Sklyanin Bracket and Deformation of the Calogero-Moser System

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Abstract

A two-dimensional integrable system being a deformation of the rational Calogero-Moser system is constructed via the symplectic reduction, performed with respect to the Sklyanin algebra action. We explicitly resolve the respective classical equations of motion via the projection method and quantize the system.

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1 Introduction

The relationship between the integrable systems and the Lie group gauge symmetries is well known [1,2,3]. However, Lie groups by no means exhaust all the symmetries and in fact there are a lot of systems whose gauge symmetries cannot be reduced to the Lie group action [4]. For this reason it is quite interesting to construct an integrable system using the non Lie group symmetries. The aim of the paper is to present the example of such integrable system. We obtain the system via the symplectic reduction from an extended phase space, where the symmetries are realized by the action of the quadratic Sklyanin algebra [3]. Let us briefly explain the key idea for realizing this action. Note that the well known coadjoint action of a Lie algebra on its dual space can be written in terms of the linear Poisson structure, naturally associated to the Lie algebra. Namely if \( p \) are coordinates in the dual space and the respective Poisson tensor is of the form

\[
(p) = f_p p;
\]
where \( f \) are the structure constants of the Lie algebra, then the coadjoint action is rewritten as
\[
p = (p) \cdot (p);
\]
where \( p \) are the infinitesimal parameters of the action. If we assume now that the Poisson tensor \( (p) \) is no longer linear in \( p \) we get the generalization of the coadjoint action to the case of an arbitrary Poisson manifold. Such a generalization of the coadjoint action has been originally proposed by Karas'ev \([7]\) and in general it cannot be reduced to the Lie algebra action in a sense that one cannot choose the global basis of the sections \( (p) \) with a Lie algebra commutation relations of the transformations \([7]\). Instead the form of the commutation relations in a general case looks as follows
\[
\{ ; \} = \cdot ;
\]
where
\[
\{ ; \} = d ( ; ) + (d ; ) + ( ; d );
\]
\[
\{ ; \} = @ ( ; ) + @ @ ( ; ) + @ @ @ ( ; ) ;
\]
\[
\{ ; \} = \frac{\partial}{\partial p} ;
\]
Thus we get an example of the manifold, equipped with the non Lie algebra symmetries. A mathematical rigor treatment of such symmetries involves a definition of the so-called Lie algebroid \([3,6]\), which is defined as a bundle \( A \) over a manifold \( M \), with a Lie bracket, given in the set of sections \( (A) \) and an anchor map \( : (A) \Rightarrow (T M ) \), being a homomorphism of the respective Lie algebras. Using this terminology we see that the above construction is an example of the Lie algebroid. Namely, the Lie algebroid is the cotangent bundle over the Poisson manifold. The Lie bracket between the sections and is given by the equation \([3]\) and the anchor map is defined by the formula \([3]\). Note that the Jacobi identity of the Lie bracket \([3]\) holds in virtue of the respective Jacobi identity for the Poisson tensor \( (p) \). In this paper we start from the Poisson manifold \( R^4 \), equipped with the standard classical Sklyanin bracket \([6]\), and thus equipped with the algebroid action, which may be regarded as the action of the Sklyanin algebra. We choose the extended phase space of the integrable system to be the cotangent bundle over \( R^4 \) with the standard symplectic structure. We lift the algebroid transformations from the base manifold to the canonical ones in the cotangent bundle and readily find the respective Hamiltonian generators, which define a natural analogue of the momentum map for the case of the non Lie group symmetries. Then we construct the phase space of the integrable system by performing the symplectic reduction on the surface of the non zero level of the generators. The underlying Hamiltonians are obtained by reduction of the Casimir functions of the Sklyanin bracket. Note that since the Sklyanin bracket \([6]\) is the deformation of the Poisson bracket, which is equivalent to the linear bracket, associated to the Lie algebra \( u_2 \) the presented symplectic reduction is similar to the one leading to the rational Calogero-Moser system \([1,4]\). In particular, one of the
exponential degenerations of the Skyrian bracket leads to the system, whose dynamics is in a sense equivalent to the dynamics of the two-particle rational Calogero-Moser system. As an aside, it is worth noting that although the integrable system is obtained via the symplectic reduction we still cannot find the respective Lax representation because the symmetries, that have been used for constructing the system are not Lie group ones. However, we show that it is possible to resolve the equations of motion for our system via the projection method. So the problem of finding the respective Lax representation is seen to be rather intriguing and we are going to comment on it in the concluding section. The organization of the paper is as follows. In the second section we present the construction of the integrable system. In the third section we develop the projection method and explicitly resolve the equations of motion. The fourth section is devoted to the quantization. We end that although it is not possible to pose the bound states problem for our system one of the Hamiltonians possesses a discrete spectrum. At last, in the concluding section we discuss some open questions. All through the paper the Greek indices take values from 0 to 3 the Latin indices run from 1 to 3 and whenever the indices $i;j;k$ are met they are assumed to take values $1;2;3$ together with their cyclic permutations. $\theta$ denotes $\frac{\partial}{\partial p}$ and $\theta$ is reserved for $\frac{\partial}{\partial x}$.

