On some properties of Tsallis hypoentropies and hypodivergences

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Abstract. Both the Kullback-Leibler and the Tsallis divergence have a strong limitation: if the value 0 appears in probability distributions \((p_1, \cdots, p_n)\) and \((q_1, \cdots, q_n)\), it must appear in the same positions for the sake of significance. In order to avoid that limitation in the framework of Shannon statistics, Ferreri introduced in 1980 the hypoentropy: “such conditions rarely occur in practice”. The aim of the present paper is to extend Ferreri’s hypoentropy to the Tsallis statistics. We introduce the Tsallis hypoentropy and the Tsallis hypodivergence and describe their mathematical behavior. Fundamental properties like nonnegativity, monotonicity, the chain rule and subadditivity are established.

Keywords : Mathematical inequality, Tsallis entropy, Tsallis hypoentropy, Tsallis hypodivergence, chain rule, subadditivity

2010 Mathematics Subject Classification : 26D15 and 94A17

1 Preliminaries

Throughout this paper, \(X\), \(Y\) and \(Z\) denote discrete random variables taking on the values \(\{x_1, \cdots, x_{|X|}\}\), \(\{y_1, \cdots, y_{|Y|}\}\) and \(\{z_1, \cdots, z_{|Z|}\}\), respectively. Where \(|A|\) denotes the number of the values of the discrete random variable \(A\). We also denote the discrete random variable following a uniform distribution by \(U\). We set the probabilities as \(p(x_i) \equiv Pr(X = x_i)\), \(p(y_j) \equiv Pr(Y = y_j)\) and \(p(z_k) \equiv Pr(Z = z_k)\). If \(|U| = n\), then \(p(u_k) = \frac{1}{n}\) for all \(k = 1, \cdots, n\). In addition, we denote by \(p(x_i, y_j) = Pr(X = x_i, Y = y_j)\), \(p(x_i, y_j, z_k) = Pr(X = x_i, Y = y_j, Z = z_k)\) the joint probabilities, by \(p(x_i | y_j) = Pr(X = x_i | Y = y_j)\), \(p(x_i | y_j, z_k) = Pr(X = x_i | Y = y_j, Z = z_k)\) the conditional probabilities and so on.

The notion of entropy was used in statistical thermodynamics by Boltzmann [2] in 1871 and Gibbs [9] in 1902, in order to quantify the diversity, uncertainty, randomness of isolated systems. Later it was seen as a measure of “information, choice and uncertainty” in the theory of communication, when Shannon [15] defined it by

\[
H(X) \equiv - \sum_{i=1}^{|X|} p(x_i) \log p(x_i). \tag{1}
\]

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In what follows we consider $|X| = |Y| = |U| = n$, unless otherwise specified.

Making use of the concavity of the logarithmic function, one can easily check that the equiprobable states are maximizing the entropy, that is

$$H(X) \leq H(U) = \log n. \quad (2)$$

The right hand side term of this inequality is known since 1928 as Hartley entropy [10].

For two random variables $X$ and $Y$ following distributions $\{p(x_i)\}$ and $\{p(y_i)\}$, the Kullback-Leibler [12] discrimination function (divergence or relative entropy) is defined by

$$D(X||Y) \equiv \sum_{i=1}^{n} p(x_i)(\log p(x_i) - \log p(y_i)) = -\sum_{i=1}^{n} p(x_i) \log \frac{p(y_i)}{p(x_i)}. \quad (3)$$

Here the conventions $1 \cdot \log \frac{a}{a} = -\infty (a > 0)$ and $0 \cdot \log \frac{b}{b} = 0 (b \geq 0)$ are used. In what follows, we use such conventions in the definitions of the entropies and divergences. However we do not state them repeatedly.

It holds that

$$H(U) - H(X) = D(X||U). \quad (4)$$

Moreover, the cross-entropy (or inaccuracy)

$$H^{(cross)}(X, Y) \equiv -\sum_{i=1}^{n} p(x_i) \log p(y_i) \quad (5)$$

satisfies the identity

$$D(X||Y) = H^{(cross)}(X, Y) - H(X). \quad (6)$$

C. Tsallis introduced a one-parameter extension of the entropy in 1988 in [13], for handling systems which appear to deviate from standard statistical distributions. It plays an important role in the nonextensive statistical mechanics of complex systems, being defined as

$$T_q(X) \equiv -\sum_{i=1}^{n} p(x_i)^q \ln_q p(x_i) = \sum_{i=1}^{n} p(x_i) \ln_q \frac{1}{p(x_i)} \quad (q \geq 0, q \neq 1). \quad (7)$$

Here the $q$–logarithmic function for $x > 0$ is defined by $\ln_q(x) \equiv \frac{x^{1-q}-1}{1-q}$, which converges to the usual logarithmic function $\log(x)$ in the limit $q \to 1$. The Tsallis divergence (relative entropy) [19] is given by

$$S_q(X||Y) \equiv \sum_{i=1}^{n} p(x_i)^q(\ln_q p(x_i) - \ln_q p(y_i)) = -\sum_{i=1}^{n} p(x_i) \ln_q \frac{p(y_i)}{p(x_i)} \quad (8)$$

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1. The relative entropy is usually defined for two probability distributions $P = \{p_i\}$ and $Q = \{q_i\}$ as $D(P||Q) \equiv -\sum_{i=1}^{n} p_i \log \frac{q_i}{p_i}$ in the standard notation of Information theory. $D(P||Q)$ is often rewritten by $D(X||Y)$ for random variables $X$ and $Y$ following the distributions $P$ and $Q$. Throughout this paper, we use the style of Eq.(3) for relative entropies to unify the notation with simple descriptions.

2. The convention is often given in the following way with the definition of $D(X||Y)$. If there exists $i$ such that $p(x_i) = 0 = p(y_i)$, then we define $D(X||Y) = +\infty$ (in this case, $D(X||Y)$ is not significant as an information measure any longer). Otherwise, $D(X||Y)$ is defined by Eq.(8) with the convention $0 \cdot \log \frac{0}{0} = 0$. This fact has been mentioned in the abstract of the paper.
2 Hypoentropy and hypodivergence

For nonnegative real numbers $a_i$ and $b_i$ ($i = 1, \ldots, n$), we define the generalized relative entropy (for incomplete probability distributions):

$$D^{(\text{gen})}(a_1, \ldots, a_n || b_1, \ldots, b_n) \equiv \sum_{i=1}^{n} a_i \log \frac{a_i}{b_i}. \tag{9}$$

Then we have the so-called “log-sum” inequality:

$$\sum_{i=1}^{n} a_i \log \frac{a_i}{b_i} \geq \left( \sum_{i=1}^{n} a_i \right) \log \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i}, \tag{10}$$

with equality if and only if $\frac{a_i}{b_i} = \text{const.}$ for all $i = 1, \ldots, n$.

If we impose the condition

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1,$$

then $D^{(\text{gen})}(a_1, \ldots, a_n || b_1, \ldots, b_n)$ is just the relative entropy,

$$D(a_1, \ldots, a_n || b_1, \ldots, b_n) \equiv \sum_{i=1}^{n} a_i \log \frac{a_i}{b_i}. \tag{11}$$

We put $a_i = \frac{1}{\lambda} + p(x_i)$ and $b_i = \frac{1}{\lambda} + p(y_i)$ with $\lambda > 0$ and $\sum_{i=1}^{n} p(x_i) = \sum_{i=1}^{n} p(y_i) = 1, p(x_i) \geq 0, p(y_i) \geq 0$. Then we find that it is equal to the hypodivergence ($\lambda$-divergence) introduced by Ferreri [5],

$$K_{\lambda}(X||Y) \equiv \frac{1}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \log \frac{1 + \lambda p(x_i)}{1 + \lambda p(y_i)}. \tag{12}$$

Clearly we have

$$\lim_{\lambda \to \infty} K_{\lambda}(X||Y) = D(X||Y). \tag{13}$$

Using the “log-sum” inequality, we have the nonnegativity

$$K_{\lambda}(X||Y) \geq 0, \tag{14}$$

with equality if and only if $p(x_i) = p(y_i)$ for all $i = 1, \ldots, n$.

The hypoentropy at the level $\lambda$ ($\lambda$-entropy) was introduced in 1980 by Ferreri [5] as an alternative measure of information in the following form:

$$F_{\lambda}(X) \equiv \frac{1}{\lambda} (\log(\lambda + 1) - 1 + \lambda) \sum_{i=1}^{n} (1 + \lambda p(x_i)) \log(1 + \lambda p(x_i)) \tag{15}$$

for $\lambda > 0$. According to Ferreri [5], the parameter $\lambda$ can be interpreted as a measure of the information inaccuracy of economic forecast. For this quantity $F_{\lambda}(X)$, we have the following fundamental relations.

