Equations over direct powers of algebraic structures in relational languages

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Abstract

We study equations over relational structures that approximate groups and semigroups. For such structures we proved the criteria, when a direct power of such algebraic structures is equationally Noetherian.

1 Introduction

Let $\mathcal{A}$ be an algebraic structure with the universe $\mathcal{A}$ of a functional language $\mathcal{L}$. In other words, there are defined functions and constants over $\mathcal{A}$ that correspond to the symbols of $\mathcal{L}$. One can define the structure $\text{Pr}(\mathcal{A})$ with the universe $\mathcal{A}$ of a pure relational language $\mathcal{L}_{\text{pred}}$ as follows:

$$R_f(x_1, \ldots, x_n, y) = \{ (x_1, \ldots, x_n, y) \mid f(x_1, \ldots, x_n) = y \} \text{ in } \text{Pr}(\mathcal{A}),$$

$$R_c(x) = \{ x \mid x = c \} \text{ in } \text{Pr}(\mathcal{A}),$$

where functional and constant symbols $f, c$ belong to the language $\mathcal{L}$. Namely, the relation $R_f \in \mathcal{L}_{\text{pred}}$ ($R_c \in \mathcal{L}_{\text{pred}}$) is the graph of the function $f \in \mathcal{L}$ (constant $c \in \mathcal{L}$).

The $\mathcal{L}_{\text{pred}}$-structure $\text{Pr}(\mathcal{A})$ is called the predication of an $\mathcal{L}$-structure $\mathcal{A}$. In particular, if $\mathcal{A}$ is a group of the language $\mathcal{L}_g = \{ \cdot, \cdot^{-1}, 1 \}$ then $\text{Pr}(\mathcal{A})$ is an algebraic structure of the language $\mathcal{L}_{g-\text{pred}}$ with $\{ I, M, E \}$. Notice that any equation over a group $\mathcal{A}$ may be rewritten in the language $\mathcal{L}_{g-\text{pred}}$ by the introducing new variables. For example, the equation $x^{-1}y^{-1}xy = 1$ has the following correspondence in the relational language $\mathcal{L}_{g-\text{pred}}$:

$$\text{Pr}(S) = \begin{cases}
I(x, x_1), \\
I(y, y_1), \\
M(x_1, y_1, z_1), \\
M(z_1, x, z_2), \\
M(z_2, y, z_3), \\
E(z_3)
\end{cases}$$

where the relations $I, M, E$ are defined by $\{ I, M, E \}$.

It is easy to see that the projection of the solution set of $S$ onto the variables $x, y$ gives the solution set of the initial equation $x^{-1}y^{-1}xy = 1$. More generally, for any finite set of group equations $S$ in variables $X$ there exists a system $\text{Pr}(S)$ of equations in the language $\mathcal{L}_{g-\text{pred}}$ such that the solution set of $S$ is the projection of the solution set $\text{Pr}(S)$ onto the variables $X$. Hence, there arises the following important problem.
**Problem.** What properties of a finite system $S$ ($\Pr(S)$) are determined by the system $\Pr(S)$ (respectively, $S$)?

This problem was originally studied in [1], where it was proved the general results for relational systems $\Pr(S)$.

The next principal problem is to describe relational structures $\Pr(A)$ with “hard” and “simple” equational properties. According to [2], an algebraic structure $\Pr(A)$ has “simple” equational properties if $\Pr(A)$ is equationally Noetherian (i.e. any system of equations is equivalent over $\Pr(A)$ to a finite subsystem). However, it was proved in [1] that any algebraic structure of a finite relational language is equationally Noetherian. Thus, the Noetherian property gives a trivial classification of “hard” and “simple” relational structures $\Pr(A)$.

Therefore, we have to propose an alternative approach in the division of relational algebraic structures into classes with “simple” and “hard” equational properties. Our approach satisfies the following:

1. we deal with lattices of algebraic sets over a given algebraic structures (a set $Y$ is algebraic over a predicatization $\Pr(A)$ if $Y$ is a solution set of an appropriate system of equations);
2. we use the common operations of model theory (direct products, substructures, ultra-products etc.);
3. the partition into “simple” and “hard” algebraic structures is implemented by a list of first-order formulas $\Phi$ such that

$$\mathcal{A} \text{ is "simple" } \iff \mathcal{A} \text{ satisfies } \Phi.$$ 

(3)

In other words, the “simple” class of algebraic structures is axiomatizable by formulas $\Phi$.

