The generic form of spacetime dynamics as a classical gauge field theory has recently been derived, based on only the action principle and on the Principle of General Relativity. It was thus shown that Einstein’s General Relativity is the special case where (i) the Hilbert Lagrangian (essentially the Ricci scalar) is supposed to describe the dynamics of the “free” (uncoupled) gravitational field, and (ii) the energy-momentum tensor is that of scalar fields representing real or complex structureless (spin-0) particles. It followed that all other source fields—such as vector fields representing massive and non-massive spin-1 particles—need careful scrutiny of the appropriate source tensor. This is the subject of our actual paper: we discuss in detail the coupling of the gravitational field with (i) a massive complex scalar field, (ii) a massive real vector field, and (iii) a massless vector field. We show that different couplings emerge for massive and non-massive vector fields. The massive vector field has the canonical energy-momentum tensor as the appropriate source term—which embraces also the energy density furnished by the internal spin. In this case, the vector fields are shown to generate a torsion of spacetime. In contrast, the system of a massless and charged vector field is associated with the metric (Hilbert) energy-momentum tensor due to its additional U(1) symmetry. Moreover, such vector fields do not generate a torsion of spacetime. The respective sources of gravitation apply for all models of the dynamics of the “free” (uncoupled) gravitational field—which do not follow from the gauge formalism but must be specified based on separate physical reasoning.

Keywords: Modified theories of gravity, gauge field theory, covariant Hamiltonian formulation

PACS numbers: 04.50.Kd, 11.15.-q, 47.10.Df

DOI: 10.1142/S0218301319500071
1. Introduction

A recent publication [1] (see also [1]) presented the formalism of extended canonical transformations in the realm of covariant Hamiltonian field theory [4–8]. This enables the description of canonical transformations of classical fields under general mappings of the spacetime geometry along the well-established procedures of gauge transformations, dating back to H. Weyl [9] and W. Fock [10]. It generalizes the canonical gauge formalism under a static spacetime background, which was worked out earlier in Refs. [8, 11, 12]. Here, the gauge formalism means the generalization of a Lorentz-covariance to a diffeomorphism covariance by introducing appropriate gauge fields. This corresponds to the transition from special relativity to general relativity. The gauge fields turned out to be given by the connection coefficients of the spacetime manifold, which contain the complete information on the spacetime curvature and torsion. With the metric and the connection emerging as independent field variables, the canonical gauge theory naturally implements the Palatini formulation [13] of General Relativity. With the canonical conjugate fields of the metric and the connection, no additional dynamical quantities need to be introduced ad hoc as is done in the Lagrangian formulation of Utiyama [14], Kibble [15], and Sciama [16].

The canonical gauge formalism has been applied [1] to a generic dynamical system of scalar and vector fields in curvilinear spacetime including non-metricity and torsion. The initially Lorentz-invariant integrand of the action functional has been gauged into a proper (world) scalar density, hence into an invariant under a dynamical spacetime geometry. It is shown in the following that two cases of invariant action functionals must be distinguished:

- If the respective field only couples to the metric, then the metric (Hilbert) energy-momentum tensor is the appropriate source term for the spacetime dynamics.
- If the field couples to both the metric and the connection, then the metric energy-momentum tensor is no longer appropriate. For the case of the Proca system, we show that the canonical energy-momentum tensor then provides the correct source of the spacetime dynamics.

As common to all gauge theories, the Hamiltonian describing the dynamics of the “free” gauge field, i.e., its dynamics in the absence of any coupling, must be inserted “by hand”. In the Hamiltonian representation of U(1) and Yang-Mills gauge theories [8], the dynamics of the “free” gauge field is described by a gauge-invariant term which is quadratic in the canonical momenta of the gauge fields. For the actual case of the free gravitational field, a Hamiltonian has been proposed [1], which is—at most—quadratic in the canonical momentum tensor of the gauge field—in analogy to the field theories mentioned above. This quadratic momentum term is actually required in order for the correlation of the momenta to the spacetime derivatives of the gauge field to be uniquely determined. It can be regarded as a correction term to the Einstein tensor.

The actual paper develops the physical implications of the covariant canonical gauge

---

*Previous attempts to set up a canonical transformation formalism for a dynamical spacetime in the realm of classical field theory were published in Refs. [2], [3], and [4]. Those works are superseded by Ref. [1].*
theory of gravitation, presented in our previous paper [1]. In Sect. 2 a brief review of its results is outlined. We proceed with the restriction of the obtained field equations to metric compatibility, hence to a covariantly constant metric. By means of two examples, namely the Klein-Gordon and the Proca system, we demonstrate the consequences of the different couplings of scalar and vector fields with spacetime. The gauge formalism will be shown to uniquely determine the appropriate energy-momentum tensor which acts as the source term for the field equation of gravitation. This is a critical issue of the theory: as the energy-momentum tensor enters directly into the field equations, it is not admissible to add a zero-divergence term to the source term—as is done, for instance, by applying the Belinfante-Rosenfeld symmetrization method [17, 18] of the energy-momentum tensor. We will show that for general vector particle fields, the canonical energy-momentum tensor provides the appropriate source term for the dynamics of the spacetime geometry—even if we neglect torsion. In contrast to Einstein’s general relativity, the spin-1 field acts also as a source for torsion of spacetime. This coupling of spin and torsion is described by a Poisson-type equation, which is why the torsion can propagate along with the gravitational wave. Hence, for vector particle fields, we encounter from gauge theory a different coupling of the source field and spacetime as from Einstein’s theory. It is thus also critical for certain astrophysical considerations. Hence, compact astrophysical objects, like neutron stars and binary neutron star mergers must be reinvestigated with the appropriate canonical energy-momentum terms for the vector repulsion effective field theory (EFT). A similar conclusion does also hold for fermions, both for protons and electrons, as well as for neutrinos, both in white dwarfs, neutron stars and in “ultra high energy cosmic ray” (UHECR) events. This will be published by the authors in a forthcoming paper.

We then discuss in Sect. 3 a system of a complex scalar field which couples minimally to a massless vector field. Such a system is commonly referred to as a U(1) gauge theory. We work out the canonical gauge theory of gravity for such a system with U(1) symmetry. It is shown that the resulting system with $U(1) \times \text{Diff}(M)$ symmetry has the metric (Hilbert) energy-momentum tensor as the source of gravity—just as in the case of scalar fields. Moreover, it turns out that the vector (Maxwell) field does not act as a source for a torsion of spacetime in this case. This also applies for the gauge vector bosons of SU($N$) (Yang-Mills) gauge theories [19].

Finally, the implications of the modified theory on the description of neutron star and black hole mergers are discussed in the Conclusions (Sect. 4).

### 2. The field equations of canonical gauge theory of gravity

#### 2.1. Action functionals invariant under the Diff(M) gauge group

The dynamics of systems of complex scalar fields $\phi$ and real vector fields $a_\nu$ in a Minkowski spacetime background—with their respective conjugates, $\bar{\pi}^\mu$ and $\bar{\rho}^{\mu\nu}$—are described in the covariant Hamiltonian formalism by a Hamiltonian scalar density $\mathcal{H}_0(\phi, \phi^*, \bar{\pi}^\mu, \bar{\rho}^{\mu\nu}, a_\nu, g_{\mu\nu})$. The particular canonical field equations follow from the vari-
ation of the action

\[ S = \int_{\mathcal{M}} \left( \frac{\partial \phi^*}{\partial \chi^\beta} \pi^\beta + \tilde{\pi}^\beta \frac{\partial \phi}{\partial \chi^\beta} + \tilde{p}^{\alpha\beta} \frac{\partial a^\alpha}{\partial \chi^\beta} + \tilde{\kappa}^{\alpha\lambda\beta} \frac{\partial \eta^\lambda}{\partial \chi^\beta} + \tilde{q}^{\alpha\xi\beta} \frac{\partial R^\eta_{\alpha\xi}}{\partial \chi^\beta} - \tilde{H}_0 \right) d^4x. \]  

(1)

Herein, \( \tilde{\pi}^\beta = \pi^\beta \sqrt{-g} \) and \( \tilde{p}^{\alpha\beta} = p^{\alpha\beta} \sqrt{-g} \) denote tensor densities formed from the (absolute) canonical momentum tensors, \( \pi^\beta \) and \( p^{\alpha\beta} \), and the determinant, \( g = \det(g_{\mu\nu}) \), of the system’s covariant metric. Accordingly, \( \tilde{H}_0 = H_0 \sqrt{-g} \) denotes the scalar Hamiltonian density constructed from the ordinary scalar Hamiltonian \( H_0 \). This ensures that the action functional is invariant under arbitrary coordinate transformations.

A closed description of the coupled dynamics of fields and spacetime geometry has been derived in Ref. [1], where the gauge formalism yields, on the basis of Eq. (1), the amended covariant action functional

\[ S = \int_{\mathcal{M}} \left( \frac{\partial \phi^*}{\partial \chi^\beta} \pi^\beta + \tilde{\pi}^\beta \frac{\partial \phi}{\partial \chi^\beta} + \tilde{p}^{\alpha\beta} a^\alpha_{\beta} + \tilde{\kappa}^{\alpha\lambda\beta} g_{\alpha\lambda \beta} - \frac{1}{2} \tilde{q}^{\alpha\xi\beta} R^\eta_{\alpha\xi} \tilde{R} - \tilde{H}_0 \right) d^4x. \]  

(2)

Here \( R^\eta_{\alpha\xi\beta} \) denotes the Riemann-Christoffel curvature tensor

\[ R^\eta_{\alpha\xi\beta} = \frac{\partial \eta^\gamma_{\alpha\beta}}{\partial \chi^\xi} - \frac{\partial \eta^\gamma_{\alpha\xi}}{\partial \chi^\beta} + \eta^\gamma_{\alpha\xi \gamma} - \eta^\gamma_{\alpha\xi} \eta^\gamma_{\xi \beta}, \]  

(3)

which is an abbreviation of this particular combination of the gauge fields \( \gamma_{\alpha\beta}^\xi \) and their spacetime derivatives. The tensor densities \( \tilde{\kappa}^{\alpha\lambda\beta} \) and \( \tilde{q}^{\alpha\xi\beta} \) denote the canonical momenta of the metric \( g_{\alpha\lambda} \) and of the connection coefficient \( \gamma^\eta_{\alpha\xi} \), respectively. From Eq. (2) we conclude that \( \tilde{\kappa}^{\alpha\lambda\beta} \) must be symmetric in \( \alpha \) and \( \lambda \) while \( \tilde{q}^{\alpha\xi\beta} \) must be skew-symmetric in \( \xi \) and \( \beta \) as only those parts contribute to the action. The canonical gauge procedure indeed reproduces the usual minimal coupling substitution, which converts the partial derivatives of tensors into covariant derivatives—with one important exception: here, the place of the (nonexistent) covariant derivative of the connection coefficient is taken over by the Riemann tensor. As common to all gauge theories, a “dynamics Hamiltonian”, \( \tilde{H}_{\text{Dyn}} = H_{\text{Dyn}}(k^{\alpha\lambda\beta}, q^{\alpha\xi\beta}, g_{\mu\nu}) \), describing the “free kinetics” of the metric and the gauge fields, must be added to the action integrand “by hand” in order to render the gauge fields dynamic quantities.

Furthermore, the torsion tensor

\[ s^\xi_{\beta\alpha} = \frac{1}{2} \left( \gamma^\xi_{\beta\alpha} - \gamma^\xi_{\alpha\beta} \right) \equiv \gamma^\xi_{[\beta\alpha]} \]  

(4)

needs to be considered if we admit non-symmetric connection coefficients. Schrödinger [20], Sciama [16], and von der Heyde [21] showed that the equivalence principle holds even for spacetime geometries with torsion—in contrast to many statements in the literature.

The complete set of field equations for the scalar field \( \phi \) and vector field \( a^\nu \), coupled to a dynamical spacetime geometry described by the metric \( g_{\mu\nu} \), the connection coefficients
and their respective conjugates, $\tilde{k}^{\alpha\beta}$ and $\tilde{q}_{\eta}^{\alpha\beta}$, then write

$$
\gamma_{\alpha\beta}, \quad \text{and their respective conjugates, } \tilde{k}^{\alpha\beta} \text{ and } \tilde{q}_{\eta}^{\alpha\beta}.
$$

$$
\phi^\eta = \frac{\partial \tilde{H}_0}{\partial q^\eta}, \quad \eta^\beta = -\frac{\partial \tilde{H}_0}{\partial \phi^\beta} + 2 \tilde{\pi}^\beta \pi^\alpha \xi_{\beta\alpha}
$$

$$
\phi^\mu = \frac{\partial \tilde{H}_0}{\partial q^{\mu}}, \quad \eta^\beta = -\frac{\partial \tilde{H}_0}{\partial \phi^\beta} + 2 \tilde{\pi}^\beta \pi^\alpha \xi_{\beta\alpha}
$$

$$
\alpha_{\nu;\mu} = \frac{\partial \tilde{H}_0}{\partial \psi_{\nu\mu}}, \quad \tilde{\pi}^\beta = -\frac{\partial \tilde{H}_0}{\partial \alpha_{\nu}} + 2 \tilde{\pi}^\beta \pi^\alpha \xi_{\beta\alpha}
$$

$$
\beta_{\lambda\mu} = \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \xi_{\lambda\mu}}, \quad \tilde{k}^{\alpha\beta} = -\frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \beta_{\lambda}} + 2 \tilde{k}^{\alpha\beta} \pi^\alpha \xi_{\beta\alpha} + 2 \tilde{q}_{\eta}^{\alpha\beta} \pi^\alpha \xi_{\beta\alpha}
$$

Equation (10) links the canonical momentum $\tilde{q}_{\eta}^{\alpha\beta}$ to the Riemann tensor $R^{\alpha\beta}_{\nu\mu}$. As $\tilde{H}_{\text{Dyn}}$ is a scalar (to be exact: a proper world scalar density), we conclude from Eq. (10) that the canonical momentum $\tilde{q}_{\eta}^{\alpha\beta}$ has the same symmetry properties as the Riemann tensor $R^{\alpha\beta}_{\nu\mu}$. The vacuum field equations, hence a subset of equations (5) to (10), was derived earlier by Nester [22] in the language of modern differential geometry by requiring a diffeomorphism-invariant action.

