A new characterization of simple $K_3$-groups using same-order type

Igor Lima and Josyane Pereira

ABSTRACT. Let $G$ be a group, define an equivalence relation $\sim$ as below:

$$\forall g, h \in G, g \sim h \iff |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of $G$ and denoted by $\alpha(G)$. And $G$ is said a $\alpha_n$-group if $|\alpha(G)| = n$. Let $\pi(G)$ be the set of prime divisors of the order of $G$. A simple group of $G$ is called a simple $K_3$-group if $|\pi(G)| = 3$. We give a new characterization of simple $K_3$-groups using same-order type. Indeed we prove that a nonabelian simple group $G$ has same-order type $\{r, m, n, k\}$ if and only if $G \cong PSL(2, q)$, with $q = 7, 8$ or $9$. This result generalizes the main results in [4, 6] and [8]. Moreover based on the main result in [8] we have the natural question: Let $S$ be a nonabelian simple $\alpha_n$-group and $G$ a $\alpha_n$-group such that $|S| = |G|$. Then $S \cong G$. In this paper with a counterexample we give a negative answer to this question.

1. Introduction and Preliminaries

In this paper all the groups we consider are finite.

Let $G$ a group and $\pi_t(G)$ be the set of element orders of $G$. Let $t \in \pi_t(G)$ and $s_t$ be the number of elements of order $t$ in $G$. Let $nse(G) = \{s_t \mid t \in \pi_t(G)\}$ the set of sizes of elements with the same order in $G$. Some authors have studied the influence of $nse(G)$ on the structure of $G$ (see [1, 5, 8] and [9]). For instance R. Shen in [6] proved that $A_4 \cong PSL(2, 3)$, $A_5 \cong PSL(2, 4) \cong PSL(2, 5)$ and $A_6 \cong PSL(2, 9)$ are uniquely determined by $nse(G)$. As a continuation in [4] was proved that if $G$ is a group such that $nse(G) = nse(PSL(2, q))$, where $q \in \{7, 8, 11, 13\}$, then $G \cong PSL(2, q)$. In [7] and [8] new characterizations of $A_5$ were given using $nse(A_5)$. The authors in [7] proved that $A_5$ is the only group such that $nse(A_5) = \{1, 15, 20, 24\}$ and the authors in [8] generalized that a nonabelian simple group $G$ has same-order type $\{r, m, n, k\}$ if and only if $G \cong A_5$ (see Th. 1.1 [8]).

Let $G$ be a group, in [8] was defined an equivalence relation $\sim$ as below:

$$\forall g, h \in G, g \sim h \iff |g| = |h|$$

the set of sizes of equivalence classes with respect to this relation is called the same-order type of $G$ and denoted by $\alpha(G)$. And $G$ is said a $\alpha_n$-group if $|\alpha(G)| = n$. Note that $\alpha(G)$ is equal to the set of sizes of elements with the same order in $G$, hence $|nse(G)| = |\alpha(G)|$.

We give a new characterization of $PSL(2, 7)$, $PSL(2, 8)$ and $PSL(2, 9)$ using same-order type.
THEOREM 1.1. Let $G$ be a simple $K_3$-group with same-order type \( \{r, m, n, k, l\} \). Then $G \cong PSL(2,7), PSL(2,8) \) or $PSL(2,9)$. 

This result generalizes the main results in [4], [6] and [8]. Combination the main results in [4] and [6] with Theorem 1.1 we have the following result

**COROLLARY 1.2.** A simple $K_3$-group $G$ has same-order type \( \{r, m, n, k, l\} \) if and only if $G \cong PSL(2,7), PSL(2,8) \) or $PSL(2,9)$. 

We see easily that the only $\alpha_1$-groups are 1 and a cyclic group of order 2. In [6] R. Shen characterized $\alpha_2$-groups as nilpotent groups and $\alpha_3$-groups as solvable groups. Moreover Taghivasani-Zarrin (see Th. 1.1 in [8]) showed that the only nonabelian simple $\alpha_4$-group is the $A_5$. As noted in [4] and [8] finite groups $G$ cannot be determined by $nse(G)$. Indeed in 1987 Thompson gave a first example as follows: Let $G_1 = (C_2 \times C_2 \times C_2 \times C_2) \rtimes A_7$ and $G_2 = PSL(3,4) \times C_2$ be the maximal subgroups of Mathieu group $M_{23}$. Then $nse(G_1) = nse(G_2)$, but $G_1 \not\cong G_2$.

Motivated by the main result in [8] about a new characterization of $A_5$ using same-order type, we have the natural question

**QUESTION 1.3.** Let $S$ be a nonabelian simple $\alpha_n$-group and $G$ a $\alpha_n$-group such that $|S| = |G|$. Then $S \cong G$.

We give a negative answer to this question in the last section.

2. **Proof of Theorem 1.1**

We need of one preliminary result to prove the main Theorem. The following result is a property very interesting of simple groups (see Lemma 2.7 in [8]).

**LEMMA 2.1.** Let $G$ be a nonabelian simple group. Then there exist two odd prime divisors $p$ and $q$ of the order of $G$ such that $p \neq q$.

In fact if $G$ is a nonabelian simple group then there exist two odd prime divisors $p$ and $q$ of the order of $G$ such that $\{1, s_2, s_p, s_q\} \subseteq \alpha(G)$ (see Corollary 2.8 in [8]).

