AUTOMORPHISMS OF THE DISK COMPLEX

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Abstract. We show that the automorphism group of the disk complex is isomorphic to the handlebody group. Using this, we prove that the outer automorphism group of the handlebody group is trivial.

1. Introduction

We show that the automorphism group of the disk complex is isomorphic to the handlebody group. Using this, we prove that the outer automorphism group of the handlebody group is trivial. These results and many of the details of the proof are inspired by Ivanov’s work [5] on the mapping class group and the curve complex.

Let $V = V_{g,n}$ be the genus $g$ handlebody with $n$ spots: a regular neighborhood of a finite, polygonal, connected graph in $\mathbb{R}^3$ with $n$ disjoint disks chosen on the boundary. See Figure 1 for a picture of $V_{2,2}$. We write $V = V_g$ when $n = 0$. Let $\partial_0 V$ denote the union of the spots. Let $\partial_+ V$ be the closure of $\partial V \setminus \partial_0 V$. So $\partial_+ V \cong S = S_{g,n}$ is a compact connected orientable surface of genus $g$ with $n$ boundary components. We write $S = S_g$ when $n = 0$. Define $e(V) = -\chi(\partial_+ V) = 2g - 2 + n$.

A simple closed curve $\alpha$ in $S = S_{g,n}$ is inessential if it cuts a disk off of $S$; otherwise $\alpha$ is essential. The curve $\alpha$ is peripheral if it cuts an
annulus off of $S$; otherwise $\alpha$ is non-peripheral. A properly embedded disk $D$ in $V = V_{g,n}$, with $\partial D \subset \partial_+ V$, is essential or non-peripheral exactly as its boundary is in $\partial_+ V$. We require any proper isotopy of $D \subset V$ to have track disjoint from the spots of $V$. This yields a proper isotopy of $\partial D$ in $\partial_+ V$.

**Definition 1.1** (Harvey [3]). The curve complex $\mathcal{C}(S)$ is the simplicial complex with vertex set being isotopy classes of essential, non-peripheral curves in $S$. The $k$–simplices are given by collections of $k + 1$ vertices having pairwise disjoint representatives.

**Definition 1.2** (McCullough [10]). The disk complex $\mathcal{D}(V)$ is the simplicial complex with vertex set being proper isotopy classes of essential, non-peripheral disks in $V$. The $k$–simplices are given by collections of $k + 1$ vertices having pairwise disjoint representatives.

Note that there is a natural inclusion $\mathcal{D}(V) \to \mathcal{C}(\partial_+ V)$ taking a disk to its boundary. This map is simplicial and injective.

If $\mathcal{K}$ is a simplicial complex then $\text{Aut}(\mathcal{K})$ denotes the group of simplicial automorphisms of $\mathcal{K}$. The elements of $\text{Aut}(\mathcal{C}(S))$ and $\text{Aut}(\mathcal{D}(V))$ are to be contrasted with mapping classes on the underlying spaces.

**Definition 1.3.** The mapping class group $\text{MCG}(S)$ is the group of homeomorphisms of $S$, up to isotopy. The handlebody group $\mathcal{H}(V)$ is the group of homeomorphisms of $V$, fixing the spots setwise, up to spot preserving isotopy.

Some authors refer to our $\text{MCG}(S)$ as the extended mapping class group, as orientation reversing homeomorphisms are allowed. Note there is a natural map $\mathcal{H}(V) \to \text{MCG}(\partial_+ V)$ which takes $f \in \mathcal{H}(V)$ to $f|_{\partial_+ V}$. Again, this map is an injective homomorphism.

Note finally that there is a natural homomorphism $\text{MCG}(S) \to \text{Aut}(\mathcal{C}(S))$ (and similarly for $V$). We will call any element of the image of this map a geometric automorphism. Our main theorem is:

**Theorem 9.3.** If a handlebody $V = V_{g,n}$ satisfies $e(V) \geq 3$ then the natural map $\mathcal{H}(V) \to \text{Aut}(\mathcal{D}(V))$ is a surjection.

In the language above: every element of $\mathcal{H}(V)$ is geometric. The plan of the proof of Theorem 9.3 is given in Section 3 and completed in Section 9. Section 4 shows that Theorem 9.3 is sharp: all handlebodies $V$ with $e(V) \leq 2$ exhibit some kind of exceptional behaviour. Theorem 9.3 has a corollary:

**Theorem 9.4.** If a handlebody $V = V_{g,n}$ satisfies $e(V) \geq 3$ then the natural map $\mathcal{H}(V) \to \text{Aut}(\mathcal{D}(V))$ is an isomorphism.
In Section 10 we use Theorem 9.3 to prove:

**Theorem 10.1.** If $e(V) \geq 3$ then the outer automorphism group of the handlebody group is trivial.

These results are inspired by work of Ivanov, Korkmaz and Luo [5, 8, 9]:

**Theorem 1.4.** If $3g - 3 + n \geq 3$, or if $(g, n) = (0, 5)$, then all elements of $\text{Aut}(\mathcal{C}(S_{g,n}))$ are geometric. Also, the outer automorphism group of $\text{MCG}(S)$ is trivial.

### 2. Background

The genus zero case of Theorem 1.4 is contained in the thesis of the first author [8, Theorem 1].

**Theorem 2.1.** If $g = 0$ and $n \geq 5$ then all elements of $\text{Aut}(\mathcal{C}(S_{0,n}))$ are geometric.

Spotted balls are the simplest handlebodies. Accordingly:

**Lemma 2.2.** The natural maps $D(V_{0,n}) \to \mathcal{C}(S_{0,n})$ and $\mathcal{H}(V_{0,n}) \to \text{MCG}(S_{0,n})$ are isomorphisms.

**Proof.** The three-manifold $V_{0,n}$ is an $n$–spotted ball. Every simple closed curve in $\partial_+ V$ bounds a disk in $V$. This proves that $D(V_{0,n}) \to \mathcal{C}(S_{0,n})$ is a surjection and thus, by the remark immediately after Definition 1.2, an isomorphism. It follows from the Alexander trick that the inclusion of mapping class groups is an isomorphism.

The genus zero case of Theorem 9.3 is an immediate corollary. We now give basic definitions.

Suppose that $V$ is a handlebody. Two disks $D, E \in D(V)$ are **topologically equivalent** if there is a mapping class $f \in \mathcal{H}(V)$ so that $f(D) = E$. The **topological type** of $D$ is its equivalence class in $D(V)$.

For any simplicial complex, $\mathcal{K}$, if $\sigma \in \mathcal{K}$ is a simplex then recall that

$$\text{link}(\sigma) = \{ \tau \in \mathcal{K} \mid \sigma \cap \tau = \emptyset, \ \sigma \cup \tau \in \mathcal{K} \}.$$

So if $\mathcal{D}$ is a simplex of $D(V)$ then $\text{link}(\mathcal{D})$ is the subcomplex of $D(V)$ spanned by disks $E$ disjoint from some $D \in \mathcal{D}$ and distinct from all $D \in \mathcal{D}$.