### 2.2. Energy-momentum balance equation

Inserting Eqs. (7), (8), and (10) into Eq. (9), it can be covariantly differentiated with respect to $x^4$ to get the consistency condition (11), which actually represents an energy-momentum balance equation

$$
2 \tilde{k}^{\alpha\beta} \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \xi_{\lambda\mu}} - 2 \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial g_{\alpha\beta}} \xi_{\lambda\mu} + \tilde{q}_{\eta}^{\alpha\beta} \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial q^{\eta}} - \tilde{q}_{\eta}^{\lambda\mu} \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial q^{\eta}} \xi_{\lambda\mu} = \frac{\partial \tilde{H}_0}{\partial \xi_{\lambda\mu}} \xi_{\lambda\mu} - \tilde{\pi}^\beta \frac{\partial \tilde{H}_0}{\partial \psi^\beta} + 2 \frac{\partial \tilde{H}_0}{\partial g_{\alpha\beta}} \xi_{\lambda\mu}
$$

The proof of this equation was originally worked out in the partial derivative representation in Appendix A. It is proved directly as a tensor equation in Appendix A.

The terms on the right-hand side of Eq. (11) are determined by the Hamiltonian $\tilde{H}_0$ of the given system of scalar fields $\phi$ and vector fields $a_{\nu}$. It will be shown that these terms constitute the canonical energy-momentum tensor of the system described by $\tilde{H}_0$.

The terms on the left-hand side emerge from the Hamiltonian $\tilde{H}_{\text{Dyn}}$, which describes the “kinetics” of spacetime. Hence, the consistency equation (11) describes the coupling of the spacetime dynamics to the dynamics of the source fields as a rank two tensor equation.

The consistency condition (11)—which follows from the set of field equations—represents a generic equation of general relativity which holds for any given system of scalar and vector fields, as described by $\tilde{H}_0$, and for any particular model for the dynamics.
of the “free” gravitational fields, as described by $\mathcal{H}_{\text{Dyn}}$. The entire set of ten field equations (5) to (10) is closed and can be integrated to yield the combined dynamics of fields and spacetime geometry only after $\tilde{H}_0$ and $\tilde{H}_{\text{Dyn}}$ have been specified. Note that Eq. (11) is not restricted to metric compatibility, hence the covariant derivative of the metric may be nonzero ($g_{\xi A,\mu} \neq 0$). Equation (11) also applies for spacetimes with torsion ($s^\xi_{\beta\alpha} \neq 0$).

In the proof of Eq. (11), the Riemann tensor is not assumed to be skew-symmetric in its first index pair. This extra symmetry of the Riemann tensor does not follow directly from its definition in Eq. (3). Without this assumption, an additional term, proportional to the weight factor $w$, appears in the Ricci formula (A.2) for the commutator of the second covariant derivatives of relative tensors—which here happens to simplify the proof.

### 2.3. Metric compatibility

If $\mathcal{H}_{\text{Dyn}}$ is defined to not depend on the conjugate of the metric, $\tilde{k}^\alpha{}_{\beta\lambda}$, then Eq. (8) establishes the metric compatibility condition

$$g_{\alpha A,\beta} = \frac{\partial \mathcal{H}_{\text{Dyn}}}{\partial \tilde{k}^\alpha{}_{\beta\lambda}} = Q_{\alpha A} = 0,$$

wherein $Q_{\alpha A}$ denotes the nonmetricity tensor. This reflects a general feature of the canonical formalism: if a Hamiltonian does not depend on a dynamical variable, then the conjugate variable is conserved. The restriction to a covariantly constant metric greatly simplifies the subsequent field equations. If the system Hamiltonian $\mathcal{H}_0$ does not depend on the metric’s conjugate momentum $\tilde{k}^\alpha{}_{\beta\lambda}$—which corresponds to a system Lagrangian $L_0$ that does not depend on the covariant derivative of the metric—the last term on the right-hand side of Eq. (11) represents Hilbert’s metric energy-momentum tensor density

$$\tilde{T}^{\alpha\beta} = \frac{1}{2} \frac{\partial \mathcal{H}_0}{\partial g_{\alpha\beta}}.$$

With a covariantly constant metric, i.e. with metric compatibility, the index $\eta$ in the field equation (9) can simply be raised. Furthermore, the covariantly constant factor $\sqrt{-g}$ (denoted by the tilde) can be eliminated to yield

$$q^{\eta\xi\beta\lambda} = -\frac{1}{2} \left( a^\eta p^\xi\lambda - a^\xi p^\eta\lambda \right) + q^{\eta\beta\alpha} s^\lambda_{\beta\alpha} + 2q^{\xi\beta\alpha} s^\eta_{\beta\alpha}.$$

As noticed above, $q^{\eta\xi\beta\lambda}$ must be skew-symmetric in $\eta$ and $\xi$ in order to obey the same symmetries as the Riemann tensor. In contrast, $k^{\xi\lambda}$ is symmetric, whereas $a^\eta p^\xi\lambda$ has no symmetry in this index pair. Hence, Eq. (14) actually splits into two equations, namely the symmetric and the skew-symmetric portion of Eq. (14) in $\eta$ and $\xi$:

$$q^{\eta\xi\beta\lambda} = -\frac{1}{2} \left( a^\eta p^\xi\lambda - a^\xi p^\eta\lambda \right) + q^{\xi\beta\alpha} s^\eta_{\beta\alpha} + 2q^{\xi\beta\alpha} s^\eta_{\beta\alpha},$$

$$2k^{\xi\lambda} = -\frac{1}{2} \left( a^\eta p^\xi\lambda + a^\xi p^\eta\lambda \right).$$

Either equation will be discussed separately in the following sections.

---

bsee Misner et al. [23], p. 324, for a discussion of this issue.
2.4. Canonical energy-momentum tensor as the source of gravity

As $\tilde{\mathcal{H}}_{\text{Dyn}}$ is assumed not to depend on $\tilde{k}^{\xi\nu}$ to achieve metric compatibility ($g_{\xi\nu}=0$), field equations do not provide an equation relating $\tilde{k}^{\xi\nu}$ to the derivative $\partial g_{\xi\nu}/\partial x^\mu$ of the metric. Nevertheless, the covariant divergence of $\tilde{k}^{\xi\nu}$ from Eq. (8) and the canonical equations (7) can be inserted into the covariant $\lambda$-derivative of Eq. (16) to yield

$$
- \frac{2}{\sqrt{-g}} \frac{\partial \tilde{\mathcal{H}}_{\text{Dyn}}}{\partial g_{\mu\nu}} = \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{H}_0}{\partial g_{\mu\nu}} - \frac{1}{2} \left( g^{\mu\alpha} p^{\nu\beta} + g^{\nu\alpha} p^{\mu\beta} \right) \frac{\partial \mathcal{H}_0}{\partial p^{\mu\beta}} + \frac{1}{2} \left( a^\mu \frac{\partial \mathcal{H}_0}{\partial a^\nu} + a^\nu \frac{\partial \mathcal{H}_0}{\partial a^\mu} \right). \tag{17}
$$

For the cases considered here, the right-hand side of Eq. (17) sums up to the symmetrized canonical energy-momentum tensor $\theta^{\mu\nu}$

$$
\theta^{\mu\nu} = T^{\mu\nu} - g^{\mu\alpha} p^{\nu\beta} \frac{\partial \mathcal{H}_0}{\partial p^{\mu\beta}} + a^\mu \frac{\partial \mathcal{H}_0}{\partial a^\nu}. \tag{18}
$$

The canonical energy-momentum tensor $\theta^{\mu\nu}$ follows for a system of complex scalar fields and real vector fields in the covariant Hamiltonian formalism from the general prescription

$$
\theta^{\mu\nu} = \pi^\mu \frac{\partial \mathcal{H}_0}{\partial \pi^\nu} + \pi^* \frac{\partial \mathcal{H}_0}{\partial \pi^\nu} + g^{\nu\beta} p^{\alpha\beta} \frac{\partial \mathcal{H}_0}{\partial p^{\alpha\beta}} + g^{\nu\beta} \left( \mathcal{H}_0 - \pi^\alpha \frac{\partial \mathcal{H}_0}{\partial \pi^\alpha} - \pi^* \frac{\partial \mathcal{H}_0}{\partial \pi^*} - p^{\alpha\beta} \frac{\partial \mathcal{H}_0}{\partial p^{\alpha\beta}} \right). \tag{19}
$$

Equation (18) with $\theta^{\mu\nu}$ from Eq. (19) is verified for a Proca system in Eq. (35). Hence, our canonical gauge theory of gravity shows that it is exactly the canonical energy-momentum tensor which constitutes the proper source term of gravity. This does not apply to a system of scalar and massless vector fields with additional U(1) symmetry, as will be shown in Sect. 3. Both energy-momentum tensors, the metric and the canonical one, differ by terms which are related to the vector field. Hence, the tensors coincide in the case of systems of pure scalar fields, as is verified in Sect. 2.6. As the energy-momentum tensor enters directly into the field equation of gravity, it is not allowed to replace the canonical energy-momentum tensor by the metric one if the system comprises a massive or non-massive vector field.

Hence, inserting Eq. (18) into Eq. (17) yields

$$
\frac{\partial \tilde{\mathcal{H}}_{\text{Dyn}}}{\partial g_{\mu\nu}} = -\frac{1}{4} (\theta^{\mu\nu} + \theta^{\nu\mu}). \tag{20}
$$

Remarkably, it is exactly the symmetric portion of the canonical energy-momentum tensor that acts as a source for the symmetric tensor on the left-hand side. Equation (20) can be regarded as a generic Einstein-type equation which holds for any model of the free (uncoupled) gravitational field, as described by $\tilde{\mathcal{H}}_{\text{Dyn}}$. It also restates the zero-energy principle (25), hence the hypothesis that the average density of matter in the universe has exactly the critical value such that the total energy of the universe is zero. Actual data suggests that this might indeed be the case (26).

A particular choice of $\tilde{\mathcal{H}}_{\text{Dyn}}$ with at most quadratic terms in the canonical momentum $\tilde{q}^{\xi\eta\lambda}$ was presented in Ref. (1). A more general case is discussed in Ref. (27).
For a scalar field $\phi$ and for restricting the resulting field equation to linear terms in the Riemann tensor, Eq. (20) yields the Einstein equation. However, in all other cases a different theory of gravity emerges.

The correlation of metric and canonical energy-momentum tensors was discussed by Belinfante [17] and Rosenfeld [18]. Their symmetrization prescription has the geometrical meaning to subtract the spin-related components from the canonical energy-momentum tensor $\theta^{\mu\nu}$—and thereby eliminates the spin-related interactions from the field equations.

### 2.5. Spin tensor as the source of spacetime torsion

The skew-symmetric part of the product $a^p p^{\xi^i}$ in $\eta$ and $\xi$ defines the canonical spin tensor

$$
\tau^{\eta\xi\lambda} = \frac{1}{2} \left( a^\eta p^{\xi\lambda} - a^\xi p^{\eta\lambda} \right),
$$

(21)

which quantifies the intrinsic angular momentum (i.e., the spin) density of the vector field $a^\mu$ [28, 29]. The tensor $\tau^{\eta\xi\lambda}$ acts as the source term as can be seen by re-writing Eq. (15) in the form of a Poisson-type equation

$$
q^{\eta\xi\lambda\beta} \gamma^{\beta\alpha\eta} - q^{\eta\xi\alpha\beta} s^{\alpha\eta}_\lambda = -\tau^{\eta\xi\lambda}.
$$

(22)

Depending on the particular model $\hat{\mathcal{H}}_{\text{Dyn}}$ for the free gravitational field, the canonical momentum $q^{\eta\xi\lambda\beta}$ is correlated in a specific way with the Riemann curvature tensor according to Eq. (10). Hence, Eq. (22) shows that the spin $\tau^{\eta\xi\lambda}$ may act as a specific source for the dynamics of a skew-symmetric part of the connection $\gamma^{\lambda\beta\alpha}$. As that torsion constitutes an intrinsic property of the Riemann tensor, it propagates with gravitational waves.