Namely, we offer to consider infinite direct powers $\Pi\Pr(A)$ of a predicatization $\Pr(A)$ and study Diophantine equations over $\Pi\Pr(A)$ instead of Diophantine equations over $\Pr(A)$ (an equation $E(X)$ is said to be Diophantine over an algebraic structure $B$ if $E(X)$ may contain the occurrences of any element of $B$). The decision rule in our approach is the following:

$$\Pr(A) \text{ is "simple" } \iff \text{all direct powers of } \Pr(A) \text{ are equationally Noetherian};$$

(4)

otherwise, an algebraic structure $\Pr(A)$ is said to be “hard”.

Some results of the type (3) and (4) were obtained in [5], where we found formulas $\Phi$ for the classes of groups, rings and monoids in functional languages. For example, a group (ring) has a “simple” equational theory in the functional language iff it is abelian (respectively, with zero multiplication).

On the other hand, we prove below that any group in the language $L_{\text{g–pred}}$ has equationally Noetherian direct powers (Corollary 3.2). Moreover, the similar result holds for the natural generalizations of groups: quasi-groups and loops (Remark 3.3).

However, the class of semigroups has a nontrivial classification (1). We find two quasi-identities (9,10) such that a semigroup $S$ satisfies (9,10) iff any direct power of $\Pr(S)$ is equationally Noetherian (Theorem 3.1).

In the class of finite semigroups the conditions (9,10) imply that the minimal ideal (kernel) of a semigroup $S$ is a rectangular band of groups, and the kernel coincides with the ideal of reducible elements of $S$ (Theorem 3.8). However, if the kernel of a finite semigroup $S$ is a group then the conditions of Theorem 3.1 become sufficient for Noetherian property of any direct power $\Pi\Pr(S)$. 

2
2 Basic notions

In the current paper we deal with relational languages that interpret functions and constants in groups and semigroups.

Let $S$ be a semigroup. One can define the language $L_{s\text{-pred}} = \{M^{(3)}\}$ and a relation

$$M(x, y, z) \leftrightarrow xy = z.$$  

Any group $G$ may be considered as an algebraic structure of the relational language $L_{g\text{-pred}} = \{M^{(3)}, I^{(2)}, E^{(1)}\}$, where

$$M(x, y, z) \leftrightarrow xy = z,$$

$$I(x, y) \leftrightarrow x = y^{-1},$$

$$E(x) \leftrightarrow x = 1.$$  

An algebraic structure of the language $L_{s\text{-pred}}$ ($L_{g\text{-pred}}$) is called the predication of a semigroup $S$ (group $G$) if the operations over $S$ ($G$) corresponds to the relations $L_{s\text{-pred}}$ ($L_{g\text{-pred}}$). The predication of a semigroup $S$ (group $G$) is denoted by $Pr(S)$ (respectively, $Pr(G)$).

Following [3], we give the main definitions of algebraic geometry over algebraic structures (below $L \in \{L_{s\text{-pred}}, L_{g\text{-pred}}\}$).

An equation over $L$ ($L$-equation) is an atomic formula over $L$. The examples of equations are the following: $M(x, x, x)$ ($L_{s\text{-pred}}$-equations); $M(x, x, y)$, $I(x, y)$, $I(x, x)$, $E(x)$ ($L_{g\text{-pred}}$-equations).

A system of $L$-equations ($L$-system for shortness) is an arbitrary set of $L$-equations. Notice that we will consider only systems in a finite set of variables $X = \{x_1, x_2, \ldots, x_n\}$. The set of all solutions of $S$ in an $L$-structure $A$ is denoted by $V_A(S) \subseteq A^n$. A set $Y \subseteq A^n$ is said to be an algebraic set over $A$ if there exists an $L$-system $S$ with $Y = V_A(S)$. If the solution set of an $L$-system $S$ is empty, $S$ is said to be inconsistent. Two $L$-systems $S_1, S_2$ are called equivalent over an $L$-structure $A$ if $V_A(S_1) = V_A(S_2)$.

