In their paper in Optica 6, 104 (2019), Mann et al. claim that time-invariant nonreciprocal cavities cannot overcome the time-bandwidth limit, based on their numerical analysis and temporal coupled-mode theory of an example structure. In this comment, we argue that their structure is actually better described as an ordinary reciprocal cavity coupled to a nonreciprocal waveguide and that while their investigation of the dynamics of their cavity is correct, their conclusion about the time-bandwidth limit cannot be generalized to all time-invariant nonreciprocal systems. In particular, “open” cavities, such as the localized wedge state they observe, can overcome the time-bandwidth limit – even in the linear, time-invariant regime of operation.

The paper by Mann et al. [1] investigates a unidirectional waveguide interacting with a cavity (Fig. 1a), concluding that the behaviour of the cavity remains unchanged by the presence of the waveguide. This conclusion is then generalised to stating that time-invariant nonreciprocal systems cannot overcome the time-bandwidth limit, which initially appears to conflict with our previous work on a terminated unidirectional waveguide (Fig. 1b) [2]. In fact, while the two systems are similar, they are crucially not identical. We previously demonstrated that under the right circumstances (e.g. Fig 1b) a linear, time-invariant, nonreciprocal optical system can exceed the traditional time-bandwidth limit.
characterizing present-day resonant and waveguiding devices. Specifically, Ref. [2] showed that the bandwidth of light – or of any wave field – stored in a nonreciprocal system can be decoupled from the decay (or storage) time of that device. In this Comment, we wish to elucidate what led to Mann et al.’s different conclusions, which cannot be generalized to open nonreciprocal systems, including ours. To be clear, we are not suggesting that there is an error in Mann et al.’s conclusions for their cavity, only that those conclusions do not apply in general to all time-invariant nonreciprocal systems and their time-bandwidth limit.

Theoretical background
A low loss, closed cavity system, i.e. one with well-defined boundaries, such as the one analysed in Ref. [1] and illustrated schematically in Fig. 1a), may be described using temporal coupled mode theory [3]. The time evolution of a field inside the cavity is then given by:

\[ \frac{da}{dt} = i\omega_0 a - (\gamma_i + \gamma_r) a + \kappa_{in} s_+ , \]  (1)

where \( a \) is the field amplitude inside the cavity, \( \omega_0 \) the resonance frequency dictated by the cavity (or individual cavity mode), \( \gamma_i \) and \( \gamma_r \) the intrinsic and radiative loss rates, respectively, \( |s_+|^2 \) the power incident onto the cavity from an external system, e.g. a waveguide, and \( \kappa_{in} \) the coupling coefficient between that external system and the cavity.

Eq. (1) is satisfied when the field inside the cavity takes the form:

\[ a(t) = a_0 e^{i\omega_0 t - t(\gamma_i + \gamma_r)} , \]  (2)

where:

\[ a_0 = \frac{\kappa_{in} s_+}{(i\omega_0 - \omega - \gamma_i - \gamma_r)} . \]  (3)

Note, here, that Eq. (3) has the exact same form as Eq. (4) of Ref. [1]. So far, we have made no assumptions about the cavity’s reciprocity, so these results hold both for reciprocal and nonreciprocal cavities. In a reciprocal cavity, the in-coupling coefficient \( \kappa_{in} \) is directly related to the radiative decay rate, \( \gamma_r \), such that \( \kappa_{in} = \sqrt{2\gamma_r} \) [3]. In nonreciprocal cavities, the in- and out-coupling energy rates (in units of \( s^{-1} \) [4, 5]) may differ by a large degree [2] (see also below). However, from Eqn. (3), we see that the in-coupling coefficient, \( \kappa_{in} \), only determines the amplitude of the field inside the cavity; the frequency-dependent behaviour is solely dependent on the denominator. Specifically, the cavity will have a half maximum when \( |\omega - \omega_0| = \gamma_i + \gamma_r \). Therefore, the bandwidth \( \Delta\omega \) of an ordinary closed cavity is directly linked to the energy decay rate \( \Delta\omega = \gamma_{tot} = \gamma_i + \gamma_r \) [2] and thus inversely proportional to the decay/storage time, resulting in the well-known time-bandwidth (T-B) limit. Note that this holds whether the cavity is connected to a reciprocal or nonreciprocal feeding waveguide such as the one presented by Mann et al.

