Abstract

We consider the problem of transfer learning in an online setting. Different tasks are presented sequentially and processed by a within-task algorithm. We propose a lifelong learning strategy which refines the underlying data representation used by the within-task algorithm, thereby transferring information from one task to the next. We show that when the within-task algorithm comes with some regret bound, our strategy inherits this good property. Our bounds are in expectation for a general loss function, and uniform for a convex loss. We discuss applications to dictionary learning and finite set of predictors. In the latter case, we improve previous $O(1/\sqrt{m})$ bounds to $O(1/m)$ where $m$ is the per task sample size.

1 INTRODUCTION

Most analyses of learning algorithms assume that the algorithm starts learning from scratch when presented with a new dataset. However, in real life, it is often the case that we will use the same algorithm on many different tasks, and that information should be transferred from one task to another. For example, a key problem in pattern recognition is to learn a dictionary of features helpful for image classification: it makes perfectly sense to assume that features learnt to classify dogs against other animals can be re-used to recognize cats. This idea is at the core of transfer learning, see (Thrun and Pratt, 1998; Balcan et al., 2015; Baxter, 1997, 2000; Cavallanti et al., 2010; Maurer, 2005; Maurer et al., 2013; Pentina and Lampert, 2014; Maurer et al., 2016) and references therein.

The setting in which the tasks are presented simultaneously is often referred to as learning-to-learn (Baxter, 2000), whereas when the tasks are presented sequentially, the term lifelong learning is often used (Thrun, 1996). In either case, a huge improvement over “learning in isolation” can be expected, especially when the sample size per task is relatively small. We will use the above terminologies in the paper.

Although a substantial amount of work has been done on the theoretical study of learning-to-learn (Baxter, 2000; Maurer, 2005; Pentina and Lampert, 2014; Maurer et al., 2016), up to our knowledge there is no analysis of the statistical performance of lifelong learning algorithms. Ruvolo and Eaton (2013) studied the convergence of
certain optimization algorithms for lifelong learning. However, no statistical guarantees are provided. Furthermore, in all the aforementioned works, the authors propose a technique for transfer learning which constrains the within-task algorithm to be of a certain kind, e.g. regularized empirical risk minimization.

The main goal of this paper is to show that it is possible to perform a theoretical analysis of lifelong learning with minimal assumptions on the form of the within-task algorithm. Given a learner with her/his own favourite algorithm(s) for learning within tasks, we propose a meta-algorithm for transferring information from one task to the next. The algorithm maintains a prior distribution on the set of representations, which is updated after the encounter of each new task using the exponentially weighted aggregation (EWA) procedure, hence we call it EWA for lifelong learning or EWA-LL.

A standard way to provide theoretical guarantees for online algorithms are regret bounds, which measure the discrepancy between the prediction error of the forecaster and the error of an ideal predictor. We prove that, as long as the within-task algorithms have good statistical properties, EWA-LL inherits these properties. Specifically in Theorem 3.1 we present regret bounds for EWA-LL, in which the regret bounds for the within-tasks algorithms are combined into a regret bound for the meta-algorithm.

We also show, using an online-to-batch analysis, that it is possible to derive a strategy for learning-to-learn, and provide risk bounds for this strategy. The bounds are generally in the order of $1/\sqrt{T} + 1/\sqrt{m}$, where $T$ is the number of tasks and $m$ is the sample size per task. Moreover, we derive in some specific situations rates in $1/\sqrt{T} + 1/m$. These rates are novel up to our knowledge and justify the use of transfer learning with very small sample sizes $m$.

The paper is organized as follows. In Section 2 we introduce the lifelong learning problem. In Section 3 we present the EWA-LL algorithm and provide a bound on its expected regret. This bound is very general, but might be uneasy to understand at first sight. So, in Section 4 we present more explicit versions of our bound in two classical examples: finite set of predictors and dictionary learning. We also provide a short simulation study for dictionary learning. At this point, we hope that the reader will have a clear overview of the problem under study. The rest of the paper is devoted to theoretical refinements: in online learning, uniform bounds are the norm rather than bounds in expectations (Cesa-Bianchi and Lugosi, 2006). In Section 5 we establish such bounds for EWA-LL. Section 6 provides an online-to-batch analysis that allows one to use a modification of EWA-LL for learning-to-learn. The supplementary material include proofs (Appendix A), improvements for dictionary learning (Appendix B) and extended results (Appendix C).

2 PROBLEM

In this section, we introduce our notation and present the lifelong learning problem.

Let $\mathcal{X}$ and $\mathcal{Y}$ be some sets. A predictor is a function $f : \mathcal{X} \rightarrow \mathcal{Y}$, where $\mathcal{Y} = \mathbb{R}$ for regression and $\mathcal{Y} = \{-1, 1\}$ for binary classification. The loss of a predictor $f$ on a pair $(x, y)$ is a real number denoted by $\ell(f(x), y)$. As mentioned above, we want to transfer the information (a common data representation) gained from the previous tasks to a new one. Formally, we let $\mathcal{Z}$ be a set and prescribe a set $\mathcal{G}$ of feature maps (also called representations) $g : \mathcal{X} \rightarrow \mathcal{Z}$, and a set $\mathcal{H}$ of functions $h : \mathcal{Z} \rightarrow \mathbb{R}$. We shall design an algorithm that is useful when there is a function $g \in \mathcal{G}$, common to
all the tasks, and task-specific functions \( h_1, \ldots, h_T \) such that

\[
f_t = h_t \circ g
\]

is a good predictor for task \( t \), in the sense that the corresponding prediction error (see below) is small.

We are now ready to describe the learning problem. We assume that tasks are dealt with sequentially. Furthermore, we assume that each task dataset is itself revealed sequentially and refer to this setting as online-within-online lifelong learning. Specifically, at each time step \( t \in \{1, \ldots, T\} \), the learner is challenged with a task, corresponding to a dataset

\[
S_t = ((x_{t,1}, y_{t,1}), \ldots, (x_{t,m_t}, y_{t,m_t})) \in (X \times Y)^{m_t}
\]

where \( m_t \in \mathbb{N} \). The dataset \( S_t \) is itself revealed sequentially, that is, at each inner step \( i \in \{1, \ldots, m_t\} \):

- The object \( x_{t,i} \) is revealed,
- The learner has to predict \( y_{t,i} \); let \( \hat{y}_{t,i} \) denote the prediction,
- The label \( y_{t,i} \) is revealed, and the learner incurs the loss \( \hat{\ell}_{t,i} := \ell(\hat{y}_{t,i}, y_{t,i}) \).

The task \( t \) ends at time \( m_t \), at which point the prediction error is

\[
\frac{1}{m_t} \sum_{i=1}^{m_t} \hat{\ell}_{t,i}. \tag{2.1}
\]

This process is repeated for each task \( t \), so that at the end of all the tasks, the average error is

\[
\frac{1}{T} \sum_{t=1}^{T} \frac{1}{m_t} \sum_{i=1}^{m_t} \hat{\ell}_{t,i}.
\]

Ideally, if for a given representation \( g \), the best predictor \( h_t \) for task \( t \) was known in advance, then an ideal learner using \( h_t \circ g \) for prediction would incur the error

\[
\inf_{h_t \in \mathcal{H}} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_t \circ g(x_{t,i}), y_{t,i}). \tag{2.2}
\]

Hence, we define the within-task-regret of the representation \( g \) on task \( t \) as the difference between the prediction error (2.1) and the smallest prediction error (2.2),

\[
\mathcal{R}_t(g) = \frac{1}{m_t} \sum_{i=1}^{m_t} \hat{\ell}_{t,i} - \inf_{h_t \in \mathcal{H}} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_t \circ g(x_{t,i}), y_{t,i}).
\]

The above expression is slightly different from the usual notion of regret [Cesa-Bianchi and Lugosi (2006)], which does not contain the factor \( 1/m_t \). This normalization is important in that it allows us to give equal weights to different tasks.

Note that an oracle who would have known the best common representation \( g \) for all tasks in advance would have only suffered, on the entire sequence of datasets, the error

\[
\inf_{g \in \mathcal{G}} \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_t \circ g(x_{t,i}), y_{t,i}).
\]
We are now ready to state our principal objective: we wish to design a procedure (meta-algorithm) that, at the beginning of each task \( t \), produces a function \( \hat{g}_t \) so that, within each task, the learner can use its own favorite online learning algorithm to solve task \( t \) on the sequence \( ((\hat{g}_t(x_{t,1}), y_{t,1}), \ldots, (\hat{g}_t(x_{t,m_t}), y_{t,m_t})) \). We wish to control the compound regret of our procedure

\[
R := \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell_{t,i} - \inf_{g \in G} \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in H} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_t \circ g(x_{t,i}), y_{t,i})
\]

which may succinctly be written as \( \sup_{g \in G} \{ \frac{1}{T} \sum_{t=1}^{T} R_t(g) \} \). This objective is accomplished in Section 3 under the assumption that a regret bound for the within-task-algorithm is available.

