Robust Unconditionally Secure Quantum Key Distribution with Two Nonorthogonal and Uninformative States

Marco Lucamarini¹, Giovanni Di Giuseppe², and Kiyoshi Tamaki³,⁴

¹CNISM UdR University of Camerino, Via Madonna delle Carceri 9, 62032 Camerino (MC), Italy.
²Physics Department, University of Camerino, Via Madonna delle Carceri 9, 62032 Camerino (MC), Italy.
³NTT Basic Research Laboratories, NTT Corporation 3-1, Kanagawa, 243-0198, Japan
⁴CREST, JST Agency, 4-1-8 Honcho, Kawaguchi, Saitama, 332-0012, Japan.

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We introduce a novel form of decoy-state technique to make the single-photon Bennett 1992 protocol robust against losses and noise of a communication channel. Two uninformative states are prepared by the transmitter in order to prevent the unambiguous state discrimination attack and improve the phase-error rate estimation. The presented method does not require strong reference pulses, additional electronics or extra detectors for its implementation.

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I. INTRODUCTION

Quantum Key Distribution (QKD) is a way to distribute secret keys between two distant parties with provable security. Despite the general principles of QKD are now well known, there is no definite answer yet about its effective use in the real world, being it dependent on practical figures of merit like the transmission rate, the working distance and even the production cost of a specific implementation. A useful criterion in such intricate situations is that of the simplest choice. In case of cryptography this is particularly desirable for it allows general and reliable security proofs together with efficient and low-cost practical realizations.

The simplest QKD protocol was conceived by C. H. Bennett in 1992 and named after him “B92” [5]. It is based on only two nonorthogonal states associated with the two values of the logical bit to-be-secretly-transmitted. Despite its simplicity, the B92 is considered as a quite impractical protocol, mainly for its low tolerance to the losses and noise of a communication channel [6]. The high dependence on channel losses can be ascribed to the so-called Unambiguous State Discrimination (USD) attack [7], which represents the principal threat against the B92, and severely limits its performances. On the other side, the high dependence on the channel noise can be imputed to the lack of a direct phase-error estimation, which entails a more than necessary privacy amplification as soon as the noise sensibly increases.

So far, no solution has been devised to improve the phase-error estimation of B92, and only one solution is available to contrast the USD attack. It was originally proposed in [8] and recently proved unconditional secure in [9] and [10]. It consists of a strong reference pulse accompanying the signal pulse and phase-related to it. The presence of the reference pulse prevents an eavesdropper (Eve) from selectively stopping the signals according to the results of a USD measurement. By adopting the technology reported in [10], this solution would allow a secure QKD over distances of about 124 Km [3]. However the high intensity of the reference pulse is expected to cause in practice noise due to the scattering inside an optical fiber [11], while if a weaker reference pulse is used [12], the maximum distance drops to about 87 Km. Furthermore the solution of a strong reference pulse cannot be applied if the B92 is realized with a single-photon source or with SPDC (Spontaneous Parametric Down Conversion), because the beam intensities cannot be modulated to the necessary extent. On the contrary, a solution functioning at the single-photon level could be at once transferred to a realistic scenario by virtue of the decoy-state technique [13, 14, 15], which allows the estimation of statistical quantities related to the single-photon pulses only.

In this paper we propose a novel solution to make the single-photon B92 robust against the losses and the noise of a communication channel. First, the transmitter prepares, besides the two conventional signal states, two additional uninformative states. This modification alone is very feasible and allows to detect the USD attack. The protocol containing this first modification will be called B92 henceforth. In contrast to the state-of-the-art, the gain of the single-photon B92 depends only linearly on the loss rate of a communication channel rather than nearly quadratically. This result is anticipated in Fig. 1.

As a second step, the receiver’s box is slightly modified so to directly measure the phase-error of the B92, whose estimation is known to be quite poor at large angles between the two signal states. The protocol containing both the first and the second modifications will be called B92. This further variant provides a higher tolerance to the channel noise, approaching, when the two signal states are nearly orthogonal, the one featured by the BB84 [16].

The main achievement of the paper is the formulation of the B92. It relies on the observation that the states of B92, being only two, are always linearly independent. Since the USD attack is effective only on sets of linearly independent states [16], it is quite natural to introduce one more state to obtain a set of linearly dependent states. This removes at the root the possibility of an USD attack. Actually the states added in the new
B92 for scales linearly with $L$ to intuitively explaining the idea of using uninformative brief review the security proof of the standard B92, protocol as well; so they will be given without repeating the conclusion of the paper. It should be noted however that no additional hardware respect to a standard B92.

