Constrained Discounted Stochastic Games

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Abstract
In this paper, we consider a large class of constrained non-cooperative stochastic Markov games with countable state spaces and discounted cost criteria. In one-player case, i.e., constrained discounted Markov decision models, it is possible to formulate a static optimisation problem whose solution determines a stationary optimal strategy (alias control or policy) in the dynamical infinite horizon model. This solution lies in the compact convex set of all occupation measures induced by strategies, defined on the set of state–action pairs. In case of $n$-person discounted games the occupation measures are induced by strategies of all players. Therefore, it is difficult to generalise the approach for constrained discounted Markov decision processes directly. It is not clear how to define the domain for the best-response correspondence whose fixed point induces a stationary equilibrium in the Markov game. This domain should be the Cartesian product of compact convex sets in a locally convex topological vector space. One of our main results shows how to overcome this difficulty and define a constrained non-cooperative static game whose Nash equilibrium induces a stationary Nash equilibrium in the Markov game. This is done for games with bounded cost functions and positive initial state distribution. An extension to a class of Markov games with unbounded costs and arbitrary initial state distribution relies on an approximation of the unbounded game by bounded ones with positive initial state distributions. In the unbounded case, we assume the uniform integrability of the discounted costs with respect to all probability measures induced by strategies of the players, defined on the space of plays (histories) of the game. Our assumptions are weaker than those applied in earlier works on discounted dynamic programming or stochastic games using the so-called weighted norm approach.

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1 Introduction

Constrained Markov decision processes arise in situations, in which a controller has many objectives. For example, when she or he wants to minimise one type of cost while keeping other costs lower than some given bounds. Such situations appear very often in computer networks and data communications, see the papers of Lazar [27], Hordijk and Spieksma [24], Ross and Chen [31] or Feinberg and Reiman [18]. The theory of constrained Markov decision processes goes back to Derman and Klein [15]. It was further developed for finite state space models by Kallenberg [26]. For the literature dealing with the discounted costs (or rewards), the reader is referred to the works of Altman [2], Borkar [12], Feinberg et al. [20], Feinberg and Shwart [19], Sennott [34] and the books by Altman [3] and Piunovskiy [29].

Unconstrained non-cooperative n-person discounted stochastic Markov games with finite state spaces were first studied by Fink [21], Takahashi [36] and Sobel [35]. Their results were extended to countable state space games by Federgruen [17]. The proofs of the existence of stationary Nash equilibria in unconstrained discounted Markov games are based on the Kakutani–Fan–Glicksberg fixed point theorem [1] and a system of Bellman’s optimality equations for dynamic programming problems associated with the players. The basic results on dynamic programming (Markov decision models) used in studying unconstrained stochastic games are very well outlined in [9, 11, 23, 37].

A number of natural examples of static constrained games come from economics [14, 30]. Constrained n-person Markov games with finite state and action spaces were first studied by Altman and Shwartz [4]. They also apply in their analysis the Kakutani fixed point theorem and formulate the problem for each player as a Markov decision process. However, by introducing stochastic constraints on the strategy choices of the players, they had to use some facts from the theory of sensitivity analysis of linear programming.

Nash equilibria in games with constraints arise quite naturally, for instance, in the context of asynchronous transfer mode networks, where users express their requirements for quality of service by bounds they wish to have on delays, etc. An audio application could therefore selfishly seek to minimise losses, subject to a maximum bound on the delay it experiences. Nash equilibria in constrained games were also studied in a dynamic environment in telecommunications and internet provisioning applications, see [6] and references cited therein. Other applications focus on selection of rate allocations in multiple access channels as well as models with asymmetric or partial information [5, 22]. Applications of constrained stochastic games to some queueing models are given in [7, 38].
The result of Altman and Shwartz [4] for discounted constrained Markov games was generalised by Alvarez-Mena and Hernández-Lerma [7], who considered compact metric action spaces. They studied first finite state space games and next, imposing a special condition on the transition probability, showed how to get a stationary Nash equilibrium in a discounted game with countably many states. The proof of their result relies on an approximation of the game with a denumerable state space by games with finitely many states.

Work [7] is devoted to a class of countable state space constrained stochastic games with bounded cost (payoff) functions and transitions satisfying a certain restrictive condition. The main result in [7] was further used by Zhang et al. [38] to establish the existence of stationary Nash equilibria in a class of discounted Markov games with countable state spaces and unbounded cost functions. The assumptions, that are made in [38], resemble conditions presented by Wessels [37], who studied dynamic programming problems with unbounded reward functions using the so-called weighted norm. Similarly as in [7], Zhang et al. [38] apply an approximation of the original game by ones with finitely many states. However, they erroneously claim that the condition on the transition probabilities used in [7] holds for all games (see Remark 2.9 in Sect. 2). Therefore, their proof for constrained discounted stochastic games with bounded costs is incorrect.

The value iteration algorithms and Bellman’s optimality equations are not sufficient tools for studying constrained Markov decision problems and constrained stochastic games. As shown by Borkar [12], some results from convex analysis and properties of so-called occupation measures induced by strategies (controls or policies) must be applied. The application of occupation measures enables to recognise the dynamic optimisation problem as a static one on some compact convex subset of probability measures on the space of state-action pairs. The compactness and convexity of the set of occupation measures in the discounted Markov decision process are closely related to the properties of the space of all probability measures on the set of trajectories of the process induced by strategies (policies) of the decision maker. For the details the reader is referred to [12, 20, 29].

The existence of stationary Nash equilibria in constrained discounted Markov games is proved by using the Kakutani–Fan–Glicksberg fixed point theorem [1]. However, the main obstacle is to define the domain for the best-response correspondence associated with an auxiliary one-shot game. In the finite state space case, studied by Altman and Shwartz [4] and Alvarez-Mena and Hernández-Lerma [7], for any player $i$, the authors take into account the set, say $\text{Pr}(K_i)$, of all probability measures on the set $K_i$ of all pairs $(x, a_i)$, where $x$ is a state and $a_i$ is an action available to player $i$ in this state. Then, $\text{Pr}(K_i)$ is convex and compact in the weak topology. The best responses of player $i$ also belong to $\text{Pr}(K_i)$ and satisfy some equations introduced in the theory of constrained Markov decision models by Borkar [12]. These equations guarantee that the fixed point of the best-response correspondence is a vector of occupation measures, from which the existence of a stationary Nash equilibrium is concluded. When the state space is countable and infinite, then the space $\text{Pr}(K_i)$ is not compact and the analysis from [4] and [7] does not work. Therefore, Alvarez-Mena and Hernández-Lerma [7] and Zhang et al. [38] approximate the stochastic game with a denumerable state space by games with finitely many states. Analogous methods
were earlier used to consider discounted Markov decision processes by Altman [3] and Cavazos-Cadena [13]. We would like to emphasise that introducing constraints in the stochastic game model and following the finite state approximations as in [7, 38] lead to several unnecessary technical considerations. Therefore, our techniques and ideas are different than in the aforementioned papers.

In this paper, we study a general class of discounted constrained stochastic Markov games with unbounded costs. In Sect. 2, we formulate our basic assumptions including the uniform integrability of discounted cost functions on the space of all trajectories (sample paths) of the process. Our assumptions are weaker than those used by Zhang et al. [38] for games and by Wessels [37] for dynamic programming. Section 2 also presents our main results and contains a few essential comments (Remarks 2.6–2.9) and remarks on earlier works [4, 7, 38]. The proof of the main theorem (Theorem 2.3) for unbounded Markov games with stochastic constraints is provided in Sect. 4. It is based on an auxiliary result (Proposition 3.15 in Sect. 3) for constrained Markov games with bounded costs and a positive initial state distribution. Then, an approximation of the general stochastic game by ones with perturbed initial state distributions and truncated costs is applied. In Sect. 5, we give examples that explain relations of our uniform integrability assumption from Sect. 2 with those of Wessels [37] and Zhang et al. [38]. Section 6 (Appendix) contains two lemmas used in the proofs in Sects. 3 and 4.

Finally, we wish to stress out that our idea applied for the study of games with bounded costs is new and relies on introducing a proper domain for the best-response correspondence associated with the auxiliary one-shot game. This domain is the Cartesian product of some appropriately constructed compact convex subsets of the spaces \( \mathcal{K}_i \) (all probability measures on \( \mathcal{K}_i \)). Our approach works in the infinite countable state space case and therefore, no finite state approximation is necessary. Instead, we apply the basic results on occupation and strategic measures from Borkar [12] and Schäl [32, 33].

## 2 The Model and the Main Results

The non-zero-sum constrained stochastic Markov game (CSG) is described by the following objects:

- \( \mathcal{N} = \{1, 2, \ldots, n\} \) is the set of players.
- \( X \) is a countable state space endowed with the discrete topology.
- \( A_i \) is a Borel action space for player \( i \in \mathcal{N} \). The set \( A_i(x) \) is a non-empty compact subset of \( A_i, x \in X, i \in \mathcal{N}. \) We put

\[
A := \prod_{i=1}^{n} A_i \quad \text{and} \quad A(x) := \prod_{i=1}^{n} A_i(x).
\]

Note that the set

\[
\mathbb{K}_i = \{ (x, a_i) : x \in X, a_i \in A_i(x) \}
\]
of feasible state–action pairs for player $i \in \mathcal{N}$ is a closed subset of $X \times A_i$. Similarly, the set

$$\mathbb{K} = \{(x, a) : x \in X, a = (a_1, \ldots, a_n) \in A(x)\}$$

of feasible state–action vectors is a closed subset of $X \times A$.

- Let $L = \{1, \ldots, l\}$ and $L_0 = L \cup \{0\}$. The real-valued functions $c^\ell_i : \mathbb{K} \to \mathbb{R}$, $i \in \mathcal{N}$, $\ell \in L_0$ are measurable. Here, $c^0_i$ denotes cost-per-stage function for player $i \in \mathcal{N}$, and for each $\ell \in L$, $c^\ell_i$ is a function used in the definition of the $\ell$-th constraint for this player.

- $p(y|x, a)$ is the transition probability from $x$ to $y \in X$, when the players choose a profile $a = (a_1, a_2, \ldots, a_n)$ of actions in $A(x)$.

- $\eta$ is the initial state distribution.

- $\alpha \in (0, 1)$ is the discount factor.

- $\kappa^\ell_i$ are constants, $i \in \mathcal{N}$, $\ell \in L$.

Let $\mathbb{N} = \{1, 2, \ldots\}$. Define $H_1 = X$ and $H_t^{t+1} = \mathbb{K} \times H^t$ for $t \in \mathbb{N}$. An element $h^t = (x^1, a^1, \ldots, x^t)$ of $H^t$ represents a history of the game up to the $t$-th stage, where $a^k = (a^k_1, \ldots, a^k_h)$ is the profile of actions chosen by the players in the state $x^k$ on the $k$-th stage of the game ($k \in \mathbb{N}$). Clearly, $h^1 = x^1$.

Strategies for the players are defined in the usual manner. A strategy for player $i \in \mathcal{N}$ is a sequence $\pi_i = (\pi^t_i)_{t \in \mathbb{N}}$, where each $\pi^t_i$ is a transition probability from $H^t$ to $A_i$ such that $\pi^t_i(A_i(x^t)|h^t) = 1$ for any history $h^t \in H^t$, $t \in \mathbb{N}$. By $\Pi_i$ we denote the set of all strategies for player $i$. Let $\Phi_i$ be the set of all transition probabilities $\varphi_i$ from $X$ to $A_i$ such that $\varphi_i(A_i(x)|x) = 1$ for all $x \in X$. A stationary strategy for player $i$ is a constant sequence $(\varphi_i^t)$, where $\varphi_i^t = \varphi_i$ for all $t \in \mathbb{N}$ and some $\varphi_i \in \Phi_i$. Furthermore, we shall identify a stationary strategy for player $i$ with the constant element $\varphi_i$ of the sequence. Thus, the set of all stationary strategies of player $i$ is $\Phi_i$. We define

$$\Pi = \prod_{i=1}^n \Pi_i \quad \text{and} \quad \Phi = \prod_{i=1}^n \Phi_i.$$ 

Hence, $\Pi$ ($\Phi$) is the set of all (stationary) multi-strategies of the players.

