SYMMETRIC PRODUCTS OF TWO DIMENSIONAL COMPLEXES

SADOK KALLEL AND PAOLO SALVATORE

Abstract. We exhibit a multiplicative and minimal cellular complex which allows explicit and complete (co)homological calculations for the symmetric products of a finite two dimensional CW complex. By considering cohomology, we observe that a classical theorem of Clifford on the dimension of various linear series on a projective curve has a purely topological statement. We give a “real” analog of this theorem for unoriented topological surfaces.

To Sam Gitler on the occasion of his 70th birthday

1. One dimensional Complexes; Constructions

The $n$-th symmetric product $\text{SP}^n X$ of a space $X$, which is assumed to be connected and based, is the quotient of $X^n$ by the permutation action of the symmetric group. We write an element of $\text{SP}^n X$ as an unordered $n$-tuple of points $<x_1,\ldots,x_n>$. When $X = S^1$ there are several ways to see that $\text{SP}^n(S^1)$ is up to homotopy the circle again. An easy such way is to replace up to homotopy $S^1$ with the punctured complex plane $\mathbb{C}^*$. Then the map

$$\text{SP}^n(\mathbb{C}^*) \longrightarrow \mathcal{H}, \quad <z_1,\ldots,z_n> \mapsto \prod_{1 \leq i \leq n} (z - z_i)$$

is a homeomorphism, where on the right we have identified the space of monic polynomials with roots avoiding the origin with $\mathcal{H}$, the complement of a hyperplane in $\mathbb{C}^n$. But then evidently $\mathcal{H} \cong \mathbb{C}^{n-1} \times \mathbb{C}^* \cong S^1$ and hence the claim. It is amusing to see for instance that $\text{SP}^2(S^1)$ is the Mobius band. In general one has the following complete description due to Morton.

Theorem 1. \cite{9} For all $n \geq 1$, the multiplication map

$$m : \text{SP}^n S^1 \longrightarrow S^1, \quad <x_1,\ldots,x_n> \mapsto x_1 \cdots x_n$$

has the structure of an $(n-1)$-dimensional disc bundle over $S^1$; trivial if $n$ is odd and non-oriented if $n$ is even.

As is clear from theorem 1, a chain complex for the symmetric product of a circle has a small retract. This suggests the following construction.

Let $X$ be any reduced CW complex of finite type, and let $\bigvee^k S^1 \hookrightarrow X$ be the one-skeleton inclusion. We choose the basepoint $* \in X$ to be the unique 0-cell and we assume that this basepoint corresponds to the identity element 1 in each circle leaf viewed as a circle group.

Define the identification space

$$\overline{\text{SP}}^n X = \text{SP}^n X / \sim$$
where \( \sim \) identifies \( (x, y, z_1, \ldots, z_{n-2}) \) to \( (*, xy, z_1, \ldots, z_{n-2}) \) whenever \( x, y \) are in the same leaf \( S^1 \). We denote by \( q \) the quotient map. The usefulness of this construction is contained in the following crucial lemma.

**Lemma 2.** The projection \( q : \text{SP}^n X \to \overline{\text{SP}}^n X \) is a homotopy equivalence.

**Proof.** We start by the case \( X = X^{(1)} \), i.e. when \( X \) is a bouquet of \( k \) circles. The space \( \text{SP}^n(X) \) is the colimit of the diagram \( (i_1, \ldots, i_k) \mapsto \text{SP}^n(S^1) \times \cdots \times \text{SP}^n(S^1) \), indexed over the full sub-poset \( I \subset \mathbb{N}^k \) containing \( k \)-tuples with \( i_1 + \cdots + i_k \leq n \). The morphisms in the diagram are induced by the obvious inclusions. The space \( \overline{\text{SP}}^n(X) \) is the colimit of a similar diagram sending \( (i_1, \ldots, i_k) \) to \( (S^1)^{e(i_1)} \times \cdots \times (S^1)^{e(i_k)} \), where \( e(x) = 1 \) if \( x > 0 \) and \( e(x) = 0 \) if \( x = 0 \). The circle multiplication induces on colimits the projection \( q \). It is not difficult to see (compare \([13]\)) that each colimit is homotopy equivalent to the associated homotopy colimit. The circle multiplication being an equivalence on each factor by Morton’s result, the lemma follows in this case by homotopy invariance.

In general we have a filtration \( E_0 \subset \cdots \subset E_n = \text{SP}^n(X) \) where \( E_i \) is the space of those unordered \( n \)-tuples containing at most \( i \) points outside the 1-skeleton. Each subspace \( E_i \) is closed in \( E_{i+1} \), it is the strong deformation retract of an open neighbourhood \( U_i \) in \( E_{i+1} \), and all this passes to the quotient. For example choose \( U_i \) to contain at most \( i \) points outside an open neighbourhood \( U \) of \( X^{(1)} \) in \( X \), with \( X^{(1)} \) deformation retract of \( U \). The projection between differences \( E_i - E_{i-1} \to q(E_i) - q(E_{i-1}) \) can be identified to the projection

\[
\text{SP}^i(X - X^{(1)}) \times \text{SP}^n(X^{(1)}) \to \text{SP}^i(X - X^{(1)}) \times \overline{\text{SP}}^{n-i}(X^{(1)})
\]

and is a homotopy equivalence by our earlier remark. We can now use the gluing lemma and induction up the filtration to show that \( E_i \simeq q(E_i) \), \( 1 \leq i \leq n \). \( \square \)

As it turns out, the construction \( \overline{\text{SP}} \) is much easier to study. For example it is straightforward to see that

**Lemma 3.** \( \overline{\text{SP}}^n(\bigvee^k S^1) \) is homeomorphic to the \( n \)-skeleton of \( (S^1)^k \).

As an immediate corollary we obtain the following main result of \([12]\) which initially made a lengthy use of the theory of hyperplane arrangements.

**Corollary 4.** For \( n \geq k \geq 1 \) there is a homotopy equivalence \( \text{SP}^n(\bigvee^k S^1) \simeq (S^1)^k \).
For \( n < k \) \( \text{SP}^n(\bigvee^k S^1) \) is homotopic to the union of \( \binom{k}{n} \) \( n \)-dimensional subtori in \( (S^1)^k \).

2. **Two Dimensional Complexes and Minimal Cell Decompositions**

Let \( X \) be a connected (based) cellular complex

\[
X = \bigvee^k \bigcup (D^2_1 \cup \cdots \cup D^2_r)
\]

obtained by attaching \( r \) 2-cells to a bouquet of \( k \) circles. A chain complex for \( X \) has 1-dimensional generators \( e_1, \ldots, e_k \) and two dimensional generators \( D_1, \ldots, D_r \). In this section we show that these classes generate multiplicatively (in a sense we define shortly) a minimal cell complex for \( \text{SP}^n X \).

The main observation in dimension two is the following standard result.
Lemma 5. Let $D^n$ be the open unit disc in $\mathbb{R}^n$, and $D^n$ its closure. Then there is a homeomorphism of pairs $\phi_n: (SP^n D^2, SP^n D^2 - SP^n D^2) \cong (D^{2n}, S^{2n-1})$.

