Generalised supersymmetry and $p$-brane actions

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Abstract

We investigate the most general $N = 1$ graded extension of the Poincaré algebra, and find the corresponding supersymmetry transformations and the associated superspaces. We find that the supersymmetry for which $\{Q, Q\} \sim P$ is not special, and in fact must be treated democratically with a whole class of supersymmetries. We show that there are two distinct types of grading, and a new class of general spinors is defined. The associated superspaces are shown to be either of the usual type, or flat with no torsion. $p$-branes are discussed in these general superspaces and twelve dimensions emerges as maximal. New types of brane are discovered which could explain many features of the standard $p$-brane theories.

1 Introduction

It is becoming apparent that twelve dimensions may have an important role to play in the formulation of theories of extended objects, and there have been many suggestions as to the possible form of this higher dimensional connection [1, 2]. Although generalising theories of extended objects to higher dimensions is a simple matter from the bosonic viewpoint, when one considers supersymmetry the picture becomes rather

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complex. Indeed, the relationship between the ten and eleven dimensional superstring and supermembrane theories is well understood, at least classically, in part due to the similarity of supersymmetry in these two dimensions. As one moves from ten and eleven dimensions to twelve dimensions, however, the supersymmetry algebras and the symmetry properties of products of gamma matrices change, and therefore the basic extensions used to generalise Green-Schwarz strings to Green-Schwarz membranes are no longer applicable: A more detailed study of the supersymmetry is required, whence it is clear that in order to formulate explicit twelve dimensional theories, the notion of supersymmetry must be generalised. Using this generalised approach it was shown in [3] that it was possible to define an $N = 1$ supersymmetric (2+2)-brane action in a spacetime of signature $(10, 2)$. This brane, however, is just one solution to one particular generalisation of supersymmetry, and the question remains as to how many different new $p$-branes and how many different new types of supersymmetry can be defined, in any given dimension. Such a question shows that there are gaps in the understanding of ‘supersymmetry’, and the purpose of this paper is fill in these gaps by studying the most general Poincaré supersymmetries (a Poincaré supersymmetry being a relationship between bosonic variables transforming as representations of the Poincaré group, and an additional set of fermionic degrees of freedom). The work shall be in essentially two parts. We shall first formulate generalised supersymmetry theories from a purely algebraic point of view and find the associated generalised superspaces. We will then formulate invariant objects in these superspaces and write down invariant actions. We will finally consider a generalisation of $p$-brane theory to include branes propagating in such general superspace backgrounds, thus presenting new types of fundamental supersymmetric objects.

1.1 Background

Historically, a theory deemed to be ‘$N = 1$ supersymmetric’ is defined on a supermanifold which has a group of isometries locally described by the following fermionic extension of the bosonic Poincaré algebra

\[
[M_{\mu\nu}, M_{\rho\sigma}] = M_{\nu\sigma}\eta_{\mu\rho} + M_{\mu\rho}\eta_{\nu\sigma} - M_{\nu\rho}\eta_{\sigma\mu} - M_{\sigma\mu}\eta_{\nu\rho},
\]

\[
[M_{\mu\nu}, P_{\rho}] = P_{\rho}\eta_{\nu\sigma} - P_{\nu}\eta_{\mu\rho},
\]

Very recently, some extended aspects of supersymmetry have been discussed in the literature [4].
where $P_\mu$ and $M_{\mu\nu}$ are the usual translation and rotation generators of the isometry group of flat $(S,T)$ space and $Q^\alpha$ is a spinorial generator. Of course, a supersymmetry is a way to include fermionic terms to the usual bosonic notion of spacetime. Experimentally we know that the bosonic part of space should locally be described by the Poincaré algebra; the inclusion of fermions to such a picture is, at present, rather more arbitrary.

The supersymmetry algebra \( (1.1) \) is the basis to the whole subject of supersymmetric string theory \cite{5}, which developed into the more general picture of $p$-branes. The $p$-branes can be thought of either as fundamental branes with a Green Schwarz action, much like the superstring, or as extended solitonic solutions to the corresponding supergravity theory. These supergravity $p$-branes must couple to the theory via $p$-form central charges, $Z_{\mu_1...\mu_p}$, which must be added as extensions to the supersymmetry algebra \( (1.1) \) as follows

\[
\{ Q, Q \} = \Gamma^\mu P_\mu + \Gamma^{\mu_1...\mu_p} Z_{\mu_1...\mu_p}.
\]  

\( (1.2) \)

Studies of such extensions of the Poincaré algebra have lead to the standard picture of which $p$-branes are and are not possible; this, of course, is all dependent on choosing \( (1.1) \) as the starting point. The question should be asked as to whether we find any new branes if this basic starting point is questioned.

As we shall explain in this paper, the usual choice of supersymmetry algebra \( (1.1) \) is actually in no way singled out; we could just as well choose our initial algebra to be

\[
\{ Q, Q \} = \Gamma^{\mu_1...\mu_p} P_{\mu_1...\mu_p},
\]  

\( (1.3) \)

and extend with a central $Z^\mu$ if we wish to define a 1-brane. Such a statement greatly enlarges the notion of supersymmetry, but as we shall see it is possible to classify all Poincaré supersymmetry theories into just two classes: those for which \( \{ Q, Q \} \sim \Gamma^{\mu\nu} M_{\mu\nu} + \ldots \), and those for which it does not. In this paper we will first discuss the formulation of a generalised Poincaré supersymmetric theory, and we shall then consider the invariant actions for $p$-branes propagating in such spaces. A new brane scan will then be presented for all branes for which an action exists.
1.2 Charge conjugation matrix

Central to the notion of supersymmetry is the Clifford algebra which may be represented by the matrices \( \{(\Gamma_{\mu})_{\alpha}^{\beta}\} \) as

\[
\{\Gamma_{\mu}, \Gamma_{\nu}\}_{\alpha}^{\beta} = 2\eta_{\mu\nu}(I)_{\alpha}^{\beta}, \tag{1.4}
\]

where \( \eta = \text{diag}(-1, \ldots, -1, 1, \ldots, 1) \) is the flat space metric for which there are \( T \) minus signs and \( S \) plus signs, corresponding to timelike and spacelike directions respectively. The spinorial indices are raised and lowered with the charge conjugation matrix \( C \) and its inverse \( C^{-1} \) respectively. The properties of the charge conjugation vary according to the signature of the spacetime under consideration. For this reason we present a brief discussion of these matrices before we begin.

For a Majorana representation, that is a purely real representation corresponding to the Clifford algebra generated over the real numbers \( \mathbb{R} \), \( C \) has the properties \( [6, 7] \) that

\[
\tilde{\Gamma}_{\mu} = (-1)^{T}\eta C \Gamma_{\mu} C^{-1},
\]

\[
C^{\dagger}C = 1
\]

\[
\tilde{C} = \epsilon \eta^{T}(-1)^{T(T+1)/2}C, \tag{1.5}
\]

where the tilde denotes matrix transpose. The possible choices of the numbers \( \eta, \epsilon = \pm 1 \) depend on the signature of the spacetime as follows

| \( \epsilon \) | \( \eta \) | \( (S - T) \) mod 8 |
|---|---|---|
| +1 | +1 | 0, 1, 2 |
| +1 | -1 | 0, 6, 7 |
| -1 | +1 | 4, 5, 6 |
| -1 | -1 | 2, 3, 4 |

(1.6)

If \( \tilde{\Gamma}_{\mu} = \pm CT_{\mu}C^{-1} \) then we define \( C = C_{\pm} \). This definition must be consistent with the conditions \( [6] \)

\[
\tilde{C}_{\pm} = (-)^{T+[(T+1)/2]}C_{\pm}, \quad \tilde{C}_{-} = (-)^{[(T+1)/2]}C_{-}, \tag{1.7}
\]

2Given a real representation of the \((S,T)\) Clifford algebra, we may always define a purely imaginary representation of the \((T,S)\) Clifford algebra. Throughout this paper we shall choose our gamma matrices always to be real.
where the brackets denote ‘integer part’. We find that these constraints imply that 
\(\epsilon = 1\). This gives us that

\[
T \mod 4 (C, \pi(C)) : \eta = 1 (C, \pi(C)) : \eta = -1
\]

\begin{array}{|c|c|c|}
\hline
T \mod 4 & (C, \pi(C)) : \eta = 1 & (C, \pi(C)) : \eta = -1 \\
\hline
0 & (C_+, 1) & (C_-, 1) \\
1 & (C_-, -1) & (C_+, 1) \\
2 & (C_+, -1) & (C_-, 1) \\
3 & (C_-, 1) & (C_+, -1) \\
\hline
\end{array}

(1.8)

where \(\pi\) is the parity operator for which \(\pi(C) = (-1)^1\) for \(C\) (anti)symmetric.

