Analysis of the immersed boundary method for a finite element Stokes problem

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Convergence results are presented for the immersed boundary (IB) method applied to a model Stokes problem. As a discretization method, we use the finite element method. First, the immersed force field is approximated using a regularized delta function. Its error in the $W^{−1},p$ norm is examined for $1 \leq p < n/(n−1)$, with $n$ representing the space dimension. Subsequently, we consider IB discretization of the Stokes problem and examine the regularization and discretization errors separately. Consequently, error estimate of order $h^{1−α}$ in the $W^{1,1} \times L^1$ norm for the velocity and pressure is derived, where $α$ is an arbitrary small positive number. The validity of those theoretical results is confirmed from numerical examples.

KEYWORDS
finite element method, immersed boundary method, Stokes equation

1 | INTRODUCTION

The immersed boundary (IB) method is a powerful method of solving a class of fluid–structure interaction problems proposed originally by Peskin [1, 2] to simulate blood flow through artificial heart valves. An earlier report [3] describes later developments. The IB method is also applied to multiphase flow problems, elliptic interface problems, and so on.

In marked contrast to the huge number of applications, few reports present results of theoretical convergence analysis. Pioneering work in the area was done by Mori in 2008 ([4]), who studied a model (stationary) Stokes problem for the velocity $u$ and pressure $q$ in an $n$-dimensional torus.
\[ U = \left[ \mathbb{R}/(2\pi \mathbb{Z}) \right]^n \subset \mathbb{R}^n, \]

as
\[ -\Delta u + \nabla q = f - g \text{ in } U, \quad \nabla \cdot u = 0 \text{ in } U, \quad (1) \]

where
\[
\begin{align*}
f &= \int_{\Theta} F(\theta) \delta_{X(\theta)} \, d\theta, \\
g &= \frac{1}{(2\pi)^n} \int_{\Theta} F(\theta) \, d\theta.
\end{align*}
\]

Herein, the IB \( \Gamma \subset U \) is parameterized as
\[ \Gamma = \{ X(\theta) = (X_1(\theta), \ldots, X_n(\theta)) \mid \theta \in \Theta \}, \]

where \( \Theta \) denotes a subset of \( \mathbb{R}^n \) or \( \mathbb{R}^{n-1} \). Subsection 2.1 provides a precise definition. Figure 1 presents a useful illustration. Function \( F = F(\theta) \) denotes the force distributed along \( \Gamma \) and \( \delta = \delta(x) \) the (scalar valued) Dirac delta function. In Ref. [5], the case \( n = 2 \) was described explicitly. Introducing the regularized delta function \( \delta_h \approx \delta \) with a parameter \( h > 0 \), he considered the regularized Stokes problem
\[ -\Delta \tilde{u} + \nabla \tilde{q} = \int_{\Theta} F(\theta) \delta_h(x - X(\theta)) \, d\theta - g \text{ in } U, \quad \nabla \cdot \tilde{u} = 0 \text{ in } U. \]

The regularized problem was discretized using the finite difference method with a uniform Eulerian grid with grid size \( h \). Then, he succeeded in deriving the maximum norm error estimate for the velocity of the form
\[ \| u - \tilde{u}_h \|_{L^\infty(U)} \leq C(h + h^\alpha) \log h \quad (\alpha > 0 \text{ suitable constant}) \]
under regularity assumptions on \( \Gamma \) and \( F \) together with structural assumptions on \( \delta_h \). Herein, \( \tilde{u}_h \) denotes the finite difference solution. Subsequently, the method and results were extended to several directions [6, 7]. For example, several \( L^p \)-error estimates, \( 1 \leq p \leq \infty \), were obtained in Ref. [7]. A typical result is given as
\[ \| u - \tilde{u}_h \|_{L^p(U)} + h\| q - \tilde{q}_h \|_{L^p(U)} \leq Ch^{1+\frac{1}{p}}|\log h|^{\eta} \quad (\eta > 0 \text{ suitable constant}). \]

Similar results for the Poisson interface problem were presented in Ref. [8]. However, the explicit formula of the Green function associated with (1) was used to derive error estimates in Refs. [5–7]. It is therefore difficult to apply those methods to more standard settings such as the Dirichlet boundary value problem. Moreover, error estimates in any (discrete) \( W^{1,p} \) norms were not undertaken.

As described herein, we take a different approach. We consider a polyhedral domain \( \Omega \subset \mathbb{R}^n \), \( n = 2 \), \( 3 \), and the Dirichlet boundary value problem for Stokes equations for velocity \( u \) and pressure \( \pi \) as
\[ -\nu \Delta u + \nabla \pi = f \text{ in } \Omega, \quad \nabla \cdot u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial \Omega \quad (3) \]
and examine the *regularization error and discretized error separately* in Sections 2 and 3. Therein, the (kinematic) coefficient of viscosity is denoted by \( \nu \). To this end, we first give interpretations of the immersed outer force \( f \) above as an \( \mathbb{R}^n \)-Lebesgue measure and as a functional over \( W^{1,p}_0(\Omega)^n \) for \( 1 \leq p < \infty \), as in Propositions 2 and 3. The meaning of mathematical symbols is presented in Subsection 2.1. Then, we introduce a regularized delta function \( \delta^\varepsilon \) with a parameter \( \varepsilon > 0 \) and examine the error between \( f \) and its regularization as

\[
f^\varepsilon = \int_{\Theta} F(\theta) \delta_X(\theta) d\theta, \quad \delta_X(\theta)(x) = \delta^\varepsilon(x - X(\theta))
\]

in the \( W^{-1,p}(\Omega)^n \) norm for \( 1 \leq p < \frac{n}{n-1} \). See Proposition 4. We then introduce the regularized problem for Equation (3),

\[
-\nu \Delta u^\varepsilon + \nabla \pi^\varepsilon = f^\varepsilon \text{ in } \Omega, \quad \nabla \cdot u^\varepsilon = 0 \text{ in } \Omega, \quad u^\varepsilon = 0 \text{ on } \partial \Omega.
\]

Because \( f^\varepsilon \) is an \( L^2(\Omega)^n \) function, Equation (5) is no more than the standard Stokes problem. Estimation for the regularization error, as explained in Proposition 10, is a direct consequence of Proposition 4 and the stability result of Ref. [9] (or Assumption 1\(_p\) below). After introducing structural assumptions (12a–d) on \( \delta^\varepsilon \), we show that the \( W^{1,p} \times L^p \) error estimate for the velocity and pressure is of order \( \varepsilon^{\frac{1}{p} - \frac{n}{n-1}} \) if \( 1 \leq p < \frac{n}{n-1} \) (see Proposition 13).

Then, we proceed to the study of discretization in Section 3. We are concerned with the finite element method rather than the finite difference method. This choice enables us to apply several sharp \( W^{1,p} \times L^p \) stability and error estimates because of Ref. [10] (or Assumption 2\(_p\)). Finally, we obtain several nearly optimal error estimates in \( W^{1,p} \times L^p \) norms and present Theorem 19 as the main result of this article. The effect of numerical integration for computing \( f^\varepsilon \) is discussed in Section 4. Actually, a simple numerical integration formula does not degrade the accuracy of the IB method, as presented in Proposition 22 and Theorem 23. The validity of those theoretical results is confirmed using numerical examples illustrated in Section 5.

We remark here that our structural assumptions on \( \delta^\varepsilon \) are simpler than those of Refs. [5–7]. We assume only that the “reference function” \( \varphi \) is continuous in \( \mathbb{R} \) with compact support and with the unit mean value; see Equation (12a–d). However, several conditions on moment and smoothing orders of \( \varphi \) were assumed in Refs. [5–7]. We can remove those restrictions.

From the viewpoint of discretization, our consideration is concentrated on the finite element method. Therefore, our results cannot be applied directly to the finite difference method, which is a popular discretization method in the IB method. Moreover, Equation (3) can be discretized directly by the finite element method with no regularization of \( \delta \). Such methods, often called the immersed finite element (IFE) method, were studied in Refs. [11–14]. Actually, formulations in those works are fundamentally the similar as those of two-phase fluid-flow problems reported in Ref. [15]. Sharpe error estimates for the IFE method applied to the Poisson interface problem are presented in Ref. [16]. Nevertheless, we study the regularization problem (5) and its finite element approximation for the following reasons:

1. Although Equation (3) is discretized directly using the finite element method with no regularization of \( \delta \) as described earlier, the implementation of the resulting discrete system is not an easy task. As a matter of fact, the computation of the IB term requires experience of programming. In contrast, the reformulation (5) provides a simple approach. Because of \( f^\varepsilon \in L^2(\Omega)^n \), it can be dealt with as the standard Stokes problem so that finite element software such as FreeFem++ [17] and FEniCS [18] are available: one need not be a specialist. Therefore, it is worthwhile to study the regularization procedure and its discretization very carefully, even if the finite element method is taken as a discretization method.
As pointed out in Ref. [4], the regularization of Dirac’s delta presents many important issues related to numerical computation of differential equations. Uniform Eulerian grid and the $L^\infty$ norm are considered in Ref. [4]. As described herein, we study the regularization and discretization errors separately using several norms such as the $W^{1,p}$ and $L'$ norms. Consequently, we can remove some structural assumptions on the delta approximation $\delta^\epsilon$. Our analysis yields another perspective for studying the regularization of Dirac’s delta.

