Analytical Derivation of Outage Correlation in Random Media Access with Application to Average Consensus in Wireless Networks

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Abstract—We study a finite and fixed relative formation of possibly mobile wireless networked nodes. The nodes apply average consensus to agree on a common value like the formation’s center. We assume framed slotted ALOHA based broadcast communication. Our work has two contributions. First, we analyze outage correlation of random media access in wireless networks under Nakagami fading. Second, the correlation terms are applied to the so called L2-joint spectral and numerical radii to analyze convergence speed of average consensus under wireless broadcast communication. This yields a unified framework for studying joint optimization of control and network parameters for consensus subject to message losses in wireless communications. Exemplary we show in this work how far outage correlation in wireless broadcast communication positively affects convergence speed of average consensus compared to consensus in the uncorrelated case.

Index Terms—Outage correlation, Nakagami fading, average consensus, L2 joint spectral radius, numerical radius, wireless networks, slotted ALOHA, broadcast.

I. INTRODUCTION

During the past decades wireless sensor networks (WSN) and distributed multi-agent systems have gained a lot of attention among both communication and control community. While there is much work on the specifics of either side, only few analytical papers consider a unifying approach where the impact of real wireless network phenomena is measured directly in terms of distributed control performance.

In this work, we follow such unifying approach. At first, we quantify outage correlations due to random media access control for finite WSN’s and for snapshots of multi-agent systems. Thereafter, we consider distributed average consensus, a basic building block of distributed control, and lay open that a thorough analysis of control performance requires knowledge of how packet losses are correlated.

When speaking of correlation, we mean outage correlation [21] in decorrelated fading channels (communication channels are typically located such that they are neither spatially nor temporally correlated). The correlation we study here comes from random media access.

We consider snapshots of node deployments and identify two components of correlation. First, correlation between all recipients of the same transmitter, and second, correlation between all other links. Typically, the former is positive, the latter is mostly negative.

The applications we have in mind all require broadcast transmission such that Request To Send/Clear To Send mechanisms cannot be applied. Examples consist of formation control [5], flocking (cf. the references in [15]) or purely computational tasks like determining the network size [20]. Literature in the former field have been considering oversimplified wireless network assumptions so far [5], [16].

We build our analysis on a framed slotted ALOHA media access protocol and measure the effects on the performance of an average consensus protocol. The simplicity of slotted ALOHA allows for exact theoretical analysis of the control performance for the given deployment. Moreover, even if carrier sensing was applied, it would still become less and less beneficial with increasing node density. For example, it was theoretically studied in [12] that the IEEE 802.11p protocol behaves like slotted ALOHA in the dense regime.

The remainder of this work is structured as follows. In the next section we relate our study to existing work on beneficial and disadvantageous effects of correlation on communication and control performance. In section III we introduce mathematical notation and the physical layer model under consideration. This is followed by section IV where we derive an expression to compute outage correlation of random media access given an underlying fading model. This result is then used in section V where we describe the effects correlation implies for consensus. We provide an exact and narrow performance region utilizing the so called $L^2$-joint spectral radius ($r_2$), e.g. [14], and $L^2$-joint numerical radius ($w_2$) [17] for mean square analysis on the convergence speed of average consensus. Both measures together quite precisely render the outage correlation in terms of (discrete) control gain, fading parameter and frame length. Compared to an uncorrelated model with the same packet loss rates, we essentially observe a beneficial impact of...
correlation on average consensus performance. We conclude our findings in section VI.

II. RELATED WORK

The performance impact of correlated communication channels is studied in an extensive field of applications. It is known that correlation in general can have positive effects, e.g. on multiple-input and multiple-output capacity [13] or for acknowledgments [1], or drawbacks, for instance for retransmissions [7]. More specifically, other correlation types can be found which we do not consider or are ruled out in this work, e.g. correlation between different antennas or temporal correlation [23].

In this work, our focus is on outage correlation, that is the correlation between packet losses along different links. In the chosen slotted ALOHA media access scheme, also other types of correlation arise which implicitly affect the term interference correlation. This includes slot correlation (i.e. correlation between different slots within the same frame), spatial correlation and correlation between different receivers, cf. [23].

Interference correlation can be studied for fixed node configurations or for networks modeled as point processes as studied in [7], [23], [11]. Recently, outage correlation has been investigated in a finite point process setting (Binomial, Poisson and Thomas processes) under Rayleigh Fading [21]. Although the authors allow for slotted ALOHA, they are mainly concerned with the effects of the spatial differences generated by the three point processes; concrete node deployments are not considered. The work [10] is set also in a point process setting where spatial interference correlation is investigated. Instead of computing the correlations exactly, the authors elaborate approximations.

