Quandles and Monodromy

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Abstract: We show that a variety of monodromy phenomena arising in geometric topology and algebraic geometry are most conveniently described in terms of homomorphisms from a(n augmented) knot quandle associated with the base to a suitable (augmented) quandle associated with the fiber. We consider the cases of the monodromy of a branched covering, braid monodromy and the monodromy of a Lefschetz fibration. The present paper is an expanded and corrected version of [Yet02].

1 Introduction

Monodromy phenomena are usually modelled by group homomorphisms from the fundamental group of a base space with a subspace of singular values or branch points deleted to a suitable group of automorphisms of a generic fiber. These homomorphisms must then satisfy side-conditions, unnatural from the point of view of group theory, which are imposed on the homomorphism by the geometry of the situation. Because of these side-conditions, some authors prefer to replace the group homomorphism with a map from a generating set on which the side-conditions are more natural (cf. [GS99, Moi81]). This latter approach, however, has the drawback of requiring the introduction of a combinatorial equivalence by moves as a replacement for algebraic homomorphisms.

It is the purpose of this paper to show that in the cases considered, all of the side-conditions can be replaced by purely algebraic conditions by describing monodromy not in terms of a group homomorphisms or maps on generating sets, but by homomorphisms between quandles associated to the base and fiber, or quandles augmented in the groups usually used to describe the monodromy.

2 Quandles, Fundamental Quandles and Knot Quandles

Quandles were originally introduced by Joyce [Joy79, Joy82] as an algebraic invariant of classical knots and links. They may be regarded as an abstraction from groups inasmuch as some of the most important examples arise by considering a group with left and right conjugation as operations.

Definition 1 A quandle is a set $Q$ equipped with two binary operations $\triangleright$ and $\triangleright$ satisfying

1. $\forall x \in Q \quad x \triangleright x = x$
2. $\forall x, y \in Q \quad (x \triangleright y) \triangleright y = x = (x \triangleright y) \triangleright y$
3. $\forall x, y, z \in Q \quad (x \triangleright y) \triangleright z = (x \triangleright z) \triangleright (y \triangleright z)$

Algebraic structures satisfying the second and third axioms only have been studied under the name “racks” by Fenn and Rourke [FR92]. Structures satisfying the third axiom only are called “right distributive semigroups” by universal algebraists.

Examples abound:

Example 2 If $G$ is any group, we can make $G$ into a quandle by letting $x \triangleright y = y^{-1}xy$ and $x \triangleright y = yxy^{-1}$. Likewise any union of conjugacy classes in a group $G$ forms a subquandle.
This example is of particular importance for the theory of quandles, as a representation theorem
due to Joyce [Joy79] shows that all free quandles embed into (free) groups as a disjoint union of
conjugacy classes, and thus that the universally quantified equations holding in all quandles are
precisely those holding in all quandles of this form.

It will also be of interest in the present investigation, as the most of the quandles considered
herein can be identified with unions of conjugacy classes in groups arising naturally in geometric
topology.

**Example 3** Given an $R$-linear automorphism $T$ of an $R$-module $V$, $V$ becomes a quandle with
\[ x \triangleright y = T(x - y) + y \quad \text{and} \quad x \triangleright y = T^{-1}(x - y) + y. \]

The following examples are of particular interest to us, as we will see a topological application
later:

**Example 4** Let $R$ be any commutative ring, and $X$ be a free $R$-module equipped with an anti-
symmetric bilinear form $\langle - , - \rangle : X \times X \to R$. (If $(R, +)$ has any two-torsion, we actually need
alternating rather than just antisymmetric.) Then $X$ is a quandle when equipped with the operations
\[ x \triangleright y = x + \langle x, y \rangle y \]

and
\[ x \triangleright y = x - \langle x, y \rangle y \]

The proof that this last example satisfies the quandle axioms is routine, but we indicated it to
give the reader unfamiliar with quandles the flavor of such things:

Observe that since $\langle - , - \rangle$ is alternating, we have $x \triangleright x = x$. Likewise by bilinearity and
alternating-ness, it follows that $\langle x + ay, y \rangle = \langle x, y \rangle$ from which the second quandle axiom follows. For the third, we calculate

\[ (x \triangleright y) \triangleright z = (x + \langle x, y \rangle y) \triangleright z \]
\[ = x + \langle x, y \rangle y + (x + \langle x, y \rangle y, z)z \]
\[ = x + \langle x, y \rangle y + \langle x, z \rangle z + \langle x, y \rangle \langle y, z \rangle z \]

while

\[ (x \triangleright z) \triangleright (y \triangleright z) = (x + \langle x, z \rangle z) \triangleright (y + \langle y, z \rangle z) \]
\[ = x + \langle x, z \rangle z + (x + \langle x, z \rangle z, y + \langle y, z \rangle z)(y + \langle y, z \rangle z) \]
\[ = x + \langle x, z \rangle z + \]
\[ [(x, y) y + \langle x, z \rangle \langle z, y \rangle + \langle y, z \rangle \langle x, z \rangle + \langle x, z \rangle \langle y, z \rangle \langle z, z \rangle](y + \langle y, z \rangle z) \]
\[ = x + \langle x, y \rangle y + \langle x, z \rangle z + \langle x, y \rangle \langle y, z \rangle z \]

where the equations follow from the definitions (twice), bilinearity and alternating-ness respectively.

We will call a quandle arising in this way an **alternating quandle**.

