Aspects of wave turbulence in preheating

José A. Crespo and H.P. de Oliveira

Universidade do Estado do Rio de Janeiro, Instituto de Física - Departamento de Física Teórica, Rio de Janeiro, RJ, CEP 20550-013 Brazil.
E-mail: jaacrespo@gmail.com, oliveira@dft.if.uerj.br

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Abstract. In this work we have studied the nonlinear preheating dynamics of several inflationary models. It is well established that after a linear stage of preheating characterized by the parametric resonance, the nonlinear dynamics becomes relevant driving the system towards turbulence. Wave turbulence is the appropriate description of this phase since the matter contents are fields instead of usual fluids. Turbulence develops due to the nonlinear interactions of waves, here represented by the small inhomogeneities of the scalar fields. We present relevant aspects of wave turbulence such as the Kolmogorov-Zakharov spectrum in frequency and wave number that indicates the energy transfer through scales. From the power spectrum of the matter energy density we were able to estimate the temperature of the thermalized system.

Keywords: inflation, physics of the early universe

1Corresponding author.
1 Introduction

One of the most significant challenges in modern Cosmology is the description of the early stages of the universe. There is a general acceptance that an inflationary phase [1] characterized by a huge expansion of the universe might have occurred that preceded the radiation dominant phase. In this context, the process of reheating plays a crucial role in the transition of the Universe from the inflationary phase into the radiation phase consequently in the creation of almost all matter constituting the present Universe.

The reheating begins at the end of inflation with a stage of parametric resonance with a rapid transfer of energy from the inflaton field into other matter fields, leading to particle production and the inflaton decay, far away from thermal equilibrium [2–5]. Several authors have explored [2, 6–13, 15–17] the entire evolution of the inflaton field until the universe has settled down in a thermalized state characterizing the radiation era. The main feature shared by these studies on the later stages of non-perturbative preheating is the ubiquity of turbulence in the process towards thermalization [18–20]. However, the matter constituents at the end of inflation are fields together with their small inhomogeneities rather conventional fluids, which makes more appropriate to deal with wave turbulence [21] instead hydrodynamic turbulence [24].

An appropriate and useful definition of wave turbulence is a state of out-of-equilibrium statistical mechanics of random nonlinear waves [21]. The most important class of solutions in wave turbulence are called Kolmogorov-Zakharov (KZ) spectra [22] that correspond to a constant flux of energy through scales, where the spectra with power-laws in frequency and wave numbers are present. In the context of preheating the small inhomogeneities associated to the fields play the role of the waves, some of them are resonant in the first phase of preheating. The nonlinear interaction of these waves results in the transfer of energy from the homogeneous inflaton field through different scales with the establishment of turbulence in this scenario.

In this paper, our main goal is to explore aspects of wave turbulence of late stages of preheating characterized by the KZ spectrum in frequency and wave number. We have considered distinct inflationary models for which the scalar field $\phi$ denotes the inflaton field. The models range from a single field inflationary model with quartic potential $V(\phi) = \frac{1}{4}\lambda\phi^4$, to those arising after introducing an interacting field $\chi$, and described by the potential [23],

$$V(\phi, \chi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\sigma_0\phi\chi^2 + \frac{1}{2}\sigma_0\phi^2\chi^2 + \frac{1}{4}\lambda\chi^4. \quad (1.1)$$
According to Dufaux et al. [23] the $\phi^2 \chi^2$ four-legs interaction and the trilinear $\phi \chi^2$ interaction are motivated by their occurrence in many theoretical models. We have organized the paper as follows. We introduce the basic equations of the model and the numerical treatment based on the collocation method in the second section. In section 1, we present the numerical results starting from a standard verification of the accuracy of the numerical method. We present the relevant aspects of wave turbulence that are common to all models, namely, the scaling laws associated to the relevant quantities in frequency and wave number. In section 4, we briefly discussed the effect of the backreaction and the evolution of the equation of state. Finally, in section 5 we conclude.

