Phantom phase power-law solution in $f(G)$ gravity

A. R. Rastkar

Department of Physics
Azarbaijan University of Tarbiat Moallem, Tabriz, Iran

M. R. Setare

Department of Science
Payame Noor University, Bijar, Iran

F. Darabi

Department of Physics
Azarbaijan University of Tarbiat Moallem, Tabriz, Iran

Corresponding Author

(Dated: January 19, 2013)

Abstract: Power-law solutions for $f(G)$ gravity coupled with perfect fluid have been studied for spatially flat universe. It is shown that despite the matter dominated and accelerating power-law solutions, the power-law solution exists for an special form of $f(G)$ when this universe enters a Phantom phase.

Keywords: Power-law, $f(G)$ gravity, Phantom phase

PACS numbers: 98.80.Cq

*Electronic address: f.darabi@azaruniv.edu
I. INTRODUCTION

Nowadays it is strongly believed that the universe is experiencing an accelerated expansion. Recent observations from type Ia supernovae [1] in association with Large Scale Structure [2] and Cosmic Microwave Background anisotropies [3] have provided main evidence for this cosmic acceleration. These observations also suggest that our universe is spatially flat, and consists of about 70% dark energy (DE) with negative pressure, 30% dust matter (cold dark matter plus baryons), and negligible radiation. On the other hand the nature of dark energy is ambiguous. The simplest candidate of dark energy is a cosmological constant with the equation of state parameter \( w = -1 \). However, this scenario suffers from serious problems like a huge fine tuning and the coincidence problem [4]. Alternative models of dark energy suggest a dynamical form of dark energy, which is often realized by one or two scalar fields. In this respect, dark energy has many dynamical components such as quintessence [5], K-essence [6], tachyon [7], phantom [8], ghost condensate and quintom [9], and so forth.

It is known that Einstein's theory of gravity may not describe gravity at very high energies. The simplest alternative to general relativity is Brans-Dicke scalar-tensor theory [10]. Modified gravity also provides the natural gravitational alternative for dark energy [11]. Moreover, thanks to the different roles of gravitational terms relevant at small and at large curvature, the modified gravity presents natural unification of the early-time inflation and late-time acceleration. It may naturally describe the transition from non-phantom phase to phantom one without necessity to introduce the exotic matter. But among the most popular modified gravities which may successfully describe the cosmic speed-up is \( f(R) \) gravity. Very simple versions of such theory like \( \frac{1}{R} \) [12] and \( \frac{1}{R} + R^2 \) [13] may lead to the effective quintessence/phantom late-time universe (to see solar system constraints on modified dark energy models refer to [14]). Another theory proposed as gravitational dark energy is scalar-Gauss-Bonnet gravity [15] which is closely related with the low-energy string effective action. In this proposal, the current acceleration of the universe may be caused by mixture of scalar phantom and (or) potential/stringy effects. The coexistence of matter dominated and accelerating power law solutions for this theory has already been shown [16]. In this paper, we extend these results to show the existence of Phantom phase power law solutions for an special form of \( f(G) \) gravity.
II. FIELD EQUATIONS FOR $[R + f(G) + L_m]$ GRAVITY

As an alternative to the $f(R)$ action considered in Ref.\[17\] we consider the following $f(G)$ action which describes Einstein’s gravity coupled with perfect fluid plus a function of the Gauss-Bonnet term \[18, 19\]

$$S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + f(G) + L_m \right],$$

where $\kappa^2 = 8\pi G_N$ and the Gauss-Bonnet invariant is defined as follows

$$G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}.\quad(2)$$

The field equations are obtained by varying the action with respect to $g_{\mu\nu}$

$$0 = \frac{1}{2\kappa^2} (-R^{\mu\nu} + \frac{1}{2}g^{\mu\nu}R) + T^{\mu\nu} + \frac{1}{2}g^{\mu\nu} f(G) - 2f_{G}R R^{\mu\nu} + 4f_{G} R_{\rho}^{\mu} R^{\nu\rho}$$

