On the Cryptomorphism between Davis’ Subset Lattices, Atomic Lattices, and Closure Systems under T1 Separation Axiom

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Abstract

In this paper we count set closure systems (also known as Moore families) for the case when all single element sets are closed. In particular, we give the numbers of such strict (empty set included) and non-strict families for the base set of size $n = 6$. We also provide the number of such inequivalent Moore families with respect to all permutations of the base set up to $n = 6$. The search in OEIS and existing literature revealed the coincidence of the found numbers with the entry for D. M. Davis’ set union lattice (A235604, up to $n = 5$) and $|L_n|$, the number of atomic lattices on $n$ atoms, obtained by S. Mapes (up to $n = 6$), respectively. Thus we study all those cases, establish one-to-one correspondences between them via Galois adjunctions and Formal Concept Analysis, and provide the reader with two of our enumerative algorithms as well as with the results of these algorithms used for additional tests. Other results include the largest size of intersection free families for $n = 6$ plus our conjecture for $n = 7$, an upper bound for the number of atomic lattices $L_n$, and some structural properties of $L_n$ based on the theory of extremal lattices.

1 Introduction

It is known that closure systems (also known as Moore families after E. H. Moore [26]) and complete lattices are closely interconnected [14, 5]. The subject of our study is the connection between a special class of Moore families on set of $n$ elements, in which every single element set is closed (w.r.t. the so-called T1 separation axiom [17, p.126]), atomic lattices [25], and union-closed families studied by D. M. Davis with a certain topological and combinatorial interest [9]. However, the main computational result of the paper is the contribution to OEIS by a new sequence containing the number of inequivalent closure systems under T1 up to $n = 6$ and extending its labelled strict and non-strict versions by their common 6th member.
During the experiments on computing the 6th member of \texttt{A334254} and \texttt{A334255} with our closure system enumeration algorithm, we found out that their inequivalent counterpart coincides with all the known members (up to \( n = 5 \)) of \texttt{A235604} except the second member of \texttt{A334255}. The literature search also revealed coincidence of the 5th and 6th members of \texttt{A334254} with that of the number of atomic lattices on five and six elements, respectively; the numbers are reported by S. Mapes [25].

Our algorithmic solution is based on one-to-one correspondences between the studied objects, namely, Davis’ lattices, atomic lattices, closure system under T1 separation axiom, and certain reduced formal contexts (which can be seen as minimal binary relations) that give rise to appropriate Galois connections and adjunctions. Galois connections and their resulting Galois lattices (or concept lattices) represented by two dually isomorphic lattices of closed sets were extensively studied in Formal Concept Analysis [14, 5, 6], an applied branch of modern lattice theory suitable for data analysis and knowledge processing, while Galois adjunctions are rather related to union-closed systems and were studied with regards to applications in category theory and topology [11, 12, 21].

Our by-product is related to the problem of the maximal size union-free set family (the asymptotic was given by D. J. Kleitman [22]), which is dually equivalent to the problem of maximal size intersection-free family or the maximal size of a reduced formal context on \( n \) objects as noted by B. Ganter and R. Wille [14]. We have found the value of the latter sequence for \( n = 6 \) and have provided a concrete lower bound for this value in the case \( n = 7 \).

The paper is organized as follows. Section 2 contains main definitions from lattice theory and studied papers by Davis [9] and Mapes [25]. Section 3 establishes correspondence between the three considered problems via Galois adjunctions. Section 4 describes two modifications of the \texttt{AddByOne} algorithm to enumerate both labelled and inequivalent families, respectively. In Section 5, we summarize the main results obtained. Section 6 discusses some alternative approaches that we used (or which can be potentially exploited) for the additional tests with contributed sequences.

## 2 Main definitions

In this section, we mainly use basic definitions and propositions from the book by Ganter and Wille on Formal Concept Analysis [14]; these basic notions and facts can also be found in classic and recent monographs on lattice theory as well [3, 6, 17].

### Lattices

A \textit{lattice} is a partially ordered set \( L := (L, \leq) \) such that for every pair of its elements \( x \) and \( y \), the supremum \( x \lor y \) and infimum \( x \land y \) always exist. \( (L, \leq) \) is called a \textit{complete lattice}, if the supremum \( \lor X \) and the infimum \( \land X \) exist for any subset \( X \) of \( L \). Every complete lattice \( L \) has its largest element \( \lor L \) called the \textit{unit element} of the lattice and denoted by \( 1_L \). Dually, the smallest element of any complete lattice \( 0_L \) is called the \textit{zero element}.
Lemma 1. Any ordered set in which the infimum exists for every subset is a complete lattice.

The upper neighbours of the zero element (if they exist) are called *atoms* of the lattice; dually, the lower neighbors of the unit element are called *coatoms*.

An *atomic lattice* (some authors prefer the term atomistic like Ganter and Wille [14]) is a complete lattice where each its element is the supremum of atoms.

**Closure systems and operators**

In what follows, to consider various set systems, without loss of generality we mainly use the set of first $n$ natural numbers instead of an arbitrary finite set of the same cardinality. We also use $[n]$ as a shorthand for the set of elements \{1, 2, \ldots n\}.

A *closure system* on a set $[n]$ is a set of its subsets which contain $[n]$ and is closed under intersection. That is $\mathcal{M} \subseteq 2^{[n]}$ is a closure system if $[n] \in \mathcal{M}$ and

\[ X \subseteq \mathcal{M} \Rightarrow \bigcap X \in \mathcal{M}. \]

If a closure system $\mathcal{M}$ contains emptyset, then $\mathcal{M}$ is *strict*.

A *closure operator* $\varphi$ on $[n]$ is a map assigning a closure $\varphi X \subseteq [n]$ to each subset $X \subseteq [n]$ under the following conditions:

1. $X \subseteq Y \Rightarrow \varphi X \subseteq \varphi Y$ \hspace{1cm} (monotony)
2. $X \subseteq \varphi X$ \hspace{1cm} (extensity)
3. $\varphi \varphi X = \varphi X$ \hspace{1cm} (idempotency)

T1 separation axiom for a closure system $\mathcal{M}$ over $[n]$ states that every single element set $\{i\} \in [n]$ is in $\mathcal{M}$, or, equivalently, is closed, i.e. $\varphi \{i\} = \{i\}$ [17].