We are now ready to conclude the proof of main Theorem.

**Proof of Theorem 1.1** As $G$ is a nonabelian simple group, it follows that $s_2 > 1$, w.l.g. $r = 1$ and $s_2 = m$. From Lemma 2.1 there exist odd prime divisors $p$ and $q$ of the order of $G$ such that $s_p \neq s_q = k$, hence $\pi(G) = \{2, p, q\}$ because $G$ is a simple $K_3$-group. Therefore $\{1, s_2, s_p, s_q\} \subseteq \alpha(G) = \{r, m, n, k, l\}$. So there exist a divisor $r \notin \pi(G)$ of order of $G$ such that $s_t = l$. It’s well known that the only nonabelian simple groups of order divisible by exactly three primes are the following eight groups: $PSL(2,q)$, where $q \in \{5,7,8,9,17\}$, $PSL(3,3), PSU(3,3)$ and $PSU(4,2)$, see Th. 1 and Th. 2 in [3]. Now we arguing as in the proof of Th. 1.1 in [8] and a GAP check yields that $|\alpha(PSL(2,7))| = 5$, $|\alpha(PSL(2,8))| = 5$, $|\alpha(PSL(2,9))| = 5$ and all others groups are $\alpha_n$-groups with $n \geq 6$ (except $A_5$ since $|\alpha(A_5)| = 4$). The result is follows.

3. **A counterexample to a Question 1.3**

Now we give a counterexample to the Question 1.3. Firstly we observed that by the main Theorem in [6], we have that $\alpha(PSL(2,7))$ is uniquely determined and we have that $\alpha(PSL(2,7)) = \{1, 21, 56, 42, 48\}$ hence $PSL(2,7)$ is a $\alpha_5$-group. Let $G = Q_8 \times (C_7 \rtimes C_8)$, where $Q_8$ is the quaternion group of order 8. As $|G| = 168$ and $G$ is a soluble group then is sufficient to prove that $|\alpha(G)| = 5$. Indeed the only 2-Sylow subgroup $Q_8$ is a normal...
A NEW CHARACTERIZATION OF SIMPLE $\mathcal{K}_3$-GROUPS USING SAME-ORDER TYPE

...using Sylow’s Theorem it follows that $s_2 = 8$. Note that a 7-Sylow subgroup of $G$ is isomorphic to $C_7$ and is a normal subgroup of $Q_8 \cdot C_7$ and $C_7 \times C_3$, hence the normalizer $N$ of $C_7$ has same order of $G$. Again from Sylow’s Theorem we have that $C_7$ is a normal subgroup of $G$ and $s_7 = 56$. As the number of 3-Sylow subgroup of $G$ is 7, then $s_3 = 14$. The number of elements of $G$ of order 2, 4 are respectively 1 and 6, hence $s_2 = 1$, $s_4 = 6$ and consequently $s_6 = 14$, $s_{12} = 84$, $s_{14} = 6$ and $s_{28} = 36$ (because of direct product in the structure of $G$). Therefore $\alpha(G) = \{1, 1, 14, 6, 14, 6, 84, 6, 36\}$ and $G$ is a $\alpha_6$-group.

Clearly $|\text{PSL}(2, 7)| = 168 = |G|$ but $\text{PSL}(2, 7) \not\cong G$.

We can obtain others groups $G$ with the computational group theory system GAP [2]: $G = C_7 \times (Q_8 \times C_3)$ or $G = C_2 \times ((C_{14} \times C_2) \times C_3)$. These groups are also counterexamples to the Question 1.3.

**Acknowledgments**: We thank L. J. Taghvasani for pointing out a mistake of ours and talking to us about Question 1.3 in the oldest version of [8].

**References**

[1] C. S. Anabanti. A counterexample to Zarrin’s Conjecture on sizes of finite nonabelian simple groups in relation to involution sizes. *Archiv der Mathematik*, 112 (3) (2019), 225–226.

[2] The GAP Group (2019). GAP – Groups, Algorithms, and Programming, Version 4.10.2. https://www.gap-system.org

[3] M. Herzog. On finite simple groups of order divisible by three primes only. *J. Algebra*, 10 (1968), 383–388.

[4] M. Khatami, B. Khosravi, Z. Akhlaghi. A new characterization for some linear groups. *Monatsh. Math.*, 163 (2011), 39-50.

[5] I. A. Malinowska. Finite groups with few normalizers or involutions. *Archiv der Mathematik*, 112 (3) (2019), 459–465.

[6] R. Shen. On groups with given same-order type. *Comm. Algebra*, 40 (2012), 2140-2150.

[7] R. Shen, C. Shao, Q. Jiang, W. Mazurov. A new characterization A5. *Monatsh. Math.*, 160 (2010), 337-341.

[8] L. J. Taghvasani and M. Zarrin. A characterization of A5 by its same-order type, *Monatsh. Math.*, 182 (2017), 731-736.

[9] M. Zarrin. A counterexample to Herzog’s Conjecture on the number of involutions. *Arch. Math.*, 111 (2018), 349–351.

**Email address**: (Lima) igor.matematico@gmail.com, (Pereira) josyanedsp@gmail.com