We emphasize that this study was conducted to ascertain the accuracy of the regularization and discretization procedures. We do not intend to propose a new computational method, as explained in Remark 7.

In relation to reason 2, it is noteworthy that the second author recently reported in Ref. [19] some convergence results for another regularization of Dirac’s delta using the characteristic function.

This article is a revised and extended version of our earlier manuscript [20], which has not been published elsewhere so far.

### 2 | IB FORMULATION

#### 2.1 | Geometry and notation

Presuming that $\Omega$ is a polyhedral domain in $\mathbb{R}^n$, $n = 2, 3$, with the boundary $\partial \Omega$, domain $\Omega$ is divided into two disjoint components $\Omega_0$ and $\Omega_1$ by a simple closed curve ($n = 2$) or surface ($n = 3$), which is designated as $\Gamma$. The curve (surface) $\Gamma$, called the IB, is presumably parameterized as $\Gamma = \{X = (X_1(\theta), \ldots, X_n(\theta)) | \theta = (\theta_1, \ldots, \theta_n) \in \Theta\}$ where $\Theta \subset \mathbb{R}^n$ is the reference configuration.

In this article, we assume the following (see Ref. [21], Section 1.4):

**Hypothesis 1.** $X$ is a $C^1$ diffeomorphism between $\Theta$ and $\Gamma$;

**Hypothesis 2.** $J_X(\theta) \equiv \det(\nabla_\theta X(\theta)) > 0$;

**Hypothesis 3.** $\text{dist}(\Gamma, \partial \Omega) > 0$.

We set $J_X^{-1}(\theta) \equiv \det(\nabla_\theta X(\theta))^{-1}$.

**Remark 1** In the case of $n = 2$, a single-chart parameterization of $\Gamma$ can be employed.

Setting $\Theta = (c, d)$ with $c < d$, we assume that $\Gamma$ is represented as $\Gamma = \{(X_1(\theta), X_2(\theta)) | \theta = (\theta_1, \ldots, \theta_n) \in \Theta\}$. Then, we assume that $X$ is a $C^1$ function: $J_X(\theta) \equiv (|\partial_\theta X_1|^2 + |\partial_\theta X_2|^2)^{1/2} > 0$; and Hypothesis 3.

Here, we collect the notation used for this study. We follow the notation of Ref. [22] for function spaces and their norms. For $1 \leq p \leq \infty$, let $p'$ be the conjugate exponent of $p$, $1 \leq p' \leq \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. We set $W^{1,p}_0(\Omega) = \{v \in W^{1,p}(\Omega) | v|_{\partial \Omega} = 0\}$ and $W^{-1,p}(\Omega)$ the topological dual of $W^{1,p}_0(\Omega)$. The dual product between $W^{-1,p}(\Omega)$ and $W^{1,p}_0(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle_{W^{-1,p},W^{1,p}_0}$. We let $L^p_0(\Omega) = \{q \in L^p(\Omega) | \int_\Omega q\, dx = 0\}$.

For a function space $X$, the space $X^n$ stands for a product space $X \times \cdots \times X$. For abbreviations, we write, for example,

$$\|u\|_{W^{1,p}} = \|u\|_{W^{1,p}(\Omega)}, \quad \|\pi\|_{L^p} = \|\pi\|_{L^p(\Omega)}.$$
Set $B(a, r) = \{ x \in \mathbb{R}^n \mid |x - a| < r \}$ for $a \in \mathbb{R}^n$ and $r > 0$.

For vectors $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, let us denote the scalar product by $a \cdot b = a_1 b_1 + \cdots + a_n b_n$.

### 2.2 IB force

We recall that the IB force field $f : \Omega \to \mathbb{R}^n$ is defined (at least formally) by Equation (2) for $F \in L^1(\Theta)^n$. We have (still formally)

$$\int_{\Omega} f(x) \cdot \varphi(x) \, dx = \int_{\Theta} F(\theta) \cdot \varphi(X(\theta)) \, d\theta \quad (\varphi \in C_0^\infty(\Omega)^n). \tag{6}$$

We state two interpretations of Equation (6).

**Proposition 2** Let $F \in L^1(\Theta)^n$. Then, $f$ defined as Equation (2) is a finite signed measure on $\Omega$, with which the integration is defined for any (vector valued) measurable function $\varphi$ on $\Omega$. In particular, if $F \in L^p(\Theta)^n$ for $1 \leq p \leq \infty$, then the integral is given as

$$\langle f, \varphi \rangle = \int_{\Omega} \varphi \, df = \int_{\Theta} F(\theta) \cdot \varphi(X(\theta)) \, d\theta$$

for any $\varphi \in W_0^1(\Omega)^n$. Moreover, $f$ is a singular measure against the Lebesgue measure on $\Omega$, and consequently, $f \not\in L^1(\Omega)^n$.

**Proof.** We identify $\delta_a(x) = \delta(x - a)$ with the Dirac measure concentrated at $a \in \mathbb{R}^n$. Then, for any measurable set $B \subset \mathbb{R}^n$ and $\Theta \in \Theta$, we have

$$\delta_{X(\theta)}(B) = 1_{X^{-1}(B)}(\theta) = \begin{cases} 1 & (X(\theta) \in B) \\ 0 & (X(\theta) \not\in B) \end{cases}$$

where $1_{X^{-1}(B)}$ denotes the indicator function of $X^{-1}(B)$ on $\Theta$. By virtue of Lebesgue’s dominated convergence theorem, we derive for any disjoint measurable sets $\{B_n\}_n$

$$f(\bigcup_{n=1}^{\infty} B_n) = \int_{\Theta} F(\theta) \sum_{n=1}^{\infty} \delta_{X(\theta)}(B_n) \, d\theta = \int_{\Theta} F(\theta) \sum_{n=1}^{\infty} 1_{X^{-1}(B_n)}(\theta) \, d\theta$$

$$= \sum_{n=1}^{\infty} \int_{\Theta} F(\theta) 1_{X^{-1}(B_n)}(\theta) \, d\theta = \sum_{n=1}^{\infty} \int_{\Theta} F(\theta) \delta_{X(\theta)}(B_n) \, d\theta$$

$$= \sum_{n=1}^{\infty} f(B_n).$$

Herein, it is noteworthy that $F(\theta) \sum_{n=1}^{N} 1_{X^{-1}(B_n)}(\theta)$ is integrable for any $N \in \mathbb{N}$ because $F \in L^1(\Theta)^n$ and $B_n$ is disjoint. It follows that $f(\theta) = 0$ from $\delta_a(\theta) = 0$ for all $a \in \mathbb{R}^n$. Consequently, $f$ is a finitely signed measure on $\Omega$ so that the integral $\int_{\Omega} \varphi \, df$ is well-defined for all measurable functions $\varphi$. According to an integral with the Dirac measure, we have $\int_{\Omega} \varphi \, df = \int_{\Theta} F(\theta) \cdot \varphi(X(\theta)) \, d\theta$, where the right-hand side is meaningful for $F \in L^p(\Theta)^n$ and $\varphi \in W_0^1(\Omega)^n$. Although the $\mathbb{R}^n$-Lebesgue measure $m(\Gamma)$ of $\Gamma$ vanishes (note that $\Gamma$ is “very thin”), we have $f(\Gamma) \not= 0$. Therefore, $f$ is singular against $m$. Finally, the fact $f \not\in L^1(\Omega)$ follows from the Lebesgue decomposition theorem.
The IB force $f$ does not belong to any $L^p(\Omega)$ spaces, as described in the preceding proposition: we are unable to apply approximation theory for problems with the right-hand side in $L^1$ (see Ref. [23]). However, it is well-defined as a functional on $W^{1,p'}_0(\Omega)^n$.