We end this section with two examples included in the framework.

**Example 2.1** (Dictionary learning). Set \( Z = \mathbb{R}^K \), and call \( g = (g_1, \ldots, g_K) \) a dictionary, where each \( g_k \) is a real-valued function on \( \mathcal{X} \). Furthermore choose \( H \) to be a set of linear functions on \( \mathbb{R}^K \), so that, for each task \( t \)

\[
h_t \circ g(x) = \sum_{k=1}^{K} \theta_{t,k} g_k(x).
\]

In practice depending on the value of \( K \), we can use least square estimators or LASSO to learn \( \theta^{(t)} \). In [Maurer et al., 2013; Ruvolo and Eaton, 2013], the authors consider \( \mathcal{X} = \mathbb{R}^d \) and \( g(x) = Dx \) for some \( d \times K \) matrix \( D \), and the goal is to learn jointly the predictors \( \theta^{(t)} \) and the dictionary \( D \).

**Example 2.2** (Finite set \( G \)). We choose \( G = \{g_1, \ldots, g_K\} \) and \( H \) any set. While this example is interesting in its own right, it is also instrumental in studying the continuous case via a suitable discretization process. A similar choice has been considered by [Crammer and Mansour (2012)] in the multitask setting, in which the goal is to bound the average error on a prescribed set of tasks.

We notice that a slightly different learning setting is obtained when each dataset \( S_t \) is given all at once. We refer to this as batch-within-online lifelong learning; this setting is briefly considered in Appendix C. On the other hand when all datasets are revealed all at once, we are in the well-known setting of learning-to-learn (Baxter, 2000). In Section 6, we explain how our lifelong learning analysis can be adapted to this setting.

## 3 ALGORITHM

In this section, we present our lifelong learning algorithm, derive its regret bound and then specify it to two popular within-task online algorithms.

### 3.1 EWA-LL Algorithm

Our EWA-LL algorithm is outlined in Algorithm 1. The algorithm is based on the exponentially weighted aggregation procedure (see e.g. [Cesa-Bianchi and Lugosi, 2006] and references therein), and updates a probability distribution \( \pi_t \) on the set of representation \( G \) before the encounter of task \( t \). The effect of Step iii is that any representation \( g \) which does not perform well on task \( t \), is less likely to be reused on the next task. We insist on the fact that this procedure allows the user to freely choose the within-task algorithm, which does not even need to be the same for each task.
Algorithm 1 EWA-LL

**Data** A sequence of datasets
\[ S_t = \{(x_{t,1}, y_{t,1}), \ldots, (x_{t,m_t}, y_{t,m_t})\}, 1 \leq t \leq T. \]
associated with different learning tasks; the points within each dataset are also given sequentially.

**Input** A prior \( \pi_1 \), a learning parameter \( \eta > 0 \) and a learning algorithm for each task \( t \) which, for any representation \( g \) returns a sequence of predictions \( \hat{y}_{t,i}^g \) and suffers a loss
\[ \hat{L}_t(g) := \frac{1}{m_t} \sum_{i=1}^{m_t} \ell\left(\hat{y}_{t,i}^g, y_{t,i}\right). \]

**Loop** For \( t = 1, \ldots, T \)
  
  i. Draw \( \hat{g}_t \sim \pi_t \).

  ii. Run the within-task learning algorithm on \( S_t \) and suffer loss \( \hat{L}_t(\hat{g}_t) \).

  iii. Update
\[ \pi_{t+1}(dg) := \frac{\exp(-\eta \hat{L}_t(g)) \pi_t(dg)}{\int \exp(-\eta \hat{L}_t(\gamma)) \pi_t(d\gamma)}. \]

3.2 Bounding the Expected Regret

Since Algorithm 1 involves a randomization strategy, we can only get a bound on the expected regret, the expectation being with respect to the drawing of the function \( \hat{g}_t \) at step i in the algorithm. Let \( \mathbb{E}_{g \sim \pi}[F(g)] \) denote the expectation of \( F(g) \) when \( g \sim \pi \). Note that the expected overall-average loss that we want to upper bound is then
\[ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_t}[\hat{L}_t(\hat{g}_t)]. \]

**Theorem 3.1.** If, for any \( g \in \mathcal{G}, \hat{L}_t(g) \in [0, C] \) and the within-task algorithm has a regret bound \( R_t(g) \leq \beta(g, m_t) \), then
\[ \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_t}\left[ \frac{1}{m_t} \sum_{i=1}^{m_t} \ell_{t,i} \right] \leq \inf_{\rho} \left\{ \mathbb{E}_{g \sim \rho}\left[ \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_t \circ g(x_{t,i}), y_{t,i}) \right. \right. \\
\left. \left. + \frac{1}{T} \sum_{t=1}^{T} \beta(g, m_t) \right] + \frac{\eta C^2}{8} + \frac{K(\rho, \pi_1)}{\eta T} \right\}, \]

where the infimum is taken over all probability measures \( \rho \) and \( K(\rho, \pi_1) \) is the Kullback-Leibler divergence between \( \rho \) and \( \pi_1 \).

The proof is given in Appendix A. Some comments are in order as the bound in Theorem 3.1 might not be easy to read. First, similar to standard analyses in online learning, the parameter \( \eta \) is a decreasing function of \( T \), hence the bound vanishes as \( T \) grows. Second, corollaries are derived in Section 4 that are easier to read, as they are more similar to usual regret inequalities (Cesa-Bianchi and Lugosi 2006), that is,
Algorithm 2 OGA

Data A task $\mathcal{S}_t = ((x_{t,1}, y_{t,1}), \ldots, (x_{t,m_t}, y_{t,m_t}))$.

Input Stepsize $\zeta > 0$, and $\theta_1 = 0$.

Loop For $i = 1, \ldots, m_t$,

1. Predict $\hat{y}_{t,i}^\theta = h_{\theta_i} \circ g(x_{t,i})$,
2. $y_{t,i}$ is revealed, update
\[ \theta_{i+1} = \theta_i - \zeta \nabla_{\theta} \ell(h_{\theta} \circ g(x_{t,i}), y_{t,i}) \big|_{\theta = \theta_i}. \]

the right hand side of the bound is of the form
\[ \inf_{g \in \mathcal{G}} \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_t \circ g(x_{t,i}), y_{t,i}) + \text{“rate”}. \tag{3.1} \]

The bound in Theorem 1 looks slightly different, but is quite similar in spirit. Indeed, instead of an infimum with respect to $g$ we have an infimum on all the possible aggregations with respect to $g$,
\[ \inf_{\rho} \mathbb{E}_{g \sim \rho} \left[ \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_t \circ g(x_{t,i}), y_{t,i}) + \text{“remainder”} \right] \]

where the remainder term depends on $K(\rho, \pi_1)$. In order to look like (3.1), we could consider a measure $\rho$ highly concentrated around the representation $g$ minimizing (3.1). When $\mathcal{G}$ is finite, this is a reasonable strategy and the bound is given explicitly in Section 4.1 below. However, in some situations, this would cause the term $K(\rho, \pi_1)$ to diverge. Studying accurately the minimizer in $\rho$ usually leads to an interesting regret bound, and this is exactly what is done in Section 4.

Finally note that the bound in Theorem 3.1 is given in expectation. In online learning, uniform bounds are usually preferred (Cesa-Bianchi and Lugosi, 2006). In Section 5 we show that it is possible to derive such bounds under additional assumptions.

3.3 Examples of Within Task Algorithms

We now specify the general bound in Theorem 1 to two popular online algorithms which we use within tasks.

3.3.1 Online Gradient

The first algorithm assumes that $\mathcal{H}$ is a parametric family of functions $\mathcal{H} = \{h_\theta, \theta \in \mathbb{R}^p, ||\theta|| \leq B\}$, and for any $(x, y, g)$, $\theta \mapsto \ell(h_\theta \circ g(x), y)$ is convex, $L$-Lipschitz, upper bounded by $C$ and denote by $\nabla_{\theta}$ a subgradient.

Corollary 3.2. The EWA-LL algorithm using the OGA within task with step size $\zeta = \frac{B}{L\sqrt{2m_t}}$ satisfies
The second algorithm is based on the EWA procedure on the space $G$. 

