CHARACTERIZATION OF HARMONIC AND SUBHARMONIC FUNCTIONS VIA MEAN–VALUE PROPERTIES

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Abstract. We give a characterization of harmonic and subharmonic functions in terms of their mean values in balls and on spheres. This includes the converse of an inequality of Beardon’s for subharmonic functions. We also obtain integral inequalities of Harnack type between these two means in general domains.

1. Introduction and results

Let \( h \) be a harmonic function in the closed ball centred at a point \( a \) and with radius \( r \), \( B(a,r) \). Then it is well–known that \( h \) satisfies the mean–value properties

\[
  h(a) = B_a(r) := \frac{1}{|B(a,r)|} \int_{B(a,r)} h(x) \, dx
\]

and

\[
  h(a) = S_a(r) := \frac{1}{|\partial B(a,r)|} \int_{\partial B(a,r)} h(x) \, ds.
\]

In both cases, there exist converse results stating that if the equality is satisfied, then \( h \) must be harmonic. In the first case, for instance, if we assume that \( h \) is locally integrable on a domain \( \Omega \) and satisfies (1) whenever \( B(a,r) \) is contained in \( \Omega \), then we have that \( h \) is harmonic in \( \Omega \). A similar result holds for (2) provided we assume now that \( h \) is continuous in \( \Omega \) — for these classical results, see [ABR, GT], for instance.

A second, lesser known, set of converse properties to these mean value results assumes that the equalities are satisfied in a set \( \Omega \) for all integrable harmonic functions defined in \( \Omega \). Then, it is possible to show that this implies that \( \Omega \) is a ball [AG, Be, ES, Ko, Ku, PS]. In particular, [Be, Ko] and [PS] consider the case where the surface and volume averages are equal.

Motivated by this, and since we could not find any reference in the literature to a converse result of the first type when it is assumed that the two averages coincide, in this paper we consider several relations between the integral means of harmonic and subharmonic functions. We begin by considering the question of what can be said when \( h \) satisfies

\[
  B_a(r) = S_a(r)
\]

for balls in a set \( \Omega \). We prove that, under suitable assumptions, this last equality also implies that \( h \) is harmonic. More precisely, we have the following

\[
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\]

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Theorem 1.1. Let \( h \) be a continuous function in \( \Omega \). Then \( h \) is harmonic in \( \Omega \) if and only if (3) is satisfied for all balls whose closure is contained in \( \Omega \).

Remark 1. Note that in the equality (3) it is not assumed a priori that the value of the averages remains fixed when \( a \) is a given point but the radius changes.

Remark 2. Continuity is essential, since otherwise any function coinciding with a harmonic function on all but a set of finite points, for instance, would clearly satisfy (3) but not be harmonic.

From the results in [Be], [Ko] and [PS] we have that the equality of the averages (3) for all harmonic functions is possible only for balls. M. Rao has shown that in fact the averages of harmonic functions on general domains must satisfy a one–sided inequality of Harnack type [R]. Here we improve on this result establishing also a lower bound by using the methods of [Be] and [PS] to obtain the following

Theorem 1.2. Let \( \Omega \) be a domain satisfying the uniform exterior sphere condition. Then there exist constants \( 0 < c_1 \leq 1 \leq c_2 < \infty \), depending only on \( \Omega \), such that
\[
\frac{c_1}{|\partial \Omega|} \int_{\partial \Omega} h \, ds \leq \frac{1}{|\Omega|} \int_{\Omega} h \, dx \leq \frac{c_2}{|\partial \Omega|} \int_{\partial \Omega} h \, ds
\]
for all non–negative harmonic functions \( h \) defined in \( \Omega \) and which take continuous values on the boundary. If either of the constants \( c_1 \) or \( c_2 \) can be taken to be equal to one, then \( \Omega \) is a ball.

The proof of the second part of the result also uses a variation of a well–known result of Serrin’s [S], which we believe to be interesting in its own right – see Lemma 2.1 for the details.

