AUSLANDER’S THEOREM FOR PERMUTATION ACTIONS ON NONCOMMUTATIVE ALGEBRAS

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ABSTRACT. When \( A = k[x_1, \ldots, x_n] \) and \( G \) is a small subgroup of \( \text{GL}_n(k) \), Auslander’s Theorem says that the skew group algebra \( A \# G \) is isomorphic to \( \text{End}_{A^G}(A) \) as graded algebras. We prove a generalization of Auslander’s Theorem for permutation actions on \((-1-)\)-skew polynomial rings, \((-1-)\)-quantum Weyl algebras, three-dimensional Sklyanin algebras, and a certain homogeneous down-up algebra. We also show that certain fixed rings \( A^G \) are graded isolated singularities in the sense of Ueyama.

1. Introduction

Throughout, \( k \) is an algebraically closed field of characteristic zero. All algebras are \( k \)-algebras and all tensor products are over \( k \). For an algebra \( A \) and \( G \) a group of algebra automorphisms of \( A \), the set of all elements of \( A \) that are invariant under the action of \( G \) (denoted by \( A^G \)) forms a subalgebra of \( A \), called the ring of invariants of \( G \) on \( A \). Many notions in commutative algebra have their roots in the study of properties of rings of invariants of finite group actions on \( k[x_1, \ldots, x_n] \).

The homological properties of \( A^G \) are often of particular interest. A classical result of Shephard-Todd and Chevalley \([10, 25]\) states that if \( A = k[x_1, \ldots, x_n] \) and \( G \) is a finite subgroup of \( \text{GL}_n(k) \) acting as graded automorphisms, then \( A^G \) has finite global dimension if and only if \( G \) is generated by reflections; an element \( g \in \text{GL}_n(k) \) is said to be a reflection if \( g \) fixes a codimension-one subspace of \( V = \bigoplus_{i=1}^n kx_i \).

When \( A \) is not the commutative polynomial ring the above definition of reflection is not appropriate, but there is a suitable generalization due to Kuzmanovich, Zhang and the second author \([16, \text{Definition 2.2}]\). For a graded algebra \( A \) of Gelfand-Kirillov (GK) dimension \( n \), a graded automorphism \( g \) is a reflection if its trace series is of the form

\[
\text{Tr}_A(g, t) := \sum_{n \geq 0} \text{Tr}(g|_{A_n}) t^n = \frac{1}{(1-t)^{n-1}q(t)} \quad \text{for } q(1) \neq 0,
\]

where \( A_n \) denotes the \( n \)th graded component of \( A \) and \( \text{Tr}(g|_{A_n}) \) denotes the usual trace of the action of \( g \) on \( A_n \). When \( A = k[x_1, \ldots, x_n] \), \( g \) is a reflection in this sense if and only if it is a reflection in the classical sense. In \([17, \text{Theorem 5.5}]\), the same authors prove that if \( A = k[p_{ij}][x_1, \ldots, x_n] \) and \( G \) is a finite group of graded automorphisms of \( A \), then \( A^G \) has finite global dimension if and only if \( G \) is generated by reflections in this more general sense.

Another classical theorem from invariant theory in which reflections play an important role is a theorem of Auslander. The skew group algebra \( A \# G \) is \( A \otimes kG \) as a vector space over \( k \), with multiplication defined by \((a \otimes g)(b \otimes h) = a(g.b) \otimes gh\).

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for all \(a, b \in A, g, h \in G\). The category of (left) modules over \(A\#G\) is equivalent to the category of (left) \(A\)-modules with compatible left \(G\)-actions and their homomorphisms. A natural algebra homomorphism (due to Auslander) from \(A\#G\) to \(\text{End}_{A^G}(A)\) (where we view \(A\) as a right \(A^G\)-module) is given by
\[
\gamma_{A,G} : A\#G \rightarrow \text{End}_{A^G}(A)
\]
\[
a\#g \mapsto \begin{pmatrix} A & \rightarrow & A \\ b & \rightarrow & ag(b) \end{pmatrix}.
\]
We call the map \(\gamma_{A,G}\) the **Auslander map** of the \(G\)-action on \(A\). In general, this map is neither injective nor surjective. Auslander showed that \(\gamma_{A,G}\) when \(|G|\) is invertible in \(k\), \(A = k[x_1, \ldots, x_n]\), and \(G\) does not contain any nontrivial reflections \([1]\). In particular, such an isomorphism identifies an \(A^G\)-module (namely \(A\)) whose endomorphism ring has finite global dimension.

Natural generalizations of Auslander’s result include replacing \(A\) with a noncommutative ring and replacing the action of the group \(G\) with the action of a Hopf algebra. Significant progress has been made in both directions, see \([2, 3, 8, 9, 11, 13, 23, 24]\). Our interest is in the former.

The second author has conjectured that the Auslander map should be an isomorphism for a noetherian AS regular domain \(A\) and a group \(G\) when \(G\) does not contain a reflection. Bao, He, and Zhang \([2, 3]\) (using some ideas that appear in an earlier paper of Buchweitz \([6]\)) prove that the Auslander map is related to an invariant of the \(G\)-action on \(A\) known as the pertinency. Let \(A\) be an affine algebra generated in degree 1 and \(G\) a finite subgroup of \(\text{GL}_n(k)\) acting on \(A_1\). The pertinency of the \(G\)-action on \(A\) \([2, \text{Definition 0.1}]\) is defined to be
\[
p(A,G) = \text{GKdim } A - \text{GKdim}(A\#G)/(f_G)
\]
where \((f_G)\) is the two sided ideal of \(A\#G\) generated by \(f_G = \sum_{g \in G} 1\#g\). In what follows, we often drop the subscript on \(f_G\) when the group \(G\) is clear from context.

**Theorem 1** ([3, Theorem 0.3]). Let \(A\) be a noetherian, connected graded, AS regular, Cohen-Macaulay \(k\)-algebra of GK dimension at least 2. Let \(G\) be a group acting linearly on \(A\). Then the Auslander map is an isomorphism if and only if \(p(A,G) \geq 2\).

Our main technique for computing the pertinency of a group action is based on an argument of Brown and Lorenz \([5, \text{Lemma 2.2}]\). They prove that for a commutative algebra \(A\), one can produce an element in \((f_G) \cap A\) by taking \(f_G\)-commutators. Two problems with directly applying their result to our situation are that commutativity is used heavily, and that the element produced has degree \(n!-1\), which is much higher than the degree where nonzero elements of \((f_G)\) first appear in \(A\). We modify their method by carefully choosing central (or normal) elements that produce elements that bound the pertinency below and prove Theorem 2.4.

Our primary interest is the \((-1)\)-skew polynomial rings \(V_n\) generated by \(x_1, \ldots, x_n\) subject to the relations \(x_ix_j + x_jx_i = 0\) for \(i \neq j\). \(V_n\) is a noetherian AS regular domain, and the symmetric group \(S_n\) acts naturally on \(V_n\) by permutations. That is, if \(\sigma \in S_n\), then \(\sigma(x_i) = x_{\sigma(i)}\), extending linearly and multiplicatively. Throughout we assume that any subgroup of \(S_n\) acts on \(V_n\) in this way unless otherwise stated. For this action on \(V_n\), \(S_n\) does not contain any reflections \([18, \text{Lemma 1.7(4)}]\). Bao, He, and Zhang proved that the Auslander map is an isomorphism for a subgroup \(S_n\) generated by an \(n\)-cycle \([3, \text{Theorem 5.7}]\). Our result (Theorem 2.4) proves, in
full generality, that the Auslander map is an isomorphism for permutation actions on $V_n$, a result quite different than for commutative polynomial rings, where the Auslander map is not an isomorphism for subgroups of $S_n$ that contain a transposition. In Section 3 we compute the pertinency of most subgroups of $S_3$ and $S_4$ acting on $V_3$ and $V_4$, respectively.

