Comparing gaussian and Rademacher cotype for operators on the space of continuous functions

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Abstract

We will prove an abstract comparison principle which translates gaussian cotype in Rademacher cotype conditions and vice versa. More precisely, let $2 < q < \infty$ and $T : C(K) \to F$ a linear, continuous operator.

1. $T$ is of gaussian cotype $q$ if and only if
   \[
   \left( \sum_{k} \left( \frac{\|T x_k\|_F}{\sqrt{\log(k + 1)}} \right)^q \right)^{1/q} \leq c_0 \sup_{t \in K} \left( \sum_{k} |x_k(t)|^q \right)^{1/2},
   \]
   for all sequences with $(\|T x_k\|_F)^q$ decreasing.

2. $T$ is of Rademacher cotype $q$ if and only if
   \[
   \left( \sum_{k} \left( \frac{\|T x_k\|_F}{\sqrt{\log(k + 1)}} \right)^q \right)^{1/q} \leq c_0 \left\| \sum_{k} \varepsilon_k x_k \right\|_{L_2(C(K))},
   \]
   for all sequences with $(\|T x_k\|_F)^q$ decreasing.

Our method allows a restriction to a fixed number of vectors and complements the corresponding results of Talagrand.

Introduction

One problem in the local theory of Banach spaces consists in the description of Rademacher cotype and gaussian cotype for operators on $C(K)$-spaces. A quite satisfactory answer for the Rademacher cotype was given by Maurey. He connected cotype conditions with summing conditions (see [MAU]):

**Theorem 0.1** [Maurey] Let $2 < q < \infty$ and $T : C(K) \to F$. Then the following are equivalent:

1. $T$ is absolutely $(q, 2)$-summing, i.e. for all $(x_k)_{k \in \mathbb{N}} \subset C(K)$ one has
   \[
   \left( \sum_{k} \|T x_k\|^q \right)^{1/q} \leq c_0 \sup_{t \in K} \left( \sum_{k} |x_k(t)|^2 \right)^{1/2}.
   \]

2. $T$ has Rademacher cotype $q$, i.e. for all $(x_k)_{k \in \mathbb{N}} \subset C(K)$ one has
   \[
   \left( \sum_{k} \|T x_k\|^q \right)^{1/q} \leq c_0 \left\| \sum_{k} \varepsilon_k x_k \right\|_{L_2(C(K))}.
   \]

3. $T$ is absolutely $(q, 1)$-summing, i.e. for all $(x_k)_{k \in \mathbb{N}} \subset C(K)$ one has
   \[
   \left( \sum_{k} \|T x_k\|^q \right)^{1/q} \leq c_0 \sup_{t \in K} \sum_{k} |x_k(t)|.\]
Later on, Pisier gave another approach to this type of results via factorization theorems. This way was pursued by Montgomery-Smith, [MSM], and Talagrand, [TAL], to give a characterization of gaussian cotype q.

**Theorem 0.2** [Talagrand] Let \( 2 < q < \infty \) and \( T : C(K) \to F \). Then the following are equivalent.

1. \( T \) has gaussian cotype \( q \), i.e. for all \( (x_k)_{k \in \mathbb{N}} \subset C(K) \) one has
   \[
   \left( \sum_k \|Tx_k\|^q \right)^{1/q} \leq c_1 \left\| \sum_k g_k x_k \right\|_{L_2(C(K))}.
   \]

2. \( T \) satisfies the following summing condition, i.e. for all \( (x_k)_{k \in \mathbb{N}} \subset C(K) \) such that \( (\|Tx_k\|)^q \) is decreasing one has
   \[
   \left( \sum_k \left( \frac{\|Tx_k\|}{\sqrt{\log(k+1)}} \right)^q \right)^{1/q} \leq c_2 \sup_{t \in K} \sum_k |x_k(t)|.
   \]

3. \( T \) factors through an Orlicz space \( L_{(e \log t)^{t/2},1}(\mu) \) for some probability measure \( \mu \) on \( K \).

The main new ingredient of this theorem is a factorization theorem for gaussian processes derived from Talagrand’s results. Independently of him we discovered the connection between gaussian cotype and summing properties with the modified \( \ell_q \) space in condition 2 of theorem 2. In order to be precise, let us give the following definition. For a maximal, symmetric sequence space \( X \) and \( T : E \to F \) we define

\[
\pi^n_{X,q}(T) := \sup \left\{ \left\| \sum_{1}^{n} \|Tx_k\|_{F} e_k \right\|_{X} \left| \sup_{a \in B_E^*} \left( \sum_{1}^{n} \langle x_k, a \rangle \right)^{1/q} \right| \leq 1 \right\},
\]

\[
rc^n_X(T) := \sup \left\{ \left\| \sum_{1}^{n} \|Tx_k\|_{F} e_k \right\|_{X} \left| \sum_{1}^{n} \varepsilon_k x_k \right\|_{L_2(E)} \leq 1 \right\},
\]

\[
gc^n_X(T) := \sup \left\{ \left\| \sum_{1}^{n} \|Tx_k\|_{F} e_k \right\|_{X} \left| \sum_{1}^{n} g_k x_k \right\|_{L_2(E)} \leq 1 \right\}.
\]

An operator is said to be (absolutely) \( (X,q) \)-summing, of Rademacher cotype \( X \), of gaussian cotype \( X \) if \( \pi^n_{X,q} := \sup_{n \in \mathbb{N}} \pi^n_{X,q} \), \( rc_X := \sup_{n \in \mathbb{N}} rc^n_X \), \( gc_X := \sup_{n \in \mathbb{N}} gc^n_X \) is finite, respectively. In contrast to Talagrand we follow Maurey’s approach and prove

**Theorem 0.3** Let \( 2 < q < \infty \), \( X \) a \( q \)-convex, maximal, symmetric sequence space and \( T : C(K) \to F \). Then the following are equivalent:

1. \( T \) is \( (X,2) \)-summing.

