KANTOROVICH TYPE TOPOLOGIES ON SPACES OF MEASURES AND CONVERGENCE OF BARYCENTERS

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Abstract. We study two topologies \( \tau_{KR} \) and \( \tau_K \) on the space of measures on a completely regular space generated by Kantorovich–Rubinshtein and Kantorovich seminorms analogous to their classical norms in the case of a metric space. The Kantorovich–Rubinshtein topology \( \tau_{KR} \) coincides with the weak topology on nonnegative measures and on bounded uniformly tight sets of measures. A sufficient condition is given for the compactness in the Kantorovich topology. We show that for logarithmically concave measures and stable measures weak convergence implies convergence in the Kantorovich topology. We also obtain an efficiently verified condition for convergence of the barycenters of Radon measures from a sequence or net weakly converging on a locally convex space. As an application it is shown that for weakly convergent logarithmically concave measures and stable measures convergence of their barycenters holds without additional conditions. The same is true for measures given by polynomial densities of a fixed degree with respect to logarithmically concave measures.

1. INTRODUCTION

The geometry and topology of spaces of measures on metric spaces have become an important direction in probability theory over the past two decades. A particular role in these studies is played by the Kantorovich metrics \( W_p \) (also called Wasserstein metrics in part of the literature) and similar Kantorovich–Rubinshtein (or Fortet–Mourier) metrics, see, for example, [2, 4, 5, 6, 9, 14, 20, 23, 24, 27, 29, 30]. These metrics are traditionally considered on probability or nonnegative measures, where they are related to the weak topology, but are also defined on all measures, although on the whole space of signed measures they induce topologies not comparable with the weak one. However, for more general spaces analogous constructions have not been studied in detail, although they were already considered by Castaing, Raynaud de Fitte and Valadier [19, Section 3.4] in the framework of Young measures. The goal of our paper is to further develop Kantorovich type seminorms in the case of completely regular spaces. We consider two natural locally convex topologies \( \tau_K \) and \( \tau_{KR} \) on the space of Radon measures, corresponding to...
the classical Kantorovich and Kantorovich–Rubinshtein (or Fortet–Mourier) norms. We show that the topology \(\tau_{KR}\) coincides with the weak topology on the cone of nonnegative measures and also on bounded uniformly tight sets of measures. For a separable space, it coincides with the weak topology on weakly compact sets. A simple sufficient compactness condition is obtained for \(\tau_{K}\), which combines the uniform tightness with some uniform integrability of quasi-metrics defining the topology. Kantorovich type seminorms can be used for analysis of barycenters. Along these lines we show convergence of the barycenters of weakly convergent logarithmically concave and stable measures. Moreover, the same is proved for measures given by polynomial densities of a fixed degree with respect to logarithmically concave measures.

2. Kantorovich and Kantorovich–Rubinstein seminorms

We recall (see details in [22]) that the topology of a completely regular space \(X\) is generated by a family of pseudo-metrics \(\Pi\) (a pseudo-metric differs from a metric by the property that it can be zero on distinct elements).

A finite nonnegative Borel measure \(\mu\) on a topological space is called Radon if for every Borel set \(B\) and every \(\varepsilon > 0\) there exists a compact set \(K \subset B\) such that \(\mu(B \setminus K) < \varepsilon\). A signed Borel measure \(\mu\) is called Radon if such is its total variation \(|\mu| = \mu^+ + \mu^-\), where \(\mu^+\) and \(\mu^-\) are the positive and negative parts of the measure \(\mu\). On Radon measures, see [11].

A set \(\mathcal{M}_r(X)\) of Radon measures on \(X\) is called uniformly tight if for every \(\varepsilon > 0\) there is a compact set \(K\) such that \(|\mu|(X \setminus K) < \varepsilon\) for all \(\mu \in \mathcal{M}_r(X)\).

Let \(\mathcal{M}_r^+\) denote the linear space of all Radon measures on \(X\) and let \(\mathcal{P}_r\) be its subsets consisting of nonnegative measures and probability measures, respectively.

Let \(\mathcal{M}_r^\Pi(X)\) denote the subset of \(\mathcal{M}_r(X)\) consisting of measures \(\mu\) for which the function \(p(x, x_0)\) belongs to \(L^1(|\mu|)\) for all \(p \in \Pi\) for some (then for all) \(x_0 \in X\). Note that this class depends on our choice of \(\Pi\); say, for \(X = (0, 1)\) with the usual metric and the family \(\Pi\) reducing to it, all measures belong to \(\mathcal{M}_r^\Pi(X)\), but the situation changes if for \(\Pi\) we take all continuous metrics. In the case of a normed space \(X\) for \(\Pi\) we shall take only the norm and in the case of a locally convex space for \(\Pi\) we shall take a collection of seminorms defining the topology (which amounts to taking the collection of all continuous seminorms); in these cases we shall write \(\mathcal{M}_r^\Pi(X)\) in place of \(\mathcal{M}_r(X)\) and speak of measures with finite first moment.

The image of a Radon measure \(\mu\) on \(X\) under a continuous mapping \(F\) to a topological space \(Y\) is the Radon measure \(\mu \circ F^{-1}\) given by the equality

\[
\mu \circ F^{-1}(B) = \mu(F^{-1}(B)).
\]

The weak topology on \(\mathcal{M}_r(X)\) is the topology of duality with the space \(C_b(X)\) of bounded continuous functions, which is generated by all seminorms of the form

\[
\mu \mapsto \left| \int_X f \, d\mu \right|, \quad f \in C_b(X).
\]

The space \(\mathcal{M}_r(X)\) of all Radon measures on a metric space \((X, d)\) can be equipped with the classical Kantorovich–Rubinstein norm

\[
\|\mu\|_{KR, d} = \sup \left\{ \int f \, d\mu : f \in \text{Lip}_1(d), \ |f| \leq 1 \right\}.
\]
where Lip$_1(d)$ is the class of 1-Lipschitz functions $f$, i.e., $|f(x) - f(y)| \leq d(x, y)$. The subspace $M^*_1(X)$ of all measures for which for some $x_0 \in X$ (then for all $x_0$) the function $d(x, x_0)$ is integrable can be equipped with the Kantorovich norm

$$\|\mu\|_{K,d} = \sup \left\{ \int f \, d\mu : f \in \text{Lip}_1(d), \ f(x_0) = 0 \right\} + |\mu(X)|. $$

The topology generated by the Kantorovich–Rubinstein norm coincides with the weak topology on the cone of nonnegative measures and also on compact sets in the weak topology, although on the whole space these two topologies are incomparable in nontrivial cases.

