Abstract

We present rigorous upper and lower bounds on the emptiness formation probability for the ground state of a spin-$1/2$ Heisenberg XXZ quantum spin system. For a $d$-dimensional system we find a rate of decay of the order $\exp(-c L^{d+1})$ where $L$ is the sidelength of the box in which we ask for the emptiness formation event to occur. In the $d = 1$ case this confirms previous predictions made in the integrable systems community, though our bounds do not achieve the precision predicted by Bethe ansatz calculations. On the other hand, our bounds in the case $d \geq 2$ are new. The main tools we use are reflection positivity and a rigorous path integral expansion which is a variation on those previously introduced by Toth, Aizenman-Nachtergaele and Ueltschi.

1 Introduction and Main Results.

In this paper we obtain mathematically rigorous bounds for a quantity that physicists have considered for some time, called the “emptiness formation probability.” This is the probability, in the ground state of the quantum Heisenberg antiferromagnet, to find a block of spins ferromagnetically aligned. In a classical model, such as the Ising model, this probability would be zero in the true ground state. It is a measure of the quantum nature of the Heisenberg antiferromagnet that this probability is not exactly zero, even in the ground state.

The expected answer in $d = 1$ is that the emptiness formation probability for a block of length $L$ is asymptotically $A \exp(-c L^2)$, in the limit $L \to \infty$. This behavior was determined by physicists for some special Bethe-ansatz solvable models, although part of their analysis is not rigorous. We prove that in any dimension, there are upper and lower bounds of the form $C_\pm \exp(-c_\pm L^{d+1})$ for constants $C_+, C_-, c_+ , c_- \in (0, \infty)$. Moreover, the general idea is simple. The reason that the large deviation rate scales as $L^{d+1}$ instead of $L^d$ (as in classical statistical mechanical models at positive temperature) is that the extra dimension is imaginary time in the graphical representation of the quantum model. This is certainly an easier explanation than the Bethe ansatz.

The mathematical analysis is still somewhat involved. We make our jobs simpler by considering reflection positive models. Luckily, many interesting physical systems do have the reflection positive property, including all the ones considered using the Bethe ansatz when viewed in the proper way.
1.1 Set-up

Let $G = (V, E)$ be a finite graph. The Hilbert space for a spin-$\frac{1}{2}$ quantum spin system on the graph is $H_V = (\mathbb{C}^2)^{|V|}$. We denote the usual Pauli spin matrices, normalized by $\frac{1}{2}$, as 

$$S^x = \frac{1}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S^y = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \text{and} \quad S^z = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$  \hfill (1)

Let $V = \{i_1, \ldots, i_{|V|}\}$ be any enumeration of the vertices in order to specify the spin matrices at the various sites. The choice of enumeration is immaterial since a re-ordering will just result in a unitarily equivalent representation. For each $n$, the spin matrices at $i_n$ are $S_n^x$, $S_n^y$ and $S_n^z$, where 

$$S_n^x = (I_2^{(n-1)}) \otimes S^x \otimes (I_2^{(|V|-n)})$$ 

Using the notation of graph theory, $E$ may be any subset of the collection of all pairs $\{i, j\}$ such that $i, j \in V$, $i \neq j$. We consider the XXZ model. There is a real parameter of the model $\Delta$ in $\mathbb{R}$, called the anisotropy parameter. With this the XXZ Hamiltonian is $H_{G, \Delta} : \mathcal{H}_V \to \mathcal{H}_V$, which is defined as 

$$H_{G, \Delta} = -\sum_{(i,j) \in E} (S_i^x S_j^x + S_i^y S_j^y + \Delta \cdot S_i^z S_j^z).$$  \hfill (2)

In this paper, we will restrict to the special case that $G$ is bipartite: $\exists A \subseteq V$ such that every edge in $E$ can be written as $\{i, j\}$ with $i \in A$ and $j \in V \setminus A$. Then defining the unitary $U_A = \prod_{i \in A} (2S_i^z)$, 

$$U_A H_{G, \Delta} U_A^* = \sum_{(i,j) \in E} (S_i^x S_j^x + S_i^y S_j^y - \Delta \cdot S_i^z S_j^z).$$

Therefore, $H_{G, \Delta}$ is ferromagnetic for $\Delta > 0$ and antiferromagnetic for $\Delta < 0$.

We define the usual thermodynamic quantities: the partition function 

$$Z_{G, \Delta}(\beta) = \text{Tr}(e^{-\beta H_{G, \Delta}}),$$

and the equilibrium state 

$$\langle X \rangle_{G, \Delta, \beta} = \frac{\text{Tr}(X e^{-\beta H_{G, \Delta}})}{Z_{G, \Delta}(\beta)}.$$  

Our results will apply to various dimensions $d \in \{1, 2, \ldots\}$. We will usually leave the dependence on $d$ implicit, in order to simplify the notation. For any $N \in \mathbb{N}$, let $B_N$ denote the box 

$$B_N = \{i = (i_1, \ldots, i_d) \in \mathbb{Z}^d : -\frac{1}{2}N < i_1, \ldots, i_d \leq \frac{1}{2}N\} = \{-[\frac{1}{2}N] + 1, \ldots, [\frac{1}{2}N]\}^d, \hfill (3)$$

where $[x]$ is the integer part of $x$, $\lfloor x \rfloor = \max\{k \in \mathbb{Z} : k \leq x\}$.

Given $N \in \mathbb{N}$, let $T_N$ denote the graph $G = (V, E)$ such that $V = B_N$ (the box of sidelength $N$) and 

$$E = E_{T_N} = \{(i, j) : i, j \in V, j - i \in \{e_1, \ldots, e_d, -(N-1)e_1, \ldots, -(N-1)e_d\}\},$$  \hfill (4)

where $e_1, \ldots, e_d$ are the usual canonical basis vectors in $\mathbb{Z}^d$. This is the discrete torus because of the periodic boundary conditions. Frequently we will abuse notation by also writing $T_N$ for the vertex set. In particular, when we write the cardinality $|T_N|$ this will denote the cardinality of the vertex set, which is $N^d$.

We write $H_{N, \Delta}$, $Z_{N, \Delta}(\beta)$ and $\langle X \rangle_{N, \Delta, \beta}$ in place of $H_{T_N, \Delta}$, $Z_{T_N, \Delta}(\beta)$ and $\langle X \rangle_{T_N, \Delta, \beta}$. We will always restrict attention to $N$ even, so that $T_N$ is bipartite.
1.2 Emptiness formation probability

The eigenvalues of the spin matrix $S_i^z$ are $\pm \frac{1}{2}$ so that the two operators $(\frac{1}{2} \pm S_i^z)$ are the projections onto the eigenspaces associated with the eigenvalues $\pm \frac{1}{2}$. As long as $N \geq L$, we may view $\mathbb{B}_L$ as a subset of the graph $T_N$ (whose vertex set is $\mathbb{B}_N$). We define the projection operator

$$Q_L = \prod_{i \in \mathbb{B}_L} \left[ \frac{1}{2} + S_i^z \right].$$

This is the projection onto the subspace spanned by all spin states on $T_N$ which have all spins up on the sub-box $\mathbb{B}_L$. The expectation of $Q_L$ in the ground state of the XXZ model is called the “emptiness formation probability” in the physics literature.

We recall that we always restrict attention to even $N$.

**Theorem 1.1.** Suppose the dimension $d$ is fixed in $\{1, 2, \ldots \}$. For each $\Delta \in [-1, 1)$, there are constants $c_1, C_1 \in (0, \infty)$ such that, whenever $L^d \leq N^d/2$

$$C_1 \exp \left(-c_1 L^d \min(L, \beta)\right) \leq \langle Q_L \rangle_{N, \Delta, \beta},$$

while if $\Delta \leq 0$ there are constants $c_2, C_2 \in (0, \infty)$ such that,

$$\langle Q_L \rangle_{N, \Delta, \beta} \leq C_2 \exp \left(-c_2 L^d \min(L, \beta)\right).$$

We mention a symmetry of the XXZ model. For each $\Delta \in \mathbb{R}$, the Hamiltonian $H_{N, \Delta}$ commutes with the operator

$$S_{\text{tot}}^z = \sum_{i \in \mathbb{B}_N} S_i^z.$$

The eigenvalues of this operator are $M \in \{-\frac{1}{2} N^d, \ldots, +\frac{1}{2} N^d\}$. Let $M_M$ denote the orthogonal projection onto the eigenspace of $S_{\text{tot}}^z$ corresponding to eigenvalue $M$. When $d = 1$ we may obtain extended results at $0T$, but we must take account of the $S_{\text{tot}}^z$ symmetry. We recall that $N$ is even.

**Theorem 1.2.** Suppose the dimension is $d = 1$. For each $\Delta < 1$, there are constants $c_1, C_1 \in (0, \infty)$ such that, whenever $L \leq N/2$

$$C_1 \exp \left(-c_1 L^2 \right) \leq \min_{M \in \{-\frac{1}{2} N, \ldots, +\frac{1}{2} N\}} \lim_{\beta \to \infty} \frac{\langle M_M \cdot Q_L \rangle_{N, \Delta, \beta}}{\langle M_M \rangle_{N, \Delta, \beta}},$$

while if $\Delta < 1$ there are constants $c_2, C_2 \in (0, \infty)$ such that

$$\lim_{\beta \to \infty} \frac{\langle M_0 \cdot Q_L \rangle_{N, \Delta, \beta}}{\langle M_0 \rangle_{N, \Delta, \beta}} \leq C_2 \exp \left(-c_2 L^2 \right).$$

Finally, at positive temperatures, if we take sufficiently large $L$ then the lower bound holds with no restrictions on $\Delta$.

**Theorem 1.3.** For any fixed $d \in \mathbb{N}$ and any $\Delta \in \mathbb{R}$, there are constants $c$ and $C$ such that whenever $0 \leq \beta \leq 4L$

$$C \exp \left(-c L^d \beta \right) \leq \langle Q_L \rangle_{N, \Delta, \beta}.$$

The background motivation for our investigation stems from a few sources. For the rest of this section, we assume $d = 1$. The name "emptiness formation probability" (EFP$_L$) seems to come from the computation of (say) density-density correlation functions in the 1-dimensional hardcore Bose gas. Leaving the dependence on $\Delta$ implicit, define

$$\text{EFP}_L(N, \beta) = \langle Q_L \rangle_{N, \Delta, \beta}.$$
One frequently takes the $\beta \to \infty$ limit, which is always well-defined, and which gives the ground state expectation to see a chain of length $L$ upspins in a row:

$$EFP_L(N) = \lim_{\beta \to \infty} EFP_L(N, \beta).$$

One would also like to take the $N \to \infty$ limit to obtain the thermodynamic limit. But proving that this limit exists is non-trivial and has not been accomplished rigorously. Nevertheless, in the physics literature it is presumed that this limit exists.

The generating functional for this and other correlation functions is $\langle e^{\alpha Q(x)} \rangle$ where

$$Q(x) = \int_0^x dy \Psi^\dagger(y) \Psi(y).$$

Here $\Psi^\dagger(y), \Psi(y)$ are field operators for the Bose gas and $\alpha \in \mathbb{C}$. As $\text{Re} \alpha \to -\infty$, $e^{\alpha Q(x)}$ converges (weakly say) to the projection operator onto the subspace with no particles present in $[0, x]$. It turns out that $\langle e^{\alpha Q(x)} \rangle$ is more easily computed using Bethe ansatz techniques than various other correlation functions [7]. More recently, it was argued that $EFP_L$ is of primary importance for the ground state correlation structure of $XXZ$ chains [3]; computing exactly, in the thermodynamic limit, $EFP_L$ for all $L$ would allow to compute many other correlation functions as well. This is prohibitively complicated once $L > 5$ or so, and in [12] the authors focused instead on the asymptotic behavior of $EFP_L$ in $L$.

Further (nonrigorous) work appears in [11, 15] to name just a few articles. In fact, there are two cases where the asymptotic computations may be made rigorous: when $\Delta \in \{0, 1/2\}$, [10, 14]. The latter is special as it corresponds to the uniform measure on 6-vertex configurations (a short explanation is given below). The former is special because, via the Jordan-Wigner transformation, it is the equivalent to the model of free fermions. One can therefore write all eigenvectors of $-H_{\Delta=0}$ as Slater determinants of 1-particle eigenfunctions of the discrete Laplace operator on a $d$ dimensional torus. In particular the groundstate is explicit.

A second motivation is provided by the fact that the quantum XY model (that is (2) with $\Delta = 0$ corresponds to free fermions. Consider for a moment a collection of $k$ independent continuous time simple random walkers on a discrete circle of $N$ vertices, conditioned not to collide. Then via the Karlin-Mcgregor formula [9], the quasi-stationary measure for this process is exactly the square of the amplitude of the groundstate wave function of the free fermion model restricted to its $k$ particle sector. In this more probabilistic language, computing the asymptotics of $EFP_L$ translates to computing the large deviation rate of decay for large gaps in a "Dyson" random walk, with density $\frac{1}{2}$ of particles. From this perspective, the generalization from $1$ to $d$ dimensions is quite natural and forms the basic heuristic explanation of the $e^{-cL^{d+1}}$ rate of decay.

To add a bit further depth to this connection, we originally learned of this problem from O. Zeitouni [19], whose interest was piqued by the resemblance between the multiple integral representation of XXZ correlation functions arising from the Bethe ansatz and certain computations from random matrix theory. Finally, a third motivation, which should be contrasted with the (non-rigorous) methods employed above is to give an explanation for these very strong rates of decay which is robust to perturbations of the model and makes intuitive sense physically. As is explained below, our proof for the second half of of Theorem 1.1 achieves this goal; for example it is possible changing the range of interaction in $-H_{\Delta}$ and obtain the same results, as long as the interaction remains reflection positive and the interaction strength decays exponentially with distance.

Finally, we wish to draw the reader’s attention to [8] of Gallavotti, Lebowitz and Mastropietro. In that paper they actually proved a large deviation principle for quantum gases, either Bosonic or Fermionic, under the condition of sufficiently low densities. More precisely, they proved bounds where the large deviation quantities in a box of volume $L^d$ in $d$-dimensions has the usual large deviation type of behavior: the probability decays as $\exp(-cL^d)$ (to leading order, meaning if
one takes the logarithm and divides by $L^d$ then one obtains $-c$ in the limit of $L \to \infty$) where $c = \beta \Delta F(\beta, \rho, \rho_0)$ for a suitably defined function $\Delta F(\beta, \rho, \rho_0)$ measuring the change in free energy density at inverse-temperature $\beta$ and density $\rho_0$, where the large deviation event is to find density $\rho$ in a prescribed large box of volume $L^d$. One of their conditions was that the intensity must be suitable regular, and not too large, depending on $\beta$. In our model, the interaction may be seen to be too strong in the $\beta \to \infty$ limit for their theorem to apply. Non-typical large deviation behavior may be seen to be one sign of truly quantum behavior in the ground state. But all of these are important questions, since the full theory of large deviations for quantum spin systems is currently underway. In other words, large deviations for quantum systems is not totally understood, yet. The reference [8] is a model result on the forefront of the theory so far.

2 The Lower Bound (5) in Theorem 1.1.

For Theorems 1.1 and 1.2 a key step for both the upper and lower bounds is the introduction of a graphical representation. This means that the quantum spin system in $d$-dimensions may be represented in terms of a stochastic process in $d + 1$ dimensions, where the last dimension is (imaginary) time.

Typically there will be a probabilistic model for the diagrams, which will enjoy some independence properties. But then there is also a multiplicative factor involved, as well, which may be thought of as a Radon-Nikodym density, or a Feynman-Kac term depending on the context. If the multiplicative factor (which may frequently be associated to the exponential of an “energy” type term) can be controlled, and if the entropy of the a priori probabilistic construction can also be controlled, then bounds follow.

2.1 The Aizenman-Nachtergaele-Toth-Ueltschi representation.

Fix $\Delta \in [-1, 1]$ and $d \in \mathbb{N}$. For this choice of $\Delta$, we now introduce a graphical representation for the XXZ Model with parameter $\Delta$. The origins of this graphical representation are [2, 17], but the synthesis of ferromagnetic and antiferromagnetic loops into the same representation only appeared in the recent paper [18], see Section 3 there. We will use that representation, which we review here.

It helps to describe the construction in the context of a general finite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. This graph is general, except that we will assume that it is bipartite.

For inverse-temperature $\beta \geq 0$, the state space for the graphical representation of Aizenman, Nachtergaele, Toth and Ueltschi is going to be certain collections of labeled graphs. The underlying space for this construction is $\mathcal{V}_\beta = \mathcal{V} \times [-\frac{1}{\beta}, \frac{1}{\beta}].$

For each edge $\{i, j\}$ in $\mathcal{E}$, we consider two independent Poisson point processes on $\mathbb{R}$, which we label as random (nonnegative integer valued-) measures $d\nu_{ij}^F(\omega)$, $d\nu_{ij}^{AF}(\omega)$ with respective rates $\frac{1}{t}(1 - u), \frac{1}{t}u$. All these Poisson point processes are taken to be independent of one another.

These Poisson processes lead to two types of edges. If $t$ is an arrival time of $d\nu_{ij}^F(\omega)$ then we create an overpass edge. If $t$ is an arrival time of $d\nu_{ij}^{AF}(\omega)$ then we create a cul-de-sac edge. An example of this is shown in Figure 1. The nature of the two types of edges is manifested in the graph labels we describe next.

The labels on the graph are classical Ising spin configurations. Let $\Sigma_\mathcal{V}$ be the set of all possible choices $\sigma = (\sigma_i)_{i \in \mathcal{V}}$ where $\sigma_i$ is in $\{+1, -1\}$ for each $i \in \mathcal{V}$.

\[\int_{\Omega} \exp \left( - \int_{-\infty}^{\infty} \sum_{\{i, j\} \in \mathcal{E}} (f_{ij}(t)d\nu_{ij}^F(\omega)(t) + g_{ij}(t)d\nu_{ij}^{AF}(\omega)(t)) \right) dP(\omega) = \exp \left( - \int_{-\infty}^{\infty} \sum_{\{i, j\} \in \mathcal{E}} \left[ (1 - e^{-f_{ij}(t)})(1 - u) + (1 - e^{-g_{ij}(t)})u \right] dt \right).\]
Let $\Sigma_{\nu, \beta}$ be the set of all piecewise constant functions $\sigma(\cdot) : [-\frac{1}{2} \beta, \frac{1}{2} \beta] \to \Sigma_\nu$. Thus, for each time $t \in [-\frac{1}{2} \beta, \frac{1}{2} \beta]$, we have $\sigma(t) = (\sigma_i(t))_{i \in \nu}$ which is an element of $\Sigma_{\nu}$. Let $\Sigma_{\nu, \beta}(\omega)$ denote the subset of functions which satisfy the following rules:

- For any $i \in \nu$, consider the union of the set of arrival times of all the Poisson processes $\nu^{PF}_{ij}(\omega)$ and $\nu^{AF}_{ij}(\omega)$ for all $j$ such that $\{i, j\} \in E$. If the set of all these times is disjoint from $[t - \epsilon, t + \epsilon]$ then $\sigma_i(\cdot)$ is constant on this time interval.

- If $t$ is an arrival time of $\nu^{PF}_{ij}(\omega)$, then $\sigma_i(t) = \sigma_j(t^-)$ and $\sigma_j(t) = \sigma_i(t^-)$.

- If $t$ is an arrival time of $\nu^{AF}_{ij}(\omega)$, then $\sigma_i(t^-) = -\sigma_j(t^-)$ and $\sigma_j(t) = -\sigma_i(t)$.

Let $\Sigma_{\nu, \beta}^{\per}(\omega)$ denote the subset of $\Sigma_{\nu, \beta}(\omega)$ consisting of those labels such that $\sigma_i(\frac{\beta}{2} - t) = \sigma_i(-\frac{\beta}{2})$ for each $i \in \nu$. These are the periodic configurations, in (imaginary) time. We note that this is equivalent to identifying $t = \frac{\beta}{2}$ with $t = -\frac{\beta}{2}$. In the periodic setting, the graph may be decomposed entirely into disjoint loops $L_\beta$. We have highlighted one loop in $L_\beta$ in Figure 1. For each loop, once $\sigma_{i_k}(t_{0_k})$ has been specified at a single space-time point $(i_k, t_{0_k}) \in \nu \beta$ on the loop, the rules above prescribe it uniquely at each other space-time point on that loop. Therefore $|\Sigma_{\nu, \beta}^{\per}(\omega)| = 2^{|L_\beta(\omega)|}$.

Let $\mathbb{P}_u(\cdot), \mathbb{E}_u[\cdot]$ denote the probability measure and expectation associated with the family of Poisson point processes $\bigotimes_{\{i,j\} \in E} [\nu^{PF}_{ij}(\omega) \otimes \nu^{AF}_{ij}(\omega)]$. We will now specialize to $G = \mathbb{T}_N$. As usual, we use the notation where $N$ replaces $G$ for this graph. Suppose $\omega$ is a configuration of edges.

Given $L \leq N$ and given $\tau \in \{1, -1\}_L^B$ let us define the event

$$E_{N,L,\beta}(\tau, \omega) = \{\sigma(\cdot) \in \Sigma_{\nu, \beta}^{\per}(\omega) : \sigma_i(0) = \tau_i \text{ for all } i \in B_L\}. \quad (10)$$

The reason we introduced the stochastic process described above is that it allows us to express various quantum spin system correlation functions in a more amenable probabilistic language.

For $\mathbb{C}^2$, let $\psi^+$ and $\psi^-$ be the standard orthonormal basis, such that the spin matrices in this basis have the form (1). Given any $\sigma \in \Sigma_\nu$, define the Ising basis vector

$$\Psi_{\nu}(\sigma) = \bigotimes_{i \in \nu} \psi_{\sigma_i}^{\tau_i}. \quad (11)$$
As usual, we write $\Psi_N(\sigma)$ as a short-hand for $\Psi_V(\sigma)$ in the special case that $V = T_N$ (which will be used mainly in §3).

