QUASI *-ALGEBRAS OF MEASURABLE OPERATORS

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ABSTRACT. Non-commutative $L^p$-spaces are shown to constitute examples of a class of Banach quasi *-algebras called CQ*-algebras. For $p \geq 2$ they are also proved to possess a sufficient family of bounded positive sesquilinear forms satisfying certain invariance properties. CQ*-algebras of measurable operators over a finite von Neumann algebra are also constructed and it is proven that any abstract CQ*-algebra $(X, \mathfrak{A}_0)$ possessing a sufficient family of bounded positive tracial sesquilinear forms can be represented as a CQ*-algebra of this type.

1. INTRODUCTION AND PRELIMINARIES

A quasi *-algebra is a couple $(X, \mathfrak{A}_0)$, where $X$ is a vector space with involution $^*$, $\mathfrak{A}_0$ is a *-algebra and a vector subspace of $X$ and $X$ is an $\mathfrak{A}_0$-bimodule whose module operations and involution extend those of $\mathfrak{A}_0$. Quasi *-algebras were introduced by Lassner [1, 2, 12] to provide an appropriate mathematical framework where discussing certain quantum physical systems for which the usual algebraic approach made in terms of C*-algebras revealed to be insufficient. In these applications they usually arise by taking the completion of the C*-algebra of observables in a weaker topology satisfying certain physical requirements. The case where this weaker topology is a norm topology has been considered in a series of previous papers [3]-[7], where CQ*-algebras were introduced: a CQ*-algebra is, indeed, a quasi *-algebra $(X, \mathfrak{A}_0)$ where $X$ is a Banach space with respect to a norm $\| \cdot \|$ possessing an isometric involution and $\mathfrak{A}_0$ is a C*-algebra with respect to a norm $\| \cdot \|_0$, which is dense in $X[\| \cdot \|]$.

Since any C*-algebra $\mathfrak{A}_0$ has a faithful *-representation $\pi$, it is natural to pose the question if this completion also can be realized as a quasi *-algebra of operators affiliated to $\pi(\mathfrak{A}_0)^\prime$. The Segal-Nelson theory [10, 11] of non-commutative integration provides a number of mathematical tools for dealing with this problem.

The paper is organized as follows. In Section 2 we consider non-commutative $L^p$-spaces constructed starting from a von Neumann algebra $\mathfrak{M}$ and a normal, semifinite, faithful trace $\tau$ as Banach quasi *-algebras. In particular if $\varphi$ is finite, then it is shown that $(L^p(\varphi), \mathfrak{M})$ is a CQ*-algebra. If $p \geq 2$, they even possess a sufficient family of positive sesquilinear forms enjoying certain invariance properties.

In Section 3, starting from a family $\mathfrak{F}$ of normal, finite traces on a von Neumann algebra $\mathfrak{M}$, we prove that the completion of $\mathfrak{M}$ with respect to a norm defined in natural

2000 Mathematics Subject Classification. Primary 46L08; Secondary 46L51, 47L60.

Key words and phrases. Banach C*-modules, Non commutative integration, Partial algebras of operators.
way by the family $\mathcal{F}$ is indeed a CQ*-algebra consisting of measurable operators, in Segal’s sense, and therefore affiliated with $\mathcal{M}$.

Finally, in Section 4, we prove that any CQ*-algebra $(X, A_0)$ possessing a sufficient family of bounded positive tracial sesquilinear forms can be continuously embedded into the CQ*-algebra of measurable operators constructed in Section 3.

In order to keep the paper sufficiently self-contained, we collect below some preliminary definitions and propositions that will be used in what follows.

Let $(X, A_0)$ be a quasi *-algebra. The unit of $(X, A_0)$ is an element $e \in A_0$ such that $xe = ex = x$, for every $x \in X$. A quasi *-algebra $(X, A_0)$ is said to be locally convex if $X$ is endowed with a topology $\tau$ which makes of $X$ a locally convex space and such that the involution $a \mapsto a^*$ and the multiplications $a \mapsto ab$, $a \mapsto ba$, $b \in A_0$, are continuous. If $\tau$ is a norm topology and the involution is isometric with respect to the norm, we say that $(X, A_0)$ is a normed quasi *-algebra and, if it is complete, we say it is a Banach quasi*-algebra.

**Definition 1.1.** Let $(X, A_0)$ be a Banach quasi *-algebra with norm $\|\cdot\|$ and involution $^*$. Assume that a second norm $\|\cdot\|_0$ is defined on $A_0$, satisfying the following conditions:

- (a.1) $\|a^*a\|_0 = \|a\|^2_0$, $\forall a \in A_0$;
- (a.2) $\|a\| \leq \|a\|_0$, $\forall a \in A_0$;
- (a.3) $\|ax\| \leq \|a\|_0\|x\|$, $\forall a \in A_0, x \in X$;
- (a.4) $A_0[\|\cdot\|_0]$ is complete.

Then we say that $(X, A_0)$ is a CQ*-algebra.

**Remark 1.2.** (1) If $A_0[\|\cdot\|_0]$ is not complete, we say that $(X, A_0)$ is a pre CQ*-algebra.

(2) In previous papers the name CQ*-algebra was given to a more complicated structure where two different involutions were considered on $A_0$. When these involutions coincide, we spoke of a proper CQ*-algebra. In this paper only this case will be considered and so we systematically omit the term proper.

The following basic definitions and results on non-commutative measure theory are also needed in what follows.

Let $\mathcal{M}$ be a von Neumann algebra and $\varphi$ a normal faithful semifinite trace defined on $\mathcal{M}_+$. Put

$$\mathcal{J} = \{X \in \mathcal{M} : \varphi(|X|) < \infty\}.$$ 

$\mathcal{J}$ is a *-ideal of $\mathcal{M}$.

We denote with $\text{Proj}(\mathcal{M})$, the lattice of projections of $\mathcal{M}$.

**Definition 1.3.** A vector subspace $D$ of $\mathcal{H}$ is said to be strongly dense (resp., strongly $\varphi$-dense) if

- $U'U \subset D$ for any unitary $U'$ in $\mathcal{M}$'
- there exists a sequence $P_n \in \text{Proj}(\mathcal{M})$: $P_nH \subset D, P_n^\perp \downarrow 0$ and $(P_n^\perp)$ is a finite projection (resp., $\varphi(P_n^\perp) < \infty$).
Clearly, every strongly \( \varphi \)-dense domain is strongly dense.

Throughout this paper, when we say that an operator \( T \) is affiliated with a von Neumann algebra, written \( T \eta \mathcal{M} \), we always mean that \( T \) is closed, densely defined and \( TU \supseteq UT \) for every unitary operator \( U \in \mathcal{M} \).

**Definition 1.4.** An operator \( T \eta \mathcal{M} \) is called

- measurable (with respect to \( \mathcal{M} \)) if its domain \( D(T) \) is strongly dense;
- \( \varphi \)-measurable if its domain \( D(T) \) is strongly \( \varphi \)-dense.

From the definition itself it follows that, if \( T \) is \( \varphi \)-measurable, then there exists \( P \in \text{Proj}(\mathcal{M}) \) such that \( TP \) is bounded and \( \varphi(P^\perp) < \infty \).

We remind that any operator affiliated with a finite von Neumann algebra is measurable [10, Cor. 4.1] but it is not necessarily \( \varphi \)-measurable.

## 2. Non-commutative \( L^p \)-spaces as CQ *-algebras

In this Section we will discuss the structure of the non-commutative \( L^p \)-spaces as quasi *-algebras. We begin with recalling the basic definitions.

