Muon capture in hydrogen

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Theoretical difficulties in reconciling the measured rates for ordinary and radiative muon capture are discussed, based on heavy-baryon chiral perturbation theory. We also examine ambiguity in our analysis due to the formation of \( p\mu p \) molecules in the liquid hydrogen target.

1 Introduction

Ordinary and radiative muon captures (OMC and RMC) on a proton, \( \mu^- p \to n\nu_\mu \) and \( \mu^- p \to n\nu_\mu \gamma \), are fundamental weak-interaction processes in nuclear physics and constitute primary sources of information on \( g_P \), the induced pseudoscalar coupling constant of the weak nucleon current \[1\].

The most accurate existing measurements of the OMC and RMC rates have been carried out using a liquid hydrogen target. The experimental OMC rate in liquid hydrogen obtained by Bardin et al. \[2\] is

\[ \Lambda^{\text{exp}}_{\text{liq}} = 460 \pm 20 \text{ [s}^{-1}] \text{ (OMC).} \] (1)

As for RMC, Jonkmans et al. \[3\] measured the absolute photon spectrum for \( E_\gamma \geq 60 \text{ MeV} \) and deduced therefrom the partial RMC branching ratio, \( R_\gamma \), which is the number of RMC events (per stopped muon) producing a photon with \( E_\gamma \geq 60 \text{ MeV} \). The measured value of \( R_\gamma \) is \[3, 4\]

\[ R^{\text{exp}}_\gamma = (2.10 \pm 0.22) \times 10^{-8} \text{ (RMC).} \] (2)

Surprisingly, the value of \( g_P \) deduced in \[3, 4\] from the RMC data is \( \sim 1.5 \) times larger than the PCAC prediction, \( g_P^{\text{PCAC}} \[5\]. By contrast, the value of \( g_P \) deduced in \[2\] from the OMC data is in good agreement with the PCAC prediction. Heavy-baryon chiral perturbation theory (HBChPT), a low-energy effective theory of QCD, allows us to go beyond the PCAC approach, but the results of detailed HBChPT calculations up to next-to-next-to-leading order \[6\] essentially agree with those obtained in the PCAC approach. Thus the theoretical framework for estimating \( g_P \) appears to be robust. What then can be the origin of the apparent conflict between the \( g_P \) values determined from OMC and RMC? In this talk we wish to address a number of issues relevant to this question.

The OMC process is described by the standard electroweak theory. The lepton current, \( j^\text{lepton}_\alpha \), interact with the very heavy \( W \) boson (\( m_W \simeq 80 \text{ GeV} \)) which propagates and interacts with the hadronic current, \( J^\text{hadron}_\beta \). Since the maximal momentum transfer between the two currents, \( q_{\text{max}} \simeq m_\mu \ll m_W \), we can for all practical purposes replace the \( W \) boson propagator with a constant. Then the interaction between the lepton and hadron currents reduces to:

\[ \sim \frac{G_F}{\sqrt{2}} j^\text{lepton}_\alpha j^\text{hadron}_\beta. \] (3)

where \( G_F \) is the effective Fermi constant. Since the leptons are considered point particles, the lepton current is given by the incoming muon and outgoing neutrino spinors: \( j^\text{lepton}_\alpha = \bar{u}_\nu \gamma_\alpha (1 - \gamma_5) u_\mu \). The hadronic current however is less well known due to the structure of the nucleons. We write \( J^\text{hadron}_\beta = V_\beta - A_\beta \), where the vector current, \( V_\beta \), and the axial current, \( A_\beta \), can be written as (based on symmetry considerations and neglecting second-class currents):

\[ V_\beta = \bar{u}_n \left[ g_V (q^2) \gamma_\beta + i \frac{g_M (q^2)}{2m_N} \sigma_{35} q^5 \right] u_p, \] (4)
\[
A_\beta = \bar{u}_n \left[ g_A(q^2)\gamma_\beta \gamma_5 + \frac{g_P(q^2)}{m_\mu} q_\beta \gamma_5 \right] u_p, \tag{5}
\]

where \( u_n \) and \( u_p \) are the outgoing neutron and incoming proton spinors; and \( q \) is the momentum transferred to the nucleon. The four form factors, \( g_V(q^2) \), \( g_M(q^2) \), \( g_A(q^2) \) and \( g_P(q^2) \), in Eqs. (4) and (5) are, at low momentum transfers relevant to the OMC and RMC reactions, given by known parameters:

\[
g_V(q^2) = 1 + \frac{1}{6} < r^2 > q^2 + \cdots; \quad g_M(q^2) = \kappa_p - \kappa_n + \cdots; \tag{6}
\]

\[
g_A(q^2) = g_A \left(1 + \frac{1}{6} < r_A^2 > q^2 + \cdots\right); \tag{7}
\]

\[
g_P(q^2) = \frac{2m_\mu f_\pi g_{\pi NN}}{m_\pi^2 - q^2} - \frac{1}{3} g_A m_\mu m_N < r_A^2 > + \cdots. \tag{8}
\]

