Maximal Privacy Without Coherence

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Privacy lies at the fundament of quantum mechanics. A coherently transmitted quantum state is inherently private. Remarkably, coherent quantum communication is not a prerequisite for privacy: there are quantum channels that are too noisy to transmit any quantum information reliably that can nevertheless send private classical information. Here, we ask how much private classical information a channel can transmit if it has little quantum capacity. We present a class of channels \( \mathcal{N}_d \) with input dimension \( d^2 \), quantum capacity \( Q(\mathcal{N}_d) \leq 1 \) and private capacity \( P(\mathcal{N}_d) = \log d \). These channels asymptotically saturate an interesting inequality \( P(\mathcal{N}) \leq \frac{1}{2}(\log d_A + Q(\mathcal{N})) \) for any channel \( \mathcal{N} \) with input dimension \( d_A \), and capture the essence of privacy stripped of the confounding influence of coherence.

Any communication link can be modeled as a (noisy) quantum channel. Given a sender, Alice, and a receiver, Bob, a quantum channel from Alice to Bob is a completely positive trace preserving map from an input space \( A \) to an output space \( B \). The capability of a quantum channel for communication is measured by various capacities, one for each type of information to be transmitted. The classical capacity \( C(\mathcal{N}) \) quantifies the capability of a quantum channel \( \mathcal{N} \) for transmitting classical information, in bits per channel use. The private capacity, \( P(\mathcal{N}) \), gives the maximum rate of private classical communication and quantifies the optimal performance for key exchange. Finally, the quantum capacity \( Q(\mathcal{N}) \), measured in qubits per channel use, establishes the ultimate limit for transmitting quantum information and the performance of quantum error correction.

The three capacities mentioned above clearly satisfy \( Q(\mathcal{N}) \leq P(\mathcal{N}) \leq C(\mathcal{N}) \). The analogy between coherent transmission and privacy, and between entanglement and a private key strongly suggest that \( Q(\mathcal{N}) = P(\mathcal{N}) \). Surprisingly, it was shown in [1] that not only can \( P \) and \( Q \) differ, there are channels too noisy to transmit any quantum information (that is, \( Q(\mathcal{N}) = 0 \)) but that can nevertheless be used to establish privacy (i.e., \( P(\mathcal{N}) > 0 \)). The central idea of [1] concerns private states that by their structure embody perfectly secure classical key, much as maximally entangled pure states embody perfectly coherent correlation.

While [1] draws a qualitative distinction between the private and the quantum capacities, it remains unclear how big the difference can be. If the capacities were always close, then privacy and coherence would still be closely related concepts and the distinction would have little practical relevance. Our contribution is to present a class of channels with \( Q(\mathcal{N}_d) \leq 1 \) and \( P(\mathcal{N}_d) = \log d \), where \( d^2 \) is the input dimension. We further establish that such a separation is tight, by proving an inequality

\[
P(\mathcal{N}) \leq \frac{1}{2}(\log d_A + Q(\mathcal{N})) \tag{1}
\]

for any channel \( \mathcal{N} \) with input dimension \( d_A \), quantifying the effect of incoherence in the channel transmission on privacy: inasmuch as a channel cannot simply transmit quantum information, it must devote half of its input space to acting as a “shield” system (as defined in [1]). While [1] can be inferred from properties of squashed entanglement of quantum states [2,3], this particular form appears to be new. Our relatively simple proof involves very different techniques.

As an aside, our channels combine features of private states from [1] and retro-correctable/random-phase-coupling channels of [4,5] (these channels have large assisted capacities but small \( C \), \( P \), and \( Q \)). In addition to finding a very tight bound on \( Q(\mathcal{N}_d) \), we can also compute both \( P(\mathcal{N}_d) \) and \( C(\mathcal{N}_d) \), a relative rarity in quantum information, especially for a highly nontrivial channel.