FIG. 1: (color online) Plots of the secure gain $G$ as a function of the channel loss rate $L$, for the standard B92 (black circles) and for the B92 (red circles). The B92 scales linearly with $L$, while the B92 scales nearly quadratically, and ceases to provide a positive gain at about $L = 0.6$. The curves are drawn assuming a depolarizing channel with $p = 0.01$ (see the text). On the top-right, the states of traditional B92 (black arrows) and the extra uninformative states (red arrows) used in the two modifications of the B92 (see the text).

protocol are two rather than one; this depends on technical reasons related to the unconditional security proof which will become clear later. The added states are uninformative and after the quantum transmission are discarded. Even so the users can measure the loss-rate pertaining to these states and obtain a signature of Eve’s presence. This solution resembles the decoy-state technique [13, 14, 15] used to realize a long-distance BB84 in a realistic scenario; so we will call the extra uninformative states “decoy states” henceforth. However it should be noted that the conventional decoy-state comes from the intensity modulation of a certain pulse, i.e. from the modulation of a degree of freedom different from the one in which information is encoded; on the contrary, in the present case, the decoy-state is encoded in the same degree of freedom of the signal states, and there is no modulation in other degrees of freedom different from that. This is a relevant peculiarity of the B92 since it entails that no additional hardware respect to a standard B92 setup is required to implement the new protocol.

In the next, we focus on the B92, leaving the B92 for the conclusion of the paper. It should be noted however that all the results obtained for the former protocol, in particular its unconditional security, hold for the latter protocol as well; so they will be given without repeating unnecessary security proofs.

Our work is structured as follows: in Section I we briefly review the security proof of the standard B92, to intuitively explaining the idea of using uninformative states to protect against the USD attack. Then, in Section II we introduce the B92 and provide the proof of its unconditional security. In Section III we show, with the help of numerical simulations, the independence of the new protocol from the losses of a communication channel. In Section IV we detail the B92, in which the users perform a direct estimate of the phase-error rate. The concluding remarks are given in Section V.

II. PRELIMINARY CONSIDERATIONS

Let us introduce the notation to explain the B92 protocol. We write the bit encoded by the transmitter (Alice) as $j = \{0, 1\}$ and the corresponding qubit as $|\psi_j\rangle \equiv \beta |0_x\rangle + (-1)^j \alpha |1_x\rangle$, where $\{ |0_x\rangle, |1_x\rangle \}$ are the eigenstates of the $X$ basis, $\beta = \cos \frac{\theta}{2}$, $\alpha = \sin \frac{\theta}{2}$, $0 < \theta < \pi/2$ (see Fig. I). The bases $X$ and $Z$ are related by $|j_x\rangle = [|0_x\rangle + (-1)^j |1_x\rangle] / \sqrt{2}$. We can also introduce the state $|\varphi_\gamma\rangle = \alpha |0_x\rangle - (-1)^j \beta |1_x\rangle$ orthogonal to $|\psi_j\rangle$. It is possible now to define a general USD measurement, parametrized by $\gamma$, as $M_j^{B_92} \equiv \{ F_0^\gamma, F_1^\gamma, F_{inc}^\gamma, F_v \}$, with $F_0^\gamma = \left( \frac{3}{4 \gamma} |\varphi_\gamma\rangle \langle \varphi_\gamma| \right)$, $F_1^\gamma = \left( \frac{1}{4 \gamma} |\varphi_{inc}\rangle \langle \varphi_{inc}| \right)$, $F_{inc}^\gamma = 1 - F_0^\gamma - F_1^\gamma$, $F_v = |v\rangle \langle v|$ is a state that includes both vacuum and multi-photon pulses, whose occurrences depend on $L$, the total loss-rate of the communication channel, which is under Eve’s control. It is maybe useful to point out that the operators $F_0^\gamma$, $F_1^\gamma$ and $F_{inc}^\gamma$ live in the subspace characterized by the single-photon projector $\Pi_s$, while $F_v$ lives in the complementary subspace $1 - \Pi_s$. Note that we are assuming for simplicity a perfect single-photon source for Alice and ideal detectors for Bob. More precisely Bob’s detectors can discriminate vacuum, single-photon and multi-photon pulses, and they are modeled like a beam-splitter with transmission $\eta_B$, controlled by Eve, followed by detectors with unit efficiency. The control by Eve on detectors is taken into account by including $\eta_B$ in the loss-rate $L$. We remark that the assumption of a perfect single-photon source can be dropped if the standard decoy-state technique [13, 14, 15] is brought into the description. Also, we define for later use $F_{conc}^\gamma = (F_0^\gamma + F_1^\gamma) = (A_{\gamma}^s)^2$, and we include in the term “vacuum” both the vacuum and the multi-photon events. Finally, we observe that the measurement $M_j^{B_92}$ is optimal when $\gamma = 1$, while it is practical when $\gamma = \beta$ [15]. While the receiver (Bob), due to his limited technology, can execute just the practical measurement $M_j^{B_92}$, Eve is supposed to be endowed with superior technology, so she can execute the optimal $M_j^{B_92}$, with even the additional condition $L = 0$. In fact we conservatively assume that Eve completely controls the channel, so all the losses and noises are caused by her.