We will not directly apply alternating quandles, but rather a particular quotient quandle which
exists for any alternating quandle:
Example 5 Given an alternating bilinear form $\langle - , - \rangle$ on an $R$-module $V$, the alternating quandle structure on $V$ induces a quandle structure on the space of orbits of the action of the multiplicative group $\{1, -1\}$ on $V$ by scalar multiplication (note: if $1 = -1$ in $R$ the action is trivial): negating $x$ negates $x \triangleright y$ and $x \trianglerightgeq y$, while negating $y$ leaves them unchanged (since the negation of the instance of $y$ in the bilinear form cancels the negation of $y$ outside).

We will call a quandle arising in this way a reduced alternating quandle.

Joyce’s principal motivation in considering this structure was to provide an algebro-topological invariant of classical knots more sensitive than the fundamental group of the complement.

We will need the corresponding notion in arbitrary dimensions. We consider pairs of a space and a subspace, equipped with a point in the complement of the subspace $(X, S, p)$. In particular we consider the “noose” or “lollipop”: $(N, \{0\}, 2)$ where $N$ is the subspace of $\mathbb{C}$ consisting of union of the unit disk and the line segment $[1, 2]$ in the real axis.

By a map of pointed pairs we mean a continuous map which preserves the base point and both the subspace and its complement. We can then make

Definition 6 The fundamental quandle $\Pi(X, S, p)$ of a pointed pair $(X, S, p)$ is the set of homotopy classes of maps of pointed pairs (where homotopies are through maps of pointed pairs), equipped with the operations $x \triangleright y$ (resp. $x \trianglerightgeq y$) induced by appending the path from the base point obtained by traversing $y([1, 2])$, followed by $y(S^1)$ oriented counterclockwise (resp. clockwise), followed by traversing $y([1, 2])$ in the opposite direction to the path $x([1, 2])$ and reparametrizing.

For the proof that this gives a quandle structure, see [Joy79, Joy82].

In the case where both the space and its subspace are smooth oriented manifolds and the subspace is of codimension 2, it is possible to identify a particularly interesting subquandle of the fundamental quandle.

Definition 7 The knot quandle $Q(M, K, p)$ of a pointed pair $(M, K, p)$, where $M$ is a smooth manifold, $K$ a smooth embedded submanifold of codimension 2 is the subquandle of $\Pi(M, K, p)$ consisting of all maps of the noose such that the bounding $S^1$ has linking number 1 with $K$. (Note: this is in the signed sense.)

Joyce [Joy79, Joy82] showed that the knot quandle of a classical knot determined the knot up to orientation.

We, however, will be concerned here with knot quandles in general. In particular, in our discussion of branched coverings, we will need to consider knot quandles in all dimensions. In the discussions of braid monodromy and of the monodromy of Lefschetz fibrations we will consider the knot quandle of an oriented set of points in a surface: Given a (path connected) oriented surface $\Sigma$, equipped with a finite set of points $S$, and a point $p$ not lying in $S$, the quandle $Q(\Sigma, S, p)$ has as elements all isotopy classes of maps of pointed pairs from the noose to $(\Sigma, S, p)$ which map the boundary of the disk with winding number $\pm 1$ with the sign given by the orientation of the point (always positive, except in the case of achiral Lefschetz fibrations).

It is easy to see that there is a relationship between $Q(\Sigma, S, p)$ and $\pi_1(\Sigma \setminus S, p)$: an action of the fundamental group $\pi_1(\Sigma \setminus S, p)$ on $Q(\Sigma, S, p)$ by quandle homomorphisms is given by appending a loop representing an element of $\pi_1$ to the initial path of the noose and rescaling.

There is, however, an more intimate relationship between $Q(\Sigma, S, p)$ and $\pi_1(\Sigma \setminus S, p)$:

Definition 8 [Joy73, Joy82] An augmented quandle is a quadruple

\begin{align*}
\end{align*}
\[(Q, G, \ell : Q \to G, \cdot : Q \times G \to Q)\]

where \(Q\) is a quandle, \(G\) is a group, \(\cdot\) is a right-action of \(G\) on \(Q\) by quandle homomorphisms, and the set-map \(\ell\) (called the augmentation) satisfies

\[
q \cdot \ell(q) = q \\
\ell(q \cdot \gamma) = \gamma^{-1} \ell(q) \gamma
\]

**Proposition 9** For any oriented manifold \(M\) with an oriented, properly embedded codimension 2 submanifold \(K\) and a point \(p \in M \setminus K\), the quadruple

\[(Q(M, K, p), \pi_1(M \setminus K, p), \ell, \cdot)\]

where \(\cdot\) is the action described above, and \(\ell(q)\) is the homotopy class of the loop at \(p\) which traverses the arc, then the boundary of the disk counterclockwise, then the arc back to \(p\), is an augmented quandle. We call the loop at \(p\) just described as a representative for \(\ell(q)\) the canonical loop of the noose \(q\).

**proof:** Having noted that the action of \(\pi_1(M \setminus K, p)\) is by quandle homomorphisms (a fact which follows essentially by conjugation in the fundamental groupoid—the reader may fill in the details), it remains only to verify that the map \(\ell\) satisfies the two conditions specified in the definition of augmented quandles.

The first reduces to the idempotence of the quandle operation. The second follows from the fact that the appended loop occurs twice in the specification of \(\ell(q \cdot \gamma)\), initially in the outgoing arc from \(p\) with positive orientation, and again in the incoming arc to \(p\) with reversed orientation. \(\blacksquare\)

We also have

**Proposition 10** If \(M\) is simply connected, the image of the augmentation, \(\ell(Q(M, K, p))\) generates \(\pi_1(M \setminus K, p)\).

**proof:** This follows from van Kampen’s Theorem: killing all of the noose boundaries kills the fundamental group, and thus the noose boundaries generate. \(\blacksquare\)

### 3 Quandles of Cords

In \([KM00]\) Kamada and Matsumoto introduce a geometric construction for a family of quandles closely related to the braid groups of surfaces. In this section we describe their construction and adaptations of it better suited to handling Moishezon’s braid monodromy (cf. \([Mo81]\)).