**Proposition 2.1.** For any $\sigma, \tau \in \Sigma_V$,
\begin{equation}
\langle \Psi_V(\tau), e^{-\beta H_O \cdot \Delta} \Psi_V(\sigma) \rangle = e^{\beta |E|/4}E_u \left[ \sum_{(\sigma ) \in \Sigma_{V, \beta}(\omega )} 1_{\{\sigma(\beta/2) = \sigma\}} 1_{\{\sigma(\beta/2) = \tau\}} \right],
\end{equation}
with the choice $u = (1 + \Delta)/2$. In particular, this means
\begin{align}
Z_{N, \Delta}(\beta) &= e^{\beta |E| (\gamma N)}/4 \sum_{\omega} 2^{L_\beta(\omega)}, \\
\text{EFP}_L(N, \beta) &= \frac{E_u[|E_{N, L, \beta}(1_L)|]}{E_u[2\{L_\beta(\omega)\}]},
\end{align}
for $1_L$ being the configuration with all 1’s on $\Lambda_L$.

**Remark 2.2.** The fact that the weight factor in (13) is $2^{|L_\beta(\omega)|}$ has important implications. The number of loops changes by at most 1 in absolute value if we add or subtract an arrival to one of the Poisson processes. Therefore, $|L_\beta(\omega)|$ has a Lipschitz property with respect to the number of arrivals of $\omega$. This is a useful property for obtaining large deviation type bounds.

**Proof.** An equivalent result is proved in [18], Section 3. This follows by considering the infinitesimal generator. Given $\sigma(i)$ in a small increment of time $0 < \Delta t \ll 1$, there is a probability $1/2 u \Delta t (1 + o(1))$ to have a cul-de-sac edge at $\{i,j\}$. When there is a cul-de-sac edge, that is represented by the operator
\begin{equation}
1 + 2Q_{\{i,j\}} = 2\left(\frac{1}{4} + (S_i^x S_j^x + S_i^y S_j^y - S_i^z S_j^z)\right)
\end{equation}
In the same time increment there is a probability $(1 - u) \Delta t (1 + o(1))$ to have an overpass edge at $\{i,j\}$. Then there is an operator
\begin{equation}
1 + 2T_{\{i,j\}} = 2\left(\frac{1}{4} + (S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z)\right).
\end{equation}
There is a probability $1 - \Delta t (1 + o(1))$ that there is no edge which is represented by the identity operator 1. The reason we have shifted the operators above by 1 is for this reason. Similarly, we scaled based on the fact that we chose the rates to be 1/2 as large. So we have
\begin{align}
Q_{\{i,j\}} &= (S_i^x S_j^x + S_i^y S_j^y - S_i^z S_j^z) - \frac{1}{4}, \\
T_{\{i,j\}} &= (S_i^x S_j^x + S_i^y S_j^y + S_i^z S_j^z) - \frac{1}{4}.
\end{align}
From this, we see that in time increment $\Delta t$ the operator representing all these effects is, to leading order, equal to
\begin{equation}
\exp \left( \Delta t \sum_{\{i,j\} \in E} [uQ_{\{i,j\}} + (1 - u)T_{\{i,j\}}] \right).
\end{equation}
Noting the relationship between $T_{\{i,j\}}, Q_{\{i,j\}}$ and the XXZ interaction for $\Delta = 1, \Delta = -1$ proves the lemma. $\square$

**2.2 Derivation of the Lower Bound**

In this section, we present the lower bound of $C \exp(-cL^{d+1})$ for $d \geq 1$. This lower bound will NOT be optimal for $\Delta \geq 1$. We do expect this bound to hold if $\Delta < -1$, but we are currently unable to prove it, though we refer the reader to § 2.2.2 for a partial result in this direction. We will derive an upper bound of $C \exp(-cL^{d+1})$ in § 3.
Lemma 2.3. There are constants $C_1, c_1 > 0$ so that, assuming $\Delta \in [-1,1]$, and $L \leq \frac{1}{2} N$,

$$\frac{Z_{N,L,\beta}(\tau)}{E[|2[\mathcal{E}_{\beta}(\omega)|]} \geq C_1 \exp(-c_1 L^d \min(\beta, L))$$

for any $\tau \in \{-1,1\}^{\Lambda_L}$.

Proof. We consider the case $d = 1$ for ease of exposition and only the most relevant case $\tau = 1_L$. The entire argument extends without difficulty to the case $d > 1$ and other choices of $\tau$. We first consider the case $\beta \geq L$. We will comment on the case $\beta < L$ at the end of the proof. We first observe the following: if $\omega, \omega_1$ are two configurations of edges which differ by $k$ space-time arrivals. Then

$$2^{-k} \leq 2[\mathcal{E}_{\beta}(\omega)] - |\mathcal{E}_{\beta}(\omega_1)| \leq 2^k. \quad (15)$$

Now let $\mathcal{W}_L$ denote the space-time window $\{-L + 1, \ldots, L\} \times [-L, L]$ and let

$$F = \{\omega : \text{there are no arrivals of } \omega \text{ in } \mathcal{W}_L\}.$$

As a simple consequence of (15) and tail bounds for large numbers of Poisson arrivals, we have constants $C, c > 0$ so that

$$E[2[\mathcal{E}_{\beta}(\omega)]] \leq C e^{cL^2} E[2[\mathcal{E}_{\beta}(\omega)]] 1\{F\}.$$ 

Next, we bound $E[2[\mathcal{E}_{\beta}(\omega)]1\{F\}]$ by $E[|E_{N,L,\beta}(1_L)|]$. Consider the event $G$ consisting of configurations of edges satisfying the following properties (see Figure 2 for an illustration):

- There are no overpass edges, i.e., no arrivals of $d_{ij}^F(\omega)$ in $\mathcal{W}_L$.
- Let $\{i,j\} = \{k-1,k\}$ be an edge with $k \in \{-L+2, \ldots, L\}$ and $k$ even. For $t \in \mathbb{Z}$ satisfying $|k| \leq 2t \leq L-1$, if there is exactly one cul-de-sac edge, i.e., one arrival of $d_{ij}^AF(\omega)$ in $(2t, 2t+1)$, and there is exactly one arrival of $d_{ij}^AF(\omega)$ in $(-2t-1, -2t)$
- Similarly, let $\{i,j\} = \{k-1,k\}$ be an edge with $k \in \{-L+2, \ldots, L\}$ and $k$ odd. For $t \in \mathbb{Z}$ satisfying $|k| \leq 2t-1 \leq L-1$, there is exactly one cul-de-sac edge, i.e., one arrival of $d_{ij}^AF(\omega)$ in $(2t-1, 2t)$, and there is exactly one arrival of $d_{ij}^AF(\omega)$ in $(-2t, -2t+1)$
- There are no other arrivals of $d_{ij}^AF(\omega)$ in $\mathcal{W}_L$.

In the case where $d > 1$ then one merely and makes this construction in the first coordinate direction for each cross section of $\mathbb{B}_L$. By spatial independence of arrivals of the Poisson point processes, there are constants $C_i, c_i > 0, i = 1, 2$ depending only on $u$ so that

$$C_1 e^{-c_1 L^{d+1}} \leq \frac{P(F)}{P(G)} \leq C_2 e^{c_2 L^{d+1}}.$$ 

Since the event $G$ only adds $O(L^{d+1})$ edges,

$$E[2[\mathcal{E}_{\beta}(\omega)]1\{F\}] \leq C_3 e^{c_3 L^{d+1}} E[2[\mathcal{E}_{\beta}(\omega)]1\{G\}].$$
Finally, it is clear that
\[ \mathbb{E}[2^{|\mathcal{L}_\beta(\omega)|} \mathbf{1}\{G\}] \leq 4L^d \mathbb{E}[|E_{\beta}(1)|] \]

since the loops induced by the event \( G \) support a labeling such that \( \sigma_i(0) = 1 \) \( \forall i \in B_L \). The factor \( 4L^d \) appears in the upper bound due to the total number of (admissible and non-admissible) ways of labeling \( B_L \times \{0\} \) with a spin configuration if we do not demand that all spins be +1. Collecting the estimates together finishes the proof in the case \( \beta \geq L \).

In the case \( \beta < L \), running through the first step of the above proof we have the existence of constants \( C, c \) so that
\[ \mathbb{E}[2^{|\mathcal{L}_\beta(\omega)|}] \leq C \exp(cL^d \beta) \mathbb{E}[2^{|\mathcal{L}_\beta(\omega)|} \mathbf{1}\{F_{\beta}\}] \]

where \( F_{\beta} \) is the event that there are no edges in \( \{-L + 1, \ldots, L\} \times \left[ -\frac{\beta}{2}, \frac{\beta}{2} \right] \). Now, because of the periodic boundary conditions, a configuration of edges \( \omega \in F_{\beta} \) supports all choices of labelings \( \tau \) on \( B_L \times \{0\} \), in particular \( \tau = 1_L \). Therefore
\[ \mathbb{E}[2^{|\mathcal{L}_\beta(\omega)|} \mathbf{1}\{F_{\beta}\}] \leq 4L^d \mathbb{E}[|E_{\beta}(1)|] \]

Gathering the estimates together, the lemma is proved in this case. \( \square \)

For more general configurations \( \tau \) than \( 1_L \), one merely reflects \(-\tau\) and connects \( \tau \) on a reflection of \( B_L \) across one of the faces. It does not matter which dimension one chooses to reflect in if \( d > 1 \). In fact, it seems reasonable that one can make an even better construction in dimensions \( d > 1 \) (reflecting something more like the true higher dimensional analogue of the dipole picture) so that one need not assume \( L \leq \frac{1}{2}N \), but instead just that \( L^d \leq \frac{1}{2}N^d \). But we will not pursue this here.
2.2.2 The Case $\Delta \notin [-1, 1]$ and $\beta < 4L$.

The above argument does not work for $\Delta \notin [-1, 1]$. Nevertheless, if $\beta < 4L$ we can use a more direct functional analysis argument, working with the operators in the expression

$$\text{EFP}_L(N, \beta) = \frac{\text{Tr} \left( \prod_{i \in \partial B_L} \left[ \frac{1}{2} S_i^z + S_i^+ e^{-\beta H'_{N,\Delta}} \right] \right)}{\text{Tr}(e^{-\beta H'_{N,\Delta}})}.$$  \hspace{1cm} (16)

where

$$-H'_{N,\Delta} = -b(\Lambda) + H_{N,\Delta},$$

(17)

where $b(\Lambda) = \frac{1}{2}|\mathcal{E}(T_N)|$, so that $\text{Tr}[\exp(-\beta H'_{N,\Delta})] = \mathbb{E}[2^{\mathcal{E}_\beta(\omega)}]$ by (13).

**Lemma 2.4.** For any $\Delta \in \mathbb{R}$, there are constants $c, C > 0$ so that the following holds. If $|\beta| \leq 4L$,

$$\text{EFP}_L(N, \beta) \geq C e^{-cL^d\beta}.$$

**Proof.** The simple and brutal idea is to compare to the system in which interactions between $T_N \setminus \mathcal{B}_L$ and $\mathcal{B}_L$ have been turned off. To this end, let us introduce the interpolating Hamiltonians

$$-H'_{N,\Delta}(a) = -H'_{T_N \setminus \mathcal{B}_L,\Delta} - H'_{\mathcal{B}_L,\Delta} - a \sum_{(i,j) \in \mathcal{E}(T_N) \setminus \mathcal{B}_L \setminus \mathcal{B}_L} S_i^z S_j^+ + S_i^+ S_j^z + \Delta S_i^z S_j^z - \frac{|\mathcal{E}_\beta(\omega)|}{4}$$

for $a \in [0, 1]$. Here $\partial \mathcal{B}_L$ denotes the edge boundary of $\mathcal{B}_L$ – the collection of edges with exactly one endpoint in $\mathcal{B}_L$. Further set

$$Z(a) = \text{Tr}(e^{-\beta H'_{N,\Delta}(a)})$$

Then $|\log Z(1) - \log Z(0)| = \left| \int_0^1 \frac{d}{da} \log Z(a) \right|$ and

$$\frac{d}{da} \log Z(a) = \frac{\beta \text{Tr}(\mathcal{I}(T_N \setminus \mathcal{B}_L, \mathcal{B}_L) e^{-\beta H'_{N,\Delta}})}{Z(a)}.$$

Because the summands of $\mathcal{I}(T_N \setminus \mathcal{B}_L, \mathcal{B}_L)$ are uniformly bounded the $\ell^1 - \ell^\infty$ Hölder inequality for operators implies $\frac{d}{da} \log Z(a) \leq c\beta L^d-1$ and thus

$$e^{-c\beta L^d-1} \leq \frac{Z(1)}{Z(0)} \leq e^{c\beta L^d-1}.$$

To handle the numerator in (16), we use a modified version of the graphical representation summarized in § 2.1 (see § 3 for more details since it is also important there). For $\Delta$ fixed, we may write $-H'_{N,\Delta} = -H'_{N,1} + \sum_{(i,j) \in \mathcal{E}(T_N)} (\Delta - 1) S_i^z S_j^z$. The term $\sum_{(i,j)} (\Delta - 1) S_i^z S_j^z$ acts as a potential over the configuration space of labeled graphs $\Sigma_{N,\beta}^\text{EFP}(\omega)$ determined by the graphical representation for $-H'_{N,1}$. (This is much the same as the outcome of the Feynman-Kac expansion for Schrödinger operators). Using this representation we have: there is a constant $C_0$ depending only on $\mathcal{E}_\beta$ and $\Delta$ so that

$$Z_{N,\Delta}(\beta) = C_0 \mathbb{E}_1 \left[ \sum_{\sigma \in \Sigma_{N,\beta}^\text{EFP}(\omega)} e^{\frac{\Delta - 1}{4} V_\beta(\sigma)} \right],$$

(18)

$$Z_{N,\Delta}(\beta) \text{EFP}_L(N, \beta) = C_0 \mathbb{E}_1 \left[ \sum_{\sigma \in \Sigma_{N,\beta}^\text{EFP}(\omega)} e^{\frac{\Delta - 1}{4} V_\beta(\sigma)} \right]$$

(19)
where \( E_1[\cdot] \) denotes the expectation associated with respect to the Poisson process for the graphical representation of \(-H_{N,1}\) and \(-V_\beta(\sigma) = \int_{-\beta/2}^{\beta/2} dt \sum_{\langle i,j \rangle} \sigma(t)\sigma_j(t)\). We use, in particular, (19). Let \( F_L \) be the event that there are no bonds connecting \( B_L \) to \( T_N \backslash B_L \). Then, of course,

\[
E_1 \left[ \sum_{\sigma \in \Sigma_{E_{N,L,\beta}(1_L)}} e^{\frac{\Delta}{1+c} V_\beta(\sigma)} \right] \geq E_1 \left[ 1\{F_L\} \sum_{\sigma \in \Sigma_{E_{N,L,\beta}(1_L)}} e^{\frac{\Delta}{1+c} U_\beta(\sigma)} \right].
\]

Combined with the bound above on the ratio of partition functions this implies (for a second pair of constants \( c_1, C_1 \))

\[
E_{FP}(N,\beta) \geq e^{-c_1 \beta L^{d-1}} \frac{E_1 \left[ 1\{F_L\} \sum_{\sigma \in \Sigma_{E_{N,L,\beta}(1_L)}} e^{\frac{\Delta}{1+c} V_\beta(\sigma)} \right]}{Z(0)} \geq C_1 e^{-c_1 \beta L^{d-1}} \frac{\text{Tr}(\prod_{i\in B_L} \left[ \frac{J}{2} + S_i^z \right] \exp(-\beta H'_{B_L,\Delta}))}{\text{Tr}(\exp(-\beta H'_{B_L,\Delta}))}.
\]

Part of the reason for the factor \( e^{-c_1 \beta L^{d-1}} \) is the potential terms spanning the boundary of \( \partial B_L \).

Since the numerator on the right-hand side of this string of inequalities is

\[
\text{Tr}(\exp(-\beta H'_{B_L,\Delta})) = \langle \Psi_{B_L}(1_L) \rangle \langle e^{-\beta H'_{B_L,\Delta}} \Psi_{B_L}(1_L) \rangle,
\]

and the denominator on the right-hand side of this string of inequalities is

\[
\sum_{\sigma \in \Sigma_{B_L}} \langle \Psi_{B_L}(\sigma) \rangle \langle \exp(-\beta H'_{B_L,\Delta}) \Psi_{B_L}(\sigma) \rangle,
\]

we have that

\[
\frac{\text{Tr}(\prod_{i\in B_L} \left[ \frac{J}{2} + S_i^z \right] \exp(-\beta H'_{B_L,\Delta}))}{\text{Tr}(\exp(-\beta H'_{B_L,\Delta}))} \geq C_2 \exp^{-c_2 \beta L^d},
\]

and the lemma follows.

### 3 The Upper Bound (6) in Theorem 1.1

The idea of the upper bound is that the projector \( Q_L \) imposes higher energy relative to the Hamiltonian \( H_N \) than in the ground state. We do not have an exact formula for the ground state or the ground state energy. But we can obtain variational upper bounds on the ground state energy.

Aside from this idea, we use two main tools for this part of the argument. The first main tool is a graphical representation for the equilibrium expectation. This is the one introduced in § 2.2.2, and used for example in equation (19).

**Lemma 3.1.** For any graph \( G = (V, E) \), and for any configurations \( \sigma, \tau \in \Sigma_V \),

\[
\langle \Psi_V(\tau), e^{-\beta H_{G,\Delta}} \Psi_V(\sigma) \rangle = e^{\beta |E|} E_1 \left[ \sum_{\sigma(\cdot) \in \Sigma_V, \sigma(\cdot) = \tau} 1_{\sigma(\cdot) = \sigma} \int_{-\beta/2}^{\beta/2} U_G(\sigma(t)) dt \right],
\]

where \( E_1 \) is the expectation defined just before Proposition 2.1, and where \( U_G : \Sigma_V \rightarrow \mathbb{R} \) is defined as the usual ferromagnetic Ising energy \( U_G(\sigma) = -\sum_{\{i,j\} \in E} \sigma_i \sigma_j \).
Comparing to Proposition 2.1 and the equations such as (19), the reader will notice that this is the same formula we have used before. We will give a proof, shortly, for completeness.

The reason Lemma 3.1 is useful is that for \( \Delta < 1 \), the exponential factor is actually the Gibbs weight factor for the antiferromagnetic Ising model. The projector \( Q_L \) is onto the ground states of the ferromagnetic Ising model on the block \( \mathbb{B}_L \). Therefore, on this block, relative to the potential energy \( U_G \), the configuration that \( Q_L \) imposes has the highest possible energy, which means that its Gibbs probability is low.

This still leaves the difficulty that the block \( \mathbb{B}_L \) is not as large as \( \mathbb{T}_N \). Moreover, the effect of imposing \( Q_L \) at one time does not last for all time \( \beta \) because the graphical representation inherent in \( \mathbb{E}_1 \) does allow the energy of \( U_G \) to change. The second main tool is reflection positivity and chessboard estimates which allows us to disseminate the event imposed by \( Q_L \) in space, as well as the generalized Hölder’s inequality to re-impose the event after a period \( \delta T \) of time. These allow us to disseminate the event to overcome this difficulty.

But the argument is still involved at this point. Every Hamiltonian satisfies the generalized Hölder’s inequality which allows to disseminate in time. But only certain Hamiltonians satisfy reflection positivity. For the XXZ model, this requires \( \Delta \leq 0 \). Moreover, the dissemination in space is antiferromagnetic in nature. The hopping allowed by the process related to \( \mathbb{E}_1 \) allows mixing of two adjacent blocks, one of which is of type \( \uparrow/\downarrow \) and the other of which is \( \downarrow/\uparrow \). But this is essentially a boundary effect if \( \delta T \) is a sufficiently small fraction of the linear size of the block \( L \). We can show this by analysis of the Poisson process related to \( \mathbb{E}_1 \).

With these guiding principles, we will now enter the details of the proof. Before doing that, let us -quickly prove Lemma 3.1.

Proof. This can be proved by the Trotter product formula or by taking derivatives. It is a typical Feynman-Kac formula for perturbing the generator of a Markov process by a potential. Defining the operator \( A_{\Delta,\beta} \) such that \( \langle \Psi_V(\tau), A_{\Delta,\beta} \Psi_V(\sigma) \rangle \) gives the right-hand-side of (20), we already know from (12) that \( \frac{d}{d\beta} A_{1,\beta} = -H_{Q,G,1}A_{1,\beta} \). So, by the rules of differentiation \( \frac{d}{d\beta} \langle \Psi_V(\tau), A_{\beta,\Delta} \Psi_V(\sigma) \rangle \) equals \( \langle \Psi_V(\tau), [-H_{Q,G,1} A_{\beta,\Delta} \Psi_V(\sigma)] \rangle \) plus \( -\frac{1}{4} (\Delta - 1) U(\tau) \langle \Psi_V(\tau), A_{\beta,\Delta} \Psi_V(\sigma) \rangle \). Then the lemma follows because \( H_{G,\Delta} - H_{Q,G,1} = - (\Delta - 1) \sum_{(i,j) \in E} S_i^z S_j^z \). I.e., \( (H_{G,\Delta} - H_{Q,G,1}) \Psi_V(\tau) = -\frac{1}{4} (\Delta - 1) U(\tau) \Psi_V(\tau) \).

\[ \] 

3.1 Reduction to Estimating the Cost of a “Universal Contour”

Recall that we always assume \( N \) is even. Recall that the vertex set of \( \mathbb{T}_N \) is the same as the vertex set \( \mathbb{B}_N \) (3), even though the edge set for \( \mathbb{T}_N \) contains extra edges beyond those in \( \mathbb{B}_N \) (4). Let us define the configuration \( \tau^{(N,L)} \) and the rank-1 projection \( \hat{Q}_{N,L} \), as follows:

\[
\tau_i^{(L,N)} = (-1)^{[(2i_1-1)/(2L)]+\cdots+[(2i_d-1)/(2L)]}, \quad \text{and} \quad \hat{Q}_{N,L} = |\Psi_N(\tau^{(L,N)})\rangle \langle \Psi_N(\tau^{(L,N)})|.
\]

The operator \( \hat{Q}_{N,L} \) is the “universal contour.” A schematic picture of it is given in Figure 3. Then, using reflection positivity, we may prove the following lemma.