2 Construction of the System

We construct the system starting with the extended phase space, which is chosen to be the cotangent bundle $T^*\mathbb{R}^4$, equipped with the canonical symplectic structure

$$! = dp \wedge dx$$

(3)

Here we use the following unusual conventions. The coordinates on the base $\mathbb{R}^4$ are denoted by $p$ and the coordinates in the cotangent space are denoted by $x$. Let us endow the base manifold $\mathbb{R}^4$ with the quadratic Skyrian bracket

$$fp_0;p_ig = _{0i}(p) = J_{jk}p_jp_k \quad fp_0;p_jg = _{ij}(p) = p_0p_k;$$

(4)

where the coefficients $J_{12}, J_{23}$ and $J_{31}$ satisfy the identity

$$J_{12} + J_{23} + J_{31} = 0;$$

(5)

which is equivalent to the Jacobi identity for the bracket [4]. The condition [5] also implies that these coefficients can be represented in the form

$$J_{ij} = J_i - J_j;$$

(6)

where the values $J_i$ are defined up to the additional constant $c$, $J_i \equiv J_i + c$. Following the steps, outlined in the introduction we first define the analogue of the coadjoint transformations on the base manifold $\mathbb{R}^4$ using the Poisson structure [4]. Namely, the infinitesimal coadjoint transformations are defined as

$$p = (p) (p);$$

(7)
where are the respective in nitesimal parameters. The transformations (7) can be easily lifted to the canonical ones in the cotangent bundle

\[ p = \langle p \rangle \quad x = \emptyset \] (8)

and the Hamiltonian generators corresponding to the transformations (3) can be also readily found

\[ M = \langle p, x \rangle = \langle p \rangle \times : \] (9)

Then in order to obtain a non-trivial dynamical system via the symplectic reduction method we consider the surface of non-zero level of the generators (3) in the extended phase space \( T \mathbb{R}^4 \). In analogy to what it is done for case of the rational Calogero-Moser system we set

\[ M_0 = J_{23}p_2 p_1 x^4 + J_{31}p_3 p_1 x^2 + J_{12}p_1 p_2 x^3 = 0 \]

\[ M_1 = p_0 (x^2 p_3 \quad x^3 p_2) \quad J_{23}p_2 p_1 x^0 = \]

\[ M_2 = p_0 (x^3 p_1 \quad x^1 p_3) \quad J_{31}p_3 p_1 x^0 = 0 \]

\[ M_3 = p_0 (x^1 p_2 \quad x^2 p_1) \quad J_{12}p_1 p_2 x^5 = 0 : \] (10)

The symplectic form (3) being reduced on the constraint surface (10) is of course degenerate and the respective kernel distribution is given by the following vector field, being tangent to the surface (10)

\[ p_0 = J_{23}p_2 p_3 \quad x^0 = (x^2 p_3 \quad x^3 p_2) \]

\[ p_1 = 0 \quad x^1 = 0 \]

\[ p_2 = p_0 p_3 \quad x^2 = (p_0 x^3 + J_{23}p_3 x^0)^1 \]

\[ p_3 = p_0 p_2 \quad x^3 = (p_0 x^2 \quad J_{12}p_2 x^0): \] (11)

Our system is then presented as a result of the symplectic reduction on the constraint surface (10) with respect to the transformations, corresponding to the vector field (11). In what follows, these transformations are called the gauge ones and the integral curves of the vector field (11) are called the gauge orbits. We perform the symplectic reduction using the gauge fixing condition \( x^2 = 0 \). In fact it is easy to see that this gauge condition has a horizon, given by the equation

\[ p_0 x^3 \quad J_{32}p_3 x^0 = 0 : \] (12)