2. \( T \) is of Rademacher cotype \( X \).

3. \( T \) is \( (X,1) \)-summing.
Furthermore, there exists a constant $c$ only depending on $q$ and $X$ such that
\[ \pi^n_{X,2}(T) \leq c \pi^n_{X,1}(T). \]

The main idea for the proof of the theorem above is a reduction to Maurey’s result via quotient formulas. These formulas are contained in chapter 2 and have already been seen to be helpful in the theory of summing operators. Their proof goes back to a joint work of Martin Defant and the author, see [DJ].

The comparison principle between gaussian and Rademacher cotype for operators on $C(K)$-spaces is formulated in

**Theorem 0.4** Let $2 < q < \infty$, $X$ a $q$-convex, maximal, symmetric sequence space. If $Y$ denotes the space of diagonal operators between $\ell_{\infty,\infty,1/2}$ and $X$ one has for all operators $T : C(K) \to F$ and $n \in \mathbb{N}$
\[ \frac{1}{c} rc^n_Y(T) \leq ge^n_X(T) \leq c rc^n_Y(T), \]
where $c$ is a constant depending on $q$ and $X$ only.

The philosophy is quite simple. The difference between gaussian and Rademacher cotype has to be corrected in the summing property with the factor $\sqrt{\log(k+1)}$. This becomes clear if we apply this first for the space $X = \ell_q$. Then we see that an operator $T : C(K) \to F$ is of gaussian cotype $q$ if and only if
\[ \left( \sum_k \left( \frac{\|Tx_k\|_F}{\sqrt{\log(k+1)}} \right)^q \right)^{1/q} \leq c \left\| \sum_{1}^{n} \varepsilon_k x_k \right\|_{L_2(C(K))}, \]
for all sequences with $(\|Tx_k\|_F)^n$ decreasing. Applying the result for $Y = \ell_q$ we see that $T$ is of Rademacher cotype $q$ if and only if
\[ \left( \sum_k \left( \frac{\|Tx_k\|_F \sqrt{\log(k+1)}}{q} \right)^q \right)^{1/q} \leq c \left\| \sum_{1}^{n} \varepsilon_k x_k \right\|_{L_2(C(K))}, \]
for all sequences with $(\|Tx_k\|_F)^n$ decreasing. Let us also note that our approach enables us to fix the number of vectors in consideration. For example, this restriction to $n$ vectors can be used to prove that for an operator of rank $n$ the gaussian cotype $q$-norm is attained on $n$ disjoint functions in $C(K)$. Another application is given in the study of weak cotype operators.

**Preliminaries**

We use standard Banach space notations. In particular, $c_0$, $c_1$, .. will denote different absolute constants and they can vary within the text. The symbols $X$, $Y$, $Z$ are reserved for sequence spaces. Standard references on sequence spaces and Banach lattices are the monograph of Lindenstrauss and Tzafriri, [LT], [LTI]. The symbols $E$, $F$ will always denote Banach sapces with unit balls $B_E$, $B_F$ and duals $E^*$, $F^*$. Basic information on operator ideals and $s$-numbers can be found in the monograph of Pietsch, [PIE]. The ideal of linear operators is denoted by $L$.

The classical sequence spaces $c_0$, $\ell_p$ and $\ell_{p,n}^n$, $1 \leq p \leq \infty$, $n \in \mathbb{N}$ are defined in the usual way. From the context it will be clear whether we mean the space $c_0$ or the absolut constant $c_0$. A generalization of the classical $\ell_p$ spaces is the class of Lorentz-Marcinkiewicz spaces. For a given continous function $f : \mathbb{N} \to \mathbb{R}_{>0}$ with $f(1) = 1$ the following two indices are defined
$$\alpha_f := \inf \{ \alpha \mid \exists M < \infty \forall t, s \geq 1 : f(ts) \leq Mf^\alpha f(s) \} ,$$

$$\beta_f := \sup \{ \beta \mid \exists c > 0 \forall t, s \geq 1 : f(ts) \geq ct^\beta f(s) \} .$$

These two indices play an important rôle in the study of the space $\ell_{f,q}$, $1 \leq q \leq \infty$ consisting of all sequences $\sigma \in \ell_\infty$ such that

$$\|\sigma\|_{f,q} := \left( \sum_n (f(n) \sigma_n^*)^q n^{-1} \right)^{1/q} < \infty .$$

For $q = \infty$ the needed modification is given by

$$\|\sigma\|_{f,\infty} := \sup_{n \in \mathbb{N}} f(n) \sigma_n^* < \infty .$$

Here and in the following $\sigma^* = (\sigma_n^*)_{n \in \mathbb{N}}$ denotes the non-increasing rearrangement of $\sigma$.