Another natural question to ask is whether a result similar to Theorem 1.1 holds for subharmonic functions. Here we call a real–valued function \( u \) defined in \( \Omega \) subharmonic, if \( u \) is continuous and for every ball \( B \) whose closure is contained in \( \Omega \) and every harmonic function \( h \) in \( B \) satisfying \( h \geq u \) on \( \partial B \) we have that \( h \geq u \) in \( B \) [GT]. We have that subharmonic functions satisfy
\[
B_a(r) \leq S_a(r),
\]
(4)
so it makes sense to ask if this inequality characterizes subharmonic functions. In this respect we have the following

Theorem 1.3. Let \( u \) be a continuous function in \( \Omega \). Then \( u \) is subharmonic in \( \Omega \) if and only if (4) holds for all balls whose closure is contained in \( \Omega \).

From the proof of Theorem 1.2 it follows that the second inequality in that result also holds for \( C^2 \) subharmonic functions and this can easily be improved to \( C^0 \) sub-harmonic functions. Also for particular domains or in dimension 1 one easily exhibits sequences of positive subharmonic functions for which the volume average is bounded and the surface average becomes unbounded. This can be improved for relatively general domains provided the geometry of the boundary allows a sufficiently rich supply of harmonic subfunctions, e.g., if the domain satisfies the uniform exterior sphere condition.
Theorem 1.4. Let $\Omega$ be as above, and $u$ be a non-negative subharmonic function in $\Omega$ and continuous in $\overline{\Omega}$. Then there exists a constant $c_2$ as in Theorem 1.2 such that
\begin{equation}
\frac{1}{|\Omega|} \int_{\Omega} u \, dx \leq \frac{c_2}{|\partial\Omega|} \int_{\partial\Omega} u \, ds.
\end{equation}

On the other hand, there exists a sequence of positive subharmonic functions $u$ for which the volume average in $\Omega$ remains bounded, while the surface average is unbounded.

Finally, we consider a property of subharmonic functions proved by Beardon [B], which states that if $u$ is subharmonic in a set $\Omega$, then
\begin{equation}
S_r(\kappa r) \leq B_a(r)
\end{equation}
for all balls whose closure is contained in $\Omega$ and where
\begin{equation}
\kappa = \begin{cases} 
\frac{1}{2}, & n = 1 \\
e^{-1/2}, & n = 2 \\
\left(\frac{2}{n}\right)^{1/(n-2)}, & n \geq 3.
\end{cases}
\end{equation}

We prove that this property again characterizes subharmonic functions. More precisely, we have

Theorem 1.5. Let $u$ be a continuous function defined on a domain $\Omega$ and assume that condition (5) is satisfied for all balls whose closure is contained in $\Omega$ and for some $\kappa$ satisfying
\begin{equation}
0 < \kappa \leq \kappa_1 := \frac{n}{4} + \frac{1}{2} \sqrt{\frac{n^2}{4} + 2n}.
\end{equation}

Then $u$ is subharmonic in $\Omega$.

Since the upper bound for $\kappa$ in the theorem is larger than (or equal to, in the case where $n$ is one) the values of $\kappa$ given in (6), this yields a converse of Beardon’s property.

2. Harmonic functions

2.1. Proof of Theorem 1.1. Assume that (3) holds for a given fixed point $a$ in $\Omega$, and all $r$ in $(0, R)$, for some positive number $R$ such that $\overline{B(a, R)}$ is contained in $\Omega$. Denoting by $\omega_n$ the volume of the unit ball in $\mathbb{R}^n$ we have that
\begin{equation}
\frac{1}{\omega_n r^n} \int_{B(a, r)} h(x) \, dx = \frac{1}{n \omega_n r^{n-1}} \int_{\partial B(a, r)} h(x) \, ds
\end{equation}
and thus
\begin{equation}
n \int_0^r \left[ \int_{\partial B(a, t)} h(x) \, ds \right] \, dt = r \int_{\partial B(a, r)} h(x) \, ds.
\end{equation}

Letting
\begin{equation}
\varphi(r) = \int_{\partial B(a, r)} h(x) \, ds,
\end{equation}

\begin{equation}
\int_{\partial B(a, r)} h(x) \, ds,
\end{equation}

\begin{equation}
\frac{n}{4} + \frac{1}{2} \sqrt{\frac{n^2}{4} + 2n}.
\end{equation}

Since the upper bound for $\kappa$ in the theorem is larger than (or equal to, in the case where $n$ is one) the values of $\kappa$ given in (6), this yields a converse of Beardon’s property.
we have that
\[ n \int_0^r \varphi(t) dt = r \varphi(r). \]

