the boundaries in the \( \Omega_M - \Omega_X \) plane between regions of expanding universes and universes which eventually collapse or have no big bang are obtained by determining where the minimum of the function

\[
\Omega_M - (\Omega_M + \Omega_X - 1)y + \Omega_X y^{3a}
\]
vanishes. This leads to the equation
\[ \frac{\Omega_X}{\Omega_M} = \frac{(3a - 1)(3a - 1)}{(3a - 1)} \left( \frac{\Omega_X}{\Omega_M} + \left( \frac{\Omega_M - 1}{\Omega_M} \right) \right)^{3a}, \]
from which \( \Omega_X \) can be found as a function of \( \Omega_M \). When \( a = 1 \), Eq. (20) gives the familiar results for the case of a cosmological constant \[23\]. For these models, the generalization of Eq. (2) is
\[ \ddot{R} = - \frac{4\pi G}{3} \rho_M - \frac{4\pi G}{3} (1 - 3a) \rho_X. \]
At the present time, this gives the acceleration parameter
\[ \frac{\dot{R}}{R_0} = (-\Omega_M + (3a - 1)\Omega_X) \frac{H_0^2}{2}, \]
and hence the universe is accelerating if
\[ \Omega_X > \frac{\Omega_M}{(3a - 1)}. \]

To gain some sense of what varying the value of \( a \) does to the quality of the fits, we performed a \( \chi^2 \) analysis of the data in Ref. [2] for the case \( a = -2/3 \) and for the case where \( a \) is varied along with the other parameters to obtain the best fit. The results are shown in Figs. [2] and [3] and the best fits are given in Table II. These examples show that the quality of the fits to the data of Ref. [2] for the cases considered – \( w = -1, -2/3, -0.73 \) – is basically the same. Even when the value of \( w \) is allowed to vary, the minimum in \( \chi^2 \) is not lowered in any significant way. One may argue that values of \( w \geq -2/3 \) are less preferred since, as suggested by Fig. [2], their \( \chi^2 \) contours encroach into the region of cosmologies which have no big bang.

The constant equations of state discussed above (\( w = -a \)) all result power law dark energy densities of the form
\[ \frac{\rho_X}{\rho_{X0}} = y^{3(a-1)}, \]
and all lead to fits with comparable \( \chi^2 \). To determine how changing the functional dependence of \( \rho_X \) on \( y \) affects the quality of the fits, we examined equations of state of the form \( w = -by \), \( b > 0 \). These equations of state lead to dark energy densities with an exponential dependence of the form
\[ \frac{\rho_X}{\rho_{X0}} = e^{3b(y-1)/y^3}. \]
In order to make direct comparisons with the constant \( w \) cases, we fit the data of Ref. [2] using \( b = 2/3 \) and \( b = 1 \). These fits are shown in Figs. [2] and [3]. The numerical details are given in Table III. Again, the \( \chi^2 \) of these fits indicates that data are not too sensitive to the \( y \)-dependence of \( w \). However, the location and size of the contours suggest that the \( w = -y \) choice is the more desirable.

The lifetime of the universe depends on the choice of \( w \) and can be used to discriminate among various models. Fig. [6] shows the \( \Omega_M - \Omega_X \) plane for four different choices of \( w \) along with lines of constant lifetime. Generally, the \( \chi^2 \) contours lie between 11.9 and 19.0 Gyr.

### 4. DARK ENERGY AS A VISCOS FLUID

As an alternative to simply modifying the equation of state of the dark component, in this section we consider the consequences of assuming the dark energy consists of a fluid having a bulk viscosity \( \zeta \). If a bulk viscosity term is introduced into the energy-momentum tensor of an ideal fluid, one effect is to replace Eq. [3] with\[21\]
\[ d(\rho_X R^3) = -p_X^* d(R^3) \equiv - \left( p_X - 3 \zeta \frac{\dot{R}}{R} \right) d(R^3). \]
In view of Eq. (14), the expression for $\dot{R}/R$ depends on the dark energy density $\rho_X$ in a nonlinear fashion, which means that, unlike the case of the multiplicative equations considered previously, Eq. (26) is not separable. In general, it must be solved numerically.

