Nambu–Poisson Dynamics with Some Applications

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Abstract—Short introduction in NPD with several applications to (in)finite dimensional problems of mechanics, hydrodynamics, M-theory and quanputing is given.

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1 The Hamiltonian mechanics (HM) is in the fundamentals of mathematical description of the physical theories [1]. But HM is in a sense blind; e.g., it does not make a difference between two opposites: the ergodic Hamiltonian systems (with just one integral of motion) [2] and (super)integrable Hamiltonian systems (with maximal number of the integrals of motion).

Nabu mechanics (NM) [3, 4] is a proper generalization of the HM, which makes explicit the difference between the dynamical systems with different numbers of integrals of motion (see, e.g. [5]).

1. HAMILTONIZATION OF DYNAMICAL SYSTEMS

Let us consider a general dynamical system described by the following system of the ordinary differential equations [6]

\[ \dot{x}_n = v_n(x), \quad 1 \leq n \leq N, \]  

(1)

\[ \dot{x}_n \] stands for the total derivative with respect to the time parameter \( t \).

When the number of the degrees of freedom is even, and

\[ v_n(x) = \varepsilon_{nm} \frac{\partial H_0}{\partial x_n} , \quad 1 \leq n, \quad m \leq 2M, \]

(2)

the system (1) is a Hamiltonian one and it can be put in the form

\[ \dot{x}_n = \{x_n, H_0\}_0, \]

(3)

where the Poisson bracket is defined as

\[ \{A, B\}_0 = \varepsilon_{nm} \frac{\partial A}{\partial x_n} \frac{\partial B}{\partial x_m} = A \frac{\partial}{\partial x_n} \varepsilon_{nm} \frac{\partial}{\partial x_m} B, \]

(4)

and summation under repeated indices is used. Let us consider the following Lagrangian

\[ L = (\dot{x}_n - v_n(x)) \psi_n \]

(5)

and the corresponding equations of motion

\[ \dot{x}_n = v_n(x), \quad \dot{\psi}_n = -\frac{\partial v_n}{\partial x_n} \psi_m. \]

(6)

The system (6) extends the general system (1) by linear equation. The extended system can be put in the Hamiltonian form [7]

\[ \dot{x}_n = \{x_n, H_1\}_1, \quad \dot{\psi}_n = \{\psi_n, H_1\}_1, \]

(7)

where the first level (order) Hamiltonian is

\[ H_1 = v_n(x) \psi_n \]

(8)

and (the first level) bracket is defined as

\[ \{A, B\}_1 = A \left( \frac{\partial}{\partial x_n} \frac{\partial}{\partial \psi_n} - \frac{\partial}{\partial \psi_m} \frac{\partial}{\partial x_n} \right) B. \]

(9)

Note that when the Grassmann gradings [8] of the conjugated variables \( x_n \) and \( \psi_n \) are different, the bracket (9) is known as Buttin bracket [9].

In the Faddeev–Jackiw formalism [10] for the Hamiltonian treatment of systems defined by first-order Lagrangians, i.e. by a Lagrangian of the form

\[ L = f_n(x) \dot{x}_n - H(x), \]

(10)

the motion equations

\[ f_{mn} \dot{x}_n = \frac{\partial H}{\partial x_m}, \]

(11)

for the regular structure function \( f_{mn} \), can be put in the explicit Hamiltonian (Poisson, Dirac) form

\[ \dot{x}_n = f_{nm}^{-1} \frac{\partial H}{\partial x_m} = \{x_n, x_m\} \frac{\partial H}{\partial x_m} = \{x_n, H\}, \]

(12)

where the fundamental Poisson (Dirac) bracket is

\[ \{x_n, x_m\} = f_{nm}^{-1}, \quad f_{mn} = \partial_m f_n - \partial_n f_m. \]

