Three-neutrino model analysis of the world’s oscillation data

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A model of neutrino oscillation experiments is constructed. The experiments incorporated are: solar neutrinos (Chlorine, Gallium, Super-K, and SNO), reactor neutrinos (Bugey and CHOOZ), beam stop neutrinos (LSND decay at rest and decay in flight), and atmospheric neutrinos. Utilizing this model and the standard three-neutrino mixing extension of the standard model, the data are analyzed. Solutions for the mixing angles and mass-squared differences are found to occur in pairs corresponding to the interchange $\Delta m^2_{12} \leftrightarrow \Delta m^2_{23}$. Two pairs of solutions are found that reasonably reproduce the data, including the LSND data. These solutions are $\theta_{12} \approx 0.5$, $\theta_{13} \approx 0.1$, $\theta_{23} \approx 0.7$, $\Delta m^2_{12} \approx 5 \times 10^{-5}$ eV$^2$ and $\Delta m^2_{23} \approx 0.2$ eV$^2$ or 2.4 eV$^2$. Other statistically significant solutions are also found which produce negligible oscillations for the LSND experiments.

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Evidence for neutrino oscillations arises from solar neutrino experiments [1, 2, 3, 4, 5, 6, 7, 8], neutrinos emitted from reactors [9, 10, 11, 12], neutrinos from the beam stop of an accelerator [13, 14], and neutrinos originating from cosmic rays impinging on the atmosphere [15, 16, 17, 18]. We here model what we believe to be the essential physics of each of these experiments. We analyze this model utilizing the standard three neutrino mixing extension of the standard model. We find a number of sets of parameters, mixing angles and mass-squared differences, which reproduce the data, some of which reproduce the entire data set including the LSND experiments. Previous examinations of three neutrino mixing have either excluded the LSND experiments [19, 20], limited the mass-squared differences [21], or used approximations and constraints in order to work analytically [22, 23, 24, 25].

Neutrino oscillations arise because neutrinos are created in flavor eigenstates, and the flavor eigenstates are not equal to the mass eigenstates. The flavor eigenstates (labeled by $\alpha$) are related to the mass eigenstates (labeled by $k$) through a unitary matrix $U_{\alpha k}$,

$$U_{\alpha k} \rightarrow \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\ s_{12}c_{23} - c_{12}s_{23}s_{13} & -c_{12}s_{23} - s_{12}c_{23}s_{13} & c_{23}c_{13} \end{pmatrix},$$

(1)

where $c_{\alpha k} = \cos \theta_{\alpha k}$, $s_{\alpha k} = \sin \theta_{\alpha k}$, and $\theta_{\alpha k}$ is real. We assume no CP violation. We order the mass eigenstates by decreasing mass, and the flavor eigenstates are ordered electron, mu, tau. The probability that a neutrino of flavor $\alpha$ will be detected a distance $L$ away as a neutrino of flavor $\beta$ is given by

$$P_{\alpha \rightarrow \beta}(L/E) = \delta_{\alpha \beta} - 4 \sum_{j,k=1}^{3} U_{\alpha j}U_{\beta j}U_{\alpha k}U_{\beta k} \sin^2 \phi_{jk}^{osc},$$

(2)

with $\phi_{jk}^{osc} = 1.27 \Delta m^2_{jk} L/E$, where the units of $L/E$ are m/MeV, and $\Delta m^2_{jk} \equiv m^2_j - m^2_k$ in units of eV$^2$.

For the neutrino flux emitted by the sun we use the standard solar model [26]. A number of detectors have measured the flux of solar neutrinos arriving here on Earth. Each has a different acceptance and thus measures different energy neutrinos. Each measures a deficit as compared to the flux predicted by the standard solar model. In order to reproduce the energy dependence of the survival rate of electron neutrinos arriving at the Earth as seen in the experiments, we must invoke the MSW effect [27, 28]. The MSW effect arises because the neutrinos created in the sun propagate through a medium with a significant electron density. The forward coherent elastic neutrino-electron scattering produces an effective change, relative to the mu and tau neutrino, in the mass of the electron neutrino given by $A(r) = \sqrt{2} G E \rho(r)/m_n$, with $\rho(r)$ the electron density at a radius $r$, $G$ the weak coupling constant, and $m_n$ the nucleon mass. In the flavor basis, the Hamiltonian then becomes