The unconditional security of a lossless B92 was proved for the first time in [20] and that of a lossy B92 in [21]. It is shown there that the Prepare-and-Measure (PM) B92, experimentally accessible to Alice and Bob, can be
obtained through a reduction argument by another protocol based on Entanglement Distillation (ED) \[22\]. The ED protocol is not really implemented by the users; it is rather a theoretical tool to find the conditions under which the B92 operations can lead to the distillation of the maximally entangled state

$$|\Phi^{+}_{AB}\rangle = \frac{1}{\sqrt{2}} (|0_{x}\rangle_{A} |0_{x}\rangle_{B} + |1_{x}\rangle_{A} |1_{x}\rangle_{B}), \quad \text{(1)}$$

starting from the state

$$|\Psi^{+}_{AB}\rangle = \frac{1}{\sqrt{2}} (|0_{x}\rangle_{A} |\varphi_{0}\rangle_{B} + |1_{x}\rangle_{A} |\varphi_{1}\rangle_{B}) = \beta |0_{x}\rangle_{A} |0_{x}\rangle_{B} + \alpha |1_{x}\rangle_{A} |1_{x}\rangle_{B}, \quad \text{(2)}$$

which explicitly contains the two signal qubits.

The reduction from the ED to the PM protocol is obtained by noting that the measurement $M^{B92}_{\varphi}$ effected by Bob in the PM protocol is equivalent in the ED protocol to a $Z$-basis measurement conditional on the successful filtering operations $Q_{x} = \{F_{x}, 1 - F_{x}\}$ and $A_{\varphi}^{\text{B92}}$. In particular, when the filters are successful, the Bell state \[11\] is obtained with probability $2\alpha^{2}\beta^{2}$ by the state \[2\], if the channel is lossless and noiseless \[23\]. Once the users share the state \[11\] they can obtain a secure key by the subsequent measurement in the $Z$ basis.

Nonetheless, in real situations, the channel is not lossless and noiseless, so the filtered states may include $n_{\text{bit}}$ bit errors, represented by the states $\{ |0_{x}\rangle_{A} |1_{x}\rangle_{B}, |1_{x}\rangle_{A} |0_{x}\rangle_{B} \}$, and $n_{\text{ph}}$ phase errors, represented by the states $\{ |0_{x}\rangle_{A} |1_{x}\rangle_{B}, |1_{x}\rangle_{A} |0_{x}\rangle_{B} \}$. Alice and Bob must then resort to an ED based on CSS codes \[24, 25\] in order to correct the errors and distill the Bell state \[11\]. This procedure needs not to be actually accomplished by the users; it suffices that they provide reliable upper bounds to the number of bit errors and phase errors present in their data. This is the main breakthrough of the security proof given in \[14\].

Bit errors can be directly estimated by sacrificing a part of the data, which are publicly revealed on the classical channel. On the contrary phase errors can not be directly estimated in B92. This is due to the fact the none of the operations of this protocol can be made equivalent to a measurement in the $X$ basis. To cope with this problem, it is useful to devise a gedanken experiment in which Alice and Bob perform a measurement of the state \[24\] in the $X$ basis. In this way the gedanken outcomes can be put in relation with measurable quantities of the protocol. Let us indicate with $|1_{i}\rangle_{A} |j_{i}\rangle_{B}$ ($i = \{0, 1\}, j = \{0, 1, v\}$) the gedanken outcomes, and with $n_{ij}$ the number of their occurrences. Then, by looking at the second of Eqs. \[24\], and by considering Bob’s measurement $M^{B92}_{\varphi}$, we can easily obtain the following relations, valid in the asymptotic limit of large $N$ \[21, 22\]:

