Conjectures on the Khovanov and Knot Floer Homologies of Legendrian and Transversely Simple Knots

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Abstract
A theorem of Kronheimer and Mrowka states that Khovanov homology is able to detect the unknot [20]. That is, if a knot has the Khovanov homology of the unknot, then it is equivalent to it. This result hold for the trefoils [8] and the figure-eight knot [7]. These are the simplest of the Legendrian simple knots. It is conjectured that Khovanov and Knot Floer homology are able to distinguish Legendrian and Transversely simple knots. Using the torus and twist knots, numerical evidence is provided for all knots up to 17 crossings.

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1 Legendrian and Transverse Knots and Links

A knot is a smooth embedding of the circle $S^1$ into $\mathbb{R}^3$. A link is a smooth embedding of $N \in \mathbb{N}$ disjoint circles in $\mathbb{R}^3$. We may impose extra structure by considering the standard contact structure of $\mathbb{R}^3$. This is an assignment of a plane to every point in $\mathbb{R}^3$ such that there is no surface $M \subset \mathbb{R}^3$ where for every
$p \in M$ the tangent plane $T_pM$ is given by the plane of the contact structure. In $\mathbb{R}^3$ this is described by the one-form $dz - y \, dx$, the plane at $(x, y, z)$ being spanned by the vectors $\partial_x + y \partial_z$ and $\partial_y$. The hyperplane distribution is shown in Fig. 1. While there are no everywhere tangent surfaces, it is possible for a curve to be everywhere tangent to this distribution of planes. A Legendrian link is a link that is everywhere tangent to the contact structure. A gentle introduction to the subject can be found in [22].

Two Legendrian links are considered to be Legendrian equivalent if there is an isotopy $H : L \times [0, 1] \to \mathbb{R}^3$ between them, $L = \bigsqcup_{k=0}^{N-1} S^1$, such that for all $t \in [0, 1]$ the link $H_t$ is Legendrian. It is possible for two links to be topologically equivalent but not Legendrian. In the other direction, any link can be made Legendrian by an appropriate isotopy (See the introduction of [26]). The two classical invariants of Legendrian links are the Thurston-Bennequin and rotation numbers. A link type is said to be Legendrian simple if any two Legendrian embeddings of it with the same Thurston-Bennequin and rotation numbers are Legendrian equivalent. That is, the classical invariants uniquely classify all Legendrian representations of the knot. Certain knot types are known to be Legendrian simple such as the unknot [15], torus knots, and the figure eight knot [16]. Not every knot is Legendrian simple, the $m_3$ twist knot (also known as the $5_2$ knot) being an example discovered by Chekanov [13].

Transverse links are links that are everywhere transverse to the contact structure. That is, at every point $p$ on the link the velocity vector and the hyperplane at $p$ span $\mathbb{R}^3$. Any Legendrian link can be made transverse by a small perturbation in the direction normal to the given plane in the contact structure. Two transverse links are transversely equivalent if there is an isotopy $H : L \times [0, 1] \to \mathbb{R}^3$ such that $H_t$ is a transverse link for all $t \in [0, 1]$. The Bennequin number of a transverse knot is defined by the algebraic crossing number.

Figure 1: Hyperplane Distribution for $dz - y \, dx$
\( e(K) \) and the braid index \( n(K) \). It is:

\[
\beta(K) = e(K) - n(K)
\]  

(1)

It is not an invariant of topological knots, but is an invariant under transverse equivalence. Similar to Legendrian simple, we define a knot (or link) type to be transversely simply if all of its transverse representations are uniquely determined by their Bennequin number (See [11]) and by whether its velocity vectors point into the half space where the contact structure is positive or not. A paper by Etnyre, Ng, and Vertesi [17] classifies when twist knots are transversely simple. In particular, infinitely many such knots are transversely simple giving us a family of knots to test conjectures with.

There are several common ways of representing topological knots, the three used in our computations are extended Gauss code, planar diagram code (PD code), and Dowker-Thistlewaite code (DT code). Given a knot diagram with \( N \) crossings, the Gauss code is a string with \( 2N \) characters, PD code is a string that is \( 4N \) long, and DT code is \( N \) characters long. Each has its benefits. Extended Gauss code can distinguish mirrors, PD and DT code cannot, PD code is the easiest to reconstruct the knot diagram, and DT code is the shortest. Because of this we will present our examples in DT code. To obtain the DT code of a knot diagram, place your finger on the knot and walk along the diagram, labelling the crossings. When you get back to your starting point each crossing will have two numbers associated with it. It is not difficult to see that each crossing will have exactly one odd number and one even number. For each even number, if that number was associated with an over crossing (that is, your finger ran over the crossing as you were labelling it), place a minus sign in front. Write out the pairs of integers as \((1, a_1), (3, a_2), ..., (2n - 1, a_n)\). The DT code is the list \( a_1, a_2, ..., a_n \). See [1] for several examples.