2 The models

Let us consider the simplest single-field model preheating that has already been studied in several works [13, 15, 26, 27]. We denote $\phi(x,t)$ as the inflaton field with potential $V(\phi) = \frac{1}{2} \lambda \phi^4$ evolving in a spatial flat FLRW universe [2, 6, 7]. The homogeneous component, or simply the homogeneous mode of the inflaton $\phi_0(t)$, is responsible by the inflationary phase. At the end of inflation, we denote $\phi_0(0) = \phi_0$ the homogeneous mode at the end of inflation. It will be useful to introduce the conformal time $\tau$ defined by $a(\tau) d\tau = \sqrt{\lambda} \phi_0(0) a(0) dt$, and the conformal scalar field $\varphi(x,\tau) = \phi_0(\tau)/\phi_0(0) a(0)$, with $a(\tau)$ being the scale factor. The evolution equation for the inflaton is written in a simple and convenient way:

$$\varphi'' - \nabla^2 \varphi - \frac{a''}{a} \varphi + \varphi^3 = 0,$$

where prime indicates derivative with respect to the conformal time $\tau$, with $\tau = 0$ signalizing the end of inflation. The beginning of preheating is characterized by coherent oscillations of the inflaton which yields an effective traceless energy-momentum tensor [40]. As a consequence, $a(\tau) \sim \tau$, therefore, allowing us to set $a'' = 0$ in eq. (2.1).

The two-fields models require another rescaling to obtain the dimensionless coordinates and field variables. We have establish that $t \to t/m$, $x_j \to x_j/m$, and redefined the scalar fields as $\phi = ma^{-3/2} \varphi$, $\chi = ma^{-3/2} X$. The scalar field equation read as,

$$\ddot{\phi} - \frac{1}{a^2} \nabla^2 \phi + \left( 1 - \frac{3\dot{a}}{2a} - \frac{3\dot{a}^2}{4a^2} \right) \varphi + \frac{g_0^2 a^{-3} X^2 \varphi + \frac{\sigma_0}{2} a^{-3/2} X^2 = 0 \right)$$

$$X'' - \frac{1}{a^2} \nabla^2 X - \frac{3}{2} \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) X + g_0^2 a^{-3} X \varphi^2 + \frac{\sigma_0}{2} a^{-3/2} X \varphi + \lambda a^{-3} X^3 = 0,$$

where the models we are going to consider depend on the dimensionless parameters $g_0, \lambda, \sigma_0 = \sigma_0/m$ in the following way: we call the four-legs model [23] if $g_0 \neq 0, \sigma_0 = \lambda = 0$, the models denoted by $A, B, C$, respectively, if $g_0, \lambda \neq 0, \sigma_0 = 0$, $g_0, \sigma_0, \lambda \neq 0$, $g_0 = 0, \sigma_0, \lambda \neq 0$. For the sake of simplicity at this moment, we shall consider the evolution in a fixed background allowing that $a = 1$.

We have integrated numerically eqs. (2.1), (2.2) and (2.3) using the collocation or pseudospectral method [33] in a two dimensional square box of size $L$ with periodic boundary conditions, but there are other numerical approaches applied to this problem [34–36]. As in any spectral method, the scalar fields are approximated as series expansions with respect to a set of basis functions. According to the boundary conditions, these basis function are
Fourier functions, and the spectral approximation establishes that,

$$\varphi(x, \tau) = \sum_{l,m=-N}^{N} a_{lm}(\tau) \psi_{lm}(x,y), \quad X(x,t) = \sum_{l,m=-N}^{N} c_{lm}(t) \psi_{lm}(x,y). \quad (2.4)$$

In the above expressions $N$ is the truncation order that limit the number of unknown modes $a_{lm}(\tau)$ (or $a_{lm}(t)$ in the case of two-field models) and $c_{lm}(t)$. The basis function are $\psi_{k} = \exp\left(\frac{2\pi i}{L} k \cdot x\right)$, and $k = (l, m)$ is the comoving momentum. Notice that the mode $a_{0}(\tau)$ corresponds to the homogeneous component of the inflaton while the remaining modes account for its small inhomogeneities.

