A GEOMETRIC INTERPRETATION AND EXPLICIT FORM FOR HIGHER-ORDER HANKEL OPERATORS

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ABSTRACT. This paper deals with group-theoretic generalizations of classical Hankel operators called higher-order Hankel operators. We relate higher-order Hankel operators to the universal enveloping algebra of the Lie algebra of vector fields on the unit disk. From this novel perspective, higher-order Hankel operators are seen to be linear differential operators. An attractive combinatorial identity is used to find the exact form of these differential operators.

1. Introduction. A classical Hankel operator is a map between Hilbert spaces whose matrix representation is constant along antidiagonals. A survey of these operators and their applications appears in [8]. Hankel operators arise naturally in the study of holomorphic function spaces. Given \( f(z) = \sum_{j \in \mathbb{Z}} f_j z^j \in L^2(S^1) \) and \( x(z) = \sum_{j=1}^{\infty} x_j z^j \), define the operators

\[
\mathcal{P}_+ f(z) = \sum_{j=0}^{\infty} f_j z^j, \quad M_x f(z) = x(z) f(z), \quad \mathcal{P}_- f(z) = \sum_{j=1}^{\infty} f_{-j} z^{-j}.
\]

The projection \( \mathcal{P}_+ \) is known as the Cauchy-Szegő projection, and \( \mathcal{P}_+ L^2(S^1) \) is the space of holomorphic functions on the open unit disk with square-integrable boundary values. The operator \( B_1(x) = \mathcal{P}_+ M_x \mathcal{P}_- \) is a Hankel operator, whose matrix representation we will write

\[
B_1(x) = \begin{pmatrix}
\vdots \\
 x_3 \\
 x_2 & x_3 \\
 x_1 & x_2 & x_3 & \ldots
\end{pmatrix}.
\]

The function \( x \) is called the symbol of \( B_1(x) \), and the map \( x \mapsto B_1(x) \) is conformally equivariant in a sense explained in Section 2. Group-theoretic generalizations of this map, which are also conformally
equivariant, were discovered in [5]. Here, we derive the following explicit expressions for these higher-order Hankel operators.

**Theorem 1.1.** The higher-order Hankel operator of order $s+1$ with symbol $x(z)(dz)^{-s}$, where $x(z) = \sum_{j=s+1}^{\infty} x_j z^{s+j}$, has the formula

$$B_{s+1}(x) = P_+ L_s(x) P_-,$$

where $L_s(x)$ is the differential operator

$$L_s(x) = \sum_{j=0}^{s} \frac{1}{s!} \binom{s}{j} \binom{s+j}{j} x^{(s-j)} \left( \frac{\partial}{\partial z} \right)^j.$$

In the case $s = 1$, this operator has the matrix representation

$$B_2(x) = \begin{pmatrix} \vdots & & \vdots \\ 4x_5 & 2x_5 & \cdot \\ 3x_4 & x_4 & 0 \\ 2x_3 & x_3 & 0 \\ x_2 & 0 & -x_4 & -2x_5 \\ 0 & -x_2 & -2x_3 & -3x_4 & -4x_5 & \ldots \end{pmatrix}.$$
into the universal enveloping algebra of the Lie algebra of vector fields on the unit disk. These embeddings determine the form of $B_{s+1}$. In Section 5, we find the image under $B_{s+1}$ of a key element of $H^{-s}_{L^2}(\Delta)$. In Section 6, we prove Theorem 1.1, using several identities for binomial coefficients. In Section 7, we present a pair of binomial coefficient identities resulting from Theorem 1.1, and in Section 8 we relate the maps $B_{s+1}$ to transvectants and higher-order Hankel forms.

2. Notation. The group

$$G = PSU(1,1) = \left\{ g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : |a|^2 - |b|^2 = 1 \right\}$$

acts on the Riemann sphere $\mathbf{\hat{C}}$ by linear fractional transformations,

$$g : z \mapsto \frac{\overline{b} + \overline{a}z}{\overline{d} + \overline{c}z}.$$ 

Let $\Delta$ and $\Delta^*$ denote the open unit disks around 0 and infinity. The decomposition

$$\mathbf{\hat{C}} = \Delta^* \sqcup S^1 \sqcup \Delta,$$

is stable under the action of $G$. Restricting its action to $\Delta$ identifies $G$ with the group of conformal automorphisms of $\Delta$.

