FINITE DETERMINATION CONJECTURE
FOR MATHER-JACOBIAN MINIMAL LOG DISCREPANCIES
AND ITS APPLICATIONS

SHIHOKO ISHII

Abstract. In this paper we study singularities in arbitrary characteristic. We propose Finite Determination Conjecture for Mather-Jacobian minimal log discrepancies in terms of jet schemes of a singularity. The conjecture is equivalent to the boundedness of the number of the blow-ups to obtain a prime divisor which computes the Mather-Jacobian minimal log discrepancy. We also show that this conjecture yields some basic properties of singularities; e.g., openness of Mather-Jacobian (log) canonical singularities, stability of these singularities under small deformations and lower semi-continuity of Mather-Jacobian minimal log discrepancies, which are already known in characteristic 0 and open for positive characteristic case. We show some evidences of the conjecture: for example, for non-degenerate hypersurface of any dimension in arbitrary characteristic and 2-dimensional singularities in characteristic not 2. We also give a bound of the number of the blow-ups to obtain a prime divisor which computes the Mather-Jacobian minimal log discrepancy.

1. Introduction

Studies of singularities with respect to “discrepancies” on a variety over a field of characteristic 0 are developed based on resolutions of singularities, generic smoothness (Strong Bertini Theorem) and vanishing theorems of cohomologies of Kodaira type. However these are not available for varieties over a field of positive characteristic. This is the reason why the study of singularities in positive characteristic case did not develop in the same direction.

On the other hand, a singularity in positive characteristic has been studied from a different view point; in terms of Frobenius map which is specific for positive characteristic case. Then, a surprising correspondence between singularities in characteristic zero with respect to “discrepancies” and singularities in positive characteristic with respect to Frobenius map started to be unveiled by N. Hara and K-i. Watanabe [4], and then many beautiful results in this direction are discovered by the contributions of many people. We do not cite all references about these results here, because it is not the main theme of this paper.

The standing point of this paper is apart from theirs. We will try to study singularities in positive characteristic in the same line as in the characteristic 0 case. More precisely, we study singularities in terms of “discrepancies” which are common for any characteristic. For this sake, lack of resolutions of the singularities, generic smoothness or vanishing theorems would cause problems. In order to avoid these problems, we propose to use jet schemes.

To explain the background, remember that there are two kinds of discrepancies at a prime divisor $E$ over a variety $X$:

---

Mathematical Subject Classification: 14B05, 14E18, 14B07
Key words: singularities in positive characteristic, jet schemes, minimal log discrepancy
The author is partially supported by Grant-In-Aid (c) 1605089 of JSPS in Japan.
• the usual log discrepancy \( a(E; X) = k_E + 1 \), where \( k_E \) is the coefficient of the relative canonical divisor at the prime divisor \( E \) over \( X \);
• Mather-Jacobian log discrepancy \( a_{MJ}(E; X) = \tilde{k}_E - j_E + 1 \), where \( \tilde{k}_E \) is the Mather discrepancy and \( j_E \) is the order of the Jacobian ideal of \( X \) at the prime divisor \( E \) over \( X \).

The usual log discrepancy is defined for a \( \mathbb{Q} \)-Gorenstein variety \( X \) and we say that \( X \) is log canonical (resp. canonical) at a point \( x \in X \) if for every prime divisor \( E \) over \( X \) with the center containing \( x \) satisfies \( a(E; X) \geq 0 \) (resp. \( a(E; X) \geq 1 \)). The minimal log discrepancy \( \text{mld}(x; X) \) is defined as the infimum of \( a(E; X) \) for every prime divisor with the center \( x \). This discrepancy plays an important role in the minimal model problem.

The second discrepancy is defined for a reduced equidimensional scheme \( X \) of finite type over the base field \( k \) and we say that \( X \) is Mather-Jacobian (MJ, for short)-log canonical (resp. MJ-canonical ) at a point \( x \in X \) if for every prime divisor \( E \) over \( X \) with the center containing \( x \) satisfies \( a_{MJ}(E; X) \geq 0 \) (resp. \( a_{MJ}(E; X) \geq 1 \)). The MJ-minimal log discrepancy \( \text{mld}_{MJ}(x; X) \) is defined as the infimum of \( a_{MJ}(E; X) \) for every prime divisor with the center \( x \).

When the base field \( k \) is of characteristic 0, the following natural properties hold:

(P1) Log canonicity, canonicity, MJ-log canonicity and MJ-canonicity are all open conditions. I.e., if \( (X, x) \) is one of these singularities, then there is an open neighborhood \( U \subset X \) of \( x \) such that \( U \) has singularities of the same type at every point.

(P2) Canonicity, MJ-log canonicity and MJ-canonicity are stable under a small deformation. (So is log canonicity if the total space is \( \mathbb{Q} \)-Gorenstein.)

(P3) The map \( X \to \mathbb{Z}; x \mapsto \text{mld}_{MJ}(x; X) \) is lower semi-continuous. (On the other hand, lower semi-continuity of \( \text{mld}(x; X) \) is not yet proved in general even in characteristic 0.)

Resolutions of singularities played essential roles in the proofs of (P1)–(P3) for characteristic 0 case. Therefore, none of them are proved in positive characteristic case in general. At present, we do not have a systematic way to prove them for the usual log canonical or canonical singularities. However, focusing on MJ-version, we propose a potentially effective way to prove them. This is based on the fact that MJ-singularities are well described in terms of local jet schemes at the singular point and do not need existence of a resolution of the singularities.

Actually in arbitrary characteristic, MJ-minimal log discrepancy of \( d \)-dimensional variety \( X \) at a point \( x \) is represented as:

\[
\text{mld}_{MJ}(x; X) = \inf_{m \in \mathbb{N}} \{(m + 1)d - \dim X_m(x)\},
\]

where \( X_m(x) \) is the local \( m \)-jet scheme of \( X \) at \( x \) (this is proved in [1] and [7] for characteristic 0 and in [9] for arbitrary characteristic). Let

\[
s_m(X, x) := (m + 1)d - \dim X_m(x).
\]

We also have the formula of \( \text{mld}_{MJ}(x; X) \) as follows ([1] and [7] for characteristic 0 case, and [9] for arbitrary characteristic case):

Let \( (X, x) \subset (A, x) \) be a closed immersion into a non-singular variety \( A \) with the codimension \( c \) and \( I_X \) the ideal of \( X \) in \( A \), then

\[
\text{mld}_{MJ}(x; X) = \text{mld}(x; A, I_X^c).
\]

In this paper we may think that this formula is the definition of \( \text{mld}_{MJ}(x; X) \).

We pose the following conjecture for every \( d \in \mathbb{N} \):
Conjecture 1.1 (C\text{d}). Let \(d\) be a positive integer. There exists \(N_d \in \mathbb{N}\) depending only on \(d\) such that for every closed point \(x \in X\) of any \(d\)-dimensional variety \(X\), there exists \(m \leq N_d\) satisfying either
\[
\begin{align*}
\begin{cases} 
  s_m(X,x) = \text{mld}_{\text{MJ}}(x;X) \geq 0, \\
  s_m(X,x) < 0, \text{ when } \text{mld}_{\text{MJ}}(x;X) = -\infty 
\end{cases}
\end{align*}
\]

This conjecture can be split into the following conjectures. Let \(\delta\) be an integer with \(\delta \leq d\). Note that \(\delta\) can be negative.

Conjecture 1.2 (C\text{d,\delta}). Let \(d\) be a positive integer and \(\delta\) an integer such that \(\delta \leq d\).

There exists \(N_{d,\delta} \in \mathbb{N}\) depending only on \(d\) and \(\delta\) such that for every closed point \(x \in X\) of any \(d\)-dimensional variety \(X\), if \(\text{mld}_{\text{MJ}}(x;X) < \delta\), then there exists \(m \leq N_{d,\delta}\) with the property \(s_m(X,x) < \delta\).

In this paper we prove the following:

Theorem 1.3. If Conjecture \(C_d\) holds (equivalently, Conjectures \(C_{d,\delta}\) hold for \(\delta \leq d\)), then the following hold:

(PMJ1) Let \(X\) be a \(d\)-dimensional variety. If \((X, x)\) is an MJ-log canonical (resp. MJ-canonical) singularity, then there is an open neighborhood \(U \subset X\) of \(x\) such that \(U\) has MJ-log canonical (resp. MJ-canonical) singularity at every point of \(U\).

(PMJ2) Let \(X \to \Delta\) be a surjective morphism to a smooth curve \(\Delta\) with the equidimensional reduced fibers of dimension \(d\). Denote the fiber of this morphism of a point \(t \in \Delta\) by \(X_t\). If \((X, x) = (X_0, x)\) is MJ-log canonical (resp. MJ-canonical), then there is an open neighborhoods \(\Delta' \subset \Delta\) and \(U \subset X\) of 0 and \(x\), respectively, such that all fibers of \(U \to \Delta'\) have MJ-log canonical (resp. MJ-canonical) singularities.

(PMJ3) For a \(d\)-dimensional variety \(X\), the map
\[
\{ \text{closed points of } X \} \to \mathbb{Z}; \ x \mapsto \text{mld}_{\text{MJ}}(x;X)
\]
is lower semi-continuous.

When we think of Minimal Model Problem over positive characteristic base field, we have to study singularities in the view point of “usual log discrepancy”. Singularities with respect to MJ-log discrepancy is different from the usual one, but useful also for “usual one”. One reason is the fact that for singularities of locally a complete intersection “usual log discrepancy” coincides with MJ-log discrepancy. So the study of singularities with respect to MJ-log discrepancy is the first step to study singularities with respect to the usual discrepancy over positive characteristic base field.

One good example is Sato-Takagi’s result ([14]) that for a quasi-projective 3-fold \(X\) with canonical singularities over an algebraically closed field of characteristic \(p > 0\), a general hyperplane section of \(X\) has also canonical singularities. They proved this by making use of a result about MJ-canonical singularities proved in [9].

Here, we propose a conjecture from another view point. A similar problem for mld in characteristic zero is considered in [12].

Conjecture 1.4 (D\text{d}). For an integer \(d \geq 1\), there exists \(M_d \in \mathbb{N}\) depending only on \(d\) such that for any \(d\)-dimensional variety \(X\) and a closed point \(x \in X\) with a closed immersion \(X \subset A\) around \(x\) into a non-singular variety \(A\) of dimension
$N \leq 2d$ there exists a prime divisor $E$ over $A$ with the center at $x$ and $k_E \leq M_d$ such that
\[
\begin{cases}
    a(E; A, I_X^d) = \operatorname{mld}(x; A, I_X^d) = \operatorname{mld}_{MJ}(x; X) \geq 0, \\
    a(E; A, I_X^d) < 0 \text{ if } \operatorname{mld}(x; A, I_X^d) = \operatorname{mld}_{MJ}(x; X) = -\infty.
\end{cases}
\]
Here, $c = N - d$.

Note that this conjecture is not yet proved in general even for characteristic zero (this is viewed as a special case of Conjecture 1.1 in [12]).

This conjecture claims the boundedness of necessary number of blow-ups to obtain a prime divisor computing the MJ-minimal log discrepancy.

In order to explain “necessary number of blow-ups”, we quote the basic theorem founded by Zariski (see, for example [11, VI.1, 1.3]).

**Proposition-Definition 1.5.** Let $X$ be an irreducible variety and $E$ a prime divisor over $X$. Then there is a sequence of blow-ups

\[ X^{(n)} \xrightarrow{\varphi_0} X^{(n-1)} \rightarrow \cdots \rightarrow X^{(1)} \xrightarrow{\varphi_1} X^{(0)} = X \]

such that

1. $E$ appears on $X^{(n)}$, i.e., the center of $E$ on $X^{(n)}$ is of codimension 1 and $X^{(n)}$ is normal at the generic point $p_n$ of $E$,
2. $\varphi_i(p_i) = p_{i-1}$ for $1 \leq i \leq n$, and
3. $\varphi_i$ is the blow-up with the center $\{p_{i-1}\}$.

The minimal such number $n$ is denoted by $\tilde{b}(E)$ and the minimal number $i$ such that $\operatorname{codim}(p_i) = 1$ is denoted by $b(E)$.

Note that $\tilde{b}(E) \geq b(E)$ in general and the equality holds if $X$ is non-singular.

**Conjecture 1.6 (U$_d$).** For every $d \in \mathbb{N}$ there is an integer $B_d \in \mathbb{N}$ depending only on $d$ such that for every singularity $(X, x)$ of dimension $d$ embedded into a non-singular variety $A$ with $\dim A = \operatorname{emb}(X, x)$, there is a prime divisor $E$ over $A$ computing $\operatorname{mld}_{MJ}(x; X)$ and satisfying $\tilde{b}(E) = b(E) \leq B_d$. Here, $\operatorname{emb}(X, x)$ is the embedding dimension of $X$ at $x$.

**Proposition 1.7.** Conjecture C$_d$, Conjecture D$_d$ and Conjecture U$_d$ are equivalent.

As evidences for Conjecture C$_d$, we obtain the following:

**Proposition 1.8.** Conjectures C$_{d,d}$ and C$_{d,d-1}$ holds for every $d \geq 1$ and $N_{d,d} = 1$ and $N_{d,d-1} = 5$.

In Section 5 we define “a singularity of maximal type” and show the following:

**Proposition 1.9.** Assume that Conjecture C$_{d-1}$ holds. Then, Conjecture C$_d$ holds in the class $\operatorname{Max}_d := \{(X, x) \mid \dim X = d, (X, x) \text{ is of maximal type}\}$.

This proposition is used in the proof of the conjecture for 2-dimensional singularities.

**Theorem 1.10.** Conjecture C$_d$ holds for every $d \geq 1$ in the category of non-degenerate hypersurfaces in arbitrary characteristic.

**Theorem 1.11.** Conjecture C$_1$ holds and we can take $N_1 = 5$, $M_1 = 4$ and $B_1 = 3$ in arbitrary characteristic.

**Theorem 1.12.** If the characteristic of the base field $k$ is not 2, then Conjecture C$_2$ holds and we can take $N_2 \leq 41$ and $M_2 \leq 58$. And the bound of the necessary number of blow-ups is $B_2 \leq 39$. (Note that these numbers may not be optimal.)

As a corollary of the theorem, we obtain the following statement about “usual minimal log discrepancies”:
Corollary 1.13. Assume the characteristic of the base field $k$ is not 2. Then, Conjecture $C_2$ holds in the category of normal locally complete intersection singularities of dimension 2 over $k$. In particular, for every normal locally complete intersection singularity $(X, x)$ of dimension 2 there is a prime divisor $E$ over $X$ computing $\mld(x; X) = \mld_{MJ}(x; X)$ such that $b(E) \leq 20$.

The paper is organized as follows: In Section 2, we introduce Mather-Jacobian log discrepancies and the notion of jet schemes and then show some basic properties which will be used in this paper. In Section 3 we give proofs of the equivalences of the conjectures. In Section 4 we show that the conjecture $(C_d)$ yields the properties (PMJ1)–(PMJ3). In Section 5 we give a proof of $(C_d)$ for non-degenerate hypersurfaces and for singularities of “maximal type” in arbitrary characteristic. In Section 6 we give a proof of $(C_1)$ in arbitrary characteristic and $(C_2)$ in characteristic $\neq 2$. For the proof of $(C_2)$ we make use of the results of Section 5.

Acknowledgement. The author would like to thank Lawrence Ein for his helpful comments and encouragements in the discussions during his stay in the University of Tokyo. The author is grateful to Kei-ichi Watanabe for his warm encouragements while the author was working on this topic. She also thanks Kohsuke Shibata for useful discussions and János Kollár for useful comments.

2. Preliminaries on Mather-Jacobian log discrepancies and jet schemes

Throughout this paper, a variety means always an equidimensional reduced connected scheme of finite type over an algebraically closed field $k$. The characteristic of $k$ is arbitrary unless otherwise stated. As our discussions are local, we take a variety $X$ as an affine variety and denote its dimension by $d$. We use the symbol $x$ for a closed point of a scheme and $\eta$ for a not-necessarily closed point of a scheme.

Mather-Jacobian log discrepancy is defined in [1] and [7] independently, for a variety over a field of characteristic zero and is easily generalized to positive characteristic case (see, for example, [9]).

Definition 2.1. We say that $E$ is a prime divisor over $X$, if there is a birational morphism $Y \to X$ such that $Y$ is normal and $E$ is a divisor on $Y$. A prime divisor $E$ over $X$ is called an exceptional prime divisor over $X$, if the morphism $Y \to X$ is not isomorphic at the generic point of $E$.

Definition 2.2. Let $E$ be a prime divisor over an arbitrary variety $X$. The Mather-Jacobian (MJ, for short) log discrepancy of $X$ at $E$ is defined by

$$a_{MJ}(E; X) = \hat{k}_E - j_E + 1,$$

where $\hat{k}_E$ is the order of vanishing of the relative Jacobian ideal $J_\varphi = \operatorname{Fitt}^i(\Omega_{Y/X})$ at $E$ for a partial resolution $\varphi : Y \to X$ on which $E$ appears. The other term $j_E$ is the order of vanishing of the Jacobian ideal $J_X$ of $X$ at $E$.

Here, we note that MJ-log discrepancy is defined on every variety, while the usual log discrepancy $a(E; X)$ is defined only for normal $\mathbb{Q}$-Gorenstein variety. We also note that for locally a complete intersection $X$ we have the coincidence $a_{MJ}(E; X) = a(E; X)$. Detailed discussions for MJ-log discrepancies can be seen in [3] and [9].