Our analytical derivations significantly differ from the aforementioned works. Here we are specifically interested in the impact of outage correlation for a finite, fixed node deployment with respect to convergence speed of average consensus using broadcast communication.

For convergence speed of average consensus under channel correlation and wireless network constraints other work exist. In [3], consensus has been considered under fast fading where correlated outages (mainly due to fading) are mentioned without being quantified. In [9], [19] abstract correlations can be handled in the consensus framework, however, neither a physical model nor a media access model is provided. Also the mean square disagreement measures used there are upper bounds only. In our work we use a different upper bound, \( w_2 \), which very well matches the shape of the exact measure \( r_2 \) characterizing (exponential) mean square stability, cf. [17].

Finally, the outage probabilities for Nakagami-m fading we plug into our model (Corollary 1) stem from [22], one of the few papers analyzing fixed node deployments. However, as opposed to our work, correlation is not considered in that reference.

III. DEFINITIONS AND MODEL ASSUMPTIONS

A. Matrix, vector and other notation

For vectors \( x \in \mathbb{R}^n \) we adopt the Euclidean norm \( ||x|| = \sqrt{\sum_{i=1}^n x_i^2} \). By \( 1 \in \mathbb{R}^n \) we denote the vector of all ones. The identity matrix on \( \mathbb{R}^n \) is denoted by \( I_n \), whereas by \( \Pi \) we denote the projection matrix \( \Pi = \frac{1}{||x||^2} x x^T \).

The Kronecker delta is denoted by \( \delta_{ij} \), i.e. \( \delta_{ij} = 1 \) if \( i = j \) and \( \delta_{ij} = 0 \) otherwise; \( \delta_{ijk} \) is short for \( 1 - \delta_{ij} \). For two matrices \( A, B \in \mathbb{R}^{n \times n} \) we denote by \( A \otimes B \) the Kronecker product of \( A \) and \( B \), i.e. for indices of the form \( I = n \cdot (i-1) + k, \ J = n \cdot (j-1) + l, \ 1 \leq i, j, k, l \leq n \), we have \((A \otimes B)_{ij} = a_{ik} b_{lj}\).

By \( \mathbb{I}(x \geq 0) \) we denote the unit step function on \( \mathbb{R} \), i.e. \( \mathbb{I}(x \geq 0) \) is equal to one if \( x \geq 0 \), and vanishes otherwise.

Let \( X \) be an arbitrary set and \( n \in \mathbb{N} \). By \( X^n \) we denote the \( n \)-fold Cartesian product \( X \times \cdots \times X \). We will use ordered tuples over sets \( X \) as follows: For a sequence \( (a_{\nu})_{\nu \in \mathbb{N}} \subset X \) and a finite index set \( J \subset \mathbb{N} \) we denote by \( [a_{\nu}]_{\nu \in J} \) the tuple \( (a_{\nu_1}, \ldots, a_{\nu_{|J|}}) \in X^{|J|} \) where \( \nu_1 = \min J \), and \( \nu_{k+1} = \min J \setminus \{ \nu_1, \ldots, \nu_k \} \). \( k = 1, \ldots, |J| - 1 \).

B. Physical layer model

We assume \( n \geq 3 \) nodes \( u_1, \ldots, u_n \) located at fixed positions \( (x_1, \ldots, x_n) \in \mathbb{R}^n \) relative to a possibly moving frame of reference. A transmission between two nodes \( u_i \) and \( u_j \) is subject to path loss with a path loss coefficient of \( \alpha_{P,L} \geq 2 \). For a transmission distance \( d \) the average received power is thus proportional to \( 1/d^{\alpha_{P,L}} \) in the far field.

Let \( \mu_{ij}^{-1}, 1 \leq i, j \leq n, i \neq j \), be the average power of a transmission from \( u_i \) received at \( u_j \). If \( u_i \) transmits with power \( q_i \), we have \( \mu_{ij} = \frac{1}{q_i} ||x_i - x_j|| / r_0^{\alpha_{P,L}} \), where \( r_0 \) is a reference distance. Denoting the wavelength of the carrier wave by \( \lambda \) we choose \( r_0 = \lambda/4\pi \) such that we arrive at Friis free space equation in case of a path loss coefficient of 2.

