ON ENDTORIVIAL MODULES FOR LIE SUPERALGEBRAS

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Abstract. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra over an algebraically closed field, $k$, of characteristic 0. An endotrivial $\mathfrak{g}$-module, $M$, is a $\mathfrak{g}$-supermodule such that $\text{Hom}_k(M,M) \cong k \oplus P$ as $\mathfrak{g}$-supermodules, where $k$ is the trivial module concentrated in degree 0 and $P$ is a projective $\mathfrak{g}$-supermodule. In the stable module category, these modules form a group under the operation of the tensor product. We show that for an endotrivial module $M$, the syzygies $\Omega^n(M)$ are also endotrivial, and for certain Lie superalgebras of particular interest, we show that $\Omega^1(k)$ and the parity change functor actually generate the group of endotrivials. Additionally, for a broader class of Lie superalgebras, for a fixed $n$, we show that there are finitely many endotrivial modules of dimension $n$.

1. Introduction

The study of endotrivial modules began with Dade in 1978 when he defined endotrivial $kG$-modules for a finite group $G$ in [14] and [15]. Endotrivial modules arose naturally in this context and play an important role in determining the simple modules for $p$-solvable groups. Dade showed that, for an abelian $p$-group $G$, endotrivial $kG$-modules have the form $\Omega^n(k) \oplus P$ for some projective module $P$, where $\Omega^n(k)$ is the $n$th syzygy (defined in Section 2) of the trivial module $k$. In general, in the stable module category, the endotrivial modules form an abelian group under the tensor product operation. It is known, via Puig in [17], that this group is finitely generated in the case of $kG$-modules and is completely classified for $p$-groups over a field of characteristic $p$ by Carlson and Thévenaz in [11] and [12]. An important step in this classification is a technique where the modules in question are restricted to elementary abelian subgroups.

Carlson, Mazza, and Nakano have also computed the group of endotrivial modules for finite groups of Lie type (in the defining characteristic) in [7]. The same authors in [8] and Carlson, Hemmer, and Mazza in [6] give a classification of endotrivial modules for the case when $G$ is either the symmetric or alternating group.

This class of modules has also been studied for modules over finite group schemes by Carlson and Nakano in [9]. The authors show that all endotrivial modules for a unipotent abelian group scheme have the form $\Omega^n(k) \oplus P$ in this case as well. For certain group schemes of this type, a classification is also given in the same paper (see Section 4). The same authors proved, in an extension of this paper, that given an arbitrary finite group scheme, for a fixed $n$, the number of isomorphism classes of endotrivial modules of dimension $n$ is finite (see [10]), but it is not known whether the endotrivial group is finitely generated in this context.

We wish to extend the study of this class of modules to Lie superalgebra modules. First we must establish the correct notion of endotrivial module in this context. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra over an algebraically closed field, $k$, of characteristic 0. A $\mathfrak{g}$-supermodule, $M$, is called endotrivial if there is a supermodule isomorphism $\text{Hom}_k(M,M) \cong k \oplus P$ where $k$ is the trivial supermodule concentrated in degree 0 and $P$ is a projective $\mathfrak{g}$-supermodule.
There are certain subalgebras, denoted \( e \) and \( f \), of special kinds of classical Lie superalgebras which are of interest. These subalgebras “detect” the cohomology of the Lie superalgebra \( g \). By this, we mean that the cohomology for \( g \) embeds into particular subrings of the cohomology for \( e \) and \( f \). These detecting subalgebras can be considered analogous to elementary abelian subgroups and are, therefore, of specific interest.

In this paper, we observe that the universal enveloping Lie superalgebra \( U(e) \) has a very similar structure to the group algebra \( kG \) when \( G \) is abelian, noncyclic of order 4 and \( \text{char } k = 2 \) (although \( U(e) \) is not commutative). With this observation, we draw from the results of [5] to prove the base case in an inductive argument for the classification of the group of endotrivial \( U(e) \)-supermodules. The inductive step uses techniques from [9] to complete the classification. For the other detecting subalgebra \( f \), even though \( U(f) \) is not isomorphic to \( U(e) \), reductions are made to reduce this case to the same proof.

The main result is that for the detecting subalgebras \( e \) and \( f \), denoted generically as \( a \), the group of endotrivial supermodules, \( T(a) \), is isomorphic to \( \mathbb{Z}_2 \) when the rank of \( a \) is one and \( \mathbb{Z} \times \mathbb{Z}_2 \) when the rank is greater than or equal to two.

We also show that for a Lie superalgebra \( g \) with only finitely many one dimensional modules, there are only finitely many endotrivial \( g \)-supermodules of a fixed dimension \( n \). This is done by considering the variety of all representations as introduced by Dade in [13]. In particular, this result holds for the detecting subalgebras \( e \) and \( f \).

The author would like to thank his Ph.D. thesis advisor Daniel Nakano for his guidance in both the formulation of questions as well as development of the tools needed for studying endotrivial supermodules. Additional thanks are due to Jon Carlson for his helpful comments and suggestions during the writing of this work.

2. Notation and Preliminaries

First, a few basic definitions are given. Only a few are included here and any others may be found in [18, Chapters 0,1]. In this paper, \( k \) will always be an algebraically closed field of characteristic 0 and a Lie superalgebra \( g \) will always be defined over \( k \).

Let \( R \) be either the ring \( \mathbb{Z} \) or \( \mathbb{Z}_2 \). The structure of the tensor product of two \( R \)-graded vector spaces is the usual one, however, given \( A \) and \( B \), \( R \)-graded associative algebras, the vector space \( A \otimes B \) is an \( R \)-graded algebra under multiplication defined by

\[
(a \otimes b)(a' \otimes b') = (-1)^{|a'||b|}(aa') \otimes (bb')
\]

where \( a \) and \( b \) are homogeneous elements. The definition is extended to general elements by linearity. This is called the graded tensor product of \( A \) and \( B \) and is denoted by \( A \otimes B \).

**Convention.** From now on, all elements are assumed to be homogeneous and all definitions are extended to general elements by linearity, as is done above.

A homomorphism is always assumed to be a homogeneous map of degree 0. Additionally, any associative superalgebra \( A \) can be given the structure of a Lie superalgebra, which is denoted \( A_L \), by defining

\[
[a, b] := ab - (-1)^{|a||b|}ba.
\]

This construction applied to the associative superalgebra \( \text{End}_k V \), where \( V = V_\uparrow \oplus V_\uparrow \) is a super vector space over \( k \), yields \( (\text{End}_k V)_L \) which is the Lie superalgebra \( \mathfrak{gl}(V) \).

An important tool for dealing with the \( \mathbb{Z}_2 \) grading of the modules in this setting is the functor which interchanges the grading of the module but does not alter any other structure.
Definition 2.2. Let $\mathfrak{g}$ be a Lie superalgebra. Define a functor
$$\Pi : \text{mod}(\mathfrak{g}) \to \text{mod}(\mathfrak{g})$$
by setting $\Pi(V) = \Pi(V)_{\bar{0}} \oplus \Pi(V)_{\bar{1}}$ where $\Pi(V)_{\bar{0}} = V_{\bar{0}}$ and $\Pi(V)_{\bar{1}} = V_{\bar{1}}$ for any $\mathfrak{g}$-module $V = V_{\bar{0}} \oplus V_{\bar{1}}$. If $\phi$ is a morphism between $\mathfrak{g}$-modules then $\Pi(\phi) = \phi$. This operation is known as the parity change functor.

Note that, since the vector spaces are the same, the endomorphisms of $\Pi(V)$ are the same as $V$ and the grading of the endomorphisms is preserved. Thus, it is clear that if $V$ is a $\mathfrak{g}$-supermodule, then the same action turns $\Pi(V)$ into a $\mathfrak{g}$-supermodule as well, however, this module does not necessarily have to be isomorphic to the original module. An explicit construction of this functor is given later (see Lemma 3.8).

Now that $\Pi$ has been defined, one particularly important module to consider is that of the trivial module $k$, concentrated in degree $0$. In order to distinguish the grading, this module will be denoted as $k_{\text{ev}}$ and $k_{\text{od}} := \Pi(k_{\text{ev}})$.

At this point, we now wish to specialize to the category of interest for the remainder of this paper. Define $\mathcal{F}_{(\mathfrak{g}, t)}$ to be the full subcategory of $\mathfrak{g}$-modules where the objects are finite dimensional modules which are completely reducible over $t$. Note that this category has enough projectives and is self injective, as detailed in [2] and [3] respectively.

Now, we can introduce a class of modules which will be of particular interest for the remainder of this paper. This definition can be used in categories with enough projectives and enough injectives.

Definition 2.3. Let $\mathfrak{g}$ be a Lie superalgebra and let $M$ be a $\mathfrak{g}$-supermodule. Let $P$ be the minimal projective which surjects on to $M$ (called the projective cover), with the map $\psi : P \to M$. The first syzygy of $M$ is defined to be $\ker \psi$ and is denoted $\Omega(M)$ or $\Omega_1(M)$. This is also referred to as a Heller shift in some literature. Inductively, define $\Omega^{n+1} := \Omega_1(\Omega^n)$.

Similarly, given $M$, let $I$ be the injective hull of $M$ with the inclusion $\iota : M \hookrightarrow I$, then define $\Omega^{-1}(M) := \text{coker} \iota$. This is extended by defining $\Omega^{-n-1} := \Omega^{-1}(\Omega^{-n})$.

Finally, define $\Omega^0(M)$ to be the largest non-projective direct summand of $M$. In other words, we can write $M = \Omega^0(M) \oplus Q$ where $Q$ is projective and maximal as a projective summand. Thus, the $n$-th syzygy of $M$ is defined for any integer $n$.

3. Endotrivial Modules

Note. We are working in the category $\mathcal{F} := \mathcal{F}_{(\mathfrak{g}, \text{ev})}$ and all modules are assumed to be $\mathbb{Z}_2$-graded (i.e. supermodules).

Definition 3.1. Given a category of modules, $\mathcal{A}$, consider the category with the same objects as the original category and an equivalence relation on the morphisms given by $f \sim g$ if $f - g$ factors through a projective module in $\mathcal{A}$. This is called the stable module category of $\mathcal{A}$.

Definition 3.2. Let $\mathfrak{g}$ be a Lie superalgebra and $M$ be a $\mathfrak{g}$-module. We say that $M$ is an endotrivial module if $\text{End}_k(M) \cong k_{\text{ev}} \oplus P$ where $P$ is a projective module in $\mathcal{F}$.

Since we have the module isomorphism $\text{Hom}_k(V, W) \cong W \otimes V^*$ for two $\mathfrak{g}$-modules $V$ and $W$, often times the condition for a module $M$ being endotrivial is rewritten as
$$M \otimes M^* \cong k_{\text{ev}} \oplus P.$$

Lemma 3.3 (Schanuel). Let $0 \to M_1 \to P_1 \to M \to 0$ and $0 \to M_2 \to P_2 \to M \to 0$ be short exact sequences of modules where $P_1$ and $P_2$ are projective, then $M_1 \oplus P_2 \cong M_2 \oplus P_1$. 

