Induced representations and harmonic analysis on finite groups

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Abstract

Given a finite group $G$ and a subgroup $K$, we study the commutant of $\text{Ind}_{K}^{G} \theta$, where $\theta$ is an irreducible $K$-representation. After a careful analysis of Frobenius reciprocity, we are able to introduce an orthogonal basis in such commutant and an associated Fourier transform. Then we translate our results in the corresponding Hecke algebra, an isomorphic algebra in the group algebra of $G$. Again a complete Fourier analysis is developed and, as particular cases, we obtain some results of Curtis and Fossum on the irreducible characters of Hecke algebras. Finally, we develop a theory of Gelfand-Tsetlin bases for Hecke algebras.

1 Introduction

Let $G$ be a finite group and denote by $\hat{G}$ a complete set of irreducible, pairwise inequivalent unitary representations of $G$ and by $L(G)$ its group algebra. The dimension of $\sigma \in \hat{G}$ is denoted by $d_{\sigma}$ and $M_{d,d}(\mathbb{C})$ is the algebra of all $d \times d$ complex matrices. One of the key facts in the representation theory of $G$ is the isomorphism

$$L(G) \cong \bigoplus_{\sigma \in \hat{G}} M_{d_{\sigma},d_{\sigma}}(\mathbb{C}),$$

(1.1)

which is given explicitly by the Fourier transform; see [4], Section 9.5. Now suppose that $K$ is a subgroup of $G$, denote by $X = G/K$ the corresponding homogeneous space and by $L(X)$ the permutation module of all complex valued functions defined on $X$. Then we have the isomorphism

$$\text{Hom}_{G}(L(X), L(X)) \cong \bigoplus_{\sigma \in J} M_{m_{\sigma}, m_{\sigma}}(\mathbb{C}),$$

(1.2)

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where \( J \) is the set of all \( \sigma \in \hat{G} \) contained in \( L(X) \) and \( m_\sigma \) the multiplicity of \( \sigma \) in \( L(X) \); see again \[4\], Section 9.4. Clearly, (1.1) is a particular case of (1.2), because \( \text{Hom}_G(L(G), L(G)) \cong L(G) \). The (spherical) Fourier transform in the setting of (1.2) has been extensively studied when \((G, K)\) is a Gelfand pair, that is when the algebra \( \text{Hom}_G(L(X), L(X)) \) is commutative, which is equivalent to say that \( L(X) \) is multiplicity free. This analysis is based on the theory of spherical functions. There are several accounts on this subject and on its many applications; see \[1, 3, 4, 14, 17, 30\]. In \[24, 25, 26\] we extended the theory of spherical Fourier transforms to homogeneous spaces with multiplicity and gave several applications, mainly to probability and statistics (an earlier example may be found in \[23\]). In particular, we showed that multiplicity freeness is not an essential tool in order to develop a satisfactory theory and to perform explicit calculations.

In the present paper we face a more general problem. Suppose that \( \theta \) is an irreducible \( K \)-representation. Then we have again

\[
\text{Hom}_G(\text{Ind}_K^G \theta, \text{Ind}_K^G \theta) \cong \bigoplus_{\sigma \in J} M_{m_\sigma,m_\sigma}(\mathbb{C}),
\]  

(1.3)

where \( J \) is the set of all \( \sigma \in \hat{G} \) contained in \( \text{Ind}_K^G \theta \) and \( m_\sigma \) is the multiplicity of \( \sigma \) in \( \text{Ind}_K^G \theta \). We introduce a Fourier transform that gives an explicit isomorphism for (1.3). The irreducible characters of the algebra \( \text{Hom}_G(\text{Ind}_K^G \theta, \text{Ind}_K^G \theta) \) were computed by C.W. Curtis and T.V. Fossum in \[9\]. Accounts of their theory may be found in \[10\] and in the recent expository paper \[27\], where it is presented as a generalization of the theory of spherical functions of finite Gelfand pairs. But the results of Curtis and Fossum can be used only for the Fourier analysis of functions in the center of the algebra. In our approach, a complete set of matrix coefficients are obtained and the results of Curtis and Fossum may be derived in a more transparent form.