Proof. For a positive integer $d$ we define a self homeomorphism $r_d: \mathbb{C} \to \mathbb{C}$ by $r_d(z) = z|z|^{1/d}$ for $z \neq 0$ and $r_d(0) = 0$. In particular if $z \geq 0$ then $r_d(z) = z^{1/d}$.

Let $f_d < z_1, \ldots, z_n >$ be the elementary degree $d$ symmetric function in $z_1, \ldots, z_n$. By the fundamental theorem of algebra the correspondence

$$< z_1, \ldots, z_n > \mapsto (f_1, \ldots, f_n)$$

gives a homeomorphism $SP^n(\mathbb{C}) \cong \mathbb{C}^n$. Let us write $\partial SP^n D^2 = SP^n D^2 - SP^n D^2$.

Via the action $t < z_1, \ldots, z_n > = < tz_1, \ldots, tz_n >, t \geq 0$, we may identify $SP^n \mathbb{C} = SP^n \mathbb{R}^2$ to the $\mathbb{R}_+$-cone on $\partial SP^n D^2$ and $SP^n D^2$ to the $I$-cone on $\partial SP^n D^2$. Of course $D^{2n}$ and $R^{2n}$ are respectively the $I$-cone and the $\mathbb{R}_+$-cone on $S^{2n-1}$. It is easy to see that the homeomorphism $\psi_n: SP^n(\mathbb{C}) \cong \mathbb{C}^n$ sending $< z_1, \ldots, z_n > \mapsto (r_1(f_1), \ldots, r_n(f_n))$ is $\mathbb{R}_+$-equivariant and preserves cone rays. On $\partial SP^n D^2$ we define $\phi_n = \psi_n/\|\psi_n\|$ and extend it to $SP^n D^2$ by cone extension. \hfill $\Box$

A quick corollary of lemma 5 is that $SP^n S$ is an orientable (resp. non-orientable) $2n$-dimensional manifold if $S$ is an orientable (resp. non-orientable) topological surface (eg. $S^1$).

Write $SP^n X = \bigsqcup_{n \geq 0} SP^n X$ where $SP^0 X = \ast$ is the basepoint and $SP^n X = X$. Note that concatenation $(S^1)^{\ast} \times (S^1)^{t} \rightarrow (S^1)^{t+\ast}$ commutes with multiplication in the circle group $S^1$ so that we have an induced pairing at the level of quotient spaces

$$X^s \times X^t \quad \longrightarrow \quad X^{s+t}$$

(2)

$$\frac{\overline{SP}^n X \times \overline{SP}^t X}{\ast} \quad \quad \overline{SP}^{n+t} X$$

This endows $\overline{SP} X$ with the structure of a commutative monoid. A cellular decomposition of $\overline{SP} X$ is called multiplicative if it is compatible with this monoid structure; i.e. if the multiplication $\ast$ is cellular. Such a decomposition specializes to a decomposition for $X$ such that the projections

$$\overline{q}: X^n \longrightarrow \overline{SP}^n X, \quad n \geq 1$$

are cellular. Given a multiplicative decomposition for $\overline{SP} X$; $c_1 \in C_\ast (\overline{SP}^s X)$ and $c_2 \in C_\ast (\overline{SP}^t X)$, we denote by the product $c_1 \ast c_2$ the image of $c_1 \otimes c_2$ under the map $C_\ast (\overline{SP}^s X) \otimes C_\ast (\overline{SP}^t X) \longrightarrow C_\ast (\overline{SP}^{s+t} X)$.

Theorem 6. Write $X = \bigvee_s S^1 \cup (D^2_1 \cup \cdots \cup D^2_r)$. Then

(a) A chain complex for $\overline{SP} X$ is multiplicatively generated under $\ast$ by a zero dimensional class $v_0$, degree one classes $e_1, \ldots, e_k$ and degree $2s$ classes $SP^s D_i$ $1 \leq i \leq r, 1 \leq s, \text{ under the relations}$

$$e_i \ast e_j = -e_j \ast e_i (i \neq j), \quad e_i \ast e_i = 0$$

$$SP^s D_i \ast SP^t D_i = \left(\begin{array}{c}s + t \\ t \end{array}\right) SP^{s+t} D_i$$
The boundaries are such that:
\[ \partial e_j = 0, \quad \partial \text{SP}^p D_i = (\partial D_i) \ast \text{SP}^{p-1} D_i \]
\[ \partial \ast \text{SP}^p D_i = \text{SP}^p (\partial D_i) \ast \text{SP}^{p-1} D_i \]
\[ \partial \ast \text{SP}^p D_i = (\partial D_i) \ast \text{SP}^{p-1} D_i \]

where \( \partial D_i \) is the boundary of \( D_i \) in \( C_1(X) \).

(b) A cellular chain complex for \( \text{SP}^p(X) \) consists of the subcomplex generated by
\[ v_0^k \ast e_{i_1} \ast \cdots \ast e_{i_t} \ast \text{SP}^q(D_{j_1}) \ast \cdots \ast \text{SP}^q(D_{j_s}) \]
with \( k + t + s_1 + \cdots + s_t = n \).

(c) \( H_*(\text{SP}^p X; \mathbb{Z}) \) and \( H_*(X; \mathbb{Z}) \) have the same prime torsion.

(d) In particular \( H_*(X) \) is torsion free if and only if \( H_*(\text{SP}^n X) \) is torsion free, in which case there is a commutative diagram
\[ H_*(\text{SP}^p X) \hookrightarrow C_*(\text{SP}^p X) \]
\[ H_*(\text{SP}^n X) \hookrightarrow C_*(\text{SP}^n X) \]
with all maps being monomorphisms. The bottom map in the diagram above is a ring map with respect to the symmetric product \( \ast \).

Proof. Start with \( X \) having one skeleton \( \bigvee^k S^1 \hookrightarrow X \) with zero-cell the wedge point, and two cells attaching to the bouquet. Write the zero cell \( \ast \), the one dimensional cells \( E_1, \ldots, E_k \) and the two dimensional cells \( D_1^2, \ldots, D_r^2 \). These cells are closed with disjoint interiors homeomorphic to discs.

Suppose \( C_1 \) and \( C_2 \) are cells of such a CW decomposition of \( X \), \( q : X \times X \longrightarrow \text{SP}^2 X \) the quotient map. The image \( q(C_1 \times C_2) \) is generally not a cell in \( \text{SP}^2 X \). We exploit the fact that this problem doesn’t occur when we map into \( \text{SP}^n X \).

Write \( \tilde{q} : X^n \longrightarrow \text{SP}^n X \) the quotient map, and denote by the product \( C_1 \ast \cdots \ast C_n \) the image of \( C_1 \times \cdots \times C_n \) under \( \tilde{q} \). We also write \( \tilde{q}(\ast) = v_0 \), and identify 1-cells and 2-cells with their image via \( \tilde{q} \). We use the following geometric properties:

(i) The \( \ast \)-product of cells of \( X \) whose interiors are disjoint is again a cell. This is because \( \tilde{q} \) is injective on the product of the interiors.

(ii) \( E_i \ast E_i = E_i \ast v_0 \) in \( \text{SP}^2 X \).

