GLOBAL REGULARITY OF THE TWO-DIMENSIONAL BOUSSINESQ EQUATIONS WITHOUT DIFFUSIVITY IN BOUNDED DOMAINS

DAOGUO ZHOU

Abstract. We address the well-posedness for the two-dimensional Boussinesq equations with zero diffusivity in bounded domains. We prove global in time regularity for rough initial data: the initial velocity has $\epsilon$ fractional derivatives in $L^q$ and the initial temperature is in $L^q$, for some $q > 2$ and $\epsilon > 0$ arbitrarily small.

1. Introduction and Main Results

In this paper, we study the initial-boundary value problem for the 2D Boussinesq equations with zero thermal diffusivity on an open bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial \Omega$. The corresponding equations reads

\[
\begin{align*}
\partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla p &= \theta e_2, \\
\partial_t \theta + u \cdot \nabla \theta &= 0, \\
\nabla \cdot u &= 0,
\end{align*}
\]

where $u = (u_1, u_2)$ is the velocity vector field, $p$ is the pressure, $\theta$ is the temperature, $\nu > 0$ is the constant viscosity, and $e_2 = (0, 1)$. This system is supplemented by the following initial and boundary conditions

\[
\begin{align*}
(u, \theta)(x, 0) &= (u_0, \theta_0)(x), \quad x \in \Omega, \\
\frac{\partial u}{\partial n}|_{\partial \Omega} &= 0.
\end{align*}
\]

Here, we have imposed the mostly used no-slip conditions on the velocity, which assume that fluid particles are adherent to the boundary due to the positive viscosity.

The Boussinesq equations play an important role in modeling large scale atmospheric and oceanic flows \cite{18}, \cite{22}. In addition, the Boussinesq equations is closely related to Rayleigh-Benard convection \cite{22}. From the mathematical view, the 2D Boussinesq equations serve as a simplified model of the 3D Euler and Navier-Stokes equations. In fact, we get the 2D Boussinesq equations when we analyze 3D axisymmetric swirling fluid in the Navier-Stokes framework. Better understanding of the 2D Boussinesq equations will undoubtedly shed light on the understanding of 3D flows \cite{19}.

Recently, the well-posedness of the 2D Boussinesq equations has attracted attention of many mathematicians, see \cite{1}, \cite{2}, \cite{3}, \cite{11}, \cite{13}, \cite{15}, \cite{16}, \cite{20}, \cite{24}, \cite{25}. In particular, when $\Omega = \mathbb{R}^2$, the Cauchy problem of (1.1) has been well

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studied. Hou and Li [9] and Chae [3] showed the global in time regularity for 
\((u_0, \theta_0) \in H^3(\mathbb{R}^2) \times H^2(\mathbb{R}^2)\). Kukavica, Wang, Ziane [13] obtained the global regularity for 
\((u_0, \theta_0) \in W^{1+s,q}(\mathbb{R}^2) \times W^{s,q}(\mathbb{R}^2)\) for 
\(s \in (0, 1), q \in [2, \infty)\) and \(sq > 2\). They also pointed out that the restriction \(sq > 2\) can be removed provided that 
the initial data have compact support or \(\Omega = \mathbb{T}^2\). Abidi and Hmidi [1] proved the global existence for 
\((u_0, \theta_0) \in L^2(\mathbb{R}^2) \cap B^{-1}_{\infty,1}(\mathbb{R}^2) \times L^2(\mathbb{R}^2)\). Danchin and Paicu [6] proved the uniqueness of weak solution for 
\((u_0, \theta_0) \in L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)\).

In real world applications, fluids often move in bounded domains, where new 
phenomena such as the creation of vorticity on the boundary appears. In such 
case, the boundary effect requires a careful analysis. The initial-boundary value 
problem of (1.1)-(1.2) was first studied by Lai, Pan, and Zhao [15], who showed the 
existence for (1.1)-(1.2) has a unique global solution which belongs to 
\(C([0, T]; D_{A_q}^{1 - \frac{1}{p},p}(\Omega) \cap W^{s,q}(\Omega))\). Here, \(D_{A_q}^{1 - \frac{1}{p},p}\) denotes some fractional domain of 
The Stokes operator whose elements have \(2 - \frac{2}{p}\) derivatives in \(L^q\) and \(s \geq 0, 1 < p, q < \infty\).