\section*{Proposition 3}

Letting $1 \leq p < \infty$ and $F \in L^p(\Theta)^n$, then, the functional
\[
\langle f, \varphi \rangle = \int_{\Theta} F(\theta) \cdot \varphi(X(\theta)) \, d\theta \quad (\varphi \in C_0^\infty(\Omega)^n)
\]
is extended by continuity to a bounded linear functional on $W^{1,p'}_0(\Omega)^n$, which will be denoted by $\langle \cdot, \cdot \rangle_{W^{-1,p},W^{1,p}'}$ below. That is, we have $f \in W^{-1,p}(\Omega)^n$.

\begin{proof}
Let $\varphi \in C_0^\infty(\Omega)^n$. Because
\[
\int_{\Theta} | \varphi(X(\theta))|^{p'} \, d\theta = \int_{\Theta} \langle \varphi(X(\theta))|^{p'} \rangle |J_X^{-1}(\theta)||J_X(\theta)| \, d\theta
\]
\[
\leq \|J_X^{-1}\|_{L^p(\Theta)} \int_{\Theta} \langle \varphi(X(\theta))|^{p'} \rangle |J_X(\theta)| \, d\theta
\]
\[
= \|J_X^{-1}\|_{L^p(\Theta)} \int_{\Gamma} \varphi |^{p'} \, d\Gamma,
\]
we have by the trace theorem
\[
|\langle f, \varphi \rangle| \leq \|F\|_{L^p(\Theta)} \left(\int_{\Theta} \langle \varphi(X(\theta))|^{p'} \rangle \, d\theta\right)^{\frac{1}{p}}
\]
\[
\leq \|F\|_{L^p(\Theta)} \|J_X^{-1}\|_{L^{p'}(\Gamma)}^{\frac{1}{p'}} \|\varphi\|_{L^{p'}(\Gamma)} \leq C\|F\|_{L^p(\Theta)} \|\varphi\|_{W^{1,p}(\Omega)}.
\]
\end{proof}

Let $\varepsilon > 0$ be a regularization parameter. Take a continuous function $\delta^\varepsilon = \delta^\varepsilon(x)$ of $\mathbb{R}^n \to \mathbb{R}$ satisfying
\[
\text{supp}\delta^\varepsilon \subset B(0, K\varepsilon)
\]
with $K > 0$. Recall that the \textit{regularized immersed force field} $f^\varepsilon : \Omega \to \mathbb{R}^n$ is defined by Equation (4). Because $\delta^\varepsilon \in L^\infty(\mathbb{R}^n)$, we have $f^\varepsilon \in L^\infty(\Omega)$ for $F \in L^1(\Theta)^n$. The following result plays the most important role in this study. It is noteworthy that the restriction on the range of $p$ such that $1 \leq p < \frac{n}{n-1}$ comes from the regularization of Dirac’s delta.

\section*{Proposition 4}

Assume that we are given a continuous function $\delta^\varepsilon$ satisfying (7). Then, for $1 \leq p < \frac{n}{n-1}$ and $F \in L^p(\Theta)^n$, we have
\[
\|f - f^\varepsilon\|_{W^{-1,p}} \leq C_0\|F\|_{L^p(\Theta)} \left[\left|1 - \int_{\mathbb{R}^n} \delta^\varepsilon(y) \, dy\right| + \|\rho \delta^\varepsilon\|_{L^1(\mathbb{R}^n)}\right],
\]
where $\rho(x) = x$ and $C_0$ denotes a positive constant depending only on $n$, $p$, $||J_X||_{L^p(\Theta)}$, and $||J_X^{-1}||_{L^{p'}(\Theta)}$.

\begin{proof}
Let $\varphi \in C_0^\infty(\Omega)^n$ and express it as
\[
\varphi(x) = \varphi(X(\theta)) + (x - X(\theta)) \cdot \int_0^1 \nabla \varphi(t(x - X(\theta)) + X(\theta)) \, dt \quad (x \in \mathbb{R}^n).
\]
Then by applying Fubini’s lemma, we have
\[
\langle f - f^\epsilon, \varphi \rangle = \int_{\Omega} F(\theta) \varphi(X(\theta)) \left( 1 - \int_{\Omega} \delta^\epsilon_{X(\theta)}(x) \, dx \right) \, d\theta
\]
\[
- \int_{0}^{1} \int_{\Omega} F(\theta) \int_{\Omega} \delta^\epsilon_{X(\theta)}(x) (x - X(\theta)) \cdot \nabla \varphi(t(x - X(\theta)) + X(\theta)) \, dx \, d\theta \, dt.
\]
For sufficiently small $\epsilon$, we have $B(X(\theta), K\epsilon) \subset \Omega$ and
\[
\int_{\Omega} \delta^\epsilon_{X(\theta)}(x) \, dx = \int_{B(X(\theta), K\epsilon)} \delta^\epsilon_{X(\theta)}(x) \, dx = \int_{B(0, K\epsilon)} \delta^\epsilon(y) \, dy = \int_{\mathbb{R}^n} \delta^\epsilon(y) \, dy.
\]
Therefore,
\[
| I_1 | \leq \left| 1 - \int_{\mathbb{R}^n} \delta^\epsilon(y) \, dy \right| \int_{\Theta} | F(\theta) | \cdot | \varphi(X(\theta)) | \, d\theta
\]
\[
\leq \| F \|_{L^p(\Theta)} \left( \int_{\Theta} | \varphi(X(\theta)) |^{p'} \, d\theta \right)^{\frac{1}{p'}} \left| 1 - \int_{\mathbb{R}^n} \delta^\epsilon(y) \, dy \right|
\]
\[
\leq \| F \|_{L^p(\Theta)} \| J^{-1}_{\Gamma} \|_{L^1(\Theta)} \| \varphi \|_{L^{p'}(\Gamma)} \left| 1 - \int_{\mathbb{R}^n} \delta^\epsilon(y) \, dy \right|
\]
\[
\leq C \| F \|_{L^p(\Theta)} \left| 1 - \int_{\mathbb{R}^n} \delta^\epsilon(y) \, dy \right| \| \varphi \|_{W^{1,p'}(\Omega)}.
\]
By virtue of Hölder’s inequality, we have
\[
| I_2 | \leq \int_{0}^{1} \int_{\Theta} | F(\theta) | \cdot |(x - X(\theta)) \delta^\epsilon_{X(\theta)}|_{L^p(\Omega)} \cdot
\]
\[
\left[ \int_{\Omega} | \nabla \varphi(t(x - X(\theta)) + X(\theta)) |^{p'} \, dx \right]^{\frac{1}{p'}} \, d\theta \, dt
\]
\[
\leq \| \rho \delta^\epsilon \|_{L^p(\mathbb{R}^n)} \int_{0}^{1} \int_{\Theta} | F(\theta) | \left[ \int_{\mathbb{R}^n} | \nabla \tilde{\varphi}(t(x - X(\theta)) + X(\theta)) |^{p'} \, dx \right]^{\frac{1}{p'}} \, d\theta
\]
\[
\leq \| \rho \delta^\epsilon \|_{L^p(\mathbb{R}^n)} \int_{0}^{1} \int_{\Theta} | F(\theta) | \, d\theta \left[ \frac{1}{t^n} \int_{\mathbb{R}^n} | \nabla \tilde{\varphi}(z) |^{p'} \, dz \right]^{\frac{1}{p'}}
\]
\[
\leq \| \rho \delta^\epsilon \|_{L^p(\mathbb{R}^n)} \| F \|_{L^1(\Theta)} \left( \int_{0}^{1} t^{-\frac{n}{p'}} \, dt \right) \| \tilde{\varphi} \|_{W^{1,p'}(\mathbb{R}^n)}
\]
\[
\leq \frac{p'}{p' - n} \| F \|_{L^1(\Theta)} \| \rho \delta^\epsilon \|_{L^p(\mathbb{R}^n)} \| \varphi \|_{W^{1,p'}}.
\]
where $\tilde{\varphi}$ denotes the zero extension of $\varphi$ into $\mathbb{R}^n$ and $z = t(x - X(\theta)) + X(\theta)$. (It is noteworthy that $n < p' \leq \infty$ by $1 \leq p < \frac{n}{n-1}$.)
\[\Box\]

**Remark 5** We take $\varphi_0 \in C^\infty(\overline{\Omega})$ satisfying
\[
\varphi_0(x) = 1 \text{ if } x \in \Gamma(\epsilon) = \{ x \in \Omega \mid \text{dist}(x, \Gamma) < \epsilon \} \cup \Omega_0.
\]
Then, $I_2$ in the proof above vanishes and
\[
\|f - f^\varepsilon\|_{L^p} \geq \frac{\langle f - f^\varepsilon, \varphi_0 \rangle}{\|\varphi_0\|_{L^p}} = \frac{1}{\|\varphi_0\|_{L^p}} \int_\varTheta F(\theta) \ d\theta \left[1 - \int_{\mathbb{R}^s} \delta^\varepsilon(x) \ dx \right].
\]
As a result,
\[
\int_{\mathbb{R}^s} \delta^\varepsilon(x) \ dx \to 1 \quad (\varepsilon \to 0)
\]
is a necessary condition for $\|f - f^\varepsilon\|_{L^p} \to 0$ to hold.