3.3.2 Exponentially Weighted Aggregation

The second algorithm is based on the EWA procedure on the space $H \circ g$ for a prescribed representation $g \in G$. Recall that a function $\varphi : \mathbb{R} \to \mathbb{R}$ is called $\zeta_0$-exp-concave if

$$\exp(-\zeta_0 \varphi)$$

is concave if $\exp(-\zeta_0 \varphi)$ is concave.

**Corollary 3.3.** Assume that $H$ is finite and that there exists $\zeta_0 > 0$ such that for any $y$, the function $\ell(\cdot, y)$ is $\zeta_0$-exp-concave and upper bounded by a constant $C$. Then the EWA-LL algorithm using the EWA within task with $\zeta = \zeta_0$ satisfies

$$\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{g_t \sim \pi_t} \left[ \frac{\ell_t, i}{m_t} \right] \leq \inf_{\rho} \left\{ \mathbb{E}_{g \sim \rho} \left[ \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in H} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_t \circ g(x_{t,i}), y_{t,i}) + \frac{1}{T} \sum_{t=1}^{T} \frac{\zeta_0 \log |H|}{m_t} \right] + \frac{\eta C^2}{8} + \frac{\mathcal{K}(\rho, \pi_1)}{\eta T} \right\}.$$

**Proof.** Apply Theorem 3.1 and use the bound $\mathcal{R}_t(g) \leq \beta(g, m_t) := \zeta_0 \log |H|/m_t$ that can be found, for example, in (Gerchinovitz, 2011, Theorem 2.2). \hfill \Box

We note that under additional assumptions on loss functions, (Hazan et al., 2007, Theorem 1) provides bounds for $\beta(g, m_t)$ that are in $\log(m_t)/m_t$. 

**Algorithm 3 EWA**

**Data** A task $S_t = \{(x_{t,1}, y_{t,1}), \ldots, (x_{t,m_t}, y_{t,m_t})\}$.

**Input** Learning rate $\eta$; a prior probability distribution $\mu_1$ on $H$.

**Loop** For $i = 1, \ldots, m_t$,

i. Predict $\hat{g}_{t,i} = \int_H h \circ g(x_{t,i}) \mu(h) dh$,

ii. $y_{t,i}$ is revealed, update

$$\mu_{t+1}(dh) = \frac{\exp(-\zeta_0 \ell(h \circ g(x_{t,i}), y_{t,i})) \mu_i(dh)}{\int \exp(-\zeta_0 \ell(h \circ g(x_{t,i}), y_{t,i})) \mu_i(dh)}.$$

A typical example is the quadratic loss function $\ell(y', y) = (y' - y)^2$. When there is some $B$ such that $|y_{t,i}| \leq B$ and $|h \circ g(x_{t,i})| \leq B$, then the exp-concavity assumption is verified with $\zeta_0 = 1/(8B)$ and the boundedness assumption with $C = 4B^2$.

Note that when the exp-concavity assumption does not hold, (Gerchinovitz, 2011) derives a bound $\beta(g, m_t) = B \sqrt{\log(|H|)/m_t}$ with $\zeta = (2/B) \sqrt{2 \log(|H|)/m_t}$. 

7
Moreover, PAC-Bayesian type bounds in various settings (including infinite $\mathcal{H}$) can be found in (Catoni, 2004; Audibert, 2006; Gerchinovitz, 2013). We refer the reader to (Gerchinovitz, 2011) for a comprehensive survey.

4 APPLICATIONS

In this section, we discuss two important applications. To ease our presentation, we assume that all the tasks have the same sample size, that is $m_t = m$ for all $t$.

4.1 Finite Subset of Relevant Predictors

We give details on Example 2.2, that is we assume that $\mathcal{G}$ is a set of $K$ functions. Note that step iii in Algorithm 1 boils down to update $K$ weights,

$$
\pi_t(g_k) = \frac{\exp(-\eta\hat{L}_t(g_k))\pi_{t-1}(g_k)}{\sum_{j=1}^{K}\exp(-\eta\hat{L}_t(g_j))\pi_{t-1}(g_j)}.
$$

**Theorem 4.1.** Under the assumptions of Theorem 3.1, if we set $\eta = \frac{2}{C}\sqrt{\frac{2\log K}{T}}$ and $\pi_1$ uniform on $\mathcal{G}$,

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{g_t \sim \pi_t} \left[ \frac{1}{m} \sum_{i=1}^{m} \hat{\ell}_{t,i} \right] \leq \min_{1 \leq k \leq K} \left\{ \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t \circ g_k(x_{t,i}), y_{t,i}) \right. \left. + \beta(g_k, m) \right\} + C \sqrt{\frac{\log K}{2T}}.
$$

**Proof.** Fix $g \in \mathcal{G}$, $\rho$ as the Dirac mass on $g$ and note that $\mathcal{K}(\rho, \pi_1) = \log K$. 

We discussed in Sections 3.3.1 and 3.3.2 that typical orders for $\beta(g, m)$ are $O(1/\sqrt{m})$, $O(\log(m)/m)$ or $O(1/m)$. We state a precise result in the finite case.

**Corollary 4.2.** Assume that $\mathcal{H}$ is finite, that for some $\zeta_0 > 0$, for any $y$, the function $\ell(\cdot, y)$ is $\zeta_0$-exp-concave and upper bounded by a constant $C$. Then the EWA-LL algorithm using the EWA within task with $\zeta = \zeta_0$ satisfies

$$
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{g_t \sim \pi_t} \left[ \frac{1}{m} \sum_{i=1}^{m} \hat{\ell}_{t,i} \right] \leq \min_{1 \leq k \leq K} \left\{ \frac{1}{T} \sum_{t=1}^{T} \min_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t \circ g_k(x_{t,i}), y_{t,i}) \right. \left. + \zeta_0 \log |\mathcal{H}| \right\} + C \sqrt{\frac{\log K}{2T}}.
$$

In Section 6, we derive from Theorem 3.1 a bound in the batch setting. As we shall see, in the finite case the bound is exactly the same as the bound on the compound regret. This allows us to compare our results to previous ones obtained in the learning-to-learn setting. In particular, our $O(1/m)$ bound improves upon (Pentina and Lampert, 2014) who derived an $O(1/\sqrt{m})$ bound.

8
4.2 Dictionary Learning

We now give details on Example 2.1 in the linear case. Specifically, we let \( \mathcal{X} = \mathbb{R}^d \), we let \( \mathcal{D}_K \) be the set formed by all \( d \times K \) matrices \( D \), whose columns have euclidean norm equal to one, and we define \( \mathcal{G} = \{ x \mapsto Dx : D \in \mathcal{D}_K \} \). Within this subsection we assume that the loss \( \ell \) is convex and \( \Phi \)-Lipschitz with respect to its first argument, that is, for every \( y \in \mathcal{Y} \) and \( a_1, a_2 \in \mathbb{R} \), it holds \( |\ell(a_1, y) - \ell(a_2, y)| \leq \Phi |a_1 - a_2| \). We also assume that for all \( t \in \{1, \ldots, T\} \) and \( i \in \{1, \ldots, m\} \), \( \|x_{t,i}\| \leq 1 \). Assume 

\[
\beta(m) := \sup_{g \in \mathcal{G}} \beta(m, g) < +\infty.
\]

We define the prior \( \pi_1 \) as follows: the columns of \( D \) are i.i.d., uniformly distributed on the \( d \)-dimensional unit sphere.

**Theorem 4.3.** Under the assumptions of Theorem 3.1, with \( \eta = \frac{2}{C}\sqrt{K}d/T \),

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_t} \left[ \frac{1}{m} \sum_{i=1}^{m} \ell_{t,i} \right] \leq \inf_{D \in \mathcal{D}_K} \left\{ \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t, Dx_{t,i}, y_{t,i}) \right. \\
+ \left. \frac{C}{4} \sqrt{\frac{Kd}{T} (\log(T) + 7) + \beta(m)} \right\} + \frac{B\Phi}{\sqrt{T}} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i}x_{t,i}^T \right)}.
\]

The proof relies on an application of Theorem 3.1. The calculations being tedious, we postpone the proof to Appendix A.

When we use OGA within tasks, we can use Corollary 3.2 with \( L = \Phi \sqrt{K} \) and so \( \beta(m) \leq \Phi B\sqrt{2K/m} \) for any \( D \in \mathcal{D}_K \). Moreover,

\[
\lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i}x_{t,i}^T \right) \leq \text{tr} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i}x_{t,i}^T \right) \leq 1 \tag{4.1}
\]

so Theorem 4.3 leads to the following corollary.