The second covariant derivative of Eq. (15) yields immediately the skew-symmetric part of the consistency equation

$$
R_{\alpha\beta\eta} q^{\eta\nu\rho\eta} - q^{\eta\nu\rho\eta} R_{\alpha\beta\eta} = \theta^{\eta\nu} - \theta^{\alpha\mu}.
$$

(23)

### 2.6. Example 1: Complex Klein-Gordon system

The Klein-Gordon Hamiltonian $\tilde{\mathcal{H}}_0(\phi, \phi^*, \tilde{\pi}^\mu, \tilde{\pi}^{\star \mu}, g_{\mu\nu})$ for a system of complex fields in a dynamic spacetime is given by

$$
\tilde{\mathcal{H}}_0 = \tilde{\pi}^{* \alpha} \tilde{\pi}^\beta g_{\alpha\beta} \frac{1}{\sqrt{-g}} + m^2 \phi^* \phi \sqrt{-g},
$$

(24)

The set of canonical equations following from (24) are

$$
\frac{\partial \phi}{\partial x^\nu} = \frac{\partial \tilde{\mathcal{H}}_0}{\partial \tilde{\pi}^\nu} = g_{\nu\beta} \tilde{\pi}^\beta \sqrt{-g} = \pi_\nu,
$$

$$
\frac{\partial \phi^*}{\partial x^\nu} = \frac{\partial \tilde{\mathcal{H}}_0}{\partial \tilde{\pi}^{\star \nu}} = g_{\alpha\nu} \tilde{\pi}^{* \alpha} \sqrt{-g} = \pi_\nu^*,
$$

$$
\frac{\partial \tilde{\pi}^\alpha}{\partial x^\nu} = - \frac{\partial \tilde{\mathcal{H}}_0}{\partial \phi^*} = -m^2 \phi^* \sqrt{-g},
$$

$$
\frac{\partial \tilde{\pi}^{* \alpha}}{\partial x^\alpha} = - \frac{\partial \tilde{\mathcal{H}}_0}{\partial \phi} = -m^2 \phi \sqrt{-g}.
$$
The canonical momenta can be eliminated by inserting the momenta into the equations for the divergence of the momenta
\[
\frac{\partial \tilde{\pi}^\alpha}{\partial x^\beta} = \frac{\partial}{\partial x^\alpha} \left( g^{\alpha\beta} \frac{\partial \phi}{\partial x^\beta} \sqrt{-g} \right) = -m^2 \phi \sqrt{-g},
\]

hence
\[
g^{\alpha\beta} \frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} + \frac{\partial \phi}{\partial x^\beta} \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\gamma} \left( g^{\alpha\beta} \sqrt{-g} \right) + m^2 \phi = 0.
\]

The second term can be expressed in terms of the connection as
\[
g^{\alpha\beta} \left( \frac{\partial^2 \phi}{\partial x^\alpha \partial x^\beta} - \frac{\partial \phi}{\partial x^\beta} \mathcal{H}_0 \right) + 2 \frac{\partial \phi}{\partial x^\beta} g^{\xi\eta} s^\xi_{\alpha\beta} + m^2 \phi = 0,
\]

with \( s^\xi_{\alpha\beta} \) the contracted torsion tensor, referred to as the torsion vector. The sum in parentheses is the covariant derivative of the covector \( \frac{\partial \phi}{\partial x^\alpha} \), which finally yields the tensor equation
\[
g^{\alpha\beta} \left( \frac{\partial \phi}{\partial x^\alpha} \mathcal{H}_0 \right) + 2 \frac{\partial \phi}{\partial x^\beta} g^{\xi\eta} s^\xi_{\alpha\beta} + m^2 \phi = 0.
\]

The term related to the torsion vector \( s^\xi_{\alpha\beta} \) states that the covariant dynamics couples the scalar field \( \phi \) to the torsion of spacetime. Yet, the scalar field does not act as a source of torsion according to Eq. (22).

The contravariant representation of the associated metric energy-momentum tensor (13) follows by virtue of
\[
\frac{\partial g}{\partial g_{\mu\nu}} = g^{\mu\nu} g \quad \Rightarrow \quad \frac{\partial}{\partial g_{\mu\nu}} = \frac{1}{2} g^{\nu\mu} \sqrt{-g}
\]
as
\[
T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\partial \mathcal{H}_0}{\partial g_{\mu\nu}} = \pi^{\nu\mu} \pi^\nu + \pi^\nu \pi^{\nu\mu} - g^{\mu\nu} \left( \pi^{\alpha\beta} \pi_\alpha \pi_\beta - m^2 \phi^* \phi \right).
\]

The canonical energy-momentum tensor for a system of complex scalar fields follows from the general prescription
\[
\theta^\mu_{\nu} = \pi^\nu \pi^\mu + \pi^\nu \pi^{\mu\nu} + \pi^\mu \pi^\nu + \delta^\mu_{\nu} \left( \mathcal{H}_0 - \pi^{\alpha\beta} \pi_\alpha \pi_\beta - m^2 \phi^* \phi \right)
\]
\[
= \pi^{\mu\nu} \pi_\nu + \pi^\nu \pi^{\mu\nu} - \delta^\mu_{\nu} \left( \pi^{\alpha\beta} \pi_\alpha \pi_\beta - m^2 \phi^* \phi \right).
\]

From Eq. (27), the contravariant representation of the canonical energy-momentum tensor for the complex Klein-Gordon system is obtained as
\[
\theta^{\mu\nu} = \pi^\nu \pi^\mu + \pi^\nu \pi^{\mu\nu} - g^{\mu\nu} \left( \pi^{\alpha\beta} \pi_\alpha \pi_\beta - m^2 \phi^* \phi \right),
\]
which is symmetric and coincides with the above metric energy-momentum tensor of this system:
\[
\theta^{\mu\nu} \equiv T^{\mu\nu}.
\]
Hence, for the system (24) there is no ambiguity with respect to the source term of the generic equations (17) and (23), which write for any $\tilde{H}_{\text{Dyn}}$

$$\partial \tilde{H}_{\text{Dyn}} \partial g_{\mu \nu} = -\partial \tilde{H}_0 \partial g_{\mu \nu},$$

$$q_\tau^{\alpha \beta \lambda} \partial \tilde{H}_{\text{Dyn}} \partial q_\xi^{\tau \beta \lambda} = -R^{\rho}_{\tau \beta \lambda} q^{\rho \tau \beta \lambda} = q^{\rho \tau \beta \lambda} R^{\rho}_{\tau \beta \lambda}.$$ 

2.7. Example 2: Proca system

The Proca Hamiltonian in static spacetime was derived from the Proca Lagrangian in Ref. [8] by means of a (regular) Legendre transformation. In a dynamic spacetime, the corresponding Proca Hamiltonian $\tilde{H}_0(a_\nu, p^\nu_{\alpha \beta}, g_{\mu \nu})$ is given by

$$\tilde{H}_0 = -\frac{1}{4} \tilde{p}^{\alpha \beta} \tilde{p}^{\xi \eta} g_{\alpha \xi} g_{\beta \eta} - \frac{1}{\sqrt{-g}} \frac{1}{2} m^2 a_\alpha a_\beta g^{\alpha \beta} \sqrt{-g},$$

with the pertaining action integral

$$S = \int_R \left[ \frac{1}{2} \tilde{p}^{\alpha \beta} (a_\alpha a_\beta - a_\beta a_\alpha) + \tilde{k}^{\alpha \beta \gamma \delta} g_{\alpha \beta} - \tilde{H}_0 \right] d^4 x. \tag{30}$$

It follows from the variation of (30) that the energy-momentum tensor is skew-symmetric, $\tilde{p}^{\alpha \beta} = -\tilde{p}^{\beta \alpha}$. The covariant derivatives indicate that the system is not closed, but depends on the connection coefficients $\gamma^\rho_{\beta \alpha}$ as external functions of spacetime. The diffeomorphism-invariant action integral of the closed system acquires a modified form compared to that of Eq. (2):

$$S = \int_R \left[ \frac{1}{2} \tilde{p}^{\alpha \beta} (a_\alpha a_\beta - a_\beta a_\alpha) + \tilde{k}^{\alpha \beta \gamma \delta} g_{\alpha \beta} - \tilde{H}_0 - \tilde{H}_{\text{Dyn}} \right] d^4 x. \tag{31}$$

By variation of (31), the canonical equations for the Proca Hamiltonian (29) follow from the general form of Eqs. (7) as

$$\frac{1}{2} \left( a_{\mu \nu} - a_{\nu \mu} \right) = \frac{\partial \tilde{H}_0}{\partial p^{\mu \nu}} = -\frac{1}{2} p^{\mu \nu},$$

$$\tilde{p}^{\alpha \beta} - 2 \tilde{k}^{\alpha \beta \gamma} g_{\gamma \delta} = -\frac{\partial \tilde{H}_0}{\partial a_\mu} = m^2 a^\mu \sqrt{-g},$$

hence, because of metric compatibility,

$$p_{\mu \nu} = \frac{\partial a_\nu}{\partial x^\mu} - \frac{\partial a_\mu}{\partial x^\nu} + 2a_\nu s^{\alpha \mu} \nu,$$

$$\tilde{p}^{\alpha \beta} = m^2 a^\mu + 2\tilde{p}^{\alpha \beta} s^{\alpha \beta}.$$ 

The vector field $a_\mu$ thus directly couples to the torsion of spacetime. The other field equations, emerging from the variation of the action integral (31), are given by Eqs. (8), (9), and (10). Hence, the consistency equation retains the form of Eq. (11). From Eq. (19), the canonical energy-momentum tensor is now obtained for this system as

$$\theta_{\mu \nu} = -\frac{1}{4} \tilde{p}^{\alpha \beta} p_{\alpha \nu} + \frac{1}{4} \tilde{k}^{\alpha \beta \gamma} \left( \tilde{p}^{\alpha \beta} p_{\alpha \beta} - 2m^2 a^\mu a_\mu \right). \tag{33}$$
Note that its covariant and contravariant representations are symmetric. The metric energy-momentum tensor (13) can be set up considering Eq. (26) and
\[
\frac{\partial g_{\alpha\beta}}{\partial g_{\mu\nu}} = -g_{\alpha\mu}g_{\nu\beta}
\]
as
\[
T_{\mu\nu} = -p_{\mu}^{\alpha}p_{\nu}^{\alpha} + m^2 d^{\alpha}a_{\alpha} + \frac{1}{4} \delta_{\mu}^{\alpha} \left( p_{\beta}^{\alpha}p_{\alpha\beta} - 2m^2 a_{\alpha} \right).
\]
(34)
The difference of canonical and metric energy-momentum tensors is now
\[
\theta_{\mu\nu} - T_{\mu\nu} = \frac{1}{2} p_{\mu}^{\alpha}p_{\nu}^{\alpha} - m^2 d^{\alpha}a_{\alpha}
\]
\[
= -p_{\mu}^{\alpha} \frac{\partial H_0}{\partial p_{\alpha\nu}} + \frac{\partial H_0}{\partial a_{\mu}} a_{\nu},
\]
hence
\[
\theta_{\mu\nu} = T_{\mu\nu} - p_{\mu}^{\alpha} \frac{\partial H_0}{\partial p_{\alpha\nu}} + \frac{\partial H_0}{\partial a_{\mu}} a_{\nu}.
\]
(35)
The canonical energy-momentum tensor (33) thus enters into the consistency equation (17) for the spacetime dynamics including torsion. The contravariant representations of Eqs. (33) and (34) for a Proca system are
\[
\theta^{\mu\nu} = \frac{1}{2} p^{\alpha\mu}p_{\nu}^{\alpha} + \frac{1}{4} \delta^{\mu\nu} \left( p_{\beta}^{\alpha}p_{\alpha\beta} - 2m^2 a_{\alpha} \right)
\]
\[
T^{\mu\nu} = -p^{\alpha\mu}p_{\nu}^{\alpha} + m^2 d^{\alpha}a_{\alpha} + \frac{1}{4} \delta^{\mu\nu} \left( p_{\beta}^{\alpha}p_{\alpha\beta} - 2m^2 a_{\alpha} \right).
\]
Our conclusion is that \(\theta^{\mu\nu}\) represents the correct source term for a Proca system. The canonical energy-momentum tensor \(\theta^{\mu\nu}\) thus entails an increased weighting of the kinetic energy over the mass as compared to the metric energy-momentum tensor \(T^{\mu\nu}\) in their roles as the source of gravity. This holds independently of the particular model for the “free” (uncoupled) gravitational field, whose dynamics is encoded in the Hamiltonian \(\tilde{H}_{\text{Dyn}}\) in the generic Einstein-type equation (17)
\[
\theta^{\mu\nu} + \frac{2}{\sqrt{-g}} \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial g_{\mu\nu}} = 0.
\]
Due to the symmetry of the canonical energy-momentum tensor \(\theta^{\mu\nu}\) of the Proca system, the Hamiltonian \(\tilde{H}_{\text{Dyn}}\) of the free gravitational field must be devised to entail a correlation of the canonical momentum \(q^{\mu\nu}\) and the Riemann tensor from the canonical equation (10), which satisfies the condition from Eq. (23)
\[
R_{\tau\beta\lambda}^{\mu} q^{\nu\rho\lambda} = q^{\mu\nu\lambda} R_{\tau\beta\lambda}^{\rho}.
\]
As will be shown in the following section, the metric energy-momentum tensor will turn out to be the appropriate source term of gravity for systems invariant under the combination of the groups \text{Diff}(M)\ and \text{U}(1).
3. System with U(1) symmetry

