MEAN-FIELD MODELING OF AN $\alpha^2$ DYNAMO COUPLED WITH DIRECT NUMERICAL SIMULATIONS OF RIGIDLY ROTATING CONVECTION

Youhei Masada$^1$ and Takayoshi Sano$^2$

$^1$ Department of Computational Science, Kobe University, Kobe 657-8501, Japan; ymasada@harbor.kobe-u.ac.jp
$^2$ Institute of Laser Engineering, Osaka University, Osaka 565-0871, Japan; sano@ile.osaka-u.ac.jp

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ABSTRACT

The mechanism of large-scale dynamos in rigidly rotating stratified convection is explored by direct numerical simulations (DNS) in Cartesian geometry. A mean-field dynamo model is also constructed using turbulent velocity profiles consistently extracted from the corresponding DNS results. By quantitative comparison between the DNS and our mean-field model, it is demonstrated that the oscillatory $\alpha^2$ dynamo wave, excited and sustained in the convection zone, is responsible for large-scale magnetic activities such as cyclic polarity reversal and spatiotemporal migration. The results provide strong evidence that a nonuniformity of the $\alpha$-effect, which is a natural outcome of rotating stratified convection, can be an important prerequisite for large-scale stellar dynamos, even without the $\Omega$-effect.

Key words: convection – stars: magnetic field – Sun: magnetic fields – turbulence

Online-only material: color figures

1. INTRODUCTION

The solar magnetism is caused by a large-scale dynamo operating in the solar interior. Its ultimate goal is to reproduce observed spatiotemporal evolution of the solar magnetic field, such as cyclic polarity reversals and butterfly-shaped migrations, in the framework of magnetohydrodynamics (MHD). Although a growing body of evidence is accumulating to reveal large-scale dynamos in numerical MHD models of Sun-like stars, unsolved questions remain to be answered if a full MHD description of the solar dynamo mechanism is to be attained (Miesch & Toomre 2010; Charbonneau 2010).

There are two approaches to simulate stellar dynamo evolution. One is global simulations that comprise the entire volume of a convection layer and the other uses local-box calculations of a small patch of the stellar interior. The first simulation of a global dynamo that succeeded in obtaining solar-like cyclic large-scale magnetic fields was performed by Ghizaru et al. (2010; see also Brown et al. 2011; Nelson et al. 2013; Masada et al. 2013). Another pioneering global simulation was done by Käpylä et al. (2012), which reproduced solar-like butterfly-shaped migration of magnetic activity belts. However, a definitive explanation on what regulates the solar-like magnetic cycle has yet to be obtained (e.g., Simard et al. 2013; Käpylä et al. 2013b). The complicated processes included in global simulations often preclude elucidating the real essence of the large-scale dynamo.

Direct numerical simulations (DNS) of the convective dynamo in local Cartesian geometry is complementary to the global model, and is expected to facilitate our knowledge of the nature of convective dynamos (e.g., Cattaneo & Hughes 2006; Favier & Bushby 2013, and references therein). Similar to global simulations, the oscillatory large-scale dynamo can also be seen in the Cartesian geometry for rigidly rotating stratified convection (Käpylä et al. 2013a; Masada & Sano 2014). Since mean velocity shear is absent in this system, only a stochastic process due to turbulent convection would contribute to the large-scale dynamo (e.g., Baryshnikova & Shukurov 1987; Rädler & Bräuer 1987). As a milestone toward a complete understanding of the solar MHD dynamo, the mechanism underlying the large-scale dynamo in the Cartesian models must be specified.

In this Letter, we quantitatively demonstrate that the $\alpha^2$ mechanism is responsible for the quasi-periodic features of this large-scale dynamo. First, using a DNS model in Cartesian geometry, we investigate the characteristic behaviors of the convective dynamo. Next, to disentangle the complicated MHD dynamo processes, we construct a mean-field (MF) $\alpha^2$ dynamo model, using the DNS results as profiles of the convective turbulent velocity and helicity. Our MF model is tested by assessing the dependence of the dynamo properties on the magnetic diffusivity. Through careful comparison between DNS results and our MF modeling, the mechanism underlying the large-scale dynamo is revealed.