In the introduction the notions of $(X,q)$-summing, Rademacher cotype $X$ and gaussian cotype $X$ are already defined. If $X = \ell_p$ we will shortly speak of $(p,q)$-summing operators or norms, Rademacher cotype $p$, etc. (possibly restricted to $n$ vectors). In this context it is convenient to use an abbreviation for the right hand side of the definition of summing operators. For a sequence $(x_k)_1^n$ in a Banach space $E$ we write

$$\omega_{q}(x_k)_1^n := \sup_{a \in B_{E^*}} \left( \sum_{1}^{n} |< x_k, a >|^q \right)^{1/q} .$$

Let us note that this expression coincides with the operator norm of

$$u := \sum_{1}^{n} \epsilon_k \otimes x_k \in \mathcal{L}(\ell^2_1, E) ,$$

where $q'$ is the conjugate index of $q$ satisfying $\frac{1}{q} + \frac{1}{q'} = 1$.

In the following $(\varepsilon_n)_{n \in \mathbb{N}}$, $(g_n)_{n \in \mathbb{N}}$ will denote a sequence of independent normalized Bernoulli (Rademacher) variables or gaussian variables respectively. They are defined on a probability space $(\Omega, \mu)$. Here Bernoulli variable means

$$\mu(\varepsilon_n = +1) = \mu(\varepsilon_n = -1) = \frac{1}{2} .$$

A very deep result in the theory of gaussian processes is Talagrand’s factorization theorem, see [TA1].

(*) There is an absolut constant $c_1$ such that for all sequence $(x_k)_1^n \in C(K)$ with

$$\left\| \sum_{1}^{n} g_k x_k \right\|_{L_2(X)} \leq 1 ,$$

there are operators $u : \ell^2_2 \rightarrow c_0$, $R : c_0 \rightarrow C(K)$ with $\|u\| \|R\| \leq c_1$ such that

$$RD_\sigma u(e_k) = x_k ,$$

where $D_\sigma$ is the diagonal operator with
\[ \sigma_k = \frac{1}{\sqrt{\log(k + 1)}}. \]

Finally some s-numbers are needed. For an operator \( T \in \mathcal{L}(E, F) \) and \( n \in \mathbb{N} \) the n-th approximation number is defined by
\[ a_n(T) := \inf \{ \| T - S \| \mid \text{rank}(S) < n \}, \]
whereas the n-th Weyl number is given by
\[ x_n(T) := \sup \{ a_n(Tu) \mid u \in \ell_2 \text{ with } \| u \| \leq 1 \}. \]

1 Maximal symmetric sequence spaces

In the following we will denote the set of all finite sequences by \( \phi \) and the sequence of unit vectors in \( \ell_\infty \) by \((e_k)_k\). For every sequence \( \sigma = (\sigma_k)_k \subset \ell_\infty \), \( n \in \mathbb{N} \) we set
\[ P_n(\sigma) := \sum_{k=1}^{n} \sigma_k e_k. \]
A maximal sequence space \((X, \| \cdot \|)\) is a Banach space satisfying the following conditions.

1. \( \ell_1 \subset X \subset \ell_\infty \) and \( \| e_k \| = 1 \) for all \( k \in \mathbb{N} \).
2. If \( \sigma \in X \) and \( \alpha \in \ell_\infty \) then the pointwise product \( \alpha \sigma \in X \) with \( \| \alpha \sigma \| \leq \| \sigma \| X \| \alpha \|_\infty \).
3. \( \sigma \in X \) if and only if \( (\| P_n \|)_n \) is bounded and in this case
\[ \| \sigma \| = \sup_{n \in \mathbb{N}} \| P_n \|. \]

For \( n \in \mathbb{N} \) and \( \sigma = (\sigma_k)_1^n \subset \mathbb{K}^n \) we set \( \| \sigma \| := \| (\sigma_k)_1^n \| := \left\| \sum_{k=1}^{n} \sigma_k e_k \right\| \). The sequence dual of \( X \) is defined by
\[ X^+ := \{ \tau \in \ell_\infty \mid \| \tau \|_+ := \sup_{\sigma \in B_X} \left\| \sum_{k} \sigma_k \tau_k \right\| < \infty \}. \]

Then \( (X^+, \| \cdot \|_+) \) is also a maximal sequence space. We observe that \( \| \tau \|_{X^*} = \| \tau \|_+ \) holds for all \( \tau \in \phi \). Thus \( X^{++} = X \) with equal norms. For two maximal sequence spaces \( X, Y \) we denote by \( \mathbb{D}\mathcal{L}(X, Y) \) the space of continuous diagonal operators from \( X \) to \( Y \) with the operator norm. A maximal sequence space is symmetric if in addition \( \sigma \in X \) if and only if \( \sigma^* \in X \) with \( \| \sigma^* \|_X = \| \sigma \|_X \).

