On a lightlike limit of entanglement

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Abstract

We study certain classes of $g_{++}$ deformations of theories arising in gauge/string realizations of nonrelativistic holography, some of which pertain to $z = 2$ Lifshitz theories while others (pertaining to hyperscaling violation) comprise certain classes of excited states. Building on previous work, we consider holographic entanglement entropy for spacelike strip subsystems in a highly boosted (lightlike) limit, where the strip is stretched along the null $x^+$-plane. The leading divergence in entanglement in this null limit for these excited states is milder than the usual area law for spacelike subsystems in ground states. For ground states, the entanglement vanishes, perhaps consistent with ultralocality. We discuss this briefly from a field theory perspective. We also present some simple free lightfront field theory examples in excited states where correlators are nonvanishing.
1 Introduction

Certain gauge/string realizations in nonrelativistic holography involve spacetimes of the form

\[ ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + dx_i^2 + dr^2) + R^2g_{++}(dx^+)^2 + R^2d\Omega^2_S. \]  

(1)

The compact space \(d\Omega^2_S\) arising in string/M-theory realizations will not play much role in what follows. For \(g_{++} = 0\), we have AdS in lightcone coordinates. When \(g_{++} > 0\), dimensional reduction along the bulk \(x^+\)-direction (regarded compact) gives interesting theories in lower dimensions, with time \(t \equiv x^-\). Explicit examples include \(z = 2\) Lifshitz in bulk \(d\)-dim \([1,2]\) arising from \(x^+\)-reduction (and on \(S\)) of non-normalizable null deformations of \(AdS_{d+1} \times S\) \([3,4]\) (also \([5,6,7,8,9,10,11]\)),

\[ ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + K^2R^2(dx^+)^2 \rightarrow ds^2 = -\frac{dt^2}{r^4} + \frac{\sum_{i=1}^{d_i} dx_i^2 + dr^2}{r^2}, \]  

(2)

and \(AdS_{d+1}\) plane waves \([13,14,15]\) giving hyperscaling violation,

\[ ds^2 = \frac{R^2}{r^2}[-2dx^+dx^- + dx_i^2 + dr^2] + R^2Qr^{d-2}(dx^+)^2 \rightarrow ds^2 = r^{\frac{d-2}{2}} \left(-\frac{dt^2}{r^{2z}} + \sum_{i=1}^{d_i} \frac{dx_i^2 + dr^2}{r^2}\right), \quad z = \frac{d-2}{2} + 2, \quad \theta = \frac{d-2}{2}, \quad d_i = d - 2. \]  

(3)

The normalizable \(g_{++}\) deformation here represents a state in the dual CFT with uniform energy-momentum density flux \(T_{++} \sim Q\). Upon \(x^+\)-reduction, we obtain a hyperscaling violating (or conformally Lifshitz) background with Lifshitz \(z\) and hyperscaling violating \(\theta\) exponents above, and \(d_i\) the boundary spatial dimension \([16,17,18,19,20,21,22]\) (and many others in the now large literature). Some of these exhibit interesting entanglement behaviour \([20,21,22]\).

The \(x^+\)-dimensional reduction implies that the lower dimensional theory has a time coordinate \(t\) identified with lightcone time \(x^-\) in the higher dimensional theory (upstairs). It is thus interesting to isolate a lightlike limit upstairs (in particular of entanglement entropy) which describes corresponding features of the lower dimensional theory. In the Ryu-Takayanagi holographic entanglement description \([23,24,25]\), this involves calculating the area of an extremal surface on a constant \(x^-\) slice (which is spacelike in the bulk since \(g^{--} < 0\)). Building on the study of entanglement entropy in \(AdS\) plane waves \([26]\), we consider a highly boosted limit of spacelike strip subsystems: in this lightlike limit, the strip is stretched along the null \(x^+\)-plane (this is somewhat different from the discussions in \([27,28]\)). This arises in a certain regime in the dual field theory upstairs involving the UV cutoff and the energy scale contained in \(g_{++}\) (sec. 2). The entanglement entropy is calculable exactly in the large \(N\) gravity approximation, the entanglement integrals mapping to those in an auxiliary \(AdS\) space. Similar structures
arise in boosted black branes and nonconformal brane plane waves \[29\]. The finite cutoff-independent part in the Lifshitz case is larger than that for \(AdS\), perhaps reflecting the IR behaviour. The leading divergence in this lightlike regime for these excited states dual to \(AdS\) plane waves is milder than the usual area law divergence for spacelike subsystems in ground states \[30\]. We give some comments from a field theory perspective (sec. 3): ultralocality \[31\] suggests vanishing entanglement for the ground state. We present some simple calculations in free lightfront quantization of nonvanishing correlation functions in excited states, suggesting nonvanishing entanglement at weak coupling. Sec. 4 has some discussion.

