Isentropic thermodynamics in the PNJL model

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We discuss the isentropic trajectories on the QCD phase diagram in the temperature and the quark chemical potential plane using the Nambu–Jona-Lasinio model with the Polyakov loop coupling (PNJL model). We impose a constraint on the strange quark chemical potential so that the strange quark density is zero, which is the case in the ultra relativistic heavy-ion collisions. We compare our numerical results with the truncated estimates by the Taylor expansion in terms of the chemical potential to quantify the reliability of the expansion used in the lattice QCD simulation. We finally discuss the strange quark chemical potential induced by the strangeness neutrality condition and relate it to the ratio of the Polyakov loop and the anti-Polyakov loop.

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I. INTRODUCTION

Thermodynamic properties of hot and dense matter out of quarks and gluons are of theoretical and experimental importance. To this end the Monte-Carlo method on the lattice has worked quite successfully to simulate the matter at high temperature from the first principle, that is, Quantum Chromodynamics (QCD) [1, 2, 3, 4, 5].

It is believed that such hot and dense matter has been created in the Relativistic Heavy Ion Collider (RHIC) located at BNL. Since the baryon stopping power in the nucleus-nucleus collision is small at the RHIC (top) energy $\sqrt{\text{s}_{\text{NN}}}=200$ GeV, the created matter is nearly free from the net baryon density, and thus, the corresponding baryon chemical potential is much smaller than the temperature.

We are facing two experimental possibilities for the future exploration of the QCD phase diagram: One is going toward even hotter matter as planned in the Large Hadron Collider (LHC) at CERN. The other one is realizing denser matter, which is accessible by collisions at smaller energy with larger baryon stopping power. The formation of baryon-rich matter is within the scope of the Facility for Antiproton and Ion Research (FAIR) at GSI and the systematic energy scan is also under discussion in the future plan for RHIC with emphasis on the QCD critical point search. The present work is focused on the latter; the thermodynamic properties of matter with a finite quark chemical potential $\mu$ whose magnitude is comparable to the temperature $T$.

The lattice QCD simulation has a serious limitation if a finite chemical potential is turned on in the Dirac operator [6, 7]. That is, the notorious sign problem prevents us from applying the Monte-Carlo method to a finite-temperature system. There are a number of proposals to tame this problem [8], among which the Taylor expansion in terms of $\mu/T$ seems to work well insofar as $\mu/T$ is under the radius of convergence [9, 10, 11]. In this way the equation of state (EoS) at finite $T$ and $\mu$ is partially available from the direct lattice QCD simulation. Actually the EoS is an indispensable input for the sake of the hydrodynamic evolution of matter. From the $s/n_B$-constant line, where $s$ is the entropy density and $n_B$ is the baryon number density, we can draw the isentropic trajectory along which the adiabatic system evolves [10, 12, 13].

An interesting progress has been made in Ref. [10]; the isentropic thermodynamics was investigated with a constraint that the strange quark density must be zero, i.e. $n_s=0$. Such a constraint is necessary to emulate matter created by the ultra relativistic heavy-ion collisions because thermalization is achieved within the time scale of the strong interaction and the system is far from $\beta$-equilibration.

In this paper we will utilize the Nambu–Jona-Lasinio model with the Polyakov loop coupling (PNJL model) [14, 15, 16, 17, 18, 19, 20, 21] to reveal the isentropic thermodynamics with $n_s=0$ imposed. In contrast to the NJL model study along the same line [22] (see also Ref. [23] for another type of approach), the PNJL model has an advantage that a part of the gluon degrees of freedom is included, which gives us a hope that the PNJL model can lead to a better EoS reflecting the gluodynamics as well as the chiral dynamics. In Ref. [24] we can already find the isentropic trajectory evaluated in the PNJL model and notice that they are qualitatively similar to the results in Ref. [22]. We will see that the constraint $n_s=0$ causes only little change in the resultant isentropic trajectory because the strange quark mass $M_s$ is still substantially large around the crossover region and the strange density is suppressed in any case.