For a general completely regular space $X$ both norms have natural analogs in the form of collections of seminorms: for every pseudo-metric $d$ one can define $\|\mu\|_{K^R,d}$ and $\|\mu\|_{K,d}$ on the corresponding spaces. It is shown below that the topology $\tau_{K^R}$ generated by all such Kantorovich–Rubinstein type seminorms coincides with the weak topology on the cone of nonnegative measures and also on weakly compact sets (for a separable space). We give a simple sufficient condition for the compactness in the topology $\tau_{K^R}$ and the similarly defined topology $\tau_K$. In the case of a locally convex space $X$ we deduce from this convergence of the barycenters of weakly convergent measures (a precise formulation is given below). It is also shown that if $X$ is a Fréchet space, then every set compact in the topology $\tau_K$ is concentrated on some separable reflexive Banach space $E$ compactly embedded into $X$ and is compact in the topology $\tau_K$ on $E$ generated by the stronger topology from $E$.

Let us connect the topology $\tau_{K^R}$ with the weak topology. The first assertion in the next theorem can be derived from [19, Lemma 1.3.3], but we include a short justification for completeness.

**Theorem 2.1.** Suppose that the topology in $X$ is generated by a family of pseudo-metrics $\Pi$. Then the weak topology on the set $\mathcal{M}^*_r(X)$ is generated by the family of seminorms $\|\cdot\|_{K^R,p}$, $p \in \Pi$.

In addition, the weak topology is generated by these seminorms on every bounded in variation and uniformly tight set in $\mathcal{M}_r(X)$.

**Proof.** Let $p \in \Pi$. We denote by $X/p$ the quotient space with the metric

$$\hat{p}([x]_p, [y]_p) = p(x, y), \quad [x]_p = \{y \in X : p(x, y) = 0\}. $$

The canonical mapping $\pi_p : x \mapsto [x]_p$ is continuous. Note that the equality

$$\|\mu \circ \pi_p^{-1}\|_{\hat{p}} = \|\mu\|_{K^R,p}$$

is true. Indeed, for every function $f \in \text{Lip}_1(\hat{p})$ on $X/p$ the composition $f \circ \pi_p$ belongs to the set $\text{Lip}_1(p)$, so $\|\mu \circ \pi_p^{-1}\|_{\hat{p}} \leq \|\mu\|_{K^R,p}$. On the other hand, if $f \in \text{Lip}_1(p)$, then we set $g([x]_p) := f(x)$. The function $g$ is well-defined, since the function $f$ is Lipschitz in the pseudo-metric $p$. Moreover, $g \in \text{Lip}_1(\hat{p})$ and we have

$$\int_X f \, d\mu = \int_{X/p} g \, d(\mu \circ \pi_p^{-1}).$$

It suffices to prove the coincidence of the weak topology and $\tau_{K^R}$ on $\mathcal{P}_r(X)$. If a net of measures $\mu_\alpha \in \mathcal{P}_r(X)$ converges weakly to a measure $\mu \in \mathcal{P}_r(X)$, then their images $\mu_\alpha$ under the mapping $\pi_p$ are Radon on the metric space $X/p$ and converge weakly to the image of $\mu$. Therefore,

$$\|\mu_\alpha \circ \pi_p^{-1} - \mu \circ \pi_p^{-1}\|_{\hat{p}} = \|\mu_\alpha - \mu\|_{K^R,p} \to 0.$$
Conversely, let $\mu_\alpha \to \mu$ with respect to all seminorms $\| \cdot \|_{K_{R,p}}$. Then we have convergence of the integrals of all functions of the form $f(x) = \min(p(x, x_0), c)$. This implies convergence $\mu_\alpha(U) \to \mu(U)$ on all open sets $U$ of the form $U = \{ x : f(x) < t \}$ with $\mu$-zero boundary, which implies weak convergence (see [14] Theorem 4.3.11).

Let us prove the second assertion. Let $S$ be a bounded and uniformly tight set in $M_p(X)$. Suppose that a net of measures $\mu_\alpha$ from $S$ converges weakly to a measure $\mu \in M_p(X)$. Let us show that for every pseudometric $p \in \Pi$ we have convergence $\|\mu_\alpha - \mu\|_{K_{R,p}} \to 0$. We can assume that $\|\mu_\alpha\| \leq 1$ for all $\alpha$. Given $\varepsilon > 0$, we can find a compact set $K$ such that $|\mu|(X \setminus K) < \varepsilon$ and $|\mu_\alpha|(X \setminus K) < \varepsilon$ for all $\alpha$. Note that the set of restrictions to $K$ of the functions from $\text{Lip}_1(p)$ bounded by 1 in absolute value is compact in the space $C(K)$ with the sup-norm by the Arzela–Ascoli theorem. Therefore, it contains a finite $\varepsilon$-net $f_1, \ldots, f_m$. Take an index $\alpha_0$ such that for all $\alpha \geq \alpha_0$ we have

$$\left| \int_X f_i d\mu_\alpha - \int_X f_i d\mu \right| < \varepsilon, \quad i = 1, \ldots, m.$$ 

Let $f \in \text{Lip}_1(p)$ and $|f| \leq 1$. There is $f_i$ with $|f - f_i| \leq \varepsilon$ on $K$. Then

$$\left| \int_X f d\mu_\alpha - \int_X f d\mu \right| < 7\varepsilon,$$

since the integrals over $K$ differ by at most $\varepsilon$ and the absolute values of the integrals over the complement of $K$ are estimated by $\varepsilon$.

**Remark 2.2.** The topologies $\tau_{KR}$ and $\tau_K$ are introduced precisely in the same way on the space of all Baire measures $M_\sigma(X)$ or on its subspace $M_p(X)$ of $\sigma$-additive measures (see [11]). The previous theorem with the same proof remains valid for $\tau$-additive measures.

**Proposition 2.3.** If a completely regular space $X$ is separable or possesses a countable collection of continuous functions separating points, then the weak topology coincides with the topology $\tau_{KR}$ on weakly compact sets in $M_p(X)$.

**Proof.** Since on a compact space every weaker topology coincides with the original one, it suffices to verify that weak convergence of a net of measures from a weakly compact set $S$ implies convergence in the topology $\tau_{KR}$ under one of our two conditions. Let $X$ be separable and $p \in \Pi$. Then the image $X_p$ of $X$ under the indicated factorization is a separable metric space with the completion $Z_p$. The image of $S$ is compact in $M_p(Z_p)$. Since $Z_p$ possesses a countable collection of continuous functions separating points, the compact image of $S$ is metrizable in the weak topology. Hence it suffices to use that any weakly convergent sequence in $M_p(Z_p)$ also converges in the Kantorovich–Rubinshtein norm (see, e.g., [26] or [14] Exercise 3.5.22). The second case is similar: here the compact set $S$ itself is metrizable (because a countable family of continuous functions separating points gives a countable family of bounded continuous functions separating measures), hence its image is also. Therefore, it suffices to verify our assertion for countable sequences of Radon measures on a metric space, which reduces to the case of a separable space.