Here \( < r^2 > \simeq 0.585 \text{ fm}^2 \) is the square of the isovector nucleon radius, \( < r_A^2 > \simeq 0.42 \text{ fm}^2 \) is the axial radius squared, and \( \kappa_p \) and \( \kappa_n \) are the proton and neutron anomalous magnetic moments, respectively. In the induced pseudoscalar form factor, \( g_P(q^2) \), the form of the dominant “pion-pole” term is dictated by the PCAC hypothesis.

In OMC the momentum transfer is \( q^2 = (m_\mu - E_\nu)^2 - E_\nu^2 = -0.88m_\mu^2 \) and, traditionally, the pseudoscalar coupling “constant”, \( g_P \), is defined as \( g_P \equiv g_P(q^2 = -0.88m_\mu^2) \), and thus the PCAC value is \( g_P^{PCAC} = 6.77 g_A(0) \). We remark that the contribution of the \( g_P(q^2) \) term to OMC is suppressed due to the fact that \( q^2 = -0.88m_\mu^2 \) is far from the pion pole. The rationale for using RMC to determine \( g_P \) is that the three-body kinematics in the final state allows \( q^2 \) to approach the pion pole, enhancing thereby the contribution of the pseudoscalar term. With the photon energy denoted by \( E_\gamma \), we have \( q^2 \simeq (2m_\mu E_\gamma - m_N^2) \), which is positive for sufficiently large values of \( E_\gamma \), and which can reach the maximal value \( \sim m_\mu^2 \). In [3, 4], the RMC process is measured for \( E_\gamma > 60 \text{ MeV} \). Meanwhile, a great challenge in observing RMC is its very low branching ratio (\( \sim 10^{-3} \) compared to OMC), and it was quite a feat that the TRIUMF group succeeded in measuring \( R_\gamma \). It should also be mentioned that the absolute photon spectrum (for \( E_\gamma \geq 60 \text{ MeV} \)) measured in [3, 4] gives information on the \( q^2 \) dependence of \( g_P(q^2) \). The afore-mentioned surprising conclusion that, to fit the TRIUMF RMC data, one needs to adopt \( g_P \sim 1.5 g_P^{PCAC} \) has triggered re-examination of the foundation of the theoretical framework used for the analysis. In particular, it motivated detailed systematic calculations based on HBChPT for RMC as well as OMC [3, 4, 10, 11, 12].

2 Chiral perturbation theory and the atomic capture rates

Chiral perturbation theory (ChPT) is an effective field theory of QCD tailored for describing low-energy hadronic interactions. Since the physical degrees of freedom here are hadrons, not quarks or gluons, we “integrate out” the quark and gluon fields to obtain an effective lagrangian, \( \mathcal{L}_{\text{eff}} \), pertinent to the hadronic degrees of freedom. \( \mathcal{L}_{\text{eff}} \) should inherit all the symmetry properties of QCD (and the patterns of their breaking, if any), including chiral symmetry. For our purposes, it is sufficient to consider only two flavors (\( u \) and \( d \)); correspondingly, \( \mathcal{L}_{\text{eff}} \) contains only the nucleon and pion fields. If chiral symmetry is an exact symmetry of \( \mathcal{L}_{\text{eff}} \), then the right- and left-handed nucleon fields decouple, and the left- and right-handed Noether currents are separately conserved: \( \partial_\alpha j_{\text{eft}}^{\alpha} = 0 \) and \( \partial_\alpha j_{\text{right}}^{\alpha} = 0 \). The QCD vacuum state, however, does not respect chiral symmetry, i.e. chiral symmetry is spontaneously broken. Then, according to the Goldstone theorem, there appear massless pseudoscalar Goldstone fields, which can be identified with (massless) pions. In reality, \( \mathcal{L}_{\text{QCD}} \) contains a tiny quark mass term (\( m_{\text{quark}} \approx 10 \text{ MeV} \)) that violates chiral symmetry and that gives rise to a finite pion mass. A consequence of the non-zero mass is that the axial current
is no longer conserved. Since $m_{\text{quark}} \ll \Lambda_{\text{chiral}} \simeq 1$ GeV, this explicit chiral symmetry breaking can be treated as a small perturbation, and ChPT consists in expanding $\mathcal{L}_{\text{eff}}$ and transition amplitudes in terms of $Q/\Lambda_{\text{chiral}}$ and $m_{\pi}/\Lambda_{\text{chiral}}$, where $Q$ is a typical scale of external momenta: $Q \sim |\vec{p}_{\pi}| \sim |\vec{p}_{\text{nucleon}}|$. A version of ChPT that is particularly useful for our present purposes is HBChPT, wherein the nucleon is treated as a heavy particle for which a Foldy-Wouthuysen-like non-relativistic expansion can be used. In HBChPT we have an expansion