**Remark 6** As usual, we set $H^n(\Omega) = W^{n,2}(\Omega)$ for $\eta > 0$. Suppose that we are given $F \in L^2(\varTheta)$ and $0 < s < 1/2$. Applying the trace theorem, we deduce $|\langle f, \varphi \rangle| \leq C\|F\|_{L^2(\varTheta)} \|\varphi\|_{H^{1/2+s}(\Omega)}$ for any $\varphi \in C_0^\infty(\Omega)^n$. This implies that $f$ is extended by continuity to a linear bounded functional on the completion of $C_0^\infty(\Omega)^n$ by the $H^{1/2+s}(\Omega)$ norm, which is designated as $H_0^{1/2+s}(\Omega)$. Consequently, the solution of Equation (3) has a regularity $(u, \pi) \in H_0^{3/2-s}(\Omega)^n \times H_1^{1/2-s}(\Omega)$ for a convex polygonal domain $\Omega$ in $\mathbb{R}^2$. Therefore, if the IB force term $\langle f, \varphi \rangle$ is discretized directly with no regularization as in Refs. [11–14, 16], we can take $p = 2$ and apply the standard error analysis of the finite element method. However, if any regularization is undertaken that is the main motivation of this study as mentioned in reasons 1 and 2 of Introduction, we cannot take $p = 2$ as revealed in Proposition 4.

### 2.3 Target and regularized problems

We proceed to formulation of the IB method. Using $f$ and $f^\varepsilon$ defined as Equations (2) and (4), we consider the IB formulation to Stokes Equation (3) and its regularized problem (5).

By a weak solution $(u, \pi) \in W^{1,\varrho}(\Omega)^n \times L^p_0(\Omega)$ of Equation (3), we mean a solution of the following variational equations: Find $(u, \pi) \in W^{1,\varrho}(\Omega)^n \times L^p_0(\Omega)$ such that
\[
a(u, v) + b(\pi, v) = \langle f, v \rangle_{W^{-1,p}, W^{1,\varrho}_0} \quad (\forall v \in W^{1,\varrho}_0(\Omega)^n),
\]
\[
b(q, u) = 0 \quad (\forall q \in L^p_0(\Omega)),
\]
where
\[
a(u, v) = \frac{v}{2} \int_\Omega \left( \frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j} \right) \left( \frac{\partial v_i}{\partial x_i} + \frac{\partial v_j}{\partial x_j} \right) \ dx,
\]
\[
b(\pi, u) = -\int_\Omega \pi(\nabla \cdot u) \ dx.
\]

**Remark 7** Problem (7a and b) can be discretized directly by the finite element method with no regularization of $f$. Such methods were studied in Refs. [11–15]; some of those works are devoted to nonstationary Navier–Stokes equations. However, our objective is to assess the accuracy of the regularization and discretization procedures, as described in reasons of 1 and 2 of Introduction.

**Remark 8** The bilinear form $a$ defined by Equation (8a) is based on the deformation-rate tensor $[(1/2)(u_{i,j} + u_{j,i})]_{1 \leq i, j \leq n}$. Another definition
\[
a(u, v) = \nu \int_\Omega \frac{\partial u_i}{\partial x_i} \frac{\partial v_j}{\partial x_j} \ dx
\]
is also available. However, with Equation (8a), our problem is fundamentally equivalent to a two-phase Stokes problem considered in Ref. [15] and other studies.

We make the following assumption for $1 \leq p < \infty$:

**Assumption 1** $p$. For a given $g \in W^{-1,p}(\Omega)^n$, there exists a unique weak solution $(w, r) \in W_0^{1,p}(\Omega) \times L_0^p(\Omega)$ of the Stokes problem,

$$ -\nu \Delta w + \nabla r = g \text{ in } \Omega, \quad \nabla \cdot w = 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega$$

satisfying

$$ \|w\|_{W^{1,p}} + \|r\|_{L^p} \leq C_1 \|g\|_{W^{-1,p}}. $$

Moreover, if $g \in W^{-1,2}(\Omega)^n \cap L^p(\Omega)$, then we have $(w, r) \in W^{2,p}(\Omega) \times W^{1,p}(\Omega)$ and

$$ \|w\|_{W^{2,p}} + \|r\|_{W^{1,p}} \leq C_2 \|g\|_{L^p}. $$

Herein, $C_1$ and $C_2$ denote positive constants depending only on $p$ and $\Omega$.

**Remark 9** If $\Omega$ is a convex Lipschitz domain and $1 < p \leq 2$, then Assumption 1 is satisfied in view of Ref. [9] (Example 5.5) and Lemma 3. However, we assume Assumption 1 directly instead of the shape condition on $\Omega$. Below, $p$ will be restricted as $p < n/(n-1)$.

The following result is a direct consequence of Lemma 4 and Assumption 1.

**Proposition 10** Letting $1 \leq p < \frac{n}{n-1}$ and assuming that (Assumption 1) is satisfied, then let $F \in L^p(\Theta)^n$. Letting $(u, \pi)$ and $(u^\varepsilon, \pi^\varepsilon)$ be the weak solutions of Equations (3) and (5), respectively, we have

$$ \|u - u^\varepsilon\|_{W^{1,p}} + \|\pi - \pi^\varepsilon\|_{L^p} \leq C_0 C_1 \|F\|_{L^p(\Theta)} \left[ 1 - \int_{\mathbb{R}^n} \delta^\varepsilon(y) \, dy \right] + \|\rho \delta^\varepsilon\|_{L^p(\mathbb{R}^n)}. $$

At this stage, we mention the structural assumptions on $\delta^\varepsilon$. That is,

$$ \delta^\varepsilon \text{ is continuous in } \mathbb{R}^n; $$

$$ \text{supp} \delta^\varepsilon \subset B(0, K\varepsilon) \text{ with some } K > 0; $$

$$ \int_{\mathbb{R}^n} \delta^\varepsilon(y) \, dy = 1; $$

$$ \int_{\mathbb{R}^n} |y|^\sigma |\delta^\varepsilon(y)|^p \, dy \leq C_3 \varepsilon^{\sigma-pn+n} \text{ for } \sigma \geq 0, $$

where $C_3$ denotes a positive constant depending only on $p, \sigma, n$, and $K$.

**Example 11** The most familiar choice of $\delta^\varepsilon$ is given as a product of one variable function:

$$ \delta^\varepsilon(x) = \frac{1}{\varepsilon^n} \prod_{i=1}^n \varphi \left( \frac{x_i}{\varepsilon} \right) \quad (x = (x_1, \ldots, x_n)), $$

where $\varphi$ is a continuous function in $\mathbb{R}$ with compact support satisfying $\int_{\mathbb{R}} \varphi(s) \, ds = 1$. In Equation (13), the function $(1/\varepsilon)\varphi(x_i/\varepsilon)$ is an approximation of the one-dimensional
Dirac delta. Then, for \( n = 3 \), we can calculate it as follows:

\[
\int_{\mathbb{R}^n} |\varphi|^{\sigma+n} \, d\nu \leq \varepsilon^{\sigma-n} \int_0^{nK} \int_0^{2\pi} \int_0^{\pi} |\varphi(s \cos \varphi \sin \theta)|^p \cdot |\varphi(s \sin \varphi \sin \theta)|^p |\varphi(s \cos \theta)|^p \sin \theta \, dsd\varphi d\theta
\]

\[
\leq \frac{2\pi^2}{\sigma+n} (\sqrt{nK})^{\sigma+n} \|\varphi\|^{3p}_{L^p(\mathbb{R})} \varepsilon^{\sigma-n}.
\]

Similarly, we can take

\[
C_3 = \frac{2\pi}{\sigma+n} (\sqrt{nK})^{\sigma+n} \|\varphi\|^{2p}_{L^p(\mathbb{R})}
\]

for \( n = 2 \).

