On relationship between conformal transformations and broken chiral symmetry.

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Abstract

Starting with the conformal transformations in the momentum space, the nonlinear $\sigma$-model and the standard model with the spontaneous broken $SU(2) \times U(1)$ symmetry are reproduced. The corresponding chiral Lagrangians are given in the five dimensional form because for the conformal transformations of the four-momentum $q_\mu$ ( $q'_\mu = q_\mu + h_\mu$, $q'_\mu = \Lambda_\mu^\nu q_\nu$, $q'_\mu = \lambda q_\mu$ and $q'_\mu = -M^2 q_\mu/q^2$) the equivalence rotations in the 6D space were used. The derived five dimensional Lagrangians consists of the parts defined in the two different regions $q_\mu q_\mu = \pm M^2$ which are connected by the inversion $q'_\mu = -M^2 q_\mu/q^2$, where $M$ is a scale parameter. For the $\sigma$-model $M$ is determined by the pion mass $M^2 = m^2_{\pi}/2$. For the 5D Lagrangian with the spontaneous broken $SU(2) \times U(1)$ symmetry the scale parameter $M^2$ is defined by the Higgs particle mass $8m^2_{Higgs} = 9M^2$.

Unlike to the usual four-dimensional formulation in the present approach the chiral symmetry breaking terms are obtained from the conformal transformations and it is demonstrated, that the corresponding interaction parts of Lagrangians have the opposite sign in the regions $q^\mu q_\mu > M^2$ and $0 \leq q^\mu q_\mu < M^2$. 
Introduction

Conformal transformations in the configuration space consist of the complete set of transformations $x'_\mu = x_\mu + a_\mu$, $x'_\mu = \Lambda_\nu^\mu x_\nu$, $x'_\mu = \lambda x_\mu$ and $x'_\mu = -\ell^2 x_\mu / x^2$. The conformal transformations of four momentum $q_\mu$ can be performed in the same manner as conformal transformations in the coordinate space. In the framework of the Dirac geometrical model [1, 2, 3, 4, 5, 6, 7, 8, 9], each of the conformal transformations can be single-valued reproduced via the appropriate 6D rotation with the invariance 6D form

$$\kappa_A \kappa^A \equiv \kappa_\mu \kappa^\mu + \kappa_5^2 - \kappa_6^2 = 0, \quad (I.1)$$

where the four momentum $q_\mu$ ($\mu = 0, 1, 2, 3$) is defined as $q_\mu = M \kappa_\mu / (\kappa_5 + \kappa_6)$ and $M$ is a scale parameter. The 6D cone $\kappa_A \kappa^A = 0$ (I.1) or corresponding surface

$$q_\mu q^\mu + M^2 \kappa_5 - \kappa_6 = 0 \quad (I.2)$$

are invariant under any conformal transformation of $q_\mu$ even when the conformal invariance is violated by the mass or other dimensional parameters of considered particle. In the present paper we use the invariance of 6D forms (I.1) or (I.2) as a constraint for the derivation of the equation of motion for a interacting massive field. In particular, we shall project conditions $\kappa_A \kappa^A = 0$ on the two 5D surface

$$q_\mu q^\mu + q_5^2 = M^2 \quad \text{and} \quad q_\mu q^\mu - q_5^2 = -M^2, \quad (I.3)$$

so that these 5D hyperboloids remains to be also invariant under the conformal transformations. This enables us to introduce the constraints $(q_\mu q^\mu \pm q_5^2 \mp M^2) \Phi(q, q_5) = 0$ for a 5D field operator $\Phi(q, q_5)$. Afterwards we shall construct corresponding 5D Lagrangians appropriate to the 5D equations of motion and consider their 4D reductions.

The 4D rotation and dilatation subgroups of the conformal group in the coordinate and in the momentum space are simply connected. Therefore only the four-momentum translation $q'_\mu = q_\mu + h_\mu$ and the four-momentum inversion $q'_\mu = -M^2 q_\mu / q^2$ require a special attention. In the homogeneous 4D momentum space invariance under the translation $q'_\mu = q_\mu + h_\mu$ for the charged particle can be interpreted as invariance under a gauge transformation and for a charged field operator we have $\Phi'_\gamma(x) = \Phi^h_\gamma(x) = e^{ih_\mu x^\mu} \Phi_\gamma(x)$. We shall show, that for a neutral field the four-momentum translation has the more complicated form $\Phi'_\gamma(x) = \Phi^h_\gamma(x)$ which does not change the creation or annihilation operator of the considered particle. Therefore the scattering $S$-matrix is invariant under translations in the four-dimensional momentum space $q'_\mu = q_\mu + h_\mu$.

The numerous applications of the conformal transformations in the coordinate space are presented in many books and review papers (see for instance [5, 7, 6, 8, 9, 10, 11, 12, 13, 14]). Conformal transformations of the field operators and corresponding equation of motion in the momentum space were considered in ref. [2, 4, 11, 15]. In these papers conformal transformations were performed in the coordinate space and followed
relations were given in the momentum space using the Fourier transform. Two particular features determine the special interest to the conformal transformations in the momentum space[11]. First, the real observables of the particle interactions, like the cross sections and the corresponding scattering amplitudes, are determined in the momentum space. Secondly, the accuracy of the measurement of the particle coordinates is in principle restricted by the Compton length of this particle. Moreover in the conformal invariant case determination of the coordinates of the massless particle generates an additional essential trouble (see [10] ch. 20 and [16]).

The 5D formulation of the quantum field theory with the invariance form \( q_\mu q^\mu + q_5^2 = M^2 \) or \( q_\mu q^\mu - q_5^2 = -M^2 \) was suggested in refs. [17, 18, 19], where the scale parameter \( M \) was interpreted as the fundamental (maximal) mass and its inverse \( 1/M \) as the fundamental (minimal) length [20, 21]. In the present formulation \( M \) has the meaning of a auxiliary mass parameter which indicates the scale of the conformal symmetry breaking. In this paper we have defined \( M \) using the well known spontaneous symmetry breaking models. In particular, for the nonlinear \( \sigma \) model \( M \) is fixed via the mass of the \( \pi \) meson and for the present 5D generalization of the standard model with the spontaneous broken \( SU(2) \times U(1) \) symmetry the scale parameter \( M \) is determined by the mass of the Higgs particle.

In the first section we consider a conformal transformations in the 4D momentum space and we shall show, that the scattering \( S \)-matrix is invariant under the four-momentum translation. Five dimensional projections in the momentum space with corresponding 5D and 4D equations of motion are considered in the sections 2 and 3. Sections 4, 5 and 6 are devoted to the 5D Lagrangian approach. Interaction part of 5D Lagrangian \( L_{INT}(x,x_5) \) in the framework of the nonlinear sigma model is constructed in sect. 5 using the boundary condition for the fifth coordinate. 5D theories with the gauge transformations and a model of the chiral \( SU(2) \times U(1) \) Lagrangian is presented in ch.6. The summary and outlook in ch. 7 is devoted to the short comparison with the other 5D field-theoretical and to the main features of the present formulation.

1. Conformal transformations in the 4D momentum space and the scattering \( S \)-matrix

Conformal group of a four-momentum \( q_\mu \) (\( \mu = 0, 1, 2, 3 \)) transformations consists of the following transformations:

translations

\[
T(h) : \quad q_\mu \rightarrow q'_\mu = q_\mu + h_\mu, \quad \text{(1.1a)}
\]

rotations

\[
R(\Lambda) : \quad q_\mu \rightarrow q'_\mu = \Lambda_\mu^\nu q_\nu, \quad \text{(1.1b)}
\]

dilatation

\[
D(\lambda) : \quad q_\mu \rightarrow q'_\mu = e^\lambda q_\mu, \quad \text{(1.1c)}
\]
and inversions
\[ I(M^2) : \quad q_\mu \rightarrow q'_\mu = -M^2 q_\mu / q^2, \quad (1.1d) \]
where a scale parameter \( M \) insures the correct dimension of \( q'_\mu \). Translation \( T(h) \) and inversions \( I(M^2) \) form the special conformal transformations
\[ \mathcal{K}(M^2, h) \equiv I(M^2)T(h)I(M^2) : \quad q_\mu \rightarrow q'_\mu = \frac{q_\mu - h_\mu q^2 / M^2}{1 - 2q_\mu h^\nu / M^2 + h^2 q^2 / M^4}, \quad (1.1e) \]

Obviously, \( q_\mu \) in (1.1a)-(1.1e) is off mass shell \((q_0 \neq \sqrt{q^2 + m^2})\). Hereafter the on mass shell 4D momenta will be denoted as \( p_\mu \) \((p^2 = m^2, \quad p_0 \geq 0)\).

Following the Dirac geometrical model \([1]\), transformations (1.1a)-(1.1e) may be realized as rotations in the 6D space with the metric \( g_{AB} = \text{diag}(+1, -1, -1, -1, +1, -1) \) and on the 6D cone
\[ \kappa^2 \equiv \kappa_A \kappa^A = \kappa_\mu \kappa^\mu + \kappa_5^2 - \kappa_6^2 = 0, \quad (1.2) \]
where according to conformal covariant formulation \([5, 22]\),
\[ q_\mu = \frac{\kappa_\mu}{\kappa_+}; \quad \kappa_+ = (\kappa_5 + \kappa_6) / M; \quad \mu = 0, 1, 2, 3; \quad (1.3) \]
where \( \kappa_+ \) is a dimensionless scale parameter and \( \kappa_\mu, q_\mu \) and \( M \) have the same dimensions in the system of units \( \hbar = c = 1 \).

**Conformal transformations (1.1a)-(1.1e) of a particle field operator \( \Phi_\gamma(x) \):**

The particle field operator \( \Phi_\gamma(x) \) with a spin-isospin quantum numbers \( \gamma \) is usually expanded in the positive and in the negative frequency parts in the 3D Fock space
\[ \Phi_\gamma(x) = \int \frac{d^3 p}{(2\pi)^3 2\omega_\mathbf{p}} [a_{p\gamma}(x_0)e^{-ipx} + b^+_{p\gamma}(x_0)e^{ipx}]; \quad p_0 \equiv \omega_\mathbf{p} = \sqrt{p^2 + m^2}, \quad (1.4a) \]

where in the asymptotic regions \( a_{p\gamma}(x_0) \) and \( b^+_{p\gamma}(x_0) \) transforms into particle (antiparticle) annihilation (creation) operators \( \lim_{x_0 \rightarrow \pm \infty} < m|a_{p\gamma}(x_0)|n > = < m|a_{p\gamma}(\text{out, in})|n >; \lim_{x_0 \rightarrow \pm \infty} < m|b^+_{p\gamma}(x_0)|n > = < m|b^+_{p\gamma}(\text{out, in})|n >, \) where \( < n, m > \) denotes some \( n,m \)-particle asymptotic states. On the other hand, \( \Phi_\gamma(x) \) may be expanded in the 4D momentum space
\[ \Phi_\gamma(x) = \int \frac{d^4 q}{(2\pi)^4} \left[ \Phi^{(+)}_\gamma(q)e^{-iqx} + \Phi^{(-)}_\gamma(q)e^{iqx} \right]. \quad (1.4b) \]

After comparison of the expressions (1.4a) and (1.4b) we get
\[ \frac{e^{-i\omega_\mathbf{p} x_0}}{2\omega_\mathbf{p}} a_{p\gamma}(x_0) = \int \frac{dq}{2\pi} \Phi^{(+)}_\gamma(q_0, \mathbf{p}) e^{-iq_0 x_0} \quad (1.5a) \]
\[ = i \frac{e^{-i\omega_\mathbf{p} x_0}}{2\omega_\mathbf{p}} \sum_\beta \int d^3 x < 0|\Phi_\beta(x)|p\gamma > \left[ \frac{\partial \Phi_\beta(x)}{\partial x_0} - i\omega_\mathbf{p} \Phi_\beta(x) \right] \quad (1.5b) \]
and

\[
\frac{e^{i\omega p_\alpha x_0}}{2\omega p_\alpha} b_{+\alpha}^\gamma(x_0) = \int \frac{dq_0}{2\pi} \Phi_\gamma^{(+)}(q_0, p_\alpha)e^{iq_0x_0} \tag{1.6a}
\]

\[
= -i e^{i\omega p_\alpha x_0} \sum_\beta \int d^3x < p_\alpha | \Phi_\beta(x) | 0 > \left[ \frac{\partial \Phi_\beta(x)}{\partial x_0} + i \omega p_\alpha \Phi_\beta(x) \right] \tag{1.6b}
\]

where we have used the following expressions for a one-particle (antiparticle) states

\[
< 0 | \Phi_\beta(x) | p_\gamma >= Z^{-1/2} \int \frac{d^3p'}{(2\pi)^3 2\omega p'} e^{ip'x} < 0 | a_{p'\beta}(x_0) | p_\gamma >= Z^{-1/2} \delta_{\beta\gamma} e^{ip_\gamma x}, \tag{1.7a}
\]

\[
< p_\alpha | \Phi_\beta(x) | 0 > = Z_a^{-1/2} \int \frac{d^3p'}{(2\pi)^3 2\omega p'} e^{ip'x} < p_\alpha | b_{+p_\alpha}^\gamma(x_0) | 0 > = Z_a^{-1/2} \delta_{\beta\gamma} e^{-ip_\alpha x}, \tag{1.7b}
\]

here index “a” denotes the antiparticle state and \(Z\) is the renormalisation constant.