Here we study communication under the effect of narrow band fast fading. This means that the instantaneous received power \( P_{ij} \) of a transmission between \( u_i \) and \( u_j \) is a non-negative random variable with expectation \( \mu_{ij}^{-1} \). Further, we assume independent block fading (IBF), i.e. \( P_{ij} \) is constant during a packet transmission and power variables belonging to different channels are mutually independent. Later we employ the Nakagami fading model, i.e., the random variable \( P_{ij} \) is Gamma distributed with probability density function (pdf)

\[
f_{ij}(x) = \mathbb{I}(x \geq 0) \frac{m_i}{m_j} \frac{m_i}{m_j} x^{m_i-1} \exp \left( -m_i x \mu_{ij} \right) \tag{1}
\]

where \( m_i \in \mathbb{N} \) are integer\(^1\) shape parameters. However, our calculations of the structure of correlation due to random media access are valid in a very general setting regardless of path loss and fading (cf. state coefficients of Theorem 1 and [18, Theorem 3]). We incorporate Nakagami fading as special case in Corollary 1 and [18, Corollary 2].

\(^1\)In general, Nakagami fading only requires \( m_i \geq 1/2 \), however since we build on [22], we can only handle integer values.
The success rate of a message transmission is modeled with outage probability under the signal to interference plus noise ratio (SINR) model. For a transmission from \( u_i \) to \( u_j \) all other currently ongoing transmissions will be considered as noise. Thus, we relate the received power over noise plus all other interfering transmissions as

\[
\text{SINR}_{ij} = \frac{P_{ij}}{N + \sum_{\nu \in \mathcal{V} \setminus \{i,j\}} P_{\nu j}}
\]

where \( N \) is a constant thermal noise term and \( \mathcal{V} \) the set of node indices of all currently transmitting nodes. Let the receiving node \( u_j \) be silent. Then we say the transmission has an outage if \( \text{SINR}_{ij} \) falls below a system specific threshold value \( \theta > 0 \) and the probability of a successful packet transmission from \( u_i \) to \( u_j \) is \( P[\text{SINR}_{ij} \geq \theta] \). The case when \( u_j \) is also transmitting will be discussed in the context of the slot model in the next section.

IV. OUTAGE CORRELATION DUE TO MEDIA ACCESS

To determine the coefficient of correlation of transmission success between different node pairs, it suffices to specify expectation and covariance, which is the concern of this section.

In addition to the physical layer model, we now assume that the nodes perform the following variant of slotted ALOHA for broadcast transmission which guarantees that each node transmits exactly once during a beacon period. Let \( \tau \) be the duration of the beacon time interval in milliseconds (ms) which is called a frame. Each frame is divided into a number \( m \) of time intervals called slots, where \( m \in \mathbb{N} \) is chosen such that each packet can be transmitted in \( \tau/m \) ms. At the beginning of each frame, each node randomly chooses exactly one slot with equal probability. Transmission from node \( u_i \) to node \( u_j \) is successful if it is successful in the chosen slot. Otherwise the packet from \( u_i \) to \( u_j \) is considered lost in this beacon interval.

Our derivation requires \( \theta \geq 1 \) and \( N > 0 \) Watt. Let us further introduce \( n \) independent random variables \( S_1, \ldots, S_n \) representing each node’s choice of slot - all equally likely - i.e. \( S_i \sim \text{unif}\{1, \ldots, m\} \), \( i = 1, \ldots, n \) (uniform distribution).

Moreover we assume the power variables \( P_{ij} \) - e.g. \( P_{ij} \sim f_{ij} \), cf. (1) - to be independent (IBF assumption) and also to be independent from the slot variables. We can now express power and interference in slot \( k \), \( 1 \leq k \leq m \):

\[
P_{ij}^k := \delta_{S_i,j} P_{ij} \quad \text{and} \quad I_{ij}^k := \sum_{l \neq j}^n P_{lj}^k.
\]

Let us introduce random variables for excess of the SINR threshold in slot \( k \) and in any slot respectively:

\[
X_{ij}^k := \left\{ \begin{array}{ll}
P_{ij}^k \left( \frac{P_{ij}^k}{N + I_{ij}^k} \geq \theta \right) & \text{and} \\
0 & \text{otherwise}
\end{array} \right.
\]

\[
X_{ij} := \sum_{k=1}^m X_{ij}^k.
\]

Clearly, from the definition of \( P_{ij}^k \) and the fact that \( \theta, (N + I_{ij}^k) > 0 \) we have \( X_{ij}^k = \delta_{S_i,j} \cdot X_{ij}^k \), thus \( X_{ij} = X_{ij}^S \).

There are two further practical aspects we have to consider: Sending nodes cannot receive (singular antenna case) and multiple receptions at some node at the same time (i.e. in the same slot) are impossible in our model since \( \theta \geq 1 \). We call the former aspect the half-duplex case. We define successful full-duplex packet transmission from node \( u_i \) to \( u_j \) to be the event \( X_{ij} = 1 \). In words: We ignore whether the receiver is sending or not when checking for excess of the SINR ratio.

