Improved leakage-equilibration-absorption scheme (ILEAS) for neutrino physics in compact object mergers

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ABSTRACT

We present a new, computationally efficient, energy-integrated approximation for neutrino effects in hot and dense astrophysical environments such as supernova cores and compact binary mergers and their remnants. Our new method, termed ILEAS for Improved Leakage-Equilibration-Absorption Scheme, improves the lepton-number and energy losses of traditional leakage descriptions by a novel prescription of the diffusion time-scale based on a detailed energy integral of the flux-limited diffusion equation. The leakage module is supplemented by a neutrino-equilibration treatment that ensures the proper evolution of the total lepton number and medium plus neutrino energies as well as neutrino-pressure effects in the neutrino-trapping domain. Moreover, we employ a simple and straightforwardly applicable ray-tracing algorithm for including re-absorption of escaping neutrinos especially in the decoupling layer and during the transition to semi-transparent conditions. ILEAS is implemented on a three-dimensional (3D) Cartesian grid with a minimum of free and potentially case-dependent parameters and exploits the basic physics constraints that should be fulfilled in the neutrino-opaque and free-streaming limits. We discuss a suite of tests for stationary and time-dependent proto-neutron star models and post-merger black-hole-torus configurations, for which 3D ILEAS results are demonstrated to agree with energy-dependent 1D and 2D two-moment (M1) neutrino transport on the level of 10–15 percent in basic neutrino properties. This also holds for the radial profiles of the neutrino luminosities and of the electron fraction. Even neutrino absorption maps around torus-like neutrino sources are qualitatively similar without any fine-tuning, confirming that ILEAS can satisfactorily reproduce local losses and re-absorption of neutrinos as found in sophisticated transport calculations.

Key words: gravitational waves — hydrodynamics — nucleosynthesis — neutrinos — stars: neutron

1 INTRODUCTION

Neutrinos play an important role in high-energy stellar astrophysics, more precisely in the context of the birth and death of neutron stars (NSs). Already in the 1990s, the neutrino-induced shock revival was theorized as a mechanism for exploding core-collapse supernovae (SNe) (Bethe & Wilson 1985, Bethe 1990, see Janka 2012, Burrows 2013 and Foglizzo et al. 2015 for recent reviews). After a successful explosion, neutrinos are essential to understand the long-term cooling of the new-born NS (e.g. Hüdepohl et al. 2010). At the other end of their lives, NSs in binaries with another compact object (CO), either a NS, a black hole (BH) or a white dwarf (WD), may be able to merge within less than a Hubble time. In such scenarios, NS matter reaches very high temperatures and densities, emitting copious neutrinos (see e.g. Shibata & Taniguchi 2011, Faber & Rasio 2012 and Rosswog 2015 for reviews). Despite being dynamically only a secondary ingredient, neutrinos, by their emission and absorption, drive the neutron-to-proton ratio of the NS as well as the ejected matter. This aspect will determine the distribution of synthesized elements in the ejecta (Goriely et al. 2015; Just et al. 2015a; Wanajo et al. 2014; Sekiguchi et al. 2015; Lippuner & Roberts 2015; Foucart et al. 2016a,b; Radice et al. 2016; Bovard et al. 2017; Martin et al. 2018; Lehner et al. 2016a) as well as the associated electromagnetic
transient, known as kilonova, powered by the radioactive decay of neutron-rich elements (Li & Paczynski 1998; Kulkarni 2005; Metzger et al. 2010; Gorio et al. 2011; Roberts et al. 2011; Hotokezaka et al. 2016; Barnes et al. 2016; Bovard et al. 2017, see Thielemann et al. 2017 and Fernández & Metzger 2016; Metzger 2017 for recent reviews on r-process in NS mergers and kilonovae, respectively). Neutrinos are also needed to understand the fate of the hypermassive NS (HMNS) remnants and the evolution of the surrounding torus (if any) (Perego et al. 2014a; Metzger & Fernández 2014; Foucart et al. 2015; Fujibayashi et al. 2017; Wu et al. 2017). Merger remnants have been envisioned as the central engines of short gamma-ray bursts (sGRB), and neutrino pair-annihilation has been shown to deposit significant amounts of energy, which might help in powering an ultrarelativistic jet (Just et al. 2016; Perego et al. 2017a). The recent detection of a NS merger through its associated signals in the form of gravitational waves (GW) and electromagnetic radiation (EM), as predicted by theoretical models, highlights the remarkable contributions of detailed numerical simulations to our understanding of the universe.

The evolution of the neutrino phase space distribution function follows the Boltzmann transport equation. In three spatial dimensions (3D), this becomes a six-dimensional, time-dependent problem for each neutrino species, which is considerably arduous to solve when no further approximations are applied to reduce its dimensionality (e.g. Lindquist 1966). Therefore, numerous schemes of varying complexity and accuracy have been developed to cope with this challenging task.

In the context of NS mergers, truncated moment schemes are the most sophisticated treatments successfully used. In such schemes, the angular dependence of the neutrino momentum distribution is removed by evolving a hierarchy of ("moment") equations, which are obtained from angular-moment integration of the Boltzmann equation. Thus one introduces moments as angular integrals of the neutrino phase-space distribution function, i.e. neutrino energy density, flux density, pressure and higher-order moments. It is necessary, in order to close the set of equations, to find a way to express the highest employed moments, which are not evolved by but appear in the moment equations. The so-called M1 schemes (e.g. Shibata et al. 2011) time-integrate the evolution equations for the zeroth- and first-order moments, closing the system with an analytical relation expressing the highest employed moment as local functions of the evolved ones, and are used in grey (e.g. Foucart et al. 2015) as well as energy-dependent versions (e.g. Just et al. 2015b). Despite their enormous current popularity for describing the transport of neutrinos, they suffer from the inability to properly handle crossing radiation beams, and thus have a tendency to overestimate the neutrino densities in the polar directions in NS-merger remnant simulations (Just et al. 2015a; Foucart et al. 2018). These limitations call for alternative treatments of neutrino re-absorption, such as ray-tracing, in order to provide an accurate description of the ejecta composition, essential for predicting EM counterparts to NS mergers. Recently, an alternative Monte Carlo closure has been suggested by Foucart (2018), but its applicability in merger simulations has not been demonstrated yet. Monte Carlo (MC) codes are still obviated from their direct use in full-scale merger simulations because of their tremendous computational costs and memory requirements, for which reason they have been set aside in favour of less expensive methods, except for some MC applications in snapshot calculations (Richers et al. 2015; Foucart et al. 2018). Ray-tracing algorithms, which solve the Boltzmann equation in one dimension, have also been employed in snapshot calculations of NS merger remnants by Just et al. (2015a) in the Newtonian framework and, recently, by Deaton et al. (2018) in a general-relativistic version, which also includes the effects of neutrino scattering in an approximative manner based on the existence of a precomputed M1 solution. Leakage schemes are a very popular and computationally simple approximation for the treatment of neutrino effects in NS merger simulations. They had been introduced first in Newtonian merger models as grey versions by Ruffert et al. (1996, 1997); Ruffert & Janka (1999, 2001) and, with a different handling of the energy integration, by Rosswog & Liebendörfer (2003); Rosswog et al. (2003); Korobkin et al. (2012); Rosswog et al. (2013); Rosswog (2013). More recently, leakage schemes have been used in many relativistic merger simulations, too (for example, Sekiguchi et al. 2011a,b; Kiuchi et al. 2012; Deaton et al. 2013; Foucart et al. 2014, 2017; Neilsen et al. 2014; Palenzuela et al. 2015; Lehner et al. 2016b,a; Bernuzzi et al. 2016). Moreover, leakage methods have been applied in evolution studies of NS merger remnants, including HMNSs (Perego et al. 2014a; Martin et al. 2015, 2018; Metzger & Fernández 2014; Lippuner et al. 2017) as well as BH-torus systems (Shibata et al. 2007; Fernández et al. 2015a,b).

The most basic version of leakage schemes accounts for neutrino energy and lepton-number losses by local source terms in the hydrodynamics equations, following the original implementations by Ruffert et al. (1996) and Rosswog & Liebendörfer (2003), possibly upgraded by gravitational redshift effects (Sekiguchi 2010; O’Connor & Ott 2010; Galeazzi et al. 2013). In more advanced versions this basic functionality of leakage schemes is supplemented by treatments of a trapped neutrino component and by neutrino absorption. Such improvements have been accomplished by hybrid methods between leakage and M1 (Sekiguchi et al. 2012, 2015, 2016; Fujibayashi et al. 2017; Shibata et al. 2017; Kiyotoku et al. 2018) or ‘M0’ (zeroth moment with a closure) (Radice et al. 2016; Radice 2017; Radice et al. 2017, 2018; Zappa et al. 2018). Alternatively, equilibrium and time-scale arguments have been used to parametrize the trapping physics, and complex propagation paths have been considered to connect the locations of neutrino production in a leakage treatment with the neutrinospheric decoupling region for describing neutrino absorption exterior to the trapping domain (see Perego et al. 2014a,b, 2016).

Comparisons by Foucart et al. (2015, 2016a) reveal major differences between results with their grey M1-based SpEC code and ‘classical’ leakage results for the first 10–15 ms after the collision of compact binary stars. In contrast, Perego et al. (2016) report very good qualitative and partially quantitative agreement in key quantities when testing their Advanced Spectral Leakage (ASL) scheme against Boltzmann transport in the context of Newtonian, spherically symmetric (1D) hydrodynamic simulations of several 100 ms of post-bounce accretion in core-collapse supernovae. However, the ASL code involves a variety of parameters that were calibrated on grounds of this considered problem. It is
not obvious that the thus determined parameter values work equally well for a broader class of conditions. Moreover, also the axisymmetric (2D) and three-dimensional (3D) simulations of stellar core-collapse and post-bounce accretion, which Perego et al. (2016) applied their ASL code to, still contain a quasi-spherical, highly opaque neutrino source that accretes mass from the collapsing star at high rates (as in the 1D simulations). These tests are not conclusive with respect to the question how well the ASL code is able to perform in the merger case, where the remnant is rotationally deformed and not accreting. Radial profiles of the neutrino quantities for the transition from the high-opacity to the low-opacity regime and tests with non-spherical neutrino sources, which could facilitate such a judgement, are not available for upgraded leakage schemes in the literature.

In this work we present a new implementation of an improved leakage treatment that does not only take into account local energy and lepton-number losses by neutrino emission, but it also accounts for the fact that neutrinos can equilibrate with matter in the optically thick regime and that they are still re-absorbed by matter when they propagate through optically thin regions. Our new method, which is termed Improved Leakage-Equilibrium-Absorption Scheme (ILEAS), is designed to fulfil a number of requirements: (1) low algorithmic complexity in order to enable easy numerical realization; (2) proper and consistent reproduction of the correct physical behaviour of the neutrino-matter system at high optical depths; (3) description of the transition to the low-opacity regime with a minimum number of free parameters and ad hoc recipes of approximation; and (4) high computational efficiency that permits the calculation of large sets of merger simulations to explore the multidimensional parameter space (system masses and mass ratios, spins, orbital parameters, NS equations of state) that describes NS-NS/BH binaries. The computational efficiency is also facilitated by the fact that ILEAS, in contrast to transport calculations with explicit schemes, is not subject to any time-stepping constraints by the Courant-Friedrichs-Lewy condition (CFL).

ILEAS is implemented on a 3D Cartesian grid and makes use of the grey description of the leakage loss terms applied by Ruffert et al. (1996). The greyness of the treatment benefits all of the mentioned requirements. Advancing beyond the original treatment by Ruffert et al. (1996), ILEAS introduces a new definition of the neutrino-loss time-scale based on the energy-integrated equation of flux-limited diffusion. This allows for a considerably improved description of the neutrino drain from regions of high optical depths. The effects of a trapped neutrino component are taken into account by considering neutrinos as part of an equilibrated neutrino-matter fluid in the trapping regime. Neutrino absorption in the transition to the optically thin limit is handled by a simplified ray-tracing method that adopts an analytical integration of the radiation attenuation along the ray paths of escaping neutrinos following Janka (2001).

To assess the quality of the ILEAS scheme, we consider different stages during the cooling evolution of a spherical proto-NS (PNS) and perform steady-state as well as time-dependent calculations (co-evolving the medium temperature and electron fraction on a fixed background density). We compare the leakage results for the neutrino emission with 1D neutrino transport results obtained with the ALCAR and VERTEX codes. Both of these codes are energy-dependent two-moment schemes employing an algebraic (M1) closure and a variable Eddington factor closure based on a solution of the Boltzmann equation, respectively. Moreover, we perform time-dependent calculations for the neutrino emission from optically thick (high-mass) as well as optically thin (low-mass) axisymmetric BH-accretion tori in direct comparison with ALCAR results. Our tests demonstrate very good compatibility between leakage and transport results (global quantities agree on the level of roughly 10 percent or better) with respect to radial luminosity profiles, neutrino luminosities evolving over periods of tens of milliseconds, mean energies, and spatial distributions of electron fraction and neutrino-energy absorption rates.

Our paper is structured as follows. In section 2 we describe the physical and algorithmic components of the ILEAS code and their numerical realization, in section 3 we present our set of neutrino transport tests for proto-NS and BH-toruses, and in section 4 we summarize our work. In the four following appendices A–D we compare results for different definitions of the diffusion time-scale used in previous literature, present an overview of the neutrino opacities and source terms employed by ILEAS, discuss different versions of implementing the $\beta$-processes in the leakage treatment, and provide test results for an alternative method to compute neutrino-number re-absorption, respectively.

2 NUMERICAL DESCRIPTION

2.1 Weak interactions with ILEAS in CFC relativistic hydrodynamics

We present a novel neutrino leakage scheme, ILEAS, that is capable of reproducing the fundamental aspects of the neutrino physics described by more sophisticated transport schemes at lower computational costs. The scheme calculates the energy and lepton number changes caused by weak interactions of three neutrino species: electron neutrinos, $\nu_e$, electron antineutrinos, $\bar{\nu}_e$, and heavy-lepton neutrinos, $\nu_x$ (which include $\mu$ and $\tau$ neutrinos and their antiparticles). Neutrinos are considered to be massless because their relevant mean energies are of order MeV, orders of magnitude larger than their rest mass ($< 1$ eV). Neutrino flavour oscillations are ignored in our treatment. The full scheme is composed of three major modules which model different aspects of the transport of neutrinos, summarized in figure 1: the leakage, the equilibration and the absorption modules. The leakage unit estimates the local number and energy loss rates associated with neutrinos which 'leak' out of the system, as an interpolation between trapping and free streaming conditions. At high optical depths, neutrinos of all species are in equilibrium with matter, which we account for explicitly with our equilibration unit. This effect is ignored in most leakage schemes with few recent exceptions (Sekiguchi et al. 2010; Sekiguchi et al. 2011a; Perego et al. 2016), but was used as initial condition for nuclear network calculations (Goriely et al. 2015). Finally, the absorption module computes the energy and number deposition rates due to interactions of the escaping neutrinos with the optically thin material, by means of a simple ray-tracing algorithm.
The leakage and absorption modules provide the neutrino cooling rates, \( Q_{\nu_e}^{-} \) and \( Q_{\nu_e}^{+} \), and heating rates, \( Q_{\bar{\nu}_e}^{+} \), respectively, for all three neutrino species. The total energy source term, which will enter the hydrodynamical evolution equations, can be calculated from them as

\[
Q_{\text{tot}} = \sum_{\nu_e, \bar{\nu}_e} Q_{\nu_e}^{+} - \sum_{\nu_e, \bar{\nu}_e} Q_{\nu_e}^{-}.
\]  

We remark that the first sum includes only contributions from \( \nu_e \) and \( \bar{\nu}_e \), namely charged-current absorptions on nucleons. We neglect here the smaller effects of neutrino-antineutrino annihilation, which affect all flavours of neutrinos but contribute little to matter heating in baryondominated regions and are difficult to treat reasonably accurately without detailed knowledge of the neutrino phase-space distribution.

Similarly, the lepton change rates, \( R_{\nu_e}^{-} \) and \( R_{\nu_e}^{+} \), can be combined to the total (electron flavour) lepton change rate as,

\[
R_{\text{tot}} = R_{\nu_e}^{+} - R_{\nu_e}^{-} \quad \text{and} \quad R_{\nu_e}^{+}.
\]  

Details for the calculation of the rates will be discussed in sections 2.2 and 2.3.

Because we want to apply ILEAS in the context of NS mergers, we provide as an example the implementation of the source terms in the evolution equations of our conformally flat (CFC\(^1\)), relativistic NS merger code. For a more detailed description of the complete scheme, we refer the reader to Oechslin et al. (2007). In the present section (2.1) we use the convention \( c = G = 1 \).

As in most hydrodynamic solvers, we define the conserved quantities, namely conserved rest-mass density, \( \rho^* \), conserved specific momentum, \( \dot{u}_i \), and conserved specific energy, \( \tau \), as a function of their primitive counterparts, rest mass density, \( \rho \), velocity, \( v_i \), and specific internal energy, \( \varepsilon \), via

\[
\rho^* = \rho \alpha u^0 \psi^6, \quad \dot{u}_i = H \dot{u}_i = H(v^i + \beta^i)\psi^4 u^0, \quad \tau = HW - \frac{P}{\rho} - \frac{1}{\psi^4}(1 + \frac{\dot{u}_i \dot{u}_j \delta^{ij}}{\psi^4}).
\]  

Here the Lorenz factor is defined as \( W = \alpha u^0 = \sqrt{1 + \gamma^2 u_\alpha u^\alpha} \), with \( \gamma^2 \) being the spatial components of the metric, \( u^0 \) and \( u^i \) are the time and space components of the 4-velocity, \( H \) represents the relativistic specific enthalpy, defined as \( H = 1 + P/\rho + \varepsilon \), \( P \) is the fluid pressure and \( \delta^{ij} \) is the Kronecker delta. In the CFC approximation, the metric can be expressed as

\[
d\mathbf{s}^2 = (-\alpha^2 + \beta^i \beta^i)dt^2 + 2\beta_i dx^i dt + \psi^4 \delta_{ij} dx^i dx^j \]

(6)

where \( \alpha, \beta_i \) and \( \psi \) are the metric potentials, i.e. lapse, shift and conformal factor, respectively. We then write the relativistic Euler equations, where we include the neutrino source term defined in equation (1), \( Q_{\text{tot}} \), in the momentum and energy equations with the pertinent corrections, as

\[
\frac{d}{dt} \rho^* = -\rho \dot{\theta}_v \dot{v}^i, \quad \frac{d}{dt} \dot{u}_i = \frac{1}{\rho} \alpha \psi^6 \dot{\theta}_v P - \alpha \psi^5 \partial \alpha + \dot{u}_i \beta^i + \frac{2 \dot{u}_i \dot{u}_k}{\psi^4} \partial_i \psi + \frac{Q_{\text{tot}} \dot{\theta}_v}{\rho \psi^4},
\]

(7)

\[
\frac{d}{dt} \tau = -\frac{\psi^6}{\rho^*} \alpha^{\nu = 3} (v^i + \beta^i) \left( 1 - \frac{HW}{\psi^4} \right) \dot{\theta}_v P - \frac{\psi^6 P}{\rho^*} \dot{\theta}_v (v^i + \beta^i) \dot{\theta}_v \partial_v P + \frac{Q_{\text{tot}} \dot{\theta}_v}{\rho \psi^4},
\]

(8)

\[
\frac{d}{dt} \omega = \frac{1}{\psi^4} \left( 1 - \frac{HW}{\psi^4} \right) \left[ \dot{u}_i \dot{u}_j \beta^i \beta^j + \frac{1}{3} \dot{u}_i \dot{u}_j \beta^i \beta^j \right] + \frac{Q_{\text{tot}} \dot{\theta}_v}{\rho} \left( 1 - \frac{\dot{u}_i \dot{u}_j \delta^{ij}}{\psi^4} \right),
\]

(9)

where \( \dot{\theta}/dt = \partial_i + \dot{v}^i \partial_i \) and \( \omega = \sqrt{1 + \dot{u}_i \dot{u}_j \delta^{ij}/\psi^4} \).

