The Cubic Equation Made Simple

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Abstract

This article introduces an intuitive function MY that simplifies solving cubic equations without venturing into the complex space. To many, it’s quite strange that cubic root(s) are expressed using trigonometric functions in the three-real-roots case versus real-radicals in the one-real-root case. Yet, the MY function provides a different perspective to this oddity and shows that the transition between the two worlds is actually smooth.

Although MY’s behavior resembles power functions such as $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$ and $x^{\frac{1}{5}}$, it cannot be expressed in real radicals. That said, we succeeded in proving an accurate closed form algebraic approximation of MY. So yes, casus irreducibilis still holds, but real radicals can get you very close!

1 Introduction

The cubic equation holds a special place in the history of mathematics. In the early 16th century, the cubic formula was discovered independently by Niccolò Fontana Tartaglia and Scipione del Ferro. Italy at the time was famed for intense mathematical duels. In 1535, Tartaglia was challenged by Antonio Fior, del Ferro’s student, with Tartaglia winning the famous contest. Gerolamo Cardano then persuaded Tartaglia to share his method with him, promising to not reveal it without giving Tartaglia time to publish. Once Cardano learned about Del Ferro’s work, which predated Tartaglia’s, he decided that his promise could be legitimately broken and published the method in his book “Ars Magna” in 1545. This led to a dramatic decade-long feud between Tartaglia and Cardano [1].

Later on, other methods were developed including a trigonometric solution for cubic equations with three real roots by François Viète (René Descartes expanded on Viète’s work) [2]. Joseph Louis Lagrange followed with a new uniform method to solve lower degree (less than 5) polynomial equations including the cubic [3]. In 1683, Ehrenfried Walther von Tschirnhaus [4] proposed a new approach using the Tschirnhaus transformation. In addition, the authors identified other more recent methods and approaches to solving the cubic equation [5][6][7][8] and developed one of their own [9].
In this article, a different approach to solving general cubic equations is discussed by introducing a new function MY that provides the real roots uniformly under various cases. MY is then expressed in closed form and hypergeometric form. We then proceed by using an algebraic iteration method that converges globally towards MY.

While casus irreducibilis [10] for cubic equations states that real-valued roots of irreducible cubic polynomials cannot be expressed in radicals without introducing complex numbers, we came really close with real radicals. Finally, the article is concluded by discussing many of the unique properties of MY.

2 Canonical function

Without loss of generality, let’s consider the depressed cubic equation:

\[ y^3 + py + q = 0 \]  (1)

Where \( p \) and \( q \) are real numbers.

Outside the trivial special cases of \( p=0 \) or \( q=0 \), equation (1) can be transformed to a canonical form:

\[ \frac{z^3 + z^2}{2} = t \]  (2)

Two transformations can be used:

1. **Transformation 1**: A change of variable \( z = \frac{y}{py} \) leads to: \( t = -\frac{q^2}{2p^3} \).

2. **Transformation 2**: When \( p < 0 \), a change of variable \( z = \frac{y}{\sqrt{-3p}} - \frac{1}{3} \) leads to:

\[ t = \frac{1}{27} - \frac{q}{2\sqrt{-27p^3}} \]

The canonical function \( f \) is defined in \( \mathbb{R} \) as:

\[ f : z \mapsto \frac{z^3 + z^2}{2} \]

For a general cubic equations: \( ax^3 + bx^2 + cx + d = 0 \) where \( a \neq 0 \), \( b \), \( c \) and \( d \) are real numbers, a change of variable \( y = x + \frac{b}{3a} \) leads to the depressed.
Using the sign of the derivative, there are three intervals where $f$ is monotonic:

1. $]-\infty,-\frac{2}{3}[$, $f$ is increasing from $-\infty$ to $\frac{2}{27}$. The point $M(-\frac{2}{3}, \frac{2}{27})$ is a local maximum.

2. $[-\frac{2}{3}, 0]$, $f$ is decreasing from $\frac{2}{27}$ to 0. The point $O(0,0)$ is a local minimum.

3. $[0, +\infty[$, $f$ is increasing from 0 and $+\infty$.

Together, these properties define the number of roots of the canonical equation $f(z) = x$ (see Figure 1):

1. Scenario 1: $x > \frac{2}{27}$, there is a unique solution that is above $\frac{1}{3}$.

2. Scenario 2: $x < 0$, there is a unique negative solution.

3. Scenario 3: $0 \leq x \leq \frac{2}{27}$ there are three real solutions (two of which may coincide). One root is positive and the other two are negative.

![Figure 1: Canonical function $f$](image-url)
3 Geometric intuition behind proposed method

To provide the intuition behind our proposed method for solving cubic equations, notice that the inflection point \( I \left(-\frac{1}{3}, \frac{1}{27}\right) \) is also a symmetry point. This can be expressed analytically as:

\[
f \left(-\frac{2}{3} - z\right) = \frac{2}{27} - f(z) \quad \text{for all } z \in \mathbb{R}
\]

Therefore, the problem of solving the equation \( f(z) = x \) for \( x \in \mathbb{R} \) is reduced to solving equations \( f_{|\mathbb{R}^+}(z) = a \) for positive real numbers \( a \). Let’s illustrate this point geometrically by considering the following example \( f(z) = x \) where \( x = 0.05 \) (see Figure 2). This equation has three real solutions \( z_1, z_2, z_3 \).

1. Construct the restricted curve \( f_{|\mathbb{R}^+} \), as well as the lines \( y = x \) and \( y = \frac{2}{27} - x \). The abscissas of the intersections of \( f_{|\mathbb{R}^+} \) with these two lines are respectively \( z_1 \), the positive root of the equation, and \( z' \).

2. Let \( z_2 \) be the reflection of \( z' \) with respect to \( -\frac{1}{3} \): \( z_2 = -2/3 - z' \). Using the symmetry property, \( z_2 \) is a root of the equation.
3. Let $z_3$ be the unique negative point such as the distance between $z_1$ and $-\frac{1}{3}$ is the same as the distance between $z'$ and $z_3$. In other words $z_3 = -1/3 - z' - z_1$. Therefore $z_1 + z_2 + z_3 = -1$. Using Vieta’s formula, $z_3$ is the third root.