**Corollary 4.4.** Algorithm EWA-LL for dictionary learning, with \( \eta = \frac{2}{C}\sqrt{K}d/T \), and using the OGA algorithm within tasks, with step \( \zeta = B/(\Phi \sqrt{2mK}) \), satisfies

\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_t} \left[ \frac{1}{m} \sum_{i=1}^{m} \ell_{t,i} \right] \leq \inf_{D \in \mathcal{D}_K} \left\{ \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t, Dx_{t,i}, y_{t,i}) \right. \\
+ \left. \frac{C}{4} \sqrt{\frac{Kd}{T} (\log(T) + 7) + \beta(m)} \right\} + \frac{B\Phi}{\sqrt{T}} \sqrt{\frac{1}{T} \sum_{t=1}^{T} \lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i}x_{t,i}^T \right)} + \frac{\Phi B\sqrt{2K}}{\sqrt{m}}.
\]

Note that the upper bound (4.1) may be lose. For example, when the \( x_{t,i} \) are i.i.d. on the unit sphere, \( \lambda_{\max} \left( \sum_{i=1}^{m} x_{t,i}x_{t,i}^T / m \right) \) is close to \( 1/d \). In this case, it is possible to improve the term \( \beta(m) \) employed in the calculation of the bound, we postpone the lengthy details to Appendix B.

### 4.2.1 Algorithmic Details and Simulations

We implement our meta-algorithm Randomized EWA in this setting. The algorithm used within each task is the simple version of the online gradient algorithm outlined in Section 3.3.1.
Algorithm 4 EWA-LL for dictionary learning

Data As in Algorithm 1.

Input A learning rate $\eta$ for EWA and a learning rate $\zeta$ for the online gradient. A number of steps $N$ for the Metropolis-Hastings algorithm.

Start Draw $\hat{g}_1 \sim \pi_1$.

Loop For $t = 1, \ldots, T$

i Run the within-task learning algorithm $S_t$ and suffer loss $\hat{L}_t(\hat{g}_t)$.

ii Set $\tilde{g} := \hat{g}_t$.

iii Metropolis-Hastings algorithm. Repeat $N$ times

   a Draw $\tilde{g}' \sim \mathcal{N}(\tilde{g}, \sigma^2 I)$ and then set $\tilde{g}' := \tilde{g}' / \|\tilde{g}'\|$. 

   b Set $\tilde{g} := \tilde{g}'$ with probability

   \[
   \min\left\{1, \exp\left[\eta \sum_{h=1}^t (\hat{L}_h(\tilde{g}) - \hat{L}_h(\tilde{g}'))\right]\right\},
   \]

   $\tilde{g}$ remains unchanged otherwise.

iv Set $\hat{g}_t := \tilde{g}$.

In order to draw $\hat{g}_t$ from $\pi_t$, we use $N$-steps of Metropolis-Hastings algorithm with a normalized Gaussian proposal (see, for example, Robert and Casella 2013). In order to ensure a short burn-in period, we use the previous drawing $\hat{g}_{t-1}$ as a starting point. The procedure is given in Algorithm 4. Note the bottleneck of the algorithm: in step b we have to compare $\tilde{g}$ and $\tilde{g}'$ on the whole dataset so far.

We now present a short simulation study. We generate data in the following way: we let $K = 2$, $d = 5$, $T = 150$ and $m = 100$. The columns of $D$ are drawn uniformly on the unit sphere, and task regression vectors $\theta_t$ are also independent and have i.i.d. coordinates in $U[-1, 1]$. We generate the datasets $S_t$ as follows: all the $x_{t,i}$ are i.i.d. from the same distribution as $\theta_t$, and $y_{t,i} = \langle \theta_t, Dx_{t,i} \rangle + \varepsilon_{t,i}$ where the $\varepsilon_{t,i}$ are i.i.d. $\mathcal{N}(0, \sigma^2)$ and $\sigma = 0.1$.

We compare Algorithm 4 with $N = 10$ to an oracle who knows the representation $D$, but not the task regression vectors $\theta_t$, and learns them using the online gradient algorithm with step size $\zeta = 0.1$. Notice that after each chunk of 100 observations, a new task starts, so the parameter $\theta_t$ changes. Thus, the oracle incurs a large loss until it learns the new $\theta_t$ (usually within a few steps). This explains the “stair” shape of the cumulative loss of the oracle in Figure 1. Figure 2 indicates that after a few tasks, the dictionary $D$ is learnt by EWA-LL: its cumulative loss becomes parallel to the one of the oracle. Due to the bottleneck mentioned above, the algorithm becomes quite slow to run when $t$ grows. In order to improve the speed of the algorithm, we also tried Algorithm 4 with $N = 1$. There is absolutely no theoretical justification for this, however, obviously the algorithm is 10 times faster. As we can see on the red line in Figure 2, this version of the algorithm still learns $D$, but it takes more steps. Note that this is not completely unexpected: the Markov chain generated by this algorithm is no longer stationary, but it can still enjoy good mixing properties. It
would be interesting to study the theoretical performance of Algorithm 4. However, this would require considerably technical tools from Markov chain theory which are beyond the scope of this paper.

5 UNIFORM BOUNDS

In this section, we show that it possible to obtain a uniform bound, as opposed to a bound in expectation as in Theorem 3.1. From a theoretical perspective, the price to pay is very low: we only have to assume that the loss function is convex with respect to its first argument. However, in practice, there is an aggregation step that might not be feasible. This is discussed at the end of the section. The algorithm is outlined in Algorithm 5.

**Theorem 5.1.** Assuming that for any $g$, $0 \leq \bar{L}_t(g) \leq C$ and that the algorithm used within-task has a regret $\mathcal{R}_t(g) \leq \beta(g, m_t)$. Assume that $\ell$ is convex with respect to its first argument. Then it holds that

$$
\frac{1}{T} \sum_{t=1}^{T} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(\hat{y}_{t,i}, y_{t,i}) \leq \inf_{\rho} \left\{ \mathbb{E}_{g \sim \rho} \left[ \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_t \circ g(x_{t,i}), y_{t,i}) \right] + \frac{1}{T} \sum_{t=1}^{T} \beta(g, m_t) + \frac{\eta C^2}{8} + \frac{\kappa(\rho, \pi_1)}{\eta T} \right\}.
$$

**Proof.** At each step $t$, the loss suffered by the algorithm is

$$
\frac{1}{m_t} \sum_{i=1}^{m_t} \ell(\hat{y}_{t,i}, y_{t,i}) = \frac{1}{m_t} \sum_{i=1}^{m_t} \ell \left( \int \hat{y}_{t,i}^g \pi_t(\text{d}g), y_{t,i} \right)
$$

Figure 1: The cumulative loss of the oracle for the first 15 tasks.
and we can just apply Theorem 3.1.

In practice, for an infinite set $G$ we are not able to run simultaneously the within-task algorithm for all $g \in G$. So, we cannot compute the prediction (5.1) exactly. A possible strategy is to draw $N$ elements of $G$ i.i.d. from $\pi_t$, say $\hat{g}_t(1), \ldots, \hat{g}_t(N)$, and to replace (5.1) by

$$\hat{y}_{t,i}^{(N)} = \frac{1}{N} \sum_{j=1}^{N} \hat{g}_{t,i}^{(j)}.$$  

An application of Hoeffding’s inequality shows for any $\delta > 0$, with probability at least $1 - \delta$, the bound in Theorem 5.1 will still hold, up to an additional term $C \sqrt{\log(T/\delta)/2N}$.

6 LEARNING-TO-LEARN

In this section, we show how our analysis of lifelong learning can be used to derive bounds for learning-to-learn. In this setting, the tasks and their datasets are generated by first sampling task distributions $P_1, \ldots, P_T$ i.i.d. from a “meta-distribution” $Q$, called environment by Baxter [2000], and then for each task $t$, a dataset $S_t = ((x_{t,1}, y_{t,1}), \ldots, (x_{t,m}, y_{t,m}))$ is sampled i.i.d. from $P_t$. We stress that in this setting, the entire data $(x_{t,i}, y_{t,i})_{1 \leq i \leq m, 1 \leq t \leq T}$ is given all at once to the learner. Note that for simplicity, we assumed that all the sample sizes are the same.
Algorithm 5 Integrated EWA-LL

Data and Input same as in Algorithm 1.

Loop For $t = 1, \ldots, T$

i Run the within-task learning algorithm on $S_t$ for each $g \in G$ and return as predictions:
\[
\hat{y}_{t,i} = \int \hat{y}^{g}_{t,i} \pi_t(dg).
\] (5.1)

ii Update $\pi_{t+1}(dg) := \exp(-\eta \hat{L}_t(g)) \pi_t(dg) \int \exp(-\eta \hat{L}_t(\gamma)) \pi_t(d\gamma)$.

