Solutions of (1+1)-dimensional Dirac equation associated with exceptional orthogonal polynomials and the parametric symmetry

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Abstract

We consider 1 + 1-dimensional Dirac equation with rationally extended scalar potentials corresponding to the radial oscillator, the trigonometric Scarf and the hyperbolic Poschl-Teller potentials and obtain their solution in terms of exceptional orthogonal polynomials. Further, in the case of the trigonometric Scarf and the hyperbolic Poschl-Teller cases, new family of Dirac scalar potentials are generated using the idea of parametric symmetry and their solutions are obtained in terms of conventional as well as exceptional orthogonal polynomials.

1 Introduction

Dirac equation plays an important role in the study of the dynamics of the relativistic systems with spin $\hbar/2$. Dirac equation has been applied to solve many problems in nuclear and high energy physics \cite{1, 2, 3}. In the quantum mechanical context, by now solutions of the Dirac equation have been obtained in the case of several scalar and vector potentials \cite{4, 5, 6, 7, 8, 9, 10} using different approaches such as supersymmetric

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quantum mechanics (SQM) approach, Nikiforov-Uvarov approach, the point canonical transformation approach, the group theoretic approach etc. Dirac equation has been solved for a broad class of potentials such as the Morse potential [11], the Coulomb potential [12], the Pöschl-Teller potential [13], the Hulthen potential [13] and the Scarf potential [15] etc. In the last few decades, it has been observed that the techniques of SQM [4, 16, 17, 18, 19, 20, 21, 22] play an important role in solving Dirac equation in $1 + 1$ space time in the case of various scalar potentials.

In recent years, the discovery of two new orthogonal polynomials namely the $X_m$ exceptional Laguerre and $X_m$ exceptional Jacobi orthogonal polynomials [23, 24, 25] (where the degree $m \geq 1$ instead of zero as in the case of usual polynomials) has lead to the discovery of rational extensions of several exactly solvable potentials in non-relativistic QM. In particular, the solution of the Schrödinger equation corresponding to the rationally extended potentials have been obtained in terms of exceptional orthogonal polynomials (EOPs) [26]-[43] or in the form of combination of usual orthogonal polynomials [44, 45].

Another development in non-relativistic QM is that of the parametric symmetry [46, 47] . It has been shown that while for some potentials the parametric symmetry leads to another set of solutions keeping the same form of the conventional potentials, but in the case of the corresponding rationally extended potentials this symmetry generates another form of the extended potentials and hence completely different solutions.

In contrast, in the relativistic case only few attempts have been made so far to solve the Dirac equation corresponding to the rationally extended scalar potentials [48, 49, 50]. Besides, the role of the parametric symmetry has not been explored in the Dirac case. The purpose of this paper is to obtain solutions of the Dirac equation in the case of few rationally extended scalar potentials and also study the role of the parametric symmetry in Dirac equation with scalar potentials.

In particular, in this paper, we consider the $(1 + 1)$-dimensional Dirac equation with three different forms of the scalar potentials $\tilde{\phi}(x)$ whose solutions are well known. We consider the corresponding rationally extended Dirac scalar potentials and obtain their solution in the form of the EOPs. Further, we show that similar to the Schrödinger case, there is also a parametric symmetry in the case of some of the Dirac equation with some of the scalar potentials. In particular, extending the idea of the parametric symmetry discussed in [46, 47] to the relativistic case, we generate a family of new form of the conventional as well as the rationally extended scalar potentials and obtain the solution of the corresponding Dirac equation.

The plan of the paper is as follows. In Section 2, we review the $1 + 1$ dimensional Dirac equation with scalar potential and discuss how its solutions can be obtained using SQM approach. In Sec. 3 we obtain the solution of the Dirac equation with rationally extended radial oscillator, trigonometric Scarf potential and the hyperbolic Pöschl-Teller potential and obtain their solutions in terms of EOPs using the SQM approach. In all these cases, for simplicity we first obtain solutions in terms of the $X_1$ Jacobi or $X_1$ Laguerre polynomials and then generalize to the general the $X_m$ case. In Sec. 4, we show that the trigonometric Scarf and the hyperbolic Poschl-Teller Dirac problems have novel parametric symmetry.
Using this symmetry, we obtain another form of the rationally extended Dirac scalar potentials and obtain their solutions. Finally, in Sec. 5 we summarize our results and point out few open problems.

2 Formalism

In this section, we review the solutions of the Dirac equation with general scalar potentials in 1+1 dimension and show how the problem can be reduced to two decoupled Schrödinger equations. In this way, using the well known SQM approach one can obtain the exact solutions of the corresponding Dirac problems in several cases.

The Dirac Lagrangian in 1 + 1\textit{D} with a Lorentz scalar potential \(\tilde{\phi}(x)\) is given by

\[
L = i\bar{\Psi} \gamma^\mu \partial_\mu \Psi - \tilde{\phi}(x) \bar{\Psi} \Psi, \quad \mu = 0, 1
\]  

(1)

where \(\Psi\) is the Dirac spinor. The Dirac equation following from Eq. (1) is

\[
i\gamma^\mu \partial_\mu \Psi(x, t) - \tilde{\phi}(x) \Psi(x, t) = 0.
\]  

(2)

Let

\[
\Psi(x, t) = \exp(-i\epsilon t) \xi(x),
\]  

(3)

so that the above Dirac equation reduces to

\[
\gamma^0 \epsilon \xi(x) + i\gamma^1 \frac{d}{dx} \xi(x) - \tilde{\phi}(x) \xi(x) = 0.
\]  

(4)

Now, we choose the following 2\textit{D} representation of the gamma matrices i.e,

\[
\gamma^0 = \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \gamma^1 = i\sigma_z = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \text{and} \quad \xi(x) = \begin{bmatrix} \tilde{\Psi}^{(1)}(x) \\ \tilde{\Psi}^{(2)}(x) \end{bmatrix},
\]  

(5)

and get two coupled equations

\[
\frac{d}{dx} \tilde{\Psi}^{(1)}(x) + \tilde{\phi}(x) \tilde{\Psi}^{(1)}(x) = \epsilon \tilde{\Psi}^{(2)}(x),
\]  

(6)

and

\[
\frac{d}{dx} \tilde{\Psi}^{(2)}(x) - \tilde{\phi}(x) \tilde{\Psi}^{(2)}(x) = -\epsilon \tilde{\Psi}^{(1)}(x).
\]  

(7)

These two equations can be decoupled easily and we obtain

\[
- \frac{d^2}{dx^2} \tilde{\Psi}^{(1)}(x) + \tilde{V}^{(1)}(x) \tilde{\Psi}^{(1)}(x) = \epsilon^2 \tilde{\Psi}^{(1)}(x)
\]  

(8)

and

\[
- \frac{d^2}{dx^2} \tilde{\Psi}^{(2)}(x) + \tilde{V}^{(2)}(x) \tilde{\Psi}^{(2)}(x) = \epsilon^2 \tilde{\Psi}^{(2)}(x)
\]  

(9)
respectively. These two equations (8) and (9) are equivalent to two independent Schrödinger equations with potentials

\[ \tilde{V}^{(1,2)}(x) = \tilde{\phi}^2(x) \mp \tilde{\phi}'(x). \] (10)

The solutions of these equations can be easily obtained for several \( \tilde{\phi}(x) \) using the well known SQM approach [16] by defining two operators \( \hat{A} \) and \( \hat{A}^\dagger \) as

\[ \hat{A} = \frac{d}{dx} + \tilde{\phi}(x) \quad \text{and} \quad \hat{A}^\dagger = -\frac{d}{dx} + \tilde{\phi}(x). \] (11)