$$H_{mat} = U M U^\dagger + A,$$

(3)

with $M$ the (diagonal) mass-squared matrix in the mass eigenstate basis and $A$ the $3 \times 3$ matrix with the interac-
tion $A(r)$ as the electron-electron matrix element and zeroes elsewhere. By diagonalizing this Hamiltonian with a unitary transformation $D_{jk}(r, E)$, we define local masses and eigenstates as a function of $r$ and $E$. Care must be taken so that $D_{jk}(r, E)$ becomes $U_{jk}$ in the limit of zero electron density. In the adiabatic limit, which we use, the electron survival probability is

$$P_{ee}^{ad}(r, E) = \sum_{k=1}^{3} D(r, E)_{ek} U_{ek}^2. \quad (4)$$

Neutrinos are produced throughout the sun by various reactions, each with its own energy spectrum. The surviving neutrinos are then detected by detectors which have a different acceptance for each energy of the neutrino. We model this by taking the survival probability for an electron neutrino in a particular experiment to be given by

$$P_{ee}^{ex} = \sum_{j=1}^{N} p_{ej}^{ex} \int_{0}^{E_{\odot}} f_{j}(r) dr \int_{E_{\text{thresh}}}^{\infty} g_{j}(E) P_{ee}^{ad}(r, E) dE. \quad (5)$$

Here, $j$ labels a particular nuclear reaction; we include three reactions – pp, $^7$Be, and $^8$B. The quantity $p_{ej}^{ex}$ is the probability that in a particular experiment the neutrino arose from nuclear reaction $j$. We take these from the analysis of Ref. 24 for the solar experiments: chlorine $^1$, gallium (GALLEX/GNO,Sage) $^2$, $^{11}$B, Super-K $^3$, and SNO $^4$. The function $f_{j}(r)$ is the probability that a neutrino is created by reaction $j$ at a radius $r$ $^20$ of the sun and is integrated from the center of the sun to the solar radius $R_{\odot}$. The function $g_{j}(E)$ is the energy distribution of the neutrinos emitted in reaction $j$. For $^7$Be this is a delta function at 0.88 MeV. The lower emission line does not contribute significantly. For the pp neutrinos, the energy distribution times the detector acceptance is a relatively narrow function of energy; we set $E$ to its average. For $^8$B neutrinos, we use the energy distribution from the standard solar model $^22$ and numerically perform the integration.

We check our treatment of solar neutrinos by performing a two neutrino analysis of only the solar neutrino experiments. We find a minimum for $\tan^2 \theta = 0.43$ with bounds $0.28 \leq \tan^2 \theta \leq 0.68$ and a minimum for $\Delta m^2 = 4.0 \times 10^{-5} \text{eV}^2$ with bounds $1.0 \times 10^{-5} \text{eV}^2 \leq \Delta m^2 \leq 8.6 \times 10^{-5} \text{eV}^2$. The analysis from Ref. 30 gives $\tan^2 \theta = 0.44$ with bounds $0.36 \leq \tan^2 \theta \leq 0.58$ and a minimum for $\Delta m^2 = 7.0 \times 10^{-5} \text{eV}^2$ with bounds $5.9 \times 10^{-5} \text{eV}^2 \leq \Delta m^2 \leq 9.7 \times 10^{-5} \text{eV}^2$. This is satisfactory for our goal of locating possible solutions that are semi-quantitatively correct. Our errors are necessarily larger than in a thorough and model-independent analysis as we do not include all of the data, such as the measured neutrino energy spectra. The experimental data for the solar experiments which we fit are given in Table I. We take values from $^29$ which differ slightly from the original analysis.

The reactor experiments that we include are Bugey $^3$, CHOOZ $^{11,11}$ and KamLAND $^{12}$. As there is an energy distribution for the neutrinos emitted from a reactor, the electron survival probability given by Eq. 2 must be integrated over this spectrum. For small values of $L/E$, the coherent limit, Eq. 2 remains correct. For sufficiently large values of $L/E$, the incoherent limit, the $\sin^2 \phi_{\text{osc}}$ term averages to $1/2$. The transition between these regions depends on the details of the energy distribution of the source neutrinos. We simplify this by using an average value for $L/E$ and by using $\sin^2 \phi_{\text{osc}}$ for $\phi_{\text{osc}} < \pi/4$ and setting $\sin^2 \phi_{\text{osc}} = 1/2$ for $\phi_{\text{osc}} > \pi/4$. KamLAND is unique among these as it measures $P_{ee}$ where the others set limits.