Lemma 3.2. Suppose that \( \Delta \leq 0 \). Let \( L \) be fixed, satisfying \( L \leq N/2 \). Then

\[
\text{EFP}_L(N,\beta) = \langle Q_L \rangle_{N,\Delta,\beta} \leq \left( \text{\hat{Q}_{N,L}} \right)_{N,\Delta,\beta}^{1/K},
\]

where \( K = 2^{d(\log_2(N/L)+1)} \).

The proof of this result is relegated to the appendix as it is a standard application of the chessboard estimates method in [5]. For the readers convenience, we give a short account of reflection positivity for quantum spin systems there; see appendix A.4.
Figure 3: On the left is a schematic view of $Q_L$, for $d = 2$ dimensions, when $L = 4$ and $N = 28$. In the right the “universal contour,” $\hat{Q}_{L,N}$ is shown. A black circle depicts a $+$ spin projector, and a white circle depicts a $-$ spin projector. A small dot represents a site without a projector.

Given Lemma 3.2 our task now is to bound $\langle \hat{Q}_{N,L} \rangle_{N,\Delta,\beta}$ from above. We note that Lemma 3.2 applies for $\Delta \leq 0$ so that an upper bound on $\langle \hat{Q}_{N,L} \rangle_{N,\Delta,\beta}$ leads to an upper bound on $\langle Q_L \rangle_{N,\Delta,\beta}$ for $\Delta \leq 0$. But we may actually obtain an upper bound on $\langle \hat{Q}_{N,L} \rangle_{N,\Delta,\beta}$ for a wider range of $\Delta$’s. We demonstrate this since the argument is the same.

**Theorem 3.3.** For each $\Delta < 1$, there exists an $L_0 \in \{1, 2, \ldots \}$ and $C, c > 0$ such that for all $L \geq L_0$ and all $N \geq 32dL$, we have

$$\langle \hat{Q}_{N,L} \rangle_{N,\Delta,\beta} \leq C e^{-cN^d \min(L, \beta)}.$$

Combined with Lemma 3.2, this theorem implies (6) of Theorem 1.1. Therefore, proving this theorem is our main goal for the rest of this section.

We note that Lemma 3.2 has allowed us to disseminate the projector $Q_L$, which projected just on spins inside $B_L$, to the projector $\hat{Q}_{N,L}$ which restricts to a specified spin configuration on all of $T_N$. This is what we described in the outline at the beginning of this section. We have also obtained the desired graphical representation in Lemma 3.1. The next step is to disseminate in time.

### 3.2 Generalized Hölder’s inequality

**Proposition 3.4.** For any positive integer $n$,

$$\langle \hat{Q}_{N,L} \rangle_{N,\Delta,\beta} \leq \left( \frac{\text{Tr}[r(\hat{Q}_{N,L} e^{-\beta H/(2n)})^{2n}]}{Z_{N,\Delta}(\beta)} \right)^{1/(2n)}.$$

This is a direct consequence of the generalized Hölder inequality for operators with $A = \hat{Q}_{N,L}$ and this choice of $n$, see Theorem A.4 of Appendix A. In that appendix we will also describe how this proposition follows.

Because we are re-imposing the rank-1 projection $\hat{Q}_{N,L}$ every $\frac{\beta}{2n}$ units of “time,” we further have

$$\text{Tr}[r(\hat{Q}_{N,L} e^{-\beta H/(2n)})^{2n}] = \left( \text{Tr}[\hat{Q}_{N,L} e^{-\beta H/(2n)}] \right)^{2n} = \left( \langle \Psi_N(\tau(N,L)), e^{-\beta H/(2n)} \Psi_N(\tau(N,L)) \rangle \right)^{2n}.$$
Therefore, (23) becomes

\[
\langle \hat{Q}_{N,L} \rangle_{N,\Delta,\beta} \leq \frac{\text{Tr}[\hat{Q}_{N,L}e^{-\beta H/(2n)}]}{(Z_{N,\Delta}(\beta))^{1/(2n)}}.
\]

(24)

Recall that \(E(T_N)\) denotes the edge-set for \(T_N\). Then we may define

\[
\begin{align*}
\text{Num}_{\beta,N,L,n} &= e^{-(\beta/(2n))(|E(T_N)|)/4} \text{Tr}[\hat{Q}_{N,L}e^{-\beta H/(2n)}], \\
\text{Den}_{\beta,N,L,n} &= e^{-(\beta/(2n))(|E(T_N)|)/4} (Z_{N,\Delta}(\beta))^{1/(2n)}.
\end{align*}
\]

(25) \hspace{1cm} (26)

We will frequently write these as just \(\text{Num}\) and \(\text{Den}\). By (24) we have the following:

**Corollary 3.5.** For any choice of \(n\)

\[
\langle \hat{Q}_{N,L} \rangle_{N,\Delta,\beta} \leq \frac{\text{Num}_{\beta,N,L,n}}{\text{Den}_{\beta,N,L,n}}.
\]

(27)

To obtain an upper bound on the left hand side of (27) we need to obtain an upper bound on \(\text{Num}\) and a lower bound on \(\text{Den}\).

The point of multiplying by \(e^{-(\beta/(2n))(|E(T_N)|)/4}\) is to cancel the multiplier in (20). This is particularly useful in obtaining a variational lower bound on \(\text{Den}\). We do this next.

### 3.3 Variational lower bound on \(\text{Den}\)

Optimally, we would calculate \(\text{Den}\) exactly. But this is difficult. It involves calculating the partition function. But even the ground state energy is difficult. However, we may make a variational calculation to obtain a bound.

This is a standard exercise in quantum formalism, and Jensen’s inequality. (It could also be deduced easily from basic results in statistical mechanics such as the Gibbs variational principle for the free energy and equilibrium state.)

**Lemma 3.6.** For any graph \(G = (V,E)\), and any \(\Delta \in \mathbb{R}, \beta \geq 0\), we have

\[
Z_{G,\Delta}(\beta) e^{-|E|\beta/4} = \text{Tr} \left[ e^{-\beta (H_{G,\Delta} + |E|/4)} \right] \geq 1.
\]

(28)

**Proof.** We define a unit vector

\[
\Phi_V = 2^{-|V|/2} \sum_{\tau \in \Sigma_V} \Psi_V(\tau).
\]

(29)

We note that \(H_{G,1} + (1/4)|E|\) equals the sum over all \(\{i,j\} \in E\) of \(\frac{1}{2}(1 - T_{i,j})\), where \(T_{i,j} \Psi_V(\tau) = \Psi_V(\tau^{(i,j)})\), where \(\tau^{(i,j)}\) is the configuration obtained from \(\tau\) by interchanging \(\tau_i^{(i,j)} = \tau_j\) and \(\tau_j^{(i,j)} = \tau_i\). But in (29), we sum over all \(\tau^{(i,j)}\)’s, uniformly. Therefore \(T_{i,j} \Phi_V = \Phi_V\) for all \(\{i,j\}\).

\[
\left( H_{G,1} + (1/4)|E| \right) \Phi_V = 0.
\]

(30)

We also know from the proof of Lemma 3.1 that \((H_{G,\Delta} - H_{G,1}) \Psi_V(\tau) = -\frac{1}{4}(\Delta - 1) U_G(\tau) \Psi_V(\tau)\). So

\[
\langle \Phi_V, (H_{G,\Delta} - H_{G,1}) \Phi_V \rangle = -\left( \frac{\Delta - 1}{4} \right) 2^{-|V|} \sum_{\tau \in \Sigma_V} U_G(\tau).
\]

But the uniform average of \(U_G(\tau)\) is the sum over \(\{i,j\} \in E\) of the uniform average of \(\sigma_i \sigma_j\), and for \(i \neq j\) the random variable \(\sigma_i\) is independent of the random variable \(\sigma_j\) in the uniform probability measure. Moreover both \(\sigma_i\) and \(\sigma_j\) have expectation zero in the uniform probability measure. So
\[ \langle \Phi_N, (H_G, \Delta - H_G, 1) \Phi_N \rangle = 0. \] Combining this with (30), we conclude that \( \langle \Phi_N, (H_G, \Delta + |E|) \Phi_N \rangle = 0. \) Finally,

\[
\begin{align*}
\text{Tr} \left[ e^{-\beta(H_G, \Delta + (1/4)|E(T_N)|)} \right] & \geq \langle \Psi, e^{-\beta(H_G, \Delta + (1/4)|E(T_N)|)} \Phi \rangle \\
& \geq \exp \left( -\beta \langle \Psi, (H_G, \Delta + (1/4)|E(T_N)|) \Phi \rangle \right) ,
\end{align*}
\]

which completes the proof.

In particular, taking the \((2n)\)th root, we obtain the following.

**Corollary 3.7.** We have the bound

\[
\text{Den} = \left( e^{-\beta |E(T_N)|/4} Z_{N, \Delta}(\beta) \right)^{1/(2n)} \geq 1.
\]

### 3.4 Upper bound on \( \text{Num} \) using large deviation bounds

Using Lemma 3.1 and the definition in (25) (and (21)), we can rewrite

\[
\text{Num}_{\beta, N, L, n} = e^{\beta |E|} E_1 \left[ \sum_{\sigma(\cdot) \in \Omega_N \beta(\omega)} 1_{\{ \sigma(-\beta/2) = \sigma(\beta/2) = \tau(N, L) \}} \exp \left( \frac{\Delta - 1}{4} \int_{-\beta/2}^{\beta/2} U_G(\sigma(t) dt) \right) \right] ,
\]

where we write \( U_N \) as a short hand notation for \( U_T \), as usual. Note that when there are no cul-desac edges, specifying \( \sigma(-\beta/2) \) completely specifies \( \sigma(t) \) for all \( t \geq -\beta/2 \). Therefore the indicators are not restricting any multiplicity of choices of \( \sigma(\cdot) \) as much as they are putting restrictions on \( \omega \).

We will prove the following.

**Proposition 3.8.** For any \( \Delta < 1 \) and \( L \geq 24 \)

\[
\text{Num}_{\beta, N, L, n} \leq e^{-(1/64)(1-\Delta)dN^d\delta T} + e^{[(1/4)(1-\Delta) - (156d^2/4n)]dN^d\delta T} ,
\]

where \( \delta T = \beta/(2n) \) and \( M = L/(1536d^2\delta T) \).

The idea is relatively straightforward. For sufficiently large \( L \), we have \( U_N(\tau(N, L)) \approx |E(T_N)| \). This is because most edges \( i, j \) have \( \tau_i^{(N, L)} = \tau_j^{(N, L)} \). This only fails if \( i \) and \( j \) span two adjoining blocks. See Figure 3. The fraction of those edges is \( 1/L \). If \( \Delta < 1 \) then \( |E(T_N)| \) is interpreted as the maximum possible energy of the antiferromagnetic Ising potential, instead of the ground state energy of the ferromagnetic Ising potential. Therefore, in (32) this leads to exponential suppression.

That reasoning works to bound \( U_N(\sigma(t)) \) for \( t = \pm \beta/(4n) \). But for \( t \) between \( -\beta/(4n) \) and \( \beta/(4n) \), one needs to account for the hopping of spins allowed by the stochastic process associated with \( E_1 \). Essentially, the stochastic process allows spins to hop between neighboring blocks. But in a short period of time, there should not be too many arrivals of \( d\nu^F_\tau(\omega) \). Therefore, with high probability, the spin configuration \( \sigma(t) \) will still have relatively low energy for the ferromagnetic Ising potential, and hence will be exponentially suppressed in the antiferromagnetic Ising Gibbs state.

In the complementary event, \( d\nu^F_\tau(\omega) \) may allow enough overpass edges to attain the ground state of the Ising antiferromagnet. Therefore, the un-normalized Gibbs weight exponential term in (32) may be exponentially large. But the large deviation bound for this rare event is nonlinear and dominates this.

**Proof.** We may decompose \( T_N \) into “blocks.” Let us define a “block” to be a maximal connected set of sites \( i \in T_N \) satisfying the condition that \( \tau_i^{(N, L)} = \tau_j^{(N, L)} \) for all pairs of points \( \{i, j\} \) in the block. Let us define the edges on the faces of blocks

\[
\mathcal{F} = \{ \{i, j\} \in \mathcal{E}(T_N) : \tau_i^{(N, L)} \neq \tau_j^{(N, L)} \} .
\]
This means that the edge \{i, j\} spans two adjacent blocks.

Some blocks are “full” having size \(L^d\). For each coordinate direction, there may also be partial blocks at a distance less than \(L\) from the two faces of \(B_N\) in the + and − side of that coordinate direction. Considering this, it is easy to see that

\[
|\mathcal{F}| \leq (L^{-1} + 2N^{-1})|\mathcal{E}(T_N)|.
\]

Since \(N\) is always at least \(L\), this implies \(|\mathcal{F}| \leq 3L^{-1}|\mathcal{E}(T_N)|\). Therefore

\[
|\mathcal{E}(T_N)| - U_N(\tau^{(N,L)}) \leq 2|\mathcal{F}| \leq 6L^{-1}|\mathcal{E}(T_N)|.
\]

At times \(t = \pm \beta/(4n)\) the only edges \{i, j\} with \(\sigma_i(t) = -\sigma_j(t)\) are the ones in \(\mathcal{F}\). We will show that with high probability at times between \(-\beta/(4n)\) and \(\beta/(4n)\) most antiferromagnetic edges are close to \(\mathcal{F}\). Because of this we note the following easy bound on the number of vertices at a short distance from \(\mathcal{F}\).

For any positive integer \(r\), let us define \(\mathcal{V}_r\) to be the set of all sites \(i\) satisfying this condition: \(i\) is in a block \(\Lambda \subset T_N\), and has distance less than or equal to \(r\) from \(T_N \setminus \Lambda\). So for instance

\[
\mathcal{V}_1 = \{i \in T_N : \exists j \in T_N \text{ such that } \{i, j\} \in \mathcal{F}\}.
\]

Because \(\mathcal{F}\) can be written as a disjoint union of coordinate “planes” in \(T_N\), it is easy to deduce from this that the formula above generalizes in the following way for distances \(r \geq 1\):

\[
|\mathcal{V}_r| \leq 2r|\mathcal{F}| \leq 6rL^{-1}|\mathcal{E}(T_N)| = 6drL^{-1}N^d.
\]

(34)

With this easy bound done, we proceed with the remainder of the proof.

For each time \(t \in [-\beta/(4n), \beta/(4n)]\) define \(X(t)\) to be the number of sites \(i\) such that \(\tau_i^{(N,L)} \neq \sigma_i(t)\). Then we have a lower bound

\[
|\mathcal{E}| - 6L^{-1}|\mathcal{E}| - U_N(\sigma(t)) \leq 4dX(t).
\]

Let \(\mathcal{A}\) be the event that \(d\nu_0^F(\omega)\) is such that for all times \(t \in [-\beta/(4n), \beta/(4n)]\) we have \(X(t) \leq N^d/8\). On this event we have

\[
1_{\{\sigma(-\beta/(4n)) = \sigma(\beta/(4n)) = \tau_i^{(N,L)}\}} \exp \left(\frac{\Delta - 1}{4} \int_{-\beta/(4n)}^{\beta/(4n)} U_N(\sigma(t)) \, dt\right) \leq e^{-(1/8)(1-\Delta)(1-12L^{-1})|\mathcal{E}|}.
\]

(35)

This is a good inequality for us, for the purpose of the proof. So now we turn our attention to bounding the probability of \(\mathcal{A}\).

Suppose that at some time \(t\) we have that \(i\) is a site in \(T_N \setminus \mathcal{V}_r\) such that \(\sigma_i(t) \neq \tau_i^{(N,L)}\). Then \(i\) is in a block and \(\sigma_i(t)\) is opposite to the spin of the block. Thus in time \([-\beta/(4n), t]\) there must be a path from some neighboring block via overpass edge arrivals of \(d\nu_0^F(\omega)\) to bring this oppositely oriented spin in to site \(i\), and in time \([t, \beta/(4n)]\) there is another path. Since \(i\) is at distance more than \(r\) from \(\mathcal{F}\), this means that there are at least \(2r\) arrivals of \(d\nu_0^F(\omega)\) associated to these two paths.

Because of this we may see that there are at least \(r(X(t) - |\mathcal{V}_r|)\) arrivals of \(d\nu_0^F(\omega)\) in the interval \([-\beta/(4n), \beta/(4n)]\). We divided by 2 since a given arrival of \(d\nu_0^F(\omega)\) could contribute to two different paths for two different vertices \(i, j\) (since an edge has two endpoints). On \(\mathcal{A}\), we have that \(X(t) > N^d/8\) for some time. So on this event, choosing \(r = L/(96d)\), we get that there are at least \(LN^d/(1536d)\) arrivals of \(d\nu_0^F(\omega)\) in the time interval \([-\beta/(4n), \beta/(4n)]\). We have used (34).

Now we recall the large deviation tail bound for a Poisson random variable \(\mathcal{N}\):

\[
P(\mathcal{N} \geq MEN) \leq e^{-(M \ln M - M + 1)EN},
\]

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for each $M \geq 1$. For $d\nu^\beta_{M,T}(\omega)$ in the time interval $[−\beta/(4n),\beta/(4n)]$, the total expectation of all the arrivals is $dN^d\delta T$, where we write $\delta T$ for the length of the time interval, $\delta T = \beta/(2n)$. Therefore

$$\mathbb{P}_1(\mathcal{A}^c) \leq e^{-(M \ln M - M + 1) dN^d \delta T},$$

where $M = L/(1536d^2 \delta T)$. Combining this with (35) and the uniform upper bound

$$1_{\{\sigma(−\beta/(4n)) = \sigma(\beta/(4n)) = \tau(N,L)\}} \exp \left( \frac{\Delta - 1}{4} \int_{−\beta/(4n)}^{\beta/(4n)} U_N(\sigma(t)) \, dt \right) \leq e^{(1/4)(1-\Delta)|\mathbb{E}|\delta T}$$

gives the result.

3.5 Completion of the proof of Theorem 3.3

We combine Corollary 3.5 with Corollary 3.7 and Proposition 3.8. Choose any fixed $\epsilon > 0$ such that $M = L/(1536d^2 \epsilon \min(\beta,L))$ is large enough that $M \ln M - M + 1 \geq (1/64)(1 - \Delta)$. Then, taking $n = \lceil \epsilon \min(\beta,L) \rceil$, we get that

$$\text{Num} \leq 2\epsilon^{(1/64)(1-\Delta)dN^d}\epsilon \min(\beta,L).$$

So taking $c = (1/64)(1-\Delta)\epsilon d$ basically gives the result. Or, choosing $\beta,L$ sufficiently large that $\epsilon \min(\beta,L) \geq 2$ we can remove the effect of the floor function by taking $c$ to be half that previous value.

4 One dimensional results using the six vertex model

In one dimension, on the 1d torus $\mathbb{G} = T_N$, the XXZ model of (2) for $\Delta < 1$ has a ground state which may be understood in terms of the six-vertex model.

In this section, we will give one description of the six-vertex model, and also explain the relation to the ground state of the XXZ model. These are well-known results, which may be found in papers of Lieb for example such as [13]. We give a brief review here, for the benefit of the reader.

The underlying lattice for the six vertex model is the usual square lattice on a discrete torus of horizontal side length $N$ and vertical sidelength which we will call $T$. When we need to, we will refer to the torus as $T_{N,T}$. Both $N$ and $T$ will be assumed even. More precisely, the configuration space for the six-vertex model is the edge set of the torus.

We will introduce some extra notation for the graph $T_{N,T}$ because the spins live on the edges. Let us use a special notation for the vertices of $T_{N,T}$. The vertices will be denoted as $V_{ij}$ for $i \in \{1,\ldots,N\}$ and $j \in \{1,\ldots,T\}$. Let $E_{N,T}$ denote the set of edges of $T_{N,T}$, both horizontal and vertical:

$$E_{N,T} = \{E_{i,j}^h : i = 1,\ldots,N, \ j = 1,\ldots,T\} \cup \{E_{i,j}^v : i = 1,\ldots,N, \ j = 1,\ldots,T\}.$$  \hspace{1cm} (36)

The edges incident to $V_{ij}$ are $E_{i,j}^h$, $E_{i,j}^v$, $E_{i-1,j}^h$ and $E_{i,j-1}^v$, where we identify $i+N \equiv i$ and $j+T \equiv j$ on the torus. In Figure 4, in the picture on the left, we show an example when $N = T = 6$ along with a few labelled edges and a labelled vertex.

The six-vertex configurations are assignments of spins to $E_{N,T}$ satisfying certain conditions. We will interpret a $+$ ($-$) spin on a horizontal edge to point right (left). As usual, a $+$ ($-$) spin on a vertical edge will point up (down). Let $S_{N,T}$ denote the set of spin configurations $\varsigma : E_{N,T} \to \{+1,-1\}$ satisfying the following six-vertex conditions: for each $i \in \{1,\ldots,N\}$ and $j \in \{1,\ldots,T\}$, considering the spins on the edges incident to $V_{ij}$, we have

$$\varsigma(E_{i,j-1}^h) - \varsigma(E_{i,j}^h) + \varsigma(E_{i,j-1}^v) - \varsigma(E_{i,j}^v) = 0. \hspace{1cm} (37)$$
These are the valid six-vertex configurations. The conditions mean that at each vertex there are two spins in and two spins out. Since there are four edges, the number of such choices is 4-choose-2, which gives rise to the name “six” vertex configuration.

In Figure 4, in the picture on the right we give an example of a valid six-vertex configuration of arrows: satisfying two-in, two-out at every vertex. The dashed edges are edges which are actually repeated on the other side of the torus due to periodic boundary conditions. Similarly, the grey arrows are arrows which are repeated from the other side.