In addition to $V_n$, we consider some related families of algebras.

- The $(−1)$-quantum Weyl algebras $W_n$, generated as $k$-algebras by $x_1, \ldots, x_n$ with relations $x_ix_j + x_jx_i = 1$ for $i \neq j$; taking the standard filtration, $\text{gr}(W_n) = V_n$.
- The three-dimensional Sklyanin algebras $S(a, b, c)$, generated by $x_1, x_2, x_3$ with parameters $(a : b : c) \in \mathbb{P}^2$ with relations
  \[
  ax_1x_2 + bx_2x_1 + cx_1^2 = 0 \\
  ax_2x_3 + bx_3x_2 + cx_2^2 = 0 \\
  ax_3x_1 + bx_1x_3 + cx_1^2 = 0.
  \]
- The graded noetherian down-up algebras $A(\alpha, \beta)$ with parameters $\alpha, \beta \in k$ with $\beta \neq 0$ (introduced in [4]) generated by $x$ and $y$ with relations
  \[
  x^2y = \alpha xyx + \beta yx^2 \\
  xy^2 = \alpha yxy + \beta y^2x.
  \]

A connected $\mathbb{N}$-graded algebra $A$ is Artin-Schelter (AS) regular provided it has finite GK dimension, finite global dimension $d$, and satisfies $\text{Ext}_A^i(A \otimes k, A) = \delta_{i \ell}k(\ell)$ for some $\ell \in \mathbb{Z}$. $A$ is Cohen-Macaulay (CM) if for every graded $A$ module $j(M) + \text{GKdim}(M) = \text{GKdim}(A)$ where $j(M) = \min\{i : \text{Ext}_A^i(M, A) \neq 0\}$; an AS regular algebra is called a quantum polynomial ring if it is a noetherian domain with Hilbert series $(1 − t)^{-n}$. By [29, Theorem 0.2], [22, Corollary 6.2], and [20], $V_n$, $S(a, b, c)$, and $A(\alpha, \beta)$ are AS regular and Cohen-Macaulay. Hence, they satisfy the hypotheses of Theorem 1. For certain groups $G$ acting linearly on these algebras, we are able to verify that $p(A, G) \geq 2$ and so prove that the Auslander map is an isomorphism. We collect all such results in the following theorem.

**Theorem 2.** The Auslander map is an isomorphism for the following groups acting on the following algebras:

1. any subgroup of $S_n$ acting on $V_n$ (Theorem 2.4),
2. any subgroup of $S_n$ acting on $W_n$ (Theorem 2.4),
3. any subgroup of $S_3$ acting on $S(1, 1, −1)$ (Theorem 2.4),
4. any subgroup of weighted permutations acting on $A(−2, −1)$ (Theorem 4.3),
5. the cyclic group $\langle (1 \ 2 \ 3) \rangle$ acting on a generic Sklyanin algebra $S(a, b, c)$ (Theorem 5.1),
6. and any subgroup of $−I_4, (1 \ 3)(2 \ 4)$ acting on $V_4$ (Theorem 5.2).

To bound $p(A, G)$, we make frequent use of the following two theorems.

**Theorem 3** ([3, Lemma 5.2]). Let $T$ be a subalgebra of an algebra $R$ such that $R_T$ and $T \cdot R$ are finitely generated. Let $R'$ be the image of the map $R \to B \to B/I$ and $T'$ be the image of $T$ in $R'$. Then $\text{GKdim}(T') = \text{GKdim}(R') = \text{GKdim}(B/I)$.

**Theorem 4** (Theorem 3.4). Let $G$ be a group acting on $A$ and let $H \leq G$ be a subgroup. Then $p(A, G) \leq p(A, H)$. 
Corollary 5 (Corollary 3.5). Let \( G \) be a finite subgroup of \( \text{Aut}_{gr}(A) \) for \( A \) a noetherian connected graded algebra, and let \( g \in G \) be a reflection. If \( A \) and \( A^g \) have finite global dimension, then the Auslander map \( \gamma_{A,G} \) is not an isomorphism.

We remark that the finite global dimension hypothesis is satisfied for reflections acting on quantum polynomial rings [16, Theorem 5.3], and hence for \( V_n \).

Auslander’s Theorem is a component of the classical McKay correspondence. In [8, 9], Chan, Walton, Zhang, and the second-named author study a quantum version of the McKay correspondence for Hopf actions on regular graded algebras with trivial homological determinant. If \( A \) is AS regular and \( g \in \text{Aut}_{gr}(A) \), then the homological determinant of \( g \), denoted \( \text{hdet}(g) \), may be computed from the Laurent series expansion in \( t^{-1} \) of the trace series using the following formula from [14, Lemma 2.6],

\[ \text{Tr}_A(g,t) = (-1)^n \text{hdet}(g)^{-1} t^{-\ell} + \text{(lower order terms)} \]

The action of a group \( G \) on the algebra \( A \) has trivial homological determinant if \( \text{hdet}(g) = 1 \) for all \( g \in G \). With the exception of certain weighted actions on the down-up algebra, all actions in Theorem 2 have trivial homological determinant and we obtain the following correspondences as a corollary. Following the terminology in [9], call an \( A \)-module \( M \) initial if \( M \) is graded, generated in degree 0 and \( M_{<0} = 0 \) correspondences. Therefore, for the algebras and groups in Theorem 2,

**Theorem 6 ([9, Theorem A]).** Let \( A \) be a noetherian AS regular algebra and \( G \) a finite group acting on \( A \) as graded automorphisms with trivial homological determinant. If \( A^G \cong \text{End}_{A^G} A \) then there are bijective correspondences between the isomorphism classes of

- indecomposable direct summands of \( A \) as left \( A^G \)-modules,
- indecomposable, finitely generated, projective, initial, left \( A^G \)-modules,
- simple left \( G \)-modules.

We conclude in Section 5 by providing examples of graded isolated singularities in the sense of Ueyama [26]. For a graded algebra \( A \), let \( \text{grmod} A \) denote the category of finitely-generated graded right \( A \)-modules. For a module \( M \in \text{grmod} A \), \( x \in M \) is called torsion if there exists a positive integer \( n \) such that \( xA^n = 0 \). The module \( M \) is called a torsion module if every element of \( M \) is torsion. We can then define the quotient category \( A = \text{grmod} A / \text{tors} A \). Following [26], we say that \( A^G \) is a graded isolated singularity if \( \text{gl.dim}(\text{tails} A^G) < \infty \). Mori and Ueyama prove that if the Auslander map is an isomorphism (equivalently if \( p(A,G) \geq 2 \)), then \( A^G \) is a graded isolated singularity if and only if \( A^G/(f_G) \) is finite-dimensional [24, Theorem 3.10]. For several algebras \( A \) and groups \( G \), we are able to show that \( A^G/(f_G) \) is finite-dimensional. These examples are of particular interest, since for a graded isolated singularity \( A^G \), the category of graded Cohen-Macaulay \( A^G \)-modules has several nice properties (we refer the reader to [27] for undefined terminology).