It is possible to go from PD code to DT code to unsigned Gauss code (i.e. the usual Gauss code, and not the extended Gauss code) and back. For certain computations, like the Alexander polynomial which is mirror insensitive, DT code is easiest since it is the shortest. For things like the Jones polynomial and Khovanov homology, invariants that distinguish mirrors, extended Gauss code is a must.

\section*{2 Khovanov Homology}

The Khovanov homology of a link is a powerful, if computationally expensive\(^1\), invariant first described by Mikhail Khovanov [19] (See also [9] for an excellent introduction). It is closely related to the Jones polynomial, but able to distinguish many more knots and links. The homology groups \( KH^*(L) \) of a link

\(^1\)The naïve algorithm is exponential in the number of crossings. Improvements by Bar-Natan [10] have sped up computations but no polynomial-time algorithm is known at the time of this writing.
(or knot) $L$ are the direct sum of homogeneous components $KH^r_\ell(L)$ and the Khovanov Polynomial (See [4]) is given by:

$$Kh(L)(q,t) = \sum_{r,\ell} t^r q^\ell \dim(KH^r_\ell(L))$$  \hspace{1cm} (2)

The Jones polynomial of $L$ is recovered via:

$$J(L)(q) = Kh(q,-1)$$  \hspace{1cm} (3)

Khovanov homology is not a perfect invariant. That is, there are distinct knots with the same Khovanov homology, but it is a powerful invariant and is capable of detecting certain knot types.

**Theorem 1** (Kronheimer and Mrowka, 2001). If a knot $K$ has the same Khovanov homology as the unknot, then $K$ is equivalent to the unknot.

The unknotted problem asks one to determine if a given knot diagram is equivalent to the unknot. Khovanov homology is a powerful enough tool to accomplish this task. The Khovanov polynomial is a generalization of the Jones polynomial and it has been conjectured that if a knot has the same Jones polynomial as the unknot, then that knot is equivalent to the unknot. At the time of this writing it has not been proven, but there is evidence for and against the claim. Morwen Thistlewaite found links with the same Jones polynomial as the unlink [24], and there is a 3-crossing virtual knot that has the same Jones polynomial as the unknot. For the claim, all knots of up to 24 crossings are either the unknot, or have a Jones polynomial different from the unknot [25].

**Theorem 2** (Baldwin and Sivek, 2022). If a knot $K$ has the same Khovanov homology as either of the trefoils, then $K$ is equivalent to one of them.

**Theorem 3** (Baldwin, Dowlin, Levine, Lidman, and Sazdanovic, 2021). If a knot $K$ has the same Khovanov homology as the figure-eight knot, then $K$ is equivalent to it.

See [8] and [7], respectively.

## 3 Knot Floer Homology

Knot Floer Homology (abbreviated KFH) is to the Alexander polynomial as Khovanov homology is to the Jones polynomial. That is, it is the *categorification* of it. It too is a powerful and deeply studied invariant and has been shown, much like Khovanov homology, to be able to distinguish certain knot types.

**Theorem 4** (Ozváth and Szabó, 2004). If $K$ is a knot with the same Knot Floer Homology as the unknot, then $K$ is equivalent to the unknot.

See [21].
**Theorem 5** (Ghiggini, 2008). *If $K$ is a knot with the same Knot Floer Homology as one of the trefoils, then $K$ is equivalent to one of them.*

**Theorem 6** (Ghiggini, 2008). *If $K$ is a knot with the same Knot Floer Homology as the figure-eight, then $K$ is equivalent to the figure-eight.*

See [18].

### 4 Conjectures on Khovanov and Knot Floer Homology

Both Khovanov and Knot Floer Homology are capable of detecting the unknot, trefoils, and figure-eight knot. The Khovanov homology with coefficients in $\mathbb{Z}/2\mathbb{Z}$ is also capable of detecting the cinquefoil knot [6], which is the $T(5, 2)$ torus knot. The Jones polynomial, on the other hand, is not capable of detecting the $T(5, 2)$ torus knot since the $10_{132}$ knot yields the same polynomial. These are the easiest of the Legendrian simple knots leading us to the following.