(13)
The system (6) is an important example of first order regular Hamiltonian systems. Indeed, in the new variables,
\begin{equation}
\begin{aligned}
y_n^1 = x_n, \quad y_n^2 = \psi_n,
\end{aligned}
\end{equation}
the Lagrangian (5) takes the following first order form
\begin{align}
L &= (x_n - \nu_n(x))\psi_n \Rightarrow \frac{1}{2} (x_n \psi_n - \psi_n x_n) - \nu_n(x)\psi_n \\
&= \frac{1}{2} y_n^a y_n^b - H(y) = f_n^a(y) y_n^a - H(y), \quad f_n^a = \frac{1}{2} \epsilon^{ab} \delta_{nm},
\end{align}
\begin{equation}
H = \nu_n(y) y_n^a = \frac{\partial f_n^a}{\partial y_n^a} - \frac{\partial f_n^a}{\partial y_m^a},
\end{equation}
and the corresponding motion equations and the fundamental Poisson bracket are
\begin{align}
y_n^a = \epsilon_{ab} \delta_{nm} \frac{\partial H}{\partial y_m^b} = \{y_n^a, H\}, \quad \{y_n^a, y_m^b\} = \epsilon_{ab} \delta_{nm}.
\end{align}

The canonical quantization of this system corresponds
\begin{align}
[\hat{y}_n^a, \hat{y}_m^b] = i\hbar \epsilon_{ab} \delta_{nm}, \quad \hat{y}_n^1 = y_n^1, \quad \hat{y}_n^2 = -i\hbar \frac{\partial}{\partial y_n^a}.
\end{align}

In this quant theory, classical part, motion equations \(y_n^1\) for remain classical.

1.1. Modified Bochner–Killing–Yano (MBKY) Structures

Now we return to our extended system (6) and formulate conditions for the integrals of motion \(H(x, \psi)\)
\begin{equation}
H = H_0(x) + H_1 + \ldots + H_N, 
\end{equation}
where
\begin{align}
H_n = A_{k_1 \ldots k_n}(x) \psi_{k_1} \ldots \psi_{k_n}, \quad 1 \leq n \leq N, \quad (19)
\end{align}
we are assuming Grassmann valued \(\psi\) and the tensor \(A_{k_1 \ldots k_n}\) is skew-symmetric. For integrals (18) we have
\begin{align}
\hat{H} = \sum_{n=0}^{N} H_n, \quad \{H_n, H_1\} = \sum_{n=0}^{N} H_n = 0. \quad (20)
\end{align}
Now we see, that each term in the sum (18) must be conserved separately. In particular for Hamiltonian systems (2), zeroth \(H_0\) and first level \(H_1(8)\), Hamiltonians are integrals of motion. For \(n = 0\)
\begin{equation}
\hat{H}_0 = H_{0k} \psi_k = 0, 
\end{equation}
for \(1 \leq n \leq N\) we have
\begin{align}
\hat{H}_n = \hat{A}_{k_1 \ldots k_n}(x) \psi_{k_1} \ldots \psi_{k_n} + A_{k_1 \ldots k_n} \psi_{k_1} \ldots \psi_{k_n} + \ldots \\
+ A_{k_1 \ldots k_n} \psi_{k_1} \ldots \psi_{k_n} = (A_{k_1 \ldots k_n} \psi_{k_1} \ldots \psi_{k_n})\psi_k \ldots \psi_k \\
- A_{k_1 \ldots k_n} \psi_{k_1} \ldots \psi_{k_n} + \ldots - A_{k_1 \ldots k_n} \psi_{k_1} \ldots \psi_{k_n} = 0, \quad (22)
\end{align}
and there is one-to-one correspondence between the existence of the integrals (19) and the existence of the nontrivial solutions of the following equations
\begin{align}
\frac{D}{Dt} A_{k_1 \ldots k_n} = A_{k_1 \ldots k_n} v_k - A_{k_1 \ldots k_n} v_k, \quad (23)
\end{align}
For \(n = 1\) the system (23) gives
\begin{align}
A_{k_1} v_k - A_{k_1} v_k = 0 \quad (24)
\end{align}
and this equation has at least one solution, \(A_k = v_k\). If we have two (or more) independent first order integrals
\begin{align}
H_1^{(1)} = A_k \psi_k; \quad H_1^{(2)} = A_k^2 \psi_k; \ldots, \quad (25)
\end{align}
we can construct corresponding (reducible) second (or higher) order MBKY tensor(s)
\begin{align}
H_2 = H_1^{(1)} H_1^{(2)} = A_k^3 \psi_k \psi_k, \quad (26)
\end{align}
where under the bracket operation, \(\{B_{k_1 \ldots k_n}\} = \{B\}\) we understand complete anti-symmetrization. The system (23) defines a generalization of the Bochner–Killing–Yano structures of the geodesic motion of the point particle, for the case of the general (1) (and extended (6)) dynamical systems. Having \(A_{k_1}, 2 \leq M \leq N\) independent structures we can construct second order Killing tensors and Nambu–Poison dynamics. In the superintegrable case, we have maximal number of the motion integrals, \(N - 1\).