Upper Bound on Privacy

Recall that any quantum channel can be expressed as an isometry followed by a partial trace, \( \hat{\mathcal{N}}(\rho) = \text{tr}_E U \rho U^\dagger \), where \( U : A \to BE \) with \( U^\dagger U = I \). The complementary channel acts as \( \hat{\mathcal{N}}^E(\rho) = \text{tr}_B U \rho U^\dagger \), and allows us to define the coherent information of a channel as

\[
Q^{(1)}(\mathcal{N}) = \max_{\phi_A} I_{\text{coh}}(\mathcal{N}, \phi_A) := \max_{\phi_A} [S(B) - S(E)]
\]

where the maximization is taken over input quantum states \( \phi_A \), and \( S(B), S(E) \) are the von Neumann entropies of \( \rho_B = \mathcal{N}(\phi_A) \) and \( \rho_E = \mathcal{N}^E(\phi_A) \) respectively. In turn, the quantum capacity is proved [8,10] to be the regularized coherent information: \( Q(\mathcal{N}) = \lim_{n \to \infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^\otimes n) \). We say that a quantum channel \( \mathcal{N} \) is degradable if \( \hat{\mathcal{N}} = D \circ \mathcal{N} \) for some channel \( D \) [11] (\( \mathcal{N} \) can be degraded to generate \( \hat{\mathcal{N}} \)). For degradable channels, \( P(\mathcal{N}) = Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) \) [12]. Degradable channels also provide a powerful tool for upper bounding the capacities of general channels [13]. If a channel \( \mathcal{N} = \mathcal{L} \circ \mathcal{M} \) is a composition of two channels \( \mathcal{L} \) and \( \mathcal{M} \) with \( \mathcal{M} \) degradable, we have

\[
Q(\mathcal{N}) \leq P(\mathcal{N}) = P(\mathcal{L} \circ \mathcal{M}) \leq P(\mathcal{M}) = Q^{(1)}(\mathcal{M}) \tag{2}
\]
We now have all the tools for proving Eq. (1). For any channel $\mathcal{N}$, define $\mathcal{M}$ as
\[
\mathcal{M}(\rho) = \frac{1}{2} [\rho \otimes |0\rangle\langle 0| + \mathcal{N}(\rho) \otimes |1\rangle\langle 1|].
\] (3)

Then, $\mathcal{N} = \mathcal{L} \circ \mathcal{M}$ where $\mathcal{L}(\sigma) = (\mathcal{N} \otimes \Pi_0 + I \otimes \Pi_1)(\sigma)$ and $\Pi_1(\mu) = \langle i | \mu | i \rangle$. To see that $\mathcal{M}$ is degradable, note that the complementary channel of $\mathcal{M}$ is
\[
\tilde{\mathcal{M}}(\rho) = \frac{1}{2} [\langle e | e \rangle \otimes |0\rangle\langle 0| + \tilde{\mathcal{N}}(\rho) \otimes |1\rangle\langle 1|],
\] (4)

where $|e\rangle\langle e|$ is an orthogonal erasure flag. Choose a degrading map $\mathcal{D}$ that first flips the flag qubit (the second register), and then conditioned on the flag being $|1\rangle$ or $|0\rangle$, applies $\tilde{\mathcal{N}}$ to the first register or resets it to $|e\rangle\langle e|$. So, $\tilde{\mathcal{M}} = \mathcal{D} \circ \mathcal{M}$. Now, applying Eq. (2),
\[
P(\mathcal{N}) \leq Q^{(1)}(\mathcal{M}) = \max_{\phi_A} \left[ S(B_1B_2) - S(E_1E_2) \right] = \max_{\phi_A} \frac{1}{2} \left[ S(\phi_A) + S(\mathcal{N}(\phi_A)) - S(\tilde{\mathcal{N}}(\phi_A)) \right] \leq \frac{1}{2} \left[ \log d_A + Q^{(1)}(\mathcal{N}) \right].
\]

This bound is in fact stronger than Eq. (1), since $Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N})$.

**Channel Construction**

The family of channels $\mathcal{N}_d$ asymptotically saturating Eq. (1) is given by:

\[
\begin{array}{c}
A_1 \\
V \\
A_2
\end{array}
\rightarrow
\begin{array}{c}
P \\
B, \text{"V"} \\
E, \text{"V"}
\end{array}
\]

(5)

The isometric extension of the channel $\mathcal{N}_d$ is given by the operations in the dashed box. $\mathcal{N}_d$ has two input registers $A_1$ and $A_2$, each of dimension $d$. A random unitary $V$ is applied to $A_2$, followed by a controlled phase gate $P = \sum_{i,j} \omega^{ij} |i\rangle\langle i| \otimes |j\rangle\langle j|$ acting on $A_1A_2$, where $\omega$ is a primitive $d$th root of unity. Bob receives only $A_1$ (now relabeled $B$) and “V”, which denotes a classical register with a description of $V$. The $A_2$ register is discarded. The complementary channel has outputs $A_2$ and “V”. More formally, let $W_V = P (I \otimes V)$, $\mathcal{N}_V(\rho) = tr_E W_V \rho W_V^\dagger$, and $\mathcal{N}_d = E_V \mathcal{N}_V \otimes |V\rangle\langle V|_{V_B}$, where the register $V_B$ holds “V”. The isometric extension is given by
\[
U_d(\psi)_{A_1A_2} = \sum_{V} \sqrt{p(V)} (W_V|\psi\rangle_{A_1A_2}) \otimes |V\rangle_{V_B} \otimes |V\rangle_{V_E}
\]