In this sense, we model successful half-duplex packet reception \( Y_{ij} \), \( 1 \leq i, j \leq n, i \neq j \), as Bernoulli variable having the property to vanish if \( u_i \) and \( u_j \) chose the same slot and to coincide with \( X_{ij} \) otherwise:

\[
Y_{ij} := (1 - \delta_{S_i,j}) \cdot X_{ij} \quad 1 \leq i, j \leq n, i \neq j.
\]

Our goal is to determine how successful transmissions are correlated. Therefore, we calculate the covariances \( \text{cov}(Y_{ij}, Y_{kl}) \) and begin with the mixed moments of the \( X_{ij} \)’s (full-duplex case) which are of special importance. By the law of total probability we can decompose the mixed moments \( \mathbb{E}[X_{ij}X_{kl}] \) into a weighted sum of the conditioned mixed moments

\[
\frac{1}{m} \cdot \mathbb{E}[X_{ij}X_{kl}|S_i = S_k] + (1 - \frac{1}{m})\mathbb{E}[X_{ij}X_{kl}|S_i \neq S_k]
\]

motivating the form of the main result of this section which needs a little technical preparation. The conditioned mixed moments of the \( X_{ij} \)’s, as well as those of the \( Y_{ij} \)’s, lead - up to normalization - to expressions of the form (5) and (6) which we will now describe: Denote by \( b_{ij} \in \{1, \ldots, m\} \) realizations of the slot random variables \( S_i \) which we call micro states. Consider for instance a transmission from node \( u_i \) to \( u_j \). The pattern of the micro states contributing to the interference for this transmission can be summarized by simpler macro state variables \( c_{ij} \in \{0, 1, 2\}, i \neq j \). Consider, e.g., a further node \( u_k \) under the condition \( S_i \neq S_k \), i.e. \( u_i \) and \( u_k \) have chosen different slots. In one possible interpretation \( c_{ij} = 0 \) may stand for all realizations of the event \( S_i \neq S_k \), the case \( c_{ij} = 1 \) might stand for all realizations of \( S_i = S_k \) and \( c_{ij} = 2 \) for all realizations of \( S_i \neq S_k \). In general, for a subset \( J \) of \( N := \{1, \ldots, m\} \) we denote by \( c \) the tuple \( [c_{ij}]_{e \in J} \), the total (macro) state. We express the number of micro states represented by this total state by functions \( \phi(c) \) and \( \psi(c) \) which we call state weights.

Now let us take a look at some potential receiver \( u_t \) of \( u_k \)’s transmission. The idea in the calculation of (un-)conditioned mixed moments of \( X_{ij} \) and \( X_{kl} \) is to condition on the slot variables until all \( S_i \) have been replaced by micro states \( b_{ij} \). The mixed moments then expand into a sum of weighted probabilities. When exploiting common interference terms, the probability summands split up into a product of two separate probabilities in virtue of the independence (IBF assumption). For either of the two factors we use the terminology state coefficient - symbolized by either \( \alpha_{ijkl}(c) \) or \( \beta_{ijkl}(c) \). These probabilities depend, of course, on the fading model (cf.
Appendix of [18]. From (3) we obtain for $i$ holds true.

For $i \neq j, k \neq l, (i, j) \neq (k, l)$, and for a finite index set $J \subset \mathcal{N}$ we now introduce expressions allowing for a quite general treatment of outage correlation expressions:

$$
\Phi^{i}_{ijk}(\phi, \alpha) := \sum_{c_{0}=0}^{1} \sum_{\nu \in J} \phi(c) \cdot \alpha_{i,j,k}(c) \cdot \alpha_{k,l}(c) 
$$

and

$$
\Psi^{i}_{ijk}(\psi, \beta) := \sum_{c_{0}=0}^{1} \sum_{\nu \in J} \psi(c) \cdot \beta^{(1)}_{ijk}(c) \cdot \beta^{(2)}_{k,l}(c),
$$

where $\phi(c), \psi(c), \alpha_{x,y,z}(c)$ and $\beta^{(d)}_{x,y,z}(c)$ are functions of $c \equiv [c_{\nu}]_{\nu \in J} \in \{0,1\}^{J} | \nu \in J \}$ (see section III-A).

For $J = \mathcal{N} \setminus \{i, k\}$ we have $[c_{\nu}]_{\nu \in J} = (c_{1}, \ldots, c_{\mu-1}, c_{\mu+1}, \ldots, c_{\nu-1}, c_{\nu+1}, \ldots, c_{\mu})$, where $\mu = \min(i, k)$ and $\mu = \max(i, k)$. In case of the calculation of $\Phi^{i}_{ijk}$ we always have $c_{\nu} \neq 2$ such that there $[c_{\nu}]_{\nu \in J} \in \{0,1\}^{J}$ holds true.