Every closure system $\mathcal{M} \subseteq 2^{[n]}$ defines a closure operator as follows:

\[ \varphi_{\mathcal{M}} X := \bigcap \{ A \in \mathcal{M} \mid X \subseteq A \}. \]

While the set of closures of a closure operator $\varphi$ is always a closure system $\mathcal{M}_{\varphi}$.

**Davis’ lattice**

We keep the original notation of Davis [9] in this subsection whenever it is possible.

**Definition 2.** If $\mathcal{M} = \{X_1, \ldots, X_n\}$ is a collection of sets, and $S \subseteq [n]$, let

\[ \mathcal{M}_S := \bigcup_{i \in S} X_i. \]

The set $\mathcal{M}$ is called *proper* if it is never the case that $X_i \subseteq X_j$ for $i \neq j$. Any $\mathcal{M}$ defines a lattice $L(\mathcal{M})$ on $2^{[n]}$ by $S \leq T$ if $\mathcal{M}_S \subseteq \mathcal{M}_T$. Lattices $L$ and $L'$ on $2^{[n]}$ are said to be equivalent if there is a permutation $\pi$ of $[n]$ under which the induced permutation of $2^{[n]}$ preserves the lattice relations; i.e., $\pi(S) \leq \pi(T)$ iff $S \leq T$. 

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As Davis states: “For a possible application to algebraic topology, we have become interested in an enumeration problem for lattices of subsets, which we have been unable to find in the literature”. Moreover, he invites: “We wish to introduce it for further investigation.”

3 Establishing cryptomorphisms

3.1 Theory

As it is shown by F. Domenach in [10], different lattice cryptomorphisms can be established by using common lattice properties and various binary relations to enable usage of the defined notions interchangeably. Below, we establish connections between the three studied algebraic structures in a similar fashion.

Theorem 3. Let $\mathcal{M} \subseteq 2^{[n]}$ be a strict closure system with T1 separation axiom fulfilled, then $(\mathcal{M}, \subseteq)$ is an atomic lattice with $\bigwedge \mathcal{X} = \bigcap \mathcal{X}$ and $\bigvee \mathcal{X} = \varphi_{\mathcal{M}} \cup \mathcal{X}$ for all $\mathcal{X} \subseteq \mathcal{M}$. Conversely, every atomic lattice is isomorphic to the lattice of all closures of a strict closure system with T1 separation axiom fulfilled.

Proof. The infimum of $\mathcal{X}$ in $(\mathcal{M}, \subseteq)$ is defined as $\bigwedge \mathcal{X} = \bigcap \mathcal{X}$. Since all single element sets are closed, i.e., $\{i\} \in \mathcal{M}$, then they are the only upper neighbors of $\emptyset$, which is the zero element of $(\mathcal{M}, \subseteq)$.

By Lemma 1, there exists the supremum of $\mathcal{X}$, which is defined as $\bigvee \mathcal{X} = \varphi_{\mathcal{M}} \cup \mathcal{X}$. Since $\bigvee \mathcal{X} = S$ for some $S \in \mathcal{M}$, then $S = \varphi_{\mathcal{M}} \{S\} = \varphi_{\mathcal{M}} \bigcup_{s \in S} \{s\} = \bigvee \{s\}$.

Let $L = (L, \leq)$ be an atomic lattice on $n$ atoms. Then the set system $\{(x) \setminus 0_L \mid x \in L\}$ is a strict closure system under T1 axiom since $\bigcap_{y \in T} (y) \setminus 0_L = (\bigwedge T) \setminus 0_L$, the system contains empty set and $n$ single-element sets obtained from each atom of $L$, respectively.

Proposition 4. Every closure system $\mathcal{M} \subseteq 2^{[n]}$ with T1 separation axiom fulfilled is strict for $n \neq 1$.

Proof. For $n = 0$ the proposition holds trivially. For $n = 1$ the system $\{\{1\}\}$ is not strict. For $n \geq 2$ any pair $i, j \in [n]$ implies $\{i\} \cap \{j\} = \emptyset$; hence $\emptyset \in \mathcal{M}$.

To deal with Davis’ lattice, which in fact combines two isomorphic lattices, let us reformulate the original definition.

Let $U = \bigcup_{i \in [n]} X_i$ and $R$ be a binary relation on $[n] \times U$ with $iRu$ if $u \in X_i$.

Consider two operators, $(\cdot)^U : 2^{[n]} \to 2^U$ and $(\cdot)^\subseteq : 2^U \to 2^{[n]}$ that are defined as follows for any $A \subseteq [n]$ and $B \subseteq U$:

$$A^U := \{u \mid iRu \text{ for some } i \in A\}$$
(the union of all $X_i$ with $i \in A$, i.e., $\mathbb{M}_A$ in Davis’ notation)

$$B^\subseteq := \{i \mid iRu \text{ implies } u \in B\}$$

(all indices $i$ such that $X_i \subseteq B$).

These two operators $(\cdot)^\cup, (\cdot)^\subseteq$ forms the so-called axialities (cf. Birkhoff’s polarities [3]), i.e., Galois adjunction [11, 12] between powersets of $[n]$ and $U$. Note that Galois adjunctions between ordered sets are also known as isotone Galois connections [21].

Before we proceed with formal definitions and proofs, let us consider properties of the composite operators $(\cdot)^\cup \cup (\cdot)^\subseteq : \mathcal{P}[n] \to \mathcal{P}[n]$ and $(\cdot)^\subseteq \cup : \mathcal{P}[U] \to \mathcal{P}[U]$.

**Proposition 5.** Let $R \subseteq [n] \times U$ is binary relation, $A, A_1, A_2 \subseteq [n]$ and $B, B_1, B_2 \subseteq U$, then

1. a) $A_1 \subseteq A_2 \Rightarrow A_1^\cup \subseteq A_2^\cup$ and b) $B_1 \subseteq B_2 \Rightarrow B_1^\subseteq \subseteq B_2^\subseteq$

2. a) $(\cdot)^\subseteq$ is conrtactive, i.e., $(A)^\subseteq \subseteq A$, while b) $(\cdot)^\cup$ is extensive

3. a) $A^\cup = A^{\cup\subseteq}$ and b) $B^\subseteq = B^{\subseteq\cup}$

4. $(\cdot)^\subseteq$ and $(\cdot)^\cup$ are idempotent

5. $(\cdot)^\cup$ and $(\cdot)^\subseteq$ are isotone.

