CONVEXITY OF THE MOMENT MAP IMAGE FOR TORUS ACTIONS ON
$b^m$-SYMPLECTIC MANIFOLDS

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ABSTRACT. We prove a convexity theorem for the image of the moment map of a Hamiltonian torus action on a $b^m$-symplectic manifold.

1. INTRODUCTION

The purpose of this paper is to prove a convexity theorem for the image of the moment map of a Hamiltonian torus action on a $b^m$-symplectic manifold. $b^m$-symplectic manifolds are Poisson manifolds where the Poisson structure is invertible on the complement of a hypersurface $Z$, and has a singularity of order $m$ on $Z$, where $m \geq 1$ is an integer. They are a generalization of $b$-symplectic (or, log-symplectic) manifolds and were introduced in the thesis of G. Scott [S].

A convexity theorem for the moment image of a Hamiltonian torus action on a $b$-symplectic manifold was proved in [GMPS2]. In this case, the moment image is governed by the singularity of the moment map in the neighborhood of $Z$, encoded in the modular weight (see [GMPS1]). If this modular weight is nonzero, the image is not only convex, but on each component of the complement of $Z$, has the form of a product of a convex polytope with a ray or the real line, possibly modified by symplectic cutting.

We show that the moment image has a similar form where $m > 1$. The argument requires some care, since in this case the moment map has an asymptotic series near $Z$ which involves $m$ modular weights. The leading term in this series gives the moment map a form as in [GMPS2]. We show that the subleading terms preserve this structure.

As a technical device to state our convexity theorem, we use the desingularization of $b^m$-symplectic forms in [GMW1]. This simplifies the statements since we can deal with families of compact manifolds, with compact moment images.

In a companion paper, we use the convexity theorem to study formal geometric quantization of $b^m$-symplectic manifolds equipped with Hamiltonian torus actions. The form of the moment image will give rise to a very simple asymptotics of this quantization.

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2. $b^m$-MANIFOLDS

Let $M$ be a compact manifold, and let $f \in C^\infty(M)$ have a transverse zero at a hypersurface $Z \subset M$. Let $m$ be a positive integer, the $m$-germ of $f$ at $Z$ gives rise to a sheaf, and therefore a vector bundle, whose sections are given by,
\[ \Gamma^{(b^mTM)} = \{ v \in \Gamma(TM) : v \text{ vanishes to order } m \text{ at } Z \}. \]

By considering sections of the wedge powers \( \Lambda^k(b^mT^*M) \) we obtain a complex \( (b^m \Omega^k(M), d) \) of differential forms with singularities at \( Z \).

The cohomology associated to this complex is given by the following:

**Theorem 1** \( (b^m\text{-Mazzeo-Melrose}, \mathbb{S}). \)

\[ b^mH^p(M) \cong H^p(M) \oplus (H^{p-1}(Z))^m. \]

This theorem comes from a Laurent expansion in a neighborhood of \( Z \) which for 2-forms has the form:

\[ (2.1) \quad \omega = \sum_{j=1}^m \frac{df}{f^j} \wedge \pi^*(\alpha_j) + \beta \]

where \( \alpha_j \) are closed one forms on \( Z \), \( \beta \) is a closed 2-form on \( M \), and \( \pi : U \to Z \) is the projection.

A two \( b \)-form \( \omega \in b^m \Omega^2(M) \) is \( b^m \)-symplectic if it is closed and nondegenerate as an element of \( \Lambda^2(b^mT^*M) \).

**Theorem 2.** There exists a neighborhood \( U = (-\epsilon, \epsilon) \times Z \) of \( Z \) and a diffeomorphism \( \psi : U \to U \) preserving \( Z \) and the \( m \)-germ of \( f \) such that

\[ (2.3) \quad \psi^*(\omega) = \sum_{j=1}^m \frac{df}{f^j} \wedge \pi^*(\alpha_j) + \pi^*(\beta) \]

where \( \pi : U \to Z \) is the projection and \( \alpha_i \) and \( \beta \) are closed.