To close the system, one needs a microphysical equation of state (EoS) as a function of \( \rho, \varepsilon \) and \( \gamma_e \), representing the thermodynamics of the fluid. In the equilibration module, we treat the regions where neutrinos are trapped and in \( \beta \)-equilibrium with the medium in a specific way by redefining the specific energy density, \( \varepsilon \), pressure, \( P \), and specific enthalpy, \( H \), to include the contributions from the combined fluid of matter plus trapped neutrinos. This means, that in order to close the set of evolution equations in those regions, we need to build an additional set of EoS tables which also incorporates the contributions from the neutrinos.

\(^1\) Conformally Flat Condition (Isenberg & Nester 1980; Wilson et al. 1996)
Without the inclusion of weak interactions, the net electron fraction, \( Y_e \), is just advected with the fluid (\( dY_e/dt = 0 \)). The leakage and absorption modules, however, provide a source term, \( R_{\text{tot}} \), as defined in equation (2), which enters the evolution equation of \( Y_e \),

\[
\frac{d}{dt} Y_e = \frac{R_{\text{tot}}}{\rho A W}.
\]

(10)

where \( A \) is Avogadro’s constant. To model the trapping conditions, we advect the trapped \( \nu_i \) and \( \bar{\nu}_e \) lepton fractions (equation 41) in addition to the \( Y_e \),

\[
\frac{d}{dt} Y_{\nu_i} = 0,
\]

(11)

\[
\frac{d}{dt} Y_{\bar{\nu}_e} = 0.
\]

(12)

The final goal of this procedure is to obtain an updated \textit{trapped lepton fraction} at the end of every time-step, defined as

\[
Y_{\nu_i} = Y_e + Y_{\nu_i}^{\text{trap}} - Y_{\bar{\nu}_e}^{\text{trap}}.
\]

(13)

We can then use this \( Y_{\nu_i} \) in an \textit{equilibration step} to recover the new equilibrium values for \( Y_e \), \( Y_{\nu_i}^{\text{erf}} \) and \( Y_{\bar{\nu}_e}^{\text{erf}} \). This requires the construction of a set of EoS tables which invert the dependence on \( \rho \), \( z \) and \( Y_{\nu_i} \) to obtain \( Y_e \). We will expand the details on the equilibration module in section 2.4.5.

### 2.2 The neutrino leakage scheme

The leakage part of our code is based on the archetypical leakage scheme from Ruffert et al. (1996). The essence of the model consists in the evaluation of the local effective neutrino production rates,

\[
R_{\nu_i}^{\text{erf}} = R_{\nu_i}^{\text{erf}}(Y_{\nu_i}^{\text{num}}, Y_{\bar{\nu}_e}^{\text{num}}),
\]

(14)

and

\[
Q_{\nu_i}^{\text{erf}} = Q_{\nu_i}^{\text{erf}}(Y_{\nu_i}^{\text{num}}, Y_{\bar{\nu}_e}^{\text{num}}),
\]

(15)

where \( R_{\nu_i}^{\text{erf}} \) and \( Q_{\nu_i}^{\text{erf}} \) are the local neutrino production rates for number and energy, respectively, as defined in equations (B33) and (B34). \( \gamma_{\nu_i}^{\text{erf}} \) and \( \gamma_{\bar{\nu}_e}^{\text{erf}} \) are obtained by means of an interpolation between the relevant time-scales in the diffusion (optically thick) regime, \( t_{\nu_i}^{\text{diff}} \) (see equations 30 and 31), and in the free streaming (optically thin) regime, \( t_{\nu_i}^{\text{prod}} \) (see equations 23 and 24):

\[
\gamma_{\nu_i}^{\text{erf}} = \left(1 + \frac{t_{\nu_i}^{\text{diff}}}{t_{\nu_i}^{\text{prod}}}\right)\gamma_{\nu_i}^{\text{erf}}.
\]

(16)

In figure 2 we provide a schematic depiction of the \textit{leakage} part of ILEAS.

Although energy-dependent leakage schemes have been developed and successfully used, a grey approximation offers advantages in connection to our treatment of the equilibration regime, while keeping the scheme at a minimum with respect to computational cost, especially in the absorption module. Therefore, we employ spectrally averaged/integrated quantities for our calculations\(^2\) (see appendix B for details). In particular, we carefully take into account the energy dependence in the calculation of the diffusion time-scale as will be explained in section 2.2.1.

We assume the neutrino spectrum to follow a Fermi-Dirac distribution with matter temperature, \( T \), expressed in energy units,

\[
f(\epsilon; T, \eta_{\nu_i}) = \frac{1}{1 + e^{(\epsilon/T - \eta_{\nu_i})/\text{Dirac distribution with matter temperature}}},
\]

(17)

for neutrinos with energy \( \epsilon \). The neutrino degeneracy parameter, \( \eta_{\nu_i} = \mu_{\nu_i}/T \) (with \( \mu_{\nu_i} \) being the neutrino chemical potential) is prescribed as an interpolation of the equilibrium degeneracy, \( \eta_{\nu_i}^\text{eq} \), at high optical depth and a vanishing value at low optical depth (Ruffert et al. 1996):

\[
\eta_{\nu_i} = \eta_{\nu_i}^\text{eq}(1 - e^{-\tau_{\nu_i}}).
\]

(18)

The equilibrium degeneracy of electron neutrinos obeys

\[
\eta_{\nu_e} = \eta_e + \eta_p - \eta_n - Q/T,
\]

(19)

where \( \eta_e \) is the electron degeneracy (including rest mass), \( \eta_p \) and \( \eta_n \) are the proton and neutron degeneracies (without rest mass) and \( Q = m_\mu c^2 - m_\mu c^2 = 1.2935\text{MeV} \) is the nucleon rest-mass energy difference. Electron antineutrinos are assumed to have an equilibrium degeneracy given by \( \eta_{\bar{\nu}_e} = -\eta_{\nu_e} \), whereas for heavy-lepton neutrinos it is considered to be zero, \( \eta_{\nu_i} = 0 \). Ensuring a behaviour of \( \eta_{\nu_i} \) compatible with the transition to low optical depth is essential when using microphysical EoSs, in order to avoid unphysical values of the Fermi integrals and their ratios. In the semi-transparent regime, however, when the neutrino phase-space distribution function begins to deviate from equilibrium, leakage schemes can only approximate \( \eta_{\nu_i} \). In the case of ILEAS we use an interpolation, which can have a non-negligible impact on the neutrino luminosities in comparison to transport schemes.

In equation (18), \( \tau_{\nu_i} \) is the optical depth for neutrino species \( \nu_i \), estimated as the minimum optical depth calculated in the six Cartesian directions \((\pm x, \pm y, \pm z)\) as

\[
\tau_{\nu_i} = \int_{-\infty}^{r_0} \frac{1}{\tilde{\kappa}_{\nu_i}}(r')dr'.
\]

(20)

Here, the energy-averaged opacity, \( \tilde{\kappa}_{\nu_i}^{-1} \), is defined as in equation (B19). We consider as opacity sources the absorption of electron neutrinos and electron antineutrinos on neutrons and protons, respectively, and the scattering of all neutrino species on nucleons, alpha particles and heavy nuclei. We employ the same absorption opacities as in Ruffert et al. (1996), with the additional inclusion of stimulated absorption\(^3\) (neutrino blocking) and nucleon rest-mass corrections. The scattering opacities are also taken from the same source but with the nucleon blocking factors from Mezzacappa & Bruenn (1993) (see appendix B2 for details). Contrary to Ruffert et al. (1996), we do not assume matter to be fully dissociated and employ the nucleon number fractions from the EoS instead, in the computation of the nucleon blocking factors (equations B4 and B5). Since \( \eta_{\nu_i} \) are necessary for the calculation of the opacities, one iteration step is performed assuming that \( \tau_{\nu_i} \) is a function of the density, \( \rho \).

\(^2\) Denoted with an overbar, when susceptible to confusion with their energy-dependent counterparts.

\(^3\) Only in the calculation of \( t_{\nu_i}^{\text{diff}} \).
We find that there is no need for multiple iterations, as the integrals of order $k$ and the multiplicity factor, $g_{\nu_i}$, is unity for $\nu_e$ and $\bar{\nu}_e$ and 4 for $\nu_x$. Now we can calculate the production time-scales for number and energy, $t_{\nu_i}^{\text{prod}}$, as

$$t_{\nu_i,\text{num}}^{\text{prod}} = \frac{E_{\nu_i}^{j=0}}{R_{\nu_i}},$$

$$t_{\nu_i,\text{en}}^{\text{prod}} = \frac{E_{\nu_i}^{j=1}}{Q_{\nu_i}}.$$  

### 2.2.1 The diffusion time-scale

At high optical depth, neutrinos are trapped and slowly diffuse through the medium on a much longer time-scale than they are produced. A simple estimate of this time-scale is obtained when considering a random walk. The average distance a particle can travel in an optically thick medium can be approximated as

$$d = \sqrt{N\lambda},$$

where $N$ is the number of times a particle scatters and $\lambda$ the mean free path between scatterings. Assuming neutrinos travel at the speed of light, one can estimate the diffusion time-scale as

$$t_{\nu_i}^{\text{diff}} \sim N\frac{\lambda}{c} = \frac{d^2}{\lambda c}.$$  

A similar expression can be obtained from the diffusion equation for the zeroth-order angular moments, ignoring neutrino source terms and considering a static background medium.

$$\frac{\partial E_{\nu_i}(\epsilon)}{\partial t} = -\nabla \cdot F_{\nu_i}(\epsilon),$$

where $E_{\nu_i}(\epsilon)$ is representative of $E_{\nu_i}^{\text{num}}$ in the previous section. The neutrino flux $F_{\nu_i}(\epsilon)$ can be obtained from Fick’s law for the diffusion case as

$$F_{\nu_i}(\epsilon) = -\frac{c}{3\kappa_{\nu_i}} \nabla E_{\nu_i}(\epsilon).$$

The factor 3 in the diffusion coefficient arises from the assumption of an isotropic neutrino distribution (see Mihalas & Weibel-Mihalas 1984). Now by a simple dimensional analysis, using $\kappa_{\nu_i} = 1/\lambda_{\nu_i}$, one gets

$$\frac{E_{\nu_i}}{t_{\nu_i}^{\text{diff}}} = \frac{c\lambda_{\nu_i} E_{\nu_i}}{3d^2},$$

easily recovering the result of equation (26) (modulo a factor 1/3).

Previous leakage schemes made different assumptions about the length-scale $d$ in order to derive a numerical value for $t_{\nu_i}^{\text{diff}}$, any of which gives a good order of magnitude approximation of neutrino losses. In appendix A, we analyse in detail some of these prescriptions and compare the corresponding results with those from more sophisticated transport calculations, as well as those obtained in the present paper. There, one can see that all such leakage approximations perform poorly when one is interested in reproducing

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**Table 1. Neutrino interactions implemented in our scheme. Appendix B provides the formulation employed for each reaction.**

| Name                  | Interaction                                      | $\nu$ species |
|-----------------------|--------------------------------------------------|---------------|
| $\beta$-react. for $\nu_e$ | $p + e^- \rightarrow n + \nu_e$            | $\nu_e$       |
| $\beta$-react. for $\bar{\nu}_e$  | $n + e^+ \rightarrow p + \bar{\nu}_e$       | $\bar{\nu}_e$ |
| $e^-e^+$ annih.       | $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$ | $\nu_e, \bar{\nu}_e, \bar{\nu}_x$ |
| Plasmon decay         | $e^- + e^+ \rightarrow \nu_e + \bar{\nu}_e$ | $\nu_e, \bar{\nu}_e, \bar{\nu}_x$ |
| N-N bremsstr.         | $N + N \rightarrow \nu_e + \bar{\nu}_e$     | $\nu_e, \bar{\nu}_e, \bar{\nu}_x$ |
| Nucleon scatt.        | $N + \nu_i \rightarrow N + \nu_i$           | $\nu_i, \bar{\nu}_e, \bar{\nu}_x$ |
| $\alpha$ part. scatt. | $\alpha + \nu_i \rightarrow \alpha + \nu_i$ | $\nu_i, \bar{\nu}_e, \bar{\nu}_x$ |
| Nuclei scatt.         | $(A, Z) + \nu_i \rightarrow (A, Z) + \nu_i$ | $\nu_i, \bar{\nu}_e, \bar{\nu}_x$ |

$^1$ $N = p, n$

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Note the change in the notation with respect to Ruffert et al. (1996).
Figure 2. Components of the leakage module and their mutual interdependences (not including neutrino re-absorption nor equilibration).

The local neutrino losses of detailed transport calculations: most neutrinos are radiated from a narrow region close to the neutrinosphere, defined as the radius where the optical depth is $\tau_{\nu_i} = 2/3$, and hardly any from the optically thick region in the deeper interior. In addition, the total luminosities can exceed those of a transport calculation by a factor of 2 or more. The reason for this behaviour is the simplistic dimensional analysis used to estimate the diffusion time-scale, which leads to a steep decrease of the time-scale with radius, inversely proportional to the mean free path, preventing the diffusion of neutrinos out from high optical depth and favouring the escape of those produced near the NS surface.

To obtain a more accurate treatment, we evaluate numerically the spatial derivatives in equations (27) and (28) using five-point stencils in order to recover the divergence of the flux. Since neutrinos with different energies diffuse at different speeds, which leads to a significant impact on the spectrally averaged diffusion time-scale in the semi-transparent regime, we retain the energy dependence in the calculation of the flux. Integration to obtain the diffusion time-scale yields,

$$t_{\text{diff}, \text{num}} = \frac{\bar{E}^{(0)}_{\nu_i}}{\nabla \cdot \int_0^\infty \frac{\Lambda_{\nu_i}^{(0)}(\epsilon)}{3 \kappa_{\nu_i}(\epsilon)} \nabla \nabla E_{\nu_i}^{(0)}(\epsilon) \, d\epsilon},$$

$$t_{\text{diff}, \text{en}} = \frac{\bar{E}^{(1)}_{\nu_i}}{\nabla \cdot \int_0^\infty -\frac{\epsilon}{3 \kappa_{\nu_i}(\epsilon)} \Lambda_{\nu_i}^{(1)}(\epsilon) \nabla \nabla E_{\nu_i}^{(1)}(\epsilon) \, d\epsilon},$$

for number and energy diffusion respectively, where the energy-dependent total opacities, $\kappa_{\nu_i}(\epsilon)$, are calculated as in equation (B12). Due to the inclusion of rest-mass corrections in the computation of the absorption opacities (see equations B2 and B3), we cannot rely on an analytical solution of the Fermi integrals. For the energy integration we employ 15 energy bins in a logarithmic spacing up to 400 MeV (with bin limits at 5.0, 6.4, 8.4, 11.2, 15.2, 20.7, 28.4, 39.2, 54.3, 75.5, 105.2, 146.7, 204.8, 286.1 and 400.0 MeV), which is the same grid employed by the M1 scheme ALCAR in the models discussed for comparison in section 3.

It is well known that in the (semi)transparent region diffusion becomes acausal because the flux diverges as $\lambda = 1/\kappa \rightarrow \infty$. In order to ensure the correct limits we employ a flux limiter, $\Lambda_{\nu_i}(\epsilon)$, as successfully used in flux limited diffusion schemes (FLD) (Wilson et al. 1975; Levermore & Pomraning 1981). Because the differences between different flux limiters are effectively small, we use the canonical expression suggested by Wilson et al. (1975), retaining the energy dependence in order to ensure causality for each of the energy bins,

$$\Lambda_{\nu_i}(\epsilon) = \left(1 + \frac{1}{3 \kappa_{\nu_i}(\epsilon)} \left| \nabla E_{\nu_i}(\epsilon) \right| \right)^{-1}. \tag{32}$$

The divergence of the flux, in equations (30) and (31), gives us information about the nature of a given region, either as a source from which neutrinos diffuse out ($\nabla \cdot F_{\nu_i}(\epsilon) > 0$) or as a sink where neutrinos flow to ($\nabla \cdot F_{\nu_i}(\epsilon) < 0$). Because the leakage model is constructed to approximate the local neutrino losses, it cannot directly deal with sinks, which would translate to negative diffusion time-scales. Sinks represent regions where more neutrinos diffuse to a volume than out of it. Therefore, no net neutrino losses occur in such regions and, in concordance, we assume the diffusion time-scale to be infinite, quenching all...
local neutrino losses. At low optical depths, however, this approach does not make sense because radiation does not obey the physics of diffusion, but the free streaming limit where $t_{\nu i}^{\text{diff}} < t_{\nu i}^{\text{prod}}$ should be recovered. Accordingly, we set $t_{\nu i}^{\text{diff}} = \infty$ only inside the neutrinosphere, where the optical depth is $\tau_\nu > 2/3$, and take its absolute value outside (which will always be smaller than $t_{\nu i}^{-1}$). In the same spirit, small regions (less than few grid cells) bounded by two sinks are treated as sinks as well, as neutrinos will diffuse to the neighbouring sinks and remain trapped. This final correction turns out to be necessary to avoid overestimated neutrino emission near the neutrinosphere in some of the PNS snapshots at later times.

Including relativistic corrections (Shibata et al. 2011) for an asymptotically flat space-time ($ds^2 = -c^2 dt^2 + \psi^2 \delta_{ij} dx^i dx^j$, where $\alpha$ is the lapse function, $\psi$ the conformal factor and we take the shift vector, $\beta$, to be negligible for simplicity), the diffusion time-scale becomes:

$$t_{\nu i}^{\text{diff}} = \psi \bar{E}_{\nu i} \left( \alpha \psi \int_0^{\infty} \frac{d\epsilon}{\kappa_{\nu i,\epsilon}} \Lambda_{\nu i}(\epsilon) \nabla E_{\nu i}(\epsilon) d\epsilon \right),$$  

(33)

for $j = 0, 1$.

### 2.3 Neutrino absorption in optically thin matter

At low optical depths, neutrinos decouple from matter and essentially stream away at the speed of light. However, before free streaming is reached, a significant fraction of these neutrinos can be re-absorbed. This neutrino energy and number deposition in semi-transparent regions is crucial for many astrophysical phenomena, such as the shock revival in SNe, the ejecta composition in CO mergers or neutrino-driven winds from the remnants of either event. Attempts to reliably simulate any of those scenarios, therefore, require to account for neutrino absorption. The `standard' leakage approach only serves the purpose of estimating neutrino losses, but does not take care of re-absorption. Therefore, a complimentary absorption scheme is needed. We present a description here based on the 1D formulation of radiation attenuation by Janka (2001), generalized to any 3D geometry by means of a simple ray-tracing algorithm.