The constructions of this section give rise to quandles associated to surfaces equipped with a finite set of (interior) points.

**Definition 11** Let \(\Sigma\) be a surface, possibly with boundary, and \(P \subset \Sigma\) a finite set of interior points. A cord in \((\Sigma, P)\) is an embedding of pairs (as defined in the previous section, but dropping base points) from \(c : ([0, 1], \{0, 1\}) \to (\Sigma, P)\).

Two cords are equivalent if there is an isotopy (rel boundary) of \(\Sigma\) which fixes \(P\) and carries one to the other.
Equivalence classes of cords, or cords labeled with integers, will form the elements of the quandles described in the section. The operations will be described using

**Definition 12** Given a cord $\alpha$ in $(\Sigma, P)$, the disk twist around $\alpha$ is the isotopy (rel boundary) class of self-diffeomorphisms of $(\Sigma, P)$ represented by any smoothing $\Phi_\alpha$ of any map $\phi_\alpha$ constructed as follows:

Choose a neighborhood $N$ of $\text{Im}(\alpha)$ diffeomorphic to a disk, whose closure is disjoint from $P \setminus \text{Im}(\alpha)$, and a chart identifying the neighborhood with the disk $\{z \mid |z| < 2\}$ and $\text{Im}(\alpha)$ with the interval $[-1, 1]$ on the real axis. Then $\phi_\alpha$ is given by the identity map outside $N$, and in local coordinates by

$$\phi_\alpha(z) = \begin{cases} z & \text{if } |z| \leq \frac{3}{2} \\ z e^{i\pi(2 - \frac{2}{3}|z|)} & \text{if } |z| > \frac{3}{2} \end{cases}$$

Observe that the isotopy class of the disk twist is independent of the choices (chart and smoothing) used in its construction, and depends only on the equivalence class of the cord.

We can then make the following definition

**Definition 13** The quandle of cords on $(\Sigma, P)$ denoted $X(\Sigma, P)$ has as elements the equivalence classes of cords on $(\Sigma, P)$, with operations given on representatives by

$$(\alpha) \triangleright [\beta] = [\Phi_\beta(\alpha)]$$

$$(\alpha) \triangleright [\beta] = [\Phi_\beta^{-1}(\alpha)]$$

In the case of a disk $D$ equipped with a finite set of $n + 1$ interior points, this quandle can be identified with the subquandle of $B_{n+1}$, the $n + 1$ strand Artin braid group under conjugation, of all conjugates of the (positive) braid generators.

Kamada and Matsumoto [KM00] give generators and relations for quandles of cords in the disk (or plane) and 2-sphere.

These quandles are almost the appropriate quandles for the discussion of Moishezon’s braid monodromy [Moi81]. However, we need a slight modification:

**Definition 14** Let $L$ be a set of non-zero integers. An $L$-cord is a cord in $(\Sigma, P)$ labeled with an element of $L$. Two $L$-cords are equivalent if they have the same label and equivalent underlying cords.

The quandle of $L$-cords on $(\Sigma, P)$, $L - C_{\Sigma, P}$ has as elements the equivalence classes of $L$-cords on $(\Sigma, P)$, with operations given on representatives by

$$(\alpha, l) \triangleright ([\beta], \lambda) = (\Phi_\beta^\lambda(\alpha), l)$$

$$(\alpha, l) \triangleright ([\beta], \lambda) = ([\Phi_\beta^{-\lambda}(\alpha)], l)$$

The quandles of cords $L$-cords in $(\Sigma, P)$ each have natural augmentations in the $|P|$-strand braid group on $\Sigma$, $B(\Sigma, P) = \pi_1((\Sigma|P| \setminus \Delta)/\mathcal{G}_r, P)$.

The augmentation maps a cord $\beta$ (resp. $L$-cord $(\beta, \lambda)$) to the braid given by fixing the other points of $P$ and moving the endpoints of the cord by the obvious isotopy from the identity to the disk twist which at no time moves any point through more than $\pi$ in the local polar coordinates used to describe the disk twist (resp. a composition of $\lambda$ copies of this isotopy). (Notice: all disk twists are isotopic to the identity once one lifts the requirement that isotopies fix $P$.)
4 Dehn Quandles

We now consider another geometric construction of quandles, related to mapping class groups of surfaces in a way weakly analogous to the relationship between knot quandles and fundamental groups.

From [Bir75] we recall

**Definition 15** If $\Sigma$ is a surface, the mapping class group of $\Sigma$ is the group $M(\Sigma) = \pi_0(F\Sigma)$, where $F\Sigma$ is the group of all orientation-preserving self-diffeomorphisms of $\Sigma$, endowed with the compact-open topology.

Birman [Bir75] actually defines more general objects depending on a set of distinguished points lying in $\Sigma$. Following the usual convention, if $\Sigma$ is of genus $g$, we denote its mapping class group by $M(g, 0)$, the 0 indicating the lack of distinguished points.

It is easy to verify that $M(0, 0)$ is trivial. It is also well-known that $M(1, 0) \cong SL(2, \mathbb{Z})$.

Birman and Hilden [BH71] gave a finite presentation for $M(2, 0)$. Building on work of McCool [McC75] and Hatcher and Thurston [HT80], Harer [Har83] gave finite presentations for the higher genus case, which were improved by Wajnryb [Waj83].