Let \( \mathcal{M} \) be a von Neumann algebra and \( \varphi \) a normal faithful semifinite trace defined on \( \mathcal{M}_+ \). For each \( p \geq 1 \), let

\[
\mathcal{J}_p = \{ X \in \mathcal{M} : \varphi(|X|^p) < \infty \}.
\]

Then \( \mathcal{J}_p \) is a *-ideal of \( \mathcal{M} \). Following [11], we denote with \( L^p(\varphi) \) the Banach space completion of \( \mathcal{J}_p \) with respect to the norm

\[
\|X\|_p := \varphi(|X|^p)^{1/p}, \quad X \in \mathcal{J}_p.
\]

One usually defines \( L^\infty(\varphi) = \mathcal{M} \). Thus, if \( \varphi \) is a finite trace, then \( L^\infty(\varphi) \subseteq L^p(\varphi) \) for every \( p \geq 1 \). As shown in [11], if \( X \in L^p(\varphi) \), then \( X \) is a measurable operator.

**Proposition 2.1.** Let \( \mathcal{M} \) be a von Neumann algebra and \( \varphi \) a normal faithful semifinite trace on \( \mathcal{M}_+ \). Then \( (L^p(\varphi), L^\infty(\varphi) \cap L^p(\varphi)) \) is a Banach quasi *-algebra.

If \( \varphi \) is a finite trace and \( \varphi(\mathbb{I}) = 1 \), then \( (L^p(\varphi), L^\infty(\varphi)) \) is a CQ*-algebra.

**Proof.** Indeed, it is easily seen that the norms \( \| \cdot \|_\infty \) of \( L^\infty(\varphi) \cap L^p(\varphi) \) and \( \| \cdot \|_p \) on \( L^p(\varphi) \) satisfy the conditions (a.1)-(a.2) of Definition [11]. Moreover, if \( \varphi \) is finite, then \( L^\infty(\varphi) \subseteq L^p(\varphi) \) and thus \( (L^p(\varphi), L^\infty(\varphi)) \) is a CQ*-algebra. \( \square \)

**Remark 2.2.** Of course the condition \( \varphi(\mathbb{I}) = 1 \) can be easily removed by rescaling the trace.

**Definition 2.3.** Let \( (\mathcal{X}, \mathcal{A}_0) \) be a Banach quasi *-algebra. We denote with \( S(\mathcal{X}) \) the set of all sesquilinear forms \( \Omega \) on \( \mathcal{X} \times \mathcal{X} \) with the following properties

(i) \( \Omega(x, x) \geq 0 \quad \forall x \in \mathcal{X} \)
(ii) \( \Omega(xa, b) = \Omega(a, x^*b) \quad \forall x \in \mathcal{X}, \forall a, b \in \mathcal{A}_0 \)
(iii) \( |\Omega(x, y)| \leq \|x\| \|y\| \quad \forall x, y \in \mathcal{X} \).

A subfamily \( \mathcal{A} \) of \( S(\mathcal{X}) \) is called sufficient if \( x \in \mathcal{X}, \Omega(x, x) = 0, \text{ for every } \Omega \in \mathcal{A} \), implies \( x = 0 \).
If \((X, \mathfrak{A}_0)\) is a Banach quasi \(*\)-algebra, then the Banach dual space \(X^\sharp\) of \(X\) can be made into a Banach \(\mathfrak{A}_0\)-bimodule with norm
\[
\|f\|_\sharp = \sup_{\|x\| \leq 1} |\langle x, f \rangle|, \quad f \in X^\sharp,
\]
by defining, for \(f \in X^\sharp\), \(a \in \mathfrak{A}_0\), the module operations in the following way:
\[
\langle x, f \circ a \rangle := \langle ax, f \rangle, \quad x \in X,
\]
\[
\langle x, a \circ f \rangle := \langle xa, f \rangle, \quad x \in X.
\]
As usual, an involution \(f \mapsto f^*\) can be defined on \(X^\sharp\) by \(\langle x, f^* \rangle = \overline{\langle x^*, f \rangle}\), \(x \in X\).

With these notations we can easily prove the following (see, also [8]):

**Proposition 2.4.** \((X, \mathfrak{A}_0)\) be a Banach quasi \(*\)-algebra and \(\Omega\) a positive sesquilinear form on \(X \times X\). The following statements are equivalent:

(i) \(\Omega \in \mathcal{S}(X)\);

(ii) there exists a bounded conjugate linear operator \(T : X \to X^\sharp\) with the properties

(ii.1) \(\langle x, Tx \rangle \geq 0\), \(\forall x \in X\);

(ii.2) \(T(ax) = (Tx) \circ a^*\), \(\forall a \in \mathfrak{A}_0\), \(x \in X\);

(ii.3) \(\|T\|_{\mathcal{B}_+(X^\sharp)} \leq 1\);

(ii.4) \(\Omega(x, y) = \langle x, Ty \rangle\), \(\forall x, y \in X\).

We will now focus our attention on the question as to whether for the Banach quasi \(*\)-algebra \((L^p(\varphi), L^\infty(\varphi) \cap L^p(\varphi))\), the family \(\mathcal{S}(L^p(\varphi))\), that we are going to describe, is or is not sufficient.

Before going forth, we remind that many of the familiar results of the ordinary theory of \(L^p\)-space hold in the very same form for the non-commutative \(L^p\)-spaces. This is the case, for instance, of Hölder’s inequality and also of the statement that characterizes the dual of \(L^p\): the form defining the duality is the extension of \(\varphi\) (this extension will be denoted with the same symbol) to products of the type \(XY\) with \(X \in L^p(\varphi)\), \(Y \in L^p(\varphi)\) with \(p^{-1} + p' = 1\) and one has \((L^p(\varphi))^\sharp \simeq L^p(\varphi)\).

In order to study \(\mathcal{S}(L^p(\varphi))\), we introduce, for \(p \geq 2\), the following notation
\[
\mathcal{B}_+^p = \{X \in L^{p/(p-2)}(\varphi), \ X \geq 0, \|X\|_{p/(p-2)} \leq 1\}
\]
where \(p/(p-2) = \infty\) if \(p = 2\).

For each \(W \in \mathcal{B}_+^p\), we consider the right multiplication operator
\[
R_W : L^p(\varphi) \to L^{p/(p-1)}(\varphi); \quad R_W X = XW, \quad X \in L^p(\varphi).
\]
Since \(L^\infty(\varphi) \cap L^p(\varphi) = \mathcal{J}_p\), we use, for shortness, the latter notation.

**Lemma 2.5.** The following statements hold.

(i) Let \(p \geq 2\). For every \(W \in \mathcal{B}_+^p\), the sesquilinear form \(\Omega(X, Y) = \varphi[X(R_W Y)^*]\) is an element of \(\mathcal{S}(L^p(\varphi))\);

(ii) If \(\varphi\) is finite, then for each \(\Omega \in \mathcal{S}(L^p(\varphi))\), there exists \(W \in \mathcal{B}_+^p\) such that
\[
\Omega(X, Y) = \varphi[X(R_W Y)^*], \quad \forall X, Y \in L^p(\varphi).
\]
Lemma 3.1. Let $M$ be a von Neumann algebra in Hilbert space $H$, $\{P_\alpha\}_{\alpha \in \mathcal{I}}$ a family of projections of $M$ with

$$\bigvee_{\alpha \in \mathcal{I}} P_\alpha = P.$$

If $A \in M$ and $AP_\alpha = 0$ for every $\alpha \in \mathcal{I}$, then $A \overline{P} = 0$.