If we assume that the pressure $p_X$ of the dark component is negligible, the differential equation determining $\rho_X$ is

$$d\rho_X = -3 \left( \rho_X - 3 \zeta \frac{\dot{y}}{y} \right) \frac{dy}{y}, \quad (27)$$

or, using Eq. (14),

$$d\left( \frac{\rho_X}{\rho_{X0}} \right) = -3 \left( \frac{\rho_X}{\rho_{X0}} - 3 \zeta \frac{H_0}{\rho_{X0}} \sqrt{\frac{\Omega_M}{y^3} + \Omega_X \frac{\rho_X}{\rho_{X0}} + \frac{\Omega_k}{y^2}} \right) \frac{dy}{y}. \quad (28)$$

It is more convenient to write a differential equation for $H^2$ since $H$ is the quantity needed to determine the luminosity distance $d_L$. This differential equation is

$$dH^2 dy = -3 \frac{H^2}{y} + 3A \frac{H}{y} + \frac{\Omega_k}{y^3}, \quad (30)$$

where $A$ is given by

$$A = 3\Omega_X \zeta H_0 \rho_{X0} = \frac{8\pi G \zeta}{H_0}. \quad (31)$$

The second equality follows from Eq. (15). Since the luminosity distance is given by

$$d_L(z) = \frac{c (1 + z)}{H_0 \sqrt{|\Omega_k|}} \left\{ \frac{\sin \theta}{1 \sinh \theta} \right\} \left[ \sqrt{|\Omega_k|} \int_0^z \frac{dz'}{H(z')} \right], \quad (32)$$

where $H(z)$ is obtained by replacing $y$ by $1/(1 + z)$, we write the differential equation in terms of $z$, and solve

$$dH^2(z) dz = 3 \frac{H^2(z) - A H(z)}{(1 + z)} - \Omega_k (1 + z). \quad (33)$$

From the form of Eq. (33), the expression for the apparent magnitude $m(z)$ depends on the parameters $A$ and $\Omega_k$, which we can vary to obtain a fit to the supernova data. We perform the integration in Eq. (32) numerically, determining $H(z')$ using a four point Runge-Kutta algorithm and the boundary condition $H(0) = 1$ to first solve the differential equation. Given $d_L(z)$, for the data point $z_i$, we then minimize $\chi^2$ as in Eqs. (12) and (13). The resulting $\chi^2$ contours are shown in Fig. 5 and the best fits are listed in Table IV. In Fig. 6 the lines corresponding to various values of $\Omega_M$ indicate the boundary of the region below which, for that $\Omega_M$, the dark energy density $\rho_X$ remains positive. This is determined by the condition

$$\Omega_X \frac{\rho_X}{\rho_{X0}} = H^2(z) - (1 + z)^3 \Omega_M - (1 + z)^2 \Omega_k \geq 0, \quad (34)$$

which is analyzed by choosing value of $\Omega_M$, fixing $A$, taking $\Omega_k$ to be large and negative and solving Eq. (33) for $1 \leq z \leq 1000$. If condition Eq. (34) is satisfied for all $z$, $\Omega_k$ is increased and the process is repeated until, for a sufficiently large value of $\Omega_k$, Eq. (34) is violated for some $z$. This determines a point $(A, \Omega_k)$ on the curve for the given $\Omega_M$ and the region above this curve is not allowed.
The range of lifetimes for this case, obtained from the relation
\[ t_0H_0 = \int_0^1 dy \frac{1}{yH(y)}, \]
(35)
is shown in Fig. 8. Here, unlike the cases of Secs. 2 and 3, the lifetime cannot be much less than 14 Gyr. The lines of constant lifetime terminate on the dark black line in this figure and Fig. 7. For \( \Omega_k \) and \( A \) above this line, the lifetime integral does not converge. In each of these figures, the dotted line separates the accelerating from the non-accelerating cosmologies, with the region above the line corresponding to accelerating cosmologies. This can be seen by noting that analog of Eq. (21) is
\[ \frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \rho_M - \frac{4\pi G}{3} (\rho_X + 3\rho_X^*), \]
(36)
which gives at the present time
\[ \frac{\ddot{R}_0}{R_0} = -\frac{4\pi G}{3} (\rho_{M0} + \rho_{X0}) + 12\pi G \zeta H_0 \]
\[ = \frac{H_0^2}{2} (3A + \Omega_k - 1). \]
(37)
Hence, accelerating cosmologies satisfy \( \Omega_k > -3A + 1 \).