$$- 2f_{G} R^{\mu\nu}\rho_{\sigma\tau} - 4f_{G} R^{\mu\nu\rho\sigma} R_{\rho\sigma} + 2(\nabla^{\mu}\nabla^{\nu} f_{G}) R - 2g^{\mu\nu} (\nabla^2 f_{G}) R - 4(\nabla_{\rho} \nabla^{\mu} f_{G}) R^{\rho\nu}$$

$$- 4(\nabla_{\rho} \nabla^{\nu} f_{G}) R^{\rho\mu} + 4(\nabla^2 f_{G}) R^{\mu\nu} + 4g^{\mu\nu} (\nabla_{\rho} \nabla_{\sigma} f_{G}) R^{\rho\sigma} - 4(\nabla_{\rho} \nabla_{\sigma} f_{G}) R^{\mu\nu\rho\sigma},$$

where $f_{G} = f'(G)$ and $f_{GG} = f''(G)$. The usual spatially-flat metric of Friedmann-Robertson-Walker (FRW) universe is chosen in agreement with observations

$$ds^2 = -dt^2 + a(t)^2 \sum_{i=1}^{3} (dx_i)^2,$$

where $a(t)$ is the scale factor as a one-parameter function of the cosmological time $t$. Using this metric in the field equations \[3\] one obtains the first FRW equation

$$-\frac{3}{\kappa^2} H^2 + G f_{G} - f(G) - 24\dot{G}H^3 f_{GG} + \rho_m = 0,$$

where $.$ denotes derivative with respect to time $t$ and Hubble parameter $H$ is defined by $H = \dot{a}/a$. In the FRW universe, the energy conservation law can be expressed as the standard continuity equation

$$\dot{\rho}_m + 3H(\rho_m + p) = \dot{\rho}_m + 3H(1 + w)\rho_m = 0,$$

where $\rho_m$ is the matter energy density and $p = w\rho_m$ is the equation of state relating pressure $p$ with energy density. From the continuity equation we obtain

$$\rho_m(t) = \rho_0 t^{-3(1+w)}.$$
For the metric (4), the Gauss-Bonnet invariant \( G \) and the Ricci scalar \( R \) may be defined as functions of the Hubble parameter

\[
G = 24(\dot{H}H^2 + H^4), \quad R = 6(\dot{H} + 2H^2).
\]

III. EXACT MATTER DOMINANT AND ACCELERATING POWER-LAW SOLUTIONS

We now assume an exact power-law solution for the field equations

\[
a(t) = a_0 t^m,
\]

where \( m \) is a positive real number. Using this assumption in (8) leads us to the following results

\[
G = \frac{24}{t^4} m^3(m - 1),
\]

\[
\dot{G} = -\frac{96}{t^5} m^3(m - 1),
\]

\[
R = \frac{6m}{t^2}(2m - 1).
\]

By substituting (7), (10) and (11) into (5) we obtain the Friedmann equation

\[
\frac{4}{m - 1} G^2 f_{GG} + G f_G - f_G - \frac{G^{1/2}}{k^2} \left( \frac{3m}{8(m - 1)} \right)^{1/2} + \rho_0 \left( \frac{G}{24m^3(m - 1)} \right)^{3/2m(1+w)} = 0.
\]

This is a differential equation for the function \( f(G) \) in \( G \) space. The general solution of this equation is obtained as

\[
f(G) = C_1 G + C_2 G^{-\frac{4}{3}(m-1)} - \frac{1}{2} \left[ \sqrt{\frac{6m(m-1)}{k^4(m+1)^2}} G^\frac{2}{3} + A_{mw} G^\frac{4}{3} m(1+w) \right],
\]

where

\[
A_{mw} = \frac{8\rho_0 (m - 1) (13824m^9 (m - 1)^3) - \frac{4}{3} m(1+w)}{4 + m [3m(w + 1)(w + 4/3) - 15w - 19]},
\]