Proof. (i): We check that the sesquilinear $\Omega(X,Y) = \varphi[X(R_wY)\ast], X,Y \in L^p(\varphi)$ satisfies the conditions (i),(ii),(iii) of Definition 2.3. For every $X \in L^p(\varphi)$ we have

$$\Omega(X,X) = \varphi[X(R_wX)\ast] = \varphi[X(XW)\ast] = \varphi[(XW)\ast X] = \varphi[W|X|^2] \geq 0.$$

Then we define $\varphi$ and assume, in addition, that the following condition (P) is satisfied:

Then we define $\varphi$ and assume, in addition, that the following condition (P) is satisfied:

(ii) Let $\Omega \in S(L^p(\varphi))$. Let $T : L^p(\varphi) \to L^p(\varphi)$ be the operator which represents $\Omega$ in the sense of Proposition 2.4. The finiteness of $\varphi$ implies that $J_p = \mathcal{M}$; thus we can put $W = T(\mathbb{I})$. It is easy to check that $R_w = T$. This concludes the proof. \hfill \Box

Proposition 2.6. If $p \geq 2$, $S(L^p(\varphi))$ is sufficient.

Proof. Let $X \in L^p(\varphi)$ be such that $\Omega(X,X) = 0$ for every $\Omega \in S(L_p(\varphi))$. By the previous lemma, since $|X|^{p-2} = L^{p-2}(\varphi)$, the right multiplication operator $R_w$ with $W = \frac{|X|^{p-2}}{\alpha}, \alpha \in \mathbb{R}$ satisfying $\|\frac{|X|^{p-2}}{\alpha}\|_{p/p-2} \leq 1$, represents a sesquilinear form $\Omega \in S(L_p(\varphi))$. By the assumption, $\Omega(X,X) = 0$. We then have

$$\Omega(X,X) = \varphi[X(R_wX)\ast] = \varphi[(XW)\ast X] = \varphi[|X|^{p-2}] = 0 \Rightarrow X = 0,$$

by the faithfulness of $\varphi$. \hfill \Box

3. CQ*-algebras over finite von Neumann algebras

Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{F} = \{\varphi_\alpha; \alpha \in \mathcal{I}\}$ be a family of normal, finite traces on $\mathcal{M}$. As usual, we say that the family $\mathcal{F}$ is sufficient if for $X \in \mathcal{M}$, $X \geq 0$ and $\varphi_\alpha(X) = 0$ for every $\alpha \in \mathcal{I}$, then $X = 0$ (clearly, if $\mathcal{F} = \{\varphi\}$, then $\mathcal{F}$ is sufficient if, and only if, $\varphi$ is faithful). In this case, $\mathcal{M}$ is a finite von Neumann algebra [15] ch.7. We assume, in addition, that the following condition (P) is satisfied:

(P) $\varphi_\alpha(\mathbb{I}) \leq 1, \forall \alpha \in \mathcal{I}.$

Then we define

$$\|X\|_{p,\mathcal{I}} = \sup_{\alpha \in \mathcal{I}} \|X\|_{p,\varphi_\alpha} = \sup_{\alpha \in \mathcal{I}} \varphi_\alpha(|X|^p)^{1/p}.$$

Since $\mathcal{F}$ is sufficient, $\|\cdot\|_{p,\mathcal{I}}$ is a norm on $\mathcal{M}$.

In the sequel we will need the following Lemmas whose simple proofs will be omitted.

Lemma 3.1. Let $\mathcal{M}$ be a von Neumann algebra in Hilbert space $H$, $\{P_\alpha\}_{\alpha \in \mathcal{I}}$ a family of projections of $\mathcal{M}$ with

$$\bigvee_{\alpha \in \mathcal{I}} P_\alpha = \mathcal{P}.$$
Lemma 3.2. Let $\mathcal{F} = \{\varphi_\alpha\}_{\alpha \in I}$ be a sufficient family of normal, finite traces on the von Neumann algebra $\mathcal{M}$ and let $P_\alpha$ be the support of $\varphi_\alpha$. Then, $\vee P_\alpha = I$, where $I$ denotes the identity of $\mathcal{M}$.

It is well-known that the support of each $\varphi_\alpha$ enjoy the following properties

(i) $P_\alpha \in Z(\mathcal{M})$, the center of $\mathcal{M}$, for each $\alpha \in I$;
(ii) $\varphi_\alpha(X) = \varphi_\alpha(XP_\alpha)$, for each $\alpha \in I$.

From the two preceding lemmas it follows that, if the $P_\alpha$’s are as in Lemma 3.2, then

$AP_\alpha = 0 \quad \forall \alpha \in I \Rightarrow A = 0$.

If Condition (P) is fulfilled, then

$\|X\|_{p,I} = \sup_{\alpha \in I} \|XP_\alpha\|_{p,\alpha} \quad \forall X \in \mathcal{M}$

Clearly, the sufficiency of the family of traces and Condition (P) imply that $\| \cdot \|_{p,I}$ is a norm $\mathcal{M}$.

Proposition 3.3. Let $\mathcal{M}(p,I)$ denote the Banach space completion of $\mathcal{M}$ with respect to the norm $\| \cdot \|_{p,I}$. Then $(\mathcal{M}(p,I)[\| \cdot \|_{p,I}], \mathcal{M}[\| \cdot \|_{B(H)}])$ is a CQ*-algebra.

Proof. Indeed, we have

(1) $\| X^* \|_{p,I} = \sup_{\alpha \in I} \| X^*P_\alpha \|_{p,\alpha} = \sup_{\alpha \in I} \| (XP_\alpha)^* \|_{p,\alpha} = \| X \|_{p,I} \quad \forall X \in \mathcal{M}$.

Furthermore, for every $X,Y \in \mathcal{M}$,

(2) $\| XY \|_{p,I} = \sup_{\alpha \in I} \| XY^*P_\alpha \|_{p,\alpha} \leq \| X \|_{B(H)} \sup_{\alpha \in I} \| YP_\alpha \|_{p,\alpha} = \| X \|_{B(H)} \| Y \|_{p,I}$.

Finally, condition (P) implies that

$\| X \|_{p,I} \leq \| X \|_{B(H)} \quad \forall X \in \mathcal{M}$.

From (1) and (2) it follows that $\mathcal{M}(p,I)$ is a Banach quasi *-algebra. It is clear that $\| \cdot \|_{B(H)}$ satisfies the conditions (a.1)-(a.3) of Section 1. Therefore $(\mathcal{M}(p,I), \mathcal{M})$ is a CQ *-algebra. $\square$

The next step consists in investigating the Banach space $\mathcal{M}(p,I)[\| \cdot \|_{p,I}]$. In particular we are interested in the question as to whether $\mathcal{M}(p,I)[\| \cdot \|_{p,I}]$ can be identified with a space of operators affiliated with $\mathcal{M}$. For shortness, whenever no ambiguity can arise, we write $\mathcal{M}_p$ instead of $\mathcal{M}(p,I)$.

Let $\mathcal{F} = \{\varphi_\alpha\}_{\alpha \in I}$ be a sufficient family of normal, finite traces on the von Neumann algebra $\mathcal{M}$ satisfying Condition (P). The traces $\varphi_\alpha$ are not necessarily faithful. Put $\mathcal{M}_\alpha = \mathcal{M}P_\alpha$, where, as before, $P_\alpha$ denotes the support of $\varphi_\alpha$. Each $\mathcal{M}_\alpha$ is a von Neumann algebra and $\varphi_\alpha$ is faithful in $\mathcal{M}P_\alpha$ [13 Proposition V. 2.10].

More precisely,

$\mathcal{M}_\alpha := \mathcal{M}P_\alpha = \{ Z = XP_\alpha, \text{ for some } X \in \mathcal{M} \}$.