The extension from the NJL model to the PNJL model is not a minor improvement, however. Our nontrivial finding is, in fact, that the PNJL model is capable of capturing the correct behavior of the induced strange quark chemical potential $\mu_s>0$ to keep $n_s=0$. Since the nonzero $\mu_s$ has its origin in confinement physics, as we will discuss later, the NJL model without any confinement effect is of no use but the PNJL model naturally provides us with $\mu_s>0$ which is related to the ratio of the Polyakov loop $\ell$ and the anti-Polyakov loop $\bar{\ell}$. As a matter of fact, although it is known that $\ell$ deviates from $\ell$ at $\mu\neq 0$ [2, 17, 25, 26], our present work is the very first demonstration to show that this difference $\ell \neq \bar{\ell}$ has a physical consequence in the best of our knowledge.

This paper is organized as follows: We explain the...
model definitions in Sec. [II] Then, in Sec. [III] we show the numerical results with the constraint $n_s = 0$, followed by the discussions on the validity of the Taylor expansion in Sec. [IV] We elucidate the physical meaning of $\mu_s \neq 0$ in Sec. [V] and the summary is in Sec. [VI]

II. MODEL SETUP

The thermodynamic potential of the three-flavor PNJL model consists of three pieces, $\Omega = \Omega_{\text{vacuum}} + \Omega_{\text{thermal}} + \Omega_{\text{Polyakov}}$, namely, the vacuum part (zero-point energy and the condensation energy), the thermal part, and the Polyakov loop potential, respectively. We shall take a close look at them in order.

The vacuum part is exactly the same as in the ordinary NJL model:

$$\Omega_{\text{vacuum}} = -6 \sum_i \int_0^\Lambda \frac{d^3 p}{(2\pi)^3} \varepsilon_i(p) + g_S \left( 2 \langle \bar{u}u \rangle + \langle \bar{d}d \rangle + \langle \bar{s}s \rangle \right) + 4g_D \langle \bar{u}u \rangle \langle \bar{d}d \rangle \langle \bar{s}s \rangle,$$

where $i$ refers to the quark flavor running over $u, d,$ and $s$. The energy dispersion relations are $\varepsilon_i(p) = \sqrt{p^2 + M_i^2}$ with $M_i = m_i - 2g_S \langle q_i q_j \rangle - 2g_D \varepsilon_i \langle \bar{q}_i q_j \rangle \langle \bar{q}_k q_k \rangle$, where $g_S$ represents the four-Fermi coupling constant and $g_D$ represents the 't Hooft interaction strength. There are three more model parameters: the light current quark mass $m_u = m_d$, the heavy current quark mass $m_s$, and the ultraviolet cutoff $\Lambda$. We here adopt the parameter set by Hatsuda-Kunihiro [27];

$$\Lambda = 631.4 \text{ MeV},$$
$$g_S \Lambda^2 = 3.67,$$
$$g_D \Lambda^2 = -9.29,$$
$$m_u = m_d = 5.5 \text{ MeV},$$
$$m_s = 135.7 \text{ MeV},$$

which fits $m_\pi, m_K, m_{K^*}, f_s$, and empirical $M_{ud}$.

The thermal part has a coupling to the Polyakov loop (spatially homogeneous $A_4$ background) in a form as

$$\Omega_{\text{thermal}} = -2T \sum_{\ell} \int \frac{d^3 p}{(2\pi)^3} \left\{ \ln \det \left[ 1 + L e^{-(\varepsilon_i(p) - \mu)}/T \right] + \ln \det \left[ 1 + L e^{-(\varepsilon_i(p) + \mu)}/T \right] \right\},$$

where the Polyakov loop is a $3 \times 3$ matrix in the fundamental representation in color space defined by

$$L(\vec{x}) = \mathcal{P} \exp \left[ -ig \int_0^\beta dx_4 A_4(\vec{x}, x_4) \right],$$

and in this paper the Polyakov loop sometimes refers to the traced one after average, that is,

$$\ell = \frac{1}{3} \langle \text{tr} L \rangle, \quad \bar{\ell} = \frac{1}{3} \langle \text{tr} L^\dagger \rangle,$$

as long as no confusion may arise. In a simple mean-field approximation for the Polyakov loop (group) integration, the determinant explicitly reads;