However, several key issues arise as Mann et al. bid to generalize their results to arbitrary nonreciprocal resonant systems: Eqs. (1)-(3) do not apply generally; they are only applicable under the following circumstances:

1. The cavity mode must be a confined, oscillatory mode with a well-defined, single resonance frequency \( \omega_0 \); in “open” resonant systems [2, 6-10] this is not the case.
2. \( \gamma_i \) must account for all internal (i.e. non-radiative) loss mechanisms; in the absence of any non-radiative losses, \( \gamma_i \) must be equal to zero.
3. \( \gamma_r \) must account for all radiative loss mechanisms, i.e. all light coupling out of the cavity, independent of the final states of the light (e.g., coupled to a waveguide mode, trapped in a
wedge state, or joining the radiation continuum). Note that $\gamma_r$ can be reconstructed into coupling to different ports if the cavity has multiple input/output ports. However, one cannot perform a theoretical analysis of a cavity on Eqs. (1)-(3) and introduce additional coupling factors afterwards.

While point 1 holds for the cavity introduced in Mann et al.’s work, it does not similarly apply to the final trapped state considered in Ref. [2]. This is because the trapped state is not a resonant mode, but instead localized in a one-way terminated waveguide, forming an open localization region [6-10] where, crucially, the absence of a hard boundary on the trailing (left) edge of the state does not give rise to a fixed-resonance or eigenfrequency ($\omega_0$) standing wave. Because it is not an ordinarily confined oscillatory mode with a particular resonance frequency, it can indeed allow for broadband performance, with a device bandwidth equal to the bandwidth of the unidirectional region [2]. A conceptually reminiscent situation occurs for Fourier-transform-unlimited chirped wave signals: By not fixing with time the instantaneous frequency $\omega$ of the signal to a constant value $\omega_0$, but allowing it to vary with time (e.g., $E(t) \propto \exp(-i\delta(t)) = \exp(-i\omega t - i\omega_0 T/2)$) over an interval $-T/2 \leq t \leq T/2$, the instantaneous frequency $\omega(t) = \delta(t) = \omega_0 + \omega_0 t$ may now span the whole band $\omega_0 - \omega_0 T/2 \leq \omega(t) \leq \omega_0 + \omega_0 T/2$. Therefore, the bandwidth $B = \omega_0 T/(2\pi)$ is no longer inversely proportional to $T$, hence the signal is not T-B limited. Such dispersive chirping unavoidably occurs during almost all propagation, including the waveguide portion of the structure considered in Mann et al.’s work (Fig. 1) and our own [2]. While the waveguide is functionally identical between the structures of Refs. [1] and [2], by adding an ordinary, closed-boundary cavity at the end of a unidirectional waveguide, Mann et al. force the frequency response of their (loss-free) cavity to be fundamentally narrowband – around $\omega_0$. Therefore, this narrowband performance is an artifact of the particular resonant configuration they have developed, rather than a general feature of all “nonreciprocal systems”.

Regarding point 2, Mann et al.’s use of a lossless (perfect electric conductor, PEC) cavity, although suitable for a temporal coupled-mode theory analysis that normally deals with lossless cavities [11], has an important corollary, which can be best appreciated using Poynting’s theorem [4, 5]: For a lossless cavity at steady state, the net flux rate of energy out of the cavity must be zero. Indeed, as was numerically shown in Ref. [2], the rate of energy change in a cavity can be written as:

$$\frac{\partial U}{\partial t} = -\Gamma_i U = -\Lambda U - \Gamma_i U,$$  \hspace{1cm} (4)