The key to approaching presentations of mapping class groups, and to our related quandles, however, predates these developments, and is due to Dehn [Deh38]. It depends upon a particular construction of self-diffeomorphisms from an embedded curve:

**Definition 16** Let $\Sigma$ be an oriented surface, and $c$ a simple closed curve lying in $\Sigma$. $c$ then admits a bicollar neighborhood $U$. If we identify this bicollar neighborhood with the annulus $A = \{z|1 < |z| < 2\}$ in $\mathbb{C}$ by an orientation preserving diffeomorphism, $\phi: U \to A$ which maps $c$ to $\{z||z| = \frac{3}{2}\}$ given in polar coordinates by $\phi = (r_\phi, \theta_\phi)$, the self-homeomorphism $t^+_c : \Sigma \to \Sigma$ given by

$$t^+_c(x) = \begin{cases} 
  x & \text{if } x \in \Sigma \setminus U \\
  \phi^{-1}(r_\phi(x), \theta_\phi(x) + 2\pi r_\phi(x)) & \text{if } x \in U
\end{cases}$$

or any self-diffeomorphism obtained by smoothing $t^+_c$ is called a positive (or left-handed) Dehn twist about $c$.

Negative (or right-handed) Dehn twists are defined similarly using

$$t^-_c(x) = \begin{cases} 
  x & \text{if } x \in \Sigma \setminus U \\
  \phi^{-1}(r_\phi(x), \theta_\phi(x) - 2\pi r_\phi(x)) & \text{if } x \in U
\end{cases}$$

It is easy to see that the positive and negative Dehn twists about a curve $c$ are inverse to each other (in the smoothed case up to isotopy).

It is well-known that the positive Dehn twists along isotopic simple closed curves are isotopic as diffeomorphisms. Thus each isotopy class of simple closed curves determines an element of the mapping class group. Similarly the images of simple closed curves under isotopic diffeomorphisms will be isotopic.

We may thus make the following definition
Definition 17 The (chiral) Dehn quandle $D(\Sigma)$ of an oriented surface $\Sigma$ is the set of isotopy classes of simple closed curves in $\Sigma$ equipped with the operations

$$x \triangleright y = t_y^-(x)$$

$$x \triangleright y = t_y^+(x)$$

where by abuse of notation we use the same symbol to denote the isotopy class and a representative curve.

This quandle was originally described by Zablow [Zab99, Zab], who did not trouble to name it, and subsequently rediscovered by the author.

By the discussion above, it is clear that the operations described are independent of the choice of representing curve and define a well-defined isotopy class of curves. We now establish

Proposition 18 The operations of Definition 17 satisfy the quandle axioms.

Proof: It is clear by the discussion above that the second quandle axiom is satisfied. Likewise observe that $t_x^\pm$ fixes the curve $x$ up to isotopy. Thus the first quandle axiom is satisfied. It thus remains only to verify the third axiom. This may be seen from the fact that any self-diffeomorphism of $\Sigma$ induces an automorphism of the algebraic structure with operations $\triangleright$ and $\triangleright$, in particular $- \triangleright y = t_y^+(-)$ is such an automorphism. □

In the next section we will use the Dehn quandle to encode the monodromy of a Lefschetz fibration (whether chiral or achiral). To provide an alternative way of handling the case of achiral Lefschetz fibration, we make

Definition 19 The achiral Dehn quandle $\tilde{D}(\Sigma)$ of an oriented surface $\Sigma$ is the quandle with underlying set $D(\Sigma) \times \{+, -\}$ and quandle operation

$$(x, \sigma) \triangleright (y, -) = (x \triangleright y, \sigma)$$

$$(x, \sigma) \triangleright (y, +) = (x \triangleright y, \sigma)$$

We consider now Dehn quandle in the case of a genus one surface, where the structure of the Dehn quandle can be completely determined. As noted above $M(1, 0) \cong SL(2, \mathbb{Z})$. Recall also that isotopy classes of essential simple closed curve are given by slopes $\frac{y}{x}$ with $x$ and $y$ relatively prime integers (and 0 is allowed in either place).

As noted in Casson and Bleiler [CB88], elements of $SL(2, \mathbb{Z})$ corresponding to powers of Dehn twists are the integer matrices of trace 2 and determinant 1. A fairly routine calculation shows that the right-hand Dehn twist along a curve of slope $\frac{y}{x}$ ($\gcd(x, y) = 1$) is given by the matrix

$$M_{\frac{y}{x}} = \begin{bmatrix}
1 - xy & x^2 \\
-y^2 & 1 + xy
\end{bmatrix}$$

Observe also that this transformation from slopes to matrices is well-defined, being independent of the choice of signs for $x$ and $y$, and one-to-one: given a matrix of the given form, $x$ and $y$ may be recovered up to sign from the off-diagonal entries, while the diagonal entries determine the sign of the product $xy$, and thus the sign of the slope $\frac{y}{x}$. 

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We can thus determine a formula for the operations of the Dehn quandle from the observation that

\[ t_{h(c)}^+ = h(t_{c}^+(h^{-1})) \quad (*). \]

Applying this fact in the case where it itself is a positive Dehn twist gives us

**Example 20** The Dehn quandle of the torus \( D(T^2) \) has underlying set

\[ \{ \frac{y}{x} \mid x, y \in \mathbb{Z}, \gcd(x, y) = 1 \} \cup \{ I \} \]

where \( I \) represents the (unique) isotopy class of contractible simple closed curves and \( \frac{y}{x} \) represents the isotopy class of essential simple closed curves of slope \( \frac{y}{x} \).

The quandle operations on \( D(T^2) \) are given by

\[
\begin{align*}
\frac{v}{u} \triangleright \frac{y}{x} &= \frac{v - vxy + uy^2}{u + uxy - vx^2} \\
I \triangleright q &= I \\
q \triangleright I &= q \\
\frac{v}{u} \triangleright \frac{y}{x} &= \frac{v + vxy - uy^2}{u - uxy + vx^2} \\
I \triangleright q &= I \\
q \triangleright I &= q
\end{align*}
\]

where \( q \) is any element of the quandle and \( x, y, u, \) and \( v \) are integers with \( \gcd(x, y) = \gcd(u, v) = 1 \).