### 2 Superspaces

The algebra (1.1) has the defining property that it reduces to the bosonic Poincaré algebra when the spinor generator \(Q_\alpha\) is set to zero. There are, however, many consistent \(N = 1\) graded extensions of the Poincaré algebra with such a property. A general (Dirac) spinor in \(D\) spacetime dimensions had \(2^\lfloor D/2 \rfloor\) complex components, where the square brackets denote ‘integer part’. A Majorana spinor has \(2^\lfloor D/2 \rfloor\) real components.

The anticommutator \(\{Q^\alpha, Q^\beta\}\) is thus a symmetric \(2^\lfloor D/2 \rfloor \times 2^\lfloor D/2 \rfloor\) matrix. In general, since \(\{\Gamma^{\mu_1\ldots\mu_p} \equiv \Gamma^{\mu_1} \ldots \Gamma^{\mu_p}\}\), for \(p = 1 \ldots D\), form a basis for the vector space of real \(2^\lfloor D/2 \rfloor \times 2^\lfloor D/2 \rfloor\) matrices, the \(\{Q^\alpha, Q^\beta\}\) anticommutator in expression (1.1), may most generally be rewritten as

\[
\{Q^\alpha, Q^\beta\} = \sum_{n \in k} (\Gamma_{\mu_1 \ldots \mu_n})^{\alpha\beta} Z^{{\mu_1} \ldots {\mu_n}}, \tag{2.1}
\]

where \(k\) is the set of all \(n\) such that \((\Gamma_{\mu_1 \ldots \mu_n})^{\alpha\beta}\) is symmetric in the spinor indices. The \(Z^{{\mu_1} \ldots {\mu_n}}\) are bosonic generators which are completely antisymmetric in the spacetime indices. If we consider the Poincaré algebra as being the infinite radius limit of the de Sitter algebra, which differs from the Poincaré algebra by the term \([P_\mu, P_\nu] = mM_{\mu\nu}\) where \(m^{-1}\) is the radius of the de Sitter space \(\mathbb{S}\), then we find that if we choose to identify \(Z^{{\mu_1} \ldots {\mu_n}} \sim M^{{\mu_1} \ldots {\mu_n}}\) and \(Z^\mu \sim P^\mu\) in the infinite radius limit, that

\[
[Z^{{\mu_1} \ldots {\mu_p}}, M^{{\mu_\nu}}] = \eta^{{\mu_1} \nu} Z^{{\mu_2} \ldots {\mu_p}} \quad \text{for} \quad p = 1 \ldots D
\]

\[
[Z^{{\mu_1} \ldots {\mu_p}}, Z^{{\nu_1} \ldots {\nu_p}}] = [Z^{{\mu_1} \ldots {\mu_p}}, Q^\alpha] = 0, \tag{2.2}
\]

and we see that the \(Z^{(p)}\) terms are added to the algebra in exactly the same way as \(Z^\mu = P^\mu\), unless \(p = 2\). In de Sitter space there will in general be non-trivial
commutation relations between the $Z^{(p)}$ and the $Z^{(q)}$, for $p, q \neq 2$. In the Poincaré space limit, however, these additional commutation relations all vanish. We thus see then the $M^{\mu\nu}$ term is singled out from the $\{Z^{(i)}\}$: all others appear on an equal footing regarding Poincaré supersymmetries. This is an interesting point since it implies that taking $\{Q, Q\} \sim P$ as in (1.1) is not a natural choice: all the other $Z^{(i)}$ appear with equal importance as $P$ in the superalgebras. Hence, the whole subject of supersymmetric $p$-branes is viewed from a biased position. It is for this reason that it is natural to consider other types of supersymmetry theory. This is an extension of the $p$-brane democracy idea \[9\] to that of a ‘superspace democracy’. By exploring general forms of the anticommutator, we hope to find new properties of supersymmetric theories, and the relationships between them. This should then be intrinsically linked to the existence of the different types of $p$-branes.

We shall consider the algebra with the graded extension

$$\{Q^\alpha, Q^\beta\} = \sum_{\tilde{n} \in k} (\Gamma_{\mu_1...\mu_n})^{\alpha\beta} Z^{\mu_1...\mu_n}, \quad (2.3)$$

for any $\tilde{k} \subseteq k$, $k$ being the full set of symmetric matrices. We shall call the Poincaré extensions with these anticommutators sio$_k$. The $Z^{\mu_1...\mu_n}$ will commute with everything except $M^{\mu\nu}$, as in (2.2). It should be noted, of course, that one cannot generally expect these supersymmetry algebras to be completely consistent from the the point of view of the super-Jacobi identities. We shall assume that the algebras will be consistent on-shell, as is the case for the sio$_1$ supersymmetry algebra (1.1). The aim is to describe $p$-branes moving in any $D$-dimensional sio$_k$ invariant background. In order to do this in a supersymmetric fashion, we need to determine the superspaces corresponding to the algebras sio$_k$, as was done in [3] for the 12 dimensional sio$_2$ case.

We shall begin with a description of the usual case of sio$_1$. This superspace is
defined to be a coset manifold $G/H$ where $G$ is some supergroup corresponding to the super-algebra sio$_1$, and $H$ some subgroup of $G$. Locally we may write $g \in G$ as

$$g(X, \theta, \omega) = \exp(X.P + \theta.Q) \exp\left(\frac{1}{2} \omega.M\right), \quad (2.4)$$

where $(X, \theta)$ are the superspace coordinates, and $\omega$ labels each coset. The effect of an infinitesimal group action on the coordinates is found by multiplication on the left of $g$ by the infinitesimal group element

$$g(\delta X, \delta \theta, 0) = \exp(\delta X.P + \delta \theta.Q). \quad (2.5)$$
To reduce the product $\delta g.g$ to an element of the form of (2.4) we use the Baker-Campbell-Hausdorff formula,

$$ \exp(\alpha A) \exp(B) = \exp\left( B + \alpha A + \alpha \sum_{n=1}^{\infty} \frac{1}{(n+1)!} [[... [A, B], B],..., B] + O(\alpha^2) \right), $$

(2.6)

where $\alpha$ is an infinitesimal super-number. Application of this formula leads to the appearance of contractions of the term $\{Q^\alpha, Q^\beta\}$ in the exponential, which produces the usual sio\(_1\) supersymmetry transformations

$$ \delta \theta^\alpha = \epsilon^\alpha, \quad \delta X^\mu = x^\mu - \frac{1}{2} \delta \theta_\alpha (\Gamma^\mu)^{\alpha\beta} \theta_\beta, $$

(2.7)

where $x^\mu$ and $\epsilon^\alpha$ are infinitesimal bosonic and fermionic superspace parameters respectively.

The sio\(_2\) supersymmetry transformations may be found for the case that $\{Q^\alpha, Q^\beta\} = \frac{1}{2} (\Gamma_{\mu\nu})^{\alpha\beta} M^{\mu\nu}$ in a similar way. However, since the anticommutator term in the algebra generates a rotation, the left action of an infinitesimal group element destroys the coset form of the group unless the following restriction on the spinors is made:

$$ \delta \theta_\alpha (\Gamma_{\mu\nu})^{\alpha\beta} \theta_\beta M^{\mu\nu} = 0. $$

(2.8)

This identity will be found whenever the anticommutator of the spinorial generators generates $M_{\mu\nu}$. In this situation, the corresponding supersymmetry transformations are trivial

$$ \delta X^\mu = x^\mu, \quad \delta \theta^\alpha = \epsilon^\alpha, $$

(2.9)

and the superspace is flat with no torsion. This is referred to as ‘simple supersymmetry’ \([11]\).

The coset procedures just detailed are very general and may be used to generate spaces which are labelled by other types of parameters. These will be investigated in the next section, and will all be shown to be of essentially sio\(_1\) or sio\(_2\) type.

### 2.1 sio\(_k\) supersymmetry

We have presented generalisations of the Poincaré supersymmetry algebra for which the translation generator $P^\mu$ is not singled out. We can think of these algebras as generating the isometries of some background superspace, which will in general be parametrised by
the coordinates \( \{ \theta^\alpha, X^\mu, X^{\mu\nu}, \ldots X^{\mu_1 \ldots \mu_D} \} \). We can consider the generalised Poincaré supersymmetric \( p \)-branes as objects propagating in these new backgrounds. In order to do this we must apply the coset procedure to produce the generalised superspaces. We shall perform the procedure for the algebras \( \tilde{sio_k} \), which have the anticommutators (2.3), to find that the general supersymmetry transformations are given by a combination of terms of the type (2.7) and (2.9). Note that the general coset theory which we shall employ works in exactly the same way as for usual non-supersymmetric groups and manifolds, at least in the case where the body of the supergroup, the part which remains when we set the spinorial terms \( Q^\alpha = 0 \), is itself a Lie algebra \([10]\), which is the case of interest to us.