We have used the standard Galerkin method to solve eq. (2.1) previously [13, 15], but with low truncation order once the equations were expressed solely in terms of the modes. The collocation method adopts a more convenient approach to deal with the nonlinearities that we illustrate considering the single field model. The starting point is to establish that the residual equation — the equation arising when the spectral approximation (2.4) is substituted into the evolution equation (2.1) — vanishes at particular points named collocation or grid points $x_{k} = \frac{2\pi k}{2N+1}$ [33]. As a consequence, we have $\text{Res}(x_{k}, \tau) = 0$, with $k = (l, m)$ and $l, m = 1, 2, \ldots, 2N+1$, and the resulting equations are expressed schematically by,

$$\varphi''_{k}(\tau) + \sum_{j} \omega_{j}^{2} a_{j}(\tau) \psi_{j}(x_{k}) + \varphi_{k}(\tau) = 0, \quad (2.5)$$

where $\omega_{j}^{2} = \frac{4\pi^{2}}{L^{2}} j^{2}$ and $\varphi_{k}(\tau)$ denotes the value of the scalar field at the collocation point $x_{k}$. It becomes very economical to express the resulting equations using both representations of the scalar field: the spectral representation through the modes $a_{k}$, and the physical representation through values of the scalar field at the collocation points. There are $(2N+1)^{2}$ independent equations that coincide with the same number of independent modes $a_{k}(\tau)$ recalling that these modes are decomposed into imaginary and real pieces, but not all are independent on the grounds of the scalar field given by eq. (2.1) is real [13]. The bridge connecting both representations consists in the following relation,

$$\varphi_{k}(\tau) = \sum_{l,m=-N}^{N} a_{lm}(\tau) \psi_{lm}(x_{k}). \quad (2.6)$$

Accordingly, the $(2N+1)^{2}$ values $\varphi_{k}(\tau)$ are related to an equal number of independent modes $a_{j}(\tau)$. Similar sets of dynamical equations are derived for the field equations (2.2) and (2.3). In all numerical simulations, we have evolved the dynamical equations (2.5) with a fourth-order Runge-Kutta integrator.

3 Numerical results

We have integrated the resulting dynamical equations with the initial conditions ($\varphi_{k}(0)$, $\varphi'_{k}(0)$, $\chi_{k}(0)$, $\chi'_{k}(0)$) at the end of inflation. With respect to the single field model the homogeneous component of the inflaton is set initially as $a_{0}(0) = 1, a'_{0}(0) = 0$, whereas all other modes and velocities, $a_{j}(0), a'_{j}(0)$, have amplitudes of order of $10^{-4}$. The initial conditions for all two-field modes satisfy the conditions $a_{0}(0) = 0, a'_{0}(0) = 1$, and the remaining modes and velocities $a_{j}(0), a'_{j}(0), c_{j}(0), c'_{j}(0)$, are set to have amplitude of order $10^{-4}$.
In the single field model, the parameter \( L \) dictates which modes with wave vector \( k = (l, m) \) undergo an initial phase of parametric resonance by considering the stability/instability chart for the Lagrangian equation that governs the evolution of the modes \( a_j(\tau) \) in the linearized regime \([2, 6, 13]\). In this case, it can be shown that if \( L = \pi \sqrt{5(l^2 + m^2)/2} \), where \( l, m \) assumes any integer value in the interval \([-N, \ldots, N]\), then those modes with \( |k| = \sqrt{l^2 + m^2} \) will grow exponentially in the first stages of preheating. In our numerical simulations, we have chosen these resonant modes with \( |k| = \sqrt{5} \), or equivalently \( L = 5\pi/\sqrt{2} \). With respect to all two-field models, we have fixed \( g_0 = 1 \) and \( \bar{\sigma}_0 = 1; \lambda = 0.5 \) for the models \( B \) and \( C \), while \( \lambda = 1 \) only for the model \( C \). This choice of parameters guarantee the regime of broad parametric resonance at the initial stage of preheating \([7]\).

Before proceeding with our numerical study, we present a numerical test that consists in verifying the conservation of the total energy of the conformal inflaton field introduced in the case of single field model, \( E_\phi(\tau) \),

\[
E_\phi(\tau) = \int_{D} \rho_\phi(\tau, x) d^2 x, 
\]

where \( D \) represents the spatial domain and \( \rho_\phi = 1/2 \phi^2 + 1/2(\nabla \phi)^2 + 1/4 \phi^4 \). As a matter of fact, this is valid only if \( a'' = 0 \) in eq. (2.1). A very useful way of checking the energy conservation is to evolve the relative variation of energy given by \( \delta E_\phi(\tau)/E_\phi(0) \) with several truncation orders and evaluate the root mean square deviation for each truncation order. In figure 1, we present the results that show a good convergence to a relative deviation of about \( 10^{-5} \). Similar results were obtained for all two-field models.

