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Spline Collocation for Multi-Term Fractional Integro-Differential Equations with Weakly Singular Kernels

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Abstract: We consider general linear multi-term Caputo fractional integro-differential equations with weakly singular kernels subject to local or non-local boundary conditions. Using an integral equation reformulation of the proposed problem, we first study the existence, uniqueness and regularity of the exact solution. Based on the obtained regularity properties and spline collocation techniques, the numerical solution of the problem is discussed. Optimal global convergence estimates are derived and a superconvergence result for a special choice of grid and collocation parameters is given. A numerical illustration is also presented.

Keywords: fractional differential equation; weakly singular kernel; caputo derivative; boundary value problem; collocation method; graded grid

1. Introduction

It is currently well established that differential and integral equations with derivatives of fractional (non-integer) order have great importance in the modeling of real-life processes. For details, including basic theory of fractional calculus and references to some applications, see the monographs [1–4] and review papers [5–7]. Some concrete examples can be found in the works [8] (modelling the behaviour of viscoelastic materials), [9,10] (kinetics of polymers), [11,12] (modelling the behaviour of humans), [13] (fractional processes in financial economics), [14] (atomic wall dynamics), [15] (viscoelastic laws for arterial wall mechanics), [16] (models of supercapacitor energy storage), [17] (transition of flow in fluid dynamics) and [18–20] (anomalous diffusion).

However, when working with problems stemming from real-world applications, it is only rarely possible to find the solution of a given fractional differential or integral equation in closed form, and even if such an analytic solution is available, it is typically too complicated to be used in practice. Therefore, in general, numerical methods are required for solving fractional differential and integral equations. As a consequence, the last few decades have witnessed a steadily increasing development and analysis of numerical methods for fractional differential equations, of which a good deal are concerned with the numerical solution of initial or boundary value problems with one fractional derivative in the equation, see, for example, the works [21–25] for initial value problems and [26–31] for boundary value problems. A comprehensive survey of the most important methods for fractional initial value problems, along with a detailed introduction to the subject and a brief summary about the convergence behaviour of the methods is given in the monograph [32]. Less attention has been paid to the numerical solution of equations with multiple fractional derivatives (the so-called multi-term equations) [33–41], fractional differential equations with non-local boundary conditions [42–46] and fractional integro-differential equations with weakly singular kernels [47–49]. This motivated us in the present paper to focus on constructing effective numerical methods for fractional weakly singular integro-differential equations with local or non-local boundary conditions.
In order to construct high-order numerical methods for solving fractional differential and integro-differential equations, one needs some information about the regularity of the exact solution of the underlying problem. This becomes even more significant if we wish to study the optimal order of convergence of the proposed algorithms. However, fractional differential and integral equations pose an extra challenge compared to integer order differential equations. For example, it is well known that, in the case of integer order differential equations, the smoothness properties of a solution are determined by certain assumptions on the given data. Very simple examples show (see also Theorem 1 below) that, in general, we can not expect this to be true for fractional differential equations—a solution of a fractional differential equation will generally exhibit non-smooth behaviour even in the case of smooth data. Thus, when constructing high order numerical methods for fractional differential and integral equations, one should take into account, in some way, the possible non-smooth behaviour of an exact solution. Numerical methods which assume smooth solutions for fractional differential equations are valid only for a tiny subclass of problems, as is made clear in [50,51]. In the numerical solution of weakly singular integrals involving Caputo fractional derivatives (for the exact problem setting see Section 3 below). To the best of our knowledge, up to now there has been no discussion about the corresponding results of [56] to a much larger class of linear multi-term fractional equations, where the terms can be Caputo fractional derivatives of arbitrary order and weakly singular integrals involving Caputo fractional derivatives (for the exact problem setting see Section 3 below). To the best of our knowledge, up to now there has been no discussion about the numerical solution of such equations. In the present paper, using special non-uniform grids reflecting the possible singular behaviour of the exact solution, we construct a high order collocation-based numerical method for these equations. Our analysis hinges on a smoothness result for the exact solution of the underlying problem, given by Theorem 1. The theoretical convergence results for the proposed algorithm are proved by Theorems 2 and 3. These theorems show how to choose grid and collocation parameters so that the best possible order of convergence is attained. Notably, we show that with a careful choice and 3. These theorems show how to choose grid and collocation parameters so that the best possible order of convergence is attained. Notably, we show that with a careful choice of collocation parameters and with a suitably graded grid, the constructed method will obtain a superconvergence rate on the whole interval of integration $[0, b]$. We also provide numerical results to verify the theoretical analysis.

Without loss of generality, in this paper we shall assume that the starting point for fractional operators is located at the origin 0.

2. Basic Notations and Definitions

By $\mathbb{N}$ we denote the set of all positive integers $\{1, 2, \ldots\}$, by $\mathbb{N}_0$ the set of all non-negative integers $\{0, 1, 2, \ldots\}$ and by $\mathbb{R}$ the set of all real numbers $(-\infty, \infty)$. By $\lfloor a \rfloor$ we denote the smallest integer greater or equal to a real number $a$.

Let $b > 0$ be a fixed real number. By $L^1(0, b)$ we denote the Banach space of measurable functions $u : [0, b] \to \mathbb{R}$ such that $\|u\|_{L^1(0, b)} = \int_0^b |u(t)| \, dt < \infty$. By $L^\infty(0, b)$ we denote the Banach space of essentially bounded measurable functions $u : [0, b] \to \mathbb{R}$ such that

$$\|u\|_{L^\infty(0, b)} = \|u\|_{\infty} = \inf_{\Omega \subset [0, b]} \sup_{t \in [0, b] \setminus \Omega} |u(t)| < \infty,$$

where $\mu(\Omega)$ is the Lebesgue measure of the set $\Omega$.

By $C[0, b]$ we denote the Banach space of continuous functions $u : [0, b] \to \mathbb{R}$ with the norm $\|u\|_{C[0, b]} = \|u\|_{\infty} = \max_{0 \leq t \leq b} |u(t)|$. By $C^m[0, b]$ we denote the Banach space of $m$ times ($m \in \mathbb{N}_0$, for $m = 0$ we set $C^0[0, b] = C[0, b]$) continuously differentiable functions $u : [0, b] \to \mathbb{R}$ with the norm $\|u\|_{C^m[0, b]} = \sum_{i=0}^m \|u^{(i)}\|_{\infty}$.
By $C^n(\Delta)$ we denote the set of $m$ times ($m \in \mathbb{N}_0$, for $m = 0$ we set $C^0(\Delta) = C(\Delta)$) continuously differentiable functions on $\Delta$, with

$$\Delta := \{(t, s) : 0 \leq s \leq t \leq b\}.$$ 

Next, we recall the definitions and some properties of Riemann–Liouville integral and Caputo fractional differential operators, see [1,2].

For given $\delta \in (0, \infty)$ by $J^\delta$ we denote the Riemann–Liouville integral operator of order $\delta$, defined by

$$(J^\delta u)(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t - s)^{\delta - 1} u(s) \, ds, \quad t \in [0, b], \quad u \in L^1(0, b),$$

where $\Gamma(\delta) := \int_0^\infty e^{-s} s^{\delta - 1} ds$ is the Euler gamma function. For $\delta = 0$ we set $J^0 := I$, where $I$ is the identity operator. If $\delta > 0$, then $(J^\delta u)(t)$ exists for almost all $t \in [0, b]$ and the function $J^\delta u$ is also an element of $L^1(0, b)$. We have for any $u \in L^\infty(0, b)$ that

$$(J^\delta u)^{(k)} \in C[0, b], \quad (J^\delta u)^{(k})(0) = 0, \quad \delta > 0, \quad k = 0, \ldots, [\delta] - 1,$$ 

where $[\delta]$ denotes the integer part of $\delta$. Moreover, it is well known (see, e.g., [57]) that the operator $J^\delta$ is linear, bounded and compact from $L^\infty(0, b)$ into $C[0, b]$.

Let $m \in \mathbb{N}$. By $Q_{m-1}[u]$ we denote the Taylor polynomial of degree $m - 1$ for the function $u \in C^m[0, b]$ at the point 0:

$$(Q_{m-1}[u])(s) = \sum_{i=0}^{m-1} \frac{u^{(i)}(0)}{i!} s^i, \quad s \in \mathbb{R}, \quad 0! = 1.$$ 

By $D^\delta_{\text{Cap}}$ we denote the Caputo fractional differential operator of order $\delta > 0$, defined by the formula

$$(D^\delta_{\text{Cap}} u)(t) = \frac{d^n}{dt^n} \left( J^{n-\delta}(u - Q_{n-1}[u]) \right)(t), \quad t \in (0, b), \quad n := [\delta].$$

In the definition (4) we assume that $u \in C^{n-1}[0, b]$ is such that the integer order derivative $(J^{n-\delta}(u - Q_{n-1}[u]))^{(n)}(t)$ exists for any $t \in (0, b)$. If $\delta \in \mathbb{N}$ and $u \in C^\delta[0, b]$ then we have $(D^\delta_{\text{Cap}} u)(t) = u^{(\delta)}(t), \quad t \in (0, b]$.

A sufficient condition for the existence of $D^\delta_{\text{Cap}} u \in C[0, b]$ is $u \in C^{[\delta]}[0, b]$. However, this is not a necessary condition. In [58], Vainikko gives a comprehensive description of the range $J^\delta C[0, b] \ (\delta > 0)$ of $J^\delta$ as an operator from $C[0, b]$ into $C[0, b]$. In particular, he has derived necessary and sufficient conditions for the existence of $D^\delta_{\text{Cap}} u \in C[0, b]$ for a function $u \in C^{[\delta]-1}[0, b], \delta > 0$.

For any $u \in L^\infty(0, b)$ we have

$$D^\delta_{\text{Cap}} J^\beta u = J^{\beta - \delta_1} u, \quad 0 < \delta_1 \leq \delta_2.$$ 

Note that if $u \in C^{[\delta]-1}[0, b] \ (\delta > 0)$ and $D^\delta_{\text{Cap}} u \in C[0, b]$, then (cf. [1])

$$u(t) = (J^\beta z)(t) + \sum_{\lambda=0}^{[\delta]-1} c_\lambda t^\lambda, \quad t \in [0, b],$$

where $z := D^\delta_{\text{Cap}} u$ and $c_\lambda \in \mathbb{R} (\lambda = 0, \ldots, [\delta] - 1)$ are some constants. Note also that for a function

$$v_\beta(t) = t^\beta, \quad t \geq 0, \quad \beta \in [0, \infty)$$

...
we have for $\delta > 0$ that
\[
(D^\delta_{\text{Cap}}y_\beta)(t) = \begin{cases} 
0 & \text{if } \beta = 0, 1, \ldots, [\delta] - 1, \\
\frac{\Gamma(1+\beta)}{\Gamma(\beta+1-\delta)} t^{\beta-\delta} & \text{if } \beta \geq [\delta],
\end{cases}
\] (7)
where $t \geq 0$.