We wish to design a strategy which, given a new task $P \sim Q$ and a new sample $(x_1, y_1), \ldots, (x_m, y_m)$ i.i.d. from $P$, computes a function $f : X \to Y$, that will predict $y$ well when $(x, y) \sim P$. For this purpose we propose the following strategy:

1. Run EWA-LL on $(x_{t,i}, y_{t,i})_{1 \leq i \leq m, 1 \leq t \leq T}$. We obtain a sequence of representations $\hat{g}_1, \ldots, \hat{g}_T$.
2. Draw uniformly $T \in \{1, \ldots, T\}$ and put $\hat{g} = \hat{g}_T$.
3. Run the within task algorithm on the sample $(x_i, y_i)_{1 \leq i \leq m}$, obtaining a sequence $h^{\hat{g}}_1, \ldots, h^{\hat{g}}_m$ of functions.
4. Draw uniformly $I \in \{1, \ldots, m\}$ and put $\hat{h} = h^{\hat{g}}_I$.

Our next result establishes that the strategy leads indeed to safe predictions.

**Theorem 6.1.** Let $\mathbb{E}$ be the expectation over all data pairs $(x_{t,i}, y_{t,i})_{1 \leq i \leq m} \sim P_t$, $(P_t)_{1 \leq t \leq T} \sim Q$, $(x_i, y_i)_{1 \leq i \leq m} \sim P$, $(x, y) \sim P$, $P \sim Q$ and also over the randomized decisions of the learner $(\hat{g}_t)_{1 \leq t \leq T}$, $T$ and $I$. Then
\[
\mathbb{E}[\ell(\hat{h} \circ \hat{g}(x), y)] \leq \inf_{P \sim Q} \mathbb{E}_{P \sim \rho} \left[ \mathbb{E}_{P \sim \rho} \inf_{h \in H} \mathbb{E}_{(x,y) \sim P} \left[ \ell(h \circ g(x), y) \right] + \beta(g, m) \right] + \frac{\eta C^2}{8} + \frac{\mathcal{K}(\rho, \pi_1)}{\eta T}.
\]

The proof is given in Appendix A. As in Theorem 3.1, the result is given in expectation with respect to the randomized decisions of the learner. Assuming that $\ell$ is convex with respect to its first argument, we can state a similar result for a non-random procedure, as was done in Section 5. Details are left to the reader.

**Remark 6.1.** In (Baxter [2000], Maurer et al. [2013], Pentina and Lampert [2014]), the results on learning-to-learn are given with large probability with respect to the observations $(x_{t,i}, y_{t,i})_{1 \leq i \leq m, 1 \leq t \leq T}$, rather than in expectation. Using the machinery in (Cesa-Bianchi and Lugosi [2006], Lemma 4.1) we conjecture that it is possible to derive a bound in probability from Theorem 6.1.
7 CONCLUDING REMARKS

We presented a meta-algorithm for lifelong learning and derived a fully online analysis of its regret. An important advantage of this algorithm is that it inherits the good properties of any algorithm used to learn within tasks. Furthermore, using online-to-batch conversion techniques, we derived bounds for the related framework of learning-to-learn.

We discussed the implications of our general regret bounds for two applications: dictionary learning and finite set $\mathcal{G}$ of representations. Further applications of this algorithm which may be studied within our framework are deep neural networks and kernel learning. In the latter case, which has been addressed by [Pentina and Ben-David (2015)] in the learning-to-learn setting, $g: \mathcal{X} \rightarrow \mathcal{Z}$ is a feature map to a reproducing kernel Hilbert space $\mathcal{Z}$, and $h_t(g(x)) = \langle z^{(t)}, g(x) \rangle_\mathcal{Z}$. In the former case, $\mathcal{X} = \mathbb{R}^d$ and $g: \mathcal{X} \rightarrow \mathbb{R}^K$ is a multilayer network, that is a vector-valued function obtained by application of a linear transformation and a nonlinear activation function. The predictor $h: \mathbb{R}^K \rightarrow \mathbb{R}$ is typically a linear function. The vector-valued function $(h_1 \circ g, \ldots, h_T \circ g)$ models a multilayer network with shared hidden weights. This is discussed in [Maurer et al. (2016)], again in the learning-to-learn setting.

Perhaps the most fundamental question is to extend our analysis to more computationally efficient algorithms such as approximations of EWA, like Algorithm [4] or fully gradient based algorithms as in [Ruvolo and Eaton (2013)].

References

Audibert, J.-Y. (2006). A randomized online learning algorithm for better variance control. In Proc. 19th Annual Conference on Learning Theory, pages 392–407. Springer.

Balcan, M.-F., Blum, A., and Vempala, S. (2015). Efficient representations for lifelong learning and autoencoding. In Proc. 28th Conference on Learning Theory, pages 191–210.

Baxter, J. (1997). A bayesian/information theoretic model of learning to learn via multiple task sampling. Machine Learning, 28(1):7–39.

Baxter, J. (2000). A model of inductive bias learning. Journal of Artificial Intelligence Research, 12:149–198.

Catoni, O. (2004). Statistical learning theory and stochastic optimization, volume 1851 of Saint-Flour Summer School on Probability Theory 2001 (Jean Picard ed.), Lecture Notes in Mathematics. Springer-Verlag, Berlin.

Cavallanti, G., Cesa-Bianchi, N., and Gentile, C. (2010). Linear algorithms for online multitask classification. Journal of Machine Learning Research, 1:2901–2934.

Cesa-Bianchi, N. and Lugosi, G. (2006). Prediction, learning, and games. Cambridge University Press.

Crammer, K. and Mansour, Y. (2012). Learning multiple tasks using shared hypotheses. In Advances in Neural Information Processing Systems 25, pages 1475–1483.
Gerchinovitz, S. (2011). *Prédiction de suites individuelles et cadre statistique classique: étude de quelques liens autour de la régression parcimonieuse et des techniques d’agrégation*. PhD thesis, Paris 11.

Gerchinovitz, S. (2013). Sparsity regret bounds for individual sequences in online linear regression. *Journal of Machine Learning Research*, 14(1):729–769.

Hazan, E., Agarwal, A., and Kale, S. (2007). Logarithmic regret algorithms for online convex optimization. *Machine Learning*, 69(2-3):169–192.

Maurer, A. (2005). Algorithmic stability and meta-learning. *Journal of Machine Learning Research*, 6:967–994.

Maurer, A., Pontil, M., and Romera-Paredes, B. (2013). Sparse coding for multitask and transfer learning. In *Proc. 30th International Conference on Machine Learning*, pages 343–351.

Maurer, A., Pontil, M., and Romera-Paredes, B. (2016). The benefit of multitask representation learning. *Journal of Machine Learning Research*, 17(81):1–32.

McAllester, D. A. (1998). Some pac-bayesian theorems. In *Proc. 11th Annual Conference on Computational Learning Theory*, pages 230–234. ACM.

Pentina, A. and Ben-David, S. (2015). Multi-task and lifelong learning of kernels. In *Proc. 26th International Conference on Algorithmic Learning Theory*, pages 194–208.

Pentina, A. and Lampert, C. (2014). A pac-bayesian bound for lifelong learning. In *Proc. 31st International Conference on Machine Learning*, pages 991–999.

Robert, C. and Casella, G. (2013). *Monte Carlo statistical methods*. Springer Science & Business Media.

Ruvolo, P. and Eaton, E. (2013). Ella: An efficient lifelong learning algorithm. In *Proc. 30th International Conference on Machine Learning*, pages 507–515.

Shalev-Shwartz, S. (2011). Online learning and online convex optimization. *Foundations and Trends in Machine Learning*, 4(2):107–194.

Thrun, S. (1996). Is learning the n-th thing any easier than learning the first? In *Advances in neural information processing systems*, pages 640–646.

Thrun, S. and Pratt, L. (1998). *Learning to Learn*. Kluwer Academic Publishers.

Vapnik, V. (1998). *Statistical Learning Theory*. Wiley.
A Proofs

Proof of Theorem 3.1. It is enough to show that the EWA strategy leads to
\[
\sum_{t=1}^{T} E_{\hat{g}_t \sim \pi_t}[\hat{L}_t(\hat{g}_t)] \leq \inf_{\rho} \left\{ \mathbb{E}_{g \sim \rho} \left[ \sum_{t=1}^{T} \hat{L}_t(g) \right] + \frac{\eta C^2 T}{8} + \frac{K(\rho, \pi_1)}{\eta} \right\}. \tag{A.1}
\]

Once this is done, we only have to use the assumption that the regret of the within-task algorithm on task \( t \) is upper bounded by \( \beta(g, m_t) \) to obtain that
\[
\sum_{t=1}^{T} \hat{L}_t(g) = \sum_{t=1}^{T} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(h_{t,i} \circ g(x_{t,i}), y_{t,i}) \leq \sum_{t=1}^{T} \left\{ \beta(g, m_t) + \inf_{h \in \mathcal{H}} \frac{1}{m_t} \sum_{i=1}^{m_t} \ell((h \circ g(x_{t,i}), y_{t,i}) \right\}
\]
and we obtain the statement of the result.