$$\text{det} \left[ 1 + L e^{-(\varepsilon_i(p))/T} \right] = 1 + e^{-3(\varepsilon_i(p))/T} + 3 \ell e^{-(\varepsilon_i(p))/T} + 3 \ell^2 e^{-(\varepsilon_i(p))/T},$$

$$\text{det} \left[ 1 + L e^{-(\varepsilon_i(p))/T} \right] = 1 + e^{-3(\varepsilon_i(p))/T} + 3 \ell e^{-(\varepsilon_i(p))/T} + 3 \ell^2 e^{-(\varepsilon_i(p))/T}. $$

It is important to note that a positive $\mu$ induces $\bar{\ell} > \ell$, while $\bar{\ell} = \ell$ at $\mu = 0$. The finite-temperature field theory tells us that the traced Polyakov loop gives the exponential of the free energy cost by a test quark, i.e., $\ell = e^{-f_q/T}$ up to normalization (or energy offset), and the anti-Polyakov loop by a test anti-quark, i.e., $\bar{\ell} = e^{-f_{\bar{q}}/T}$.

Therefore, $f_q \to \infty$ and thus $\bar{\ell} \to 0$ signifies quark confinement, so that the Polyakov loop serves as an order parameter for the color deconfinement phase transition. In the presence of dynamical quarks in the color fundamental representation, however, neither $\ell$ nor $\bar{\ell}$ can be strictly zero due to screening. If the color screening is stronger, the free energy cost is smaller, and the Polyakov loop is larger accordingly. In a medium with $\mu > 0$ the test anti-quark is screened more efficiently than the test quark, that means $\ell > \bar{\ell}$ [9, 17, 25, 26].

We shall choose the Polyakov loop potential as [21]

$$\Omega_{\text{Polyakov}} = -b \cdot T \left\{ 54 e^{-a/T} \bar{\ell} \ell + \ln \left[ 1 - 6 \bar{\ell} \ell - 3(\bar{\ell}^2 + 4(\bar{\ell}^3 + \ell^3)) \right] \right\}. $$

Here, there are two parameters $a$ and $b$ in the above ansatz. We fix $a = 664$ MeV to reproduce $T_c \simeq 270$ MeV in the pure gluonic sector and $b = 0.03$ to yield $T_c \simeq 200$ MeV for the simultaneous crossovers of deconfinement and chiral restoration.

III. RESULTS

We are now ready to proceed to the numerical calculations using the PNJL model. Here let us focus on the entropy per baryon density ratio, $s/n_B$. This is because the adiabatic hydrodynamic expansion conserves $s/n_B$ along the time evolution. It is easy to confirm that $s/n_B$ is a constant indeed from conservation of the entropy current and the baryon current, that is, $(d/d\tau)(s/n_B) = u^\mu \partial_\mu (s/n_B) = 0$ readily follows from $\partial_\mu (su^\mu) = 0$ and $\partial_\mu (nu^\mu) = 0$. Hence, one does not have to integrate the hydrodynamic equation to draw the time-evolution path, which is simply inferred from an $s/n_B$-constant line. This is the case insofar as the expansion is fast enough to make the system thermally isolated and the entropy production due to dissipation is negligible.
at the PNJL model can reproduce the pressure behavior near the critical point [29, 30, 31].
We already know, however, that the soft-mode fluctuations, the thermodynamic functions such as the entropy
fluctuations.
In what follows we shall draw the isentropic trajectories (categorized by the bright colors on the density plot in Fig. 1).
A narrow region surrounding to the critical point (indicated by the blue triangle marks the location of the critical point.
and the baryon number of our current interest obtain by means of the thermodynamic relations:
\[ s = \frac{\partial \Omega}{\partial T}, \quad n_B = -\frac{1}{3} \frac{\partial \Omega}{\partial \mu}. \]
Figure 2 shows the numerical results for the isentropic trajectories for various values of $s/n_B$ without imposing constraint on $n_s$.
We note that Fig. 2 is reasonably consistent with Fig. 9 in Ref. 24 which employs the two-flavor PNJL model.
Moreover, the trajectories look qualitatively similar to the results in the NJL model as in Ref. 22, while the value of $s/n_B$ associated with each trajectory is greater in our case than in the NJL model study. This has an intuitive interpretation. The PNJL model is composed of quasi-quarks and a part of gluons, so that $s$ has steeper behavior near $T_c$ and grows larger above $T_c$, as compared to the NJL model. As for $n_B$, because of the Polyakov loop average, $n_B$ has steeper behavior as well, but it does not exceed the NJL model value. The ratio $s/n_B$ in the PNJL model, therefore, results in mild sensitivity to the steepness of $s$ and $n_B$ near $T_c$, leading to the similar trajectory curves to the NJL model results. In contrast, the magnitude of $s/n_B$ corresponding to the trajectory becomes larger in the PNJL model as a consequence of the gluon degrees of freedom contributing to $s$.
Comparing our results to the lattice QCD simulation with the Taylor expansion 13, we see that our estimate of $s/n_B$ (for instance comparing the $s/n_B = 40$ curve in Fig. 2 and the $s/n_B = 45$ curve in Ref. 13) improves an agreement.
We remark that all the trajectories in the low density side must go to $\mu > M_u$ as $T$ decreases because $n_B \to 0$ when $\mu < M_u$.