Let $H^\infty = \mathbb{K} \times \mathbb{K} \times \cdots$ be the space of all infinite histories of the game endowed with the product $\sigma$-algebra. For any multi-strategy $\pi \in \Pi$, a probability measure $\mathbb{P}_\eta^\pi$ and a stochastic process $(x^t, a^t)_{t \in \mathbb{N}}$ are defined on $H^\infty$ in a canonical way, see the Ionescu-Tulcea theorem, e.g., Proposition 7.28 in [9]. The measure $\mathbb{P}_\eta^\pi$ is induced by $\pi$, the transition probability $p$ and the initial distribution $\eta$. The expectation operator with respect to $\mathbb{P}_\eta^\pi$ is denoted by $\mathbb{E}_\eta^\pi$.

Let $\pi \in \Pi$ be any multi-strategy. For each $i \in \mathcal{N}$ and $\ell \in L_0$, the discounted cost functionals (see [9, 11]) are defined as follows:

$$J^\ell_i(\pi) = (1 - \alpha) \mathbb{E}_\eta^\pi \left[ \sum_{t=1}^\infty \alpha^{t-1} c^\ell_i(x^t, a^t) \right]. \quad (2.1)$$
Below we provide conditions that guarantee that the functionals are well-defined. We assume that \( J_i^\ell(\pi) \) is the expected discounted cost of player \( i \in \mathcal{N} \), who wishes to minimise it over \( \pi_i \in \Pi_i \) in such a way that the following constraints are satisfied

\[
J_i^\ell(\pi) \leq \kappa_i^\ell \quad \text{for all } \ell \in L. \tag{2.2}
\]

A multi-strategy \( \pi \) is feasible, if (2.2) holds for each \( i \in \mathcal{N}, \ell \in L \). We denote by \( \Delta \) the set of all feasible multi-strategies in CSG.

As usual, for any \( \pi \in \Pi \), we denote by \( \pi_{-i} \) the multi-strategy of all players but player \( i \). More precisely, \( \pi_{-1} = (\pi_2, \ldots, \pi_n) \), \( \pi_{-n} = (\pi_1, \ldots, \pi_{n-1}) \), and for \( i \in \mathcal{N} \setminus \{1, n\} \),

\[
\pi_{-i} = (\pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n).
\]

We identify \( [\pi_{-i}, \pi_i] \) with \( \pi \). For each \( \pi \in \Pi \), we define the set of feasible strategies for player \( i \) with \( \pi_{-i} \) as

\[
\Delta_i(\pi_{-i}) = \{\pi_i \in \Pi_i : J_i^\ell(\pi) = J_i^\ell([\pi_{-i}, \pi_i]) \leq \kappa_i^\ell \quad \text{for all } \ell \in L\}. \nonumber
\]

Hence, \( \pi \in \Delta \) if and only if \( \pi_i \in \Delta_i(\pi_{-i}) \) for all \( i \in \mathcal{N} \).

Let \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \in \Pi \) and \( \sigma_i \in \Pi_i \). By \( [\pi_{-i}, \sigma_i] \) we denote the multi-strategy, where player \( i \) uses \( \sigma_i \) and every player \( j \neq i \) uses \( \pi_j \).

**Definition 2.1** A multi-strategy \( \pi \in \Pi \) is a Nash equilibrium in CSG, if \( \pi_i \in \Delta_i(\pi_{-i}) \) and

\[
J_i^0(\pi) = \inf_{\sigma_i \in \Delta_i(\pi_{-i})} J_i^0([\pi_{-i}, \sigma_i])
\]

for every player \( i \in \mathcal{N} \).

**Assumption A** (i) The function \( p(y|x, \cdot) \) is continuous on \( A(x) \) for all \( x, y \in X \).

(ii) The function \( c_i^\ell(x, \cdot) \) is continuous on \( A(x) \) for all \( x \in X, i \in \mathcal{N} \) and \( \ell \in L_0 \).

**Assumption B** (i) There exists a function \( w : X \to [1, \infty) \) such that \( |c_i^\ell(x, a)| \leq w(x) \) for all \((x, a) \in \mathcal{K}, \ell \in \bar{L}_0 \) and \( i \in \mathcal{N} \).

(ii) It holds

\[
\lim_{k \to \infty} \sup_{\pi \in \Pi} (1 - \alpha) \mathbb{E}_\pi^\eta \left[ \sum_{t=k}^{\infty} \alpha^{t-1} w(x^t) \right] = 0, \tag{2.3}
\]

and, for each \( t \in \mathbb{N} \),

\[
\lim_{k \to \infty} \sup_{\pi \in \Pi} \mathbb{E}_\pi^\eta \left[ w(x^t) 1_{[w(x^t) \geq k]} \right] = 0, \tag{2.4}
\]

where \( 1_K \) denotes the indicator function of the set \( K \).
A weaker version of Assumption B(i) was used in [20] to study constrained Markov decision processes on a Borel state space.

**Lemma 2.2** Assumption B implies that all expectations in (2.1) are finite.

**Proof** Fix an $\epsilon > 0$. From (2.3), there exists $N_0 \in \mathbb{N}$ such that for all $k > N_0$, we have

$$|J_\ell^i(\pi)| \leq (1 - \alpha)E_\pi^\pi \left[ \sum_{t=1}^{\infty} \alpha^{t-1} w(x^t) \right] = (1 - \alpha)E_\pi^\pi \left[ \sum_{t=1}^{k} \alpha^{t-1} w(x^t) \right]$$

$$+ (1 - \alpha)E_\pi^\pi \left[ \sum_{t=k+1}^{\infty} \alpha^{t-1} w(x^t) \right] \leq (1 - \alpha)E_\pi^\pi \left[ \sum_{t=1}^{k} \alpha^{t-1} w(x^t) \right] + \epsilon. \quad (2.5)$$

for all $\pi \in \Pi$ and $i \in \mathcal{N}$ and $\ell \in L_0$. Now let us consider the first term on the right-hand side in (2.5). By (2.4), we may choose $m \in \mathbb{N}$ such that for each $t = 1, \ldots, k$ we obtain the bound

$$(1 - \alpha)E_\pi^\pi \left[ \sum_{t=1}^{k} \alpha^{t-1} w(x^t) \right] \leq (1 - \alpha) \left[ \sum_{t=1}^{k} \alpha^{t-1} m \right]$$

$$+(1 - \alpha)E_\pi^\pi \left[ \sum_{t=1}^{k} \alpha^{t-1} w(x^t) 1_{[w(x^t) \geq m]} \right]$$

$$< m + (1 - \alpha) \left[ \sum_{t=1}^{k} \alpha^{t-1} \epsilon \right] < m + \epsilon.$$

Summing up, we get that

$$|J_\ell^i(\pi)| < m + 2\epsilon < \infty,$$

which terminates the proof. \hfill \Box

Similar assumptions to study non-stationary Markov decision processes with unbounded payoffs were used and thoroughly discussed in [16]. Further details and comments, the reader can find in Sect. 5.

The next assumption is called in the literature the Slater condition, see [4], Assumption 3.3(c) in [7] and Assumption 2 in [38].

**Assumption C** For each stationary multi-strategy $\varphi \in \Phi$ and for each player $i \in \mathcal{N}$, there exists $\pi_i \in \Pi_i$ such that

$$J_\ell^i ([\varphi_{-i} \cdot \pi_i]) < \kappa_\ell^i \text{ for all } \ell \in L.$$

We are ready to state our main result.

**Theorem 2.3** Assume A, B and C. Then, the CSG possesses a Nash equilibrium in the set $\Phi$.
The proof of this result is given in Sect. 4.
Below we describe a sufficient condition for assumption \( B \) to hold.

**Assumption \( W \)**

(i) There exists a function \( w : X \to [1, \infty) \) such that \( B(i) \) holds and

\[
\sum_{y \in X} w(y) p(y|x, a) \leq \delta w(x) \quad \text{for all } (x, a) \in \mathbb{K},
\]

and for some constant \( \delta \geq 1 \) satisfying \( \delta \alpha < 1 \).

(ii) The function \( \sum_{y \in X} w(y) p(y|x, \cdot) \) is continuous on \( A(x) \) for each \( x \in X \).

(iii) \( \sum_{y \in X} w(y) \eta(y) < \infty \).

The above conditions were first introduced in [37] to deal with unbounded payoffs in Markov decision processes. They gained recognition and were broadly applied to several models, see for instance [23, 25].

By Lemma 9 in [20], it follows that, if there exists a function \( w \) that satisfies \( W \), then \( B \) holds as well. However, Example 4 in [20] warns that the violation of \( W(ii) \) entails that the uniform integrability condition in (2.4) fails. Moreover, this example (case II on p. 10 in [20]) also illustrates that, if \( B \) is satisfied with any value of a discount factor, then \( W(i) \) holds only for \( \alpha < 4/5 \).

From Theorem 2.3 we can deduce two conclusions.

**Corollary 2.4** Assume \( A, W \) and \( C \). Then, the CSG possesses a Nash equilibrium in the set \( \Phi \).

**Corollary 2.5** Assume \( A, C \) and that every function \( c^e_i \) is bounded for \( i \in \mathbb{N}, \ell \in L_0 \). Then, the CSG possesses a Nash equilibrium in the set \( \Phi \).

**Remark 2.6** Discounted constrained stochastic games with countable state spaces and unbounded functions \( c^e_i \) were studied by Zhang et al. [38]. However, the assumptions imposed in [38] are stronger than ours. Indeed, they require that there exists an unbounded function \( w \) that satisfies \( B(i) \) and such that \( w^2 \) is integrable with respect to the initial state distribution \( \eta \) and with respect to the transition probability. More precisely, Assumption 1(e) in [38] says that there exists a constant \( \beta \geq 1 \) such that

\[
\alpha \beta^2 < 1 \quad \text{and} \quad \sum_{y \in X} w^2(y) p(y|x, a) \leq \beta^2 w^2(x) \quad \text{for all } (x, a) \in \mathbb{K}. \tag{2.6}
\]

In addition, \( w \) is a moment function, i.e., there exists an increasing sequence of finite sets \( (Z_m)_{m \in \mathbb{N}} \) such that \( \bigcup_{m \in \mathbb{N}} Z_m = X \) and \( \lim_{m \to \infty} \inf_{x \notin Z_m} w(x) = \infty \). This condition enforces that the function \( w \) must be necessarily unbounded, if the game is played on the infinite countable state space. However, the consideration of an unbounded function \( w \) implies that one has to restrict the classes of transition probabilities and initial distributions (e.g., to exclude those for which the integral of \( w \) with respect to \( \eta \) is infinite).
Note that (2.6) can be written in the form
\[ \sum_{y \in X} \hat{w}(y) p(y|x, a) \leq \delta \hat{w}(x) \quad \text{for all } (x, a) \in \mathbb{K}, \]
where \( \hat{w} = w^2 \), \( \delta = \beta^2 \), \( \delta \alpha < 1 \). This is exactly condition \( W(i) \). But then \( B(i) \) (assumed in \( W(i) \)) holds as well, that is,
\[ |c^l_i(x, a)| \leq \sqrt{\hat{w}(x)} \quad \text{for all } (x, a) \in \mathbb{K}, \ell \in L_0, i \in \mathcal{N}. \]  
(2.7)

Thus, the class of games satisfying our assumption \( B \) is essentially larger than the class studied in [38]. For example, condition (2.7) excludes linear functions \( c^l_i \), when \( \hat{w} \) is linear as it happens in many Markov decision processes, for examples consult with [20, 25]. Therefore, Corollary 2.4 extends Theorem 1 in [38].