**Theorem 7 ([27, Theorem 3.10 and Example 3.13]).** Let \( A \) be a noetherian AS regular algebra of dimension \( d \geq 2 \) and let \( G \) be a group acting linearly on \( A \) with trivial homological determinant. If \( A^G \) is a graded isolated singularity, then

- \( A^G \) is an AS Gorenstein algebra of dimension \( d \geq 2 \),
A ∈ CM^{gr}(A^G) is a (d − 1)-cluster tilting module, and

for M ∈ CM^{gr}(A^G) the k vector spaces Ext^1_{A^G}(A, M) and Ext^1_{A^G}(M, A)
are finite-dimensional.

We show that A^G is a graded isolated singularity in the following cases.

Theorem 8. For the following groups G acting on the following k-algebras A, A^G
is a graded isolated singularity:

1. (1 2)(3 4), (1 3)(2 4) acting on V_4 (Proposition 3.8),
2. (1 2)(3 4) · · · (2n − 1 2n) acting on V_{2n} (Proposition 3.9),
3. the cyclic group ⟨1 2 · · · 2^n⟩ acting on V_{2^n} (3, Theorem 5.7),
4. the cyclic group ⟨1 2 3⟩ acting on a generic Sklyanin algebra S(a, b, c)
   (Theorem 5.1),
5. and any subgroup of (−I_4, (1 3)(2 4)) acting on V_4 (Theorem 5.2).

2. (−1)-skew polynomials and related algebras

In this section we prove that the pertinency of G a subgroup of S_n acting on
V_n as permutations is bounded below by 2, thus proving that, in this case, the
Auslander map is an isomorphism. We also consider related algebras including
the three-dimensional Sklyanin algebra S(1, 1, −1) and the (−1)-quantum Weyl
algebras W_n.

Let A be a quotient of the free algebra k(x_1, . . . , x_n) such that S_n acts on A.
Denote by C(A) the center of A. We assume x^2_{1} ∈ C(A) for ℓ ∈ {1, 2, . . . n},
which clearly holds for V_n and W_n, and is well-known for S(1, 1, −1) (see [28, Section
8.2]).

Let G be a subgroup of S_n and set f = \sum_{σ ∈ G} 1#σ. For i < j, we define the
elements

\begin{align*}
    f_{i, j} = (x_i − x_j) \prod_{(a, b) \neq (i, j)} (x^2_a − x^2_b).
\end{align*}

Lemma 2.1. f_{i, j} ∈ (f) ∩ A for 1 ≤ i < j ≤ n.

Proof. Let U = \{(a, b) | 1 ≤ a < b ≤ n, (a, b) \neq (i, j)\} and let m = |U|.
Enumerate the elements of U: \{(a_1, b_1), . . . , (a_m, b_m)\}. Set p_0 = f. For 1 < k ≤ m, we define
p_k recursively. Let p_k = x^2_{a_k}p_{k−1} − p_{k−1}x^2_{b_k}. Since p_0 = f, it is clear that p_k ∈ I
for all k. Because the x^2 are in C(A) by hypothesis, a direct computation shows

\begin{align*}
    p_m = \sum_{σ ∈ G} \left(x^2_{a_1} − x^2_{σ(a_1)}) \cdots (x^2_{a_m} − x^2_{σ(a_m)}\right)#σ.
\end{align*}

Further, since U was taken over all a < b except (a, b) = (i, j), if σ \neq e and
σ \neq (i j), there exists some a_k, b_k such that σ(b_k) = a_k. Hence, p_m vanishes
on all components except for the identity and possibly the transposition (i j), if
(i j) ∈ G. We now observe that x_i p_m − p_m x_j = f_{i, j} ∈ (f), completing the proof. □

We define VdM(T) the Vandermonde determinant on the elements y_1, . . . , y_n in
T = k[y_1, . . . , y_n] by VdM(T) = \prod_{i < j}(y_i − y_j). If A = V_n or A = S(1, 1, −1) (with
n = 3), we set y_i = x^2_i ∈ C(A) and T ⊂ C(A). By Lemma 2.1, f_{i, j} ∈ (f) ∩ A. It
follows that VdM(T) = x_1 f_{1, 2} + f_{1, 2} x_2 ∈ (f) ∩ A. Moreover, for 1 ≤ i < j ≤ n we
define $\hat{f}_{i,j} = \frac{1}{2}(x_i f_{i,j} + f_{i,j} x_i) \in (f) \cap A$. Then
\begin{equation}
\hat{f}_{i,j} = \begin{cases} 
y_i \prod_{(a,b) \neq (i,j)} (y_a - y_b) & \text{if } A = V_n \\
(2y_i - y_j) \prod_{(a,b) \neq (i,j)} (y_a - y_b) & \text{if } A = S(1,1,-1).
\end{cases}
\end{equation}

Let $J$ be the ideal generated by the $\binom{n}{2} + 1$ elements $\hat{f}_{i,j}$ of degree $\binom{n}{2}$.

**Proposition 2.3.** The GK dimension of the algebra $T/J$ is at most $n - 2$.

**Proof.** Set $\hat{f} = \sum_{1 \leq i < j \leq n} \hat{f}_{i,j}$. We will show that $\hat{f}$ and VdM($T$) are relatively prime, and hence they form a regular sequence. It follows that the grade of $J$ (that is, the length of the longest regular sequence in $J$) is at least two, and hence the dimension of $T/J$ is at most $n - 2$ by the depth inequality.

Let $1 \leq a < b \leq n$. Consider the image of $\hat{f}$ in $T_{a,b} = k[y_1, \ldots, y_n]/(y_a - y_b)$. Since each $\hat{f}_{i,j}$ for $(i,j) \neq (a,b)$ has $y_a - y_b$ as a factor, the image of $\hat{f}$ and $\hat{f}_{a,b}$ in $T_{a,b}$ agree. Since $T_{a,b}$ is a domain and the image of all the irreducible factors of $\hat{f}_{a,b}$ are nonzero, the image of $\hat{f}_{a,b}$ is nonzero as well. Therefore $\hat{f}$ does not have $y_a - y_b$ as a factor for any such $a, b$, and hence $\hat{f}$ and VdM($T$) are relatively prime. As $T/J$ is commutative, its GK dimension is equal to its Krull dimension, so $\text{GKdim} \, T/J \leq n - 2$. \hfill \square

**Theorem 2.4.** The Auslander map $\gamma_{A,G} : A\#G \to \text{End}_{A^G}(A)$ is an isomorphism for $G$ a subgroup of $S_n$ acting on $A = V_n$, $W_n$, or $S(1,1,-1)$.

**Proof.** First, let $A = V_n$ or $A = S(1,1,-1)$. We assume $n = 3$ in the second case. Then $A$ is finitely generated over the central subalgebra $T$. Let $A'$ be the image of the map $A \to A\#G \to (A\#G)/(f)$ and $T'$ the image of $T$ in $A'$. By [3, Lemma 5.2], $\text{GKdim} \, T' = \text{GKdim} \, A' = \text{GKdim} (A\#G)/(f)$. Clearly, $\text{GKdim} \, T' \leq \text{GKdim} \, T/J$ and $\text{GKdim} \, T/J \leq n - 2$ by the above argument. Hence
\[
\text{p}(A,G) = \text{GKdim} \, A - \text{GKdim} (A\#G)/(f) \geq n - (n - 2) = 2.
\]
Applying [3, Theorem 0.3] now completes the proof for $V_n$ and $S(1,1,-1)$.

