ON THE SPHERICAL SLICE TRANSFORM

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ABSTRACT. We study the spherical slice transform $\mathcal{S}$ which assigns to a function $f$ on the unit sphere $S^n$ in $\mathbb{R}^{n+1}$ the integrals of $f$ over cross-sections of $S^n$ by $k$-dimensional affine planes passing through the north pole $(0,\ldots,0,1)$. These transforms are known when $k = n$. We consider all $2 \leq k \leq n$ and obtain an explicit formula connecting $\mathcal{S}$ with the classical $(k-1)$-plane Radon-John transform $R_{k-1}$ on $\mathbb{R}^n$. Using this connection, known facts for $R_{k-1}$, like inversion formulas, support theorems, representation on zonal functions, and some others, are reformulated for $\mathcal{S}$.

1. Introduction

Let $S^n$ be the unit sphere in $\mathbb{R}^{n+1}, n \geq 2$. Given a point $a \in \mathbb{R}^{n+1}$ and an integer $k, 2 \leq k \leq n$, let $\mathcal{T}_a$ be the manifold of all $k$-dimensional affine planes $\tau$ in $\mathbb{R}^{n+1}$ passing through $a$ and intersecting $S^n$. The spherical slice transform assigns to a function $f$ on $S^n$ the collection of integrals of $f$ over cross-sections $\tau \cap S^n$. Specifically, we set

$$(\mathcal{S}_a f)(\tau \cap S^n) = \int_{\tau \cap S^n} f(\eta) \, dS_\tau(\eta), \quad \tau \in \mathcal{T}_a, \quad (1.1)$$

where integration is performed with respect to the standard surface area measure on the subsphere $\tau \cap S^n$.

Depending on the configuration of cutting planes, one can distinguish different kinds of spherical slice transforms. The classical case when $a = o$, the origin of $\mathbb{R}^{n+1}$, has more than one hundred years’ history. Such operators are known as the Funk transforms (or Minkowski-Funk transforms, or spherical Radon transforms) and play an important role in different branches of analysis, geometry, and tomography; see, e.g., [6–8, 10–12, 16–18, 24].

The study of the operators $\mathcal{S}_a$ with $a \neq o$, arising in spherical tomography and spherical integral geometry is quite recent; see, e.g., [11]. If $a$ lies strictly inside or outside of $S^n$, the operators (1.1) were studied

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in a series of papers by Agranovsky and Rubin [2–4, 26], Quellmalz [19, 20], Salman [28, 29].

It turns out that the cases \( a \in S^n \) and \( a \notin S^n \) are essentially different. Specifically, in the second case, \( S_a \) is non-injective (i.e., has a non-trivial kernel) on standard function spaces, like \( C(S^n) \) or \( L^p(S^n) \), while in the first case the injectivity is available. The reason for this striking phenomenon is that if \( a \notin S^n \), then \( S_a \) is equivalent to the classical Funk transform which annihilates odd functions; see [2–4, 26] for details. On the contrary, if \( a \in S^n \), then \( S_a \) inherits properties of the Radon-John transform over affine planes (see (1.2) below), and therefore it is injective.

In the present paper, we focus on the case \( a \in S^n \). Without loss of generality, we assume that \( a \) is the north pole \( N = (0, \ldots, 0, 1) \) and write \( S \) in place of \( S_a \).

The transformation \( S \) for \( n = 2 \) was introduced by Abouelaz and Daher [1] who obtained inversion formulas for \( Sf \), provided that \( f \) is continuous, vanishes identically in a neighborhood of \( N \), and zonal (i.e. invariant under rotations about the last coordinate axis). In this case, \( Sf \) is represented by a one-dimensional Abel-type integral that can be explicitly inverted using the tools of fractional differentiation. A general (not necessarily zonal) case for \( n = 2 \) was studied by Helgason [12, p. 145] who proved the support theorem and injectivity of \( S \) on smooth functions vanishing at \( N \) with all derivatives. These results were extended by the author [24, Section 7.2] to all \( k = n \geq 2 \) under minimal assumptions for functions in Lebesgue spaces.

The term spherical slice transform was introduced in [12, p. 145] specifically for \( Sf \). However, it can be equally adopted to all Radon-like transforms over plane sections of the sphere. These include operators \( S_a \) with any \( a \in \mathbb{R}^{n+1} \), the limiting case \( a = \infty \) when all cross-sections are parallel [4, 13, 27], the spherical section transforms over geodesic subspheres of \( S^n \) [21], and many others. The list of references at the end of the article may be helpful to the reader who is willing to try his/her hand at some of such intriguing transforms.

