Finite generation of adjoint rings after Lazić: an introduction*

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1 Introduction

This note is an introduction to all the key ideas of Lazić’s recent proof of
the theorem on the finite generation of adjoint rings [Laz09]. (The theorem
was first proved in [BCHM09].) I try to convince you that, despite technical
issues that are not yet adequately optimised, nor perhaps fully understood,
Lazić’s argument is a self-contained and transparent induction on dimension
based on lifting lemmas and relying on none of the detailed general results

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of Mori theory. On the other hand, it is shown in [CL10] that all the fundamental theorems of Mori theory follow easily from the finite generation statement discussed here: together, these results give a new and more efficient organisation of higher dimensional algebraic geometry.

The approach presented here is ultimately inspired by a close reading of the work of Shokurov [Sho03], I mean specifically his proof of the existence of 3-fold flips. Siu was the first to believe in the possibility of a direct proof of finite generation, and believing that something is possible is, of course, a big part of making it happen. All mathematical detail is taken from [Laz09]; my contribution is merely exegetic. I begin with a few key definitions leading to the statement of the main result.

### 1.1 Basic definitions

**Convention 1.1.** Throughout this paper, I work with nonsingular projective varieties over the complex numbers.

Let $V$ is a (finite dimensional) real vector space defined over the rationals. By a *cone* in $V$ I always mean a convex cone, that is a subset $C \subset V$ such that $0 \in C$ and:

\[
t \geq 0, \ v \in C \Rightarrow t v \in C,
\]

\[
v_1, v_2 \in C \Rightarrow v_1 + v_2 \in C.
\]

A *finite rational cone* is a cone $C \subset V$ generated by a finite number of rational vectors.

If $U \subset V$ an affine subspace, then I denote by $U(\mathbb{R})$ and $U(\mathbb{Q})$ the sets of real and rational points of $U$.

The end of a proof of a statement, or the absence of a proof, is denoted by $\square$.

**Notation 1.2.** I denote by $\mathbb{R}$ one of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$. If $X$ is a normal variety, I denote by $\text{Div}_{\mathbb{R}} X$ the group of (Weil) divisors on $X$ with coefficients in $\mathbb{R}$, and by $\text{Div}_{\mathbb{R}}^+ X$ the sub-monoid of effective divisors. In this note, I almost always work with actual divisors, *not* divisors modulo linear equivalence. For instance, when I write $K_X$, I mean that I have chosen a specific divisor in the canonical class; the choice is made at the beginning and fixed throughout the discussion.
**Notation 1.3.** If $X$ is a normal variety and $D \in \text{Div}_R X$ is a divisor on $X$, I write:

$$D = \text{Fix } D + \text{Mob } D,$$

where \text{Fix } $D$ and \text{Mob } $D$ are the fixed and the mobile part of $D$. The definition makes sense even when $D$ is not integral or effective. Indeed the sheaf $\mathcal{O}_X(D)$ is defined as

$$\Gamma(U, \mathcal{O}_X(D)) = \{ f \in k(X) \mid \text{div} U f + D|_U \geq 0 \}$$

and then, by definition:

$$\text{Mob } D = \sum m_E E \quad \text{where} \quad m_E = -\inf\{ \text{mult}_E f \mid f \in H^0(X, \mathcal{O}_X(D)) \}.$$ 

In applications $D$ is almost always integral and effective. If $D$ is not integral, then the definition says that $\mathcal{O}_X(D) = \mathcal{O}_X([D])$; if $D$ is integral, then $\text{Fix } D = \text{Fix } |D|$ is the fixed part of the complete linear system $|D|$.

Throughout this paper, I use without warning the following elementary fact, often called Gordan’s lemma.

**Lemma 1.4.** Let $C \subset \mathbb{R}^r$ a finite rational cone. The monoid $\Lambda = C \cap \mathbb{Z}^r$ is finitely generated.

**Definition 1.5.** Let $X$ be a nonsingular projective variety and $\Lambda = C \cap \mathbb{Z}^r$ where $C \subset \mathbb{R}^r$ is a finite rational cone.

1. A divisorial ring on $X$ is a $\Lambda$-graded ring of the form

$$R(X; D) = \bigoplus_{\lambda \in \Lambda} H^0(X; D(\lambda)) \quad \text{where} \quad D: \Lambda \to \text{Div}_R X$$

is a map such that $M(\lambda) = \text{Mob } D(\lambda)$ is superadditive, i.e., $M(\lambda_1 + \lambda_2) \geq M(\lambda_1) + M(\lambda_2)$ for all $\lambda_1, \lambda_2 \in \Lambda$. The map $D: \Lambda \to \text{Div}_R X$ is called the characteristic system of the ring. When I wish to emphasise the grading by $\Lambda$, I write $R(X; \Lambda)$ instead of $R(X; D)$.

2. A divisorial ring $R(X; D)$ is adjoint if in addition

- $D: \Lambda \to \text{Div}_R X$ is rational PL, i.e., it is the restriction of a rational piecewise linear map that, abusing notation, I still denote by

$$D: C \to \text{Div}_R X.$$
This means that there is a finite decomposition $C = \bigcup_{i=1}^{m} C_i$ into finite rational cones $C_i \subset \mathbb{R}^r$ such that $D|_{C_i}$ is (the restriction of) a rational linear function.

- there is a rational PL function $r: C \to \mathbb{R}_+$ and for all $\lambda \in \Lambda$ $D(\lambda) = r(\lambda)(K_X + \Delta(\lambda))$ with $\Delta(\lambda) \geq 0$.

3. An adjoint ring is big if there is an ample $\mathbb{Q}$-divisor $A$ and

$$\Delta(\lambda) = A + B(\lambda)$$

with all $B(\lambda) \geq 0$.

4. A big adjoint ring is klt (dlt) if there is a fixed simple normal crossing (snc) divisor $\sum_{j=1}^{r} B_j \subset X$ such that all $\text{Supp } B(\lambda) \subset \cup_{j=1}^{r} B_j$ and all $(X, B(\lambda))$ are klt (dlt).

Remark 1.6. 1. If $D: \Lambda \to \text{Div}_{\mathbb{Q}}^+X$ is superadditive and rational PL, then it is also concave.

2. If $R(X; D)$ is an adjoint ring then in particular: $\Delta: C \to \mathbb{R}_+$ is homogeneous of degree 0; that is, $\Delta(tw) = t\Delta(w)$ for all $t \in \mathbb{R}_+, w \in C$.

3. Note that in the definition of dlt (klt) big adjoint ring, all divisors in sight are contained in a fixed snc divisor $\sum_{j=1}^{r} B_j$. In this context, a (big) adjoint ring is dlt (klt) if and only if

$$B(\lambda) = \sum_{j=1}^{r} b_j(\lambda)B_j$$

where all $0 \leq b_j(\lambda) \leq 1$ ($< 1$) for all $\lambda \in \Lambda, i = j, \ldots, r$. In this paper we never need the definitions, results and techniques of the general theory of singularities of pairs.

1.2 The main result

Theorem 1.7 (Theorem A). [BCHM09] Let $X$ be nonsingular projective, and $R = R(X; D)$ a dlt big adjoint ring on $X$. Assume, in addition, that $D: \Lambda \to \text{Div}_{\mathbb{Q}}^+X$, that is, $D(\lambda) \geq 0$ for all $\lambda \in \Lambda$. Then $R = R(X; D)$ is finitely generated.
Remark 1.8. The additional assumption can be removed. The statement is written here in the form that best suits the logic of proof described below in section 1.3. Once theorem A, and theorems B and C of section 1.3, are proved, then the additional assumption is removed by a straightforward application of theorem C.

Corollary 1.9. If $X$ is nonsingular projective of general type, then the canonical ring $R(X, K_X)$ is finitely generated.

Remark 1.10. In fact, by work of Fujino and Mori [FM00], the results here imply the stronger statement that if $X$ is nonsingular, then the canonical ring of $X$ is finitely generated. I don’t know if theorem 1.7 can similarly be strengthened: it would be extremely useful—see [CL10]—if it could.

Remark 1.11. • This theorem is proved in [BCHM09]. That proof uses all that is known about the minimal model program; in particular, I mention [Kol92, Sho03, HM06, Cor07, BCHM09, HM09].

• The proof by Lazić is a self-contained induction on dimension based on lifting lemmas [Siu98], [Cor07, Chapter 5], [HM09], etcetera. On the other hand it is shown in [CL10] that theorem A readily implies all the fundamental theorems of Mori theory.

