The origin of separable states and separability criteria from entanglement-breaking channels

Bang-Hai Wang\textsuperscript{1, 2}, Qin Li\textsuperscript{1} and Dongyang Long\textsuperscript{1}

\textsuperscript{1} Department of Computer Science, Sun Yat-sen University, Guangzhou 510006, People’s Republic of China
\textsuperscript{2} Faculty of Computer, Guangdong University of Technology, Guangzhou 510006, People’s Republic of China

E-mail: wangbanghai@gmail.com, liqin805@163.com and issldy@mail.sysu.edu.cn

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Abstract

In this paper, we show that an arbitrary separable state can be the output of a certain entanglement-breaking channel corresponding exactly to the input of a maximally entangled state. A necessary and sufficient separability criterion and some sufficient separability criteria from entanglement-breaking channels are given.

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1. Introduction

Quantum entanglement, which is applied to various types of quantum information processing such as quantum computation \cite{1}, quantum dense coding \cite{2}, quantum teleportation \cite{3}, quantum cryptography \cite{4}, etc, lies at the heart of quantum information theory. However, quantum entanglement is not yet fully understood, and many problems on entanglement remain open \cite{5}. It is one of the central problems to check whether or not a state is entangled. Some necessary \cite{6–13} or sufficient \cite{14, 15} separability criteria, as well as the necessary and sufficient separability criterion for low dimension states \cite{16}, have been found. Unfortunately, there have not been any effective, necessary and sufficient separable conditions for an arbitrary state yet.

Entanglement underlines the intrinsic order of statistical relations between subsystems of a composite quantum system \cite{5, 17}. In this paper, we show that there exists a correlation between the separable state and a maximally entangled state in any quantum systems of arbitrary dimensions from the entanglement-breaking channel (EBC). This paper is organized as follows. Section 2 demonstrates that all separable states can be the output of certain EBCs corresponding exactly to the input of a maximally entangled state. The separability criteria from EBCs are investigated in section 3. Section 4 summarizes our main results.
2. The origin of all separable states can be a maximally entangled state

Let $\mathcal{H}_A, \mathcal{H}_B$ be two Hilbert spaces, and $\mathcal{H}_{A,B}$ the tensor product space $\mathcal{H}_A \otimes \mathcal{H}_B$. Denote $\mathcal{B}(\mathcal{H}_{A,B})$ as the set of operators on $\mathcal{H}_{A,B}$, and $\mathcal{B}(\mathcal{H}_{A,B})^+$ as the subset of positive semidefinite operators (i.e. the unnormalized density operators) on $\mathcal{H}_{A,B}$. Let $\dim \mathcal{H}_A = d_A$ and $\dim \mathcal{H}_B = d_B$. A mixed state $\rho \in \mathcal{B}(\mathcal{H}_{A,B})^+$ is a called separable state if it can be written as

$$\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k| \otimes |\phi_k\rangle \langle \phi_k|, \quad (1)$$

where $\{p_k\}$ is a probability distribution, and $|\psi_k\rangle$ and $|\phi_k\rangle$ are the pure states of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

A quantum channel is a completely positive (CP) linear map [18, 19]. The EBC is the class of quantum channels, for which $(I \otimes \Phi)(\rho)$ is always separable for any state $\rho$, where $\rho$ is separable or not [20–22]. By [22], an EBC can be written in either of the following equivalent forms:

$$\Phi(\sigma) = \sum_k R_k \text{Tr} F_k \sigma \quad (2)$$

$$= \sum_k |\psi_k\rangle \langle \psi_k| \langle \phi_k| \sigma |\phi_k\rangle, \quad (3)$$

where $\sigma$ is any density operator, $R_k$ is a density operator and $F_k$ is a positive semi-definite operator. The channel $\Phi$ is entanglement-breaking and trace-preserving (EBT) if and only if $\sum_k F_k = \sum_k |\phi_k\rangle \langle \phi_k| = I$, where the set $\{F_k\}$ form a POVM. The properties of EBT were investigated by many people [20–23]. The action of an EBT can be substituted by a measurement and state-preparation protocol [24]. The sender makes a measurement on the input state $\sigma$ by means of a POVM $\{F_k\}$, and sends the outcome $k$ via a classical channel to the receiver. Then the receiver prepares an agreed-upon state $R_k$ [21, 25].