In the present paper, our main concern is the afore-mentioned operator \( S \) for arbitrary \( 2 \leq k \leq n \). Using the stereographic projection, it will be shown that \( S \) inherits injectivity and many other properties of the classical Radon-John transform

\[
(R_{k-1}g)(\zeta) = \int_{\zeta} g(x) \, d_\zeta x, \quad \zeta \in G(n, k-1),
\]

(1.2)
where $G(n, k-1)$ is the manifold of all $(k-1)$-dimensional affine planes in $\mathbb{R}^n$ and $d_\zeta x$ stands for the standard Euclidean measure on $\zeta$. Our aim is to make the connection between $\mathcal{S}$ and $R_{k-1}g$ precise and transfer known results for $R_{k-1}g$ to $\mathcal{S}f$.

Most of the results of the paper were obtained several years ago, but remained unpublished. Later developments in [2–4], related to $\mathcal{S}_a f$ with $a \notin S^n$, inspired me to make my notes available to a wide audience.

Section 2 contains Preliminaries. In Section 3 we prove the main theorem (see Theorem 3.1) establishing a connection between $\mathcal{S}f$ and $R_{k-1}g$. Section 4 contains corollaries of Theorem 3.1. It includes sharp existence conditions for $\mathcal{S}f$ in integral terms (Proposition 4.1), representation on zonal functions by Abel-type integrals (Proposition 4.3), support theorem (Theorem 4.4), and inversion formulas (Subsection 4.4).

For convenience of the reader, we collected some auxiliary facts about Radon-John transforms in the Appendix. Here the Support Theorem 5.2 is new. It contains a slight, but important, improvement of the remarkable discovery by Kurusa [14, Theorem 3.1], establishing essential difference between the cases $k = n - 1$ and $k < n - 1$. It turns out that in the case $k < n - 1$, the well-known rapid decrease condition, as, e.g., in [12, p. 33], can be eliminated. Some results from [14] were generalized in [15].

2. Preliminaries

2.1. Notation. In the following, $e_1, \ldots, e_{n+1}$ are the coordinate unit vectors,

$N = (0, \ldots, 0, 1) = e_{n+1}, \quad \mathbb{R}^n = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_n, \quad S^{n-1} = S^n \cap \mathbb{R}^n$,

$\mathcal{T}$ is the manifold of all $k$-dimensional affine planes $\tau$ in $\mathbb{R}^{n+1}$ passing through $N$ and intersecting $\mathbb{R}^n$. Abusing notation, we write $|\tau|$ for the Euclidean distance from the origin to $\tau$; $dS(\cdot)$ (sometimes with a subscript) stands for the surface area measure on the surface under consideration. A function $f$ on $S^n$ is called zonal if it is invariant under rotations about the last coordinate axis.

2.2. Stereographic Projection. Consider the bijective mapping

$$\mathbb{R}^n \ni x \xrightarrow{\nu} \eta \in S^n \setminus \{N\}, \quad \nu(x) = \frac{2x + (|x|^2 - 1)e_{n+1}}{|x|^2 + 1}. \quad (2.1)$$

The inverse mapping $\nu^{-1} : S^n \setminus \{N\} \to \mathbb{R}^n$ is the stereographic projection from the north pole $N$ onto $\mathbb{R}^n$. 

The following lemma is well known; see, e.g., [24, Lemma 7.6]. We present it here for the sake of completeness and further references.

Lemma 2.1.
(i) If \( f \in L^1(S^n) \), then
\[
\int_{S^n} f(\eta) d\eta = 2^n \int_{\mathbb{R}^n} (f \circ \nu)(x) \frac{dx}{(|x|^2 + 1)^n}.
\] (2.2)

(ii) If \( g \in L^1(\mathbb{R}^n) \), then
\[
\int_{\mathbb{R}^n} g(x) dx = \int_{S^n} (g \circ \nu^{-1})(\eta) \frac{d\eta}{(1 - \eta_{n+1})^n}.
\] (2.3)

Proof. (i) Passing to spherical coordinates, we have
\[
l.h.s. = \int_0^{\pi} \left( \sin^{n-1} \varphi \, d\varphi \right) \int_{S^{n-1}} f(\omega \sin \varphi + e_{n+1} \cos \varphi) d\omega
= 2^n \int_0^{\infty} s^{n-1} ds \int_{S^{n-1}} f \left( \frac{2s\omega + (s^2 - 1)e_{n+1}}{s^2 + 1} \right) d\omega = r.h.s.
\]

(ii) We set \( g(x) = 2^n(|x|^2 + 1)^{-n}(f \circ \nu)(x) \) in (2.2). Since
\[
|x|^2 + 1 = s^2 + 1 = \cot^2(\varphi/2) + 1 = \frac{2}{1 - \cos \varphi} = \frac{2}{1 - \eta_{n+1}},
\] (2.4)
the result follows. \( \square \)
3. Connection with the Radon-John Transform

**Theorem 3.1.** Let $2 \leq k \leq n$, $\tau \in \mathcal{T}$. Then

$$\mathcal{S}(f)(\tau \cap S^n) = (R_{k-1}g)(\tau \cap \mathbb{R}^n), \quad g(x) = \frac{2^{k-1}(f \circ \nu)(x)}{(|x|^2 + 1)^{k-1}}, \quad (3.1)$$

provided that either side of the first equality in (3.1) exists in the Lebesgue sense.