By [2, Proposition 3.6 and Corollary 3.7], $\text{p}(W_n, G) \geq \text{p}(V_n, G)$ and so the Auslander map is an isomorphism for $W_n$ as well. \hfill \square

3. Pertinency Computations

In this section we give more precise pertinency computations for subgroups of $S_3$ and $S_4$ acting on $V_3$ and $V_4$, respectively. In addition, we provide techniques for computing bounds on pertinency distinct from the method in the previous section. Pertinency values for $n$-cycles acting on $V_n$ were bounded, and for $n = 2^d$ computed exactly, by Bao, He, and Zhang in a result we recall below.

**Theorem 3.1** ([3, Theorem 5.7]). Let $G = ((1 2 \ldots n))$, $n \geq 2$ and let $\phi(n)$ be Euler’s phi function. If $n = 2^d$ then $\text{p}(V_n, G) = n$. In general, $\text{p}(V_n, G) \geq \frac{\phi(n)}{2}$ when $n$ is even and $\text{p}(V_n, G) \geq \phi(n)$ when $n$ is odd.

**Remark 3.2.** Let $n \geq 2$ and suppose $\sigma$ is a $k$-cycle acting on $V_n$. Let $H = \langle \sigma \rangle$. If $k = n$, then $\text{p}(V_n, H)$ is as in Theorem 3.1. However, if $k < n$, then by reindexing we may assume $\sigma = (1 2 \ldots k)$. We recognize $V_n$ as an Ore extension of $V_k$ with
By the computation above, this is equal to $A$ but with the $n$ on the right hand side replaced by $k$. We use this result frequently in what follows without mention.

Let $G$ be a finite subgroup of $\text{Aut}_\ell(A)$. For a finite-order element $g \in G$, the reflection number of $g$, denoted $r(A, g)$, is defined to be the difference between $\text{GKdim} A$ and the order of the pole of $\text{Tr}(g, t)$ at $t = 1$. The reflection number of the $G$-action on $A$ is defined to be

$$r(A, G) = \min\{r(A, g) : e \neq g \in G\}.$$ 

Bao, He, and Zhang conjecture that $p(A, G) \geq r(A, G)$ [3, Conjecture 0.9]; if true this conjecture would imply that the Auslander map is an isomorphism for groups that contain no reflections. For all of the permutation actions considered in this paper, we show that the conjectured inequality holds.

Next we will bound the pertinency of the group action above by the pertinency of the action of a subgroup. Suppose $G$ is a group acting on an algebra $A$ and let $H \subseteq G$ be a subgroup. Let

$$f_H = \sum_{\sigma \in H} 1#\sigma \quad \text{and} \quad f_G = \sum_{\sigma \in G} 1#\sigma.$$ 

**Lemma 3.3.** If $\alpha\#e \in (f_G) \cap A \subseteq A\#G$, then $\alpha\#e \in (f_H) \cap A \subseteq A\#H$.

**Proof.** Since $\alpha\#e \in (f_G) \cap A$, there exist $a_i, b_i \in A$ and $\sigma_i, \sigma_i' \in G$, $1 \leq i \leq n$, such that

$$\alpha\#e = \sum_{i=1}^{n} (a_i\#\sigma_i)(f_G)(b_i\#\sigma_i').$$

Since, for all $i$, $(1\#\sigma_i)f_G = f_G = f_G(1\#\sigma_i')$, by replacing $b_i$ with $(\sigma_i')^{-1}(b_i)$, we may assume $\sigma_i = \sigma_i' = e$ for all $i$. Therefore,

$$\alpha\#e = \sum_{i=1}^{n} (a_i\#e)(\sum_{\sigma \in G} 1#\sigma)(b_i\#e) = \sum_{i=1}^{n} \sum_{\sigma \in G} a_i\sigma(b_i)#\sigma.$$

That is,

$$\sum_{i=1}^{n} a_i\sigma(b_i) = \begin{cases} \alpha & \text{if } \sigma = e \\ 0 & \text{otherwise} \end{cases}$$

Now consider

$$\sum_{i=1}^{n} (a_i\#e)(f_H)(b_i\#e) = \sum_{i=1}^{n} (a_i\#e)\left(\sum_{\sigma \in H} 1#\sigma\right)(b_i\#e) = \sum_{i=1}^{n} \sum_{\sigma \in H} a_i\sigma(b_i)#\sigma.$$ 

By the computation above, this is equal to $\alpha\#e$, which proves our claim. \hfill \Box

The next theorem resolves a conjecture of Bao, He, and Zhang in the group case [3, Remark 5.6(2)].

**Theorem 3.4.** Let $G$ be a group acting on $A$ and let $H \subseteq G$ be a subgroup. Then

$$p(A, G) \leq p(A, H).$$

**Proof.** By [3, Lemma 5.2], $\text{GKdim} A\#G/(f_G) = \text{GKdim} A/((f_G) \cap A)$. Lemma 3.3 implies that $\text{GKdim} A/((f_G) \cap A) \geq \text{GKdim} A/((f_H) \cap A)$ and the result follows. \hfill \Box
Corollary 3.5. Let $G$ be a finite subgroup of Aut$_{gr}(A)$ for $A$ a noetherian connected graded algebra, and let $g \in G$ be a reflection. If $A$ and $A^{(g)}$ have finite global dimension, then the Auslander map $\gamma_{A,G}$ is not an isomorphism.

Proof. Let $H = \langle g \rangle$ be the subgroup of $G$ generated by the reflection $g$. By [16, Lemmas 1.10 and 1.11], the graded $A^H$-module $A$ is finitely generated and free so $\text{End}_{A^H}(A)$ is a matrix ring over $A^H$. This implies that $A^H \# H$ is not isomorphic to $\text{End}_{A^H}(A)$ as $A^H \# H$ is concentrated in nonnegative degree and there must be a map of negative degree in $\text{End}_{A^H}(A)$. One therefore has $p(A,G) \leq p(A,H) < 2$ and so $\gamma_{A,G}$ is not an isomorphism. \hfill \Box \\

Corollary 3.6. For $n \geq 2$, $p(V_n, S_n) = 2$.

Proof. By Theorem 2.4, $p(V_n, S_n) \geq 2$. The cyclic subgroup generated by a 2-cycle is a subgroup of $S_n$ with pertinency exactly equal to 2 (Theorem 3.1), so $p(V_n, S_n) \leq 2$ by Theorem 3.4. \hfill \Box \\

The above theorems, combined with prior results, are summarized in the table below as they apply to nontrivial subgroups of $S_3$ acting on $V_3$. It is clear that pertinency is stable under conjugation, so we list the subgroups up to conjugacy.

| subgroup | $p(V_3, G)$ | reason | $r(V_3, G)$ |
|----------|-------------|---------|-------------|
| $(1 2)$  | 2           | Theorem 3.1 | 2           |
| $(1 2 3)$ | 2 or 3      | Theorem 2.4 | 2           |
| $(1 2), (2 3)$ | 2 | Theorems 2.4, 3.4 | 2           |

Question 3.7. What is $p(V_3, (123))$? We conjecture that the value is 2.