However, a general channel need not be trace-preserving, and a channel is trace-preserving if there is no loss of the particle [18, 19]. Any finite-dimensional CP trace-preserving (CPT) linear map $\Phi$ can be represented as

$$\Phi(\sigma) = \sum_k E_k \sigma E_k^\dagger, \quad (4)$$

where the $E_k$ are the complex matrices satisfying

$$\sum_k E_k E_k^\dagger = I. \quad (5)$$

In this paper, we consider

$$\sum_k E_k E_k^\dagger \leq I, \quad (6)$$

i.e. the channel has only the CP property and need not be trace-preserving. Some properties hold for the channel without being trace-preserving but not for the channel which is trace-preserving, and vice versa.

We need two easily proved lemmas.

**Lemma 1** ([26]). A pure bipartite state $|\beta\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} (d_B \geq d_A)$ is a maximally entangled if and only if $E(|\beta\rangle) = \log_2 d_A$, where $E(|\beta\rangle) = -\text{Tr} (\rho_{A|B} \log_2 \rho_{A|B})$ and $\rho_{A|B} = \text{Tr}_{B(A)}(|\beta\rangle \langle \beta|)$ is the reduced density operator.
Lemma 2 ([26]). A pure bipartite state $|\beta\rangle \in \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ ($d_B \geq d_A$) is a maximally entangled state if and only if there exists an orthonormal basis in $\mathbb{C}^{d_A}$ for any given orthonormal complete basis $\{|i_A\rangle\}$ in $\mathbb{C}^{d_A}$, such that $|\beta\rangle$ can be written in the following form:

$$|eta\rangle = \frac{1}{\sqrt{d_A}} \sum_{i=1}^{d_A} |i_A\rangle \otimes |i_B\rangle.$$  

(7)

Theorem 1. An arbitrary separable state can be the output of a certain EBC corresponding exactly to the input of a maximally entangled state.

Proof. Without loss of generality, suppose an arbitrary separable state $\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k| \otimes |\phi_k\rangle \langle \phi_k|$ in $\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}$ and $d_B \geq d_A$. Since $|\psi_k\rangle$ can be represented as a linear combination of the orthonormal basis $\{|i_A\rangle\}$ in $\mathbb{C}^{d_A}$, i.e. $|\psi_k\rangle = \sum_{i} |i_A\rangle \langle i_A| \psi_k\rangle$, we have

$$\rho = \sum_k p_k \left( \sum_i |i_A\rangle \langle i_A| \psi_k\rangle \langle j_A| \psi_k\rangle \right) \left( \sum_j \langle j_A| \langle j_A| \psi_k\rangle \langle j_A| \psi_k\rangle \right) |\phi_k\rangle \langle \phi_k|$$

(8)

$$= \sum_{i,j,k} |i_A\rangle \langle i_A| \otimes |\phi_k\rangle \langle \phi_k| p_k |i_A\rangle \langle i_A| \psi_k\rangle \langle j_A| \psi_k\rangle$$

(9)

$$= \sum_{i,j} |i_A\rangle \langle j_A| \otimes \sum_k |\phi_k\rangle \langle \phi_k| \text{Tr}(|i_A\rangle \langle j_A| p_k |\psi_k\rangle \langle \psi_k|)$$

(10)

$$= \sum_{i,j} |i_A\rangle \langle j_A| \otimes \sum_k |\phi_k\rangle \langle \phi_k| \text{Tr}(|i_B\rangle \langle j_B| p_k |\psi_k\rangle \langle \psi_k|)$$

