HOMOLOGICAL COMBINATORICS AND EXTENSIONS OF THE CD-INDEX

ABSTRACT OF DISSERTATION

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the College of Arts and Sciences at the University of Kentucky

By
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Director: Dr. Richard Ehrenborg, Professor of Mathematics
Lexington, Kentucky 2008

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ABSTRACT OF DISSERTATION

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Many combinatorial proofs rely on induction. When these proofs are formulated in traditional language, they can be bulky and unmanageable. Coalgebras provide a language which can reduce many inductive proofs in graded poset theory to comprehensible size. As a bonus, the visual form of the resulting recursive proofs suggests combinatorial interpretations for constants appearing in the longer arguments. We use the techniques of coalgebras to compute invariants of toric and affine arrangements as well as of poset products. In additional chapters we prove structure theorems for acyclic orientations and critical groups of graphs.

KEYWORDS: cd-index, polytopes, coalgebras, posets, spanning trees

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By
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I have many people to thank. I would like to single out Dora Ahmadi, Jimmy Booth, Tom Chapman, Vivian Cyrus, Scott Davison, Richard Ehrenborg, Jennifer Eli, Edgar Enochs, Charles P. Fairchild, Claire A. Foley, Brauch Fugate, Scott Godefroy, Trish Hall, Brad Hamlin, Mike Hammond, David Johnson, Eric Kahn, Daniel Kiteck, Carl Lee, David Little, Kathryn Lybarger, Penny Pajel McCollum, Neil Moore, Mark Motley, Mary Motley, Tricia Muldoon, Sunil Nanwani, Carlos M. Nicolás, Rebecca Novak, Wendell O’Brien, Sonja Petrović, Pat Quillen, Margaret Readdy, Josh Roberts, Robert D. Royar, Jack Schmidt, Yuho Shin, Aekyoung Shin Kim, R. Duane Skaggs, Bethany Slone, Cephas Slone, Donald J. Spickler, Erik Stokes, Brett Strassner, Jack Weir, and Yu Xiang.

The chapter “Affine and toric arrangements” is based on joint work with Richard Ehrenborg and Margaret Readdy. Ehrenborg and Slone were partially supported by National Security Agency grant H98230-06-1-0072.

The chapter “A geometric approach to acyclic orientations” is based on joint work with Richard Ehrenborg.
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For any collection of mathematical objects, two questions have fundamental importance.

1. Can we enumerate the objects in the collection?
2. Can we classify the objects in the collection?

This dissertation deals primarily with the question of enumeration in the field of algebraic combinatorics.

Here “enumerate” is intended in both its common senses: counting objects and listing objects. We should be able to count objects so we have a rough idea of the complexity of the task of organizing them. But we should also be able to give representative examples of the objects. In particular, if we can construct representative examples in a recursive way, then we can teach a computer to perform operations on the objects. Moreover, in spending the time to find appropriate recursively-defined representations of objects, we generally discover properties of the objects which will be useful when we turn to the question of classification.

The chapters of this dissertation can be read independently. However, there are strong connections between some of the chapters. Here we indicate some of the connections and briefly explain the topics to be discussed.

Chapters 1 and 2 deal with the cd-index, which is a polynomial invariant encoding the flag structure of polytopes and similar objects. With the cd-index of a polytope available, one can quickly answer questions such as:

- How many vertices does this polytope have? or
- How many ways can one select a connected chain of a vertex, an edge, and a face in this polytope?

The cd-index is not fully understood. In particular, even in cases where the coefficients are known to be nonnegative it is not always known what they count.

In Chapter 1 we examine the behavior of the cd-index (and more generally, the ab-index) on non-spherical manifolds. This viewpoint allows combinatorial questions for polytopes, which are spheres, to be transported to other manifolds. We start this by handling the simplest possible case, that of the n-dimensional torus, via the notion of toric hyperplane arrangement.

In Chapter 2 we streamline computation of and proofs regarding the cd-index. Recursive formulas are already known for the effects of some natural geometric operations on the cd-index. However, some of these rely on delicate chain-counting arguments, since their proofs are expressed in poset-theoretic rather than cd-theoretic terms. By importing the arguments into the cd-language, we are able to simplify many arguments. We are also able to interpret the coefficients of the cd-index in a
special case as counting lattice paths. Several results in this chapter were discovered with the assistance of GAP [32].

Chapters 3 and 4 deal, in one way or another, with chip-firing games on graphs. Chip-firing games arise out of statistical mechanics, where they are called abelian sandpile models. There are also connections to Kirchhoff’s fundamental work in circuit theory.

In Chapter 3 we use chip-firing games as a tool to give a geometric proof of the result of Propp that acyclic orientations of a graph with a fixed sink have the structure of a distributive lattice.

Finally, in Chapter 4 we study the critical group, which is the group of configurations of a chip-firing game. It is known that the order of this group is equal to the number of spanning trees of the graph. However, the structure of the critical group is only known for a few classes of graphs. We can shed a little light on the structure of the critical group of uniformly cleft graphs, which are introduced in this dissertation. We can also count the spanning trees of non-uniformly cleft trees.

Some work in this dissertation is jointly authored. In particular, Chapter 1 is joint work with Richard Ehrenborg and Margaret Readdy, while Chapter 3 is joint work with Richard Ehrenborg. We have submitted Chapter 1 to the journal Discrete and Computational Geometry. It has been refereed, and we are preparing a new version for resubmission. The chapter is based on a snapshot of that new version. None of the other chapters have yet been submitted for publication.
1 Affine and toric arrangements

1.1 Introduction

Traditionally combinatorialists have studied topological objects that are spherical, such as polytopes, or which are homeomorphic to a wedge of spheres, such as those obtained from shellable complexes. In this chapter we break from this practice and study hyperplane arrangements on the \( n \)-dimensional torus.

It is classical that the convex hull of a finite collection of points in Euclidean space is a polytope and its boundary is a sphere. The key ingredient in this construction is convexity. At the moment there is no natural analogue of this process to obtain a complex whose geometric realization is a torus.

In this chapter we are taking a zonotopal approach to working with arrangements on the torus. Recall that a zonotope can be defined without the notion of convexity, that is, it is a Minkowski sum of line segments. Dually, a central hyperplane arrangement gives rise to a spherical cell complex. By considering an arrangement on the torus, we are able to obtain a subdivision whose geometric realization is indeed the torus. We will see later in Section 1.3 that this amounts to restricting ourselves to arrangements whose subspaces in the Euclidean space \( \mathbb{R}^n \) have coefficient matrices with rational entries. Under the quotient map \( \mathbb{R}^n \rightarrow \mathbb{R}^n/\mathbb{Z}^n = T^n \) these subspaces are sent to subtori of the \( n \)-dimensional torus \( T^n \).

Zaslavsky initiated the modern study of hyperplane arrangements in his fundamental treatise [64]. For early work in the field, see the references given in Grünbaum’s text [37, Chapter 18]. Zaslavsky showed that evaluating the characteristic polynomial of a central hyperplane arrangement at \(-1\) gives the number of regions in the complement of the arrangement. For central hyperplane arrangements, Bayer and Sturmfels [7] proved the flag \( f \)-vector of the arrangement can be determined from the intersection lattice; see Theorem 1.2.3. However, their result is stated as a sum of chains in the intersection lattice and hence it is hard to apply. Billera, Ehrenborg, and Readdy improved the Bayer–Sturmfels result by showing that it is enough to know the flag \( f \)-vector of the intersection lattice to compute the flag \( f \)-vector of a central arrangement. Recall that the \text{cd}-index of a regular cell complex is an efficient tool to encode its flag \( f \)-vector without linear redundancies [6]. The Billera–Ehrenborg–Readdy theorem gives an explicit way to compute the \text{cd}-index of the arrangement, and hence its flag \( f \)-vector [10].

We generalize Zaslavsky’s theorem on the number of regions of a hyperplane arrangement to the toric case. Although there is no intersection lattice per se, one works with the intersection poset. From the Zaslavsky result we obtain a toric version of the Bayer–Sturmfels result for hyperplane arrangements, that is, there is a natural poset map from the face poset to the intersection poset, and furthermore, the cardinality of the inverse image of a chain under this map is described.

As in the case of a central hyperplane arrangement, our toric version of the Bayer–Sturmfels result determines the flag \( f \)-vector of the face poset of a toric arrangement.
in terms of its intersection poset. However, this is far from being explicit. Using the coalgebraic techniques from [25], we are able to determine the flag $f$-vector explicitly in terms of the flag $f$-vector of the intersection poset. Moreover, the answer is given by a \text{cd} type of polynomial. The flag $f$-vector of a regular spherical complex is encoded by the \text{cd}-index, a non-commutative polynomial in the variables $c$ and $d$, whereas the $n$-dimensional toric analogue is a \text{cd}-polynomial plus the \text{ab}-polynomial $(a - b)^{n+1}$.

Zaslavsky also showed that evaluating the characteristic polynomial of an affine arrangement at 1 gives the number of bounded regions in the complement of the arrangement. Thus we return to affine arrangements in Euclidean space with the twist that we study the \textit{unbounded} regions. The unbounded regions form a spherical complex. In the case of central arrangements, this complex is exactly what was studied previously by Billera, Ehrenborg, and Readdy [10]. For non-central arrangements, we determine the \text{cd}-index of this complex in terms of the lattice of unbounded intersections of the arrangement.

Interestingly, the techniques for studying toric arrangements and the unbounded complex of non-central arrangements are similar. Hence, we present these results in the same chapter. For example, the toric and non-central analogues of the Bayer–Sturmfels theorem only differ by which Zaslavsky invariant is used. The coalgebraic translations of the two analogues involve exactly the same argument, and the resulting underlying maps $\varphi_t$ (in the toric case) and $\varphi_{ub}$ (in the non-central case) differ only slightly in their definitions.

We end with many open questions about subdivisions of manifolds.

1.2 Preliminaries

All the posets we will work with are graded, that is, posets having a unique minimal element $\hat{0}$, a unique maximal element $\hat{1}$, and rank function $\rho$. For two elements $x$ and $z$ in a graded poset $P$ such that $x \leq z$, let $[x, z]$ denote the interval $\{y \in P : x \leq y \leq z\}$. Observe that the interval $[x, z]$ is itself a graded poset. Given a graded poset $P$ of rank $n + 1$ and $S \subseteq \{1, \ldots, n\}$, the $S$-\textit{rank-selected poset} $P(S)$ is the poset consisting of the elements $P(S) = \{x \in P : \rho(x) \in S\} \cup \{\hat{0}, \hat{1}\}$. The partial orders of $[x, y]$ and $P(S)$ are each inherited from that of $P$. The \textit{dual poset} of $P$, written $P^*$, is the poset having the same underlying set as $P$ but with the order relation reversed: $x <_P y$ if and only if $y <_P x$. For standard poset terminology, we refer the reader to Stanley’s work [59].

The Möbius function $\mu(x, y)$ on a poset $P$ is defined recursively by $\mu(x, x) = 1$ and for elements $x, y \in P$ with $x < y$ by $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$; see Section 3.7 in [59]. For a graded poset $P$ with minimal element $\hat{0}$ and maximal element $\hat{1}$ we write $\mu(P) = \mu_P(\hat{0}, \hat{1})$.

We now review important results about hyperplane arrangements, the \text{cd}-index, and coalgebraic techniques. All are essential for proving the main results of this chapter.
1.2.1 Hyperplane arrangements

Let \( \mathcal{H} = \{ H_1, \ldots, H_m \} \) be a hyperplane arrangement in \( \mathbb{R}^n \), that is, a finite collection of affine hyperplanes in \( n \)-dimensional Euclidean space. For brevity, throughout this chapter we will often refer to a hyperplane arrangement as an arrangement. We call an arrangement essential if the normal vectors to the hyperplanes in \( \mathcal{H} \) span \( \mathbb{R}^n \). In this chapter we are only interested in essential arrangements.

Observe that the intersection \( \bigcap_{i=1}^{m} H_i \) of all of the hyperplanes in an essential arrangement is either the empty set \( \emptyset \) or a singleton point. We call an arrangement central if the intersection of all the hyperplanes is one point. We may assume that this point is the origin \( \mathbf{0} \) and hence all of the hyperplanes are subspaces of codimension 1. If the intersection is the empty set, we call the arrangement non-central.

The intersection lattice \( L \) is the lattice formed by ordering all the intersections of hyperplanes in \( \mathcal{H} \) by reverse inclusion. If the intersection of all the hyperplanes in a given arrangement is empty, then we include the empty set \( \emptyset \) as the maximal element in the intersection lattice. If the arrangement is central, the maximal element is \( \{ \mathbf{0} \} \). In all cases, the minimal element of \( L \) will be all of \( \mathbb{R}^n \).

For a hyperplane arrangement \( \mathcal{H} \) with intersection lattice \( L \), the characteristic polynomial is defined by

\[
\chi(\mathcal{H}; t) = \sum_{x \in L} \mu(\hat{0}, x) \cdot t^{\dim(x)},
\]

where \( \mu \) denotes the Möbius function. The characteristic polynomial is a combinatorial invariant of the arrangement. The fundamental result of Zaslavsky [64] is that this invariant determines the number and type of regions in the complement of the arrangement.

**Theorem 1.2.1 (Zaslavsky).** For a hyperplane arrangement \( \mathcal{H} \) in \( \mathbb{R}^n \) the number of regions in the complement of the arrangement is given by \((-1)^n \cdot \chi(\mathcal{H}; -1)\). Furthermore, the number of bounded regions is given by \((-1)^n \cdot \chi(\mathcal{H}; 1)\).

For a graded poset \( P \), define the two Zaslavsky invariants \( Z \) and \( Z_b \) by

\[
Z(P) = \sum_{\hat{0} \leq x \leq \hat{1}} (-1)^{\rho(x)} \cdot \mu(\hat{0}, x),
\]

\[
Z_b(P) = (-1)^{\rho(P)} \cdot \mu(P).
\]

In order to work with Zaslavsky’s result, we need the following reformulation of Theorem 1.2.1.

**Theorem 1.2.2.**

(i) For a central hyperplane arrangement the number of regions is given by \( Z(L) \), where \( L \) is the intersection lattice of the arrangement.

(ii) For a non-central hyperplane arrangement the number of regions is given by \( Z(L) - Z_b(L) \), where \( L \) is the intersection lattice of the arrangement. The number of bounded regions is given by \( Z_b(L) \).
Given a central hyperplane arrangement $\mathcal{H}$ there are two associated lattices, namely, the intersection lattice $\mathcal{L}$ and the lattice $T$ of faces of the arrangement. The minimal element of $T$ is the empty set $\emptyset$ and the maximal element is the whole space $\mathbb{R}^n$. The lattice of faces can be seen as the face poset of the cell complex obtained by intersecting the arrangement $\mathcal{H}$ with a small sphere centered at the origin. Each hyperplane corresponds to a great circle on the sphere. An alternative way to view the lattice of faces $T$ is that the dual lattice $T^*$ is the face lattice of the zonotope corresponding to $\mathcal{H}$.

Let $\mathcal{L} \cup \{\hat{0}\}$ denote the intersection lattice with a new minimal element $\hat{0}$ adjoined. Define an order- and rank-preserving map $z$ from the dual lattice $T^*$ to the augmented lattice $\mathcal{L} \cup \{\hat{0}\}$ by sending a face of the arrangement, that is, a cone in $\mathbb{R}^n$, to its affine hull. Note that under the map $z$ the minimal element of $T^*$ is mapped to the minimal element of $\mathcal{L} \cup \{\hat{0}\}$. Observe that $z$ maps chains to chains. Hence we view $z$ as a map from the set of chains of $T^*$ to the set of chains of $\mathcal{L} \cup \{\hat{0}\}$. Bayer and Sturmfels [7] proved the following result about the inverse image of a chain under the map $z$.

**Theorem 1.2.3** (Bayer–Sturmfels). Let $\mathcal{H}$ be a central hyperplane arrangement with intersection lattice $\mathcal{L}$. Let $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ be a chain in $\mathcal{L} \cup \{\hat{0}\}$. Then the cardinality of the inverse image of the chain $c$ under the map $z : T^* \rightarrow \mathcal{L} \cup \{\hat{0}\}$ is given by the product

$$|z^{-1}(c)| = \prod_{i=2}^{k} Z([x_{i-1}, x_i]).$$

### 1.2.2 The cd-index

Let $P$ be a graded poset of rank $n+1$ with rank function $\rho$. For $S = \{s_1 < \cdots < s_{k-1}\}$ a subset of $\{1, \ldots, n\}$ define $f_S$ to be the number of chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ that have elements with ranks in the set $S$, that is,

$$f_S = |\{c : \rho(x_1) = s_1, \ldots, \rho(x_{k-1}) = s_{k-1}\}|.$$

Observe that $f_S$ is the number of maximal chains in the rank-selected poset $P(S)$. The flag $h$-vector is obtained by the relation (here we also present its inverse)

$$h_S = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot f_T \quad \text{and} \quad f_S = \sum_{T \subseteq S} h_T.$$

Recall that by Philip Hall’s theorem, the Möbius function of $P(S)$ is $\mu(P(S)) = (-1)^{|S|-1} \cdot h_S$.

Let $a$ and $b$ be two non-commutative variables of degree 1. For $S$ a subset of $\{1, \ldots, n\}$ let $u_S$ be the monomial $u_S = u_1 \cdots u_n$ where $u_i = b$ if $i \in S$ and $u_i = a$ if $i \notin S$. Then the **ab-index** is the noncommutative polynomial defined by

$$\Psi(P) = \sum_S h_S \cdot u_S.$$
where the sum is over all subsets $S \subseteq \{1, \ldots, n\}$. The \textbf{ab}-index of a poset $P$ of rank $n + 1$ is a homogeneous polynomial of degree $n$.

A poset $P$ is \textit{Eulerian} if every interval $[x, y]$, where $x < y$, satisfies the Euler-Poincaré relation, that is, there are the same number of elements of odd as even rank. Equivalently, the Möbius function of $P$ is given by $\mu(x, y) = (-1)^{\rho(x, y)}$ for all $x \leq y$ in $P$. The quintessential result is that the \textbf{ab}-index of an Eulerian poset has the following form.

**Theorem 1.2.4.** The \textbf{ab}-index of an Eulerian poset $P$ can be expressed in terms of the noncommutative variables $c = a + b$ and $d = ab + ba$.

This theorem was originally conjectured by Fine and proved by Bayer and Klapper [6]. Stanley provided an alternative proof for Eulerian posets [61]. There are proofs which have both used and revealed the underlying algebraic structure. See for instance [19, 27]. When the \textbf{ab}-index $\Psi(P)$ is written in terms of $c$ and $d$, the resulting polynomial is called the \textbf{cd}-index. There are linear relations among the entries of the flag $f$-vector of an Eulerian poset, known as the generalized Dehn-Sommerville relations; see [4]. The importance of the \textbf{cd}-index is that it removes all of these linear redundancies among the flag $f$-vector entries.

Observe that the variables $c$ and $d$ have degrees 1 and 2, respectively. Thus the \textbf{cd}-index of a poset of rank $n + 1$ is a homogeneous polynomial of degree $n$ in the noncommutative variables $c$ and $d$. Define the reverse of an \textbf{ab}-monomial $u = u_1u_2\cdots u_n$ to be $u^* = u_n\cdots u_2u_1$ and extend by linearity to an involution on $\mathbb{Z}\langle a, b \rangle$. Since $c^* = c$ and $d^* = d$, this involution applied to a \textbf{cd}-monomial simply reverses the \textbf{cd}-monomial. Finally, the \textbf{ab}-index respects this involution. For any graded poset $P$ we have $\Psi(P)^* = \Psi(P^*)$.

A direct approach to describe the \textbf{ab}-index of a poset $P$ is to give each chain a weight and then sum over all chains. For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in the poset $P$, define its \textit{weight} to be

$$
\text{wt}(c) = (a - b)^{\rho(x_0, x_1) - 1} \cdot b \cdot (a - b)^{\rho(x_1, x_2) - 1} \cdot b \cdots b \cdot (a - b)^{\rho(x_{k-1}, x_k) - 1},
$$

where $\rho(x, y)$ denotes the rank difference $\rho(y) - \rho(x)$. Then the \textbf{ab}-index of $P$ is the polynomial

$$
\Psi(P) = \sum_c \text{wt}(c),
$$

where the sum is over all chains $c$ in the poset $P$.

Finally, a third description of the \textbf{ab}-index is Stanley’s recursion for the \textbf{ab}-index of a graded poset [61, Equation (7)]. It is:

$$
\Psi(P) = (a - b)^{\rho(P) - 1} + \sum_{\hat{0}<x<\hat{1}} (a - b)^{\rho(x) - 1} \cdot b \cdot \Psi([x, \hat{1}]).
$$

The initial condition for this recursion is the unique poset of rank 1, $B_1$, where $\Psi(B_1) = 1$. 

1.2.3 Coalgebraic techniques

A coproduct $\Delta$ on a free $\mathbb{Z}$-module $C$ is a linear map $\Delta : C \rightarrow C \otimes C$. In order to be explicit, we use the Heyneman–Sweedler sigma notation \[38\] for writing the coproduct. To explain this notation, notice that $\Delta(w)$ is an element of $C \otimes C$ and thus has the form

$$\Delta(w) = \sum_{i=1}^{k} w_1^{i} \otimes w_2^{i},$$

where $k$ is the number of terms and $w_1^{i}$ and $w_2^{i}$ belong to $C$. Since all the maps that are applied to $\Delta(w)$ treat each term the same, the sigma notation drops the index $i$ and instead one writes

$$\Delta(w) = \sum_{w} w_{(1)} \otimes w_{(2)}.$$  

Informally, this sum should be thought of as all the ways of breaking the element $w$ in two pieces, where the first piece is denoted by $w_{(1)}$ and the second by $w_{(2)}$. The Sweedler notation for the expression $(\Delta \otimes \text{id}) \circ \Delta$, where $\text{id}$ denotes the identity map, is the following

$$(\Delta \otimes \text{id}) \circ \Delta((w)) = \sum_{w} \sum_{w_{(1)}} w_{(1,1)} \otimes w_{(1,2)} \otimes w_{(2)}.$$  

The right-hand side should be thought of as first breaking $w$ into the two pieces $w_{(1)}$ and $w_{(2)}$ and then breaking $w_{(1)}$ into the two pieces $w_{(1,1)}$ and $w_{(1,2)}$. See Joni and Rota for a more detailed explanation \[41\].

The coproduct $\Delta$ is coassociative if $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$. The sigma notation expresses coassociativity as

$$\sum_{w} \sum_{w_{(1)}} w_{(1,1)} \otimes w_{(1,2)} \otimes w_{(2)} = \sum_{w} \sum_{w_{(2)}} w_{(1)} \otimes w_{(2,1)} \otimes w_{(2,2)}.$$  

Informally coassociativity states that all the possible ways to break $w$ into two pieces and then breaking the first piece into the two pieces is equivalent to all the ways to break $w$ into two pieces and then break the second piece into two pieces. Compare coassociativity with associativity of a multiplication map $m : A \otimes A \rightarrow A$ on an algebra $A$.