We need a few additional results to handle the subgroups of $S_4$ acting on $V_4$.

Proposition 3.8. Let $G$ be the Klein 4 subgroup $\langle (1 2)(3 4), (1 3)(2 4) \rangle$ acting on $A = V_4$. Then $p(A, G) = 4$.

Proof. Let $f = \sum_{\sigma \in G} 1 \# \sigma$. We will show that for $1 \leq i \leq 4$, $x_i^4 \# e \in (f)$, so $\dim_k A/((f) \cap A) < \infty$ and hence $\text{GKdim} A^G/f = \text{GKdim} A/((f) \cap A) = 0$. Observe that

$$q_{(1 2)(3 4)} := \frac{1}{2} [(x_1 + x_2 - x_3 - x_4) [(x_1 + x_2 - x_3 - x_4)f + f(x_1 + x_2 - x_3 - x_4)] = (x_1^2 + x_2^2 + x_3^2 + x_4^2)\# e + (x_1^2 + x_2^2 + x_3^2 + x_4^2)\# (1 2)(3 4) \in (f).$$

Similarly, for each $\sigma \in G$, $q_{\sigma} = (x_1^2 + x_2^2 + x_3^2 + x_4^2)\# e + (x_1^2 + x_2^2 + x_3^2 + x_4^2)\# \sigma \in (f)$. Additionally, $(x_1^2 + x_2^2 + x_3^2 + x_4^2)f \in (f)$, so by subtracting $\sum_{\sigma \in G} q_{\sigma}$, we have

$$r = (x_1^2 + x_2^2 + x_3^2 + x_4^2) \in (f) \cap A.$$

We also have

$$s = (x_1^2 + x_2^2 - x_3^2 - x_4^2)f + f(x_1^2 + x_2^2 - x_3^2 - x_4^2)$$

$$= (x_1^2 + x_2^2 - x_3^2 - x_4^2)\# e + (x_1^2 + x_2^2 - x_3^2 - x_4^2)\# (1 2)(3 4) \in (f).$$

Therefore, $(x_1 - x_2)r + x_1s - sx_2 = (x_1 - x_2)(x_1^2 + x_2^2) \in (f) \cap A$. Indeed, by the same argument, for any $1 \leq i \neq j \leq 4$,

$$f_{i,j} = (x_i - x_j)(x_i^2 + x_j^2) \in (f) \cap A.$$
Now, we see that \( x_i f_{i,j} + f_{i,j} x_i = 2(x_i^4 + x_j^2 x_i^2) \in (f) \cap A \) and \((x_i - x_j) f_{i,j} = x_i^4 + 2x_i^2 x_j^2 + x_j^4 \in (f) \cap A \). Subtracting, it follows that \( x_i^4 - x_j^4 \in (f) \cap A \). Therefore, we have that \( x_i^2 x_j^2 + x_k^4 \in (f) \cap A \) for any \( 1 \leq i \neq j \leq 4 \), and \( 1 \leq k \leq 4 \).

Now \( r^2 \) is congruent to \(-8x_i^4\) modulo the other 6 relations, and therefore \( x_i^4 \in (f) \cap A \) for all \( i \), completing our proof. \( \square \)

Using Molien’s Theorem one can show that the Hilbert series of \( A^G \) above is

\[
H_{A^G}(t) = \frac{1 - 3t + 5t^2 - 3t^3 + t^4}{(1 + t^2)(1-t)^4},
\]

and so \( A^G \) is a rather complicated AS Gorenstein ring. Hence, Theorem 7 provides useful information for graded isolated singularities.

Proposition 3.9. Let \( \sigma = (1 \ 2)(3 \ 4) \cdots (2n-1 \ 2n), \ G = \langle \sigma \rangle, \) and \( A = V_{2n}. \) Then \( p(A, G) = 2n. \)

Proof. Let \( f = 1#e + 1#\sigma. \) Then for \( i \) odd, one has \( x_i f - f x_i+1 = (x_i - x_i+1). \) Now skew-commuting \( x_i - x_i+1 \) with either \( x_i \) or \( x_i+1 \) shows that \( x_i^2 \) and \( x_i^2+1 \) are in \( (f) \cap A. \) Hence, \( \dim_k A/(\langle f \rangle \cap A) < \infty \) and \( p(A, G) = 2n. \) \( \square \)

Below we list the pertinencies of subgroups of \( S_4 \) acting on \( V_4 \), along with the corresponding reflection numbers.

| subgroup                                   | \( p(V_4, G) \) | reason         | \( r(V_4, G) \) |
|--------------------------------------------|----------------|----------------|-----------------|
| \( \langle (1 \ 2) \rangle \)              | 2              | Theorem 3.1     | 2               |
| \( \langle (1 \ 2)(3 \ 4) \rangle \)       | 4              | Proposition 3.9 | 4               |
| \( \langle (1 \ 2 \ 3) \rangle \)          | 2 or 3         | Theorems 2.4, 3.4 | 2               |
| \( \langle (1 \ 2 \ 3 \ 4) \rangle \)      | 4              | Theorem 3.1      | 4               |
| \( \langle (1 \ 2), (3 \ 4) \rangle \)     | 2              | Theorems 2.4, 3.4 | 2               |
| \( \langle (1 \ 2)(3 \ 4), (1 \ 3)(2 \ 4) \rangle \) | 4 | Proposition 3.8  | 4               |
| \( \langle (1 \ 2 \ 3), (2 \ 4) \rangle \)  | 2              | Theorems 2.4, 3.4 | 2               |
| \( \langle (1 \ 2 \ 3), (1 \ 2 \ 4) \rangle \) | 2 or 3         | Theorems 2.4, 3.4 | 2               |
| \( \langle (1 \ 2 \ 3), (1 \ 2) \rangle \)  | 2              | Theorems 2.4, 3.4 | 2               |
| \( \langle (1 \ 2 \ 3 \ 4), (1 \ 2) \rangle \) | 2              | Theorems 2.4, 3.4 | 2               |

If \( p(V_{2n}, (1 \ 2 \ 3)) = 2 \), then \( p(V_{2n}, G) = 2 \) for \( G = \langle (1 \ 2 \ 3) \rangle \) and \( \langle (1 \ 2 \ 3), (1 \ 2 \ 4) \rangle \).

We now give an alternate way to realize an upper bound on the pertinence in the case of \( \langle (1 \ 2), (3 \ 4) \rangle \).

Let \( A \) and \( B \) be algebras with multiplication maps \( \mu_A \) and \( \mu_B \), respectively. A \( k \)-linear homomorphism \( \tau : B \otimes A \to A \otimes B \) is a twisting map provided \( \tau(b \otimes 1_A) = 1_A \otimes b \) and \( \tau(1_B \otimes a) = a \otimes 1_B \), \( a \in A, b \in B \). A multiplication on \( A \otimes B \) is then given by \( \mu_\tau := (\mu_A \otimes \mu_B) \circ (\text{id}_A \otimes \tau \otimes \text{id}_B) \). By [7, Proposition 2.3], \( \mu_\tau \) is associative if and only if \( \tau \circ (\mu_B \otimes \mu_A) = \mu_\tau \circ (\tau \otimes \tau) \circ (\text{id}_B \otimes \tau \otimes \text{id}_A) \) as maps \( B \otimes B \otimes A \otimes A \to A \otimes B \). The triple \( (A \otimes B, \mu_\tau) \) is a twisted tensor product of \( A \) and \( B \), denoted by \( A \otimes_\tau B \).

