(Θₙ, slₙ) - GRADED LIE ALGEBRAS (n = 3, 4)

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Abstract. Let \( F \) be a field of characteristic zero and let \( g \) be a non-zero finite-dimensional split semisimple Lie algebra with root system \( \Delta \). Let \( \Gamma \) be a finite set of integral weights of \( g \) containing \( \Delta \) and \( \{0\} \). Following [2, 10], we say that a Lie algebra \( L \) over \( F \) is generalized root graded, or more exactly \( (\Gamma, g) \)-graded, if \( L \) contains a semisimple subalgebra isomorphic to \( g \), the \( g \)-module \( L \) is the direct sum of its weight subspaces \( L_\alpha \ (\alpha \in \Gamma) \) and \( L \) is generated by all \( L_\alpha \) with \( \alpha \neq 0 \) as a Lie algebra. Let \( g \cong sl_n \) and

\[
\Theta_n = \{0, \pm \varepsilon_i \pm \varepsilon_j, \pm \varepsilon_i, \pm 2\varepsilon_i \mid 1 \leq i \neq j \leq n\}
\]

where \( \{\varepsilon_1, \ldots, \varepsilon_n\} \) is the set of weights of the natural \( sl_n \)-module. In [9], we classify \( (\Theta_n, sl_n) \)-graded Lie algebras for \( n > 4 \). In this paper we describe the multiplicative structures and the coordinate algebras of \( (\Theta_n, sl_n) \)-graded Lie algebras \( (n = 3, 4) \). In \( n = 3 \), we assume that

\[
[V(2\omega_1) \otimes C, V(2\omega_1) \otimes C] = [V(2\omega_2) \otimes C', V(2\omega_2) \otimes C'] = 0
\]

where \( V(\omega) \) is the simple \( g \)-module of highest weight \( \omega \), \( C = \text{Hom}_g(V(2\omega_1), L) \) and \( C' = \text{Hom}_g(V(2\omega_2), L) \).

Contents

1. Introduction
2. Tensor product decompositions for the modules in \( \Theta_3^+ \) (\( n = 3, 4 \)).
3. Multiplication in \( (\Theta_n, sl_n) \)-graded Lie algebras \( (n = 3, 4) \)
   3.1. Multiplication in \( \Theta_3 \)-graded Lie algebras
   3.2. Multiplication in \( \Theta_4 \)-graded Lie algebras
4. Coordinate algebra of \( (\Theta_n, sl_n) \)-graded Lie algebras \( (n = 3, 4) \)
   4.1. Unital algebra \( a \)
   4.2. Coordinate algebra \( b \)

References

1. Introduction

In 1992, Berman and Moody introduced root-graded Lie algebras to study toroidal Lie algebras and Slodowy matrix algebras intersection. Nonetheless, this concept previously appeared in the study of simple Lie algebras by Seligman [9]. Root graded Lie algebras of simply-laced finite root systems were classified up to central isogeny by Berman and Moody in [7]. The case of double-laced finite root systems was settled by Benkart and Zelmanov [6]. Non-reduced systems \( BC_n \) were considered by Allison, Benkart and Gao [11] (for \( n \geq 2 \)) and by Benkart and Smirnov [5] (for \( n = 1 \)). The aim of this paper is to
describe the multiplicative structures and the coordinate algebras of \((\Theta_n, sl_n)\)-graded Lie algebras where \(n = 3, 4\) and

\[
\Theta_n = \{0, \pm \varepsilon_i \pm \varepsilon_j, \pm \varepsilon_i, \pm 2\varepsilon_i \mid 1 \leq i \neq j \leq n\}
\]

and \(\{\varepsilon_1, \ldots, \varepsilon_n\}\) is the set of weights of the natural \(sl_n\)-module. Throughout the paper, the ground field \(\mathbb{F}\) is of characteristic zero and the grading subalgebra \(\mathfrak{g} \cong sl_n\). We denote by \(\Theta_n^+\) the set of dominant weights in \(\Theta_n\) and the corresponding simple \(sl_n\)-modules. Thus,

\[
\Theta_n^+ = \{\omega_1 + \omega_3 = \varepsilon_1 - \varepsilon_4, \omega_1 = \varepsilon_1, \omega_3 = -\varepsilon_4, 2\omega_1 = 2\varepsilon_1, 2\omega_3 = -2\varepsilon_4, \omega_2 = \varepsilon_1 + \varepsilon_2, 0\},
\]

\[
\Theta_n^3 = \{\omega_1 + \omega_3 = \varepsilon_1 - \varepsilon_4, \omega_1 = \varepsilon_1, \omega_3 = -\varepsilon_4, 2\omega_1 = 2\varepsilon_1, 2\omega_3 = -2\varepsilon_4, 0\}.
\]

These are the highest weights of the irreducible \(g\)-modules. First we compute tensor product decompositions for the modules in \(\Theta_n^+\). Then we describe the multiplicative structures of \((\Theta_n, sl_n)\)-graded Lie algebras. The coordinate algebra of these Lie algebras are described in Section 4.

2. TENSOR PRODUCT DECOMPOSITIONS FOR THE MODULES IN \(\Theta_n^+ (n = 3, 4)\).

We begin with the general definition of Lie algebras graded by finite weight systems.

**Definition 2.1.** \([2]\) Let \(\Delta\) be a root system and let \(\Gamma\) be a finite set of integral weights of \(\Delta\) containing \(\Delta\) and \(\{0\}\). A Lie algebra \(L\) is called \((\Gamma, \mathfrak{g})\)-graded (or simply \(\Gamma\)-graded) if

- \((\Gamma 1)\) \(L\) contains as a subalgebra a non-zero finite-dimensional split semisimple Lie algebra

\[
\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,
\]

whose root system is \(\Delta\) relative to a split Cartan subalgebra \(\mathfrak{h} = \mathfrak{g}_0\);

- \((\Gamma 2)\) \(L = \bigoplus_{\alpha \in \Gamma} L_\alpha\) where \(L_\alpha = \{x \in L \mid [h, x] = \alpha(h) x \text{ for all } h \in \mathfrak{h}\}\);

- \((\Gamma 3)\) \(L_0 = \sum_{\alpha, -\alpha \in \Gamma \setminus \{0\}} [L_\alpha, L_{-\alpha}]\).

The subalgebra \(\mathfrak{g}\) is called the grading subalgebra of \(L\). If \(\mathfrak{g}\) is the split simple Lie algebra and \(\Gamma = \Delta \cup \{0\}\) then \(L\) is said to be root-graded.

We fix a base \(\Pi = \{\varepsilon_i - \varepsilon_{i+1} \mid i = 1, 2, \ldots, n-1\}\) of simple roots for the root system \(\Delta_{n-1} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \neq j \leq n\}\). Let \(L\) be a Lie algebra containing a non-zero split simple subalgebra \(\mathfrak{g}\). Using the same arguments of \([10]\) Lemma 3.1.2 and Proposition 3.2.2] we get the following:

- (1) A Lie algebra \(L\) is \((\Theta_3, \mathfrak{g})\)-graded if and only if \(L\) is generated by \(\mathfrak{g}\) as an ideal and the \(\mathfrak{g}\)-module \(L\) decomposes into copies of the adjoint module, its symmetric \(S^2V\), its exterior squares \(\wedge^2V\), their duals and the one dimensional trivial \(\mathfrak{g}\)-module.