Assuming coassociativity, the sigma notation simplifies to

$$\Delta^2(w) = \sum_{w} w_{(1)} \otimes w_{(2)} \otimes w_{(3)},$$

where $\Delta^2$ is defined as $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$, and the three pieces have been renamed as $w_{(1)}$, $w_{(2)}$ and $w_{(3)}$. Coassociativity allows one to define the $k$-ary coproduct $\Delta^{k-1} : C \rightarrow C^{\otimes k}$ by the recursion $\Delta^0 = \text{id}$ and $\Delta^k = (\Delta^{k-1} \otimes \text{id}) \circ \Delta$. The sigma notation for the $k$-ary coproduct is

$$\Delta^{k-1}(w) = \sum_{w} w_{(1)} \otimes w_{(2)} \otimes \cdots \otimes w_{(k)}.$$
Let \(\mathbb{Z}(a, b)\) denote the polynomial ring in the non-commutative variables \(a\) and \(b\). We define a coproduct \(\Delta\) on the algebra \(\mathbb{Z}(a, b)\) by letting \(\Delta\) satisfy the following identities: \(\Delta(1) = 0\), \(\Delta(a) = \Delta(b) = 1 \otimes 1\) and the Leibniz condition

\[
\Delta(u \cdot v) = \sum u_{(1)} \otimes u_{(2)} \cdot v + \sum v_{(1)} \otimes v_{(2)}.
\]

(1.2.3)

For an \(ab\)-monomial \(u = u_1 u_2 \cdots u_n\) we have that

\[
\Delta(u) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n.
\]

The fundamental result for this coproduct is that the \(ab\)-index is a coalgebra homomorphism \([25]\). We express this result as the following identity.

**Theorem 1.2.5** (Ehrenborg–Readdy). For a graded poset \(P\) with \(ab\)-index \(w = \Psi(P)\) and for any \(k\)-multilinear map \(M\) on \(\mathbb{Z}(a, b)\), the following coproduct identity holds:

\[
\sum_{c} M(\Psi([x_0, x_1]), \Psi([x_1, x_2]), \ldots, \Psi([x_{k-1}, x_k])) = \sum_{w} M(w_{(1)}, w_{(2)}, \ldots, w_{(k)}),
\]

where the first sum is over all chains \(c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}\) of length \(k\) and the second sum is over the \(k\)-ary coproduct of \(w\), that is, over \(\Delta^{k-1}\).

**1.2.4 The cd-index of the face poset of a central arrangement**

We recall the definition of the omega map \([10]\).

**Definition 1.2.6.** The linear map \(\omega\) from \(\mathbb{Z}(a, b)\) to \(\mathbb{Z}(c, d)\) is formed by first replacing every occurrence of \(ab\) in a given \(ab\)-monomial by \(2d\) and then replacing the remaining letters by \(c\).

For a central hyperplane arrangement \(\mathcal{H}\) the \(cd\)-index of the face poset is computed as follows \([10]\).

**Theorem 1.2.7** (Billera–Ehrenborg–Readdy). Let \(\mathcal{H}\) be a central hyperplane arrangement with intersection lattice \(\mathcal{L}\) and face lattice \(T\). Then the \(cd\)-index of the face lattice \(T\) is given by

\[
\Psi(T) = \omega(a \cdot \Psi(\mathcal{L})).
\]

We review the basic ideas behind the proof of this theorem. We will refer back to them when we prove similar results for toric and affine arrangements in Sections 1.3 and 1.4.

Define three linear operators \(\kappa\), \(\beta\) and \(\eta\) on \(\mathbb{Z}(a, b)\) by

\[
\kappa(v) = \begin{cases} (a - b)^m & \text{if } v = a^m \text{ for some } m \geq 0, \\ 0 & \text{otherwise}, \end{cases}
\]
\[ \beta(v) = \begin{cases} (a - b)^m & \text{if } v = b^m \text{ for some } m \geq 0, \\ 0 & \text{otherwise}, \end{cases} \]

and

\[ \eta(v) = \begin{cases} 2 \cdot (a - b)^{m+k} & \text{if } v = b^m a^k \text{ for some } m, k \geq 0, \\ 0 & \text{otherwise}. \end{cases} \]

Observe that \( \kappa \) and \( \beta \) are both algebra maps. The following relations hold for a poset \( P \). See [10, Section 5].

\[ \kappa(\Psi(P)) = (a - b)^{\rho(P) - 1}, \quad (1.2.4) \]
\[ \beta(\Psi(P)) = Z_b(P) \cdot (a - b)^{\rho(P) - 1}, \quad (1.2.5) \]
\[ \eta(\Psi(P)) = Z(P) \cdot (a - b)^{\rho(P) - 1}. \quad (1.2.6) \]

For \( k \geq 1 \) the operator \( \varphi_k \) is defined by the coalgebra expression

\[ \varphi_k(v) = \sum_v \kappa(v(1)) \cdot b \cdot \eta(v(2)) \cdot b \cdot \cdots \cdot b \cdot \eta(v(k)), \]

where the coproduct splits \( v \) into \( k \) parts. Finally \( \varphi \) is defined as the sum

\[ \varphi(v) = \sum_{k \geq 1} \varphi_k(v). \]

Note that in this expression only a finite number of terms are nontrivial. The connection with hyperplane arrangements is given by the following proposition.

**Proposition 1.2.8.** The \( ab \)-index of the lattice of faces of a central hyperplane arrangement is given by

\[ \Psi(T) = \varphi(\Psi(\mathcal{L} \cup \{\hat{0}\}))^*. \]

The function \( \varphi \) satisfies the functional equation

\[ \varphi(v) = \kappa(v) + \sum_v \varphi(v(1)) \cdot b \cdot \eta(v(2)). \]

From this functional equation it follows that the function \( \varphi \) satisfies the initial conditions \( \varphi(1) = 1 \) and \( \varphi(b) = 2 \cdot b \) and the recurrence relations:

\[ \varphi(v \cdot a) = \varphi(v) \cdot c, \quad (1.2.7) \]
\[ \varphi(v \cdot bb) = \varphi(v \cdot b) \cdot c, \quad (1.2.8) \]
\[ \varphi(v \cdot ab) = \varphi(v) \cdot 2d. \quad (1.2.9) \]

for an \( ab \)-monomial \( v \); see [10, Section 5]. These recursions culminate in the following result.

**Proposition 1.2.9.** The maps \( \varphi \) and \( \omega \) agree on \( ab \)-monomials that begin with \( a \), that is, if \( w = a \cdot v \), then \( \varphi(w) = \omega(w) \).

Theorem 1.2.7 follows from the fact that \( \Psi(\mathcal{L} \cup \{\hat{0}\}) = a \cdot \Psi(\mathcal{L}) \) by applying Proposition 1.2.9.
1.2.5 Regular subdivisions of manifolds

The face poset $P(\Omega)$ of a cell complex $\Omega$ is the set of all cells in $\Omega$ together with a minimal element $\hat{0}$ and a maximal element $\hat{1}$. One partially orders two cells $\tau$ and $\sigma$ by requiring that $\tau < \sigma$ if the cell $\tau$ is contained in $\sigma$, the closure of $\sigma$. In order to define a regular cell complex, consider the cell complex $\Omega$ embedded in Euclidean space $\mathbb{R}^n$. This condition is compatible with toric cell complexes since the $n$-dimensional torus can be embedded in $2n$-dimensional Euclidean space. Let $B^n$ denote the ball $\{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$ and let $S^{n-1}$ denote the sphere $\{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 = 1\}$. A cell complex $\Omega$ is regular if (i) $\Omega$ consists of a finite number of cells, (ii) for every cell $\sigma$ of $\Omega$ the pair $(\sigma, \sigma - \sigma)$ is homeomorphic to a pair $(B^k, S^{k-1})$ for some integer $k$, and (iii) the boundary $\sigma - \sigma$ is the disjoint union of smaller cells in $\Omega$. See Section 3.8 in [59] for more details. For a discussion of regular cell complexes not embedded in $\mathbb{R}^n$, see [12].

The face poset of a regular subdivision of the sphere is an Eulerian face poset and hence has a cd-index. For regular subdivisions of compact manifolds, a similar result holds. This was independently observed by Swartz [63].

Theorem 1.2.10. Let $\Omega$ be a regular cell complex whose geometric realization is a compact $n$-dimensional manifold $M$. Let $\chi(M)$ denote the Euler characteristic of $M$. Then the ab-index of the face poset $P$ of $\Omega$ has the following form.

(i) If $n$ is odd then $P$ is an Eulerian poset and hence $\Psi(P)$ can be written in terms of $c$ and $d$.

(ii) If $n$ is even then $\Psi(P)$ has the form

$$\Psi(P) = \left(1 - \frac{\chi(M)}{2}\right) \cdot (a - b)^{n+1} + \frac{\chi(M)}{2} \cdot c^{n+1} + \Phi,$$

where $\Phi$ is a homogeneous cd-polynomial of degree $n+1$ and $\Phi$ does not contain the term $c^{n+1}$.

Proof. Observe that the poset $P$ has rank $n + 2$. By [59, Theorem 3.8.9] we know that every interval $[x, y]$ strictly contained in $P$ is Eulerian. When the rank of $P$ is odd this implies that $P$ is also Eulerian; see [59, Exercise 69c]. Hence in this case the ab-index of $P$ can be expressed as a cd-index. When $n$ is even, we use [19, Theorem 4.2] to conclude that the ab-index of $P$ belongs to $\mathbb{R}\langle c, d, (a - b)^{n+1}\rangle$. Since $\Psi(P)$ has degree $n + 1$, the ab-index $\Psi(P)$ can be written in the form

$$\Psi(P) = c_1 \cdot (a - b)^{n+1} + c_2 \cdot c^{n+1} + \Phi,$$

where $\Phi$ is a homogeneous cd-polynomial of degree $n + 1$ that does not contain any $c^{n+1}$ terms. By looking at the coefficients of $a^{n+1}$ and $b^{n+1}$, we have $c_1 + c_2 = 1$ and $c_2 - c_1 = \mu(P) = \chi(M) - 1$, where the last identity is again [59, Theorem 3.8.9]. Solving for $c_1$ and $c_2$ proves the result.

For the $n$-dimensional torus Theorem [1.2.10] can be expressed as follows.
Corollary 1.2.11. Let $\Omega$ be a regular cell complex whose geometric realization is the $n$-dimensional torus $T^n$. Then the $ab$-index of the face poset $P$ of $\Omega$ has the following form:

$$\Psi(P) = (a - b)^{n+1} + \Phi,$$

where $\Phi$ is a homogeneous $cd$-polynomial of degree $n+1$ and $\Phi$ does not contain the term $c^{n+1}$.

Proof. When $n$ is even this is Theorem 1.2.10. When $n$ is odd this is Theorem 1.2.10 together with the two facts that $\chi(T^n) = 0$ and $(a - b)^{n+1} = (c^2 - 2d)^{(n+1)/2}$. $\square$

1.3 Toric arrangements

1.3.1 Toric subspaces and arrangements

The $n$-dimensional torus $T^n$ is defined as the quotient $\mathbb{R}^n / \mathbb{Z}^n$. Recall that the torus $T^n$ is an abelian group. When identifying the torus $T^n$ with the set $[0, 1)^n$, the group structure is componentwise addition modulo 1.

Lemma 1.3.1. Let $V$ be a $k$-dimensional affine subspace in $\mathbb{R}^n$ with rational coefficients. That is, $V$ has the form

$$V = \{\vec{v} \in \mathbb{R}^n : A\vec{v} = \vec{b}\},$$

where the matrix $A$ has rational entries and the vector $\vec{b}$ is allowed to have real entries. Then the image of $V$ under the quotient map $\mathbb{R}^n \to \mathbb{R}^n / \mathbb{Z}^n$, denoted by $\overline{V}$, is a $k$-dimensional torus.

Proof. By translating $V$, we may assume that the vector $\vec{b}$ is the zero vector, and therefore $V$ is a subspace. In this case, the intersection of $V$ with the integer lattice $\mathbb{Z}^n$ is a subgroup of the free abelian group $\mathbb{Z}^n$. Since the matrix $A$ has all rational entries, the rank of this subgroup is $k$, that is, the subgroup is isomorphic
to $\mathbb{Z}^k$. Hence the image $\overline{V}$ is the quotient $V/(V \cap \mathbb{Z}^n)$, which is isomorphic to the quotient $\mathbb{R}^k/\mathbb{Z}^k$, that is, a $k$-dimensional torus.

We call the image $\overline{V}$ a toric subspace of the torus $T^n$ because it is homeomorphic to some $k$-dimensional torus. When we remove the condition that the matrix $A$ is rational, the image is not necessarily homeomorphic to a torus.

The intersection of two toric subspaces is in general not a toric subspace, but instead is the disjoint union of a finite number of toric subspaces. For two affine subspaces $V$ and $W$ with rational coefficients, we have that $\overline{V \cap W} \subseteq \overline{V} \cap \overline{W}$. In general, this containment is strict.

Define the translate of a toric subspace $U$ by a point $x$ on the torus to be the toric subspace $U + x = \{ u + x : u \in U \}$. Alternatively, one may lift the toric subspace to an affine subspace in Euclidean space, translate it and then map back to the torus. Then for two toric subspaces $V$ and $W$, their intersection has the form

$$V \cap W = \bigcup_{p=1}^{r} (U + x_p),$$

where $U$ is a toric subspace, $r$ is a non-negative integer and $x_1, \ldots, x_r$ are points on the torus $T^n$.

A toric hyperplane arrangement $\mathcal{H} = \{H_1, \ldots, H_m\}$ is a finite collection of toric hyperplanes. Define the intersection poset $\mathcal{P}$ of a toric arrangement to be the set of all connected components arising from all possible intersections of the toric hyperplanes, that is, all connected components of $\bigcap_{i \in S} H_i$ where $S \subseteq \{1, \ldots, m\}$, together with the empty set. Order the elements of the intersection poset $\mathcal{P}$ by reverse inclusion, that is, the torus $T^n$ is the minimal element of $\mathcal{P}$ corresponding to the empty intersection, and the empty set is the maximal element. A toric subspace $V$ is contained in the intersection poset $\mathcal{P}$ if there are toric hyperplanes $H_{i_1}, \ldots, H_{i_k}$ in the arrangement such that $V \subseteq H_{i_1} \cap \cdots \cap H_{i_k}$ and there is no toric subspace $W$ satisfying $V \subset W \subseteq H_{i_1} \cap \cdots \cap H_{i_k}$. In other words, $V$ has to be a maximal toric subspace in some intersection of toric hyperplanes from the arrangement.
The notion of using the intersection poset can be found in work of Zaslavsky, where he considers topological dissections \[65\]. In this setting there is not an intersection lattice, but rather an intersection poset.

To every toric hyperplane arrangement \( \mathcal{H} = \{ H_1, \ldots, H_m \} \) there is an associated periodic hyperplane arrangement \( \tilde{\mathcal{H}} \) in the Euclidean space \( \mathbb{R}^n \). Namely, the inverse image of the toric hyperplane \( H_i \) under the quotient map \( \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n \) is the union of parallel integer translates of a real hyperplane. Let \( \tilde{\mathcal{H}} \) be the collection of all these integer translates. Observe that every face of the toric arrangement \( \mathcal{H} \) can be lifted to a parallel class of faces in the periodic real arrangement \( \tilde{\mathcal{H}} \).

As in the case of real arrangements, a toric arrangement subdivides the torus into a number of regions. Let \( T_t \) denote the poset of regions in the induced subdivision of the torus.

For a toric hyperplane arrangement \( \mathcal{H} \) define the \textit{toric characteristic polynomial} to be

\[
\chi(\mathcal{H}; t) = \sum_{x \in P, x \neq \emptyset} \mu(\hat{0}, x) \cdot t^{\dim(x)}.
\]

**Example 1.3.2.** Consider the line arrangement consisting of the two lines \( y = 2 \cdot x \) and \( x = 2 \cdot y \) in the plane \( \mathbb{R}^2 \). In \( \mathbb{R}^2 \) they intersect in one point, namely the origin, whereas on the torus \( T^2 \) they intersect in three points, namely \((0,0), (2/3, 1/3), \) and \((1/3, 2/3)\). The characteristic polynomial is given by \( \chi(\mathcal{H}; t) = t^2 - 2 \cdot t + 3 \).

However, this arrangement is not regular, since the induced subdivision of \( T^2 \) is not regular. The boundary of each region is a wedge of two circles. See Figure 1.1.

**Example 1.3.3.** Consider the line arrangement consisting of the three lines \( y = 3 \cdot x, x = 2 \cdot y, \) and \( y = 1/5 \). It subdivides the torus into a regular cell complex. The subdivision and the associated intersection poset are shown in Figure 1.2. The characteristic polynomial is given by \( \chi(\mathcal{H}; t) = t^2 - 3 \cdot t + 8 \). Furthermore, the \( ab \)-index of the subdivision of the torus is given by \( \Psi(T_t) = (a - b)^3 + 7 \cdot dc + 8 \cdot cd \), as the following calculation shows.

| \( S \) | \( f_S \) | \( h_S \) | \( u_S \) | \( (a - b)^3 \) | \( 7 \cdot dc \) | \( 8 \cdot cd \) |
|-------|-------|-------|-------|-------------|--------------|--------------|
| \( \emptyset \) | 1     | 1     | aaa   | 1           | 0            | 0            |
| \( \{1\} \) | 7     | 6     | baa   | -1          | 7            | 0            |
| \( \{2\} \) | 15    | 14    | aba   | -1          | 7            | 8            |
| \( \{3\} \) | 8     | 7     | aab   | -1          | 0            | 8            |
| \( \{1,2\} \) | 30    | 9     | bba   | 1           | 0            | 8            |
| \( \{1,3\} \) | 30    | 16    | bab   | 1           | 7            | 8            |
| \( \{2,3\} \) | 30    | 8     | abb   | 1           | 7            | 0            |
| \( \{1,2,3\} \) | 60    | -1    | bbb   | -1          | 0            | 0            |

Recall that \( dc = aba + abb + baa + bab \) and \( cd = aab + aba + bab + bba \). Here in the last three columns we indicate the contribution of a given term to each \( ab \)-monomial. Observe that the sum of the last three columns gives the flag \( h \)-vector entries.
We now give a natural interpretation of the toric characteristic polynomial. Recall that the intersection of toric subspaces is the disjoint union of toric subspaces that are translates of each other. Let $G$ be the collection of finite intersections of toric subspaces of the $n$-dimensional torus $T^n$, that is, $G$ consists of sets of the form $V = W_1 \cap \cdots \cap W_q$, where $W_1, \ldots, W_q$ are toric subspaces. Such a set $V$ can be written as a union $V = \bigcup_{p=1}^r (U + x_p)$, where $U$ is a toric subspace, $r$ a non-negative integer, and $x_1, \ldots, x_r$ are points on the torus. Observe that the empty set $\emptyset$ and the torus $T^n$ belong to $G$. Furthermore, $G$ is closed under finite intersections. Let $L$ be the distributive lattice consisting of all subsets of the torus $T^n$ that are obtained from the collection $G$ by finite intersections, finite unions and complements. The set $G$ is the generating set for the lattice $L$. A valuation $v$ on the lattice $L$ is a function on $L$ to an abelian group satisfying $v(\emptyset) = 0$ and \( v(A) + v(B) = v(A \cap B) + v(A \cup B) \) for all sets $A, B \in L$.

The next theorem is analogous to Theorem 2.1 in [26]. The proof here is more involved due to the fact that the collection of toric subspaces is not closed under intersections.

**Theorem 1.3.4.** There is a valuation $v$ on the distributive lattice $L$ to integer polynomials in the variable $t$ such that for a $k$-dimensional toric subspace $V$ its valuation is $v(V) = t^k$.

**Proof.** Define the function $v$ on the generating set $G$ by
\[
v \left( \bigcup_{p=1}^r (U + x_p) \right) = r \cdot t^k,\]
where we assume that $U$ is a $k$-dimensional toric subspace and the $r$ translates $U + x_1, \ldots, U + x_r$ are pairwise disjoint. Observe that the function $v$ is additive with respect to disjoint unions, that is, for elements $V_1, \ldots, V_m$ in $G$ which are pairwise disjoint and $V_1 \cup \cdots \cup V_m \in G$. In this case, each $V_i$ is a disjoint union of translates of the same affine subspace $U$ and both sides of the identity $v(V_1) + \cdots + v(V_m) = v(V_1 \cup \cdots \cup V_m)$ count the number of translates of $U$ times $t^{\dim(U)}$.

Groemer’s integral theorem [36] (see also [47, Theorem 2.2.1]) states that a function $v$ defined on a generating set $G$ extends to a valuation on the distributive lattice generated by $G$ if for all $V_1, \ldots, V_m$ in $G$ such that $V_1 \cup \cdots \cup V_m \in G$, the inclusion-exclusion formula holds:
\[
v(V_1 \cup \cdots \cup V_m) = \sum_i v(V_i) - \sum_{i<j} v(V_i \cap V_j) + \cdots. \tag{1.3.1}\]

To verify this relation for our generating set $G$, first consider the case when the union $V_1 \cup \cdots \cup V_m$ is a toric subspace. This case implies that $V_1 \cup \cdots \cup V_m = V_i$ for some index $i$. It then follows that the inclusion-exclusion formula (1.3.1) holds trivially.

Before considering the general case, we introduce some notation. For $S$ a non-empty subset of the index set $\{1, \ldots, m\}$, let $V_S = \bigcap_{i \in S} V_i$. Equation (1.3.1) can then be written as
\[
v(V_1 \cup \cdots \cup V_m) = \sum_S (-1)^{|S|-1} \cdot v(V_S),\]

15
where the sum ranges over non-empty subsets $S$ of $\{1, \ldots, m\}$. Now assume that $V_1 \cup \cdots \cup V_m$ is the disjoint union $(U + x_1) \cup \cdots \cup (U + x_r)$. Let $V_{S,p}$ denote the intersection $V_S \cap (U + x_p)$. Observe that $U + x_p = \bigcup_{i=1}^m V_{\{i\},p}$ and since $U + x_p$ is itself a toric subspace, we have already proved that the inclusion-exclusion formula (1.3.1) holds for this union. Hence we have

$$v(V_1 \cup \cdots \cup V_m) = \sum_{p=1}^r v(U + x_p)$$

$$= \sum_{p=1}^r \sum_S (-1)^{|S|-1} \cdot v(V_{S,p})$$

$$= \sum_S (-1)^{|S|-1} \cdot \sum_{p=1}^r v(V_{S,p})$$

$$= \sum_S (-1)^{|S|-1} \cdot v(V_S),$$

where $S$ ranges over all non-empty subsets of $\{1, \ldots, m\}$. The last step follows since the terms in the union $V_S = \bigcup_{p=1}^r V_{S,p}$ are pairwise disjoint.

By Möbius inversion we directly have the following theorem. The proof is standard. See the references [1, 14, 26, 42].