When \( \rho \) and \( p \) are related by an equation of state of the form \( p = w \rho \), the cosmology evolves at constant entropy. In the present case, there is entropy production. This can be seen by noting that [24]
\[ \frac{dS}{dt} = 2\pi^2 \rho_X R^3 \left( \frac{d\rho_X R^3}{dt} + p_X \frac{dR^3}{dt} \right) = \frac{2\pi^2}{T(R)} \left( \frac{d\rho_X R^3}{dt} + \left( p_X + 3\zeta \frac{R}{R} \right) \frac{dR^3}{dt} \right), \]
(38)
where \( T(R) \) is the temperature of the dark energy. On using Eq. (26), we have
\[ \frac{dS}{dt} = 9\zeta \frac{2\pi^2 R^3}{T(R)} \left( \frac{\dot{R}^2}{R^2} \right). \]
(39)
Considering \( S \) as a function of \( y = R/R_0 \), the evolution of \( S(y) \) is given by
\[ \frac{dS}{dy} = 9\zeta V_0 \left( \frac{\dot{y}}{y} \right) y^2, \]
(40)
where \( V_0 = 2\pi^2 R_0^3 \). With the aid of Eq. (24), the entropy is
\[ S(y) - S_0 = 9\zeta V_0 H_0 \int_1^y dy \frac{y^2 \mathcal{H}(y')}{T(y')} \]
(41)
where \( \mathcal{H}(y) \) is obtained by solving Eq. (30). Setting
\[ \mathcal{H}^2(y) = y^{-3}u(y) \]
(42)
and using Eq. (31) to eliminate \( \zeta \), the change in the entropy density at present is given by
\[ \frac{\Delta S_0}{V_0} = 9AH_0^2 \int_0^1 dy \sqrt{y} \sqrt[3]{u(y)} \left( \frac{\dot{y}}{T(y)} \right) = 3A\rho_{crit} \int_0^1 dy \sqrt{y} \sqrt[3]{u(y)} \left( \frac{\dot{y}}{T(y)} \right), \]
(43)
where \( \rho_{crit} = 3H_0^2/8\pi G \) is the critical density and \( u(y) \) satisfies
\[ \frac{du}{dy} = 3A\sqrt{y} \sqrt[3]{u} + \Omega_k. \]
(44)
To complete the calculation of $\Delta S_0$, the functional form of $T(y)$ is needed. The choice of $T(y)$ must be consistent with the integrability of the entropy, which, for $T$ taken to be a function of $V$ and $\rho_X$, requires

$$- V \left( \frac{\partial T}{\partial V} \right)_{\rho_X} + (\rho_X + p_X) \left( \frac{\partial T}{\partial \rho_X} \right)_V = T \left( \frac{\partial p_X}{\partial \rho_X} \right)_V .$$

(45)

Using Eq. (45) and Eq. (25), it is possible to obtain the relation

$$\frac{dT}{T} = -3 \left( \frac{\partial p_X}{\partial \rho_X} \right)_V \frac{dR}{R} + 9 \zeta H \left( \frac{\partial T}{\partial \rho_X} \right)_V \frac{dR}{R} .$$

(46)

Assuming $p_X = 0$, Eq. (45) implies that $T(V, \rho_X)$ has the general form

$$T(V, \rho_X) = T(V \rho_X) = \int d\beta A(\beta)(V \rho_X)^\beta ,$$

(47)

and, for the purpose of assessing the entropy growth, we examine a single term of the form

$$T = C(V \rho_X)^\beta ,$$

(48)

where $C$ is a constant. In this case, Eq. (46) becomes

$$\frac{dT}{T} = \beta \frac{9 \zeta H}{\rho_X} \frac{dR}{R} = \beta \frac{3 A \sqrt{y u(y)} dy}{u(y) - \Omega_M - y \Omega_k} = \beta \frac{d[u(y) - \Omega_M - y \Omega_k]}{[u(y) - \Omega_M - y \Omega_k]} ,$$

(49)

where Eqs. (24), (12) and (14) have been used. Integrating Eq. (46) gives

$$\frac{T(y)}{T(1)} = \frac{[u(y) - \Omega_M - y \Omega_k]^{\beta}}{\Omega_X^{\beta}} = \frac{(V \rho_X)^{\beta}}{(V_0 \rho_{X0})^{\beta}} ,$$

(50)

which is consistent with Eq. (45). Using Eq. (50), Eq. (43) is also integrable and the entropy density at present is

$$\frac{S_0}{V_0} = \frac{\rho C \Omega_X}{T(1) - 1 - \beta} \left[ 1 - \left( \frac{u(0) - \Omega_M}{\Omega_X} \right)^{1-\beta} \right] .$$

(51)

As with the luminosity distance above, we evaluate $u(0)$ numerically using a simple four-point algorithm to integrate Eq. (14). The ratio of the present value of the entropy density produced by the viscous medium to the entropy density of the microwave background is plotted in Fig. 9 for several values of $\beta$ and $T(1)$ taken to be the temperature of the microwave background, $T_\gamma = 2.73$ K. The numerical results scale as $T_\gamma/T(1)$. Clearly, a side effect of the viscosity being adequate to produce an accelerating cosmology is the production of large amounts of entropy.