2. LARGE-SCALE DYNAMO IN THE REFERENCE MODEL

We use the same model (model B) as studied by Masada & Sano (2014, hereafter MS14a) as a reference model, in which the large-scale dynamo was successfully operated. What follows is a brief review of our numerical MHD model.

In MS14a, a convective dynamo was solved by Cartesian domain (see Figure 1(a)). This computational domain comprises three layers: a top cooling layer (depth 0.15$d$), a middle convection layer (depth $d$), and a bottom stably stratified layer of depth 0.85$d$. The horizontal size is assumed to be $4d$ ($\times$ $4d$) in $x$ ($\times$ $y$). The basic equations are compressible MHD equations in the rotating frame of reference, with a constant angular velocity $\Omega = -\Omega_0 \hat{e}_z$.

The initial hydrostatic balance is described by a polytropic distribution with the polytropic index $m$,

$$\frac{d\epsilon}{dz} = g_0/[(\gamma - 1)(m + 1)] ,$$

where $\epsilon$ is the specific internal energy, $\gamma$ is the adiabatic index, and $g_0$ is the constant gravity. Here, when $m < 1.5$, it becomes convectively unstable. We choose $m = 1$ for the convection zone, and $m = 3$ for the stable zone. The density contrast between the top and bottom of the domain is $\approx 10$. 

1 Department of Computational Science, Kobe University, Kobe 657-8501, Japan; ymasada@harbor.kobe-u.ac.jp
2 Institute of Laser Engineering, Osaka University, Osaka 565-0871, Japan; sano@ile.osaka-u.ac.jp
number), and \( Ra = 4 \times 10^6 \) (Rayleigh number), constant angular velocity of \( \Omega_0 = 0.4 \), and the spatial resolution of \((N_x, N_y, N_z) = (256, 256, 128)\) were adopted in the reference model (see MS14a for definitions of \( Pr \), \( Pm \), and \( Ra \)).

Initially, small random perturbations are added to the velocity and magnetic fields. Typically after the magnetic diffusion time, a saturated turbulent state is achieved. The convective motion there provides \( u_{cv} = 0.02 \), \( \text{Co} = 40 \), \( B_{cv} = 0.045 \), and \( \tau_{cv} = 50 \). The surface visualization in Figure 1(a) indicates the vertical velocity at \( t = 400\tau_{cv} \), with the red (blue) tone denoting downflow (upflow). The convective motion is characterized by cellular upflows surrounded by downflow networks. Since there is no symmetry breaking in the horizontal direction, the mean horizontal shear flow, and thus the \( \Omega \)-effect, are absent in our model.

The time-depth diagram of \( \langle B_z \rangle_h \) is shown in Figure 2(a). The orange and blue tones represent positive and negative \( \langle B_z \rangle_h \) in units of \( B_{cv} \), respectively. The time is normalized by \( \tau_{cv} \). As seen from this figure, oscillatory large-scale magnetic field spontaneously organized in our reference model. The \( \langle B_z \rangle_h \) has a peak in the middle of the convection zone and propagates from there to the top and base of the zone. Note that \( \langle B_z \rangle_h \) shows a similar cyclic behavior with \( \langle B_z \rangle_h \) yet with a phase delay of \( \pi/2 \) (see also Figure 3).

It is well known that, in the \( \alpha \Omega \) dynamo solution, \( B_\phi \) lags \( B_z \) by \( \pi / 4 \) (for \( \partial \Omega / \partial r > 0 \)), while \( B_\phi \) advances \( B_z \), by \( 3\pi / 4 \) (for \( \partial \Omega / \partial r < 0 \)) (e.g., Brandenburg & Subramanian 2005; Käpylä et al. 2013b). In contrast, our DNS model provides a phase relation similar to the S-parity solution of the MF \( \alpha^2 \) dynamo model of Brandenburg et al. (2009), wherein the vertical field condition is imposed on the top boundary (note that \( z \)-axis points upward in Brandenburg et al. 2009). If the top perfect conductor condition is adopted in our model, it is expected that \( \langle B_z \rangle_h \) advances \( \langle B_\phi \rangle_h \) by \( \pi / 2 \) (\( A \)-parity solution).