In this way, the Eqs. (8) and (9) are reduced to

\[ \hat{A}^\dagger \hat{A} \tilde{\Psi}^{(1)} = \varepsilon^2 \tilde{\Psi}^{(1)} \quad \text{and} \quad \hat{A} \hat{A}^\dagger \tilde{\Psi}^{(2)} = \varepsilon^2 \tilde{\Psi}^{(2)}. \] (12)

respectively. On comparing with the well known formalism of SQM [16], we see that there is a supersymmetry in the problem and the scalar potential \( \tilde{\phi}(x) \) is just the superpotential of the Schrödinger formalism. Further \( \tilde{\Psi}^{(1)} \) and \( \tilde{\Psi}^{(2)} \) are the eigenfunctions of the Hamiltonians \( H_1 \equiv \hat{A}^\dagger \hat{A} \) and \( H_2 \equiv \hat{A} \hat{A}^\dagger \) respectively with \( \tilde{V}^{(1,2)}(x) \) being the partner potentials. Thus the eigenvalues and the eigenfunctions of the two Hamiltonians are related except that one of them has an extra bound state at zero energy so long as \( \tilde{\phi}(x \to \pm \infty) \) have opposite signs. Without loss of generality we shall always choose \( \tilde{\phi}(x) \) such that the ground state energy of \( H_1 \) is zero. In that case the eigenfunctions and eigenvalues (\( \tilde{E}^{(1)}_n \) and \( \tilde{E}^{(2)}_n \)) corresponding to these two Hamiltonians are related to each other as follows [16]

\[ \tilde{\Psi}^{(2)}_n(x) = [\tilde{E}^{(1)}_n]^\frac{1}{2} \hat{A} \tilde{\Psi}^{(1)}_{n+1}(x), \] (13)

\[ \tilde{\Psi}^{(1)}_{n+1}(x) = [\tilde{E}^{(2)}_n]^\frac{1}{2} \hat{A}^\dagger \tilde{\Psi}^{(2)}_n(x) \] (14)

and

\[ \tilde{E}^{(2)}_n = \tilde{E}^{(1)}_{n+1}, \quad \tilde{E}^{(1)}_0 = 0. \] (15)

Here \( n = 0, 1, 2, \ldots \). Thus once we have the eigenfunctions \( \tilde{\Psi}^{(1)}_n(x) \) and the energy eigenvalues \( \tilde{E}^{(1)}_n \), we can easily obtain \( \tilde{\Psi}^{(2)}_n(x) \) and \( \tilde{E}^{(2)}_n \) using Eq. (13) and (15) respectively.

### 3 Rational Dirac Potentials

We shall now discuss three examples of the rational scalar Dirac potentials \( \tilde{\phi}(x) \), i.e. the radial oscillator, trigonometric Scarf and the Poschl-Teller potentials. To motivate the discussion we first mention the well known results about the corresponding conventional Dirac scalar potentials and then obtain the solution of the corresponding rational cases. For simplicity, we first discuss the \( X_1 \) case and then generalize to the general \( X_m \) case.
3.1 Radial oscillator

3.1.1 The conventional case

Let us consider the scalar potential defined on the half line \((0 \leq r \leq \infty)\) of the form

\[
\bar{\phi}(r) \rightarrow \bar{\phi}_{\text{con}}(r) = \frac{1}{2} \omega r - \frac{\ell + 1}{r}.
\]  

(16)

On using Eq. (10) it gives rise to

\[
\tilde{V}^{(1)} \rightarrow \tilde{V}^{(1)}_{\text{con}}(r) = \bar{\phi}_{\text{con}}^2(r) - \bar{\phi}'_{\text{con}}(r)
\]

\[
= \frac{\omega r^2}{4} + \frac{\ell(\ell + 1)}{r^2} - \omega(\ell + \frac{3}{2}),
\]  

(17)

which is the well known radial oscillator potential \((\omega > 0, \ell > 0)\) whose solutions \([16]\) are given in terms of the classical Laguerre polynomials \(L^{(\ell + \frac{1}{2})}_{n}(z)\)

\[
\tilde{\Psi}^{(1)}_{\text{con},n}(r) = N_{l,1,1,1} \bar{\phi}_{\text{con},n} \exp \left( -\frac{z(r)}{2} \right) L^{(\ell + \frac{1}{2})}_{n}(z(r)), \quad n = 0, 1, 2, ...
\]  

(18)

where \(z(r) = \frac{\omega r^2}{2}\) and the normalization constant

\[
N_{l,1,1,1} = \left[ \frac{n! \omega^{(\ell + \frac{1}{2})}}{2^{(\ell + \frac{1}{2})}(\ell + n + \frac{1}{2})\Gamma(\ell + n + \frac{1}{2})} \right]^{1/2}.
\]  

(19)

The corresponding energy eigenvalue are

\[
\tilde{E}^{(1)} \rightarrow \tilde{E}^{(1)}_{\text{con},n} = \varepsilon^2 = \omega(2n + \ell + \frac{3}{2}).
\]  

(20)

3.1.2 The Rationally Extended Case

(a) The \(X_1\) Case

In this case, we consider the scalar potential \(\tilde{\phi}(r)\) which is defined as the sum of the conventional scalar potential \(\tilde{\phi}_{\text{con}}(r)\) as given by Eq. (10) and a rational term \(\tilde{\phi}_{\text{rat}}(r)\) i.e.,

\[
\tilde{\phi}(r) \rightarrow \tilde{\phi}_{\text{ext}}(r) = \tilde{\phi}_{\text{con}}(r) + \tilde{\phi}_{\text{rat}}(r),
\]  

(21)

where

\[
\tilde{\phi}_{\text{rat}}(r) = \frac{4 \omega r}{(2z(r) + 2\ell + 1)(2z(r) + 2\ell + 3)}.
\]  

(22)

On using Eq. (21) in Eq. (10), we get the rationally extended radial oscillator potential

\[
\tilde{V}_{\text{rat}}^{(1)}(r) = \tilde{V}_{\text{con}}^{(1)}(r) + \tilde{V}_{\text{rat}}^{(1)}(r),
\]  

(23)
with
\[
\tilde{V}_{rat}^{(1)}(r) = 4\omega \left( \frac{1}{(2z(r) + 2\ell + 1)} - \frac{2(2\ell + 1)}{(2z(r) + 2\ell + 1)^2} \right),
\]
while \(\tilde{V}_{con}^{(1)}(r)\) is as given by Eq. (17). The solution of the corresponding Schrödinger equation is \[26\]
\[
\Psi_{ext,n}^{(1)}(r) = N_{n,ext}^{\ell} r^{\ell+1} \exp \left( -\frac{z(r)}{2} \right) L_{n+1}^{(\ell+\frac{1}{2})}(z(r)),
\]
where \(L_{n+1}^{(\alpha)}(z(r))\) is \(X_1\) exceptional Laguerre Polynomials while the normalization constant is \[26\]
\[
N_{ext,n}^{\ell} = \left[ \frac{n! \omega^{(\ell+\frac{1}{2})}}{2^{(\ell+1)}(\ell + n + 1 + \frac{1}{2}) \Gamma(\ell + n + \frac{1}{2})} \right]^{1/2}.
\]
Notice that the energy eigenvalues are same as that of the conventional one and are given by
\[
\tilde{E}_{n,ext}^{(1)} = \varepsilon_{ext}^{2} = 2n\omega.
\]