4. **MY function definition**

The restriction, $f_{|\mathbb{R}^+}$, of $f$ to $\mathbb{R}^+$ is strictly increasing, continuous with $f_{|\mathbb{R}^+}(0) = 0$ and $\lim_{z \to +\infty} f_{|\mathbb{R}^+}(z) = +\infty$. Therefore it is bijective from $\mathbb{R}^+$ to $\mathbb{R}^+$ and admits a reciprocal function:

$$MY : x \mapsto MY(x) = f_{|\mathbb{R}^+}^{-1}(x) \quad x \in \mathbb{R}^+$$

the graph of $MY$ is symmetrical to the graph of $f_{|\mathbb{R}^+}$ with respect to the line $y = x$ (See Figure 3).

![Figure 3: Left: MY inverse of f Right: MY versus power functions](image)

$MY$ is continuous, strictly increasing and infinitely differentiable. Its behavior resembles power functions (See Figure 3): When $x$ is close to 0, $MY(x)$ is equivalent to $\sqrt[2]{x}$. For large values of $x$, $MY(x)$ behaves like $\sqrt[3]{2x}$ (see section 8, Properties of $MY$). $x^{\frac{1}{3}}$ is an upper bound for $MY$. 

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Thanks to Cardano and Vieta’s trigonometric formulas for cubic roots, \( MY \) can be expressed in closed form. Define:

\[
u = x - \frac{1}{27}\]

1. For \( x \in \left[\frac{2}{27}, +\infty\right] \)

\[
MY(x) = -\frac{1}{3} + \sqrt[3]{u + \sqrt{u^2 - \left(\frac{1}{27}\right)^2}} + \sqrt[3]{u - \sqrt{u^2 - \left(\frac{1}{27}\right)^2}}
\]

2. For \( x \in \left[0, \frac{2}{27}\right] \)

\[
MY(x) = -\frac{1}{3} + \frac{2}{3} \cos \left(\frac{\arccos(27u)}{3}\right)
\]

Notice that \( MY \) is continuous and smooth at \( x = \frac{2}{27} \) despite two totally different closed form expressions!

## 5 Cubic roots expressed in \( MY \)

### 5.1 Solving the canonical equation

Roots of the canonical equation \( f(z) = x \) can be expressed in a simple form using \( MY \):

1. For \( x > \frac{2}{27} \), there is a unique real solution \( z_1 = MY(x) \).

2. For \( x < 0 \), there is a unique real solution \( z_1 \). Using the symmetry property:

\[
f \left( -\frac{2}{3} - z_1 \right) = \frac{2}{27} - x \quad \text{it follows that} \quad z_1 = -\frac{2}{3} - MY \left( \frac{2}{27} - x \right)
\]

3. For \( 0 \leq x \leq \frac{2}{27} \) there are three real solutions. \( z_1 = MY(x) \) is the positive solution. The two other negative solutions are obtained, first, by symmetry:

\[
 z_2 = -\frac{2}{3} - MY \left( \frac{2}{27} - x \right)
\]
second, by using Vieta’s formula\(^2\), \(z_1 + z_2 + z_3 = -1\):

\[ z_3 = MY \left( \frac{2}{27} - x \right) - MY(x) - \frac{1}{3} \]

Given the sign of the three roots: \(z_2, z_3 \leq 0 \leq z_1\). In addition:

\[ z_3 - z_2 = \frac{1}{3} + 2MY \left( \frac{2}{27} - x \right) - MY(x) \]

Since \(MY(x) \leq MY\left(\frac{2}{27}\right) = \frac{1}{3}\):

\[ z_2 \leq z_3 \leq z_1 \quad (4) \]

## 5.2 Solving the depressed cubic

Using the results from section 2 and section 5.1, we can express the roots of the depressed equation:

\[ y^3 + py + q = 0 \quad (1) \]

Indeed, equation (1) can be transformed to a canonical form using either Transformation 1 or Transformation 2. Interestingly, each transformation leads to a different expression of the roots. In particular, when \(p < 0\) both transformations can be applied which leads to useful equalities. For this purpose, let’s define:

\[ \xi = \frac{3q}{2p} \sqrt{-\frac{3}{p}} \]

We distinguish 4 cases:

**Case 1:** \(p = 0\). There is a unique real solution \(\alpha = -\sqrt[3]{q}\).

**Case 2:** \(p > 0\). There is a unique real solution:

\[ \alpha = \frac{q}{p \left( -\frac{2}{3} - MY \left( \frac{2}{27} + \frac{q^2}{2p^3} \right) \right)} \]

\(^2\text{Also, for } x \neq 0 \text{, } z_3 = 2x/(z_1 z_2)\)
**Case 3:** $p < 0$ and $|\xi| > 1$. There is a unique real solution with two expressions:

$$\alpha = -\sqrt{-\frac{p}{3}} \left( 3MY \left( \frac{1 + |\xi|}{27} \right) + 1 \right) = \frac{q}{pMY \left( -\frac{q^2}{2p^3} \right)}$$

**Case 4:** $p < 0$ and $-1 \leq \xi \leq 1$. There are three real solutions that can be expressed using Transformation 2 as:

$$\alpha = \sqrt{-\frac{p}{3}} \left( 3MY \left( \frac{1 + \xi}{27} \right) + 1 \right)$$

$$\beta = 3\sqrt{-\frac{p}{3}} \left( MY \left( \frac{1 - \xi}{27} \right) - MY \left( \frac{1 + \xi}{27} \right) \right)$$

$$\gamma = -\sqrt{-\frac{p}{3}} \left( 3MY \left( \frac{1 - \xi}{27} \right) + 1 \right)$$

As a result of (4):

$$\gamma \leq \beta \leq \alpha$$

When $q \neq 0$, these roots can be expressed differently using Transformation 1:

$$\alpha' = \frac{q}{pMY \left( -\frac{q^2}{2p^3} \right)}$$

$$\beta' = \frac{q}{p \left( MY \left( \frac{2}{27} + \frac{q^2}{2p^3} \right) - MY \left( -\frac{q^2}{2p^3} \right) - \frac{1}{3} \right)}$$

$$\gamma' = \frac{-q}{p \left( \frac{2}{3} + MY \left( \frac{2}{27} + \frac{q^2}{2p^3} \right) \right)}$$