**Proof.** STEP I. Every $k$-plane $\tau \in \mathcal{T}$ is uniquely represented as

$$\tau \equiv \tau(\tau_0, u) = \tau_0 + u, \quad \tau_0 \in G_{n+1,k}, \quad u \in \tau_0^\perp,$$

where $G_{n+1,k}$ is the Grassmann manifold of $k$-dimensional linear subspaces of $\mathbb{R}^{n+1}$ and $\tau_0^\perp$ is the orthogonal complement of $\tau_0$. Let $\psi$ be the angle between $u$ and $e_{n+1}$. Then

$$u = (\theta \sin \psi + e_{n+1} \cos \psi) \cos \psi, \quad \theta \in S^{n-1} \subset \mathbb{R}^n, \quad |u| = \cos \psi. \quad (3.2)$$

We also set

$$r = \sin \psi, \quad t = \cot \psi.$$

If $\zeta = \tau \cap \mathbb{R}^n$ and $G_{n,k-1}$ is the Grassmann manifold of $(k-1)$-dimensional linear subspaces of $\mathbb{R}^n$, then

$$\zeta = \zeta_0 + v, \quad \zeta_0 \in G_{n,k-1}, \quad v \in \zeta_0^\perp \cap \mathbb{R}^n, \quad (3.3)$$

where $\zeta_0^\perp$ is the orthogonal complement of $\zeta_0$ in $\mathbb{R}^{n+1}$. By (3.2),

$$|v| = \frac{|u|}{\sin \psi} = \frac{|u|}{\sqrt{1 - |u|^2}} = \cot \psi = t. \quad (3.4)$$

Let

$$\mathbb{R}_{k-1} = \mathbb{R}e_{n-k+1} \oplus \cdots \oplus \mathbb{R}e_{n-1}$$

be the coordinate $(k-1)$-subspace and choose a rotation

$$\alpha = \begin{bmatrix} \alpha_0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \alpha_0 \in SO(n),$$

satisfying $\alpha_0(\mathbb{R}_{k-1}) = \zeta_0$, $\alpha_0(e_n) = \theta$; cf. (3.2). Then $\tau = \alpha \tau'$ for some $\tau' \in \mathcal{T}$ satisfying

$$\tau' = \tau_0' + u', \quad \tau_0' \in G_{n+1,k}, \quad u' \in (\tau_0')^\perp, \quad \tau_0 = \alpha \tau_0', \quad (3.5)$$

$u = \alpha u'$. Similarly, for $\zeta = \tau \cap \mathbb{R}^n$ we have

$$\zeta = \alpha \zeta', \quad \zeta' = \tau' \cap \mathbb{R}^n = \mathbb{R}_{k-1}^c + te_n.$$

Clearly,

$$u' = (e_n \sin \psi + e_{n+1} \cos \psi) \cos \psi; \quad (3.6)$$

cf. (3.2). Let

$$\gamma = \tau \cap S^n, \quad \gamma' = \tau' \cap S^n.$$
be the corresponding spherical cross-sections. In this notation, $\mathcal{S}f$ reads
\[
(\mathcal{S}f)(\alpha \gamma') = \int_{\gamma'} (f \circ \alpha)(\eta') \, dS_{\gamma'}(\eta').
\] (3.7)

STEP II. Let $P_{\tau'_0} \gamma'$ be the orthogonal projection of $\gamma'$ onto the subspace $\tau'_0 \subset G_{n+1,k}$, as in (3.5). We rotate $\tau'_0$ about the subspace $\tilde{R}^{k-1} = \mathbb{R}e_{n-k+1} \oplus \cdots \oplus \mathbb{R}e_{n-1}$, making $\tau'_0$ coincident with the $k$-dimensional coordinate plane $\tilde{R}^k = \tilde{R}^{k-1} \oplus \mathbb{R}e_{n+1}$. This gives
\[
P_{\tau'_0} \gamma' \equiv \gamma' - u' = \rho \tilde{\gamma},
\]
where $u'$ has the form (3.6), $\tilde{\gamma}$ is the $(k - 1)$-sphere of radius $r = \sin \psi$ in the “vertical” $k$-subspace $\tilde{R}^k$, and the rotation $\rho$ has the form
\[
\rho = \begin{bmatrix}
I_{n-1} & 0 \\
0 & \rho_{\psi}
\end{bmatrix}, \quad \rho_{\psi} = \begin{bmatrix}
\sin \psi & -\cos \psi \\
\cos \psi & \sin \psi
\end{bmatrix}.
\]