**Proposition 2.1** For $\lambda > 0$, we have the following inequalities:

$$0 \leq F_{\lambda}(X) \leq F_{\lambda}(U). \tag{16}$$

The equality in the first inequality holds if and only if $p(x_j) = 1$ for some $j$ (then $p(x_i) = 0$ for all $i \neq j$). The equality in the second inequality holds if and only if $p(x_i) = 1/n$ for all $i = 1, \ldots, n$. 

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Proof: From the nonnegativity of the hypodivergence Eq. (14), we get

\[ 0 \leq K_\lambda(X||U) \]
\[ = \frac{1}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \log(1 + \lambda p(x_i)) - \frac{1}{\lambda} (n + \lambda) \log \left(1 + \frac{\lambda}{n}\right). \]

(17)

Thus we have

\[ -\frac{1}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \log(1 + \lambda p(x_i)) \leq -\frac{1}{\lambda} (n + \lambda) \log \left(1 + \frac{\lambda}{n}\right). \]

(19)

Adding \(\frac{1}{\lambda}(\lambda + 1)\log(\lambda + 1)\) to both sides, we have

\[ F_\lambda(X) \leq F_\lambda(U), \]

(20)

with equality if and only if \(p(x_i) = 1/n\) for all \(i = 1, \cdots, n\).

For the first inequality it is sufficient to prove:

\[ (1 + \lambda) \log(1 + \lambda) - \sum_{i=1}^{n} (1 + \lambda p(x_i)) \log(1 + \lambda p(x_i)) \geq 0. \]

(21)

Since \(\sum_{i=1}^{n} p(x_i) = 1\), the above inequality is written as

\[ \sum_{i=1}^{n} \{p(x_i)(1 + \lambda) \log(1 + \lambda) - (1 + \lambda p(x_i)) \log(1 + \lambda p(x_i))\} \geq 0, \]

(22)

so that we have only to prove

\[ p(x_i)(1 + \lambda) \log(1 + \lambda) - (1 + \lambda p(x_i)) \log(1 + \lambda p(x_i)) \geq 0, \]

(23)

for any \(\lambda > 0\) and \(0 \leq p(x_i) \leq 1\). Lemma 2.2 below shows this inequality and the equality condition.

Lemma 2.2 For any \(a > 0\) and \(0 \leq x \leq 1\), we have

\[ x(1 + a) \log(1 + a) \geq (1 + ax) \log(1 + ax). \]

(24)

Proof: We set \(f(x) \equiv x(1 + a) \log(1 + a) - (1 + ax) \log(1 + ax)\). For any \(a > 0\) we then have \(\frac{d^2f(x)}{dx^2} = -\frac{a^2}{(1+ax)^2} < 0\) and \(f(0) = f(1) = 0\). Thus we have the inequality.

It is a known fact that \(F_\lambda(X)\) is monotonically increasing as a function of \(\lambda\) and

\[ \lim_{\lambda \to \infty} F_\lambda(X) = H(X), \]

(25)

whence its name. Thus the hypoentropy appears as a generalization of Shannon’s entropy. One can see that the hypoentropy also equals zero as the entropy does, in the case of certainty (i.e., for a so-called pure state when all probabilities vanish but one).

It also holds that

\[ F_\lambda(U) - F_\lambda(X) = K_\lambda(X||U). \]

(26)
It is of some interest for the reader to look at the hypoentropy which arises for equiprobable states,

$$F_\lambda(U) = \left(1 + \frac{1}{\lambda}\right) \log (1 + \lambda) - \left(1 + \frac{n}{\lambda}\right) \log \left(1 + \frac{\lambda}{n}\right).$$  \hspace{1cm} (27)

Seen as a function of two variables, $n$ and $\lambda$, it increases in each variable \[5\]. Since

$$\lim_{\lambda \to \infty} F_\lambda(U) = \log n,$$  \hspace{1cm} (28)

we shall call it Hartley hypoentropy\[3\]. We have the cross-hypoentropy

$$F_\lambda^{\text{(cross)}}(X, Y) \equiv \left(1 + \frac{1}{\lambda}\right) \log (1 + \lambda) - \frac{1}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \log (1 + \lambda p(y_i)).$$  \hspace{1cm} (29)

It holds

$$K_\lambda(X||Y) = F_\lambda^{\text{(cross)}}(X, Y) - F_\lambda(X) \geq 0,$$  \hspace{1cm} (30)

therefore we have $F_\lambda^{\text{(cross)}}(X, Y) \geq F_\lambda(X)$. This enables us to state the following lemma.

**Lemma 2.3** We have the following inequality

$$- \sum_{i=1}^{n} (1 + \lambda p(x_i)) \log (1 + \lambda p(x_i)) \leq - \sum_{i=1}^{n} (1 + \lambda p(x_i)) \log (1 + \lambda p(y_i))$$  \hspace{1cm} (31)

for all $\lambda > 0$.

As direct consequences we have some interesting inequalities as follows.

**Proposition 2.4** It holds that

$$\left(1 + \frac{\lambda}{n}\right)^{n} \geq \prod_{i=1}^{n} (1 + \lambda p(y_i)),$$

for all $\lambda > 0$.

**Proof**: From Lemma 2.3 for $X = U$ we get

$$- n \log \left(1 + \frac{\lambda}{n}\right) \leq - \sum_{i=1}^{n} \log (1 + \lambda p(y_i))$$  \hspace{1cm} (32)

and the conclusion follows.  \hspace{1cm} \[\blacksquare\]

An upper bound for $F_\lambda(X)$ can be found as follows:

**Proposition 2.5** The following inequality holds.

$$F_\lambda(X) \leq (1 - p_{\text{max}}) \log (1 + \lambda),$$

for all $\lambda > 0$, where $p_{\text{max}} \equiv \max \{p(x_1), \cdots , p(x_n)\}$.

\[3\]Throughout the paper we add the name Hartley to the name of mathematical objects whenever they are considered for the uniform distribution. In the same way we proceed with the name Tsallis which we add to the name of some mathematical objects which we define, to emphasize that they are used in the framework of Tsallis statistics. This means that we will have Tsallis hypoentropies, Tsallis hypodivergences and so on.
Proof: In Lemma 2.3 if for a fixed $k$ one takes the probability of the $k$-th component of $Y$ to be $p(y_k) = 1$, then
\[- \sum_{i=1}^{n} (1 + \lambda p(x_i)) \log (1 + \lambda p(x_i)) \leq - (1 + \lambda p(x_k)) \log (1 + \lambda). \tag{33}\]

This implies that
\[F_\lambda(X) \leq \left(1 + \frac{1}{\lambda}\right) \log (1 + \lambda) - \frac{1}{\lambda} (1 + \lambda p(x_k)) \log (1 + \lambda) \tag{34}\]
\[= (1 - p(x_k)) \log (1 + \lambda). \tag{35}\]

Since $k$ is arbitrarily fixed, the conclusion follows.

Remark 2.6 It is of interest to notice now that, for the particular case $X = U$, we have
\[F_\lambda(U) \leq \left(1 - \frac{1}{n}\right) \log (1 + \lambda). \tag{36}\]

We add here one more detail: the inequality (36) can be verified using Bernoulli’s inequality.

3 Tsallis hypoentropy and hypodivergence

Now we turn our attention to the Tsallis statistics. We extend the definition of hypodivergences as follows:

Definition 3.1 The Tsallis hypodivergence ($q$-hypodivergence, Tsallis relative hypoentropy) is defined by
\[D_{\lambda,q}(X||Y) \equiv - \frac{1}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \ln_q \frac{1 + \lambda p(y_i)}{1 + \lambda p(x_i)} \tag{37}\]
for $\lambda > 0$ and $q \geq 0$.

Then we have the relation:
\[\lim_{\lambda \to \infty} D_{\lambda,q}(X||Y) = S_q(X||Y) \tag{38}\]
which is the Tsallis divergence, and
\[\lim_{q \to 1} D_{\lambda,q}(X||Y) = K_\lambda(X||Y) \tag{39}\]
which is the hypodivergence.

Remark 3.2 This definition can be also obtained from the generalized Tsallis relative entropy (for incomplete probability distributions $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$)
\[D_q^{(gen)}(a_1, \ldots, a_n||b_1, \ldots, b_n) \equiv - \sum_{i=1}^{n} a_i \ln_q \frac{b_i}{a_i}, \tag{40}\]
by putting $a_i = \frac{1}{\lambda} + p(x_i)$ and $b_i = \frac{1}{\lambda} + p(y_i)$ for $\lambda > 0$. 
The generalized relative entropy (9) and the generalized Tsallis relative entropy (40) can be written as the generalized $f$-divergence (for incomplete probability distributions):

$$D_f^{(gen)}(a_1, \ldots, a_n || b_1, \ldots, b_n) \equiv \sum_{i=1}^{n} a_i f \left( \frac{b_i}{a_i} \right)$$

(41)

for a convex function $f$ on $(0, \infty)$ and $a_i \geq 0, b_i \geq 0$ ($i = 1, \ldots, n$).

