SOLITON EQUATIONS IN 2+1 DIMENSIONS: REDUCTIONS,
BILINEARIZATIONS AND SIMPLEST SOLUTIONS

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Abstract

In the preceding paper [13], we presented, in particular, the some
(2+1)-dimensional integrable reductions of the M-IX equation, moreover,
their gauge equivalent counterparts. In this paper, we first construct the
some new (1+1)-, and/or (2+0)-dimensional reductions - the $\sigma$- model
equations with potentials. We next establish the gauge equivalence be-
tween the (1+1)-dimensional integrable classical compressible Heisenberg
ferromagnet models and the Yajima-Oikawa and Ma equations. The bi-
linear forms of the M-IX and Zakharov equations and their simplest one
soliton solutions are found. Also it is shown that the Zakharov and Fokas
equations are formally equivalent to each other.
1 Introduction

Considerable effort has been given recently to investigate (2+1)-dimensional soliton equations (see, for example, [1-5]). The study of these equations has thrown up new ideas in soliton theory and in other related branches of mathematics and physics. Integrable spin systems are an important both from the mathematical and physical points of view, subclass of soliton equations [6-15]. In this paper we consider the Myrzakulov IX (M-IX) equation [10]

\[ S_t = S \wedge M_1 S - iA_2 S_x - iA_1 S_y \]  

(1a)

\[ M_2 u = 2\alpha^2 S(S_x \wedge S_y) \]  

(1b)
where $\alpha, b, a = \text{consts}$, $S = (S_1, S_2, S_3)$, $S^2 = E = \pm 1$ and

$$M_1 = \alpha^2 \frac{\partial^2}{\partial y^2} + 4\alpha(b - a) \frac{\partial^2}{\partial x \partial y} + 4(a^2 - 2ab - b) \frac{\partial^2}{\partial x^2},$$

$$M_2 = \alpha^2 \frac{\partial^2}{\partial y^2} - 2\alpha(2a + 1) \frac{\partial^2}{\partial x \partial y} + 4a(a + 1) \frac{\partial^2}{\partial x^2},$$

$$A_1 = i\{\alpha(2b + 1)u_y - 2(2ab + a + b)u_x\},$$

$$A_2 = i\{4\alpha^{-1}(2a^2b + a^2 + 2ab + b)u_x - 2(2ab + a + b)u_y\}.$$

Equation (1) is integrable in the sense that it admits the Lax representation (LR) [10] and has different types solutions [23]. In general we will distinguish the two integrable cases: the M-IXA equation as $\alpha^2 = 1$ and the M-IXB equation as $\alpha^2 = -1$. Besides, equation (1) contains several interesting particular cases: (i) $a = b = -1$, yields the M-VIII equation; (ii) $a = b = -\frac{1}{2}$, yields the celebrated Ishimori equation and so on. Equation (1) is the (2+1)-dimensional integrable generalisation of the Landau-Lifshitz equation (LLE)

$$S_t = S \wedge S_{xx}$$

and in 1+1 dimensions reduces to it.

This paper is a sequel to the preceding paper [13]. In [13], we have presented some results on equation (1), in particular, the some (2+1)-dimensional reductions. The main goal of the present paper is to continue the studies started in [13] and the key questions which we would like to address here are the following ones. First of all we want to find the other reductions in 1+1 and/or 2+0 dimensions. Secondly, we will present the bilinear form of equation (1) that allows us to construct the different exact solutions of (1) and we demonstrate its work, presenting the simplest 1-soliton solution (1-SS).

The outline of this paper is as follows. After recalling in section 2, some basic facts related to the M-IX equation, in section 3 we will present some (2+0)-dimensional reductions of equation (1) - the some $\sigma$ -models with potentials and their equivalent counterparts. In section 4 we show how derive the M-XXXIV equation from equation (1), which describe nonlinear dynamics of compressible magnets. Also we establish the gauge equivalence between the M-XXXIV equation and the Yajima-Oikawa equation (YOE). In sections 5 and 6, we present the bilinear forms of equation (1) and its gauge equivalent counterpart - the Zakharov equation (ZE), and use it to construct their simplest 1-SS. A connection between the ZE and the Fokas equation (FE) is discussed in section 7. The last section is devoted to the concluding remarks.

2 Some basic facts on the M-IX equation

In this section we briefly recall the some basic facts related to the M-IX equation (1).
2.1 The Lax representation

As integrable, equation (1) has the following LR [10]

\[ \alpha \Phi_y = [S + (2a + 1)I]\Phi_x \quad (3a) \]

\[ \Phi_t = 2i[S + (2b + 1)I]\Phi_{xx} + W\Phi_x \quad (3b) \]

with

\[ W = 2i\{(2b+1)(F^+ + F^-)S) + (F^+ S + F^-) + (2b-a+\frac{1}{2})SS_x + \frac{1}{2}S_x + \alpha S_S_y \}, \]

\[ S = \begin{pmatrix} S_3 & rS^- \\ rS^+ & -S_3 \end{pmatrix}, \quad S^\pm = S_1 \pm iS_2, \quad S^2 = EI, \quad E = \pm 1, \quad r^2 = \pm 1, \]

\[ F^\pm = A \pm D, \quad A = i[u_y - \frac{2a}{\alpha}u_x], \quad D = i\left[\frac{2(a+1)}{\alpha}u_x - u_y \right]. \]

The compatibility condition of these linear equations gives

\[ iS_t + \frac{1}{2}[S,M_1S] + A_2S_x + A_1S_y = 0 \quad (4a) \]

\[ M_2u = \frac{\alpha^2}{2i}tr(S[S_x,S_y]) \quad (4b) \]

which is the matrix form of equation (1) within to change \( t \) to \(-t\).