FIG. 1: Chiral susceptibility with respect to light ($u$ and $d$) quarks defined by $-\partial^2 \Omega/\partial m^2_s$ in the unit of $\Lambda$ as a function of $\mu$ and $T$.

FIG. 2: Isentropic trajectories on the $\mu$-$T$ plane in the case without constraint on $n_s$. The (blue) triangle marks the location of the critical point.

A. Case without constraint on $n_s$

We first take a quick look at the results without constraint on $n_s$. The quark density is specified by the quark chemical potential (or one third of the baryon chemical potential) which is common to all three flavors. In the results presented in this subsection, thus, the net strange quark density $n_s$ is nonzero.
We solve the following gap equations self-consistently;
\[ \frac{\partial \Omega}{\partial \langle\bar{d}d\rangle} = \frac{\partial \Omega}{\partial \langle\bar{s}s\rangle} = \frac{\partial \Omega}{\partial \mu} = \frac{\partial \Omega}{\partial T} = 0 \]
with assuming isospin symmetry $\langle \bar{d}d \rangle = \langle \bar{u}u \rangle$. In this way we have the chiral condensates, the Polyakov loop, and the anti-Polyakov loop as functions of $T$ and $\mu$. To illustrate the phase structure, we show the light-quark chiral susceptibility in a density plot in Fig. 1.
We can perceive from Fig. 1 that the critical region extends in the vicinity of the second-order critical point located at $(T, \mu) = (315$ MeV, 100 MeV) on top of the enhanced strip along the chiral crossover. Because our approximation neglects the soft-mode fluctuations, the EoS obtained in this work may miss the singular contribution to thermodynamic quantities near the critical point 23, 31, 32. We already know, however, that the PNJL model can reproduce the pressure behavior at $\mu = 0$, which implies that the soft-mode contribution is not significant. We can then anticipate that the singular contribution would become important only in a narrow region surrounding to the critical point (indicated by the bright colors on the density plot in Fig. 1). In what follows we shall draw the isentropic trajectories onto this phase structure not taking account of soft-mode fluctuations.

By substituting the solution of the gap equations into $\Omega$, the thermodynamic functions such as the entropy

model works well near and above $T_c$ and $\mu_c$ but not far below them.

**B. Case with constraint on $n_s$**

We next proceed to the case with imposing $n_s = 0$ to emulate the situation in the high-energy heavy-ion collisions. We should determine $\mu_s$ self-consistently solving 

$$n_s = -\frac{\partial \Omega}{\partial \mu_s} = 0,$$

(11)

together with other gap equations in Eq. (9).