**Example 12** An alternate choice of \( \delta^\varepsilon \) is given as

\[
\delta^\varepsilon(x) = \frac{1}{\varepsilon^n} \varphi \left( \frac{x}{\varepsilon} \right) \quad (r = \sqrt{x_1^2 + \cdots + x_n^2}),
\]

where \( \varphi \) is the function, as in the previous example.

Therefore, our error estimate for the regularized problem is given as explained below.

**Proposition 13** Letting \( 1 \leq p < \frac{n}{n-1} \) and assuming that Assumption 1, is satisfied, then let \( F \in L^p(\Theta)^n \). Letting \((u, p)\) and \((u^\varepsilon, p^\varepsilon, \varphi^\varepsilon)\), respectively, be the weak solutions of Equations (3) and (5) with Equation (12a–d), then we have

\[
\|u - u^\varepsilon\|_{W^{1,p}} + \|p - p^\varepsilon\|_{L^p} \leq C \|F\|_{L^p(\Theta)} \varepsilon^{1-n+p^\varepsilon},
\]

(14)

where \( C \) denotes a positive constant depending only on \( n, p, \|J_X\|_{L^\infty(\Theta)}\), \( \|J_X^{-1}\|_{L^\infty(\Theta)} K, \|\varphi\|_{L^\infty(\mathbb{R})} \) and \( \Omega \).

**Remark 14** Proposition 13 remains valid for a bounded Lipschitz domain \( \Omega \).

### 3 DISCRETIZATION USING THE FINITE ELEMENT METHOD

This section is devoted to a study of the finite element approximation applied to Equation (5). We introduce a family of regular triangulations \( \mathcal{T}_h \) of \( \Omega \) ([24], (4.4.16)). Hereinafter, we set \( h = \max \{h_T \mid T \in \mathcal{T}_h\} \), where \( h_T \) denotes the diameter of \( T \). For any \( T \in \mathcal{T}_h \), let \( \mathcal{P}_1(T) \) be the set of all polynomials defined on \( T \) of degree \( \leq 1 \) and let \( \mathcal{B}(T) = [\mathcal{P}_1(T) \oplus \text{span} \{\lambda_1, \lambda_2, \ldots, \lambda_{n+1}\}]^n \), where \( \lambda_i \) are the barycentric coordinates of \( T \). Below, we consider the P1-b/P1 element (MINI element) approximation: set

\[
V_h = \{v_h \in C(\bar{\Omega})^n \cap W^{1,2}_{0,0}(\Omega)^n \mid v_h|_T \in \mathcal{B}(T) \ (\forall T \in \mathcal{T}_h)\},
\]

\[
Q_h = \{q_h \in C(\bar{\Omega}) \cap L^2_0(\Omega) \mid q_h|_T \in \mathcal{P}_1(T) \ (\forall T \in \mathcal{T}_h)\}.
\]

A pair of \( V_h \) and \( Q_h \) is well known ([25], Lem. II.4.1) to satisfy the uniform Babuška–Brezzi (inf–sup) condition as

\[
\sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_{W^{1,2}}} \geq \beta \|q_h\|_{L^2} \quad (q_h \in Q_h),
\]

where \( \beta > 0 \) is independent of \( h \).
Remark 15  We deal with the P1-b/P1 element only for the sake of simple presentation. Another pair of conforming finite element spaces $V_h \subset W^{1,2}_0(\Omega)^n$ and $Q_h \subset L^2_0(\Omega)$ satisfying the uniform Babuška–Brezzi condition is available.

We state the finite element approximation to Equation (5): Find $(u_h, \pi_h^e) \in V_h \times Q_h$ such that

\begin{align}
    a(u_h, v_h) + b(\pi_h^e, v_h) &= (f^e, v_h)_{L^2} \quad (\forall v_h \in V_h), \\
    b(q_h, u_h) &= 0 \quad (\forall q_h \in Q_h). 
\end{align}

The finite element approximation $(w_h, r_h) \in V_h \times Q_h$ of Equation (9) is defined similarly.

We make the following assumption for $1 \leq p \leq \infty$, which is called the $W^{1, p} \times L^p$ stability of the Stokes projection:

**Assumption 2_p**  For a given $(w, r) \in W^{1, p}_0(\Omega)^n \times L^p_0(\Omega)$, let $(w_h, r_h) \in V_h \times Q_h$ solve

\begin{align}
    a(w_h, v_h) + b(r_h, v_h) &= a(w, v_h) + b(r, v_h) \quad (\forall v_h \in V_h), \\
    b(q_h, r_h) &= 0 \quad (\forall q_h \in Q_h).
\end{align}

Then, we have

$$
\|w_h\|_{W^{1, p}} + \|r_h\|_{L^p} \leq C_4(\|w\|_{W^{1, p}} + \|r\|_{L^p}),
$$

where $C_4$ denotes a positive constant depending only on $p$ and $\Omega$.

Remark 16  If $\Omega$ is a convex polyhedral domain in $\mathbb{R}^n$ with $n = 2, 3$ and $\{T_h\}_h$ is quasi-uniform ([24], (4.4.15)), then Assumption 2_p is actually satisfied for $1 < p \leq \infty$, as supported by Corollaries 4, 5 and Remark 4 of one earlier report [10]. However, we assume Assumption 2_p directly instead of the shape condition on $\Omega$ as before.

**Proposition 17**  Let $1 \leq p < \infty$. Suppose that Assumptions 1_p and 2_p are satisfied. Let $(u^e, \pi^e)$ and $(u_h^e, \pi_h^e)$, respectively, represent solutions of Equations (5) and (15a and b). Then, we have

$$
\|u^e - u_h^e\|_{W^{1, p}} + \|\pi^e - \pi_h^e\|_{L^p} \leq C\|f^e\|_{L^p},
$$

where $C$ denotes a positive constant depending only on $p$ and $\Omega$.

**Proof.**  Letting $z_h \in V_h$ and $\chi_h \in X_h$ be arbitrary, then because

\begin{align}
    a(u_h^e - z_h, v_h) + b(\pi_h^e - \chi_h, v_h) &= a(u^e - z_h, v_h) + b(\pi^e - \chi_h, v_h) \quad (\forall v_h \in V_h), \\
    b(q_h, \pi_h^e - \chi_h) &= 0 \quad (\forall q_h \in Q_h),
\end{align}

we have by Assumption 2_p

$$
\|u_h^e - z_h\|_{W^{1, p}} + \|\pi_h^e - \chi_h\|_{L^p} \leq C_4(\|u^e - z_h\|_{W^{1, p}} + \|\pi^e - \chi_h\|_{L^p}).
$$

Therefore, by the triangle inequality

$$
\|u^e - u_h^e\|_{W^{1, p}} + \|\pi^e - \pi_h^e\|_{L^p} \leq (1 + C_4)(\|u^e - z_h\|_{W^{1, p}} + \|\pi^e - \chi_h\|_{L^p}).
$$

At this stage, we let $z_h$ be the Lagrange interpolation of $u^e$ and $\chi_h$ be the $L^2$ projection of $\pi^e$. Then, applying the standard interpolation–projection error estimates together with Assumption 1_p, we obtain Equation (16).

Putting together those results, we deduce the following error estimate.
Proposition 18 Let \( 1 \leq p < \frac{n}{n-1} \) and assume that Assumptions 1\(_p\) and 2\(_p\) are satisfied. Also assume \( F \in L^p(\Theta)^n \). Letting \((u, \pi)\) and \((u_h^\varepsilon, \pi_h^\varepsilon)\), respectively, represent solutions of Equations (3) and (15a and b) with Equation (12a–d), we have

\[
\|u - u_h^\varepsilon\|_{W^{1, p}} + \|\pi - \pi_h^\varepsilon\|_{L^p} \leq C \varepsilon^{-n+\frac{2}{p}}(\varepsilon + h),
\]

where \( C \) denotes a positive constant depending only on \( n, p, \|J_\varepsilon\|_{L^n(\Theta)}, \|J_X^{-1}\|_{L^n(\Theta)}, \text{ meas}(\Theta), K, \|\varphi\|_{L^\infty(\Omega)}, \|F\|_{L^p(\Theta)} \) and \( \Omega \).