- (2) A Lie algebra \(L\) is \((\Theta_4, \mathfrak{g})\)-graded if and only if \(L\) is generated by \(\mathfrak{g}\) as an ideal and the \(\mathfrak{g}\)-module \(L\) decomposes into copies of the adjoint module (we will denote it by the same letter \(\mathfrak{g}\)), the natural module \(V\), its exterior squares \(\wedge^2V\), its symmetric \(S^2V\), their duals and the one dimensional trivial \(\mathfrak{g}\)-module.
Thus, by collecting isotypic components, we get the following decomposition of the $g$-module $L$:

$$\text{(2.1)} \quad L = (g \otimes A) \oplus (S \otimes C) \oplus (S' \otimes C') \oplus (\Lambda \otimes E) \oplus (\Lambda' \otimes E') \oplus D \quad \text{for } n = 3;$$

$$\text{(2.2)} \quad L = (g \otimes A) \oplus (V \otimes B) \oplus (V' \otimes B') \oplus (S \otimes C) \oplus (S' \otimes C') \oplus (\Lambda \otimes E) \oplus D \quad \text{for } n = 4;$$

where $A, B, B', C, C', E$ are vector spaces,

$$g := V(\omega_1 + \omega_{n-1}), \quad V := V(\omega_1), \quad V' := V(\omega_{n-1}),$$

$$S := V(2\omega_1), \quad S' := V(2\omega_{n-1}), \quad \Lambda := V(\omega_2), \quad \Lambda' := V(\omega_{n-2})$$

and $D$ is the sum of the trivial $g$-modules. Note that $\Lambda \cong \Lambda'$ for $n = 4.$

Alternatively, these spaces can also be viewed as the corresponding $g$-mod Hom-spaces: $A = \text{Hom}_g(g, L), B = \text{Hom}_g(V, L),$ etc, so for each simple $g$-module $M$, the space $M \otimes \text{Hom}_g(M, L)$ is canonically identified with the $M$-isotypic component of $L$ via the evaluation map

$$\text{(2.2)} \quad M \otimes \text{Hom}_g(M, L) \rightarrow L, \quad m \otimes \varphi \mapsto \varphi(m),$$

see [8, Proposition 4.1.15].

We can write the following examples of $\Theta_n$-graded Lie algebras ($n = 3, 4$):

1) Let $L = sl_{n+k}$ and let $g$ be the copy of $sl_n$ in the northwest corner. We consider the adjoint action of $g$ on $L$. Then the $g$-module $L$ decomposes into $k$ copies of the natural module $V = \mathbb{F}^n$, $k$ copies of the dual module $V' = \text{Hom}(V, \mathbb{F})$, an adjoint module $g$ and one dimensional trivial $g$-modules in its southeast corner. Then

$$L = g \oplus V^\oplus k \oplus V'^\oplus k \oplus D$$

where $D$ is the sum of the trivial $sl_n$-modules. As a result, we may write

$$L = g \oplus (V \otimes B) \oplus (V' \otimes B') \oplus D$$

where $B \cong B' \cong \mathbb{F}^k.$ Then $L$ is $(A_{n-1}, g)$-graded. Bahturin and Benkart [3] (for $n > 3$) and Benkart and Elduque [1] (for $n = 3$) described the multiplicative structure of this type of Lie algebras.

2) Any Lie algebra which is $(A_{n-1}, sl_n)$-graded is also $\Theta_n$-graded. For such a Lie algebra, the space $L_\alpha = \{0\}$ for all $\alpha$ not in $A_{n-1}.$

3) Let $L = sl_{2n+1}$ and $g = \left\{ \begin{bmatrix} x & 0 & 0 \\ 0 & -x^t & 0 \\ 0 & 0 & 0 \end{bmatrix} \mid x \in sl_n \right\} \subset L$ where $n = 3, 4.$ We consider the adjoint action of $g$ on $L.$ Then $L$ is $(\Theta_n, g)$-graded.

**Definition 2.2.** (1) We identify the $g$-modules $V$ and $V'$ with the space $\mathbb{F}^n (n = 3, 4)$ of column vectors with the following actions:

$$x.v = xv \quad \text{for } x \in sl_n, v \in V,$$

$$x.v' = -x^tv' \quad \text{for } x \in sl_n, v' \in V'.$$
(2) We identify $S$ and $S'$ (resp. $\Lambda$ and $\Lambda'$) with symmetric (resp. skew-symmetric) $n \times n$ matrices over $\mathbb{F}$ $(n = 3, 4)$. Then, $S$, $S'$, $\Lambda$ and $\Lambda'$ are $\mathfrak{g}$-modules under the actions:

$$x.s = xs + sx^t \quad \text{for} \quad x \in sl_n, \ s \in S,$$

$$x.\lambda = x\lambda + \lambda x^t \quad \text{for} \quad x \in sl_n, \ \lambda \in \Lambda,$$

$$x.s' = -s'x - x^ts' \quad \text{for} \quad x \in sl_n, \ s' \in S,$$

$$x.\lambda' = -\lambda'x - x^t\lambda' \quad \text{for} \quad x \in sl_n, \ \lambda' \in \Lambda'.$$

Since the subalgebra $\mathfrak{g}$ of $L$ is a $\mathfrak{g}$-submodule, there exists a distinguished element $\mathbf{1}$ of $A$ such that $\mathfrak{g} = \mathfrak{g} \otimes \mathbf{1}$. In particular,

\[(2.3) \quad [x \otimes 1, y \otimes b] = x.y \otimes b.\]

where $x \otimes 1$ is in $\mathfrak{g} \otimes 1$, $y \otimes b$ belongs to one of the components in $\mathfrak{g}$, and $x.y$ is as in Definition 2.2.