Definition 2.3. For a (not necessarily closed) point $\eta \in X$ and for a proper closed subset $W$, we define the minimal MJ-log discrepancy at $\eta$ as follows:
(1) When \( \dim X \geq 2 \),
\[
\mld_{MJ}(\eta; X) = \inf \{ a_{MJ}(E; X) \mid E : \text{prime divisor with center } \overline{\eta} \},
\]
\[
\mld_{MJ}(W; X) = \inf \{ a_{MJ}(E; X) \mid E : \text{prime divisor with center in } W \}.
\]

(2) When \( \dim X = 1 \), define \( \mld_{MJ}(\eta; X) \) and \( \mld_{MJ}(W; X) \) by the same definitions as above if the right hand sides of the above definitions are non-negative and otherwise define \( \mld_{MJ}(\eta; X) = -\infty \) and \( \mld_{MJ}(W; X) = -\infty \), respectively.

Here, we emphasize that “with center \( \{\eta\} \)” means that the center coincides with \( \{\eta\} \), and different from “with center ‘in’ \( \{\eta\} \).”

The following is well known (see, for example, [9]):

**Proposition 2.4.** The inequality
\[
\mld_{MJ}(\eta; X) \leq \operatorname{codim}(\overline{\eta}, X)
\]
holds for a point \( \eta \in X \), and the equality holds if and only if \((X, \eta)\) is non-singular. In particular, if \( x \in X \) is a closed point, the inequality
\[
\mld_{MJ}(x; X) \leq \dim X
\]
holds, and the equality holds if and only if \((X, x)\) is non-singular.

**Definition 2.5.** A variety \( X \) is called \( MJ \)-canonical (resp. \( MJ \)-log canonical) at a (not necessarily closed) point \( \eta \in X \), if
\[
a_{MJ}(E; X) \geq 1 \quad (\text{resp.} \geq 0)
\]
for every exceptional prime divisor \( E \) over \( X \) whose center on \( X \) contains \( \eta \).

**Proposition 2.6.**
1. A variety \( X \) is \( MJ \)-log canonical at a point \( \eta \) if and only if \( \mld_{MJ}(\eta; X) \geq 0 \).
2. If a variety \( X \) is \( MJ \)-canonical at \( \eta \), then \( \mld_{MJ}(\eta; X) \geq 1 \) holds by definition. But the converse does not hold in general.

The \( MJ \)-version of the singularities has a good description in terms of jet schemes. Here, we introduce jet schemes. The precise descriptions about jet schemes and the arc spaces are found, for example in [2], [6].

**Definition 2.7.** Let \( X \) be a variety and \( K \supset k \) a field extension. For \( m \in \mathbb{Z}_{\geq 0} \), a \( k \)-morphism \( \text{Spec } K[[t]]/(t^{m+1}) \to X \) is called an \( m \)-jet of \( X \) and \( k \)-morphism \( \text{Spec } K[[t]] \to X \) is called an arc of \( X \).

Let \( X_m \) be the space of \( m \)-jets or the \( m \)-jet scheme of \( X \). There exists the projective limit
\[
X_{\infty} := \lim_m X_m
\]
and it is called the space of arcs or the arc space of \( X \).

**Definition 2.8.** Denote the canonical truncation morphisms induced from \( k[[t]] \to k[t]/(t^{m+1}) \) and \( k[t]/(t^{m+1}) \to k \) by \( \psi_m : X_{\infty} \to X_m \) and \( \pi_m : X_m \to X \), respectively. In particular we denote the morphism \( \psi_0 = \pi_{\infty} : X_{\infty} \to X \) by \( \pi \). We also denote the canonical truncation morphism \( X_{m'} \to X_m \) \((m' > m)\) induced from \( k[t]/(t^{m'+1}) \to k[t]/(t^{m+1}) \) by \( \psi_{m', m} \). To specify the space \( X \), we sometimes write \( \psi_{m', m}^X \).

For a point \( \eta \in X \), the fiber scheme \( \pi_{m}^{-1}(\eta) \) is denoted by \( X_m(\eta) \) and the dimension \( \dim X_m(\eta) \) is defined by \( \dim \pi_m^{-1}(\eta) \). Note that it is different from \( \dim \pi_m^{-1}(\{\eta\}) \).
We define the subsets “contact loci” in the arc space as follows: let $\gamma^*: \phi_{X,0} \to K[[t]]$ be the corresponding ring homomorphism of $\gamma$, where $0 \in \text{Spec } K[[t]]$ is the closed point. Then, we define $\text{ord}_\gamma(a) = \sup\{r \in \mathbb{Z}_{\geq 0} \mid \gamma^*(a) \subset (t^r)\}$.

We define the subsets “contact loci” in the arc space as follows:

\[ \text{Cont}^m(a) = \{ \gamma \in X_\infty \mid \text{ord}_\gamma(a) = m \} \]

In a similar manner, we define

\[ \text{Cont}^m(a) = \{ \gamma \in X_\infty \mid \text{ord}_\gamma(a) \geq m \} \]

By this definition, we can see that

\[ \text{Cont}^m(a) = \psi_{m-1}^{-1}(Z(a)_{m-1}), \]

where $Z(a)$ is the closed subscheme defined by the ideal $a$ in $X$.

The following fact was already used in [3], [8], [9] in the proofs of some statements. Here, we give the proof for the reader’s convenience as we will use it several times in this paper. About the basic terminologies appear in the proof, we refer to [6].

**Proposition 2.10.** Denote $\mathbb{A}_m^N = \text{Spec } k[x_1, \ldots, x_N]$ just by $\mathbb{A}^N$ in order to avoid confusion with the “$k$-jet scheme”. Let $X \subset \mathbb{A}^N$ be defined by an ideal $I \subset k[x_1, \ldots, x_N]$ and contain the origin $0$ and let $Z$ be defined by the ideal generated by the $m$-truncations of the elements of $I$. Here the $m$-truncation of a polynomial $f = \sum_{i=0}^r f_i$ ( $f_i$ is homogeneous of degree $i$) is $\sum_{i=0}^m f_i$. Then for every $j \leq m$, the local $j$-jet schemes coincide

\[ X_j(0) = Z_j(0). \]

In particular, if for every element $f \in I_X$ has order greater than $m$, then for every $j \leq m$,

\[ X_j(0) = \mathbb{A}_m^N(0). \]

**Proof.** The $m$-jet scheme $\mathbb{A}_m^N$ of the affine space $\mathbb{A}^N$ is given as

\[ \mathbb{A}_m^N = \text{Spec } k \left[ x^{(j)}_i \mid i = 1, \ldots, N, j = 0, 1, \ldots, m \right] \simeq \mathbb{A}^{N(m+1)}. \]

Here, the closed point $P \in \mathbb{A}_m^N$ with the coordinates $(a_i^{(j)})_{i=1, \ldots, N, j=0, 1, \ldots, m}$ corresponds to the $m$-jet Spec $k[t]/(t^{m+1}) \to \mathbb{A}^N$ whose associated $k$-algebra homomorphism is

\[ k[x_1, \ldots, x_N] \to k[t]/(t^{m+1}), \quad x_i \mapsto \sum_{j=0}^m a_i^{(j)} t^j. \]

Let $I \subset k[x_1, \ldots, x_N]$ be the defining ideal of $X$ in $\mathbb{A}^N$. Then the defining ideal $I_m \subset k \left[ x^{(j)}_i \mid i = 1, \ldots, N, j = 0, 1, \ldots, m \right]$ of $X_m$ in $\mathbb{A}_m^N$ is generated by

\[ \left\{ f^{(j)} \mid j = 0, 1, \ldots, m, f \in I \right\}, \]

where $f^{(j)} \in k \left[ x^{(j)}_i \mid i = 1, \ldots, N, l = 0, 1, \ldots, m \right]$ is defined from $f \in I$ as follows:

\[ f \left( \sum_{j=0}^m x^{(j)}_i t^j, \ldots, \sum_{j=0}^m x^{(j)}_N t^j \right) \equiv \sum_{j=0}^m f^{(j)} t^j \pmod{t^{m+1}}. \]

Here, we define the weight of the variable $x^{(j)}_i$ by $j$ and the weight of a monomial by the sum of the weights of variables appearing in the monomial. Then the polynomial $f^{(j)}$ is homogeneous of weight $j$. 

\[ f \left( \sum_{j=0}^m x^{(j)}_i t^j, \ldots, \sum_{j=0}^m x^{(j)}_N t^j \right) \equiv \sum_{j=0}^m f^{(j)} t^j \pmod{t^{m+1}}. \]
We note that \( f^{(j)} \) may have another homogeneity. If \( f \) is homogeneous with respect to the usual degree, then \( f^{(j)} \) is also homogeneous with respect to the usual degree. For a general \( f \), let \( f = f_0 + f_1 + \cdots + f_r \) be the homogeneous decomposition with respect to the degree and \( f_i \) is the homogeneous part of degree \( i \). Then
\[
 f^{(j)} = f_0^{(j)} + f_1^{(j)} + \cdots + f_r^{(j)},
\]
where \( f_i^{(j)} \) is defined in the same way as in (1) for the homogeneous polynomial \( f_i \).

Now we consider the fibers \( \mathbb{A}^N_m(0) \) and \( X_m(0) \) by \( \pi_m^N \) and \( \pi_m^X \), respectively. These are defined by \( x_i^{(0)} = 0 \) for all \( i = 1, \ldots, N \), in \( \mathbb{A}^N_m \) and in \( X_m \), respectively.

Therefore, \( X_m(0) \) is defined by \( \{ f^{(j)} \mid j = 0, 1, \ldots, m, \ f \in I \} \) in \( \mathbb{A}^N_m \) and in \( X_m \), respectively.

where \( f^{(j)} \) is the polynomial substituted \( x_i^{(0)} = 0 \) \((i = 1, \ldots, N)\) into \( f^{(j)} \). Under the notation in (2), we note that
\[
 f_i^{(j)} = 0 \quad \text{if} \quad i > j.
\]
This is because the monomials of \( f_i^{(j)} \) are homogeneous of degree \( i \) of the form \( x_1^{(j_1)} \cdots x_r^{(j_r)} \) with \( j_1 + j_2 + \cdots + j_r = j \). Here, by \( i > j \), there is some \( l, (1 \leq l \leq i) \) such that \( j_l = 0 \). This yields that every monomial of \( f_i^{(j)} \) contains \( x_l^{(0)} \) as a factor for some \( s_l \), therefore by the substitution \( x_l^{(0)} = 0 \), the monomial becomes 0. Thus it follows (4).

Since \( X_m(0) \) is defined by \( \{ f_i^{(j)} \mid j \leq m, \ f \in I \} \), the vanishing (3) implies that the homogeneous parts \( f_i \) \((i > m)\) do not affect the defining ideal of \( X_m(0) \). Therefore \( X_m(0) = Z_m(0) \).

\[ \text{Theorem 2.11} \quad (\text{[1], [7], [9]}). \quad \text{Let} \ X \text{ be a variety over an algebraically closed field} \ k \text{ of an arbitrary characteristic and} \ A \text{ a smooth variety containing} \ X \text{ as a closed subscheme of codimension} \ c. \text{Denote the ideal of} \ X \text{ in} \ A \text{ by} \ I_X. \text{Then, for a proper closed subset} \ W \text{ of} \ X, \text{we have}
\]
\[
 (4) \quad \text{mld}_{\mathbb{A}^N}(W; X) = \text{mld}(W; A, I_X).
\]

For a point \( \eta \in X \), we have
\[
 (5) \quad \text{mld}_{\mathbb{A}^N}(\eta; X) = \text{mld}(\eta; A, I_X).
\]

By the above theorem we may think that the right hand sides of (4) and (5) are the definitions of \( \text{mld}_{\mathbb{A}^N}(W; X) \) and \( \text{mld}_{\mathbb{A}^N}(\eta; X) \), respectively.

\[ \text{Proposition 2.12} \quad (\text{[1], [7], [9]}). \quad \text{Let} \ X \text{ be a variety of dimension} \ d \text{ embedded into a non-singular variety} \ A \text{ with codimension} \ c \text{ and let} \ \eta \in X \text{ be a point, then we have}
\]
\[
 \text{mld}_{\mathbb{A}^N}(\eta; X) = \inf_{m \in \mathbb{N}} \left\{ \text{codim} \left( \text{Cont}^{\geq m+1}(I_X) \cap (\pi^A)^{-1}(\eta), A_\infty \right) - c \cdot (m + 1) \right\}
\]
\[
 = \inf_{m \in \mathbb{N}} \left\{ (m + 1)d - \dim X_m(\eta) \right\},
\]
where \( \dim X_m(\eta) = \dim X_m(\eta) \). Note that it does not coincide with \( \dim X_m(\eta) \) in general.

\[ \text{Definition 2.13}. \ \text{Under the assumption of the previous proposition, for a point} \ \eta \in X \text{ we define the function} \ s_m(X, \eta) \text{ in} \ m \text{ as follows:}
\]
\[
 s_m(X, \eta) = (m + 1)d - \dim X_m(\eta).
\]

\[ \text{Definition 2.14.} \quad \text{For a variety} \ X, \ \text{fix a closed immersion} \ X \subset A \text{ into a non-singular variety} \ A.\]
We say that a prime divisor $E$ over $A$ with the center $\{\eta\}$ computes $\text{mld}_{MJ}(\eta; X)$ if either
$$a(E; A, I_X^\eta) = \text{mld}(x; A, I_X^\eta) = \text{mld}_{MJ}(\eta; X) \geq 0,$$
or
$$a(E; A, I_X^\eta) < 0, \text{ when } \text{mld}_{MJ}(\eta; X) = -\infty.$$ 

2. We say that $s_m(X, \eta)$ computes $\text{mld}_{MJ}(\eta; X)$ if either
$$s_m(X, \eta) = \text{mld}_{MJ}(\eta; X) \geq 0,$$
or
$$s_m(X, \eta) < 0, \text{ when } \text{mld}_{MJ}(\eta; X) = -\infty.$$ 

3. THE CONJECTURES AND THEIR RELATIONS

In this section we prove the equivalence of Conjecture $C_d$ with the other conjectures.

**Definition 3.1.** Let $X$ be a $d$-dimensional variety and $\eta \in X$ a point. Let $A$ be a non-singular variety containing $X$ as a closed subscheme. We define the following invariants:
$$\nu(X, \eta) := \min\{r \mid s_{r-1}(X, \eta) \text{ computes } \text{mld}_{MJ}(\eta, X)\}.$$ 

Then, in terms of $\nu$, Conjecture $C_d$ (Conjecture [1]) is represented as follows:

**Conjecture 3.2 ($C_d$).** For an integer $d \geq 1$, there exists $N_d > 0$ depending only on $d$ such that the bound $\nu(X, x) - 1 \leq N_d$ holds for any $d$-dimensional variety $X$ and a closed point $x \in X$.

First we pose another conjecture which seems stronger than ($C_d$). Actually Conjecture $C_d$ is the statement for closed points, while the following conjecture is for any points in $d$-dimensional varieties.

**Conjecture 3.3 ($C_d$).** For an integer $d \geq 1$, there exists $N_d > 0$ depending only on $d$ such that the bound $\nu(X, \eta) - 1 \leq N_d$ holds for any $d$-dimensional variety $X$ and a point $\eta \in X$.

**Proposition 3.4.** For an integer $d \geq 1$, Conjecture $C_d$ and Conjecture $\tilde{C}_d$ are equivalent.

**Proof.** The implication $\tilde{C}_d \Rightarrow C_d$ is obvious. To show the converse implication, take any non-closed point $\eta \in X$. Let $N_d$ be as in Conjecture $C_d$. Let $\nu = \nu(X, \eta)$, and we will show that $\nu - 1 \leq N_d$. Note that for every $m \in \mathbb{N}$, there exists an open dense subset $U_m \subset \{\eta\}$ such that
$$\dim X_m(\eta) = \dim X_m(x) + \dim \{\eta\}$$
holds for every closed point $x \in U_m$. Then for every closed point $x \in U_m$ we have
$$s_m(X, \eta) = d(m + 1) - \dim X_m(\eta) = s_m(X, x) - \dim \{\eta\}.$$ 

Then, for a closed point
$$x \in U_\nu \cap \left(\bigcap_{n \leq N_d} U_n\right),$$
we obtain
$$\text{mld}_{MJ}(\eta, X) = s_{\nu-1}(X, \eta) = s_{\nu-1}(X, x) - \dim \{\eta\} \geq s_{\nu(X, x)-1}(X, x) - \dim \{\eta\} = s_{\nu(X, \eta)-1}(X, \eta) \geq \text{mld}_{MJ}(\eta, X).$$ 

Therefore all inequalities in (5) become equalities. By the minimality of $\nu = \nu(X, \eta)$, we obtain that $\nu - 1 \leq \nu(X, x) - 1 \leq N_d$. \qed
By this proposition, we reduce the problem in \( (\tilde{C}_d) \) into the problem on closed points. Henceforth, we will consider the conjectures only for closed points.

**Lemma 3.5.** Let \( X \) be a variety and \( x \in X \) a closed point. We assume that \( X \) is embedded into a non-singular variety \( A \) with codimension \( c \). Then, we have

\[
\nu(X, x) = \min \left\{ m \in \mathbb{N} \mid k_E + 1 - cm = \mld_{MJ}(x; X) \text{ for a prime divisor } E \text{ over } A \right\},
\]

where in case \( \mld_{MJ}(x; X) = -\infty \), the condition \( k_E + 1 - cm = \mld_{MJ}(x; X) \) means that \( k_E + 1 - cm < 0 \).

In case \( \mld_{MJ}(x; X) \geq 0 \), the statement gives

\[
\nu(X, x) = \min \{ \val_E(I_X) \mid \text{for } E \text{ computing } \mld_{MJ}(x; X) \}.
\]

**Proof.** The last statement is obvious, once we prove the main statement. Because the condition \( k_E + 1 - cm = \mld_{MJ}(x; X) = \mld(x; A, I_X^c) \) implies \( m = \val_E(I_X) \) under the assumption \( \mld_{MJ}(x; X) \geq 0 \).