If $X \subset Y$ is a properly embedded submanifold then we write $\text{neigh}(X)$ and $\text{neigh}(X)$ to denote open and closed regular neighborhoods of $X$ in $Y$. If $X$ is codimension zero then the **frontier** of $X$ in $Y$ is the closure of $\partial X \setminus \partial Y$.
A simplex $D \in \mathcal{D}(V)$ is a cut system if $V \setminus \text{neigh}(D)$ is a spotted ball. Note that every disk of $D$ yields two spots of $V \setminus \text{neigh}(D)$.

Recall that for simple curves $\alpha, \beta$ properly embedded in $S$ the geometric intersection number $i(\alpha, \beta)$ is the minimum possible intersection number between proper isotopy representatives.

Two disks $D, E \in \mathcal{D}(V)$ are dual if $i(\partial D, \partial E) = 2$; equivalently, after a suitable proper isotopy $D$ and $E$ intersect along a single arc; equivalently, after a suitable proper isotopy a regular neighborhood of $D \cup E$ is a four-spotted ball with all spots essential in $V$. See Figure 2.

![Figure 2](image-url)

**Figure 2.** Every spot of the $V_{0.4}$ containing a pair of dual disks is essential in $V$.

If $D = \{D_i\}$ is a cut system we define $\text{dual}_i(D)$ to be the subcomplex spanned by the disks $E \in \mathcal{D}(V)$ which are dual to $D_i$ and disjoint from $D_j$ for all $j \neq i$. We take $\text{dual}(D)$ to be the complex spanned by $\bigcup_i \text{dual}_i(D)$.

### 3. The Proof of Theorem 9.3

Let $V = V_{g,n}$ be a genus $g$ handlebody with $n$ spots. We suppose that $g \geq 1$ and $e(V) \geq 3$. Let $\phi$ be any automorphism of $\mathcal{D}(V)$. Lemma 5.1 proves that $\phi$ preserves the topological types of disks. In addition, $\phi$ sends cut systems to cut systems (Claim 5.6). Next Lemma 7.2 shows that $\phi$ preserves duality. Also, for any cut system $D = \{D_i\}$, the complex $\text{dual}_i(D)$ is connected (Lemma 7.3).

Pick any geometric automorphism $f_{\text{cut}}$ so that $f_{\text{cut}}(D) = \phi(D)$, vertex-wise; $f_{\text{cut}}$ exists by Claim 5.6. Define $\phi_{\text{cut}} = f_{\text{cut}}^{-1} \circ \phi$. Thus

$$\phi_{\text{cut}}|D = \text{Id}.$$ 

Let $V' \cong V_{0,2g+n}$ be the spotted ball obtained by cutting $V$ along a regular neighborhood of $D$. Now, since $\phi_{\text{cut}}$ preserves $\text{link}(D) \cong \mathcal{D}(V')$, by Theorem 2.1 and Lemma 2.2 there is a homeomorphism $f : V' \to V'$ so that the induced automorphism $f \in \text{Aut}(\mathcal{D}(V'))$ satisfies $f =$
\(\phi_{\text{cut}}\mid \text{link}(\mathbb{D})\). Section 5 proves that \(f\) preserves the \(g\) pairs of spots of \(V'\) coming from \(\mathbb{D}\). Thus \(f\) can be glued to give a homeomorphism \(f_{\text{link}}: V \to V\) as well as an induced geometric automorphism \(f_{\text{link}} \in \text{Aut}(\mathcal{D}(V))\). Define \(\phi_{\text{link}} = f_{\text{link}} \circ \phi_{\text{cut}}\). Thus

\[
\phi_{\text{link}}|\mathbb{D} \cup \text{link}(\mathbb{D}) = \text{Id}.
\]

Recall that \(\phi_{\text{link}}\) preserves duals by Lemma 7.2. For every \(D_i \in \mathbb{D}\) pick some dual \(E_i \in \text{dual}_i(\mathbb{D})\). By Lemma 8.1 there is an integer \(m_i \in \mathbb{Z}\) so that \(T^{m_i}_i(E_i) = \phi_{\text{link}}(E_i)\), where \(T_i\) is the Dehn twist about \(D_i\). Define \(f_{\text{dual}} = \prod T^{m_i}_i\) and define \(\phi_{\text{dual}} = f_{\text{dual}}^{-1} \circ \phi_{\text{link}}\). Letting \(E = \{E_i\}\) we have

\[
\phi_{\text{dual}}|\mathbb{D} \cup \text{link}(\mathbb{D}) \cup E = \text{Id}.
\]

Recall that Lemma 7.3 proves that \(\text{dual}_i(\mathbb{D})\) is connected. Therefore, a crawling argument, given in Lemma 8.2 proves that

\[
\phi_{\text{dual}}|\mathbb{D} \cup \text{link}(\mathbb{D}) \cup \text{dual}(\mathbb{D}) = \text{Id}.
\]

Wajnryb [13] proves that the cut system complex is connected. Thus we may likewise crawl through \(\mathcal{D}(V)\) and prove (Section 9) that

\[
\phi_{\text{dual}} = \text{Id}
\]

and so prove that

\[
\phi = f_{\text{cut}} \circ f_{\text{link}} \circ f_{\text{dual}}.
\]

Thus \(\phi\) is geometric.

4. Small handlebodies

In this section we deal with the small cases, where \(e(V) = 2g-2+n \leq 2\). We start with genus zero. If \(n \leq 3\) then \(\mathcal{D}(V_0,n)\) is empty. By Lemma 2.2 the mapping class groups of \(V\) and \(\partial V\) are equal. Thus

\[
\mathcal{H}(V_0), \mathcal{H}(V_{0,1}) \cong \mathbb{Z}/2\mathbb{Z}
\]

while

\[
\mathcal{H}(V_{0,2}) \cong K_4 \quad \text{and} \quad \mathcal{H}(V_{0,3}) \cong \mathbb{Z}/2\mathbb{Z} \times \Sigma_3.
\]

Here \(K_4\) is the Klein four-group and \(\Sigma_3\) is the symmetric group on three objects [12 Appendix A].