**Remark 2.7** Discounted constrained stochastic games with finite state and action spaces were first studied by Altman and Shwartz [4]. An extension to games with compact metric action spaces was given by Alvarez-Mena and Hernández-Lerma [7]. The existence of stationary Nash equilibrium in the finite state space framework with constraints is established by a fixed point argument, but the approach from the unconstrained case as in [17, 21, 36] cannot be applied. The main difficulty is to determine a domain for the best-response correspondence, sometimes called the Nash correspondence. Unlike the standard case [17, 21, 36], the Cartesian product of the sets of stationary strategies is not appropriate in the constrained setting. The authors consider the Cartesian product of the sets \( \text{Pr}(\mathbb{K}_i) \) of all probability measures on \( \mathbb{K}_i (i \in \mathcal{N}) \) and an auxiliary one-shot game. The sets \( \text{Pr}(\mathbb{K}_i) \) are actually too large and, therefore, some functional equations, characterising so-called “occupation measures” on \( \mathbb{K}_i \), are requested in the definition of the Nash correspondence. These equations play a fundamental role in discounted constrained decision processes and games [7, 12, 29]. In the finite state space case, the sets \( \text{Pr}(\mathbb{K}_i) \) are compact in the weak topology and obviously they are convex. Consequently, the Kakutani–Fan–Glicksberg fixed point theorem [1] can be applied. When the state space \( X \) is countable and infinite, then the Cartesian product of the spaces \( \text{Pr}(\mathbb{K}_i) \) cannot be used as a domain for the Nash correspondence, because \( \text{Pr}(\mathbb{K}_i) \) is non-compact in the weak topology. Therefore, to study discounted \( CSG \)s with an infinite countable state space \( X \), Alvarez-Mena and Hernández-Lerma [7] and Zhang et al. [38] use an approximation of the original game by games with finite state spaces. The proof in [7] strongly exploits Assumption 3.4 (see [7,p. 267]) that entails the convergence of discounted costs in the approximating models to the discounted cost in the original model (consult with the proof Theorem 3.6(c) in [7]). Using our notation, Assumption 3.4 from [7] sounds as follows: there exists an increasing sequence of finite sets \((Z_m)_{m \in \mathbb{N}}\) such that \( \bigcup_{m \in \mathbb{N}} Z_m = X \) and

\[ \lim_{m \to \infty} \max_{x \in Z_m} \max_{a \in A(x)} p(X \setminus Z_{m+1} | x, a) = 0. \]  
(2.8)

The cost and constraint functions in [7], however, are bounded and additionally, the condition in (2.8) looks restrictive. Zhang et al. [38] also approximate the original
discounted CSG by appropriately defined auxiliary finite state space games and show that their stationary Nash equilibria converge to a Nash equilibrium in the original game. As mentioned in Remark 2.6, they allow the functions $c_i^f$ to be unbounded and drop Assumption 3.4 from [7]. Their proof, on the other hand, is inspired by an estimation technique developed in [13] and Chapter 16 in [3].

**Remark 2.8** The proof of Theorem 2.3 proceeds along different lines than those in [7, 38]. First of all, we do not apply the finite state space approximations. Our proof is more direct. The idea is based on studying first auxiliary stochastic games with bounded cost functions and with positive initial state distributions. An important new feature of our approach is to define the Nash correspondence using some compact convex subsets of the spaces $\Pr(K_i)$ being projections of the set $Y$ of occupation measures induced by correlated strategies of the players (for any state $x \in X$, the action profile is selected according to a probability measure on $A(x)$). For a formal definition, the reader is referred to Subsection 3.1. The set $Y$ is a compact convex subset of the non-compact set of all probability measures on $\mathcal{K}$. This idea combined with basic results from convex analysis in Markov decision processes [12, 29] establishes in Sect. 3 the existence of stationary Nash equilibria in the auxiliary discounted CSGs. In Sect. 4, we show how to approximate the original game by the aforementioned auxiliary games with bounded costs. We prove that there is a sequence of Nash equilibria in auxiliary games converging to a Nash equilibrium in the original discounted CSG.

**Remark 2.9** Corollary 2.5 states that there exists a stationary Nash equilibrium in games with bounded cost and constraint functions. The same result is formulated as Corollary 1 in [38]. However, the proof of Corollary 1 in [38] is incorrect, because in this proof they erroneously claim that every transition probability satisfies condition (2.8) (formulated as Assumption 3.4 in [7]) with $Z_m = \{1, 2, \ldots, m\}$. This is not true. For example, (2.8) fails to hold when $p(x + 2|x, a) = 1$ for all $x \in X$ and $a \in A(x)$. Claiming that (2.8) holds in general, Zhang et al. [38] conclude their Corollary 1 from Theorem 3.6(c) in [7].

### 3 Stochastic Games with Bounded Costs and Positive State Distribution

In this section, we state an auxiliary result (Proposition 3.15) using basic theorems on occupation and strategic measures obtained by Borkar [12] and Schäl [32, 33].

#### 3.1 Occupation Measures and Their Important Properties

Let $Y$ be a metric space with the Borel $\sigma$-algebra $\mathcal{B}(Y)$. Let $\Pr(Y)$ ($\mathcal{M}(Y)$) be the set of all probability (finite signed) measures on $Y$ and $C(Y)$ be the space of all bounded uniformly continuous functions on $Y$. A sequence $(\nu^k)_{k \in \mathbb{N}}$ in $\mathcal{M}(Y)$ is said to converge weakly to $\nu \in \mathcal{M}(Y)$ if

$$\int_Y f d\nu^k \to \int_Y f d\nu \quad \text{for all} \quad f \in C(Y).$$
If $Y$ is compact metric, then so is $\Pr(Y)$ equipped with the topology of weak convergence, see Theorem 6.4 in [28]. If $Y$ is a Borel space, then we may equip $\mathbb{M}(Y)$ with the metric

$$d(\mu, \nu) = \sum_{k=1}^{\infty} \frac{1}{2^k} \left| \int_Y f_k d\mu - \int_Y f_k d\nu \right|, \quad \mu, \nu \in \mathbb{M}(Y).$$

Here, $\mathcal{F} = \{ f_k \}_{k \in \mathbb{N}}$ is a countable family of functions, which is dense in the unit ball in $C(Y)$ and such that for each different points $y, y' \in Y$, there exists $f_k \in \mathcal{F}$ such that $f_k(y) \neq f_k(y')$ (see p. 47 in [28]). It is obvious that the topology induced by this metric is equivalent to the weak topology in $\mathbb{M}(Y)$. Thus, $\mathbb{M}(Y)$ is a Hausdorff locally convex space.

Since $A_i(x)$ is compact metric for each $x \in X$, so is $\Pr(A_i(x))$ in the weak topology. The set $\Phi_i$ can be identified with the product space $\prod_{x \in X} \Pr(A_i(x))$. By the Tychonoff theorem the spaces

$$\Phi_i = \prod_{x \in X} \Pr(A_i(x)) \quad \text{and} \quad \Phi = \prod_{i=1}^{n} \Phi_i$$

are compact, when endowed with the product topologies. Moreover, these spaces are metrisable.

A sequence $(\varphi^k)_{k \in \mathbb{N}}$ in $\Phi_i$ converges to $\varphi_i \in \Phi_i$, if the sequence $(\varphi^k(\cdot|x))_{k \in \mathbb{N}}$ in $\Pr(A_i(x))$ converges weakly to $\varphi_i(\cdot|x)$ for each $x \in X$. A sequence $(\varphi^k)_{k \in \mathbb{N}}$ in $\Phi$ converges to $\varphi \in \Phi$, if $(\varphi^k)_{k \in \mathbb{N}}$ converges to $\varphi_i$ in $\Phi_i$ for every $i \in N$.

For some technical reasons we also introduce $\hat{\Pi}$ as the set of all correlated strategies $\pi = (\pi^t)_{t \in \mathbb{N}}$ of the players. Here, $\pi^t$ is a transition probability from $H^t$ to $A$ such that $\pi^t(A(x^t)|h^t) = 1$ for any history $h^t \in H^t$. Using a correlated strategy the players act like one decision maker in the Markov decision process with the action spaces $A(x)$, $x \in X$.

Let $\mathcal{M}$ be the set of probability occupation measures on $\mathbb{K}$ induced by all correlated strategies $\pi \in \hat{\Pi}$ and the initial distribution $\eta$, i.e., $\rho \in \mathcal{M}$ is defined as follows

$$\rho(K) = (1 - \alpha) \mathbb{E}_\eta^\pi \left[ \sum_{t=1}^{\infty} \alpha^{t-1} 1_K(x^t, a^t) \right] \quad \text{for} \quad K \in \mathcal{B}(\mathbb{K}). \quad (3.1)$$

The expectation operator $\mathbb{E}_\eta^\pi$ is taken with respect to the unique probability measure $\mathbb{P}_\eta^\pi$ on $H^\infty$, called a strategic measure.

For each $x \in X$, the integral of any bounded measurable function $f : A(x) \to \mathbb{R}$ with respect to $\rho([x] \times \cdot)$ is denoted by $\int_{A(x)} f(a) \rho([x] \times da)$.

From (3.1), it follows that for any bounded measurable function $c : \mathbb{K} \to \mathbb{R}$,

$$\int_{\mathbb{K}} c \rho = \sum_{x \in X} \int_{A(x)} c(x, a) \rho([x] \times da) = (1 - \alpha) \mathbb{E}_\eta^\pi \left[ \sum_{t=1}^{\infty} \alpha^{t-1} c(x^t, a^t) \right]. \quad (3.2)$$
Hence, for any \( i \in \mathcal{N}, \ell \in L_0 \) and \( \pi \in \tilde{\Pi} \),
\[
\int_{\mathbb{K}} c_i^\ell \, d \rho = \sum_{x \in X} \int_{A_i(x)} c_i^\ell (x, a) \rho(\{x\} \times da) = J_i^\ell (\pi).
\]

**Lemma 3.1** Under assumption \( A(i) \), \( \mathcal{M} \) is convex and compact in \( \text{Pr}(\mathbb{K}) \) equipped with the weak topology.

**Proof** The set \( \{\mathbb{P}_\eta^{\pi} : \pi \in \tilde{\Pi}\} \) of all strategic measures induced by all correlated strategies is weakly compact in \( \text{Pr} (H^\infty) \). It is also convex. These facts are well-known in the literature and together with (3.2) imply the lemma. For a detailed discussion see: [29], Sects. 7.1 and 7.4 in [33], Theorem 5.6 in [32], or Theorem 3.1 in [12]. \( \square \)

Let \( \mathcal{M}_i \) be the set of all probability measures on \( \mathbb{K}_i \) defined as follows. A measure \( \mu \) belongs to \( \mathcal{M}_i \), if there exists a probability measure \( \rho \in \mathcal{M} \) such that, for each \( x \in X \), \( \mu (\{x\} \times \cdot) \) is the projection of \( \rho (\{x\} \times \cdot) \) on \( A_i(x) \). More detailed, if \( x \in X \), \( B \in B(A_i(x)) \) and \( pr \) is the projection from \( A(x) \) on \( A_i(x) \), then
\[
\mu (\{x\} \times B) = \rho (\{x\} \times pr^{-1}(B)).
\]
If \( f \) is a bounded measurable real-valued function on \( A_i(x) \), then \( \int_{A_i(x)} f(a_i) \mu (\{x\} \times da_i) \) means the integral of \( f \) with respect to the probability measure \( \mu (\{x\} \times \cdot) \). Moreover, every bounded measurable function \( f \) on \( \mathbb{K}_i \) can be recognised as a function on \( \mathbb{K} \). Then,
\[
\int_{\mathbb{K}} f \, d \rho = \int_{\mathbb{K}_i} f \, d \mu = \sum_{x \in X} \int_{A_i(x)} f(x, a_i) \mu (\{x\} \times da_i).
\]
(3.3)

Hence, if \( (\rho^k)_{k \in \mathbb{N}} \) is a sequence of measures converging weakly to some \( \rho^0 \) in \( \mathcal{M} \) and \( \mu^k, \mu^0 \) are projections of \( \rho^k \) and \( \rho^0 \), respectively, defined as above, then by (3.2) and (3.3), \( (\mu^k) \) converges weakly to \( \mu^0 \) in \( \mathcal{M}_i \). This fact and convexity of the set \( \mathcal{M} \) imply the following result.

**Lemma 3.2** If \( A(i) \) holds, then \( \mathcal{M}_i \) is a convex and compact subset in \( \text{Pr}(\mathbb{K}_i) \) equipped with the weak topology.