We can now write the superspace as a coset \( G/\mathcal{H} \). Until now we have naturally considered any super-extension of the bosonic Poincaré algebra. In order to keep the coset procedure natural, the question must now be asked as to which subgroup \( \mathcal{H} \) should we choose to quotient by. Since every bosonic generator has a non-trivial commutation of qualitatively the same form with \( Z^{\mu\nu} \), it is natural to quotient out by this term. We can then write down the supergroup elements parametrised by \( \{ \omega^{\mu\nu}; X^p : p \in P \subset \{ 1, 3, 4, \ldots, D \} \} \) for each \( P \) as as

\[
g(X, \theta, \omega) \equiv \exp \left( \sum_{p \in P} X^{\mu_1 \ldots \mu_p} Z_{\mu_1 \ldots \mu_p} + \theta^\alpha Q_\alpha \right) \exp \left( \frac{1}{2} \omega^{\mu\nu} M_{\mu\nu} \right), \tag{2.10}
\]

Notice that we allow any choice of the set \( P \), even if it includes values of \( p \not\in \tilde{k} \). This is an important point: Even though the spinorial terms do not generate a particular \( p \)-form, there is no reason not to include it as a parameter of the underlying supermanifold; we simply add the purely bosonic commutator \( [Z^{(p)}, M] \sim Z^{(p)} \) to the algebra. The coordinates of the superspaces invariant under the action of \( \tilde{sio_k} \) will be given by \( \{ \{ X^{\mu_1 \ldots \mu_p} \}, \theta^\alpha \} \), where \( p \in P \); the cosets will be labelled by the parameters \( \omega^{\mu\nu} \).

We now evaluate the effects of the left action of an infinitesimal group element which is constant in each coset, in that \( \delta \omega_{\mu\nu} = 0 \), on (2.10). We require that the supermanifold have isometries generated by the appropriate superalgebra, hence that the supermanifold should be invariant under such a left shift. After simplification using the BCH formula, we find that

\[
g(\delta X, \delta \theta, 0)g(X, \theta, \omega) = \exp (\delta X.Z + X.Z + (\delta \theta + \theta)^\alpha Q_\alpha + C + D) \exp \left( \frac{\omega^{\mu\nu} M_{\mu\nu}}{2} \right)
\]

\[
C = \frac{1}{2} [\delta X.Z + \delta \theta^\alpha Q_\alpha, X.Z + \theta^\beta Q_\beta]
\]
\[ X.Z \equiv \sum_{p \in P} X_{\mu_1 \cdots \mu_p} Z_{\mu_1 \cdots \mu_p}, \]  

(2.11)

where the \([ , ]\) denotes the super-commutator. The \(D\) term is formed by repeated commutation of the exponent of \(g(\delta X, \delta \theta, 0)\) with \(C\). In order to evaluate these terms we refer to the super-commutation relations (2.3) and (2.2), from which it is clear that

\[ C = \delta \theta^\alpha \{ Q_\alpha, Q_\beta \} \theta^\beta = \frac{1}{2} \delta \theta^\alpha \sum_{p \in \tilde{k}} (\Gamma_{\mu_1 \cdots \mu_p})_{\alpha \beta} Z_{\mu_1 \cdots \mu_p} \theta^\beta. \]  

(2.12)

If the anticommutator of the supersymmetries does not generate the rotations \(M_{\mu \nu}\) then \(D\) is zero, since \([C, \delta X.Z + \delta \theta^\alpha Q_\alpha] = 0\). We can now read off the supersymmetry transformations as being the shift in the coordinates induced by the infinitesimal transformation. In this case, we therefore find that the action of the supergroup yields the following supersymmetry transformations

\[ \delta \theta^\alpha = \epsilon^\alpha, \quad \delta X_{\mu_1 \cdots \mu_p} = x_{\mu_1 \cdots \mu_p} + \sigma_p \epsilon^\alpha (\Gamma_{\mu_1 \cdots \mu_p})^{\alpha \beta} \theta^\beta \quad \forall p \in \tilde{k}, \]  

(2.13)

where \(\sigma_p = 1\) if \(p \in \tilde{k}\) and \(\sigma_p = 0\) otherwise. Hence if \(Z^{(p)}\) is generated by \(\{Q, Q\}\) then \(\sigma_p = 1\), and the supersymmetry transformation is of an exactly analogous form to the usual supersymmetry for the \(p = 1\) case. If \(Z^{(p)}\) is not generated by \(\{Q, Q\}\) then \(\sigma_p = 0\), and the corresponding supersymmetry transformation is trivial, as we would expect.

In the situation where the anticommutation of the supersymmetries does generate \(M_{\mu \nu}\) we find that \(D \neq 0\) in general, since the commutation of \(C\) with \(Q\) produces \(M_{\mu \nu}\), which does not trivially commute with any of the generators. This destroys the coset construction since we are unable to factor out the \(M\)-dependence to give an expression of the form

\[ g(\delta X, \delta \theta, 0)g(X, \theta, \omega) = \exp \left( \tilde{X}.Z + \tilde{\theta}.Q \right) \exp \left( \frac{\tilde{\omega}^{\mu \nu} M_{\mu \nu}}{2} \right), \]  

(2.14)

for some \(\tilde{X}, \tilde{\theta}, \tilde{\omega}\). The only way to solve this problem is to impose the constraint (2.8) that \(C = 0\):

\[ \delta \theta_\alpha (\Gamma^{\mu \nu})^{\alpha \beta} \theta_{\beta} M_{\mu \nu} = 0 \quad \forall \mu, \nu. \]  

(2.15)

Of course, since \(\delta \theta\) is an arbitrary infinitesimal spinor we require that the equation must be satisfied for \textit{any general pair of spinors} \(\psi, \phi\). If we do not wish to place any restriction on the \(M_{\mu \nu}\) then we must have

\[ \psi_\alpha (\Gamma^{\mu \nu})^{\alpha \beta} \phi_{\beta} = 0 \quad \forall \mu, \nu. \]  

(2.16)
This is the defining relation of the $\text{so}_2$ superspace spinors, or indeed any superspace for which $2 \in \tilde{k}$: The expression (2.16) must hold for the coset construction to be well defined. If the identity is satisfied, then the supersymmetry transformations (2.13) are unchanged, and (2.16) merely serves to restrict the number of spin variables.

We are now in a position to calculate the $\text{so}_\tilde{k}$ invariant forms from which we may build invariant actions. From the transformations (2.13) we can write down forms which are invariant under the action of the supergroup $\text{SO}_\tilde{k}$, and therefore are supersymmetric

$$
\Pi^\alpha = d\theta^\alpha, \quad \Pi^{\mu_1 \ldots \mu_p} = dX^{\mu_1 \ldots \mu_p} + \sigma_p \theta_\alpha (\Gamma_{\mu_1 \ldots \mu_p})^{\alpha \beta} d\theta_\beta, \quad (2.17)
$$

where $d$ is the exterior derivative. This shows that there are in fact precisely two general forms of supersymmetry transformation associated with the Poincaré group: those for which $\sigma_p = 0$ and those for which $\sigma_p = 1$. The $\sigma_p = 0$ cases, those for which $\{Q, Q\} \sim Z^{(p)}$, are essentially the trivial cases, corresponding to a flat supersymmetry with no torsion. If, however, the superspace is of $\text{so}_2$ type, for which $\{Q, Q\} \sim M$, then the superspace identity (2.16) must also hold. The supersymmetry transformations (2.13) are unchanged, but we must have extra constraints on the spinors in the theory.

To conclude, we reiterate the result. We have found the general supersymmetry transformations corresponding to any $N = 1$ grading of the Poincaré algebra. These gradings all appear on nearly an equal footing. The supersymmetry transformations are given by the expression (2.13). In addition, if the $\{Q, Q\}$ term generates $M_{\mu\nu}$ then the expression (2.15) must hold in order to preserve the coset construction, although the supersymmetry transformations are thus unaffected by this. For each form $\Pi^{\mu_1 \ldots \mu_p}$ there are four possibilities, depending on whether or not the superspace demands a projection of the spinors and whether $\sigma_p = 0$ or 1.