We start with the premise that neutrinos are produced in the centre of a given cell and approximately escape in the direction of the local gradient of the neutrino energy density, $-\nabla \bar{E}_{\nu i}^{\rho=1}$, following a straight ray. This is a fair assumption in spherical symmetry, and although in complex geometries neutrinos will scatter and change direction along their way out (and also be subject to gravitational ray bending), it is a reasonable first-order approximation\(^7\). We use a 3D slab formalism (Kay & Kajiya 1986) to find all cells of our 3D Cartesian grid crossed by a given ray and estimate the deposited energy and number as a function of the distance traversed in the cell, reducing the escaping luminosity accordingly.

The luminosity produced by a cell, $s_i$, is generally calculated in leakage schemes as

$$L_{\nu i}^{\text{ray}}(s) \approx Q_{\nu i}(\psi^4 V)_{\text{cell,em}},$$  

(34)

including metric corrections to the volume of the emitting cell. In our absorption scheme, we propagate along a ray the luminosity produced by a given cell, defined in equation (34). In general, according to equation (72) of Janka (2001), the luminosity is attenuated due to absorption along a path, from $s^-$ to $s^+$, as given by

$$L_{\nu i}^{\text{ray}}(s^+) = L_{\nu i}^{\text{ray}}(s^-) \exp \left( - \int_{s^-}^{s^+} \frac{\kappa_{\nu i,\alpha}(s') ds'}{(\chi_{\nu i})} \right),$$  

(35)

where $L_{\nu i}^{\text{ray}}(s^-)$ is the incoming luminosity, $\kappa_{\nu i,\alpha}$ the spectrally averaged absorption opacity (equations B13 and B14) and $(\chi_{\nu i})$ is the energy-averaged flux factor (see below). Radiation attenuation is described by equation (35), where we take the path from $s^-$ to $s^+$ as the one traversed by the ray in each absorbing cell.\(^8\)

In equation (35) $(\chi_{\nu i})$ is defined as the ratio of the neutrino flux to the neutrino energy density times the speed of light. In leakage schemes, however, there is no notion of local neutrino number/energy density outside the diffusive regime, and therefore, an approximate expression is required. In free streaming conditions, the average neutrino flux factor, $\langle \chi_{\nu i} \rangle$, approaches the value of 1 as the radiation becomes forward peaked far away from the source, while at high optical depths $\langle \chi_{\nu i} \rangle$ becomes very small. Its exact behaviour between both extremes, however, remains strongly dependent on the geometry of the neutrino emitting object. In the case of a spherical cooling PNS (e.g. Janka 1991), $\langle \chi_{\nu i} \rangle$ is known to be about 1/4 at the neutrinosphere, and we adopt for such a case the interpolation suggested by O’Connor & Ott (2010), $\langle \chi_{\nu i} \rangle_{\text{PNS}} = 4.275 \tau_{\nu} + 1.15$. For more complex geometries, such as a BH-torus system or a binary NS merger, more sophisticated models for the streaming factor, which encode the geometric effects, would be necessary. However, we take the aforementioned linear interpolation to be sufficiently good for the present work, as shown in the tested scenarios in section 3.

We include gravitational redshift of the luminosities between the emitting and absorbing cells following O’Connor & Ott (2010), but we omit Doppler effects for simplicity:

$$L_{\nu i}^{\text{ray}}(s_2) = L_{\nu i}^{\text{ray}}(s_1) \exp \left( - \int_{s_1}^{s_2} \frac{\kappa_{\nu i,\alpha}(s') ds'}{(\chi_{\nu i})} \right) \frac{\alpha(x_1)^2}{\alpha(x_2)^2},$$  

(36)

for neutrinos that are emitted at ray coordinate\(^9\) $s_1$, which corresponds to the center of the emitting cell at Cartesian coordinate $x_1$, and absorbed at $s_2$, which corresponds to

---

\(^7\) Perego et al. (2014a,b) spent significant effort on designing recipes to construct radiation paths for their ray-tracing treatment. We refrain from adding complications to our code in this aspect, first to save computer time, second because our simple scheme works well in near-surface or low optical depth regions that dominate the neutrino emission and absorption (as proven in practise by our test results), and third because any complicated path definition will still remain an approximation whose general validity cannot be guaranteed without verification by comparison to detailed transport.

\(^8\) We also include re-absorption of neutrinos produced by the cell itself, along a path from the center of the emitting cell to its boundary.

\(^9\) We note that the luminosity attenuation is calculated along the ray coordinate, $s$, as a function of the distance traversed, whereas all the thermodynamical and metric variables are assumed to be homogeneous within a cell located at Cartesian coordinate $x$. 
the intersection of the incoming ray with the absorbing cell at Cartesian coordinate \( \mathbf{x}_2 \). Equation (36) also includes the attenuation due to neutrino re-absorption in the traversed cells between \( s_1 \) and \( s_2 \), described by (35).

Since the attenuation corresponds to an energy deposition per cell expressed by \( (L_{\nu}^{\text{ray}}(s^+)/\langle \psi^0 V \rangle_{\text{cell}} \text{abs}) \), we obtain the absorption rate, \( Q_{\nu}^\text{abs} \), of a given cell with Cartesian coordinate \( \mathbf{x} \), from the superposition of all the rays crossing this cell, and for homogeneous conditions in the cell, as

\[
Q_{\nu}^+(\mathbf{x}) = \gamma_{\text{eff}} \sum_{\text{rays}} \frac{L_{\nu}^{\text{ray}}(s^-)}{\langle \psi^0 V \rangle_{\text{cell}} \text{abs}} \left[ 1 - \exp \left( -\kappa_{\nu,\text{abs}}(s^+ - s^-) \right) \right].
\]  

(37)

where \( s^- \) and \( s^+ \) delimit the path the ray travels inside the cell. The factor \( \gamma_{\text{eff}} \) ensures that the absorption is mainly applied in the optically thin regime (see equation 16 for the definition of \( \gamma_{\nu,\text{eff}} \)). The mean absorption opacity of a given cell, \( \kappa_{\nu,\text{abs}} \), is calculated as in equations (B13) and (B14) with the corresponding spectrum of the neutrinos as seen by the fluid at a given location (see below).

In the framework of the leakage scheme, neutrinos are assumed to instantaneously leak out of the system, which would imply that they carry their production spectrum along the ray. Physically, however, neutrinos slowly diffuse out of the hot NS, thermalizing with the medium in the process, until the optical depth becomes small enough for them to freely stream away. In order to account for this behaviour, we determine the spectrum of neutrinos (produced at \( s_1 \) and irradiating the fluid at \( s_2 \)) by differentiating between the three following cases:

- Neutrinos emitted anywhere are re-absorbed by cells which are inside of the neutrinosphere (\( \tau_{\nu,\text{es}} > 2/3 \)) with a Fermi spectrum characterized by the local matter temperature and neutrino degeneracy (equation 18) of the re-absorption cell (thermal spectrum, \( f(c; T_{\nu,\text{es}}, \eta_{\nu,\text{es}}) \)).
- Neutrinos emitted from inside of the neutrinosphere (\( \tau_{\nu,\text{es}} > 2/3 \)) are re-absorbed by cells outside of the neutrinosphere (\( \tau_{\nu,\text{es}} < 2/3 \)) with a Fermi spectrum characterized by the matter temperature and neutrino degeneracy of the last cell crossed by the ray which is inside of the neutrinosphere (neutrinosphere spectrum, \( f(c; T_{\nu,\text{es}}, \eta_{\nu,\text{es}}) \)).
- Neutrinos produced outside of the neutrinosphere (\( \tau_{\nu,\text{es}} < 2/3 \)) are re-absorbed by cells also outside of the neutrinosphere (\( \tau_{\nu,\text{es}} < 2/3 \)) with the same Fermi spectrum with which they were produced, i.e. characterized by the matter temperature and neutrino degeneracy of the production cell (production spectrum, \( f(c; T_{\nu,\text{es}}, \eta_{\nu,\text{es}}) \)).

As in other grey absorption schemes (e.g. O’Connor & Ott 2010), we then estimate the lepton number deposition as

\[
R_{\nu}^+ = \frac{Q_{\nu}^+}{\bar{e}_{\nu}}.
\]

(38)

We calculate the mean neutrino energy, \( \bar{e}_{\nu} \), of neutrinos being absorbed in \( \beta \)-processes by considering Fermi spectra:

\[
\bar{e}_{\nu} = \frac{T_{\nu} F_{\nu}(\eta_{\nu})}{\bar{e}_{\nu}} \frac{\alpha(\tilde{s}_1)}{\alpha(s_2)}.
\]

(39)

where the matter temperature and the neutrino degeneracies are consistently taken as above for each of the described cases. The redshift, \( \alpha(\tilde{s}_1)/\alpha(s_2) \), is then applied only for absorption on cells outside of the neutrinosphere and only between the neutrinosphere (if the neutrinos are produced inside, i.e. \( \tau_{\nu,\text{es}} > 2/3 \)), \( \tilde{s}_1 = s(\tau_{\nu,\text{es}} = 2/3) \), or the production cell (if they are produced outside), \( \tilde{s}_1 = s_1 \), and the absorbing cell, \( s_2 \). Neutrinos absorbed inside of the neutrinosphere locally thermalize with matter, thus eliminating any trace of the prior neutrino spectrum\(^{10}\).

As an alternative approach, one can calculate the neutrino-number re-absorption rates, \( R_{\nu}^+ \), by the same procedure described for the treatment of the neutrino energy re-absorption, i.e. independently of the neutrino-energy absorption rates. Following equation (37), but using the neutrino-number luminosities and spectrally-averaged opacities over the neutrino number spectra, defined in equations (B13) and (B14) (with \( j = 0 \)), we obtain directly the neutrino-number absorption rates. The comparison between the results obtained by both approaches on a PNS snapshot calculation, see appendix D, does not reveal any significant differences in the mean energies of radiated neutrinos (see equation 45), thus demonstrating the robustness of our treatment to methodical variations in details.

We employ a Gaussian smoothing filter with standard deviation \( \sigma = 1 \) over the absorption rates in the three-dimensional spatial domain. This ensures the conservation of the total absorption rates and mitigates the drawbacks of employing a limited number of rays in a ray-tracing approach. Thus smoothing out high local rates over neighbouring spatial points moderately boosts the computational performance of the scheme.

2.4 Neutrino equilibration in optically thick matter

At the typical densities and temperatures achieved during NS mergers or SNe, a part of the neutrinos is expected to remain trapped in optically thick conditions. Under such circumstances, they will achieve local beta equilibrium with the surrounding matter within a very short time, carrying lepton number, and contributing to the energy and pressure of the stellar medium.

In order to account for this important effect, we developed an equilibration scheme which ensures that the fluid remains in beta equilibrium with the trapped neutrinos in the optically thick regime, by two measures. First, we employ a set of EoS tables which include the contributions of trapped neutrinos to the specific internal energy and pressure of the medium, to be used for the hydrodynamical evolution of the system. Second, we perform an equilibration step after each hydro step, ‘reshuffling’ the trapped leptons and recovering the equilibrium values of the corresponding thermodynamical quantities. This last step requires the advection of the trapped lepton fraction \( Y_{\nu}^{\text{lep}} \), which can be expressed as the sum of the individual trapped neutrino fractions, \( Y_{\nu_i}^{\text{lep}} \), and

\(^{10}\) Effectively, for redshift effects, neutrinos absorbed inside of the neutrinosphere are seen in the absorption frame as if they were emitted at the same location where absorption occurs.
the electron fraction, $Y_e$, as we described in section 2.1 (see equations 10-12 there).

We treat each of the three different neutrino species independently, describing overlapping equilibration regions, which requires us to build an additional EoS table for each possible combination of trapped species. This amounts to a total of eight different possibilities, listed in table 2. Even though we opted for the most general implementation of the neutrino equilibration regions, it is also possible to reduce the number of different equilibration zones by assuming a hierarchy in the minimum densities for which neutrinos of the different species remain trapped. In most relevant astrophysical scenarios, $\nu_e$ will decouple from matter at higher densities than the other two species, followed by $\bar{\nu}_e$, and finally $\nu_x$ at lower densities. Therefore, a simpler equilibration treatment could be achieved with only the inclusion of regions 1, 2, 5, and 8, yet capturing all important physical effects under most circumstances.

| Equilibration region | Trapped $\nu$ species |
|----------------------|-----------------------|
| 1                    | $\nu_e$, $\bar{\nu}_x$, $\nu_x$ |
| 2                    | $\nu_e$, $\bar{\nu}_x$, $\nu_x$ |
| 3                    | $\nu_e$, $\bar{\nu}_x$, $\nu_x$ |
| 4                    | $\nu_e$, $\bar{\nu}_x$, $\nu_x$ |
| 5                    | $\nu_x$ |
| 6                    | $\bar{\nu}_x$ |
| 7                    | $\nu_x$ |
| 8                    | none |

The neutrino contribution to the specific internal energy $\varepsilon_{\nu_i}$, can be obtained from the neutrino equilibration number density $E_{\nu_i}^1$ (equation 22 with $j = 1$, as

$$\varepsilon_{\nu_i} = \frac{E_{\nu_i}^1}{\rho c^2}$$

assuming all quantities are expressed in cgs units. The neutrino fraction $Y_{\nu_{\text{trap}}}$ can be obtained from the neutrino equilibration number density $E_{\nu_i}^0$ (equation 22 with $j = 0$, as

$$Y_{\nu_{\text{trap}}} = \frac{E_{\nu_i}^0}{\rho A}$$

where $A$ is Avogadro’s constant. The Fermi integrals for the equilibrium energy density of $\nu_i$, $\bar{\nu}_i$, $\nu_x$ pairs are computed by the analytical expression from Bhudman & van Riper (1978), whereas for $\nu_i$ and $\bar{\nu}_i$ analytically approximate the Fermi integrals in equation (22) (with $j = 0$ for number and $j = 1$ for energy) following Takahashi et al. (1978). Then, one can calculate the pressure of each neutrino species as

$$P_{\nu_i} = \frac{1}{3} E_{\nu_i}^1.$$

We apply our equilibration treatment for a given neutrino species, $\nu_i$, down to optical depths $\tau_{\nu_i} \geq 1$. At lower optical depths, the deviations from the equilibrium energy density become significant (>20 per cent), and thus the assumption of beta equilibrium is not suitable.

Each equilibration region listed in table 2 employs a different EoS table, which depends on the composition of matter in that region via $Y_{\nu_{\text{trap}}}$ and the fluid (stellar medium plus trapped neutrinos) specific internal energy, $\varepsilon$. During the dynamical evolution of a system, SPH particles or grid cells will switch between the different equilibration regions. In order to ensure energy conservation of material crossing these boundaries, we add or subtract from that cell’s or particle’s $\varepsilon$, the corresponding neutrino contribution. This requires the recovery of the neutrino specific energy component, $\varepsilon_{\nu_i}$, from the EoS tables at every time step. Because outside of a given equilibration region, we assume that $\nu_i$ are not in equilibrium with matter but do not leave the system immediately if marginally outside of their equilibration region, we simply advect $\varepsilon_{\nu_i}$.
region, but in the whole domain (see equations 11 and 12). This advection serves the purpose of avoiding non-physical energy and lepton losses by material oscillating around a given boundary. When matter flows inside an equilibration region, its advected $Y_{\nu_i}$ and $\varepsilon_{\nu_i}$ will contribute again to the fluid’s total lepton fraction and specific internal energy, respectively. This boundary treatment is sufficiently good under the assumption that material re-entering a given regime spent too little time outside to experience significant losses of its neutrino content. Since hardly ever material re-enters the equilibrium domain from having been far outside, this treatment is sufficiently good. We remind the reader again that neutrino losses from the stellar medium are accounted for by the leakage module, both in the trapping and free-streaming domains. In figure 3 we summarize the quantification module for a given equilibration region and the transition from this domain to its neighbouring regions, where equilibrium is not fulfilled.

2.5 Extraction of neutrino properties from ILEAS

It is often desirable to extract some relevant neutrino-related quantities from numerical simulations, to be used for post-processing, in nucleosynthesis calculations or to treat neutrino oscillations, to predict the detectability of a signal by neutrino detectors or just for diagnostics. In the present section, we describe how we calculate the neutrino luminosities and the radiated mean neutrino energies in ILEAS.

Given the neutrino loss and absorption rates, the net neutrino “luminosities” that reach distant observers (in the rest frame of the source’s center of mass) can be approximately written (neglecting Doppler effects, gravitational ray bending, time retardation and the shift vector) as:

$$L_{\nu_i}^{\text{en}}(\mathbf{x}_{\text{obs}}) = \int (Q_{\nu_i} - Q_{\nu_i}^+) \frac{\alpha(\mathbf{x})^2}{\alpha(\mathbf{x}_{\text{obs}})^2} \psi(\mathbf{x})^6 dV,$$  

for energy, and

$$L_{\nu_i}^{\text{num}}(\mathbf{x}_{\text{obs}}) = \int R_{\nu_i} - R_{\nu_i}^+ \frac{\alpha(\mathbf{x})}{\alpha(\mathbf{x}_{\text{obs}})} \psi(\mathbf{x})^6 dV,$$  

for number, where $\mathbf{x}$ and $\mathbf{x}_{\text{obs}}$ are the positions of the emitting/absorbing cell and of the observer respectively, and the integral runs over our grid domain. Trivially, for an observer at rest at an infinite distance, $\alpha(\mathbf{x}_{\text{obs}}) = \alpha(\infty) = 1$.

The natural way to estimate the mean neutrino energies in the leakage framework is simply by the ratio of net energy and number luminosities,

$$\langle \epsilon_{\nu_i}^{\text{leak}} \rangle(\mathbf{x}_{\text{obs}}) = \frac{L_{\nu_i}^{\text{en}}(\mathbf{x}_{\text{obs}})}{L_{\nu_i}^{\text{num}}(\mathbf{x}_{\text{obs}})}.$$  

This approach, however, does not yield very good agreement with transport results for the mean energies of radiated neutrinos (see section 3.1), because it bears the deficiency mentioned earlier, namely that it ignores the thermalization of neutrinos produced at high optical depths on their way out of the star. As detailed in section 2.3, in our absorption module we do not follow this leakage ansatz, but instead work with the local neutrino spectra inside the neutrinosphere, $\tau_{\nu_i} > 2/3$, and assume that neutrinos in the optically thin region ($\tau_{\nu_i} < 2/3$) carry either their neutrinospheric spectra, if produced in the optically thick region, or their production ones.

In order to provide a more meaningful value for the radiated mean neutrino energies, we make the following approximations in a post processing step. First, we differentiate the optically thick and optically thin regimes, as introduced earlier, separated by the neutrinosphere at $\tau_{\nu_i} = 2/3$. The mean energy for neutrinos produced in the optically thin regime is calculated in the fashion of leakage schemes, but accounting independently for the absorption of energy and number, as

$$\langle \epsilon_{\nu_i}^{\text{thin}} \rangle(\mathbf{x}_{\text{obs}}) = \frac{Q_{\nu_i}^+(s_1) \exp \left(-\int_{s_1}^{s_{\nu_i}} \tilde{\kappa}_{\nu_i}^{\text{num}}(s)/\langle \chi_{\nu_i} \rangle ds \right) \alpha(s_1)}{R_{\nu_i}(s_1) \exp \left(-\int_{s_1}^{s_{\nu_i}} \tilde{\kappa}_{\nu_i}^{\text{num}}(s)/\langle \chi_{\nu_i} \rangle ds \right) \alpha(s_{\text{obs}})}$$  

(46)

Here the spectrally averaged opacities for energy and number absorption, $\tilde{\kappa}_{\nu_i}^{\text{num}}$ and $\tilde{\kappa}_{\nu_i}^{\text{num}}$, are calculated as in equations (B13) and (B14) with the neutrino production spectrum, $f(c; T_{\nu_i}, \eta_{\nu_i}, s_1)$, of the emitting cell. We remind the reader that $s$ is the ray coordinate, as used in section 2.3, and the ray origin, $s_1$, corresponds to the Cartesian coordinate $x$. Because we want to evaluate the mean energies as observable from outside of our domain, we include the gravitational redshift from the production cell to an observer positioned at $s_{\text{obs}} = x_{\text{obs}}$. Given the steep density distribution typical of the environments of PNSs or HMNSs, it is a fairly accurate approximation that most neutrinos will be re-absorbed near their emission location. In this spirit, we approximate the path in the line integral in equation (46) by the total distance the ray would travel if it crossed the whole production cell, which we consider as a proxy for the absorption along the whole outgoing ray. Note that in the absorption module (section 2.3), we only take the path from the centre to the edge of the cell for self-absorption of a production cell, but follow the whole paths of outgoing rays.