Setting
\[
S^{k-1} = S^n \cap \tilde{R}^k
\]
(cf. Figure 2 for $k = n$), from (3.7) we obtain
\[
(\mathcal{S}f)(\alpha \gamma') = (\mathcal{S}f)(\alpha [u' + \rho \tilde{\gamma}]) = r^{k-1} \int_{S^{k-1}} (f \circ \alpha)(u' + r \rho \sigma) \, dS(\sigma).
\] (3.8)

![Figure 2](image)

**Figure 2.** $\gamma = o' + \rho \tilde{\gamma}$, $o' = u'$, $r = \sin \psi$.

Denote $f_\alpha(\cdot) = f(\alpha(\cdot))$. By Lemma 2.1 (i), with $n$ replaced by $k - 1$, we continue:
\[
(\mathcal{S}f)(\alpha \gamma') = (2r)^{k-1} \int_{\tilde{R}^{k-1}} f_\alpha(u' + r \rho \nu(y)) \frac{dy}{(|y|^2 + 1)^{k-1}}.
\] (3.8)
The argument of $f_\alpha$ can be transformed as follows:
\[
u' + r \rho \nu(y) = u' + \begin{bmatrix} I_{n-1} & 0 \\ 0 & \rho \psi \end{bmatrix} \frac{2y + (|y|^2 - 1) e_{n+1}}{|y|^2 + 1} \sin \psi \\
= (e_n \sin \psi + e_{n+1} \cos \psi) \cos \psi \\
+ \frac{\sin \psi}{|y|^2 + 1} \left[2y - e_n(|y|^2 - 1) \cos \psi + e_{n+1}(|y|^2 - 1) \sin \psi \right] = \frac{A}{|y|^2 + 1},
\]
where simple calculation yields
\[A = 2y \sin \psi + e_n \sin 2\psi + e_{n+1}(|y|^2 + \cos 2\psi).
\]
Hence,
\[(\mathcal{S}f)(\gamma) = (2 \sin \psi)^{k-1} \times \int_{\tilde{\mathbb{R}}^{k-1}} f_\alpha \left( \frac{2y \sin \psi + e_n \sin 2\psi + e_{n+1}(|y|^2 + \cos 2\psi)}{|y|^2 + 1} \right) \frac{dy}{(|y|^2 + 1)^{k-1}}.
\]
Changing variable $y = z \sin \psi$ and setting $t = \cot \psi$, we write this expression as
\[2^{k-1} \int_{\tilde{\mathbb{R}}^{k-1}} f_\alpha \left( \frac{2(z+t e_n) + (|z+t e_n|^2 - 1) e_{n+1}}{|z+t e_n|^2 + 1} \right) \frac{dz}{(|z+t e_n|^2 + 1)^{k-1}}
\]
\[= \int_{\tilde{\mathbb{R}}^{k-1}} g(\alpha(z + t e_n)) \, dz, \quad g(x) = \frac{2^{k-1} (f \circ \nu)(x)}{(|x|^2 + 1)^{k-1}}.
\]
This is exactly the Radon-John transform
\[(R_{k-1}g)(\alpha \tilde{\mathbb{R}}^{k-1} + t e_n) = (R_{k-1}g)(\zeta_0 + v) = (R_{k-1}g)(\zeta),
\]
where (cf. (3.4))
\[\zeta_0 = \tau_0 \cap \mathbb{R}^n, \quad v = t \theta = \frac{|u| \theta}{\sqrt{1 - |u|^2}}, \quad \zeta = \tau \cap \mathbb{R}^n. \tag{3.9}
\]

4. Existence Conditions, Zonal Functions, Support Theorems, and Inversion

Theorem 3.1 makes it possible to study the spherical slice transform $\mathcal{S}$ using known properties of the Radon-John transform $R_{k-1}$. Below we consider several corollaries of this theorem.
4.1. Existence Conditions. We recall Theorem 3.2 from [23], which characterizes convergence of the Radon-John transform (see also Theorem 4.24 in [24] for $k = n$). In our notation, this theorem reads as follows.

**Theorem 4.1.** If
\[
\int_{|x| > a} \frac{|g(x)|}{|x|^{n-k+1}} \, dx < \infty, \quad \forall \, a > 0, \quad (4.1)
\]
then $(R_{k-1}g)(\tau)$ is finite for almost all $\tau \in \mathcal{T}$. If $g$ is nonnegative, radial, and (4.1) fails, then $(R_{k-1}g)(\tau) = \infty$ for every $\tau$ in $\mathcal{T}$.

The next statement extends Theorem 7.8 from [24] from $k = n$ to all $2 \leq k \leq n$.