2 Entanglement entropy in a null limit

We want to consider a lightlike limit of holographic entanglement entropy for the spacetimes \([1]\), building on that for the \(AdS_5\) plane wave in \[26\]. Consider a strip-like subsystem with width \(\Delta x = l\) along some \(x \in \{x_i\}\) (the others labelled \(y_i\)), and extended along the \(x^+\)-direction,

\[
x^+ = \alpha \chi, \quad x^- = -\beta \chi, \quad -\frac{l}{2} < x \leq \frac{l}{2}, \quad -\infty < \chi, y_i < \infty,
\]

\(i.e.\) wrapping the \(y_i, \chi\) directions completely. For \([5]\), \([1]\), the holographic entanglement is

\[
S = \frac{V_{d-2} R^{d-1}}{4 G_{d+1}} \int \frac{dr}{r^{d-1}} \sqrt{2 \alpha \beta + \alpha^2 g_{++} r^2} \sqrt{1 + (\partial_r x)^2} \implies \frac{l}{2} = 2 \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \sqrt{2 \alpha \beta + \alpha^2 g_{++} r^2}, \quad S = \frac{V_{d-2} R^{d-1}}{4 G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2 \alpha \beta + \alpha^2 g_{++} r^2}{\sqrt{2 \alpha \beta + \alpha^2 g_{++} r^2 - A^2 r^{2d-2}}},
\]

the second line obtained by extremization. \(\epsilon\) is the UV cutoff, \(V_{d-2} = \int (\prod_{i=1}^{d-3} dy_i) dx\), and \(r_*\) the extremal surface turning point where \(\frac{dr}{dr} r_* = 0\). We have \(\frac{r_*^{d-1}}{G_{d+1}} \sim N^2, \frac{N^3}{2}, \frac{N^3}{\alpha}\) for D3- \((AdS_5)\), M2- \((AdS_4)\) and M5-branes \((AdS_7)\) respectively. The usual entanglement entropy corresponds to a spacelike strip \(\alpha = \beta = 1\) on a time slice \(T = x^+ + x^- = 0\) and stretched along \(x^3 = \frac{x^+ - x^-}{\sqrt{2}}\), with the familiar area law divergence \(\frac{V_{d-2}}{\epsilon^{d-2}}\) and subleading terms.

The lightlike limit corresponding to null time \(x^-\) slices, is obtained with \(\alpha = 1, \beta = 0\),

\[
l = \Delta x = 2 \int_\epsilon^{r_*} \frac{dr}{\sqrt{g_{++} r^2 - A^2 r^{2d-2}}}, \quad S = \frac{V_{d-2} R^{d-1}}{4 G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{g_{++} r^2}{\sqrt{g_{++} r^2 - A^2 r^{2d-2}}}. \tag{7}
\]

This is natural from the point of view of the lower dimensional theory obtained by \(x^+\)-reduction, where \(x^-\) becomes time \(t\) below. It can be defined as a highly boosted limit of a spacelike subsystem: consider \(x^\pm \rightarrow \lambda^{\pm 1} x^\pm\), transforming \([5]\) with \(\alpha = 1 = \beta\) to \(\alpha = \lambda \gg 1, \beta = \frac{1}{\lambda} \ll 1\), giving

\[
\frac{l}{2} = \int_\epsilon^{r_*} \frac{dr}{\sqrt{2 + \lambda^2 g_{++} r^2 - A^2 r^{2d-2}}}, \quad S = \frac{V_{d-2} R^{d-1}}{4 G_{d+1}} \int_\epsilon^{r_*} \frac{dr}{r^{d-1}} \frac{2 + \lambda^2 g_{++} r^2}{\sqrt{2 + \lambda^2 g_{++} r^2 - A^2 r^{2d-2}}}. \tag{8}
\]
Since $r > r_{\text{min}} = \epsilon$, the expression (8) is well approximated by the null one (7) in the regime

$$\lambda^2 g_{++}(\epsilon) \epsilon^2 \gtrsim 1, \quad (9)$$

i.e. the ultraviolet cutoff is comparable to the scale contained in the $g_{++}$ deformation: roughly the boundary $r = \epsilon$ feels the $g_{++}$-deformation. In this limit, the strip subsystem (5) parametrized as $x^+ = 0$, $-\frac{1}{2} < x \leq \frac{1}{2}$, $-\infty < x^+, y_i < \infty$, is stretched along the null $x^+$-plane. The leading divergence here is $\int_{\epsilon} dr \sqrt{\frac{x^2 g_{++} r^2}{r^d-1}} \sim \sqrt{\frac{\lambda^2 g_{++}(\epsilon)}{\epsilon^{d-2}}}$. We will see this in greater detail below.

### 2.1 AdS null deformations, $g_{++} = K^2 \rightarrow z = 2$ Lifshitz

Consider first the $AdS_{d+1}$ null deformations [2] which give $z = 2$ Lifshitz in bulk $d$-dim upon $x^+$-reduction [3, 4], with $x^- \equiv t$. The $g_{++}$ mode independent of $r$ is sourced by other matter: e.g. an axion source $c_0 = K x^+$ gives $g_{++} \sim (\partial_+ c_0)^2 = K^2$, with $K$ a constant of mass dimension one [10, 9, 12]. The SL(2,Z) duality $\tau \rightarrow \tau + 1$, i.e. $c_0 \rightarrow c_0 + 1$, of IIB string theory means the axion profile is effectively $x^+$-periodic with periodicity $\frac{1}{K}$. For noncompact $x^+$, the $x^-, x_i$-subspace has $z = 2$ Lifshitz scaling, which in the bulk is $(x^-, x_i, r) \rightarrow (\lambda^2 x^-, \lambda x_i, \lambda r)$.