The phase structure is only slightly changed by the constraint. The critical point moves from $(\mu, T) = (315 \text{ MeV}, 100 \text{ MeV})$ to $(\mu, T) = (317 \text{ MeV}, 100 \text{ MeV})$. The resultant isentropic trajectory as shown in Fig. 3 thus takes a very similar shape to the case without constraint $n_s = 0$. We further calculate the pressure along the isentropic trajectories on Fig. 3 and make a plot of them into the series like Ref. [32] as

\[ s = \lim_{N \to \infty} \sum_{n=0}^{M} \frac{d^{2n}s(0)}{d\mu^{2n}} \mu^{2n}, \]

(12)

\[ n_B(\mu) = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{d^{2n+1}n_B(0)}{d\mu^{2n+1}} \mu^{2n+1}, \]

(13)

which is to be validated if no singularity associated with the first-order phase transition lies along the $\mu$-direction. It should be mentioned that the derivative in Eqs. (12) and (13) is the total derivative in a sense that it acts on the implicit $\mu$-dependence in the mean-fields. In this way the mixing effect can be correctly taken into account in the model treatment [32]. From symmetry $s$ is an even function of $\mu$ and $n_B$ is an odd function.

Let us explain the numerical procedure in details to make a comparison between the results with and without truncation of the degree in the Taylor expansion. We first approximate $s(\mu)$ and $n_B(\mu)$ by the Taylor expansion

![FIG. 3: Isentropic trajectories with $n_s = 0$ imposed, which is relevant to the relativistic heavy-ion collisions. The (blue) triangle marks the location of the critical point in this case.](image1)

![FIG. 4: Pressure along the respective isentropic trajectories given in Fig. 3.](image2)

We could have placed a plot here for the induced $\mu_s$ as a function of $T$. For later convenience, however, we postpone showing it and let us turn to the validity of the Taylor expansion as utilized in the lattice QCD simulation.

**IV. VALIDITY OF THE TAYLOR EXPANSION**

All of our calculations result from the PNJL model in the mean-field approximation and do not rely on the Taylor expansion unlike the lattice QCD simulation. Although we have no need to carry the expansion out, it is interesting to compare “full” results of our numerical calculations and “truncated” ones to verify how nicely the Taylor expansion works.

Within the framework of the PNJL model we have solved $s$ and $n_B$ as functions of $\mu$. Now we shall expand them into the series like Ref. [32] as

\[ s = \lim_{M \to \infty} \sum_{n=0}^{M} \frac{d^{2n}s(0)}{d\mu^{2n}} \mu^{2n}, \]

(12)

\[ n_B(\mu) = \lim_{N \to \infty} \sum_{n=0}^{N} \frac{d^{2n+1}n_B(0)}{d\mu^{2n+1}} \mu^{2n+1}, \]

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with sufficiently large number of $M$ and $N$ so that the coefficients in the first several terms barely change with an increment of $M$ and $N$.

We should further specify the fit range of $\mu \in [0, \mu_0]$ to read the Taylor expansion coefficients. In principle, if the “exact” calculation were possible, $\mu_0$ could be zero or there is no $\mu_0$ dependence at all. Even in the mean-field level, however, we are far from the exact calculation. Here we have chosen $\mu_0 = 50$ MeV and $\mu_0 = 100$ MeV. In fact, $\mu_0$ is a parameter which controls the precision in the determination of the Taylor expansion coefficients in the same sense as using the multiple-point formula for the numerical differentiation. One might have thought that $\mu_0 = 500$ MeV, for instance, can cover the whole density region in Fig. 3, but such a choice would bring artifact from outside the radius of convergence.

Once we fix the Taylor expansion coefficients of $s(\mu)$ and $n_B(\mu)$, then we cut the series at smaller $M$ and $N$. We will elucidate the leading-order case ($M = 2$ and $N = 1$) and the next to leading-order case ($M = 3$ and $N = 2$) below.