(iii) The \( n \)-fold product of \( D_2^2 \) via \( \ast \) is a cell that is covered \( n! \) times by \( (D_2^2)^n \) according to lemma 4. We will denote by \( \text{SP}^n(D_2^2) \) this symmetric product cell.

This yields a CW decomposition of \( \text{SP}^n X \) for all \( n \geq 1 \) such that \( \tilde{q} : X^n \longrightarrow \text{SP}^n X \) is cellular. In other words, a multiplicative CW decomposition for \( \text{SP}^n X := \coprod_{n \geq 0} \text{SP}^n X \) is obtained from \( v_0 \), the \( E_i \)'s, the \( D_2^2 \)'s and all possible products under \( \ast \) among these with the restriction that there are only a finite number of non-zero cells in the product. If we agree to identify cells when they differ by multiples of \( v_0 \), then the CW structure on \( \text{SP}^n X \) is obtained by taking all \( m \)-fold products with \( m \leq n \).

We can now pass to the chain complex level. To \( E_i \) corresponds the algebraic generator \( e_i \in C_1(\text{SP}^p X) \), to \( D_2^2 \) corresponds \( D_j \) of degree two with well defined boundary. This determines \( C_*(X) \). We extend this complex multiplicatively to all
of \( C_*(\overline{SP}X) \) using the diagram obtained from (2)

\[
C_i(X^s) \otimes C_j(X^t) \quad \longrightarrow \quad C_{i+j}(X^{s+t})
\]

\[
\begin{array}{c}
C_i(\overline{SP}^s X) \otimes C_j(\overline{SP}^t X) \quad \longrightarrow \quad C_{i+j}(\overline{SP}^{s+t} X)
\end{array}
\]

(4)

First of all, since \( E_i \ast E_i = E_i \), we set \( c_i \ast c_i = 0 \). For chains \( c_1 \) and \( c_2 \) supported by geometric cells \( C_1 \) and \( C_2 \) with disjoint interiors, the diagram implies that \( c_1 \ast c_2 \) is the cell supported by \( C_1 \ast C_2 \). More interestingly note that \( \overrightarrow{q}_s(D_i^{s=}) = s!SP^s(D_i) \) so that tracing through (1) we must have that \( SP^s(D_i) \ast SP^t(D_i) = (s+t)!SP^{s+t}D_i \). Finally by commutativity we have that \( c_i \ast c_j = (-1)^{|i||j|}c_j \ast c_i \).

We next analyze the boundaries. It is clear that \( \partial e_i = 0 \) while \( \partial D_2 \) is determined by the attaching maps of \( X \). We need understand the boundary on the “symmetric product” cell \( SP^s(D) \). This is described geometrically as the image under the symmetric quotient \( X \ast \longrightarrow SP^sX \) of

\[
\partial(D_2^s)^s = \bigcup_{j=1} (D_2^s)^{j-1} \times \partial(D_2^s) \times (D_2^s)^{s-j}
\]

and each term in the union maps to \( (\partial D_2^s) \ast SP^{s-1}(D_2^j) \) (in an orientation preserving manner). But the degree of the projection of the right hand side of (5) into \( (\partial D_2^s) \ast SP^{s-1}(D_2^j) \) is \( s \cdot (s-1)! \cdot s! \) (taking into account \( s \)-terms in the union and then the degree of the projection \( (D_2^s)^{s-1} \longrightarrow SP^{s-1}(D_2^j) \)). On the other hand \( s! \) is precisely the degree of the projection \( \partial(D_2^s) \longrightarrow \partial SP^s(D_2) \) so that in the chain complex we must have \( \partial SP^s(D_i) = (\partial D_i) \ast SP^{s-1}(D_i) \).

We observe that \( \partial \) is a derivation because the cellular decomposition is multiplicative. The remaining claims follow by construction.

Parts (c) and (d) follow immediately from parts (a) and (b).

\[\square\]

**Remark 7.** We have embeddings \( \overline{SP}^{s−1}X \hookrightarrow \overline{SP}^sX \) (adjunction of basepoint) and \( C_*(\overline{SP}^{s−1}X) \hookrightarrow C_*(\overline{SP}^sX) \) (multiplication by \( v_0 \)). We can then assign a bidegree to cells so that \( c \) has bidegree \( (s,*) \) if \( c \in C_*(\overline{SP}^sX,\overline{SP}^{s-1}X) \). We refer to \( s \) as the filtration degree. For example \( SP^s(D) \) has bidegree \( (s,2s) \) and the product of distinct terms \( e_1 \ast \cdots \ast e_s \) has bidegree \( (s,s) \). The useful feature here is that the boundary operator \( \partial \) preserves filtration degrees so that we have a decomposition

\[
H_*(\overline{SP}^sX) \cong \bigoplus_{1 \leq i \leq n} H_*(\overline{SP}^sX,\overline{SP}^{s-i}X)
\]

which is a special case of a more general splitting result of Steenrod [3].

**Remark 8.** It is often convenient to consider as in the literature the infinite symmetric product \( SP^\infty(X) \) obtained as the direct limit of the basepoint inclusions \( SP^sX \longrightarrow SP^{s+1}X \). Let \( \overline{SP}^\infty(X) \) be the induced quotient, \( \overline{SP}^\infty(X) \cong SP^\infty(X) \). A chain complex for \( \overline{SP}^\infty(X) \) is obtained from a chain complex for \( \overline{SP}(X) \) by identifying cells differing by a multiple of \( v_0 \).

**Examples.** We illustrate theorem (6) with a few examples.

1. First when \( X = S^2 = \ast \cup D^2 \) and there are no one cells. Here \( \partial D^2 = 0 \) and the homology of \( SP^nX = \overline{SP}^nX \) is generated in dimension \( 2i \) by the unique
cell in that dimension $SP^i(D)$, $i \leq n$. This is of course in accordance with
the identification $SP^n S^2 = \mathbb{CP}^n$.

(2) If instead we write $S^2 = S^1 \cup D_1^2 \cup D_2^2$ with $\partial D_1 = e_1$ and $\partial D_2 = -e_1$, then $H_* (\underline{SP}^\infty(X)) = H_* (SP^\infty(X))$ has generator $D_1 + D_2$ in dimension 2, $SP^2(D_1) + D_1 * D_2 + SP^2(D_2)$ in dimension 4 and more generally
$
\sum_{s+t=n} SP^s(D_1) * SP^t(D_2)$ in dimension $2n$.

(3) Write $\mathbb{RP}^2 = S^1 \cup_f D^2$ where the attaching map is of degree two. A chain complex for $\mathbb{RP}^2$ has generators $e, D$ with $\partial D = 2e$. A chain complex for $\underline{SP}^n X$ has even generators $SP^i(D)$ and odd generators $eSP^i(D)$ with
\[
\partial (SP^i(D)) = 2eSP^{i-1}(D),
\partial (SP^i(D)e) = 2e^2SP^{i-1}D = 0.
\]
One sees immediately that $H_* (\underline{SP}^n(\mathbb{RP}^2)) = H_* (\mathbb{RP}^{2n})$ in accordance with the identification $SP^n(\mathbb{RP}^2) \cong \mathbb{RP}^{2n}$ (lemma 27).