3. Problem Setting

In the present paper, we will consider a general class of linear multi-term fractional weakly singular integro-differential equations in the form
\[
(D^\alpha_{\text{Cap}}y)(t) + \sum_{i=0}^{p-1} d_i(t)(D^\beta_{\text{Cap}}y)(t) + \sum_{i=0}^{q} \int_{0}^{t} (t-s)^{-\kappa_i(t,s)}(D^\gamma_{\text{Cap}}y)(s)ds = f(t), \quad 0 \leq t \leq b,
\] (8)
subject to the conditions
\[
\sum_{j=0}^{n_0} \beta_{ij0} y^{(j)}(0) + \sum_{k=0}^{n_1} \beta_{ijk} y^{(j)}(b_k) + \beta_j \int_{0}^{b_j} y(s)ds = \gamma_i, \quad i = 0, \ldots, n-1, \quad n := [a_p].
\] (9)

Here $y = y(t)$ is the unknown function and by $D^\alpha_{\text{Cap}}y$ ($i = 0, \ldots, p$) and $D^\beta_{\text{Cap}}y$ ($j = 0, \ldots, q$) are denoted the Caputo fractional derivatives of $y$ of orders $\alpha_i$ and $\theta_j$, respectively. We assume that the following conditions (10) are fulfilled:
\[
\begin{align*}
0 &\leq a_0 < a_1 < \cdots < a_p \leq n, \quad p \in \mathbb{N}, \quad n = [a_p], \\
0 &< \theta_j < a_p, \quad 0 \leq \kappa_j < 1, \quad j = 0, \ldots, q, \quad q \in \mathbb{N}, \\
0 &< b_1 < \cdots < b_l \leq b, \quad l \in \mathbb{N}, \quad 0 < b_i \leq b, \quad i = 0, \ldots, n-1, \\
n_0, n_1 &\in \mathbb{N}, \quad n_0 < n, \quad n_1 < n, \quad \beta_{ij0}, \beta_{ijk}, \beta_j, \gamma_i \in \mathbb{R}, \\
d_i &\in C[0, b] \ (i = 0, \ldots, p-1), \quad f \in C[0, b], \\
K_j &\in C(\Lambda), \quad j = 0, \ldots, q.
\end{align*}
\] (10)

Note that for certain values of coefficients $\beta_{ij0}, \beta_{ijk}$ and $\beta_j$ the problem (8)–(10) takes the form of an initial value problem or a multi-point boundary value problem.

Following some ideas of [39,48,56], we will construct a high-order method for the numerical solution of problem (8) and (9). We first introduce an integral equation reformulation of the underlying problem (Section 4) and prove some results about the existence, uniqueness and regularity of the exact solution of (8) and (9) (Section 5). Using this information and spline collocation techniques, the numerical solution of the problem is discussed (Section 6). Optimal global convergence estimates are derived and a superconvergence result for a special choice of collocation parameters is established (Section 7). Finally, numerical illustrations confirming the convergence estimates are given (Section 8).

4. Integral Equation Reformulation

Let $n = [a_p] \in \mathbb{N}$ and let $y \in C^{n-1}[0, b]$ be an arbitrary function such that $D^\alpha_{\text{Cap}}y \in C[0, b]$. We denote
\[
z := D^\alpha_{\text{Cap}}y.
\]
Then, due to (6),
\[
y(t) = (f^\alpha z)(t) + \sum_{\lambda=0}^{n-1} c_\lambda t^\lambda, \quad t \in [0, b],
\] (11)
where $c_\lambda \in \mathbb{R} \ (\lambda = 0, \ldots, n-1)$ are some constants. From properties (2) and (5) we see that for $y$ in the form (11) we can write
\[ y^{(j)}(0) = f^j c, \quad y^{(j)}(c) = (f^{a_j-1} z)(c) + \sum_{\lambda=j}^{n-1} \frac{\lambda!}{(\lambda-j)!} c^{\lambda-j}, \quad c \in [0, b], \]

where \( j = 0, \ldots, n-1 \), and, using (3), we have that

\[ \int_0^a y(s) ds = (f^{a_j+1} z)(a) + \sum_{\lambda=0}^{n-1} \frac{c_{\lambda}}{\lambda+1} a^{\lambda+1}, \quad a \in [0, b]. \]

Thus, a function \( y \) in the form (11) satisfies the conditions (9) if and only if

\[
\sum_{j=0}^{n_0} \beta_{i j 0} j! c_j + \sum_{k=1}^{n_1} \sum_{j=0}^{n_1} \beta_{j k i} \left[ (f^{a_j-1} z)(b_k) + \sum_{\lambda=j}^{n-1} \frac{\lambda!}{(\lambda-j)!} b_k^{\lambda-j} c_{\lambda} \right] \\
+ \bar{\beta}_i \left[ (f^{a_j+1} z)(\bar{b}_i) + \sum_{\lambda=0}^{n-1} \frac{c_{\lambda}}{\lambda+1} \bar{b}_i^{\lambda+1} \right] = \gamma_i, \tag{12}
\]

where \( i = 0, \ldots, n-1 \). By setting \( \beta_{i j 0} = 0 \) for \( j = n_0 + 1, \ldots, n-1 \) and \( \beta_{i j k} = 0 \) for \( j = n_1 + 1, \ldots, n-1 \) \((k = 1, \ldots, l)\), we can write

\[
\sum_{j=0}^{n_0} \beta_{i j 0} j! c_j = \sum_{j=0}^{n-1} \beta_{i j 0} j! c_j \quad (i = 0, \ldots, n-1)
\]

and

\[
\sum_{j=0}^{n_1} \beta_{i j k} \left[ (f^{a_j-1} z)(b_k) + \sum_{\lambda=j}^{n-1} \frac{\lambda!}{(\lambda-j)!} b_k^{\lambda-j} c_{\lambda} \right] \\
= \sum_{j=0}^{n_1} \beta_{i j k} (f^{a_j-1} z)(b_k) + \sum_{j=0}^{n_1} \sum_{\lambda=0}^{j} \beta_{i j k} \frac{j!}{(j-\lambda)!} b_k^{j-\lambda} c_j, \quad (i = 0, \ldots, n-1), \tag{13}
\]

for \( k = 1, \ldots, l, i = 0, \ldots, n-1 \). The conditions (12) can thus be rewritten in the form

\[
\sum_{j=0}^{n-1} \left[ j! \beta_{i j 0} + \sum_{k=1}^{l} \sum_{\lambda=0}^{j} \beta_{i j k} \frac{j!}{(j-\lambda)!} b_k^{j-\lambda} + \bar{\beta}_i \frac{j!}{j+1} \right] c_j \\
= \gamma_i - \sum_{k=1}^{n_1} \beta_{i j k} (f^{a_j-1} z)(b_k) - \beta_i (f^{a_j+1} z)(\bar{b}_i), \quad i = 0, \ldots, n-1,
\]

giving us an algebraic linear system of \( n \) equations with respect to \( c_0, \ldots, c_{n-1} \). Let

\[
M := \left( j! \beta_{i j 0} + \sum_{k=1}^{l} \sum_{\lambda=0}^{j} \beta_{i j k} \frac{j!}{(j-\lambda)!} b_k^{j-\lambda} + \bar{\beta}_i \frac{j!}{j+1} \right)_{i,j=0}^{n-1}
\]

be the matrix of the system (13).

In the sequel, we assume that the matrix \( M \) is regular. Observe that the matrix \( M \) is regular if and only if from all polynomials \( y \) of degree \( n-1 \) only \( y = 0 \) satisfies the homogeneous conditions

\[
\sum_{j=0}^{n_0} \beta_{i j 0} y^{(j)}(0) + \sum_{j=0}^{n_1} \beta_{i j 0} y^{(j)}(b_k) + \beta_i \int_0^{b_i} y(s) ds = 0, \quad i = 0, \ldots, n-1, \tag{14}
\]

corresponding to the conditions (9) by \( \gamma_i = 0, \quad i = 0, \ldots, n-1 \).

Indeed, substituting (11) with \( z = 0 \) into (14) we obtain a homogeneous system of algebraic equations with respect to \( c_0, c_1, \ldots, c_{n-1} \). This system coincides with (13) by \( \gamma_i = 0 \) \((i = 0, \ldots, n-1)\) and \( z = 0 \). Therefore, the homogeneous system corresponding
to (13) has only the trivial solution \( c_0 = c_1 = \cdots = c_{n-1} = 0 \) (and thus \( M \) is regular) if and only if from all polynomials \( y \) of degree \( n - 1 \) only \( y = 0 \) satisfies (14).

Let

\[
M^{-1} = (p_{ij})_{i,j=0}^{n-1}
\]

be the inverse of \( M \). Using \( M^{-1} \), the solution of the system (13) can be written in the form

\[
c_{\lambda} = \phi_{\lambda} - \xi_{\lambda}, \quad \lambda = 0, \ldots, n - 1,
\]

with

\[
\phi_{\lambda} := \sum_{\mu=0}^{n-1} p_{\lambda \mu} \gamma_{\mu}, \quad \xi_{\lambda} := \sum_{k=1}^{n_1} \sum_{j=0}^{n_1} \psi_{\lambda j k} (f_j^{n_j} z)(b_k) + \omega_{\lambda} (f_j^{n_j+1} z)(b_k),
\]

where

\[
\psi_{\lambda j k} := \sum_{\mu=0}^{n-1} p_{\lambda \mu} \beta_{\mu j k}, \quad \omega_{\lambda} := \sum_{\mu=0}^{n-1} p_{\lambda \mu} \beta_{\lambda}.
\]

Therefore, a function \( y \) in the form (11) satisfies the conditions (9) if and only if it can be expressed by the formula

\[
y = G z + Q,
\]

where

\[
(G z)(t) := (f_j^{n_j+1} z)(t) - \sum_{\lambda=0}^{n-1} \xi_{\lambda} t^\lambda, \quad t \in [0, b],
\]

\[
Q(t) := \sum_{\lambda=0}^{n-1} \phi_{\lambda} t^\lambda, \quad t \in [0, b].
\]

Suppose now that \( y \in C^{n-1}[0, b] \) is a solution of problem (8) and (9) such that \( D_{\text{Cap}}^{\eta} y \in C[0, b] \). Then, it follows from the observations above that \( y \) has the form (17), where \( z = D_{\text{Cap}}^{\eta} y \in C[0, b] \), and \( G \) and \( Q \) are defined by the Formulas (18) and (19), respectively. Inserting (17) into (8), we see that

\[
z(t) + \sum_{i=0}^{p-1} d_i(t)(D_{\text{Cap}}^{\eta_i} [G z + Q])(t) + \sum_{i=0}^{q-1} \int_0^t (t - s)^{-\kappa_i} K_i(t, s)(D_{\text{Cap}}^{\eta_i} [G z + Q])(s) ds = f(t),
\]

for \( t \in [0, b] \). Therefore, \( z = D_{\text{Cap}}^{\eta} y \) satisfies the integral equation

\[
z = T z + g,
\]

with

\[
(T z)(t) := -\sum_{i=0}^{p-1} d_i(t)(D_{\text{Cap}}^{\eta_i} G z)(t) - \sum_{i=0}^{q-1} \int_0^t (t - s)^{-\kappa_i} K_i(t, s)(D_{\text{Cap}}^{\eta_i} G z)(s) ds,
\]

\[
g(t) := f(t) - \sum_{i=0}^{p-1} d_i(t)(D_{\text{Cap}}^{\eta_i} Q)(t) - \sum_{i=0}^{q-1} \int_0^t (t - s)^{-\kappa_i} K_i(t, s)(D_{\text{Cap}}^{\eta_i} Q)(s) ds,
\]