**Proof.** 1. a) $A_2^\cup = \{u \mid iRu \text{ for some } i \in A_1 \cup (A_2 \setminus A_1)\} = A_1^\cup \cup \{u \mid iRu \text{ for some } i \in A_2 \setminus A_1\}$. 1. b) If $i \in B^\subseteq_1$ then $iRu$ implies $u \in B_1$, i.e., also $u \in B_2$ since $B_1 \subseteq B_2$. 2. a) If $j \in A^\subseteq_2$, then $jRu$ implies $u \in A^\cup$, i.e., there exists $i \in A$ such that $iRu$. In short, $jRu$ implies $iRu$. 2. b) Let $u \in B$, then all $i$ such that $iRu$ are in $B^\subseteq$. It means that $u$ is also in $B^{\subseteq\cup}$ since $(\cdot)^\cup$ collects all $v$ incident to $i$ in $R$. 3. a) $A^{\cup\subseteq} \subseteq A^\cup$ by 2a and 1a, while $A^\cup \subseteq A^{\cup\subseteq\cup}$ follows immediately from 2b. 4. follows from 3a and 3b, respectively. 5. follows from 1. \qed

**Definition 6** ([21]). A pair $(\alpha, \beta)$ of maps $\alpha : P \to Q$, $\beta : Q \to P$ is called a Galois adjunction between the posets $(P, \leq)$ and $(Q, \leq)$ provided that

for all $p \in P$ and $q \in Q$, we have $\beta q \leq p \iff q \leq \alpha p$.

Given Galois adjunction $(\alpha, \beta)$, $\alpha$ is called the upper adjoint of $\beta$ and $\beta$ the lower adjoint of $\alpha$.

**Theorem 7.** The pair of operators $(\cdot)^\cup, (\cdot)^\subseteq$ forms the Galois adjunction between powersets of $[n]$ and $U$ related by $R \subseteq [n] \times U$. 

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Proof. For \(A \subseteq [n]\) and \(B \subseteq U\) we need to prove \(B^{\subseteq} \subseteq A \iff B \subseteq A^{\cup}\).

\[\Rightarrow\] Let \(B^{\subseteq} \subseteq A\), then due to isotony of \((\cdot)^{\cup}\) (Proposition 5.1a) we have \(B^{\subseteq U} \subseteq A^{\cup}\) and by extensity of \((\cdot)^{\subseteq U}\) we get \(B \subseteq B^{\subseteq U}\).

\[\Leftarrow\] Similarly, due to isotony of \((\cdot)^{\subseteq}\) (Proposition 5.1b) we have \(B^{\subseteq} \subseteq A^{\subseteq U}\) and by contraction of \((\cdot)^{\subseteq U}\) we get \(A^{\subseteq U} \subseteq A\).

\[\square\]

Note that any contractive, monotone, and idempotent operator \(\psi\) on a set \(S\) is called kernel (interior) operator. Its fixed points, i.e., \(X \subseteq S\) such that \(\psi X = X\) are called open sets or dual closures.

**Theorem 8 ([6],[12]).** Let \((P, \leq)\) and \((Q, \leq)\) be two ordered sets, \((\alpha, \beta)\) is the Galois adjunction between them. The following properties hold:

1. \(\alpha \beta \alpha = \alpha\) and \(\beta \alpha \beta = \beta\).
2. The composition map \(\varphi = \beta \alpha\) is a kernel operator on \(P\) and the composition map \(\psi = \alpha \beta\) is a closure operator on \(Q\).
3. The ordered subset \(\varphi(P)\) of open elements of \(\varphi\) in \(P\) is equal to \(\beta(Q)\) and the ordered subset \(\psi(Q)\) of closed elements of \(\psi\) in \(Q\) is equal to \(\alpha(P)\). The ordered subsets \(\varphi(P)\) and \(\psi(Q)\) are isomorphic, by the restrictions to the latter of \(\alpha\) and \(\beta\).

Particular cases of Statements 1. and 2. of Theorem 8 are proven in Proposition 5 for the pair of operators \(((\cdot)^{\cup}, (\cdot)^{\subseteq})\). Statement 3 implies the following corollary.

**Corollary 9.** For a given binary relation \(R \subseteq [n] \times U\) and the Galois adjunction \(((\cdot)^{\cup}, (\cdot)^{\subseteq})\), the ordered set of dual closures \(K_{U^{\subseteq}} = \{A^{\subseteq} \mid A \subseteq [n]\}\) is equal to \(K_{\subseteq} = \{B^{\subseteq} \mid B \subseteq U\}\) and the ordered set of closures \(M_{\subseteq U} = \{B^{\subseteq U} \mid B \subseteq U\}\) is equal to \(M_{U} = \{A^{\cup} \mid A \subseteq [n]\}\). \(K_{U^{\subseteq}}\) and \(M_{\subseteq U}\) are isomorphic subject to restrictions of \((\cdot)^{\cup}\) and \((\cdot)^{\subseteq}\).

Let us consider \(L(M)\) on a proper set \(M = \{X_1, X_2, \ldots, X_k\}\) (i.e., the antichain of \(X_i, i \in \{1, \ldots, k\}\) ) and the related incidence relation \(R \subseteq [n] \times U\) with the Galois adjunction \(((\cdot)^{\cup}, (\cdot)^{\subseteq})\).

**Theorem 10.** \(L(M) = (M_{\subseteq U}, \subseteq)\) is atomic lattice. Conversely, every atomic lattice is isomorphic to some \((M_{\subseteq U}, \subseteq)\).

**Proof.** \(\Rightarrow\) \(L(M) = (M_{\subseteq U}, \subseteq)\) since, by Corollary 9, \((M_{\subseteq U}, \subseteq) = (M_{U}, \subseteq)\) and \(A^{\cup} = M(A)\) for \(A \subseteq M\), by the definitions of operators \((\cdot)^{\cup}\) and \(M(\cdot)\). The zero element of \(L(M)\) is \(0_{L(M)} = 0_{M_{\subseteq U}} = \emptyset\). The upper neighbors of \(0_{L(M)}\) are closed single-element sets \(\{i\} = \{i\}^{\subseteq U} = M(\{i\})\) for \(i \in [n]\) and every \(A \in M_{\subseteq U}\) is equal to \(\bigcup_{i \in A}\{i\}\).