The positive cone $\mathcal{M}_\alpha^+$ of $\mathcal{M}_\alpha$ equals the set

$\{ Z = XP_\alpha, \text{ for some } X \in \mathcal{M}^+ \}$. 
For $Z = XP_\alpha \in \mathcal{M}_\alpha^+$, we put:

$$\sigma_\alpha(Z) := \varphi_\alpha(XP_\alpha).$$

The definition of $\sigma_\alpha(Z)$ does not depend on the particular choice of $X$. Each $\sigma_\alpha$ is a normal, finite, faithful trace on $\mathcal{M}_\alpha$. It is then possible to consider the spaces $L^p(\mathcal{M}_\alpha, \sigma_\alpha), p \geq 1$, in the usual way. The norm of $L^p(\mathcal{M}_\alpha, \sigma_\alpha)$ is indicated as $\| \cdot \|_{p, \alpha}$.

Let now $(X_k)$ be a Cauchy sequence in $\mathcal{M}[[\| \cdot \|_p]]$. For each $\alpha \in I$, we put $Z_k^{(\alpha)} = X_kP_\alpha$. Then, for each $\alpha \in I$, $(Z_k^{(\alpha)})$ is a Cauchy sequence in $\mathcal{M}_\alpha[[\| \cdot \|_{p, \alpha}]]$. Indeed, since $|Z_k^{(\alpha)} - Z_h^{(\alpha)}|^p = |X_k - X_h|^p P_\alpha$,

$$\|Z_k^{(\alpha)} - Z_h^{(\alpha)}\|_{p, \alpha} = \sigma_\alpha(|Z_k^{(\alpha)} - Z_h^{(\alpha)}|^p)^{1/p} = \varphi_\alpha(|X_k - X_h|^p P_\alpha)^{1/p} = \varphi_\alpha(|X_k - X_h|^p)^{1/p} \to 0.$$

Therefore, for each $\alpha \in I$, there exists an operator $Z^{(\alpha)} \in L^p(\mathcal{M}_\alpha, \sigma_\alpha)$ such that:

$$Z^{(\alpha)} = \| \cdot \|_{p, \alpha} - \lim_{k \to \infty} Z_k^{(\alpha)}.$$

It is now natural to ask the question as to whether there exists an operator $X$ closed, densely defined, affiliated with $\mathcal{M}$ which reduces to $Z^{(\alpha)}$ on $\mathcal{M}_\alpha$. To begin with, we assume that the projections $\{P_\alpha\}$ are mutually orthogonal. In this case, putting $\mathcal{H}_\alpha = P_\alpha \mathcal{H}$, we have

$$\mathcal{H} = \bigoplus_{\alpha \in I} \mathcal{H}_\alpha = \{(f_\alpha) : f_\alpha \in \mathcal{H}_\alpha, \sum_{\alpha \in I} \|f_\alpha\|^2 < \infty\}.$$

We put

$$D(X) = \{ (f_\alpha) \in \mathcal{H} : f_\alpha \in D(Z^{(\alpha)}); \sum_{\alpha \in I} \|Z^{(\alpha)} f_\alpha\|^2 < \infty \}$$

and for $f = (f_\alpha) \in D(X)$ we define

$$Xf = (Z^{(\alpha)} f_\alpha).$$

Then

(i) $D(X)$ is dense in $\mathcal{H}$.

Indeed, $D(X)$ contains all $f = (f_\alpha)$ with $f_\alpha = 0$ except that for a finite subset of indices.

(ii) $X$ is closed in $\mathcal{H}$.

Indeed, let $f_n = (f_{n, \alpha})$ be a sequence of elements of $D(X)$ with $f_n \to g = (g_\alpha) \in \mathcal{H}$ and $Xf_n \to h$. Since

$$f_n \to g \iff f_{n, \alpha} \to g_\alpha \in \mathcal{H}_\alpha, \forall \alpha \in I$$

and

$$Xf_n \to h \iff (Xf_n)_\alpha \to h_\alpha \in \mathcal{H}_\alpha, \forall \alpha \in I,$$

by $(Xf_n)_\alpha = Z^{(\alpha)} f_{n, \alpha}$ and from the closedness of each $Z^{(\alpha)}$ in $\mathcal{H}_\alpha$, we get

$$g_\alpha \in D(Z^{(\alpha)}) \text{ and } h_\alpha = Z^{(\alpha)} g_\alpha.$$
It remains to check that \( \sum_{\alpha \in I} \| Z^{(\alpha)} g_\alpha \| < \infty \) but this is clear, since both \( (Z^{(\alpha)} g_\alpha) \) and \( h = (h_\alpha) \in \mathcal{H} \).

(iii) \( X \eta \mathcal{M} \).

Let \( Y \in \mathcal{M}' \). Then, \( \forall f \in \mathcal{H}, Yf = (YP_\alpha f) \) and \( YP_\alpha \in (\mathcal{MP}_\alpha)' = \mathcal{M}' \).

Therefore
\[
XYf = ((XY)P_\alpha f) = (YP_\alpha f) = Yf.
\]

In conclusion, \( X \) is a measurable operator.

Thus, we have proved the following

**Proposition 3.4.** Let \( \mathfrak{F} = \{ \varphi_\alpha \}_{\alpha \in I} \) be a sufficient family of normal, finite traces on the von Neumann algebra \( \mathcal{M} \). Assume that Condition \((P)\) is fulfilled and that the \( \varphi_\alpha \)'s have mutually orthogonal supports. Then \( \mathcal{M}_p, p \geq 1, \) consists of measurable operators.

The analysis of the general case would really be simplified if, from a given sufficient family \( \mathfrak{F} \) of normal finite traces, one could extract (or construct) a *sufficient* subfamily \( \mathcal{G} \) of traces with mutually orthogonal supports. Apart from quite simple situations (for instance when \( \mathfrak{F} \) is finite or countable), we do not know if this is possible or not. There is however a relevant case where this can be fairly easily done. This occurs when \( \mathfrak{F} \) is a convex and \( \tau^*-\)compact family of traces on \( \mathcal{M} \).

**Lemma 3.5.** Let \( \mathfrak{F} \) be a convex \( \tau^*-\)compact family of normal, finite traces on a von Neumann algebra \( \mathcal{M} \); assume that, for each central operator \( Z \), with \( 0 \leq Z \leq 1 \), and each \( \eta \in \mathfrak{F} \) the functional \( \eta_Z(X) := \eta(XZ) \) belongs to \( \mathfrak{F} \). Let \( \mathcal{E} \mathfrak{F} \) be the set of extreme elements of \( \mathfrak{F} \). If \( \eta_1, \eta_2 \in \mathcal{E} \mathfrak{F}, \eta_1 \neq \eta_2, \) and \( P_1 \) and \( P_2 \) are their respective supports, then \( P_1 \) and \( P_2 \) are orthogonal.