The proof is straightforward and can be found in [1]. This lemma is useful because it shows that, in the stable category, it is not necessarily to use the projective cover to obtain the syzygy of a module. Indeed, any projective module will suffice because the kernel of the projection maps will only differ by a projective summand.

**Proposition 3.4.** The category $\mathcal{F}$ is self injective. That is, a module $M$ in $\mathcal{F}$ is projective if and only if it is injective.

The proof can be found in [3] Proposition 2.2.2. Now we state several results on syzygies.

**Proposition 3.5.** Let $M$ and $N$ be modules in $\mathcal{F}$, and let $m, n \in \mathbb{Z}$. Then

(a) $\Omega^0(M) \cong \Omega^{-1}(\Omega^1(M)) \cong \Omega^1(\Omega^{-1}(M))$;

(b) $\Omega^n(\Omega^m(M)) \cong \Omega^{n+m}(M)$ for any $n, m \in \mathbb{Z}$;

(c) $(\Omega^n(M))^* \cong \Omega^{-n}(M^*)$ for any $n \in \mathbb{Z}$;

(d) if $P$ is a projective module in $\mathcal{F}$, then $P \otimes N$ is also projective in $\mathcal{F}$;

(e) $\Omega^n(M) \otimes N \cong \Omega^n(M \otimes N) \oplus P$ for some projective module in $\mathcal{F}$, $P$;

(f) $\Omega^m(M) \otimes \Omega^n(N) \cong \Omega^{m+n}(M \otimes N) \oplus P$ for some projective module in $\mathcal{F}$, $P$;

(g) $\Omega^n(M) \oplus \Omega^n(N) \cong \Omega^n(M \oplus N)$.

**Proof.** (a) By the tensor identity found in [2] Lemma 2.3.1,

$$\text{Ext}^n_{\mathcal{F}}(S, P \otimes N) \cong \text{Ext}^n_{\mathcal{F}}(S \otimes N^*, P) = 0$$

for $n > 0$ since $P$ is projective. Hence, $P \otimes N$ is also projective.

The other proofs are omitted as they are very similar to the case of modules for group rings of finite groups found in [1].

We now have enough tools to prove the following.

**Proposition 3.6.** If a $g$-module, $M$, is endotrivial, then so is $\Omega^n(M)$ for any $n \in \mathbb{Z}$.

**Proof.** By assumption, $M \otimes M^* \cong k_{ev} \oplus P$ for some projective module $P$. Applying Proposition 3.5 parts (c) and (e) yields

$$\Omega^n(M) \otimes (\Omega^n(M))^* \cong \Omega^n(M) \otimes \Omega^{-n}(M^*) \cong \Omega^0(M \otimes M^*) \oplus P' \cong k_{ev} \oplus P'.$$

Given a fixed Lie superalgebra, $g$, we can consider the endotrivial $g$-modules. We define the set

$$T(g) := \{ M \in \text{Stmod}(g) \mid M \otimes M^* \cong k_{ev} \oplus P_M \}.$$

**Proposition 3.7.** Let $g$ be a Lie superalgebra. Then $T(g)$ forms an abelian group under the operation $M + N = M \otimes N$.

**Proof.** Since tensoring with a projective module yields another projective module, if $M, N \in T(g)$, then

$$(M \otimes N) \otimes (M \otimes N)^* = (k_{ev} \oplus P_M) \otimes (k_{ev} \oplus P_N) = k_{ev} \oplus P_{M \otimes N}$$

and so $M \otimes N \in T(g)$ as well, and the set is closed under the operation $\cdot$. This operation is associative by the associativity of the tensor product and commutative by the canonical isomorphism $M \otimes N \cong N \otimes M$.

The isomorphism class of the trivial module $k$ is the identity element and, since $M$ is endotrivial, $M^{-1} = M^*$ which is also necessarily in $T(g)$ since $M$ is finite dimensional.
The following lemma simplifies computations involving both syzygies and the parity change functor and will be useful throughout this work.

**Lemma 3.8.** Let $k$ be either the trivial supermodule, $k_{ev}$, or $\Pi(k_{ev}) = k_{od}$ in $\mathcal{F}$, then

$$\Pi(\Omega^n(k)) = \Omega^n(\Pi(k))$$

for all $n \in \mathbb{Z}$.

**Proof.** The case where $n = 0$, the claim is trivial.

The parity change functor, $\Pi$, can be realized by the following. Let $M$ be a $g$-supermodule, then

$$\Pi(M) \cong M \otimes k_{od}$$

and if $N$ is another $g$-supermodule and $\phi : M \to N$ is a $g$-invariant map, then

$$\Pi(\phi) : M \otimes k_{od} \to N \otimes k_{od}$$

$$m \otimes c \mapsto \phi(m) \otimes c$$

defines the functor $\Pi$. Let

$$0 \to \Omega^1(k) \to P \to k \to 0$$

be the exact sequence defining $\Omega^1(k)$. Then $\Pi(P)$ is the projective cover of $\Pi(k)$ and since the tensor product is over $k$, the following sequence

$$0 \to \Pi(\Omega^1(k)) \to \Pi(P) \to \Pi(k) \to 0$$

is exact. Thus, $\Pi(\Omega^1(k)) = \Omega^1(\Pi(k))$ as desired. We can easily dualize this argument to see that $\Pi(\Omega^{-1}(k)) = \Omega^{-1}(\Pi(k))$ and the proof is completed by an induction argument. \qed

### 4. Computing $T(\mathfrak{g})$ for Rank 1 Detecting Subalgebras

Determining $T(\mathfrak{g})$ for different Lie superalgebras will be the main goal for the next three sections. We will begin by discussing the detecting subalgebras, $e$ and $f$ as introduced in [2, Section 3]. Let $a$ denote an arbitrary detecting subalgebra (either $e$ or $f$).

First, we define the rank of a detecting subalgebra $a$. Let $e_n := q(1) \times q(1) \times \cdots \times q(1)$ with $n$ products of $q(1)$ (defined below), and $f_n := sl(1|1) \times sl(1|1) \times \cdots \times sl(1|1)$ with $n$ products of $sl(1|1)$ (defined below). Define the rank of a detecting subalgebra to be $\dim(a)$, the dimension of the odd degree. The rank of $e_n$ is $n$ and the rank of $f_n$ is $2n$ and, in general, a detecting subalgebra of rank $r$ is denoted $a_r$. We start by considering rank 1 subalgebras.

Recall the definition of $q(n) \subseteq gl(n|n)$. The Lie superalgebra $q(n)$ consists of $2n \times 2n$ matrices of the form

$$
\begin{pmatrix}
A & B \\
B & A
\end{pmatrix}
$$

where $A$ and $B$ are $n \times n$ matrices over $k$. We are interested in $q(1)$ which has a basis of

$$t = \begin{pmatrix}1 & 0 \\0 & 1\end{pmatrix} \quad e = \begin{pmatrix}0 & 1 \\1 & 0\end{pmatrix}.$$

Note that $t$ spans $q(1)_{\Omega}$ and $e$ spans $q(1)_{\Pi}$. The brackets are easily computed using Equation 2.1

$$[t, t] = tt - tt = 0, \quad [t, e] = te - et = 0, \quad [e, e] = ee + ee = 2t.$$
It is possible to determine the indecomposable endotrivial $q(1)$-modules by the classification of all indecomposable $q(1)$-modules found in [2, Section 5.2]. We know that $k \times \mathbb{Z}_2$ parameterizes the simple $q(1)$-supermodules and the set $\{L(\lambda), \Pi(L(\lambda)) \mid \lambda \in k\}$ is a complete set of simple supermodules. For $\lambda \neq 0$, $L(\lambda)$ and $\Pi(L(\lambda))$ are two dimensional and projective. Thus, the only simple modules that are not projective are $L(0)$, the trivial module, and $\Pi(L(0))$. The projective cover of $L(0)$ is obtained as

$$P(0) = U(q(1)) \otimes_{U(q(1))} L(0)_{\mid q(1)}.$$ 

Since $U(q(1))$ has a basis (according to the PBW theorem) of

$$\{e^r t^s \mid r \in \{0, 1\}, s \in \mathbb{Z}_{\geq 0}\},$$

we see that $P(0)$ has a basis of $\{1 \otimes 1, e \otimes 1\}$ and there is a 1 dimensional submodule spanned by $e \otimes 1$ which is isomorphic to $k_{od}$, the trivial module under the parity change functor. The space spanned by $1 \otimes 1$ is not closed under the action of $U(q(1))$ since $e(1 \otimes 1) = e \otimes 1$. It is now clear that the structure of $P(0)$ is

$$\begin{pmatrix}
  k_{ev} & \\
  e & \\
  k_{od} & 
\end{pmatrix}$$

and the reader may check that $P(\Pi(L(0))) \cong \Pi(P(0))$. Thus, directly computing all indecomposable modules shows that the only indecomposables which are not projective are $k_{ev}$ and $k_{od}$ which are clearly endotrivial modules.

The group $T(e_1)$ can be computed in terms of the syzygies. The kernel of the projection map from $P(0)$ to $L(0)$ is $\Omega^1(k_{ev}) = k_{od}$. In order to compute $\Omega^2(k_{ev})$, consider the projective cover of $k_{od}$, which is $\Pi(P(0))$. The kernel of the projection map is again $k_{ev}$, the trivial module. The situation is the same for $\Pi(L(0))$ only with the parity change functor applied.

Now we have the following complete list of indecomposable endotrivial modules,

$$\Omega^n(k_{ev}) = \begin{cases} 
  k_{ev} & \text{if } n \text{ is even} \\
  k_{od} & \text{if } n \text{ is odd}
\end{cases} \quad \Omega^n(k_{od}) = \begin{cases} 
  k_{od} & \text{if } n \text{ is even} \\
  k_{ev} & \text{if } n \text{ is odd}
\end{cases}$$

and an application of Proposition 3.5 (f) proves the following proposition.

**Proposition 4.1.** Let $e_1$ be the rank one detecting subalgebra of type $e$. Then $T(e_1) \cong \mathbb{Z}_2$.

Before considering the base case for $f_1$, recall the following definition. The Lie superalgebra $\mathfrak{sl}(m|n) \subseteq \mathfrak{gl}(m|n)$ consists of $(m + n) \times (m + n)$ matrices of the form

$$\begin{pmatrix}
  A & B \\
  C & D
\end{pmatrix}$$

where $A$ and $D$ are $m \times m$ and $n \times n$ matrices respectively, which satisfy the condition $\text{tr}(A) - \text{tr}(D) = 0$.

Since $\mathfrak{sl}(1|1)$ is of primary interest, note that $\mathfrak{sl}(1|1)$ consists of $2 \times 2$ matrices and has a basis of

$$t = \begin{pmatrix}
  1 & 0 \\
  0 & 1
\end{pmatrix} \quad x = \begin{pmatrix}
  0 & 1 \\
  0 & 0
\end{pmatrix} \quad y = \begin{pmatrix}
  0 & 0 \\
  1 & 0
\end{pmatrix}.$$
A direct computation shows that \( [x,y] = xy + yx = t \) and that all other brackets in \( \mathfrak{sl}(1|1) \) are 0. By the PBW theorem, a basis of \( U(\mathfrak{sl}(1|1)) \) is given by

\[
\{ x^{r_1} y^{r_2} t^s \mid r_i \in \{0,1\} \text{ and } s \in \mathbb{Z}_{\geq 0} \}.
\]

Not all \( U(\mathfrak{sl}(1|1)) \)-supermodules will be classified yet since this is a rank 2 detecting subalgebra. First consider \( U(\mathfrak{sl}(1|1)) \)-modules which are endotrivial when restricted to one element of \( \mathfrak{sl}(1|1)_T \).