Note that a similar assertion is true for the space $M_p(X)$ of Baire measures on a separable space $X$. Recall (see [14] Theorem 4.8.3) that a set $M$ in $M_\sigma(X)$ is contained in a weakly compact set precisely when for every sequence of functions $f_n \in C_b(X)$ pointwise decreasing to zero (in the formulation of the cited theorem it
is mistakenly said “converging” in place of “decreasing”, but the proof deals with decreasing sequences; the case of nonnegative measures is covered by Theorem 4.5.10 with a correct formulation) one has

$$\lim_{n \to \infty} \sup_{\mu \in \mathcal{M}} \left| \int_X f_n d\mu \right| = 0.$$  

This criterion extends at once to weakly complete sets in the space of Radon measures (or in the case where all Baire measures on $X$ have Radon extensions).

There is a sufficient condition of convergence in the topology $\tau_K$.

**Proposition 2.4.** Suppose that a net $\{\mu_\alpha\} \subset \mathcal{M}_r^r(X)$ converges in the topology $\tau_{KR}$ to a measure $\mu \in \mathcal{M}_r^r(X)$ (for nonnegative measures or from measures in a bounded and uniformly tight family this is equivalent to weak convergence). If every pseudo-metric $p$ from $\Pi$ satisfies the condition of uniform integrability

$$\lim_{R \to \infty} \sup_{\alpha \in \mathcal{P} \geq R} p(x, x_0) |\mu_\alpha| (dx) = 0,$$

then $\{\mu_\alpha\}$ converges in the topology $\tau_K$. In the case of probability measures this is also a necessary condition.

Finally, in the case of a countable sequence of measures, convergence in $\tau_{KR}$ can be replaced with weak convergence.

**Proof.** Let $p \in \Pi$ and $\varepsilon > 0$. There is $R > 0$ such that the integral of $p(x, x_0)$ over the set $\{p \geq R\}$ is less than $\varepsilon$ for every measure $|\mu_\alpha|$. Next we take an index $\alpha_0$ such that $\|\mu - \mu_\alpha\|_{K^p, R} < \varepsilon(1 + R)^{-1}$ for all $\alpha \geq \alpha_0$ and $|\mu(X) - \mu_\alpha(X)| < \varepsilon$. Let $f \in \text{Lip}_1(p)$ and $f(x_0) = 0$. Set $f_R = \max(-R, \min(f, R))$. Then $f_R = f$ on $\{p < R\}$, $f_R \in \text{Lip}_1(p)$ and $|f_R| \leq R$. Hence the integrals of $f_R$ against $\mu$ and $\mu_\alpha$ with $\alpha \geq \alpha_0$ differ in absolute value by at most $\varepsilon$. Clearly, $|f| \leq p$ and $|f_R| \leq p$, so the integrals of $f$ and $f_R$ against $\mu_\alpha$ differ in absolute value by at most $2\varepsilon$. Then the same is true for $\mu$. Therefore, the difference of the integrals of $f$ against $\mu$ and $\mu_\alpha$ with $\alpha \geq \alpha_0$ does not exceed $3\varepsilon$. Hence $\|\mu - \mu_\alpha\|_{K^p, R} \leq 4\varepsilon$. For nonnegative measures or measures from a bounded uniformly tight family weak convergence is equivalent to convergence in the topology $\tau_{KR}$.

It is readily seen that for probability measures the converse is also true. Finally, for a countable sequence of measures $\mu_\alpha$, as above, it suffices to consider the case of a complete metric space, but then we arrive at the case of a uniformly tight family. □

As one can see from Example 3.2 below, for nets of signed measures, convergence in the topology $\tau_{KR}$ cannot be replaced with weak convergence.

Let us give a sufficient condition for the compactness of sets in $\mathcal{M}_r(X)$ in the topology $\tau_{KR}$ and for sets in $\mathcal{M}_r^r(X)$ in the topology $\tau_K$.

**Proposition 2.5.** Let $S \subset \mathcal{M}_r(X)$ be a bounded and uniformly tight set. Then $S$ has compact closure in the topology $\tau_{KR}$.

Moreover, if $S \subset \mathcal{M}_r^r(X)$ and every pseudo-metric $p$ from $\Pi$ satisfies the condition of uniform integrability

$$\lim_{R \to \infty} \sup_{\mu \in S} \int_{\{p \geq R\}} p(x, x_0) |\mu| (dx) = 0$$

for some $x_0 \in X$, then $S$ is contained in a compact set in the topology $\tau_K$. 

Proof. It follows from the assumption that $S$ has compact closure in the weak topology, and the previous theorem states that on $S$ it coincides with the topology $\tau_{KR}$. The second assertion follows by the previous proposition. Indeed, every net in $S$ contains a subnet $\{\mu_\alpha\} \subset S$ converging weakly and in the topology $\tau_{KR}$. The limiting measure $\mu$ belongs to $\mathcal{M}_\Pi(X)$. Indeed, for every $p \in \Pi$ and $R > 0$, letting $f_R(x) = \min(p(x, x_0), R)$ for a fixed point $x_0$, we have weak convergence of the measures $f_R \cdot \mu_\alpha$ to $f_R \cdot \mu$, which yields the bound
\[
\|f_R \cdot \mu\| \leq \sup_{\alpha} \|f_R \cdot \mu_\alpha\| \leq \sup_{\alpha} \int_X p(x, x_0) |\mu_\alpha|(dx).
\]
So the function $p(x, x_0)$ is $|\mu|$-integrable.

We now prove that a uniformly tight family of Radon measures on a Banach space with a uniformly integrable norm remains uniformly tight with some stronger norm, and this norm is also uniformly integrable (so that this family is contained in some compact set in the Kantorovich norm). More precisely, this family turns out to be uniformly tight on a compactly embedded separable reflexive Banach space with a uniformly integrable norm. The result for a single Borel probability measure on a separable Banach space was proved in [10], extending Buldygin’s theorem [17]. The proof employs the known Grothendieck’s construction (see [15 §2.5]). Let $B$ be a bounded absolutely convex set in a locally convex space $X$. Denote by $E_B$ the linear span of $B$ equipped with the norm
\[
p_B(x) = \inf\{t > 0 : t^{-1}x \in B\},
\]
which is the Minkowski functional of the set $B$. If $X$ is sequentially complete, then $E_B$ is a Banach space.

**Theorem 2.6.** Let $X$ be a Fréchet space and let $\mathcal{M}$ be a uniformly tight family of Radon measures on $X$ such that all seminorms $p_n$ generating the topology of $X$ are uniformly integrable with respect to the measures from $\mathcal{M}$, i.e.,
\[
\lim_{m \to \infty} \sup_{\mu \in \mathcal{M}} \int_{\{x : p_n(x) > m\}} p_n(x) |\mu|(dx) = 0.
\]
Then there is a linear subspace $E \subset X$ with the following properties:

(i) the space $E$ with some norm $\| \cdot \|_E$ is a separable reflexive Banach space whose closed unit ball is compact in $X$;

(ii) the family $\mathcal{M}$ is concentrated and uniformly tight on $E$ and $\| \cdot \|_E$ is also uniformly integrable.