**Remark 3.3** Introducing the sets \( \tilde{\Pi} \) and \( \mathcal{M} \) is crucial in our proof. It enables us to use the compact and convex sets \( \mathcal{M}_i \) in our definition of the best-response correspondence \( S \) given below. In contrast to the finite state space case [4, 7], the sets \( \text{Pr}(\mathbb{K}_i), i \in \mathcal{N} \), in games with the infinite countable state spaces cannot be utilised, because they need not be compact.

In this subsection, we add the following condition.

**Assumption D** For all \( x \in X \), \( \eta(x) > 0 \).

Let \( \mu \) denote the projection of \( \mu \in \mathcal{M}_i \) on \( X \), i.e., \( \mu(x) = \mu (\{x\} \times A_i(x)) \), \( x \in X \).
Lemma 3.4 Assume A(i) and D. If $\mu \in \mathcal{M}_i$, then $\hat{\mu}(x) > 0$ for all $x \in X$ and there exists a unique $\varphi_i \in \Phi_i$ such that

$$\mu([x] \times B) = \varphi_i(B|x)\hat{\mu}(x), \quad \text{for all } B \in \mathcal{B}(A_i(x)), \ x \in X. \quad (3.4)$$

Proof Let $\mu \in \mathcal{M}_i$ be a projection of $\rho \in \mathcal{M}$ induced by some $\pi \in \tilde{\Pi}$ according to (3.1). Then, for any $x \in X$,

$$\hat{\mu}(x) = \mu([x] \times A_i(x)) = \rho([x] \times A(x)) = (1 - \alpha) \sum_{t=1}^{\infty} \alpha^{t-1} \mathbb{E}_{\eta} (1_{\{x\} \times A(x)}(x^t, a^t))$$

$$\geq (1 - \alpha) \mathbb{E}_{\eta} (x^1 = x) = (1 - \alpha) \eta(x) > 0.$$ 

Therefore, $\varphi_i$ defined by

$$\varphi_i(B|x) := \frac{\mu([x] \times B)}{\hat{\mu}(x)} \quad \text{for all } B \in \mathcal{B}(A_i(x)), \ x \in X,$$

is the unique transition probability satisfying (3.4). \hfill \square

Remark 3.5 Lemma 3.4 and assumption D allow to omit the study of so-called equivalence classes of functions in $\Phi_i$, which are equal on the set $Z \subset X$ with $\eta(Z) = 1$. They were considered in [7]. In our case, condition D implies the uniqueness of $\varphi_i \in \Phi_i$ in the above lemma and this fact simplifies our proofs in the sequel.

Further, we shall write (3.4) in the abbreviated form

$$\mu = \hat{\mu} \varphi_i.$$

Lemma 3.6 Let $\mu^k = \hat{\mu}^k \varphi^k_i$, $k \in \mathbb{N} \cup \{0\}$. Under assumptions A(i) and D, if $\mu^k \rightarrow \mu^0$ weakly in $\mathcal{M}_i$, then, for each $x \in X$, $\hat{\mu}^k(x) \rightarrow \hat{\mu}^0(x)$ and $\varphi^k_i(\cdot|x) \rightarrow \varphi^0_i(\cdot|x)$ weakly in $\text{Pr}(A_i(x))$ as $k \rightarrow \infty$.

Proof Since $\mu^k \rightarrow \mu^0$ weakly in $\mathcal{M}_i$, it follows that $\hat{\mu}^k(x) \rightarrow \hat{\mu}^0(x)$ for every $x \in X$. Therefore, by Lemma 3.4, for every $x \in X$,

$$\varphi^k_i(\cdot|x) = \frac{\mu^k([x] \times \cdot)}{\hat{\mu}^k(x)} \rightarrow \varphi^0_i(\cdot|x) = \frac{\mu^0([x] \times \cdot)}{\hat{\mu}^0(x)} \quad \text{in } \text{Pr}(A_i(x)),$$

which finishes the proof. \hfill \square

3.2 The Existence of Nash Equilibria in Games with Bounded Costs

In this subsection, we add the following assumption.

Assumption B’ The functions $c^\ell_i$ are bounded for all $i \in \mathcal{N}$ and $\ell \in L_0$. 

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Under $\mathbf{B'}$ all functionals are bounded: $|J_i^\ell(\pi)| \leq \hat{c}$ for some $\hat{c} > 0$ and for all $i \in \mathcal{N}$, $\ell \in L_0$ and $\pi \in \Pi$. Moreover, $\mathbf{B(i)}$ and $\mathbf{B(ii)}$ are trivially satisfied by taking $w(x) = \hat{c}$ for all $x \in X$.

Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in \Phi$. We denote by $[\varphi_{-i}(x), a_i]$ the profile of actions used in state $x \in X$, where player $i$ chooses $a_i \in A_i(x)$ and every player $j \neq i$ uses $\varphi_j(\cdot|x)$. Then, for $i \in \mathcal{N}$, $\ell \in L_0$, $x \in X$,

$$c_i^\ell(x, [\varphi_{-i}(x), a_i]):= \int_{A_i(x)} c_i^\ell(x, a_1, \ldots, a_n)\psi_1(da_1|x)\ldots\psi_n(da_n|x)$$

and

$$p(y|x, [\varphi_{-i}(x), a_i]):= \int_{A_i(x)} p(y|x, a_1, \ldots, a_n)\psi_1(da_1|x)\ldots\psi_n(da_n|x),$$

where $\psi_i([a_i]|x) = 1$ and $\psi_j = \varphi_j$ for all $j \neq i$.

**Definition 3.7 (Optimisation problem)** Let $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \in \Phi$. For each player $i \in \mathcal{N}$ consider the following constrained optimisation problem $COP(\varphi_{-i})$:

Minimise

$$\sum_{x \in X} \int_{A_i(x)} c_i^0(x, [\varphi_{-i}(x), a_i])\mu(\{x\} \times da_i)$$

subject to $\mu \in \Pr(\mathbb{K}_i)$ and

$$\sum_{x \in X} \int_{A_i(x)} c_i^\ell(x, [\varphi_{-i}(x), a_i])\mu(\{x\} \times da_i) \leq \kappa_i^\ell$$

for all $\ell \in L$, (3.6)

and

$$\tilde{\mu}(x) = \mu(\{x\} \times A_i(x)) = (1 - \alpha)\eta(x) + \alpha \sum_{z \in X} \int_{A_i(z)} p(x|z, [\varphi_{-i}(z), a_i])\mu(\{z\} \times da_i)$$

for all $x \in X$.

We denote the set of all solutions to the problem $COP(\varphi_{-i})$ by $\mathcal{O}_i(\varphi_{-i})$.

**Remark 3.8** In the constrained optimisation problem $COP(\varphi_{-i})$ player $i$ acts as the decision maker in a constrained discounted Markov decision model. The transition probability, cost function and constraint functions are as follows: $p(x|z, [\varphi_{-i}(z), a_i])$, $c_i^0(x, [\varphi_{-i}(x), a_i])$ and $c_i^\ell(x, [\varphi_{-i}(x), a_i])$, $\ell \in L$, respectively. Here, $a_i \in A_i(x)$ and $x$, $z \in X$. Equation (3.7) implies that $\mu$ is an occupation measure defined for this Markov decision process. For details, see [12], Lemma 25 in [29] or Remark 6.3.1 in [23]. Assumption C assures that the set of all occupation measures in $COP(\varphi_{-i})$ satisfying (3.6) is non-empty.
Lemma 3.9 If A, B' and C hold, then problem $COP(\varphi_{-i})$ has a solution and the set $O_i(\varphi_{-i})$ is compact and convex.

Proof The set of all occupation measures, i.e., the measures satisfying (3.7) is convex and compact (see for instance Theorem 3.1 in [12]). Moreover, since $\mu \to \int_{\mathbb{R}_+} c^\ell(x, [\varphi_{-i}(x), a_i])\mu([x] \times da_i)$ is continuous on $Pr(\mathbb{K}_i)$ for all $\ell \in L_0$, it follows that the subset of occupation measures for which (3.6) holds is closed, and consequently compact. By assumption C it is non-empty. Hence, there exists an occupation measure that minimises (3.5) subject to (3.6) and (3.7). Thus, $O_i(\varphi_{-i})$ is non-empty and compact. The convexity of $O_i(\varphi_{-i})$ is obvious. □

Lemma 3.10 Assume A, B', C and D. Let $\varphi \in \Phi$. Then, $O_i(\varphi_{-i}) \subset M_i$ for every player $i \in \mathcal{N}$.

Proof If $\mu \in O_i(\varphi_{-i})$, then, by Lemma 3.4, there exists $\phi_i \in \Phi_i$ such that $\mu = \hat{\mu}\phi_i$ where $\hat{\mu}$ is the marginal of $\mu$ on $X$. Furthermore, $\mu([x] \times \cdot)$ is the projection of $\rho \in M$ defined as follows

$$
\rho([x] \times da) := \psi_1(da_1|x) \cdots \psi_n(da_n|x)\hat{\mu}(x),
$$

where $\psi_i = \phi_i$ and $\psi_j = \varphi_j$ for all $j \neq i$. To see that $\rho$ is indeed an occupation measure from the set $M$, it is sufficient to show (3.7) for $\rho$. From the Fubini theorem and the fact that (3.7) holds for $\mu$, we have

$$
\hat{\rho}(x) = \hat{\mu}(x) = (1 - \alpha)\eta(x) + \alpha \sum_{z \in X} \int_{A_i(x)} p(x|z, [\varphi_{-i}(z), a_i])\mu([z] \times da_i)
\quad = (1 - \alpha)\eta(x) + \alpha \sum_{z \in X} \int_{A_i(z)} \cdots \int_{A_1(z)} p(x|z, a_1, \ldots, a_n)\psi_1(da_1|z) \cdots \psi_n(da_n|z)\hat{\mu}(z)
\quad = (1 - \alpha)\eta(x) + \alpha \sum_{z \in X} \int_{A(z)} p(x|z, a)\rho([z] \times da).
$$

This finishes the proof. □

The next result is valid under assumption B and will be used in the Appendix and in the proofs of Theorem 2.3 and Proposition 3.15.

Lemma 3.11 Assume A(i) and B. Let $\varphi \in \Phi$ and $\pi_i \in \Pi_i$ for $i \in \mathcal{N}$. Then, there exists a stationary strategy $\sigma_i \in \Phi_i$ such that $J^\ell_i([\varphi_{-i}, \pi_i]) = J^\ell_i([\varphi_{-i}, \sigma_i])$ for all $\ell \in L_0$.

Proof Let $\mu_{\pi_i}$ be the occupation measure defined as follows

$$
\mu_{\pi_i}(K) = (1 - \alpha)\mathbb{P}_{\pi_i}^{\pi_i} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} 1_K(x^t, a^t_i) \right] \text{ for each } K \in B(\mathbb{K}_i).
$$
Here, $\mathbb{E}^{\pi_i}_\eta$ denotes the expectation operator taken with respect to the unique probability measure defined on the history space of the Markov decision process governed by the transition probability $p(x | z, [\varphi_{-i}(z), a_i])$, the initial distribution $\eta$, and a strategy $\pi_i$ of the decision maker (player $i$). (Note that $\mathbb{E}^{\pi_i}_\eta = \mathbb{E}^{[\varphi_{-i}, \pi_i]}_\eta$.) From Proposition D.8 in [23], it follows that there exists a strategy $\sigma_i \in \Phi_i$ such that $\mu_{\pi_i} = \widehat{\mu}_{\pi_i} \sigma_i$, where $\widehat{\mu}_{\pi_i}$ is the projection of $\mu_{\pi_i}$ on $X$. Moreover, from Lemma 3.1 in [12], it follows that $\mu_{\pi_i} = \mu_{\sigma_i}$, where $\mu_{\sigma_i}$ is an occupation measure defined as above with $\mathbb{E}^{\pi_i}_\eta$ replaced by $\mathbb{E}^{\mu_{\sigma_i}}_\eta$. Therefore, using Lemma 2.2 we may write

$$J_i^\ell ([\varphi_{-i}, \pi_i]) = \sum_{x \in X} \int_{A_i(x)} c_i^\ell (x, [\varphi_{-i}(x), a_i]) \mu_{\pi_i} (\{ x \} \times da_i)$$

$$= \sum_{x \in X} \int_{A_i(x)} c_i^\ell (x, [\varphi_{-i}(x), a_i]) \mu_{\pi_i} (\{ x \} \times da_i) = J_i^\ell ([\varphi_{-i}, \pi_i])$$

for all $\ell \in L_0$.