We are now ready to formulate our covariance result for the $Y_{ij}$’s. An analogous result for the $X_{ij}$’s can be found in the Appendix of [18]. From (3) we obtain for $i \neq k$ and $k \neq l$

$$
\mathbb{E}[Y_{ij} Y_{kl}] = \left(1 - \frac{1}{m}\right)^{2-\delta_{ij} \delta_{kl}} \mathbb{E}[X_{ij} X_{kl} | S_{i} \neq S_{j}; S_{k} \neq S_{l}].
$$

In the following Theorem we treat the cases $S_{i} = S_{k}$ and $S_{j} \neq S_{l}$ separately, cf. (4).

**Theorem 1 (Half-duplex covariances).**

Let $n \geq 3, \theta \geq 1, N > 0, i \neq j, k \neq l$, and let $(i, j) \neq (k, l)$.

(a) We have for $j \neq k$ and $i \neq l$

$$
\mathbb{E}[X_{ij} X_{kl} | S_{i} = S_{k}; S_{j} \neq S_{l}; S_{k} \neq S_{l}] = \left(1 - \delta_{ij}\right) \cdot \Psi^{N}_{ij,k,l}(\phi, \alpha) / m^{n-4+\delta_{k,l}},
$$

where the state weight of $c \equiv [c_{\nu}]_{\nu \in \mathcal{N} \setminus \{i, j, k, l\}}$ is given by

$$
\phi(c) = \prod_{\nu \in \mathcal{N} \setminus \{i, j, k, l\}} (m - 1)^{1-c_{\nu}},
$$

the state coefficients $\alpha_{x,y,z}(c)$ are given by

$$
\mathbb{P} \left( \frac{P_{xy}}{N + \sum_{\nu \neq x, y, z} c_{\nu} P_{xy} + (1 - \delta_{xz}) P_{zx}} \geq \theta \right)
$$

In particular, $\mathbb{E}[Y_{ij} Y_{il}]$ can be expressed this way, hence for $j \neq l$

$$
\text{cov}(Y_{ij}, Y_{il}) = \left(1 - \frac{1}{m}\right)^{2} \cdot \Phi^{N}_{ij,k,l}(\phi, \alpha) - \mathbb{E}[Y_{ij}] \mathbb{E}[Y_{il}].
$$

(b) Let $J' = \{i, j, k, l\}$, let $J'' = \{i, k\}$ if $|J'| = 4$ and $J'' = J'$ else. Moreover, let $J''' = \{i, j, k\}$ if $|J'| = 4$ and $J''' = J'$ otherwise. Then, for $k \neq i$, we have

$$
\mathbb{E}[X_{ij} X_{kl} | S_{i} \neq S_{j}; S_{k} \neq S_{l}] = \frac{\Psi^{N}_{ij,k,l}(\psi, \beta)}{M},
$$

where $M = m^{n-|J'|}$ if $|J'| = 2, 3$ and $M = m^{n-3} \cdot (m - 1)$ if $|J'| = 4$. Moreover, for $k \neq i$,

$$
cov(Y_{ij}, Y_{kl}) = (1 - \frac{1}{m})^{2-\delta_{ij} \delta_{kl}} \left[ \frac{\delta_{J''|A}(m - 1) \Phi^{N}_{ij,k,l}(\phi, \alpha)}{m^{n-3}} \right]
$$

$$
+ \left[ \frac{m - 1}{m} \delta_{J''|4} + \frac{m - 2}{m - 1} \delta_{i,j} + \delta_{i \neq j} \delta_{J''|4} \right] \frac{\Psi^{N}_{ij,k,l}(\psi, \beta)}{M}
$$

and $\Psi^{N}_{ij,k,l}(\psi, \beta)$ depends on the state weight $\psi(c) = \delta_{i}, \delta_{j} \cdot (m - 1)^{\delta_{ij} \delta_{j}}, i, j = 1, 2$.

and on the state coefficients

$$
\beta^{(d)}_{x,y,z}(c) = \mathbb{P} \left( \frac{P_{xy}}{N + \sum_{\nu \neq x, y, z} c_{\nu} P_{xy} + (1 - \delta_{x,z}) P_{zx}} \geq \theta \right),
$$

where $c \equiv [c_{\nu}]_{\nu \in \mathcal{N} \setminus J''}, d = 1, 2$.

**Proof.** The proof utilizes the conditioning and macro state techniques sketched above. For details see [18, Appendix].

**Remark 1.**

(i) The requirement $\theta \geq 1$ is used in the case $l = j$ and yields the $(1 - \delta_{ij})$ factor.