3.1. U(1) gauge theory

The action integral of a complex scalar field $\phi$ in conjunction with a “free” real 4-vector field $a_\mu$—which later acts as a “gauge field”— writes

$$S = \int \left[ \pi^\alpha \frac{\partial \phi}{\partial x^\alpha} + \frac{\partial \phi^*}{\partial x^\alpha} \pi^{*\alpha} + p^{\alpha\beta} \frac{\partial a_\alpha}{\partial x^\beta} - \mathcal{H}_0 \right] d^4x,$$

with the initially uncoupled Hamiltonian $\mathcal{H}_0$ given by

$$\mathcal{H}_0 = \pi^\alpha \pi^{*\alpha} + m^2 \phi^* \phi - \frac{1}{4} p^{\alpha\beta} p_{\alpha\beta}.$$  

The action integral (36) is obviously invariant under the global ($\Lambda = \text{const.}$) symmetry transformation

$$p^{\nu\mu} = P^{\nu\mu}, \quad a_\mu = A_\mu, \quad \phi = \Phi e^{i\Lambda}, \quad \phi^* = \Phi^* e^{i\Lambda}.$$  

The corresponding local ($\Lambda \neq \text{const.}$) symmetry transformation is defined by means of the generating function

$$F_{2}^\mu = \Pi^\mu \phi e^{i\Lambda(x)} + \phi^* \Pi^\mu e^{-i\Lambda(x)} + p^{\alpha\beta} \left( a_\alpha + \frac{1}{q} \frac{\partial \Lambda}{\partial x^\alpha} \right).$$

In this context, the notation local refers to the fact that the generating function (38) depends explicitly on $x^\nu$. The general transformation rules applied to the actual generating function yield for the fields

$$p^{\nu\mu} = P^{\nu\mu}, \quad a_\mu = A_\mu - \frac{1}{q} \frac{\partial \Lambda}{\partial x^\mu},$$

$$\pi^\mu = \Pi^\mu e^{-i\Lambda(x)}, \quad \phi = \Phi e^{-i\Lambda(x)},$$

$$\pi^{*\mu} = \Pi^{*\mu} e^{i\Lambda(x)}, \quad \phi^* = \Phi^* e^{i\Lambda(x)}.$$  

The transformation rule for the Hamiltonians follows from the explicit $x^\nu$-dependence of the generating function

$$\mathcal{H}' = \mathcal{H} + \frac{\partial F_{2}^\beta}{\partial x^\beta} \bigg|_{\text{expl}},$$

which means for the particular generating function (38)

$$\frac{\partial F_{2}^\beta}{\partial x^\beta} \bigg|_{\text{expl}} = i \left( \Pi^\alpha \phi e^{i\Lambda(x)} - \phi^* \Pi^\alpha e^{-i\Lambda(x)} \right) \frac{\partial \Lambda}{\partial x^\alpha} + \frac{1}{q} p^{\alpha\beta} \frac{\partial^2 \Lambda}{\partial x^\alpha \partial x^\beta}.$$
Inserting the inhomogeneous rule for the vector field $A_\mu$ yields
\[
\left. \frac{\partial A_\mu}{\partial x^\alpha} \right|_{\text{expl}} = iq (\Pi^\alpha \Phi - \Phi^\alpha \Pi) A_\mu - iq (\pi^\alpha \phi - \phi^\alpha \pi) a_\alpha \\
+ \frac{1}{2} P_{\alpha\beta} \left( \frac{\partial A_\alpha}{\partial x^\beta} + \frac{\partial A_\beta}{\partial x^\alpha} \right) - \frac{1}{2} P_{\alpha\beta} \left( \frac{\partial a_\alpha}{\partial x^\beta} + \frac{\partial a_\beta}{\partial x^\alpha} \right).
\]

According to the canonical transformation formalism [1], it follows from Eq. (40) that the uncoupled Hamiltonian (37) is now replaced by the amended Hamiltonian $H_1$, defined as
\[
H_1 = \pi_\alpha^\alpha \pi^\alpha + iq (\pi^\alpha \phi - \phi^\alpha \pi) a_\alpha + m^2 \phi^\alpha \phi - \frac{1}{4} p_{\alpha\beta} p_{\alpha\beta} + \frac{1}{2} p_{\alpha\beta} \left( \frac{\partial a_\alpha}{\partial x^\beta} + \frac{\partial a_\beta}{\partial x^\alpha} \right).
\]

This Hamiltonian $H_1(\phi, \phi^*, \pi^\mu, \pi^{*\mu}, a_\alpha, p^{\mu})$ is mapped under the canonical transformation rules (39) of the fields and (40) into a Hamiltonian $H_1(\Phi, \Phi^*, \Pi^\mu, \Pi^{*\mu}, A_\alpha, P^{\mu})$ of exactly the same form in the transformed fields.

The Hamiltonian (41) can finally be combined with the corresponding terms in the initial action integral from Eq. (36). We thus end up with the particular action integral
\[
S = \int \left[ \pi^\alpha \left( \frac{\partial \Phi}{\partial x^\alpha} - iq a_\alpha \right) + \left( \frac{\partial \Phi^*}{\partial x^\alpha} + i q a_\alpha \right) \right] dx^\alpha + \frac{1}{2} p_{\alpha\beta} \left( \frac{\partial a_\alpha}{\partial x^\beta} - \frac{\partial a_\beta}{\partial x^\alpha} \right) - H_0 d^3x.
\]

The action integral (42) is form-invariant under the local canonical transformation rules (39) and (40), which are generated by $F_2$ from Eq. (38). It follows directly from the action integral (42) that $p^{\alpha\beta}$ is skew-symmetric. This property of the canonical momentum tensor conjugate to the vector field $a_\alpha$ follows here from the gauge formalism and need not to be postulated. Notice that the derivative of the vector field in Eq. (36) is now replaced by a skew-symmetric tensor yielding a U(1) gauge invariant action integral.

3.2. Extension to the $U(1) \times \text{Diff}(M)$ symmetry group

The system of complex scalar and real vector fields described by the action (42) is form-invariant under a local U(1) symmetry transformation, hence under phase transformations of the scalar field and the shift transformation of the real vector field. In Appendix B, we derive the combined transformation rules for the U(1) symmetry transformation and the symmetry under the diffeomorphism group Diff(M) of the spacetime manifold $M$. The Hamiltonian $\mathcal{H}_{\text{Dyna}}(\bar{k}, \bar{q}, g)$ stands for the model which describes both the dynamics of the momenta $\bar{k}$ of the metric $g$, and $\bar{q}$ the momenta of the connection coefficients $\gamma$. On the basis of the action (42), the generally covariant action which describes in addition the interaction of the massive complex scalar field $\phi$ and the massless real (Maxwell) vector field $a_\mu$ with the spacetime geometry is obtained as
\[
S = \int d^4x \left[ \bar{k}^\alpha \left( \frac{\partial \phi}{\partial x^\alpha} - iq a_\alpha \phi \right) + \left( \frac{\partial \phi^*}{\partial x^\alpha} + i q a_\alpha \phi^* \right) \bar{k}^\alpha \\
+ \frac{1}{2} \bar{q}^{\alpha\beta} \left( \frac{\partial a_\alpha}{\partial x^\beta} - \frac{\partial a_\beta}{\partial x^\alpha} \right) + \bar{k}^\alpha \gamma_{\alpha\beta} \bar{q}^{\alpha\beta} - \frac{1}{2} \bar{q}^{\alpha\beta} \bar{q}^{\lambda\gamma} R_{\lambda\gamma}^\alpha - \mathcal{H}_0 - \mathcal{H}_{\text{Dyna}} \right] d^3x.
\]
Clearly, this system is form-invariant under the combined symmetry group $U(1) \times \text{Diff}(M)$. Comparing the invariant action functional (43) with that of the Proca system (Eq. (31)) shows that now the gauge field $a_\mu$ does no longer couple directly to the connection $\gamma^\xi_{\alpha\beta}$.

The Proca action contains the additional term
\[ \frac{1}{2} \tilde{p}^{\alpha\beta} \left( \gamma^\xi_{\alpha\beta} - \gamma^\xi_{\beta\alpha} \right) a_\xi, \]
which is invariant under the $\text{Diff}(M)$ symmetry group, but not under the group $U(1) \times \text{Diff}(M)$.

With $\tilde{\mathcal{H}}_0$ in particular the sum of the Klein-Gordon Hamiltonian (24) and the massless Proca Hamiltonian (29), Eq. (43) represents the generic action of the Einstein’s “unified field theory” of electromagnetics and gravitation. The Hamiltonian $\tilde{\mathcal{H}}_{\text{Dyn}}(\tilde{k}, \tilde{q}, \tilde{g})$ stands for all possible descriptions of the free gravitational field—which are not restricted to metric compatibility and zero torsion. The final “gauged” Hamiltonian is thus with $\mathcal{H}_0$ from Eq. (37)
\[ \tilde{\mathcal{H}}_3 = \mathcal{H}_0 + \mathcal{H}_{\text{Dyn}} + i q (\tilde{\pi}^{\alpha} a^{\alpha} - \phi^{\ast} \tilde{\pi}^{\alpha}) a_\alpha \]
\[ + \frac{1}{2} \tilde{p}^{\alpha\beta} \left( \frac{\partial a_\alpha}{\partial \chi^\mu} + \frac{\partial a_\beta}{\partial \chi^\mu} \right) + \left( \tilde{k}^{\alpha\beta} \partial_\xi + \tilde{k}^{\alpha\beta} \partial_\xi \right) \gamma^\xi_{\alpha\beta} \]
\[ + \frac{1}{2} \tilde{q}^{\alpha\beta} \left( \frac{\partial \gamma^\eta_{\alpha\xi}}{\partial \chi^\beta} + \frac{\partial \gamma^\eta_{\alpha\beta}}{\partial \chi^\xi} + \gamma^\eta_{\alpha\beta} \gamma^\xi_{\eta\xi} - \gamma^\eta_{\alpha\xi} \gamma^\xi_{\eta\beta} \right), \]
from which the canonical field equations follow now as tensor equations.

### 3.3. Canonical Field equations and the consistency equation

In this section, we derive the set of canonical field equations emerging from the gauge-invariant Hamiltonian $\tilde{\mathcal{H}}_3$ of Eq. (44). To begin with, the field equations for the complex scalar field $\phi$ and its conjugates follow as
\[ \frac{\partial \phi}{\partial \chi^\mu} = \frac{\partial \tilde{\mathcal{H}}_3}{\partial \tilde{\pi}^{\alpha} a_\alpha} = \frac{\partial \mathcal{H}_0}{\partial \tilde{\pi}^{\alpha} a_\alpha} - i q a_\alpha \phi, \]
\[ \frac{\partial \phi^*}{\partial \chi^\mu} = \frac{\partial \tilde{\mathcal{H}}_3}{\partial \phi^*} = \frac{\partial \mathcal{H}_0}{\partial \phi^*} + i q \phi a_\alpha, \]
\[ \frac{\partial \tilde{\pi}^{\alpha}}{\partial \chi^\alpha} = - \frac{\partial \phi}{\partial \phi^*} = - \frac{\partial \mathcal{H}_0}{\partial \phi^*} + i q \tilde{\pi}^{\alpha} a_\alpha, \]
\[ \frac{\partial \tilde{\pi}^{\alpha} a_\alpha}{\partial \phi^*} = - \frac{\partial \mathcal{H}_0}{\partial \phi^*} + i q \tilde{\pi}^{\alpha} a_\alpha. \]

The field equations related to the real vector field $a_\mu$ emerge as
\[ \frac{\partial a_\mu}{\partial \chi^\nu} = \frac{\partial \tilde{\mathcal{H}}_3}{\partial \tilde{p}^{\mu\nu}} = \frac{\partial \mathcal{H}_0}{\partial \tilde{p}^{\mu\nu}} + \frac{1}{2} \left( \frac{\partial a_\nu}{\partial \chi^\nu} + \frac{\partial a_\mu}{\partial \chi^\nu} \right) \]
\[ - \frac{\partial \mathcal{H}_0}{\partial a_\nu} - i q (\tilde{\pi}^{\nu} \phi - \phi^* \tilde{\pi}^{\nu}). \]
hence
\[ \frac{\partial \tilde{H}_0}{\partial \tilde{p}^{\nu\mu}} = \frac{1}{2} \left( \frac{\partial a_\nu}{\partial x^\alpha} - \frac{\partial a_\mu}{\partial x^\nu} \right), \]
\[ \frac{\partial \tilde{H}_0}{\partial a_\nu} = \frac{\partial \tilde{p}^{\nu\mu}}{\partial x^\beta} - iq (\tilde{n}^\nu \phi - \phi^\nu \tilde{n}). \]