For each half-integer $m$, the action of $G$ on $\mathbf{\hat{C}}$ lifts to an action of $SU(1,1)$ on $\kappa^m$, the $m$th (tensor) power of the canonical bundle. This induces an action of $SU(1,1)$ on the space of holomorphic sections of $\kappa^m|_\Delta$, the holomorphic differentials of degree $m$ on $\Delta$. We denote such a differential by $f(z)(dz)^m$, and the space of all such differentials by $H^m(\Delta)$. The action of $g \in SU(1,1)$ on $H^m(\Delta)$ is

$$g : f(z)(dz)^m \mapsto f \left( \frac{-\overline{b} + \overline{a}z}{\overline{d} - \overline{c}z} \right)(\overline{a} - \overline{b}z)^{-2m}(dz)^m.$$

The action of $SU(1,1)$ on $H^m(\Delta)$ is essentially unitary for $m > 0$; the dense Hilbert subspace of $H^m(\Delta)$ with Hermitian inner product

$$\langle f(z)(dz)^m, g(z)(dz)^m \rangle_m = \begin{cases} \int_\Delta f(z)\overline{g}(z)(1 - |z|^2)^{2m-2}dz \overline{dz} & m = 1, 3/2, 2, \ldots, \\ \int_{S^1} f(z)\overline{g}(z)dz, & m = 1/2 \end{cases}$$
will be denoted $H_{L^2}^m(\Delta)$. Thus, $f(z)(dz)^m \in H_{L^2}^m(\Delta)$ if and only if $f(z)$ belongs to the weighted Bergman space $A_{2m-2}^2(\Delta)$.

The space of sections of $\kappa^{1/2}|_{S^1}$ will be denoted $\Omega^{1/2}(S^1)$. The actions of $SU(1,1)$ and the $SU(1,1)$-invariant Hermitian inner product on this space are also given by (2) and (3), with $m = 1/2$. We will denote the Hilbert subspace by $\Omega_{L^2}^{1/2}(S^1)$, so that $f(z)(dz)^{1/2} \in \Omega_{L^2}^{1/2}(S^1)$ if and only if $f(z) \in L^2(S^1)$.

The decomposition (1) corresponds to an $SU(1,1)$-stable decomposition

$$
\Omega_{L^2}^{1/2}(S^1) = H_{L^2}^{1/2}(\Delta) \oplus H_{L^2}^{1/2}(\Delta^*)
$$

We abuse notation slightly by using $\mathcal{P}_+$ to denote the projection onto $H_{L^2}^{1/2}(\Delta)$, and $\mathcal{P}_-$ to denote the projection onto $H_{L^2}^{1/2}(\Delta^*)$. As noted, $\mathcal{P}_+ f(z)(dz)^{1/2} = f_+(z)(dz)^{1/2}$, where $f_+(z)$ is the Cauchy-Szegő projection (Cauchy transform) of $f(z)$.

If $\theta = f(z)(dz)^{1/2} \in H_{L^2}^{1/2}$ and $\eta = g(z)(dz)^{1/2} \in H_{L^2}^{1/2}(\Delta^*)$, then $\theta\eta = fg\,dz$ is a one density on $S^1$ that can be integrated to a nontrivial constant, so $(H_{L^2}^{1/2}(\Delta))^* = H_{L^2}^{1/2}(\Delta^*)$. The decomposition (4) induces a diagonal action of $SU(1,1)$ on Hilbert-Schmidt operators sending $H_{L^2}^{1/2}(\Delta)$ to $H_{L^2}^{1/2}(\Delta)$, via the identifications

$$
\mathcal{L}_2 \left( H_{L^2}^{1/2}(\Delta^*), H_{L^2}^{1/2}(\Delta) \right) = H_{L^2}^{1/2}(\Delta) \otimes \left( H_{L^2}^{1/2}(\Delta) \right)^*
$$

$$
= H_{L^2}^{1/2}(\Delta) \otimes H_{L^2}^{1/2}(\Delta).
$$

Calculating the character of the rotation group $S^1 \subset PSU(1,1)$, one sees that

$$
H_{L^2}^{1/2}(\Delta) \otimes H_{L^2}^{1/2}(\Delta) = H_{L^2}^{1}(\Delta) \oplus H_{L^2}^{2}(\Delta) \oplus H_{L^2}^{3}(\Delta) \oplus \cdots
$$

so for each $m \in \mathbb{N}$ there is an intertwining map

$$
H_{L^2}^{m}(\Delta) \to \mathcal{L}_2(H_{L^2}^{1/2}(\Delta^*), H_{L^2}^{1/2}(\Delta)).
$$

Let $x \in H^0(\Delta)/\mathbb{C}$. Then $x$ acts on $\Omega^{1/2}(S^1)$ by multiplication. With respect to the decomposition (4), the multiplication operator $M_x$ can be written

$$
M_x = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.
$$
\[ B = B_1(x) = \mathcal{P}_+ M_x \mathcal{P}_- \in L_2(H_{L^2}^{1/2}(\Delta^*), H_{L^2}^{1/2}(\Delta)) \] is the Hankel operator associated to \( x \). The action of \( G \) on \( H^0(\Delta)/\mathbb{C} \) intertwines with the diagonal action of \( SU(1, 1) \) on \( L_2(H_{L^2}^{1/2}(\Delta^*), H_{L^2}^{1/2}(\Delta)) \), and \( B_1(x) \) is Hilbert-Schmidt precisely when \( \theta = x'dz \in H_{L^2}^{1} (\Delta) \). Thus \( B_1 \) is an \( SU(1, 1) \)-equivariant map from \( H_{L^2}^{1} (\Delta) \) to \( L_2(H_{L^2}^{1/2}(\Delta^*), H_{L^2}^{1/2}(\Delta)) \). This motivates the following.

