ON BRANCHING RULES OF DEPTH-ZERO REPRESENTATIONS

MONICA NEVINS

Abstract. We present tools for analysing the restriction of a smooth irreducible representation of a $p$-adic group $G$ to a maximal compact subgroup $K$, without recourse to case-by-case analysis. Using these, we prove the coincidence of branching rules of certain Deligne-Lusztig supercuspidal representations. Furthermore, we show that under mild compatibility conditions, the restriction to $K$ of a Deligne-Lusztig supercuspidal representation of $G$ intertwines with the restriction of a depth-zero principal series representation in infinitely many distinct components of arbitrarily large depth. We illustrate the results with examples.

1. Introduction

The branching rules considered here are those arising from the restriction of a complex admissible representation of a $p$-adic group $G$ to a maximal compact open subgroup $K$. The ultimate goal of this analysis is to examine the interplay between the admissible duals of $G$ and $K$, as well as to illuminate their respective structures. Aspects of this question for $G$ include the theory of types and the study of newforms. On the other hand, the representation theory of $K$ is still in its infancy, and branching rules provide a framework in which to search for results.

In this paper, we consider the restriction of certain depth-zero supercuspidal representations (those induced from inflations of Deligne-Lusztig cuspidal representations of associated finite groups of Lie type) to a hyperspecial maximal compact subgroup (denoted $G_y$) under the hypothesis that $G$ is connected, simply connected, semisimple and split over a local non-archimedean ground field $k$ of odd residual characteristic. Our particular focus is the set of “atypical” representations, that is, those which are not types, and are common to the branching rules of several representations. To this end we prove two main results.

The first concerns Deligne-Lusztig supercuspidal representations (recalled in Section 6). We parametrize their coarse decomposition into Mackey components by a set $X^+_{x,y}$ in Sections 4 and 5. In Theorem 6.4 we prove that whenever two Deligne-Lusztig supercuspidal representations arise from the same minisotropic torus and have the same central character, then a large portion of their branching rules are identical, namely, those parametrized by $\text{int}(X^+_{x,y})$. Moreover, in certain circumstances
we prove that their complete restrictions to $G_y$ coincide. This is Corollary 6.5; we illuminate its hypotheses with some examples.

The second main result, stated in Theorem 7.4, concerns the intertwining between restrictions of Deligne-Lusztig supercuspidal representations and principal series. We prove that the restriction of a Deligne-Lusztig supercuspidal representation $\pi$ intertwines in infinitely many distinct components with any compatible depth-zero principal series representation $\text{Ind}_G^B \chi$. The compatibility condition relates to the central character of the cuspidal representation inducing to $\pi$. Further refinements of this result, relating to the depths at which these intertwinings occur, are given as a sequence of corollaries in Section 7.

One of the main methods underlying the proofs of these results, beyond Mackey theory, is the analysis of subgroups of $G$ which are stabilizers of subsets of an apartment $A$. A result of independent interest is given in Proposition 3.3, where we relate certain stabilizer subgroups with Moy-Prasad filtration subgroups. We use this in Theorem 5.4 and Proposition 7.2 to glean information about the depths of the representations of $G_y$ which arise.

Proving results on branching rules at this level of generality is a new and novel step, and anticipates the development of a general theory out of the case-by-case analysis achieved to date. In this sense the current work complements a series by the author on branching rules of $\text{SL}(2, k)$ [11, 12, 13] and, with P. Campbell, $\text{GL}(3, k)$ [2, 3]. Recently, U. Onn and P. Singla in [15] determined the complete decomposition into irreducible representations of the blocks of representations of $\text{GL}(3, k)$. We use their results in our example in Section 8 and anticipate that in fact the complete branching rules for $\text{GL}(3, k)$ are attainable, using the above results and ideas inspired from the present paper.

The branching rules for $\text{GL}(2, k)$ and $\text{PGL}(2, k)$ were previously studied by W. Casselman, K. Hansen, A. Silberger and others. K. Maktouf and P. Toreasso recently considered the branching rules of the Weil representation of a general symplectic group in [7], a particular case of which had been studied by D. Prasad in [16].

We assume that $G$ is semisimple; this simplifies the exposition, particularly in Section 2. It is feasible and would be interesting to extend the results to $G$ reductive, so that the Levi components of the proper parabolic subgroups are also in this class. This could allow an inductive analysis of branching rules including all parabolically induced representations.

Our proofs of the main theorems Theorem 6.4 and Theorem 7.4 rely on showing that certain double cosets support nonzero intertwining operators. These questions reduce to computations with Deligne-Lusztig characters. To determine which other double cosets also support intertwining operators would seem to require restricting representations to subgroups which are stabilizers of subsets of the Bruhat-Tits building not contained in any single apartment, and there is currently a dearth of literature on such subgroups. Moreover, the classification of the double coset spaces which arise
is expected to be highly nontrivial: for $\text{GL}(n,k)$, $n \geq 3$, it was shown by U. Onn, A. Prasad and L. Vaserstein in [14] to contain a wild classification problem in the limit.

The methods for decomposing supercuspidal representations in Section 5 and for defining a filtration on principal series representations in Section 7.1 extend easily to the positive depth case. The complication in extending the remaining results arises in the computation of intertwining: as predicted by calculations for $\text{SL}(2,k)$ [13] the intertwining will occur on less accessible double cosets and, as above, an analysis of the intersection of subgroups arising as stabilizers of subsets in different apartments is required.

The characters of Deligne-Lusztig cuspidal representations have a uniform description and are well-known; we make use of these in several computations. It would be useful to extend our results to other families of cuspidal representations: in fact for $\text{SL}(2,k)$, the non-Deligne-Lusztig cuspidal representations give all atypical irreducible positive-depth components of all representations [13]. In general we expect they exhaust the atypical components of all supercuspidal representations.

An eventual goal is the complete decomposition of supercuspidal or principal series representations into irreducible $G_y$-representations. As we see in Section 5, this would imply describing the branching rules for the (simple) restriction of cuspidal representations to a parabolic subgroup. In Section 6 we relate this in the Deligne-Lusztig case to questions about the intersection of minisotropic tori with split Levi subgroups. These are interesting open problems in the representation theory of finite groups of Lie type which have been solved only in special cases using CHEVIE [5], for example.

1.1. Outline. In Section 2 we provide a survey of the background required, including several results from Bruhat-Tits theory. In Section 3 we present various properties of pointwise stabilizers of bounded subsets of an apartment, and prove that with few exceptions, the Moy-Prasad filtration subgroups are just stabilizer subgroups of certain convex subsets, up to a toral factor. Section 4 is devoted to determining a set of double coset representatives $X^{+}_{x,y}$ for the Mackey components of the supercuspidal representations of $G$ for a special vertex $y$ and any vertex $x$, and describing the structure of this set.

In Section 5 we prove general results about the restriction of any depth-zero supercuspidal representation of $G$ to a (hyper)special maximal compact subgroup $G_y$. In Section 6 we specialize to the case of Deligne-Lusztig representations, proving the coincidence of their branching rules in many cases. We address principal series representations, proving their extensive intertwining over $G_y$ with Deligne-Lusztig supercuspidal representations, in Section 7. We conclude in Section 8 with an example illustrating the use of the many related results in this paper for the group $G = \text{SL}(3,k)$. 
Acknowledgments. This research was conducted during a wonderful visiting year at l’Institut de Mathématiques et Modélisation de Montpellier, Université de Montpellier II, at the invitation of Ioan Badulescu. This work also flourished through conversations with Anne-Marie Aubert, Corinne Blondel and Cédric Bonnafé. It is a pleasure to thank all these people.

2. Background: Summary

The main references for the background material in this section are [1, 20].

2.1. Notation and conventions. Let $k$ be a local nonarchimedean field of residual characteristic $p \neq 2$. Its characteristic may be 0 or $p$. Its residue field $\kappa$ is a finite field of order $q$. For the sake of brevity we will refer to our field as a $p$-adic field and our group as a $p$-adic group.

Let the integer ring of $k$ be $\mathcal{R}$ and its maximal ideal $\mathcal{P}$. Let $\varpi$ be a uniformizer, and normalize the valuation on $k$ so that $\text{val}(\varpi) = 1$. The units of $\mathcal{R}$ admit a filtration by subgroups $U_n$ where $U_0 = \mathcal{R}^\times$ and $U_n = 1 + \mathcal{P}^n$ if $n > 0$.

Given a subgroup $H$ of a group $G$ we denote its center by $Z(H)$ and for any $g \in G$ write $g H g^{-1}$. Whenever defined, a representation $(\sigma, V)$ of $H$ is smooth and $V$ is a complex vector space. We write $V^H$ for the fixed points of $H$ on $V$. If $g \in G$ then we write $g \sigma$ for the corresponding representation of $g H g^{-1}$. Whenever defined, the group $G$ acts on the normalized induced representation $\text{Ind}^G_H \sigma$, or the compactly induced representation $c-\text{Ind}^G_H \sigma$, by right translation.

Define $\mathbb{R} = \mathbb{R} \cup (\mathbb{R}+) \cup \{\infty\}$ as in [1, 6.4.1]. For $r \in \mathbb{R}$ we denote by $[r]$ the least integer $k$ satisfying $k \geq r$ and $[r+]$ the least integer $k$ with $k > r$. For $r \in \mathbb{R}$ we also set $[r] = -[-r]$.

2.2. Structure theory. Let $\mathbb{G}$ be a connected, simply connected, semisimple algebraic group which is defined and split over $k$. We write $G = \mathbb{G}(k)$. Let $\mathbb{S}$ be a maximal torus of $\mathbb{G}$, split over $k$, and denote the associated root system $\Phi$. Choose positive roots $\Phi^+ \subset \Phi$ and simple roots $\Delta \subset \Phi^+$. Let $\mathbb{B}$ be the Borel subgroup of $\mathbb{G}$ defined by $(\mathbb{S}, \Phi^+)$ and $\mathbb{N}$ the normalizer of $\mathbb{S}$ in $\mathbb{G}$. We set $S = \mathbb{S}(k)$, $B = \mathbb{B}(k)$ and $N = \mathbb{N}(k)$. The corresponding finite Weyl group is $W_0 = N/S$.

Denote by $X_*(S) = \text{Hom}_k(\mathbb{G}_m, \mathbb{S})$ the group of $k$-rational cocharacters of $\mathbb{S}$, and $X^*(S) = \text{Hom}_k(\mathbb{S}, \mathbb{G}_m)$ the group of $k$-rational characters. Set $S_0 = \{t \in S \mid \forall \chi \in X^*(S), \text{val}(\chi(t)) = 0\}$; this is the maximal compact subgroup of $S$.