The field operators \(a_{p\gamma}(x_0)\) and \(b_{+p_\gamma}(x_0)\) are simply defined via the corresponding source operator \(\partial/\partial x_0 a_{p_\gamma}(x_0) = i \int d^3x e^{ipx} j_\beta(x)\), where \(\left(\partial^2/\partial x_\mu \partial x^\mu + m^2\right) \Phi_\beta(x) = j_\beta(x)\). Moreover, these operators determine the transition \(S\)-matrix

\[
S_{mn} \equiv< out; p_1^{\alpha_1},..., p_m^{\alpha_m} | p_1^{\alpha_1},..., p_n^{\alpha_n}; in > \equiv \prod_{i=1}^m \left[ \int dx_0^i \frac{d}{dx_0^i} \right] \prod_{j=1}^n \left[ \int dx_0^j \frac{d}{dx_0^j} \right] < 0 | T(a_{p_m^{\alpha_m}},..., a_{p_1^{\alpha_1}}; a_{p_n^{\alpha_n}},..., a_{p_1^{\alpha_1}}) | 0 >. \tag{1.8}
\]

Next we shall consider the transformations of \(\Phi_\beta(x)\) according to the conformal transformations of \(\Phi_\gamma^{(\pm)}(q)\)

\[
\Phi_\gamma^{(\pm)}(q) \rightarrow \Phi_\gamma^{(\pm)}(q') = U(g) \Phi_\gamma^{(\pm)}(q) U^{-1}(g) = T_\gamma^{\beta} \Phi_\gamma^{(\pm)}(g^{-1}q), \tag{1.9}
\]

where \(g\) indicates one of the (1.1a)-(1.1e) transformations \(g \equiv \left(T(h), R(\Lambda), D(\lambda), K(M, h)\right)\), \(T_\gamma^{\beta}\) is the spin-isospin matrix and \(U(g)\) are defined through the generators of the corresponding transformations in the well known form:

\[
T(h) : \quad U(h) = e^{ih\mu \chi^\mu} ; \quad [\chi^\mu, \Phi_\gamma^{(\pm)}(q)] = -i \frac{\partial}{\partial q^\mu} \Phi_\gamma^{(\pm)}(q), \tag{1.10a}
\]

\[
R(\Lambda) : \quad U(\Lambda) = e^{i\Lambda_{\mu\nu} \mathcal{M}_{\mu\nu}} ; \quad [\mathcal{M}_{\mu\nu}, \Phi_\gamma^{(\pm)}(q)] = -i \left(q_\mu \frac{\partial}{\partial q^\nu} - q_\nu \frac{\partial}{\partial q^\mu} - i\Sigma_{\mu\nu}\right) \Phi_\gamma^{(\pm)}(q) \tag{1.10b}
\]
where $\Sigma_{\mu\nu} = 0$ for scalars, $\Sigma_{\mu\nu} = i/4[\gamma_{\mu}, \gamma_{\nu}]$ for fermions and $(\Sigma_{\mu\nu}V_{\rho}) = ig_{\mu\rho}V_{\nu} - ig_{\nu\rho}V_{\mu}$ for the vectors $V_{\rho}$.

\[ D(\lambda) : \quad U(\lambda) = e^{i\lambda D}; \quad [D, \Phi_{\gamma}^{(\pm)}(q)] = -i\left(q_{\mu}\frac{\partial}{\partial q^\mu} + id_m\right)\Phi_{\gamma}^{(\pm)}(q), \quad (1.10c) \]

where $d_m$ indicates the scale dimension of field. For example, in the scale-invariant case $d_m = -3$.

\[ K(M, \bar{h}) : \quad U(\bar{h}) = e^{ih_{\mu}K_{\mu}^\gamma}; \quad [K_{\mu}, \Phi_{\gamma}^{(\pm)}(q)] = -\left(2q_{\mu}D - q^2X_{\mu} + 2iq^{\nu}\Sigma_{\mu\nu}\right)\Phi_{\gamma}^{(\pm)}(q). \quad (1.10d) \]

According to (1.4b) the conformal transformations of the operators $\Phi_{\gamma}^{(\pm)}(q)$ (1.9) generates the corresponding transformations of $\Phi_{\gamma}^{(\pm)}(x)$

\[ \Phi_{\gamma}^{(\pm)}(x) = \frac{T_{\gamma}}{\beta}(g^{-1}q)e^{-iqx} + \Phi_{\gamma}^{(\pm)}(g^{-1}q)e^{iqx}. \quad (1.11) \]

In particular, eq.(1.11) consists of the following transformations:

**Four-momentum translation:**

For a charged particle a four-momentum translation is equivalent to

\[ q'_{\mu} = q_{\mu} + h_{\mu} \quad \Rightarrow \quad i\frac{\partial}{\partial x'} = i\frac{\partial}{\partial x} + h_{\mu}, \quad (1.12a) \]

which implies the well known gauge transformation of the charged particle field operator

\[ \Phi_{\gamma}'(x) = e^{ih_{\gamma}x}\Phi_{\gamma}(x). \quad (1.12b) \]

In order to get the the gauge transformation formula (1.12b) we introduce the following transformations of $\Phi_{\gamma}^{(\pm)}(q)$

\[ \Phi_{\gamma}^{(\pm)}(q) = \Phi_{\gamma}(q + h); \quad \Phi_{\gamma}^{(+)}(q) = \Phi_{\gamma}^{(\pm)}(q + h) \quad (1.13a) \]

\[ \Phi_{\gamma}^{(-)}(q) = \Phi_{\gamma}(q - h); \quad \Phi_{\gamma}^{(-)}(q) = \Phi_{\gamma}^{(-)}(q - h). \quad (1.13b) \]

After substitution of (1.13a,b) in (1.4b) we get

\[ \Phi_{\gamma}'(x) = \int \frac{dq}{(2\pi)^4} \left[ \Phi_{\gamma}^{(\pm)}(q + h)e^{-iqx} + \Phi_{\gamma}^{(-)}(q - h)e^{iqx} \right] = e^{ih_{\gamma}x}\Phi_{\gamma}(x) \quad (1.14). \]

In the same way we obtain

\[ \int \frac{dq}{2\pi}\Phi_{\gamma}^{(\pm)}(q_{o} + h_{o}, p + h)e^{-iq_{o}x_{o}} = \int \frac{dq}{2\pi}\Phi_{\gamma}^{(\pm)}(q_{o}, p + h)e^{-i(q_{o} + h_{o})x_{o}} \]
where the operator

\[ a'_\gamma(x_o) = i \sum_\beta \int d^3x <0|\Phi'_\beta(x)|p \rangle \frac{\partial}{\partial x^\beta} \Phi'_\beta(x) \]  

(1.15b)

coincides with the operator (1.5b)

\[ a'_\gamma(x_o) = a_\gamma(x_o). \]  

(1.15c)

Thus the gauge transformation (1.12a,b) generates the following transformation of the S-matrix (1.8)

\[ S'_{mn} = \langle \text{out}; p'_1 + h \alpha'_1, ..., p'_m + h \alpha'_m|p_1 + h \alpha_1, ..., p_n + h \alpha_n; \text{in} > \]  

(1.16)

This expression differs from \( S_{mn} \) by a shift of the position of the origin in the 3D momentum space. Therefore \( S'_{mn} = S_{mn} \) because only the relative momenta are physically meaningful. A more complicated shift of a four-momentum operator \( \hat{P}_\mu = \hat{P}_\mu - 0|\hat{P}_\mu|0 > \) is often used in quantum field theory concerning the so-called zero-mode problem (see for example ch. 12 of [23]).

**For a neutral particle** field operator \( \phi(x) \) the translation \( q'_\mu = q_\mu + h_\mu \) (1.12a) has a more complicated form due to absence of the antiparticle degree of freedom. In particular, using (1.13a) we obtain

\[ \phi'_\gamma(x) = \int \frac{d^4q}{(2\pi)^4} \left[ \phi^{(+)}_\gamma(q)e^{-i(q-h)x} + \phi^{(-)}_\gamma(q)e^{i(q-h)x} \right] \neq e^{ihx}\phi_\gamma(x) \]  

(1.17a)

or

\[ \phi'_\gamma(x) = \int \frac{d^4q}{(2\pi)^3} \delta((q_o - h_o)^2 - (q - h)^2 - m^2)\theta(q_o - h_o) \]

\[ [a'_{q\gamma}(x_0)e^{-i(q_\mu - h_\mu)x_0 + i(q-h)x} + a'^+_{q\gamma}(x_0)e^{i(q_\mu - h_\mu)x_0 + i(q-h)x}], \]

(1.17b)

where

\[ a'_{p\gamma}(x_o) = i \sum_\beta \int d^3x <0|\phi'_\beta(x)|p \rangle \frac{\partial}{\partial x^\beta} \phi'_\beta(x) \]  

(1.18)

According to (1.17a,b) \( \phi'_\gamma(x) \) remains to be hermitian after translations \( q'_\mu = q_\mu + h_\mu \). On the other hand these translations generate the nontrivial dependence of \( \phi'_\gamma(x) \) on \( h_\mu \). The similar dependence on the additional parameter \( h_\mu \) appears in the real fields \( \phi_{1,2}(x) \).
constructed from the charged pion fields $\pi_{\pm}(x)$ after their gauge transformation (1.12a,b)
\[ \phi'_{1}(x) = 1/\sqrt{2} \exp (-ix)\pi_{+}(x) + \exp (ix)\pi_{+}^{+}(x) \]
and
\[ \phi'_{2}(x) = i/\sqrt{2} \exp (-ix)\pi_{+}(x) - \exp (ix)\pi_{+}^{+}(x) \]. It must be noted, that a splitting of $\phi_{\gamma}(x)$ on the positive and the negative frequency parts $\phi_{\gamma}(x) = \phi_{\gamma}^{+(x)} + \phi_{\gamma}^{+(x)^{\dagger}}$ can be realized with arbitrary parameter $\alpha$ [33] as $\phi_{\gamma}(x) = e^{i\alpha}\phi_{\gamma}^{+(x)} + e^{-i\alpha}\phi_{\gamma}^{+(x)^{\dagger}}$. In our case the additional dependence of $\phi_{\gamma}(x)$ on $h_{\mu}$ is result of the condition (1.13a) which is necessary for the gauge transformations rule (1.12a,b) of the charged field operators.
Using the normalization condition for functions $f_{p-h}(x) = e^{i(p_{o} - h_{o})x - i(p - h)x}$
\[ i \int f'_{p-h}(x) \frac{\partial}{\partial x^{0}} f_{p-h}(x) d^{3}x = 2(p_{o} - h_{o})(2\pi)^{3} \delta(p' - p), \tag{1.19a} \]
\[ i \int f'_{p-h}(x) \frac{\partial}{\partial x^{0}} f_{p-h}(x) d^{3}x = i \int f_{p-h}(x) \frac{\partial}{\partial x^{0}} f'_{p-h}(x) d^{3}x = 0 \tag{1.19b} \]
It is easy to obtain
\[ a'_{p}(x_{o}) = a_{p}(x_{o}), \tag{1.20} \]
where $a_{p}(x_{o}) = i \sum_{\beta} f d^{3}x < 0|\phi_{\beta}(x)|p_{\gamma} > \frac{\partial}{\partial x^{0}} \phi_{\beta}(x)$. Relation (1.20) is similar to the relation (1.15c) for the charged fields. This means, that $S$-matrix transforms according to the same relation (1.16) for the charged and neutral particles after translation in the 4D momentum space $q'_{\mu} = q_{\mu} + h_{\mu}$. The dependence on the dummy variables $q_{o}$ and $q_{o} + h_{o}$ disappears in the $S$-matrix after the appropriate integration in (1.5a), (1.6a) and (1.15a,b). Thus for the $S$-matrix and other observables the translation of $q \equiv (q_{o}, p)$ is reduced to the 3D translations $p' = p + h$ which does not affect these observables.

**Rotation (1.1b) and dilatation (1.1c) of $q_{\mu}$** for the particle field operator $\Phi_{\gamma}(x)$ (1.11) may be performed using the rotations (1.10b) and scale transformations (1.10c) of $\Phi_{\gamma}^{(\pm)}(q)$ operators. In particular, rotations $q'_{\mu} = \Lambda_{\mu\nu}q_{\nu}$ generates the following transformation of the field operators in the configuration space
\[ R(\Lambda) : \quad \Phi'_{\gamma}(x_{\mu}) = \Phi_{\gamma}(\Lambda^{-1}_{\mu\nu}x'_{\nu}), \tag{1.21} \]
and for dilatation $q'_{\mu} = e^{\lambda} q_{\mu}$ we have
\[ D(\lambda) : \quad \Phi'_{\gamma}(x) = e^{4\lambda} \Phi_{\gamma}(e^{-\lambda} x). \tag{1.22} \]
Therefore the rotations and dilatation of $\Phi_{\gamma}(q)$ generate the analogical transformations of $\Phi_{\gamma}(x)$.

**Special conformal transformation and inversion**: Special conformal transformation of $q_{\mu}$ (1.1e) for $\Phi_{\gamma}^{(\pm)}(q)$ has the form
\[ \Phi_{\gamma}^{(+)}(q) = \Phi_{\gamma}^{(+)}((q' + h)'^{\dagger}) ; \quad \Phi_{\gamma}^{(+)^{+}}(q) = \Phi_{\gamma}^{(+)^{+}}((q' + h)'^{\dagger}) \tag{1.23a} \]
\[ \Phi_{\gamma}^{(-)}(q) = \Phi_{\gamma}^{(-)}((q' - h)'^{\dagger}) ; \quad \Phi_{\gamma}^{(-)^{+}}(q) = \Phi_{\gamma}^{(-)^{+}}((q' - h)'^{\dagger}), \tag{1.23b} \]
where the index $I$ relates to the inversion of $q_\mu$. According to (1.11) we get

$$
\Phi'_\gamma(x) = \int \frac{d^4q}{(2\pi)^4} \left[ \Phi^{(+)}_\gamma((q^I + h)^I) e^{-iqx} + \Phi^{(-)}_\gamma((q^I - h)^I) e^{iqx} \right].
$$

(1.24)

Arbitrary operator $\Phi^{(\pm)}_\gamma(q)$ may be divided into inversion invariant and inversion anti-invariant parts

$$
\Phi^{(\pm)}_{\gamma \text{ inv.}}(q) = \frac{1}{2} \left[ \Phi^{(\pm)}_\gamma(q) + \Phi^{(\pm)}_\gamma(q^I) \right]
$$

(1.25a)

and

$$
\Phi^{(\pm)}_{\gamma \text{ anti-inv.}}(q) = \frac{1}{2} \left[ \Phi^{(\pm)}_\gamma(q) - \Phi^{(\pm)}_\gamma(q^I) \right].
$$

(1.25b)

Expressions (1.25a,b) simplifies use of the special conformal transformation.