By the concavity of the $q$-logarithmic function, we have the following “ln$q$-sum” inequality

$$- \sum_{i=1}^{n} a_i \ln_q \frac{b_i}{a_i} \geq - \left( \sum_{i=1}^{n} a_i \right) \ln_q \left( \frac{\sum_{i=1}^{n} b_i}{\sum_{i=1}^{n} a_i} \right),$$

(42)

with equality if and only if $\frac{a_i}{b_i} = \text{const.}$ for all $i = 1, \ldots, n$. Using the “ln$q$-sum” inequality, we have the nonnegativity of the Tsallis hypodivergence:

$$D_{\lambda,q}(X||Y) \geq 0,$$

(43)

with equality if and only if $p(x_i) = p(y_i)$ for all $i = 1, \ldots, n$. (The equality condition comes from the equality condition of the “ln$q$-sum” inequality and the condition $\sum_{i=1}^{n} p(x_i) = \sum_{i=1}^{n} p(y_i) = 1$.)

**Definition 3.3** For $\lambda > 0$ and $q \geq 0$, the Tsallis hypoentropy ($q$-hypoentropy) is defined by

$$H_{\lambda,q}(X) \equiv \frac{h(\lambda, q)}{\lambda} \left\{ -(1 + \lambda) \ln_q \frac{1}{1 + \lambda} + \sum_{i=1}^{n} (1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)} \right\}$$

(44)

where the function $h(\lambda, q) > 0$ satisfies two conditions,

$$\lim_{q \to 1} h(\lambda, q) = 1$$

(45)

and

$$\lim_{\lambda \to \infty} \frac{h(\lambda, q)}{\lambda^{1-q}} = 1.$$  

(46)

These conditions are equivalent to

$$\lim_{q \to 1} H_{\lambda,q}(X) = F_{\lambda}(X) = \text{Hypoentropy}$$

(47)

and, respectively,

$$\lim_{\lambda \to \infty} H_{\lambda,q}(X) = T_q(X) = \text{Tsallis entropy}.$$  

(48)

Some interesting examples are $h(\lambda, q) = \lambda^{1-q}$ and $h(\lambda, q) = (1 + \lambda)^{1-q}$.

**Remark 3.4** It may be remarkable to discuss the Tsallis cross-hypoentropy. The first candidate for the definition of the Tsallis cross-hypoentropy is

$$H_{\lambda,q}^{(cross)}(X, Y) \equiv \frac{h(\lambda, q)}{\lambda} \left\{ -(1 + \lambda) \ln_q \frac{1}{1 + \lambda} - \sum_{i=1}^{n} (1 + \lambda p(x_i))^{q} \ln_q (1 + \lambda p(y_i)) \right\}$$

(49)
which recovers the cross-hypoentropy defined in Eq. (29) in the limit \( q \to 1 \). Then we have

\[
H^{(\text{cross})}_{\lambda,q}(X,Y) - H_{\lambda,q}(X) = \frac{h(\lambda, q)}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i))^q \{ \ln_q (1 + \lambda p(x_i)) - \ln_q (1 + \lambda p(y_i)) \}
\]

\[
= - \frac{h(\lambda, q)}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \ln_q \frac{1 + \lambda p(y_i)}{1 + \lambda p(x_i)}
\]

\[
= h(\lambda, q) D_{\lambda,q}(X||Y) \geq 0.
\]

The last inequality is due to the nonnegativity given in Eq. (30). Since \( \lim_{q \to 1} h(\lambda, q) = 1 \) by the definition of the Tsallis hypoentropy (see Eq. (45)), the above relation recovers the inequality (30) in the limit \( q \to 1 \).

The second candidate for the definition of the Tsallis cross-hypoentropy is

\[
\tilde{H}^{(\text{cross})}_{\lambda,q}(X,Y) = \frac{h(\lambda, q)}{\lambda} \left\{ -(1 + \lambda) \ln_q \frac{1}{1 + \lambda} + \sum_{i=1}^{n} (1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(y_i)} \right\}
\]

(50)

which also recovers the cross-hypoentropy defined in Eq. (29) in the limit \( q \to 1 \). Then we have

\[
\tilde{H}^{(\text{cross})}_{\lambda,q}(X,Y) - H_{\lambda,q}(X) = - \frac{h(\lambda, q)}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \left\{ \ln_q \frac{1}{1 + \lambda p(x_i)} - \ln_q \frac{1}{1 + \lambda p(y_i)} \right\}
\]

\[
= h(\lambda, q) \tilde{D}_{\lambda,q}(X||Y),
\]

where the alternative Tsallis hypodivergence has to be defined by

\[
\tilde{D}_{\lambda,q}(X||Y) \equiv - \frac{1}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \left\{ \ln_q \frac{1}{1 + \lambda p(x_i)} - \ln_q \frac{1}{1 + \lambda p(y_i)} \right\}.
\]

We have \( \tilde{D}_{\lambda,q}(X||Y) \neq D_{\lambda,q}(X||Y) \) and \( \lim_{q \to 1} \tilde{D}_{\lambda,q}(X||Y) = K_{\lambda}(X||Y) \). However, the nonnegativity of \( D_{\lambda,q}(X||Y) \), \( (q \geq 0) \) does not hold in general, as the following counter-examples show. Take \( \lambda = 1, n = 2, p(x_1) = 0.9, p(y_1) = 0.8, q = 0.5 \), then \( D_{\lambda,q}(X||Y) \simeq -0.0137586 \). In addition, take \( \lambda = 1, n = 3, p(x_1) = 0.3, p(x_2) = 0.4, p(y_1) = 0.2, p(y_2) = 0.7 \) and \( q = 1.9 \), then \( D_{\lambda,q}(X||Y) \simeq -0.0195899 \). Therefore we may conclude that Eq. (49) is to be given the preference over Eq. (30).

We turn to show the nonnegativity and maximality for the Tsallis hypoentropy.

**Lemma 3.5** For any \( a > 0, q \geq 0 \) and \( 0 \leq x \leq 1 \), we have

\[
x(1 + a) \ln_q \frac{1}{1 + a} \leq (1 + ax) \ln_q \frac{1}{1 + ax}.
\]

(51)

**Proof:** We set \( g(x) = x(1 + a) \ln_q \frac{1}{1 + a} - (1 + ax) \ln_q \frac{1}{1 + ax} \). For any \( a > 0 \) and \( q \geq 0 \) we then have \( \frac{d^2 g(x)}{dx^2} = qa^2 \left( \frac{1}{1 + ax} \right)^{2-q} \geq 0 \) and \( g(0) = g(1) = 0 \). Thus we have the inequality.

\( \blacksquare \)

**Proposition 3.6** For \( \lambda > 0, q \geq 0 \) and \( h(\lambda, q) > 0 \) satisfying (45) and (46), we have the following inequalities:

\[
0 \leq H_{\lambda,q}(X) \leq H_{\lambda,q}(U).
\]

The inequality in the first inequality holds if and only if \( p(x_j) = 1 \) for some \( j \) (then \( p(x_i) = 0 \) for all \( i \neq j \)). The equality in the second inequality holds if and only if \( p(x_i) = 1/n \) for all \( i = 1, \ldots, n \).

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Proof: In a similar way to the proof of Proposition 21 for the first inequality it is sufficient to prove

$$- \sum_{i=1}^{n} \left\{ p(x_i)(1 + \lambda) \ln_q \frac{1}{1 + \lambda} - (1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)} \right\} \geq 0,$$

(53)

so that we have only to prove

$$p(x_i)(1 + \lambda) \ln_q \frac{1}{1 + \lambda} \leq (1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)},$$

(54)

for any $\lambda > 0$, $q \geq 0$ and $0 \leq p(x_i) \leq 1$. Lemma 3.5 shows this inequality with equality condition.

The second inequality is proven by the use of the nonnegativity of the Tsallis hypodivergence in the following way:

$$0 \leq D_{\lambda,q}(X||U) = -\frac{1}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \ln_q \frac{1 + \frac{1}{n}}{1 + \lambda p(x_i)}$$

(55)

which implies (by the use of the formula, $\ln_q \frac{b}{a} = b^{1-q} \ln_q \frac{1}{a} + \ln_q b$)

$$\frac{1}{\lambda} \sum_{i=1}^{n} (1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)} \leq \frac{n + \lambda}{\lambda} \ln_q \frac{n}{n + \lambda}.$$

(56)

The equality condition of the second inequality follows from the equality condition of the nonnegativity of the Tsallis hypodivergence [13].

We may call

$$H_{\lambda,q}(U) = \frac{h(\lambda, q)}{\lambda} \left\{ -(1 + \lambda) \ln_q \frac{1}{1 + \lambda} + (n + \lambda) \ln_q \frac{1}{1 + \frac{1}{n}} \right\}$$

the Hartley-Tsallis hypoentropy. We study the monotonicity of the Hartley-Tsallis hypoentropy $H_{\lambda,q}(U)$ and the Tsallis hypoentropy $H_{\lambda,q}(X)$.