$$n_{\text{ph}} = \beta^{2}n_{01} + \alpha^{2}n_{10}, \quad \text{(3)}$$

$$\alpha^{2}N = n_{10} + n_{11} + n_{1v}, \quad \text{(4)}$$

Eq. \[3\] quantifies, in terms of gedanken quantities, the number of phase errors; it contains $n_{01}$ and $n_{10}$ which can be put in relation to measurable quantities through an argument based on quantum theory \[20\]. Eq. \[4\] is a direct consequence of the fact that neither Eve nor the channel can access Alice’s qubit as long as it is in Alice’s hands; it contains the crucial parameter $n_{1v}$, which quantifies the effective nonorthogonality of the signal states in presence of losses. In fact, consider the case of a lossless channel, i.e. $n_{1v} = 0$. From Eq. \[4\] we can see that the quantity $\Delta = n_{10} + n_{11}$ measures the degree of nonorthogonality of the two signal states, being it equal to $\alpha^{2}N$, and being $1 - 2\alpha^{2}$ equal to the scalar product of the two signal states. When losses are taken into account, $\Delta$ still represents the nonorthogonality of the two states on a loss-free channel, but this time it is equal to $\alpha^{2}N - n_{1v}$, smaller than before. This means that losses have increased the orthogonality of the two states, making the B92 more prone to USD and other loss-based attacks. So it is important to include $n_{1v}$ in the phase estimation process, for its optimization is directly related to the amount of information leaked to Eve through a loss-based mechanism. By executing the numerical search of the phase error upper bound, it turns out that the optimal value for $n_{1v}$ is zero in most of the cases. Below, we show that it suffices to consider the USD attack to intuitively explain this value.

The USD attack is performed by Eve via the optimal measurement $M^{B92}_{\varphi}$, executed on the qubits prepared by Alice. Let us consider the signal states $\{|\varphi_{0}\rangle, |\varphi_{1}\rangle\}$ and the $X$-basis states $\{|0_{x}\rangle, |1_{x}\rangle\}$. A conclusive outcome is obtained by Eve for these four states with the following probabilities:

$$P^{USD}_{\varphi_{0}} = P^{USD}_{\varphi_{1}} = 1 - \cos \theta, \quad \text{(5)}$$

$$P^{USD}_{x_{0}} = 0, \quad P^{USD}_{x_{1}} = 1. \quad \text{(6)}$$

If the result is conclusive, Eve forwards the qubit (in the correct state) to Bob; otherwise she stops the qubit and creates a loss in the channel. So, while in case of no attack the four states are expected to arrive at Bob’s detectors with the same probability, in case of USD attack, the users would detect $N_{s}(1 - \cos \theta)$ signal states, $N_{d}$ states $|1_{x}\rangle$ and 0 states $|0_{x}\rangle$, where $N_{s}$ and $N_{d}$ are respectively the total number of signal and decoy states prepared by Alice. Note that this implies in case of USD attack the following setting:

$$n_{1v} = 0. \quad \text{(7)}$$

In standard B92 the users can not measure the loss-rates of Eq. \[6\], so they conservatively assume that Eve executed an USD attack even if she actually did not, whence Eq. \[7\]. But the loss-rates of Eq. \[6\] become measurable in B92, because of the presence of two uninformative states which are chosen on purpose equal to $|0_{x}\rangle$ and $|1_{x}\rangle$. This is the main feature that makes the new protocol independent of losses.
III. PROTOCOL AND UNCONDITIONAL SECURITY

In the PM B92, Alice prepares the two signal states \( |\phi_0\rangle, |\phi_1\rangle \) plus two additional uninformative decoy states \( |\phi_d\rangle, |\phi_d'\rangle \), which are chosen respectively equal to the states \( |1_x\rangle, |0_x\rangle \). Here we consider two decoy states only, but the use of three or more decoy states could be useful to adverse other kinds of USD-based attacks, like those described in [24, 25]. Notice that the addition of two decoy states makes the overall states prepared by Alice linearly dependent; hence is impossible for Eve to unambiguously discriminate them. More importantly, the presence of two decoy states, prepared with suitable probabilities, makes the signal and decoy density matrices identical. This is fundamental for the unconditional security proof of the protocol, because it prevents Eve from behaving differently with signals and decoys, and legitimates the use of random sampling arguments between the two classes of pulses prepared by Alice. Let us write explicitly the density matrix of the signal states:

\[
\rho = (|\phi_0\rangle\langle\phi_0| + |\phi_1\rangle\langle\phi_1|)/2 = \beta^2(0_x\langle 0_x | + \alpha^2)1_x\langle 1_x|.
\]

The above equation explains why we chose the decoy states equal to the X-basis states \( |1_x\rangle, |0_x\rangle \). Moreover, the requirement of equal density matrices for signal and decoy states fixes the preparation probability of the states \( |1_x\rangle \) (i.e. \( |\phi_d\rangle \)) and \( |0_x\rangle \) (i.e. \( |\phi_d'\rangle \)) respectively to \( \alpha^2 \) and \( \beta^2 \). After the whole quantum transmission the decoy instances are discarded. Even so, they allow the direct estimate of the crucial quantity \( n_{1v} \) previously introduced.