(11)

$$= \sum_{i,j} |i_A\rangle \langle j_A| \otimes \Phi(|i_B\rangle \langle j_B|)$$

(12)

$$= (I \otimes \Phi)(|I\rangle \langle I|),$$

(13)

where $\Phi(\sigma) = \sum_k |\phi_k\rangle \langle \phi_k| \text{Tr}(\sigma F_k) (F_k = p_k |\psi_k\rangle \langle \psi_k|)$ is an EBC, and $|I\rangle = \sum |i_A\rangle \otimes |i_B\rangle$ is an unnormalized maximally entangled state. \hfill \Box

Now we give an interpretation of EBC with the input of a maximally entangled state. Since $\sum_k F_k = \sum_k p_k |\psi_k\rangle \langle \psi_k| \leq I$, $\Phi(\cdot)$ is non-trace-preserving and does not provide a complete description of the processes that may occur in the system [27], the output separable state may be acquired with some probability. Let the separable state, for example, $\rho = |00\rangle \langle 00|$. With the input of the two-qubit maximally entangled state (Bell state) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, we have

$$I \otimes \Phi(\frac{1}{\sqrt{2}}|00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11|)$$

$$= |00\rangle \langle 00| + (1 + |11\rangle \langle 00|)\text{Tr}(\frac{1}{\sqrt{2}}|00\rangle \langle 00| + |01\rangle \langle 01| + |10\rangle \langle 10|)$$

$$= \frac{1}{\sqrt{2}}|00\rangle \langle 00| \otimes |00\rangle$$

$$= \frac{1}{\sqrt{2}}|00\rangle \langle 00|,$$

i.e. the result $|00\rangle \langle 00|$ by EBC occurs with the probability of $\frac{1}{2}$.

Rewriting equation (11), we have

$$\rho = \frac{1}{d_A} \sum_{i,j} |i_A\rangle \langle j_A| \otimes d_A \sum_k |\phi_k\rangle \langle \phi_k| \text{Tr}(|i_B\rangle \langle j_B| p_k |\psi_k\rangle \langle \psi_k|)$$

(14)
\[
\begin{align*}
\frac{1}{d_A} \sum_{i,j} |i_A \rangle \langle j_A | \otimes \Phi'(i_B \langle j_B |) \\
= (I \otimes \Phi')(|\beta \rangle \langle \beta |),
\end{align*}
\]
where \( \Phi'(\sigma) = d_A \sum_k |\phi_k \rangle \langle \phi_k | \text{Tr}(\sigma F_k'), F_k' = d_A p_k |\psi_k \rangle \langle \psi_k |, \) and \( |\beta \rangle = d_A^{-1/2} \sum_i |i_A \rangle \otimes |i_B \rangle \) is the normalized maximally entangled state. Clearly, only if one reduced state and all reduced states of the separable state are maximally mixed, \( \Phi'(\cdot) \) is EBT, otherwise \( \Phi'(\cdot) \) is non-trace-preserving. The theoretical and experimental analysis of non-trace-preserving processes have been carried out [28–30].

3. The separability criteria from entanglement-breaking channels

By the definition of EBC, we can easily obtain the separability criteria from EBC.

**Theorem 2.** A state is entangled if and only if it is not any output of the EBC.

It is impossible to travel through all EBCs, and therefore the separability criterion above is not operational. However, from the definition of EBC, we can obtain a class of sufficient separability criteria in the matrix form.