Since \( h \) is continuous, we can differentiate this with respect to \( r \) which gives that \( \varphi \) satisfies the differential equation
\[ r \varphi'(r) + (1 - n) \varphi(r) = 0. \]

Hence \( \varphi(r) = cr^{n-1} \) for some constant \( c \) and
\[ \frac{c}{n \omega_n} = \frac{\phi(r)}{n \omega_n r^{n-1}} = S_a(r), \]
from which it follows that, for \( a \) and \( r \) as above, the average \( B_a(r) \) is independent of \( r \). Since \( h \) is continuous, the value taken by this average must be attained by \( h \) for some point in the ball \( B(a, r) \). As this holds for arbitrarily small \( r \), we have that this value must coincide with \( h(a) \). Applying now the converse result to the mean–value equality (1) completes the proof of Theorem 1.1.

### 2.2. The integral Harnack inequality.

Let \( v \) be the solution of the equation
\[ \begin{cases} \Delta v + 1 = 0, & x \in \Omega \\ v = 0, & x \in \partial \Omega. \end{cases} \]

We have that
\[ \frac{1}{|\Omega|} \int_\Omega h \, dx = -\frac{1}{|\Omega|} \int_\Omega h \Delta v \, dx = \frac{1}{|\Omega|} \int_{\partial \Omega} h \left( -\frac{\partial v}{\partial \nu} \right) \, ds. \]

Since there exist constants \( c_1 \) and \( c_2 \) such that
\[ 0 < c_1 < -\frac{\partial v}{\partial \nu} < c_2 < \infty, \]
the inequalities follow.

If \( c_1 \) can be taken to be equal to 1, we have that
\[ \int_{\partial \Omega} h \left( \frac{1}{|\partial \Omega|} + \frac{1}{|\Omega|} \frac{\partial v}{\partial \nu} \right) \, ds \leq 0. \]

Let \( c \) be a positive real number such that
\[ h_0 = c + \frac{1}{|\partial \Omega|} + \frac{1}{|\Omega|} \frac{\partial v}{\partial \nu} \]
is positive, and choose now \( h \) to be equal to \( h_0 \) on the boundary. Since
\[ \int_{\partial \Omega} \frac{1}{|\partial \Omega|} + \frac{1}{|\Omega|} \frac{\partial v}{\partial \nu} \, ds = 0, \]
replacing \( h \) in (8) gives
\[ \int_{\partial \Omega} \left( \frac{1}{|\partial \Omega|} + \frac{1}{|\Omega|} \frac{\partial v}{\partial \nu} \right)^2 \, ds \leq 0, \]
from which it follows that \( \partial v/\partial \nu \) is constant on the boundary. By Serrin’s result [5], \( \Omega \) must be a ball.

If \( c_2 \) equals 1, we now have
\[ \frac{1}{|\Omega|} \int_{\partial \Omega} h \left( -\frac{\partial v}{\partial \nu} \right) \, ds \leq \frac{1}{|\partial \Omega|} \int_{\partial \Omega} h \, ds \]
and taking \( h = -\partial v / \partial \nu \) on the boundary yields
\[
\frac{1}{|\Omega|} \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} \right)^2 \, ds \leq \frac{1}{|\partial \Omega|} \int_{\partial \Omega} \left( - \frac{\partial v}{\partial \nu} \right) \, ds = \frac{|\Omega|}{|\partial \Omega|}.
\]
The result now follows from the following lemma, which is a variation on Serrin’s result [\S].

**Lemma 2.1.** Let \( v \) be the solution of equation (7). Then
\[
\frac{1}{|\partial \Omega|} \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} \right)^2 \, ds \geq \left( \frac{|\Omega|}{|\partial \Omega|} \right)^2, \tag{9}
\]
with equality if and only if \( \Omega \) is a ball.