2.2 Gauge equivalent counterpart

It is well known that the gauge equivalent counterpart of equation (1) is the following ZE [5]

\[ iq_t + M_1q + vq = 0 \quad (5a) \]

\[ ip_t - M_1p - vp = 0 \quad (5b) \]

\[ M_2v = -2M_1(pq). \quad (5c) \]

This equation contains many interesting particular cases such as the Davey-Stewartson (DS) equation, the YOE and so on. In 1+1 dimensions equation (5) reduces to the nonlinear Schrödinger equation (NLSE)

\[ iq_t + q_{xx} + 2E |q|^2 q = 0. \quad (6) \]

The LR of the ZE (5) is given by [5]

\[ \alpha \Psi_y = 2B_1\Psi_x + B_0\Psi \quad (7a) \]

\[ \Psi_t = 4iC_2\Psi_{xx} + 2C_1\Psi_x + C_0\Psi \quad (7b) \]

with

\[ B_1 = \begin{pmatrix} a+1 & 0 \\ 0 & a \end{pmatrix}, \quad B_0 = \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} \]

\[ C_2 = \begin{pmatrix} b+1 & 0 \\ 0 & b \end{pmatrix}, \quad C_1 = \begin{pmatrix} 0 & iq \\ ip & 0 \end{pmatrix}, \quad C_0 = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \]
\[ c_{12} = i[2(2b - a + 1)q_x + i\alpha q_y], \quad c_{21} = i[2(a - 2b)p_x - i\alpha p_y]. \]

Here \( c_{jj} \) is the solution of the following equations

\[
\begin{align*}
2(a + 1)c_{11x} - \alpha c_{11y} &= i[2(2b - a + 1)(pq)_x + \alpha(pq)_y] \quad (8a) \\
2ac_{22x} - \alpha c_{22y} &= i[2(a - 2b)(pq)_x - \alpha(pq)_y]. \quad (8b)
\end{align*}
\]

Note that the ZE (5) admits the Painleve property and is integrable in this P-sense [15]. Also it has the different types solutions (solitons, dromions and so on) [15, 27]

### 2.3 The (2+1)-dimensional reductions

As above mentioned, equations (1) and (5) contain the several important particular cases. Let us recall the some of these cases.

#### 2.3.1 The Myrzakulov VIII equation

First, let us consider the reduction of the M-IX equation (4) as \( a = b = -1 \). We have

\[
\begin{align*}
iS_t + \frac{1}{2}[S, S_{YY}] + iwS_Y &= 0 \quad (9a) \\
w_X + w_Y + \frac{1}{4i}tr(S[S_X, S_Y]) &= 0 \quad (9b)
\end{align*}
\]

where \( X = x/2, \quad Y = y/\alpha, \quad w = -\alpha^{-1}u_Y \). This equation is the Myrzakulov VIII (M-VIII) equation [1]. The gauge equivalent counterpart of equation (9) is given by

\[
\begin{align*}
iq_t + q_{YY} + vq &= 0 \quad (10a) \\
 ip_t - p_{YY} - vp &= 0 \quad (10b) \\
v_X + v_Y + 2(pq)_Y &= 0 \quad (10c)
\end{align*}
\]

which we denote as the M-VIII \( q \) equation. The LR of these equations we can get from equations (3) and (7), respectively, as \( a = b = -1 \) [10]. Equations (9)-(10) admit the different types exact solutions, such as solitons, dromions, vortices and so on [10, 24-25].

#### 2.3.2 The Ishimori equation

Now let \( a = b = -\frac{1}{2} \). Then equation (4) reduces to the known Ishimori equation [9]

\[
\begin{align*}
iS_t + \frac{1}{2}[S, (S_{xx} + \alpha^2 S_{yy})] + iu_yS_x + iu_xS_y &= 0 \quad (11a) \\
\alpha^2 u_{yy} - u_{xx} &= \frac{\alpha^2}{2i}tr(S[S_x, S_y]). \quad (11b)
\end{align*}
\]

From equation (5) we get the gauge equivalent counterpart of equation (11)

\[
\begin{align*}
iq_t + q_{xx} + \alpha^2 q_{yy} + vq &= 0 \quad (12a)
\end{align*}
\]
\[ \alpha^2 v_{yy} - v_{xx} = -2(\alpha^2(pq)_{yy} + (pq)_{xx}) \]  

which is the famous DS equation. This fact was for first time established in [2]. The LR of (11) and (12) we can get from (3) and (7), respectively, as \( a = b = -\frac{1}{2} \).