For each $\alpha \in \Phi \subset X^*(S)$ we denote by $\alpha^\vee \in \Phi^\vee \subset X_*(S)$ the corresponding coroot. Since $G$ is simply connected the lattice $X_*(S)$ is spanned by $\Phi^\vee$. 
Denote by $\mathcal{A} = \mathcal{A}(G, S, k)$ the apartment corresponding to $(G, S, \Phi, k)$, which we think of as the affine space under $E = X_*(S) \otimes \mathbb{R}$. The set of affine roots $\Phi_{af}$ is the set of affine functions $\{\alpha_k = \alpha + k \mid \alpha \in \Phi, k \in \mathbb{Z}\}$ on $\mathcal{A}$; $\alpha$ is the gradient of $\alpha_k$. The set of hyperplanes $\{\beta = 0 \mid \beta \in \Phi_{af}\}$ define the walls of a polysimplicial complex structure on $\mathcal{A}$. Let $D$ denote the positive cone $\{x \in \mathcal{A} \mid \forall \alpha \in \Phi^+, \alpha(x) > 0\}$ and let $C$, the fundamental chamber, be the unique chamber (also called alcove) in $D$ containing $0 \in E$ in its closure.

The affine Weyl group $W$ is generated by the affine reflections $r_\beta$ for $\beta \in \Phi_{af}$, where $r_\beta$ denotes the reflection in the hyperplane $\beta = 0$. Since $G$ is simply connected, $W \cong X_*(S) \rtimes W_0$ and coincides with the extended affine Weyl group $N/S_0$. Here, $W_0$ acts as the stabilizer of $0 \in E$ and $X_*(S)$ acts by translations. For each $\ell \in X_*(S)$ let $t(\ell) \in W$ be its representative in $W$, which we identify with an element of $S \subset N$ when appropriate. For each $w \in W$ and $\ell \in X_*(S)$ we have $wt(\ell)w^{-1} = t(w\ell)$.

For any $x \in \mathcal{A}$, set $\Phi_x = \{\beta \in \Phi_{af} \mid \beta(x) = 0\}$ and $W_x = \langle r_\beta \in W \mid \beta \in \Phi_x \rangle$. Let $\Phi^{lin}_x$ be the set of gradients of elements of $\Phi_x$; since $G$ is split this is itself a root system. Choose a base $\Delta_x$ of $\Phi^{lin}_x$ so that the positive roots $\Phi^{lin,+}_x$ coincide with $\Phi^{lin}_x \cap \Phi^+$. Let $W^{lin}_x \subset W_0$ be the subgroup generated by the linear reflections in elements of $\Phi^{lin}_x$. Then the map of $W^{lin}_x$ into $W_x$ given by

$$w \mapsto t(x - wx)w = wt(w^{-1}x - x) \tag{2.1}$$

is a group isomorphism. If $W^{lin}_x = W_0$, then the point $x$ is a vertex and is called a special vertex [20, 1.9]; not all vertices of $\mathcal{A}$ are special. Since $G$ is split over $k$, $x$ is special if and only if $\alpha(x) \in \mathbb{Z}$ for all $\alpha \in \Phi$.

### 2.3. Filtrations and special subgroups.

Following [9], we associate to each $x \in \mathcal{A}, \alpha \in \Phi$ and $r \in \mathbb{R}$ subgroups $G_{\alpha}(k)_{x,r}$ of the corresponding root subgroup and, for $r \geq 0$, subgroups $S_r$ of $S$. Then the Moy-Prasad filtration group for $r \in \mathbb{R}_{\geq 0}$ is

$$G_{x,r} = \langle S_r, G_{\alpha}(k)_{x,r} \mid \alpha \in \Phi \rangle.$$ 

By a choice of pinning we simply have $G_{\alpha}(k)_{x,r} = G_{\alpha}(P^{[r - \alpha(x)]})$ and $S_r = S(U_{[r]})$. Note that given $\ell \in X_*(S)$, we have

$$t(\ell)G_{\alpha}(k)_{x,r} = G_{\alpha}(k)_{x + \ell, r} = G_{\alpha}(k)_{x, r - \alpha(\ell)}.$$ 

Let $B = B(G, k)$ denote the (reduced) Bruhat-Tits building for $G$ over $k$ as in [11, 7.4.1]. Given any point $y \in B$, there exist $g \in G$ and $x \in \mathcal{A}$ such that $y = g \cdot x$. For any $r \in \mathbb{R}_{\geq 0}$, one defines $G_{y,r} := gG_{x,r}$; this is independent of choices [9]. Since $G$ is semisimple and simply connected, for any $x \in B$, $G_{x,0}$ coincides with the stabilizer $G_x$ of $x$ in $G$ [20, §3.1] and is the parahoric subgroup of $G$ associated to $x$. If $x$ is in an (open) alcove $\Gamma$ then $G_{x}$ is called an Iwahori subgroup.

In our setting, the maximal compact open subgroups of $G$ are exactly the stabilizers of vertices of $B$. If $x$ is a special vertex, then $G_x$ is a good maximal compact subgroup, in the sense that $G$ admits decompositions $G = G_xSG_x$ (Cartan decomposition) and $G = G_xB$ (Iwasawa decomposition).
Given any \( x \in B \) the group \( G_{x,+} := G_{x,0,+} \) is the unipotent radical of the parahoric subgroup \( G_x \). The quotient group \( G_x/G_{x,+} \) is the group of \( \kappa \)-points of a connected reductive group \( \mathbb{M}_x \) defined over \( \kappa \) (as in [9]). Set \( S := S(\kappa) \subseteq \mathbb{M}_x(\kappa) \). If \( x \) is a hyperspecial vertex (as defined in [20, 1.10]) then \( \mathbb{M}_x = \mathbb{G} \). Since in our setting \( G \) is split over \( k \), hyperspecial vertices exist and coincide with the special vertices.

The maximal compact subgroups which are stabilizers of hyperspecial vertices are distinguished among all maximal compact subgroups in two ways. First, from their definition it follows that they are isomorphic to \( G(\mathcal{R}) \). Secondly, they have maximal volume from among all maximal compact open subgroups [20, 3.8]. In this paper we choose to restrict to a maximal compact subgroup which is the stabilizer of a (hyper)special vertex, always denoted \( y \).

To reduce notational burden, we write \( G_x = G_x/G_{x,+} \) for \( \mathbb{M}_x(\kappa) \) and refer to parabolic subgroups \((P \text{ and } B)\) and tori \((T)\) of \( G_x \) without reference to the algebraic group \( \mathbb{M}_x \). This is unfortunate in one case arising in Section 6; let us define the needed terms here. Let \( s \in G_x \) be semisimple and let \( C_s \) denote its centralizer, which is a reductive subgroup of \( \mathbb{M}_x \), and \( C_s^0 \) its connected component subgroup. Then define \( C_{G_x}^0(s) = C_s^0(\kappa) \). Note that if \( s \in Z(G_x) \) then \( C_s^0 = \mathbb{M}_x \) and so \( C_{G_x}^0(s) = G_x \).

### 2.4. Representations of \( G \)

Given an irreducible admissible representation \( \pi \) of \( G \) on a complex vector space \( V \), the depth of \( \pi \) is a rational number defined as the least \( r \in \mathbb{R}_{\geq 0} \) such that there exists \( x \in B(G,k) \) for which \( V \) contains vectors invariant under \( G_{x,r,+} \). Where appropriate, we also refer to the depth of a representation of \( G_x \), for fixed \( x \). If \( x \) is a special vertex then the depth of any representation of \( G_x \) is a nonnegative integer.

By Jacquet’s theorem, every irreducible admissible representation of \( G \) occurs as a subrepresentation of \( \text{Ind}_{P}^{G} \sigma \), for some parabolic subgroup \( P \) with Levi decomposition \( MN \) and supercuspidal representation \( \sigma \) of \( M \) (extended trivially across \( N \)). In case \( P = B \), a Borel subgroup, the representation \( \sigma \) is simply a character \( \chi \) of a split torus \( S \) and the representation \( \text{Ind}_{B}^{G} \chi \), which may fail to be irreducible, is called a principal series representation.

The classification of (irreducible) supercuspidal representations is not yet complete. It is a lasting conjecture, proven now in many cases, that all supercuspidal representations of depth \( r \) are compactly induced from a compact open subgroup. In case \( r = 0 \) this has been proven; more precisely L. Morris [8] and A. Moy and G. Prasad [10] proved that all depth-zero supercuspidal representations of \( G \) are given by

\[
\pi = c\text{-Ind}_{G_x}^{G} \tau
\]

for some vertex \( x \in B \) and inflation \( \tau \) of a cuspidal representation of \( G_x \). Among these cuspidal representations \( \tau \) are the Deligne-Lusztig cuspidal representations, whose characters are well-known; see Section 6.
3. Stabilizers of subsets of $\mathcal{A}$

Let $\Omega$ be a bounded subset of $\mathcal{B}$. Its convex closure $\overline{\Omega}$ is the union of all the facets of $\mathcal{B}$ meeting $\Omega$. The pointwise stabilizer of $\Omega$ is $G_\Omega = \cap_{x \in \Omega} G_x$ and it coincides with $G_I$ [2 Prop 2.4.13]. Given two points $x, y \in \mathcal{B}$, we have $G_x \cap G_y = G_{[x,y]}$, where $[x, y]$ is the unique geodesic joining $x$ and $y$, which is a line in any apartment containing both points [2 Prop 2.5.4]. From these facts one concludes that if $F$ is a facet such that $[x, y] \cap F \neq \emptyset$, then $G_{[x,y]} \subseteq G_F$.

F. Bruhat and J. Tits give the following description of $G_\Omega$ if $\Omega \subseteq \mathcal{A}$ [2 §6.4].

**Proposition 3.1.** Suppose $\Omega$ is a bounded subset of $\mathcal{A}$. For each $\alpha \in \Phi$, define
\[
f_\Omega(\alpha) = \max\{-\alpha(x) \mid x \in \Omega\}.
\]
Then $G_\Omega = S_0 U_\Omega$ where $U_\Omega = \langle G_\alpha(P^{f_\Omega(\alpha)}) \mid \alpha \in \Phi \rangle$. Furthermore, if $\Omega$ contains an open set of $\mathcal{A}$ then for any order on $\Phi$ the product map
\[
S_0 \times \prod_{\alpha \in \Phi} G_\alpha(P^{f_\Omega(\alpha)}) \rightarrow G_\Omega
\]
is a bijection.

As a particular consequence we note the following. Write $\text{int}(\Omega)$ for the interior of a set.

**Corollary 3.2.** Let $\Omega \subseteq \mathcal{A}$ be a bounded set such that $x \in \text{int}(\Omega)$. Then in the factorization $G_\Omega = S_0 U_\Omega$ we have $U_\Omega \subseteq G_{x,+}$.

**Proof.** Since $G_{x,+}$ is generated by $S_1$ and the groups $G_\alpha(P^{[-\alpha(x)+1]})$, by Proposition 3.1 it suffices to show that for all $\alpha \in \Phi$, $f_\Omega(\alpha) > -\alpha(x)$. Since $x \in \Omega$, this is immediate if $\alpha(x) \notin \mathbb{Z}$. Otherwise, since $x \in \text{int}(\Omega)$ there exists some $z \in \Omega$ such that $\alpha(z) < \alpha(x)$, whence $f_\Omega(\alpha) \geq [-\alpha(z)] > -\alpha(x)$.