Essentially for the following is the definition of p-convex sequence spaces. Let \( 1 \leq p < \infty \). A maximal sequence space is \( p \)-convex if there is a constant \( c > 0 \) such that for all \( n \in \mathbb{N} \) and \( (x_k)_1^n \subset X \)
\[ \left\| \left( \sum_{k=1}^{n} |x_k|^p \right)^{1/p} \right\| \leq c \left( \sum_{k=1}^{n} \| x_k \|^p \right)^{1/p}. \]

The best constant \( c \) satisfying the above condition will be denoted by \( M^p(X) \). Obviously, every maximal sequence space is 1-convex. On the other hand we observe
\[ X^+ = \mathbb{D}\mathcal{L}(X, \ell_1) \quad \text{and thus} \quad X = \mathbb{D}\mathcal{L}(X^+, \ell_1). \]

More generally, one has
Proposition 1.1 Let $1 \leq p < \infty$ and $X$ a maximal sequence space. Then the following are equivalent:

1. $X$ is $p$-convex.

2. The homogenous expression $\left\| |\sigma|^{1/p} \right\|_X^p$ is equivalent to a norm $\| \cdot \|_p$ with
   \[ \frac{1}{c} \| \sigma \|_X \leq \left\| |\sigma|^{1/p} \right\|_X^p \leq \| \sigma \|_X . \]

3. There exists a maximal sequence space $Y$ such that
   \[ X \cong \mathcal{DL}(Y, \ell_p) . \]
   Moreover, in this case we can choose $Y = \mathcal{DL}(X, \ell_p)$ and have
   \[ \frac{1}{M^p(X)} \| \sigma \|_X \leq \| D \sigma \| \leq \| \sigma \|_X . \]

Proof: The equivalence between 1. and 2. is classical and can be found for example in [LTII]. Now we proof 2. $\Rightarrow$ 3. We denote by $X_p$ the maximal sequence space defined by the norm $\| \cdot \|_p$. We set $Y = \mathcal{DL}(X, \ell_p)$. Clearly, we have $X \subset \mathcal{DL}(Y, \ell_p)$. By the observations above we have

\[
\frac{1}{c} \| \sigma \| \leq \| |\sigma|^{1/p} \|_p^{1/p} \\
= \sup_{\tau \in B(X_p^+)} \left( \sum_k |\sigma_k|^{1/p} \tau_k \right)^{1/p} \\
\leq \| \sigma \|_{\mathcal{DL}(Y, \ell_p)} \sup_{\tau \in B(X_p^+)} \left( \| \tau \|_{\mathcal{DL}(X, \ell_p)} \right)^{1/p} \\
= \| \sigma \|_{\mathcal{DL}(Y, \ell_p)} \sup_{\rho \in B_X} \left( \sum_k |\tau_k|^{1/p} \right)^{1/p} \\
= \| \sigma \|_{\mathcal{DL}(Y, \ell_p)} \sup_{\rho \in B_X} \| |\rho|^{1/p} \|_p^{1/p} \\
\leq \| \sigma \|_{\mathcal{DL}(Y, \ell_p)} . 
\]

For the proof of 3. $\Rightarrow$ 1. we can assume that $X = \mathcal{DL}(Y, \ell_p)$ with equal norms. The definition of the norm implies for $(x_j)_{j=1}^n \subset X$

\[
\left( \sum_{j=1}^n |x_j|^p \right)^{1/p} = \sup_{\tau \in B_Y} \left( \sum_k \sum_{j=1}^n |x_j(k)|^p \tau_k \right)^{1/p} \\
= \sup_{\tau \in B_Y} \left( \sum_k \sum_{j=1}^n |x_j(k)\tau_k| \right)^{1/p} \\
\leq \left( \sum_{j=1}^n \sup_{\tau \in B_Y} \left( \sum_k |x_j(k)\tau_k| \right) \right)^{1/p} \\
= \left( \sum_{j=1}^n \| x_k \|_p \right)^{1/p} . 
\]

$\square$
**Remark 1.2** i) An Orlicz sequence space

\[ \ell_\phi := \{ \sigma \in \ell_\infty \mid \sum_k \phi(\sigma_k) < \infty \} \]

is \( p \)-convex if and only if \( \phi(t\lambda) \leq c\lambda^p \phi(t) \).

ii) The criterion above is very useful to study the \( p \)-convexity of a Lorentz-Marcinkiewicz sequence space \( \ell_{f,q} \). It was observed in [COB] that for \( p \leq q \) and \( 0 < \beta_f \leq \alpha_f < 1/p \) one has

\[ \left\| \sigma \right\|_{f,q}^{1/p} \sim \left\| \left( \frac{1}{n} \sum_{k=1}^{n} \sigma_k \right) \right\|_{f^{p,q}/p} . \]

Since the right hand side is a norm, see again [COB], the conditions above imply the \( p \)-convexity of \( \ell_{f,q} \).

## 2 Quotient formulas for summing properties

We will start with a quotient formula for \((X, q)\)-summing operators.