### 3 σ-models with potentials

It is interesting to note that equation (1) admits some (1+1)-dimensional (and/or (2+0)-dimensional) reductions. In this section we present the σ-model with potential, which are the stationary limit of the M-IX equation.

Consider the (2+1)-dimensional LLE

\[ S_t = S \wedge \Delta S \]  

It is well known that the LLE (13) in the stationary limit coincide with the σ-model equation

\[ \Delta S + (\nabla S)^2 S = 0. \]  

Here

\[ S^2 = S^2_x + r^2(S^2_1 + S^2_2) = E = \pm 1, \quad \Delta = \frac{\partial^2}{\partial x^2} + \alpha^2 \frac{\partial^2}{\partial y^2}, \]

\[ \nabla = i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y}, \quad i^2 = 1, \quad j^2 = \alpha^2 = \pm 1, \quad r^2 = \pm 1, \quad E = \pm 1. \]

From these results naturally arises the following question: what σ-model equation is the stationary limit of equation (1)? Let us find it. In the stationary limit, i.e. when the spin vector \( S \) independent of \( t \), equation (4) takes the form

\[ M_1 S + \{a_1 S_x^2 + a_2 S_x S_y + a_3 S_y^2\} S + A_2 SS + A_1 SS = 0 \]  

\[ M_2 u = \frac{\alpha^2}{2i} tr(S[S_x, S_y]) \]

where \( M_1 \) we write in the form

\[ M_1 = a_3 \frac{\partial^2}{\partial y^2} + a_2 \frac{\partial^2}{\partial x \partial y} + a_1 \frac{\partial^2}{\partial x^2}, \quad a_1 = 4(a^2 - 2ab - b), \quad a_2 = 4\alpha(b-a), \quad a_3 = \alpha^2, \]

which is the Myrzakulov XXXII (M-XXXII) equation [10]. At the same time, the stationary limit of the ZE (5) looks like

\[ M_1 q + vq = 0, \quad M_1 p + vp = 0, \quad M_2 v = -2M_1(pq). \]

which is the some complex modified Klein-Gordon equation (mKGE).

Also we would like note that the M-XXXII equation (15), in turn contains the several particular cases. So, for example, in the case \( a = b = -\frac{1}{2} \), we have the following σ-model

\[ S_{xx} + \alpha^2 S_{yy} + (S_x^2 + \alpha^2 S_y^2) S + iu_y SS + iu_x SS = 0 \]
\[ \alpha^2 u_{yy} - u_{xx} = \frac{\alpha^2}{2t} tr(S[S_x, S_y]) \]  
(17b)

which is the Myrzakulov XIII (M-XIII) equation [10]. In this case, from the mKGE (16) we obtain the corresponding equivalent equation

\[ q_{xx} + \alpha^2 q_{yy} + vq = 0 \]  
(18a)

\[ \alpha^2 v_{yy} - v_{xx} = -2\{\alpha^2(pq)_{yy} + (pq)_{xx}\}. \]  
(18b)

4 Reductions in 1+1 dimensions: the gauge equivalence of the Myrzakulov XXXIV equation and the YOE

Now let us consider the case, when \( X = t \). Then the M-VIII equation (9) pass to the following Myrzakulov XXXIV (M-XXXIV) equation

\[ iS_t + \frac{1}{2}[S, S_{YY}] + iwS_Y = 0 \]  
(19a)

\[ w_t + w_Y + \frac{1}{4}\{tr(S^2)\}_Y = 0. \]  
(19b)

The M-XXXIV equation (19) was proposed in [10] to describe nonlinear dynamics of compressible magnets. It is integrable and has the different soliton solutions [10, 26].

In our case equation (10) becomes

\[ iq_t + q_{YY} + vq = 0 \]  
(20a)

\[ ip_t - p_{YY} - v_p = 0 \]  
(20b)

\[ v_t + v_Y + 2(pq)_Y = 0 \]  
(20c)

that is the YOE [16]. So, we have proved that the M-XXXIV equation (19) and the YOE (20) is gauge equivalent to each other. The LR of (19) and (20) we can get from (3) and (7), respectively, as \( a = b = -1 \) (see, for example, the ref.[10]). Note that our LR for the YOE (20) is different than that which was presented in [16].

Also we would like note that the M-VIII equation (9) we usually write in the following form

\[ iS_t + \frac{1}{2}[S, S_{\xi\xi}] + iwS_{\xi} = 0 \]  
(21a)

\[ w_\eta - \frac{1}{4i} tr(S[S_{\xi}, S_\eta]) = 0 \]  
(21b)

where

\[ \xi = \frac{x}{2} + \frac{a + 1}{\alpha} y, \quad \eta = -\frac{x}{2} - \frac{a}{\alpha} y. \]  
(22)

This equation also admits the some soliton solutions [24]. The gauge equivalent counterpart of this equation is given by

\[ iq_t + q_{\xi\xi} + vq = 0 \]  
(23a)
\[ v_\eta + 2r^2(\bar{q}q)\xi = 0 \]  
(23b)

that is the other ZE [5]. As \( \eta = t \) equation (21) take the other form of the
M-XXXIV equation

\[ iS_t + \frac{1}{2}[S, S_{\xi \xi}] + iwS_\xi = 0 \]  
(24a)

\[ w_t + \frac{1}{4}(\text{tr}S^2_\xi)\xi = 0. \]  
(24b)

Some soliton solutions of this equation were found in [26], using the Hirota
bilinear method. The gauge equivalent counterpart of the M-XXXIV equation
(24) we get from (23)

\[ iq_t + q\xi + vq = 0, \]  
(25a)

\[ v_t + 2r^2(\bar{q}q)\xi = 0, \]  
(25b)

which is called the Ma equation (ME) and was considered in [17]. The LR of
equations (24) and (25) were found in [10]. Note that our LR of equation (25)
is different than the LR, which was presented in [17].