From equation 2.1 and Definition 2.2 and the properties that $V \sim \Lambda^\prime$ and $V' \sim \Lambda$ for $sl_3$, we get the following decomposition of the $\mathfrak{g}$-module $L$:

\[(2.4) \quad L = (\mathfrak{g} \otimes A) \oplus (S \otimes C) \oplus (S' \otimes C') \oplus (\Lambda \otimes E) \oplus (\Lambda' \otimes E') \oplus D.\]

From equation 2.1 and Definition 2.2 and the property that $\Lambda \sim \Lambda'$ for $sl_4$, we get the following decomposition of the $\mathfrak{g}$-module $L$:

\[(2.5) \quad L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus (V' \otimes B') \oplus (S \otimes C) \oplus (S' \otimes C') \oplus (\Lambda \otimes E) \oplus D.\]

Using [10, Section 3.4], in Tables 1 and 2 we get the following $\Theta$-components of all tensor product decompositions for the modules in $\Theta^+(n = 3, 4)$. If the cell in row $X$ and column $Y$ contains $Z$ this means that $\Theta(X \otimes Y) \cong Z$.

| $\otimes$ | $\mathfrak{g}$ | $S$ | $S'$ | $V \cong \Lambda'$ | $V' \cong \Lambda$ |
|-----------|----------------|-----|------|-------------------|-------------------|
| $\mathfrak{g}$ | $\mathfrak{g} + \mathfrak{T}$ | $S + \Lambda$ | $S' + \Lambda'$ | $S' + \Lambda$ | $S + \Lambda$ |
| $S$ | $S + \Lambda$ | $S'$ | $\mathfrak{g} + \mathfrak{T}$ | $\mathfrak{g}$ | $\Lambda'$ |
| $S'$ | $S' + \Lambda'$ | $\mathfrak{g} + \mathfrak{T}$ | $S$ | $\Lambda$ | $\mathfrak{g}$ |
| $V$ | $S' + \Lambda'$ | $\mathfrak{g}$ | $\Lambda$ | $S + \Lambda$ | $\mathfrak{g} + \mathfrak{T}$ |
| $V'$ | $S + \Lambda'$ | $\Lambda'$ | $\mathfrak{g}$ | $\mathfrak{g} + \mathfrak{T}$ | $S' + \Lambda'$ |

Table 1. $\Theta$-component of tensor product decompositions for $sl_3$

| $\otimes$ | $\mathfrak{g}$ | $S$ | $\Lambda \cong \Lambda'$ | $S'$ | $V$ | $V'$ |
|-----------|----------------|-----|-------------------|------|-----|-----|
| $\mathfrak{g}$ | $\mathfrak{g} + \mathfrak{T}$ | $S + \Lambda$ | $S + \Lambda + S'$ | $S' + \Lambda$ | $V$ | $V'$ |
| $S$ | $S + \Lambda$ | $\mathfrak{g}$ | $\mathfrak{g} + \mathfrak{T}$ | $\mathfrak{g}$ | $0$ | $V$ |
| $\Lambda$ | $S + \Lambda + S'$ | $\mathfrak{g}$ | $\mathfrak{g} + \mathfrak{T}$ | $\mathfrak{g}$ | $0$ | $V'$ |
| $S'$ | $S' + \Lambda$ | $\mathfrak{g} + \mathfrak{T}$ | $\mathfrak{g}$ | $0$ | $V'$ | $0$ |
| $V$ | $V$ | $0$ | $V'$ | $V'$ | $S + \Lambda$ | $\mathfrak{g} + \mathfrak{T}$ |
| $V'$ | $V'$ | $V$ | $V$ | $0$ | $\mathfrak{g} + \mathfrak{T}$ | $S' + \Lambda$ |

Table 2. $\Theta$-component of tensor product decompositions for $sl_4$
In Tables 3 and 4 below, if the cell in row \( X \) and column \( Y \) contains \( Z \), this means that there is a bilinear map \( X \otimes Y \to Z \) given by \( x \otimes y \mapsto (x, y)_Z \). For simplicity of notation, we will write \( dy \) instead of \( (d, y)_D \) if \( X = Z = D \) and we will write \( \langle x, y \rangle \) instead of \( (x, y)_D \) if \( X, Y \neq D \) and \( Z = D \). In the case \( X = Y = Z = A \), we have two bilinear products \( a_1 \otimes a_2 \mapsto a_1 \circ a_2 \) and \( a_1 \otimes a_2 \mapsto [a_1, a_2] \) for \( a_1, a_2 \in A \). Note that some of the cells are empty. The corresponding products \( X \otimes Y \to Z \) will be defined later by extending the existing maps \( Y \otimes X \to Z \). This will make the table symmetric.

### Table 3. Bilinear products for \( n = 3 \)

| . | \( A \) | \( C \) | \( C' \) | \( E \) | \( E' \) | \( D \) |
|---|---|---|---|---|---|---|
| \( A \) | \( A, \circ, [ ] \), \( D \) | \( C, E \) | \( C, E \) | | | |
| \( C \) | | | \( 0 \) | \( A, D \) | \( E' \) | \( A \) |
| \( C' \) | \( C', E' \) | | | | \( A \) | \( E \) |
| \( E \) | | \( E' \) | | \( C', E' \) | \( A, D \) | |
| \( E' \) | \( C', E' \) | | \( E \) | \( C, E \) | | |
| \( D \) | | \( A \) | \( C \) | \( C' \) | \( E \) | \( E' \) | \( D \) |

### Table 4. Bilinear products for \( n = 4 \)

| . | \( A \) | \( B \) | \( B' \) | \( C \) | \( C' \) | \( E \) | \( D \) |
|---|---|---|---|---|---|---|---|
| \( A \) | \( A, \circ, [ ] \), \( D \) | \( B \) | | \( C, E \) | \( C, E, C' \) | | |
| \( B \) | | \( C, E \) | \( A, D \) | | \( 0 \) | | |
| \( B' \) | | \( A \) | \( C', E \) | \( B \) | \( 0 \) | \( B \) | |
| \( C \) | | | \( 0 \) | \( 0 \) | \( A, D \) | \( 0 \) | |
| \( C' \) | \( C', E \) | \( B' \) | \( 0 \) | \( 0 \) | \( A \) | | |
| \( E \) | | | \( 0 \) | \( 0 \) | \( 0 \) | | |
| \( D \) | \( A \) | \( B \) | \( B' \) | \( C \) | \( C' \) | \( E \) | \( D \) |

Let \( x \) and \( y \) be \( n \times n \) matrices. We will use the following products:

\[
[x, y] = xy - yx,
\]

\[
x \circ y = xy + yx - \frac{2}{n} \text{tr}(xy)I,
\]

\[
x \circ y = xy + yx,
\]

\[
(x \mid y) = \frac{1}{n} \text{tr}(xy).
\]

3. Multiplication in \( (\Theta_n, sl_n) \)-graded Lie algebras \( (n = 3, 4) \)

### 3.1. Multiplication in \( \Theta_3 \)-graded Lie algebras

Let \( L \) be a \( \Theta_3 \)-graded Lie algebra with the grading subalgebra \( \mathfrak{g} \cong sl_3 \) and the properties that

\[
[V(2\omega_1) \otimes C, V(2\omega_1) \otimes C] = [V(2\omega_2) \otimes C', V(2\omega_2) \otimes C'] = 0.
\]
where \( V(\omega) \) is the simple \( g \)-module of highest weight \( \omega \), \( C = \text{Hom}_g(V(2\omega_1), L) \) and \( C' = \text{Hom}_g(V(2\omega_2), L) \). In [2.4] we show that \( L \) is the direct sum of finite-dimensional irreducible \( g \)-modules whose highest weights are in \( \Theta^+_3 \), i.e. as a \( g \)-module,

\[
L = (g \otimes A) \oplus (S \otimes C) \oplus (S' \otimes C') \oplus (\Lambda \otimes E) \oplus (\Lambda' \otimes E') \oplus D.
\]

Note that \( V' = U(sl_3)e_3 \) and \( \Lambda = U(sl_3)(E_{1,2} - E_{2,1}) \) are highest weight modules with highest weight \( \omega_2 \) where \( E_{i,j} \) denote the matrix units and \( e_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \) (resp. \( V = U(sl_4)e_1 \) and \( \Lambda' = U(sl_4)(E_{3,4} - E_{4,3}) \) are highest weight module with highest weight \( \omega_1 \) where \( e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \). Define \( f: \Lambda \rightarrow V \) by \( f(x.(E_{3,4} - E_{4,3})) = x.e_1 \) and \( g: \Lambda \rightarrow V' \) by \( g(x.(E_{1,2} - E_{2,1})) = x.e_3 \) for all \( x \in U(sl_4) \). This allows us to identify \( \Lambda \) with \( V' \) and \( \Lambda' \) with \( V \).