For the security proof of B92 we adopt the same argument as in [21]; we introduce below the PM B92 and show that it can be obtained by an ED protocol through a reduction argument. Then we show the unconditional security of the ED protocol by applying it to a Shor-Preskill security proof [4].

**PM B92** – (i) Alice randomly and uniformly prepares 2N signal qubits in the state \( |\phi_0\rangle \) or \( |\phi_1\rangle \), \( \alpha^2N \) decoy qubits in the state \( |\phi_d\rangle \), and \( \beta^2N \) decoy qubits in the state \( |\phi_d'\rangle \). (ii) Bob executes the measurement \( M_{B92}^d \) in case he obtains \( |v\rangle \) he labels the outcome as ‘vacuum’; in case he obtains \( |\phi_0\rangle \) or \( |\phi_1\rangle \) he labels the outcomes as ‘inconclusive’; in case he obtains \( |\phi_d\rangle \) (\( |\phi_d'\rangle \)) he labels the outcome as ‘conclusive’ and decides as ‘1’ (‘0’). – (iii) After the quantum transmission, the users randomly permute their bits. Then Bob tells Alice the positions of vacuum, inconclusive and conclusive counts. Alice calculates the number \( n_{1v} \) of joint occurrences \( \{ |\phi_d\rangle, |v\rangle \} \) \( (k = \{ 0, 1, d, d' \}) \) between her preparation and Bob’s measurement, the number \( n_{v} = \sum_k n_{kv} \) of total vacuum counts, and the number \( n_{conc} \) of conclusive counts. She announces the estimated quantities to Bob together with the positions of decoy bits. – (iv) The data corresponding to decoy or inconclusive outcomes are removed by the users. – (v) The first half of the remaining bits (check bits) are used for estimating the number of errors \( n_{err} \), in which Alice prepared \( |\phi_0\rangle \) \( (|\phi_1\rangle) \) and Bob decoded as ‘1’ (‘0’). – (vi) From \( n_v, n_{err}, n_{conc}, \) and \( n_{dv} \) the users estimate the number of bit errors \( n_{bit} \), and an upper bound on the number of phase errors \( n_{ph} \) in the second half of the remaining bits (data bits). – (vii) The users perform error correction and privacy amplification on the data bits according to the values of \( n_{bit} \) and \( n_{ph} \) respectively, thus obtaining an \( n_{key} \)-bit shared secret key.

In the following we provide the ED protocol that we show to be unconditionally secure, and that will eventually reduce to the PM B92.

ED B92 – (i) Alice prepares 3N copies of the bipartite state given in Eq. [2] and sends the 3N systems \( B \) to Bob over the quantum channel. – (ii) Alice and Bob randomly permute by public discussion the positions of all their pairs. – (iii) Bob performs the QND measurement described by \( Q_v = \{ F_v, 1 - F_v \} \), and publicly announces the outcomes [29]; let \( n_v \) be the number of outcomes associated with \( F_v \). – (iv) For the first N pairs (decoy pairs), Alice measures system \( A \) in the Z basis, and publicly announces the positions of the decoy pairs. Since neither Eve, Bob nor the channel can touch Alice’s qubits, we can infer from Eq. [2] that she will obtain \( |0_x\rangle \) with probability \( \beta^2 \) and \( |1_x\rangle \) with probability \( \alpha^2 \). Alice counts the number \( n_{1v} \) of joint occurrences \( \{ |1_x\rangle, F_v \} \) in the outcomes. Then the users discard these results. – (v) For half of the remaining pairs (check pairs), Alice measures system \( A \) in the Z basis, and Bob performs the measurement \( M_{B92}^d \) on his system. By public discussion, they determine the number \( n_{err} \) of errors in which Alice found ‘0’ (‘1’) and Bob’s outcome was ‘1’ (‘0’). – (vi) For the other half of the remaining pairs (data pairs), Bob performs the filtering \( A_{1v}^d \) on each of his qubits, and announces the positions and the total number \( n_{fal} \) of the qubits that have passed the filtering. – (vii) From \( n_v, n_{err}, n_{fal}, n_{1v} \) the users estimate an upper bound for the number of bit errors \( n_{bit} \) and phase errors \( n_{ph} \), in the \( n_{fal} \) pairs. – (viii) They run an ED protocol that can produce \( n_{key} \) nearly perfect EPR pairs if the estimation is correct. – (ix) Alice and Bob each measures the EPR pairs in Z basis to obtain an \( n_{key} \)-bit shared secret key.