For the proof of the main statement, we define

\[
\mu(X, x) := \min \left\{ m \in \mathbb{N} \mid k_E + 1 - cm = \mld_{MJ}(x; X) \text{ for a prime divisor } E \text{ over } A \right\},
\]

and will prove that \( \nu(X, x) = \mu(X, x) \).

For simplicity of notation on proving the main statement of the lemma, we denote \( \mu(X, x) \) and \( \nu(X, x) \) by just \( \mu \) and \( \nu \), respectively.

**Case 1.** \( \delta := \mld_{MJ}(x; X) \geq 0 \).

In this case,

\[
\delta = s_{\nu-1}(X, x) = \codim(\Cont^{\geq \nu}(I_X) \cap (\pi^A)^{-1}(x), A_{\infty}) - c \cdot \nu
\]

Let \( C \) be an irreducible component of \( \Cont^{\geq \nu}(I_X) \cap (\pi^A)^{-1}(x) \) that gives the codimension. Then, \( C \) is a maximal divisorial set \( C_A(q \cdot \val_E) \) for some \( q \in \mathbb{N} \) and a prime divisor \( E \) over \( A \) with the center \( x \) (see, for example, [9 Corollary 3.16]). As \( C \subset \Cont^{\geq \nu}(I_X) \), it follows that

\[
q \cdot \val_E(I_X) \geq \nu.
\]

On the other hand, by [16 Lemma 2.7] (see also [9 Theorem 3.13]) it follows

\[
\codim(C, A_\infty) = \codim(C_A(q \cdot \val_E), A_\infty) = q(k_E + 1).
\]

Therefore, we obtain

\[
\delta = s_{\nu-1}(X, x) = q(k_E + 1) - c\nu \geq q(k_E + 1 - c \cdot \val_E(I_X)) \geq k_E + 1 - c \cdot \val_E(I_X) \geq \delta.
\]

Then, all inequalities become equalities. Therefore, the last equality shows that \( E \) computes \( \mld_{MJ}(x; X) \) and \( \val_E(I_X) \leq \nu \). (More precisely, \( \val_E(I_X) = \nu \) if \( \delta > 0 \) and \( \val_E(I_X) = \nu/q \leq \nu \) if \( \delta = 0 \).) Hence, by the minimality of \( \mu \), we obtain

\[
\mu \leq \nu.
\]

Next we prove the converse inequality \( \mu \geq \nu \). Let \( E \) be a prime divisor over \( A \) with the center \( x \) computing \( \mld_{MJ}(x; X) = \delta \) and \( \val_E(I_X) = \mu \). As \( C_A(\val_E) \subset \Cont^{\geq \mu}(I_X) \cap (\pi^A)^{-1}(x) \),

\[
\codim(\Cont^{\geq \mu}(I_X) \cap (\pi^A)^{-1}(x), A_{\infty}) \leq \codim(C_A(\val_E), A_{\infty}) = k_E + 1.
\]

Therefore, we obtain

\[
s_{\mu-1}(X, x) = \codim(\Cont^{\geq \mu}(I_X) \cap (\pi^A)^{-1}(x), A_{\infty}) - c \cdot \mu
\]

\[
\leq k_E + 1 - c \cdot \val_E(I_X) = \delta.
\]

Hence the equality holds, which yields \( \nu \leq \mu \) by the minimality of \( \nu \).
Case 2. \(\text{mld}_{MJ}(x; X) = -\infty\). Then, by the same argument as in Case 1 we can write

\[
s_{\nu-1}(X, x) = \text{codim} \left( \text{Cont}^{2\nu}(I_X) \cap \pi^{-1}(x), A_{\infty} \right) - c\nu
\]

\[
= \text{codim}C(q \cdot \text{val}_E) - c\nu = q(k_E + 1) - c\nu < 0,
\]

for some \(q \in \mathbb{N}\) and a prime divisor \(E\) over \(A\) with the center \(x\) such that

\[
q \cdot \text{val}_E(I_X) \geq \nu.
\]

Therefore,

\[
q(k_E + 1 - c \cdot \text{val}_E(I_X)) \leq q(k_E + 1) - c\nu < 0,
\]

which implies that \(E\) computes \(\text{mld}_{MJ}(x; X) = -\infty\). Then, by the minimality of \(\mu\), it follows that \(\mu \leq \left\lceil \frac{\nu}{c} \right\rceil \leq \nu\).

Now, to show the converse inequality, take a prime divisor \(E\) over \(A\) computing \(\text{mld}_{MJ}(x; X) = -\infty\) and satisfies \(k_E + 1 - c \cdot \mu < 0\). By the minimality of \(\mu\) we have \(\mu \leq \text{val}_E(I_X)\), therefore \(C(\text{val}_E) \subset \text{Cont}^{2\mu}(I_X) \cap \pi^{-1}(x)\). By the same argument as the corresponding part in Case 1, we observe

\[
s_{\mu-1}(X, x) \leq k_E + 1 - c \cdot \mu < 0.
\]

Then, by the minimality of \(\nu\), we obtain \(\nu \leq \mu\). \(\square\)

As \(\nu\) does not depend on the choice of a closed immersion \(X \subset A\) into a non-singular variety \(A\), we obtain the following:

**Corollary 3.6.** Let \(X\) be embedded into a non-singular variety \(A\) with codimension \(c\). The invariant

\[
\mu(X, x) = \min \left\{ m \in \mathbb{N} \mid k_E + 1 - cm = \text{mld}_{MJ}(x; X) \right\}
\]

for a prime divisor \(E\) over \(A\) computing \(\text{mld}_{MJ}(x; X)\), is independent of the choice of a closed immersion into a non-singular variety.

Now we can interpret the conjecture into a more birational theoretic Conjecture \(D_d\) (Conjecture \([14]\)) as follows:

**Proposition 3.7.** Conjecture \(C_d\) and Conjecture \(D_d\) are equivalent.

**Proof.** First we show that \((C_d)\) implies \((D_d)\).

Let \(x\) be a closed point of a \(d\)-dimensional variety \(X\) embedded into a non-singular variety \(A\) of dimension \(N \leq 2d\). Let a prime divisor \(E\) over \(A\) compute \(\text{mld}_{MJ}(x; X)\) and satisfy

\[
k_E - c \cdot \mu(X, x) + 1 = \text{mld}_{MJ}(x; X) \leq d,
\]

where \(c = \text{codim} (X, A) \leq d\). Here, in case \(\text{mld}_{MJ}(x; X) = -\infty\), the above equality implies \(k_E - c \cdot \mu(X, x) + 1 < 0\), as in Lemma \([3.5]\). Then, by the assumption \((C_d)\) and Lemma \([3.5]\) we have

\[
k_E \leq d + d(N_d + 1) - 1,
\]

therefore Conjecture \(D_d\) (Conjecture \([14]\)) holds and we can take

\[
M_d \leq d(N_d + 2) - 1.
\]

Next we show that \((D_d)\) implies \((C_d)\). First we consider the case that there is a closed immersion \(X \subset A\) into a non-singular variety \(A\) of dimension \(N \leq 2d\) around \(x\), so that we can apply the statement of \((D_d)\).

When \(\text{mld}_{MJ}(x; X) \geq 0\), by the condition \((D_d)\), there is a prime divisor \(E\) over \(A\) such that

\[
k_E \leq M_d \quad \text{and}
\]

\[
k_E - c \cdot \text{val}_E I_X + 1 = \text{mld}_{MJ}(x; X) \geq 0.
\]
This yields the following bound:

\[ \mu(X, x) = \operatorname{val}_{E}I_{X} \leq \frac{k_{E} + 1}{c} \leq \frac{M_{d} + 1}{c} \leq M_{d} + 1. \]

When \( \operatorname{mld}_{\text{MJ}}(x; X) = -\infty \), by the condition \((D_{d})\), there is a prime divisor \( E \) over \( A \) computing \( \operatorname{mld}_{\text{MJ}}(x; X) \) such that

\[ k_{E} \leq M_{d} \quad \text{and} \quad k_{E} - c \cdot \mu(X, x) + 1 < 0. \]

By the definition, \( \mu(X, x) \) is the minimal integer satisfying the last inequality, which yields that

\[ \mu(X, x) \leq \frac{k_{E} + 2}{c} \leq M_{d} + 2. \]

Next we assume that \( \operatorname{emb}(X, x) > 2d \), then we have \( \dim X_{1}(0) > 2d \) which implies \( s_{1}(X, x) < 0 \). In this case, automatically \( \operatorname{mld}_{\text{MJ}}(x; X) = -\infty \) and

\[ \mu(X, x) = \nu(X, x) = 2. \]

Hence, for all cases, we obtain \((C_{d})\) and we can take

\[ N_{d} \leq M_{d} + 1. \]

These conjectures are also equivalent to the following conjecture implying the boundedness of the number of blow-ups to obtain a prime divisor computing the MJ-minimal log discrepancy.

**Conjecture 3.8** \((U_{d})\). For every \( d \in \mathbb{N} \) there is an integer \( B_{d} \in \mathbb{N} \) depending only on \( d \) such that for every singularity \((X, x)\) of dimension \( d \) embedded into a non-singular variety \( A \) with \( \dim A = \operatorname{emb}(X, x) \), there is a prime divisor \( E \) over \( A \) computing \( \operatorname{mld}_{\text{MJ}}(x; X) \) and \( \tilde{b}(E) = b(E) \leq B_{d} \).

**Proposition 3.9.** Conjecture \( U_{d} \) is equivalent to Conjecture \( C_{d} \) and \( D_{d} \).

**Proof.** First we will show the implication \((D_{d}) \Rightarrow (U_{d})\).

When \( \operatorname{emb}(X, x) = \dim A = N > 2d \), then the exceptional divisor \( E_{1} \) obtained by the blow-up of \( A \) at a point \( x \) computes \( \operatorname{mld}_{\text{MJ}}(x; X) = -\infty \). Indeed,

\[ a(E_{1}; A, I_{X}) = k_{E_{1}} + 1 - c \cdot \operatorname{val}_{E}I_{X} \leq N - 2(N - d) < 0. \]

So we may assume that \( N \leq 2d \). By the assumption \((D_{d})\), there is a prime divisor \( E \) over \( A \) computing \( \operatorname{mld}_{\text{MJ}}(x; X) \) such that \( k_{E} \leq M_{d} \). Let \( c_{i} \) be the codimension in \( A \) of the center of the \( i \)-th blow-up \((1 \leq i \leq \tilde{b}(E))\). Then \( c_{1} = \operatorname{emb}(X, x) \geq d + 1 \) and \( c_{i} \geq 2 \). Therefore it follows

\[ M_{d} \geq k_{E} \geq \sum_{i=1}^{\tilde{b}(E)}(c_{i} - 1) \geq d + (\tilde{b}(E) - 1) = \tilde{b}(E) + d - 1. \]

Hence, we obtain \( \tilde{b}(E) \leq M_{d} - d + 1 \), which yields the positive answer to Conjecture \( U_{d} \) and

\[ B_{d} \leq M_{d} - d + 1. \]

Next we prove the converse \((U_{d}) \Rightarrow (D_{d})\). Let \((X, x)\) be any \( d \)-dimensional singularity embedded into a non-singular variety \( A \) of dimension \( \leq 2d \). Let \( E \) be a prime divisor over \( A \) computing \( \operatorname{mld}_{\text{MJ}}(x; X) \) such that \( \tilde{b}(E) \leq B_{d} \). As \( \operatorname{ord}_{E}K_{A_{i}/A_{i-1}} \leq 2d - 1 \), where the left hand side is the coefficient of the divisor \( K_{A_{i}/A_{i-1}} \) at \( E_{i} \), we obtain

\[ k_{E} \leq 2^{\tilde{b}(E)-1}(2d - 1). \]
This implies the Conjecture $D_d$ and $M_d \leq 2^{B_d - 1}(2d - 1)$.

When one tries to prove Conjecture $C_d$, it may be useful to split it into small conjectures $C_{d,\delta}$ for every integer $\delta \leq d$ as follows:

Conjecture 3.10 ($C_{d,\delta}$). Let $d$ and $\delta$ be as above. There exists $N_{d,\delta} \in \mathbb{N}$ depending only on $d$ and $\delta$ such that for every closed point $x \in X$ of any $d$-dimensional variety $X$ with $\text{mld}_{M3}(x; X) < \delta$, there exists $m \leq N_{d,\delta}$ with the property $s_m(X, x) < \delta$.

Actually we have the following equivalence:

Proposition 3.11. The following are equivalent:

1. Conjecture $C_d$.
2. Conjecture $C_{d,\delta}$ for every integer $\delta$ such that $0 \leq \delta \leq d$.
3. Conjecture $C_{d,\delta}$ for every integer $\delta \leq d$.

Proof. For the proof (1) $\Rightarrow$ (2), we can take $N_{d,\delta} := N_d$ for every $\delta$ with $0 \leq \delta \leq d$. For the converse (1) $\Leftarrow$ (2), we can take $N_d := \max_{\delta} N_{d,\delta}$. The implication (3) $\Rightarrow$ (2) is obvious. For the proof of (2) $\Rightarrow$ (3), it is sufficient to show that for every $\delta = -i < 0$, we can take $N_{d,-i} := (i + 1)(N_{d,0} + 1) - 1$. Indeed, by the assumption, there is $m \leq N_{d,0}$ such that

$$\text{mld}(X, x) = \text{codim} \left( \text{Cont}^{\geq m + 1}(I_X) \cap (\pi^A)^{-1}(\eta), A_\infty \right) - c \cdot (m + 1) \leq -1,$$

which implies that there is an integer $q$ and a prime divisor $E$ over $A$ with the center $x$ such that $q \cdot \text{val}_E I_X \geq m + 1$ and

$$q(k_E + 1) - c(m + 1) = \text{mld}(X, x) \leq -1.$$

Then, since the maximal divisorial set $C_A((i + 1)q \cdot \text{val}_E)$ satisfies the following

$$C_A((i + 1)q \cdot \text{val}_E) \subset \text{Cont}^{\geq (i + 1)(m + 1)}I_X \cap (\pi^A)^{-1}(x),$$

it follows

$$s_{(i + 1)(m + 1) - 1}(X, x) \leq ((i + 1)q(k_E + 1) - (i + 1)c(m + 1) = (i + 1)s_m(X, x) \leq -(i + 1),$$

therefore we can take $N_{d,-i} := (i + 1)(N_{d,0} + 1) - 1$.

□

4. Applications of the conjecture

In this section we prove the properties (PMJ1)-(PMJ3) under the assumption that Conjecture $C_d$ holds. First we prove (PMJ3). The following is a relative version of (PMJ3) and the absolute version follows immediately as a special case.

Proposition 4.1 (Lower semi-continuity). Assume that Conjecture $C_d$ holds for an integer $d \geq 1$. Let $\rho : X \to Y$ be a surjective morphism of varieties with the $d$-dimensional varieties as fibers. Let us denote the fiber $\rho^{-1}(y)$ of $y \in Y$ by $X_y$. Consider the map $X \to \mathbb{Z}$ associating a closed point $x \in X$ to $\text{mld}_{M3}(x; X_{\rho(x)})$. Then the map is lower semi-continuous, i.e., if

$$\text{mld}_{M3}(x; X_{\rho(x)}) = \delta,$$

then there is an open neighborhood $U \subset X$ of $x$ such that for all closed point $x' \in U$,

$$\text{mld}_{M3}(x'; X_{\rho(x')}) \geq \delta.$$

In particular, in case $Y = \text{Spec} k$, then the map $X \to \mathbb{Z}; x \mapsto \text{mld}_{M3}(x; X)$ is lower semi-continuous.
is lower semi-continuous for all $m$ and observe that distinguish them from the fibers $X_t$ of $p$. The definition/construction of the relative $m$-jet scheme is given in [3] Proof of Theorem 4.9 and also [7, Proof of Proposition 2.3]. The relative $m$-jet scheme $L_m(X/Y)$ is a closed subscheme of $L_m(X)$ such that $\tilde{\pi}_m(X_y) = L_m(X_y)$, where $\tilde{\pi}_m : L_m(X/Y) \to X$ is the canonical truncation morphism.

For every $m \in \mathbb{N}$, $X \to \mathbb{Z}$, $x \mapsto \dim \pi_m^{-1}(x) = \dim \left(\pi_{m}^{\rho(x)}\right)^{-1}(x)$ is upper semi-continuous (see, for example [3, 4.11]). Here, $\pi_{m}^{\rho(x)} : L_m(X_{\rho(x)}) \to X_{\rho(x)}$ is the canonical truncation morphism and also the restriction of $\pi_m$ on $L_m(X_{\rho(x)})$. Therefore

$$s_m(X_{\rho(x)}, x) = (m + 1)d - \dim \pi_m^{-1}(x) = (m + 1)d - \dim \left(\pi_{m}^{\rho(x)}\right)^{-1}(x)$$

is lower semi-continuous for all $m \in \mathbb{N}$. The Conjecture $C_d$ implies that $\mathrm{mld}_{MJ}(x; X_{\rho(x)}) = \min\{s_m(X_{\rho(x)}, x) \mid m \leq N_d\}$.

Hence, $\mathrm{mld}_{MJ}(x; X_{\rho(x)})$ is lower semi-continuous. \hfill $\square$

**Remark 4.2.** Assume that Conjecture $C_d$ holds. For $\delta = -\infty, 0, 1, \ldots, d$, let $X(\delta)$ be the locally closed subset formed by the closed points $x \in X$ such that $\mathrm{mld}_{MJ}(x; X) = \delta$. Then $\{X(\delta)\}_\delta$ is a finite stratification of $X$. We call this the MJ-stratification. In the similar way, for a morphism $\rho : X \to Y$ as in the previous proposition, we can also define the locally closed subset

$$X/Y(\delta) = \{x \in X \mid \mathrm{mld}_{MJ}(x; X_{\rho(x)}) = \delta\},$$

and observe that $\{X/Y(\delta)\}_\delta$ is a finite stratification of $X$.