Let \( G \) and \( H \) be groups acting on \( A \) and \( B \), respectively. Then \( G \times H \) acts naturally on \( A \otimes B \) by \((g, h)(a \otimes b) = g(a) \otimes h(b)\).

Lemma 3.10. Let \( G \) and \( H \) be finite subgroups of automorphisms acting on algebras \( A \) and \( B \). Let \( \tau : B \otimes A \to A \otimes B \) be a twisting map. Then \( G \times H \) acts on \( A \otimes_\tau B \).
provided
(3.11) \( \tau((h, g)(b \otimes a)) = (g, h)\tau(a \otimes b) \).

**Proof.** As observed above, \( G \times H \) acts naturally on \( A \otimes B \) and similarly \( H \times G \) acts on \( B \otimes A \). Let \( \mu_A \) and \( \mu_B \) be the multiplication maps on \( A \) and \( B \), respectively, and \( \mu_\tau \) the (twisted) multiplication on \( A \otimes B \). Recall that \( A \otimes B \) has the same basis as \( A \otimes B \).

Let \( a \otimes b, a' \otimes b' \in A \otimes B \) and \( \sum a''_i \otimes b''_i = \tau(b \otimes a') \). Assuming (3.11),
\[
\mu_\tau((g, h)(a \otimes b) \otimes (g, h)(a' \otimes b')) = \mu_\tau((g(a) \otimes h(b)) \otimes (g(a') \otimes h(b')))
\]
\[
= (\mu_A \otimes \mu_B)(g(a) \otimes \tau((h, g)(b \otimes a')) \otimes h'(b'))
\]
\[
= (\mu_A \otimes \mu_B)(g(a) \otimes (g, h)\tau(a' \otimes b) \otimes h(b'))
\]
\[
= \sum g(a)g(a''_i) \otimes h(b''_i)h(b')
\]
\[
= (g, h)((\mu_A \otimes \mu_B)(a \otimes \tau(b \otimes a) \otimes b))
\]
\[
= (g, h)(\mu_\tau((a \otimes b) \otimes (a' \otimes b'))) \). □

**Example 3.12.** Suppose \( A = \mathbb{k}_{-1}[x_1, \ldots, x_n] \) and \( B = \mathbb{k}_{-1}[y_1, \ldots, y_m] \). Define a twisting map \( \tau : B \otimes A \rightarrow A \otimes B \) by \( b \otimes a \mapsto (-1)^{|b|}(a \otimes b) \) for \( a \in A_k \) and \( b \in B_t \) and extending \( \tau \) linearly. Then \( A \otimes B \cong V_{n+m} \). If \( G \) and \( H \) are any groups acting linearly as automorphisms on \( A \) and \( B \) respectively then they preserve degree and hence (3.11) holds in this case.

The proof of the next theorem should be compared to [2, Lemma 6.4].

**Theorem 3.13.** Let \( A \) and \( B \) be affine algebras generated in degree 1 and a twisting map \( \tau : B \otimes A \rightarrow A \otimes B \). Assume \( A \otimes B \) is Cohen-Macaulay. Let \( G \) and \( H \) be finite subgroups of automorphisms acting on \( A \) and \( B \), respectively. If (3.11) holds for \( G, H, \) and \( \tau \), then one has the inequality
(3.14) \( \max(p(A, G), p(B, H)) \geq p(A \otimes \tau B, G \times H) \).

**Proof.** It suffices to prove that \( p(A \otimes B, G \times \{e_H\}) = p(A, G) \) and \( p(A \otimes B, \{e_G\} \times H) = p(B, H) \). The result will then follow from Theorem 3.4. We will prove the first and the second follows similarly.

Since \( G \times H \) satisfies (3.11), then clearly so does its subgroup \( G \times \{e_H\} \). By a modification of [21, Proposition 3.11], \( \text{GKdim}(A \otimes B) = \text{GKdim}(A) + \text{GKdim}(B) \).

Set \( F = \sum_{g \in G} 1_\#(g, e_H) \). Then \( F \) commutes with \( B \) and so
\[
((A \otimes B) \#(G \times \{e_H\}) / (F) \cong ((A \# G) \otimes \tau B) / (F) \cong (A \# G) / (f_G) \otimes \tau B.
\]

Now,
\[
p(A \otimes B, G \times \{e_H\}) = \text{GKdim}(A \otimes B) - \text{GKdim}((A \otimes B) \#(G \times \{e_H\})) / (F)
\]
\[
= (\text{GKdim}(A) + \text{GKdim}(B)) - \text{GKdim}((A \# G) / (f_G) \otimes \tau B).
\]

**Question 3.15.** Under what hypotheses does equality hold in (3.14)?

**Example 3.16.** Let \( A = \mathbb{k}_{-1}[y_1, y_2, y_3, y_4] \) and \( B = \mathbb{k}_{-1}[z_1, z_2, z_3, z_4] \). Let \( \tau \) be as in Example 3.12. Then \( A \otimes \tau B \cong V_8 \). Let \( G \) be the subgroup of \( S_4 \) generated by \((1 2 3 4)\). Then \( G \) acts on both \( A \) and \( B \) and by Theorem 3.1 we have \( p(A, G) = p(B, G) = 4 \). By Theorems 2.4 and 3.13 we have
\[
2 \leq p(V_8, \{(1 2 3 4), (5 6 7 8)\}) = p(A \otimes \tau B, G \times G) \leq p(A, G) = 4.
\]
4. THE DOWN-UP ALGEBRA $A(-2, -1)$

A more general version of Theorem 2.4 would apply to groups of weighted permutations acting on $V_n$. In this section, we demonstrate that our methods can be applied to certain groups of weighted permutations. As a consequence, we prove that for any group acting on the down-up algebra $A(-2, -1)$ the Auslander map is an isomorphism (Theorem 4.3). Bao, He, and Zhang proved that the Auslander map is an isomorphism for noetherian $A$ and any finite group of graded automorphisms when $\beta \neq -1$, and also for the case $A(2, -1)$ [2, Theorem 0.6]. Thus, our theorem solves one of the remaining cases.

Recall that $A = A(-2, -1)$ is the $k$-algebra generated by $x$ and $y$ subject to the two cubic relations $x^2y + yx^2 + 2xyx = xy^2 + y^2x + 2yxy = 0$. The element $z = xy + yx$ is central in $A$.