### 3 The superspace identity

We now discuss the superspace identity (2.16) for which we must define a restricted subclass of spinors such that

$$
\psi_\alpha (\Gamma^{\mu\nu})^{\alpha \beta} \phi_\beta = 0 \quad \forall \mu, \nu. \quad (3.1)
$$

These equations must be satisfied for all pairs $\phi, \psi$ and may be thought of as defining a class of spinors. This defining relationship is a completely covariant expression, and a
theory involving such spinors would therefore be Lorentz invariant. It is an interesting
question to ask which subsets of the space of spinors satisfy this equation. We are
used to dealing with Dirac or Majorana spinors, so we now investigate the relationship
between these and the new class of spinors. Can a spinor satisfying (3.1) be obtained
via a projection of a Dirac spinor

$$\theta_D^\alpha \rightarrow \psi^\alpha = \mathcal{P}^{\alpha \beta} \theta_D^\beta, \quad (3.2)$$

where $\theta_D$ is a Dirac spinor? Since the equation (3.1) must be satisfied for every $\phi$ and
$\psi$ we discover that the problem is equivalent to finding projectors such that

$$\bar{\mathcal{P}} C(\Gamma^{\mu\nu}) \mathcal{P} = 0 \quad \forall \mu, \nu, \quad (3.3)$$

since a matrix is orthogonal with respect to all vectors of the appropriate dimension if
and only if it is the zero matrix, even if the vectors are Grassmann-odd valued.

To see that (3.3) must hold we merely need choose a non-zero spinor $\phi$ in the
expression $\mathcal{P}^{\alpha \beta} \psi^\beta C_{\alpha \rho} (\Gamma^{\mu\nu})^\rho \mathcal{P}^{\gamma \delta} \phi^\delta$. We can then always construct a spinor $\psi$ for which
this expression is not zero. We must therefore impose the restriction (3.3).

We may ask what the rank of the projectors $\mathcal{P}$ must be. To answer this question
we note that the equation (3.1) is in fact a linear constraint. Since the identity must
hold for all spinors, the spinors $\lambda_1 \psi + \lambda_2 \phi$ must also be orthogonal to $\phi$ and $\psi$ for
any real numbers $\lambda_1, \lambda_2$. We therefore must restrict the spinors to lie in some vector
subspace of the full Dirac spin space. This implies that the number of spin degrees of
freedom will be some multiple of 2, hence the projectors $\mathcal{P}$ will be of rank $\left(\frac{1}{2}\right)^n$ for
integer values of $n$. A similar set of identities to (2.16) are used to define the so called
pure spinors [12]. The pure spinor relationship, however, admits non-linear solutions,
which leads to unusual degrees of freedom, unlike the case here.

We now wish to investigate the construction of some of the projectors which satisfy
(3.3), to discover for which spaces we may define the new spinors. There are two sub-
cases to consider: those for which $\mathcal{P}$ is invariant under $SO(S,T)$ rotations, which we
shall call Lorentz invariant, and those for which it is not.

### 3.1 $SO(S,T)$ invariant cases

There is essentially only one projector which is invariant under $SO(S,T)$ rotations: the
Weyl projector

$$\mathcal{P} = \frac{1}{2} (1 + \Gamma^{D+1}), \quad (3.4)$$
where $\Gamma^{D+1} = \Gamma^1 \ldots \Gamma^D$. In order that $\mathcal{P}^2 = \mathcal{P}$ we must have $(\Gamma^{D+1})^2 = +1$. We must additionally choose $D$ to be even, so that $\Gamma^{D+1}$ is not proportional to the identity matrix, to ensure that $\mathcal{P}$ is a non-trivial projection matrix. However, $\mathcal{P}$ does not always satisfy the identity (3.3), and it is not always possible, therefore, to construct an $SO(S, T)$ invariant superspace if $\{Q, Q\}$ generates $M_{\mu\nu}$. In fact, we have the following result

**Lemma**

Given an irreducible Majorana representation of the $D$ dimensional Clifford algebra, the identity (3.3) is satisfied for $\mathcal{P} = \frac{1}{2}(1 + \Gamma^{D+1})$ if and only if $T$ is odd, $D \mod 4 = 2$ and $(S - T) \mod 8 = 0$.

**Proof**

Since we have a Majorana representation of the Clifford algebra we take the gamma matrices to be real. Suppose that $\tilde{\Gamma}^\mu = k C \Gamma^\mu C^{-1}$. We may always choose a basis for the gamma matrices such that $(\Gamma^\mu)^\dagger = -\Gamma^\mu$ for $\mu$ a timelike index and $(\Gamma^\nu)^\dagger = +\Gamma^\nu$ for $\nu$ spacelike [6]. Then we have that

$$
\Gamma^\mu C = -k C \Gamma^\mu, \quad \Gamma^\nu C = +k C \Gamma^\nu,
$$

Using these expressions we find that

$$
C \Gamma^1 \ldots \Gamma^D = (-k)^T(k)^S \Gamma^1 \ldots \Gamma^D C.
$$

We also have

$$
(\Gamma^1 \ldots \Gamma^D)^\pi = \tilde{\Gamma}^D \ldots \tilde{\Gamma}^1 = (-1)^T \pi \Gamma^1 \ldots \Gamma^D
$$

\[\begin{align*}
\pi &= \frac{1}{2}((D-1)+(D-2)+\ldots+2+1) \\
&= \begin{cases} 
+1 & \text{if } D \mod 4 = 0, 1 \\
-1 & \text{if } D \mod 4 = 2, 3
\end{cases}
\end{align*}
\] (3.7)

We now consider the matrix equation

$$
\hat{\mathcal{P}} \Gamma^{\mu\nu} C \mathcal{P} = \frac{1}{2} \left(1 + (-1)^T \pi \Gamma^1 \ldots \Gamma^D\right) \left(1 + (-k)^T(k)^S \Gamma^1 \ldots \Gamma^D\right) \Gamma^{\mu\nu} C
$$

$$
= \frac{1}{2} \left(1 + ((-k)^T(k)^S + (-1)^T \pi)\Gamma^1 \ldots \Gamma^D + \pi^2(-1)^T(k)^D\right) \Gamma^{\mu\nu} C
$$

$$
= 0.
$$

(3.9)
If this equation holds, then so does (3.3). For (3.9) to be true, we must have that, assuming that $\Gamma^{1\ldots D}$ is not proportional to the identity matrix,

$$(-k)^T(k)^S + (-1)^T \pi = \pi^2(-1)^T(k)^D + 1 = 0.$$  \hspace{1cm} (3.10)

Evaluating all the possibilities we find that these equations are satisfied only if

(i) $T$ is even and $S$ is odd, $D \mod 4 = 1$ and $C = C_-$

(ii) $T$ is odd and $S$ is even, $D \mod 4 = 3$ and $C = C_+$

(iii) Both $T$ and $S$ odd, $D \mod 4 = 2$ and $C = C_+$ or $C = C_-$. 

We must check which of these possibilities are consistent with the Majorana condition, by referring to (1.6) and (1.8). We find that item (i) is always inconsistent and items (ii) and (iii) only hold if $(S - T) \mod 8$ equals 1 and 0 respectively. We thus have that the equation (3.9) is satisfied for the projector $P$ if and only if $T$ is odd, $D \mod 4 = 2, 3$ and $(S - T) \mod 8 = 0$ or 1. We now recall that in odd dimensions $\Gamma^{1\ldots D}$ is proportional to the identity matrix, in which case the spinor identity is never solved. This requires us to choose $D$ to be even. Listing all the possibilities provides the result \(\Box\).

### 3.2 Non-SO\((S, T)\) invariant cases

We now search for some general types of projector which satisfy the superspace identity in a non-covariant fashion. Of course, the underlying theory is still Lorentz invariant: The choice of projector is analogous to a gauge choice. We need not, therefore, worry that we lose manifest Lorentz invariance.

#### 3.2.1 Product projectors

To begin with, we consider acting on the spinors with a product of projectors of the form $P_{(s,t)} = \frac{1}{2}(1 + \Gamma_{(s,t)})$ where $\Gamma_{(s,t)}$ is the product of $s$ spacelike and $t$ timelike gamma matrices. Since $P$ is to be a projector, we require that $\Gamma^2_{(s,t)} = 1$ and that $\Gamma_{(s,t)}$ is not proportional to the identity. This is possible iff $(s - t) \mod 8 = 0$.

For a product of $n$ such projectors, the left hand side of the superspace identity becomes

$$L = (1 + \tilde{\Gamma}_{(s_1,t_1)}) \ldots (1 + \tilde{\Gamma}_{(s_n,t_n)})C^{-1}_\pm \Gamma^{\mu\nu}(1 + \Gamma_{(s_n,t_n)}) \ldots (1 + \Gamma_{(s_1,t_1)}),$$  \hspace{1cm} (3.11)
which becomes

\[
L_+ = C_+^{-1}(1 + (-1)^{t_1} \Gamma_{(s_1,t_1)}) \cdots (1 + (-1)^{t_n} \Gamma_{(s_n,t_n)}) \Gamma^{\mu \nu} (1 + \Gamma_{(s_n,t_n)}) \cdots (1 + \Gamma_{(s,t)}) \\
L_- = C_-^{-1}(1 + (-1)^{s_1} \Gamma_{(s,t)}) \cdots (1 + (-1)^{s_n} \Gamma_{(s_n,t_n)}) \Gamma^{\mu \nu} (1 + \Gamma_{(s_n,t_n)}) \cdots (1 + \Gamma_{(s,t)})
\]  

(3.12)

for the two choices of the charge conjugation matrix. Evaluating all the possibilities, we find that \( L = 0 \) if and only if

1. The sets of gamma matrices labelled by \((s_i, t_i)\) for \(i = 1, \ldots, n\) form a partition of \(\{\Gamma^1, \ldots, \Gamma^S, \Gamma^{S+1}, \ldots, \Gamma^{(S+T)}\}\).