An $L$-structure $A$ is $L$-equationally Noetherian if any infinite $L$-system $S$ is equivalent over $A$ to a finite subsystem $S' \subseteq S$.

Let $A$ be an $L$-structure. By $L(A)$ we denote the language $L \cup \{a \mid a \in A\}$ extended by new constants symbols which correspond to elements of $A$. The language extension allows us to use constants in equations. The examples of equations in the extended languages are the following: $M(x, y, a)$ ($L_{s\text{-pred}}$-equation and $a \in S$); $M(a, x, b)$, $I(x, a)$, $E(a)$ ($L_{g\text{-pred}}$-equations and $a, b \in G$). Obviously, the class of $L(A)$-equations is wider than the class of $L$-equations, so an $L$-equationally Noetherian algebraic structure $A$ may lose this property in the language $L(A)$.

Since the algebraic structures $A$ and $Pr(A)$ have the same universe, we will write below $V_A(S)$ ($L(A)$) instead of $V_{Pr(A)}(S)$ (respectively, $L(Pr(A))$).

Let $A$ be a relational $L$-structure. The direct power $\Pi A = \prod_{i \in I} A$ of $A$ is the set of all sequences $[a_i | i \in I]$ and any relation $R \in L$ is defined as follows

$$R([a_i^{(1)} | i \in I], [a_i^{(2)} | i \in I], \ldots, [a_i^{(n)} | i \in I]) \leftrightarrow R(a_i^{(1)}, a_i^{(2)}, \ldots, a_i^{(n)})$$  

for each $i \in I$.

A map $\pi_k: \Pi A \to A$ is called the projection onto the $i$-th coordinate if $\pi_k([a_i | i \in I]) = a_k$.

Let $E(X)$ be an $L(\Pi A)$-equation over a direct power $\Pi A$. We may rewrite $E(X)$ in the form $E(X, \vec{C})$, where $\vec{C}$ is an array of constants occurring in the equation $E(X)$. One can introduce the projection of an equation onto the $i$-th coordinate as follows:

$$\pi_i(E(X)) = \pi_i(E(X, \vec{C})) = E(X, \pi_i(\vec{C})),$$
where \( \pi_i(\overline{C}) \) is an array of the \( i \)-th coordinates of the elements from \( \overline{C} \). For example, the \( L_{s-pred}(ILA) \)-equation \( M(x, [a_1, a_2, a_3, \ldots], [b_1, b_2, b_3, \ldots]) \) has the following projections

\[
M(x, a_1, b_1), \\
M(x, a_2, b_2), \\
M(x, a_3, b_3), \\
\ldots
\]

Obviously, any projection of an \( L(ILA) \)-equation is an \( L(A) \)-equation.

Let us take an \( L(ILA) \)-system \( S = \{ E_j(X) \mid j \in J \} \). The \( i \)-th projection of \( S \) is the \( L(A) \)-system defined by \( \pi_i(S) = \{ \pi_i(E_j(X)) \mid j \in J \} \). The projections of an \( L(ILA) \)-system \( S \) allow to describe the solution set of \( S \) by

\[
V_{ILA}(S) = \{ [P_i \mid i \in I] \mid P_i \in V_A(\pi_i(S)) \}. \tag{8}
\]

In particular, if one of the projections \( \pi_i(S) \) is inconsistent, so is \( S \).

The following statement immediately follows from the description \( S \) of the solution set over a direct powers.

**Lemma 2.1.** Let \( S = \{ E_j(X) \mid j \in J \} \) be an \( L(ILA) \)-system over \( ILA \). If one of the projections \( \pi_i(S) \) is inconsistent, so is \( S \). Moreover, if \( A \) is \( L \)-equationally Noetherian, then an inconsistent \( L(ILA) \)-system \( S \) is equivalent to a finite subsystem.