3.1. The separability criteria from entanglement-breaking channels with trace-preserving

Observe the CP linear map \( \Pi_1 \):

\[
\sigma \mapsto \Pi_1[\sigma] = \sum_{n=1}^{d_B} |e_n \rangle \langle e_n | = \sum_{n=1}^{d_B} |e_n \rangle \langle e_n | \otimes |e_n \rangle \langle e_n |,
\]

where the set \( \{|e_n\rangle\} \) form the orthonormal basis in \( \mathcal{H}_B \) and \( \sum_n |e_n \rangle \langle e_n | = I \). Let \( P_n = |e_n \rangle \langle e_n | \), we can obtain the following linear map:

\[
\sigma \mapsto \Pi_1[\sigma] = \sum_{n=1}^{d_B} P_n \sigma P_n,
\]

which is the mathematical description of the wave-packet reduction [31]. It is not hard to see that equation (17) is EBT.

**Corollary 1.** A density operator \( \rho \) is separable if its entries \( \rho_{i,k,l} (j \neq l) = 0, \) where \( \rho_{i,j,k,l} = \langle i|\rho|k,l \rangle, \{|i\rangle\} \) and \( \{|k,l\rangle\} \) in \( \mathcal{H}_A, \{|j\rangle\} \) in \( \mathcal{H}_B. \) are computational bases.

**Proof.** An arbitrary density operator \( \rho \in B(\mathcal{H}_A, \mathcal{H}_B)^+ \) can be defined as

\[
\rho = \sum_{ij,kl} \langle i|\rho|k,l \rangle \langle i| \langle k| \otimes |j \rangle \langle l |
\]

by computational (real orthonormal) bases \( \{|i\rangle\} \) and \( \{|k,l\rangle\} \) in \( \mathcal{H}_A, \{|j\rangle\} \) in \( \mathcal{H}_B. \) By equations (17) and (19), we have

\[
(I \otimes \Pi_1)[\rho] = \sum_{ij,kl} \langle i|\rho|k,l \rangle \langle i| \langle k| \otimes \sum_{n=1}^{d_B} |e_n \rangle \langle e_n | \langle (j| \langle l| \otimes |e_n \rangle \langle e_n |).
\]

Let the orthonormal basis \( \{|e_n\rangle\} \) be the same as \( \{|l|\} \); it follows that

\[
(I \otimes \Pi_1)[\rho] = \sum_{ik,l} \langle i|\rho|k,l \rangle \langle i| \langle k| \otimes \langle (l| \langle l|) = \rho_{i,k,l}.
\]
This result establishes a class of the representations of matrix form for the separable state output from the EBT.

According to corollary 1, for example, a two-qubit quantum state

\[
\rho = \begin{pmatrix}
\rho_{00,00} & 0 & \rho_{00,10} & 0 \\
0 & \rho_{01,01} & 0 & \rho_{01,11} \\
\rho_{10,00} & 0 & \rho_{10,10} & 0 \\
0 & \rho_{11,01} & 0 & \rho_{11,11}
\end{pmatrix}
\]  

(22)

is separable.

By a simple calculation, we can obtain different sufficient separability criteria in the matrix form with different \([|e_i\rangle\] in equation (20). For example, let \(|e_0\rangle = \frac{1}{1}(\sqrt{2}|0\rangle - |1\rangle + |2\rangle)\), \(|e_1\rangle = \frac{1}{1}((\sqrt{2}|0\rangle + |1\rangle - |2\rangle)\) and \(|e_2\rangle = \frac{1}{12}((1) + |2\rangle)\) for qutrits, if