**Theorem 1.3.5.** The characteristic polynomial of a toric arrangement is given by

$$\chi(\mathcal{H}) = v \left( T^n - \bigcup_{i=1}^m H_i \right).$$

When each region is an open ball we can now determine the number of regions in a toric arrangement. The proof is analogous to the proofs in [26, 24]. Recall that the Euler characteristic can be viewed as a valuation. Here we use the notation $\varepsilon$ to indicate that we are viewing the Euler valuation as a valuation.

**Theorem 1.3.6.** Let $\mathcal{H}$ be a toric hyperplane arrangement on the $n$-dimensional torus $T^n$ that subdivides the torus into regions that are open $n$-dimensional balls. Then the complement of the arrangement has $(-1)^n \cdot \chi(\mathcal{H}; 0)$ regions.

**Proof.** Observe that the Euler valuation $\varepsilon$ of a $k$-dimensional torus is given by the Kronecker delta $\delta_{k,0}$. Hence for a toric subspace $V$ of the $n$-dimensional torus, the Euler valuation of $V$ is obtained by setting $t = 0$ in the valuation, that is, $\varepsilon(V) = v(V)|_{t=0}$. Since the two valuations $\varepsilon$ and $v|_{t=0}$ are additive with respect to disjoint unions, they agree for any member of the generating set $G$. Hence they also agree for any member in the distributive lattice $L$. In particular,

$$\varepsilon \left( T^n - \bigcup_{i=1}^m H_i \right) = v \left( T^n - \bigcup_{i=1}^m H_i \right)|_{t=0}. \quad (1.3.2)$$
Since the Euler valuation of an open ball is \((-1)^n\) and \(T^n - \bigcup_{i=1}^m H_i\) is a disjoint union of open balls, the left-hand side of (1.3.2) is \((-1)^n\) times the number of regions. The right-hand side is \(\chi(H; t = 0)\) by Theorem 1.3.5.

Continuation of Example 1.3.2. Setting \(t = 0\) in the characteristic polynomial in Example 1.3.2 we obtain 3, which is indeed the number of regions of this arrangement.

We call a toric hyperplane arrangement \(H = \{H_1, \ldots, H_m\}\) rational if each hyperplane \(H_i\) is of the form \(\vec{a}_i \cdot \vec{x} = b_i\) where the vector \(\vec{a}_i\) has integer entries and \(b_i\) is an integer for \(1 \leq i \leq m\). This is equivalent to assuming every constant \(b_i\) is rational since every vector \(\vec{a}_i\) was already assumed to be rational. In what follows it will be convenient to assume every coefficient is integral in a given rational arrangement.

Define \(N(H)\) to be the least common multiple of all the \(n \times n\) minors of the \(n \times m\) matrix \((\vec{a}_1, \ldots, \vec{a}_m)\). We can now give a different interpretation of the toric chromatic polynomial by counting lattice points.

**Theorem 1.3.7.** For a rational hyperplane arrangement \(H\) there exists a constant \(k\) such that for every \(q > k\) where \(q\) is a multiple of \(N(H)\), the toric characteristic polynomial evaluated at \(q\) is given by the number of lattice points in \(\left(\frac{1}{q} \mathbb{Z}\right)^n / \mathbb{Z}^n\) that do not lie on any of the toric hyperplanes \(H_i\), that is,

\[
\chi(H; q) = \left| \left(\frac{1}{q} \mathbb{Z}\right)^n / \mathbb{Z}^n - \bigcup_{i=1}^m H_i \right|.
\]

The condition that \(q\) is a multiple of \(N(H)\) implies that every subspace \(x\) in the intersection poset \(\mathcal{P}\) intersects the toric lattice \(\left(\frac{1}{q} \mathbb{Z}\right)^n / \mathbb{Z}^n\) in exactly \(q^{\dim(x)}\) points. Theorem 1.3.7 now follows by Möbius inversion. This theorem is the toric analogue of the finite field method of Athanasiadis. See [2] Theorem 2.1] in particular.

In the case when \(N(H) = 1\), the toric arrangement \(H\) is called unimodular. Novik, Postnikov, and Sturmfels [54] state Theorem 1.3.6 in the special case of unimodular arrangements. Their first proof is based upon Zaslavsky’s result on the number of bounded regions in an affine arrangement. The second proof, due to Reiner, is equivalent to our proof for arbitrary toric arrangements. See also the paper [65] by Zaslavsky, where more general arrangements are considered.

### 1.3.2 Graphical arrangements

We digress in this subsection to discuss an application to graphical arrangements, which are hyperplane arrangements arising from graphs. For a graph \(G\) on the vertex set \(\{1, \ldots, n\}\) define the **graphical arrangement** \(H_G\) to be the collection of hyperplanes of the form \(x_i = x_j\) for each edge \(ij\) in the graph \(G\).

**Corollary 1.3.8.** For a connected graph \(G\) on \(n\) vertices the regions in the complement of the graphical arrangement \(H_G\) on the torus \(T^n\) are each homotopy equivalent to the 1-dimensional torus \(T^1\). Furthermore, the number of regions is given by \((-1)^{n-1}\) times the linear coefficient of the chromatic polynomial of \(G\).
Proof. The chromatic polynomial of the graph $G$ is equal to the characteristic polynomial of the graphical arrangement $\mathcal{H}_G$. Furthermore, the intersection lattice of the real arrangement $\mathcal{H}_G$ is the same as the intersection poset of the toric arrangement $\mathcal{H}_G$. Translating the graphic arrangement in the direction $(1, \ldots, 1)$ leaves the arrangement on the torus invariant. Since $G$ is connected this is the only direction that leaves the arrangement invariant. Hence each region is homotopy equivalent to $T^1$. By adding the hyperplane $x_1 = 0$ to the arrangement we obtain a new arrangement $\mathcal{H'}$ with the same number of regions, but with each region homeomorphic to a ball. Since the intersection lattice of $\mathcal{H'}$ is just the Cartesian product of the two-element poset with the intersection lattice of $\mathcal{H}_G$, we have

$$\chi(\mathcal{H'}, t) = (t - 1) \cdot \chi(\mathcal{H}_G, t)/t.$$  

The number of regions is obtained by setting $t = 0$ in this equality. $\Box$

A similar statement holds for graphs that are disconnected. The result follows from the fact that the complement of the graphical arrangement is the product of the complements of each connected component.

**Corollary 1.3.9.** For a graph $G$ on $n$ vertices consisting of $k$ components, the regions in the complement of the graphical arrangement $\mathcal{H}_G$ on the torus $T^n$ are each homotopy equivalent to the $k$-dimensional torus $T^k$. The number of regions is given by $(-1)^{n-k}$ times the coefficient of $t^k$ in the chromatic polynomial of $G$.

Stanley [58] proved the celebrated result that the chromatic polynomial of a graph evaluated at $t = -1$ is $(-1)^n$ times the number of acyclic orientations of the graph. A similar interpretation for the linear coefficient of the chromatic polynomial is due to Greene and Zaslavsky [35]:

**Theorem 1.3.10 (Greene–Zaslavsky).** Let $G$ be a connected graph and $v$ a given vertex of the graph. The linear coefficient of the chromatic polynomial is $(-1)^{n-1}$ times the number of acyclic orientations of the graph such that the only sink is the vertex $v$.

Proof. It is enough to give a bijection between regions in the complement of the graphical arrangement on the torus $T^n$ and acyclic orientations with the vertex $v$ as the unique sink. For a region $R$ of the arrangement intersect it with the hyperplane $x_v = 0$ to obtain the face $S$. Let $\mathcal{H}'$ be the arrangement $\mathcal{H}_G$ together with the hyperplane $x_v = 0$. Lift $S$ to a face $\tilde{S}$ in the periodic arrangement $\tilde{\mathcal{H}'}$ in $\mathbb{R}^n$. Observe that $\tilde{S}$ is the interior of a polytope. When minimizing the linear functional $L(x) = x_1 + \cdots + x_n$ on the closure of the face $\tilde{S}$, the optimum is a lattice point $k = (k_1, \ldots, k_n)$. Pick a point $x = (x_1, \ldots, x_n)$ in $\tilde{S}$ close to the optimum, that is, such that each coordinate $x_i$ lies in the interval $[k_i, k_i + \epsilon)$ for some small $\epsilon > 0$.

Let $y = (y_1, \ldots, y_n)$ be the image of the point $x$ on the torus $T^n$, that is, $y_i = x_i \mod 1$. Note that each entry $y_i$ lies in the half open interval $[0, 1)$ and that $y_v = 0$. Construct an orientation of the graph $G$ by letting the edge $ij$ be oriented $i \rightarrow j$ if $y_i > y_j$. Note that this orientation is acyclic and has the vertex $v$ as a sink.

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To show that the vertex \( v \) is the unique sink, assume that the vertex \( i \) is also a sink, where \( i \neq v \). In other words, for all neighbors \( j \) of the vertex \( i \) we have that \( y_i < y_j \). We can continuously move the point \( x \) in \( \tilde{S} \) by decreasing the value of the \( i \)th coordinate \( x_i \). Observe that there is no hyperplane in the periodic arrangement blocking the coordinate \( x_i \) from passing through the integer value \( k_i \) and continuing down to \( k_i - 1 + \epsilon \). This contradicts the fact that we chose the original point \( x \) close to the optimum of the linear functional \( L \). Hence the vertex \( i \) cannot be a sink.

It is straightforward to verify that this map from regions to the set of acyclic orientations with the unique sink at \( v \) is a bijection. \( \Box \)

The technique of assigning a point to every region of a toric arrangement using a linear functional was used by Novik, Postnikov and Sturmfels in their paper [54]. See their first proof of the number of regions of a toric arrangement.

### 1.3.3 The toric Bayer–Sturmfels result

Define the \textit{toric Zaslavsky invariant} of a graded poset \( P \) having \( \hat{0} \) and \( \hat{1} \) by

\[
Z_t(P) = \sum_{x \coatom \text{ of } P} (-1)^{\rho(\hat{0}, x)} \cdot \mu(\hat{0}, x) = (-1)^{\rho(P)} \cdot \sum_{x \coatom \text{ of } P} \mu(\hat{0}, x).
\]

We reformulate Theorem 1.3.6 as follows.

**Theorem 1.3.11.** For a toric hyperplane arrangement \( \mathcal{H} \) on the torus \( T^n \) that subdivides the torus into open \( n \)-dimensional balls, the number of regions is given by \( Z_t(P) \), where \( P \) is the intersection poset of the arrangement \( \mathcal{H} \).

As a corollary of Theorem [1.3.11], we can describe the \( f \)-vector of the subdivision \( T_t \) of the torus. For similar results for more general manifolds see [65, Section 3].

**Corollary 1.3.12.** The number of \( i \)-dimensional regions in the subdivision \( T_t \) of the \( n \)-dimensional torus is given by the sum

\[
f_{i+1}(T_t) = (-1)^i \cdot \sum_{x \leq y, \dim(x) = i \atop \dim(y) = 0} \mu(x, y),
\]

where \( \mu(x, y) \) denotes the Möbius function of the interval \([x, y]\) in the intersection poset \( P \).

**Proof.** Each \( i \)-dimensional region is contained in a unique \( i \)-dimensional subspace \( x \). By restricting the arrangement to the subspace \( x \) and applying Theorem [1.3.6] we have that the number of \( i \)-dimensional regions in \( x \) is given by

\[
(-1)^i \cdot \sum_{x \leq y, \dim(y) = 0} \mu(x, y).
\]

Summing over all \( x \), the result follows. \( \Box \)
For the remainder of this section we will assume that the induced subdivision of the torus is a regular cell complex. Let $T$ be the face poset of the subdivision of the torus induced by the toric arrangement. Define the map $z_t : T_\ast \rightarrow \mathcal{P} \cup \{\hat{0}\}$ by sending each face to the smallest toric subspace in the intersection poset that contains the face and sending the minimal element in $T_\ast$ to $\hat{0}$. Observe that the map $z_t$ is order- and rank-preserving, as well as being surjective. As in the central hyperplane arrangement case, we view the map $z_t$ as a map from the set of chains of $T_\ast$ to the set of chains of $\mathcal{P} \cup \{\hat{0}\}$.

Let $x$ be an element in the intersection poset $\mathcal{P}$ of a toric hyperplane arrangement $H$. Then the interval $[x, \hat{1}]$ is the intersection poset of a toric arrangement in the toric subspace $x$. The atoms of the interval $[x, \hat{1}]$ are the toric hyperplanes in this smaller toric arrangement.

More interesting is the geometric interpretation of the interval $[\hat{0}, x]$. It is the intersection lattice of a central hyperplane arrangement in $\mathbb{R}^{n - \dim(x)}$. Without loss of generality we may assume that $x$ contains the zero point $(0, \ldots, 0)$, that is, when we lift the toric subspace $x$ to an affine subspace $V$ in $\mathbb{R}^n$ we may assume that $V$ is a subspace of $\mathbb{R}^n$. Any toric subspace $y$ in the interval $[\hat{0}, x]$, that is, a toric subspace containing $x$, can be lifted to a subspace $W$ containing the subspace $V$. In particular, the toric hyperplanes in $[\hat{0}, x]$ lift to hyperplanes in $\mathbb{R}^n$ containing $V$. This lifting is a poset isomorphism and we obtain an essential central arrangement of dimension $n - \dim(x)$ by quotienting out by the subspace $V$. We conclude by noticing that an interval $[x, y]$ in $\mathcal{P}$, where $y < \hat{1}$, is the intersection lattice of a central hyperplane arrangement.

The toric analogue of Theorem 1.2.3 is as follows.

**Theorem 1.3.13.** Let $\mathcal{P}$ be the intersection poset of a toric hyperplane arrangement whose induced subdivision is regular. Let $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ be a chain in $\mathcal{P} \cup \{\hat{0}\}$ with $k \geq 2$. Then the cardinality of the inverse image of the chain $c$ is given by the product

$$|z_t^{-1}(c)| = \prod_{i=2}^{k-1} Z([x_{i-1}, x_i]) \cdot Z_t([x_{k-1}, x_k]).$$

**Proof.** We need to count the number of ways we can select a chain $d = \{\hat{0} = y_0 < y_1 < \cdots < y_k = \hat{1}\}$ in $T_\ast$ such that $z_t(y_i) = x_i$. The number of ways to select the element $y_{k-1}$ in $T_\ast$ is the number of regions in the arrangement restricted to the toric subspace $x_{k-1}$. By Theorem 1.3.11 this can be done in $Z_t([x_{k-1}, x_k])$ ways. Observe now that all other elements in the chain $d$ contain the face $y_{k-1}$.

To count the number of ways to select the element $y_{k-2}$, we follow the original argument of Bayer–Sturmfels. We would like to pick a face $y_{k-2}$ such that it contains the face $y_{k-1}$ and it is a region in the toric subspace $x_{k-2}$. This is equal to the number of regions in the central arrangement having the intersection lattice $[x_{k-2}, x_{k-1}]$, which is given by $Z([x_{k-2}, x_{k-1}])$. By iterating this procedure until we reach the element $y_1$, the result follows. \qed
Corollary 1.3.14. The flag f-vector entry $f_S(T_t)$ of the face poset $T_t$ of a toric arrangement whose induced subdivision is regular subdivision of $T^n$ is divisible by $2^{|S|-1}$ for $S \subseteq \{1, \ldots, n+1\}$ with $S \neq \emptyset$.

Proof. The proof follows from the fact that the Zaslavsky invariant $Z$ is an even integer and that a given flag f-vector entry is the appropriate sum of products appearing in Theorem 1.3.13. □

1.3.4 The connection between posets and coalgebras

For an ab-monomial $v$ define the linear map $\lambda_t$ by letting

$$\lambda_t(v) = \begin{cases} (a-b)^m & \text{if } v = b^m \text{ for some } m \geq 0, \\
(a-b)^{m+1} & \text{if } v = b^ma \text{ for some } m \geq 0, \\
0 & \text{otherwise.} \end{cases}$$

Define the linear operator $H'$ on $\mathbb{Z}\langle a, b \rangle$ to be the one which removes the last letter in each ab-monomial, that is, $H'(a \cdot a) = H'(a \cdot b) = w$ and $H'(1) = 0$. We use the prime in the notation to distinguish it from the $H$ map defined in [10, Section 8] which instead removes the first letter in each ab-monomial. From [10] we have the following lemma.

Lemma 1.3.15. For a graded poset $P$ with $\hat{1}$ of rank greater than or equal to 2, the following identity holds:

$$H'(\Psi(P)) = \sum_{x \text{ coatom of } P} \Psi([\hat{0}, x]).$$

The next lemma gives the relation between the toric Zaslavsky invariant $Z_t$ and the map $\lambda_t$.

Lemma 1.3.16. For a graded poset $P$ with $\hat{1}$ of rank greater than or equal to 1, the following identity holds:

$$\lambda_t(\Psi(P)) = Z_t(P) \cdot (a-b)^{\rho(P)-1}.$$

Proof. When $P$ has rank 1, both sides are equal to 1. For an ab-monomial $v$ different from 1, we have that $\lambda_t(v) = \beta(H'(v)) \cdot (a-b)$. Hence

$$\lambda_t(\Psi(P)) = \beta(H'(\Psi(P))) \cdot (a-b)
= \sum_{x \text{ coatom of } P} \beta(\Psi([\hat{0}, x])) \cdot (a-b)
= (-1)^{\rho(P)} \cdot \sum_{x \text{ coatom of } P} \mu(\hat{0}, x) \cdot (a-b)^{\rho(P)-1},$$

which concludes the proof. □
Define a sequence of functions \( \varphi_{t,k} : \mathbb{Z}^n \to \mathbb{Z}^n \) by \( \varphi_{t,1} = \kappa \), and for \( k \geq 2 \),

\[
\varphi_{t,k}(v) = \sum_v \kappa(v_1) \cdot b \cdot \eta(v_2) \cdot b \cdot \eta(v_3) \cdot b \cdots b \cdot \eta(v_{k-1}) \cdot b \cdot \lambda_t(v_k).
\]

Finally, let \( \varphi_t(v) \) be the sum \( \varphi_t(v) = \sum_{k \geq 1} \varphi_{t,k}(v) \).

**Theorem 1.3.17.** The **ab-index** of the face poset \( T_t \) of a toric arrangement is given by

\[
\Psi(T_t)^* = \varphi_t(\Psi(\mathcal{P} \cup \{\hat{0}\})).
\]

**Proof.** The **ab-index** of the poset \( T_t \) is given by the sum \( \Psi(T_t) = \sum_c |z_t^{-1}(c)| \cdot \text{wt}(c) \).

Fix \( k \geq 2 \) and sum over all chains \( c = \{0 = x_0 < x_1 < \cdots < x_k = 1\} \) of length \( k \). We then have

\[
\sum_c \frac{1}{z_t^{-1}(c)} \cdot \text{wt}(c)
= \sum_{c} \prod_{i=2}^{k-1} Z([x_{i-1}, x_i]) \cdot Z_t([x_{k-1}, x_k]) \cdot (a - b)^{\rho(x_0, x_1) - 1} \cdot b \cdots b \cdot (a - b)^{\rho(x_{k-1}, x_k) - 1}
= \sum_c \kappa(\Psi([x_0, x_1])) \cdot \prod_{i=2}^{k-1} (b \cdot \eta(\Psi([x_{i-1}, x_i]))) \cdot b \cdot \lambda_t(\Psi([x_{k-1}, x_k]))
= \sum_w \kappa(w_1) \cdot \prod_{i=2}^{k-1} (b \cdot \eta(w_i)) \cdot b \cdot \lambda_t(w_k)
= \varphi_{t,k}(w),
\]

where we let \( w \) denote the **ab-index** of the augmented intersection poset \( \mathcal{P} \cup \{\hat{0}\} \).

For \( k = 1 \) we have that \( (a - b)^{\rho(T_t) - 1} = \varphi_{t,1}(\Psi(\mathcal{P} \cup \{\hat{0}\})) \). Summing over all \( k \geq 1 \), we obtain the result. \( \square \)

### 1.3.5 Evaluating the function \( \varphi_t \)

**Proposition 1.3.18.** For an **ab**-monomial \( v \), the following identity holds:

\[
\varphi_t(v) = \kappa(v) + \sum_v \varphi(v_1) \cdot b \cdot \lambda_t(v_2).
\]

**Proof.** Using the coassociative identity \( \Delta^{k-1} = (\Delta^{k-2} \otimes \text{id}) \circ \Delta \), for \( k \geq 2 \) we have that

\[
\varphi_{t,k}(v) = \sum_v \kappa(v_1) \cdot b \cdot \eta(v_2) \cdot b \cdot \eta(v_3) \cdot b \cdots b \cdot \eta(v_{k-1}) \cdot b \cdot \lambda_t(v_k)
= \sum_v \sum_{v_1} \kappa(v_{1,1}) \cdot b \cdot \eta(v_{1,2}) \cdot b \cdots b \cdot \eta(v_{1,k-1}) \cdot b \cdot \lambda_t(v_2)
= \sum_v \varphi_{k-1}(v_{1,1}) \cdot b \cdot \lambda_t(v_{2}).
\]
By summing over all \( k \geq 1 \), the result follows. \( \square \)

**Lemma 1.3.19.** Let \( v \) be an \( ab \)-monomial that begins with \( a \) and let \( x \) be either \( a \) or \( b \). Then
\[
\varphi_t(v \cdot a \cdot x) = \kappa(v \cdot a \cdot x) + 1/2 \cdot \omega(v \cdot ab).
\]

**Proof.** Using Proposition 1.3.18 we have
\[
\varphi_t(v \cdot a \cdot x) = \kappa(v \cdot a \cdot x) + \varphi(v \cdot a) \cdot b \cdot \lambda_t(1) + \varphi(v) \cdot b \cdot \lambda_t(x)
+ \sum_{v} \varphi(v(1)) \cdot b \cdot \lambda_t(v(2) \cdot b \cdot x)
= \kappa(v \cdot a \cdot x) + \varphi(v) \cdot c \cdot b + \varphi(v) \cdot b \cdot (a - b)
= \kappa(v \cdot a \cdot x) + \omega(v) \cdot d
= \kappa(v \cdot a \cdot x) + 1/2 \cdot \omega(v \cdot ab),
\]

since \( \lambda_t(v(2) \cdot b \cdot x) = 0 \). \( \square \)

**Lemma 1.3.20.** Let \( v \) be an \( ab \)-monomial that begins with \( a \), let \( k \) be a positive integer, and let \( x \) be either \( a \) or \( b \). Then
\[
\varphi_t(v \cdot ab^k \cdot x) = \kappa(v \cdot ab^k \cdot x) + 1/2 \cdot \omega(v \cdot ab^{k+1}).
\]

**Proof:** Using Proposition 1.3.18 we have
\[
(\varphi_t - \kappa)(v \cdot ab^k \cdot x) = \varphi(v \cdot ab^k) \cdot b \cdot \lambda_t(1) + \varphi(v \cdot a) \cdot b \cdot \lambda_t(b^{k-1} \cdot x)
+ \varphi(v) \cdot b \cdot \lambda_t(b^k \cdot x)
+ \sum_{i+j=k-2} \varphi(v \cdot ab^{i+1}) \cdot b \cdot \lambda_t(b^j \cdot x)
= \varphi(v) \cdot \left( 2dc^{k-1} \cdot b + c \cdot b \cdot (a - b)^k + b \cdot (a - b)^{k+1}
+ \sum_{i+j=k-2} 2dc^i \cdot b \cdot (a - b)^{j+1} \right). \quad (1.3.3)
\]

In order to simplify this expression, consider the butterfly poset of rank \( k \). This is the poset consisting of two rank \( i \) elements, for \( i = 1, \ldots, k - 1 \), adjoined with a minimal and maximal element. Each of the rank \( i \) elements covers the rank \( i - 1 \) element(s) for \( i = 1, \ldots, k - 1 \). The butterfly poset is the unique poset having the \( cd \)-index \( c^{k-1} \). It is also Eulerian. Applying (1.2.2) to the butterfly poset, we have
\[
c^{k-1} = (a - b)^{k-1} + 2 \cdot \sum_{i+j=k-2} c^i \cdot b \cdot (a - b)^j.
\]
Using this relation to simplify equation (1.3.3), we obtain
\[ \varphi_t(v \cdot ab^k \cdot x) - \kappa(v \cdot ab^k \cdot x) = \varphi(v) \cdot d \cdot c^k = 1/2 \cdot \omega(v \cdot ab^{k+1}). \]

This completes the proof. \(\square\)

By combining Lemmas 1.3.19 and 1.3.20 we have the following proposition.