In the upstairs description, we consider holographic entanglement entropy for the $AdS$-Lifshitz deformation: structurally this is similar to [26] for $AdS$ plane waves. For the strip subsystem (5), the entanglement entropy is (6) with $g_{++} = K^2$. The spacelike subsystem (5) has $\alpha = 1 = \beta$. With $\epsilon \ll \frac{1}{K}$ the leading divergence is the area law $\sim \frac{V_{d-2}}{\epsilon^{d-2}}$ in boundary $d$-dim, i.e. in $AdS_{d+1}$. The turning point is $2 + K^2 r_*^2 - A^2 r_*^{2d-2} = 0$. For large width $l$ and large $K$, this is $K^2 \sim A^2 r_*^{2d-4}$, giving $l \sim r_* \sim \int_{\epsilon} r_* \frac{dr}{\sqrt{1 - (A/K)^2 r_*^{2d-2}}}$. The subsystem (5) with $\alpha = 1 = \beta$ under the boost $x^\pm \rightarrow \lambda^\pm 1 x^\pm$, with large $\lambda$, leads to the regime (9) governing the lightlike limit. In the unboosted theory, this is the regime

$$K^2 \epsilon^2 \gtrsim 1, \quad i.e. \quad K \gtrsim \epsilon^{-1}, \quad (10)$$

i.e. the Lifshitz-deformation scale $K$ is comparable to the UV scale $\epsilon^{-1}$. Equivalently, we have redefined $\lambda K \rightarrow K$ in (9). Since $r > r_{\text{min}} = \epsilon$, the entanglement becomes

$$S \sim \frac{V_{d-2} R^{d-1}}{4G_{d+1}} \int_{\epsilon}^{r_*} dr \frac{K}{r^{d-2} \sqrt{1 - (A/K)^2 r^{2d-2}}} \sim \frac{R^{d-1}}{4G_{d+1}} \frac{V_{d-2} K}{d-3} \left( \frac{1}{\epsilon^{d-3} - c_d \frac{1}{\epsilon^{d-3}}} \right). \quad (11)$$

The expression (11), identical to that in some auxiliary $AdS_d$ space, is a null limit, $\alpha \sim 1, \beta \sim 0$, (7) of (8). The $AdS_d$ deformed theory gives $V_1 K N^{3/2} \log \frac{1}{\epsilon}$. This resembles the entanglement in the lower dimensional Lifshitz$_d$ theory: thus in the regime (9), (10), on length scales longer than the axion variation length scale $\frac{1}{K}$ the $AdS_{d+1}$ deformed theory resembles the Lifshitz$_d$ theory, the leading divergence being the area law in boundary $d - 1$ dimensions. The finite
entanglement $S_{\text{finite}} \sim \frac{R^{d-1} / G_{d+1} V_{d-2}}{3 - d} \frac{V_{d-2} K^2}{3^{d-3}}$ (and $N^{3/2} V_1 K \log(lK)$ for $d = 3$) in a sense contains the IR degrees of freedom encoding entanglement: as the width $l$ increases, the extremal surface dips deeper into the interior ($r_s \sim l$). Comparing with $AdS_{d+1}$, the finite part in the Lifshitz case is larger for $l \gg \frac{1}{K}$. Perhaps this is a reflection of more soft modes in the Lifshitz theory (compared with a relativistic CFT) responsible for the IR singularities [1, 32], stemming from the $\omega \sim k^z$ dispersion relation in field theory (see also [33]).

2.2 $AdS_{d+1}$ plane waves: $g_{++} = Q r^{d-2}$

Now let us consider $AdS_{d+1}$ plane waves [3]. Using (8), we have the turning point equation $2 + \lambda^2 Q r_s^d - A^2 r_s^{2d-2} = 0$, which for $\lambda^2 Q r_s^d \gg 1$ is well approximated by $\lambda^2 Q r_s^d - A^2 r_s^{2d-2} = 0$. Here the regime (9) is

$$\lambda^2 Q e^d \gtrsim 1, \quad \text{i.e.} \quad P_+^d \gtrsim \epsilon^{-1},$$

i.e. the elemental lightcone momentum $P_+ = T_{++} \Delta x^+ \Delta x^{d-2} \epsilon$ at scale $\epsilon$ in the boosted frame is comparable to the UV cutoff $\epsilon^{-1}$ (the transverse area $\Delta x^{d-2} \Delta x$ is boost invariant). In this regime, $\lambda^2 Q r_s^d \gtrsim 1$ for all $r$-values appearing in (8) since $r > \epsilon$: the entanglement (8) for the $AdS_{d+1}$ plane wave is well-approximated by the corresponding expressions (7) at null slicing

$$\frac{l}{2} \sim \int_0^{r_s} dr \frac{(A/\sqrt{\lambda^2 Q}) r^{d/2-1}}{\sqrt{1 - (A^2/\lambda^2 Q) r^{2d-2}}} \sim r_s, \quad S \sim \frac{V_{d-2} R^{d-1}}{4 G_{d+1}} \int_{\epsilon}^{r_s} dr \frac{\sqrt{\lambda^2 Q}}{r^{d/2-1} \sqrt{1 - (A^2/\lambda^2 Q) r^{2d-2}}}.$$ 