For the example of a six-vertex configuration in Figure 4, we give some extra decorations, to help to see certain features. For each vertex \( V_{ij} \) we call it a “sink” if

\[
(\varsigma(E^h_{i-1,j}) = \varsigma(E^v_{ij}) = -\varsigma(E^h_{i,j}) = -\varsigma(E^v_{i,j-1})) = 1.
\]

In the picture on the right in Figure 4 the sinks are denoted by black (filled) circles. A source is a vertex \( V_{ij} \) such that \( \varsigma(E^h_{i-1,j}) = \varsigma(E^v_{ij}) = -\varsigma(E^h_{i,j}) = -\varsigma(E^v_{i,j-1}) = -1 \). Sources are indicated in Figure 4 by white (open) circles.

In the six vertex model there is also a weight for each \( \varsigma \in \Sigma_{N,T} \). For each \( i \in \{1, \ldots, N\} \), \( j \in \{1, \ldots, T\} \) let us define

\[
m_{ij}(\varsigma) = 1\{\varsigma(E^h_{i-1,j}) = \varsigma(E^v_{ij}) = -\varsigma(E^h_{i,j}) = -\varsigma(E^v_{i,j-1})\},
\]

which is the indicator of having either a sink or a source at \( V_{ij} \). Given a parameter \( \kappa \in \mathbb{R} \), which will play the role of an inverse-temperature, the weight of any \( \varsigma \in \Sigma_{N,T} \) is

\[
W^\kappa_{N,T}(\varsigma) = \prod_{i=1}^{N} \prod_{j=1}^{T} w_{ij}^\kappa(\varsigma), \quad w_{ij}^\kappa(\varsigma) = e^{\kappa m_{ij}(\varsigma)}.
\]

The relation between the XXZ model and the six-vertex model was originally noted by Lieb, for example in [13]. The row-to-row transfer matrix of the six-vertex model may be viewed as an operator on \( \mathcal{H}_N \), the Hilbert space for the circle, \( \mathbb{T}_N \) for \( d = 1 \). This operator has the same eigenvectors as \( \mathcal{H}_N^\Delta \) if \( K \) and \( \Delta \) satisfy a certain relation.

Let us be more precise. Given the parameter \( K \), the row-to-row transfer operator for the six vertex model may be defined as \( A_{N,K} : \mathcal{H}_N \rightarrow \mathcal{H}_N \) (where again \( \mathcal{H}_N \) is quantum spin Hilbert space for the graph \( G = \mathbb{T}_N \) with \( d = 1 \)). It is defined as follows. Let \( \Sigma_N \) denote \( \Sigma_\mathcal{V} \) for \( \mathcal{V} = \mathbb{T}_N \). For
any $\sigma, \sigma' \in \Sigma_N$ define

$$\langle \Psi_N(\sigma), A_{N,\kappa} \Psi_N(\sigma') \rangle = \sum_{\tau \in \Sigma_N} \prod_{i=1}^{N} \left( \mathbf{1}\{\tau_{i-1} = \tau_i\} \mathbf{1}\{\sigma'_i = \sigma_i\} + e^\kappa \mathbf{1}\{\tau_{i-1} = -\tau_i = -\sigma_i\} \right).$$

where $N + 1 \equiv 1$ because of periodic boundary conditions, as usual. This represents the product of indicators for (37) for $i = 1, \ldots, N$ for a fixed $j$, and the product of the weights (39), if we define a partial configuration only on edges incident to the vertices $V_{ij}$ for $i = 1, \ldots, N$ and a fixed $j$, where $\varsigma(E_{i,j-1}^h) = \sigma_i$, $\varsigma(E_{i,j}^e) = \sigma'_i$ and $\varsigma(E_{i,j}^h) = \tau_i$. See Figure 5 for an example. Therefore, for example, defining the partition function for the six vertex model on $T_{N,T}$ with inverse temperature $\kappa$, we have

$$Z_{N,T}(\kappa) = \sum_{\varsigma \in \mathcal{S}_{N,T}} W_N^{N,T}(\varsigma) = \text{Tr}[\{A_{N,\kappa}\}^T].$$

The following is an important theorem due to Sutherland in [16]:

**Proposition 4.1.** For each $N$, the operators $A_{N,\kappa}$ and $H_{N,\Delta}$ (for $d = 1$) commute if

$$\Delta = 1 - \frac{1}{2} e^{2\kappa}.$$  \hspace{1cm} (42)

An important observation for us is that $A_{N,\kappa}$ commutes with $M_M$ for each $M \in \{-\frac{1}{2}N, \ldots, \frac{1}{2}N\}$, just like $H_{N,\Delta}$. This might not be totally obvious at first, but it follows using periodic boundary conditions of $T_N$ (for $d = 1$). Let us state a bit more precisely why, since the lemma we will introduce will be useful to us for our argument to prove the upper bounds (8), later.

**Lemma 4.2.** Suppose that $\varsigma \in \mathcal{S}_{N,T}$ is any valid six-vertex configuration. For any $j \in \{1, \ldots, T\}$, the sinks and sources on the vertices $V_{ij}$, $i = 1, \ldots, N$ must alternate. Similarly, for any $i \in \{1, \ldots, N\}$, the sinks and sources on the vertices $V_{ij}$, $j = 1, \ldots, T$ must alternate.

**Proof.** Suppose that $V_{ij}$ is a source. Then on the edges $E_{ij}^h$ and $E_{i-1,j}^h$, the arrows are flowing away from $V_{ij}$. This means $\varsigma(E_{ij}^h) = +1$ and $\varsigma(E_{i,j}^h)$. Similarly, we will have $\varsigma(E_{i+1,j}^h) = +1$, and so on, until we come to a vertex $V_{i+k,j}$ which is a sink. Similarly $\varsigma(E_{i-2,j}^h) = -1$ and so on until we come to a vertex $V_{i-k,j}$ which is a sink. If we start with a source there is a symmetric argument. This shows that sinks and sources must alternate in rows. The argument for columns is exactly symmetric. \hfill $\square$

**Corollary 4.3.** For any $\varsigma \in \mathcal{S}_{N,T}$, looking at any row or column, the number of sinks must equal the number of sources.

**Proof.** This follows from Lemma 4.2 and periodic boundary conditions. \hfill $\square$

**Corollary 4.4.** For any $N$ and any $M \in \{-\frac{1}{2}N, \ldots, \frac{1}{2}N\}$, the operator $A_{N,\kappa}$ commutes with $M_M$. 

---

**Figure 5:** An example of the spins $\sigma$, $\sigma'$ and $\tau$ involved in (40).
Given Corollary 4.6. in Appendix A.6. For us now, the following is the most important implication. We will discuss both of the important and well-known results Proposition 4.1 and Proposition 4.5.

Proof. Suppose that $\sigma, \sigma' \in \Sigma_N$ are two spin configurations and $\Psi_\sigma$ is in the range of $M_M$. Suppose that there is some $\tau \in \Sigma_N$ such that

$$\prod_{i=1}^{N} \left(1\{\tau_i-1 = \tau_i\}1\{\sigma'_i = \sigma_i\} + e^{\kappa}1\{\tau_i-1 = -\tau_i = -\sigma_i\}\right) > 0.$$ 

Then note that the sinks and sources of $\tau$ must be equal. For each sink at a position $i$ we have $\sigma_i = -\sigma'_i = -1$. If there is a source, then $\sigma_i = -\sigma'_i = 1$. Otherwise $\sigma_i = \sigma'_i$. Thus, since the number of sinks equals the number of sources by Corollary 4.3, $\sum_{i=1}^{N} \sigma_i = \sum_{i=1}^{N} \sigma'_i$. \hfill $\square$

Lieb used the Bethe ansatz [13] and the Perron-Frobenius theorem to prove the following:

**Proposition 4.5.** For any $N$ and for any $M \in \{-\frac{1}{2}, N, \ldots, \frac{1}{2}, N\}$ restrict attention to the subspace which is the range of $M_M$. Then the eigenspace of $H_{N,\Delta}$ restricted to this subspace with minimal eigenvalue is one dimensional, and one may choose a unique normalized vector $\Psi_{N,M}^{(\Delta)}$ in this eigenspace such that $\langle \Psi_{N,M}^{(\Delta)}(\Psi_N(\sigma))$ is nonnegative for every $\sigma \in \Sigma_N$, and positive whenever $\Psi_N(\sigma)$ is in the range of $M_M$. The similar property holds for $A_{\Delta,\kappa}$ looking for the eigenspace with the largest eigenvalue, including that it is spanned by the same eigenvector $\Psi_{N,M}^{(\Delta)}$, as long as the relation (42) is satisfied.

Using Sutherland’s theorem, this proposition may be proved without using the Bethe ansatz. We will discuss both of the important and well-known results Proposition 4.1 and Proposition 4.5 in Appendix A.6. For us now, the following is the most important implication.

**Corollary 4.6.** Given $N$ and $M \in \{-\frac{1}{2}, N, \ldots, \frac{1}{2}, N\}$, suppose $\sigma \in \Sigma_N$ is such that $\Psi_N(\sigma)$ is in the range of $M_M$. Then for any $\Delta$ and $\kappa$ satisfying (42)

$$|\langle \Psi_{N,M}^{(\Delta)}(\Psi_N(\sigma))|^2 = \lim_{T \to \infty} \frac{\sum_{\varsigma \in \mathcal{S}_{N,T}} W_{N,T}^{\varsigma}(\varsigma)1\{\forall i \in \{1, \ldots, N\}, \varsigma(E_i^{\eta}) = \sigma_i\} \{1\{s_{1,1} + \cdots + s_{N,1} = 2M\}\}.$$

Proof. Let $a_{N,M,\kappa}$ be the eigenvalue such that $A_{\Delta,\kappa}^{(\Delta)} = a_{N,M,\kappa} \Psi_{N,M}^{(\Delta)}$. Let $\Psi_\alpha$ and $a_\alpha$, for $\alpha = 1, \ldots, 2^N - 1$ be the remaining ortho-normalized eigenvectors and eigenvalues for the real symmetric operator $A_{N,\kappa}$. Then

$$\sum_{\varsigma \in \mathcal{S}_{N,T}} W_{N,T}^{\varsigma}(\varsigma)1\{\forall i \in \{1, \ldots, N\}, \varsigma(E_i^{\eta}) = \sigma_i\} = a_{N,M,\kappa}^{T} |\langle \Psi_{\sigma}, \Psi_{N,M}^{(\Delta)}|2 + \sum_{\alpha = 1}^{2^N-1} a_\alpha^{T} |\langle \Psi_N(\sigma), \Psi_\alpha\rangle|^2.$$ 

A similar formula holds for the trace, so we see that since $|a_\alpha| < a_{N,M,\kappa}$ for each $\alpha$, we have that the right hand side of (43) is

$$\lim_{T \to \infty} \frac{|\langle \Psi_{\sigma}, \Psi_{N,M}^{(\Delta)}\rangle|^2 + \sum_{\alpha = 1}^{2^N-1} a_\alpha^{T} |\langle \Psi_N(\sigma), \Psi_\alpha\rangle|^2}{1 + \sum_{\alpha = 1}^{2^N-1} a_\alpha^{T} |\langle \Psi_N(\sigma), \Psi_\alpha\rangle|^2} = |\langle \Psi_{\sigma}, \Psi_{N,M}^{(\Delta)}\rangle|^2,$$

which is evidently the same as the left hand side of (43). \hfill $\square$

Let us denote

$$\mathcal{M}(N, M) = \{s \in \mathcal{S}_{N,T} : s_{1,1} + \cdots + s_{N,1} = 2M\}.$$

An immediate consequence is the following.
Corollary 4.7. Suppose that $d = 1$, as we have been assuming throughout this section. Given $N$ and $M \in \{-\frac{1}{2}, N, \frac{1}{2}\}$ and $\Delta < 1$ we have the following. For any $L \leq N$,

$$
\lim_{\beta \to \infty} \frac{\langle M_M, Q L \rangle_{N, \Delta, \beta}}{\langle M_M \rangle_{N, \Delta, \beta}} = \lim_{T \to \infty} \frac{\sum_{s \in S_{N, T}} W_{N, T}^c(s) \{ \forall i \in \mathbb{B}_L, \ s(E_{i,1}) = +1 \} 1_{M(M,M)}}{\sum_{s \in S_{N, T}} W_{N, T}^c(s) 1_{M(M,M)}},
$$

(45)

where $\kappa \in \mathbb{R}$ is the unique parameter satisfying (42) for $\Delta$.

Proof. Use the fact that $\langle M_M X \rangle_{N, \Delta, \beta}$ is asymptotic to $\langle \Psi_{N,M}^{(\Delta)} \rangle$ as $\beta \to \infty$ for any operator $X$ commuting with $M_M$, and then in (43) sum over all $\sigma \in \Sigma_N$ such that $\sigma_i = +1$ for all $i \in \mathbb{B}_L$. \qed

5 The Upper Bound (8) in Theorem 1.2

For the 1d problem the upper bound (8) is easier than the lower bound (7). So we start with the proof of (8). For both bounds, a key will be to use Corollary 4.7. To begin with, we use Lemma 4.2 to bound the number of changes of downspins in any interval when going from one row to the next.

Lemma 5.1. Suppose that $L \leq N$. For any $\varsigma \in S_{N, T}$, if $\sum_{i=1}^{L} \varsigma(E_{i,1}) = L - 2k$ then $\sum_{i=1}^{L} \varsigma(E_{i,2}) = L - 2k'$ where $k'$ is in $\{k - 1, k, k + 1\}$.

Proof. This is the same basic argument as in the proof of Corollary 4.3. What changes is that, when $L$ is less than $N$, the number of sinks and sources on the vertices $V_{ij}, i \in \{1, \ldots, L\}$ need not be equal. But they can only differ by at most one, by Lemma 4.2. Hence $k' - k$ must be in $\{-1, 0, 1\}$. \qed

An immediate corollary is this.

Corollary 5.2. Suppose that for a fixed $j \in \{1, \ldots, N\}$, we have $\sum_{i=1}^{L} \varsigma(E_{i,j}) = L$. Then for any real number $\epsilon > 0$, and any integer $\ell$ such that $\ell + 1 \leq \epsilon L$, we also have $\sum_{i=1}^{L} \sum_{j' = j - \ell}^{j + \ell} \varsigma(E_{i,j'}) \geq L(2\ell + 1)(1 - \epsilon)$.

Proof. Iterate Lemma 5.1 to obtain $\sum_{j'=j-\ell}^{j+\ell} \varsigma(E_{i,j'}) = L - 2k'$ with $|k'| \leq |j' - j|$. Summing over $j' \in \{j - \ell, \ldots, j + \ell\}$ (and using the formula for the sum of an arithmetic sequence) gives $\sum_{i=1}^{L} \sum_{j'=j-\ell}^{j+\ell} \varsigma(E_{i,j'}) = L(2\ell + 1) - 2K$ where $K \leq \ell(\ell + 1)$. Using $\ell + 1 \leq \epsilon \cdot L$ and $2\ell \leq 2\ell + 1$, we see that $2K \leq \epsilon L(2\ell + 1)$. \qed

Corollary 5.3. With the same hypotheses as in the last corollary, let $S_{L,\ell}^{(j)}$ be the set of all valid six-vertex configurations $\varsigma$ on the set of edges incident to all vertices $V_{ij}, i \in \{1, \ldots, L\}, j' \in \{j - \ell + 1, \ldots, j + \ell\}$, subject to the constraint $\sum_{i=1}^{L} \varsigma(E_{i,j}) = L$. Then

$$
\frac{\ln |S_{L,\ell}^{(j)}|}{L(2\ell + 1)} \leq -\epsilon \ln (\epsilon) - (1 - \epsilon) \ln (1 - \epsilon) + \frac{\ln(2)}{L}.
$$

(46)

Note: the set $S_{L,\ell}^{(j)}$ is the set of all assignments of $\varsigma(E_{i,j'})$ for $i \in \{1, \ldots, L\}$ and $|j - j'| \leq \ell$, and $\varsigma(E_{k,j'})$ for $i \in \{0, \ldots, L + 1\}$ and $j' \in \{j - \ell + 1, \ldots, j + \ell\}$. Nevertheless we have divided $\ln |S_{L,\ell}^{(j)}|$ by $L(2\ell + 1)$ because that is the total number of vertical edges, and fits with the last corollary.

Proof. Let us first focus just on the spins on the vertical edges: $\varsigma(E_{i,j'})$ for $i \in \{1, \ldots, L\}$ and $|j - j'| \leq \ell$. By corollary 5.2, we know that the total number of $\downarrow$ spins among these vertical edges is at most an integer $K$ where $K \leq \epsilon M$, where we define $M = L(2\ell + 1)$ for notational convenience.

Then the standard binomial large deviation bound gives that the total number of choices of placing $K$ $\downarrow$ spins among $M$ vertical edges is bounded by $\exp(-K \ln(\epsilon) - (M - K) \ln(1 - \epsilon))$. For
example, taking $p = \epsilon$ in the Binomial $(M, p)$ distribution, we see that the total probability is bounded by 1, but this is at least equal to $p^K(1 - p)^{(M - K)}$ times the total number of ways to choose at most $K$ objects from a group of $M$ because $1 - p \geq p$ by hypothesis. By monotonicity, it is easy to see that $\exp(-K \ln(\epsilon) - (M - K) \ln(1 - \epsilon))$ is at most $\exp(M[-\epsilon \ln(\epsilon) - (1 - \epsilon) \ln(1 - \epsilon)])$, as long as $\epsilon \leq 1 - \epsilon$.

The extra $\ln(2)/L$ arises from the fact that for any row of horizontal edges with no sources or sinks (an unusual occurrence) all spins will point in the same direction, and there are two choices $\leftarrow$ or $\rightarrow$. This is the maximum number of choices for a given row of horizontal edges, once the neighboring rows of vertical edges have been prescribed. That occurs if and only if both of the two neighboring rows of vertical edges have precisely the same spin configurations. Otherwise, there is either 1 choice, or no valid choices, of prescribing all spins on the row of horizontal edges to satisfy the six-vertex configuration given the spins on the neighboring vertical rows.

Equation (46) is a basic entropy-type bound. The upper bound (8) will use entropy-type bounds to bound the likelihood of having a long row of vertical edges with all spins aligned.

### 5.1 Chessboard estimate and proof strategy for (8) in Theorem 1.2

Fröhlich, Israel, Lieb and Simon proved that the six-vertex model is a reflection positive classical spin system, much like the classical Ising model [4]. See section 6.1 of that reference culminating in Theorem 6.1, there. We state the following very minor generalization of their result, here.

Let us define the Boltzmann-Gibbs state, as

$$\langle f(\varsigma) \rangle_{N,T,\kappa} = \frac{\sum_{\varsigma \in \mathcal{S}_{N,T}} W_{N,T}^\kappa(\varsigma) f(\varsigma)}{\sum_{\varsigma \in \mathcal{S}_{N,T}} W_{N,T}^\kappa(\varsigma)},$$

and the state constrained to the zero magnetization, as

$$\langle f(\varsigma) \rangle_{N,T,\kappa,0} = \frac{\langle f(\varsigma) 1_{M(N,0)}(\varsigma) \rangle_{N,T,\kappa}}{\langle 1_{M(N,0)}(\varsigma) \rangle_{N,T,\kappa}}.$$  

Then we have the following result.

**Theorem 5.4.** For any $n$ and $t$, let $A \subset \mathcal{S}_{N,T}$ be an event depending only on the spins at edges incident to vertices $V_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq T$. Then, for any $a$ and $b$ such that $2^an \leq N$ and $2^bt \leq T$,

$$\langle 1_A \rangle_{N,T,\kappa,0} \leq \left( \prod_{I=0}^{2^n-1} \prod_{J=0}^{2^t-1} \theta_{n,I}^{(1)}(\theta_{t,J}^{(2)}(A)) \right)^{1/2^{a+b}},$$

where $\theta_{n,I}^{(1)}$ is defined by

- if $I$ is even then $\theta_{n,I}^{(1)}$ is translation in the horizontal direction by $nI$;
- if $I$ is odd then $\theta_{n,I}^{(1)}$ involves translating by $nI$, then mapping spins on the horizontal and vertical edges by reflecting (in the horizontal directions) all edges, but keeping the orientation of the spins the same on the horizontal edges, and then flipping the orientation of all spins on vertical edges;

and $\theta_{t,J}^{(2)}$ is defined similarly by switching the notion of horizontal and vertical directions.

This is a direct translation of Theorem 6.1 in [4], except for the fact that we have conditioned on the event $M(N,0)$. But this is a simple modification, which is well-known to work with reflection positivity. We will comment briefly on this generalization in appendix A.6.
Figure 6: For \( n = 3 \) and \( t = 1 \), if \( A \) is the set of spin configurations with the prescribed spins at the edges pictured on the left picture then \( A \cap \theta^{(1)}_{n,1}(A) \) is pictured on the right picture.

An example of \( \theta^{(1)}_{1,1} \) is pictured in fig. 6. Note that an alternative description of \( \theta^{(1)}_{n,1} \) in the odd case is as follows: one may assign spin configurations to half-edges, with the consistency condition that on two half-edges forming a full edge, there is the same spin. Then one may reflect all half-edge spins across the vertical line, which changes \( \leftarrow \) spins into \( \rightarrow \) spins and vice-versa, but which leaves \( \uparrow \) spins and \( \downarrow \) spins unchanged. Then one reverses all the spins in the new block, including spins on both the horizontal and vertical edges. This has the effect of flipping the \( \rightarrow \) and \( \leftarrow \) spins again, so that now the horizontal half-edges adjacent to the vertical flip line are consistently labelled across the whole edge. But since a global flip preserves the 6-vertex condition, the whole configuration still does satisfy the 6-vertex conditions, everywhere.

Theorem 5.4 should be compared to Lemma 3.2. Actually, at this point there is no “loss” in Theorem 5.4. We will provide the lossy version in the context of the actual proof of the upper bound.