Note that, since \( [x,x] = [y,y] = 0 \), an \( \langle x \rangle \)-supermodule or a \( \langle y \rangle \)-supermodule will also fall under the classification given in \([2\text{ Section 5.2}]\). For modules of this type, there are only four isomorphism classes of indecomposable modules, \( k_{ev}, k_{od}, U(\langle x \rangle), \) and \( \Pi(U(\langle x \rangle)) \). It can be seen by direct computation that \( U(\langle x \rangle) \) is the projective cover of \( k_{ev} \) (and the kernel of the projection map is \( k_{od} \)) and \( \Pi(U(\langle x \rangle)) \) is the projective cover of \( k_{od} \) (and the kernel of the projection map is \( k_{ev} \)).

Alternatively, let \( z = ax + by \) where \( a, b \in k \setminus \{0\} \). Then \( U(\langle z \rangle) \cong U(q(1)) \). Thus, we have the same result and proof as in Proposition 4.1.

**Proposition 4.2.** Let \( f_1|\langle z \rangle \) be a rank 1 subalgebra of \( \mathfrak{sl}(1|1) \) generated by \( z \), an element of \( \mathfrak{sl}(1|1)_T \). Then \( T(f_1|\langle z \rangle) \cong \mathbb{Z}_2 \).

5. **Computing \( T(\mathfrak{g}) \) for Rank 2 Detecting Subalgebras**

The main result of this section is the classification of \( T(\mathfrak{a}_2) \), stated in Theorem 5.10. Given this goal, the first case to consider is that of \( q(1) \times q(1) \), which is denoted \( \mathfrak{e}_2 \). The matrix realization of this Lie superalgebra has basis vectors

\[
t_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
e_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]

and a direct computation shows that the only nontrivial brackets are \( [e_1,e_1] = 2t_1 \) and \( [e_2,e_2] = 2t_2 \). By the PBW Theorem, a basis for \( U(\mathfrak{e}_2) \) is given by

\[
\{ e_1^{r_1} e_2^{r_2} t_1^{s_1} t_2^{s_2} \mid r_i \in \{0,1\} \text{ and } s_i \in \mathbb{Z}_{\geq 0} \}.
\]

Note that if \( \mathfrak{g} \) and \( \mathfrak{g}' \) are two Lie superalgebras, and \( \sigma : \mathfrak{g} \to U(\mathfrak{g}) \) and \( \sigma' : \mathfrak{g}' \to U(\mathfrak{g}') \) are the canonical inclusions, then \([18]\) gives an isomorphism between \( U(\mathfrak{g} \times \mathfrak{g}') \) and the graded tensor product \( U(\mathfrak{g}) \hat{\otimes} U(\mathfrak{g}') \) where the mapping

\[
\tau : \mathfrak{g} \times \mathfrak{g}' \to U(\mathfrak{g}) \hat{\otimes} U(\mathfrak{g}'')
\]

\[
\tau(g,g') = \sigma(g) \hat{\otimes} 1 + 1 \hat{\otimes} \sigma'(g')
\]

corresponds to the canonical inclusion of \( \mathfrak{g} \times \mathfrak{g}' \) into \( U(\mathfrak{g} \times \mathfrak{g}') \). It will be useful to think of the universal enveloping algebra in both contexts.

Following a similar procedure as in the previous case, all indecomposable endotrivial modules are classified by considering the simple \( \mathfrak{e}_2 \) modules and their projective covers. Take the induced simple modules from the previous case, \( L(\lambda) \) and \( L(\mu) \) for \( \lambda, \mu \in k \), and consider
the outer tensor product of pairs of these modules, $L(\lambda) \boxtimes L(\mu)$. This is defined by taking two $U(q(1))$ modules $L(\lambda)$ and $L(\mu)$ and letting $x \otimes y \in U(q(1)) \boxtimes U(q(1))$ act by

$$x \otimes y(v \boxtimes w) = (-1)^{|y||v|} x(v) \boxtimes y(w).$$

According to a theorem of Brundan from [4, Section 4], the simple $e_n = q(1) \times q(1) \times \cdots \times q(1)$ ($n$ products) modules, denoted $u(\lambda)$ for $\lambda \in \mathbb{Z}^n$, have characters given by

$$\text{ch} u(\lambda) = 2^{\lfloor h(\lambda) + 1/2 \rfloor} x^\lambda$$

where $h(\lambda)$ denotes the number of $t_i$ which do not act by 0. This is significant because it indicates that all modules are even dimensional except in the case when $h(\lambda) = 0$. Another computation of the induced module shows that the projective cover of the module $L(0) \boxtimes L(0)$, the trivial module, is

$$\begin{array}{c}
\begin{array}{ccc}
e_1 & & n_ev \\
e_2 & & n_od \\
\end{array}
\end{array}$$

which is also even dimensional. So all projective $e_2$ modules must be even dimensional.

Now we consider $\mathfrak{sl}(1|1)$ and prove the same result for this setting. By following the same line of reasoning found in [2, Section 5.2], all projective indecomposable $U(\mathfrak{sl}(1|1))$-supermodules can be found by inducing up from $U(\mathfrak{sl}(1|1)\mathfrak{m})$-supermodules. This is done by considering the basis given in [3] and thinking of $U(\mathfrak{sl}(1|1))$-supermodules as left $U(\mathfrak{sl}(1|1))$-supermodules and right $U(\mathfrak{sl}(1|1)\mathfrak{m})$-supermodules and defining

$$P(\lambda) = U(\mathfrak{sl}(1|1)) \otimes_{U(\mathfrak{sl}(1|1)\mathfrak{m})} k^\lambda$$

where $k$ is either even or odd degree. It is clear that each of these modules will be 4 dimensional. Furthermore, if $\lambda \neq 0$, then $P(\lambda)$ has a two dimensional socle and a two dimensional, simple head, denoted $L(\lambda)$. When $\lambda = 0$, then $P(0)$, the projective cover of the trivial module $k_{ev}$, and $\Pi(P(0))$, the projective cover of $k_{od}$, have simple heads, $k_{ev}$ and $k_{od}$ respectively.

Given these results, we may now prove the following lemma which will greatly restrict our search for endotrivial modules.

**Lemma 5.1.** Let $M$ be an indecomposable $e_2$-supermodule or an $\mathfrak{sl}(1|1)$-supermodule. If $M$ is an endotrivial supermodule, then $M$ must be in the principal block, i.e. all of the even elements must act on $M$ by 0.

**Proof.** Since $M$ is endotrivial, $M \otimes M^* \cong k_{ev} \oplus P$ for some projective module $P$. Since $\dim P = 2m$ for $m \in \mathbb{N}$ by the previous observations, $\dim M \otimes M^* = \dim M^2 \equiv 1 \pmod{2}$. Since all modules outside of the principal block are even dimensional, $M$ must be in the principal block. \hfill \Box

This greatly simplifies the search for endotrivial modules and we also can conclude that the only simple endotrivial modules are $k_{ev}$ and $k_{od}$. Now we wish to show that the only endotrivials are $\{\Omega^n(k_{ev}),\Pi(\Omega^n(k_{ev}))|n \in \mathbb{Z}\}$. 
Note. Since endotrivial \( a \)-modules are restricted to the principal block, the even elements act via the zero map on any module. With this in mind, it is convenient to think of endotrivial \( a \)-supermodules in a different way. Since \( t_i \) acts trivially for all \( i \), considering \( a \)-modules as \( a_1 \)-modules with trivial bracket yields an equivalent representation. The representations of these superalgebras are equivalent and the notation for this simplification is \( V(a) := \Lambda(a_T) \).

Given the above simplification, a basis for \( V(e_2) \) is given by
\[
\left\{ e_1^{r_1} e_2^{r_2} \mid r_i \in \{0, 1\} \right\}.
\]
If we consider the left regular representation of \( V(e_2) \) in itself with this basis, we have a 4-dimensional module with the structure
\[
\begin{array}{ccc}
\text{1} & \text{e}_1 & \text{e}_2 \\
\text{e}_1 & \text{e}_2 & \text{e}_1 e_2 \\
\text{e}_2 & \text{e}_1 e_2 & \text{e}_1 \\
\end{array}
\]
which is isomorphic to the projective cover of \( L(0) \boxtimes L(0) \) mentioned previously.

In the \( \mathfrak{sl}(1|1) \) case, now that the search has been restricted to the principal block, \( V(\mathfrak{sl}(1|1)) \) has the same structure. \( V(\mathfrak{sl}(1|1)) \) has a basis given by
\[
\left\{ x^{r_1} y^{r_2} \mid r_i \in \{0, 1\} \right\}
\]
and \( V(\mathfrak{sl}(1|1)) \) has the structure
\[
\begin{array}{ccc}
\text{x} & \text{1} & \text{y} \\
\text{x} & \text{y} & \text{xy} \\
\end{array}
\]
which is isomorphic to that of \( V(e_2) \).

In general, as previously observed, endotrivial \( a_r \)-modules are simply endotrivial modules for an abelian Lie superalgebra of dimension \( r \) concentrated in degree \( T \). For simplicity, denote a basis for \( (a_r)_T \) by \( \{a_1, \ldots, a_r\} \). Then it is clear that \( V(a_r) = \langle 1, a_1, \ldots, a_r \rangle \).

For the rank 2 case, the algebra \( V(a_2) \) is similar to the group algebra \( k(\mathbb{Z}_2 \times \mathbb{Z}_2) \) and endotrivial in the superalgebra case will be classified using a similar approach to Carlson in [5]. First we give analogous definitions and constructions to those in Carlson’s paper.

Let \( M \) be a \( g \)-supermodule. The rank of \( M \), denoted \( \text{Rk}(M) \), is defined by \( \text{Rk}(M) = \dim_k(M/\text{Rad}(M)) \). The socle of \( M \) has the standard definition (largest semi-simple submodule) and can be identified in the case of the principal block by \( \text{Soc} M = \{m \in M|u.m = 0 \text{ for all } u \in \text{Rad}(V(a_r))\} \). If \( h \) is a subalgebra of \( a_r \) with a basis of \( h_T \) given by \( \{h_1, \ldots, h_s\} \), then \( \tilde{h} := \bigotimes_{i=1}^{s} h_i \) is a useful element of \( V(h) \). This is because in \( M|_{V(h)}, \tilde{h}.M \subseteq \text{Soc} M|_{V(h)} \).

Now we prove a lemma in the same way as [5].

**Lemma 5.2.** Let \( M \) be an endotrivial \( a_r \)-supermodule for any \( r \in \mathbb{N} \). Then
\[
\dim \text{Ext}^1_{V(a_r)}(M, \Omega^1(M)) = 1
\]
and $M$ is the direct sum of an indecomposable endotrivial module and a projective module.