Alternatively, $\beta'$ could be derived from any of Vieta’s formulas for equation (1). For example:

$$\alpha' + \beta' + \gamma' = 0$$

In addition, using the order of the three roots, when $q < 0$:

$$\alpha' = \alpha \quad \beta' = \gamma \quad \gamma' = \beta$$

and when $q > 0$:

$$\alpha' = \gamma \quad \beta' = \alpha \quad \gamma' = \beta$$
Naturally, the roots coincide with the trigonometric roots provided by François Viète:

\[ t_k = 2 \sqrt{-\frac{p}{3}} \cos(\theta_k) \quad \text{where} \quad \theta_k = \frac{\arccos(\xi)}{3} - \frac{2k\pi}{3} \quad \text{for} \quad k = 0, 1, 2 \]

Since \( \theta_0 \in [0, \pi/3], \theta_1 \in [-2\pi/3, -\pi/3], \theta_2 \in [-4\pi/3, -\pi] \):

\[ t_2 \leq t_1 \leq t_0 \]

Therefore

\[ \alpha = t_0 \quad \beta = t_1 \quad \gamma = t_2 \]

6 Approximation of \( MY \) using radicals

6.1 Casus irreducibilis

One of the oddities of solving general cubic equations using radicals is the absolute requirement to use complex numbers in the irreducible case. In other words, despite the three roots being real for irreducible cubic polynomials, they cannot be expressed as radicals of real numbers \([10]\). We note of course the alternate solution provided by Viète, which bypasses the use of complex numbers by introducing trigonometric functions. This is reflected indirectly in the provided closed form of the function \( MY \), which is a hybrid of an algebraic function for \( x \geq \frac{2}{\pi} \) and a trigonometric function for \( x < \frac{2}{\pi} \).

In the following section, we provide an accurate closed form algebraic approximation of \( MY \). In doing so, we offer a new perspective on solving the cubic equation including in the irreducible case. Please note that the proposed algorithm below is scalable to certain higher degree polynomial equations. In a forthcoming paper, the authors will apply the same method to solve the general Quintic equation and other type of equations in the complex space.

6.2 Fixed point iteration

Assume that \( z = MY(x) \). \( z \) satisfies the equation:

\[ z^3 + z^2 = 2x \quad (5) \]
This equation can be exploited in two ways. First, by factorizing (5) we obtain:

\[ z = \sqrt{\frac{2x}{1 + z}} \quad (6) \]

Second, we can complete the cubic:

\[ \left(z + \frac{1}{3}\right)^3 = 2x + \frac{1}{27} + \frac{z}{3} \quad (7) \]

When substituting \( z \) in the right hand side of (7) by using (6):

\[ z = G(x, z) \quad (8) \]

Where the function \( G \) is defined by:

\[ G(x, y) = \sqrt[3]{2x + \frac{1}{27} + \frac{1}{3} \sqrt[3]{\frac{2x}{1 + y}} - \frac{1}{3}} \quad (9) \]

Therefore \( z \) is a fixed point of \( G(x, \cdot) \), inspiring a fixed point iteration.

The following theorem proves the convergence of this method towards \( MY \). More importantly, it provides right away an accurate closed form algebraic approximation of \( MY \) across \( \mathbb{R}^+ \).

**Convergence theorem:**

Define the sequence of functions \( (M_n(\cdot))_{n \in \mathbb{N}} \) defined by:

\[ M_0(x) = G(x, x^\frac{2}{3}) \quad \text{and} \quad M_{n+1}(x) = G(x, M_n(x)) \]

1. For all positive real numbers \( x \):

\[ |M_0(x) - MY(x)| < C_0 \quad \text{where} \quad C_0 = 1.516.10^{-3} \]

2. For all positive real numbers \( x \):

\[ |M_n(x) - MY(x)| < \frac{C_0}{K^n} \quad \text{where} \quad K = 25.05 \]

3. This sequence converge uniformly to \( MY \) over \( \mathbb{R}^+ \).
In order to demonstrate the convergence theorem, we start with the following lemma:

**Lemma:**

1. For all positive real numbers $x$ and $y$:

$$\left| \frac{\partial G}{\partial y}(x, y) \right| < C_1 \quad \text{where} \quad C_1 \approx \frac{1}{21.2398}$$

2. For all positive real numbers $x$:

$$\left| \frac{\partial G}{\partial y}(x, MY(x)) \right| < C_2 \quad \text{where} \quad C_2 \approx \frac{1}{30.5475}$$

**Proof of lemma:**

$G(x, \cdot)$ is decreasing with respect to $y$ and $G(\cdot, y)$ is increasing with respect to $x$. Define:

$$z = MY(x) \quad (10)$$

$z$ satisfies:

$$G(x, z) = z \quad (11)$$

For any $x > 0$, $G(x, \cdot)$ is derivable and for $y \geq 0$:

$$\frac{\partial G}{\partial y}(x, y) = -\frac{\sqrt{2x}}{18} \left( \left( 2x + \frac{1}{27} \right) (1+y)^{\frac{3}{4}} + \frac{\sqrt{2x}}{3} (1+y) \right)^{-\frac{3}{4}} \quad (12)$$

1. Upper bound for $\left| \frac{\partial G}{\partial y}(x, \cdot) \right|$:

$\left| \frac{\partial G}{\partial y}(x, \cdot) \right|$ is strictly decreasing and convex. Therefore its maximum $C_1$ is reached at $y = 0$.