**Proposition 4.2.** Let $\eta \in S^n$, $\tau \in \mathcal{T}$, $2 \leq k \leq n$. If
\[
\int_{\eta_{n+1} > 1-\varepsilon} \frac{|f(\eta)|}{(1 - \eta_{n+1})^{(n+1-k)/2}} \, dS(\eta) < \infty, \quad \forall \, 0 < \varepsilon \leq 2, \quad (4.2)
\]
then $(\mathcal{S}f)(\tau \cap S^n)$ is finite for almost all $\tau \in \mathcal{T}$. If $f$ is nonnegative, zonal, and (4.2) fails, then $(\mathcal{S}f)(\tau \cap S^n) = \infty$ for all $\tau \in \mathcal{T}$. If $f$ is continuous on $S^n \setminus \{N\}$, then $(\mathcal{S}f)(\tau \cap S^n)$ is finite for each $\tau \in \mathcal{T}$ provided that
\[
\sup_{\eta \in S^n} (1 - \eta_{n+1})^{\mu} |f(\eta)| < \infty \quad \text{for some} \quad \mu < \frac{k-1}{2}, \quad (4.3)
\]
where the bound $(k - 1)/2$ is sharp.

**Proof.** By (3.1), the integral in (4.1) can be written as
\[
\int_{|x| > a} \left( \frac{2}{|x|^2 + 1} \right)^{k-1} \frac{|(f \circ \nu)(x)|}{|x|^{n-k+1}} \, dx.
\]
Convergence of the latter is equivalent to convergence of the integral
\[
I = \int_{|x| > a} \frac{|(f \circ \nu)(x)| \, dx}{(|x|^2 + 1)^{(n+k-1)/2}}.
\]
Hence, by Lemma 2.1 and (2.4), (4.1) is equivalent to (4.2). The result for continuous functions easily follows if we write $I$ as
\[
I = c \int_{\eta_{n+1} > 1-\varepsilon} f(\eta)(1 - \eta_{n+1})^{(k-n-1)/2} \, d\eta
\]
and switch to spherical coordinates; cf. [24, formula (1.12.10)]. □
4.2. Zonal Functions. The next Proposition is a generalization of Theorem 7.11 from [24] for \( k = n \).

**Proposition 4.3.** Let \( 2 \leq k \leq n \), and let \( f \) be a zonal function satisfying (4.2). We write \( \eta \in S^n \) in spherical coordinates as

\[
\eta = \omega \sin \varphi + e_{n+1} \cos \varphi, \quad \omega \in S^{n-1}, \quad 0 < \varphi \leq \pi, \quad (4.4)
\]

and set \( f(\eta) = f_0(\cot \varphi/2) \); (cf. Figure 1). Then

\[
(\mathcal{S}f)(\tau \cap S^n) = F_0(t), \quad t = \frac{|\tau|}{\sqrt{1 - |\tau|^2}}, \quad (4.5)
\]

where

\[
F_0(t) = 2^{k-1} \sigma_{k-2} \int_t^\infty \frac{f_0(s)}{(1 + s^2)^{k-1}} (s^2 - t^2)^{(k-3)/2} \, s \, ds,
\]

\( \sigma_{k-2} \) being the area of the \((k-2)\)-dimensional unit sphere.

**Proof.** The statement follows from the similar fact for the Radon-John transform, according to equalities

\[
(\mathcal{S}f)(\tau \cap S^n) = (R_{k-1}g)(\tau \cap \mathbb{R}^n), \quad g(x) = \frac{2^{k-1}(f \circ \nu)(x)}{(|x|^2 + 1)^{k-1}},
\]

see (3.1). Because \( f \) is zonal on \( S^n \), \( f \circ \nu \) and \( g \) are radial functions on \( \mathbb{R}^n \). We set \( |x| = s \) and recall that \( |x| = \cot \varphi/2 \); see (2.4), (4.4). Then

\[
(f \circ \nu)(x) = f \left( \frac{2se_n + (s^2-1)e_{n+1}}{s^2 +1} \right) = f_0(s), \quad g(x) = \frac{2^{k-1}f_0(s)}{(s^2 + 1)^{k-1}}.
\]

If \( \tau \cap \mathbb{R}^n = \zeta_0 + v \), \( \zeta_0 \in G_{n,k-1} \), \( v \in \zeta_0^\perp \cap \mathbb{R}^n \), then, by [22, Formula (2.6)],

\[
(R_{k-1}g)(\tau \cap \mathbb{R}^n) \equiv (R_{k-1}g)(\zeta_0 + v) = \sigma_{k-2} \int_t^\infty \tilde{g}(s) (s^2 - t^2)^{(k-3)/2} \, s \, ds,
\]

where

\[
t = |v| = \frac{|\tau|}{\sqrt{1 - |\tau|^2}}, \quad \tilde{g}(s) = \frac{2^{k-1}f_0(s)}{(s^2 + 1)^{k-1}}.
\]

cf. (3.9). This gives the result. \( \square \)

\( ^1 \)We recall (see Notation) that \( |\tau| \) stands for the smallest Euclidean distance of any point of the affine plane \( \tau \) to the origin \((0, \ldots, 0)\).
4.3. Support Theorems. Our next aim is to establish connection between supports of \( f \) and \( \mathcal{G}f \). We set \( b \in (-1, 1), b_\ast = \sqrt{(1 + b)/2} \), \( \Omega_b = \{ \eta \in S^n : \eta_{n+1} > b \} \), \( \tilde{\Omega}_b = \{ \tau \in \mathcal{T} : |\tau| > b_\ast \} \).