In (3.2) we list bases for all non-zero \( g \)-module homomorphism spaces \( \text{Hom}_g(X \otimes Y, Z) \) where \( X, Y, Z \in \{ g, V, V', S, S', T \} \) and \( X \) and \( Y \) are both non-trivial. Note that all of them are 1-dimensional except the first one (which is 2-dimensional).

\[
\text{Hom}_g(g \otimes g, g) = \text{span}\{x \otimes y \mapsto xy - xy, x \otimes y \mapsto xy + yx - \frac{2}{3} \text{tr}(xy)I\},
\]

\[
\text{Hom}_g(\Lambda \otimes \Lambda', g) = \text{span}\{\lambda \otimes \lambda' \mapsto \lambda\lambda' - \frac{\text{tr}(\lambda\lambda')}{}I\},
\]

\[
\text{Hom}_g(\Lambda \otimes \Lambda, \Lambda') = \text{span}\{\lambda_1 \otimes \lambda_2 \mapsto f(\lambda_1')(f(\lambda_2))' - f(\lambda_2')(f(\lambda_1))'\},
\]

\[
\text{Hom}_g(\Lambda \otimes \Lambda, S') = \text{span}\{\lambda_1 \otimes \lambda_2 \mapsto f(\lambda_1')(f(\lambda_2))' + f(\lambda_2')(f(\lambda_1))'\},
\]

\[
\text{Hom}_g(\Lambda \otimes \Lambda, \Lambda) = \text{span}\{\lambda_1 \otimes \lambda_2 \mapsto g(\lambda_1)(g(\lambda_2))' - g(\lambda_2)(g(\lambda_1))'\},
\]

\[
\text{Hom}_g(S \otimes \Lambda', g) = \text{span}\{s \otimes \lambda' \mapsto s\lambda'\},
\]

\[
\text{Hom}_g(S' \otimes \Lambda, g) = \text{span}\{s' \otimes \lambda \mapsto s'\lambda\},
\]

\[
\text{Hom}_g(S \otimes S', g) = \text{span}\{s \otimes s' \mapsto ss' - \frac{\text{tr}(ss')}{3}I\},
\]

\[
\text{Hom}_g(S \otimes S', g) = \text{span}\{}\{\lambda' \otimes x \mapsto \lambda'x + x'\lambda'\},
\]

\[
\text{Hom}_g(S \otimes S', g) = \text{span}\{}\{s \otimes x \mapsto sx + x's\},
\]

\[
\text{Hom}_g(S' \otimes S, g) = \text{span}\{}\{s' \otimes x \mapsto s'x + x's\},
\]

\[
\text{Hom}_g(S' \otimes S', g) = \text{span}\{}\{\lambda' \otimes x \mapsto \lambda'x - x'\lambda'\},
\]
Following the methods in [1, 9, 7, 6], using the results of (3.2) and Table I, we may suppose that the multiplication in $L$ is given as follows. For all $x, y \in sl_3, s \in S, \lambda_1, \lambda_2 \in \Lambda, s' \in S', \lambda_1', \lambda_2' \in \Lambda'$ and for all $a, a_1, a_2 \in A, c \in C, c' \in C', e \in E, e' \in E'$ and $d, d_1, d_2 \in D$.

(3.3)

$$[x \otimes a_1, y \otimes a_2] = (x \circ y) \otimes \frac{[a_1, a_2]}{2} + [x, y] \otimes \frac{a_1 \circ a_2}{2} + (x \mid y)(a_1, a_2),$$

$$[\lambda \otimes e, \lambda' \otimes e'] = (\lambda\lambda' - (\lambda \mid \lambda')I) \otimes (e, e')_{A} + (\lambda \mid \lambda')\langle e, e' \rangle = -[\lambda \otimes e', \lambda \otimes e],$$

$$[\lambda_1 \otimes e_1, \lambda_2 \otimes e_2] = (g(\lambda_1)(g(\lambda_2))^t + g(\lambda_2)(g(\lambda_1))^t) \otimes \frac{(e_1, e_2)_C}{2} + (g(\lambda_1)(g(\lambda_2))^t - g(\lambda_2)(g(\lambda_1))^t) \otimes \frac{(e_1, e_2)_E}{2},$$

$$[\lambda'_1 \otimes e'_1, \lambda'_2 \otimes e'_2] = (f(\lambda'_1)(f(\lambda'_2))^t + f(\lambda'_2)(f(\lambda'_1))^t) \otimes \frac{(e'_1, e'_2)_{C'}}{2} + (f(\lambda'_1)(f(\lambda'_2))^t - f(\lambda'_2)(f(\lambda'_1))^t) \otimes \frac{(e'_1, e'_2)_{E'}}{2},$$

$$[s \otimes c, s' \otimes c'] = (ss' - (s \mid s')I) \otimes (c, c')_{A} + (s \mid s')(c, c') = -[s \otimes c', s \otimes c],$$

$$[x \otimes a, s \otimes c] = (xs + sx^t) \otimes \frac{(a, c)_C}{2} + (xs - sx^t) \otimes \frac{(a, c)_E}{2} = -[s \otimes c, x \otimes a],$$

$$[x \otimes a, \lambda \otimes e] = (x\lambda + \lambda x^t) \otimes \frac{(a, e)_E}{2} + (x\lambda - \lambda x^t) \otimes \frac{(a, e)_C}{2} = -[\lambda \otimes e, x \otimes a],$$

$$[s' \otimes c', x \otimes a] = (s'x + x^ts') \otimes \frac{(c', a)_{C'}}{2} + (s'x - x^ts') \otimes \frac{(c', a)_{E'}}{2} = -[x \otimes a, s' \otimes c'],$$

$$[\lambda' \otimes e', x \otimes a] = (\lambda'x + x^t\lambda') \otimes \frac{(e', a)_{E'}}{2} + (\lambda'x - x^t\lambda') \otimes \frac{(e', a)_{C'}}{2} = -[x \otimes a, \lambda' \otimes e'],$$

$$[s \otimes c, \lambda' \otimes e'] = s\lambda' \otimes (c, e')_{A} = -[\lambda' \otimes e', s \otimes c],$$

$$[s' \otimes c', \lambda \otimes e] = s'\lambda \otimes (c', e')_{A} = -[\lambda \otimes e, s' \otimes c'],$$

