Tight Chang’s-lemma-type bounds for Boolean functions

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Abstract
Chang’s lemma (Duke Mathematical Journal, 2002) is a classical result in mathematics, with applications spanning across additive combinatorics, combinatorial number theory, analysis of Boolean functions, communication complexity and algorithm design. For a Boolean function $f$ that takes values in $\{-1, 1\}$ let $r(f)$ denote its Fourier rank (i.e., the dimension of the span of its Fourier support). For each positive threshold $t$, Chang’s lemma provides a lower bound on $\delta(f) := \Pr[f(x) = -1]$ in terms of the dimension of the span of its characters with Fourier coefficients of magnitude at least $1/t$. In this work we examine the tightness of Chang’s lemma with respect to the following three natural settings of the threshold:

- the Fourier sparsity of $f$, denoted $k(f)$,
- the Fourier max-supp-entropy of $f$, denoted $k'(f)$, defined to be the maximum value of the reciprocal of the absolute value of a non-zero Fourier coefficient,
- the Fourier max-rank-entropy of $f$, denoted $k''(f)$, defined to be the minimum $t$ such that characters whose coefficients are at least $1/t$ in magnitude span a $r(f)$-dimensional space.

In this work we prove new lower bounds on $\delta(f)$ in terms of the above measures. One of our lower bounds, $\delta(f) = \Omega\left(\frac{r(f)^2}{k(f) \log^2 k(f)}\right)$, subsumes and refines the previously best known upper bound on $r(f)$ in terms of $k(f)$ by Sanyal (Theory of Computing, 2019). Another lower bound, $\delta(f) = \Omega\left(\frac{r(f)}{k''(f) \log k(f)}\right)$, is based on our improvement of a bound by Chattopadhyay, Hatami, Lovett and Tal (ITCS, 2019) on the sum of absolute values of level-1 Fourier coefficients in terms of $\mathbb{F}_2$-degree. We further show that Chang’s lemma for the above-mentioned choices of the threshold is asymptotically outperformed by our bounds for most settings of the parameters involved.

Next, we show that our bounds are tight for a wide range of the parameters involved, by constructing functions witnessing their tightness. All the functions we construct are modifications of the Addressing function, where we replace certain input variables by suitable functions. Our final contribution is to construct Boolean functions $f$ for which

- our lower bounds asymptotically match $\delta(f)$, and
- for any choice of the threshold $t$, the lower bound obtained from Chang’s lemma is asymptotically smaller than $\delta(f)$.

Our results imply more refined deterministic one-way communication complexity upper bounds for XOR functions. Given the wide-ranging application of Chang’s lemma, we strongly feel that our refinements of Chang’s lemma will find many more applications.

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1 Introduction

Chang’s lemma [Cha02, Gre04] is a classical result in additive combinatorics. Informally, the lemma states that all the large Fourier coefficients of the indicator function of a large subset of an Abelian group reside in a low dimensional subspace. The discovery of this lemma was motivated by an application to improve Frieman’s theorem on set additions [Cha02]. The lemma has subsequently found many applications in additive combinatorics and combinatorial number theory. Chang’s lemma and the ideas developed in Chang’s paper [Cha02] have been used to prove theorems about arithmetic progressions in sumsets [Gre02, San08], structure of Boolean functions with small spectral norm [GS08], and improved bounds for Roth’s theorem on three-term arithmetic progressions in the integers [San11, Blo16, BS20]. Green and Ruzsa [GR07] used the ideas of Chang’s lemma to prove a generalization of Frieman’s theorem for arbitrary Abelian groups. The Chang’s lemma is known to be sharp for various settings of parameters for the group $\mathbb{Z}_N$ [Gre03].

In this paper, our focus is a specialization of Chang’s lemma for the Boolean hypercube. Let $f : \{-1,1\}^n \to \{-1,1\}$ be a Boolean function. For any positive real number $t$ (which we refer to as the threshold) define $S_t := \{S \subseteq [n] : |\hat{f}(S)| \geq \frac{1}{t}\}$. Viewing elements of $S_t$ as vectors in $\mathbb{F}_2^n$, Chang’s lemma gives a lower bound on $\delta(f) := \Pr_x[f(x) = -1]$ (called the weight of $f$), in terms of $t$ and the dimension of the span of $S_t$ (denoted by $\dim(S_t)$). Formally, we have the following lemma, referred to as Chang’s lemma in this paper.

**Lemma 1.1** (Chang’s lemma [Cha02]). There exists a universal constant $c > 0$ such that the following is true for every integer $n > 0$. Let $f : \{-1,1\}^n \to \{-1,1\}$ be any function and $t$ be any positive real number. Let $\delta(f) := \Pr_x[f(x) = -1]$ and $d = \dim(S_t) > 1$. If $\delta(f) < c$, then

$$\delta(f) = \Omega\left(\frac{\sqrt{d}}{t\sqrt{\log (t^2/d)}}\right).$$

**Remark 1.2.** In the literature Chang’s lemma is generally stated as an upper bound on $d$ in terms of $\delta(f)$ and $t$. In Section A we state the more commonly seen form of Chang’s lemma (Lemma A.1) and prove that it is equivalent to Lemma 1.1.

This lemma has found numerous applications in complexity theory and algorithms [BRTW14, CLRS16], analysis of Boolean functions [GS08, TWXZ13], communication complexity [TWXZ13, HLY19] and extremal combinatorics [FKKK18]. See [IMR14] for a proof of Lemma 1.1.

In this paper, we investigate the tightness of Lemma 1.1 for three natural choices of the threshold $t$ based on the Fourier spectrum of the function (see Section 1.1 for details about these thresholds). We prove additional lower bounds on $\delta(f)$, and compare relative performances of all the bounds under consideration. Our results imply that the bounds given by Chang’s lemma for the choices of the threshold that we consider are asymptotically outperformed by one of the bounds we prove for a broad range of the parameters involved. For most regimes of the parameters we are able to construct classes of functions that witness the tightness of our bounds.

Interestingly, for each choice of threshold that we consider, $\dim(S_t)$ equals the Fourier rank of $f$ (denoted by $r(f)$, see Definition 4.23). In particular, setting $t$ to be the Fourier sparsity of $f$...
Granularity is another widely-studied measure that is closely associated for a detailed discussion.

Throughout this paper, we assume that $f$ is not a constant function or a parity or a negative parity (unless mentioned otherwise). In other words, $k(f), r(f) \geq 2$.

### 1.1 Thresholds considered for Chang’s lemma

For a Boolean function $f$, let $\text{supp}(f)$ denote the Fourier support of $f$ (Definition 4.3). In this section, we discuss and motivate the choices of the threshold $t$ considered in this work.

**The Fourier sparsity of $f$.** It was shown in [GOS+11, Theorem 3.3] that for all $S \in \text{supp}(f)$, $|\hat{f}(S)| \geq \frac{1}{k(f)}$. It follows that $S_{k(f)} = \text{supp}(f)$ and hence $\dim(S_{k(f)}) = r(f)$. Moreover, there exist functions (e.g. $f = \text{AND}_n$) for which $\dim(S_t) = 0$ for $t = o(k(f))$, justifying the choice of threshold $k(f)$.

This choice also leads us to a fundamental structural problem of bounding the weight of a Boolean function $f$ from below, in terms of its Fourier sparsity and Fourier rank. The *uncertainty principle* (see, for example, [GT13] for a statement and a proof) asserts that $\delta(f) = \Omega\left(\frac{1}{k(f)}\right)$. Chang’s lemma with $t = k(f)$ and the fact that $\log \left(\frac{k(f)^2}{r(f)}\right) = \Theta(\log k(f))$ (Lemma 4.27 (part 1)) implies that

$$\delta(f) = \Omega\left(\frac{1}{k(f)}\sqrt{\frac{r(f)}{\log k(f)}}\right),$$

thereby subsuming the uncertainty principle (note that $r(f)/\log k(f) \geq 1$) and refining it by incorporating $r(f)$ into the bound.

**The Fourier max-supp-entropy of $f$.** The next choice of the threshold that we consider is the *Fourier max-supp-entropy* of $f$, denoted by $k'(f)$, which we define to be $\max_{S \in \text{supp}(f)} \frac{1}{|\hat{f}(S)|}$ (Definition 4.26). By its definition $k'(f)$ is the smallest value of $t$ such that $S_t = \text{supp}(f)$. Since $k'(f) \leq k(f)$ (see the discussion in the last item), the knowledge of $k'(f)$ can potentially offer us a more fine-grained lower bound on $\delta(f)$ than as in the last item; Chang’s lemma with $t = k'(f)$ and $\log \left(\frac{k'(f)^2}{r(f)}\right) = \Theta(\log k'(f))$ (Lemma 4.27 (part 2)) implies

$$\delta(f) = \Omega\left(\frac{1}{k'(f)}\sqrt{\frac{r(f)}{\log k'(f)}}\right).$$

Notice that Equation (2) subsumes the bound in Equation (1).

In [HKP11] an equivalent statement of the well-known sensitivity conjecture was presented in terms of $k'(f)$.

Granularity is another widely-studied measure that is closely associated with Fourier max-supp-entropy.

\footnote{In [HKP11] $\log(k'(f)^2)$ is called the Fourier max-entropy while we refer to $k'(f)$ as the Fourier max-supp-entropy.}
**The Fourier max-rank-entropy of** \( f \). Our final choice of the threshold is the Fourier max-rank-entropy of \( f \), denoted by \( k''(f) \), which we define to be the smallest positive real number \( t \) such that \( \dim(\mathcal{S}_t) = r(f) \) (Definition 4.26). We have that \( k''(f) \leq k'(f) \leq k(f) \) by their definitions. Amongst all settings of the threshold \( t \) for which \( \dim(\mathcal{S}_t) = r(f) \), the value \( t = k''(f) \) yields the best lower bound from Chang’s lemma. Chang’s lemma with \( \dim(\mathcal{S}_t) = r(f) \) implies

\[
\delta(f) = \Omega \left( \frac{1}{k''(f)} \sqrt{\frac{r(f)}{\log (k''(f)2/r(f))}} \right),
\]

which subsumes the bounds in Equations (2) and (1).

1.2 Our contributions

We prove the following results regarding the three natural instantiations of the threshold \( t \) (mentioned in the preceding section) for Chang’s lemma.

1. **The Fourier sparsity of** \( f \): Recall that Chang’s lemma with threshold \( t = k(f) \) (Equation (1)) implies that \( \delta(f) = \Omega \left( \frac{1}{k(f)} \sqrt{\frac{r(f)}{\log k(f)}} \right) \). It was shown in [ACL+19] that \( \delta(f) = \Omega \left( \frac{1}{k(f)} \left( \frac{r(f)}{\log k(f)} \right) \right) \), improving upon this bound asymptotically (note that \( r(f)/\log k(f) \geq 1 \)). In this work we improve their bound further.

**Theorem 1.3.** Let \( f : \{-1,1\}^n \to \{-1,1\} \) be any function such that \( k(f) > 1 \). Then

\[
\delta(f) = \Omega \left( \frac{1}{k(f)} \left( \frac{r(f)}{\log k(f)} \right)^2 \right).
\]

Observe that the statement of Theorem 1.3 is equivalent to \( r(f) = O(\sqrt{k(f)\delta(f)} \log k(f)) \). This bound subsumes the bound \( r(f) = O(\sqrt{k(f)} \log k(f)) \) shown by Sanyal [San19]. We prove Theorem 1.3 by incorporating \( \delta(f) \) in Sanyal’s arguments and thereby refining his proof. See Section 5.1 for the proof of Theorem 1.3.

We also show that Theorem 1.3 is tight. For nearly all admissible values of \( \rho \) and \( \kappa \) we construct many Boolean functions \( f \) with \( k(f) = O(\kappa) \), \( r(f) = O(\rho) \) and \( \delta(f) = O \left( \frac{1}{\kappa} \left( \frac{\rho}{\log \kappa} \right)^2 \right) \) (Theorem 1.5 and Claim 6.15).

**Comparison with Sanyal’s bound:** The bound \( r(f) = O(\sqrt{k(f)} \log k(f)) \) proven by Sanyal is a special case of Theorem 1.3 for \( \delta(f) = \Theta(1) \). It is not known whether the \( \log k(f) \) term is required in Sanyal’s upper bound on \( r(f) \) (when \( f \) equals the Addressing function, \( r(f) = \Omega(\sqrt{k(f)}) \)), see Definition 4.19 and Observation 4.30). For all the functions we construct witnessing the tightness of the bound in Theorem 1.3, \( \delta(f) = o(1) \).

We prove Theorem 1.3 by generalizing Sanyal’s proof. As stated before, our bound is tight in this generality, i.e. the logarithmic factor is required in the upper bound on \( r(f) \). This sheds light on the presence of the logarithmic term in the bound \( r(f) = O(\sqrt{k(f)} \log k(f)) \).

2. **The Fourier max-supp-entropy of** \( f \): Recall from Section 1.1 that the Fourier max-supp-entropy of \( f \), denoted \( k'(f) \), is defined as \( k'(f) = \max_{S \subseteq \supp(f)} \frac{1}{|f(S)|} \). It can be shown that \( \sqrt{k(f)} \leq k'(f) \leq k(f)/2 \) (Lemma 4.27 (part 2)). We prove the following lower bound.
**Theorem 1.4.** Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function such that \( k(f) > 1 \). Then,

\[
\delta(f) = \Omega \left( \max \left\{ \frac{1}{k(f)} \left( \frac{r(f)}{\log k(f)} \right)^2, \frac{k(f)}{k'(f)^2} \right\} \right).
\]

As is evident from the statement, Theorem 1.4 presents two lower bounds, one of which is Theorem 1.3. The other lower bound \( \delta(f) \geq \frac{k(f)}{k'(f)^2} \) is Claim 4.28.

Chang’s lemma with the threshold \( t \) set to \( k'(f) \) (Equation (2)), together with the observation that \( \log k(f) = \Theta(\log k'(f)) \), implies \( \delta(f) = \Omega \left( \frac{1}{k'(f)} \sqrt{\frac{r(f)}{\log k(f)}} \right) \). Theorem 1.4 subsumes this bound since

\[
\delta(f) = \Omega \left( \frac{1}{k(f)} \left( \frac{r(f)}{\log k(f)} \right)^2 \cdot \frac{k(f)}{k'(f)^2} \right)^{1/2} = \Omega \left( \frac{1}{k'(f)} \left( \frac{r(f)}{\log k(f)} \right) \right) = \Omega \left( \frac{1}{k'(f)} \sqrt{\frac{r(f)}{\log k(f)}} \right),
\]

where the equality follows from \( r(f)/\log k(f) \geq 1 \).

In addition, observe from the last equality above that the bound of Theorem 1.4 is asymptotically larger than the bound obtained from Chang’s lemma for \( t = k'(f) \) (Equation (2)) except when \( r(f)/\log k(f) = \Theta(1) \). Theorem 1.5 complements Theorem 1.4 by showing that for nearly all admissible values of \( r(f), k(f) \) and \( k'(f) \), there exists a function for which the larger of the two bounds presented in Theorem 1.4 is tight.

**Theorem 1.5.** For all \( \rho, \kappa, \kappa' \in \mathbb{N} \) such that \( \kappa \) is sufficiently large, for all constants \( \epsilon > 0 \) such that \( \log \kappa \leq \rho \leq \kappa^{\frac{2}{3}+\epsilon} \) and \( \kappa^{\frac{2}{3}} \leq \kappa' \leq \kappa \), there exists a Boolean function \( f_{\rho, \kappa, \kappa'} \) such that \( r(f_{\rho, \kappa, \kappa'}) = \Theta(\rho) \), \( k(f_{\rho, \kappa, \kappa'}) = \Theta(\kappa) \), \( k'(f_{\rho, \kappa, \kappa'}) = \Theta(\kappa') \) and

\[
\delta(f_{\rho, \kappa, \kappa'}) = \Theta \left( \max \left\{ \frac{1}{\kappa} \left( \frac{\rho}{\log \kappa} \right)^2, \frac{\kappa}{\kappa'^2} \right\} \right).
\]

The range of parameters considered in Theorem 1.5 is justified by Lemma 4.27. We prove Theorem 1.5 in two parts. Fix any \( \rho, \kappa \) such that \( \log \kappa \leq \rho \leq \kappa^{\frac{2}{3}+\epsilon} \) for some constant \( \epsilon > 0 \). First, for each value of \( \kappa' \in [\frac{\kappa \log \kappa}{\rho}, \kappa] \) we construct a function \( f \) for which the first lower bound on \( \delta(f) \) from Theorem 1.4 is tight (Claim 6.15). Next, for each value of \( \kappa' \in [\kappa^{\frac{2}{3}}, \frac{\kappa \log \kappa}{\rho}] \) we construct a function \( f \) for which the second lower bound on \( \delta(f) \) from Theorem 1.4 is tight (Claim 6.16). See Figure 1 for a graphical visualization of the bounds in Theorem 1.4 for any fixed values of \( \rho \) and \( \kappa \).

3. The Fourier max-rank-entropy of \( f \):

Recall from Section 1.1 that the Fourier max-rank-entropy of \( f \), denoted \( k''(f) \), is the smallest positive real number \( t \) such that \( \dim(S_t) = r(f) \). It can be shown that max \( \left\{ \sqrt{r(f)}, \frac{r(f)}{\log k(f)} \right\} \leq k''(f) \leq k(f) \) (Lemma 4.27 (part 2)). We prove the following lower bound.

**Theorem 1.6.** Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function such that \( k(f) > 1 \). Then,

\[
\delta(f) = \Omega \left( \max \left\{ \frac{1}{k(f)} \left( \frac{r(f)}{\log k(f)} \right)^2, \frac{r(f)}{k''(f) \log k(f)} \right\} \right).
\]
Theorem 1.6 yields a better lower bound than Chang’s lemma with the threshold $t = k''(f)$ (Equation (3)), except when $r(f) < (\log k(f))^2$ (see the caption of Figure 2). Theorem 1.6 presents two lower bounds: the first one is Theorem 1.3, and the second one is Lemma 5.4. We prove Lemma 5.4 by strengthening a bound due to [CHLT19] on the sum of absolute values of level-1 Fourier coefficients of a Boolean function in terms of its $\mathbb{P}_2$-degree. A proof of Theorem 1.6 can be found in Section 5.2.

We also show that for nearly all admissible values of $r(f), k(f)$ and $k''(f)$, there exist functions for which the larger of the two bounds presented in Theorem 1.6 is nearly tight.

**Theorem 1.7.** For all $\rho, \kappa, \kappa'' \in \mathbb{N}$ such that $\kappa$ is sufficiently large, for all $\epsilon > 0$ such that $\log \kappa \leq \rho \leq \kappa^{2-\epsilon}$ and $\rho \leq \kappa'' \leq \kappa$ there exists a Boolean function $f_{\rho,\kappa,\kappa''}$ such that $r(f_{\rho,\kappa,\kappa''}) = \Theta(\rho), k(f_{\rho,\kappa,\kappa''}) = \Theta(\kappa), k''(f_{\rho,\kappa,\kappa''}) = \Theta(\kappa'')$ and

$$\delta(f_{\rho,\kappa,\kappa''}) = \Theta \left( \max \left\{ \frac{1}{\kappa} \left( \frac{\rho}{\log \kappa} \right)^2, \frac{\rho}{\kappa'' \log(\kappa''/\rho)} \right\} \right).$$

The range of parameters considered in Theorem 1.7 is justified by Lemma 4.27. Theorem 1.7 is proved in two parts. Fix any $\rho, \kappa$ such that $\log \kappa \leq \rho \leq \kappa^{2-\epsilon}$ for some constant $\epsilon > 0$. First, for each value of $\kappa'' \in [\frac{\log \kappa}{\rho}, \kappa]$ we construct a function $f$ for which the first lower bound on $\delta(f)$ from Theorem 1.6 is tight (Claim 6.18). In fact these are the same functions that are used to prove the first bound in Theorem 1.5. Next, for each value of $\kappa'' \in [\epsilon \rho, \frac{\log \kappa}{\rho}]$ we construct a function $f$ for which $\delta(f) = \Theta\left( \frac{\rho}{\kappa'' \log(\kappa''/\rho)} \right)$ (Claim 6.17). From the above discussion one may verify that for every $\rho, \kappa$ that we consider and for every $\kappa'' \geq \rho \cdot \kappa^{O(1)}$, the function that we construct witnesses tightness of the lower bound in Theorem 1.6.

In general, for all settings of $\rho, \kappa$ and $\kappa''$ that we consider, the upper bound on $\delta(f)$ from Theorem 1.7 is off by a factor of at most $O(\log \kappa)$ from the lower bound in Theorem 1.6.
See Figure 2 for a graphical visualization of the bounds in Theorem 1.6 for any fixed values of $\rho$ and $\kappa$.

For any fixed values of $\rho, \kappa$, we will refer to this plot as $(\rho, \kappa)$-plot. Chang’s lemma implies that Boolean functions lie above the CL-$k''$-curve. Theorem 1.6 improves upon Chang’s lemma and shows that Boolean functions lie above both the $k$-line and the $k''$-curve, highlighted by the dark grey region in the figure. Although the picture indicates that the CL-$k''$-curve is better than the $k''$-curve for certain ranges of $\kappa''$, this is actually only possible for certain values of $\rho$ and $\kappa$. This is because the CL-$k''$-curve and the $k''$-curve intersect at $\frac{\sqrt{\rho \kappa}}{\sqrt{\rho}}$, which is less than $\sqrt{\rho}$ if $\rho \geq (\log \kappa)^2$. By Lemma 4.27 we know that for any function $f$ on this plot, the range of $k''(f)$ is between $\max\{\sqrt{\rho}, \rho/\log \kappa\}$ and $\kappa$. Thus our bounds in Theorem 1.6 dominate those given by the CL-$k''$-curve in all $(\rho, \kappa)$-plots where $\rho \geq \log^2 \kappa$.

**Dominating Chang’s lemma for all thresholds.** Our final contribution is to show that there exists a function for which: our lower bounds (Theorem 1.4 and 1.6) asymptotically match its weight, but for any choice of the threshold the lower bound obtained from Chang’s lemma (Lemma 1.1) is asymptotically smaller than its weight (Claim 7.1).

### 1.3 Applications of our results

An application of our result is an enhanced understanding of the bound $r(f) = O(\sqrt{k(f)} \log k(f))$ proven by Sanyal [San19]. This bound is a special case of Theorem 1.3 for $\delta(f) = \Theta(1)$. It is not known whether the $\log k(f)$ term is required in Sanyal’s upper bound on $r(f)$ (when $f$ equals the Addressing function, $r(f) = \Omega(\sqrt{k(f)})$, see Definition 4.19 and Observation 4.30). For all the functions we construct witnessing the tightness of the bound in Theorem 1.3, $\delta(f) = o(1)$. We prove Theorem 1.3 by generalizing Sanyal’s proof. As stated before, our bound is tight in this generality, i.e. the logarithmic factor is required in the upper bound on $r(f)$. This sheds light on the presence of the logarithmic term in the bound $r(f) = O(\sqrt{k(f)} \log k(f))$.