We next wish to describe the relationship between subgroups $G_\Omega$, with $\Omega \subseteq \mathcal{A}$, and Moy-Prasad filtration subgroups $G_{x,r}$. We begin by setting some notation.

Given an irreducible root system $\bar{\Phi}$ let $\bar{\Phi}^l$ denote the set of its long roots. If $\bar{\Phi}$ has two root lengths let $\bar{\Phi}^s = \bar{\Phi} \setminus \bar{\Phi}^l$ be its short roots; otherwise, set $\bar{\Phi}^s = \bar{\Phi}$. More generally, given a root system $\Phi$ with irreducible components $\Phi_i$, for $1 \leq i \leq m$, define $\Phi^l = \cup_i \Phi^l_i$ and $\Phi^s = \cup_i \Phi^s_i$. Note that $\Phi$, $\Phi^l$ and $\Phi^s$ all have the same rank.

Given $x \in \mathcal{A}$ and $r \in \mathbb{R}_{\geq 0}$, define
\[
\Omega_x(\mathcal{A}, r) = \{z \in \mathcal{A} \mid \forall \alpha \in \Phi, |\alpha(x) - \alpha(z)| \leq r\}.
\]
Define $\Omega_x^l(\mathcal{A}, r)$ and $\Omega_x^s(\mathcal{A}, r)$ by replacing $\Phi$ in (3.2) with $\Phi^l$ and $\Phi^s$, respectively.

**Proposition 3.3.** Let $x \in \mathcal{A}$ and $r \in \mathbb{R}_{\geq 0}$. Then
\[
G_{\Omega_x^l(\mathcal{A}, r)} \subseteq S_0 G_{x,r} \subseteq G_{\Omega_x^s(\mathcal{A}, r)} = G_{\Omega_x(\mathcal{A}, r)}.
\]
Moreover, whenever the root system $\Phi$ does not contain an irreducible component of type $G_2$ the second inclusion is an equality, that is, $S_0 G_{x,r} = G_{\Omega^s_x(A,r)}$.

Proof. First note that $\Omega^s_x(A,r) = \Omega_x(A,r)$. Namely, given $z \in \Omega^s_x(A,r)$, choose a positive system $\Phi^{(+)}$ for which $z-x$ is in the closure of the positive cone and let $\theta^{(+)} \in \Phi^l$ be the corresponding highest (long) root. Then for each $\beta \in \Phi$, $|\beta(x-z)| \leq \theta^{(+)}(z-x) \leq r$, so $z \in \Omega_x(A,r)$. Clearly also $\Omega^s_x(A,r) \supseteq \Omega_x(A,r)$. Hence $G_{\Omega^s_x(A,r)} = G_{\Omega_x(A,r)}$.

If $r = 0$ the groups appearing in (3.3) are all equal and there is nothing to show, so suppose $r > 0$. Each group is generated by $S_0$ and certain subgroups of the root groups; thus it suffices to show the inclusions on each root subgroup.

Let $z \in \Omega_x(A,r)$. Then for each $\alpha \in \Phi$ we have $-\alpha(z) \leq r - \alpha(x)$, whence $\mathbb{G}_\alpha(\mathcal{P}_{[\alpha(z)]}) \subseteq \mathbb{G}_\alpha(\mathcal{P}_{-\alpha(z)})$. It follows that $G_{x,r} \subseteq \cap_{z \in \Omega_x(A,r)} G_z = G_{\Omega_x(A,r)}$, and the second inclusion holds.

Now consider the first inclusion. It suffices to show that for all $\alpha \in \Phi$ there exists $z_\alpha \in \Omega^s_x(A,r)$ such that $-\alpha(z_\alpha) \geq r - \alpha(x)$.

First suppose $\alpha \in \Phi^s$. Then $\alpha$ lies in a unique irreducible component $\Phi^{s'}$ of $\Phi^s$, corresponding to a subspace $E'$ of $E = X_s(S) \otimes \mathbb{Z} \mathbb{R}$. Let $\Delta'$ be the base of $\Phi^{s'}$ with respect to which $\alpha$ is the highest root and let $H'_{a,r}$ denote the (nonempty) intersection of the hyperplane $a = r$ with the positive cone $D'$ defined by $\Delta'$. Choose $v \in H'_{a,r} \subseteq E'$. Then for all $\beta \in \Phi^{s'}$ we have $|\beta(v)| \leq \alpha(v) = r$, and for all $\beta \in \Phi^s \setminus \Phi^{s'}$ we have $\beta(v) = 0 \leq r$. Therefore the element $z_\alpha = x - v$ satisfies our requirements.

Now let $\alpha \in \Phi \setminus \Phi^s$. Let $\Phi''$ denote the irreducible component of $\Phi$ containing $\alpha$, $\Delta''$ the base with respect to which $\alpha$ is the highest root, and $D''$ the corresponding positive cone. Let $\alpha_0 \in \Phi^s \cap \Phi''$ be the corresponding highest short root and define $H'_{\alpha_0,r}$ as in the preceding paragraph. For any $v$ in the nonempty intersection $H'_{\alpha_0,r} \cap D''$ we have $\alpha(v) \geq \alpha_0(v) = r$ and, as argued above, for all $\gamma \in \Phi^s$, $|\gamma(v)| \leq r$. Therefore $z_\alpha = x - v \in \Omega^s_x(A,r)$ and satisfies $-\alpha(z_\alpha) \geq r - \alpha(x)$, as required.

Now consider the final assertion. If $\Phi$ is simply-laced then equality holds because $\Omega^s_x(A,r) = \Omega_x(A,r)$. Otherwise by the preceding arguments it suffices to show that in each non-simply-laced irreducible root system except $G_2$, there exists a short root $\alpha$ and a vector $v$ such that $\alpha(v) = r$ and for all $\beta \neq \alpha$, $|\beta(v)| \leq r$. This is easily verified case-by-case. \qed

We remark that equality fails on the simple system of type $G_2$ because the boundary of $\Omega^s_x(A,r)$ does not intersect the boundary of $\Omega_x(A,r)$. 


4. The double coset space \( G_y \backslash G / G_x \)

We begin by recalling a result about generalized BN-pairs [1, Proposition 7.4.15].

**Proposition 4.1.** For \( i = 1, 2 \) let \( \Omega_i \) denote a nonempty subset of \( \mathcal{A} \), \( G_i \) its pointwise stabilizer in \( G \), \( N_i \) the pointwise stabilizer of \( \Omega_i \) in \( N \), and \( \hat{W}_i \) its image in \( W = N / S_0 \). Then the natural map

\[
\hat{W}_1 \backslash W / \hat{W}_2 \rightarrow G_1 \backslash G / G_2
\]

is bijective.

**Corollary 4.2.** Let \( x, y \) be vertices of \( \mathcal{A} \). Then

\[
G_y \backslash G / G_x \cong W_y \backslash W / W_x.
\]

**Proof.** By Proposition 4.1 it suffices to note that for any vertex \( z \in \mathcal{A} \), the group \( \hat{W}_z = (N \cap G_z)/S_0 \) coincides with \( W_z \), the group generated by the reflections in the affine hyperplanes through \( z \). This follows in our case from [1 7.1.3]. \( \square \)

Let \( D_x := \{ z \in \mathcal{A} \mid \forall \alpha \in \Phi^{lin,+}_x, \alpha(z) > 0 \} \) denote the positive cone for \( \Phi^{lin,+}_x \).

**Proposition 4.3.** Suppose \( y \) is special. A set of double coset representatives for \( W_y \backslash W / W_x \) is given by

\[
X_{x,y}^+ = X_+(S) \cap (y - x + D_x) = \{ \ell \in X_+(S) \mid \forall \alpha \in \Phi^{lin,+}_x, \alpha(\ell) \geq \alpha(y - x) \}.
\]

**Proof.** Each \( w \in W \) can be written uniquely as \( w = w_0 t(v) \) for some \( w_0 \in W_0 \) and \( v \in X_+(S) \). Since \( y \) is special, \( w_0 \in W^{lin}_y \) and thus by (2.1) \( w_y := w_0 t(w_0^{-1} y - y) \in W_y \). Therefore we can factor \( w = w_y t(v_1) \) with \( v_1 = v + y - w_0^{-1} y \in X_+(S) \) and it remains to show that there exists \( \ell \in X^+_{x,y} \) such that \( t(v_1) \in W_y t(\ell) W_x \).

Choose \( w_1 \in W^{lin}_x \) such that \( w_1 (v_1 + x - y) \in D_x \). Set \( \ell := w_1 (v_1 + x - y) + y - x = (y - w_1 y) + w_1 v_1 + (w_1 x - x) \). Since \( w_1 \in W^{lin}_x \cap W^{lin}_y \), each summand lies in \( X_+(S) \). Thus \( \ell \in X^+_{x,y} \) and we have

\[
t(\ell) = t(y - w_1 y) t(w_1 v_1) t(w_1 x - x)
\]

\[
= t(y - w_1 y) w_1 t(v_1) w_1^{-1} t(w_1 x - x)
\]

\[
= w'_y t(v_1) w_x
\]

where \( w'_y := t(y - w_1 y) w_1 \in W_y \) and \( w_x := w_1^{-1} t(w_1 x - x) \in W_x \). Hence \( X^+_{x,y} \) exhausts \( W_y \backslash W / W_x \).

Now suppose that \( \ell, \ell' \in X^+_{x,y} \) are such that there exist \( w_y \in W_y \) and \( w_x \in W_x \) for which \( t(\ell') = w_y t(\ell) w_x \). Since the composition is a translation, the linear parts of \( w_y \) and \( w_x \) are mutually inverse. Therefore there is some \( w_0 \in W^{lin}_x \cap W^{lin}_y = W^{lin}_x \) such that \( w_y = t(y - w_0 y) w_0 \) and \( w_x = w_0^{-1} t(w_0 x - x) \). Using (4.1) we conclude...
\ell' = w_0(\ell + x - y) + (y - x). \quad \text{Since both } \ell + x - y \text{ and } \ell' + x - y = w_0(\ell + x - y) \text{ lie in } D_x \text{ and are conjugate by } W^\text{lin}_x \text{ they are equal.} \quad \square

For example, if \( y = x \) is special then \( X^+_{y,y} = X_+ \), the set of dominant cocharacters.

**Remark 4.4.** If \( x \neq y \), then there is some \( \alpha \in \Phi^\text{lin}_x \) for which \( \alpha(x - y) \neq 0 \), so that \( X^+_{x,y} \neq X_+ \). More generally \( X^+_{x,y} = X_+ + (y - x) \) if and only if \( x - y \in X_* (S) \), which will not arise if \( x, y \) are chosen in distinct orbits under \( G \), for example.