1.3 The logic of the proof

In [Laz09] theorem A is proved by a bootstrap induction on dimension together with two other theorems that I state shortly following some preparations.

Asymptotic fixed part

I summarise some facts on asymptotic invariants of divisors, mostly following [ELM+06].

For a projective normal variety $X$, I denote by $\text{Eff}(X; \mathbb{R}) \subset N^1(X; \mathbb{R})$ the cone of (numerical equivalence classes of) effective divisors with coefficients in $\mathbb{R}$, and by $\overline{\text{Eff}}(X; \mathbb{R}) \subset N^1(X; \mathbb{R})$ the cone of pseudo-effective divisors, that is, the closure of the cone of effective divisors (it only makes sense to do this with real coefficients). Similarly, I denote by $\text{Big}(X; \mathbb{R}) \subset N^1(X; \mathbb{R})$ the cone of big divisors; $\text{Big}(X; \mathbb{R})$ is the interior of $\overline{\text{Eff}}(X; \mathbb{R})$. 
If $D \in \text{Div}_\mathbb{Q} X$ is a $\mathbb{Q}$-divisor, then the stable base locus of $D$ is the subset

$$B(D) = \bigcap_{0 < p \in \mathbb{Z}, pD \in \text{Div}_\mathbb{Z} X} \text{Bs}|pD| \subset X$$

(if $|pD| = \emptyset$ for all $0 < p \in \mathbb{Z}$, then I say by convention that $B(D) = X$).

If $D \in \text{Div}_\mathbb{Q} X$ is a big $\mathbb{Q}$-divisor, then the asymptotic fixed part of $D$ is

$$F(D) = \inf_{0 < n \in \mathbb{Z}, nD \in \text{Div}_\mathbb{Z} X} \frac{1}{n} \text{Fix} nD \in \text{Div}_\mathbb{R}^+ X.$$  

It is clear that $F(-)$ is a degree 1 homogeneous convex function of the numerical equivalence class of $D$; thus, it extends to a degree 1 homogeneous convex function:

$$F: \text{Big}(X; \mathbb{R}) \to \text{Div}_\mathbb{R}^+ X.$$  

In his book [Nak04], Nakayama defines, as follows, a canonical extension of this function to the closure $\overline{\text{Big}}(X; \mathbb{R}) = \overline{\text{Eff}}(X; \mathbb{R})$:

$$F(D) = \lim_{\varepsilon \to 0^+} F(D + \varepsilon A) \quad \text{for} \quad D \in \overline{\text{Eff}}(X; \mathbb{R})$$

where $A$ is an ample divisor (the definition is independent of the choice of $A$). A subtle point is that $F(-)$ is continuous on $\text{Big}(X; \mathbb{R})$ (because it is convex), but not necessarily on $\overline{\text{Eff}}(X; \mathbb{R})$.

The paper [ELM+06] promotes the view that certain asymptotic invariants defined on $\overline{\text{Eff}}(X; \mathbb{R})$, for instance the asymptotic fixed part, are reasonably well-behaved under surprisingly general conditions. The assertions that follow demonstrate how these invariants are much better behaved on the subcone of adjoint divisors.

**General setup**

In what follows, $X$ is a nonsingular projective variety, $A$ an ample $\mathbb{Q}$-divisor on $X$, and $\sum_{j=1}^r B_j$ a snc divisor on $X$. I denote by $\mathbb{R}^r \simeq V \subset \text{Div}_\mathbb{R} X$ the vector subspace spanned by the components $B_j$. I write:

$$\mathcal{L}_V = \{ B \in V \mid K_X + B \text{ is log canonical} \} = \prod_{j=1}^r [0, 1]B_j;$$

$$\mathcal{E}_{V,A} = \{ B \in \mathcal{L}_V \mid K_X + A + B \in \overline{\text{Eff}} X \} \subset \mathcal{L}_V.$$
In addition, if \( S \) is a component of \( \sum_{j=1}^r B_j \), I write:

\[
\mathcal{B}_{V,A}^S = \{ B \in \mathcal{E}_{V,A} \mid S \not\subset B(K_X + A + B) \} \subset \mathcal{E}_{V,A};
\]

\[
\mathcal{B}_{V,A}^{S=1} = \{ B = S + B' \in \mathcal{E}_{V,A} \mid S \not\subset B(K_X + A + S + B') \}.
\]

**Statements**

**Theorem 1.12** (Theorem B). \( \mathcal{B}_{V,A}^{S=1} \) is a rational polytope; moreover:

\[
\mathcal{B}_{V,A}^{S=1} = \{ B = S + B' \in \mathcal{E}_{V,A} \mid \text{mult}_S F(K_X + A + B) = 0 \}.
\]

\[\Box\]

**Theorem 1.13** (Theorem C). \( \mathcal{E}_{V,A} \) is a rational polytope. More precisely, \( \mathcal{E}_{V,A} \) is the convex hull of finitely many rational vectors \( K_X + A + B_i \) where, for all \( i \): \( B_i \geq 0 \) is a \( \mathbb{Q} \)-divisor, and there is a positive integer \( p_i > 0 \) such that \( |p_i(K_X + A + B_i)| \neq \emptyset \).

\[\Box\]

**Theorem 1.14** (Theorem \( B^+ \)). \( \mathcal{B}_{V,A}^S \) is a rational polytope. Furthermore one can say the following. There is an integer \( r > 0 \) such that:

- Suppose that \( B \in \mathcal{L}_V \) and no component of \( B \) is in \( B(K_X + A + B) \). If \( p(K_X + A + B) \) is an integral divisor, then no component of \( B \) is in \( \text{Fix}(rp(K_X + A + B)) \);
- Suppose that \( B \in \mathcal{E}_{V,A} \). If \( p(K_X + A + B) \) is an integral divisor, then \( |rp(K_X + A + B)| \neq \emptyset \).

**The logic of the proof**

Here is a table of logical dependencies in Lazić’s paper where, e.g., \( A_n \) signifies ‘Theorem A in dimension \( n \):’

| \( A_n \) + \( C_n \) \( \Rightarrow \) \( B_n^+ \) | \( A_{n-1} \) + \( C_{n-1} \) \( \Rightarrow \) \( B_n \) | \( A_{n-1} \) + \( C_{n-1} \) + \( B_n \) \( \Rightarrow \) \( C_n \) | \( B_{n-1}^+ \) + \( B_n \) \( \Rightarrow \) \( A_n \) |
|---|---|---|---|
| (1) | (2) | (3) | (4) |

In this note, I outline all the key steps and ideas in the proof of the last implication \( B_{n-1}^+ + B_n \Rightarrow A_n \), stopping somewhat short of a complete proof. The other implications have similar and easier proofs. For a direct analytic proof of a weaker form of theorem C, see also [Pau08].
1.4 The key ideas of the proof

The proof is based on two key ideas that I explain in a bit more detail below and then fully in the rest of the note. The first is to prove finite generation of strictly dlt rings (see below) by restriction to a boundary divisor using lifting lemmas and induction on dimension. The second idea is what I call below the “main construction.” Starting with a klt big adjoint ring with characteristic system $D : C \to \text{Div}_R X$, I inflate the cone $C$ and semigroup $\Lambda$ to a larger cone $C'$ and semigroup $\Lambda'$, and then “chop” into finitely many smaller $C_j$ and $\Lambda_j$ such that the rings $R(X; \Lambda_j)$ are strictly dlt. This is a version of constructions that are ubiquitous in the proofs of all the fundamental theorems of Mori theory. Finite generation of $R(X; \Lambda)$ follows easily from finite generation of the $R(X; \Lambda_j)$.

Restriction of strictly dlt rings

**Definition 1.15.** I say that a dlt big adjoint ring $R(X; D)$ with characteristic system

$$D(\lambda) = r(\lambda)(K_X + \Delta(\lambda)), \quad \text{where} \quad \Delta(\lambda) = A + B(\lambda)$$

is strictly dlt if there is a prime divisor $S$ that appears in all $B(\lambda)$ with multiplicity one:

$$B(\lambda) = S + B'(\lambda).$$

I say that $R(X; D)$ is plt if it is strictly dlt and all $(X, S + B(\lambda))$ are plt.

When $R(X; D)$ is a strictly dlt adjoint ring, it is natural to want to study the restriction homomorphisms:

$$\rho_\lambda : H^0(X; r(\lambda)(K_X + A + S + B(\lambda))) \to H^0(S; r(\lambda)(K_S + \Omega(\lambda))).$$

where $\Omega(\lambda) = (A + B(\lambda))|_S$. The $\rho_\lambda$ are not surjective, but, with small additional assumptions, lifting lemmas give us a good control on the images.