It is easy to see that Dehn twist on contractible curves are isotopic to the identity, and likewise that the isotopy class of contractible curves is fixed by any Dehn twist. The first of the remaining two relations may be obtained by using the equation (*) above in the case where \( h = \frac{t_{c}^+}{\mathcal{N}} \) and \( c \) has slope \( \frac{v}{u} \), computing the conjugate \( M_{\frac{x}{u}}^{-1} M_{\frac{y}{u}} M_{\frac{x}{u}} \) and identifying the numerator and denominator which give rise to the resulting matrix. The last remaining relation may be verified by observing that it provides the inverse operation to that just computed.

As in the case of the fundamental and knot quandles, the Dehn quandle admits an augmentation in the obvious related group:

**Proposition 21** There is an obvious right action of \( M(\Sigma) \) on \( D(\Sigma) \) by quandle homomorphisms given by \([q] \cdot [h] = [h(q)]\). Let \( \ell : D(\Sigma) \to M(\Sigma) \) be given by mapping an isotopy class of simple closed curve in \( \Sigma \) it the isotopy class of the positive Dehn twist about any of its representatives. Then

\[ (D(\Sigma), M(\Sigma), \ell, \cdot) \]

is an augmented quandle. We call it the augmented Dehn quandle of \( \Sigma \).

**proof:** The proof is routine.

As was the case with the augmented knot quandle for a simply connected underlying manifold, so with augmented Dehn quandles we have
Proposition 22 The image of the augmentation \( \ell(D(\Sigma)) \) generates \( M(\Sigma) \).

**proof:** This is simply a restatement of the classical theorem that the mapping class group is generated by (right-hand) Dehn twists \([\text{Deh38}, \text{Lic64}]\). \( \square \)

Similarly the \( M(\Sigma) \) admits a right action on \( D(\Sigma) \) by \( (q, \sigma) \cdot h = (h(q), \sigma) \), and there is an augmentation map \( \ell : D(\Sigma) \to M(\Sigma) \) given by mapping \( (q, -) \) (resp. \( (q, +) \)) to the negative (resp. positive) Dehn twist along \( q \). We call this the **augmented achiral Dehn quandle** of \( \Sigma \).

As of this writing, the structure of the Dehn quandle for higher genus surfaces has yet to be determined. One thing which can be read off from the well-known presentation for the mapping class group for a surface \( \Sigma_2 \) of genus two is

\[ \text{Proposition 23} \quad \text{The Dehn quandle}\ D(\Sigma_2) \quad \text{of a surface of genus two admits a quotient to a seventeen element quandle, two of whose elements act trivially and the other fifteen of which form the quandle of all transpositions in} \ G_6. \]

**proof:** First pass by the augmentation map to the subquandle of \( M(\Sigma_2) \) under conjugation, then to the subquandle of \( \mathbb{Z}/10 \times \mathcal{S}_6 \) under the quandle map induced by the group homomorphism which maps the generator \( \zeta_i \) to \( (1, (i \ i + 1)) \). The image is then the subset \( \{(0, e), (2, e)\} \cup \{(1, (a b))|1 \leq a < b \leq 6\} \). The element \( (2, e) \) is the image of \( (0, 1) \) under conjugation, which induces the quandle structure describe in the proposition. \( \square \)

It is also possible in general to find interesting quotients of chiral and achiral Dehn quandles by considering the (reduced) alternating quandle associated to \( H_1(\Sigma, R) \) with the intersection form, where \( R \) is any quotient of \( \mathbb{Z} \). We call the alternating quandle associated to the intersection form the **\( R \)-homology quandle** of \( \Sigma \) and denote it by \( HQ_R(\Sigma) \), omitting the \( R \) when \( R = \mathbb{Z} \). (As an aside, by the same construction, we can put a quandle structure on \( H_2_{2n+1}(X, R) \) for \( X \) any \( 4n + 2 \) manifold.)

Since the Dehn quandle \( D(\Sigma) \) has as elements isotopy classes of **unoriented** simple closed curves, they can be more naturally related to the reduced alternating quandle associated to the intersection form, which we call the **\( R \)-homology Dehn quandle** of \( \Sigma \) and denote by \( HD_R(\Sigma) \), as before omitting the subscript \( R \) when \( R = \mathbb{Z} \).

Any unoriented simple closed curve represents an element of \( HD_R(\Sigma) \), with isotopic simple closed curves representing the same element. We thus have a map \( D(\Sigma) \to HD_R(\Sigma) \) for any surface \( \Sigma \).

To see that this map is a quandle map we must relate the geometric construction of the operations in \( D(\Sigma) \) to the algebraic construction of the operation on \( HD_R(\Sigma) \) from the intersection form. Consider a pair \( a, b \) of unoriented simple closed curves in an oriented surface \( \Sigma \). Depending on how they are oriented, their intersection number (if it is non-zero) may be given either sign. Since we are really concerned with isotopy classes of curves, we may assume the curves intersect tranversely.

Now choose an orientation on \( a \). We may induce an orientation on \( b \) as follows: orient \( b \) so that at each intersection point, the “turn right” rule defining a right-handed Dehn twist about \( b \) causes the curve representing \( a \triangleright b \) in \( D(\Sigma) \) to traverse \( b \) with the same sign as the intersection point.