**Definition 4.5.** Let \( \text{int}(X^+_{x,y}) = X_* (S) \cap (y - x + D_x) \) and \( \partial(X^+_{x,y}) = X^+_{x,y} \setminus \text{int}(X^+_{x,y}) \), which we call the interior and the boundary of \( X^+_{x,y} \), respectively.

We record some key properties of the interior of \( X^+_{x,y} \) in two lemmas.

**Lemma 4.6.** Let \( \Upsilon_x = \{ w \in W_0 \mid wD \subseteq D_x \} \). Then we have

\[
\text{int}(X^+_{x,y}) = \bigcup_{w \in \Upsilon_x} X^+_{x,y} \cap (y - x + wD).
\]

**Proof.** Since \( \Phi^\text{lin}_x \subseteq \Phi \), \( D_x = \bigcup_{w \in \Upsilon_x} wD \) and thus \( X^+_{x,y} \subseteq \bigcup_{w \in \Upsilon_x} (y - x + wD) \). Fix \( w \in \Upsilon_x \) and suppose \( \ell \in X^+_{x,y} \cap (y - x + w(D \setminus D)) \). Then \( x - y + \ell \in w(D \setminus D) \) so there exists \( \alpha \in \Phi \) such that \( \alpha(x - y + \ell) = 0 \). But as \( y \) is special and \( \ell \in X_* (S) \), this implies \( \alpha(x) \in \mathbb{Z} \), whence \( \alpha \in \Phi^\text{lin}_x \). Consequently, \( x - y + \ell \in \partial(X^+_{x,y}) \). \( \square \)

**Lemma 4.7.** If \( \ell \in \text{int}(X^+_{x,y}) \) then the convex closure of \( [y, x + \ell] \) in \( \mathcal{A} \) contains unique alcoves adjacent to each endpoint.

**Proof.** For \( z \in \{ y, x + \ell \} \), let \( \mathcal{F}_z \) be the set of facets of \( \mathcal{A} \) containing \( z \) in their closure. A nontrivial line segment with an endpoint at \( z \) has nonzero intersection with a unique element \( F_z \) of \( \mathcal{F}_z \setminus \{ z \} \). We claim that in our case, \( F_z \) is an alcove. If not, then \( F_z \), and consequently also \( [y, x + \ell] \), is contained in the hyperplane \( \alpha = k \) for some \( \alpha \in \Phi \) and \( k \in \mathbb{Z} \). In particular, we have \( \alpha(x) = k - \alpha(\ell) \in \mathbb{Z} \) so \( \alpha \in \Phi^\text{lin}_x \). But since \( \alpha(y) = \alpha(x + \ell) = k \), we have \( \alpha(\ell) = \alpha(y - x) \) whence \( \ell \in \partial(X^+_{x,y}) \), a contradiction. \( \square \)

5. Restrictions of Supercuspidal Representations to \( \mathcal{G}_y \)

For reference we cite a consequence of Mackey theory for compactly induced representations derived from [6].

**Lemma 5.1.** Let \( G \) be the \( k \)-points of a linear algebraic group defined over \( k \), with a compact open subgroup \( K \) and a compact-mod-center subgroup \( H \). Let \( \rho \) be a smooth representation of \( H \) such that \( \pi = \text{c-Ind}^G_H \rho \) is admissible. For any \( t \in K \setminus G/H \), the subspace of \( \text{c-Ind}^G_H \rho \) consisting of vectors supported on the double coset \( Ht^{-1}K \)
is $K$-invariant, and as a representation of $K$ is isomorphic to $\text{Ind}^K_{K\cap \ell H}^t \sigma$. Thus we have

\begin{equation}
\text{Res}_K c\text{-Ind}^G_H \sigma \cong \bigoplus_{t \in K \setminus G/H} \text{Ind}^K_{K\cap \ell H}^t \sigma.
\end{equation}

In our case, let $H = G_x$ and $K = G_y$, for vertices $x, y \in \mathcal{A}$ with $y$ special. Given an irreducible supercuspidal representation $\pi = c\text{-Ind}^G_{G_x} \tau$ we therefore have

\[ \text{Res}_{G_y} \pi = \text{Res}_{G_y} c\text{-Ind}^G_{G_x} \tau \cong \bigoplus_{t \in G_y \setminus G/G_x} \text{Ind}^G_{G_y \cap G_x} t \tau. \]

By Proposition 4.3 we may choose the representatives of $G_y \setminus G/G_x$ to be $\{t(\ell) \mid \ell \in X^+_{x,y}\}$, whence $G_y \cap t(\ell) G_x = G_y \cap G_{x+\ell} = G_{[y,x+\ell]}$. Thus we may rewrite the sum above as

\begin{equation}
\text{Res}_{G_y} \pi \cong \bigoplus_{\ell \in X^+_{x,y}} \text{Ind}^G_{G_{[y,x+\ell]}} t(\ell) \tau.
\end{equation}

We refer to the representation $\pi_{\ell} = \text{Ind}^G_{G_{[y,x+\ell]}} t(\ell) \tau$ as a Mackey component of $\text{Res}_{G_y} \pi$. Note that this is not an irreducible representation in general.

Suppose from now on that $\tau$ has depth zero, and let us record some basic properties of the Mackey components $\pi_{\ell}$.

**Proposition 5.2.** Suppose $\ell \in \text{int}(X^+_{x,y})$ and set $\pi_{\ell} = \text{Ind}^G_{G_{[y,x+\ell]}} t(\ell) \tau$. Let $\Phi^\dagger = \{\alpha \in \Phi \mid \alpha(\ell) > \alpha(y - x)\}$ and set

\[ \eta(x - y + \ell) = \sum_{\alpha \in \Phi^\dagger} (\alpha(\ell) + \lceil \alpha(x - y) \rceil - 1). \]

Then

\[ \deg(\pi_{\ell}) = \deg(\tau) q^{\eta(x - y + \ell)} |G_y / \mathcal{B}|, \]

where $\mathcal{B}$ is a Borel subgroup of $G_y$. If $x$ is also special then $\eta(x - y + \ell) = 2\rho(x - y + \ell) - |\Phi^\dagger|$, where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^\dagger} \alpha$.

**Proof.** We suppose $\ell \in \text{int}(X^+_{x,y})$ and compute $[G_y : G_{[y,x+\ell]}]$.

By Lemma 4.7 the convex closure of $[y, x + \ell]$ contains an alcove $\Gamma$ adjacent to $y$ so $G_{[y,x+\ell]} \subseteq G_\Gamma \subseteq G_y$. Since $G_\Gamma$ is an Iwahori subgroup of $G$ contained in $G_y$, its image in $G_y \cong G(\kappa)$ is a Borel subgroup $\mathcal{B}$ of $G_y$. This Borel subgroup is defined by a choice of positive system of $\Phi$, namely the one consisting of the gradients of those affine roots in $\Phi_y$ which are positive on $x + \ell$. This is the set $\Phi^\dagger$.

We have $[G_y : G_\Gamma] = |G_y / \mathcal{B}|$; note that this factor is independent of the choice of $\mathcal{B}$.

We use Proposition 3.1 to compute the remaining factor $[G_\Gamma : G_{[y,x+\ell]}]$. Set $\Omega = [y, x + \ell]$; then for each $\alpha \in \Phi$ we have $f_\Omega(\alpha) = \max\{-\alpha(y), \lceil -\alpha(x + \ell) \rceil\}$. If $\alpha \in \Phi^\dagger$
we have \( f_\Omega(\alpha) = -\alpha(y) = f_\Gamma(\alpha) \) whereas \( f_\Omega(-\alpha) = \lceil \alpha(x+\ell) \rceil \) and \( f_\Gamma(-\alpha) = \alpha(y)+1 \). It thus follows from (3.11) that

\[
|G_\Gamma/G_{[y,x+\ell]}| = \prod_{\alpha \in \Phi^+} q^{\lceil \alpha(x-y+\ell)-1 \rceil} = q^{n(x-y+\ell)}.
\]

If \( x \) is special, then \( \alpha(x-y+\ell) \in \mathbb{Z} \) for all roots \( \alpha \) and furthermore we deduce that \( \Phi^\dagger = \Phi^+ \).

\[\square\]

**Remark 5.3.** The computation of the degree of \( \pi_\ell \) in the case that \( \ell \in \partial(X_{x,y}^x) \) is entirely analogous, using slightly more detailed results from [1, §6.4]. The factor \( |G_y/\mathcal{B}| \) is replaced by \( |G_y/P| \) for a parabolic subgroup \( P \) of \( G_y \).

**Theorem 5.4.** Let \( \ell \in X_{x,y}^+ \). Set

\[
r_0 = \max\{ \beta(x-y+\ell) \mid \beta \in \Delta_x \}
\]

and

\[
s_0 = \max\{ |\alpha(x-y+\ell)| \mid \alpha \in \Phi \}.
\]

Then the depth \( d \) of an irreducible subrepresentation of \( \text{Ind}^{G_y}_{G_{[y,x+\ell]}} t(\ell) \tau \) satisfies \( r_0 \leq d \leq s_0 \).

**Proof.** Let \( \ell \in X_{x,y}^+ \) and set \( \pi_\ell = \text{Ind}^{G_y}_{G_{[y,x+\ell]}} t(\ell) \tau \). If the space of \( \tau \) is denoted \( V_\tau \) then the space of \( \pi_\ell \) is \( V_\ell = \{ f : G_y \to V_\tau \mid \forall h \in G_{[y,x+\ell]}, \forall g \in G_y, f(hg) = t(\ell) \tau(h)f(g) \} \).

We prove that \( V_\ell^{G_y, r_0} = \{ 0 \} \) and \( V_\ell^{G_y, s_0+} = V_\ell \), whence the result.

By construction, \( \tau \) is trivial on \( G_{x,+} \), and thus \( t(\ell) \tau \) is trivial on \( t(\ell) G_{x,+} = G_{x+\ell,+} \). Given a nonnegative integer \( s \), the subgroup \( G_{y, s+} \) is contained in \( G_{x+\ell,+} \) if and only if for each \( \alpha \in \Phi \), we have \( \lceil (s - \alpha(y)) + 1 \rceil \geq \lceil -\alpha(x+\ell) + 1 \rceil \). As \( \alpha(y), \alpha(\ell) \in \mathbb{Z} \) this condition is equivalent to \( s \geq \lceil \alpha(y-x-\ell) \rceil \). Let \( s_0 = \max \{ |\alpha(x-y+\ell)| \mid \alpha \in \Phi \} \); this is nonnegative since \( \alpha(x-y+\ell) \geq 0 \) for \( \alpha \in \Phi^{\text{lin}, \tau} \). Thus \( G_{y, s_0+} \) is a normal subgroup of \( G_y \) contained in the kernel \( G_{x+\ell,+} \) of \( t(\ell) \tau \), whence \( V_\ell^{G_y, s_0+} = V_\ell \).