**Proof.** Because \( f^\varepsilon \) is defined in terms of \( \delta^\varepsilon \) given as Equation (12a–d), we have by Equation (12d)

\[
\|f^\varepsilon\|_{L^p} \leq \text{meas}(\Theta)^{1/p}\|f\|_{L^p(\mathbb{R}^n)}\|\delta^\varepsilon\|_{L^p(\mathbb{R}^n)} \\
\leq C\text{meas}(\Theta)^{1/p}\|f\|_{L^p(\Theta)}\varepsilon^{-n+\frac{2}{p}},
\]

where \( C > 0 \) is a constant depending only on \( p, n, K, \) and \( \|\varphi\|_{L^\infty(\mathbb{R})} \). Moreover, note that \( \|f\|_{L^p(\Theta)} \leq \text{meas}(\Theta)(2-p)/p\|F\|_{L^p(\Theta)} \). Therefore, in view of Lemmas 13 and 17,

\[
\|u - u_h^\varepsilon\|_{W^{1, p}} + \|\pi - \pi_h^\varepsilon\|_{L^p} \leq \|u - u^\varepsilon\|_{W^{1, p}} + \|\pi - \pi^\varepsilon\|_{L^p} \\
+ \|u^\varepsilon - u_h^\varepsilon\|_{W^{1, p}} + \|\pi^\varepsilon - \pi_h^\varepsilon\|_{L^p} \\
\leq C\varepsilon^{-n+\frac{2}{p}} + C h \cdot C \varepsilon^{-n+\frac{2}{p}}.
\]

We usually take \( \varepsilon = ch \) in the IB method, where \( c > 0 \) is a suitable constant. Therefore, applying Proposition 18 with \( p = 1 \), we obtain the optimal order error estimate as

\[
\|u - u_h^\varepsilon\|_{W^{1, 1}} + \|\pi - \pi_h^\varepsilon\|_{L^1} \leq C h.
\]

It is noteworthy that this estimate is available only if Assumptions 1\(_p\) and 2\(_p\) are true. However, the case \( p = 1 \) is excluded both in Refs. [9, 10], as described in Remarks 9 and 16. In conclusion, we offer the following theorem as the final error estimate in this article.

**Theorem 19** Assume that \( \Omega \) is a convex polyhedral domain in \( \mathbb{R}^n \) with \( n = 2, 3 \) and that \( \{\mathcal{T}_h\}_h \) is a family of quasi-uniform triangulations of \( \Omega \). Let \( 0 < \alpha < 1 \) be arbitrary. Letting \( F \in L^{n/\alpha}(\Theta) \), suppose that \((u, \pi)\) and \((u_h^\varepsilon, \pi_h^\varepsilon)\), respectively, represent solutions of Equations (3) and (15a and b) with Equation (12a–d). Furthermore, let \( \varepsilon = \gamma h \) with a positive constant \( \gamma \). Then, there exists a positive constant \( C \) depending only on \( \gamma, n, \alpha, \Omega, K, \|\varphi\|_{L^\infty(\mathbb{R})}, \|J_\varepsilon\|_{L^n(\Theta)}, \|J_X^{-1}\|_{L^n(\Theta)}, \text{ meas}(\Theta) \), and \( \|F\|_{L^{n/\alpha}(\Theta)} \) such that

\[
\|u - u_h^\varepsilon\|_{W^{1, q}} + \|\pi - \pi_h^\varepsilon\|_{L^q} \leq C h^{1-\alpha} q \leq 1 \leq q \leq \frac{n}{n-\alpha}
\]

**Proof.** Let \( 0 < \alpha < 1 \) be arbitrary and set \( p = \frac{n}{n-\alpha} \). Then, \( p' = \frac{n}{\alpha} \) and \( \alpha = n \left( 1 - \frac{1}{p'} \right) \). As pointed out in Remarks 9 and 16, Assumptions 1\(_p\) and 2\(_p\) are true for a convex polyhedral domain. Because \( 1 < p < \frac{n}{n-1} \), we can apply Equation (17) and an elementary inequality \( \|\varphi\|_{L^q} \leq C \|\varphi\|_{L^p} \) for \( 1 \leq q \leq p \) to deduce (19).

**Remark 20** Our \( W^{1, q} \times L^q \) error estimate (19) is nearly optimal and seems to be new. On the other hand, we are unable to derive any sharp estimates for the velocity in the \( L^q \) norm at present. Actually, applying the Aubin–Nitsche duality argument, we obtain

\[
\|u^\varepsilon - u_h^\varepsilon\|_{L^q} \leq Ch\|u^\varepsilon - u_h^\varepsilon\|_{W^{1, q}}.
\]
In this section, we study the error caused by numerical integrations for computing interpolations of $u$. For the partition of $\Theta = (c, d)$, we choose points arbitrarily, we choose points $\hat{\theta}_i$ $(\theta_{i-1} < \theta_i < \cdots < \theta_M = d$ and set $\zeta_i = \theta_i - \theta_{i-1}$ for $1 \leq i \leq M$. Arbitrarily, we choose points $\hat{\theta}_i$ in $[\theta_{i-1}, \theta_i]$ for $1 \leq i \leq M$. For example, we can take $\hat{\theta}_i = (\theta_{i-1} + \theta_i)/2$. Furthermore, we set $\zeta = \max_{1 \leq i \leq M} \zeta_i$. Then, we apply the rectangle rule to compute $f^\epsilon$. That is,

$$f^\epsilon \zeta(x) = \sum_{i=1}^{M} F(\hat{\theta}_i) \delta^\epsilon x(\hat{\theta}_i)(x) \zeta_i = \sum_{i=1}^{M} F(\hat{\theta}_i) \delta^\epsilon (x - X(\hat{\theta}_i)) \zeta_i.$$  

(21)

It is useful to express $f^\epsilon \zeta$ as

$$f^\epsilon \zeta(x) = \int_{\Theta} \hat{F}^\zeta(\theta) \delta^\epsilon (x - \hat{X}^\zeta(\theta)) d\theta,$$  

(22)

where $\hat{F}^\zeta(\theta)$ and $\hat{X}^\zeta(\theta) = (\hat{X}^\zeta_1(\theta), \hat{X}^\zeta_2(\theta))$ are piecewise constant functions such that

$$\hat{F}^\zeta(\theta) = F(\hat{\theta}_i), \quad \hat{X}^\zeta(\theta) = X(\hat{\theta}_i) \quad (\theta_{i-1} < \theta \leq \theta_i, \quad 1 \leq i \leq M).$$

From the standard theory, it is known that

$$\|F - \hat{F}^\zeta\|_{L^\infty(\Theta)} \leq C\zeta |F|_{W^{1,\infty}(\Theta)},$$  

(23a)

$$\|X - \hat{X}^\zeta\|_{L^\infty(\Theta)} \leq C\zeta |X|_{W^{1,\infty}(\Theta)} r,$$  

(23b)

where $|F|_{W^{1,\infty}(\Theta)}$ denotes the seminorm in $W^{1,\infty}(\Theta)$.

**Remark 21** When $\hat{\theta}_i = (\theta_{i-1} + \theta_i)/2$ for $1 \leq i \leq M$, Equation (23a) is improved as

$$\|F - \hat{F}^\zeta\|_{L^\infty(\Theta)} \leq C\zeta^2 |F|_{W^{2,\infty}(\Theta)}.$$  

However, Equation (23a) is sufficient to derive our error estimates.

For the case of $n = 3$, $f^\epsilon \zeta$ is defined similarly. Letting $\hat{F}^\zeta(\theta)$ and $\hat{X}^\zeta(\theta)$ be piecewise constant interpolations of $F(\theta)$ and $X(\theta)$, respectively, then $f^\epsilon$ is approximated by $f^\epsilon \zeta$ defined as Equation (22). For the partition of $\Theta$, we only assume it so that Equation (23a and b) hold true.
Proposition 22  Let $\delta^\epsilon$ be a continuous function satisfying Equation (7). Assuming that $F \in W^{1,\infty}(\Theta)$, then, for $1 \leq p < \frac{n}{n-1}$, we have

$$\|f - f^{\epsilon,\zeta}\|_{W^{1,p}} \leq C \left[ 1 - \int_{\mathbb{R}^n} \delta^\epsilon(y) \, dy \right] + \|\rho\delta^\epsilon\|_{L^p(\mathbb{R}^n)} + \zeta^{1-n+\frac{2}{p}}$$

where $\rho(x) = x$ and $C$ denotes a positive constant depending only on $n$, $p$, $\|J_X\|_{L^\infty(\Theta)}$, $\|\tilde{F}\|_{L^\infty(\Theta)}$, $|F|_{W^{1,\infty}(\Theta)}$, and $|X|_{W^{1,\infty}(\Theta)}$.