Namely, for the points, satisfying (12) the coordinate \( x^2 \) is unchanged under the gauge transformations (11), and hence for these points the value of the coordinate \( x^2 \) in general cannot be reduced to zero with the help of the gauge transformations. However, in the

\[ ^{\text{thus we have constructed the so-called Ham iltonian algebraoid [12] over } \mathbb{R}^4; \text{ in the paper [12] it is shown that the natural construction of the Ham iltonian algebraoid exists for an arbitrary Lie algebraoid.}} \]
In this section we show that in spite of the more complicated nature of the gauge transformations the equations of motion for the presented system (15) can be explicitly resolved via the projection method. Let us briefly recall the main idea of the method. It is
well known that each point of the reduced phase space corresponds to a gauge orbit on the
constraint surface, where the dynamics is described by the gauge invariant Hamiltonians
\(^{(14)}\) in our case. We integrate the equations of motion defined by the Hamiltonians \(^{(14)}\)
and construct the surface, which consists of the gauge orbits, passing through to all the
points of the evolution curves. The intersections of the surface with the level of the gauge
xing condition are just the evolution curves of the reduced system. Thus the explicit
solutions of the evolution equations for the reduced system can be derived if the explicit
description of the gauge orbits is known. In order to describe the gauge orbits we have to
integrate the system of differential equations given by the kernel vector field \(^{(11)}\).

\[
\frac{d}{ds}p_0 = J_{23}p_2p_3, \quad \frac{d}{ds}x^0 = (x^2p_3 + x^3p_2)
\]

\[
\frac{d}{ds}p_1 = 0, \quad \frac{d}{ds}x^1 = 0
\]

\[
\frac{d}{ds}p_2 = p_0p_3, \quad \frac{d}{ds}x^2 = (p_0x^2 + J_{23}p_3x^0)
\]

\[
\frac{d}{ds}p_3 = p_0p_2, \quad \frac{d}{ds}x^3 = (p_0x^3 + J_{23}p_2x^0);
\]

where \(s\) parameterizes the gauge orbit. Note that the r.h.s. of the differential equations
for the momenta contains the momenta only. Hence we can integrate these equations
separately. Then using the fact that the gauge transformations \(^{(11)}\) are canonical ones
we can resolve the rest equations for the coordinates with the help of the following formula

\[
x(s) = \frac{\partial p^0}{\partial p}x^0;
\]

where \(x^0, p^0\) denote the initial conditions for the system \(^{(17)}\). Note also that the phase
coordinates \(x^1; p_1\) are not changed under the gauge transformations so in what follows
we are going to omit them. It is easy to see that the system of differential equation for
the momenta is similar to the system of equations for the ordinary \(so_3\) top \(^{(13)}\). Using
the analogy we can readily nd two independent integrals of motion for the system \(^{(17)}\).

\[
A = q \frac{(p_2)^2 + (p_3)^2}{(p_3)^2 + J(p_2)^2}, \quad B = q \frac{(p_0)^2 + J(p_3)^2}{(p_0)^2 + J(p_2)^2};
\]

where \(J = J_{32}\) (in the rest of the paper we assume that \(J > 0\)). Let us consider the
integrals \(^{(13)}\) as the new independent momenta. Then the rest third coordinate in the
momentum space can be chosen in two different ways. First, one can introduce the
following angle variable

\[
' = \arctg(p_3/p_2);
\]

for which the momenta \(p_0, p_2, p_3\) take the following form

\[
p_0 = B^2 \frac{J}{A^2 \sin^2 '}, \quad p_2 = A \cos ' \quad p_3 = A \sin ';
\]
Second, the third coordinate can be chosen as
\[ ' \equiv \arctg \left( \frac{p_2}{Jp_3} = p_0 \right) \] (22)
And in this case we get
\[ p_0 = B \cos ' \quad p_2 = \frac{B^2}{J} \sin^2 ' \quad p_3 = \frac{B}{J} \sin ' \] (23)