The structures defined by the system (23) we call the Modified Bochner–Killing–Yano structures or MBKY structures for short, [11].

1.2. Point Vortex Dynamics (PVD)

PVD can be defined (see e.g. [12, 13]) as the following first order system
\begin{align}
\dot{z}_n = i \sum_{m \neq n} \gamma_{mn} z_n^* - z_m^*, \quad z_n = x_n + iy_n, \quad 1 \leq n \leq N. \quad (27)
\end{align}
The corresponding first order Lagrangian, Hamiltonian, momenta, Poisson brackets and commutators are respectively
\begin{align}
L = \sum_{n=0}^{N} \gamma_{n} (x_n z_n^* - z_n x_n) - \sum_{n,m} \gamma_{nm} \ln |z_n - z_m|,
\end{align}
\begin{align}
H = \sum_{n,m} \gamma_{nm} \ln |z_n - z_m| = \sum_{n,m} \gamma_{nm} \ln |z_n - z_m| + \ln (p_n - p_m), \quad (28)
\end{align}
\begin{align}
p_n = \frac{\partial L}{\partial \dot{z}_n} = \frac{-i}{2} \gamma_{n} z_n^*, \quad p_n = \frac{\partial L}{\partial \dot{z}_n} = \frac{-i}{2} \gamma_{n} z_n^*,
\end{align}
\[ \{p_n, z_m\} = \delta_{nm}, \quad \{p_n^a, z_m^a\} = \delta_{nm}, \quad \{x_n, y_m\} = \delta_{nm}, \]
\[ [p_n, z_m] = -i\hbar \delta_{nm} \Rightarrow [x_n, y_m] = -i \frac{\hbar}{\gamma_n} \delta_{nm}. \]

So, the quantum vortex dynamics is realized in a non-commutative space. It is natural to assume that vortex parameters are quantized as
\[ \gamma_n = \frac{\hbar}{a^n}, \quad n = \pm 1, \pm 2, \ldots \]
and \( a \) is a characteristic (fundamental) length.

2. NAMBU DYNAMICS

In the canonical formulation, the equations of motion of a physical system are defined via a Poisson bracket and a Hamiltonian, [6]. In Nambu’s formulation, the Poisson bracket is replaced by the Nambu–Poisson bracket and a Hamiltonian, [6]. In Nambu’s formula-

motion of a physical system are defined via a Poisson bracket and can be presented in the Nambu–Poisson form, [14].

\[ i\dot{V}_i = \Delta V - \frac{V^2}{2}, \]
\[ i\dot{\psi}_i = -\Delta \psi + V \psi. \]

An interesting solution to the equation for the potential (33) is
\[ V = \frac{4(4 - d)}{r^2}, \]
where \( d \) is the dimension of the space. In the case of \( d = 1 \), we get the potential of conformal quantum mechanics.

The variational formulation of the extended quantum theory, is given by the following Lagrangian
\[ L = \left( iV_i - \Delta V + \frac{1}{2}V^2 \right) \psi_i. \]

The momentum variables are
\[ P_v = \frac{\partial L}{\partial \psi_i}, \quad P_{\psi} = 0. \]

As Hamiltonians of the Nambu-theoretic formulation, we take the following integrals of motion
\[ H_1 = \int d^d x \left( \Delta V - \frac{1}{2}V^2 \right) \psi_i, \]
\[ H_2 = \int d^d x (P_v - i\psi_i), \]
\[ H_3 = \int d^d x P_{\psi_i}. \]