Furthermore, we define an absorption correction factor of the mean energy for neutrinos coming from inside the neutrinosphere,

$$c_{\text{abs}} = \left[ \frac{\exp \left(-\int_{s_{\nu_i}}^{s_{\nu_i}^+} \tilde{\kappa}_{\nu_i}^{\text{num}}(s)/\langle \chi_{\nu_i} \rangle ds \right)}{\exp \left(-\int_{s_{\nu_i}}^{s_{\nu_i}^+} \tilde{\kappa}_{\nu_i}^{\text{num}}(s)/\langle \chi_{\nu_i} \rangle ds \right)} \right]^{\tau_{\nu_i} = 2/3}. $$  

(47)

This correction factor is evaluated in the cells immediately adjacent to the neutrinosphere, using their local equilibrium neutrino spectrum. Rays escaping from inside the neutrinosphere and crossing such cells, will then have their mean energies corrected by means of $c_{\text{abs}}$, representing the whole absorption outside the neutrinosphere. Each ray emerging from the optically thick regime will thus contribute to the final average with a mean energy

$$\langle \epsilon_{\nu_i}^{\text{thick}} \rangle(\mathbf{x}_{\text{obs}}) = \left[ c_{\text{abs}} T F_1(\eta_{\nu_i}) F_2(\eta_{\nu_i}) \right]^{\tau_{\nu_i} = 2/3} \frac{\alpha(\tau_{\nu_i} = 2/3)}{\alpha(s_{\text{obs}})} \frac{\alpha(\tau_{\nu_i} = 2/3)}{\alpha(s_{\text{obs}})},$$  

calculated where the ray crosses the neutrinosphere, and including the aforementioned correction factor.
Table 3. Neutrino interactions employed in section 3 for our tests with the ILEAS code in comparison to ALCAR and VERTEX calculations ($N = p, n$). Note that for the BH-Torus models in section 3.2 we neglected $\nu_e$ altogether.

| Name              | Interaction                        | $\nu$ species |
|-------------------|------------------------------------|----------------|
| $\beta$-react. for $\nu_e$ | $p + e^- \leftrightarrow n + \nu_e$ | $\nu_e$         |
| $\beta$-react. for $\bar{\nu}_e$ | $n + e^+ \leftrightarrow p + \bar{\nu}_e$ | $\bar{\nu}_e$  |
| $e^-e^+$ annihil. | $e^- + e^+ \leftrightarrow \nu_e + \bar{\nu}_e$ | $\nu_e, \bar{\nu}_e$ |
| N-N bremsstr.†  | $N + N \rightarrow N + N + \nu_i + \bar{\nu}_i$ | $\nu_e, \bar{\nu}_e, \nu_x$ |
| Nucleon scatt.    | $N + \nu_i \rightarrow N + \nu_i$ | $\nu_e, \bar{\nu}_e, \nu_x$ |

† $N = p, n$

Finally, we obtain the total radiated mean neutrino energy by means of a weighted average of all rays, using the neutrino energy luminosities $\langle \nu \rangle$ the corresponding cells either at the neutrinosphere for the optically thick rays and from production cells in the optically thin region:

$$\langle \epsilon_{\nu_i}^{\text{tot}} \rangle = \frac{\sum_{k \mid r_{\nu_i} > 2/3} \langle \epsilon_{\nu_i, k}^{\text{thick}} \rangle \Delta I_{\nu_i, k}(x)}{\sum_{k \in V} \Delta I_{\nu_i, k}(x)} + \frac{\sum_{k \mid r_{\nu_i} < 2/3} \langle \epsilon_{\nu_i, k}^{\text{thin}} \rangle \Delta I_{\nu_i, k}(x)}{\sum_{k \in V} \Delta I_{\nu_i, k}(x)} \quad (49)$$

Here the summations in the numerator go over all rays $k$ which are emitted from cells inside ($r_{\nu_i} > 2/3$) or outside ($r_{\nu_i} < 2/3$) the neutrinosphere, and the one in the denominator over the whole volume.

### 3 ASTROPHYSICAL TEST APPLICATIONS: COOLING PNS AND BH-TORUS SYSTEMS

All the ILEAS calculations presented in this work (section 3) were performed on a three-dimensional Cartesian grid with $\sim 0.7$ km in resolution in all three coordinate directions. This grid expands $\sim 100$ km in all six Cartesian directions ($\pm x, \pm y, \pm z$) from the centre-of-mass of the system, covering the astrophysical objects and their immediate surroundings. The same grid can also be employed for NS merger simulations, providing full coverage of the late inspiral phase as well as the initial merger remnant and the absorption-dominated regions along polar directions.

#### 3.1 Snapshot calculations of a cooling proto-neutron star

In order to assess the quality of our new ILEAS code, we need to test it against more sophisticated transport schemes and in different regimes. Given that our ultimate goal is the application of ILEAS in the context of NS mergers, cooling PNSs present a relevant test scenario. At the explosion of a massive star in a SN, its core contracts to high densities and temperatures, giving birth to a young NS. The hot, dense interior of such a newly formed PNS is a perfect representation of an optically thick regime where the diffusion treatment can be tested. Additionally, the star is surrounded by a less dense envelope, where absorption of the neutrinos emitted from the NSs surface will apply. In between, the transition region around the neutrinosphere poses the most challenging conditions for treatments based on an interpolation of diffusive and free streaming regimes, such as in our ILEAS method.

We apply our scheme to several snapshots from a hydrodynamical simulation performed by Hiedepohl et al. (2010), who used the 1D version of the PROMETHEUS-VERTEX code with energy-dependent two-moment neutrino transport including Boltzmann-closure. We take the hydrodynamical and thermodynamical data at different times post bounce from the model labelled Sr (reduced opacities), and map it to our 3D Cartesian grid. The motivation behind the chosen model is the similarity of our opacities and production reactions with the ones included in the original setup. Figure 4 shows the density, temperature and electron fraction profiles of the corresponding snapshots.

For the sake of more detailed comparisons, we also employed the energy-dependent M1 scheme ALCAR (Just et al. 2015b) to calculate the neutrino luminosities for one of the snapshots. Starting the evolution from an earlier timestep (0.3 s post-bounce) of the VERTEX simulation and evolving it hydrodynamically for 0.1 s, ALCAR was able to reproduce the results of VERTEX at 0.5 s with remarkable accuracy. We use this evolved ALCAR background (at 0.5 s post-bounce) for our direct, detailed comparison of the results obtained by ILEAS and ALCAR. The neutrino interactions employed by both schemes for the tests are summarized in table 3. We must point out that the prescriptions for $\nu_e$ production rates (pair processes and bremsstrahlung) differ between both codes, therefore a bigger disagreement is to be expected in the luminosities of these heavy-lepton neutrinos (see Rampp & Janka 2002 and appendix B for the exact definitions of the rates employed by ALCAR and ILEAS, respectively). Moreover, some differences will unavoidably arise from the fact that ILEAS is in essence a grey scheme while ALCAR is fully energy-dependent. Finally, ILEAS is implemented on a 3D Cartesian grid, whereas ALCAR uses a spherical (polar) grid, offering advantages with respect to resolving radial gradients.

As a foreword to the comparison, it is important to note that there are still some noteworthy differences between ALCAR and the standard formulation of ILEAS in the derivation of the neutrino production rates. The former calculates the rates for $\beta$-production of $\nu_e$ and $\bar{\nu}_e$ from a formulation that ensures detailed balance, based on blocking-corrected absorption opacities, $\kappa^*$, defined in equations (C12) and (C13), following Rampp & Janka (2002). On the other hand, ILEAS employs the emissivity, $j_{\nu_i}$, defined as in equations (C4) and (C6) (Bruenn 1985), to compute the rates. In appendix C we show the derivation of the rates in both schemes, and provide a detailed comparison of the effects of each prescription on the neutrino luminosities with ILEAS. In order to enable a more accurate comparison, ILEAS employs the prescription of the $\beta$-production rates from ALCAR in the results shown in this section, also using the same energy binning as described in section 2.2.1.

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12 This cooling PNS is the remnant of an 8.8 $M_\odot$ (zero-age main sequence stellar mass) electron capture SN.
Figures 5 and 6 show the luminosity profiles of each neutrino species obtained by ILEAS for the selected time snapshots from the VERTEX simulation, in comparison to the original transport results. In the bottom panels of figure 5 we present the results obtained on the background evolved with ALCAR, where the results obtained by both transport codes are also plotted for comparison. Note that in this panel, for a better comparison with ALCAR, we do not include redshift in the calculations with ILEAS. In order to obtain the results presented in this section, we have relaxed the background using ILEAS to adjust the temperature and electron fraction to their new steady-state values. After a brief transient of a few ms, all quantities settle into a quasi-stationary state. We will discuss the details of the scheme employed for relaxation as well as for the longer time evolution of one of these snapshots in section 3.3.

In all the tested snapshots, from 0.2 s to 1.5 s post bounce, ILEAS is able to reproduce the transport results for $\nu_\tau$ and $\bar{\nu}_\tau$ with better than 10 per cent accuracy. The slightly bigger discrepancies for $\nu_\tau$ are very likely associated with the different prescriptions of nucleon bremsstrahlung employed by the different codes. Moreover, ILEAS as a leakage scheme
calculates only neutrino losses and, therefore, it is unable to model the negative neutrino fluxes observed with transport treatments at $\sim 10$ km.

The performance of ILEAS in the optically thick region is remarkable, especially for the ALCAR background, in which case both codes use exactly the same opacities for $\nu_e$ and $\bar{\nu}_e$. The good agreement arises from the definition of our diffusion time-scale, which is derived from an energy integration of the zeroth-order moment of the transport equation with a flux-limited neutrino diffusion

$$L_{\nu e} = \int_{r_{\text{inner}}}^{r_{\text{outer}}} \frac{dL_{\nu e}}{dr} dr$$
Figure 6. Radial neutrino luminosity profiles obtained by ILEAS on a quasi-stationary background for $\nu_e$, $\bar{\nu}_e$ and one representative of $\nu_x$, as measured in the local frame at each radius $r$, compared to the results of more sophisticated transport schemes. Snapshots are taken from the results of PNS cooling simulations by Hudepohl et al. (2010) at different post-bounce times, which are noted in the lower right corner of each panel. The temperature and $Y_e$ profiles were relaxed for 5 ms with ILEAS to obtain stationary results. The VERTEX results shown are the original luminosities from the source model.

This effectively translates in a local source term calculated as $Q_{\nu_i} \simeq - \int_{0}^{\infty} \nabla \cdot F_{\nu_i} \, d\epsilon$, which, in the case of quasi-stationarity, $\partial E_{\nu_i}(\epsilon)/\partial t \sim 0$, is essentially the same result as with ALCAR. As we approach the semi-transparent region, however, the results start to differ slightly due to the deviations from $\beta$-equilibrium of the neutrino spectrum, which we approximated using our interpolation of the neutrino degeneracies (equation 18). This is one of the most delicate aspects of our scheme, as the diffusion time-scale depends sensitively on the neutrino spectrum, which cannot be properly determined by a leakage method. Finally, in the optically thin regime, our 3D absorption model successfully captures the essential features of energy and lepton-number deposition in the PNS envelope. This is visible from a very good agreement of the $Y_e(r)$ profiles obtained with ILEAS and ALCAR/VERTEX, respectively (figure 7).
section 3.2 we will discuss in further detail the features of our 3D absorption scheme in the context of a BH-torus system. In the profiles of $\bar{\nu}_e$ and $\nu_e$, negative luminosities can be observed at around 10 km for the transport schemes. They are a consequence of the local temperature maximum at about 12 km, which leads to a net neutrino diffusion flux directed towards the centre of the PNS, i.e. neutrinos in this region diffuse inward. Because ILEAS is unable to reproduce such an effect by construction, the diffusion time-scale in those regions, which would become negative, is set to infinity, preventing any leakage of neutrinos out of the star.\footnote{See section 2.2.1 for details on our treatment of negative diffusion time-scales.}

It is interesting to note the small differences in the relaxed electron fraction profile. Figure 7 shows the original profile from the 0.5 s PNS snapshot evolved by VERTEX in comparison to the profiles obtained by ALCAR, and the one further relaxed using ILEAS. It catches the eye that there is a slight but systematic shift of the rising flank of the $Y_e$ “trough”, which is located close to the PNS surface, to slightly larger radii for the ILEAS model. In fact, this effect is generic because of the poor ability of any leakage scheme to accurately model the semi-transparent regime regardless of the absorption or equilibration parts. However, we emphasize that our implementation of ILEAS with its novel definition of the diffusion time-scale performs extremely well also in this respect compared to other schemes presented in the literature (e.g. Perego et al. 2016), as can be seen by our test results obtained with other definitions of the diffusion time-scale, summarized in appendix A. Tentatively, the remaining moderate overestimation of the loss of neutrino-lepton number from a narrow layer around the neutrinosphere could be mitigated by a further improved handling of the neutrino spectrum out of equilibrium.

Table 4 lists a summary of the luminosities and mean energies of the three neutrino species, as seen by a local observer in the rest frame of the neutrino source at the edge of our grid, $\sim$100 km, obtained by ILEAS for all our tested conditions, in comparison to the original results obtained by the corresponding transport codes. All ILEAS results are extracted after a few milliseconds of relaxation, employing the formulations described in section 2.5. As mentioned earlier, the neutrino luminosities obtained by ILEAS for all tested PNS snapshots provide a very good approximation of the luminosities obtained by the transport calculations.

The mean neutrino energies calculated in the leakage approach, however, exhibit a greater disagreement with the transport results, especially at later times of the PNS evolution (table 4). For $\nu_e$, the leakage mean energies are increasingly lower compared to the M1 results (up to $\sim$4 MeV at 1.5 s), whereas the $\bar{\nu}_e$ show the opposite trend, but to a smaller extent (up to $\sim$3 MeV). This energy discrepancy does not significantly improve when we compute our neutrino number and energy absorption rates in an analogous way but independently (see section 2.3 and appendix D). This systematic and consistent disagreement with transport results is probably linked to the approximative treatment of the semi-transparent regime by the flux-limiting approach. Figure D1 in appendix D reveals that, particularly in the case of $\nu_e$, a dominant component of the neutrino-number luminosity is emitted from the semi-transparent region (right below the neutrinosphere), whereas the neutrino-energy luminosity comes from deeper inside the NS. For this reason, the calculation of the neutrino-number luminosity is more sensitive to the approximations applied in our flux-limiting prescription for the diffusion timescale, thus impacting the neutrino mean energies computed by the leakage method.

To offer an alternative measure of the radiated mean energies, which is also more compatible with the mean energies of neutrinos used in our absorption module, we provide the approximate diagnostic mean energies defined by equation (49). We find that, as expected, these post-processed energies considerably improve our mean energy estimates for $\nu_e$, with just moderate corrections for $\bar{\nu}_e$, providing an agreement better than typically $\sim$15 per cent ($\sim$1.5 MeV difference in the worst case). The larger differences observed in the $\bar{\nu}_e$ mean neutrino energies stem from the different prescriptions of bremsstrahlung employed by ILEAS, ALCAR, and VERTEX.

3.2 Snapshot calculations: black hole-torus system

In order to assess the performance of our scheme on a possible remnant of a CO merger, we calculate the neutrino luminosities for two different BH-torus systems evolved previously using the ALCAR code (Just et al. 2015b). The neutrino reactions employed for these cases are the same as for the PNS (table 3), except for heavy-lepton neutrinos, which are switched off in both calculations because of their minor relevance for this setup. The results obtained by ALCAR and ILEAS presented in this section do not include the effects of redshift.

In table 4 we also include the neutrino luminosities and mean energies for all neutrino species as obtained by ILEAS, applied to the two BH-torus systems. Because tori are optically thinner than PNSs, their cooling time-scale is much shorter, and the temperature can change considerably during the relaxation of the background. We took this into account by providing the results of both ALCAR and ILEAS after 3 ms of evolution starting from the original snapshots.
is optically thin during the relaxation (the optical depth the BH-torus models considered in this work, matter be-
case of BH-torus systems, for two simple reasons. First, in
imation via equation (45) should be more accurate in the
by equation (49), the ones obtained by the leakage approx-
Even though we also provide the mean energies calculated
provide the mean energies calculated by equation (45), while
mean energies for diagnostics are obtained via equation (49). All values are taken for a local observer in the rest frame of the source at
the edge of the grid (100 km). \( \nu_e \) luminosities refer to a single representative of the four species of heavy-lepton neutrinos.