The main stages of the dynamics of the inflaton have been described with details in refs. \([8–13, 15, 26, 27]\). We do not intend to repeat the description of these stages towards turbulence here, but illustrate them in the single field and four legs models by displaying in figure 2 the long time behavior of the homogeneous component of the inflaton, \( a_0 \), and \( \sigma^2 \).
Figure 2. Evolution of the homogeneous inflaton mode $a_0$, and the spatial average of the variance, $\sigma^2$ (cf. eq. (3.2)), for the single field (upper graphs) and the four-legs model. Both homogeneous modes display the same qualitative behavior with an initial phase of coherent oscillations. The transfer of energy from the homogeneous inflaton’s energy, triggered by the nonlinear interactions with all other modes, produces the decrease of the homogeneous inflaton amplitude followed by chaotic behavior. The collective evolution of all modes $a_k$ expressed by $\sigma^2$ also exhibits the same qualitative behavior.

The quantity $\text{var}(\varphi) = (\varphi - \langle \varphi \rangle)^2$ is the variance of the field $\varphi$ with $\langle \ldots \rangle = 1/L^2 \int_D \ldots d^2x$ being the average over the spatial domain. Using expression (2.4) it follows that homogeneous mode is the spatial average of the inflaton field, $\langle \varphi \rangle = a_0$.

The numerical experiments have indicated that, in all models the structure of the time signals of the homogeneous inflaton component, $a_0$, and $\sigma^2$ are qualitatively the same. The main difference is the time scale that separates approximately the stages until the turbulent phase is established. We have depicted in figure 2 the evolution of these quantities corresponding to the single field and the four-legs models. One may notice the accentuated and rapid decay of the homogeneous inflaton in the four legs model if compared with the single
field model. This fact is explained due to the broad resonance regime at the early stages of preheating in the four-legs models, where the turbulent phase commences at $t \approx 230$, and for the single field model at $\tau \approx 1,300$ (with $\lambda = 10^{-4}$, for $\lambda = 10^{-8}$, $\tau \approx 14,000$). We have tested the effect of decreasing the value of $g_0$ that produces a considerable delay of the beginning of turbulence. We call attention to the decay of the homogeneous mode amplitude in both models which starts when the resonant modes grow beyond the linear regime. In particular, for the single model, we have verified that the amplitude decays approximately as $\tau^{-1/3}$ in agreement with previous studies [8, 9] until a certain time, beyond which the amplitude remains approximately constant. In what follows we have considered this phase to exhibit some aspects of wave turbulence.

One of the most relevant features of any turbulent signal concerns, not to its detailed structure, but to some property that is reproducible and related to the statistical description of turbulence. We exhibit this property by constructing a sequence of histograms of $\sigma^2(\tau)$ for the single field model considering intervals of time, $\Delta \tau = 100, 300, 600, 800$, about $\tau = 2,350$. As shown in figure 3 the histograms are similar, and the same axisymmetric Gaussian distribution fits them.

The aspect of wave turbulence relevant for the preheating is the cascade or transfer of energy among distinct scales. In particular, for preheating this is crucial to distribute

\textbf{Figure 3.} Histograms of $\sigma^2(\tau)$ for the single-field model constructed with the same bin using intervals of time $\Delta \tau = 100, 300, 600, 800$ (from left to right and top to bottom), about $\tau = 2,350$. It is clear that the same distribution is found no matter is the interval of time. This reproducible property of the turbulent signal indicates its self-similar nature.
the energy content from the homogeneous component of the inflaton to all perturbative modes. In this way, turbulence offers a very elegant mechanism to describe the transition from an empty and cold post-inflationary universe to a hot universe dominated by radiation in accordance with the hot big bang model. The imprint of such mechanism results in the so-called Kolmogorov-Zakharov spectrum [21] that have the form of power-law in space and time domains.