where $U = |\alpha|^2$ is the total field energy in the cavity, $\Gamma_i$ the energy decay rate, $\Lambda = \nabla \cdot S/U$ the flux rate that measures the outflow of energy, and $\Gamma_i$ the power loss rate. In a lossless ($\Gamma_i = 0$) cavity at steady state ($\partial U/\partial t = 0$), one always gets $\Lambda = 0$. Rewritten in Mann et al.'s framework of analysis, this becomes:

$$|s_i|^2 - |s_a|^2 = d|\alpha|^2/\partial t + 2\gamma_r|\alpha|^2,$$  \hspace{1cm} (5)

where, by imposing $\gamma_r = 0$ at steady state ($d|\alpha|^2/\partial t = 0$), we obtain $|s_i| = |s_a|$ (cf. $\Lambda = 0$). Thus, not only is the cavity considered by Mann et al. reciprocal by design, since power can reciprocally flow between the waveguide and the cavity, as shown in Fig. 1a, but it is also fundamentally reciprocal on the basis of it being lossless, limiting how generally the conclusions based on this configuration can be applied to all time-invariant non-reciprocal systems.

It is potentially point 3 that most clearly reveals the distinction between the device and conclusions reached by Mann et al. as compared to our work in Ref. [2]: Their overarching analysis of the system of Fig. 1a does not explicitly account for light trapped in the wedge state observed in their simulations. This localized wedge state (Fig. 1a) has decoupled bandwidth and decay rates, meaning that it is not time-bandwidth limited. This wedge state is equivalent to the trapped state at the terminated edge of
the waveguide in Ref. [2], where light stops and is restricted from counter propagating. Mann et al. write that Ref. [2] does "demonstrate broadband focusing of photons in an ultrasmall volume near the termination, whose decay rate is unrelated to, and can be much slower than, the excitation", but their description does not fully appreciate that the accumulation of light at the termination, or in the wedge mode of their own system, is, in fact, an open localization region capable of broadband performance and a key element to overcoming the time-bandwidth limit.

Instead, Mann et al. associate the wedge state with plasmonic nanofocusing [12]. This assertion is misleading as the one-way propagation in Ref. [2] is fundamentally (topologically) different from adiabatically tapered plasmonic nanoguides, which are ordinary (topologically trivial) waveguides. In this way, it is the photonic analogue to the quantum Hall effect [13-15]. Importantly, in plasmonic nanoguides there are always back-reflections occurring near their tip region [16], meaning that they allow bi-directional propagation, unlike e.g. the structures considered in Refs. [2, 6-10]: There is continuous propagation and immediate back-reflection [16] from the end of the guide, rather than localization and formation of open-cavity "confinement" [6-10].

Examining these three requirements for their validity, Eqs. (1)-(3) thus show that while the conclusions of Ref. [1] about non-reciprocal cavities are correct when these requirements hold, they cannot be applied generally to all time-invariant, non-reciprocal systems, and particularly not to the system presented in Ref [2]. Rather than comparing the terminated unidirectional waveguide of Ref. [2] to a cavity, the wedge mode described by Mann et al. not only appears to be a more apt comparison but indeed shows the same broadband behavior, and thus ability to overcome the time-bandwidth limit even in a linear, time-invariant structure, that Ref. [2] also revealed.

Decoupling bandwidth from decay in the wedge mode

In order to investigate the nature of this wedge mode further, a subtle but crucial distinction must be drawn between a multi-port cavity, such as a Fabry-Perot cavity with two mirrors each coupled to a single-mode waveguide, and a single-port cavity, such as a Fabry-Perot cavity coupled to a single multimode waveguide. While Mann et al. treat the system in Ref. [1] (cf. Fig. 1a) as the former, it would be more accurate to use the later description. In a multi-port cavity, each port can be assigned a separate radiative loss rate \( \gamma_{r1}, \gamma_{r2}, \ldots, \gamma_{rn} \), determined by the coupling coefficient through that port. By contrast, in the case of a single port cavity coupled to multiple systems, the coupling rate to each optical system must be determined in an inherently different way: energy leaves the cavity through the single coupling port, and does so with a decay rate \( \gamma_r \); note that this depends on the coupling port, not on the final mode(s) to which the energy out-couples. Thus, the ratio with which energy enters into the different external optical systems should, in this case, be determined by the overlap integrals [3] between the output mode of the cavity (the port) and the modes of the external systems.