5 Bilinearization of the M-IX equation

The M-IX equation contains rich class exact solutions such as solitons, dromions,
vortices, lumps and so on [10, 23]. To construct solutions of soliton equations
there are exist several powerful methods [1-5]. One of beatfull and constructive
among them is the Hirota method. Here we make use of Hirota’s method to get
exact 1-SS of equation (1), which satisfies the boundary condition \( S = (0, 0, 1) \)
as \( x, y = \pm \infty \). The use of the IST method is left in the future. By putting [23]

\[ S^+ = \frac{2\bar{g}g}{ff + gg}, \quad S_3 = \frac{\bar{f}f - \bar{g}g}{ff + gg}, \]  
(26a)

\[ u_x = 2i\alpha(2a + 1)D_x(\bar{f} \circ f + \bar{g} \circ g) + 2i\alpha^2D_y(\bar{f} \circ f + \bar{g} \circ g) \]  
(26b)

\[ u_y = 8i\alpha(a + 1)D_x(\bar{f} \circ f + \bar{g} \circ g) - 2i\alpha(2a + 1)D_y(\bar{f} \circ f + \bar{g} \circ g) \]  
(26c)

the M-IX equation (1) is transformed into the bilinear equations [23]

\[ [iD_t - 4(a^2 - 2ab - b)D_x^2 - 4\alpha(b - a)D_xD_y - \alpha^2D_y^2](\bar{f} \circ g) = 0 \]  
(27a)

\[ [iD_t - 4(a^2 - 2ab - b)D_x^2 - 4\alpha(b - a)D_xD_y - \alpha^2D_y^2](\bar{f} \circ f - \bar{g} \circ g) = 0 \]  
(27b)

In addition, we have the following condition

\[ D_y\{2\alpha(2a + 1)D_x(\bar{f} \circ f + \bar{g} \circ g) - 2\alpha^2D_y(\bar{f} \circ f + \bar{g} \circ g)\} \circ (\bar{f}f + \bar{g}g) = \]

\[ D_x\{8\alpha(a + 1)D_x(\bar{f} \circ f + \bar{g} \circ g) - 2\alpha(2a + 1)D_y(\bar{f} \circ f + \bar{g} \circ g)\} \circ (\bar{f}f + \bar{g}g), \]  
(28)

which follows from the compatibility condition \( u_{xy} = u_{yx} \).
5.1 Particular cases

5.1.1 The M-VIII equation

In this case \(a = b = -1\) and the expressions (26b,c) take the form

\[
\begin{align*}
\left[D_x (f \circ f + \bar{g} \circ g) \right]_{ff + gg} & = -2iD_x (f \circ f + \bar{g} \circ g) \frac{D_y (f \circ f + \bar{g} \circ g)}{ff + gg}, \\
\left[D_y (f \circ f + \bar{g} \circ g) \right]_{ff + gg} & = -2iD_y (f \circ f + \bar{g} \circ g) \frac{D_y (f \circ f + \bar{g} \circ g)}{ff + gg}
\end{align*}
\]

or

\[
\begin{align*}
\left[D_x (f \circ f + \bar{g} \circ g) \right]_{ff + gg} & = -2iD_x (f \circ f + \bar{g} \circ g) \frac{D_y (f \circ f + \bar{g} \circ g)}{ff + gg}, \\
\left[D_y (f \circ f + \bar{g} \circ g) \right]_{ff + gg} & = -2iD_y (f \circ f + \bar{g} \circ g) \frac{D_y (f \circ f + \bar{g} \circ g)}{ff + gg}
\end{align*}
\]

where \(w = -\alpha^{-1}u_y\). Equation (27) becomes

\[
[iD_t - \alpha^2 D^2_y (f \circ g) = 0, \quad [iD_t - \alpha^2 D^2_y (f \circ f - \bar{g} \circ g) = 0]
\]

or

\[
[iD_t - D^2_t ](f \circ g) = 0, \quad [iD_t - D^2_t ](f \circ f - \bar{g} \circ g) = 0
\]

In addition, we have from (28)

\[
\begin{align*}
D_y \{2\alpha D_x (\tilde{f} \circ f + \bar{g} \circ g) + 2\alpha^2 D_y (\tilde{f} \circ f + \bar{g} \circ g) \} \circ (\tilde{f} f + \bar{g} g) & = -2\alpha D_x \{D_y (\tilde{f} \circ f + \bar{g} \circ g) \} \circ (\tilde{f} f + \bar{g} g) \\
D_Y \{D_X (\tilde{f} \circ f + \bar{g} \circ g) + 2D_Y (\tilde{f} \circ f + \bar{g} \circ g) \} \circ (\tilde{f} f + \bar{g} g) & = -D_X \{D_Y (\tilde{f} \circ f + \bar{g} \circ g) \} \circ (\tilde{f} f + \bar{g} g)
\end{align*}
\]

which are biquadratic.