Remark 1.4. While preparing the manuscript, the author becomes to know that 
Proposition 1.3 was obtained very recently by Ju [12] independently. However, our 
method is completely different from that of Ju, which exploited Brezis-Gallouet 
type inequalities and spectral decomposition.
Remark 1.5. The proof of Theorem 1.2 is based on the maximal regularity of the Stokes operator and some interpolation inequalities. Our method is elementary and can be carried over to the whole space case $\mathbb{R}^2$ without difficulty.

Remark 1.6. Since the “regularity index” $2 - 2/p$ in Theorem 1.2 can be arbitrarily close to zero, our result improves the previous works of Lai et al. [15], Hu et al. [10] and Kukavica et al. [13] by requiring much less regularity for the initial data.

The rest of this paper is organized as follows. In Section 2, we recall the maximal regularity of Stokes equations as well as some elementary inequalities. In Section 3, we present the detailed proofs of the main results.

2. Preliminaries

Notations:

(1) Let $\Omega$ be a bounded domain in $\mathbb{R}^2$. For $1 < q < \infty$, denote by $L^q_\sigma(\Omega)$ the completion in $L^q(\Omega)$ of the set of solenoidal vector-fields with coefficients in $C_0^\infty(\Omega)$. If $k$ is an integer, we denote by $W_k,q(\Omega)$ the set of $L^q(\Omega)$ functions whose derivatives up to order $k$ belong to $L^q(\Omega)$. For $s \in (0, 1)$, the Sobolev space $W^{s,q}(\Omega)$ is defined as

$$W^{s,q}(\Omega) = \left\{ f \mid \|f\|_{W^{s,q}} = \left( \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^q}{|x - y|^{q+2}} \, dx \, dy \right)^{\frac{1}{q}} < \infty \right\}.$$ 

(2) For $\alpha \in (0, 2)$ and $1 < p, q < \infty$, denote by $B_{\alpha,q,p}$ the Besov space which is defined as the real interpolation space between $L^q(\Omega)$ and $W^{m,q}(\Omega)$ ($m > \alpha$):

$$B_{\alpha,q,r}(\Omega) = (L^q(\Omega), W^{m,q}(\Omega))_{\frac{\alpha}{m}, r}.$$ 

Denote by $\hat{B}_{\alpha,q,p}$ the completion of $C_0^\infty(\Omega)$ in $B_{\alpha,q,p}$. See Adams and Fournier [3] for more about the Besov space.

(3) For $T > 0$ and a function space $X$, denote by $L^p(0, T; X)$ the set of Bochner measurable $X$-valued time dependent functions $f$ such that $t \to \|f\|_X$ belongs to $L^p(0, T)$.

First we give the definition of the fractional domains of the Stokes operator in $L^q$.

Definition 2.1. For $\alpha \in (0, 1)$ and $s, q \in (1, \infty)$, we set

$$\|u\|_{D_{A_q}^\alpha} = \|u\|_{L^q} + \left( \int_0^\infty \|t^{1-\alpha} A_q e^{-tA_q} u\|_{L^q}^s \frac{dt}{t} \right)^{\frac{1}{s}},$$

where $A_q = -\mathbb{P}\Delta$ has domain $D(A_q) = W^{2,q}(\Omega) \cap W^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$. Here, $\mathbb{P}$ denotes the Leray projector.

Roughly, the vector fields of $D_{A_q}^{\alpha,s}$ have $2\alpha$ derivatives in $L^q$, are divergence-free, and vanish on $\partial \Omega$. In fact, we have the following imbedding (cf. Proposition 2.5 in Danchin [5]).

Lemma 2.2. $\hat{B}_{q,s}^{2\alpha} \cap L^q_\sigma \hookrightarrow D_{A_q}^{\alpha,s} \hookrightarrow B_{q,s}^{2\alpha} \cap L^q_\sigma$, for $\alpha \in (0, 1)$, $1 < q, s < \infty$. Moreover, if $2\alpha \leq \frac{1}{q}$, then the three spaces are the same (with equivalent norms).