Proof. By definition, $\text{Hom}_k(M, M) \cong k e_v \oplus P$ for some projective module $P$. It is clear from the definitions, that $\text{Hom}_{V(a_r)}(M, M) = \text{Soc}(\text{Hom}_k(M, M))$. We have observed that $\tilde{a}.M \subseteq \text{Soc}(M)$ and in the case of a projective module, equality holds. So then

$$\tilde{a}. \text{Hom}_k(M, M) = \tilde{a}.(k \oplus P) = \text{Soc}(P).$$

Since $\text{Soc}(\text{Hom}_k(M, M)) = k \oplus \text{Soc}(P)$, we can see that $\tilde{a}.\text{Hom}_k(M, M)$ is a submodule of $\text{Hom}_{V(a_r)}(M, M)$ of codimension one.

Let $P'$ be the projective cover of $M$. Apply $\text{Hom}_k(M, -)$ and the long exact sequence in cohomology to the short exact sequence defining $\Omega^1(M)$ to get the following commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}_k(M, \Omega^1(M)) & \longrightarrow & \text{Hom}_k(M, P') & \longrightarrow & \text{Hom}_k(M, M) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}_{V(a_r)}(M, \Omega^1(M)) & \longrightarrow & \text{Hom}_{V(a_r)}(M, P') & \psi^* & \longrightarrow & \text{Hom}_{V(a_r)}(M, M) \\
& & & & & & \longrightarrow & \text{Ext}^1_{V(a_r)}(M, \Omega^1(M)) & \longrightarrow & 0
\end{array}
$$

where the vertical maps are multiplication by $\tilde{a}$. Since the diagram commutes and the map into $\text{Hom}_k(M, M)$ is surjective, the image of $\psi^*$ contains $\tilde{a}.\text{Hom}_k(M, M)$, and we conclude that the dimension of $\text{Ext}^1_{V(a_r)}(M, \Omega^1(M))$ is at most 1. The dimension is nonzero since the extension between a non-projective module and the first syzygy does not split. Since $\text{Ext}^1_{V(a_r)}$ splits over direct sums, the claim is established. \qed

Lemma 5.3. Let $M$ be a $V(a_r)$-module and let $b$ be a subalgebra of $a_r$. Then

$$\Omega^n_{a_r}(M)|_b \cong \Omega^n_b(M|_b) \oplus P$$

for all $n \in \mathbb{Z}$, where $P$ is a projective $V(b)$-module.

Proof. The case when $n = 0$ is proven by considering the rank varieties of $a_r$ and $b$. Let $M$ be as above and

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1_{a_r}(M) & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0
\end{array}
$$

be the short exact sequence of $V(a_r)$-supermodules defining $\Omega^1(M)$. Then by the rank variety theory of [2, Section 6], the module $P|_b$ is a projective $V(b)$-module (although perhaps not the projective cover of $M|_b$) and

$$
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^1_{a_r}(M)|_b & \longrightarrow & P|_b & \longrightarrow & M|_b & \longrightarrow & 0
\end{array}
$$

is still exact. Then by definition, $\Omega^1_{a_r}(M)|_b \cong \Omega^1_b(M|_b) \oplus P$. This argument applies to $\Omega^{-1}(M)$ as well and so by induction, $\Omega^n_{a_r}(M)|_b \cong \Omega^n_b(M|_b) \oplus P$ for all $n \in \mathbb{Z}$ \qed

The previous lemma indicates that the syzygies of a module commute with restriction up to a projective module (and truly commute in the stable module category). With this in mind, the subscripts on the syzygies will be suppressed as it will be clear from context which syzygy to consider.

In proving Theorem 5.10, we work with 2 conditions on a $V(a_2)$-supermodule, $M$. For notation, recall that $V(a_2) = \langle 1, a_1, a_2 \rangle$. The conditions are
(1) \( \text{Rk}(M) > \text{Rk}(\Omega^{-1}(M)) \);
(2) for any \( a_i \in (a_2)_\mathbb{T} \), \( M|_{(a_i)} \cong \Omega^t(k|_{(a_i)}) \oplus P_t \) where \( t = 0 \) or \( 1 \) and \( P_t \) is a projective \( \langle a_i \rangle \)-supermodule.

The technique is to show that some endotrivial modules have these properties and use them to classify all endotrivial modules.

Let \( M \) be an endotrivial \( V(a_2) \) module. Because the complexity of an \( M \) is nonzero (see \cite[Corollary 2.71]{3}), \( \Omega^n(M) \) satisfies (1) for some \( n \in \mathbb{Z}_{\geq 0} \), so we can simply replace \( M \) with \( \Omega^n(M) \) initially and proceed. Since the endotrivils for rank 1 supermodules have been classified, and \( M|_{(a_i)} \) is isomorphic to a \( V(a_1) \)-supermodule, \( M \) satisfies (2) as well.

The classification of \( T(a_2) \) is approached in the same way as in \cite{5}, but the techniques are altered to suit the rank 2 detecting subalgebra case. We begin by establishing several supplementary results.

**Lemma 5.4.** A \( V(a_2) \)-supermodule \( M \) satisfies (2) if and only if \( \Omega^n(M) \) satisfies (2) for all \( n \in \mathbb{Z} \).

*Proof.* Since \( M|_{(a_i)} \) is endotrivial, then \( \Omega^n(M)|_{(a_i)} \) is as well by Lemma 5.3 and so by the classification of the \( V(a_1) \) endotrivils, \( \Omega^n(M) \) satisfies (2) for all \( n \) if and only if \( M \) satisfies (2). \( \square \)

**Lemma 5.5.** Let \( M \) be a \( V(a_2) \)-supermodule which has no projective submodules and which satisfies condition (2). Let \( v \) be a generator for the \( \Omega^t(k|_{(a_i)}) \) component of some decomposition of \( M|_{(a_i)} \), then \( M \) satisfies condition (1) if and only if every such element \( v \) satisfies \( a_j.v = 0 \) for \( j \neq i \).

*Proof.* Let \( M \) be as above and assume that \( a_i.v \neq 0 \) for \( j \neq i \). We will show that \( M \) satisfies condition (1) by considering two exact sequences.

Define \( \text{Ann}_{a_i}(M) = \{ m \in M \mid a_i.m = 0 \} \). There is an exact sequence of \( V(a_2) \)-modules

\[
0 \longrightarrow k \longrightarrow M/a_i.M \overset{a_i}{\longrightarrow} a_i.M \longrightarrow 0
\]

where the map on the right is defined by multiplication by \( a_i \), and \( k \) (either even or odd) is the kernel of that map. By construction, \( v \) generates \( k \) as a vector space.

There is another exact sequence

\[
0 \longrightarrow a_i.M \overset{\iota}{\longrightarrow} \text{Ann}_{a_i}(M) \longrightarrow k \longrightarrow 0
\]

where \( a_i.M \) is included into the annihilator and \( k \) is the cokernel, generated by \( v \) as well. Since \( a_i \) acts trivially on all the modules in these sequences, the modules are all direct sums of cyclic \( \langle a_j \rangle \)-supermodules, where \( i \neq j \). Because they are cyclic, the rank of each module is the number of \( \langle a_j \rangle \)-components, or equivalently, the dimension of the socle.

With this in mind note that, since \( M \) contains no projective submodules,

\[
\text{Rk}(M) = \text{Rk}(M/a_i.M) = \dim(\text{Soc}(M/a_i.M))
\]

and

\[
\text{Rk}(\Omega^{-1}(M)) = \text{Rk}(I) = \dim(\text{Soc} \text{Ann}_{a_i}(M))
\]

where \( 0 \rightarrow M \rightarrow I \rightarrow \Omega^{-1}(M) \rightarrow 0 \) is the exact sequence which defines \( \Omega^{-1}(M) \). Since \( M \) contains no projective submodules, \( a_i.M \subseteq \text{Soc}(M) \) and so the ranks of \( M \) and \( \Omega^{-1}(M) \) differ at most by one dimension, depending on where the \( k \) lies relative to the structure of these modules. By assumption, \( a_j.v \neq 0 \) so \( v \notin \text{Soc} \text{Ann}_{a_i}(M) \). Also, since \( M|_{(a_i)} \cong \)
\[ \Omega^t(M_{\langle a_i \rangle}) \oplus P_i, \text{ and } a_j.v \in \text{Soc}(M), \text{ then } a_j.v \subseteq a_i.M \text{ which implies that } v \in \text{Soc}(M/a_i.M). \]

Thus, \( \text{Rk}(M) = \text{Rk}(\Omega^{-1}(M)) + 1 \) establishing the claim.

Now assume that \( \text{Rk}(M) > \text{Rk}(\Omega^{-1}(M)) \). The same exact sequences can be constructed and by the same observations, the rank will differ at most by 1. In order to satisfy the assumption, by the previous work, \( v \) must satisfy \( a_j.v \subseteq a_i.M \) and \( a_j.v \notin \text{Soc Ann}_{a_i}(M) \) implying that \( a_j.v \neq 0 \).

\[ \square \]

A new condition is introduced to encapsulate the previous lemma.

(3) Let \( M \) satisfy condition (2). If \( v_i \) is a generator for the \( \Omega^t(k|_{\langle a_i \rangle}) \) component of \( M|_{\langle a_i \rangle} \), then \( v_i \) is a generator for \( M \), for \( i = 1 \) and 2. If \( m \) is any generator for \( M \) then either \( a_i.m \neq 0 \) or \( a_2.m \neq 0 \).

**Lemma 5.6.** Let \( M \) be a \( V(a_2) \)-supermodule which has no projective submodules and which satisfies condition (2), then \( M \) satisfies (1) if and only if \( M \) satisfies (3).

**Proof.** First, assume \( M \) satisfies (3). Since \( M \) satisfies (2), let \( v' \) be some generator for the \( \Omega^t(k|_{\langle a_i \rangle}) \) component in some decomposition of \( M|_{\langle a_i \rangle} \). Since \( a_i.v' = 0, a_j.v' \neq 0 \) (for \( j \neq i \)) by (3) and by Lemma 5.5, \( M \) satisfies (1).

Next, assume that \( M \) satisfies (1). By Lemma 5.3, \( a_j.v_i \neq 0 \) for \( j \neq i \). Assume that \( v_i \) is not a generator for \( M \). Then \( v_i = a_1.m_1 + a_2.m_2 \) for some \( m_1, m_2 \in M \). By assumption

\[ a_j.v_i = a_j.(a_1.m_1 + a_2.m_2) = \tilde{a}.m_i \neq 0 \]

but then \( m_i \) generates a projective submodule which is a contradiction, so \( v_i \) is also a generator of \( M \). The only thing left to show is that if \( m \) is any generator for \( M \), either \( a_1.m \neq 0 \) or \( a_2.m \neq 0 \). If \( m \) is any such generator, then \( m \) must also be a generator for \( M|_{\langle a_i \rangle} \cong \Omega^t(k|_{\langle a_i \rangle}) \oplus P_i \). The case when \( m \) generates \( \Omega^t(k|_{\langle a_i \rangle}) \) has been handled, so assume that \( m \) is a generator for \( P_i \). Since \( P_i \) is a projective \( \langle a_i \rangle \)-module, and \( m \) is a generator, then \( a_i.m \neq 0 \).

\[ \square \]

**Proposition 5.7.** Let \( M \) be a \( V(a_2) \)-supermodule which has no projective submodules satisfying (1), (2), and (3), then \( \Omega^n(M) \) does as well for all \( n \geq 0 \).