Figure 5 shows the $s/n_B$-constant curves with and without truncation on the $\mu$-$T$ plane. The solid curves represent the full results without truncation. We have drawn the (green) dotted curves by choosing $N = 2$ for $s(\mu)$ and $M = 1$ for $n_B(\mu)$ and the fit range as $\mu_0 = 50$ MeV. We see that the truncated series can approximate the curve for $s/n_B = 40$ well, and the curve for $s/n_B = 30$ is still close to the full estimate, but the dotted curve for $s/n_B = 20$ has a huge deviation from the corresponding solid curve. If we determine the Taylor expansion coefficients in a wider range as $\mu_0 = 100$ MeV, as shown by the (blue) dashed curves, the agreement becomes better, of course. Not only the curve for $s/n_B = 40$ but also for $s/n_B = 30$ agrees quite well with the full results. Besides, the curve for $s/n_B = 20$ is significantly improved at $T \gtrsim 200$ MeV, while it does not fit the full results at lower $T$. It seems that the curve for $s/n_B = 10$ is too far from $\mu_0$ to perceive the effect of changing $\mu_0$.

It is intriguing to increase the truncation degree of the Taylor expansion to discuss how much the approximation is improved. We leave the terms up to the $\mu^6$ order in the expansion of $s(\mu)$ (i.e. $M = 3$) and the $\mu^5$ order in the expansion of $n_B(\mu)$ (i.e. $N = 2$) and have found the curves in Fig. 6. Although the results at low $T$ become slightly better than Fig. 5, it is unexpected that the agreement at high $T$ goes worse! This undesirable poor convergence turns out to stem solely from the Taylor expansion of $n_B(\mu)$.

To examine the problem concretely, we present a plot for $n_B(\mu)$ in Fig. 7. The truncated results at $N = 1$ (green dotted curve) can well describe the bold solid curve which is the full data. As we go to the higher order, however, the truncation leads to a larger deviation from the full results, implying that the expansion seems to fail. If the numerical accuracy is arbitrarily good as needed, or equivalently, $\mu_0 \to \infty$, the Taylor expansion would work with very small coefficients for the higher order terms in the expansion series. In practice, however, the available accuracy is limited, and the artifact may enter, which eventually goes wrong for $\mu$ away from the fitted region. Here, at the same time, we should emphasize that there is no such weird behavior observed in the expansion of $s(\mu)$: the higher order we take account of, the better convergence of $s(\mu)$ we can reach, as is naturally expected. In view of this, hence, it could be conceivable that there may be some profound reason why only the expansion of $n_B(\mu)$ is dangerous.
We can thus learn the following important lessons from these analyses using the model.

1) The Taylor expansion method does not work in the low-\( T \) and high-\( \mu \) region. The adiabatic path for \( s/n_B \lesssim 20 \) deduced from the expansion may be totally different from the true results. This value of \( s/n_B \) happens to be close to the threshold below which the pressure starts moving apart from that at \( \mu = 0 \) as seen in Fig. 4. This fact implies that the Taylor expansion may well work only in the regime where the density effect is not such appreciable.

2) Even at high \( T \) where the Taylor expansion is believed to work nicely, the expansion of the baryon density \( n_B \) may be problematic suffering from the uncertainty in the higher order terms than \( \mathcal{O}(\mu^3) \). One of the simplest remedies for this pathological expansion is the Padé-improvement as addressed in Ref. [11]. We can see in Fig. 7 that the Padé-improved results, \( R_2[1,1] \) for \( N = 2 \) and \( R_3[2,1] \) for \( N = 3 \), certainly come close to the full curve. \([R_2[1,1]]\) denotes \( c_1\mu(1+c_2\mu^2)/(1+c_3\mu^2) \) with \( c_1, c_2, c_3 \) fixed to yield the original series up to \( N = 2 \) and \( R_3[2,1] \) should be understood likewise.] We note that the entropy density \( s \) does not have such a kind of expansion problem at all.

V. STRANGE QUARK CHEMICAL POTENTIAL

As promised in Sec. III, the final topic discussed in this paper is the induced chemical potential for strange quarks to keep the strangeness neutrality. In general the condition of \( n_s = 0 \) requires a positive \( \mu_s \). This can be intuitively understood as follows.

The strong interaction does not change the number of strange quarks but can make strange particles in a process, for example, such as \( \pi^- + p \rightarrow K^0 + \Lambda \) and \( \pi^- + p \rightarrow K^+ + \Sigma^- \) etc. The strangeness of \( \Lambda \) and \( \Sigma^0, \Sigma^+ \) is negative one, meaning that they contain positive one strange quark. Therefore, if \( p \) and \( n \) are abundant at finite baryon chemical potential, the strong interaction pushes strange quarks into strange baryons, that results in \( \mu_s > 0 \). [For more phenomenological details see Ref. [34] and references therein, and see also Ref. [33].]