(13)

In this highly boosted limit, we are effectively probing entanglement at scales $P_+^d \gtrsim \epsilon$. Equivalently, the regime (9) moves us into the hyperscaling violating regime.

From (13) we see that the null entanglement integral for the $AdS_{d+1}$ plane wave excited states is identical to that for a spacelike strip subsystem in an auxiliary pure $AdS$ space in $(\frac{d}{2} + 1)$-dim (e.g. using (10) with $\alpha = \beta = 1$, $g_{++} = 0$): it can be evaluated exactly as for $AdS$

$$l \sim r_s, \quad S \sim \frac{R^{d-1}}{4 G_{d+1}} \frac{V_{d-2} \sqrt{\lambda^2 Q}}{d - 4} \left( \frac{1}{\epsilon^{d/2-2}} - c_d \frac{1}{l^{d/2-2}} \right),$$

(14)

using the turning point $\lambda^2 Q r_s^d = A^2 r_s^{2d-2}$, with $c_d$ a constant. The lightlike limit of entanglement (13), (14) in the $AdS_{d+1}$ plane wave in the regime (9) is essentially the entanglement with ordinary time slicing in a theory in $\frac{d}{2}$-dim: this is in fact the lower dimensional theory with hyperscaling violation (after $x^+$-reduction), living in an effective dimension $d_{eff} = d - 1 - \theta = \frac{d}{2}$, using (4). Since $d_{eff} < d - 1$ for $d > 2$, the leading divergence is milder than the usual area law (see [33]). For the $AdS_5$ plane wave dual to 4d SYM CFT excited states, (13) reduces to

$$\frac{l}{2} = \int_0^{r_s} dr \frac{r (A/\sqrt{\lambda^2 Q})}{\sqrt{1 - A^2/\lambda^2 Q} r^{2d-2}} \sim (#) r_s, \quad S \sim \frac{V_2 R^3}{4 G_5} \int_{\epsilon}^{r_s} dr \frac{\sqrt{\lambda^2 Q}}{r \sqrt{1 - A^2/\lambda^2 Q} r^2} \sim N^2 V_2 \sqrt{\lambda^2 Q} \log \frac{l}{\epsilon},$$ 

(15)

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discussed previously in [15, 26]. This is reminiscent of a 2-dim CFT with central charge $N^2V_2\sqrt{Q}$. From (15), the $AdS_4$ plane wave (with just one transverse dimension) gives $S \sim V_1N^{3/2}\sqrt{\lambda^2Q(\sqrt{1}\sqrt{\epsilon})}$, with no divergence, valid if $\epsilon \geq \frac{1}{\lambda^2Q}$. (The change in entanglement $\Delta S$ is as in the lower dimensional theory.) Pure $AdS$ corresponds to the CFT ground state: for null slicing (7), entanglement entropy vanishes since $g_{++} = 0$. From (1), we see the entangling surface degenerates.

### 2.3 Boosted black branes

Consider black D3-branes $ds^2 = \frac{R^2}{r^2}[-(1-r_0^4t^4)dx^2 + dx_3^2 + \sum_{i=1}^2 dx_i^2] + \frac{R^2dt^2}{r^2(1-r_0^4t^4)}$ and lightcone coordinates $x^\pm$ as $T = \frac{x^+ - x^-}{\sqrt{2}}$, $x^3 = \frac{x^+ - x^-}{\sqrt{2}}$. Boosting as $x^\pm \rightarrow \lambda^\pm x^\pm$ [3] gives

$$ds^2 = \frac{R^2}{r^2}(-2dx^+dx^- + \frac{\lambda^4r^4}{2}(\lambda dx^+ + \lambda^{-1}dx^-)^2 + \sum_{i=1}^2 dx_i^2) + \frac{R^2dt^2}{r^2(1-r_0^4t^4)}.$$  