2. Five dimensional projection

The invariant form of the $O(2, 4)$ group $\kappa_A\kappa^A = 0$ (1.2) can be represented in the five dimensional form with $q_\mu$ (1.3) variables

$$
q_\mu q^\mu + M^2 \frac{\kappa_+}{\kappa_-} = 0,
$$

(2.1a)

where

$$
q_\mu = \frac{\kappa_\mu}{\kappa_-}; \quad \kappa_\pm = \frac{\kappa_5 \pm \kappa_6}{M}.
$$

(2.1b)

It is convenient to introduce new fifth momentum $q_5$ instead of two variables $\kappa_\pm$ (or $\kappa_5$, $\kappa_6$) in (2.1a). This procedure implies a projection of the 6D rotational invariant form $\kappa_A\kappa^A = 0$ into corresponding 5D forms. There exists only two 5D De Sitter spaces with the constant curvature which have the invariant forms of the $O(2, 3)$ and $O(1, 4)$ rotational groups [9, 14, 18]

$$
q_\mu q^\mu + q_5^2 = M^2
$$

and

$$
q_\mu q^\mu - q_5^2 = -M^2
$$

(2.2a)

and

$$
q_5^2 = M^2 \frac{2\kappa_5}{\kappa_5 + \kappa_6},
$$

(2.2b)

In the literature often is considered the stereographic projection of the 6D cone $\xi_A\xi^A = 0$ into 4D Minkowski space with coordinates $x_\mu = \xi_\mu (\xi_5 - \xi_6)/\xi_5$, where at the intermediate stage are used projections on the 5D hyperboloid $\eta_\mu \eta^\mu - \eta_5^2 = -\ell^2$ (see for example ch. 13 of [24]) with $\eta_\mu = \xi_\mu \ell/\xi_5$; $\eta_5 = \xi_6 \ell/\xi_5$ and $\eta_\mu = 2x_\mu/(1 - x^2/\ell^2)$; $\eta_5 = \ell(1+x^2/\ell^2)/(1-x^2/\ell^2)$. Here $x^2 = \ell^2(\eta_5/\ell - 1)/(\eta_5/\ell + 1)$ and at first sight $x_\mu$ is not restricted by the 5D condition $\eta_\mu \eta^\mu - \eta_5^2 = -\ell^2$ like $q^2$ (2.2a,b) in table 1 or 2. Nevertheless, the 6D invariant form can be rewritten as $x^\mu x_\mu + \ell^2(\xi_5 - \xi_6)/(\xi_5 + \xi_6) = 0$ and the appropriate projection into 5D hyperboloid $x^\mu x_\mu \pm x_5^2 = \pm \ell^2$ with $x_5^2 = 2\xi_5$ (or $\ell_6 \ell^2)/(\xi_5 + \xi_6)$ generates the corresponding restrictions.
The real variables $q_\mu$ and $q_5$ are defined in the regions $(-\infty, +\infty)$ and $[0, +\infty)$ respectively. In the considered formulation $q_\mu$ and $q_5$ are disposed on hyperboloids (2.2a,b). In particular, we can place $0 < q^2 \leq M^2$ and $q^2 > M^2$ on the hyperboloids (2.2a) and (2.2b) respectively. Thus the conformal transformations for the whole $q^2$ values may be performed using both hyperboloid (2.2a,b). The values of $q_\mu$ and $q_5$ on these hyperboloids are singlevalued connected with each other via inversion $q_\mu' = -M^2 q_\mu/q^2$ (1.1d). On the 6D cone $\kappa_A \kappa^A = 0$ (1.2) inversion (1.1d) can be carried out using the reflection of the $\kappa_6$ variable

$$I(M^2) : \quad \kappa_6^I = \kappa_5, \quad \kappa_6^I = -\kappa_6, \quad \kappa_+^I = \kappa_-, \quad \kappa_-^I = \kappa_+, \quad (2.3)$$

which generates $q_\mu^I = -M^2 q_\mu/q^2$ according to (2.1a,b). The advantage of the 6D representation (2.3) of the 4D transformation $q_\mu^I = -M^2 q_\mu/q^2$ is that it determines the transparent realization of the nonlinear 4D transformation using the simple reflection in 6D space. In particular, for $0 < q_\mu \leq M^2$ on the hyperboloid $q^2 + q_5^2 = M^2$, we have $q^2I = M^4/q^2 \geq M^2$ and $(-q^2_5 + M^2)^I = M^2 \kappa_+^I/\kappa_6^I = M^2/(\kappa_-^I/\kappa_+^I) = M^4/(q_5^2 - M^2)$. Therefore, if $q_\mu^I$ belongs to hyperboloid $q^2I + (-q^2_5 + M^2)^I = 0$ (2.2b), then $q_\mu$ will be placed on the other hyperboloid, because $q^2I + (-q^2_5 + M^2)^I = M^4/(q_5^2 - M^2)((q_5^2 + q_5^2 - M^2) = 0$. The distribution of the regions of the 5D hyperboloid $q^2 \pm q_5^2 = \pm M^2$ (2.2a,b) which covers the whole values of $q_\mu$ and $q_5$ ($-\infty < q^2 < \infty$ and $q_5^2 \geq 0$) is given in table 1.

| Table 1 |
|-----------------|-----------------|-----------------|-----------------|
| $q^2 + q_5^2 = M^2$ | $q^2 - q_5^2 = -M^2$ | $q^2 + q_5^2 = M^2$ | $q^2 - q_5^2 = -M^2$ |
| $q^2$ | $0 \leq q^2 \leq M^2$ | $M^2 < q^2 < \infty$ | $-\infty < q^2 < -M^2$ |
| $q_5^2$ | $0 \leq q_5^2 \leq M^2$ | $2M^2 < q_5^2 < \infty$ | $2M^2 < q_5^2 < \infty$ |

Momentum $q_\mu$ from the region I is singlevalued connected with $q_\mu$ in the region II via inversion $\{q_\mu\}_{II \text{ region}} = -M^2 \{q_\mu/q^2\}_{I \text{ region}}$ and vice versa $\{q_\mu\}_{I \text{ region}} = -M^2 \{q_\mu/q^2\}_{II \text{ region}}$. In the same manner are connected the four momenta in the regions III and IV with $q^2 < 0$. For $M \to \infty$ 5D spaces transforms into ordinary Minkowski space with the domains I and IV. For $M \to 0$ we get again a Minkowski space with remained regions II and III.

The scale transformation have the different form in the different areas in table 1. In 6D space the scale transformation $q_\mu^I = e^\lambda q_\mu$ (1.1c) implies the rotation in the (6,5) plane. For $q^2 \geq 0$ rotation $\kappa_5 = M \sinh(\lambda)$; $\kappa_6 = M \cosh(\lambda)$ generates the following transformation of $q^2$: $q_\mu q^\mu = -M^2 (\kappa_5 - \kappa_6)/(\kappa_5 + \kappa_6) = M^2 e^{-2\lambda}$. For negative $q^2 < 0$ we take $\kappa_5 = M \cosh(\lambda)$; $\kappa_6 = M \sinh(\lambda)$ (i.e. $\kappa_+ = e^\lambda$) which gives $q_\mu q^\mu = -M^2 e^{2\lambda}$. The corresponding transformation of $q_5^2$ with the related $\lambda$, is given in table 2.

\footnote{The border point $q^2 = 0$ with $q_5^2 = M^2$ is included in the domain $q_\mu q^\mu + q_5^2 = M^2$, because it belongs to the physical spectrum of the massless particles. After inversion $q^2 = 0$ transforms into $q^2 = \infty$ of the hyperboloid $q_\mu q^\mu - q_5^2 = -M^2$ in the region II.}
In table 2 it is shown, that the scale transformation parameter $\lambda$ (or $\kappa_+ = e^{-\lambda}$) single-valued determines $q_5^2$ and $q^2$. In particular, in the region I with $q^2 + q_5^2 = M^2$ the scale transformation is realizable by $\lambda > 0$, which implies the compression of $q^2$. In opposite to this, in the region II with $q^2 - q_5^2 = -M^2$ we have $\lambda < 0$, i.e. the same scale transformation generates the stretching of $q^2$. A similar scale transformation can be observed for $q^2 < 0$, where in the region III dilatation generates stretching and in the region IV we have compression. In other words, projections of the 5D cone $\kappa_A \kappa^A = 0$ on the 5D hyperboloid for $q^2 > 0$ implies

\begin{equation}
\kappa_A \kappa^A = 0 \implies q_\mu q^\mu + q_5^2 = M^2, \quad \text{for } 0 \leq q_\mu q^\mu \leq M^2 \text{ with } \lambda \geq 0 \tag{2.4a}
\end{equation}

\begin{equation}
\kappa_A \kappa^A = 0 \implies q_\mu q^\mu - q_5^2 = -M^2, \quad \text{for } q_\mu q^\mu > M^2 \text{ with } \lambda \leq 0. \tag{2.4b}
\end{equation}

Inversion $q^2 = M^4/q^2$ replaces the points from the internal region $0 \leq q^2 < M^2$ (section I) with the points from the external region $q^2 > M^2$ (section II) and vice versa. Thereby the hyperboloid $q^2 + q_5^2 = M^2$ we shall denote as the “internal” and the hyperboloid $q^2 + q_5^2 = M^2$ we shall call as the “external”.

By translation $q'_\mu = q_\mu + h_\mu$ the 6D cone $\kappa_A \kappa^A = 0$, as well as the 5D forms (2.2a,b) $q^2 \pm q_5^2 = \pm M^2$ are preserved. In particular, after the appropriate 6D rotations $\kappa'_\mu = \kappa_\mu + h_\mu \kappa_+; \ k'_+ = \kappa_+; \ k'_- = \kappa_- + 2/M^2 h_\mu \kappa^\mu + h^2/M^2 \kappa_+$, we get

\begin{equation}
q''_\mu = q^2 + 2h_\mu q^\mu + h^2 = -M^2 \frac{h'_\mu}{\kappa_+}; \quad q''_5 = q_5^2 \pm (2h_\mu q^\mu + h^2), \tag{2.5}
\end{equation}

where the sign $-$ corresponds to $q^2 + q_5^2 = M^2$ and $+$ relates to $q^2 - q_5^2 = -M^2$. Using (2.5) we have $q''_\mu \pm q''_5 = q^2 \pm q_5^2$. Nevertheless, the transformation $q'_\mu = q_\mu + h_\mu$ can generate a transition from the time-like region $q''_\mu \geq 0$ into space-like region $q''_5 < 0$. Transition between the $q^2 > 0$ and $q^2 < 0$ regions can be compensated by inversion or by transposition of $\kappa_6$ and $\kappa_5$ variables.

It must be noted, that inversion transforms the generators of the conformal group (1.10a) - (1.10d) in the following way

\begin{equation}
X_\mu = I(M^2) K_\mu I(M^2); \quad M_{\mu\nu} = I(M^2) M_{\mu\nu} I(M^2); \quad D' = I(M^2) D I(M^2) = -D; \quad K_\mu = I(M^2) X_\mu I(M^2), \tag{2.6}
\end{equation}
Therefore one can perform the conformal transformations only in the “internal” regions $I$ and $III$ and obtain the corresponding transformations in the “external” regions $II$ and $IV$ using the inversion.

**5D reduction of the field operators:** Next we have to connect a 6D field operator $\phi^{(\pm)}(\kappa_A) \, d (A = \mu; 5, 6 \equiv 0, 1, 2, 3; 5,6)$ with the 5D operators $\phi^{(\pm)}_{\text{inr}}(q, q_5)$ and $\phi^{(\pm)}_{\text{ext}}(q, q_5)$. Here index $(\pm)$ corresponds to positive or negative frequency, the subscripts $\text{inr}$ or $\text{ext}$ indicate the surfaces $q^2 + q_5^2 = M^2$ and $q^2 + q_5^2 = -M^2$ correspondingly.

Afterwards for the sake of simplicity we omit the spin-isospin indices $\gamma$.

According to the manifestly covariant construction of the $O(2, 4)$ conformal group [5, 22], we shall use following independent variables

$$q_{\mu} = \kappa_{\mu}/\kappa_+; \quad \kappa_+ = (\kappa_5 + \kappa_6)/M; \quad \kappa^2 = \kappa^A \kappa_A.$$ \hspace{1cm} (2.7)

Only this choice of the variables makes independent the generators of the conformal group $O(2, 4)$ on $\partial/\partial \kappa^2$. In particular, for a spinless particle these generators are

$$\mathcal{X}_{\mu} = i \frac{\partial}{\partial q_{\mu}}; \quad \mathcal{M}_{\mu\nu} = i (q_{\mu} \frac{\partial}{\partial q_{\nu}} - q_{\nu} \frac{\partial}{\partial q_{\mu}});$$

$$D = i (q_{\mu} \frac{\partial}{\partial q_{\mu}} - k_+ \frac{\partial}{\partial k_+}); \quad \mathcal{K}_\mu = 2q_{\mu}D - q^2 \mathcal{X}_\mu.$$ \hspace{1cm} (2.8)

Using the variables (2.7) the 6D field operator takes the form

$$\phi^{(\pm)}(\kappa_A) \equiv \phi^{(\pm)}(q_{\mu}, \kappa_+, \kappa^2).$$ \hspace{1cm} (2.9)

The homogeneous over the scale variable $\kappa_+$ operator $\phi^{(\pm)}(q_{\mu}, \kappa_+, \kappa^2)$ may be rewritten as

$$\phi^{(\pm)}(q_{\mu}, \kappa_+, \kappa^2) = (\kappa_+)^d \phi^{(\pm)}(q_{\mu}, \kappa^2),$$ \hspace{1cm} (2.10)

and for the 4D physical field operator, in analogue to [5, 22] we get

$$\Phi^{(\pm)}(q_{\mu}) = (\kappa_+)^{-d} \mathcal{O} \phi^{(\pm)}(\kappa_A),$$ \hspace{1cm} (2.11)

where $d$ defines the scale dimension of the considered operator, and $\mathcal{O}$ acts on the spin-isospin variables.