Also, Fourier sparsity and Fourier rank of $f$ have intimate connections with the communication complexity of functions of the form $F := f \circ \text{XOR}$. The Fourier sparsity of $f$ equals the real rank $(\text{rank}(M_F))$ of the communication matrix $M_F$ of $F$, and the Fourier rank of $f$ equals the deterministic (and even exact quantum) one-way communication complexity of $F$ [MO09]. Theorem 1.3
thus implies an improved upper bound of $O(\sqrt{k(f)} \delta(f) \log k(f))$ on the one-way communication complexity of $F$ in these models, which asymptotically beats the best known upper bound of $O(\sqrt{\text{rank}(M_F)})$ even for two-way protocols [TWXZ13, Lov16], for the special case of functions of this form (when $\delta(f) = o(1/ \log k)$).

Given the wide-ranging application of Chang’s lemma to areas like additive combinatorics, learning theory and communication complexity, we strongly feel that our refinements of Chang’s lemma will find many more applications.

## 2 Proof techniques for lower bound results

Our lower bound results on $\delta(f)$ can be divided into two parts: lower bounds in terms of $r(f)$, $k(f)$, and $k'(f)$ (Theorem 1.4), and lower bounds in terms of $r(f)$, $k(f)$, and $k''(f)$ (Theorem 1.6).

Theorem 1.4 consists of two lower bounds. The second bound, $\delta(f) = \Omega\left(\frac{k(f)}{\sqrt{f(f)}}\right)$, is a direct application of Parseval’s identity (Claim 4.28). The first bound follows from Theorem 1.3, one of the main technical contributions of this paper. Similarly, Theorem 1.6 consists of two lower bounds: the first bound is Theorem 1.3 and the second one is Lemma 5.4.

The formal proofs of Theorems 1.4 and 1.6 are given in Section 5. We discuss the outline of the proofs of Theorem 1.3 and Lemma 5.4 in Sections 2.1 and 2.2, respectively.

### 2.1 Overview of the proof of Theorem 1.3

The lower bound on $\delta(f)$ in Theorem 1.3 can also be viewed as an upper on $r(f)$ in terms of $\delta(f)$ and $k(f)$. The best known upper bound on the Fourier rank of a Boolean function in terms of the sparsity of the function was given by Sanyal [San19]. They showed that for any Boolean function $f$, $r(f) = O(\sqrt{k(f)} \log k(f))$. Theorem 1.3 improves upon this upper bound on the Fourier rank by adding a dependence on the weight of the function: $r(f) = O(\sqrt{\delta(f)} k(f) \log k(f))$.

The outline of the proof is similar to the proof by Sanyal ([San19, Theorem 1.2]). We give an algorithm which takes a Boolean function as an input and outputs $O(\sqrt{\delta(f)} k(f) \log k(f))$ parities, such that, any assignment of these parities makes the function constant. This gives an upper bound on Fourier rank of the function since the Fourier support of the function must be contained in the span of this set of parities (Observation 4.25). The central ingredient in the algorithm is a lemma in [TWXZ13, Lemma 28].

**Lemma 2.1** ([TWXZ13]). Let $f : \{-1,1\}^n \to \{-1,1\}$ a function. There is an affine subspace $V \subseteq \{-1,1\}^n$ of co-dimension at most $3\sqrt{\delta(f)} k(f)$ such that $f$ is constant on $V$.

The above lemma is stated slightly differently in [TWXZ13]; they use $\|\hat{f}\|_1$ to bound the co-dimension instead of $3\sqrt{\delta(f)} k(f)$ (Claim 4.29). Lemma 2.1 allows us to fix small number of parities, such that, the sparsity of every possible restriction (for all possible assignments to these parities) is halved. The algorithm is formally stated in Section 5.1 (Algorithm 1); we give an outline here.

**Outline of the algorithm:** Our iterative algorithm incrementally constructs a set of parities such that, finally, the function becomes constant for every assignment of these set of parities. Every iteration, implemented as a while loop in Algorithm 1, is essentially an application of Lemma 2.1.

Let $\Gamma$ be the set of parities fixed after a certain number of iterations of the while loop. For the next iteration of the loop, we “greedily” pick a function, out of all possible restrictions corresponding
to $2^{|\Gamma|}$ possible assignments, of $\Gamma$. We then find a set of parities such that the greedily picked function becomes constant under some assignment of these parities; a small set of such parities exist (Lemma 2.1) and we include these parities in $\Gamma$. The algorithm finishes once all possible restrictions of $f$, corresponding to $\Gamma$, become constant. The termination condition implies that the algorithm outputs a set of parities satisfying the required condition.

**Completing the proof of Theorem 1.3:** It remains to show is that the number of parities fixed in Algorithm 1 is small. Given a Boolean function $f$ and a set of parities $\Gamma$ over the set of the variables of $f$, following equivalence relation over supp($f$) arises naturally:

$$\forall \gamma_1, \gamma_2 \in \text{supp}(f), \gamma_1 \equiv \gamma_2 \iff \gamma_1 + \gamma_2 \in \text{span}(\Gamma).$$

Let us denote $\Gamma$ after the $i$-th iteration of the while loop by $\Gamma^{(i)}$ ($\Gamma^{(0)} = \emptyset$). Let $f_{\min}^{(i)}$ be the selected function $f_{\min}$ after the $i$-th iteration ($f_{\min}^{(0)} = f$).

To bound the total number of parties fixed in Algorithm 1, we would like to bound the number of parities included in the $i$-th iteration of the while loop. In Step (a), the algorithm chooses the minimum number, say $q_i$, of parities such that $f_{\min}^{(i-1)}$ becomes constant after fixing these parities to some assignment. $\Gamma$ is updated with these parities to obtain $\Gamma^{(i)}$. Let $\ell_i$ be the number of partitions of the Fourier support of $f$ with respect to the equivalence relation corresponding to $\Gamma^{(i)}$.

In Step (b) the algorithm considers all possible assignments of parities in $\Gamma^{(i)}$ and the corresponding restrictions of $f$. A non-constant restriction with the smallest weight-to-sparsity ratio is chosen to be $f_{\min}$.

The main idea for the analysis of Algorithm 1 is to upper bound the ratio of $q_i$ and $(\ell_{i-1} - \ell_i)$ for every iteration $i$. On one hand $\frac{q_i}{(\ell_{i-1} - \ell_i)}$ is at most the square root of weight-to-sparsity ratio of the chosen $f_{\min}$ (Lemma 5.3 and its proof). On the other hand, Lemma 5.2 (main technical lemma in this proof, outline of the proof in the next paragraph) ensures that the weight-to-sparsity ratio of $f_{\min}$ can be upper bounded by $O\left(\frac{\delta(f_{\min})k(f_{\min})}{\ell_{i-1}}\right)$. Thus, we show that for every iteration $i$, $q_i$ is upper bounded by

$$O\left(\sqrt{\frac{\delta(f_{\min})k(f_{\min})}{\ell_{i-1}}(\ell_{i-1} - \ell_i)}\right).$$

Using standard arguments, summing over the iterations, we get a bound of $O(\sqrt{\delta(f)k(f)\log \ell_0})$ on $|\Gamma|$. Since $\ell_0 = k$, the desired upper bound on $r(f)$ in Theorem 1.3 follows from Observation 4.25.

**Outline of the proof of Lemma 5.2:** Given a Boolean function $f$ and a set of parities $\Gamma$, let $\ell$ be the number of equivalence classes for the equivalence relation corresponding to $\Gamma$. Define $f|_{(\Gamma,b)} := f|_{\{x \in \{-1,1\}: \forall \gamma \in \Gamma, \chi_{\gamma}(x) = b_{\gamma}\}}$ to be the restricted function when parities of $\Gamma$ are set to assignment $b$ in function $f$. Lemma 5.2 states that there exists an assignment $b$ of $\Gamma$ such that the restricted function $g_b = f|_{(\Gamma,b)}$ satisfies

$$\frac{\delta(g_b)}{k(g_b)} \leq \frac{4k(f)\delta(f)}{\ell^2}.$$

In contrast to the proof in [San19], where we only need to find a restriction with large sparsity, we need to balance both $\delta(g_b)$ and $k(g_b)$ here.\footnote{For technical reasons, we consider sparsity without the empty Fourier coefficient in this proof.}
We show that $E_b[\delta(g_b)] = \delta(f)$ and $E_b[k(g_b)] \geq \ell^2/4k(f)$, where $b$'s are picked uniformly from $\{-1, 1\}^\Gamma$. A careful manipulation of these expected values gives the required $b$.

The equality $E_b[\delta(g_b)] = \delta(f)$ follows by the observation that the set of inputs of $f$ is partitioned by the set of inputs of $g_b$. For the expectation of the sparsity of $g_b$, observe that the Fourier coefficients of $g_b$ are non-zero polynomials over the parities of $\Gamma$. By the uncertainty principle (Lemma 4.7), any Fourier coefficient is non-zero for a large number of $g_b$'s. Summing up these lower bounds for all Fourier coefficients, we get that the total number of non-zero Fourier coefficients (for all $g_b$) is large. This shows the required lower bound on the expectation of the sparsity of $g_b$, finishing the proof of Lemma 5.2.

### 2.2 Overview of the proof of Lemma 5.4 (for Theorem 1.6)

The crucial ingredient to prove the lower bound in Lemma 5.4 is the following lemma.

**Lemma 2.2.** For any Boolean function $f$, $\sum_{i=1}^n |\hat{f}(i)| = O(\delta(f)\deg_{\mathbb{F}_2}(f))$.

This lemma is a refinement of a similar theorem proved in [CHLT19] (Theorem 5.5) which does not contain the factor of $\delta(f)$. The proof of Lemma 2.2 for a Boolean function $f$ essentially applies Theorem 5.5 on the XOR of disjoint copies of $f$.

Lemma 2.2 shows a bound on the sum of absolute values of level-1 Fourier coefficients for the standard basis of the Fourier support of $f$; we extend this bound for any basis span of the Fourier support of $f$ (Corollary 5.6). The proof essentially constructs another function $h$ by doing a basis change on parities, and then applies Lemma 2.2 on the function $h$.

Lemma 5.4 is a direct implication of Corollary 5.6; observe that every Fourier coefficient on the left hand side of Corollary 5.6 is bigger than $1/k''(f)$ (from the definition of $k''(f)$).

### 3 Proof techniques for upper bound results

In this section we give the overview of our two upper bound results, Theorems 1.5 and 1.7. For presenting the overview of the proofs of these theorems we will use $(\rho, \kappa)$-$k'$-plots (Figure 1) and $(\rho, \kappa)$-$k''$-plots (Figure 2), respectively. In a $(\rho, \kappa)$-$k'$-plot ($(\rho, \kappa)$-$k''$-plot, respectively) we will refer to the “intersection point” as the point of intersection between the $k$-line and $k'$-curve (the point of intersection between the $k$-line and $k''$-curve, respectively). Which intersection point we are referring to will be clear from context.

#### 3.1 Proof techniques for Theorem 1.5

To prove Theorem 1.5, we split our goal into two natural parts: constructing functions on the $k$-line and constructing functions on the $k'$-curve. Both the classes of functions are modifications of the Addressing function (Definition 4.19). In these modifications, all or some of the target variables of the Addressing function are replaced with an AND function or a Bent function or a combination of them. We first provide a description of some functions that lie on the intersection point. While we do not require this, we choose to describe these functions in order to provide more intuition.

**Construction of functions at the intersection point in any $(\rho, \kappa)$-$k'$-plot:** Note that a function lies at the intersection point when

$$k'(f) = \frac{k(f) \log(k(f))}{r(f)}. \quad (4)$$
Thus, we want to construct a function \( f \) with \( k(f) = \Theta(\kappa) \), \( r(f) = \Theta(\rho) \), \( k'(f) = \Theta \left( \frac{\kappa \log \kappa}{\rho} \right) \) and \( \delta(f) = \rho^2/\kappa(\log^2 \kappa) \). In particular, we want to construct such functions for all \( \rho, \kappa \) satisfying \( \log \kappa \leq \rho \leq \kappa^{1/2} \). Note that, the Addressing function \( \text{AD}_t : \{0,1\}^{\log t + t} \rightarrow \{0,1\} \) has sparsity \( t^2 \), rank \( (t + \log t) \), max-supp-entropy \( t \) and weight \( 1/2 \) (Observation 4.30) and thus, \( \text{AD}_t \) satisfies Equation (4). This only gives functions on the intersection point on all \((\rho, \kappa)\)-\(k'\)-plots where \( \rho = \Theta(\sqrt{\kappa}) \), while we have to exhibit such functions for all \((\rho, \kappa)\)-\(k'\)-plots where \( \log \kappa \leq \rho = O(\sqrt{\kappa}) \).

Our next step is to tweak \( \text{AD}_t \) in such a way that the rank of the new function \( f \) does not change significantly while the sparsity and max-supp-entropy both increase by the same multiplicative factor. This would ensure that the resulting function satisfies Equation (4). If the resulting function’s weight decreases to the required value, we would have a function at the intersection point.

In order to tweak \( \text{AD}_t \), we consider a special kind of composed function \( f := \text{AD}_t \circ_{\text{target}} g \), obtained by replacing each target variable in the addressing function with a function \( g \) where each copy of \( g \) acts on a set of new variables. Lemma 6.8 gives the properties of such composed functions. Due to the structure of the Fourier spectrum of the Addressing function, Lemma 6.8 gives us \( r(f) \approx t \cdot r(g) \), \( k(f) \approx t^2 \cdot k(g) \), \( k'(f) = t \cdot k'(g) \) and \( \delta(f) = \delta(g) \).

So, if \( g \) is a function on a small number of variables (say \( \log t' \)) with near-maximal sparsity and max-supp-entropy (\( \Theta(t') \)), then the resulting function satisfies Equation (4). The \( \text{AND} \) function is a natural choice for \( g \). We denote the resulting function by \( \text{AD}_{t,t'} \) (Definition 6.1), and this is a function at the intersection point for all plots by suitably varying \( t \) and \( t' \).

**Constructing functions on the \( k \)-line:** We start with \( \text{AD}_{t,t'} \), the function at the intersection point in \((\rho, \kappa)\)-\(k'\)-plots. We modify \( \text{AD}_{t,t'} \) in such a way that its sparsity, rank and weight do not change much, while the max-supp-entropy increases. We replace a single \( \text{AND}_{\log t'} \) in \( \text{AD}_{t,t'} \) by \( \text{AND}_{\log a} \) for some suitable \( a > t \), denote the new function by \( \text{AD}_{t,t',a} \) (Definition 6.2). A suitable setting of the parameters \( t, t', a \) yields functions on the \( k \)-line for all plots (Claim 6.15).

**Constructing functions on the \( k' \)-curve of the \((\rho, \kappa)\)-\(k'\)-plot:** We start with \( \text{AD}_{t,t'} \) at the intersection point on \((\rho, \kappa/\ell)\)-\(k'\)-plot (for some parameter \( \ell > 0 \)). We modify \( \text{AD}_{t,t'} \) in such a way that its rank and weight do not change, the sparsity increases by a multiplicative factor of \( \ell \) and the max-supp-entropy increases by a factor of \( \sqrt{\ell} \). The new function \( f \) will be on the \( k' \)-curve in the \((\rho, \kappa)\)-\(k'\)-plot because \( \frac{k(f)}{k'(f)} = \frac{k(\text{AD}_{t,t'})}{k'(\text{AD}_{t,t'})} = \delta(\text{AD}_{t,t'}) = \delta(f) \). Note that \( k'(f) \approx \frac{\kappa \log(\kappa)}{\rho \sqrt{\ell}} \), thus making \( \ell \) suitably large yields functions on the \( k' \)-curve for all \( \rho \leq \kappa' \leq \frac{\kappa \log(\kappa)}{\rho \sqrt{\ell}} \) for all plots.

We now change \( \text{AD}_{t,t'} \) to have the properties mentioned above. We modify each \( \text{AND}_{\log t'} \) in \( \text{AD}_{t,t'} \) as follows: replace a single variable \( x \) by \( x \cdot B \), where \( B \) is a bent function on \( \log \ell \) new variables. We denote this new inner function by \( \text{AB}_{t',\ell} \) (Definition 6.3), and \( \text{AD}_t \circ_{\text{target}} \text{AB}_{t',\ell} \) by \( \text{AAB}_{t,t',\ell} \) (Definition 6.4). The effect of changing \( \text{AND}_{\log t'} \) to \( \text{AB}_{t',\ell} \) keeps its rank and weight roughly the same, while increasing its sparsity by a factor of \( \ell \) and increasing its max-supp-entropy by a factor of \( \sqrt{\ell} \) (Claim 6.10). In Claim 6.11 we show, using our composition lemma (Lemma 6.8), that the properties of \( \text{AD}_t \circ_{\text{target}} \text{AND}_{\log t'} \) and \( \text{AD}_t \circ_{\text{target}} \text{AB}_{t',\ell} \) change in a similar fashion. Thus, a suitable setting of the parameters \( t, t', \ell \) yields functions on the \( k' \)-curve for all plots (Claim 6.16).

\(^{5}\text{see Definition 4.22 for a precise definition}\)
3.2 Proof techniques for Theorem 1.7

We split our goal into two parts: constructing functions on the $k$-line when $\frac{p}{2} \log \kappa \leq \kappa'' \leq \kappa$, and constructing functions on the $k''$-curve when $\kappa \leq \frac{p}{2} \log \kappa$. To construct functions on the $k$-line, we use the functions $\text{AD}_{t,t',a}$ constructed for the proof of Theorem 1.5, since $k'(\text{AD}_{t,t',a}) = k''(\text{AD}_{t,t',a})$.

For constructing functions on the $k''$-curve, we need to construct functions $f$ such that

$$\delta(f) = \Theta \left( \frac{r(f)}{k''(f) \log (k''(f)/r(f))} \right).$$

(5)

We will use a similar technique as in our construction of functions on the $k'$-curve in Theorem 1.5. We start from the function $\text{AD}_{t,t'}$ at the intersection point. Note that $\text{AD}_{t,t'}$ satisfies Equation (5). We modify $\text{AD}_{t,t'}$ such that the rank, weight and max-rank-entropy changes very little but the sparsity increases by a multiplicative parameter $2^p$. We achieve this by replacing a variable (say $x$) in $\text{AD}_{t,t'}$ with $x \cdot \text{AND}(y_1, \ldots, y_p)$, where $x$ and $y_i$s are all variables in $\text{AD}_{t,t'}$, but for any $i$, $x$ and $y_i$ do not appear in the same monomial (Claim 6.17). The new function $f$ still satisfies Equation (5). This places $f$ on the $k''$-curve in a plot corresponding to the same rank as that of $\text{AD}_{t,t'}$, but where the sparsity increases by a factor of $2^p$. By suitably setting $p$, $t$ and $t'$, we obtain functions on the $k''$-curve for all plots. This proves the second bound in Theorem 1.7.

4 Preliminaries

All logarithms in this paper are taken to be base 2. We use the notation $[n]$ to denote the set $\{1, 2, \ldots, n\}$. When necessary, we assume $t$ is a power of 2. We use the notation $1^n$ (respectively, $(-1)^n$) to denote the $n$-bit string $(1, 1, \ldots, 1)$ (respectively, $(-1, -1, \ldots, -1)$).

For a function $f : \{-1, 1\}^n \to \{-1, 1\}$, its $\mathbb{F}_2$-degree, denoted by $\deg_{\mathbb{F}_2}(f)$, is the degree of its unique $\mathbb{F}_2$-polynomial representation. Throughout this paper, we often identify subsets of $[n]$ with their corresponding characteristic vectors in $\mathbb{F}_2^n$. Thus when we refer to linear algebraic measures of a collection of subsets of $[n]$, we mean the measure on the corresponding subset of $\mathbb{F}_2^n$ (where $\mathbb{F}_2^n$ is viewed as an $\mathbb{F}_2$-vector space).

Throughout this paper, we assume that $f$ is not a constant function or a parity or a negative parity, unless mentioned otherwise.

4.1 Fourier analysis of Boolean functions

Consider the vector space of functions from $\{-1, 1\}^n$ to $\mathbb{R}$ equipped with the following inner product.

$$\langle f, g \rangle := \frac{1}{2^n} \sum_{x \in \{-1, 1\}^n} f(x)g(x).$$

For a set $S \subseteq [n]$, define a parity function (which we also refer to as characters) $\chi_S : \{-1, 1\}^n \to \{-1, 1\}$ by $\chi_S(x) = \prod_{i \in S} x_i$. The set of parity functions $\{\chi_S : S \subseteq [n]\}$ forms an orthonormal basis for this vector space. Hence, every function $f : \{-1, 1\}^n \to \mathbb{R}$ has a unique representation as

$$f = \sum_{S \subseteq [n]} \hat{f}(S)\chi_S,$$
where \( \hat{f}(S) = \langle f, \chi_S \rangle \) for all \( S \subseteq [n] \). The coefficients \( \{ \hat{f}(S) : S \subseteq [n] \} \) are called the Fourier coefficients of \( f \). Define the Fourier \( \ell_1 \)-norm of a function \( f : \{-1, 1\}^n \to \mathbb{R} \) by \( \|f\|_1 := \sum_{S \subseteq [n]} |\hat{f}(S)| \).

**Definition 4.1** (Weight of a Boolean function). Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function. The weight of \( f \), denoted by \( \delta(f) \), is defined as

\[
\delta(f) = \Pr_{x \in \{-1, 1\}^n} [f(x) = -1].
\]

The following observation follows from the fact that \( \hat{f}(\emptyset) = \frac{1}{2^n} \sum_{x \in \{-1, 1\}^n} f(x) \).

**Observation 4.2.** Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function. Then,

\[
\hat{f}(\emptyset) = 1 - 2\delta(f).
\]

**Definition 4.3** (Fourier Support). Let \( f : \{-1, 1\}^n \to \mathbb{R} \) be any function. The Fourier support of \( f \), denoted by \( \text{supp}(f) \), is defined as

\[
\text{supp}(f) = \{ S \subseteq [n] : \hat{f}(S) \neq 0 \}.
\]

**Remark 4.4.** In the literature, Fourier support is generally denoted by \( \text{supp}(\hat{f}) \). For ease of notation we drop the hat symbol above \( f \). A similar convention has been adopted in Definitions 4.5, 4.23, and 4.26.

For ease of notation, we sometimes abuse notation and say that the elements of the Fourier support of \( f \) are the characters \( \{ \chi_S : S \subseteq [n], \hat{f}(S) \neq 0 \} \), rather than the corresponding sets as given in Definition 4.3.