We need the well-known Sobolev embedding, Ladyzhenskaya inequality and Gagliardo-Nirenberg inequality (see Adams and Fournier [3], Ladyzhenskaya [14] and Nirenberg [21]).
Lemma 2.4. Let $C$ holds true:

1. $H^1(\Omega) \hookrightarrow L^q(\Omega)$, for all $q \in (1,\infty)$.
2. $\|f\|_{L^4} \leq \sqrt{2}\|f\|_{L^2}^\frac{1}{2}\|\nabla f\|_{L^2}^\frac{1}{2}$, for $f \in H^1_0(\Omega)$.
3. $\|\nabla u\|_{L^\infty} \leq C\|\nabla^2 u\|_{L^q}^\alpha\|u\|_{L^r}^\beta + C\|u\|_{L^s}$, for all $u \in W^{2,q}(\Omega)$, with $q \in (2,\infty)$, $\alpha = \frac{1}{2} + \frac{1}{q}$ and $C$ is a constant depending on $q, \Omega$.

We also need the following interpolation inequality (cf. Lemma 4.1 in Danchin [5]).

Lemma 2.5. Let $1 < p, q < \infty$ satisfy $0 < \frac{1}{p} - \frac{1}{q} < \frac{1}{p}$. The following inequality holds true:

$$\|\nabla u\|_{L^p(0,T;L^\infty)} \leq CT^{\frac{1}{2} - \frac{1}{q}}\|f\|_{L^\infty(0,T;D^\frac{1}{2}_{x\partial_t}L^1)}^{\frac{1}{2}}\|f\|^{\theta}_{L^p(0,T;W^{2,q})},$$

for $C = (p,q,\Omega)$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{2} - \frac{1}{q}$.

Now we recall the following standard result for linear transport equations.

Lemma 2.6. Let $\Omega$ be a Lipschitz domain of $\mathbb{R}^2$ and $u \in L^1(0,T;W^{1,\infty})$ such that $\text{div} u = 0$ and $u \cdot n = 0$ on $\partial \Omega$. Let $a_0 \in W^{s,q}$ with $q \in [1,\infty)$ and $s \in [0,1]$. Then the equation

$$\begin{cases}
\partial_t a + u \cdot \nabla a = 0, \\
a(x,t)|_{t=0} = a_0,
\end{cases} \quad (2.1)$$

has a unique solution in $C([0,T];W^{s,q})$. Moreover, the following estimate holds true for all $t \in [0,T]$

$$\|a(t)\|_{W^{s,q}} \leq \|a_0\|_{W^{s,q}} e^{C\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau},$$

with $C = C(s,q)$. If in addition $a(t)$ belongs to $L^p$ for some $p \in [1,\infty]$ then for all $t \in [0,T]$

$$\|a(t)\|_{L^p} = \|a_0\|_{L^p}.$$

The result for $s = 0$ and $s = 1$ is well-known (cf. Proposition 3.1 in Danchin [5]). The case $s \in (0,1)$ seems to be folklore, but I cannot locate the proof. Here we provide a sketched proof.

**Proof.** We only establish a priori estimates and refer the reader to Desjardins [7] for the existence and uniqueness parts.

Denote by $\psi_t(x)$ the flow of $u$, which is defined by

$$\partial_t \psi_t(x) = u(\psi_t(x), t), \quad \psi_t(x)|_{t=0} = x \in \Omega,$$

It follows that (2.1) has the formal solution

$$a(x,t) = a_0(\psi_t^{-1}(x)).$$

From the assumptions $\text{div} u = 0$ and $u \in L^1(0,T;W^{1,\infty})$, we obtain

$$|\det \psi_t(x)| = 1, \quad \text{and} \quad |x - y| \leq |\psi_t(x) - \psi_t(y)| e^{\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau}. \quad (2.2)$$

See Chapter 4 in Majda [18] for more details.
Using the definition of fractional Sobolev space and (2.2), we can compute \( \|a(t)\|_{W^{s,q}} \) as follows:

\[
\|a(t)\|_{W^{s,q}} = \left( \int_{\Omega} \int_{\Omega} \left| a_0(\psi_t^{-1}(x)) - a_0(\psi_t^{-1}(y)) \right|^q \frac{dx}{|x-y|^{s+2}} dy \right)^{\frac{1}{q}} \\
= \left( \int_{\Omega} \int_{\Omega} \left| a_0(u) - a_0(v) \right|^q \frac{|\det \nabla \psi_t(u)||\det \nabla \psi_t(v)|}{|u-v|^{s+2}} du dv \right)^{\frac{1}{q}} \\
= \left( \int_{\Omega} \int_{\Omega} \left| a_0(u) - a_0(v) \right|^q \frac{|u-v|}{|\psi_t(u) - \psi_t(v)|^{s+2}} du dv \right)^{\frac{1}{q}} \\
\leq \|a_0\|_{W^{s,q}} e^{(s+\frac{3}{2}) \int_0^T \|\nabla u(\tau)\|_{L^\infty} d\tau}.
\]

This completes the proof of Lemma 2.5. \( \square \)

We conclude this section by recalling the maximal regularity of the Stokes equations (cf. Theorem 3.2 in Danchin [5] or Theorem 1.1 in Solonnikov [23]), which will be used in the proof of Theorem 1.2.