**Proposition 1.3.21.** For an ab-monomial \(v\) that begins with the letter \(a\),
\[ \varphi_t(v) = \kappa(v) + 1/2 \cdot \omega(H'(v) \cdot b). \]

We now obtain the main result for computing the ab-index of the face poset of a toric arrangement.

**Theorem 1.3.22.** Let \(\mathcal{H}\) be a toric hyperplane arrangement on the \(n\)-dimensional torus \(T^n\) that subdivides the torus into a regular cell complex. Then the ab-index of the face poset \(T_t\) can be computed from the ab-index of the intersection poset \(\mathcal{P}\) as follows:
\[ \Psi(T_t) = (a - b)^{n+1} + 1/2 \cdot \omega(a \cdot H'(\Psi(\mathcal{P})) \cdot b)^*. \]

Observe that in Lemmas 1.3.19 and 1.3.20, Proposition 1.3.21 and Theorem 1.3.22 no rational coefficients were introduced. Only the ab-monomial \(a^n\) is mapped to a cd-polynomial with an odd coefficient, hence \(1/2 \cdot \omega(v \cdot b)\) has all integer coefficients.

**Continuation of Example 1.3.3.** The flag \(f\)-vector of the intersection poset \(\mathcal{P}\) in Example 1.3.3 is given by \((f_0, f_1, f_2, f_{12}) = (1, 3, 7, 15)\), the flag \(h\)-vector by \((h_0, h_1, h_2, h_{12}) = (1, 2, 6, 6)\), and so the ab-index is \(\Psi(P) = a^2 + 2 \cdot ba + 6 \cdot ab + 6 \cdot b^2\). Thus
\[
\begin{align*}
\Psi(T_t) &= (a - b)^3 + 1/2 \cdot \omega(a \cdot H'(a^2 + 2 \cdot ba + 6 \cdot ab + 6 \cdot b^2) \cdot b)^* \\
&= (a - b)^3 + 1/2 \cdot \omega(a \cdot (7 \cdot a + 8 \cdot b) \cdot b)^* \\
&= (a - b)^3 + 1/2 \cdot \omega(7 \cdot a^2 b + 8 \cdot ab^2)^* \\
&= (a - b)^3 + 7 \cdot dc + 8 \cdot cd,
\end{align*}
\]
which agrees with the calculation in Example 1.3.3.

Theorem 1.3.22 gives a different approach from Corollary 1.3.12 for determining the \(f\)-vector of \(T_t\). For notational ease, for positive integers \(i\) and \(j\), let \([i, j] = \{i, i+1, \ldots, j\}\) and \([j] = \{1, \ldots, j\}\).
Corollary 1.3.23. The number of $i$-dimensional regions in the subdivision $T_t$ of the $n$-dimensional torus is given by the following sum of flag $h$-vector entries from the intersection poset $P$:

$$f_{i+1}(T_t) = h_{[n-i,n]}(P) + h_{[n-i,n-1]}(P) + h_{[n-i+1,n]}(P) + h_{[n-i+1,n-1]}(P),$$

for $1 \leq i \leq n-1$. The number of vertices is given by $f_1(T_t) = 1 + h_n(P)$ and the number of maximal regions by $f_{n+1}(T_t) = h_{[n-1]}(P) + h_{[n]}(P)$.

Proof. Let $\langle \cdot | \cdot \rangle$ denote the inner product on $\mathbb{Z}\langle a, b \rangle$ defined by $\langle u | v \rangle = \delta_{u,v}$ for two $ab$-monomials $u$ and $v$. For $1 \leq i \leq n-1$ we have

$$f_{i+1}(T_t) = 1 + h_{i+1}(T_t)$$

$$= 1 + \langle a^i b a^{n-i} | \Psi(T_t) \rangle$$

$$= \frac{1}{2} \cdot \langle a^i b a^{n-i} | \omega(a \cdot H'(\Psi(P)) \cdot b)^* \rangle$$

$$= \frac{1}{2} \cdot \langle c^i d c^{n-i} | \omega(a \cdot H'(\Psi(P)) \cdot b)^* + \frac{1}{2} \cdot \langle c^i d c^{n-i} | \omega(a \cdot H'(\Psi(P)) \cdot b)^* \rangle$$

$$= \langle a^{n-i} \cdot ab \cdot b^{i-1} + a^{n-i-1} \cdot ab \cdot b^i | a \cdot H'(\Psi(P)) \cdot b \rangle$$

$$= \langle a^{n-i-1} \cdot (a + b) \cdot b^{i-1} | H'(\Psi(P)) \rangle$$

$$= \langle a^{n-i-1} \cdot (a + b) \cdot b^{i-1} | (a + b) \Psi(P) \rangle.$$ 

Expanding in terms of the flag $h$-vector the result follows. The expressions for $f_1$ and $f_{n+1}$ are obtained by similar calculations. 

The fact that Corollaries 1.3.12 and 1.3.23 are equivalent follows from the coalgebra techniques in Theorem 1.2.5.

1.4 The complex of unbounded regions

1.4.1 Zaslavsky and Bayer–Sturmfels

The unbounded Zaslavsky invariant is defined by

$$Z_{ub}(P) = Z(P) - 2 \cdot Z_{b}(P).$$

As the name suggests, the number of unbounded regions in a non-central arrangement is given by this invariant. By taking the difference of the two statements in Theorem 1.2.2 part (ii), we immediately obtain the following result.

Lemma 1.4.1. For a non-central hyperplane arrangement $H$ the number of unbounded regions is given by $Z_{ub}(L)$, where $L$ is the intersection lattice of the arrangement $H$.

Let $H$ be a non-central hyperplane arrangement in $\mathbb{R}^n$ with intersection lattice $L$ having the empty set $\emptyset$ as the maximal element. Let $L_{ub}$ denote the unbounded intersection lattice, that is, the subposet of the intersection lattice consisting of all
affine subspaces with the points (dimension zero affine subspaces) omitted but with the empty set \( \emptyset \) continuing to be the maximal element. Equivalently, the poset \( L_{ub} \) is the rank-selected poset \( L([1, n-1]) \), that is, the poset \( L \) with the coatoms removed.

Let \( T \) be the face lattice of the arrangement \( \mathcal{H} \) with the minimal element \( \hat{0} \) denoting the empty face and the maximal element denoted by \( \hat{1} \). Similarly, let \( T_{ub} \) denote the set of all faces in the face lattice \( T \) which are not bounded. Observe that \( T_{ub} \) includes the minimal and maximal elements of \( T \) and that \( T_{ub} \) is the face poset of an \((n-1)\)-dimensional sphere. Pick \( R \) large enough so that all of the bounded faces are strictly inside a ball of radius \( R \). Intersect the arrangement \( \mathcal{H} \) with a sphere of radius \( R \). The resulting cell complex has face poset \( T_{ub} \). Our goal is to compute the \textbf{cd}-index of \( T_{ub} \) in terms of the \textbf{ab}-index of \( L_{ub} \).

The collection of unbounded faces of the arrangement \( \mathcal{H} \) forms a lower order ideal in the poset \( T^* \). Let \( Q \) be the subposet of \( T^* \) consisting of this ideal with a maximal element \( \hat{1} \) adjoined. We define the rank of an element in \( Q \) to be its rank in the original poset \( T^* \), that is, for \( x \in Q \) let \( \rho_Q(x) = \rho_{T^*}(x) \). This rank convention will simplify the later arguments. As posets, \( T_{ub}^* \) and \( Q \) are isomorphic. However, since their rank functions differ, their \textbf{ab}-indexes satisfy \( \Psi(T_{ub})^* \cdot (a - b) = \Psi(Q) \).

Restrict the zero map \( z : T^* \to L \cup \{\hat{0}\} \) to form the map \( z_{ub} : Q \to L \cup \{\hat{0}\} \). The map \( z_{ub} \) is order- and rank-preserving. However, it is not necessarily surjective. As before we view the map \( z_{ub} \) as a map from the set of chains of \( Q \) to the set of chains of \( L \cup \{\hat{0}\} \). The following theorem is a toric deformation of Theorem 1.2.3.
**Theorem 1.4.2.** Let $H$ be a non-central hyperplane arrangement with intersection lattice $L$. Let $c = \{ \hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1} \}$ be a chain in $L \cup \{ \hat{0} \}$ with $k \geq 2$. Then the cardinality of the inverse image of the chain $c$ under $z_{ub}$ is given by

$$|z_{ub}^{-1}(c)| = \prod_{i=2}^{k-1} Z([x_{i-1}, x_i]) \cdot Z_{ub}([x_{k-1}, x_k]).$$

**Proof.** We need to count the number of ways we can select a chain $d = \{ \hat{0} = y_0 < y_1 < \cdots < y_k = \hat{1} \}$ in the poset of unbounded regions $Q$ such that $z_{ub}(y_i) = x_i$. The number of ways to select the element $y_{k-1}$ in $Q$ is the number of unbounded regions in the arrangement restricted to the subspace $x_{k-1}$. By Lemma 1.4.1 this can be done in $Z_{ub}([x_{k-1}, x_k])$ ways. Since $y_{k-1}$ is an unbounded face of the arrangement and all other elements in the chain $d$ contain the face $y_{k-1}$, the other elements must be unbounded.

The remainder of the proof is the same as that of Theorem 1.3.13. \qed

**Corollary 1.4.3.** The flag $f$-vector entry $f_S(T_{ub})$ is divisible by $2^{|S|}$ for any index set $S \subseteq \{1, \ldots, n\}$.

**Proof.** The proof is the same as Corollary 1.3.14 with the extra observation that the Zaslavsky invariant $Z_{ub}$ is even. \qed

### 1.4.2 The connection between posets and coalgebras

Define $\lambda_{ab}$ by $\lambda_{ab} = \eta - 2 \cdot \beta$. By equations (1.2.5) and (1.2.6), for a graded poset $P$ we have

$$\lambda_{ab}(\Psi(P)) = Z_{ab}(P) \cdot (a - b)^{\rho(P)} - 1.$$

Define a sequence of functions $\varphi_{ab,k}: \mathbb{Z}^a \cdot \mathbb{Z}^b \to \mathbb{Z}^a \cdot \mathbb{Z}^b$ by $\varphi_{ab,1} = \kappa$ and for $k > 1$,

$$\varphi_{ab,k}(v) = \sum_{v} \kappa(v(1)) \cdot b \cdot \eta(v(2)) \cdot b \cdot \eta(v(3)) \cdot b \cdots b \cdot \eta(v(k-1)) \cdot b \cdot \lambda_{ab}(v(k)) \cdot b.$$

Finally, let $\varphi_{ab}(v)$ be the sum $\varphi_{ab}(v) = \sum_{k \geq 1} \varphi_{ab,k}(v)$.

Similar to Theorem 1.3.17 we have the next result. The proof only differs in replacing the map $z_t: T^* \to \mathcal{P} \cup \{ \hat{0} \}$ with $z_{ab}: Q \to \mathcal{L} \cup \{ \hat{0} \}$ and the invariant $Z_t$ by $Z_{ab}$.

**Theorem 1.4.4.** The $ab$-index of the poset $Q$ of unbounded regions of a non-central arrangement is given by

$$\Psi(Q) = \varphi_{ab}(\Psi(\mathcal{L} \cup \{ \hat{0} \})).$$
1.4.3 Evaluating the function $\varphi_{ub}$

In this subsection we analyze the behavior of $\varphi_{ub}$.

Lemma 1.4.5. For any $ab$-monomial $v$,
\[
\varphi_{ub}(v) = \varphi(v) - 2 \cdot \sum_{v} \varphi(v(1)) \cdot b \cdot \beta(v(2)).
\]

Proof. Using the coassociative identity $\Delta^{k-1} = (\Delta^{k-2} \otimes id) \circ \Delta$, we have for $k \geq 2$
\[
\varphi_{ub,k}(v) = \varphi_k(v) - 2 \cdot \sum_{v} k(v(1)) \cdot b \cdot \eta(v(2)) \cdot b \cdots b \cdot \eta(v(k-1)) \cdot b \cdot \beta(v(k))
\]
\[
= \varphi_k(v) - 2 \cdot \sum_{v} k(v(1,1)) \cdot b \cdot \eta(v(1,2)) \cdot b \cdots b \cdot \eta(v(1,k-1)) \cdot b \cdot \beta(v(2))
\]
\[
= \varphi_k(v) - 2 \cdot \sum_{v} \varphi_{k-1}(v(1)) \cdot b \cdot \beta(v(2)).
\]
The result then follows by summing over all $k \geq 2$ and adding $\varphi_{ub,1}(v) = k(v) = \varphi_1(v)$.

Lemma 1.4.6. Let $v$ be an $ab$-monomial. Then
\[
\varphi_{ub}(v \cdot a) = \varphi(v) \cdot (a - b).
\]

Proof. By Lemma 1.4.5 and the Leibniz relation (1.2.3) we have
\[
\varphi_{ub}(v \cdot a) = \varphi(v \cdot a) - 2 \cdot \varphi(v) \cdot b \cdot \beta(1) - 2 \cdot \sum_{v} \varphi(v(1)) \cdot b \cdot \beta(v(2) \cdot a).
\]

By equation (1.2.7), $\varphi(v \cdot a) = \varphi(v) \cdot c$. The summation above is zero because $\beta(v(2) \cdot a)$ is always zero. Hence $\varphi_{ub}(v \cdot a) = \varphi(v) \cdot (c - 2b) = \varphi(v) \cdot (a - b)$.

Lemma 1.4.7. Let $v$ be an $ab$-monomial. Then
\[
\varphi_{ub}(v \cdot bb) = \varphi_{ub}(v \cdot b) \cdot (a - b).
\]

Proof. Let $u = v \cdot b$. Applying Lemma 1.4.5 and the Leibniz relation (1.2.3) to $u$ gives
\[
\varphi_{ub}(u \cdot b) = \varphi(u \cdot b) - 2 \cdot \varphi(u) \cdot b \cdot \beta(1) - 2 \cdot \sum_{u} \varphi(u(1)) \cdot b \cdot \beta(u(2) \cdot b)
\]
\[
= \varphi(u) \cdot (c - 2b) - 2 \cdot \sum_{u} \varphi(u(1)) \cdot b \cdot \beta(u(2)) \cdot (a - b)
\]
\[
= \left( \varphi(u) - 2 \cdot \sum_{u} \varphi(u(1)) \cdot b \cdot \beta(u(2)) \right) \cdot (a - b)
\]
\[
= \varphi_{ub}(u) \cdot (a - b).
\]

Here we have used the facts that $\varphi(u \cdot b) = \varphi(u) \cdot c$ and $\beta(u(2) \cdot b) = \beta(u(2)) \cdot (a - b)$.  

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Lemma 1.4.8. Let \( v \) be an \( ab \)-monomial. Then \( \varphi_{ub}(v \cdot ab) = 0 \).

Proof. Directly we have

\[
\varphi_{ub}(v \cdot ab) = \varphi(v \cdot ab) - 2 \cdot \varphi(v) \cdot b \cdot \beta(b) \\
- 2 \cdot \varphi(v \cdot a) \cdot b \cdot \beta(1) \\
- 2 \cdot \sum_v \varphi(v_{(1)}) \cdot b \cdot \beta(v_{(2)} \cdot ab) \\
= \varphi(v) \cdot 2d - 2 \cdot \varphi(v) \cdot b \cdot (a - b) - 2 \cdot \varphi(v) \cdot cb \\
= 2 \cdot \varphi(v) \cdot (d - b(a - b) - cb) \\
= 0,
\]

where we have used the facts that \( \varphi(v \cdot ab) = \varphi(v) \cdot 2d \) and \( \beta(v_{(2)} \cdot ab) = 0 \). \( \square \)

The previous three lemmas enable us to determine \( \varphi_{ub} \). In order to obtain more compact notation, define a map \( r : \mathbb{Z}\langle a, b \rangle \rightarrow \mathbb{Z}\langle a, b \rangle \) by \( r(1) = 0 \), \( r(v \cdot a) = v \), and \( r(v \cdot b) = 0 \). By using the chain definition of the \( ab \)-index, it is straightforward to see that \( \Psi(L_{ub}) = r(\Psi(L)) \).

Proposition 1.4.9. Let \( w \) be an \( ab \)-polynomial homogeneous of degree greater than zero. Then

\[
\varphi_{ub}(a \cdot w) = \omega(a \cdot r(w)) \cdot (a - b).
\]

Proof. The case \( w = v \cdot a \) follows from Lemma 1.4.6. The remaining case is \( w = v \cdot b \). Note that \( a \cdot v \cdot b \) can be factored as \( u \cdot ab \cdot b^k \) for a monomial \( u \). Hence \( \varphi_{ub}(u \cdot ab \cdot b^k) = \varphi_{ub}(u \cdot ab) \cdot (a - b)^k = 0 \) by Lemmas 1.4.7 and 1.4.8. \( \square \)

We combine all of these results to conclude that the \( cd \)-index of the poset of unbounded regions \( T_{ub} \) can be computed in terms of the \( ab \)-index of the unbounded intersection lattice \( L_{ub} \).

Theorem 1.4.10. Let \( \mathcal{H} \) be a non-central hyperplane arrangement with the unbounded intersection lattice \( L_{ub} \) and poset of unbounded regions \( T_{ub} \). Then the \( ab \)-index of \( T_{ub} \) is given by

\[
\Psi(T_{ub}) = \omega(a \cdot \Psi(L_{ub}))^*.
\]

Proof. We have that

\[
\Psi(T_{ub})^* \cdot (a - b) = \Psi(Q) \\
= \varphi_{ub}(a \cdot \Psi(L)) \\
= \omega(a \cdot r(\Psi(L))) \cdot (a - b) \\
= \omega(a \cdot \Psi(L_{ub})) \cdot (a - b).
\]

The result follows by cancelling \( a - b \) from both sides of the identity. \( \square \)
Example 1.4.11. Consider the non-central hyperplane arrangement consisting of the six hyperplanes $x = 0, 1, y = 0, 1$ and $z = 0, 1$. See Figure 1.3. After intersecting this arrangement with a sphere of large enough radius we obtain the cell complex in Figure 1.4. The polytopal realization of this complex is known as the rhombicuboctahedron. The dual of the face lattice of this spherical complex is not realized by a zonotope. However, one can view the dual lattice as the face lattice of a $2 \times 2 \times 2$ pile of cubes.

The intersection lattice $\mathcal{L}$ is the face lattice of the three-dimensional crosspolytope, in other words, the octahedron. Hence the lattice of unbounded intersections $\mathcal{L}_{ub}$ has the flag $f$-vector $(f_0, f_1, f_2, f_{12}) = (1, 6, 12, 24)$ and the flag $h$-vector $(h_0, h_1, h_2, h_{12}) = (1, 5, 11, 7)$. The $ab$-index is given by $\Psi(\mathcal{L}_{ub}) = a^2 + 5 \cdot ba + 11 \cdot ab + 7 \cdot b^2$. Hence the $cd$-index of $T_{ub}$ is

$$
\Psi(T_{ub}) = \omega(a^3 + 5 \cdot aba + 11 \cdot a^2b + 7 \cdot ab^2)^* = c^3 + 22 \cdot dc + 24 \cdot cd.
$$

1.5 Concluding remarks

For regular subdivisions of manifolds questions abound.

(i) What is the right analogue of a regular subdivision in order that it be polytopal? Can flag $f$-vectors be classified for polytopal subdivisions?

(ii) Is there a Kalai convolution for manifolds that will generate more inequalities for flag $f$-vectors? [44]

(iii) Is there a lifting technique that will yield more inequalities for higher dimensional manifolds? [21]
(iv) Are there minimization inequalities for the cd-coefficients in the polynomial Ψ?
As a first step, can one prove the non-negativity of Ψ? [1, 23]

(v) Is there an extension of the toric g-inequalities to manifolds? [5, 43, 45, 60]

(vi) Can the coefficients for Ψ be minimized for regular toric arrangements as was done in the case of central hyperplane arrangements? [10]

The most straightforward manifold to study is n-dimensional projective space $P^n$.
We offer the following result in obtaining the ab-index of subdivisions of $P^n$.

**Theorem 1.5.1.** Let $\Omega$ be a centrally symmetric regular subdivision of the n-dimensional
sphere $S^n$. Assume that when antipodal points of the sphere are identified, a regular
subdivision $\Omega'$ of the projective space $P^n$ is obtained. Then the ab-index of $\Omega'$ is
given by

$$\Psi(\Omega') = \frac{c_{n+1}}{2} + \frac{(a - b)^{n+1}}{2} + \Phi,$$

where the cd-index of $\Omega$ is $\Psi(\Omega) = c^{n+1} + \Phi$.

**Proof.** Each chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ with $k \geq 2$ in $\Omega'$ corresponds
to two chains in $\Omega$ with the same weight $\text{wt}(c)$. The chain $c = \{\hat{0} = x_0 > x_1 = \hat{1}\}$
corresponds to exactly one chain in $\Omega$ and has weight $(a - b)^{n+1}$. Hence $\Psi(\Omega) = 2 \cdot \Psi(\Omega') - (a - b)^{n+1}$, proving the result.

The results in this chapter have been stated for hyperplane arrangements. In true
generality one could work with the underlying oriented matroid, especially since there
are nonrealizable ones such as the non-Pappus oriented matroid. All of these can be
represented as pseudo-hyperplane arrangements. We chose to work with hyperplane
arrangements to preserve the geometric intuition.