$$\mathcal{L}_{\text{eff}} = \sum \mathcal{L}_\beta = \mathcal{L}_1 + \mathcal{L}_2 + \cdots. \quad (9)$$

Here the “chiral index” $\beta$ is given by $\beta = d+n/2-1$, where $n$ is the number of nucleon fields involved in a given vertex and $d$ the number of derivatives or powers of $m_\pi$ involved. For example, the lowest chiral order lagrangian is given as

$$\mathcal{L}_1 = \bar{N} \left[ i \frac{\partial}{\partial t} + \frac{g_A}{2f_\pi} \tau \cdot (\vec{\sigma} \cdot \vec{\nabla} \pi) \right] N + \frac{1}{2} \left( \partial_\beta \pi \cdot \partial^\beta \pi \right) - \frac{1}{2} m_\pi^2 \pi^2 + \cdots, \quad (10)$$

while the next order Lagrangian is given by

$$\mathcal{L}_2 = \bar{N} \left[ \frac{\vec{\nabla}^2}{2m_N} + \cdots \right] N + \cdots \quad (11)$$

It is to be noted that in HBChPT the Schroedinger operator $\vec{\nabla}^2/(2m_N)$ for the nucleon kinetic energy is treated as a “recoil” correction to $\mathcal{L}_1$. In muon capture (both OMC and RMC), $Q \sim m_\mu = 105.7$ MeV or $Q/\Lambda_{\chi} \sim 0.1$ and hence the chiral expansion is expected to converge rapidly. The explicit calculations in HBChPT 9, 12 corroborates this expectation.

For OMC, leading-order (LO) contributions in chiral counting come from two tree diagrams of order $(Q/\Lambda_{\text{chiral}})^0$. Next-to-leading order (NLO) contributions are again given by two tree diagrams which however are of order $(Q/\Lambda_{\text{chiral}})^1$. These NLO diagrams arising from $\mathcal{L}_2$ define the nucleon-weak current (or pion) vertices which include the nucleon recoil correction of order $\sim m_N^{-1}$. To next-to-next-to-leading order (NNLO), $(Q/\Lambda_{\text{chiral}})^2$, we have both tree and one-pion-loop diagrams. The tree diagrams here involve vertices coming from $\mathcal{L}_3$, which contains three low energy constants (LEC). Two of them can be determined from $< r^2 >$ and $< r^2_A >$, while the third LEC can be constrained by the Goldberger-Treiman discrepancy. The one-pion-loop diagrams effectively introduce a form factor at the nucleon-pion vertex. It should be stressed that there are no free parameters in this ChPT calculation; all the LEC’s are given by $g_A$, $\kappa_p - \kappa_n$, $f_\pi \simeq 93$ MeV, $g_{\pi N}$, $< r^2 >$, $< r^2_A >$, and the Goldberger-Treiman discrepancy $\Delta_\pi$ defined by $m_N g_A = f_\pi g_{\pi N} (1 - \Delta_\pi)$. Bernard et al. 11 have shown the very rapid convergence of these ChPT calculations for the spin-averaged OMC rate: $\Lambda = (247 - 62 - 4 + \cdots) s^{-1}$, where the the first, second and third terms correspond to the LO, NLO and NNLO contributions, respectively. We can see that the NLO term is $\sim 25\%$ of the LO term and that the NNLO term is $\sim 1.6\%$ of the LO term. Meanwhile, the OMC rate from the $\mu-p$ singlet atomic state exhibits a more subtle convergence behavior, indicating that a systematic calculation based on well-defined chiral expansion is indeed needed to get accurate predictions:

$$\Lambda_s (s^{-1}) = 957 - \frac{245 \text{GeV}}{m_N} + \left[ \frac{30.4 \text{GeV}^2}{m_N} - 43.17 \right] + \cdots \simeq 687 s^{-1}. \quad (12)$$