### Table 4. Neutrino luminosities and mean energies obtained by ILEAS applied to several snapshots of a PNS cooling simulation at different times and two BH-torus models, in comparison to the results from transport calculations with different codes. All leakage quantities are computed as described in section 2.5. Leakage mean energies provide the mean energies calculated by equation (45), while mean energies for diagnostics are obtained via equation (49). All values are taken for a local observer in the rest frame of the source at the edge of the grid (100 km). \( \nu_e \) luminosities refer to a single representative of the four species of heavy-lepton neutrinos.

| Model         | \( \nu_e \) | Transport \( (10^{51} \text{ erg s}^{-1}) \) | Leakage \( (10^{51} \text{ erg s}^{-1}) \) | Transport mean energy (MeV) | Leakage mean energy (MeV) | Mean energy for diagnostics (MeV) | Transport code |
|---------------|-------------|---------------------------------|---------------------------------|-------------------|-------------------|---------------------------------|----------------|
| PNS 0.2 s     | \( \nu_e \) | 16.2                            | 16.4                            | 9.72              | 7.85              | 11.04                           | VERTEX         |
| PNS 0.2 s     | \( \nu_e \) | 17.3                            | 19.1                            | 12.42             | 12.43             | 12.72                           | VERTEX         |
| PNS 0.2 s     | \( \nu_e \) | 13.7                            | 14.5                            | 14.32             | 21.38             | -                               | VERTEX         |
| PNS 0.3 s     | \( \nu_e \) | 9.8                             | 9.1                             | 9.43              | 7.20              | 10.65                           | VERTEX         |
| PNS 0.3 s     | \( \nu_e \) | 10.8                            | 11.5                            | 12.18             | 11.98             | 12.27                           | VERTEX         |
| PNS 0.3 s     | \( \nu_e \) | 10.2                            | 11.0                            | 13.80             | 20.14             | -                               | VERTEX         |
| PNS 0.4 s     | \( \nu_e \) | 7.4                             | 6.7                             | 9.31              | 6.96              | 10.52                           | VERTEX         |
| PNS 0.4 s     | \( \nu_e \) | 8.1                             | 8.7                             | 12.00             | 11.88             | 12.30                           | VERTEX         |
| PNS 0.4 s     | \( \nu_e \) | 8.4                             | 9.1                             | 13.51             | 19.19             | -                               | VERTEX         |
| PNS 0.5 s     | \( \nu_e \) | 6.2                             | 5.5                             | 9.26              | 7.11              | 10.58                           | VERTEX         |
| PNS 0.5 s     | \( \nu_e \) | 6.7                             | 6.9                             | 11.86             | 11.32             | 12.28                           | VERTEX         |
| PNS 0.5 s     | \( \nu_e \) | 7.3                             | 8.0                             | 13.33             | 18.75             | -                               | VERTEX         |
| PNS 0.5 s     | \( \nu_e \) | 7.0                             | 6.7                             | 9.93              | 7.95              | 11.62                           | ALCAR          |
| PNS 0.5 s     | \( \nu_e \) | 7.6                             | 8.1                             | 13.32             | 12.62             | 13.13                           | ALCAR          |
| PNS 0.5 s     | \( \nu_e \) | 9.0                             | 10.4                            | 15.67             | 21.46             | -                               | ALCAR          |
| PNS 0.8 s     | \( \nu_e \) | 4.6                             | 4.4                             | 9.24              | 7.42              | 10.44                           | VERTEX         |
| PNS 0.8 s     | \( \nu_e \) | 4.9                             | 5.1                             | 11.64             | 11.39             | 12.45                           | VERTEX         |
| PNS 0.8 s     | \( \nu_e \) | 5.7                             | 6.3                             | 13.02             | 18.07             | -                               | VERTEX         |
| PNS 1.1 s     | \( \nu_e \) | 3.8                             | 3.6                             | 9.24              | 6.56              | 10.10                           | VERTEX         |
| PNS 1.1 s     | \( \nu_e \) | 4.0                             | 4.0                             | 11.45             | 12.54             | 12.64                           | VERTEX         |
| PNS 1.1 s     | \( \nu_e \) | 4.9                             | 5.3                             | 12.76             | 17.48             | -                               | VERTEX         |
| PNS 1.2 s     | \( \nu_e \) | 3.7                             | 3.5                             | 9.24              | 6.16              | 10.02                           | VERTEX         |
| PNS 1.2 s     | \( \nu_e \) | 3.8                             | 3.7                             | 11.43             | 12.98             | 12.73                           | VERTEX         |
| PNS 1.2 s     | \( \nu_e \) | 4.7                             | 5.0                             | 12.69             | 17.17             | -                               | VERTEX         |
| PNS 1.3 s     | \( \nu_e \) | 3.5                             | 3.3                             | 9.24              | 5.79              | 10.08                           | VERTEX         |
| PNS 1.3 s     | \( \nu_e \) | 3.6                             | 3.5                             | 11.38             | 13.46             | 12.85                           | VERTEX         |
| PNS 1.3 s     | \( \nu_e \) | 4.4                             | 4.7                             | 12.61             | 16.89             | -                               | VERTEX         |
| PNS 1.5 s     | \( \nu_e \) | 3.2                             | 3.0                             | 9.22              | 5.18              | 10.01                           | VERTEX         |
| PNS 1.5 s     | \( \nu_e \) | 3.3                             | 3.2                             | 11.27             | 14.26             | 12.88                           | VERTEX         |
| PNS 1.5 s     | \( \nu_e \) | 4.1                             | 4.3                             | 12.43             | 16.46             | -                               | VERTEX         |
| BH-torus 0.3 M\(_o\) | \( \nu_e \) | 23.3                           | 21.5                            | 12.13             | 12.66             | 14.19                           | ALCAR          |
| BH-torus 0.3 M\(_o\) | \( \nu_e \) | 18.4                           | 16.7                            | 14.97             | 15.89             | 17.16                           | ALCAR          |
| BH-torus 0.1 M\(_o\) | \( \nu_e \) | 6.5                             | 6.5                             | 12.02             | 12.69             | 14.85                           | ALCAR          |
| BH-torus 0.1 M\(_o\) | \( \nu_e \) | 5.2                             | 4.8                             | 14.20             | 14.50             | 16.28                           | ALCAR          |

Even though we also provide the mean energies calculated by equation (49), the ones obtained by the leakage approx-
imation via equation (45) should be more accurate in the case of BH-torus systems, for two simple reasons. First, in
the BH-torus models considered in this work, matter be-
comes optically thin during the relaxation (the optical depth is \( \tau_{\nu_e} < 2/3 \) almost everywhere after a few milliseconds of evolution) or optically thick material encloses a very small volume, so that the leakage ansatz, namely that neutrinos stream away with the mean energy obtained from their local production, is a reasonable approximation. Second, the gra-
dients in the hydrodynamical and thermodynamical quantities are considerably flatter than in the PNS case. Therefore,
the reasoning that most absorption occurs in the production cell, which is employed to estimate the mean energies in equation (49), is a less accurate approximation. Because the leakage mean energies employ a more accurate description of absorption in optically thin regions, which are the far
dominant conditions in the tori, we advise the reader to con-
sider the leakage mean energies for any diagnostic analysis or comparison.

Figure 8 shows the performance of our absorption scheme on the snapshot of a thick torus (initial torus mass 0.3 M\(_o\)) around a 3 M\(_o\) BH. Despite the ray patterns caused by the ray-tracing approach, the qualitative resemblance in the top and middle plots between ALCAR (left half-panels) and ILEAS (right half-panels) is remarkable. More-
over, the bottom panel in figure 8 displays the ratio of the net energy-exchange rates obtained by ALCAR and ILEAS in the absorption-dominated regions, which also highlights the overall quantitatively satisfactory agreement between both schemes, within a factor of \(~2\) accuracy. We refrain from performing a comparison of the rates immediately above the BH and in the close vicinity of the z-axis, because a consistent treatment of general relativistic and special relativistic effects would be needed to describe the influence of the BH
Results of our neutrino absorption scheme for an 0.3 $M_\odot$ torus around a 3 $M_\odot$ BH, top plot for $\nu_e$, middle plot for $\bar{\nu}_e$. Colour coding displays the net energy-exchange rate $Q_{\nu_i}^{\text{net}} = Q_{\nu_i} - Q_{\bar{\nu}_i}$ ($\nu_i = \nu_e, \bar{\nu}_e$) in the absorption-dominated region, where this net rate is positive. The left half-panel of each of these plots shows the results obtained by the ALCAR scheme, while the right ones are the results with ILEAS. The white contours depict the neutrinospheres, where $\nu_e$, $\bar{\nu}_e = 2/3$. The bottom plot displays the ratio of the net energy-exchange rates between ALCAR and ILEAS in regions where both rates are positive (left panel for $\nu_e$, right panel for $\bar{\nu}_e$).

Figure 8. Results of our neutrino absorption scheme for an 0.3 $M_\odot$ torus around a 3 $M_\odot$ BH, top plot for $\nu_e$, middle plot for $\bar{\nu}_e$. Colour coding displays the net energy-exchange rate $Q_{\nu_i}^{\text{net}} = Q_{\nu_i} - Q_{\bar{\nu}_i}$ ($\nu_i = \nu_e, \bar{\nu}_e$) in the absorption-dominated region, where this net rate is positive. The left half-panel of each of these plots shows the results obtained by the ALCAR scheme, while the right ones are the results with ILEAS. The white contours depict the neutrinospheres, where $\nu_e$, $\bar{\nu}_e = 2/3$. The bottom plot displays the ratio of the net energy-exchange rates between ALCAR and ILEAS in regions where both rates are positive (left panel for $\nu_e$, right panel for $\bar{\nu}_e$).

For a more direct assessment of the impact of differences in the absorption rates between ALCAR and ILEAS on possible outflows originating from the torus, we performed another test. For this purpose we defined parametrized outflows in the polar directions and compared the evolution of $Y_e$ under the influence of the emission and absorption rates from ALCAR and ILEAS.

In a steady-state situation, the evolution of the electron fraction of an outflow can be approximated by (McLaughlin et al. 1996)

$$\frac{d}{dt} Y_e = v(z) \frac{d}{dz} Y_e = \frac{R_{\text{tot}}(z)}{\Delta p(z)},$$

with the total lepton-number exchange rate, $R_{\text{tot}}$, defined as in equation (2). While $\rho(z)$ is adopted from the hydrodynamic solution of the ALCAR calculation, we assume the unbound material to move in the $z$-direction with the parametrized velocity, $v(z)$,

$$v(z) = \min \left[ \frac{c}{\Delta z}, v_0, v_{\text{lim}} \right],$$

accelerating along the $z$-direction to a terminal velocity $v_{\text{lim}}$. In equation (51), $\Delta z$ determines the length-scale over which the velocity reaches its limiting value. We set it to 30 km for the presented tests. $z_0$ defines the position of the surface from which the outflow is launched. We locate this surface at the position where the ALCAR net lepton-number-exchange rates, $R_{\text{tot}}^{\text{net}} = R_{\nu_e} - R_{\bar{\nu}_e}$, of $\nu_e$ as well as $\bar{\nu}_e$, become absorption-dominated. We then use exactly the same surface location to launch the outflow in both models, in one case employing the ALCAR rates and in the other one employing the ILEAS rates. The last free parameter of this toy model is $v_0$, which determines the initial velocity of the outflow. In order to assess the influence of the parameters $v_0$ and $v_{\text{lim}}$, which determine the duration of time the ejecta remain in the near-surface region, where the absorption rates are higher, we test two different sets of values corresponding to a slow wind, $v_0 = 100$ km/s and $v_{\text{lim}} = c/30$, and a fast wind, $v_0 = 1000$ km/s and $v_{\text{lim}} = c/10$ (see left and right columns in figure 9, respectively).

Integrating equation (50), the $Y_e$ of the outflow at a given position is simply determined by

$$Y_e(z) = Y_e(z_0) + \int_{z_0}^{z} \frac{R_{\text{tot}}(z')}{\Delta p(z')} dz'.$$

Figure 9 shows the evolution of the $Y_e$ as described by equation (52) along the paths travelled by the outflow in the polar directions. The left half of each panel illustrates the results obtained when employing the rates calculated by ALCAR, whereas the right half shows the results with the ILEAS rates. The two wind descriptions (slow wind in

or ultrarelativistic GRB jets. Furthermore, Foucart et al. (2018), for example, compared Monte Carlo results and M1 results in the context of a HMNS surrounded by a torus and pointed out that the inexact M1 closure strongly overestimates the number density in the polar regions, by $\sim 50$ per cent for $\nu_e$ and $\bar{\nu}_e$, which leads to significant boosting of the absorption rates by charged-current reactions and excess heating. Just et al. (2015a) also reported similar behaviour when comparing BH-torus calculations with a ray-tracing Boltzmann solver against their M1 results. Therefore, a detailed quantitative comparison between ILEAS and M1 results in the vicinity of the polar axis could be misleading.
it travels along a straight path in the z-direction, exposed to the neutrino rates calculated by ALCAR.

Figure 9. Evolution of the $Y_e$ of a prescribed outflow launched from the torus (initial torus mass $0.3 \, M_\odot$) surrounding a BH ($3 \, M_\odot$) as it travels along a straight path in the z-direction, exposed to the neutrino rates calculated by ALCAR (left half-panels) and ILEAS (right half-panels). The plots in the top row display the results obtained for the same BH-torus snapshot also used in figure 8, while those in the bottom row show the results for a snapshot obtained after 5 ms of evolution with ALCAR and ILEAS, respectively (see section 3.3). The two plots in the left column provide the results obtained for a slow wind, with an initial outflow velocity $v_0 = 100 \, \text{km/s}$ and a limiting velocity $v_{\text{lim}} = c/30$, whereas the ones in the right-column provide the results obtained by a fast wind, with $v_0 = 1000 \, \text{km/s}$ and $v_{\text{lim}} = c/10$ (see text for details).

The left column of plots, fast wind in the right), agree qualitatively for the snapshot at 0 ms (top row) to the extent that $Y_e > 0.5$ is achieved in the same spatial region close to the polar axis. There differences in $Y_e$ are on the level of $\sim 20$ per cent for the slow wind, showing the limitations of our scheme, and of $\sim 10$ per cent for the fast wind ($\sim 0.55$ for ALCAR compared to $\sim 0.5-0.6$ for ILEAS), with local variations due to the ray patterns associated with ILEAS. The regions where $Y_e < 0.5$ display even better quantitative agreement within $\sim 5$ per cent. We warn the reader that, as we already pointed out, Just et al. (2015a) and Foucart et al. (2018) found possible deficiencies in the calculation of the absorption rates by M1 schemes in the polar region when contrasting with ray-tracing and MC results, respectively, and thus any comparison in such regions should be taken with caution. Furthermore, as we will see in section 3.3, the long-term evolution of the BH-torus model with ILEAS produces a transient at the beginning of the simulation as the system relaxes to a new quasi-steady state (figure 11), which could affect the discussed results. For this reason, we also provide in figure 9 (bottom row) the results of outflow calculations applied on snapshots obtained after 5 ms of evolution with ALCAR and ILEAS, respectively (see section 3.3, figure 11). At these later times, the evolution of the composition of the ejecta predicted by ALCAR and ILEAS agrees very well, especially in the fast wind scenario.

It must be noted that ILEAS assumes the flux factor to follow the simple interpolation $\langle \chi_{\nu_i} \rangle_{\text{Ileas}} = 4.275 \nu_i + 1.15$, suggested by O’Connor & Ott (2010). This is an acceptable approximation for the case of a cooling PNS, but fails to
capture the geometry of the BH-torus system. More sophisticated prescriptions of the flux factor, which account for geometric effects, would certainly improve the accuracy of the absorption scheme in the area around the inner edge of the torus. This also concerns the results for $Y_e$ in outflows and is likely to reduce differences between ILEAS and transport calculations, in particular also for slow winds. However, such improvement is beyond the scope of this work, and we consider the obtained results with the presented approximations as satisfactory.

3.3 Time evolution of a proto-neutron star and two black hole-torus systems

In order to test the performance of our scheme in evolving systems and to relax the thermodynamical background, we have coupled ILEAS to a simple time evolution scheme. As we want to focus on the neutrino effects, we only evolve the temperature (via the internal energy density of the fluid, $E_{\text{fluid}}$) and the electron fraction, keeping the matter density fixed and ignoring the velocity terms\textsuperscript{14}.

We can calculate the changes in the electron fraction from the rate equation as

$$
\frac{dY_e}{dt}_{\text{source}} = \frac{R_{\text{tot}}}{Ap},
$$

where $R_{\text{tot}}$ is given in equation (2) and $A$ is the Avogadro constant. For the fluid energy density, following the first law of thermodynamics for a quasi-static system with fixed density, we can express its evolution equation as

$$
\frac{dE_{\text{fluid}}}{dt}_{\text{source}} = Q_{\text{tot}},
$$

where $Q_{\text{tot}}$ is given in equation (1). We solve these simple equations explicitly with a forward integration, allowing for changes on either quantity of up to 2 per cent in a single time-step. Then, we only need to call the EoS to obtain the temperature from the energy density, matter density and $Y_e$ (via bisection) and then the chemical potentials, which we use in the next leakage step. We also include equilibration as described in this paper.

We initialize the system by computing the fluid energy density and chemical potentials from the EoS using the density, temperature and electron fraction from the initial snapshot, then calculating the equilibration surfaces and initializing the lepton fractions in their pertinent regimes.

Figure 10 displays the time evolution of the neutrino luminosities for the PNS snapshot evolved by ALCAR (0.5 s post bounce). The time axis starts at the time of the original snapshot, where the evolution is started. After a brief transient of a few milliseconds, the electron fraction and the temperature relax to their equilibrium values, and the system slowly evolves in a quasi-steady state. Thus the results listed in table 4 correspond to the results of the plot at 5 ms. ILEAS is capable of reproducing the results obtained by ALCAR with ~10 per cent accuracy throughout the 50 ms simulated. In figure 10 we also provide the evolution of the mean neutrino energies obtained by ALCAR and ILEAS, using both equations (45) and (49) described in section 2.5, for the same PNS model. We find an agreement within 1–2 MeV between both codes, with a tendency of improvement at later times and better results for the diagnostic mean neutrino energies as given by equation (49).

It is important to note that, in the BH-torus models, the unavoidable transient which occurs when switching on ILEAS, proceeds to swiftly cool the optically thin disc before a stationary state can be reached. The natural consequence is, therefore, that the ILEAS luminosities become smaller than those obtained by ALCAR, whose background remains hotter. Nevertheless, for most of the time the results obtained by ILEAS agree to less than 10 per cent with the ones obtained by ALCAR, as can be seen in figure 11. Note that these plots start at the time of the original snapshots from where the 11 display the evolution of the mean

\textsuperscript{14} Radial velocities are small for the PNS case and not very high for the BH-torus systems.

Figure 10. Top panel: time evolution of the neutrino luminosities obtained by ILEAS (solid lines) and ALCAR (dashed lines) produced by a cooling PNS with fixed density background, but evolved $T$ and $Y_e$ profiles. Bottom panel: time evolution of the mean neutrino energies obtained by ALCAR (dashed lines) and ILEAS using the leakage approach (equation 45, solid lines) or the one for diagnostics (equation 49, dotted lines) for the same PNS model. The luminosities and mean energies are computed at later times and better results for the diagnostic mean neutrino energies as given by equation (49).
Figure 11. Top and middle panels: time evolution of the neutrino luminosities obtained by ILEAS (solid lines) and ALCAR (dashed lines) produced by a thick BH-torus (left panels, initial torus mass 0.3 M⊙ around a 3 M⊙ BH) and by a thin BH-torus (right panels, initial torus mass 0.1 M⊙ around a 3 M⊙ BH) with fixed density background, but evolved T and Y_e distributions. The top panels show the first 10 ms of evolution, while the middle ones show the full duration of the simulations with the luminosities on a logarithmic scale for better visibility. Bottom panels: time evolution of the mean neutrino energies obtained by ALCAR (dashed lines) and ILEAS (equation 45, solid lines) for the same BH-torus model. The luminosities and mean energies are computed for a local observer in the centre-of-mass frame of the neutrino source at the edge of our grid (100 km).
neutrino energies obtained by ALCAR and ILEAS (using equation 45) for the same two BH-torus models. As for the PNS case, our results agree within ~1 MeV or, for the thin torus case, even better than 1 MeV during most of the simulation.

4 SUMMARY

The detection of a NS-NS merger using GW interferometers and its associated EM transients (Abbott et al. 2017a; Abbott et al. 2017b) has opened the doors to a new era of multi-messenger astronomy. The vivid discussions about the nature of the observed kilonova (e.g. Cowperthwaite et al. 2017; Nicholl et al. 2017; Chornock et al. 2017; Kasen et al. 2017; Smartt et al. 2017; Pian et al. 2017; Perego et al. 2017b; Waxman et al. 2017b) has brought to light the need of a reliable understanding not only of the composition of the ejected material, but also its dependence on the direction of ejection. At the same time, the error-bars associated with the measured NS masses, together with the underlying uncertainty of the EoS of NS matter, call for the exploration of a wide distribution of initial conditions for numerical simulations. In order to reconcile these two needs we have introduced ILEAS, an improved leakage scheme which accounts for the basic physical effects of neutrino transport at a moderate computational cost. ILEAS is ideal for exploring wide parameter spaces in three dimensions, where ~10 per cent of accuracy is enough to capture the essential impact of weak interactions.