Proof.  It is merely a modification of the proof of Proposition 4. Letting $\varphi \in C_0^\infty(\Omega)^n$ and expressing it as

$$\varphi(x) = \varphi(\tilde{X}(\theta)) + (x - \tilde{X}(\theta)) \cdot \int_0^1 \nabla \varphi(t(x - \tilde{X}(\theta))) \, dt$$

for $x \in \mathbb{R}^n$, we have

$$\langle f - f^{\epsilon,\zeta}, \varphi \rangle_{W^{1,p},W^{1,p}_0} = \int_{\Theta} F(\theta)\varphi(X(\theta)) \, d\theta - \int_{\Theta} \varphi(\tilde{X}(\theta)) \left( \int_{\tilde{X}(\theta)} \delta^\epsilon(x - \tilde{X}(\theta)) \, dx \right) \, d\theta$$

$$- \int_0^1 \int_{\Omega} \tilde{F}(\theta) \varphi(\tilde{X}(\theta)) \nabla \varphi(t(x - \tilde{X}(\theta))) \cdot (x - \tilde{X}(\theta)) \, dx \, d\theta \, dt.$$

To estimate $|I_1|$, we divide it further as

$$I_1 = \int_{\Theta} \left[ F(\theta) - \tilde{F}(\theta) \right] \varphi(X(\theta)) \, d\theta + \int_{\Theta} \tilde{F}(\theta) \left[ \varphi(X(\theta)) - \varphi(\tilde{X}(\theta)) \right] \, d\theta$$

$$+ \int_{\Theta} \tilde{F}(\theta) \varphi(\tilde{X}(\theta)) \left( 1 - \int_{\tilde{X}(\theta)} \delta^\epsilon(x - \tilde{X}(\theta)) \, dx \right) \, d\theta$$

As in the proof of Proposition 4, we derive

$$|I_{13}| \leq C \|\tilde{F}\|_{L^p(\Theta)} \left[ 1 - \int_{\mathbb{R}^n} \delta^\epsilon(y) \, dy \right] \|\varphi\|_{W^{1,p}(\Omega)}.$$  

By substituting Equation (23a and b), we have

$$|I_{11}| \leq C \|F - \tilde{F}\|_{L^p(\Theta)} \|\varphi\|_{L^\infty(\Theta)}$$

$$\leq C \zeta \|F\|_{W^{1,\infty}(\Theta)} \|\varphi\|_{W^{1,p}(\Omega)}.$$  

We apply Morrey’s inequality to obtain

$$|I_{12}| \leq C \int_{\Theta} \|\tilde{F}\|_{L^p(\Theta)} \|X(\theta) - \tilde{X}(\theta)\|^{1-n/p'} \|\varphi\|_{W^{1,p}(\Omega)} \, d\theta$$

$$\leq C \zeta^{1-n/p'} \|\tilde{F}\|_{L^p(\Theta)} \|X\|_{W^{1,\infty}(\Omega)} \|\varphi\|_{W^{1,p}(\Omega)}.$$
Estimation for $|I_2|$ is done in exactly the same manner as that used for the proof of Proposition 4. That is, we deduce

$$|I_2| \leq C \|\hat{F}\|_{L^1(\Theta)} \|\rho \delta^\varepsilon\|_{L^p(\mathbb{R}^n)} \|\varphi\|_{W^{1,p}}.$$ 

Finally, noting $\|\hat{F}\|_{L^1(\Theta)} \leq C \|F\|_{L^\infty(\Theta)}$, we obtain the desired inequality. □

Applying Proposition 22 instead of Proposition 4, we obtain the following result.

**Theorem 23** Letting $(u, \pi)$ and $(u_h, \pi_h)$ denote solutions of Equations (3) and (15a and b) with Equation (12a–d), where $f^\varepsilon$ is replaced by $f^\varepsilon$ defined as Equation (22), then in addition to assumptions of Theorem 19, we assume that $F \in W^{1, \infty}(\Theta)$. Furthermore, let $\zeta = \gamma_2 h$ with a positive constant $\gamma_2$. Then, error estimates (19) and (20) remain true.

### 5 | NUMERICAL EXAMPLES

In this section, we confirm the validity of our results obtained using numerical examples. To do so, the exact solution (24) proposed by Ref. [13] (Section 7.1) is useful. We let $\Omega = (0, 1)^2 \subset \mathbb{R}^2$ and $\Gamma = B((1/2, 1/2), R)$, where $0 < R < 1/2$. We consider the Stokes problem (3) with Equation (2) and employ a single-chart parameterization of $\Gamma$ given as $X(\theta) = (R \cos(\theta/R), R \sin(\theta/R))$ ($\theta \in \Theta = (0, 2\pi R)$).

Set

$$F(\theta) = \frac{\partial X}{\partial \theta} = (\cos(\theta/R), \sin(\theta R)) \quad (\theta \in \Theta).$$

Then, the exact solution

$$u = 0, \quad p(x) = \begin{cases} \frac{1}{R} - \frac{\pi R}{|x|} & (|x| < R) \\ -\pi R & (|x| > R) \end{cases}$$

is available ([13], subsection 7.1). Below, let $R = 1/4$.

Triangulation $\mathcal{T}_h$ is taken as a uniform mesh composed of $2N^2$ congruent right-angle triangles. Each side of $\Omega$ is divided into $N$ intervals of the same length. Then each small square is decomposed into two equal triangles by a diagonal. Then, we apply the finite element method and numerical integral formula described in the previous sections. The parameters are set as $h = \sqrt{2}/N$, $\zeta = 2\pi R/M$, and $M = N$. All computations were performed using FreeFem++ [17].

The delta approximation $\delta^\varepsilon$ is given as Equation (13) with

$$\varphi(s) = \begin{cases} \frac{1}{2}(1 + \cos(\pi s)) & (|s| \leq 1) \\ 0 & (\text{otherwise}) \end{cases}$$

It must be recalled that $\varepsilon$ is defined as $\varepsilon = \gamma_1 h$, as in Theorems 19 and 23. Therefore, one must choose a suitable $\gamma_1$ first. In Figures 2 and 3, we display computed pressures $\pi_h$ for several $(\varepsilon, h)$. Numerical oscillations are observed for the cases of $\varepsilon = h$ and $\varepsilon = 0.5$. Based on these results, we take $\varepsilon = 2h$ or $\varepsilon = 3h$ below.

To confirm convergence results described in Theorems 19 and 23, we examine

$$e_q(h) = \|u - u_h\|_{L^q}, \quad E_q(h) = \|u - u_h\|_{W^{1,q}}, \quad \text{and} \quad E_q(h) = \|\pi - \pi_h\|_{L^q},$$

for $q = 1, 2$. The convergence results of Theorems 19 and 23 are confirmed.
where \((u, \pi)\) denotes the exact solution (24). Moreover, we compute, for example,

\[
\text{Rate} = \frac{\log e_q(2h) - \log e_q(h)}{\log 2}.
\]

Results are reported in Tables 1–6. We observe from these tables that convergence rates of \(E_1(h)\) and \(\mathcal{E}_1(h)\) are very close to 1. It is also observed that each convergence rate becomes worse as \(q\) becomes larger. Those numerical results support our theoretical results. Moreover, we infer from Tables 3 and 6

\[\text{TABLE 1} \quad \text{Convergence in } L^1, W^{1,1} \text{ norms for } \varepsilon = 2h\]

| \(h\)     | \(e_1(h)\)   | Rate | \(E_1(h)\)  | Rate | \(\mathcal{E}_1(h)\) | Rate |
|----------|---------------|------|--------------|------|----------------------|------|
| 0.282843 | 6.37755 \cdot 10^{-4} | –    | 1.98156 \cdot 10^{-2} | –    | 0.956690             | –    |
| 0.141421 | 2.09457 \cdot 10^{-4} | 1.61 | 1.35270 \cdot 10^{-2} | 0.55 | 0.526213             | 0.86 |
| 0.070710 | 6.26596 \cdot 10^{-5} | 1.74 | 6.59742 \cdot 10^{-3} | 1.04 | 0.269020             | 0.97 |
| 0.035355 | 2.38774 \cdot 10^{-5} | 1.39 | 3.31870 \cdot 10^{-3} | 0.99 | 0.135075             | 0.99 |
### TABLE 2  Convergence in $L^{3/2}$, $W^{1,3/2}$ norms for $\epsilon = 2h$

| $h$  | $e_{3/2}(h)$        | Rate | $E_{3/2}(h)$       | Rate | $\mathcal{E}_{3/2}(h)$     | Rate |
|------|---------------------|------|-------------------|------|---------------------------|------|
| 0.282843 | 1.84851 · 10^{-5} | –    | 3.21967 · 10^{-3} | –    | 1.161270                  | –    |
| 0.141421 | 3.80345 · 10^{-6} | 1.52 | 2.09202 · 10^{-3} | 0.41 | 0.570142                  | 0.68 |
| 0.070710 | 7.55398 · 10^{-7} | 1.55 | 9.46287 · 10^{-4} | 0.76 | 0.286601                  | 0.66 |
| 0.035355 | 1.77253 · 10^{-7} | 1.39 | 4.60730 · 10^{-4} | 0.69 | 0.142560                  | 0.67 |