**Theorem 1.16.** Assume theorems $B_{n-1}^+$ and $B_n$; let $\dim X = n$ and let $R(X; D)$ be a strictly dlt big adjoint ring on $X$. The restricted ring:

$$R_S(X; D) = \sum_{\lambda \in \Lambda} \text{Image}(\rho_\lambda) \subset R(S; D(\lambda)|_S)$$

is a klt big adjoint ring.
Remark 1.17. In fact, $R_s(X; D)$ is a klt big adjoint ring not on $S$, but on some birational model $T \to S$. This technicality is relevant in the proof of the theorem, but it is otherwise unimportant.

The techniques of Lazić’s proof of theorem 1.16 are subtle but generally well understood by the experts. I sketch the key ideas in section 5 below.

The main construction

In section 3, I give a complete proof that theorem 1.16 implies theorem A. Roughly speaking, here is the outline: I want to show that a given dlt adjoint ring $R(X; \Lambda)$, where $\Lambda = \mathcal{C} \cap \mathbb{Z}^r \subset \mathbb{R}^r$, satisfying the additional assumption of theorem A, is finitely generated.

First, I inflate $\mathcal{C}$ to a larger cone $\mathcal{C} \subset \mathcal{C}' \subset \mathbb{R}^r$ and extend $D: \Lambda \to \text{Div}_Q^+ X$ to an appropriate $D': \Lambda' = \mathcal{C}' \cap \mathbb{Z}^r \to \text{Div}_Q^+ X$.

Next, I construct a decomposition into subcones:

$$\mathcal{C}' = \bigcup_{j=1}^r \mathcal{C}_j$$

such that, for all $j = 1, \ldots, r$, writing:

$$D_j = D'_{|\Lambda_j}: \Lambda_j = \mathcal{C}_j \cap \mathbb{Z}^r \to \text{Div}_Q^+,$$

the ring:

$$R_j = R(X; \Lambda_j)$$

is strictly dlt with

$$D_j(\lambda) = r_j(\lambda)(B_j + B'_j(\lambda)).$$

Finally, each of the $R_j$ has a surjective restriction homomorphism to a restricted ring:

$$\rho_j: R_j(X; \Lambda_j) \to R_{B_j}(X; \Lambda_j),$$

and a relatively straightforward argument shows, assuming—as I may by theorem 1.16 and induction on dimension—that the restricted rings $R_{B_j}(X; \Lambda_j)$ are finitely generated, that the ring $R(X; \Lambda')$ also is finitely generated, and then ultimately so is the ring $R(X; \Lambda)$.

The construction is explained in detail in section 3.

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2 Natural operations with divisorial rings

I briefly discuss the behaviour of divisorial and adjoint rings under natural operations. These properties are elementary and mostly well-known [AH06, ADHL10].

2.1 Veronese subrings

Definition 2.1. If $R = \oplus_{\lambda \in \Lambda} R_\lambda$ is a $\Lambda$-graded ring (e.g., $R$ could be a divisorial ring), $L \subset \mathbb{Z}^r$ is a finite index subgroup and $\Lambda' = \Lambda \cap L$, then I say that

$$R' = \bigoplus_{\lambda \in \Lambda'} R_\lambda \subset R$$

is a Veronese subring.

Remark 2.2. If $R' \subset R$ is a Veronese subring, then $R$ is finitely generated if and only if $R'$ is. Indeed, $R' \subset R$ is the ring of invariants under the action of the finite group $G = \mathbb{Z}^r / L$, so the statement is a special case of a well-known theorem of E. Noether.

2.2 Inflating

A more general version of the next lemma can be found in [ADHL10, Proposition 1.1.6].

Lemma 2.3. Consider an inclusion of finite rational cones:

$$C \subset C' \subset \mathbb{R}^r;$$

and write $\Lambda' = C' \cap \mathbb{Z}^r$, $\Lambda = C \cap \mathbb{Z}^r$. Let $R' = \oplus_{\lambda \in \Lambda'} R'_{\lambda}$ be a $\Lambda'$-graded ring, and write

$$R = \bigoplus_{\lambda \in \Lambda} R'_{\lambda}.$$ 

If $R'$ is finitely generated, then so is $R$.

Proof. This is elementary and well-known, so I only give a very brief sketch of the proof. The cone $C \subset C'$ is cut out by finitely many rational hyperplanes; working one hyperplane at a time, I may assume that

$$C = \{ w \in C' \mid f(w) \geq 0 \},$$

for a linear map $f: \mathbb{R}^r \to \mathbb{R}$.
with \( f(\mathbb{Z}^r) \subset \mathbb{Z} \). Now the \( \Lambda' \)-grading on \( R' \) means that \( R' \) has a \( T = \mathbb{C}^\times \)-action, and \( f: \mathbb{Z}^r \to \mathbb{Z} \) corresponds to a 1-parameter \( \mathbb{C}^\times \to T \), in turn endowing \( R' \) with a \( \mathbb{Z} \)-grading, and then, tautologically:

\[
R = R'_+ = \bigoplus_{n \geq 0} R'_n.
\]

Now \( R'_+ \subset R' \) is finitely generated, because it is the subring of invariants for the action of the reductive group \( \mathbb{C}^\times \) on

\[
R'[z] = \bigoplus_{n \geq 0} z^n R' = \bigoplus_{n \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}} z^n R'_m
\]

acting on \( z^n R'_m \) with weight \(-n + m\).

2.3 Injective characteristic systems

In general, the characteristic system \( D: \Lambda \to \text{Div}_{\mathbb{Q}} X \) of a divisorial ring is not injective.

**Lemma 2.4.** Consider a characteristic system \( D: \Lambda \to \text{Div}_{\mathbb{Q}} X \) where \( \Lambda = \mathcal{C} \cap \mathbb{Z}^r \) for a finite rational cone \( \mathcal{C} \subset \mathbb{R}^r \). Assume that \( D \) is the restriction of a rational linear function, still denoted \( D: \mathbb{R}^r \to \text{Div}_{\mathbb{R}} X \). Write \( \overline{\mathcal{C}} = D(\mathcal{C}) \subset \text{Div}_{\mathbb{Q}} X \), the image of \( \mathcal{C} \) under \( D \), and \( \overline{\Lambda} = \overline{\mathcal{C}} \cap \text{Div}_{\mathbb{Z}} X \subset \text{Div}_{\mathbb{Z}} X \). Then \( R(X; \Lambda) \) is finitely generated if and only if \( R(X; \overline{\Lambda}) \) is finitely generated.

**Proof.** A simple application of all the above.

2.4 \( \mathbb{Q} \)-linear equivalence

**Definition 2.5.** Let \( X \) be a projective normal variety. Denote by \( \text{Div}_{\mathbb{Q}}^0 X \subset \text{Div}_{\mathbb{R}} X \) the subgroup of divisors that are \( \mathbb{R} \)-linearly equivalent to 0. Two characteristic systems on \( X \):

\[
D: \Lambda \to \text{Div}_{\mathbb{Q}} X \quad \text{and} \quad D': \Lambda \to \text{Div}_{\mathbb{Q}} X
\]

are \( \mathbb{Q} \)-linearly equivalent if there exists an additive map \( \text{div}: \Lambda \to \text{Div}_{\mathbb{Q}}^0 X \) such that

\[
D(\lambda) = D'(\lambda) + \text{div}(\lambda) \quad \text{for all} \ \lambda \in \Lambda.
\]
Remark 2.6. If $D$ and $D'$ are $\mathbb{Q}$-linearly equivalent, then $R(X; D)$ and $R(X; D')$ have isomorphic Veronese subrings. In particular, one is finitely generated if and only if the other is.

In some arguments, I use this device to replace the ample $\mathbb{Q}$-divisor $A$ by a $\mathbb{Q}$-linearly equivalent $\mathbb{Q}$-divisor $A'$ such that $A'$ is “general,” in the sense that $A' \geq 0$, the coefficients of $A'$ are as small as I care for them to be, and $A'$ meets every divisor and locally closed locus in sight as generically as possible.

Lemma 2.7. Let $X$ be nonsingular projective and $R(X; D)$ a big adjoint ring on $X$.

- If $R(X; D)$ is dlt, then there exists a $\mathbb{Q}$-linearly equivalent system $D'$ such that $R(X; D')$ is a klt big adjoint ring.

- If $R(X; D)$ is strictly dlt, then there exists a $\mathbb{Q}$-linearly equivalent system $D'$ such that $R(X; D')$ is a plt big adjoint ring.