From now on, we shall denote an element of $O_i (\varphi_{-i})$ by $\mu_i$. When $\mu_i \in O_i (\varphi_{-i})$, we take $\phi_i \in \Phi_i$ such that $\mu_i = \widehat{\mu}_i \phi_i$. Then, $\phi_i$ is an optimal stationary strategy for player $i$ in the constrained Markov decision process associated with $C O P (\varphi_{-i})$.

**Definition 3.12** Under assumption D, define the correspondence $S : \prod_{i=1}^n M_i \rightarrow \prod_{i=1}^n M_i$ by

$$S(\mu_1, \mu_2, \ldots, \mu_n) = \prod_{i=1}^n O_i (\varphi_{-i}),$$

where $\varphi_j \in \Phi_j$ is the unique strategy for player $j \in N$ such that $\mu_j = \widehat{\mu}_j \varphi_j$.

By Lemma 3.10 the correspondence $S$ is well-defined. We equip $\prod_{i=1}^n M_i$ with the product topology.

The next result was proved in Lemma 2.1 in [17].

**Lemma 3.13** Let A and B’ hold. The function $J_i^\ell (\cdot)$ is continuous on $\Phi$ for every $\ell \in L_0$ and $i \in N$.

**Lemma 3.14** Assume A, B’, C and D. The correspondence $S$ is non-empty compact convex-valued and is upper semicontinuous.

**Proof** Since the spaces $M_i$ are compact, to show the upper semicontinuity of the correspondence $S$, it is enough to prove that $S$ has a closed graph. The other properties follow from Lemma 3.9. Assume that $\mu_k \rightarrow \mu_0$ in $\prod_{i=1}^n M_i$, where $\mu_k = (\mu_{1k}, \ldots, \mu_{nk})$ for every $k \in N \cup \{0\}$. Then, by Lemma 3.4, there exists a unique strategy $\varphi_{-i}^k \in \Phi_i$ such that $\mu_{ik} = \widehat{\mu}_{ik} \varphi_{-i}^k$ for every $i \in N$ and $k \in N \cup \{0\}$. Let $\nu_{ik} \in O_i ([\varphi_{-i}^k], k \in N, i \in N$. Suppose that $\nu_{ik} \rightarrow \nu_{ik}^0$ weakly for every $i \in N$. By Lemma 3.4, for any $i \in N$ and $k \in N \cup \{0\}$, there exists a unique strategy $\phi_i^k \in \Phi_i$ such that $\nu_{ik} = \widehat{\nu}_{ik} \phi_i^k$, where $\widehat{\nu}_{ik}^k$ is the marginal of $\nu_{ik}$ on $X$. We have to show that

$$v_i^0 \in O_i (\varphi_{-i}^0) \quad \text{for every} \quad i \in N.$$

(3.8)
From $\nu_i^k \in \mathcal{O}_i(\varphi_{-i}^k)$, for all $k \in \mathbb{N}$, $i \in \mathcal{N}$, it follows that

$$J_i^\ell((\varphi_{-i}^k, \phi_i^k)) \leq \kappa_i^\ell \quad \text{for all} \quad \ell \in L, \ i \in \mathcal{N}.$$  

By Lemma 3.6, we know that $\varphi_i^k \to \varphi_i^0$ and $\phi_i^k \to \phi_i^0$ in $\Phi_i$ for every $i \in \mathcal{N}$. Thus, by Lemma 3.13, we conclude that

$$J_i^\ell((\varphi_{-i}^0, \phi_i^0)) \leq \kappa_i^\ell \quad \text{for all} \quad \ell \in L, \ i \in \mathcal{N}. \quad (3.9)$$

Moreover, we have

$$\lim_{k \to \infty} J_i^0((\varphi_{-i}^k, \phi_i^k)) = J_i^0((\varphi_{-i}^0, \phi_i^0)) \quad \text{for all} \quad i \in \mathcal{N}. \quad (3.10)$$

Inequality (3.9) proves that the correspondence $\varphi_{-i} \to \Delta_i(\varphi_{-i}) \cap \Phi_i$ has a closed graph. Since all spaces $\Phi_j$, $j \in \mathcal{N}$, are compact, this correspondence is upper semi-continuous. By Lemma 6.2 (see Appendix), we conclude that $\varphi_{-i} \to \Delta_i(\varphi_{-i}) \cap \Phi_i$ is continuous. From the Berge maximum theorem, see pp. 115–116 in [8], it follows that the function

$$\varphi_{-i} \to \min_{\sigma_i \in \Delta_i(\varphi_{-i}) \cap \Phi_i} J_i^0((\varphi_{-i}, \sigma_i))$$

is continuous for any $i \in \mathcal{N}$. Hence, we have

$$J_i^0 \left( \left[ \varphi_{-i}^k, \phi_i^k \right] \right) = \min_{\sigma_i \in \Delta_i(\varphi_{-i}^k) \cap \Phi_i} J_i^0 \left( \left[ \varphi_{-i}^k, \sigma_i \right] \right)$$

$$\to \min_{\sigma_i \in \Delta_i(\varphi_{-i}^0) \cap \Phi_i} J_i^0 \left( \left[ \varphi_{-i}^0, \sigma_i \right] \right) \quad \text{as} \quad k \to \infty. \quad (3.11)$$

Expressions (3.10) and (3.11) imply that

$$J_i^0 \left( \left[ \varphi_{-i}^0, \phi_i^0 \right] \right) = \min_{\sigma_i \in \Delta_i(\varphi_{-i}^0) \cap \Phi_i} J_i^0 \left( \left[ \varphi_{-i}^0, \sigma_i \right] \right) \quad \text{for all} \quad i \in \mathcal{N}.$$  

This equality can be expressed in terms of problem COP($\varphi_{-i}^0$). Hence, (3.8) follows.

\[\square\]

**Proposition 3.15** Assume A, B’, C and D. Then, the CSG possesses a Nash equilibrium in $\Phi$.

**Proof** The set $\mathcal{M}_i$ can be viewed as a compact and convex subset of the set, denoted by $\mathbb{M}_i$, of all signed finite measures on $X \times A_i$ equipped with the weak topology. $\mathbb{M}_i$ is a locally convex topological Hausdorff space. Hence, the set $\prod_{i=1}^n \mathbb{M}_i$ endowed with the product topology is also a locally convex topological Hausdorff space. From
Lemma 3.13 and the Kakutani–Fan–Glicksberg theorem (see Corollary 17.55 in [1]), it follows that there exists

\[
(\mu_1^*, \mu_2^*, \ldots, \mu_n^*) \in S(\mu_1^*, \mu_2^*, \ldots, \mu_n^*).
\]

Now using Lemma 3.4, take \( \varphi_i^* \in \Phi_i \) such that \( \mu_i^* = \hat{\mu}_i^* \varphi_i^* \) for all \( i \in \mathcal{I} \). We claim that \( \varphi^* = (\varphi_1^*, \ldots, \varphi_n^*) \) is a Nash equilibrium in the CSG. We immediately have

\[
J_i^0(\varphi^*) \leq J_i^0([\varphi_{-i}^*, \sigma_i]) \quad \text{for all} \quad \sigma_i \in \Phi_i \cap \Delta_i(\varphi_{-i}^*). \tag{3.12}
\]

Suppose that there exists some \( \pi_i \in \Pi_i \) such that

\[
J_i^0([\varphi_{-i}^*, \pi_i]) < J_i^0(\varphi^*) \quad \text{and} \quad \pi_i \in \Delta_i(\varphi_{-i}^*). \tag{3.13}
\]

Then, by Lemma 3.11, we conclude the existence of \( \sigma_i \in \Phi_i \) for which

\[
J_i^0([\varphi_{-i}^*, \sigma_i]) = J_i^0([\varphi_{-i}^*, \pi_i]) \quad \text{and} \quad \sigma_i \in \Delta_i(\varphi_{-i}^*). \tag{3.14}
\]

From (3.13) and (3.14) we get

\[
J_i^0([\varphi_{-i}^*, \sigma_i]) < J_i^0(\varphi^*),
\]

which contradicts (3.12). \( \square \)

### 4 The Proof of Theorem 2.3

In this section, we introduce an approximation of the original game by ones with truncated cost and constraint functions and slightly perturbed initial state distributions. We apply Proposition 3.15 and some auxiliary results from this section to obtain Nash equilibria in the truncated CSGs and show that their limit is a Nash equilibrium in the original game.

Let \( X = X_0 \cup X_1 \), where \( \eta(x) = 0 \) for \( x \in X_0 \) and \( \eta(x) > 0 \) for \( x \in X_1 \). Let \( \tilde{\eta} \) be a probability measure on \( X_0 \) such that \( \tilde{\eta}(x) > 0 \) for every \( x \in X_0 \). For any \( m \in \mathbb{N} \), define a perturbed initial state distribution on \( X \) as follows

\[
\eta(m)(x) := \left(1 - \frac{1}{m}\right) \eta(x) + \frac{1}{m} \tilde{\eta}(x). \tag{4.1}
\]

Clearly, \( \eta(m)(x) > 0 \) and \( \eta(m)(x) \to \eta(x) \) for every \( x \in X \) as \( m \to \infty \).

Moreover, for any \( (x, a) \in \mathbb{K}, i \in \mathcal{I}, \ell \in L_0 \) and \( m \in \mathbb{N} \), we set

\[
c_i^{\ell, m}(x, a) := \begin{cases} 
-\sqrt{m}, & \text{if } c_i^{\ell}(x, a) < -\sqrt{m} \\
\sqrt{m}, & \text{if } c_i^{\ell}(x, a) > \sqrt{m} \\
c_i^{\ell}(x, a), & \text{if } |c_i^{\ell}(x, a)| \leq \sqrt{m}
\end{cases} \tag{4.2}
\]
Before proving the theorem, we define a few functionals used in the proof. Let a multi-strategy \( \pi \in \Pi \) be fixed. For every \( i \in \mathcal{N}, \ell \in L_0 \) and \( m \in \mathbb{N} \), put

\[
J_{\ell,m}^{\eta}(\pi) := (1 - \alpha)E_{\pi}^{\eta} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} c_{\ell,m}^{\eta}(x^t, a^t) \right], \quad (4.3)
\]

\[
J_{\ell,\eta(m)}^{\eta(m)}(\pi) := (1 - \alpha)E_{\pi}^{\eta(m)} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} c_{\ell,m}^{\eta(m)}(x^t, a^t) \right]. \quad (4.4)
\]

Note that in (4.3) the initial state distribution is \( \eta \), while in (4.4) we use its perturbation \( \eta(m) \). Both \( J_{\ell,m}^{\eta}(\pi) \) and \( J_{\ell,\eta(m)}^{\eta(m)}(\pi) \) are defined with the aid of truncated functions \( c_{\ell,m}^{\eta} \).