and \( C_1, C_2 \) are arbitrary constants of integration. This solution is in agreement with the one obtained in [16] and, as is explained there, we can without any loss of generality assume the constants \( C_1 = C_2 = 0 \). Hence, the required form of the function \( f(G) \) becomes

\[
f(G) = -\frac{1}{2} \left[ \sqrt{\frac{6m(m-1)}{k^4(m+1)^2}} G^\frac{2}{3} + A_{mw} G^\frac{4}{3} m(1+w) \right].
\]
First, we note that a real valued solution for $f(G)$ requires the values $m \leq 0$ or $m \geq 1$. While the former leads to a contracting universe and also causes a divergence at $m = -1$ via the first term in the bracket, the latter is cosmologically desirable for an expanding universe as follows.

The case $m = 1$ leads to $G = 0$ and $R = 6t^{-2}$ which is the general relativity limit with the power-law solution

$$a(t) = a_0 t,$$

(17)

and the energy density

$$\rho_m(t) = \rho_0 t^{-3(1+w)}.$$  

(18)

According to (7), it is easy to see that $m = 2/[3(1 + w)]$ indicates the reduction to general relativity, so for $m = 1$ the equation of state parameter is fixed by $w = -1/3$ which accounts for a negative pressure but not yet an accelerating universe.

The case $m > 1$, leads to a nonzero real Gauss-Bonnet term $G$ and a positive Ricci scalar $R$. However, in order to avoid divergence in the Gauss-Bonnet term we have to keep $m$ and $w$ away from the values for which $A_{mw}$ diverges according to the following equation

$$4 + m [3m(w+1)(w+4/3) - 15w - 19] = 0.$$  

(19)

This case with $m > 1$ predicts an accelerating universe. Thus, power-law solutions of the type $a(t) = a_0 t^m$ or $H = \frac{m}{t}$ exist for the actions of the type $[R + f(G) + L_m]$ with $f(G)$ given by (14) except for those values of $m$ which satisfy (19).

**IV. EXACT PHANTOM PHASE POWER-LAW SOLUTION**

One may also study the power-law solutions where the universe enters a phantom phase leading to a Big Rip singularity. For this case, the general class of Hubble parameters and cosmological solutions are defined as

$$H(t) = \frac{m}{t_s - t},$$  

(20)

$$a(t) = a_0 (t_s - t)^{-m},$$  

(21)

where $t_s$ is the so called “Rip time” at future singularity. Again, using the above solution and repeating the similar calculations we obtain the following results

$$\rho_m(t) = \rho_0 t^{3m(1+w)},$$  

(22)
\[ G = \frac{24}{(t_s-t)^4}m^3(m+1), \]  
\[ \dot{G} = \frac{96}{(t_s-t)^5}m^3(m+1), \]  
\[ R = \frac{6m}{(t_s-t)^2}(2m+1). \]  

Substituting (22), (23) and (24) into the first FRW equation (5) we obtain

\[ -\frac{4}{m+1}G^2 f_{GG} + G f_G - f_G - \frac{G^{1/2}}{\kappa^2} \left( \frac{3m}{8(m+1)} \right)^{1/2} + \rho_0 \left( \frac{G}{24m^3(m+1)} \right)^{-\frac{3}{2}m(1+w)} = 0. \]  

This equation is easily recovered by the map \( m \rightarrow -m \) in the previous equation (13). Therefore its solution is obtained by using the same map in (14) as

\[ f(G) = C_1 G + C_2 G^{\frac{1}{2}}(m+1) - \frac{1}{2} \left[ \sqrt{\frac{6m(m+1)}{k^4(m-1)^2}} G^{\frac{1}{2}} + A_{mw} G^{-\frac{3}{2}m(1+w)} \right], \]  

where

\[ A_{mw} = -\frac{8 \rho_0 (m+1)(13824m^9(m+1)^3)}{4+m[3m(w+1)(w+4/3)+15w+19]} \]  