where \( t \in [0, b] \). Conversely, it turns out that if \( z \in C[0, b] \) is a solution of Equation (20) then \( y \) defined by (17) belongs to \( C^{n-1}[0, b] \) and is a solution to (8) and (9). In this sense, Equation (20) is equivalent to the problem (8) and (9).
In the following, we find for (21) and (22) a different form. First, for \( i = 0, \ldots, q \) we denote
\[
L_i(t, s) := \int_0^1 t^{a_p - \theta_i - 1}(1 - \tau)^{-\alpha_i} K_i(t, (t - s)\tau + s)d\tau, \quad 0 \leq s \leq t \leq b, \quad (23)
\]
\[
K_{i, \lambda}(t) := \int_0^1 (1 - s)^{-\alpha_i}s^{\lambda - \theta_i} K_i(t, ts)ds, \quad 0 \leq t \leq b, \quad \lambda \geq 0. \quad (24)
\]
From (5), (7) and (18) it follows that for \( i = 0, \ldots, p - 1 \) we have
\[
(D_{Cap}^{\alpha_i} Gz)(t) = (f^{a_p - \alpha_i}z)(t) - \sum_{n=1}^{n-1} \frac{\xi_{\lambda}}{\Gamma(\lambda + 1 - \alpha_i)} t^{\lambda - \alpha_i}, \quad t \in [0, b],
\]
and for \( i = 0, \ldots, q \) we have
\[
\int_0^t (t - s)^{-\alpha_i} K_i(t, s)(D_{Cap}^{\alpha_i} Gz)(s)ds = \int_0^t (t - s)^{-\alpha_i} K_i(t, s)(f^{a_p - \alpha_i}z)(s)ds \nonumber
\]
\[
= \sum_{\lambda = [\theta_i]}^{\lambda - \alpha_i} \frac{\lambda!}{\Gamma(\lambda + 1 - \theta_i)} \int_0^t (t - s)^{-\alpha_i} K_i(t, s)s^{\lambda - \theta_i}ds \nonumber
\]
\[
= \frac{1}{\Gamma(\alpha_p - \theta_i)} \int_0^t (t - s)^{a_p - \alpha_i} L_i(t, s)z(s)ds \nonumber
\]
\[
- \sum_{\lambda = [\theta_i]}^{\lambda - \alpha_i} \frac{\lambda!}{\Gamma(\lambda + 1 - \theta_i)} t^{1 + \lambda - \theta_i - \alpha_i} K_i(t, s), \quad t \in [0, b];
\]
from (5), (7) and (19) it follows that for \( i = 0, \ldots, p - 1 \) we have
\[
(D_{Cap}^{\alpha_i} Q)(t) = \sum_{\lambda = [\theta_i]}^{\lambda - \alpha_i} \frac{\lambda!}{\Gamma(\lambda + 1 - \alpha_i)} \phi_{\lambda} t^{\lambda - \alpha_i}, \quad t \in [0, b]
\]
and for \( i = 0, \ldots, q \) we have
\[
\int_0^t (t - s)^{-\alpha_i} K_i(t, s)(D_{Cap}^{\alpha_i} Q)(s)ds = \sum_{\lambda = [\theta_i]}^{\lambda - \alpha_i} \frac{\lambda!}{\Gamma(\lambda + 1 - \theta_i)} t^{1 + \lambda - \theta_i - \alpha_i}, \nonumber
\]
where \( t \in [0, b] \). Thus, on the basis of (21) and (22) we can write
\[
(Tz)(t) = -\sum_{i=0}^{p-1} d_i(t) \left[ (f^{a_p - \alpha_i}z)(t) - \sum_{\lambda = [\theta_i]}^{\lambda - \alpha_i} \frac{\lambda!}{\Gamma(\lambda + 1 - \alpha_i)} t^{\lambda - \alpha_i} \right] \nonumber
\]
\[
- \sum_{i=0}^{q} \frac{1}{\Gamma(\alpha_p - \theta_i)} \int_0^t (t - s)^{a_p - \theta_i - \alpha_i} L_i(t, s)z(s)ds \nonumber
\]
\[
+ \sum_{i=0}^{q} \sum_{\lambda = [\theta_i]}^{\lambda - \alpha_i} \frac{\lambda!}{\Gamma(\lambda + 1 - \theta_i)} t^{1 + \lambda - \theta_i - \alpha_i} K_i(t, s), \quad t \in [0, b] \quad (25)
\]
and
\[ g(t) = f(t) - \sum_{i=0}^{p-1} d_i(t) \sum_{\lambda=|a_i|}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - a_i)} \phi_{a_i} t^{\lambda - a_i} \]
\[ - \sum_{i=0}^{q} \sum_{\lambda=|\theta_i|}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - \theta_i)} \phi_{\theta_i} K_{\lambda}(t) t^{1+\lambda-\theta_i}, \]  \[ t \in [0,b]. \] (26)

**Remark 1.** If \( K_0, \ldots, K_q \in C^m(\Delta), m \in \mathbb{N}_0, \) then it follows from (10), (23) and (24) that \( L_0, \ldots, L_q \in C^m(\Delta) \) and \( K_{0,\lambda}, \ldots, K_{q,\lambda} \in C^m(0,b), \lambda \geq 0. \)

**Remark 2.** A special case of problem (8)–(10) is the initial value problem
\[ (D^a_{\text{Cap}} y)(t) + \sum_{i=0}^{p-1} d_i(t) (D^a_{\text{Cap}} y)(t) + \sum_{i=0}^{q} \int_0^t (t-s)^{-\alpha_i} K_i(t,s) (D^a_{\text{Cap}} y)(s) ds = f(t), \]
\[ y^{(i)}(0) = \gamma_i, \quad i = 0, \ldots, n-1, \quad n = \lceil a_p \rceil, \] (27)
where \( t \in [0,b]. \) The solution of (27) can be expressed by the formula (see (17))
\[ y = Gz + Q, \]
where
\[ z = D^a_{\text{Cap}} y, \quad G = f^a_{\nu}, \quad Q(t) = \sum_{i=0}^{n-1} \frac{\gamma_i}{i!} t^i. \]
In the integral equation \( z = Tz + g \) (see (20)) corresponding to the initial value problem (27) the expressions for \( Tz \) and \( g \) are for \( t \in [0,b] \) given by
\[ (Tz)(t) = - \sum_{i=0}^{p-1} d_i(t) (f^a_{\nu} z)(t) - \sum_{i=0}^{q} \frac{1}{\Gamma(\alpha_i - \theta_i)} \int_0^t (t-s)^{\alpha_i - \theta_i - \gamma_i} L_i(t,s) z(s) ds \]
and
\[ g(t) = f(t) - \sum_{i=0}^{p-1} d_i(t) \sum_{\lambda=|a_i|}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - a_i)} \phi_{a_i} t^{\lambda - a_i} \]
\[ - \sum_{i=0}^{q} \sum_{\lambda=|\theta_i|}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - \theta_i)} \phi_{\theta_i} K_{\lambda}(t) t^{1+\lambda-\theta_i}, \]
with \( L_i \) and \( K_{\lambda} \) \((\lambda = 0, \ldots, n-1)\) defined by (23) and (24), respectively.

5. Existence, Uniqueness and Smoothness of the Solution

In order to characterize the possible singular behaviour of a solution of a fractional differential equation, we introduce a weighted space \( C^{m,\nu}(0,b) \) of smooth functions on \( (0,b) \) (cf. [57,59]).

For given \( b \in \mathbb{R}, b > 0, m \in \mathbb{N} \) and \( \nu \in \mathbb{R}, \nu < 1, \) by \( C^{m,\nu}(0,b) \) we denote the set of continuous functions \( u : [0,b] \to \mathbb{R} \) which are \( m \) times continuously differentiable in \( (0,b) \) such that
\[ |u^{(i)}(t)| \leq c \begin{cases} 1 & \text{if } i < 1 - \nu, \\ 1 + |\log t| & \text{if } i = 1 - \nu, \\ t^{1-\nu-i} & \text{if } i > 1 - \nu \end{cases}, \quad t \in (0,b], \quad i = 1, \ldots, m, \]
where $c$ is a positive constant. In other words, $u \in C^{m,v}(0,b]$ if $u \in C[0,b] \cap C^m(0,b]$ and

$$|u|_{m,v} := \sum_{i=1}^{m} \sup_{0 < t \leq b} \omega_{i-1+i}(t) |u^{(i)}(t)| < \infty,$$

where, for $t > 0$,

$$\omega_{\rho}(t) := \begin{cases} 1 & \text{if } \rho < 0, \\ \frac{1}{\rho^{1/\log |t|}} & \text{if } \rho = 0, \\ \frac{1}{\rho} & \text{if } \rho > 0. \end{cases}$$

Equipped with the norm

$$\|u\|_{C^{m,v}(0,b]} := \|u\|_\infty + |u|_{m,v}, \quad u \in C^{m,v}(0,b],$$

the set $C^{m,v}(0,b]$ becomes a Banach space.

Note that $C^m[0,b]$ ($m \in \mathbb{N}$) belongs to $C^{m,v}(0,b]$ for arbitrary $v < 1$. Some other examples are given by $y_1(t) = t^2$, $y_2(t) = t^3$ and $y_3(t) = t \log t$ with $y_3(0) = 0$. Clearly, $y_1 \in C^{m-\frac{2}{3}}(0,b]$, $y_2 \in C^{m+\frac{1}{3}}(0,b]$ and $y_3 \in C^{m,\varepsilon}(0,b]$. Moreover, a function of the form

$$y(t) = g_1(t) t^\delta + g_2(t) \quad (\delta > 0)$$

belongs to $C^{m,v}(0,b]$ for all $v \in [1-\delta,1)$ and $g_1, g_2 \in C^m[0,b]$, $m \in \mathbb{N}$. Note also that

$$C^q[0,b] \subset C^{q,v}(0,b] \subset C^{m,v}(0,b] \subset C[0,b], \quad q \geq m \geq 1, \quad v < 1.$$  \tag{28}

Observe that as $v$ increases so do the possible singularities of the derivatives of the functions in $C^{q,v}(0,b]$. Next we introduce some auxiliary results. Their proofs can be found in [57].

**Lemma 1.** If $y_1, y_2 \in C^{m,v}(0,b], m \in \mathbb{N}, v < 1$, then $y_1 y_2 \in C^{m,v}(0,b]$, and

$$\|y_1 y_2\|_{C^{m,v}(0,b]} \leq c \|y_1\|_{C^{m,v}(0,b]} \|y_2\|_{C^{m,v}(0,b]},$$

with a constant $c$ which is independent of $y_1$ and $y_2$.

**Lemma 2.** Let $\eta \in \mathbb{R}$, $\eta < 1$ and let $K \in C(\Delta)$. Then operator $S$ defined by

$$(Sy)(t) := \int_0^t (t-s)^{-\eta} K(t,s)y(s)ds, \quad t \in [0,b],$$  \tag{29}

is compact as an operator from $L^\infty(0,b)$ into $C[0,b]$. If, in addition, $K \in C^m(\Delta), m \in \mathbb{N}$, then $S$ is compact as an operator from $C^{m,v}(0,b]$ into $C^{m,v}(0,b]$, where $\eta \leq v < 1$.

The following theorem characterizes the existence, uniqueness and regularity properties of the solution of (8) and (9).

**Theorem 1.** (i) Suppose that assumptions (10) hold. Moreover, assume that problem (8) and (9) with $f = 0$ and $\gamma_i = 0$ ($i = 0, \ldots, n-1$) has in $C[0,b]$ only the trivial solution $y = 0$, and from all polynomials $y$ of degree $n-1$ only $y = 0$ satisfies the conditions (14).

Then, problem (8) and (9) has a unique solution $y \in C^{m-1}[0,b]$ with $D_C^{p_\mu}y \in C[0,b]$.

(ii) Assume that (i) holds and let $d_i \in C^{m,\mu}(0,b]$ ($i = 0, \ldots, p-1$), $f \in C^{m,\mu}(0,b]$, $K_j \in C^m(\Delta) (j = 0, \ldots, q)$, where $m \in \mathbb{N}, \mu \in \mathbb{R}, \mu < 1$.

Then, problem (8) and (9) possesses a unique solution $y \in C^{n-1}[0,b]$ such that $y \in C^{m,v}(0,b]$ and $D_C^{p_\mu}y \in C^{m,v}(0,b]$, where

$$v := \max\{\mu, v_1, v_2, v_3, v_4\},$$  \tag{30}
with

\[ v_1 := \max\{1 - (\alpha_p - \alpha_i) : \alpha_p - \alpha_i \in \mathbb{N}, i = 0, \ldots, p-1\}, \]
\[ v_2 := \max\{1 - (|\alpha| - |\alpha_i|) : \alpha < n-1, \alpha_i \notin \mathbb{N}_0, i = 0, \ldots, p-1\}, \]
\[ v_3 := \max\{k_i - (|\theta_1| - \theta_i) : \theta_i \leq n-1, i = 0, \ldots, q\}, \]
\[ v_4 := \max\{k_i - (\alpha_p - \theta) : i = 0, \ldots, q\}. \]

If for all indices \( i = 0, \ldots, p-1 \) we have \( \alpha_p - \alpha_i \in \mathbb{N} \), then we may set \( v = \max\{v_1, v_2, v_3, v_4\} \).