**Definition 4.5** (Fourier sparsity). Let \( f : \{-1, 1\}^n \to \mathbb{R} \) be any function. The Fourier sparsity of \( f \), denoted by \( k(f) \), is defined as

\[
k(f) = |\text{supp}(f)|.
\]

For simplicity we assume that \( k(f) \geq 2 \) for all Boolean functions \( f \) considered in this paper (unless explicitly mentioned otherwise). We often simply refer to the Fourier sparsity as sparsity.

**Theorem 4.6** (Parseval’s identity). Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function. Then,

\[
\sum_{S \subseteq [n]} \hat{f}(S)^2 = 1.
\]

We require the following lemma (see, for example, [GT13]).

**Lemma 4.7** (Uncertainty Principle). Let \( f : \{-1, 1\}^n \to \mathbb{R} \) be a polynomial and let \( U_n \) denote the uniform distribution on \( \{-1, 1\}^n \). Then,

\[
\Pr_{x \sim U_n} [f(x) \neq 0] \geq \frac{1}{k(f)}.
\]
Lemma 4.8 ([GOS+11, Theorem 8.1]). Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function. Then, for all \( S \subseteq [n] \), \( |\hat{f}(S)| \) is an integral multiple of \( 2^{1−\log k(f)} \).

We also require the following lemma relating the \( \mathbb{F}_2 \)-degree of a Boolean function and its Fourier sparsity (see, for example, [BC99]).

Lemma 4.9. Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function with \( k(f) > 1 \). Then,
\[
\deg_{\mathbb{F}_2}(f) \leq \log k(f).
\]

The next claim shows that \( \deg_{\mathbb{F}_2}(f) \) does not change under a change of basis over the Fourier domain.

Claim 4.10. Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function and let \( B \in \mathbb{F}_2^{n \times n} \) be an invertible matrix. Define the function \( f_B : \{-1, 1\}^n \to \mathbb{R} \) as
\[
\hat{f}_B(\alpha) = \hat{f}(B\alpha) \quad \text{for all} \quad \alpha \in \mathbb{F}_2^n,
\]
then \( f_B \) is Boolean valued and \( \deg_{\mathbb{F}_2}(f_B) = \deg_{\mathbb{F}_2}(f) \).

Proof. Viewing \( f_B \) and \( f \) as functions over the domain \( \{0, 1\}^n \) instead of \( \{-1, 1\}^n \), we get that this basis change over the Fourier domain amounts to applying \((B^{-1})^T\) on the input space (see [ACL+19, Lemma 4]). In other words \( f_B \) is Boolean valued, and if \( p_{f_B} \) and \( p_f \) are the \( \mathbb{F}_2 \)-polynomials representing \( f_B \) and \( f \), respectively, then \( p_{f_B}(x) = p_f((B^{-1})^T x) \).

For all \( x \in \mathbb{F}_2^n \), let \( p_f(x) = \sum_{\gamma \in \mathbb{F}_2^n} \hat{p}_f(\gamma) \prod_{i: \gamma_i=1} x_i \). If \((B^{-1})^T_j\) denotes the \( j \)-th row of \((B^{-1})^T\) (for \( j \in [n] \)), then \( p_{f_B} \) has the unique representation
\[
p_{f_B}(x) = \hat{p}_f(\gamma) \prod_{i: \gamma_i=1} \langle (B^{-1})^T_j, x \rangle.
\]
So, every variable appearing in the polynomial representation of \( p_f \) is replaced by a linear combination (over \( \mathbb{F}_2 \)) of \( x \)'s in \( p_{f_B} \). In particular, the degree of any monomial in the polynomial representation of \( p_f \) is at least as large as the degree of its expansion in \( p_{f_B} \), and hence \( \deg(p_{f_B}) \leq \deg(p_f) \).

Since \( B \) is invertible, the same argument shows \( \deg(p_f) \leq \deg(p_{f_B}) \). Thus \( \deg(p_{f_B}) = \deg(p_f) \), which implies \( \deg_{\mathbb{F}_2}(f_B) = \deg_{\mathbb{F}_2}(f) \).

The following corollary follows from [CHLT19, Theorem 13] and Lemma 4.9.

Corollary 4.11. Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function, and let \( S \subseteq \text{supp}(f) \) be a basis of \( \text{span}(\text{supp}(f)) \). Then,
\[
\sum_{S \in \mathcal{S}} |\hat{f}(S)| \leq 4 \log k(f).
\]

We now define notions of restriction of a function \( f : \{-1, 1\}^n \to \{-1, 1\} \) to a subset \( A \subseteq \{-1, 1\}^n \).

Definition 4.12 (Restriction). Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) and \( A \subseteq \{-1, 1\}^n \). The restriction of \( f \) to \( A \) is the function \( f|_A : A \to \{-1, 1\} \) defined as \( f|_A(x) = f(x) \) for all \( x \in A \).

Definition 4.13 (Affine Restriction). Let \( f : \{-1, 1\}^n \to \{-1, 1\} \), let \( \Gamma \) be a set of parities and \( b \in \{-1, 1\}^\Gamma \) be an assignment to these parities. Define the function \( f|_{(\Gamma, b)} \) to be the restriction of \( f \) to the affine subspace obtained by fixing parities in \( \Gamma \) according to \( b \). That is,
\[
f|_{(\Gamma, b)} := f|_{\{x \in \{-1, 1\}^n : \chi_\gamma(x) = b, \text{ for all } \gamma \in \Gamma\}}.
\]
4.2 Fourier expansions and properties of some standard functions

For any integer \( n > 0 \), define the function \( \text{AND}_n : \{-1,1\}^n \rightarrow \{-1,1\} \) by \( \text{AND}_n(x) = -1 \) if \( x = (-1)^n \), and 1 otherwise. We drop the subscript \( n \) when it is clear from the context. We state the Fourier expansion of \( \text{AND} \) below without proof.

**Fact 4.14** (Fourier expansion of \( \text{AND} \)). Let \( n \geq 1 \) be any positive integer. Then

\[
\hat{\text{AND}}_n(S) = \begin{cases} 
1 - \frac{2}{2^{(\lvert S \rvert + 1)}} & S = \emptyset, \\
\frac{2^{\lvert S \rvert + 1}}{2^n} & \text{otherwise}.
\end{cases}
\]

**Definition 4.15** (Bent functions). A function \( f : \{-1,1\}^n \rightarrow \{-1,1\} \) is said to be a bent function if \( \lvert \hat{f}(S) \rvert = \lvert \hat{f}(T) \rvert \) for all \( S, T \subseteq [n] \).

Using Parseval’s identity (Theorem 4.6) we get the following observation.

**Observation 4.16.** Let \( f : \{-1,1\}^n \rightarrow \{-1,1\} \) be a bent function. Then,

\[
\lvert \hat{f}(S) \rvert = \frac{1}{\sqrt{2^n}} \text{ for all } S \subseteq [n].
\]

**Definition 4.17** (Indicator function). For any integer \( n \geq 1 \) and \( b \in \{-1,1\}^n \), define the function \( I_b : \{-1,1\}^n \rightarrow \{0,1\} \) by

\[
I_b(x) = \begin{cases} 
1 & x = b, \\
0 & \text{otherwise}.
\end{cases}
\]

We require the following observation about the Fourier expansion of Indicator functions, which we state without proof.

**Observation 4.18** (Fourier expansion of Indicator functions). For any integer \( n \geq 1 \) and \( b \in \{-1,1\}^n \), let \( I_b \) be as in Definition 4.17. Then,

\[
\hat{I}_b(S) = \prod_{i \in S} b_i 2^n \text{ for all } S \subseteq [n].
\]

**Definition 4.19** (Addressing function). For any integer \( t \geq 2 \), define the Addressing function \( \text{AD}_t : \{-1,1\}^{\log t} \times \{-1,1\}^t \rightarrow \{-1,1\} \) by

\[
\text{AD}_t(x, y) = y_{\text{bin}(x)},
\]

where \( x \in \{-1,1\}^{\log t} \) and \( y \in \{-1,1\}^t \), and \( \text{bin}(x) \) denotes the integer in \([t]\) whose binary representation is given by \( x \) (where -1’s are viewed as 1 in the string \( x \), and 1’s are viewed as 0). We refer to the \( x \)-variables as addressing variables, and the \( y \)-variables as target variables.

The following combinatorial observation is useful to us.

**Observation 4.20.** For any integer \( n \geq 1 \) and non-empty subset \( S \subseteq [n] \),

\[
\sum_{b \in \{-1,1\}^n} \prod_{i \in S} b_i = 0.
\]

We require the following representation of Addressing functions.
Observation 4.21. For any integer $t \geq 2$, $x \in \{-1,1\}^{\log t}$ and $y \in \{-1,1\}^t$, we have
\[
\text{AD}_t(x, y) = \sum_{b \in \{-1,1\}^{\log t}} y_b \|_b(x).
\]

We next define a way of modifying the Addressing function that is of use to us. In this modification, we replace target variables by functions, each acting on disjoint variables.

Definition 4.22 (Composed addressing functions). Let $t \geq 2$, $\ell_1, \ldots, \ell_t \geq 1$ be any integers. Let $g_i : \{-1,1\}^{\ell_i} \to \{-1,1\}$ be any functions for $i \in [t]$. Define the function $\text{AD}_t \circ_{\text{target}} (g_1, \ldots, g_t) : \{-1,1\}^{\log t} \times \{-1,1\}^{\ell_1 + \cdots + \ell_t} \to \{-1,1\}$ by
\[
\text{AD}_t \circ_{\text{target}} (g_1, \ldots, g_t)(x, y_1, \ldots, y_t) = \text{AD}_t(x, g_1(y_1), \ldots, g_t(y_t)),
\]
where $x \in \{-1,1\}^{\log t}$ and $y_i \in \{-1,1\}^{\ell_i}$ for all $i \in [t]$.

For any function $g : \{-1,1\}^s \to \{-1,1\}$, we use the notation $\text{AD}_t \circ_{\text{target}} g$ to denote the function $\text{AD}_t \circ_{\text{target}} (g, g, \ldots, g) : \{-1,1\}^{\log t} \times \{-1,1\}^{ts} \to \{-1,1\}$.

4.3 Fourier-analytic measures of Boolean functions

We now introduce a few Fourier-analytic measures on Boolean functions that we use throughout the rest of the paper, and state some important relationships between them. Recall that we use the notation $\dim(S)$ to denote the dimension of the span of the set $S$.

Definition 4.23 (Fourier rank). Let $f : \{-1,1\}^n \to \{-1,1\}$ be any function. Define the Fourier rank of $f$, denoted $r(f)$, by
\[
r(f) = \dim(\text{supp}(f)).
\]

We often refer to Fourier rank as simply rank. Sanyal [San19] showed the following upper bound on the rank of Boolean functions in terms of their sparsity.

Theorem 4.24 ([San19, Theorem 1.2]). Let $f : \{-1,1\}^n \to \{-1,1\}$ be any function. Then
\[
r(f) = O(\sqrt{k(f)} \log k(f)).
\]

We require the following observation which gives a simple upper bound on the rank of a Boolean function.

Observation 4.25. Let $f : \{-1,1\}^n \to \{-1,1\}$ be any function and $\Gamma$ be a set of parities. If for all $b \in \{-1,1\}^\Gamma$ the restricted function $f|_{(\Gamma, b)}$ is constant then $r(f) \leq |\Gamma|$.

Recall that for any function $f : \{-1,1\}^n \to \{-1,1\}$ and any real $t > 0$, we define $S_t := \{S \subseteq [n] : |\hat{f}(S)| \geq 1/t\}$ (we suppress the dependence of $S_t$ on $f$ as the underlying function will be clear from context).

Definition 4.26. Let $f : \{-1,1\}^n \to \{-1,1\}$ be any function. Define the Fourier max-supp-entropy of $f$, denoted $k'(f)$, by
\[
k'(f) := \argmin_{t} \{S_t = \text{supp}(f)\}.
\]
Equivalently,
\[ k'(f) := \max_{S \in \text{supp}(f)} \left\{ \frac{1}{|\hat{f}(S)|} \right\}. \]

Define the Fourier max-rank-entropy of \( f \), denoted \( k''(f) \), by
\[ k''(f) := \arg\min_t \{ \dim(S_t) = r(f) \}. \]

We often refer to the Fourier max-supp-entropy and Fourier max-rank-entropy as simply max-supp-entropy and max-rank-entropy, respectively.

**Lemma 4.27** (Relationships between parameters). Let \( f : \{-1,1\}^n \to \{-1,1\} \) be any function. Then the following inequalities hold.

1. \( \log k(f) \leq r(f) = O(\sqrt{k(f)} \log k(f)) \).
2. \( \sqrt{k(f)} \leq k'(f) \leq k(f)/2 \).
3. \( \max \left\{ \sqrt{r(f)}, r(f)/(4 \log k(f)) \right\} \leq k''(f) \leq k'(f) \).

**Proof.**

1. The first inequality holds since \( k(f) \leq 2^{r(f)} \), and the second inequality follows from Theorem 4.24.

2. Recall from Definition 4.26 that \( k'(f) = \arg\min_t \{ \dim(S_t) = r(f) \} \). This means for all \( S \in \text{supp}(f) \), \( |\hat{f}(S)| \geq \frac{1}{k'(f)} \). We have from Parseval’s identity (Theorem 4.6) that
\[ \sum_{S \subseteq [n]} \hat{f}(S)^2 = 1 \implies k(f)/k'(f)^2 \leq 1 \implies \sqrt{k(f)} \leq k'(f). \]

By Lemma 4.8,
\[ \frac{1}{k'(f)} \geq 2^{1-\log k(f)} = \frac{2}{2^{\log k(f)}} \geq \frac{2}{k(f)}. \]

3. Recall from Definition 4.26 that \( k''(f) = \arg\min_t \{ \dim(S_t) = r(f) \} \). Observe that for \( t = k'(f) \), we have \( \dim(S_t) = \dim(\text{supp}(f)) = r(f) \). Hence \( k''(f) \leq k'(f) \).

Since \( \text{rank}(S_{k''(f)}) = r(f) \), there exists \( \mathcal{B} \subseteq S_{k''(f)} \) such that \( |\mathcal{B}| = r(f) \) and \( \mathcal{B} \) is a basis of \( \text{span}(\text{supp}(f)) \). Moreover \( |\hat{f}(S)| \geq 1/k''(f) \) for all \( S \in \mathcal{B} \). Choose such a set \( \mathcal{B} \).

By Theorem 4.6,
\[ 1 \geq \sum_{S \in \mathcal{B}} \hat{f}(S)^2 \geq \frac{r(f)}{(k''(f))^2} \implies k''(f) \geq \sqrt{r(f)}. \]

By Corollary 4.11,
\[ \sum_{S \in \mathcal{B}} |\hat{f}(S)| \leq 4 \log k(f) \]
\[
\Rightarrow \frac{r(f)}{k''(f)} \leq 4 \log k(f)
\]
\[
\Rightarrow k''(f) \geq r(f)/(4 \log k(f)).
\]

Therefore \(k''(f) \geq \max \left\{ \sqrt{r(f)}, r(f)/(4 \log k(f)) \right\} \).

\[\square\]

**Claim 4.28.** Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) a function with \( k(f) \geq 2 \). Then

\[
\delta(f) = \Omega \left( \frac{k(f)}{k'(f)^2} \right).
\]

**Proof.** Recall from Definition 4.26 that \( k'(f) = \arg\min_t \{S_t = \text{supp}(f)\} \). This means for all \( S \in \text{supp}(f) \), \( |\hat{f}(S)| \geq \frac{1}{k'(f)} \). Therefore

\[
(k(f) - 1)/(k'(f))^2 \leq \sum_{S \subseteq [n], S \neq \emptyset} \hat{f}(S)^2
\]
\[
= 1 - \hat{f}(\emptyset)^2
\]
\[
= 1 - (1 - 2\delta(f))^2
\]
\[
= 4\delta(f) - 4\delta(f)^2 \leq 4\delta(f)
\]
\[
\Rightarrow \delta(f) \geq \frac{(k(f) - 1)}{4(k'(f))^2} \geq \frac{k(f)}{8(k'(f))^2},
\]

since \( k(f) \geq 2 \)

\[\square\]

**Claim 4.29.** Let \( f : \{-1, 1\}^n \to \{-1, 1\} \) be any function. Then

\[
\|\hat{f}\|_1 \leq 3 \sqrt{k(f)\delta(f)}.
\]

**Proof.** Since \( \hat{f}(\emptyset) = 1 - 2\delta(f) \) we have \( \|\hat{f}\|_1 = |1 - 2\delta(f)| + \sum_{S \neq \emptyset} |\hat{f}(S)| \). The term \( \sum_{S \neq \emptyset} |\hat{f}(S)| \) can be bounded as follows:

\[
\left( \sum_{S \neq \emptyset} |\hat{f}(S)| \right)^2 \leq k(f) \left( \sum_{S \neq \emptyset} \hat{f}(S)^2 \right)
\]
\[
= k(f)(1 - (1 - 2\delta(f))^2) \leq 4k(f)\delta(f).
\]

Thus we have,

\[
\|\hat{f}\|_1 = |1 - 2\delta(f)| + \sum_{S \neq \emptyset} |\hat{f}(S)|
\]
\[
\leq |1 - 2\delta(f)| + 2\sqrt{k(f)\delta(f)}
\]
\[
\leq 1 + 2\sqrt{k(f)\delta(f)} \leq 3\sqrt{k(f)\delta(f)},
\]

since \( k(f)\delta(f) \geq 1 \) by Lemma 4.7

\[\square\]
We require the following observation about the rank, sparsity, max-supp-entropy, max-rank-entropy and weight of \( \text{AND}, \text{AD}_t \) and bent functions, which follows immediately from definitions and first principles. We omit its proof.

**Observation 4.30.** Let \( t \geq 2 \) and \( \ell, t' \geq 4 \) be any positive integers, and let \( B_\ell : \{-1, 1\}^{\log \ell} \to \{-1, 1\} \) be any bent function. Then the rank, sparsity, max-supp-entropy, max-rank-entropy and weight of \( \text{AND}_{\log t'}, B_\ell \) and \( \text{AD}_t \) are as in the following table.

| \( f \) | \( r(f) \) | \( k(f) \) | \( k'(f) \) | \( k''(f) \) | \( \delta(f) \) |
|-------|-------|------|-----|------|-------|
| \( \text{AND}_{\log t'} \) | \( \log t' \) | \( t' \) | \( t'/2 \) | \( t'/2 \) | \( \frac{1}{t} \) |
| \( B_\ell \) | \( \log \ell \) | \( \ell \) | \( \sqrt{\ell} \) | \( \sqrt{\ell} \) | \( \frac{1}{2} \pm \frac{1}{2\sqrt{t}} \) |
| \( \text{AD}_t \) | \( t + \log t \) | \( t^2 \) | \( t \) | \( t \) | \( \frac{1}{t} \) |

5 Lower bound proofs

For lower bounds on \( \delta(f) \) of a Boolean function \( f \), we need to prove two theorems: Theorems 1.4 and 1.6. The proof of Theorem 1.4 is given in Section 5.1 and the proof of Theorem 1.6 is given in Section 5.2.

5.1 Proof of Theorem 1.4 (and Theorem 1.3)

Remember that we defined the *Fourier max-supp-entropy* of a Boolean function \( f \), denoted by \( k'(f) \), to be

\[
\max_{S \in \text{supp}(f)} \frac{1}{|\hat{f}(S)|}.
\]

The main aim of this section is to give a lower bound on \( \delta(f) \) with respect to \( k'(f) \) for a Boolean function \( f \) (Theorem 1.4).

We first prove Theorem 1.3 which implies Theorem 1.4 (together with Claim 4.28). See Section 2.1 for an overview of the proof of Theorem 1.3.

Theorem 1.3 can be viewed as an upper bound of \( O(\sqrt{k(f)\delta(f)} \log k(f)) \) on the Fourier rank of \( f \). In order to prove Theorem 1.3, we give an algorithm (Algorithm 1) which takes a Boolean function \( f \) as input and outputs a set of \( O(\sqrt{\delta(f)k(f)} \log k(f)) \) parities such that any assignment of these parities makes the function constant. From Observation 4.25, this implies an upper bound of \( O(\sqrt{\delta(f)k(f)} \log k(f)) \) on Fourier rank of the function. We start by formally describing this algorithm.

Recall that for a function \( f : \{-1, 1\}^n \to \{-1, 1\} \), a set of parities \( \Gamma \) and an assignment \( b \in \{-1, 1\}^\Gamma \), we define the restriction

\[
f|_{(\Gamma,b)} := f|_{\{x \in \{-1, 1\}^n : \chi_\gamma(x) = b_\gamma \text{ for all } \gamma \in \Gamma\}}.
\]
Also let $\mathcal{B}_\Gamma := \{b \in \{-1,1\}^\Gamma : f|_{(\Gamma,b)} \text{ is not constant}\}$.

Algorithm 1:

**Input:** A function $f : \{-1,1\}^n \to \{-1,1\}$.

**Output:** A set $\Gamma$ of parities whose evaluation determines $f$.

**Initialization:** $f_{\text{min}} \leftarrow f$, $\Gamma \leftarrow \emptyset$.

**while $\mathcal{B}_\Gamma$ is non-empty do**

(a) **Update $\Gamma$:** Let $\Gamma'$ be the smallest set of parities, such that, there exists $b \in \{-1,1\}^\Gamma$ for which $f_{\text{min}}|_{(\Gamma',b)}$ is constant, $\Gamma \leftarrow \Gamma \cup \Gamma'$.

(b) **Update $f_{\text{min}}$:** Define $b^* := \text{argmin}_{b \in \mathcal{B}_\Gamma} \left\{ \frac{\delta(f|_{(\Gamma,b)})}{k(f|_{(\Gamma,b)})} \right\}$, and update $f_{\text{min}} \leftarrow f|_{(\Gamma,b^*)}$.

**end**

Return $\Gamma$.

Since number of parities are finite and we fix at least one parity at each iteration of Step (a) of the while loop, the algorithm terminates. The termination condition implies that the algorithm outputs a set of parities $\Gamma$ such that for any assignment $b \in \{-1,1\}^\Gamma$ of $\Gamma$, the restricted function $f|_{(\Gamma,b)}$ becomes constant.

The only remaining step is to show that the number of parities fixed in Algorithm 1 is $O(\sqrt{\delta(f)k(f) \log k(f)})$. For this we first need to recall the notion of equivalence relation defined in Section 2.1 and few properties of restricted functions (restricted according to an assignment of a set of parities).