**Lemma 2.6.** Let \( \Omega \) be a bounded domain with a \( C^{2+\epsilon} \) boundary in \( \mathbb{R}^2 \) and \( 1 < p, q < \infty \). Assume that \( u_0 \in D^{1-\frac{1}{p},p}_{A_q}, f \in L^p(0,\infty; L^q) \). Then the system

\[
\begin{cases}
\partial_t u - \nu \Delta u + \nabla p = f, \\
\nabla \cdot u = 0, \\
|u(x,t)|_{\partial \Omega} = 0, \quad u(x,t)|_{t=0} = u_0,
\end{cases}
\]

has a unique solution \((u,p)\) satisfying the following inequality for all \( T > 0 \):

\[
\left\| u \right\|_{L^\infty(0,T;D^{1-\frac{1}{p},p}_{A_q})} + \left\| u \right\|_{L^p(0,T;W^{2,s,q})} + \left\| \partial_t u \right\|_{L^p(0,T;L^s)} + \left\| p \right\|_{L^p(0,T;W^{1,s})} \\
\leq C \left( \left\| u_0 \right\|_{D^{1-\frac{1}{p},p}_{A_q}} + \left\| f \right\|_{L^p(0,T;L^q)} \right),
\]

with \( C = C(p, q, \nu, \Omega) \).

3. PROOF OF MAIN RESULTS

In this section, we prove Theorem 1.2 and Proposition 1.3. To do so, we make two preparations. The first is a local existence result for system (1.1)-(1.2).

**Lemma 3.1.** Let the conditions in Theorem 1.2 hold. Then there exists a \( T_0 = T_0(|u_0|_{D^{1-\frac{1}{p},p}_{A_q}}, |\theta_0|_{L^q}) > 0 \) such that system (1.1)-(1.2) has a unique solution in \( M^{p,q,s}_{T_0} \).

*Proof:* The proof consists of several steps, including constructing the approximate solutions, obtaining the uniform local in time estimates, showing the convergence, and proving the uniqueness.

**First step: Construction of approximate solutions.** We initialize the construction of approximate solutions by smoothing out the initial data \((u_0, \theta_0)\) and get a sequence of smooth initial data \((u^0_n, \theta^0_n)_{n \in \mathbb{N}}\) which is bounded in \( D^{1-\frac{1}{p},p}_{A_q} \times L^q \). In addition, these smooth data belong to the Sobolev space \( H^3 \). Hence, applying the result of Lai, Pan, and Zhao [15] provides us a sequence of smooth global solutions.
(u^n, p^n, \theta^n)_{n \in \mathbb{N}}, \text{ which satisfy that } (u^n, \theta^n) \in C([0, \infty); H^3) \cap C^1([0, \infty); H^2) \text{ and } p^n \in C([0, \infty); H^3).

**Second step: Uniform estimate for some small fixed time** $T_0$. We aim at finding a positive time $T_0$ independent of $n$ for which $(u^n, p^n, \theta^n)_{n \in \mathbb{N}}$ is uniformly bounded in the space $M_{T_0}^{p,q,s}$.

Applying Lemma 2.5 to the temperature equation, we find that for all $t \geq 0$ and $s \in [0, 1]$

\begin{equation}
\|\theta^n\|_{L^\infty(0, t; L^q)} \leq \|\theta_0\|_{L^s},
\end{equation}

and

\begin{equation}
\|\theta^n\|_{L^\infty(0, t; W^{p,q,s})} \leq \|\theta_0\|_{W^{s,q}} C \int_0^t \|\nabla u^n(\tau)\|_{L^\infty} d\tau.
\end{equation}