First, let us give a brief outline of how the proof will proceed.

5.1.1 Proof outline for the upper bound:

For the upper bound in 1d we need to prove (8). Theorem 5.4 is a key to this inequality, much as Lemma 3.2 was key to (6). However, now because of some differences in the type of model we use a slightly different argument than before.

The beginning of the upper bound is still to consider a long string of aligned spins. Now we consider vertical edges \( E_{ij}^v \) for say \( i \in \{1, \ldots, L\} \), and any \( j \). We will let \( \mathcal{A}^{(j)}_{L} \) be the event that all the spins on these \( L \) edges are \( \uparrow \) spins. Using Corollary 5.3 we know that there are relatively few choices of spin configurations for a box of height \( \ell \), with \( \ell + 1 \leq \epsilon L \), surrounding this row. Using reflection positivity as in Theorem 5.4 we will disseminate the event \( \mathcal{A}^{(j)}_{L} \), tiling \( \mathcal{T}_{N,T} \) with tiles of length \( n = L \) and height \( t = 2\ell + 1 \). More precisely, then we choose \( j = \ell \) so that the first tile is in the lower left corner before periodic boundary conditions.

Because the number of choices of configurations in each tile, satisfying \( \theta^{(1)}_{n,1}(\theta^{(2)}_{n,1}(\mathcal{A}^{(\ell)}_{L})) \), is so low, the actual weights \( w^{\kappa}_{ij}(\varsigma) \) will not be able to make up for this, even if \( \kappa \) is negative, so that configurations with spins aligned result in higher weights. This is closely related to the fact that \( \epsilon \ln(\epsilon) \) is right continuous at \( \epsilon = 0 \), but the derivative is infinite. In other words, we may say that we are exploiting the fact that the all-aligned spin states suffer instabilities due to entropy. Thinking of \( w^{\kappa}_{ij}(\varsigma) \) as being like a classical Boltzmann-Gibbs weight related to a Hamiltonian, we may say that the energy cannot balance the entropy, when the entropy is so extreme. (The local interaction for the Hamiltonian is the indicator of having a source or sink spin configuration at a given vertex.)

We will need to show that the total number of configurations times weights is small in comparison to the unconstrained partition function. In principle, the partition function of the six-vertex model is calculable using the Bethe ansatz. But we prefer to just use easy bounds. For this purpose we will consider configurations which still have a small density of mis-aligned spins on vertical edges. But we will make the density significantly larger than the density \( \epsilon \) associated to the events \( \mathcal{A}^{(\ell)}_{L} \) disseminated as above. Then we can get easy binomial type lower bounds on the partition function.
5.2 The Partition Function: Denominator Bound

Let us define
\[ Z_{N,T}(\kappa, 0) = \sum_{\varsigma \in S_{N,T}} W_{N,T}^\kappa(\varsigma) \mathbf{1}_{M(N,0)}(\varsigma). \] (50)

Then
\[ \langle f(\varsigma) \rangle_{N,T,\kappa,0} = \frac{\sum_{\varsigma \in S_{N,T}} W_{N,T}^\kappa(\varsigma)f(\varsigma)\mathbf{1}_{M(N,0)}(\varsigma)}{Z_{N,T}(\kappa, 0)}. \] (51)

We could apply this formula to the function \( f(\sigma) \) appearing inside the expectation on the right hand side of (49). (We will actually apply it to a slightly different function, but this example gives the right idea.) Therefore to get an upper bound on the left hand side of (51), we need an upper bound on the numerator of the right hand side of (51), and a lower bound on the denominator. We are going to start with a lower bound on the denominator, meaning a lower bound on \( Z_{N,T}(\kappa, 0) \).

Let \( R \) be any positive integer. Recall that \( N \) and \( T \) are both assumed to be even for the torus \( \mathbb{T}_{N,T} \). We divide \( N \) into \( 2 \lfloor N/(2R) \rfloor \) intervals of length \( R \). There may be a remainder of sites, \( N - 2 \lfloor N/(2R) \rfloor \), which is even. We split these remaining sites into two subintervals of equal size as well.

Now we have a “background” configuration. This is a configuration that we will perturb. On rows of vertical edges, on alternating tiles, we make the spins all \( \uparrow \) and all \( \downarrow \). That is the background configuration.

The rows of vertical edges are also numbered by \( j \). On even rows, in each full tile (intervals of length \( R \), not remainder intervals), we choose one vertical edge to have a reversed spin. An example is shown in fig. 7. For each reversed spin, if it is \( \downarrow \), then we have a source at the top vertex (a filled circle) and a sink at the bottom vertex (an empty circle). It is \( \uparrow \), then it is reversed.

We have \( 2 \lfloor N/(2R) \rfloor \) sinks and sources on each line of horizontal edges. Therefore for each \( \varsigma \) obtained this way, we have that \( W_{N,T}^\kappa(\varsigma) \) is equal to \( \exp(2\kappa \lfloor N/(2R) \rfloor T) \).

There is also a choice with \( R \) possible outcomes, for each of the \( 2 \lfloor N/(2R) \rfloor \) full tiles per row. But only for half of the rows. Therefore, the total number of \( \varsigma \)'s obtained in this way is \( \exp(\lfloor N/(2R) \rfloor T \ln(R)) \).

Hence, putting the energy and entropy bounds together, we obtain the lower bound
\[ \frac{\ln Z_{N,T}(\kappa, 0)}{NT} \geq \frac{\lfloor N/(2R) \rfloor}{N} (2\kappa + \ln(R)). \] (52)

This is not a particularly good bound, in general. But it is convenient for comparing to the upper bound we will obtain in the next subsection, when \( R \) is large but much smaller than \( 1/\epsilon \).

5.3 The Chessboard Estimate with Some Loss: Numerator Bound

We begin with a more direct analogue of Lemma 3.2. Let \( \mathcal{A}(L) \) denote the event \( \{ \varsigma \in S_{N,T} : \forall i \in B_L, \varsigma(E_{i,1}^\kappa) = +1 \} \).
For integers $L \leq N/2$ and $\ell \leq T/2$, let us also define the following disseminated event: 
\( \tilde{A}(N, T, L, \ell) \). This will be the event described as follows. For the horizontal direction, make a decomposition of the length $N$ into $2\lfloor N/(2L) \rfloor$ tiles each of length $L$, as well as a number of remainder vertices, numbering less than $2\ell$. Split the remainder vertices into two partial tiles each with equal numbers of vertices. This can be done since $N - 2\lfloor N/(2L) \rfloor$ is even.

For the vertical direction, make a decomposition of the height $T$ into $2\lfloor T/(2\ell) \rfloor$ tiles each of length $\ell$, as well as a number of remainder vertices, numbering less than $2\ell$. Split the remainder vertices into two partial tiles each with equal numbers of vertices. This can be done since $T - 2\lfloor T/(2\ell) \rfloor$ is even.

Disseminate the event $A(L)$, viewed as occupying the first tile, into these $2\lfloor N/(2L) \rfloor \cdot 2\lfloor T/(2\ell) \rfloor$ other tiles, using reflections associated to $\theta_{n,1}^{(1)}$ and $\theta_{t,2}^{(2)}$ for $n = L$ and $t = \ell$. (These involutions are associated to the algebra of functions. But it is easily seen that they map indicator functions to indicator functions. So they may also be seen as acting on events of configurations.) On the partial tiles, continue the same pattern, except only project onto $\zeta = +1$ or $\zeta = -1$ for those sites in the partial tile.

**Lemma 5.5.** Assume that $L$ is fixed with $L \leq N/2$ and $\ell$ is fixed with $\ell \leq T/2$. Then

\[
\langle 1_{A(L)}(\varsigma) \rangle_{N,T,\kappa,0} \leq \left( \langle 1_{\tilde{A}(N,T,L,\ell)} \rangle_{N,T,\kappa,0} \right)^{1/K},
\]

where $K = 2^{\log_2(N/L) + \log_2(T/\ell) + 2}$.

This lemma follows almost immediately from Theorem 5.4. It requires just one more reflection in each direction, through any appropriate plane. We will comment a bit more on this in appendix A.6.

Now we will try to bound the right hand side of (53). It will suffice to obtain a good bound on

\[
\langle 1_{\tilde{A}(N,T,L,\ell)} \rangle_{N,T,\kappa,0}.
\]

For this, we use (51) with $f = 1_{\tilde{A}(N,T,L,\ell)}$. We then need to bound the right hand side of (51). We already have a lower bound on the denominator, $Z_{N,T}(\kappa,0)$. So now we just need an upper bound on the numerator:

\[
\sum_{\varsigma \in S_{N,T}} W^{\varsigma}_{N,T} \langle 1_{M(N,0)}(\varsigma) \rangle_{N,T,L,\ell} \langle 1_{\tilde{A}(N,T,L,\ell)}(\varsigma) \rangle.
\]

This is what we will describe next.

We will choose $\ell$ such that $\ell + 1 \leq \epsilon L$. Then on each full tile, we will know by corollary 5.3 that the number of choices of valid spin configurations for that tile $S_{L,\ell,1,1}$ is such that

\[
\frac{|S_{L,\ell,1,1}|}{\ell L} \leq -\epsilon \ln(\epsilon) - (1 - \epsilon) \ln(1 - \epsilon) + \frac{\ln(2)}{L}.
\]

This is not precisely what was proved in corollary 5.3, but it is easy to see that this is implied by that corollary. Moreover, in each full tile, a source or a sink must have the two incident vertical edges of opposite spins. Therefore, the density of sources and sinks is no greater than $2\epsilon$, where we use corollary 5.2 to bound the number of reversed spins (relative to the majority in that tile). So the product of the $w^{\varsigma}_{\epsilon}(\varsigma)$’s for a single full tile is no greater than $\exp(2\epsilon|\varsigma|\ell L)$.

If one only had to worry about full tiles, then this would lead one to a bound such as

\[
\frac{1}{NT} \ln \left( \sum_{\varsigma \in S_{N,T}} W^{\varsigma}_{N,T} \langle 1_{M(N,0)}(\varsigma) \rangle_{N,T,L,\ell} \langle 1_{\tilde{A}(N,T,L,\ell)}(\varsigma) \rangle \right) \leq -\epsilon \ln(\epsilon) - (1 - \epsilon) \ln(1 - \epsilon) + \frac{\ln(2)}{L} + 2\epsilon|\varsigma|.
\]

But we do also need to worry about partial tiles.
However, since the partial tiles are smaller than the full tiles, we have that there is no greater a number of configurations satisfying the conditions than for the full tiles, and we still have that the product of all the $w^\kappa(\varsigma)$’s is no greater than $\exp(2|\kappa|\ell L)$. Finally, the number of partial tiles is no greater than the number of full tiles. So we may obtain:

**Corollary 5.6.** With the set-up as above

$$\frac{1}{NT} \ln \left( \sum_{\varsigma \in S_{N,T}} W^\varsigma_{N,T}(\varsigma) 1_{M(N,0)}(\varsigma) 1_{\tilde{A}(N,T,L,\ell)}(\varsigma) \right) \leq -2\epsilon \ln(e) - 2(1 - \epsilon) \ln(1 - \epsilon) + 4\epsilon|\kappa|$$

$$+ \frac{2\ln(2)}{L} + \left( \frac{2L - 1}{N} + \frac{2\ell - 1}{T} \right) (|\kappa| + 2\ln(2)).$$

(54)

**Proof.** This follows from the argument above and the realization that imposing correct boundary conditions at the edge of each tile (instead of considering each tile as having free boundary conditions as above) only further reduces the total number of valid configurations. \[
\square
\]

**5.4 Completion of the proof of (8)**

We will use (52). Let us denote $\eta = 1/(8R)$. Then we may rewrite it as

$$\frac{\ln Z_{N,T}(\kappa,0)}{NT} \geq \frac{|4\eta N|}{N} (-2|\kappa| - \ln(\eta) - 3\ln(2)).$$

(55)

Note that for $N$ sufficiently large, we have $|4\eta N|/N \geq 2\eta$. Also, for $\eta$ sufficiently small, we have $-\ln(\eta) - 2|\kappa| - 3\ln(2)$ is positive. So, combining (55) with (51) and (54), we have

$$\frac{1}{NT} \ln(1_{\tilde{A}(N,T,L,\ell)})_{N,T,\kappa,0} \leq -2\epsilon \ln(e) + 2\eta \ln(\eta)$$

$$+ 4(\epsilon + \eta)|\kappa| - 2(1 - \epsilon) \ln(1 - \epsilon) + 3\eta \ln(2)$$

$$+ \frac{2\ln(2)}{L}.$$

(56)

We have tried to arrange the terms on the right hand side as: most important first, on the first line, second most important on the second line, and least important on the last line. The term on the last line goes to 0 as $L \rightarrow \infty$.

We will choose basically $\epsilon > 0$ very small but fixed, and say $\eta = 2\epsilon$. Then the first line is $-2\epsilon(\ln(e)) + 4\epsilon \ln(2)$. Note that the term $4\epsilon \ln(2)$ may be moved to the second line. With that, all the terms on the second line are bounded linearly in $\epsilon$. They are $O(\epsilon)$. But the term on the first line $-2\epsilon(\ln(e))$ is much larger in absolute value than $O(\epsilon)$ when $\epsilon$ is small. This is because $|\ln(e)|$ diverges as $\epsilon \downarrow 0$. (Again, as we have alluded to before, this is the entropic instability near low densities because $x \ln(x)$ is right continuous at 0, but its derivative diverges.) From all this we see that by the proper choice of $\epsilon$ and $\eta$ we can arrange that the left hand side of (56) is bounded by a strictly negative quantity $-\alpha$.

One then sees that

$$\ln(1_{\tilde{A}(N,T,L,\ell)})_{N,T,\kappa,0} \leq -\alpha NT.$$

So

$$\frac{1}{K} \ln(1_{\tilde{A}(N,T,L,\ell)})_{N,T,\kappa,0} \leq -\frac{\alpha NT}{K}.$$

But $K \leq 4NT/(\ell L)$. So we have

$$\frac{1}{K} \ln(1_{\tilde{A}(N,T,L,\ell)})_{N,T,\kappa,0} \leq -\frac{\alpha \ell L}{4}.$$
Finally, we chose \( \ell \) just so that \( \ell + 1 \leq \epsilon L \), and \( \epsilon > 0 \) is small but fixed. We may choose \( \ell \) such that \( \ell \geq \frac{1}{2} \epsilon L \) for sufficiently large \( L \). So we get

\[
\frac{1}{K} \ln \langle \mathcal{A}(N,T,L,\ell) \rangle_{N,T,\kappa,0} \leq -\frac{\alpha \epsilon}{8} L^2.
\]

Putting this together with Lemma 5.5 and Corollary 4.7 (and the definition of \( \mathcal{A}(L) \) as the event that \( \varsigma(E^{i}_{k}) = +1 \) for all \( i \in B_L \)), we have the desired result.

6 The Lower bound (7) in Theorem 1.2

The lower bound argument uses the six-vertex configuration again, and Corollary 4.7. Now the argument becomes more combinatorial. Let us give an indication how. Firstly, recall that for the \( d \geq 1 \) dimensional system there were two bounds. The upper bound relied on reflection positivity and the Aizenman-Nachtergaele representation. We were able to show essentially that having all spins aligned was energetically unfavorable. The lower bound was variational. By a “dipole” construction, we were able to set up an event which was not entropically too unfavorable, where the entropy refers to the large deviation of the event with respect to the Poisson process of “overpass” and “cul-de-sac” edges.

For the \( d = 1 \) dimensional system, the argument is a bit different. We just completed the upper bound. It relies on reflection positivity. But the argument turned on the entropic instability of the all-aligned state, not the energetic instability.

Now we want to obtain the lower bound. The idea is to try show that given any typical configuration we may perturb it in a square in such a way as to obtain a large interval of all aligned spins. Because configurations are discrete in both space and time, we merely need to construct a configuration. In essence entropy is cut-off below in a discrete system. Note that a “typical configuration,” here will mean that we sample a configuration according to the Gibbs state, and we show that with a probability \( p \) close to 1 this works. In fact, this part of the argument will also use reflection positivity a bit, too. We will comment on this a bit, later.

6.1 Osculating paths

We switch our perspective on six-vertex configurations, now using osculating paths which is an equivalent formulation. This is well known, and is also easily understood. Start with a reference configuration such as: all vertical edges \( \uparrow \) and all horizontal edges \( \rightarrow \). Then for any given edge, color the interior of the edge black if it differs from the reference configuration. Then connect up the edges in the natural way: use rounded corners whenever there is a corner. This way, two paths in an osculating path configuration are allowed to bounce off each other at corners. But they are not allowed to cross at a vertex.

In fig. 8, we show the translation from the 6 valid types of vertices in the six-vertex model and the 6 types of valid vertices for an osculating path configuration.

Let us abbreviate “opc” for “osculating path configuration.”

For later reference, we note that when there is a valid black opc, then if we use the reverse rule for constructing an opc from the 6-vertex configuration, referring to \( \downarrow \) and \( \leftarrow \) spins, then we may get a gray opc. The black and the gray opc’s occupy complementary edges. Two paths of different colors may cross at a vertex. But two paths of the the same colors will not.

The black opc has all of the information. But there are times when the picture of the gray opc is helpful to have for some descriptions.

We have shown how to convert a six-vertex configuration into an osculating paths configuration in a particular example in fig. 9.
The idea of this section is going to be to start with a valid osculating paths configuration on a large $L \times R$ block inside $\mathbb{T}_{N,T}$. (Later we will choose $R$ to be a large, but fixed, multiple of $L$.) We call this block $\Gamma_{L,R} \subset \mathbb{T}_{N,T}$. For $\Gamma_{L,R}$, we do not have periodic boundary conditions. Instead we will have fixed boundary conditions. We consider a given fixed opc on $\mathbb{T}_{N,T}$, let us call it $X$, and we record the pattern of black and gray edges for each edge on the boundary $\partial \Gamma_{L,R}$. We consider all possible valid opc’s of $\Gamma_{L,R}$ with this prescribed pattern on $\partial \Gamma_{L,R}$. Let us call such a one $x$. Then we could cut-out the $\Gamma_{L,R}$ part of $X$ and paste in $x$. Our goal is going to be to seek an $x$ that we can paste in which will have a long row of aligned spins somewhere in $\Gamma_{L,R}$.

6.2 Local “isotopy” moves

Suppose that we have a valid opc $x$ on $\Gamma_{L,R}$ with a prescribed pattern of black and gray edges on $\partial \Gamma_{L,R}$. We will construct a different $y$ on $\Gamma_{L,R}$, with the same pattern of black and gray edges on $\partial \Gamma_{L,R}$ by performing a sequence of “local isotopy moves.”

These are based on “flippable plaquettes.” In fig. 10, we have shown two different squares. These are the two possible configurations for a “flippable plaquette.”

Figure 10: We show two possible squares which may be moved by a local move. We call the square on the left a corner of type $C_-$ and the square on the right a corner of type $C_+$. 

---

Figure 8: This is a translation of the 6 valid vertex configurations, and 6 vertex configurations for the osculating paths. We enumerate them for later reference.

Figure 9: This picture is an example of a valid 6-vertex configuration and its mapping to an osculating path configuration (opc). (The middle picture is an intermediate step to guide the eye.)
Figure 11: On the left, this picture is the same configuration in fig. 9 again. Now on the right we have shown the configuration which is the highest opc with the same boundary conditions.

**Definition 6.1.** We define a + move to be one which switches a type $C_-$ corner square to a type $C_+$ one. We define a − move to be the opposite.

Given an opc $x$ on $\Gamma_{L,R}$, we may define the height of $x$ as $h(x)$, as follows. Recall that an opc $x$ consists black paths and gray paths. Let us only consider black paths. For a given black path, it is an up-right path. Consider all the vertices to the lower right of the path. The height of the opc is equal to the sum of all these vertices, summed over all black paths on $x$.

Then it is easy to see that a + move increases $h(x)$ by 1.

Given $x$, we perform + moves to each $C_-$-type square, one at a time, until there are no more $C_-$ squares. It turns out that, even if there were some choices to be made of which $C_-$ square to turn into a $C_+$ square at some stage, the final configuration is unique. It is called the highest opc relative to the boundary conditions on $\partial \Gamma_{L,R}$. It is characterized by the property that there is no $C_-$-type square.

We claim that once this has been done, the resulting configuration will have a long interval of aligned spins, in a row of vertical edges, unless the boundary edge pattern is unusual, which happens with a low probability. So the unusual event of a bad boundary edge pattern will be controllable, a priori, in terms of the Boltzmann-Gibbs measure.

### 6.3 Blockades

We assume that $\Gamma_{L,R}$ consists of all vertices $V_{ij}$ for $1 \leq i \leq L$ and $1 \leq j \leq R$. The boundary of $\Gamma_{L,R}$ are the edges $E_{0,j}^h$ and $E_{L,j}^h$ for $j \in \{1, \ldots, R\}$ and $E_{i,0}^v$ and $E_{i,R}^v$ for $i \in \{1, \ldots, L\}$.

**Lemma 6.2.** Assume that $x$ is a highest opc in $\Gamma_{L,R}$. Then the following is true in reference to $x$. Suppose that some vertex $V_{ij}$ in $\Gamma_{L,R}$ is either of type 4 or 5 from fig. 8, and suppose $i > 1$ and $j < R$. Then either all vertices $V_{1,j}, \ldots, V_{i-1,j}$ are of type 4 or all vertices $V_{i,j+1}, \ldots, V_{i,R}$ are of type 4, or possibly both.

**Proof.** This proof will rely on pictures, in part. Let us denote the event that a vertex $V_{ij}$ is of type in $\{4,5\}$ by filling in $E_{i-1,j}^h$ and $E_{i,j}^v$ colored black, but not filling in the other two edges $E_{i,j}^h$ and $E_{i,j-1}^v$ with either color, black or gray, since we do not know which color it may be filled in. See fig. 13.