$$[s' \otimes c', \lambda' \otimes e'] = s'f(\lambda') \otimes (c', e')_{E} = -[\lambda' \otimes e', s' \otimes c'],$$

$$[\lambda \otimes e, s \otimes c] = sg(\lambda) \otimes (e, c)_{E'} = -[s \otimes c, \lambda \otimes e],$$

$$[d, x \otimes a] = x \otimes da = -[x \otimes a, d],$$

$$[d, s \otimes c] = s \otimes dc = -[s \otimes c, d],$$
Multiplication in $\Theta_4$-graded Lie algebras. Let $L$ be a $\Theta_4$-graded Lie algebra with the grading subalgebra $\mathfrak{g} \cong \mathfrak{sl}_4$. In [2,4] we show that $L$ is the direct sum of finite-dimensional irreducible $\mathfrak{g}$-modules whose highest weights are in $\Theta_3^+$, i.e. as a $\mathfrak{g}$-module,

$$L = (\mathfrak{g} \otimes A) \oplus (V \otimes B) \oplus (V' \otimes B') \oplus (S \otimes C) \oplus (S' \otimes C') \oplus (\Lambda \otimes E) \oplus D.$$

Note that $\Lambda = U(sl_4)(E_{1,2} - E_{2,1})$ and $\Lambda' = U(sl_4)(E_{3,4} - E_{4,3})$ are highest weight modules with highest weight $\omega_2$. Define $f: \Lambda' \to \Lambda$ by $f(x.(E_{3,4} - E_{4,3})) = x.(E_{1,2} - E_{2,1})$ for all $x \in U(sl_4)$. This allows us to identify $\Lambda$ with $\Lambda'$.

In (3.4) we list bases for all non-zero $\mathfrak{g}$-module homomorphism spaces $\text{Hom}_\mathfrak{g}(X \otimes Y, Z)$ where $X, Y, Z \in \{\mathfrak{g}, V, V', S, \Lambda, S', T\}$ and $X$ and $Y$ are both non-trivial. Note that all of them are 1-dimensional except the first one (which is 2-dimensional).

(3.4) \begin{align*}
\text{Hom}_\mathfrak{g}(\mathfrak{g} \otimes \mathfrak{g}, \mathfrak{g}) &= \text{span}\{x \otimes y \mapsto xy - yx, \ x \otimes y \mapsto xy + yx - \frac{2}{4} \text{tr}(xy)I\}, \\
\text{Hom}_\mathfrak{g}(V \otimes V', \mathfrak{g}) &= \text{span}\{u \otimes v' \mapsto uv' - \frac{1}{4} \text{tr}(uv')I\}, \\
\text{Hom}_\mathfrak{g}(S \otimes \Lambda, \mathfrak{g}) &= \text{span}\{s \otimes \lambda \mapsto sf^{-1}(\lambda)\}, \\
\text{Hom}_\mathfrak{g}(S' \otimes \Lambda, \mathfrak{g}) &= \text{span}\{s' \otimes \lambda \mapsto s'\lambda\}, \\
\text{Hom}_\mathfrak{g}(\Lambda \otimes \Lambda, \mathfrak{g}) &= \text{span}\{\lambda \otimes \lambda' \mapsto \lambda f^{-1}(\lambda') - \frac{1}{4} \text{tr}(\lambda f^{-1}(\lambda'))I\}, \\
\text{Hom}_\mathfrak{g}(S \otimes S', \mathfrak{g}) &= \text{span}\{s \otimes s' \mapsto ss' - \frac{1}{4} \text{tr}(ss')I\}, \\
\text{Hom}_\mathfrak{g}(\mathfrak{g} \otimes V, V) &= \text{span}\{x \otimes v \mapsto xv\}, \\
\text{Hom}_\mathfrak{g}(\Lambda \otimes V', V) &= \text{span}\{\lambda \otimes v' \mapsto \lambda v'\}, \\
\text{Hom}_\mathfrak{g}(S \otimes V', V) &= \text{span}\{s \otimes v' \mapsto sv'\}, \\
\text{Hom}_\mathfrak{g}(\mathfrak{g} \otimes V', V') &= \text{span}\{x \otimes v' \mapsto x'v'\}, \\
\text{Hom}_\mathfrak{g}(S' \otimes V, V') &= \text{span}\{s' \otimes v \mapsto s'v\}, \\
\text{Hom}_\mathfrak{g}(\Lambda \otimes V, V') &= \text{span}\{\lambda \otimes v \mapsto f^{-1}(\lambda)v\}, \\
\text{Hom}_\mathfrak{g}(\mathfrak{g} \otimes S, S) &= \text{span}\{x \otimes s \mapsto xs + sx'\}, \\
\text{Hom}_\mathfrak{g}(V \otimes V, S) &= \text{span}\{u \otimes v \mapsto uv' + vu'\}, \\
\text{Hom}_\mathfrak{g}(\mathfrak{g} \otimes \Lambda, S) &= \text{span}\{x \otimes \lambda \mapsto x\lambda - \lambda x'\}, \\
\text{Hom}_\mathfrak{g}(\mathfrak{g} \otimes \Lambda, S') &= \text{span}\{x \otimes \lambda \mapsto f^{-1}(\lambda)x - x'f^{-1}(\lambda)\}, \\
\text{Hom}_\mathfrak{g}(S' \otimes \mathfrak{g}, S') &= \text{span}\{s' \otimes x \mapsto s'x + x's'\}, \end{align*}
Following the methods in [1, 9, 7, 6], using the results of (3. 4) and Table 2, we may suppose that the multiplication in $L$ is given as follows. For all $x, y \in sl_4$, $u, v \in V$, $u', v' \in V'$, $s \in S$, $\lambda \in \Lambda$, $s' \in S'$, $\lambda' \in \Lambda'$ and for all $a, a_1, a_2 \in A$, $b, b_1, b_2 \in B$, $b', b'_1, b'_2 \in B'$, $c \in C$, $c' \in C'$, $e \in E$ and $d, d_1, d_2 \in D$,}