**Lemma 4.3.** Let $X$ be a variety of dimension $d$. Let $V \subset W$ be two irreducible proper closed subsets of $X$ and $\eta_V$ and let $\eta_W$ be the generic points of $V$ and $W$, respectively. Then the following inequality holds:

$$(7) \quad \mathrm{mld}_{MJ}(\eta_V; X) \leq \mathrm{mld}_{MJ}(\eta_W; X) + \mathrm{codim}(V, W),$$

Here, if either char $k = 0$ or Conjecture $C_d$ holds, then we have the equality in (7) for general $V$ in $W$. I.e., there exists an open subset $U \subset W$ such that

if $\eta_V \in U$ holds for an irreducible closed subset $V \subset W$, then the equality in (7) holds.

**Proof.** In [3] Corollary 3.27, (ii)] the inequality (7) is proved for an arbitrary characteristic. The equality in (7) for general $V$ in $W$ for char $k = 0$ is also proved in [3 Corollary 3.27]. So it is sufficient to show the second statement under the assumption that Conjecture $C_d$ holds. For $m \in \mathbb{N}$ let $d_mV$ and $d_mW$ be the dimensions of a general fibers of $\pi_m : \pi_m^{-1}(V) \to V$ and $\pi_m : \pi_m^{-1}(W) \to W$, respectively. Remember for every $m \in \mathbb{N}$ the following hold:

$$s_m(X, \eta_V) = (m + 1)d - \dim X_m(\eta_V) = (m + 1)d - (\dim V + d_mV)$$

$$s_m(X, \eta_W) = (m + 1)d - \dim X_m(\eta_W) = (m + 1)d - (\dim W + d_mW).$$

Then, as $d_mV \geq d_mW$ in general, we have

$$s_m(X, \eta_V) - s_m(X, \eta_W) = \mathrm{codim}(V, W) + d_mW - d_mV \leq \mathrm{codim}(V, W).$$
Let $U_m \subset W$ be the open subset such that the dimension of the fibers of a closed point by $\pi_m$ is the minimum. Note that $\eta_V \in U_m$ if and only if $d_{mV} = d_{mW}$, which is equivalent to
\[ s_m(X, \eta_V) = s_m(X, \eta_W) + \text{codim}(V, W). \]
Take the number $N_d$ in Conjecture $C_d$ and let $U = \cap_{m \geq 1}^N U_m$. Then we should note that $U$ is an open dense subset of $W$, because $U$ is the intersection of finite number of open dense subsets. Then for every $V \subset W$ with $\eta_V \in U$ we have
\[ \text{mld}_{MJ}(\eta_V, X) = \min_{m \leq N_d} s_m(X, \eta_V) = \min_{m \leq N_d} s_m(X, \eta_W) + \text{codim}(V, W). \]

**Lemma 4.4.** Let $\eta \in X$ be a point. If there exists a stratification $\{X(\delta)\}$ as in Remark 4.2, then the following hold:

(i) If $\delta \geq \dim X(\delta)(j)$ (resp. $\delta \geq \dim X(\delta)(j) + 1$) for every irreducible component $X(\delta)(j)$ of the stratum $X(\delta)$ such that the closure of $X(\delta)(j)$ contains $\eta$, then $X$ is $MJ$-log canonical (resp. $MJ$-canonical) at $\eta$.

(ii) If either the base field $k$ is uncountable, or Conjecture $C_d$ holds, then the converse of (i) also holds.

**Proof.** First we note that a special case of the formula in Lemma 4.3 implies the following:

For a closed point $x$ of an irreducible closed subset $W \subset X$ with the generic point $\eta_W$,
\[ \text{mld}_{MJ}(x; X) \leq \text{mld}_{MJ}(\eta_W, X) + \dim W. \]

Let $W \subset X$ be an irreducible closed subset containing $\eta$ and let $\eta_W$ be the generic point of $W$. For (i), we assume $\delta \geq \dim X(\delta)(j)$ and will show $\text{mld}_{MJ}(\eta_W, X) \geq 0$. Take an irreducible component $X(\delta)(j)$ of the stratum $X(\delta)$ containing $\eta_W$ in its closure, then $\dim W \leq \dim X(\delta)(j)$. As a closed point $x \in X(\delta)$ has $\text{mld}_{MJ}(x; X) = \delta$, it follows from (8)
\[ \text{mld}_{MJ}(\eta_W; X) \geq \delta - \dim W \geq \delta - \dim X(\delta)(j) \geq 0. \]

The proof for $MJ$-canonicity is similar, as we can replace the inequalities $\geq 0$ by $\geq 1$. (Here, we note that for $MJ$-log canonicity, we have only to prove $\text{mld}_{MJ}(\eta, X) \geq 0$. The reason why we dare to prove $\text{mld}_{MJ}(\eta_W, X) \geq 0$ for $W$ containing $\eta$ is because this proof works for $MJ$-canonicity by just shifting the number.)

For the proof of (ii), we should note that there are closed points $x \in X(\delta)(j)$ contained by the closure of $\eta$ such that the following equality holds:
\[ \text{mld}_{MJ}(x; X) = \text{mld}_{MJ}(\eta', X) + \dim X(\delta)(j), \]
where $\eta'$ is the generic point of $X(\delta)(j)$. Actually the first case (uncountable base field case) is proved in [9] and the second case (Conjecture $C_d$ holds) is proved in Lemma 4.3. By the formula (9), the assumption that $X$ is $MJ$-log canonical yields that
\[ \delta - \dim X(\delta)(j) = \text{mld}_{MJ}(x; X) - \dim X(\delta)(j) = \text{mld}_{MJ}(\eta', X) \geq 0. \]
The proof for $MJ$-canonicity is similar, as we can replace the inequalities $\geq 0$ by $\geq 1$. \qed

The following global statement follows immediately from the local statement, Lemma 4.3.
Corollary 4.5. Assume a variety $X$ has the stratification as in Remark 4.2. If the base field $k$ is uncountable or Conjecture $C_d$ holds, then a variety $X$ has MJ-log canonical (resp. MJ-canonical) singularities if and only if

$$\delta \geq \dim X(\delta) \ (\text{resp.} \ \delta \geq \dim X(\delta) + 1).$$

Now we will prove (PMJ1).

**Proposition 4.6** (Openness of MJ-log canonicity/MJ-canonicity). Assume Conjecture $C_d$ holds. Let $X$ be a $d$-dimensional variety. If $(X, \eta)$ is an MJ-log canonical (resp. MJ-canonical) singularity, then there is an open neighborhood $U \subset X$ of $\eta$ such that $U$ has MJ-log canonical (resp. MJ-canonical) singularities at every point of $U$.

**Proof.** As we assume $(C_d)$, there is the stratification $\{X(\delta)\}$ on $X$ as in Remark 4.2. Let $Z$ be the union of the closures of irreducible components $X(\delta)^{(j)}$ of strata $X(\delta)$'s such that $\eta \notin (X(\delta)^{(j)}).$ Then $U = X \setminus Z$ is an open neighborhood of $\eta$. If $(X, \eta)$ is MJ-log canonical, then the MJ-stratification $\{U_{\delta} = U \cap X(\delta)\}$ satisfies $\delta \geq \dim U_{\delta}$ for every stratum $U_{\delta}$ by Lemma 4.4. By Corollary 4.5 this shows that $U$ has MJ-log canonical singularities. For MJ-canonicity, the proof is similar. □

Next we prove (PMJ2) in the following:

**Proposition 4.7** (Stability under a deformation). Assume Conjecture $C_d$ holds. Let $\rho : X \to \Delta$ be a surjective morphism of a variety to a smooth curve $\Delta$ with the equidimensional reduced fibers of dimension $d$. Denote the fiber of this morphism of a point $t \in \Delta$ by $X_t$. If $(X_t, x_0) = (X_0, x_0)$ for $0 \in \Delta$ is MJ-log canonical (resp. MJ-canonical), then there are open neighborhoods $\Delta' \subset \Delta$ and $U \subset X$ of $0$ and $x_0$, respectively, such that all fibers of $U \to \Delta'$ have MJ-log canonical (resp. MJ-canonical) singularities.

**Proof.** By the assumption and Proposition 4.6, we may assume that the fiber $X_0$ has MJ-log canonical (resp. MJ-canonical) singularities, by replacing $X$ by a sufficiently small neighborhood around $x_0$. We use the notation in Remark 4.2. As the relative MJ-stratification $\{X_t/\Delta(\delta)\}_t$ has a finite number of irreducible strata, the set $B = \{t \in \Delta \mid t \neq 0\}$, there is an irreducible component $Z$ in $X_t/\Delta(\delta)$ such that $Z \subset X_t$ is a finite set. Replacing $\Delta$ by $\Delta' = \Delta \setminus B$, we may assume that every irreducible component of the strata $X_t/\Delta(\delta)$ is dominating $\Delta$. For the statement of MJ-log canonicity, we have only to prove that

$$\dim X_t(\delta) \leq \delta$$

for every $\delta = -\infty, 0, 1, \ldots, d$ and $t \in \Delta$. Here, we note that $X_t(\delta) = (X_t/\Delta(\delta)) \cap X_t$. Take an irreducible component $Z \subset X_t/\Delta(\delta)$. If $Z \cap X_0 = \emptyset$, then we replace $X_t$ by an open subset $X_t \setminus Z$. By this procedure we may assume that $Z \cap X_0 \neq \emptyset$ for every irreducible component $Z \subset X_t/\Delta(\delta)$.

Then consider the restriction $\rho : Z \to \Delta$. Take an irreducible component $Z^{(i)}$ of $\rho^{-1}(0) = Z \cap X_0$. Then the generic point of $Z^{(i)}$ is contained in $X_0(\delta_i)$ for some $\delta_i \leq \delta$ by the lower semi-continuity of $\mld_{MJ}$ proved in Proposition 4.1. By the assumption that $X_0$ has MJ-log canonical singularities, we obtain

$$\dim Z^{(i)} \leq \dim X_0(\delta_i) \leq \delta_i \leq \delta.$$

Now deleting finite number of points from $\Delta$ we may assume that

$$\dim \rho^{-1}(t) \leq \dim \rho^{-1}(0) \leq \delta.$$
For each irreducible component of $X/\Delta(\delta)$ we have the same inequality as above by deleting finite points of $\Delta$ and the number of such irreducible components are finite, we obtain a non-empty open subset $\Delta' \subset \Delta$ such that

$$\dim X_t(\delta) \leq \delta$$

holds for every $\delta = -\infty, 0, \ldots, d$ and for every $t \in \Delta'$.

For the MJ-canonicity, the proof goes parallel as above. □

5. Some affirmative cases

In this section we will show some affirmative cases for our conjectures. We start with a simple observation:

**Proposition 5.1.** Conjecture $C_{d,d}$ holds for every $d \geq 1$ and $C_{d,d-1}$ holds for $d \geq 2$. ($C_{1,0}$ will be treated in the next section.) We can take $N_{d,d} = 1$ and $N_{d,d-1} = 5$.

**Proof.** For a variety $X$ of dimension $d$ and a closed point $x \in X$, the inequality $\mld_{MJ}(x;X) \geq d$ holds if and only if $s_m(X,x) \geq d$ for every $m \in \mathbb{N}$. In particular for $m = 1$, the condition $s_1(X,x) = 2d - \dim X_1(x) \geq d$ implies that $\text{emb}(X,x) = \dim X_1(x) \leq d$. Then the equality

$$\text{emb}(X,x) = \dim X_1(x) = d$$

must hold, which yields that $(X,x)$ is non-singular. On the other hand, we know that $\mld_{MJ}(x;X) = d$ for non-singular $(X,x)$ (see for example [9, Corollary 3.17]). This shows Conjecture $C_{d,d}$ holds and $N_{d,d} = 1$. For $d \geq 2$, Conjecture $C_{d,d-1}$ was proved and $N_{d,d-1} = 5$ is showed in [9]. □

Now we will show some affirmative cases for Conjecture $C_d$.

**Non degenerate hypersurfaces.**

**Definition 5.2.** Let $M = \mathbb{Z}^{d+1}$ and $M_\mathbb{R} = M \otimes \mathbb{R} \cong \mathbb{R}^{d+1}$. For a finite set $\Lambda \subset M$, we define the Newton polygon generated by $\Lambda$ as follows

$$\Gamma_{\Lambda} := \text{convex hull of } \left( \bigcup_{m \in \Lambda} \left( m + \mathbb{R}^{d+1}_{\geq 0} \right) \right)$$

Let $\Lambda_0 \subset \Lambda$ be the subset consisting of the vertices of $\Gamma_\Lambda$, then, $\Gamma_\Lambda = \Gamma_{\Lambda_0}$. We call $\Lambda_0$ the minimal generator set of $\Gamma_\Lambda$.

In particular, for a polynomial $f \in k[X_0, \ldots, X_d]$ we define $\Gamma_+(f)$, the Newton polygon of $f$ as follows: For $m = (m_0, \ldots, m_d) \in M$ denote $X^m = X_0^{m_0} \cdots X_d^{m_d}$. By using this expression we represent $f$ as $f = \sum_{m \in M} a_m X^m$, $(a_m \in k)$. Let $\Lambda = \{m \in M | a_m \neq 0 \}$ and define $\Gamma_+(f) = \Gamma_\Lambda$.

**Definition 5.3.** Under the notation above, $f$ is called non-degenerate if every face $\gamma$ of $\Gamma_+(f)$ the equations $\frac{\partial f_\gamma}{\partial X_i} = 0$ ($i = 0, \ldots, d$) do not have common zeros on $\{a_k \neq 0\}^{d+1}$, where $f_\gamma := \sum_{m \in \gamma} a_m X^m$.

We say that $f$ is non-degenerate with respect to compact faces if for every compact face $\gamma$ the condition above holds.
Remark 5.4. Note that non-degenerate hypersurfaces are general among all hypersurfaces with a fixed Newton polygon.

It is well known that a non-degenerate hypersurface has an embedded log-resolution by a toric birational transformation in any characteristic. (See, for example, [13 III, Proposition 1.3.1].) This proposition is stated under the base field is $\mathbb{C}$, however the proof works for any algebraically closed field.) We also note that if a hypersurface $X$ has an isolated singularity at 0 and defined by a non-degenerate polynomial with respect to compact faces, then it also has an embedded log-resolution by a toric birational transformation in any characteristic.

By [5] Lemma 5.4 we have the following formula:

Proposition 5.5. Let $N$ be the dual of $M$, $\sigma \subset \mathbb{R} \otimes \mathbb{Z}$ $\mathbb{R}^{d+1}$ be the positive quadrant $(\mathbb{R}_{\geq 0})^{d+1}$ and $\sigma^o$ be the interior of $\sigma$. Let $X \subset \mathbb{A}^{d+1}$ be the hypersurface defined by a non-degenerate polynomial $f$. For an element $p \in N$ we define $(p, \Gamma_{+}(f)) = \min\{p, m \mid m \in \Gamma_{+}(f)\}$. We denote the point $(1, 1, \ldots, 1, 1) \in M$ by $1$. Then,

$$mld_{M}(0, X) = mld(0, X) = \inf_{p \in \sigma^o \cap N} ((p, 1) - (p, \Gamma_{+}(f))).$$

Let us denote by $E_p$ the toric prime divisor over $A = \mathbb{A}^{d+1}$ corresponding to the 1-dimensional cone $p\mathbb{R}_{\geq 0}$. Then, $(p, 1) = k_{E_p} + 1$ and $(p, \Gamma_{+}(f)) = val_{E_p}(f)$.

Definition 5.6. By the proposition above, for a non-degenerate hypersurface $X \subset \mathbb{A}^{d+1}$, $mld_{M}(0, X)$ depends only on the Newton polygon $\Gamma_{+}(f)$. Therefore we define the minimal log discrepancy for a Newton polygon $\Gamma$ by

$$mld\Gamma = \inf_{p \in \sigma^o \cap N} ((p, 1) - (p, \Gamma)).$$

We say that $p$ computes $mld\Gamma$ if

$$(p, 1) - (p, \Gamma) = mld\Gamma \geq 0, \quad \text{or}$$

$$(p, 1) - (p, \Gamma) < 0, \quad \text{when} \ mld\Gamma = -\infty.$$  

Here, we note that for a given Newton polygon $\Gamma$ we can find a non-degenerate polynomial $f$ such that $\Gamma = \Gamma_{+}(f)$, and “$p$ computes $\Gamma$” is the same as “$E_p$ computes $mld_{M}(0, X)$” for the hypersurface $X$ defined by $f$.

Theorem 5.7. For every $d \geq 1$ Conjecture $C_d$ holds for non-degenerate hypersurfaces $X$ and the origin $0 \in X$.

Proof. In the proof we use the notation of Proposition 5.5. As the possible values of $mld_{M}(0, X)$ are finite, it is sufficient to fix $\delta (\delta = -\infty, 0, \ldots, d)$ and to prove a contradiction under the assumption that $\nu(X, 0)$ is unbounded among non-degenerate hypersurface singularities $(X, 0)$ satisfying $mld_{M}(0, X) = \delta$.

Let $\{\Gamma_j\}$ be an infinite sequence of Newton polygon with $mld(\Gamma_j) = \delta$, such that $\nu_j := \nu(X_j, 0) \rightarrow \infty$ $(j \rightarrow \infty)$, where $\Gamma_j$ is the Newton polygon of a non-degenerate hypersurface $X_j$.

For points $a, b \in \sigma \cap M$ we define the relation $a < b$, if either

$|a| < |b|$, where $|a|$ is the sum of all coordinates of $a$, or

$|a| = |b|$ and $a < b$ lexicographically.