(ii) In case $m = 1$ there is no outage correlation in the full-duplex case, i.e. $\text{cov}(X_{ij}, X_{kl}) = 0$ for $(i, j) \neq (k, l)$.

(iii) Typically, the covariance component given by (a) seems to be stronger than the one given by (b). For the correlations this observation does not hold; we may merely assert that the component due to (a) is typically positively correlated while the one due to (b) is correlated mainly negatively, cf. Figure 1.

(iv) Since the summations can get large it is important to efficiently implement the sums. E.g. $\sum_{\nu \neq x, y, z} c_{\nu} P_{xy}$ can be implemented using a loop from 1 to $2^{n-2}$ over a single integer variable $i$ and accessing the “components” $c_{\nu}$ of $i$ by bitwise operations (bit-shift).

**Remark 2.** Using Rayleigh fading in the slot model we recover the success probabilities of [22] (full-duplex) and [17] (half-duplex). The calculation is provided in [18]. Moreover, a similar argument is part of Corollary 1.

To incorporate Nakagami fading (cf. (1)) we introduce for convenience a further notation, which is the link to [22]: For a finite index set $J \subset \mathcal{N}$ and an $R$-valued tuple $\xi = [\xi_{\nu}]_{\nu \in J}$ let
\[
\gamma_{xy}^J(\xi) = e^{-\theta \cdot m_x \cdot \mu_{xy} N} \sum_{s=0}^{m_x-1} (\theta \cdot m_x \cdot \mu_{xy} N)^s \cdot \sum_{t=0}^{s} \left( \frac{q_N^t}{(s-t)!} \cdot \prod_{\ell_i \geq 0 \atop \nu \in J, \ell_i = i} (1 - \xi_{\nu}) \delta_{0\ell_i} \right) \left( \sum_{\nu \in J} \xi_{\nu} \cdot \frac{1}{m_x \cdot \mu_{xy} + 1} \right)^{m_x+t}.
\]

**Corollary 1** (Nakagami fading expectations and covariances). In the half-duplex model under Nakagami fading with pdf (1) the expectation for the link \(u_i u_j\), \(i \neq j\), is given by
\[
E[Y_{ij}] = \left( 1 - \frac{1}{m} \right) \cdot \gamma_{ij}^N(i,j)(1/m).
\]

For \(k\) the covariance of the link \(u_i u_j\) with link \(u_k u_l\) is given by (8) using (7) and the state coefficients
\[
\alpha_{xyzw}(c) = \gamma_{xy}^N(x,y,w)(c),
\]
where \(c = [\xi_{\nu}]_{\nu \in N \setminus \{i,j\}}\).

For \(i \neq k\), it is given by (9) using (7), (10) and the state coefficients
\[
\alpha_{xyzw}(c') = \gamma_{xy}^N(x,y,w)(\xi), \ c' = [\xi_{\nu}]_{\nu \in N \setminus \{i,j,k,l\}},
\]
where \(\xi = [\xi_{\nu}]_{\nu \in N \setminus \{x,y,w,u\}}, \ \xi_z = 1, \ \xi_{\nu} = c'_{\nu}, \ \nu \neq z, \) and
\[
\beta_{xyzw}(c'') = \gamma_{xy}^N(x,y,z)(\xi), \ c'' = [\xi_{\nu}]_{\nu \in N \setminus \{i,k\}},
\]
where \(\xi = [\xi_{\nu}]_{\nu \in N \setminus \{x,y,z\}}, \ \xi_z = \delta_{c''d}, \ d = 1, 2, \)
\[
\text{Proof.} \quad \text{This is a consequence of Theorem 1} \) and \([22]\). \[\Box\]

In Fig. 1 we show an example plot resulting from Corollary 1 with Nakagami fading shape parameter 2 and 2 slots. The plot shows the correlation matrix for all possible communication links among 6 nodes arranged on a 2 \times 3 grid (cf. Section V-C).