The unconditional security of our protocol follows the proof given in [21] after minor modifications. Actually, all the operations and measurements in our entanglement-based protocol, with exception of Step (iv), are purposely chosen equal to [21] to maximally exploit the results obtained there.

The first step is to show the equivalence of the two protocols given above. For that, it suffices to note that Eve can not distinguish the preparation of the decoy states \( \{ |\phi_d\rangle, |\phi_d'\rangle \} \) in the PM protocol from that of \( \{ |1_x\rangle, |0_x\rangle \} \) in the ED protocol, effected through a X basis measurement, since the resulting states are the same, they
are prepared with the same probabilities and the time at which Alice performs the $X$ basis measurement can not have an influence on the results. This also implies that the quantity $n_{\text{PM}}$ of the PM protocol corresponds to the quantity $n_{1v}$ of the ED protocol. For the same reason Eve cannot distinguish the preparation of the signal states in the PM protocol from that in the ED one. Furthermore, from the definitions of $F^{3\text{conc}}$ and $A^{3\text{fil}}_i$, it can be easily seen that the sequence of filters $\{(1-F_i), A^{3\text{fil}}_i\}$ is equivalent to the operator $F^{3\text{conc}}$ measured with a perfect detector. This also implies that the quantity $n_{\text{conc}}$ of the PM protocol corresponds to the quantity $n_{fil}$ of the ED protocol.

Regarding the security of the ED $\text{B92}$, the only point that deserves some care is the estimation of the quantity $n_{1v}$, by means of the decoy states. In particular we must show that this estimation is exponentially reliable. For that we take inspiration from the estimation of the bit-error rate, which follows closely the standard $\text{B92}$ [21,2]. The number $n_{\text{bit}}$ of bit errors in the data pairs can be deduced from the number $n_{\text{err}}$ of errors in the check pairs obtained in the above ED-$\text{B92}$, Step (v). The argument is that in order to obtain $n_{\text{bit}}$, Alice and Bob should perform $Z$ basis measurements on the data pairs. Despite these measurements are not really performed on data pairs, they are performed on the check pairs. This is because the measurement $\mathcal{M}^{3\text{fil}}_{\text{B92}}$ is equivalent to a $Z$ measurement conditional on the outcome $(1-F_i)$ of $\mathcal{Q}_e$ and on the successful filtering $A^{3\text{fil}}_i$. Since in Step (ii) all the pairs are randomly permuted, the check pairs can be seen as a classical random sample of all the pairs remained after Step (iv). This leads to the inequality $|n_{\text{bit}} - n_{\text{err}}| \leq N\varepsilon$ which is exponentially reliable for large $N$.

We can apply the same argument to the decoy pairs. In order to obtain $n_{1v}$, Alice and Bob should perform $X$ basis and QND measurements on their data pairs. Although these measurements are not really performed on data pairs, they are performed on decoy pairs in Step (iv) of the ED protocol. Then, because of the random permutation of Step (ii), the decoy pairs can be regarded as a classical random sample of the $3N$ pairs prepared by Alice. Hence we obtain that the estimation of $n_{1v}$ in our modified $\text{B92}$ is exponentially reliable and can be used in the numerical optimization of the phase-error upper bound. In the next Section we will show the practical advantages of such a direct estimation.

### IV. NUMERICAL SIMULATIONS

To see the practical advantages of the $\text{B92}$ consider a channel with total loss rate $L$. If there is no eavesdropping in the line it is natural to expect (see Eq. (11)) that:

$$n_{1v} = \alpha^2 NL.$$  

Eq. (9) can be experimentally verified in the $\text{B92}$; this represents the main advantage of the new protocol. On the contrary, as already mentioned, Alice and Bob can by no means verify Eq. (9) in the standard $\text{B92}$, so they must choose $n_{1v} = 0$, according to the worst-case scenario described by Eq. (7).