This sort of dynamics is completely missing in the NJL model without color confinement. If one solves \( n_s = 0 \) in the three-flavor NJL model, it ends up with \( \mu_s = 0 \), which is unphysical. In fact a positive \( \mu_s \) has been concluded in Ref. [10] as it should be. In this section we will see that the PNJL model has a crucial advantage in describing the induced \( \mu_s \) correctly because it encompasses the confinement physics.

Let us first show our numerical results in Fig. 8. A non-zero and positive \( \mu_s \) certainly appears in the PNJL model unlike the NJL model. The numerical values are qualitatively consistent with the lattice results in Ref. [10], though the quantitative comparison is not straightforward. We would say that this demonstration of \( \mu_s > 0 \) adds another example to the successful PNJL model applications besides the bulk thermodynamics. The rest of this section is devoted to explaining how the Polyakov loop makes it possible to accommodate the induced \( \mu_s \).

We recall that the Polyakov loop coupling takes a form of

\[
\ln \det \left[ 1 + L e^{-\varepsilon-\mu)/T} \right] + \ln \det \left[ 1 + L^\dagger e^{-(\varepsilon+\mu)/T} \right], \quad (14)
\]

in each flavor sector. As long as \( \mu \) is small Eq. (14) is well approximated as

\[
\simeq 3 e^{-\varepsilon/T} (\ell e^{\mu/T} + \bar{\ell} e^{-\mu/T}). \quad (15)
\]

For light flavors at \( \mu > 0 \), therefore, the source weight for anti-quarks is larger than that for quarks which yields \( \ell > \bar{\ell} \), as is consistent with the argument given below.
Eq. (17). It might be a bit confusing but \( \ell e^{\mu/T} \) is the source for anti-quarks because the derivative of \( \ell \ell \) with respect to \( \ell \) gives the anti-Polyakov loop \( \ell \). Then, in the heavy flavor sector, the neutrality condition means in the same approximation,

\[
\frac{\partial}{\partial \mu} (\ell e^{\mu/T} + \ell e^{-\mu/T}) = 0, \tag{16}
\]

which immediately leads to

\[
\mu_s = \frac{T}{2} \log(\ell/\ell). \tag{17}
\]

Interestingly enough, the above equation (17) holds approximately in the entire \( \mu-T \) plane with only a 3\% violation at worst! This is an intriguing relation discovered in a heuristic manner in the PNJL model and could be tested in the future lattice simulation.

One might wonder why the PNJL model could yield a positive \( \mu_s \) though it does not properly describe the confined baryons such as \( p \) and \( n \). As a matter of fact, at sufficiently high temperature, the thermal system may well consist of mesons and quarks rather than baryons, in which \( \pi^- + p \rightarrow K^0 + \Lambda \), for instance, should be replaced by \( \pi^- + u \rightarrow K^0 + s \) in terms of quarks. The Polyakov loop mediates the mesonic correlation through the color average, so that this kind of process is to be taken into account in the PNJL model. Moreover this process via quarks is rather realistic because in the hadronic phase the cross section of kaons is small and \( n_s = 0 \) would no longer hold at the later stage of evolution [35].

From Eq. (17) it is easy to confirm that \( \mu_s \rightarrow 0 \) when \( T \) grows large. If \( T \) exceeds \( T_c \), both the Polyakov loop and the anti-Polyakov loop come close to unity, and thus their ratio is nearly one, the logarithm of which is zero. This naturally coincides with the intuition that deconfined quarks have no correlation and \( \mu_s = 0 \) corresponds to \( n_s = 0 \) just like in the NJL model. In contrast to the high-\( T \) situation, the confined phase at small \( T \) has \( \ell \approx \ell \approx 0 \). The ratio of \( \ell \) and \( \ell \) could be any number. If \( \ell \) goes to zero much slower than \( \ell \), the ratio can become arbitrarily large.