**Proof.** The first assertion directly follows from \( S \). Suppose \( A \) is \( L \)-equationally Noetherian, and \( \pi_i(S) \) is inconsistent. Hence, \( \pi_i(S) \) is equivalent to its finite inconsistent subsystem \( \{ \pi_i(E_j(X)) \mid j \in J' \} \), \( |J'| < \infty \), and the finite subsystem \( S' = \{ E_j(X) \mid j \in J' \} \subseteq S \) is also inconsistent. \( \square \)

### 3 Predicatization of semigroups and groups

**Theorem 3.1.** Let \( Pr(S) \) be the predicatization of a semigroup \( S \). A direct power of \( Pr(S) \) is equationally Noetherian iff the following quasi-identities

\[
\forall a \forall b \forall a \forall \beta ((a \alpha = a \beta) \rightarrow (b \alpha = b \beta)), \tag{9}
\]

\[
\forall a \forall b \forall a \forall \beta ((\alpha a = \beta a) \rightarrow (ab = \beta b)) \tag{10}
\]

hold in \( S \).

**Proof.** First, we prove the "if" part of the theorem. Suppose \( S \) satisfies \( \text{(9)} \text{(10)} \) and consider an infinite \( L_{s-pred}(ILS) \)-system \( S \). One can represent \( S \) as a finite union of the following systems

\[
S = \bigcup_{1 \leq i, j \leq n} S_{cij} \bigcup_{1 \leq i, j \leq n} S_{icj} \bigcup_{1 \leq i \leq n} S_{ijc} \bigcup_{1 \leq i \leq n} S_{cic} \bigcup_{1 \leq i \leq n} S_{icc} \bigcup_{1 \leq i \leq n} S_0, \tag{11}
\]

where each equation of \( S_0 \) is one of the following types:

1. \( x_i = x_j \);
2. \( x_i = c_j \);
3. \( c_i = c_j \);
4. \( M(x_i, x_j, x_k); \)
and \( S_{cij} = \{ M(c_k, x_i, x_j) \mid k \in K \}, S_{icj} = \{ M(x_i, c_k, x_j) \mid k \in K \}, S_{ijc} = \{ M(x_i, x_j, c_k) \mid k \in K \}, S_{cci} = \{ M(c_k, d_k, x_i) \mid k \in K \}, S_{cic} = \{ M(c_k, x_i, d_k) \mid k \in K \} \) \( (c_k, d_k \in \Pi Pr(S)) \), where each system above has its own index set \( K \).

Clearly, the system \( S_0 \) is equivalent to its finite subsystem. So it is sufficient to prove that each of other systems is equivalent to a finite subsystem over \( \Pi S \). According to Lemma 2.1, we may assume that any system below is consistent.

Thus, we have the following cases.

1. Let \( S_{icc} = \{ M(x_i, c_k, d_k) \mid i \in I \} \) and \( M(x_i, c_1, d_1) \) be an arbitrary equation of \( S_{icc} \). Since \( S_{icc} \) is consistent then one can choose \( \bar{\alpha} \in \Pi S(S_{icc}), \bar{\beta} \in \Pi S(M(x_i, c_1, d_1)) \). We have \( \bar{\alpha} c_1 = \bar{\beta} c_1 = d_1 \). By the quasi-identities \( \langle 9, 10 \rangle \), \( \bar{\alpha} c_k = \bar{\beta} c_k \) for any \( c_k \). Hence, \( \bar{\beta} \) satisfies all equations from \( S_{icc} \), and \( S_{icc} \) is equivalent to the equation \( M(x_i, c_1, d_1) \). The proof for the systems \( S_{cic}, S_{cci} \) is similar.

2. Let \( S_{icj} = \{ M(x_i, c_k, x_j) \mid i \in I \} \) (the proof for \( S_{cij}, S_{ijc} \) is similar). Since \( S_{icj} \) is consistent, there exist a point \( (\bar{\alpha}, \bar{\beta}) \in \Pi S(S_{icj}) \) and the equalities \( \bar{\alpha} c_k = \bar{\beta} c_k \) hold for any \( k, l \in K \). By \( \langle 9, 10 \rangle \), for any \( \bar{\gamma} \in \Pi S \) it holds \( \bar{\gamma} c_k = \bar{\gamma} c_l \). Thus, the solution set of \( S_{icj} \) is \( Y = \{ (\bar{\gamma}, \bar{\gamma} c_1) \mid \bar{\gamma} \in \Pi S \} \) and \( S_{icj} \) is equivalent to the equation \( x c_1 = y \).