(b) The \(X_m\) Case

The above results for the \(X_1\) case are immediately generalized to the general \(X_m\) case. In this case the scalar potential is defined as \(\phi(r) \rightarrow \phi_{ext,m}(r)\), given by
\[
\phi_{ext,m}(r) = \phi_{con}(r) + \phi_{m,rat}(r); \quad (m = 0, 1, 2, ...),
\]
where \(\phi_{con}(r)\) is again as given by Eq. (16) while the rational term \(\phi_{m,rat}(r)\) is given by
\[
\phi_{m,rat}(r) = \omega r \left[ L_{m-1}^{(\ell+\frac{1}{2})}(-z(r)) - L_{m}^{(\ell+\frac{1}{2})}(-z(r)) \right].
\]
On using \(\phi_{ext,m}(r)\) instead of \(\phi_{con}(r)\) in Eq. (10), we get \[28, 35\]
\[
\tilde{V}_{ext,m}^{(1)}(r) = \tilde{V}_{con}^{(1)}(r) + \tilde{V}_{rat,m}^{(1)}(r)
\]
where \(\tilde{V}_{con}^{(1)}(r)\) is again as given by Eq. (17) while
\[
\tilde{V}_{rat,m}^{(1)}(r) = -2\omega z(r) \left( \frac{L_{m-\frac{1}{2}}^{(\ell+\frac{1}{2})}(-z(r))}{L_{m}^{(\ell+\frac{1}{2})}(-z(r))} \right) + \frac{\omega(2z(r) + 2\ell - 1)}{L_{m-\frac{1}{2}}^{(\ell+\frac{1}{2})}(-z(r))} \left( \frac{L_{m-\frac{1}{2}}^{(\ell+\frac{1}{2})}(-z(r))}{L_{m}^{(\ell+\frac{1}{2})}(-z(r))} \right)^2 - 2m\omega, \quad 0 < r < \infty.
\]
The solutions of the corresponding Schrödinger equation are in terms of \(X_m\) Laguerre Polynomials \(L_{m,n}^{(\ell+\frac{1}{2})}(z)\) and are given by
\[
\tilde{\Psi}_{ext,n,m}^{(1)}(r, \omega, \ell) = N_{ext,n,m}^{\ell,1} \frac{r^{\ell+1}}{L_{m}^{(\ell+\frac{1}{2})}(-z(r))} \left( \frac{L_{m}^{(\ell+\frac{1}{2})}(-z(r))}{L_{m+n}^{(\ell+\frac{1}{2})}(z(r))} \right), \quad m = 1, 2, ...
where
\[
\hat{L}^{(\alpha)}_{n+m}(z) = L^{(\alpha)}_m(-z)L^{(\alpha-1)}_n(z) + L^{(\alpha-1)}_m(-z)L^{(\alpha)}_{n-1}(z); \quad n \geq m
\]  
(33)

with the normalization constant being
\[
N^{\ell,1}_{ext,n,m} = \left[ \frac{n!\omega^{(\ell+\frac{3}{2})}}{2^{(\ell+\frac{1}{2})}(\ell + n + m + \frac{1}{2})\Gamma(\ell + n + \frac{1}{2})} \right]^{1/2}.
\]  
(34)

As a check on our calculations, for \( m = 0 \) and 1, we recover the results corresponding to the conventional and the \( X_1 \) extended rational scalar Dirac potentials respectively. The energy spectrum is again the same as of the conventional case and given by Eq. (20).

The plots of the Dirac scalar potentials \( \tilde{\phi}_{ext,m}(r) \) and the corresponding normalized ground state eigen functions \( \tilde{\Psi}^{(1)}_{ext,0,m}(r, \omega, \ell) \) are given for \( m = 0, 1, 2 \) in figs. 1(a) and 1(b) respectively in case \( \omega = 2, \ell = 1 \).

\[ \text{Fig.1: (a) Rationally extended Dirac scalar potentials for } m = 0, 1 \text{ and 2.} \]
Fig.1: (b) Normalized ground-state wave functions for $m = 0, 1$ and 2.

3.2 Trigonometric Scarf case

3.2.1 The Conventional Case

In this case, the scalar potential $\tilde{\phi}(x) \rightarrow \tilde{\phi}_{\text{con}}(x, A, B)$ (defined on $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$) is given by

$$\tilde{\phi}_{\text{con}}(x, A, B) = A \tan x - B \sec x; \quad -\frac{\pi}{2} < x < \frac{\pi}{2}, \quad 0 < B < A - 1. \quad (35)$$

On using this $\tilde{\phi}_{\text{con}, A, B}(x)$ in Eq. (10), we get the well known trigonometric Scarf potential

$$\hat{V}^{(1)}(x) \rightarrow \hat{V}_{\text{con}}^{(1)}(x, A, B) = [(A - 1)A + B^2] \sec^2 x - B(2A - 1) \sec x \tan x - A^2. \quad (36)$$

The solution of the Schrödinger equation corresponding to this potential is well known and is given in terms of the classical Jacobi polynomial $P_n^{(\alpha, \beta)}(z)$ as

$$\tilde{\Psi}_{\text{con}, n}^{(1)}(x, A, B) = N_{\text{con}, n}^{(1)}(\alpha, \beta)(1 - z(x))^{-\frac{(A-B)}{2}}(1 + z(x))^{-\frac{(A+B)}{2}} P_n^{(\alpha, \beta)}(z(x)). \quad (37)$$

Here $\alpha = A - B - \frac{1}{2}$, $\beta = A + B - \frac{1}{2}$, $z(x) = \sin x$ and the normalization constant is given by

$$N_{\text{con}, n}^{(1)}(\alpha, \beta) = \left[\frac{n!(\alpha + \beta + 2n + 1)\Gamma(n + \alpha + \beta + 1)}{2^{\alpha+\beta+1}(n + \beta)\Gamma(n + \alpha + 1)\Gamma(n + \beta)}\right]^\frac{1}{2}. \quad (38)$$

The energy eigenvalues are

$$\varepsilon^2 = \tilde{E}_{\text{con}, n} = (A + n)^2 - A^2. \quad (39)$$
3.2.2 The Rationally Extended Case

(a) The $X_1$ Case: In the extended $X_1$ case, the function

$$\tilde{\phi}(x) \rightarrow \tilde{\phi}_{ext}(x, A, B) = \tilde{\phi}_{con}(x, A, B) + \tilde{\phi}_{rat}(x, A, B)$$

(40)

where $\tilde{\phi}_{con}(x, A, B)$ is as given by Eq. (35) while

$$\tilde{\phi}_{rat}(x, A, B) = -2Bz'(x)\left[\frac{1}{2A - 1 - 2Bz(x)} - \frac{1}{2A + 1 - 2Bz(x)}\right].$$

(41)

Here $z'(x)$ is the first derivative of $z(x)$ with respect to $x$. Using Eq. (10), the corresponding rationally extended trigonometric Scarf potential $\tilde{V}_{ext}^{(1)}(x, A, B)$ turns out to be

$$\tilde{V}_{ext}^{(1)}(x, A, B) = \tilde{V}_{con}^{(1)}(x, A, B) + \tilde{V}_{rat}^{(1)}(x, A, B)$$

(42)

where $\tilde{V}_{con}^{(1)}(x, A, B)$ is as given by Eq. (36) while

$$\tilde{V}_{rat}^{(1)}(x, A, B) = 2\left(\frac{(2A - 1)}{(2A - 1 - 2Bz(x))} - \frac{(2A - 1)^2 - B^2}{(2A - 1 - 2Bz(x))^2}\right).$$

(43)

The solutions of the Schrödinger equation corresponding to this potential are given in the form of exceptional Jacobi Polynomials ($\tilde{P}^{(\alpha, \beta)}_{n+1}(g(x))$) as

$$\tilde{\psi}_{ext,n}^{(1)}(x) = N_{ext,n}^{(1)}(\alpha, \beta)\frac{(1 - z(x))^{(\frac{1}{2} - \alpha)}(1 + z(x))^{(\frac{1}{2} + \beta)}}{P^{(-\alpha - 1, \beta - 1)}_1(z(x))}\tilde{\psi}^{(\alpha, \beta)}(z(x))$$

(44)

where the normalization constant is

$$N_{ext,n}^{(1)}(\alpha, \beta) = \left[\frac{n!(n + \alpha + 1)(\alpha + \beta + 2n + 1)\Gamma(n + \alpha + \beta + 1)}{2^{\alpha + \beta + 1}(n + \alpha)(n + 1 + \beta)\Gamma(n + \alpha + 1)\Gamma(n + \beta)}\right]^{\frac{1}{2}}.$$

(45)

Here $P^{(-\alpha - 1, \beta - 1)}_1(z)$ is the classical Jacobi Polynomial for $n = 1$.

