Heavy quarkonia

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Abstract

Two complementary approaches to the theory of heavy quarkonia are discussed. The nonrelativistic potential models give amazingly accurate predictions, but lack a theoretical justification. The expansion in powers of \( v/c \) is theoretically very acceptable, but is not as good in giving numerical predictions. The importance of combining these two approaches is stressed.

1 INTRODUCTION

The subject of this presentation are bound states in the \( \bar{b}b, \bar{b}c (\bar{c}b) \) and \( \bar{c}c \) systems. In the old days it was usual to write that the heavy quarkonia containing the \( t \)-quark, the \( \bar{t} \)-antiquark or both will be the most interesting ones. Now, however, it is known that the lifetime of the \( t \)-(anti)quark is too short for hadronization, so that the quarkonia containing \( t \)-(anti)quarks do not exist. We limit our discussion to bound states below the thresholds for strong decays, i.e. below \( 2M_B = 10.558 \) GeV for the \( \bar{b}b \) system, below \( M_B + M_D = 7.146 \) GeV for the \( \bar{b}c (\bar{c}b) \) system and below \( 2M_D = 3.690 \) GeV for the \( \bar{c}c \) system. We also ignore purely relativistic effects like hyperfine splittings, fine splittings etc.

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Plotting the excitation energies $M - M(^3S_1)$ for the three families of quarkonia one makes the following two observations. Firstly, the spectrum of excitations in the range below the thresholds for strong decays ($M_{th}$) depends little on the quark masses. Secondly, $M_{th} - M(^3S_1)$ increases with the quark masses. Thus the number of bound states below $M_{th}$, ignoring the fine and hyperfine splittings, is three for $cc$, four for $bc(\bar{c}b)$ and ten for $bb$. Actually, many of these states have not yet been observed, but the predictions of the nonrelativistic potential models are so reliable that there is little doubt about the correctness of this counting.

We shall discuss two complementary approaches. Nonrelativistic potential models give for many quantities predictions in excellent agreement with experiment. On the other hand their relation to accepted theory is unknown and they have serious consistency problems. For this reason they are often interpreted as recipes rather than as theoretical models. Nevertheless, they are generally used, when realistic predictions are necessary. The recently proposed method of expanding in powers of $v/c$ (the heavy quarks are slow) \cite{nobody}, \cite{other}, \cite{yet} has much better theoretical foundations. On the other hand its predictive power cannot (yet?) compete with that of the nonrelativistic potential models in the range of their applicability. Nevertheless, this approach has produced some general results of great interest. We shall show how it resolved a problem, which has resisted attempts to clarify it for almost twenty years.

\section{2 \hspace{1cm} NONRELATIVISTIC POTENTIAL MODELS}

There are many nonrelativistic potential models for heavy quarkonia and each of them has been introduced for some good reason. Thus e.g., among the more than ten published models using potentials of the form

$$V(r) = ar^\alpha - br^{-\beta} + c,$$

(1)

where the constants $a, b, \alpha, \beta$ are nonnegative, the Cornell potential $\alpha = \beta = 1$ \cite{cornell} becomes Coulombic for $r \to 0$ and stringy for $r \to \infty$ as expected from QCD; the logarithmic potential of Quigg and Rosner $\alpha = \beta \to 0$ \cite{log} gives an excitation spectrum, which does not depend on the quark masses; Martin’s
potential $\alpha = 0$, $\beta = 0.1$ makes it possible to perform a particularly elegant mathematical analysis. Since the point we want to stress is the predictive power of the nonrelativistic potential models, we choose the potential

$$V(r) = 0.706380 \left[ \sqrt{r - 0.460442} \right] + 8.81715,$$

(2)

where $r^{-1}$ and $V(r)$ are in units of GeV. The corresponding mass of the $b$ quark is $m_b = 4.80303$ GeV. This potential and this quark mass have been chosen so as to optimize the agreement of the predictions with the following measured quantities characterizing the $\bar{b}b$ system: the masses $M(1S)$, $M(1P)$, $M(2S)$, $M(2P)$, $M(3S)$; the squared moduli of the wave functions at zero separation between the quark and the antiquark $|\psi_{1S}(0)|^2$, $|\psi_{2S}(0)|^2$, $|\psi_{3S}(0)|^2$; the absolute values of the dipole electric transition matrix elements $|\langle 1P|r|2S\rangle|$, $|\langle 2P|r|3S\rangle|$ and the ratio of such absolute values $|\langle 1S|r|2P\rangle|/|\langle 2S|r|2P\rangle|$. Actually, instead of fitting the eleven observables with four free parameters, we fitted eight observables with one parameter and fixed the other three parameters so as to reproduce exactly the remaining three observables. This has little effect on the quality of the fit and makes the analysis simpler and more meaningful statistically. Another advantage of this procedure is that the poorly known corrections in the evaluation of the squares of the wave functions at the origin from the directly measured leptonic decay widths almost drop out.