In the present paper we shall use other recipe of projection of the 6D cone $\kappa^2 = 0$ into 5D surfaces $q_{\mu}q^\mu + q_5^2 = \pm M^2$ and in the 4D momentum space. In particular, we shall treat the condition $\kappa^2 = 0$ as the dynamical restriction, i.e. we shall require the validity of the following constraint

$$\left(\kappa^A \kappa_A\right) \phi^{(\pm)}(q_{\mu}, \kappa_+, \kappa^2) = \kappa_+^2 \left(q_{\mu}q^\mu + M^2 \frac{\kappa_-}{\kappa_+}\right) \phi^{(\pm)}(q_{\mu}, \kappa_+, \kappa^2) = 0,$$ \hspace{1cm} (2.12)

\footnote{A similar transformation can be performed using the Weyl reflection, i.e. rotation through 90° in the (0, 5) plane [11].}
Projection of this equation on the 5D surfaces \( q_\mu q^\mu \pm q_5^2 = \pm M^2 \) gives

\[
(q_\mu q^\mu + q_5^2 - M^2)\phi^{(\pm)}_{\text{inv}}(q_\mu, \kappa+, \kappa+^2(q_\mu q^\mu + q_5^2 - M^2)) = 0, \quad (2.13a)
\]

\[
(q_\mu q^\mu - q_5^2 + M^2)\phi^{(\pm)}_{\text{ext}}(q_\mu, \kappa+, \kappa+^2(q_\mu q^\mu - q_5^2 + M^2)) = 0. \quad (2.13b)
\]

As it was show in table 2, the magnitude of the scale parameter \( \kappa_+ = e^\lambda \) is unambiguously defined via \( q^2 \) or \( q_5^2 \) variables. Therefore we can introduce the 5D fields

\[
\varphi^{(\pm)}_{\text{inv}}(q_\mu, q_5) \equiv (\kappa_+)^{-d}O \phi^{(\pm)}_{\text{inv}}(q_\mu, \kappa+, \kappa+^2(q_\mu q^\mu + q_5^2 - M^2)), \quad (2.14a)
\]

\[
\varphi^{(\pm)}_{\text{ext}}(q_\mu, q_5) \equiv (\kappa_+)^{-d}O \phi^{(\pm)}_{\text{ext}}(q_\mu, \kappa+, \kappa+^2(q_\mu q^\mu - q_5^2 + M^2)). \quad (2.14b)
\]

Then using eq.(2.13a,b) we get

\[
(q_\mu q^\mu + q_5^2 - M^2)\varphi^{(\pm)}_{\text{inv}}(q_\mu, q_5) = 0; \quad (2.15a)
\]

\[
(q_\mu q^\mu - q_5^2 + M^2)\varphi^{(\pm)}_{\text{ext}}(q_\mu, q_5) = 0,. \quad (2.15b)
\]

Equations (2.15a,b) present the desired 5D projections of the 6D constraint (2.12). These relations can be treated also as the 4D equations, because the fifth momentum on \( q^2 \pm q_5^2 = \pm M^2 \) shell is given \( q_5 = \pm \sqrt{|M^2 \mp q^2|} \).

### 3. 4D and 5D equations of motion.

It is well known that the systems with any dimensional parameters are conformal non-invariant. Nevertheless one can always perform the conformal transformations for the conformal-non-invariant systems. Conformal transformations in momentum space can be realized as 6D rotations with invariant form \( \kappa^A\kappa_A \) which generates the requirement \( \kappa^A\kappa_A\phi(\kappa) = 0 \) (2.12). We shall consider this condition and the appropriate 5D projections \( (q_\mu q^\mu \pm q_5^2 \mp M^2)\varphi^{(\pm)}_{\text{inv,ext}}(q_\mu, q_5) = 0 \) (2.15a,b) as restrictions which can be taken into account by construction of the 4D equation of motion

\[
\left(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + m^2\right)\Phi(x) = J(x), \quad (3.1a)
\]

where

\[
J(x) = 1/(2\pi)^4 \int d^4q \left[ e^{-iqx}J^{(+)}(q) + e^{iqx}J^{(-)}(q) \right], \quad (3.1b)
\]

or

\[
J(x) = \int \frac{d^3p}{(2\pi)^32\omega_p} \left[ e^{-ipx} \frac{\partial}{\partial x_0} a_p\gamma(x_0) + e^{ipx} \frac{\partial}{\partial x_0} b^+_p\gamma(x_0) \right]; \quad p_0 \equiv \omega_p = \sqrt{p^2 + m^2}. \quad (3.1c)
\]
In order to determine the relation between 4D and 5D formulations we introduce the following boundary conditions over the fifth coordinate \( x_5 \)

\[
\Phi(x) = \Phi(x, x_5 = t_5) = \varphi_{\text{inr}}(x, x_5 = t_5) + \varphi_{\text{ext}}(x, x_5 = t_5)
\]  

(3.2a)

\[
\frac{i}{M} \frac{\partial}{\partial x_5} \Phi(x, x_5)|_{x_5=t_5} = \sum_{a=1,2} \frac{i}{M} \frac{\partial}{\partial x_5} \varphi_a(x, x_5)|_{x_5=t_5}, \quad \text{where} \; a = 1, 2 \equiv \text{inr, ext}
\]  

(3.2b)

and for \( t_5 = x_5 \) it is convenient to choose boundary condition as \( t_5 = \tau = \sqrt{x_0^2 - x^2} \) or \( t_5 = 0 \).

Using (2.15a,b) we get

\[
\left( \frac{\partial^2}{\partial x^\mu \partial x^\mu} + \frac{\partial^2}{\partial x^5 \partial x_5} + M^2 \right) \varphi_{\text{inr}}(x, x_5) = 0
\]  

(3.3a)

for the internal hyperboloid with the regions I, III in table 1 and

\[
\left( \frac{\partial^2}{\partial x^\mu \partial x^\mu} - \frac{\partial^2}{\partial x^5 \partial x_5} - M^2 \right) \varphi_{\text{ext}}(x, x_5) = 0
\]  

(3.3b)

for the “external” regions II, IV.

Besides of (3.2a,b) we introduce the following boundary condition for the fifth coordinate \( x_5 \)

\[
\frac{i}{M} \frac{\partial}{\partial x_5} \varphi_a(x, x_5) = \eta_a \varphi_a(x, x_5) + l_a(x, x_5); \quad a = 1, 2 \equiv \text{inr, ext},
\]  

(3.4)

where

\[
\eta_{\text{inr}} = \sqrt{1 - m^2/M^2}; \quad \eta_{\text{ext}} = \sqrt{1 + m^2/M^2}.
\]  

(3.5)

Acting with \( M^2 (i/M \partial/\partial x_5 + \eta_a) \) on the relation (3.4) we get

\[
\left( \frac{\partial^2}{\partial x_5 \partial x_5} + M^2 - m^2 \right) \varphi_{\text{inr}}(x, x_5) = -M^2 \left( \frac{i}{M} \frac{\partial}{\partial x_5} + \eta_{\text{inr}} \right) l_{\text{inr}}(x, x_5)
\]  

(3.6a)

and

\[
\left( \frac{\partial^2}{\partial x_5 \partial x_5} + M^2 + m^2 \right) \varphi_{\text{ext}}(x, x_5) = -M^2 \left( \frac{i}{M} \frac{\partial}{\partial x_5} + \eta_{\text{ext}} \right) l_{\text{ext}}(x, x_5).
\]  

(3.6b)

The similar boundary conditions can be introduced using the conformal transformation group in the coordinate space. In particular, the 6D invariant form \( \xi_\lambda \xi_\lambda = 0 \) with \( x_\mu = \xi_\mu \xi^5/\xi_5 + \xi_6 \) and \( x'^\mu x_\mu = -x^\ell (\xi_\mu - \xi_\ell)/(\xi_5 + \xi_6) \) can be projected into two 5D hyperboloid \( x'^\mu x_\mu = \pm x_5^2 \) with \( x_5^2 = 2 \xi_5 \) or \( \xi_6 \xi^6/\xi_5 \xi_6 \). Then we get the internal and the external 5D regions with the boundary values at \( x^2 = 0, \pm 1 \) as it is given for \( q^2, q_5^2 \) variables in table 1. For \( \Phi(x) \) we can introduce an analogue to (3.2a,b) boundary conditions \( \Phi(x) = \varphi_{\text{inr}}(x, x_5 = t) + \varphi_{\text{ext}}(x, x_5 = t) \). In the such constructions the operator [6] \( \mathcal{M}_{\mu\nu}(x) = g_{\mu\nu} - 2x_{\mu}x_{\nu}/x^2 = \mathcal{M}_{\mu\nu}(1/x) \) has the properties of the metric tensor \( \mathcal{M}_{\mu\nu}(x) \mathcal{M}^{\sigma\tau}(x) = \delta^\sigma_\mu, \mathcal{M}_{\mu\nu}(x)x^\nu = -x_\mu \) and \( \partial/\partial x^\mu = 1/x^2 \mathcal{M}_{\mu\nu}(x') \partial/\partial x'^\nu \).
Combining these equations with (3.3a,b) we obtain

\[
\left( \frac{\partial^2}{\partial x_\mu \partial x_\mu} + m^2 \right) \phi_{inr}(x, x_5) = M^2 \left( \frac{i}{M} \frac{\partial}{\partial x_5} + \eta_{inr} \right) l_{inr}(x, x_5) = j_{inr}(x, x_5) \tag{3.7a}
\]

\[
\left( \frac{\partial^2}{\partial x_\mu \partial x_\mu} + m^2 \right) \phi_{ext}(x, x_5) = -M^2 \left( \frac{i}{M} \frac{\partial}{\partial x_5} + \eta_{ext} \right) l_{ext}(x, x_5) = j_{ext}(x, x_5), \tag{3.7b}
\]

where \( j_a(x, x_5) \) are determined via \( l_a(x, x_5) \).

Solutions of eq. (3.7a,b) determine the solutions of 4D equation (3.1a) with

\[
J(x) \equiv J(x, x_5 = t_5) = j_{inr}(x, x_5 = t_5) + j_{ext}(x, x_5 = t_5) \tag{3.8}
\]

On the other hand the boundary condition (3.4) can be presented in the integral form

\[
\phi_a(x, x_5) = e^{-iM(x_5 - t_5)} \left\{ \phi_a(x, x_5 = t_5) - \frac{1}{\eta_a} l_a(x, x_5 = t_5) [e^{2iM\eta_a(x_5 - t_5)} - 1] \right. \\
\left. - \frac{1}{2iM\eta_a} \int_{t_5}^{x_5} dz_5 j_a(x, z_5) e^{-iM\eta_a(z_5 - t_5)} [e^{2iM\eta_a(x_5 - t_5)} - e^{2iM\eta_a(z_5 - t_5)}] \right\}. \tag{3.9}
\]

For noninteracting particles, when \( l_a(x, x_5) = 0 \) and \( j_a(x, x_5) = 0 \), equations (3.7a,b), (3.3a,b) and (3.4) coincide with the similar equations and constraints from ref. [18, 19].

**Consistency condition for the 5D equation of motion (3.7a,b) and the boundary conditions (3.3a,b) and (3.4):**

Combining eq. (3.7a,b) and (3.6a,b) we find

\[
\left( \frac{\partial^2}{\partial x_\mu \partial x_\mu} - \frac{\partial^2}{\partial x_5 \partial x_5} + M^2 \right) j_{inr}(x, x_5) = 0, \tag{3.10a}
\]

\[
\left( \frac{\partial^2}{\partial x_\mu \partial x_\mu} + \frac{\partial^2}{\partial x_5 \partial x_5} + M^2 \right) j_{ext}(x, x_5) = 0. \tag{3.10b}
\]

According to this relation, 5D equations of motion (3.7a,b) are consistent with the boundary conditions (3.3a,b) and (3.4) if \( j_a(x, x_5) \) and \( l_a(x, x_5) \) are embedded in hyperboloid \( q^2 \pm q_5^2 = \pm M^2 \), i.e. in analogue to \( \phi_a(x, x_5) \) the operators \( j_a(x, x_5) \) and \( l_a(x, x_5) \) must satisfy the conditions

\[
\left( \frac{\partial^2}{\partial x_\mu \partial x_\mu} + \frac{\partial^2}{\partial x_5 \partial x_5} + M^2 \right) j_{inr}(x, x_5) = 0, \tag{3.10b}
\]

\[
\left( \frac{\partial^2}{\partial x_\mu \partial x_\mu} - \frac{\partial^2}{\partial x_5 \partial x_5} - M^2 \right) j_{ext}(x, x_5) = 0. \tag{3.10c}
\]

Using the conditions (3.3a,b) \( \phi_a(x, x_5) \) may be represented as
\[ \varphi_{\text{inr}}(x, x_5) = \frac{2M}{(2\pi)^4} \int d^5q e^{-iqx^5} \delta(q^2 + q_5^2 - M^2)[\theta(q^2)\theta(M^2 - q^2) + \theta(-q^2)\theta(-M^2 + q^2)] \]

\[ [e^{-iqx}\varphi^{(+)}_{\text{inr}}(q, q_5) + e^{iqx}\varphi^{(-)}_{\text{inr}}(q, q_5)], \quad (3.11a) \]

and

\[ \varphi_{\text{ext}}(x, x_5) = \frac{2M}{(2\pi)^4} \int d^5q e^{-iqx^5} \delta(q^2 - q_5^2 + M^2)[\theta(q^2)\theta(-M^2 + q^2) + \theta(-q^2)\theta(M^2 + q^2)] \]

\[ [e^{-iqx}\varphi^{(+)}_{\text{ext}}(q, q_5) + e^{iqx}\varphi^{(-)}_{\text{ext}}(q, q_5)]. \quad (3.11b) \]

From (3.10b,c) we get the same representation for source operator

\[ j_{\text{inr,ext}}(x, x_5) = \frac{2M}{(2\pi)^4} \int d^5q e^{-iqx^5} \delta(q^2 \pm q_5^2 \mp M^2)[\theta(q^2)\theta(\pm M^2 \mp q^2) + \theta(-q^2)\theta(\mp M^2 \pm q^2)] \]

\[ [e^{-iqx}j^{(+)}_{\text{inr,ext}}(q, q_5) + e^{iqx}j^{(-)}_{\text{inr,ext}}(q, q_5)]. \quad (3.12) \]

The same representation is valid for \( l_{\text{inr,ext}}(x, x_5) \).