For metric compatibility, hence for \( \tilde{\mathcal{H}}_{\text{Dyn}} \) not depending on \( \tilde{k}^{\alpha\beta} \), the field equations related to the (symmetric) metric tensor \( g_{\mu\nu} \) are
\[ \frac{\partial g_{\alpha\lambda}}{\partial x^\beta} = \frac{\partial \tilde{H}_3}{\partial \tilde{q}_{\eta}^{\alpha\beta}} = g_{\epsilon \lambda} \gamma_{\alpha \beta} + g_{\alpha \epsilon} \gamma_{\beta \lambda} \quad \Leftrightarrow \quad g_{\alpha \lambda \beta} = 0 \]
\[ \frac{\partial \tilde{k}^{\alpha\beta}}{\partial x^\epsilon} = -\frac{\partial \tilde{H}_3}{\partial g_{\epsilon \lambda}} = -\tilde{k}^{\alpha \beta} \gamma_{\epsilon \lambda} + \tilde{k}^{\epsilon \lambda} \gamma_{\alpha \beta} - \frac{\partial \tilde{H}_0}{\partial g_{\epsilon \lambda}} \frac{\partial \tilde{\mathcal{H}}_{\text{Dyn}}}{\partial g_{\epsilon \lambda}}. \]

Finally, the field equations for the connection \( \gamma_{\alpha \xi} \) and its conjugate, \( \tilde{q}_{\eta}^{\alpha \beta} \), are set up:
\[ \frac{\partial \gamma_{\alpha \xi}}{\partial t} = \frac{\partial \tilde{H}_3}{\partial \tilde{q}_{\eta}^{\alpha \beta}} + \frac{1}{2} \left( \frac{\partial \gamma_{\alpha \xi}}{\partial \xi^\beta} + \frac{\partial \gamma_{\alpha \beta}}{\partial \xi^\xi} + \gamma_{\alpha \xi} \gamma_{\beta \xi} - \gamma_{\beta \xi} \gamma_{\alpha \xi} \right), \]
which yields the negative Riemann tensor \( R_{\eta}^{\alpha \beta \xi} \)
\[ 2 \frac{\partial \tilde{\mathcal{H}}_{\text{Dyn}}}{\partial \tilde{q}_{\eta}^{\alpha \beta}} = \frac{\partial \gamma_{\alpha \xi}}{\partial \xi^\beta} - \frac{\partial \gamma_{\alpha \beta}}{\partial \xi^\xi} + \gamma_{\alpha \xi} \gamma_{\beta \xi} - \gamma_{\beta \xi} \gamma_{\alpha \xi} = -R_{\eta}^{\alpha \beta \xi}. \]
(45)

As \( \tilde{H}_0 \) and \( \tilde{\mathcal{H}}_{\text{Dyn}} \) do not depend on the connection \( \gamma_{\alpha \beta} \), the equation for its conjugate, \( \tilde{q}_{\eta}^{\alpha \beta \lambda} \), is fully determined
\[ \frac{\partial \tilde{q}_{\eta}^{\alpha \beta \lambda}}{\partial x^\xi} = -\frac{\partial \tilde{H}_3}{\partial \gamma_{\alpha \beta}} = 2 \tilde{k}_{\lambda \alpha} g_{\xi \lambda} \tilde{q}_{\eta}^{\alpha \beta \lambda} + \tilde{q}_{\eta}^{\alpha \beta \lambda} \gamma_{\xi \lambda} + \tilde{q}_{\eta}^{\alpha \beta \xi} \gamma_{\lambda \eta}. \]
(46)

Similarly to the corresponding field equation of the pure diffeomorphism symmetry from Eq. (9), we can rewrite this equation in terms of a covariant divergence
\[ \tilde{q}_{\eta}^{\xi \beta \lambda} \gamma_{\alpha \beta} = -2 g_{\alpha \mu} \tilde{k}^{\beta \lambda \epsilon} + \tilde{q}_{\eta}^{\epsilon \lambda \mu} \gamma_{\beta \lambda} + 2 \tilde{q}_{\eta}^{\xi \beta \mu} \gamma_{\alpha \lambda}. \]
(47)

The term \( a_{\eta} \tilde{p}^{\epsilon \lambda} \) is now missing as the vector field does not couple to the connection in the Hamiltonian \( \tilde{\mathcal{H}}_3 \). Consequently, the consistency equations for metric compatibility now follow as
\[ -\frac{\partial \tilde{\mathcal{H}}_{\text{Dyn}}}{\partial g_{\mu\nu}} = \frac{\partial \tilde{H}_0}{\partial g_{\mu\nu}}, \]
(48)
and

\[ q_\tau \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \tilde{q}_\tau^\alpha} = \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \tilde{q}_\tau^\beta} \partial \tilde{H}_{\text{Dyn}}^{\beta\alpha} q_\xi^\beta \]

In contrast to the \text{Diff}(M) gauge theory of the Proca system, for which the canonical energy-momentum tensor emerges as the source in the consistency equation (11), the \text{U}(1) \times \text{Diff}(M) gauge theory has the metric energy-momentum tensor as its source. For the conventional case of \( \tilde{H}_0 \) not depending on the conjugate of the metric, \( \tilde{k}^{\alpha\lambda\mu} \), this tensor is generally obtained according to Eq. (13). For the Klein-Gordon-Maxwell Hamiltonian (41), hence for the \text{U}(1) gauge theory, this tensor has the explicit Hamiltonian form

\[ T^{\mu\nu} = \pi^{\mu} \pi^\nu + \pi^{\nu} \pi^\mu - p^{\mu\rho} p^{\nu\sigma} g_{\rho\sigma} - g^{\mu\nu} \left( \pi^\alpha \pi^\alpha - m^2 \phi^2 \phi - \frac{1}{4} p^{\alpha\beta} p_{\alpha\beta} \right). \] (49)

It can be shown similarly that the metric energy momentum tensor also acts as the source term for \text{SU}(N) \times \text{Diff}(M) gauge theories, hence for the general relativistic extension of Yang-Mills gauge theories. This again holds independently of the particular \( \tilde{H}_{\text{Dyn}} \) describing the dynamics of the “free” gravitational field.

4. Conclusions

Any (globally) Lorentz-invariant Lagrangian/Hamiltonian system can be converted into an amended Lagrangian or Hamiltonian, which is form-invariant under the \text{Diff}(M) symmetry group, following the well-established reasoning of gauge theories. The integrand of the final action integral (2) was shown to represent a proper (world) scalar density, thereby meeting the requirement of Einstein’s Principle of General Relativity, i.e. form-invariance under local chart transitions (diffeomorphisms). For simplicity, metric compatibility was imposed later in Eq. (12).

The gauge formalism reveals that scalar and vector source fields couple differently to a dynamic spacetime. In the context of this paper, we must distinguish three cases:

(1) Massive or massless, real or complex scalar fields \( \phi \) are associated with the metric (Hilbert) energy-momentum tensor \( T^{\mu\nu} \) as the source term of the Einstein-type equation (17). Yet, in this case, \( T^{\mu\nu} \) coincides with the canonical energy-momentum tensor \( \theta^{\mu\nu} \). Scalar fields do not act as source for a torsion of spacetime.

(2) Systems of massive or non-massive vector fields \( a_\mu \) with no other symmetries than the \text{Diff}(M) group—such as the Proca system—require the canonical energy-momentum tensor \( \theta^{\mu\nu} \) as the source term. Such systems do couple to a torsion of spacetime.

(3) A system consisting of a complex (charged) scalar field \( \phi \), which couples minimally to a massless vector field \( a_\mu \) (Maxwell field)—hence a system with additional \text{U}(1) symmetry—has the metric energy-momentum tensor \( T^{\mu\nu} \) as the source term. This can be generalized to systems with \text{SU}(N) symmetry [19]. These systems do not couple to torsion of spacetime.

The general prescription to promote a Lorentz-invariant action into a generally covariant action is thus to replace the partial derivatives of all non-scalar objects in the action integral...
Spacetime coupling of spin-0 and spin-1 particle fields

by covariant derivatives. Exceptions to this recipe are twofold. The first exception is en-
countered regarding the invariant action integral (42) of the U(1) gauge theory. The direct
coupling \( \tilde{p}_{\alpha\beta} a_\xi a_\xi \) to the dynamical spacetime geometry is exactly compensated due to
the inhomogeneous transformation rule (B.6) in Eq. (B.16). The remaining derivative of
\( \tilde{H}_0 \) with respect to the metric \( g_{\alpha\beta} \) then yields the metric energy-momentum tensor as the
appropriate source term for the spacetime dynamics of a system with U(1) resp. SU(\( N \))
symmetry. Moreover, due to the missing coupling term, the canonical spin tensor \( \tau_{\eta\xi\lambda} \)
does not show up on the right-hand side of Eq. (22). Hence, the system does not generate torsion
of spacetime.

The second exception is the partial derivative of the affine connection in the action
integral. As the affine connection is no tensor, its partial derivatives cannot directly be con-
verted into covariant derivatives in the initial action integral (1). Yet, by virtue of the gauge
formalism, a term quadratic in the affine connection \( \gamma \) emerges, which is shown to make the
partial derivatives of \( \gamma_{\alpha\beta} \) into one-half the Riemann tensor—and hence into a generally co-
variant object. The action integral (2) thus complies with the Principle of General Relativ-
ity. Its subsequent field equations are then tensor equations, which quantify the interaction
of the source fields with the spacetime geometry, the latter being described by the metric
and the affine connection as separate geometrical objects. The canonical transformation
approach to spacetime dynamics thus naturally implements the Palatini formalism [13].

As the gauge formalism determines only the coupling of matter and spacetime dynami-
cics, an additive Hamiltonian \( \tilde{H}_{\text{Dyn}} \) of the “free” gravitational field is to be postulated. An
action with a quadratic term in the canonical momenta of the gauge field is required to
obtain a closed system of field equations for the coupled dynamics of fields and spacetime
geometry [1].

We have shown in this paper that the correct energy-momentum tensor for the massive
vector is the canonical one, whereas the metric one is correct for massless vector fields,
hence for systems with additional U(1) or, more general, SU(\( N \)) symmetry. As a conse-
quence, compact astrophysical objects, like neutron stars and binary neutron star mergers
must be reinvestigated with the appropriate canonical energy-momentum terms for the vec-
tor repulsion from an effective field theory (EFT). A similar conclusion does also hold for
fermions, both for protons and electrons, as well as for neutrinos, both in white dwarfs,
neutron stars and in “ultra high energy cosmic ray” (UHECR) events. The particular cou-
lpling of spinor fields with spacetime will be the topic of separate papers [30, 31]. It will be
shown that the coupling of a spinor with the spacetime dynamics gives rise to an effective
mass term in the generally covariant Dirac equation.

The theory of geometrodynamics with quadratic Riemann tensor and a modified cou-
lpling of vector fields developed here entails important astrophysical and cosmological con-
sequences according to the new general equation (61) of Ref. [1]. Details will be addressed
in a forthcoming article, yet qualitative consequences are discussed briefly in the following.
Reference [1] demonstrated that Eq. (61) is solved as well as the classical Einstein equa-
tion by the Schwarzschild and the Kerr metrics. However, due to the canonical quadratic
curvature terms, a new understanding emerges of both Friedman cosmology, as well as for
the interior of compact stellar objects, pulsar dynamics and binary neutron star mergers. The canonical energy-momentum tensor replaces the Einstein’s metric tensor, resulting in structural changes of compact astrophysical objects and relativistic collapse dynamics.

The interior structure of the neutron stars can be calculated using the QCD-motivated relativistic mean-field (RMF) theory with meson fields interacting with neutrons, other baryons, and with leptons. This is done e.g. in the Walecka [32,33] and in non-linear Boguta-Bodmer RMF theories [34,35] by introducing the repulsive \( \omega^\mu \) and \( \rho^\mu \) interactions, as well as the attractive scalar \( \sigma \) field. For both \( N = Z \) nuclear matter, and for the \( N \)-dominated neutron matter, the nucleons and heavier baryons acquire substantial relativistic effects in the dense medium, including both, strong attraction \( (U_s \approx -450 \text{ MeV}) \) induced by scalar \( \sigma \) mesons with mass \( m_\sigma \approx 550 \text{ MeV} \), and strong repulsion \( (U_v \approx 350 \text{ MeV}) \) due to the \( \omega^\mu \) and \( \rho^\mu \) vector mesons with masses \( m_\omega \approx m_\rho \) of the order of 800 MeV. Due to scalar-meson interactions, the effective in-medium nucleon mass drops from the value \( m_N \approx 940 \text{ MeV} \) to about \( m_N^* \approx 550 \text{ MeV} \) at densities around the nuclear ground state density \( \rho_0 \). In this case contributions of massive meson vector (spin-1) fields to the energy density are of the same order of magnitude as the energy density of neutrons and other baryons, exceeding the baryon kinetic energy density above baryon densities of \( 2\rho_0 \approx 0.3 \text{ fm}^{-3} \).