**Proof.** By the Cauchy–Schwarz inequality we have that
\[
\int_{\partial \Omega} \left( - \frac{\partial v}{\partial \nu} \right) \, ds \leq \frac{1}{|\partial \Omega|} \left( \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} \right)^2 \, ds \right)^{1/2} = \left( \frac{|\Omega|}{|\partial \Omega|} \right)^2 \int_{\Omega} \Delta v \, dx.
\]
Thus
\[
\frac{1}{|\partial \Omega|} \int_{\partial \Omega} \left( \frac{\partial v}{\partial \nu} \right)^2 \, ds \geq \left( \frac{1}{|\Omega|} \int_{\partial \Omega} \frac{\partial v}{\partial \nu} \, ds \right)^2 = \left( \frac{1}{|\partial \Omega|} \int_{\Omega} \Delta v \, dx \right)^2 = \left( \frac{|\Omega|}{|\partial \Omega|} \right)^2.
\]
On the other hand, equality holds in (9) if and only if \( \partial v / \partial \nu \) is constant on the boundary and thus, by Serrin’s result, \( \Omega \) must be a ball.

3. **Subharmonic Functions**

3.1. **Proof of Theorems 1.3 and 1.4.** We begin by proving the following

**Lemma 3.1.** If \( u \) is a continuous function in \( \Omega \) such that \( B_a(r) \leq S_a(r) \) holds for all balls whose closure is contained in \( \Omega \), then \( u \) satisfies the maximum principle in \( \Omega \).

**Proof.** Let \( M \) be the maximum of \( u \) in \( \overline{\Omega} \), and assume that this is attained at an interior point. Define
\[
\mathcal{A} = \{ x \in \Omega : u(x) = M \}
\]
From the continuity of \( u \) we have that \( \mathcal{A} \) is closed in \( \Omega \) and that \( S \) is a continuous function of the radius. Let \( x_0 \) be a point in \( \mathcal{A} \), and take \( r_1 \) such that \( B(x_0, r_1) \) is in \( \Omega \). We have that either there exists \( \mathbf{r} \) in \((0, r_1]\) such that \( S_{x_0}(\mathbf{r}) < M \) or not. In the second case, \( S_{x_0}(r) = M \) for all \( r \) in \((0, r_1]\), which immediately yields that \( x_0 \) is an interior point of \( \mathcal{A} \).

In the first case, let
\[
r_* = \inf \{ r \in (0, \mathbf{r}) : S_{x_0}(r) = S_{x_0}(\mathbf{r}) \}.
\]
Again by continuity, we have that \( r_* \) is positive and also that \( S_{x_0}(r_*) = S_{x_0}(\mathbf{r}) < M \).

Hence
\[
\omega_n r_*^n B_{x_0}(r_*) = \int_{B(x_0, r_*)} v \, dx = n \omega_n \int_0^{r_*} r^{n-1} S_{x_0}(r) \, dr > \omega_n r_*^n S_{x_0}(r_*)
\]
and thus \( B_{x_0}(r_*) > S_{x_0}(r_*) \), contradicting the hypothesis.
Proof of Theorem 1.3: Assume that \( w \) satisfies the inequality \( \mathcal{B}_w(r) \leq \mathcal{S}_w(r) \) but that it is not subharmonic. Then, there exists a ball \( B \) and a function \( h \) harmonic in \( \mathcal{B} \) such that \( \mathcal{B} \) is contained in \( \Omega \) and \( h \geq w \) on \( \partial B \), but \( h(x_0) < w(x_0) \) for some \( x_0 \) in \( B \). Since the function \( w - h \) also satisfies the inequality but not the maximum principle, we have a contradiction and hence \( w \) must be subharmonic.

Proof of Theorem 1.4: Let \( v \) be subharmonic. Let \( h \) be the harmonic solution of the Dirichlet problem with boundary values specified by \( v \). Then \( v - h \) is sub harmonic and the maximum principle shows that \( v \leq h \). Hence Theorem 1.2 applied to \( h \) implies

\[
\frac{1}{|\Omega|} \int_{\Omega} v \, dx \leq \frac{1}{|\Omega|} \int_{\Omega} h \, dx \leq \frac{c_2}{|\partial\Omega|} \int_{\partial\Omega} h \, ds = \frac{c_2}{|\partial\Omega|} \int_{\partial\Omega} v \, ds.
\]

For the second part of the statement consider first the trivial one dimensional case. Assume (without loss of generality) \( \Omega = (0, 1) \) and let, for \( k \) a positive integer, \( v_k(x) = \max\{1, k - k^2x, k + k^2(x - 1)\} \). Then the surface average is \( k \) and the volume average is bounded.