The spectrum $\varepsilon^2 = E_{ext,n}^{(1)}$ is however unchanged compared to the conventional Scarf case and is given by Eq. (39).

(b) The $X_m$ Case:

Here, we replace $\tilde{\phi}(x) \rightarrow \tilde{\phi}_{m,ext}(x, A, B)$ given by

$$\tilde{\phi}_{m,ext}(x, A, B) = \tilde{\phi}_{con}(x, A, B) + \tilde{\phi}_{m,rat}(x, A, B),$$

(46)

where $\tilde{\phi}_{con}(x, A, B)$ is again given by Eq. (35) while $\tilde{\phi}_{m,rat}(x, A, B)$ is given in terms of the Jacobi polynomials by

$$\tilde{\phi}_{m,rat}(x, A, B) = -\frac{(\beta - \alpha + m - 1)}{2}z'(x)\left[P^{(-\alpha - 1, \beta + 1)}_{m-1}(z(x)) - P^{(-\alpha - 2, \beta)}_m(z(x))\right].$$

(47)
Using Eq. (10), the corresponding potential $\tilde{V}^{(1)}_{ext,m}(x, A, B)$ turns out to be
\[
\tilde{V}^{(1)}_{ext,m}(x, A, B) = \tilde{V}_{con}(x, A, B) + \tilde{V}_{m, rat}(x, A, B),
\]
where $\tilde{V}_{con}(x, A, B)$ is again given by Eq. (36) while the $m$ dependent rational potential is
\[
\tilde{V}^{(1)}_{m, rat}(x, A, B) = (2B - m - 1)[2A - 1 + (-2B + 1)z(x)] \left( \frac{P^{(-\alpha, \beta)}_{m-1}(z(x))}{P^{(-\alpha-1, \beta-1)}_{m}(z(x))} \right)
+ \frac{(-2B - m + 1)^2}{2}(z'(x))^2 \left( \frac{P^{(-\alpha, \beta)}_{m-1}(z(x))}{P^{(-\alpha-1, \beta-1)}_{m}(z(x))} \right)^2
- 2m(-2B - m - 1); \quad -\pi/2 < x < \pi/2, \quad 0 < B < A - 1. \tag{49}
\]

The eigen functions $\tilde{\Psi}^{(1)}_{ext,n,m}(x, A, B)$ of the Schrödinger equation with this potential turn out to be
\[
\tilde{\Psi}^{(1)}_{ext,n,m}(x, A, B) = N^{(1)}_{ext,n,m}(\alpha, \beta) \frac{(1 - z(x))^{(A-B)/2}(1 + z(x))^{(A+B)/2}}{P^{(-\alpha-1, \beta-1)}_{m}(z(x))} \hat{P}^{(\alpha, \beta)}_{n+m}(z(x)) \tag{50}
\]
where
\[
N^{(1)}_{ext,n,m}(\alpha, \beta) = \left[ \frac{n!(n + \alpha + 1)^2(\alpha + \beta + 2n + 1)\Gamma(n + \alpha + \beta + 1)}{2^{\alpha+\beta+1}(n + \alpha - m + 1)(n + m + \beta)\Gamma(n + \alpha + 2)\Gamma(n + \beta)} \right]^{\frac{1}{2}}, \tag{51}
\]
while the $X_m$ exceptional Jacobi polynomials satisfy
\[
\hat{P}^{(\alpha, \beta)}_{n+m}(z) = (-1)^m \left[ \frac{1 + \alpha + \beta + n}{2(1 + \alpha + n)}(g - 1)P^{(-\alpha-1, \beta-1)}_{m}(g)P^{(\alpha+2, \beta)}_{n}(g)
+ \frac{1 + \alpha - m}{\alpha + 1 + n}P^{(-2-\alpha, \beta)}_{m}(g)P^{(\alpha+1, \beta-1)}_{n}(g) \right]; \quad n, m \geq 0. \tag{52}
\]
The energy spectrum is again same as that of the conventional or $X_1$ case and is given by Eq. (39).

The plots of the Dirac scalar potentials $\tilde{\phi}_{ext,m}(x, A, B)$ and the corresponding normalized ground state eigen functions $\tilde{\Psi}^{(1)}_{ext,0,m}(x, A, B)$ are shown for $m = 0, 1, 2$ in figs. 2(a) and 2(b) respectively in case $A = 3$ and $B = 1$. 

3.3 Hyperbolic Pöschl-Teller case

3.3.1 The Conventional Case

In this case we define

\[ \tilde{\phi}(r) \rightarrow \tilde{\phi}_{con}(r, A, B) = A \coth r - B \coth cr, \quad 0 \leq r \leq \infty \] (53)
with $B > A + 1 > 1$ which gives rise to the conventional hyperbolic Pöschl-Teller potential

\[ \tilde{V}_{\text{con}}^{(1)}(r) = [(A + 1)A + B^2] \cosech^2 r - B(2A + 1) \cosech r \coth r + A^2. \]  

(54)

The corresponding eigen functions of the Schrödinger equation are

\[ \tilde{\Psi}_{\text{con}}^{(1)}(r, A, B) = N_{\text{con}, n}^{(1)}(\alpha, \beta) \left( z(r) - 1 \right) \left( z(r) + 1 \right)^{-\frac{(B-A)}{2}} P_n^{(\alpha, \beta)}(z(r)). \]

(55)

where $\alpha = -A + B - \frac{1}{2}$, $\beta = -A - B - \frac{1}{2}$, $z(r) = \cosh r$ and the normalization constant is

\[ N_{\text{con}}^{(\alpha, \beta)} = \left[ \frac{n!(-\alpha - \beta - 2n - 1)(\alpha + n + 1)(\alpha + n + 1)\Gamma(-\beta - n)}{2^{\alpha + \beta + 1}(\alpha + 1)^2 \Gamma(\alpha + n + 1)\Gamma(-\alpha - \beta - n)} \right]^{1/2}. \]

(56)

The energy eigenvalue spectrum turns out to be

\[ \varepsilon^2 = \tilde{E}_{\text{con}, n}^{(1)} = A^2 - (A - n)^2, \quad n = 0, 1, 2, ..., n_{\text{max}} < A. \]

(57)

### 3.3.2 The Extended Case

(a) The $X_1$ Case:

In this case, the Dirac scalar potential is defined as

\[ \tilde{\phi}_{\text{ext}}(r, A, B) = \tilde{\phi}_{\text{con}}(r, A, B) + \tilde{\phi}_{\text{rat}}(r, A, B) \]

(58)

where $\tilde{\phi}_{\text{con}}(r, A, B)$ is given by Eq. (53) while

\[ \tilde{\phi}_{\text{rat}}(r, A, B) = 2Bz'(r) \left[ \frac{1}{2Bz(r) - 2A - 1} - \frac{1}{2Bz(r) - 2A + 1} \right]. \]

(59)