\[
\rho = \begin{pmatrix}
\rho_{00,00} & \rho_{00,01} & \rho_{00,02} & \rho_{00,10} & \rho_{00,11} & \rho_{00,12} & \rho_{00,20} & \rho_{00,21} & \rho_{00,22} \\
\rho_{01,00} & \rho_{01,01} & \rho_{01,02} & \rho_{01,10} & \rho_{01,11} & \rho_{01,12} & \rho_{01,20} & \rho_{01,21} & \rho_{01,22} \\
\rho_{10,00} & \rho_{10,01} & \rho_{10,02} & \rho_{10,10} & \rho_{10,11} & \rho_{10,12} & \rho_{10,20} & \rho_{10,21} & \rho_{10,22} \\
\rho_{11,00} & \rho_{11,01} & \rho_{11,02} & \rho_{11,10} & \rho_{11,11} & \rho_{11,12} & \rho_{11,20} & \rho_{11,21} & \rho_{11,22} \\
\rho_{12,00} & \rho_{12,01} & \rho_{12,02} & \rho_{12,10} & \rho_{12,11} & \rho_{12,12} & \rho_{12,20} & \rho_{12,21} & \rho_{12,22} \\
\rho_{20,00} & \rho_{20,01} & \rho_{20,02} & \rho_{20,10} & \rho_{20,11} & \rho_{20,12} & \rho_{20,20} & \rho_{20,21} & \rho_{20,22} \\
\rho_{21,00} & \rho_{21,01} & \rho_{21,02} & \rho_{21,10} & \rho_{21,11} & \rho_{21,12} & \rho_{21,20} & \rho_{21,21} & \rho_{21,22} \\
\rho_{22,00} & \rho_{22,01} & \rho_{22,02} & \rho_{22,10} & \rho_{22,11} & \rho_{22,12} & \rho_{22,20} & \rho_{22,21} & \rho_{22,22}
\end{pmatrix}
\]  

(23)

is an arbitrary density operator (entangled or not), then

\[
\rho' = \begin{pmatrix}
\rho^{00} & \rho^{01} & \rho^{02} \\
\rho^{10} & \rho^{11} & \rho^{12} \\
\rho^{20} & \rho^{21} & \rho^{22}
\end{pmatrix}
\]  

(24)

is separable, where

\[
\rho^{i,j} = \begin{pmatrix}
\rho_{00} & \rho_{01} & \rho_{02} \\
\rho_{10} & \rho_{11} & \rho_{12} \\
\rho_{20} & \rho_{21} & \rho_{22}
\end{pmatrix}
\]  

(25)

and

\[
\rho^{i,j,k}_{00} = \frac{2\rho_{i,j,0} + \rho_{i,j,1} - \rho_{i,j,2} + \rho_{i,j,3}}{4},
\]

(26)

\[
\rho^{i,j,k}_{01} = \frac{\rho_{i,j,0} + \rho_{i,j,1} - \rho_{i,j,2} + \rho_{i,j,3}}{4},
\]

(27)

\[
\rho^{i,j,k}_{02} = \frac{-\rho_{i,j,0} + \rho_{i,j,1} + \rho_{i,j,2} + \rho_{i,j,3}}{4},
\]

(28)

\[
\rho^{i,j,k}_{10} = \frac{-\rho_{i,j,0} + \rho_{i,j,1} + \rho_{i,j,2} + \rho_{i,j,3}}{4},
\]

(29)

\[
\rho^{i,j,k}_{11} = \frac{2\rho_{i,j,0} + 3\rho_{i,j,1} - \rho_{i,j,2} + \rho_{i,j,3}}{8},
\]

(30)

\[
\rho^{i,j,k}_{12} = \frac{-2\rho_{i,j,0} + \rho_{i,j,1} + 3\rho_{i,j,2} + \rho_{i,j,3}}{8},
\]

(31)
\[
\rho_{20} = \frac{-\rho_{i0,j1} + \rho_{i0,j2} - \rho_{i1,j0} + \rho_{i1,j2}}{4},
\]
(32)
\[
\rho_{21} = \frac{-2\rho_{i0,j0} + \rho_{i1,j1} + 3\rho_{i1,j2} + 3\rho_{i2,j1}}{8},
\]
(33)
\[
\rho_{22} = \frac{2\rho_{i0,j0} + \rho_{i1,j1} + \rho_{i2,j1} + 3\rho_{i2,j2}}{8},
\]
(34)
for all \(0 \leq i_A, j_A \leq 2\).

Clearly, we can obtain different sufficient separability criteria in the matrix form from different EBTs.