**Definition 2.1.** The Hankel operator of order \( m \) with symbol \( \theta \in H_{L^2}^{m}(\Delta) \) is the image of \( \theta \) under the intertwining map

\[ H_{L^2}^{m}(\Delta) \longrightarrow L_2(H_{L^2}^{1/2}(\Delta^*), H_{L^2}^{1/2}(\Delta)) = H_{L^2}^{1/2}(\Delta) \otimes H_{L^2}^{1/2}(\Delta). \]

The space of Hankel operators of order \( m \) is the image of \( H_{L^2}^{m}(\Delta) \) under this map.

3. **An outline of the proof.** Our geometric interpretation of higher-order Hankel operators relies upon two facts. First, for \( m > 0 \), the action of \( SU(1, 1) \) on \( H_{L^2}^{m}(\Delta) \) is essentially irreducible. But if \( m = -s \) for \( s \in \mathbb{N} \), there is a short exact sequence

\[ 0 \longrightarrow \mathbb{C}^{2s+1} = \text{span} \left\{ (dz)^{-s}, \ldots , z^{2s}(dz)^{-s} \right\} \longrightarrow H^{-s}(\Delta) \xrightarrow{I_s} H^{s+1}(\Delta) \longrightarrow 0. \]

\( H^{-s}(\Delta)/\mathbb{C}^{2s+1} \) can be given a Hilbert space structure via the intertwining map \( I_s \), and we refer to this Hilbert space as \( H_{L^2}^{-s}(\Delta) \). We study the maps \( B_{s+1} : H_{L^2}^{-s}(\Delta) \rightarrow H_{L^2}^{1/2}(\Delta) \otimes H_{L^2}^{1/2}(\Delta) \).

Second, in the case \( s = 1 \), \( H^{-1}(\Delta) \) is the Lie algebra of complex vector fields on the unit disk (note \( (dz)^{-1} = \partial/\partial z \)). These vector fields act on \( \Omega^{1/2}(S^1) \) as differential operators of order 1 via the Lie derivative. Composing this action with \( \mathcal{P}_+, \mathcal{P}_- \) gives us a Hankel operator of order two. For \( \theta = f(z)(dz)^{1/2} \in H_{L^2}^{1/2}(\Delta^*) \) and \( x(z)(\partial/\partial z) \in H_{L^2}^{-1}(\Delta) \), a formal calculation leads to

\[ B_2(x)\theta := \mathcal{P}_+ \left( \frac{1}{2} I_{x(z)(\partial/\partial z)} f(z)(dz)^{1/2} \right) = \mathcal{P}_+ \left( \frac{1}{2} x' f + xf' \right)(dz)^{1/2}. \]

The key to the higher-order Hankel operators is that the above action of \( H^{-1}(\Delta) \) determines an action of \( H^{-s}(\Delta) \) on \( \Omega^{1/2}(S^1) \), as differential
operators of order \( \leq s \). Each representation of \( H^{-1}(\Delta) \) corresponds to a representation of the universal enveloping algebra \( \mathcal{U}(H^{-1}(\Delta)) \). The elements of \( \mathcal{U}(H^{-1}(\Delta)) \) act as linear differential operators on \( \Omega^{1/2}(S^1) \).

Let \( S^s(H^{-1}(\Delta)) \) be the space of symmetric tensors of order \( \leq s \) over \( H^{-1}(\Delta) \). \( S(H^{-1}(\Delta)) = \bigoplus_{s=0}^{\infty} S^s(H^{-1}(\Delta)) \) is isomorphic as a filtered vector space to \( \mathcal{U}(H^{-1}(\Delta)) \), so elements of \( S^s(H^{-1}(\Delta)) \) can be mapped to linear differential operators of order \( \leq s \) on \( \Omega^{1/2}(S^1) \). In Section 3, we show the existence of a \( G \)-equivariant embedding of \( H^{-s}(\Delta) \) into \( S^s(H^{-1}(\Delta)) \). Thus, the image of \( H^{-s}(\Delta) \) under \( B_{s+1} \) contains linear differential operators of order \( \leq s \).

\( B_{s+1} \) is unique up to multiplication by a constant and must map the lowest-weight vector in \( H^{-s}(\Delta)/\mathcal{C}^{2s+1} \), namely, \( z^{2s+1}(\partial/\partial z)^s \), to the lowest-weight vector \( l_s \) of weight \( 2(s+1) \) in \( H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta) \). We find the form of \( l_s \) in Section 4 and use it to find \( B_{s+1} \) in Section 5.