If \(n = 4\) then \(\mathcal{D}\) is a countable collection of vertices with no higher dimensional simplices. Thus \(\text{Aut}(\mathcal{D}) = \Sigma_\infty\) is uncountable. However, there are only countably many geometric automorphisms. In fact, by Lemma 2.2 the mapping class group \(\mathcal{H}(V_{0,4})\) is isomorphic to \(K_4 \rtimes \text{PGL}(2,\mathbb{Z})\) [12 Appendix A].
For genus one, if \( n = 0 \) or \( 1 \) then \( \mathcal{D} \) is a single point and \( \text{Aut}(\mathcal{D}) \) is trivial. On the other hand

\[
\mathcal{H}(V_1), \; \mathcal{H}(V_{1,1}) \cong \mathbb{Z} \rtimes K_4.
\]

For \( V = V_{1,2} \) matters are more subtle. The subcomplex \( \text{NonSep}(V) \subset \mathcal{D}(V) \), spanned by non-separating disks, is a copy of the Bass-Serre tree for the meridian curve in \( S_{1,1} = \partial_+ V_{1,1} \) [2]. Thus \( \text{NonSep}(V) \) is a copy of \( T_\infty \): the regular tree with countably infinite valance. Now, if \( E \in \mathcal{D}(V) \) is separating then there is a unique disk \( D \) disjoint from \( E \); also, \( D \) is necessarily non-separating. It follows that \( \mathcal{D}(V) \) is a copy of \( \text{NonSep}(V) \) with countably many leaves attached to every vertex. Thus \( \text{Aut}(\mathcal{D}) \) contains a copy of \( \text{Aut}(T_\infty) \) as well as countably many copies of \( \Sigma_\infty \) and is therefore uncountable. As usual \( \mathcal{H}(V) \) is countable and so \( \text{Aut}(\mathcal{D}) \) contains non-geometric elements. However, following Luo’s treatment of \( C(S) \) [9] suggests the following problem:

**Problem 4.1.** Suppose that \( V = V_{1,2} \). Let \( \mathcal{G} \) be the subgroup of \( \text{Aut}(\mathcal{D}(V)) \) consisting of automorphisms preserving duality: if \( \phi \in \mathcal{H} \) and \( D, E \) are dual then so are \( \phi(D), \phi(E) \). Is every element of \( \mathcal{G} \) geometric?

Note that this approach of recording duality is precisely correct for the four-spotted ball; the complex where simplices record duality in \( V_{0,4} \) is the *Farey tessellation*, \( \mathcal{F} \), and every element of \( \text{Aut}(\mathcal{F}) \) is geometric. See [9, Section 3.2].

The last exceptional case is \( V = V_2 \). Let \( \text{NonSep}(V) \) be the subcomplex of \( \mathcal{D}(V) \) spanned by non-separating disks. Then \( \text{NonSep}(V) \) is an increasing union, as follows: \( \mathcal{N}_0 \) is a single triangle, \( \mathcal{N}_{i+1} \) is obtained by attaching (to every free edge of \( \mathcal{N}_i \)) a countable collection of triangles, and \( \text{NonSep}(V) \) is the increasing union of the \( \mathcal{N}_i \). A careful discussion of \( \text{NonSep}(V) \) is given by Cho and McCullough [2, Section 4]

We obtain \( \mathcal{D}(V) \) by attaching a countable collection of triangles to every edge of \( \text{NonSep}(V) \). To see this note that every separating disk \( E \) divides \( V \) into two copies of \( V_{1,1} \). These copies of \( V_{1,1} \) have meridian disks, say \( D \) and \( D' \). Thus \( \text{link}(E) \) is an edge and the triangle \( \{E, D, D'\} \) has two free edges in \( \mathcal{D}(V) \), as indicated. Finally, there is a countable collection of separating disks lying in \( V \setminus (D \cup D') \), again as indicated.

It follows that \( \text{Aut}(\mathcal{D}(V_2)) \) is uncountable. Again, as in Problem 4.1 we may ask: are all “duality-respecting” elements \( f \in \text{Aut}(\mathcal{D}(V_2)) \) geometric? We end with another open problem:

**Problem 4.2.** Suppose that \( V \) is a handlebody with \( e(V) \) and genus both sufficiently large. Show that \( \text{Aut}(\text{NonSep}(V)) = \mathcal{H}(V) \).
A solution to Problem [12] may lead to a simplified proof of Theorem [10].

5. Topological types

The goal of this section is:

Lemma 5.1. Suppose that $\phi \in \text{Aut}(D(V))$. Then $\phi$ preserves topological types of disks.

The complexity of $V_{g,n}$ is $\xi(V) = 3g - 3 + n$. If $\xi(V) \geq 1$ then $\xi(V)$ is the number of vertices of a maximal simplex of $D(V)$. Note that $V_1, V_{1,1}$ and $V_{0,4}$ are the only handlebodies where $D(V)$ has dimension zero. (When $D(V)$ is empty its dimension is $-1$.) Further $V_1$ and $V_{1,1}$ are the only handlebodies where $D(V)$ is a single point.

We will call $V_{0,3}$, the three-spotted ball, a solid pair of pants. Thus $\xi(V)$ is the number of disks in a pants decomposition of $V$ while $e(V) = 2g - 2 + n$ is the number of solid pants in the decomposition. We will call $V_{1,1}$ a solid handle. Suppose now that $E$ is separating with $V \setminus \text{neigh}(D) = X \cup Y$. If $X$ or $Y$ is a solid pants then we call $E$ a pants disk. If $X$ or $Y$ is a solid handle then we call $E$ a handle disk.

Recall that if $K$ and $L$ are non-empty simplicial complexes with disjoint vertex sets then $K \lor L$, their join, is the complex

$$K \cup \{\sigma \cup \tau | \sigma \in K, \tau \in L\} \cup L.$$ 

Claim 5.2. For any handlebody $V$ the complex $D(V)$ is not a join.

Proof. When $e(V) \leq 2$ this can be checked case-by-case, following Section [4]. The remaining handlebodies all admit disks $D, E$ that fill: every disk $F$ meets at least one of $D$ or $E$. It follows that any edge-path in $D^{(1)}(V)$ connecting $D$ to $E$ has length at least three. However, the diameter of the one-skeleton of a join is either one or two. 

The complex $D(V)$ is flag: minimal non-faces have dimension one.

Observe that $\phi$ preserves the combinatorics of $D(V)$. Thus any topological property of $V$ that has a combinatorial characterization will be preserved by $\phi$. We proceed with a sequence of claims.

Claim 5.3. The disk $E$ is a separating disk yet not a pants disk if and only if $\text{link}(E)$ is a join. Furthermore, in this case $\text{link}(E)$ is realized as a join in exactly one way, up to permuting the factors.

Proof. Suppose that $V \setminus \text{neigh}(E) = X \cup Y$, where neither $X$ nor $Y$ is a solid pants. Since $E$ is essential and non-peripheral both $D(X)$ and $D(Y)$ are non-empty. It follows that $\text{link}(E) = D(X) \lor D(Y)$, and neither factor is empty. Furthermore, this join is realized uniquely,
because $D(X)$ is never itself a join (by Claim 5.2), $D(X)$ is flag and join is associative.

On the other hand, if $E$ is non-separating then link($E$) is isomorphic to $D(V_{g-1,n+2})$. If $E$ is a pants disk then link($E$) $\cong D(V_{g,n-1})$. Neither of these is a join by Claim 5.2. □

A cone is the join of a point with some non-empty simplicial complex.