(3.5)

\[
[x \circ a_1, y \circ a_2] = (x \circ y) \otimes \frac{[a_1, a_2]}{2} + [x, y] \otimes \frac{a_1 \circ a_2}{2} + (x | y)(a_1, a_2),
\]

\[
[u \otimes b, v' \otimes b'] = (uv'' - \frac{\text{tr}(uv'')}{n} I) \otimes (b, b')_A + \frac{2}{n} \text{tr}(uv'')(b, b') = -[v' \otimes b', u \otimes b],
\]

\[
[s \otimes c, s' \otimes c'] = (ss' - (s | s')I) \otimes (c, c')_A + (s | s')(c, c') = -[s' \otimes c', s \otimes c],
\]

\[
[\lambda_1 \otimes e_1, \lambda_2 \otimes e_2] = (\lambda_1 f^{-1}(\lambda_2) - (\lambda_1 | f^{-1}(\lambda_2))I) \otimes (e_1, e_2)_A + (\lambda_1 | f^{-1}(\lambda_2))(e_1, e_2),
\]

\[
[u \otimes b_1, v \otimes b_2] = (uv'' + v''u') \otimes \frac{(b'_1, b'_2)_C}{2} + (uv'' - v''u') \otimes \frac{(b_1, b_2)_E}{2},
\]

\[
[u' \otimes b'_1, v' \otimes b'_2] = (u'v'' + v'u'') \otimes \frac{(b'_1, b'_2)_C}{2} + f(u'v'' - v'u'') \otimes \frac{(b'_1, b'_2)_E}{2},
\]

\[
[x \otimes a, s \otimes c] = (xs + sx') \otimes \frac{(a, c)_C}{2} + (xs - sx') \otimes \frac{(a, c)_E}{2} = -[s \otimes c, x \otimes a],
\]

\[
[x \otimes a, \lambda \otimes e] = (x\lambda + \lambda x') \otimes \frac{(a, e)_E}{2} + (x\lambda - \lambda x') \otimes \frac{(a, e)_C}{2} + (f^{-1}(\lambda)x - x'f^{-1}(\lambda)) \otimes \frac{(a, e)_C}{2} = -[\lambda \otimes e, x \otimes a]
\]

\[
[s' \otimes c', x \otimes a] = (s'x + x's') \otimes \frac{(c', a)_C}{2} + f(s'x - x's') \otimes \frac{(c', a)_E}{2} = -[x \otimes a, s' \otimes c'],
\]

\[
[s \otimes c, \lambda \otimes e] = sf^{-1}(\lambda) \otimes (c, e)_A = -[\lambda \otimes e, s \otimes c],
\]

\[
[s \otimes c, \lambda \otimes e] = sf^{-1}(\lambda) \otimes (c, e)_A = -[\lambda \otimes e, s \otimes c],
\]
All other products of the homogeneous components of the decomposition \((2.1)\) are zero.

\[ [s' \otimes c', \lambda \otimes e] = s' \lambda \otimes (c', e)_A = -[\lambda \otimes e, s' \otimes c'], \]
\[ [x \otimes a, u \otimes b] = xu \otimes (a, b)_B = -[u \otimes b, x \otimes a], \]
\[ [s' \otimes c', u \otimes b] = s' u \otimes (c', b)_{B'} = -[u \otimes b, s' \otimes c'], \]
\[ [\lambda \otimes e, u \otimes b] = f^{-1}(\lambda)u \otimes (e, b)_{B'} = -[u \otimes b, \lambda \otimes e], \]
\[ [u' \otimes b', x \otimes a] = x u' \otimes (b', a)_{B'} = -[x \otimes a, u' \otimes b'], \]
\[ [u' \otimes b', s \otimes c] = su' \otimes (b', c)_B = -[s \otimes c, u' \otimes b'], \]
\[ [u' \otimes b', \lambda \otimes c] = -\lambda u' \otimes (b', e)_B = -[\lambda \otimes e, u' \otimes b'], \]
\[ [d, x \otimes a] = x \otimes da = -[x \otimes a, d], \]
\[ [d, u \otimes b] = u \otimes db = -[u \otimes b, d], \]
\[ [d, s \otimes c] = s \otimes dc = -[s \otimes c, d], \]
\[ [d, \lambda \otimes e] = \lambda \otimes de = -[\lambda \otimes e, d], \]
\[ [d, s' \otimes c'] = s' \otimes dc' = -[s' \otimes c', d], \]
\[ [d, u' \otimes b'] = u' \otimes db' = -[u' \otimes b', d], \]
\[ [d_1, d_2] \in D, \]

4. Coordinate algebra of \((\Theta_n, sl_n)\)-graded Lie algebras \((n = 3, 4)\)

Let \(L\) be a \(\Theta_n\)-graded Lie algebra with the grading subalgebra \(g \cong sl_n\). Throughout this section we assume that \(n = 4\) or \(n = 3\) and the conditions \((3.1)\) hold. Let \(g^\pm = \{x \in sl_n \mid x^t = \pm x\}\). Then

\[ (4.1) \quad g \otimes A = (g^+ \oplus g^-) \otimes A = (g^+ \otimes A) \oplus (g^- \otimes A) = (g^+ \otimes A^-) \oplus (g^- \otimes A^+) \]

where \(A^\pm\) is a copy of the vector space \(A\). Recall that we identify \(g\) with \(g \otimes 1\) where 1 is a distinguished element of \(A\). We denote by \(a^\pm\) the image of \(a \in A\) in the space \(A^\pm\).

Denote
\[ a := A^+ \oplus A^- \oplus C \oplus C' \oplus E \oplus E' \quad \text{for} \quad n = 3. \]
\[ a := A^+ \oplus A^- \oplus C \oplus E \oplus C' \quad \text{and} \quad b := a \oplus B \oplus B' \quad \text{for} \quad n = 4. \]

In Section \(3\) we described the multiplicative structures of these Lie algebras. In this section we describe the coordinate algebras of them, and we show that the product in \(L\) induces an algebra structure on both \(a\) and \(b\).

4.1. Unital algebra \(a\). We are going to define Lie and Jordan multiplication on \(a\) by extending the bilinear products given in Tables \(5\) and \(6\) in a natural way. It can be shown that all products \((\alpha_1, \alpha_2)_Z\) with \(\alpha_1, \alpha_2 \in a\) are either symmetric or skew-symmetric, except in the cases \((\alpha_1 = e\) and \(\alpha_2 = e\) or \(\alpha_1 = c'\) and \(\alpha_2 = e')\) for \(n = 3\). This is why we will write \((\alpha_1 \circ \alpha_2)_Z\) or \([\alpha_1, \alpha_2]_Z\), respectively, instead of \((\alpha_1, \alpha_2)_Z\). The aim of this subsection is to show that \(a\) is a unital algebra with respect to the new multiplication given by \(\alpha_1 \alpha_2 := \frac{[\alpha_1, \alpha_2]}{2} + \frac{\alpha_1 \circ \alpha_2}{2}\) for all homogeneous \(\alpha_1, \alpha_2 \in a\) with the
products \([\ ]\) and \(\circ\) given by Table 5, except in the cases \((\alpha_1 = c\) and \(\alpha_2 = e\) or \(\alpha_1 = c'\) and \(\alpha_2 = e')\) for \(n = 3\) and \(a^\pm \lambda\) for \(n = 4\), see (4.2).