The objective of player \( i \in \mathcal{N} \) in the modified game is to minimise \( J_{0,\eta(m)}^{\eta(m)}(\pi) \) over \( \pi_i \in \Pi_i \) with respect to the following constraints

\[
J_{\ell,\eta(m)}^{\eta(m)}(\pi) \leq \kappa_{\ell,m}^{\eta}, \quad \text{for all } \ell \in L, \ i \in \mathcal{N}, \ m \in \mathbb{N}. \quad (4.5)
\]

Denote by \( J_{\ell,m}^{\eta}(x, \varphi) \) the functional defined in (4.3) with the initial distribution \( \eta \) replaced by the Dirac delta \( \delta_x \). Observe that assumption C holds with \( \kappa_{\ell,m}^{\eta} \) instead of \( \kappa_{\ell}^{\eta} \), since for any multi-strategy \( \varphi \in \Phi \) there exists \( \pi_i \in \Pi_i \) such that

\[
J_{\ell,\eta(m)}^{\eta(m)}([\varphi_{-i}, \pi_i]) = \left( 1 - \frac{1}{m} \right) \sum_{x \in X_1} J_{\ell,m}^{\eta}(x, [\varphi_{-i}, \pi_i]) \eta(x) \\
+ \frac{1}{m} \sum_{x \in X_0} J_{\ell,m}^{\eta}(x, [\varphi_{-i}, \pi_i]) \tilde{\eta}(x) \\
< \left( 1 - \frac{1}{m} \right) \kappa_{\ell}^{\eta} + \frac{1}{m} \sqrt{m} = \kappa_{\ell,m}^{\eta}
\]

for all \( \ell \in L \).

For any \( \pi \in \Pi \) and \( m \in \mathbb{N} \), define

\[
\Delta_{\ell}^{m}(\pi_{-i}) := \left\{ \pi_i \in \Pi_i : J_{\ell,\eta}^{\eta(m)}(\pi) \leq \kappa_{\ell,m}^{\eta}, \text{ for all } \ell \in L \right\}.
\]

**Definition 4.1** The constrained discounted stochastic game with the initial distribution (4.1), the cost and constraint functions as in (4.2), the cost functionals as in (4.4) and constants as in (4.5) is called an \( m-CSG \).
Lemma 4.2  Assume A and B. Then, the following holds.

(a) For every $\ell \in L_0$ and $i \in \mathbb{N}$

$$\sup_{\varphi \in \Phi} | J_i^{\ell, \eta(m)}(\varphi) - J_i^\ell(\varphi) | \to 0 \quad \text{as} \quad m \to \infty.$$ 

(b) $J_i^\ell(\cdot)$ is continuous on $\Phi$ for every $\ell \in L_0$ and $i \in \mathbb{N}$.

Proof  (a) From the triangle inequality we have

$$\sup_{\varphi \in \Phi} | J_i^{\ell, \eta(m)}(\varphi) - J_i^\ell(\varphi) | \leq \sup_{\varphi \in \Phi} | J_i^{\ell, \eta(m)}(\varphi) - J_i^{\ell, m}(\varphi) |$$

$$+ \sup_{\varphi \in \Phi} | J_i^{\ell, m}(\varphi) - J_i^\ell(\varphi) | =: I + II.$$

Then, we obtain

$$I = \sup_{\varphi \in \Phi} | J_i^{\ell, \eta(m)}(\varphi) - J_i^{\ell, m}(\varphi) | \leq \sup_{\varphi \in \Phi} \sup_{y \in X} | J_i^{\ell, m}(y, \varphi) | \cdot \sum_{x \in X} | \eta(m)(x) - \eta(x) |$$

$$\leq \sqrt{m} \left( \sum_{x \in X_0} \frac{\tilde{\eta}(x)}{m} + \sum_{x \in X_1} \frac{\eta(x)}{m} \right) = \frac{2}{\sqrt{m}} \to 0 \quad \text{as} \quad m \to \infty.$$ 

Now let us consider the second term. By assumption B(i) and (2.3), for any $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$, it holds

$$\sup_{\varphi \in \Phi} (1 - \alpha)^\beta \left[ \sum_{t=k}^{\infty} \alpha^{t-1} c_i^\ell(x^t, a^t) \right] \leq \sup_{\varphi \in \Phi} (1 - \alpha)^\beta \left[ \sum_{t=k}^{\infty} \alpha^{t-1} w(x^t) \right] \leq \frac{\varepsilon}{4} \quad \text{(4.6)}$$

Similarly, for all $m \in \mathbb{N}$, $k > N_1$, $\ell \in L_0$ and $i \in \mathbb{N}$

$$\sup_{\varphi \in \Phi} (1 - \alpha)^\beta \eta \left[ \sum_{t=k}^{\infty} \alpha^{t-1} c_i^{\ell, m}(x^t, a^t) \right] \leq \frac{\varepsilon}{4} \quad \text{(4.7)}$$

Note that

$$| c_i^\ell(x, a) - c_i^{\ell, m}(x, a) | \leq w(x) 1_{[|c_i^\ell(x, a)| > \sqrt{m}]}(x, a) \leq w(x) 1_{[w(x) > \sqrt{m}]}(x) \quad \text{(4.8)}$$
Consequently, by (4.6)–(4.8), we obtain

$$II = \sup_{\varphi \in \Phi} |J_{\ell, \eta}(\varphi) - J_{\ell}^i(\varphi)|$$

$$\leq \sup_{\varphi \in \Phi} \left| (1 - \alpha) \mathbb{E}_{\eta} ^{\varphi} \left[ \sum_{t=1}^{N_1} \alpha^t - 1 c_{i,t}(x^t, a^t) \right] - (1 - \alpha) \mathbb{E}_{\eta} ^{\varphi} \left[ \sum_{t=1}^{N_1} \alpha^t - 1 c_{i,t}(x^t, a^t) \right] \right| + \frac{\varepsilon}{2}$$

$$\leq \sup_{\varphi \in \Phi} (1 - \alpha) \mathbb{E}_{\eta} \left[ \sum_{t=1}^{N_1} \alpha^t - 1 w(x^t) 1_{[w(x^t) > \sqrt{m}]} \right] + \frac{\varepsilon}{2}$$

for every $\ell \in L_0$ and $i \in N$. From (2.4) for sufficiently large values of $m \in \mathbb{N}$, it follows that

$$\sup_{\varphi \in \Phi} \mathbb{E}_{\eta} \left[ w(x^t) 1_{[w(x^t) > \sqrt{m}]} \right] \leq \frac{\varepsilon}{2}$$

for all $t = 1, \ldots, N_1$. Therefore,

$$\sup_{\varphi \in \Phi} (1 - \alpha) \mathbb{E}_{\eta} \left[ \sum_{t=1}^{N_1} \alpha^t - 1 w(x^t) 1_{[w(x^t) > \sqrt{m}]} \right] \leq (1 - \alpha) \sum_{t=1}^{N_1} \alpha^t - 1 \frac{\varepsilon}{2} < \frac{\varepsilon}{2}.$$ 

Hence, $II < \varepsilon$. This finishes the proof of part (a).

(b) By Lemma 3.13, the functional $J_{i, \eta}(m)()$ is continuous on $\Phi$ for every $m \in \mathbb{N}$, $\ell \in L_0$ and $i \in N$. This fact and the uniform convergence proved in point (a) imply the assertion.$\square$

**Proof of Theorem 2.3** From Proposition 3.15, it follows that each $m$-CSG possesses a Nash equilibrium $\phi^m \in \Phi$. Let $(\phi^m)_{m \in \mathbb{N}}$ be a sequence in $\Phi$ of Nash equilibria in the $m$-CSGs. From the compactness of $\Phi$, without loss of generality, we may assume that $\phi^m$ converges to some $\phi \in \Phi$ as $m \to \infty$. We claim that $\phi$ is a Nash equilibrium in the original CSG. Since

$$J_{i, \eta}(\phi^m) \leq \kappa_{i,m}^{\ell,m} \quad \text{for all } m \in \mathbb{N}, \ell \in L, i \in N,$$

by Lemma 4.2 and the fact that $\kappa_{i,m}^{\ell,m} \to \kappa_{i}^{\ell}$ as $m \to \infty$, it follows for $\ell \in L$ and $i \in N$ that

$$J_{i, \eta}(\phi^m) \to J_{i, \eta}(\phi) \leq \kappa_{i}^{\ell} \quad \text{as } m \to \infty.$$ 

These facts immediately entail that $\phi \in \Phi$ is feasible in the original CSG. The rest will follow, if we show that

$$J_i^0(\phi) = \min_{\pi_i \in \Delta_i(\phi_{-i})} J_i^0([\phi_{-i}, \pi_i]) \quad \text{for every } i \in N.$$
On the contrary, assume that there exists player \( i \in \mathcal{N} \) and a strategy \( \pi_i \in \Pi_i \) such that \( \pi_i \in \Delta_i(\phi_{-i}) \) and

\[
J_i^0([\phi_{-i}, \pi_i]) < J_i^0(\phi).
\]

By Lemma 3.11, we may replace \( \pi_i \) by a strategy \( \gamma_i \in \Phi_i \) in the sense that \( \gamma_i \in \Delta_i(\phi_{-i}) \) and

\[
J_i^0([\phi_{-i}, \pi_i]) = J_i^0([\phi_{-i}, \gamma_i]) < J_i^0(\phi).
\]

By Lemma 6.1 (see Appendix), one can select a sequence \((\gamma_i^m)_{m \in \mathbb{N}}\) with \( \gamma_i^m \in \Delta_i^m(\phi_{-i}) \) such that

\[
\lim_{m \to \infty} J_i^0(\phi^m) = J_i^0([\phi_{-i}, \gamma_i]).
\]

Since \( \phi_i^m \) is the \( i \)-th coordinate of the Nash equilibrium profile \( \phi^m \) in the \( m \)-C\(S\)G, we have

\[
J_i^0(\phi^m) = \min_{\sigma_i \in \Delta_i^m(\phi_{-i})} J_i^0(\phi^m, \sigma_i) \leq J_i^0(\phi^m, \gamma_i).
\]

Taking the limit as \( m \to \infty \) in the above display and applying Lemma 4.2 and (4.10), we get

\[
J_i^0(\phi) = J_i^0([\phi_{-i}, \phi_i]) \leq J_i^0([\phi_{-i}, \gamma_i]).
\]

This inequality contradicts (4.9).

In the above proof we tacitly assumed that \( X_0 \neq \emptyset \). If \( X_0 = \emptyset \), then our proof can be simplified in an obvious manner.

### 5 Additional Remarks on Assumptions

In this section, we give some examples and comments on assumptions \( B \) and \( W \). For simplicity, we consider a one-person game, i.e., a constrained discounted Markov decision process, where the player is called a decision maker. Therefore, \( A_1(x) = A(x) \) for all \( x \in X \) and an element of \( A(x) \) will be denoted by \( a \) instead of \( a \).

The following example is inspired by the example given by Blackwell [10]. The function \( w \) satisfying assumption 2(ii) does not exist.

**Example 5.1** We consider a simple Markov decision process. Let \( X = \mathbb{N} \cup \mathbb{N}^* \), where \( \mathbb{N}^* := \{1^*, 2^*, 3^*, \ldots\} \). The action sets are: \( A(n) = \{c, s\} \) and \( A(n^*) = \{s\} \) for \( n \in \mathbb{N}, n^* \in \mathbb{N}^* \). Here, \( c \) means *continue* and \( s \) means *stop*. State \( 1^* \) is absorbing and \( p(1^*|n^*, s) = p((n + 1)^*|n, s) = 1 \) for \( n \in \mathbb{N}, n^* \in \mathbb{N}^* \). Moreover, \( p(1^*|n, c) = 1 - p(n + 1|n, c) = q, n \in \mathbb{N}, n^* \in \mathbb{N}^* \), where \( q \in [0, 1] \). The cost functions
are non-negative and satisfy inequalities: \( c(1^*, s) = 0 \leq c(n, s) = c(n, c) \leq 1 \) and \( c(n^*, s) \leq n \) for \( n \in \mathbb{N}, n^* \in \mathbb{N}^* \).