(4) If $S$ is a closed Riemann surface, then $H_* (\underline{SP}^n S)$ is torsion-free for all $n \geq 1$ (classical, cf. [11]).

**Remark 9.** Notice that the largest cells in $C_* (\underline{SP}^n X)$ have dimension $2n$ and are of the form $SP^n (D)$ for some two cell $D \in C_2(X)$. This implies that if $X$ is a two complex, then $H_* (\underline{SP}^n X; \mathbb{Z})$ is trivial for $* \geq 2n + 1$. More generally if $X$ is an $m$-dimensional complex, then $\underline{SP}^n X$ is $nm$-dimensional.

**Corollary 10.** $\pi_1 (\underline{SP}^n X)$ is abelian for $n > 1$.

**Proof.** By theorem [8] $\underline{SP}^n X$ and $\underline{SP}^\infty X$ have the same two skeleton and hence the same $\pi_1$. But $\underline{SP}^\infty X$ is an abelian monoid and hence has abelian fundamental group.

The next two corollaries are of good use in applications and combine remark [8] with theorem [6] (compare [11]).

**Corollary 11.** Let $X$ be a 2-complex and suppose that $\partial D = 0$ for some two cell $D$. Then the cells $SP^s (D)$ for $s \geq 0$ generate a divided power algebra $\Gamma(D)$ in $H_* (\underline{SP}^\infty(X))$.

**Corollary 12.** Let $X = (\sqrt{2g} S^1) \cup D^2$ be a Riemann surface of genus $g$. Then a minimal multiplicative chain complex for $\underline{SP}^\infty (X)$ is given by
\[
E(e_1, \ldots, e_{2g}) \otimes \Gamma(D).
\]
At every finite stage $\underline{SP}^n X$ has a minimal cell complex consisting of all chains of filtration $\leq n$ in the bigraded complex (6) above, where $e_i$ is of bidegree $(1,1)$ and $D$ of bidegree $(1,2)$ (remark [7]). The class $SP^s (D)$ corresponds in homology to the orientation class of the manifold $SP^s (X)$.

2.1. **Comparison with methods of Dold and Milgram.** Dold [3] and then Milgram [11] gave an effective recipe to compute the homology of symmetric products of CW complexes. The idea is that if $X$ is any complex of the homology type of a wedge of Moore spaces $\bigvee_{i=1}^r M_i$, then as graded abelian groups

$H_* (\underline{SP}^n X, SP^{n-1} X) \cong$
\[
\bigoplus_{i_1 + \cdots + i_r = n} H_* (SP^{i_1} M_1, SP^{i_1-1} M_1) \otimes \cdots \otimes H_* (SP^{i_r} M_r, SP^{i_r-1} M_r)
\]

(7)
with the tensor product on the right corresponding to the symmetric product pairing * in homology on the left. For example in the case of a Riemann surface $S$ of genus $g$, $H_*(S) \cong H_*(\mathbb{V}^{2g} S^1 \vee S^2)$ and hence according to (10) we recover corollary (12).

The decomposition in (10) however does not shed light on neither cup products nor cohomology operations in $\text{SP}^nX$. We will deal with this in the next section.

3. Cohomology Structure

Let $X$ be a two dimensional complex and suppose that $H_*(\text{SP}^nX)$ is torsion free. Then the transfer shows that

$$H^*(\text{SP}^nX) = \left( H^*(X)^{\otimes n} \right)^{\Sigma_n}$$

is the submodule of invariants. In particular the induced map in cohomology $\pi^* : H^*(\text{SP}^nX) \longrightarrow H^*(X^n)$ is injective. This is the method adopted by MacDonald in (10) to determine the cohomology ring of the symmetric product of an orientable surface $S$ (i.e. theorem 16).

The situation for more general $X$ is harder to track down as $\pi^*$ is not necessarily injective (eg. this is already not the case for $X = \mathbb{R}P^2$ and $n = 2$). To remedy to this problem, we need use other arguments based on the multiplicative cell complex introduced in section 2.

Let $\Delta$ be the diagonal map and $\delta : X \longrightarrow X \times X$ a cellular approximation. Write $H : \Delta \simeq \delta$ for the homotopy. For reasons that will soon be clear, we would like to choose $H$ so that $H_t, t \in [0,1]$ sends each leaf of the bouquet $\bigvee S^1 \subset X$ to its square. Start with a standard approximation for the diagonal on $S^1$. This can be done on each leaf to yield an approximation $\delta^\vee$ for the diagonal $\Delta^\vee$ on the bouquet. The relative cellular approximation theorem (9, theorem 4.8) states that it is possible to extend $\delta^\vee$ to a cellular map $\delta$ on all of $X$.

We wish to understand the cup product of $H^*(\text{SP}^nX)$ starting from

$$\delta_* : C_*(X) \longrightarrow C_*(X) \otimes C_*(X).$$

The first step is to consider the coproduct for $C_*(X^n)$ which is obtained up to suitable shuffle from the map $\delta^{\otimes n}$. More explicitly, if $\chi$ is the shuffle map

$$\chi : (x_1, \ldots, x_n) \times (y_1, \ldots, y_n) \mapsto (x_1, y_1) \times (x_2, y_2) \times \ldots \times (x_n, y_n)$$

then we can write the diagonal $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n; x_1, \ldots, x_n)$ as a composite of $(x_1, \ldots, x_n) \mapsto (x_1, x_1, \ldots, x_n, x_n)$ followed by $\chi^{-1}$. A diagonal approximation for $X^n$ is then given by

$$X^n \xrightarrow{\delta^n} (X^2)^n \xrightarrow{\chi^{-1}} X^n \times X^n.$$

Suppose now that $\bigsqcup \text{SP}^nX$ is given a multiplicative cell decomposition (as in section 2) so that in particular the quotient $\pi : X^n \longrightarrow \bigsqcup \text{SP}^nX$ is cellular.

Lemma 13. There is a commutative diagram

$$\begin{array}{ccc}
C_*(X)^{\otimes n} & \xrightarrow{\chi^{-1}g^n} & C_*(X)^{\otimes n} \otimes C_*(X)^{\otimes n} \\
\downarrow{\pi_*} & & \downarrow{\pi_* \otimes \pi_*} \\
C_*(\text{SP}^nX) & \xrightarrow{\lambda_*} & C_*(\text{SP}^nX) \otimes C_*(\text{SP}^nX)
\end{array}$$

where $\lambda_*$ induces in cohomology the cup product.
Proof. The cellular approximation of the diagonal of $X^n$ induces a map

$$\lambda : \text{SP}^n X \longrightarrow \text{SP}^n X \times \text{SP}^n X$$

homotopic to the diagonal, but not a map $\text{SP}^n X \to \text{SP}^n X \times \text{SP}^n X$ because the approximation on a leaf $S^1 \to S^1 \times S^1$ is not a homomorphism. Let us filter $\text{SP}^n X$ by the inverse images of the skeleton $\text{SP}^n X$. The proof of lemma 2 shows that the projection from an inverse image of a skeleton to the skeleton is a homotopy equivalence. Thus the Leray spectral sequence of our filtration has as $E_1$ term the chain complex $C_\ast$ of $\text{SP}^n X$ and collapses at the $E_2$ term. If we filter similarly the product $\text{SP}^n X \times \text{SP}^n X$ then $\lambda$ is a filtration preserving map inducing $\lambda_\ast$ on the $E_1$-term. The commutative diagram in the statement lives at the level of Leray $E_1$-terms, where $X^n$ and its square are filtered by skeleta. □