Although we are assuming that the matter and dark energy densities constitute non-interacting components of the total energy density, it is nevertheless necessary to check the compatibility of this assumption with the existence of matter density fluctuations which grow at large $z$ (or small $y$). For an inviscid fluid in the Newtonian limit, a matter density fluctuation $\delta$ satisfies the equation

$$\ddot{\delta} + 2 \frac{\dot{R}}{R} \delta - 4\pi G \rho_M \delta = 0 ,$$

(52)

and the influence of the viscous dark energy enters through the $\dot{R}/R$ term. Using Eqs. (14) and (20), the differential equation Eq. (33), and assuming that $\rho_M \propto (1 + z)^3$, the equation for $\delta$ can be written

$$(1 + z)H^2(z) \delta'' + \frac{1}{2} \left( H^2(z) - 3A \dot{H}(z) - \Omega_k(1 + z)^2 \right) \delta' - \frac{3}{2} \Omega_M (1 + z)^2 \delta = 0 ,$$

(53)
where the ' denotes a derivative with respect to $z$. To solve this equation for large $z$, we note that Eqs. (42) and (44) imply

$$\lim_{y \to 0} H^2(y) \sim y^{-3} (u(0) + \Omega_k y).$$

This gives for large $z$

$$H^2(z) \sim (1 + z)^3 u(0) + (1 + z)^2 \Omega_k.$$  \hfill (55)

Keeping only the leading terms for large $z$, Eq. (55) becomes

$$2(1 + z)^2 \delta'' + (1 + z) \delta' - 3 \frac{\Omega_M}{u(0)} \delta = 0,$$

and a solution of the form $\delta = (1 + z)^a$ gives

$$a_{\pm} = \frac{1}{4} \left( 1 \pm \sqrt{1 + 24 \frac{\Omega_M}{u(0)}} \right).$$

Hence, the solution which grows as $z$ decreases is

$$\delta = \left( \frac{1}{1 + z} \right)^x, \quad x = \frac{1}{4} \left( \sqrt{1 + 24 \frac{\Omega_M}{u(0)}} - 1 \right).$$

In general, $x \leq 1$ because the condition that $\rho_X$ be positive, Eq. (54), implies, in view of Eq. (55), that $u(0) \geq \Omega_M$. The curves of constant exponent $x$ for several values of $\Omega_M$ are shown in Fig. 10. The regions below the $x = 1$ curves have matter density fluctuations which grow, the trend being toward slower growth for larger negative values of $\Omega_k$.

5. CONCLUSIONS

When analyzed in terms of a universe consisting of matter and dark energy in the form of a cosmological constant, the supernova data of Refs. [1, 2] favor an accelerating universe. The best fits in this case do not correspond to a flat cosmology, but a flat solution is well within the likelihood contours with a $\chi^2$ relatively close to the minimum value of $\chi^2$. Our ability to reproduce the results of Refs. [2] is indicated in Table 1.

As a means of assessing the sensitivity of the currently available supernova data to the form of the equation of state $p = w(y)\rho$, we repeated the analysis using equations of state of the form

$$w(y) = -a - by, \quad a, b > 0.$$  \hfill (59)

For $b = 0$ and a number of different values of $a$, with $0 < a < 1$, we find best fits with $\chi^2$ very nearly identical to those obtained for a cosmological constant. This includes a determination of the best value of $a$ by treating $a$ as one of the search parameters. In these cases, the sum $\Omega_M + \Omega_X$ tends to be a bit larger than $\Omega_M + \Omega_\Lambda$, and the flat cosmology value of $\Omega_M$ is somewhat smaller than the corresponding value obtain for a cosmological constant. When $a = 0$ and $b$ is varied, the quality of the fits in terms of $\chi^2$ is again virtually unchanged. The case $w = -y$ is very similar to $w = -1$ in all respects, as can be seen by comparing Figs. 1 and 5. This tendency of the minimum value of $\chi^2$ to be the same as long as the equation of state describes a dark energy with negative pressure is another indication that the precise form of $w$ is ill determined by the current supernova data [26].