**Proof.** By Lemmas 5.4 and 5.6 it is sufficient to show that (3) holds for \( \Omega^n(M) \) for all \( n \geq 0 \). This proposition it trivial when \( n = 0 \) since \( \Omega^0(M) = M \) by assumption.

Let \( m \) be a generator for \( \Omega^1(M) \), then \( m \) is also a generator for \( \Omega^1(M)|_{\langle a_i \rangle} \cong \Omega^t(k|_{\langle a_i \rangle}) \oplus P_i \) in some decomposition of \( \Omega^1(M)|_{\langle a_i \rangle} \). Assume that \( m \) is a generator for \( P_i \). Since \( P_i \) is a projective \( \langle a_i \rangle \)-module, \( a_i.m \neq 0 \).

All that remains to be shown is that if \( w_i \) is a generator for the \( \Omega^t(k|_{\langle a_i \rangle}) \) component of \( \Omega^1(M)|_{\langle a_i \rangle} \), then \( w_i \) is a generator for \( \Omega^1(M) \) and \( a_j.w_i \neq 0 \).

Let \( 0 \rightarrow \Omega^1(M) \rightarrow P \xrightarrow{\psi} M \rightarrow 0 \) be the exact sequence defining \( \Omega^1(M) \). Let \( v_i \) be a generator for the \( \Omega^t(k|_{\langle a_i \rangle}) \) component of \( M|_{\langle a_i \rangle} \). This generator can be chosen so that \( v_i = \psi(p) \) where \( p \) is a generator of \( P \). We claim that \( a_i.p \) is a generator for \( \Omega^1(M) \subseteq P \). If not, then write

\[ a_i.p = a_i.l + a_j.m \]

for some elements \( l, m \in \Omega^1(M) \). Then

\[ a_ja_i.p = a_ja_i.l. \]
Define $\omega := a_j.p - a_j.l$. Since $a_i.\omega = a_j.\omega = 0$, by definition, $\omega \in \text{Soc}(P) = \hat{a}.P \subseteq \Omega^1(M)$. So then $a_j.p = \omega + a_j.l \in \Omega^1(M) = \ker \psi$ (as in Definition 5.3). Now

$$\psi(\omega + a_j.l) = a_j.\psi(p) = \psi(a_j.p) = a_j.v_1 \neq 0$$

by assumption since $M$ satisfies (3). This is a contradiction, thus, $a_i.p$ is a generator for $\Omega^1(M)$.

Note that in $\Omega^1(M)|_{(a_i)}$, $a_i.p$ generates a trivial $\langle a_i \rangle$-module. Thus, any generator for the $\Omega^1(k|_{(a_i)})$ component of $\Omega^1(M)|_{(a_i)} \cong \Omega^1(k|_{(a_i)}) \oplus Q$ is equivalent to $a_i.p$ modulo $\text{Soc}(Q) = a_i.Q$. So if $w_i$ is any such generator, then it is possible to write $w_i = c.a_i.p + a_i.\nu$ for some $0 \neq c \in k$ and $\nu \in \text{Soc}(Q)$. Then

$$a_j.w_i = a_j.(c.a_i.p + a_i.\nu) = c\hat{a}.p \neq 0.$$ 

These computations show that condition (3) holds for $\Omega^1(M)$, hence condition (1) does as well. An inductive argument completes the proof of the proposition. $\square$

**Proposition 5.8.** Let $M$ be a $V(a_2)$-supermodule which has no projective submodules satisfying (1), (2), and (3), then either $\Omega^{-1}(M)$ satisfies all three conditions or $\Omega^{-1}(M)$ has a summand which is isomorphic to $k$ (either even or odd).

**Proof.** Let $0 \rightarrow M \rightarrow I \xrightarrow{\psi} \Omega^{-1}(M) \rightarrow 0$ be the exact sequence of $V(a_2)$-supermodules defining $\Omega^{-1}(M)$. By Lemma 5.4, $\Omega^{-1}(M)$ already satisfies (2).

Let $v \in M$ be a generator for the $\Omega^1(k|_{(a_i)})$ component of some decomposition of $M|_{(a_i)}$. Our previous work shows that we may choose $v$ so that $v = a_i.p$ for some $p \in I$. We also know that $\hat{a}.p = a_j.v \neq 0$. Then $p$ is a generator for $I$ and $\psi(p)$ is a generator for $\Omega^{-1}(M)$. There are two cases to consider.

First, if $a_j.\psi(p) = 0$, then $\psi(p) \in \text{Soc}(\Omega^{-1}(M))$, since $a_i.\psi(p) = \psi(a_i.p) = \psi(v) = 0$ because $v \in M$. If this happens, then $k.\psi(p) \cong k$ is a direct summand of $\Omega^{-1}(M)$.

For the rest of the proof, assume that for any such element $p$, $a_j.p \not\in M$ and we will show that $\Omega^{-1}(M)$ satisfies all three conditions. Note that, by Lemma 5.6, we only need to establish (3). Indeed, it has been observed seen that $\psi(p)$ is a generator for $\Omega^{-1}(M)$ and by the assumption, $a_j.p \not\in M$ yields that $a_j.\psi(p) \neq 0$ in $\Omega^{-1}(M)$.

Now let $m$ be a generator for $\Omega^{-1}(M)$ such that $a_i.m = 0$. By (2), $\Omega^{-1}(M)|_{(a_i)} \cong \Omega^1(k|_{(a_i)}) \oplus P$ where $P$ is a projective $\langle a_i \rangle$-module. Since $a_i.\psi(p) = 0$, any generator of the $\Omega^1(k|_{(a_i)})$ component will be equivalent to $\psi(p)$ modulo $\text{Soc}(P)$. This case has been covered since we assumed $a_j.p \not\in M$ so assume that $m$ does not generate the $\Omega^1(k|_{(a_i)})$ component and, thus, must be a generator for the projective summand. However, since $a_i.m = 0$, $m \in \text{Soc}(P)$ and we conclude that $m$ cannot be a generator for $P$, a contradiction. So if $m \not\equiv \psi(p) \mod \text{Soc}(P)$ is any generator, then $a_i.m \neq 0$. Thus, $\Omega^{-1}(M)$ satisfies (3) and thus (1) in this case, and the proof is complete. $\square$

**Theorem 5.9.** Let $M$ be an endotrivial $V(a_2)$-supermodule in $\mathcal{F}$. Then $M \cong \Omega^n(k) \oplus P$ for some $n \in \mathbb{Z}$ and where $k$ is either the trivial module $k_{ev}$ or $\Pi(k_{ev}) = k_{ad}$ and $P$ is a projective module in $\mathcal{F}$.

**Proof.** It has been observed that if $M$ is endotrivial, then $M$ satisfies condition (2). Additionally, for some $r \geq 0$, $\Omega^r(M)$ satisfies (1). So by Lemma 5.6, $\Omega^r(M)$ satisfies (3) as well.
By (1), we can see that $\Omega^r(M)$ has no summand isomorphic to $k$. Assume that $\Omega^{-s}(\Omega^r(M))$ has no such summand for for all $s > 0$. By Proposition 5.8

$$\text{Rk}(M) > \text{Rk}(\Omega^{-1}(M)) > \text{Rk}(\Omega^{-2}(M)) > \cdots$$

which is clearly impossible since $\text{Rk}(M)$ is finite for any module in $\mathcal{F}$. Thus, $\Omega^{-n-r}(\Omega^r(M)) \cong k \oplus Q$ for some $n \in \mathbb{Z}$. Since $\Omega^{-n-r}(\Omega^r(M))$ satisfies (2), $Q_{|\langle a_i \rangle}$ is a projective $\langle a_i \rangle$-module for $i = 1$ and 2 and by considering the rank variety of $V(a_2)$, $Q$ is a projective $V(a_2)$-module. The $k$ summand may either be contained in $\Omega^{-n}(M)_{\Pi}$ or $\Omega^{-n}(M)_{\f}$ and since $\Omega^{-n}(M)$ contains no projective submodules, $\Omega^{-n}(M) \cong k$ and $\Omega^0(M) \cong \Omega^n(k)$. By Lemma 5.2

$$M \cong \Omega^n(k) \oplus P$$

where $P$ is a projective $V(a_2)$-supermodule and $k$ is either $k_{ev}$ or $k_{od}$.

Given this theorem, it is now possible to identify the group $T(a_2)$.

**Theorem 5.10.** Let $a_2$ be a rank 2 detecting subalgebra of $\mathfrak{g}$, then $T(a_2) \cong \mathbb{Z} \times \mathbb{Z}_2$ and is generated by $\Omega^1(k_{ev})$ and $k_{od}$.

**Proof.** Let $M$ be an endotrivial $V(a_2)$-supermodule. By Theorem 5.9, in the stable module category, $M \cong \Omega^n(k_{ev})$ or $M \cong \Omega^n(\Pi(k_{ev}))$. By Lemma 3.8 this can be rewritten as $M \cong \Omega^n(k_{ev})$ or $M \cong \Pi(\Omega^n(k_{ev}))$. Since the group operation in $T(a_2)$ is tensoring over $k$ and by Proposition 5.5 (i),

$$M \cong \begin{cases} \Omega^1(k_{ev}) \otimes_k (k_{od})^\otimes t & \text{if } n > 0 \\ \Omega^{-1}(k_{ev}) \otimes_k (k_{od})^\otimes t & \text{if } n < 0 \\ \Omega^1(k_{ev}) \otimes_k \Omega^{-1}(k_{ev}) \otimes_k (k_{od})^\otimes t & \text{if } n = 0 \end{cases}$$

where $t \in \{1, 2\}$. Thus, there is an isomorphism, $\phi$, from $T(a_2)$ to $\mathbb{Z} \times \mathbb{Z}_2$ given by

$$\phi(M) := \begin{cases} (n, t) & \text{if } M \cong \Omega^1(k_{ev}) \otimes_k (k_{od})^\otimes t \text{ for } n > 0 \\ (n, t) & \text{if } M \cong \Omega^{-1}(k_{ev}) \otimes_k (k_{od})^\otimes t \text{ for } n < 0 \\ (0, t) & \text{if } M \cong k_{ev} \otimes_k (k_{od})^\otimes t \end{cases}$$

and it is now clear that $T(a_2)$ is generated by $\Omega^1(k_{ev})$ (and its inverse) and $k_{od}$.

### 6. Computing $T(\mathfrak{g})$ for All Ranks Inductively

Now we wish to proceed by induction to classify endotrivial modules for the general case $\mathfrak{a}_r$ where $r > 2$. We begin by briefly reviewing the structure of $\mathfrak{e}_n \cong q(1) \times \cdots \times q(1) \subseteq \mathfrak{gl}(n|n)$ and $\mathfrak{f}_n \cong \mathfrak{sl}(1|1) \times \cdots \times \mathfrak{sl}(1|1) \subseteq \mathfrak{gl}(n|n)$ where there are $r$ copies of $q(1)$ and $\mathfrak{sl}(1|1)$ respectively.