The settings of Eqs. (7),(9) lead to the two curves of Fig.1, which represent the gain $G$ as a function of the loss rate $L$ for the traditional $\text{B92}$ and for the $\text{B92}$. The gain is given by $G = n_{fil} [1 - h(n_{\text{bit}}/n_{fil}) - h(\eta_{\text{ph}}/n_{fil})]$, where $\eta_{\text{ph}}$ is the phase-error upper-bound, and $h$ is the Shannon entropy [30]. The curves are drawn assuming, as in [21], a depolarizing channel with losses $\rho \rightarrow L(1-p)|V\rangle\langle V| + (1-L)|0\rangle\langle 0| + (1-p)\rho + \sum_{i=x,y,z} \sigma_i \rho \sigma_i \rangle$, where $|V\rangle$ is the vacuum state, $\sigma_i$ are the Pauli matrices and $p$, taken equal to 0.01 in our simulations, is the depolarizing rate. The plot pertaining to the $\text{B92}$ is related to Eq. (8); on the contrary that pertaining to the standard $\text{B92}$ comes from Eq. (7), and contains the same results given in [21] for a $\text{B92}$ on a lossy channel. From the two plots is apparent that our technique leads to a linear decrease of the gain with the loss rate of the channel, rather than that, nearly quadratic, of the standard $\text{B92}$. Only for $L = 0$ the standard $\text{B92}$ features a higher gain. This is due to the presence of decoy states that go discarded in the $\text{B92}$.

For the lossy depolarizing channel given above, the bit-error rate can be easily found to be equal to $(1-L)p/3$, while that of filtered states is $(1-L)(4p + 3 + (4p - 3)\cos(2\theta))/12$; hence their ratio, i.e., the relative bit-error rate, is independent of losses. In the $\text{B92}$ even the upper-bounded relative phase-error, $\Lambda_{\text{ph}} = \eta_{\text{ph}}/n_{fil}$, is independent of losses. This can be realized by observing the top diagram of Fig. 2 where we plotted $\Lambda_{\text{ph}}$ ver-
sus the squared scalar product of the two signal states, $|\langle \phi_0 | \phi_1 \rangle|^2$, for several values of the total transmittance $\eta = 1 - L$. The points pertaining to different values of the transmittance follow all the same curve, thus demonstrating the independence of $\Phi_{ph}$ from losses. From the bottom diagram of Fig. 2 containing the gain as a function of $|\langle \phi_0 | \phi_1 \rangle|^2$, we can learn that, regardless of the transmittance, the maximum gain is obtained when the two signal states are separated by an angle $\theta \simeq 55.4^\circ$, while the minimum phase-error upper-bound is obtained when $\theta \simeq 42.8^\circ$. This is in sharp contrast with the standard theory of B92 [21, 12], which assigns very small values to $\theta$ to prevent the USD attack, thus reducing substantially the final rate.

Since the relative bit-error and phase-error rates are loss-independent, we can use the results of the lossless B92 [20] to give an estimate of the single-photon B92 working distance when practical devices are taken into account. It is found in [21] that the maximum depolarizing rate $p^*$ for which the gain of the lossless single-photon B92 is still positive is $p^* = 0.033$. In real apparatuses, the depolarizing rate is given essentially by detectors dark counts, which become dominant when the quantum signal becomes too low. In [11] the dark count probability is $p_{dark} = 1.7 \times 10^{-6}$, the attenuation of the fiber is $\xi = 0.21$ dB/Km, and the mean detectors efficiency is 0.045 and the single-photon detection probability is $p_x(l) = 0.045 \times 10^{-l/10}$, with $l$ the distance between the users. From the inequality $p = p_{dark}/p_x(l) \leq p^*$ we can easily obtain a working distance equal to about 140 Km. This value can be compared with the ones achieved by other protocols under similar circumstances. For example it is known that the maximum depolarizing rate of BB84 is $p^* = 0.165$ [4], which implies a working distance of about 173 Km, while that of SARG04 [31] is $p^* = 0.080$ [32] with a corresponding distance of 158 Km. The difference between the protocols depends crucially on their tolerance of the channel noise, as exemplified by the above values of the depolarizing rate. One way to make a protocol more tolerant to noise is to improve its phase-error rate estimation. We accomplish this task in the next section by slightly modifying the measuring apparatus of the receiver Bob.