**Proof.** We can assume that all measures $\mu \in \mathcal{M}$ are nonnegative. We need the following technical assertion. Let $p_n \leq p_{n+1}$ for all $n$. Then there is a sequence of continuous seminorms $q_n$, which generates the original topology of $X$, and there is sequence of positive numbers $\alpha_n$ decreasing to zero such that $q_n \leq q_{n+1}$ and
\[
\lim_{n \to \infty} \sup_{\mu \in \mathcal{M}} \sum_{k=n}^{\infty} \mu(x : q_k(x) > k\alpha_k) = 0, \tag{2.1}
\]
\[
\lim_{n \to \infty} \alpha_n^{-1} \sup_{\mu \in \mathcal{M}} \int_{\{x : q_n(x) > n\alpha_n\}} q_n(x) \mu(dx) = 0. \tag{2.2}
\]
Indeed, there are increasing numbers \( N_n \) such that \( N_{n+1} > 2^n N_n \) and

\[
\sup_{\mu \in \mathcal{M}} \sum_{k=N_n}^{\infty} \mu(x: p_n(x) > k) \leq \sup_{\mu \in \mathcal{M}} \int_{\{x: p_n(x) > N_n\}} p_n(x) \mu(dx) < 4^{-n}.
\]

Using these numbers, we set \( \alpha_k = 1 \) and \( q_k = p_1 \) if \( k \leq N_2 \), \( \alpha_k = 2^{-1} \) and \( q_k = p_2 \) if \( N_2 < k \leq N_3 \), \( \alpha_k = 2^{-n} \) and \( q_k = p_n \) if \( N_{n+1} < k \leq N_{n+2} \). Then if \( N_{n+1} < k \leq N_{n+2} \) we have

\[
\alpha_k^{-1} \sup_{\mu \in \mathcal{M}} \int_{\{x: q_k(x) > k \alpha_k\}} q_k(x) \mu(dx) \leq 2^n \sup_{\mu \in \mathcal{M}} \int_{\{x: p_n(x) > N_n\}} p_n(x) \mu(dx) < 2^{-n}
\]

and

\[
\sup_{\mu \in \mathcal{M}} \sum_{k>N_{n+1}}^{\infty} \mu(x: q_k(x) > k \alpha_k) = \sup_{\mu \in \mathcal{M}} \sum_{j=1}^{\infty} \sum_{N_{n+j} < k \leq N_{n+j+1}} \mu(x: q_k(x) > k \alpha_k)
= \sup_{\mu \in \mathcal{M}} \sum_{j=1}^{\infty} \sum_{N_{n+j} < k \leq N_{n+j+1}} \mu(x: p_{n+j-1}(x) > k 2^{-(n+j-1)})
\leq \sup_{\mu \in \mathcal{M}} \sum_{j=1}^{\infty} 2^{n+j-1} \sum_{k=N_{n+j-1}}^{\infty} \mu(x: p_{n+j-1}(x) > k) < 2^{1-n}.
\]

For every \( n \in \mathbb{N} \) there is a compact set \( K_n \) in the set \( U_n := \{x: q_n(x) \leq n\} \) such that for all \( \mu \in \mathcal{M} \) we have

\[
\mu(\alpha_n U_n \setminus \alpha_n K_n) < 2^{-n}.
\]

Then

\[
\mu \left( \mathbb{X} \setminus \bigcup_{n=1}^{\infty} \alpha_n K_n \right) = 0 \quad \forall \mu \in \mathcal{M},
\]

since

\[
\mu(\mathbb{X} \setminus \alpha_n K_n) = \mu(\alpha_n U_n \setminus \alpha_n K_n) + \mu(\mathbb{X} \setminus \alpha_n U_n) < 2^{-n} + \mu(\mathbb{X} \setminus \alpha_n K_n) < 2^{-n} + \mu(\mathbb{X} \setminus \alpha_n U_n).
\]

Note that the set

\[
K = \bigcup_{n=1}^{\infty} c_n K_n, \quad c_n := \alpha_n n^{-1},
\]

is totally bounded. Indeed, given \( \varepsilon > 0 \), take \( n_0 \in \mathbb{N} \) and \( \delta > 0 \) such that \( \alpha_{n_0} < \delta \) and \( \{x: q_{n_0}(x) \leq \delta\} \) lies in the open ball of radius \( \varepsilon \) in the metric of \( \mathbb{X} \) centered at zero. Then, since the sequence \( \{\alpha_n\} \) decreases, we obtain that the compact sets \( c_n K_n \) are contained in this ball for all \( n \geq n_0 \). The remaining compact sets are also covered by finitely many balls of radius \( \varepsilon \). The closed absolutely convex hull \( V \) of \( K \) is also precompact (see [15, Proposition 1.8.2]), so \( V \) is compact. Then \( (E_V, p_V) \) is a Banach space (see [15, Proposition 2.5.1]). The function \( p_V \) is Borel measurable, since \( \{x: p_V(x) \leq c\} = c V \). Moreover,

\[
\alpha_n K_n \subset n K \subset n V,
\]

whence it follows that

\[
\mu(\mathbb{X} \setminus \alpha_n K_n) < 2^{-n} + \mu(\mathbb{X} \setminus \alpha_n U_n).
\]
Hence

\[
\int_{\{p_V > n\}} p_V \, d\mu = n \mu(x: p_V(x) > n) + \int_n^\infty \mu(x: p_V(x) > t) \, dt \\
\leq n(2^{-n} + \mu(x: q_n(x) > n\alpha_n)) + \sum_{k=n}^{\infty} \int_{k+1}^{\infty} \mu(x: p_V(x) > t) \, dt \\
\leq n2^{-n} + \alpha_n^{-1} \int_{\{q_n(x) > n\alpha_n\}} q_n(x) \, d\mu(x) + \sum_{k=n}^{\infty} (2^{-k} + \mu(x: q_k(x) > k\alpha_k)).
\]

The right-hand side of the last inequality tends to zero as \( n \to \infty \) uniformly in \( \mu \in \mathcal{M} \), which implies the uniform integrability of the function \( p_V \) with respect to the family of measures \( \mathcal{M} \).