**Conjecture 1.** *If a link type $L$ is Legendrian simple, then the Khovanov and Knot Floer homologies of $L$ distinguish it. That is, if $\tilde{L}$ is another link with the same Khovanov or Knot Floer homology, then $\tilde{L}$ is equivalent to $L$.*

Numerical evidence has been tallied for all Torus knots with up to 50 crossings against all knots of up to 17 crossings. There are many torus knots that have the same Jones polynomial as a non-torus knot ($T(2, 5)$ matches a 10 and 17 crossing knots, $T(2, 7)$ matches a 12 crossing knot, and $T(2, 11)$ matches a 14 crossing knot) so we cannot generalize the Jones unknot conjecture. Nevertheless, in all cases the Khovanov homologies were different (see Numerical Results section).

The computation was done as follows. There are libraries for Python and Sage for working with knot polynomials. In particular, we used Regina [12], SnapPy [14], the Sage knot library [23], and our own ever-growing C library. The need for four different libraries was for the sake of sanity. One library alone is sufficient for the computation of the Jones polynomial but it never hurts to double check. The Jones polynomials of all torus knots up to 50 crossings were computed using the formula:

$$J(T(m, n))(q) = q^{(m-1)(n-1)/2} \frac{1 - q^{m+1} - q^{n+1} + q^{m+n}}{1 - q^2}$$  \hspace{1cm} (4)$$

Using any of the aforementioned libraries, the Jones polynomial of all knots up to 17 crossings were computed and compared against this table of torus knot Jones polynomials (Eqn. 4). If a match was found the regina library was used to determine if the knots were actually identical. That is, if the knot whose Jones polynomial was being compared against the torus knots was indeed a torus knot itself. If the knots were distinct, this knot was saved in a text file for later examination. At the end of the computation 4 non-torus knots had the same Jones
polynomial as a torus knot (the 4 mentioned previously). Since the Khovanov polynomial contains the Jones polynomial in it (recall $J(L)(q) = Kh(L)(q, -1)$) the only possible non-torus knots with the same Khovanov homology as a torus knot were these 4.

Using the Java library JavaKh\textsuperscript{2} we found that these four knots with the same Jones polynomials as some torus knot all had different Khovanov homologies. Thus, we have the following claim:

**Theorem 7.** If a knot $K$ has less than or equal to 17 crossings and has the Khovanov homology of a torus knot $T$ with less than 50 crossings, then $K$ is equivalent to $T$.

A similar search for Knot Floer Homology has been performed for up to 17 crossings. First, the Alexander polynomial was computed. In comparison to the Jones polynomial (which has no known classical polynomial-time algorithm)\textsuperscript{3}, the naïve algorithm for the Alexander polynomial is cubic in time (one need only compute the determinant of a particular matrix), but improvements have been made here as well. Needless to say, this greatly improves performance in the search for a match for Knot Floer homology. Much like the Jones polynomial, the Alexander polynomial of a torus knot $T(p, q)$ has a well-known closed form solution:

$$\Delta_{T(p, q)}(t) = t^{-\frac{(p-1)(q-1)}{2}} \frac{(t^{pq} - 1)(t - 1)}{(tp - 1)(tq - 1)} \quad (5)$$

While the computation of the Alexander polynomial is more efficient than the Jones’ polynomial, this comes at a cost. Only four non-torus knots had the same Jones polynomial as a torus knot, but thousands of non-torus knots had the same Alexander polynomial as a torus knot. The Knot Floer Homologies of these matching knots were computed using the SnapPy python library. In the end none of these knots had the same KFH as a torus knot, giving us the following.

**Theorem 8.** If a knot with less than or equal to 17 crossings has the same Knot Floer homology as a torus knot with less than or equal to 50 crossings, then the knot is equivalent to the torus knot.

Lastly, a search through the twist knots yielded some more results. The Jones polynomials of the twist knots are known, with the formula:

$$J(m_n)(q) = \begin{cases} 
(1 + q^{-2} + q^{-n} + q^{-n-3})/(1 + q), & n \text{ odd} \\
(1 + q - q^3 - n + q^{-n})/(1 + q), & n \text{ even} 
\end{cases} \quad (6)$$

The Alexander polynomial for this family is known as well.

$$\Delta_{m_n}(t) = \begin{cases} 
\frac{n+1}{2} t - n + \frac{n+1}{2} t^{-1}, & n \text{ odd} \\
-\frac{n}{2} t + (n + 1) - \frac{n}{2} t^{-1}, & n \text{ even} 
\end{cases} \quad (7)$$

\textsuperscript{2}Thanks must be paid to Nikolay Pultsin who made edits to JavaKh-v2 so that it may run on a GNU/Linux machine using OpenJDK 17.