By [15, Proposition 1.1],

$$\text{Aut}_{gr}(A) = \left\{ \begin{bmatrix} a_{11} & 0 & a_{12} \\ 0 & a_{22} & 0 \\ a_{21} & 0 & a_{22} \end{bmatrix} : a_{11}, a_{12}, a_{21}, a_{22} \in k^\times \right\}.$$ 

There exists a filtration $\mathcal{F} = \{ F_n \}$ defined by

$$F_n A = (k \oplus kx \oplus ky \oplus kz)^n \subset A$$

for all $n \geq 0$ and $gr_n(\mathcal{F}) (A) \cong V_3$ [19, Lemma 7.2(2)]. Note that $\mathcal{F}$ is stable with respect to the action of $\text{Aut}_{gr}(A)$ on $A$. Moreover, $V_3$ is a connected graded algebra and the group of automorphisms of $V_3$ induced by the filtration is

$$G := \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & ab \end{bmatrix} , \begin{bmatrix} 0 & c & 0 \\ d & 0 & 0 \\ 0 & cd \end{bmatrix} : a, b, c, d \in k^\times \right\}.$$ 

As a $G$-module, $A \cong V_3$ so the $G$-action is inner faithful and homogeneous. Let $H$ be a finite subgroup of $\text{Aut}_{gr}(A)$. We identify $H$ with the corresponding subgroup of $G$ which, by an abuse of notation, we also call $H$. By [2, Proposition 3.6], $p(A, H) \geq p(V_3, H)$. Hence, it suffices to prove $p(V_3, H) \geq 2$.

There are a few special cases of such groups. If $H$ is diagonal, then the Auslander map is an isomorphism by [2, Theorem 5.5]. In particular, $H$ is small in the commutative sense and the action of $H$ commutes with the graded twist sending $V_3$ to $k[x_1, x_2, x_3]$. Also, if $H$ is small (in the commutative sense) when acting on $T = k[y_1, y_2, y_3] \subset C(V_3)$ where $y_i = x_i^2 \in C(V_3)$, then the Auslander map is an isomorphism by classical results.

Set $f = \sum_{h \in H} 1\#h$ and write $H = H_d \cup H_i$ where the $H_d$ are diagonal and the $H_i$ are not. Denote elements of the first type as $M(a, b)$ and elements of the second type as $N(c, d)$.

Let $M = \text{diag}(a, b, ab) \in H_d$. If two of $a, b, ab$ are equal to 1 then so is the third. Hence, we conclude that for every nonidentity element of $H_d$, at least two of the entries are not 1. Define $S_t = \{ h_{2,1} \mid h \in H_t \}$, $S_d = \{ h_{1,1} \mid h \in H_d, h_{1,1} \neq 1 \}$, and $S'_d = \{ h_{2,2} \mid h \in H_d, h_{1,1} = 1, h_{2,2} \neq 1 \}$.

**Lemma 4.1.** $V = x_1^2|S_d|_2 \prod_{d \in S_t} (x_1^2 - d^{-2}x_2^2) \in (f) \cap T$.

**Proof.** Set $p_0 = f$. Enumerate the elements of $S_t$: $d_1, \ldots, d_n$. Inductively define $p_k = x_1^{2|S_d|} - x_2^{2|S_d|}$. Set $\hat{V} = p_0$. All components corresponding to $H_i$ are now zero in $\hat{V}$ and the coefficient of the identity component is $\prod_{d \in S_t} (x_1^2 - d^{-2}x_2^2)$.
Enumerate the elements of $S_d$: $a_1, \ldots, a_\ell$, and the elements of $S_d'': b_1, \ldots, b_r$. Set $q_0 = \hat{V}$ and inductively define $q_k = \begin{cases} x_1(x_1 q_{k-1} - a_k^{-1} q_{k-1} x_1) & 0 \leq k \leq \ell \\ x_2(x_2 q_{k-1} - b_{k-1}^{-1} q_{k-1} x_2) & \ell + 1 \leq k \leq \ell + r. \end{cases}$ Then $V = q_{r+\ell}$.

**Lemma 4.2.** There exists $\mu \in (f) \cap T$ such that $V$ and $\mu$ are relatively prime.

**Proof.** We claim that for each factor $v \in V$, there exists $\mu_v \in (f) \cap T$ such that $v \nmid \mu_v$ and $u \mid \mu_v$ for all other factors $u$ of $V$ relatively prime to $v$. It then follows that $V$ and $\mu = \sum_v \mu_v$ are relatively prime.

Suppose $v = x_1^{2i}$ and let $\hat{V}$ be as in Lemma 4.1. The proof follows almost identically to that lemma after replacing $S_d, S_d''$ with $\hat{S}_d = \{h_{2,2} : h \in H_d, h_{2,2} \neq 1\}$ and $\hat{S}_d'' = \{h_{3,3} : h \in H_d, h_{2,2} = 1, h_{3,3} \neq 1\}$. Then $\gamma_v = q_{r+s} \in (f) \cap T$. The proof is similar when $v = x_2^{2j}$.

Suppose $v = (x_1^{2j} - d^{-2} x_2^{2k})$ for some $d$. Define $H'_d$ as the set of $h \in H_d$ that are killed by $d^{-2}$ and not by any other constant. We may assume $H'_d \neq \emptyset$. Set $p_0 = f$. Enumerate the elements of $S_d$: $d = d_1, \ldots, d_n$, and assume $d = d_1$. Inductively define $p_k = x_1^{2k} p_{k-1} - d_1^{2k} p_{k-1} x_2^2$. Hence, all components in $p_{n-1}$ corresponding to $H_d$ are zero except those in $H'_d$.

Let $\hat{S} = \{h_{3,3} : h \in H_d \cup H'_d, h_{3,3} \neq 1\}$. Enumerate the elements of $\hat{S} = \beta_1, \ldots, \beta_m$. Set $q_0 = p_{n-1}$. Inductively define $q_k = x_3(x_3 q_{k-1} - \beta_k^{-1} q_{k-1} x_3)$. Hence, the only remaining nonzero components in $q_m$ have one of the following forms (for fixed, $a$ arbitrary): $M_a = M(a, a^{-1}), N_1 = N(d^{-1}, d), N_2 = N(-d^{-1}, -d)$.

Consider $q = x_1 q_{m} - d^{-1} q_{m} x_2$, which kills the $N_1$ component. The coefficient in the $M_a$ component is $(x_1 - (ad)^{-1} x_2)$, and the coefficient in the $N_2$ component is $2x_1$. Set $r_0 = x q - d q x_1$. Since $(x_2(x_1) - d(x_1)(-d^{-1} x_2) = x_2 x_1 + x_1 x_2 = 0$, then the $N_2$ component is zero in $r_0$. What remains in the $M_a$ component is $2x_2 x_1 - ad(x_1^2 + (ad)^{-2} x_2^2)$.

Enumerate the distinct $(1,1)$-entries of the $M_a$ components whose coefficient is not zero: $a_1, \ldots, a_n$, with $a_1 = 1$. Inductively define $r_k = \begin{cases} x_1(x_1 r_{k-1} + a_k^{-1} r_{k-1} x_1) & k \text{ odd} \\ x_2(x_2 r_{k-1} + a_k^{-1} r_{k-1} x_2) & k \text{ even}. \end{cases}$

All coefficients in $r_n$ are central and it remains only to kill the remaining nonidentity $M_a$ components. For this we can proceed as in Lemma 4.1.

**Theorem 4.3.** Let $A = A(-2, -1)$. The Auslander map $\gamma_{A, G} : A \# G \to \text{End}_{A^G}(A)$ is a graded isomorphism for any finite subgroup $G$ of $\text{Aut}_{gr}(A)$.

**Proof.** As noted previously, it suffices by [2, Proposition 3.6] to prove this for $V_3$ and the corresponding group $H$ acting on $V_3$. By Lemmas 4.1 and 4.2, $(f) \cap T$ contains two relatively prime elements (see Proposition 2.3). We may now proceed as in Theorem 2.4.