We now prove the existence of a separable reflexive Banach space \( E \) satisfying conditions (i) and (ii). There is a convex balanced compact set \( W \) such that \( V \subset W \), the Banach space \( (E_W, p_W) \) is separable and reflexive and \( K \) is also compact in the norm \( p_W \) (see [13, Corollary 2.5.12]). Then \( p_W \leq p_V \), which shows that \( p_W \) is also uniformly integrable. Moreover, all Borel sets in \( E_W \) are Borel in \( X \), since the image of a Borel set under a continuous injective mapping from a Polish to a metric space is Borel (see [11, Theorem 6.8.6]). Therefore, the measures from \( \mathcal{M} \) can be restricted to the Borel \( \sigma \)-algebra of the Banach space \( E_W \) and they are concentrated on this space and uniformly tight, since \( V \) is compact in \( E_W \) and

\[
\mu(E_W \setminus nV) \leq \mu(X \setminus \alpha_n K_n) < 2^{-n} + \mu(x: q_n(x) > n\alpha_n),
\]

which tends to zero as \( n \to \infty \) for each measure \( \mu \) in \( \mathcal{M} \). \( \square \)

**Remark 2.7.** Let \( (X, d) \) be a metric space, \( x_0 \in X \) a fixed point, \( q \geq 1 \) and \( \mathcal{M}_q(X) \) the space of Radon measures with finite moment of order \( q \), that is, measures \( \mu \) such that the function \( d(x, x_0)^q \) is \( |\mu| \)-integrable for some \( x_0 \). Recall that for any \( q \geq 1 \) the subspace of probability measures \( \mathcal{P}_q(X) \) can be equipped with the \( q \)-Kantorovich metric \( d_{K,d,q} \) defined by

\[
d_{K,d,q}^q(\mu, \nu) = \inf_{\sigma \in \Pi(\mu, \nu)} \int d(x, y)^q \, \sigma(dx \, dy),
\]

where \( \Pi(\mu, \nu) \) is the set of probability measures on \( X \times X \) with projections \( \mu \) and \( \nu \) on the factors. The metric \( d_{K,d,q} \) with \( q > 1 \) is not generated by a norm (unlike the case \( q = 1 \)), where \( d_{K,d,1}(\mu, \nu) = \|\mu - \nu\|_K \), but the norm

\[
K_{d,q}(\mu) = \|(1 + d(\cdot, x_0)^q)\mu\|_{KR}
\]

generates the same topology on \( \mathcal{P}_q(X) \) as \( d_{K,d,q} \) (see [14, Corollary 3.3.7]), where there is a misprint in the formula: the norm \( \|\cdot\|_K \) should be replaced with \( \|\cdot\|_{KR} \). So in the general case of a family of pseudo-metrics \( \Pi \) we can introduced the Kantorovich topology \( \tau_{K,q} \) on \( \mathcal{M}_q(X) \) generated by the seminorms \( K_{p,q} \) with \( p \in \Pi \).

The same reasoning as above leads to the following result for \( \tau_{K,q} \).

**Proposition 2.8.** Suppose that a net of measures \( \mu_\alpha \in \mathcal{M}_q(X) \), where \( q \geq 1 \), converges to a measure \( \mu \in \mathcal{M}_\tau(X) \) in the topology \( \tau_{KR} \) (for nonnegative measures or measures from a bounded and uniformly tight family this is equivalent to weak convergence). If for every pseudo-metric \( p \) from \( \Pi \) we have

\[
\lim_{R \to \infty} \sup_\alpha \int_{\{p \geq R\}} p(x, x_0)^q |\mu_\alpha|(dx) = 0,
\]
then \( \mu \in \mathcal{M}^q_r(X) \) and \( \{\mu_\alpha\} \) converges to \( \mu \) in the topology \( \tau_{K,q} \).

3. Convergence of barycenters

We say that a Borel measure \( \mu \) on a locally convex space \( X \) has a mean (or barycenter) \( m_\mu \in X \) if \( X^* \subset L^1(\mu) \) and for every \( f \in X^* \) we have

\[
f(m_\mu) = \int_X f(x) \mu(dx).
\]

In the case of a Banach space \( X \) with a Radon measure \( \mu \), the mean exists if the norm is \( \mu \)-integrable. In this case \( m_\mu \) is the Bochner integral

\[
m_\mu = \int_X x \mu(dx).
\]

A similar statement is true in any quasi-complete locally convex space (see [15, Corollary 5.6.8]): for the existence of the mean, it is sufficient to have the integrability of all seminorms from a family generating the topology of this space (which is equivalent to the integrability of all continuous seminorms).

It is worth noting (although we do not use it below) that consideration of convergence of barycenters in locally convex spaces reduces to Banach spaces by means of the factorizations used above and the following simple observation: if a net of elements \( v_\alpha \) in a locally convex space \( X \) and an element \( v \in X \) are such that \( Tv_\alpha \to Tv \) for every continuous linear operator \( T \) on \( X \) with values in a normed space, then \( v_\alpha \to v \) in \( X \). Indeed, for each continuous seminorm \( p \) on \( X \) the linear subspace \( Y = p^{-1}(0) \) is closed, so the quotient space \( X/Y \) is normed with the norm

\[
\| [x] \| = p(x), \quad [x] = x + Y, \ x \in X,
\]

and the natural projection \( x \mapsto [x] \) is linear and continuous. It follows from this observation that if a net of measures \( \mu_\alpha \in \mathcal{M}_r(X) \) and a measure \( \mu \in \mathcal{M}_r(X) \) with barycenters in a locally convex space \( X \) are such that for each normed space \( Y \) and each continuous linear operator \( T: X \to Y \) the barycenters of \( \mu_\alpha \circ T^{-1} \) converge to the barycenter of \( \mu \circ T^{-1} \), then \( m_{\mu_\alpha} \to m_\mu \). Indeed, the barycenter of \( \mu_\alpha \circ T^{-1} \) is \( Tm_{\mu_\alpha} \).

Note also that if a Borel measure \( \mu \) has a barycenter and a continuous seminorm \( q \) is \( \mu \)-integrable, then from the Hahn–Banach theorem and the definition of the barycenter we obtain

\[
q(m_\mu) = \sup \left\{ \left\| \int_X f \ d\mu \right\| : f \in X^*, \ |f| \leq q \right\} \leq \| \mu \|_{K,q}.
\]

As a consequence of the results of the previous section (see Proposition 2.4), we obtain the following sufficient condition for convergence of barycenters.

**Corollary 3.1.** Suppose that a sequence of Radon measures \( \mu_n \) and a Radon measure \( \mu \) on a locally convex space \( X \) have barycenters, the measures \( \mu_n \) converge weakly to \( \mu \) and every continuous seminorm is uniformly integrable with respect to the sequence \( \mu_n \). Then \( m_{\mu_n} \to m_\mu \).

In the case of probability measures, the same is true for nets.