Poset transformations related to the $\omega$ map have been considered in [20, 28, 40].
Are there toric or affine analogues of these poset transforms?

Another way to encode the flag f-vector data of a poset is to use the quasisymmetric function of a poset [18]. In this language the $\omega$ map is translated to Stembridge’s $\vartheta$ map; see [11, 62]. Would the results of Theorems 1.3.22 and 1.4.10 be appealing in the quasisymmetric function viewpoint?

Richard Stanley has asked if the coefficients of the toric characteristic polynomial
are alternating. If so, is there any combinatorial interpretation of the absolute values
of the coefficients.

A far reaching generalization of Zaslavsky’s results for hyperplane arrangements
is by Goresky and MacPherson [34]. Their results determine the cohomology groups
of the complement of a complex hyperplane arrangement. For a toric analogue of the
Goresky–MacPherson results, see work of De Concini and Procesi [16]. For algebraic
considerations of toric arrangements, see [17, 52, 51, 53].

In Section 1.3 we restricted ourselves to studying arrangements that cut the torus
into regular cell complexes. In a future paper [29], two of the authors are developing
the notion of a cd-index for non-regular cell complexes.
2 Mixing operators

2.1 Introduction

Kalai [44] showed that a basis for flag $f$-vectors of polytopes is given by the flag $f$-vectors of polytopes constructed from simplices by repeatedly taking joins or products. Ehrenborg and Readdy [25] studied how the cd-index changes under these operations. They discovered bilinear operators on the Newtonian coalgebra $\mathbb{Z}(c,d)$ which they called the mixing operator (for joins of polytopes) and the diamond operator (for products of polytopes). Later, Ehrenborg and Fox [22] analyzed these operators further, obtaining recursive coalgebraic formulas for the cd-indices of joins and products of polytopes. Using these formulas, they obtained a cd-index inequality relating the product of a join with the join of a product, providing evidence for Stanley’s Gorenstein* conjecture, which was only settled later [23].

It is difficult to use the join and product operations to study non-spherical manifolds, such as tori, since both preserve Eulerianness and take spheres to spheres. To remedy this difficulty, we introduce the manifold product. This is defined on manifolds as the Cartesian product of the underlying cell complexes, and yields a bilinear operator on ab-indices. A manifold product of Eulerian manifolds is not globally Eulerian, but it is locally Eulerian. We extend inequalities proved by Ehrenborg and Fox to the case of manifold products.

The mixing and diamond operators are nonnegative operators on cd-indices. Therefore, it makes sense to ask if there is something the coefficients count. We prove that the coefficients of the cd-index of the diamond product of two butterfly posets, which have pure c-power cd-indices, can be interpreted as a weighted sum of restricted lattice paths. This also extends to a lattice-path interpretation for the coefficients of the mixing operator applied to pure c-power terms. We also extend this interpretation to the manifold operator in the situation where the manifold operator yields a near cd-index which is nonnegative.

2.2 Preliminaries

For any cell complex $X$, let $\mathcal{L}(X)$ denote its face poset. The empty face $\hat{0}$ and the total complex $\hat{1}$ are faces in $\mathcal{L}(X)$. If $X$ is a polytope, then $\mathcal{L}(X)$ is a lattice.

A graded poset is a poset $P$ with distinct minimum and maximum elements $\hat{0}$ and $\hat{1}$ which is equipped with a rank function $\rho: P \to \mathbb{N}$. The rank function must preserve covers and send the minimum element of $P$ to 0. In other words, $\rho(\hat{0}) = 0$, and if $x < y$ in $P$, then $\rho(x) + 1 = \rho(y)$. The face poset of a finite regular cell complex, such as a polytope, is graded by dimension.

Fix once and for all a collection $\mathcal{G}$ which has exactly one representative of each isomorphism class of finite graded posets. From now on, we identify each graded poset with its isomorphic representative in $\mathcal{G}$. 
Fix a ground ring $k$. All modules, algebras, and coalgebras we discuss will be over this ground ring.

An algebra is a module $A$ together with linear structure maps $∇ : A ⊗ A → A$, called the product, and $η : k → A$, called the unit, such that the diagrams

$$
\begin{align*}
A ⊗ A ⊗ A & \xrightarrow{\nabla ⊗ \text{id}} A ⊗ A & \quad \text{and} & \quad k ⊗ A & \xrightarrow{\eta ⊗ \text{id}} A ⊗ A \quad \text{and} \quad \text{id} ⊗ A \xrightarrow{\nabla} A
\end{align*}
$$

are commutative. If $A$ and $B$ are algebras, then an algebra morphism from $A$ to $B$ is a linear map $ϕ : A → B$ which respects the product and unit. That is, $∇_B \circ (ϕ ⊗ ϕ) = ϕ \circ ∇_A$ and $ϕ \circ η_A = η_B$.

Dually, a coalgebra is a module $C$ together with linear structure maps $Δ : C → C ⊗ C$, called the coproduct, and $ε : C → k$, called the counit, such that the diagrams

$$
\begin{align*}
C ⊗ C ⊗ C & \xrightarrow{\Delta ⊗ \text{id}} C ⊗ C & \quad \text{and} & \quad k ⊗ C & \xrightarrow{\epsilon ⊗ \text{id}} C ⊗ C \quad \text{and} \quad \text{id} ⊗ C \xrightarrow{\Delta} C
\end{align*}
$$

are commutative. Coalgebras will generally not be assumed to have a counit. If $C$ and $D$ are coalgebras, then a coalgebra morphism from $C$ to $D$ is a linear map $ϕ : C → D$ which respects the coproduct and counit, that is, the equations $(ϕ \otimes ϕ) \circ Δ_C = Δ_D \circ ϕ$ and $ε_D \circ ϕ = ε_C$ hold.

Just as taking a product can be thought of assembling something out of smaller pieces, taking a coproduct can be thought of as disassembling something into its constituent pieces. Following this analogy, we define a piece of $c$ to be any term $c_{(1)}$ or $c_{(2)}$ which appears in the expansion $Δ(c) = \sum c_{(1)} \otimes c_{(2)}$.

We will generally suppress the notation $∇$ for product, writing $ab$ or $a · b$ instead of $∇(a \otimes b)$. The sigma notation for coproducts was introduced by Heyneman and Sweedler [38] and is now widely used. We adopt a variant of sigma notation, writing the coproduct of $c$ as

$$
Δ(c) = \sum c_{(1)} \otimes c_{(2)}.
$$

If the coproduct is understood, we will generally suppress $Δ$, writing

$$
Δ(c) = \sum c_{(1)} \otimes c_{(2)}.
$$

Using sigma notation, the coassociativity condition can be written as the equation

$$
\sum (c_{(1,1)} \otimes c_{(1,2)}) \otimes c_{(2)} = \sum c_{(1)} \otimes (c_{(2,1)} \otimes c_{(2,2)}) = \sum c_{(1)} \otimes c_{(2)} \otimes c_{(3)}.
$$

while the counital condition can be written as the equation

$$
c = \sum \epsilon(c_{(1)})c_{(2)} = \sum c_{(1)}\epsilon(c_{(2)}).
$$
A bialgebra is a module with compatible algebra and coalgebra structure maps. In other words, the algebra structure maps are coalgebra morphisms, while the coalgebra structure maps are algebra morphisms. If $B$ is a bialgebra with product $\nabla$ and coproduct $\Delta$, then $\text{Hom}_k(B, B)$ is an algebra with the convolution product, defined by $f \ast g = \nabla \circ (f \otimes g) \circ \Delta$. Using sigma notation, the convolution of linear maps $f$ and $g$ is written

$$(f \ast g)(b) = \sum \Delta f(b_{(1)})g(b_{(2)}).$$

Observe that the composition $\eta \circ \varepsilon: B \to B$ of the unit and counit (if there is one) is the identity under convolution.

A Hopf algebra is a bialgebra $H$ for which the identity map $\text{id}: H \to H$ has a convolution inverse $S: H \to H$, that is, such that

$$(\eta \circ \varepsilon)(h) = \sum \Delta S(h_{(1)})h_{(2)} = \sum \Delta h_{(1)}S(h_{(2)}),$$

for all $h$ in $H$. The map $S$, which is always an antihomomorphism, is called the antipode of $H$.

A Newtonian coalgebra is a module $N$ with both algebra and coalgebra structure maps such that the Leibniz condition

$$\Delta(u \cdot v) = \Delta(u) \cdot v + u \cdot \Delta(v)$$

holds for all $u$ and $v$. In other words, the coproduct is a derivation over the product. Newtonian coalgebras were introduced by Joni and Rota [41], who called them infinitesimal coalgebras. A Newtonian coalgebra can have a unit or a counit, but not both.

Now we indicate the algebras of interest and briefly describe each.

- **$G$**, the Newtonian coalgebra of graded posets;
- **$A$**, the Newtonian coalgebra of ab-polynomials;
- **$G^*$**, the Hopf algebra of graded posets; and
- **$A^*$**, the nonassociative bialgebra of ab-polynomials.

### 2.2.1 The Newtonian coalgebra of graded posets

Let $G = kG$ be the free module generated by $G$. The star product of two posets $P$, $Q$ in $G$, denoted by $P \ast Q$, is the poset with ground set $(P \setminus \{0_P\}) \cup (Q \setminus \{1_Q\})$ and order relation

$$x \leq_{P\ast Q} y \text{ if and only if } \begin{cases} x \leq_P y \\ x \leq_Q y \\ x \in P \text{ and } y \in Q. \end{cases}$$
The star product $\star$ makes $G$ into an algebra with the Boolean algebra on a one-element set as the unit. Ehrenborg and Hetyei showed in unpublished work that $G$ is a Newtonian coalgebra. The coproduct of a poset $P$ is defined by the formula

$$\Delta(P) = \sum_{0 < x < 1} [0, x] \otimes [x, 1].$$

It is straightforward to verify the Leibniz condition:

$$\Delta(P \star Q) = \Delta(P) \star Q + P \star \Delta(Q).$$

Since $\Delta$ is a derivation over the unital product $\star$, there is no counit.

### 2.2.2 The Newtonian coalgebra of $ab$-polynomials

The noncommutative polynomial algebra $A = k\langle a, b \rangle$ also has the structure of a Newtonian coalgebra. The coproduct is defined on a monomial $u_1 \cdots u_n$ by the formula

$$\Delta(u_1 \cdots u_n) = \sum_{i=1}^{n} u_1 \cdots u_{i-1} \otimes u_{i+1} \cdots u_n.$$

The $ab$-index $\Psi(P)$ of a graded poset $P$ is an invariant of the poset. Ehrenborg and Readdy showed that $\Psi$ can be viewed as a morphism $\Psi : G \to A$ of Newtonian coalgebras. Moreover, $\Psi$ is surjective.

Stanley developed a recursive formula for the $ab$-index of a poset which is amenable to computation and best expressed using coalgebraic notation. Define an algebra endomorphism $\kappa$ on $A$ by setting $\kappa(a) = a - b$ and $\kappa(b) = 0$. Stanley proved that the $ab$-index satisfies the recursive formula

$$\Psi(P) = \kappa(\Psi(P)) + \sum \kappa(\Psi(P_{(1)})) \cdot b \cdot \Psi(P_{(2)})$$

$$= \kappa(\Psi(P)) + \sum \Psi(P_{(1)}) \cdot b \cdot \kappa(\Psi(P_{(2)})).$$

Applying the surjectivity of $\Psi$, the same recursive formula holds for every $ab$-polynomial:

$$u = \kappa(u) + \sum \kappa(u_{(1)}) \cdot b \cdot u_{(2)} = \kappa(u) + \sum u_{(1)} \cdot b \cdot \kappa(u_{(2)}).$$

This can also be proved inductively for $ab$-polynomials, or be viewed as a consequence of the Ehrenborg-Readdy theorem that $\Psi$ is a morphism of Newtonian coalgebras. In any case, the $\kappa$ morphism is fundamental for the study of the $ab$-index.

The map $\kappa$ preserves $a$ and kills $b$. In a similar way we can define a map $\lambda$ which preserves $b$ and kills $a$. Let $\tau : k\langle a, b \rangle \to k\langle a, b \rangle$ denote the map which swaps $a$ and $b$. Define an algebra endomorphism $\lambda$ on $k\langle a, b \rangle$ by $\lambda(u) = \kappa(\tau(u))$. Then for any $ab$-polynomial $u$,

$$u = \lambda(u) + \sum \lambda(u_{(1)}) \cdot a \cdot u_{(2)} = \lambda(u) + \sum u_{(1)} \cdot a \cdot \lambda(u_{(2)}).$$
Note that the maps $\kappa$ and $\lambda$ act as near-counits in $A$.

The Newtonian coalgebra $A$ has an important Newtonian subcoalgebra $C = k\langle c, d \rangle$, which is generated by the monomials $c = a + b$ and $d = ab + ba$. If a poset is Eulerian, its $ab$-index lives in the subcoalgebra $C$. In general, if $\Psi(P)$ is in $k\langle c, d \rangle$, then we say that $P$ has a $cd$-index. The existence of the $cd$-index was conjectured by Fine. Bayer and Klapper [6] showed that a poset has a $cd$-index if and only if it satisfies the generalized Dehn-Sommerville relations, while Stanley [61] provided an alternative proof for Eulerian posets and established that the $cd$-index of a polytope has nonnegative coefficients. Several proofs of the existence of the $cd$-index have been given [6], [19], [27], [61].

2.2.3 The Hopf algebra of graded posets

We will also need to make use of the Hopf algebra structure on graded posets. Let $\mathcal{G} = \mathcal{G} \cup \{\bullet\}$, where $\bullet$ is the one-point poset. Then the module $G^* = k\mathcal{G}$ has the structure of a Hopf algebra, with product coming from the Cartesian product and coproduct defined by

$$
\Delta^*(P) = \sum_{x \in P} \Delta^* P(1) \otimes P(2) = \sum_{x \in P} [\hat{0}, x] \otimes [x, \hat{1}].
$$

Schmitt [57] derived an explicit formula for the antipode. Ehrenborg showed [18] that the antipode plays the role of the Möbius function, since if we define $\phi : G^* \to k$ by $\phi(P) = 1$ for each poset $P$, then $\mu(P) = \phi(S(P))$.

2.2.4 The nonassociative bialgebra of $ab$-polynomials

In a similar way, we can extend the Newtonian coalgebra $A$ to a nonassociative bialgebra $A^* = k\langle a, b \rangle \oplus k\xi$ via the formulas

$$a\xi = b\xi = \xi a = \xi b = 1 \quad \text{and} \quad \xi^2 = 0.$$

Note that $\xi$ does not usually associate, so one must exercise care with its use. If $\xi$ is flanked by two copies of $a$ or $b$, then it does associate, yielding the identities $a\xi a = a$ and $b\xi b = b$. However, $(d\xi)d = cd$ while $d(\xi d) = dc$. This bialgebra was introduced by Ehrenborg and Fox [22].

The Stanley recursion for the $ab$-index may be expressed more briefly in this bialgebra:

$$u = \sum_{x \in P} \Delta^* \kappa(u(1)) \cdot b \cdot u(2) = \sum_{x \in P} \Delta^* u(1) \cdot b \cdot \kappa(u(2))$$

$$= \sum_{x \in P} \Delta^* \lambda(u(1)) \cdot a \cdot u(2) = \sum_{x \in P} \Delta^* u(1) \cdot a \cdot \lambda(u(2))$$

Observe that $\kappa$ and $\lambda$ act as near-counits in $A^*$.

Just as $A$ has a subcoalgebra $C$ of $cd$-polynomials, $A^*$ has a subbialgebra $C^*$ of $cd$-polynomials with $\xi$. 

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2.3 Binary operations on posets

Kalai [44] constructed a basis of polytopes obtained from simplices by repeatedly taking joins and direct sums. He showed that the face lattice of a join of polytopes is the Cartesian product of the respective face lattices, and the face lattice of a direct sum is the diamond product of the face lattices. That is,

\[
\mathcal{L}(X \bigvee Y) = \mathcal{L}(X) \times \mathcal{L}(Y)
\]

\[
\mathcal{L}(X \times Y) = \mathcal{L}(X) \diamond \mathcal{L}(Y),
\]

where \( \bigvee \) denotes the join operation and \( \diamond \) denotes the diamond product. Recall that the diamond product (or lower truncated product) of posets \( P \) and \( Q \) is defined by

\[
P \diamond Q = (P \setminus \{\hat{0}\}) \times (Q \setminus \{\hat{0}\}) \cup \{\hat{0}\}.
\]

There is also a dual diamond product (or upper truncated product), which we denote by \( \diamond^* \):

\[
P \diamond^* Q = (P \setminus \{\hat{1}\}) \times (Q \setminus \{\hat{1}\}) \cup \{\hat{1}\}.
\]

The diamond product and dual diamond product are related by the identity

\[
P \diamond^* Q = (P^* \diamond Q^*)^*,
\]

where \( P^* \) denotes the dual of the poset \( P \).

The geometric operations of pyramid and prism arise from \( \times \) and \( \diamond \) on the poset level, since

\[
\mathcal{L}({\text{Pyr}}(P)) = \mathcal{L}(P) \times B_1 \quad \text{and} \quad \mathcal{L}({\text{Pri}}(P)) = \mathcal{L}(P) \diamond B_2,
\]

where \( B_i \) denotes the Boolean algebra on \( i \) elements. Since the \( ab \)-index encodes the flag \( f \)-vector, it is of interest to study the effects of \( \times \) and \( \diamond \) on the \( ab \)-index. Ehrenborg [15] used quasisymmetric functions to show that \( \Psi(P \times Q) \) is a function of \( \Psi(P) \) and \( \Psi(Q) \). Ehrenborg and Readdy [25] derived recursive formulas for \( \Psi(P \times Q) \) which were improved by Ehrenborg and Fox [22].

In preparation for the study of the manifold product, we present a completely coalgebraic derivation of the recursive formulas for \( \Psi(P \times Q) \) and \( \Psi(P \diamond Q) \). We need two basic facts. First, we need the Stanley recursion discussed above. Second, we need to know the coproduct of a Cartesian product of posets.

Since the Cartesian product is the product in the Hopf algebra of graded posets,

\[
\Delta^*(P \times Q) = \Delta^*(P) \times \Delta^*(Q) = \sum \Delta^* (P_1 \times Q_1) \otimes (P_2 \times Q_2).
\]

Hence the coproduct of a Cartesian product is

\[
\Delta^*(u \times v) = \sum \Delta^* (u_1 \times v_1) \otimes (u_2 \times v_2).
\]
Using Stanley’s recursion for the ab-index, we obtain the following recursive formula for the mixing operator \( \times \) applied to the ab-polynomials \( u \) and \( v \):

\[
\begin{align*}
 u \times v &= \sum \Delta^* \kappa(u(1) \times v(1)) \cdot b \cdot (u(2) \times v(2)) \\
 &= \kappa(u \times v) + \kappa(u) \cdot b + \kappa(v) \cdot b \cdot u \\
 &\quad + \sum \kappa(u(1)) \cdot b \cdot (u(2) \times v) + \sum \kappa(v(1)) \cdot b \cdot (u \times v(2)) \\
 &\quad + \sum \kappa(u \times v(1)) \cdot b \cdot v(2) + \sum \kappa(u(1) \times u(2)) \cdot b \cdot (u(2) \times v(2)).
\end{align*}
\]

2.3.1 Computing the cd-index of a Cartesian product

For any graded poset \( P \), the coefficient of the pure a term is always 1. Hence \( \kappa(P) \) depends only on the rank of \( P \), that is, \( \kappa(P) = (a - b)^{\rho(P) - 1} \). If \( P \) and \( Q \) are graded posets, their Cartesian product has rank \( \rho(P) + \rho(Q) + 1 \). So

\[
\kappa(\Psi(P \times Q)) = \kappa(\Psi(P) \cdot a \cdot \Psi(Q))
\]

Hence for any ab-polynomials \( u \) and \( v \),

\[
\kappa(u \times v) = \kappa(u) \cdot \kappa(v) \cdot (a - b)
\]

Analogously,

\[
\lambda(u \times v) = \lambda(u) \cdot \lambda(v) \cdot (b - a)
\]

We use these facts to prove the following lemma.

**Lemma 2.3.1** (Ehrenborg–Readdy [25, Proposition 4.2]). For any ab-polynomial \( u \),

\[
\begin{align*}
 u \times 1 &= \sum \Delta^* u(1) \cdot ba \cdot u(2) = a \cdot u + u \cdot b + \sum u(1) \cdot ba \cdot u(2) \quad (2.3.1) \\
 &= \sum \Delta^* u(1) \cdot ab \cdot u(2) = b \cdot u + u \cdot a + \sum u(1) \cdot ab \cdot u(2). \quad (2.3.2)
\end{align*}
\]

Since the formula for \( u \times 1 \) is invariant under the action of the involution \( \tau \) which swaps \( a \) and \( b \), if \( u \) is a cd-polynomial, then so is \( u \times 1 \).

**Proof.** Since 1 is the ab-index of the Boolean algebra \( B_1 \), the expression \( 1 \times 1 \) is the ab-index of the product \( B_1 \times B_1 = B_2 \), that is, \( 1 \times 1 = \Psi(B_2) = c \). Equations \( (2.3.1) \) and \( (2.3.2) \) both hold when \( u = 1 \).

To complete the proof, assume for induction that Equation \( (2.3.1) \) holds for all pieces of \( u \), that is, for any polynomial \( u(1) \) or \( u(2) \) appearing in the coproduct of \( u \).
Since $\Delta^*(1) = 1 \otimes \xi + \xi \otimes 1$,
\[
    u \times 1 = \sum_{u \in \Delta^*} \kappa(u(1) \times 1) \cdot b \cdot u(2) + \sum_{u \in \Delta^*} \kappa(u(1)) \cdot b \cdot (u(2) \times 1)
    = b \cdot u + \kappa(u) \cdot b \\
    + \sum_{u \in \Delta} (a - b) \cdot \kappa(u(1)) \cdot b \cdot u(2) + \sum_{u \in \Delta} \kappa(u(1)) \cdot b \cdot u(2) \cdot b \\
    + \sum_{u \in \Delta} \kappa(u(1)) \cdot ba \cdot u(2) + \sum_{u \in \Delta} \kappa(u(1)) \cdot b \cdot u(2) \cdot ba \cdot u(3)
\]

Use the identity $\kappa(u \times 1) = \kappa(u) \cdot (a - b)$ to combine two of the isolated terms, and apply the induction hypothesis to expand the second summation. The Stanley recursion permits the terms above to be expressed in a much simpler way.

\[
    u \times 1 = b \cdot u + \kappa(u) \cdot a \\
    + (a - b) \cdot (u - \kappa(u)) + (u - \kappa(u)) \cdot b \\
    + \sum_{u \in \Delta} u_{(1)} \cdot ba \cdot u_{(2)}
    = a \cdot u + u \cdot b + \sum_{u \in \Delta} u_{(1)} \cdot ba \cdot u_{(2)}.
\]

Equation (2.3.2) could be proved by imitating the one just given, replacing $\kappa$ with $\lambda$ and making other appropriate changes. However, it is more direct to apply the fact that the star involution is a Newtonian coalgebra anti-isomorphism. Hence

\[
    u \times 1 = \left( u^* \times 1^* \right)^*
    = \left[ a \cdot u^* + u^* \cdot b + \sum_{u \in \Delta} u^*_{(2)} \cdot ba \cdot u^*_{(1)} \right]^*
    = b \cdot u + u \cdot a + \sum_{u \in \Delta} u_{(1)} \cdot ab \cdot u_{(2)},
\]

which completes the proof. \hfill $\square$

**Lemma 2.3.2** (Ehrenborg–Fox [22, Proposition 5.8]). For any ab-polynomials $u$ and $v$, the identities

\[
    u \times (v \cdot a) = \sum_{u \in \Delta^*} (u(1) \times v) \cdot ab \cdot u(2)
    = v \cdot ab \cdot u + (u \times v) \cdot a + \sum_{u \in \Delta} (u(1) \times v) \cdot ab \cdot u_{(2)} \quad (2.3.3)
\]

\[
    u \times (v \cdot b) = \sum_{u \in \Delta^*} (u(1) \times v) \cdot ba \cdot u(2)
    = v \cdot ba \cdot u + (u \times v) \cdot b + \sum_{u \in \Delta} (u(1) \times v) \cdot ba \cdot u_{(2)} \quad (2.3.4)
\]

hold.