To examine whether the introduction of a negative pressure component by means of a multiplicative equation of state was essential, we explored the notion that the dark energy is a fluid with bulk viscosity. The bulk viscosity provides a negative pressure and, again, a fit to the supernova data gives a $\chi^2$ that is indistinguishable from the cosmological constant case (compare Tables IV and I). We have shown that such a dark energy can provide a fit which is consistent with an accelerating flat cosmology having reasonable values for the lifetime
and $\Omega_M$ and density fluctuations with the appropriate behavior. There remains a question of whether the use of a simple linear bulk viscosity term $\zeta \dot{y}/y$ is valid throughout the range of $y(t)$’s probed by the supernova data. There are known non-linear effects, usually associated with viscosity-driven inflation models, which could modify our results \cite{27, 28, 29}. We have not attempted to include them in this investigation. The model presented here is different from those which suggest that viscosity effects associated with cold dark matter could be the source of the observed acceleration \cite{30, 31}. Viscosity effects have also been discussed in models with matter creation \cite{32}.

This approach to the inclusion of bulk viscosity is highly testable in the sense that future measurements could rule it out. As mentioned above, the lifetime cannot be much less than 14 Gyr. Further, if the universe is flat, or nearly so, as seems likely, then we require $\Omega_M$ to be rather small. A close examination of Fig. 4 near $\Omega_k = 0$ reveals that the present supernova data rule out $\Omega_M = 0.3$ at the 90% level. On the other hand, improving the supernova contours is unlikely to rule out this model, even for a flat universe, as long as $\Omega_M \leq 0.1$ is allowed.

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\begin{tabular}{lcc}
\hline
 & General & Flat \\
\hline
$\Omega_M$ & 0.79 & $0.28^{+0.09}_{-0.08}$ \\
$\Omega_\Lambda$ & 1.41 & $1 - \Omega_M$ \\
$\mathcal{M}$ & 23.9 & 23.9 \\
$\chi^2$ & 56.9 & 57.7 \\
\hline
\end{tabular}

TABLE I: The first pair of entries corresponds to the fit obtained using Eq. (13) and the second pair are the results from Ref. [2].

\begin{tabular}{lcc}
\hline
 & $w = -2/3$ & \\
 & General & Flat \\
\hline
$\Omega_M$ & 0.54 & $0.08^{+0.10}_{-0.08}$ \\
$\Omega_\Lambda$ & 2.33 & $1 - \Omega_M$ \\
$\mathcal{M}$ & 23.9 & 24.0 \\
$\chi^2$ & 56.81 & 58.2 \\
\hline
\end{tabular}

\begin{tabular}{lcc}
\hline
 & $w = -0.73$ & \\
 & General & Flat \\
\hline
$\Omega_M$ & 0.63 & $0.13^{+0.11}_{-0.10}$ \\
$\Omega_\Lambda$ & 2.10 & $1 - \Omega_M$ \\
$\mathcal{M}$ & 23.9 & 24.0 \\
$\chi^2$ & 56.80 & 58.1 \\
\hline
\end{tabular}

TABLE II: Fits to the data of Ref. [2] for two constant values of $w$ obtained using Eq. (13) are shown.

\begin{tabular}{lcc}
\hline
 & $w = -2y/3$ & \\
 & General & Flat \\
\hline
$\Omega_M$ & 0.18 & $0.08^{+0.13}_{-0.18}$ \\
$\Omega_\Lambda$ & 2.33 & $1 - \Omega_M$ \\
$\mathcal{M}$ & 23.9 & 24.0 \\
$\chi^2$ & 56.9 & 58.0 \\
\hline
\end{tabular}

\begin{tabular}{lcc}
\hline
 & $w = -y$ & \\
 & General & Flat \\
\hline
$\Omega_M$ & 0.55 & $0.21\pm0.08$ \\
$\Omega_\Lambda$ & 1.39 & $1 - \Omega_M$ \\
$\mathcal{M}$ & 23.9 & 23.9 \\
$\chi^2$ & 56.9 & 57.6 \\
\hline
\end{tabular}

TABLE III: Fits to the data of Ref. [2] for the cases $w = -2y/3$ and $w = -y$ obtained using Eq. (13) are shown.
| Bulk Viscosity | General | Flat |
|---------------|---------|------|
| $\Omega_k$    | -1.36   | 0.0  |
| $A$           | 1.47    | 0.66 |
| $M$           | 23.9    | 24.0 |
| $\chi^2$     | 56.9    | 58.0 |

TABLE IV: Fits to the data of Ref. [2] for the case of a dark component consisting of an ideal fluid with a bulk viscosity $\zeta$ are shown.