The large-scale magnetic field with spatiotemporal coherence was a remarkable feature of the convective dynamo achieved in our DNS. This feature is reproducible using a mean-field dynamo model with the velocity and helicity profiles consistently extracted from DNS results.

3. MEAN-FIELD DYNAMO MODEL

3.1. Governing Equation and Link to DNS

To explore the mechanism underlying the large-scale dynamo in our DNSs, we construct a one-dimensional MF dynamo model wherein velocity profiles are adopted from the DNS results of the saturated convective turbulence and determine the coefficients required for MF modeling. See Simard et al. (2013) for the similar approach.

Since the \( \Omega \)-effect is excluded from our MHD simulations, the \( \alpha^2 \) dynamo rather than the \( \alpha \Omega \) dynamo will be realized. The MF equation for the \( \alpha^2 \) dynamo is obtained from the induction equation, by dividing the variables into the horizontal mean values and fluctuating components, \( u = \langle u \rangle_h + u' \) and \( B = \langle B \rangle_h + B' \), and taking the horizontal average:

\[
\frac{\partial \langle B \rangle_h}{\partial t} = \nabla \times \left[ \mathcal{E} - \eta_0 \nabla \times \langle B \rangle_h \right],
\]

with

\[
\mathcal{E} = \alpha \langle B \rangle_h + \gamma \mathcal{E}_x \times \langle B \rangle_h - \eta \nabla \times \langle B \rangle_h,
\]

where \( \eta_0 \) is the microscopic magnetic diffusivity, \( B_h = \langle B \rangle_h \), the horizontal field, and \( \mathcal{E} \) is the turbulent electromotive forces.
force (e.g., Ossendrijver et al. 2002). The coefficients $\alpha$, $\gamma$, and $\eta$ represent the $\alpha$-effect, turbulent pumping, and turbulent magnetic diffusivity, respectively. All the terms related to $\langle u \rangle_h$ and $\langle B \rangle_h$ can be ignored in considering the symmetry of the system. All the variables, except for $\eta_0$, depend on the time ($t$) and depth ($z$).

The MF dynamo described by Equation (2) falls into the $\alpha^2$-type category. The MF theory predicts that the $\alpha^2$ mode can generate a large-scale magnetic field with an oscillatory nature (e.g., Baryshnikova & Shukurov 1987; Rädler & Brüer 1987; Brandenburg et al. 2009). A key ingredient for the oscillatory $\alpha^2$-effect across the equator.

In the first-order smoothing approximation (FOSA), the unquenched coefficients $\alpha_k$, $\gamma_k$, and $\eta_k$ in anisotropic forms are given by (e.g., Käpylä et al. 2006, 2009b)

$$\alpha_k(z) = -\tau_c [\langle u_x \partial_z u_x \rangle_h + \langle u_x u_z \rangle_h] \equiv -\tau_c \mathcal{H}_{\text{eff}}, \quad (7)$$

$$\gamma_k(z) = -\tau_c \partial_z \langle u_x^2 \rangle_h \equiv -\tau_c u_{\text{rms}}^2, \quad (8)$$

$$\eta_k(z) = \tau_c \langle u_x^2 \rangle_h \equiv \tau_c u_{\text{rms}}^2, \quad (9)$$