To generalize the one dimensional case we will need for each element of the sequence a maximum of a finite number of harmonic functions which is large in a neighborhood of \( \partial\Omega \) and small elsewhere. To deal with a fairly general geometry of \( \partial\Omega \) requires using in the construction harmonic functions other than affine functions and will require a covering argument to control the oscillation of these on \( \partial\Omega \). For domains satisfying the uniform exterior sphere condition translating and rescaling fundamental solutions will be enough. Details follow below.

Let \( k > 2 \) be an integer and \( n > 1 \). For each \( y \in \partial\Omega \), let \( B_{\delta_k}(\overline{y}) \) denote a closed ball such that \( \overline{B_{\delta_k}(y)} \cap \overline{\Omega} = \{y\} \) and \( \delta_k > 0 \). Here \( \delta_k \) is a sequence of positive numbers smaller than the radius in the uniform exterior sphere condition and verifying \( \delta_k \to 0 \).

Denote the fundamental solution of the Laplace operator by \( \Psi(|x|) \). Define, for each \( k > 2 \) and \( y \in \partial\Omega \)

\[
v^k_y(x) = -k^2\Psi(|x - \overline{y}|/\delta_k) + k^2\Psi(|y - \overline{y}|/\delta_k) + k.
\]

Clearly \( v^k_y(y) = k \) and \( v^k < k \) on \( \partial\Omega \setminus \{y\} \). By continuity and compactness there is a finite number (dependent on \( k \)) of points \( y_j^k \in \partial\Omega \) and corresponding balls \( B_{r_{y_j^k}}(y_j^k) \) forming a covering of \( \partial\Omega \) such that each \( v^k_{y_j^k} > k - 1 \) in \( B_{r_{y_j^k}}(y_j^k) \).

Let \( v^k = \max\{\max_j v^k_{y_j}, 1\} \). Then \( v^k \) is subharmonic, \( k \geq v^k \geq 1 \) in \( \overline{\Omega} \) and \( v^k > k - 1 \) on \( \partial\Omega \).

Let \( \Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) \geq \delta\} \). If \( z \in \Omega_\delta \) then \( |z - \overline{y_j^k}| \geq 2\delta_k \) for all \( \overline{y_j^k} \) and \( v^k_{y_j^k}(z) \leq -k^2\Psi(2) + k^2\Psi(1) + k \) implying that, for sufficiently large \( k \), \( v^k = 1 \) in \( \Omega_\delta \). This allows, for sufficiently large \( k \), the estimates

\[
\int_{\partial\Omega} v^k \, ds > |\partial\Omega|(k-1),
\]

\[
\int_{\Omega} v^k \, dx = \int_{\Omega_\delta} 1 + \int_{\Omega \setminus \Omega_\delta} v^k \, dx \leq |\Omega| + k|\Omega \setminus \Omega_\delta|.
\]

The result follows by choosing the sequence \( (\delta_k)_{k>2} \) in such a way that it implies boundedness of the last expression.
3.2. **Beardon’s property.** The converse of Beardon’s property follows from three facts: a standard mollification argument that shows the result holds if it holds for smooth functions, a computation for smooth functions that shows that Beardon’s inequalities holding for balls centered at a point \( x_0 \) imply the Laplacian is non-negative at \( x_0 \), provided \( \kappa < \kappa_1 \), and a well known device to extend the latter result to \( \kappa \leq \kappa_1 \). We present these as three lemmas.

**Lemma 3.2.** Assume \( u \) is a continuous function for which (1.2) holds and is not subharmonic. Then a smooth subharmonic function exists for which the same properties hold.

**Proof.** Let \( u \) be such a continuous function. Then, in some point \( \partial \Omega \) in the interior of some ball \( B \subseteq \Omega \), \( u \) is bigger then the solution \( h \) of Dirichlet’s problem for the the Laplace equation in \( B \) having \( u \) as boundary data. Denote \( \gamma \equiv u(\partial \Omega) - h(\partial \Omega) > 0 \).

Let \( \delta = \text{dist}(B, \partial \Omega) \) and \( 0 < \epsilon < \delta/4 \) and consider the usual mollifiers \( \rho_* \) supported in \( B_r(0) \), \( \varphi \) a continuous cut-off function which is 0 in \( \Omega \setminus \Omega_{\delta/4} \) and 1 in \( \Omega_{\delta/2} \) and extend \( u \) by 0 in the complement of \( \Omega \).