Define $t = \sqrt{2x}$. $C_1$ is also the maximum of:

$$h(t) = \frac{t}{18} \left( t^2 + \frac{1}{27} + \frac{t}{3} \right)^{-\frac{3}{4}} = \frac{1}{18} \left( t^\frac{1}{2} + \frac{t^{-\frac{3}{2}}}{27} + \frac{t^{-\frac{1}{2}}}{3} \right)^{-\frac{3}{4}}$$
Let \( v = t^{−\frac{2}{3}} \):

\[
h(t) = g(v) = \frac{1}{18} \left( \frac{1}{v} + \frac{v^3}{27} + \frac{v}{3} \right)^{-\frac{2}{3}}
\]

\( C_1 \), is obtained by setting the derivative of \( g \) to 0, reached at \( v_0 \):

\[
v_0 = \sqrt{-1 + \sqrt{5}}
\]

and

\[
C_1 \approx \frac{1}{21.2398}
\]

2. Upper bound for \( \left| \frac{\partial G}{\partial y} (x, MY(x)) \right| \)

For \( x > 0 \) define \( z = MY(x) \):

\[
\left| \frac{\partial G}{\partial y} (x, z) \right| = \frac{\sqrt{2x}}{18(1 + z)^{\frac{2}{3}}} \left( \left( 2x + \frac{1}{27} \right) + \frac{1}{3} \sqrt{\frac{2x}{1 + z}} \right)^{-\frac{2}{3}}
\]

Recall that:

\[
z = \sqrt{\frac{2x}{1 + z}} \quad \text{and} \quad (z + \frac{1}{3})^3 = 2x + \frac{1}{27} + \frac{z}{3}
\]

Therefore

\[
\left| \frac{\partial G}{\partial y} (x, z) \right| = \frac{z}{18(1 + z) \left( z + \frac{1}{3} \right)^2}
\]  \hspace{1cm} (13)

Or:

\[
\left| \frac{\partial G}{\partial y} (x, z) \right| = \frac{1}{18} \left( z^2 + \frac{5}{3} z^3 + \frac{7}{9} + \frac{1}{9z} \right)^{-1}
\]

Setting the derivative to 0 leads to the maximum \( C_2 \), reached at \( z \) such that:

\[
2z + \frac{5}{3} - \frac{1}{9z^2} = 0
\]

Notice that:

\[
2z + \frac{5}{3} - \frac{1}{9z^2} = \frac{(3z + 1)(6z^2 + 3z - 1)}{9z^2}
\]

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The only positive solution is:

\[ z = -3 + \sqrt{33} \]

leading to:

\[ C_2 \approx \frac{1}{30.5475} \]

**Proof of Convergence theorem**

1. For \( x > 0 \), define \( z = MY(x) \).
   Since \( M_0(x) = G(x, x^{\hat{a}}) \) and \( z = G(x, z) \):

\[ M_0(x) - MY(x) = |G(x, x^{\hat{a}}) - z| \]

As proven below in the Properties of \( MY \) section \( z \leq x^{\hat{a}} \). Since \( \left| \frac{\partial G}{\partial y} (x, .) \right| \) is decreasing with respect to \( y \) over the interval \([z, x^{\hat{a}}]\):

\[ |M_0(x) - MY(x)| = |G(x, x^{\hat{a}}) - G(x, z)| \leq U(z) = \left| \frac{\partial G}{\partial y} (x, z) \right| (x^{\hat{a}} - z) \quad (14) \]

Recall:

\[ x = \frac{z^3 + z^2}{2} \quad \text{and} \quad \left| \frac{\partial G}{\partial y} (x, z) \right| = \frac{z}{18(1 + z) \left( z + \frac{1}{3} \right)^2} \]

Therefore

\[ U(z) = \frac{z \left( \frac{z^3 + z^2}{2} - z \right)}{18(1 + z) \left( z + \frac{1}{3} \right)^2} \]

Leading to:

\[ U(z) = \frac{\left( \frac{z^{1/2} + z^{-1/2}}{2} \right)^2 - 1}{18(1 + z) \left( 1 + \frac{1}{3z} \right)^2} < \frac{\left( \frac{z^{1/2} + z^{-1/2}}{2} \right)^2 - 1}{18(1 + z)} \]

Define: \( w = \frac{z^{1/2} + z^{-1/2}}{2} \geq 1 \quad \text{which implies} \quad z = 2w^2 - 1 + 2\sqrt{w^4 - w^2} \)
\[ U(z) < \frac{w^2 - 1}{36(w^2 + \sqrt{w^4 - w^2})} \]

Inequality \( w^2 - 1 \leq \frac{2}{3}(w - 1) \ (w^2 \text{ is concave}) \) leads to:
\[ U(z) < \frac{w - 1}{90(w^2 + \sqrt{w^4 - w^2})} \]

Define: \( \xi = w - 1 \)
\[ U(z) < \frac{1}{90(\xi + 1)^2 + \sqrt{(\xi + 1)^4 - (\xi + 1)^2}} \]

Then
\[ U(z) < \frac{1}{90 \left( \frac{(\xi+1)^2}{\xi} + \frac{\sqrt{(\xi+1)^4-(\xi+1)^2}}{\xi} \right)} \] (15)

Let’s find lower bounds for the two terms of the denominator:
(a) First:
\[ \sqrt{(\xi + 1)^4 - (\xi + 1)^2} = \sqrt{\xi^2 + 4\xi + 5 + \frac{2}{\xi}} \]

Setting the derivative to zero leads to:
\[ 2\xi + 4 - \frac{2}{\xi^2} = 0 \quad \text{or} \quad \frac{(\xi + 1)(\xi^2 + \xi - 1)}{\xi^2} = 0 \]

Therefore the minimum is reached at \( \xi = \frac{-1 + \sqrt{5}}{2} \) and:
\[ \frac{\sqrt{(\xi + 1)^4 - (\xi + 1)^2}}{\xi} \geq 3.33019 \] (16)

(b) Second, notice that:
\[ \frac{(\xi + 1)^2}{\xi} \geq 4 \] (17)

Therefore
\[ U(z) \leq \frac{1}{90 \left( \frac{1}{4 + 3.33019} \right)} = \frac{1}{659.7171} \approx 1.516 \times 10^{-3} \]
Going back to (14), for all positive real numbers $x$:

$$|M_0(x) - MY(x)| < C_0 \quad (18)$$

Where

$$C_0 = \frac{1}{659.7171} \approx 1.516 \times 10^{-3}$$

2. For $n \in \mathbb{N}$ and any positive real number $x$, define $z = MY(x)$ and $y = M_n(x)$. Since $M_{n+1}(x) = G(x, y)$ and $z = G(x, z)$:

$$|M_{n+1}(x) - z| = \int_y^z |\frac{\partial G}{\partial y}(x, u)| du \quad (19)$$