\textsuperscript{3}A quantum algorithm has been discovered [5].
A search through all knots up to 17 crossings against all twist knots up to 40 crossings provided many matches for the Jones polynomial, but none for Khovanov homology. Similarly for the Alexander polynomial and KFH.

5 Numerical Results

In our search we found that four torus knots had the same Jones polynomial as a non-torus knot. The $T(2, 5)$ knot, which is the cinquefoil, matches $10_{132}$. This result has already been known, and it has also been known for some time that Khovanov homology distinguishes these two (See [4]). The resulting Khovanov polynomials are given in the following tables. The coefficient of $q^rt^r$ is given by the $(r, t)$ slot in the tables. Empty represents a zero coefficient. The DT code of $10_{132}$ is 4, 8, $-12$, 2, $-16$, $-6$, $-20$, $-18$, $-10$, $-14$.

\[
\begin{array}{ccccccc}
q/t & -5 & -4 & -3 & -2 & -1 & 0 \\
-15 & 1 & & & & & \\
-13 & & & & & & \\
-11 & 1 & 1 & & & & \\
-9 & & & & & & \\
-7 & & & & 1 & & \\
-5 & & & & & 1 & \\
-3 & & & & & & 1
\end{array}
\]

Table 1: Khovanov Polynomial for $T(5, 2)$

\[
\begin{array}{ccccccc}
q/t & -7 & -6 & -5 & -4 & -3 & -2 & -1 & 0 \\
-15 & 1 & & & & & & & \\
-13 & & & & & & & & \\
-11 & 1 & 1 & & & & & & \\
-9 & & & 1 & 1 & & & & \\
-7 & & & 1 & & & & & \\
-5 & & & & 1 & 2 & & & \\
-3 & & & & & & 1 & & \\
-1 & & & & & & & 1 & 1
\end{array}
\]

Table 2: Khovanov Polynomial for $10_{132}$

The cinquefoil also has the same Jones polynomial as a 17 crossing knot, but once again the Khovanov homologies were able to distinguish between them. The DT code of this 17 crossing knot is 18, $-28$, $-16$, 24, $-32$, $-20$, 34, $-6$, 30, $-22$, $-12$, 26, 8, $-2$, $-4$, 14, $-10$. 

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Table 3: Khovanov Polynomial for the 17 Crossing Knot

The $T(7, 2)$ knot, also the $7_1$ knot, and occasionally called the septafoil, has the same Jones polynomial as a 12 crossing knot. The Khovanov polynomials are distinct. The DT of this 12 crossing knot is given by 6, 10, 14, $-18$, 2, $-20$, 4, 22, 24, $-8$, $-12$, 16.

Table 4: Khovanov Polynomial for $T(7, 2)$

Table 5: Khovanov Polynomial for the 12 Crossing Knot
Lastly, the $T(11,2)$ torus knot has the same Khovanov homology as a 14 crossing knot. The table for the Khovanov polynomial of this knot is quite large and has been omitted. Nevertheless, it differs from the Khovanov polynomial of $T(11,2)$. The DT code of this knot is $8, -12, -16, -20, 22, -2, 24, -4, 26, -6, 28, 10, 14, 18$.

For twist knots there were 8 matches for the Jones polynomial. In all cases the Khovanov homologies were distinct. The following table shows the DT code of the matching knots.

| Twist Knot | Non-Twist Knot |
|------------|----------------|
| $m_2$      | 4 8 10 -16 2 -18 -20 -22 -6 -12 -14 |
| $m_3$      | 4 8 -14 2 -18 -18 -20 -6 -10 -22 -12 |
| $m_3$      | 4 10 -16 -18 2 20 22 24 -8 -6 12 14 |
| $m_3$      | 4 12 16 -22 14 -20 2 8 24 26 -10 -6 18 |
| $m_5$      | 4 10 12 16 18 2 -20 6 8 -22 -14 |
| $m_6$      | 4 8 -14 2 -18 -20 -6 -22 -12 -10 -16 |
| $m_6$      | 4 10 -16 -24 -18 2 -20 -22 -26 -12 -14 -8 -6 |
| $m_7$      | 4 8 10 16 2 -18 -20 6 -22 -12 -14 |

Table 6: Non-Twist Knots with the Same Jones Polynomial as a Twist Knot

The $m_2$ twist knot is the figure eight knot. The matching 11 crossing knot is the K11n19 knot from the Hoste-Thistlewaite table of 11 crossing knots. That the Jones polynomial of these knots match has been known, and this can be found on the knot atlas (See [2] and [3]). Also found on the knot atlas is that Khovanov homology does indeed distinguish these knots.

To conclude, for up to 17 crossings, Khovanov and Knot Floer homologies have thus far been able to distinguish the twist knots.

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