Recall, $\mathfrak{e}_n$ has a basis of

$$\{e_1, \ldots, e_n, t_1, \ldots, t_n\}$$

and there are matrix representations of $t_i$ and $e_i$ which are $2n \times 2n$ matrices with blocks of size $n \times n$. Let $d_i$ be the $n \times n$ matrix with a 1 in the $i$th diagonal entry and 0 in all other entries. Then

$$t_i = \begin{pmatrix} d_i & 0 \\ 0 & d_i \end{pmatrix} \quad e_i = \begin{pmatrix} 0 & d_i \\ d_i & 0 \end{pmatrix}$$
is a representation of \( \mathfrak{e}_n \). The only nontrivial bracket operations on \( \mathfrak{e}_n \) are \([e_i, e_j] = 2t_i\), thus all generating elements in \( U(\mathfrak{e}_n) \) commute except for \( e_i \) and \( e_j \) anti-commute when \( i \neq j \) and by the PBW theorem,
\[
\{ e_{k_1}^{l_1} \cdots e_{k_n}^{l_n} | k_i \in \{0, 1\} \text{ and } l_i \in \mathbb{Z}_{\geq 0} \}
\]
is a basis for \( U(\mathfrak{e}_n) \).

For \( f_n \), the set
\[
\{ x_1, \cdots, x_n, y_1, \cdots, y_n, t_1, \cdots, t_n \}
\]
forms a basis. The matrix representation of each element are also \( 2n \times 2n \) matrices with blocks of size \( n \times n \). If \( d_i \) is as above then
\[
t_i = \begin{pmatrix} d_i & 0 \\ 0 & d_i \end{pmatrix} \quad x_i = \begin{pmatrix} 0 & d_i \\ 0 & 0 \end{pmatrix} \quad y_i = \begin{pmatrix} 0 & 0 \\ d_i & 0 \end{pmatrix}
\]
gives a representation of \( f_n \). The only nontrivial brackets are \([x_i, y_i] = t_i \). So, in \( U(f_n) \), \( x_i \otimes y_j = -y_j \otimes x_i \) when \( i \neq j \) and \( t_i \) commutes with \( x_j \) and \( y_k \) for any \( i, j, \) and \( k \). Observe that \( x_i \otimes y_i = -y_i \otimes x_i + t_i \) for each \( i \). Finally, the PBW theorem shows that
\[
\{ x_1^{k_1} \cdots x_n^{k_n} y_1^{l_1} \cdots y_n^{l_n} | k_i, l_i \in \{0, 1\} \text{ and } m_i \in \mathbb{Z}_{\geq 0} \}
\]
is a basis for \( U(\mathfrak{sl}(1|1)) \).

As noted, since any endotrivial module for a detecting subalgebra is in the principal block, we can consider (equivalently) endotrivial representations of \( V(\mathfrak{e}_n) = \Lambda((\mathfrak{e}_n)_\mathbb{T}) \) and \( V(f_n) = \Lambda((f_n)_\mathbb{T}) \). The support variety theory of [2, Section 6] will be used as well. For an endotrivial module \( M \), since \( M \otimes M^* \cong k_{ev} \oplus P \), we have \( \mathcal{V}_{(a, a^\partial)}(M) = \mathcal{V}_{(a, a^\partial)}(k_{ev}) \cong \mathbb{A}^r \) (which is also equivalent to the rank variety of \( M \)).

Our first step in the classification comes by following [9, Theorem 4.4].

**Theorem 6.1.** Let \( M \) be an endotrivial \( V(\mathfrak{a}_r) \)-supermodule, where \( \mathfrak{a}_r \) is a rank \( r \) detecting subalgebra. Let \( v = c_1 a_1 + \cdots + c_r a_r \in V(\mathfrak{a}_r)_\mathbb{T} \) with \( c_i \neq 0 \) for some \( i < r \) and let \( A = \langle v, a_r \rangle \) be the subsuperalgebra of dimension 4 generated by \( v \) and \( a_r \). Then \( M|_A \cong \Omega^r(k|_A) \oplus P \) for some \( \mathfrak{a}_r \)-projective module \( P \) where \( k|_A \) is either the trivial module \( k_{ev} \) or \( \Pi(k_{ev}) = k_{ov} \).

**Proof.** First, note that since \( \Lambda(\mathfrak{a}_r) \) is a purely odd, abelian Lie superalgebra, \( v \otimes v = \frac{v \cdot v}{2} = 0 \) and \( a_r \otimes a_r = 0 \) but \( v \otimes a_r = -a_r \otimes v \neq 0 \) and so \( A \cong V(\mathfrak{a}_r) \). Also note that if \( v' = c_1 a_1 + \cdots + c_{r-1} a_{r-1} \), since \( c_i \neq 0 \) for some \( i < r \), then \( \langle v, a_r \rangle \cong \langle v', a_r \rangle = \langle cv', a_r \rangle \) by a change of basis for any \( 0 \neq c \in k \). So without loss of generality, redefine \( v = c_1 a_1 + \cdots + c_{r-1} a_{r-1} \) and \( A = \langle v, a_r \rangle \) for the new \( v \) and identify all such \( v \) with the points in \( \mathbb{A}^{n-1} \setminus \{0\} \).

By the previous classification, \( \Omega^0(M|_A) \cong \Omega^m(k) \) where \( k \) is either even or odd. We now show that \( m_v \) is independent of the choice of \( v \). Note, it is immediate that \( m_v = m_{ev} \) for any \( 0 \neq c \in k \).

Next, since \( \dim \Omega^m(k|_A) = \dim \Omega^{-m}(k|_A) > \dim M \) for large enough \( m \), then there exist \( b, B \in \mathbb{Z} \) such that \( b \leq m_v \leq B \) for any \( v \in \mathbb{A}^{r-1} \setminus \{0\} \). Moreover, we can choose \( b \) and \( B \) such that equality holds for some \( v \) and \( v' \). Now replace \( M \) by \( \Omega^{-b}(M) \). Once this is done, we assume \( b = 0 \), and for all \( v \in \mathbb{A}^{r-1} \setminus \{0\} \), \( 0 \leq m_v \leq B \) where the bounds are actually attained.

Let \( C \in \mathbb{Z} \) be such that \( 0 \leq C < B \) and let
\[
S_C = \{ v \in \mathbb{A}^{r-1} \setminus \{0\} | m_v > C \}
\]
We claim that \( S_C \) is closed in the Zariski topology of \( \mathbb{A}^{r-1} \setminus \{0\} \).
Since \( m_v = 0 \) for some \( v \), it follows that \( \dim M \equiv 1 \pmod{4} \). This implies that \( m_v \) is even for all \( v \). It is also true that for any \( v \), \( \dim \Omega^{2s}(k|_A) = 1 + 4s \) for any \( s \geq 0 \). With this in mind, define

\[
t = (\dim M - \dim \Omega^{2r}(k|_A))/4
\]

where \( c = C/2 \) if \( C \) is even and \( c = (C - 1)/2 \) if \( C \) is odd. In either case, \( 2c \leq C < 2c + 2 \). This construction is done to ensure that for any \( v \), the statement that \( m_v \leq C \) means that the dimension of the projective part of \( M|_A \) is

\[
\dim M - \dim \Omega^{m_v}(k|_A) \geq 4t.
\]

In other words, if \( m_v \leq C \), then \( M|_A \) has an \( A \)-projective summand of rank at least \( t \) so the rank of the matrix of the element \( \omega_v = v \otimes a_r \) (which generates the socle of \( A \)) acting on \( M \) is at least \( t \). Otherwise, if \( m_v > C \), then \( M|_A \) has no \( A \)-projective summand of rank \( t \). Consequently, the rank of the matrix of \( \omega_v \) is strictly less than \( t \).

Let \( d = \dim M \) and let \( S \) be the set of all subsets of \( \mathcal{R} = \{1, \ldots, d\} \) having exactly \( t \) elements. For any \( S, T \in \mathcal{S} \) define \( f_{ST} : \mathbb{A}^{r-1} \setminus \{0\} \rightarrow k \) by

\[
f_{ST}(v) = \text{Det}(M_{ST}(\omega_v))
\]

where \( M_{ST} \) is the \( t \times t \) submatrix of the matrix of \( \omega_v \) acting on \( M \) having rows indexed by \( S \) and columns indexed by \( T \). The functions \( f_{ST} \) are polynomial maps thus,

\[
f = \prod_{ST \in \mathcal{S}} f_{ST} : \mathbb{A}^{r-1} \setminus \{0\} \rightarrow k^{(t)^2}
\]

is a polynomial map. If \( M|_A \) has no \( A \)-projective summand of rank \( t \), then each determinant must always be 0, hence 0 in the product, and otherwise, at least one of the \( f_{ST}(v) \) will be nonzero. Thus we have constructed a polynomial such that \( f(v) = 0 \) if and only if \( v \in S_C \). We conclude that \( S_C \) is closed in \( \mathbb{A}^{r-1} \setminus \{0\} \).

It is also true that for any \( C \), \( S_C \) is open in \( \mathbb{A}^{r-1} \setminus \{0\} \). First, replace \( M \) with \( M^* \) (which is also endotrivial). Since \( (\Omega^n(M^*))^* \cong \Omega^{-n}(M) \), for \( M^* \), the bounds are \(-B < m_v < 0\). Replacing \( M^* \) with \( \Omega^B(M^*) \) again yields \( 0 < m_v < B \). However, now we have that for any \( v \),

\[
M|_A \cong \Omega^{m_v}(k|_A) \oplus P,
\]

and by the above computation, we also have

\[
(\Omega^B(M^*))|_A \cong \Omega^{B-m_v}(k|_A) \oplus P.
\]

Thus, \( S_C = (S_{B-C})^c \) and so \( S_C \) is open. Since \( S_C \) is both open and closed and \( \mathbb{A}^{r-1} \setminus \{0\} \) is connected, we conclude that \( S_C \) is either the empty set or all of \( \mathbb{A}^{r-1} \setminus \{0\} \). By assumption, there is a \( v \) such that \( m_v = 0 \), so \( S_0 \) is nonempty. Thus, \( S_0 = \mathbb{A}^{r-1} \setminus \{0\} \) and \( B = 0 \) as well (since the bounds are attained). Thus, the number \( m_v \) is constant over all \( v \in \mathbb{A}^{r-1} \setminus \{0\} \) and

\[
M|_A \cong \Omega^s(k|_A) \oplus P
\]

for any subsuperalgebra \( A \cong V(\mathfrak{a}_2) \) where \( k \) is either \( k_{ee} \) or \( k_{od} \), by the classification of \( T(\mathfrak{a}_2) \).