The LSND experiments use neutrinos produced from muons created in the LAMPF beam stop. There are two experiments. The decay at rest experiment $^{13}$ measures the oscillation of an muon antineutrino into an electron antineutrino, while the decay in flight experiment $^{14}$ measures the oscillation of a muon neutrino into an electron neutrino. These experiments are unique in that they measure the appearance of a different flavor neutrino rather than the disappearance of electron neutrinos. We treat the $\sin^2 \phi_{\text{osc}}$ in Eq. 2 just as we did for the reactor neutrinos.

The Super-Kamiokande experiment $^{12,16,17,18}$ has

| Experiment | Measured $L/E$ (m/MeV) | Data |
|------------|------------------------|------|
| LSND-DAR   | $P_{ee}^{ex}$          | .73  | $3.1 \pm 1.3 \times 10^{-4}$ |
| LSND-DIF   | $P_{ee}^{ex}$          | .40  | $2.6 \pm 1.1 \times 10^{-4}$ |
| CHOOZ      | $P_{ee}$               | .300 | $> 0.96$ |
| Bugey      | $P_{ee}$               | 10.3 | $> 0.96$ |
| KamLAND    | $P_{ee}$               | $4.1 \times 10^4$ | $611 \pm 594$ |
| Super-K    | $P_{ee}$               | $2.2 \times 10^6$ | $465 \pm 594$ |
| SNO        | $P_{ee}$               | $2.2 \times 10^8$ | $348 \pm 573$ |
| Chlorine   | $P_{ee}$               | $4.0 \times 10^{10}$ | $337 \pm 565$ |
| Gallium    | $P_{ee}$               | $35. \times 10^{10}$ | $550 \pm 0.48$ |

TABLE I: The experiments, quantity measured, the average value of $L/E$, and experimental data for those quantities included in the model are presented. For the atmospheric data, the quantity $\mathcal{R}$ is defined in the text, and the upper (lower) value of $L/E$ is for upward (downward) going neutrinos. Note that Super-K plus SNO combined provide a measurement of $P_{ee}$ and $P_{ee} + P_{en} + P_{er}$. 


also measured neutrinos that originate from cosmic rays hitting the upper atmosphere. The detector distinguishes between $e$-like (electron and anti-electron) neutrinos and $\mu$-like (muon and anti-muon) neutrinos. The rate of $e$-like neutrinos of energy $E$ arriving at the detector from a source a distance $L$ away is

$$R_e(L, E) = \mathcal{P}_{ee}(L, E) + n(E)\mathcal{P}_{ep}(L, E),$$

(6)

and for $\mu$-like neutrinos

$$R_\mu(L, E) = \mathcal{P}_{\mu\mu}(L, E) + \frac{1}{n(E)}\mathcal{P}_{ep}(L, E),$$

(7)

where $n(E)$ is the ratio of $\mu$-like neutrinos to $e$-like neutrinos at the source. We separate the data into neutrinos with an energy less than or greater than 1 GeV. An average value for the energy is calculated from results given in Ref. [16, 17] which uses the model of the neutrino fluxes from Ref. [31]. We approximate the energy distribution of these neutrinos by fitting a simple “teepee” shaped function to the distributions given in Ref. [16, 17]. This allows us to do the energy integral analytically. The energy averaged values of $R_e$ we call $R_\alpha(r, \langle E \rangle)$. The high-energy $\mu$-like events are classified as “fully contained” or “partially contained” events and each of these arises from a different energy spectrum. We combined these as 0.24 fully contained and 0.76 partially contained [18].

To remove the dependence on the overall normalization of neutrino flux, the ratio of measured fluxes for upward going (coming from the far side of the earth) neutrinos to downward going (coming from overhead) neutrinos is taken. This ratio is

$$\mathcal{R}_\alpha = \frac{R_\alpha(r_{up}, \langle E \rangle)}{R_\alpha(r_{down}, \langle E \rangle)}. $$

We utilize a definition of upward/downward going neutrinos as those with the scattered lepton direction in the detector of no more than $\pi/5$ radians off-axis. The downward-going neutrinos were assumed to travel 15 km from the top of the atmosphere, whereas the upward ones travel one earth diameter, 13,000 km. The experiment measures the neutrino fluxes as a function of the azimuthal angle. We utilize only the endpoints of these measurements. Assuming that if we fit the endpoints, the curve between would equally well be fit, we divide the error associated with the endpoint by $\sqrt{5/2}$ (5/2 = the number of experimental points over the number used) to more properly weight this experiment with respect to the others. The data and the parameters used for all the experiments are given in Table II.