We give numbers to all vertices of each Newton polygon $\Gamma_j$ according to the order:

$$a_1(\Gamma_j) \leq a_2(\Gamma_j) \leq \cdots.$$  

Then we may assume that $|a_j(\Gamma_j)| \leq d+1$ for infinitely many $j$. Indeed, if there is an infinite subsequence such that $|a_j(\Gamma_j)| \geq d+2$, then the multiplicity of the
defining function $f_j$ of $X_j$ at 0 is bigger than $d + 1$, therefore by Lemma 2.10 we obtain

$$s_{d+1}(X_j, 0) = d(d + 2) - \dim(\mathcal{A}^{d+1})_{d+1}(0) = d(d + 2) - (d + 1)^2 = -1 < 0$$

for each such $j$. This implies that $\delta = \text{mld}(\Gamma_j) = -\infty$ and $\nu_j$ is bounded by $d + 2$ for the subsequence, which is a contradiction to the assumption that $\nu_j := \nu(X_j, 0) \to \infty \ (j \to \infty)$. Therefore, we may assume that there is an infinite subsequence such that $a_1(\Gamma_j) \leq d + 1$ for all $j$ in the sequence.

Here, as there are only finitely many points $a \in \sigma \cap M$ with $|a| \leq d + 1$, there is an infinite subsequence of $\{\Gamma_j\}$ such that all $a_1(\Gamma_j)$ are common. Let them be $a_1$.

Next, if $|a_2(\Gamma_j)|$ is bounded for infinitely many $j$ among the subsequence obtained above, then, in the same way as above we take the infinite subsequence with the common $a_2(\Gamma_j) =: a_2$. We perform this procedure successively. But this procedure stops at a finite stage. Because, if not, then we obtain an infinite strictly increasing sequence of polygons:

$$P_1 \subset P_2 \subset \cdots,$$

where $P_i$ is the polygon generated by $a_1, \ldots, a_i$. This means that there is an infinite strictly increasing sequence of monomial ideals in a Noetherian algebra $k[\sigma \cap M]$, which is a contradiction.

Now, we can assume that there exists $m \in \mathbb{N}$ such that there is an infinite subsequence $\{\Gamma_j\}$ with the common $a_1, a_2, \ldots, a_m$ but $a_{m+1}(\Gamma_j)$ is not bounded for any infinite subsequences. Then, for any $D > 0$ there is an infinite subsequence $\{\Gamma_j\}$ such that $|a_{m+1}(\Gamma_j)| > D$ for all $j$. Let $\Gamma$ be the polygon generated by $a_1, a_2, \ldots, a_m$ and $n = \nu(\Gamma)$. Then, we may assume that, for all $j$, $|a_{m+1}(\Gamma_j)| \geq n$ so that there is no lattice point on the faces of $\Gamma_j$ containing $a_{m+1}(\Gamma_j)$. Let $f_j$ and $f$ be non-degenerate polynomials with the Newton polygon $\Gamma_j$ and $\Gamma$, respectively. We may assume that the monomials of these polynomials are only on the Newton boundary, because the minimal log discrepancies depend only on the Newton boundaries. By Lemma 2.10 we have

$$s_{n-1}(f_j) = s_{n-1}(f) = \text{mld}(f)$$

for all $j$. Therefore $\text{mld}(\Gamma_j) \leq \text{mld}(\Gamma)$. But $\Gamma \subset \Gamma_j$ yields $\text{mld}(\Gamma) \leq \text{mld}(\Gamma_j)$ by the Definition 5.6. Therefore, we obtain that $\text{mld}(\Gamma_j) = \text{mld}(\Gamma) = s_{n-1}(\Gamma_j)$ for all $j$, which implies that $\nu_j \leq n$, a contradiction.

\[\square\]

**Remark 5.8.** (1) The proof of [12, Theorem 1.4] can be interpreted into the discussion of Newton polygon and we can prove the above theorem in the same way as [12, Theorem 1.4] using Maclagan’s result.

(2) When the base field is of characteristic 0, by the following lemma we can reduce Conjecture $D_d$ to the hypersurface case. Moreover, by Theorem 5.7 the conjecture for characteristic 0 case is reduced to degenerate hypersurfaces.

**Proposition 5.9 ([10, Theorem 3.1]).** Let $0 \in X \subset A$ be a closed point on a closed subvariety $X$ in a non-singular affine variety $A$ over a base field of characteristic zero. Then, there exists a hypersurface $0 \in H \subset A$ such that

$$\text{mld}_{\text{MJ}}(0, X) = \text{mld}_{\text{MJ}}(0, H).$$

Moreover, there exists a log-resolution $A' \to A$ of $X$ and $H$ such that the prime divisors on $A'$ computing $\text{mld}_{\text{MJ}}(0, X)$ are the same as those computing $\text{mld}_{\text{MJ}}(0, H)$.

In order to study a general hypersurface singularity, the following lemma is useful to prove the conjecture in a special case:
Lemma 5.10. Let $X \subset \mathbb{A}^{d+1}$ be a hypersurface defined by a polynomial $f$ with a singularity at $0$. Let $\Gamma$ be a Newton polygon containing $\Gamma_+(f)$. If $1 = (1,1,\ldots,1) \notin \Gamma$, then $\mld_{\MJ}(0;X) = -\infty$ and a toric divisor $E_p$ computing $\mld_{\Gamma}$ computes $\mld_{\MJ}(0;X)$. In particular, if $f$ is a non-degenerate function with the Newton polygon $\Gamma_+(f) = \Gamma$ and $\tilde{X}$ is the hypersurface defined by $f$, then $\nu(X,0) \leq \nu(\tilde{X},0)$ which is uniformly bounded on the dimension $d$.

Proof. By the formula in Proposition 5.5, the assumption $1 = (1,1,\ldots,1) \notin \Gamma$ implies $\mld_{\MJ}(0;\tilde{X}) = \mld_{\Gamma} = -\infty$. Let $p$ compute $\mld_{\Gamma}$, then, by $\Gamma_+(f) \subset \Gamma$, we have

$$\val_{E_p} f \geq \langle p, \Gamma_+(f) \rangle \geq \langle p, \Gamma \rangle \geq \langle p, 1 \rangle,$$

which implies that $\mld_{\MJ}(0;\tilde{X}) = \mld_{\Gamma} = -\infty$ and $E_p$ computes it.

For the second statement, note that $\nu(X,0) = \mu(X,0)$ and $\nu(\tilde{X},0) = \mu(\tilde{X},0)$. Take a prime divisor $E_p$ computing $\mld_{\MJ}(0;\tilde{X})$ such that

$$k_{E_p} - \mu(\tilde{X},0) + 1 < 0.$$

Then, as $E_p$ also computes $\mld_{\MJ}(0;X)$ by the discussion above, we can see that the inequality (10) implies

$$\mu(X,0) \leq \mu(\tilde{X},0)$$

by the minimality of $\mu(X,0)$.

Singulartoies of maximal type.

The second case that $(C_d)$ holds is a $d$-dimensional singularity $(X,x)$ of maximal type under the assumption that Conjecture $C_{d-1}$ holds for $(d-1)$-dimensional “scheme”. Here, note that we use a bit stronger conjecture $C_{d-1}$ for not only $(d-1)$-dimensional varieties but also $(d-1)$-dimensional schemes. This result is used in the next section.

First, we give the definition of a singularity of maximal type.

Definition 5.11. Let $X$ be a $d$-dimensional variety over $k$, $x \in X$ a closed point with $\text{emb}(X,x) = N$ and let $c = N - d$. Let $X \subset A$ be a closed immersion around $x$ into a non-singular variety $A$ with $\dim A = N$ and $I_X$ the defining ideal of $X$ in $A$. We define

$$\ord_x I_X = \min\{\mult_x f \mid f \in I_X\}.$$

We say that the singularity $(X,x)$ is of maximal type if

$$N = c \cdot \ord_x I_X.$$

The following shows the status of a singularity of maximal type.

Lemma 5.12. Under the notation in the above definition, if a singularity $(X,x)$ is MJ-log canonical, then

$$N \geq c \cdot \ord_x I_X.$$

If the singularity $(X,x)$ is also of maximal type, then $\mld_{\MJ}(x;X) = 0$.

Proof. Take the blow-up $A' \to A$ at the closed point $x$ and let $E$ be the exceptional divisor. Then, the log discrepancy $a(E;A,I_X^{e}) = N - c \cdot \ord_x I_X \geq 0$, because $(A,I_X^{e})$ is log canonical by the assumption that $(X,x)$ is MJ-log canonical. This shows the first statement. If the singularity is moreover of maximal type, then $a(E;A,I_X^{e}) = 0$, which shows that $\mld_{\MJ}(x;X) = 0$.

In order to give the basic lemma, we need to give a little generalization of definitions.
Definition 5.13. Fix $d \in \mathbb{N}$. Let $Y$ be a $d$-dimensional scheme over $k$ and $\eta \in Y$ a point. For $m \in \mathbb{N}$, define a function $s_m(Y, \eta)$ as follows:

$$s_m(Y, \eta) = (m + 1)d - \dim Y_m(\eta).$$

And also we define minimal MJ-log discrepancy

$$\text{mld}_{MJ}(\eta; Y) = \inf_{m \in \mathbb{N}} s_m(Y, \eta).$$

We say that $(Y, \eta)$ is MJ-log canonical at a point $y \in Y$, if $\text{mld}_{MJ}(\eta; Y) \geq 0$ for every $\eta \in Y$ such that $[\eta] \ni y$. Note that this is equivalent to

$$\text{mld}_{MJ}(\eta; Y) \geq 0.$$

The conjectures $C_d$, $D_d$ and $U_d$ are for pairs of $d$-dimensional varieties and closed points on them. Here, we extend these conjectures for pairs of $d$-dimensional schemes and closed points.

We say that Conjecture $C_{d,i}$ holds for $d$-dimensional schemes if there exists $N_{d,i} \in \mathbb{N}$ depending only on $d$ and $i$ such that

for every pair $(Y, y)$ consisting of a $d$-dimensional scheme $Y$ and a closed point $y \in Y$, if $\text{mld}_{MJ}(y; Y) < i$, then there exists $m \leq N_{d,i}$ with the property $s_m(Y, y) < i$.

We extend $(C_d)$, $(D_d)$ and $(U_d)$ to this class of singularities in similar ways.

Proposition 5.14. Let $Y$ be a $d$-dimensional scheme.

(i) If $Y$ is embedded into a non-singular variety $A$ of dimension $N$ as a closed subscheme, then for a closed point $y \in Y$,

$$\text{mld}_{MJ}(y; Y) = \text{mld}(y; A, I_Y^{N-d}).$$

(ii) If a $d$-dimensional scheme $Y$ is MJ-log canonical at every point of $Y$, then,

$$\dim Y_m \leq (m + 1)d.$$

Proof. The statement (i) follows by interpreting the definition of $\text{mld}_{MJ}(y; Y)$ in terms of contact loci. For (ii), take an irreducible component $W$ of $Y_m$ which gives the dimension of $Y_m$. Let $\eta \in Y$ be the generic point of the image $\pi_m(W)$. As $Y$ is MJ-log canonical at $\eta$ we obtain

$$0 \leq \text{mld}_{MJ}(\eta; Y) \leq (m + 1)d - \dim(\pi_m)^{-1}(\eta),$$

which yields

$$\dim Y_m \leq (m + 1)d.$$

The following is a slight generalization of [35, Lemma 3.4] and will be used to reduce the conjecture for singularities of maximal type to the case of singularities defined by homogeneous ideals.

Lemma 5.15. Let $(X, x)$ be a singularity on a $d$-dimensional variety $X$ of maximal type with the embedding $X \subset A$ into a non-singular variety $A$ of dimension $N = \text{emb}(X, x)$. Let $I_X = (f_1, \ldots, f_s)$ be the defining ideal of $X$ in $A$. Let $\{f_1, \ldots, f_s\}$ $(s \leq r)$ be the subset of the generators such that $\text{mult}_x f_i = \alpha := \text{ord}_x I_X$ for $i = 1, \ldots, s$. Define $J = (\text{inf} f_1, \ldots, \text{inf} f_s)$ and let $J$ define a closed subscheme $Y \subset A^N = \text{Spec} k[x_1, \ldots, x_N]$, where $\{x_1, \ldots, x_N\}$ is a regular system of parameters of $A$ around $x$.

Then, for any integer $m \geq \alpha$,

$$s_m(X, x) \geq s_m(Y, 0) \geq s_{(\alpha + 1)m - \alpha^2}(X, x).$$
Claim 1. Let \( I_m \) be the defining ideal of \( X_m(x) \) in \( A_m(x) \) and let \( J_m \) be the defining ideal of \( Y_m(0) \) in \( A^N_m(0) \). We can identify \( A^N_m(x) \) and \( A^N_m(0) \) as follows:
\[
A_m(x) = \mathbb{A}^N_m(0) = \text{Spec} \, k[x^{(1)}, x^{(2)}, \ldots, x^{(m)}],
\]
where \( x^{(j)} \) denotes the collection of the coordinates \( x_1^{(j)}, \ldots, x_N^{(j)} \). Under this identification, we have \( J_m \subset \text{in} I_m \), therefore \( \text{ht} J_m \leq \text{ht} (\text{in} I_m) = \text{ht} I_m \), which yields the first inequality in the theorem.

For the second inequality, we give a technical definition. For any integers \( l \geq 2 \) and \( m > l \) and a polynomial \( f \in k[x^{(1)}, x^{(2)}, \ldots, x^{(m)}] \), the symbol \( f^{(l)} \) denotes the polynomial in \( k[x^{(l)}, x^{(l+1)}, \ldots, x^{(m)}] \) substituting \( x^{(1)} = \cdots = x^{(l-1)} = 0 \) into \( f \).

Claim 1. For a monomial \( h \in k[x_1, \ldots, x_N] \) of degree \( \alpha + 1 \), it follows that
\[
\overline{ht}^{(i)} = 0, \quad (1 \leq i \leq (\alpha + 1)l - 1),
\]
where \( \overline{ht}^{(i)} \) is as in [1] in Proposition 2.10 i.e.,
\[
(12) \quad \overline{ht}(\sum_{j=0}^{m} x_1^{(j)} t^j, \ldots, \sum_{j=0}^{m} x_N^{(j)} t^j) \equiv \sum_{j=0}^{m} h^{(j)} t^j \pmod{t^{m+1}}.
\]

For the proof of the Claim 1, remind us that \( \deg h^{(i)} \geq \alpha + 1 \) and the weight \( i \) of \( h^{(i)} \) is less than \( (\alpha + 1)l \), there is a variable \( x_v^{(u)} \) in \( h^{(i)} \) with the weight \( u \leq l - 1 \). Therefore, substituting \( x^{(1)} = \cdots = x^{(l-1)} = 0 \) into \( h^{(i)} \), we obtain \( h^{(i)} = 0 \).

Claim 2. Let \( h \in k[x_1, \ldots, x_N] \) be a polynomial with \( \text{mult}_h \geq \alpha \), where \( \text{mult}_h \) is the degree of initial term of \( h \) in variables \( x_1, \ldots, x_N \). Let \( \text{inh} \) be the initial term of \( h \). Then
\[
\overline{ht}^{(i)} = \overline{(\text{inh})^{(i)}}, \quad (1 \leq i \leq (\alpha + 1)l - 1).
\]
Indeed, if \( \text{mult}_h \geq \alpha + 1 \), then the both hand sides of above are zero by Claim 1. If \( \text{mult}_h = \alpha \), then again by Claim 1, we have that only the parts of degree \( \alpha \) can survive by substituting \( x^{(i)} = \cdots = x^{(l-1)} = 0 \). This completes the proof of Claim 2.

Now let the defining ideal of \( X_n(x) \) in \( A_n(x) \) be \( I_n \) for \( n \in \mathbb{N} \). Then, for \( n = (\alpha + 1)l - 1 \), we have
\[
I_{(\alpha+1)l-1} \subset (x^{(1)}, \ldots, x^{(l-1)}, f_1^{(\alpha)}, \ldots, f_r^{(\alpha)}, \ldots, f_1^{((\alpha+1)l-1)}, \ldots, f_r^{((\alpha+1)l-1)}).
\]

Here, we denote
\[
I(\alpha, l) = (f_1^{(\alpha)}, \ldots, f_r^{(\alpha)}, \ldots, f_1^{((\alpha+1)l-1)}, \ldots, f_r^{((\alpha+1)l-1)}) \subset k[x^{(l)}, \ldots, x^{((\alpha+1)l-1)}].
\]

Then we obtain
\[
(13) \quad \text{ht} I_{(\alpha+1)l-1} \leq \text{ht} I(\alpha, l) + (l-1)N.
\]

Here, by Claim 2, we obtain that
\[
I(\alpha, l) = (\text{in} f_1^{(\alpha)}, \ldots, \text{in} f_r^{(\alpha)}, \ldots, \text{in} f_1^{((\alpha+1)l-1)}, \ldots, \text{in} f_r^{((\alpha+1)l-1)})
= (\text{in} f_1^{(\alpha)}, \ldots, \text{in} f_s^{(\alpha)}, \ldots, \text{in} f_1^{((\alpha+1)l-1)}, \ldots, \text{in} f_s^{((\alpha+1)l-1)}).
\]

By the shift \( x_v^{(u)} \mapsto x_v^{(u-l-1)} \) of variables, the ideal \( I(\alpha, l) \) becomes
\[
(\text{in} f_1^{(\alpha)}, \ldots, \text{in} f_s^{(\alpha)}, \ldots, \text{in} f_1^{((\alpha+l-1)}, \ldots, \text{in} f_s^{((\alpha+l-1)}) = J_{\alpha+l-1}.
\]
Hence, \( \text{ht} I(\alpha, l) = \text{ht} J_{\alpha+l-1} \). By these interpretations, the inequality (13) yields
\[
\text{ht} I_{(\alpha+1)l-1} \leq \text{ht} J_{\alpha+l-1} + (l-1)N.
\]

If we put \( m = \alpha + l - 1 \), then we obtain
of MJ Lemma 5.12 and Lemma 5.15, we obtain that \( mld \) greater than
\[ \text{ht} I_{(\alpha+1)m-\alpha^2} \leq \text{ht} J_m + (m - \alpha)N, \]
which will give the required inequality. Actually, to see this, remind us the definition of \( s_m(X, x) \) and \( s_m(Y, 0) \) to obtain:
\[ s_m(Y, 0) = (m + 1)d - mN + \text{ht} J_m = -cm + d + \text{ht} J_m, \]
\[ s_{(\alpha+1)m-\alpha^2}(X, x) = -c((\alpha+1)m - \alpha^2) + d + \text{ht} I_{(\alpha+1)m-\alpha^2}. \]
Substituting these into (14) and by noting that \( N = \alpha \cdot c \), we finally obtain
\[ s_m(Y, 0) \geq s_{(\alpha+1)m-\alpha^2}(X, x). \]

\[ \square \]

**Corollary 5.16.** Under the same notation and the assumptions as in the previous lemma, the following are equivalent:

1. \((X, x)\) is MJ-log canonical;
2. \(Y\) is of dimension \(d\) at 0 and \(mld_{MJ}(0; Y) = 0\).