**Proof.** The proof is a double induction on the lengths of $u$ and $v$. By explicitly constructing appropriate posets, one can compute that

\[
    1 \times a = c^2 - b^2 \quad \text{and} \quad 1 \times b = (1 \times c) - (1 \times a) = (c^2 + d) - (c^2 - b^2) = c^2 - a^2.
\]
Thus Equations (2.3.3) and (2.3.4) are both satisfied if $u = v = 1$.

Now assume for induction that Equation (2.3.4) holds for $v = 1$ and any piece of $u$. Expand $u \times b$ via the general recursion for products, keeping in mind that $\kappa(w \times b) = 0$ for any $w$.

\[
u \times b = \sum_{\Delta^*} \kappa(u(1) \times 1) \cdot b \cdot (u(2) \times 1) + \sum_{\Delta^*} \kappa(u(1)) \cdot b \cdot (u(2) \times b).
\]

Apply Lemma 2.3.1 to the first summation and the induction hypothesis to the second summation.

\[
u \times b = \sum_{\Delta^*} \kappa(u(1) \times 1) \cdot b \cdot (u(2) \times \xi) \cdot ba \cdot u(3) + \sum_{\Delta^*} \kappa(u(1) \times \xi) \cdot b \cdot (u(2) \times 1) \cdot ba \cdot u(3).
\]

The part of the above expression preceding $ba$ is recognizable as an expansion of the product $u(1) \times 1$.

\[
u \times b = \sum_{\Delta^*} (u(1) \times 1) \cdot ba \cdot u(2),
\]

which is what needed to be shown.

To complete the double induction, assume that Equation (2.3.4) holds for any piece of $u$ or $v$. Since $\Delta^*(v \cdot b) = \Delta^*(v) \cdot b + v \cdot b \otimes \xi$,

\[
u \times (v \cdot b) = \sum_{\Delta^*} \kappa(u(1) \times v(1)) \cdot b \cdot (u(2) \times (v(2) \cdot b)) + \sum_{\Delta^*} \kappa(u(1) \times (v \cdot b)) \cdot b \cdot (u(2) \times \xi).
\]

The second summation vanishes because $\kappa$ kills $b$. Apply the induction hypothesis to expand the first summation. As in the case $v = 1$, this results in a recognizable expansion of a product. No parentheses are needed below because $u(2) \times v(2)$, the only expression which could be $\xi$, is flanked by copies of $b$.

\[
u \times (v \cdot b) = \sum_{\Delta^*} \kappa(u(1) \times v(1)) \cdot b \cdot (u(2) \times v(2)) \cdot ba \cdot u(3) = \sum_{\Delta^*} (u(1) \times v) \cdot ba \cdot u(2).
\]

This completes the proof of Equation (2.3.4).

Equation (2.3.3) can be proved in a similar way, replacing $\kappa$ with $\lambda$ and making other appropriate changes.

In the previous lemmas identities appeared in pairs differing only by the action of the involution $\tau$. This suggests that $\times$ respects the action of $\tau$. This is a consequence of the identities proved in Lemma 2.3.2, but it is more fundamentally a consequence of the existence of the paired recursive formulas

\[
u = \sum_{\Delta^*} \kappa(u(1)) \cdot b \cdot u(2) = \sum_{\Delta^*} \lambda(u(1)) \cdot a \cdot u(2).
\]

Ehrenborg and Fox proved that $\times$ respects the involution $\tau$. We offer the following alternative proof.
Proposition 2.3.3 (Ehrenborg–Fox [22, Lemma 5.5]). For any ab-polynomials $u$ and $v$, the identity
\[ u \times v = \overline{u} \times \overline{v} \]
holds.

Proof. If $u = v = 1$, there is nothing to prove. Suppose the claim holds for pieces of $u$ and $v$. By the recursive formula for the product $u \times v$,
\[
\begin{align*}
\sum_{\Delta^*} \lambda(u(1) \times v(1)) \cdot a \cdot \overline{u(2)} \times \overline{v(2)}.
\end{align*}
\]
Now apply the induction hypothesis and the fact that $\overline{\cdot}$ is a coalgebra morphism.
\[
\begin{align*}
\sum_{\Delta^*} \lambda(u(1) \times v(1)) \cdot a \cdot \overline{u(2)} \times \overline{v(2)}.
\end{align*}
\]
This completes the proof.

Corollary 2.3.4 (Ehrenborg–Fox [22, Theorem 5.1]). For any cd-polynomials $u$ and $v$, the identities
\[
\begin{align*}
u \times (v \cdot c) &= \sum_{\Delta^*} (u(1) \times v) \cdot d \cdot u(2) \\
u \times (v \cdot d) &= \sum_{\Delta^*} (u(1) \times v) \cdot d \cdot \text{Pyr}(u(2))
\end{align*}
\]
hold.

Proof. Expand $c$ and $d$, then apply Lemma 2.3.2. Thus
\[
\begin{align*}
u \times (v \cdot c) &= \sum_{\Delta^*} (u(1) \times v) \cdot (ab + ba) \cdot u(2) \\
&= \sum_{\Delta^*} (u(1) \times v) \cdot d \cdot u(2).
\end{align*}
\]
To compute $u \times (v \cdot d)$, the lemma must be invoked twice. We have
\[
\begin{align*}
u \times (v \cdot ab) &= \sum_{\Delta^*} (u(1) \times (v \cdot a)) \cdot ba \cdot u(2) \\
&= \sum_{\Delta^*} (u(1) \times v) \cdot ab \cdot u(2) \cdot \text{Pyr}(u(3)).
\end{align*}
\]
By Lemma 2.3.1, we can collapse $u(2) \cdot \text{Pyr}(u(2))$ into Pyr($u(2)$). Similarly,
\[
\begin{align*}
u \times (v \cdot ba) &= \sum_{\Delta^*} (u(1) \times v) \cdot ba \cdot \text{Pyr}(u(2)),
\end{align*}
\]
from which the recursive formula for $u \times (v \cdot d)$ follows.\hfill\qed
2.3.2 Computing the cd-index of a diamond product

Just as with the Cartesian product, the algebra maps $\kappa$ and $\lambda$ interact nicely with the $ab$-index of a diamond product of posets. If $P$ is a graded poset, then $\kappa(P)$ is given by $\kappa(\Psi(P)) = (a - b)^{\rho(P) - 1}$. If $P$ and $Q$ are graded posets, their diamond product has rank $\rho(P) + \rho(Q)$. Thus

$$\kappa(\Psi(P \diamond Q)) = \kappa(\Psi(P) \cdot \Psi(Q))$$

Hence for any $ab$-polynomials $u$ and $v$,

$$\kappa(u \diamond v) = \kappa(u) \cdot \kappa(v).$$

Analogously,

$$\lambda(u \diamond v) = \lambda(u) \cdot \lambda(v).$$

These formulas describe the situation in the algebra $A$. For simplicity, we require that the above formulas hold in $A^\bullet$, even if $u$ or $v$ is $\xi$, subject to the constraint that $\kappa(\xi) = 0$. In particular, $\kappa(u \diamond \xi) = 0$ for any $u$, which implies that $u \diamond \xi = 0$ for any $u$. This may conflict with the intuition that

$$P \diamond \bullet = (P \setminus \{\hat{0}\}) \times \emptyset \cup \{\hat{0}\} = \emptyset \cup \{\hat{0}\} = \bullet,$$

but has the advantage of maintaining homogeneity of degree in the recursive formulas that follow. Since an $ab$-index of a poset is always homogeneous in degree, we accept failure of intuition in exchange for correctness of formulas.

We summarize the basic properties of the diamond product with the following result from Ehrenborg and Fox.

**Proposition 2.3.5** (Ehrenborg–Fox [22, Corollary 6.3]). The diamond product $\diamond$ makes $A$ into an abelian monoid with unit 1 and makes $A^\bullet$ into a commutative semigroup satisfying the rules

$$u \diamond 1 = u \text{ for any } u \text{ in } A,$$

$$u \diamond \xi = 0 \text{ for any } u \text{ in } A^\bullet.$$ 

The diamond product obeys the coalgebraic recursive formula

$$u \diamond v = \sum_{\Delta^1} \kappa(u_{(1)} \diamond v_{(1)}) \cdot b \cdot (u_{(2)} \times v_{(2)})$$

as well as the analogous formulas obtained by moving $\kappa$ or replacing $\kappa$ and $b$ with $\lambda$ and $a$.

The following lemma is the diamond version of Lemma 2.3.2.
Lemma 2.3.6. For any \( ab \)-polynomials \( u \) and \( v \), the identities

\[
\begin{align*}
    u \diamond (v \cdot a) &= \sum \Delta^* \left( (u_{(1)} \diamond v) \cdot ab \cdot u_{(2)} \right) \\
                        &= (u \diamond v) \cdot a + \sum (u_{(1)} \diamond v) \cdot ab \cdot u_{(2)} \quad (2.3.5) \\
    u \diamond (v \cdot b) &= \sum \Delta^* \left( (u_{(1)} \diamond v) \cdot ba \cdot u_{(2)} \right) \\
                        &= (u \diamond v) \cdot b + \sum (u_{(1)} \diamond v) \cdot ba \cdot u_{(2)} \quad (2.3.6)
\end{align*}
\]

hold.

Proof. This lemma is essentially a corollary of Lemma 2.3.2. Here we demonstrate Equation (2.3.6). Since \( \Delta^*(v \cdot b) = \Delta^*(v) \cdot b + v \cdot b \otimes \xi \),

\[
\begin{align*}
    u \diamond (v \cdot b) &= \sum \Delta^* \kappa(u_{(1)} \diamond v_{(1)}) \cdot b \cdot (u_{(2)} \times (v_{(2)} \cdot b)) \\
                      &\quad + \sum \Delta^* \kappa(u_{(1)} \diamond (v \cdot b)) \cdot b \cdot (u_{(2)} \times \xi).
\end{align*}
\]

Expand the first summation using the recursion for \( \times \), and notice that the second summation vanishes. Finally, recognize the left factor of the expression as an expansion of the diamond product.

\[
\begin{align*}
    u \diamond v &= \sum \Delta^* \kappa(u_{(1)} \diamond v_{(1)}) \cdot b \cdot (u_{(2)} \times v_{(2)}) \cdot ba \cdot u_{(3)} \\
               &= \sum \Delta^* (u_{(1)} \diamond v) \cdot ba \cdot u_{(2)}.
\end{align*}
\]

The proof of Equation (2.3.5) is similar. \( \square \)

The diamond product also respects the involution \( \overline{\cdot} \). Combining the recursive formulas for \( u \diamond (v \cdot a) \) and \( u \diamond (v \cdot b) \) produces recursive formulas for \( u \diamond (v \cdot c) \) and \( u \diamond (v \cdot d) \).

Corollary 2.3.7 (Ehrenborg–Fox [22, Theorem 7.1]). For any \( cd \)-polynomials \( u \) and \( v \), the identities

\[
\begin{align*}
    u \diamond (v \cdot c) &= \sum \Delta^* \left( (u_{(1)} \diamond v) \cdot d \cdot u_{(2)} \right) \\
                      &= (u \diamond v) \cdot c + \sum \Delta \left( (u_{(1)} \diamond v) \cdot d \cdot u_{(2)} \right) \quad (2.3.7) \\
    u \diamond (v \cdot d) &= \sum \Delta^* \left( (u_{(1)} \diamond v) \cdot d \cdot \text{Pyr}(u_{(2)}) \right) \\
                      &= (u \diamond v) \cdot d + \sum \Delta \left( (u_{(1)} \diamond v) \cdot d \cdot \text{Pyr}(u_{(2)}) \right) \quad (2.3.8)
\end{align*}
\]

hold.
2.4 Lattice-path interpretation of mixing operators

Equation (2.3.7) can be used to give an explicit recursive formula for \( c^p \diamond c^q \). In this section we display this formula and show how to interpret its coefficients as counting weighted lattice paths.

First we define the algebra of lattice paths. Consider the noncommutative polynomial algebra on the generators \( D, R, \) and \( U \), where \( D \) has degree 2 and \( R \) and \( U \) have degree 1. The generators correspond to the steps

\[
\text{Diagonal} = (1, 1) \\
\text{Right} = (1, 0) \\
\text{Up} = (0, 1).
\]

This algebra admits a bigrading into homogeneous parts indexed by \( p \) and \( q \) and generated by monomials with \( p \) occurrences of \( D \) or \( R \) and \( q \) occurrences of \( D \) or \( U \). Note that \( D \), which represents a diagonal step, counts toward both \( p \) and \( q \).

The \((p, q)\) summand of this algebra represents lattice paths in \( \mathbb{N} \times \mathbb{N} \) from the origin to \((p, q)\) which use only \( D, R, \) and \( U \) steps.

To avoid overcounting in what follows, we need to restrict to a submodule. Let \( \Lambda \) denote the submodule generated by monomials which do not contain \( UR \) as a contiguous subword. It inherits a grading \( \Lambda = \bigoplus_{p,q} \Lambda_{p,q} \) from the grading of the polynomial algebra.

Example 2.4.1. By direct computation, one can verify that

\[
c^3 \diamond c^2 = c^5 + 2c^3d + 4c^2dc + 4cde^2 + 2d^3 + 4dcd + 4d^2c.
\]

Compare this polynomial with Figure 2.1, which displays each \( DRU \)-word in \( \Lambda_{3,2} \) together with its associated path. The coefficients of the terms in \( c^3 \diamond c^2 \) can be obtained by weighting \( R \) and \( U \) steps by \( c \) and weighting \( D \) steps by \( 2d \). Note that the pair of terms \( RRDU \) and \( RUDR \) contribute to the same term of \( c^3 \diamond c^2 \), as do the pair of terms \( RDRU \) and \( UDRR \). Hence \( c^3 \diamond c^2 \) has only eight terms, even though there are ten \( DRU \)-words in \( \Lambda_{3,2} \).

The following proposition shows that this situation is general.

Proposition 2.4.2. Let \( \text{wt}: \Lambda \to \mathbb{C} \) be the linear map determined by

\[
\text{wt}(D) = 2d \quad \text{and} \quad \text{wt}(R) = \text{wt}(U) = c.
\]

Then for any natural numbers \( p \) and \( q \), the \( cd \)-index \( c^p \diamond c^q \) is given by the formula

\[
c^p \diamond c^q = \sum_{P \in \Lambda_{p,q}} \text{wt}(P).
\]
Figure 2.1: Paths in $\Lambda_{3,2}$ correspond to terms of $c^3 \cdot c^2$. 

Proof. The proof proceeds by induction on $p$ and $q$. For $p = q = 0$ there is nothing to show. Suppose the weighted lattice path interpretation is correct for $(p', q')$ strictly smaller than $(p, q)$ in at least one coordinate. As a consequence of Corollary 2.3.7

$$c^p \cdot (c^{q-1} \cdot c) = (c^p \cdot c^{q-1}) \cdot c + (c^{p-1} \cdot c^{q-1}) \cdot 2d + \sum_{k=0}^{p-2} (c^k \cdot c^{q-1}) \cdot 2d \cdot c^{p-1-k}.$$ 

Applying the induction assumption, the first summand is

$$c^p \cdot (c^{q-1} \cdot c) = \sum_{P \in \Lambda_{p,q-1}} \text{wt}(P \cdot U),$$

and the second summand is

$$(c^{p-1} \cdot c^{q-1}) \cdot 2d = \sum_{P \in \Lambda_{p-1,q-1}} \text{wt}(P \cdot D).$$

The summation corresponds to lattice paths to which an $R$ can be appended, that is,

$$\sum_{k=0}^{p-2} (c^k \cdot c^{q-1}) \cdot 2d \cdot c^{p-1-k} = \sum_{P \in \Lambda_{p-1,q-1, \text{P does not end in } U}} \text{wt}(P \cdot R).$$

But the module $\Lambda_{p,q}$ decomposes as

$$\Lambda_{p,q} = \{P \cdot D: P \in \Lambda_{p-1,q-1}\} \oplus \{P \cdot U: P \in \Lambda_{p,q-1}\} \oplus \{P \cdot R: P \in \Lambda_{p-1,q} \text{ and } P \text{ does not end in } U\}.$$ 

This completes the proof. \qed
As an application, we prove that the cd-polynomial $c^p \diamond c^q$ is always symmetric.

**Proposition 2.4.3.** For any natural numbers $p$ and $q$,

$$(c^p \diamond c^q)^* = c^p \diamond c^q.$$ 

**Proof.** We prove the claim by constructing an involution on the lattice paths in $\Lambda_{p,q}$. Suppose $\alpha = \alpha_1 \ldots \alpha_n$ is a UR-avoiding path from $(0,0)$ to $(p,q)$. Following the steps of $\alpha$ in reverse order yields the path $\alpha^* = \alpha_n \ldots \alpha_1$. Now, $\alpha^*$ is a path from $(0,0)$ to $(p,q)$, but it could contain UR as a contiguous subword. Adjust $\alpha^*$ to $\varphi(\alpha)$ by replacing each instance of $U^k R^i$ with $R^i U^k$. In other words, we push in any “bumps” we find in the path. The map $\alpha \mapsto \varphi(\alpha)$ is an involution, and since $U$ and $R$ have the same weight,

$$\text{wt}(\varphi(\alpha)) = \text{wt}(\alpha)^*.$$ 

This completes the proof. \qed

Since we can interpret the coefficients of $c^p \diamond c^q$ as counting lattice paths, it is natural to ask whether we can interpret the coefficients of $c^p \times c^q$ in a similar way. First, recall the recursive formula for $c^p \times c^q$:

$$c^p \times c^q = (c^p \times c^{q-1}) \cdot c + c^q \cdot d \cdot c^p + \sum_{j+k=p-1} (c^j \times c^{q-1}) \cdot 2d \cdot c^k.$$ 

If the coefficients are to represent lattice paths in a straightforward way, then it seems natural that the term $(c^p \times c^{q-1}) \cdot c$ represents lattice paths which pass through $(p, q-1)$ and end in $U$, so that they pass through $(p, q-1)$, while a term of the form $(c^j \times c^{q-1}) \cdot 2d \cdot c^k$ represents lattice paths which pass through $(i, q-1)$ and end in $DR^k$. But how are we to interpret the term $c^q \cdot d \cdot c^p$? It seems to require a lattice path of the form $U^q \cdot R^p$, which contains the forbidden subpath UR.

We can avoid forbidden subpaths by introducing another step $S = (0,0)$. Thus $S$ represents standing still for a moment to avoid UR. It can also be thought of as marking a particular point on a lattice path.

Now we develop our argument more formally. Consider the noncommutative polynomial algebra on the generators $D$, $R$, $U$, and $S$, where $D$ has degree 2 and the other generators have degree 1. The generators correspond to the steps

- Diagonal = $(1,1)$
- Right = $(1,0)$
- Up = $(0,1)$
- Stand = $(0,0)$.

For natural numbers $p$ and $q$, let $\Lambda'_{p,q}$ be the module generated by monomials of degree $p+q+1$ with $p$ occurrences of $\hat{D}$ or $R$ and $q$ occurrences of $D$ or $U$ which do not contain UR as a contiguous subword. In this context, we can prove a proposition analogous to Proposition [2.4.3] for the Cartesian product.

We can prove that the diamond product is unimodal.
Proposition 2.4.4 (Unimodality of diamond product). The sequence

\[ 1 \diamond c^{2n}, c \diamond c^{2n-1}, \ldots, c^n \diamond c^n, c^{n+1} \diamond c^{n-1}, \ldots, c^{2n} \diamond 1 \]

is unimodal.

2.5 Concluding remarks

In addition to the mixing operators studied above, there is also the manifold product
(or doubly-truncated product), denoted by \( P \boxtimes Q \) and defined by

\[ P \boxtimes Q = (P \setminus \{0_P, 1_P\}) \times (Q \setminus \{0_Q, 1_Q\}) \cup \{0, 1\}. \]

The name comes from its relation with manifolds. For example, if \( P \) and \( Q \) are face lattices of polytopes, then \( P \boxtimes Q \) is the face poset of the torus which is the Cartesian product of the boundary complexes of the polytopes.

While the Cartesian product \( \times \) increases degree by 1 and the diamond product \( \diamond \) preserves degree, the manifold product \( \boxtimes \) decreases degree by 1. For any posets \( P \) and \( Q \) with rank at least 2,

\[ \kappa(\Psi(P) \boxtimes \Psi(Q)) = \kappa(\Psi(P)) \cdot \kappa(\Psi(Q))/(a - b). \]

Hence

\[ \kappa(u \boxtimes v) = \kappa(u \cdot v)/(a - b) + \sum_{1}^{\Delta^*} (u_{(1)} \diamond v_{(1)}) \cdot b \cdot \kappa(u_{(2)} \diamond^* v_{(2)}) \]

whenever \( u \) and \( v \) have sufficiently large degree.

While the operations \( \times \) and \( \diamond \) have the cd-polynomials \( \xi \) and 1 respectively as units, the unit of \( \boxtimes \) is \( a \). Hence the manifold product does not preserve the cd-index.

There are still recursive rules for computing \( u \boxtimes v \). In particular,

\[ u \boxtimes (v \cdot (a - b)) = (u \boxtimes v) \cdot (a - b) \text{ and } u \boxtimes (v \cdot d) = \sum_{1}^{\Delta^*} (u_{(1)} \diamond v) \cdot 2d \cdot u_{(2)} \]

Although the manifold product does not generally preserve cd-polynomial or nonnegativity, there are some special cases where it does. In particular, \( c^p \boxtimes c^q \) is a cd-polynomial if \( p + q \) is odd, and \( c^n \boxtimes c^{n+1} \) is a nonnegative cd-polynomial for any \( n \). Increasing the difference in degree between the arguments rapidly introduces negative terms. Since these expressions denote ab-indices of products of spheres of different dimensions, we would like to give conditions which guarantee nonnegativity of the coefficients.