3.1. A calculation. As an illustration of the method and for later use, we determine the cohomology of $\text{SP}^n(S^1 \cup_m D^2)$, where $X = S^1 \cup_m D^2$ is the complex obtained by attaching $D^2$ along a degree $m$ map. The chain complex for $X$ has generators $e, D$ with $\partial D = me$. The cup product structure is only interesting with $\mathbb{Z}_m$ coefficients. In this case, $D$ is primitive if $m$ is odd, and otherwise (see [6], example 3.9)

$$(8) \quad D = \lambda \rightarrow D \otimes 1 + ke \otimes e + 1 \otimes D \quad , \quad m = 2k$$

To determine $\lambda_\ast(\text{SP}^2 D)$ we look at the coproduct upstairs in lemma 13

$$(D \otimes D) \quad \rightarrow \quad \chi^{-1} ((D \otimes 1 \otimes 1 + ke \otimes e \otimes 1 \otimes 1 + 1 \otimes D \otimes 1 \otimes 1)
\quad + (1 \otimes 1 \otimes D \otimes 1 + k1 \otimes 1 \otimes e \otimes e + 1 \otimes 1 \otimes 1 \otimes D))
\quad = \quad D \otimes D \otimes 1 + 1 \otimes D \otimes e + D \otimes 1 \otimes 1 \otimes D
\quad + ke \otimes D \otimes e \otimes 1 - ke \otimes 1 \otimes e \otimes D
\quad +1 \otimes D \otimes D \otimes 1 + k1 \otimes e \otimes D \otimes e + 1 \otimes 1 \otimes D \otimes D$$

We can now apply $\pi_\ast$ to the left and $\pi_\ast \otimes \pi_\ast$ to the right to obtain (after dividing by 2)

$$(9) \quad \text{SP}^2 D \longrightarrow \lambda \rightarrow \text{SP}^2 D \otimes 1 + kDe \otimes e + D \otimes D + ke \otimes eD + 1 \otimes \text{SP}^2 D$$

Note that $H^\ast(\text{SP}^n X; \mathbb{Z}_m)$ has one generator per dimension. Denote by $b := D^\ast$ the dual of $D$ and by $f = e^\ast$ the dual of $e$. Then (8) implies that $f^2 = kb$ while (9) implies that $b^2 = (\text{SP}^2 D)^\ast$ and $f^2 b = k(\text{SP}^2 D)^\ast = kb^2$. Carrying this game to the remaining classes shows that $b$ generates a truncated polynomial algebra where $b^k$ is dual to $\text{SP}^k D$ and $b^{n+1} = 0$. On the other hand, $f b^k$ is dual to $e \text{SP}^k D$ (compare (10)). This yields

Lemma 14. If $m$ is even, $m = 2k$, then $H^\ast(\text{SP}^n(S^1 \cup_m D); \mathbb{Z}_m)$ is generated by $e$ in dimension one and $b$ in dimension two subject to $e^2 = kb$ and $b^{n+1} = eb^n = 0$. If $m$ is odd, we have to change the first relation to $e^2 = 0$.

4. Surfaces

4.1. Orientable Surfaces. In corollary 12 we have determined the homology of $\text{SP}^n S$ for $S$ a Riemann surface of genus $g \geq 0$. This was based on the construction of a chain complex for $\text{SP}^n S$ based on cells $e_1, \ldots, e_{2g}$ and $D$. The coproduct at the chain level is such that the $e_i$’s are primitive and

$$(10) \quad D \rightarrow D \otimes 1 + \sum e_i \otimes e_{i+g} - \sum e_{i+g} \otimes e_i + 1 \otimes D \quad , \quad 1 \leq i \leq g$$
If \( f_1 = e_1^*, \ldots, f_{2g} = e_{2g}^* \) are the dual cohomology classes, then their cup product satisfies \( f_1 f_{i+g} = b \), where \( b \) as before is dual to the orientation class \( D \). There are no other relations in the cohomology of \( S \).

Consider the diagram in lemma 13. We propose to determine for \( i \neq j \) \( \lambda_*(e_i \otimes e_j) \in C_*(X^2)^{\otimes 2} \). Here \( \pi_*(e_i \otimes e_j) = e_i e_j \). The effect \( \chi^{-1} \delta_2^* \) on \( e_i \otimes e_j \) is as follows

\[
e_i \otimes e_j \mapsto \chi^{-1}((e_i \otimes 1 \otimes 1 \otimes 1 \otimes e_j \otimes 1 \otimes 1 \otimes 1)(1 \otimes 1 \otimes e_j \otimes 1 \otimes 1 \otimes 1 \otimes 1 \otimes e_j)) = e_i \otimes e_j \otimes 1 + e_i \otimes 1 \otimes 1 \otimes e_j - 1 \otimes e_j \otimes e_i \otimes 1 + 1 \otimes 1 \otimes e_i \otimes e_j
\]

Applying \( \pi_* \) to the left and \( \pi_* \pi_* \) to the right of this expression we obtain the coproduct

\[
e_i e_j \mapsto e_i e_j \otimes 1 + e_i \otimes e_j - e_j \otimes e_i + 1 \otimes e_i e_j
\]

Both of (11) and (10) pass to the coproduct in homology. We will use throughout the same symbol for a cycle in the chain complex and the homology class it generates.

By looking at dual classes in (11) and (10) and pulling back, we see right away that \( e_i^* e_j^* = (e_i e_j)^* + b \); i.e.

**Lemma 15.** \( (e_i e_j)^* = e_i^* e_j^* - b \).

It turns out that this relation together with truncation by filtration degree generate all relations in the cohomology of \( \text{SP}^n S \). The original calculation of \( H^*(\text{SP}^n S) \) in (10) seems somewhat long wounded. It can be phrased in the following easier way. Note that since \( D \) generates a divided power algebra in \( H_*(\text{SP}^\infty(X)) \) (corollary (10)), then its dual \( b \) generates a polynomial algebra \( \mathbb{Z}[b] \). On the other hand, the dual of an exterior algebra is exterior and hence

\[
H^*(\text{SP}^\infty S) \cong E(f_1, \ldots, f_{2g}) \otimes \mathbb{Z}[b]
\]

Since \( H_*(\text{SP}^n S) \rightarrow H_*(\text{SP}^\infty S) \) is injective, then \( H^*(\text{SP}^\infty S) \rightarrow H^*(\text{SP}^n S) \) is surjective, and \( H^*(\text{SP}^n S) \) is a quotient of (12) by some relations.

Consider the MacDonald relation on the cohomology classes \( f_1, \ldots, f_{2g} \) and \( b \)

If \( i_1, \ldots, i_a, j_1, \ldots, j_b, k_1, \ldots, k_c \) are distinct integers from 1 to \( g \) inclusive, then provided that \( a + b + 2c + q = n + 1 \) we have

\[
f_{i_1} \cdots f_{i_a} f_{j_1+g} \cdots f_{j_b+g} (f_{k_1} f_{k_1+b} - b) \cdots (f_{k_c} f_{k_c+b} - b) b^q = 0
\]

**Theorem 16.** \( H^*(\text{SP}^n S) \) is the quotient of \( E(f_1, \ldots, f_{2g}) \otimes \mathbb{Z}[b] \) by the MacDonald relation.