Similar to the solutions in the previous section, we assume \( C_1 = C_2 = 0 \). Then, the required form of the function \( f(G) \) becomes

\[ f(G) = -\frac{1}{2} \left[ \sqrt{\frac{6m(m+1)}{k^4(m-1)^2}} G^{\frac{1}{2}} + A_{mw} G^{-\frac{3}{2}m(1+w)} \right]. \]  

Actually, \( m > 0 \) leads to a real valued function \( f(G) \) according to (27). However, demanding a Big Rip during the phantom phase, as the cosmic time \( t \) approaches \( t_s \), requires \( m \geq 1 \) in (21). However, \( m = 1 \) causes a divergence in \( f(G) \) through the first term of (27) in the bracket. Moreover, the Gauss-Bonnet term diverges through \( A_{mw} \) for those values of \( m \) for which the following equation is satisfied

\[ 4 + m[3m(w+1)(w+4/3)+15w+19] = 0. \]  

Therefore, power-law solutions in the phantom phase of the type \( a(t) = a_0(t_s-t)^{-m} \) or \( H(t) = \frac{m}{t_s-t} \) exist for the actions of the type \( [R + f(G) + L_m] \) with \( f(G) \) given by (27) except for those values of \( m \) which satisfy (30).
v. THE STABILITY ISSUE

The stability issue of a large class of modified gravitational models has been discussed
with particular emphasis to de Sitter solutions [20–25], [26], [27], [28], [29]. In the present
modified gravity, namely \([R + f(G)]\), the stability issue leads to the following conditions [22]

\[
G_0 f'_0 - f_0 = 6H_0^2, \tag{31}
\]

\[
1 < \frac{9}{R_0^3 f''_0}, \tag{32}
\]

where the critical points are defined by

\[
R_0 = 12H_0^2, \quad G_0 = 24H_0^4, \tag{33}
\]

and \(f_0, f'_0 = (df/dG)_0, f''_0 = (d^2f/dG^2)_0, H_0\) are suitable constants corresponding to the de
Sitter solutions. In the case of Phantom phase power law solution, namely [29], the first
condition reads as

\[
\frac{1}{2} \sqrt{6m(m+1)\frac{1}{k^4(m-1)^2}} + \left(1 - \frac{3}{2}m(1 + \omega)\right) A_{mw} \times G_0^{-\frac{3}{2}m(1+\omega)-\frac{1}{2}} = \sqrt{6}, \tag{34}
\]

which implies that

\[
\left(1 - \frac{3}{2}m(1 + \omega)\right) A_{mw} > 0. \tag{35}
\]

By using (35), the second condition (32) reads as

\[
(3/2)^{3/2} \sqrt{6m(m+1)\frac{1}{k^4(m-1)^2}} + \left(-\frac{3}{4}m(1 + \omega)\right) \tag{36}
\]

\[
- \left(\frac{3}{4}m(1 + \omega) + 1\right) \left(1 - \frac{3}{2}m(1 + \omega)\right)^{-1} \times \left[\sqrt{6} - \frac{1}{2} \sqrt{6m(m+1)\frac{1}{k^4(m-1)^2}}\right] < 9.
\]

Then, the model is stable around de Sitter solution if the arbitrary parameters also satisfy
both the conditions (35) and (36).
VI. CONCLUSION

In the present paper we have considered an $f(G)$ action which describes Einstein’s gravity plus a function of the Gauss-Bonnet term. Then, by considering an exact power-law solution for the field equations we have obtained the Friedmann equation in spatially flat universe. The Friedmann equation appears as a differential equation for function $f(G)$. We could obtain the solution of this equation and show that our model with this solution for $f(G)$ has power-law solution of the type $a(t) = a_0 t^m$ expect for those values of $m$ for which $f(G)$ diverges. These solutions are in agreement with those obtained in [16]. We have also studied the power-law solutions when the universe enters a Phantom phase. By considering such power-law solution for the field equations we have obtained the corresponding Friedmann equation. The solution $f(G)$ of this differential equation is obtained and it is shown that the power-law solution in the phantom phase of the type $a(t) = a_0 (t_s - t) ^{-m}$ exists for this $f(G)$ except for those values of $m$ for which this function diverges.