**Proposition 2.1** Let \( 1 \leq r \leq q \leq \infty \), \( Y \) a maximal, symmetric sequence space and \( X \cong DL(Y, \ell_q) \). Then we have for all \( n \in \mathbb{N} \) and \( T \in \mathcal{L}(E, F) \)

\[ \pi_{x,r}^n(T) = \sup \{ \pi_{q,r}^n(D_\sigma RT) \mid R \in \mathcal{L}(F, \ell_q), D_\sigma \in \mathcal{L}(\ell_\infty, \ell_\infty), \text{ with } \|R\|, \|\sigma\|_Y \leq 1 \} . \]

**Proof:** " \( \leq \) " Let \((x_k)_k^p \subset E \) with. For \( \varepsilon > 0 \) there exists a \( \sigma \in B_Y \) with

\[ \left\| \sum_{k=1}^{n} \|Tx_k\| e_k \right\|_X \leq (1 + \varepsilon) \left( \sum_{k=1}^{n} \|Tx_k\| \sigma_k \right)^{1/q} . \]

Let \( y_k^* \in B_{F^*} \) with \( < y_k^*, Tx_k > = \|Tx_k\| \). If we define \( R := \sum_{k=1}^{n} y_k^* \otimes e_k \in \mathcal{L}(F, \ell_\infty) \) we obtain

\[ \frac{1}{1 + \varepsilon} \left\| \sum_{k=1}^{n} \|Tx_k\| e_k \right\|_X \leq \left( \sum_{k=1}^{n} \left| < y_k^*, Tx_k > \sigma_k \right| \right)^{1/q} \]

\[ \leq \left( \sum_{j=1}^{n} \sup_{k} \left| < y_j^*, Tx_k > \sigma_k \right| \right)^{1/q} \]

\[ \leq \pi_{q,r}^n(D_\sigma RT) \omega_r(x_k)^n . \]

" \( \geq \) " Let \( \sigma \in B_Y \) and \( R \in \mathcal{L}(F, \ell_\infty) \) with \( \|R\| \leq 1 \). By the maximality of \((X, r)\)-summing operators there is no restriction to assume \( R \in \mathcal{L}(F, \ell_{m,\infty}) \) for some \( m \in \mathbb{N} \). Now we will use a duality argument. Following the proof of theorem 1. in [DJ] there is an operator \( S \in \mathcal{L}(\ell_{m,\infty}^m, E) \) with

\[ \pi_{q,r}^n(D_\sigma RT) = \text{trace}(S D_\sigma RT) \quad \text{and} \quad S = BD_\tau P \]

where \( B \in \mathcal{L}(\ell_{m,\infty}^m, E) \) with \( \|B\| \leq 1 \), \( \tau \in B_{\ell_{m,\infty}^m} \) and there is an increasing sequence \((l_k)_k^m \subset \{1, .., m\} \) such that

\[ P = \sum_{k=1}^{n} e_{l_k} \otimes e_k \in \mathcal{L}(\ell_{m,\infty}^m, \ell_{m,\infty}^m) . \]
Therefore we deduce
\[
\text{trace}(SD_\sigma RT) = \text{trace}(D_\sigma PD_\sigma RTB) \\
= \sum_1^n \tau_k < e_{lk}, D_\sigma RTB(e_k) > \\
\leq \sum_1^n |\tau_k \sigma_{lk}| \|RTB(e_k)\| \\
\leq \left( \sum_1^n (|\sigma_{lk}| \|RTB(e_k)\|)^q \right)^{1/q} \\
\leq \|\sigma\|_Y \pi^n_{X,r}(RT) \|B\| \\
\leq \pi^n_{X,r}(T). \tag{\*}
\]

We can now prove the generalized Maurey theorem.

**Theorem 2.2** Let \(1 \leq r < q \leq \infty\), \(X\) a \(q\)-convex maximal, symmetric sequence space and \(n \in \mathbb{N}\). Then for all operators \(T \in \mathcal{L}(C(K), F)\) one has
\[
\pi^n_{X,r}(T) \leq c_0 M^q(X) \frac{1}{r} \left( \frac{1}{r} - \frac{1}{q} \right)^{-1/q'} \pi^n_{X,1}(T).
\]

**Proof** : By proposition \(\text{[I]}\) we can assume that there exists a maximal, symmetric sequence space \(Y\) with \(X \cong \mathcal{DL}(Y, \ell_q)\). By the classical Maurey theorem, for the constants see \([TJM]\), we deduce from proposition \(\text{[2]}\)
\[
\pi^n_{X,r}(T) \leq M^q(X) \sup \{ \pi^n_{q,r}(D_\sigma RT) \mid R \in \mathcal{L}(F, \ell_\infty), D_\sigma \in \mathcal{L}(\ell_\infty, \ell_\infty), \text{ with } \|R\|, \|\sigma\|_Y \leq 1 \} \\
\leq M^q(X) c_0 \frac{1}{r} \left( \frac{1}{r} - \frac{1}{q} \right)^{-1/q'} \times \\
\sup \{ \pi^n_{q,1}(D_\sigma RT) \mid R \in \mathcal{L}(F, \ell_\infty), D_\sigma \in \mathcal{L}(\ell_\infty, \ell_\infty), \text{ with } \|R\|, \|\sigma\|_Y \leq 1 \} \\
= c_0 M^q(X) \pi^n_{X,1}(T). \tag{\*}
\]

**Remark 2.3** Now it is again well-known, see \([MAU]\), how to derive from the above theorem the equivalence between Rademacher cotype conditions and summing properties as stated in the introduction as theorem 3, namely
\[
\pi^n_{X,1}(T) \leq c_0 M^q(X) \left( \frac{1}{2} - \frac{1}{q} \right)^{-1/q'} \pi^n_{X,1}(T).
\]

At the end of this chapter we will prove another quotient formula which is more adapted for operators on \(C(K)\)-spaces.