The curve representing \( a \triangleright b \) in \( D(\Sigma) \), oriented to agree with the orientation on \( a \), then represents the homology class \( a + (a, b)b \) in \( H_1(\Sigma, R) \). Passing to the quotient \( HD_R(\Sigma) \) then removes any dependence on orientation, and we see that the map carrying a simple closed curve to the \( \{\pm 1\} \)-orbit of its homology class is a quandle homomorphism.
One thing which should be observed is that for genus 1, $D(T^2) \cong HD(T^2)$ since each homology class is represented by a unique isotopy class of oriented simple closed curve. In higher genus, $HD(\Sigma)$ will be a proper quotient of $D(\Sigma)$, as different isotopy classes of curves can represent the same homology class. For example, both a curve which bounds a disk and a curve which separates a surface of genus two into two genus one surfaces with boundary are both null-homologous, but they represent different isotopy classes.

It might naively be thought that the achiral Dehn quandle should have an analogous relationship with the homology quandle with the signs in the pairs defining the elements becoming orientations on the curves. This, however, is not the case: $(b, -)$ and $(b, +)$ act differently on elements of the achiral Dehn quandle, while in the homology quandle $a \triangleright b = a \triangleright -b$.

5 Monodromy

5.1 Branched Coverings

The simplest and most classical example of monodromy phenomena we will consider is that of the monodromy of a branched covering space. The branch set $S$ is a codimension two subspace. We will consider the case in which it is nonsingular. Classically it is described by considering the homomorphism from the knot group of the singular set, $\pi_1(B \setminus S, p)$, to the group of permutations of the (generic) fiber over $p$, $\mathcal{S}_d$, where $d$ is the number of sheets. (cf. for example [Fox57, IP02]).

However, as the local model of a branch point is given by the self-map of $D^2 \times D^{n-2}$ by $(z, \vec{x}) \mapsto (z^k, \vec{x})$, not all group homomorphisms arise: only those sending meridians of $S$, considered as elements of $\pi_1(B \setminus S, p)$, to cyclic permutations actually arise. It is also common to consider simple branched coverings in which the monodromy of a meridian is restricted to be a transposition.

Both conditions are easily imposed by considering the monodromy as a quandle homomorphism rather than a group homomorphism:

**Definition 24** The quandle monodromy of a $d$-sheeted branched covering with base $B$ and branch locus $S \subset B$ is the quandle homomorphism $\mu : Q(B, S, p) \rightarrow \mathcal{C}_d$, given by mapping any noose to the monodromy around its boundary, where $\mathcal{C}_d$ is the subquandle of $\mathcal{S}_d$ under conjugation consisting of non-trivial cyclic permutations. For simple branched covers, the monodromy may be considered as taking values in $\mathcal{I}_d$, the subquandle of $\mathcal{S}_d$ under conjugation consisting of transpositions.

In the case where $B$ is simply connected, this suffices to recover the monodromy in the classical sense, since the meridians generate the knot group of the singular set, and thus the branched covering.

For a more general base, it is still necessary to consider the classical notion of monodromy, but here the restriction on the monodromy of meridians can be imposed algebraically rather than combinatorially by considering the augmentations from $Q(B, S, p)$ to $\pi_1(B \setminus S, p)$ and from $\mathcal{C}_d$ or $\mathcal{I}_d$ to $\mathcal{S}_d$.

**Definition 25** The augmented quandle monodromy of a branched covering is the homomorphism of augmented quandles whose components are the quandle monodromy and the monodromy of the branched covering.

5.2 Braid Monodromy

Our second example, Moisheson’s braid monodromy, takes a bit more description. It is an invariant of complex curves in $\mathbb{CP}^2$, although it can be adapted to more general surfaces in $\mathbb{CP}^2$, and when
used as an invariant of the branch locus, together with the type of monodromy just discussed, is important in the theory of symplectic 4-manifolds (cf. [AR]).

Given a complex projective plane curve $V$, one can change coordinates so that the curve does not pass through $[0 : 0 : 1]$. The curve then lies in the tautological line bundle over the exceptional locus $\mathbb{CP}^1 = \{(x : y : 0) | (x, y) \in \mathbb{C} \setminus \{(0, 0)\}\}$.

The inverse image of a generic point under the projection is then a set of $d$ points, where $d$ is the degree of the curve. At a finite set of points, however, the curve is either tangent to the fiber or singular, and the inverse image will have fewer points.

Now, if one travels in a loop in the base $\mathbb{CP}^1$ around a singular point (one where the curve is either singular or tangent to the fiber), the monodromy is of the curve as an embedded object is an element of the $d$-th Artin braid group, $B_d$.

Consideration of local models show that the monodromy around a singular point is restricted to certain conjugacy classes of $B_d$ by the geometry of the singularity: points of tangency have conjugate of the braid generators as monodromy, nodes have conjugates of squares of generators, cusps have conjugates of cubes of generators, and so on.

Selecting a non-singular fiber, and choosing coordinates on $\mathbb{CP}^1$ so that it is the point at infinity, one then has an affine plane curve fibered over $\mathbb{C}$.

We will restrict our attention to curves with at worst cuspidal singularities, and assume (by a small perturbation if necessary) that the tangencies and singularities all occur in different fibers.

Moishezon then makes

**Definition 26**  The braid monodromy of the (affine) plane curve $V$ is the group homomorphism

$$\mu : \pi_1(\mathbb{C} \setminus S, p) \rightarrow B_d$$

obtained by identifying the braid group of the fiber over a non-singular point $p$ with $B_d$ and mapping generating loops to the induced monodromy. Here $S$ is the set of singular values for the projection.

There are two restrictions on the homomorphisms thus arising. The first, arising from the geometry near the singular fibers, is readily handled by considering quandles.