In the best of our knowledge Eq. (17) is the very first demonstration that the discrepancy between \( \ell \) and \( \ell \) at finite \( \mu \) has a physically significant consequence. The importance of \( \ell \neq \ell \) has been overlooked maybe because the difference, \( \ell - \ell \), is negligibly small (only a few \% at most) as compared to \( \frac{1}{2}(\ell + \ell) \). The essential point is that we consider not the difference but the ratio, \( \ell/\ell \), which may take a huge value when \( \ell \approx 0 \) and \( \ell \approx 0 \).

We plot the induced \( \mu_s \) on the \( \mu-T \) plane in Fig. 9. The functional shape is quite characteristic at small temperature. Let us disclose the origin of this triangle-peak structure at low \( T \) separating the \( \mu \)-region into three pieces:

i) Quark-regime — At small \( \mu \) up to the peak position, the induced \( \mu_s \) rises linearly along with \( \mu \). In this region the mesonic correlation is the governing dynamics as already explained above.

ii) Diquark-regime — The turning point of \( \mu \) where \( \mu_s \) starts decreasing corresponds to the chemical potential with which the diquark excitation \( \ell e^{-2(\epsilon-\mu)/T} \) is energetically more favorable than the quark excitation \( \ell e^{-(\epsilon-\mu)/T} \). Using Eq. (17) we can derive the condition \( \mu > \frac{1}{4} \epsilon \) for the diquark excitation overcoming the quark excitation. Since the constituent quark mass is 336 MeV in this model, the threshold should be \( \mu \approx 110 \) MeV. This estimate is really consistent with our numerical results shown in Fig. 9. In this \( \mu \)-region \( \mu_s \) decreases because diquarks behave like anti-quarks in color space. In other words the mesonic (quark–anti-quark) correlation is taken over by the baryonic (quark–diquark) correlation which carries nonzero baryon number and so \( \mu_s \) is partially canceled by this effect.

iii) Baryonic-regime — If \( \mu \) is greater than \( \epsilon \) the color singlet contribution, \( e^{-3(\epsilon-\mu)/T} \), is dominant, which is interpreted as the baryonic excitation. In this regime the Polyakov loop is decoupled from the dynamics and there is no confinement effect, even though the Polyakov loop is zero. (Such a state is recently named the “quarkyonic phase” [36, 37, 38, 39].) Then \( \mu_s = 0 \) suffices for \( n_s = 0 \).

VI. SUMMARY

We have calculated the isentropic trajectories on the phase diagram in the \( \mu-T \) plane using the PNJL model. Our results are quantitatively consistent with the lattice results in the high-\( T \) and low-\( \mu \) region where the lattice data is available by means of the Taylor expansion.

To test whether the Taylor expansion is under theoretical control or not, we have expanded our numerical results of the entropy \( s \) and the baryon density \( n_B \) into polynomial series in terms of \( \mu \). We have confirmed
that the Taylor expansion can access the trajectories for \( s/n_B = 40, 30, \) and 20, but cannot for \( s/n_B = 10 \) at which the isentropic thermodynamics differs substantially from that at zero density. We have also realized that \( n_B(\mu) \) has pathological behavior if expanded at high temperature, which can be cured by the Padé approximation.

Finally we have discussed the induced strange quark chemical potential \( \mu_s \) to keep \( n_s = 0 \) in the system ruled by the strong interaction. We have found the interesting relation between \( \mu_s \) and the ratio of the Polyakov loop \( \ell \) and the anti-Polyakov loop \( \ell \). This prediction could be confirmed in the lattice QCD simulation.

In this work we neglected the effect of the soft-mode fluctuations around the critical point. This is one important direction of the future extension. Another interesting direction is the origin of the poor convergence in the expansion of \( n_B(\mu) \). This is not fully understood in the present work; other thermodynamic quantities like \( s(\mu) \) have a smooth expansion but only \( n_B(\mu) \) fails. It would be also interesting, if the baryon number susceptibility does not have such a problem of expansion (we guess so), to validate the idea that the QCD critical point is to be deduced from the radius of convergence using the model. Although these issues are all beyond our current scope, we believe that the present research contributes to opening these extensions.

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