**Equivalence relation for a set of parities** Let $f$ be the input to Algorithm 1, first we define an equivalence relation given a set of parities over the variables of $f$. Given a set of parities $\Gamma$, define the following equivalence relation among parities in $\text{supp}(f)$.

$$\forall \gamma_1, \gamma_2 \in \text{supp}(f), \gamma_1 \equiv \gamma_2 \text{ iff } \gamma_1 + \gamma_2 \in \text{span}(\Gamma).$$

(6)

Let $\ell$ be the number of equivalence classes according to the equivalence relation for $\Gamma$. For $j \in [\ell]$, let $k_j$ be the size of the $j$-th equivalence class. Since the equivalence classes form a partition of $\text{supp}(f)$, we have

**Observation 5.1.** Following the notation of the paragraph above, $\sum_{j=1}^{\ell} k_j = k(f)$.

Let $\beta_1, \ldots, \beta_\ell \in \text{supp}(f)$ be some representatives of the equivalence classes. For $j \in [\ell]$, let $\beta_j + \alpha_{j,1}, \ldots, \beta_j + \alpha_{j,k_j}$ be the elements of the $j$-th equivalence class. This notation gives a compact representation of $f$ in terms of these equivalence classes. For all $x \in \{-1,1\}^n$,

$$f(x) = \sum_{j=1}^{\ell} P_j(x) \chi_{\beta_j}(x),$$

(7)

where

$$P_j(x) = \sum_{r=1}^{k_j} \hat{f}(\beta_j + \alpha_{j,r}) \cdot \chi_{\alpha_{j,r}}(x).$$

(8)
Note that $P_j$ are non-zero multilinear polynomials and depend only on the parities in $\Gamma$. So, fixing parities in $\Gamma$ collapses all the parities in an equivalence class to their representative, thereby making $P_j$’s constant.

We will denote $\Gamma$ after the $i$-th iteration of the while loop by $\Gamma^{(i)}$ (so $\Gamma^{(0)} = \emptyset$). Let $f^{(i)}_{\min}$ be the selected function $f_{\min}$ after the $i$-th iteration (thus $f^{(0)}_{\min} = f$).

With the above properties of restricted functions we are ready to prove the main technical lemma needed to show Theorem 1.3.

**Lemma 5.2.** Let $f : \{-1,1\}^n \to \{-1,1\}$ a function. Suppose $\Gamma$ be a set of parities and $\ell$ be the number of equivalence classes of $\text{supp}(f)$ under the equivalence relation defined by in Equation (6), Then, there exists a $b \in \{-1,1\}^\Gamma$ such that $f|_{(\Gamma,b)}$ is non-constant and

$$\frac{\delta(f|_{(\Gamma,b)})}{k(f|_{(\Gamma,b)})} \leq \frac{4k(f)\delta(f)}{\ell^2}.$$  

**Proof.** For the sake of succinctness, when $\Gamma$ is clear from the context, let $V_b = \{x \in \{-1,1\}^n : \forall \gamma \in \Gamma, x_\gamma = b_\gamma\}$, for all $b \in \{-1,1\}^\Gamma$, and $f|_b = f|_{\{x : x \in V_b\}}$.

Since we are interested in a non-constant $f|_b$, define $k_{(\emptyset)}(f)$ to be the number of non-zero non-empty monomials in Fourier representation of $f$. We first need to prove the following two bounds on the expected values of $\delta(f|_b)$ and $k_{(\emptyset)}(f|_b)$.

- $\mathbb{E}_b[\delta(f|_b)] = \delta(f)$,
- $\mathbb{E}_b[k_{(\emptyset)}(f|_b)] \geq \frac{\ell^2}{4\delta(f)}$.

**Expected value of $\delta(f|_b)$:** Since $\{V_b : b \in \{-1,1\}^\Gamma\}$ form a partition on $\{-1,1\}^n$ and all partitions are of the same size, we get the expected value of $\delta(f|_b)$.

$$\mathbb{E}_b[\delta(f|_b)] = \delta(f). \tag{9}$$

**Expected value of $k_{(\emptyset)}(f|_b)$:** From Equation (7), for all $b \in \{-1,1\}^\Gamma$ and for all $x \in \{-1,1\}^n$,

$$f|_b(x) = \sum_{j=1}^{\ell} P_j(b)\chi_{\beta_j}(x). \tag{10}$$

For each $j \in [\ell]$ and $b \in \{-1,1\}^\Gamma$, let $I_j(b)$ be the indicator function for $P_j(b) \neq 0$,

$$I_j(b) = \begin{cases} 1 & \text{if } P_j(b) \neq 0 \\ 0 & \text{otherwise}. \end{cases}$$

From Equation (8), each $P_j$ is a polynomial having monomials $\{\chi_{\alpha_{j,r}} : r \in [k_j]\}$ with Fourier sparsity of $P_j$ being equal to $k_j$. Since each $P_j$ is a non-zero polynomial, by Lemma 4.7

$$\mathbb{E}_b[I_j(b)] = \text{Pr}_{b \sim \{-1,1\}^{\Gamma}}[P_j(b) \neq 0] \geq \frac{1}{k_j}. \tag{11}$$

20
We calculate the expectation of $k_{\emptyset}^{\epsilon}(f|b)$.

$$
\mathbb{E}_b \left[ k_{\emptyset}^{\epsilon}(f|b) \right] = \mathbb{E}_b \left[ \sum_{j=1}^{\ell-1} I_j(b) \right] \\
= \sum_{j=1}^{\ell-1} \mathbb{E}_b \left[ I_j(b) \right] \quad \text{by linear expectation}
$$

$$
\geq \sum_{j=1}^{\ell-1} \frac{1}{k_j} \quad \text{by Equation (11)}
$$

$$
\geq \frac{(\ell - 1)^2}{\ell \sum_{j=1}^{\ell-1} k_j} \quad \text{by Cauchy-Schwarz inequality}
$$

$$
\geq \frac{\ell^2}{4k(f)} \quad \text{by Observation 5.1}
$$

To finish the proof of the theorem, we use bounds on the two expected values,\footnote{this part of our proof is inspired by a proof of the Cheeger’s inequality in spectral graph theory. See, for example, the proof of Fact 2 in https://people.eecs.berkeley.edu/~luca/expanders2016/lecture04.pdf.}

$$
\frac{\mathbb{E}_b \left[ \delta(f|b) \right]}{\mathbb{E}_b \left[ k_{\emptyset}^{\epsilon}(f|b) \right]} \leq \frac{4k(f)\delta(f)}{\ell^2}
$$

$$
\iff \mathbb{E}_b \left[ \delta(f|V_b) - \frac{4k(f)\delta(f)}{\ell^2} k_{\emptyset}^{\epsilon}(f|V_b) \right] \leq 0. \quad \text{by linearity of expectation}
$$

If $\delta(f|V_b) - \frac{4k(f)\delta(f)}{\ell^2} k_{\emptyset}^{\epsilon}(f|V_b) = 0$ for all $b$, then pick any non-constant $f|b$. Otherwise, there exists a $b_0$ such that

$$
\delta(f|V_{b_0}) - \frac{4k(f)\delta(f)}{\ell^2} k_{\emptyset}^{\epsilon}(f|V_{b_0}) < 0.
$$

Since this equation can only be satisfied when $k_{\emptyset}^{\epsilon}(f|V_{b_0}) > 0$, $f|V_{b_0}$ is not constant. Dividing by $k_{\emptyset}^{\epsilon}(f|V_{b_0})$,

$$
\frac{\delta(f|b_0)}{k(f|b_0)} \leq \frac{\delta(f|b_0)}{k_{\emptyset}^{\epsilon}(f|b_0)} \leq \frac{4k(f)\delta(f)}{\ell^2},
$$

and $f|b_0$ is non-constant.

Lemma 5.2 allows us to bound the number of parities fixed in the $i$-th iteration (in terms of the decrease in number of equivalence classes).

**Lemma 5.3.** Suppose $f$ is given as input to Algorithm 1. Consider the $i$-th iteration of Algorithm 1. Let $q_i$ be the be number of parities fixed in Step (a) of the $i$-th iteration of the while loop, and $\ell_i$ be the number of equivalence classes after Step (a) of the $i$-th iteration. Then

$$
\frac{q_i}{(\ell_{i-1} - \ell_i)} \leq \frac{6\sqrt{\delta(f)k(f)}}{\ell_{i-1}}.
$$
Proof. Recall that $\Gamma = \Gamma^{(i)}$ after the $i$-th of Step (a) of Algorithm 1. Again, for the sake of succinctness, let $V_b = \{ x \in \{-1, 1\}^n : \forall \gamma \in \Gamma^{(i)}, x_\gamma = b_\gamma \}$, for all $b \in \{-1, 1\}^{\Gamma^{(i)}}$, and $f|_b = f|_{\{x : x \in V_b\}}$.

Let $f_{\text{min}}$ be the function chosen after the $i$-th iteration of Step (b) of Algorithm 1. Since Step (b) of Algorithm 1 chooses $f_{\text{min}}$ to be a non-constant function such that weight-to-sparsity ratio is minimized, from Lemma 5.2 we have,

$$\frac{\delta(f_{\text{min}})}{k(f_{\text{min}})} \leq \frac{4k(f)\delta(f)}{\ell_{i-1}^2}. \tag{12}$$

Write every $f|_b$ as in Equation (7), and define $S^{(i)} := \bigcup_{b \in \{-1, 1\}^{\Gamma^{(i)}}} \text{supp}(f|_b)$. We now prove that $|S^{(i)}| = \ell_i$.

- $|S^{(i)}| \leq \ell_i$: Follows from the representation in Equation (7), since each $\text{supp}(f|_b)$ is a subset of $\{\chi_{b(j)}^{(i)} \mid j \in [\ell_i]\}$.

- $|S^{(i)}| \geq \ell_i$: Since $P_j^{(i)}$ is a non-zero polynomial, there exists an assignment to parities in $\Gamma^{(i)}$, such that, $P_j^{(i)}$ is non-zero. Thus, for all $j \in [\ell_i]$, we have $\chi_{b(j)}^{(i)} \in S^{(i)}$.

Since $|S^{(i)}| = \ell_i$, Lemma 2.1 guarantees that $q_i \leq 3\sqrt{k(f_{\text{min}})\delta(f_{\text{min}})}$. Since $f_{\text{min}}$ becomes constant after fixing these $q_i$ parities, every parity in $\text{supp}(f_{\text{min}})$ is paired with at least one other parity in $\text{supp}(f_{\text{min}})$ for the equivalence class with respect to $\Gamma^{(i)}$.\footnote{There is a boundary case ($k(f) = 1$) which can be dealt with separately, as in [San19, Lemma 3.4]. For readability, we assume $k(f) \geq 2$.} This implies that $\ell_{i-1} - \ell_i \geq \frac{k(f_{\text{min}})}{2}$. Combining the two inequalities in the last paragraph we have,

$$\frac{q_i}{(\ell_{i-1} - \ell_i)} \leq 6\sqrt{\frac{\delta(f_{\text{min}})}{k(f_{\text{min}})}}.$$  

From Equation (12),

$$\frac{q_i}{(\ell_{i-1} - \ell_i)} \leq \frac{6\sqrt{\delta(f)k(f)}}{\ell_{i-1}}. \tag{13}$$  

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** We only need to show that the parities fixed in Algorithm 1 is $O(\sqrt{\delta(f)k(f)} \log k(f))$ (Observation 4.25). Suppose the while loop runs for $t$ iterations. Let $q_i$ be the number of queries made in Step (a) of Algorithm 1 in the $i$-th iteration. From Lemma 5.2, we have

$$q_i \leq \frac{6\sqrt{\delta(f)k(f)}}{\ell_{i-1}}(\ell_{i-1} - \ell_i)$$
Thus when Algorithm 1 is run of $f$, the total number of queries made by the algorithm is

$$\sum_{i=1}^{t} q_i \leq 6\sqrt{\delta(f)k(f)} \sum_{i=1}^{t} \left(\frac{1}{\ell_{i-1}} + \frac{1}{\ell_{i-1} - 1} + \cdots + \frac{1}{\ell_i} + 1\right)$$

$$\leq 6\sqrt{\delta(f)k(f)} \ell_0 \sum_{i=1}^{t} \frac{1}{i}$$

$$\leq 6\sqrt{\delta(f)k(f)} \log \ell_0$$

$$= 6\sqrt{\delta(f)k(f)} \log k(f).$$

Observation 4.25 implies $r(f) = O(\sqrt{\delta(f)k(f)} \log k(f))$. \hfill \Box

Along with Theorem 1.3, this proves Theorem 1.4.

Proof of Theorem 1.4. The bound $\delta(f) = \Omega\left(\frac{r(f)}{\log k(f)}\right)^2$ follows from Theorem 1.3 and the bound $\delta(f) = \Omega\left(\frac{k(f)}{\log k(f)}\right)$ from Claim 4.28. \hfill \Box

5.2 Proof of Theorem 1.6

Recall from Definition 4.26 that we defined max-rank-entropy of a Boolean function $f$, denoted by $k''(f)$, to be

$$\arg\min_{t} \{\dim(S_t)\} = r(f).$$

The main aim of this section is to give a lower bound on $\delta(f)$ with respect to $k''(f)$ for a Boolean function $f$ (Theorem 1.6). The second bound of Theorem 1.6 is given by the following lemma.

Lemma 5.4. Let $f : \{-1,1\}^n \to \{-1,1\}$ be any function such that $k(f) > 1$. Then,

$$\delta(f) = \Omega\left(\frac{r(f)}{k''(f) \log k(f)}\right).$$

Together with Theorem 1.3 proved in Section 5.1, Lemma 5.4 implies Theorem 1.6. We now give the proof of Lemma 5.4. See Section 2.2 for an overview of the proof of Lemma 5.4.

Lemma 5.4 gives a lower bound of $\Omega\left(\frac{r(f)}{k''(f) \log k(f)}\right)$ on $\delta(f)$. The crucial ingredient for this lower bound is Lemma 2.2, which is a refinement of the following theorem.

Theorem 5.5 ([CHLT19, Theorem 13]). Let $f : \{-1,1\}^n \to \{-1,1\}$ be any function such that $\deg_{F_2}(f) = d$. Then,

$$\sum_{i \in [n]} |\hat{f}(\{i\})| \leq 4d.$$

The only difference in the statement of Lemma 2.2 and Theorem 5.5 is that the right hand side becomes $O(\delta(f) \cdot \deg_{F_2}(f))$ instead of $4\deg_{F_2}(f)$.
Proof of Lemma 2.2. Assume \( \delta(f) \leq 1/4 \) (otherwise Theorem 5.5 implies \( \sum_{i=1}^{n} |\hat{f}(i)| = O(\delta(f)d) \)). Define \( F : \{-1,1\}^{nt} \rightarrow \{-1,1\} \) to be
\[
F(x^{(1)}, \ldots, x^{(t)}) = f(x^{(1)}) \times \ldots \times f(x^{(t)}),
\]
where \( t \) is a parameter to be fixed later, and \( x^{(i)} \in \{-1,1\}^{n} \) for all \( i \in [t] \). Since \( \deg_{\mathbb{F}_2}(F) = \deg_{\mathbb{F}_2}(f) \), Theorem 5.5 implies
\[
\sum_{S \subseteq [nt], |S| = 1} |\hat{F}(S)| = O(d). \tag{14}
\]
Since \((1 - x)^{1/x} \) is a decreasing function in \( x \) for \( x \in (0,1/2] \), we have
\[
(1 - x)^{1/x} \geq 1/4 \quad \text{for all } x \in (0,1/2]. \tag{15}
\]
Expressing the Fourier coefficients of \( F \) in terms of the Fourier coefficients of \( f \),
\[
\sum_{S \subseteq [nt], |S| = 1} |\hat{F}(S)| = t \cdot |\hat{f}(\emptyset)| - 1 \sum_{i=1}^{n} |\hat{f}(i)|
\]
\[
= \left(1 + \frac{1}{2\delta(f)}\right) \cdot (1 - 2\delta(f))^{\frac{1}{2\delta(f)}} \sum_{i=1}^{n} |\hat{f}(i)|
\]
Choosing \( t = 1 + \frac{1}{2\delta(f)} \), and by Observation 4.2
\[
\geq \left(1 + \frac{1}{2\delta(f)}\right) \cdot \left(\frac{1}{4}\right) \sum_{i=1}^{n} |\hat{f}(i)|
\]
by Equation (15)
\[
\geq \frac{1}{8\delta(f)} \cdot \sum_{i=1}^{n} |\hat{f}(i)|.
\]
Now, Equation (14) implies the desired bound, \( \sum_{i=1}^{n} |\hat{f}(i)| = O(\delta(f)d) \). \( \square \)

We would like to extend the upper bound of Lemma 2.2 to any basis of \( \text{span}(\text{supp}(f)) \) instead of just the standard basis of the set of parities.

Corollary 5.6. Let \( f : \{-1,1\}^{n} \rightarrow \{-1,1\} \) be any function with \( \deg_{\mathbb{F}_2}(f) = d \). Suppose \( S \subseteq \text{supp}(f) \) is a basis of \( \text{span}(\text{supp}(f)) \), then
\[
\sum_{S \in \mathcal{S}} |\hat{f}(S)| = O(\delta(f)d) = O(\delta(f) \log k(f)).
\]

Proof. The main idea of the proof is to do a basis change on parities and construct another function \( h \), the corollary will follow by applying Lemma 2.2 on \( h \).

Recall that we denote both a subset of \([n]\) and the corresponding indicator vector in \( \mathbb{F}_2^n \), by the same notation.

Let \( S = \{S_1, \ldots, S_{r(f)}\} \), extend \( S \) to \( S' = \{S_1, \ldots, S_{r(f)}, S_{r(f)+1}, \ldots, S_n\} \), a complete basis of \( \mathbb{F}_2^n \). Observe that \( \hat{f}(S_i) = 0 \), for \( i \in \{r(f) + 1, \ldots, n\} \) (since \( S \) spans \( \text{supp}(f) \)). Fix the change of basis matrix \( B \in \mathbb{F}_2^{n \times n} \) with \( i \)-th column as \( S_i, i \in [n] \).
Consider the function $h : \{-1, 1\}^n \to \mathbb{R}$ satisfying $\hat{h}(\alpha) = \hat{f}(B\alpha)$, for all $\alpha \in \mathbb{F}_2^n$. By Claim 4.10, $h$ is Boolean and $\deg_{\mathbb{F}_2}(h) = \deg_{\mathbb{F}_2}(f)$. Using Lemma 2.2,
\[ \sum_{i \in [n]} |\hat{h}(\{i\})| = O(\delta(f)d). \]
From the definition of $h$, $\hat{h}(e_i) = \hat{f}(S_i)$ for $i \in [r(f)]$ and $\hat{h}(e_i) = 0$ for $i \in \{r(f) + 1, \ldots, n\}$.
\[ \sum_{S \in S} |\hat{f}(S)| = O(\delta(f)d). \]
The second equality in the statement of the lemma follows from Lemma 4.9.

**Proof of Lemma 5.4.** Observe that every summand on the left hand side of Corollary 5.6 is at least $1/k''(f)$, giving the following lower bound on $\delta(f)$ and finishing the proof of Lemma 5.4.

**Proof of Theorem 1.6.** From Lemma 5.4 we have $\delta(f) = \Omega\left(\frac{r(f)^2}{k'(f) \log k(f)}\right)$, and from Theorem 1.3 we have $\delta(f) = \Omega\left(\frac{r(f)^2}{k''(f) \log k(f)}\right)$.

The following corollary combines the lower bounds on $\delta(f)$ from Theorem 1.6 and Lemma 1.1 by setting $k''(f)$ as the threshold.

**Corollary 5.7.** Let $f : \{-1, 1\}^n \to \{-1, 1\}$ be any function such that $k(f) > 1$. Then,
\[ \delta(f) = \Omega\left(\max\left\{\frac{r(f)^2}{k'(f) \log k(f)}, \frac{r(f)}{k''(f) \log k(f)}, \frac{\sqrt{r(f)}}{k''(f) \log (k''(f)^2 / r(f))}\right\}\right). \]

### 6 Upper bound proofs

In this section we prove Theorems 1.5 and 1.7. Recall that these theorems require us to exhibit functions $f$ witnessing certain upper bounds on $\delta(f)$. The descriptions of these functions are given in Section 6.1. In Section 6.2 we compute certain properties of interest of these functions. Finally in Section 6.3 we instantiate these functions with suitable parameters to yield the proofs of Theorems 1.5 and 1.7.

#### 6.1 Defining some functions

The functions we consider are all modifications of the Addressing function defined in Definition 4.19. The main technique we use to define our functions is given in Definition 4.22. That is, we first consider an Addressing function on $t + \log t$ input bits. Next, we replace each target bit by a suitable function. Different choices of the various functions substituted yield our upper bounds.

In our first modification, we replace the target bits by AND functions on disjoint variables, each having the same arity.

**Definition 6.1 (AND-Target-Addressing Function).** For any integers $t, t' \geq 2$, define the function $AD_{t,t'} : \{-1, 1\}^{\log t} \times \{-1, 1\}^{\log t'} \to \{-1, 1\}$ by $AD_{t,t'} = AD_t \circ_{\text{target}} AND_{\log t'}$. 

25
Our next modification is similar to the previous one, except for the fact that one of the AND functions used in the replacement above has larger arity than the others.

**Definition 6.2 (AND-Target-Addressing Function with a Huge AND).** For any integers \( t \geq 2 \) and \( a \geq t' \geq 2 \), define the function \( \text{AD}_{t,t',a} : \{-1,1\}^{\log t} \times \{-1,1\}^{\log a} \times \{-1,1\}^{(t-1) \log t'} \rightarrow \{-1,1\} \) by

\[
\text{AD}_{t,t',a} = \text{AD}_{t} \circ_{\text{target}} (\text{AND}_{\log a}, \text{AND}_{\log t'}, \ldots, \text{AND}_{\log t}).
\]

That is, for all \((x, z) \in \{-1,1\}^{\log t} \times \{-1,1\}^{\log a} \times \{-1,1\}^{(t-1) \log t'}\), where \( z_{1 \log t} \in \{-1,1\}^{\log a} \) and \( z_{b} \in \{-1,1\}^{\log t'} \) for all \( 1^{\log t} \neq b \in \{-1,1\}^{\log t}, \)

\[
\text{AD}_{t,t',a}(x, z) = \begin{cases} 
\text{AND}(z_{1 \log t}, \ldots, z_{1 \log t', \log a}) & \text{if } x = 1^{\log t} \\
\text{AND}(z_{x}, \ldots, z_{x, \log t'}) & \text{otherwise.}
\end{cases}
\]

We require the following function to define our next modification.

**Definition 6.3 (AND-of-Bent).** For any integers \( t', \ell \geq 2 \), let \( B : \{-1,1\}^{\log \ell} \rightarrow \{-1,1\} \) be a bent function on \( \log \ell \) input bits. Define the function \( \text{AB}_{t',\ell} : \{-1,1\}^{\log t'} \times \{-1,1\}^{\log \ell} \rightarrow \{-1,1\} \) by

\[
\text{AB}_{t',\ell}(y, z) = \text{AND}(y_{1} B(z), y_{2}, \ldots, y_{\log t'}),
\]

where \( y \in \{-1,1\}^{\log t'} \), \( z \in \{-1,1\}^{\log \ell} \).