Finally, a technical point. The elements of \( H^{-1}(\Delta) \) may not extend to holomorphic vector fields on \( S^1 \), so neither \( H^{-1}(\Delta) \) nor \( \mathcal{U}(H^{-1}(\Delta)) \) act naturally on \( \Omega^{1/2}(S^1) \) a priori. However, the polynomial sections of \( \kappa^{-1}|_{S^1} \), denoted by \( H_{poly}^{-1}(\Delta) \), do extend to \( S^1 \), and the space of polynomial sections in \( H^{-s}(\Delta) \), \( H_{poly}^{-s}(\Delta) \), is mapped into \( \mathcal{U}(H_{poly}^{-1}(\Delta)) \) by the embedding from Section 3. Since \( H_{poly}^{-s}(\Delta) \) is dense in \( H_{L^2}^{-s}(\Delta) \), the action of \( H_{poly}^{-s}(\Delta) \) on \( \Omega^{1/2}(S^1) \) extends to an action of \( H_{L^2}^{-s}(\Delta) \).

4. The equivariant cross-section of \( S^s(H^{-1}(\Delta)) \to H^{-s}(\Delta) \).

\( S^s(H^{-1}(\Delta)) \) sits inside \( T^s(H^{-1}(\Delta)) \), the space of tensors of order \( s \). We will write a monomial in \( S^s(H^{-1}(\Delta)) \) as

\[
\bigotimes_{i=1}^{s} f_i(z) \frac{\partial}{\partial z} = \frac{1}{s!} \sum_{\sigma \in S_s} \left( f_{\sigma(1)}(z) \frac{\partial}{\partial z} \otimes \cdots \otimes f_{\sigma(s)}(z) \frac{\partial}{\partial z} \right),
\]

where \( S_s \) is the symmetric group on \( s \) elements. For each \( s \), \( S^s(H^{-1}(\Delta)) \) projects onto \( H^{-s}(\Delta) \) via the map

\[
P_s : \bigotimes_{i=1}^{s} f_i(z) \frac{\partial}{\partial z} \mapsto \prod_{i=1}^{s} f_i(z) \left( \frac{\partial}{\partial z} \right)^s.
\]

Let \( d_p = z^p(\partial/\partial z) \). We refer to \( p \) as the power of \( d_p \). The vectors

\[
\left\{ \bigotimes_{i=1}^{s} d_{p_i} \mid p_1 \geq p_2 \geq \cdots \geq p_s, \quad p_i \in \mathbb{N} \right\}
\]
are a basis for $S^s(H^{-1}(\Delta))$. We refer to $\sum_{i=1}^{s} p_i$ as the *total power* of such a basis vector.

The infinitesimal action of $\mathfrak{sl}(2, \mathbb{C})$ on $H^m(\Delta)$, in terms of the coordinate $f$, is

$$A^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mapsto -\frac{\partial}{\partial z},$$
$$A^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mapsto z^2 \frac{\partial}{\partial z} + 2mz,$$
$$E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mapsto 2z \frac{\partial}{\partial z} + 2m.$$

The action of $\mathfrak{sl}(2, \mathbb{C})$ on $T^s(H^{-1}(\Delta))$ is by a Liebniz rule and preserves the subspace of symmetric tensors. We denote this action by $\pi \otimes (X)$.

The following lemma is key.

**Lemma 4.1.** $(\pi \otimes (A^+))^{2s}(d_0) \otimes^s = C_s(d_2) \otimes^s$, for some constant $C_s \in \mathbb{R}$.

**Proof.** The raising operator $\pi \otimes (A^+)$ maps a monomial of total power $p$ into a linear combination of monomials having total power $p + 1$. Thus,

$$\left(\pi \otimes (A^+)\right)^{2s}(d_0) \otimes^s = \sum_{p \in \mathcal{P}^s} a_p \bigodot_{i=1}^{s} d_{p(i)},$$

where $\mathcal{P}_{2s}$ is the set of partitions of $2s$ into $s$ or fewer parts, arranged so that $p(i) \geq p(i + 1)$. At the same time, $A^+d_2 = 0$, so none of the parts $p(i)$ can be greater than 2. But there is only one partition of $2s$ into $s$ or fewer parts less than or equal to 2, namely $p(i) = 2$ for all $i$. □

**Proposition 4.2.** For each $s$, there is a $PSU(1, 1)$-equivariant cross-section of $S^s(H^{-1}(\Delta)) \to H^{-s}(\Delta)$.