Since $|\frac{\partial G}{\partial y}(x, .)|$ is positive and convex, the integral in (19) is lower than the area of the trapezoid:

$$|M_{n+1}(x) - z| \leq \frac{1}{2} |y - z| \left( |\frac{\partial G}{\partial y}(x, y)| + |\frac{\partial G}{\partial y}(x, z)| \right)$$

Using results from the lemma:

$$|M_{n+1}(x) - z| \leq |M_n(x) - z| \frac{C_1 + C_2}{2} \quad (20)$$

Or

$$|M_{n+1}(x) - z| \leq \frac{|M_n(x) - z|}{K} \quad (21)$$

Where

$$K = \frac{2}{C_1 + C_2} \approx 25.0572$$

It follows from (A.8) and (A.11) that:

$$|M_n(x) - MY(x)| \leq \frac{C_0}{K^n} \quad (22)$$

3. As a natural consequence of (22), the sequence $(M_n)_{n \in \mathbb{N}}$ converges uniformly to $MY$.
6.3 Examples:

6.3.1 Numerical examples for the approximation of MY

To showcase the efficiency of the process described above, we consider two examples:

**Example 1:** \( x = 0.01 \), \( MY(x) \) closed form approximate value to 10th decimal digit is 0.1328694292.

| Iteration | \( M_n(x) \) | \( \delta_n = |M_n(x) - MY(x)| \) | \( r_n = \frac{\delta_n}{\delta_{n+1}} \) |
|-----------|-------------|----------------------------------|----------------------------------|
| 0         | 0.1321129198 | 7.6 \( 10^{-04} \)             |                                  |
| 1         | 0.1328921191 | 2.3 \( 10^{-05} \)             | 33.34                           |
| 2         | 0.1328687489 | 6.8 \( 10^{-07} \)             | 33.36                           |
| 3         | 0.1328694495 | 2.0 \( 10^{-08} \)             | 33.36                           |
| 4         | 0.1328694285 | 6.1 \( 10^{-10} \)             | 33.36                           |
| 5         | 0.1328694292 | 1.8 \( 10^{-11} \)             | 33.36                           |

**Example 2:** \( x = 1000 \), \( MY(x) \) closed form approximate value to 10th decimal digit is 12.2745406200.

| Iteration | \( M_n(x) \) | \( \delta_n = |M_n(x) - MY(x)| \) | \( r_n = \frac{\delta_n}{\delta_{n+1}} \) |
|-----------|-------------|----------------------------------|----------------------------------|
| 0         | 12.2735762826 | 9.6 \( 10^{-04} \)             |                                  |
| 1         | 12.2745409317 | 3.1 \( 10^{-07} \)             | 3093.87                         |
| 2         | 12.2745406200 | 6.9 \( 10^{-11} \)             | 4552.52                         |

Note that when \( x < 2/27 \) (example 1), the closed form expression of \( MY \) is trigonometric, and when \( x \geq \frac{2}{27} \) (example 2), the closed form expression is in radicals. Yet, the global algebraic (in radicals) approximation provided works well for both scenarios.

6.3.2 Numerical examples for solving cubic equations

**Example 1:** \( x^3 + x + 1 = 0 \).

Here \( p = 1 \) and \( q = 1 \). The results of section 5.2, case 2 apply. There is a
unique real solution $\alpha$. Using $MY$ closed form:

$$\alpha \approx -0.6823278038$$

$\alpha_n$ is the estimate of $\alpha$ using $n$ iterations of the fixed point algorithm of $MY$.

| $n$ | $\alpha_n$     | $|\alpha_n - \alpha|$ |
|-----|----------------|---------------------|
| 0   | -0.6823458163 | 1.8 $10^{-05}$      |
| 1   | -0.6823274572 | 3.5 $10^{-07}$      |
| 2   | -0.6823278105 | 6.7 $10^{-09}$      |
| 3   | -0.6823278037 | 1.3 $10^{-10}$      |

**Example 2:** $x^3 - 3x + 1 = 0$.

Here $p = -3$ and $q = 1$ which leads to $\xi = q\sqrt{-\frac{27}{4p^3}} = \frac{1}{2}$. The results of section 5.2, case 4 apply. There are three real roots $\alpha$, $\beta$ and $\gamma$. Using the closed form of $MY$:

$$\alpha \approx 1.5320888862 \quad \beta \approx 0.3472963553 \quad \gamma \approx -1.8793852416$$

$\alpha_n$, $\beta_n$ and $\gamma_n$ are the respective estimates of $\alpha$, $\beta$ and $\gamma$ using $n$ iterations of the fixed point algorithm of $MY$.

| $n$ | $\alpha_n$   | $|\alpha_n - \alpha|$ | $\beta_n$   | $|\beta_n - \beta|$ | $\gamma_n$   | $|\gamma_n - \gamma|$ |
|-----|---------------|------------------------|--------------|-----------------------|--------------|------------------------|
| 0   | 1.5296764368 | 2.4 $10^{-03}$         | 0.3476559549 | 3.6 $10^{-04}$         | -1.8773323917 | 2.1 $10^{-03}$         |
| 1   | 1.5321663348 | 7.7 $10^{-05}$         | 0.3472848043 | 1.2 $10^{-05}$         | -1.8794511391 | 6.6 $10^{-05}$         |
| 2   | 1.5320864010 | 2.5 $10^{-06}$         | 0.3472967260 | 3.7 $10^{-07}$         | -1.8793831270 | 2.1 $10^{-06}$         |
| 3   | 1.5320889600 | 8.0 $10^{-08}$         | 0.3472963434 | 1.2 $10^{-08}$         | -1.8793853094 | 6.8 $10^{-08}$         |
| 4   | 1.5320888837 | 2.6 $10^{-09}$         | 0.3472963557 | 3.8 $10^{-10}$         | -1.8793852394 | 2.2 $10^{-09}$         |
| 5   | 1.5320888863 | 8.2 $10^{-11}$         | 0.3472963553 | 1.2 $10^{-11}$         | -1.8793852416 | 7.0 $10^{-11}$         |