Then Beardon’s inequality holds for \( \rho_*(w \varphi) \) in \( \Omega_{\delta/4} \). To check this statement let \( B_r(a) \in \Omega_{\delta/4} \) and notice that

\[
\frac{1}{r^{n-1}K^n \omega_n} \int_{\partial B_r(a)} \rho_*(w \varphi) \, dS = \frac{1}{r^{n-1}K^n \omega_n} \int_{\partial B_r(a)} \left( \int_{B_r(0)} \rho_e(y) u(x-y) \varphi(x-y) \, dy \right) \, dS(x)
\]

\[
= \frac{1}{r^{n-1}K^n \omega_n} \int_{B_r(0)} \left( \int_{\partial B_r(a)} \rho_e(y) u(x-y) \, dS(x) \right) \, dy
\]

\[
\leq \frac{n}{r^n \omega_n} \int_{B_r(0)} \left( \int_{B_r(a)} \rho_e(y) u(x-y) \, dx \right) \, dy
\]

\[
= \frac{n}{r^n \omega_n} \int_{B_r(a)} \left( \int_{B_r(0)} \rho_e(y) u(x-y) \, dy \right) \, dx
\]

\[
= \frac{n}{r^n \omega_n} \int_{B_r(a)} \rho_*(w \varphi) \, dx.
\]

As \( \rho_*(w \varphi) \to u \) uniformly on \( \partial B \cup (\partial \Omega) \) as \( \epsilon \to 0 \), we have, for sufficiently small \( \epsilon > 0 \), \( \rho_*(w \varphi) - \gamma/2 < h \) on \( \partial B \) and \( \rho_*(w \varphi)(\partial \Omega) - \gamma/2 > h(\partial \Omega) \).

Hence \( \rho_*(w \varphi) - \gamma/2 \) has all the desired properties in \( \Omega_{\delta/4} \). \( \square \)

**Lemma 3.3.** Let \( u \in C^2(B_R(a)) \) and assume (1.3) holds for \( u \) for some \( \kappa \) satisfying \( 0 < \kappa < \kappa_1 \) and all \( r \) satisfying \( 0 < r < R \). Then \( \Delta u(a) \geq 0 \).

**Proof.** Assume \( a = 0 \). Beardon’s inequality can be rewritten as

\[
0 \leq \frac{n}{r^n} \int_0^r \left( \int_{\partial B_t(0)} u(x) \, dS(x) \right) \, dt - \frac{1}{r^{n-1}K^n \omega_n} \int_{\partial B_{rt}(0)} u(x) \, dS(x)
\]