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3 A geometric approach to acyclic orientations

The set of acyclic orientations of a connected graph with a given sink has a natural poset structure. We give a geometric proof of a result of Propp: this poset is the disjoint union of distributive lattices.

Let $G$ be a connected graph on the vertex set $[n] = \{0\} \cup [n]$, where $[n]$ denotes the set $\{1, \ldots, n\}$. Let $P$ denote the collection of acyclic orientations of $G$, and let $P_0$ denote the collection of acyclic orientations of $G$ with $0$ as a sink. If $\Omega$ is an orientation in $P$ with the vertex $i$ as a source, we can obtain a new orientation $\Omega'$ with $i$ as a sink by firing the vertex $i$, reorienting all the edges adjacent to $i$ towards $i$. The orientations $\Omega$ and $\Omega'$ agree away from $i$.

A firing sequence from $\Omega$ to $\Omega'$ in $P$ consists of a sequence $\Omega = \Omega_1, \ldots, \Omega_{m+1} = \Omega'$ of orientations and a function $F : [m] \to [n]$ such that for each $i \in [m]$, the orientation $\Omega_{i+1}$ is obtained from $\Omega_i$ by firing the vertex $F(i)$. We will abuse language by calling $F$ itself a firing sequence. We make $P$ into a preorder by writing $\Omega \leq \Omega'$ if and only if there is a firing sequence from $\Omega$ to $\Omega'$. From the definition it is clear that $P$ is reflexive and transitive. While $P$ is only a preorder, $P_0$ is a poset.

By finiteness, antisymmetry can be verified by showing that firing sequences in $P_0$ cannot be arbitrarily long. This is a consequence of the fact that neighbors of the distinguished sink $0$ cannot fire. The proof depends on the following lemma.

Lemma 3.0.1. Let $F : [m] \to [n]$ be a firing sequence for the graph $G$. If $i$ and $j$ are adjacent vertices in $G$, then

$$|F^{-1}(i)| \leq |F^{-1}(j)| + 1.$$ 

Proof. A vertex can fire only if it is a source. Firing $i$ reverses the orientation of its edge to $j$. Hence $i$ cannot fire again until the orientation is again reversed, which can only happen by firing $j$. 

As a corollary, firing sequences have bounded length, implying that $P_0$ is a poset.

Corollary 3.0.2. The preorder $P_0$ of acyclic orientations with a distinguished sink is a poset.

Proof. Let $F : [m] \to [n]$ be a firing sequence. By iterating the lemma, $|F^{-1}(i)| \leq d(0, i) - 1$, so

$$m = \sum_{i \in [n]} |F^{-1}(i)| \leq \sum_{i \in [n]} d(0, i) - 1$$

Hence firing sequences cannot be arbitrarily long, implying that $P_0$ is antisymmetric.

For a real number $a$ let $[a]$ denote the largest integer less than or equal to $a$. Similarly, let $[a]$ denote the least integer greater than or equal to $a$. Finally, let $\{a\}$ denote the fractional part of the real number $a$, that is, $\{a\} = a - [a]$. Observe that
the range of the function \( x \mapsto \{ x \} \) is the half open interval \([0, 1)\). In this chapter we use \( \{ a \} \) only to denote the fractional part and never to denote a singleton set.

Let \( \mathcal{H} = \mathcal{H}(G) \) be the periodic graphic arrangement of the graph \( G \), that is, \( \mathcal{H} \) is the collection of all hyperplanes of the form

\[
x_i = x_j + k,
\]

where \( ij \) is an edge in the graph \( G \) and \( k \) is an integer. This hyperplane arrangement cuts \( \mathbb{R}^{n+1} \) into open regions. Note that each region is translation-invariant in the direction \((1, \ldots, 1)\). Let \( C \) denote the complement of \( \tilde{\mathcal{H}} \), that is,

\[
C = \mathbb{R}^{n+1} \setminus \bigcup_{H \in \tilde{\mathcal{H}}} H.
\]

Define a map \( \varphi : C \to P \) from the complement of the periodic graphic arrangement to the preorder of acyclic orientations as follows. For a point \( x = (x_0, \ldots, x_n) \) and an edge \( ij \) observe that \( \{ x_i \} \neq \{ x_j \} \) since the point does not lie on any hyperplane of the form \( x_i = x_k + k \). Hence orient the edge \( ij \) towards \( i \) if \( \{ x_i \} < \{ x_j \} \) and towards \( j \) if the inequality is reversed. This defines the orientation \( \varphi(x) \). Also note that this is an acyclic orientation, since no directed cycles can occur.

Let \( H_0 \) be the coordinate hyperplane \( \{ x \in \mathbb{R}^{n+1} : x_0 = 0 \} \). The map \( \varphi \) sends points of the intersection \( C_0 = C \cap H_0 \) to acyclic orientations in \( P_0 \).

The real line \( \mathbb{R} \) is a distributive lattice; meet is minimum and join is maximum. Since \( \mathbb{R}^{n+1} \) is a product of copies of \( \mathbb{R} \), it is also a distributive lattice, with meet and join given by componentwise minimum and maximum. That is, given two points in \( \mathbb{R}^n \), say \( x = (x_0, \ldots, x_n) \) and \( y = (y_0, \ldots, y_n) \), their meet and join are given by

\[
x \land y = (\min(x_0, y_0), \ldots, \min(x_n, y_n))
\]

and

\[
x \lor y = (\max(x_0, y_0), \ldots, \max(x_n, y_n))
\]

respectively.

**Lemma 3.0.3.** Each region \( R \) in the complement \( C \) of the periodic graphic arrangement \( \mathcal{H} \) is a distributive sublattice of \( \mathbb{R}^{n+1} \). Hence the intersection \( R \cap H_0 \), which is a region in \( C_0 \), is also a distributive sublattice of \( \mathbb{R}^{n+1} \).

**Proof.** Since each region \( R \) is the intersection of slices of the form

\[
T = \{ x \in \mathbb{R} : x_i + k < x_j < x_i + k + 1 \},
\]

it is enough to prove that each slice is a sublattice of \( \mathbb{R}^{n+1} \). Let \( x \) and \( y \) be two points in the slice \( T \). Then \( \min(x_i, y_i) + k = \min(x_i + k, y_i + k) < \min(x_i + k + 1, y_i + k + 1) = \min(x_i, y_i) + k + 1 \), implying that \( x \land y \) also lies in the slice \( T \). A dual argument shows that the slice \( T \) is closed under the join operation. Thus the region \( R \) is a sublattice. Since distributivity is preserved under taking sublattices, it follows that \( R \) is a distributive sublattice of \( \mathbb{R}^{n+1} \). \( \square \)
In the remainder of this chapter we let $R$ be a region in $C_0$.

**Lemma 3.0.4.** Consider the restriction $\varphi|_R$ of the map $\varphi$ to the region $R$. The inverse image of an acyclic orientation in $P_0$ is of the form:

$$R \cap \left( \{0\} \times \prod_{i=1}^{n} [a_i, a_i + 1] \right),$$

where each $a_i$ is an integer. That is, the inverse image of an orientation is the intersection of the region $R$ with a half-open lattice cube. Hence the inverse image is a sublattice of $\mathbb{R}^{n+1}$.

**Proof.** Assume that $x$ and $y$ lie in the region $R$. Define the integers $a_i$ and $b_i$ by $a_i = \lfloor x_i \rfloor$ and $b_i = \lfloor y_i \rfloor$. Hence the coordinate $x_i$ lies in the half-open interval $[a_i, a_i + 1)$ and the coordinate $y_i$ lies in the half-open interval $[b_i, b_i + 1)$. Lastly, assume that $\varphi|_R$ maps $x$ and $y$ to the same acyclic orientation. The last condition implies that for every edge $ij$ that $0 \leq x_i - a_i < x_j - a_j < 1$ is equivalent to $0 \leq y_i - b_i < y_j - b_j < 1$. Consider an edge that is directed from $j$ to $i$. Since $x$ and $y$ both lie in the region $R$, there exists an integer $k$ such that $x_i + k < x_j < x_i + k + 1$ and $y_i + k < y_j < y_i + k + 1$. Now we have that $a_j - a_i < x_j - x_i < k + 1$. Furthermore, observe that $x_j - a_j - 1 < 0 \leq x_i - a_i$. Hence $a_j - a_i > x_j - x_i - 1 > k - 1$. Since $a_j - a_i$ is an integer, the two bounds implies that $a_j - a_i = k$. By similar reasoning we obtain that $b_j - b_i = k$.

Hence for every edge $ij$ we know that $a_j - a_i = b_j - b_i$. Since $a_0 = b_0 = 0$ and the graph $G$ is connected we obtain that $a_i = b_i$ for all vertices $i$. \hfill $\square$

**Lemma 3.0.5.** The restriction $\varphi|_R : R \rightarrow P_0$ is a poset map.

**Proof.** Assume that $y$ and $z$ belong to the region $R$ and that $y \leq z$. Since the region $R$ is convex, the line segment from $y$ to $z$ is contained in $R$. Let a point $x$ move continuously from $y$ to $z$ along this line segment and consider what happens with the associated acyclic orientations $\varphi(x)$. Note that each coordinate $x_i$ is non-decreasing. When the point $x$ crosses an hyperplane of the form $x_i = p$ where $p$ is an integer, observe that the value $\{x_i\}$ approaches 1 and then jumps down to 0. Hence the vertex $i$ switches from being a source to being a sink, that is, the vertex $i$ fires.

Observe that two adjacent nodes $i$ and $j$ cannot fire at the same time, since the intersection of the two hyperplanes $x_i = p$ and $x_j = q$ is contained in the hyperplane $x_i = x_j + (p - q)$ which is not in the region $R$.

Hence we obtain a firing sequence from the acyclic orientation $\varphi(y)$ to $\varphi(z)$, proving that $\varphi(y) \preceq \varphi(z)$. \hfill $\square$

**Lemma 3.0.6.** Let $x$ be a point in the region $R$. Let $\Omega'$ be an acyclic orientation comparable to $\Omega = \varphi(x)$ in the poset $P_0$. Then there exists a point $z$ in the region of $R$ as $x$ such that $\varphi(z) = \Omega'$.

**Proof.** It is enough to prove this for cover relations in the poset $P$. We begin by considering the case when $\Omega'$ covers $\Omega$ in $P$. Thus $\Omega'$ is obtained from $\Omega$ by firing a vertex $i$.  

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First pick a positive real number \( \lambda \) such that \( \{x_j\} < 1 - \lambda \) for each nonzero vertex \( j \). Let \( y \) be the point \( y = x + \lambda \cdot (0,1,\ldots,1) \). Observe that \( y \) belongs to the same region \( R \) and that \( \varphi \) maps \( y \) to the same acyclic orientation as the point \( x \).

Since \( i \) is a source in \( \Omega \), the value \( \{y_i\} \) is larger than any other value \( \{y_j\} \) for vertexes \( j \) adjacent to the vertex \( i \). Let \( z \) be the point with coordinates \( z_j = y_j \) for \( j \neq i \) and \( z_i = \lceil y_i \rceil + \lambda/2 \). Observe that moving from \( y \) to the point \( z \) we do not cross any hyperplanes of the form \( x_i = x_j + k \). Hence the point \( z \) also belongs to region \( R \).

However, we did cross a hyperplane of the form \( x_i = p \), corresponding to firing the vertex \( i \). Hence we have that \( \varphi(z) = \Omega' \). Now we can iterate this to extend to the general case when \( \Omega < \Omega' \).

The case when \( \Omega' \) is covered by \( \Omega \) is done similarly. However this case is easier since one can skip the middle step of defining the point \( y \). Hence this case is omitted.

A connected component of a finite poset is a weakly connected component of its associated comparability graph. That is, a finite poset is the disjoint union of its connected components.

**Lemma 3.0.7.** Let \( Q \) be a connected component of the poset of acyclic orientations \( P_0 \). Then there exists a region \( R \) in \( C_0 \) such that the map \( \varphi \) maps \( R \) onto the component \( Q \).

*Proof.* Let \( \Omega \) be an orientation in the component \( Q \). Since \( \varphi \) is surjective we can lift \( \Omega \) to a point \( x \) in \( C_0 \). Say that the point \( x \) lies in the region \( R \). It is enough to show that every orientation \( \Omega' \) in \( Q \) can be lifted to a point in \( R \). The two orientations \( \Omega \) and \( \Omega' \) are related by a sequence in \( Q \) of orientations \( \Omega = \Omega_1, \Omega_2, \ldots, \Omega_k = \Omega' \) such that \( \Omega_i \) and \( \Omega_{i+1} \) are comparable. By iterating Lemma 3.0.6 we obtain points \( x_i \) in \( R \) such that \( \varphi(x_i) = \Omega_i \). In particular, \( \varphi(x_k) = \Omega' \).

**Proposition 3.0.8.** Let \( Q \) be a connected component of the poset of acyclic orientations \( P_0 \). Then the component \( Q \) as a poset is a lattice. Moreover, let \( R \) be a region of \( C_0 \) that maps onto \( Q \) by \( \varphi \). Then the poset map \( \varphi|_R : R \rightarrow Q \) is a lattice homomorphism.

*Proof.* The previous discussion showed that we can lift the component \( Q \) to a region \( R \). Consider two acyclic orientations \( \Omega \) and \( \Omega' \). We can lift them to two points \( x \) and \( y \) in \( R \), that is, \( \varphi(x) = \Omega \) and \( \varphi(y) = \Omega' \). Since \( \varphi|_R \) is a poset map we obtain that \( \varphi(x \lor y) \) is a lower bound for \( \Omega \) and \( \Omega' \). It remains to show that the lower bound is unique.

Assume that \( \Omega'' \) is a lower bound of \( \Omega \) and \( \Omega' \). By Lemma 3.0.6 we can lift \( \Omega'' \) to an element \( z \) in \( R \) such that \( z \leq x \). Similarly, we can lift \( \Omega'' \) to an element \( w \) in \( R \) such that \( w \leq y \). That is we have that \( \varphi(z) = \varphi(w) = \Omega'' \). Now by Lemma 3.0.4 we have that \( \varphi(z \lor w) = \Omega'' \). But since \( z \lor w \) is a lower bound of both \( x \) and \( y \) we have that \( z \lor w \leq x \lor y \). Now applying \( \varphi \) we obtain that \( \varphi(x \lor y) \) is the greatest lower bound, proving that the meet is well-defined. A dual argument shows that the join is well-defined, hence \( Q \) is a lattice.
Finally, we have to show that $\varphi|_R$ is a lattice homomorphism. Let $x$ and $y$ be two points in the region $R$. By Lemma 3.0.6 we can lift the inequality $\varphi(x) \wedge \varphi(y) \leq \varphi(x)$ to obtain a point $z$ in $R$ such that $z \leq x$ and $\varphi(z) = \varphi(x) \wedge \varphi(y)$. Similarly, we can lift the inequality $\varphi(x) \wedge \varphi(y) \leq \varphi(y)$ to obtain a point $w$ in $R$ such that $w \leq y$ and $\varphi(w) = \varphi(x) \wedge \varphi(y)$. By Lemma 3.0.4 we know that $\varphi(z \wedge w) = \varphi(x) \wedge \varphi(y)$. But $z \wedge w$ is a lower bound of both $x$ and $y$, so $\varphi(x) \wedge \varphi(y) = \varphi(z \wedge w) \leq \varphi(x \wedge y)$. But since $\varphi(x \wedge y)$ is a lower bound of both $\varphi(x)$ and $\varphi(y)$ we have $\varphi(x \wedge y) \leq \varphi(x) \wedge \varphi(y)$. Thus the map $\varphi|_R$ preserves the meet operation. The dual argument proves that $\varphi|_R$ preserves the join operation, proving that it is a lattice homomorphism.

Combining these results we can now prove the result of Propp [56].

**Theorem 3.0.9.** Each connected component of the poset of acyclic orientations $P_0$ is a distributive lattice.

**Proof.** It is enough to recall that $\mathbb{R}^{n+1}$ is a distributive lattice and each region $R$ is a sublattice. Furthermore, the image under a lattice morphism of a distributive lattice is also distributive. 

Observe that the minimal element in each connected component $Q$ is an acyclic orientation with the unique sink at the vertex 0. Greene and Zaslavsky [35] proved that the number such orientations is given by the sign $-1$ to the power one less than the number of vertices times the linear coefficient in the chromatic polynomial of the graph $G$. Gebhard and Sagan gave several proofs of this result [33]. A geometric proof of this result can be found in Chapter 1 of this dissertation.

That the connected component are confluent, that is, each pair of elements has a lower and an upper bound, can also be shown to follow from a special case of chip-firing games [13]. Is there a geometric way to prove the confluency of chip-firing? More discussions relating these distributive lattice with chip-firing can be found in [48, 49].
4 Critical groups of cleft graphs

4.1 Introduction

The number of spanning trees of an undirected graph is an important invariant of the graph. The matrix tree theorem reduces the problem of determining the tree number to linear algebra. (The problem of listing all spanning trees for a specific graph was solved by Feussner \[30, 31\] using what is essentially deletion-contraction.)

**Theorem 4.1.1** (Kirchhoff’s matrix tree theorem \[46\]). *Let* $X$ *be a graph on* $n$ *vertices with Laplacian* $L$. *Suppose* $\lambda_1 \leq \cdots \leq \lambda_n$ *are the eigenvalues of* $L$. *Then the tree number of* $X$ *is*

$$\kappa(X) = \frac{1}{n} \prod_{i=2}^{n} \lambda_i.$$ 

*Equivalently, $\kappa(X)$ is the value of any cofactor of $L$.*

Kirchhoff developed this theorem with the theory of electrical networks in mind.

More than a hundred years later, the physicists Bak, Tang, and Wiesenfeld \[3\] developed the apparently unrelated abelian sandpile model in an attempt to explain flicker noise, an effect which appears in widely varying physical systems. In the abelian sandpile model, grains of sand are added one at a time to small piles of sand. Since this is inherently unstable, eventually a pile will collapse, distributing grains to neighboring piles. They called a configuration critical if it is stable but becomes unstable if a single grain is added anywhere.

The problem of characterizing critical configurations was studied by graph theorists and other combinatorialists in the 1990s under the guise of chip-firing games. A chip-firing game, in the sense of Björner, Lovász, and Shor \[13\], involves firing vertices in a finite graph $G$ with a nonnegative number of chips on each vertex. A vertex fires by distributing a chip to each of its neighbors, and cannot fire unless it has sufficiently many chips. Only one vertex can fire at a time, so it might be expected that the decision of which vertex to fire at a particular step is of major importance. However, Björner, Lovász, and Shor showed that a chip-firing game on a graph is a confluent system. Hence if an initial configuration is not recurrent, its terminal stable configuration of chips does not depend on the order in which vertices are fired. Biggs \[8\] developed a variant of this called the dollar game, which includes one vertex which can fire if and only if no other vertex can fire, even if it would have a negative number of chips after doing so. Biggs proved that the number of critical configurations of a graph is equal to the order of the critical group, which is the torsion part of the cokernel of the Laplacian. Thus the problem of counting spanning trees is subsumed by the problem of understanding the critical group of a graph.

The critical group is only known for a few classes of graphs. In this chapter, we study cleft graphs, which are graphs obtained from a base graph by replacing
each vertex with an anticlique, that is, a collection of nonadjacent vertices. This construction is the vertex analogue of that of Lorenzini [50], who studied the effect of replacing all edges in a graph with paths of uniform length. We derive an exact sequence relating the critical group of a uniformly cleft graph with that of its base graph. Moreover, we also have results in the non-uniform case. By studying the spectrum of the Laplacian we are able to determine the tree number of a non-uniformly cleft tree.

4.2 Preliminaries

All graphs we consider are simple, loopless, undirected graphs with no parallel edges. Our discussion will be greatly simplified if we imagine a graph as being endowed with an orientation. None of our results depend on which orientation is used. With this in mind, we define an oriented graph to be a structure \( \mathcal{X} = (\mathcal{E}, \mathcal{V}) \) consisting of a set of edges \( \mathcal{E} \) and a set of vertices \( \mathcal{V} \) which are related by a pair of structure maps from edges to vertices, called \( s \) for source and \( t \) for target. If \( \mathcal{X} = (\mathcal{E}, \mathcal{V}) \) and \( B = (\mathcal{E}', \mathcal{V}') \) are oriented graphs, then a morphism \( \varphi: \mathcal{X} \rightarrow B \) consists of two functions \( \varphi: \mathcal{E} \rightarrow \mathcal{E}' \) and \( \varphi: \mathcal{V} \rightarrow \mathcal{V}' \) such that an oriented edge with source \( u \) and target \( v \) is mapped to an oriented edge with source \( \varphi(u) \) and target \( \varphi(v) \). We let \( \delta(v) \) denote the neighborhood of a vertex \( v \) in the unoriented graph. The degree of \( v \) is denoted by \( \deg(v) \) and is the size of the neighborhood, that is, \( \deg(v) = |\delta(v)| \). In an oriented graph, the neighborhood of a vertex \( v \) decomposes as \( \delta(v) = \delta^+(v) \sqcup \delta^-(v) \), where \( \delta^+(v) \) is the out-neighborhood of \( v \), the set of vertices reachable from \( v \) in one step, and \( \delta^-(v) \) is the in-neighborhood of \( v \), the set of vertices from which \( v \) can be reached in one step.

An oriented graph \( X \) can be viewed as an oriented 1-dimensional cell complex. Hence \( X \) comes equipped with a chain complex \( C(X) \), where

\[
C_1(X) = \bigoplus_{e \in \mathcal{E}} \mathbb{Z}e \quad \text{and} \quad C_0(X) = \bigoplus_{v \in \mathcal{V}} \mathbb{Z}v.
\]

The boundary map \( \partial: C_1(X) \rightarrow C_0(X) \) is defined on an edge \( e \) by \( \partial(e) = t(e) - s(e) \). Hence the boundary map is the same as the incidence matrix of the graph. The Laplacian of \( X \) is the map \( L = \partial \partial^T \), where \( \partial^T \) is the transpose of \( \partial \). Thus \( \partial^T \) represents the coboundary of the graph. If \( X \) has \( n \) vertices, we can view \( X \) as an \( n \times n \) matrix. For vertices \( u \) and \( v \), one can compute that the \((u,v)\) entry of \( L \) is

\[
L(u,v) = \begin{cases} 
\deg(u), & u = v \\
-|[u,v]|, & u \neq v,
\end{cases}
\]

where the notation \([u,v]\) indicates the set of edges with endpoints \( u \) and \( v \) in either orientation. Some authors use this as the definition of the Laplacian matrix. Thus \( L = D - A \), where \( D \) is the diagonal matrix whose diagonal gives the degree sequence of \( X \) and \( A \) is the incidence matrix of \( X \). The critical group of \( X \) is the torsion part of the cokernel of \( L \). The cokernel can be found by reducing \( L \) to its
Smith normal form, which can be done using row and column operations which are invertible over the integers.