Analogously, if we have \( \alpha_i \in \mathbb{N}_0 \) for all indices \( i = 0, \ldots, p-1 \) such that \( \alpha_i < n-1 \), then we may set \( v = \max\{v_1, v_1, v_1, v_4\} \). Note also that in the case \( k_i = 0, i = 0, \ldots, q \) we may set \( v = \max\{v_1, v_1, v_2\} \).

**Proof.** (i) First, we observe that the forcing function \( g \) of equation \( z = Tz + g \) (see (20) and (26)) belongs to \( C[0, b] \). This follows from \( f \in C[0, b], d_i \in C[0, b] \) \((i = 0, \ldots, p-1)\) and from Remark 1 with \( m = 0 \).

Next, due to (25), operator \( T \) can be rewritten in the form

\[
T = -\sum_{i=0}^{p-1} D_i \left[ f^{a_p-a_i} - \sum_{\lambda=|\alpha|}^{n-1} \frac{\lambda!}{\Gamma(\lambda + 1 - \alpha_i)} A_{\lambda-a_i} M_{\lambda} \right] - \sum_{i=0}^{q} B_{i, L} - \sum_{\theta_i \leq n-1} A_{1+\lambda-\theta_i-\kappa} M_{\lambda} K_{\ell, \lambda},
\]

(31)

with \( D_i, A_{\sigma}, M_{\lambda}, B_{i, L} \) and \( K_{\ell, \lambda} \) defined by

\[
(D_i x)(t) := d_i(t) x(t), \quad i = 0, \ldots, p-1, \quad (A_{\sigma} x)(t) := t^{\sigma} x(t), \quad \sigma \in \mathbb{R}, \sigma \geq 0,
\]
\[
(M_{\lambda} x)(t) := \sum_{k=1}^{l} \sum_{j=0}^{n_k} \phi_{\lambda, j} (f^{a_p-a_i} (b_k) + \omega_{\lambda} (f^{a_p+1} (b_k)), \quad \lambda = 0, \ldots, n-1,
\]
\[
(B_{i, L} x)(t) := \frac{1}{\Gamma(\alpha_p - \theta)} \int_0^t (t-s)^{\alpha_p-\theta} L_i(t,s) x(s) ds, \quad i = 0, \ldots, q,
\]
\[
(K_{\ell, \lambda} x)(t) := K_{\ell, \lambda}(t) x(t), \quad \lambda = 0, \ldots, n-1, \quad i = 0, \ldots, q,
\]

where \( t \in [0, b] \) and \( x \in C[0, b] \). Here (see Remark 1), \( L_i \in C(\Delta) \) and \( K_{\ell, \lambda} \in C[0, b] \) are given by (23) and (24), respectively. Using Lemma 2 we obtain that \( f^\delta \) \((\delta > 0)\), \( M_{\lambda} \) \((\lambda = 0, \ldots, n-1)\) and \( B_{i, L} \) \((i = 0, \ldots, q)\) are compact as operators from \( C[0, b] \) into \( C[0, b] \). Clearly \( D_i \) \((i = 0, \ldots, p-1)\), \( K_{\ell, \lambda} \) \((\lambda = 0, \ldots, n-1, i = 0, \ldots, q)\) and \( A_{\sigma} \) \((\sigma \in \mathbb{R}, \sigma \geq 0)\) are bounded as operators from \( C[0, b] \) into \( C[0, b] \). This yields that \( T \), given by (31), is compact as an operator from \( C[0, b] \) into \( C[0, b] \).

Note that if \( f = 0 \) and \( \gamma_i = 0 \) \((i = 0, \ldots, n-1)\), then \( \phi_{\lambda} = 0 \) \((\lambda = 0, \ldots, n-1)\) (see (15)) and thus \( g = 0 \) (see (26)). From this, we obtain that if the homogeneous equation corresponding to problem (8) and (9) has only the trivial solution \( y = 0 \), then \( z = Tz \) has in \( C[0, b] \) only the trivial solution \( z = 0 \). Since \( g \in C[0, b] \), we obtain by Fredholm alternative theorem that equation \( z = Tz + g \) possesses a unique solution \( z \in C[0, b] \). This together with (17) yields that problem (8) and (9) has a unique solution \( y \in C^{n-1}[0, b] \) such that \( D_{C, \alpha_p}^{\nu} y = z \in C[0, b] \).

(ii) Let us prove that \( z = D_{C, \alpha_p}^{\nu} y \) belongs to \( C^{m, \mu}[0, b] \) \((m \in \mathbb{N} \) and \( \nu \) given by (30)) for \( K_i \in C^m(\Delta) \) \((i = 0, \ldots, q)\), \( d_i \in C^{m, \mu}[0, b] \) \((i = 0, \ldots, p-1)\), \( f \in C^{m, \mu}[0, b], \mu \in \mathbb{R}, \mu \geq 0 \).
\( \mu < 1 \). To this end we first establish that \( g \), the forcing function of equation \( z = Tz + g \), belongs to \( C^{m,p}(0, b] \). Indeed, it follows from (26) that \( g = g_1 + g_2 + g_3 \),

\[
g_1(t) := f(t), \quad g_2(t) := -\sum_{i=0}^{p-1} d_i(t) \sum_{\lambda = [\theta_i]}^{\lambda - a_i} \frac{\lambda!}{\Gamma(\lambda + 1 - \theta_i)} \phi_\lambda t^{\lambda - a_i},
\]

\[
g_3(t) := -\sum_{i=0}^{\theta_i} \lambda - [\theta_i] \frac{\lambda!}{\Gamma(\lambda + 1 - \theta_i)} \phi_\lambda K_{i,\lambda}(t) t^{\lambda + \alpha_i - \theta_i},
\]

where \( t \in [0, b] \). Clearly \( g_1 = f \in C^{m,p}(0, b] \subset C^{m,\nu}(0, b] \). Note that, if \( \delta \in \mathbb{N}_0 \), then for all \( \lambda \geq \lceil \delta \rceil \), \( \lambda \in \mathbb{N}_0 \) we have \( t^{\lambda - \delta} \in C^{m}(0, b] \subset C^{m,\nu}(0, b] \) for arbitrary \( m \in \mathbb{N} \) and \( \nu < 1 \). If \( \delta \notin \mathbb{N}_0 \), then for all \( \lambda \geq \lceil \delta \rceil \), \( \lambda \in \mathbb{N}_0 \) we have \( t^{\lambda - \delta} \in C^{m,1-\lceil \delta \rceil + \delta}(0, b] \). Thus, since \( d_i \in C^{m,p}(0, b] \) \( (i = 0, \ldots, p - 1) \), by using Lemma 1 we can write \( g_2 \in C^{m,p}(0, b] \) with \( \nu \) defined by (30). Finally, since for all \( \lambda \geq \lceil \delta_i \rceil \) \( (i = 0, \ldots, q) \) it holds that \( t^{1+\lambda-\delta_i - \xi_i} \in C^{m,\nu}(0, b] \subset C^{m,p}(0, b] \) and \( K_{i,\lambda} \in C^{m,\nu}(0, b] \) (see (24)), we have \( g_3 \in C^{m,p}(0, b] \) and hence \( g = g_1 + g_2 + g_3 \in C^{m,p}(0, b] \).

If there exists \( i \in \{0, \ldots, p - 1\} \) such that \( \alpha_i \notin \mathbb{N}_0 \), then it follows from the definition of \( \nu \) that \( 1 - (\alpha_p - \alpha_i) \leq \nu \) and therefore from Lemma 2 we have that \( f^{m,p-\alpha_i} \) is compact as an operator from \( C^{m,p}(0, b] \) into \( C^{m,\nu}(0, b] \). If \( \alpha_p \in \mathbb{N}, \alpha_i \notin \mathbb{N}_0 \), then \( f^{m,p-\alpha_i} \) is compact as an operator from \( C^{m,\nu}(0, b] \) into \( C^{m,p}(0, b] \). Clearly \( D_1 \) \( (i = 0, \ldots, p - 1) \) and \( A_{\lambda-\alpha_i} (\lambda = [\alpha_i], \ldots, n - 1, \alpha_i \leq n - 1, i = 0, \ldots, p - 1) \) are linear and bounded as operators from \( C^{m,\nu}(0, b] \) into \( C^{m,p}(0, b] \). Linear operators (functionals) \( M_\lambda : C^{m,\nu}(0, b] \to \mathbb{R}(\lambda = 0, \ldots, n - 1) \) are bounded and consequently compact in \( C^{m,\nu}(0, b] \). Thus, as the composition of a compact and bounded operator is compact, we see that \( D_1 f^{m,p-\alpha_i} \) \( (i = 0, \ldots, p - 1) \) and \( D_1 A_{\lambda-\alpha_i} M_\lambda (\lambda = [\alpha_i], \ldots, n - 1, \alpha_i \leq n - 1, i = 0, \ldots, p - 1) \) are linear and compact as operators from \( C^{m,\nu}(0, b] \) into \( C^{m,p}(0, b] \).

Similarly, we see that operators \( \mathcal{K}_{i,\lambda} A_{1+\lambda-\theta_i-\xi_i} (\lambda = 0, \ldots, n - 1, i = 0, \ldots, q) \) are linear and bounded as operators from \( C^{m,\nu}(0, b] \) into \( C^{m,p}(0, b] \), thus the operators \( A_{1+\lambda-\theta_i-\xi_i} M_\lambda \mathcal{K}_{i,\lambda} (\lambda = 0, \ldots, n - 1, i = 0, \ldots, q) \) are linear and compact as operators from \( C^{m,\nu}(0, b] \) into \( C^{m,p}(0, b] \). Finally, since \( \xi_i + \theta_i - \alpha_p \leq \nu \) \( (i = 0, \ldots, q) \), it follows from Lemma 2 that \( B_1 \) is compact as an operator from \( C^{m,\nu}(0, b] \) into \( C^{m,p}(0, b] \). Thus, \( T \) defined by (31) is linear and compact as an operator from \( C^{m,\nu}(0, b] \) into \( C^{m,p}(0, b] \). Since the homogeneous equation \( z = Tz \) has in \( C^{m,\nu}(0, b] \subset C[0, b] \) only the trivial solution \( z = 0 \), it follows from Fredholm alternative theorem that equation \( z = Tz + g \) has a unique solution \( z \in C^{m,\nu}(0, b] \).

Note that the inclusion \( z \in C^{m,\nu}(0, b] \) together with (17) and Lemma 2 yields that problem (8) and (9) possesses a unique solution \( y \in C^{m,\nu}(0, b] \) such that \( D_1 f y = z \in C^{m,\nu}(0, b] \).

Theorem 1 states that the regularity properties of \( y \), the solution of problem (8) and (9), depend on the smoothness of functions \( f, d_i (i = 0, \ldots, p - 1) \) and \( K_i (i = 0, \ldots, q) \). However, even when we have \( f, d_i \in C^m(0, b] \) \( (i = 0, \ldots, p - 1) \) and \( K_i \in C^m(\Lambda) \) \( (i = 0, \ldots, q) \) for some \( m \in \mathbb{N} \), we cannot claim that \( y \in C^m(0, b] \)—we may only say that \( y \in C^{m,\max\{v_1,v_2,v_3,v_4\}}(0, b] \). That is, the solution of the problem (8) and (9) can, in general, exhibit singular behaviour even when the data of the problem is smooth. This complicates the construction of high order methods for solving such equations numerically.

6. Numerical Method

In order to take into account the potential non-smoothness of the exact solution \( y = y(t) \) of (8) and (9) at the origin \( t = 0 \), we introduce on the interval \([0, b] \) a graded grid \( \Pi_N (N \in \mathbb{N}) \). More precisely, let \( N \in \mathbb{N} \), then \( \Pi_N := \{t_0, \ldots, t_N\} \) is a partition (a graded grid) of the interval \([0, b] \) with the grid points
where the grading exponent $r \in \mathbb{R}$, $r \geq 1$. If $r = 1$, then the grid points (32) are distributed uniformly; for $r > 1$ the points (32) are more densely clustered near the left endpoint of the interval $[0, b]$.