Sketch of Proof. The idea is, of course, to “absorb” into $A$ a small amount of $B(\lambda)$ where it has coefficient 1. I briefly discuss a very special case that illustrates the key issue. Assume that $\Lambda = N^2 = Ne_1 + Ne_2$ and

$$D(e_1) = K_X + A + S_1,$$
$$D(e_2) = K_X + A + S_2$$

where $S_1$, $S_2$ are smooth and meet transversally. The ring $R(X; D)$ is dlt.

For $N \gg 0$ we can write:

$$NA \sim S_1 + T_2 \sim S_2 + T_1$$

where $S_1 + S_2 + T_1 + T_2$ is an snc divisor. Choose a rational function $\varphi \in k(X)$ such that $-S_1 + T_1 = -S_2 + T_2 + \text{div}_X \varphi$. For $0 \ll \varepsilon \ll 1$ we have:

$$D = -\frac{\varepsilon}{N} S_1 + \frac{\varepsilon}{N} T_1 = -\frac{\varepsilon}{N} S_2 + \frac{\varepsilon}{N} T_2 + \frac{\varepsilon}{N} \text{div}_X (\varphi).$$

Then $A' = A - D$ is ample and, setting $B_i = (\varepsilon/N)T_i$:

$$K_X + A + S_1 = K_X + A' + S_1 + D = K_X + A' + (1 - \frac{\varepsilon}{N}) S_1 + B_1$$
$$K_X + A + S_2 = K_X + A' + (1 - \frac{\varepsilon}{N}) S_2 + B_2 + \frac{\varepsilon}{N} \text{div}_X (\varphi).$$
Next, define a new characteristic system $D'$: $\mathbb{N}^2 \to \text{Div}_\mathbb{Q} X$ by

\[
D'(e_1) = K_X + A' + \left(1 - \frac{\varepsilon}{N}\right)S_1 + B_1 \\
D'(e_2) = K_X + A' + \left(1 - \frac{\varepsilon}{N}\right)S_2 + B_2
\]

By construction, $D'$ is $\mathbb{Q}$-linearly equivalent to $D$ and the ring $R(X; D')$ is klt, which proves the first part of the statement in this case. \(\square\)

2.5 Proper birational morphisms

Adjoint rings behave well under proper birational morphisms; when working with the restriction of strictly dlt rings, it is useful to blow up $X$ to simplify singularities in order to satisfy the assumptions of the lifting lemma.

**Lemma 2.8.** Let $X$ be nonsingular projective and $R(X; D)$ a plt big adjoint ring on $X$:

\[
D(\lambda) = r(\lambda)(K_X + \Delta(\lambda)) \quad \text{where} \quad \Delta(\lambda) = A + S + B(\lambda).
\]

Let $B \subset X$ be a snc divisor such that all $\text{Supp} B(\lambda) \subset B$. There is a proper birational morphism $f: Y \to X$, and a plt big adjoint ring $R(Y; D')$:

\[
D'(\lambda) = r(\lambda)(K_Y + \Delta'(\lambda)) \quad \text{where} \quad \Delta'(\lambda) = A' + T + B'(\lambda)
\]

with $T \subset Y$ the proper transform of $S$, with the following properties:

- The $f$-exceptional set $E$ is a divisor. Also, denoting by $B' \subset Y$ the proper transform of $B \subset X$, $B' \cup E$ is a snc divisor and all $\text{Supp} B'(\lambda) \subset B' \cup E$;

- for all $\lambda \in \Lambda$,

\[
K_Y + T + A' + B'(\lambda) = f^*\left(K_X + S + A + B(\lambda)\right) + E(\lambda);\]

where $E(\lambda) \geq 0$ is $f$-exceptional. This implies that $R(X; D) = R(Y; D')$, and;

- for all $\lambda \in \Lambda$, the pair $(T, B'(\lambda)|_T)$ is terminal.
Proof in a special case. I prove the statement in a special case that contains all the ideas: assume that $\Lambda = \mathbb{N}$, that is:

$$R(X; D) = \bigoplus_{n \geq 0} H^0(X; n(K_X + S + A + B))$$

where $(X, S + B)$ is a plt pair.

For $f: Y \to X$ a proper birational morphism with exceptional divisors $E_i$, I write

$$K_Y + T + f_*^{-1}B = f^*(K_X + S + B) + \sum a_iE_i$$

with all $a_i > -1$.

Here $a_i = a(E_i; K_X + S + B)$ is the discrepancy along $E_i$ of the divisor $K_X + S + B$: it only depends on the geometric valuation $\nu = \nu(E_i)$ measuring order of vanishing along $E_i$. Next, setting $B_Y = f_*^{-1}B - \sum_{a_i < 0} a_iE_i$, I get:

$$K_Y + T + B_Y = f^*(K_X + S + B) + E$$

where $f_*B_Y = B$ and $E = \sum_{a_i \geq 0} a_iE_i \geq 0$ is exceptional.

Pick a good resolution $f: \tilde{Y} \to X$ with the property that all geometric valuations $\nu$ with $a(\nu; K_X + S + B) < 0$ are divisors on $Y$ (the set of these valuations is finite hence such a resolution exists); it is a simple matter to check that the pair $(T, B_Y|_T)$ is terminal.

Finally, choose an ample $\mathbb{Q}$-divisor

$$A' = f^*A - \sum \varepsilon_i E_i$$

on $Y$, where $0 < \varepsilon_i \ll 1$. Setting $B' = B_Y + \sum \varepsilon_i E_i$, it is still true that $(T, B'|_T)$ is terminal, and:

$$K_Y + T + A' + B' = f^*(K_X + S + A + B) + E.$$  

□

3 The main construction

In this section I give a complete proof of theorem A assuming theorem 1.16.
Lemma 3.1. Let $X$ be a nonsingular projective variety, $\sum_{i=1}^{r} B_i$ a snc divisor on $X$, and $B = \sum_{i=1}^{r} b_i B_i$ a klt divisor (that is, $0 \leq b_i < 1$ for $i = 1, \ldots, r$).

Let $A$ be an ample $\mathbb{Q}$-divisor on $X$ and assume that for some integer $p > 0$, $\lvert p(K_X + A + B) \rvert \neq 0$.

Consider the parallelepiped:

$$B = \prod_{i=1}^{r} [b_i, 1]B_i \subset \text{Div}_\mathbb{R} X,$$

and the cone and monoid:

$$C = \mathbb{R}_+(K_X + A + B) \subset \text{Div}_\mathbb{R} X; \quad \Lambda = C \cap \text{Div}_\mathbb{Z} X.$$

Then, assuming theorem 1.16, the ring $R(X; \Lambda)$ is finitely generated.

Proof. In the course of the proof, I work in the vector subspace $V = \mathbb{R}^r \subset \text{Div}_\mathbb{R} X$ spanned by the prime divisors $B_i$; I denote by $D: V \rightarrow \text{Div}_\mathbb{R} X$ the canonical inclusion. By suitably enlarging (lemma 2.3) the set $\{B_i\}$, and an appropriate choice of the canonical divisor (lemma 2.7), I can assume that

$$K_X + A + \sum_{i=1}^{r} b_i B_i = \sum_{i=1}^{r} p_i B_i,$$

where all $p_i \geq 0$. In addition, perhaps by blowing up $X$ and using lemma 2.8, I can assume that, even after enlargement, the divisor $\sum_{i=1}^{r} B_i$ is still snc.

The key is to “chop up” $R = R(X; \Lambda)$ into finitely many strictly dlt subrings. Consider the $r$ ‘back faces’ of the parallelepiped $B$:

$$B_j = \{B = B_j + \sum_{i \neq j} c_i B_i \mid \text{all } b_i \leq c_i \leq 1\} \quad \text{for } j = 1, \ldots, r.$$

It is clear that

$$C = \bigcup_{j=1}^{r} C_j, \quad \text{where} \quad C_j = \mathbb{R}_+(K_X + A + B_j)$$

(this uses in a crucial way that $C \subset \sum_{i=1}^{r} \mathbb{R}_+ B_i$) hence, setting $\Lambda_j = C_j \cap \mathbb{Z}^r$, $R = \sum_{j=1}^{r} R_j$ where each $R_j = R(X; \Lambda_j)$ is a strictly dlt adjoint ring. For $j = 1, \ldots, r$, denote by

$$\rho_j: R_j \rightarrow R_{B_j} = R_{B_j}(X; \Lambda_j)$$
the surjective ring homomorphisms to the restricted rings. By theorem 1.16 and by induction on dimension, the \( R_{B_j} \) are finitely generated.