We observe that by [16, Proposition 6.4], the group $G$ in Theorem 4.3 contains no reflections.
5. Graded isolated singularities

Recall that under the setting of this paper, $A^G$ is a graded isolated singularity if and only if $A#G/(f_G)$ is finite-dimensional. Thus, we have previously shown that $V^G_4$ is a graded isolated singularity when $G = \langle (1 \ 2 \ 3 \ 4) \rangle$ or $\langle (1 \ 2)(3 \ 4) \rangle$. In this section we present several additional examples of graded isolated singularities using our methods. For these examples, we therefore obtain the conclusions of Theorem 7 as a corollary.

5.1. Generic three-dimensional Sklyanin algebras. Let $A = S(a, b, c)$ for a generic choice of $a, b, c \in \mathbb{k}$. Then $C_3 = \langle \sigma \rangle$ acts on $A$ with $\sigma = (1 \ 2 \ 3)$ permuting the variables. Let $\xi$ be a primitive third root of unity, and set $X = x_1 + \xi x_2 + \xi^2 x_3$, $Y = x_1 + \xi^2 x_2 + \xi x_3$ and $Z = x_1 + x_2 + x_3$.

Then $A$ is also generated by $X, Y$ and $Z$, and $X, Y, Z$ satisfy Sklyanin relations with parameters

$$(c + \xi a + \xi^2 b, c + \xi^2 a + \xi b, c + a + b) =: (\alpha, \beta, \gamma).$$

Note also that if one applies this change of variable again, one obtains the same presentation of $A$ (up to scaling) that we started with. One benefit of working in these coordinates is that $\sigma(X) = \xi X$, $\sigma(Y) = \xi^2(Y)$ and $\sigma(Z) = Z$.

First, note that $X^2#e$ is in the ideal generated by $f = f_G$. Indeed, one has that

$$Xf - \xi fX = (1 - \xi)X#e + (1 - \xi^2)X#\sigma,$$

hence $g = X#e + (1 - \xi)X#\sigma$ is in $(f)$. The computation $Xg - \xi^2 gX = (1 - \xi^2)X^2#e$ proves the desired claim. A similar calculation shows that $Y^2#e$ is in $(f)$.

Let $I$ be the ideal generated by the Sklyanin relations on $X, Y$ and $Z$, and $X^2, Y^2$. Using the generators $X^2$ and $Y^2$ we may simplify these relations to:

$$\alpha XY + \beta YX + \gamma Z^2, \alpha YZ + \beta ZX, X^2, \alpha ZX + \beta XZ, Y^2.$$

We now use the diamond lemma with $X > Y > Z$ to produce a Gröbner basis of this ideal. In the computations, we will assume that $\alpha, \beta, \gamma$ are nonzero, and that $\alpha^3 \neq \beta^3$.

First, we compute the overlap between $\alpha XY + \beta YX + \gamma Z^2$ and $X^2$ to get

$$X (\alpha XY + \beta YX + \gamma Z^2) - X^2(\alpha Y) \sim \beta YX + \gamma XZ^2 \sim$$

$$\gamma XZ^2 - \frac{\beta^2}{\alpha} YX^2 - \frac{\beta \gamma}{\alpha} Z^2X \sim - \frac{\alpha \gamma}{\beta} Z^2X - \frac{\beta \gamma}{\alpha} Z^2X \sim$$

$$\frac{(\alpha \gamma^2)}{\beta^2} \frac{\beta \gamma}{\alpha} Z^2X = \left(\frac{\gamma (\alpha^3 - \beta^3)}{\alpha \beta^2}\right) Z^2X.$$

Note that in the above calculations, we use $\sim$ to denote equality in the tensor algebra and $\sim$ to indicate a reduction has taken place using the generators of the Gröbner basis of the defining ideal that have been found thus far. Therefore $Z^2X$ is in the ideal as long as $\alpha, \beta, \gamma$ is nonzero, and $\alpha^3 \neq \beta^3$. A similar calculation shows that $Z^2Y$ is in the ideal under the same conditions. Computing the overlap between $Z^2X$ and $\alpha XY + \beta YX + \gamma Z^2$ shows that $Z^4$ is in the ideal as well.

This gives us that under the hypotheses on the parameters mentioned above, the ideal $I$ contains elements with lead terms $X^2, XY, Y^2, YZ, ZX, Z^2X, Z^2Y, Z^4$. A straightforward calculation shows that this monomial ideal determines a finite-dimensional quotient with basis given by $1, X, Y, Z, ZX, ZY, Z^2, ZYX, Z^3$. We therefore have proven the following theorem:
Thus, E. Kirkman was partially supported by grant #208314 from the Simon’s Foundation. The authors would like to thank Andrew and many computations for this paper were performed with the computer algebra system Macaulay2 [12]. The authors would like to thank Andrew Conner and James Zhang for many stimulating conversations related to this project. E. Kirkman was partially supported by grant #208314 from the Simons Foundation.

5.2. Weighted action on $V_4$. Consider the following action of the Klein-4 group on $V_4$. Let $K = \langle \alpha, \beta \rangle \subseteq \text{Aut}_\text{gr}(A)$ where $\alpha(x_i) = -x_i$ for all $i$ and $\beta = (1\, 3)(2\, 4)$.

Theorem 5.2. Let $K$ be the group above. Then $\mathfrak{p}(V_4, K) = 4$ and thus $V_4^K$ is a graded isolated singularity, as is $V_4^H$ for any subgroup $H$ of $K$.

Proof. Let $f = 1#\alpha + 1#\beta + 1#\alpha\beta$ and set
\[ p_1 = (x_1 + x_3)f + f(x_1 + x_3) = 2(x_1 + x_3)#e + 2(x_1 + x_3)#\beta \]
\[ p_2 = x_1f - f(x_3) = 2(x_1 - x_3)#e + 2(x_1 - x_3)#\alpha + 2(x_1 - x_3)#\alpha\beta. \]

Then $x_2p_1 + p_1x_4 = 2(x_2 - x_4)(x_1 + x_3)#e$ and $(x_2 + x_4)p_2 - p_2(x_2 + x_4) = 2(x_2 + x_4)(x_1 - x_3)#e$. Combining these gives $x_2x_3 - x_4x_1, x_2x_1 - x_4x_3 \in (f) \cap V_4$. Next we produce degree three elements in $(f) \cap V_4$. Set
\[ p_3 = (x_1x_3)f + f(x_1x_3) = 2x_1x_3#e + (2x_1x_3)#\alpha. \]

Thus, $x_1p_3 - p_3x_1 = 4x_1^2x_3#e \in (f) \cap V_4$. Conjugation and symmetry now give $x_1x_3^2, x_2^2x_4, x_2x_3^2 \in (f) \cap V_4$. Finally we set
\[ p_4 = x_1^2f + f(x_3^2) = (x_1^3 - x_3^3)#e + (x_1^3 - x_3^3)#\alpha. \]

Then $x_1p_4 + p_4x_1 = 2x_1(x_1^3 - x_3^3)#e$ and it follows that $x_1^3 \in (f) \cap V_4$. Similarly, $x_2^3, x_3^3, x_4^3 \in (f) \cap V_4$ hence $V_4/((f) \cap V_4)$ is finite-dimensional. The remaining claim now follows from Theorem 3.4. \qed

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