Considering the velocity equation, we obtain

\begin{align}
\|u^n\|_{L^\infty(0, t; D_{Aq}^{1-\frac{1}{p}, p})} + \|\partial_t u^n\|_{L^p(0, t; L^s)} + \|u^n\|_{L^p(0, t; W^{2,q})} + \|p^n\|_{L^p(0, t; W^{1,q})} & \\
\leq C \left( \|u_0\|_{D_{Aq}^{1-\frac{1}{p}, p}} + \|u^n \cdot \nabla u^n\|_{L^p(0, t; L^s)} + \|\theta^n\|_{L^p(0, t; L^s)} \right) \\
\leq C \left( \|u_0\|_{D_{Aq}^{1-\frac{1}{p}, p}} + \|u^n\|_{L^\infty(0, t; L^s)} \|\nabla u^n\|_{L^p(0, t; L^q)} + \|\theta^n\|_{L^p(0, t; L^s)} \right).
\end{align}

If $\frac{2}{p} + \frac{2}{q} > 1$, Lemma 2.3 yields for $\theta = 1 - \frac{2}{q}(1 - \frac{2}{q})$

\begin{equation}
\|\nabla u^n\|_{L^p(0, t; L^\infty)} \leq Ct^{\frac{1}{2q}} \|\frac{\theta}{u^n}\|_{L^p(0, t; W^{2,q})} \|u^n\|_{L^\infty(0, t; D_{Aq}^{1-\frac{1}{p}, p})}.
\end{equation}

If $\frac{2}{p} + \frac{2}{q} < 1$, we have $D_{Aq}^{1-\frac{1}{p}, p} \hookrightarrow W^{1, \infty}$ so that

\begin{equation}
\|\nabla u^n\|_{L^p(0, t; L^\infty)} \leq Ct^{\frac{\theta}{q}} \|u^n\|_{L^\infty(0, t; D_{Aq}^{1-\frac{1}{p}, p})}.
\end{equation}

If $\frac{2}{p} + \frac{2}{q} = 1$, applying Hölder’s inequality, we arrive at the following inequality,

\begin{equation}
\|u^n \cdot \nabla u^n\|_{L^p(0, t; L^s)} \leq \|u^n\|_{L^\infty(0, t; L^s)} \|\nabla u^n\|_{L^p(0, t; L^q^+)}.
\end{equation}

where $q^+$ is slightly bigger than $q$. Noticing that $D_{Aq}^{1-\frac{1}{p}, p} \hookrightarrow W^{1, \frac{q^+}{q^+}}$ and $D_{Aq}^{1-\frac{1}{p}, p} \hookrightarrow L^{q^+}$, we eventually get

\begin{equation}
\|u^n \cdot \nabla u^n\|_{L^p(0, t; L^s)} \leq Ct^{\frac{1}{q}} \|u^n\|_{L^\infty(0, t; D_{Aq}^{1-\frac{1}{p}, p})}^2.
\end{equation}

On the other hand, combining Hölder’s inequality and (3.1), we have

\begin{equation}
\|\theta^n\|_{L^p(0, t; L^s)} \leq t^{\frac{1}{q}} \|\theta^n\|_{L^\infty(0, t; L^s)} \leq t^{\frac{1}{q}} \|\theta_0\|_{L^s}.
\end{equation}

Define

\begin{equation}
U^n(t) = \|u^n\|_{L^\infty(0, t; D_{Aq}^{1-\frac{1}{p}, p})} + \|\partial_t u^n\|_{L^p(0, t; L^s)} + \|u^n\|_{L^p(0, t; W^{2,q})},
\end{equation}

and

\begin{equation}
\|U^n\| = \|u_0\|_{D_{Aq}^{1-\frac{1}{p}, p}} + \|\theta_0\|_{L^q}.
\end{equation}
Inserting (3.4)-(3.8) into (3.3), we deduce that

\[ U^n(t) \leq C \left( \|u_0\|_{D^{n-1/4}_{A_q}} + \max(t^{1/2-1/4}, t^{1/2}) (U^n(t))^2 + t^{1/2} \|\theta_0\|_{L^2} \right). \]

Set

\[ T_0 = \min \left\{ 1, \left( \frac{1}{4C_0} \right)^p, \left( \frac{1}{4CU_0} \right)^{2\gamma_2} \right\} \]