I show that \( R \) is finitely generated. I can’t prove this directly, so let \( \sigma_j \in H^0(X; B_j) \) be a section vanishing on \( B_j \) (\( \sigma_j \) is determined up to multiplication by a nonzero constant); I show instead that the ring

\[
R[\sigma_1, \ldots, \sigma_r], \quad \text{graded by } \mathbb{N}^r,
\]

is finitely generated. By lemma 2.3 again, this implies that \( R \) itself is finitely generated. Fix the total degree function

\[
\tau: \mathbb{N}^r \to \mathbb{N}, \quad \tau(m_1, \ldots, m_r) = \sum_{j=1}^r m_j.
\]

Let \( N \gg 0 \) be large enough that the following holds:

If \( m = (m_1, \ldots, m_r) \in C_j \cap \mathbb{Z}^r \) and \( \tau(m) > N \),

then \( m - B_j = (m_1, \ldots, m_j - 1, \ldots, m_r) \in C \cap \mathbb{Z}^r. \) (5)

(It is pretty obvious that you can find \( N \gg 0 \) with this property; draw a picture!) Prepare now the following finite sets:

- A basis \( G_0 \) of \( \oplus_{\tau(\lambda) \leq N} R_\lambda \);
- For all \( j = 1, \ldots, r \), a set \( G_j \subset R_j \subset R \) lifting a set of generators of \( R_{B_j} \).

I conclude the argument by showing that, using equation 3, the union \( G = \bigcup_{j=0}^r G_j \cup \{\sigma_0, \ldots, \sigma_r\} \) generates the ring \( R[\sigma_1, \ldots, \sigma_r] \). It is enough to show that \( R \subset \mathbb{C}[G] \).

Assume by induction that \( M \geq N \) and: for \( \tau(\lambda) \leq M \) all \( R_\lambda \subset \mathbb{C}[G] \); let \( \tau(\lambda) = M + 1 \), and \( x \in R_\lambda \). Now, for some \( j = 1, \ldots, r \), \( \lambda \in C_j \), so consider the restriction homomorphism:

\[
\rho_j: R_j \to R_{B_j}.
\]

It is clear that there is \( x_j \in \mathbb{C}[G_j] \) such that \( \rho_j(x - x_j) = 0 \). This means that

\[
x - x_j = \sigma_j y
\]

where \( y \in H^0(X; D(\lambda - B_j)) \). By property 3 \( \lambda - B_j \in C \), therefore

\[
y \in R_{\lambda-B_j} \quad \text{has total degree } \tau(y) = \tau(x) - 1,
\]

hence, by induction, \( y \in \mathbb{C}[G] \), and hence also \( x \in \mathbb{C}[G] \). \( \square \)
Proof of theorem A. I need to show that a big klt adjoint ring \( R = R(X; D) \), where \( D(C) \subset \text{Div}_X^+ \), is finitely generated. By a simple application of lemma \(^2.3\) I may assume that \( D: (C \subset \mathbb{R}^n) \rightarrow \text{Div}_X^+ \) is rational linear. By lemma \(^2.4\) I may also assume that \( D \) is injective. As before for \( v \in C \) I write

\[
D(v) = r(v)(K_X + A + B(v)) = r(v) \left( K_X + A + \sum_{i=1}^{r} b_i(v)B_i \right).
\]

Let \( e_l \) for \( l = 1, \ldots, m \) be generators of the cone \( C \). I will shortly need the quantity

\[
\delta = \min_{i=1, \ldots, r} \min_{v \in C} \{1 - b_i(v)\} = \min_{i=1, \ldots, r} \min_{l=1, \ldots, m} \{1 - b_i(e_l)\} > 0.
\]

For a rational vector \( v \in \mathbb{Q} \) lying on the hyperplane \( \Pi = \{v \mid r(v) = 1\} \), consider the parallelepiped \( B(v) = \prod [b_i(v), 1] \subset V \) and the cone \( C(v) = \mathbb{R}_+(K_X + A + B(v)) \). Because all sides of all parallelepipeds \( B(v) \) have length \( \geq \delta \), there are finitely many vectors \( v_1, \ldots, v_n \in C \cap \Pi \) such that \( \mathcal{C} \subset \cup_{k=1}^{n} C(v_k) \). Write \( \Lambda(v) = \mathcal{C}(v) \cap \mathbb{Z}^r \). Now \( R \subset \sum_{k=1}^{n} R(X; \Lambda(v_k)) \) is finitely generated by lemma \(^2.3\) and lemma \(^3.1\). \( \square \)

4 Lifting lemmas

A quick internet search will turn up several papers on lifting lemmas. The prototype can be traced back to \cite{Siu98}; the best place to start learning the material is \cite[Theorem 11.5.1]{Laz04}; the lifting theorem \(^4.2\), theorem \(^4.5\) and corollary \(^4.7\) are all due to Hacon and M\textsuperscript{c}Kernan.

4.1 General initial set-up

All variants and improvements of the lifting lemma have a common initial set-up that I now summarise:

- \( X \) is nonsingular projective; \( S \subset X \) is a nonsingular divisor;
- \( (X, \Delta = S + A + B) \) is a plt pair; here \( A \) is an ample \( \mathbb{Q} \)-divisor; I always assume that \( A \) meets transversally everything in sight, and I sometimes assume that the coefficients of \( A \) are sufficiently small;
- Write \( \Omega := (A + B)|_S \); I assume that the pair \( (S, \Omega) \) is terminal.
The purpose of the lifting lemma is always this: Fix a strictly positive integer $p$ such that $p\Delta \in \text{Div}_Z X$, then study the restricted adjoint linear system:

$$|p(K_X + \Delta)|_S.$$ 

Now, of course, $p(K_X + \Delta)|_S = p(K_S + \Omega)$, and, in general, I don’t expect the restricted linear system to be the complete linear system $|p(K_S + \Omega)|_S$. Indeed, simple examples show that, for basic reasons, the restricted linear system can have a fixed part. The key point of the lifting lemma is to choose divisors $\Theta, \Phi$ on $S$ with

$$0 \leq \Theta \leq \Omega \quad \text{and} \quad \Theta + \Phi = \Omega$$

and compare the restricted linear system $|p(K_X + \Delta)|_S$ with the linear system with fixed part $|p(K_S + \Theta)|_S + \Phi$.

**Notation 4.1.** Let $E_i$ be prime divisors on $X$. For divisors

$$D_1 = \sum d_1^i E_i, \quad D_2 = \sum d_2^i E_i,$$

I write

$$D_1 \wedge D_2 = \sum \min\{d_1^i, d_2^i\} E_i.$$

### 4.2 Simple lifting

This is the simplest statement that one can make:

**Theorem 4.2.** Fix an integer $p > 0$ such that $p\Delta$ is integral. Assume that $S \not\subset B(K_X + \Delta)$.

Write

$$F_p = \frac{1}{p} \text{Fix} |p(K_X + \Delta)|_S; \quad \Phi_p = \Omega \wedge F_p; \quad \Theta_p = \Omega - \Phi_p.$$

Then

$$|p(K_X + \Delta)|_S = |p(K_S + \Theta_p)|_S + p\Phi_p.$$

$\square$
4.3 Sharp lifting

Next I state a subtle but crucial improvement of the lifting lemma.

**Definition 4.3.** Let $X$ be normal projective, $S \subset X$ a codimension 1 subvariety, and $D$ a $\mathbb{Q}$-Cartier divisor on $X$. If $S \not\subset B(D)$, then the *restricted asymptotic fixed part* is

$$F_S(D) = \inf_{0 < n \in \mathbb{Z}, nD \in \text{Div}_{\mathbb{R}} X} \frac{1}{n} \text{Fix}(|nD|_S) \in \text{Div}_{\mathbb{K}}^+ S.$$ 

It is important to appreciate that, in general, even though $D$ is a divisor with rational coefficients, $F_S(D)$ can have nonrational real coefficients.

**Remark 4.4.** It is crucial to be aware that $F_S(D)$, $F(D)|_S$, and $F(D|_S)$ are three distinct divisors in general.

**Theorem 4.5.** Fix an integer $p > 0$ such that $p\Delta$ is integral. Assume that

$$S \not\subset B(K_X + \Delta + A/p).$$

(Note that this holds in particular if $S \not\subset B(K_X + \Delta + \epsilon A)$ for some rational $0 \leq \epsilon \leq 1/p$.)