\(\Leftarrow\) By Theorem 3. Since every single-element set is closed in \(M_{\subseteq U}\) (T1 separation axiom is fulfilled) and \(\emptyset^{\subseteq U} = \emptyset\) (the system \(M_{\subseteq U}\) is strict). Note that for \(n = 1\), we have the only system \(M_{\subseteq U} = \{\emptyset, \{1\}\}\).
Theorem 10 allows us to transfer atomicity to Moore families. Then let us call any Moore family containing all single element sets from its base set atomic or, equivalently, atomic closure system.

Corollary 9 implies that kernel system $\mathcal{K}_{\cup \subseteq}$ is also isomorphic to $L(M)$. Actually, one can pair the fixed points of $\cdot)^{\cup \subseteq}$ and $\cdot)^{\subseteq \cup}$ via the Galois adjunction as follows.

For a given Galois adjunction $(\alpha, \beta)$ between two ordered sets $(P, \leq)$ and $(Q, \leq)$, consider a pair $(p, q)$, where $p \in P$ and $q \in Q$ and $p = \beta q$ and $q = \alpha p$.

In case of the adjunction on a binary set $R \subseteq [n] \times U$ for Davis’ lattice, such pairs are called upper concepts [29, 6], and we have the lattice of upper concepts $(\mathcal{A}, \sqsubseteq)$ such that

$$\mathcal{A} = \{(A, B) \mid A^{\cup} = B \text{ and } B^{\subseteq} = A \text{ for } A \subseteq [n], B \subseteq U\}$$

and

$$(A, B) \sqsubseteq (C, D) \iff A \subseteq C \text{ and } B \subseteq D \text{ for } (A, B), (C, D) \in \mathcal{A}.$$  

We provide several examples of such lattices in the next subsection.

One more isomorphism exists between atomic and LCM lattices, where LCM stands for least common multiplier; see, for example, S. Mapes work [25].

### 3.2 Examples

Let us consider several binary relations and the lattices of their lower concepts. In Fig. 3.2, one can see three $3 \times 3$ exemplary binary relations often used in Formal Concept Analysis for data scaling [14].

![Figure 1: Example relations for order, nominal, and contranominal scales.](image)

The line (or Hasse) diagrams of the relations are shown in Fig. 3.2. The shaded nodes depict atoms of the lattices. The leftmost lattice is not atomic since it is a chain of four elements.

From Mapes [25], we know that $L_n$, the lattice of all atomic lattices is also an atomic lattice. The central diagram in Fig. 3.2 shows the unit element (the Boolean lattice on three elements) of $L_3$, while the rightmost diagram shows its zero element (the diamond $M_3$), w.r.t. to the established isomorphism of lattices.
Let $R \subseteq [n] \times U$ be a binary relation with the Galois adjunction $(\langle \cdot \rangle^U, \langle \cdot \rangle^\subseteq)$.  
1) For every pair $i, j \in [n]$ such that $i \neq j$, it fulfills $\{i\}^U \not\subseteq \{j\}^U$ (antichain condition) $\iff$
 2) for $i \in [n]$ $\{i\}$ is closed w.r.t. set intersection ($T1$ separation axiom) $\iff$
 3) for $i \in [n]$ $\{i\}$ is dually closed, i.e. $\{i\}^\subseteq = \{i\}$.

Proof. $1 \Rightarrow 3$. If $\{i\}^U \not\subseteq \{j\}^U (i \neq j)$, then $\{i\}^\subseteq = \{k | \{k\}^U \subseteq \{i\}^U\} = \{i\}$.

$3 \Rightarrow 2$. If $\{i\}^\subseteq = \{i\}$, then there is no $j \neq i$ such that $\{j\}^U \subseteq \{i\}^U$. Hence, there is a unique $S \subseteq [n]$ such that $\{i\}^U = \bigcap_{k \in S} \{k\}^U$, namely, $S = \{i\}$.

$2 \Rightarrow 1$. If $\{i\}$ cannot be intersection of any $S \subseteq [n]$ except $S = \{i\}$, then $\{i\}^U \not\subseteq \bigcap_{j \in S} \{j\}^U$ for every $S \subseteq [n]$ such that $i \notin S$. Hence, there is no $j \neq i$ such that $\{j\}^U \subseteq \{i\}^U$ and $\{i\}^U$ cannot be union of sets $T \subseteq U$ except $T = \{i\}^U$, which implies $\{i\}^\subseteq = \{i\}$.

4 Algorithms

This section introduces the algorithm to traverse and count all binary relations resulting in unique atomic lattices.

First, we note that system of sets is proper (in the sense of Davis) if and only if each element in closed.

**Theorem 11.** Let $R \subseteq [n] \times U$ be a binary relation with the Galois adjunction $(\langle \cdot \rangle^U, \langle \cdot \rangle^\subseteq)$.  
1) For every pair $i, j \in [n]$ such that $i \neq j$, it fulfills $\{i\}^U \not\subseteq \{j\}^U$ (antichain condition) $\iff$
 2) for $i \in [n]$ $\{i\}$ is closed w.r.t. set intersection ($T1$ separation axiom) $\iff$
 3) for $i \in [n]$ $\{i\}$ is dually closed, i.e. $\{i\}^\subseteq = \{i\}$.

Proof. $1 \Rightarrow 3$. If $\{i\}^U \not\subseteq \{j\}^U (i \neq j)$, then $\{i\}^\subseteq = \{k | \{k\}^U \subseteq \{i\}^U\} = \{i\}$.

$3 \Rightarrow 2$. If $\{i\}^\subseteq = \{i\}$, then there is no $j \neq i$ such that $\{j\}^U \subseteq \{i\}^U$. Hence, there is a unique $S \subseteq [n]$ such that $\{i\}^U = \bigcap_{k \in S} \{k\}^U$, namely, $S = \{i\}$.

$2 \Rightarrow 1$. If $\{i\}$ cannot be intersection of any $S \subseteq [n]$ except $S = \{i\}$, then $\{i\}^U \not\subseteq \bigcap_{j \in S} \{j\}^U$ for every $S \subseteq [n]$ such that $i \notin S$. Hence, there is no $j \neq i$ such that $\{j\}^U \subseteq \{i\}^U$ and $\{i\}^U$ cannot be union of sets $T \subseteq U$ except $T = \{i\}^U$, which implies $\{i\}^\subseteq = \{i\}$.

Note the antichain condition also implies absence of duplicate rows in $R$ being represented as an incidence table since $\{i\}^U$ is “the row” of related elements to $i$ by $R$.