The second term in this expression is the nucleon recoil correction. Of the two terms in the square brackets representing the NNLO corrections, the first represents the $m_N^{-2}$ correction term while the second term originates from the $q^2$-dependence in the hadronic form factors given by the ChPT terms of one-loop order. It is noteworthy that these two terms cancel each other to a significant
Table 1: Comparison of calculated atomic OMC rates. $\Lambda_s$ ($\Lambda_t$) is the atomic capture rate (in sec$^{-1}$) calculated for the initial singlet (triplet) hyperfine state including terms up to NLO or NNLO, as indicated. The entries for the columns labeled “BHM” and “AMK” are taken from [11] and [10], respectively. The “AMK” results have been obtained with the use of $g_A = 1.267$ and $g_A^{\pi N} = 13.4$.

Apart from small chiral corrections, the numerical results of the classic works by Primakoff and Opat would be close to those of AMK(NNLO), if the updated value of $g_A = 1.267$ is adopted; Primakoff and Opat used $g_A = 1.24$ and $g_A = 1.22$, respectively.

degree, a feature that cannot be studied reliably without a systematic expansion scheme such as ChPT. We compare in Table 1 the OMC rates obtained in two independent ChPT calculations, BHM [11] and AMK [10]. The variance in the results of BHM and AMK are ascribable to the slight differences in the approaches used.

The RMC atomic capture rates have been calculated with the same $L_{\text{eff}}$. The use of $L_{\text{eff}}$ ensures that the photon coupling in our RMC calculation automatically satisfies gauge invariance. The leading order (LO) diagram representing the emission of a photon by a muon gives a dominant contribution. In LO we also have Feynman diagrams of the following types: (i) a photon couples to an intermediate pion propagator; (ii) a photon couples to a pion-nucleon vertex; (iii) a photon couples to a $W^-$-pion vertex. In the Coulomb gauge, there are five LO Feynman diagrams, ten NLO diagrams, and more than twenty NNLO diagrams including pion-loop diagrams [9].

It has been pointed out [14] that, with the use of the atomic RMC rates calculated model-independently with the use of HBChPT, it is extremely difficult to reproduce $R_{\gamma}^{\text{exp}}$ obtained in the TRIUMF experiment. We describe below some salient features of this difficulty.

3 Muonic states in liquid hydrogen

To make a comparison between theory and experiment, one needs to relate the theoretically calculated atomic OMC and RMC rates to the capture rates measured in a liquid $H_2$ target, $\Lambda_{\text{liq}}$ and $R_\gamma$, respectively. It is also important to know the temporal behavior of each of the various $\mu$-capture components (capture from the atomic states and capture from $p\mu p$ molecular states). Fig. 1 schematically depicts various competing atomic and molecular processes in liquid $H_2$. A muon stopped in liquid hydrogen quickly forms a muonic atom ($\mu p$) in the lowest Bohr state. The atomic hyperfine-triplet state ($S=1$) decays extremely rapidly to the singlet state ($S=0$), with a transition rate $\lambda_{10} \simeq 1.7 \times 10^{10}$ s$^{-1}$. In the liquid hydrogen target a muonic atom and a hydrogen molecule collide with each other to form a $p\mu p$ molecule with the molecule predominantly in its ortho state. The transition rate from the atomic singlet state to the ortho $p\mu p$ molecular state, $\lambda_{\text{ppp}}$, has an averaged value $\lambda_{\text{ppp}} \sim 2.5 \times 10^{6}$ s$^{-1}$, which is comparable to the muon decay rate, $\lambda_0 = 0.455 \times 10^{6}$ s$^{-1}$. The ortho $p\mu p$ state further decays to the para $p\mu p$ molecular state with a rate $\lambda_{\text{op}} \sim (4 - 7) \times 10^4$ s$^{-1}$.

We now discuss very briefly the time structure relevant to the OMC experiment [2]. We denote by $N_s(t)$, $N_{om}(t)$ and $N_{pm}(t)$ the numbers of muons at time $t$ in the atomic singlet, ortho-molecular, and para-molecular states, respectively. They satisfy coupled differential equations (the kinetic equations), see Eq. (54a) in Ref. [13]. For illustration purposes, let us consider a case in which there is one muon in the atomic singlet state at $t = 0$; i.e., $N_s(0) = 1$ and $N_{om}(0) = N_{pm}(0) = 0$. $N_s(t)$,
Figure 1: Atomic and molecular states relevant to muon capture in liquid hydrogen; \( \lambda_{pp\mu} \) is the transition rate from the atomic singlet state to the ortho \( p-\mu-p \) molecular state, and \( \lambda_{op} \) is that from the ortho to para molecular state.