2. \((-1)^{t_1} \cdots (-1)^{t_n} = 1\) for \(C_+\) and \((-1)^{s_1} \cdots (-1)^{s_n} = 1\) for \(C_-\).

3. All of the matrices \(\Gamma_{(s_i,t_i)}\) for \(i = 1, \ldots, n\) commute with each other.

4. The matrices \(\Gamma_{(s_i,t_i)}\) are all independent.

Studying these constraints for \(D \leq 14\) and \(T \leq 3\) provides a single non-Weyl solution. This is given by the decomposition \((8,0)(1,1)(1,0)\) in signature \((10,1)\). Although many other decompositions appear to fit this picture, they are inconsistent for the final reason on the previous list. By substituting an explicit representation of the \((10,1)\) gamma matrices, it is easy to show that the rank of the projector corresponding to \((8,0)(1,1)(1,0)\) is \(\frac{1}{4}\).

### 3.2.2 Tensor product projectors

We now try to determine the rank of the solution of the superspace identity in certain signatures, by considering tensor product projectors of the form

\[
\mathcal{P} = \mathcal{P}_1 (I_2 \otimes \mathcal{P}_2) ,
\]  

(3.13)

for a projector \(\mathcal{P}_1\) of dimension \(2^{[D/2]} \times 2^{[D/2]}\) and a smaller projector \(\mathcal{P}_2\) of dimension \(2^{[D/4]} \times 2^{[D/4]}\). Of course, since we wish to deal with Majorana spinors, we must require that Majorana spinors exist on the subspace acted on by \(\mathcal{P}_2\) as well as in the full space.

We consider spacetimes with both spatial and temporal directions. We may then write

\[
\begin{align*}
\Gamma_0 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\
\Gamma_D &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
\Gamma_p &= \begin{pmatrix} \gamma_p & 0 \\ 0 & -\gamma_p \end{pmatrix} \quad p = 1 \ldots D - 1;
\end{align*}
\]  

(3.14)
the charge conjugation matrices are given by

\[ C_+ = \begin{pmatrix} 0 & c_- \\ c_- & 0 \end{pmatrix}, \quad C_- = \begin{pmatrix} 0 & c_+ \\ -c_+ & 0 \end{pmatrix}, \] (3.15)

with \(c_\pm\) being the charge conjugation matrices for the \(\gamma_1 \ldots, \gamma_{D-1}\). We now reduce the problem to a similar one in a lower dimension by employing the projector

\[ \mathcal{P} = \frac{1}{2}(1 + \Gamma_0 \Gamma_D)(I_2 \otimes \mathcal{P}_2), \] (3.16)

where \(\mathcal{P}_2\) is a matrix projector of the same dimension as the \(\gamma_p\), and \(I_2\) is the two dimensional identity matrix. We then find that the problem is equivalent to solving

\[ \tilde{\mathcal{P}}_2 c_\pm \gamma^p \mathcal{P}_2 = 0, \] for \(C = C_\pm\) respectively . (3.17)

If \(\mathcal{P}_2 = \frac{1}{2}(1 + \gamma_a \ldots \gamma_{D-1})\) is a projection operator, which is the case if \((S - T) \mod 8 = 0\), then the equation (3.17) has a solution iff \(T\) is an even number (and non-zero, of course). This gives the total rank of the projector \(\mathcal{P}\) to be one quarter in the cases \(T \mod 8 = S \mod 8\).

We may now repeat the reduction procedure on the matrix identity (3.17) involving the lower dimensional gamma matrices \(\gamma_p\), if we have that \(T, S \geq 2\). In an exactly analogous way, if we consider the projector

\[ \mathcal{P} = \frac{1}{2}(1 + \Gamma_0 \Gamma_D)I_2 \otimes \left( \frac{1}{2} (1 + \gamma_t \gamma_s) I_2 \otimes \mathcal{P}_4 \right) \] (3.18)

we find that the surviving piece of the identity is

\[ \tilde{\mathcal{P}}_4 CP_4 = 0, \] (3.19)

where \(\mathcal{P}_4\) is a projection matrix of dimension \(2^{[D/8]} \times 2^{[D/8]}\), and \(C\) is the appropriate charge conjugation matrix. The equation (3.19) always has a solution for a rank \(\frac{1}{2}\) projector \(\mathcal{P}_4\). Thus the complete projector \(\mathcal{P}\) required to satisfy the full superspace identity is of rank \(\frac{1}{8}\).

We have presented various methods for constructing projectors which satisfy the superspace identity (2.15). There may, of course, be other projectors which satisfy the identity (3.3), although we have considered the obvious constructions of such objects. We present the signatures with \(S \geq T\) and \(D \leq 14\) for which a solution has been found.
for real spinors, as well as the rank of the appropriate projection matrix

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
\(D\) & \((S,T)\) & \text{Rank} \\
\hline
2 & (1,1) & 1/2 \\
6 & (3,3) & 1/2 \\
10 & (9,1), (5,5) & 1/2 \\
11 & (10,1) & 1/4 \\
12 & (10,2), (6,6) & 1/4 \\
\hline
\end{tabular}
\end{table}

(3.20)

In addition, we also have that there exist projectors of rank \(1/8\) in the cases that a space of signature \((S,T)\) admits Majorana spinors, and that \(S,T \geq 2\). Some examples of interest in twelve, thirteen and fourteen dimensions are \((S,T) = (9,3), (10,3), (11,2), (11,3)\).

4 \(p\)-brane actions

We now turn to the question of \(p\)-branes moving in general \(\text{so}_k\) superspaces. From studies of the usual \(\text{so}_1\) Poincaré supersymmetry the existence of different types of \(p\)-brane has emerged. The hope is that certain branes will have a more natural description in the new types of superspace and that we will discover more branes by employing this more comprehensive description of supersymmetry. A \(p\)-brane in \(D\) dimensions is a \(p+1\)-dimensional brane manifold embedded in a \(D\) dimensional spacetime manifold. The \(p\)-branes we shall discuss correspond to fundamental objects, such as Green-Schwarz superstrings, as opposed to solitonic \(p\)-branes. Usually the spacetime is taken to be of signature \((D-1,1)\), and the brane to be of signature \((p,1)\) but we shall lift these restrictions. The ‘spacetime’ and the \(p\)-brane will be of signature \((S,T)\) and \((s,t)\) respectively, so that \(D = S + T\) and \(p + 1 = s + t\) to fit in with the usual definition of a \(p\)-brane for \(t = 1\); to be more precise one could call the brane an \((s + t)\)-brane. In order that the construction be classically stable, which corresponds to the absence of ghosts quantum mechanically, we require that \(t = T\). This removes the possibility of negative norm states propagating in directions transverse to the brane.

We now assume the principle of least action for a \(p\)-brane moving in a general Poincaré superspace to produce a set of \(p\)-brane actions. Recall that the bosonic part of the superspace can be parametrised by the terms \(\{X^{\mu_1 \ldots \mu_n}\}\) where \(n \in P \subseteq \{1,3,4,\ldots,D\}\). We thus may write down the actions

\[
S_P = \int d^{p+1}x \left[ \det \left( \sum_{n \in P} \Pi^{\mu_1 \ldots \mu_n} \Pi^{\nu_1 \ldots \nu_n} \eta_{\mu_1 \nu_1} \ldots \eta_{\mu_n \nu_n} \right) \right]^{1/2},
\]  

(4.1)
where $\Pi_{\mu_1\ldots\mu_n}^i$ are the pullback of the forms (2.17) to the worldvolume of the brane, which has coordinates $\xi_1, \ldots, \xi_{p+1}$. These actions are manifestly spacetime supersymmetric under the action of the supergroups corresponding to $\text{so}_k$, since they are constructed from invariant forms, and are generalisations of the standard Dirac type actions to include higher spin gauge fields on the worldsheet.