**Proof.** Our strategy is to map the basis elements of $H^{-s}(\Delta)$, namely, the set $\{z^p(\partial/\partial z)^s\}_{p=0}^{\infty}$, into $S^s(H^{-1}(\Delta))$, in such a way that the resulting cross-section is equivariant.
It is clear from the action of $\pi_\odot (E)$ on $S^s(H^{-1}(\Delta))$ that $z^p(\partial/\partial z)^s$ must be mapped to an element having total power $p$. Thus, we map $(\partial/\partial z)^s$ to the only monomial of total power zero, the vector $(d_0)^\odot s$. We map $z^{2s+1}(\partial/\partial z)^s$ into a vector $v_{2s+1}$ of total power $2s + 1$. The images of all other basis vectors are obtained by applying the raising operator. The resulting cross-section will be equivariant, provided that

\begin{equation}
(5) \quad \pi_\odot (A^-) v_{2s+1} = (\pi_\odot (A^+))^s (d_0)^\odot s = C_s (d_2)^\odot s.
\end{equation}

In fact, we take

\[ v_{2s+1} = -\frac{C_s}{3} d_3 \odot (d_2)^\odot (s-1) - \sum_{4 \leq p \leq 2s+1} d_p \odot \sum_{m+n \leq s-1 \atop 2m+n=2s+1-p} A_{pn} (d_2)^\odot m \odot (d_1)^\odot n \odot (d_0)^\odot (s-m-n-1). \]

For example, for $s = 2$, one has

\[ -\frac{3}{C_2} v_5 = d_3 \odot d_2 - \frac{1}{2} d_4 \odot d_1 + \frac{1}{10} d_5 \odot d_0. \]

The monomials in $v_{2s+1}$ all have total power $2s + 1$. The largest power $p$ ranges from 3 to $2s + 1$. Among basis elements with highest power $p$, the sum includes all those where the remaining powers are no greater than two. Thus, the inner sum is really over partitions of $2s + 1 - p$ into $s - 1$ or fewer parts, each part being 1 or 2. The coefficients $A_{pn}$ are indexed by the highest power and $n$, the (symmetric tensor) exponent of $d_1$. If $p$ is even, $n$ is odd, and vice-versa. Modulo parity, $n$ ranges from 0 to the minimum of $p - 3$ and $2s + 1 - p$. Thus, the range of $n$ increases until $p = s + 2$, and then decreases to $n = 0$ when $p = 2s + 1$. For all other $n, p$, take $A_{pn} = 0$.

Because the monomials in $v_{2s+1}$ are all products of $d_p, d_2, d_1$ and $d_0$, applying $\pi_\odot (A^-)$ to each monomial results in at most three terms. In fact,

\[ \pi_\odot (A^-) v_{2s+1} = C_s (d_2)^\odot (s) + \left( \frac{2C_s(s-1)}{3} + 4A_{41} \right) d_3 \odot (d_2)^\odot (s-2) \odot d_1 \]
\[ + \sum_{4 \leq q \leq 2s} d_q \odot \sum_{l+k \leq s-1 \atop 2l+k=2s-q} B_{qk}(d_2)^\odot l \odot (d_1)^\odot k \odot (d_0)^\odot (s-l-k-1), \]

where

\[ B_{qk} = (q+1)A_{q+1}k + (2s-q-k+2)A_{q(k-1)} + (k+1)A_{q(k+1)}. \]

Thus, choosing \( A_{41} = -[C_s(s-1)]/6, \) and defining

\[ A_{(p+1)n} = -\frac{1}{p+1} \left[ (2s-p-n+2)A_{p(n-1)} + (n+1)A_{p(n+1)} \right] \]

for all other \( n \) and \( p, \) one obtains (5). This recursion relation is linear and terminates at \( p = 2s+1. \) Thus, it can be solved, and \( v_{2s+1} \) exists and satisfies (5). This proves the proposition. \( \square \)

5. Lowest weight vectors in \( L_2(H^{1/2}_{L^2} (\Delta^*), H^{1/2}_{L^2} (\Delta)) \). The action of \( sl(2, \mathbb{C}) \) on \( L_2(H^{1/2}_{L^2} (\Delta^*), H^{1/2}_{L^2} (\Delta)) = H^{1/2}_{L^2} (\Delta) \otimes H^{1/2}_{L^2} (\Delta) \) is again by a Liebniz rule and will be denoted \( \pi\otimes(X). \) The irreducible subspaces of \( H^{1/2}_{L^2} (\Delta) \otimes H^{1/2}_{L^2} (\Delta) \) are lowest-weight representations of \( SU(1,1). \) We now identify the vectors in \( H^{1/2}_{L^2} (\Delta) \otimes H^{1/2}_{L^2} (\Delta) \) annihilated by \( \pi\otimes(A^-). \)

**Proposition 5.1.** The set of vectors

\[ \left\{ l_s := \sum_{i=0}^{s} (-1)^i \binom{s}{i} z^{s-i}(dz)^{1/2} \otimes z^i(dz)^{1/2} \right\}_{s=0}^{\infty} \]

are annihilated by the operator \( \pi\otimes(A^-). \) The vector \( l_s \) has weight \( 2(s+1). \)