Direct computations show that for \( t \in [0, T_0] \)

\[ U^n(t) \leq I_1(t) = \frac{1 - \sqrt{1 - 4C \max(t^{1/2-1/4}, t^{1/2}) U_0}}{2 \max(t^{1/2-1/4}, t^{1/2})} \leq 2CU_0, \]

or

\[ U^n(t) \geq I_2(t) = \frac{1 + \sqrt{1 - 4C \max(t^{1/2-1/4}, t^{1/2}) U_0}}{2 \max(t^{1/2-1/4}, t^{1/2})} \geq \frac{1}{2 \max(t^{1/2-1/4}, t^{1/2})}. \]

We show that (3.10) holds for \( t \in [0, T_0] \). Since \( u^n \in C([0, \infty); H^2) \) and \( \lim_{t \to 0} I_2(t) = \infty \), there exists some time \( T_1 > 0 \) such that (3.10) holds for \( t \in [0, T_1] \). By contradiction, suppose that (3.10) does not hold for all \( t \in [0, T_0] \), then there exists a first time \( T_2 > 0 \) such that (3.10) holds. It follows that \( \lim_{t \to T_2} U^n(t) \leq I_1(T_2) \) and \( U^n(T_2) \geq I_2(T_2) \), which contradicts the fact \( u^n \in C([0, \infty); H^2) \). Hence we have for all \( t \in [0, T_0] \)

\[ U^n(t) \leq 2CU_0. \]

Coming back to (3.2), noting that

\[ \int_0^t \|\nabla u^n(\tau)\|_{L^\infty} d\tau \leq \int_0^t \|u^n\|_{W^{2,q}} d\tau \leq t^{1-\frac{1}{2}} \|u^n\|_{L^p(0,t;W^{2,q})}, \]

we derive that

\[ \theta \in C([0,T]; W^{s,q}), \quad \partial_t \theta \in L^p(0,T; W^{-1,q}). \]

**Third step: Passing to the limit.** Since \((u^n, p^n, \theta^n)_{n \in \mathbb{N}}\) is uniformly bounded in the space \( M^{p,q,s}_{T_0} \), applying Aubin-Lions lemma yields the solution to system (1.1)-(1.2) which belongs to \( M^{p,q,s}_{T_0} \).

**Fifth step: Uniqueness.** The uniqueness is implied by the result in He [8], which says that the energy weak solution to system (1.1)-(1.2) is unique. This completes the proof of Lemma 3.2. \( \square \)

The following lemma is the main ingredient of the proof of Theorem 1.2.

**Lemma 3.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^2 \) with \( C^{2+\epsilon} \) boundary for some \( \epsilon > 0 \). Let \( p \in (1, \infty), \ q \in (2, \infty) \) and \( s \in [0, 1] \). Suppose that \( u_0 \in D^{1-\frac{1}{2}p}_{A_q} \cap H^1 \) and \( \theta_0 \in W^{s,q} \). Then system (1.1)-(1.2) has a unique global solution \((u, p, \theta)\) which belongs to \( M^{p,q,s}_T \) for all \( T > 0 \). Furthermore, we have

\[ u \in L^\infty(0,T; H^1) \cap L^2(0,T; H^2). \]

We divide the proof of Lemma 3.2 into three steps. First, we recall some elementary energy estimates. Next, we derive global \( H^1 \) estimate for the velocity. Finally, we use the maximal regularity of the Stokes operator to improve the regularity for both the velocity and the temperature.
Proof. **Step 1 Energy Estimates.**

Let $T > 0$ be any fixed given time. Reasoning as in Lemma 3.1, we get from the temperature equation for all $r \in [1, \infty)$
\begin{equation}
\theta \in L^\infty(0, T; L^r).
\end{equation}
The basic energy estimate for the velocity equation yields that
\[ \frac{1}{2} \frac{d}{dt} \| u \|^2_{L^2} + \nu \| \nabla u \|^2_{L^2} \leq \| u \|_{L^2} \| \theta \|_{L^2}. \]
Applying Gronwall’s inequality, we have
\begin{equation}
u \| u \|^2_{L^2} \leq \| u \|_{L^2} \| \theta \|_{L^2}.
\end{equation}