**Proposition 2.4** Let \(Y, Z\) be maximal, symmetric sequence spaces and \(X = \mathcal{DL}(Y, Z)\). then we have for all \(T \in \mathcal{L}(E, F)\) and \(n \in \mathbb{N}\)
\[
\pi^n_{X,1}(T) = \sup \{ \pi^n_{Z,1}(TRD_\sigma) \mid R \in \mathcal{L}(\ell_\infty, E), D_\sigma \in \mathcal{L}(\ell_\infty, \ell_\infty), \text{ with } \|R\|, \|\sigma\|_Y \leq 1 \}.
\]
Proof: ”≤” can be proved exactly as in proposition [2.1].

”≥” Again by maximality we can assume $R \in \mathcal{L}(\ell^m_\infty, E)$ and $D_\sigma \in \mathcal{L}(\ell^n_\infty, \ell^n_\infty)$ with $\|R\|, \|\sigma\|_Y \leq 1$. We have to show that for all $S \in \mathcal{L}(\ell^n_\infty, \ell^m_\infty)$ with $\|S\| \leq 1$ we have

$$\left\| \sum_{1}^{n} \|T R D_\sigma S(e_k)\|_F e_k \right\|_Z \leq \pi^n_{X,1}(T).$$

By a lemma of Maurey, calculating essentially the extreme points of operators from $\ell^n_\infty$ to $\ell^m_\infty$, see [MAU], and using the convexity of $Z$ we can assume that $S$ has the form

$$S = \sum_{1}^{n} e_k \otimes g^k.$$

Here the $(g^k)$'s have disjoint support and satisfy $0 < \|g^k\|_{\ell^\infty_\infty} \leq 1$. Now we define

$$J := R \left( \sum_{1}^{n} e_k \otimes \frac{D_\sigma g^k}{\|D_\sigma g^k\|_{\ell^\infty_\infty}} \right) \in \mathcal{L}(\ell^n_\infty, E)$$

and $\tau := (\|D_\sigma g^k\|_{\ell^\infty_\infty})^n_1$. We observe that $\|R\| \leq 1$ and there is a subsequence $(l_k)^n_1 \subset \{1, \ldots, m\}$ such that $\|D_\sigma g^k\|_{\ell^\infty_\infty} = |< e_{l_k}, D_\sigma g^k >|$. From the rearrangement invariance of $Y$ we deduce

$$\|\tau\|_Y = \left\| \left( |< \sigma_{l_k} < e_{l_k}, g^k >| \right)^n_1 \right\|_Y$$

$$\leq \left\| \sum_{1}^{n} \sigma_{l_k} e_{l_k} \right\|_Y$$

$$\leq \|\sigma\|_Y \leq 1.$$

Hence we obtain

$$\left\| \sum_{1}^{n} \|T R D_\sigma S(e_k)\|_F e_k \right\|_Z = \left\| \sum_{1}^{n} \left( \|T J(e_k)\|_F \|D_\sigma g^k\|_{\ell^\infty_\infty} \right) e_k \right\|_Z \leq \pi^n_{X,1}(T) \|\tau\|_Y \leq \pi^n_{X,1}(T).$$

\[\square\]

3 Gaussian cotype conditions

As a consequence of Talagrand’s factorization theorem for gaussian processes cotype conditions on $C(K)$-spaces can be reformulated with a quotient formula. This was remarked by Pisier and Montgomery-Smith, see [MSM]. We will give a prove for an arbitrary maximal, symmetric sequence space. Let us recall that $\ell_{\infty, \infty, 1/2}$ is the space of sequences $\sigma \in \ell_{\infty}$ with

$$\|\sigma\|_{\ell_{\infty, \infty, 1/2}} := \sup_{k \in \mathbb{N}} \sqrt{\log(k+1)} \sigma^*_k < \infty.$$

Lemma 3.1 Let $X$ be a maximal, symmetric sequence space, $T \in \mathcal{L}(C(K), F)$ and $n \in \mathbb{N}$. Then we have for an absolut constant $c_1$

$$gc^n_k(T) \sim c_1 \sup \left\{ \pi^n_{X,2}(T R D_\sigma) \mid R \in \mathcal{L}(c_o, E), D_\sigma \in \mathcal{L}(c_o, c_o) \text{ with } \|R\|, \|\sigma\|_{\ell_{\infty, \infty, 1/2}} \leq 1 \right\}.$$
Proof: ” ≥ ” W.l.o.g. we can assume that \( \sigma_k = (\log(k + 1))^{-1/2} \). Then it follows from \([\text{LIP}]\) that for all \( u \in \mathcal{L}(\ell_2^q, c_0) \) we have
\[
\left\| \sum_{1}^{n} g_k RD_\sigma u(e_k) \right\|_{L_2(C(K))} \leq c_1 \| R \| \| u \| .
\]
With a glance on definition of \( ge_X^0 \) we see that the first inequality is proved.