and the complementary channel acts as $\tilde{\mathcal{N}}_d(\rho) = tr_{BV_B} U_d \rho U_d^\dagger = E_V tr_B W_V \rho W_V^\dagger \otimes |V\rangle\langle V|_{V_B}$.

Here is the intuition behind the construction: The classical capacity of this channel is at least $\log d$, since the $d$ computation basis states of $A_1$ are transmitted without error. Furthermore, we will see that inserting a maximally mixed state into $A_2$ keeps the environment ignorant of the transmitted message so $P(\mathcal{N}_d) \geq \log d$. However, as the classical capacity is no greater than the output entropy, and “V” is independent of the input, $C(\mathcal{N}_d) \leq \log d$, so $C(\mathcal{N}_d) = P(\mathcal{N}_d) = \log d$. However, the channel is constructed to suppress the quantum capacity, since without knowing $V$, Alice cannot avoid the $P$ gate from entangling $A_1$ with $A_2$, thereby dephasing $A_1$. We will prove $P(\mathcal{N}_d) \leq 1$.

Our proofs of the above statements hold for any $V$ drawn from a so-called exact unitary $2$-design, and thus, $V$ can be a random Clifford gate (14). In our work to lower bound $Q(\mathcal{N})$, a Haar distributed $V$ is analyzed as a first attempt. We expect a similar conclusion for random Clifford gate $V$.

**Private Capacity**

For an ensemble $\mathcal{E} = \{p_i, \phi_i\}$ and channel $\mathcal{N}$, the private information is defined as
\[
P^{(1)}(\mathcal{N}, \mathcal{E}) = \chi(\mathcal{N}, \mathcal{E}) - \chi(\tilde{\mathcal{N}}, \mathcal{E}),
\] (6)

with Holevo information $\chi(\mathcal{N}, \mathcal{E}) = S(\rho) - \sum_i p_i S(\rho_i)$ evaluated on the induced ensemble $\mathcal{N}(\mathcal{E}) = \{p_i, \rho_i = N(\phi_i)\}$ and average state $\rho = \sum_i p_i \rho_i$ (similarly for $\chi(\tilde{\mathcal{N}}, \mathcal{E})$). For any ensemble $\mathcal{E}$, $P^{(1)}(\mathcal{N}, \mathcal{E})$ is an achievable rate for private communication and thus a lower bound on $P(\mathcal{N})$ (10).

For our channel $\mathcal{N}_d$, choosing probabilities $p_i = \frac{1}{d}$ and states $\phi_i = |i\rangle\langle i|_{A_1} \otimes (\frac{1}{\sqrt{d}})_{A_2}$ for $i = 1, \ldots, d$, gives $\chi(\mathcal{N}_d, \mathcal{E}) = \log d$ and $\chi(\tilde{\mathcal{N}}_d, \mathcal{E}) = 0$, so $P(\mathcal{N}_d) \geq \log d$, as claimed. Readers familiar with (11) will notice that the Choi-state of $\mathcal{N}_d$ with Alice holding $R_1, R_2$ is an exact private state of key system $R_1B$ and shield $R_2$.

**Upper Bound on Quantum Capacity**

To get an upper bound on $Q(\mathcal{N}_d)$, we consider the asymptotic behavior of the coherent information, $Q^{(1)}(\mathcal{N}^{\otimes n}_d)$ for arbitrarily large $n$. We first define suitable notations. We group together the first input $A_1$ from all $n$ channel uses, call it $A^n_1$, and we similarly define $A^n_2, B^n, V^n_B$, and $V^n_E$. We use $x$ to denote an $n$-tuple of integers $(x_1, x_2, \ldots, x_n)$ where each $x_i$ has range $[0, 1, \ldots, d-1]$, and similarly for $y$. Finally, a random $V$ is drawn from each channel use, and we denote the tensor product of $n$ such independent and identically drawn unitaries by $V^n$.