Neutron star formations in supernovae collapses, and creation of hypermassive neutron stars (HMNS) in binary neutron star mergers, may produce even larger relativistic effects due to strong magnetic fields in pulsars, in rapidly rotating neutron stars, and even more in violently moving binary HMNSs with high temperatures, higher densities and strong magnetic fields \( H \approx 10^{18} \text{ Gauss} \). The high angular frequencies may align spins of both, nucleons (spin-1/2) and vector mesons, with similar masses. Hence, the terms proportional to \( p^{\beta}p^\nu \) and \( \bar{e}^a a^\nu \) on the right-hand side of Eq. (35) can be of the same order of magnitude as the metric energy-momentum tensor \( T^{\mu\nu} \) of the Einstein equation. On this account, the theory of neutron star structure and dynamics need to be fundamentally revised.

The recent detection of the gravitational wave from a binary neutron star merger by the LIGO-VIRGO collaboration (GW170817) [36,37] opens various interesting astrophysical scenarios: analysis of the gravitational wave data, in combination with the independently detected gamma-ray burst (GRB 170817A) [38] and further electromagnetic radiation [39] results in a neutron star merger scenario which is in good agreement with numerical simulations of binary neutron star mergers performed in full general relativistic hydrodynamics. As a result of the binary merger, a fast, differentially rotating HMNS was produced, which lived for \( \approx 1 \) second before it collapsed to a rotating Kerr black hole. Matter in the interior of the HMNS reaches densities of up to several times normal nuclear matter, and temperatures could reach \( T \sim 50 – 100 \text{ MeV} \). However, for such high densities, the equation of state (EOS) is still poorly constrained. High energy heavy ion collision data are compatible with a hadron-quark phase transition, which then shall also be present in the interior of the HMNS [40].

As predicted by Csernai and coworkers [41], a strongly rotating quark-gluon plasma has been detected experimentally in non-central ultra-relativistic heavy ion collisions [42]. The coupling of the internal spin component of the quark-gluon plasma phase (in the interior of the HMNS) may act as a source of torsion of spacetime. Therefore, numerical
Simulations of the post-merger phase of a neutron star merger performed within the canonical theory of gravitation will differ fundamentally from classical Einstein simulations.

The replacement of the metric energy-momentum-tensor by the canonical energy-momentum-tensor thus results in two important changes in the general relativistic descriptions of neutron stars and binary neutron star mergers: Both the static Tolman-Oppenheimer-Volkov equation and the dynamic theory of binary neutron stars become invalid, because not only the EOS of hot, dense, and strongly rotating matter, but also the metric itself will look quite differently as compared to the Einstein general relativity.

Acknowledgements

The authors are deeply indebted to the “Walter Greiner-Gesellschaft zur Förderung der physikalischen Grundlagenforschung e.V.” in Frankfurt for its support. H. Stoecker acknowledges the Judah M. Eisenberg Professor Laureatus Chair at the Institute or Theoretical Physics of the Goethe University Frankfurt.

Appendix A. Proof of the consistency equation (11)

If a dynamical quantity $T^\alpha_1...\alpha_n_{\beta_1...\beta_m}(x)$ at the spacetime location $x$ transforms to $T^{\xi_1...\xi_n}_{\eta_1...\eta_m}(X)$ at $X$ according to

$$T^{\xi_1...\xi_n}_{\eta_1...\eta_m} = \rho^{\alpha_1...\alpha_n}_{\beta_1...\beta_m} \frac{\partial X^{\xi_1}}{\partial x^{\beta_1}} ... \frac{\partial X^{\xi_n}}{\partial x^{\beta_m}} \frac{\partial x^{\beta_1}}{\partial X^{\eta_1}} ... \frac{\partial x^{\beta_m}}{\partial X^{\eta_m}} | \frac{\partial x}{\partial X} |$$

with $| \frac{\partial x}{\partial X} |$ the determinant of the Jacobi matrix (“Jacobian”) of the transformation $x \mapsto X$.

$$\frac{\partial x}{\partial X} = \frac{\partial (x^0, ..., x^3)}{\partial (X^0, ..., X^3)}$$

then $T$ is referred to as a relative $(n,m)$-tensor of weight $w$. The difference of the second covariant derivatives of this kind of tensor is given by the Ricci formula

$$c_{\mu\nu} (T^{\xi_1...\xi_n}_{\eta_1...\eta_m}) \equiv \left( T^{\xi_1...\xi_n}_{\eta_1...\eta_m} \right)_{,\mu;\nu} - \left( T^{\xi_1...\xi_n}_{\eta_1...\eta_m} \right)_{,\nu;\mu}$$

$$= \sum_{k=1}^n R^{\xi_1...\xi_n}_{\eta_1...\eta_m;\mu;\nu} T^{\xi_k}_{\eta_1...\eta_{k-1}...\eta_n} - \sum_{k=1}^n R^{\xi_1...\xi_{k-1}\eta_k}_{\tau_{\mu\nu}} T^{\xi_k}_{\tau_{\eta_1...\eta_k}} - w R^{\xi_1...\xi_n}_{\eta_1...\eta_m;\mu;\nu} + 2 s^{\tau}_{\mu\nu} \left( T^{\xi_1...\xi_n}_{\eta_1...\eta_m} \right)_{,\tau}$$

where $R^{\xi_1...\xi_n}_{\eta_1...\eta_m;\mu;\nu}$ denotes the Riemann curvature tensor and $s^{\tau}_{\mu\nu}$ the torsion tensor. In the first and second term on the right-hand side of Eq. (A.2), the index $\tau$ is located at the $k$-th position of the index list, respectively.

Note that Ref. [43] in its Eq. (3.10) uses the opposite sign convention in the definition of the weight factor $w$, as compared to Eq. (A.1). In our case, $\sqrt{g}$ represents a relative scalar of weight $w = +1$. So all momentum fields in our paper follow as relative tensors of weight $w = +1$. 

Spacetime coupling of spin-0 and spin-1 particle fields
If the Riemann tensor is assumed to be skew-symmetric in its first index pair, then
\[ R^\tau_{\tau \beta \lambda} = R_{\tau \beta \lambda} = 0. \]

Yet, we do not pursue the assumption at this point in order to maintain the consistency of our derivation, where we did not discuss symmetry properties of Eq. (9).

With the Ricci formula in the general form of Eq. (A.2), we calculate the second covariant derivative \( \tilde{q}_\eta^{\,\epsilon \lambda \beta} \) of the (3, 1)-tensor \( \tilde{g} \) of weight \( w = 1 \) from Eq. (9). Owing to the skew-symmetry of \( \tilde{q}_\eta^{\,\epsilon \lambda \beta} \) in its last index pair, \( \lambda \) and \( \beta \), the second covariant derivative can be expressed as a difference of two second covariant derivatives with the differentiation sequence reversed. Then the Ricci formula (A.2) can be applied
\[ 2 \tilde{q}_\eta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} = \tilde{q}_\eta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} - \tilde{q}_{\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} = \tilde{q}_\eta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} - \tilde{q}_{\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} = 2 \tilde{q}_\eta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} - \tilde{q}_{\epsilon \lambda \beta} \tilde{g}_{\beta \lambda}, \]
\[ = R^\tau_{\tilde{g} \beta \lambda} \tilde{q}_\eta^{\,\epsilon \lambda \beta} - \tilde{q}_\eta^{\,\epsilon \lambda \beta} R^\tau_{\tilde{g} \beta \lambda} - R^\lambda_{\tilde{g} \beta \lambda} \tilde{q}_\eta^{\,\epsilon \lambda \beta} - R^\beta_{\tilde{g} \beta \lambda} \tilde{q}_\eta^{\,\epsilon \lambda \beta}, \]
\[ + R^\tau_{\tilde{g} \beta \lambda} \tilde{q}_\eta^{\,\epsilon \lambda \beta} - 2 \tilde{s}_\beta^{\,\epsilon \lambda \beta} \tilde{q}_\eta^{\,\epsilon \lambda \beta} = R^\tau_{\tilde{g} \beta \lambda} \tilde{q}_\eta^{\,\epsilon \lambda \beta} - 2 \tilde{s}_\beta^{\,\epsilon \lambda \beta} \tilde{q}_\eta^{\,\epsilon \lambda \beta} = \tilde{q}_\eta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} - 2 \tilde{s}_\beta^{\,\epsilon \lambda \beta} \tilde{q}_\eta^{\,\epsilon \lambda \beta}, \]
\[ + (R^\beta_{\tilde{g} \beta \lambda} + R^\beta_{\tilde{g} \beta \lambda} + R^\beta_{\tilde{g} \beta \lambda}) \tilde{q}_\eta^{\,\epsilon \lambda \beta} = \tilde{q}_\eta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} - 2 \tilde{s}_\beta^{\,\epsilon \lambda \beta} \tilde{q}_\eta^{\,\epsilon \lambda \beta}. \]
\[ + 4 \left( \tilde{q}_\eta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} + \frac{1}{2} \tilde{s}_\beta^{\,\epsilon \lambda \beta} + \frac{1}{2} \tilde{s}_\beta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} \right) \tilde{q}_\eta^{\,\epsilon \lambda \beta}. \] (A.4)

In the last step, the contracted representation of the Bianchi identity for spaces with torsion was inserted (43)
\[ R^\tau_{\tilde{g} \beta \lambda} + R^\tau_{\tilde{g} \beta \lambda} + R^\tau_{\tilde{g} \beta \lambda} = -2 \left( \tilde{s}_{\epsilon \beta \lambda \tau \alpha} + \tilde{s}_{\beta \lambda \tau \alpha} + \tilde{s}_{\epsilon \beta \lambda \tau} \right), \]
\[ -4 \left( \tilde{s}_{\epsilon \beta \lambda \tau \alpha} + \tilde{s}_{\epsilon \beta \lambda \tau} \right), \] hence
\[ R^\beta_{\tilde{g} \beta \lambda} + R^\beta_{\tilde{g} \beta \lambda} + R^\beta_{\tilde{g} \beta \lambda} = -2 \left( \tilde{s}_{\epsilon \beta \lambda \tau \alpha} + \tilde{s}_{\beta \lambda \tau \alpha} + \tilde{s}_{\epsilon \beta \lambda \tau} \right), \]
\[ -4 \left( \tilde{s}_{\epsilon \beta \lambda \tau \alpha} + \tilde{s}_{\epsilon \beta \lambda \tau} \right). \] (A.5)

Equation (A.4) can now be equated with the covariant \( \lambda \)-derivative of the right-hand side of Eq. (9):
\[ \frac{1}{2} R^\tau_{\tilde{g} \beta \lambda} \tilde{q}_\eta^{\,\epsilon \lambda \beta} - \frac{1}{2} \tilde{q}_\eta^{\,\epsilon \lambda \beta} R^\tau_{\tilde{g} \beta \lambda} - 2 \tilde{s}_\beta^{\,\epsilon \lambda \beta} \tilde{q}_\eta^{\,\epsilon \lambda \beta} = -2 \tilde{q}_\eta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} + \frac{1}{2} \tilde{s}_\beta^{\,\epsilon \lambda \beta} + \frac{1}{2} \tilde{s}_\beta^{\,\epsilon \lambda \beta} \tilde{g}_{\beta \lambda} + 2 \tilde{s}_\beta^{\,\epsilon \lambda \beta} \tilde{q}_\eta^{\,\epsilon \lambda \beta} \]
Inserting the canonical field equations (7), (8), and (10) yields

\[
\tilde{q}_\eta \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \tilde{q}_\xi} - \tilde{q}_\eta \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \tilde{q}_\tau} + \tilde{q}_\eta \frac{\partial \tilde{H}_{\text{Dyn}}}{\partial \tilde{q}_\sigma} = 2 \left( \eta^{\sigma} \eta_{\tau} + \frac{1}{2} \eta^{\sigma} \eta_{\tau} + \frac{1}{2} \eta^{\sigma} \eta_{\tau} \right) \tilde{q}_\eta
\]

\[
= a_\eta \left( \frac{\partial \tilde{H}_0}{\partial a_\xi} - \frac{2 s^\rho s^a}{\rho a} \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\rho} \right) + \tilde{q}_\eta \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\sigma} 2 \tilde{q}_\eta
\]

\[
+ \frac{s^\xi}{\rho a} \tilde{q}_\eta \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\sigma} + \frac{s^\lambda}{\rho a} \tilde{q}_\eta \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\sigma} 2 \frac{s^a}{\rho a} \tilde{q}_\eta \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\sigma} \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\sigma} - 2 \frac{p_{\xi\eta\lambda}}{\tilde{q}_\lambda}
\]

with

\[
B_\eta^\xi = s_{\alpha,\lambda} (a_\eta \frac{p_{\xi\eta}}{\rho a} - \frac{2 s^\rho s^a}{\rho a} \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\rho} - 2 \tilde{q}_\eta \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\sigma})
\]

\[
(A.6)
\]

Inserting the covariant divergence of \( \tilde{q}_\eta \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\sigma} \) from Eq. (9) into Eq. (A.6) all terms but one cancel

\[
B_\eta^\xi = 2 s_{\alpha,\lambda} (a_\eta \frac{p_{\xi\eta}}{\rho a} - \frac{2 s^\rho s^a}{\rho a} \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\rho} - 2 \tilde{q}_\eta \frac{\partial \tilde{H}_0}{\partial \tilde{q}_\sigma}) \]

Yet, this term vanishes as the product of the torsion tensors is symmetric in \( \beta \) and \( \lambda \), which completes the proof of the consistency equation (11).