Now let \( \mathcal{H} \) be the unipotent radical of a proper parabolic subgroup \( P \) of \( G_x \). Since \( \tau \) is a cuspidal representation of the finite group \( G_x \), \( V_\tau^{\mathcal{H}} = \{ 0 \} \). Let \( H \subseteq G_x \) be a subgroup satisfying \( H/(H \cap G_{x,+}) = \mathcal{H} \). Using elementary arguments, and the normality of \( G_{y,r} \) in \( G_y \), one can show that if \( t(\ell) H \subseteq G_{y,r} \) then \( V_\ell^{G_{y,r}} = \{ 0 \} \).

Now each proper subset \( \Delta' \) of \( \Delta_x \) defines two proper parabolic subgroups of \( G_x \): the standard parabolic \( P_{\Delta'} \) and its opposite \( P_{\Delta'}^\text{opp} \). Let \( \mathcal{H} \) be the unipotent radical of \( P_{\Delta'} \). If \( \Phi' \) is the subrootsystem of \( \Phi^{\text{lin}}_x \) generated by \( \Delta' \), then \( \mathcal{H} \) is spanned by the root subgroups of \( G_x \) corresponding to \( \{ -\alpha \mid \alpha \in \Phi' = \Phi^{\text{lin}, \tau}_x \} \). We may choose \( H = \langle G_{-\alpha}(k)_{x,0} \mid \alpha \in \Phi' \rangle \subseteq G_x \) as our lift of \( \mathcal{H} \). Note that if \( x \) is not special then \( H \) is not necessarily contained in the unipotent radical of a parabolic subgroup of \( G_x \).

We have \( t(\ell) H = \langle G_{-\alpha}(k)_{x,\alpha(\ell)} \mid \alpha \in \Phi' \rangle \). Thus \( t(\ell) H \subseteq G_{y,r} \) if and only if for each \( \alpha \in \Phi', \lceil r + \alpha(y) \rceil \leq \lceil \alpha(x+\ell) \rceil \). Since \( \alpha \) takes integral values on \( x, y \) and \( \ell \), this simplifies to \( r \leq \alpha(x-y+\ell) \). Each simple root \( \beta \in \Delta_x \) takes nonnegative values on
on some simple root $\beta \in \Delta_x \setminus \Delta' \subseteq \tilde{\Phi}$, whence we conclude $V_\ell^G_{\beta(x-y+\ell)} = \{0\}$.

Conversely, given $\beta \in \Delta_x$, choosing $\Delta' = \Delta_x \setminus \{\beta\}$ above ensures that $V_\ell^G_{\beta(x-y+\ell)} = \{0\}$. We conclude that $r_0 = \max\{\beta(x-y+\ell) \mid \beta \in \Delta_x\}$ has the property required. \qed

Example 1. For $G = \text{SL}(2, k)$, with $y = 0$, one always has $r_0 = s_0$. Indeed, the depths of the irreducible components of $\pi_\ell$ were shown to be exactly $\delta(\ell) = \alpha(x-y+\ell) = x + \alpha(\ell)$ in [13, §5].

Example 2. For $G = \text{Sp}(4, k)$, with $\Phi^+ = \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta\}$, if $x$ is the non-special vertex of $C$ then $\Delta_x = \{\beta, 2\alpha + \beta\}$. Since the highest root of $\Phi$ is a simple root of $\Delta_x$, the depth of each irreducible subrepresentation of $\pi_\ell$ is exactly $\max\{\beta(x-y+\ell), (2\alpha + \beta)(x-y+\ell)\} \in \mathbb{Z}$. If $x$ is special, however, then the lower and upper bounds given in Theorem 5.4 cannot coincide.

More generally we note that if $x$ is special, then $s_0 = hr_0$ where $h$ is the height of the highest root of $\Phi$.

Theorem 5.4 gives an immediate criterion for disjointness of representations of $G_y$ occurring as factors of different Mackey components.

Corollary 5.5. For $i = 1, 2$ let $x_i$ be vertices of $\mathcal{A}$ and $\tau_i$ cuspidal representations of $G_{x_i}$. Suppose $\ell_i \in X_{x_i,y}$ satisfy

$$\max\{\alpha(x_1-y+\ell_1) \mid \alpha \in \Phi\} < \max\{\alpha(x_2-y+\ell_2) \mid \alpha \in \Delta_{x_2}\}.$$ 

Then the two Mackey components

$$\text{Ind}_{G_{[y,x_1+\ell_1]}}^{G_y} t^{(\ell_1)} \tau_1 \text{ and } \text{Ind}_{G_{[y,x_2+\ell_2]}}^{G_y} t^{(\ell_2)} \tau_2$$

are disjoint representations of $G_y$.

6. Case of Deligne-Lusztig cuspidal representations

Our main reference for this section is [4]. Recall that a minisotropic (maximal) torus $T$ of $G_x = G_x/G_{x,+}$ is one which is contained in no proper parabolic subgroup [13] II.1.11. Writing $rk(\mathcal{H})$ for the $k$-rank of the group $\mathcal{H}$ we set $\varepsilon = (-1)^{rk(G) - rk(Z(G_x))}$. 

Let $T$ be a minisotropic maximal torus of $G_x$ and $\theta$ a character of $T$. From this data P. Deligne and G. Lusztig constructed a virtual representation of $G_x$ whose character we denote $R_T^G(\varepsilon)(\theta)$. If $\theta$ is in general position [4, §7.3], then $\varepsilon R_T^G(\varepsilon)(\theta)$ is irreducible and cuspidal, and the corresponding representation $\tau$ is called a Deligne-Lusztig cuspidal representation. This character is given on an element $h \in G_x$ with Jordan decomposition $h = su$ by $\varepsilon R_T^G(\varepsilon)(\theta)(h) = 0$ if $s$ is not conjugate of an element of $T$, and otherwise, by

$$\varepsilon R_T^G(\varepsilon)(\theta)(h) = \frac{1}{|C^0_{G_x}(s)|} \sum_{g \in G_x, gs g^{-1} \in T} \theta(gsg^{-1}) Q^G_{\varepsilon,\tau}(s)(u).$$ (6.1)
where \( Q^\mathcal{G}_x(s) \) denotes the Green function, which takes values in \( \mathbb{Z} \) \([1, \S 7.6]\). It is known that

\[
\deg(\varepsilon R^\mathcal{G}_x(\theta)) = Q^\mathcal{G}_x(1) = \frac{|\mathcal{G}_x|}{|\mathcal{U}_x||T|}
\]

where \( \mathcal{U}_x \) denotes the unipotent radical of a Borel subgroup \( \mathcal{B}_x \) of \( \mathcal{G}_x \).

Let us now work towards understanding the Mackey components of the corresponding supercuspidal representation \( \pi = c\text{-Ind}_{\mathcal{G}_x}^G \tau \). We begin with a general lemma.

**Lemma 6.1.** Let \( \tau \) be a depth-zero representation of \( \mathcal{G}_x \) and \( \ell \in X^+_x \). Let \( F \neq \{x\} \) be the facet of \( \mathcal{A} \) which contains \( x \) in its closure and meets \( [y - \ell, x] \). Let \( \mathcal{P} = G_F/G_{x,+} \) be the parabolic subgroup of \( \mathcal{G}_x \) whose inflation to \( \mathcal{G}_x \) is \( \mathcal{G}_F \). Then the irreducible components of \( \text{Res}_{\mathcal{G}_x} \tau \) coincide with those of \( \text{Res}_\mathcal{P} \tau \).

**Proof.** The existence of \( F \) follows as in the proof of Lemma 4.7. Since \( x \in \overline{F} \) and \( F \) is a facet we have \( G_{x,+} \subseteq G_F \) and \( G_F/G_{x,+} \) is indeed a parabolic subgroup of \( G_x/G_{x,+} \).

Let \( \Omega \in \{F, [y - \ell, x]\} \). Since \( \tau \) is the inflation of a representation, say for the moment \( \mathcal{T} \), which is trivial on \( G_{x,+} \), \( \text{Res}_{G_\Omega}\mathcal{G}_x \tau \) and \( \text{Res}_{G_\Omega} \mathcal{T} \) have the same irreducible components.

Now if \( \beta \) is an affine root such that \( \beta(t) \geq 0 \) for all \( t \in \Omega \) and \( \beta(x) > 0 \) then necessarily \( \beta(t) > 0 \). It follows that the quotient \( G_\Omega/(G_\Omega \cap G_{x,+}) \) is uniquely determined by the set of affine roots vanishing on \( \Omega \). As these coincide for \( \Omega = F \) and \( \Omega = [y - \ell, x] \), the lemma follows. \( \square \)

Thus to determine the decomposition into irreducible subrepresentations of each Mackey component, one should first determine the restriction of a cuspidal representation of \( \mathcal{G}_x \) to a parabolic subgroup — a highly nontrivial open problem in general. Nevertheless, one can deduce some results in an important special case.

Let \( \mathcal{B}_x = \mathcal{S}\mathcal{U}_x \) be a standard Borel subgroup of \( \mathcal{G}_x \). Then the Jordan decomposition of any \( h \in \mathcal{B}_x \) is \( h = su \) with \( s \in \mathcal{S} \) and \( u \in \mathcal{U}_x \). But such an \( s \) is conjugate to an element of the minisotropic torus \( \mathcal{T} \) if and only if \( s \in Z(\mathcal{G}_x) \), since \( \mathcal{T} \) cannot contain a split subtorus outside of the center. Consequently \( C^\mathcal{G}_x(s) = \mathcal{G}_x \). The Green function depends only on the conjugacy class of \( \mathcal{T} \) within \( C^\mathcal{G}_x(s) \) and so in this case, is simply \( Q^\mathcal{G}_x \). Since \( s \) is central, \( gsg^{-1} = s \), and the character formula from (6.1) simplifies to

\[
\text{Res}_{\mathcal{B}_x} \varepsilon R^\mathcal{G}_x(\theta)(su) = \begin{cases} 0 & \text{if } s \notin Z(\mathcal{G}_x), \\ \theta(s)Q^\mathcal{G}_x(u) & \text{otherwise.} \end{cases}
\]

An immediate consequence of this calculation is the following proposition.

**Lemma 6.2.** The restriction of a Deligne-Lusztig cuspidal representation \( \varepsilon R^\mathcal{G}_x(\theta) \) to a Borel subgroup of \( \mathcal{G}_x \) depends only on the choice of minisotropic torus \( \mathcal{T} \) and the restriction of \( \theta \) to the center \( Z(\mathcal{G}_x) \).
Remark 6.3. In general we do not expect $\text{Res}_{B_x} \varepsilon R_T^G(\theta)$ to be irreducible; in fact we compute its self-intertwining number to be

$$\langle \varepsilon R_T^G(\theta), \varepsilon R_T^G(\theta) \rangle_{B_x} = \frac{1}{|B_x|} \sum_{s \in Z(G_x), u \in U_x} |\theta(s)|^2 |Q_T^G(u)|^2$$

$$= \frac{|Z(G_x)|}{|B_x|} \sum_{u \in U_x} Q_T^G(u)^2.$$

Example 3. If $G_x = \text{SL}(2, \kappa)$ then we determined in [13] that this intertwining number is $2 = |Z(G_x)|$.