In the next modification, we replace each target bit by the function \( \text{AB}_{t',\ell} \) on \( \log t' + \log \ell \) input bits, as in Definition 6.3.

**Definition 6.4 ((AND-of-Bent)-Target-Addressing Function).** For any integers \( t, t', \ell \geq 2 \), define the function \( \text{AAB}_{t,t',\ell} : \{-1,1\}^{\log t} \times \{-1,1\}^{(t \log \ell + \log t')} \rightarrow \{-1,1\} \) by

\[
\text{AAB}_{t,t',\ell} = \text{AD}_{t} \circ_{\text{target}} \text{AB}_{t',\ell}.
\]

We define an auxiliary function, which is a modification of the AND function where the first variable is replaced by that variable times another AND on a disjoint set of variables.

**Definition 6.5 (Modified AND).** For any integers \( t' \geq 2, p \geq 1 \), define the function \( \text{mAND}_{t',p} : \{-1,1\}^{\log t' + p} \rightarrow \{-1,1\} \) by

\[
\text{mAND}_{t',p}(y, u) = \text{AND}_{\log t'}(y_{1} \text{AND}_{p}(u), y_{2}, y_{3}, \ldots, y_{\log t'}),
\]

where \( y \in \{-1,1\}^{\log t'} \) and \( u \in \{-1,1\}^{p} \).

In the next modification we replace one of the variables in the first block of \( \text{AD}_{t,t'} \) (where the variables in the first block refer to those variables on which \( \text{AND}_{\log t} \) is evaluated when the addressing variables equal \( 1^{\log t} \)) with that variable times the AND of some \( p \) variables from the the other blocks.

**Definition 6.6 (Modified \( \text{AD}_{t,t'} \) with Modified AND).** Let \( t, t' \geq 2 \) be any integers and let \( p \) be an integer such that \( (t-1)(\log t') \geq p \geq 1 \). Let \( x \in \{-1,1\}^{\log t} \), for each \( b \in \{-1,1\}^{\log t} \), let \( y_{b} \in \{-1,1\}^{\log t'} \). Let \( u = \{y_{b}, b \in \{-1,1\}^{\log t} \setminus \{1^{\log t}\}, i \in \{\log t'\}\} \). Fix an arbitrary ordering on the variables in \( u \) and let \( u_{\leq p} \) be the the first \( p \) variables in \( u \) according to that order. Define the function \( \text{mAD}_{t,t',p} : \{-1,1\}^{\log t + t(\log t')} \rightarrow \{-1,1\} \) by

\[
\text{mAD}_{t,t',p}(x, y) = \begin{cases} 
\text{mAND}_{t',p}(y_{1 \log t}, u_{\leq p}) & \text{if } x = 1^{\log t} \\
\text{AND}_{\log t'}(y_{x}) & \text{otherwise.}
\end{cases}
\]

26
The table in the following claim summarizes various properties of interest of the functions defined above: rank, sparsity, max-supp-entropy, max-rank-entropy and weight. The first row is used to show that our lower bounds on weight beat those obtained from Chang’s lemma (Lemma 1.1) for the function \( \text{AD}_{t,t'} \), no matter what threshold is chosen (Claim 7.1).

The second and third rows of the table will be crucial to prove Theorem 1.5 (Claims 6.15 and 6.16). The second and the last row of the table are required to prove Theorem 1.7 (Claims 6.17 and 6.18).

**Claim 6.7.** The rank, sparsity, max-supp-entropy, max-rank-entropy and weight of the functions \( \text{AD}_{t,t'}, \text{AD}_{t,t',a}, \text{AAB}_{t,t',\ell} \) and \( \text{mAD}_{t,t',p} \) are as follows.\(^8\)

| \( f \) | \( r(f) \) | \( k(f) \) | \( k'(f) \) | \( k''(f) \) | \( \delta(f) \) |
|---|---|---|---|---|---|
| \( \text{AD}_{t,t'} \) | \( \Theta(t \log t') \) | \( \Theta(t^2t') \) | \( \Theta(tt') \) | \( \Theta(tt') \) | \( \frac{1}{t} \) |
| \( \text{AD}_{t,t',a} \) | \( \Theta(t \log t' + \log a) \) | \( \Theta(t^2t' + ta) \) | \( \Theta(at) \) | \( \Theta(at) \) | \( \frac{1}{t} + \frac{1}{at} - \frac{1}{tt'} \) |
| \( \text{AAB}_{t,t',\ell} \) | \( \Theta(t(log t' + \log \ell)) \) | \( \Theta(t^2\ell') \) | \( \Theta(tt'\sqrt{\ell}) \) | \( \Theta(tt'\sqrt{\ell}) \) | \( \frac{1}{t} \) |
| \( \text{mAD}_{t,t',p} \) | \( \Theta (t \log t') \) | \( \Theta(2^ptt' + t^2t') \) | \( \Theta(2^ptt') \) | \( \Theta(tt') \) | \( \frac{1}{t} \) |

### 6.2 Proof of properties of our constructed functions

In this section, we prove Claim 6.7 by computing the properties of interest i.e., rank, sparsity, max-supp-entropy, max-rank-entropy and weight for each of the functions \( \text{AD}_{t,t'}, \text{AD}_{t,t',a}, \text{AAB}_{t,t',\ell} \) and \( \text{mAD}_{t,t',p} \). We prove a composition lemma (Lemma 6.8) that relates the rank, sparsity, max-supp-entropy, max-rank-entropy and weight of \( \text{AD}_t \circ \text{target} g \) to those of \( g \).

Since \( \text{AD}_{t,t'} = \text{AD}_t \circ \text{target} \ AND_{\log t'} \) and \( \text{AAB}_{t,t',\ell} = \text{AD}_t \circ \text{target} \ \text{AAB}_{t',\ell} \), we are able to use the composition lemma to prove the properties of interest of \( \text{AD}_{t,t'} \) and \( \text{AAB}_{t,t',\ell} \) (Claims 6.9 and 6.11, respectively). This proves the bounds corresponding to two of the rows in Table 6.7. To conclude the proof of Claim 6.7 we prove bounds on the rank, sparsity, max-supp-entropy, max-rank-entropy and weight of \( \text{AD}_{t,t',a} \) and \( \text{mAD}_{t,t',p} \) from first principles (Claims 6.12 and 6.13, respectively).

We begin by stating a composition lemma.

**Lemma 6.8** (Composition lemma). *Let \( t \geq 2, m \geq 1 \) be any positive integers, and let \( g : \{-1,1\}^m \to \{-1,1\} \) be a non-constant function such that there exists a non-empty set \( S \subseteq [m] \) with \( 0 \neq |\hat{g}(S)| \leq |\hat{g}(\emptyset)| \). Let \( f : \{-1,1\}^{\log t + mt} \to \{-1,1\} \) be defined as \( f = \text{AD}_t \circ \text{target} g \). Then*

\[
  r(f) = t \cdot r(g) + \log t, \quad (18)
\]

\[
  k(f) = 1 + t^2(k(g) - 1), \quad (19)
\]

\[
  k'(f) = t \cdot k'(g), \quad (20)
\]

\[
  k''(f) = t \cdot k''(g), \quad (21)
\]

\[
  \delta(f) = \delta(g). \quad (22)
\]

\(^8\)Precise statements along with quantifications on \( t, t', t, a, p \) are stated in Section 6.3 (Claims 6.9, 6.12, 6.11 and 6.13). We do not formally prove the claimed bound on the max-supp-entropy of \( \text{mAD}_{t,t',p} \) since we do not require it; this bound can be observed from the proof of Claim 6.13.
We defer the proof of the composition lemma (Lemma 6.8) to Section 6.2.1 and proceed to compute the properties of $AD_{t,t'}$ using the composition lemma.

The other functions we consider in this section can be viewed as modifications of $AD_{t,t'}$.

We prove the properties of $AD_{t,t'}$ below to provide insight into our other proofs. Moreover, we use this function to show that Theorem 1.4 gives a tight lower bound on $\delta(AD_{t,t'})$ as opposed to Chang’s lemma (Lemma 1.1) applied with any threshold parameter, which only gives a weaker bound (Claim 7.1).

**Claim 6.9 (Properties of $AD_{t,t'}$).** Fix any integers $t \geq 2$, $t' > 4$ let $AD_{t,t'} : \{-1,1\}^{\log t} \times \{-1,1\}^{t \log t'} \rightarrow \{-1,1\}$ be as in Definition 6.1. Then,

- $r(AD_{t,t'}) = t \log t' + \log t$,
- $k(AD_{t,t'}) = 1 + t^2 (t' - 1)$,
- $k'(AD_{t,t'}) = k''(AD_{t,t'}) = \frac{t'}{2}$, and
- $\delta(AD_{t,t'}) = \frac{1}{t}$.

**Proof.** Recall from Definition 6.1 that $AD_{t,t'} = AD \circ_{target} AND$ where AND is on $\log t'$ bits. Since $t' > 4$, by Observation 4.30, $|\text{AND}(\emptyset)| = 1 - \frac{2}{t'} > \frac{2}{t} = |\text{AND}(S)|$ for all $S \neq \emptyset$. Therefore the claim follows by Lemma 6.8 and Observation 4.30. \(\square\)

We next compute the properties of $AAB_{t,t',\ell}$. Since $AAB_{t,t',\ell} = AD \circ_{target} AB_{t',\ell}$, we first state the properties of $AB_{t',\ell}$ in Claim 6.10 and then deduce the properties of $AAB_{t,t',\ell}$ using composition lemma (Lemma 6.8) and Claim 6.10.

**Claim 6.10 (Properties of $AB_{t',\ell}$).** For any integers $t' > 3, \ell \geq 2$, let $AB_{t',\ell} : \{-1,1\}^{\log t'} \times \{-1,1\}^{\log \ell} \rightarrow \{-1,1\}$ be as in Definition 6.3. Then,

- $r(AB_{t',\ell}) = \log t' + \log \ell$,
- $k(AB_{t',\ell}) = 1 + \frac{t'}{2} + \frac{\ell t'}{2}$,
- $k'(AB_{t',\ell}) = k''(AB_{t',\ell}) = \frac{t' \sqrt{2}}{2}$ \ and 
- $\delta(AB_{t',\ell}) = \frac{1}{t'}$.

We defer the proof of Claim 6.10 to Section 6.2.2. The following claim gives the properties of $AAB_{t,t',\ell}$.

**Claim 6.11 (Properties of $AAB_{t,t',\ell}$).** For any integers $t \geq 2, t' > 3, \ell \geq 2$, let $AAB_{t,t',\ell} : \{-1,1\}^{\log t} \times \{-1,1\}^{t(\log \ell + \log t')} \rightarrow \{-1,1\}$ be as in Definition 6.4. Then,

- $r(AAB_{t,t',\ell}) = t(\log t' + \log \ell) + \log t$,
- $k(AAB_{t,t',\ell}) = 1 + t^2 (\ell + 1) t'$,
- $k'(AAB_{t,t',\ell}) = k''(AAB_{t,t',\ell}) = \frac{t' \sqrt{2}}{2}$, and
- $\delta(AAB_{t,t',\ell}) = \frac{1}{t'}$.  

28
Proof. Recall from Definition 6.4 that $\text{AAB}_{t,t',\ell} = \text{AD} \circ_{\text{target}} \text{AB}_{t',\ell}$ where $\text{AB}_{t',\ell}$ is on log $\ell + \log t'$ bits. Since $t' > 3$, $\ell \geq 2$, by Claim 6.10, $|\text{AAB}_{t,t',\ell}(0)| = 1 - \frac{2}{\ell} \geq \frac{2}{\ell^2} = \frac{1}{k(\text{AAB}_{t,t',\ell})}$. Therefore the claim follows by Lemma 6.8 and Claim 6.10.

In the following claim, we deduce the properties of $\text{AD}_{t,t',a}$ from its Fourier expansion.

Claim 6.12 (Properties of $\text{AD}_{t,t',a}$). Fix any integers $t \geq 2$, $t' \geq 2$ and $a \geq 2t'$. Let $\text{AD}_{t,t',a} : \{-1,1\}^{\log t} \times \{-1,1\}^{\log a} \times \{-1,1\}^{(t-1)\log t'} \to \{-1,1\}$ be as in Definition 6.2. Then,

- $r(\text{AD}_{t,t',a}) = (t-1)\log t' + \log a + \log t$,
- $k(\text{AD}_{t,t',a}) = (t-1)(t'-1)t + t\alpha$,
- $k'(\text{AD}_{t,t',a}) = k''(\text{AD}_{t,t',a}) = \frac{t\alpha}{2}a$, and
- $\delta(\text{AD}_{t,t',a}) = \frac{1}{t} + \frac{1}{\alpha} - \frac{1}{t\alpha}$.

We prove Claim 6.12 in Section 6.2.3. In the following claim, we deduce the properties of $m\text{AD}_{t,t',p}$ from first principles.

Claim 6.13 (Properties of $m\text{AD}_{t,t',p}$). Let $t, t' \geq 2$ be any integers and let be an integer such that $(t-1)(\log t') \geq p$. Let $m\text{AD}_{t,t',p} : \{-1,1\}^{\log t + t\log t'} \to \{-1,1\}$ be as in Definition 6.6. Then,

- $r(m\text{AD}_{t,t',p}) = t \cdot \log t' + \log t$,
- $k(m\text{AD}_{t,t',p}) = \Theta(2^p t' t^2 t')$,
- $k''(m\text{AD}_{t,t',p}) = \Theta(t')$, and
- $\delta(m\text{AD}_{t,t',p}) = \frac{1}{p}$.

We prove Claim 6.13 in Section 6.2.4. Finally, the proof of Claim 6.7 follows from some of the claims above.

Proof of Claim 6.7. The proof follows from Claims 6.9, 6.11, 6.12 and 6.13.

6.2.1 Proof of Composition lemma (Lemma 6.8)

Let $t \geq 2$ and $m \geq 1$ be any integers. For the purpose of the following proof, we introduce the following notation. For any $b \in \{-1,1\}^{\log t}$, $T \subseteq \log t$ and non-empty $S_b \subseteq \{m\}$, define characters $\chi_{b,S_b,T} : \{-1,1\}^{\log t} \times \{-1,1\}^{tm} \to \{-1,1\}$ by $\chi_{b,S_b,T}(x,z) = \prod_{j \in S_b} z_{b,j} \prod_{i \in T} x_i$. Here $z = (\ldots, z_b, \ldots)$, where $b \in \{-1,1\}^{\log t}$ and $z_b \in \{-1,1\}^m$ for all $b \in \{-1,1\}^{\log t}$.

Proof of Lemma 6.8. Let $x \in \{-1,1\}^{\log t}$ and $z \in \{-1,1\}^{tm}$. By Definition 4.22 and Observation 4.21,

$$f(x,z) = \sum_{b \in \{-1,1\}^{\log t}} g(z_b) \cdot \hat{u}_b(x)$$

$$= \sum_{b} \left( \sum_{S_b \subseteq \{m\}} \hat{g}(S_b) \chi_{S_b}(z_b) \right) \left( \sum_{T \subseteq \log t} \hat{u}_b(T) \chi_T(x) \right)$$
\[ A_1 = \sum_b g(\emptyset) \frac{1}{t} = g(\emptyset), \]
\[ A_2 = \sum_b g(\emptyset) \sum_{T \neq \emptyset} \frac{\prod_{i \in T} b_i}{t} \chi_T(x) \]
\[ = \sum_b g(\emptyset) \sum_{T \neq \emptyset} \frac{\prod_{i \in T} b_i}{t} \chi_T(x) = 0, \quad \text{by Observation 4.20} \]
\[ A_3 = \sum_{b, S_b \neq \emptyset} \sum_T \frac{\hat{g}(S_b) \cdot \prod_{i \in T} b_i}{t} \chi_{S_b}(z_b) \chi_T(x) \]
\[ = \sum_{b, S_b \neq \emptyset, T} c_{b, S_b, T} \cdot \chi_{b, S_b}(x, z), \]
where \( |c_{b, S_b, T}| = \left| \frac{\hat{g}(S_b)}{t} \right| \) for all \( b \in \{-1, 1\}^{\log t}, T \subseteq [\log t] \) and non-empty \( S_b \subseteq [m] \). From Equation (23) and the above expressions for \( A_1, A_2 \) and \( A_3 \), we obtain the following Fourier expansion for \( f \):
\[ f = \hat{g}(\emptyset) + \sum_{b, \emptyset \neq S_b \subseteq \supp(g), T} c_{b, S_b, T} \cdot \chi_{b, S_b}(x, z), \quad (24) \]

since \( |c_{b, S_b, T}| = \left| \frac{\hat{g}(S_b)}{t} \right| \), \( c_{b, S_b, T} \) is non-zero iff \( \hat{g}(S_b) \) is non-zero. Therefore
\[ \supp(f) = \{\chi_{b, S_b, T} \mid b \in \{-1, 1\}^{\log t}, T \subseteq [\log t] \}. \quad (25) \]

- **Rank**: Fix a Fourier basis \( B_g \) of \( g \) such that \( B_g \subseteq \supp(g) \) and a character \( \chi_U \in B_g \). Consider the set of characters
\[ B_f = \left\{ \chi_{b, S_b, 0} \mid b \in \{-1, 1\}^{\log t}, S_b \in B_g \right\} \cup \left\{ \chi_{1, U, (i)} \mid i \in [\log t] \right\}. \]

By Equation (25), \( B_f \subseteq \supp(f) \) and \( \supp(f) \subseteq \text{span}(B_f) \). Therefore,
\[ r(f) = |B_f| = t|B_g| + \log t = t \cdot r(g) + \log t. \]

- **Sparsity**: By Equation (25),
\[ k(f) = |\supp(f)| = 1 + t^2(k(g) - 1). \]
• Max-supp-entropy: Recall from Definition 4.26 that $k'(f)$ equals the smallest non-zero Fourier coefficient of $f$ in absolute value. From the Fourier expansion of $f$ given in Equation (24),

$$k'(f) = \max \left\{ \frac{1}{|g(\emptyset)|} : \max \left\{ \frac{t}{|g(S)|} : \emptyset \neq S \in \text{supp}(g) \right\} \right\}$$

$$= \max \left\{ \frac{t}{|g(S)|} : \emptyset \neq S \in \text{supp}(g) \right\} \text{ since } |\hat{g}(\emptyset)| \geq \frac{1}{k'(g)} \text{ and } t \geq 1 \text{ by assumption}$$

$$= t \cdot k'(g).$$

• Max-rank-entropy: Recall from Definition 4.26 that $k''(f) = \arg \min_g \{ \dim(S_g) = r(f) \}$, where $S_g = \left\{ S : |\hat{f}(S)| \geq \frac{1}{\log^{r(f)}(g)} \right\}$.

From the Fourier expansion of $f$ given in Equation (24), the following set $B_f$ is a spanning set for the Fourier support of $f$. Let $B_g$ be a Fourier basis for $g$ such that $|\hat{g}(S)| \geq \frac{1}{k''(g)}$ for all $S \in B_g$. Define

$$B_f = \left\{ \chi_{b,S,T} : b \in \{-1,1\}^{|\log t|}, S \in B_g, T \subseteq [\log t] \right\} \quad (26)$$

One may verify that $B_f$ indeed is a spanning set for $\text{supp}(f)$. By Equation (24), $|c_{b,S,T}| = \frac{|\hat{g}(S)|}{t}$ for all $b \in \{-1,1\}^{|\log t|}, T \subseteq [\log t]$ and non-empty $S \subseteq [m]$. Hence $k''(f) \leq t \cdot k''(g)$.

It now remains to show that $k''(f) \geq t \cdot k''(g)$. Towards a contradiction, consider a basis $T_f \subseteq \left\{ \chi_{b,S,T} : b \in \{-1,1\}^{|\log t|}, S \in \text{supp}(g) \right\}$ for $\text{supp}(f)$, with $|\hat{f}(S)| > \frac{1}{t \cdot k''(g)}$ for all $S \in T_f$.

Fix any $b \in \{-1,1\}^{|\log t|}$. Observe that the set $\left\{ \chi_{S_b} : \chi_{b,S,T} \in T_f \right\}$ forms a spanning set for $\text{supp}(g)$. Moreover, since $|c_{b,S,T}| = \frac{|\hat{g}(S)|}{t}$ for all $b \in \{-1,1\}^{|\log t|}, T \subseteq [\log t]$ and non-empty $S \subseteq [m]$ by Equation (24), the set $\left\{ \chi_{S_b} : \chi_{b,S,T} \in T_f \right\}$ is such that each of its Fourier coefficients (i.e. $\hat{g}(S_b)$) has absolute value strictly larger than $\frac{1}{k''(g)}$, which is a contradiction by the definition of $k''(g)$.

• Weight: By Observation 4.2,

$$\delta(f) = \frac{1 - \hat{f}(\emptyset)}{2} = \frac{1 - \hat{g}(\emptyset)}{2} = \delta(g),$$

where the second equality follows by Equation (24).

\[ \square \]

6.2.2 Properties of AND of Bent (Claim 6.10)

For the purpose of this proof, define the characters $\chi_{S,T} : \{-1,1\}^{|\log t'|} \times \{-1,1\}^{|\log \ell|} \to \{-1,1\}$ by $\chi_{S,T}(y,z) = \prod_{i \in S} y_i \cdot \prod_{j \in T} z_j$ for all $S \subseteq [\log t'], T \subseteq [\log \ell]$.

Proof of Claim 6.10. Recall that from Definition 6.3, for all $y \in \{-1,1\}^{|\log t'|}$ and $z \in \{-1,1\}^{|\log \ell|}$,

$$AB_{t',\ell}(y,z) = \text{AND}(y_1 B(z), y_2, \ldots, y_{|\log \ell|}).$$
where $B : \{-1, 1\}^{|\log \ell|} \to \{-1, 1\}$ is a bent function. Since $\text{AND}(y_1, \ldots, y_{\log t'}) = 1 - 2 \prod_{i=1}^{\log t'} \left( \frac{1-\hat{y}_i}{2} \right)$,

$$AB_{v, \ell}(y, z) = \text{AND}(y_1 B(z), y_2, \ldots, y_{\log t'})$$

$$= 1 - (1 - y_1 B(z)) \cdot \prod_{i=2}^{\log t'} \frac{1 - y_i}{2}$$

$$= 1 - \left( \frac{2}{t'} \sum_{1 \not\in S \subseteq [\log t']} (-1)^{|S|} \chi_S(y) \right)$$

$$= 1 - \left( \frac{2}{t'} \sum_{1 \not\in S \subseteq [\log t']} (-1)^{|S|} \chi_S(y) \right) + 2 \cdot \frac{1}{t'} y_1 \sum_{T \subseteq [\log \ell]} \sum_{1 \in S \subseteq [\log t']} (-1)^{|S|} \hat{B}(T) \chi_S(y) \chi_T(z)$$

Hence, the Fourier expansion of $AB_{v, \ell}$ is given by

$$AB_{v, \ell} = \left( 1 - \frac{2}{t'} \right) \chi_{\emptyset, \emptyset} - \frac{2}{t'} \left( \sum_{1 \not\in S \subseteq [\log t']} (-1)^{|S|} \chi_S, \emptyset \right) + 2 \cdot \frac{1}{t'} y_1 \left( \sum_{1 \in S \subseteq [\log t']} (-1)^{|S|} \hat{B}(T) \chi_S, T \right). \quad (27)$$

Since $\hat{B}(T) \neq 0$ for all $T \subseteq [\log \ell]$ by Definition 4.15, Equation (27) implies

$$\text{supp}(AB_{v, \ell}) = \{ (\emptyset, \emptyset) \} \cup \{ (S, \emptyset) : S \neq \emptyset, 1 \not\in S \subseteq [\log t'] \} \cup \{ (S, T) : 1 \in S \subseteq [\log t'], T \subseteq [\log \ell] \}. \quad (28)$$

- Rank: Consider $B_B = \{(i, \emptyset) : i \in [\log t'] \} \cup \{(1, j) : j \in [\log \ell] \}$.