Using Eq. (10), this leads to the rationally extended hyperbolic Pöschl-Teller potential

\[ \tilde{V}_{\text{ext}}^{(1)}(r, A, B) = \tilde{V}_{\text{con}}^{(1)}(r, A, B) + \tilde{V}_{\text{rat}}^{(1)}(r, A, B) \]

(60)

where $\tilde{V}_{\text{con}}^{(1)}(r, A, B)$ is given by Eq. (54) while

\[ \tilde{V}_{\text{rat}}^{(1)}(r, A, B) = 2 \left[ \frac{(2A + 1)}{(2Bz(r) - 2A - 1)} - \frac{(4B^2 - (2A + 1)^2)}{(2Bz(r) - 2A - 1)^2} \right] + A^2. \]

(61)

The corresponding eigen functions of the Schrödinger equation turn out to be

\[ \tilde{\Psi}_{\text{ext}, n}^{(1)}(r, A, B) = N_{\text{ext}, n}^{(1)}(\alpha, \beta) \left( z(r) - 1 \right) \left( z(r) + 1 \right)^{-\frac{(B-A)}{2}} P_{n+1}^{(\alpha, \beta)}(z(r)), \]

(62)

where the normalization constant is

\[ N_{\text{ext}, n}^{(1)}(\alpha, \beta) = \left[ \frac{n!(-\alpha - \beta - 2n - 1)(\alpha + n + 1)\Gamma(-\beta - n + 1)}{2^{\alpha + \beta + 1}(-\beta - n - 1)(\alpha)^2 \Gamma(\alpha + n)\Gamma(-\alpha - \beta - n)} \right]^{1/2}. \]

(63)
The energy spectrum is same as that of the conventional case and given by Eq. (57).

(b) **The $X_m$ Case:**

In this case the Dirac scalar potential (for any arbitrary $m$) is given by

$$
\tilde{\phi}_{ext,m}(r, A, B) = \tilde{\phi}_{con}(r, A, B) + \tilde{\phi}_{m, rat}(r, A, B),
$$

where $\tilde{\phi}_{con}(r, A, B)$ is again given by Eq. (53) while

$$
\tilde{\phi}_{m, rat}(r, A, B) = -\frac{(\beta - \alpha + m - 1)}{2} z'(r) \left[ \frac{P^{(-\alpha-1,\beta+1)}_{m-1}(z(r))}{P^{(-\alpha-2,\beta)}_m(z(r))} - \frac{P^{(-\alpha,\beta)}_{m-1}(z(r))}{P^{(-\alpha-1,\beta-1)}_m(z(r))} \right].
$$

Using Eq. (10) this leads to the potential which is now $m$-dependent [28, 34] and is given as

$$
\tilde{V}_{ext,m}(r, A, B) = \tilde{V}_{con}(r, A, B) + \tilde{V}_{rat,m}(r, A, B),
$$

where $\tilde{V}_{con}(r, A, B)$ is again given by Eq. (54) while

$$
\tilde{V}_{rat,m}(r, A, B) = (2B - m + 1)[2A + 1 - (2B + 1)z(r)] \left( \frac{P^{(-\alpha,\beta)}_{m-1}(z(r))}{P^{(-\alpha-1,\beta-1)}_m(z(r))} \right)
$$

$$
+ \frac{(2B - m + 1)^2}{2} (z'(r))^2 \left( \frac{P^{(-\alpha,\beta)}_{m-1}(z(r))}{P^{(-\alpha-1,\beta-1)}_m(z(r))} \right)^2
$$

$$
+ 2m(-2B - m - 1); \quad 0 \leq x \leq \infty, \quad B > A + 1 > 1.
$$

The corresponding eigen functions of the Schrödinger equation turn out to be

$$
\tilde{\Psi}_{ext,n,m}(r, A, B) = N_{ext,n,m}(\alpha, \beta) \frac{(z - 1)^{(\frac{B-A}{2})} (z + 1)^{(\frac{B+A}{2})}}{P^{(-\alpha-1,\beta-1)}_m(z(r))} \hat{P}_{n+m}(\alpha, \beta)(z(r)),
$$

where the normalization constant is given by

$$
N_{ext,n,m}(\alpha, \beta) = \frac{N_{ext,n}(\alpha, \beta)}{\left( n + \alpha - m + 1 \right) \left( \alpha + n \right)}.
$$

The energy eigenvalue spectrum is again same as in the conventional case and is given by Eq. (57).

The plots of the Dirac scalar potentials $\tilde{\phi}_{ext,m}(r, A, B)$ and the corresponding normalized ground state eigen functions $\tilde{\Psi}_{ext,n,m}(r, A, B)$ are shown for $m = 0, 1, 2$ in figs. 3(a) and 3(b) respectively in case $A = 1$ and $B = 3$. 

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Fig.3: (a) *Rationally extended Dirac scalar potentials for* $m = 0, 1$ *and 2.*

Fig.3: (b) *Normalized ground-state wave functions for* $m = 0, 1$ *and 2.*

### 4 Parametric symmetry and new forms of Dirac Scalar Potentials

Recently, the role of the parametric symmetry has been discussed in the case of Schrödinger equation \[46, 47\]. It is then worthwhile to discuss the role of the parametric symmetry in the context of the Dirac equation. It turns out that out of three examples discussed in this paper, this symmetry only exist in the two cases, i.e. the trigonometric scarf and...
the hyperbolic Pöschl-Teller potentials which we discuss one by one. We shall see that in the conventional cases, this symmetry generates new Dirac scalar potentials keeping the corresponding Schrödinger potential \( \tilde{V}_{\text{con}}^{(1)} \) unchanged but having a new partner \( \tilde{V}_{\text{con}}^{(2)} \). On the other hand, in the extended cases, one finds that the corresponding \( \tilde{V}_{\text{ext}}^{(1)} \) and \( \tilde{V}_{\text{ext}}^{(2)} \) are both modified.

4.1 Trigonometric Scarf Case

4.1.1 The Conventional Case

If we replace the parameters \( B \leftarrow A - \frac{1}{2} \) in the conventional scalar potential \( \tilde{\phi}_{\text{con}}(x, A, B) \) as given in [46], we get a new form of \( \tilde{\phi}_{\text{con}}^{(p)}(x, A, B) \) generated due to this parametric transformation i.e.,

\[
\tilde{\phi}(x) \rightarrow \tilde{\phi}_{\text{con}}^{(p)}(x, A, B) = \tilde{\phi}_{\text{con}}(x, A \rightarrow B + \frac{1}{2}, B \rightarrow A - \frac{1}{2}) = \left( B + \frac{1}{2} \right) \tan x - \left( A - \frac{1}{2} \right) \sec x; \quad B > A - 1 > 0
\] (70)

which is different from the \( \tilde{\phi}_{\text{con}}(x, A, B) \) as given by Eq. (35). Remarkably, this scalar potential leads to the same potential \( \tilde{V}_{\text{con}}^{(1,p)}(x, A, B) = \tilde{V}_{\text{con}}^{(1)}(x, A, B) \) as given by Eq. (36) but different \( \tilde{V}^{(2,p)}_{\text{con}}(x) = \tilde{V}_{\text{con}}(x, A, B \rightarrow B + 1) \). In other words, \( \tilde{V}_{\text{con}}^{(1)}(x, A, B) \) has two different SUSY partners. Thus we have another set of \( \tilde{\phi}(i.e, \tilde{\phi}_{\text{con}}^{(p)}(x, A, B)) \) leading to different eigenvalues and eigen functions, i.e,

\[
\tilde{\Psi}_{\text{con,n}}^{(1,p)}(x, A, B) = \tilde{\Psi}_{\text{con,n}}^{(1)}(x, A \rightarrow B + \frac{1}{2}, B \rightarrow A - \frac{1}{2}); \quad B > A - 1 > 0
\] (71)

and

\[
\tilde{\Psi}_{n}^{(2,p)}(x, A, B) = \tilde{\Psi}_{\text{con,n}}^{(1,p)}(x, A, B \rightarrow B + 1),
\] (72)

with the energy eigenvalues

\[
\varepsilon^2 = \tilde{E}_{n}^{(1,p)} = (B + n + \frac{1}{2})^2, \quad n = 0, 1, 2, ...
\] (73)