We consider the $n$-shot coherent information $Q^{(1)}(\mathcal{N}^{\otimes n}_d) = \max_{\phi_1 \ldots \phi_n} [S(B^n|V^n_B) - S(E^n|V^n_E)]$. Since Bob and the environment receives the same classical description “V”, $Q^{(1)}(\mathcal{N}^{\otimes n}_d) = \max_{\phi_1 \ldots \phi_n} [S(B^n|V^n_B) - S(E^n|V^n_E)]$. First, we show that the optimal input state has a special form.
Lemma 1. For the channel $\mathcal{N}_d$ of Eq (2), the coherent information $I_{coh}(\mathcal{N}^{\otimes n}_d, \phi_{A_1^\otimes A_2^\otimes})$ is maximized on states of the form
\[
\phi_{A_1^\otimes A_2^\otimes} = \sum_x p_x |x\rangle\langle x|_{A_1^\otimes} \otimes |\varphi^x\rangle\langle \varphi^x|_{A_2^\otimes},
\]
where $x = (x_1, \ldots, x_n)$ and $|x\rangle = \otimes_{i=1}^n |x_i\rangle$ is a standard basis state on $A_1^\otimes$.

Proof. First, we show that the optimal state has the form
\[
\sigma_{A_1^\otimes A_2^\otimes} = \sum_x p_x |x\rangle\langle x|_{A_1^\otimes} \otimes \varphi_{A_2^\otimes},
\]
where $\varphi_{A_2^\otimes}$ is potentially mixed. To see this, let $\psi_{A_1^\otimes A_2^\otimes}$ be an arbitrary input state, and let $\sigma_{A_1^\otimes A_2^\otimes}^\otimes = (\mathcal{P} \otimes I_{A_2^\otimes}) (\psi_{A_1^\otimes A_2^\otimes})$ where $\mathcal{P}(\rho) = \frac{1}{d} \sum_{i=0}^{d-1} Z_i \rho Z_i^\dagger$ is the completely dephasing map. So, $\sigma_{A_1^\otimes A_2^\otimes}^\otimes$ indeed has the form given by Eq. (3).

Now,
\[
I_{coh}(\mathcal{N}^{\otimes n}_d, \sigma_{A_1^\otimes A_2^\otimes}^\otimes) = I_{coh}(\mathcal{N}^{\otimes n}_d \circ (\mathcal{P} \otimes I_{A_2^\otimes}), \psi_{A_1^\otimes A_2^\otimes}) = I_{coh}(\mathcal{P} \otimes I_{A_2^\otimes}, \psi_{A_1^\otimes A_2^\otimes}) \geq I_{coh}(\mathcal{N}^{\otimes n}_d, \psi_{A_1^\otimes A_2^\otimes})
\]
since $\mathcal{P}$ commutes with $\mathcal{N}_d$; and $\mathcal{P}$ is unital, thus the entropy cannot decrease. Meanwhile the reduced state on $E^n V_E^{\otimes}$ remains the same, so the coherent information cannot decrease.