It remains to prove (A.1). To this end, we follows the same guidelines as in the proof of Theorem 1 in [Audibert, 2006]. First, note that
\[
\pi_t(g) = \frac{\exp \left[ -\eta \sum_{u=1}^{t-1} \hat{L}_u(g) \right] \pi_1(dg)}{\int \exp \left[ -\eta \sum_{u=1}^{t-1} \hat{L}_u(\gamma) \right] \pi_1(d\gamma)} = \frac{\exp \left[ -\eta \sum_{u=1}^{t-1} \hat{L}_u(g) \right] \pi_1(dg)}{W_t} \tag{A.2}
\]
where we introduce the notation \( W_t \) for the sake of shortness. Put \( E_t = \int \hat{L}_t(g) \pi_t(dg) = E_{\hat{g}_t \sim \pi_t}[\hat{L}_t(g)] \). Using Hoeffding’s inequality on the bounded random variable \( \hat{L}_t(g) \in [0, C] \) we have, for any \( t \), that
\[
\mathbb{E}_{\hat{g}_t \sim \pi_t} \left[ \exp \left\{ \eta (E_t - \hat{L}_t(g)) \right\} \right] = \int \exp \left\{ \eta (E_t - \hat{L}_t(g)) \right\} \pi_t(dg) \leq \exp \left\{ \frac{C^2 \eta^2}{8} \right\}
\]
which can be rewritten as
\[
\exp \left\{ -\eta \mathbb{E}_{\hat{g}_t \sim \pi_t}[\hat{L}_t(g_t)] \right\} \geq \exp \left( -\frac{C^2 \eta^2}{8} \right) \mathbb{E}_{\hat{g}_t \sim \pi_t} \left[ \exp \left\{ -\eta \hat{L}_t(g_t) \right\} \right]. \tag{A.3}
\]
Next, we note that
\[
\exp \left\{ -\eta \sum_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_t}[\hat{L}_t(g_t)] \right\} = \prod_{t=1}^{T} \exp \left\{ -\eta \mathbb{E}_{\hat{g}_t \sim \pi_t}[\hat{L}_t(g_t)] \right\} \geq \exp \left( -\frac{TC^2 \eta^2}{8} \right) \prod_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_t} \left[ \exp \left\{ -\eta \hat{L}_t(g_t) \right\} \right] (\text{using (A.3)})
\]
\[
= \exp \left\{ -\frac{TC^2 \eta^2}{8} \right\} \prod_{t=1}^{T} \int \exp \left\{ -\eta \hat{L}_t(g) \right\} \pi_t(dg)
\]
\[
= \exp \left\{ -\frac{TC^2 \eta^2}{8} \right\} \prod_{t=1}^{T} \int \frac{\exp \left\{ -\eta \sum_{u=1}^{t} \hat{L}_u(g) \right\} \pi_1(dg)}{W_t}
\]
(\text{using (A.2)})
\[
= \exp \left\{ -\frac{TC^2 \eta^2}{8} \right\} \prod_{t=1}^{T} \frac{W_{t+1}}{W_t} = \exp \left\{ -\frac{TC^2 \eta^2}{8} \right\} W_{T+1}.
\]

16
So
\[
\sum_{t=1}^{T} \mathbb{E}_{g_t \sim \pi_t} [\hat{L}_t(g_t)] \leq -\frac{\log W_{T+1}}{\eta} + \frac{TC^2 \eta}{8}
\]
\[
= -\frac{\log \int \exp \left[ -\eta \sum_{t=1}^{T} \hat{L}_t(g_t) \right] \pi_1(\text{d}g) + TC^2 \eta}{8}
\]
and finally we use (Catoni, 2004, Equation (5.2.1)) which states that
\[
-\frac{\log \int \exp \left[ -\eta \sum_{t=1}^{T} \hat{L}_t(g_t) \right] \pi_1(\text{d}g) + TC^2 \eta}{8} = \inf_{\rho} \left\{ \mathbb{E}_{g \sim \rho} \left[ \sum_{t=1}^{T} \hat{L}_t(g_t) + \beta(m) \right] + K(\rho, \pi_1) \right\}.
\]

Proof of Theorem 4.3. Let $D^*$ denote a minimizer to the optimization problem
\[
\min_{D \in D_K} \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} \ell(\langle h_t, D x_{t,i} \rangle, y_{t,i}).
\]
We apply Theorem 3.1 and upper bound the infimum with respect to any $\rho$ by an infimum with respect to $\rho$ in the following parametric family
\[
\rho_c(dD) \propto 1\{\forall j = 1, \ldots, K : \|D \cdot j - D^*_j\| \leq c\} \pi_1(dD).
\]
where $c$ is a positive parameter. Note that when $c$ is small, $\rho_c$ highly concentrates around $D^*$, but we will show this is at a price of an increase in $K(\rho_c, \pi_1)$. The proof then proceeds in optimizing with respect to $c$.

We have that
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{g_t \sim \pi_t} \left[ \frac{1}{m} \sum_{i=1}^{m} \hat{\ell}_{t,i} \right]
\]
\[
\leq \inf_{c} \left\{ \mathbb{E}_{D \sim \rho_c} \left[ \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} \ell(\langle h_t, D x_{t,i} \rangle, y_{t,i}) + \beta(m) \right] + \frac{\eta C^2}{8} + \frac{K(\rho_c, \pi_1)}{\eta T} \right\}.
\]
Now, we have
\[
K(\rho_c, \pi_1) = -\log \pi_1(\{\forall j = 1, \ldots, K : \|D \cdot j - D^*_j\| \leq c\}),
\]
and
\[
\pi_1(\{\forall j = 1, \ldots, K : \|D \cdot j - D^*_j\| \leq c\}) \geq \prod_{j=1}^{K} \left( \frac{\pi^{(d-1)/2}(c/2)\Gamma(d-1/2)}{\Gamma(d+1/2)} \right) \geq \prod_{j=1}^{K} \left( \frac{c^{d-1}}{2^{1/2}} \right)
\]
where the first inequality follows by observing that, since $\pi_1$ is the uniform distribution on the unit $d$-sphere, the probability to be calculated is greater or equal to the ration
between the volume of the \((d-1)\)-ball with radius \(c/2\) and the surface area of the unit \(d\)-sphere. So we get

\[
\mathcal{K}(\rho_c, \pi_1) \leq K d \log(1/c) + 3Kd.
\]

Furthermore, using the notation

\[
h_t^* := \arg \inf_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t, D^* x_{t,i}),
\]

we get

\[
\inf_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t, D x_{t,i}), y_{t,i}) - \frac{1}{m} \sum_{i=1}^{m} \ell(h_t^*, D^* x_{t,i}), y_{t,i})
\]

\[
\leq \frac{1}{m} \sum_{i=1}^{m} \ell(h_t^*, D x_{t,i}), y_{t,i}) - \frac{1}{m} \sum_{i=1}^{m} \ell(h_t^*, D^* x_{t,i}), y_{t,i}).
\]

Under the condition on the loss, we have

\[
\ell(h_t^*, D x_{t,i}), y_{t,i}) - \ell(h_t^*, D^* x_{t,i}), y_{t,i}) \leq \Phi \|h_t^*, (D - D^*) x_{t,i})\|
\]

where \(\| \cdot \|_F\) denotes the Frobenius norm. We obtain an upper-bound

\[
\mathbb{E}_{D \sim \rho_c} \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t, D x_{t,i}), y_{t,i}) 
\]

\[
\leq \inf_{D \in \mathcal{D}_\kappa} \left\{ \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t, D x_{t,i}), y_{t,i}) + \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} \Phi \|h_t^*, (D - D^*) x_{t,i})\| \right\}.
\]