4.1.2 The extended Case

(a) The \( X_1 \) case:

In the extended case as \( B \leftrightarrow A - \frac{1}{2} \), we have

\[
\tilde{\phi}(x) \rightarrow \tilde{\phi}_{\text{ext}}^{(p)}(x, A, B) = \tilde{\phi}_{\text{ext}}(x, A \rightarrow B + \frac{1}{2}, B \rightarrow A - \frac{1}{2})
\]

\[
= \tilde{\phi}_{\text{con}}^{(p)}(x, A, B) + \tilde{\phi}_{\text{rat}}^{(p)}(x, A, B),
\] (74)
where \( \tilde{\phi}_{con}^{(p)}(x, A, B) \) is as given by Eq. (70) while

\[
\tilde{\phi}_{rat}^{(p)}(x, A \rightarrow B + \frac{1}{2}, B \rightarrow A - \frac{1}{2}) = 2 \left( A - \frac{1}{2} \right) z'(x) \left[ \frac{1}{2B + 2 - (2A - 1)z(x)} - \frac{1}{2B - (2A - 1)z(x)} \right].
\] (75)

Note that this scalar extended Dirac potential Eq. (74) is different from (40) and unlike the conventional case it leads to both \( \tilde{V}_{ext}^{(1,p)}(x, A, B) \) and \( \tilde{V}_{ext}^{(2,p)}(x, A, B) \) being different and are given as

\[
\tilde{V}_{ext}^{(1,p)}(x, A, B) = \tilde{V}_{ext}^{(1)}(x, A \rightarrow B + \frac{1}{2}, B \rightarrow A - \frac{1}{2})
\] (76)

and

\[
\tilde{V}_{ext}^{(2,p)}(x, A, B) = \tilde{V}_{ext}^{(1,p)}(x, A, B \rightarrow B + 1)
\] (77)

The corresponding eigen functions of the Schrödinger equation can be written as

\[
\tilde{\Psi}_{ext,n}^{(1,p)}(x, A, B) = \tilde{\Psi}_{ext,n}^{(1)}(x, A \rightarrow B + \frac{1}{2}, B \rightarrow A - \frac{1}{2}); \quad B > A - 1 > 0
\]

\[
= N_{ext}^{(1,p)}(\gamma, \delta) \left( 1 - z(x) \right)^{\frac{(B - A + 1)}{2}} \left( 1 + z(x) \right)^{\frac{(A + B)}{2}} \hat{P}_{n+1}^{(\gamma, \delta)}(z(x))
\] (78)

and

\[
\tilde{\Psi}_{ext,n}^{(2,p)}(x, A, B) = \tilde{\Psi}_{ext,n}^{(1,p)}(x, A, B \rightarrow B + 1),
\] (79)

where \( \gamma = B - A + \frac{1}{2} \) and \( \delta = A + B - \frac{1}{2} \).

The corresponding energy eigenvalues are however unchanged from the conventional case and are again given by Eq. (73).

(b) The \( X_m \) Case:

For the more general \( X_m \) case, in the extended case, as \( B \leftrightarrow A - \frac{1}{2} \), the Dirac scalar potential is

\[
\tilde{\phi}_{ext,m}^{(p)}(x, A, B) = \tilde{\phi}_{con}^{(p)}(x, A, B) + \tilde{\phi}_{rat,m}^{(p)}(x, A, B)
\] (80)

where \( \tilde{\phi}_{con}^{(p)}(x, A, B) \) is again given by Eq. (70) while

\[
\tilde{\phi}_{rat,m}^{(p)}(x, A, B) = \tilde{\phi}_{rat,m}^{(p)}(x, A \rightarrow B + \frac{1}{2}, B \rightarrow A - \frac{1}{2})
\]

\[
= - \left( \frac{2A + m - 2}{2} \right) z'(x) \left[ \frac{P^{(-\alpha - 1, \beta + 1)}_{m-1}(z(x))}{P^{(-\alpha - 2, \beta)}_{m}(z(x))} - \frac{P^{(-\alpha, \beta)}_{m-1}(z(x))}{P^{(-\alpha - 1, \beta - 1)}_{m}(z(x))} \right].
\] (81)

This scalar extended potential leads to both \( \tilde{V}_{ext}^{(1,p)}(x, A, B) \) and \( \tilde{V}_{ext}^{(2,p)}(x, A, B) \) being different.
The corresponding eigen functions of the Schrödinger equation can be written as

\[ \tilde{\Psi}^{(1,p)}_{ext,m,n}(x, A, B) = \tilde{\Psi}^{(1)}_{ext,m,n}(x, A \rightarrow B + \frac{1}{2}, B \rightarrow A - \frac{1}{2}); \quad B > A - 1 > 0 \]  \hspace{1cm} (82)

and

\[ \tilde{\Psi}^{(2,p)}_{ext,m,n}(x, A, B) = \tilde{\Psi}^{(1,p)}_{ext,m,n}(x, A, B \rightarrow B + 1). \]  \hspace{1cm} (83)

The energy eigenvalues are however unchanged and are again given by Eq. (73).

The plots of the scalar potential \( \phi^{(p)}_{ext,m}(x, A, B) \) and the normalized ground state eigen function \( \tilde{\Psi}^{(1,p)}_{ext,0,m}(x, A, B) \) for \( A = \frac{3}{2}, B = \frac{5}{2} \) and different values of \( m(=0, 1 \text{ and } 2) \) corresponding to the conventional, \( X_1 \) and \( X_2 \) respectively are given in Figs. 4(a) and 4(b) respectively.

**Fig.4:** (a) Rationally extended parametric Dirac scalar potentials for \( m = 0, 1 \text{ and } 2 \).
4.2 Hyperbolic Pöschl-Teller Case

4.2.1 The Conventional Case

Similar to the trigonometric Scarf case, in this case under the parametric transformation $B \leftrightarrow A + \frac{1}{2}$ the scalar potential $\tilde{\phi}_{con}(r, A, B)$ as given by Eq. (53) becomes

$$
\tilde{\phi}_{con}^{(p)}(r, A, B) = \tilde{\phi}_{con}(r, A \rightarrow B - \frac{1}{2}, B \rightarrow A + \frac{1}{2})
$$

$$
= (B - \frac{1}{2}) \coth r - (A + \frac{1}{2}) \coth r; \quad 0 < r < \infty.
$$

Remarkably, under this transformation, the potential $\tilde{V}_{con}^{(1,p)}(r, A, B) = \tilde{V}_{con}^{(1)}(r, A, B)$ remains the same as given by Eq. (54), however the partner potential $\tilde{V}_{con}^{(2,p)}(r, A, B)$ gets changed i.e,

$$
\tilde{V}_{con}^{(1,p)}(r, A, B) = \tilde{V}_{con}^{(1)}(r, A \rightarrow B - \frac{1}{2}, B \rightarrow A + \frac{1}{2}) = \tilde{V}_{con}^{(1)}(r, A, B)
$$

and

$$
\tilde{V}_{con}^{(2,p)}(r, A, B) = \tilde{V}_{con}^{(1,p)}(r, A, B \rightarrow B - 1)
$$

In other words, $\tilde{V}_{con}^{(1)}(x, A, B)$ has two different SUSY partners. The eigen functions $\tilde{\Psi}_{con,n}^{(1,p)}(r, A, B)$ and $\tilde{\Psi}_{con,n}^{(2,p)}(r, A, B)$ are different from the conventional case and related to them by

$$
\tilde{\Psi}_{con,n}^{(1,p)}(r, A, B) = \tilde{\Psi}_{con,n}^{(1)}(r, A \rightarrow B - \frac{1}{2}, B \rightarrow A + \frac{1}{2}), \quad A > -\frac{1}{2}, \quad B > 0
$$

Fig.4: (b) Normalized ground-state wave functions for $m = 0, 1$ and $2$. 