**Theorem 4.4.** Let \( 2 \leq k \leq n \) and suppose that \( f \) satisfies (4.2).

(i) If \( f(\eta) = 0 \) for almost all \( \eta \in \Omega_b \), then \( (\mathcal{G}f)(\tau \cap S^n) = 0 \) for almost all \( \tau \in \tilde{\Omega}_b \).

(ii) Conversely, suppose additionally that \( f \) is continuous on \( S^n \setminus \{ N \} \) and
\[
\sup_{\eta \in S^n} (1 - \eta_{n+1})^{k-1-m/2} |f(\eta)| < \infty \text{ for all } m > 0. \tag{4.6}
\]
Then the following implication holds:
\[
(\mathcal{G}f)(\tau \cap S^n) = 0 \quad \forall \, \tau \in \tilde{\Omega}_b \implies f(\eta) = 0 \quad \forall \, \eta \in \Omega_b. \tag{4.7}
\]
If \( k < n \), then (4.7) holds under the less restrictive assumption (4.3), which is sharp.

**Proof.** The result follows from the similar one for the Radon-John transform in accordance with (3.1). It is known [12, p. 33] that if \( g \in \mathcal{C}(\mathbb{R}^n) \) and \( |x|^m g(x) \) is bounded on \( \mathbb{R}^n \) for all \( m > 0 \), then, for any fixed \( a > 0 \), the equality \( (R_{k-1}g)(\zeta) = 0 \quad \forall \, |\zeta| > a \) implies \( g(x) = 0 \quad \forall \, |x| > a. \) By (3.1) and (2.4),
\[
|x|^m g(x) = \frac{2^{k-1} |x|^m (f \circ \nu)(x)}{(|x|^2 + 1)^{k-1}} = f(\eta)(1 - \eta_{n+1})^{k-1-m/2}(1 + \eta_{n+1})^{m/2}.
\]
Suppose \( |x| \equiv |\nu(\eta)| > a > 0 \), which is equivalent to
\[
\frac{2}{1 - \eta_{n+1}} > a^2 + 1 \text{ or } \eta_{n+1} > b > -1, \text{ where } b = \frac{a^2 - 1}{a^2 + 1} < 1.
\]
Hence, if we set \( b_\ast = \sqrt{(1 + b)/2} \), \( \tilde{\Omega}_b = \{ \tau \in \mathcal{T} : |\tau| > b_\ast \} \), and assume \( (\mathcal{G}f)(\tau \cap S^n) = 0 \quad \forall \, \tau \in \tilde{\Omega}_b \), we get \( (R_{k-1}g)(\zeta) = 0 \quad \forall \, |\zeta| > a. \) The latter implies \( g(x) = 0 \quad \forall \, |x| > a \) or \( f(\eta) = 0 \quad \forall \, \eta \in \Omega_b \). The statement (i) can be proved similarly. If \( k \) is strictly less than \( n \) then, by Theorem 5.2, the number \( m \) in the above reasoning can be replaced by \( \lambda > k - 1 \). Setting \( \mu = k - 1 - \lambda/2 \), we obtain the second statement in (ii). \( \square \)

**Remark 4.5.** In the case \( k = n \), the condition (4.6) cannot be dropped; cf. [12, Remark 2.9 on p. 15], [24, Remark 4.119]. However, the continuity requirement can be substituted as follows.

**Theorem 4.6.** (cf. [24, Theorem 7.13]) Let \( k = n \). If
\[
\int_{\Omega_b} (1 - \eta_{n+1})^{-1-m/2} |f(\eta)| \, dS(\eta) < \infty \quad \text{for all } m \in \{1, 2, \ldots\}
\]
and \((\mathcal{S}f)(\tau \cap S^n) = 0\) for almost all \(\tau \in \tilde{\Omega}_b\), then \(f(\eta) = 0\) for almost all \(\eta \in \Omega_b\).

The proof of this fact relies on the theory of spherical harmonics and the structure of the null space of the hyperplane Radon transform. The description of the null space of the \(k\)-plane transform in \(\mathbb{R}^n\) with \(k < n - 1\) is more complicated (see [5] for discussion), and the lower-dimensional analogue of Theorem 4.6 is unknown.