For large boost $\lambda$ and low temperature $r_0$, with $Q = \frac{\lambda^2r_0^4}{2}$, this approaches the (near extremal) $AdS_5$ plane wave [3]. For the subsystem [5], we have the entanglement area functional [26]

$$S = \frac{V_2R^3}{4G_5} \int_\epsilon^{r_s} \frac{dr}{r^3} \frac{2\alpha\beta + \frac{\lambda^2r^4}{2}(\alpha - \frac{\beta}{\lambda^2})^2}{\sqrt{1-r_0^4t^4}\sqrt{2\alpha\beta + \frac{\lambda^2r^4}{2}(\alpha - \frac{\beta}{\lambda^2})^2 - A^2r^6}},$$

and $\Delta x = l \sim r_s$, after extremizing. For large boost $\lambda$, we have $\frac{\beta}{\lambda^2} \rightarrow 0$ and in the regime $\lambda^2r_0^4e^4 \gtrsim 1$, the second term in the radical (containing $\lambda^2r_0^4t^4$) is always greater than the first: thus this resembles the entanglement $S \sim \frac{V_2R^3}{4G_5} \int_\epsilon^{r_s} \frac{dr}{r^3} \frac{(\lambda^2r_0^4/2)}{\sqrt{1-r_0^4t^4}\sqrt{(\lambda^2r_0^4/2)-A^2r^2}}$ for a null subsystem (setting $\alpha \sim 1$). The boost parameter introduces a separation of scales. The entanglement (16) in the regime $\frac{1}{\sqrt{Mr}} \lesssim \epsilon \ll r_s \ll \frac{1}{r_0}$, becomes $S \sim N^2V_2\sqrt{Q} \log \frac{l}{\epsilon}$, as for the $AdS_5$ plane wave [15] which is regulated by the boosted black brane. For an unboosted brane $\lambda = 1$, this null entanglement ($\alpha = 1, \beta = 0$) is $S \sim N^2V_2\sqrt{Q} \log \frac{l}{\epsilon} + \frac{N^2V_2r_0^4}{R^2} + \ldots$. Thus the 4-dim SYM CFT thermal state with null time slicing in the regime $\epsilon \lesssim \frac{1}{r_0}$ exhibits entanglement [16] with a leading logarithmic divergence. In the far IR limit $l \sim r_s \sim \frac{1}{r_0}$, this is extensive, resembling the usual thermal entropy, expected from the lower dimensional theory.

### 2.4 Nonconformal D-brane plane waves

The string metric and dilaton for nonconformal $Dp$-brane plane waves [23] (see also [14]) are

$$ds^2_{st} = \frac{r^{(7-p)/2}}{R_p^{(7-p)/2}} dx^2 + \frac{G_{10}Q_p}{R_p^{(7-p)/2}} \frac{(dx^+)^2}{r^{(7-p)/2}} + \frac{R_p^{(7-p)/2}}{r^{(7-p)/2}} \frac{dr^2}{r^{(7-p)/2}} + \frac{R_p^{-(7-p)/2}r^{-(p-3)/2}dQ_{8-p}^2}{},$$

$$e^\Phi = g_s \frac{R_p^{-(7-p)/2}}{r} \left( \frac{3p+2}{2p} \right) ; \quad g_{YM} \sim g_s \alpha^{(p-3)/2} ; \quad R_p^{-(p-3)/2} \sim g_s N \alpha^{(7-p)/2}.$$ (17)
(The ultraviolet is towards large \( r \) here.) As in the conformal cases, the \( g_{++} \) term is obtained from the non-conformal finite temperature solutions \([35]\) in a large boost \( \lambda \), low temperature limit holding the energy-momentum density \( \lambda \) gives hyperscaling violating spacetimes \([4]\) with exponents \( \lambda \). These describe strongly coupled Yang-Mills theories with energy flux \( T_{++} \). The Einstein metric \( ds^2_E = e^{-\Phi/2} ds^2_{st} \), upon dimensional reduction on \( S^{d-2} \) and the \( x^+ \)-direction, gives hyperscaling violating spacetimes \([4]\) with exponents \( \theta = 5/2 - \theta_{++)} = 2(p-6)/p-5 \). With \( r_{UV} \) the UV cutoff and \( r_s \) the turning point, the entanglement entropy \([29]\) for the subsystem \([5]\) is

\[
S_A \sim \frac{V_{p-1} R_{p}^{7-p}}{G_{10}} \int_{r_s}^{r_{UV}} dr \frac{A R_{p}^{7-p}}{r^{3-p}/2} \sim \frac{V_{p-1} Q N^2}{3-p} \left( \frac{\sqrt{g_{YM} N}}{\epsilon^d \epsilon_{eff}^{d-2}} - \frac{(\sqrt{g_{YM} N})^{d_{eff} - 2}}{\epsilon_{eff}^{d-2}} \right).
\]

We have used \([17]\) to recast the bulk result in field theory variables \( l \sim \frac{R_{UV}^{7-p}/2}{r_{UV}^{3-p}/2} \), \( \epsilon \sim \frac{R_{UV}^{7-p}/2}{r_{UV}^{3-p}/2} \).

The effective dimension here, after \( x^+ \)-reduction, works out to \( d_{eff} = \frac{7-p}{5-p} = p - \theta \), with \( d_{eff} < p \) for the cases \( p = 2, 3, 4 \) of relevance. For ground states \( Q = 0 \), the entanglement vanishes.

In terms of the scale-dependent number of degrees of freedom \( N_{eff}(s) = N^2 (g_{YM} N)^{d_{eff} - 2} \) \([36]\), the finite part of entanglement can be written \([29]\) as \( \frac{\sqrt{N_{eff}(l)}}{3-p} \frac{V_{p-1} Q}{(p-3)/2} \) : the form \([19]\) makes manifest the lightlike limit as equivalent to the lower dimensional theory.