Now we prove the “only if” part of the theorem. Suppose the quasi-identity \( \langle 9 \rangle \) does not hold in \( S \) (for the formula \( \langle 10 \rangle \) the proof is similar). It follows there exist elements \( a, b, \alpha, \beta \) such that \( a a = a \beta = c, b \alpha \neq b \beta \). Let us consider the system

\[
S = \{ M(a_n, x, c_n) \mid n \in \mathbb{N} \},
\]

where

\[
a_n = [a, \ldots, a, b, b, \ldots] \quad \text{and} \quad c_n = [c, \ldots, c, b \alpha, b \alpha, \ldots].
\]

One can directly check that the point

\[
a = [\beta, \ldots, \beta, \alpha, \alpha, \ldots]
\]

satisfies the first \( n \) equations of \( S \). However the \((n + 1)\)-th equation of \( S \) gives \( a_{n+1} a \neq c_{n+1} \), since its \((n + 1)\)-th projection defines the equation \( b x = b \alpha \), but \( b \alpha \neq b \beta \). Thus, \( S \) is not equivalent to any finite subsystem.

**Corollary 3.2.** Let \( \Pr(G) \) be the predicatization of a group \( G \). Then any direct power of \( \Pr(G) \) is \( L_{g-pred}(\Pi G) \)-equationally Noetherian.

**Proof.** Since the equality \( a a = a \beta \) (\( aa = \beta a \)) implies \( \alpha = \beta \) in any group, the quasi-identities \( \langle 9, 10 \rangle \) obviously hold in \( G \). Thus, any infinite system of the form \( \{ M(\ast, \ast, \ast) \mid i \in I \} \) is equivalent to a finite subsystem.

One can directly prove that for any group infinite systems of the form \( \{ I(\ast, \ast) \mid i \in I \} \) and \( \{ E(\ast) \mid i \in I \} \) are also equivalent to their finite subsystems over \( \Pi G \).

Thus, any system of \( L_{g-pred}(\Pi G) \)-equations is equivalent over \( \Pi G \) to its finite subsystem.
Remark 3.3. The last corollary also holds for quasi-groups. Notice that a quasi-group is a non-associative analogue of a group. Any quasi-group admits the analogues of the group divisibility, hence the quasi-identities (9,10) obviously hold in any quasi-group. Thus, any direct power of a quasi-group $G$ is $\mathcal{L}_{s\text{-pred}}(H^G)$-equationally Noetherian (notice here we consider quasi-groups and loops in the language $\mathcal{L}_{s\text{-pred}}$, since not any quasi-group admits the relations $I(x,y)$ and $E(x)$).

Below we study finite semigroups $S$ that satisfy Theorem 3.1

A subset $I \subseteq S$ is called a left (right) ideal if for any $s \in S$, $a \in I$ it holds $sa \in I$ (as $\in I$). An ideal which is right and left simultaneously is said to be two-sided (or an ideal for shortness).

A semigroup $S$ with a unique ideal $I = S$ is called simple. Let us remind the classical Sushkevich-Rees theorem for finite simple semigroups.

Theorem 3.4. For any finite simple semigroup $S$ there exists a finite group $G$ and finite sets $I, \Lambda$ such that $S$ is isomorphic to the set of triples $(\lambda, g, i)$, $g \in G$, $\lambda \in \Lambda$, $i \in I$. The multiplication over the triples $(\lambda, g, i)$ is defined by

$$(\lambda, g, i)(\mu, h, j) = (\lambda, gp_{\mu h}, j),$$

where $p_{\mu h} \in G$ is an element of a matrix $P$ such that

1. $P$ consists of $|I|$ rows and $|\Lambda|$ columns;
2. the elements of the first row and the first column equal $1 \in G$ (i.e. $P$ is normalized).

Following Theorem 3.4 we denote any finite simple semigroup $S$ by $S = (G, P, \Lambda, I)$.

The minimal ideal of a semigroup $S$ is called a kernel and denoted by $Ker(S)$ (any finite semigroup always has a unique kernel). Obviously, if $S = Ker(S)$ the semigroup is simple. If $Ker(S)$ is a group then $S$ is said to be a homogroup. The next theorem contains the necessary information about homogroups.

Theorem 3.5. In a homogroup $S$ the identity element $e$ of the kernel $Ker(S)$ is idempotent ($e^2 = e$) and belongs to the center of $S$ (i.e. $e$ commutes with any $s \in S$).