**Proof.** Let \( b_p = z^p(dz)^{1/2}. \) Applying \( \pi\otimes(A^-) \) to \( b_{s-i} \otimes b_i \) results in two terms unless \( i = 0 \) or \( i = s, \) so we pull these cases out of the sum \( l_s. \) Thus,

\[ -\pi\otimes(A^-)[l_s] = sb_{s-1} \otimes b_0 \]

\[ + \sum_{i=1}^{s-1} (-1)^i \binom{s}{i} \pi\otimes(A^-)[b_{s-i} \otimes b_i] \]

\[ + (-1)^s sb_0 \otimes b_{s-1}. \]
The middle term is
\[
\sum_{i=1}^{s-1} (-1)^i \binom{s}{i} \left[ (s - i)b_{s-i-1} \otimes b_i + ib_{s-i} \otimes b_{i-1} \right]
\]
\[
= \sum_{i=1}^{s-1} (-1)^i \frac{s \cdots (s - i)}{i!} b_{s-i-1} \otimes b_i
\]
\[
+ \sum_{i=1}^{s-1} (-1)^i \frac{s \cdots (s - i + 1)}{(i - 1)!} b_{s-i} \otimes b_{i-1}
\]
\[
= \sum_{i=1}^{s-1} (-1)^i \frac{s \cdots (s - i)}{i!} b_{s-i-1} \otimes b_i
\]
\[
+ \sum_{j=0}^{s-2} (-1)^{j+1} \frac{s \cdots (s - j)}{j!} b_{s-j-1} \otimes b_j
\]
\[
= -sb_{s-1} \otimes b_0
\]
\[
+ \sum_{j=1}^{s-2} \left[ (-1)^j + (-1)^{j+1} \right] \frac{s \cdots (s - j)}{j!} b_{s-j-1} \otimes b_j
\]
\[
+ (-1)^s b_0 \otimes b_{s-1}
\]
\[
= -sb_{s-1} \otimes b_0 + (-1)^s b_0 \otimes b_{s-1}.
\]

Thus,
\[
A^{-\left[ l_s \right]} = sb_{s-1} \otimes b_0 - sb_{s-1} \otimes b_0 + (-1)^{s-1} b_0 \otimes b_{s-1} + (-1)^s b_0 \otimes b_{s-1} = 0.
\]

To prove the second claim, simply notice that
\[
E \left[ b_{s-i} \otimes b_i \right] = 2(s - i + i + 1)b_{s-i} \otimes b_i = 2(s + 1)b_{s-i} \otimes b_i. \quad \Box
\]

6. An explicit formula for $B_{s+1}$. We will use the following lemma to prove Theorem 1.1. For $x(z)(\partial/\partial z)^s \in H_{L^2}^s(\Delta)$, let
\[
\mathcal{O}_j(x) = \mathcal{P}_x^{(s-j)} \left( \frac{\partial}{\partial z} \right)^j \in \mathcal{L}_2(H_{L^2}^{1/2}(\Delta^*), H_{L^2}^{1/2}(\Delta)).
\]
Lemma 6.1. Let $k > s - j$. As an element of $H_{L^2}^{1/2}(\Delta) \otimes H_{L^2}^{1/2}(\Delta)$,

$$
\mathcal{O}_j(z^k) = \sum_{i=0}^{k-s-1} (-1)^j \frac{(i + j)!}{i!} \frac{k!}{(k - s + j)!} z^{(k-s-1)-i} (dz)^{1/2} \otimes z^i (dz)^{1/2}.
$$

Proof. Let $f(z)(dz)^{1/2} \in H_{poly}^{1/2}(\Delta^*)$, with $f(z) = \sum_{n=1}^N f_n z^{-n}$ for $N > k - s$. Then,

$$
\left( \frac{\partial}{\partial z} \right)^j f(z) = \sum_{n=1}^N (-1)^j \frac{(n + j - 1)!}{(n - 1)!} f_n z^{-(n+j)}.
$$

Also,

$$
\left( \frac{\partial}{\partial z} \right)^{s-j} z^k = \frac{k!}{(k - s + j)!} z^{k-s+j}.
$$

Thus,

$$
\left[ \left( \frac{\partial}{\partial z} \right)^{s-j} z^k \right] \left[ \left( \frac{\partial}{\partial z} \right)^j f(z) \right] = \sum_{n=1}^N (-1)^j \frac{(n + j - 1)!}{(n - 1)!} \frac{k!}{(k - s + j)!} f_n z^{k-s-n},
$$

and so

$$
\mathcal{O}_j(z^k) f(z)(dz)^{1/2} = \sum_{n=1}^{k-s} (-1)^j \frac{(n + j - 1)!}{(n - 1)!} \frac{k!}{(k - s + j)!} f_n z^{k-s-n} (dz)^{1/2}.
$$