5.1.2 The Ishimori equation

Let \(a = b = -\frac{1}{2}\). Then the expressions (26) for the derivatives of the potential \(u\) become

\[
\begin{align*}
\left[D_x (f \circ f + \bar{g} \circ g) \right]_{ff + gg} & = -2iD_x (f \circ f + \bar{g} \circ g) \frac{D_y (f \circ f + \bar{g} \circ g)}{ff + gg}, \\
\left[D_y (f \circ f + \bar{g} \circ g) \right]_{ff + gg} & = -2iD_y (f \circ f + \bar{g} \circ g) \frac{D_y (f \circ f + \bar{g} \circ g)}{ff + gg}
\end{align*}
\]

At the same time, equation (27) is transformed into the bilinear equations [9]

\[
\begin{align*}
[iD_t - D^2_t - \alpha^2 D^2_y (f \circ g) = 0 \quad (33a) \]
\end{align*}
\]

or

\[
\begin{align*}
[iD_t - D^2_t - \alpha^2 D^2_y (f \circ f - \bar{g} \circ g) = 0 \quad (33b) \]
\end{align*}
\]

In addition, we have the following biquadratic condition

\[
\alpha^2 D_y \{D_y (f \circ f + \bar{g} \circ g) \} \circ (\tilde{f} f + \bar{g} g) = D_x \{D_x (f \circ f + \bar{g} \circ g) \} \circ (\tilde{f} f + \bar{g} g)
\]

which follows from (28). Note that the equations (32)-(34) are the same as in [9].
5.1.3 The M-XXXIV equation

At last we consider the case \(a = b = -1, x = 2X = 2t, y = \alpha Y, w = -\alpha^{-1}u\). In this case we have

\[
\begin{align*}
\frac{u_t}{ff + \alpha g g} &= -2i\alpha D_t (f \circ f + \alpha g \circ g) - i\alpha D_Y (f \circ f + \alpha g \circ g), \\
\frac{w}{ff + \alpha g g} &= -\alpha^{-1}u_Y - 2i\alpha D_Y (f \circ f + \alpha g \circ g).
\end{align*}
\]

(35a)

(35b)

The corresponding bilinear equations are given by [26]

\[
[iD_t - D_Y^2](f \circ g) = 0, \quad [iD_t - D_Y^2](f \circ f - \alpha g \circ g) = 0.
\]

(36)

The biquadratic condition has the form

\[
D_Y \{D_t (f \circ f + \alpha g \circ g) + 2D_Y (f \circ f + \alpha g \circ g) \circ (f f + \alpha g g)\} = -D_t \{D_Y (f \circ f + \alpha g \circ g) \circ (f f + \alpha g g)\}.
\]

(37)

5.2 Simplest soliton solution

In this section we get, the simplest soliton solution of equation (1) (for details, see, e.g. [23]). As example, we present only the 1-SS of the M-IXA equation, i.e. as \(\alpha^2 = 1\). The bilinear equation (27) represents the starting point to obtain interesting classes of solutions for the equation (1). The construction of the soliton solutions is standard. One expands the functions \(g\) and \(f\) as a series

\[
g = e^1g_1 + e^3g_3 + e^5g_5 + \cdots,
\]

(38a)

\[
f = 1 + e^2f_2 + e^4f_4 + e^6f_6 + \cdots.
\]

(38b)

Substituting these expansions into (27) and equating the coefficients of \(\epsilon\), in the 1-SS case, one obtains the following system of equations:

\[
e^1 : L(1 \circ g_1) = 0
\]

(39a)

\[
e^3 : L(\bar{f}_2 \circ g_1) = 0
\]

(39b)

\[
e^2 : L(1 \circ f_2 + \bar{f}_2 \circ 1 - \bar{g}_1 \circ g_1) = 0
\]

(39c)

\[
e^4 : L(\bar{f}_2 \circ f_2) = 0
\]

(39d)

where

\[
L = iD_t - 4(a^2 - 2ab - b)D_x^2 - 4\alpha(b - a)D_xD_y - \alpha^2D_y^2)(f \circ g).
\]

(40)

We now ready to construct the 1-SS of equation (1). In order to construct exact 1-SS of equation (1), we take the ansatz

\[
g_1 = \exp \chi_1, \quad \chi_1 = m_1x + n_1y + c_1t + e_1
\]

(41)
where \( m_1, n_1, c_1 \) and \( e_1 \) are complex constants. By substituting the above value of \( g_1 \) in eq.(38a), we get

\[
c_1 = i(a_1 m_1^2 + a_2 m_1 n_1 + a_3 n_1^2).
\] (42)