Lemma 3.7 The function

$$f(x) = (x + 1) \ln_q \frac{x}{x + 1} \quad (x > 0)$$

is monotonically increasing in $x$, for any $q \geq 0$.

Proof: By direct calculations, we have

$$\frac{df(x)}{dx} = \frac{1}{1 - q} \left\{ \left(1 + \frac{1}{x}\right)^{q-1} \left(1 + \frac{1 - q}{x}\right) - 1 \right\}$$

and

$$\frac{d^2 f(x)}{dx^2} = -q x^{-3} \left(1 + \frac{1}{x}\right)^{q-2} \leq 0.$$

Since $\lim_{x \to \infty} \frac{df(x)}{dx} = 0$, we have $\frac{df(x)}{dx} \geq 0$. 

Proposition 3.8 The Hartley-Tsallis hypoentropy

\[ H_{\lambda,q}(U) = \frac{h(\lambda, q)}{\lambda} \left\{ -\left(1 + \frac{1}{\lambda}\right) \ln_q \frac{1}{1 + \lambda} + (n + \lambda) \ln_q \frac{1}{1 + \lambda} \right\} \]

is a monotonically increasing function of \( n \), for any \( \lambda > 0 \) and \( q \geq 0 \).

Proof: Note that

\[ H_{\lambda,q}(U) = h(\lambda, q) \left\{ -\left(1 + \frac{1}{\lambda}\right) \ln_q \frac{1}{1 + \lambda} + \left(1 + \frac{n}{\lambda}\right) \ln_q \frac{1}{1 + \frac{n}{\lambda}} \right\}. \]

Putting \( x = \frac{n}{\lambda} > 0 \) for \( \lambda > 0 \) fixed in Lemma 3.7, we get the function

\[ g(n) = \left(1 + \frac{n}{\lambda}\right) \ln_q \frac{1}{1 + \frac{n}{\lambda}}, \]

which is a monotonically increasing function of \( n \). Thus we have the present proposition.\[ \square \]

Remark 3.9 We have the relation

\[ \lim_{n \to \infty} H_{\lambda,q}(U) = h(\lambda, q) \left\{ -\left(1 + \frac{1}{\lambda}\right) \ln_q \frac{1}{1 + \lambda} - 1 \right\}. \]

We notice from the condition (46) that

\[ \lim_{\lambda \to \infty} \left( \lim_{n \to \infty} H_{\lambda,q}(U) \right) = \lim_{\lambda \to \infty} h(\lambda, q) \cdot \lambda^{1-q} \left\{ -1 - \left(1 + \frac{1}{\lambda}\right) \ln_q \frac{1}{1+\lambda} \right\} \]

\[ = \frac{1}{1-q} \lim_{\lambda \to \infty} \frac{1+q\lambda-(1+\lambda)^q}{\lambda^q} = \begin{cases} 0 & (q = 0) \\ \infty & (0 < q < 1) \\ \frac{1}{q-1} & (q > 1), \end{cases} \]

and conclude that the result is independent of the choice of \( h(\lambda, q) \).

For the limit \( \lambda \to 0 \) we consider two cases.

(1) In the case of \( h(\lambda, q) = \lambda^{1-q} \), we have

\[ \lim_{\lambda \to 0} \left( \lim_{n \to \infty} H_{\lambda,q}(U) \right) = \lim_{\lambda \to 0} \lambda^{1-q} \left\{ -1 - \left(1 + \frac{1}{\lambda}\right) \ln_q \frac{1}{1+\lambda} \right\} \]

\[ = \frac{1}{1-q} \lim_{\lambda \to 0} \frac{1+q\lambda-(1+\lambda)^q}{\lambda^q} = \begin{cases} \infty & (q > 2) \\ 1 & (q = 2) \\ 0 & (0 \leq q < 2), \end{cases} \]

as one obtains using l’Hôpital’s rule.

(2) In the case of \( h(\lambda, q) = (1 + \lambda)^{1-q} \), we have for all \( q \geq 0 \)

\[ \lim_{\lambda \to 0} \left( \lim_{n \to \infty} H_{\lambda,q}(U) \right) = \lim_{\lambda \to 0} (1 + \lambda)^{1-q} \left\{ -1 - \left(1 + \frac{1}{\lambda}\right) \ln_q \frac{1}{1+\lambda} \right\} \]

\[ = \frac{1}{1-q} \lim_{\lambda \to 0} \frac{1+q\lambda-(1+\lambda)^q}{\lambda^q} = \frac{q}{1-q} \lim_{\lambda \to 0} \frac{1-(1+\lambda)^{q-1}}{(1+\lambda)^{q-1}+q(1+\lambda)^{q-2}} = 0. \]

These results mean that our Hartley-Tsallis hypoentropy with \( h(\lambda, q) = \lambda^{1-q} \) or \( (1 + \lambda)^{1-q} \) has the same limits as the Hartley hypoentropy, \( F_{\lambda}(U) \) (see also [5]), in the case \( 0 < q < 1 \).
We study here the monotonicity of \(H_{\lambda,q}(X)\) for \(h(\lambda,q) = (1+\lambda)^{1-q}\). The other case \(h(\lambda,q) = \lambda^{1-q}\) is studied in the next section, see Lemma B.3.

**Proposition 3.10** We assume \(h(\lambda,q) = (1+\lambda)^{1-q}\). Then \(H_{\lambda,q}(X)\) is a monotone increasing function of \(\lambda > 0\) when \(0 \leq q \leq 2\).

**Proof:** Note that
\[
H_{\lambda,q}(X) = \sum_{i=1}^{n} S_{n\lambda,q}(p(x_i)),
\]
where
\[
S_{n\lambda,q}(x) \equiv \frac{(1+\lambda)^{1-q}}{\lambda(1-q)} \{(1+\lambda x)^q - (1+\lambda)^q x + x - 1\}
\]
is defined on \(0 \leq x \leq 1, 0 \leq q \leq 2\) and \(\lambda > 0\). Then we have
\[
\frac{dH_{\lambda,q}(X)}{d\lambda} = \sum_{i=1}^{n} \frac{dS_{n\lambda,q}(p(x_i))}{d\lambda} = \sum_{i=1}^{n} s_{\lambda,q}(p(x_i)),
\]
where
\[
s_{\lambda,q}(x) \equiv \frac{q\lambda(1-x) \left\{1 - (1+\lambda x)^{q-1}\right\} + 1 - x + (1+\lambda)^q - (1+\lambda x)^q}{(1-q)\lambda^2(1+\lambda)^q}
\]
is defined on \(0 \leq x \leq 1, 0 \leq q \leq 2\) and \(\lambda > 0\). By some computations, we have
\[
\frac{d^2s_{\lambda,q}(x)}{dx^2} = -q(1+\lambda x)^{q-3} \left[1 + \lambda \{(x-1)(q-1) + 1\}\right] \leq 0,
\]
since \((x-1)(q-1) + 1 \geq 0\) for \(0 \leq x \leq 1\) and \(0 \leq q \leq 2\). We easily find \(s_{\lambda,q}(0) = s_{\lambda,q}(1) = 0\). Thus we have \(s_{\lambda,q}(x) \geq 0\) for \(0 \leq x \leq 1\), \(0 \leq q \leq 2\) and \(\lambda > 0\). Therefore we have \(\frac{dH_{\lambda,q}(X)}{d\lambda} \geq 0\) for \(0 \leq q \leq 2\) and \(\lambda > 0\).

This result agrees with the known fact that the usual (Ferreri) hypentropy is increasing as a function of \(\lambda\).

Closing this subsection, we give a \(q\)-extended version for Proposition 2.5 and Proposition 2.4.

**Proposition 3.11** Let \(p_{\max} \equiv \max\{p(x_1), \ldots, p(x_n)\}\). Then we have the following inequality.
\[
H_{\lambda,q}(X) \leq \frac{h(\lambda,q)}{\lambda} \{(1+\lambda)^q - (1+\lambda p_{\max})^q\} \ln_q(1+\lambda) \tag{57}
\]
for all \(\lambda > 0\) and \(q \geq 0\).