By Equation (28), $B_B \subseteq \text{supp}(AB_{v, \ell})$. Moreover $B_B$ is a linearly independent set and generates all the characters. Therefore $B_B$ forms a Fourier basis of $AB_{v, \ell}$. Hence,

$$r(AB_{v, \ell}) = |B_B| = \log t' + \log \ell.$$ 

- Sparsity: From Equation (28),

$$k(AB_{v, \ell}) = |\text{supp}(AB_{v, \ell})| = \frac{t'}{2} + \frac{\ell t'}{2}.$$ 

- Max-supp-entropy: Recall from Definition 4.26 that $k'(AB_{v, \ell})$ equals the smallest non-zero Fourier coefficient of $AB_{v, \ell}$ in absolute value. From the Fourier expansion of $AB_{v, \ell}$ given in Equation (27),

$$k'(AB_{v, \ell}) = \max \left\{ \frac{t'}{t' - 2}, \frac{t'}{2}, \max \left\{ \frac{t'}{2|\hat{B}(T)|} : T \subseteq [\log \ell] \right\} \right\}$$

$$= \frac{t' \sqrt{\ell}}{2}. \quad \text{by Observation 4.16}$$

32
• Max-rank-entropy: Recall from Definition 4.26, \( k''(f) = \arg \min_{g} \{ \dim(S_g) = r(f) \} \), where 
\( S_g = \left\{ S : |\hat{f}(S)| \geq \frac{1}{g} \right\} \). Observe from the Fourier expansion of \( \mathbf{A}_{t', \ell} \) in Equation (27) that the only monomials which containing \( z \)-variables are \( \chi_{S,T} \) such that \( T \neq \emptyset \). Any such monomial has coefficient whose absolute value is \( \frac{2}{\ell' \sqrt{\ell}} \). So, \( k''(\mathbf{A}_{t', \ell}) \geq \frac{t' \sqrt{\ell}}{2} \). Furthermore by Lemma 4.27 \( k''(\mathbf{A}_{t', \ell}) \leq k'(\mathbf{A}_{t', \ell}) = \frac{t' \sqrt{\ell}}{2} \).

\[
k''(\mathbf{A}_{t', \ell}) = \frac{t' \sqrt{\ell}}{2}.
\]

• Weight: Observation 4.2 and Equation (27) imply

\[
\delta(\mathbf{A}_{t', \ell}) = \frac{1 - \mathbf{A}_{\ell, \ell}(\emptyset, \emptyset)}{2} = \frac{1}{t'}. 
\]

\[\square\]

6.2.3 Properties of \( \mathbf{A}_{t, t', a} \) (Claim 6.12)

Let \( t \geq 2 \), \( t' \geq 2 \) and \( a \geq 2t' \) be any integers. Consider the function \( \mathbf{A}_{t, t', a} : \{-1, 1\}^{\log t} \times \{-1, 1\}^{\log a} \times \{-1, 1\}^{(t-1) \log t'} \rightarrow \{-1, 1\} \) as in Definition 6.2. For the purpose of the following proof, we introduce the following notation. Let \( 1 := 1^{\log t} \). For any \( 1 \neq b \in \{-1, 1\}^{\log t} \), \( T \subseteq [\log t] \) and non-empty \( S_b \subseteq [\log t'] \), define characters \( \chi_{b, S_b, T} : \{-1, 1\}^{\log t} \times \{-1, 1\}^{\log a} \times \{-1, 1\}^{(t-1) \log t'} \rightarrow \{-1, 1\} \) by

\[
\chi_{b, S_b, T}(x, z) = \prod_{j \in S_b} z_{b,j} \prod_{i \in T} x_i.
\]

Here \( z = (\ldots, z_b, \ldots) \), where \( b \in \{-1, 1\}^{\log t} \), \( z_1 \in \{-1, 1\}^{\log a} \) and \( z_b \in \{-1, 1\}^{\log t'} \) for all \( b \in \{-1, 1\}^{\log t} \setminus \{1\} \). Also for \( b = 1 \), \( T \subseteq [\log t] \) and non-empty \( S_1 \subseteq [\log a] \), define characters \( \chi_{1, S_1, T} : \{-1, 1\}^{\log t} \times \{-1, 1\}^{\log a} \times \{-1, 1\}^{(t-1) \log t'} \rightarrow \{-1, 1\} \) by

\[
\chi_{1, S_1, T}(x, z) = \prod_{j \in S_1} z_{1,j} \prod_{i \in T} x_i.
\]

For any set \( U \subseteq [\log t] \), define characters \( \chi_{0, U} : \{-1, 1\}^{\log t} \times \{-1, 1\}^{\log a} \times \{-1, 1\}^{(t-1) \log t'} \rightarrow \{-1, 1\} \) by

\[
\chi_{0, U}(x, z) = \prod_{i \in U} x_i.
\]

Proof of Claim 6.12. Let \( x \in \{-1, 1\}^{\log t} \) and \( z \in \{-1, 1\}^{\log a} \times \{-1, 1\}^{(t-1) \log t'} \) be such that \( z_1 \in \{-1, 1\}^{\log a} \) and \( z_b \in \{-1, 1\}^{\log t'} \) for \( 1 \neq b \in \{-1, 1\}^{\log t} \). By Definition 6.2 and Observation 4.21,

\[
\mathbf{A}_{t, t', a}(x, z) = \mathbf{AND}(z_1) \cdot \mathbb{I}_1(x) + \sum_{\substack{1 \neq b \in \{-1, 1\}^{\log t} \\ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ B}} \mathbf{AND}(z_b) \cdot \mathbb{I}_b(x). \tag{29}
\]
We first analyze $A$ from Equation (29). In this analysis, AND is on log $a$ variables.

$$A = \left( \sum_{S_1 \subseteq [\log a]} \widehat{\text{AND}}(S_1) \chi_{S_1}(z_1) \right) \left( \sum_{T \subseteq [\log t]} \hat{\mathbb{I}}_1(T) \chi_T(x) \right)$$

$$= \left( \widehat{\text{AND}}(\emptyset) + \sum_{S_1 \neq \emptyset} \widehat{\text{AND}}(S_1) \chi_{S_1}(z_1) \right) \left( \hat{\mathbb{I}}_1(\emptyset) + \sum_{T \neq \emptyset} \hat{\mathbb{I}}_1(T) \chi_T(x) \right)$$

$$= \left( \widehat{\text{AND}}(\emptyset) \hat{\mathbb{I}}_1(\emptyset) + \widehat{\text{AND}}(\emptyset) \sum_{T \neq \emptyset} \hat{\mathbb{I}}_1(T) \chi_T(x) + \sum_{S_1 \neq \emptyset} \widehat{\text{AND}}(S_1) \hat{\mathbb{I}}_1(T) \chi_{S_1}(z_1) \chi_T(x) \right)$$

$$= \frac{\widehat{\text{AND}}(\emptyset) \hat{\mathbb{I}}_1(\emptyset)}{A_1} + \frac{\widehat{\text{AND}}(\emptyset) \sum_{T \neq \emptyset} \hat{\mathbb{I}}_1(T) \chi_T(x)}{A_2} + \frac{\sum_{S_1 \neq \emptyset} \widehat{\text{AND}}(S_1) \hat{\mathbb{I}}_1(T) \chi_{S_1}(z_1) \chi_T(x)}{A_3}.$$  

(30)

By Fact 4.14 and Observation 4.18,

$$A_1 = \left( 1 - \frac{2}{a} \right) \frac{1}{t},$$  

(31)

$$A_2 = \left( 1 - \frac{2}{a} \right) \frac{\sum_{T \neq \emptyset} \chi_T(x)}{t} = \frac{1}{t} \left( 1 - \frac{2}{a} \right) \sum_{T \neq \emptyset} \chi_T(x),$$  

(32)

$$A_3 = \sum_{S_1 \neq \emptyset} \sum_{T} \frac{2(-1)^{|S_1|+1}}{at} \chi_{S_1}(z_1) \chi_T(x).$$  

(33)

We next analyze $B$ from Equation (29). In this analysis, AND is on log $t'$ variables.

$$B = \sum_{b \neq 1} \left( \sum_{S_b \subseteq [\log t']} \widehat{\text{AND}}(S_b) \chi_{S_b}(z_b) \right) \left( \sum_{T \subseteq [\log t]} \hat{\mathbb{I}}_b(T) \chi_T(x) \right)$$

$$= \sum_{b \neq 1} \left( \widehat{\text{AND}}(\emptyset) + \sum_{S_b \neq \emptyset} \widehat{\text{AND}}(S_b) \chi_{S_b}(z_b) \right) \left( \hat{\mathbb{I}}_b(\emptyset) + \sum_{T \neq \emptyset} \hat{\mathbb{I}}_b(T) \chi_T(x) \right)$$

$$= \sum_{b \neq 1} \left( \widehat{\text{AND}}(\emptyset) \hat{\mathbb{I}}_b(\emptyset) + \widehat{\text{AND}}(\emptyset) \sum_{T \neq \emptyset} \hat{\mathbb{I}}_b(T) \chi_T(x) + \sum_{S_b \neq \emptyset} \widehat{\text{AND}}(S_b) \hat{\mathbb{I}}_b(T) \chi_{S_b}(z_b) \chi_T(x) \right)$$

$$= \sum_{b \neq 1} \frac{\widehat{\text{AND}}(\emptyset) \hat{\mathbb{I}}_b(\emptyset)}{B_1} + \sum_{b \neq 1} \frac{\widehat{\text{AND}}(\emptyset) \sum_{T \neq \emptyset} \hat{\mathbb{I}}_b(T) \chi_T(x)}{B_2} + \sum_{b \neq 1, S_b \neq \emptyset} \frac{\sum_{T \neq \emptyset} \widehat{\text{AND}}(S_b) \hat{\mathbb{I}}_b(T) \chi_{S_b}(z_b) \chi_T(x)}{B_3}.$$  

(34)

By Fact 4.14 and Observation 4.18,

$$B_1 = \sum_{1 \neq b \in \{-1,1\}^{\log t}} \left( 1 - \frac{2}{t'} \right) \left( \frac{1}{t} \right) = \left( 1 - \frac{2}{t'} \right) \left( 1 - \frac{1}{t} \right),$$  

(35)

$$B_2 = \left( 1 - \frac{2}{t'} \right) \sum_{T \neq \emptyset} \sum_{b \neq 1} \frac{\prod_{i \in T} b_i}{t} \chi_T(x).$$
Finally, the only terms contributing to $B_3 = \sum_{b \neq 1} \sum_{S_b \neq \emptyset} \sum_T \frac{2(-1)^{|S_b|+1}}{tt'} \prod_{i \in T} b_i \chi_{S_b(z_i)} \chi_T(x)$, where Equation (36) follows since $\sum_{b \neq 1} \prod_{i \in T} b_i = -1$ for any non-empty $T \subseteq [\log t]$ by Observation 4.20. From Equation (29),

$$AD_{t,t',a}(x,z) = A + B = A_1 + B_1 + A_2 + B_2 + A_3 + B_3.$$  \hspace{1cm} (38)

Observe that the only terms from Equation (38) that contribute to $c_0$ are $A_1$ and $B_1$. Moreover, we have from Equations (31) and (35) that

$$c_0 = A_1 + B_1 = \left(1 - \frac{2}{a}ight) \frac{1}{t} + \frac{1}{t'} \left(1 - \frac{1}{t}\right) = 1 + \frac{2}{tt'} - \frac{2}{a} - \frac{2}{t'},$$  \hspace{1cm} (39)

Next observe that the only terms contributing to $c_U$ for $\emptyset \neq U \subseteq [\log t]$ appear in $A_2$ and $B_2$. Matching coefficients we obtain from Equations (32) and (36) that for any non-empty $U \subseteq [\log t]$,

$$c_U = \frac{1}{t} \left(1 - \frac{2}{a}\right) - \frac{1}{t'} \left(1 - \frac{2}{t'}\right) = \frac{2}{tt'} - \frac{2}{at}.$$  \hspace{1cm} (40)

Next, the only terms contributing to $c_{b,S_b} = 1 \neq b \in \{-1,1\}^{[\log t]}$, non-empty $S_b \subseteq [\log t']$ and $T \subseteq [\log t]$ arise from $B_3$. By comparing coefficients we obtain from Equation (37) that for any $1 \neq b \in \{-1,1\}^{[\log t]}$, non-empty $S_b \subseteq [\log t']$ and $T \subseteq [\log t]$,

$$|c_{b,S_b}| = \frac{2}{tt'},$$  \hspace{1cm} (41)

Finally the only term that contributes to $c_{1,S_1} = 1 \neq S_1 \subseteq [\log a]$ and $T \subseteq [\log t]$ is $A_3$. Matching coefficients, we obtain from Equation (33) that for any non-empty $S_1 \subseteq [\log a]$ and $T \subseteq [\log t]$,

$$|c_{1,S_1}| = \frac{2}{at}.$$  \hspace{1cm} (42)

Moreover, all terms that appear in Equation (38) appear in the cases covered above. This proves the claim.

Thus the Fourier expansion of $AD_{t,t',a}$ is given by

$$AD_{t,t',a} = c_0 + \sum_{\emptyset \neq U \subseteq [\log t]} c_U \chi_{\emptyset,U} + \sum_{1 \neq b \in \{-1,1\}^{[\log t]}, \emptyset \neq \emptyset \subseteq [\log t'], \emptyset \neq S_b \subseteq [\log t']}, c_{b,S_b} \cdot \chi_{b,S_b,T} + \sum_{T \subseteq [\log t]} c_{1,S_1} \cdot \chi_{1,S_1,T},$$  \hspace{1cm} (43)

where $c_0$ is as in Equation (39), $c_U$ is as in Equation (40) for all $\emptyset \neq U \subseteq [\log t]$, $c_{b,S_b} = 1 \neq b \in \{-1,1\}^{[\log t]}$, $\emptyset \neq S_b \subseteq [\log t']$ and $T \subseteq [\log t]$, and $c_{1,S_1} = 1 \neq S_1 \subseteq [\log a]$ and $T \subseteq [\log t]$.

In the Fourier expansion of $AD_{t,t',a}$ from Equation (43), coefficients in the third and fourth summands are non-zero from Equations (42) (41), coefficients in the second summand are non-zero
by Equation (40) since \( a \geq 2t' \). Finally, by Equation (39), \( c_0 \neq 0 \) since \( 1 + \frac{\theta}{tt'} - \frac{2}{at} - \frac{2}{tt'} \geq 1 + \frac{1}{tt'} - \frac{2}{tt'} \geq \frac{1}{tt'} \geq 0 \) as \( a \geq 2t' \geq 4 \). From Equation (43),

\[
\text{supp}(\text{AD}_{t,t',a}) = \left\{ \chi_{0,0} \right\} \cup \left\{ \chi_{0,U} | \emptyset \not= U \subseteq [\log t] \right\} \cup \left\{ \chi_{1,S_1,T} | \emptyset \not= S_1 \subseteq [\log a], T \subseteq [\log t] \right\} \\
\cup \left\{ \chi_{b,S_b,T} | b \in \{-1,1\}^{\log t} \setminus \{1\}, \emptyset \not= S_b \subseteq [\log t'], T \subseteq [\log t] \right\}. \quad (44)
\]

- **Rank:** Consider the set of characters

\[
\mathcal{B} = \left\{ \chi_{b,(i),\emptyset} : b \in \{-1,1\}^{\log t} \setminus \{1\}, i \in [\log t'] \right\} \cup \left\{ \chi_{1,(i),\emptyset} : i \in [\log a] \right\} \cup \left\{ \chi_{1,(1),j} : j \in [\log t] \right\}.
\]

These characters can be seen to be linearly independent and span all monomials. Moreover, by Equation (44), \( \mathcal{B} \subseteq \text{supp}(\text{AD}_{t,t',a}). \) Therefore,

\[
r(\text{AD}_{t,t',a}) = |\mathcal{B}| = (t - 1) \log t' + \log a + \log t.
\]

- **Sparsity:** By Equation (44),

\[
k(\text{AD}_{t,t',a}) = |\text{supp}(\text{AD}_{t,t',a})| \\
= 1 + t - 1 + (a - 1)t + (t - 1)(t' - 1)t \\
= (t - 1)(t' - 1)t + ta.
\]

- **Max-supp-entropy:** Recall from Definition 4.26 that \( k'(\text{AD}_{t,t',a}) \) equals the inverse of the smallest non zero Fourier coefficient in absolute value. From the Fourier expansion of \( \text{AD}_{t,t',a} \) given in Equation (43), the candidates for smallest nonzero coefficient in absolute value are

\[
\left\{ \frac{1}{2} + \frac{2}{at} - \frac{2}{tt'} - \frac{2}{tt'} \right\} \cup \left\{ \frac{2}{tt'} \cdot \frac{2}{at} \right\} \cup \left\{ \frac{2}{tt'} - \frac{2}{at} \right\} \cup \left\{ \frac{2}{tt'} \cdot \frac{2}{at} \right\} \cup \left\{ \frac{2}{tt'} - \frac{2}{at} \right\} \cup \left\{ \frac{2}{tt'} \cdot \frac{2}{at} \right\}. \quad (45)
\]

Since \( t' \geq 3 \), \( 1 + \frac{\theta}{tt'} - \frac{2}{at} - \frac{2}{tt'} \geq \frac{2}{tt'} - \frac{2}{at} \). Since, \( a \geq 2t' \), \( \frac{2}{at} - \frac{2}{tt'} \geq \frac{1}{at} - \frac{2}{tt'} \). Therefore

\[
k'(\text{AD}_{t,t',a}) = \frac{ta}{2}.
\]

- **Max-rank-entropy:** Recall from Definition 4.26 that \( k''(f) = \arg\min_{\theta} \{ \dim(S_{\theta}) = r(f) \} \), where \( S_{\theta} = \left\{ S : \|\hat{f}(S)\| \geq \frac{1}{tt'} \right\} \). From the Fourier expansion of \( \text{AD}_{t,t',a} \) given in Equation (43), observe that every monomial which involves a variable from \( z_1 \) has coefficient whose absolute value equals \( \frac{2}{at} \). Thus, if \( \theta < \frac{at}{2} \), \( S_{\theta} \) does not include any monomial containing a variable from \( z_1 \). Therefore \( k''(\text{AD}_{t,t',a}) \geq \frac{at}{2} \). By Lemma 4.27, \( k''(\text{AD}_{t,t',a}) \leq k'(\text{AD}_{t,t',a}) = \frac{at}{2} \). Therefore, by Equation (45)

\[
k''(\text{AD}_{t,t',a}) = \frac{at}{2}.
\]

- **Weight:** From Observation 4.2 and Equation (39),

\[
\delta(\text{AD}_{t,t',a}) = \frac{1 - \text{AD}_{t,t',a}(\theta, \theta)}{2} \\
= \frac{1}{tt'} + \frac{1}{at} - \frac{1}{tt'}.
\]
6.2.4 Properties of \( m_{AD_t, t', p} \) (Claim 6.13)

Recall that we constructed the Modified AND function (Definition 6.5) by replacing one variable by that variable times the product of an AND function of other variables. The next claim computes the Fourier coefficients of \( m_{AD_t, t', p} \).

For the purpose of the following claim, for any \( S \subseteq [\log t'] \) and \( T \subseteq [p] \), define characters \( \chi_{S,T} : \{-1,1\}^{\log t' + p} \to \{-1,1\} \) by \( \chi_{S,T}(y,u) = \prod_{i \in S} y_i \prod_{j \in T} u_j \).

**Claim 6.14.** Let \( t' \geq 2, p \geq 2 \) be any integers and let \( f = m_{AD_t, t', p} : \{-1,1\}^{\log t' + p} \to \{-1,1\} \) be as in Definition 6.5. Then \( f = \sum_{S \subseteq [\log t'], T \subseteq [p]} \hat{f}(S,T) \chi_{S,T} \), where

\[
\hat{f}(S,T) = \begin{cases} 
1 - \frac{2}{t'} & S = T = \emptyset \\
\frac{2(\log |S|)}{p} - \frac{2}{t'} & T = \emptyset, 1 \notin S \subseteq [\log t'], S \neq \emptyset \\
\frac{4(\log |S| + |T| + 1)}{2^{pt'}} & \emptyset \neq T \subseteq [p], 1 \in S \subseteq [\log t'] \end{cases}
\]

For the purpose of the proof, recall that we view inputs to \( m_{AD_t, t', p} \) as \((y,u)\), where \( y \in \{-1,1\}^{\log t'} \) and \( u \in \{-1,1\}^p \).

**Proof of Claim 6.14.** We have \( \text{AND}_{\log t'}(y_1, \ldots, y_{\log t'}) = 1 - 2 \prod_{i=1}^{\log t'} \frac{(1-y_i)}{2} \). Thus, by Definition 6.5,

\[
m_{AD_t, t', p}(y,u) = \text{AND}_{\log t'}(y_1 \text{AND}_p(u), y_2, \ldots, y_{\log t'})
\]

\[
= 1 - (1 - y_1 \text{AND}_p(u)) \prod_{i=2}^{\log t'} \frac{(1-y_i)}{2}
\]

\[
= 1 - \left(1 - y_1 \sum_{T \subseteq [p]} \text{AND}_p(T) \chi_T(u) \right) \left(\frac{2}{t'} \sum_{1 \notin S \subseteq [\log t']} (-1)^{|S|} \chi_S(y) \right)
\]

\[
= 1 - \frac{2}{t'} \left(\sum_{1 \notin S \subseteq [\log t']} (-1)^{|S|} \chi_S(y) \right) + \frac{2}{t'} y_1 \left(\sum_{T \subseteq [p]} \sum_{1 \in S \subseteq [\log t']} (-1)^{|S|} \text{AND}_p(T) \chi_S(y) \chi_T(u) \right)
\]

\[
= 1 - \frac{2}{t'} \left(\sum_{1 \notin S \subseteq [\log t']} (-1)^{|S|} \chi_S(y) \right) + \frac{2}{t'} \left(1 - \frac{2}{2^{pt'}} \right) y_1 \left(\sum_{1 \in S \subseteq [\log t']} (-1)^{|S|} \chi_S(y) \right)
\]

\[
+ \frac{4}{2^{pt'}} \left(\sum_{\emptyset \neq T \subseteq [p]} \sum_{1 \in S \subseteq [\log t']} (-1)^{|S|+|T|+1} \chi_S(y) \chi_T(u) \right)
\]

This proves the claim. \(\square\)

We now prove the required properties of \( m_{AD_t, t', p} \).