**Proof.** We outline an alternative proof based on theorem 6 and lemma 14. We know that \( H_*(\text{SP}^n S) \) is rationally generated by \( m \) fold products of generators in \( H_*(S) \), \( m \leq n \). It follows that any element \( x \) of filtration degree \( m \geq n + 1 \) cannot be in the image of \( t_* : H_*(\text{SP}^n X) \rightarrow H_*(\text{SP}^\infty X) \) and hence \( \iota^*(x^*) = 0 \in H^*(\text{SP}^n S) \). Choose a generator of filtration \( (n + 1) \) which we write in the form

\[
x = e_{i_1} \cdots e_{i_k} \text{SP}^i S \quad , \quad i_1 \neq i_i, \quad t + k = n + 1
\]

We can show by writing coproduct formulae that if no pair of the form \( (e_{i_1}, e_{i_2}^* \text{SP}^j S) \) figures among the \( e_{i_j} \)'s above, then the dual class verifies

\[
(e_{i_1} \cdots e_{i_k} \text{SP}^i S)^* = f_{i_1} \cdots f_{i_k} b^j
\]

where again the \( f \)'s are dual to the \( e \)'s. If say \( e_{i_1} = e_s \) and \( e_{i_2} = e_{s+g} \), then

\[
(e_{i_1} \cdots e_{i_k} \text{SP}^i S)^* = (e_s e_{s+g} \cdots e_{k} \text{SP}^i S)^*, \text{ and } (e_s e_{s+g} \cdots e_{k} \text{SP}^i S)^* \text{ as in lemma 14}
\]
In light of this, the condition $\ell^* (e_{i_1} \cdots e_{i_k} \text{SP}^4 S)^* = 0$ translates directly to the MacDonald’s relation and this is the only such relation. \hfill \square

4.2. Non orientable Surfaces. As far as we know the non-orientable case is not in the literature. Let $U$ be the non-orientable surface of genus $g$. Then $U$ is the connected sum of $g$-copies of the real projective plane

\begin{equation}
U := \mathbb{RP}^2 \# \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2 \quad (g\text{-times})
\end{equation}

We can write $U$ as a wedge of $g$-circles with a single disc $D^2$ attached along the sum of degree two maps on each leaf. If we denote as before the cellular generators by $e_1, \ldots, e_g$ and $D \in C_2(U)$, then

\begin{equation}
\partial D = 2e_1 + \cdots + 2e_g
\end{equation}

The homology $H_*(\text{SP}^n U; \mathbb{Z})$ is completely determined by theorem 6. In particular of course $H_1(U) = \mathbb{Z}_2 \oplus \mathbb{Z}^{g-1}$ and $H_2(U) = 0$.

Remark 17. Note that $U$ has as oriented two cover a Riemann surface $S$ of genus $g - 1$. This covering is obtained by embedding $S$ in $\mathbb{R}^3$ so that the origin $O$ is a center of symmetry. The central symmetry with respect to $O$ is a $\mathbb{Z}_2$ free action on $S$ and the quotient is $U$.

To determine the cohomology ring, we proceed as in the orientable case and write the coproduct in $C_*(U)$. Evidently the $e_i$’s are primitive while

\begin{equation}
D \mapsto D \otimes 1 + e_1 \otimes e_1 + \cdots + e_g \otimes e_g + 1 \otimes D
\end{equation}

Since $D$ is a cycle modulo two, \[15\] gives the coproduct in homology modulo two as well. Moreover the coproduct on classes $e_i e_j$ is as in \[14\]. Since $H_2(\text{SP}^n U; \mathbb{Z}_2)$ is generated by $D$ and $e_i e_j$, $i < j$, both formulae yield the relation

Lemma 18. $b = f_1^2 = \cdots = f_g^2$.

We now show that this is the only relation in $H^*(\text{SP}^n U; \mathbb{Z}_2)$ together with the filtration relation which demands that all $n + 1$-products be trivial.

Lemma 19. In $H^*(\text{SP}^n U; \mathbb{Z}_2)$, $(e_{i_1} \cdots e_{i_r} \text{SP}^D)^* = f_{i_1} \cdots f_{i_r} b^t$ where $r + t \leq n$ and $i_l \neq i_j$ for $l \neq j$.

Proof. Let’s go back to the integral chain complex, $\pi_* : C_*(U^n) \rightarrow C_*(\text{SP}^n U)$. We have $\pi_*(e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes D^2 \otimes \cdots \otimes D) = tle_{i_1} \cdots e_{i_r} \text{SP}^D$. Writing the coproduct for this general class is notationally very involved. We get precise enough of an idea by working out the case of $e_1 \text{ SP}^2 D$. We first write down $\delta_1^2 : C_*(X^3) \rightarrow C_*(X^0)$ on $e_1 \otimes D \otimes D$. This consists of 18 terms obtained from the product

\begin{align*}
(e_1 \otimes 1^{\otimes 2} + 1 \otimes e_1 \otimes 1^{\otimes 2}) \\
(1^{\otimes 2} \otimes D \otimes 1^{\otimes 3} + \sum 1^{\otimes 2} \otimes e_1 \otimes e_1 \otimes e_1 \otimes 1^{\otimes 2} + 1^{\otimes 3} \otimes D \otimes 1^{\otimes 2}) \\
(1^{\otimes 4} \otimes D \otimes 1 + \sum 1^{\otimes 4} \otimes e_1 \otimes e_1 + 1^{\otimes 5} \otimes D)
\end{align*}
As in the example of §3.1 we develop this expression, shuffle by \(\chi^{-1}\) and then apply \(\pi_* \otimes \pi_*\) to all terms to obtain the coproduct in \(C_\ast(\text{SP}^3U)\)

\[
e_1 \text{SP}^2D \xrightarrow{\lambda_*} e_1 \text{SP}^2D \otimes 1 + e_1 \otimes \text{SP}^2D + \text{SP}^2D \otimes e_1 + 1 \otimes e_1 \text{SP}^2D + \sum_{i<j} e_1e_i \otimes e_j + \sum_{i<j} e_i D \otimes e_j + e_i \otimes e_1 D + e_i \otimes e_1 e_j
\]

Reducing modulo two, and since all classes involved represent homology classes, we arrive at the coproduct on \(e_1 \text{SP}^2D\) in \(H_\ast(\text{SP}^3U; \mathbb{Z}_2)\). The class \(e_1 \text{SP}^2D\) is the only basis element whose image under \(\lambda_*\) involves the basis element \(e_1 D \otimes D\).

Consequently

\[(e_1 \text{SP}^2D)^* = \lambda^* (e_1 D \otimes D)^* = \lambda^* ((e_1 D)^* \otimes D^*) = \lambda^*(f_1 b \otimes b) := f_1 b^2\]

The general case is completely analogous.