We start with the power spectrum of $\sigma^2$ in the time or frequency domain shown in figure 4 corresponding to the following models: single field, four-legs, B and C. A remarkable feature is that all power spectra present a similar structure. We have recognized three typical frequencies delimiting approximately four regions with scaling laws. The first frequency is $\omega \approx 0.5656$, which is the smallest natural frequency associated to the modes with $|k| = 1$. The second frequency, $\omega \approx 0.8695$, 1.0, is the initial frequency of the homogeneous component of the inflaton, $a_0$, and the third frequencies, $\omega \approx 15.99 (N = 20), 14.40 (N = 18)$ for the single-field and two-field models, respectively, are the highest frequencies associated to those modes with $|k|_{\text{max}} = N\sqrt{2}$. It is necessary to comment about the first region of low frequencies $\omega \leq \omega_{\text{min}}$. These frequencies originate from the period bifurcations which take place in the route to turbulence [13, 15, 37]. For the single-field model we have found that in the intervals $\omega \leq \omega_{\text{min}}$ and $\omega \geq \omega_{\text{max}}$, we found that $P(\omega) \sim \omega^{-2.51}$ while for $\omega_{\text{min}} \leq \omega \leq \omega_{\text{max}}$ there are two regions for which $P(\omega) \sim \omega^{-6.52}$. The power spectra of all two-field models exhibit similar scaling laws. These scaling laws are a direct consequence of turbulence in the late stages of preheating. They are representative of KZ spectra for a steady-state turbulent system with typical cascade process which a universal feature for turbulent fluids and plasmas [28].

Besides the power spectrum of $\sigma^2$ in the time domain, we can calculate the power spectrum of the variance in the space domain. It is convenient to expand the variance with respect to the basis functions,

$$\text{var}(\varphi) = (\varphi - \langle \varphi \rangle)^2 = \sum_k b_k \psi_k(x), \quad (3.3)$$

where the modes $b_k$ are associated to the wavenumber vector $k$; in particular the $0th$-mode is the spatial average of the variance, $b_0 = \sigma^2$. From the above expansion, we can construct the power spectrum in wave numbers $k = |k|$ at several times. In the first plot of figure 5, we show the power spectrum in the space domain evaluated at $\tau = 15,000$ with $\lambda = 10^{-8}$ corresponding to the single field model. The structure of the power spectrum almost does not change if calculated at any instant during the turbulent phase, in this case $\tau \geq 4,000$. We were able to fit the whole spectrum by a curve that contains both the power law and exponential factors,

$$P(k) \propto k^{-\nu_1} e^{-\nu_2 k^{\nu_3}}, \quad (3.4)$$

where $\nu_1 \approx 0.51$, $\nu_2 \approx 0.0211$ and $\nu_3 \approx 1.54$. This type of decay in wave numbers occurs in magnetohydrodynamic turbulence [14]. Parenthetically, as indicated by the plots of figure 5, the power spectra of $\sigma^2$ corresponding to all two-field models are described by the above expression.

As established by the spectral approximation of eq. (2.4), the classical modes $a_k = \alpha_k + i\tilde{\alpha}_k$ are interpreted as c-number amplitudes associated to the processes of creation and annihilation of quantum fluctuations of the inflaton field in the mode $k$. In this case, a relevant quantity is the occupation number of created particles, $n_k$, given by,

$$n_k = \frac{1}{\omega_k} |\tilde{\alpha}_k|^2 + \frac{\omega_k}{2} |\alpha_k|^2 \quad (3.5)$$
where $\omega_k = 2\pi|k|/L$. We have considered $\lambda = 10^{-8}$ and evaluated the power spectrum of $n_k$ with respect to $k = |k|$ at $\tau = 15,000$. The power spectrum showed in figure 6 was also found by Micha and Tkachev [26, 27] using a different numerical approach in 3D cubic lattice with a grid of $256^3$ points. Accordingly, the scaling law $n(k) \sim k^{-s}$ for small $k$ (straight line) with $s \approx 1.69$, is in agreement of the Micha’s result.