**Theorem 12.** Let $R \subseteq [n] \times U$ be a binary relation with the Galois adjunction $(\langle \cdot \rangle^U, \langle \cdot \rangle^\subseteq)$.  
$A \subseteq [n]$ is open (dually closed), $\{A\}^\subseteq = \{A\}$ $\iff$ $A$ is closed w.r.t. set intersection in the complementary binary relation $\overline{R}$ such that $(i, u) \in \overline{R}$ $\iff$ $(i, u) \notin R$.

Proof. Let $\{i\}' = \{i\}^U$, i.e. $m \in \{i\}' \iff m \notin \{i\}^U$.

Taking into account contraposition, for any dually closed $A \subseteq [n]$ we get
$A^\cup = \bigcup_{i \in A} \{i\}^\cup = \bigcap_{i \in A} \{i\}^\cup = \bigcap_{i \in A} \{i\}'$, i.e.,

the intersection of rows in $\overline{R}$ with all row indices from $A$. There are no other $j \in [n] \setminus A$ with $\bigcap_{i \in A} \{i\}' \subseteq \{j\}'$ since $A$ is open, i.e. $\bigcup_{i \in A} \{i\}^\cup \not\supseteq \{j\}^\cup$.

$\Leftarrow$ Since $A$ is closed, then there exists $B = \bigcap_{i \in A} \{i\}'$, i.e. intersection of all rows in $\overline{R}$ with indices from $A$. By contraposition, we get

$B = \bigcap_{i \in A} \{i\}' = \bigcup_{i \in A} \{i\}' = \bigcup_{i \in A} \{i\}^\cup$, i.e.,

the union of rows in $R$ with all row indices from $A$.

There are no other $j \in [n] \setminus A$ with $\{j\}^\cup \subseteq B$ since $A$ is closed, i.e. $\bigcap_{i \in A} \{i\}' \not\subseteq \{j\}'$.

Theorems 11 and 12 allow us working with enumeration of all closure systems and that of all kernel systems or all systems of sets closed under union interchangeably, given fixed $n$ and the smallest $R$ w.r.t the size of $U$. Similar replacement of Moore families by set systems closed under union was exploited by Brinkmann and Deklerck [4].

Since different binary relations $R \subseteq [n] \times U$ with a fixed $n$ can produce the same Moore families on $[n]$ (e.g., by removing a full column $[n] \times \{u\}$ in $R$ if $[n] \times \{u\} \subseteq R$), we need to identify valid ways to reduce $U$ and thus $R$ without affecting the resulting Moore family.

**Definition 13** (adopted from Ganter and Wille [14]). We call a binary relation $R \subseteq [n] \times U$ column reduced if 1) it is clarified, i.e. $R$ does not contain duplicate rows and columns ($\forall i, j \in [n] : \{i\}^\cup = \{j\}^\cup \Rightarrow i = j$; similarly, for $u, v \in U$) and 2) there is no $u \in U$, which can be obtained by intersection of other columns $X \subseteq U$, i.e. $u \notin X$ and $\bigcap_{x \in X} \overline{x} \neq u$.

A row reduced binary relation is defined similarly. If $R$ is both row and column reduced, $R$ is called reduced.

In practice, we cannot simultaneously eliminate all the rows and the columns that are reducible, but this is no problem if we add rows (or columns) in a lexicographic order and check reducibility.

By Sperner theorem [28] the largest set antichain in $2^{[n]}$ contains $\binom{n}{\lfloor n/2 \rfloor}$ sets. It makes it possible to deduce the exact lower bound for the number of elements in $\overline{U}$ for Davis’ lattice and the associated relation.

**Theorem 14.** The smallest size of $U$ in $R \subseteq [n] \times U$ such that the associated closure system is $T1$-separated (or atomic) is the minimal $k$ under which $\binom{k}{\lfloor k/2 \rfloor} \geq n$.

In our algorithms, we use binary set representation and work with integer types. For example, $7_{10} = 111_2$ represents the set $\{1,2,3\}$ (or $\{a,b,c\}$) since the first three bits are on.
The **Atomic AddByOne** algorithm is inspired by the **CloseByOne** algorithm proposed by S. O. Kuznetsov [24].

**Algorithm 1. Atomic AddByOne**

**Input:** the number of atoms \( n \in \mathbb{N} (n > 1) \)

**Output:** the number of Moore families fulfilling T1 separation axiom

1. Generate all combinations \( \binom{2^{[n]} \setminus \{\emptyset, [n]\}}{k_{\text{min}}} \) in lexicographic order.
2. Check each combination represented by a tuple \( t = (i_1, \ldots, i_k) \) whether it is a column reduced binary relation and fulfills T1 axiom. If yes, store 1 in \( \text{cnt}[t] \).
3. Extend each valid tuple \( t \) (the column reduced binary relation) from step 2 by a next integer \( i_{k+1} \) after \( i_k \) from \( \{i_k + 1, \ldots, 2^n - 2\} \) and check whether the new tuple \( t^* = (i_1, \ldots, i_{k+1}) \) is a reduced binary relation and fulfills T1 axiom (if T1 was fulfilled for \( t \), then skip T1-check). If yes, increment \( \text{cnt}[t] \) and repeat step 3 with \( t^* \) recursively.
4. Return the sum of all \( \text{cnt}-s \).

Step 1 excludes combinations with empty set since \( \emptyset \) should be present in the resulting system as intersection of atoms by Theorem 3. Since full rows and full columns are reducible, \( 2^n - 1 \) is always excluded (every closure system on \([n]\) contains \([n]\) by definition). Note that all subsets of \( 2^{[n]} \) of size \( k_{\text{min}} \), which elements has \( \lfloor k_{\text{min}}/2 \rfloor \) (or \( \lceil k_{\text{min}}/2 \rceil \)) bits each, forms the antichain of \( k_{\text{min}} \) elements by Theorem 14 and our previous work on Boolean matrix factorization of contranominal scales [20]. So, Step 1 can be further improved accordingly for \( n \) larger than 6 (\( k_{\text{min}} = 4 \)).

To enumerate inequivalent atomic Moore families, we apply all the permutations \( \pi \in \Pi(n) \) on the set \([n]\) to every subset of a concrete Moore family represented by tuple \( t \), i.e. we compute all \( \pi(t) = (\pi(i_1), \ldots, \pi(i_k)) \). We call \( t \) **canonic** if it is lexicographically smallest among all permuted tuples \( \pi(t) \). Algorithm 2 counts each canonic representative per an equivalence class w.r.t. \( \Pi(n) \).