For the parameters quoted above one finds $\chi^2 = 6.5$ for seven degrees of freedom, which means an excellent fit. This is nontrivial, because the accuracy of the experimental data is high. In particular, the four masses are measured with uncertainties ranging from 0.2 MeV to 0.5 MeV, i.e. with an error below 0.01%! Incidentally, our parameters are given with six digit accuracy, only in order to enable the reader to check our computer calculations.

One of the early successes of the nonrelativistic quark models was that the potentials tuned to describe the $\bar{c}c$ quarkonia have also given an acceptable description of the $\bar{b}b$ quarkonia, when these were discovered several years later. This fact, which had been expected in the nonrelativistic quark models ”QCD is flavour blind”, is not predicted any more in the modern approach based on the expansion in powers of velocity. Therefore, it is of interest to check, how this universality is satisfied with the present data. Introducing one more parameter $m_c = 1.3959$ GeV and assuming that the constant in
the potential changes by $2(m_c - m_b)$, we have calculated for the $\tau c$ quarkonia the values of the observables $M(1S)$, $M(1P)$, $M(2S)$, $|\psi_{1S}(0)|^2$, $|\psi_{2S}(0)|^2$ $|\langle 1S| r |1P\rangle|$ and $|\langle 1P| r |2S\rangle|$. The masses agree with experiment within 4 MeV. This margin is about an order of magnitude more than the experimental uncertainties, but on the absolute scale it is not much. The wave functions and the electric dipole matrix elements have to be rescaled (a rescaling factor of 1.3 for the wave functions and a rescaling factor of 0.73 for the matrix elements), but then they agree with experiment. Some rescaling is expected, because when extracting the quantities to be compared with the model from the directly measurable ones we used the same correction factors as for the $\overline{b}b$ system, which in the case of the wave functions is certainly, and in the case of the matrix elements very probably, a rather crude approximation only. We conclude that deviations from the $\overline{b}b - \tau c$ universality are clearly seen, but that they are not very large.

Let us conclude by stressing that the nonrelativistic potential model cannot be interpreted naïvely as a consistent theory. E.g. within this model one can calculate the kinetic energies of the quarks and from that their mean square velocities. The results are $\langle v^2 \rangle \approx 0.25c^2$ for the ground state of the $\tau c$ system and $\langle v^2 \rangle \approx 0.08c^2$ for the ground state of the $\overline{b}b$ system. This is inconsistent with the assumption that high precision results can be derived assuming that the motion is nonrelativistic. To be sure, one can speculate that relativistic corrections can be absorbed into redefinitions of the nonrelativistic parameters of the model, but before this is demonstrated, the main argument in favour of the nonrelativistic potential models remains their amazing success in predicting experimental results.

Let us consider now a more formal approach.

3 EXPANSION IN POWERS OF VELOCITY

Since confinement is a nonperturbative effect, there is little hope of obtaining a good description of heavy quarkonia by summing Feynman diagrams. Another obvious idea – to try an expansion in inverse powers of the heavy quark mass – is also unlikely to work. Arguments derived from quantum field theory can be found in ref [2]. Here we shall qualitatively discuss the high
mass limit using the Cornell potential

\[ V(r) = ar - \frac{b}{r} + c, \quad (3) \]

where \( a > 0, \ b > 0 \). It is plausible that in the high mass limit the system becomes Coulombic. A Coulombic system with particle masses \( M \) has a radius of order \( \frac{1}{M} \), so that in the high mass limit it is consistent to neglect the \( ar \) term. Let us try, however, to consider this term as a perturbation. For \( a < 0 \) the potential is unbounded from below. There is no ground state and the perturbation series cannot converge. Theorems about the convergence radii of power series guarantee that if the perturbation series in \( a \) is divergent for some \( a_0 < 0 \), it will also diverge for all positive \( a \) from the range \( 0 < a < -a_0 \). Thus \( a = 0 \) is a singular point. This could be harmless. E.g. the perturbative expansion in QCD is believed to be an (asymptotic) power series expansion around a singular point. The reason for its success is that the first terms of this expansion approach the correct result so rapidly that they are enough for most practical applications and, therefore, the convergence or divergence of the whole series is irrelevant. In the quarkonium problem, however, the high mass limit — with the wave function localized in an infinitesimal region around \( r = 0 \) — is so remote from any plausible description of a quarkonium that there is little hope that the first few terms of the expansion will make it realistic. Thus, one has to look for another idea.