Since

$$
f_n z^{k-s-n} (dz)^{1/2} = \left( z^{k-s-n} (dz)^{1/2} \otimes z^{n-1} (dz)^{1/2} \right) f(z)(dz)^{1/2},
$$

the required formula is obtained after reindexing. But this formula depends only upon the first $k - n$ coefficients of $f$, so it applies to all of $H_{L^2}^{1/2}(\Delta^*)$.  \[\square\]
Proof of Theorem 1.1. By Proposition 4.2, we know $B_{s+1}(v) = P_+ L_s(v)$, where

$$L_s(v) = \sum_{j=0}^{s} c_j(v) \left( \frac{\partial}{\partial z} \right)^j.$$ 

The cases $s = 0$ and $s = 1$ suggest that $c_j(v) = a_j v^{(s-j)}$. Since $B_{s+1}$ is unique, we need only find coefficients $a_j$ which satisfy

(6) $$P_+ L_s \left(z^{2s+1}\right) = l_s.$$ 

By Proposition 5.1 and Lemma 6.1, (6) is equivalent to

$$\sum_{j=0}^{s} \sum_{i=0}^{s} (-1)^j \frac{(i+j)!}{i!} \frac{(2s+1)!}{(s+j+1)!} z^{s-i} (dz)^{1/2} \otimes z^i (dz)^{1/2} = \sum_{i=0}^{s} (-1)^i \binom{s}{i} z^{s-i} (dz)^{1/2} \otimes z^i (dz)^{1/2}.$$ 

In other words, the $a_j$'s must solve the linear system

$$M_s \vec{a}_s := \left( (-1)^j \frac{(i+j)!}{i!} \frac{(2s+1)!}{(s+j+1)!} \right)_{i,j=0}^{s} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_s \end{pmatrix} = \begin{pmatrix} 1 \\ -s \\ \vdots \\ (-1)^s \end{pmatrix} := \vec{l}_s.$$ 

Pleasantly, the matrix $M_s$ is easily factored. First, we rewrite $M_s$ as

$$N_s D_s := \left( \frac{(i+j)!}{i! j!} \right)_{i,j=0}^{s} \text{diag} \left( \left( (-1)^j \frac{(2s+1)!}{(s+j+1)!} \right)_{j=0}^{s} \right).$$ 

Now, a beautiful combinatorial identity comes into play, and one has

$$N_s = \left( \binom{i+j}{i} \right)_{i,j=0}^{s} = \left( \binom{j}{i} \right)_{i,j=0}^{s} = L_s U_s,$$

using (5.23) from [4]. Let

$$\vec{L}_s = \left( (-1)^{i+j} \binom{i+j}{i} \right)_{i,j=0}^{s},$$
and let $P = L_s \tilde{L}_s$. Then the $(i,j)$th entry of $P$ is

$$p_{ij} = \sum_{k=j}^{i} (-1)^{k+j} \binom{i}{k} \binom{k}{j}$$

$$= \binom{i}{j} \sum_{k=j}^{i} (-1)^{k+j} \binom{i-j}{i-k}$$

$$= \binom{i}{j} \sum_{k=0}^{i-j} (-1)^{k} \binom{i-j}{k}$$

$$= \delta_{ij},$$

since $\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} = (1 - 1)^{n} = \delta_{n0}$. Thus, $\tilde{L}_s = L_s^{-1}$. Analogously,

$$U_{s}^{-1} = \left( (-1)^{i+j} \binom{j}{i} \right)_{i,j=0}^{s}.$$

Now,

$$L^{-1}_s \tilde{i}_s = \left( \sum_{k=0}^{s} (-1)^{j+k} (-1)^{k} \binom{j}{k} \binom{s+k}{k} \right)_{j=0}^{s}$$

$$= \left( (-1)^{j} \binom{s+j}{j} \right)_{j=0}^{s},$$

again using (5.23) from [4]. Next,

$$U_{s}^{-1} L_{s}^{-1} \tilde{i}_s = \left( (-1)^{j} \sum_{k=0}^{s} \binom{k}{j} \binom{s+k}{k} \right)_{j=0}^{s}$$

$$= \left( (-1)^{j} \binom{s+j}{j} \sum_{k=j}^{s} \binom{s+k}{s+j} \right)_{j=0}^{s}$$

$$= \left( (-1)^{j} \binom{s+j}{j} \binom{2s+1}{s+j+1} \right)_{j=0}^{s},$$

using upper summation ([4, (5.10)]), and finally,

$$\tilde{a}_s = M_{s}^{-1} l_s = \left( \frac{(s+j+1)!}{j!(2s+1)!} \binom{s+j}{j} \binom{2s+1}{s+j+1} \right)_{j=0}^{s}$$

$$= \left( \frac{1}{s!} \binom{s}{j} \binom{s+j}{j} \right)_{j=0}^{s}.$$
7. Binomial coefficient identities. The equivariance of $B_{s+1}$ implies a pair of identities relating sums of products of binomial coefficients.