The first way to choose the new momenta is applicable when
\[ k = \frac{p - JA}{B} < 1; \] (24)

because it is the case when the expression \( B^2 \frac{JA^2}{B^2} \sin^2 ' \) do not change the sign. Otherwise, one should use the second way of choosing coordinates in the momentum space. In both cases one can develop the projection method and find the evolution equations of the reduced system proceeding similarly at each stage. So in our paper we are going to discuss the projection method only for the first case (24) omitting another possibility. In what follows, it is more convenient to use instead of \( x^0; x^2; x^3 \) the coordinates, which are canonically conjugated to \( A; B \) and \('\). They can be readily found in the following form
\[
\begin{align*}
x_A &= x^2 \cos ' + x^3 \sin ' \\
x_B &= \frac{B x^0}{B^2} \\
x_\nu &= A x^2 \sin ' + A x^3 \cos ' \\
\end{align*}
\] (25)

In terms of \( A; B; '; x_A; x_B \) and \( x_\nu \), the solution for the system (27) looks as follows
\[
\begin{align*}
x_A (s) &= x_A^0 + \frac{p}{J} q \left[ \frac{1}{B} k^2 \sin^2 ' (s + A \frac{p}{J} q) \right] x_\nu^0 \\
x_B (s) &= x_B^0 + \frac{p}{J} q \left[ \frac{1}{B} k^2 \sin^2 ' (s + A \frac{p}{J} q) \right] x_\nu^0 \\
\end{align*}
\] (26)

where \( u = am (t;k) \) is the elliptic Jacobi function being the inverse function to the first order elliptic integral
\[
F (u;k) = \frac{1}{k^2 \sin^2 t} \int_0^u \frac{dt}{1 - k^2 \sin^2 t};
\]

\( F_k (u;k) \) denotes the corresponding derivative
\[
F_k (u;k) = \frac{1}{(1 - k^2 \sin^2 t)^{3/2}} \int_0^u k \sin t dt.
\]
and $A_0; B_0; x_A^0; x_B^0$ are the initial conditions of the respective phase coordinates. Now suppose we are given the initial conditions $u^1; v_1; u^2; v_2$ for the equations of motion of the reduced system (13):

$$
\theta_{u^1} u^1 = \frac{2 u^2}{(v_1 u^2 - J v_2 u^1)^3} \quad \theta_{v_1} u^1 = v_1 \\
\theta_{u^2} u^2 = v_2 + \frac{J 2 u^1}{(v_1 u^2 - J v_2 u^1)^2} \quad \theta_{v_2} u^2 = J v_2 \\
\theta_{u^1} v_1 = \frac{J v_2}{(v_1 u^2 - J v_2 u^1)^3} \quad \theta_{v_1} v_1 = 0 \\
\theta_{u^2} v_2 = \frac{2 v_1}{(v_1 u^2 - J v_2 u^1)^3} \quad \theta_{v_2} v_2 = 0.
$$

(27)

The initial point of the evolution curve (27) is mapped via the embedding (13) to the point in the extended phase space with the phase coordinates $x, p$, such that $x^1 = p_1 = 0$ and the gauge fixing condition $x^2 = 0$ is satisfied. Here we assume for simplicity that the inequality (24) is also satisfied. The gauge fixing condition and the gauge invariant Hamiltonians (14) in terms of the phase coordinates (19), (20), (25) take the following form:

$$
A x_A \cos' = x \cdot \sin';
$$

(28)

$$
C_1 = \frac{(p_1)^2}{2} + \frac{A^2}{2} \quad C_2 = J \frac{(p_1)^2}{2} + \frac{B^2}{2}.
$$

(29)

Solving the equations of motion with respect to the Hamiltonians (29) we get that the momenta (13, 20) are conserved $x^1$ and $p_1$ do not change their zero values and the coordinates (29) are linear functions of the times $t_1$ and $t_2$, corresponding to the Hamiltonians $C_1$ and $C_2$ (29), respectively, namely

$$
x_0^1 = 0 \quad p_0^1 = 0 \\
x_0^2 = x, \quad '0 = ' \\
x_A^0 = x_A + A t_1 \quad A_0 = A \\
x_B^0 = x_B + B t_2 \quad B_0 = B.
$$

(30)