Next, for a given integer $k \geq 0$, by $S_k^{(-1)}(\Pi_N)$ we denote the standard space of piecewise polynomial functions:

$$S_k^{(-1)}(\Pi_N) := \{ v : v|_{[t_{j-1}, t_j]} \in \pi_k, j = 1, \ldots, N \}.$$  

Here, $v|_{[t_{j-1}, t_j]}$ is the restriction of $v : [0, b] \to \mathbb{R}$ onto the subinterval $[t_{j-1}, t_j] \subset [0, b]$ and $\pi_k$ denotes the set of polynomials of degree not exceeding $k$. Note that the elements of $S_k^{(-1)}(\Pi_N)$ may have jump discontinuities at the interior points $t_1, \ldots, t_{N-1}$ of the grid $\Pi_N$.

Let $m \in \mathbb{N}$. In every subinterval $[t_{j-1}, t_j]$ ($j = 1, \ldots, N$) we define $m$ collocation points $t_{jk}, \ldots, t_{jm}$ by

$$t_{jk} := t_{j-1} + \eta_k (t_j - t_{j-1}), \quad k = 1, \ldots, m, \quad j = 1, \ldots, N,$$

where $\eta_1, \ldots, \eta_m$ are some fixed parameters which do not depend on $j$ and $N$ and satisfy

$$0 \leq \eta_1 < \eta_2 < \ldots < \eta_m \leq 1.$$  

Approximations $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ ($m, N \in \mathbb{N}$) to the solution $z$ of (20) we find by collocation conditions

$$z_N(t_{jk}) = (Tz_N)(t_{jk}) + g(t_{jk}), \quad k = 1, \ldots, m, \quad j = 1, \ldots, N,$$

where $T, g$ and $\{t_{jk}\}$ are defined by (25), (26) and (33), respectively. Note that conditions (35) for finding $z_N \in S_{m-1}^{(-1)}(\Pi_N)$ have an operator equation representation

$$z_N = \mathcal{P}_N Tz_N + \mathcal{P}_N g,$$

where $\mathcal{P}_N = \mathcal{P}_{N,m} : C[0, b] \to S_{m-1}^{(-1)}(\Pi_N)$ is defined by

$$(\mathcal{P}_N v)(t_{jk}) = v(t_{jk}), \quad k = 1, \ldots, m, \quad j = 1, \ldots, N, \quad v \in C[0, b].$$

If $\eta_1 = 0$, then by $(\mathcal{P}_N v)(t_{jk})$ we denote the right limit $\lim_{t \to t_{j-1}, t > t_{j-1}} (\mathcal{P}_N v)(t)$. If $\eta_m = 1$, then by $(\mathcal{P}_N v)(t_{jm})$ we denote the left limit $\lim_{t \to t_{j-1}, t < t_{j-1}} (\mathcal{P}_N v)(t)$. The collocation conditions (35) form a system of equations whose exact form is determined by the choice of a basis in the space $S_{m-1}^{(-1)}(\Pi_N)$. If $\eta_1 > 0$ or $\eta_m < 1$ then we can use the Lagrange fundamental polynomial representation:

$$z_N(t) = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu} \Phi_{\lambda\mu}(t), \quad t \in [0, b],$$

where, for $\mu = 1, \ldots, m, \lambda = 1, \ldots, N$,

$$\Phi_{\lambda\mu}(t) := \left\{ \begin{array}{ll} 0 & \text{for } t \notin [t_{\lambda-1}, t_\lambda], \\
\prod_{i=1, i \neq \mu}^{m} \frac{t-t_i}{t_{\lambda\mu}-t_{\lambda i}} & \text{for } t \in [t_{\lambda-1}, t_\lambda]. \end{array} \right.$$
Then, \( z_N \in S^{[-1]}_{m-1}(\Pi_N) \) and \( z_N(t_j) = c_{jk}, \, k = 1, \ldots, m, \, j = 1, \ldots, N \). Substituting \( z_N \) in the form (38) to (35), we obtain a system of linear algebraic equations with respect to the coefficients \( \{c_{jk}\} \):

\[
c_{jk} = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} (T\Phi_{\lambda\mu})(t_j)c_{\lambda\mu} + g(t_j), \quad k = 1, \ldots, m, \, j = 1, \ldots, N.
\]  

(40)

Solving this system of equations, we obtain the coefficients \( \{c_{jk}\} \) and thus have found the approximation \( z_N \) in the form (38). Note that for the computation of \( (T\Phi_{\lambda\mu})(t_j) \) we need to find the weakly singular integrals \( (I^\delta\Phi_{\lambda\mu})(t_j) \) \( (\delta > 0) \), which can be found exactly.

Approximation \( y_N \) to \( y \) we find by the formula (cf. (17))

\[
y_N = Gz_N + Q,
\]

(41)

where \( G \) and \( Q \) are defined by (18) and (19), respectively. By substituting \( z_N \) in the form (38) into (41), we obtain the following expression for the approximate solution \( y_N \):

\[
y_N(t) = \sum_{\lambda=1}^{N} \sum_{\mu=1}^{m} c_{\lambda\mu}(G\Phi_{\lambda\mu})(t) + Q(t).
\]

(42)

7. Convergence Analysis

In this section, we study the convergence and convergence order of the proposed algorithms. For this we need Lemmas 3–6 below. The proofs of Lemmas 3–5 follow from the results of [57,59]. The proof of Lemma 6 can be found in [60]. In what follows, for Banach spaces \( E \) and \( F \), by \( \mathcal{L}(E,F) \) is denoted the Banach space of linear bounded operators \( A : E \to F \) with the norm \( \|A\|_{\mathcal{L}(E,F)} = \sup\{\|Au\|_F : u \in E, \|u\|_E \leq 1\} \).

**Lemma 3.** Let \( \mathcal{P}_N : C[0,b] \to S^{[-1]}_{m-1}(\Pi_N) \) \( (N \in \mathbb{N}) \) be defined by (37). Then \( \mathcal{P}_N \in \mathcal{L}(C[0,b], L^\infty(0,b)) \) and the norms of \( \mathcal{P}_N \) are uniformly bounded: \( \|\mathcal{P}_N\|_{\mathcal{L}(C[0,b], L^\infty(0,b))} \leq c \), with a positive constant \( c \) which is independent of \( N \). Moreover, for every \( u \in C[0,b] \) we have

\[
\|u - \mathcal{P}_Nu\|_{L^\infty(0,b)} \to 0 \quad \text{as} \quad N \to \infty.
\]

**Lemma 4.** Let \( S : L^\infty(0,b) \to C[0,b] \) be a linear compact operator. Let \( \mathcal{P}_N : C[0,b] \to S^{[-1]}_{m-1}(\Pi_N) \) \( (N \in \mathbb{N}) \) be defined by (37). Then

\[
\|S - \mathcal{P}_NS\|_{\mathcal{L}(L^\infty(0,b), L^\infty(0,b))} \to 0 \quad \text{as} \quad N \to \infty.
\]

**Lemma 5.** Let \( u \in C^{m,p}(0,b], \, m \in \mathbb{N}, \, p \in (-\infty,1) \). Let \( \mathcal{P}_N : C[0,b] \to S^{[-1]}_{m-1}(\Pi_N) \) \( (N \in \mathbb{N}) \) be defined by (37). Then

\[
\|u - \mathcal{P}_Nu\|_\infty \leq c \begin{cases} 
N^{-m} & \text{for} \quad m < 1 - \nu, \quad r \geq 1, \\
N^{-m}(1 + \log N) & \text{for} \quad m = 1 - \nu, \quad r = 1, \\
N^{-m} & \text{for} \quad m = 1 - \nu, \quad r > 1, \\
N^{-r(1-\nu)} & \text{for} \quad m > 1 - \nu, \quad 1 \leq r < \frac{m}{1-\nu}, \\
N^{-m} & \text{for} \quad m > 1 - \nu, \quad r \geq \frac{m}{1-\nu},
\end{cases}
\]

where \( r \in [1,\infty) \) is the grading exponent in (32) and \( c \) is a positive constant independent of \( N \).

**Lemma 6.** Let \( u \in C^{m+1,p}(0,b], \, m \in \mathbb{N}, \, p \in (-\infty,1) \). Let \( N \in \mathbb{N}, \, \alpha > 0, \, r \in [1,\infty) \). Let \( I^\alpha \) and \( \mathcal{P}_N \) \( (N \in \mathbb{N}) \) be defined by (1) and (37), respectively. Assume that the collocation points (33)
Fractal Fract. 2021, 5, 90

There exists an integer \( N \) such that for all \( m \geq 0 \) the following holds:

\[
\int_0^1 F(x)dx \approx \sum_{k=1}^{m} w_k F(\eta_k), \quad 0 \leq \eta_1 < \eta_2 < \ldots < \eta_m \leq 1,
\]

with appropriate weights \( \{ w_k \} \) is exact for all polynomials \( F \) of degree \( m \).

Then, the following estimate holds:

\[
\| f^\ast(P_Nz - z) \|_\infty \leq c \left\{ \begin{array}{ll}
E_N(m, \alpha, \nu, r) & \text{if } 0 < \alpha < 1 \\
E_N^\ast(m, \nu, r) & \text{if } \alpha = 1
\end{array} \right.
\]

(43)

Here, \( c \) is a constant independent of \( N \),

\[
E_N(m, \alpha, \nu, r) := \left\{ \begin{array}{ll}
N^{-m-\nu} & \text{for } m < 1 + \alpha - \nu, \quad r \geq 1, \\
N^{-m-\nu}(1 + \log N) & \text{for } m = 1 + \alpha - \nu, \quad r = 1, \\
N^{-m-\nu} & \text{for } m = 1 + \alpha - \nu, \quad r > 1, \\
N^{-r(1+\alpha-\nu)} & \text{for } m > 1 + \alpha - \nu, \quad 1 \leq r < \frac{m+\nu}{1+\alpha-\nu}, \\
N^{-m-\nu} & \text{for } m > 1 + \alpha - \nu, \quad r \geq \frac{m+\nu}{1+\alpha-\nu},
\end{array} \right.
\]

(44)

and

\[
E_N^\ast(m, \nu, r) := \left\{ \begin{array}{ll}
N^{-m-1} & \text{for } m < 2 - \nu, \quad r \geq 1, \\
N^{-m-1}(1 + \log N)^2 & \text{for } m = 2 - \nu, \quad r = 1, \\
N^{-m-1}(1 + \log N) & \text{for } m = 2 - \nu, \quad r > 1, \\
N^{-r(2-\nu)} & \text{for } m > 2 - \nu, \quad 1 \leq r < \frac{m+1}{2-\nu}, \\
N^{-m-1} & \text{for } m > 2 - \nu, \quad r \geq \frac{m+1}{2-\nu}.
\end{array} \right.
\]

(45)

The following Theorems 2 and 3 characterize the convergence rate of the proposed method.

**Theorem 2.** (i) Let \( m, N \in \mathbb{N}, \ r \geq 1 \) and assume that the grid points \( \{ \eta_i \} \) with collocation points \( \{ \eta_i \} \) and arbitrary parameters \( \eta_1, \ldots, \eta_m \) satisfying (34) are used. Assume that conditions (10) are satisfied. Moreover, assume that problem (8) and (9) with \( f = 0 \) and \( \gamma_i = 0 \) \( (i = 0, \ldots, n-1) \) has only the trivial solution \( y = 0 \) and from all polynomials \( y \) of degree \( n - 1 \) only \( y = 0 \) satisfies the conditions (14).