Present formulation has a number of common properties with the other 5D field-theoretical approaches based on the proper time method [24, 26, 27, 28, 29], where \( x_5 \equiv \tau = x_5^0 - x^2 \equiv x, x^\mu \). From this point of view the boundary conditions (3.4) has a form of an evolution equation. But \( l_a \) as well as \( j_a \) are variation of Lagrangians over the interacted fields, i.e. eq.(3.4) can not be treated as an evolution equation. On the other hand the fifth momentum \( q_5 \) is singlevalued determined via the scale parameter \( \lambda \) (see table 2). Therefore \( x_5 \) may be provided with the a scale interpretation if we take \( \lambda^{-1} = ln(Mx_5) \). Unlike other 5D approaches [26, 30, 31, 32], in the present formulation field operators and the source operators are defined in the 5D space with the invariant forms (2.2a,b).

### 4. 5D Lagrangian approach

The 5D operators \( \varphi_{\text{inr}}(x, x_5) \) and \( \varphi_{\text{ext}}(x, x_5) \) are independent because they are defined in the different domains of \( q^2 \equiv q_\mu q^\mu \) and \( q_5^2 \) variables. Therefore the sought 5D Lagrangian \( \mathcal{L} \equiv \mathcal{L}(x, x_5) \) must be constructed using the two sets of the independent fields \( \varphi_{\text{inr}}(x, x_5) \) and \( \varphi_{\text{ext}}(x, x_5) \). A complete 5D Lagrangian has a form

\[ \mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{INT}} + \mathcal{L}_c, \quad (4.1a) \]

where \( \mathcal{L}_0 \) stands for the noninteracting part

\[ \mathcal{L}_0 = \sum_{a=1,2} \left[ \frac{\partial \varphi_a(x, x_5)}{\partial x_\mu} \frac{\partial \varphi^+_a(x, x_5)}{\partial x^\mu} - m^2 \varphi_a(x, x_5) \varphi^+_a(x, x_5) \right], \quad (4.1b) \]
\( \mathcal{L}_{\text{INT}} = \mathcal{L}_{\text{INT}}(\varphi_a, \phi_a^+, \partial \varphi_a / \partial x_\mu, \partial \phi_a^+ / \partial x^\mu; \partial \varphi_a / \partial x_5, \partial \phi_a^+ / \partial x^5) \)

is the interacting part of Lagrangian and \( \mathcal{L}_c \) generates the constraint (3.4)

\[
\mathcal{L}_c = M^2 \sum_{a=1,2} \left| \frac{i}{M} \frac{\partial \varphi_a}{\partial x_5} - \eta_a \varphi_a - l_a(x, x_5) \right|^2,
\]

(4.1c)

where \( l_a(x, x_5) \equiv l_a(\varphi_a, \phi_a^+, \partial \varphi_a / \partial x_\mu, \partial \phi_a^+ / \partial x^\mu; \partial \varphi_a / \partial x_5, \partial \phi_a^+ / \partial x^5) \) is in response to the interaction in the constraint (3.4).

Next we consider action

\[
\mathcal{S}(x_5) = \int d^4x \mathcal{L}(x, x_5)
\]

(4.2)

and its variation under the conformal transformations (1.1a)-(1.1e)

\[
\delta q_\mu = \delta h_\mu + \delta \Lambda_{\mu\nu} q^\nu + \delta \lambda q_\mu + (q^2 \delta \bar{h}_{\mu} - 2q^\nu \delta \bar{h}_{\nu} q_\mu) / M^2,
\]

(4.3a)

where \( \delta h_\mu(\delta h_\mu), \delta \Lambda_{\mu\nu} q^\nu = -\delta \Lambda_{\nu\mu} q^\nu \) \( \delta \lambda \) stands for the infinitesimal parameters of the corresponding transformations.

Translation \( q_\mu' = q_\mu + h_\mu \) does not change \( x_\mu \). Therefore the variation of coordinates \( \delta x_\mu = x'_\mu - x_\mu \), generated by variation of four momenta \( q_\mu \) (4.3), includes only the rotation and the scale transformations

\[
\delta x_\mu = \delta \Lambda_{\mu\nu} ^{-1} x^\nu - \delta \lambda x_\mu
\]

(4.3b)

In the considered formulation \( x_5 \) is independent variable. Therefore we take

\[
\frac{\delta x_5}{\delta x_\mu} = \frac{\delta dx_5}{\delta x_\mu} = 0
\]

(4.4)

which is consistent with our choice of action (4.2).

Now we have

\[
\delta \mathcal{S}(x_5) = \sum_{a=1,2} \left\{ \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \varphi_a^+(x, x_5)} - \frac{\partial}{\partial x_\mu} \left( \frac{\partial \mathcal{L}}{\partial (\varphi_a^+(x, x_5)/\partial x^\mu)} \right) \right] \delta \varphi_a^+(x, x_5)
\]

\[
+ \int d^4x \frac{d}{dx_\mu} \left[ \frac{\partial \mathcal{L}}{\partial (\varphi_a^+(x, x_5)/\partial x^\mu)} \right] \delta \varphi_a^+(x, x_5) + \mathcal{L}(x, x_5) \delta x^\mu \right\}
\]

\[
+ \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial (\varphi_a^+(x, x_5)/\partial x^5)} \delta \left( \frac{\partial}{\partial x_5} \varphi_a^+(x, x_5) \right) + \frac{d \mathcal{L}}{dx_5} \delta x^5 \right]
\]

\[
+ \text{hermitian conjugate}
\]

(4.5)

where \( \delta \) denotes a variation of form of the corresponding expression. Substituting \( d \mathcal{L} / dx_5 \) in (4.5) we obtain

\[
\delta \mathcal{S}(x_5) = \sum_{a=1,2} \left\{ \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial \varphi_a^+(x, x_5)} - \frac{\partial}{\partial x_\mu} \left( \frac{\partial \mathcal{L}}{\partial (\varphi_a^+(x, x_5)/\partial x^\mu)} \right) \right] \delta \varphi_a^+(x, x_5) + \frac{d \mathcal{L}}{dx_5} \delta x^5 \right\}
\]

(4.6a)
\[ + \int d^4x \frac{d}{dx_\mu} \left[ \frac{\partial L}{\partial \varphi^+_a(x, x_5)/\partial x^\mu} \delta \varphi^+_a(x, x_5) + L(x, x_5) \delta x^\mu \right] \quad (4.6b) \]

\[ + \int d^4x \frac{\partial L}{\partial [\partial \varphi^+_a(x, x_5)/\partial x^\mu]} \left[ \delta \left( \frac{\partial}{\partial x_5} \varphi^+_a(x, x_5) \right) + \frac{\partial^2 \varphi^+_a(x, x_5)}{\partial x_5^2} \delta x_5 \right] \quad (4.6c) \]

\[ + \int d^4x \frac{d}{dx_\mu} \left[ \frac{\partial L}{\partial \varphi^+_a(x, x_5)/\partial x_\mu} \right] \delta x_\mu \right] \quad (4.6d) \]

\[ + \text{hermitian conjugate} \]

In order to get \( \delta S(x_5) = 0 \) we shall suppose that every term of eq.(4.6) vanishes. Then for the every term separately we obtain the following equations:

1. The first term (4.6a) represents the equation of motion for \( \varphi^+_a(x, x_5) \) and \( \varphi^-_a(x, x_5) \):

\[
\frac{\partial L}{\partial \varphi^+_a(x, x_5)} = \frac{d}{dx_\mu} \left( \frac{\partial L}{\partial \varphi^+_a(x, x_5)/\partial x^\mu} \right); \quad \frac{\partial L}{\partial \varphi^-_a(x, x_5)} = \frac{d}{dx_\mu} \left( \frac{\partial L}{\partial \varphi^-_a(x, x_5)/\partial x^\mu} \right),
\]

or

\[
\left( \frac{\partial^2}{\partial x_\mu \partial x^\mu} + m_0^2 \right) \varphi_a(x, x_5) = \frac{\partial L_{INT}}{\partial \varphi^+_a(x, x_5)/\partial x_\mu} - \frac{d}{dx_\mu} \left( \frac{\partial L_{INT}}{\partial \varphi^-_a(x, x_5)/\partial x^\mu} \right) \equiv j_a(x, x_5)
\]

which coincides with (3.7a,b).

2. The next term (4.6b) relates to the 4D current conservation condition

\[
\mathcal{J}^\mu(x) = \sum_{a=1,2} \mathcal{J}_a^\mu(x, x_5 = t_5), \quad (4.11a)
\]

where

\[
\mathcal{J}_a^\mu(x, x_5) = \frac{\partial L}{\partial [\partial \varphi^+_a(x, x_5)/\partial x_\mu]} \delta \varphi^+_a(x, x_5) + L(x, x_5) \delta x^\mu
\]

\[ + \text{hermitian conjugate}. \quad (4.7b) \]

3. Third term (4.6c) contains \( \partial \varphi_a(x, x_5)/\partial x^5 \). This field may be treated as independent due to fifth degrees of freedom [18, 19]. Therefore we can introduce a new kind of fields

\[
\chi_a(x, x_5) = \frac{i}{M} \left( \frac{\partial \varphi_a(x, x_5)}{\partial x_5} \right).
\]

Using the variation principle and the independence of the fields \( \chi_a(x, x_5) \) we get

\[
\frac{\partial L}{\partial [\partial \varphi^+_a(x, x_5)/\partial x^5]} = \frac{\partial L}{\partial [\partial \varphi^-_a(x, x_5)/\partial x^5]} = \frac{\partial L_c}{\partial [\partial \varphi_a(x, x_5)/\partial x^5]} = 0 \quad (4.9)
\]

which implies
\[
\chi_a - \eta_a \varphi_a - l_a(x, x_5) = -\frac{1}{M^2} \frac{\partial L_{INT}}{\partial \chi^+_a(x, x_5)} + \frac{\partial l^+_a}{\partial \chi^+_a(x, x_5)}(\chi_a - \eta_a \varphi_a - l_a(x, x_5)) + \frac{\partial l_a}{\partial \chi^+_a(x, x_5)}(\chi^+_a - \eta_a \varphi^+_a - l_a(x, x_5)^+). (4.10)
\]

Afterwards we restrict our formulation with such \( L_{INT} \) which are independent on \( \partial \varphi_a/\partial x_5 \) and \( \partial \varphi^+/\partial x_5 \). Then instead of (4.10) we get

\[
\chi_a(x, x_5) - \eta_a \varphi_a(x, x_5) - l_a(x, x_5) = 0, \quad (4.11)
\]

which coincides with (3.4).

Combining (3.2a,b) and (4.11) we get the connections between \( l_a(x, x_5) \) and \( j_a(x, x_5) \)

\[
j_a(x, x_5) = (-1)^{a-1} M^2 \left( \frac{i}{M} \frac{\partial}{\partial x_5} + \eta_a \right) l_a(x, x_5) \quad (4.12)
\]

which was presented in eq. (3.7a,b).

4. The fourth term (4.6d) contains the current operator

\[
J^\mu_a(x, x_5) = \frac{\partial \mathcal{L}}{\partial [\partial \varphi_a^+(x, x_5)/\partial x^\mu]} \frac{\partial \varphi_a^+(x, x_5)}{\partial x_5} \delta x_5 + \text{hermitian conjugate} \quad (4.13).
\]

in the asymptotic region, where \( j_a = 0 \) and \( l_a = 0 \), \( J^\mu_a(x, x_5) \) has the same form as the electro-magnetic current operator

\[
J^\mu_a(x, x_5) = -i \eta_a \delta x_5 \left\{ \frac{\partial \mathcal{L}_0}{\partial [\partial \varphi_a(x, x_5)/\partial x^\mu]} \varphi_a(x, x_5) - \frac{\partial \mathcal{L}_0}{\partial [\partial \varphi_a^+(x, x_5)/\partial x^\mu]} \varphi_a^+(x, x_5) \right\} \quad (4.14)
\]

and \( d/dx^\mu J^\mu_a(x, x_5) = 0 \). For the interacting fields expression (4.13) vanishes if we require that \( \delta x_5 = 0 \).

Thus we have derived the equation of motion (3.7a,b), constraint (3.4) and the expression for the conserved currents (4.7b) and (4.13) using the variation principle. Combining these equations one can verify, that \( dS(x_5)/dx_5 = 0 \), i.e. \( S(x_5) \) (4.2) is not dependent on \( x_5 \). In particular, using the 5D equations of motion (4.7a,b) we get \( dS(x_5)/dx_5 = \int d^4x d\mu \left\{ \partial \mathcal{L}/\partial \varphi_a^+(x, x_5)/\partial x^\mu] \varphi_a^+(x, x_5)/\partial x_5 + \ldots \right\} \) which vanishes according to fifth current conservation condition (4.13).
5. Construction interaction part of Lagrangian $\mathcal{L}_{\text{INT}}$ via constrain (3.4).