**Proof of Claim 6.13.** Define \( 1 := 1^{\log t} \). Recall that on input \((x,y) \in \{-1,1\}^{\log t + t' \log t'}\), we define the set of variables \( u = \{ y_{b,i} | b \in \{-1,1\}^{\log t} \setminus \{1\}, i \in [\log t'] \} \). We also fix an arbitrary ordering
on the variables in \( u \) and let \( u_{\leq p} \) be the the first \( p \) variables in \( u \) according to that order. By Definition 6.6, we have

\[
m_{\text{AD}}_{t,t',p}(x, y) = \mathbb{I}_1(x) \cdot m_{\text{AND}}_{t',p}(y_1, u_{\leq p}) + \sum_{1 \neq b \in \{-1,1\}^{\log t'}} \mathbb{I}_b(x) \cdot \text{AND}_{\log t'}(y_b) \tag{46}
\]

We use Claim 6.14 to expand \( T_1 \) as

\[
T_1 = \mathbb{I}_1(x) \left[ 1 - \frac{2}{t'} + \sum_{\emptyset \neq S \subseteq \log t'} \frac{2 \cdot (-1)^{|S|}}{t'} \prod_{j \in S} y_{1,j} + \sum_{1 \leq |S| \leq \log t'} \frac{2 \cdot (-1)^{|S|}}{t'} \prod_{j \in S} y_{1,j} \right] + \sum_{\emptyset \neq T \subseteq \log t'} \frac{4 \cdot (-1)^{|S|+|T|+1}}{2^{|p|} t'} \prod_{j \in S, \ell \in T} y_{1,j} u_{\ell} \tag{47}
\]

and Fact 4.14 to expand \( T_b \), for \( b \neq 1 \), as

\[
T_b = \mathbb{I}_b(x) \left[ 1 - \frac{2}{t'} + \frac{2}{t'} \sum_{S \neq \emptyset} (-1)^{|S|} \prod_{j \in S} y_{b,j} \right]. \tag{48}
\]

- Rank: For all \( b \in \{-1,1\}^{\log t} \), define

\[
\mathcal{B}_b = \{ y_{b,j} | j \in [\log t'] \}.
\]

From Equations (47) and (48) the monomials from \( \mathcal{B}_1 \) occur only in the term \( T_1 \). Since \( \mathbb{I}_1(\emptyset) = \frac{1}{t'} \) by Observation 4.18, Equation (47) yields that the absolute value of the coefficient of \( y_{1,1} \) is \( (1 - \frac{2}{t'}) \frac{1}{t'} \), and the absolute value of the coefficient of \( y_{1,j} \) is \( \frac{2}{t'} \) for all \( j \in [\log t'] \setminus \{1\} \).

Similarly, for \( 1 \neq b \in \{-1,1\}^{\log t} \), from Equations (47) and (48) the monomials from \( \mathcal{B}_b \) occur only in the term \( T_b \). Since \( \mathbb{I}_b(\emptyset) = \frac{1}{t'} \) by Observation 4.18, Equation (48) yields that the absolute value of the coefficient of \( y_{b,j} \) is \( \frac{2}{t'} \) for all \( j \in [\log t'] \) and \( 1 \neq b \in \{-1,1\}^{\log t} \).

Fix \( 1 \neq c \in \{-1,1\}^{\log t} \), and define

\[
\mathcal{B}_c = \{ y_{c,1} \cdot x_i | i \in [\log t] \}.
\]

From Equations (47) and (48) the monomials from \( \mathcal{B}_c \) occur only in the term \( T_c \). Since \( \mathbb{I}_c(\emptyset) = \frac{1}{t'} \) by Observation 4.18, Equation (48) yields that the absolute value of the coefficient of \( y_{c,1} \cdot x_i \) is \( \frac{2}{t'} \) for all \( i \in [\log t] \).

Therefore \( \bigcup_{b \in \{-1,1\}^{\log t}} \mathcal{B}_b \cup \mathcal{B}_c \subseteq \text{supp}(m_{\text{AD}}_{t,t',p}) \). Since \( \bigcup_{b \in \{-1,1\}^{\log t}} \mathcal{B}_b \cup \mathcal{B}_c \) generate all monomials,

\[
r(m_{\text{AD}}_{t,t',p}) = t \log t' + \log t.
\]

38
• Sparsity: By Equation (47), all monomials appearing in $T_1$, except for those purely in $x$-variables, contain at least one variable from $y_1$. Moreover, from Equation (48), no monomial appearing in $T_b$ for $b \neq 1$ contains a variable from $y_1$. Since all Fourier coefficients of $I_1$ are non-zero by Observation 4.18, Equation (47) yields that these monomials contribute

$$t \left( \frac{t'}{2} - 1 + \frac{t'}{2} + (2^p - 1) \frac{t'}{2} \right) = t \left( \frac{t'}{2} - 1 + 2^{p-1} t' \right) = \Theta(2^p t'). \quad (49)$$

to the sparsity of $mAD_{t,t',p}$.

By Equation (48), all monomials all monomials appearing in $T_b$, for any $b \neq 1$, except for those purely in $x$-variables, contain at least one variable from $y_b$. Moreover, from Equations (47) and (48), no monomial appearing in $T_b$ for $b' \neq b$ contains a variable from $y_b$. Since all Fourier coefficients of $I_b$ are non-zero by Observation 4.18, Equation (48) yields that these monomials contribute at least (including contributions from each $T_b$ for $b \neq 1$)

$$(t - 1) \cdot t \cdot (t' - 1) = \Omega(t^2 t') \quad (50)$$
to the sparsity of $mAD_{t,t',p}$. By Equations (49) and (50),

$$k(mAD_{t,t',p}) = \Omega(2^p t' + t^2 t').$$

By Equation (46),

$$k(mAD_{t,t',p}) \leq k(I_1)k(mAND_{t',p}) + \sum_{1 \neq b \in \{-1,1\}^{\log t}} k(I_b)k(AND_{\log t'}) \leq t \cdot k(mAND_{t',p}) + (t - 1)t \cdot t' \quad \text{by Observations 4.18 and 4.30} = O(2^p t' + t^2 t'). \quad \text{since $mAND_{t',p}$ is a function on $(\log t' + p)$ variables}$$

Therefore,

$$k(mAD_{t,t',p}) = \Theta(2^p t' + t^2 t').$$

• Max-rank-entropy: Recall from the argument for rank that $B = \bigcup_{b \in \{-1,1\}^{\log t}} B_b \cup B_x \subseteq supp(mAD_{t,t',p})$ generate all the monomials of $mAD_{t,t',p}$. Moreover the absolute values of the coefficients of these monomials take values in the set $\{\frac{2}{tt'}, (1 - \frac{2}{tt'}) \frac{2}{tt'}\}$. Since $p \geq 2$, $1 \geq 1 - \frac{2}{tt'} \geq \frac{1}{2}$. Therefore

$$k''(mAD_{tt'}) = O(tt'). \quad (51)$$

Recall that no monomial arising from the terms $T_b$ for $1 \neq b \in \{-1,1\}^{\log t}$ (see Equation (48)) contain variables from $y_1$. Thus, monomials which contain the variable $y_{1,1}$ only appear in Equation (47). Moreover, by Observation 4.18 we conclude that the absolute value of coefficient of any such monomial takes values in the set $\{\frac{2}{tt'}, (1 - \frac{2}{tt'}) \frac{4}{tt'}\}$.

Recall from Definition 4.26 that $k''(f) = \text{argmin}_b \{\dim(S_b) = r(f)\}$, where $S_b = \left\{S : |\hat{f}(S)| \geq \frac{1}{2}\right\}$. Therefore if $\theta < \frac{2}{tt'}$, then $S_b$ does not include any monomial containing $y_{1,1}$. Therefore

$$k''(mAD_{t,t',p}) \geq \left(1 - \frac{2}{tt'}\right)^{-1} \frac{tt'}{2} \geq \frac{tt'}{2}.$$ 

Hence by Equation (51),

$$k''(mAD_{t,t',p}) = \Theta(tt').$$
• Weight: By Equation (46),

\[
m\overrightarrow{\text{AD}}_{t,t',p}(\emptyset) = \overrightarrow{\text{I}}_1(\emptyset)m\overrightarrow{\text{AND}}_{t',p}(\emptyset) + \sum_{1 \neq b \in \{-1,1\}^{\log t}} \overrightarrow{\text{I}}_b(\emptyset)m\overrightarrow{\text{AND}}_{t',p}(\emptyset)
\]

by Observation 4.18 and Claim 6.14

\[
= \frac{1}{t} \left( 1 - \frac{2}{t'} \right) + \left( t - 1 \right) \frac{1}{t} \left( 1 - \frac{2}{t'} \right).
\]

\[
= 1 - \frac{2}{t'}
\]

Thus by Observation 4.2,

\[
\delta(m\overrightarrow{\text{AD}}_{t,t',p}) = \frac{1 - m\overrightarrow{\text{AD}}_{t,t',p}(\emptyset)}{2} = \frac{1}{t'}.
\]

\[\square\]

6.3 Setting parameters in our constructed functions

In this section we prove Theorems 1.5 and 1.7. Recall that these theorems require us to exhibit functions which achieve certain bounds. Claims 6.15 and 6.16 correspond to the bounds in Theorem 1.5, and describe the required functions. Claims 6.17 and 6.18 correspond to the bounds in Theorem 1.7, and describe the required functions.

Claim 6.15. For all \(\rho, \kappa, \kappa' \in \mathbb{N}\) such that \(\kappa\) is sufficiently large, for all \(\epsilon > 0\) such that \(\log \kappa \leq \rho \leq \kappa^{\frac{1}{2} - \epsilon}\) and \(\frac{\kappa \log \kappa}{\rho} \leq \kappa' \leq \kappa\), for \(t = \frac{2\rho}{\log \kappa}\), \(t' = \frac{\kappa \log \kappa}{\rho^2}\) and \(a = \frac{2\kappa' \log \kappa}{\rho}\),

- \(\Omega(\epsilon \rho) = r(\overrightarrow{\text{AD}}_{t,t',a}) = O(\rho)\).
- \(k(\overrightarrow{\text{AD}}_{t,t',a}) = \Theta(\kappa)\).
- \(k'(\overrightarrow{\text{AD}}_{t,t',a}) = \Theta(\kappa')\).
- \(\delta(\overrightarrow{\text{AD}}_{t,t',a}) = \Theta \left( \frac{1}{\kappa} \left( \frac{\rho}{\log \kappa} \right)^2 \right)\).

We prove Claim 6.15 in Section 6.3.1.

Claim 6.16. For all \(\rho, \kappa, \kappa' \in \mathbb{N}\) such that \(\kappa\) is sufficiently large, for all constants \(\epsilon > 0\), such that \(\kappa^{1/2} \leq \kappa' \leq (\kappa \log \kappa)/\rho\) and \(\log \kappa \leq \rho \leq \kappa^{\frac{1}{2} - \epsilon}\) for \(t = \frac{2\rho}{\log \kappa}\), \(t' = \frac{4\kappa'^2}{\kappa}\) and \(\ell = \frac{\kappa \log \kappa}{\kappa' \rho}\),

- \(\Omega(\epsilon \rho) = r(\overrightarrow{\text{AAB}}_{t,t',\ell}) = O(\rho)\).
- \(k(\overrightarrow{\text{AAB}}_{t,t',\ell}) = \Theta(\kappa)\).
- \(k'(\overrightarrow{\text{AAB}}_{t,t',\ell}) = \Theta(\kappa')\).
- \(\delta(f) = O \left( \frac{\kappa}{\kappa'^2} \right)\).

We prove Claim 6.16 in Section 6.3.2.
Claim 6.17. For all \(\rho, \kappa, \kappa'' \in \mathbb{N}\) such that \(\kappa\) is sufficiently large, for all constants \(\epsilon > 0\) such that \(\log \kappa \leq \rho \leq \kappa^{1/2-\epsilon}\), \(\epsilon \rho \leq \kappa'' \leq \frac{\kappa \log \kappa}{\rho}\), for \(t = \frac{2\rho}{\log(\kappa''/\rho)}\), \(t' = \frac{\kappa''}{\rho} \log (\kappa''/\rho)\), \(p = \log \left(\frac{4\kappa}{\kappa''}\right)\),

- \(r(\text{mAD}_{t,t',p}) = \Theta(\rho)\).
- \(\Omega(\kappa) = k(\text{mAD}_{t,t',p}) = O(\kappa/\epsilon)\).
- \(k''(\text{mAD}_{t,t',p}) = \Theta(\kappa'')\).
- \(\delta(\text{mAD}_{t,t',p}) = \frac{\rho}{\kappa'' \log(\kappa''/\rho)}\).

We prove Claim 6.17 in Section 6.3.3.

Claim 6.18. For all \(\rho, \kappa, \kappa' \in \mathbb{N}\) such that \(\kappa\) is sufficiently large, for all \(\epsilon > 0\) such that \(\log \kappa \leq \rho \leq \kappa^{1/2-\epsilon}\) and \(\frac{\kappa \log \kappa}{\rho} \leq \kappa'' \leq \kappa\), there exists a constant \(c \geq 1\) such that the following holds for \(t = \frac{2\rho}{\log \kappa}\), \(t' = \frac{c\kappa'}{\log(\kappa'/\rho)}\) and \(a = \frac{2\kappa'' \log \kappa}{\rho}\).

- \(\Omega(\epsilon \rho) = r(\text{AD}_{t,t',a}) = O(\rho)\).
- \(k(\text{AD}_{t,t',a}) = \Theta(\kappa)\).
- \(k''(\text{AD}_{t,t',a}) = \Theta(\kappa'')\).
- \(\delta(\text{AD}_{t,t',a}) = \Theta\left(\frac{1}{\kappa} \left(\frac{\rho}{\log \kappa}\right)^2\right)\).

Proof. It follows by Claim 6.15 and the fact that \(k'(\text{AD}_{t,t',a}) = k''(\text{AD}_{t,t',a})\) (Claim 6.12). \(\square\)

6.3.1 Proof of Claim 6.15

In this section we prove Claim 6.15, which gives us Fourier-analytic properties of \(\text{AD}_{t,t',a}\) for particular settings of \(t, t', a\).

Proof of Claim 6.15. Given any \(\rho, \kappa, \kappa'\) as in the assumptions of the claim, recall that we fix the following values.

\[
\begin{align*}
t &= \frac{2\rho}{\log \kappa}, \quad (52) \\
t' &= \kappa \left(\frac{\log \kappa}{\rho}\right)^2, \quad (53) \\
a &= \frac{2\kappa' \log \kappa}{\rho}. \quad (54)
\end{align*}
\]

Since \(\rho \geq \log \kappa\),

\[
t = \frac{2\rho}{\log \kappa} \geq 2.
\]

Since \(\rho < \sqrt{\kappa}\) by assumption,

\[
t' = \kappa \left(\frac{\log \kappa}{\rho}\right)^2 \geq \log^2 \kappa \geq 2.
\]
for large enough $\kappa$. Since $\kappa' \geq \frac{\kappa \log \kappa}{\rho}$,

$$a = \frac{2\kappa' \log \kappa}{\rho} \geq 2\kappa \left( \frac{\log \kappa}{\rho} \right)^2 = 2t'. \quad (55)$$

Hence the assumptions in Claim 6.12 are satisfied with these values of $t, t', a$. By Equations (52), (53) and (54), we obtain the following bound which is of use to us later.

$$\frac{at}{t'} = \frac{2\kappa' \log \kappa}{\rho} \cdot \frac{2 \rho}{\log \kappa} \cdot \frac{\rho^2}{\kappa \log^2 \kappa} = \frac{4\kappa'}{\kappa} \left( \frac{\rho}{\log \kappa} \right)^2 \leq \left( \frac{2 \rho}{\log \kappa} \right)^2 \leq t^2. \quad (56)$$

We have the following properties of $AD_{t, t', a}$.

- **Rank:**

  $$r(AD_{t, t', a}) = (t - 1) \log t' + \log a + \log t$$
  by Claim 6.12

  $$= t \log t' + \log \left( \frac{at}{t'} \right) \leq 3t \log t'$$
  by Equation (56)

  $$= \frac{6 \rho}{\log \kappa} (\log(\kappa/\rho^2) + 2 \log \log \kappa)$$
  by Equations (52) and (53)

  $$\leq \frac{6 \rho}{\log \kappa} (\log \kappa + 2 \log \kappa)$$

  $$= O(\rho).$$

For our setting of parameters (Equations (52), (53) and (54)), $at = 4\kappa'$. Since $\rho \geq \log \kappa$ and $\kappa' \geq \frac{\kappa \log \kappa}{\rho}$,

$$\frac{at}{t'} = \frac{4\kappa'}{\kappa} \left( \frac{\rho}{\log \kappa} \right)^2 \geq \frac{4 \rho}{\log \kappa} > 1. \quad (57)$$

Thus,

$$r(AD_{t, t', a}) = (t - 1) \log t' + \log a + \log t$$

$$= t \log t' + \log \left( \frac{at}{t'} \right) \geq t \log t'$$

$$= \frac{2 \rho}{\log \kappa} (\log(\kappa/\rho^2) + 2 \log \log \kappa)$$

$$\geq \frac{2 \rho}{\log \kappa} (\log(\kappa/\rho^2)) \geq \frac{2 \rho}{\log \kappa} (\log \kappa^{2e})$$

$$\geq \frac{2 \rho}{\log \kappa} (2e \log \kappa)$$

$$= \Omega(e \rho).$$

- **Sparsity:** By our choice of parameters (Equations (52), (53) and (54)), we have

  $$ta = 4\kappa' \leq 4\kappa = t^2 t'. \quad (58)$$

Thus,

$$k(f_{\rho, \kappa, \kappa'}) = (t - 1)(t' - 1)t + ta$$

by Claim 6.12
= \Theta(t^2 t') \quad \text{by Equation (58) and since } t, t' \geq 2
\]
\]
\[
= \Theta\left(\left(\frac{2\rho}{\log \kappa}\right)^2 \kappa \left(\frac{\log \kappa}{\rho}\right)^2\right) \quad \text{by Equations (52) and (53)}
\]
\[
= \Theta(\kappa).
\]

- Max-supp-entropy: Since \( ta = 4\kappa' \) from Equation (58), we have by Claim 6.12 that
\[
k'(f_{\rho, \kappa, \kappa'}) = \frac{ta}{2} = \Theta(\kappa').
\]

- Weight:
\[
d(f_{\rho, \kappa, \kappa'}) = \frac{1}{t'} + \frac{1}{a} - \frac{1}{tt'}
\]
\[
= \frac{1}{t'} + \frac{1}{t} \left(\frac{1}{a} - \frac{1}{t'}\right)
\]
\[
\in \left[\frac{1}{t'} - \frac{1}{tt'} - \frac{1}{tt'} - \frac{1}{2tt'}\right] \quad \text{since } 2t' \leq a \text{ by Equation (55)}
\]
\[
= \Theta\left(\frac{1}{t'}\right) \quad \text{since } t \geq 2
\]
\[
= \Theta\left(\frac{1}{\kappa} \left(\frac{\rho}{\log \kappa}\right)^2\right). \quad \text{by Equation (53)}
\]

\[
\]

6.3.2 Proof of Claim 6.16

In this section we prove Claim 6.16, which gives us Fourier-analytic properties of \( \text{AAB}_{t,t',\ell} \) for particular settings of \( t, t', \ell \).

Proof of Claim 6.16. Given any \( \rho, \kappa, \kappa' \) as in the assumptions of the claim, recall that we fix the following values.

\[
t = \frac{2\rho}{\log \kappa}, \quad \text{Equation (59)}
\]
\[
t' = \frac{4\kappa'^2}{\kappa}, \quad \text{Equation (60)}
\]
\[
\ell = 2 \left(\frac{\kappa \log \kappa}{\kappa' \rho}\right)^2. \quad \text{Equation (61)}
\]

Since \( \rho \geq \log \kappa \), we have \( t = \frac{2\rho}{\log \kappa} \geq 2 \). Since \( \kappa' \geq \sqrt{\kappa} \), we have \( t' = \frac{4\kappa'^2}{\kappa} \geq 4 \). Finally since \( \kappa' \leq (\kappa \log \kappa)/\rho \), \( \ell = 2 \left(\frac{\kappa \log \kappa}{\kappa' \rho}\right)^2 \geq 2 \). Hence the assumptions in Claim 6.11 are satisfied with these values of \( t, t' \) and \( \ell \).
By Equations (60) and (61), we obtain the following bound which is of use to us later.

\[ \ell t' = 2 \left( \frac{\kappa \log \kappa}{\kappa' \rho} \right)^2 \cdot 4\kappa' \kappa = 8\kappa \left( \frac{\log \kappa}{\rho} \right)^2. \]  \hspace{1cm} (62)

We have the following properties of \( \text{AAB}_{t,t',\ell}. \)

- **Rank:**

  \[
  r(\text{AAB}_{t,t',\ell}) = t(\log t' + \log \ell) + \log t \\
  = t \log(\ell t') + \log t \leq 2t \log(\ell t') \hspace{1cm} \text{by Claim 6.11} \\
  = \frac{4\rho}{\log \kappa} \left( \log 8 + \log (\kappa/\rho^2) + 2 \log \log \kappa \right) \hspace{1cm} \text{by Equation (62)} \\
  \leq \frac{4\rho}{\log \kappa} \left( \log \kappa + 2 \log \kappa \right) \hspace{1cm} \text{since} \kappa \text{is sufficiently large} \\
  = O(\rho).
\]

For the lower bound, we have

\[
  r(\text{AAB}_{t,t',\ell}) = t(\log t' + \log \ell) + \log t \\
  \geq t \log(\ell t') \hspace{1cm} \text{since} \ t \geq 1 \\
  = \frac{2\rho}{\log \kappa} \cdot (\log 8 + \log (\kappa/\rho^2) + 2 \log \log \kappa) \hspace{1cm} \text{by Equations (59) and (62)} \\
  \geq \frac{2\rho}{\log \kappa} \cdot (\log (\kappa/\rho^2)) \geq \frac{2\rho}{\log \kappa} \cdot (\log \kappa^{2\epsilon}) \hspace{1cm} \text{since} \ \rho \leq \kappa^{\frac{1}{2} - \epsilon} \\
  = \frac{2\rho}{\log \kappa} (2\epsilon \log \kappa) \\
  = \Omega(\epsilon \rho).
\]

- **Sparsity:**

  \[
  k(\text{AAB}_{t,t',\ell}) = 1 + \frac{1}{2} t^2 (\ell + 1)t' \\
  = \Theta \left( t^2 \ell t' \right) \\
  = \Theta \left( \left( \frac{2\rho}{\log \kappa} \right)^2 \cdot 8\kappa \left( \frac{\log \kappa}{\rho} \right)^2 \right) \hspace{1cm} \text{by Equations (59) and (62)} \\
  = \Theta(\kappa).
\]

- **Max-supp-entropy:**

  \[
  k'(\text{AAB}_{t,t',\ell}) = \frac{tt'\sqrt{\ell}}{2} \hspace{1cm} \text{by Claim 6.11} \\
  = \frac{1}{2} \cdot \frac{2\rho}{\log \kappa} \cdot (2\kappa')^2 \cdot \sqrt{2} \kappa \log \kappa \hspace{1cm} \text{by Equations (59), (60) and (61)} \\
  = \Theta(\kappa').
\]
• Weight:

\[
\delta(AAB_{t,t',t}) = \frac{1}{t'}
\]

by Claim 6.11

\[
= \Theta\left(\frac{\kappa}{\kappa''^2}\right).
\]

by Equation (60)

\[
\nabla
\]

\[
\nabla
\]

6.3.3 Proof of Claim 6.17

In this section we prove Claim 6.17, which gives us Fourier-analytic properties of \(mAD_{t,t',p}\) for particular settings of \(t, t', p\).