We also claim that for any such \( A \), the parity of \( k|_A \) is constant as well. This can be seen by assuming that there are \( A \) and \( A' \) such that \( M|_A \cong \Omega^s(k_{ev}) \oplus P \) and \( M|_{A'} \cong \Omega^s(k_{od}) \oplus P' \). Now consider the dimensions of \( M_{ST}^\sigma \) and \( M_{ST}^\tau \). Since \( \dim \Omega^s(k_{ev}) = \dim \Omega^s(k_{od}) \), it follows that \( \dim P = \dim P' \). Note that \( \dim P_{ST}^\sigma = \dim P_{ST}^\tau \) and consequently, \( \dim P_{ST}^\sigma = \dim P_{ST}^\tau \) and
\[ \dim P_T = \dim P'_T. \] Finally, recall that \( \dim \Omega^s(k_{ev})_{\overline{\Pi}} \neq \Omega^s(k_{ev})_T \). Without loss of generality, assume that \( \dim \Omega^s(k_{ev})_{\overline{\Pi}} > \dim \Omega^s(k_{ev})_T \), i.e. \( s \) is an even integer. Then
\[ \dim \Omega^s(k_{od})_{\overline{\Pi}} = \dim \Omega^s(\Pi(k_{ev}))_{\overline{\Pi}} = \dim \Pi(\Omega^s(k_{ev})_{\overline{\Pi}}) = \dim \Omega^s(k_{ev})_T \]
and similarly,
\[ \dim \Omega^s(k_{od})_T = \dim \Omega^s(\Pi(k_{ev}))_T = \dim \Pi(\Omega^s(k_{ev})_{\overline{\Pi}}) = \dim \Omega^s(k_{ev})_T. \]

This implies that \( \dim \Omega^s(k_{od})_{\overline{\Pi}} < \dim \Omega^s(k_{od})_T \). These different decompositions combine to yield that \( \dim M_{\overline{\Pi}} > \dim M_T \) by considering \( M|_A \) and \( \dim M_{\overline{\Pi}} < \dim M_T \) by considering \( M|_{A'} \). This is a contradiction and so the parity of the \( k \) in the decomposition of \( M|_A \) is constant for any choice of \( A \) as well.

**Theorem 6.2.** Let \( M \) be an endotrivial \( V(\mathfrak{a}_r) \)-supermodule, then \( M \cong \Omega^n(k) \oplus P \) for some \( n \in \mathbb{Z} \) where \( k \) is either the trivial module \( k_{ev} \) or \( \Pi(k_{ev})_L = k_{od} \) and \( P \) is a projective module in \( \mathcal{F} \).

**Proof.** Let \( M \) be an endotrivial \( V(\mathfrak{a}_r) \)-supermodule and let \( A = \langle a_{r-1}, a_r \rangle \). By Theorem \ref{thm:dim_P_1} \( M|_A \cong \Omega^m(k|_A) \oplus P \). The goal is to prove that \( M \cong \Omega^m(k) \oplus P' \) or, equivalently, \( \Omega^{-m}(M) \cong k \). For simplicity, replace \( M \) by \( \Omega^{-m}(M) \) and assume that \( M|_A \cong k|_A \oplus P \).

The first step is to show that the module \( \hat{M} = a_r.M \) is a projective \( \hat{V} = V(\mathfrak{a}_r)/\langle a_r \rangle \) module. We do this by considering the rank variety \( \mathcal{V}^{\text{rank}}(\hat{M}|_{\hat{V}}) \) (see \cite[Section 6.3]{[2]}).

Now, let \( v = c_1 a_1 + \cdots + c_{r-1} a_{r-1} \) for some \( (c_1, \ldots, c_{r-1}) \in \mathbb{A}^{r-1} \setminus \{0\} \). Let \( B = \langle v, a_r \rangle \). As noted before, we have that \( M|_B \cong k|_B \oplus P_B \) where \( P_B \) is a projective \( B \)-module. Then \( a_r(M|_B) \cong a_r(k|_B) \oplus a_r(P_B) \cong a_r(P_B) \). Now, the action of \( a_r \) on these modules is trivial, so think of them now as \( v \)-modules. We also know that \( a_r(P_B) \) is still projective as a \( v \)-module, hence \( \hat{M}|_v = a_r(M)|_v \) is projective for all \( v \in \mathbb{A}^{r-1} \setminus \{0\} \). This tells us that \( \mathcal{V}^{\text{rank}}(\hat{M}|_{\hat{V}}) = \{0\} \) and so \( \hat{M} \) is a projective \( \hat{V} \)-module.

The socle of \( \hat{V} \cong V(\mathfrak{a}_{r-1}) \) is generated by \( \hat{a}_{r-1} = a_1 \times \cdots \times a_{r-1} \). Hence,
\[ \dim \hat{M} = 2^{r-1} \dim \hat{a}_{r-1}.\hat{M}. \]

Also, \( \hat{a}_r = \hat{a}_{r-1} \times a_r \) is a generator for the socle of \( V(\mathfrak{a}_r) \) and \( \hat{a}_{r-1}.\hat{M} = \hat{a}_r.M \) by construction. Therefore, \( M \) has a projective submodule, \( Q \) of dimension \( 2^r \dim \hat{a}_r.M = 2^r \dim \hat{a}_{r-1}.\hat{M} \). Thus,
\[ 2 \dim \hat{M} = \dim M - 1 \]
and we conclude that \( M \cong k \oplus Q \). Note, since this is a direct sum decomposition, as super vector spaces, \( k = k|_A \). Thus \( k \) has the same parity as \( k|_A \) (which was uniquely determined by \( M \)). Thus, the claim is proven.

We can now classify endotrivial \( \mathfrak{a}_r \)-modules for all \( r \).

**Theorem 6.3.** Let \( \mathfrak{a}_r \) be a rank \( r \) detecting subalgebra where \( r \geq 2 \), then \( T(\mathfrak{a}_r) \cong \mathbb{Z} \times \mathbb{Z}_2 \) and is generated by \( \Omega^1(k_{ev}) \) and \( k_{od} \).

**Proof.** The proof is exactly the same as in Theorem \ref{thm:dim_P_10}.

\[ \square \]
7. A Finiteness Theorem for $T(\mathfrak{g})$

Let $\mathfrak{g} = \mathfrak{g}_T \oplus \mathfrak{g}_\tau$ be a classical Lie superalgebra. Just as in the case of finite group schemes, it is not known if $T(\mathfrak{g})$ is finitely generated, but we can show that in certain cases, there are finitely many endotrivial modules of a fixed dimension. This is done by extending a proof in [10] to superalgebras.

The variety of all representations of a fixed dimension $n$ is denoted $\mathcal{V}_n$ and is defined by considering a set of generators $g_1, \ldots, g_r$ for the superalgebra $U(\mathfrak{g})$. A representation is a homomorphism of superalgebras $\phi : U(\mathfrak{g}) \to \text{End}_k(V)$ where $\dim(V) = n$, and if a basis for $V$ is fixed, we can think of this homomorphism as a superalgebra homomorphism $\phi : U(\mathfrak{g}) \to M_n(k)$. Since $g_1, \ldots, g_r$ generate $U(\mathfrak{g})$, the map $\phi$ is completely determined by $\phi(g_i)$ which is an $n \times n$ matrix with entries $(g_{i, st})$ in $k$ where $1 \leq i \leq r$ and $1 \leq s, t \leq n$.

Consider the polynomial ring $R = k[x_{i, st}]$, where $1 \leq i \leq r$ and $1 \leq s, t \leq n$, which has $rn^2$ variables. The information of each representation can be encoded in the form of a variety by defining a map $\overline{\phi} : U(\mathfrak{g}) \to M_n(R)$ by $\overline{\phi}(g_i) = (x_{i, st})$ for $1 \leq i \leq r$. Since $U(\mathfrak{g}) = T(\mathfrak{g})/\mathcal{I}$, the relations in $\mathcal{I}$ must be imposed on $M_n(R)$ by constructing the following ideal of $R$. By using $\overline{\phi}$, a relation in $\mathcal{I}$ is transferred the same relation in $M_n(R)$ by creating a relation on the rows and columns in the corresponding matrix multiplication. For example, take the relation $g_1g_2 = 0$ in $U(\mathfrak{g})$. This would correspond to the relation $\sum_{u=1}^n x_{1, su}x_{2, ut} = 0$ for each $1 \leq s, t \leq n$. Then the matrices have the same algebra structure as $U(\mathfrak{g})$ does (and now $\overline{\phi}$ is actually a homomorphism), but they are expressed as zero sets in the polynomial ring $R$. Thus, the ideal $\mathcal{I}$ uniquely corresponds to an ideal $\mathcal{J} \subseteq R$. Note that, there may be more relations, i.e. equations which must be 0, for any given $n$ dimensional representation of $U(\mathfrak{g})$, but that just means a particular representation would be a subvariety of the variety $\mathcal{V}_n$. Thus, $\mathcal{V}_n$ contains all $n$ dimensional representations of $U(\mathfrak{g})$ as subvarieties.

With this setup, we may now show [10, Lemmas 2.2 and 2.3] and a version of [10, Theorem 2.4], since $U(\mathfrak{g})$-modules satisfy a similar result to that of [10, Lemma 2.1].

**Lemma 7.1.** Let $M$ be a $U(\mathfrak{g})$-supermodule and $P$ a projective indecomposable supermodule and $m \in \mathbb{N}$. Let $\mathcal{U}$ be the subset of $\mathcal{V}_n$ of all representations, $\overline{\phi}$, of $U(\mathfrak{g})$ such that $M \otimes L_{\overline{\phi}}$ has no submodule isomorphic to $P^m$, where $L_{\overline{\phi}}$ is the module given by the representation $\overline{\phi}$. Then $\mathcal{U}$ is closed in $\mathcal{V}_n$.

**Proof.** Using a similar idea as in the proof of Theorem 6.1 consider the rank of the matrix of $\mathfrak{g}$ (see the discussion preceding Lemma 5.2 for the definition) on $M \otimes L_{\overline{\phi}}$. Denote this matrix by $M_{\mathfrak{g}}$ and let $r$ be the rank of the matrix. Since $\mathfrak{g}$ is a polynomial in the generators of $U(\mathfrak{g})$, and since the matrix of the action of $\overline{\phi}(g_i)$ on $L_{\overline{\phi}}$ has entries in the polynomial ring $R$, then $\mathfrak{g}$ acting on $L_{\overline{\phi}}$ has entries in $R$ as well. Furthermore, for a fixed representation of $M$, the matrix of the action of $\mathfrak{g}$ on $M$ has entries in $k$, so the entries of $M_{\mathfrak{g}}$ are all polynomials in $R$.

By the same reasoning in Theorem 6.1, the condition that the rank of $M_{\mathfrak{g}}$ be less than $r$ is the same condition that any $r \times r$ submatrix have determinant zero which we can then translate to a condition that certain polynomials in $R$ be zero. Hence, the subset $\mathcal{U}$ is closed in $\mathcal{V}_n$.

**Lemma 7.2.** Let $M$ be an endotrivial $U(\mathfrak{g})$-supermodule of dimension $n$. Let $\mathcal{U}$ be the subset of representations, $\overline{\phi}$ of $\mathcal{V}_n$ such that $L_{\overline{\phi}}$ is not isomorphic to $M \otimes \lambda$ for any one dimensional module $\lambda$, where $L_{\overline{\phi}}$ is the module given by $\overline{\phi}$. Then $\mathcal{U}$ is closed in $\mathcal{V}_n$. 