Then, problem (8) and (9) has a unique solution \( y \in C^{n-1}[0, b] \) such that \( D^\nu_{\text{Cap}} y \in C[0, b] \). There exists an integer \( N_0 \) such that for all \( N \geq N_0 \) Equation (36) possesses a unique solution \( z_N \in S^{m-1}_m(\Pi_N) \), determining by (41) a unique approximation \( y_N \) to \( y \), the solution of (8) and (9), and

\[
\| y_N - y \|_\infty \to 0 \quad \text{as} \quad N \to \infty.
\]

(46)

(ii) If, in addition to (i), we assume that \( d_i \in C^{m\mu}(0, b) \) \( (i = 0, \ldots, p-1) \), \( f \in C^{m\mu}(0, b] \), \( K_i \in C^{m}(\Delta)(i = 0, \ldots, q) \), where \( \mu \in \mathbb{R}, \ \mu < 1 \), then for all \( N \geq N_0 \) the following error estimate holds:

\[
\| y - y_N \|_\infty \leq c \left\{ \begin{array}{ll}
N^{-m} & \text{for } m < 1 - \nu, \quad r \geq 1, \\
N^{-m}(1 + \log N) & \text{for } m = 1 - \nu, \quad r = 1, \\
N^{-m} & \text{for } m = 1 - \nu, \quad r > 1, \\
N^{-r(1-\nu)} & \text{for } m > 1 - \nu, \quad 1 \leq r < \frac{m}{1-\nu}, \\
N^{-m} & \text{for } m > 1 - \nu, \quad r \geq \frac{m}{1-\nu},
\end{array} \right.
\]

(47)

where \( \nu \) is determined by the Formula (30), \( r \geq 1 \) is the grading parameter in (32) and \( c \) is a constant independent of \( N \).
Proof. (i) First, we prove the convergence (46). To this end, we need to show that equation $z = Tz + g$ (see (20)), with $T$ and $g$ given by (21) and (22), is uniquely solvable in $L^\infty(0, b)$. We observe that $T$ is compact as an operator from $L^\infty(0, b)$ to $C[0, b]$, thus also from $L^\infty(0, b)$ to $L^\infty(0, b)$. Further, $g \in C[0, b] \subset L^\infty(0, b)$ and the homogeneous equation $z = Tz$ has in $C[0, b]$ only the trivial solution $z = 0$. This together with $T \in L(L^\infty(0, b), C[0, b])$ yields that $z = Tz$ possesses also in $L^\infty(0, b)$ only the trivial solution $z = 0$. Consequently, by Fredholm alternative theorem, equation $z = Tz + g$ with $g \in L^\infty(0, b)$ possesses a unique solution $z \in L^\infty(0, b)$. In other words, operator $I - T$ is invertible in $L^\infty(0, b)$ and its inverse $(I - T)^{-1}$ is bounded: $(I - T)^{-1} \in L(L^\infty(0, b), L^\infty(0, b))$. From Lemma 4 and from the boundedness of $(I - T)^{-1}$ in $L^\infty(0, b)$ we obtain that $I - P_N T$ is invertible in $L^\infty(0, b)$ for all sufficiently large $N$, say $N \geq N_0$, and

$$
\| (I - P_N T)^{-1} \|_{L(L^\infty(0, b), L^\infty(0, b))} \leq c, \quad N \geq N_0,
$$

where $c$ is a constant independent of $N$. Thus, for $N \geq N_0$, Equation (36) provides a unique solution $z_N \in S^{-1}_{m-1}(I_N)$. Note that for $z$, the solution of equation $z = Tz + g$, it holds $P_Nz = P_N Tz + P_N z_N$. We have for $z$ and $z_N$ that

$$(I - P_N T)(z - z_N) = z - z_N - P_N Tz + P_N Tz_N = z - P_Nz, \quad N \geq N_0.$$  

Therefore, by (48),

$$
\| z - z_N \|_\infty \leq c \| z - P_N z \|_\infty, \quad N \geq N_0,
$$

where $c$ is a positive constant independent of $N$. Using (41) we obtain that

$$
y(t) - y_N(t) = (G(z - z_N))(t), \quad N \geq N_0, \quad t \in [0, b].
$$

From this and (49) it follows that

$$
\| y - y_N \|_\infty \leq c \| z - z_N \|_\infty \leq c_1 \| z - P_N z \|_\infty, \quad N \geq N_0,
$$

where $c$ and $c_1$ are some positive constants independent of $N$. This together with $z \in C[0, b]$ and Lemma 3 yields the convergence (46).

(ii) If $K \in C^\infty(\Delta)$, $h, f \in C^{m, \mu}(0, b)$, $m \in \mathbb{N}$, $\mu \in \mathbb{R}$, $\mu < 1$, then it follows from the part (ii) of Theorem 1 that $z \in C^{m, \mu}(0, b)$, with $v$ given by (30). This together with (51) and Lemma 5 yields the estimate (47). □

It follows from Theorem 2 that in case of sufficiently smooth $f, d_i (i = 0, \ldots, p - 1)$ and $K_i (i = 0, \ldots, q)$ by using sufficiently large values of $r$ for every choice of collocation parameters $0 \leq \eta_1 < \cdots < \eta_m \leq 1$ a convergence of order $O(N^{-m})$ can be expected. From Theorem 3 below, we see that by a careful choice of parameters $\eta_1, \ldots, \eta_m$ and by a slightly more restrictive smoothness requirement on $f, d_i (i = 0, \ldots, p - 1)$ and $K_i (i = 0, \ldots, q)$ it is possible to establish a faster convergence.

Theorem 3. Let $m \in \mathbb{N}$ and let the following conditions be fulfilled:

(i) the assumption of Theorem 2 hold with $d_i \in C^{m+1, \mu}(0, b) (i = 0, \ldots, p - 1)$, $f \in C^{m+1, \mu}(0, b)$, $K_i \in C^{m+1}(\Delta) (i = 0, \ldots, q)$, where $\mu \in \mathbb{R}$, $\mu < 1$;

(ii) the quadrature approximation

$$
\int_0^1 F(x) \, dx \approx \sum_{k=1}^m w_k F(\eta_k),
$$

with the knots $\{\eta_k\}$ satisfying (34) and appropriate weights $\{w_k\}$ is exact for all polynomials $F$ of degree $m$.

Then, problem (8)-(9) has a unique solution $y \in C^{m+1, \nu}(0, b)$ such that $D^{\nu}_x y \in C^{m+1, \nu}(0, b)$. There exists an integer $N_0$ such that, for all integers $N \geq N_0$, Equation (36) possesses a unique
solution $z_N \in S_{m-1}^{-1}(\mathbb{N})$, determining by (41) a unique approximation $y_N$ to $y$, the solution of (8) and (9), and the following error estimates hold:

$$
\|y - y_N\|_{\infty} \leq c \left\{ \begin{array}{ll}
N^{-m-a^*} & \text{for } m < 1 + \alpha^*, r \geq 1, \\
N^{-m-a^*}(1 + \log N) & \text{for } m = 1 + \alpha^*, r = 1, \\
N^{-m-a^*} & \text{for } m = 1 + \alpha^*, r > 1, \\
N^{-r(1+a^*+\nu)} & \text{for } m > 1 + \alpha^*, 1 \leq r < \frac{m+\alpha^*}{1+\alpha^*}, \\
N^{-m-a^*} & \text{for } m > 1 + \alpha^*, r \geq \frac{m+\alpha^*}{1+\alpha^*},
\end{array} \right.
$$

for $\alpha^* := \min\{\alpha_\rho - \alpha_{\rho-1}, \alpha_{\rho} - n_1\} < 1$, and

$$
\|y - y_N\|_{\infty} \leq c_1 \left\{ \begin{array}{ll}
N^{-m-1} & \text{for } m < 2 - \nu, r \geq 1, \\
N^{-m-1}(1 + \log N)^2 & \text{for } m = 2 - \nu, r = 1, \\
N^{-m-1}(1 + \log N) & \text{for } m = 2 - \nu, r > 1, \\
N^{-r(2-\nu)} & \text{for } m > 2 - \nu, 1 \leq r < \frac{m+1}{2-\nu}, \\
N^{-m-1} & \text{for } m > 2 - \nu, r \geq \frac{m+1}{2-\nu},
\end{array} \right.
$$

for $\alpha^* \geq 1$. Here $\nu$ is determined by (30) (see Theorem 1), $r$ is the grid parameter in (32) and $c, c_1$ are positive constants independent of $N$.

**Proof.** From Theorem 2, we know that problem (8) and (9) has a unique solution $y \in C^{n-1}[0, b]$ such that $z = D_{c_{\text{cap}}}^\rho y \in C[0, b]$ and there exists an integer $N_0$ such that for all $N \geq N_0$ Equation (36) has a unique solution $z_N$ for which (49) is valid. Denote

$$
\tilde{z}_N := Tz_N + g, \quad N \geq N_0,
$$

where $T$ and $g$ are defined by (21) and (22), respectively. From (36) we see that $z_N = P_N \tilde{z}_N$. Substituting this expression of $z_N$ into (55) we obtain that

$$
\tilde{z}_N = TP_N \tilde{z}_N + g, \quad N \geq N_0.
$$

From (20) and (56) follows the identity

$$(I - TP_N)(\tilde{z}_N - z) = T(P_N z - z), \quad N \geq N_0.
$$

Since

$$(I - TP_N)^{-1} = I + T(I - P_N T)^{-1} P_N, \quad N \geq N_0,
$$

we obtain, with the help of (48), that

$$
\|\tilde{z}_N - z\|_{\infty} \leq c \|T(P_N z - z)\|_{\infty}, \quad N \geq N_0.
$$

This together with (25) yields

$$
\|\tilde{z}_N - z\|_{\infty} \leq c \sum_{i=0}^{p-1} \|f^p - a_i(P_N z - z)\|_{\infty} + c_1 \sum_{i=0}^{q} \|\tilde{B}_{i,L}(P_N z - z)\|_{\infty}
$$

$$
+ c_2 \sum_{i=0}^{p-1} \|f^p - (P_N z - z)(b_i)\| \|a_i\| + c_3 \sum_{i=0}^{p-1} \|f^{p+1}(P_N z - z)(\tilde{b}_i)\|, \quad N \geq N_0,
$$

where

$$(\tilde{B}_{i,L}x)(t) := \int_0^t (t - s)^{a_p - b^i} L_i(t, s)x(s)ds, \quad x \in L^\infty(0, b),$$

For the upper bound of $\alpha_\rho = \rho \frac{\Gamma(\rho - \theta)}{\Gamma(\rho)}$, we take $\rho = \theta = 0$ in the first term of (55) and $\rho = 1$ in the second term of (56). We then obtain, respectively, that

$$
\|\tilde{z}_N - z\|_{\infty} \leq \frac{c}{\Gamma(\theta + 1)} \|T(z_N - z)\|_{\infty}, \quad N \geq N_0
$$

and

$$
\|\tilde{z}_N - z\|_{\infty} \leq c \|T(z_N - z)\|_{\infty}, \quad N \geq N_0.
$$

Since $\theta < 0$, the right-hand side of the first inequality is more precise than the second. We then obtain that for $\nu > 1$

$$
\|\tilde{z}_N - z\|_{\infty} \leq \frac{c}{\Gamma(\theta + 1)} \|T(z_N - z)\|_{\infty}, \quad N \geq N_0
$$

and for $\nu = 1$

$$
\|\tilde{z}_N - z\|_{\infty} \leq c \|T(z_N - z)\|_{\infty}, \quad N \geq N_0.
$$
where \(\alpha\) with \(L\) defined by (23). Here and below \(c, c_1, c_2\) and \(c_3\) are generic positive constants which are independent of \(N\). With the help of (3) and from the boundedness of the Riemann–Liouville integral operator we obtain for \(N \geq N_0\) the following estimates:

\[
\|z_N - z\|_{\infty} \leq c \|f^\ast (\mathcal{P}Nz - z)\|_{\infty}, \quad N \geq N_0,
\]

where \(\alpha^* = \min\{\alpha_p - \alpha_{p-1}, \alpha_p - n_1\}\). Since \(z_N = \mathcal{P}Nz_N\), we obtain with the help of (17) and (41) that

\[
\|y_N - y\|_{\infty} = \|G(z_N - z)\|_{\infty} \leq \|G\mathcal{P}N(z_N - z)\|_{\infty} + \|G(\mathcal{P}Nz - z)\|_{\infty}, \quad N \geq N_0.
\]

Using (18), (59) and (60) we obtain

\[
\|G(\mathcal{P}Nz - z)\|_{\infty} \leq \|J_{\alpha}(\mathcal{P}Nz - z)\|_{\infty} + c \|\mathcal{P}Nz - z\|_{\infty} \leq c_1 \|J_{\alpha}(\mathcal{P}Nz - z)\|_{\infty}, \quad N \geq N_0.
\]

This together with (62) and (63) yields

\[
\|y_N - y\|_{\infty} \leq c \|J_{\alpha}(\mathcal{P}Nz - z)\|_{\infty}, \quad N \geq N_0.
\]

Because of Theorem 1 we have \(z \in C^{m+1,\nu}(0, b]\) and due to (3)

\[
\|J_{\alpha}(\mathcal{P}Nz - z)\|_{\infty} \leq c \|J_1(\mathcal{P}Nz - z)\|_{\infty}, \quad N \geq N_0,
\]

for \(\alpha^* \geq 1\). Therefore, it follows from (64) and Lemma 6 that the estimates (53) and (54) are true.