FIG. 1: The $\Delta \chi^2$ contours for the fit to the 54 supernovae analyzed in Ref. [2] are shown. The contours correspond to the 68.3%, 90% and 95.4% confidence levels. The cross indicates the location of the minimum and the dot indicates the location of the minimum assuming a flat cosmology. The dashed line separates accelerating from non-accelerating cosmologies.
FIG. 2: The $\Delta \chi^2$ contours for the fit to the 54 supernovae analyzed in Ref. 2, assuming a constant equation of state of the form $w = -2/3$, are shown. The contours correspond to the 68.3%, 90% and 95.4% confidence levels. The cross indicates the location of the minimum and the dot indicates the location of the minimum assuming a flat cosmology. The dashed line separates accelerating from non-accelerating cosmologies.

FIG. 3: The $\Delta \chi^2$ contours for the best fit to the 54 supernovae analyzed in Ref. 2, assuming a constant equation of state of the form $w = -a$, are shown. The contours correspond to the 68.3%, 90% and 95.4% confidence levels. The cross indicates the location of the minimum and the dot indicates the location of the minimum assuming a flat cosmology. The dashed line separates accelerating from non-accelerating cosmologies.
FIG. 4: The $\Delta \chi^2$ contours for the best fit to the 54 supernovae analyzed in Ref. [2], assuming an equation of state of the form $w = -2y/3$, are shown. The contours correspond to the 68.3%, 90% and 95.4% confidence levels. The cross indicates the location of the minimum and the dot indicates the location of the minimum assuming a flat cosmology. The dashed line separates accelerating from non-accelerating cosmologies.

FIG. 5: The $\Delta \chi^2$ contours for the best fit to the 54 supernovae analyzed in Ref. [2], assuming an equation of state of the form $w = -y$, are shown. The contours correspond to the 68.3%, 90% and 95.4% confidence levels. The cross indicates the location of the minimum and the dot indicates the location of the minimum assuming a flat cosmology. The dashed line separates accelerating from non-accelerating cosmologies.
FIG. 6: The $\chi^2$ contours and (dashed) lines of constant lifetime are shown. Starting from the lower right and moving to the upper left of each panel, the dashed green lines correspond to 8.0, 9.5, 11.9, 14.3 and 19.0 Gyr. In computing the lifetimes, we used $H_0=63$ km/sec/Mpc. In the $w=-2/3$ panel, the 19.0 Gyr line is not visible on the portion of the plane shown.
FIG. 7: The $\Delta \chi^2$ contours for the best fit to the 54 supernovae analyzed in Ref. [2], assuming a viscous medium, are shown. The contours correspond to the 68.3%, 90% and 95.4% confidence levels. The cross indicates the location of the minimum $\chi^2$ and the dot indicates the location of the minimum $\chi^2$ assuming a flat cosmology. The dashed lines are the boundaries of the region below which for the given value of $\Omega_M$ the dark energy density $\rho_X$ is positive. The region above the dotted line corresponds to accelerating cosmologies.

FIG. 8: The $\Delta \chi^2$ contours for the best fit to the 54 supernovae analyzed in Ref. [2], assuming a viscous medium, are shown. The contours correspond to the 68.3%, 90% and 95.4% confidence levels. The cross indicates the location of the minimum $\chi^2$ and the dot indicates the location of the minimum $\chi^2$ assuming a flat cosmology. The dashed lines are the lines of constant lifetime. The region above the dotted line corresponds to accelerating cosmologies.
FIG. 9: The ratio of the present entropy associated with the viscous medium to the present entropy of the microwave background is plotted for the 68% $\Delta \chi^2$ contour corresponding to the best fit to the 54 supernovae analyzed in Ref. [2] and $\Omega_M = 0.1$. Results for the indicated values of the parameter $\beta$ are shown. The cross indicates the location of the minimum $\chi^2$ and the dot indicates the location of the minimum $\chi^2$ assuming a flat cosmology. The dashed portions of the curves correspond to the $\Omega_M = 0.1$ boundary in Fig. 7.

FIG. 10: Curves of constant values of the exponent $x$ which controls the growth of the matter density fluctuations $\delta$, $\delta = 1/(1+z)^x$, are shown for several values of $\Omega_M$. 