Very much as in the proof of theorem 10 we can now deduce

**Proposition 20.** \(H^\ast(\text{SP}^3U; \mathbb{Z}_2)\) is generated by classes \(f_1, \ldots, f_g\) and \(b\) under the relations

(i) \(f_i^2 = b\)

(ii) \(f_i \cdots f_i b^t = 0, r + t = n + 1, i_i \neq i_j \) for \(l \neq j\).

**Example 21.** Suppose \(g = 1\) and \(U = \mathbb{R}P^2\). Then \(H^\ast(\text{SP}^3U; \mathbb{Z}_2) \cong \mathbb{Z}_2[f_1]/(f_1^3)\) which is the cohomology of \(\mathbb{R}P^4\) (see lemma 27).

**Remark 22.** We can invoke the theorem of Dold and Thom to the effect that

\[\text{SP}^\infty(X) \simeq \prod K(\tilde{H}_i(X; \mathbb{Z}), i)\]

for any finite type connected CW complex \(X\). Applying this to \(S\) and \(U\) we find

\[\text{SP}^\infty(S) \simeq (S^1)^{2g} \times \mathbb{C}P^\infty, \quad \text{SP}^\infty(U) \simeq (S^1)^{g-1} \times \mathbb{R}P^\infty\]

This is well in accordance with our homological calculations since from proposition 20 we can deduce that \(H^\ast(\text{SP}^\infty U; \mathbb{Z}_2) \cong E(h_1, \ldots, h_{g-1}) \otimes \mathbb{Z}_2[c]\) where \(h_i := f_i + f_i\) and \(c := f_1\).

5. **Clifford’s theorem and Analogs**

Let us consider the case when \(S = C\) is a smooth complex projective curve, or equivalently a compact Riemann surface. One then defines the \(n\)-th Abel-Jacobi map which is a holomorphic map

\[\mu_n : \text{SP}^n(C) \longrightarrow J(C)\]

where \(J(C)\), the “Jacobian” of \(C\), is a complex torus of dimension the genus of \(C\).

The maps \(\mu_n\) are additive in the sense that the following commutes

\[\text{SP}^r(C) \times \text{SP}^s(C) \longrightarrow \text{SP}^{r+s}(C)\]

\[\mu_r \times \mu_s \quad \mu_{r+s} \]

the bottom map being addition in the abelian torus \(J(C)\), and the top map concatenation of points. If \(C\) is an elliptic curve for example, then \(J(C) \cong C\). The
inversely preimages of $\mu_n$ are complex projective spaces $\mathbb{C}P^m$, where $m$ is related to the dimension of some complete linear series on $C$(cf. [1]). The dimension of the preimages $\mu_n^{-1}(x), x \in J(C)$ is an upper semi-continuous function of $x$. The following is classical.

**Theorem 23.** [1] Write $\mu_n : SP^nC \rightarrow J(C), n \geq 1, g$ genus of $C$.

(i) If $n < 2g$, and $y \in SP^nC$, then $\mu_n^{-1}(\mu_n(y)) = \mathbb{C}P^m(y)$ for some $m(y) \leq \frac{g}{2}$.

(ii) If $n \geq 2g$ then $\mu_n^{-1}(x) = \mathbb{C}P^{n-g}$ for all $x$.

The bounds in the theorem are sharp and are attained for hyperelliptic curves. Notice that part (ii) has a much more elaborate version due to Mattuck and asserting that $\mu_n$ is a projectivized analytic bundle projection with fiber $\mathbb{C}P^{n-g}$. The second part of this theorem is due to Clifford and the proof is algebra-geometric in nature. We now make the observation that theorem 23 is in fact a purely topological statement.

**Proposition 24.** (Clifford’s theorem : topological version) Let $S$ be a closed oriented topological surface of genus $g > 0$, and $\mathbb{C}P^m \rightarrow SP^nS$ a map that is non-trivial in homology.

(i) If $n < 2g$, then necessarily $m \leq \left\lfloor \frac{n}{2} \right\rfloor$;

(ii) if $n \geq 2g$, then $m \leq n - g$.

**Proof.** Let $h : \mathbb{C}P^m \rightarrow SP^nS$ be a map such that $h_*[\mathbb{C}P^k] \neq 0$ for some $1 \leq k \leq m$, where $[\mathbb{C}P^k]$ is the generator of $H_{2k}(\mathbb{C}P^m)$. This says that if $u \in H^2(\mathbb{C}P^m)$ is the generator, then there must be a class $x \in H^{2k}(SP^n(C))$ such that $h^*(x) = u^k$. But by the cohomology structure of $SP^n(C)$ (theorem [19]), $x$ is decomposable into a product of one dimensional generators and a single two dimensional class $b$. Write $I$ the ideal generated by the one dimensional generators. Since $h^*(x) = u^k \neq 0$, necessarily $x$ is $\pm b^k$ modulo terms in $I$, and hence $h^*(b) = \pm u$. Now $h^*(f_i) = 0$ and hence

$$\pm u^{\left\lfloor \frac{n}{2} \right\rfloor + 1} = h^*(b^{\left\lfloor \frac{n}{2} \right\rfloor + 1}) = h^*\left( \prod_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor + 1} (b - f_k f_{k+g}) \right) = 0$$

using the MacDonald relation (theorem [19]). This implies $m \leq \left\lfloor \frac{n}{2} \right\rfloor$. One uses a similar argument for (ii). \hfill \Box

The next corollary recovers the original Clifford theorem.

**Corollary 25.** Choose a complex structure on $S$. If $f : \mathbb{C}P^m \rightarrow SP^nS$ is a non-constant holomorphic map, then the conditions (i) and (ii) of proposition 24 hold.

**Proof.** It suffices to argue that a holomorphic map $f : \mathbb{C}P^m \rightarrow SP^nS$ is trivial in homology if and only if it is constant. Assume then that $f_*$ is trivial, and choose an embedding $e : SP^n(S) \rightarrow \mathbb{C}P^N$ realizing $SP^nS$ as a projective variety [1]. The composite $e \circ f : \mathbb{C}P^m \rightarrow \mathbb{C}P^N$ is also trivial in homology. If we show that $e \circ f$ is necessarily constant, then since $e$ is an embedding it follows that $f$ is constant as well and hence the claim.

Let $g : \mathbb{C}P^m \rightarrow \mathbb{C}P^N$ be a non-constant holomorphic map. Then $g$ is a finite ramified covering over its image $g(\mathbb{C}P^m)$ which is a subvariety of $\mathbb{C}P^N$. If $g(\mathbb{C}P^m)$ is not reduced to point, then it has dimension $\geq 1$ and its fundamental cycle is non-trivial in $H_*(\mathbb{C}P^N)$. This fundamental cycle is covered by a non-zero homology class in $\mathbb{C}P^m$ (by a transfer argument over $\mathbb{Q}$ for example), and hence $g_*$ cannot be trivial. So if $g_*$ is trivial, $g(\mathbb{C}P^m)$ must be reduced to point and $g$ is constant. \hfill \Box
Similarly there is a “real analog” of proposition 24 for unoriented surfaces.

**Proposition 26. (Clifford’s theorem: real version)** Let $U$ be a closed non-orientable topological surface of genus $g > 0$, and $\tau : \mathbb{R}P^n \longrightarrow SP^n U$ a map that is non-zero on homology. Then $m \leq 2n - g + 1$.