**Proof.** Let \( P_1, P_2 \) be, respectively, the supports of \( \eta_1 \) and \( \eta_2 \). We begin with proving that either \( P_1 = P_2 \) or \( P_1 P_2 = 0 \). Indeed, assume that \( P_1 P_2 \neq 0 \). We define
\[
\eta_{1,2}(X) = \eta_1(XP_2) \quad X \in \mathcal{M}.
\]

Were \( \eta_{1,2} = 0 \), then, in particular \( \eta_{1,2}(P_2) = 0 \), i.e. \( \eta_1(P_2) = 0 \) and therefore, by definition of support, \( P_2 \leq 1 - P_1 \). This implies that \( P_1 P_2 = 0 \), which contradicts the assumption. We now show that the support of \( \eta_{1,2} \) is \( P_1 P_2 \). Let, in fact, \( Q \) be a projection such that \( \eta_{1,2}(Q) = 0 \). Then
\[
\eta_1(QP_2) = 0 \Rightarrow QP_2 \leq 1 - P_1 \Rightarrow QP_2 (1 - P_1) = QP_2 \Rightarrow QP_2 P_1 = 0.
\]

Then the largest \( Q \) for which this happens is \( 1 - P_2 P_1 \). We conclude that the support of the trace \( \eta_{1,2} \) is \( P_1 P_2 \). Finally, by definition, one has \( \eta_{1,2}(X) = \eta_1(XP_2) \), and, since \( XP_2 \leq X \),
\[
\eta_{1,2}(X) = \eta_1(XP_2) \leq \eta_1(X) \quad \forall X \in \mathcal{M}.
\]

Thus \( \eta_1 \) majorizes \( \eta_{1,2} \). But \( \eta_1 \) is extreme in \( \mathfrak{F} \). Therefore \( \eta_{1,2} \) has the form \( \lambda \eta_1 \) with \( \lambda \in [0,1] \). This implies that \( \eta_{1,2} \) has the same support as \( \eta_1 \); therefore \( P_1 P_2 = P_1 \) i.e. \( P_1 \leq P_2 \). Starting from \( \eta_{2,1}(X) = \eta_2(XP_1) \), we get, in similar way, \( P_2 \leq P_1 \). Therefore, \( P_1 P_2 \neq 0 \) implies \( P_1 = P_2 \). However, two different traces of \( \mathcal{E} \mathfrak{F} \) cannot have the same support. Indeed, assume that there exist \( \eta_1, \eta_2 \in \mathfrak{F} \) having the same support \( P \). Since \( P \) is central, we can consider the von Neumann algebra \( \mathcal{M}P \). The restrictions of \( \eta_1, \eta_2 \)
to $\mathcal{M}P$ are normal faithful semifinite traces. By [14, Prop. V.2.31] there exist a central element $Z$ in $\mathcal{M}P$ with $0 \leq Z \leq P$ ($P$ is here considered as the unit of $\mathcal{M}P$) such that

\begin{equation}
\eta_1(X) = (\eta_1 + \eta_2)(ZX) \quad \forall X \in (\mathcal{M}P)_+.
\end{equation}

Then $Z$ also belongs to the center of $\mathcal{M}$, since for every $V \in \mathcal{M}$

$$ZV = Z(VP + VP^\perp) = VZP = VZ.$$

Therefore the functionals

$$\eta_{1,Z}(X) := \eta_1(XZ) \quad \eta_{2,Z}(X) := \eta_2(XZ) \quad X \in \mathcal{M}$$

belong to the family $\mathcal{F}$ and are majorized, respectively, by the extreme elements $\eta_1, \eta_2$. Then, there exist $\lambda, \mu \in [0, 1]$ such that

$$\eta_1(XZ) = \lambda \eta_1(X) \quad \eta_2(XZ) = \mu \eta_1(X), \quad \forall X \in \mathcal{M}.$$

If $\lambda = 1$ we would have, from (3), $\eta_2(ZX) = 0$, for every $X \in (\mathcal{M}P)_+$; in particular, $\eta_2(|Z|^2) = 0$; this implies that $Z = 0$. Thus $\lambda \neq 1$. Analogously, $\mu \neq 0$; indeed, if $\mu = 0$, then $\eta_1(X) = \lambda \eta_1(X)$ and thus $\lambda = 1$. Therefore there exist $\lambda, \mu \in (0, 1)$ such that

$$\eta_1(X) = \lambda \eta_1(X) + \mu \eta_2(X) \quad \forall X \in \mathcal{M}P,$$

which, in turn, implies

$$\eta_1(X) = \lambda \eta_1(X) + \mu \eta_2(X) \quad \forall X \in \mathcal{M}$$

Hence,

$$(1 - \lambda) \eta_1(X) = \mu \eta_2(X) \quad \forall X \in \mathcal{M}.$$

From the last equality, dividing by $\max\{1 - \lambda, \mu\}$ one gets that one of the two elements is a convex combination of the other and of 0; which is absurd. In conclusion, different supports of extreme traces of $\mathcal{F}$ are orthogonal.

\[\square\]

Since, for every $X \in \mathcal{M}$, $\|X\|_{p,Z}$ remains the same if computed either with respect to $\mathcal{F}$ or to $\mathcal{E}_{\mathcal{F}}$, we can deduce the following

**Theorem 3.6.** Let $\mathcal{F}$ be a convex and $w^\ast$-compact sufficient family of normal, finite traces on the von Neumann algebra $\mathcal{M}$. Assume that $\mathcal{F}$ satisfies Condition (P) and that for each central operator $Z$, with $0 \leq Z \leq I$, and each $\eta \in \mathcal{F}$ the functional $\eta_Z(X) := \eta(XZ)$ belongs to $\mathcal{F}$. Then the completion $\mathcal{M}_p[\|\cdot\|_{p,Z}]$, consists of measurable operators.

Families of traces satisfying the assumptions of Theorem 3.6 will be constructed in the next section.
4. A REPRESENTATION THEOREM

Once we have constructed in the previous section some CQ*-algebras of operators affiliated to a given von Neumann algebra, it is natural to pose the question under which conditions can an abstract CQ*-algebra \((X, \mathfrak{A})\) be realized as a CQ*-algebra of this type.

Let \((X, \| \cdot \|; \mathfrak{A}_0[\| \cdot \|_0])\) be a CQ*-algebra with unit \(e\) and let
\[
T(X) = \{ \Omega \in S(X) : \Omega(x, x) = \Omega(x^*, x^*), \forall x \in X \}.
\]
We remark that if \(\Omega \in T(X)\) then, by polarization, \(\Omega(y^*, x^*) = \Omega(x, y)\), \(\forall x, y \in X\).

It is easy to prove that the set \(T(X)\) is convex.

For each \(\Omega \in T(X)\), we define a linear functional \(\omega_\Omega\) on \(\mathfrak{A}_0\) by
\[
\omega_\Omega(a) := \Omega(a, e) \quad a \in \mathfrak{A}_0.
\]
We have
\[
\omega_\Omega(a^*a) = \Omega(a^*a, e) = \Omega(a, a) = \Omega(a^*, a^*) = \omega_\Omega(aa^*) \geq 0.
\]
This shows at once that \(\omega_\Omega\) is positive and tracial.

We put
\[
\mathcal{M}_T(\mathfrak{A}_0) = \{ \omega_\Omega; \Omega \in T(X) \}.
\]
From the convexity of \(T(X)\) it follows easily that \(\mathcal{M}_T(\mathfrak{A}_0)\) is convex too. If we denote with \(\| f \|_2\) the norm of the bounded functional \(f\) on \(\mathfrak{A}_0\), we also get
\[
\| \omega_\Omega \|_2^2 = \omega_\Omega(e, e) \leq \| e \|_2^2.
\]
Therefore
\[
\mathcal{M}_T(\mathfrak{A}_0) \subseteq \{ \omega \in \mathfrak{A}_0^\# : \| \omega \|_2^2 \leq \| e \|_2^2 \},
\]
where \(\mathfrak{A}_0^\#\) denotes the topological dual of \(\mathfrak{A}_0[\| \cdot \|_0]\).