Claim 5.4. Suppose that $V \neq V_{1,2}$. Then $E \in D(V)$ is a handle disk if and only if link($E$) is a cone.

Proof. Suppose that $E$ cuts off a solid handle $X$ with meridian $D$. Let $Y$ be the other component of $V \setminus \text{neigh}(E)$. Since $V \neq V_{1,2}$ we have that $D(Y)$ is non-empty; in particular $E$ is not a pants disk. By Claim 5.3 we have link($E$) $\cong D(X) \vee D(Y)$. As $D(X) = \{D\}$ we are done with the forward direction.

Now suppose that link($E$) is a cone from $D$. Since a cone is the join of the apex with the base, by Claim 5.3 the disk $E$ is separating. Let $V \setminus \text{neigh}(E) = X \cup Y$. Thus link($E$) $\cong D(X) \vee D(Y)$. However, by Claim 5.3 the decomposition of link($E$) is unique; breaking symmetry we may assume that $D(X) = \{D\}$. Thus $X$ is a solid handle and we are done. □

It immediately follows that:

Claim 5.5. Suppose that $V \neq V_{1,2}$. Then $D \in D(V)$ is non-separating if and only if there is an $E \in D(V)$ so that link($E$) is a cone with apex $D$. □

Claim 5.6. Suppose that $e(V) \geq 3$. A simplex $D \in D(V)$ is a cut system if and only if the following properties hold:

- for every pair of disks $D, E \in \text{link}(D)$ the complex $\text{link}(E) \cap \text{link}(D)$ is not a cone with apex $D$ and
- for every proper subset $\sigma \subset D$ there is a pair of disks $D, E \in \text{link}(\sigma)$ so that the complex $\text{link}(E) \cap \text{link}(\sigma)$ is a cone with apex $D$.

Proof. The forward direction follows from Claim 5.5 and the definition of a cut system. (When $V$ is a spotted ball the only cut system is the empty set; the empty set has no proper subsets.)

Now for the backwards direction: From the first property and by Claim 5.5 deduce that $V' = V \setminus \text{neigh}(D)$ is a collection of spotted balls. If $V'$ has at least two components then there is a proper subset $\sigma \subset D$ which is a cut system for $V$. Thus $V \setminus \text{neigh}(\sigma)$ is a spotted ball and this contradicts the second property. □
Lemma 5.7. Suppose that \(V, W\) are handlebodies with \(D(V) \cong D(W)\). Then either:
\[
\begin{align*}
&\text{• } V \cong W \text{ or } \\
&\text{• } V, W \in \{V_1, V_{1,1}\} \text{ or } \\
&\text{• } V, W \in \{V_0, V_{0,1}, V_{0,2}, V_{0,3}\}.
\end{align*}
\]

This is the handlebody version of \([8, \text{Lemma 4.5}]\) and \([9, \text{Lemma 2.1}]\).

Proof of Lemma 5.7. When \(e(V) \leq 2\) this can be checked case-by-case, following Section 4. When \(V\) has \(e(V) \geq 3\) then \(\xi(V) = \xi(W)\). By Claim 5.6 the handlebodies \(V\) and \(W\) have cut systems of the same size. It follows that \(V, W\) have the same genus and thus the same number of spots. \(\square\)

We now have:

Proof of Lemma 5.1. Let \(V = V_{g,n}\) and fix \(\phi \in \text{Aut}(D(V))\). When \(e(V) \leq 2\), Lemma 5.1 can be checked case-by-case, following Section 4. So suppose that \(e(V) \geq 3\).

The automorphism \(\phi\) must preserve the set of non-separating disks by Claim 5.5.

Suppose that \(E \in D(V)\) is a separating disk yet not a pants disk. Writing \(V \setminus \text{neigh}(E) = X \cup Y\) we have \(\text{link}(E) = D(X) \cap D(Y)\). By Claim 5.3 this join is realized uniquely and so we can recover \(D(X)\) and \(D(Y)\). By Lemma 5.7 we may deduce, combinatorially, the genus and number of spots of \(X\) and \(Y\). Thus \(\phi\) preserves the topological type of \(E\).

The only topological type remaining is the set of pants disks. Since all other types are preserved, so are the pants disks. We are done. \(\square\)

6. Regluing

Suppose that \(\phi_D \in \text{Aut}(D(V))\) fixes \(D\). By Lemma 2.2 there is a homeomorphism \(f\) of \(V' = V \setminus \text{neigh}(D)\) so that the induced geometric automorphism equals \(\phi_D|\text{link}(D)\). We must show that \(f\) gives a homeomorphism of \(V\): that is, for every \(i\) the spots \(D_i^\pm\) are preserved by \(f\).

Let \(\text{handle}_i(D) \subset \text{link}(D)\) be the collection of handle disks \(E \in D(V)\) such that
\[
\begin{align*}
&\text{• one component of } V \setminus \text{neigh}(E) \text{ is a solid handle containing } D_i \\
&\text{and} \\
&\text{• } E \text{ is disjoint from all of the } D_j.
\end{align*}
\]
Let pants$_i(D) \subset \mathcal{D}(V')$ be the collection of pants disks $E$ such that one component of $V' \setminus \text{neigh}(E)$ is a solid pants meeting the spots $D^+_i$.

By the claims in the previous section the set handle$_i(D)$ is, for all $i$, combinatorially characterized and so preserved by $\phi_D$. It follows that the homeomorphism $f \in \text{Homeo}(V')$ preserves the set pants $i(D)$, for all $i$. Now, suppose that $f(D^+_1), f(D^-_1) = A, B$ where $A, B$ are spots of $V'$. Let $E \in \text{pants}_1(D)$ be any pants disk. Then $f(E)$ is a pants disk cutting off $A$ and $B$. It follows that the spots $A, B$ (in some order) equal the spots $D^+_1$ as desired.

7. Duality

Recall that two disks $D, E \in \mathcal{D}(V)$ are dual if $i(\partial D, \partial E) = 2$ (see Figure 2). A pentagon $P \subset \mathcal{D}(V_{0,5})$ is a collection of five disks $P = \{E_i\}_{i=0}^4$ so that $E_i$ and $E_{i+1}$ are disjoint, for all $i$ (modulo five). We say that the disks $E_i, E_{i+2}$ are non-adjacent in $P$, for all $i$ (modulo five).

Lemma 7.1 (Pentagon Lemma). Suppose that $V = V_{0,5}$. Two disks $D, E \in \mathcal{D}(V)$ are dual if and only if there is a pentagon $P$ so that $D, E \in P$ and $D, E$ are non-adjacent in $P$.