### 3 On null entanglement in field theory excited states

We have discussed entanglement entropy in the strongly coupled regime for strip subsystems \([5]\) in a lightlike limit using the Ryu-Takayanagi prescription. We will now make some comments on field theory in excited states with energy-momentum density \( T_{++} = Q \) nonzero. We recall \([14]\) for CFT\(_d\) excited states, i.e.

\[
S_{div} \sim N^2 \sqrt{Q} \frac{V_{d-2}}{\epsilon^{d_{eff} - 2}} = N^2 \sqrt{Q \epsilon^d} \frac{V_{d-2}}{\epsilon^{d-2}}, \quad d_{eff} = \frac{d}{2}.
\]

The leading divergence \([20]\) is less severe than the usual area law \( \frac{V_{d-2}}{\epsilon^{d-2}} \) for the ground state \([30]\).

For \( d = 4 \), we obtain a logarithmic divergence, the entanglement scaling as \( \log \frac{1}{\epsilon} \). Heuristically, the short distance “entangling degrees of freedom” or “partons” with this null slicing are fewer
equivalently at points with spacelike separation $\Delta x_i > 0$, vanish (see also [27, 28]).

By ultralocality, correlation functions vanish and so entanglement must also vanish. While this is true for ground states, it may not hold for excited states, consistent with the entanglement exhibiting the milder divergence we have discussed so far. In this regard, we now heuristically discuss some simple free field correlation functions in lightfront quantization (see e.g. [37, 38, 39] and references therein; our notation here uses $x^-$ as lightcone time however) which corroborate this. Consider a 4-dim massless scalar with mode expansion and commutation relations

$$\phi = \int d^2 k_i \int_0^\infty \frac{dk_+}{(2\pi)^3 2k_+} \left( a_{k_+,k_i} e^{-ik_+ x^+ - ik_i x^-} + a^\dagger_{k_+,k_i} e^{ik_+ x^+ + ik_i x^-} \right),$$

$$[a_{k_+,k_i}, a^\dagger_{k'_+,k_i}] = (2\pi)^3 \delta(k_+ - k'_+) \delta^2(k_i - k'_i), \quad a_{k_+,k_i} |0\rangle = 0.$$  \hfill (21)

A positive frequency mode has $k_- = \frac{k^2}{2k_+} > 0$. This imposes $k_+ \geq 0$ in the sum over modes, and excitations must have positive lightcone momentum $k_+$: small $k_+$ modes are high energy (we will not worry about zero mode issues here). With $T_{++} = (\partial_+ \phi)^2$, the operator $P_+ = \int dx^+ d^2 x_i T_{++}$ measuring lightcone momentum is

$$P_+ = \int dk_+ d^2 k_i \ k_+ a^\dagger_{k_+,k_i} a_{k_+,k_i}$$  \hfill (22)

dropping normal ordering terms. A simple excited state contributing to nonzero $T_{++}$ and so $P_+$ is

$$|k_+ \neq 0\rangle = a^\dagger_{k_+,k_i} |0\rangle, \quad P_+ |k_+\rangle = k_+ |k_+\rangle.$$  \hfill (23)

It can be checked that a 2-point function $\langle 0 \langle \partial_+ \phi(x_1) \partial_+ \phi(x_2) | 0 \rangle$ on an $x^- = \text{const}$ surface, with $x_1^i$ and $x_2^i$ distinct, in the vacuum or ground state vanishes (as do $n$ point functions): these contain terms of the form $\int d^2 k_i e^{-ik_i \Delta x^i} \sim \delta(\Delta x_i)$ which vanish for $\Delta x_i \neq 0$. For excited states of the form above, such a 2-point function is

$$\langle k_+ | \partial_+ \phi(x_1^i, x_1^i) \partial_+ \phi(x_2^i, x_2^i) | k_+\rangle \sim \langle 0 | a_{k_+} \partial_+ \phi(x_1^i, x_1^i) \partial_+ \phi(x_2^i, x_2^i) a^\dagger_{k_+,k_i} | 0 \rangle \neq 0.$$  \hfill (24)

The term $\langle 0 | \int k_1 \int k_2 (i k_1^i)(-ik_2^i) a_{k_1,k_i} a^\dagger_{k_1,k_i} e^{ik_1 x^1 - ik_2 x^2} | 0 \rangle$, with $\int k_1 \equiv \int d^2 k_i \int_0^\infty \frac{dk_+}{(2\pi)^3 2k_+}$ and $\delta(k_1, k_2) \equiv \delta(k_1^1 - k_2^1) \delta(k_1^2 - k_2^2)$ gives $\langle 0 | \int k_1 \int k_2 (i k_1^i)(-ik_2^i) \delta(k_1^2, 2) \delta(1, k_+) e^{ik_1^2 x^2 - ik_1^1 x^1} | 0 \rangle \sim k_1 e^{i k_+ \Delta x_1^2}$. Another similar term arises with $1 \leftrightarrow 2$. These terms are nonzero for generic
This is quite different from a 4-point function in the ground state, which also vanishes. Similar calculations apply for higher point correlation functions (in states like $\prod_i a_{k_i+}^\dagger |0\rangle$). Gauge fields in lightcone gauge $A_+ = 0$ have some structural similarities. It would be interesting to study correlation functions systematically by using smearing functions, constructing normalizable states (23) and so on: in this case, $\partial_+ \phi$ gives a good operator-valued distribution (possible UV divergences in its correlators, at small $k_+$ or equivalently large $k_-$, can be controlled by smearing along the lightfront directions). Entanglement entropy is related to 2-point correlation functions using correlation matrices (see e.g. [41]), suggesting nonvanishing entanglement entropy at weak coupling: this would be interesting to study in detail.