It is a matter of interpretation to decide which actions are physically relevant to structureless (fundamental) $p$-brane propagation. Under the superspace rescaling $X^{\mu_1\ldots\mu_n} \rightarrow \Omega^n X^{\mu_1\ldots\mu_n}$, each term in the sum

$$\left( \sum_{n \in P} \Pi_{\mu_1\ldots\mu_n}^i \Pi_{\nu_1\ldots\nu_n}^j \eta_{\mu_1\nu_1} \ldots \eta_{\mu_n\nu_n} \right),$$

scales differently. We therefore suppose that the fundamental actions are written as

$$S_n = \int d^{p+1}\xi [\det(\Pi_{\mu_1\ldots\mu_n}^i \Pi_{\nu_1\ldots\nu_n}^j \eta_{\mu_1\nu_1} \ldots \eta_{\mu_n\nu_n})]^{\frac{1}{2}} \equiv \int d^{p+1}\xi [\det(\Pi_{i}^{(n)} \Pi_{j}^{(n)})]^{\frac{1}{2}}. \quad (4.3)$$

These actions can be rewritten in the Howe and Tucker form [14] as follows

$$S_n = \int d^{p+1}\xi \left( \sqrt{|g|} g^{ij} \Pi_{i}^{(n)} \Pi_{j}^{(n)} - \frac{1}{2} (p - 1) \right), \quad (4.4)$$

where $g_{ij}$ is the induced metric on the worldsheet. Thus for $n > 0$ we have a natural way in which to define the action for one of the $n$-form charges propagating on the worldsheet. This idea may be of use if we wish to add degrees of freedom to the $p$-brane problem by adding higher spin fields on the brane. To see that this is consistent, we note that the equation of motion of the $p$-form is simply

$$dF = 0, \quad F = d\Pi^{(p)}, \quad (4.5)$$

as one would expect. This is a pleasing result, since it was obtained via the principle of least action for a manifold invariant under the action of a generalised Poincaré supergroup.

### 4.1 Wess-Zumino terms

#### 4.1.1 $S_1$ case

The standard $p$-brane actions may be augmented with an additional piece, called the Wess-Zumino action. This term is added to the basic action $S_1$ because it contains
an antisymmetric tensor field which provides a coupling of the brane to the local supergravity theory. We define the Wess-Zumino term to be of the form

$$S_{WZ} = - \int d^{p+1} \xi \left( \frac{1}{(p+1)!} \epsilon^{i_1 \ldots i_{p+1}} B_{i_1 \ldots i_{p+1}} \right),$$

(4.6)

where $B_{i_1 \ldots i_{p+1}}$ are the components of the pullback to the brane of a super $(p+1)$-form, $B$, which is the potential for a super $(p+2)$-form $H = dB$. For the action $S_1$, parametrised by $(X^\mu, \theta)$, consistency requires the Wess-Zumino integral to scale in the same way as $S_1$ under the superspace rescaling $X^\mu \rightarrow \Omega X^\mu, \theta^\alpha \rightarrow \Omega^{\frac{1}{2}} \theta^\alpha$. For a given $p$ we can then define $H$ to be

$$H = \Pi^{\mu_1} \ldots \Pi^{\mu_p} d\bar{\theta} \Gamma_{\mu_1 \ldots \mu_p} d\theta,$$

(4.7)

where the forms $\Pi^\mu$ are defined in (2.17). For this not to vanish identically, $(\Gamma_{\mu_1 \ldots \mu_p})_{\alpha\beta}$ must be symmetric in the spinor indices, since the $d\theta$ are commuting variables as the $\theta$ are Grassmann odd. Taking the exterior derivative of the forms $\Pi^\mu$ we find that

$$d\Pi^\mu = \sigma_1 d\theta_\alpha (\Gamma^\mu)_{\alpha\beta} d\theta_\beta.$$

(4.8)

Since $\sigma_1 = 0$ automatically if $\Gamma^\mu$ is antisymmetric in the spinor indices, $d\Pi^\mu = 0$ iff $\sigma_1 = 0$. As $H = dB$ we must check that $dH = 0$ for consistency. If $\sigma_1 = 1$ then for $H$ to be closed it is well known that then we must have that

$$D - p - 1 = \frac{nN}{4},$$

(4.9)

where $n$ is the number of spin degrees of freedom of the spinors and $N$ is the number of supersymmetry generators. If $\sigma_1 = 0$ then $H$ is automatically closed, due to the triviality of the superspace forms. This differing character of the superspace forms for the two different types of supersymmetry also gives rise to two different types of $B$.

Up to total derivatives, for the standard $\sigma_1 = 1$ case, $B$ may be shown to be

$$B = \frac{(-1)^p}{2(p+1)!} (d\bar{\theta} \Gamma_{\mu_1 \ldots \mu_p} \theta) \left[ \sum_{r=0}^{p} (-1)^r \frac{(p+1)}{r+1} \Pi_1^{\mu_1} \ldots \Pi_r^{\mu_r} + (d\bar{\theta} \Gamma^{\mu_1} \theta) \ldots (d\bar{\theta} \Gamma^{\mu_p} \theta) \right].$$

(4.10)

For the $\sigma_1 = 0$ superspaces the situation is much simpler, since all the forms in the problem are exact. We find that

$$B = \frac{(-1)^p}{(p+1)!} dX^{\mu_1} \ldots dX^{\mu_p} (d\bar{\theta} \Gamma_{\mu_1 \ldots \mu_p} \theta),$$

(4.11)
which may be obtained from (4.10) by setting \( d\bar{\theta}\Gamma^\mu \theta = 0 \). It is well known that the action \( S_1 + S_{WZ} \) is invariant under a local \( \kappa \)-symmetry transformation for the standard \( \text{sio}_1 \) superspace. We need to investigate whether or not this is the case for all the superspaces for the action \( S_1 \).

### 4.1.2 \( S_n \) case, for \( n > 1 \)

One may always define the Wess-Zumino term for the action \( S_1 \), but this is not always the case with actions for higher values of \( n \). If one considers a theory with simply the coordinates \( \{ X^{\mu_1\cdots\mu_n}, \theta \} \) then the Wess-Zumino term may only be constructed by the scaling argument if \( p \) is some integer multiple of \( n \). We would then find that

\[
H = \Pi^{\mu_1\cdots\mu_n} \cdots \Pi^{\mu_{(kn-n+1)}\cdots\mu_{kn}} d\bar{\theta}\Gamma_{\mu_1\cdots\mu_{kn}} d\theta. \tag{4.12}
\]

If we have a theory parametrised by a larger set of coordinates, then it may possible to construct more general Wess-Zumino terms, which are not necessarily unique. If, on the other hand, \( p \) is not an integer power of \( n \) then we cannot construct a Wess-Zumino term without introducing fractional powers of the integrand to get the correct scaling behaviour.

We shall henceforth mainly consider the traditional type \( S_1 \) action, since it incorporates the basic coordinates of spacetime, although we shall look at those which are invariant under the general types of Poincaré supersymmetry.

## 5 Generalised supersymmetry \( p \)-branes

### 5.1 \( \kappa \)-symmetry for the \( n = 1 \) \( \text{sio}_1 \), \( \text{sio}_2 \) and \( \text{sio}_{1,2} \) invariant actions

The \( \kappa \)-symmetry is a hidden symmetry which arises in the standard formulation of \( p \)-brane actions. This symmetry, which is an interrelationship between \( S_1 \) and the Wess-Zumino term in the full \( p \)-brane action, is a special local spinor transformation. In order to discuss \( \kappa \)-symmetry in general, it is instructive to first detail the transformation in the standard context. For \( \text{sio}_1 \) superspace, the \( \kappa \) symmetry is defined to be a local version of the supersymmetry transformations

\[
\delta X^\mu = \bar{\theta}\Gamma^\mu \delta \theta, \tag{5.1}
\]
for some local spinor $\delta \theta$. If this is so then the variation in the forms $\Pi^\mu$ are given by

$$
\delta \Pi^\mu_i = -2\delta \bar{\theta}^\alpha \Gamma^\mu_{\alpha \beta} \partial_i \theta^\beta, \quad \delta \Pi^\alpha_i = \partial_i \delta \theta^\alpha.
$$

(5.2)

In order to obtain invariance of the full action $S_1 + S_{WZ}$ under these changes, we must relate the Wess-Zumino term, which has no metric dependence, to the integral $S_1$. To do this we work with the pullback to the brane of the gamma matrices

$$
\Gamma_i = \Pi^\mu_i \Gamma^\mu, \quad \{\Gamma_i, \Gamma_j\} = 2g_{ij}.
$$

(5.3)

We then obtain the relationship

$$
\epsilon^{i\ldots jk} \Gamma_{1\ldots j} = 2\sqrt{-g} g^{kl} \Gamma_l \Gamma,
$$

(5.4)

with

$$
\Gamma = \frac{(-1)^{(p+1)(p+2)/4}}{(p+1)! \sqrt{|g|}} \Gamma_{i_1 \ldots i_{p+1}} \epsilon^{i_1 \ldots i_{p+1}}.
$$

(5.5)

The matrix $\Gamma$ has the properties that $\text{Tr}(\Gamma) = 0$, iff $p + 1 \neq D$, and that $\Gamma^2$ is the identity matrix. The full action is then seen to be invariant under the $\kappa$-transformation [13] if we make the choice

$$
\delta \theta = (1 + \Gamma) \kappa.
$$

(5.6)

We may thus gauge away half the spinor degrees of freedom. It has not been proven that this is the only hidden symmetry of these $p$-branes actions, but it is difficult to think of any other possibility.