Next, we show that $\varphi_{A_2^\otimes}$ in Eq. (3) can be taken to be pure. Fix an arbitrary $x$. Let $\varphi_{A_2^\otimes}^w = \sum_w q(w|x) |\mu_{w}^x\rangle\langle \mu_{w}^x|$, and for each $w$, let
\[
\eta_{A_1^\otimes A_2^\otimes}^{x,w} = p_x |x\rangle\langle x|_{A_1^\otimes} \otimes |\mu_{w}^x\rangle\langle \mu_{w}^x|_{A_2^\otimes} + \sum_{y \neq w} p_y |y\rangle\langle y|_{A_2^\otimes} \otimes \varphi_{A_2^\otimes}^w.
\]
We now show that
\[
\exists w' s.t. I_{coh}(\mathcal{N}^{\otimes n}_d, \eta_{A_1^\otimes A_2^\otimes}^{x,w'}) \geq I_{coh}(\mathcal{N}^{\otimes n}_d, \sigma_{A_1^\otimes A_2^\otimes}^\otimes).
\]
To see this, note that for each $x$ and $w$, $\mathcal{N}^{\otimes n}_d (\sigma_{A_1^\otimes A_2^\otimes}^\otimes) = \mathcal{N}^{\otimes n}_d (\eta_{A_1^\otimes A_2^\otimes}^{x,w'})$, so those states have the same entropy. For the complementary channel, observe that by construction,
\[
\sigma_{A_1^\otimes A_2^\otimes}^\otimes = \sum_w q(w|x) \eta_{A_1^\otimes A_2^\otimes}^{x,w} \otimes |\mu_{w}^x\rangle\langle \mu_{w}^x|
\]
so $\mathcal{N}^{\otimes n}_d (\sigma_{A_1^\otimes A_2^\otimes}) = \sum_w q(w|x) \mathcal{N}^{\otimes n}_d (\eta_{A_1^\otimes A_2^\otimes}^{x,w})$, and by concavity of entropy,
\[
S(E^n V_E^{\otimes}) (\sigma_{A_1^\otimes A_2^\otimes}) \geq \sum_w q(w|x) S(E^n V_E^{\otimes}) (\eta_{A_1^\otimes A_2^\otimes}^{x,w})
\]
so Eq. (9) holds. Iterating this process gives an optimal state of the form given by Eq. (7).
We thus have
\[
  \mathbb{E}_V \text{tr} \left( \rho_{E_n}^V \right)^2 \leq \sum_{w=0}^{n} \frac{1}{(d-1)^w} \sum_{|x|=w} \sum_{x} p_x^2.
\]
\[
  = \sum_{w=0}^{n} \frac{1}{(d-1)^w} \left( \binom{n}{w} (d-1)^w \sum_{x} p_x^2 = 2^n \sum_{x} p_x^2 \right).
\]
To prove the second statement Eq. (13), consider the case that we input \( m \) copies of the state given by Eq. (7) into \( mn \) copies of the channel \( \mathcal{N}_d \). Conditioning on the random unitaries of the channels, this results in an output state \( \rho_{E_{nm}}^V = \otimes_{i=1}^m \rho_{E_n}^V \) on the environment. The state \( \rho_{E_{nm}}^V \) involves a mixture over \( (X)^m \), which is \( m \) i.i.d. copies of \( X \). Let \( \tilde{X}_m \) be the restriction of \( (X)^m \) to its typical set \( \{x^m : -\frac{1}{m} \log p_{x^m} \leq S(X) \} \). Then the state \( \rho_{E_{nm}}^V \) can be well approximated by \( \rho_{E_{nm}}^V \), where the mixture is only taken over \( \tilde{X}_m \). Specifically, using the properties of typical sets (cf. [13], for example), we can easily check that for \( m \) big enough \( \| \rho_{E_{nm}}^V - \rho_{E_{nm}}^V \|_1 \leq 2\epsilon \). Thus we have
\[
m \mathbb{E}_V S(\rho_{E_{nm}}^V) = \mathbb{E}_V S(\rho_{E_{nm}}^V) \geq \epsilon m n \log d - H((\epsilon, 1-\epsilon)),
\]
where for the inequality we have used the strengthened Fannes inequality [10]. Furthermore, Eq. (12) applies to \( \rho_{E_{nm}}^V \), so that
\[
  \mathbb{E}_V S(\rho_{E_{nm}}^V)^2 \leq 2mn \sum_{x^m \in \tilde{X}_m} \left( \sum_{x^m \in \tilde{X}_m} p_{x^m} \right)^2 \sum_{x^m \in \tilde{X}_m} p_{x^m}^2.
\]
By convexity of \( -\log \), Eq. (15) translates to
\[
  \mathbb{E}_V S(\rho_{E_{nm}}^V)^2 \geq S_2(\tilde{X}_m) - mn,
\]
where the Rényi-2 entropy \( S_2 \) is defined as \( S_2(\rho) := -\log \text{tr} \rho^2 \). Since \( \tilde{X}_m \) only has typical sequences, \( S_2(\tilde{X}_m) \approx m S(X) \). Here we only need an inequality, specifically, for \( m \) sufficiently large
\[
  S_2(\tilde{X}_m) \geq m S(X) - 3m \epsilon + 2 \log(1-\epsilon),
\]
which can be easily confirmed using the properties of typical sets. Finally, connecting Eqs. (14) and (16) by the well-known relation between Rényi entropies, \( S \geq S_2 \), and also using Eq. (17), we have
\[
  \mathbb{E}_V S(\rho_{E_{nm}}^V) \geq S(X) - n - \epsilon n \log d - 3\epsilon 
  - \frac{1}{m} H((\epsilon, 1-\epsilon)) + \frac{2}{m} \log(1-\epsilon).
\]
Taking \( \epsilon \to 0 \) gives the desired result.