The expression (20) is recovered if the field theory is taken to live on a space

$$ds^2 = -2dx^+ dx^- + g^2(dx^+)^2 + \sum_{i=1}^{d-2} dx_i^2 , \quad g^2 = T_{++}\epsilon^d \gtrsim 1 .$$

The dimensionless parameter $g^2$ is proportional to the energy density $T_{++}$ and contains appropriate powers of the UV cutoff $\epsilon$. In the ground state, $T_{++} = 0 \Rightarrow g^2 = 0$ so the $x^-$ direction is null and represents lightcone time. For excited states with $T_{++} = Q$, the $x^+$ direction formerly null “puffs up”. Discussions of lightfront quantization often use the space (25) with $g^2$ a regulator (see e.g. [39, 40]): in the present case $g^2$ is in a sense physical, as reflected in entanglement entropy. Now a constant $x^-$ surface is spacelike (with a timelike normal since $g^{--} = -g^2 < 0$). The relation between $x^-$ and lightcone time $X^-$ is $x^- = X^- + \frac{g^2}{2} x^+$, so that $x^-$ is not null but timelike in the boundary theory if $g^2 \sim O(1)$. This corroborates with the bulk: for $AdS$ plane waves (3), the field theory lives on the boundary $r = \epsilon$ with metric (25). The condition $g^2 \sim O(1)$ is equivalent to the lightlike regime (9) with new divergence behaviour in the bulk entanglement.

We consider entanglement in this weak coupling field theory on the space (25) for a strip subsystem (5) with width $l$. When $g^2 \ll 1$ and we use time $T = \frac{x^+ + x^-}{\sqrt{2}}$, the entanglement leading divergence is the usual area law $S_{div}^t \sim N^2 \frac{V_{x^+} V_{y^+}}{\epsilon^{d-2}}$. Now with $g^2 \sim O(1)$ in the regime (9), we consider $x^-$ as time, with entanglement on constant $x^-$ slices. For a strip (5) stretched along $x^+$, the usual area law divergence for ground states in the space (25) is

$$S_{div} \sim N^2 \frac{V_{x^+} V_{y^+}}{\epsilon^{d-2}} = N^2 V_{d-2} \frac{\sqrt{T_{++}\epsilon^d}}{\epsilon^{d-2}} = N^2 \sqrt{\frac{Q}{V_{d-2}}} \frac{V_{d-2}}{\epsilon^{d_{eff}-2}} , \quad d_{eff} = \frac{d}{2} .$$

This is in agreement with the bulk scaling (14) (20) in the lightlike regime, with an effective scaling dimension (in the hyperscaling violating regime) arising from the energetic “puffing up” of the originally null $x^+$ direction. The $g^2$ term in a sense encodes the backreaction of the excited state in the original space (with $g^2 = 0$ and $x^+$ null): this shows up as the usual area law (26) for the new ground state in the backreacted space (25).
For nonconformal theories, the space on which the field theory lives is again \( (25) \), with

\[
g = \sqrt{\frac{G_{10} Q}{r_{UV}^{7-p}}} = \frac{\sqrt{\Omega} \epsilon_{d_{eff}}^{d_{eff}}}{N(g_{YM}^2 N)^{d_{eff}-2}} , \quad \epsilon \sim \frac{R_p^{(7-p)/2}}{r_{UV}^{(5-p)/2}} , \quad d_{eff} = \frac{7 - p}{5 - p} ,
\]

using \((17)\). The usual area law divergence for nonconformal theories on such a space, using \((27)\), gives \( S_{div} \sim N_{eff}(\epsilon) \frac{V_{++} V_{ii}}{\epsilon_{d_{eff}}} = N(g_{YM}^2 N)^{(d_{eff}-2)/2} \sqrt{\Omega} \epsilon_{d_{eff}}^{d_{eff}-2} \), of the form \((19)\) in the bulk.

### 4 Discussion

We have described a null limit of entanglement entropy for theories with a \( g_{++} \) deformation \((1)\) arising in gauge/string realizations of Lifshitz and hyperscaling violating nonrelativistic holography, the strip subsystem stretched along a null \( x^+ \)-plane. This shows a milder leading divergence, perhaps consistent with ultralocality in a field theory perspective, and implied in the gauge/string realizations \((1)\) by entanglement behaviour in hyperscaling violating backgrounds \([20, 21, 22]\).