**Proof**

We know by [CMPT] proposition 3.1 that

\[ \omega = \sum_{j=1}^m \frac{df}{f^j} \wedge \pi^*(\alpha_j) + \beta. \]

Let \( i : Z \to U \) be the restriction. Then

\[ \pi^* i^* \alpha_i - \alpha_i = da_i \]
\[ \pi^* i^* \beta - \beta = db \]

and for \( \epsilon \) sufficiently small, letting

\[ \omega_0 = \sum_{j=1}^m \frac{df}{f^j} \wedge \pi^* i^*(\alpha_j) + \pi^* i^*(\beta) \]

that \( \omega_t = t\omega_0 + (1 - t)\omega \) is symplectic for all \( t \in [0, 1] \).

Thus

\[ \frac{d}{dt}\omega_t = db_t \]

where

\[ b_t = \nu_t \omega_t \]

for \( \nu_t \) a section of \( b^mTM \).
By shrinking \( \epsilon \) we may choose \( \nu_t \) compactly supported on \( U \) and \( ||\nu_t|| < \delta \) for any fixed \( \delta > 0 \). Thus the existence theorem for ODE shows that \( \nu_t \) integrates to a family of diffeomorphisms, \( \psi_t \), vanishing to order \( m \) on \( Z \) such that \( \psi_t^* (\omega_0) = \omega \).

3. TORUS ACTIONS ON \( b^m \)-MANIFOLDS

Now let us assume a torus \( T \) acts on \((M, Z, f)\) preserving \( \omega \). We denote 
\[ b^m C^\infty(M) = \{ a \log |f| + g, g \in C^\infty(M) \} \]
the space of smooth functions with logarithmic singularity at \( Z \) and write 
\[ b^m C^\infty(M) = C^\infty(M) \oplus \bigoplus_{j=1}^{m-1} f^{-j} C^\infty(M) \oplus b C^\infty(M). \]

**Definition 3.1.** The action of \( T \) is Hamiltonian if there exists a moment map \( \mu \in b^m C^\infty(M) \otimes t^* \) with 
\[ \langle d\mu, X \rangle = \iota_{X^M} \omega \]
for any \( X \in t \) where \( X^M \) is the fundamental vector field generated by \( X \) on \( M \).

We will prove the following

**Lemma 3.2.** There exists a neighborhood \( U = Z \times (-\epsilon, \epsilon) \) where the moment map \( \mu : M \rightarrow t^* \) is given by 
\[ \mu = a_1 \log |f| + \sum_{i=2}^{m} a_i \frac{f^{-(i-1)}}{i-1} + \mu_0 \]
with \( a_i \in \mathfrak{v}_L \), and \( \mu_0 \) is the moment map for the \( T_L \)-action on the symplectic leaves of the foliation.

**Proof**
Note that theorem 2 holds equivariantly so we may assume that \( \omega \) can be written
\[ \omega = \sum_{j=1}^{m} \frac{df}{f_j} \wedge \pi^* (\alpha_j) + \pi^* (\beta) \]
where \( \pi : U \rightarrow Z \) is the projection and \( \alpha_j \) and \( \beta \) are closed and \( T \)-invariant.

The moment map \( \mu \) therefore has the form,
\[ \mu = a_1 \log |f| + \sum_{i=2}^{m} a_i \frac{f^{-(i-1)}}{i-1} + \mu_0 \]
with \( \langle a_i, X \rangle = \alpha_i (X^M) \) and \( \mu_0 \) is the moment map for the action of \( T \) on the regular Poisson manifold \( Z \).