Here we suspend the phase space coordinates by 0 and in order to show that they should be considered as the initial conditions for the system of differential equations (17) because in general a point of the phase curve (31) does not satisfy the gauge fixing condition (28) but just define the orbit that should intersect the level of the gauge fixing condition in another point. Substituting now the phase space coordinates of the curve (24) with the initial data (30) into the gauge fixing condition (28) we get the following equation

$$
A (x_A + A t_1) = x, \quad \theta \left[ \frac{1}{k^2 \sin^2' (s)} \right] = \frac{1}{k^2 \sin^2' (s)} \theta' (s).
$$
where \( k = \frac{JA}{B} \). Since the r.h.s. of the equation (31) does not explicitly depend on the parameter \( s \) we can forget about it and treat \( 31 \) as the equation, defining \( \tau \) as an implicit function of \( t_1 \). Really if we extract the dependence on \( \tau \) in the r.h.s. of (31) we get the function

\[
\tau' = \frac{1}{k^2 \sin^2 \tau} \tau \frac{F_k(\tau';k)}{F_k(\tau';k)}
\]

with the positive derivative

\[
\tau^0 = \frac{1}{\cos^2 \tau - \frac{1}{k^2 \sin^2 \tau}} > 0.
\]

Thus the intersection of the surface, which consists of the gauge orbits, passing through the points of the phase curve (30), with the level of the gauge fixing condition (28) is given by the following equations

\[
x_\perp(t_1; t_2) = x_\perp(t_1) \frac{1}{k^2 \sin^2 \tau(t_1)} \quad x_\perp(t_1; t_2) = \frac{\tan'(t_1)}{A} x_\perp(t_1) \frac{1}{k^2 \sin^2 \tau(t_1)}
\]

\[
x_x(t_1; t_2) = x_x + B t_2 \quad x_x = x_\perp \frac{1}{k^2 \sin^2 \tau(t_1)} (F(\tau'; k) F(\tau'; k))
\]

\[
+ \frac{A}{B^2} (F_k(\tau'; k) F_k(\tau'; k))
\]

\[
A = A \quad B = B \quad \tau = \tau(t_1);
\]

where \( \tau(t_1) \) is the implicit function given by (31). Using the equations (32) we get the explicit solutions of the differential equations (27) in the following form

\[
u_1(t_1; t_2) = \frac{1}{B^2 \sin^2 \tau(t_1)} x_x(t_1; t_2)
\]

\[
u_2(t_1; t_2) = x_x(t_1; t_2) \sin'(t_1) + \frac{x_x(t_1; t_2)}{A} \cos'(t_1) + \frac{JA}{B} x_x(t_1; t_2) \sin'(t_1)
\]

\[
u_3(t_1; t_2) = \frac{1}{B^2 \sin^2 \tau(t_1)} v_x(t_1; t_2) \quad v_2(t_1; t_2) = A \sin'(t_1);
\]
where the initial data in the extended phase space can be obtained with the help of the following formulas

\[ A = \frac{v_2}{u_1} + \frac{2}{(v_1 u_2^2 + J v_2 u_1^2)^2} \quad B = \frac{q}{(v_1)^2 + J (v_2)^2} \]

\[ \alpha = \arctan \frac{v_2 (v_1 u_2^2 - J v_2 u_1^2)}{u_1} \]

\[ x_A = u_2^2 \sin' \frac{J A \sin^2'}{B^2 - J A^2 \sin^2'} u_1 \]

\[ x_B = u_1^2 \frac{B u_1}{B^2 - J A^2 \sin^2'} \]

\[ x' = A u_2^2 \cos' \frac{J A^2 \sin' \cos'}{B^2 - J A^2 \sin^2'} u_1 \]

4 Quantization

In this section we consider the canonical quantization of the presented system restricting ourselves to the case of the positive \( J = J_{32} > 0 \), that is to the case when the quantum Hamiltonian \( H_2 \) is an elliptic operator. We realize the Hamiltonians \( H_1 \) at the quantum level in such a way that the strong involution relation between them is not destroyed by quantum corrections. Then, given these commuting Hamiltonians, we solve the respective eigenvalue problem by requiring that the wave functions have no branch points. The surprising thing is that the spectrum of the Hamiltonian \( H_2 \) turns out to be discrete and the coupling constant \( J^2 \) turns out to be negative. Our starting point is the classical Hamiltonians

\[ H_1 = \frac{(v_2)^2}{2 J} + \frac{2}{2 J (v_1 u_2^2 + J v_2 u_1^2)^2} \quad H_2 = \frac{(v_1)^2}{2} + \frac{(v_2)^2}{2} ; \]

which are achieved from \( H_1 \) by the obvious rescaling

\[ v_1 \leftrightarrow v_1 \quad u_1 \leftrightarrow u_1 \]

\[ v_2 \leftrightarrow \frac{v_2}{ \sqrt{J} } \quad u_2 \leftrightarrow \frac{P}{J u_2} \]