The following example shows that the second assertion of Corollary 3.1 can fail for a net of signed measures.
Example 3.2. In the Banach space $X = l^1$ there is a net of uniformly bounded signed measures concentrated on the unit ball such that it converges weakly to zero, but their barycenters have unit norms. We fix a finite collection of bounded continuous functions $f_1, \ldots, f_n$ on $X$. Consider the following vectors in $\mathbb{R}^n$:

$$v_j = (f_1(e_j), \ldots, f_n(e_j)), \quad j = 1, \ldots, n + 1,$$

where $\{e_j\}$ is the standard basis in $l^1$. The vectors $v_j$ are linearly dependent, so there are numbers $c_1, \ldots, c_{n+1}$ not vanishing simultaneously such that

$$\sum_{j=1}^{n+1} c_j v_j = 0,$$

or, equivalently,

$$\sum_{j=1}^{n+1} c_j f_i(e_j) = 0, \quad i = 1, \ldots, n.$$

By normalization, we can achieve the equality $\sum_j |c_j| = 1$. Now for the basic neighborhood of zero in the weak topology

$$U = U_{f_1, \ldots, f_n, \varepsilon} = \left\{ \mu \in \mathcal{M} : \left| \int_X f_i \, d\mu \right| < \varepsilon, \ i = 1, \ldots, n \right\}$$

we define a measure by the formula

$$\mu_U := \sum_{j=1}^{n+1} c_j \delta_{e_j}.$$

Then by construction $\mu_U \in U$, and this measure is concentrated on the unit sphere. The set of basic neighborhoods of zero is directed with respect to the inverse inclusion: a neighborhood $V$ is declared to be larger than a neighborhood $U$ if $V \subset U$. By definition, the constructed net of measures $\mu_U$ converges weakly to zero. Finally, the mean of the measure $\mu_U$ equals $\sum_j c_j e_j$, therefore, we have the equality

$$\|m_{\mu_U}\| = \sum_{j=1}^{n+1} |c_j| = 1.$$

In this example all measures in the net are absolutely continuous with respect to the measure $\sum_n 2^{-n} \delta_{e_n}$. Note that similarly one can construct a net converging in the stronger topology of duality with the space of all bounded Borel functions.

In the general case, weak convergence of measures $\mu_n$ to $\mu$ and weak convergence of measures $\nu_n = f_n \cdot \mu_n$ to a measure $\nu$ do not imply that $\nu$ is absolutely continuous with respect to $\mu$. For example, the measures $(1 - n^{-1}) \delta_0 + n^{-1} \delta_1$ converges weakly (and even in the total variation norm) to $\delta_0$, but the measure $\delta_1$ is mutually singular with $\delta_0$. However, under the following additional condition this implication is true.

Lemma 3.3. Suppose that Radon probability measures $\mu_\alpha$ on a completely regular space $X$ converge weakly to a Radon measure $\mu$ and the measures $\nu_\alpha = f_\alpha \cdot \mu_\alpha$ converge weakly to a Radon measure $\nu$. Assume also that

$$\lim_{R \to \infty} \sup_{\alpha} |\nu_\alpha|(x : |f_\alpha(x)| \geq R) = 0.$$

Then $\nu \ll \mu$. In particular, this is true if $\sup_\alpha \|f_\alpha\|_{L^2(\mu_\alpha)} < \infty$. 
Proof. Let $K$ be a compact set such that $\mu(K) = 0$. Suppose that $\nu(K) = \delta > 0$ (the case $\nu(K) < 0$ is similar). Pick $R > 1$ such that
\[
\sup_{\alpha} |\nu_\alpha|(x) : |f_\alpha(x)| \geq R < \delta/4.
\]
We can find an open set $U$ such that $K \subset U$ and $\mu(U) = \mu(U_0) < \delta(2R)^{-1}$, where $U_0$ is the closure of $U$. This is possible, since we can take some open set $U_0$ with $K \subset U_0$ and $\mu(U_0) < \delta(2R)^{-1}$, then find a continuous function $f$ with values in $[0, 1]$ for which $f|K = 1$ and $f|X \setminus U_0 = 0$, finally, for $U$ we can take the set $\{f > c\}$, where $c \in (0, 1)$ is picked such that $\mu(f^{-1}(c)) = 0$. Then $\mu_\alpha(U) \rightarrow \mu(U)$ by Alexandrov’s criterion (see [14, Corollary 4.3.5]), hence $\mu_\alpha(U) < \delta(2R)^{-1}$ for all $\alpha$ large enough. For such $\alpha$ we finally obtain
\[
|\nu_\alpha|(U) = \int_U |f_\alpha| \, d\mu_\alpha \leq R \mu_\alpha(U) + \delta/4 < 3\delta/4,
\]
which gives the estimate $|\nu(U)| \leq \delta$, hence $\nu(K) \leq \delta$, which contradicts our assumption. □

4. Logarithmically concave and stable measures

Now we investigate convergence of logarithmically concave measures and their means. Recall that a Radon probability measure $\mu$ on a locally convex space $X$ is called logarithmically concave if $\mu$ satisfies the inequality
\[
\mu(tA + (1 - t)B) \geq t \mu(A)^t \mu(B)^{1-t}
\]
for all compact sets $A$ and $B$. This definition is also equivalent to the property that for every continuous linear operator $T$ from $X$ to $\mathbb{R}^n$ the measure $\mu \circ T^{-1}$ has a density of the form $\exp(-V)$ with respect to Lebesgue measure on some affine subspace with a convex function $V$ (see [16], [12]).

The class of logarithmically concave measures contains all Gaussian measures, i.e., measures for which all continuous linear functionals are Gaussian random variables.

We need the following estimate due to C. Borell (see [16] or [12, Theorem 4.3.7]). Let $\mu$ be a logarithmically concave measure on a locally convex space $X$ and let $A$ be an absolutely convex Borel set with $\theta := \mu(A) > 0$. Then
\[
\mu(X \setminus tA) \leq \left(\frac{1 - \theta}{\theta}\right)^{t/2}, \quad t \geq 1.
\]
This estimate implies that, for any Borel seminorm $q$ such that $\mu(q > 1) = 1 - \theta < 1/2$, one has
\[
\int_X \exp(\alpha(\theta)q) \, d\mu \leq M(\theta)
\]
with some constants $\alpha(\theta)$ and $M(\theta)$ depending only on $\theta > 1/2$. Therefore,
\[
\|q\|_{L^p(\mu)} \leq C(p, \theta)
\]
with some constants $C(p, \theta)$ depending only on $p$ and $\theta > 1/2$.

We apply inequality (4.1) to prove the following sufficient condition for convergence of means of logarithmically concave measures. Note that all Radon Gaussian measures have barycenters, but for logarithmically concave measures this is known only under the assumption of sequential completeness of the space, as for general measures with finite first moment.
Theorem 4.1. If a net of logarithmically concave measures \( \mu_\alpha \) converges weakly to a measure \( \mu \), then for every continuous seminorm \( q \) there is \( \kappa > 0 \) such that

\[
\lim_{\alpha} \int_X \exp(\kappa q) \, d\mu_\alpha = \int_X \exp(\kappa q) \, d\mu.
\]

Therefore, for every \( r > 0 \) we have

\[
\lim_{\alpha} \int_X q^r \, d\mu_\alpha = \int_X q^r \, d\mu.
\]

Finally, if \( \mu_\alpha, \mu \) have barycenters, then \( m_{\mu_\alpha} \to m_\mu \).