Together, \( Q(1) (\mathcal{N}_d^m) \leq n \), so, \( Q(\mathcal{N}_d) \leq 1 \).

When proving the upper bound on \( Q \), we cannot assume apriori the entropy of \( B_n^m \) is maximal for the optimal input, ruling out the simpler path to show that the entropy of \( E_n^m \) is maximal. Instead we have to show that \( S(B_n^m) - S(E_n^m) \) is small for all distributions. Perhaps our technique has other applications. Also Lemma 3 effectively converts a statement concerning the Rényi-2 entropy into an analogue for the entropy for a large family of states, which may be of interest elsewhere.

**Achievable Quantum Rate**

We have shown that \( Q(\mathcal{N}_d) \leq 1 \), but could it actually be 0? It turns out not: by choosing a specific input state and evaluating the associated coherent information, we can obtain the explicit lower bound \( Q(\mathcal{N}_d) \geq (1-\gamma) \log_2 e \approx 0.61 \) as \( d \to \infty \), where \( \gamma = \lim_{d \to \infty} \sum_{k=1}^{d} \frac{1}{k} - \ln(d) \) is the Euler-Mascheroni constant. In the appendix, we derive this lower bound of the coherent information by considering the input \( \phi_{A_1 A_2} = (\frac{1}{2})_A \otimes |0\rangle \langle 0|_B \).

**Discussion**

In [1] it was shown that privacy and distillable entanglement can be different, indeed privacy can be nonzero even for bound-entangled states. What we have shown is similar, but somewhat incomparable. Our result is stronger in that the separation is maximal, saturating Eq. (1), but it only applies to the channel case, implicitly not allowing classical communication. The two-way assisted quantum capacity \( Q(\mathcal{N}_d) \) is maximal (not zero!) and equal to the private capacity \( \log d \). An open question is how big the separation can be in the two-way setting?

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Appendix A: Low quantum capacity

Lemma A.1. Define, on $A^n_2$, the state
\[ \rho_{E^n} = Z^x V |\varphi^x\rangle \langle \varphi^x | V^\dagger Z^{-x}. \] (A1)

Then,
\[ \mathbb{E}_V \text{tr} \rho_{E^n}^{XV} \rho_{E^n}^{YV} \leq \frac{1}{(d-1)^{d_H(x,y)}}, \] (A2)

where $d_H(x,y) = |\{i | x_i \neq y_i\}|$ is the Hamming distance between $x$ and $y$.

Proof. First note that
\[ \mathbb{E}_V \text{tr} \rho_{E^n}^{XV} \rho_{E^n}^{YV} \]
\[ = \mathbb{E}_V \text{tr} [Z^x V |\varphi^x\rangle \langle \varphi^x | V^\dagger Z^{-x} Z^x V |\varphi^x\rangle \langle \varphi^x | V^\dagger Z^{-y} Y^\dagger Z^y V |\varphi^y\rangle \langle \varphi^y | ] \]
\[ = \mathbb{E}_V \langle \varphi^x | V^\dagger Z^{-y} Z^x V |\varphi^x\rangle \langle \varphi^x | V^\dagger Z^{-y} Z^x V |\varphi^y\rangle \]
\[ = \mathbb{E}_V \langle \varphi^x | V^\dagger Z^{-y} Z^x V |\varphi^x\rangle \langle \varphi^x | ] \]
\[ = \langle \varphi^x | (S^{x-y} |\varphi^x\rangle |\varphi^x\rangle \rangle, \] (A3)

where
\[ S^{x-y} = \mathbb{E}_V [V^\dagger \otimes V^\dagger (Z^{x-y} \otimes Z^{y-x}) V \otimes V] . \] (A4)

We now evaluate $S^{x-y}$ which equals $\otimes_{i=1}^n S_{x_i - y_i}$ for
\[ S_{x_i - y_i} = E_{V^\dagger} \otimes V^\dagger (Z^{x_i - y_i} \otimes Z^{y_i - x_i}) V \otimes V . \]

If $x_i - y_i = 0$, then $S_0 = I$. If $x_i - y_i = a \neq 0$, then,
\[ S^a = E_{V^\dagger} V^\dagger \otimes V^\dagger (Z^a \otimes Z^{-a}) V \otimes V . \]

\[ = \text{tr} [\Pi_{\text{sym}} Z^a \otimes Z^{-a}] \frac{\Pi_{\text{sym}}}{d_{\text{sym}}} + \text{tr} [\Pi_{\text{anti}} Z^a \otimes Z^{-a}] \frac{\Pi_{\text{anti}}}{d_{\text{anti}}} \]

where $\Pi_{\text{sym}}$ and $\Pi_{\text{anti}}$ are the projectors onto the symmetric and the antisymmetric subspaces. More specifically, let $P = \sum_{i,j} |i\rangle \langle j| (j\langle i|$ be the swap operator.