Obviously, the goal here is not to conduct a deep numerical analysis, nor is it to provide the most optimal root-finding algorithm (see Newton method, Secant method, Steffensen method, Halley method, Laguerre method, Aberth-Ehrlich method, Durand-Kerner method, etc.). Instead, our aim is to merely shed light on an algorithm that approximates with real-radicals $MY$ and indirectly the roots, especially in casus irreducibilis.
7 Hypergeometric representation

The objective of this section is to express $MY$ using hypergeometric functions. Recall that $MY(x)$ is the unique positive solution of the equation:

$$z^3 + z^2 - 2x = 0$$

For $x > 0$ we consider $y = 1/z$, this implies that:

$$y^3 - \frac{1}{6x}y - \frac{1}{4x} = 0$$

We use the method provided by Zucker in [8]. Define:

$$p' = -\frac{1}{2x}, \quad q' = -\frac{1}{2x} \quad \text{and} \quad \Delta' = q'^2 + p'^3$$

The Cardano formula expresses the root as $y = u + v$ with:

$$u = \left(-q' + \sqrt{\Delta'}\right)^{-\frac{1}{3}} \quad v = \left(-q' - \sqrt{\Delta'}\right)^{-\frac{1}{3}}$$

Or:

$$u + v = (-q')^{-\frac{1}{3}} \left(\left(1 + \sqrt{\frac{\Delta'}{q'^2}}\right)^{\frac{1}{3}} + \left(1 - \sqrt{\frac{\Delta'}{q'^2}}\right)^{\frac{1}{3}}\right)$$

Using the identity:

$$(1 + z)^{-2a} + (1 - z)^{-2a} = 2F(a, a + \frac{1}{2}; \frac{1}{2}; z^2) \quad \text{for} \quad a = -\frac{1}{6}$$

where $F$ is the Gaussian hypergeometric function (analytically continued), we obtain:

$$u + v = (-q')^{-\frac{1}{3}} F\left(-\frac{1}{6}; \frac{1}{3}, \frac{1}{2}; \frac{\Delta'}{q'^2}\right)$$

Using Kummer’s transformation [11]: $F(a, b, c, z) = (1-z)^{-b}F(c-a, b, c; \frac{z}{z-1})$:

$$u + v = \left(\frac{2q'}{p'}\right)^{-\frac{1}{3}} F\left(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}; \frac{\Delta'}{p'^3}\right)$$

Or:

$$u + v = 3F\left(\frac{2}{3}, \frac{1}{3}, \frac{1}{2}; 1 - \frac{27x}{2}\right) \quad (23)$$
Using Kummer’s transformation a second time:

\[ u + v = 3 \left( \frac{27x}{2} \right)^{-\frac{2}{3}} F \left( \frac{1}{6}, \frac{2}{3}, \frac{1}{2}, 1 - \frac{2}{27x} \right) \]  

(24)

If \( x \leq \frac{2}{27} \), using formula (23), \( u + v \) is positive. Likewise when \( x \geq \frac{2}{27} \), formula (24) shows that \( u + v \) is positive.

Therefore

\[ \text{MY}(x) = \frac{1}{3F \left( \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, 1 - \frac{27x}{2} \right)} \]

Notice that

\[ \lim_{z \to 1} F \left( \frac{1}{3}, \frac{2}{3}, \frac{1}{2}, z \right) = +\infty \]

which is coherent with \( \text{MY}(0) = 0 \).

8 Properties of \( \text{MY} \)

In Annex we prove all of the following properties:

Equalities:

1. For \( x \in \left[ \frac{2}{27}, +\infty \right] \):

\[ \text{MY}(x) = \sqrt[3]{\frac{2x \left( x - \sqrt{x \left( x - \frac{2}{27} \right)} \right)}{\frac{1}{3} + \sqrt{\left( x - \frac{1}{27} \right) - \sqrt{x \left( x - \frac{2}{27} \right)}}}} \]

2. For all positive real numbers \( x \), \( \text{MY} \) satisfies the identity:

\[ \text{MY}(x) \left( 3\text{MY} \left( \sqrt{\frac{x}{54} + \frac{1}{27}} \right) + 1 \right) = \sqrt{6x} \]

3. For all positive real numbers \( x \neq 0 \):

\[ \text{MY}(x) = \frac{1}{\text{MY} \left( \frac{x}{(\text{MY}(x))^6} \right)} \]
4. For $0 \leq x \leq \frac{2}{27}$, the three roots of the equation $z^3 + z^2 = 2x$ are:

$$z_1 = MY(x)$$

$$z_2 = -\frac{2}{3} - MY\left(\frac{2}{27} - x\right) = -\frac{1 + MY(x)}{2} \left(1 + \sqrt{\frac{1 - 3MY(x)}{1 + MY(x)}}\right)$$

$$z_3 = MY\left(\frac{2}{27} - x\right) - MY(x) - \frac{1}{3} = -\frac{1 + MY(x)}{2} \left(1 - \sqrt{\frac{1 - 3MY(x)}{1 + MY(x)}}\right)$$

Consequently

$$MY\left(\frac{2}{27} - x\right) = \frac{1 + MY(x)}{2} \left(1 + \sqrt{\frac{1 - 3MY(x)}{1 + MY(x)}}\right) - \frac{2}{3}$$

5.

$$\lim_{x \to 0} \frac{MY(x)}{\sqrt{2x}} = 1 \quad \text{and} \quad \lim_{x \to +\infty} \frac{MY(x)}{\sqrt{2x}} = 1$$

Consequently for any $x \geq 0$

$$\sqrt{x} = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} MY\left(\frac{x\epsilon^2}{2}\right) \quad \text{and} \quad \sqrt[3]{x} = \lim_{\epsilon \to 0^+} \epsilon MY\left(\frac{x}{2\epsilon^3}\right)$$

**Inequalities:**

1. For all positive real numbers $x$:

$$\sqrt{\frac{2x}{1 + x^2}} \leq MY(x) \leq x^\frac{1}{2}$$

2. For all positive real numbers $x \neq 0$:

   (a) If $a$ is a real number such that $0 \leq a \leq 1$:

   $$MY(x^a) \geq (MY(x))^a$$

   (b) If $a$ is a real number such that $a \leq 0$ or $a \geq 1$:

   $$MY(x^a) \leq (MY(x))^a$$
3. For $x \in [0, 1]$:
\[ \sqrt{x} \leq MY(x) \leq 3\sqrt{x} \]