Write $F_S = F_S(K_X + \Delta + A/p)$. Consider a $\mathbb{Q}$-divisor $\Phi$ on $S$ such that $p\Phi$ is integral and $\Omega \wedge F_S \leq \Phi \leq \Omega$; write $\Theta = \Omega - \Phi$.

Then

$$|p(K_X + \Delta)|_S \supset |p(K_X + \Theta)| + p\Phi.$$  

\[\square\]

**Remark 4.6.** Sharp lifting improves simple lifting in two ways: it relaxes the assumption and it strengthens the conclusion.

**It relaxes the assumption** Here I just require that

$$S \not\subset B(K_X + \Delta + A/p).$$

**It strengthens the conclusion** The conclusion now holds for $\Omega \wedge F_S \leq \Phi$ whereas earlier I required $\Omega \wedge F_p = \Phi_p$: note that

$$F_S = F_S(K_X + \Delta + A/p) \leq F_S(K_X + \Delta) \leq$$

$$\leq \frac{1}{p} \text{Fix}(|p(K_X + \Delta)|_S) = F_p,$$

hence $\Phi$ is allowed potentially to be smaller than $F_p$. 

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4.4 Tinkering lifting

It is possible still to tinker with the statement of the lifting lemma:

**Corollary 4.7.** Fix an integer $p > 0$ such that $p\Delta$ is integral.
Assume that $S \not\subset B(K_X + \Delta)$. Write $F_S = F_S(K_X + \Delta)$, and fix a rational $\varepsilon > 0$ such that $\varepsilon(K_X + \Delta) + A$ is ample.
Consider a $\mathbb{Q}$-divisor $\Phi$ on $S$ such that

$$p\Phi \text{ is integral and } \Omega \wedge \left(1 - \frac{\varepsilon}{p}\right)F_S \leq \Phi \leq \Omega; \text{ write } \Theta = \Omega - \Phi.$$  

Then

$$|p(K_X + \Delta)|_S \supset |p(K_S + \Theta)| + p\Phi.$$  

**Proof.** Corollary 4.7 follows from theorem 4.3:

$$K_X + \Delta + \frac{1}{p}A = \left(1 - \frac{\varepsilon}{p}\right)(K_X + \Delta) + \frac{1}{p}(\varepsilon(K_X + \Delta) + A),$$

hence:

$$F_S\left(K_X + \Delta + \frac{1}{p}A\right) \leq \left(1 - \frac{\varepsilon}{p}\right)F_S(K_X + \Delta) + \frac{1}{p}F_S(\text{ample}) = \left(1 - \frac{\varepsilon}{p}\right)F_S(K_X + \Delta);$$

thus, the assumptions (those pertaining to $\Phi$) of theorem 4.3 are satisfied, hence its conclusion holds. \qed

5 Restriction of strictly dlt rings

In this final section, I sketch the proof of theorem 1.16. Let me briefly recall the set-up. $X$ is nonsingular projective, $\dim X = n$, and $R(X; D)$ is a strictly dlt big adjoint ring on $X$ with characteristic system:

$$D(\lambda) = r(\lambda)(K_X + \Delta(\lambda)) \text{ where } \Delta(\lambda) = S + A + B(\lambda).$$

The aim, remember, is to show that the restricted ring

$$R_S(X; D) = \bigoplus_{\lambda \in \Lambda} \text{Image}(\rho_\lambda), \text{ where } \rho_\lambda: H^0\left(X; r(\lambda)(K_X + \Delta(\lambda))\right) \to H^0\left(S; r(\lambda)(K_S + \Omega(\lambda))\right)$$

\[20\]
is the restriction map, is a klt adjoint ring.

After some simple manipulations, I may in addition assume the following:

1. All \((S, B(\lambda)|_S)\) are terminal pairs. (This can be achieved by using lemmas 2.7 and 2.8 in tandem.)

2. \(A\) has small coefficients and meets everything in sight as generically as possible; in particular, for instance, I assume that all pairs
\[
\left(S, \Omega(\lambda) = (A + B(\lambda))|_S\right)
\]
are terminal. (See remark 2.6 for this.)

3. For \(\lambda \in \Lambda\), \(S \not\subset B(D(\lambda))\). This can be achieved as an application of theorem \(B_n: \mathcal{C}' = \mathcal{C} \cap \mathbb{R}_+ (B^{\mathbb{R})}_S)\) is a finite rational cone; now work with \(\Lambda' = \Lambda \cap \mathcal{C}'\) and \(R(X; \Lambda')\) in place of \(\Lambda\) and \(R(X, \Lambda)\): the point is that \(R_S(X; \Lambda') = R_S(X; \Lambda)\).

4. Denote by \(e_i \in \mathbb{R}^r\) the standard basis vectors. Then \(\mathcal{C} = \sum_{i=1}^r \mathbb{R}_+ e_i \subset \mathbb{R}^r\) is a simplicial cone, \(\Lambda = \mathbb{N}^r \subset \mathbb{R}^r\), and \(D: \Lambda \to \text{Div}_{\mathbb{Q}} X\) is the restriction of a linear function that, abusing notation, I still denote by \(D: \mathbb{R}^r \to \text{Div}_\mathbb{R} X\). This can be achieved by finding a triangulation of \(\mathcal{C}\) on which \(D\) is linear.

**Notation 5.1.** Below I denote by \(\Pi \subset \mathbb{R}^r\) the affine hyperplane spanned by the basis vectors \(e_i\).

By what I said, \(\Delta, B: \Lambda \to \text{Div}_{\mathbb{Q}}^+ X\) are restrictions of functions that, abusing notation, I still denote by \(\Delta, B: \mathbb{R}^r \to \text{Div}_\mathbb{R} X\). These are degree 0 homogeneous; hence, they are determined by their restrictions to the affine hyperplane \(\Pi \subset \mathbb{R}^r\); note that these restrictions are affine.

Similarly, \(\Omega: \mathbb{R}^r \to \text{Div}_\mathbb{R} S\) is degree 0 homogeneous, and \(\Omega|_\Pi\) is affine.

**Lemma 5.2.** For \(\lambda \in \Lambda\), write \(F_S(\lambda) = F_S(D(\lambda))\) (N.B. by construction if \(Z \ni n > 0\), then \(F_S(n\lambda) = nF_S(\lambda)\)).

Then \(F_S(-)\) can be uniquely extended to a degree 1 homogeneous convex function that, abusing notation, I still denote by
\[
F_S: \mathcal{C} \to \text{Div}_\mathbb{R}^+ S.
\]

\(F_S\) is continuous on the interior \(\text{Int} \mathcal{C}\) but not necessarily on \(\mathcal{C}\).
**Proof.** By homogeneity I extend to $F_S : C \cap \mathbb{Q}^r \to \text{Div}^+_R S$; this function is homogeneous convex hence locally Lipschitz hence locally uniformly continuous hence it can uniquely be extended to a function on $C$ continuous on $\text{Int} C$.

After some further blowing up, I may in addition assume:

5. there is a fixed snc divisor $F$ on $S$ such that, for all $\lambda \in \Lambda$, $\text{Supp} F_S(\lambda) \subset F$.

Now for $w \in C$ write:

$\Omega(w) = (A + B(w))|_S$; $\Phi(w) = \Omega(w) \wedge F_S(w)$; $\Theta(w) = \Omega(w) - \Phi(w)$.

By construction, for all $w \in C$, $0 \leq \Theta(w) \leq \Omega(w)$ and $\Theta(w) + \Phi(w) = \Omega(w)$.

**Lemma 5.3.** For all $\lambda \in \Lambda$, there is $Z \ni n = n(\lambda) > 0$ such that

$$\Phi(\lambda) = \Omega(\lambda) \wedge \frac{1}{n} \text{Fix}(|D(n\lambda)|_S).$$

In particular, for all $\lambda \in \Lambda$, $\Theta(\lambda) \in \text{Div}^+_Q S$ is a rational divisor (and so is $\Phi(\lambda)$).

**Proof.** The proof is explained very well in [HM09, Theorem 7.1]; it is an application of tinkering lifting; it is simpler than and based on the same idea of the proof of lemma 5.4 below.

The proof of theorem 1.16 follows easily if I show that

$$\Theta|_{\Pi \cap C} : \Pi \cap C \to \text{Div}^+_R S$$

is **piecewise affine**. (Indeed, write $D_S(\lambda) = r(\lambda)(K_S + \Theta(\lambda))$. By lemma 5.3 and sharp lifting, the restricted ring $R_S(X; D)$ and the adjoint ring $R(S, D_S)$ have a common Veronese subring.) I don’t prove the statement completely. Instead, in the remaining part of this section, I prove:

**Lemma 5.4.** Let $x \in \Pi(\mathbb{R})$ and assume that the smallest rationally defined affine subspace $U \subset \mathbb{R}^r$ containing $x$ is $\Pi$. Then, $\Theta|_{\Pi \cap C}$ is affine in a neighbourhood of $x$. 