**Proof:** From the “\(\ln_q\)-sum” inequality, we have \(D_{\lambda,q}(X\|Y) \geq 0\). Since \(\lambda > 0\), we have
\[
-\sum_{i=1}^{n} (1+\lambda p(x_i)) \ln_q \frac{1+\lambda p(y_i)}{1+\lambda p(x_i)} \geq 0 \tag{58}
\]
which is equivalent to
\[
\sum_{i=1}^{n} (1+\lambda p(x_i))^q \{\ln_q(1+\lambda p(x_i)) - \ln_q(1+\lambda p(y_i))\} \geq 0. \tag{59}
\]
Thus we have
\[ - \sum_{i=1}^{n} (1 + \lambda p(x_i))^q \ln_q (1 + \lambda p(x_i)) \leq - \sum_{i=1}^{n} (1 + \lambda p(x_i))^q \ln_q (1 + \lambda p(y_i)), \] (60)
which extends the result of Lemma 2.3. For arbitrarily fixed \( k \), we set \( p(y_k) = 1 \) (and \( p(y_i) = 0 \) for \( i \neq k \)) in the above inequality, then we have
\[ - \sum_{i=1}^{n} (1 + \lambda p(x_i))^q \ln_q (1 + \lambda p(x_i)) \leq -(1 + \lambda p(x_k))^q \ln_q (1 + \lambda). \] (61)
Since \( x^q \ln_q x = -x \ln_q \frac{1}{x} \), we have
\[ \sum_{i=1}^{n} (1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)} \leq -(1 + \lambda p(x_k))^q \ln_q (1 + \lambda). \] (62)
Multiplying both sides by \( \frac{h(\lambda, q)}{\lambda} > 0 \) and then adding
\[ - \frac{h(\lambda, q)}{\lambda} (1 + \lambda) \ln_q \frac{1}{1 + \lambda} = \frac{h(\lambda, q)}{\lambda} (1 + \lambda)^q \ln_q (1 + \lambda) \] (63)
to both sides, we have
\[ H_{\lambda, q}(X) \leq \frac{h(\lambda, q)}{\lambda} \{(1 + \lambda)^q - (1 + \lambda p(x_k))^q\} \ln_q (1 + \lambda). \] (64)

Since \( k \) is arbitrary, we have this proposition.

Letting \( q \to 1 \) in the above proposition, we recover Proposition 2.5.

We give some notations before we state the next proposition. For any \( x, y > 0 \) satisfying \( x^{1-q} + y^{1-q} - 1 > 0 \), we define the \( q \)-product [16] by
\[ x \otimes_q y \equiv (x^{1-q} + y^{1-q} - 1)^{\frac{1}{1-q}}. \]
Then we have \( \lim_{q \to 1} x \otimes_q y = xy \) and \( \ln_q (x \otimes_q y) = \ln_q x + \ln_q y \). We also use the notation \( x^{\otimes_q n} = x \otimes_q \cdots \otimes_q x \) and \( \otimes_q (x_j) = x_1 \otimes_q \cdots \otimes_q x_n \).

**Proposition 3.12**
\[ \left( 1 + \frac{\lambda}{n} \right)^{\otimes_q n} \geq \otimes_q (1 + \lambda p(y_i)) \]
for all \( \lambda > 0 \) and \( 0 \leq q < 1 \).

*Proof:* In the inequality [30], we put \( p(x_i) = \frac{1}{n} \) for all \( i = 1, \cdots, n \). Then we have
\[ n \ln_q \left( 1 + \frac{\lambda}{n} \right) \geq \sum_{i=1}^{n} \ln_q (1 + \lambda p(y_i)), \]
which implies this proposition.

The limit \( q \to 1 \) in the above proposition recovers Proposition 2.4. In addition, it is known that \( \lim_{q \to 1} (1 + \frac{\lambda}{n})^{\otimes_q n} = \exp_q(\lambda) \), where \( \exp_q(x) \) is the inverse function of \( \ln_q(x) \) and defined as \( \exp_q(x) \equiv \{1 + (1-q)x\}^{\frac{1}{1-q}} \) for the case \( 1 + (1-q)x > 0 \).
4 The subadditivities of the Tsallis hypoentropies

Throughout this section we assume \(|X| = n, |Y| = m, |Z| = l\). We define the joint Tsallis hypoentropy at the level \(\lambda\) by

\[
H_{\lambda,q}(X, Y) \equiv \frac{h(\lambda, q)}{\lambda} \left\{ -(1 + \lambda) \ln_q \frac{1}{1 + \lambda} + \sum_{i=1}^{n} \sum_{j=1}^{m} (1 + \lambda p(x_i, y_j)) \ln_q \frac{1}{1 + \lambda p(x_i, y_j)} \right\}.
\]  

(65)

Note that \(H_{\lambda,q}(X, Y) = H_{\lambda,q}(Y, X)\).

For all \(i = 1, \ldots, n\) for which \(p(x_i) \neq 0\), we define the Tsallis hypoentropy of \(Y\) given \(X = x_i\), at the level \(\lambda p(x_i)\), by

\[
H_{\lambda p(x_i),q}(Y|x_i) = \frac{h(\lambda p(x_i), q)}{\lambda p(x_i)} \left\{ -(1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)} + \sum_{j=1}^{m} (1 + \lambda p(x_i)p(y_j|x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)p(y_j|x_i)} \right\}.
\]

(66)

For \(n = 1\), this coincides with the hypoentropy \(H_{\lambda,q}(Y)\). As for the particular case \(m = 1\), we get \(H_{\lambda p(x_i),q}(Y|x_i) = 0\).

**Definition 4.1** The Tsallis conditional hypoentropy at the level \(\lambda\) is defined by

\[
H_{\lambda,q}(Y|X) \equiv \sum_{i=1}^{n} p(x_i)^q H_{\lambda p(x_i),q}(Y|x_i).
\]

(67)

(As a usual convention, the corresponding summand is defined as 0, if \(p(x_i) = 0\).)

Throughout this section we consider the particular function \(h(\lambda, q) = \lambda^{1-q}\) for \(\lambda > 0, q \geq 0\).

**Lemma 4.2** We assume \(h(\lambda, q) = \lambda^{1-q}\). The chain rule for the Tsallis hypoentropy holds:

\[
H_{\lambda,q}(X, Y) = H_{\lambda,q}(X) + H_{\lambda,q}(Y|X).
\]

\[
H_{\lambda,q}(X) = \frac{\lambda^{1-q}}{\lambda} \left\{ -(1 + \lambda) \ln_q \frac{1}{1 + \lambda} + \sum_{i=1}^{n} (1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)} \right\} + \sum_{i=1}^{n} \frac{(\lambda p(x_i))^{1-q}}{\lambda p(x_i)} p(x_i)^q \left\{ -(1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)} + \sum_{j=1}^{m} (1 + \lambda p(x_i, y_j)) \ln_q \frac{1}{1 + \lambda p(x_i, y_j)} \right\} = H_{\lambda,q}(X, Y).
\]
In the limit \( \lambda \to \infty \), the identity \((68)\) becomes \( T_q(X,Y) = T_q(X) + T_q(Y|X) \), where \( T_q(Y|X) \equiv \sum_{i=1}^{n} p(x_i)^q T_q(Y|x_i) = -\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i,y_j)^q \ln p(y_j|x_i) \) is the Tsallis conditional entropy and \( T_q(X,Y) \equiv \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i,y_j) \ln \frac{1}{p(x_i,y_j)} \) is the Tsallis joint entropy (see also \([6\text{ p.3}]\)).

In the limit \( q \to 1 \) in Lemma \([4.2\text{]}\) we also obtain the identity \( F_\lambda(X,Y) = F_\lambda(X) + F_\lambda(Y|X) \), which naturally leads to the definition of \( F_\lambda(Y|X) \) as conditional hypoentropy.

In order to obtain the subadditivity for the Tsallis hypoentropy, we prove the monotonicity of the Tsallis hypoentropy.

**Lemma 4.3** We assume \( h(\lambda,q) = \lambda^{1-q} \). The Tsallis hypoentropy \( H_{\lambda,q}(X) \) is a monotonically increasing function of \( \lambda > 0 \) when \( 0 \leq q \leq 2 \) and a monotonically decreasing function of \( \lambda > 0 \) when \( q \geq 2 \) (or \( q \leq 0 \)).

**Proof:** Note that

\[
H_{\lambda,q}(X) = \sum_{i=1}^{n} L_{\lambda,q}(p(x_i)),
\]

where

\[
L_{\lambda,q}(x) \equiv \frac{(1 + \lambda x)^q - (1 + \lambda)^q x + x - 1}{\lambda^q (1 - q)}
\]

is defined on \( 0 \leq x \leq 1 \) and \( \lambda > 0 \). Then we have

\[
\frac{dH_{\lambda,q}(X)}{d\lambda} = \sum_{i=1}^{n} \frac{dL_{\lambda,q}(p(x_i))}{d\lambda} = \sum_{i=1}^{n} l_{\lambda,q}(p(x_i)),
\]

where

\[
l_{\lambda,q}(x) \equiv \frac{q}{2^q (1 - q)} \left\{ \left( \frac{1}{\lambda} + 1 \right)^{q-1} x - \left( \frac{1}{\lambda} + x \right)^{q-1} - \frac{(x-1)^{q-1}}{\lambda^{q-1}} \right\}
\]

is defined on \( 0 \leq x \leq 1 \) and \( \lambda > 0 \). By elementary computations, we obtain

\[
\frac{d^2 l_{\lambda,q}(x)}{dx^2} = q(q-2)\lambda^{1-q}(1+\lambda x)^{q-3}.
\]

Since we have \( l_{\lambda,q}(0) = l_{\lambda,q}(1) = 0 \), we find that \( l_{\lambda,q}(x) \geq 0 \) for \( 0 \leq x \leq 2 \) and any \( \lambda > 0 \). We also find that \( l_{\lambda,q}(x) \leq 0 \) for \( q \geq 2 \) (or \( q \leq 0 \)) and any \( \lambda > 0 \). Therefore we have \( \frac{dH_{\lambda,q}(X)}{d\lambda} \geq 0 \) when \( 0 \leq q \leq 2 \), and \( \frac{dH_{\lambda,q}(X)}{d\lambda} \leq 0 \) when \( q \geq 2 \) (or \( q \leq 0 \)).