Cleft graphs are similar to graph fibrations, but they obey a weaker unique lifting condition. Hence we will adopt some of the language, including the notions of total graph and base graph. Before presenting the technical definition of cleft graph we offer the following way to visualize cleaving a single vertex in two. Suspend the graph by the vertex to be cleft, so that the edges which connect it to the rest of the graph are hanging downwards. Carefully drape these edges and the vertex on a chopping block. Then take a very sharp (and infinitely thin) cleaver and cut through the vertex and its incident edges. Thus the vertex is cleft into two vertices, and each of the edges incident with the vertex is cleft into two edges, one for each half of the cleft vertex. Thus the vertex to be cleft has been replaced with two nonadjacent vertices, each of which has the same neighborhood as the cleft vertex. See Figure 4.1. In a similar way, we can cleave a vertex \(m\)-fold, replacing the vertex with an anticlique of \(m\) vertices, each with the same neighborhood as the cleft vertex.

The structure of a graph after multiple vertices have been cleft does not depend on the order in which the cleavings were performed. So given a graph \(B\) and a weight vector on the vertex set of \(B\), there is a unique graph \(X\) which is obtained from \(B\) by cleaving each vertex of \(B\) according to its weight. Moreover, there is a natural projection morphism \(p: X \to B\) which assigns each vertex in \(X\) to the vertex in \(B\) from which it was cleft. Hence we can define a cleft graph to be an oriented graph morphism \(p: X \to B\) which satisfies the following two properties:

- **(weak unique lifting)** For any vertices \(\tilde{u}, \tilde{v} \in VX\), if \(e \in EB\) is an edge from \(p(\tilde{u})\) to \(p(\tilde{v})\), then the edge \(e\) has a unique lift \(\tilde{e} \in EX\) with source \(\tilde{u} = s(\tilde{e})\) and target \(\tilde{v} = t(\tilde{e})\).

- **(cleaving)** Each fibre of \(p\) is a nonempty anticlique.

Observe that since \(p\) is a graph morphism, the edge \(\tilde{e}\) mentioned above is a lift of \(e\). We say that the cleft graph is induced by the weight vector \((m_v)_{v \in VB}\), where the weight of a vertex is the size of its fibre, that is, \(m_v = |p^{-1}(v)|\). A cleft graph is \(m\)-**uniform** if every fibre has the same size \(m\). An example of a non-uniform cleft graph appears in Figure 4.2.

Suppose we know the tree number or critical group of a base graph \(B\). It is natural to ask how much we can deduce about the tree number or critical group...
of the total graph of a cleft graph over \( B \). It turns out that this is not difficult if the cleaving is uniform or if the base graph is a tree. In Section 4.3, we determine the tree number of a uniformly-cleft graph. In Section 4.4, we determine the tree number of a non-uniformly-cleft tree.

4.3 The exact sequence of a uniformly-cleft graph

Let \( p: X \to B \) be a cleft graph. Since the projection \( p \) is a graph morphism, it commutes with the boundary map, that is, \( \partial p = p \partial \). The interaction between the coboundary map \( \partial^T \) and the projection is slightly more complex, and is described by the following lemma.

Lemma 4.3.1. Let \( p: X \to B \) be a cleft graph with weight vector \( m \). Define a linear map \( \varphi: C_0(B) \to C_1(B) \) by \( \varphi(v) = \sum_e \varphi(e, v) \cdot e \), where

\[
\varphi(e, v) = \begin{cases} 
  m_{s(e)} & t(e) = v \\
  -m_{t(e)} & s(e) = v \\
  0 & \text{otherwise}.
\end{cases}
\]

Then the diagram

\[
\begin{array}{ccc}
C_0(X) & \xrightarrow{\partial^T} & C_1(X) \\
\downarrow p & & \downarrow p \\
C_0(B) & \xrightarrow{\varphi} & C_1(B)
\end{array}
\]

is commutative. Moreover, if \( X \) is an \( m \)-uniformly cleft graph, then \( \varphi = m \partial^T \).

Proof. The composite map \( \varphi p \) is given by

\[
\varphi p(e, \tilde{v}) = \sum_{u \in VB} \varphi(e, u) p(u, \tilde{v}) = \varphi(e, p(\tilde{v})).
\]

Figure 4.2: A bipartite graph viewed as a cleft path.
On the other hand, the composite map $p\partial^T$ is given by

$$p\partial^T(e, \tilde{v}) = \sum_{f \in EX} p(e, \tilde{f})\partial^T(\tilde{f}, \tilde{v}) = \sum_{\tilde{e} \in p^{-1}(e)} \partial^T(\tilde{e}, \tilde{v}).$$

The sum vanishes unless $e$ is incident with $p(\tilde{v})$. The number of lifts of $e$ which have a given endpoint is given by the weight of the vertex at the other endpoint, and the sign of the term $\partial^T(\tilde{e}, \tilde{v})$ is determined by whether $p(\tilde{v})$ is the source or target of $e$. Hence $p\partial^T = \varphi\partial$, as claimed.

Combining Lemma 4.3.1 with the commutativity relation $\partial p = p\partial$, we can define the compressed Laplacian of a cleft graph $X$ with respect to its base graph $B$ as the composite map $C = \partial\varphi$. If we define a vector $(M_v)_{v \in VB}$ by

$$M_v = \sum_{u \in \delta(v)} m_u,$$

then it follows directly that

$$C(u, v) = \begin{cases} M_u, & u = v \\ -m_u \cdot \#[u, v] & u \neq v \end{cases}$$

for any vertices $u, v \in VB$.

**Corollary 4.3.2.** Let $p: X \to B$ be an $m$-uniformly cleft graph. Then the compressed Laplacian of $X$ is $m \cdot L(B)$.

For the rest of this section we will specialize to the case of an $m$-uniformly cleft graphs $p: X \to B$. Let $(M_v)$ be the vector defined above. Thus $M_v$ is the degree in $X$ of any lift $\tilde{v}$ of $v$. Let $S$ denote the transpose of $p$. The map $S$ sends a vertex $v$ to the sum of its lifts, that is, $S(v) = \sum_{\tilde{v} \in p^{-1}(v)} \tilde{v}$. Since both $L(X)$ and $C = m \cdot L(B)$ are symmetric matrices, it follows from Lemma 4.3.1 that the diagram

$$\begin{array}{ccc}
C_0(B) & \xrightarrow{C} & C_0(B) \\
s \downarrow & & \downarrow s \\
C_0(X) & \xrightarrow{L} & C_0(X)
\end{array}$$

is commutative. Since $SC = LS$, there is an injection $\text{coker} C \to \text{coker} L$. We can use the fact that $X$ is a uniformly cleft graph to determine the factor by which splitting increases the tree number. But first we need a lemma.

**Lemma 4.3.3.** Let $p: X \to B$ be an $m$-uniformly cleft graph, and define a vector $(M_v)_{v \in VB}$ by $M_v = m \cdot |\delta(v)|$. If $B$ is connected, then there is an exact sequence

$$0 \to \text{coker} C \to K(X) \oplus \mathbb{Z} \to \bigoplus_{v \in VB} \mathbb{Z}^{m-1}_{M_v} / M_v \to 0$$

of abelian groups.
Proof. Since $B$ is connected, so is $X$. Thus $\text{coker } L = K(X) \oplus \mathbb{Z}$. Applying the snake lemma to the commutative diagram

\[
\begin{array}{c}
0 \rightarrow C_0(B) \xrightarrow{S} C_0(X) \xrightarrow{\text{coker } S} 0 \\
\downarrow C \quad \quad \downarrow L \quad \quad \downarrow \tau \\
0 \rightarrow C_0(B) \xrightarrow{S} C_0(X) \xrightarrow{\text{coker } S} 0
\end{array}
\]

with exact rows yields the exact sequence

\[
\ker \text{L} \rightarrow \text{coker } C \rightarrow K(X) \oplus \mathbb{Z} \rightarrow \text{coker } \text{L} \rightarrow 0,
\]

where the map $\text{L}: \text{coker } S \rightarrow \text{coker } S$ is induced by $L$. For any vertex $v$ in $B$, the sum of the lifts of $v$ in $X$ is a representative of zero in $\text{coker } S$, but there are no other relations among the vertices of $X$. The Laplacian sends a lift $\tilde{v}$ of $v$ to

\[
L(\tilde{v}) = M_v \tilde{v} - m \sum_{u \in \delta^+(v)} S(u),
\]

which by our observation represents $M_v \tilde{v}$ in $\text{coker } S$. Hence we can represent $\text{L}$ by the block matrix $\bigoplus_{v \in V_B} M_v I_{m-1}$, which is injective and has the desired cokernel. \qed

To make this exact sequence useful for enumeration, we need to kill the infinite factors in $\text{coker } C$ and $K(X) \oplus \mathbb{Z}$. The following observation allows us to do this.

Lemma 4.3.4. Let $M$ be an $n \times n$ integer matrix with corank 1. Let $H$ be the submodule of $\mathbb{Z}^n$ generated by all vectors whose coordinates sum to 0. If $\text{im } M \subseteq H$, then each standard basis vector $e_i$ represents an infinite generator of $\text{coker } M$, possibly with nonzero torsion part.

Proof. First observe that $\mathbb{Z}^n$ is isomorphic to $\mathbb{Z}$ and is generated by any standard basis vector $e_i$. Lifting $e_i$ to $\text{coker } M$ yields an element of the form $n \cdot \gamma + r$, where $\gamma$ is the infinite generator of $\text{coker } M$ and $r$ is a torsion element. But this implies that $\gamma$ is mapped to $n^{-1}$ times the generator of $\mathbb{Z}^n / H$ under the canonical surjection. Hence $n$ is a unit. \qed

Proposition 4.3.5. Let $p: X \rightarrow B$ be an $m$-uniformly cleft graph with Laplacian $L$ and compressed Laplacian $C$. If $B$ has $n$ vertices, then the tree number of $X$ is given by the formula

\[
\kappa(X) = \kappa(B) \cdot m^{n-2} \cdot \prod_{v \in V_B} (m \cdot \text{deg}(v))^{m-1}.
\]

Moreover, if the Smith normal form of $L(B)$ has the form $\text{diag}(d_1, \ldots, d_{n-1}, 0)$, then the critical group of $X$ fits into the exact sequence

\[
0 \rightarrow \bigoplus_{i=1}^{n-1} \mathbb{Z}_{m-d_i} \rightarrow K(X) \rightarrow \left( \bigoplus_{u} \mathbb{Z}_{m-\text{deg}(u)}^{m-1} / \mathbb{Z}_m \right) \rightarrow 0.
\]
Proof. We may assume \( B \) is connected. The map \( \text{coker } C \to \text{coker } L \cong K(X) \oplus \mathbb{Z} \) sends the element \( v + \text{im } C \) to \( S(v) + \text{im } L \). By Lemma 4.3.4 this element can be rewritten as \( m \cdot \tilde{v} + \text{im } L \) plus a torsion element, where \( \tilde{v} \) is a lift of \( v \) in \( X \). Hence the map must send the infinite generator of \( \text{coker } C \) to \( m \) times the infinite generator of \( \text{coker } L \). This allows us to embed \( K(X) \oplus \mathbb{Z} \) in the commutative diagram

\[
\begin{array}{ccccccccc}
0 & 
\downarrow & 
0 & 
\downarrow & 
0 & 
\downarrow & 
0 \\
0 & 
\rightarrow & 
\mathbb{Z} & 
\rightarrow & 
\mathbb{Z} & 
\rightarrow & 
\mathbb{Z}_m & 
\rightarrow & 
0 \\
0 & 
\rightarrow & 
\text{coker } C & 
\rightarrow & 
K(X) \oplus \mathbb{Z} & 
\rightarrow & 
\text{coker } L & 
\rightarrow & 
0 \\
0 & 
\rightarrow & 
\text{coker } C/\mathbb{Z} & 
\rightarrow & 
K(X) & 
\rightarrow & 
\text{coker } L/\mathbb{Z}_m & 
\rightarrow & 
0 \\
0 & 
\downarrow & 
0 & 
\downarrow & 
0 & 
\downarrow & 
0 \\
\end{array}
\]

with exact rows and columns. Since \( X \) is \( m \)-uniformly cleft, its compressed Laplacian is \( C = mL(B) \). Hence \( C \) has Smith normal form \( \text{diag}(m \cdot d_1, \ldots, m \cdot d_n, 0) \). This completes the proof.

The above proposition measures the growth in tree number produced by uniform splitting. We get the following corollary in the case where the base graph is a tree.

**Corollary 4.3.6.** Let \( p: X \to T \) be an \( m \)-uniformly cleft graph whose base graph \( T \) is a tree on \( n \) vertices. Then the tree number of \( X \) is

\[
\kappa(X) = m^{n-2} \cdot \prod_{v \in VT} (m \cdot \deg(v))^{m-1}.
\]

We would like to extend this method to the case of non-uniformly cleft graphs. Since the compressed Laplacian need not be symmetric, it is unclear how to do it. In the next section, we will extend Corollary 4.3.6 to the case of non-uniformly cleft trees. However, the proof we give makes necessary use of the fact that the base graph is a tree, and it is unclear how to generalize it.

### 4.4 Tree numbers of cleft trees

In this section we count spanning trees of a cleft tree using a weighted analogue of the following classical theorem.

**Theorem 4.4.1** (Poincaré [55], Chuard [15]). Let \( X \) be a graph on \( n \) vertices with incidence matrix \( A \), and let \( A' \) be an \( n-1 \times n-1 \) submatrix of \( A \). The matrix \( A' \) is nonsingular (in fact, \( \det(A') = \pm 1 \)) if and only if the columns of \( A' \) represent the edges of a spanning tree of \( X \).
To motivate the main ideas behind our argument, we study a recursive function on a special class of trees we call weighted marked trees. A weighted marked tree is a tree $T$ together with a weight vector $m = (m_v)_{v \in V_T}$ and two special vertices, the root $r$ and a marked vertex $q$, which could also be the root. We define a function $F(T, m, r, q)$ according to the following recursive procedure.

1. If $T$ has no edges, then $F(T, m, r, q) = 1$.

2. Otherwise:
   a) Let $v$ be a leaf of $T$. Do not select the marked vertex $q$ unless it is the only leaf.
   b) Let $w$ be the parent of $T$.
   c) Let $T'$ be the tree obtained from $T$ by collapsing the edge connecting $v$ and $w$ to $w$. Let $m'$ be the restriction of the weight vector of $T$ to the vertices of $T'$.
   d) Define a tuple $(q', w')$ by
      \[
      (q', w') = \begin{cases} 
      (q, w), & v \neq q \\
      (w, q), & v = q.
      \end{cases}
      \]
   e) With the above notation, $F(T, m, r, q) = m_w \cdot F(T', m', r, q')$.

We illustrate this algorithm by applying it to the tree in Figure 4.3.

![Figure 4.3: A weighted tree $T$ with root $r$ and marked vertex $b$.](image)

In order, we select the vertices $c$, $f$, $d$, and $a$, collapsing the edges $ac$, $bf$, $ad$, and $ra$, and picking up the weights $m_a$, $m_b$, $m_d$, and $m_r$. After these collapses, the tree has been reduced to the tree $T'$ displayed in Figure 4.4. Now the marked vertex $b$ is the only leaf, so we must select it. Thus we collapse $rb$ to $r$ and move the marker from $b$.

![Figure 4.4: The tree $T'$ obtained from $T$ by collapsing several edges.](image)
to $r$. Since $b$ was marked, we pick up its weight, $m_b$, rather than the weight of its parent. The collapsed tree has no more edges, so there are no more steps to perform. The value of $F$ on the tree $T$ is $m_a^2 \cdot m_b^2 \cdot m_r$. Notice that for each non-marked vertex $v$, the factor $m_v$ appears in $F$ a total of $\text{deg}(v) - 1$ times. The factor $m_b$ appears twice. This property holds for any weighted marked tree, as we now show.

**Lemma 4.4.2.** Let $(T, m, r, q)$ be a weighted marked tree, and let $F$ be the function defined above. Then

$$F(T, m, r, q) = m_q \cdot \prod_{v \in VT} m_v^{\text{deg}(v) - 1}.$$  

**Proof.** Let $v$ be a vertex of $T$. There are three cases, depending on the position of the marked vertex.

*Case 1.* Neither $v$ nor any of its children is marked. Each child of $v$ contributes a factor of $m_v$ to the value of $F$. Since $v$ is not marked, it contributes the weight of its parent to the value of $F$ when selected as a leaf. Hence $v$ contributes a total of $m_v^{\text{deg}(v) - 1}$ to the value of $F$.

*Case 2.* The vertex $v$ is marked. If $v$ is marked, there is a contribution of $m_v$ for each of its children as well as a contribution of $m_v$ when it is selected as a leaf. Hence $v$ contributes a total of $m_v^{\text{deg}(v)}$ to the value of $F$.

*Case 3.* The vertex $v$ has a marked descendant. Hereditarily unmarked children of $v$ behave as in Case 1. Hence we may assume that $v$ has the marked vertex as its unique child. When the child of $v$ is selected, it contributes nothing to the exponent of $m_v$, but then the mark is passed from the child to $v$. So when $v$ is selected as a leaf, it contributes a weight of $m_v$ to the value of $F$. Hence $v$ contributes a total of $m_v^{\text{deg}(v) - 1}$ to the value of $F$. 

The next step is to observe that the function $F$ is, up to a sign, the result of computing a determinant by cofactor expansion. Recall that the compressed Laplacian $C$ of a cleft graph factors as $C = \partial \varphi$, where $\partial: C_1(B) \to C_0(B)$ is the boundary map and $\varphi: C_0(B) \to C_1(B)$ is the map defined in Lemma 4.3.1.

**Lemma 4.4.3.** Let $p: X \to T$ a cleft tree with weight vector $m$. Select a root $r$ for $T$ and orient all edges away from the root. Let $M$ be a matrix representing $\varphi$, and let $K$ be a matrix representing $\partial$. Then the determinant of $MK$ is

$$\det(MK) = \left( \sum_{q \in VT} m_q \right) \cdot \prod_{v \in VT} m_v^{\text{deg}(v) - 1}.$$  

**Proof.** By the Binet–Cauchy theorem, the determinant of $MK$ is given by the sum

$$\det(MK) = \sum_{q \in VT} \det(M_q) \cdot \det(K_q),$$  

where $M_q$ is obtained from $M$ by striking the column corresponding to $q$, and $K_q$ is defined similarly. It follows from Theorem 4.4.1 that $\det(K_q) = \pm 1$. To evaluate $\det(M_q)$, select a leaf $v$ of the tree $T$, let $w$ be the parent of $v$, and let $e$ be the
edge from \( w \) to \( v \). If \( v \neq q \), then by cofactor expansion about the \((v, e)\) entry of \( M_q \),

\[
det(M_q) = \pm m_w \cdot det(M'_q),
\]

where \( M'_q \) is the submatrix of \( M_q \) obtained by striking the column corresponding to \( v \) and the row corresponding to its unique incident edge. If \( v = q \), then by cofactor expansion about the \((w, e)\) entry of \( M_q \),

\[
det(M_q) = \pm m_v \cdot det(M'_q).
\]

Up to a sign, this recursive computation of \( det(M_q) \) agrees with the recursive computation of the function \( F(T, \mathbf{m}, r, q) \). By computing the determinant of \( K_q \) in the same way we see that \( det(K_q) \) is equal to the sign of \( det(M_q) \). Applying Lemma 4.4.2, we conclude that

\[
det(M_q) \cdot det(K_q) = m_q \cdot \prod_{v \in VT} m_{deg(v)} - 1.
\]

Summing over all \( q \in VT \) completes the proof.

We need the following technical lemma.

**Lemma 4.4.4** (Horn–Johnson [39, Theorem 1.3.20]). Suppose \( r \leq n \). Let \( P \) be an \( n \times r \) matrix and \( Q \) be an \( r \times n \) matrix. Then the eigenvalues of \( PQ \) are also eigenvalues of \( PQ \), with (at least) the same multiplicity. All other eigenvalues of \( PQ \) are \( 0 \).

Now we use the above results to count the spanning trees of a cleft graph.

**Theorem 4.4.5.** Let \( p : X \to T \) be a cleft graph with weight vector \((m_v)_{v \in VT}\), and define a vector \((M_v)_{v \in VT}\) by \( M_v = \sum_{u \in \delta(v)} m_u \). If \( T \) is a tree, then the tree number of \( X \) is

\[
\kappa(X) = \prod_{v \in VT} (M_v^{m_v - 1} \cdot m_{deg(v)} - 1).
\]

**Proof.** The graph \( X \) has Laplacian matrix \( L \) and compressed Laplacian \( C = \partial \varphi \). Suppose \( T \) has \( n \) vertices, and let \( N \) denote the sum

\[
N = \sum_{v \in VT} m_v,
\]

that is, \( N \) is the number of vertices of \( X \). By Theorem 4.1.1 the tree number of \( X \) is

\[
\kappa(X) = \frac{1}{N} \prod_{i=2}^{N} \lambda_i,
\]

where \( \lambda_1 \leq \cdots \leq \lambda_N \) are the eigenvalues of \( L \). The diagonal entries of \( L \) have the form \( M_v \), each such entry occurring \( m_v \) times. Hence for each \( v \in VT \), the Laplacian of \( X \) has eigenvalue \( M_v \) occurring with multiplicity \( m_v - 1 \). This leaves \( n \) eigenvalues to be determined. Since the rows and columns of \( L \) sum to zero, one of these eigenvalues is \( \lambda_1 = 0 \).
From the fact that \( Lp^T = p^TC^T \) we conclude that every eigenvalue of \( C^T \) (hence also \( C \)) is an eigenvalue of \( L \). Since \( T \) is a tree, it has one more vertex than it has edges, so while \( C = \partial \varphi \) is an \( n \times n \) matrix, its companion \( \varphi \partial \) is an \((n - 1) \times (n - 1)\) matrix. Applying Lemma 4.4.4, we conclude that the product of the remaining eigenvalues of \( L \) is \( \det(\varphi \partial) \). But it follows from Lemma 4.4.3 that

\[
\det(\varphi \partial) = \left( \sum_{q \in VT} m_q \right) \cdot \prod_{v \in VT} m_v^{\deg(v)-1} = N \cdot \prod_{v \in VT} m_v^{\deg(v)-1},
\]

Hence

\[
\kappa(X) = \prod_{v \in VT} M_v^{m_v-1} \cdot \prod_{v \in VT} m_v^{\deg(v)-1},
\]

which is what we wanted to show.

4.5 Concluding remarks

The arguments used to study uniformly-cleft graphs and non-uniformly-cleft trees are different enough that it is unclear what form a possible common generalization would take. We can compute the critical group explicitly in some simple cases, such as a uniformly-cleft path. However, the available techniques for working with these structures do not yet generalize even to the case of uniformly-cleft trees. We would like to have a leaf-cutting procedure, similar to the weighted analogue of the Poincaré–Chuard theorem, which operates on the critical group level.
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