4.4. Inversion Formulas. Theorem 3.1 yields explicit inversion formulas for \(\mathcal{S}f\). Specifically, let us write (3.1) as

\[
\mathcal{S}f = A R_{k-1} B f,
\]

where the operators \(A\) and \(B\) are defined by

\[
(Ah)(\tau \cap S^n) = (h \circ \nu)(\tau \cap S^n), \quad (Bf)(x) = \frac{2^{k-1}(f \circ \nu)(x)}{(|x|^2 + 1)^{k-1}}.
\]

Hence, if \(\mathcal{S}f = F\), then

\[
f = B^{-1} R^{-1}_{k-1} A^{-1} F,
\]

where \(A^{-1}\) and \(B^{-1}\) act by the rule

\[
(A^{-1} F)(\tau \cap \mathbb{R}^n) = (F \circ \nu^{-1})(\tau \cap \mathbb{R}^n), \quad (B^{-1} g)(\eta) = (1 - \eta_{n+1})^{1-k}(g \circ \nu^{-1})(\eta),
\]

\[
\nu^{-1}(\eta) \equiv \nu^{-1}(\omega \sin \varphi + e_{n+1} \cos \varphi) = \omega \cot(\varphi/2);
\]

see Figure 1.

A plenty of inversion formulas for \(R_{k-1}\) are available in the literature; see, e.g., [10], [12, p. 33], [22–24], and references therein. The choice of the analytic expression for \(R_{k-1}\) in (4.9) depends on the class of functions \(f\) or, equivalently, on the class of functions \(g = B f\). In particular, many inversion formulas for \(R_{k-1} g\) are known when \(g \in L^p(\mathbb{R}^n)\); see an example in Appendix. By Lemma 2.1 and (2.4), the condition \(g \in L^p(\mathbb{R}^n)\) is equivalent to

\[
(1 - \eta_{n+1})^{k-1-n/p} f(\eta) \in L^p(S^n), \quad 1 \leq p < \frac{n}{k-1}.
\]

The restriction \(p < n/(k-1)\) is sharp because it is inherited from the relevant sharp restriction for \(R_{k-1} g\).

It is also known that if \(k < n\), then the inversion problem for the Radon-John transform \((R_{k-1} g)(\zeta)\) is overdetermined if \(\zeta\) varies over the set of all \((k-1)\)-planes in \(\mathbb{R}^n\). Gel’fand’s celebrated question [9] asks how to eliminate this overdeterminedness. In our case, it means that we want to define an \(n\)-dimensional submanifold \(M\) of the set
of all \((k - 1)\)-planes \(\zeta\) in \(\mathbb{R}^{n+1}\) so that \(g\) could be reconstructed from \((R_{k-1}g)(\zeta)\) when the latter is known only for \(\zeta\) belonging to \(M\).

There are several ways to do this. For instance, one can choose \(M\) to be the subset of all \((k - 1)\)-planes which are parallel to a fixed \(k\)-dimensional coordinate plane; see \[25\] and references therein for details. Thus, to reconstruct \(f\) from \(\left(\mathcal{S}_f\right)(\tau \cap S^n)\), \(\tau \in \mathcal{T}\), it suffices to restrict \(\tau\) to the subset \(\nu(M) \subset \mathcal{T}\).

In conclusion we observe that an idea of factorization of a given operator in terms of auxiliary operators with known properties (cf. (4.8)), is widely used in Analysis; see, e.g., \[17, Section 3.1\], where it is applied to a wide class of Funk-type transforms.

5. Appendix

Below we recall some known facts for the \(k\)-plane transforms; see, e.g., \[12, 22\]. For the sake of convenience, we replace \(k - 1\) in (1.2) by \(k\) and set

\[
\varphi(\zeta) \equiv (R_k g)(\zeta) = \int g(x) \, d_\zeta x, \quad \zeta \in G(n, k),
\]

where \(G(n, k)\) is the Grassmannian bundle of all \(k\)-dimensional affine planes in \(\mathbb{R}^n\) and \(d_\zeta x\) stands for the Euclidean volume element in \(\zeta\). The dual \(k\)-plane transform \(R_k^*\) averages functions \(\varphi\) on \(G(n, k)\) over all \(k\)-planes passing through \(x \in \mathbb{R}^n\):

\[
(R_k^* \varphi)(x) = \int_{O(n)} \varphi(\gamma \zeta_0 + x) \, d\gamma. \quad (5.1)
\]

Here \(\zeta_0\) is an arbitrary fixed \(k\)-plane through the origin and \(d\gamma\) denotes the Haar probability measure on the orthogonal group \(O(n)\).