V. APPLICATION TO AVERAGE CONSENSUS

A. Discrete probabilistic average consensus

In the discrete, linear, first order probabilistic average consensus protocol [15], [6], [19], a group of \(n\) nodes periodically exchanges broadcast messages each containing a sender’s data set. For simplicity we assume all nodes have scalar, real valued data. It is an iterative random process whose goal is that all nodes agree on the same value in the limit with probability one. Further this value, the agreement value, should be close to the average of the initial data. Nonetheless, due to asymmetries in packet loss between outgoing and incoming transmissions, the agreement value will differ from the true average which, however, shall not be our concern here. Instead we focus on a probabilistic description of all nodes’ deviation from the current average at every iteration. We measure this deviation in terms of the root mean square (RMS) error (or disagreement) after \(k \in \mathbb{N}\) iterations (or steps) which we now introduce: If the (random) vector \(x_k = (x_k^{(1)}, \ldots, x_k^{(n)}) \in \mathbb{R}^n\) represents the nodes’ states after \(k\) consensus iterations having started from the deterministic initial state \(x_0 \in \mathbb{R}^n\), then by
\[
\sum_{i=1}^{n} \left( x_k^{(i)} - \frac{1}{n} \bar{x}^T \right)^2
\]

is the RMS error after \(k\) steps is defined as \(\sqrt{\text{Var}(x_0)}\).

Finally, the maximum root mean square (mRMS) error is the RMS error for the worst possible choice of initial state inside the closed unit ball, \(\max_{\|x\| \leq 1} \sqrt{\text{Var}(x)}\). Note that the mRMS error attains its maximum for normalized initial values, i.e. those on the unit sphere.

B. Performance measure

In this subsection we use slightly modified results from the literature to bound the mRMS error from above [17] and below [14]. The mRMS error stochastically describes the consensus disagreement at the \(k\)-th step for the worst possible, step dependent choice of the initial state. It constitutes a performance measure capable of well capturing the outage correlation effects. In [17] an uncorrelated half-duplex Bernoulli model (UHBM) with symmetric packet losses was assumed. This assumption, however, does not affect the generality of the proof of the \(m\)-step estimates presented there, although some adaptions in notation are required.

To quantify the impact of outage correlation on consensus, let us introduce \(n^2 - n\) uncorrelated Bernoulli variables \(Z_{ij}, i \neq j\), rendering an asymmetric UHBM having the property
\[E[Z_{ij}] = E[Y_{ij}], \]
where \(Y_{ij}\) are defined in section IV. In particular \(\text{cov}(Z_{ij}, Z_{kl}) = 0\), for \(i \neq j, k \neq l\) and \((i,j) \neq (k,l)\).

The symmetric UHBM of [17] has the alternative property \(E Z_{ij} = (E Y_{ij} + E Y_{ji})/2\) which we do not use here.
To apply the UHBM and our correlated half-duplex Bernoulli model (CHBM) in the consensus context, we need to link the uncorrelated variables $Z = (Z_{ij})_{i\neq j}$ and their correlated counterparts $Y = (Y_{ij})_{i\neq j}$ to so called Laplacian valued random matrices $L^W$ and $L^\epsilon$ respectively: For any ensemble of Bernoulli variables $W = (W_{ij}), 1 \leq i, j \leq n, i \neq j$, define $L^W = (l^W_{ij})$ by

$$l^W_{ij} = \begin{cases} W_{ki}, & \text{if } i = j, \\ -W_{ji}, & \text{if } i \neq j. \end{cases}$$

Note that for any $\epsilon > 0$ and any sequence $W^{(1)}, W^{(2)}, \ldots$ of realizations of $W$ the discrete consensus protocol [15], [19], (also cf. [17]) is then given by the matrix iteration

$$x_{k+1} = (I_n - \epsilon \cdot L^W(k)) \cdot x_k,$$

provided the values $W_{ij} = 0, 1$ are interpreted as unsuccessful/successful links $u_{ij}, \omega_{ij}$, i.e. $W$ represents the off-diagonal entries of a (transposed) adjacency matrix of an (in general undirected) graph (without loops); $x_0 \in \mathbb{R}^n$ is some arbitrary initial value. We call the number $\epsilon$ (discrete) gain. We assume the realizations $W^{(1)}, W^{(2)}, \ldots$ to be mutually independent, which reflects the IBF assumption. Note that from the communication perspective, this constitutes a sequence of frames; the following theorem introduces an upper and a lower bound on the upper bound dominates

Theorem 2 (cf. [17] and e.g. [14]). Let $W = Y, Z$ represent the CHBM or the UHBM. Then the mRMS disagreement after $k$ iterations is bounded as follows:

$$\frac{r_k(W, \epsilon)}{\sqrt{n}} \leq \max_{\|x\|_2 \leq 1} \sqrt{E\delta^2_k(x)} \leq \sqrt{2n-1} \cdot w_k(W, \epsilon),$$

Proof. The left-hand inequality follows from the infimum property of $r_2, \text{e.g.} [14, \text{Lem.} 2.7]$, the right-hand one from [17] up to a factor of $\sqrt{(n-1)/n}$. See [18] for details.