The argument is a simple induction argument. As the first step, we claim that if $V_{ij}$ has type in $\{4,5\}$, and assuming that $x$ is a highest opc (meaning it admits no more + type flips) as we always assume, then either $V_{i-1,j}$ or $V_{i,j+1}$ has type 4, or possibly both. In fig. 14 we have plotted these two vertices next to $V_{ij}$, assuming the type of $V_{ij}$ is in $\{4,5\}$. 29
There is no straight segment spanning the long dimension (which would block black paths from crossing the small dimension), nor any black blockade spanning the long dimension (which would block gray paths from crossing the small dimension). Roughly, heuristically, the length scale for straight segments as well as for blockades is set by the smaller dimension. (In the actual argument we have a tall, thin rectangle instead. But it is easier to draw it laying down.)

Figure 13: This is the graphical representation for the event that the opc \(x\) satisfies that vertex \(V_{ij}\) has type from fig. 8 in \(\{4, 5\}\).

In order to reach a contradiction, suppose that neither \(V_{i-1,j}\), \(V_{i,j+1}\) has type 4. For \(V_{i-1,j}\), since the edge \(E_{i-1,j}^h\) is colored black, this means that its type must be in \(\{2, 4, 6\}\) according to fig. 8. But by our contradiction hypothesis, we assume that \(V_{i-1,j}\) is not of type 4. So its type is in \(\{2, 6\}\).

In particular, this means that the edge \(E_{i-1,j}^v\) is colored gray, as it is in both types 2 and 6 applied to \(V_{i-1,j}\). A similar argument shows that if \(V_{i,j+1}\) is not of type 4, then \(E_{i-1,j+1}^v\) is colored gray. See fig. 15 But then that means that the plaquette joining vertices \(V_{i-1,j}, V_{i,j}, V_{i,j+1}, V_{i-1,j+1}\) is a \(-\) type plaquette according to fig. 10. But then we could apply a \(+\) type plaquette flip to that plaquette, according to definition 6.1. This contradicts our assumption that \(x\) is a highest opc.

So, by a contradiction argument, we have shown that if \(V_{ij}\) has type in \(\{4, 5\}\) then at least one of \(V_{i-1,j}, V_{i,j+1}\) has type equal to 4. But certainly 4 is in \(\{4, 5\}\). So we may repeat the argument. This yields the conclusion that there are a sequence of vertices \(V_{i(1),j(1)}, \ldots, V_{i(K),j(K)}\) for some \(K\), such that for each \(k \in \{1, \ldots, K - 1\}\), \(i(k + 1) = i(k)\) and \(j(k + 1) = j(k) + 1\), or else \(i(k + 1) = i(k - 1)\) and \(j(k + 1) = j(k)\). The number \(K\) is the first time that this path hits either the left face, so that

Figure 14: Assuming vertex \(V_{ij}\) has type in \(\{4, 5\}\), we may look at vertices \(V_{i-1,j}\) and \(V_{i,j+1}\). According to fig. 8, vertex \(V_{i-1,j}\) has type in \(\{2, 4, 6\}\) and vertex \(V_{i,j+1}\) has type in \(\{3, 4, 6\}\).
Figure 15: Assuming vertex $V_{ij}$ has type in $\{4, 5\}$, and assuming that neither $V_{i-1,j}$ nor $V_{i,j+1}$ has type 4, we conclude that the edges $E^v_{i-1,j}$ and $E^b_{i-1,j+1}$ are colored gray. Hence the plaquette joining vertices $V_{i-1,j}, V_{i,j}, V_{i,j+1}, V_{i-1,j+1}$ is a $-$ type plaquette according to fig. 10.

$i(K) = 1$, or the top face so that $j(K) = R$. Moreover, each of these vertices is of type 4.

But now note that since each of these vertices is of type 4, that means that all the 4 edges incident to each vertex $V_{i(k), j(k)}$ is colored black, for each $1 \leq k \leq K$. So no edge incident to any of these vertices may be gray. In other words, this connected chain of vertices creates a blockade against any gray path crossing it. In particular since gray paths are up-right paths, that means that the gray paths must stay out of the rectangle joining $V_{i(K), j(K)}$ to $V_{i(1), j(1)}$. Since $V_{i(K), j(K)}$ is either on the left face or the top face, this implies the stated result. (We recommend the reader to draw a picture if they wish to have some intuition for the last part of this argument.)

6.4 Conclusion of the proof

We will take $R$ to be a large but fixed multiple of $L$. Namely, suppose that $\rho \in (1, \infty)$ is fixed. Then we assume that $R$ is even and

$$\frac{1}{2} \rho \leq \frac{R}{L} \leq 2\rho.$$  \hfill (57)

We consider the rectangle $\Gamma_{2L,R}$. We will prove the following.

**Lemma 6.3.** We consider the six-vertex model with parameter $\kappa \in \mathbb{R}$ fixed. For $\rho$ sufficiently large, there are constants $c_0, C_0 \in (0, \infty)$, depending on $\rho$ and $\kappa$, such that the following happens. Suppose $N, T$ are even and $N \geq 2L$ and $T \geq R$ (for $R$ an even integer satisfying (57)). Fix a rectangle $\Gamma_{2L,R}$ inside $\mathbb{T}_{N,T}$. Let $\mathcal{A}(\Gamma_{2L,R})$ be the event consisting of the set of all $\varsigma \in \mathcal{S}_{N,T}$ such that, taking the boundary conditions of $\varsigma$ on $\partial \Gamma_{2L,R}$, the highest opc on $\Gamma_{2L,R}$ with these boundary conditions has a sequence of $L$ consecutive aligned spins at the edges in some horizontal row of vertical edges somewhere in $\Gamma_{2L,R}$. Then

$$1 - \langle 1_{\mathcal{A}(\Gamma_{2L,R})}(\varsigma) \rangle_{N,T,\kappa,0} \leq C_0 e^{-c_0 L^2}.$$  \hfill (58)

Let us make a remark, in several parts.

- Although we can make the probability of $\mathcal{A}(\Gamma_{2L,R})$ quite close to 1. We cannot eliminate all bad configurations $\varsigma$. For example, consider a striped phase, such that we consider vertical lines of vertical edges in $\mathbb{T}_{N,T}$ and on each line all spins are aligned, but the orientation of the lines alternates. Suppose all the horizontal spins are $\rightarrow$. Then it turns out that on any rectangle $\Gamma_{L,R}$, this is already a highest opc. But obviously no interval of vertical edges is aligned since all vertical edges have spins which alternate in rows.

- Considering the last remark, the reason that this is part of an event with probability bounded as in (58) is precisely because of the condition that all horizontal spins are $\rightarrow$. (Note that if we had said to take the horizontal spins alternating, then it would turn out that this is not a highest opc, and in fact in this case we would obtain the positive conclusion that $\varsigma$ is in $\mathcal{A}_{L,R}$ for sufficiently large $L$.)
• The probability to have all horizontal spins aligned is a very improbable event, relative to the 6-vertex Gibbs measure, for precisely the reasons described in § 5, although now applied to horizontal edges instead of vertical edges. This is a key to the proof.

Let us now state this as a lemma.

Lemma 6.4. For any $\kappa \in \mathbb{R}$, there exists a $\delta > 0$, depending on $\kappa \in \mathbb{R}$, such that the following holds. We consider $\mathbb{T}_{N,T}$ with even $N$ and $T$. For any $L \leq \min\{N/2,T/2\}$, let us consider a prescribed interval $I(L)$ of $L$ consecutive edges in $\mathbb{T}_{N,T}$: either $L$ consecutive horizontal edges in a vertical column, or else $L$ consecutive vertical edges in a horizontal row. The spins on $I(L)$ are all in $\{\leftarrow, \rightarrow\}$ if $I(L)$ is a horizontal interval of horizontal edges, and are all in $\{\uparrow, \downarrow\}$ if $I(L)$ is a horizontal interval of vertical edges. Given this, we let $D_{I(L),\delta}$ be the event that these spins have a density of $\leftarrow$ spins or $\uparrow$ spins in the respective cases in $[0,\delta) \cup (1-\delta,1]$. Then there are constants $c,C \in (0,\infty)$ depending only on $\kappa$ and $\delta$ such that

$$\langle 1_{D_{I(L),\delta}}(\varsigma) \rangle \leq Ce^{-cL^2}.$$

We will now prove this lemma. But we will not include all details since many details are exactly as in § 5. We will explain what changes to get the present result from that one.

Proof. We note that in § 5 we gave an argument that applied to horizontal intervals. The argument was based on having a low density of misaligned spins. We then showed that for entropic reasons, if the density of misaligned spins was low enough, then it was better (in terms of Gibbs probability) to allow sets of configurations with slightly higher density.

Nothing in that proof required the density on the initial interval to be precisely $\delta = 0$. Indeed, even though the density on the initial interval in that section was taken to be zero, in our argument we saw that, as one goes up row-by-row in the block, the density may become $\epsilon/2$ in the middle row, if the height of the block is $\ell = \epsilon L$. That is why we took $\ell = \epsilon L$ with $\epsilon > 0$ small but fixed. If we start with an initial density of $\delta$, then the maximum density will just be $\delta + \frac{\epsilon}{2}$ instead of $\frac{\epsilon}{2}$. If we choose $\delta$ and $\epsilon$ sufficiently small but fixed, then the same argument goes through, essentially unchanged.

Next, to deal with vertical intervals instead of just horizontal intervals, we note that the model is both reflection positive in the vertical and horizontal directions. There is a difference in how we treat the horizontal and vertical dimensions. For instance, in $\langle \cdot \rangle_{N,T,\kappa,0}$ we condition on having an equal number of $\uparrow$ and $\downarrow$ spins on each horizontal row of vertical edges. We do not make any conditioning for the vertical columns of horizontal edges. But this was just to make contact with the XXZ model. This model is reflection positive in both directions, and that is all that was used in § 5. (Indeed, we did use reflection positivity in both directions in § 5 in order to disseminate the initial reference block in both directions.)

We did need the following condition: $L \leq N/2$ and $\ell = \epsilon L \leq T/2$. The reason we needed $L \leq N/2$ is obvious: we needed to reflect the initial block at least 1 full time in the horizontal direction. We took the height of the block to be $\ell = \epsilon L$ for some $\epsilon > 0$ small, but fixed. We may easily demand that $\epsilon \leq 1$. Then we are okay as long as we have $L \leq \min\{N/2,T/2\}$. This is a condition which is unchanged under switching the vertical and horizontal coordinates. So the same argument will work for an interval $I(L)$ consisting of consecutive edges in a vertical column of horizontal edges.

Finally, we note that we can replace the condition of having density in $[0,\delta)$ of one type of arrow with a density in $(1-\delta,1]$ for that same type of arrow, due to spin-flip symmetry of the six-vertex model.

\end{proof}
6.4.1 Proof of Lemma 6.3

We recall that we are assuming the conditions on $R$ from (57). We will choose $\rho$ later. But it is supposed to be large and finite, as $L \to \infty$. Without loss of generality, we will assume that $\Gamma_{2L,R}$ consists of the vertices $V_{ij}$ for $i = 1, \ldots, 2L$ and $j = 1, \ldots, R$. Then that means that the edges in $\partial \Gamma_{2L,R}$ are $E_{i,j}^0$ and $E_{i,j}^1$ for $j = 1, \ldots, R$ and $E_{i,0}^0$ and $E_{i,R}^0$ for $i = 1, \ldots, 2L$.

We are going to identify several events $G_i$ which are “good events” contributing to $\mathcal{A}(\Gamma_{2L,R})$. Then we will identify events $B_i$ which comprise the complement of the good events. But we will be able to bound each $B_i$, using Lemma 6.4.

**Step 1:** Consider the bottom row of vertical edges, $\{E_{i,0}^v : i = 1, \ldots, 2L\}$, which comprises the bottom face in $\partial \Gamma_{2L,R}$. Let us consider the second half of these edges: $E_{i,0}^v$ for $i \in \{L + 1, \ldots, 2L\}$.

$$G_1 = \{ \zeta \in \mathcal{S}_{N,T} : \text{all edges } E_{i,0}^v \text{ are gray for } i = L + 1, \ldots, 2L \}.$$ 

Clearly $G_1 \subseteq \mathcal{A}(\Gamma_{2L,R})$ because the interval $\mathcal{I}_L = \{E_{i,0}^v : i = L + 1, \ldots, 2L\}$ has one color.

Henceforth, we consider $\zeta \in G_1$. This means that there is at least one $i \in \{L + 1, \ldots, 2L\}$ such that $E_{i,0}^v$ is colored black. Let $i_1$ be the smallest such $i$. We may follow the path connected to this black edge. Let us call this path $\gamma$.

$$B_1 = \{ \zeta \in \mathcal{S}_{N,T} : \gamma \text{ passes through } E_{i_1,R/2}^v \}.$$ 

Event $B_1$ would block any black paths in the bottom half of $\Gamma_{2L,R}$ from crossing the thinner direction of the square, $2L$. So we will show later that it has a small probability.

**Step 2:** Let us now consider ourselves to be in the complement of $G_1 \cup B_1$. Then there is some $j_1 \in \{1, \ldots, R/2\}$, such that $\gamma$ passes through $E_{i_1,j_1}^v$ and $\gamma$ turns right there. A right turn is okay for a highest up-right path. It does not necessarily induce a blockade. In other words $V_{i_1,j_1}$ is either of type 4 or 6 in fig. 8. But vertices of type 6 do not induce blockades.

We let

$$G_2 = \{ \zeta \in \mathcal{S}_{N,T} : E_{i,j_1}^v \text{ is gray for each } i \in \{L + 1, \ldots, 2L\} \}.$$ 

Then, once again, $G_2 \subseteq \mathcal{A}(\Gamma_{2L,R})$ because the interval $\mathcal{I}_L = \{E_{i,j_1}^v : i = L + 1, \ldots, 2L\}$ has one color.

**Step 3:** So now we assume we are in the complement of $G_1 \cup B_1 \cup G_2$. This puts us in the situation that there is some vertex $V_{i_2,j_2}$, for some $i_2 \in \{L + 1, \ldots, 2L\}$ and $j_2 \in \{1, \ldots, j_1\}$ which is of type 4 or 5. Let us explain why this must happen.

If it does not happen then $\gamma$ must turn right at $V_{i_1,j_1}$ and then go straight to $V_{2L,j_1}$. Otherwise the path would have to turn up at some $V_{i_2,j_2}$ (since it starts going right at $V_{i_1,j_1}$) and then this vertex would be of type 4 or 5. This forces all edges $E_{i,j_1}^v$ for $i \in \{i_1, \ldots, 2L\}$ to be gray, since the straight horizontal segment of $\gamma$ blocks any other black edges from crossing it. But then we may argue that also all edges $E_{i,j_1}^v$ for $i \in \{L + 1, \ldots, i_1 - 1\}$ are also gray. For if some $E_{i_2,j_2}$ is black for some $i_2 \in \{L + 1, \ldots, i_1 - 1\}$ then it is part of some path $\gamma'$. But following $\gamma'$ down-and-left, it cannot pass through $E_{i_2,0}^v$ because $i_1$ is the smallest possible $i \in \{L + 1, \ldots, 2L\}$ such that $E_{i,0}^v$ is black. Therefore, $\gamma'$ must turn left (following it backwards, so it is going down-and-left) at some vertex $V_{i_2,j_2}$ for $j_2 \in \{1, \ldots, j_1\}$. But that is the eventuality we wanted to occur: so it is part of the event we were hypothetically considering ourselves to be in the complement of. So we reach a contradiction.

This means if we are in the complement of $G_1 \cup B_1 \cup G_2$ then there is some vertex $V_{i_2,j_2}$, for some $i_2 \in \{L + 1, \ldots, 2L\}$ and $j_2 \in \{1, \ldots, j_1\}$ which is of type 4 or 5. Then because of Lemma 6.2 we either have the final two events $G_3$ or $B_2$ where

$$G_3 = \{ \zeta \in \mathcal{S}_{N,T} : V_{i_2,j_2} \text{ is of type 4, for each } i \in \{1, \ldots, i_2 - 1\} \},$$

and

$$B_2 = \{ \zeta \in \mathcal{S}_{N,T} : V_{i_2,j} \text{ is of type 4, for each } j \in \{j_2 + 1, \ldots, R\} \}.$$ 

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In $G_3$ we have that $E^v_j$ is colored black for all $i \in \{1, \ldots, i_2 - 1\}$. But since $i_2 \in \{L + 1, \ldots, 2L\}$, this means that $I_L = \{E^v_i : i = 1, \ldots, L\}$ has all edges black. So $G_3 \subseteq A(\Gamma_{2L, R})$.

Explanation of bad events: In fig. 12 we have shown a highest OPC for a “typical” boundary configuration. Combined with corollary 4.7 and the definitions in (47) and (48), this proves (7).

Moreover, the weight of the configuration bounded above and below by $\exp(\kappa|\cdot|E(\Gamma_{2L,R}))$ is at most $2$ times the number of edges $|E_{2L,j}|$. This blocks any black path from crossing this line. In particular, this means that for any black path incident to $E^h_{2L,j}$ for $j \in \{1, \ldots, R/2\}$ the path connected to it must not terminate on the bottom. So it must terminate on the upper face of $\Gamma_{2L,R}$. This means that there is at most $2L$ such paths. So, among the $(R/2)$ edges $E^h_{2L,j}$ for $j \in \{(R/2) + 1, \ldots, R\}$, at most $2L$ of them may be black. So the density is at most $2L/R$.

Let us now consider $B_2$. In this case we know that there is a black blockade running along the vertices $V_{i_2,j}$ for $j \in \{j_2 + 1, \ldots, R\}$. This blocks any gray paths from crossing it. In particular for any edge $E^h_{i_2,j}$ for $i \in \{(R/2) + 1, \ldots, R\}$ if it is gray, then following it up and right it must terminate on the upper face of $\Gamma_{2L,R}$. This means that there are at most $2L$ such paths. So, among the $(R/2)$ edges $E^h_{i_2,j}$ for $j \in \{(R/2) + 1, \ldots, R\}$, at most $2L$ of them may be gray. So the density is at most $4L/R$.

By choosing $\rho$ appropriately, we may make the density $4L/R$ small enough to apply Lemma 6.4. In particular, given the $\delta$ from Lemma 6.4, let us choose $\rho = 8/\delta$. Then $4/\rho = \delta/2$ so that we may definitely apply Lemma 6.4. Then we see that the two bad scenarios may be included in a subset of events where Lemma 6.4 applies with $L$ replaced by $R/2$ or $R$, which is at least $\frac{4}{\rho} = \frac{1}{\rho}L$. So $c_0 = \frac{4}{\rho}L$, and $C_0 = 3C$.

6.4.2 Final argument

We have the following now. We choose $\Gamma_{2L,R}$ as described before. From Lemma 6.3, there is an event $A(\Gamma_{2L,R})$ such that

$$1 - \langle 1_{A(\Gamma_{2L,R})}(\varsigma) \rangle_{N, T, \kappa, 0} \leq C_0 e^{-c_0 L^2}.$$ 

For each $\varsigma \in A(\Gamma_{2L,R})$, we may alter $\varsigma$ inside $\Gamma_{2L,R}$, by fixing the boundary conditions of $\varsigma$ on $\partial \Gamma_{2L,R}$, and replacing $\varsigma$ by the highest opc in $\Gamma_{2L,R}$ relative to these boundary conditions. Let us call this $\varsigma'$. It is easy to see that the mapping $\varsigma \mapsto \varsigma'$ is at most $n$-to-1 where $n$ is the cardinality of all 6-vertex configurations in $\Gamma_{2L,R}$ which is not more than $2$-to-the-power $|E(\Gamma_{2L,R})|$ the number of edges in $\Gamma_{2L,R}$ which is $4L \leq 8\rho L^2$. So the multiplicity is not more than $\exp(8\rho \ln(2)L^2)$. Moreover, the weight of the configuration $\varsigma'$ is a multiple of the weight of $\varsigma$, where the ratio $\alpha$ is bounded above and below by $\exp(\pm |\kappa| \cdot |E(\Gamma_{2L,R})|)$. This is again of the form $e(c_2 L^2)$ for some $c_2$.

Moreover, the reason for doing all this is that either $\varsigma'$ or $-\varsigma'$ does satisfy for some $i(0), j(0)$ with $V_{i(0), j(0)} \in \Gamma_{2L,R}$: $\forall i \in \mathbb{B}_L$, $\varsigma(E^v_{i(0), j(0)}) = +1$. Let $\varsigma''$ be the mapping where we shift $\varsigma'$ by $-i(0)$ in the x-direction and $1 - j(0)$ in the y-direction, and if necessary we then multiply the resulting configuration everywhere by $-1$. This means that now the mapping $\varsigma \mapsto \varsigma''$ is at most $n'$-to-1, where $n' = 2n(2L)R$. In other words, $n' \leq 8\rho L e^{8\rho \ln(2)L^2}$.

So, letting $A'$ be the event consisting of all $\varsigma$'s such that $\forall i \in \mathbb{B}_L$, $\varsigma(E^v_{i, j}) = +1$, we have

$$\langle 1_{A'}(\varsigma) \rangle_{N, T, \kappa, 0} \geq (8\rho L)^{-1} e^{-8\rho \ln(2)L^2} e^{-8\rho |\kappa| L^2} \langle 1_{A(\Gamma_{2L,R})}(\varsigma) \rangle_{N, T, \kappa, 0} \geq (8\rho L)^{-1} e^{-8\rho \ln(2)L^2} e^{-8\rho |\kappa| L^2} (1 - C_0 e^{-c_0 L^2}).$$

It is easily seen from this that there is some $C_1$ and $c_1$ such that

$$\langle 1_{A'}(\varsigma) \rangle_{N, T, \kappa, 0} \geq C_1 e^{-c_1 L^2}.$$ 

Combined with corollary 4.7 and the definitions in (47) and (48), this proves (7).
7 Outlook and extensions

We now mention some extensions. Nothing in our proof really relied upon the fact that in \( \Lambda_L \) there needs to be a totally empty box. Instead our upper and lower bounds rely on the fact that in an associated graphical representation the density is sufficiently low. We could easily extend the proofs here to the case that the density of oppositely oriented spins in \( \Lambda_L \) is sufficiently low. Of course, since we do not have good control of various bounds, sufficiently low density means density less than \( \epsilon_0 \) for some unknown but fixed, positive \( \epsilon_0 > 0 \). We also have given proofs just for \( \beta \to \infty \). The special \( d = 1 \) bounds require this in an essential way because Lieb’s mapping between the ground state of the XXZ model and the infinite height \( T \to \infty \) limit of the six-vertex model on an \( N \times T \) torus only works for the ground state. It does not work for any other \( \beta \). But for the general results for \( d \geq 1 \), we could take \( \beta \) finite. The difference is that this introduces a cut-off for the imaginary time dimension.