V. IMPROVED PHASE-ERROR ESTIMATE

With reference to the above-described ED protocol, in order to directly measure $n_{ph}$, Alice and Bob should perform $X$ basis measurements on their data pairs, and publicly compare their results on the classical channel. However in this way their data pairs could be no more used to distill a secret key, so this procedure is usually substituted by the estimation of the phase-error on a subsampling of the data pairs, the check pairs, which are representative of the whole sample. This is what happens in the BB84 for example [3, 17]. Nevertheless such an estimation procedure cannot be done in the standard B92 and neither in our modified version B92. The reason is that although Alice prepares with a certain probability the states in the $X$ basis, i.e. the decoy states, Bob never measures them in the $X$ basis. By consequence, even in B92 the phase-error can not be directly estimated and must be indirectly upper bounded from the values of other quantities like $n_c$, $n_{err}$, $n_{fid}$ and $n_{1e}$, through a numerical optimization algorithm.

As already mentioned, it is possible to further modify the B92 to introduce such a direct estimation of the phase-error rate, and make it more resistant to noise. The resulting protocol is the B92. It consists in a random switch of Bob’s measurement between the $M_{B92}^0$ and a measurement in the $X$ basis. From the instances related to the $X$ basis measurement the users can obtain a direct estimation of the phase-error of the channel, as it happens in the BB84. So, in particular, this is achieved by modifying the following Steps of the given PM protocol:

(ii') Bob executes the measurement $M_{B92}^0$ with probability 1/2 and the measurement in the $X$ basis with probability 1/2 and takes note of outcomes; $[-]$. 

(iii') After the quantum transmission, Bob tells Alice the positions of vacuum, conclusive and inconclusive counts and those of his measurements in the $X$ basis; $[-]$. She announces the estimated quantities to Bob together with the positions and the values of decoy bits. All the instances in which Alice prepared signal states and Bob measured in the $X$ basis are discarded.

(iv') Bob estimates the number of errors $n_{ph}$ in the decoy instances in which Alice prepared $|1_x\rangle$ ($|0_x\rangle$) and Bob detected $|0_x\rangle$ ($|1_x\rangle$); $[-]$.

(vi') From $n_{err}$ the users estimate the number of bit errors $n_{bit}$.

The slight increase in the complexity of Bob’s measurement, and the decrease in the final rate entailed by the new Step (iii'), are compensated by the benefits of a better tolerance to the channel noise. By assuming again a depolarizing channel, it can be easily seen that the tolerable depolarizing rate $p^*$ depends directly on the angle $\theta$ between the two signal states: the greater $\theta$ the larger $p^*$. This is summarized in Table I. It can be seen that the tolerable depolarizing rate can be increased up to the BB84 level, and well above the SARG04 threshold. However we note incidentally that a similar solution for a direct phase-error estimation can be applied to the SARG04 protocol by modifying only its classical data processing. Let us also notice that the unconditional security of the B92 follows closely that of the B92; in fact

| $\theta$ | $10^\circ$ | $20^\circ$ | $30^\circ$ | $40^\circ$ | $50^\circ$ | $60^\circ$ | $70^\circ$ | $80^\circ$ | $90^\circ$ |
|--------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $p^*_{045}$ | 0.4     | 1.5     | 3.4     | 5.9     | 8.6     | 11.5    | 14.0    | 15.8    | 16.5    |

TABLE I: The maximum depolarizing rate tolerable by the B92, and the corresponding angle $\theta$ between the two signal states. The depolarizing rates for BB84 [17] and SARG04 [31] are respectively 16.5% [4] and 8.04% [32].
the decoy instances are used as a classical sampling to provide an exponentially reliable bound to the phase errors.

VI. CONCLUSION

In conclusion we have shown that a pair of uninformative states can be introduced in the single-photon B92 protocol in order to remove its high dependence on the losses and noise of a communication channel. In particular, without modifying the receiver’s box, the technique can prevent at the root an USD attack by Eve. Furthermore, with a slight modification of Bob’s measurement, the technique allows a direct estimation of the number of phase errors, thus increasing the robustness of the protocol against external sources of noise. The results are of theoretical interest since they solve at the root the long-standing problem of the unambiguous state discrimination of the two single-photon B92 signal states. In fact, other protocols based on similar principles could benefit of our analysis [33, 34]. Furthermore, although the results are limited to the single-photon case, they are easily exportable to a realistic scenario by applying the well established decoy-state technique [13, 14, 15]. The presented method can be extended to more than two uninformative states, with the potential of diverting other USD-based attacks [22, 27, 28], possibly related to a non-ideal equipment of the users. Another option is to adapt the proposed solution to the B92 with a not-so-strong reference pulse [12]; this would allow to use in that protocol a wider angle between the signal states, thus reducing the problem of a precise phase stabilization.

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