3.2. The separability criteria from entanglement-breaking channels without trace-preserving

Observe the depolarizing channel [27]
\[
D_\epsilon(\rho) = \left(1 - \epsilon\right) \frac{I_{d_A,B}}{d_{A,B}} + \epsilon \rho.
\]
(35)

Concretely, let us consider first the case of states of two qubits. According to [14], an arbitrary density operator \(\rho\) for two qubits can be written as
\[
\rho = \frac{1}{4} \left( (\omega_{i0} \omega_{j0} + \epsilon (c_{i0} \omega_{j0} + \omega_{j0} c_{i0} + c_{ij})) P_i \otimes P_j + (\omega_{i0} \omega_{j0} - c_{i0} \omega_{j0} + \omega_{j0} c_{i0} - c_{ij}) \overline{P}_i \otimes P_j 
\right.
\]
\[
+ (\omega_{i0} \omega_{j0} + c_{i0} \omega_{j0} - \omega_{j0} c_{i0} - c_{ij}) P_i \otimes \overline{P}_j + (\omega_{i0} \omega_{j0} - c_{i0} \omega_{j0} - \omega_{j0} c_{i0} + c_{ij}) \overline{P}_i \otimes \overline{P}_j \right],
\]
(36)
and the maximally mixed density operator \(I_2\) for two qubits may be written as
\[
\frac{1}{4} (\omega_{i0} \omega_{j0}) [P_i \otimes P_j + \overline{P}_i \otimes P_j + P_i \otimes \overline{P}_j + \overline{P}_i \otimes \overline{P}_j],
\]
(37)
where \(I_2\) is the \(2 \times 2\) identity matrix, \(\omega_{i(j)} = 1/3, -1 \leq c_{i(j)0} \leq 1, -1 \leq c_{ij} \leq 1\),
\[
P_{i(j)} = \frac{1}{2} (I_2 + \sigma_{i(j)}),
\]
\[
\overline{P}_{i(j)} = \frac{1}{2} (I_2 - \sigma_{i(j)}),
\]
\(\sigma_{i(j)}\) are the Pauli matrices and \(i(j) = 1, 2, 3\).

Let
\[
q_{0}^{ij} = \frac{1}{4}(\omega_{i0} \omega_{j0} + \epsilon (c_{i0} \omega_{j0} + \omega_{j0} c_{i0} + c_{ij})),
\]
\[
q_{1}^{ij} = \frac{1}{4}(\omega_{i0} \omega_{j0} + \epsilon (-c_{i0} \omega_{j0} + \omega_{j0} c_{i0} - c_{ij})),
\]
\[
q_{2}^{ij} = \frac{1}{4}(\omega_{i0} \omega_{j0} + \epsilon (c_{i0} \omega_{j0} - \omega_{j0} c_{i0} - c_{ij})),
\]
\[
q_{3}^{ij} = \frac{1}{4}(\omega_{i0} \omega_{j0} + \epsilon (-c_{i0} \omega_{j0} - \omega_{j0} c_{i0} + c_{ij})).
\]

We have
\[
D_\epsilon(\rho) = (1 - \epsilon) \frac{I_A}{d_{A,B}} + \epsilon \rho
\]
(38)
\[
= q_{0}^{ij} P_i \otimes P_j + q_{1}^{ij} \overline{P}_i \otimes P_j + q_{2}^{ij} P_i \otimes \overline{P}_j + q_{3}^{ij} \overline{P}_i \otimes \overline{P}_j
\]
(39)
\[
= (I \otimes \Phi)((|00\rangle + |11\rangle)(|00\rangle + |11\rangle)),
\]
(40)
where \(\Phi(\sigma) = P_i \text{Tr}(\sigma q_{0}^{ij} P_i) + P_j \text{Tr}(\sigma q_{1}^{ij} \overline{P}_i) + \overline{P}_i \text{Tr}(\sigma q_{2}^{ij} P_i) + \overline{P}_j \text{Tr}(\sigma q_{3}^{ij} \overline{P}_i).
\]

Clearly, \(\sum_{i,j} q_{ij}^{ij} = 1\). Since \(\Phi(\cdot)\) is an EBC if \(q_{ij}^{ij} \geq 0\), i.e. \(\epsilon \leq \frac{1}{15}\), \(D_\epsilon(\rho)\) is separable.