A Appendix

A.1 Equilibrium quantum statistical mechanics

The framework for classical statistical mechanics is probability theory. In quantum statistical mechanics, one starts with a complex Hilbert space \( \mathcal{H} \). In the examples we will consider in this paper, the dimension will be finite. The set of observables, analogous to the random variables in probability, is \( \mathcal{B}(\mathcal{H}) \) the set of bounded linear operators \( X : \mathcal{H} \to \mathcal{H} \). The adjoint of \( X \) is the usual definition \( X^* \) such that \( \langle X \psi, \phi \rangle = \langle \psi, X^* \phi \rangle \) for all \( \psi, \phi \in \mathcal{H} \). The norm is the operator norm, and then \( \mathcal{B}(\mathcal{H}) \) becomes a \( C^* \)-algebra, since \( \|X^*\| = \|X\| \) and \( \|XY\| \leq \|X\| \cdot \|Y\| \). A quantum state is a function \( (\cdot) : \mathcal{B}(\mathcal{H}) \to \mathbb{C} \) satisfying

- **Linearity:** \( (A + cB) = (A) + c(B) \) for any \( A, B \in \mathcal{B}(\mathcal{H}) \) and \( c \in \mathbb{C} \).
- **Positivity:** \( (X^*X) \geq 0 \) for every \( X \in \mathcal{B}(\mathcal{H}) \).
- **Normalization:** \( (I) = 1 \).

**Example:** Suppose that \( \Omega \) is a finite sample space, and let \( p : \Omega \to \mathbb{R} \) be a probability mass function. Let \( \mathcal{H} = l^2(\Omega) \) be the Hilbert space of all functions \( f : \Omega \to \mathbb{C} \) with the norm \( \|f\|^2 = \sum_{\omega \in \Omega} |f(\omega)|^2 \). Then we may define an orthonormal basis \( \chi_{\omega} \) for \( \omega \in \Omega \) given by \( \chi_{\omega}(\omega') = \delta_{\omega,\omega'} \). A quantum state is defined by \( (A) \) for each operator \( A : \mathcal{H} \to \mathcal{H} \)

\[
(A) = \sum_{\omega \in \Omega} A\chi_{\omega}(\omega)p(\omega).
\]

It is easy to see that this is linear and normalized, and moreover

\[
(A^*A) = \sum_{\omega \in \Omega} |f_A(\omega)|^2 p(\omega), \quad f_A(\omega) = \sum_{\omega'} A\chi_{\omega}(\omega'),
\]

for each \( A \in \mathcal{B}(\mathcal{H}) \), from which positivity easily follows. A commutative sub-algebra of \( \mathcal{B}(\mathcal{H}) \) consists of all diagonal operators. Let \( \mathcal{C}(\Omega) \) be the set of all functions on \( \Omega \). This is a commutative \( C^* \)-algebra with: the usual pointwise product \( fg(\omega) = f(\omega)g(\omega) \), adjoint equal to the pointwise complex conjugation \( f \mapsto \bar{f} \), and norm equal to the max-norm \( \|f\|_{\infty} = \max_{\omega \in \Omega} |f(\omega)| \). Given \( f \in \mathcal{C}(\Omega) \) define an operator \( \hat{f} : \mathcal{H} \to \mathcal{H} \) such that given any \( g \in \mathcal{H} \) we have \( \hat{f}g(\omega) = f(\omega)g(\omega) \) for each \( \omega \in \Omega \). Then \( f \mapsto \hat{f} \) is a \( C^* \)-algebra isomorphism between \( \mathcal{C}(\Omega) \) and the diagonal operators in \( \mathcal{B}(\mathcal{H}) \). Moreover, \( \langle \hat{f} \rangle = \sum_{\omega \in \Omega} \hat{f}(\omega)p(\omega) \). From this example, we see that the notion of a quantum
state on a finite dimensional Hilbert space generalizes the notion of the expectation with respect to a probability measure on a finite sample space. For any quantum state $\langle \cdot \rangle$, for an $n$-dimensional Hilbert space $\mathcal{H}$, there is a probability mass function $p_1, \ldots, p_n$, as well as an orthonormal basis $\varphi_1, \ldots, \varphi_n$, such that

$$\langle A \rangle = \sum_{k=1}^{n} p_k \langle \varphi_k, A \varphi_k \rangle.$$  

(59)

So the extra data required for a quantum state, beyond that required for a classical probability measure, is a choice of an ordered orthonormal basis. Suppose that $H : \mathcal{H} \rightarrow \mathcal{H}$ is any self-adjoint operator, called a Hamiltonian operator. Then given any $\beta \in \mathbb{R}$, we may define a quantum state as $\langle X \rangle_\beta$ for each $X \in \mathcal{B}(\mathcal{H})$

$$\langle X \rangle_\beta = \frac{\text{Tr}[X e^{-\beta H}]}{\text{Tr}[e^{-\beta H}]}.$$  

(60)

This is the equilibrium state at inverse-temperature $\beta$, associated to $H$. Let $\text{spec}(H)$ be the set of eigenvalues of $H$. These are all real because $H$ is self-adjoint. Let $\min \text{spec}(H)$ be the minimum eigenvalue, and let $\mathcal{G}_H$ denote the eigenspace of $H$ for the eigenvalue $\min \text{spec}(H)$. Then for any $X \in \mathcal{B}(\mathcal{H})$

$$\lim_{\beta \rightarrow \infty} \langle X \rangle_\beta = \frac{\text{Tr}[X \text{Proj}(\mathcal{G}_H)]}{\text{dim}(\mathcal{G}_H)},$$

where $\text{Proj}(\mathcal{G}_H)$ denotes the orthogonal projection onto $\mathcal{G}_H$. We denote this state as $\langle \cdot \rangle_\infty$.

A.2 Trace inequalities I

Trace inequalities are important in quantum statistical mechanics. Frequently, these follow the classical arguments, but with the addition of a chosen orthonormal basis.

**Theorem A.1.** (1) Suppose that $f : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function, and $H$ is a self-adjoint operator on $\mathcal{H}$. For any vector $\psi \in \mathcal{H}$ with norm $\|\psi\|^2 = 1$,

$$\langle \psi, f(H)\psi \rangle \geq f(\langle \psi, H\psi \rangle).$$

(2) Under the same hypotheses as (1) if $\psi_1, \ldots, \psi_n$ is any orthonormal basis for $\mathcal{H}$ then

$$\sum_{k=1}^{n} f(\langle \psi_k, H\psi_k \rangle) \leq \text{Tr}[f(H)].$$

If $f$ is strictly convex then the inequality is strict unless $\psi_1, \ldots, \psi_n$ are eigenvectors of $H$.

(3) Under the same hypotheses, if $H = tA + (1-t)B$ for self-adjoint operators $A$ and $B$ on $\mathcal{H}$ and $t \in (0,1)$ then

$$\text{Tr}[f(H)] \leq t \text{Tr}[f(A)] + (1-t) \text{Tr}[f(B)],$$

and the inequality is strict if $f$ is strictly convex unless $A = B = H$.

**Proof.** (1) Let $\varphi_1, \ldots, \varphi_n$ be an orthonormal basis of eigenvectors of $H$. Then, defining $p_k = |\langle \varphi_k, \varphi \rangle|^2$, we have a mass function $p_1, \ldots, p_n$. So Jensen’s inequality gives

$$\langle \psi, f(H)\psi \rangle = \sum_{k=1}^{n} p_k f(\langle \varphi_k, H\varphi_k \rangle) \geq f \left( \sum_{k=1}^{n} p_k \langle \varphi_k, H\varphi_k \rangle \right) = f(\langle \psi, H\psi \rangle).$$

(2) Continuing with the same set-up, let $p_{k,j} = |\langle \varphi_k, \varphi_j \rangle|^2$. Then the matrix $(p_{k,j})_{j,k=1}^{n}$ is doubly stochastic. So Jensen’s inequality gives

$$\sum_{k=1}^{n} f(\langle \psi_k, H\psi_k \rangle) = \sum_{k=1}^{n} f \left( \sum_{j=1}^{n} p_{k,j} \langle \varphi_j, H\varphi_j \rangle \right) \leq \sum_{j=1}^{n} \sum_{k=1}^{n} p_{k,j} f(\langle \varphi_j, H\varphi_j \rangle) = \text{Tr}[f(H)].$$
Under the assumption of strict convexity, the inequality is strict unless for each \( k \), the mass function \( p_{k,1}, \ldots, p_{k,n} \) concentrates on \( j \)’s such that \( \langle \phi_j, H \phi_j \rangle \) all have the same value. But this is a way of saying that \( \psi_k \) is an eigenvector for \( H \) with that same eigenvalue.

(3) With the set-up of (3),

\[
\text{Tr}[f(H)] = \sum_{k=1}^{n} f(\langle \phi_k, H \phi_k \rangle) = \sum_{k=1}^{n} f(t(\langle \phi_k, A \phi_k \rangle) + (1-t)\langle \phi_k, B \phi_k \rangle) \\
\leq t \sum_{k=1}^{n} f(\langle \phi_k, A \phi_k \rangle) + (1-t) \sum_{k=1}^{n} f(\langle \phi_k, B \phi_k \rangle)
\]

The inequality is strict unless

\[
\langle \phi_k, A \phi_k \rangle = \langle \phi_k, B \phi_k \rangle = \langle \phi_k, H \phi_k \rangle,
\]

for all \( k \). Then using part (2) with the orthonormal basis \( \phi_1, \ldots, \phi_n \) we obtain

\[
\text{Tr}[f(H)] \leq t \text{Tr}[f(A)] + (1-t) \text{Tr}[f(B)].
\]

Moreover, if \( f \) is strictly convex then the inequality is strict unless \( \phi_1, \ldots, \phi_n \) are eigenvectors of both \( A \) and \( B \). But in that case, (61) implies the eigenvalues for \( \phi_k \) for \( A, B \) and \( H \) are all the same. Therefore, \( A = B = H \). □

An important consequence of Theorem A.1 is the Gibbs variational principle for quantum statistical mechanics. Let \( \mathcal{D}(\mathcal{H}) \) denote the set of all operators \( D \) on \( \mathcal{H} \) such that \( D \) is self-adjoint, and the eigenvalues of \( D \) form a mass function. These are the density matrices. Any state on \( \mathcal{H} \) is of the form \( \langle \cdot \rangle = \text{Tr}[\cdot D] \) for a unique \( D \in \mathcal{D}(\mathcal{H}) \). So these play the role in quantum mechanics, analogous to the role of densities (or mass functions) relative to the counting measure, in classical probability theory on finite sample spaces. The set \( \mathcal{D}(\mathcal{H}) \) is convex. Define \( u(x) = -x \ln(x) \) for \( x \in (0, \infty) \) and \( u(0) = 0 \) so that \( u \) is continuous on \( [0, \infty) \). This is strictly concave. For any \( D \in \mathcal{D}(\mathcal{H}) \) the von Neumann entropy is \( S(D) = \text{Tr}[u(D)] \). This is \( \sum_{k=1}^{n} u(p_k) \) if \( p_1, \ldots, p_n \) are the eigenvalues of \( D \).

**Theorem A.2.** For any self-adjoint operator \( H \) on \( \mathcal{H} \) and \( \beta \in \mathbb{R} \),

\[
\ln \text{Tr}[e^{-\beta H}] = \max_{D \in \mathcal{D}(\mathcal{H})} (S(D) - \beta \text{Tr}[DH]),
\]

and the maximum is attained uniquely at the point \( D_\beta = e^{-\beta H} / \text{Tr}[e^{-\beta H}] \).

**Proof.** Note that \( u(D_\beta) = \beta HD_\beta + \ln \text{Tr}[e^{-\beta H}]D_\beta \). So \( S(D_\beta) = \beta \text{Tr}[D_\beta H] + \ln \text{Tr}[e^{-\beta H}] \). This shows, defining \( \Phi_\beta(D) = S(D) - \beta \text{Tr}[DH] \) on \( \mathcal{D}(\mathcal{H}) \), \( \Phi_\beta(D_\beta) = \ln \text{Tr}[e^{-\beta H}] \). The argument that \( D_\beta \) is the unique arg-max of \( \Phi_\beta \) follows the classical argument. One may take some care for the quantum version of perturbation theory. The function \( \Phi_\beta \) on \( \mathcal{D}(\mathcal{H}) \) equals the strictly concave function \( S \) minus the linear function \( \beta \text{Tr}[\cdot H] \). Therefore, it is strictly concave, and the only possible critical point is the unique maximizer. It remains only to check that \( D_\beta \) is a critical point of \( \Phi_\beta \) and this is perturbative. Since the entropy involves the logarithm, one may perturb the exponential map, working backwards. Suppose that \( X \) is any element of \( \mathcal{B}(\mathcal{H}) \), then a calculation shows that

\[
\frac{d}{dt} e^{-\beta H + tX} \bigg|_{t=0} = \frac{d}{dt} \lim_{n \to \infty} \left( I + \frac{tX - \beta H}{n} \right)^n \bigg|_{t=0} = \int_0^1 e^{-\beta \theta H} X e^{-\beta (1-\theta) H} \, d\theta.
\]

Defining this as \( \mathcal{L}(X) \), cyclicity of the trace implies that

\[
\text{Tr}[\mathcal{L}(X)] = \text{Tr}[XD_\beta] \text{Tr}[e^{-\beta H}] \quad (63)
\]
Also, letting \( \psi_1, \ldots, \psi_n \) be an orthonormal basis of \( \mathcal{H} \) consisting of eigenvectors of \( H \), with associated eigenvalues \( E_1, \ldots, E_n \), and defining \( X_{jk} = \langle \psi_j, \psi_k \rangle \) (using the physicists' convention), \( \mathcal{L}(X_{jk}) = \left( \int_0^1 e^{-\beta t} E_k e^{-\beta(1-\theta)E_k} \, d\theta \right) X_{jk} \). So there is a complete family of eigenvectors of \( \mathcal{L} \) all with positive eigenvalues. So \( \mathcal{L} \) is invertible which expresses the fact that there is a branch of logarithm which is a diffeomorphism in a neighborhood of \( D_\beta \). For any \( D \in \Delta(\mathcal{H}) \) take \( D(t) = (1-t)D_\beta + tD \). Then \( D(t) = \exp(Y(t)) \) for sufficiently small \( t \) where \( Y(t) = -\beta H - \ln(\Tr[e^{-\beta H}])I + tX + O(t^2) \), \( X = \Tr[e^{-\beta H}] \mathcal{L}^{-1}(D - D_\beta) \).

\[
S(D(t)) = - \Tr[Y(t)D(t)] = - \Tr[(-\beta H - \ln(\Tr[e^{-\beta H}])I + tX)((1-t)D_\beta + tD)] + O(t^2)
= (1-t)S(D_\beta) + (1-t) \Tr[XD_\beta] + t(\Tr[\beta HD] + \ln(\Tr[e^{-\beta H}])) + O(t^2).
\]

But (63) implies \( \Tr[XD_\beta] = \Tr[\mathcal{L}(X)]/\Tr[e^{-\beta H}] = \Tr[D - D_\beta] = 0 \). Putting this together with \( \Phi_\beta = S - \beta \Tr[\mathcal{H}] \) and the fact that \( \Phi_\beta(D_\beta) = \ln(\Tr[e^{-\beta H}]) \) implies

\[
\lim_{t \to 0^+} \frac{d}{dt} \Phi_\beta(D(t)) = 0,
\]

which is true for all \( D \in \Delta(\mathcal{H}) \) so that \( D_\beta \) is a critical point of \( \Phi_\beta \).

We stated part (1) of Theorem A.1 in order to also obtain the following. Recall that \( A \) is a positive semi-definite operator if and only if \( A \) is self-adjoint and \( \langle \psi, A\psi \rangle \) is nonnegative for all \( \psi \in \mathcal{H} \).

**Theorem A.3.** Suppose that \( H, H' \) are both self-adjoint operator such that \( H - H' \) is positive semi-definite. If \( Q \) is an orthogonal projection onto a subspace \( \mathcal{H}' \subset \mathcal{H} \) such that \( Q \) commutes with \( H \), then

\[
\Tr[Qe^{-\beta H}] \leq \Tr[Qe^{-\beta H'}].
\]

**Proof.** Since \( Q \) and \( H \) commute, we may find simultaneous eigenvectors. Let \( \psi_1, \ldots, \psi_k \) be an orthonormal basis of \( \mathcal{H}' \) which are eigenvectors of \( H \). Then

\[
\sum_{j=1}^k \langle \psi_j, e^{-\beta H} \psi_j \rangle \overset{(1)}{=} \sum_{j=1}^k \exp(-\beta \langle \psi_j, H \psi_j \rangle) \overset{(2)}{\leq} \sum_{j=1}^k \exp(-\beta \langle \psi_j, H' \psi_j \rangle) \overset{(3)}{\leq} \sum_{j=1}^k \langle \psi_j, e^{-\beta H'} \psi_j \rangle,
\]

where (1) follows because the \( \psi_k \)'s are eigenvectors of \( H \), (2) follows because \( H - H' \) is positive semi-definite, and (3) follows from Theorem A.1 part (1).

---

### A.3 Trace inequalities II: Generalized Hölder Inequality, first version

**Theorem A.4.** Suppose that \( H \) is a self-adjoint operator and define \( Z(\beta) = \Tr[e^{-\beta H}] \). Then for any \( A \in \mathcal{B}(\mathcal{H}) \)

\[
|Z(\beta)^{-1} \Tr[A e^{-\beta H}]| \leq \left( Z(\beta)^{-1} \Tr \left[ (e^{-\beta H/4n} A e^{-\beta H/2n} A^* e^{-\beta H/4n})^n \right] \right)^{1/2n},
\]

for each \( n \in \{1, 2, \ldots\} \). In particular, if \( A = A^* \) then by cyclicity of the trace this implies

\[
|Z(\beta)^{-1} \Tr[A e^{-\beta H}]| \leq \left( Z(\beta)^{-1} \Tr \left[ (e^{-\beta H/2n} A^*)^n \right] \right)^{1/2n}.
\]

Fröhlich and Lieb proved the generalized Hölder inequality in [5]. In the context that we are proving, they showed that, defining a multilinear form, \( \alpha : (\mathcal{B}(\mathcal{H}))^{2n} \to \mathbb{C} \),

\[
\alpha(A_1, \ldots, A_{2n}) = \Tr \left[ (e^{-\beta H/4n} A_1 e^{-\beta H/4n}) \cdots (e^{-\beta H/4n} A_{2n} e^{-\beta H/4n}) \right],
\]

38
then
\[ |\alpha(A_1, \ldots, A_{2n})|^{2n} \leq \prod_{k=1}^{2n} \alpha(A_k, A^*_k, A_k, A^*_k, \ldots, A_k, A^*_k). \]

But we will only need the special case we have proved, which has \( A_1 = A \) and \( A_k = I \) for all \( k = 2, \ldots, 2n \). Cyclicity of the trace is the property that, for all \( A, B \in \mathcal{B}(\mathcal{H}) \),

\[ \text{Tr}[AB] = \text{Tr}[BA]. \quad (64) \]

Another important property is the Cauchy-Schwarz inequality. The bilinear form, \( \langle \cdot, \cdot \rangle_{\text{HS}} : \mathcal{B}(\mathcal{H}) \times \mathcal{B}(\mathcal{H}) \to \mathbb{C} \) defined by \( \langle A, B \rangle_{\text{HS}} = \text{Tr}[A^*B] \) is a positive-definite form known as the Hilbert-Schmidt inner-product. The Cauchy-Schwarz inequality says

\[ |\text{Tr}[A^*B]|^2 \leq \text{Tr}[A^*A] \text{Tr}[B^*B]. \quad (65) \]

For Frohlich and Lieb’s multilinear form (65) implies

\[ |\alpha(A_1, \ldots, A_{2n})|^2 \leq \alpha(A_1, \ldots, A_n, A^*_n, \ldots, A^*_1)\alpha(A^*_2, \ldots, A^*_n+1, A_{n+1}, \ldots, A_{2n}). \quad (66) \]

This is reflection positivity. Later, we will see that it applies in more general contexts. Another important property of \( \alpha \) is cyclicity. Defining \( \tau : (\mathcal{B}(\mathcal{H}))^{2n} \to (\mathcal{B}(\mathcal{H}))^{2n} \) by \( \tau(A_1, A_2, \ldots, A_{2n}) = (A_2, \ldots, A_{2n}, A_1) \), equation (64) gives

\[ \alpha(A_1, A_2, \ldots, A_{2n}) = \alpha(\tau(A_1, A_2, \ldots, A_{2n})) = \alpha(A_2, \ldots, A_{2n}, A_1). \quad (67) \]

Let us now prove Theorem A.4.