The form \( \alpha_m \) is nowhere vanishing and determines the symplectic foliation of \( Z \). We now make the following two assumptions,

- **Assumption 1** There exists \( \xi \in t \) such that \( a_m(\xi) \neq 0 \). Remark: Without loss of generality, we may assume also that \( \xi \) generates a circle subgroup \( S^1_{\xi} \subset T \).
- **Assumption 2** The foliation given by \( \alpha_m \) has a compact leaf \( L \). Thus \((L, \beta)\) is a compact symplectic manifold acted on by the Hamiltonian torus action of \( T_L = T/S^1_{\xi} \), with moment map \( \mu_0|_{\mathcal{L}} \). This would follow from integrality of \( \omega \); see [GMW3].
4. Desingularization

Theorem 3. Given a $b^m$-symplectic structure $\omega$ on a compact manifold $M^{2n}$ let $Z$ be its critical hypersurface.

- If $m$ is even, there exists a family of symplectic forms $\omega_\epsilon$ which coincide with the $b^m$-symplectic form $\omega$ outside an $\epsilon$-neighborhood of $Z$ and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the $C^{m-1}$-topology to the Poisson structure $\omega^{-1}$ as $\epsilon \to 0$.
- If $m$ is odd, there exists a family of folded symplectic forms $\omega_\epsilon$ which coincide with the $b^m$-symplectic form $\omega$ outside an $\epsilon$-neighborhood of $Z$.

If a torus acts, this family of forms $\omega_\epsilon$ may be chosen equivariantly.

Remark 4.1. Observe that even if the initial action is Hamiltonian in the $b^m$-sense, the desingularized action need not be Hamiltonian in the standard sense because its moment map might be circle valued.

5. The Local Convexity Theorem

Theorem 4. Let $Z_i$ be a connected component of $Z$. Then there exists a neighborhood $(-\epsilon, \epsilon) \times Z_i$ where the image of the moment map for the $T$-action on the desingularized family $(M, \omega_\epsilon)$ is

- $\Delta_i \times (-a_\epsilon, a_\epsilon)$ for even $m$.
- $\Delta_i \times (-a_\epsilon, a_\epsilon)/\psi$ for odd $m$.

where $\Delta_i$ is the image of the moment map for the $T_{L_i}$-action on $L_i$, a symplectic leaf on $Z$ and

- $a_\epsilon \to \infty$ as $\epsilon \to 0$.
- $\psi: (-a_\epsilon, a_\epsilon) \mapsto (-a_\epsilon, a_\epsilon)$ is the involution $x \mapsto -x$.

Remark 5.1. This implies that the image of the moment map is locally convex.

Proof. Recall the moment map is given by the expression (3.4) with $a_i \in t^*$ constant. We claim that in $\mathbb{C}^n$, $\langle a_i, \xi \rangle = 0$ for all $\xi \in t_{L_i}$. To see this, suppose there exists $\xi \in t_{L_i}$ such that $\langle a_i, \xi \rangle \neq 0$. Since $T_L$ is a torus action on a compact Hamiltonian $T_L$-space, $L$, it must have a fixed point $p \in Z$. At this fixed point $e^M_{\rho|p} = 0$ so $\langle a_i, \xi \rangle = \alpha_i(e^M_{\rho|p}) = 0$. Thus $\langle a_i, \xi \rangle = 0$. 

6. Global Convexity Theorem

In this section we prove,

Theorem 5. Let $(M, Z, f)$ be a $b^m$-symplectic manifold. Let $M \setminus Z = \bigsqcup_{i=1}^r M_i$. Then the image of the moment map for the desingularized symplectic form $\omega_\epsilon$ on $M_i$ is given by either:

1. A product $\Delta \times [-a_\epsilon, a_\epsilon]$, where $a_\epsilon \to \infty$ as $\epsilon \to 0$; or,
2. A convex polytope which has the form of a product $\Delta \times [0, a_\epsilon]$ in the neighborhood of $Z$; in other words, a polytope of the form
   $\Delta \times [-a_\epsilon, a_\epsilon] \cap H_1 \cap \cdots \cap H_n$.