where $\tau_c$ is the correlation time, $\mathcal{H}_{\text{eff}}$ is the effective helicity, and $u_{\text{rms}}$ is the rms velocity. The vertical profiles of $\mathcal{H}_{\text{eff}}$ and $u_{\text{rms}}^2$ in the reference DNS model are shown in Figure 1(b) by solid and dashed lines, respectively.
The correlation time should be zero in the top cooling and bottom stable layers since the convective turbulence is not fully developed; thus \( \alpha_k = \gamma_k = \eta_k = 0 \) there. Assuming the Strouhal number is unity in the convection zone (St = \( \tau_c u_{rms} k_c = 1 \)), the vertical profile of \( \tau_c \) is given by
\[
\tau_c(z) = \frac{1}{u_{rms} k_c} \left[ 1 + \text{erf} \left( \frac{z - z_b}{h} \right) \right] \left[ 1 + \text{erf} \left( \frac{z_i - z}{h} \right) \right],
\]
where \( z_i \) (\( i = t, b \)) represents the location of the boundaries between regions with and without fully developed turbulence. We define \( z_t \) and \( z_b \) as the depth where \( \mathcal{H}_{\text{eff}} \) achieves the maximum and minimum values, respectively (see Figure 1(b)).

The transition width \( h \) is an arbitrary parameter and assumed here to be \( h = 2 \Delta z \) with \( \Delta z = 2d/N_z \). The uncertainty of \( h \) is discussed in the next section. All the coefficients (\( \tau_c, B_{eq}, \mathcal{H}_{\text{eff}}, \alpha_k, \gamma_k, \eta_k \)) required for the MF modeling can subsequently be computed from the DNS results.

### 3.2. Comparison with DNS

Given all the coefficients in Equations (2)–(10) from the reference DNS model, the MF equations can be solved using the second-order central difference. For time integration, the fourth-order Runge-Kutta method is used. We adopt the same parameters used in the DNS: the calculation domain of \( 0 \leq z \leq 2d \), the resolution of \( N_z = 128 \), and the magnetic diffusivity providing \( P_m = 4 \).

The time-depth diagram of \( \langle B_z \rangle_h \) normalized by \( B_{eq} \) in the MF model is shown in Figure 2(b). The time is normalized by turbulent magnetic diffusion time defined by \( \tau_{\text{diff}} \equiv 1/[(\langle \eta \rangle) k_d^2] \), where \( k_d \) is the typical wavenumber of the dynamo wave and is chosen here as \( k_d = \pi/2d \) (cf. Brandenburg et al. 2009). The large-scale field, which is of similar amplitude and spatiotemporal structure as the DNS, is generated and sustained in the bulk of the convection zone for the MF model.

Qualitative agreement between the MF model and DNS can be seen in Figure 3, which shows the time series of \( \langle B_z \rangle_t \) and \( \langle B_z \rangle_v \). The orange [cyan] solid line denotes \( \langle B_z \rangle_t \) [\( \langle B_z \rangle_v \)] normalized by \( B_{eq} \) in the DNS and the red dashed [blue dash-dotted] line is that in the MF model. The time of the DNS is rescaled by \( \tau_{\text{diff}} \) of \( \langle \eta \rangle \), evaluated from the MF model and \( k_d = \sqrt{\pi}/(2d) \). The longer wavelength required for DNS would be due to the geometrical effect. The time of the MF model shifts to match the DNS phase.

The cycle and amplitude of the large-scale magnetic field in the MF model coincide with those in the DNS. Furthermore, the phase difference between \( \langle B_z \rangle_t \) and \( \langle B_z \rangle_v \) seen in the DNS model is also reproduced perfectly. This indicates that the oscillatory \( \alpha^2 \) dynamo wave is regulated by the turbulent magnetic diffusivity and is responsible for the spatiotemporal evolution of the large-scale magnetic field in the DNS.

### 3.3. Validation of our MF Model

To demonstrate the validity of our MF model, we apply it to other DNS models with varying parameters. Here, we focus on the effect of magnetic diffusivity (\( \eta_0 \)). The setup is identical to that used in the reference model except for \( \eta_0 \) or the magnetic Prandtl number. The models with \( P_m = 2 \) and 8, which adopt two times and half of \( \eta_0 \) assumed in the reference model, are simulated by both DNS and our MF model. Note that \( u_{eq} \) and \( B_{eq} \) remain unchanged when varying \( P_m \).