In these cases, \((Y, y)\) is also of maximal type.

**Proof.** If \((X, x)\) is MJ-log canonical, then \(s_m(X, x) \geq 0\), for every \(m \in \mathbb{N}\). Then, by Lemma 5.12 and Lemma 5.15 we obtain that \(mld_{MJ}(0; Y) = 0\). By the definition of \(Y\), we have \(\dim Y \geq d\). Here, if \(Y\) has an irreducible component \(Y_1\) of dimension greater than \(d\), then
\[ mld_{MJ}(0; Y) = mld(0; \mathbb{A}^N, I_{Y_1}) \leq mld(0; \mathbb{A}^N, I_{Y_1}) = -\infty, \]
which is a contradiction to Lemma 5.15.

Conversely, we assume that \(Y\) is a scheme of dimension \(d\) and \(mld_{MJ}(0; Y) = 0\). Then we have \(s_m(Y, 0) \geq 0\) for every \(m \in \mathbb{N}\). By Lemma 5.15 we obtain \(mld_{MJ}(x; X) \geq 0\).

\[ \square \]

The following shows a reduction step on \(C_{d,i}\) for a singularity of maximal type defined by homogeneous polynomials with the same degree. Here, we note that \(i\) can be negative.

**Lemma 5.17.** Fix integers \(d \geq 2\), \(i < d\) and \(\alpha \geq 2\). Let \(S\) be the set of pairs \((Y, 0)@\) consisting of a \(d\)-dimensional scheme \(Y\) over \(k\) and a closed point \(0 \in Y\) satisfying

(a) \((Y, 0) \subset (\mathbb{A}^N, 0)\) is of maximal type and

(b) \((Y, 0) \subset (\mathbb{A}^N, 0)\) is defined by homogeneous polynomials of a common degree \(\alpha\).

Assume Conjecture \(C_{d-1,i-1}\) holds true for the set of pairs consisting of \((d - 1)\)-dimensional schemes over \(k\) and closed point on it.

Then, Conjecture \(C_{d,i}\) holds true for \(S\) and \(N_{d,i} \leq \max(\alpha, N_{d-1,i-1})\).

**Proof.** Take \((Y, 0)\) from \(S\). Then, by (a), we have \(N = (N - d)\alpha\). Therefore it follows
\[ N = (\alpha/(\alpha - 1))d \leq 2d \text{ and also } \alpha \leq N \leq 2d. \]

In order to prove \(C_{d,i}\), assume \(mld_{MJ}(0; Y) < i\). By (a), we see that the exceptional divisor \(E_i\) of the blow-up \(A' \to A = \mathbb{A}^N\) at the origin has log-discrepancy \(a(E_i; A, I_{Y_1}) = 0\), which implies \(mld_{MJ}(0; Y) \leq 0\). Therefore, it is sufficient to prove \(C_{d,i}\) for \(i \leq 0\). Henceforth in the proof we assume that \(i \leq 0\).

**Case 1.** When \(Y \setminus \{0\}\) is MJ-log canonical, then we will show that there is
\[ m \leq \alpha \text{ such that } s_m(Y, 0) < i. \]
Indeed, let $f_1, \ldots, f_r$ be the homogeneous generators of $I_Y$ of degree $\alpha$, then, for $j \geq \alpha$,
\[
\phi^{(j)}(0, x^{(1)}, \ldots, x^{(j)}) \text{ corresponds to } \phi^{(j-\alpha)}(x^{(0)}, \ldots, x^{(j-\alpha)}),
\]
by the shift of variables $x_u^{(u)} \mapsto x_v^{(v-1)}$, because $f_l$ ($l = 1, \ldots, r$) are homogeneous of degree $\alpha$. Since $Y_m(0)$ is defined by $f_l^{(j)}(0, x^{(1)}, \ldots)$ ($j = \alpha, \ldots, m$) in Spec $k[x^{(1)}, \ldots, x^{(m)}]$, we obtain that
\[
Y_m(0) \simeq \text{Spec } k[x^{(0)}, \ldots, x^{(m-1)}]/ \left( \phi^{(j-\alpha)}(x^{(0)}, \ldots, x^{(j-\alpha)}) \right)_{j-\alpha = 0, \ldots, m-\alpha}
\]
\[
\simeq \left[ \text{Spec } k[x^{(0)}, \ldots, x^{(m-\alpha)}]/ \left( \phi^{(j-\alpha)}(x^{(0)}, \ldots, x^{(j-\alpha)}) \right) \right] \times A_k^{\alpha-1} N
\]
\[
\simeq Y_{m-\alpha} \times A_k^{\alpha-1} N.
\]
Hence we have
\[
\dim Y_m(0) = \dim Y_{m-\alpha} + (\alpha - 1) N,
\]
Now, take the smallest number $m$ satisfying $s_m(Y, 0) < i$. Then
\[
\dim Y_m(0) > d(m + 1) - i.
\]
If $m > \alpha$, then (15), (17) and (18) yield
\[
\dim Y_{m-\alpha} > d(m - \alpha + 1) - i.
\]
As $Y \setminus \{0\}$ is MJ-log canonical, dim $(Y_{m-\alpha} \setminus Y_{m-\alpha}(0)) \leq d(m - \alpha + 1)$ by Proposition 5.14, which yields dim $Y_{m-\alpha}(0) > d(m - \alpha + 1) - i$ and therefore
\[
s_m(Y, 0) < i,
\]
which is a contradiction to the minimality of $m$. Therefore we obtain (16).

**Case 2.** When $Y \setminus \{0\}$ is not MJ-log canonical, then there is a number $m \leq N_{d-1,i-1}$ such that $s_m(Y, 0) < i$.

Indeed, since $Y$ is an affine cone, for any open neighborhood $U$ of the vertex $0 \in Y$, there is a non-MJ-log canonical closed point $y \in U \setminus \{0\}$. We may think that $(Y, y)$ is isomorphic to $(Z \times A^1, z \times 0)$ around $y$ for some $(d-1)$-dimensional scheme $Z$ and its closed point $z$. As we assume that Conjecture $C_{d-1,i-1}$ holds, there is a number $m \leq N_{d-1,i-1}$ such that $s_m(Z, z) < i - 1$.

Therefore we obtain
\[
s_m(Y, y) = s_m(Z \times A^1, z \times 0) = s_m(Z, z) + 1 < i.
\]
Note that the vertex 0 is in the closure of $z \times A^1$ and $s_m(Y, 0)$ is lower semicontinuous, we obtain
\[
s_m(Y, y) < i, \quad \text{for } m \leq N_{d-1,i-1}
\]
as required.

As the conclusion of Case 1 and Case 2, we obtain that $C_{d,i}$ holds true and $N_{d,i} \leq \max\{\alpha, N_{d-1,i-1}\}$.

**Proposition 5.18.** For an integer $i \leq d$. Assume Conjecture $C_{d-1,i-1}$ holds true. Then, Conjecture $C_{d,i}$ holds true for the category $S_\alpha$ of singularities $(X, x) \subset (A, x)$ on varieties $X$ of maximal type with $\alpha = \text{ord}_X I_X$, then in this category we can take
\[
N_{d,i} = \max\{\alpha, (\alpha + 1)(d-1-i) - \alpha^2, (\alpha + 1)N_{d-1,i-1} - \alpha^2\}
\]
In particular, if Conjecture $(C_{d-1})$ holds, then in the category of $d$-dimensional singularities $(X, x)$ on varieties $X$ of maximal type, Conjecture $C_d$ holds.
Proof. Let \((X, x)\) be a singularity of maximal type with the dimension \(d\). Let \(\dim A = N = \text{emb}(X, x)\). Let \((Y, 0) \subset \mathbb{A}^N\) be the scheme defined by homogeneous polynomials of degree \(\alpha\) as introduced in Lemma 5.15.

When \(\dim Y > d\), then \(\dim Y_m(0) \geq m \cdot \dim Y \geq m(d+1)\). Then, it follows that
\[
s_m(Y, 0) = (m+1)d - \dim Y_m(0) \leq (m+1)d - m(d+1) = d - m.
\]
Therefore, for \(m = d - i + 1\), we obtain \(s_m(Y, 0) < i\), which shows \((C_{d,i})\) for the class of such \(Y\) by taking \(N_{d,i} = d - i + 1\).

Next consider the case \(\dim Y = d\). Note that \((Y, 0)\) satisfies the condition \((a), (b)\) in Lemma 5.17. By the assumption of the proposition and Lemma 5.17 there is \(N'_{d,i}\) such that for some \(m \leq N'_{d,i}\), \(s_m(Y, 0) < i\) if \(\text{mld}_{MJ}(0; Y) < i\). Here, by Lemma 5.17 we can take \(N'_{d,i} = \max\{\alpha, d - i + 1, N_{d-1,i-1}\}\).

By Lemma 5.15 \((C_{d,i})\) holds in the category of singularities of maximal type with \(\alpha = \text{ord}_x I_X\) and we can take
\[
N_{d,i} = (\alpha+1)N'_{d,i} - \alpha^2 = \max\{(\alpha+1)\alpha - \alpha^2, (\alpha+1)(d-i+1) - \alpha^2, (\alpha+1)N_{d-1,i-1} - \alpha^2\}.
\]

For the last statement, note that the bound \(N_{d,i}\) in \(S_\alpha\) depends on \(\alpha\). But \(d\)-dimensional singularities of maximal type with \(\alpha = \text{ord}_x I_X\) have a bound \(\alpha \leq 2d\).

6. Curve and surface cases

Theorem 6.1. In the category of connected schemes of dimension 1 over the base field \(k\) of arbitrary characteristic, Conjecture \(C_1\) (therefore Conjecture \(D_1\) and \(U_1\) also) holds and we can take \(N_1 = 5\), \(M_1 = 4\) and \(B_1 = 3\). More precisely, we can take \(N_{1,1} = 1\), \(N_{1,0} = 5\) and \(N_{1,-1} = 11\).

Proof. Once we obtain the bounds \(N_{1,1}\) and \(N_{1,0}\), then the statement about \(N_{1,-1}\) follows from Proposition 3.11. We can prove \((C_1)\) by direct calculations of the jet schemes of one dimensional schemes. But the simplest way for a proof of the theorem is to show \((D_1)\). Let \(X\) be a 1-dimensional scheme over \(k\) embedded into a non-singular variety \(A\) of dimension \(\leq 2d = 2\). Then, \(X\) is a plane curve defined by one equation \(f = 0\).

Conjecture \((C_{1,1})\) is already proved and \(N_{1,1} = 1\). So, we may assume that \(\text{mult}_x f \geq 2\).

Case 1. If \(\text{mult}_x f \geq 3\), the blow-up \(A' \to A\) at the closed point \(x \in X \subset A\) provides with the prime divisor \(E_1\) over \(A\) which computes \(\text{mld}_{MJ}(x; A, I_X) = -\infty\). In this case \(k_{E_1} = 1\).

Case 2. Assume that \(\text{mult}_x f = 2\). If \(X\) is a normal crossing double point at \(x\), i.e., \(f = x_1 \cdot x_2\) in \(\mathcal{O}_{A,x} = k[[x_1, x_2]]\), then the blow-up \(A' \to A\) at the closed point \(x \in X \subset A\) provides with the prime divisor \(E_1\) over \(A\) which computes \(\text{mld}_{MJ}(x; A, I_X) = 0\).

If \(X\) is not normal crossing double at \(x\), then the exceptional divisor \(E_3\) of the third blow-up computes \(\text{mld}_{MJ}(x; A, I_X) = -\infty\). In this case, \(k_{E_3} = 4\) and \(b(E_3) = b(E_3) = 3\).

As a conclusion we obtain \((D_1)\) and \(M_1 = 4\), \(B_1 = 3\). On the other hand, by the proof of Proposition 3.7 we can take
\[
N_1 = M_1 + 1 = 5.
\]

Next we are going to prove Conjecture \(C_2\). Let us see some examples used in the proof of \(C_2\).
Example 6.2. We observe some examples of Newton polygons $\Gamma$ such that $\text{mld} \, \Gamma = -\infty$. If a hypersurface $X \subset A^3$ is defined by a (not necessarily non-degenerate) polynomial $f \in k[x_1, x_2, x_3]$ whose Newton polygon $\Gamma_f(f)$ contained in the following $\Gamma_i$, then $\text{mld}_{MJ}(0, X) = -\infty$. In the following, by applying Lemma 6.10 we obtain a bound of $\nu(X, 0)$ that is the minimal value $m$ such that $s_{m-1}(X, 0) < 0$.

1. Let $\Gamma_1$ be the Newton polygon generated by three points

$$(2, 0, 0), (0, 5, 0), (0, 0, 5) \in M = \mathbb{Z}^3.$$ 

Then, for $p = (5, 2, 2) \in N = \mathbb{M}^*$, we obtain

$$\langle p, 1 \rangle = 9, \quad \langle p, \Gamma_1 \rangle = 10.$$ 

Therefore,

$$0 > \langle p, 1 \rangle - \langle p, \Gamma_1 \rangle \geq \text{mld} \, \Gamma_1 = -\infty.$$ 

If $\Gamma_1(f) \subset \Gamma_1$ for a polynomial defining a hypersurface $X \subset A^3$, we have $k_{E_0} = 8$ and $\text{val}_{E_0}(f) \geq \langle p, \Gamma_1 \rangle = 10$, hence $\text{mld}_{MJ}(0, X) = -\infty$. We also have $\nu(X, 0) \leq 10$.

2. We use the same notation as in (1). Let $\Gamma_2$ be generated by $(2, 0, 0), (0, 3, 0)$ and $(0, 0, 7)$. Then, for $p = (21, 14, 6) \in N$, we obtain

$$\langle p, 1 \rangle = 41, \quad \langle p, \Gamma_2 \rangle = 42, \quad \text{mld}_{MJ}(0, X) = -\infty, \quad \nu(X, 0) \leq 42.$$ 

3. Let $\Gamma_3$ be generated by $(2, 0, 0), (0, 3, 1), (0, 0, 5)$. Then, for $p = (15, 8, 6) \in N$, we obtain

$$\langle p, 1 \rangle = 29, \quad \langle p, \Gamma_3 \rangle = 30, \quad \text{mld}_{MJ}(0, X) = -\infty, \quad \nu(X, 0) \leq 30.$$ 

4. Let $\Gamma_4$ be generated by $(2, 0, 0), (0, 4, 0), (0, 0, 5)$. Then, for $p = (10, 5, 4) \in N$, we obtain

$$\langle p, 1 \rangle = 19, \quad \langle p, \Gamma_4 \rangle = 20, \quad \text{mld}_{MJ}(0, X) = -\infty, \quad \nu(X, 0) \leq 20.$$ 

Note that the Newton polygon in (1) is contained in the polygon in (4). In this sense, the example (1) seems redundant. The reason why we take (1) as an example is because the valuation of $I_X$ at the prime divisor computing $\text{mld}_{MJ}$ in (1) is smaller than that of (4).

Theorem 6.3. If $\text{chark} \neq 2$, then Conjecture C2 (therefore Conjecture D2 and $U_2$ also) holds for 2-dimensional schemes and we can take $N_2 = 41, M_2 = 58$ and $B_2 = 39$.

Proof. As we saw, Conjecture C2.2 and C2.1 hold, the only problems are to show C2.0 and to obtain the bound numbers, $N_2, M_2$ and $B_2$.

For $\text{chark} = 0$, aside the bound numbers, Conjecture C2.0 is proved in [12]. Actually (C2.0) is translated into the following:

For a non-singular point $(A, x)$ of $\text{dim} \, A = N = 3, 4$, there is an integer $M \in \mathbb{N}$ independent of the choice of 2-dimensional subscheme $X \subset A$ such that if the inequality

$$a(E; A, I_X^{N-2}) \geq 0$$

holds for every prime divisor $E$ over $A$ with the center at $x$ satisfying $k_E \leq M$, then

$$\text{mld}(x; A, I_X^{N-2}) \geq 0$$

holds.
This is proved in [12] Proposition 3.3 under a more general setting and it completes the proof of (C2) for characteristic 0.

Here we give a proof which works for any characteristic except for 2 and give bound values of $N_2$, $M_2$ and $B_2$. (As one sees in the proof, the values may not be optimal.) We divide the problem into 6 cases and will check each. Among these cases only the 6th case needs the condition that $\text{char} k \neq 2$ for the proof.

**Case 1.** Assume $\text{emb}(X, 0) \geq 5$.

In this case, $s_1(X, 0) = 2 \cdot 2 - \dim X_1(0) = 4 - \text{emb}(X, 0) < 0$, therefore $\nu(X, 0) \leq 2$. On the other hand the exceptional divisor $E_1$ of the first blow-up computes $\text{mld}_{MJ}(x; X) = -\infty$.

**Case 2.** Assume $\text{emb}(X, 0) = 4$ and $\text{ord}_0 I_X \geq 3$.

In this case, $s_2(X, 0) = 3 \cdot 2 - \dim X_2(0) = 6 - 8 < 0$, therefore $\nu(X, 0) \leq 3$. There is a prime divisor $E$ computing $\text{mld}_{MJ}(x; X) = -\infty$ such that $k_E \leq 4$ and $b(E) \leq 2$.