Then,
\[ \Pi_{\text{sym}} = \frac{1}{2} (I + F) \] (A5)

\[ \Pi_{\text{anti}} = \frac{1}{2} (I - F) . \] (A6)

Note that $\text{tr} [Z^a \otimes Z^{-a}] = 0$, and
\[ \text{tr} [F(Z^a \otimes Z^{-a})] = \sum_{i,j} \omega^{a_i - a_j} \text{tr} [\{i\langle j| \otimes |j\langle i|$ = $\sum_1 \gamma = d . \]

So, $\text{tr} [\Pi_{\text{sym}} Z^a \otimes Z^{-a}] = \frac{d}{2}$, $\text{tr} [\Pi_{\text{anti}} Z^a \otimes Z^{-a}] = -\frac{d}{2}$.

Consequently,
\[ S^a = \frac{d}{2} \left[ \frac{\Pi_{\text{sym}}}{d_{\text{sym}}} - \frac{\Pi_{\text{anti}}}{d_{\text{anti}}} \right] = \frac{\Pi_{\text{sym}}}{d + 1} - \frac{\Pi_{\text{anti}}}{d - 1} . \] (A7)

With Eqs. (A3) and (A7), we can finish the proof in two steps. From Eq. (A7), $\|S^a\|_{\infty} = \frac{1}{d-1}$ and $\|S^{x-y}\|_{\infty} = (d-1)^{-d_H(x,y)}$.

Furthermore, for any hermitian operator $H$ and unit vectors $|\psi\rangle$ and $|\phi\rangle$, $|\langle \psi | H | \phi \rangle| \leq \|H\|_{\infty}$ (see Lemma A.2). Applying these to Eq. (A3) gives
\[ \mathbb{E}_V \text{tr} \rho_{E^n}^{XV} \rho_{E^n}^{YV} \leq \|S^{x-y}\|_{\infty} = \frac{1}{(d-1)^{d_H(x,y)}} . \]

Appendix B: Average entropy of dephased random states

The following lemma lets us evaluate the entropy of the environment when the input state is $\phi_{A_1 A_2} = (\frac{1}{2})_{A_1} \otimes |0\rangle \langle 0|_{A_2}$.

Lemma B.1. Let $|\varphi\rangle$ be a normalized quantum state drawn randomly from the Hilbert space $\mathbb{C}^d$, according to the unitarity-invariant probability measure, i.e., $|\varphi\rangle = \cos \theta |\psi\rangle $.
Let $U$ be a Haar-random unitary operator. Let $\mathcal{P}$ be the completely dephasing quantum operation, namely, $\mathcal{P}(\rho) := \sum_{k=0}^{d-1} |k\rangle \langle k| |k\rangle \langle k|$, we have

$$E_\varphi S(\mathcal{P}(|\varphi\rangle \langle \varphi|)) = (\log \epsilon)(H_d - 1),$$

where $H_d := \sum_{k=1}^d \frac{1}{k}$ is the $d$th harmonic number.

Proof. Write $|\varphi\rangle = \sum_{k=0}^{d-1} (x_k + iy_k)|k\rangle$, with $\vec{v} := (x_0, x_1, \ldots, x_{d-1}, y_0, y_1, \ldots, y_{d-1})^T \in \mathbb{R}^{2d}$ being a vector on the $(2d - 1)$-dimensional unit sphere $S(2d-1)$. For any unitary operator $V = A + iB$ acting on $\mathbb{C}^d$, with $A$ and $B$ being the real part and imaginary part, respectively, it is straightforward to check that

$$V|\varphi(\vec{v})\rangle = |\varphi(O\vec{v})\rangle,$$

where

$$O = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$$

is an orthogonal operator acting on $\mathbb{R}^{2d}$. Thus, letting $\mu$ be the unitarily-invariant probability measure on the set of normalized pure states, we find that, on the parameter set $\{\vec{v} | \vec{v} \in S(2d-1)\}$, it translates to the orthogonally-invariant measure, which in turn is proportional to the Euclidean volume of the corresponding portion on $S(2d-1)$. Denote the volume of $S^{(t)}$ as $S^{(t)}$, and the volume elements on $S^{(t)}$ as $dS^{(t)}$. As a result of the above argument we have

$$\mu(d\varphi) \equiv \mu(d\vec{v}) = \frac{1}{S(2d-1)} dS^{(2d-1)}. \quad (B1)$$