In present formulation all field operators in equation of motion (3.7a,b), in constrain (3.4) and in Lagrangian (4.1a,b,c) are defined on the surface $q^2 \pm q_5^2 = \pm M^2$. But the product of $\varphi_a(x, x_5)$ is not on $q^2 \pm q_5^2 = \pm M^2$ shell. Therefore in this section we consider off $q^2 \pm q_5^2 = \pm M^2$ shell representations of equation of motion (3.7a,b) with corresponding off shell constrain (3.4) and off shell Lagrangian. Afterwards using the simple projection procedure, we put these Lagrangians, equation of motions with related constrains on $q^2 \pm q_5^2 = \pm M^2$ surface. We introduce off $q^2 \pm q_5^2 = M^2$ shell operator $\tilde{\varphi}_a(x, x_5)$ as follows

$$
\varphi_a(x, x_5) = \int d^5 y \tilde{\varphi}_a(x - y, x_5 - y_5) D_a(y, y_5),
$$

(5.1)

where

$$
\varphi_{a=1}(x, x_5) = \frac{2M}{(2\pi)^4} \int d^5 q e^{-iqx^5} \left[ \theta(q^2) \theta(M^2 - q^2) + \theta(-q^2) \theta(-M^2 - q^2) \right]
\left[ e^{-iqx} \varphi_{1}^{(+)}(q, q_5) + e^{iqx} \varphi_{1}^{(-)}(q, q_5) \right],
$$

(5.2a)

and

$$
\varphi_{a=2}(x, x_5) = \frac{2M}{(2\pi)^4} \int d^5 q e^{-iqx^5} \left[ \theta(q^2) \theta(-M^2 + q^2) + \theta(-q^2) \theta(M^2 + q^2) \right]
\left[ e^{-iqx} \varphi_{2}^{(+)}(q, q_5) + e^{iqx} \varphi_{2}^{(-)}(q, q_5) \right].
$$

(5.2b)

and

$$
D_a(x, x_5) = \frac{1}{(2\pi)^5} \int d^5 q e^{-iqx^5} \delta(q^2 \pm q_5^2 = M^2).
$$

(5.3)

Substituting (5.2a,b) and (5.3) into (5.1) we obtain expressions (3.11a,b) after Fourier transforms.

In the same manner as for $\tilde{\varphi}_a(x, x_5)$ (5.1) we introduce

$$
\tilde{j}_a(x, x_5) = \int d^5 y \tilde{j}_a(x - y, x_5 - y_5) D_a(y, y_5),
$$

(5.4)

$$
\tilde{l}_a(x, x_5) = \int d^5 y \tilde{l}_a(x - y, x_5 - y_5) D_a(y, y_5),
$$

(5.5)

$$
\tilde{\mathcal{L}}_a(x, x_5) = \int d^5 y \tilde{\mathcal{L}}_a(x - y, x_5 - y_5) D_a(y, y_5),
$$

(5.6)

Then from the equations of motion

$$
\left[ \frac{\partial^2}{\partial x_\mu \partial x^\mu} + m^2 \right] \tilde{\varphi}_a(x - y, x_5 - y_5) = \tilde{j}_a(x - y, x_5 - y_5),
$$

(5.7)

we obtain the equations of motion (3.7a,b) using integration over $y, y_5$ variables according to (5.1) and (5.4).
In the same way as eq.(4.7a,b) with the constraint (4.11) we can derive equation of motion (5.7) and the off shell constraint for the off \( q^2 \pm q_5^2 \mp M^2 \) shell Lagrangian

\[
\tilde{\mathcal{L}}_a = (\tilde{\mathcal{L}}_a)_0 + (\tilde{\mathcal{L}}_a)_{INT} + (\tilde{\mathcal{L}}_a)_c,
\]

where \((\tilde{\mathcal{L}}_a)_0\) stands for the noninteracting part, \((\tilde{\mathcal{L}}_a)_{INT}\) is the interaction part and \((\tilde{\mathcal{L}}_a)_c\) generate the constraint

\[
\tilde{\chi}_a(x,x_5) - \eta_a \tilde{\varphi}_a(x,x_5) - \tilde{l}_a(x,x_5) = 0
\]

and have the form

\[
(\tilde{\mathcal{L}})_c = M^2 \sum_{a=1,2} \frac{i}{M} \frac{\partial \tilde{\varphi}_a}{\partial x_5} - \eta_a \tilde{\varphi}_a - \tilde{l}_a(x,x_5)^2.
\]

Thus relations (5.1), (5.4), (5.5) and (5.6) enables us to obtain the straightforward off shell representations of eq.(3.7a,b) and the constraint (3.4) using the similar as (4.1a,b,c) Lagrangian but with off \( q^2 \pm q_5^2 \mp M^2 \) shell operators \( \tilde{\varphi}_a \). Now we consider some example of \( \tilde{l}_a(x,x_5) \) and the corresponding interaction Lagrangians:

**\( \varphi^4 \) model:** The simplest \( \tilde{l}_a \) which is not dependent on \( \tilde{\chi}_a(x,x_5) \equiv i/M \partial \tilde{\varphi}_a(x,x_5)/\partial x_5 \) is

\[
\tilde{l}_a = g_a \tilde{\varphi}_a^2.
\]

Using the constraint (5.9a) we get

\[
\frac{i}{M} \frac{\partial \tilde{\varphi}_a(x,x_5)}{\partial x_5} = \eta_a \tilde{\varphi}_a(x,x_5) + g_a \tilde{\varphi}_a^2(x,x_5).
\]

\[
\tilde{j}_a(x,x_5) = \partial \tilde{\mathcal{L}}_{INT}/\partial \tilde{\varphi}_a^+(x,x_5) = (-1)^{a-1} M^2 \left( \frac{i}{M} \frac{\partial \tilde{\varphi}_a}{\partial x_5} + \eta_a \right) \tilde{l}_a(x,x_5)
\]

\[
= (-1)^{a-1} M^2 \left( 3g_a \eta_a \tilde{\varphi}_a^2(x,x_5) + 2g_a^2 \tilde{\varphi}_a^3(x,x_5) \right).
\]

The corresponding equation of motion can be derived using the following Lagrangians

\[
\mathcal{L} = \frac{1}{2} \sum_{a=1,2} \left[ \frac{\partial \tilde{\varphi}_a}{\partial x_\mu} - m_a \tilde{\varphi}_a \right]^2 + M^2 \sum_{a=1,2} \left| \tilde{\chi}_a - \eta_a \tilde{\varphi}_a - g_a \tilde{\varphi}_a^2 \right|^2 + \tilde{\mathcal{L}}_{INT},
\]

where

\[
(\tilde{\mathcal{L}}_a)_{INT}(x,x_5) = (-1)^{a-1} M^2 \left( g_a \eta_a \tilde{\varphi}_a^3 + \frac{g_a^2}{2} \tilde{\varphi}_a^4 \right)
\]

Lagrangian (5.13) has the following attractive properties

I. The considered model is renormalizable, because \( \tilde{\mathcal{L}}_{INT} \) and \( \mathcal{L} \) contains \( \tilde{\varphi}_a \) in the third and in the fourth power.
II. $(\tilde{L}_{\text{inv}})_{\text{INT}} (a = 1)$ and $(\tilde{L}_{\text{ext}})_{\text{INT}} (a = 2)$ have the opposite sign.

III. The Lagrangian (5.14) has a local minimum at $-2\eta_a/g_a$ and a local maximum at $-\frac{2}{3}\eta_a/g_a$.

Nonlinear $\sigma$ model: Here we have $\pi$-meson fields $\pi^{\pm}, \pi^0$ instead of $\varphi$. We choose $l_a$ depending on the auxiliary fields $\chi$

$$\tilde{l}_a^\alpha = \frac{1}{4f_\pi^2}(\tilde{\chi}_a^\gamma \tilde{\chi}_a^\gamma)\tilde{\pi}_a^\alpha \equiv \frac{1}{4f_\pi^2}\tilde{\chi}_a^\gamma \tilde{\pi}_a^\alpha,$$  \hspace{1cm} (5.15)

where we have used the well known isospin redefinition of the $\pi$-meson fields $\pi^{\pm} \equiv 1/2(\pi^1 \pm i\pi^2)$; $\pi^0 \equiv \pi^3$; $\alpha, \beta, \gamma = 1, 2, 3$, $f_\pi = 93\text{MeV}$ is the $\pi$-meson decay constant and Lagrangian is choosing in the form

$$\tilde{\mathcal{L}} = \sum_{a=1,2} \left( \frac{1}{2} \frac{\partial}{\partial x_\mu} \tilde{\pi}_a^\alpha \frac{\partial}{\partial x_\mu} \tilde{\pi}_a^\alpha + M^2 \left[ \tilde{\chi}_a^\alpha - \tilde{\pi}_a^\alpha - \frac{1}{4f_\pi^2} \tilde{\chi}_a^\gamma \tilde{\pi}_a^\gamma \right]^2 \right) + \tilde{\mathcal{L}}_{\text{chir}} + \tilde{\mathcal{L}}_{\text{INT}},$$  \hspace{1cm} (5.16)

where the second term generates the constraint between the auxiliary field $\tilde{\chi}_a^\alpha(x, x_5) = i/M\partial \tilde{\pi}_a^\alpha(x, x_5)/\partial x_5$ and the $\pi$ meson field $\tilde{\pi}_a^\alpha$

$$\tilde{\chi}_a^\alpha - \tilde{\pi}_a^\alpha - \frac{1}{4f_\pi^2} \tilde{\chi}_a^\gamma \tilde{\pi}_a^\gamma = 0.$$  \hspace{1cm} (5.17a)

This constraint coincides with the relation between the $\pi$ meson field and the interpolating field in the nonlinear $\sigma$-model [34, 7]

$$\tilde{\pi}_a^\alpha = \frac{1}{1 + \frac{\chi_a^\alpha}{4f_\pi^2}}.$$  \hspace{1cm} (5.17b)

Third term of (5.16) $\tilde{\mathcal{L}}_{\text{chir}}$ reproduces the constraint between $\pi$-meson fields and the auxiliary $\sigma$-meson fields

$$\tilde{\pi}_a^2 + \tilde{\sigma}_a^2 = f_\pi^2$$  \hspace{1cm} (5.18)

and correspondingly

$$(\tilde{\mathcal{L}}_a)_{\text{chir}}(x, x_5) = \left( \tilde{\pi}_a^2 + \tilde{\sigma}_a^2 - f_\pi^2 \right)^2.$$  \hspace{1cm} (5.19)

In the usual $\sigma$ model the chiral symmetry is weakly broken with the additional Lagrangian $\mathcal{L}' = -f_\pi m^2 \sigma$. In the considered model the chiral symmetry breaking terms arise in $[\tilde{\mathcal{L}}_a]_{\text{INT}}$ as the result of connection between $\tilde{l}_a$ (5.15) with $j_a$ and $[\tilde{\mathcal{L}}_a]_{\text{INT}}$. Thus the source operator $\tilde{j}_a^\alpha$ is defined via operator (5.15) as

$$\tilde{j}_a^\alpha(x, x_5) = \partial(\tilde{\mathcal{L}}_a)_{\text{INT}}/\partial \tilde{\pi}_a^\alpha(x, x_5) = (-1)^{a-1}M^2 \left( \frac{i}{M} \frac{\partial}{\partial x_5} + 1 \right) \tilde{l}_a^\alpha(x, x_5)$$
\[ (-1)^{a-1} M^2 \left( \frac{f_\pi + \tilde{\sigma}_a}{\tilde{\sigma}_a} \right) \left[ 1 + f_\pi \left( \frac{3f_\pi - \tilde{\sigma}_a}{f_\pi - \tilde{\sigma}_a} \right)^2 \right] \tilde{\pi}^a. \] (5.20)

The corresponding Lagrangian is

\[ \tilde{\mathcal{L}}_{\text{INT}} = - \sum_{a=1,2} (-1)^{a-1} M^2 \left( f_\pi \tilde{\sigma}_a + \frac{1}{2} \tilde{\sigma}_a^2 + f_\pi \frac{(f_\pi + \tilde{\sigma}_a)^2}{(f_\pi - \tilde{\sigma}_a)} \right) \] (5.21a)

which after expansion in \( \pi_a^2 \) power series takes the form

\[ \tilde{\mathcal{L}}_{\text{INT}} = - \sum_{a=1,2} (-1)^{a-1} M^2 \left( -\frac{9}{2} f_\pi^2 + 8 \frac{f_\pi^4}{\pi_a^2} - \tilde{\pi}_a^2 - \frac{1}{4} f_\pi^2 \tilde{\pi}_a^4 - \ldots \right) \] (5.21b)

Afterwards \( \tilde{\mathcal{L}}_{\text{INT}} \) obtains the real \( \pi \) meson mass term for the \( \text{ext} = a \equiv 2 \) if \( M \) is fixed as

\[ M = \frac{m_\pi}{\sqrt{2}}. \] (5.22)

and for the internal \( \pi \) meson field \( \pi_{a=1} \) in (5.21b) appear only negative \( m_\pi^2 \), i.e. \( \pi_{a=1} \) remains to be massless.