Substituting (41) in (39b,c,d), we obtain the expression for \( f_2 \) as

\[
f_2 = B \exp(\chi_1 + \chi_1^*) \] (43)

where \( B = B_R + iB_I \) with

\[
B_R = \frac{1}{4} \{ [2c_1 R B_I - (a_1 m_1^2 + a_2 m_1 n_1 + a_3 n_1^2)] [a_1 m_1^2 R + a_2 m_R n_1 R + a_3 n_1^2 R]^{-1} - 1 \}. 
\]

\[
B_I = \frac{1}{2} [\alpha^2 n_1 R n_1 I - \alpha^2(2a + 1) (m_1 R n_1 R + m_1 R n_1 I) + 4a(a + 1) m_1 R n_1 I] [2\alpha(2a + 1) m_1 R n_1 R - \alpha^2 n_1^2 R - 4a(a + 1) m_1^2 R]^{-1}. 
\]

The derivatives of potential have the forms

\[
u_x = \frac{K_1}{\exp(-2\chi_1 R + (2B_R + 1) + |B|^2 \exp(2\chi_1 R)} \] (44a)

\[
u_y = \frac{K_2}{\exp(-2\chi_1 R + (2B_R + 1) + |B|^2 \exp(2\chi_1 R)} \] (44b)

where

\[
K_1 = 4\alpha(2a + 1)(2B_1 m_1 R + m_1 I) - 4\alpha^2(2B_1 n_1 R + n_1 I)
\]

\[
K_2 = 16a(a + 1)(2B_1 m_1 R + m_1 I) - 4\alpha(2a + 1)(2B_1 n_1 R + n_1 I)
\]

From the biquadratic condition (28) is obtained

\[
n_1 R K_1 = m_1 R K_2. \] (45)

By substituting the above values of \( g_1 \) and \( f_2 \) in equations (26), we obtain the expressions for the spin components and for the potential. In detail, the different types of solutions (solitons, dromions, lumps, vortices) of the M-IX equation were presented in [10, 23].

6 Bilinearization of the Zakharov equation and its 1-SS

To find the bilinear form of the ZE (5), we introduce the following transformation

\[
q = \frac{G}{\phi}, \quad p = \frac{P}{\phi}. \] (46)

Inserting this transformation in equation (5), we get the Hirota bilinear form of the ZE (5) as [27]

\[
[i D_t - 4(a^2 - 2ab - b) D_x^2 - 4\alpha (b - a) D_x D_y - \alpha^2 D_y^2] (G \circ \phi) = 0 \] (47a)

\[
[i D_t - 4(a^2 - 2ab - b) D_x^2 - 4\alpha (b - a) D_x D_y - \alpha^2 D_y^2] (P \circ \phi) = 0 \] (47b)

\[
[4\alpha(a + 1) D_x^2 - 2\alpha(2a + 1) D_x D_y + \alpha^2 D_y^2] (\phi \circ \phi) = -2PG \] (47c)

with

\[
v = 2M_2 \log \phi. \] (48)
6.1 Limiting cases

6.1.1 The M-VIII equation (10)

In this case $a = b = -1$ and the bilinear equation (47) takes the form [25]

\[ [iD_t - \alpha^2 D_y^2](G \circ \phi) = [iD_t - D_Y^2](G \circ \phi) = 0 \] (49a)
\[ [iD_t - \alpha^2 D_y^2](P \circ \phi) = [iD_t - D_Y^2](P \circ \phi) = 0 \] (49b)
\[ (2\alpha D_x D_y + \alpha^2 D_y^2)(\phi \circ \phi) = (D_X D_Y + D_Y^2)(\phi \circ \phi) = -2PG \] (49c)

with
\[ v = 2[\alpha^2 \partial_y^2 + 2\alpha \partial_{xy}] \log \phi = 2[\partial_Y^2 + \partial_{XY}^2] \log \phi. \] (50)

6.1.2 The Davey-Stewartson equation

Let $a = b = -\frac{1}{2}$. In this case the bilinear equations become

\[ [iD_t - D_x^2 - \alpha^2 D_y^2](G \circ \phi) = 0 \] (51a)
\[ [iD_t - D_x^2 - \alpha^2 D_y^2](P \circ \phi) = 0 \] (51b)
\[ (-D_x^2 + \alpha^2 D_y^2)(\phi \circ \phi) = -2PG \] (51c)

where $v = M'_1 \log \phi, \ M'_1 = M_1$, as $a = b = -\frac{1}{2}$.

6.1.3 The YOE

Now consider the case $a = b = -1, x = 2X = 2t$. In this case we have

\[ [iD_t - D_Y^2](G \circ \phi) = 0 \] (52a)
\[ [iD_t - D_Y^2](P \circ \phi) = 0 \] (52b)
\[ (D_tD_Y + D_Y^2)(\phi \circ \phi) = -2PG. \] (52c)

Note that in this case the potential is equal to
\[ v = 2[\partial_Y^2 + \partial_{XY}^2] \log \phi. \] (53)