Proof. Recall that $\mathcal{D}(V_{0,5}) \cong \mathcal{C}(S_{0,5})$, by Lemma 2.2. The pentagon lemma for $S_{0,5}$ (see [8, Theorem 3.2] or [9, Lemma 4.2]) implies that there is only one pentagon in $\mathcal{D}(V_{0,5})$, up to the action of the handlebody group.

Lemma 7.2. Suppose that $V = V_{g,n}$ has $e(V) \geq 3$. Two disks $D, E \in \mathcal{D}(V)$ are dual if and only if there is a simplex $\sigma \in \mathcal{D}(V)$ with

- link($\sigma$) $\cong \mathcal{D}(V_{0,5})$,
- $D, E$ are non-adjacent in some pentagon of link($\sigma$).

It follows that every $\phi \in \text{Aut}(\mathcal{D}(V))$ preserves duality. We will say that a handlebody $W \subset V$ is cleanly embedded if:

- all spots of $W$ are essential in $V$ and
- if a spot of $W$ is peripheral in $V$ then it is also a spot of $V$.

Proof of Lemma 7.2 Suppose that $D, E$ are dual. Let $X$ be the four-spotted ball containing them. Isotope $X$ to be cleanly embedded. Let $E$ be a pants decomposition of $V' = V \setminus \text{neigh}(X)$. Now, there is at least one solid pants $P$ in $V' \setminus \text{neigh}(E)$ which has a spot, say $F$, which is parallel to a spot of $X$. If not then $e(V) \leq 2$, a contradiction.

Let $Y = X \cup \text{neigh}(F) \cup P$ and notice that this is a five-spotted ball containing $D$ and $E$, our original disks. Isotope $Y$ to be cleanly embedded. Let $E'$ be any pants decomposition of $V \setminus \text{neigh}(Y)$. Add to $E'$ any spots of $Y$ which are non-peripheral in $V$. This then is the
desired simplex \( \sigma \in \mathcal{D}(V) \). Since \( D \) and \( E \) are dual the pentagon lemma implies that there is a pentagon in \( \mathcal{D}(Y) \) making \( D, E \) non-adjacent.

The backwards direction follows from Lemma 5.7, the combinatorial characterization of genus and number of spots, and from the pentagon lemma. \( \square \)

We now discuss the dual complex. Fix a cut system \( \mathbb{D} = \{ D_i \} \). Recall that dual \( i(\mathbb{D}) \) is the subcomplex of \( \mathcal{D}(V) \) spanned by the disks \( E \in \mathcal{D}(V) \) which are dual to \( D_i \) and disjoint from \( D_j \) for all \( j \neq i \).

Define \( V_i \) to be the spotted solid torus obtained by cutting \( V \) along all disks of \( \mathbb{D} \) except \( D_i \). Note that \( V_i \) has exactly \( e(V) \)–many spots, and this is at least three. Also, \( D_i \) is a meridian disk for \( V_i \). Note that dual \( i(\mathbb{D}) \subset \mathcal{D}(V_i) \). A disk \( E \in \text{dual}_i(\mathbb{D}) \) is a simple dual if \( E \) is a pants disk in \( V_i \).

Let \( \mathcal{A}_i(\mathbb{D}) \) be the complex where vertices are isotopy classes of arcs \( \alpha \subset \partial_+ V_i \) so that
- \( \alpha \) meets \( \partial D_i \) exactly once, transversely, and
- \( \partial \alpha \) meets distinct spots of \( V_i \).

A collection of vertices spans a simplex if they can be realized disjointly.

If an arc \( \alpha \in \mathcal{A}_i(\mathbb{D}) \) meets spots \( A, B \in \partial_0 V_i \) then the frontier of \( \text{neigh}(A \cup \alpha \cup B) \) is a simple dual, \( E_\alpha \).

Lemma 7.3. If \( e(V) \geq 3 \) then the complex dual \( i(\mathbb{D}) \) is connected.

It suffices to check this for \( i = 1 \). To simplify notation we write \( D = D_1, U = V_1, \text{dual}(D) = \text{dual}_1(\mathbb{D}) \) and \( \mathcal{A}(D) = \mathcal{A}_1(\mathbb{D}) \). We will prove Lemma 7.3 via a sequence of claims.

Claim. For any pair of arcs \( \alpha, \gamma \in \mathcal{A}(D) \) there is a sequence \( \{ \alpha_k \}_{k=0}^N \subset \mathcal{A}(D) \) so that:
- the arcs \( \alpha_k, \alpha_{k+1} \) are disjoint, for all \( k < N \),
- \( \alpha_0 = \alpha \) and \( \alpha_N = \gamma \), and
- there is at most one spot in common between the endpoints of \( \alpha_k \) and \( \alpha_{k+1} \), for all \( k < N \).

Proof. Fix, for the remainder of the proof, an arc \( \beta \in \mathcal{A}(D) \) so that \( \alpha \) and \( \beta \) are disjoint and so that the endpoints of \( \alpha \) and \( \beta \) share at most one spot. This is possible as \( U \) has at least three spots. Define the complexity of \( \gamma \) to be \( c(\gamma) = i(\alpha, \gamma) + i(\beta, \gamma) \). Notice if \( c(\gamma) = 0 \) then we are done: one of the sequences

\[ \{ \alpha, \gamma \} \quad \text{or} \quad \{ \alpha, \beta, \gamma \} \]

has the desired properties.
Now induct on $c(\gamma)$. Suppose, breaking symmetry, that $\alpha$ meets a spot, say $A \in \partial_0 U$, so that $\gamma \cap A = \emptyset$. If $i(\alpha, \gamma) = 0$ then the sequence \{\alpha, \gamma\} has the desired properties. If not, then let $x$ be the point of $\alpha \cap \gamma$ that is closest, along $\alpha$, to the endpoint $\alpha \cap A$. Let $\alpha' \subset \alpha$ be the subarc connecting $x$ and $\alpha \cap A$. Let $N$ be a regular neighborhood, taken in $\partial_+ U$, of $\gamma \cup \alpha'$. The frontier of $N$, in $\partial_+ U$, is a union of three arcs: one arc properly isotopic to $\gamma$ and two more arcs $\gamma', \gamma''$.

The arcs $\gamma'$ and $\gamma''$ are disjoint from $\gamma$ and satisfy $c(\gamma') + c(\gamma'') \leq c(\gamma) - 1$. Also, since $\gamma'$ and $\gamma''$ each have one endpoint on the spot $A$ the arcs $\gamma'$ and $\gamma''$ have exactly one spot in common with $\gamma$. Now, if $\alpha' \cap \partial D = \emptyset$ then one of $\gamma', \gamma''$ meets $\partial D$ once and the other is disjoint. On the other hand, if $\alpha' \cap \partial D \neq \emptyset$ then $\alpha'$ meets $\partial D$ once. Thus one of $\gamma', \gamma''$ meets $\partial D$ once and the other meets $\partial D$ twice. In either case we are done. \hfill $\square$

Recall that if $\alpha \in \mathcal{A}(D)$ is an arc then $E_\alpha$ is the associated simple dual.