Leakage models have been used to emulate neutrino losses in the context of NS mergers since the 1990’s (Ruffert et al. 1996), but little work has been devoted to assess their accuracy (exceptions are Foucart et al. 2016a; Perego et al. 2016). In fact, we show that, in their standard formulations, leakage schemes have a tendency to overproduce neutrinos in the region close to the neutrinosphere, which could lead to numerical artefacts in near-surface regions and incorrect estimates of the ejecta composition. Moreover, the traditional leakage schemes were only a simple ansatz to estimate the local neutrino losses, ignoring other important physical effects inherent to neutrino transport, such as equilibration or re-absorption. In the recent years, truncated moment schemes have been developed and successfully used in the context of NS mergers, providing a more sophisticated alternative to leakage schemes. These approaches, however, also possess disadvantages and shortcomings of their own, such as problems with crossing flows (see, e.g., Foucart et al. 2018) or the need of more computational resources, in particular when combined with a ray-tracing scheme for computing high-resolution neutrino distributions. While not a proper neutrino transport scheme, ILEAS is able to capture all the aforementioned physical effects, yet retaining the simple and inexpensive aspects of leakage schemes. Its improved diffusion time-scale, obtained directly from the flux-limited diffusion equation, provides a much better estimate of the neutrino losses in optically thick regions. This is reinforced by the inclusion of equilibration: the equilibration step ensures the recovery of the correct lepton fractions in the $\beta$-equilibrium regime, and the EoS also includes the energy and pressure contributions of the trapped neutrinos. Finally, by means of a simple multi-dimensional ray-tracing algorithm, we account for the re-absorption in optically thin conditions of neutrinos leaking out from the system. In order to keep the absorption module computationally efficient, we decided to resort to a grey approximation for ILEAS. However, our spectral calculation of the diffusive flux allows us to approximately capture the energy-dependent decoupling of neutrinos from matter in the integrated diffusion time-scale. Our results show that, despite the inherent approximations, ILEAS is sufficiently good to reproduce the results of more sophisticated transport schemes on the level of 10 per cent, locally and globally.

Motivated by its future application in the context of NS mergers, we tested the performance of ILEAS by comparison to available simulations representing some of the typical conditions encountered during NS mergers. We presented the results obtained with ILEAS applied on 3D mappings of several PNS cooling snapshots from the 1D VERTEX-PROMETHEUS simulation (Sr) with energy-dependent neutrino transport performed by H"udepoli et al. (2010). For all tested snapshots, ranging from 0.2 s until 1.5 s post-bounce, ILEAS was able, after a short relaxation of the medium (i.e., after evolving temperature and $Y_e$ for 5 ms), to reproduce not only the total VERTEX luminosities, but also the complete radial luminosity profiles within ~10 per cent accuracy. In order to provide a more detailed comparison, we also tested ILEAS on a snapshot obtained from the evolution of the same PNS performed by ALCAR (Just et al. 2015b), which includes an energy-dependent M1 transport solver and exactly the same neutrino reactions as ILEAS for $\nu_e$ and $\bar{\nu}_e$. As with the VERTEX cases, ILEAS reached an agreement within 10 per cent accuracy with the ALCAR results. Furthermore, we evolved the temperature and $Y_e$ of the ALCAR snapshot (keeping the density fixed) for 50 ms with both ALCAR and ILEAS, and the good agreement was maintained throughout the simulation.

As possible remnants of CO mergers, BH-torus systems provide a useful scenario for testing the performance of our scheme in the low optical depth limit. Snapshot calculations allowed us to attest the capability of our absorption treatment to capture the qualitative features of neutrino absorption in comparison with the results attained by ALCAR. Furthermore, we evolved two BH-torus models for 50 ms using ALCAR and ILEAS, in the same fashion as the PNS snapshot. In spite of the initial over-cooling caused by the unavoidable transient produced when switching on ILEAS, the neutrino luminosities of both models preserved an agreement of ~10 per cent.

In conclusion, ILEAS has been shown to reproduce within ~10 per cent accuracy basic results of more sophisticated transport schemes also in multi-dimensional scenarios. Albeit not as accurate as full-fledged 3D transport, ILEAS includes all the relevant physical effects of neutrino transport and surpasses the quality of previous, conventional leakage schemes while retaining most of their efficiency and simplicity. These features make ILEAS an appropriate description of neutrinos for numerical simulations of NS mergers, where the relatively short evolution time-scales may not require a full-scale 3D neutrino transport to still obtain a consistent picture of the composition evolution of merger medium and ejecta. The exploration of the vast parameter space of possible binary configurations demands computationally efficient
solution. However, in the tested BH-torus scenario, the results had already been reported in the context of BH-torus treatments for CO mergers and their remnants. The work by Foucart et al. (2018) underlines that M1 solutions have their own shortcomings when applied to the highly aspherical environments of merger remnants. For this reason our tests with BH-torus systems, comparing ILEAS to M1 results from the ALCAR code, cannot be considered as finally conclusive regarding the accuracy of ILEAS. Direct comparisons of ILEAS and MC results would be desirable.

ACKNOWLEDGEMENTS

The authors are grateful to N. Rahman for helpful discussions. The project was supported by the Deutsche Forschungsgemeinschaft through Sonderforschungsbereich SFB-1258 “Neutrinos and Dark Matter in Astro- and Particle Physics (NDM)” and the Excellence Cluster Universe (EXC 153; http://www.universe-cluster.de/) and by the European Research Council through grant ERC-AdG No. 341157-COCO2CASA. Calculations were performed on Hydra and Draco of the Max Planck Computing and Data Facility (MPCDF). OJ acknowledges support by the Special Postdoctoral Researchers (SPDR) program and iTHEMS cluster of RIKEN. AB acknowledges support by the Klaus Tschira Foundation and by the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme under grant agreement No. 759253.

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but sufficiently accurate codes. ILEAS is intended to serve this requirement.

Nevertheless, we caution the reader that ILEAS cannot be perceived as a perfect replacement for a full neutrino transport scheme, especially not in situations where the transfer of energy and leptons from one location to another by neutrino diffusion is crucial to describe the interior evolution of an object, as, for example, in the case of the long-time neutrino-cooling of PNNe. ILEAS is a suitable alternative to neutrino transport codes particularly in conditions where the dynamical changes of the system happen on a time-scale shorter than or comparable to the neutrino diffusion time-scale. Examples are the rapid evolution of two merging NSs or the cooling of (semi-)transparent tori around merger remnants. Finally, for its use in consistent GR simulations, ILEAS needs to be extended to include effects such as Doppler shift, time retardation and general relativistic gravitational ray bending, which were not essential in the test calculations presented in this work. In future versions, ILEAS will also be supplemented by a module describing the effects of neutrino-antineutrino annihilation.

A computationally considerably more expensive alternative to ILEAS is advertised by Foucart et al. (2018), who plan to upgrade their grey M1 scheme by an Eddington tensor closure obtained from a Monte Carlo (MC) solution of the Boltzmann equation exterior to regions of high optical depths, instead of using the analytical closure relations applied so far. Whether such a hybrid code yields stable results with acceptable numerical “noise” for affordable low-resolution MC calculations will have to be demonstrated. For reasons of accuracy, Foucart et al. (2018) recommend to evolve the neutrino number densities as well as the neutrino energy densities in order to obtain reasonably accurate local estimates of the average neutrino energies. In order to evaluate errors of M1 results with analytical closure relations, they performed a time-dependent calculation over 4.5 ms with their relativistic MC solver for a HMNS as a representative remnant of a NS-NS merger, using the time-dependent fluid quantities from the M1 radiation-hydrodynamics run and not feeding back the MC results into fluid or M1 transport solutions.

They found relative differences of 10–30 per cent in the average neutrino energies between the M1 and the MC transport results and concluded that this implies that the absorption and scattering opacities can be off by ~30 per cent up to close to a factor of 2, dependent on positions closer to the polar axis (i.e. the rotation axis of the remnant) or farther away from it. Moreover, because of artificial shocks associated with the use of a non-linear, algebraic closure relation, they diagnosed that the M1 code accumulates neutrinos close to the polar axis, leading to an excess of the neutrino density in the polar regions by about 50 per cent for $\nu_e$ and $\bar{\nu}_e$ and by nearly a factor of 2 for $\nu_x$. The $\nu\bar{\nu}$ pair-annihilation rate above the poles of the HMNS is underestimated by factors of 2–3 by the M1 description. Similar results had already been reported in the context of BH-torus systems by Just et al. (2015a) (in the appendix there), who also found an overestimation of the neutrino densities in the polar regions when comparing their energy-dependent M1 scheme, ALCAR, with a ray-tracing Boltzmann solution. However, in the tested BH-torus scenario, the results by Just et al. (2015a) imply an over-estimation of the pair-annihilation rate around the polar axis. While the spectral and opacity differences may be handled better by fully energy dependent (and considerably more complex and costly) transport codes such as ALCAR, the overestimated number densities and underestimated pair-annihilation rates have to await their cure through a replacement of the analytic closure by a Boltzmann transport solution, possibly based on MC results.

In view of the considerable error margins associated with a grey M1 approximation and considering the high computational demands of future hybrid schemes, our ILEAS method constitutes itself as an interesting option for the next generation of NS-NS/BH merger simulations surveying the huge multi-dimensional parameter space of possibilities. ILEAS is not only computationally very efficient but also appears to be competitive concerning its accuracy compared to other forefront developments of neutrino transport treatments for CO mergers and their remnants. The work by Foucart et al. (2018) underlines that M1 solutions have their own shortcomings when applied to the highly aspherical environments of merger remnants. For this reason our tests with BH-torus systems, comparing ILEAS to M1 results from the ALCAR code, cannot be considered as finally conclusive regarding the accuracy of ILEAS. Direct comparisons of ILEAS and MC results would be desirable.
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APPENDIX A: COMPARATIVE ANALYSIS OF DIFFUSION TIME-SCALE PRESCRIPTIONS USED IN THE LITERATURE

Although leakage schemes have been around for more than two decades, not many comparisons between the different realizations can be found in the literature. Here we want to briefly compare the most common leakage implementations used in the context of neutrino physics in NS mergers, in particular, the schemes from Ruffert et al. (1996) (RJS) and Rosswog & Liebendörfer (2003) (RL).

There are three main differences between both schemes: the definition of the diffusion time-scale, the energy averaging and the prescription of the neutrino chemical potential to describe the neutrino spectra.

As we discussed in section 2.2.1, in a first approximation the diffusion time-scale can be obtained from a dimensional analysis of the diffusion equation as
\[
t_{\text{diff}} = \frac{3d^2}{c\lambda_{\nu}}. \tag{A1}\]

where \(d\) is simply a characteristic length-scale of the system. RJS took the approximation of a homogeneous sphere to define
\[
\lambda_{\nu} = \frac{d}{c}. \tag{A2}\]

By plugging equation (A2) into (A1) we are left with one factor of the length-scale, which is chosen as the integration path for the optical depth, taken as the minimum distance to the neutrinosphere \((r(\tau_{\nu} = 2/3)):\)
\[
t_{\text{diff, RJS}} = \frac{3d}{c} \tau_{\nu}. \tag{A3}\]

Similarly, RL proceeded to further approximate the remaining length-scale as \(d \sim \tau_{\nu} \lambda_{\nu} \) (with \(\lambda_{\nu} = 1/\kappa_{\nu}\)) to obtain:
\[
t_{\text{diff, RL}} = \frac{3\lambda_{\nu}}{c} \tau_{\nu}^2. \tag{A4}\]

It is worth noting that other groups have suggested alternative prescriptions for the definition of \(d\), such as using the pressure scale-height \(d \sim P/\nabla P\) (Metzger & Fernández 2014). This approach could be generalized to using the scale-height of any convenient scalar quantity that defines the medium in which neutrinos diffuse.

One of the caveats of grey schemes is the ambiguity associated with the energy averaging of the neutrino quantities. In RJS, the diffusion time-scales are computed from spectrally averaged opacities (and optical depths),
\[
\kappa_{\nu} = \frac{\int_{0}^{\infty} \kappa_{\nu}(\epsilon)\epsilon^2 f(\epsilon; T, \eta_{\nu}) d\epsilon}{\int_{0}^{\infty} \epsilon^2 f(\epsilon; T, \eta_{\nu}) d\epsilon}. \tag{A5}\]

In RL, on the contrary, the (roughly) \(\epsilon^2\) dependence of the opacities is factored out and carried on to the calculation of the “integrated diffusion rates”, defined as
\[
R_{\nu;\text{diff}} = \int_{0}^{\infty} \frac{E_{\nu}^{\text{i, diff}}(\epsilon)}{\kappa_{\nu}(\epsilon)} d\epsilon = \int_{0}^{\infty} \frac{E_{\nu}^{\text{i, diff}}(\epsilon)}{\kappa_{\nu,\text{RL}}(\epsilon)} d\epsilon, \tag{A6}\]

for lepton number diffusion and equivalently for the energy diffusion rate,
\[
Q_{\nu;\text{diff}} = \int_{0}^{\infty} \frac{E_{\nu}^{\text{i, diff}}(\epsilon)}{\kappa_{\nu}(\epsilon)} d\epsilon = \int_{0}^{\infty} \frac{E_{\nu}^{\text{i, diff}}(\epsilon)}{\kappa_{\nu,\text{RL}}(\epsilon)} d\epsilon. \tag{A7}\]

In both schemes, the effective loss term is then calculated as an interpolation between diffusion rates (equations A6 and A7) and production rates (equations B3 and B34):
\[
R_{\nu} = R_{\nu;\text{diff}} \left(1 + \frac{R_{\nu}}{R_{\nu;\text{diff}}}\right)^{-1}, \tag{A8}\]
and
\[
Q_{\nu} = Q_{\nu;\text{diff}} \left(1 + \frac{Q_{\nu}}{Q_{\nu;\text{diff}}}\right)^{-1}. \tag{A9}\]

for number and energy, respectively, which are equivalent to equations (14) and (15) in the case of the RJS averaging.

Moreover, without an actual energy-dependent transport scheme, it is impossible to determine the correct neutrino phase-space distribution. In the optically thick regime, neutrinos remain in \(\beta\)-equilibrium with the medium, thus their spectrum is a Fermi distribution with the chemical potential being easily obtained from the EoS. RL assume...
that this behaviour will remain a good approximation even in the
optically thin regime, where neutrinos decouple from matter. RJS on the other hand also use a Fermi distribution throughout, but interpolate between the equilibrium chemical potential and an expected value at free streaming conditions of $\mu_\nu = 0$ (see equation (18)).

We apply both schemes to one of our PNS cooling snapshots and compare the results to the ILEAS model presented in this work. To further disentangle the contribution of each approximation, we also test different permutations, combining the prescriptions for the diffusion time-scale, the spectral averaging and the neutrino chemical potential out of equilibrium. A list of the models and the prescriptions employed are provided in Table 1. In order to focus on the impact of the leakage module alone, we let the system relax without including equilibration or neutrino absorption in any of the calculations. Relativistic corrections in the diffusion time-scale are omitted as well. Figure A1 shows the radial profiles of the electron fraction and luminosity profiles for all the relaxed models (after 5 ms) together with the ones obtained by ALCAR and ILEAS (including all modules).

The first aspect that catches the eye is the substantial improvement of our prescription (model 7) with respect to all previous models. For the luminosities this is particularly true in the high-optical-depth regime, where the diffusion time-scale dominates. This is no surprise, as we define $t^{\text{diff}}_\nu$ directly from the diffusion equation, which encodes much more information about the way neutrinos are transported than the simple ansatz of equation (A1). Due to the slightly slower increase of RL’s $t^{\text{diff}}_\nu$ with growing optical depth (models 2,3,6) in comparison to RJS’s $t^{\text{diff}}_\nu$ (models 1,4,5), neutrinos escape from further inside the star in the former models. As a consequence, they resemble a bit closer the transport profile, but overproduce neutrinos of all species at lower optical depths. The consequences of the energy averaging are much less straightforward. Differences between a few percent up to a factor 3 can be seen for the different species.

The choice of neutrino chemical potential does not significantly affect the results of the leakage scheme in the chosen snapshot. The differences could become more significant in a scenario where most neutrino luminosities are produced in the semi-transparent region. Based on the current results, however, the interpolated $\mu_\nu$ (equation 18) should be preferred, because it fulfills the correct limit at high optical depth and a well controlled behaviour at low optical depths, thus avoiding an undesirable behaviour of the analytical solutions of the Fermi integrals and their ratios at low optical depth.

It is impossible to draw a definitive conclusion from the presented data, but for the studied case, the standard formalism from RJS seems, overall, to be in somewhat better agreement with the transport results.

It is worth noticing that, as can be seen in the first panel of figure A1, all leakage versions produce a similar effect on the electron fraction after relaxation. Namely, the low $Y_e$ trough near the NS surface expands outwards and the matter becomes more neutron-rich. The cause for this effect is simply the inability of any leakage scheme to accurately describe the semi-transparent region, which comes as no surprise being a model constructed as an interpolation between pure diffusion and pure free streaming. A similar effect, albeit to a much smaller extent due to the more accurate diffusion time-scale, can be observed when applying ILEAS. This comparison further highlights the advantages of the scheme presented in this work with respect to some of the leakage versions widely used in the literature.

**APPENDIX B: NEUTRINO REACTIONS**

In this appendix we collect the formulae for the different neutrino reactions (opacities and production rates) of all three neutrino species, employed in our scheme. Most reactions and their constants are extracted from Ruffert et al. (1996) and references therein. In this section, unlike in the body of this work, we employ only the superscript of $Q^j_\nu$, with $j = 0, 1$ to denote number and energy rates, respectively, for reasons of compactness in the formulation. All production rates and opacities for $\nu_\tau$ include the contributions of all four species ($\nu_\tau, \bar{\nu}_\mu, \nu_\mu$ and $\bar{\nu}_\tau$).