6.2 Simplest soliton solution of the ZE (5)

Let us, as example, we present the 1-SS of the ZE (5) in the case $\alpha^2 = 1$ and $P = E \bar{G}$. Note that equations (47) allow us to obtain the interesting classes of solutions for the ZE (5) [27, 15]. The construction of the solutions is standard. One expands the functions $G$ and $\phi$ as a series of $\epsilon$

\[ G = \epsilon G_1 + \epsilon^3 G_3 + \epsilon^5 G_5 + \cdots, \] (54a)
\[ \phi = 1 + \epsilon^2 \phi_2 + \epsilon^4 \phi_4 + \epsilon^6 \phi_6 + \cdots. \] (54b)

Substituting these expansions into (47) and equating the coefficients of $\epsilon$, in the 1-soliton case, one obtains the following system of equations:
\[ e^1 : L(1 \circ G_1) = 0 \]  
\[ e^3 : L(\bar{\phi}_2 \circ G_1) = 0 \]  
\[ e^2 : L(1 \circ \phi_2 + \phi_2 \circ 1 + 2 \bar{G}_1 \circ G_1) = 0 \]  
\[ e^4 : L(\phi_2 \circ \phi_2) = 0 \]  

where \( L \) is given by (40). Using these equations we can construct the 1-SS of equation (5). In order to construct exact 1-SS of equation (5), as above, we take the ansatz

\[ G_1 = \exp \chi_1, \quad \chi_1 = m_1 x + n_1 y + c_1 t + e_1 \]  

where \( m_1, n_1, c_1 \) and \( e_1 \) are complex constants. By substituting the above value of \( G_1 \) in equation (55a), we get

\[ c_1 = i(a_1 m_1^2 + a_2 m_1 n_1 + a_3 n_1^2). \]

From (56) and (55b,c,d), we obtain the expression for \( \phi_2 \) as

\[ \phi_2 = B' \exp (\chi_1 + \chi_1^*), \]  

where

\[ B' = -\frac{E}{3(b_1 m_1^2 + b_2 m_1 n_1 + b_3 n_1^2)}. \]

By substituting the above values of \( G = G_1 \) and \( \phi = 1 + \phi_2 \) in equations (46), we obtain the expressions for the field \( q \) and for the potential \( v \). This 1-SS and its generalizations and also dromions, lumps, vortices types solutions of the ZE (and the corresponding solutions of the M-IX equation) we have considered, in detail, in [15, 10, 27, 23]).

7 Integrability: a connection between the ZE and the FE

In order to see whether equation (5) [and hence equation (1)], is in general integrable, in [15] we have carried out the singularity structure analysis of equation (5) and shown that it [and (1)] has the Painleve property. Here this statement we prove using the following observation: the ZE and the FE are formally equivalent to each other. For this purpose we write equation (4) in terms of the coordinates \( \xi, \eta \) as

\[ iS_t + \frac{1}{2}[S, (b + 1)S_{\xi\xi} - bS_{\eta\eta}] + iw_{\eta}S_{\eta} + i(b + 1)w_{\xi}S_{\xi} = 0 \]  

\[ w_{\xi\eta} = \frac{1}{4i} tr(S[S_{\xi}, S_{\eta}]). \]

This equation, for convenience, in [10] we called the M-XX equation. Its gauge equivalent equation looks like

\[ iq_t + (1 + b)q_{\xi\xi} - bq_{\eta\eta} + vq = 0 \]  

\[ 12 \]
\[ iq_t - (1 + b)p_{\xi \xi} + bq_{\eta \eta} - vq = 0 \quad (59b) \]
\[ v_{\xi \eta} = -2\{(1 + b)(pq)_{\xi \xi} - b(pq)_{\eta \eta}\}. \quad (59c) \]

Equations (58) - (59) are of course integrable in the sense that admit the LR and have the different types solutions (solitons, dromions, lumps and so on) [15, 23, 27]. In fact these equations are not new, and are only the new forms of equations (4) and (5), respectively in terms of \(\xi, \eta\), which are given by (22).

Now we make the simplest scaling transformation: from \((t, \xi, \eta, q, p, v)\) to \((Ft, C\xi, D\eta, Aq, Bp, F^{-1}v)\). Then, for example, equation (59) takes the form

\[ iq_t - (\gamma - \beta)q_{\xi \xi} + (\gamma + \beta)q_{\eta \eta} + vq = 0 \quad (60a) \]
\[ ip_t + (\gamma - \beta)p_{\xi \xi} - (\gamma + \beta)p_{\eta \eta} - vq = 0 \quad (60b) \]
\[ v_{\xi \eta} = -2\lambda[(\gamma + \beta)(pq)_{\eta \eta} - (\gamma - \beta)(pq)_{\xi \xi}] \quad (60c) \]

where

\[ \lambda = ABCD, \quad C^2 = \frac{(\gamma + \beta)(b + 1)}{(\gamma - \beta)b}, \quad F = \frac{\beta - \gamma}{b + 1}C^2 \]
\[ \gamma = -\frac{1}{2}F[(b + 1)D^2 + bC^2]C^{-2}D^{-2}, \quad \beta = \frac{1}{2}F[(b + 1)D^2 - bC^2]C^{-2}D^{-2}. \]