Example 4. If $G_x = \text{SL}(3, \kappa)$ then we compute directly that $\sum_{u \in U_x} Q_T^G(u)^2 = q^4(q-1)^2$ and hence that the intertwining number of $\text{Res}_{B_x} R_T^G(\theta)$ with itself is $|Z(G_x)|q$, where $|Z(G_x)| = 3$ if $3$ divides $q - 1$.

Theorem 6.4. Let $x, y$ be vertices of $A$ with $y$ special. Let $\tau_1$ and $\tau_2$ be two Deligne-Lusztig cuspidal representations of $G_x$, induced from the same minisotropic torus and with the same central character. Let $\pi_i = c \text{-Ind}_{G_x}^{G_{11}} \tau_i$ be the corresponding depth-zero supercuspidal representations of $G$, for $i = 1, 2$. Then for each $\ell \in \text{int}(X_{x,y}^+)$, the Mackey components of $\text{Res}_{G_y} \pi_i$ corresponding to $\ell$ coincide for $i = 1, 2$. That is, we have

$$\text{Ind}_{G_{[y,x+\ell]}}^{G_y} t(\ell) \tau_1 \cong \text{Ind}_{G_{[y,x+\ell]}}^{G_y} t(\ell) \tau_2.$$

Proof. The induced representation $\text{Ind}_{G_{[y,x+\ell]}}^{G_y} t(\ell) \tau_i$ is determined by $\text{Res}_{G_{[y,x+\ell]}} t(\ell) \tau_i$. Conjugating by $t(\ell)^{-1}$ we deduce that (6.4) would follow from

$$\text{Res}_{G_{[y-x,\ell]}} \tau_1 \cong \text{Res}_{G_{[y-x,\ell]}} \tau_2.$$

By Lemma 4.7, the geodesic $[y, x + \ell]$ meets a unique alcove $\Gamma'$ adjacent to $x + \ell$; thus $\Gamma = \Gamma' - \ell$ is an alcove adjacent to $x$ meeting $[y - \ell, x]$. It follows that the group $G_{\Gamma}/G_{x, +}$ is a Borel subgroup $B_x$ of $G_x$, and thus by Lemma 6.4, (6.5) is equivalent to the condition that $\text{Res}_{B_x} \tau_1 \cong \text{Res}_{B_x} \tau_2$; this follows from Lemma 6.2 by our hypotheses on $\tau_1$ and $\tau_2$. \hfill $\square$

We can say much more, under certain circumstances.

Corollary 6.5. Suppose we are in the setting of Theorem 6.4 and suppose additionally that $T$ has the property that $T \cap P = Z(G_x)$ for all proper parabolic subgroups $P$ of $G$. Then if $y$ and $x$ are not conjugate under $G$ we have

$$\text{Res}_{G_y} \pi_1 \cong \text{Res}_{G_y} \pi_2,$$

whereas if $y = x$ then there exists a representation $W$ of $G_y$ such that we can write

$$\text{Res}_{G_y} \pi_i \cong \tau_i \oplus W$$

for $i = 1, 2$, with $W$ common to both.
Proof. Under the given hypotheses, the method of the proof of Lemma 6.2 applies equally to the restriction of $\tau_i$ to any proper parabolic subgroup $P$, and we conclude that $\text{Res}_P \tau_1 \cong \text{Res}_P \tau_2$. Therefore, for each $\ell \in X^+_x,y$ for which $y - \ell \neq x$, we may apply Lemma 6.1 to conclude that $\text{Res}_{G[y-\ell,x]} \tau_1 \cong \text{Res}_{G[y-\ell,x]} \tau_2$ and hence that the corresponding Mackey components coincide, that is, for all such $\ell$, (6.4) holds. If $x$ and $y$ lie in distinct orbits under $W$, then $y - \ell \neq x$ holds for all $\ell \in X^+_x,y$. Otherwise, we may without loss of generality assume $y = x$, in which case the single non-shared Mackey component is simply $\text{Ind}^{G_y}_{G_y} \tau_i = \tau_i$. The result follows. 

We conclude with some examples to explore the hypotheses of Corollary 6.5.

Example 5. The group $G = \text{SL}(3, \kappa)$ has a unique maximal anisotropic torus $T$ of order $q^2 + q + 1$ [18, II.1.10]. Up to conjugacy there is only one proper parabolic subgroup which is not a Borel, and its Levi component $L$ is isomorphic to $\text{GL}(2, \kappa)$. Were $t \in T \cap L$, it would be a semisimple element of $L$, hence lie in a torus of $\text{GL}(2, \kappa)$. However these tori have order $(q - 1)^2$ and $q^2 - 1$; in each case the gcd with $q^2 + q + 1$ is 3 and thus the order of $t$ is either 1 or 3. Hence $t \in Z(\text{SL}(3, \kappa))$ and the hypotheses of Corollary 6.5 hold.

Example 6. Let $G = \text{Sp}(4, \kappa)$. Then $G$ has two maximal anisotropic tori up to conjugacy [17, II.4–8]. The Coxeter torus $T_{w_0}$ has order $q^2 + 1$ and one concludes as in Example 5 that it cannot meet a proper parabolic subgroup except in the center of $G$.

The other anisotropic torus $T_{-1}$ corresponds to the element $w = -1$ in the Weyl group. In [19] this torus is the subgroup $H_4 = \langle a_4 \rangle \times \langle b_4 \rangle$, which is isomorphic to $N_1(\kappa')^2$ where $N_1(\kappa')$ is the group of norm-one elements of a quadratic extension $\kappa'$ of $\kappa$. One can show that the generator $a_4$ lies in a parabolic subgroup with Levi component isomorphic to $\text{SL}(2, \kappa) \times \text{GL}(1, \kappa)$.

Therefore the hypotheses of Corollary 6.5 hold for $T = T_{w_0}$ but not for $T = T_{-1}$.

7. INTERTWINING WITH PRINCIPAL SERIES

Let $\chi$ be a depth-zero character of $S$. Construct the parabolically induced representation $\text{Ind}^G_B \chi$; this is a depth-zero (possibly reducible) principal series representation of $G$. We denote by $V$ the space of $\text{Ind}^G_B \chi$.

Let $y$ be a special vertex. Then as $G = BG_y$ we have

\begin{equation}
\text{Res}_{G_y} \text{Ind}^G_B \chi \cong \text{Ind}^G_{B \cap G_y} \chi_0
\end{equation}

where $\chi_0$ denotes the restriction of $\chi$ to $B \cap G_y$. Note that twisting $\chi$ by any unramified character produces the same restriction to $G_y$; this holds in particular for the modular character which appears in our normalized induction $\text{Ind}^G_B \chi$. 

7.1. Some subrepresentations. The nature of parabolic induction is such that it is easier to construct a filtration of $V$ by $G_y$-invariant subspaces than a direct sum decomposition.

Lemma 7.1. Let $\Omega$ be a bounded convex closed subset of $\mathcal{A}$ satisfying $\overline{C} \subseteq \Omega \subseteq \overline{D}$. Then $\chi_0$ extends trivially to a character of $G_{y+\Omega}$ and $\text{Ind}_G^{G_{y+\Omega}} \chi_0$ is a subrepresentation of $\text{Ind}_B^{G_y} \chi_0$. Let $V_{y+\Omega}$ denote the space of this representation; it is finite-dimensional. If $\Omega' \supseteq \Omega$ is another such set, then $V_{y+\Omega'} \supseteq V_{y+\Omega}$.

Proof. By Proposition 3.1 we can write $G_{y+C}$ as $S_0 U_{y+C}$ where $S_0$ normalizes $U_{y+C}$. Since $\chi_0$ is trivial on $S_0 \cap U_{y+C} = S_1$, it extends to a character of $G_{y+C}$, trivial on $U_{y+C}$, which coincides with $\chi_0$ on $B \cap G_y$. Denote again by $\chi_0$ the restriction of this character to any subgroup of $G_{y+C}$. Since $\overline{C} \subseteq \Omega \subseteq \overline{D}$, we have $B \cap G_y \subset G_{y+\Omega} \subseteq G_{y+C}$. The rest follows.

We have the following estimates relating to the depth and degree of $V_{y+\Omega}$.

Proposition 7.2. Suppose $n$ is a positive integer. If $y + \Omega \subseteq \Omega_{y}(\mathcal{A}, n)$ then $V_{y+\Omega} \subseteq V_{G_{y,n}}^{G_y}$ and the depth of any irreducible subrepresentation of $V_{y+\Omega}$ is strictly less than $n$. Moreover, $\dim(V_{G_{y,n}}^{G_y}) = |G_y/|B| q^{(n-1)|\Phi^+|}$ for any Borel subgroup $B$ of $G_y$.

Proof. We may restrict to integral values since $y$ is special. If $y + \Omega \subseteq \Omega_{y}(\mathcal{A}, n)$ then $G_{y,n} \subseteq G_{y+\Omega}$ by Proposition 3.3. Since $n > 0$ we further have $G_{y,n} \subseteq S_1 U_{y+\Omega} = \ker(\chi_0)$, so it acts trivially on the induced representation, yielding $V_{G_{y,n}}^{G_y} = V_{y+\Omega}$. In fact, this defines an isomorphism $V_{G_{y,n}}^{G_y} \cong \text{Ind}_{B \cap G_y}^{G_y} \chi_0$, whence the dimension formula.

Remark 7.3. Let $r \in \mathbb{R}_{>0}$. If $\Phi$ does not contain an irreducible component of type $G_2$ then from Proposition 3.3 we may deduce that $(B \cap G_y)G_{y,r} = G_{\Omega_r}$ where $\Omega_r = \overline{D} \cap \Omega_y(\mathcal{A}, r)$. In general, however, the partially ordered filtration of subrepresentations $V_{y+\Omega}$ does not necessarily include the subrepresentations $V_{G_{y,r}}^{G_y}$ of $G_y$-fixed vectors. Although not needed here, note that one can obtain a much finer filtration (which in particular includes the $V_{G_{y,r}}^{G_y}$) by replacing the subgroups $G_{y+\Omega}$ with groups $G_f$ where $f$ is a concave function $\Pi \text{ [6.4]}$ satisfying $f(\alpha) = -\alpha(y)$ and $f(-\alpha) > \alpha(y)$ for all $\alpha \in \Phi^+$, as in $\Pi$.