8. Numerical Examples

**Example 1.** Consider the following boundary value problem:

\[
(D_{\text{Cap}}^{1/2} y)(t) + d_0(t)y(t) + \int_0^t (t-s)^{-\frac{1}{2}} y(s)ds + \int_0^t (t-s)^{-\frac{1}{2}} (D_{\text{Cap}}^{1/2} y)(s)ds = f(t),
\]

\[
y(0) + y(1) + \int_0^1 y(z)dz = 1 + \frac{4}{7},
\]

with

\[
d_0(t) = t^2, \quad f(t) = \frac{3\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})} t^3 + \frac{5}{2} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right) t,
\]
where $0 \leq t \leq 1$. We see that (65) and (66) is a special problem of (8) and (9) with

$$a_0 = 0, \quad a_1 = \frac{1}{2}, \quad p = 1, \quad n = 1,$$

$$\theta_0 = 0, \quad \theta_1 = \frac{1}{4}, \quad \kappa_0 = \frac{3}{4}, \quad \kappa_1 = \frac{1}{2}, \quad g = 1,$$

$$b_1 = b = 1, \quad l = 1, \quad b_0 = 1,$$

$$n_0 = n_1 = 0, \quad \beta_{000} = \beta_{001} = \beta_0 = 1, \quad \gamma_0 = 1 + \frac{4}{7},$$

$$d_0 \in C^{m, \frac{1}{2}}(0, 1), \quad f \in C^{m, \frac{2}{3}}(0, 1) \text{ for all } m \in \mathbb{N},$$

$$K_0 = K_1 = 1.$$  

Clearly, $d_0, f \in C^{m, \mu}(0, 1]$ with $\mu = \frac{3}{4}$ and arbitrary $m \in \mathbb{N}$. Therefore, by (30),

$$\nu = \max \{ \mu, \nu_1, \nu_3, \nu_4 \} = \max \{ \mu, 1 - \alpha_1, \kappa_0, \max \{ \kappa_0 - \alpha_1, \kappa_1 - (\alpha_1 - \theta_1) \} \} = \frac{3}{4}.$$  

To solve (65) and (66) by method ((36), (41)) we set $z := D_{\text{cap}}^{1/2}y$. We have for $z$ Equation (20) with $T$ and $g$ given by (25) and (26), respectively. Approximations $z_N \in S_m^{(-1)}(\Pi_N), \ N \in \mathbb{N}$, to the solution $z$ of Equation (20) on the interval $[0, 1]$ are found by (35) using (33) with

$$\eta_1 = \frac{3 - \sqrt{3}}{6}, \quad \eta_2 = 1 - \eta_1 \quad (m = 2)$$

and

$$\eta_1 = \frac{5 - \sqrt{15}}{10}, \quad \eta_2 = \frac{1}{2}, \quad \eta_3 = 1 - \eta_1 \quad (m = 3),$$

the knots of the Gaussian quadrature approximation (52) for $m = 2$ and $m = 3$, respectively. The coefficients $c_{jk} = z_N(t_{jk}) \ (k = 1, \ldots, m, \ j = 1, \ldots, N)$ and the function $z_N(t) \ (t \in [0, b])$ are determined by (40) and (38), respectively. After that, by using Formula (42) we can find the approximate solution $y_N$ for (65) and (66).

In Tables 1 and 2, some results of numerical experiments for different values of the parameters $m, N$ and $r$ are presented. The errors $\varepsilon_N$ in Tables 1 and 2 are calculated as follows:

$$\varepsilon_N := \max_{j=1, \ldots, N} \max_{k=0, \ldots, 10} |y(t_{jk}) - y_N(t_{jk})|,$$  

where $y(t) = t^2$ ($t \in [0, 1]$)

is the exact solution to (65) and (66) and

$$t_{jk} := t_{j-1} + k(t_j - t_{j-1})/10, \quad k = 0, \ldots, 10, \quad j = 1, \ldots, N,$$

with $t_j$ defined by (32). The ratios

$$\Theta_N := \frac{\varepsilon_N/2}{\varepsilon_N},$$  

characterizing the observed convergence rate, are also presented.

In the case $m = 2$, it follows from (53) with $\alpha_1 = \frac{1}{2}$ and $\nu = \frac{3}{4}$ that, for sufficiently large $N$,

$$\varepsilon_N \leq c_0 \begin{cases} N^{-0.75r} & \text{if } 1 \leq r < \frac{10}{3} = 3.33(3), \\ N^{-2.5} & \text{if } r \geq \frac{10}{3}, \end{cases}$$  

(69)
where \( c_0 \) is a positive constant independent of \( N \). Due to (69), the ratios \( \Theta_N \) for \( r = 1 \), \( r = 2 \), \( r = 3 \) and \( r \geq \frac{10}{3} \) ought to be approximately \( 2^{0.75} \approx 1.68 \), \( 2^{1.5} \approx 2.83 \), \( 2^{2.25} \approx 4.76 \) and \( 2^{3.5} \approx 5.66 \), respectively. These values are given in the last row of Table 1.

In the case \( m = 3 \), it follows from (53) with \( \alpha_1 = \frac{1}{2} \) and \( \nu = \frac{3}{4} \) that, for sufficiently large \( N \),

\[
\varepsilon_N \leq c_1 \begin{cases} N^{-0.75}r & \text{if } 1 \leq r < \frac{14}{3} = 4.66(6), \\ N^{-3.5} & \text{if } r \geq \frac{14}{3}, \end{cases}
\]

(70)

where \( c_1 \) is a positive constant independent of \( N \). Due to (70), the ratios \( \Theta_N \) for \( r = 3 \), \( r = 4 \) and \( r \geq \frac{14}{3} \) ought to be approximately \( 2^{2.25} \approx 4.76 \), \( 2^{3} = 8.00 \) and \( 2^{3.5} \approx 11.31 \), respectively. These values are given in the last row of Table 2.

As we can see from Tables 1 and 2, the numerical results are in good accord with the theoretical estimates given by Theorem 3 (by estimates (69) and (70)).

**Table 1. Numerical results for problem (65) and (66) with \( m = 2 \).**

| \( N \) | \( r = 1 \) \( \varepsilon_N \) | \( \Theta_N \) | \( r = 2 \) \( \varepsilon_N \) | \( \Theta_N \) | \( r = 3 \) \( \varepsilon_N \) | \( \Theta_N \) | \( r = \frac{10}{3} \) \( \varepsilon_N \) | \( \Theta_N \) |
|------|--------|--------|--------|--------|--------|--------|--------|--------|
| 4    | \( 1.05 \times 10^{-2} \) | \( 3.89 \times 10^{-3} \) | \( 2.92 \times 10^{-3} \) | \( 3.62 \times 10^{-3} \) |
| 8    | \( 7.11 \times 10^{-3} \) | \( 1.48 \times 10^{-3} \) | \( 2.30 \times 10^{-4} \) | \( 5.03 \times 10^{-4} \) | \( 6.46 \times 10^{-4} \) | \( 5.61 \times 10^{-4} \) |
| 16   | \( 4.63 \times 10^{-3} \) | \( 1.54 \times 10^{-3} \) | \( 2.62 \times 10^{-4} \) | \( 5.29 \times 10^{-4} \) | \( 1.18 \times 10^{-4} \) | \( 5.46 \times 10^{-4} \) |
| 32   | \( 2.93 \times 10^{-3} \) | \( 1.58 \times 10^{-3} \) | \( 2.74 \times 10^{-4} \) | \( 5.17 \times 10^{-4} \) | \( 2.08 \times 10^{-5} \) | \( 5.69 \times 10^{-5} \) |
| 64   | \( 1.81 \times 10^{-3} \) | \( 1.62 \times 10^{-3} \) | \( 2.80 \times 10^{-4} \) | \( 4.33 \times 10^{-4} \) | \( 3.62 \times 10^{-4} \) | \( 5.76 \times 10^{-4} \) |
| 128  | \( 1.10 \times 10^{-3} \) | \( 1.64 \times 10^{-3} \) | \( 2.82 \times 10^{-4} \) | \( 4.48 \times 10^{-4} \) | \( 6.29 \times 10^{-4} \) | \( 5.75 \times 10^{-4} \) |
| 256  | \( 6.65 \times 10^{-4} \) | \( 1.66 \times 10^{-4} \) | \( 2.82 \times 10^{-4} \) | \( 4.57 \times 10^{-4} \) | \( 1.09 \times 10^{-7} \) | \( 5.74 \times 10^{-4} \) |
| 512  | \( 3.99 \times 10^{-4} \) | \( 1.67 \times 10^{-4} \) | \( 2.83 \times 10^{-4} \) | \( 4.62 \times 10^{-4} \) | \( 1.91 \times 10^{-8} \) | \( 5.72 \times 10^{-4} \) |

| 1.68 | 2.83 | 4.76 | 5.66 |

**Table 2. Numerical results for problem (65) and (66) with \( m = 3 \).**

| \( N \) | \( r = 3 \) \( \varepsilon_N \) | \( \Theta_N \) | \( r = 4 \) \( \varepsilon_N \) | \( \Theta_N \) | \( r = \frac{14}{3} \) \( \varepsilon_N \) | \( \Theta_N \) | \( r = 5 \) \( \varepsilon_N \) | \( \Theta_N \) |
|------|--------|--------|--------|--------|--------|--------|--------|--------|
| 4    | \( 9.01 \times 10^{-4} \) | \( 1.05 \times 10^{-3} \) | \( 1.31 \times 10^{-3} \) | \( 1.44 \times 10^{-3} \) |
| 8    | \( 1.73 \times 10^{-4} \) | \( 5.20 \times 10^{-4} \) | \( 7.78 \times 10^{-4} \) | \( 9.41 \times 10^{-4} \) | \( 1.54 \times 10^{-4} \) | \( 9.34 \times 10^{-4} \) |
| 16   | \( 3.48 \times 10^{-5} \) | \( 4.98 \times 10^{-5} \) | \( 8.81 \times 10^{-5} \) | \( 1.11 \times 10^{-5} \) | \( 1.29 \times 10^{-5} \) | \( 12.0 \times 10^{-5} \) |
| 32   | \( 7.17 \times 10^{-6} \) | \( 4.85 \times 10^{-6} \) | \( 8.69 \times 10^{-6} \) | \( 11.9 \times 10^{-6} \) | \( 1.01 \times 10^{-6} \) | \( 12.8 \times 10^{-6} \) |
| 64   | \( 1.50 \times 10^{-6} \) | \( 4.79 \times 10^{-7} \) | \( 8.45 \times 10^{-7} \) | \( 11.8 \times 10^{-7} \) | \( 8.60 \times 10^{-8} \) | \( 11.7 \times 10^{-7} \) |
| 128  | \( 3.14 \times 10^{-7} \) | \( 4.77 \times 10^{-8} \) | \( 8.29 \times 10^{-8} \) | \( 11.7 \times 10^{-8} \) | \( 7.38 \times 10^{-9} \) | \( 11.6 \times 10^{-9} \) |
| 256  | \( 6.58 \times 10^{-8} \) | \( 4.76 \times 10^{-9} \) | \( 8.18 \times 10^{-9} \) | \( 11.6 \times 10^{-10} \) | \( 6.38 \times 10^{-10} \) | \( 11.6 \times 10^{-10} \) |
| 512  | \( 1.38 \times 10^{-8} \) | \( 4.76 \times 10^{-10} \) | \( 8.13 \times 10^{-11} \) | \( 11.4 \times 10^{-11} \) | \( 6.04 \times 10^{-11} \) | \( 10.6 \times 10^{-11} \) |

| 4.76 | 8.00 | 11.3 | 11.3 |

Note that the conditions imposed by Theorem 2 are, in general, not sufficient for obtaining a global superconvergence rate. To show this, we have solved (65) and (66) by method (36), (41)) with collocation parameters