Now let us consider the FE [4]

\[ iq_t - (\gamma - \beta)q_{\xi \xi} + (\gamma + \beta)q_{\eta \eta} - 2\lambda q[(\gamma + \beta)(\int_{-\infty}^{\xi}(pq)_{\eta}d\xi') + v_1(\eta, t)] - (\gamma - \beta)(\int_{-\infty}^{\eta}(pq)_{\xi}d\eta' + v_2(\xi, t))] = 0 \quad (61a) \]
\[ ip_t + (\gamma - \beta)p_{\xi \xi} - (\gamma + \beta)p_{\eta \eta} + 2\lambda p[(\gamma + \beta)(\int_{-\infty}^{\xi}(pq)_{\eta}d\xi') + v_1(\eta, t)] - (\gamma - \beta)(\int_{-\infty}^{\eta}(pq)_{\xi}d\eta' + v_2(\xi, t))] = 0 \quad (61b) \]

with \(p = \bar{q}\) and in contrast with the equation (60), in our case \(\xi, \eta\) are the characteristic coordinates defined by

\[ \xi = x + y, \quad \eta = x - y. \quad (62) \]

This equation also contains several interesting particular cases. Let us recall these cases.

(i) \(\gamma = \beta = \frac{1}{2}, v_1 = v_2 = 0\), yields equation

\[ iq_t + q_{xx} - 2\lambda q\int_{-\infty}^{y}(pq)x dy' = 0, \quad \lambda = \pm 1. \quad (63) \]

As noted by Fokas, equation (63) is perhaps the simplest complex scalar equation in 2+1 dimensions, which can be solved by the IST method. It is also worth pointing out that when \(x = y\) this equation reduces to the NLSE (6).
(ii) $\gamma = 0, \beta = 1$, yields the celebrated DSI equation
\[
i q_t + q_{\xi \xi} + q_{\eta \eta} - 2\lambda q[(\int_{-\infty}^{\xi} (pq)_{\eta} d\xi' + v_1(\eta, t)) + (\int_{-\infty}^{\eta} (pq)_{\xi} d\eta' + v_2(\xi, t))] = 0. \tag{64}
\]
This equation has the Painleve property and admits exponentially localized solutions including dromions for nonvanishing boundaries.

(iii) $\gamma = 1, \beta = 0$ yields the DSIII equation
\[
i q_t - q_{\xi \xi} + q_{\eta \eta} - 2\lambda q[(\int_{-\infty}^{\xi} (pq)_{\eta} d\xi' + v_1(\eta, t)) - (\int_{-\infty}^{\eta} (pq)_{\xi} d\eta' + v_2(\xi, t))] = 0. \tag{65}
\]
Equation (65) also supports certain localized solutions.

Now let us return to equation (61) and introduce the potential $V$ by
\[
V = -2\lambda[(\gamma + \beta)(\int_{-\infty}^{\xi} (pq)_{\eta} d\xi' + v_1(\eta, t)) - (\gamma - \beta)(\int_{-\infty}^{\eta} (pq)_{\xi} d\eta' + v_2(\xi, t))]. \tag{66}
\]
Then the FE (61) takes the form
\[
\begin{align*}
i q_t - (\gamma - \beta)q_{\xi \xi} + (\gamma + \beta)q_{\eta \eta} + Vq &= 0 \tag{67a} \\
i p_t + (\gamma - \beta)p_{\xi \xi} - (\gamma + \beta)p_{\eta \eta} + Vp &= 0 \tag{67b} \\
V_{\xi \eta} = -2\lambda[(\gamma + \beta)(pq)_{\eta \eta} - (\gamma - \beta)(pq)_{\xi \xi}]. \tag{67c}
\end{align*}
\]
Comparing the ZE in the form (60) and the FE in the form (67), we see that they have formally the same forms. Recently it was proved by Radha and Lakshmanan [3] that the FE (61) satisfies the Painleve property and hence it is expected to be integrable. From these results follow that the ZE and hence its equivalent counterpart the M-IX equation also satisfy the Painleve property and are integrable and in this sense (also, see [15]). Of course, strictly speaking, this statement is correct in the case when $\xi, \eta$ are real, i.e. when $\alpha$ is real. In particular, this is why the ZE contains and at the same time the FE not contains the DSII equation.

8 Conclusion

The some new (1+1)-, and/or (2+0)-dimensional integrable reductions of the M-IX equation and their equivalent counterparts are considered. In particular, we have established the gauge equivalence between the (1+1)-dimensional integrable inhomogeneous continuous Heisenberg ferromagnets (the M-XXXIV equation) and the YOE and ME.

We have also constructed the bilinear forms of the M-IX equation and the ZE and of their reductions. Moreover the simplest 1-SS of the ZE and the M-IX equation are found. Of course, these equations admit the generalizations of these solutions and other interesting solutions such as dromions, vortices, lumps and so on [15, 23, 27].

Also we have shown that the ZE and the FE are formally equivalent to each other. As shown by Radha and Lakshmanan [3], the FE satisfies the Painleve
property, i.e. it is expected to be integrable. Hence and from the results of [15] follow that the ZE and its equivalent the M-IX equation are integrable in the Painleve property sense.

Concluding, we note that between the some above considered equations as well as between the other spin systems and the NLSE-type equations take place the so-called Lakshmanan equivalence or L-equivalence. This problem we will consider elsewhere (see, for example, the refs.[18-22]).

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