Setting
\[
f_\Omega(a) := \frac{\omega_\Omega(a)}{\| a \|_2}
\]
we get
\[
f_\Omega \in \{ \omega \in \mathfrak{A}_0^\# : \| \omega \|_2^2 \leq 1 \}.
\]
By the Banach - Alaglou theorem, the set \(\{ \omega \in \mathfrak{A}_0^\# : \| \omega \|_2^2 \leq 1 \}\) is a \(w^*\)-compact subset of \(\mathfrak{A}_0^\#\). Then, the set \(\{ \omega \in \mathfrak{A}_0^\# : \| \omega \|_2^2 \leq \| e \|_2^2 \}\) is also \(w^*\)-compact.

**Proposition 4.1.** \(\mathcal{M}_T(\mathfrak{A}_0)\) is \(w^*\)-closed and, therefore, \(w^*\)-compact.

**Proof.** Let \((\omega_\Omega_\alpha)\) be a net in \(\mathcal{M}_T(\mathfrak{A}_0)\) \(w^*\)-converging to a functional \(\omega \in \mathfrak{A}_0^\#\). We will show that \(\omega = \omega_\Omega\) for some \(\Omega \in T(X)\). Let us begin with defining \(\Omega_\alpha(a, b) = \omega(b^*a), a, b \in \mathfrak{A}_0\). By the definition itself, \((\omega_\Omega_\alpha)(a) \longrightarrow \omega(a) = \Omega_\alpha(a, e)\). Moreover, for every \(a, b \in \mathfrak{A}_0\),
\[
\Omega_\alpha(a, b) = \omega(b^*a) = \lim_\alpha \omega_\Omega_\alpha(b^*a) = \lim_\alpha \Omega_\alpha(a, b).
\]
Therefore
\[
\Omega_\alpha(a, a) = \lim_\alpha \Omega_\alpha(a, a) \geq 0.
\]
We also have
\[
\| \Omega_\alpha(a, b) \| = \lim_\alpha \| \Omega_\alpha(a, b) \| \leq \| a \| \| b \|.
\]
Hence \( \Omega_0 \) can be extended by continuity to \( \mathfrak{X} \times \mathfrak{X} \). Indeed, let
\[
x = \| \cdot \| - \lim_n a_n \quad y = \| \cdot \| - \lim_n b_n \quad (a_n, b_n) \subseteq \mathfrak{A}_0
\]
then
\[
| \Omega_o(a_n, b_n) - \Omega_o(a_m, b_m) | = | \Omega_o(a_n, b_n) - \Omega_o(a_m, b_n) + \Omega_o(a_m, b_n) - \Omega_o(a_m, b_m) | \leq \\leq | \Omega_o(a_n - a_m, b_n) | + | \Omega_o(a_m, b_n - b_m) | \leq \| a_n - a_m \| \| b_n \| + \| a_m \| \| b_n - b_m \| \to 0,
\]
since \( (\| a_n \|) \) ed \( (\| b_n \|) \) are bounded sequences. Therefore we can define
\[
\Omega(x, y) = \lim_n \Omega_o(a_n, b_n).
\]
Clearly, \( \Omega(x, x) \geq 0 \quad \forall x \in \mathfrak{X} \).
It is easily checked that \( \Omega \in \mathcal{T}(\mathfrak{X}) \). This concludes the proof. \( \square \)

Since \( \mathfrak{M}_T(\mathfrak{A}_o) \) is convex and \( w^* \)-compact, by the Krein-Milman theorem it follows that it has extreme points and it coincides with the \( w^* \)-closure of the convex hull of the set \( \mathfrak{M}_T(\mathfrak{A}_o) \) of its extreme points.

By the Gelfand - Naimark theorem each \( C^* \)-algebra is isometrically *-isomorphic to a \( C^* \)-algebra of bounded operators in Hilbert space. This isometric *-isomorphism is called the universal *-representation.

Thus, let \( \pi \) be the universal *-representation of \( \mathfrak{A}_0 \) and \( \pi(\mathfrak{A}_0)'' \) the von Neumann algebra generated by \( \pi(\mathfrak{A}_0) \).

For every \( \Omega \in \mathcal{T}(\mathfrak{X}) \) and \( a \in \mathfrak{A}_0 \), we put
\[
\varphi_\Omega(\pi(a)) = \omega_\Omega(a).
\]

Then, for each \( \Omega \in \mathcal{T}(\mathfrak{X}) \), \( \varphi_\Omega \) is a positive bounded linear functional on the operator algebra \( \pi(\mathfrak{A}_0) \).

Clearly,
\[
\varphi_\Omega(\pi(a)) = \omega_\Omega(a) = \Omega(a, e)
\]
\[
| \varphi_\Omega(\pi(a)) | = | \omega_\Omega(a) | = | \Omega(a, e) | \leq \| a \| \| e \| \leq \| a \| \| e \|^2 = \| \pi(a) \| \| e \|^2.
\]
Thus \( \varphi_\Omega \) is continuous on \( \pi(\mathfrak{A}_0) \).

By \cite{[10]} Theorem 10.1.2], \( \varphi_\Omega \) is weakly continuous and so it extends uniquely to \( \pi(\mathfrak{A}_0)'' \). Moreover, since \( \varphi_\Omega \) is a trace on \( \pi(\mathfrak{A}_0) \), the extension \( \varphi_\Omega \) is a trace on \( \mathfrak{M} := \pi(\mathfrak{A}_0)'' \) too.

The norm \( \| \varphi_\Omega \| \) of \( \varphi_\Omega \) as a linear functional on \( \mathfrak{M} \) equals the norm of \( \varphi_\Omega \) as a functional on \( \pi(\mathfrak{A}_0) \).

We have:
\[
\| \varphi_\Omega \| = \varphi_\Omega(\pi(e)) = \omega_\Omega(e) \leq \| e \|^2.
\]

The set
\[
\mathfrak{M}_T(\mathfrak{A}_o) = \{ \varphi_\Omega; \Omega \in \mathcal{T}(\mathfrak{X}) \}
\]
is convex and \( w^* \)-compact in \( \mathfrak{M}^2 \), as can be easily seen by considering the map
\[
\omega_\Omega \in \mathfrak{M}_T(\mathfrak{A}_o) \to \varphi_\Omega \in \mathfrak{M}_T(\mathfrak{A}_o)
\]
which is linear and injective and taking into account the fact that, if \( a_\alpha \to a \) in \( \mathfrak{A}_o[\| \cdot \|] \), then \( \varphi_\Omega(\pi(a_\alpha) - \pi(a)) = \omega_\Omega(a_\alpha - a) \to 0. \)
Let $\mathcal{N}_T(\mathfrak{A}_o)$ be the set of extreme points of $\mathfrak{N}_T(\mathfrak{A}_o)$; then $\mathfrak{N}_T(\mathfrak{A}_o)$ coincides with $w^*$-closure of the convex hull of $\mathcal{E}N_T(\mathfrak{A}_o)$. The extreme elements of $\mathfrak{N}_T(\mathfrak{A}_o)$ are easily characterized by the following

**Proposition 4.2.** $\tilde{\varphi}_\Omega$ is extreme in $\mathfrak{N}_T(\mathfrak{A}_o)$ if, and only if, $\omega_\Omega$ is extreme in $\mathfrak{M}_T(\mathfrak{A}_o)$.

**Definition 4.3.** A Banach quasi $*$-algebra $(\mathfrak{X}[\| \cdot \|], \mathfrak{A}_o[\| \cdot \|_o])$ is said to be strongly regular if $T(\mathfrak{X})$ is sufficient and

$$\| x \| = \sup_{\Omega \in T(\mathfrak{X})} \Omega(x, x)^{1/2}, \quad \forall x \in \mathfrak{X}.$$ 

**Example 4.4.** If $\mathfrak{M}$ is a von Neumann algebra possessing a sufficient family $\tilde{\mathfrak{F}}$ of normal finite traces, then the CQ*-algebra $(\mathfrak{M}_p, \mathfrak{M})$ constructed in Section 3 is strongly regular. This follows from the definition itself of the norm in the completion.