### B1 Opacities for diffusion

We define the energy-dependent absorption opacities, $\kappa_{\nu_\tau}^j(\epsilon)$, following Bruenn (1985), with the correction of stimulated absorption (neutrino phase space blocking) from Rampp & Janka (2002),

$$\kappa_{\nu_\tau}^j(\epsilon) = \kappa_{\nu_\tau}(\epsilon)[1 - f(\epsilon; T, \eta^j_\nu)]^{-1}. \tag{B1}$$

Here, $f(\epsilon; T, \eta^j_\nu) = [1 + \exp((\epsilon/T) - \eta^j_\nu)]^{-1}$ is the distribution function of fermions with degeneracy parameter $\eta^j_\nu = \mu^j_\nu/T$ and energy $\epsilon$. The superscript ‘eq’, in this case, denotes the usage of the equilibrium neutrino degeneracy instead of the interpolated one (see section 2.2). The opacity for $\nu_\tau$ absorption on neutrons, $n$, is given by

$$\kappa_{\nu_\tau,n}^j(\epsilon) = \frac{1 + 3g_A^2}{4} \frac{\sigma_\nu \xi_{\nu n}}{A} \left[1 - \frac{f(\epsilon + Q; T, \eta^j_\nu)}{1 - f(\epsilon; T, \eta^j_\nu)} \frac{\epsilon + Q}{m_e c^2} \right]^{-2} \frac{1}{1 - \left(\frac{m_e c^2}{\epsilon + Q}\right)^2} \frac{1}{\epsilon} \left[1 - \left(\frac{m_e c^2}{\epsilon - Q}\right)^2\right]^{\frac{1}{2}}, \tag{B2}$$

and $\bar{\nu}_\tau$ absorption on protons, $p$, by

$$\kappa_{\bar{\nu}_\tau,p}^j(\epsilon) = \frac{1 + 3g_A^2}{4} \frac{\sigma_\nu \xi_{\bar{\nu} p}}{A} \left[1 - \frac{f(\epsilon - Q; T, \eta^j_\nu)}{1 - f(\epsilon; T, \eta^j_\nu)} \frac{\epsilon - Q}{m_e c^2} \right]^{-2} \frac{1}{1 - \left(\frac{m_e c^2}{\epsilon - Q}\right)^2} \frac{1}{\epsilon} \left[1 - \left(\frac{m_e c^2}{\epsilon + Q}\right)^2\right]^{\frac{1}{2}} \Theta(\epsilon - Q - m_e c^2). \tag{B3}$$

Here $c$ is the speed of light, $g_A \approx 1.25$, $\sigma_\nu = 1.76 \times 10^{-44} \text{cm}^2$ and $m_e$ the electron mass. $\xi_{\nu p}$ and $\xi_{\bar{\nu} p}$ (Bruenn 1985) are related to the nucleon blocking factors $Y_{np}$ and $Y_{pn}$ (Ruffert et al. 1996) as

$$\xi_{np} = (n + p)Y_{np} = \frac{d}{dQ} \frac{Y_p - Y_n}{e^{\eta_\nu - \eta_\pi} - 1}, \tag{B4}$$

and

$$\xi_{pn} = (n + p)Y_{pn} = \frac{d}{dQ} \frac{Y_p - Y_n}{e^{\eta_\pi - \eta_\nu} - 1}, \tag{B5}$$

where $Y_p$ and $Y_n$ are the proton and neutron number fractions, respectively, and $A$ is the Avogadro constant (Bruenn 1985). As we pointed out in section 2.2, this formulation of the blocking factors assumes nucleons to be well represented.
by a free Fermi gas. In order to avoid unphysical behaviour, we make use of the free Fermi gas nucleon chemical potentials, which we calculate by inverting the relation (Rampp 2000; Hecht 1989),

\[ \eta_N = \frac{4\pi}{(hc)^2} (2m_N c^2 T)^{3/2} F_{1/2}(\eta_N). \] (B6)

where \( N \) refers to the nucleon type, \( p \) or \( n \).

If one assumes complete dissociation of matter in protons and neutrons, the nucleon fractions can be expressed as \( Y_p = Y_e \) and \( Y_n = (1 - Y_e) \), as in Ruffert et al. (1996). However, for more consistent comparison to ALCAR, we relaxed this assumption and employed the nucleon number densities obtained from the EoS. The Heaviside step function \( \Theta(\epsilon - Q - m_N c^2) \) in equation (B3) ensures that the opacity remains defined and positive, setting the rest-mass difference between particles on both sides of the interaction as the minimum energy for \( \bar{\nu}_e \) absorption.

The transport opacities for neutrino-nucleon scattering of all three neutrino species are defined as

\[ \kappa_{\nu_i,\alpha}(\epsilon) = C_N \sigma_0 \xi_{NN} \left( \frac{\epsilon}{m_N c^2} \right)^2, \] (B7)

where \( C_p = [4(C_V - 1)^2 + 5g^2_A]/24 \) and \( C_n = (1 + 5g^2_A)/24 \) with \( C_V = 1/2 + 2\sin^2\theta_W \) and \( \sin^2\theta_W = 0.23 \). We define the nucleon Pauli blocking factor, \( Y_{NN} \), following Mezzacappa &Bruenn (1993), as an interpolation between (non-relativistic) degenerate and non-degenerate limits:

\[ \xi_{NN} = (n_N + n_p)Y_{NN} = A_0 Y_N \frac{\xi_N}{\sqrt{1 + \xi_N}}, \] (B8)

with \( \xi_N = \frac{3T}{2E^F_N} \),

(B9)

where \( E^F_N \) is the Fermi energy of nucleon \( N \),

\[ E^F_N = \frac{h^2}{8\pi^2 m_N} \left( 3\pi^2 n_N \right)^{2/3}. \] (B10)

Similarly, scattering on nuclei of mass number \( A \) can be expressed as

\[ \kappa_{\nu_i,\alpha}(\epsilon; A) = \frac{1}{6} A^2 \left[ C_A - 1 + \frac{Z}{A} (2 - C_A - C_V) \right]^2 \cdot \sigma_0 n_A \left( \frac{\epsilon}{m_N c^2} \right)^2, \] (B11)

where \( C_A = 1/2 \), \( Z \) is the proton number of nuclei and \( n_A \) the nuclei number density. This equation is used both for scattering on heavy nuclei of average mass and proton numbers \( \bar{A} \) and \( \bar{Z} \) and for scattering on \( \alpha \)-particles (\( A = 4 \) and \( Z = 2 \)).

The total opacities for each neutrino species, both for energy and number transport, are simply

\[ \kappa_{\nu_i}(\epsilon) = \kappa_{\nu_i,\alpha}(\epsilon) + \kappa_{\nu_i,\beta}(\epsilon; n) + \kappa_{\nu_i,\beta}(\epsilon; p) + \kappa_{\nu_i,\gamma}(\epsilon; \alpha) + \kappa_{\nu_i,\gamma}(\epsilon; \bar{A}), \]

(B12)

These opacities are used for the calculation of the diffusion time-scales (equations 30 and 31) as explained in section 2.2.1.

### B2 Opacities for absorption and optical depth

We use spectrally averaged opacities to estimate the optical depth (equation 20) for the interpolation of the neutrino degeneracies (equation 18), as well as in the absorption module. For consistency with our production rates, we do not correct these opacities for stimulated absorption (equation B1, see also the discussion in appendix C). Following...
Ruffert et al. (1996), we average the absorption opacities as

\[
\bar{\kappa}_{\nu_e,\alpha}^j = \frac{\int_0^{\infty} \kappa_{\nu_e,\alpha}(\epsilon) E_{\nu_e}^j(\epsilon) \, d\epsilon}{\int_0^{\infty} E_{\nu_e}^j(\epsilon) \, d\epsilon}
\]

\[
= \frac{1 + 3g_A^2}{4(m_e c^2)^2} \sigma_0 \epsilon \epsilon_{\nu_e}(1 - f(\epsilon_{\nu_e}; T, \epsilon_{\nu_e}))
\]

\[
\times T^2F_{4j}^j(\eta_{\nu_e}) + 2QTF_{3j}^j(\eta_{\nu_e}) + Q^2F_{2j}^j(\eta_{\nu_e}),
\]

(B13)

and

\[
\bar{\kappa}_{\bar{\nu}_e,\alpha}^j = \frac{\int_0^{\infty} \kappa_{\bar{\nu}_e,\alpha}(\epsilon) E_{\bar{\nu}_e}(\epsilon) \Theta(\epsilon - Q) \, d\epsilon}{\int_0^{\infty} E_{\bar{\nu}_e}(\epsilon) \, d\epsilon}
\]

\[
= \frac{1 + 3g_A^2}{4(m_e c^2)^2} \sigma_0 \epsilon \epsilon_{\bar{\nu}_e}(1 - f(\epsilon_{\bar{\nu}_e}; T, \epsilon_{\bar{\nu}_e}))
\]

\[
\times T^2F_{4j}^j(\eta_{\bar{\nu}_e}) + 2QTF_{3j}^j(\eta_{\bar{\nu}_e} - Q/T) + \frac{(1 + 2j)Q^2F_{2j}^j(\eta_{\bar{\nu}_e} - Q/T)}{F_{2j}^j(\eta_{\bar{\nu}_e})}
\]

\[
+ \frac{jQ^3T^{-1}F_{1j}^j(\eta_{\bar{\nu}_e} - Q/T)}{F_{2j}^j(\eta_{\bar{\nu}_e})},
\]

(B14)

where \( T \) is the matter temperature, \( E_{\nu_e}^j(\epsilon) \) is defined as in equation (21) and \( F_i = \int_0^{\infty} x^i f(x; T, \eta_{\nu_e}, \epsilon_{\nu_e}) \) dx are the Fermi integrals of order \( k \) of particle \( i \), with \( F_i(\eta_{\nu_e} - Q/T) \) evaluated including the nucleon rest-mass correction to the lepton energy. In this averaging procedure, we consider the correction of the electron rest mass to the neutrino energy to be negligible. As in Ruffert et al. (1996), we also approximate the lepton blocking factors, \( (1 - f(\epsilon_{\nu_e}; T, \eta_{\nu_e})) \), assuming that the mean electron and positron production energies are equal to those of the absorbed \( \nu_e \) and \( \bar{\nu}_e \) with an additional correction for the nucleon rest mass difference\(^{15}\)

\[
\epsilon_{\nu_e}^- = \frac{T F_{4j}^j(\eta_{\nu_e})}{F_{4j}(\eta_{\nu_e})} + Q,
\]

(B15)

\[
\epsilon_{\bar{\nu}_e}^- = \frac{T F_{4j}^j(\eta_{\bar{\nu}_e})}{F_{4j}(\eta_{\bar{\nu}_e})},
\]

(B16)

One can easily recover the results from Ruffert et al. (1996) by assuming the nucleon rest mass to be negligible, \( \bar{\epsilon}_{\nu_e} \approx \epsilon_{\nu_e} \) and \( \bar{\epsilon}_{\bar{\nu}_e} \approx \epsilon_{\bar{\nu}_e} \). Note that equations (B13) and (B14) with \( j = 0 \) are only used to calculate the mean neutrino energies for diagnostics (equation 46).

Following the same procedure, the spectrally averaged scattering opacities read,

\[
\tilde{\kappa}_{\nu_e,\alpha}^j = C_N \sigma_0 \xi_{NN} \left( \frac{T}{m_e c^2} \right)^2 \frac{F_{4j}^j(\eta_{\nu_e})}{F_{2j}^j(\eta_{\nu_e})},
\]

(B17)

for scattering on nucleons and,

\[
\tilde{\kappa}_{\bar{\nu}_e,\alpha}^j(A) = \frac{1}{6} A^2 \left[ C_A - 1 + \frac{Z}{A} (2 - C_A - C_V) \right]^2 \sigma_{\bar{\nu}_e A} \left( \frac{T}{m_e c^2} \right)^2 \frac{F_{4j}^j(\eta_{\bar{\nu}_e})}{F_{2j}^j(\eta_{\bar{\nu}_e})},
\]

(B18)

for scattering on \( \alpha \)-particles and heavy nuclei. Like for the energy-dependent opacities (equation B12), we define the

\(^{15}\) Note that produced \( e^- \) will have a minimum energy \( \epsilon_{e^-}^{\min} = Q \).
...total number \((j = 0)\) and energy \((j = 1)\) averaged opacities as

\[
\begin{align*}
\kappa^j_{\nu_e} &= \bar{\kappa}^j_{\nu_e, \nu} + \bar{\kappa}^j_{\nu_e, \nu_e}(n) + \bar{\kappa}^j_{\nu_e, \nu_e}(p) + \bar{\kappa}^j_{\nu_e, \nu_e}(\alpha) + \\
\kappa^j_{\bar{\nu}_e} &= \bar{\kappa}^j_{\bar{\nu}_e, \nu} + \bar{\kappa}^j_{\bar{\nu}_e, \nu_e}(n) + \bar{\kappa}^j_{\bar{\nu}_e, \nu_e}(p) + \bar{\kappa}^j_{\bar{\nu}_e, \nu_e}(\alpha) + \\
\kappa^j_{\nu_\mu} &= \bar{\kappa}^j_{\nu_\mu, \nu} + \bar{\kappa}^j_{\nu_\mu, \nu_e}(n) + \bar{\kappa}^j_{\nu_\mu, \nu_e}(p) + \bar{\kappa}^j_{\nu_\mu, \nu_e}(\alpha) + \\
\kappa^j_{\bar{\nu}_\mu} &= \bar{\kappa}^j_{\bar{\nu}_\mu, \nu} + \bar{\kappa}^j_{\bar{\nu}_\mu, \nu_e}(n) + \bar{\kappa}^j_{\bar{\nu}_\mu, \nu_e}(p) + \bar{\kappa}^j_{\bar{\nu}_\mu, \nu_e}(\alpha) + \\
\kappa^j_{\nu_\tau} &= \bar{\kappa}^j_{\nu_\tau, \nu} + \bar{\kappa}^j_{\nu_\tau, \nu_e}(n) + \bar{\kappa}^j_{\nu_\tau, \nu_e}(p) + \bar{\kappa}^j_{\nu_\tau, \nu_e}(\alpha) + \\
\kappa^j_{\bar{\nu}_\tau} &= \bar{\kappa}^j_{\bar{\nu}_\tau, \nu} + \bar{\kappa}^j_{\bar{\nu}_\tau, \nu_e}(n) + \bar{\kappa}^j_{\bar{\nu}_\tau, \nu_e}(p) + \bar{\kappa}^j_{\bar{\nu}_\tau, \nu_e}(\alpha).
\end{align*}
\]

(B19)

Finally, in the calculation of the neutrino absorption rates, spectrally averaged absorption opacities are calculated as in equations (B13) and (B14), but employing the neutrino spectrum from the corresponding ray, as defined in section 2.3.

### B3 Production rates

The \(\beta\)-processes are the main (far dominant) production sources of \(\nu_e\) and \(\bar{\nu}_e\). From the emissivities obtained by Bruenn (1985), we define the corresponding spectrally averaged production rates (including nucleon rest-mass corrections but without electron rest-mass terms) as

\[
Q^j_{\nu_e, \beta} = \frac{1 + 3g^2}{8} \frac{\sigma_0 c}{(m_e c^2)^2} \xi_{\nu_e} (1 - f(\hat{\epsilon}_{\nu_e}^j; T, \eta_{\nu_e}))
\]

\[
\times \frac{8\pi}{(hc)^3} \left[ \omega^{\nu_e}\nu_e F_{\nu_e}(\eta_e - Q/T) + 2Q^{\nu_e}\nu_e F_{\nu_e}(\eta_e - Q/T) + Q^{2\nu_e}\nu_e F_{\nu_e}(\eta_e - Q/T) \right],
\]

\[(B20)\]

for \(\nu_e\).

\[
Q^j_{\bar{\nu}_e, \beta} = \frac{1 + 3g^2}{8} \frac{\sigma_0 c}{(m_e c^2)^2} \xi_{\bar{\nu}_e} (1 - f(\hat{\epsilon}_{\bar{\nu}_e}^j; T, \eta_{\bar{\nu}_e}))
\]

\[
\times \frac{8\pi}{(hc)^3} \left[ \omega^{\bar{\nu}_e}\bar{\nu}_e F_{\bar{\nu}_e}(\eta_{\bar{\nu}_e} + Q/T) + (2 + j)Q^{\bar{\nu}_e}\bar{\nu}_e F_{\bar{\nu}_e}(\eta_{\bar{\nu}_e} + Q/T) + jQ(T)^2\bar{\nu}_e F_{\bar{\nu}_e}(\eta_{\bar{\nu}_e} + Q/T) \right],
\]

\[(B21)\]

for \(\bar{\nu}_e\). All quantities and constants are defined as in section B1.

The mean \(\nu_e\) and \(\bar{\nu}_e\) production energies are approximated in analogy of equations (B15) and (B16), assuming they are equal to the mean energies of the captured electrons and positrons, respectively, with a correction for the nucleon rest mass difference\(^{16}\)

\[
\hat{\epsilon}_{\nu_e}^\beta = \max\left(\frac{T F_{\bar{\nu}_e}(\eta_{\bar{\nu}_e})}{F_{\nu_e}(\eta_{\nu_e})} - Q, 0\right),
\]

\[(B22)\]

\[
\hat{\epsilon}_{\bar{\nu}_e}^\beta = \frac{T F_{\nu_e}(\eta_{\nu_e})}{F_{\bar{\nu}_e}(\eta_{\bar{\nu}_e})},
\]

\[(B23)\]

The mean neutrino and electron rest mass terms were implemented following Rampp & Janka (2002), as is explained in detail in appendix C.

Thermal processes such as electron-positron pair-annihilation are also an important source of neutrino pairs of all three species. Following Cooperstein et al. (1986, 1987), the \(\nu_e\) and \(\bar{\nu}_e\) production rates read

\[
Q^j_{\nu_e, \bar{\nu}_e, \nu_e} = \frac{(C_1 + C_2)_{\nu_e, \bar{\nu}_e}}{18} \frac{\sigma_0 c}{(m_e c^2)^2} \left(1 - f(\hat{\epsilon}_{\nu_e}^j; T, \eta_{\nu_e})\right)^2 \left(1 - f(\hat{\epsilon}_{\bar{\nu}_e}^j; T, \eta_{\nu_e})\right) + \left(\frac{8\pi}{(hc)^3}\right)^2 \left[ T^{4\nu_e} F_{\nu_e}(\eta_{\nu_e} - Q/T) T^4 F_{\nu_e}(\eta_{\nu_e}) \right] + \]

\[
T^4 F_{\nu_e}(\eta_{\nu_e} - Q/T) T^{4\bar{\nu}_e} F_{\bar{\nu}_e}(\eta_{\bar{\nu}_e}) \right],
\]

\[(B24)\]

with the constants \((C_1 + C_2)_{\nu_e, \bar{\nu}_e} = (C_V - C_A)^2 + (C_V + C_A)^2\), with \(C_A\) and \(C_V\) as defined in section B1. Again, the mean neutrino energy in the neutrino phase space blocking is approximated as Ruffert et al. (1996).

\[
\hat{\epsilon}_{\nu_e} = \frac{T}{2} \left(\frac{F_{\nu_e}(\eta_e)}{F_{\nu_e}(\eta_{\nu_e})} + 1\right).
\]

\[(B25)\]

It is worth noting that the rates above are for each individual neutrino species. When comparing with the source material (Cooperstein et al. 1986, 1987), one should keep in mind that the energy production rate of \(\nu_e\) is half of the energy of the produced pair, whereas the number production rate of \(\nu_e\) is the same as the pair production rate.