Acknowledgment

This work has been supported by the Research office of Azarbaijan University of Tarbiat Moallem, Tabriz, Iran.

[1] Riess A. G. et al. [Supernova Search Team Collaboration], Astron. J. 116, 1009 (1998); Perlmutter S. et al. [Supernova Cosmology Project Collaboration], Astrophys. J. 517, 565 (1999); Astier P. et al., Astron. Astrophys. 447, 31 (2006).

[2] Abazajian K. et al. [SDSS Collaboration], Astron. J. 128, 502 (2004); Abazajian K. et al. [SDSS Collaboration], Astron. J. 129, 1755 (2005).

[3] Spergel D. N. et al. [WMAP Collaboration], Astrophys. J. Suppl. 148, 175 (2003); Spergel D. N. et al., astro-ph/0603449.

[4] Shani. V. and Starobinsky. A., Int. J. Mod. Phys. D 9, 373 (2000); Carroll. S. M., Living Rev. Rel. 4, 1 (2001); Copeland. E. J., Sami. M. and Tsujikawa. S., Int. J. Mod. Phys. D 15, 1753 (2006); Sahni. V. and Starobinsky. A. A., Int. J. Mod. Phys. D 15, 2105 (2006).
[5] Ratra. B. and Peebles. P. J. E. , Phys. Rev. D 37, 3406 (1988); Wetterich. C., Nucl. Phys. B 302, 668 (1988); Caldwell. R. R., Dave. R. and Steinhardt. P. J., Phys. Rev. Lett. 80, 1582 (1998); Zlatev. I., Wang. L. M. and Steinhardt. P. J., Phys. Rev. Lett. 82, 896 (1999).

[6] Armendariz-Picon. C., Mukhanov. V. F., and Steinhardt. P.J., Phys. Rev. Lett. 85, 4438 (2000).

[7] Sen. A., JHEP 0207, 065 (2002); Padmanabhan. T., Phys. Rev. D 66, 021301 (2002); Setare. M. R., Phys. Lett. B 653, 116 (2007).

[8] Caldwell. R. R., Phys. Lett. B 545, 23 (2002); Nojiri. S. and Odintsov. S. D., Phys. Lett. B 562, 147 (2003); Wei. Y. H. and Tian. Y., Class. Quant. Grav. 21, 5347 (2004); Onemli. V. K., and Woodard. R. P., Phys. Rev. D 70, 107301 (2004); Setare. M. R., Eur. Phys. J. C 50, 991 (2007).

[9] Feng. B., Wang. X. L., and Zhang. X. M., Phys. Lett. B 607, 35 (2005); Guo. Z. K., Piao. Y. S., Zhang. X. M., and Zhang. Y. Z., Phys. Lett. B 608, 177 (2005); Cai. Y-F., Li. H., Piao. Y. S. and Zhang. X., Phys. Lett. B 646, 141 (2007); Setare. M. R., Sadeghi. J., and Amani. A. R., Phys. Lett. B 660, 299 (2008); Setare. M. R., and Saridakis. E. N., Phys. Lett. B 668, 177 (2008); Setare. M. R., and Saridakis. E. N., [arXiv:0807.3807[hep-th]]; Setare. M. R., and Saridakis. E. N., JCAP 09, 026 (2008); Cai. Y-F, Saridakis. E. N., Setare. M. R., and Xia. J-Q., Phys. Rep. 493, 1 (2010).

[10] Brans. C. and Dicke. C. H., Phys. Rev. 124, 925 (1961).