Here $\Delta$ is the image of the moment map for the $T_L$-action on $L$ and $H_1, \ldots, H_n$ are half-spaces. In particular the image polytopes $\Delta_i$ coincide.

Proof. By the local convexity theorem, we know that if $\partial M_i = Z_i \cup Z_{i+1}$, then in a neighborhood of $Z_i$, respectively $Z_{i+1}$ the image is of the form $\Delta_i \times [-a_\epsilon, c]$ resp $\Delta_{i+1} \times [c', a_\epsilon]$ where $a_\epsilon \to \infty$ as $\epsilon \to 0$ and we may take $c$ and $c'$ positive.
Let $\nu = \mu|M_i|$ denote the restriction of $\mu$ to $M_i$ and let $P = \nu(M_i)$ be the image of the moment map. By the convexity theorem for torus actions on symplectic manifolds, the image $P$ of the moment map restricted to $M_i$ is convex.

Therefore $P \cap \nu^{-1}([-a_\epsilon, c]) = [-a_\epsilon, c] \times \Delta_i$ and $P \cap \nu^{-1}([c', a_\epsilon]) = [c', a_\epsilon] \times \Delta_{i+1}$ for all $c, c'$ sufficiently close to $a_\epsilon$.

Let us see that this implies $P \cap \nu^{-1}([-a_\epsilon, a_\epsilon]) = [-a_\epsilon, a_\epsilon] \times \Delta_i$.

We first prove $P \cap [-a_\epsilon, a_\epsilon] \subset [-a_\epsilon, a_\epsilon] \times \Delta_i$. Suppose $p \in P \cap \nu^{-1}([-a_\epsilon, a_\epsilon])$ but $p \notin [-a_\epsilon, a_\epsilon] \times \Delta_i$; then $p = (t, \tau)$ where $t \in [-a_\epsilon, a_\epsilon]$ with $\tau \notin \Delta_i$.

Let $\sigma$ be a point of $\Delta_i$ at minimal distance from $\tau$. The line $\ell$ connecting $(t, \tau)$ and $(-\alpha, \sigma)$ must lie in $P$. Thus for a sufficiently close to $-\alpha$, $\ell \cap [-\alpha, a] \times \mathbb{R}^{d-1}$ must lie in $[-\alpha, a] \times \Delta_i$. Hence $\ell$ must intersect the plane $(a, x)_{x \in \mathbb{R}^{d-1}}$ at a point $\sigma' \in \Delta_i$ closer to $\tau$ than $\sigma$ (see picture below). This is a contradiction.

Thus we have proved that $P \subset [-a_\epsilon, a_\epsilon] \times \Delta_i$; also $P \subset [-a_\epsilon, a_\epsilon] \times \Delta_{i+1}$. Therefore, $P \subset [-a_\epsilon, a_\epsilon] \times (\Delta_i \cap \Delta_{i+1})$. In particular for $c$ sufficiently close to $a_\epsilon$, $[-a_\epsilon, c] \times \Delta_i \subset [-a_\epsilon, a_\epsilon] \times (\Delta_i \cap \Delta_{i+1})$ and $[c, a_\epsilon] \times \Delta_i \subset [c, a_\epsilon] \times (\Delta_i \cap \Delta_{i+1})$ so $\Delta_i \subset (\Delta_i \cap \Delta_{i+1})$ and $\Delta_{i+1} \subset (\Delta_i \cap \Delta_{i+1})$; thus $\Delta_i = \Delta_{i+1}$. Since $P$ contains all lines $[-a_\epsilon, a_\epsilon] \times \Delta_i$, it must be equal to $[-a_\epsilon, a_\epsilon] \times \Delta_i$. On the other hand, if $\partial M_i$ is connected the image of the moment map must be of the form (1) or (2) and can be written as $\Delta \times [-a_\epsilon, a_\epsilon] \cap H_1 \cap \cdots \cap H_n$ for some half-spaces $H_1, \ldots, H_n$.

\[\square\]

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