**Proposition 7.1.** For $s \in \mathbb{N}$, $k \geq 2s + 1$, and $l = 0, \ldots, k - s$,

$$\sum_{j=0}^{s} (-1)^j \binom{s + j}{j} \binom{k}{s - j} \left[ \binom{l + j}{j} (k - s) - \binom{l + j - 1}{j - 1} l \right] = \sum_{j=0}^{s} (-1)^j \binom{s + j}{j} \binom{k + 1}{s - j} \binom{l + j}{j} (k - 2s),$$

and, for $s \in \mathbb{N}$, $i + j \geq s$,

$$\sum_{l=0}^{s} (-1)^l \binom{s + l}{l} \binom{i + j + s}{s - l} = \sum_{l=0}^{s} (-1)^l \binom{s + 1}{l} \binom{i}{s - l}.$$

**Proof.** To prove the first identity, expand the equation $B_{s+1}(A^+ z^k) = \pi \otimes (A^+) B_{s+1}(z^k)$ in the basis $\{z^i(dz)^{1/2} \otimes z^j(dz)^{1/2}\}$; to prove the second, expand the equation $B_{s+1}((A^+)^{(i+j-s)} z^{2s+1}) = (\pi \otimes (A^+))^{(i+j-s)} B_{s+1}(z^{2s+1})$.

8. Connections with prior work. The transvectant $\tau_{s+1}^s = \tau_{s+1}$ is the essentially unique equivariant map $\tau_{s+1} : H_{L^2}^{1/2}(\Delta) \otimes H_{L^2}^{1/2}(\Delta) \to H_{L^2}^{s+1}(\Delta)$. Let $\theta = f(z)(dz)^{1/2}, \eta = g(z)(dz)^{1/2} \in H_{L^2}^{1/2}(\Delta)$. Then

$$\tau_{s+1}(\theta \otimes \eta) = \tau_{s+1}(f, g)(dz)^s + 1 = \left( \sum_{j=1}^{s} (-1)^j \binom{s}{j}^2 f^{(s-j)} g^{(j)} \right) (dz)^s + 1.$$

In [5], this map is used to construct higher-order Hankel bilinear forms on the space $H_{L^2}^{1/2}(\Delta) \otimes H_{L^2}^{1/2}(\Delta)$. If $\nu = x(z)(\partial/\partial z)^s \in H_{L^2}^{-s}(\Delta)$ and
\( \mu = y(z)(dz)^{s+1} \in H^{s+1}_{L^2}(\Delta) \), then \( \nu \mu = \overline{x}y \, dz \) is a one-density on \( S^1 \), and so \( (H^{s+1}_{L^2}(\Delta))^* = H^{-s}_{L^2}(\Delta) \). As in [5], define the Hankel form of order \( s + 1 \) with symbol \( x \) to be

\[
K_{s+1}(x)[f, g] = \int_{S^1} \nu \tau_{s+1}(\theta \otimes \eta) = \int_{S^1} \overline{\nu} \tau_{s+1}(f, g) \, dz,
\]

where \( \nu = x(z)(\partial / \partial z)^s \). Since \( B_{s+1}(x)\overline{\theta} \in H^{1/2}_{L^2}(\Delta) \), another bilinear form is defined by

\[
\tilde{K}_{s+1}(x)[f, g] = \langle B_{s+1}(x)\overline{\theta}, \eta \rangle_{1/2} = \int_{S^1} B_{s+1}(x)\overline{f} \, g \, d\theta,
\]

and \( K_{s+1}(x) \) and \( \tilde{K}_{s+1}(x) \) are easily seen to be equivalent. Another expression for \( \tilde{K}_{s+1}(x) \) is

\[
\tilde{K}_{s+1}(x)[f, g] = \langle B_{s+1}(x), \theta \otimes \eta \rangle_{\otimes} = \text{tr} \left( B_{s+1}(x)(\theta \otimes \eta)^* \right),
\]

where \( \langle \, , \rangle_{\otimes} \) is the inner product on \( H^{1/2}_{L^2}(\Delta) \otimes H^{1/2}_{L^2}(\Delta) \). Thus, \( B_{s+1} \) is the adjoint of the map \( \tau_{s+1} \), and we have the diagram

\[
\begin{array}{ccc}
H_{L^2}(-s)(\Delta) & \xrightarrow{B_{s+1}} & H^{s+1}_{L^2}(\Delta) \\
\downarrow \tau_{s+1} & & \downarrow \tau_{s+1} \\
H_{L^2}^{1/2}(\Delta^*) & \leftarrow L_2(H^{1/2}_{L^2}(\Delta^*), H^{1/2}_{L^2}(\Delta)).
\end{array}
\]

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