Remark 4.1. By using the same arguments of [10] Remark 4.1), some of the products in (3.5) can be rewritten in terms of symmetric and skew-symmetric elements. For example for \(n = 4\) these products can be written as follows: For all \(x^\pm, x_1^\pm, x_2^\pm \in g^\pm, u, v \in V, u', v' \in V', s \in S, \lambda, \lambda_1, \lambda_2 \in \Lambda, s' \in S'\) and for all \(a^\pm, a_1^\pm, a_2^\pm \in A, b, b_1, b_2 \in B, b_1', b_2' \in B', c \in C, c' \in C', e \in E, d \in D\),

\[
[x_1^+ \otimes a_1^-, x_2^+ \otimes a_2^-] = x_1^+ \circ x_2^+ \otimes \frac{[a_1^-, a_2^-]_{A^-}}{2} + [x_1^+, x_2^+] \otimes \frac{(a_1^- \circ a_2^-)_{A^+}}{2} + (x_1^+ | x_2^+)\langle a_1^-, a_2^- \rangle,
\]

\[
[x_1^- \otimes a_1^+, x_2^- \otimes a_2^+] = x_1^- \circ x_2^- \otimes \frac{[a_1^+, a_2^+]_{A^-}}{2} + [x_1^-, x_2^-] \otimes \frac{(a_1^+ \circ a_2^+)_{A^+}}{2} + (x_1^- | x_2^-)\langle a_1^+, a_2^+ \rangle,
\]

\[
[x_1^+ \otimes a_1^-, x_1^- \otimes a_1^-] = x_1^+ \circ x_1^- \otimes \frac{[a_1^-, a_1^-]_{A^+}}{2} + [x_1^+, x_1^-] \otimes \frac{(a_1^- \circ a_1^-)_{A^-}}{2}.
\]

The mappings \(\alpha \otimes \beta \mapsto (\alpha \circ \beta)_{Z_1}\) and \(\alpha \otimes \beta \mapsto [\alpha, \beta]_{Z_2}\) can be extended to \(Y \otimes X\) in a consistent way by defining \((\beta \circ \alpha)_{Z_1} = (\alpha \circ \beta)_{Z_1}\) and \([\beta, \alpha]_{Z_2} = -[\alpha, \beta]_{Z_2}\). For \(n = 3\), \(\alpha \in C\) and \(\beta \in E\) we will write \(\alpha \beta\) (resp. \(\beta \alpha\)) instead of \((\alpha, \beta)_{Z}\) (resp. \((\beta, \alpha)_{Z}\)). The map can be extended by defining \((\beta \circ \alpha)_{E'} = -\beta \circ \alpha\) (resp. \((\beta \circ \alpha)_{E'} = -\beta \circ \alpha\)). In Table 5 and 6 below, if the cell in row \(X\) and column \(Y\) contains \((Z_1, \circ), (Z_2, [\ ]))\) this means that there is a symmetric bilinear map \(X \times Y \rightarrow Z_1\), given by \(\alpha \otimes \beta \mapsto (\alpha \circ \beta)_{Z_1}\), and a skew symmetric bilinear map \(X \times Y \rightarrow Z_2\), given by \(\alpha \otimes \beta \mapsto [\alpha, \beta]_{Z_2}\), \(\alpha \in X, \beta \in Y\).
for all homogeneous $\alpha$ respect to multiplication defined as follows:

\[ (\Theta_n, sl_n) - \text{GRADED LIE ALGEBRAS} \ (n = 3, 4) \]

We are going to show that $a = A^+ \oplus A^- \oplus C \oplus E \oplus C'$ is an associative algebra with respect to multiplication defined as follows:

\[ \alpha_1 \alpha_2 := \frac{[\alpha_1, \alpha_2]}{2} + \frac{\alpha_1 \circ \alpha_2}{2} \]

for all homogeneous $\alpha_1, \alpha_2 \in a$ with the products $[\ ]$ and $\circ$ given by Table 5 except in the case $a^\pm \lambda$ which is equal to

\[
\begin{align*}
\alpha_1 \alpha_2 & := \frac{[\alpha_1, \alpha_2]}{2} + \frac{\alpha_1 \circ \alpha_2}{2} \\
(4.2) a^- \lambda &= \frac{[a^- , e]_E}{2} + \frac{(a^- \circ e)_C}{2} + \frac{(a^- \circ e)_{C'}}{2} \\
&= \frac{[a^+ , e]_C}{2} + \frac{(a^+ \circ e)_E}{2} + \frac{[a^+ , e]_{C'}}{2}.
\end{align*}
\]

| . | $A^+$ | $A^-$ | $C$ | $C'$ | $E$ | $E'$ |
| --- | --- | --- | --- | --- | --- | --- |
| $A^+$ | $(A^+, \circ)$ | $(A^-, \circ)$ | $(C, \circ)$ | $(C', \circ)$ | $(E, \circ)$ | $(E', \circ)$ |
| $A^-$ | $(A^-, \circ)$ | $(A^+, \circ)$ | $(A^-, [\ )$ | $(A^+, [\ )$ | $(E', \circ)$ | $(C', \circ)$ |
| $C$ | $(C, \circ)$ | $(C, [\ ]$ | $(0)$ | $(A^+, \circ)$ | $(A^-, \circ)$ | $(E', \circ)$ |
| $C'$ | $(C', \circ)$ | $(C', [\ )$ | $(A^+, \circ)$ | $(0)$ | $(A^-, \circ)$ | $(E)$ |
| $E$ | $(E, \circ)$ | $(E, [\ )$ | $(E')$ | $(A^+, \circ)$ | $(A^-, \circ)$ | $(C, [\ ]$ |
| $E''$ | $(E', \circ)$ | $(C', \circ)$ | $(A^-, \circ)$ | $(A^+, \circ)$ | $(E, \circ)$ | $(C, [\ ]$ |

Table 5. Products of homogeneous components of $a$ for $n = 3$

| . | $A^+$ | $A^-$ | $C$ | $E$ | $C'$ |
| --- | --- | --- | --- | --- | --- |
| $A^+$ | $(A^+, \circ)$ | $(A^-, \circ)$ | $(C, \circ)$ | $(E, \circ)$ | $(C', \circ)$ |
| $A^-$ | $(A^-, \circ)$ | $(A^+, \circ)$ | $(E, \circ)$ | $(C, \circ)$ | $(E', \circ)$ |
| $C$ | $(C, \circ)$ | $(E, \circ)$ | $(0)$ | $(A^+, \circ)$ | $(A^-, \circ)$ |
| $E$ | $(E, \circ)$ | $(C', \circ)$ | $(A^-, \circ)$ | $(A^+, \circ)$ | $(A^-, \circ)$ |
| $C'$ | $(C', \circ)$ | $(E, \circ)$ | $(A^+, \circ)$ | $(A^-, \circ)$ | $(0)$ |

Table 6. Products of homogeneous components of $a$ for $n = 4$
Proof. 1) This is similar to the proof of [10, Theorem 4.1.3].

(2) and (3) can be deduce from tensor product decompositions for $sl_n$ ($n = 3, 4$), see Tables [1] and [2].

**Theorem 4.3.** The linear transformation $\gamma : a \rightarrow a$ defined by
\[\gamma(a^-) = -a^-, \gamma(a^+) = a^+, \gamma(c) = -c, \gamma(e) = e, \gamma(c') = -c',\]
is an antiautomorphism of order 2 of the algebra $a$.