” ≤ ” Let \( (x_k)^n_k \in C(K) \) with
\[
\left\| \sum_{1}^{n} g_k x_k \right\|_{L_2(C(K))} \leq 1 .
\]
By Talagrand’s factorization theorem, see (*) in the preliminaries, there are \( u \in \mathcal{L}(\ell_2^q, c_0) \) and \( R \in \mathcal{L}(c_0, C(K)) \) with \( \| u \| \leq c_1, \| R \| \leq 1 \) such that
\[ RD_\sigma u(e_k) = x_k , \]
and \( \sigma_k = (\log(k + 1))^{-1/2} \). Hence we deduce that
\[
\left\| \sum_{1}^{n} \| Tx_k \|_F e_k \right\|_X = \left\| \sum_{1}^{n} \| TRD_\sigma u(e_k) \|_F e_k \right\|_X \leq \pi_{X,2}^n (TRD_\sigma) \| u \| \leq c_1 \pi_{X,2}^n (TRD_\sigma) \left\| \sum_{1}^{n} g_k x_k \right\|_{L_2(C(K))} .
\]
Taking the supremum over all sequences \( (x_k)^n_k \) yields the assertion. \( \square \)

Now we are able to prove the comparison theorem for gaussian and Rademacher cotype.

Theorem 3.2 Let \( 2 < q < \infty \) and \( X \) a \( q \)-convex maximal, symmetric sequence space. We set \( Y = \mathbb{D} \mathcal{L}(\ell_{\infty,1/q}, X) \). Then we have for all \( T \in \mathcal{L}(C(K), F) \) and \( n \in \mathbb{N} \)
\[
1. \pi_{Y,1}^n(T) \leq rc_Y^n(T) \leq \sqrt{2} \pi_{Y,2}^n(T) \leq c_0 M^q(X) \left( \frac{1}{2} - \frac{1}{q} \right)^{-1/q'} \pi_{Y,1}^n(T) ,
\]
\[
2. ge_X^0(T) \sim_{c_q} re_Y^n(T) .
\]

Proof: First we note that the \( q \)-convexity of \( X \) implies the \( q \)-convexity of the maximal, symmetric sequence space \( Y \). This can be seen exactly as in the proof of proposition \([3,1]\). Therefore the first assertion follows from theorem \([2,2]\), more precisely remark \([2,3]\), applied for \( Y \). With the help of the previous Lemma \([2,1]\) applying theorem \([2,2]\) for \( X \) and with the second quotient formula \([2,4]\) we obtain
\[
ge X^0(T) \sim_{c_1} \sup \{ \pi_{X,2}^n(TRD_\sigma) | R \in \mathcal{L}(c_0, E), D_\sigma \in \mathcal{L}(c_0, c_0) \text{ with } \| R \| , \| \sigma \|_{\ell_{\infty,1/2}} \leq 1 \} \sim_{c_q(X)} \sup \{ \pi_{X,1}^n(TRD_\sigma) | R \in \mathcal{L}(c_0, E), D_\sigma \in \mathcal{L}(c_0, c_0) \text{ with } \| R \| , \| \sigma \|_{\ell_{\infty,1/2}} \leq 1 \} = \pi_{Y,1}^n(T) .
\]

Using the first assertion we see that the proof of the second assertion is completed. \( \square \)
Remark 3.3: Probably the most important applications of the above theorem are given for gaussian cotype \( q \) and Rademacher cotype \( q \) operators when \( q > 2 \).

1. In the case when \( X = \ell_q \) it turns out that \( Y \) is in fact the Lorentz-Marcinkiewicz space \( \ell_{q,q,-1/2} \). This space consists of all sequences \( \sigma \in \ell_\infty \) such that
\[
\left( \sum_k \left( \frac{\sigma_k^*}{\sqrt{\log (k+1)}} \right)^q \right)^{1/q} < \infty.
\]

2. If we want to calculate the cotype conditions for \((q,1)\)-summing operators or Rademacher cotype \( q \) operators we have to solve the equation
\[
\ell_q = \mathbb{D}\mathcal{L}(\ell_\infty, \infty, 1/2, Y).
\]

Again this is easy with the help of Lorentz-Marcinkiewicz spaces. The space \( \ell_{q,q,-1/2} \) with the norm
\[
\|\sigma\|_{\ell_{q,q,-1/2}} := \left( \sum_k \left( \sigma_k^* \sqrt{\log (k+1)} \right)^q \right)^{1/q}
\]
solves the problem up to some constant. In order to apply theorem 3.2 we have to check the \( r \)-convexity of \( \ell_{q,q,-1/2} \) for some \( r > 2 \). If we identify \( \ell_{q,q,-1/2} \) with a space \( \ell_{f,q} \) this easily follows from remark 1.2. Indeed, \( f \) is given by
\[
f(t) := t^{1/q} \sqrt{\log (t+1)},
\]
which satisfies \( \beta_f = \alpha_f = \frac{1}{q} \).

In the following we will state further applications of theorem 3.2.

Corollary 3.4: Let \( 2 < q < \infty \) and \( X \) a \( q \)-convex maximal, symmetric sequence space then there is a constant \( c \) depending on \( q \) and \( X \) only such that for all \( n \in \mathbb{N} \) and \( T \in \mathcal{L}(C(K), F) \) with \( \text{rank}(T) \leq n \) one has
\[
gc_X(T) \leq c g\ell^2_X(T).
\]
Moreover, the gaussian cotype constant is, up to \( c \), attained on \( n \) disjoints functions in \( C(K) \).