But then note that

\[
\frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} \Phi \|h_t^*, (D - D^*) x_{t,i})\|
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} \Phi \sqrt{\langle h_t^*, (D - D^*) x_{t,i}\rangle^2}
\]

\[
\leq \Phi \sqrt{\frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} \langle h_t^*, (D - D^*) x_{t,i}\rangle^2} \quad \text{(Jensen)}
\]

\[
= \Phi \sqrt{\frac{1}{T} \sum_{t=1}^{T} \langle h_t^*, (D - D^*) (\frac{1}{m} \sum_{i=1}^{m} x_{t,i} x_{t,i}^T) (D - D^*)^T h_t^* \rangle}
\]

\[
\leq \Phi \sqrt{\frac{1}{T} \sum_{t=1}^{T} \lambda_{\max} \langle \frac{1}{m} \sum_{i=1}^{m} x_{t,i} x_{t,i}^T \rangle \| (D - D^*)^T h_t^* \|^2}
\]

\[
\leq \Phi c B \sqrt{\frac{1}{T} \sum_{t=1}^{T} \lambda_{\max} \langle \frac{1}{m} \sum_{i=1}^{m} x_{t,i} x_{t,i}^T \rangle}.
\]
So Theorem 3.1 leads to

\[
\frac{1}{T} \sum_{t=1}^{T} E_{q_t \sim \eta_t} \left[ \frac{1}{m} \sum_{i=1}^{m} \ell_{t,i} \right] - \inf_{D \in D_K} \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_{t,i}, Dx_{t,i}, y_{t,i}) \\
\leq \inf_{c} \left\{ c \Phi \left( \frac{1}{T} \sum_{t=1}^{T} \lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i} x_{t,i}^T \right) \right) + Kd \frac{Kd}{\eta T} \log(1/c) \right\} + 3Kd \frac{Kd}{\eta T} + \beta(m) + \frac{\eta C^2}{8}.
\]

The choices \( c = \sqrt{\frac{1}{T}} \) and \( \eta = \frac{2}{c} \sqrt{\frac{Kd}{T}} \) lead to the result. \( \Box \)

**Proof of Theorem 6.7.** The proof relies on an application of the well-known online-to-batch trick, discussed pedagogically in Section 5 page 186 in [Shalev-Shwartz 2011]. Still, it is very cumbersome, and it is easy to get confused. For these reasons, we think it is important to write it completely. We use the following notation for any random variable \( V, E_V \) is the expectation with respect to \( V \). This is very important as the online-to-batch trick relies essentially on inverting the order of the random variables in the integration. We have:

\[
E[\ell(\hat{h} \circ \hat{g}(x), y)]
\]

\[
= E_T E_T E_{P_1,...,P_T} E_{x_i,y_j,i,j \leq T,i \leq m} E_{P} E_{x,y} \left[ \ell(\hat{h} \circ \hat{g}(x), y) \right]
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} E_{P_1,...,P_T} E_{x_i,y_j,i,j \leq T,i \leq m} E_{P} E_{x,y} \left[ \ell(\hat{h}^{q_i} \circ \hat{g}_t(x), y) \right]
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} E_{P_1,...,P_T} E_{x_i,y_j,i,j \leq T,i \leq m} E_{P} \left[ \ell(\hat{h}^{q_i} \circ \hat{g}_t(x), y) \right]
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} E_{P_1,...,P_T} E_{x_i,y_j,i,j \leq T,i \leq m} E_{P} \left[ \ell(\hat{h}^{q_i} \circ \hat{g}_t(x_i), y_i) \right]
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} E_{P_1,...,P_T} E_{x_i,y_j,i,j \leq T,i \leq m} E_{P} \left[ \frac{1}{m} \sum_{i=1}^{m} \ell(\hat{h}^{q_i} \circ \hat{g}_t(x_i), y_i) \right]
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} E_{P_1,...,P_T} E_{x_i,y_j,i,j \leq T,i \leq m} E_{P} \left[ \frac{1}{m} \sum_{i=1}^{m} \ell(\hat{h}^{q_i} \circ \hat{g}_t(x_i), y_i) \right]
\]

\[
= \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} E_{P_1,...,P_T} E_{x_i,y_j,i,j \leq T,i \leq m} E_{P} \left[ \frac{1}{m} \sum_{i=1}^{m} \ell(\hat{h}^{q_i} \circ \hat{g}_t(x_t,i), y_t,i) \right]
\]

\[
\leq \frac{1}{T} \sum_{t=1}^{T} \frac{1}{m} \sum_{i=1}^{m} \ell(\hat{h}^{q_i} \circ \hat{g}(x_{t,i}), y_{t,i})
\]

\[
\leq E_{P_1,...,P_T} E_{x_i,y_j,i,j \leq T,i \leq m} \inf_{\rho} \left\{ E_{q \sim \rho} \left[ \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t \circ g(x_{t,i}), y_{t,i}) \right] \right\}
\]
\[ + \frac{1}{T} \sum_{t=1}^{T} \beta(g, m) \right] + \frac{\eta C^2}{8} + \frac{K(\rho, \pi_1)}{\eta T} \right\}, \text{using Theorem 3.1,}
\]
\[
\leq \inf_{\rho} \left\{ \mathbb{E}_{g \sim P} \left[ \mathbb{E}_{P \sim Q} \inf_{h_t \in H} \mathbb{E}_{(x,y) \sim P} \ell(h_t \circ g(x), y) + \beta(g, m) \right] + \frac{\eta C^2}{8} + \frac{K(\rho, \pi_1)}{\eta T} \right\}.
\]

\[B \quad \text{Better Bounds for Dictionary Learning}\]

We now state a refined version of the bounds for dictionary learning in Section 4. As pointed out in that section, while in general the bound
\[
\lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i}x_{t,i}^T \right) \leq 1
\]

is unimprovable, if the input vectors \(x_{t,i}\) are i.i.d. random variables from uniform distribution on the unit sphere, then
\[
\frac{1}{m} \sum_{i=1}^{m} x_{t,i}x_{t,i}^T \xrightarrow{a.s.} \text{Cov}(x_{t,i}, x_{t,i}) = \frac{1}{d} I
\]

where \(I\) is the identity matrix. Consequently,
\[
\lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i}x_{t,i}^T \right) \xrightarrow{a.s.} \frac{1}{d}
\]

We can take advantage of this fact in order to improve the term \(\beta(m) = \sup_{g \in G} \beta(g, m)\), but only if we assume that we know in advance that \(\lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i}x_{t,i}^T / m \right)\) is not too large. This is the meaning of the following theorem.

**Theorem B.1.** Assume that we know in advance that for all \(t \in \{1, \ldots, T\}\),
\[
\lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i}x_{t,i}^T \right) \leq \Lambda
\]

for some \(\Lambda > 0\). Assume the same assumptions as in Theorem 4.3, still with \(\eta = \frac{2}{\varepsilon} \sqrt{\frac{Kd}{T}}\). Use within tasks Algorithm 2 (online gradient) with a fixed gradient step \(\zeta = B/(L\sqrt{2mK})\). Then we have
\[
\frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{g_t \sim \pi_t} \left[ \frac{1}{m} \sum_{i=1}^{m} \hat{\ell}_{t,i} \right] - \inf_{g \in G} \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in H} \frac{1}{m} \sum_{i=1}^{m} \ell(h_t \circ g(x_{t,i}), y_{t,i})
\]
\[
\leq C \sqrt{\frac{Kd}{T}} (\log(T) + 7) + \frac{2BL \sqrt{2K}}{\sqrt{m}} + \frac{B\Phi \sqrt{\Lambda}}{\sqrt{T}}.
\]

In particular, note that when \(\Lambda = 1/d\) the bound becomes
\[
\frac{C}{4} \sqrt{\frac{Kd}{T}} (\log(T) + 7) + \frac{2BL \sqrt{2K}}{\sqrt{md}} + \frac{B\Phi}{\sqrt{dT}}.
\]

20
Proof. We apply Theorem 4.3 so we only have to upper bound the term $\beta(g, m)$ for the online gradient algorithm with the prescribed step size. Note that in (Corollary 2.7 Shalev-Shwartz 2011) we actually have the following regret bound for Algorithm 2 with fixed step size $\eta > 0$:

$$\beta(g, m) = \frac{B^2}{2\eta m} + \frac{\eta}{m} \sum_{i=1}^{m} \|\nabla_{\theta=\theta_t}(\langle \theta, gx_{t,i}\rangle, y_{t,i})\|^2.$$ 

By the $L$-Lipschitz assumption on $\ell$, $\|\nabla_{\theta=\theta_t}(\langle \theta, gx_{t,i}\rangle, y_{t,i})\|^2 \leq L^2\|gx_{t,i}\|^2$. So we have

$$\sum_{t=1}^{m} \|\nabla_{\theta=\theta_t}(\langle \theta, gx_{t,i}\rangle, y_{t,i})\|^2 \leq L^2 \sum_{i=1}^{m} \|gx_{t,i}\|^2 = L^2 \sum_{i=1}^{m} \sum_{k=1}^{K} (g_k, x_{t,i})^2 \leq L^2 \sum_{i=1}^{m} \sum_{k=1}^{K} g_k^T x_{t,i} x_{t,i}^T g_k.$$ 

$$\leq m L^2 \sum_{k=1}^{K} g_k^T \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i} x_{t,i}^T \right) g_k.$$ 

$$\leq m K L^2 \lambda_{\max} \left( \frac{1}{m} \sum_{i=1}^{m} x_{t,i} x_{t,i}^T \right) \|g_k\|^2 \leq m K L^2 \Lambda.$$ 

Consequently, $\beta(m) = \sup_g \beta(g, m) \leq B^2/(2\eta m) + \eta K L^2 \Lambda$ and The choice $\eta \leq B/(L \sqrt{2mK\Lambda})$ leads to

$$\beta(m) = 2BL \sqrt{2K\Lambda/m}.$$ 

C Batch-Within-Online Lifelong Learning

In this last section of the appendix, we present an alternative approach for the batch-within-online setting discussed in Section 2. In this setting, the tasks are presented sequentially, but, for each task $t \in \{1, \ldots, T\}$ the dataset $S_t$ is presented all at once and we assume it is obtained i.i.d. from a distribution $P_t$. Unlike to the reasoning in Section 6 where we assumed that the $P_t$ were i.i.d. from a distribution $Q$, here we make no assumptions on the generation process underlying the $P_t$’s, which may even be adversarial chosen.