**Claim.** If $\alpha, \beta \in \mathcal{A}(D)$ are disjoint arcs, with at most one spot in common between their endpoints, then there is an edge-path in dual($D$) of length at most four between $E_\alpha$ and $E_\beta$.

**Proof.** If $\alpha$ and $\beta$ share no spots then \{E_\alpha, E_\beta\} is a path of length one. Suppose that $\alpha$ and $\beta$ share a single spot. Let $A, B, C$ be the three spots that $\alpha$ and $\beta$ meet, with both meeting $C$. Let $\alpha', \beta'$ be the subarcs of $\alpha, \beta$ connecting $C$ to $\partial D$. There are two cases: either $\alpha'$ and $\beta'$ are incident on the same side of $\partial D$ or are incident on opposite sides.

Suppose that $\alpha'$ and $\beta'$ are incident on the same side of $\partial D$. Then $\alpha'$ and $\beta'$, together with subarcs of $\partial C$ and $\partial D$ bound a disk $\Delta \subset \partial U$. Note that $\Delta$ may contain spots, but it meets $A \cup B \cup C$ only along the subarc in $\partial C$. It follows that the disk $F$, defined to be the frontier of $$\text{neigh} ((A \cup B \cup C) \cup (\alpha \cup \beta) \cup \Delta),$$
is dual to $D$. The disk $F$ is also essential as it separates at least three spots from a solid handle. So \{E_\alpha, F, E_\beta\} is the desired path.

Suppose that $\alpha'$ and $\beta'$ are incident on opposite sides of $\partial D$. Let $d \subset \partial D$ be either component of $\partial D \setminus (\alpha \cup \beta)$. Let $\alpha'' = \overline{\alpha' \alpha}$ and define $\beta''$ similarly. Define $\gamma \in \mathcal{A}(D)$ by forming the arc $\alpha'' \cup d \cup \beta''$ and using an proper isotopy of $\partial_+ U$ to make $\gamma$ transverse to $\partial D$. Now apply the previous paragraph to the pairs \{\alpha, \gamma\} and \{\gamma, \beta\} to obtain the desired path of length four. \hfill $\square$
Claim. For every dual $E \in \text{dual}(D)$ there is a simple dual connected to $E$ by an edge-path of length at most two.

Proof. The graph $\partial E \cup \partial D$ cuts $\partial U$ into a pair of disks $B, C$ and an annulus $A$. Each of $B, C$ contain at least one spot.

Suppose $E$ is separating. Then the disks $B, C$ are adjacent along an subarc of $\partial D$. Connect a spot in $B$ to a spot in $C$ by an arc $\alpha$ that meets $\partial D$ once and that is disjoint from $\partial E$. Thus $E_{\alpha}$ is disjoint from $E$.

Suppose $E$ is non-separating. Then the two disks $B, C$ meet only at the points of $\partial D \cap \partial E$. Now, if the annulus $A$ contains a spot then we may connect a spot in $B$ to a spot in $A$ by an arc $\alpha$ meeting $\partial D$ once and $\partial E$ not at all. In this case we are done as in the previous paragraph.

If $A$ contains no spots then, breaking symmetry, we may assume that $B$ contains at least two spots while $C$ contains at least one. Let $\delta$ be an arc connecting some spot, say $B' \subset B$, to $E$. Let $N$ be a regular neighborhood of $E \cup \delta \cup B'$. Then the frontier of $N$ contains two disks. One of these is isotopic to $E$ while the other, say $E'$, is non-separating, dual to $D$, and divides the spots as described in the previous paragraph.

Equipped with these claims we have:

Proof of Lemma [7.3]. The first two claims imply that the set of simple duals in $\text{dual}(D)$ is contained in a connected set. The third claim shows that every vertex in $\text{dual}(D)$ is distance at most two from the set of simple duals. This completes the proof. □

8. Crawling through the complex of duals

Lemma 8.1. Suppose that $\phi_{\text{link}}$ fixes $D$ and link$(D)$. For any $E \in \text{dual}_i(D)$ the disks $E$ and $\phi(E)$ differ by some power of $T_i$, the Dehn twist about $D_i$.

Proof. As usual, it suffices to prove this for $D = D_1$. Let $U = V_1$.

Let $X \subset U$ be the four-spotted ball filled by $D$ and the dual disk $E$. Isotope $X$ to be cleanly embedded. Let $F$ be the components of $\partial X$ which are not spots of $U$. Note that $\phi_{\text{link}}$ fixes $D$ as well as every disk of $F$. This, together with Lemma [7.2], implies that $\phi_{\text{link}}$ preserves the set of disks that are contained in $X$ and dual to $D$.

Since $D(X)$ equipped with the duality relation is a copy of $\mathcal{F}$, the Farey graph, it follows that $E$ and $F = \phi_{\text{link}}(E)$ differ by some number of half-twists about $D$. If $E$ and $F$ differ by an odd number of half-twists then $E$ and $F$ have differing topological types, contradicting
Lemma 5.1 applied to \( \phi_{\text{link}}|D(U) \). Thus \( E \) and \( F \) differ by an even number of half-twists, as desired. \( \square \)

Lemma 8.2. Suppose that \( \phi_{\text{dual}} \) fixes \( D \), \( \text{link}(D) \), and \( E \), a collection of duals (that is, \( E_i \in \text{dual}_i(D) \)). Then \( \phi_{\text{dual}} \) fixes every vertex of \( \text{dual}_i(D) \), for all \( i \).

Proof. As usual, it suffices to prove this for \( D = D_1 \). Let \( E = E_1 \) and let \( U = V_1 \). We crawl through \( \text{dual}(D) = \text{dual}_1(D) \), as follows.

Suppose that \( F, G \in \text{dual}(D) \) are adjacent vertices and suppose that \( \phi_{\text{dual}}(F) = F \). By Lemma 8.1, the disks \( G \) and \( G' = \phi_{\text{dual}}(G) \) differ by some number of Dehn twists about \( D \). Also, as \( \phi_{\text{dual}} \) is a simplicial automorphism the disks \( F \) and \( G' \) are disjoint. Let \( X \) be the four-spotted ball filled by \( D \) and \( F \). If \( G \) and \( G' \) are not equal then \( G \cap X \) and \( G' \cap X \) are also not equal and in fact differ by some non-zero number of twists; thus one of \( G \cap X \) or \( G' \cap X \) must cross \( F \), a contradiction.

Recall that \( \phi_{\text{dual}}(E) = E \). Suppose that \( G \) is any vertex of \( \text{dual}(D) \). Since \( \text{dual}(D) \) is connected (Lemma 7.3) there is a path \( P \subset \text{dual}(D) \) connecting \( E \) to \( G \). Induction along \( P \) completes the proof. \( \square \)

9. Crawling through the disk complex

Before continuing we will need the following complex:

Definition 9.1 (Wajnryb [13]). The cut system graph \( \mathcal{C}G(V) \) is the graph with vertex set being isotopy classes of unordered cut systems in \( V \). Edges are given by pairs of cut systems with \( g - 1 \) disks in common and the remaining pair of disks disjoint.