Thus the simple form of \( \tilde{\ell}_a \) (5.15) and constraint (5.17b) allows us to interpret \( \tilde{\chi} = i/M \partial / \partial x_5 \tilde{\pi} \) as the auxiliary interpolating pion field according to the nonlinear \( \sigma \)-model [34, 7]. From the other hand the same \( \tilde{\ell}_a \) (5.15) determines the source operator (5.20) and corresponding Lagrangian (5.21a) which consists from the usual chiral symmetry breaking term of \( \sigma \) models \(-m_\pi^2 f_\pi \tilde{\sigma}_a \) and other terms which break chiral symmetry more strongly. The 4D \( \pi \)-meson field operator \( \tilde{\pi}^a(x) = \tilde{\pi}^a_{a=1}(x, x_5 = t_5) + \tilde{\pi}^a_{a=2}(x, x_5 = t_5); \) \((t_5 = 0 \ or \ \sqrt{x_0^2 - x^2}) \) is similar to the pion field of the nonlinear \( \sigma \) model in the region \( q^2 > M^2 = m_\pi^2/2, \) where \( \tilde{\pi}^a(x) = \tilde{\pi}^a_{a=2}(x, x_5 = t_5). \) In the region \( 0 < q^2 < M^2 = m_\pi^2/2, \) where \( \tilde{\pi}^a(x) = \tilde{\pi}^a_{a=1}(x, x_5 = t_5), \) \((\tilde{\mathcal{L}}_{a=1})_{\text{INT}} \) has the opposite to \((\tilde{\mathcal{L}}_{a=2})_{\text{INT}} \) sign. In the limit \( m_\pi \to 0 \) (i. e. \( M^2 \to 0, \)) the above Lagrangian transforms into the free Lagrangian for the massless pion. Note that the chiral symmetry breaking mechanism allowed us to fix the scale parameter \( M \) of the conformal transformation group.

The considered 5D nonlinear \( \sigma \) model as well as the ordinary 4D nonlinear \( \sigma \) model is unrenormalizable. In order to make this model renormalizable we must introduce the auxiliary scalar \( \sigma \) field according to the constraint (5.18) and the other auxiliary field which compensates the \( 1/\pi^2 \) nonlinearity of the last term in (5.21a). On the other hand inversion procedure transforms the internal region with \( q^2 < M^2 \) and the corresponding inversion-invariant(anti-invariant) field operators (1.25a,b) into external region \( q^2 > M^2 \) with the same operators (1.25a,b). Therefore we might hope that this approach will lead to specific regularization in the ultraviolet region, where \( q^2 >> M^2. \)
6. Models with the gauge transformations

**Gauge transformation in the 4D and 5D coordinate space.**

Here we shall treat gauge transformations \( q'_\mu = q_\mu - eA_\mu(q) \) as the special form of 4D momentum translation which can be performed in the same manner as the usual 4D translation \( q'_\mu = q_\mu + h_\mu \) (1.1a) in the 6D space with the invariant form \( \kappa_A\kappa^A = 0 \). In analogue to eq.(2.5), translations \( q'_\mu = q_\mu - eA_\mu(q) \) imply the following transformations of the 6D variables

\[
\kappa'_\mu = \kappa_\mu - ea_\mu(\kappa_A)\kappa_+ \tag{6.1a}
\]

with only four auxiliary fields \( a_\mu(\kappa_A) \) and

\[
\kappa'_+ = \kappa_+; \quad \kappa'_- = \kappa_- - e/M^2(a_\nu(\kappa_A)\kappa^\nu + \kappa^\nu a_\nu(\kappa_A)) + e^2/M^2\kappa_+a_\nu(\kappa_A)a^\nu(\kappa_A) \tag{6.1b}
\]

Afterwards we get

\[
q^{2'} = q^2 - e(A_{a\nu}(q, q_5)q^\nu + q^\nu A_{a\nu}(q, q_5)) + e^2A_{a\nu}(q, q_5)A^\nu_a(q, q_5) \tag{6.1c}
\]

and

\[
q_5^{2'} = q_5^2 \mp e(A_{a\nu}(q, q_5)q^\nu + q^\nu A_{a\nu}(q, q_5)) + e^2A_{a\nu}(q, q_5)A^\nu_a(q, q_5), \tag{6.1d}
\]

where the sign \( - \) corresponds to \( q^2 + q_5^2 = M^2 \) and the sign \( + \) relates to \( q^2 - q_5^2 = -M^2 \). \( A_{a\nu}(q, q_5) \) is constructed by \( a_\nu(\kappa_A) \) in the same way as in eq.(2.14a,b). As a result of these gauge transformations we get \( q^{2'} \pm q_5^{2'} = q^2 \pm q_5^2 \).

We can derive equations of motions for the off \( q^2 \pm q_5^2 = \pm M^2 \) shell operator \( \tilde{\varphi}_a(x, x_5) \) using gauge transformations \( \partial/\partial x'_\mu = \partial/\partial x_\mu + ieA_{a\mu}(x, x_5) \) in the Klein-Gordon equations . \( (\partial^2/\partial x_\mu\partial x^\mu + m^2)\tilde{\varphi}_a(x, x_5) = 0 \). In particular we get

\[
\left( \frac{\partial^2}{\partial x_\mu\partial x^\mu} + m^2 \right)\tilde{\varphi}_a(x, x_5) = \tilde{j}_a(x, x_5), \tag{6.2a}
\]

where

\[
\tilde{j}_a(x, x_5) = (ie\frac{\partial}{\partial x_\mu}\tilde{A}_{a\mu}(x, x_5) + ie\tilde{A}_{a\mu}(x, x_5)\frac{\partial}{\partial x_\mu}) - e^2\tilde{A}_{a\mu}(x, x_5)\tilde{A}^\mu_a(x, x_5)\tilde{\varphi}_a(x, x_5). \tag{6.2b}
\]

The on \( q^2 \pm q_5^2 = \pm M^2 \) shell operators \( \varphi_a(x, x_5) \) and \( j_a(x, x_5) \) are determined via \( \tilde{\varphi}_a(x, x_5) \) and \( \tilde{j}_a(x, x_5) \) according to (5.1) and (5.4). Next we can find \( \tilde{l}_a(x, x_5) \) via \( \tilde{j}_a(x, x_5) \) as

\[
\tilde{l}_a(q, q_5) = (-1)^{a-1} \frac{\tilde{j}_a(q, q_5)}{M(q_5 + M\eta_a)} \tag{6.3a}
\]

\[\text{In other 5D and 6D formulations with gauge transformations usually were introduced five} \left(A_{a}(x_\mu, x_5), A_5(x_\mu, x_5)\right) \text{or six} \left(a_\mu(\xi_\mu, \xi_5), \xi_6\right) \text{auxiliary gauge fields [2, 26, 19]. Such type gauge transformations violates invariance of 6D} \left(\kappa_A\kappa^A = 0\right) \text{or 5D} \left(q^2 \pm q_5^2 = \pm M^2\right) \text{forms, i.e. these gauge transformations destroy the necessary condition of the conformal transformations.} \]
and reproduce the corresponding constraint

\[ \frac{i}{M} \frac{\partial}{\partial x_5} \tilde{\varphi}_a(x, x_5) = \eta_a \tilde{\varphi}_a(x, x_5) + \tilde{t}_a(x, x_5). \] (6.3b)

In addition we can build the off \( q^2 \pm q_5^2 = \pm M^2 \) shell Lagrangian from the source operator (6.2b)

\[ \mathcal{L} = \sum_{a=1,2} \left[ \frac{\partial \tilde{\varphi}_a(x, x_5)}{\partial x_{5\mu}} \frac{\partial \tilde{\varphi}_a^+(x, x_5)}{\partial x^\mu} - m^2 \tilde{\varphi}_a(x, x_5)\tilde{\varphi}_a^+(x, x_5) + M^2 \left| \frac{i}{M} \frac{\partial \tilde{\varphi}_a}{\partial x_5} - \eta_a \tilde{\varphi}_a - \tilde{t}_a(x, x_5) \right|^2 \right] + \sum_{a=1,2} \left[ -ie \tilde{\varphi}_a(x, x_5) \frac{\partial}{\partial x_{5\mu}} \tilde{\varphi}_a^+(x, x_5)\tilde{A}_a^\mu(x, x_5) + e^2 \tilde{A}_{a\mu}(x, x_5) \tilde{A}_a^\mu(x, x_5) \tilde{\varphi}_a(x, x_5)\tilde{\varphi}_a^+(x, x_5) \right]. \] (6.4)

Finally we shall build 4D equation of motion

\[ \left( \frac{\partial^2}{\partial x_{\mu} \partial x^\mu} + m^2 \right) \Phi(x) = J(x), \] (6.5a)

where \( \Phi(x) \) and \( J(x) \) consists from the two parts

\[ J(x) = j_{a=1}(x, t_5) + j_{a=2}(x, t_5), \quad \Phi(x) = \phi_{a=1}(x, t_5) + \phi_{a=2}(x, t_5) \] (6.5b)

with arbitrary \( M \) and \( t_5 = \tau = \sqrt{x_5^2 - x^2} \) or \( t_5 = 0 \).

It is important to note, that source operator (6.5b) in 4D equation (6.5a) differs from the ordinary source operator in quantum electrodynamical \( J(x) = (ie\partial/\partial x_\mu A_{\mu}(x) + iA_{\mu}(x)\partial/\partial x_\mu - e^2 A_{\mu}(x)A^\mu(x)) \Phi(x) \) which is generated by the usual 4D gauge transformation \( q_\mu = q_\mu - eA_\mu(q) \). In the present 5D formulation (6.4) Lagrangian and the corresponding source operators are divided into two sets of fields which are defined on the different 5D hyperboloids \( q^2 \pm q_5^2 = \pm M^2 \) (see table 1). But in the limit \( M \rightarrow 0 \) or \( M \rightarrow \infty \) one of the sets of the fields disappear and we obtain the conventional QED formulation.

In the same way we can construct the equation of motion for the fermion field operator

\[ \left( i\gamma_\mu \frac{\partial}{\partial x_\mu} - m_F \right) \Psi(x) = J(x) \] (6.6a)

using the appropriate 5D equation

\[ \left( i\gamma_\mu \frac{\partial}{\partial x_\mu} - m_F \right) \tilde{\psi}_a(x, x_5) = \tilde{j}_a(x, x_5) = e\gamma_\mu \tilde{A}_a^\mu(x, x_5) \tilde{\psi}_a(x, x_5), \] (6.6b)

where \( J(x) = j_{a=1}(x, t_5) + j_{a=2}(x, t_5) \) and \( \Psi(x) = \psi_{a=1}(x, t_5) + \psi_{a=2}(x, t_5) \). Here \( M \) is not fixed and \( \tilde{\psi}_a(x, x_5) \) satisfies the constraint \( \left( i/M \partial/\partial x_5 - \eta_a \right) \tilde{\psi}_a(x, x_5) = \tilde{t}_a(x, x_5) \) and \( \tilde{j}_a(x, x_5) = (-1)^a M^2 \left( i/M \partial/\partial x_5 + \eta_a \right) \tilde{t}_a(x, x_5) \).
The geometrical realization of the scalar electrodynamics with conformal and gauge transformations in the curved 7D configuration space was given by Chang and Chodos [25]. This generalization of the Kaluza-Klein model provides a specific dynamical mechanism for the breakdown of conformal symmetry and mass generation. The resulting equations of motion are also defined in the projective five-dimensional space and afterwards the corresponding 5D equations were reduced into usual 4D equation of motion for a charged scalar fields. But the translation and inversion procedure in the momentum and in the coordinate space are independent for the quantum fields. Therefore one can extend Chang and Chodos model using the additional mechanisms of the mass generation and the conformal symmetry breaking coming from the considered conformal group of transformations of quantum fields in the momentum space. The goal of this extension is to fix the scale parameter $M$ and to try to reproduce the observed mass spectrum of scalar or pseudo-scalar particles. But this investigation is out of the scope of present paper.

**Gauge SU(2) × U(1) theory** can by formulated in the 5D form with off $q^2 ± q_s^2 = ±M^2$ shell Lagrangian following notations in [35]

$$\tilde{L}(x, x_5) = \sum_{a=1,2} (\tilde{L}_a)V(x, x_5) + (\tilde{L}_a)_{sk}(x, x_5) + (\tilde{L}_a)F(x, x_5),$$  \hspace{1cm} (6.7)

where $(\tilde{L}_a)_V$ denotes the vector part of Lagrangian with Yang-Mills fields $(\tilde{A}_a)_{\mu}(x, x_5)$ and Abelian fields $(\tilde{B}_a)_{\mu}(x, x_5)$

$$ (\tilde{L}_a)_V = -\frac{1}{4}(F_a)_{\mu\nu}(F_a)^{\mu\nu} - \frac{1}{4}(G_a)_{\mu\nu}(G_a)^{\mu\nu}$$

$$-M^2|\frac{i}{M}\frac{\partial}{\partial x_5}(\tilde{A}_a)_{\mu} - (\tilde{A}_a)^{\alpha}_{\mu} - (\tilde{l}_a^{A})_{\mu}|^2 - M^2|\frac{i}{M}\frac{\partial}{\partial x_5}(\tilde{B}_a)_{\mu} - (\tilde{l}_a^{B})_{\mu}|^2,\hspace{1cm} (6.8a)$$

where

$$ (F_a)_{\mu\nu} = \frac{\partial}{\partial x^\mu}(\tilde{A}_a)^{\alpha}_{\nu} - \frac{\partial}{\partial x^\nu}(\tilde{A}_a)^{\alpha}_{\mu} + g\varepsilon^{\alpha\beta\gamma}(\tilde{A}_a)^{\beta}_{\mu}(\tilde{A}_a)^{\gamma}_{\nu},\hspace{1cm} (6.8b)$$

$$ (G_a)_{\mu\nu} = \frac{\partial}{\partial x^\mu}(\tilde{B}_a)_{\nu} - \frac{\partial}{\partial x^\nu}(\tilde{B}_a)_{\mu}.\hspace{1cm} (6.8c)$$

We can determine $(\tilde{l}_a^{A})_{\mu}, (\tilde{l}_a^{B})_{\mu}$ via $(\tilde{j}_a^{A})_{\mu}, (\tilde{j}_a^{B})_{\mu}$ in the same way as in eq. (6.3a). The interacting parts of Lagrangian with the gauge fields $(\tilde{B}_a)_{\mu}$ and $(\tilde{A}_a)^{\alpha}_{\mu}$ are included in the fermion and in the scalar Lagrangians $(\tilde{L}_a)_{F}$ and $(\tilde{L}_a)_{sk}$.