This result also agrees with the known fact that the usual (Ferreri) hypoentropy is increasing as a function of \( \lambda \).

**Theorem 4.4** We assume \( h(\lambda,q) = \lambda^{1-q} \). It holds \( H_{\lambda,q}(Y|X) \leq H_{\lambda,q}(Y) \) for \( 1 \leq q \leq 2 \).

**Proof:** We note that \( L_{\lambda,q}(x) \) is a nonnegative and concave function in \( x \), when \( 0 \leq x \leq 1 \), \( \lambda > 0 \) and \( q \geq 0 \). Here we use the notation for the conditional probability as \( p(y_j|x_i) = \frac{p(x_i,y_j)}{p(x_i)} \) when \( p(x_i) \neq 0 \). By the concavity of \( L_{\lambda,q}(x) \), we have

\[
\sum_{i=1}^{n} p(x_i) L_{\lambda,q}(p(y_j|x_i)) \leq L_{\lambda,q} \left( \sum_{i=1}^{n} p(x_i) p(y_j|x_i) \right)
\]

\[
= L_{\lambda,q} \left( \sum_{i=1}^{n} p(x_i,y_j) \right) = L_{\lambda,q}(p(y_j)).
\]
Summing both sides of the above inequality over \( j \), we have
\[
\sum_{i=1}^{n} p(x_i) \sum_{j=1}^{m} \lambda_{\lambda,q} (p(y_j|x_i)) \leq \sum_{j=1}^{m} \lambda_{\lambda,q}(p(y_j)). \tag{76}
\]
Since \( p(x_i)^q \leq p(x_i) \) for \( 1 \leq q \leq 2 \) and \( \lambda_{\lambda,q}(x) \geq 0 \) for \( 0 \leq x \leq 1 \), \( \lambda > 0 \) and \( q \geq 0 \), we have
\[
p(x_i)^q \sum_{j=1}^{m} \lambda_{\lambda,q} (p(y_j|x_i)) \leq p(x_i) \sum_{j=1}^{m} \lambda_{\lambda,q} (p(y_j|x_i)). \tag{77}
\]
Summing both sides of the above inequality over \( i \), we have
\[
\sum_{i=1}^{n} p(x_i)^q \sum_{j=1}^{m} \lambda_{\lambda,q} (p(y_j|x_i)) \leq \sum_{i=1}^{n} p(x_i) \sum_{j=1}^{m} \lambda_{\lambda,q} (p(y_j|x_i)). \tag{78}
\]
By the two inequalities (76) and (78), we have
\[
\sum_{i=1}^{n} p(x_i)^q \sum_{j=1}^{m} \lambda_{\lambda,q} (p(y_j|x_i)) \leq \sum_{j=1}^{m} \lambda_{\lambda,q}(p(y_j)). \tag{79}
\]
Here we can see that \( \sum_{j=1}^{m} \lambda_{\lambda,q} (p(y_j|x_i)) \) is the Tsallis hypoentropy for fixed \( x_i \) and the Tsallis hypoentropy is a monotonically increasing function of \( \lambda \) in the case \( 1 \leq q \leq 2 \), due to Lemma 4.3. Thus we have
\[
\sum_{j=1}^{m} \lambda_{\lambda,p(x_i),q} (p(y_j|x_i)) \leq \sum_{j=1}^{m} \lambda_{\lambda,q}(p(y_j)). \tag{80}
\]
By the two inequalities (79) and (80), we finally have
\[
\sum_{i=1}^{n} p(x_i)^q \sum_{j=1}^{m} \lambda_{\lambda,p(x_i),q} (p(y_j|x_i)) \leq \sum_{j=1}^{m} \lambda_{\lambda,q}(p(y_j)), \tag{81}
\]
which implies (since \( p(y_j|x_i) = \frac{p(x_i,y_j)}{p(x_i)} \))
\[
\sum_{i=1}^{n} p(x_i)^q \lambda_{\lambda,p(x_i),q} (Y|x_i) \leq \sum_{j=1}^{m} \lambda_{\lambda,q}(p(y_j)), \tag{82}
\]
since we have for all fixed \( x_i \),
\[
\lambda_{\lambda,p(x_i),q} (Y|x_i) = \frac{1}{\lambda^q p(x_i)^q} \sum_{j=1}^{m} \left\{ -p(y_j|x_i) (1 + \lambda p(x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)} \\
+ (1 + \lambda p(x_i)p(y_j|x_i)) \ln_q \frac{1}{1 + \lambda p(x_i)p(y_j|x_i)} \right\} = \sum_{j=1}^{m} \lambda_{\lambda,p(x_i),q}(p(y_j|x_i)).
\]
Therefore we have \( \lambda_{\lambda,q}(Y|X) \leq \lambda_{\lambda,q}(Y) \).
\]
\[\text{Corollary 4.5} \quad \text{We have the following subadditivity for the Tsallis hypoentropies:}
\]
\[
\lambda_{\lambda,q}(X,Y) \leq \lambda_{\lambda,q}(X) + \lambda_{\lambda,q}(Y) \tag{83}
\]
in the case \( h(\lambda, q) = \lambda^{1-q} \) for \( 1 \leq q \leq 2 \).
Proof: The proof is easily done by Lemma 4.2 and Theorem 1.4.

We are now in a position to prove the strong subadditivity for the Tsallis hypoentropies. The strong subadditivity for entropy is one of interesting subjects in entropy theory [13]. For this purpose, we firstly give a chain rule for three random variables \( X, Y \) and \( Z \).

**Lemma 4.6** We assume \( h(\lambda, q) = \lambda^{1-q} \). The following chain rule holds:

\[
H_{\lambda,q}(X,Y,Z) = H_{\lambda,q}(X|Y,Z) + H_{\lambda,q}(Y,Z).
\] 

Proof: The proof can be done following the recipe used in Lemma 4.2.

\[
H_{\lambda,q}(X|Y,Z) = \sum_{j=1}^{l} \sum_{k=1}^{m} p(y_j, z_k) \lambda^{q} \left( \frac{1}{\lambda p(y_j, z_k)} \right)^{q} \left( 1 + \lambda p(y_j, z_k) \right) \ln_{q} \frac{1}{1 + \lambda p(y_j, z_k)} \right) \right.
\]

\[
+ \frac{1}{\lambda} \left\{ - (1 + \lambda) \ln_{q} \frac{1}{1 + \lambda} + \sum_{j=1}^{l} \sum_{k=1}^{m} (1 + \lambda p(y_j, z_k)) \ln_{q} \frac{1}{1 + \lambda p(y_j, z_k)} \right\}
\]

\[
= H_{\lambda,q}(X,Y,Z).
\]

**Theorem 4.7** We assume \( h(\lambda, q) = \lambda^{1-q} \). The strong subadditivity for the Tsallis hypoentropies,

\[
H_{\lambda,q}(X,Y,Z) + H_{\lambda,q}(Z) \leq H_{\lambda,q}(X,Z) + H_{\lambda,q}(Y,Z),
\] 

holds for \( 1 \leq q \leq 2 \).