**Proof.** We take the description in lemma 20. Write $\tau$ such that $\tau^* = u$. Under the hypothesis, and proceeding as in the first part of the proof of proposition 24, there is $f_i$ such that $\tau^*(f_i) = u$. Since $f_1^n = \cdots = f_g^n = b$, we have also that $\tau^*(f_1) = \cdots = \tau^*(f_g) = u$. Note that $f_1 \cdots f_g b^{n-g+1} = 0$ in $H^*(SP^n U; \mathbb{Z}_2)$. But

$$\tau^*(f_1 \cdots f_g b^{n-g+1}) = u^g u^{2n-2g+2} = u^{2n-g+2} = 0$$

which means necessarily that $m + 1 \leq 2n - g + 2$ and hence the claim. \qed

We can see that the bound in proposition 26 is best possible since in the genus 1 case (i.e. $U = \mathbb{R}P^2$, covered by $S^2$) we have the following result [4].

**Lemma 27.** There is a homeomorphism $SP^n(\mathbb{R}P^2) \cong \mathbb{R}P^{2n}$.

**Proof.** Write $\mathbb{R}P^2 = S^2/\langle T \rangle$ where $T$ is the antipodal involution acting on $S^2$. Note that $T$ extends to an action on $SP^n(S^2)$ by acting componentwise. Now $T$ has no fixed points which implies that the fixed point set of the action on $SP^n(S^2)$ is $SP^n(S^2/\langle T \rangle) = SP^n(\mathbb{R}P^2)$. We need analyze this fixed point set.

First of all if we write $S^2 = C \cup \{\infty\}$, then the action of $T$ is $T(z) = -\bar{z}$. If on the other hand we identify $S^2$ with $\mathbb{C}P^1$ then in homogeneous coordinates we have $T([a:b]) = [-\bar{b}:a]$.

More generally identify $SP^n S^2$ with $\mathbb{C}P^n$ as in lemma 5. That is identify first $\mathbb{C}P^n$ with polynomials of degree at most $n$, modulo scalar multiplication, by sending $[a_0 : \cdots : a_n]$ to $a_0 + a_1 z + \cdots + a_n z^n$. We can then check that the map

$$SP^n S^2 \longrightarrow \mathbb{C}P^n, \quad <z_1, \cdots, z_n> \mapsto (z + z_1) \cdots (z + z_n)$$

is a homeomorphism (cf. [6], chapter 4). Note that if $z_i$ coincides with $+\infty$, then the factor “$z + \infty$” is omitted from the product. The action of $T$ on $SP^{2n} S^2$ translates to an action on polynomials $(z + z_1)(z + z_2) \cdots (z - 1/z_2) \cdots (z - 1/z_2n)$.

If $(a_0 = z_1 \cdots z_{2n}, a_1 = \sum z_1 \cdots z_i \cdots z_{2n}, \cdots, a_{2n-1} = z_1 + \cdots + z_{2n}, a_{2n} = 1)$ are the coefficients of $p(z) = (z + z_1)(z + z_2n)$, modulo scalar, then $(1, (1) a_{2n-1}, \cdots, (1) a_1, \cdots, (1) a_0)$ are the coefficients of $Tp(z)$, modulo scalars as well. After identification with $\mathbb{C}P^{2n}$, the antipodal action in homogeneous coordinates becomes

$$T([a_0 : \cdots : a_{2n}]) \longmapsto [a_{2n} : \cdots : (1) a_i : \cdots : a_0]$$

The fixed point set $F \subset \mathbb{C}P^{2n}$ of this action consists of all $[a_0 : \cdots : a_{2n}]$ such that $a_i = (1)a_{2n-i}$ up to usual scalar multiplication. By splitting into real and imaginary parts, it is easy to see that $F$ is a copy of $\mathbb{R}P^{2n}$.

6. The Dold-Thom Homotopy Splitting

Finally we point out how our previous constructions can be used to give an elementary proof of the Dold-Thom splitting of $SP^n X$ into a product of Eilenberg-MacLane spaces (remark 22).

For $X$ a two dimensional complex, theorem 6 shows that the homology of $SP^n X$ only depends on the homology of $X$. So set

$$H_2(X) = \mathbb{Z}^b, \quad H_1(X) = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r} \oplus \mathbb{Z}^a$$
We can assume that $C_r(X)$ is the chain complex $\mathbb{Z}^{r+b} \xrightarrow{\partial} \mathbb{Z}^{r+a} \xrightarrow{} 0$ sending basis elements $D_1, \ldots, D_{r+b}$ to basis elements $e_1, \ldots, e_{r+a}$ according to
\begin{equation}
\partial D_j = n_j e_j, \quad 1 \leq j \leq r, \quad \partial D_j = 0, \quad r < j \leq r + b
\end{equation}
We argue that the chain complex of $\text{SP}^\infty X$ in theorem 6 is a tensor product of a certain number of copies of the chain complexes for $\text{SP}^\infty S^1 = S^1$, $\text{SP}^\infty S^2 = \mathbb{C}P^\infty$ and $L_{n_j} = S^\infty/\mathbb{Z}_{n_j}$ the infinite Lens spaces.

It is evident indeed that the free generators in dimension one generate a subchain complex of which homology is that of $(S^1)^a$, while the free generators in dimension two contribute the homology of $(\mathbb{C}P^\infty)^b$. The subchain complex generated by $\text{SP}^i(D_j)$ and $e_i \text{SP}^i(D_j)$ for fixed $j$ and $i \geq 0$ has the homology of $L_{n_j}$. If $X$ satisfies (17), then $\text{SP}^\infty(X)$ is homologous to $Y = (S^1)^a \times (\mathbb{C}P^\infty)^b \times L_{n_1} \times \cdots \times L_{n_r}$. But $Y$ is a generalized Eilenberg-MacLane space and it classifies the cohomology of $X$. So there is a map $X \longrightarrow Y$ which extends to $\text{SP}^\infty X$ since $Y$ is an abelian monoid:
\[
\Psi : \text{SP}^\infty(X) \longrightarrow (S^1)^a \times (\mathbb{C}P^\infty)^b \times L_{n_1} \times \cdots \times L_{n_r}
\]
By the very construction of $C_r(\text{SP}^\infty X)$ and the fact that $\Psi_*$ is a multiplicative map, we readily see that $\Psi$ induces a homology isomorphism. Since both spaces involved are monoids again, they have abelian fundamental groups and are simple. The map $\Psi$ is necessarily a homotopy equivalence. We deduce

**Proposition 28.** Let $X$ be a two dimensional complex, and $H_*(X)$ as in (17). Then there is a homotopy equivalence
\[\text{SP}^\infty X \simeq (S^1)^a \times (\mathbb{C}P^\infty)^b \times L_{n_1} \times \cdots \times L_{n_r}.
\]

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Laboratoire Painlevé, Université de Lille I, Villeneuve d’Ascq, France
E-mail address: sadok.kallel@math.univ-lille1.fr

Dipartimento di matematica, Università di Roma “Tor Vergata”, Italy
E-mail address: salvator@mat.uniroma2.it