**Appendix B. Derivation of the invariant action of the U(1)×Diff(M) symmetry group**

The sets of local coordinates referring to two coordinate charts of the spacetime manifold \( M \) are denoted by \( x \) and \( X \). The extension of the U(1) symmetry from Sect. 3.1 to the additional symmetry under the diffeomorphism group Diff(M) is derived from the following extended generating function of type \( \tilde{F}_{\tilde{Z}}^\mu (\phi, \bar{\Pi}, x, \bar{\Pi}, \alpha, \lambda, \eta, P, g, \bar{K}, y, \bar{Q}, X) \):

\[
\tilde{F}_{\tilde{Z}}^\mu = \left[ \bar{\Pi}^\alpha (x) \phi (x) e^{iA(x)} + \phi^\alpha (x) \bar{\Pi}^{\alpha} (x) e^{-iA(x)} \right.
\]

\[
+ P^\alpha (x) \left( a_\xi (x) + \frac{1}{q} \frac{\partial A(x)}{\partial x^\xi} \right) \frac{\partial x^\xi}{\partial X^\eta}
\]

\[
+ K^{\alpha \beta} (x) g_{\xi \xi} (x) \frac{\partial x^\xi}{\partial X^\eta} \frac{\partial x^\xi}{\partial X^\xi}
\]

\[
+ Q^\alpha_\eta (x) \left( \gamma^\alpha_{\tau \sigma} (x) \frac{\partial X^\eta}{\partial x^\xi} \frac{\partial x^\eta}{\partial x^\xi} + \frac{\partial X^\eta}{\partial x^\xi} \frac{\partial^2 x^\xi}{\partial X^\eta \partial X^\xi} \right) \frac{\partial x^\xi}{\partial X^\eta} \frac{\partial x^\xi}{\partial X^\xi}^{-1}
\]

\[
(B.1)
\]
According to the general rules for extended canonical transformations \cite{1}, the particular rules from the generating function (B.1) follow for the scalar fields and their conjugates as

\begin{equation}
\delta^\mu_\nu \Phi(x) = \frac{\partial \tilde{F}_2}{\partial \Gamma^{x\nu}} \frac{\partial X^\mu}{\partial x^c} \left| \frac{\partial x^c}{\partial X^\lambda} \right| = \delta^\mu_\nu \phi(x)e^{i\Lambda(x)} \tag{B.2}
\end{equation}

\begin{equation}
\delta^\mu_\nu \Phi^*(x) = \frac{\partial \tilde{F}_2}{\partial \Gamma^{x\nu}} \frac{\partial x^c}{\partial X^\lambda} \left| \frac{\partial x^c}{\partial X^\lambda} \right| = \delta^\mu_\nu \phi^*(x)e^{-i\Lambda(x)} \tag{B.3}
\end{equation}

\begin{equation}
\bar{\rho}^\mu(x) = \frac{\partial \tilde{F}_2}{\partial \rho} = \bar{\Gamma}(x) e^{-i\Lambda(x)} \frac{\partial x^c}{\partial X^\lambda} \left| \frac{\partial x^c}{\partial X^\lambda} \right|^{-1} \tag{B.4}
\end{equation}

\begin{equation}
\bar{\rho}^{\ast\mu}(x) = \frac{\partial \tilde{F}_2}{\partial \rho^*} = \bar{\Gamma}^*(x) e^{i\Lambda(x)} \frac{\partial x^c}{\partial X^\lambda} \left| \frac{\partial x^c}{\partial X^\lambda} \right|^{-1}. \tag{B.5}
\end{equation}

For the vector field \(a_\xi\), one encounters the \textit{inhomogeneous} rule

\begin{equation}
\delta^\mu_\nu a_\xi(x) = \frac{\partial \tilde{F}_2}{\partial \bar{\rho}^{\ast\nu}} \frac{\partial X^\mu}{\partial x^c} \left| \frac{\partial x^c}{\partial X^\lambda} \right| = \delta^\mu_\nu \left( a_\xi(x) + \frac{1}{q} \frac{\partial \Lambda(x)}{\partial x^c} \right) \frac{\partial x^c}{\partial X^\lambda} \tag{B.6}
\end{equation}

\begin{equation}
\bar{\rho}^{\ast\nu}(x) = \frac{\partial \tilde{F}_2}{\partial \bar{\rho}} = \bar{P}(x) a_\xi \frac{\partial x^c}{\partial X^\lambda} \frac{\partial x^c}{\partial X^\lambda} \left| \frac{\partial x^c}{\partial X^\lambda} \right|^{-1}. \tag{B.7}
\end{equation}

The rules for the metric \(g_{\mu\nu}\) and the connection \(\gamma^k_{\tau\sigma}\), in conjunction with their respective conjugates, \(k_{\tau\sigma}^{\ast\mu}\) and \(\bar{Q}_\eta\), are

\begin{equation}
\delta^\mu_\nu G_{\alpha\lambda} = \frac{\partial \tilde{F}_2}{\partial k_{\tau\sigma}^{\ast\mu}} \frac{\partial X^\mu}{\partial x^c} \left| \frac{\partial x^c}{\partial X^\lambda} \right| = g_{\xi\xi} \frac{\partial x^c}{\partial X^\lambda} \frac{\partial x^c}{\partial X^\lambda} \delta^\mu_\nu \tag{B.8}
\end{equation}

\begin{equation}
\bar{G}^{\xi\eta}_{\mu\lambda} = \frac{\partial \tilde{F}_2}{\partial \bar{Q}_\eta} = \bar{K}_{\xi\xi}^{\ast\mu} \frac{\partial x^c}{\partial X^\lambda} \frac{\partial x^c}{\partial X^\lambda} \left| \frac{\partial x^c}{\partial X^\lambda} \right|^{-1} \tag{B.9}
\end{equation}

\begin{equation}
\delta^\mu_\nu \Gamma^{\xi}_{\alpha\lambda} = \frac{\partial \tilde{F}_2}{\partial \bar{Q}_\eta} \frac{\partial X^\mu}{\partial x^c} \left| \frac{\partial x^c}{\partial X^\lambda} \right| \tag{B.10}
\end{equation}

\begin{equation}
\delta^\mu_\nu \Gamma^{\xi}_{\alpha\lambda} = \delta^\mu_\nu \left( \gamma^k_{\tau\sigma} \frac{\partial x^c}{\partial X^\lambda} \frac{\partial x^c}{\partial X^\lambda} + \frac{\partial X^\eta}{\partial x^c} \frac{\partial x^c}{\partial X^\lambda} \right), \tag{B.11}
\end{equation}

\begin{equation}
\delta^\mu_\nu \Gamma^{\xi}_{\eta\epsilon} = \delta^\mu_\nu \left( \bar{Q}_\eta a_{\xi\epsilon} \frac{\partial x^c}{\partial X^\lambda} \frac{\partial x^c}{\partial X^\lambda} \frac{\partial x^c}{\partial X^\lambda} \left| \frac{\partial x^c}{\partial X^\lambda} \right|^{-1} \right). \tag{B.12}
\end{equation}

Finally, the transformation rule for the Hamiltonian follows from the \textit{explicit} dependence of the generating function,

\begin{equation}
\delta \tilde{H}_{\left| \Lambda \right| X} = \left( \tilde{H}_{\left| \Lambda \right| X} + \frac{\partial \tilde{F}_2}{\partial \gamma_{\tau\sigma}^{\ast\mu}} \right) \left| \frac{\partial x^c}{\partial X^\lambda} \right|, \tag{B.13}
\end{equation}

Owing to Eq. (A3) of Ref. \cite{1},

\begin{equation}
\frac{\partial}{\partial x^\alpha} \left( \frac{\partial x^\alpha}{\partial X^\beta} \left| \frac{\partial x^c}{\partial X^\lambda} \right|^{-1} \right) \equiv 0,
\end{equation}
the divergence of the explicitly $x$-dependent coefficients of $\tilde{\mathcal{F}}_{\mu}^{\lambda}$ simplifies to
\[
\frac{\partial \tilde{\mathcal{F}}^{\lambda}_{\mu}}{\partial x^\beta}_{\text{expl}} = \left\{ \frac{\partial x}{\partial X^\alpha} \right\}^{-1} \left( \left( \tilde{\Pi}^{\alpha}_{\phi} e^{iA} - \phi^* \tilde{\Pi}^{\alpha} e^{-iA} \right) \frac{\partial \Lambda}{\partial X^\alpha} \right.
\]
\[\left. + \tilde{\mathcal{P}}_{\mu \beta} \left[ \left( \alpha_\Lambda + \frac{1}{q} \frac{\partial \Lambda}{\partial x^\xi} \right) \frac{\partial^2 x^\xi}{\partial X^\alpha \partial X^\beta} + \frac{1}{q} \frac{\partial^2 \Lambda}{\partial x^\xi \partial x^\beta} \frac{\partial x^\xi}{\partial x^\alpha} \right] \right] \frac{\partial \Lambda}{\partial X^\alpha} \right\}
\]
\[\left. + \tilde{\mathcal{K}}^{\alpha \beta} \xi_\Lambda \left( \frac{\partial^2 x^\xi}{\partial X^\alpha \partial X^\beta} \frac{\partial x^\xi}{\partial X^\alpha} + \frac{\partial^2 x^\xi}{\partial X^\beta \partial X^\alpha} \frac{\partial x^\xi}{\partial X^\beta} \right) \right\}
\]
\[\left. + \tilde{\mathcal{Q}}_{\mu \alpha} \left( \gamma_\tau^\alpha \tau_{\mu \beta} \left( \frac{\partial X^\alpha}{\partial X^\beta} \frac{\partial x^\tau}{\partial X^\alpha} \frac{\partial x^\tau}{\partial X^\beta} \right) + \frac{\partial}{\partial X^\beta} \left( \frac{\partial X^\alpha}{\partial X^\beta} \frac{\partial^2 x^\xi}{\partial X^\alpha \partial X^\beta} \right) \right) \right\} \frac{\partial \Lambda}{\partial X^\alpha} \right].
\]
(B.14)

All terms depending on the phase $\Lambda(x)$ can now be expressed in terms of the system’s fields according to the canonical transformation rules (B.2) to (B.6). The first line of Eq. (B.14) yields
\[
\left( \tilde{\Pi}^{\alpha}_{\phi} e^{iA} - \phi^* \tilde{\Pi}^{\alpha} e^{-iA} \right) \frac{\partial \Lambda}{\partial X^\alpha} \left\{ \frac{\partial x}{\partial X^\alpha} \right\}^{-1} \left( A_{\alpha} - \alpha_\Lambda \frac{\partial x^\alpha}{\partial X^\alpha} \right)
\]
\[= i q \left( \tilde{\Pi}^{\alpha}_{\phi} e^{iA} - \phi^* \tilde{\Pi}^{\alpha} e^{-iA} \right) \frac{\partial x}{\partial X^\alpha} \left\{ \frac{\partial x}{\partial X^\alpha} \right\}^{-1} \left( \tilde{\Lambda}^{\alpha}_{\phi} - \alpha_\Lambda \frac{\partial x^\alpha}{\partial X^\alpha} \right)
\]
(B.15)

Hence all $\Lambda$-dependent terms are eliminated and replaced in a symmetric way by the fields of the original and the transformed system. The second line of Eq. (B.14) is treated similarly by inserting Eqs. (B.6) and (B.7)
\[
\tilde{\mathcal{P}}_{\mu \beta} \left[ \left( \alpha_\Lambda + \frac{1}{q} \frac{\partial \Lambda}{\partial x^\xi} \right) \frac{\partial^2 x^\xi}{\partial X^\alpha \partial X^\beta} + \frac{1}{q} \frac{\partial^2 \Lambda}{\partial x^\xi \partial x^\beta} \frac{\partial x^\xi}{\partial x^\alpha} \right] \frac{\partial \Lambda}{\partial X^\alpha} \right\}
\]
\[\left. + \tilde{\mathcal{Q}}_{\mu \alpha} \left( \gamma_\tau^\alpha \tau_{\mu \beta} \left( \frac{\partial X^\alpha}{\partial X^\beta} \frac{\partial x^\tau}{\partial X^\alpha} \frac{\partial x^\tau}{\partial X^\beta} \right) + \frac{\partial}{\partial X^\beta} \left( \frac{\partial X^\alpha}{\partial X^\beta} \frac{\partial^2 x^\xi}{\partial X^\alpha \partial X^\beta} \right) \right) \right\} \frac{\partial \Lambda}{\partial X^\alpha} \right].
\]
(B.16)

The last line of Eq. (B.16) vanishes by virtue of the identity
\[
\frac{\partial}{\partial X^\beta} \left( \frac{\partial X^\alpha}{\partial x^\xi} \frac{\partial x^\xi}{\partial X^\alpha} \right) \equiv \frac{\partial}{\partial X^\beta} \left( \delta^\alpha_\beta \right) \equiv 0,
\]
and thus
\[
\frac{\partial X^\alpha}{\partial x^\xi} \frac{\partial^2 x^\xi}{\partial X^\alpha \partial X^\beta} = -\frac{\partial^2 X^\alpha}{\partial x^\xi \partial X^\alpha} \frac{\partial x^\xi}{\partial X^\beta} \frac{\partial x^\xi}{\partial X^\beta}.
\]
(B.17)

Hence, the coupling of the term $\tilde{\mathcal{P}}_{\mu \beta} a_\alpha$ to the dynamical spacetime geometry is exactly compensated due to the inhomogeneous transformation rule (B.6), which is the U(1) symmetry.
transformation rule for the gauge field $a_\mu$. This is confirmed by setting $\Lambda \equiv 0$ in Eq. (B.1) and in the subsequent transformation rules (B.2) to (B.14)—which amounts to eliminating the U(1) symmetry transformation. In this case, the connection coefficients pertaining to the spacetime manifold enter as gauge quantities, which converts the partial derivative of the gauge field $a_\mu$ into a covariant derivative and thus renders the action integral generally covariant.