Proof: This theorem is proven in a similar way as Theorem 4.4. By the concavity of the function \( \ln_{\lambda p(z_k), q}(x) \) in \( x \), we have

\[
\sum_{j=1}^{m} p(y_j|z_k) \ln_{\lambda p(z_k), q}(p(x_i|y_j, z_k)) \leq \ln_{\lambda p(z_k), q} \left( \sum_{j=1}^{m} p(y_j|z_k)p(x_i|y_j, z_k) \right).
\]

Multiplying both sides by \( p(z_k)^q \) and summing over \( i \) and \( k \), we have

\[
\sum_{j=1}^{m} \sum_{k=1}^{l} p(z_k)^q p(y_j|z_k) \ln_{\lambda p(z_k), q}(p(x_i|y_j, z_k)) \leq \sum_{k=1}^{l} \sum_{i=1}^{n} p(z_k)^q \ln_{\lambda p(z_k), q}(p(x_i|y_j, z_k)),
\] 

since \( \sum_{j=1}^{m} p(y_j|z_k)p(x_i|y_j, z_k) = p(x_i|z_k) \). By \( p(y_j|z_k)^q \leq p(y_j|z_k) \) for all \( j, k \) and \( 1 \leq q \leq 2 \), and by the nonnegativity of the function \( \ln_{\lambda p(z_k), q} \), we have

\[
p(y_j|z_k)^q \sum_{i=1}^{n} \ln_{\lambda p(z_k), q}(p(x_i|y_j, z_k)) \leq p(y_j|z_k) \sum_{i=1}^{n} \ln_{\lambda p(z_k), q}(p(x_i|y_j, z_k)).
\]
Multiplying both sides by $p(z_k)^q$ and summing over $j$ and $k$ in the above inequality, we have

$$
\sum_{j=1}^{m} \sum_{k=1}^{l} p(z_k)^q p(y_j|z_k)^q \sum_{i=1}^{n} Ln_{\lambda p(z_k),q}(p(x_i|y_j, z_k)) \leq \sum_{j=1}^{m} \sum_{k=1}^{l} p(z_k)^q p(y_j|z_k) \sum_{i=1}^{n} Ln_{\lambda p(z_k),q}(p(x_i|y_j, z_k)).
$$

From the two inequalities \[86\] and \[87\] we have

$$
\sum_{j=1}^{m} \sum_{k=1}^{l} p(y_j, z_k)^q \sum_{i=1}^{n} Ln_{\lambda p(y_j, z_k),q}(p(x_i|y_j, z_k)) \leq \sum_{k=1}^{l} p(z_k)^q \sum_{i=1}^{n} Ln_{\lambda p(z_k),q}(p(x_i|z_k)),
$$

which implies

$$
\sum_{j=1}^{m} \sum_{k=1}^{l} p(y_j, z_k)^q \sum_{i=1}^{n} Ln_{\lambda p(y_j, z_k),q}(p(x_i|y_j, z_k)) \leq \sum_{k=1}^{l} p(z_k)^q \sum_{i=1}^{n} Ln_{\lambda p(z_k),q}(p(x_i|z_k)),
$$

since $p(y_j, z_k) \leq p(z_k)$ (because of $\sum_{j=1}^{m} p(y_j, z_k) = p(z_k)$) for all $j$ and $k$ and the function $Ln_{\lambda p(z_k),q}$ is monotonically increasing in $\lambda p(z_k) > 0$, when $1 \leq q \leq 2$. Thus we have $H_{\lambda,q}(X|Y, Z) \leq H_{\lambda,q}(X|Z)$ which is equivalent to the inequality

$$
H_{\lambda,q}(X,Y,Z) - H_{\lambda,q}(Y,Z) \leq H_{\lambda,q}(X,Z) - H_{\lambda,q}(Z)
$$

by Lemma 4.2 and Lemma 4.6.

\[\square\]

**Remark 4.8** Passing to the limit $\lambda \to \infty$ in Corollary 4.5 and Theorem 4.7 we recover the subadditivity and the strong subadditivity\[7\] for the Tsallis entropy:

$$
T_q(X,Y) \leq T_q(X) + T_q(Y) \quad (q \geq 1)
$$

and

$$
T_q(X,Y,Z) + T_q(Z) \leq T_q(X,Z) + T_q(Y,Z) \quad (q \geq 1).
$$

Thanks to the subadditivities, we may define the Tsallis mutual hypoentropies for $1 \leq q \leq 2$ and $\lambda > 0$.

**Definition 4.9** Let $1 \leq q \leq 2$ and $\lambda > 0$. The Tsallis mutual hypoentropy is defined by

$$
I_{\lambda,q}(X;Y) \equiv H_{\lambda,q}(X) - H_{\lambda,q}(X|Y)
$$

and the Tsallis conditional mutual hypoentropy is defined by

$$
I_{\lambda,q}(X;Y|Z) \equiv H_{\lambda,q}(X|Z) - H_{\lambda,q}(X|Y,Z).
$$

From the chain rule given in Lemma 4.2 we find that the Tsallis mutual hypoentropy is symmetric, that is,

$$
I_{\lambda,q}(X;Y) \equiv H_{\lambda,q}(X) - H_{\lambda,q}(X|Y) = H_{\lambda,q}(X) + H_{\lambda,q}(Y) - H_{\lambda,q}(X,Y) = H_{\lambda,q}(Y) - H_{\lambda,q}(Y|X) = I_{\lambda,q}(Y;X).
$$

(88)
In addition, we have
\[ 0 \leq I_{\lambda,q}(X;Y) \leq \min \{ H_{\lambda,q}(X), H_{\lambda,q}(Y) \} \] (89)
from the subadditivity given in Theorem 4.4 and nonnegativity of the Tsallis conditional hypentropy. We also find \( I_{\lambda,q}(X;Y|Z) \geq 0 \) from the strong subadditivity given in Theorem 4.7.

Moreover we have the chain rule for the Tsallis mutual hypentropy in the following.
\[
I_{\lambda,q}(X;Y|Z) = H_{\lambda,q}(X|Z) - H_{\lambda,q}(X|Y, Z) = H_{\lambda,q}(X) - I_{\lambda,q}(X;Y|Z).
\] (90)

From the strong subadditivity, we have \( H_{\lambda,q}(X|Y, Z) \leq H_{\lambda,q}(X|Z) \), thus we have
\[
I_{\lambda,q}(X;Z) \leq I_{\lambda,q}(X;Y, Z).
\]
for \( 1 \leq q \leq 2 \) and \( \lambda > 0 \).

## 5 Jeffreys and Jensen-Shannon hypodivergences

In what follows we indicate extensions of two known information measures.

**Definition 5.1** ([4],[11]) The Jeffreys divergence is defined by
\[
J(X||Y) \equiv D(X||Y) + D(Y||X)
\] (91)
and the Jensen-Shannon divergence is defined as
\[
JS(X||Y) \equiv \frac{1}{2} \left\{ D \left( X|| \frac{X + Y}{2} \right) + D \left( Y|| \frac{X + Y}{2} \right) \right\}
\] (92)
\[
= H \left( \frac{X + Y}{2} \right) - \frac{1}{2} \left( H(X) + H(Y) \right). \] (93)

The Jensen-Shannon divergence was introduced in 1991 in [13], but its roots can be older, since one can see some analogous formulae used in thermodynamics under the name entropy of mixing [17, p.598], for the study of gaseous, liquid or crystalline mixtures.

Jeffreys and Jensen-Shannon divergences have been extended to the context of Tsallis theory in [8].

**Definition 5.2** The Jeffreys-Tsallis divergence is
\[
J_q(X||Y) \equiv S_q(X||Y) + S_q(Y||X)
\] (94)
and the Jensen-Shannon-Tsallis divergence is
\[
JS_q(X||Y) \equiv \frac{1}{2} \left\{ S_q \left( X|| \frac{X + Y}{2} \right) + S_q \left( Y|| \frac{X + Y}{2} \right) \right\}.
\] (95)

Note that
\[
JS_q(X||Y) \neq T_q \left( \frac{X + Y}{2} \right) - \frac{1}{2} \left( T_q(X) + T_q(Y) \right).
\]
This expression was used in [11] as Jensen-Tsallis divergence.

In accordance with the above definition, we define the directed Jeffreys and Jensen-Shannon \( q \)- hypodivergence measures between two distributions and emphasize the mathematical significance of our definitions.
Definition 5.3 The Jeffreys-Tsallis hypodivergence is

\[ J_{\lambda,q}(X||Y) \equiv D_{\lambda,q}(X||Y) + D_{\lambda,q}(Y||X) \] (96)

and the Jensen-Shannon-Tsallis hypodivergence is

\[ J_{S,\lambda,q}(X||Y) \equiv \frac{1}{2} \left\{ D_{\lambda,q} \left( X || \frac{X + Y}{2} \right) + D_{\lambda,q} \left( Y || \frac{X + Y}{2} \right) \right\}. \] (97)

Here we point out that again one has

\[ J_{S,\lambda}(X||Y) = \frac{1}{2} K_{\lambda} \left( X || \frac{X + Y}{2} \right) + \frac{1}{2} K_{\lambda} \left( Y || \frac{X + Y}{2} \right) \] (98)

\[ = F_{\lambda} \left( \frac{X + Y}{2} \right) - \frac{1}{2} \left( F_{\lambda}(X) + F_{\lambda}(Y) \right), \] (99)

where

\[ J_{S,\lambda}(X||Y) = \lim_{q \to 1} J_{S,\lambda,q}(X||Y). \]

Lemma 5.4 The following inequality holds:

\[ D_{\lambda,q} \left( X || \frac{X + Y}{2} \right) \leq \frac{1}{2} D_{\lambda,\frac{1+q}{2}}(X||Y) \]

for \( q \geq 0 \) and \( \lambda > 0 \).