The time-depth diagram of \( \langle B_z \rangle_h \) is shown in Figure 4. Panels (a) and (b) correspond to DNSs with \( P_m = 2 \) and 8. Regardless of \( P_m \), the large-scale oscillatory magnetic field is organized in the bulk of the convection zone. The red squares in Figure 5 indicate the \( \eta_0 \)-dependence of (a) the dynamo period \( \tau_{\text{cyc}} \) and (b) the saturated field strength \( B_M \), where \( \tau_{\text{cyc}} \) is the statistically averaged value and \( B_M \equiv \langle \langle (B_z)^2 \rangle \rangle^{1/2} \). Each axis is normalized by the value of the reference DNS model \( \langle \eta_0 \rangle \). The longer wavelength required for DNS is also built up in the MF model. The blue circles in Figure 5 represent \( \tau_{\text{cyc}} \) and \( B_M \) for the MF model. Normalization units are those of the reference MF model, \( \tau_{\text{cyc,R}} = 2380\sqrt{d/g_0} \) and \( B_{eq,R} = 0.028 \). The slope and amplitude of the \( \eta_0 \)-dependence is well reproduced by the MF model. These results confirm that the oscillatory large-scale magnetic field observed in the DNS is a consequence of the turbulent \( \alpha^2 \) dynamo.

### 4. SUMMARY AND DISCUSSION

In this Letter, the mechanism controlling the large-scale dynamo in rotating stratified convection was examined by DNS in Cartesian geometry and the MF dynamo model with the information of turbulent velocity extracted from DNS. We then quantitatively demonstrated that the oscillatory \( \alpha^2 \) dynamo wave, excited and sustained in the convection zone, was responsible for the large-scale dynamo with cyclic polarity reversals and spatiotemporal migrations observed in the DNS. Our MF model was validated by evaluating the dependence of the large-scale dynamo on the magnetic diffusivity. It is concluded that the nonuniformity of the \( \alpha \)-effect is a key ingredient for the large-scale dynamo with an oscillatory nature.

The oscillatory \( \alpha^2 \) dynamo mode is attaining a greater level of attention in solar dynamo modeling. Recently, Mitra et al. (2010) reported an intriguing numerical finding in their forced helical turbulence that the \( \alpha^2 \) dynamo can yield solar-like equatorward migration of magnetic activity belts (see also Schrinner et al. 2011). The superiority of the \( \alpha^2 \) mode over \( \alpha \Omega \) mode at the nonlinear stage was found by Hubbard et al. (2011). Furthermore, a connection between \( \alpha^2 \) dynamo mode and solar magnetism was suggested in some recent results of global MHD dynamos (e.g., Simard et al. 2013; Käpylä et al. 2013b). The crucial factor is the nonuniformity of the \( \alpha \)-effect. Therefore, accurate numerical modeling of the solar internal \( \alpha \) profile will offer a way to unveil the mystery of solar magnetism.

There is an arbitrary parameter in the MF model, the thickness of the transition layer \( h \) in Equation (10). The MF solution is actually dependent on this parameter. If the thickness \( h \) increases, the cycle period of the MF dynamo becomes shorter and its spatiotemporal pattern deviates from that of DNS. The value \( h = 2\Delta z \) adopted here is not based on physical reason but is best suited for reproducing the spatiotemporal pattern of the large-scale dynamo in the DNS. Several methods, such as
imposed- and test-field methods, have been previously proposed to directly calculate the MF coefficients from DNS without the use of a statistical turbulence model (e.g., Ossendrijver et al. 2002; Hubbard et al. 2009; Simard et al. 2013). In contrast, our model is based on FOSA for small-scale turbulence. Not only the quenching functions, but also the applicability of FOSA to the anisotropic turbulence remains a matter of debate (e.g., Käpylä et al. 2006). Although there is room for improvement in our MF model, it appears to appropriately describe various aspects of the large-scale dynamo in the DNS.