We perform the quantization of the system \( H_2 \) in the momentum representation using the polar coordinates on the momentum plane

\[ v_1 = r \sin \quad v_2 = r \cos : \]

In this setting

\[ \hat{a}_1 = \hbar (\sin \theta + \frac{\cos \theta}{r}) \quad \hat{a}_2 = \hbar (\cos \theta - \frac{\sin \theta}{r}) \]
\[ \psi_1 = r \sin \theta \quad \psi_2 = r \cos \theta; \]  
and the quantum Hamiltonians take the following form
\[ \hat{H}_1 = \frac{r^2 \cos^2 \theta}{2J} + \frac{2}{2JM^2} \quad \hat{H}_2 = \frac{r^2}{2}; \]  
where the operator \( \hat{M} \) is defined as
\[ \hat{M} = \psi_1 \hat{a}_2 \quad \psi_2 \hat{a}_1 = \sin \theta; \]
In order to determine the inverse operator to \( \hat{M}^2 \) we use the following anzats for the basis wave functions
\[ (r; \theta) = R(r) (\theta); \]  
and represent the factor \((\theta)\) as the respective Fourier series
\[ (\theta) = \sum_{m=-1}^{\infty} e^{im \theta}; \]
Then the operator is naturally defined on the functions \((\theta)\) by linearity if we set that the action of \( \hat{M}^2 \) on the eigenfunctions \( e^{im \theta} \) of \( \hat{M} \) looks as follows
\[ \hat{M}^2 e^{im \theta} = \frac{1}{(\theta m)^2} e^{im \theta}; \]
Of course we assume here that the zero mode in the expansion \((\theta)\) is vanishing \(0\) = 0. It is easy to see that thus defined quantum Hamiltonians \((38)\) are commuting operators and one may pose the respective eigenvalue problem
\[ \hat{H}_1 = E_1 \quad \hat{H}_2 = E_2; \]
Using the same anzats \((39)\) for the wave function we see that this problem \((42)\) reduces to a single iteration equation for the coefficients \(m^2 (m+2 + m^-2 \quad c_m) + b_m = 0 \) 

\[ m^2 (m+2 + m^-2 \quad c_m) + b_m = 0; \]
since the dependence of the wave function \((39)\) on \(r\) is obviously defined by the second equation in \((42)\)
\[ R(r) = (r^{q} 2E2); \]
The iteration equation \((43)\) is equivalent to the following differential equation for the generating function \((z) = \sum_{m} z^m\)
\[ (z \theta z)^2 (\theta z^2 + \frac{1}{z^2} \quad c) (z) + b (z) = 0; \]
\[ (z \theta z)^2 (\theta z^2 + \frac{1}{z^2} \quad c) (z) + b (z) = 0; \]
where
\[ c = 2 \left( \frac{J_1 E_1}{E_2} \right) \quad b = \frac{2}{E_2 h^2} \]

In what follows it is more convenient to work with the unfolded form of this equation (44), namely
\[ z^2 (z^4 + c z^2 + 1)^0 (z) + z (5z^4 + c z^2 + 3)^0 (z) + (4z^4 + 4 + b) (z) = 0 \quad (45) \]

It goes without saying that one can hardly find the solutions for the equation (45) in the explicit form. However, we may use the standard results of the complex analysis and conclude that the solutions are analytical functions on C besides may be the points where the polynomial \( P (z) = z^2 (z^4 + c z^2 + 1) \) is vanishing. Hence, we can qualitatively analyze asymptotical the behaviour of the solutions of (45) in the vicinity of the roots of the polynomial \( P (z) = z^2 (z^4 + c z^2 + 1) \). And thus, we can derive the properties of the desired functions ( ), obtained by restricting the solutions to the unit circle \( f(z = e^{i \theta}) \) for the coupling constant (44), namely
\[ \frac{2}{E_2 h^2} \]