We invent unifying vector notation, \( \phi = (\phi_1, \phi_2, \phi_3, \phi_4) = (\psi, P_{\psi}, V, P_v) \). Then it may be verified that the equations of the extended quantum theory can be put in the following Nambu-theoretic form
\[ \phi_i(x) = \{\phi(x), H_1, H_2, H_3\}, \]
where the bracket is defined as
\[ \{A_1, A_2, A_3, A_4\} = i\epsilon_{ijkl} \int \Delta A_i \delta A_j \delta A_k \delta A_l dy \]
\[ = i \int \frac{\delta A_1}{\delta \phi_i(y)} \frac{\delta A_2}{\delta \phi_j(y)} \frac{\delta A_3}{\delta \phi_k(y)} \frac{\delta A_4}{\delta \phi_l(y)} dy \]
\[ = i \det \left( \frac{\delta A_i}{\delta \phi_j} \right). \]

2.3. M Theory

The basic building blocks of the M theory are membranes and M5—branes. Membranes are fundamental objects carrying electric charges with respect to the 3-form C-field, and M5-branes are magnetic solitons. The Nambu-Poisson 3-algebras appear as gauge symmetries of superconformal Chern–Simons nonabelian theories in 2 + 1 dimensions with the maximum allowed number of \( N = 8 \) linear supersymmetries.

The Bagger and Lambert [16] and, Gustavsson [17] (BLG) model is based on a 3-algebra,
\[ [T^a, T^b, T^c] = f^{abc}_d T^d, \]
where \( T^a \) are generators and \( f_{abcd} \) is a fully anti-symmetric tensor. Given this algebra, a maximally super-symmetric Chern–Simons Lagrangian is:

\[
L = L_{\text{CS}} + L_{\text{matter}},
\]

\[
L_{\text{CS}} = \frac{1}{2} \varepsilon^{\mu \nu \lambda} \left( f_{abcd} A_{\mu}^{ab} \partial_{\nu} A_{\lambda}^{cd} + \frac{2}{3} f_{edg} f_{\mu}^{g} A_{\nu}^{ab} A_{\lambda}^{cd} A_{\rho}^{ef} \right),
\]

\[
L_{\text{matter}} = \frac{1}{2} B_{\mu}^{ab} B_{\nu}^{ab} - B_{\mu}^{ab} D_{\nu}^{a} X_{a}^{I} + \frac{i}{2} \psi^{a} \Gamma^{\nu} D_{\nu} \psi_{a} + \frac{i}{4} \bar{\psi}^{a} \psi^{a},
\]

where \( A_{\mu}^{ab} \) is a gauge boson, \( \psi^{a} \) and \( X_{a}^{I} \) are field elements. If \( a = 1, 2, 3, 4 \), then we can obtain an \( SO(4) \) gauge symmetry by choosing \( f_{abcd} = f_{\varepsilon_{abcd}}, f^{a} \) being a constant. It turns out to be the only case that gives a gauge theory with manifest unitarity and \( N = 8 \) super-symmetry.

The action has the first order form so we can use the formalism of the first section. The motion equations for the gauge fields

\[
f_{nm}^{mm} A_{nt}^{cd}(t, x) = \frac{\delta H}{\delta A_{nt}^{cd}(t, x)}, \quad f_{nm}^{nm} = \varepsilon_{nm} f_{abcd}
\]

(43)

can be written in the canonical form

\[
A_{nt}^{ab} = \int_{nm} f_{nm}^{abcd} \frac{\delta H}{\delta A_{nt}^{cd}} = (A_{nt}^{ab}, A_{m}^{cd}) \frac{\delta H}{\delta A_{m}^{cd}} = (A_{nt}^{ab}, H),
\]

\[
\{A_{nt}^{ab}(t, x), A_{m}^{cd}(t, y)\} = \varepsilon_{nm} f_{abcd} \delta^{(2)}(x - y).
\]

(44)

### 3. DISCRETE DYNAMICAL SYSTEMS

Computers are physical devices and their behavior is determined by physical laws. The Quantum Computation [18, 19], Quantum Computing, Quanputing [20], is a new interdisciplinary field of research, which benefits from the contributions of physicists, computer scientists, mathematicians, chemists and engineers.