Proof of Claim 6.17. Given \(\rho, \kappa\) and \(\kappa''\) choose:

\[
p = \log\left(\frac{4\kappa}{\kappa''}\right)
\]

(63)

\[
t' = \frac{\kappa''}{\rho} \log\left(\frac{\kappa''}{\rho}\right)
\]

(64)

\[
t = \frac{2\rho}{\log(\kappa''/\rho)}
\]

(65)

Since \(\rho \geq \log \kappa \geq \log(\kappa''/\rho)\), we have \(t = \frac{2\rho}{\log(\kappa''/\rho)} \geq 2\). Since \(\kappa'' \geq 2\rho\), we have \(t' = \frac{\kappa''}{\rho} \log\left(\frac{\kappa''}{\rho}\right) \geq \frac{\kappa''}{\rho} \geq 2\). Note that \(p \geq 2\) since \(\log(\frac{4\kappa}{\kappa''}) \geq \log 4 = 2\).

Finally we show that \(p \leq (t/2) \log t' \leq (t - 1) \log t'\):

\[
(t - 1) \log t' \geq (t/2) \log t'
\]

\[
= \frac{\rho}{\log(\kappa''/\rho)} \log\left(\frac{\kappa''}{\rho} \log\left(\frac{\kappa''}{\rho}\right)\right)
\]

\[
= \rho + \rho \cdot \frac{\log \log(\kappa''/\rho)}{\log(\kappa''/\rho)}
\]

\[
\geq \rho
\]

\[
\geq \log \kappa \geq \log \left(4\kappa/\kappa''\right)
\]

since \(\kappa'' \geq e \log \kappa \geq 4\) as \(\kappa\) is sufficiently large

\[
= p.
\]

Hence the assumptions in Claim 6.13 are satisfied with these values of \(t, t', p\).

We first state and prove some auxiliary claims which we require. The derivative of \(\frac{\log(\kappa''/\rho)}{\kappa''}\) with respect to \(\kappa''\) equals

\[
\frac{1 - \ln (\kappa''/\rho)}{\ln 2 \cdot (\kappa'')^2}.
\]

This value is negative since \(\kappa'' > e \cdot \rho\). Thus, \(\frac{\log(\kappa''/\rho)}{\kappa''}\) is a decreasing function in \(\kappa''\) for \(\kappa'' > e \cdot \rho\). Consider the expression

\[
\frac{2p}{t} = \frac{4\kappa \log(\kappa''/\rho)}{2\rho}
\]
\[ \geq 2\kappa \cdot \log \left( \frac{\log \kappa}{\rho'} \right) \]

Since \( \frac{\log (\kappa''/\rho)}{\kappa''} \) is a monotone decreasing function in \( \kappa'' \) and \( \kappa'' \leq \frac{\kappa \log \kappa}{\rho} \).

\[ = 2 \left( \frac{\log \kappa}{\rho^2} \right) + \log \log \kappa \]

\[ \geq 2 \frac{\log^2 \epsilon + \log \log \kappa}{\log \kappa} \quad \text{since } \rho \leq \kappa^{1/2-\epsilon} \]

\[ \geq 4\epsilon. \]

Therefore

\[ 2^p \geq 4\epsilon t. \quad (66) \]

Next, observe that

\[ tt' = \frac{2\rho}{\log (\kappa''/\rho)} \left( \frac{\kappa'' \log (\kappa''/\rho)}{\rho} \right) = 2\kappa''. \quad (67) \]

We have the following properties of \( \text{mAD}_{t,t',p} \).

- Rank:

\[ r(\text{mAD}_{t,t',p}) = \Theta \left( t \log t' \right) \quad \text{by Claim 6.13} \]

\[ = \Theta \left( \frac{\rho}{\log (\kappa''/\rho)} \left( \log \kappa'' + \log \log \left( \frac{\kappa''}{\rho} \right) \right) \right) \quad \text{by Equations (65) and (64)} \]

\[ = \Theta \left( \frac{\rho}{\log (\kappa''/\rho)} \left( \log \kappa'' \right) \right) \]

\[ = \Theta(\rho). \]

- Sparsity: For the upper bound, we have

\[ k(\text{mAD}_{t,t',p}) = O(2^p t t' + t^2 t') \quad \text{by Claim 6.13} \]

\[ = O \left( 2^p t t' + \frac{2^p t t'}{4\epsilon} \right) \quad \text{by Equation (66)} \]

\[ = O \left( \frac{1}{\epsilon} \cdot 2^p t t' \right) \quad \text{since } 0 < \epsilon < 1/2 \]

\[ = O \left( \frac{1}{\epsilon} \cdot \frac{\kappa}{\kappa''} \cdot \kappa'' \right) \quad \text{by Equations (67) and (63)} \]

\[ = O \left( \frac{\kappa}{\epsilon} \right). \]

For the lower bound, we have

\[ k(\text{mAD}_{t,t',p}) = \Omega(2^p t t') \quad \text{by Claim 6.13} \]

\[ = \Omega \left( \frac{\kappa}{\kappa''} \cdot \kappa'' \right) \quad \text{by Equations (67) and (63)} \]

\[ = \Omega(\kappa). \]
• Max-rank-entropy:

\[
\begin{align*}
\kappa''(m\text{AD}_{t,t'},p) &= \Theta(tt') \\
&= \Theta(\kappa'').
\end{align*}
\]

by Claim 6.13

\[
\begin{align*}
\kappa'' &= \Theta(\kappa'' / \rho).
\end{align*}
\]

by Equation (67)

• Weight:

\[
\begin{align*}
\delta(m\text{AD}_{t,t'},p) &= \frac{1}{t'} \\
&= \frac{\rho}{\kappa'' \log(\kappa'' / \rho)}.
\end{align*}
\]

by Claim 6.13

\[
\begin{align*}
\delta &= \frac{1}{t}.
\end{align*}
\]

by Equation (64)

7 Dominating Chang’s lemma for all thresholds

In this section, we show that there exists a function such that for any choice of threshold, the lower bound on weight of that function that we obtain from Theorems 1.4 and 1.6 are stronger than the lower bound obtained from Chang’s lemma (Lemma 1.1).

Claim 7.1 (Beating Chang’s lemma for all thresholds for \(\text{AD}_{t,t}\)). Consider any integer \(t > 4\) and define the function \(f = \text{AD}_{t,t} : \{-1,1\}^{\log t} \times \{-1,1\}^{t \log t} \to \{-1,1\}\) as in Definition 6.1. Then,

• \(\delta(f) = \frac{1}{t}\).

• For all real \(x > 0\) for which \(\dim(S_x) > 1\), we have

\[
\frac{\sqrt{\dim(S_x)}}{x \sqrt{\log(x^2 / \dim(S_x))}} = O\left(\frac{1}{x^{3/2}}\right).
\]

• \(\frac{1}{k(f)} \left(\frac{r(f)}{\log k(f)}\right)^2 = \Omega\left(\frac{1}{t}\right), \quad k(f) k'(f)^2 = \Omega\left(\frac{1}{t}\right) \quad \text{and} \quad \frac{r(f)}{k''(f) \log k(f)} = \Omega\left(\frac{1}{t}\right)\).

In particular, Claim 7.1 shows that our bounds can be strictly stronger than those given by Chang’s lemma, in the following sense.

• All the lower bounds on \(\delta(f)\) from Theorems 1.4 and 1.6 are tight, as witnessed by \(f = \text{AD}_{t,t}\).

• No matter what threshold \(x\) is chosen in Lemma 1.1, the best possible lower bound on \(\delta(\text{AD}_{t,t})\) that we get can get from Lemma 1.1 is \(\Omega\left(\frac{1}{x^{3/2}}\right)\), which is polynomially smaller than \(1/t\), the actual weight of \(\text{AD}_{t,t}\).

Proof of Claim 7.1. From Claim 6.9, we have

\[
\delta(f) = \frac{1}{t}.
\]
Thus from Observation 4.2,

$$\hat{f}(\emptyset) = 1 - \frac{2}{t}.$$  \hspace{1cm} (68)

First, we show that except for the Fourier coefficient of the empty set, all other Fourier coefficients of \( f \) have magnitude equal to \( \frac{2}{t^2} \).

Towards a contradiction assume that there exists \( T \subseteq [\log t] \cup [t \log t] \) such that \( \left| \hat{f}(T) \right| > \frac{2}{t} \).

We have,

$$1 = \sum_{S \subseteq [\log t] \cup [t \log t]} \hat{f}^2(S) \quad \text{from Theorem 4.6}$$

$$= \left(1 - \frac{2}{t} \right)^2 + \sum_{S \subseteq [\log t] \cup [t \log t], S \neq \emptyset} \hat{f}^2(S) \quad \text{by Equation (68)}$$

$$= \left(1 - \frac{2}{t} \right)^2 + \hat{f}^2(T) + \sum_{S \subseteq [\log t] \cup [t \log t], S \neq \emptyset, S \neq T} \hat{f}^2(S)$$

$$> \left(1 - \frac{2}{t} \right)^2 + \frac{4}{t^4} + \sum_{S \subseteq [\log t] \cup [t \log t], S \neq \emptyset, S \neq T} \hat{f}^2(S) \tag{69}$$

From Claim 6.9, \( k(f) = 1 + t^2(t - 1) = t^3 - t^2 + 1 \) and \( k'(f) = t^2/2. \) Using these, along with the definition of \( k'(f) \) (Definition 4.26), in Equation (69),

$$1 > \left(1 - \frac{2}{t} \right)^2 + \frac{4}{t^4} + \frac{4(k(f) - 2)}{t^4}$$

$$= \left(1 - \frac{2}{t} \right)^2 + \frac{4}{t^4} + \frac{4(t^3 - t^2 + 1 - 2)}{t^4}$$

$$= \left(1 - \frac{2}{t} \right)^2 + \frac{4}{t^4} + \frac{4(t^3 - t^2 - 1)}{t^4}$$

$$= \left(1 - \frac{2}{t} \right)^2 + \frac{4}{t^4} + \left(\frac{4}{t} - \frac{4}{t^2} - \frac{4}{t^3}\right)$$

$$= \left(1 + \frac{4}{t^2} - \frac{4}{t}\right) + \frac{4}{t^4} + \left(\frac{4}{t} - \frac{4}{t^2} - \frac{4}{t^3}\right)$$

$$= 1.$$

Thus,

$$\left| \hat{f}(S) \right| = \frac{2}{t^2} \quad \text{for all non-empty } S \subseteq ([\log t] \cup [t \log t]). \tag{70}$$

We now prove the second part of the claim. If \( x < \frac{t^2}{2} \) then \( S_x = \{\emptyset\} \) and has dimension 0. On the other hand, for any \( x \geq \frac{t^2}{2} \), we have \( S_x = \text{supp}(f) \) by Equation (70), and hence \( \dim(S_x) = r(f) = (t + 1)(\log t) \) by Claim 6.9. Hence, in this case,

$$\frac{\sqrt{\dim(S_x)}}{x \sqrt{\log(x^2/\dim(S_x))}} = O \left( \frac{\sqrt{t \log t}}{t^2 \sqrt{\log t}} \right) = O \left( \frac{1}{t^{3/2}} \right),$$
On the other hand, by Claim 6.9, \( r(f) = \Theta(t \log t) \), \( k(f) = \Theta(t^2) \), \( k'(f) = k''(f) = \Theta(t^2) \). The third part of the claim follows.

\[ \square \]

8 Conclusions

In this paper, for Boolean functions \( f \), we study the relationship between weight and other Fourier-analytic measures namely rank, sparsity, max-supp-entropy and max-rank-entropy. For a threshold \( t > 0 \), Chang’s lemma gives a lower bound on the weight of a Boolean function \( f \) in terms of \( \dim \left( \{ S \subseteq [n] : |\hat{f}(S)| \geq \frac{1}{t} \} \right) \). We consider three natural thresholds \( t \) in Chang’s lemma, namely \( k(f) \), \( k'(f) \) and \( k''(f) \), yielding three lower bounds on weight in terms of these measures. We prove new lower bounds on weight in Theorems 1.4 and 1.6, and our bounds dominate all the above-mentioned bounds from Chang’s lemma for a wide range of parameters.

When \( \log k(f) = \Theta(r(f)) \), the function \( f = \text{AND} \) already shows that all the above lower bounds are tight. To consider all other feasible relationships between \( k(f) \) and \( r(f) \), we divide our investigation of these lower bounds into two different parts. In the first part, we vary over all feasible settings of \( r(f), k(f) \) and \( k'(f) \), and construct functions that witness tightness of our lower bounds in Theorem 1.4 for nearly all such feasible settings (Theorem 1.5). In the second part, we vary over all feasible settings of \( r(f), k(f) \) and \( k''(f) \), and construct functions that witness near-tightness of our lower bounds in Theorem 1.6 for nearly all such feasible settings (Theorem 1.7). These functions are constructed by carefully composing the Addressing function with suitable inner functions. We show a composition lemma (Lemma 6.8), which relates the properties of the composed function with those of the inner functions; this allows us to come up with functions that match our lower bounds.

We also construct functions for which our lower bounds are asymptotically stronger than the lower bounds obtained from Chang’s lemma for all choices of threshold (Claim 7.1). The functions that we construct in this work might be of independent interest.

Open Problems. Claim 6.15 shows tightness of our upper bound on rank in terms of sparsity and weight (Theorem 1.3). Since our proof of Theorem 1.3 is a generalization of the proof of the upper bound \( r(f) = O(\sqrt{k(f)} \log k(f)) \) due to Sanyal [San19], it sheds light on the presence of the \( \log k \) factor in Sanyal’s upper bound. This still leaves the following question open: do there exist Boolean functions \( f \) for which \( r(f) = \omega(\sqrt{k(f)}) \)?

There are some ranges of parameters where we were not able to construct functions with upper bounds matching our lower bounds from Theorem 1.6. It will be interesting to see if our techniques can be extended to cover these ranges as well.

All thresholds \( t \) considered for Chang’s lemma in this work satisfy \( \dim(\{ S \subseteq [n] : |\hat{f}(S)| \geq \frac{1}{t} \}) = r(f) \). It is an interesting problem to obtain Chang’s-lemma-type bounds for thresholds for which this dimension is strictly less than \( r(f) \).

Acknowledgements: R.M. thanks DST (India) for grant DST/INSPIRE/04/2014/001799. S.S. is supported by an ISIRD Grant from SRIC, IIT Kharagpur. N.S.M. is supported by the NWO through QuantERA project QuantAlgo 680-91-034. T.M. would like to thank Prahladh Harsha and Ramprasad Saptharishi for helpful discussions.
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A Equivalence of the two forms of Chang’s lemma

Recall that for any Boolean function $f : \{-1,1\}^n \to \{-1,1\}$ and positive real number $t$, we define $S_t := \{S \subseteq [n] : |\hat{f}(S)| \geq \frac{1}{t}\}$. Chang’s lemma for the Boolean hypercube is usually stated in the literature as an upper bound on the dimension of $S_t$, as in Lemma A.1.

**Lemma A.1** (Common form of Chang’s lemma). Let $f : \{-1,1\}^n \to \{-1,1\}$ be a Boolean function and $t$ be any positive real number. Let $d = \dim(S_t) > 1$. Then

$$d = O(t^2 \delta(f)^2 \log(1/\delta(f))).$$
In this section we show that Lemma A.1 and Lemma 1.1 can be easily derived from each other. For convenience, we restate Lemma 1.1 below.

**Lemma A.2** (Restatement of Lemma 1.1). There exists a universal constant $c > 0$ such that the following is true. Let $f : \{-1, 1\}^n \rightarrow \{-1, 1\}$ be a Boolean function and $t$ be any positive real number. Let $d = \text{dim}(S_t) > 1$. If $\delta(f) < c$, then

$$
\delta(f) = \Omega \left( \frac{\sqrt{d}}{t \sqrt{\log \left( \frac{t^2}{d} \right)}} \right).
$$

We need the following claim.

**Claim A.3.** Define a function $h : (0, 1/e] \rightarrow \mathbb{R}$ by $h(\eta) = \eta \log(1/\eta)$. Then $h$ is monotonically non-decreasing.

**Proof.** Define $g(\eta) := \eta \ln(1/\eta)$, where $\ln$ denotes the natural logarithm (i.e., logarithm with base $e$). Since, $h(\eta) = \log e \cdot g(\eta)$, it is sufficient to show that $g$ is monotonically non-decreasing for $\eta \in (0, 1/e]$. Let $g'(\cdot)$ denote the derivative of $g$ with respect to $\eta$. We have that,

$$
g'(\eta) = \ln(1/\eta) - 1,
$$

which is non-negative for $\eta \in (0, 1/e]$. The claim follows. \hfill \Box

**Proof of equivalence of Lemma A.1 and Lemma A.2.**

**Lemma A.1 $\implies$ Lemma A.2.** Assume Lemma A.1 and let $f, t, \delta(f)$ and $d$ be as in the statement of Lemma A.2. From Lemma A.1 we have that $d = O(t^2 \delta(f)^2 \log(1/\delta(f)))$. Now,

$$
d = O(t^2 \delta(f)^2 \log \left( \frac{1}{\delta(f)} \right)) \quad (71)
$$

$$
\implies \frac{t^2}{d} = \Omega \left( \frac{1}{\delta(f)^2 \log \left( \frac{1}{\delta(f)} \right)} \right). \quad (72)
$$

Equation (72) implies that there exists a universal constant $c > 0$ (that depends on the constant hidden in the asymptotic notation) such that $t^2/d > 1$ whenever $\delta(f) < c$. Assuming $\delta(f) < c$ and taking logarithm of both sides of Equation (72) we have that

$$
0 < \log \left( \frac{t^2}{d} \right) = \Omega \left( \log(1/\delta(f)^2) - \log \log(1/\delta(f)) \right) = \Omega \left( \log(1/\delta(f)) \right). \quad (73)
$$

Equations (71) and (73) yield

$$
\frac{d}{t^2 \log \left( \frac{t^2}{d} \right)} = O \left( \frac{t^2 \delta(f)^2 \log(1/\delta(f))}{t^2 \log(1/\delta(f))} \right) = O(\delta(f)^2)
$$

$$
\implies \delta(f) = \Omega \left( \frac{\sqrt{d}}{t \sqrt{\log \left( \frac{t^2}{d} \right)}} \right).
$$

52
Lemma A.2 $\implies$ Lemma A.1. Assume Lemma A.2. Let $f, t, \delta(f)$ and $d$ be as in the statement of Lemma A.1 and $c$ be the universal constant in the statement of Lemma A.2. We assume without loss of generality (by replacing $f$ by $-f$ if necessary) that $\delta(f) \leq 1/2$. Now, we have that
\[
\frac{d}{t^2} \leq \frac{|S_t|}{t^2} \leq \sum_{S \in S_t} \hat{f}(S)^2 \leq 1,
\]
where the first inequality holds by the definition of $S_t$ and $d$, the second inequality holds from the definition of $S_t$, and the last inequality follows from Parseval’s identity (Theorem 4.6).

We conclude that
\[
d \leq t^2. \tag{74}
\]
We split the proof into two cases.

**Case 1:** $c \leq \delta(f) \leq 1/2$. By Claim A.3 and the observation that $c > 0$ and $\delta(f)^2 \leq 1/4 < 1/e$ we have that
\[
\delta(f)^2 \log(1/\delta(f)^2) \geq c^2 \log(1/c^2). \tag{75}
\]

By Equation (74) we have that
\[
d \leq t^2 = O(t^2 c^2 \log(1/c^2)) = O(t^2 \delta(f)^2 \log(1/\delta(f)^2)) = O(t^2 \delta(f)^2 \log(1/\delta(f))),
\]
where the third step follows from Equation (75).

**Case 2:** $0 \leq \delta(f) < c$. From Lemma A.2 we have that $\delta(f)^2 = \Omega\left(\frac{d}{t^2 \log(t^2/d)}\right) > 0$. Recall that by our assumption, $\delta(f) \leq 1/2$ and hence $\log (1/\delta(f)) \geq 1$. Now, if $t^2/d < 2$, we have from Lemma A.2 that $d = O(\delta(f)^2 t^2 \log (t^2/d)) = O(\delta(f)^2 t^2) = O(\delta(f)^2 t^2 \log (1/\delta(f)))$ and the proof is complete. Thus, assume henceforth that $t^2/d \geq 2$ and hence $\log (t^2/d) \geq 1$. By Lemma A.2, Claim A.3 and the observation that $\delta(f)^2 \leq \frac{1}{4} < \frac{1}{e}$ we have that
\[
t^2 \delta(f)^2 \log(1/\delta(f)^2) = \Omega\left(t^2 \left(\frac{d}{t^2 \log(t^2/d)}\right) \cdot \log \left(\frac{t^2 \log(t^2/d)}{d}\right)\right)
= \Omega\left(\left(\frac{d}{\log(t^2/d)}\right) \cdot \log \left(\frac{t^2 \log(t^2/d)}{d}\right)\right)
= \Omega\left(\frac{d}{\log(t^2/d)} \cdot \log (t^2/d)\right) \quad \text{since } \log (t^2/d) \geq 1
= \Omega(d).
\]

This completes the proof.

---

9This case is vacuous if $c > 1/2$. 

53