\[
= \frac{n}{r^n} \int_0^r \left( \int_{\partial B_t(0)} u(tx) \, dS(x) \right) \, dt - \int_{\partial B_t(0)} u(\kappa t x) \, dS(x).
\]
Let now
\[ \psi(r) = n \int_0^r \left( t^{n-1} \int_{\partial B_1(0)} u(tx) \, dS(x) \right) \, dt - r^n \int_{\partial B_1(0)} u(k rx) \, dS(x). \]
As \( \psi \geq 0 \) we have that
\[ \lim_{r \to 0} \frac{\psi(r)}{r^{n+2}} \geq 0, \]
which in turn equals
\[ \lim_{r \to 0} \frac{\psi'(r)}{r^{n+1}}, \]
provided the latter limit exists. Now
\[ \psi'(r) = nr^{n-1} \int_{\partial B_1(0)} u(tx) \, dS(x) - nr^{n-1} \int_{\partial B_1(0)} u(k rx) \, dS(x) \]
\[ - r^n \int_{\partial B_1(0)} \kappa \nabla u(k rx) \cdot x \, dS(x) \]
\[ = nr^{n-1} \int_{\partial B_1(0)} \left[ \int_{\kappa r}^r \frac{d}{dt} (u(tx)) \, dt \right] \, dS(x) - r^n \int_{\partial B_1(0)} \kappa \nabla u(k rx) \cdot x \, dS(x) \]
\[ = nr^{n-1} \int_{\partial B_1(0)} \left[ \int_{\kappa r}^r \nabla u(tx) \cdot x \, dt \right] \, dS(x) - r^n \int_{\partial B_1(0)} \kappa \nabla u(k rx) \cdot x \, dS(x) \]
\[ = nr^{n-1} \int_{\kappa r}^r \int_{\partial B_1(0)} \nabla u(tx) \cdot x \, dS(x) \, dt - r^n \int_{\partial B_1(0)} \kappa \nabla u(k rx) \cdot x \, dS(x) \]
\[ = nr^{n-1} \int_{\kappa r}^r \int_{B_1(0)} t \Delta u(tx) \, dx \, dt - r^{n+1} \int_{B_1(0)} \kappa^2 \Delta u(k rx) \, dx, \]
where the last step follows by applying the divergence theorem. Thus,
\[ \lim_{r \to 0} \frac{\psi'(r)}{r^{n+1}} = \lim_{r \to 0} \left[ \frac{n}{r^2} \int_{\kappa r}^r \int_{B_1(0)} t \Delta u(tx) \, dx \, dt - \kappa^2 \int_{B_1(0)} \Delta u(k rx) \, dx \right] \]
\[ = \lim_{r \to 0} \left[ \frac{n}{r^2} \int_{\kappa r}^r \int_{B_1(0)} t \Delta u(tx) \, dx \, dt - \kappa^2 \int_{B_1(0)} \Delta u(0) \, dx \right]. \]
Since
\[ \lim_{r \to 0} \left[ \frac{n}{r^2} \int_{\kappa r}^r \int_{B_1(0)} t \Delta u(tx) \, dx \, dt \right] = \frac{n}{2} \int_{B_1(0)} (1 - \kappa) \Delta u(0) \, dx, \]
we finally obtain that
\[ \left[ \frac{n}{2} (1 - \kappa) - \kappa^2 \right] \Delta u(0) \geq 0, \]
and thus if \( \kappa \in (0, \kappa_1) \), it follows that \( \Delta u(0) \geq 0 \).

The next lemma is only relevant in establishing the converse of Beardon’s property in the case \( n = 1 \).

**Lemma 3.4.** The previous lemma holds if \( \kappa = \kappa_1 \).
Proof. Again assume \( a = 0 \). Assume Beardon’s inequality holds for some smooth function \( u \) with \( \kappa = \kappa_1 \). Let \( \kappa = \kappa_1 \). Let \( v(x) = \|x\|^2 \). Then
\[
\frac{1}{\omega_n |\kappa|^{n-1} r^{n-1}} \int_{\partial B_r} v(x) \, dS = \kappa^2 r^2
\]
and
\[
\frac{n}{\omega_n r^n} \int_{B_r} v(x) \, dx = \int_0^r \left( \int_{\partial B_{r^p}} |\rho|^2 \, dS \right) \, d\rho = \frac{n}{n+2} r^2.
\]
Hence Beardon’s property holds for \( v \) with \( \kappa = \kappa_0 \), hence it holds for \( u + \epsilon v \) for any \( \epsilon > 0 \) and with \( \kappa = \kappa_0 \) and consequently \( \Delta u(0) + 2n\epsilon = \Delta (u + \epsilon v)(0) \geq 0 \) for any \( \epsilon > 0 \). But then \( \Delta u(0) \geq 0 \).

From the proof of Lemma 3.3 we have that if \( B_\kappa(r) \leq S_\kappa(\kappa r) \) for \( \kappa > \kappa_1 \) then
\[
\left[ \frac{n}{2} (1 - \kappa) - \kappa^2 \right] \Delta u(a) \leq 0,
\]
and we again conclude that \( \Delta u(a) \geq 0 \). This of course already followed from Theorem 1.3 (of which this argument gives a different proof). On the other hand, we have that for \( u(x) = \|x\|^{2p} \) (\( p \) integer)
\[
B_0(r) = \frac{nr^{2p}}{2p + n}, \quad \text{while} \quad S_0(r) = \kappa^{2p} r^{2p},
\]
giving that \( B_0(r) \leq S_0(\kappa r) \) provided
\[
\kappa \geq \left( \frac{n}{2p + n} \right)^{1/(2p)},
\]
which converges to one as \( p \to \infty \). Thus we see that the best possible value of \( \kappa \) for this property of subharmonic functions is one.

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