Proof: Using the inequality between the arithmetic and geometric mean, one has

\[ D_{\lambda,q} \left( X || \frac{X + Y}{2} \right) = -\frac{1}{\lambda} \sum_{i=1}^{n} \left( 1 + \lambda p(x_i) \right) \ln_q \left( \frac{1+\lambda p(x_i)+1+\lambda p(y_i)}{2} \right) \] (100)

\[ \leq -\frac{1}{\lambda} \sum_{i=1}^{n} \left( 1 + \lambda p(x_i) \right) \ln_q \left( \frac{1+\lambda p(y_i)}{1+\lambda p(x_i)} \right) \] (101)

\[ = -\frac{1}{2\lambda} \sum_{i=1}^{n} \left( 1 + \lambda p(x_i) \right) \left( \frac{1+\lambda p(y_i)}{1+\lambda p(x_i)} \right)^{1-\frac{1+q}{2}} - 1 \] (102)

\[ = \frac{1}{2} D_{\lambda,\frac{1+q}{2}}(X||Y). \] (103)

Thus the proof is completed.

In the limit \( \lambda \to \infty \), Lemma 5.4 recovers Lemma 3.4 in [8].

Lemma 5.5 ([8]) The function

\[ f(x) = -\ln_q \frac{1+\exp_q x}{2} \]

is concave for \( 0 \leq r \leq q \).

The next two results of the present paper are stated in order to establish the counterpart of Theorem 3.5 in [8] for hypodivergences.
Proposition 5.6 It holds
\[ JS_{\lambda,q}(X||Y) \leq \frac{1}{4} J_{\lambda}^{1+q}(X||Y) \]
for \( q \geq 0 \) and \( \lambda > 0 \).

Proof: By the use of Lemma 5.4, one has
\[
2JS_{\lambda,q}(X||Y) = D_{\lambda,q} \left( \frac{X+Y}{2} \right) + D_{\lambda,q} \left( \frac{Y}{2} \right) \leq \frac{1}{2} D_{\lambda,\frac{1+q}{2}}(X||Y) + \frac{1}{2} D_{\lambda,\frac{1+q}{2}}(Y||X) = \frac{1}{2} J_{\lambda,\frac{1+q}{2}}(X||Y).
\]
This completes the proof.

Proposition 5.7 It holds that
\[ JS_{\lambda,r}(X||Y) \leq - \frac{n + \lambda}{\lambda} \ln r + \exp_q \left( -\frac{1}{2} \cdot \frac{\lambda}{n+\lambda} \cdot J_{\lambda,q}(X||Y) \right) \]
for \( 0 \leq r \leq q \) and \( \lambda > 0 \).

Proof: According to Lemma 5.5
\[
JS_{\lambda,r}(X||Y) = - \frac{n + \lambda}{\lambda} \left\{ \sum_{i=1}^{n} \ln r + \exp_q \left( -\frac{1}{2} \cdot \frac{\lambda}{n+\lambda} \cdot D_{\lambda,q}(X||Y) \right) \right\}
\]
Then
\[
JS_{\lambda,r}(X||Y) \leq - \frac{n + \lambda}{\lambda} \ln r + \exp_q \left( -\frac{1}{2} \cdot \frac{\lambda}{n+\lambda} \cdot D_{\lambda,q}(Y||X) \right)
\]
Thus the proof is completed.

We further define the dual symmetric hypodivergences.

Definition 5.8 The dual symmetric Jeffreys-Tsallis hypodivergence is defined by
\[ J_{\lambda,q}^{(ds)}(X||Y) \equiv D_{\lambda,q}(X||Y) + D_{\lambda,2-q}(Y||X) \]
and the dual symmetric Jensen-Shannon-Tsallis hypodivergence is defined by
\[ JS_{\lambda,q}^{(ds)}(X||Y) \equiv \frac{1}{2} \left\{ D_{\lambda,q} \left( X||\frac{X+Y}{2} \right) + D_{\lambda,2-q} \left( Y||\frac{X+Y}{2} \right) \right\} \].

20
Using Lemma [5.4] we have the following inequality.

**Proposition 5.9** It holds

\[ JS^{(ds)}_{\lambda,q}(X||Y) \leq \frac{1}{4} J^{(ds)}_{\lambda} (X||Y) \]

for \(0 \leq q \leq 2\) and \(\lambda > 0\).

In addition, we have the following inequality.

**Proposition 5.10** It holds

\[ JS^{(ds)}_{\lambda,q}(X||Y) \leq -\frac{n + \lambda}{\lambda} \ln_q \left( 1 + \exp_q \left( -\frac{\lambda}{2(n+\lambda)} J_{\lambda,q}(X||Y) \right) \right) \]

for \(1 < r \leq 2, r \leq q \) and \(\lambda > 0\).

**Proof:** The proof can be done by similar calculations with Proposition [5.7] applying the facts (see Lemma 3.9 and 3.10 in [8]) that \(\exp_q(x)\) is a monotonically increasing function in \(q\) for \(x \geq 0\), and the inequality \(-\ln_{2-r} x \leq -\ln_r x\) holds for \(1 < r \leq 2\) and \(x > 0\).

\[ \blacksquare \]

### 6 Concluding remarks

In this paper, we introduced the Tsallis hypoentropy \(H_{\lambda,q}(X)\) and studied some properties of \(H_{\lambda,q}(X)\). We named \(H_{\lambda,q}(X)\) Tsallis hypoentropy because of the relation \(H_{\lambda,q}(X) \leq T_{q}(X)\) which follows from the monotonicity given in Proposition [3.10] and Lemma [4.3] for the case \(h(\lambda, q) = (1 + \lambda)^{1-q}\) and the case \(h(\lambda, q) = \lambda^{1-q}\), respectively (this relation can be also proven directly). In this naming we follow Ferreri, as he has termed \(F_{\lambda}(X)\) hypoentropy due to the relation \(F_{\lambda}(X) \leq H(X)\).

The monotonicity of the hypoentropy and the Tsallis hypoentropy for \(\lambda > 0\), indeed, is an interesting feature. It may be remarkable to examine the monotonicity of the Tsallis entropy for the parameter \(q \geq 0\). We find that the Tsallis entropy \(T_{q}(X)\) is monotonically decreasing with respect to \(q \geq 0\). Indeed, we find \(\frac{dT_{q}(X)}{dq} = \sum_{j=1}^{n} \frac{p_{i} v_{q}(p_{i})}{(1-q)^{2}}\), where \(v_{q}(x) \equiv 1 - x^{1-q} + (1-q) \log x\) \((0 \leq x \leq 1)\). Since \(x^{q} v_{q}(x) = 0\) for \(x = 0\) and \(q > 0\), we prove \(v_{q}(x) \leq 0\) for \(0 < x \leq 1\). We find \(\frac{dv_{q}(x)}{dx} = \frac{1-q(1-x^{1-q})}{x} \geq 0\) when \(0 < x \leq 1\), thus we have \(v_{q}(x) \leq v_{q}(1) = 0\) which implies \(\frac{dT_{q}(X)}{dq} \leq 0\). This monotonicity implies the relations \(H(X) \leq T_{q}(X)\) for \(0 \leq q < 1\) and \(T_{q}(X) \leq H(X)\) for \(q > 1\). (These relations are also proven by the inequalities \(\log \frac{1}{x} \leq \ln_{q} \frac{1}{x}\) for \(0 \leq q < 1, x > 0\) and \(\log \frac{1}{x} \geq \ln_{q} \frac{1}{x}\) for \(q > 1, x > 0\).)

As another important results, we also gave the chain rules, subadditivity and the strong subadditivity of the Tsallis hypoentropies in the case of \(h(\lambda, q) = \lambda^{1-q}\). For the case of \(h(\lambda, q) = (1 + \lambda)^{1-q}\), we can prove \(H_{\lambda,q}(Y|X) \leq H_{\lambda,q}(X)\) and \(H_{\lambda,q}(X|Y, Z) \leq H_{\lambda,q}(X|Z)\) for \(1 \leq q \leq 2\) in a similar way to the proofs of Theorem [4.3] and [4.7] since the function \(S_{n,\lambda,q}(x)\) defined in the proof of Proposition [5.10] is also nonnegative, monotone increasing and concave in \(x \in [0,1]\) and we have \(H_{\lambda,\rho(x)}(Y|X) = \sum_{j=1}^{n} S_{n,\lambda,\rho(x)}(p(y_{j}|x_{j}))\) for all fixed \(x_{i}\). However we cannot obtain the inequalities

\[ H_{\lambda,q}(X,Y) \leq H_{\lambda,q}(X) + H_{\lambda,q}(Y) \] \((1 \leq q \leq 2)\)

\[ H_{\lambda,q}(X,Y,Z) + H_{\lambda,q}(Z) \leq H_{\lambda,q}(X,Z) + H_{\lambda,q}(Y,Z) \] \((1 \leq q \leq 2)\)

for \(h(\lambda, q) = (1 + \lambda)^{1-q}\), because the similar proof for the chain rules does not work well in the case \(h(\lambda, q) = (1 + \lambda)^{1-q}\).
Acknowledgement

The author (S.F.) was partially supported by JSPS KAKENHI Grant Number 24540146.

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