[11] Nojiri. S., Odintsov. S. D., Stefancic. H., Phys. Rev. D 74, 086009 (2006); Nojiri. S., Odintsov. S. D., J. Phys. A 40, 6725 (2007); Cognola. G., Elizalde. E., Nojiri. S., Odintsov. S. D., and Zerbini. S., Phys. Rev. D 75, 086002 (2007); Nojiri. S. and Odintsov. S. D., J. Phys. Conf. Ser. 66, 012005 (2007); Setare. M. R., Int. J. Mod. Phys. D 17, 2219 (2008); Setare. M. R., Astrophys. Space Sci. 326, 27 (2010).

[12] Capozziello. S., Int. J. Mod. Phys. D 11, 483 (2002); Capozziello. S., Carloni. S., and Troisi. A., [arXiv:astro-ph/0303041]. Carroll. S. M., Duvvuri. V., Trodden. M., and Turner. S., Phys. Rev. D 70, 043528 (2004).

[13] Nojiri. S. and Odintsov. S. D., Phys. Rev. D 68, 123512 (2003).

[14] Nojiri. S., Odintsov. S. D., [arXiv:0707.1941v2[hep-th]]; Nojiri. S., Odintsov. S. D., [arXiv:0710.1738v2[hep-th]]; Cognola. G., Elizalde. E., Nojiri. S., Odintsov. S. D., Sebastiani. L., Zerbini. S., [arXiv:0712.4017v1[hep-th]].
[15] Nojiri. S., Odintsov. S. D., and Sasaki. M., Phys. Rev. D 71, 123509 (2005); Nojiri. S., Odintsov. S. D., and Sami. M., Phys. Rev. D 74, 046004 (2006); Carter. B. M. N., Neupane. I. P., Phys. Lett. B 638, 94 (2006); Carter. B. M. N., and Neupane. I. P., JCAP 0606, 004 (2006); Moffat. J. W., and Toth. V. T., arXiv:0710.0364[astro-ph]; Elizalde. E., Myrzakulov. R., Obukhov. V. V., Sez-Gmez. D., Class. Quant. Grav. 27, 095007 (2010); Myrzakulov. R., Sez-Gmez. D., Tureanu. A., Gen. Rel. Grav., 2011 (in press) arXiv:1009.0902[gr-qc].
[16] Goheer. N., Goswami. R., Dunsby. P., and Ananda. K., Phys. Rev. D 79, 121301(R) (2009).
[17] Goheer. N., Larena. J., Dunsby. P. K. S., Phys. Rev. D 80, 061301 (2009).
[18] Nojiri. S., Odintsov. S. D., Phys. Lett. B631, 1 (2005).
[19] Nojiri. S., Odintsov. S. D., Gorbunova. O. G., J. Phys. A39, 6627 (2006).
[20] Cognola. G., Elizalde. E., Nojiri. S., Odintsov. S. D., and Zerbini. S., JCAP, 0502, 010 (2005).
[21] Cognola. G., Zerbini. S, J. Phys. A39, 6245, (2006).
[22] Cognola. G., Castaldi. M., Zerbini. S., Int. J. Theor. Phys., 47, 898 (2008).
[23] Cognola. G., Zerbini. S, The Problems of Modern Cosmology, on the occasion of the 50th birthday of Prof. S. D. Odintsov Tomsk State Pedagogical University, 153-163 (2009).
[24] Capozziello. S., Nojiri. S., Odintsov. S. D., and Troisi. A., Phys. Letts. B639, 135 (2006).
[25] Elizalde. E., Nojiri. S., Odintsov. S. D., Sebastiani. L., Zerbini. S., Phys. Rev. D83, 086006 (2011).
[26] Faraoni. V., Ann. Phys., 317, 366 (2005).
[27] Faraoni. V., Phys. Rev. D72, 061501 (2005).
[28] Nojiri. S., Odintsov. S. D., Phys. Rev. D68, 123512 (2003).
[29] Nojiri. S., Odintsov. S. D., Phys. Rev. D74, 086006 (2006).