The question of a $\kappa$-symmetry for a superspace without torsion is not so obvious. At the most basic level, in constructing a local fermionic symmetry of a $p$-brane action, we require that the fermionic variation is cancelled by the bosonic variation, with an additional constraint on the spinors used in the variation. The Wess-Zumino term for an invariant action with no torsion is the integral over the brane of the $(p + 1)$-form $B$, given by (4.11). Since the superspace forms are trivial, there is no link between the bosonic and fermionic coordinates, and the interplay which occurs in the usual definition of $\kappa$-symmetry, (5.1), does not occur. Writing down the variation of the action for the $n = 1$ case, and using the expression (5.3), we find that

$$
\delta S = \int d^{p+1}\xi \sqrt{g} g^{ij} \left(2\partial_i (\delta X^\mu) \partial_j X^\nu + \delta \bar{\theta} \Gamma_{j} \partial_i \theta\right) - \int d\bar{\theta} \Gamma_{\mu_1 \mu_2 \ldots \mu_p} d\theta (\delta X^{\mu_1} dX^{\mu_2} \ldots dX^{\mu_p}).
$$

(5.7)
In order for this variation to vanish, we must have that $\delta X^\mu = 0$, in which case the constraint on the variation for the $\theta$ terms becomes

$$\delta \theta \Gamma_i \partial_j \theta = 0.$$  \hfill (5.8)

If $\theta$ is a general Dirac or Majorana spinor, then this equation is satisfied iff $\delta \theta = 0$, in which case we find that there is no $\kappa$-symmetry. For the sio$_2$ superspace, however, we must work with a spinor projection, in which case (5.8) is satisfied for non-zero $\delta \theta$ iff

$$\tilde{\mathcal{P}} \mathcal{C} \Gamma_i \mathcal{P} = 0.$$  \hfill (5.9)

For such sio$_2$ superspaces we already have that

$$\tilde{\mathcal{P}}^\pm \mathcal{C} \Gamma^\mu \mathcal{P}^\pm = 0,$$  \hfill (5.10)

for some orthogonal projectors $\mathcal{P}^\pm$. This implies that

$$\tilde{\mathcal{P}}^\pm \mathcal{C} \Gamma^\mu \mathcal{P}^\pm = \mathcal{C} \Gamma^\mu \mathcal{P}^\mp \mathcal{P}^\pm,$$  \hfill (5.11)

Since $\Gamma_i$ is a sum of independent products of $p+2$ gamma matrices, then by considering the expressions

$$\tilde{\mathcal{P}}^\pm \mathcal{C} \Gamma^\mu \Gamma^\nu \Gamma^{\rho_1} \ldots \Gamma^{\rho_p} \mathcal{P}^\pm = \mathcal{C} \Gamma^\mu \Gamma^\nu \mathcal{P}^\mp \Gamma^{\rho_1} \ldots \Gamma^{\rho_p} \mathcal{P}^\pm,$$  \hfill (5.12)

we find that

1. If $\mathcal{P}$ is Weyl then (5.9) is satisfied iff $p$ is even or zero.

2. If $\mathcal{P}$ is not the Weyl projector then (5.9) is only satisfied if we take $p = 0$.

It is therefore often the case that there is no $\kappa$-symmetry for the superspaces with $\sigma_1 = 0$. This should be no surprise. For a supersymmetry of the form $\{Q, Q\} \sim P$ there exist local supergravity theories for dimensions eleven or less. The $\kappa$-symmetry is a local version of the rigid supersymmetry transformations. For a supersymmetry of the form $\{Q, Q\} \sim M$ there is no local supergravity theory since the $P$ terms generate the diffeomorphisms which enable us to couple supersymmetry to general relativity. Such a coupling does not occur for sio$_2$ theories, hence we should expect the related $\kappa$-symmetry problem to be more subtle.
5.2 Worldvolume supersymmetry

In the previous section we presented superspacetime symmetric actions. We shall now construct Green-Schwarz $p$-brane actions for all possible values of $p$ and $(S, T)$. To do this we enforce supersymmetry on the brane, in addition to spacetime supersymmetry. We consider supersymmetric theories generated by $\{P, M, Q\}$ for which the brane fields form scalar supermultiplets, given by $(\theta^\alpha, X^\mu)$. For this type of worldvolume supersymmetry to occur we require that the brane bosonic and fermionic physical degrees of freedom match up. We have many possibilities for the degrees of freedom counting, depending on whether there exists a $\kappa$-symmetry for the brane action and whether or not it is necessary to take a projection of the spinors. In addition to these considerations, we are interested in physical spinor degrees of freedom. We must therefore go ‘on-shell’. The spinor equations of motion are really second class constraints, and thus half the number of degrees of freedom. The final spinorial degrees of freedom must match the $D - p - 1$ transverse bosonic degrees of freedom, to produce the degrees of freedom matching formula

$$D - p - 1 = \kappa R \frac{n}{2} N. \quad (5.13)$$

In this expression, $\frac{n}{2}$ corresponds to the physical degrees of freedom of an on-shell Majorana spinor, $n$ being the real dimension of the Majorana spinor. $N$ is the number of supersymmetry generators, which we shall take to be 1. The value of $R$ is the rank of any projection made on the spinors, and $\kappa$ is equal to one half or unity, depending on whether there is or is not a $\kappa$-symmetry of the $p$-brane action. We shall investigate branes for which a Wess-Zumino action exists, to provide a coupling of the brane to local theories. For a given $p$-brane this requires $\Gamma^{\mu_1,...,\mu_p}$ to be symmetric in the spinor indices. The parity of such matrices is determined by the equation [3]

$$\pi = \epsilon \eta^T (-1)^{\frac{TT+1}{2}} \left((-1)^T \eta\right)^i (-1)^{\frac{i(i-1)}{2}}, \quad (5.14)$$

where the choices of $\epsilon$ and $\eta$ are given in (1.6), from which we find that $\pi(\Gamma^{\mu_1,...,\mu_4})$ is given by

$$\begin{array}{ccc}
T = 0 & \epsilon \eta & -\epsilon \\
T = 1 & \epsilon & \epsilon \eta
\end{array} \quad (5.15)$$

This table is anti-periodic modulo 2 for both $i$ and $T$. For a Majorana representation we must have that $\epsilon = 1$ and $(S - T) \mod 8 = 0, 1, 2, 6, 7$. For such cases we give the
sets $k_\pm = \{i : (\Gamma^{\mu_1...\mu_i} C_\pm)^{\alpha\beta} = (\Gamma^{\mu_1...\mu_i} C_\pm)^{\beta\alpha}\}$ for each choice of the charge conjugation matrix

| $T \mod 4$ | $k_+$ | $k_-$ |
|------------|-------|-------|
| 0          | $\{1, 4, 5, 8, 9, 12,...\}$ | $\{3, 4, 7, 8, 11, 12,...\}$ |
| 1          | $\{1, 4, 5, 8, 9, 12,...\}$ | $\{1, 2, 5, 6, 9, 10,...\}$ |
| 2          | $\{2, 3, 6, 7, 10, 11,...\}$ | $\{1, 2, 5, 6, 9, 10, 13,...\}$ |
| 3          | $\{2, 3, 6, 7, 10, 11,...\}$ | $\{3, 4, 7, 8, 11, 12,...\}$ |

(5.16)

5.3 Brane scans

We shall say that a $p$-brane action exists if, for a given anticommutator (2.3), there exists $p$, $D$ and signature such that (5.13) holds. We also require that $\Gamma^{(p)}$ be symmetric in its spinor indices so that we may define a Wess-Zumino action. Finally, all of these requirement must be consistent with the definition of a charge conjugation matrix for real spinors in the given signature. In the cases for which a projection of the spinors is made we present all of the branes which are allowed from a Bose-Fermi matching point of view. The explicit construction of possible projectors is found previously.