We begin with showing that our assumption \( B \) holds. We define the function \( w \) in the simplest way, i.e.,

\[
w(n) = 1, \quad w(n^*) = n, \quad \text{for } n \in \mathbb{N}, \ n^* \in \mathbb{N}^*.
\]

The initial distribution is geometric and is given on the set \( \mathbb{N} \): \( \eta(m) = (1 - g)g^{m-1} \) for each \( m \in \mathbb{N} \), where \( g \in (0, 1) \) is fixed. We now prove \( B(ii) \). Fix any strategy \( \pi \) of the decision maker. Denote by \( \mathbb{E}_m^\pi \) the expectation operator on the trajectories of the process governed by \( \pi \) and starting at state \( m \in \mathbb{N} \). We note that

\[
\mathbb{E}_m^\pi \left( \sum_{k=n}^{\infty} \alpha^{k-1} w(x^k) \right) \leq \sum_{m=1}^{\infty} \mathbb{E}_m^\pi \left( \sum_{k=n}^{\infty} \alpha^{k-1} w(x^k) \right) (1 - g)g^{m-1} \\
\leq \sum_{m=1}^{\infty} \mathbb{E}_m^\pi \left( \sum_{k=n}^{\infty} \alpha^{k-1} (k + m - 1) \right) (1 - g)g^{m-1} \\
= \frac{\alpha^{n-1}(n - 1)(1 - \alpha) + \alpha^n}{(1 - \alpha)^2} + \frac{\alpha^{n-1}}{g(1 - \alpha)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

Hence, (2.3) is satisfied. The inequality in the above display is due to the observation that \( w(x^k) \leq k + (m - 1) \) if the initial state is \( m \) and no matter which strategy the decision maker uses. Now we show that (2.4) holds. Fix \( t \in \mathbb{N} \) in (2.4) and any strategy \( \pi \). Then,

\[
\mathbb{E}_m^\pi \left( w(x^t) 1_{w(x^t) \geq k} \right) \leq \sum_{m=1}^{\infty} \left( (t + m - 1) \mathbb{E}_m^\pi 1_{w(x^t) \geq k} \right) g^{m-1}(1 - g).
\]

For any \( \varepsilon > 0 \), there exists \( M \in \mathbb{N} \) such that

\[
\max \left\{ (t - 1) \sum_{m=M}^{\infty} g^{m-1}(1 - g), \sum_{m=M}^{\infty} mg^{m-1}(1 - g) \right\} \leq \frac{\varepsilon}{2}.
\]

Set \( k := t + M - 1 \) and note that

\[
\mathbb{E}_m^\pi 1_{w(x^t) \geq k} \leq 1_{m + (m - 1) \geq k} = 1_{m \geq M}.
\]

Hence, for any strategy \( \pi \) of the decision maker, we have

\[
\mathbb{E}_m^\pi \left( w(x^t) 1_{w(x^t) \geq k} \right) \leq \sum_{m=M}^{\infty} (t + m - 1)g^{m-1}(1 - g) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

This proves (2.4).

Now it can be easily seen that the inequality in \( W(i) \) does not hold. Indeed, there is no \( \delta > 1 \) such that the inequality

\[\text{...}\]
\[
\sum_{y \in X} w(y) p(y|n, s) = w((n+1)^* ) = n + 1 \leq \delta w(n) = \delta
\]

holds for all \( n \in \mathbb{N} \).

The second possibility is to change a function \( w \) in such a way that \( w(n) = n \). Then, we obtain the special conditions on the discount factor \( \alpha \) through inequalities from \( W(i) \):

\[
\sum_{y \in X} w(y) p(y|n, s) = w((n+1)^* ) = n + 1 \leq n\delta \Rightarrow 1 + \frac{1}{n} \leq \delta \Rightarrow \delta = 2 \text{ and } \alpha < \frac{1}{2},
\]

and

\[
\sum_{y \in X} w(y) p(y|n, c) = q + (1-q)(n+1) \leq n\delta \Rightarrow 1 - q + \frac{1}{n} \leq \delta \Rightarrow \delta = 2 - q \text{ and } \alpha < \frac{1}{2} - q.
\]

Thus, \( W \) holds, but only for \( \alpha < 1/2 \). This is a serious restriction for the discount factor. Other inequalities in \( W(i) \) are automatically satisfied and we do not consider them here. Finally, the third possibility is to modify \( w \) by adding some constant \( d > 0 \) (but we keep \( w(1^*) = 1 \)). This idea was first discussed in [25]. The above inequalities are as follows:

\[
\sum_{y \in X} w(y) p(y|n, s) = w((n+1)^* ) = n + 1 + d \leq (n+d)\delta \Rightarrow 1 + \frac{1}{n+d} \leq \delta
\]

and

\[
\sum_{y \in X} w(y) p(y|n, c) = q + (1-q)(n+1+d) \leq (n+d)\delta \Rightarrow 1 - q + \frac{1}{n+d} \leq \delta
\]

for all \( n \in \mathbb{N} \). Hence, for any \( \alpha \in (0, 1) \), we may choose a constant \( d \) such that \( \alpha \delta < 1 \) with \( \delta := 1 + \frac{1}{1+d} \). This forces us to select carefully an appropriate function \( w \). The second disadvantage is that the function \( w \) is less natural than the original \( w \) given above, i.e., when \( w(n) = 1 \) for all \( n \in \mathbb{N} \).

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interests.
6 Appendix

First note that from (4.5), it follows that for sufficiently large values of \( m \) we have

\[
\kappa_i^\ell \leq \kappa_i^{\ell,m} \quad \text{for all} \quad \ell \in L, \ i \in \mathcal{N}. \tag{6.1}
\]

Indeed, this inequality is equivalent to \( \kappa_i^\ell \leq \sqrt{m} \).

The following result is crucial in the proofs of Lemma 3.14 and Theorem 2.3.

**Lemma 6.1** Let \( A, B \) and \( C \) hold and let \( \phi \in \Phi \) be feasible in the CSG. Assume that \( \phi^m \to \phi \) in \( \Phi \) as \( m \to \infty \). For any \( i \in \mathcal{N}, \gamma_i \in \Delta_i(\phi_{-i}) \), there exists a sequence of strategies \( (\gamma_i^m)_{m \in \mathbb{N}} \) in \( \Phi_i \) such that \( \gamma_i^m \in \Delta_i^m(\phi_{-i}^m) \) for every \( m \in \mathbb{N} \) and

\[
J_i^{0,n(m)}([\phi_{-i}^m, \gamma_i^m]) \to J_i^0([\phi_{-i}, \gamma_i]) \quad \text{as} \quad m \to \infty. \tag{6.2}
\]

**Proof** Recall that, if \( \phi^m = (\phi_1^m, \ldots, \phi_n^m) \) and \( \phi = (\phi_1, \ldots, \phi_n) \), then \( \phi^m \to \phi \) means that \( \phi_j^m \to \phi_j \) for all \( j \in \mathcal{N} \) as \( m \to \infty \).

Observe that from Lemma 4.2, it follows that, for every \( \ell \in L, i \in \mathcal{N} \),

\[
J_i^\ell([\phi_{-i}, \gamma_i]) = \lim_{m \to \infty} J_i^{\ell,n(m)}([\phi_{-i}^m, \gamma_i^m]) \leq \kappa_i^\ell. \tag{6.3}
\]

We consider two cases.

1. If \( J_i^\ell([\phi_{-i}, \gamma_i]) < \kappa_i^\ell \) for all \( \ell \in L \), then by (6.1) there exists \( N \in \mathbb{N} \) such that, for all \( m > N \) and for all \( \ell \in L \),

\[
J_i^{\ell,n(m)}([\phi_{-i}^m, \gamma_i]) \leq \kappa_i^\ell \leq \kappa_i^{\ell,m}.
\]

Hence, \( \gamma_i \in \Delta_i^m(\phi_{-i}^m) \) for all \( m > N \). Therefore, it suffices to put \( \gamma_i^m := \gamma_i \) for \( m > N \). For \( m = 1, \ldots, N \) from assumption \( C \), it is enough to take any \( \gamma_i^m \in \Delta_i^m(\phi_{-i}^m) \cap \Phi_i \). Then (6.2) follows immediately from (6.3).

2. Let \( i \in \mathcal{N} \) be fixed. Assume now that for at least one \( \ell \in L, J_i^\ell([\phi_{-i}, \gamma_i]) = \kappa_i^\ell \).

By assumption \( C \) and Lemma 3.11, there exists a strategy \( \xi_i \in \Phi_i \) for which

\[
J_i^\ell([\phi_{-i}, \xi_i]) < \kappa_i^\ell \quad \text{for all} \quad \ell \in L.
\]

Hence, there is a constant \( \kappa > 0 \) such that

\[
J_i^\ell([\phi_{-i}, \xi_i]) \leq \kappa_i^\ell - \kappa \tag{6.4}
\]

and by Lemma 4.2

\[
\lim_{m \to \infty} J_i^{\ell,n(m)}([\phi_{-i}^m, \gamma_i]) \leq \kappa_i^\ell \quad \text{and} \quad J_i^\ell([\phi_{-i}, \xi_i]) = \lim_{m \to \infty} J_i^{\ell,n(m)}([\phi_{-i}^m, \xi_i]) \tag{6.5}
\]

for all \( \ell \in L \).
Let $\mu^m_{\gamma_i}$ be the occupation measure defined on $K_i$, when the Markov process is induced by the initial distribution $\eta(m)$, the transition probability $p(x|z, [\phi^m_{-i}(z), a_i])$ and the strategy $\gamma_i$ of the decision maker (player $i$). By definition

$$
\mu^m_{\gamma_i}(K) = (1 - \alpha) \mathbb{E}^m_{\eta(m)} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} 1_K(x^t, a^t_i) \right] \quad \text{for any } K \in \mathcal{B}(K_i).
$$

Analogously, we define $\mu^m_{\xi_i}$. Thus, from (6.4) and (6.5), we deduce that for every $\tau \in (0, \kappa)$, there exists $N_\tau \in \mathbb{N}$ such that

$$
\sum_{x \in X} \int_{A_i(x)} c_{i,m}^{\ell,m} (x, [\phi^m_{-i}(x), a_i]) \mu^m_{\gamma_i} (\{x\} \times da_i) < \kappa_i^{\ell} - \kappa + \tau \quad (6.6)
$$

and

$$
\sum_{x \in X} \int_{A_i(x)} c_{i,m}^{\ell,m} (x, [\phi^m_{-i}(x), a_i]) \mu^m_{\xi_i} (\{x\} \times da_i) < \kappa_i^{\ell} + \tau \quad (6.7)
$$

and

$$
0 < \kappa_i^{\ell,m} - \kappa_i^{\ell} < \tau \quad (6.8)
$$

for all $m > N_\tau$ and $\ell \in L$.

Put

$$
\lambda(\tau, m) := \frac{\tau + \kappa_i^{\ell} - \kappa_i^{\ell,m}}{\kappa}
$$

and note by (6.8) that $\lambda(\tau, m) \in (0, 1)$ and $\lambda(\tau, m) \searrow 0$ as $\tau \searrow 0$. Moreover, for all $m > N_\tau$ and $\ell \in L$, it follows from (6.6) and (6.7) that

$$
(1 - \lambda(\tau, m)) \sum_{x \in X} \int_{A_i(x)} c_{i,m}^{\ell,m} (x, [\phi^m_{-i}(x), a_i]) \mu^m_{\gamma_i} (\{x\} \times da_i) \\
+ \lambda(\tau, m) \sum_{x \in X} \int_{A_i(x)} c_{i,m}^{\ell,m} (x, [\phi^m_{-i}(x), a_i]) \mu^m_{\xi_i} (\{x\} \times da_i) \\
< (1 - \lambda(\tau, m)) (\kappa_i^{\ell} + \tau) + \lambda(\tau, m) (\kappa_i^{\ell} - \kappa + \tau) = \kappa_i^{\ell} \\
+ \tau - \lambda(\tau, m) \cdot \kappa = \kappa_i^{\ell,m}. \quad (6.9)
$$

Let $(\epsilon_k)_{k \in \mathbb{N}}$ be a sequence of numbers in $(0, \kappa)$ such that $\epsilon_k \searrow 0$ as $k \to \infty$. For each $\tau = \epsilon_k$, there exists $N_k$ such that (6.6)–(6.8) hold for all $m > N_k$. We may assume that the sequence $(N_k)_{k \in \mathbb{N}}$ is increasing, so $\lim_{k \to \infty} N_k = \infty$. Note that, for each $m > N_1$, there exists a unique $k$ such that $N_k < m \leq N_{k+1}$. Using this
positive integer $k$, we put $\tau(m) = \epsilon_k$. Observe that $\tau(m) = \epsilon_k$ for all $m$ such that $N_k < m \leq N_{k+1}$.