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This is compelling, but note that it stops short of proving theorem 1.16; the statement implies that there is a decomposition of $C$ in rational subcones such that $\Theta$ is affine on each subcone, but there is no guarantee that the decomposition is locally finite, nor indeed that the subcones themselves are finite. The proof of the lemma contains all the ideas of Lazić’s proof of theorem 1.16.

**Lemma 5.5** (Diophantine approximation). Let $x \in \mathbb{R}^n$; denote by $U$ the smallest rationally defined affine subspace containing $x$, and let $\dim U = m - 1$.

Fix $\varepsilon > 0$ and an integer $M > 0$. There exist vectors $w_1, \ldots, w_m \in U(\mathbb{Q})$ with the following properties:

1. For $i = 1, \ldots, m$, there are real numbers $0 < r_i < 1$ with $\sum_{i=1}^{m} r_i = 1$, $x = \sum_{i=1}^{m} r_i w_i$;
2. there is an $m$-tuple $(p_1, \ldots, p_m)$ of strictly positive integers, all $p_i$ divisible by $M$, such that all $p_i w_i \in \mathbb{Z}^n$ are integral, and $\|x - w_i\| < \varepsilon/p_i$.

**Proof of lemma 5.4.** There is a nagging difficulty with the proof:

**A nagging difficulty and an additional assumption**

The point is this: by definition,

$$\Phi(-) = \Omega(-) \lor F_S(-) \quad \text{and} \quad \Theta(-) = \Omega(-) - \Phi(-);$$

hence, although $\Omega: C \to \text{Div}_{\mathbb{R}}^+ S$ is linear, and $F_S: C \to \text{Div}_{\mathbb{R}}^+ S$ is convex, $\Phi(-)$ is not necessarily convex, and $\Theta(-)$ is not necessarily concave. To be more specific, if for some prime divisor $P \subset S$, $\text{mult}_P \Omega(x) = \text{mult}_P F_S(x)$, then $\Theta(-)$ may fail to be concave in a neighbourhood of $x$. I first run the proof under the following additional assumption:

For all prime divisors $P \subset S$, $\text{mult}_P \Omega(x) \neq \text{mult}_P F_S(x)$.

In the proof below, I only use the additional assumption to ensure that $\Theta(-)$ is concave in a neighbourhood of $x$. At the end, I briefly explain how to get around this difficulty.
The strategy of the proof

The idea of the proof is to choose a real $\varepsilon > 0$, an integer $M > 0$ and a rational Diophantine approximation

$$(w_i, \Theta_i) \in \Pi(Q) \times \text{Div}_Q(S)$$

of the vector $(x, \Theta(x)) \in \Pi(R) \times \text{Div}_R(S)$, such that

i. For $i = 1, \ldots, r$, there are real numbers

\[0 < \mu_i < 1\]

with $\sum_{i=1}^{r} \mu_i = 1$, and

\[
\begin{cases}
x = \sum_{i=1}^{r} \mu_i w_i, \\
\Theta(x) = \sum_{i=1}^{r} \mu_i \Theta_i.
\end{cases}
\]

In particular this implies that $\Delta(x) = \sum \mu_i \Delta(w_i)$. (All this is guaranteed by lemma 5.5.) In addition I require that:

ii. $\Theta_i \leq \Theta(w_i)$.

Indeed, once I know this, then, by concavity of $\Theta$:

$\sum \mu_i \Theta(w_i) \leq \Theta(x) = \sum \mu_i \Theta_i$.

I deduce $\Theta(x) = \sum \mu_i \Theta(w_i)$ and, by concavity again, this implies that $\Theta$ is affine on the convex span of the $w_i$.

Choice of $\varepsilon$

I choose $\varepsilon > 0$ small enough that it has the following features:

(a) $\Theta$ is concave in $\{w \mid \|w - x\| < \varepsilon\}$. (This is OK by the additional assumption.)

(b) There is a local Lipschitz constant $C = C_x$ such that:

If $\|w - x\| < \varepsilon$, then $\|\Theta(w) - \Theta(x)\| < C \|w - x\|$.

(All concave functions are locally Lipschitz.)

(c) There is a constant $0 < \delta < 1$ with the following property: For all prime divisors $P \subset S$:

If $\text{mult}_P(\Omega(x) - \Theta(x)) > 0$, and $\|w - x\| < \varepsilon$, then $\text{mult}_P(\Omega(w) - \Theta(w)) > \delta$,

and I also assume that $\varepsilon < \delta$. 

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(d) If $p \geq 1$ and $\|w - x\| < \frac{\varepsilon}{p}$, then the $\mathbb{Q}$-divisor

$$\Delta(w) - \Delta(x) + \frac{A}{p}$$

is ample.

(e) If $|w - x| < \varepsilon$, then the $\mathbb{Q}$-divisor

$$(C + 1)\frac{\varepsilon}{\delta}(K_X + \Delta(w)) + A$$

is ample.

(f) If $\Theta \in \text{Div}_\mathbb{Q} S$ and $\|\Theta - \Theta(x)\| < \varepsilon$, then no component of $\Theta$ is in the asymptotic fixed part

$F(K_S + \Theta)$.

It is a simple consequence of Theorem $B_{n-1}^+$ that I can arrange for this to hold.

**Choice of $M > 0$**

Next I choose $M > 0$ such that the rational Diophantine approximation given by lemma 5.5

$$(w_i, \Theta_i) \in \Pi(\mathbb{Q}) \times \text{Div}_\mathbb{Q}(S)$$

of the vector $$(x, \Theta(x)) \in \Pi(\mathbb{R}) \times \text{Div}_\mathbb{R}(S),$$

satisfies the following **conditions**:

1. For $i = 1, \ldots, r$, there are real numbers

$$0 < \mu_i < 1$$

with $\sum_{i=1}^{r} \mu_i = 1$, and

$$\begin{cases}
x = \sum_{i=1}^{r} \mu_i w_i, \\
\Theta(x) = \sum_{i=1}^{r} \mu_i \Theta_i.
\end{cases}$$

In particular this implies that $\Delta(x) = \sum \mu_i \Delta(w_i)$.

2. There is an $r$-tuple $(p_1, \ldots, p_r) \in \mathbb{N}^r$ of positive integers such that:

- $(p_i w_i; p_i \Theta_i) \in \mathbb{N}^r \times \text{Div}_\mathbb{Z}(S)$ is integral;

- For all $i$, $\|x - w_i\| < \frac{\varepsilon}{p_i}$ and $\|\Theta(x) - \Theta_i\| < \frac{\varepsilon}{p_i}$.
3. For all prime divisors $P \subset S$:

- If $\text{mult}_P \Theta(x) < \text{mult}_P \Omega(x)$, then also $\text{mult}_P \Theta_i < \text{mult}_P \Omega(w_i)$ (this is automatic from feature (c));
- If $\text{mult}_P \Theta(x) = \text{mult}_P \Omega(x)$, then also $\text{mult}_P \Theta_i = \text{mult}_P \Omega(w_i)$.
- If $\text{mult}_P \Theta(x) = 0$, then also $\text{mult}_P \Theta_i = 0$.

(Although the second bullet point doesn’t strictly speaking follow from a blind usage of lemma 5.5, it is easy to arrange for it to hold. Indeed in this case $\text{mult}_P \Theta(x) = \sum \mu_i \text{mult}_P \Omega(w_i)$ and it pays to declare from the start that $\text{mult}_P \Theta_i = \text{mult}_P \Omega(w_i)$.

The third bullet point is similar and easier.)

4. $M$ is large enough that the $p_i$ are large enough and divisible enough that:

$$F(K_S + \Theta_i) = \text{Fix } |p_i(K_S + \Theta_i)|.$$

(This can easily be arranged using theorem $B_{n-1}^+$.)

The key inclusion

For all $i = 1, \ldots, r$ I show the key inclusion:

$$\left| p_i \left( K_X + \Delta(w_i) \right) \right|_S \supset \left| p_i \left( K_S + \Theta_i \right) \right| + p_i \left( \Omega(w_i) - \Theta_i \right). \quad (6)$$

The key inclusion allows me to control the restricted algebra in a neighbourhood of $x$: as I show below, it readily implies that $\Theta_i \leq \Theta(w_i)$. If you get bored with the details of the proof, you may want to press forward to the conclusion.