4. For $x \in [1, +\infty]$:
\[ 3\sqrt{x} \leq MY(x) \leq \sqrt{x} \]

**Derivative and primitive:**

1. For $x > 0$ the derivative of $M$ is given by the expression:
\[ MY'(x) = \frac{2}{3MY^2(x) + 2MY(x)} \]

2. For $x \geq 0$ a primitive of $MY$ is given by the expression:
\[ \frac{3}{4} x MY(x) - \frac{x}{12} + \frac{MY^2(x)}{24} \]

**Proof of the properties of $MY$**

**Equalities:**

1. The objective is to prove:
\[ MY(x) = \frac{v}{1 - u} \quad (25) \]

Where
\[
\begin{align*}
    u &= 3^\frac{3}{2} \left( \frac{1}{27} - x \right) + \sqrt{x \left( x - \frac{2}{27} \right)} \\
    v &= 3^\frac{3}{2} 2x \left( x - \sqrt{x \left( x - \frac{2}{27} \right)} \right) \\
    \frac{v}{1 - u} &= \frac{v(1 + u + u^2)}{1 - u^3} = \frac{v}{1 - u^3} + \frac{vu}{1 - u^3} + \frac{vu^2}{1 - u^3} \quad (26)
\end{align*}
\]

Then notice:

(a)
\[ \frac{1}{u} = 3^\frac{3}{2} \left( \frac{1}{27} - x \right) - \sqrt{x \left( x - \frac{2}{27} \right)} \]
(b) \[ v^2 = -6xu \]

(c) \[ \frac{v^3}{1 - u^3} = 2x \]

Therefore, the three terms in (26) can be expressed:

\[
\begin{align*}
\frac{uv}{1 - u^3} &= \frac{v^2}{1 - u^3} = -\frac{1}{6x} \frac{v^3}{1 - u^3} = -\frac{2x}{6x} = -\frac{1}{3} \\
\frac{u^2v}{1 - u^3} &= u \frac{vu}{1 - u^3} = -\frac{u}{3} = 3 \sqrt{\left(x - \frac{1}{27}\right) - \sqrt{x \left(x - \frac{2}{27}\right)}} \\
\frac{v}{1 - u^3} &= \frac{1}{u} \frac{vu}{1 - u^3} = -\frac{1}{3u} = 3 \sqrt{\left(x - \frac{1}{27}\right) + \sqrt{x \left(x - \frac{2}{27}\right)}}
\end{align*}
\]

Which adds up to:

\[ MY(x) = \frac{v}{1 - u} \]

In other words:

\[ MY(x) = \frac{\sqrt[3]{2x \left(x - \sqrt{x \left(x - \frac{2}{27}\right)}\right)}}{\frac{1}{3} + \frac{1}{3} \sqrt{\left(x - \frac{1}{27}\right) - \sqrt{x \left(x - \frac{2}{27}\right)}}} \]

2. For any \( x > 0 \), define \( z = MY(x) \):

\[ \frac{z^3 + z^2}{2} = x \]

A change of variable \( y = 1/z \) leads to:

\[ y^3 - \frac{1}{2x} y - \frac{1}{2x} = 0 \]

Let's introduce a second change of variable:

\[ y = 3w \sqrt{\frac{1}{6x} + \sqrt{\frac{1}{6x}}} \]
which leads to:
\[
\frac{w^3 + w^2}{2} = \sqrt{\frac{x}{54} + \frac{1}{27}}
\]
so:
\[
w = MY \left( \sqrt{\frac{x}{54} + \frac{1}{27}} \right)
\]
Since \( y = 1/z = \frac{1}{MY(x)} \):
\[
MY(x) \left( 3MY \left( \sqrt{\frac{x}{54} + \frac{1}{27}} \right) + 1 \right) = \sqrt{6x}
\]
3. Let \( z = MY(x) \):
\[
\frac{z^3 + z^2}{2} = 2x
\]
By multiplying both sides by \( z^{-5} \):
\[
\frac{z^{-3} + z^{-2}}{2} = 2xz^{-5}
\]
Or
\[
\frac{1}{z} = M \left( \frac{x}{z^5} \right)
\]
Therefore
\[
MY(x) = \frac{1}{\frac{1}{MY\left( \frac{x}{MY(x)} \right)}}
\]
4. For \( 0 \leq x \leq \frac{2}{27} \), \( z_1 = MY(x) \) is root of the equation \( z^3 + z^2 = 2x \). The other two roots, \( z_2 \) and \( z_3 \), can be found by symmetry and using Vieta’s formula as provided in section 5.1. Alternatively:
\[
z_2 + z_3 = -1 - z_1 \quad \text{and} \quad z_2z_3 = 2x/z_1 = z_1(1 + z_1)
\]
\( z_2 \) and \( z_3 \) are therefore roots of a quadratic equation. The discriminant is:
\[
\delta = (1 + z_1)^2 \frac{1 - 3z_1}{1 + z_1}
\]
Therefore
\[
z_2 = -(1 + z_1) \left( 1 + \sqrt{\frac{1 - 3z_1}{1 + z_1}} \right)
\]
\[ z_3 = -(1 + z_1) \left( 1 - \sqrt{\frac{1 - 3z_1}{1 + z_1}} \right) \]

Notice \( z_2 \leq z_3 \). Using the results from section 5.1:

\[ z_2 = -\frac{2}{3} - MY \left( \frac{2}{27} - x \right) = -\frac{1 + MY(x)}{2} \left( 1 + \sqrt{\frac{1 - 3MY(x)}{1 + MY(x)}} \right) \]

\[ z_3 = MY \left( \frac{2}{27} - x \right) - MY(x) \frac{1}{3} = -\frac{1 + MY(x)}{2} \left( 1 - \sqrt{\frac{1 - 3MY(x)}{1 + MY(x)}} \right) \]

Consequently

\[ MY \left( \frac{2}{27} - x \right) = \frac{1 + MY(x)}{2} \left( 1 + \sqrt{\frac{1 - 3MY(x)}{1 + MY(x)}} \right) - \frac{2}{3} \]