Figure 2: Number of muons at time \( t \) in each state in liquid hydrogen.

\[ N_{\text{om}}(t) \] and \( N_{\text{pm}}(t) \) corresponding to this case are plotted in Fig. 2. It turns out to be crucially important to take proper account of the \( t \)-dependence of these populations in analyzing the OMC data in [2], since data taking in Ref. [2] starts at \( t \neq 0 \).

In the OMC experiment (see Fig. 4 in Ref. [3]), \( \mu^- \) beams arrive at the target in a (on the average) 3 \( \mu \)s-long burst with repetition rate 3000 Hz. The data collection typically starts 1 \( \mu \)s after the end of the 3 \( \mu \)s-long beam burst, and the measurement lasts until 306 \( \mu \)s after the end of the beam burst. To proceed, we assume that the quoted average time intervals represent the actual values (ignoring fluctuations). Then, provided all the muons arrive at the same time, we can choose with no ambiguity that arrival time as the origin of time \( (t = 0) \) and let \( t = t_i \), the starting time for data collection, refer to that origin. However, the finite duration (3 \( \mu \)s) of the beam burst causes uncertainty in the value of \( t = t_i; \) \( t_i \) can be anywhere between 1.0 \( \mu \)s and 4.0\( \mu \)s. We choose here to average over the muon burst duration time for deducing \( \Lambda_{\text{liq}} \), see Ref. [14] for details.

For the RMC experiment [3, 4], the muons essentially arrive one by one and data taking begins at \( t_i = 365 \) ns. We therefore can neglect the beam burst duration time in calculating \( R_{\gamma} \), see Ref. [14].

4 Discussion

The value of \( \Lambda_{\text{liq}} \) obtained with the use of the atomic OMC rates calculated in HBChPT up to NNLO is \( \Lambda_{\text{liq}} \approx 459 \text{ s}^{-1} \) [2]. This value is in good agreement with \( \Lambda_{\text{exp}} \). By contrast, it is not possible to reproduce \( R_{\gamma} \) in Eq. (1) in the existing theoretical framework and with the use of the standard set of input parameters, see Ref. [14] for a more detailed discussion.

The important question is: Have we exhausted all possible ways for reconciling the measured OMC and RMC rates? One thing worth studying as a speculative possibility [14] is the sensitivity of \( \Lambda_{\text{liq}} \) and \( R_{\gamma} \) to a so-far neglected possible change in the value of the molecular mixing parameter \( \xi \). As first discussed by Weinberg [15], \( \xi \) parametrizes a possible mixing of the spin-3/2 and spin-1/2 states in the \( p-\mu-p \) molecule, and this mixing changes the molecular capture rate to

\[
\Lambda'_{\text{om}} = \xi \Lambda_{\text{om}}(1/2) + (1 - \xi) \Lambda_{\text{om}}(3/2),
\]

where \( \Lambda_{\text{om}}(1/2) = \Lambda_{\text{om}} \) and \( \Lambda_{\text{om}}(3/2) = 2\gamma_0 \Lambda_{\gamma} \) for both OMC and RMC; \( \Lambda_{\text{om}} = 2\gamma_0(0.75\Lambda_{\gamma} + 0.25\Lambda_{\xi}) \) and \( 2\gamma_0 = 1.009 \) [13]. Although the existing theoretical estimate suggests \( \xi \approx 1 \) [13],
we treat $\xi$, as we did in Ref. [10], as a parameter to fit the data. Our study [14] demonstrates that, even with this extra adjustable parameter, it is impossible to fit the OMC and the RMC data simultaneously. In Ref. [14] we have also found that both the OMC and RMC capture rates are sensitive to the value of $\lambda_{op}$. Obviously, the results of a more precise measurement of $\lambda_{op}$ at TRIUMF [17] will be very important for both OMC and RMC. Note that since HBChPT constrains the value of $g_P$ with high accuracy, there is not much room for adjusting the value of $g_P$.

Our findings reported in [14] and briefly summarized in this talk are mostly the reconfirmation of the conclusions stated in one way or another in the literature, but we hope that the coherent treatment of OMC and RMC in liquid hydrogen as described in Ref. [14] would be useful. Although we have presented examples of simulation of the experimental conditions, they are only meant to serve illustrative purposes. Definitive analyses can be done only by the people who carried out the relevant experiments. Finally, we remark that a precise measurement of the OMC rate in hydrogen gas is planned at PSI [16]. This experiment will allow us to avoid the molecular complexity discussed above and directly test the HBChPT prediction [10, 11].

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