The turbulent phase of preheating produces a thermalized universe. A possible way of evaluating the temperature of this thermalized phase is to calculate the wave number power spectrum of the total energy density $a^4\rho_\phi(x, \tau)/\lambda\phi_0^4$ (cf. eq. (4.3)) for the single-field model, and $a^3\rho_\phi(x, t)/m^4$ for the two-field models. Both expressions are expanded with respect to the basis functions as $\sum_k E_k \psi_k(x)$. With this expansion, we have evaluated the spectrum of energy $E(k)$ in wave numbers $k$, where the energy $E(k)$ is the root mean squared of all energies $E_k$ whose corresponding $k$ has modulus $k$. In figure 7 we show the power spectrum evaluated at $\tau = 15,000$ and $\lambda = 10^{-8}$ for the single field, and $t = 1,100$ for the models four-legs, A and B. The structure of these the power spectra have a similar structure displaying two components separated by a gap of energy [15]. Most of the points lie in the second component that corresponds to the energy distribution for large wavenumbers, or small scales, which can be interpreted as the inertial range.
Figure 5. Power spectrum $P(k)$ of the variance (3.3) for following models (upper to down, left to right): single-field ($\lambda = 10^{-8}$), A, B and C. The continuous lines described by eq. (3.4) fits the whole spectrum for all models. A similar combination of exponential and power decay can be found in magnetohydrodynamic turbulence.

We can understand the second piece of the spectrum as resulting from the energy transfer from the homogeneous mode to small scale modes as expected in turbulence. The distribution at large wave numbers might correspond to a thermalized system consistent with a distribution of incoherent radiation satisfying the Planck distribution given by,

$$\mathcal{E}_k = \mathcal{E}_0 \frac{k^3}{ebk - 1},$$

(3.6)

where $\mathcal{E}_0$ and $b$ being constants. For the single field model, the best fit of (3.6) represented by the continuous line has $b \approx 0.18$ and $\mathcal{E}_0 \approx 10^{-4.9}$. Another interesting feature is that a considerable part of the first region of the distribution can be fitted by the above distribution with the same value of $b$, but $\mathcal{E}_0 \approx 10^{-3.9}$ (cf. figure 7). Since the parameter $b$ depends on the temperature of the distribution, both pieces of the distribution are thermalized with the same temperature. Turning into the two-field models the best fit gives $b \approx 0.174$ (four-legs), $b \approx 0.155$ (model A) and $b \approx 0.16$ (model B). We have noticed that contrary to the single field model, the first region of the distribution does not satisfy the distribution (3.6).

In order to extract the value of the temperature of the distribution, it is necessary to recover the physical variables present in the argument of the exponential from the dimensionless coordinates $x$ and the momenta $k$. Let us consider first the single field model,
where we have $b k = \hbar c k_{\text{phys}} / k_B T$, where $k_{\text{phys}}$ is physical momentum, $k_B$ and $T$ are the Boltzmann constant and temperature, respectively. From the relation $k = L k_{\text{phys}} / \sqrt{\lambda} \phi_e a_0$, with $\phi_e = 0.1 M_{\text{pl}}, a_0 = 1$ [1] corresponding to the scalar field, and scale factor at the end of inflation, the temperature of the distribution in natural units is estimated as,

$$T = \frac{\hbar c \sqrt{\lambda} \phi_e a_0}{k_B b L} \approx 10^{-5} M_{\text{pl}} \approx 10^{14} \text{GeV.}$$

(3.7)

This temperature is consistent with the beginning of the radiation era [38]. Repeating the above procedure for the two-field models, it follows that $k = L k_{\text{phys}} / m$ and the temperature is estimated as,

$$T = \frac{\hbar c m}{k_B b L} \approx 10^{-6} M_{\text{pl}} \approx 10^{13} \text{GeV,}$$

(3.8)

after assuming that $m \approx 10^{-6} M_{\text{pl}}$.