**Case 3.** Assume $\text{emb}(X, 0) = 4$ and $\text{ord}_0 I_X = 2$.

In this case, $(X, 0) \subset (\mathbb{A}^4, 0)$ is of maximal type, because

$$\text{codim}(X, \mathbb{A}^4) \cdot \text{ord}_0 I_X = 2 \cdot 2 = 4 = \dim \mathbb{A}^4.$$  

As we know in Theorem [5.1] that Conjecture $C_1$ holds for the category of schemes, applying Proposition [5.18] we obtain that $C_{2,0}$ holds and in this case,

$$N_{2,0} \leq 3 \cdot 11 - 4 = 29.$$  

Therefore, $\nu(X, 0) \leq 30$. There is a prime divisor $E$ over $A$ computing $\text{mld}_{MJ}(x; X) = -\infty$ such that $k_E \leq 58$ and $b(E) \leq 28$.

**Case 4.** Assume $\text{emb}(X, 0) = 3$ and $\text{ord}_0 I_X \geq 4$.

In this case, $s_3(X, 0) = 4 \cdot 2 - \dim X_3(0) = 8 - 9 < 0$, therefore $\nu(X, 0) \leq 4$. There is a prime divisor $E$ over $A$ computing $\text{mld}_{MJ}(x; X) = -\infty$ such that $k_E \leq 3$ and $b(E) \leq 3$.

**Case 5.** Assume $\text{emb}(X, 0) = 3$ and $\text{ord}_0 I_X = 3$.

In this case, $(X, 0) \subset (\mathbb{A}^3, 0)$ is of maximal type, because

$$\text{codim}(X, \mathbb{A}^3) \cdot \text{ord}_0 I_X = 1 \cdot 3 = 3 = \dim \mathbb{A}^3.$$  

We know Conjecture $C_1$ holds. Therefore, applying Proposition [5.18] we obtain that $C_{2,0}$ holds and in this case,

$$N_{2,0} \leq (3 + 1) \cdot 11 - 9 = 35.$$  

Therefore, $\nu(X, 0) \leq 36$. There is a prime divisor $E$ over $A$ computing $\text{mld}_{MJ}(x; X) = -\infty$ such that $k_E \leq 35$ and $b(E) \leq 34$.

**Case 6.** Assume $\text{emb}(X, 0) = 3$ and $\text{ord}_0 I_X = 2$.

In this case, the singularity is a hypersurface double point. So we let it be defined by $f \in k[[x, y, z]] = \mathbb{A}^3_{x,y,z}$ such that $\text{mult}_0 f = 2$. For simplicity, we denote $\mathbb{A}^3$ by $A$.

By Weierstrass Theorem for the formal power series ring $k[[x, y, z]]$ one has the presentation of $f$ as follows:

$$f = x^2 + xy(y, z) + h(y, z),$$

where $g, h \in k[[y, z]]$. As $\text{char} k \neq 2$, we can make the Tschirnhausen transformation $x + (1/2)g = x'$ to have

$$f = x'^2 + h'(y, z).$$

Hence, by the coordinate change of the formal power series ring $k[[x, y, z]]$, we may write

$$f = x^2 + h(y, z),$$

(19)
where \( h \in k[[y, z]] \). Here, we may assume that \( \text{mult}_0 h \geq 3 \). Because if \( \text{mult}_0 h = 2 \), then by [9], \( X \) has the singularity with \( \text{mld}_{X,0}(0; X) = 1 \) and \( C_2 \) is already proved in this case and \( \nu(X, 0) \leq 6 \).

On the other hand, we may also assume that \( \text{mult}_0 h \leq 4 \). Because if \( \text{mult}_0 h \geq 5 \), then the toric divisor \( E_p \) corresponding to \( p = (5, 2, 2) \) satisfies \( a(E_p, A, I_X) < 0 \) as we have seen in Example 6.2 (1). Therefore by Lemma 5.10 we have \( \nu(X, 0) \leq 10 \).

Now we have to consider only two classes, \( \text{mult}_0 h = 3 \) and \( \text{mult}_0 h = 4 \). Each class will be divided into several classes according to the form of the initial term of \( h \). Therefore, \( \text{inh} \) be the initial term of \( h \), then it is a homogeneous polynomial of two variables. Therefore, \( \text{inh} \) is presented as the product of linear forms. Now we divide all \( f = x^2 + h(y, z) \), that must be considered, into the following 8 classes:

**Class A** \( \text{mult}_0 h = 3 \). The following \( l \) and \( l_i \) \((i = 1, 2, 3)\) are linear forms with \( l_i \neq l_j \) \((i \neq j)\).

- (Class A-1) \( \text{inh} = l_1 l_2 l_3 \). (Class A-2) \( \text{inh} = l_1^2 l_2 \). (Class A-3) \( \text{inh} = l_1^3 \).

**Class B** \( \text{mult}_0 h = 4 \). The following \( l \) and \( l_i \) \((i = 1, \ldots, 4)\) are linear forms with \( l_i \neq l_j \) \((i \neq j)\).

- (Class B-1) \( \text{inh} = l_1 l_2 l_3 l_4 \). (Class B-2) \( \text{inh} = l_1^2 l_2 l_3 \). (Class B-3) \( \text{inh} = l_1^2 l_2^2 \).
- (Class B-4) \( \text{inh} = l_1^3 l_2 \). (Class B-5) \( \text{inh} = l_1^4 \).

Our strategy for the proof of \( C_2 \) is to show one of the following for each class among (A-1)–(B-5):

(i) \( f \) is non-degenerate with respect to all faces of \( \Gamma(f) \).

(ii) \( X \) has an isolated singularity at 0 and \( f \) is non-degenerate with respect to all compact faces of the Newton polygon \( \Gamma(f) \).

(iii) We find a prime divisor \( E \) over \( A \) with center at 0 such that \( a(E, A, I_X) = 0 \) and \( \text{val}_X I_X \leq n \) for some fixed \( n \) and prove that \((A, I_X)\) is log canonical by constructing a log-resolution of \((A, I_X)\).

(iv) We find a Newton polygon \( \Gamma \) such that \( \text{mld}(\Gamma) = -\infty \) and \( \Gamma(f) \subset \Gamma \).

Indeed, if we prove one of the above (i)–(iv) for every \( f \) described as in [19], then the proof of Conjecture \( C_2 \) for Case 6 will be complete. Because if we prove (i) or (ii), we can apply Proposition 5.7 to get \( C_2 \). If we prove (iii), then it shows that \( E \) computes \( \text{mld}(0; A, I_X) = 0 \) and \( \text{val}_X f \leq n \), i.e., \( \nu(X, 0) = \mu(X, 0) \leq n \). Then, \( C_2 \) holds in this class. If we prove (iv), a toric divisor \( E \) which computes \( \text{mld}(\Gamma) = -\infty \) also computes \( \text{mld}(0; A, I_X) = -\infty \). Therefore by Proposition 5.7 we obtain \( C_2 \) for this class. Now we start to pursue the strategy.

(Class A-1) \( \text{inh} = l_1 l_2 l_3 \).

In this case, we will show (ii). First we can see that \( h \) is reduced, since the initial term of \( h \) is reduced. Therefore \( X \) has an isolated singularity at 0. Next we will show the non-degeneracy of \( f \) with respect to the compact faces of the Newton polygon \( \Gamma(f) \).

By a coordinates transformation in \( k[[y, z]] \) we may assume that
\[
l_1 = y, \quad l_2 = z, \quad l_3 = y + z.
\]

Then, looking at the Newton polygon \( \Gamma(f) \), we can see that a compact face \( \sigma \) of \( \Gamma(f) \) is either a compact face \( \tau \) of \( \Gamma(h) \) or the convex hull \( \sigma = \langle \langle (2, 0, 0) \rangle, \tau \rangle \rangle \) generated by \( (2, 0, 0) \) and a compact face \( \tau \) of \( \Gamma(h) \). Here, we denote the convex hull of the set \( S \) by \( \langle \langle S \rangle \rangle \).
If a compact face $\sigma$ is of type $\langle \langle (2, 0, 0), \tau_1 \rangle \rangle$, then $f$ is non-degenerate with respect to $\sigma$. Because in this case, $f_\sigma$ is represented as $f_\sigma = x^2 + h_\sigma(y, z)$ and the singular locus of the hypersurface defined by $f_\sigma$ must be in the zero locus of $x$ (here, we use the assumption that $chark \neq 2$). Therefore, the rest of the faces which we should check the non-degeneracy of $f$ are the compact faces $\tau$ of $\Gamma(h)$. (This argument will work for all classes in Class A and B.)

Now we check the non-degeneracy of $f$ with respect to the compact faces of $\Gamma(h)$. The compact face generated by in $h = yz(y + z)$ is

$$\gamma = \langle \langle (2, 1), (1, 2) \rangle \rangle \subset \Gamma(h) \subset \mathbb{R}^2$$

and $f_\gamma = h \gamma = yz(y + z)$ is clearly non-degenerate. Here, we list the other possible compact faces of $\Gamma(h)$ and check the non-degeneracy of $f$ there.

- $\tau_1 = \langle \langle (2, 1), (a, 0) \rangle \rangle$ ($a \geq 4$), $f_{\tau_1} = h_{\tau_1} = y^2(z - \alpha y^{a-2})$, ($a \in k$)
- $\tau_2 = \langle \langle (0, b), (1, 2) \rangle \rangle$ ($b \geq 4$), $f_{\tau_2} = h_{\tau_2} = z^2(y - \beta z^{b-2})$, ($\beta \in k$)

By the form of $f_{\tau_1}$, it is clear that $f$ is non-degenerate with respect to the face $\tau_1$ ($i = 1, 2$). This completes the proof of the fact that $f$ is non-degenerate with respect to all compact faces of $\Gamma(f)$, which yields the proof of (i). In this case, by the formula in Proposition 5.5 we obtain $\mld_{MJ}(0; X) = 1$ and the prime divisor $E_p$ ($p = (3, 2, 2)$) computes it. As $\val_{E_p}f = (p, \Gamma(f)) = 6$, we obtain $\nu(X, 0) \leq 6$ and $k_{E_p} = 6$

(Class A-2) $inh = l_1^2l_2$

In this case, we will prove that either (i) or (ii) holds and $\mld_{MJ}(0; X) = 1$ and $\nu(X, 0) \leq 6$. By a coordinate change, we may assume that $l_1 = y$ and $l_2 = z$.

First we consider the case that $h$ is reduced. In this case $f$ has an isolated singularity at $0$. According to the argument as in A-1, we have only to check the non-degeneracy of $f$ with respect to the compact faces of $\Gamma(h)$. The compact face generated by in $h = y^2z$ is

$$\gamma = \langle \langle (2, 1) \rangle \rangle \subset \Gamma(h) \subset \mathbb{R}^2$$

and $f_\gamma = h \gamma = y^2z$ which is clearly non-degenerate. Here, we list the other possible compact faces of $\Gamma(h)$ and check the non-degeneracy of $f$ there.

- $\tau_1 = \langle \langle (0, b), (1, 2) \rangle \rangle$ ($a \geq 4$), $f_{\tau_1} = h_{\tau_1} = y^2(z - \alpha y^{a-2})$, ($\alpha \in k$)
- $\tau_2 = \langle \langle (1, 1), (b, 0) \rangle \rangle$ ($b \geq 3$), $f_{\tau_2} = h_{\tau_2} = z^2(y - \beta z^{b-2})$, ($\beta \in k$)
- $\tau_1 = \langle \langle (1, b), (0, c) \rangle \rangle$ ($e \geq b + 2$), $f_{\tau_1} = h_{\tau_1} = z^2(y - \lambda z^{c-2})$, ($\lambda \in k$)
- $\tau_2 = \langle \langle (2, 1), (0, d) \rangle \rangle$ ($d \geq 4$), $f_{\tau_2} = h_{\tau_2} = z(y^2 - \mu z^{c-1})$, ($\mu \in k$)

By the form of $f_{\tau_1}$ and $f_{\tau_2}$, it is clear that $f$ is non-degenerate with respect to each face. Here, we note that we use $chark \neq 2$ only for the proof of $\tau_2$. This completes the proof of the fact that $f$ is non-degenerate with respect to all compact faces of $\Gamma(f)$, which yields the proof of (ii). In this case, by the formula Proposition 5.5 we obtain $\mld_{MJ}(0; X) = 1$ and the same prime divisor $E_p$ ($p = (3, 2, 2)$) as in A-1 computes it. As $\val_{E_p}f = 6$, we obtain $\nu(X, 0) \leq 6$.

Next we consider the case that $h$ is not reduced. In this case, $h$ is decomposed as

$$h = h_1^2 \cdot h_2,$$

where $inh_1 = y$ and $inh_2 = z$. Then by a coordinate change in $k[[y, z]]$, we can put $h_1 = y$ and $h_2 = z$. Hence we obtain

$$f = x^2 + y^2z,$$

which gives that $(X, 0)$ is the pinch point and we already know in [3] that $\mld_{MJ}(0; X) = 1$ and $\nu(X, 0) \leq 6$. 


(Class A-3) \( \text{inh} = l^3 \).

Under this situation, we will show (iii) in some cases, (iv) in some of the other cases and reduce to the case A-1 and A-2 in the rest of the cases. We may assume that \( l = y \).

(A-3-1) First, if \( \Gamma(h) \subset \Gamma((3, 0), (0, 7)) \), then \( \Gamma(f) \) is contained in \( \Gamma((2, 0, 0), (0, 3, 0), (0, 0, 7)) \) generated by \( (2, 0, 0), (0, 3, 0), (0, 0, 7) \) in Example 6.2 (2). This shows that \( \text{mld}_{M}(0; X) = -\infty \) and it is computed by the prime divisor \( E_p \), where \( p = (21, 14, 6) \in N \), therefore we obtain \( \nu(X, 0) \leq 42 \). In this case \( k_{E_p} = 40 \), and therefore by Proposition 3.9 it follows \( b(E_p) \leq 39 \).

Now we may assume that there is an integer point \( P \) on the boundary of \( \Gamma(h) \) such that \( P \not\in \Gamma((3, 0), (0, 7)) \). Then the possible coordinates of \( P \) are

\[
(20) \quad (2, 2), (1, 3), (1, 4), (0, 4), (0, 5) \quad \text{and} \quad (0, 6).
\]

(A-3-2) Assume that \( h \) is reduced, then \( X \) has an isolated singularity at 0.

When, in particular either \( (0, 4) \) or \( (0, 5) \) is on the boundary of \( \Gamma(h) \), then every other point in the list in (20) is not on the boundary and

\[
f = x^2 + \alpha y^3 + \beta z^4 + \text{(higher term)} \quad (i = 4, 5, \alpha, \beta \in \mathbb{C})
\]

is non-degenerate with respect to the compact faces of \( \Gamma(f) \) and \( \text{mld}_{M}(0; X) = 1 \) and this case we already know that \( \nu(X, 0) \leq 6 \).

Next, when \((1, 3)\) is on the boundary of \( \Gamma(h) \), then every other point in the list (20) is not on the boundary and \( f = x^2 + \alpha y^3 + \beta yz^3 + \text{(higher term)} \) is non-degenerate with respect to the compact faces of \( \Gamma(f) \) and \( \text{mld}_{M}(0; X) = 1 \) and this case we already know that \( \nu(X, 0) \leq 6 \).

Next, note that the remaining points \((2, 2), (1, 4)\) and \((0, 6)\) are lying on the segment connecting \((3, 0), (0, 6)\). If some of these three points are lying on the boundary of \( \Gamma(h) \), denote the face generated by \((3, 0)\) and these points by \( \gamma \). Then, decompose \( h \) as follows:

\[
h = h_{\gamma} + h',
\]

where \( \Gamma(h') \subset \Gamma((3, 0), (0, 7)) \). Here, \( h_{\gamma} \) is a homogeneous polynomial of degree 3 in the variables \( y \) and \( Z = z^2 \). Therefore it is decomposed into the products of linear forms in \( y \) and \( Z \) as follows:

\[
h_{\gamma} = L_1 L_2 L_3, \quad \text{or} \quad L_1^2 L_2, \quad \text{or} \quad L_3.
\]

By an appropriate coordinate change, we may assume that \( L_1 = L = y, L_2 = z^2 \) and \( L_3 = y + z^2 \). Then in the last case we have the expression:

\[
h = y^3 + h''
\]

where \( \Gamma(h'') \subset \Gamma((3, 0), (0, 7)) \). Therefore, we can reduce this case to (A-3-1).

In the first two cases for \( h_{\gamma} \), we can see that \( f \) is non-degenerate with respect to \( \gamma \) and also non-degenerate with respect to the other possible cases:

- \( \tau_1 = \langle \langle (2, 2), (1, a) \rangle \rangle, a \geq 5, \quad f_{\tau_1} = h_{\tau_1} = yz^2(y + \alpha z^{-a}) \).
- \( \tau_2 = \langle \langle (1, a), (0, b) \rangle \rangle, b - 2 \geq a \geq 5, \quad f_{\tau_2} = h_{\tau_2} = z^a(y + \beta z^{-a}) \).
- \( \tau_3 = \langle \langle (1, 0), (0, c) \rangle \rangle, c \geq 7, \quad f_{\tau_3} = h_{\tau_3} = z^b(y + \lambda z^{-c}) \).

Therefore in this case \( \text{mld}_{M}(0; X) = 0 \) by the formula in Proposition 6.3 and this value is computed by the prime divisor \( E_p \), where \( p = (3, 2, 1) \). We have \( \nu(X, 0) \leq 6 \) and \( k_{E_p} = 5 \).

(A-3-3) Assume that \( h \) is not reduced.