We fit the mixing angles and the mass-squared differences to the data by minimizing chi-squared per degree of freedom $\chi^2_{dof}$. In Table II, we present the parameters for the ten best fits. We have also found local minima for $\chi^2_{dof}$ near 2.3 and 2.7. Notice that the solutions come in pairs, (solutions 1 and 4, 2 and 3, 5 and 6, 7 and 8, 9 and 10) corresponding to the interchange $\Delta m^2_{12} \leftrightarrow \Delta m^2_{23}$ and an appropriate redefinition of the mixing angles. We derive this symmetry elsewhere [32].

The experimental data and the theoretical results for these fits are presented in Table III. Note that solutions 1 through 4 produce non-negligible and reasonable results for the LSND experiments while fitting the remainder of the experiments. We can see how this comes about by first considering the solar experiments. We find a mass-squared difference on the order of $10^{-5}$ eV$^2$, which is the magnitude needed for the MSW effect to produce the energy dependence of the solar neutrinos. Quantitatively, this is different from the $2 \times 2$ case as the MSW effect is producing an energy dependence for a case where $P_{ee}(\infty)$ is about 0.6. In fact, we find that KamLAND is measuring $P_{ee}(\infty)$.

Considering LSND and the reactor experiments Bugey and CHOOZ, we note that the term in Eq. 2 with $\Delta m^2 \approx 10^{-5}$ eV$^2$ does not contribute. The coefficients of $\sin^2 \phi_{\text{osc}}$ for $P_{ep}$ for the large mass-squared difference terms are each of the order $10^{-2}$. They are of opposite sign and equal to each other to about ten percent. This produces the LSND results for $P_{ee}$ of order $10^{-3}$. For $P_{ee}$ these coefficients are of the same sign but are individually of the order of $10^{-3}$ and thus do not give results which contradict Bugey or CHOOZ. The coefficient of the $\sin^2 \phi_{\text{osc}}$ term corresponding to the mass-squared difference of $10^{-5}$ eV$^2$ term is about 0.25. This term then contributes significantly to the asymptotic value of $P_{ee}$ which is important for fitting KamLAND and the solar experiments. For solutions 1 and 4, $\phi_{\text{osc}}$ for the large mass-squared difference terms for LSND experiments are in the coherent region, while for solutions 2 and 3 they are near $\pi/2$.

Finally we look at the atmospheric data. We find that for solutions 2 and 3, the solutions with the largest mass-squared difference, all the values of $\phi_{\text{osc}}$ are greater than $\pi/2$. This does not give a perfect fit to the atmospheric data, but it is sufficiently close that when combined with an excellent fit to the LSND data, a good $\chi^2_{dof}$ results.

| fit | $\chi^2_{dof}$ | $\theta_{12}$ | $\theta_{13}$ | $\theta_{23}$ | $\Delta m^2_{12}$ (eV$^2$) | $\Delta m^2_{23}$ (eV$^2$) |
|-----|---------------|---------------|---------------|---------------|----------------|----------------|
| 1   | 1.4           | .55           | .16           | .79           | 5.6 x 10$^{-7}$ | .24            |
| 2   | 1.5           | .58           | .06           | .66           | 4.3 x 10$^{-19}$ | 2.3            |
| 3   | 1.6           | .59           | .99           | 2.3           | 4.6 x 10$^{-7}$ | .24            |
| 4   | 1.7           | .58           | .99           | 2.4           | 5.0 x 10$^{-7}$ | .12            |
| 5   | 1.9           | .58           | .10           | .78           | 4.2 x 10$^{-5}$ | .12            |
| 6   | 1.9           | .58           | .10           | .78           | 4.2 x 10$^{-5}$ | .12            |
| 7   | 2.0           | .63           | .001          | .37           | 1.4 x 10$^{-5}$ | 1.2 x 10$^{-1}$ |
| 8   | 2.0           | $\pi/2$       | .63           | .12           | 1.4 x 10$^{-5}$ | 1.4 x 10$^{-5}$ |
| 9   | 2.0           | .63           | .05           | .64           | 1.4 x 10$^{-5}$ | .11            |
| 10  | 2.0           | $\pi/2$       | .63           | .96           | 1.4 x 10$^{-5}$ | 1.4 x 10$^{-5}$ |

TABLE II: The value of $\chi^2_{dof}$ and the fit parameters for the ten best fits.
For solutions 1 and 4, the large mass-squared difference is smaller in order to better fit the atmospheric data, but at a cost to the fit of LSND.