5.3.1 $(\kappa, R) = (\frac{1}{2}, 1)$

This is the usual case for a $p$-brane with $\text{sio}_1$ supersymmetry, producing the usual brane scan. It is noteworthy that the scan is not invariant under the interchange of $S$ and $T$, given a fixed metric convention. The reason for this is that the symmetry properties of products of gamma matrices vary according to whether $T \mod 4 = 0, 1, 2, 3$, as may be seen in (5.13). For example, we may define a 2-brane in signature $(S, T) = (3, 1)$, but no 2-brane exists in signature $(S, T) = (1, 3)$. This type of behaviour is present in all the brane scans. For some Minkowski dimensions of importance, namely $(1,1)$, $(5,1)$, $(9,1)$, this problem does not arise. Conversely, we see that supersymmetric physics in $(10,1)$ is not strictly equivalent to that in $(1,10)$, since $10 \mod 4 \neq 1 \mod 4$ implies a different superalgebra structure.

5.3.2 $(\kappa, R) = (\frac{1}{4}, \frac{1}{4})$

Majorana-Weyl branes in $\text{sio}_1$ superspace, such as the $N = 2$ type IIA and type IIB superstrings fall into this category. These values of $\kappa$ and $R$ would include Majorana-Weyl
branes propagating in the $\text{sio}_2$ superspace for which a $\kappa$-symmetry exists: however, this requires $p$ to be even or zero, and there are no such solutions.

5.3.3 $(\kappa, R) = (1, \frac{1}{4})$

These are branes which propagate in the $\text{sio}_2$ superspace. The solutions with no $\kappa$-symmetry are

\[
\begin{array}{c|cccc}
S = 10 & 6^- & 3^+ & 6^\pm & 3^+ \\
9 & 5^- & 6^- & 3^- & 5^+ \\
8 & 5^- & 6^- & 3^- & 5^+ \\
7 & 5^- & 6^- & 3^- & 5^+ \\
T & 0 & 1 & 2 & 3 & 4 \\
\end{array}
\] (5.17)

This table repeats for $(S, T) \rightarrow (S - 4, T + 4)$. Projectors for the signatures $(S, T) = (9, 1), (10, 1), (10, 2), (6, 6)$ and $(1, 10)$ have been explicitly constructed in the previous sections. The other values are the only others which are allowed by symmetry and degrees of freedom matching. These would certainly occur in an $N = 2$ brane scan with spinors projected by a rank $\frac{1}{8}$ projector. The three-brane in $(10, 2)$ corresponds to the super $(2 + 2)$-brane $\mathbb{3}$, which reproduces the type IIB string and the M-theory 2-brane after dimensional reduction. The $(10,1)$ six-brane is a new type of brane in eleven dimensions. Although $\text{sio}_1$ supersymmetry does not permit such an object, it naturally occurs within the context of the generalised supersymmetry. This brane could then provide a higher dimensional origin to the type IIA six-brane.

5.3.4 $(\kappa, R) = (1, \frac{1}{2})$

These are $\text{sio}_2$ branes with no $\kappa$-symmetry defined in an explicitly Lorentz invariant way using Majorana-Weyl spinors. This requires us to choose $p$ to be odd. The solutions are given in the following table, which is symmetric in $(S, T)$

\[
\begin{array}{c|ccc}
S = 9 & 1^\pm & 3^\pm & 3^\pm \\
3 & & & \\
T & 0 & 1 & 2 & 3 \\
\end{array}
\] (5.18)

The 1-brane we obtain in this picture is very interesting, as it is the simple supersymmetry analogue of the Green-Schwarz string in $\text{sio}_1$ superspace. The corresponding string action is constructed from flat one-forms, and is thus in some sense a trivially
supersymmetric object. Of course, there is no reason why the sio\textsubscript{2} string should be the same object as the traditional string, and merely points to the existence of a new type of 1-brane.

5.3.5 \((\kappa, R) = (\frac{1}{2}, \frac{1}{2})\)

There are no solutions for these degrees of freedom

5.3.6 \(R = \frac{1}{4}\)

These degrees of freedom only give us a \(7^+\) brane in signature \((9,3)\), if we have no \(\kappa\)-symmetry. Otherwise, we find that there are no consistent branes.

5.4 Discussion

We have presented all the possible branes with worldvolume scalar supermultiplets propagating in general Poincaré superspaces. It should be possible to extend the analysis to consider \(p\)-branes with higher spin fields on the worldsheet, which would render the generalised supersymmetry arguments applicable to \(D\)-branes, although we do not consider this problem here. Given the existence of a particular \(p\)-brane in the previous tables it is a simple matter to construct associated action, using the results from the previous sections. All the underlying theories to these branes are fully covariant, although some of the actions may need a non-covariant projector for their explicit description. We present the new branes for which we can construct explicit superspace projections for low \(T\). They all occur for superspaces with \(\{Q, Q\} = M_{\mu\nu}\Gamma^{\mu\nu}\)

| \((S, T)\) | \(p\) | \((\kappa, R)\) |
| --- | --- | --- |
| \((10, 2)\) | \(3^+\) | \((1, \frac{1}{4})\) |
| \((9, 3)\) | \(7^+\) | \((1, \frac{1}{4})\) |
| \((10, 1)\) | \(6^-\) | \((1, \frac{1}{4})\) |
| \((9, 1)\) | \(1^\pm\) | \((1, \frac{1}{2})\) |
| \((3, 3)\) | \(3^\pm\) | \(\) |

Several interesting points are raised by these results, not least that it is possible to formulate brane theories using a simple supersymmetry, which arises from the most natural fermionic extension of the Poincaré algebra. In such a scenario we find a description of a superstring in a flat superspace with no torsion. We also find that
generalised supersymmetry provides only a small number of additional fundamental branes which must propagate in a maximal spacetime of dimension twelve. Of course this maximal dimension arises if one considers scalar supersymmetry on the brane, and may be altered in more general scenarios. Twelve dimensional theories have been employed recently to answer some problems associated with lower dimensional physics [1, 2]. Dimensional reduction of the $\text{so}_2$ threebrane and six-brane should in principle be able reproduce many of the fundamental branes in lower dimensional string theory and M-theory, the necessary superspace torsion being introduced upon compactification as in [3]. It is interesting to note that in dimensions ten and eleven, Minkowskian signature branes are singled out for all the types of supersymmetry, whereas in the maximal twelfth dimension we see a Kleinian space with two timelike directions appearing. This ties in well with much of the other work on twelve dimensions for which a signature of (10,2) is necessary. In order to dimensionally reduce the Kleinian branes down to Minkowski signature branes we would need to reduce on a Lorentzian torus, as in the case of the twelve dimensional F-theory. It has been suggested that the F-theory construction merely makes use of auxiliary extra dimensions and that only 10 of them are ‘real’. Generalised supersymmetry, however, points to intrinsically twelve dimensional theories, independent of any lower dimensional considerations.

6 Conclusion

We have considered the most general supersymmetric extensions of the Poincaré algebra of spacetime and constructed the associated superspaces. This analysis reveals that there are in fact two distinct classes of supersymmetry. It also shows that we must in general work with a new class of spinors, of which Weyl spinors are a special case. Constructing $p$-branes in these new superspace backgrounds produces additional points on the brane scan which are not present for the usual description of supersymmetry. The methods used are natural and seem to provide some new ideas in string theory. The new class of supersymmetry does not provide many new points on the brane scan, but produces a different set of $p$-branes to the ones usually obtained. These $p$-branes create, in a sense, a complete brane scan in dimensions ten and eleven, and give a higher dimensional origin for some of the branes in such dimensions. This could have corollaries for the supergravity theories in which the $p$-branes are solutions. It should therefore be of interest to investigate the implications of generalised supersymmetry in
other aspects of string theory. In particular, it would seem to be an interesting problem to investigate the local version of the \text{si}_2 supersymmetry. This type of theory would be a twelve dimensional ‘supergravity’ theory. Possible forms of such a theory have been discussed previously \cite{17}. Obviously these could not be of the same form as the true supergravity theories in lower dimensions since the anticommutator of the spinor generator with itself does not produce a translation. The existence of such a local theory in twelve dimensions would doubtless be related in some sense to supergravities in lower dimensions. The discovery of such a theory could very well, therefore, aid our understanding of the Minkowski space signature theories we are ultimately interested in.

Although some of the corollaries of generalised supersymmetry presented in this paper may seem to be at odds with standard \text{p}-brane folklore, such as the existence of an eleven dimensional six-brane and the twelve dimensional three-branes, they all stem from the assumption that supersymmetry is a fermionic extension of the Poincaré algebra, and that all such extensions should basically be equivalent. Reassuringly, however, these new branes do seem to provide a unifying scheme for many of the previously known \text{p}-brane theories. We have become accustomed to the idea that no one particular brane should be singled out as fundamental in theories of extended objects; perhaps the same idea should now be applied to supersymmetry.

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