7.2. Calculations on intertwining. Now let $\pi = c \text{-Ind}_{G_{y+\Omega}}^{G_y} \tau$ be a depth-zero supercuspidal representation of $G$. Let $\ell \in X_{x,y}^+$ and denote the corresponding Mackey component by $\pi_{\ell} = \text{Ind}_{G_{[y,x+\ell]}}^{G_{[y,x+\ell + \ell^0]}} \tau$.
Then for each set \( \Omega \) as in Lemma 7.1 we have

\[
\text{Hom}_{G_y}(\pi_\ell, \text{Ind}_{G_{y+\Omega}}^{G_y} \chi_0) \cong \text{Hom}_{G_{[y,x+\ell]}}(\ell(\ell)^\tau, \text{Res}_{G_{[y,x+\ell]}} \text{Ind}_{G_{y+\Omega}}^{G_y} \chi_0)
\]

\[
\cong \text{Hom}_{G_{[y,x+\ell]}}(\ell(\ell)^\tau, \bigoplus_{c \in \Psi_{x,y,\Omega}} \text{Ind}_{G_{[y,x+\ell]}}^{G_y} (\ell(\tau)^c \chi_0))
\]

\[
\cong \bigoplus_{c \in \Psi_{x,y,\Omega}} \text{Hom}_{G_{[y,x+\ell]}}(\ell(\tau)^c \chi_0)
\]

where \( \Psi_{x,y,\Omega} = G_{[y,x+\ell]} \setminus G_y / G_{y+\Omega} \).

Determining a set of representatives for \( \Psi_{x,y,\Omega} \) is a large subset of the problem of classifying \( B \cap G_y \) double cosets in \( G_y \), which for some groups is known to contain the matrix pair problem, that is, be wild [14]. Furthermore, while \( t(-\ell)c \cdot (y+\Omega) \) will be a convex closed subset of an apartment \( \mathcal{A}' \), meeting \( \mathcal{A} \) in at least the point \( y-\ell \), it is not to be expected that there exists a choice of such \( \mathcal{A}' \) which also contains \( x \). Thus in general the convex closure of \( [x, y-\ell] \cup t(-\ell)c \cdot (y+\Omega) \) is not contained in any apartment of \( \mathcal{B} \), and therefore its stabilizer is much more difficult to describe.

Nevertheless, there remain some tractable cases to consider, which suffice for proving the following theorem. Let \( Z_x \subseteq G_x \) denote the full preimage of \( Z(G_x) \subseteq G_x \).

**Theorem 7.4.** Let \( \tau \) be a Deligne-Lusztig cuspidal representation of \( G_x \) with central character \( \theta \). Let \( \widehat{\theta} \) denote the inflation of \( \theta \) to \( Z_x \). Let \( \chi \) be a character of \( S \) such that for some \( w \in W_0 \), \( \text{Res}_{Z_x}^w \chi = \widehat{\theta} \). Then the restrictions to \( G_y \) of

\[
\pi^s = c \cdot \text{Ind}_{G_x}^G \tau \quad \text{and} \quad \pi^p = \text{Ind}_{B}^G \chi
\]

have infinitely many distinct irreducible representations in common, of arbitrarily large depth.

**7.3. Proof of Theorem 7.4.** We begin by proving that each Mackey component of a Deligne-Lusztig supercuspidal representation corresponding to an element of \( \text{int}(X_{x,y}^+) \) intertwines with any compatible principal series representation.

**Proposition 7.5.** Let \( \tau, \theta \) and \( \widehat{\theta} \) be as above. Let \( \ell \in \text{int}(X_{x,y}^+) \) and define \( w \in \mathcal{Y}_x \) by \( \ell + x - y \in wD \). Let \( \chi \) be a character of \( S \) such that \( \text{Res}_{Z_x}^w \chi = \widehat{\theta} \) and denote by \( \chi_0 \) the trivial extension of \( \chi \) to any subgroup of \( G_{y+C} \). Then the representations

\[
\text{Ind}_{G_{[y,x+\ell]}}^{G_y} t(\ell)^\tau \quad \text{and} \quad \text{Ind}_{G_{y+\Omega}}^{G_y} \chi_0
\]

intertwine, for all bounded convex closed subsets \( \Omega \) with \( \overline{C} \subseteq \Omega \subseteq \overline{D} \) for which \( x - y + \ell \in w \Omega \).

**Proof.** Note that as \( S_1 = w^{-1}S_1 \subseteq Z_x \), the hypotheses imply that \( \chi \) has depth zero. It therefore suffices to show that there exists a nonzero summand in \( (7.2) \).

The existence and uniqueness of \( w \in \mathcal{Y}_x \subseteq W_0 \) follows from Lemma 4.6. By (2.1), \( w_y := t(y-wy)w \in W_y \), which we lift to an element of \( G_y \). Set \( \Omega' = t(-\ell)w_y \cdot (y+\Omega) = y - \ell + w \Omega \). When \( x - y + \ell \in w \Omega \) both \( x \) and \( y - \ell \) lie in \( \Omega' \), so \( G_{[x,y-\ell]} \cap G_{\Omega'} = G_{\Omega'} \).
Defining $U_{\Omega'}$ as in Proposition 3.1, we deduce that the summand for $c = w_y$ in (7.2) is

\[(7.3) \quad \text{Hom}_{S_0U_{\Omega'}}(\tau, t(-\ell)w_y\chi_0).\]

By hypothesis we have $x \in \text{int}(\Omega')$ so by Corollary 3.2 $U_{\Omega'} \subseteq G_{x,+} \subseteq \ker(\tau)$. On the other hand, note that

\[w^{-1}_y t(\ell)(S_0 U_{\Omega'}) = S_0 U_{w^{-1}_y t(\ell) \Omega'} = S_0 U_{y+\Omega}\]

and that $\chi_0$ was defined to be trivial on $U_{y+\Omega}$. Therefore $t(-\ell)w_y\chi_0$ is trivial on $U_{\Omega'}$. Moreover, on $S_0$ the character $t(-\ell)w_y\chi_0$ coincides with $w\chi$. Thus (7.3) is isomorphic to

\[\text{Hom}_{S_0}(\tau, w\chi).\]

Using the character formula from (6.3), the intertwining of the character $\varepsilon R^G_T(\theta)$ of $\tau$ with $w\chi$ is given on $S_0$ by

\[
\langle \varepsilon R^G_T(\theta), w\chi \rangle_{S_0} = \frac{1}{|S_0|} \int_{S_0} \varepsilon R^G_T(\theta)(s)w\chi(s) \, ds
\]

\[= \frac{1}{|S_0|} \int_{S_0} \int_{Z(G_0)} \int_{S_1} \deg(\tau)\theta(z)w\chi(zs_1) \, dz \, ds_1
\]

\[= \begin{cases} 0 & \text{if } \text{Res}_{Z_z} w\chi = \hat{\theta} \\ \deg(\tau) |Z(G_0)| / |S| & \text{otherwise.} \end{cases}
\]

Consequently $\text{Hom}_{S_0}(\tau, w\chi) \neq \{0\}$ exactly when the restriction of $w\chi$ to $Z_z$ coincides with $\hat{\theta}$. The proposition follows. \qed

We now do away with the apparent dependence on $w$ in Proposition (7.3).

**Corollary 7.6.** Let $\text{Ind}_{G_y}^{G_x}$ be a depth-zero principal series representation. Suppose $\tau$ is a Deligne-Lusztig cuspidal representation of $G_x$ with central character $\theta$ with inflation $\hat{\theta}$ to $G_{x,+}$. Let $w \in W_0$ and suppose $\text{Res}_{Z_z} w\chi = \hat{\theta}$. Then for every $\ell \in \text{int}(X^+_{x,y})$, there exists a subrepresentation of the Mackey component $\pi_\ell$ of $\text{Res}_{G_y} \text{Ind}_{G_x}^{G} \tau$ which is isomorphic to a subrepresentation of $\text{Res}_{G_y} \text{Ind}_{B}^{G} X$.

**Proof.** For any $\ell \in \text{int}(X^+_{x,y})$, we define $w_0 \in \mathbf{Y}_x$ as in Proposition 7.3. Thus $x - y + \ell \in w_0D$. Choose a bounded closed convex set $\Omega$ satisfying $C \cup \{w^{-1}_0(x - y + \ell)\} \subset \Omega \subset \overline{D}$. Since $\text{Res}_{Z_z}(w_0^{-1}w\chi) = \text{Res}_{Z_z} w\chi = \hat{\theta}$, Proposition 7.5 implies that $\pi_\ell$ intertwines with the subrepresentation of $\text{Res}_{G_y} \text{Ind}_{B}^{G} (w_0^{-1}w\chi)$ induced from $G_{y+\Omega}$. Consequently $\text{Res}_{G_y} \text{Ind}_{B}^{G} (w_0^{-1}w\chi)$ contains a subrepresentation of $G_y$ which is isomorphic to a subrepresentation of $\pi_\ell$. Finally, since $w^{-1}_0 w \in W_0$, $\text{Ind}_{B}^{G} (w_0^{-1}w\chi) \cong \text{Ind}_{G_y}^{G} \chi$ as representations of $G$ and therefore their restrictions to $G_y$ must also be isomorphic. \qed

Although the subrepresentations arising in Corollary 7.6 are not necessarily distinct, we have the following result.
Corollary 7.7. Let $\pi^s$ be a Deligne-Lusztig supercuspidal representation and $\pi^p$ a depth-zero principal series representation, which are compatible in the sense of Corollary 7.6. Then $\text{Res}_{G_y}^G \pi^s$ and $\text{Res}_{G_y}^G \pi^p$ have infinitely many distinct components in common, and the set of depths of these components is unbounded.

Proof. The first part follows from Corollary 7.6 by the Pigeonhole Principle since there are infinitely many $\ell \in \text{int}(X_{x,y}^+)$ and the admissibility of each supercuspidal representation implies each $G_y$-subrepresentation occurs with finite multiplicity.

More explicitly, we may restrict $\ell$ to an infinite subset of $X_{x,y}^+ \cap (y - x + D)$ in which every pair of elements satisfy the conditions of Corollary 5.5 thereby ensuring that their components are distinct. By Theorem 5.4 the set of depths of these representations is unbounded above. □

Remark 7.8. Given a depth-zero principal series representation, one may ask if for each vertex $x$ and minisotropic maximal torus $T \subseteq G_x$ there exists a Deligne-Lusztig cuspidal character $R_{G_x}^T(\theta)$ such that the corresponding supercuspidal representation is compatible with $\chi$. This is equivalent to the question of the existence of a character $\theta$ of $T$, coinciding with $\chi$ on $Z(G_x)$, and which is in general position, that is, not fixed by any nontrivial element of $W_x$. For $q$ sufficiently large, this follows from the arguments in [4, Lemma 8.4.2] with minor modification.

8. An example

We now illustrate the use of the results of Sections 5 to 7 with an example.

Let $G = \text{SL}(3, k)$. Suppose that $p \neq 3$ and $3 \nmid (q - 1)$, whence we have simply $\text{GL}(3, \mathcal{R}) = Z(\text{GL}(3, \mathcal{R})) \text{SL}(3, \mathcal{R})$ and the irreducible representations of $\text{GL}(3, \mathcal{R})$ and $\text{SL}(3, \mathcal{R})$ coincide. Since all vertices of $B$ are special and are conjugate by $\text{GL}(3, \mathcal{R})$, we may without loss of generality set $x = y = 0$.