Wajnryb also gives a two-skeleton, but we will only require:

Theorem 9.2 (Wajnryb [13]). The cut system graph \( \mathcal{C}G(V) \) is connected.

For the remainder of this section suppose that \( \Phi = \phi_{\text{dual}} \) is an automorphism of \( \mathcal{D}(V) \) and \( D \) is a cut system so that \( \Phi \) fixes \( D \) and \( \text{dual}(D) \).

For the crawling step, suppose that \( E, F \) are adjacent in \( \mathcal{C}G(V) \) and that \( \Phi \) fixes \( E \), \( \text{link}(E) \) and \( \text{dual}(E) \). Let \( G \) be a pants decomposition obtained by adding the new disk of \( F \) to \( E \) and then adding non-separating disks until we have \( 3g - 3 + n \) disks. Let \( \{P_k\} \) enumerate the solid pants of \( G \). Let \( X_i = P_k \cup P_{k+1} \) be the four-spotted ball containing \( G_i \) in its interior.

Let \( H, I \subset \mathcal{D}(V) \) be collections of disks so that \( H_i, I_i \) are contained in \( X_i \) and \( G_i, H_i, I_i \) are pairwise dual in \( X_i \). Now, all of these disks \( G \cup H \cup I \) lie in \( E \cup \text{link}(E) \cup \text{dual}(E) \). Thus \( \Phi \) fixes all of them.
Thus $\Phi$ fixes $\mathbb{F}$. Consider $\Phi|\text{link}(\mathbb{F})$. By Theorem 2.1 the automorphism $f = \Phi|\text{link}(\mathbb{F})$ is geometric. Let $f$ also denote the given homeomorphism of $V' = V \setminus \text{neigh}(\mathbb{F})$. Let $\mathcal{G}' = \mathcal{G}\setminus \mathbb{F}$ and $\mathcal{H}', \mathcal{I}'$ be the disks of $\mathcal{H}, \mathcal{I}$ contained in $V'$. Thus $f$ fixes all disks of $\mathcal{G}', \mathcal{H}', \mathcal{I}'$. It follows that $f$ permutes the solid pants $\{P_k\}$.

If $f$ nontrivially permutes $\{P_k\}$ then, since each $G_i$ is fixed, we find that adjacent solid pants are interchanged. This implies that $V' = P_1 \cup P_2$, a contradiction. So $f$ fixes every $P_k$. Since all disks in $\mathcal{G}'$ are fixed, $f$ is either orientation reversing, isotopic to the identity, or isotopic to a half-twist on each of the $P_k$. Let $G_i \in \mathcal{G}'$ be any disk meeting $P_k$. Then $f|P_k$ cannot be orientation reversing because the triple $G_i, H_i, I_i$ determines an orientation on $X_i$ and hence on $P_k$. If $f|P_k$ is a half-twist then $P_k$ meets two spots of $V'$. Thus $G_i$ meets two solid pants $P_k, P_\ell$ so that $X_i = P_k \cup P_\ell$. Now, as $e(V') \geq 3$, the solid pants $P_\ell$ meets at most one spot of $V'$. Thus $f|P_\ell$ is isotopic to the identity. So if $f|P_k$ is a half-twist then $f(H_i) \neq H_i$, a contradiction. Deduce that $f$, when restricted to any solid pants, is isotopic to the identity. Now, since $f$ fixes all of the $H_i$, $f$ is isotopic to the identity on $V'$, as desired.

Deduce that $\Phi|\text{link}(\mathbb{F})$ is the identity. As $\Phi$ fixes duals to $\mathbb{F}$ by Lemma 8.2 the automorphism $\Phi$ fixes all of dual($\mathbb{F}$). This completes the crawling step and so completes the proof of:

**Theorem 9.3.** If a handlebody $V = V_{g,n}$ satisfies $e(V) \geq 3$ then the natural map $\mathcal{H}(V) \rightarrow \text{Aut}(\mathcal{D}(V))$ is a surjection.

As a corollary:

**Theorem 9.4.** If a handlebody $V = V_{g,n}$ satisfies $e(V) \geq 3$ then the natural map $\mathcal{H}(V) \rightarrow \text{Aut}(\mathcal{D}(V))$ is an isomorphism.

Note that Theorems 9.3 and 9.4 are sharp: when $e(V) \leq 2$ the conclusions are false. See Section 4.

**Proof of Theorem 9.4.** Theorem 9.3 shows that the natural map is surjective. Suppose that the mapping class $f$ lies in the kernel. As in the discussion of crawling through $\mathcal{CG}(V)$ given above, let $\mathcal{G} = \{G_i\}$ be a pants decomposition of $V$ so that all of the $G_i$ are non-separating. Let $\{P_k\}$ enumerate the solid pants of this decomposition. Let $X_i = P_j \cup P_k$ be the four-spotted ball containing $G_i$ in its interior. Let $\mathcal{H}, \mathcal{I} = \{H_i\}, \{I_i\}$ be collections of disks so that $H_i, I_i$ are contained in $X_i$ and $G_i, H_i, I_i$ are pairwise dual in $X_i$. All of these disks are fixed by $f$. It follows that $f$ is isotopic to the identity. \qed
10. AN APPLICATION

Theorem 10.1. If \( e(V) \geq 3 \) then the outer automorphism group of the handlebody group is trivial.

This may be restated as: \( \text{Aut}(\mathcal{H}) \cong \mathcal{H} \). When \( g = 0 \) then Theorem 10.1 follows from Lemma 2.2 and the first author’s thesis [8, Theorem 3]. For the rest of this section we restrict to the case \( g \geq 1 \).

The idea of the proof is to turn an element \( \phi \in \text{Aut}(\mathcal{H}) \) into an automorphism of the disk complex \( D(V) \). We do this, following [4], by giving an algebraic characterization of first Dehn twists about non-separating disks and then Dehn twists generally. We then apply Theorem 9.3 to \( \phi \) to find the corresponding geometric automorphism. An algebraic trick then gives the desired result.

A finite index subgroup \( \Gamma < \mathcal{H} \) is pure if every reducible class in \( \Gamma \) fixes every component of every reducing set. For example, the kernel of \( \mathcal{H} \to \text{Aut}(H_1(\partial_+ V, \mathbb{Z}/3\mathbb{Z})) \) is pure.