A semigroup $S$ is called a rectangular band of groups if $S = (G, P, \Lambda, I)$ and $p_{i\lambda} = 1$ for any $i \in I$, $\lambda \in \Lambda$.

Lemma 3.6. Suppose a finite simple semigroup $S$ satisfies (9,10). Then $S$ is a rectangular band of groups.

Proof. By Theorem 3.4 $S = (G, P, \Lambda, I)$ for some finite group $G$, matrix $P$ and finite sets of indexes $\Lambda, I$.

Assume that $|\Lambda| > 1$ and $p_{i\lambda} \neq 1$ for some $i, \lambda$.

Let $a = (1, 1, 1)$, $\alpha = (\lambda, 1, 1)$, $\beta = (1, 1, 1)$ and hence

$$a\alpha = (1, 1, 1)(\lambda, 1, 1) = (1, 1, 1) = (1, 1, 1)(1, 1, 1) = a\beta. \quad (12)$$

However, for $b = (1, 1, 1)$ we have

$$b\alpha = (1, 1, 1)(\lambda, 1, 1) = (1, p_{\lambda \cdot 1}, 1) \neq (1, 1, 1) = (1, 1, 1)(1, 1, 1) = b\beta. \quad (13)$$

We obtain that the equalities (12,13) contradict (9,10).

Thus, either $p_{i\lambda} = 1$ for all $i, \lambda$ or $|\Lambda| = |I| = 1$. In any case $S$ is a rectangular band of groups. \qed
An element \( s \) of a semigroup \( S \) is called \textit{reducible} if there exist \( a, b \in S \) with \( s = ab \). Clearly, the set of all reducible elements \( \text{Red}(S) \) is an ideal of a semigroup \( S \).

**Lemma 3.7.** Let \( S \) be a finite semigroup satisfying (9,10). Then \( \text{Ker}(S) \) is the set of all reducible elements.

**Proof.** Let \( b \in S \). We have \( \lambda, g, i)b = (\lambda, g, i)(1, 1, i)b = (\lambda, g, i)r \), where \( r = (1, 1, i)b \in \text{Ker}(S) \). By (9), we obtain \( ab = ar \) for any \( a \in S \). Since \( ar \in \text{Ker}(S) \), so is \( ab \). Thus, any product of elements belongs to \( \text{Ker}(S) \). Thus, \( \text{Red}(S) = \text{Ker}(S) \).

**Theorem 3.8.** If a direct power \( \Pi Pr(S) \) of a finite semigroup \( S \) is equationally Noetherian, then \( \text{Ker}(S) = \text{Red}(S) \) and \( \text{Ker}(S) \) is a rectangular band of groups.

**Proof.** The proof immediately follows from Lemmas 3.6, 3.7.

However, homogroups satisfy the converse statement of Theorem 3.8.

**Theorem 3.9.** If \( \text{Ker}(S) = \text{Red}(S) \) for a homogroup \( S \), then the direct power \( \Pi S \) is \( \mathcal{L}_{a-\text{pred}}(\Pi S) \)-equationally Noetherian.

**Proof.** Let us take \( a, b, \alpha, \beta \) such that \( a\alpha = a\beta \), and \( e \) be the identity of \( \text{Ker}(S) \). We have

\[
\begin{align*}
  a\alpha &= a\beta | e \\
  ea\alpha &= ea\beta \\
  (ea)\alpha &= (ea)\beta | (ea)^{-1} \text{ since } ea \text{ belongs to the group } \text{Ker}(S) \\
  ea &= e\beta | e \text{ is a central element} \\
  ae &= \beta e.
\end{align*}
\]

We have (below we use \( b\beta \in \text{Ker}(S) = \text{Red}(S) \)):

\[
  b\alpha = (b\alpha)e = b(\alpha e) = b(\beta e) = (b\beta)e = b\beta
\]

Thus, the quasi-identity (9) holds for \( S \). The proof for the quasi-identity (10) is similar.

One can directly check that for a rectangular band of groups \( S = (G, \Pi, \Lambda, I) \) the converse statement of Theorem 3.8 also holds.

Thus, one can formulate the following conjecture.

**Conjecture.** If a finite semigroup \( S \) has a rectangular band of groups \( \text{Ker}(S) = \text{Red}(S) \) then \( S \) satisfies the quasi-identities (9,10).
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