**Algorithm 2. Atomic IneqAddByOne**

**Input:** the number of atoms \( n \in \mathbb{N} (n > 1) \)

**Output:** the number of inequivalent Moore families fulfilling T1 separation axiom

The only modification of Algorithm 1 is done at step 3.

3’. We additionally check whether the new tuple \( t^* \) is canonic and count only such tuples.

All the implementations are coded in Python, speeded up with Cython extension and multiprocess library, and available on the author’s Github\(^1\) along with the results of experiments recorded in Jupyter notebooks.

\(^1\)https://github.com/dimachine/ClosureSeparation
Table 1: Studied sequences with the found extensions in italic.

| n  | A334254 | A334255 | A235604 | A355517 |
|----|---------|---------|---------|---------|
| 0  | 1       | 1       | 1       | 1       |
| 1  | 2       | 1       | 1       | 2       |
| 2  | 1       | 1       | 1       | 1       |
| 3  | 8       | 8       | 4       | 4       |
| 4  | 545     | 545     | 50      | 50      |
| 5  | 702 525 | 702 525 | 7 443   | 7 443   |
| 6  | 66 096 965 307 | 66 096 965 307 | 95 239 971 | 95 239 971 |

5 Resulting numbers and sequences

By means of **Atomic AddByOne**, we obtained the new 6th member of A334254 and A334255, i.e. enumerated 66960965307 atomic Moore families for $n = 6$. The number of strict inequivalent atomic Moore families coincides with all the known members (up to $n = 5$) of A235604 and, as implied by theorems 3 and 10, its sixth member 95239971 obtained by us as well.

Since for the 2nd members A334255 and A334254 are different, as well as 2nd member of A235604 differs from $a(2)$ of the sequence with the numbers of all inequivalent atomic and strict Moore families, we suggest adding a new sequence A355517, which contains the numbers of strict non-isomorphic atomic Moore families for $n$ up to 6.

In addition to that, we provide a new realization of A305233. Since by Theorem 14, A305233 contains exactly the minimal number $k$ of elements needed to exceed a given number of atoms $n$ with the value of middle binomial coefficient $\binom{k}{\lfloor k/2 \rfloor}$ to form a proper antichain of $n$ sets with $\lfloor k/2 \rfloor$ elements each out of $k$.

Another our contribution is related to the maximal size of the reduced contexts. The first five members for $n = 1, \ldots, 5$ are known in the literature [14]: 1, 2, 4, 7, 13. While we found the sixth term 24 by full enumeration with Algorithms 1 and 2.

For $n = 6$, one out of ten Moore families of length 24 in its equivalence class is as follows:

$$\{7, 11, 13, 14, 19, 21, 22, 25, 26, 29, 30, 37, 38, 39, 41, 42, 43, 44, 49, 50, 51, 52, 56, 60\}.$$  

For the 7th member, we state that it is not less than 41, since by combinatorial (though non-exhaustive) search we found the largest set system of size 41 to form the reduced context of the Moore family:  \{7, 11, 13, 14, 19, 21, 22, 25, 26, 28, 35, 37, 38, 41, 42, 44, 49, 50, 52, 56, 67, 69, 70, 73, 74, 76, 81, 82, 84, 88, 97, 98, 100, 104, 113, 114, 116, 121, 122, 123, 124\}. In total, 420 Moore families of size 41 are in the found equivalence class.

Note that the last results for $n = 6$ and 7 are also valid for Moore systems without additional constraints.
6 Other approaches

There are algorithms to systematically enumerate closed set of any closure operator, in particularly given in terms of Galois connections on a binary relation. To do so in our case for the family of atomic closure systems, which is isomorphic $L_6$, we need to know its representation as a binary relation.

In FCA, Ganter and Wille [14] showed that for any finite lattice $L := (L, \leq)$, there exists a unique binary relation on join and meet irreducible elements, $J(L)$ and $M(L)$, respectively, such that the lattice formed by all its rows (or columns) closures under intersection is isomorphic to the original lattice; this relation is called a standard context and defined as the restriction of $\leq$, i.e. as $\leq \cap J(L) \times M(L)$. We use $K(L) := (J(L), M(L), L, \leq)$ to denote the standard context of a lattice $L$. A similar approach to represent and analyze finite lattices based on a poset of irreducibles is employed by G. Markowsky (e.g., to answer the question from genetics: “What is the smallest number of factors that can be used to represent a given phenotype system?”) [8].

From Mapes we know about two important theorems on the number of atoms and meet irreducible elements of $L_n$. The first theorem on the number of atoms is attributed to Phan [27] by Mapes [25]. So, we add the description of atoms to the statement of this theorem, refine the range of valid $n$ and provide its shorter proof here.

**Theorem 15.** The number of atoms of the lattice $L_n$ formed by all atomic closure families on $n > 1$ is equal to $2^n - n - 2$ and each atom has the form $\{\varnothing, \{1\}, \ldots, \{n\}, \sigma, [n]\}$, where $\sigma \subseteq [n]$ and $2 \leq |\sigma| < n$.

**Proof.** For $n = 2$ we have no atoms since $0_{L_2} = 1_{L_2} = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$. The difference of $A_\sigma = \{\varnothing, \{1\}, \ldots, \{n\}, \sigma, [n]\}$ and $0_{L_n} = \{\varnothing, \{1\}, \ldots, \{n\}, [n]\}$ is $\{\sigma\}$ for some $\sigma \subset [n]$ (such that $2 \leq |\sigma| < n$). Hence, all $A_\sigma$ are the upper neighbors $0_{L_n}$ since all other families in $L_n$ greater than $0_{L_n}$ are also greater than some $A_\sigma$. The number of all $\sigma$ is $2^n - n - 2$.

Note that for $n = 1$, $1_{L_1} = \{\varnothing, \{1\}\}$ is the only atom, i.e. the upper neighbor of $0_{L_n} = \{1\}$.

The statement of the second theorem is also enriched by us with the description of join-irreducible elements taken directly from the original proof by Mapes [25] except $n = 1$.

**Theorem 16.** Each meet irreducible element in $L_n$ for $n > 2$ has the form $2^{[n]} \setminus [\sigma, [n] \setminus i]$, where $\sigma \subset [n]$ and $2 \leq |\sigma| < n$ and $i \in [n]$. The number meet irreducible elements for $n \neq 1$ is $n(2^n - 1 - n)$ and $1$ for $n = 1$.