To formulate the inversion result for \(R_k g\) in the \(L^p\) setting, we invoke Riesz fractional derivatives, which can be defined as hypersingular integrals of the form

\[
(D^k h)(x) \equiv \frac{1}{d_{n, \ell}(k)} \int_{\mathbb{R}^n} \frac{(\Delta^\ell_y h)(x)}{|y|^{n+k}} \, dy
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{d_{n, \ell}(k)} \int_{|y| > \varepsilon} \frac{(\Delta^\ell_y h)(x)}{|y|^{n+k}} \, dy. \quad (5.3)
\]
In our case, \( h = R_k^* \varphi \),

\[
(\Delta^\ell_y h)(x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} h(x - jy)
\]

is the finite difference of \( h \) of order \( \ell \),

\[
d_{n,\ell}(k) = \frac{\pi^{n/2}}{2^k \Gamma((n + k)/2)} \times \begin{cases} 
\Gamma(-k/2)B_t(k) & \text{if } k \neq 2, 4, 6, \ldots, \\
\frac{2(-1)^{k/2-1}}{(k/2)!} \left[ \frac{d}{d\alpha} B_t(\alpha) \right]_{\alpha=k} & \text{if } k = 2, 4, 6, \ldots, 
\end{cases}
\]

\[
B_t(\alpha) = \sum_{j=0}^{\ell} (-1)^j \binom{l}{j} j^\alpha.
\]

The integer \( \ell \) is arbitrary with the choice \( \ell = k \) if \( k = 1, 3, 5, \ldots \) and any \( \ell > 2[k/2] \) (the integer part of \( k/2 \)), otherwise. Further details can be found in [24, Section 3.5] and [30, Chapter 3].

**Theorem 5.1.** Let \( 1 \leq k \leq n - 1 \). If \( g \in L^p(\mathbb{R}^n) \), \( 1 \leq p < n/k \), and \( R_k g = \varphi \), then,

\[
g = c_{k,n}^{-1} \mathbb{D}^k R_k^* \varphi, \quad c_{k,n} = \frac{2^k \pi^{k/2} \Gamma(n/2)}{\Gamma((n - k)/2)},
\]

where the Riesz fractional derivative \( \mathbb{D}^k \) is defined by (5.2)-(5.3). The limit in (5.3) can be interpreted in the \( L^p \)-norm and in the almost everywhere sense. If, moreover, \( g \) is continuous, then the convergence in (5.3) is uniform on \( \mathbb{R}^n \).

The following support theorem is a slight, but important, improvement of the remarkable discovery by Kurusa [14, Theorem 3.1] who established essential difference between the cases \( k = n - 1 \) and \( k < n - 1 \).

**Theorem 5.2.** Let \( (R_k g)(\zeta) \) be the \( k \)-plane transform of a continuous function \( g \) on \( \mathbb{R}^n \), satisfying

\[
|x|^\lambda |g(x)| \leq c \quad \text{for some } \lambda > k, \quad c = \text{const}^2.
\]

We assume \( 1 \leq k < n - 1 \) and denote by \( |\zeta| \) the Euclidean distance from \( \zeta \) to the origin. Then for any \( r > 0 \) the following implication holds:

\[
(R_k g)(\zeta) = 0 \quad \forall |\zeta| > r \quad \implies \quad g(x) = 0 \quad \forall |x| > r.
\]

\(^2\)In [14] it was assumed \( \lambda = k + 1 \).
Proof. Fix any \((k+1)\)-plane \(T\) at a distance \(t > r\) from the origin and let \(\rho_T \in SO(n)\) be a rotation satisfying
\[
\rho_T : \mathbb{R}^{k+1} + te_n \to T, \quad \mathbb{R}^{k+1} = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_{k+1}.
\]
Suppose that \(\zeta \subset T\) and denote \(\zeta_1 = \rho_T^{-1}\zeta \subset \mathbb{R}^{k+1} + te_n\), \(g_T(x) = g(\rho_T x)\). By the assumption, \((R_kg)(\zeta) = 0\) for all \(\zeta \subset T\), and therefore \((R_kg_T)(\zeta_1) = 0\) for all \(k\)-planes \(\zeta_1\) in \(\mathbb{R}^{k+1} + te_n\). Every such plane has the form \(\zeta_1 = \zeta_2 + te_n\), where \(\zeta_2\) is a \(k\)-plane in \(\mathbb{R}^{k+1}\).

Hence,
\[
0 = (R_kg_T)(\zeta_2 + te_n) = \int_{\zeta_2} g_T(y + te_n) \, dy = (R_k\tilde{g})(\zeta_2),
\]
which is the \(k\)-plane transform transform of \(\tilde{g}(y) = g_T(y + te_n), y \in \mathbb{R}^{k+1}\). By (5.5), \(|\tilde{g}(y)| = c|y|^{-\lambda}\). Indeed,
\[
|y|^\lambda|\tilde{g}(y)| = |y|^\lambda|g_T(y + te_n)| \leq \frac{c|y|^\lambda}{|y + te_n|^\lambda} \leq c.
\]
By the injectivity of the \(k\)-plane (see [23, Theorem 3.4]), the above inequality implies \(\tilde{g}(y) = g_T(y + te_n) = 0\) for all \(y \in \mathbb{R}^{k+1}\). Because \(T\) and \(t > r\) are arbitrary, the result follows.

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