In particular, the multi-port analysis applied by Mann et al. to their terminating cavity cannot be similarly applied to the wedge mode that they observe. The wedge mode arises independently of the presence, as in Ref. [1], or absence, as in Ref. [2], of a terminating closed cavity. It is a broadband mode, without a well-defined resonance frequency, \( \omega_0 \), i.e. it is an open, localized state and not an ordinary (closed) cavity. Since it cannot be described by Eqs. (1)-(3), it instead is more appropriate to treat it using Poynting’s theorem [4, 5]. At steady state, there is no net flow of energy into – or out of – the wedge mode, in agreement with thermodynamics, i.e. the energy coupled into the wedge mode per unit time is equal to the one leaving it:

\[
\rho_{\text{in}} E_{\text{in}} = (\rho_r + \rho_t) U_{\text{stored}} ,
\]
where $\rho_{\text{in}}$ is the in-coupling rate for the wedge mode, $E_{\text{in}}$ the incident energy, $U_{\text{stored}}$ the energy contained within the wedge mode, and $\rho_r, \rho_l (= \Gamma_l)$ the rates at which energy is lost from the wedge mode through radiative (coupling to the waveguide and cavity) or non-radiative processes, respectively [4, 5]. Note from Eq. (6) that the in-coupling rate, $\rho_{\text{in}}$, is not necessarily equal to the out-coupling rate $\rho_{\text{out}} = (\rho_r + \rho_l)$, as the amplitude of the wedge mode can greatly exceed that of the incident wave, i.e. $U_{\text{stored}} \gg E_{\text{in}}$ [2].

If we remove the cavity from Ref. [1], the time-bandwidth product of its wedge mode and the termination point of Ref. [2] can be considered the same way. Within the unidirectional region, at steady state, light incident onto the terminated waveguide is coupled into the wedge mode. Since $\rho_r = 0$ (because of the unidirectionality), the storage time is only determined by $\rho_l$, i.e. by the material absorption, and is thus completely independent of the energy in-coupling rate, $\rho_{\text{in}}$, and of the bandwidth of one-way propagation, $\Delta \omega_{\text{CUP}}$. Crucially, we note that the wedge mode, being inherently broadband, has no intrinsic “cavity linewidth”, and thus its time-bandwidth product should be calculated using its “acceptance bandwidth”, i.e. the frequency range that couples to the wedge mode. This is given by $\Delta \omega_{\text{acc}} = \rho_{\text{in}}$ and for the wedge mode is determined by $\Delta \omega_{\text{CUP}}$. By contrast, the traditional cavity linewidth is given by outcoupling: $\Delta \omega_{\text{cav}} = \rho_{\text{out}} = 1/\tau_{\text{out}}$, where $\tau_{\text{out}}$ is the total decay time. In time-invariant cavities, such as the one investigated by Mann et al., $\rho_{\text{in}} = \rho_{\text{out}}$ and thus the two bandwidths coincide giving a time-bandwidth limit of $\Delta \omega_{\text{acc}} \tau_{\text{out}} = \Delta \omega_{\text{acc}} / \Delta \omega_{\text{cav}} = \rho_{\text{in}} / \rho_{\text{out}} = 1$. However, in this wedge mode and similar instances, the two bandwidths may differ vastly, giving rise to a time-bandwidth product of $\rho_{\text{in}} / \rho_{\text{out}} > 1$. Thus, while a cavity system as described by Eq. (3) and as designed by Mann et al. remains time-bandwidth limited [17, 18], a nonreciprocal wedge state overcomes the time-bandwidth limit.

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