4.2 Hyperbolic Pöschl-Teller Case

4.2.1 The Conventional Case

Similar to the trigonometric Scarf case, in this case under the parametric transformation $B \leftrightarrow A + \frac{1}{2}$ the scalar potential $\tilde{\phi}_{con}(r, A, B)$ as given by Eq. (53) becomes

$$
\tilde{\phi}_{con}^{(p)}(r, A, B) = \tilde{\phi}_{con}(r, A \rightarrow B - \frac{1}{2}, B \rightarrow A + \frac{1}{2})
$$

$$
= (B - \frac{1}{2}) \coth r - (A + \frac{1}{2}) \coth r; \quad 0 < r < \infty.
$$

Remarkably, under this transformation, the potential $\tilde{V}_{con}^{(1,p)}(r, A, B) = \tilde{V}_{con}^{(1)}(r, A, B)$ remains the same as given by Eq. (54), however the partner potential $\tilde{V}_{con}^{(2,p)}(r, A, B)$ gets changed i.e,

$$
\tilde{V}_{con}^{(1,p)}(r, A, B) = \tilde{V}_{con}^{(1)}(r, A \rightarrow B - \frac{1}{2}, B \rightarrow A + \frac{1}{2}) = \tilde{V}_{con}^{(1)}(r, A, B)
$$

and

$$
\tilde{V}_{con}^{(2,p)}(r, A, B) = \tilde{V}_{con}^{(1,p)}(r, A, B \rightarrow B - 1)
$$

In other words, $\tilde{V}_{con}^{(1)}(x, A, B)$ has two different SUSY partners. The eigen functions $\tilde{\Psi}_{con,n}^{(1,p)}(r, A, B)$ and $\tilde{\Psi}_{con,n}^{(2,p)}(r, A, B)$ are different from the conventional case and related to them by

$$
\tilde{\Psi}_{con,n}^{(1,p)}(r, A, B) = \tilde{\Psi}_{con,n}^{(1)}(r, A \rightarrow B - \frac{1}{2}, B \rightarrow A + \frac{1}{2}), \quad A > -\frac{1}{2}, \quad B > 0
$$
and
\[ \tilde{\Psi}_{\text{con, } n}^{(2,p)}(r, A, B) = \tilde{\Psi}_{\text{con, } n}^{(1,p)}(r, A \to B - 1) \] (88)

The corresponding energy eigenvalue are
\[ \tilde{E}_n^{(1,p)} = \varepsilon^2 = -(B - n - \frac{1}{2})^2, \quad n = 0, 1, 2, \ldots, n_{\text{max}} < B - \frac{1}{2}. \] (89)

4.2.2 The extended Case

(a) The X₁ Case:

In this case, under the transformation \( B \leftrightarrow A + \frac{1}{2} \) the extended scalar potential is given by
\[ \tilde{\phi}_{\text{ext}}^{(p)}(r, A, B) = \tilde{\phi}_{\text{ext}}^{(p)}(r, A \to B - \frac{1}{2}, B \to A + \frac{1}{2}) = \tilde{\phi}_{\text{con}}^{(p)}(r, A, B) + \tilde{\phi}_{\text{rat}}^{(p)}(r, A, B) \] (90)

where \( \tilde{\phi}_{\text{con}}^{(p)}(r, A, B) \) is as given by Eq. (33) while
\[ \tilde{\phi}_{\text{rat}}^{(p)}(r, A, B) = 2 \left( A + \frac{1}{2} \right) \frac{z'(x)}{2(A + \frac{1}{2})z(r) - 2B - 2} \] (91)

The extended potentials under this transformation are completely different and are given by
\[ \tilde{V}_{\text{ext}}^{(1,p)}(r, A, B) = \tilde{V}_{\text{ext}}^{(1)}(r, A \to B - \frac{1}{2}, B \to A + \frac{1}{2}) \] (92)

and
\[ \tilde{V}_{\text{ext}}^{(2,p)}(r, A, B) = \tilde{V}_{\text{ext}}^{(1,p)}(r, A \to B - 1). \] (93)

The associated eigen functions of the Schrödinger equation are
\[ \tilde{\Psi}_{\text{ext, } n}^{(1,p)}(r, A, B) = \tilde{\Psi}_{\text{ext, } n}^{(1)}(r, A \to B - \frac{1}{2}, B \to A + \frac{1}{2}) \] (94)

and
\[ \tilde{\Psi}_{\text{ext, } n}^{(2,p)}(r, A, B) = \tilde{\Psi}_{\text{ext, } n}^{(1,p)}(r, A, B \to B - 1). \] (95)

The energy eigenvalues though are unchanged and are given by Eq. (89).

(b) The Xₘ Case:

For the Xₘ-case, we define
\[ \tilde{\phi}_{\text{m, ext}}^{(p)}(r, A, B) = \tilde{\phi}_{\text{con}}^{(p)}(r, A, B) + \tilde{\phi}_{\text{m, rat}}^{(p)}(r, A, B) \] (96)
where \( \tilde{\phi}(p)_{\text{con}}(r, A, B) \) is again given by Eq. (53) while

\[
\tilde{\phi}(p)_{m, \text{rat}}(r, A, B) = \frac{[2(A + 1) - m]}{2} \left[ \frac{P_m^{(-\eta - 1, \zeta + 1)}(z(r))}{P_m^{(-\eta - 2, \zeta)}(z(r))} - \frac{P_m^{(-\eta - 1, \zeta - 1)}(z(r))}{P_m^{(-\eta - 1, \zeta)}(z(r))} \right],
\]

(97)

here \( \eta = A - B + \frac{1}{2} \) and \( \zeta = -A - B - \frac{1}{2} \). The extended potentials under this transformation are again completely different. The associated eigenfunctions of the Schrödinger equation are

\[
\tilde{\Psi}^{(1,p)}_{\text{ext}, m, n}(r, A, B) = \tilde{\Psi}^{(1)}_{\text{ext}, m, n}(r, A \rightarrow B - \frac{1}{2}, B \rightarrow A + \frac{1}{2}); \quad A + 1 > B > 0
\]

(98)

and

\[
\tilde{\Psi}^{(2,p)}_{\text{ext}, m, n}(r, A, B) = \tilde{\Psi}^{(1,p)}_{\text{ext}, m, n}(r, A, B \rightarrow B - 1).
\]

(99)

The energy eigenvalues are remain unchanged and given by Eq. (89). The plots of the Dirac scalar potential \( \tilde{\phi}(p)_{\text{ext}, m, n}(r, A, B) \) and the normalized ground state eigen functions \( \tilde{\Psi}^{(1,p)}_{\text{ext}, 0, m}(r, A, B) \) for the parameters \( A = \frac{5}{2}, B = \frac{3}{2} \) and \( m = 0, 1, 2 \) are shown in Figs. 5(a) and 5(b) respectively.