4 Backreaction

We have studied the dynamics of the inflaton field in several preheating models without taking into account the backreaction of all perturbative modes. The primary importance of the back-reaction is the generation of an effective energy-momentum of a fluid. By considering the single-field model, a perfect fluid emerges with an effective equation of state,

$$w = \frac{\langle p_\phi \rangle}{\langle \rho_\phi \rangle},$$

(4.1)

where $\langle \ldots \rangle$ represents the spatial averaging as mentioned before, and the pressure and energy density associated to the scalar field are, respectively,

$$p_\phi = \frac{\lambda \phi_e^4}{a^4} \left[ \frac{1}{2} \left( \frac{\phi'}{\phi} - \frac{\phi'}{\phi} \right)^2 - \frac{1}{6} (\nabla \phi)^2 - \frac{1}{4} \phi^2 \right],$$

(4.2)

$$\rho_\phi = \frac{\lambda \phi_e^4}{a^4} \left[ \frac{1}{2} \left( \frac{\phi'}{\phi} - \frac{\phi'}{\phi} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{4} \phi^2 \right].$$

(4.3)
Figure 7. Power spectrum $E(k)$ of the energy density of the inflaton field for the single field model followed by the models four-legs, $A$ and $B$ (from left to right, up and down). Notice the two components separated with a gap of energy as a common feature to all models. The continuous lines (blue) represent the Planck distribution with $b = 0.18, 0.174, 0.155$ and $0.16$, for the single-field, and the models four-legs, $A$ and $B$, respectively. Only for the single-field model both regions can be fitted by the similar distributions with distinct values of $\mathcal{E}_0$.

In order to provide a consistent framework for including the back-reaction, we have integrated the following field equations,

$$a'^2 = \frac{\phi_e^2}{3M_{pl}^2} \left\langle \frac{1}{2} \left( \phi' - \frac{a'}{a} \right)^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{4} \phi^4 \right\rangle$$

(4.4)

$$\varphi'' - \nabla^2 \varphi - \frac{a''}{a} \varphi + \varphi^3 = 0.$$  

(4.5)

We have set $\phi_e = 0.1M_{pl}$, $a(0) = 1$, $N = 20$, $\lambda = 10^{-4}$ and the initial conditions for the conformal scalar field were the same used in the previous section. We have integrated the field equations in a squared box of size $L = 5\pi/\sqrt{2}$. At each time step the integrals resulting from the space average were calculated using Gauss quadrature formulae [33]. In figure 8 we show the evolution of $w(t)$. Notice that at the initial phase of preheating the homogeneous modes dominates producing an effective equation of state of radiation, $w \approx 0.33 \approx 1/3$ as expected in the present model [40]. During the evolution $w(t)$ fluctuates about 0.33 indicating that at late stages of preheating the universe will be effectively dominated by radiation.
5 Discussion

In this paper, we have studied the late stages of preheating in several inflationary models. Although some of them are not in agreement, within an acceptable level of confidence with current observational data [25], the stages towards the turbulence and thermalization seem to be robust and model independent.

We have evolved the field equations using collocation method. The complex modes $a_k(\tau)$ (or $a_k(t)$) and $c_k(t)$ in the spectral approximations of the scalar fields (cf. eq. (2.4)) can be viewed as the amplitudes of plane waves of wave number vector $k$. Turbulence develops with the nonlinear interaction of these waves together with the transfer of energy from the homogeneous mode (zeroth mode) to other modes or scales. We have obtained the power spectrum in space of the variance (cf. eq. (3.3)) and energy density of the inflaton field, and the power spectrum in time of $\sigma^2 = \langle \text{var}(\phi) \rangle$. It is noteworthy that all power spectra are described by similar scaling laws. This is a strong evidence of the universal character of turbulence in all models. Also, the KZ spectra are indicators of the energy transfer from the homogeneous inflaton field to all inhomogeneous modes producing a state of steady turbulence. Therefore, this mechanism plays a fundamental role in the establishment of thermalization allowing the universe to enter in the radiation era. In particular, by analyzing the power spectrum of the total energy density, we were able to determine the temperature of the thermalized configuration by assuming the spectrum of the energy density is consistent with the Plank distribution.

We intend to explore further the interplay between wave turbulence processes in the early universe in the two-field models taking into consideration the backreaction of the inhomogeneities. In the realm of single field models, we shall investigate the model proposed recently by Kallosh and Linde [39] that consists in a scalar field non-minimally coupled with curvature with potential $V(\phi) = \lambda \phi^4/4$. This model is good agreement with the current observational data [25]. Finally, another relevant topic is the possibility of the parametric growth of gravitational perturbations during the preheating [41]. Therefore, it might be possible that the nonlinear phase undergoes what could be called gravitational wave turbulence.
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