Proof. Take \( c > 0 \) such that \( \mu(q < c) > \theta > 1/2 \). Since \( \{ \mu_\alpha \} \) converges weakly to \( \mu \) and the set \( \{ q < c \} \) is open, by Alexandrov’s theorem we have \( \mu_\alpha(q < c) > \theta \) for all \( \alpha \) larger than some \( \alpha_0 \). Then by (4.1) we obtain the inequality

\[
\mu_\alpha(q \geq ct) \leq \left( \frac{1 - \theta}{\theta} \right)^{t/2}, \quad t \geq 1.
\]

Set

\[
\tau := \left( \frac{1 - \theta}{\theta} \right)^{1/2}, \quad \tau \in (0, 1).
\]

For all \( \alpha \geq \alpha_0 \) we have

\[
\int_X \exp(\kappa q) \, d\mu_\alpha = 1 + c\kappa \int_0^\infty \exp(ckt)\mu_\alpha(q \geq ct) \, dt \\
\leq \exp(ck) + c\kappa \int_1^\infty \exp(ckt)\tau^t \, dt.
\]

The integral

\[
\int_1^\infty \exp(ckt)\tau^t \, dt
\]

is finite for \( \kappa < -\ln \tau/c \). Thus, there exists \( \kappa_0 > 0 \) such that for all \( \alpha \geq \alpha_0 \) the inequality

\[
\int_X \exp(\kappa_0 q) \, d\mu_\alpha \leq I(\theta, c)
\]

holds, whence for all \( \kappa < \kappa_0 \) we have the estimate

\[
\int_{\{q \geq R\}} \exp(\kappa q) \, d\mu_\alpha \leq \exp((\kappa - \kappa_0)R)I(\theta, c).
\]

Consequently, the integrals of \( \exp(\kappa q) \) converge (see [14, Theorem 4.3.15]), which yields convergence of the integrals of \( q^r \). Convergence of means follows from Corollary 3.1.

Corollary 4.2. In the previous theorem one has also convergence in the topology \( \tau_{K,q} \) with any \( q \geq 1 \) introduced in Remark 2.7.

For a Radon probability measure \( \mu \) on a locally convex space, we denote by \( \mathcal{P}^d(\mu) \) the set of all \( \mu \)-measurable polynomials of degree \( d \geq 0 \), i.e., \( \mu \)-measurable functions possessing versions that are polynomials of degree \( d \) on \( X \) in the usual algebraic sense (this is equivalent to the property that the restrictions to all affine lines are polynomials of degree \( d \)). For a Gaussian measure, every measurable polynomial of degree \( d \) is the limit almost everywhere and in \( L^2 \) of a sequence of polynomials of the form \( f(l_1, \ldots, l_n) \), where \( f \) is a polynomial of degree \( d \) on \( \mathbb{R}^n \) and \( l_j \) are continuous
linear functionals. It is not known whether this is true for all logarithmically concave measures; about measurable polynomials, see [13].

For measurable polynomials on a space equipped with a logarithmically concave measure two very important estimates are known with constants independent of the measure. The first one (obtained in [18], [25] in the finite-dimensional case and extended in [3] to the infinite-dimensional case) gives an estimate for small values:

$$\mu(x: |f(x)| \leq r)\|f\|_{L^1(\mu)}^{1/d} \leq cdr^{1/d}, \quad f \in \mathcal{P}^d(\mu). \quad (4.4)$$

The second one (see [7], [8], [3]) gives the equivalence of all $L^p$-norms on $\mathcal{P}^d(\mu)$:

$$\|f\|_{L^p(\mu)} \leq C(p,d)\|f\|_{L^1(\mu)}, \quad f \in \mathcal{P}^d(\mu).$$

**Corollary 4.3.** (i) Let $\{\mu_\alpha\}$ be a net of logarithmically concave measures on a sequentially complete locally convex space $X$ and let $\nu_\alpha = f_\alpha \cdot \mu_\alpha$ be probability measures, where $f_\alpha \in \mathcal{P}^d(\mu_\alpha)$ with a common degree $d$. If the measures $\nu_\alpha$ converge weakly to a measure $\nu$, then their means $m_\alpha$ converge to $m_\nu$.

In addition, if $\{\nu_\alpha\}$ is uniformly tight, which holds automatically in the case of a weakly convergent countable sequence on a Fréchet space, then $\{\mu_\alpha\}$ is also uniformly tight and has a limit point $\mu$ that is a logarithmically concave measure, moreover, the measures $\mu$ and $\nu$ are equivalent.

(ii) Suppose that $\{\mu_\alpha\}$ is a uniformly tight family of logarithmically concave measures on a locally convex space and for each $\alpha$ there is a measure $\nu_\alpha = f_\alpha \cdot \mu_\alpha$ with $f_\alpha \in \mathcal{P}^d(\mu_\alpha)$. If the family $\{\nu_\alpha\}$ is bounded in variation, then it is uniformly tight.

**Proof.** (i) Let us verify that every continuous seminorm $q$ on $X$ is uniformly integrable with respect to the measures $|\nu_\alpha|$ and also with respect to the measures $\mu_\alpha$. It suffices to verify the uniform boundedness of the norms $\|q\|_{L^2(\mu_\alpha)}$, since the equality $\|f_\alpha\|_{L^1(\mu_\alpha)} = 1$ implies the uniform follows boundedness of the norms $\|f_\alpha\|_{L^2(\mu_\alpha)}$. Let $\varepsilon > 0$. By (4.4) there holds the estimate

$$\mu_\alpha(f_\alpha \leq r) \leq cdr^{1/d}.$$

Pick $r \in (0,1)$ such that $cdr^{1/d} < \varepsilon/2$. Next we find $t > 1$ such that $2^{-t/2} < \varepsilon r/2$. There is a number $R > 0$ for which $\nu(\{q < R\}) > 2/3$. By weak convergence $\nu_\alpha(\{q < R\}) > 2/3$ for all $\alpha$ large enough. We can assume that this is true for all $\alpha$. By inequality (4.4) we have

$$\nu_\alpha(\{q \geq tR\}) < \varepsilon r/2.$$

Then

$$\mu_\alpha(\{q \geq tR\} \cap \{f_\alpha > r\}) \leq r^{-1}\nu_\alpha(\{q \geq tR\}) < \varepsilon/2,$$

whence

$$\mu_\alpha(\{q \geq tR\}) \leq \varepsilon/2 + \mu_\alpha(\{f_\alpha \leq r\}) < \varepsilon.$$

If the family $\{\nu_\alpha\}$ is uniformly tight, then the uniform tightness of $\{\mu_\alpha\}$ is obvious from (4.4).