Thus, the above result coincides with the result of [14].
By equation (35), we have

\[ D_\epsilon(\rho) = (1 - \epsilon) \frac{I}{d_{A,B}} + \epsilon \rho \]  

(41)

\[ = (I \otimes \Phi)(|I\rangle\langle I|) \]  

(42)

\[ = \sum_{i,j} |i_A\rangle\langle j_A| \otimes \Phi(|i_B\rangle\langle j_B|) \]  

(43)

\[ = \sum_{i_A, j_A} |i_A\rangle\langle j_A| \otimes \sum_{i_B, j_B} \Phi_{i_A, j_A}(|i_B\rangle\langle j_B|), \]  

(44)

where

\[ \Phi_{i_A, j_A}(|i_B\rangle\langle j_B|) = \begin{cases} 
\sum_{i_B, j_B} (\epsilon \langle i_Aj_B|\rho|j_Aj_B\rangle |i_B\rangle\langle j_B|) + \frac{1 - \epsilon}{d_{A,B}} |i_B\rangle\langle j_B|, & i = j \\
\sum_{i_B, j_B} (\epsilon \langle i_Aj_B|\rho|j_Aj_B\rangle |i_B\rangle\langle j_B|), & i \neq j.
\end{cases} \]  

(45)

Since the depolarizing channel in equation (35) is entanglement breaking if \( \epsilon \leq \frac{1}{d^2} \) [32], \( \Phi_{i_A, j_A}() \) is EBC for all \( 0 \leq i_A, j_A \leq d_A - 1 \) if \( \epsilon \leq \frac{1}{d^2} \).

Furthermore, for an arbitrary density operator \( \rho \) (separable or not),

\[ \rho = (I \otimes \Psi)(|I\rangle\langle I|) \]  

(46)

\[ = \sum_{i_A, j_A} |i_A\rangle\langle j_A| \otimes \sum_{i_B, j_B} \Psi_{i_A, j_A}(|i_B\rangle\langle j_B|), \]  

(47)

where

\[ \Psi_{i_A, j_A}(|i_B\rangle\langle j_B|) = \sum_{i_B, j_B} (i_Ai_B|\rho|j_Aj_B)i_B\rangle\langle j_B| \]  

(48)

is a map for all \( 0 \leq i_A, j_A \leq d_A - 1 \). By equation (48), we have

\[ \Psi_{i_A, j_A}(\sigma) = \Psi_{i_A, j_A} \left( \sum_{i_B} |i_B\rangle\sigma\langle j_B| |i_B\rangle\langle j_B| \right) \]  

(49)

\[ = \sum_{i_B} (i_Ai_B|\rho|j_Aj_B) |i_B\rangle\sigma|j_B\rangle |i_B\rangle\langle j_B|, \]  

(50)

where \( \sigma \) is any density operator in \( \mathbb{C}^{d_B} \) and its entries \( \sigma_{i_B, j_B} = \langle i_B|\sigma|j_B\rangle |i_B\rangle\langle j_B| \).

An arbitrary bipartite state \( \rho \) in \( \mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B} \) may be considered as an \( d_A \times d_A \) matrix with the entries being \( d_B \times d_B \) matrices.