**Step 2 $H^1$ Estimate for the Velocity.**

Taking $L^2$-inner product of the velocity equation with $-\mathbb{P} \Delta u$, where $\mathbb{P}$ is the Leray projector. we deduce that
\begin{equation}
\frac{1}{2} \frac{d}{dt} \| \nabla u \|^2_{L^2} + \nu \| \Delta u \|^2_{L^2} = \int_{\Omega} u \cdot \nabla u \mathbb{P} \Delta u dx - \int_{\Omega} \theta e_2 \mathbb{P} \Delta u dx.
\end{equation}
We now estimate the right hand side of (3.19). For the first term, using Hölder’s inequality, Gagliardo-Nirenberg’s inequality and Young’s inequality, we get:
\begin{equation}
\left| \int_{\Omega} u \cdot \nabla u \mathbb{P} \Delta u dx \right| \leq C \| u \|_{L^4} \| \nabla u \|_{L^4} \| \mathbb{P} \Delta u \|_{L^2}
\end{equation}
\begin{equation}
\leq C \| u \|^\frac{3}{2} \| \nabla u \|^\frac{3}{2} \| \mathbb{P} \Delta u \|^\frac{3}{2}_{L^2}
\end{equation}
\begin{equation}
\leq C \| u \|^2 \| \nabla u \|^4 + \frac{1}{4} \| \mathbb{P} \Delta u \|^2_{L^2}.
\end{equation}
For the second term, it follows from the Cauchy-Schwarz inequality that
\begin{equation}
\left| \int_{\Omega} \theta e_2 \mathbb{P} \Delta u dx \right| \leq C \| \theta \|^2_{L^2} + \frac{1}{4} \| \mathbb{P} \Delta u \|^2_{L^2}
\end{equation}
Substituting (3.17) and (3.18) into (3.16), we find that
\[ \frac{d}{dt} \| \nabla u \|^2_{L^2} + \nu \| \Delta u \|^2_{L^2} \leq C \left( \| u \|^2_{L^2} \| \nabla u \|^2_{L^2} \right) \| \nabla u \|^2_{L^2} + \| \theta \|^2_{L^2}.
\]
Then, from (3.14), (3.15) and Gronwall’s inequality, we obtain
\begin{equation}
\theta \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2),
\end{equation}
which, by the Sobolev embedding (Lemma 2.3), implies that for all $q \in (2, \infty)$
\begin{equation}
\theta \in L^\infty(0, T; L^q).
\end{equation}

**Step 3 Bootstrap Argument.**

We derive $W^{2,p}$ estimate for the velocity by the maximal regularity of the Stokes operator. To this end, we rewrite the velocity equation as follows,
\[ \partial_t u - \nu \Delta u + \nabla p = -u \cdot \nabla u + \theta e_2, \]
\[ \nabla \cdot u = 0. \]
Using Lemma 2.6, we see that for $p \in (1, \infty)$, $q \in (2, \infty)$,
\begin{equation}
\| u \|_{L^\infty(0, T; D^{1-\frac{1}{p}}_{A_q})} + \| \nabla u \|_{L^p(0, T; L^q)} + \| u \|_{L^p(0, T; W^{2,q})} + \| p \|_{L^p(0, T; W^{1,q})}
\leq C \left( \| u_0 \|_{D^{1-\frac{1}{p}}_{A_q}} + \| u \cdot \nabla u \|_{L^p(0, T; L^q)} + \| \theta \|_{L^p(0, T; L^q)} \right).\]
We now estimate the term $\|u \cdot \nabla u\|_{L^p(0,T;L^q)}$. Applying the interpolation inequality in Lemma 2.2, Hölder's inequality and Young's inequality, we find for any $\varepsilon > 0$

$$\|u \cdot \nabla u\|_{L^p(0,T;L^q)} \leq \|\nabla u\|_{L^p(0,T;L^q)} \|u\|_{L^{\infty}(0,T;L^s)}$$

$$\leq C \left( \|\nabla^2 u\|_{L^p(0,T;L^q)}^{\alpha} \|u\|_{L^{\infty}(0,T;L^s)}^{1-\alpha} + \|u\|_{L^p(0,T;L^q)} \right) \|u\|_{L^{\infty}(0,T;L^s)}$$

$$\leq C (\varepsilon \|\nabla^2 u\|_{L^p(0,T;L^q)} + C(\varepsilon) \|u\|_{L^p(0,T;L^q)}) \|u\|_{L^{\infty}(0,T;L^s)}$$

$$\leq C(\varepsilon) \|u\|_{L^\infty(0,T;L^s)} \|\nabla^2 u\|_{L^p(0,T;L^q)} + C(\varepsilon) T^\frac{q}{p} \|u\|_{L^{\infty}(0,T;L^s)}^2.$$ 