**Proof.** See [9, Theorem 4.1.6].

4.2. **Coordinate algebra** $b$. Let $L$ be an $\Theta_3$-graded Lie algebra with the grading subalgebra $g \cong sl_4$. Recall that we denote
\[a := A^+ \oplus A^- \oplus C \oplus E \oplus C' \quad \text{and} \quad b := a \oplus B \oplus B'.\]
The aim of this subsection is to show that $b$ is an algebra with identity $1^+$ with respect to the multiplication extending that on $a$ given in Table [7]. It can be shown that all products $(\beta_1, \beta_2)_Z$ with $Z = B \oplus B'$ are either symmetric or skew-symmetric. This is why we will write $(\beta_1 \circ \beta_2)_Z$ or $[\beta_1, \beta_2]_Z$, respectively, instead of $(\beta_1, \beta_2)_Z$. For $\alpha \in a$ and $\beta \in B \oplus B'$ we will write $\alpha\beta$ (resp. $\beta\alpha$) instead of $(\alpha, \beta)_Z$ (resp. $(\beta, \alpha)_Z$). Let $b \in B$ and $b' \in B$. We define $ba := \gamma(\alpha)b$ and $ab' := b' \gamma(\alpha)$. We will show that $B \oplus B'$ is an $a$-bimodule.

Recall that
\[x \otimes a = \frac{(x + x^t)}{2} \otimes a + \frac{(x - x^t)}{2} \otimes a \in g^+ \otimes A + g^- \otimes A.\]

Let $u \otimes b \in V \otimes B$ and $v' \otimes b' \in V' \otimes B'$. We need the following formula from (3.5):
\[\{u \otimes b, v' \otimes b'\} = (uv^n - \frac{\text{tr}(uv^n)}{4} I) \otimes (b, b')_A + \frac{2 \text{tr}(uv^n)}{4} (b, b'}).\]

By splitting $(b, b')_A$ into symmetric and skew-symmetric parts and using the equations
\[\begin{align*}
(uv^n - \frac{\text{tr}(uv^n)}{4} I) + (uv^n - \frac{\text{tr}(uv^n)}{4} I)^t &= uv^n + v'u^t - \frac{2 \text{tr}(uv^n)}{4} I, \\
(uv^n - \frac{\text{tr}(uv^n)}{4} I) - (uv^n - \frac{\text{tr}(uv^n)}{4} I)^t &= uv^n - v'u^t,
\end{align*}\]
we get
\[\{u \otimes b, v' \otimes b'\} = (uv^n + v'u^t - \frac{2 \text{tr}(uv^n)}{4} I) \otimes \frac{[b, b']_A-}{2} + \\
(4.3) \quad (uv^n - v'u^t) \otimes \frac{(b \circ b')_A+}{2} + \frac{2 \text{tr}(uv^n)}{4} (b, b').\]
Proof. (1) This is similar to [10, Theorem 4.2.1].

Then \(\mathbf{b} = \mathfrak{a} \oplus \mathfrak{B} \oplus \mathfrak{B}'\) is an algebra with multiplication extending that on \(\mathfrak{a}\). The following table describes the products of homogeneous elements of \(\mathbf{b}\) (use Table 5 for the products on \(\mathfrak{a}\)).

|     | \(A^+ + A^-\) | \(C + E\) | \(C'\) | \(B\) | \(B'\) |
|-----|----------------|-----------|--------|------|--------|
| \(A^+ + A^-\) | \(A^+ + A^-\) | \(C + E + C'\) | \(C + E\) | \(B\) | \(B'\) |
| \(C + E\) | \(C + E + C'\) | 0 | \(A^+ + A^-\) | 0 | \(B\) |
| \(C'\) | \(C + E\) | \(A^+ + A^-\) | 0 | \(B'\) | 0 |
| \(B\) | \(B'\) | 0 | \(B'\) | \(C + E\) | \(A^+ + A^-\) |
| \(B'\) | \(B\) | 0 | \(A^+ + A^-\) | \(C' + E\) |

**Table 7. Products in \(\mathbf{b}\)**

Using the same arguments of [10, Section 4.2] we get the following properties:

**Theorem 4.4.**

(1) The linear transformation \(\eta : \mathbf{b} \to \mathbf{b}\) defined by \(\eta(\alpha) = \gamma(\alpha)\), \(\eta(b) = b\) and \(\eta(b') = b'\) for all \(\alpha \in \mathfrak{a}\), \(b \in \mathfrak{B}\) and \(b' \in \mathfrak{B}'\) is an antiautomorphism of order 2 of the algebra \(\mathbf{b}\).

(2) \(1^+\) is the identity element of \(\mathfrak{b}\).

(3) Let \(b \in \mathfrak{B}, b' \in \mathfrak{B}'\) and \(\alpha \in \mathfrak{a}\). Then

\[
[z \otimes \alpha, u \otimes b] = zu \otimes \alpha b = -[u \otimes b, z \otimes \alpha],
\]

\[
[u' \otimes b', z \otimes \alpha] = z' u' \otimes b' \alpha = -[z \otimes \alpha, u' \otimes b'].
\]

(4) \(\mathfrak{B} \oplus \mathfrak{B}'\) is an \(\mathfrak{a}\)-bimodule.

(5) \(\mathfrak{B}\) and \(\mathfrak{B}'\) are \(\mathfrak{A}\)-bimodules.

**Proof.** (1) This is similar to [10, Theorem 4.2.1].

(2) This is similar to [10, Theorem 4.2.2].

(3) We deduce this by using (3.3) and Table 7.

(4) See [10, Proposition 4.2.4].
(5) We deduce this from the properties that $B$ and $B'$ are invariant under multiplication by $\mathcal{A}$, see Table 7.

The mapping $\langle , \rangle : X \otimes X' \to D$ with $X = B, C$ can be extended to $X' \otimes X$ in a consistent way by defining $\langle x', x \rangle := -\langle x, x' \rangle$. Let $X,Y \in \{ A^+, A^-, B, B', C, C', E \}$. Recall also the maps $\langle , \rangle : A^\pm \otimes A^\pm \to D$ described previously (see Remark 4.1(a)). For the convenience, we extend the mappings to the whole space $\mathfrak{b}$ by defining the remaining $\langle X, Y \rangle$ to be zero. Hence

$$\langle \mathfrak{b}, \mathfrak{b} \rangle = \langle A^+, A^+ \rangle + \langle A^-, A^- \rangle + \langle B, B' \rangle + \langle C, C' \rangle + \langle E, E \rangle.$$

It follows from condition (Γ3) in Definition 2.1 that

$$D = \langle \mathfrak{b}, \mathfrak{b} \rangle.$$

**Proposition 4.5.**

1. $[d, \langle \alpha, \beta \rangle] = \langle d\alpha, \beta \rangle + \langle \alpha, d\beta \rangle$ for all $\alpha, \beta \in \mathfrak{b}$ and $d \in D$.
2. $\langle A^+, A^+ \rangle, \langle A^-, A^- \rangle, \langle B, B' \rangle, \langle C, C' \rangle$ and $\langle E, E \rangle$ are ideals of the Lie algebra $D$.
3. $D$ acts by derivations on $\mathfrak{b}$ and leaves all subspaces $A^+, A^-, B, \ldots, E$ invariant.

**Proof.** This is similar to [10, Proposition 4.2.8]. □

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