**Example 4.5.** If $\varphi$ is a normal faithful finite trace on $\mathfrak{M}$, then $T(L^p(\varphi))$, for $p \geq 2$, is sufficient. To see this, we start with defining $\Omega_0$ on $\mathfrak{M} \times \mathfrak{M}$ by

$$\Omega_0(X, Y) = \varphi(Y^* X), \quad X, Y \in \mathfrak{M}.$$ 

Then

$$|\Omega_o(X, Y)| = |\varphi(Y^* X)| \leq \|X\|_p \|Y\|_{p'}, \quad \forall X, Y \in \mathfrak{M}.$$ 

Since $p \geq 2$, then $L^p(\varphi)$ is continuously embedded into $L^p(\varphi)$. Thus, there exists $\gamma > 0$ such that $\|Y\|_{p'} \leq \gamma \|Y\|_p$ for every $Y \in \mathfrak{M}$. Let us define

$$\Omega(X, Y) = \frac{1}{\gamma} \Omega_o(X, Y), \quad \forall X, Y \in \mathfrak{M}.$$ 

Then

$$|\Omega(X, Y)| \leq \|X\|_p \|Y\|_{p'}, \quad \forall X, Y \in \mathfrak{M}.$$ 

Hence, $\Omega$ has a unique extension, denoted with the same symbol, to $L^p(\varphi) \times L^p(\varphi)$. It is easily seen that $\Omega \in T(L^p(\varphi))$.

Were, for some $X \in L^p(\varphi)$, $\Omega(X, X) = 0$, for every $\Omega \in T(L^p(\varphi))$, we would then have $\Omega(X, X) = \|X\|_2^2 = 0$. This, clearly, implies $X = 0$. The equality $\Omega(X, X) = \|X\|_2^2$ also shows that $L^2(\varphi)$ is strongly regular.

Let now $(\mathfrak{X}[\| \cdot \|], \mathfrak{A}_o[\| \cdot \|_o])$ be a CQ*-algebra with unit $e$ and sufficient $T(\mathfrak{X})$. Let $\pi : \mathfrak{A}_o \rightarrow \mathcal{B}(\mathcal{H})$ be the universal representation of $\mathfrak{A}_o$. Assume that the $C^*$-algebra $\pi(\mathfrak{A}_o) := \mathfrak{M}$ is a von Neumann algebra. In this case, $\mathfrak{M}_T(\mathfrak{A}_o) = \mathfrak{M}_{T}(\mathfrak{A}_o)$ and $\mathfrak{N}_T(\mathfrak{A}_o)$ is a family of traces satisfying Condition (P). Therefore, by Proposition 3.3, we can construct for $p \geq 1$, the CQ*-algebras $(\mathfrak{M}_p[\| \cdot \|_p, \mathfrak{N}_T(\mathfrak{A}_o)], \mathfrak{M}[\| \cdot \|])$. Clearly, $\mathfrak{A}_o$ can be identified with $\mathfrak{M}$. It is then natural to pose the question if also $\mathfrak{X}$ can be identified with some $\mathfrak{M}_p$. The next Theorem provides the answer to this question.

**Theorem 4.6.** Let $(\mathfrak{X}[\| \cdot \|], \mathfrak{A}_o[\| \cdot \|_o])$ be a CQ*-algebra with unit $e$ and and sufficient $T(\mathfrak{X})$.

Then there exist a von Neumann algebra $\mathfrak{M}$ and a monomorphism $\Phi : x \in \mathfrak{X} \rightarrow \Phi(x) := \tilde{X} \in \mathfrak{M}_2$ with the following properties:
Then the family of traces \( \Omega \) is a von Neumann algebra. By Proposition 4.1, the family of traces \( \{ \Omega \} \) is strongly regular, then

\[ (i) \quad \Phi \text{ extends the universal } *\text{-representation } \pi \text{ of } \mathfrak{A}_0; \]

\[ (ii) \quad \Phi(x^*) = \Phi(x^*), \quad \forall x \in \mathfrak{X}; \]

\[ (iii) \quad \Phi(xy) = \Phi(x)\Phi(y) \text{ for every } x, y \in \mathfrak{X} \text{ such that } x \in \mathfrak{A}_0 \text{ or } y \in \mathfrak{A}_0. \]

Then \( \mathfrak{X} \) can be identified with a space of operators affiliated with \( \mathfrak{M} \).

Proof. Let \( \pi \) be the universal representation of \( \mathfrak{A}_0 \) and assume first that \( \pi(\mathfrak{A}_0) =: \mathfrak{M} \) is a von Neumann algebra. By Proposition 4.1, the family of traces \( \mathfrak{M}_T(\mathfrak{A}_0) \) is convex and \( w^* \)-compact. Moreover, for each central positive element \( Z \) with \( 0 \leq Z \leq 1 \) and for \( \varphi \in \mathfrak{M}_T(\mathfrak{A}_0) \), the trace \( \varphi_Z(X) := \varphi(ZX) \) yet belongs to \( \mathfrak{M}_T(\mathfrak{A}_0) \). Indeed, starting from the form \( \Omega \in T(\mathfrak{X}) \) which generates \( \varphi \), one can define the sesquilinear form

\[ \Omega_Z(x, y) := \Omega(x\pi^{-1}(Z^{1/2}), y\pi^{-1}(Z^{1/2})) \quad \forall x, y \in \mathfrak{X}. \]

We check that \( \Omega_Z \in T(\mathfrak{X}) \).

\[ (i) \quad \Omega_Z(x, x) = \Omega(x\pi^{-1}(Z^{1/2}), x\pi^{-1}(Z^{1/2})) \geq 0, \quad \forall x \in \mathfrak{X} \]

\[ (ii) \quad \text{We have, for every } x \in \mathfrak{X} \text{ and for every } a, b \in \mathfrak{A}_0, \]

\[ \Omega_Z(xa, b) = \Omega(xa\pi^{-1}(Z^{1/2}), b\pi^{-1}(Z^{1/2})) \]

\[ = \Omega(a\pi^{-1}(Z^{1/2}), x^*b\pi^{-1}(Z^{1/2})) \]

\[ = \Omega_Z(a, x^*b). \]

\[ (iii) \quad \text{We have, for every } x, y \in \mathfrak{X}, \]

\[ | \Omega_Z(x, y) | = | \Omega(x\pi^{-1}(Z^{1/2}), y\pi^{-1}(Z^{1/2})) | \]

\[ \leq || x\pi^{-1}(Z^{1/2}) || || \pi^{-1}(Z^{1/2})y || \]

\[ \leq || x || || \pi^{-1}(Z^{1/2}) ||_0 || y || || \pi^{-1}(Z^{1/2}) ||_0 \]

\[ \leq || x || || y ||. \]

\[ (iv) \quad \text{For every } x \in \mathfrak{X}, \]

\[ \Omega_Z(x^*, x^*) = \Omega(x^*\pi^{-1}(Z^{1/2}), x^*\pi^{-1}(Z^{1/2})) \]

\[ = \Omega(x\pi^{-1}(Z^{1/2}), x\pi^{-1}(Z^{1/2})) = \Omega_Z(x, x). \]

Moreover, \( \Omega_Z \) defines, for every \( A = \pi(a) \in \mathfrak{M} = \pi(\mathfrak{A}_0) \), the following trace

\[ \varphi_{\Omega_Z}(A) = \Omega_Z(a, e) = \Omega(a\pi^{-1}(Z^{1/2}), \pi^{-1}(Z^{1/2})) \]

\[ = \Omega(a\pi^{-1}(Z), e) = \Omega(\pi^{-1}(AZ), e) = \varphi_{\Omega}(AZ) \]

Then, the family of traces \( \mathfrak{M}_T(\mathfrak{A}_0) (= \mathfrak{M}_T(\mathfrak{A}_0)) \) satisfies the assumptions of Lemma 3.5 therefore, if \( \eta_1, \eta_2 \in \mathfrak{M}_T(\mathfrak{A}_0) \), denoting with \( P_1 \) and \( P_2 \) their respective supports, one has \( P_1P_2 = 0 \).