Lemma 10.2. Suppose \( \Gamma < \mathcal{H} \) is pure and finite index. Then \( \{f_i\} \subset \mathcal{H} \) is a collection of Dehn twists along a pants decomposition of non-separating disks in \( V \) if and only if

- the subgroup \( A = \langle f_i \rangle \) is free Abelian of rank \( \xi(V) \),
- \( f_i \) and \( f_j \) are conjugate in \( \mathcal{H} \), for all \( i, j \),
- \( f_i \) is primitive in \( C_\mathcal{H}(A) \): \( f_i \) is not a proper power of any \( h \in C_\mathcal{H}(A) \), and
- the center of the centralizer of the class \( f_i^n \) in \( \Gamma \) is infinite cyclic (for all \( i \) and for all \( n \) so that \( f_i^n \in \Gamma \)):

\[
C(C_\Gamma(f_i^n)) \cong \mathbb{Z}.
\]

Proof. The forwards direction is identical to the forwards direction of [4, Theorem 2.1]. The backwards direction is similar in spirit to the backwards direction of [4, Theorem 2.1] but some details differ. Accordingly we sketch the backwards direction.

The mapping class \( f_i \) can not be periodic or pseudo-Anosov as that would contradict the first property. Let \( \Theta \subset S = \partial_+ V \) be the canonical reduction system for the Abelian group \( A \) [4]. Let \( \{X_j\} \) be the components of \( S \setminus \text{neigh}(\Theta) \) and let \( \{Y_k\} \) be the collection of annuli \( \text{neigh}(\Theta) \). By [4, Lemma 3.1(2)] the number of annuli in \( \{Y_k\} \) plus the number of non-pants in \( \{X_j\} \) equals \( \xi(V) \). It follows that every non-pants \( X_j \) has complexity one (so is homeomorphic to \( S_{0,4} \) or \( S_{1,1} \)).

Fix a power \( n \) (independent of \( i \)) to ensure that \( f_i^n \in \Gamma \). For each \( X_j \) of complexity one there is some \( f_i^n \) so that \( f_i^n|X_j \) is pseudo-Anosov.
Suppose that \( f = f_1^n \), \( X = X_1 \) has complexity one, and \( f|X \) is pseudo-Anosov. Let \( \lambda^\pm \) be the stable and unstable laminations of \( f|X \). For every \( i \), the mapping \( f_i^n|X \) is either the identity or pseudo-Anosov. Note that in the latter case the stable and unstable laminations of \( f_i^n|X \) agree with \( \lambda^\pm \); otherwise a ping-pong argument gives a rank two free group in \( A \), a contradiction. Thus, perhaps taking a larger power \( n \), we may assume that for each \( i \) either \( f_i^n|X = f|X \) or identical.

Continuing in this manner we find a free Abelian group \( B < A \cap \Gamma \) of rank at least \( \xi(V) \) where all elements are supported inside of the union of annuli \( \{Y_k\} \). Since \( B \) is pure, it follows that all elements of \( B \) are compositions of powers of Dehn twists along disjoint curves. A theorem of McCullough [11] implies that every curve in \( \Theta \) either bounds a disk or cobounds an annulus with some other curve of \( \Theta \). However, each annulus reduces the possible rank of \( B \) by one; it follows that every curve in \( \Theta \) bounds a disk.

Let \( \gamma \) be any essential non-peripheral component of \( \partial X \). It follows that \( f \) commutes with \( T_\gamma \), that \( T_\gamma \) lies in \( \mathcal{H} \) by the above paragraph, and that \( T_\gamma \) to some power lies in \( C(C_T(f)) \). But this contradicts the fourth property. It follows that every component \( X_j \) is a pants and that \( \Theta = \xi(V) \). Thus every \( f_i \) is a compositions of powers of disjoint twists. Again, by the fourth property each \( f_i \) is some power of a single twist. By the third property (following [4]) \( f_i \) is in fact a twist. Finally, by the second property, each twist is supported on a disk of the same topological type. As every pants decomposition of \( V \) must contain a non-separating disk all of the twists \( f_i \) are supported by non-separating disks. \( \square \)

We now give the general characterization:

**Lemma 10.3.** Suppose \( \Gamma < \mathcal{H} \) is pure and finite index. Then \( \{f_i\} \subset \mathcal{H} \) is a collection of Dehn twists along a pants decomposition of \( V \) if and only if

- the subgroup \( A = \langle f_i \rangle \) is free Abelian of rank \( \xi(V) \),
- \( f_i \) is primitive in \( C_\mathcal{H}(A) \),
- for all \( i \) and for all \( n \) so that \( f_i^n \in \Gamma \) either \( C(C_T(f_i^n)) \cong \mathbb{Z} \) or there is a \( j \) so that \( C(C_T(f_j^n)) \cong \mathbb{Z}^2 \) with the latter given by \( \langle f_i, f_j \rangle \) and \( f_j \) is a twist on a non-separating disk.

**Proof.** Suppose that \( \{D_i\} \) is a pants decomposition and \( f_i \) is the positive twist on \( D_i \). Then \( A = \langle f_i \rangle \) is free Abelian of the correct rank. The second property follows as \( A = C_\mathcal{H}(A) \). The third property follows...
from Ivanov’s discussion \[4\] except if \(D_i\) is a handle disk. In this case
the meridian of the handle, say \(D_j\), gives a twist \(f_j\) which lies in the
center of the centralizer.

The backwards direction is similar to that of the proof of Lemma \[10.2\]
The only change occurs when \(f\) is pseudo-Anosov: when the center
of the centralizer has rank two then the additional element is a twist
about a separating disk and this contradicts the third property. \(\square\)

The following lemmas follow from the identical statements for the
mapping class group of \(S\) \([6]\):

**Lemma 10.4.** Suppose \(D\) and \(E\) are essential disks. The twists \(T_D, T_E\)
commute if and only if \(D\) and \(E\) can be made disjoint via proper isotopy.
\(\square\)

**Lemma 10.5.** For any twist \(T_D\) and for any homeomorphism \(h\) we
have \(hT_Dh^{-1} = T_{h(D)}\). \(\square\)

**Lemma 10.6.** For any pair of disks \(D\) and \(E\) and any pair of integers
\(n\) and \(m\), if \(T^n_D = T^m_E\) then \(D = E\) and \(n = m\). \(\square\)

The proof of Theorem \[10.1\] now follows, essentially line-by-line, the
proof of either \([5, \text{Theorem 2}]\) or \([8, \text{Theorem 3}]\). \(\square\)

To extend our algebraic characterization of twists in \(\mathcal{H}(V)\) (Lem-
mas \[10.2\] and \[10.3\]) to a characterization of powers of twists inside of
finite index pure subgroups \(\Gamma < \mathcal{H}(V)\) appears to be a delicate matter.
Solving this problem would, following Ivanov \([5]\), solve:

**Problem 10.7.** Show that the abstract commensurator of \(\mathcal{H}(V)\) is
\(\mathcal{H}(V)\) itself. Show that \(\mathcal{H}(V)\) is not arithmetic.

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