I plan to prove this using sharp lifting. To begin with, I remark that I am in the general initial set-up of section 4.1.

For all $i$, I now check that the specific assumptions of sharp lifting are satisfied; that is:

$$\Omega(w_i) \land F_i \leq \Omega(w_i) - \Theta_i$$

where $F_i = F_S(K_X + \Delta(w_i) + A/p_i)$. For all prime divisors $P \subset S$ I check that

$$\text{mult}_P(\Omega(w_i) \land F_i) \leq \text{mult}_P(\Omega(w_i) - \Theta_i).$$

The discussion breaks down in two cases:
Case 1: \( \text{mult}_P \Theta(x) = \text{mult}_P \Omega(x) \). By condition 3, \( \text{mult}_P \Theta_i = \text{mult}_P \Omega(w_i) \).

By condition (d), \( \Delta(w_i) - \Delta(x) + A/p_i \) is ample, therefore:

\[
\text{mult}_P FS(K_X + \Delta(w_i) + \frac{A}{p_i}) = \]

\[
= \text{mult}_P FS(K_X + \Delta(x) + \left( \Delta(w_i) - \Delta(x) + \frac{A}{p_i} \right)) \leq \]

\[
\leq \text{mult}_P FS(K_X + \Delta(x)) = 0.
\]

Case 2: \( \text{mult}_P \Theta(x) < \text{mult}_P \Omega(x) \). Using feature (e):

\[
\text{mult}_P F_i = \text{mult}_P FS(K_X + \Delta(w_i) + \frac{A}{p_i}) \leq \]

\[
\leq \left(1 - \frac{(C+1)\varepsilon}{p_i} \right) \text{mult}_P FS(K_X + \Delta(w_i)) = \]

\[
= \left(1 - \frac{(C+1)\varepsilon}{p_i} \right) \text{mult}_P FS(w_i).
\]

This implies that

\[
\text{mult}_P \Omega(w_i) \wedge F_i \leq \left(1 - \frac{(C+1)\varepsilon}{p_i} \right) \text{mult}_P (\Omega(w_i) - \Theta(w_i))
\]

(indeed, by definition \( \Omega(w_i) - \Theta(w_i) = \Omega(w_i) \wedge F_S(w_i) \));

and, finally, using feature (c) and:

\[
\|\Theta(w_i) - \Theta_i\| \leq \|\Theta(w_i) - \Theta(x)\| + \|\Theta(x) - \Theta_i\| \leq \frac{C\varepsilon}{p_i} + \frac{\varepsilon}{p_i} = (C + 1)\frac{\varepsilon}{p_i},
\]

in tandem, I get:

\[
\left(1 - \frac{(C+1)\varepsilon}{p_i} \right) \text{mult}_P (\Omega(w_i) - \Theta(w_i)) \leq \]

\[
\leq \text{mult}_P (\Omega(w_i) - \Theta(w_i)) - \frac{C+1}{p_i}\varepsilon \leq \]

\[
\leq \text{mult}_P (\Omega(w_i) - \Theta(w_i)) + \text{mult}_P (\Theta(w_i) - \Theta_i) = \]

\[
= \text{mult}_P (\Omega(w_i) - \Theta_i).
\]
Conclusion

I show that the key inclusion of equation 6 implies the statement. By construction of the function $\Theta: C \to \text{Div}_{\mathbb{R}}^+ S$ and the lifting lemma, I know that:

$$|p_i(K_X + \Delta(w_i))|_S = |p_i(K_S + \Theta(w_i))| + \Phi(w_i).$$

Thus, the key inclusion readily implies that:

$$\text{Mob} |p_i(K_S + \Theta_i)| \leq \text{Mob} |p_i(K_S + \Theta(w_i))|. \quad (7)$$

Now, by feature (f), no component of $\Theta_i$ is in the fixed part of $|p_i(K_X + \Theta_i)|$; thus, the last equation implies

$$\Theta_i \leq \Theta(w_i).$$

\[\square\]

How to remove the additional assumption

Assume that for some prime $P \subset S \, \text{mult}_P \Omega(x) = \text{mult}_P F_S(x)$. Consider an effective divisor

$$G = \sum_{j=1}^r \varepsilon_j B_j.$$ 

If the coefficients $0 \leq \varepsilon_j$ are small enough, then:

- Writing $B'(\lambda) = B(\lambda) + G > B(\lambda)$, all the $(S, B'(\lambda)|_S)$ are terminal, and:

- $A - G$ is still ample, so I can choose $A' \sim_{\mathbb{Q}} A - G$ meeting everything in sight transversally, and such that, upon setting

$$\Delta'(\lambda) = S + A' + B'(\lambda); \quad \Omega'(\lambda) = (A' + B'(\lambda))|_S,$$

then all the pairs $(S, \Omega'(\lambda))$ are terminal.

Note that, from the definition, for all $\lambda \in \Lambda$:

$$F_S(\lambda) = F_S(D(\lambda)) = F_S(D'(\lambda)).$$

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By choosing $A'$ generically, I can arrange that the additional assumption for $D'$ is satisfied, and conclude as above that $\Theta'(-)$ is rational affine in a neighbourhood of $x$. By construction:

In a neighbourhood of $x$, \[ \text{mult}_P(\Omega'(-) - \Theta'(-)) = \text{mult}_P F_S(-), \]
that is, in a neighbourhood of $x$, $\text{mult}_P F_S(-)$ also is rational affine. But then

\[ x \in U = \{ w \mid \text{mult}_P \Omega(w) = \text{mult}_P F_S(w) \} \]
implies that, in a neighbourhood of $x$, $\text{mult}_P \Omega(w) = \text{mult}_P F_S(w)$ ($U$ is affine and defined over $\mathbb{Q}$, hence $\Pi \subset U$ by minimality of $\Pi$), that is, $\Theta(-)$ is concave in a neighbourhood of $x$ after all. 

\[ \square \]

References

[ADHL10] Ivan Arzhantsev, Ulrich Derenthal, Juergen Hausen, and Antonio Laface. Cox rings, arXiv:1003.4229.

[AH06] Klaus Altmann and Jürgen Hausen. Polyhedral divisors and algebraic torus actions. Math. Ann., 334(3):557–607, 2006.

[BCHM09] Caucher Birkar, Paolo Cascini, Christopher D. Hacon, and James McKernan. Existence of minimal models for varieties of log general type. J. Amer. Math. Soc., posted on November 13, 2009. PII: S 0894-0347(09)00649-3 (to appear in print).

[CL10] Alessio Corti and Vladimir Lazić. Finite generation implies the Minimal Model Program, arXiv:1005.0614.

[Cor07] Alessio Corti, editor. Flips for 3-folds and 4-folds, volume 35 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2007.

[ELM+06] Lawrence Ein, Robert Lazarsfeld, Mircea Mustaţă, Michael Nakamaye, and Mihnea Popa. Asymptotic invariants of base loci. Ann. Inst. Fourier (Grenoble), 56(6):1701–1734, 2006.

[FM00] Osamu Fujino and Shigefumi Mori. A canonical bundle formula. J. Differential Geom., 56(1):167–188, 2000.
[HM06] Christopher D. Hacon and James McKernan. Boundedness of pluricanonical maps of varieties of general type. *Invent. Math.*, 166(1):1–25, 2006.

[HM09] Christopher D. Hacon and James McKernan. Existence of minimal models for varieties of log general type II. *J. Amer. Math. Soc.*, posted on November 13, 2009. PII: S 0894-0347(09)00651-1 (to appear in print).

[Kol92] János Kollár, editor. *Flips and abundance for algebraic threefolds*. Société Mathématique de France, Paris, 1992. Papers from the Second Summer Seminar on Algebraic Geometry held at the University of Utah, Salt Lake City, Utah, August 1991, Astérisque No. 211 (1992).

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.

[Laz09] Vladimir Lazić. Adjoint rings are finitely generated, arXiv:0905.2707.

[Nak04] Noboru Nakayama. *Zariski-decomposition and abundance*, volume 14 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2004.

[Pău08] Mihai Păun. Relative critical exponents, non-vanishing and metrics with minimal singularities, arXiv:0807.3109.

[Sho03] Vyacheslav V. Shokurov. Prelimiting flips. *Tr. Mat. Inst. Steklova*, 240(Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry):82–219, 2003.

[Siu98] Yum-Tong Siu. Invariance of plurigenera. *Invent. Math.*, 134(3):661–673, 1998.