Note $S(\mathcal{P}(|\varphi\rangle \langle \varphi|)) = - \sum_{k=0}^{d-1} (|k\rangle \langle k|)^2 \log (|k\rangle \langle k|)^2$. By symmetry and Eq. (B1) we have

$$E_\varphi S(\mathcal{P}(|\varphi\rangle \langle \varphi|)) = -d \frac{1}{S(2d-1)} \int (x_0^2 + y_0^2) \log (x_0^2 + y_0^2) dS^{(2d-1)}.$$ 

Let $\cos \theta = \sqrt{x_0^2 + y_0^2}$, with $\theta \in [0, \frac{\pi}{2}]$. Then

$$\sin \theta = \sqrt{\sum_{k=1}^{d-1} (x_k^2 + y_k^2)}.$$ 

Fixing $\theta$, the changing of $(x_0, y_0)$ forms a circle (one-dimensional sphere) of radius $\cos \theta$, and we denote it as $S_1$; the changing of $(x_1, x_2, \ldots, x_{d-1}, y_1, y_2, \ldots, y_{d-1})$ forms a $(2d - 3)$-dimensional sphere $S_2$ of radius $\sin \theta$. The volume elements of these two spheres are thus, respectively,

$$dS_1 = \cos \theta \ dS^{(1)}, \quad dS_2 = \sin^{2d-3} \theta \ dS^{(2d-3)}.$$ 

On the other hand, fixing the other spherical coordinates of $S_1$ and $S_2$, the changing of $\theta$ forms a quarter of a unit circle, $S_3 := \{(\cos \theta, \sin \theta)|0 \leq \theta \leq \frac{\pi}{2}\}$. The volume element of $S_3$ is obviously

$$dS_3 = d\theta.$$ 

We further observe that $dS_1$, $dS_2$ and $dS_3$ are mutually orthogonal on the big sphere $S(2d-1)$, because $dS_1$ and $dS_2$ fall in two distinct orthogonal subspaces, respectively, and $dS_3$ falls in the radial directions of both $S_1$ and $S_2$. Hence we have

$$dS^{(2d-1)} = dS_1 dS_2 dS_3 \quad (B3)$$

$$= \cos \theta \ sin^{2d-3} \theta \ d\theta \ dS^{(1)} \ dS^{(2d-3)}.$$ 

Using Eq. (B3), and substituting $\cos^2 \theta$ for $x_0^2 + y_0^2$, we find that the right hand side of Eq. (B2) equals

$$-dS^{(1)} \int_{\theta = 0}^{\frac{\pi}{2}} \cos^3 \theta \ sin^{2d-3} \theta \ \log (\cos^2 \theta) \ d\theta.$$ 

The volume of the $(\ell - 1)$-dimensional sphere is given by $\Gamma(\ell - 1) = 2 \pi^{\ell/2} / \Gamma(\frac{\ell}{2})$. We are only concerned with even $\ell$, thus $\Gamma(\frac{\ell}{2}) = (\frac{\ell}{2} - 1)!$ and $S^{(1)} S^{(2d-3)} / S(2d-1) = 2 (d-1)$. We also perform the change of variable $t = \cos^2 \theta$, so Eq. (B4) is equal to

$$- (\log \epsilon) d(d-1) \int_{0}^{1} t (1 - t)^{d-2} \log t \ dt.$$ 

Observing that a primitive of $t (1 - t)^{d-2}$ is

$$\frac{1}{d(d-1)} \sum_{k=0}^{d-1} (1 - t)^k - \frac{1}{d-1} t (1 - t)^{d-1}$$

and the derivative of $\log t$ is $\frac{1}{t}$, and integrating by parts, we eventually obtain

$$E_\varphi S(\mathcal{P}(|\varphi\rangle \langle \varphi|)) = (\log \epsilon) \int_{0}^{1} \left( \sum_{k=0}^{d-1} (1 - t)^k - d(1 - t)^{d-1} \right) dt$$

$$= (\log \epsilon) \sum_{k=2}^{d} \frac{1}{k},$$

concluding the proof.