There are two possibilities: \( h = h_1^{\nu} \) and \( h = h_2^{\nu} h_2 \). In both cases \( \text{inh}_1 = \text{inh}_2 = y \). Therefore, by the coordinate change, we may assume that \( h_1 = y \) in both cases. Then, in the first case we have:

\[
f = x^2 + y^3,
\]
which implies $\Gamma(f) \subset \Gamma((2,0,0),(0,3,0),(0,0,7))$, which can be reduced to the case (A-3-1).

While, in the second case we have

$$f = x^3 + y^2(\alpha y + h_2'), \quad (\alpha \in k)$$

where $\text{mult}_0 h_2' \geq 2$. Here, if $h_2'$ does not contain the monomial $z^2$ as a summand, then $\Gamma(f) \subset \Gamma((2,0,0),(0,3,0),(0,0,7))$, which is again reduced to the case (A-3-1).

When $h_2$ contains the monomial $z^2$ as a summand, then the singular locus $\text{Sing}(X)$ is defined by $x = y = 0$. Let $A' \to A$ be the blow-up with the center $\text{Sing}(X)$ and then let $A'' \to A'$ be the blow-up with the center at the origin $0' \in \text{Spec} k[y,x/y,z] \subset A'$. Then the composite $A'' \to A' \to A$ becomes a log-resolution of $(A, I_X)$ and the log-discrepancies $a(E; A, I_X) \geq 0$ for every prime divisor $E$ appearing on $A''$. Hence $(A, I_X)$ is log canonical, i.e., $X$ is MJ-log canonical.

On the other hand we have $a(\mathcal{E}_{(3,2,1)}, A, I_X) = 0$, which implies that the prime divisor $E_{(3,2,1)}$ computes $\text{mld}_{MJ}(0; X) = 0$, therefore $\nu(X,0) \leq 6$ and $k_{E_6} = 5$.

(Class B-1) $\text{inh} = l_1 l_2 l_3 l_4$.

In this case $h$ is reduced, and therefore $X$ has an isolated singularity at 0. By a coordinate change, we may assume that $l_1 = y, l_2 = z, l_3 = y + z, l_4 = y - z$. Then, we can see that $h$ is non-degenerate with respect to the face $\gamma$ corresponding to $\text{inh}$. On the other hand, also with respect to the other possible faces $\tau_1 = \langle \langle (a, 0), (3,1) \rangle \rangle$ and $\tau_2 = \langle \langle (1,3),(0,b) \rangle \rangle$, $h$ is non-degenerate. This can be checked in the same way as in (A-1). Therefore, by the formula in Proposition[53] we obtain $\text{mld}_{MJ}(0; X) = 0$ and the prime divisor $E_{(2,1,1)}$ computes it. Hence, $\nu(X,0) \leq 4$.

(Class B-2) $\text{inh} = l_1^2 l_2 l_3$. In this case, by a coordinate transformation we may assume that $l_1 = y, l_2 = z, l_3 = y + z$.

(B-2-1) Assume that $h$ is reduced. Then $X$ has an isolated singularity at 0. Let $\gamma$ be the compact face corresponding to $\text{inh}$, then $\gamma = \langle \langle (3,1),(2,2) \rangle \rangle$. We can see that $h$ is non-degenerate with respect to $\gamma$, as $h_{\gamma} = \text{inh} = y^2 z(y + z)$. On the other hand, also with respect to the other possible faces

$\tau_1 = \langle \langle (a,0),(3,1) \rangle \rangle (a \geq 5),$

$\tau_2 = \langle \langle (2,2),(1,b) \rangle \rangle (b \geq 4),$

$\tau_3 = \langle \langle (1,b),(0,c) \rangle \rangle (c \geq 6) \quad \text{and}$

$\tau_2' = \langle \langle (2,2),(0,d) \rangle \rangle (d \geq 5),$

$h$ is non-degenerate. This can be proved in the same way as in (A-2). Here, we note that we use $\text{char} k \neq 2$ for the proof of non-degeneracy with respect to $\tau_2'$. In this case we also have $\text{mld}_{MJ}(0; X) = 0$ and the prime divisor $E_{(2,1,1)}$ computes it. Hence, $\nu(X,0) \leq 4$.

(B-2-2) Assume that $h$ is not reduced. Then, by a coordinate transformation of $k[[y,z]]$, we can take $h = h_1 h_2$, such that $h_1 = y$ and $\text{inh}_2 = z(y + z)$. In this case $\text{Sing}(X)$ is defined by $x = y = 0$. As in (A-3-3), let $\varphi : X' \to A$ be the blow-up with the center $\text{Sing}(X)$, then the proper transform $X' \subset X$ of $X$ has an isolated singularity at a point, say $0' \in X'$. Compose $\varphi$ with the blow-up $A'' \to A'$ with the center $0'$, then $A'' \to A' \to A$ becomes a log-resolution of $(A, I_X)$. The log-discrepancies $a(E; A, I_X) \geq 0$ for every prime divisor $E$ appearing on $A''$. Hence $(A, I_X)$ is log canonical, i.e., $X$ is MJ-log canonical. On the other hand we have $a(\mathcal{E}_{(2,1,1)}, A, I_X) = 0$, which implies that the prime divisor $E_{(2,1,1)}$ compute the $\text{mld}_{MJ}(0; X) = 0$, therefore $\nu(X,0) \leq 4$.

(Class B-3) $\text{inh} = l_1^2 l_2^2$. By a coordinate change, we may assume that $l_1 = y$ and $l_2 = z$. 

(B-3-1) Assume that $h$ is reduced, then $X$ has an isolated singularity at 0. The possible compact faces of $\Gamma(h)$ are:
the same $\gamma_2, \gamma_3$ and $\gamma'_2$ as in (B-2-1) and
the symmetric faces $\gamma_2, \gamma_3$ and $\gamma'_2$ of them with respect to $y$ and $z$.
Therefore, $f$ is non-degenerate with respect to all compact faces of $\Gamma(f)$ and $\mld_{MJ}(0; X) = 0$ with $E_{(2,1,1)}$ as a computing prime divisor. Hence, $\nu(X, 0) \leq 4$.

(B-3-2) Assume that $h$ is not reduced. In this case, there are two possibilities:
h = $h_1^2 h_2$, where $h_2$ is reduced, and $h = h_1^2 h_2^2$.
In the first case, $h = h_1^2 h_2$, where $h_2$ is reduced, we can put $h_1 = y$ and $ih_2 = z^2$ by the coordinate change, as we may assume that $ih_1 = y$. In this case the singular locus $\Sing(X)$ of $X$ is defined by $x = y = 0$. Let $\varphi : A^{(1)} \to A$ be the blow-up of $A$ with the center $\Sing(X)$. Then the proper transform $X^{(1)}$ of $X$ in $A^{(1)}$ has an isolated singularity at a point $0_1$ which is $A_n$-singularity. The composite of the successive blow-ups at the singularities and $\varphi$:
\[
A^{(m)} \to A^{(m-1)} \to \cdots A^{(1)} \to A,
\]
gives a log-resolution of $(A, I_X)$. Here, we observe that every exceptional divisor $E$ has log-discrepancy $\nu(E; A, I_X) = 0$, therefore $\mld_{MJ}(0; X) = 0$ and $E_2$ computes it. As $\val_{E, I_X} = 4$, we can see that $\nu(X, 0) \leq 4$.
In the second case, $h = h_1^2 h_2^2$, we can put $h_1 = y$ and $h_2 = z$ by the coordinate change, as we may assume that $ih_1 = y$ and $ih_2 = z$. Hence, we obtain
\[
f = x^2 + y^2 z^2
\]
which is non-degenerate with respect to all faces of $\Gamma(f)$. In this case $\mld_{MJ}(0; X) = 0$ and the divisor $E_{(2,1,1)}$ computes it. Therefore, $\nu(X, 0) \leq 4$.

(Class B-4) $ih = l_1^3 l_2$. By the coordinate change, we may assume that $l_1 = y$ and $l_2 = z$. Then,
\[
\Gamma(f) \subset \Gamma((2, 0, 0), (0, 3, 1), (0, 0, 5)),
\]
which yields that $\mld_{MJ}(0; X) = -\infty$ and the prime divisor $E_{(15,8,6)}$ computes it and $\nu(X, 0) \leq 30$ by Example 6.2 (3).

(Class B-5) $ih = l^4$. By the coordinate change, we may assume that $l = y$. Then $f$ is of the form:
\[
f = x^2 + y^4 + (\text{terms of degree } \geq 5).
\]
Then,
\[
\Gamma(f) \subset \Gamma((2, 0, 0), (0, 0, 4), (0, 0, 5)),
\]
which yields that $\mld_{MJ}(0; X) = -\infty$ and the prime divisor $E_{(10,5,4)}$ computes it and $\nu(X, 0) \leq 20$ by Example 6.2 (4).

Now we obtain $\nu(X, 0)$ for all cases. In each case, we can calculate also the bounds of $k_E$ and $b(E)$ for a prime divisor $E$ computing $\mld_{MJ}(x; X)$. As conclusions, $\nu(X, 0) = \mu(X, 0) \leq 42, k_E \leq 58$ and $b(E) \leq 39$ for all $(X, 0)$ and a prime divisor $E$ computing $\mld_{MJ}(x; X)$.

Corollary 6.4. Assume the characteristic of the base field $k$ is not 2. Then, Conjecture $C_2$ holds in the category of normal locally complete intersection singularities of dimension 2 over $k$.

In particular, for every singularity $(X, x)$ in this category there is a prime divisor $E$ over $X$ computing $\mld(x; X) = \mld_{MJ}(x; X)$ such that $b(E) \leq 20$.

Proof. The first statement follows from Theorem 6.3 because the equality $\mld(x; X) = \mld_{MJ}(x; X)$ holds for locally a complete intersection singularity $(X, x)$. For the
second statement, let $X \subset A$ be a closed immersion into a non-singular variety $A$ of dimension $N = \text{emb}(X, x)$. Let

$$A^{(n)} \xrightarrow{\varphi_0} A^{(n-1)} \rightarrow \cdots \rightarrow A^{(1)} \xrightarrow{\varphi_1} A^{(0)} = A$$

be the minimal sequence of blow-ups to obtain a prime divisor $\widetilde{E} \subset A^{(n)}$ computing $\text{mld}_M(x; X)$ and let $E_i$ be the exceptional divisor dominant to the center of $\varphi_i$ and $\tilde{E}^{(n)} = \tilde{E}$.

We already know the following:

(i) If $N \geq 5$, then $n = 1$ by Proposition \textbf{3.9}.

(ii) if $N = 4$, then $k_{\tilde{E}} \leq 58$ and $n \leq 29$ (cf. Case 3 in the proof of Theorem \textbf{6.3}), and

(iii) if $N = 3$, then $k_{\tilde{E}} \leq 40$ and $n \leq 39$ (cf. Case 6, A-3-1).

**Claim.** For an irreducible component $E^{(n)}$ of $\tilde{E}^{(n)} \cap X^{(n)}$ there exists a prime divisor $E$ over $X$ such that $E$ computes $\text{mld}(x; X)$ and has the center $E^{(n)}$ on $X^{(n)}$.

Once the claim is proved, then the required statement of the corollary follows. Indeed, first we know $b(E) \leq n = b(\tilde{E})$, by the definition of $b(E)$. Then the sequence \textbf{(21)} consists of

- $b(E)$-times blow-ups with 0-dimensional centers and
- $(b(\tilde{E}) - b(E))$-times blow-ups with 1-dimensional centers.

Therefore, it follows

$$k_{\tilde{E}} \geq (N - 2)(b(\tilde{E}) - b(E)) + (N - 1)b(E), \text{ hence we have }$$

$$b(E) \leq \frac{k_{\tilde{E}}}{N - 1}.$$ 

Now in the case (i), we have $b(E) = 1$. In the case (ii) and (iii), we have

$$b(E) \leq \frac{58}{3} \text{ and } b(E) \leq \frac{40}{2} \text{ respectively,}$$

which shows the required statement.

Now we are going to prove the claim. Let $\varphi_{n+1} : A^{(n+1)} \rightarrow A^{(n)}$ be the blow-up with the center $E^{(n)}$ which is contained in $X^{(n)}$. Let $p_n \in E^{(n)}$ be the generic point and let $\tilde{E}^{(n+1)}$ be the exceptional divisor of $\varphi_{n+1}$ dominating $E^{(n)}$. Then, we have

$$\text{mld}(x; A, I_X) \leq k_{\tilde{E}^{(n+1)}} - c \cdot \text{val}_{\tilde{E}^{(n+1)}} I_X + 1$$

$$\leq (k_{\tilde{E}^{(n)}} + c) - c \cdot \text{val}_{\tilde{E}^{(n)}} I_X = \text{mult}(X^{(n)}, p_n) + 1$$

$$\leq k_{\tilde{E}^{(n)}} - c \cdot \text{val}_{\tilde{E}^{(n)}} I_X + 1 = \text{mld}(x; A, I_X).$$

Here, the middle inequality follows from $\text{codim}(E^{(n)}, A^{(n)}) = c + 1$. Hence, we obtain that $\tilde{E}^{(n+1)}$ also computes $\text{mld}(x; A, I_X)$.

Now, let

$$A^{(m)} \xrightarrow{\varphi_m} A^{(m-1)} \rightarrow \cdots \rightarrow A^{(n+1)} \xrightarrow{\varphi_{n+1}} A^{(n)}$$

be the sequence of blow-ups so that $X_m$ is non-singular and crossing $\tilde{E}^{(m)}$ normally at the generic point $p_m$ of an irreducible component $E^{(m)}$ of $\tilde{E}^{(m)} \cap X_m$, where $E^{(m)}$ is dominant to $E^{(n)}$. Then, applying the same discussion as in \textbf{(22)} to each blow-up of the sequence \textbf{(23)}, we obtain that $\tilde{E}^{(m)}$ computes $\text{mld}(x; A, I_X)$.

Now we have

$$\text{mld}(x; X) \leq k_{E^{(m)}} + 1 \leq k_{E^{(m)}} + 1 - c \cdot \text{val}_{E^{(m)}, I_X} = \text{mld}(x; A, I_X).$$

Therefore $E^{(m)}$ is a required prime divisor over $X$ in the claim. \hfill \Box
Remark 6.5. The minimal value $\hat{b}(E)$ such that $E$ computes $\text{mld}_{MJ}(x; X)$ is not bounded for all locally complete intersection singularities. Actually for a singularity $(X, 0) \subset \mathbb{A}^2$ defined by $x^2 - y^m = 0$ for odd $m$. Then, in order to obtain a variety normal at the generic point of the prime divisor computing $\text{mld}_{MJ}(0; X) = -\infty$, the necessary number of blow-ups tends to infinity, when $m \to \infty$.

Remark 6.6. In the theorem for surfaces we assume that $\text{chark} \neq 2$. The only case we assume this condition is for hypersurface double points. The proof of $C_2$ for $\text{chark} = 2$ will be treated in another paper. This is because we should take care of more cases for $\text{chark} = 2$ than considered here and the volume of the proof may exceed the capacity of a paper of a reasonable size.

Remark 6.7. In this paper, we concentrate only on the singularities of $X$, i.e., the singularity of the trivial pair $(X, \mathcal{O}_X)$. Because it is the skeleton of the structure and seeing this first would help the further work on singularities of general pairs $(X, a^n)$.

References

[1] T. De Fernex, R. Docampo, Jacobian discrepancies and rational singularities, J. Eur. Math. Soc. 16 (2014), 165–199.
[2] L. Ein and M. Mustaţă, Jet schemes and singularities, Proc. Symp. Pure Math. 80, 2, (2009) 505–546.
[3] L. Ein and S. Ishii, Singularities with respect to Mather–Jacobian discrepancies, MSRI Publications, 67 (2015) 125–168.
[4] N. Hara and K-i. Watanabe, $F$-regular and $F$-pure rings vs. log terminal and log canonical singularities, J. Algebraic Geom., 11 (2002) 363–392.
[5] S. Ishii, A resolution of singularities of a toric variety and non-degenerate hypersurface, Proc. Trieste Singularity Summer School and Workshop 2005, World Scientific (2007) 354–369.
[6] S. Ishii, Jet schemes, arc spaces and the Nash problem, C.R.Math. Rep. Acad. Canada, 29 (2007) 1–21.
[7] S. Ishii, Mather discrepancy and the arc spaces, Ann. de l’Institut Fourier, 63, (2013) 89–111.
[8] S. Ishii and A. Reguera, Singularities with the highest Mather minimal log discrepancy, Math. Zeitsschrift., 275, Issue 3-4, (2013) 1255–1274.
[9] S. Ishii and A. Reguera, Singularities in arbitrary characteristic via jet schemes Preprint, [arXiv:1510.05210] to appear in Hodge theory and $L^2$ analysis (2016).
[10] S. Ishii and W. Niu, A strongly geometric general residual intersection preprint, 2016, [arXiv:1611.01581] to appear in Contemporary Math..
[11] J. Kollár, Rational Curves on Algebraic Varieties, Springer-Verlag, Ergebnisse der Math. 32, (1995).
[12] M. Mustaţă and Y. Nakamura, Boundedness conjecture for minimal log discrepancies on a fixed germ. Preprint, arXiv: 1502.00837v3.
[13] M. Oka, Non-Degenerate Complete Intersection Singularity, Actualités Math. , Herman, (1997) 309 pages.
[14] K. Sato and S. Takagi, General hyperplane sections of threefold in positive characteristic, preprint,
[15] K. Shibata and N. D. Tam, Characterization of 2-dimensional normal Mather-Jacobian log canonical singularities, preprint 2015, to appear in Tohoku Math. J.
[16] Z. Zhu, Log canonical thresholds in positive characteristic, preprint, [arXiv:1308.5445]

Shihoko Ishii,
Department of Mathematics, Tokyo Woman’s Christian University, 2-6-1 Zenpukuji, Suginami, 167-8585 Tokyo, Japan.