It follows from Lemma 3.3 that the measure $\nu$ is absolutely continuous with respect to $\mu$. The absolute continuity of $\mu$ with respect to $\nu$ follows from the same lemma applied to the probability measures $f_\alpha \cdot \mu_\alpha$ and the measures $\mu_\alpha$ given by the densities $f_\alpha^{-1}$ with respect to $f_\alpha \cdot \mu_\alpha$ (note that $f_\alpha(x) > 0$ for $\mu_\alpha$-a.e. $x$). These densities satisfy the hypotheses of the lemma, since

$$\mu_\alpha(x: f_\alpha(x)^{-1} \geq R) = \mu_\alpha(x: f_\alpha(x) \leq R^{-1}) \leq cdr^{-1/d}$$

by estimate (4.4).
(ii) The norms $\|f_\alpha\|_{L^2(\mu_\alpha)}$ are uniformly bounded by some number $M$ as explained above. Hence for every Borel set $B$ we have $|\nu_\alpha|(B) \leq M^{1/2}\mu_\alpha(B)^{1/2}$, which yields the claim. \hfill \qed

It remains unclear in the considered situation with a countable sequence whether the measures $\mu_n$ must converge (even if they are uniformly tight, it is not clear whether the limit point is unique). This is unclear even in the case of Gaussian measures $\mu_n$ (if they are different).

Stable measures form another important class of probability distributions (see [11], [28], [31]). Recall that a Radon probability measure $\mu$ on a locally convex space $X$ is called stable of order $p$ if for every $\alpha > 0$ and $\beta > 0$ there is a vector $v$ such that the image of $\mu$ under the mapping $x \mapsto (\alpha^p + \beta^p)^{1/p}x + v$ equals the convolution of the images of $\mu$ under the homotheties with the coefficients $\alpha$ and $\beta$. In other words, if $\xi$ and $\eta$ are independent random vectors with distribution $\mu$, then $\alpha\xi + \beta\eta$ has the same law as $(\alpha^p + \beta^p)^{1/p}\xi + v$. The case $p = 2$ corresponds to Gaussian measures, and this is the only intersection with the class of logarithmically concave measures. Stable measures of order $p > 1$ possess barycenters. Indeed, as shown in [1], in this case all measurable seminorms are integrable, hence the barycenter exists in the case of a complete space $X$, so it exists in the completion, but it is readily seen from the definition that it must belong to the original space. Note also that if a net of measures $\mu_\alpha$ that are stable of orders greater than some $p_1 > 1$ converges weakly to a Radon measure $\mu$, then $\mu$ is also stable of some order $p \geq p_1$. Indeed, it is known (see [21]) that if all one-dimensional projections of a measure $\nu$ are stable, then they are stable of the same order $\gamma$, and if $\gamma > 1$, then $\nu$ is stable of order $\gamma$. In order to apply this result from [21] we can take a linear topological embedding of $X$ into a suitable power $\mathbb{R}^T$ of the real line and obtain that $\nu$ is stable on $\mathbb{R}^T$, but then it remains stable on $X$, because $X$ is $\nu$-measurable in $\mathbb{R}^T$ by the Radon property (there is a sequence of compacts sets $K_n$ in $X$ with $\nu(K_n) \to 1$ and these sets are also compact in $\mathbb{R}^T$). Hence it suffices to consider the one-dimensional case, where there is a countable sequence $\mu_{\alpha_n}$ of elements of the original net converging to $\mu$. Passing to a subsequence we can assume that the orders $p_n$ of $\mu_{\alpha_n}$ converge to some $p \in [p_1, 2]$. The Fourier transform of $\mu_{\alpha_n}$ has the form (see [31])

$$\exp[i\pi a_n - c_n|t|^{p_n}(1 - ib_n\text{sign}(t)\tan(\pi p_n/2))]$$

It follows that $a_n \to a$, $b_n \to b$, $c_n \to c$, so that the Fourier transform of $\mu$ has the same form with $(a, c, p, b)$, hence is stable of order $p$.

**Theorem 4.4.** Suppose that a net of measures $\mu_\alpha$ that are stable of orders greater than some $p_1 > 1$ converges weakly to a Radon measure $\mu$. Then they converge in the Kantorovich topology. Hence their barycenters converge to the barycenter of $\mu$.

**Proof.** As explained above, the measure $\mu$ is also stable of some order in $[p_1, 2]$. It suffices to show that for every continuous seminorm $q$ there is a number $r > 1$ such that the integrals of $q^r$ with respect to $\mu_\alpha$ are uniformly bounded for $\alpha$ larger than some $\alpha_0$.

We fix a number $\delta$ such that $2^{-1/2} < \delta < 1$ and then pick $k > 1$ such that $(2^{1/2}\delta)^k > 3$. Next, set

$$q = \prod_{n=1}^{\infty}(1 + \delta^n).$$
It is seen from the proof of [1, Lemma 3.3] that if \( \mu \) is a stable measure of order \( p \geq p_1 \) and \( q \) is a measurable seminorm such that \( \mu(q < 1) > 3/4 \), then
\[
\sum_{n=1}^{\infty} \mu(q > \beta^n) \leq 4k; \quad \beta = 2^{1/p} \delta.
\]
Indeed, in the notation of the cited lemma one has \( \beta(\beta^k - 2^{1/p}) > 1 \), because \( 2^{1/p} \leq 2 \) and \( \beta \geq 2^{1/2} \delta \) due to the bound \( p \leq 2 \). Hence
\[
\gamma = \mu(q \leq \beta(\beta^k - 2^{1/p})) > 3/4,
\]
which yields the bound \( 2\gamma/(2\gamma - 1) < 4 \), but in the cited lemma the series above is estimated by \( k2\delta/(2\delta - 1) \).

Next, an easy inspection of the proof of [1, Theorem 3.1] shows that for a strictly stable measure one has
\[
\mu(q > t) \leq Ct^{-p}, \quad t > 0,
\]
where
\[
C \leq 2C_n^\mu q^p = 2q^p/C'_n = 2q^p/\prod_{n=1}^{\infty} \mu(q \leq \beta^n)
= 2q^p/\prod_{n=1}^{\infty} (1 - \mu(q > \beta^n)) \leq 2q^p \exp\left(2\sum_{n=1}^{\infty} \mu(q > \beta^n)\right) \leq 2q^p \exp(8k).
\]
Finally, in the general case
\[
\mu(q > t) \leq 2C(t-1)^{-p} \leq 4q^p \exp(8k)(t-1)^{-p}, \quad t > 1.
\]
Clearly, in our situation we can assume that \( \mu(q < 1) > 3/4 \), so \( \mu_\alpha(q < 1) > 3/4 \) for all \( \alpha \) sufficiently large. It follows that we have
\[
\mu_\alpha(q > t) \leq 8q^{p_\alpha} \exp(8k)t^{-p_1}, \quad t > 2.
\]
This estimate completes the proof. \( \square \)

**Corollary 4.5.** In the previous theorem one has also convergence in the topology \( \tau_{K,q} \) with any \( q < p_1 \) introduced in Remark 2.7.

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