The scalar part of Lagrangian (6.7) is

$$ (\tilde{L}_a)_{sk} = (\tilde{D}_a^{\mu}\tilde{F}_a)^\dagger (\tilde{D}_a^{\mu}\tilde{F}_a) - M^2|\frac{i}{M}\frac{\partial}{\partial x_5}(\tilde{F}_a) - (\tilde{F}_a) - (\tilde{l}_a^{B})_{\mu}|^2 + (\tilde{L}_a)_{INT},\hspace{1cm} (6.9a)$$

where $(\tilde{L}_a)_{INT}$ contains the self-interaction term of the scalar particle and

$$\tilde{D}_a^{\mu}\tilde{F}_a = (\frac{\partial}{\partial x^\mu} - ig\frac{\tau^a}{2}(\tilde{A}_a)_{\mu} - ig'(\tilde{B}_a)_{\mu})\tilde{F}_a.\hspace{1cm} (6.9b)$$
Unlike the standard $SU(2) \times U(1)$ theory, we start from the massless $\Phi_a(x,x_5)$ field and the Higgs mechanism we shall reproduce using

$$\tilde{l}_a(x,x_5) = -\frac{f}{M} \Phi_a(x,x_5) \left( \tilde{\Phi}_a^*(x,x_5) \Phi_a(x,x_5) \right)^{1/2}. \quad (6.10)$$

In the unitary gauge

$$\tilde{\Phi}_a(x,x_5) = \mathcal{U} \left( \tilde{\zeta}_a(x,x_5) \left( \begin{array}{c} 0 \\ \tilde{\phi}_a(x,x_5) \end{array} \right) \right), \quad (6.11a)$$

where

$$\mathcal{U} \left( \tilde{\zeta}_a(x,x_5) \right) = \exp \left( i \tilde{\zeta}_a(x,x_5)/v \right). \quad (6.11b)$$

We shall assume, that $\tilde{\zeta}_a$ is independent on $x_5$

$$\frac{\partial}{\partial x_5} \tilde{\zeta}_a(x,x_5) = 0. \quad (6.12)$$

Afterwards $\tilde{l}_a$ (6.10) takes a form

$$\tilde{l}_a(x,x_5) = -\frac{f}{M} \mathcal{U} \left( \tilde{\zeta}_a(x,x_5) \left( \begin{array}{c} 0 \\ \tilde{\phi}_a(x,x_5) \end{array} \right) \right) \sqrt{\left( \tilde{\phi}_a(x,x_5) \right)^2}, \quad (6.13)$$

and for $i/M \partial/\partial x_5 \tilde{\phi}_a$ we get

$$i \frac{\partial}{M \partial x_5} (\tilde{\phi}_a) = \tilde{\phi}_a - \frac{f}{M} \tilde{\phi}_a \sqrt{\left( \tilde{\phi}_a \right)^2}. \quad (6.14)$$

Next for $\tilde{j}_a$ and for $(\tilde{\mathcal{L}}_a)_{INT}$ we have

$$\tilde{j}_a = (-1)^{a-1} \mathcal{U}(\tilde{\zeta}_a) M^2 \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \left( -3fM\tilde{\phi}_a \sqrt{\left( \tilde{\phi}_a \right)^2} + 2f^2\tilde{\phi}_a^3 \right) \quad (6.15)$$

and

$$(\tilde{\mathcal{L}}_a)_{INT} = (-1)^{a-1} \left( -fM\tilde{\phi}_a^2 \sqrt{\left( \tilde{\phi}_a \right)^2} + \frac{f^2}{2} \tilde{\phi}_a^4 \right). \quad (6.16)$$

For the positive $f$ Lagrangian $\tilde{\mathcal{L}}_{a=1=\text{inr}}$ is similar to the self-interaction potential

$$V(\Phi) = -\mu^2 \Phi^2 + \lambda \Phi^4. \quad (6.17)$$

In particular $\tilde{\mathcal{L}}_1$ has zero at $\tilde{\phi}_1 = 0$ and $\tilde{\phi}_1 = \pm 2M/f$ and $\tilde{\mathcal{L}}_1$ has minima at $\tilde{\phi}_1 = \pm 3M/2f$.

It is important to note, that $[\tilde{\mathcal{L}}_{a=2}]_{INT} = -[\tilde{\mathcal{L}}_{a=1}]_{INT}$. Therefore in the mass term of Lagrangian the real positive $m^2$ can arise only in $\tilde{\mathcal{L}}_1$. In order to taken into account the spontaneous symmetry breaking mechanism we introduce
< 0|φ_a(x,x_5)|0 > = \frac{1}{2} \begin{pmatrix} 0 \\ v_a \end{pmatrix}, \quad (6.18)

where v_{a=1} = v and v_{a=2} = 0. Afterwards we get

\tilde{\phi}_1(x,x_5) = \frac{1}{2} \begin{pmatrix} 0 \\ v + \tilde{\phi}_1'(x,x_5) \end{pmatrix}; \quad \tilde{\phi}_2(x,x_5) = \frac{1}{2} \begin{pmatrix} 0 \\ \tilde{\phi}_2'(x,x_5) \end{pmatrix}. \quad (6.19)

Therefore Lagrangian (6.16) takes the form

(\tilde{\mathcal{L}}_a)_{INT} = (-1)^{a-1} \left( -\frac{fM}{4}(\tilde{\phi}_a + v_a)^2\sqrt{(\tilde{\phi}_a' + v_a)^2} + \frac{f^2}{16}(\tilde{\phi}_a' + v_a)^4 \right). \quad (6.20)

Thus instead of the mass term in the usual 4D self interacting potential (6.17) (see for instance ch. 8 and 11 of [35]) in the 5D Lagrangian (6.20) arise the following mass terms

\mu^2 \left( \frac{1}{\sqrt{2}}(\Phi' + v) \right)^2 \Longrightarrow \frac{fM}{4}(\tilde{\phi}_1 + v)^2\sqrt{(\tilde{\phi}_1' + v)^2} - \frac{fM}{4}(\tilde{\phi}_2')^2\sqrt{(\tilde{\phi}_2')^2}, \quad (6.21)

which determine the mass spectrum. \quad (6.21)

The effective Lagrangian (6.21) has minima at \tilde{\phi}_1 = \pm 3M/f. Therefore \tilde{\mathcal{L}}_{a=1} does not contain the linear terms in \tilde{\phi}_1, i.e. \tilde{\mathcal{L}}_1 does not include the constant terms. If we put

v = \sqrt{2}v = 3\sqrt{2}M/f, \quad \text{then all expressions for the fermion masses in the considered model were reproduced, i.e.} \quad m_k = f_kv/\sqrt{2}, \quad k = e, u, d \quad \text{and the} \quad W, Z \quad \text{meson masses remain be the same} \quad m_Z = m_W/\cos\theta_W. \quad \text{But, for the Higgs boson mass term in Lagrangian (6.21) we get} \quad m_{higgs}^2 = 9/8M^2. \quad \text{Thus the principal difference between the suggested 5D formulation and the standard} \quad SU(2) \times U(1) \quad \text{gauge field theory is generated by the symmetry breaking terms (6.21). In the present model the scale parameter} \quad M \quad \text{is determined via the mass of the Higgs boson and it indicates the border of regions} \quad q^2 = 0, \pm M^2 \quad \text{where the interaction of the scalar fields change the sign.}

\section{Summary and outlook}

In this paper we have considered conformal transformations of the interacted quantum fields in the momentum space. In the quantum field theory translations and inversions in the coordinate and in the momentum spaces are independent each from other. Therefore we can separate two different \( O(2,4) \) groups of the conformal transformations: one for the well known conformal transformations in the coordinate space and the other in the momentum space. This offers an additional possibilities by applications of the conformal groups in the quantum field theory. Here we have applied these conformal transformations of quantum fields in the momentum space to the chiral symmetry breaking models with and without gauge fields.

\footnote{According to the definition of action (4.2) and field operators (3.11a,b) present formulation is four-dimensional in the momentum space. Therefore Lagrangians (6.16) and (6.20) remain to be renormalizable.}
Unlike to the other papers devoted to the conformal transformations, we operate from the beginning with the interacting fields with mass and other scale parameters. Therefore any mass generation procedure can be incorporated in the considered scheme. The key point of the present formulation is the invariance of the 6D cone \( \kappa_\mu \kappa^\mu + \kappa_5^2 - \kappa_6^2 = 0 \) under the conformal transformations. Therefore the 5D forms \( q^2 \pm q_5^2 = \pm M^2 \) (2.2a,b), arising after projection of this 6D cone into 5D momentum space with \( q_\mu = \kappa_\mu / \kappa_+; \ k_\pm = (\kappa_5 \pm \kappa_6) / M; \ \mu = 0, 1, 2, 3 \) (1.3), are also invariant. This invariance of 5D forms was obviously preserved by derivation of the 5D equation of motion \((\partial^2 / \partial x_\mu \partial x^\mu + m^2)\varphi_a(x, x_5) = j_a(x, x_5), (a = 1, 2)\) and derivation of the boundary condition for the fifth coordinate \( i / M \partial / \partial x_5 \varphi_a(x, x_5) = \eta_a \varphi_a(x, x_5) + l_a(x, x_5) \) (3.4), where \( j_a(x, x_5) = (-1)^{(a-1)} M^2 (i / M \partial / \partial x_5 + \eta_a) l_a(x, x_5) \). In addition, operators \( \varphi_a(x, x_5), l_a(x, x_5), j_a(x, x_5) \) are embedded in the different 5D space with two different invariant forms \( q^2 \pm q_5^2 = \pm M^2 \). Therefore they satisfy boundary conditions \((\partial^2 / \partial x_\mu \partial x^\mu \pm \partial^2 / \partial x_5 \partial x_5 \pm M^2)O_a(x, x_5) = 0, \) where \( O_a = \varphi_a, l_a, j_a \) (see (3.3a,b) and (3.10a,b,c)). The 4D field operator \( \Phi(x) \) is determined via \( \varphi_a(x, x_5) \) as \( \Phi(x) = \Phi(x, x_5 = t_5) = \varphi_1(x, x_5 = t_5) + \varphi_2(x, x_5 = t_5) \), where \( t_5 = \sqrt{x_5^2 - x^2} \) or \( t_5 = 0 \). Therefore from the above set of the 5D equations we obtain the ordinary 4D equation of motion \((\partial^2 / \partial x_\mu \partial x^\mu + m^2)\Phi(x) = J(x) \) with \( J(x) = J_1(x, x_5 = t_5) + J_2(x, x_5 = t_5) \).

Separation of the boundary conditions over the fifth coordinates \( i / M \partial / \partial x_5 \varphi_a(x, x_5) = \eta_a \varphi_a(x, x_5) + l_a(x, x_5) \) with \( j_a(x, x_5) = (-1)^{(a-1)} M^2 (i / M \partial / \partial x_5 + \eta_a) l_a(x, x_5) \) makes it possible to interpret \( l_a(x, x_5) \) as a translation operator of the fifth coordinate. But unlike to the other five-dimensional formulations [26, 27, 28, 29], these one dimensional equation can not be interpreted as the evolution equations, because \( l_a \) are the determined as the second variation from the Lagrangians.

The considered scheme of 5D projections of the 6D conformal invariant forms can be realized also in the coordinate space. Thus the 6D invariant form \( \xi_\alpha \xi^\alpha = 0 \) with \( x_\mu = \xi_\mu / (\xi_5 + \xi_6) \) and \( x^\mu x_\mu = - \xi^2 (\xi_5 - \xi_6) / (\xi_5 + \xi_6) \) can be projected into two 5D hyperboloids \( x^\mu x_\mu \pm x_5^2 = \pm \ell^2 \) with \( x_5^2 = 2 \xi_5 (\text{or} \xi_6) \ell^2 / (\xi_5 + \xi_6) \). This leads to separation of the internal and the external 5D regions with the boundary values at \( x^2 = 0, \pm \ell^2 \). Besides, in this approach arise an additional boundary condition for the fifth coordinate for construction of the 4D field operator \( \Phi(x) = \varphi_{\text{inv}}(x, x_5 = t) + \varphi_{\text{ext}}(x, x_5 = t) \) at \( t = 0 \) or \( t = \sqrt{x^2} \). The advantage of this scheme is that there is introduced a new scale parameter \( \ell \) which can be used by description of observables.

Considered above field operators are defined in the 5D space with the invariant forms \( q^2 + q_5^2 = M^2 \) or \( q^2 - q_5^2 = -M^2 \) correspondingly. These areas are connected with inversion \( q_\mu = -M^2 q_\mu / q^2 \). Present formulation contains an additional mass parameter \( M \) which indicates the scale of the broken conformal symmetry. Moreover the interaction 5D Lagrangians and the corresponding source operators change their sign at the border \( q_5^2 = 0 \) and \( q^2 = \pm M^2 \). We have shown, that the simple choice of \( q_\mu \) (5.10) and (6.10) leads to the 5D generalization of the non-linear \( \sigma \)-model and of the standard \( SU(2) \times U(1) \).
theory. Corresponding 5D Lagrangians contains the specific mechanism of the breakdown of the chiral symmetry and allows us to determine $M$ through the mass of pion or Higgs boson. Unlike to the usual $\sigma$-model, the present 5D Lagrangian contains the standard weak chiral symmetry breaking term $\lambda \sigma$ together with other terms which broke the chiral symmetry more strongly. For the gauge theories without the chiral symmetry like QED, where $M$ is not fixed, it was shown, that in the limit $M \to \infty$ or $M \to 0$ we obtain the conventional 4D formulation of QED.

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