For ease of notation, let $G_{abc}$ denote the subgroup which is the intersection with $G$ of the set

$$
\left[ \begin{array}{ccc}
\mathcal{R} & \mathcal{R} & \mathcal{R} \\
\mathcal{P}^a & \mathcal{R} & \mathcal{R} \\
\mathcal{P}^c & \mathcal{P}^b & \mathcal{R}
\end{array} \right].
$$

Since $Z(\text{SL}(3, \kappa)) = \{1\}$ and there is a unique anisotropic torus in $\text{SL}(3, \kappa)$, the compatibility condition in Theorem 6.4 trivially holds for any two Deligne-Lusztig cuspidal representations of $\text{SL}(3, \kappa)$. Furthermore, as noted in Example 5, the hypotheses of Corollary 6.5 hold, implying that all the components of positive depth in the restriction to $\text{SL}(3, \mathcal{R})$ of any two such supercuspidal representations coincide. So let us fix one choice of Deligne-Lusztig cuspidal representation $\tau$ and set $\pi = \text{c-Ind}_{G_y}^G \tau$.

We next fix one Mackey component $\pi_\ell$ and determine its decomposition into irreducible representations of $G_0$. 
Let $\Delta = \{\alpha, \beta\}$; then $X^+_{xy} = X_+ = \{\ell \in X_+(S) \mid \alpha(\ell) \geq 0 \text{ and } \beta(\ell) \geq 0\}$. The smallest nonzero element in $X_+$ is $\ell = (\alpha + \beta)^V$; it lies in $\text{int}(X_+)$. We have $G_{[0, \ell]} = G_{112}$. By Theorem 5.4 the degrees of the irreducible subrepresentations of $\pi_\ell$ are either 1 or 2. By Corollary 5.5, $\pi_\ell$ is disjoint from every other $\pi_{\ell'}$ except possibly $\pi_{2\ell'}$: if $\ell' \in X_+ \setminus \{0, \ell, 2\ell\}$ then its irreducible subrepresentations have depth at least 3. The degree of $\tau$ is $(q - 1)(q^2 - 1)$ from (6.2), so by Proposition 5.2 the degree of $\pi_\ell$ is $q(q + 1)(q^2 - 1)(q^3 - 1)$.

By Remark 6.3 and as computed in Example 4 the intertwining number of $\text{Res}_{G_0} \tau$ with itself is $q$. To decompose it into irreducible subrepresentations, we begin by restricting $\tau$ to the unipotent radical $U^{op}$ of $B^{op}$, which is simply a Heisenberg group over $\kappa$ with center $G_{-\alpha - \beta}(\kappa)$. Using character computations, one determines that the restriction of $\tau$ to $U^{op}$ consists of $(q - 1)$ copies of each of the $(q - 1)$ distinct Stone-Von Neumann representations $H_\psi$ (corresponding to a nontrivial central character $\psi$) together with the $(q - 1)^2$ characters of $U^{op}$ arising from the characters $\psi_{-\alpha} \otimes \psi_{-\beta} \otimes 1$ of $G_{-\alpha}(\kappa) \times G_{-\beta}(\kappa) \times G_{-\alpha - \beta}(\kappa)$ where neither $\psi_{-\alpha}$ nor $\psi_{-\beta}$ is trivial. It is then straightforward to determine that $\text{Res}_{G^{op}} \tau$ decomposes as $q$ distinct irreducible representations: the $(q - 1)$ components of $\rho = \text{Ind}_{U^{op}_{\psi}} H_\psi$, each of dimension $q(q - 1)$, and the representation $\phi = \text{Ind}_{U^{op}_{\psi}} \psi_{-\alpha} \otimes \psi_{-\beta} \otimes 1$ (for any choice of nontrivial $\psi_{-\alpha}$ and $\psi_{-\beta}$), of dimension $(q - 1)^2$.

We obtain a corresponding decomposition $\pi_\ell = \rho' \oplus \phi'$, where $\rho' := \text{Ind}_{G^{[0, \ell]}}^{G_0} t(\ell) \rho$ and $\phi' = \text{Ind}_{G^{[0, \ell]}}^{G_0} t(\ell) \phi$. One can show this induction is irreducible (that is, $\phi'$ is irreducible and $\rho'$ has exactly $q - 1$ irreducible components) by computing directly that of the seven double cosets of $G_{[0, \ell]}$ in $G_0$, only the trivial one supports intertwining operators.

Now let us consider the intertwining of $\pi_\ell$ with a principal series representation.

Since $Z(G_0) = \{1\}$, the compatibility condition of Theorem 7.4 holds for any depth zero character of $S_0$, so without loss of generality let $\chi = 1$ and consider $\text{Ind}_{G_0}^{G_1}$. By Proposition 7.2 all intertwining of $\pi_\ell$ with $V = \text{Ind}_{B^{op} \cap G_0}^{G_0} 1$ must already occur with the $q^2(q^2 + 1 + 1)(q + 1)$-dimensional subrepresentation $V^{G_0, 3} = V_{\Omega_3} = \text{Ind}_{G_{\Omega_3}}^{G_0} 1$ where $\Omega_3 = \mathcal{D} \cap \Omega_0(\mathcal{A}, n)$.

By the proof of Proposition 7.5 the intertwining number of $\pi_\ell$ with $V_{\Omega_3}$ is at least $\deg(\tau)/|S| = q + 1$, which suggests the possibility that $\pi_\ell$ can be embedded into $V$ as a subrepresentation. This is in fact the case, as follows.

Using the same arguments as in the proof of Proposition 7.5 one can show directly that the irreducible representation $\phi'$ intertwines already with $V_{\Omega_2}$; in particular this implies that $\phi'$ has depth 1, which follows readily from its construction. Furthermore, we can compute that this intertwining occurs for no larger subgroup than $G_{\Omega_2}$, whence it lives on the highest-dimensional quotient of $V_{\Omega_2}$. By 2, this component is irreducible of dimension $q(q^2 - 1)(q^3 - 1)$, whence another proof of the irreducibility of $\phi'$. 
On the other hand, $\rho'$ has no intertwining on the identity double coset with $V_{G_2}$, but the remaining $q$ intertwining operators map to $V_{223} = \text{Ind}_{G_{223}}^{G_0} 1$. In [2] it is shown that $V_{223}$ contains two isomorphic irreducible representations $U_{123} \cong U_{213}$ of dimension $q^2(q+1)(q^3-1)$ and in [15] it is shown that the quotient $U_{223}$ of $V_{223}$ by the sum of all its proper subrepresentations $V_{ijk}$ decomposes into a direct sum of $q-1$ distinct irreducible representations of this same degree, each distinct from $U_{123}$. These exhaust all irreducible subrepresentations of $V_{223}$ of this degree. We deduce that each subrepresentation of $\rho'$ occurs in $U_{123} \oplus U_{223} \subseteq V$, and hence that (cf. Corollary 7.6) $\pi_\ell$ embeds in $V$ (nonuniquely!).

Consequently, all the irreducible components of $\pi_\ell$ of positive depth are atypical, in the sense that they occur also in principal series representations. However, there exist several irreducible components of depth 1 in $\text{Ind}_{B \cap G_0}^{G_0} 1$ [2] which do not occur in $\pi_\ell$, and therefore by our analysis we may conclude that they are not atypical with any Deligne-Lusztig supercuspidal representation.

References

[1] F. Bruhat and J. Tits, Groupes réductifs sur un corps local, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251.
[2] Peter S. Campbell and Monica Nevins, Branching rules for unramified principal series representations of $GL(3)$ over a $p$-adic field, J. Algebra 321 (2009), no. 9, 2422–2444.
[3] ———, Branching rules for ramified principal series representations of $GL(3)$ over a $p$-adic field, Canad. J. Math. 62 (2010), no. 1, 34–51.
[4] Roger W. Carter, Finite groups of Lie type, Wiley Classics Library, John Wiley & Sons Ltd., Chichester, 1993, Conjugacy classes and complex characters, Reprint of the 1985 original, A Wiley-Interscience Publication.
[5] Meinolf Geck, Gerhard Hiss, Frank Lübeck, Gunter Malle, and Görtz Pfeiffer, CHEVIE—a system for computing and processing generic character tables, Appl. Algebra Engrg. Comm. Comput. 7 (1996), no. 3, 175–210, Computational methods in Lie theory (Essen, 1994).
[6] P. C. Kutzko, Mackey’s theorem for nonunitary representations, Proc. Amer. Math. Soc. 64 (1977), no. 1, 173–175.
[7] Khemais Makouch and Pierre Torasso, Restriction de la représentation de Weil à un sous-groupe compact maximal ou à un tore maximal elliptique, Preprint arxiv.org arXiv:1101.0560, 2011.
[8] Lawrence Morris, Level zero $G$-types, Compositio Math. 118 (1999), no. 2, 135–157.
[9] Allen Moy and Gopal Prasad, Unrefined minimal $K$-types for $p$-adic groups, Invent. Math. 116 (1994), no. 1-3, 393–408.
[10] ———, Jacquet functors and unrefined minimal $K$-types, Comment. Math. Helv. 71 (1996), no. 1, 98–121.
[11] Monica Nevins, Branching rules for principal series representations of $SL(2)$ over a $p$-adic field, Canad. J. Math. 57 (2005), no. 3, 648–672.
[12] ———, Patterns in branching rules for irreducible representations of $SL_2(k)$, for $k$ a $p$-adic field, Harmonic analysis on reductive, $p$-adic groups, Contemp. Math., vol. 543, Amer. Math. Soc., Providence, RI, 2011, pp. 185–199.
[13] ———, Branching rules for supercuspidal representations of $SL_2(k)$, for $k$ a $p$-adic field, J. Algebra 377 (2013), 204–231.
[14] Uri Onn, Amritanshu Prasad, and Leonid Vaserstein, A note on Bruhat decomposition of $GL(n)$ over local principal ideal rings, Comm. Algebra 34 (2006), no. 11, 4119–4130.
[15] Uri Onn and Pooja Singla, On the unramified principal series of $GL(3)$ over non-archimedean local fields, Preprint arxiv.org arXiv:1210.5640, 2012.
[16] Dipendra Prasad, *A brief survey on the theta correspondence*, Number theory (Tiruchirapalli, 1996), Contemp. Math., vol. 210, Amer. Math. Soc., Providence, RI, 1998, pp. 171–193.

[17] T. A. Springer, *Characters of special groups*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics, Vol. 131, Springer, Berlin, 1970, pp. 121–166.

[18] T. A. Springer and R. Steinberg, *Conjugacy classes*, Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics, Vol. 131, Springer, Berlin, 1970, pp. 167–266.

[19] Bhama Srinivasan, *The characters of the finite symplectic group* \( \text{Sp}(4, q) \), Trans. Amer. Math. Soc. **131** (1968), 488–525.

[20] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and \( L \)-functions (Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69.

Department of Mathematics and Statistics, University of Ottawa, Ottawa, Canada K1N 6N5

E-mail address: mnevins@uottawa.ca