For the case of a system with additional U(1) symmetry, hence $\Lambda \neq 0$, Eq. (B.14) yields inserting Eqs. (B.15) and (B.16)

$$\frac{\partial \hat{F}_2}{\partial x^\alpha} = i q \left( \hat{F}_2^{\alpha} \Phi - \Phi^* \hat{F}_2^{\alpha} \right) A_\alpha \left| \frac{\partial x^{-1}}{\partial X} \right| - i q (\hat{\pi}^\alpha \phi - \phi^* \hat{\pi}^\alpha) a_\alpha$$

$$+ \frac{1}{2} \rho^{\alpha \beta} \left( \frac{\partial A_\alpha}{\partial x^\beta} + \frac{\partial A_\beta}{\partial x^\alpha} \right) \left| \frac{\partial x^{-1}}{\partial X} \right| - \frac{1}{2} \rho^{\alpha \beta} \left( \frac{\partial a_\alpha}{\partial \theta^\beta} + \frac{\partial a_\beta}{\partial \theta^\alpha} \right)$$

$$+ \left( \tilde{K}^{\alpha \beta} G_{\xi \lambda} + \tilde{K}^{\lambda \alpha} g_{\xi \beta} \right) \partial_x^{\xi \lambda} \left( \frac{\partial X}{\partial x} \right) - \left( \tilde{K}^{\alpha \beta} g_{\xi \lambda} + \tilde{K}^{\lambda \alpha} g_{\xi \beta} \right) \gamma^{\xi \lambda}$$

$$+ \frac{1}{2} Q_\eta^{\alpha \beta} \left( \gamma^{\eta \alpha \xi} + \gamma^{\eta \alpha \xi} + \Gamma^{\eta \alpha \beta} \gamma^{\eta \lambda \kappa} - \gamma^{\eta \alpha \xi} \gamma^{\eta \lambda \kappa} - \gamma^{\eta \alpha \xi} \gamma^{\eta \lambda \kappa} \right).$$

The terms in Eq. (B.14) proportional to $\tilde{K}^{\alpha \beta}$ and $Q_\eta^{\alpha \beta}$ were already discussed in Ref. [1]. We observe that the divergence of the spacetime dependent coefficients of $\hat{F}_2^{\mu}$ in Eq. (B.1) is expressed symmetrically in the original and the transformed fields. The particular Hamiltonian

$$\hat{H}_3 = \hat{H}_0 + i q (\hat{\pi}^\alpha \phi - \phi^* \hat{\pi}^\alpha) a_\alpha + \frac{1}{2} \rho^{\alpha \beta} \left( \frac{\partial a_\alpha}{\partial x^\beta} + \frac{\partial a_\beta}{\partial x^\alpha} \right)$$

$$+ \left( \tilde{K}^{\alpha \beta} g_{\xi \lambda} + \tilde{K}^{\lambda \alpha} g_{\xi \beta} \right) \gamma^{\xi \lambda} = \gamma^{\xi \lambda}$$

is thus converted by Eq. (B.18) according to the general transformation rule (B.13) into a Hamiltonian $\hat{H}_3$ of the same form. Combining the initial action integral [1] with the “gauged” Hamiltonian (B.19), the form-invariant action integral follows as

$$S = \int \left[ \hat{\pi}^\alpha \left( \frac{\partial \phi}{\partial x^\alpha} - i q a_\alpha \phi \right) + \left( \frac{\partial \phi^*}{\partial x^\alpha} + i q \phi^* a_\alpha \right) \hat{\pi}^\alpha \right.$$

$$+ \frac{1}{2} \rho^{\alpha \beta} \left( \frac{\partial a_\alpha}{\partial x^\beta} + \frac{\partial a_\beta}{\partial x^\alpha} \right) + \tilde{K}^{\alpha \beta} \left( \frac{\partial \gamma^{\eta \alpha \beta}}{\partial x^\beta} - \gamma^{\eta \alpha \beta} \gamma^{\eta \lambda \kappa} - \gamma^{\eta \alpha \beta} \gamma^{\eta \lambda \kappa} \right)$$

$$+ \frac{1}{2} Q_\eta^{\alpha \beta} \left( \frac{\partial \gamma^{\eta \alpha \xi}}{\partial x^\beta} - \frac{\partial \gamma^{\eta \alpha \xi}}{\partial x^\beta} - \gamma^{\eta \alpha \xi} \gamma^{\eta \lambda \kappa} + \gamma^{\eta \alpha \xi} \gamma^{\eta \lambda \kappa} \right) - \hat{H}_0 \right] d^4 x.$$
the gauge covariant derivatives and hence implement the minimum coupling principle—which is not postulated here but emerges from the canonical transformation formalism. Furthermore, the vector field $a_\mu$ does not couple directly to the spacetime geometry as the respective term cancels in Eq. (B.16). The covariant derivative of $a_\mu$ in the generally covariant action integral is replaced due to the U(1) symmetry by the curl of $a_\mu$ in Eq. (B.20), which already has tensor property. The coupling thus occurs only via the related energy-momentum tensor and not via the connections coefficients. This statement also holds for the SU(N) symmetry group.

References

1. J. Struckmeier, J. Muench, D. Vasak, J. Kirsch, M. Hanauske and H. Stoecker, *Phys. Rev. D* **95** (June 2017) 124048, [arXiv:1704.07246](https://arxiv.org/abs/1704.07246).
2. J. Struckmeier, *J. Phys. G: Nucl. Phys.* **40** (2013) 015007.
3. J. Struckmeier, *Phys. Rev. D* **91** (2015) 085030.
4. J. Struckmeier, D. Vasak and H. Stoecker, *Astron. Nachr.* **336** (2015) 731.
5. T. De Donder, *Théorie Invariantive Du Calcul des Variations* (Gaulthier-Villars & Cie., Paris, 1930).
6. H. Weyl, *Ann. Math.* **36** (1935) 607.
7. J. Struckmeier and A. Redelbach, *Int. J. Mod. Phys. E* **17** (2008) 435, [arXiv:0811.0508](https://arxiv.org/abs/0811.0508).
8. J. Struckmeier and H. Reichau, General U(N) gauge transformations in the realm of covariant Hamiltonian field theory, in *Exciting Interdisciplinary Physics*, ed. W. Greiner, Proceedings of the “Symposium on Exciting Physics: Quarks and gluons/atomic nuclei/biological systems/networks”, Makutsi Safari Farm, South Africa, 13–20 November 2011, (Springer International Publishing Switzerland, 2013), p. 367. [arXiv:1205.5754](https://arxiv.org/abs/1205.5754).
9. H. Weyl, *Sitzungsberichte der Königlich-Preussischen Akademie der Wissenschaften* **1918** (1918) 465.
10. W. Fock, *Z. Phys.* **39** (1926) 226.
11. A. Koenigstein, J. Kirsch, H. Stoecker, J. Struckmeier, D. Vasak and M. Hanauske, *International Journal of Modern Physics E: Nuclear Physics* **25** (2016) 1642005.
12. J. Struckmeier, D. Vasak and H. Stoecker, Covariant Hamiltonian Representation of Noether’s Therorem and its Application to SU(N) Gauge Theories, in *New Horizons in Fundamental Physics*, eds. S. Schramm and M. SchaeferFLAS Interdisciplinary Science Series (Springer International Publishing Switzerland, 2017).
13. A. Palatini, *Rendiconti del Circolo Matematico di Palermo* **43** (1919) 203.
14. R. Utiyama, *Phys. Rev.* **101** (March 1956) 1597.
15. T. W. B. Kibble, *J. Math. Phys.* **2** (March 1961) 212.
16. D. W. Sciama, The analogy between charge and spin in general relativity, in *Recent Developments in General Relativity*, (Pergamon Press, Oxford; PWN, Warsaw, 1962), pp. 415–439. Festschrift for Infeld.
17. F. J. Belinfante, *Physica* **6** (1939) 887.
18. L. Rosenfeld, *Mem. Acad. Roy. Belgique, cl. sc.* **18** (1940) 1.
19. J. Struckmeier and W. Greiner, *Extended Lagrange and Hamilton Formalisms for Point Dynamics and Field Theory* (World Scientific, to be published).
20. E. Schrödinger, *Space-Time Structure* (Cambridge University Press, 2002).
21. P. von der Heyde, *Nuovo Cimento* **14** (1975) 250.
22. J. Nester, *Progress of Theoretical Physics Supplement No.172* (2008) 30.
23. C. W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation* (W. H. Freeman and Company, New York, 1973).
24. D. Hilbert, Nachr. Ges. Wiss. Göttingen, Math. Phys. Klasse 1915 (1915) 395.
25. R. Feynman, W. Morinigo and W. Wagner, Feynman Lectures On Gravitation (Frontiers in Physics) (Westview Press, Boulder, Colorado, 2002).
26. P. Ade et al., Astron. Astrophys. A13 (2016).
27. D. Benisty, E. Guendelman, D. Vasak, J. Struckmeier and H. Stoecker, arXiv:1809.10447 (2018).
28. F. W. Hehl, Rep. Math. Phys. 9 (1976) 55.
29. W. Greiner and J. Reinhardt, Field Quantization (Springer, Berlin, Heidelberg, 1996).
30. D. Vasak and J. Struckmeier, to be published (2019).
31. J. Struckmeier and D. Vasak, to be published (2019).
32. J. Walecka, Ann. Phys. (N.Y.) 83 (1974) 491.
33. B. Serot and J. Walecka, The Relativistic Many-body Problem (Advances in Nuclear Physics) (Springer, 1986).
34. J. Boguta and A. Bodmer, Nucl. Phys. A 292 (1977) 413.
35. J. Steinheimer, S. Schramm and H. Stoecker, Phys. Rev. C 84 (2011) 045208.
36. D. A. Coulter, R. J. Foley, C. D. Kilpatrick, M. R. Drout, A. L. Piro, B. J. Shappee, M. R. Siebert, J. D. Simon, N. Ulloa, D. Kasen, B. F. Madore, A. Murguia-Berrich, Y.-C. Pan, J. X. Prochaska, E. Ramirez-Ruiz, A. Rest and C. Rojas-Bravo, Science 358 (2017) 1556, http://science.sciencemag.org/content/358/6370/1556.full.pdf.
37. B. P. Abbott et al., Physical Review Letters 119 (2017) 161101.
38. B. Abbott, R. Abbott, T. Abbott, F. Acernese, K. Ackley, C. Adams, T. Adams, P. Addesso, R. Adhikari, V. Adya et al., The Astrophysical Journal Letters 848 (2017) L13.
39. B. Abbott et al., Astrophysical Journal Letters 848 (2017) L12.
40. Hanauske, J. Steinheimer, L. Bovard, A. Mukherjee, S. Schramm, K. Takami, J. Papenfort, N. Wechscherberger, L. Rezzolla and H. Stöcker, Journal of Physics: Conference Series 878 (2017) 012031.
41. L. Csernai, V. Magas and D. Wang, Phys. Rev. C 87 (2013) 034906.
42. L. Adamczyk, A. Lebedev, I. Kisiel, T. Todoroki, J. Chen, M. Nasim, K. Krueger, H. Spinka, G. Wang, F. Liu et al., Nature 548 (2017).
43. J. Plebanski and A. Krasiński, An Introduction to General Relativity and Cosmology (Cambridge University Press, 2006).