**Proof.** See Mapes [25]. For $n = 1$ the coatom is $0_{L_1} = \{1\}$.

**Corollary 17.** The standard context for the atomic closure system $L_n$ with $n > 2$ is given by
In Fig. 3, one can see the standard context (10 \times 16) of the lattice of atomic lattices \( L_4 \) obtained for the base set \( U = \{a, b, c, d\} \).

We computed the resulting values of \( A_{334254} \) and \( A_{334255} \) for \( n = 3, 4, 5, 6 \) with our implementation of parallel \textsc{NextClosure} algorithm (originally proposed by Ganter and Reuter [13]) and thus confirmed the results of \textsc{Atomic AddByOne}. The total computational time is hard to summarise properly per process, but it took about four days for our approach and five days and 17 hours on a laptop with 12 core Intel i-9 processor in parallel mode (seven days and six hours in sequential mode) for \textsc{NextClosure}.

Another interesting approach for enumeration of atomic lattices on \( n \) atoms was implemented by S. Mapes in Haskell [25]; even though its running time for \( n = 6 \) is not reported, it took less than a second for \( n = 5 \).

If we consider the algorithms and results on the number of Moore families without additional constraints \( A_{193674} \), then three notable algorithmic approaches pop up. The first one by Habib and Nourine [18] for \( n = 6 \) is based on a bijection between Moore families and ideal color sets of the colored poset based on the sum of \( n \) Boolean lattices with \( n - 1 \) atoms. The second one by Colomb, Irlande, and Raynaud [7] for \( n = 7 \) converts the original problem to union-closed sets due to computational efficiency issues, while the third one by Brinkmann and Deklerck [4] uses sophisticated automorphism enumeration techniques to obtain the solution for inequivalent union-closed families in the case \( n = 7 \). The result for
| $n$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ | $6$ |
|-----|-----|-----|-----|-----|-----|-----|-----|
| A193674, $a(n)$ | 1 | 2 | 7 | 61 | 2 480 | 1 385 552 | 75 973 751 474 |
| A334254, $b(n)$ | 1 | 2 | 1 | 8 | 545 | 702 525 | 66 096 965 307 |
| $a(n)/b(n)$ | 1 | 1 | $\approx 0.14$ | $\approx 0.13$ | $\approx 0.22$ | $\approx 0.51$ | $\approx 0.87$ |

Table 2: Ratio between $n$th members of sequences A193674 and A334254.

$n = 5$ was obtained by Higuchi [19] using depth-first search.

An interesting approach by Belohlavek and Vychodil [2] to generate non-isomorphic lattices employs the rejection of candidates to full isomorphism test using the vectors of essential pairs from the corresponding order relation of a lattice. However, this approach is devised for enumeration of lattices with a fixed number of elements.

The direct computation by the NEXTCLOSURE algorithm is possible up to $n = 6$ using the standard context of the lattice of all Moore families as reported by Ganter and Obiedkov [15].

Let us have a look at the ratio of members A193674 and that of A334254.

From Table 6, one can see that after reaching minimum $8/61$ at $n = 3$, the ratio $a(n)/b(n)$ rises up to $\approx 0.87$ at $n = 6$, which may be a good starting point for analysing asymptotic behaviour of the latter series.

At the moment we propose the following theorem on a weak upper bound for the number of atomic closure systems.

**Theorem 18.** Let $\mathcal{L}_n$ and $\mathcal{M}_n$ be the lattices of all atomic Moore families and all Moore families on a set $[n]$ $(n > 1)$, respectively, then

$$|\mathcal{L}_n| \leq |\mathcal{M}_n| - 2^n - n.$$  

**Proof.** Let us count the size of principal order ideal of $0_{\mathcal{L}_n}$ taken in $\mathcal{M}_n$, i.e., at first, to count all Moore subfamilies of $\{\emptyset, \{1\}, \ldots, \{n\}, [n]\}$. There are $2^n - 2$ subsets of $\{\{1\}, \ldots, \{n\}\}$ to be added to $\{\emptyset, [n]\}$ to form a valid Moore family. Every single element set $\{i\} \subset [n]$ gives rise to one out $n$ Moore families in the form $\{\{i\}, [n]\}$. Two remaining Moore families are $0_{\mathcal{M}_n} = \{[n]\}$ and $\{\emptyset, [n]\}$. We conclude with visiting another interesting venue, namely, extremal lattice theory, where questions “Why finite lattices described by standard contexts are large?” are studied based on the notion of VC-dimension [1]. As it was shown by Albano and Chornomaz [1], the reason to have a huge number of elements of a lattice is the presence in its standard contexts of the so-called contranominal scales, i.e. induced subcontexts (subrelations) of the form $N_c(k) := (\{1, \ldots, k\}, \{1, \ldots, k\}, \neq)$ (e.g., the rightmost binary relation in Fig. 1 is $N_c(3)$).

For example, the closure systems generated by a contranominal scale on $n$ elements taken as a standard context has $2^n$ closed sets.

The **breadth** of a complete lattice is the number of atoms of the largest Boolean lattice that the lattice contains as a suborder, i.e. the size of the largest contranominal scale subrelation of its standard context (valid for all finite contexts) as noted by Ganter [16].
Table 3: Estimated breadths of lattices $\mathcal{L}_n$ and $\mathcal{M}_n$ for $n = 3$ up to 6 and 7, respectively.

| $n$ | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|
| $|J(\mathcal{L}_n)|$ | 3 | 10 | 25 | 56 | 119 |
| $|J(\mathcal{M}_n)|$ | 7 | 15 | 31 | 63 | 127 |
| Estimated breadth of $\mathcal{L}_n$ | 3 | 5 | 7 | 11 | ? |
| Estimated breadth of $\mathcal{M}_n$ | 3 | 5 | 7 | 10 | 16 |

| Breadth of $\mathcal{L}_n$ | 3 | 7 | 13 | $\geq 18$ | $\geq 25$ |

Theorem 19 (Albano and Chornomaz [1]). Let $\mathbb{K} := ([n], U, I \subseteq [n] \times U)$ be an $\mathbb{N}^c(k)$-free formal context, then

$$|\text{L}(\mathbb{K})| \leq \sum_{i=0}^{k-1} \binom{n}{i},$$

In what follows, we denote the sum from Theorem 19 by $f_{AC}(n, k)$.