One can also consider null intervals, \( i.e. \) spacelike strip subsystems with width along the \( x^+ \)-direction, \( \Delta x^+ = -\Delta x^- = \frac{1}{\sqrt{2}} \), \(-\infty < y_i < \infty\), (although this has no natural interpretation after \( x^+ \)-reduction). This gives

\[
\frac{\Delta x^+}{2} = \int_0^{r_+} \frac{dr}{\sqrt{A^2 B^2 + g_{++} r^{2d} - 2B r^{2(d-1)}}} , \quad \frac{\Delta x^-}{2} = \int_0^{r_+} \frac{dr}{\sqrt{A^2 B^2 + g_{++} r^{2d} - 2B r^{2(d-1)}}} ,
\]

\[
S = \frac{2 R^{d-1} V_{d-2}}{4 G_{d+1}} \int_c^{r_+} \frac{d r}{\sqrt{A^2 B^2 + g_{++} r^{2d} - 2B r^{2(d-1)}}} AB ,
\]

from the entanglement area functional \( S = \frac{V_{d-2} R^{d-1}}{2 G_{d+1}} \int \frac{d r}{\sqrt{1 - 2(\partial_r x^+)(\partial_r x^-) + g_{++} r^2 (\partial_r x^+)^2}}. \)

For \( AdS \) plane waves, this was studied in \([26]\), and a phase transition was observed: for large width \( l \), there is no connected extremal surface. The \( \Delta x^+ \) and \( \Delta x^- \) integrals have widely different scaling and it is not possible to represent a spacelike subsystem (with \( \Delta x^- < 0 \)) here. Equivalently, the boost \( x^\pm \rightarrow \lambda^{\pm 1} x^\pm \) scales up the \( g_{++} \) term in the entanglement area functional and \((28)\) as before, but this eventually makes \( \Delta x^- \) positive. This critical value occurs at \( g_{++} r^2 |_{r_c} \sim B \) for \( AdS \) plane waves \( r_c \sim Q^{-1/d} \).

Unlike \([25, 27]\), the null time \( x^- \) slicing does not arise as a highly boosted lightlike limit \( \Delta x^+ \sim l \), \( \Delta x^- \rightarrow 0 \), of entanglement \((28)\) for spacelike subsystems, perhaps not surprising from the lower dimensional description. The boosted black 3-brane (which near extremality is the regulated \( AdS_5 \) plane wave) exhibits similar behaviour. A null limit arises \([28]\) under a boost plus dilation, scaling down all non-vacuum terms with the \( g_{++} \) term being the most dominant: this then becomes similar to the \( AdS_5 \) plane wave. The entanglement can now be expanded and
evaluated relative to the vacuum contribution \[28\]: \( g_{++} \sim r^{d-2}T_{++} \) gives \( \Delta S \sim \int g(x^+)\langle T_{++} \rangle \) with some function \( g(x^+) \).

For the \( AdS \)-Lifshitz deformation, the entanglement is of the form \[28\], with \( g_{++} = K^2 \). As for \( AdS \) plane waves, there is a phase transition here, with critical value \( r_c \sim l_c \sim \frac{1}{K} \). Under the boost plus dilation \( x^+ \rightarrow x^+, \ x^- \rightarrow \eta^2x^-, \ r \rightarrow \eta r \), with \( \eta \rightarrow 0 \) as in \[28\], the \( g_{++} \) term does not scale down here, structurally similar to the \( AdS_3 \) plane wave \[3\]. The Lifshitz \( g_{++} \) term is a non-normalizable deformation, sourced by e.g. the lightlike dilaton/axion. Treating this as a small perturbation to \( AdS \) (with \( \epsilon \ll \frac{1}{K} \)), we can expand \[28\], as in \[42\] for \( AdS \) plane waves (and more generally \[43-47\] for excited states), except this is not an excited state but a nontrivial CFT deformation. In this Lifshitz case, \( r \sim l \), and \( AB = r^{d-1}_s\sqrt{2B - K^2r^2_s} \) from the turning point equation. Expanding about \( B = 1, K = 0 \) (as in \( AdS \)) gives

\[
S \sim \frac{V_{d-2}R^{d-1}}{G_{d+1}} \int_{\epsilon}^{r_s} \frac{dr}{r^{d-1}} \frac{\sqrt{1 - K^2r^2_s/2B}}{\sqrt{1 - (r/r_s)^2(d-1)}} \Rightarrow \Delta S \sim \frac{R^{d-1}}{2G_{d+1}} V_{d-2}K^2l^{d-4}M,
\]

where the constant can be shown to be \( M > 0 \). There is no simple relation here between the \( g_{++} \) deformation and the holographic stress tensor. It might be interesting to explore this.

**Acknowledgments:** I thank A. Laddha, J. Maldacena and T. Takayanagi for helpful conversations, and N. Sircar and S. Trivedi for many discussions on entanglement and correlation matrices. I also thank the Organizers of the Strings 2014 conference, Princeton/IAS for hospitality while this work was in progress.

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