**Fig.5:** (a) Rationally extended parametric Dirac scalar potentials for \( m = 0, 1 \) and 2.
5 Summary and Possible Open Problems

In this paper we have obtained exact solutions of the 1 + 1-dimensional Dirac equation for three different extended scalar potentials, i.e. radial oscillator, trigonometric Scarf and hyperbolic Poschl-Teller potentials in terms of exceptional orthogonal polynomials by connecting them to the corresponding Schrödinger problems. Further, using the idea of the parametric symmetry in the case of the trigonometric Scarf and the hyperbolic Pöschl-Teller Dirac scalar potentials we have generated a new class of conventional as well as rational scalar potentials and have obtained their exact solutions in terms of conventional as well as exceptional orthogonal polynomials.

This paper raises few obvious questions. For example, are there other exactly solvable Dirac scalar potentials whose solutions are also in terms of the exceptional orthogonal polynomials. Secondly, are there other Dirac scalar potentials admitting parametric symmetry and if yes can one obtain the solutions of the newly generated Dirac scalar problem?

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References

[1] R.F. Furnstahl, J.J. Rusnak, B.D. Serot, *Nucl. Phys. A* 632 (1998) 607.
[2] J.N. Ginocchio, A. Leviatan, *Phys. Lett. B* 425 (1998) 1.

[3] J.N. Ginocchio, *Phys. Rep.* 414 (2005) 165.

[4] F. Cooper, A. Khare, R. Musto, A. Wipf, *Ann. Phys.* 187 (1988) 1-28.

[5] G. Soff, B. Müller, J. Rafelski, W. Greiner, *Zeitschrift fur Naturforschung a* 28a (1973).

[6] D. F. Lima et al, *The Eur. Phys. Journal. C* 79 (2019) 596.

[7] E. S. Rodrigues, A. F. de Lima, R. de Lima Rodrigues, ”Dirac Equation with vector and scalar potentials via Supersymmetry in Quantum Mechanics”, arXiv:1301.6148v1 [math-ph] (2013).

[8] W. A. Yahya and K. J. Oyewumi, *J. Math. Phys.* 54 (2013) 013508.

[9] A. D. Alhaidari, *Phys. Rev. A* 65 (2002) 042109.

[10] Shi-Hai Dong, *J. Math. Phys.* 44 (2003) 4467.

[11] M.G.Garcia, A. S. de Castro, P. Alberto, L.B.Castro, *Phys. Lett. A* 381 (2017) 2050-2054.

[12] B. Goodman, S. R. Ignjatovic, *Am. J. of Phys.* 65 (1997) 214.

[13] C.S. Jia, Tao Chen, Li-Gong Cui, *Phys. Lett. A* 373 (18-19) (2009) 1621.

[14] B. Roy, R. Roychoudhury, *J. Phys. A: Math. Gen.* 23 (1990) 5095.

[15] Axel Schulze-Halberg and Pinaki Roy, *J. Math. Phys.* 58 (2017) 113507.

[16] F. Cooper, A. Khare, U. Sukhatme, *Phys. Rep.* 251 (1995) 267; *Supersymmetry in Quantum Mechanics, World Scientific* (2001).

[17] A. Khare and U. P. Sukhatme, *J. Phys. A: Math. Gen.* 22 (1989) 2847.

[18] C. V. Sukumar, *J. Phys. A: Math. Gen.* 18 (1985) L 697.

[19] R. J. Hughes, V. A. Kostelecky, and M. M. Nieto, *Phys. Rev. D* 34 (1986) 1100.

[20] A. Hirshfeld, *The Supersymmetric Dirac Equation*, *World Scientific* (2011).

[21] B. Bagchi, and R. Ghosh, *J. Math. Phys.* 62 (2021) 072101.

[22] G. Junker, *Eur. Phys. J. Plus* 135 (2020) 464.

[23] D. Gomez-Ullate, N. Kamran, R. Milson, *J. Math. Anal. Appl.* 359 (2009) 352.

[24] D. Gomez-Ullate, N. Kamran, R. Milson, *J. Phys. A* 43 (2010) 434016.
[25] D. Gomez-Ullate, N. Kamran, R. Milson, *Contemp. Math.* 563 (2012) 51.

[26] C. Quesne, *J.Phys.A* 41 (2008) 392001.

[27] B. Bagchi, C. Quesne, R. Roychoudhary, *Pramana J. Phys.* 73 (2009) 337.

[28] S. Odake, R. Sasaki, *Phys. Lett. B* 684 (2010) 173; ibid 679 (2009) 414. *J. Math. Phys.*, 51 (2010) 053513.

[29] S. Odake, R. Sasaki, *Phys. Lett. B* 702 (2011) 164.

[30] Y. Grandati, *J. Math. Phys.* 52 (2011) 103505.

[31] Y. Grandati, *Ann. Phys.* 326 (2011) 2074.

Y. Grandati, *Ann. Phys.* 327 (2012) 2411.

Y. Grandati, *Ann. Phys.* 327 (2012) 185.

[32] B Midya, B Roy, *J. Phys. A: Math. Theor.* 46 (17) (2013) 175201.

[33] R. K. Yadav, A. Khare, B. P. Mandal, *Ann. Phys.* 331 (2013) 313.

[34] R. K. Yadav, A. Khare, B. P. Mandal, *Phys. Lett. B* 723 (2013) 433.

R. K. Yadav, A. Khare, B. P. Mandal, *Phys. Lett. A* 379 (2015) 67.

[35] R. K. Yadav, B. P. Mandal, A. Khare, *Acta Polytechnica* 57(6) (2017) 477.

[36] R. K. Yadav, N. Kumari, A. Khare, B. P. Mandal, *Ann. Phys.* 359 (2015) 46.

[37] N. Kumari, R. K. Yadav, A. Khare, B. Bagchi, B. P Mandal, *Ann. Phys.* 373 (2016) 163.

[38] A. Ramos et al., *Ann. Phys.* 382 (2017) 143.

[39] R. K. Yadav, A. Khare, B. Bagchi, N. Kumari, B. P. Mandal, *J. Math. Phys.* 57 (2016) 062106.

[40] N. Kumari, R. K. Yadav, A. Khare, B. P. Mandal, *Ann. Phys.* 385 (2017) 57.

N. Kumari, R. K. Yadav, A. Khare, B. P. Mandal, *J. Math. Phys.* 59 (2018) 062103-1.

[41] B. Basu-Mallick, B. P. Mandal, P. Roy, *Ann. Phys.* 380 (2017) 206.

[42] R. K. Yadav, A. Khare, N. Kumari, B. P. Mandal, *Ann. Phys.* 400 (2019) 189.

[43] R.K. Yadav, S. Banerjee, N. Kumari, A. Khare, B.P. Mandal, *Ann. Phys.* 436 (2022) 168679.

[44] C. Quesne, *SIGMA* 8 (2012) 080.

[45] Y. Grandati, C. Quesne, *SIGMA* 11 (2015) 061.
[46] B. Bagchi, C. Quesne, *Phys. Lett. A* **273** (2000) 285.

[47] R. K. Yadav, A. Khare, B. Bagchi, N. Kumari, B. P. Mandal, *Jour. of Math. Phys.* **57** (2016) 062106.

[48] A. Schulze-Halberg, B. Roy, *Ann. Phys.* **349** (2014) 159.

[49] A. Schulze-Halberg, O. Yesiltas, *Eur. Phys. J. Plus.* **26** (2018) 133.

[50] K. Haritha, K.V.S. Shiv Chaitanya, *Pramana J. Phys.* **94** (2020) 102.

[51] R. Jackie, C. Rebbi, *Phys. Rev. D* **13** (1976) 3358.

[52] F. Darabi, S. K. Moayedi, A. R. Ahmadi, *Int. J. Theor. Phys.* **49** (2010) 1232.