For heavy-lepton neutrinos, the production rate via electron-positron annihilation for all 4 \(\nu_e\) species in total is expressed as

\[
Q^j_{\nu_e, \bar{\nu}_e, \nu_\mu} \approx \frac{\pi^3}{3\alpha^3} C_V^2 \frac{\sigma_0 c}{(m_e c^2)^2} T^8 \frac{8\pi}{(hc)^3} \eta_e^6 e^{-\gamma(1 + \gamma)} \left(1 - f(\hat{\epsilon}_{\nu_e}^j; T, \eta_{\nu_e})\right) \left(1 - f(\hat{\epsilon}_{\bar{\nu}_e}^j; T, \eta_{\bar{\nu}_e})\right) \left[ \frac{1}{2} T \left(2 + \frac{\gamma^2}{1 + \gamma}\right) \right]^j,
\]

\[(B27)\]

for \(\nu_e\) and \(\bar{\nu}_e\), and

\[
Q^j_{\nu_\mu, \bar{\nu}_e} \approx \frac{4\pi^3}{3\alpha^3} (C_V - 1)^2 \frac{\sigma_0 c}{(m_e c^2)^2} T^8 \frac{8\pi}{(hc)^3} \eta_e^6 e^{-\gamma(1 + \gamma)} \left(1 - f(\hat{\epsilon}_{\nu_\mu}^j; T, \eta_{\nu_\mu})\right) \left(1 - f(\hat{\epsilon}_{\bar{\nu}_e}^j; T, \eta_{\bar{\nu}_e})\right) \left[ \frac{1}{2} T \left(2 + \frac{\gamma^2}{1 + \gamma}\right) \right]^j,
\]

\[(B28)\]

for all \(\nu_e\) together. \(\alpha^* = 1/137.036\) is the fine structure constant and \(\gamma = 5.565 \times 10^{-2}\). The mean energy of neutrinos produced via plasmon decay is taken as

\[
\hat{\epsilon}_{\nu_e} = \frac{1}{2} T \left(2 + \frac{\gamma^2}{1 + \gamma}\right).
\]

\[(B29)\]

\(\text{Note that electrons must have a minimum energy } e_{\text{min}} = Q \text{ to be absorbed.}\)

---

16 Note that electrons must have a minimum energy \(e_{\text{min}} = Q\) to be absorbed.
Finally, in addition to the reactions included in Rufert et al. (1996), we incorporate the production of $\nu_e$ via nucleon-nucleon bremsstrahlung, which has been shown to have a significant contribution to the production of heavy-lepton neutrinos. For the total energy production rate (four $\nu_e$ species together), we employ the prescription of Thompson et al. (2000):

$$Q_{\nu_e,\text{brems}}^{1+1} = 2.08 \times 10^2 \xi_{\text{brems}} \left(Y_e^2 + Y_p^2 + \frac{28}{3} Y_n Y_p\right) \rho T^{2.5}.$$  

(B30)

Following Burrows et al. (2006), we set the constant $\xi_{\text{brems}} = 0.5$, which is its approximate value at the typical neutrinosphere conditions in PNSs. The factor of 2 higher numerical value of equation (B30) compared to Burrows et al. (2006) comes from the fact that we include two neutrino pairs ($\nu_e$, $\bar{\nu}_e$ and $\nu_\mu$, $\bar{\nu}_\mu$) in $\nu_e$. In order to estimate a number production rate, we make an assumption of the average neutrino energy (Hannestad & Raffelt 1998),

$$\frac{\xi_{\text{brems}}}{\bar{E}_{\nu_e,\text{brems}}} \sim 3T,$$

namely, that all the kinetic energy of the nucleons is transferred to the created neutrinos. Then, the total number production rate is simply,

$$Q_{\nu_e,\text{brems}}^{1+0} = \frac{Q_{\nu_e,\text{brems}}^{1+1}}{\bar{E}_{\nu_e,\text{brems}}},$$

(B32)

for $\nu_e$, of all kinds in total.

The total neutrino production rates are then written as

$$R_\nu = Q_{\nu_e,\beta}^{0+0} + Q_{\nu_e,\bar{\nu}_e,\bar{e}e}^{0+0} + Q_{\nu_e,\bar{\nu}_e,\bar{\gamma}e}^{0+0},$$

$$R_{\bar{\nu}} = Q_{\nu_e,\beta}^{0+0} + Q_{\nu_e,\bar{\nu}_e,\bar{e}e}^{0+0} + Q_{\nu_e,\bar{\nu}_e,\bar{\gamma}e}^{0+0},$$

$$R_{\nu_\mu} = Q_{\nu_e,\bar{\nu}_e,\bar{e}e}^{0+0} + Q_{\nu_e,\bar{\nu}_e,\bar{\gamma}e}^{0+0} + Q_{\nu_e,\nu_\mu,\gamma}^{0+0},$$

for numbers of $\nu_e$, $\bar{\nu}_e$, and all kinds of $\nu_e$, respectively, and

$$R_{\bar{\nu}_\mu} = Q_{\nu_e,\bar{\nu}_e,\bar{e}e}^{0+0} + Q_{\nu_e,\bar{\nu}_e,\bar{\gamma}e}^{0+0} + Q_{\nu_e,\bar{\nu}_e,\gamma}^{0+1},$$

(B34)

for energy of $\nu_e$, $\bar{\nu}_e$, and all kinds of $\nu_e$, respectively.

**APPENDIX C: PRODUCTION RATES AND OPACITIES USED FOR TESTS**

As we noted in section 3.1, there exists certain ambiguity in the derivation of the neutrino $\beta$-production rates, which could become a source of uncertainty in our neutrino treatment.

Following the derivation by Bruenn (1985), the rate of change of the neutrino production function due to $\beta$-interactions (emission and absorption), $Q_{\nu_e}^{\text{net}}$, is proportional to

$$Q_{\nu_e}^{\text{net}} \propto j_{\nu_e}(e) \left[1 - f(e; T, \eta_{\nu_e}) \right] - f(e; T, \eta_{\nu_e}) j_{\nu_e}(e).$$

(C1)

On one hand, one can derive the neutrino production rates from the neutrino emissivities as

$$j_{\nu_e}(e) = \frac{4\pi c}{(hc)^2} \int_0^\infty \epsilon^{2+j} j_{\nu_e}(e) | 1 - f(e; T, \eta_{\nu_e}) | \, de,$$

(C2)

where $| 1 - f(e; T, \eta_{\nu_e}) |$ accounts for the neutrino final state blocking, and the neutrino emissivities are defined as in Bruenn (1985),

$$j_{\nu_e}(e) = \frac{\sigma_0 (1 + 3g_A^2)}{4m_e^2 c^4} \xi_{\nu_e} f(e + Q: T, \eta_{-}) (\epsilon + Q)^2$$

(C3)

for $\nu_e$ and

$$j_{\bar{\nu}_e}(e) = \frac{\sigma_0 (1 + 3g_A^2)}{4m_e^2 c^4} \xi_{\nu_e} f(e - Q: T, \eta_{+}) (\epsilon - Q)^2$$

(C5)

for $\bar{\nu}_e$. This formula corresponds to the one adopted for the presented ILEAS scheme, as detailed in appendix B.

On the other hand, one can use the Kirchhoff-Planck relation,

$$\kappa_{\nu_e, a}(e) = j_{\nu_e}(e) \left[1 - f(e; T, \eta_{\nu_e}^{\text{eq}}) \right],$$

(C7)

to define a corrected absorption opacity, which includes the effects of stimulated absorption (Rampp & Janka 2002),

$$\kappa^{*}_{\nu_e, a}(e) = 1 - f(e; T, \eta_{\nu_e}^{\text{eq}}) \kappa_{\nu_e, a}(e)$$

(C8)

can be rewritten as

$$Q_{\nu_e}^{\text{net}} \propto \kappa^{*}_{\nu_e, a}(e) \left[ f(e; T, \eta_{\nu_e}^{\text{eq}}) - f(e; T, \eta_{\nu_e}) \right].$$

(C10)

This expression is exactly equivalent to equation (C1), and can be interpreted as a redefinition of emission and absorption, with $\kappa^{*}_{\nu_e, a}(e)$ as the new opacity and $j^{*}_{\nu_e}(e) = \kappa^{*}_{\nu_e, a}(e) f(e; T, \eta_{\nu_e}^{\text{eq}}) [ 1 - f(e; T, \eta_{\nu_e}) ]^{-1}$ as the new emissivity. The production rates can be calculated accordingly with equation (C2) employing $j^{*}_{\nu_e}$ instead of $j_{\nu_e}$ as

$$Q_{\nu_e}^{*} = \frac{4\pi c}{(hc)^2} \int_0^\infty \epsilon^{2+j} j^{*}_{\nu_e}(e) f(e; T, \eta_{\nu_e}^{\text{eq}}) \, de.$$  

(C11)

This formulation is adopted by many truncated moment schemes (Rampp & Janka 2002; Just et al. 2015b) because it simplifies the computation of neutrino interactions to $\kappa^{*}_{\nu_e, a}(e)$, instead of calculating the emissivities and opacities separately. Both codes employed in the comparisons presented in section 3. ALCAR and VERTEX, make use of this formulation. For this reason, we have also implemented this formulation of the $\beta$-production rates in ILEAS and used it for such tests. In order to be fully consistent, this reformulation requires to employ the corrected absorption opacities for the neutrino absorption scheme as well. This translates to redefining the opacities in equations (B13) and (B14) as

$$\kappa^{*}_{\nu_e, a} = \frac{\int_0^\infty \xi_{\nu_e} f(e; T, \eta_{\nu_e}) \sigma_0 E^2_{\nu_e}(e) \, de}{\int_0^\infty E^2_{\nu_e}(e) \, de}$$

$$= 1 + 3g_A^2 \frac{\int_0^\infty \frac{\xi_{\nu_e} f(e; T, \eta_{\nu_e}) \sigma_0 E^2_{\nu_e}(e) \, de}{\int_0^\infty E^2_{\nu_e}(e) \, de}}{4(m_e c^2)^4}$$

$$\cdot \frac{T^2 F_{\nu_e, \gamma}^2(\eta_{\nu_e}) + 2Q^2 F_{\nu_e, \bar{\nu}_e}^2(\eta_{\nu_e}) + Q^2 F_{\nu_e, \nu_\mu}^2(\eta_{\nu_e})}{F_{\nu_e, \nu_\mu}^2(\eta_{\nu_e})}.$$  

(C12)
and

\[
\kappa_{\nu_e,\nu_e}^{\nu_e,\nu_e} = \int_0^\infty \kappa_{\nu_e,\nu_e}^2(\epsilon) E_{\nu_e}^2(\epsilon) \Theta(\epsilon - Q) d\epsilon \\
= \frac{1 + 2g_\alpha^2}{2(\mu_e c^2)^2} \sigma_{\nu e} \left( \frac{1 - f(\nu_e; T, \eta_e)}{1 - f(\bar{\nu}_e; T, \eta_e)} \right) \\
T^2 F_{2+}\left(\eta_{\bar{\nu}_e} - Q/T\right) + (1 + 2j)Q F_{2+}\left(\eta_{\nu_e} - Q/T\right) \\
+ jQ^2 T^{-1} F_{2+}\left(\eta_{\nu_e} - Q/T\right) \\
F_{2+}\left(\eta_{\nu_e}\right)
\]

(C13)

and using them instead of \(\kappa_{\nu_e}^2\) for equations (35) and (37) in the absorption algorithm. The electron, positron, \(\nu_e\) and \(\bar{\nu}_e\) mean energies are calculated as in equations (B15), (B16), (B22) and (B23), respectively.

We remind the reader that for the tests presented in section 3 we employed only the reactions included in table 3 in order to consistently compare our results with those obtained by the ALCAR and VERTEX codes. The total neutrino production rates used for such tests are,

\[
\begin{align*}
R_{\nu_e} &= Q_{\nu_e,\beta}^{\nu_e,\beta}e^{\gamma_{\nu_e,\nu_e}}, \\
R_{\bar{\nu}_e} &= Q_{\nu_e,\beta}^{\nu_e,\beta}e^{\gamma_{\nu_e,\nu_e}}, \\
R_{\bar{\nu}_e} &= Q_{\nu_e,\beta}^{\nu_e,\beta} + Q_{\nu_e,\nu_e}^{\nu_e,\nu_e},
\end{align*}
\]

and

\[
\begin{align*}
Q_{\nu_e} &= Q_{\nu_e,\beta}^{\nu_e,\beta}, \\
Q_{\bar{\nu}_e} &= Q_{\nu_e,\beta}^{\nu_e,\beta}, \\
Q_{\bar{\nu}_e} &= Q_{\nu_e,\nu_e}^{\nu_e,\nu_e} + Q_{\nu_e,\nu_e}^{\nu_e,\nu_e},
\end{align*}
\]

for number and energy, respectively, with the \(\beta\)-production rates described in equation (C11).

For a given nucleon species, \(n\) or \(p\), equation (C2) and equation (C11) employ different nucleon blocking factors, either (B4) or (B5). These two blocking factors obey the relation (Bruenn 1985),

\[
\xi_{np} = e^{\gamma_{\nu_e,\nu_e} - \gamma_{\bar{\nu}_e,\bar{\nu}_e}}\xi_{\bar{\nu}_e}.
\]

(C16)

Employing this equation, and the general mathematical property of Fermi functions, \(f(x) = [1 + \exp(x)]^{-1}\),

\[
e^x = \frac{1 - f(x)}{f(x)}
\]

(C17)

the two definitions of the \(\beta\)-production rates in equations (C2) and (C11) are apparently equivalent except for a factor \([1 - f(\epsilon; T, \eta_{\bar{\nu}_e})]^{-1}\) in the integrand of equation (C11). This factor, which corresponds to the neutrino phase space blocking, amounts to a 1 to 2 per cent correction on the local rates (on average \([1 - f(\epsilon_{\bar{\nu}_e}; T, \eta_{\bar{\nu}_e})] \approx 0.98\) with \(\epsilon_{\bar{\nu}_e}\) defined as in equation B22). This is accounted for in the corresponding handling of absorption by the correct redefinition of the absorption opacities in equations (C12) and (C13).

However, figure C1 shows a larger difference between the rates described by equations (C2) (labelled em.) and (C11) (labelled \(\kappa\)) in the optically thick region \((r \lesssim 20 \text{ km})\) of a PNS snapshot (ALCAR snapshot at 0.5 s post-bounce). This is due to a more subtle, yet important difference between equations (C2) and (C11). Our definitions of the nucleon phase space blocking (equations B4 and B5) assume nucleons behave like a free Fermi gas and, accordingly, employ the free Fermi gas nucleon chemical potentials in their calculation (equation B6). As we already pointed out in section 2.2, it is essential to use the free Fermi gas chemical potentials, instead of the ones provided by the EoS, in order to avoid unphysical behaviour of the blocking factors (becoming negative, bigger than unity or not fulfilling the non-degenerate limits). This inconsistency between the nucleon degeneracies employed in the nucleon blocking factors (free Fermi gas, see equations B4 and B5) and in the calculation of the neutrino blocking factors (see equations 18 and 19), is a source of discrepancy between the two approaches.

Because we employ the free Fermi gas nucleon chemical potentials in the blocking factors, \(\eta_{\nu_e}^{eq}\) in equation C16 is different than the one obtained from the high-density EoS (which considers nucleons as interacting particles) by equation (19). In order to be able to recover (C2) from (C11) (except for a factor \([1 - f(\epsilon; T, \eta_{\bar{\nu}_e})]^{-1}\), or vice versa, the same nucleon chemical potentials would need to be used consistently. Because in our formulation this is not the case, there appear notable differences between both rates in the regimes where the nucleon interactions (and thus deviations between the two values of \(\eta_{\nu_e}^{eq}\)) become important.

However, at the high optical depth at which this discrepancy exists, the neutrino transport is dominated by diffusion. As a consequence, even though the production rates differ considerably in the two approaches, neutrino luminosities are vastly produced by the diffusion behaviour, and the impact of the prescription chosen for the production rates is small. In figure C2 this small effect in the luminosities (less than 5 per cent) can be inspected. Thus we conclude that the choice of the formulation of the \(\beta\)-production rates is not significantly relevant for the present work, as its associated uncertainty lies well within the differences between the results obtained by ILEAS in comparison to more sophisticated transport schemes (section 3). If higher accuracy were desired, mean-field effects of the interacting nucleons should be taken into account in the nucleon-neutrino interactions.
We refer the reader to Reddy et al. (1998) for details on the original formulation, and Roberts (2012); Roberts et al. (2012); Martínéz-Pinedo et al. (2012) for the implementation and application of such corrections in the context of SN and PNS cooling simulations.

APPENDIX D: ALTERNATIVE METHOD TO COMPUTE NEUTRINO-NUMBER RE-ABSORPTION

In some of the tests presented in section 3 we observe a disagreement of up to ∼45 per cent (∼4 MeV difference) for νe and up to ∼20 per cent (∼2 MeV) for νx when comparing the mean energies of radiated neutrinos obtained by ILEAS (as described by equation (45)) with the ones obtained by the M1 scheme ALCAR. Since it is possible that some approximations introduced in the absorption scheme described in section 2.3 could have an impact on the neutrino mean energies, we tested an alternative version of the absorption treatment in this appendix.

In a procedure analogous to the one employed in equation (37) for the energy treatment, we calculated the neutrino number absorption rates, \( R_{\nu_i}^{n} \), independently of the energy absorption rates. Instead of the neutrino energy luminosities (from the modified opacities (solid lines) and from the emissivities (dashed lines), for the ALCAR relaxed snapshot at 0.5 s. The results obtained by ALCAR are also shown for comparison.

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Table D1. Neutrino mean energies obtained by the alternative version of neutrino-number re-absorption in ILEAS (described in appendix D) applied to two of the snapshots of a PNS cooling simulation and one of the BH-torus models discussed in section 3. These data correspond to results after relaxation by 5 ms for PNS cases and 3 ms for BH-torus models. For comparison also the results from transport calculations with ALCAR and VERTEX are listed. The columns labelled Alternative leakage luminosity and Alternative leakage mean energy provide the neutrino luminosities and mean energies obtained from the alternative version of neutrino-number re-absorption. In addition, the columns labelled Standard leakage luminosity and Standard leakage mean energy provide the neutrino luminosities and mean energies obtained by the standard ILEAS version copied from table 4. In both cases the mean energies are calculated by equation (45). All values are taken for a local observer in the rest frame of the source at the edge of the grid (100 km).

| Model       | ν-species | Transport luminosity \(10^{51} \text{ erg}\,\text{s}^{-1}\) | Alternative leakage luminosity \(10^{51} \text{ erg}\,\text{s}^{-1}\) | Standard leakage luminosity \(10^{51} \text{ erg}\,\text{s}^{-1}\) | Transport mean energy (MeV) | Alternative leakage mean energy (MeV) | Standard leakage mean energy (MeV) | Transport code |
|-------------|-----------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|-------------------------------|-------------------------------|-------------------------------|----------------|
| PNS 0.5 s   | \(\nu_e\) | 7.0                                             | 6.3                                             | 6.7                                             | 9.93                          | 8.19                          | 7.95                          | ALCAR          |
| PNS 0.5 s   | \(\bar{\nu}_e\) | 7.6                                             | 8.6                                             | 8.1                                             | 13.32                         | 16.00                         | 12.62                         | ALCAR          |
| PNS 1.2 s   | \(\nu_e\) | 3.7                                             | 3.5                                             | 3.5                                             | 9.24                          | 6.39                          | 6.16                          | VERTEX         |
| PNS 1.2 s   | \(\bar{\nu}_e\) | 3.8                                             | 3.7                                             | 3.7                                             | 11.43                         | 14.05                         | 12.98                         | VERTEX         |
| BH-torus 0.1 M\(_\odot\) | \(\nu_e\) | 6.5                                             | 6.1                                             | 6.5                                             | 12.02                         | 10.83                         | 12.69                         | ALCAR          |
| BH-torus 0.1 M\(_\odot\) | \(\bar{\nu}_e\) | 5.2                                             | 5.2                                             | 4.8                                             | 14.20                         | 14.29                         | 14.50                         | ALCAR          |

Figure D1. Neutrino energy luminosity (top row) and lepton-number luminosity (bottom row) profiles of \(\nu_e\) (left column) and \(\bar{\nu}_e\) (right column) obtained by ILEAS after 5 ms relaxation of the ALCAR relaxed snapshot at 0.5 s. Solid red lines display the standard formulation of neutrino re-absorption described in section 2.3 and solid blue lines show the alternative version described in appendix D. The results obtained by ALCAR and VERTEX (only neutrino energy luminosities) are also shown for comparison. While the VERTEX calculations take gravitational redshift effects into account, ALCAR and ILEAS do not. The vertical black lines in each panel illustrate the position of the neutrinosphere \((\tau_{\nu} = 2/3)\) for the corresponding neutrino species.