Proof. Let μ be a one dimensional \( U(\mathfrak{g}) \)-supermodule. Since \( M \) is endotrivial, so is \( M \otimes \mu \). Then

\[
(M \otimes \mu) \otimes (M \otimes \mu)^* \cong k \oplus \bigoplus_{i=1}^{l} P_{i}^{n_{i}}
\]

where \( P_{i} \) is a projective indecomposable module and \( n_{i} \in \mathbb{N} \). For each \( i \), let \( \mathcal{U}_{i} \) be the subset of \( \mathcal{V} \subseteq \mathcal{V}_{n} \) such that \( L_{\mathcal{V}} \otimes M^{*} \otimes \mu^{*} \) does not contain a submodule isomorphic to \( P_{i}^{n_{i}} \). By the previous lemma, each \( \mathcal{U}_{i} \) is closed and so is \( \mathcal{U}_{\mu} = \mathcal{U}_{1} \cup \cdots \cup \mathcal{U}_{l} \).

Clearly, for any \( \phi \subseteq \mathcal{U} \), \( L_{\phi} \) is not isomorphic to \( M \otimes \mu \) since they have different projective indecomposable summands. Now we will consider some \( \nu \nsubseteq \mathcal{U}_{\mu} \) and show that \( L_{\phi} \cong M \otimes \lambda \) for some one dimensional module \( \lambda \). Since \( \phi \nsubseteq \mathcal{U}_{\mu} \),

\[
L_{\phi} \otimes M^{*} \otimes \mu^{*} \cong \nu \oplus \bigoplus_{i=1}^{l} P_{i}^{n_{i}}
\]

for some supermodule \( \nu \). However, \( \dim L_{\phi} \otimes M^{*} \otimes \mu^{*} = n_{i} = \dim (M \otimes \mu) \otimes (M \otimes \mu)^* \) so \( \nu \) is one dimensional. Since \( \nu \) is one dimensional,

\[
\nu^{*} \otimes L_{\phi} \otimes M^{*} \otimes \mu^{*} \cong k \oplus \bigoplus_{i=1}^{l} (\nu^{*} \otimes P_{i}^{n_{i}}).
\]

Tensoring both sides by \( \nu \otimes M \otimes \mu \) yields

\[
L_{\phi} \oplus \bigoplus_{i=1}^{l} P_{i}^{n_{i}} \cong (M \otimes \mu \otimes \nu) \oplus \bigoplus_{i=1}^{l} (P_{i}^{n_{i}} \otimes M \otimes \mu)
\]

and so \( L_{\phi} \cong M \otimes \nu \otimes \mu \cong M \otimes \lambda \) where \( \lambda = \nu \otimes \mu \) by comparing the nonprojective summands. Then \( \mathcal{U}_{\mu} \) is exactly the subset of \( \mathcal{V}_{n} \) such that \( L_{\phi} \) is not isomorphic to \( M \otimes \mu \) where \( \mu \) is a fixed one dimensional module. The proof is concluded by observing that

\[
\mathcal{U} = \bigcap_{\lambda} \mathcal{U}_{\lambda}
\]

where the intersection is over all one dimensional modules \( \lambda \). Thus, \( \mathcal{U} \) is closed. \( \square \)

Theorem 7.3. Let \( \mathfrak{g} = \mathfrak{g}_{0} \oplus \mathfrak{g}_{1} \) be a classical Lie superalgebra such that there are only finitely many one dimensional representations in \( F_{\mathfrak{g}} \). Let \( n \in \mathbb{N} \), then there are only finitely many isomorphism classes of endotrivial modules of dimension \( n \).

Proof. Let \( M \) be an indecomposable endotrivial \( U(\mathfrak{g}) \)-supermodule of dimension \( n \). Let \( \mathcal{U}_{M} \) be the subset of \( \mathcal{V}_{n} \) of representations \( \phi \) such that the associated module \( L_{\phi} \) is isomorphic to \( M \otimes \lambda \) for some one dimensional module \( \lambda \).

Since \( U(\mathfrak{g}) \) has only finitely many isomorphism classes of one dimensional modules, there are finitely many isomorphism classes in \( \mathcal{U}_{M} \). By the previous lemma, \( \mathcal{U}_{M} \) is open in \( \mathcal{V}_{n} \) and so \( \mathcal{U}_{M} \) is a union of components in \( \mathcal{V}_{n} \).

Now consider another endotrivial module \( N \) such that \( N \) is not isomorphic to \( M \otimes \lambda \) for any one dimensional module \( \lambda \). Consider the open set \( \mathcal{U}_{N} \) of representations \( \phi \) such that the associated module \( L_{\phi} \) is isomorphic to \( N \otimes \eta \) for some one dimensional module \( \eta \). This is an open set as well and since \( N \) is not isomorphic to \( M \otimes \lambda \) for any \( \lambda \), for any \( \eta \), \( N \otimes \eta \) is not isomorphic to \( M \otimes \lambda \) either.
Since the representations in the sets cannot be isomorphic to each other, their closures cannot be the same set. Since there are finitely many components of \( V_n \), there are only a finite number of possible closures of these open subsets \( U_M \) where \( M \) is endotrivial and only a finite number of isomorphism classes in each such set. Hence, we conclude that there are only finitely many endotrivial modules of dimension \( n \).

\[ \square \]

It is interesting to note that, while some superalgebras have finitely many one dimensional modules (e.g. detecting subalgebras—which only have two one dimensional modules, \( k_{ev} \) and \( k_{od} \)), the condition may be more subtle than it initially seems. It is not hard to find examples which fail this condition. When \( \mathfrak{g} = \mathfrak{gl}(1|1) \), there are infinitely many one dimensional \( \mathfrak{g} \)-modules. The matrix realization \( \mathfrak{gl}(1|1) \) has basis vectors \( x \) and \( y \) as in \( \mathfrak{sl}(1|1) \), but has two toral basis elements \( t_1 \) and \( t_2 \) which are given by

\[ t_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

and the weights of the simple modules in \( \mathfrak{g} \) are given by \((\lambda|\mu)\) where \( \lambda, \mu \in k \) and \( t_1 \) and \( t_2 \) act on \( k \) via multiplication by \( \lambda \) and \( \mu \) respectively. If \( \lambda = -\mu \) then the representation of the simple \( \mathfrak{g} \)-module is one dimensional. Thus there are infinitely many one dimensional modules given by the representations \((\lambda| -\lambda)\).

In general, the condition that \( \mathfrak{g} \) has finitely many one dimensional modules in \( \mathcal{F} \) is equivalent to the condition that \( \mathfrak{g}_\Omega/((\mathfrak{g}, \mathfrak{g}) \cap \mathfrak{g}_\Omega) \) has finitely many one dimensional modules in \( \mathcal{F} \).

**Proposition 7.4.** Let \( \mathfrak{g} \) be a classical Lie superalgebra. Then there are finitely many one dimensional \( \mathfrak{g} \)-modules in \( \mathcal{F} \) if and only if \( \mathfrak{g}_\Omega \subseteq [\mathfrak{g}, \mathfrak{g}] \).

**Proof.** Assume that \( \mathfrak{g} \) is a Lie superalgebra such that \( \mathfrak{g}_\Omega \subseteq [\mathfrak{g}, \mathfrak{g}] \). A one dimensional representation of \( \mathfrak{g} \) corresponds to a Lie superalgebra homomorphism \( \phi : \mathfrak{g} \rightarrow k_{ev} \) since \( \text{End}_k(k) \) (for \( k \) either even or odd) is always isomorphic to \( k_{ev} \). Since \( k_{ev} \) is concentrated in degree 0 and \( \phi \) is an even map, any element of \( \mathfrak{g}_\Omega \) necessarily maps to 0. Furthermore, since \( k_{ev} \) is abelian as a Lie superalgebra, then \([\mathfrak{g}, \mathfrak{g}]\) must be mapped to zero and so by assumption \( \mathfrak{g}_\Omega \) maps to 0 as well and \( \phi \) is the 0 map. This forces the one dimensional module to be either \( k_{ev} \) or \( k_{od} \).

Now, assume that \( \mathfrak{g} \) has only finitely many one dimensional modules but that \( \mathfrak{g}_\Omega \not\subseteq [\mathfrak{g}, \mathfrak{g}] \). Let \( g \in \mathfrak{g}_\Omega \setminus [\mathfrak{g}, \mathfrak{g}] \). As noted, since \( k_{ev} \) is abelian, if \( \phi \) is a representation of \( \mathfrak{g} \), then \( \tilde{\phi} : \mathfrak{g}_\Omega/((\mathfrak{g}, \mathfrak{g}) \cap \mathfrak{g}_\Omega) \rightarrow k_{ev} \) yields another representation which agrees on nonzero elements. They are also equivalent in the sense \( \phi \) can be obtained uniquely from \( \tilde{\phi} \) and vice versa. Since \( \mathfrak{g} \) is classical, \( \mathfrak{g}_\Omega \) is a reductive Lie algebra and given that \( g \notin [\mathfrak{g}, \mathfrak{g}] \), \( g \) must be in the center of \( \mathfrak{g}_\Omega \) and therefore in the torus of \( \mathfrak{g}_\Omega \) as well. Thus, \( g \) is a semisimple element, and in \( \mathcal{F} \), \( g \) must act diagonally on any one dimensional module. If \( \mathcal{F} \) is the image of \( g \) in \( \mathfrak{g}_\Omega/((\mathfrak{g}, \mathfrak{g}) \cap \mathfrak{g}_\Omega) \), then \( \mathcal{F} \) is nonzero and \( \langle \mathcal{F} \rangle \) is a one dimensional abelian Lie superalgebra. Since \( \mathcal{F} \) acts diagonally and \( k \) is infinite, this yields infinitely many distinct one dimensional modules resulting from the diagonal action of \( \mathcal{F} \). These one dimensional modules lift (possibly non-uniquely) to \( \mathfrak{g}_\Omega/((\mathfrak{g}, \mathfrak{g}) \cap \mathfrak{g}_\Omega) \) and consequently \( \mathfrak{g} \) as well. This is a contradiction and the assumption that \( \mathfrak{g}_\Omega \not\subseteq [\mathfrak{g}, \mathfrak{g}] \) is false. \( \square \)

**Corollary 7.5.** Let \( \mathfrak{g} \) be a simple classical Lie superalgebra. Then there are finitely many one dimensional \( \mathfrak{g} \)-modules in \( \mathcal{F} \).
Proof. By the necessary condition of simplicity given in Proposition 1.2.7 of \cite{Kac1977} that $\mathfrak{g}_0 = \mathfrak{g}_1$, the lemma is proven.

Note that in the case where there are finitely many one dimensional modules in $\mathcal{F}$, there are in fact only two, $k_{ev}$ and $k_{od}$, as in the case of the detecting subalgebras.

In regard to the $\mathfrak{g} = \mathfrak{gl}(1|1)$ example, $[\mathfrak{g}, \mathfrak{g}] = \langle x, y, t_1 + t_2 \rangle$. Then $\mathfrak{g}_0 / ([\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{g}_0) = \langle t_1, t_2 \rangle / (t_1 + t_2)$ and it is clear that if $t_1$ has any weight $\lambda$, then $\mu$ is determined to be $-\lambda$. Thus, there are infinitely many one dimensional representations resulting from the free parameter $\lambda$.

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