Let us note an important implication [1] of the observation that in the high mass limit the quarkonium is approximately a Coulombic system. Since the kinetic and the potential energies should be comparable, we expect for high masses

\[ \frac{\alpha_s(\frac{1}{R})}{R} \approx Mv^2, \quad (4) \]

where \( R = \frac{1}{Mv} \) is an estimate of the quarkonium radius. Since \( \alpha_s(\mu) \) is a decreasing function of its argument and since \( v \) (which is in units of the velocity of light \( c \)) is less than one, this implies

\[ v > \alpha_s(M). \quad (5) \]

Thus, it is inconsistent to include radiative corrections of order \( O(\alpha_s^k) \) without also including the relativistic corrections of order at least \( v^k \).
The new and promising idea [1], [2], [3] is to combine factorization and an expansion in powers of the quark velocity (in the rest frame of the quarkonium and in units of $c$) $v$. Thus, any probability amplitude, e.g. the probability amplitude for the decay of quarkonium $\overline{Q}Q$ into a gluon pair, is represented as a sum of terms. Each of these terms is a product of a soft matrix element, which in principle is obtainable from a lattice calculation, but at present is not known, and of a hard matrix element, which can be calculated perturbatively. Using suitable scaling rules it is possible to ascribe to each term an order $n$, which means that for $v \to 0$ this term is of order $O(v^n)$. At each order there is a finite number of terms only. The leading term approximation consists of all the terms of lowest order. It may be systematically improved by including higher and higher order terms.

In this approach the soft matrix element depends on the quark mass in a way, which is beyond our control. Thus, the simple universality from the nonrelativistic quark model is lost. Nevertheless, many approximations known from other approaches, like spin symmetry, vacuum saturation or some of the relations between the production and decay amplitudes can in many cases be rigorously justified at sufficiently low orders of the expansion in powers of $v$. Moreover, the results have often simple physical interpretations.

Perhaps the most impressive success of this approach is the analysis of the decays of $P$-wave quarkonia into light hadrons [8], [1]. This process had been studied for a long time in the framework of the nonrelativistic potential models. One finds that, since for $P$-states $|\psi(0)|^2 = 0$, $Q$ and $\overline{Q}$ have small probability to get so close to each other as to be able to annihilate with a significant probability. As a result, the decay probability amplitude is reduced (compared to the decay probabilities of the $S$-states) by a factor of order $O(v)$. The problem is, however, that this comparatively small amplitude has infinite QCD corrections! The infrared divergences in the calculation do not cancel.

In the spirit of the expansion in powers of $v$, we must look for other contributions to the decay amplitude, which are of the same order in $v$. There is one more such term, where the quarkonium component consists of a $\overline{Q}Q$ system in a colour octet $S$-state accompanied by a "dynamical" gluon so that the whole quarkonium is a colour singlet as it should. This component is small – of order $O(v)$ as compared to the main term. Its $\overline{Q}Q$ part, however, being an $S$-state annihilates easily, so that the contribution of this component to the decay amplitude of the quarkonium into light hadrons...
is of the same order in $v$ as the decay amplitude of the $P$-wave term. Thus, both terms must be included in a correct leading order calculation. After this is done, the infrared singularities in the QCD corrections cancel \[8\] and a finite, acceptable result is obtained. Thus, at least in principle, the problem posed in 1976 has been solved.

4 CONCLUSIONS

The expansion in powers of the velocity $v$ is a respectable physical approach. It can be used to prove statements, which in the nonrelativistic potential approach had to be, often implicitly, conjectured. It also introduces important corrections beyond the scope of the potential approach, like the contribution of the $|\overline{Q}Q, \text{gluon}\rangle$ state to the annihilation of the $P$-wave quarkonia into light hadrons. Nevertheless, for making practically useful predictions it cannot, at present, replace the potential models. The key problem seems to be, how to combine the advantages of the two approaches? How to calculate the soft matrix elements of the expansion in powers of the velocity $v$ from potential models, or how to justify the potential models using the powerful formalism of the expansion in powers of $v$?

References

[1] G.T. Bodwin, E. Braaten and G.P Lepage, Phys. Rev. D51(1995) 1125.
[2] Th. Mannel and G.A. Schuler, Z. Phys. C67 (1995) 159.
[3] H. Khan and P. Hoodboy, Phys. Rev. D53 (1996) 2534.
[4] E. Eichten et al., Phys. Rev. Letters 34 (1975) 369.
[5] C. Quigg and J.L. Rosner, Phys. Letters 71B (1977) 153.
[6] A. Martin, Phys. Letters 100B (1981) 511.
[7] L. Motyka and K. Zalewski, Z. Phys C69 (1995) 343.
[8] G.T. Bodwin, E. Braaten and G.P Lepage, Phys. Rev. D46 (1995) R1914.
[9] R. Barbieri, R. Gatto and G. Remiddi, Phys. Letters B61 (1976) 465.