In fact we have to require that the restriction ( ) is the function on the circle. It means that the respective analytical function ( ) have no branch points in the interior of the unit disc \( f(z = 2) \) for \( j \) or the factors that arise from these branches should combine into the trivial factor 1 if the branch point is not the only one. In addition, we claim that the function ( ) belongs to \( L^1 \). The roots of the polynomial \( P (z) = z^2 (z^4 + c z^2 + 1) \) that are worthy of notice are \( z_1 = 0 \) for arbitrary values of the parameters \( c \) and \( b \) if \( c = 2 \); \( b \neq 8 \) and \( z_{1,5} = i \) for \( c = 2 \) and \( b \neq 8 \). It is easy to show that the solutions in the vicinity of the point \( z_1 = 0 \) behaves like ( ) if \( b = 4 \) and behaves analytically if \( b = 2 \). Hence \( z_1 = 0 \) is the branch point for our solutions if \( b \neq n^2 ; n \in \mathbb{N} \). It turns out that there are no other branch points in the interior of the unit disc and thus we conclude that the admissible values for \( b \) are \( n^2 \) with natural numbers \( n \). Considering the points \( z_{2,3} = 1 \) for \( c = 2 \); \( b \neq 8 \) and \( z_{4,5} = i \) for \( c = 2 \) and \( b \neq 8 \) it is easy to verify that the solutions of (45) behaves like ( ) if \( b = n^2 \).

If \( w = z_{2,3} \) or \( w = z_{4,5} \) respectively and \( \frac{p}{b} \in N \) we conclude that for an arbitrary \( b = n^2 \); \( n \in \mathbb{N} \) there exists a solution ( ) \( 2 L^1 \). Note that in virtue of the relation
\[ b = \frac{2}{E_2 h^2} = n^2 \]

the coupling constant \( c \) turns out to be negative since \( E_2 \), being the eigenvalue of the elliptic operator \( H \) is necessarily positive. Thus, we get the following spectrum for the Hamiltonians [33]
\[ E_1 > 0 \quad E_2 = \frac{2}{h^2 n^2} ; \quad n \in \mathbb{N} \]

The discrete spectrum of the Hamiltonian \( H \) is not so surprising because if we consider the exponential degeneration of our system (46) and quantize it in the representation of
the wave functions $\psi = (u^2; v_1)$ we note that the momentum $v_1$ defining the coupling constant of the two-particle Calogero-Moser system is quantized and thus the respective quantum Hamiltonian $H_2$ also has a discrete spectrum.

5 Concluding Remarks

The main lesson we have learned from the above considerations is that we may not restrict ourselves to the Lie group symmetries in constructing the integrable system via the symplectic reduction method. Of course, if we go far beyond the Lie group symmetries, the explicit description of the gauge orbits cannot be guaranteed and we cannot be sure that we are able to resolve the equations of motion via the projection method. Nevertheless, we suspect that the non-Lie group symmetries may be useful for constructing the mechanisms of the symplectic reduction for well-known integrable systems. For example, although the mechanism of the symplectic reduction for the Calogero-Moser system associated to the root systems of the simple Lie algebras are known, the projection method for these systems has not been developed. We suppose that the projection method for the systems can be presented if we construct the mechanisms of the symplectic reduction in the context of the non-Lie group symmetries. It should be also very interesting to perform analogous symplectic reduction starting with the Odesskii bracket, which generalizes the Sklyanin bracket to the case of the arbitrary elliptic R-matrix. We suppose that the projection method for the respective systems should be related to solving the equations of motion for some higher-dimensional tops as is the projection method for our system related to the equations of motion for the ordinary $so_3$ top. As it has been noted in the introduction, the problem of finding the Lax representation for our system seems to be rather intriguing. Here we mention that this problem raises the question of whether it is possible to get the quadratic Sklyanin bracket via the reduction from the linear Poisson bracket. As far as we know, the latter question is not answered yet even for the quadratic Poisson brackets, related to the constant triangular R-matrix. Finally, it is worth mentioning that in our paper we quantize the system starting with the reduced phase space of the model. However, it would be very interesting to obtain the same quantum mechanics via the respective quantum reduction, which may be performed with the help of the BRST approach. We suspect that this quantum reduction should be somehow related to the representations of the quantum Sklyanin algebra as are the quantum Calogero-Moser systems related to the representation of the respective Lie algebras.

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