\( \eta_1 = 0.1, \eta_2 = 0.9 \) \( (m = 2) \)

for which the condition (ii) in Theorem 3 is not satisfied. The results of the numerical experiments are presented in Table 3, with the ratios (68) corresponding to the estimate (69) given in the last row of the table. As we can see, although the numerical convergence rate for \( r = 1 \) and \( r = 2 \) is still in accord with Theorem 3, for \( r = 3 \) and \( r = \frac{10}{3} \) the numerical results do not attain the predicted superconvergence rate. Instead, the highest attained numerical convergence rate is close to \( 2^2 = 4 \), which is the maximal convergence rate predicted by Theorem 2.
Consider the following boundary value problem:

$$\begin{align*}
(D_{C_{\text{Cap}}}^\alpha y)(t) + d_0(t)y(t) + \int_0^t (t-s)^{-\frac{\alpha}{2}} (D_{C_{\text{Cap}}}^\beta y)(s)ds &= f(t), \\
y(0) &= 0, \quad y'(0) = 0,
\end{align*}$$

(71)

with

$$d_0(t) = t^\frac{1}{5}, \quad f(t) = \frac{\Gamma\left(\frac{11}{10}\right)}{\Gamma\left(\frac{11}{10}\right)} t^\frac{1}{10} + t^\frac{1}{5} + \frac{\Gamma\left(\frac{11}{10}\right)}{\Gamma\left(\frac{3}{2}\right)} t^\frac{3}{5},$$

where $0 \leq t \leq 1$. We see that (71) and (72) is an initial value problem (27) with

$$\alpha_0 = 0, \quad \alpha_1 = \frac{11}{10}, \quad p = 1, \quad n = 2, \quad b = 1,$$

$$\theta_0 = \frac{1}{10}, \quad \kappa_0 = \frac{1}{2}, \quad q = 0,$$

$$\gamma_0 = 0, \quad \gamma_1 = 0, \quad d_0 \in C^{m, \frac{1}{2}}(0, 1), \quad f \in C^{m, \frac{3}{10}}(0, 1) \text{ for all } m \in \mathbb{N},$$

$$K_0 = 1.$$

Clearly, $d_0, f \in C^{m, \mu}(0, 1]$ with $\mu = \frac{9}{10}$ and arbitrary $m \in \mathbb{N}$. Therefore, by (30),

$$\nu = \max\{\mu, \nu_1, \nu_3, \nu_4\} = \max\{\mu, 1 - \alpha_1, \kappa_0 - (1 - \theta_0), \kappa_0 - (\alpha_1 - \theta_0)\} = \frac{9}{10}.$$

We solve (71) and (72) by method ((36), (41)) in a similar way as in Example 1. In Tables 4 and 5 some results of numerical experiments for different values of the parameters $m, N$ and $r$ are presented. The errors $\varepsilon_N$ in Tables 4 and 5 are calculated by the Formula (67), where

$$y(t) = t^\frac{3}{5} \quad (t \in [0, 1])$$

is the exact solution to (71) and (72).

In the case $m = 2$ it follows from (54) with $\alpha_1 = \frac{11}{10}$ and $\nu = \frac{9}{10}$ that, for sufficiently large $N$,

$$\varepsilon_N \leq \varepsilon_0 \begin{cases} N^{-1.1r} & \text{if } 1 \leq r < \frac{30}{11} = 2.72(72), \\
N^{-3} & \text{if } r \geq \frac{30}{11}, \end{cases}$$

(73)
where $c_0$ is a positive constant independent of $N$. Due to (73), the ratios $\Theta_N$ for $r = 1, r = 2$ and $r \geq \frac{30}{11}$ ought to be approximately $2^{1.1} \approx 2.14, 2^{2.2} \approx 4.60$ and $2^3 = 8$, respectively. These values are given in the last row of Table 4.

In the case $m = 3$, it follows from (54) with $\alpha_1 = \frac{11}{11}$ and $\nu = \frac{9}{11}$ that, for sufficiently large $N$,

$$\varepsilon_N \leq c_1 \left\{ \begin{array}{ll} N^{-1.1r} & \text{if } 1 \leq r < \frac{40}{11} = 3.63(63), \\ N^{-4} & \text{if } r \geq \frac{40}{11}, \end{array} \right.$$  

(74)

where $c_1$ is a positive constant independent of $N$. Due to (74), the ratios $\Theta_N$ for $r = 1, r = 2, r = 3$ and $r \geq \frac{40}{11}$ ought to be approximately $2^{1.1} \approx 2.14, 2^{2.2} \approx 4.60, 2^3 \approx 9.85$ and $2^4 = 16$, respectively. These values are given in the last row of Table 5.

As we can see from Tables 4 and 5, the numerical results are in accord with the theoretical estimates given by Theorem 3 (by estimates (73) and (74)).

**Table 4.** Numerical results for problem (71) and (72) with $m = 2$.

| $r = 1$ | $r = 2$ | $r = \frac{30}{11}$ | $r = 3$ |
|---------|---------|---------------------|---------|
| $N$ | $\varepsilon_N$ | $\Theta_N$ | $\varepsilon_N$ | $\Theta_N$ | $\varepsilon_N$ | $\Theta_N$ |
| 4 | $2.15 \times 10^{-3}$ | $5.21 \times 10^{-4}$ | $4.04 \times 10^{-4}$ | $4.27 \times 10^{-4}$ |
| 8 | $9.65 \times 10^{-4}$ | $2.23 \times 10^{-4}$ | $5.00 \times 10^{-5}$ | $7.51 \times 10^{-5}$ | $8.19$ |
| 16 | $4.26 \times 10^{-4}$ | $2.03 \times 10^{-5}$ | $5.13 \times 10^{-6}$ | $7.90 \times 10^{-6}$ | $8.65$ |
| 32 | $1.86 \times 10^{-4}$ | $2.29 \times 10^{-6}$ | $4.89 \times 10^{-7}$ | $8.06 \times 10^{-7}$ | $8.97$ |
| 64 | $8.13 \times 10^{-5}$ | $8.74 \times 10^{-7}$ | $4.75 \times 10^{-7}$ | $8.12 \times 10^{-7}$ | $9.05$ |
| 128 | $3.54 \times 10^{-5}$ | $1.87 \times 10^{-7}$ | $4.68 \times 10^{-8}$ | $8.14 \times 10^{-8}$ | $9.07$ |
| 256 | $1.54 \times 10^{-5}$ | $4.03 \times 10^{-8}$ | $4.64 \times 10^{-9}$ | $8.15 \times 10^{-9}$ | $9.06$ |
| 512 | $6.80 \times 10^{-6}$ | $8.72 \times 10^{-9}$ | $4.62 \times 10^{-9}$ | $8.15 \times 10^{-10}$ | $9.03$ |

\[ \begin{array}{c}
2.14 \\
4.16 \\
8.00 \\
8.00
\end{array} \]

**Table 5.** Numerical results for problem (71) and (72) with $m = 3$.

| $r = 1$ | $r = 2$ | $r = \frac{30}{11}$ | $r = 3$ |
|---------|---------|---------------------|---------|
| $N$ | $\varepsilon_N$ | $\Theta_N$ | $\varepsilon_N$ | $\Theta_N$ | $\varepsilon_N$ | $\Theta_N$ |
| 4 | $9.34 \times 10^{-4}$ | $1.79 \times 10^{-4}$ | $7.89 \times 10^{-5}$ | $8.54 \times 10^{-5}$ |
| 8 | $4.11 \times 10^{-4}$ | $3.41 \times 10^{-5}$ | $5.26 \times 10^{-6}$ | $5.34 \times 10^{-6}$ | $16.0$ |
| 16 | $1.79 \times 10^{-4}$ | $7.05 \times 10^{-6}$ | $4.84 \times 10^{-7}$ | $4.67 \times 10^{-7}$ | $16.6$ |
| 32 | $7.82 \times 10^{-5}$ | $1.52 \times 10^{-6}$ | $4.64 \times 10^{-8}$ | $4.57 \times 10^{-8}$ | $16.8$ |
| 64 | $3.40 \times 10^{-5}$ | $3.30 \times 10^{-7}$ | $4.61 \times 10^{-9}$ | $5.55 \times 10^{-9}$ | $16.9$ |
| 128 | $1.48 \times 10^{-5}$ | $7.18 \times 10^{-8}$ | $4.60 \times 10^{-10}$ | $5.48 \times 10^{-10}$ | $16.5$ |
| 256 | $6.79 \times 10^{-6}$ | $1.56 \times 10^{-8}$ | $4.60 \times 10^{-11}$ | $5.47 \times 10^{-11}$ | $16.5$ |
| 512 | $3.17 \times 10^{-6}$ | $3.40 \times 10^{-9}$ | $4.60 \times 10^{-12}$ | $5.50 \times 10^{-12}$ | $16.4$ |

\[ \begin{array}{c}
2.14 \\
4.16 \\
9.85 \\
16.0
\end{array} \]

9. Concluding Remarks

In this work, we have introduced and analyzed a high order numerical method for solving a wide class of linear multi-term fractional weakly singular integro-differential equations with Caputo fractional derivatives for local or non-local boundary conditions. For certain values of coefficients, the considered problem is an initial value problem or a multi-point boundary value problem. We have reformulated the proposed problem as an integral equation with respect to the highest order Caputo derivative in the fractional integro-differential equation. Using this reformulation, we have first studied the existence, uniqueness and regularity of the problem. We have shown that, in general, the exact solution of the problem is non-smooth, even if the initial data is smooth. On the basis of the results obtained regarding the smoothness of the exact solution, with the help of special graded grids and spline collocation techniques we have constructed an effective
numerical method that recovers its optimal convergence order. Moreover, we have shown that, by a careful choice of grid and collocation parameters, the method will obtain a global superconvergence rate. Note that (see [45,56]) the superconvergence advantage might not hold if the reformulated integral equation is obtained with respect to the exact solution of the original problem. The main conclusions of the paper extend known ones and are formulated in Theorems 1–3. From these results, an optimal choice of grid and collocation parameters can be made for finding a numerical solution to any problem given in the form (8) and (9) that satisfies the conditions of Theorems 1–3. More precisely, in order to apply Theorems 2 or 3, we first of all have to calculate the smoothness parameter \(\nu\), characterizing the regularity of the exact solution of the underlying problem. After finding the value of \(\nu\) and choosing an order \(m - 1\) for the polynomials used in the piecewise polynomial method, one can select a grid parameter \(r\) large enough so that the corresponding conditions set for the convergence rates in Theorems 2 or 3 are fulfilled. Note that while the conditions given by Theorems 2 and 3 do not set an upper bound for the parameter \(r\), for the optimal convergence rate the smallest possible value of \(r\) satisfying the corresponding inequality tends to give a more precise result.

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