We also note that our solution 7 corresponds to that found in Ref. [20], in which LSND was not included. This helps give us confidence that our simplified model of the experiments is capable of locating possible solutions.

We have found a set of mixing angles and mass-squared differences that, within a model, can produce results that fit the world’s data, including the LSND experiments. The future requires a thorough and model-independent analysis to see whether this is actually so. We have also found a new symmetry that arises from the interchange of the mass-squared differences.

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| Experiment | Data | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|------------|------|---|---|---|---|---|---|---|---|---|----|
| LSND-DAR ($\times 10^{-3}$) | $3.1 \pm 1.3$ | $2.4$ | $2.8$ | $2.7$ | $1.5$ | $2.4$ | $1.3$ | $0.0$ | $0.0$ | $0.4$ | $0.1$ |
| LSND-DIF ($\times 10^{-3}$) | $2.6 \pm 1.1$ | $0.74$ | $2.8$ | $2.7$ | $0.45$ | $0.07$ | $0.04$ | $0.00$ | $0.00$ | $0.01$ | $0.00$ |
| Chooz | $> 1.96$ | $0.95$ | $0.99$ | $0.99$ | $0.96$ | $0.98$ | $0.99$ | $1.0$ | $1.0$ | $0.99$ | $1.0$ |
| Bugsey | $> 1.95$ | $0.95$ | $0.99$ | $0.99$ | $0.96$ | $0.98$ | $0.99$ | $1.0$ | $1.0$ | $0.99$ | $1.0$ |
| KamLAND | $0.81 \pm 0.094$ | $0.58$ | $0.57$ | $0.57$ | $0.57$ | $0.57$ | $0.60$ | $0.60$ | $0.59$ | $0.59$ |
| Super-K | $0.48 \pm 0.094$ | $0.33$ | $0.34$ | $0.34$ | $0.34$ | $0.34$ | $0.35$ | $0.35$ | $0.35$ | $0.35$ |
| SNO | $0.348 \pm 0.075$ | $0.33$ | $0.34$ | $0.34$ | $0.34$ | $0.34$ | $0.35$ | $0.35$ | $0.35$ | $0.35$ |
| Chlorine | $0.337 \pm 0.085$ | $0.39$ | $0.39$ | $0.39$ | $0.38$ | $0.38$ | $0.37$ | $0.37$ | $0.37$ | $0.37$ |
| Gallium | $0.550 \pm 0.048$ | $0.54$ | $0.54$ | $0.54$ | $0.52$ | $0.53$ | $0.49$ | $0.49$ | $0.49$ | $0.49$ |
| $R^e_{\mu\tau}$ | $1.13 \pm 0.09$ | $1.17$ | $1.27$ | $1.26$ | $1.15$ | $1.12$ | $1.12$ | $1.12$ | $1.12$ | $1.06$ | $1.06$ |
| $R^e_{\tau\mu}$ | $0.85 \pm 0.16$ | $1.04$ | $1.06$ | $1.07$ | $1.05$ | $1.05$ | $1.05$ | $1.01$ | $1.01$ | $1.01$ | $1.01$ |
| $R^\mu_{\mu\tau}$ | $0.73 \pm 0.06$ | $0.79$ | $0.85$ | $0.86$ | $0.82$ | $0.73$ | $0.73$ | $0.73$ | $0.73$ | $0.73$ | $0.72$ |
| $R^\mu_{\tau\mu}$ | $0.62 \pm 0.09$ | $0.64$ | $0.69$ | $0.70$ | $0.66$ | $0.62$ | $0.63$ | $0.62$ | $0.62$ | $0.65$ | $0.66$ |

TABLE III: The experimental results and the predictions of the model for the ten fits given in Table II.