Let us recap the setting. At each time $t \in \{1, \ldots, T\}$, a task is presented to the learner in the following manner:

1. nature chooses $P_t$, no assumption is made on this choice. This $P_t$ is not revealed to the forecaster.

2. nature draws the sample $S_t = \{(x_{t,1}, y_{t,1}), \ldots, (x_{t,m_t}, y_{t,m_t})\}$ i.i.d. from $P_t$, and this sample is revealed to the forecaster.

3. based on her/his current guess $\hat{g}_t$ of $g$ and on the sample $S_t$, the forecaster has to run her/his favourite learning algorithm $\hat{h}$ on $(\hat{g}_t, S_t)$ to get an estimate
\[ \hat{h}_t = \hat{h}(\tilde{g}_t, S_t) \] based on an algorithm of his choice. Note that the forecaster observes \( \tilde{r}_t := r_t(\hat{h}_t \circ \tilde{g}_t) \) where
\[
r_t(f) = \frac{1}{m_t} \sum_{i=1}^{m_t} \ell(f(x_{t,i}, y_{t,i})).
\]

4. the forecaster incur the loss \( R_t(\hat{h}_t \circ \tilde{g}_t) \) where
\[
R_t(f) = \mathbb{E}_{(x,y) \sim R_t} [\ell(f(x), y)].
\]
Unfortunately, this quantity is not known to the forecaster.

At the end of time, we are interested in a strategy such that the compound regret
\[
\mathcal{R} := \frac{1}{T} \sum_{t=1}^{T} R_t(\hat{h}_t \circ \tilde{g}_t) - \inf_{g \in G} \frac{1}{T} \sum_{t=1}^{T} \inf_{h_t \in H} R_t(h_t \circ g)
\]
is controled. The situation is similar to the setting discussed in the core of the paper: we will propose an EWA algorithm for transfer learning, EWA-TL, for which the regret will be controlled, on the condition that the learner chooses a suitable within task algorithm. In the online case, the within tasks algorithm was either EWA or OGA. In Subsection [C.1] we discuss briefly the within task algorithm. In Subsection [C.2] we present the EWA-TL algorithm and its theoretical analysis.

C.1 Within-task Algorithms

We make an additional assumption, that is that the estimator \( \hat{h} \) satisfies a bound in probability:
\[
P \left[ \forall g \in G, |r(\hat{h}(g, S_t) \circ g) - R_t(\hat{h}(g, S_t) \circ g)| \leq \delta(g, m_t, \varepsilon) \right] \quad \text{and} \quad |R_t(\hat{h}(g, S_t) \circ g) - \inf_{h_t \in H} R_t(h_t \circ g)| \leq 2\delta(g, m_t, \varepsilon) \geq 1 - \varepsilon. \quad (C.1)
\]

In classification, when \( \ell \) is the 0-1 loss function, and for any \( g \), the family \( \{h \circ g, h \in H\} \) has a Vapnik-Chervonenkis dimension bounded by \( V \), then the empirical risk minimizer (ERM)
\[
\hat{h}(g, S_t) = \arg \min_{h_t \in H} r_t(h_t \circ g)
\]
satisfies the above condition with
\[
\delta(g, m_t, \varepsilon) = 2 \sqrt{\frac{V \log \left( \frac{2m_t}{V} \right) + \log \left( \frac{4}{\varepsilon} \right)}{m_t}},
\]
see e.g. (Chapter 4, page 94 Vapnik 1998). Similar rates can be obtained with PAC-Bayesian bounds [McAllester 1998, Catoni 2004], but we postpone the details to future work.
C.2 EWA-TL

Algorithm 6 EWA-TL

Data A sequence of datasets
\[ S_t = \{ (x_{t,1}, y_{t,1}), \ldots, (x_{t,m_t}, y_{t,m_t}) \} \], 1 \leq t \leq T, associated with different learning tasks; the datasets are revealed sequentially, but the points within each dataset \( S_t \) are revealed all at once.

Input A prior \( \pi_1 \), a learning parameter \( \eta > 0 \) and a learning algorithm \( \hat{h} \) which satisfies (C.1).

Loop For \( t = 1, \ldots, T \)

i Draw \( \hat{g}_t \sim \pi_t \).

ii Run the within-task learning algorithm \( \hat{t} \) on \( S_t \) to get \( \tilde{h}_t = \hat{h}(\hat{g}_t, S_t) \).

iii Update
\[
\pi_{t+1}(dg) \propto \exp \left\{ -\eta \left( r_t(\tilde{h}_t \circ g) + \delta(g, m_t, \varepsilon/T) \right) \right\} \pi_t(dg).
\]

We now provide a bound on the regret of EWA-TL.

**Theorem C.1.** Under (C.1), and assuming that there is a constant \( C \) such that
\[ 0 \leq r_t(\tilde{h}(S_t, g) \circ g) + \delta(g, m_t, \varepsilon/T) \leq C, \]
with probability at least \( 1 - \varepsilon \),
\[
\sum_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_{t-1}} \left[ R_t(\tilde{h}_t \circ \hat{g}_t) \right] \leq \inf_{\rho} \left\{ \mathbb{E}_{g \sim \rho} \left[ \frac{1}{T} \sum_{t=1}^{T} \inf_{h \in \mathcal{H}} R_t(h \circ g) + \frac{4}{T} \sum_{t=1}^{T} \delta(g, m_t, \varepsilon/T) \right] + \frac{\eta C^2}{8} + \frac{K(\rho, \pi_1)}{\eta T} \right\}.
\]

**Sketch of the proof.** First, follow the proof of Theorem 3.1 to get:
\[
\sum_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_{t-1}} \left[ r_t(\tilde{h}_t \circ \hat{g}_t) + \delta(\hat{g}_t, m_t, \varepsilon/T) \right] \leq \inf_{\rho} \left\{ \mathbb{E}_{g \sim \rho} \left[ r_t(\tilde{h}_t \circ g) + \delta(g, m_t, \varepsilon/T) \right] + \frac{\eta TC^2}{8} + \frac{K(\rho, \pi)}{\eta} \right\}.
\]

So, with probability at least \( 1 - \varepsilon \),
\[
\sum_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_{t-1}} \left[ R_t(\tilde{h}_t \circ \hat{g}_t) \right] \leq \sum_{t=1}^{T} \mathbb{E}_{\hat{g}_t \sim \pi_{t-1}} \left[ r_t(\tilde{h}_t \circ \hat{g}_t) + \delta(\hat{g}_t, m_t, \varepsilon/T) \right] \leq \inf_{\rho} \left\{ \sum_{t=1}^{T} \mathbb{E}_{g \sim \rho} \left[ r_t(\tilde{h}_t \circ g) + \delta(g, m_t, \varepsilon/T) \right] + \frac{\eta TC^2}{8} + \frac{K(\rho, \pi_1)}{\eta} \right\}.
\]
\[ \leq \inf_{\rho} \left\{ \sum_{t=1}^{T} \mathbb{E}_{g \sim \rho} \left[ R_t(\hat{h}_t(g, S_t) \circ g) + 2\delta(g, m_t, \varepsilon / T) \right] + \frac{\eta TC^2}{8} + \frac{K(\rho, \pi_1)}{\eta} \right\} \]

\[ \leq \inf_{\rho} \left\{ \mathbb{E}_{g \sim \rho} \left[ \sum_{t=1}^{T} \inf_{h \in \mathcal{H}} R_t(h \circ g) + 4 \sum_{t=1}^{T} \delta(g, m_t, \varepsilon / T) \right] + \frac{\eta TC^2}{8} + \frac{K(\rho, \pi_1)}{\eta} \right\}. \]