A contemporary digital computer and its logical elements can be considered as a spatial type of discrete dynamical systems [21]

\[
S_n(k + 1) = \Phi_n(S(k)),
\]

(45)

where

\[
S_n(k), \quad 1 \leq n \leq N(k),
\]

(46)

is the state vector of the system at the discrete time step \( k \). Vector \( S \) may describe the state and the \( \Phi \) transition rule of some Cellular Automata [22]. The systems of the type (45) appears in applied mathematics as an explicit finite difference scheme approximation of the equations of the physics [23].

**Definition:** We assume that the system (45) is time-reversible if we can define the reverse dynamical system

\[
S_n(k) = \Phi_n^{-1}(S(k + 1)).
\]

(47)

In this case the following matrix

\[
M_{nn} = \frac{\partial \Phi_n(S(k))}{\partial S_n(k)}
\]

(48)

is regular, i.e. has an inverse. If the matrix is not regular, this is the case, for example, when \( N(k + 1) \neq N(k) \), we have an irreversible dynamical system (usual digital computers and/or corresponding irreversible gates).

Let us consider an extension of the dynamical system (45) given by the following action function

\[
A = \sum_{kn} l_n(k)(S_n(k + 1) - \Phi_n(S(k))),
\]

(49)

and corresponding motion equations

\[
S_n(k + 1) = \Phi_n(S(k)) = \frac{\partial H}{\partial l_n(k)},
\]

(50)

\[
l_n(k) = l_m(k) \frac{\partial \Phi_m(S(k))}{\partial S_m(k)} = l_m(k) M_{nn}(S(k)) = \frac{\partial H}{\partial S_n(k)},
\]

(51)

where

\[
H = \sum_{kn} l_n(k) \Phi_n(S(k)),
\]

is a discrete Hamiltonian. In the regular case, we put the system (50) in explicit form

\[
S_n(k + 1) = \Phi_n(S(k)),
\]

(52)

\[
l_n(k + 1) = l_m(k) M_{nn}^{-1}(S(k + 1)).
\]

From this system it is obvious that, when the initial value \( l_n(k_0) \) is given, the evolution of the vector \( l(k) \) is defined by the evolution of the state vector \( S(k) \). The equation of motion for \( l_n(k) \)—Elenka is linear and has the important property that a linear superpositions of the solutions are also solutions.

**Statement:** Any time-reversible dynamical system (e.g. a time-reversible computer) can be extended by corresponding linear dynamical system (quantum—like processor) which is controlled by the dynamical system and has a huge computational power, [20, 21, 24, 25].

#### 3.1 (de)Coherence Criterion

For motion equations (50) in the continual approximation, we have

\[
S_n(k + 1) = x_n(t_k + \tau) = x_n(t_k) + \dot{x}_n(t_k) \tau + O(\tau^2),
\]

\[
\dot{x}_n(t_k) = v_n(x(t_k)) + O(\tau), \quad t_k = k \tau,
\]

\[
v_n(x(t_k)) = (\Phi_n(x(t_k)) - x_n(t_k))/\tau,
\]

(53)

\[
M_{mn}(x(t_k)) = \delta_{mn} + \tau \frac{\partial v_n(x(t_k))}{\partial x_n(t_k)}.
\]
(de) **Coherence criterion:** the system is reversible, the linear (quantum, coherent, soul) subsystem exists, when the matrix $M$ is regular,

$$\det M = 1 + \varepsilon \sum \frac{\partial V_n}{\partial x_n} + O(\varepsilon^2) \neq 0.$$  \hspace{1cm} (54)

For the Nambu–Poisson dynamical systems (see e.g. [5])

$$V_n(x) = e_{nm_1...m_p} \frac{\partial H_1}{\partial x_{m_1}} \frac{\partial H_2}{\partial x_{m_1}} ... \frac{\partial H_p}{\partial x_{m_p}}, \quad p = 1, 2, 3, ..., N - 1,$$

$$\sum_n \frac{\partial V_n}{\partial x_n} = \text{div} V = 0.$$ \hspace{1cm} (55)

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