5. \( MY \) is the inverse of \( f \). Since \( \lim_{x \to 0} f(x) = 0 \) and \( \lim_{x \to +\infty} f(x) = +\infty \):

\[ \lim_{x \to 0} MY(x) = 0 \quad \text{and} \quad \lim_{x \to +\infty} MY(x) = +\infty \]

Also \( MY(x)^2 \left( 1 + MY(x) \right) = 2x \) which leads to:

\[ \frac{MY(x)}{\sqrt{2x}} = \sqrt{\frac{1}{1 + MY(x)}} \]

Therefore:

\[ \lim_{x \to 0} \frac{MY(x)}{\sqrt{2x}} = 1 \]

Similarly, since \( MY(x)^3 \left( 1 + \frac{1}{MY(x)} \right) = 2x \):

\[ \frac{MY(x)}{\sqrt{2x}} = \sqrt[3]{\frac{1}{1 + \frac{1}{MY(x)}}} \]

Therefore

\[ \lim_{x \to \infty} \frac{MY(x)}{\sqrt{2x}} = 1 \]

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Consequently for any $x > 0$

$$\sqrt{x} = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} MY\left(\frac{x\epsilon^2}{2}\right) \quad \text{and} \quad \sqrt{x} = \lim_{\epsilon \to 0^+} \epsilon MY\left(\frac{x}{2\epsilon^3}\right)$$

This holds also for $x = 0$.

**Inequalities:**

1. For all positive real numbers $x$

   $$f\left(x^{\frac{a}{2}}\right) = \frac{1}{2} \left(x^{\frac{a}{2}} + x^{\frac{1}{2}}\right) = x \left(1 + \frac{1}{2} \left(x^{\frac{1}{2}} - x^{-\frac{1}{2}}\right)^2\right) \geq x$$

   Since $f$ is strictly increasing

   $$MY(x) \leq x^{\frac{a}{2}}$$

   In addition

   $$MY(x) = \sqrt{\frac{2x}{1 + MY(x)}} \geq \sqrt{\frac{2x}{1 + x^{\frac{a}{2}}}}$$

2. Let’s consider the function $h(x) = x^a$ and define $z = MY(x)$:

   (a) For $0 \leq a \leq 1$, $h$ is concave and:

   $$\frac{h(z)^3 + h(z^2)}{2} \leq h\left(\frac{z^3 + z^2}{2}\right) = h(x)$$

   Since $h(z^3) = (h(z))^3$ and $h(z^2) = (h(z))^2$:

   $$\frac{(h(z))^3 + (h(z))^2}{2} \leq h(x) \quad \text{and} \quad h(z) \leq MY(h(x))$$

   Therefore

   $$MY\left(x^a\right) \geq (MY(x))^a$$

   (b) Similarly, for $a \leq 0$ or $a \geq 1$, $h$ is convex and we have:

   $$MY\left(x^a\right) \leq (MY(x))^a$$
3. For all \( x \in [0, 1] \):
\[
\begin{align*}
  f(\sqrt{x}) &= x \frac{\sqrt{x} + 1}{2} \leq x \\
  f\left(\sqrt[3]{x}\right) &= x \frac{x^{-1/3} + 1}{2} \geq x
\end{align*}
\]
Since \( f \) is strictly increasing:
\[
\sqrt{x} \leq MY(x) \leq \sqrt[3]{x}
\]

4. For \( x \in [1, +\infty] \):
\[
\begin{align*}
  f(\sqrt{x}) &= x \frac{\sqrt{x} + 1}{2} \geq x \\
  f\left(\sqrt[3]{x}\right) &= x \frac{x^{-1/3} + 1}{2} \leq x
\end{align*}
\]
Since \( f \) is strictly increasing:
\[
\sqrt[3]{x} \leq MY(x) \leq \sqrt{x}
\]

**Derivative and primitive:**

1. Deriving the identity \( f(MY(x)) = x \) leads to \( MY'(x)f'(MY(x)) = 1 \)
\[
MY'(x) \left( \frac{3}{2} MY^2(x) + MY(x) \right) = 1
\]
Which means:
\[
MY'(x) = \frac{2}{3MY^2(x) + 2MY(x)}
\]

2. Since \( MY \) is continuous over \( \mathbb{R}^+ \), a primitive of \( MY \) is given by:
\[
\int_0^x MY(t)dt
\]
Let's use a change of variable \( MY(t) = w \) (or \( t = f(w) \) and \( dt = f'(w)dw \)):
\[
\int_0^x MY(t)dt = \int_0^{MY(x)} w \left( \frac{3w^2 + 2w}{2} \right) dw
\]
Using \( x = (MY^3(x) + MY^2(x))/2 \), the integral can be simplified to:
\[
\int_0^x MY(t)dt = \frac{3}{4} x MY(x) - \frac{x}{12} + \frac{MY^2(x)}{24}
\]
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References

[1] https://arxiv.org/abs/1308.2181

[2] Nickalls, R. W. D. (July 2006), ”Viète, Descartes and the cubic equation” (PDF), Mathematical Gazette, 90 (518): 203-208, doi:10.1017/S0025557200179598

[3] J.L. Lagrange. Réflexions sur la résolution algébrique des équations. Nouveaux Mémoires de l’Academie royale des Sciences et Belles-Lettres de Berlin, 3:205–421, 1869.

[4] Tschirnhaus’s 1683 paper ”A method for removing all intermediate terms from a given equation”, translation by RF Green.

[5] Cubic Equations— Another Solution, Math Forum (Drexel).

[6] R W D Nickalls, A new approach to solving the cubic: Cardan’s solution revealed, Mathematical Gazette, Vol.77, 1993.

[8] Zucker, I.J. (July 2008). ”The cubic equation — a new look at the irreducible case”. Mathematical Gazette. 92: 264–268.

[9] A. Missa, C. Youssfi. An Alternative Formula For The Cubic Equation.

[10] Wantzel, Pierre (1843), ”Classification des nombres incommensurables d’origine algébrique” (PDF), Nouvelles Annales de Mathématiques (in French), 2: 117–127

[11] A. Erdelyi, W. Magnus, F. Oberrettinger and F. G. Tricomi, Higher transcendental functions (Vol 1), McGraw-Hill (1953) Chapter 2.