**Proof.** For \( k = 0, 1, \ldots, 2n \), let us define a complex number \( c_k \) as

\[ c_k = \alpha(A_{k,1}, \ldots, A_{k,n}), \quad \text{where} \quad A_{k,j} = \begin{cases} A & \text{if } j \leq k \text{ and } k - j \text{ is even}, \\ A^* & \text{if } j \leq k \text{ and } k - j \text{ is odd}, \\ I & \text{if } j > k. \end{cases} \]

Then, considering the terms on the right hand side of (66), and using \( \tau \) defined in (67),

\[ (A_{k,1}, \ldots, A_{k,n}, A^*_k, n, \ldots, A^*_k, 1) = \begin{cases} \tau^{-k}(A_{2k,1}, \ldots, A_{2k,2n}) & \text{if } k \leq n, \\ \tau^{-k}(A_{2n,1}, \ldots, A_{2n,2n}) & \text{if } n \leq k \leq 2n, \end{cases} \]

\[ (A^*_k, 2n, \ldots, A^*_k, n+1, A_{k,n+1}, \ldots, A_{2k,2n}) = \begin{cases} (A_{0,1}, \ldots, A_{0,2n}) & \text{if } k \leq n, \\ \gamma^{2n-k}(A_{2k-2n,1}, \ldots, A_{2k-2n,2n}) & \text{if } n \leq k \leq 2n. \end{cases} \]

Therefore, using this with (66) and (67),

\[ |c_k|^2 \leq c_{2k}c_0, \quad \text{when } k \leq n; \quad (68) \]

\[ |c_k|^2 \leq c_{2n}c_{2k-2n}, \quad \text{when } n \leq k \leq 2n. \quad (69) \]

Then a maximum principle applies to \( \gamma_k = (c_k/c_0)^{1/(k)} \) defined for \( k = 2, 4, \ldots, 2n \). Equation (68) implies that \( \gamma_k \leq \gamma_{2k} \) for all even \( k \) such that \( k \leq n \). Therefore, \( \max_{k=2, \ldots, 2n} \gamma_{2k} \) must equal \( \gamma_k \) for some even \( \kappa \) such that \( n \leq \kappa \leq 2n \). Then (69) implies

\[ \gamma^{\gamma_k}_{\kappa} \leq \gamma^{\gamma_{2k}, \gamma_{2k-2n}}_{\kappa}. \quad (70) \]

But \( \gamma_{2k-2n} \leq \max_{k=2, \ldots, 2n} \gamma_k = \gamma_\kappa \). So (70) implies \( \gamma^{\gamma_k}_{\kappa} \leq \gamma^{\gamma_{2n}, \gamma_{2n-2n}}_{\kappa} \) which in turn implies \( \gamma_{2n} \geq \gamma_\kappa \). In other words \( \gamma_{2n} \geq \gamma_k \) for all \( k = 2, 4, \ldots, 2n \). So finally by (68) again

\[ |c_1| \leq c_1^{1/2}c_0^{-1/2} = c_0^{1/2} \leq c_0\gamma_{2n} = c_0(c_{2n}/c_0)^{1/(2n)} \]

which is the desideratum, writing out \( c_0, c_1 \) and \( c_{2n} \).
The full generality of Frohlich and Lieb’s generalized Hölder’s inequality will be described later. The special case of the inequality just proved allows for a special application to reflections in (imaginary) time.

### A.4 Reflection Positivity for Quantum Spin Systems

Suppose that $N$ is even. Then we may decompose $\mathbb{T}_N^d$ into two halves:

$$
\Lambda^+ = \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in \{0, \ldots, N - 1\}, \ x_\nu \in \{(N/2), \ldots, N - 1\}\}, \\
\Lambda^- = \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in \{0, \ldots, N - 1\}, \ x_\nu \in \{0, \ldots, (N/2) - 1\}\}.
$$

Moreover, let us define

$$
\mathcal{H}_+ = \ell^2(\Omega(\Lambda^+)) \quad \text{and} \quad \mathcal{H}_- = \ell^2(\Omega(\Lambda^-)),
$$

where as usual $\Omega(\Lambda)$ is the set of all function $\sigma = (\sigma_x)_{x \in \Lambda}$. We may identify $\mathcal{H}(\mathbb{T}_N^d) = \mathcal{H}_- \otimes \mathcal{H}_+$. For example, the simple tensor product may be understood as follows: suppose that $f_-$ and $f_+$ are functions in $\mathcal{H}_-$ and $\mathcal{H}_+$, respectively. Then we may define $(f_- \otimes f_+) \in \mathcal{H}(\mathbb{T}_N^d)$ as follows: given $\sigma \in \Omega(\mathbb{T}_N^d)$ define $(\sigma)^-$ and $(\sigma)^+$ to be the restrictions:

$$(\sigma)^\pm = (\sigma_x)_{x \in \Lambda^\pm}.$$ Then

$$(f_- \otimes f_+)(\sigma) = f_-((\sigma)^-)f_+((\sigma)^+).$$

Let us define a reflection $R : \Lambda_+ \to \Lambda_-$ by

$$R(x_1, \ldots, x_{\nu-1}, x_\nu, x_{\nu+1}, \ldots, x_d) = (x_1, \ldots, x_{\nu-1}, N - 1 - x_\nu, x_{\nu+1}, \ldots, x_d).$$

Then we define an isomorphism $\mathcal{R} : \Omega(\Lambda_-) \to \Omega(\Lambda_+)$ as

$$\mathcal{R}((\sigma_x)_{x \in \Lambda_-}) = (\tau_x)_{x \in \Lambda_+}, \quad \tau_x = \sigma_{R(x)}.$$ Let us define a unitary transformation $F : \mathcal{H}_+ \to \mathcal{H}_-$ by

$$Ff((\sigma_x)_{x \in \Lambda_-}) = f((\tau_x)_{x \in \Lambda_+}), \quad (\tau_x)_{x \in \Lambda_+} = \mathcal{R}((\sigma_x)_{x \in \Lambda_-}).$$ Finally, we define a $C^*$-algebra isomorphism $\mathcal{F} : \mathcal{B}(\mathcal{H}_-) \to \mathcal{B}(\mathcal{H}_+)$ by

$$\mathcal{F}(A) = FAF^*.$$ Let us define $\mathcal{A}^-$ to be the $C^*$ subalgebra of $\mathcal{B}(\mathbb{T}_N^d)$ which is equivalent to set of all operators of the form $A \otimes I_{\mathcal{H}_+}, \ A \in \mathcal{B}(\mathcal{H}_-)$. Note that this is a $C^*$-subalgebra because it is closed under all the algebra operations, as well as the adjoint: $(A \otimes I_{\mathcal{H}_+})^* = A^* \otimes I_{\mathcal{H}_+}$, and $A^*$ is in $\mathcal{B}(\mathcal{H}_-)$ for each $A$ in $\mathcal{B}(\mathcal{H}_-)$. We define $\mathcal{A}^+$ similarly as the set of all operators $I_{\mathcal{H}_-} \otimes A$ for $A \in \mathcal{B}(\mathcal{H}_+)$. We use the symbol $\mathcal{F}$ for the $C^*$ algebra isomorphism between $\mathcal{A}^-$ and $\mathcal{A}^+$:

$$\mathcal{F}(A \otimes I_{\mathcal{H}_+}) = I_{\mathcal{H}_-} \otimes \mathcal{F}(A).$$

An important consideration for the further part of the definition will be the introduction of an orthonormal basis. We take the basis previously stated as the appropriate orthonormal basis. Given an operator $A : \mathcal{H}_- \to \mathcal{H}_-$ we will say that it is “real” if

$$\langle \chi_{\{\sigma\}}, A \chi_{\{\sigma'\}} \rangle \in \mathbb{R},$$

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for all \( \sigma, \sigma' \in \Omega(\Lambda_-) \). Note that this does depend on the choice of orthonormal basis, and even depends on the choice of the phase for the ortho-normal basis. This is a definition with less flexibility than one usually associates to the framework of quantum mechanics, as defined on complex Hilbert spaces. But it is needed. We define \( \mathcal{B}_R(\mathcal{H}_-) \) to be the set of all “real” operators \( A \) in \( \mathcal{B}(\mathcal{H}_-) \). Similarly, we say that \( A \in \mathcal{B}(\mathcal{H}_+) \) is “real” if

\[
\langle \chi(\sigma), A\chi(\sigma') \rangle \in \mathbb{R},
\]

for all \( \sigma, \sigma' \in \Omega(\Lambda_+) \), and we define \( \mathcal{B}_R(\mathcal{H}_+) \) to be the set of all such “real” operators. Finally, we define \( \mathcal{A}^-_R \) to be the subspace of operators in \( \mathcal{A}^- \) of the form \( A \otimes I_{\mathcal{H}_+} \) for \( A \) in \( \mathcal{B}_R(\mathcal{H}_-) \), and we define \( \mathcal{A}^+_R \), similarly. It is straightforward to check that \( \mathcal{F} \) actually also maps \( \mathcal{A}_R^- \) to \( \mathcal{A}_R^+ \).

**Definition A.5.** A linear functional \( \alpha : \mathcal{B}(\mathcal{H}(\mathbb{T}^d_N)) \to \mathbb{C} \) is said to be reflection positive if, for every operator \( A \in \mathcal{A}^-_R \) we have

\[
\alpha(A\mathcal{F}(A)) \geq 0.
\]

**Lemma A.6.** The tracial state, defined as \( \langle \cdot \rangle_0 = \text{Tr}[\cdot]/\text{Tr}[I] \), is reflection positive.

**Proof.** For every operator \( A \in \mathcal{A}_R^- \), there is some \( \tilde{A} \in \mathcal{B}_R(\mathcal{H}_-) \) such that \( A = \tilde{A} \otimes I_{\mathcal{H}_+} \). Note that then \( \text{Tr}_{\mathcal{H}_-}(A) \) is real, because \( \tilde{A} \) is “real,” and the trace may be calculated in the basis used for the definition of “real” operators. Then we have the string of identities

\[
\text{Tr}[A\mathcal{F}(A)] = \text{Tr}[\tilde{A} \otimes \mathcal{F}(\tilde{A})] = \sum_{\sigma \in \Omega(\mathbb{T}^d_N)} \langle \chi(\sigma), \tilde{A} \otimes \mathcal{F}(\tilde{A})\chi(\sigma) \rangle = \sum_{\sigma \in \Omega(\mathbb{T}^d_N)} \langle \chi(\sigma), \tilde{A}\chi(\sigma) \rangle + \langle \mathcal{F}(\tilde{A})\chi(\sigma), \chi(\sigma) \rangle.
\]

But considering the definition of \( F^* \) on \( \chi(\sigma) \), we realize that (because \( R \) is a bijection) the second factor is equal to the first. In other words,

\[
\text{Tr}[A\mathcal{F}(A)] = \left( \text{Tr}_{\mathcal{H}_-}(\tilde{A}) \right)^2.
\]

Because the trace of \( \tilde{A} \) is real, this is nonnegative. \( \square \)

**Definition A.7.** A linear functional \( \alpha : \mathcal{B}(\mathcal{H}(\mathbb{T}^d_N)) \to \mathbb{C} \) is said to be generalized reflection positive if, for every \( n \) and all operators \( A_1, \ldots, A_n \in \mathcal{A}_R^- \) we have

\[
\alpha(A_1\mathcal{F}(A_1)A_2\mathcal{F}(A_2)\cdots A_n\mathcal{F}(A_n)) \geq 0.
\]

**Theorem A.8.** The tracial state is also generalized reflection positive.

**Proof.** Note that \( \mathcal{A}^- \) commutes with \( \mathcal{A}^+ \). So we are actually trying to prove

\[
\langle A_1A_2\cdots A_n\mathcal{F}(A_1)\mathcal{F}(A_2)\cdots \mathcal{F}(A_n) \rangle \geq 0.
\]

Because \( \mathcal{F} \) is a \( C^* \)-algebra homomorphism, this may be rewritten as

\[
\langle A_1A_2\cdots A_n\mathcal{F}(A_n\cdots A_2A_1) \rangle \geq 0.
\]

Then this reduces to the definition of reflection positivity proved in Lemma A.6. \( \square \)
The key theorem for obtaining reflection positive examples is as follows.

**Theorem A.9.** Suppose that we have a generalized reflection positive linear functional which we will denote \( \alpha_0 : \mathcal{B}(\mathcal{H}(\mathbb{T}_N^d)) \to \mathbb{C} \). Suppose that for some \( K \) there are operators \( B, C_1, \ldots, C_K \in \mathcal{A}_R^- \) such that a Hamiltonian \( H \in \mathcal{B}(\mathcal{H}(\mathbb{T}_N^d)) \) may be written as

\[
H = B + \mathcal{F}(B) - \sum_{k=1}^{K} C_k \mathcal{F}(C_k).
\]

Then, for each \( \beta \in [0, \infty) \), defining the linear functional \( \alpha_{\beta} \) as

\[
\alpha_{\beta}(\cdot) = \frac{\alpha_0(e^{-\beta H})}{\alpha_0(e^{-\beta H})},
\]

this is also generalized reflection positive.

**Proof.** This follows from the Trotter product formula. Given any \( A_1, \ldots, A_n \in \mathcal{A}_R^- \), the Trotter product formula implies

\[
\alpha_0(A_1 \mathcal{F}(A_1) \cdots A_n \mathcal{F}(A_n)e^{-\beta H}) = \lim_{R \to \infty} \alpha_0 (A_1 \mathcal{F}(A_1) \cdots A_n \mathcal{F}(A_n) e^{-\beta H / R} C_1 \mathcal{F}(C_1) \cdots (I + \beta / R C_K \mathcal{F}(C_K)))^R).
\]

For any fixed \( R \) the quantity on the right hand side may be expanded as a nonnegative combination of terms of the form \( \alpha_0(D_1 \mathcal{F}(D_1) \cdots D_L \mathcal{F}(D_L)) \) for finite \( L \)'s and operators \( D_1, \ldots, D_L \in \mathcal{A}_R^- \). So, since \( \alpha_0 \) is generalized reflection positive, we see that this term is nonnegative as well. The property of being nonnegative survives the limit \( R \to \infty \). Therefore,

\[
\alpha_0(A_1 \mathcal{F}(A_1) \cdots A_n \mathcal{F}(A_n)e^{-\beta H}) \geq 0.
\]

A similar calculation shows \( \alpha_0(e^{-\beta H}) \geq 0 \). So the theorem follows.

Since the tracial state is generalized reflection positive, if \( H \) satisfies the form (71) then the equilibrium state \( \langle \cdot \rangle_\beta \) is also generalized reflection positive by this theorem.

**A.5 The XXZ model for \( \Delta \leq 0 \)**

The XXZ Hamiltonian (2) with \( \Delta \leq 0 \) is not generalized reflection positive. But it is unitarily equivalent to a Hamiltonian which is. Let \( U : \mathcal{H}_+ \to \mathcal{H}_+ \) be the unitary transformation on \( \mathcal{H}_+ = \ell^2(\Omega(\Lambda_+)) \) given by

\[
Uf(\sigma) = f(-\sigma),
\]

where \(-\sigma = (-\sigma_x)_{x \in \Lambda_+}\) for \( \sigma = (\sigma_x)_{x \in \Lambda_+} \). Note that \( U = U^* = U^{-1} \). Importantly, we also have

\[
US_x^{(1)}U = S_x^{(1)}, \quad US_x^{(2)}U = -S_x^{(2)}, \quad \text{and}\quad US_x^{(3)}U = -S_x^{(3)},
\]

for each \( x \in \Lambda_+ \). Let us define \( \mathcal{U} = I_{\mathcal{H}_-} \otimes \sigma \), which is a unitary transformation on \( \mathcal{H}(\mathbb{T}_N^d) \) also satisfying \( \mathcal{U} = \mathcal{U}^* = \mathcal{U}^{-1} \). Then, we claim that if \( x \) is in \( \Lambda_- \) and \( y \) is in \( \Lambda_+ \) then

\[
\mathcal{U}h_{xy}^\Delta \mathcal{U} = -S_x^{(1)}S_y^{(1)} + S_x^{(2)}S_y^{(2)} + \Delta S_x^{(3)}S_y^{(3)}.
\]

This is important because \( S_x^{(2)} \) and \( S_y^{(2)} \) are not real operators because of the presence of a factor \( i \). But \( iS_x^{(2)} \) and \( iS_y^{(2)} \) are real operators. Therefore, we may write

\[
\mathcal{U}h_{xy}^\Delta \mathcal{U} = -S_x^{(1)}S_y^{(1)} - (iS_x^{(2)})(iS_y^{(2)}) + \Delta S_x^{(3)}S_y^{(3)}.
\]
If we have the condition $\Delta \leq 0$ then this means that if $y = Rx$ then

$$U h_{xy}^\Delta U = - \sum_{j=1}^{3} C_{x}^{(j)} \mathcal{F}(C_{x}^{(j)}) ,$$

where $C^{(1)} = S_{x}^{(1)}$, $C^{(2)} = iS_{x}^{(2)}$ and $C^{(3)} = \sqrt{-\Delta} S_{x}^{(3)}$. We note that if $x$ and $y$ are both in $\Lambda_-$ then we do have $h_{xy}^\Delta$ in $A_{\Xi}^{-}$ because even though $S_{x}^{(2)}$ and $S_{y}^{(2)}$ involve $i$, when we multiply them both together we only get a real factor, $i^2 = -1$. Also, we observe that for $x, y \in \Lambda_+$ we have $U h_{xy}^\Delta U = h_{xy}^\Delta$ (because the $-1$ factors are squared in $S_{x}^{(2)} S_{y}^{(2)}$ and $S_{x}^{(3)} S_{y}^{(3)}$). Therefore, if we enumerate the pairs $\{x, y\}$ with $y = Rx$ and $\{x, y\} \in E(T_{N}^{d})$ as $\{x_1, y_1\}, \ldots, \{x_K, y_K\}$, then we have

$$U H_{N,d}^\Delta U = B + \mathcal{F}(B) - \sum_{k=1}^{K} \sum_{j=1}^{3} C_{x_k}^{(j)} \mathcal{F}(C_{x_k}^{(j)}) ,$$

where

$$B = \sum_{\{x, y\} \in E(\Lambda_-)} h_{xy}^\Delta .$$

Therefore, we conclude:

**Corollary A.10.** For $\Delta \leq 1$, the equilibrium state associated to the Hamiltonian $U H_{N,d}^\Delta U$ is generalized reflection positive.

**Proof.** From (72), the Hamiltonian satisfies the condition (71) needed to apply Theorem A.9. Then by the discussion immediately following the theorem, the corollary follows. \qed

### A.5.1 Chessboard estimate with some loss

**Proof of Lemma 3.2.** This proof is similar to the proof of Theorem A.4. In the present context, Fröhlich and Lieb refer to this type of inequality as a chessboard estimate. Consider $P_{L}^{+}$ which projects onto all spins $+$ on $\mathbb{B}_{L}^{d}$. Consider the face of $\mathbb{B}_{L}^{d}$ whose outward pointing normal points in the direction $e_{k}$. One may reflect in the plane separating that face from the neighboring spin sites outside the box $\mathbb{B}_{L}^{d}$. In order to use reflection positivity, we must conjugate by $U$, introduced in the last subsection, on one of the halves. But we can actually undo this conjugation if we consider the effect this has on the operators whose expectation we take. Since $U P_{L}^{+} U = P_{L}^{+}$ and vice-versa, the Cauchy-Schwarz inequality implies

$$\langle P_{L}^{+}\rangle_{N,\beta,\Delta} \leq \left( \prod_{x \in \mathbb{B}_{2L}^{d}} \left( 1 + \frac{z_{x}^{(L)} S_{x}^{2}}{2} \right) \right)_{N,\beta,\Delta}^{(2)^{-d}} .$$

More precisely, each time the Cauchy-Schwarz inequality is applied the operator on one of the two halves is just the identity operator. Since the identity operator has expectation 1, this does not contribute. The effect of having to conjugate by $U$ after each reflection gives rise to the factor $z_{i}^{(L)}$. Note also that one must reflect in each of the $d$ directions. This increases the cardinality of the box by a factor of $2^{d}$. The effect of taking the square-root for each application of the Cauchy-Schwarz inequality results in taking the $2^{d}$th root. One may repeat this procedure $n = \lfloor \log_{2}\left(\frac{N}{L}\right) \rfloor$ times with no essential change to the procedure. The last time (in each direction) however will result in cutting-off what is already a projection covering more than half of the torus. That is okay, then in the Cauchy-Schwarz inequality the other factor will be an expectation of a projection which is always less than or equal to 1. We bound it by 1. This results in some loss of sharpness in our
inequality, which is why we call this “Chessboard estimates with some loss.” But this is not an essential point for us, because we will frequently be considering the case that $N$ is much larger than $L$. So an extra 1 added to the already large term $\log_2(N/L)$ does not concern us.

A.6 The Six Vertex Model

We wish to mention that Sutherland’s result Proposition 4.1 follows from the Yang-Baxter relation. Let us not mention more about this, here. One may also understand Proposition 4.1 from a diagrammatic standpoint. But the real goal, to generalize to higher dimensions, eludes us. The reason this is so useful is that Proposition 4.1 along with the fact that both the 6-vertex model and the XXZ model satisfy the good signs condition of the Perron-Frobenius theorem implies that the ground state of the XXZ model is the eigenvector of the 6-vertex model with largest eigenvector.

The six-vertex model is reflection positive, using the result from [4]. We may also condition on the event $\mathcal{M}(N,0)$. This was not treated in [4]. But the idea has been used before for other reflection positive models. See the paper [1] of Aizenman, et. al., particularly Appendix B. Given any vertical or horizontal hyperplane, we may enforce $\mathcal{M}(N,0)$ in a reflection positive way. For a horizontal hyperplane of vertical edges, we merely restrict to configurations on the plane with the same number of ↑’s as ↓’s. For a vertical hyperplane of horizontal edges, we can write $1_{\mathcal{M}(N,0)}(\varsigma) = \sum_{\mathcal{M}(N/2,k)} 1_{\mathcal{M}(N/2,k)}(\varsigma^{(L)}) 1_{\mathcal{M}(N/2,k)}(\varsigma^{(R)})$ where we decompose $\varsigma = (\varsigma^{(L)}, \varsigma^{(R)})$. This is the type of condition one needs for reflection positivity.

With this, the chessboard estimate with loss Lemma 5.5 follows just like in appendix A.5.1.

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