Concretely, let

\[
\rho = \begin{pmatrix}
\rho_{00} & \rho_{01} & \ldots & \rho_{0(d_A-1)} \\
\rho_{10} & \rho_{11} & \ldots & \rho_{1(d_A-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{(d_A-1)0} & \rho_{(d_A-1)1} & \ldots & \rho_{(d_A-1)(d_A-1)}
\end{pmatrix},
\]  

(51)

where

\[
\rho_{i_A, j_A}^{i_B, j_B} = \begin{pmatrix}
\rho_{00}^{i_A, j_A} & \rho_{01}^{i_A, j_A} & \ldots & \rho_{0(d_B-1)}^{i_A, j_A} \\
\rho_{10}^{i_A, j_A} & \rho_{11}^{i_A, j_A} & \ldots & \rho_{1(d_B-1)}^{i_A, j_A} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{(d_B-1)0}^{i_A, j_A} & \rho_{(d_B-1)1}^{i_A, j_A} & \ldots & \rho_{(d_B-1)(d_B-1)}^{i_A, j_A}
\end{pmatrix},
\]  

(52)

for all \( 0 \leq i_A, j_A \leq d_A - 1 \). Therefore, we can obtain the following theorem.
Theorem 3. A density operator $\rho$ is separable if all blocks $\rho_{iA,jA}(0 \leq iA, jA \leq d_A - 1)$ of $\rho$ are positive.

Proof. By equations (50)–(52), we have

$$\Psi_{iA,jA}(\sigma) = \begin{pmatrix}
\rho_{00}^{iA,jA} \cdot \sigma_{00} & \rho_{01}^{iA,jA} \cdot \sigma_{01} & \cdots & \rho_{0(dA-1)}^{iA,jA} \cdot \sigma_{0(dA-1)} \\
\rho_{10}^{iA,jA} \cdot \sigma_{10} & \rho_{11}^{iA,jA} \cdot \sigma_{11} & \cdots & \rho_{1(dA-1)}^{iA,jA} \cdot \sigma_{1(dA-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_{(dA-1)0}^{iA,jA} \cdot \sigma_{(dA-1)0} & \rho_{(dA-1)1}^{iA,jA} \cdot \sigma_{(dA-1)1} & \cdots & \rho_{(dA-1)(dA-1)}^{iA,jA} \cdot \sigma_{(dA-1)(dA-1)}
\end{pmatrix}$$

(53)

$$= \rho^{iA,jA} \circ \sigma,$$

(54)

where $\rho^{iA,jA} \circ \sigma$ denotes the Hadamard product of $\rho^{iA,jA}$ and $\sigma$. By Schur product theorem [33], since $\sigma$ is a density operator in $\mathbb{C}^{dA}$ and positive, $\Psi_{iA,jA}(\sigma)$ is positive if $\rho^{iA,jA}$ is positive. Since all $\rho^{iA,jA}$ are positive, all $\Psi_{iA,jA}(\sigma)$ are positive. It is not difficult to see that all $\Psi_{iA,jA}(\sigma)$ can be written in the form of equation (2), and therefore $\rho$ is separable. $\square$

Note that the identity map up to a common factor on a subsystem does not mean the identity map on a composite system for channel without trace-preserving. For example,

$$\rho = \frac{1}{4} \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1
\end{pmatrix}$$

(55)

is separable. Clearly, the state in theorem 3 is a family of completely new separable PPT states (states which are positive under partial transposition) [16]. ByFejer’s theorem [33], we have the following.

Corollary 2. A density operator $\rho$ is separable if

$$\sum_{iA,jA} \rho_{iA,jA}^{iA,jA} \cdot \sigma_{iA,jA} \geq 0$$

(56)

for all $0 \leq iA, jA \leq d_A - 1$ and any density operator $\sigma$ in $\mathbb{C}^{dA}$.

4. Conclusion

In conclusion, we have demonstrated that the origin of an arbitrary separable state in arbitrary composite quantum systems of arbitrary dimensions can originate from a maximally entangled state by the EBC. A class of separability criteria can be obtained from the EBC and a family of completely new separable PPT states is given. The separability criteria from EBC without trace-preserving are under investigation.

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