Choosing $\varepsilon$ small such that $C(\varepsilon) \|u\|_{L^\infty(0,T;L^s)} \leq \frac{1}{2C}$, we get

$$\|u \cdot \nabla u\|_{L^p(0,T;L^q)} \leq \frac{1}{2C} \|\nabla^2 u\|_{L^p(0,T;L^q)} + CT^\frac{q}{p} \|u\|_{L^{\infty}(0,T;L^s)}^2.$$

Substituting (3.22) into (3.21), together with (3.11) and (3.20), we deduce that

$$\|u\|_{L^\infty(0,T;D^{1-\frac{1}{p},1}_{A_q})} + \|\partial_t u\|_{L^p(0,T;L^q)} + \|u\|_{L^p(0,T;W^{2,q})} + \|p\|_{L^p(0,T;W^{1,q})}$$

$$\leq C(\|u_0\|_{D^{1-\frac{1}{p},1}_{A_q}}, \|\theta_0\|_{L^q}, T).$$

Finally, reasoning similarly as in Lemma 3.1 we obtain estimate for the temperature in $L^\infty(0,T;W^{1,q})$. This completes the proof of Lemma 3.2.

With Lemma 3.1 and 3.2 at hand, we are at a position to prove Theorem 1.2. If the initial velocity is smooth such that $u_0 \in H^1$, then the proof is easy due to the global bound $u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$ for all $T > 0$ (see Lemma 3.2). However, if the initial velocity is rough such that $u_0 \notin H^1$, then the global $H^1$ bound for the velocity is absent, which makes it difficult to improve the regularity for the velocity by bootstrap argument. To solve this issue, we shall exploit the continuation argument due to Danchin (cf. Section 7 in [3]).

**Proof of Theorem 1.2**

We treat two cases $p \geq 2$ and $1 < p < 2$ differently.

1. **The Case of Smooth Data** $p \geq 2$. Combining the embedding $D^{1-\frac{1}{p},1}_{A_q} \hookrightarrow H^1$ (see Lemma 2.2) and Lemma 3.2 yields the result.

2. **The Case of Rough Data** $1 < p < 2$. First, Lemma 3.1 gives us a local smooth solution $(u,p,\theta)$ with the initial data $(u_0,\theta_0)$. Let $T^* \in (0,\infty)$ be the existence time such that $(u,p,\theta)$ belongs to $M^{p,q}_{T^*}$. Then we shall prove that $T^*$ can be arbitrarily large by adapting the method of Danchin [5].

Since $u \in L^p(0,T^*;W^{2,q})$ and $\theta \in L^\infty(0,T^*;W^{1,q})$, there exists some $t_0 \in (0,T^*)$ such that $u(t_0) \in W^{2,q} \cap L^2_\sigma$ and $\theta(t_0) \in W^{1,q}$. Noticing that $W^{2,q} \cap L^2_\sigma \hookrightarrow D^{1-\frac{1}{p},p}_{A_q}$ (see Lemma 2.2), we obtain $u(t_0) \in D^{1-\frac{1}{p},p}_{A_q} \cap H^1$. Due to Lemma 3.2, we can find a unique global smooth solution $(\tilde{u},\tilde{\theta},\tilde{p})$ to system (1.1)-(1.2) with initial data $(u(t_0),\theta(t_0))$.

On the other hand, because the smooth solution to system (1.1)-(1.2) is unique, we get that $(u,p,\theta) \equiv (\tilde{u},\tilde{\theta},\tilde{p})$ on $(t_0,T^*)$. Thus, $(\tilde{u},\tilde{\theta},\tilde{p})$ is a global smooth continuation of $(u,p,\theta)$. This completes the proof of Theorem 1.2.

**Remark 3.3**. The above argument does not give any information about the possible growth of the solution with respect to time. But it does work.

The proof of Proposition 1.3 is a direct consequence of Lemma 3.2.
Proof of Proposition 1.3. Choosing $p, q$ such that $\frac{1}{p} + \frac{1}{q} > 1$ in Lemma 2.2, we have $H^1 \cap L^2_\sigma \hookrightarrow D_{A_q}^{1-\frac{1}{p}}$. Then, it follows from Lemma 3.2 that 

$$u \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2), \nabla u \in L^1(0, T; L^\infty).$$

Applying Lemma 2.5 yields that $\theta \in L^\infty(0, T; H^1)$. 

This completes the proof of Proposition 1.3. □

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School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo, Henan 454000, China

E-mail address: daoguozhou@hpu.edu.cn