In analogy to the geometric mean we now define the following lower and upper performance bounds for the average per step mRMS error, $\max_{\|x\|_2 \leq 1} \sqrt{E\delta^2_k(x)}$:

$$\begin{align*}
\text{lb}(\epsilon) &= \frac{r_2(W, \epsilon)}{\sqrt{n}} \\
\text{ub}(\epsilon) &= \sqrt{2n-1} \cdot w_2(W, \epsilon),
\end{align*}$$

(11)

where $\text{lb}(\epsilon) = \text{lb}(\epsilon; k, W)$ and $\text{ub}(\epsilon) = \text{lb}(\epsilon; k, W)$.

Remark 3. (i) Asymptotically, the lower bound is tight since $r_2(W, \epsilon) = \lim_{k \to \infty} \max_{\|x\|_2 \leq 1} \sqrt{E\delta^2_k(x)}$, cf. [18].

(ii) The number of steps $k$ required to obtain a prescribed precision increases only logarithmically with the number of nodes $n$ (apart from the spread of $r_2$ and $w_2$).

(iii) The upper bound used in [19] is simply the squared mRMS disagreement for the first step ($k = 1$), cf. [18].

C. Exemplary effects for a selected deployment and parameters

We adopt network parameters similar to the IEEE 802.11p standard: A carrier frequency of 5.9 GHz and a bandwidth $B$ of 10 MHz. We choose a reference bitrate $R_0$ of 40% of the 802.11p maximum bitrate of 27 Mbit/s. To obtain comparable consensus performance results, we let the beacon time interval be constant in terms of slots, more precisely we set $R$ according to the number of slots $m$ in use: $R = m \cdot R_0$. The threshold $\theta$ is then computed according to Shannon capacity formula: $\theta = 2R/B - 1$. For $m = 1$ we obtain $\theta = 2.23$ such that the requirement $\theta \geq 1$ is always fulfilled. The thermal noise parameter $N$ is obtained from the Boltzmann constant $k_B$ at a reference temperature $T$ of 293K: $N = B \cdot k_B \cdot T$.

Our nodes are deployed on a regular 2 x 3 grid having a horizontal and vertical distance of 1500m between neighbors, a common transmit power of 500 mW, a common varying Nakagami shape parameter $m_i = m_S, i = 1, \ldots, n$, and a path loss coefficient of 2.0.

Figures 2a and 2b show the consensus performance (11) over the gain parameter which is scaled in terms of the minimizer $\epsilon$ of the essential spectral radius $g(\Pi - \epsilon E[L^W])$, e.g. [17].The minimizer, denoted by $\epsilon_\theta(m, m_S)$, depends on the number of slots and the shape parameter utilized, but is the same for the correlated and uncorrelated model, $W = Y, Z$. We set $k$ to 250 steps.

In Figure 2a the number of slots is varied, in Figure 2b the shape parameter. We observe two effects. (i) In both figures the correlated model shows significantly better performance and also different locations of the minimizing gain setting $\epsilon$ (the minimizer of the correlated model is closer to $\epsilon_\theta$). (ii) Remarkably, in Figure 2b the order of the curves in terms of performance is inverted: Curves with lower shape parameter perform better in the uncorrelated model while they perform worse in the correlated one. While we observed effect (i) for various parameter settings (Figure 2a, $m = 6$ constitutes a borderline case), effect (ii) is rather unusual.
Fig. 2: Comparison of the consensus performance for uncorrelated ($W = Z$, dashed curves) and correlated model ($W = Y$, solid curves) in terms of the upper and lower bounds (11) for $k = 250$ steps. On the abscissa we lay off the gain parameter scaled by the optimal essential spectral radius for the pair $(m, m_S)$, on the ordinate the dimensionless bounds (11). Smaller values correspond to better performance, a value below (above) 1 for both bounds implies (in-)stability in the exponential mean square sense [14]. Further, at a scope of $k$ steps the average per step mRMS error lies within the colored regions. In Figure 2a the shape parameter $m_S$ is set to 1 (Rayleigh fading) while the number of slots per frame $m$ varies. In Figure 2b the frame length $m$ is fixed to 3 while the shape parameter $m_S$ varies.

VI. CONCLUSION

We have derived closed form expressions for the outage/success correlation in a framed slotted ALOHA scheme under the Nakagami fast fading model for fixed relative node positions. The found expressions enable a thorough performance analysis of distributed control applications in wireless networks for which we provide bounds in an appropriate form.

Although inaccurate compared to the correlated model, the uncorrelated one could still serve as rough but simpler structured bound for the control performance. How far it is sufficient to use the simpler bound requires a broader study.

Our method can in principle be carried over to related slotted ALOHA protocols, e.g. spatial and opportunistic spatial ALOHA [2]. Apart from the broadcast case, an application in interference cancellation [4] is also thinkable.

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