### TABLE 3  Convergence in $L^2$, $H^1$ norms for $\epsilon = 2h$

| $h$  | $e_2(h)$          | Rate | $E_2(h)$          | Rate | $\mathcal{E}_2(h)$      | Rate |
|------|-------------------|------|-------------------|------|--------------------------|------|
| 0.282843 | 5.69726 · 10^{-7} | –    | 5.78648 · 10^{-4} | –    | 1.548920                  | –    |
| 0.141421 | 7.65079 · 10^{-8} | 1.45 | 3.64338 · 10^{-4} | 0.33 | 0.668649                  | 0.61 |
| 0.070710 | 1.05183 · 10^{-8} | 1.43 | 1.60292 · 10^{-4} | 0.59 | 0.329591                  | 0.51 |
| 0.035355 | 1.60145 · 10^{-9} | 1.36 | 7.83419 · 10^{-5} | 0.52 | 0.162474                  | 0.51 |

### TABLE 4  Convergence in $L^1$, $W^{1,1}$ norms for $\epsilon = 3h$

| $h$  | $e_2(h)$          | Rate | $E_2(h)$          | Rate | $\mathcal{E}_2(h)$      | Rate |
|------|-------------------|------|-------------------|------|--------------------------|------|
| 0.282843 | 1.43132 · 10^{-4} | –    | 4.64748 · 10^{-3} | –    | 1.160000                  | –    |
| 0.141421 | 1.14356 · 10^{-4} | 0.32 | 7.87403 · 10^{-3} | – 0.76 | 0.753044                  | 0.63 |
| 0.070710 | 3.20908 · 10^{-5} | 1.83 | 4.00349 · 10^{-3} | 0.98 | 0.394694                  | 0.93 |
| 0.035355 | 8.22480 · 10^{-6} | 1.96 | 1.89386 · 10^{-3} | 1.08 | 0.198714                  | 0.99 |

### TABLE 5  Convergence in $L^{3/2}$, $W^{1,3/2}$ norms for $\epsilon = 3h$

| $h$  | $e_{3/2}(h)$        | Rate | $E_{3/2}(h)$       | Rate | $\mathcal{E}_{3/2}(h)$     | Rate |
|------|---------------------|------|-------------------|------|---------------------------|------|
| 0.282843 | 2.01798 · 10^{-6} | –    | 3.80099 · 10^{-4} | –    | 1.510000                  | –    |
| 0.141421 | 1.47635 · 10^{-6} | 0.30 | 8.83507 · 10^{-4} | – 0.81 | 0.854271                  | 0.55 |
| 0.070710 | 2.53338 · 10^{-7} | 1.70 | 3.99117 · 10^{-4} | 0.76 | 0.424394                  | 0.67 |
| 0.035355 | 4.48127 · 10^{-8} | 1.67 | 1.85223 · 10^{-4} | 0.74 | 0.211014                  | 0.67 |

### TABLE 6  Convergence in $L^2$, $H^1$ norms for $\epsilon = 3h$

| $h$  | $e_2(h)$          | Rate | $E_2(h)$          | Rate | $\mathcal{E}_2(h)$      | Rate |
|------|-------------------|------|-------------------|------|--------------------------|------|
| 0.282843 | 3.07409 · 10^{-8} | –    | 3.47150 · 10^{-5} | –    | 2.166200                  | –    |
| 0.141421 | 2.10332 · 10^{-8} | 0.27 | 1.09180 · 10^{-5} | – 0.83 | 1.054740                  | 0.52 |
| 0.070710 | 2.29385 · 10^{-9} | 1.60 | 4.49813 · 10^{-5} | 0.64 | 0.492575                  | 0.55 |
| 0.035355 | 2.89764 · 10^{-10}| 1.49 | 2.10016 · 10^{-5} | 0.55 | 0.241028                  | 0.52 |
that convergence rates of $e_2(h)$, $E_2(h)$, and $\mathcal{E}_2(h)$ are, respectively, close to 1.5, 0.5, and 0.5, which accords with the result reported in Ref. [13] where IFE method with no regularization of Dirac’s delta was studied. The convergence rate of $e_q(h)$ is strictly larger than 1, which implies that our $L'$ estimate (20) is not sharp; recall Remark 20. However, we leave that derivation of better estimates as a subject for future study.

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REFERENCES

[1] C. S. Peskin. Flow patterns around heart valves: A digital computer method for solving the equations of motion. Ph.D. Thesis, Albert Einstein College of Medicine of Yeshiva University. 1972.
[2] C. S. Peskin, Numerical analysis of blood flow in the heart, J. Comput. Phys. vol. 25(3) (1977) pp. 220–252.
[3] C. S. Peskin, The immersed boundary method, Acta Numer vol. 11 (2002) pp. 479–517.
[4] A. K. Tornberg and B. Engquist, Numerical approximations of singular source terms in differential equations, J. Comput. Phys. vol. 200(2) (2004) pp. 462–488.
[5] Y. Mori, Convergence proof of the velocity field for a Stokes flow immersed boundary method, Comm. Pure Appl. Math vol. 61(9) (2008) pp. 1213–1263.
[6] Y. Liu and Y. Mori, Properties of discrete delta functions and local convergence of the immersed boundary method, SIAM J. Numer. Anal vol. 50(6) (2012) pp. 2986–3015.
[7] Y. Liu and Y. Mori, $L^p$ convergence of the immersed boundary method for stationary Stokes problems, SIAM J. Numer. Anal vol. 52(1) (2014) pp. 496–514.
[8] Z. Li, On convergence of the immersed boundary method for elliptic interface problems, Math. Comput vol. 84(293) (2015) pp. 1169–1188.
[9] V. Maz’ya and J. Rossmann, $L^p$ estimates of solutions to mixed boundary value problems for the Stokes system in polyedral domains, Math. Nachr vol. 280(7) (2007) pp. 751–793.
[10] V. Girault, R. H. Nochetto, and L. R. Scott, Max-norm estimates for Stokes and Navier-Stokes approximations in convex polyhedra, Numer. Math. vol. 131(4) (2015) pp. 771–822.
[11] D. Boffi and L. Gastaldi, A finite element approach for the immersed boundary method, Comput. Struct. vol. 81(8–11) (2003) pp. 491–501.
[12] D. Boffi, L. Gastaldi, and L. Heltai, Numerical stability of the finite element immersed boundary method, Math. Models Methods Appl. Sci vol. 17(10) (2007) pp. 1479–1505.
[13] D. Boffi, L. Gastaldi, L. Heltai, and C. S. Peskin, On the hyper-elastic formulation of the immersed boundary method, Comput. Methods Appl. Mech. Engrg vol. 197(25–28) (2008) pp. 2210–2231.
[14] L. Heltai and F. Costanzo, Variational implementation of immersed finite element methods, Comput. Methods Appl. Mech. Engrg vol. 229/232 (2012) pp. 110–127.
[15] M. Tabata, Finite element schemes based on energy-stable approximation for two-fluid flow problems with surface tension, Hokkaido Math. J vol. 36(4) (2007) pp. 875–890.
[16] J. Guzmán, M. A. Sánchez, and M. Sarkis, On the accuracy of finite element approximations to a class of interface problems, Math. Comp vol. 85(301) (2016) pp. 2071–2098.
[17] F. Hecht, New development in freefem++, J. Numer. Math vol. 20(3–4) (2012) pp. 251–265.
[18] A. Logg, K. A. Mardal, and G. N. Wells, Automated solution of differential equations by the finite element method, volume 84 of Lecture Notes in Computational Science and Engineering, Springer-Verlag, Heidelberg, Germany, 2012.
[19] Y. Sugitani, Numerical analysis of a Stokes interface problem based on formulation using the characteristic function, Appl. Math vol. 62(5) (2017) pp. 459–476.
[20] N. Saito and Y. Sugitani, Convergence of the immersed-boundary finite-element method for the Stokes problem, arXiv:1611.07172, 2016.
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