Proof: We set \( Y = \mathbb{D}\mathcal{L}(\ell_\infty, \infty, 1/2, X) \). By theorem 3.2 we have
\[
gc_X(T) \sim c \pi_Y(T).
\]

Therefore it remains to show that the \( (Y,1) \)-summing norm is attained on \( n \) vectors. Using Maurey’s lemma about the extreme points of operators from \( \ell_\infty^n \) to \( C(K) \) (already used in the proof of proposition 2.4), see [MAU], it is then clear from that a restriction to \( n \) disjoint blocs is possible.

In theorem 3.2 it was also observed that \( Y \) is \( q \)-convex. By proposition 1.1 there is a maximal, symmetric sequence space \( Z \) with \( Y \)
\[
\cong \mathbb{D}\mathcal{L}(Z, \ell_q).
\]
Furthermore, it is known that for the computation of the \((q,2)\)-summing norm of an
operator with rank n only n vectors are needed, see for example [DJ]. Hence we can deduce from proposition 2.3 and theorem 2.2

\[
\pi_{Y,1}(T) \leq \sup_{k \in \mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} a_k(Tu) \leq c \ell(u).
\]

The best constant c will be denoted by \(\omega_{c_q}(T)\). It was essentially remarked by Mascioni, see [MAS], that for \(q > 2\) another definition would have been possible. An operator \(T \in \mathcal{L}(E, F)\) is said to be a weak cotype q operator, if there exists a constant \(c > 0\) such that

\[
\sup_{k \in \mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} \|Tx_k\|_F \leq c \left\| \sum_k g_k x_k \right\|_{L^q(E)}
\]

for each sequence \((x_k)_k \subseteq E\) such that \(\|Tx_k\|\) is non-increasing (for further information see also [DJ1]).

The next proposition gives a characterization of weak cotype operators on \(C(K)\)-spaces in terms of Weyl numbers.

**Corollary 3.5** Let \(2 < q < \infty\). An operator \(T \in \mathcal{L}(C(K), F)\) is of weak cotype q if and only if

\[
\sup_{k \in \mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} x_k(T) < \infty.
\]

**Proof**: By remark 1.2 the space \(X := \ell_{q,\infty} := \ell_{f,\infty}\) with \(f(t) = t^{1/q}\) is \(r\)-convex for all \(2 < r < q\).

We observe that \(Y := \mathbb{H} \mathcal{L}(\ell_{q,\infty,1/2}, X)\) coincides with \(\ell_{g,\infty}\) where \(g(t) = t^{1/q}/\sqrt{\log(t+1)}\). Using Mascioni’s observation above we deduce from theorem 3.2 that \(T\) is of weak cotype q if and only if \(T\) is \((Y,2)\)-summing.

If \(T\) is \((Y,2)\)-summing and \(u \in \mathcal{L}(\ell_2, C(K))\) we can apply a lemma due to Lewis, see [PIE], which guarantees for all \(\varepsilon > 0\) the existence of an orthonormal system \((o_k)_k \subset \ell_2\) with \(\|Tu(o_k)\|_F\) decreasing and

\[
a_k(Tu) \leq (1 + \varepsilon) \|Tu(o_k)\|_F.
\]

Therefore we deduce

\[
\sup_{k \in \mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} a_k(Tu) \leq (1 + \varepsilon) \sup_{k \in \mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} \|Tu(o_k)\|_F
\]

\[
\leq (1 + \varepsilon) \pi_{Y,2}(T) \omega_2(u(o_k))_k
\]

\[
\leq (1 + \varepsilon) \pi_{Y,2}(T) \|u\|.
\]

Taking the infimum over all \(\varepsilon\) and the supremum over all \(u \in \mathcal{L}(\ell_2, C(K))\) with norm less than 1 we obtain
\[
\sup_{k \in \mathbb{N}} \frac{k^{1/q}}{\sqrt{\log(k+1)}} x_k(T) \leq \pi_{Y,2}(T).
\]

Vice versa, let us assume that the sequence of Weyl numbers is in \(Y\). Let \((x_k)_k \in C(K)\) with \(\omega_2(x_k)_k \leq 1\). There is no restriction to assume that \(\|Tx_k\|_F\) is decreasing. If we define \(u_n := \sum_{k=1}^{n} e_k \otimes x_k\) we can deduce from an inequality of König, see [PIE],

\[
n^{1/2} \|Tx_n\| \leq \left(\sum_{k=1}^{n} \|Tx_k\|^2\right)^{1/2} \leq \pi_2(Tu_n)
\]

\[
\leq c_1 \sum_{k=1}^{n} \frac{a_k(Tu_n)}{\sqrt{k}}
\]

\[
\leq c_1 \sum_{k=1}^{n} \frac{(\log(k+1))^{1/2}}{k^{1/2 + 1/q}} \left\| \sum_{k=1}^{n} x_k(T) e_k \right\|_Y \|u\|
\]

\[
\leq c_1 \sqrt{\log(n+1)} \left(\sum_{k=1}^{n} \frac{1/2 - 1/q}{1/2 - 1/q} \sum_{k=1}^{n} x_k(T) e_k \right)_Y.
\]

Taking the supremum over all \(n \in \mathbb{N}\) we have shown that \(T\) is \((Y,2)\)-summing.

\[\square\]

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