**Definition 27**  The quandle monodromy of an affine cuspidal plane curve $V$ is the quandle homomorphism from the knot quandle $Q(\mathbb{C}, S, p)$ to $\{1, 2, 3\} - C(\pi^{-1}(p), \pi|_{V^{-1}(p)})$, the quandle of $\{1, 2, 3\}$-cords on $(\pi^{-1}(p), \pi|_{V^{-1}(p)})$ obtained by mapping each noose to the $\{1, 2, 3\}$-cord whose action describes the monodromy around the noose boundary.

As has been observed, each of the quandles admits an augmentation: from the knot quandle to the fundamental group of $\mathbb{C} \setminus S$ and from the quandle of L-cords to the braid group of the plane $\pi^{-1}(p)$. The augmented quandle monodromy of a cuspidal affine plane curve is the augmented quandle homomorphism with the quandle monodromy and braid monodromy as components.

The second restriction required in the projective case, however, lives more comfortably at the level of groups: because the tautological bundle is non-trivial, there is a monodromy around the (non-singular) point at infinity. In particular, travelling around the point at infinity, the sheets undergo a full twist (corresponding to the twist in the line bundle). Thus certain suitable products of the monodromies around the singular fibers must be the full-twist in $B_d$, usually denoted $\Delta^2$.

The inclusion $\iota : \mathbb{C} \rightarrow \mathbb{CP}^1$ induces quotient maps $q : Q(\mathbb{C}, S, p) \rightarrow Q(\mathbb{CP}^1, S, p)$, and quotient maps on the fundamental groups $\pi_1(\iota) : \pi_1(\mathbb{C} \setminus P) \rightarrow \pi_1(\mathbb{CP}^1 \setminus P)$. The kernel of this group homomorphism is free on a single generator represented by a counterclockwise loop $\lambda$ in $\mathbb{C}$ which has
\[ L - C_{\mathbb{C}, \pi^{-1}(p)} \quad \rightarrow \quad B_{\pi^{-1}(p)} \quad \rightarrow \quad \pi_1(\mathbb{C} \setminus S, p) \]

Figure 1: The square of augmented quandle homomorphisms giving the braid monodromy of a projective plane curve with the induced map on group kernels

\[ S \quad \text{lying in the bounded region. Because of the behavior of the monodromy at infinity, we consider also an augmentation into the quotient of the braid group, } \delta : B_{\pi^{-1}(p)} \rightarrow B_{\pi^{-1}(p)} / < \Delta^2 >. \]

From this, given a projective cuspidal plane curve, lying in the tautological bundle over the exceptional locus, we obtain a commutative square of augmented quandle homomorphisms

The restriction on the monodromy at infinity then becomes the requirement that the induced map on the kernels of the group homomorphisms maps \( \lambda \) to \( \Delta^2 \). Dropping this restriction will give monodromies corresponding to surfaces in various line bundles over \( \mathbb{C}P^1 \)—in particular, those squares in which the induced map on kernels maps \( \lambda \) to \( (\Delta^2)^k \) will correspond to cuspidal surfaces in the line bundle with first Chern class \( k \).

We may then make

**Definition 28** The augmented quandle monodromy of a projective plane curve is the augmented quandle monodromy of the associated affine plane curve. It will necessarily satisfy the condition that the element \( \lambda \) of \( \pi_1(\mathbb{C} \setminus S) \) is mapped to \( \Delta^2 \).

### 5.3 Lefschetz Fibrations

Our third example is the monodromy of a Lefschetz fibration.

We briefly recall the relevant facts about Lefschetz fibrations, following Gompf and Stipsicz [GS99]:

**Definition 29** A Lefschetz fibration of a smooth, compact oriented 4-manifold \( X \) (possibly with boundary) is a smooth map \( f : X \rightarrow \Sigma \), where \( \Sigma \) is a compact connected oriented surface, \( f^{-1}(\partial \Sigma) = \partial X \) and such that each critical point of \( f \) lies in the interior of \( X \) and has an oriented local coordinate chart modelled (in complex coordinates) by \( f(z, w) = z^2 + w^2 \).

We moreover require that each singular fiber have a unique singular point.

An achiral Lefschetz fibration is defined similarly, except that the prescribed local coordinate chart at singularities may be orientation reversing.

Now the generic fiber \( F \) of \( f \) is a compact, canonically oriented surface. The genus of \( F \) is called the genus of the fibration \( f \).

As is pointed out in [GS99], the choice of a regular point of the fibration \( p \in \Sigma \) and an identification of the fiber over \( p \) with a standard surface \( F \) of the appropriate genus gives rise to
a group homomorphism $\Psi : \pi_1(\Sigma \setminus S, p) \to M(F)$, where $S$ is the set of critical values of $f$, called the monodromy representation of $f$.

In the case of genus $g \geq 2$, this group homomorphism completely determines the structure of the Lefschetz fibration by a theorem of Matsumoto [Mat96]. There are, however, restrictions on which group homomorphisms can occur. In particular the image of any loop linking exactly one critical value with linking number one must be a positive Dehn twist about the vanishing cycle of the singularity—the simple closed curve which collapses to a point at the singular point [GS99].

Due to the awkwardness of imposing such a condition while trying to work in a group theoretic context, when discussing Lefschetz fibrations over the disk $D^2$ and the sphere $S^2$, Gompf and Stipsicz [GS99] work instead with the monodromy of the fibration: the $|S|$-tuple of Dehn twists given by a family of generating loops each of which links a single critical value with linking number one.

This, then, has the drawback that the $|S|$-tuple is determined only up to an overall conjugation by an element of $M(F)$, cyclic permutation, and combinatorial moves given by swapping two of the Dehn twists while conjugating one of them by its partner in a suitable sense.