By using this theorem and our knowledge on the number of meet-irreducible element for the lattice of all atomic Moore families and Moore families, respectively, we obtain two series (Table 3) for the estimated breadth of those two lattices for known $|\mathcal{L}_n|$ and $|\mathcal{M}_n|$. Note that $|J(\mathcal{M}_n)| = 2^n - 1$ follows from Definition 17 and Proposition 18 in Caspard and Monjardet [5].

For example, for $n = 6$ we know $|J(\mathcal{L}_6)| = 56$. For $k = 11$ we get $f_{AC}(56, 11) = \sum_{i=0}^{10} \binom{56}{i} = 44872116214 < |\mathcal{L}_6| = 66096965307$, while for $k = 12$ we have $f_{AC}(n, k)(56, 12) = 193774331494 > |\mathcal{L}_6|$. So, $\mathcal{L}_6$ contains a 2048-elements Boolean lattice and the breadth of $\mathcal{L}_6$ is at least 11.

The actual breadth of $\mathcal{L}_3$ is indeed 3 since the standard context of $\mathcal{L}_3$ coincides with $\mathbb{N}^c(3)$, while the actual breadth of $\mathcal{L}_4$ is 7 since there are 80 embedded $\mathbb{N}^c(7)$ and that of $\mathcal{L}_5$ is 13 since there are 10980 embedded $\mathbb{N}^c(13)$ (the last line in Table 3 is found by a combinatorial search).

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References

[1] A. Albano and B. Chornomaz, Why concept lattices are large: extremal theory for generators, concepts, and VC-dimension, *Int. J. of General Systems*, 46:5 (2017), 440–457.

[2] R. Belohlavek and V. Vychodil, Residuated Lattices of Size \( \leq 12 \), *Order* 27 (2010), 147–161.

[3] G. Birkhoff, *Lattice Theory*, American Mathematical Society, 1967.

[4] G. Brinkmann and R. Deklerck, Generation of Union-Closed Sets and Moore Families, *Journal of Integer Sequences* 18.1.7 (2018), 1–9.

[5] N. Caspard and B. Monjardet, The lattices of closure systems, closure operators, and implicational systems on a finite set: a survey, *Discrete Applied Math.* 127 (2003), 241–269.

[6] N. Caspard, B. Lecrec, and B. Monjardet, *Finite Ordered Sets: Concepts, Results and Uses*, Cambridge University Press, 2012.

[7] P. Colomb, A. Irlande, and O. Raynaud, Counting of Moore families for \( n = 7 \), in *Formal Concept Analysis*, Vol. 5986 of *Lecture Notes in Computer Science*, Springer, 2010, pp. 72–87.

[8] G. Markowsky and L. Markowsky, Lattice Data Analytics: The Poset of Irreducibles and the MacNeille Completion, in. 10th Int. Conf. on Intelligent Data Acquisition and Advanced Computing Systems, IEEE, 2019, pp. 263–268.

[9] D. M. Davis, Enumerating lattices of subsets, preprint, 2013. Available at [http://arxiv.org/abs/1311.6664v2](http://arxiv.org/abs/1311.6664v2).

[10] F. Domenach, CryptoLat – a pedagogical software on lattice cryptomorphisms and lattice properties, in *Proceedings of the Tenth Int. Conf. on Concept Lattices and Their Applications*, 2013, pp. 93–103.

[11] M. Erne, J. Koslowski, A. Melton, and G. Strecker, A primer on Galois connections, in S. Andima et al. (eds.), *Papers on General Topology and its Applications*, Annals New York Acad. Sci. 704, 1993, pp. 103–125.
[12] M. Erne, Adjunctions and Galois Connections: Origins, History and Development, in *Galois Connections and Applications*, Springer, Springer, 2004, pp. 1–138.

[13] B. Ganter and K. Reuter, Finding all closed sets: A general approach. *Order* **8** (1991), 283–290.

[14] B. Ganter and R. Wille, *Formal Concept Analysis: Mathematical Foundations*, Springer, 1999.

[15] B. Ganter, S. Obiedkov, *Conceptual Exploration*, Springer, 2017.

[16] B. Ganter, Notes on Integer Partitions, in Francisco J. Valverde-Albacete and Martin Trnecka (eds.), *Proceedings of the 15th Int. Conf. on Concept Lattices and Their Applications*, 2020, pp. 19–31.

[17] G. Grätzer, *Lattice Theory: First Concepts and Distributive Lattices*, Dover, 2009.

[18] M. Habib and L. Nourine, The number of Moore families on $n = 6$, *Discrete Math.* **294** (2005), 291–296.

[19] A. Higuchi, Lattices of closure operators, *Discrete Math.* **179** (1998), 267–272.

[20] D. I. Ignatov, A. Yakovleva, On Suboptimality of GreConD for Boolean Matrix Factorisation of Contranominal Scales, in S. O. Kunetsov, A. Napoli, S. Rudolph (eds.) *Proc. of the 9th Int. Workshop “What can FCA do for Artificial Intelligence?”*, 2021, pp. 87–98.

[21] K. Keimel and J. Lawson, Continuous and Completely Distributive Lattices, in G. Grätzer, F. Wehrung (eds.), *Lattice Theory: Special Topics and Applications: Volume 1*, Springer, 2014, pp. 5–54.

[22] D. J. Kleitman, Extremal properties of collections of subsets containing no two sets in their union. *Journal of Combinatorial Theory*, **20** (1976), 390–392.

[23] S. O. Kuznetsov and S. A. Obiedkov, Comparing performance of algorithms for generating concept lattices. *J. Exp. Theor. Artif. Intell.* **14**(2–3) (2002), 189–216.

[24] S. O. Kuznetsov, A fast algorithm for computing all intersections of objects in a finite semilattice, *Nauchno-Tekhnicheskaya Informatika, Ser. 2* **1** (1993), 17–22.

[25] S. Mapes, Finite atomic lattices and resolutions of monomial ideals, *Journal of Algebra* **379** (2013), 259–276.

[26] E. H. Moore, *Introduction to a Form of General Analysis*, vol. 2. Yale University Press, 1910.
[27] J. Phan, Properties of Monomial Ideals and their Free Resolutions. PhD thesis, Columbia University, 2006.

[28] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, *Math. Zietschrift* 27 (1928), 544–548.

[29] M. Wolski, Galois Connections and Data Analysis, *Fundamenta Informaticae* 60 (2004), 401–415.

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