By the sufficiency of \( T(\mathfrak{X}) \) we get

\[ \| X \|_{2, \pi, \mathfrak{M}_T(\mathfrak{A}_0)} := \sup_{\varphi \in \mathfrak{M}_T(\mathfrak{A}_0)} \| X \|_{2, \varphi} = \sup_{\varphi \in \mathfrak{M}_T(\mathfrak{A}_0)} \| X \|_{2, \varphi} \quad \forall X \in \pi(\mathfrak{A}_0). \]

By Proposition 3.3 the Banach space \( \mathfrak{M}_2 \), completion of \( \mathfrak{M} \) with respect to the norm \( \| \cdot \|_{2, \pi, \mathfrak{M}_T(\mathfrak{A}_0)} \), is a CQ*-algebra. Moreover, since the supports of the extreme traces
Indeed, if \( N \) converges to \( T \) then, owed to the sufficiency of \( T \) implies that \( \| T \| \leq \| \cdot \| \).

The condition (iii) can be easily proved. If \( T \), then there exists a sequence \( \{a_n\} \) of elements of \( \mathfrak{A}_0 \) converging to \( x \) with respect to the norm \( \mathfrak{X}(\| \cdot \|) \). Put \( X_n = \pi(a_n), \ n \in \mathbb{N} \). Then,

\[
\| X_n - X_m \|_{2,\mathfrak{M}_2(\mathfrak{A}_0)} := \sup_{\varphi \in \mathfrak{M}_2(\mathfrak{A}_0)} \| \pi(a_n) - \pi(a_m) \|_{2,\varphi}
\]

\[
= \sup_{\Omega \in \mathcal{T}(X)} [\Omega((a_n - a_m)^*(a_n - a_m), e)]^{1/2}
\]

\[
= \sup_{\Omega \in \mathcal{T}(X)} [\Omega(a_n - a_m, a_n - a_m)]^{1/2} \leq \| a_n - a_m \| \to 0.
\]

Let \( \tilde{X} \) be the \( \| \cdot \|_{2,\mathfrak{M}_2(\mathfrak{A}_0)} \)-limit of the sequence \( \{X_n\} \) in \( \mathfrak{M}_2 \). We define \( \Phi(x) := \tilde{X} \).

For each \( x \in \mathfrak{X} \), we put

\[
p_{T(x)}(x) = \sup_{\Omega \in \mathcal{T}(x)} \Omega(x,x)^{1/2}.
\]

Then, owed to the sufficiency of \( T(\mathfrak{X}) \), \( p_{T(x)} \) is a norm on \( \mathfrak{X} \) weaker than \( \| \cdot \| \). This implies that

\[
\| \tilde{X} \|_{2,\mathfrak{M}_2(\mathfrak{A}_0)}^2 = \lim_{n \to \infty} \sup_{\Omega \in \mathcal{T}(x)} \Omega(a_n, a_n) = \lim_{n \to \infty} p_{T(x)}(a_n)^2 = p_{T(x)}(x)^2.
\]

From this equality it follows easily that the linear map \( \Phi \) is well defined and injective. The condition (iii) can be easily proved. If \( (\mathfrak{X}, \mathfrak{A}_0) \) is strongly regular, then, for every \( x \in \mathfrak{X} \), \( p_{T(x)}(x) = \| x \| \). Thus \( \Phi \) is isometric. Moreover, in this case, \( \Phi \) is surjective; indeed, if \( T \in \mathfrak{M}_2 \), then there exists a sequence \( T_n \) of bounded operators of \( \pi(\mathfrak{A}_0) \) which converges to \( T \) with respect to the norm \( \| \cdot \|_{2,\mathfrak{M}_2(\mathfrak{A}_0)} \). The corresponding sequence \( \{T_n\} \subseteq \mathfrak{A}_0 \), \( T_n = \Phi(T_n) \), converges to \( T \) with respect to the norm of \( \mathfrak{X} \) and \( \Phi(T) = T \) by definition. Therefore \( \Phi \) is an isometric \(*\)-isomorphism.

To complete the proof, it is enough to prove that the given \( \mathrm{CQ}^* \)-algebra \( (\mathfrak{X}, \mathfrak{A}_0) \) can be embedded in a \( \mathrm{CQ}^* \)-algebra \( (\mathfrak{K}, \mathfrak{B}_0) \) where \( \mathfrak{B}_0 \) is a \( \mathrm{W}^* \)-algebra. Of course, we may directly work with \( \pi(\mathfrak{A}_0) \) with the universal representation of \( \mathfrak{A}_0 \). The family of traces \( \mathfrak{M}_T(\mathfrak{A}_0) \) defined on \( \pi(\mathfrak{A}_0)' \) is not necessarily sufficient. Let \( P_{\Omega}, \ \Omega \in \mathcal{T}(\mathfrak{X}) \), denote the support of \( \tilde{\varphi}_\Omega \) and let

\[
P = \mathop{\bigvee}_{\Omega \in \mathcal{T}(\mathfrak{X})} P_{\Omega}.
\]

Then \( \mathfrak{B}_0 := \pi(\mathfrak{A}_0)''P \) is a von Neumann algebra, that we can complete with respect to the norm

\[
\| X \|_{2,\mathfrak{M}_2(\mathfrak{A}_0)} = \sup_{\Omega \in \mathcal{T}(\mathfrak{X})} \tilde{\varphi}_\Omega(X^*X), \ \ X \in \pi(\mathfrak{A}_0)''P.
\]

We obtain in this way a \( \mathrm{CQ}^* \)-algebra \( (\mathfrak{K}, \mathfrak{B}_0) \) with \( \mathfrak{B}_0 \) a \( \mathrm{W}^* \)-algebra. The faithfulness of \( \pi \) on \( \mathfrak{A}_0 \) implies that

\[
\pi(a)P = \pi(a), \ \ \forall a \in \mathfrak{A}_0.
\]

It remains to prove that \( \mathfrak{X} \) can be identified with a subspace of \( \mathfrak{K} \). But this can be shown in the very same way as we did in the first part: for each \( x \in \mathfrak{X} \) there exists
a sequence \( \{ a_n \} \subset \mathfrak{A}_0 \) such that \( \| x - a_n \| \to 0 \) as \( n \to \infty \). We now put \( X_n = \pi(a_n) \). Then, proceeding as before, we determine the element \( \hat{X} \in \mathfrak{K} \), where

\[
\hat{X} = \| \cdot \|_{2,\mathfrak{K}(\mathfrak{A}_0)} - \lim \pi(a_n)P.
\]

It is easy to see that the map \( x \in \mathcal{X} \to \hat{X} \in \mathfrak{K} \) is injective. If \( (\mathcal{X}, \mathfrak{A}_0) \) is regular, but \( \pi(\mathfrak{A}_0) \subset \pi(\mathfrak{A}_0)'' \), then \( \Phi \) is an isometry of \( \mathcal{X} \) into \( \mathfrak{M}_2 \), but needs not be surjective. □

**Acknowledgment** We acknowledge financial support of MIUR through national and local grants.

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