Both drawbacks—the use of geometric side-conditions in what would otherwise be the purely group theoretic setting monodromy representations, and the ambiguity of definition inherent in the notion of the monodromy, are removed by considering

**Definition 30** The quandle monodromy of a Lefschetz fibration $f : X \to \Sigma$ with critical set $S \subset \Sigma$, relative to a regular point $p$, is the quandle homomorphism

$$\mu : Q(\Sigma, S, p) \to D(F)$$

given by mapping each element of $Q(\Sigma, S, p)$ to the monodromy around the canonical loop of any representing noose. Here all points of $S$ are given the positive orientation.

The quandle monodromy of an achiral Lefschetz fibration is defined in the same way, except that the points of $S$ are oriented positively if the local chart modelling the corresponding singularity is orientation preserving, and negatively if it is orientation reversing.

The augmented quandle monodromy of a Lefschetz fibration (resp. achiral Lefschetz fibration) $f : X \to \Sigma$ relative to $p$ is the map of augmented quandles $(\mu, \Psi)$, where $\mu$ is the quandle monodromy and $\Psi$ is the monodromy representation.

We then have

**Proposition 31** The quandle monodromy of a Lefschetz fibration determines the monodromy representation. Conversely the monodromy representation determines the quandle monodromy.

**proof:** The fundamental group of the complement of the singular set in the base is generated by the image of the knot quandle under the augmentation, while the mapping class group of the fiber is generated by the image of the Dehn quandle under the augmentation. Thus the quandle monodromy induces the monodromy representation. Conversely, the monodromy representation satisfies the side condition that positively oriented noose boundaries (or noose boundaries oriented according to the sign of the singular point in the achiral case) are mapped to positive Dehn twists, and thus the restriction of the monodromy representation to appropriately oriented noose boundaries is the quandle monodromy. $\square$

This then yields the following:
Theorem 32  The isomorphism type of the augmented quandle monodromy determines the isomorphism class of any Lefschetz fibration of genus \( g \geq 2 \). Moreover, if \( g \geq 2 \) and the base \( \Sigma \) is \( D^2 \) or \( S^2 \), the isomorphism class of the quandle monodromy determines the isomorphism class of the Lefschetz fibration.

proof: The first statement follows \textit{a fortiori} from the theorem of Matsumoto [Mat98]. The second statement follows from the first, Propositions 10 and 22, and the fact that in either case \( \pi_1(\Sigma \setminus S, p) \) is free. \( \square \)

Observe that the first statement of this formulation includes the restriction on which homomorphisms \( \Psi : \pi_1(\Sigma \setminus S, p) \to M(F) \) actually occur as an algebraic rather than combinatorial condition.

In the case of \( S^2 \), the second statement has an analogous deficiency to the classical formulations. Not all quandle homomorphisms extend to augmented quandle homomorphisms: a suitably ordered product of the Dehn twists (images of curves under the augmentation) must be the identity in \( M(F) \).

In the case of Lefschetz fibrations over the disk \( D^2 \) or the sphere \( S^2 \), we have

Proposition 33  The quandle monodromy determines the monodromy up to equivalence. Conversely, the monodromy determines the quandle monodromy.

proof: Given the quandle monodromy, the image of any minimal generating set of nooses for the knot quandle of the base, when ordered by some linear restriction of the cyclic ordering induced on the generators by their crossing the boundary of a sufficiently small disk neighborhood of the base point is a monodromy in the sense of [GS99]. \( \square \)

We can also define another type of quandle monodromy in the achiral case:

Definition 34  The achiral quandle monodromy of an achiral Lefschetz fibration is the quandle homomorphism from \( Q(\Sigma, S, p) \), where all points of \( S \) are oriented positively, to \( \tilde{D}(\Sigma) \), which assigns to each element of \( Q(\Sigma, S, p) \) the monodromy around the boundary of its noose.

6 Prospects for Quandle Invariants of Monodromy

The foregoing suggests that a fruitful approach to studying various monodromy phenomena could be begun by finding invariants of quandle homomorphism and augmented quandle homomorphisms which are effectively computable from a presentation by generators and relations.

We briefly outline several places where one might begin:

- Simple counting invariants: count the number of homomorphisms of (augmented) quandle maps (that is, commuting squares of (augmented) quandle maps) from the (augmented) quandle monodromy to a fixed (augmented) quandle map between finite (augmented) quandles. Variants of this include counting factorizations of a fixed quandle map from \( Q(\Sigma, S, p) \) to a finite quandle through the quandle monodromy.

- Quandles map valued invariants: Joyce [Joy79] considered quandles satisfying additional axioms (e.g. involutory quandles where \( x \triangleright y = y \triangleright x \)), and abelian quandles which satisfy \( (w \triangleright x) \triangleright (y \triangleright z) = (w \triangleright y) \triangleright (x \triangleright z) \). We may consider the induced map between (universal) quotient quandles as an invariant of the branched covering, plane curve or Lefschetz fibration. Similarly, in the last case, the map \( \eta : Q(\Sigma, S, p) \to HD(F, R) \), the “\( R \)-homology quandle monodromy” is plainly an invariant of the Lefschetz fibration. This particular invariant, being constructed out of homology and intersection theory, seems likely to have some geometric significance.
• Invariants based on the quandle (co)homology of Carter, Jelsovsky, Kamada, Langford and Saito \cite{CJK+99, CJKS01}: this structure may be considered in two ways—first as a variant of counting invariants in their guise as counting “colorings”, and second homologically: the quandle monodromy giving rise to a (co)chain map between the quandle (co)chain complexes, the (co)homology of whose cone is then an invariant of the branched covering, projective curve or Lefschetz fibration.

Geometric interpretation of this latter invariant would then depend upon understanding the geometric significance of the quandle (co)homology of the variously geometrically described quandles.

Pursuit of any of these is beyond the scope of the present work.

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