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REFERENCES

Baryshnikova, I., & Shukurov, A. 1987, AN, 308, 89
Blackman, E. G., & Brandenburg, A. 2002, ApJ, 579, 359
Brandenburg, A., Candelaresi, S., & Chatterjee, P. 2009, MNRAS, 398, 1414
Brandenburg, A., & Subramanian, K. 2005, PhR, 417, 1
Brown, B. P., Miesch, M. S., Browning, M. K., Brun, A. S., & Toomre, J. 2011, ApJ, 731, 69
Cattaneo, F., & Hughes, D. W. 2006, JFM, 553, 401
Charbonneau, P. 2010, LRSP, 7, 3 Clarke, D. A. 1996, ApJ, 457, 291
Evans, C. R., & Hawley, J. F. 1988, ApJ, 332, 659 Favier, B., & Bushby, P. 2013, JFM, 723, 529
Ghizaru, M., Charbonneau, P., & Smolarkiewicz, P. K. 2010, ApJL, 715, L133
Hubbard, A., Del Sordo, F., Käpylä, P. J., & Brandenburg, A. 2009, MNRAS, 398, 1891
Hubbard, A., Rheinhardt, M., & Brandenburg, A. 2011, A&A, 535, A48
Käpylä, P. J., Korpi, M. J., & Brandenburg, A. 2009, A&A, 500, 633
Käpylä, P. J., Korpi, M. J., Ossendrijver, M., & Stix, M. 2006, A&A, 455, 401
Käpylä, P. J., Mantere, M. J., & Brandenburg, A. 2012, ApJL, 755, L22
Käpylä, P. J., Mantere, M. J., & Brandenburg, A. 2013a, GApFD, 107, 244
Käpylä, P. J., Mantere, M. J., Cole, E., Warnecke, J., & Brandenburg, A. 2013b, ApJ, 778, 41
Masada, Y., Yamada, K., & Kageyama, A. 2013, ApJ, 778, 11
Masada, Y., & Sano, T. 2014, arXiv:1403.6221
Miesch, M. S., & Toomre, J. 2009, AnRFM, 41, 317
Mitra, D., Tavakol, R., Käpylä, P. J., & Brandenburg, A. 2010, ApJL, 719, L1
Nelson, N. J., Brown, B. P., Brun, A. S., Miesch, M. S., & Toomre, J. 2013, ApJ, 762, 73
Ossendrijver, M., Stix, M., Brandenburg, A., & Rüdiger, G. 2002, A&A, 394, 735
Rädler, K.-H., & Brüher, H.-J. 1987, AN, 308, 101
Rogachevskii, I., & Klecorin, N. 2001, PhRvE, 64, 056307
Sano, T., Inutsuka, S., & Miyama, S. M. 1998, ApJL, 506, L57
Schrinner, M., Petitdemange, L., & Dormy, E. 2011, A&A, 530, A140
Simard, C., Charbonneau, P., & Bouchat, A. 2013, ApJ, 768, 16

Figure 4. Time-depth diagram of $\langle B_x \rangle_h$ for the DNSs with (a) $P_m = 2$ and (b) $P_m = 8$. The MF models corresponding to the DNS models with $P_m = 2$ and 8 are shown in panels (c) and (d). The orange (blue) tone denotes the positive (negative) $\langle B_x \rangle_h$ in units of $B_{cv}$ in all the panels.

(A color version of this figure is available in the online journal.)

Figure 5. (a) Dynamo period $\tau_{\text{cyc}}$ and (b) the saturated field strength $B_M \equiv [\langle (B_x)^2 \rangle + \langle (B_y)^2 \rangle]^{1/2}$ as a function of $\Gamma \equiv \eta_{\text{dr}} / \eta_0$. The red squares denote the DNS results and the blue circles are the MF models. The vertical axis is normalized by the values of the reference model, $\tau_{\text{cyc}}$, $R$ and $B_M$. (A color version of this figure is available in the online journal.)