Let

$$\lambda(\tau(m), m) := \frac{\tau(m) + \kappa^\ell i - \kappa^\ell m}{\kappa}.$$ 

Note that, if $m \to \infty$, then $N_k \to \infty$ and $\tau(m) \to 0$. Thus, $\lambda(\tau(m), m) \to 0$ as $m \to \infty$.

Now we define a new occupation measure as follows: for each $m$ such that $N_k < m \leq N_{k+1}$, we set

$$\nu^m_i = (1 - \lambda(\tau(m), m)) \mu^m_{\gamma_i} + \lambda(\tau(m), m) \mu^m_{\xi_i},$$  

(6.10)

where $\tau(m) = \epsilon_k$. Observe that $\mu^m_{\gamma_i}$ and $\mu^m_{\xi_i}$ belong to $\mathcal{M}_i$. By Lemma 3.2, $\nu^m_i \in \mathcal{M}_i$. Moreover, (6.9) holds for $\tau = \tau(m) = \epsilon_k$ and for all $m$ such that $N_k < m \leq N_{k+1}$.

By Lemma 3.4, for each $m \in \mathbb{N}$, there exists a unique strategy $\gamma^m_i \in \Phi_i$ such that $\nu^m_i = \gamma^m_i$. Note that by (6.9), (6.10) and Lemma 3.11, we have

$$J^{\ell, \eta(m)}_i (\Phi^m_{-i}, \gamma^m_i) \leq \kappa^\ell m$$

for all $\ell \in L$ and for all $m$ such that $N_k < m \leq N_{k+1}$. Hence, $\gamma^m_i \in \Delta^m(\Phi^m_{-i}) \cap \Phi_i$ for all $m > N_1$. Clearly, for $m = 1, \ldots, N_1$ assumption C enables us to choose any $\gamma^m_i \in \Delta^m(\Phi^m_{-i}) \cap \Phi_i$.

Let $\ell = 0$. Making use of Lemma 4.2, we infer that

$$J^{0, \eta(m)}_i (\Phi^m_{-i}, \gamma^m_i) = \lim_{m \to \infty} J^{0, \eta(m)}_i (\Phi^m_{-i}, \gamma^m_i)$$

$$= \lim_{m \to \infty} \sum_{x \in X} \int_{A_i(x)} c^{0,m}_i (x, [\Phi^m_{-i}(x), a_i]) \mu^m_{\gamma_i} (\{x\} \times da_i).$$

(6.11)

Since $\lambda(\tau(m), m) \to 0$ in (6.10) as $m \to \infty$, we have

$$\lim_{m \to \infty} \sum_{x \in X} \int_{A_i(x)} c^{0,m}_i (x, [\Phi^m_{-i}(x), a_i]) \mu^m_{\gamma_i} (\{x\} \times da_i)$$

$$= \lim_{m \to \infty} \sum_{x \in X} \int_{A_i(x)} c^{0,m}_i (x, [\Phi^m_{-i}(x), a_i]) \nu^m_i (\{x\} \times da_i).$$

(6.12)

However, $\nu^m_i = \gamma^m_i$ and $\nu^m_i$ is a convex combination of two occupation measures $\mu^m_{\gamma_i}$ and $\mu^m_{\xi_i}$ determined (among others) by the disturbed initial state distribution $\eta(m)$.

Therefore,

$$\sum_{x \in X} \int_{A_i(x)} c^{0,m}_i (x, [\Phi^m_{-i}(x), a_i]) \nu^m_i (\{x\} \times da_i) = J^{0, \eta(m)}_i (\Phi^m_{-i}, \gamma^m_i).$$

(6.13)
Consequently, combing together (6.11), (6.12) and (6.13), we conclude that
\[
\lim_{m \to \infty} J_{i}^{0, \eta(m)}(\phi_{-i}^{m}, \gamma_{i}^{m}) = J_{i}^{0}(\phi_{-i}, \gamma_{i}).
\]

This finishes the proof. \(\square\)

The next result can be proved in a similar manner as the above lemma with some necessary amendments. Lemma 6.2 is used in Lemma 3.14, where we assume that the cost and constraint functions are bounded and the support of the initial distribution is the whole state space \(X\).

**Lemma 6.2** Let \(A\) and \(C\) hold. Assume that for each \(i \in \mathcal{N}\) and \(\ell \in L\) the function \(c_{i}^{\ell}\) is bounded and \(\eta(x) > 0\) for all \(x \in X\). Then the correspondence \(\phi_{-i} \to \Delta_{i}(\phi_{-i}) \cap \Phi_{i}\) from \(\Phi_{-i} = \prod_{j \neq i} \Phi_{j}\) to \(\Phi_{i}\) is lower semicontinuous for each player \(i \in \mathcal{N}\).

**Proof** We have to prove that, if \(\phi_{-i} \in \Phi_{-i}, \gamma_{i} \in \Delta_{i}(\phi_{-i}) \cap \Phi_{i}\) and \(\phi_{-i}^{m} \to \phi_{-i}\) in \(\Phi_{-i}\) as \(m \to \infty\), then there exists a sequence \((\gamma_{i}^{m})_{m \in \mathbb{N}}\) in \(\Phi_{i}\) such that \(\gamma_{i}^{m} \in \Delta_{i}(\phi_{-i}^{m})\) for every \(m \in \mathbb{N}\) and \(\gamma_{i}^{m} \to \gamma_{i}\) as \(m \to \infty\).

Observe that from Lemma 3.13, it follows that, for every \(\ell \in L, i \in \mathcal{N}\),
\[
\lim_{m \to \infty} J_{i}^{\ell}(\phi_{-i}^{m}, \gamma_{i}) = J_{i}^{\ell}(\phi_{-i}, \gamma_{i}) \leq \kappa_{i}^{\ell}.
\]

We consider two cases.

1. If \(J_{i}^{\ell}(\phi_{-i}, \gamma_{i}) < \kappa_{i}^{\ell}\) for all \(\ell \in L\), then there exists \(N \in \mathbb{N}\) such that, for all \(m > N\) and for all \(\ell \in L\),
\[
J_{i}^{\ell}(\phi_{-i}^{m}, \gamma_{i}) \leq \kappa_{i}^{\ell}.
\]

Hence, \(\gamma_{i} \in \Delta_{i}(\phi_{-i}^{m})\) for all \(m > N\). Therefore, it suffices to put \(\gamma_{i}^{m} := \gamma_{i}\) for \(m > N\). For \(m = 1, \ldots, N\) from assumption \(C\), it is enough to take any \(\gamma_{i}^{m} \in \Delta_{i}(\phi_{-i}^{m}) \cap \Phi_{i}\). Then, the convergence of \(\gamma_{i}^{m}\) to \(\gamma_{i}\) is obvious.

2. Fix \(i \in \mathcal{N}\). Assume that for at least one \(\ell \in L\), it holds that \(J_{i}^{\ell}(\phi_{-i}, \gamma_{i}) = \kappa_{i}^{\ell}\). From assumption \(C\) and Lemma 3.11, there exists a strategy \(\xi_{i} \in \Phi_{i}\) for which
\[
J_{i}^{\ell}(\phi_{-i}, \xi_{i}) < \kappa_{i}^{\ell}\quad \text{for all} \quad \ell \in L.
\]

Hence, there is a constant \(\kappa > 0\) such that
\[
J_{i}^{\ell}(\phi_{-i}, \xi_{i}) \leq \kappa_{i}^{\ell} - \kappa
\]
and by Lemma 3.13
\[
\lim_{m \to \infty} J_{i}^{\ell}(\phi_{-i}^{m}, \gamma_{i}) \leq \kappa_{i}^{\ell}\quad \text{and}\quad J_{i}^{\ell}(\phi_{-i}, \xi_{i}) = \lim_{m \to \infty} J_{i}^{\ell}(\phi_{-i}^{m}, \xi_{i})
\]
for all \(\ell \in L\).
Define the occupation measures $\mu_{\gamma_i}^m$ and $\mu_{\xi_i}^m$ on $\mathbb{K}_i$ as in case 2 in the proof of Lemma 6.1, but with $\eta$ instead of $\eta(m)$. In addition, replace $c_i^\ell, c_i^\ell \kappa^\ell, m$ by $c_i^\ell, \kappa^\ell$ and $\lambda(\tau, m)$ by $\lambda(\tau) := \tau/\kappa$ in the proof of Lemma 6.1. Then, we introduce the occupation measures $\nu_i^m$ in a similar way as in (6.10). Namely,

$$\nu_i^m = (1 - \lambda(\tau(m))) \mu_{\gamma_i}^m + \lambda(\tau(m)) \mu_{\xi_i}^m,$$

if $N_k < m \leq N_{k+1}$, $k = \tau(m)$ (6.14)

The definition of $\tau(m)$ is the same as in the proof of Lemma 6.1, but $\lambda(\tau(m)) = \tau(m)/\kappa$. From the proof of Lemma 6.1 (or Lemma 3.4), we also conclude that, for any $m \in \mathbb{N}$, there exists a unique $\gamma_i^m \in \Phi_i$ such that $\nu_i^m = \hat{\nu}_i^m m^m$. Moreover, $\gamma_i^m \in \Delta_i(\phi_i^m)$ for every $m \in \mathbb{N}$. It remains to show that $\gamma_i^m \rightarrow \gamma_i$ as $m \rightarrow \infty$.

Let $\mu_{\gamma_i}$ be the occupation measure defined on $\mathbb{K}_i$, when the Markov process is induced by the initial distribution $\eta$, the transition probability $p(x|z, [\phi_i(z), a_i])$ and $\gamma_i$. For any bounded continuous function $f$ on $\mathbb{K}_i$, we put

$$\hat{f}(x) = \int_{A_i(x)} f(x, a_i) \gamma_i (da_i|x).$$

Then, we have

$$\sum_{x \in X} \int_{A_i(x)} f(x, a_i) \mu_{\gamma_i}([x] \times da_i) = (1 - \alpha) E_{\phi_i \sim \gamma_i} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} f(x^t, a_i^t) \right]$$

$$= (1 - \alpha) E_{\phi_i \sim \gamma_i} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} \hat{f}(x^t) \right]$$

$$= \sum_{x \in X} \int_{A_i(x)} \hat{f}(x) \mu_{\gamma_i}([x] \times da_i)$$

$$= \sum_{x \in X} \hat{f}(x) \mu_{\gamma_i}(x)$$

$$= \sum_{x \in X} \int_{A_i(x)} f(x, a_i) \gamma_i (da_i|x) \mu_{\gamma_i}(x).$$

Since $f$ is arbitrary, it follows that $\mu_{\gamma_i} = \hat{\mu}_{\gamma_i} \gamma_i$. By Lemma 2.1 in [17] (or Lemma 3.13 with $c_i^\ell = f$), for every bounded continuous function $f$ on $\mathbb{K}_i$, we have

$$\sum_{x \in X} \int_{A_i(x)} f(x, a_i) \mu_{\gamma_i}^m ([x] \times da_i) = (1 - \alpha) E_{\phi_i \sim \gamma_i} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} f(x^t, a_i^t) \right]$$

$$\rightarrow (1 - \alpha) E_{\phi_i \sim \gamma_i} \left[ \sum_{t=1}^{\infty} \alpha^{t-1} \hat{f}(x^t, a_i^t) \right] = \sum_{x \in X} \int_{A_i(x)} f(x, a_i) \mu_{\gamma_i}^m ([x] \times da_i).$$

Thus, $\mu_{\gamma_i}^m$ converges weakly to $\mu_{\gamma_i}$ as $m \rightarrow \infty$. This fact and (6.14) imply that $\nu_i^m = \hat{\nu}_i^m \gamma_i^m$ converges weakly to $\mu_{\gamma_i} = \hat{\mu}_{\gamma_i} \gamma_i$ as $m \rightarrow \infty$. Since $\eta(x) > 0$ for all $x \in X$, by Lemma 3.6, $\gamma_i^m \rightarrow \gamma_i$ in $\Phi_i$ as $m \rightarrow \infty$. This finishes the proof. \( \square \)
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