The plan of the paper is the following. Section 2 is devoted to fix notation and to introduce one of the key ideas of the paper: suitable scalar products are used not only in the representation spaces but also in the space of intertwining operators (normalized Hilbert-Schmidt scalar products). This leads to several natural orthogonality relations: in Section 3 these are obtained by a detailed analysis of Frobenius reciprocity. The results may be summarized in a commutative diagram of isomorphisms that are either isometries or multiples of isometries. In particular, for \( \sigma \in J \), the explicit isomorphism between \( \text{Hom}_K(\theta, \text{Res}_K^G \sigma) \) and \( \text{Hom}_G(\sigma, \text{Ind}_K^G \theta) \) and a particular choice of an orthonormal basis in \( \text{Hom}_K(\theta, \text{Res}_K^G \sigma) \) lead to an explicit decomposition of the \( \sigma \)-isotypic component in \( \text{Ind}_K^G \theta \). In the particular case of Gelfand pairs, this corresponds to the choice of a \( K \)-invariant vector in each spherical representation and to the use of the spherical functions to decompose the permutation representation; see Section 4.6 in \[4\]. In Section 4 the results on Frobenius reciprocity are used to get a natural orthogonal basis in \( \text{Hom}_G(\text{Ind}_K^G \theta, \text{Ind}_K^G \theta) \). The associated Fourier transform is the first explicit form of (1.3). In \[9, 10, 27\] the Hecke algebra was introduced as a subalgebra of \( L(G) \), then, using the theory of idempotents
in group algebras, $\text{Ind}_K^G \theta$ was identified with a subspace of $L(G)$. In Section 5 of the present paper we use a different approach: the theory developed in Section 3 naturally yields an isometric immersion of $\text{Ind}_K^G \theta$ in $L(G)$ and this isometry may be used as a tool to translate the harmonic analysis in Section 4 into a harmonic analysis of the Hecke algebra. Adapted bases for the irreducible representations of $G$ involved in the decomposition of $\text{Ind}_K^G \theta$ yield a complete set of matrix coefficients. In Section 6 we develop a theory of Gelfand-Tsetlin bases: when it is applicable, it leads to a natural orthonormal basis for $\text{Hom}_K(\theta, \text{Res}_K^G \sigma)$ and to a corresponding basis for the $\sigma$-isotypic component of $\text{Ind}_K^G \theta$. The first one is obtained by means of iterated restrictions, while the second one is obtained by iterated inductions.

It should be interesting to examine the case in which $K$ is a normal subgroup using Clifford theory (see [6]). Another direction of research might be the extension of our results for permutation representations of wreath products (see [2, 7] and [24]) to induced representations. A parallel theory was developed by D’Angeli and Donno in [11, 12, 13] by generalizing some constructions that arise in the setting of association schemes; from the point of view of special functions see [18].

All the prerequisites for this paper may found in our books [4, 8] and in our survey papers [5, 27]. Motivations may be found in our preceding papers [23, 24, 25, 26], where only permutation representations were studied and applied. Concrete examples of spherical functions associated with induced representations are in [19, 28], but the authors of these papers consider only multiplicity free induced representations.

## 2 Preliminaries

In this section, in order to fix notation, we recall some basic facts on linear operators on finite dimensional spaces and on the representation theory of finite groups. The scalar product on a finite dimensional Hermitian vector space $V$ is denoted by $\langle \cdot, \cdot \rangle_V$ and the associated norm by $\| \cdot \|_V$; we usually omit the subscript if it is clear from the context the vector space considered. All the vector spaces will be Hermitian, and therefore we will omit this adjective. Given two finite dimensional vector spaces $W, U$ we denote by $\text{Hom}(W, U)$ the vector space of all linear maps from $W$ to $U$ and for $T \in \text{Hom}(W, U)$ we denote by $T^*$ the adjoint of $T$. We define a (normalized Hilbert-Schmidt) scalar product on $\text{Hom}(W, U)$ by setting

$$\langle T_1, T_2 \rangle_{\text{Hom}(W,U)} = \frac{1}{\dim W} \text{tr}(T_2^* T_1)$$

for all $T_1, T_2 \in \text{Hom}(W, U)$. Since $\text{tr}(T_2^* T_1) = \text{tr}(T_1 T_2^*)$ we have
\[
\langle T_1, T_2 \rangle = \frac{\dim U}{\dim W} \langle T_2^*, T_1^* \rangle
\]

In particular, the map \( \text{Hom}(W, U) \ni T \mapsto \sqrt{\frac{\dim U}{\dim W}} T^* \in \text{Hom}(U, W) \) is a bijective isometry. Finally, note that if \( I_W : W \to W \) is the identity operator then \( \|I_W\|_{\text{Hom}(W,W)} = 1 \).

We consider only \textit{unitary representations} of finite groups and the adjective unitary will be usually omitted. If \( \sigma \) is a representation of a finite group \( G \) its dimension will be denoted by \( d_\sigma \). If \( (\sigma, W) \) and \( (\rho, U) \) are two representations of \( G \) we denote by \( \text{Hom}_G(W, U) = \{ T \in \text{Hom}(W, U) : T\sigma(g) = \rho(g)T, \forall g \in G \} \), the space of all \textit{intertwining operators}. Observe that if \( T \) belongs to \( \text{Hom}_G(W, U) \) then \( T^* \) belongs to \( \text{Hom}_G(U, W) \). Indeed, for all \( g \in G \) we have

\[
T^* \rho(g) = T^* \rho(g^{-1})^* = (\rho(g^{-1})T)^* = (T\sigma(g^{-1}))^* = \sigma(g^{-1})^* T^* = \sigma(g) T.
\]

If \( (\sigma, W) \) is irreducible and \( m = \dim \text{Hom}_G(W, U) \) then \( U \) contains \( m \) copies of \( W \). In this case we say that \( T_1, T_2, \ldots, T_m \in \text{Hom}_G(W, U) \) give rise to an \textit{isometric orthogonal decomposition} of the \( W \)-isotypic component \( mW \) of \( U \) if for every \( w_1, w_2 \in W \) and \( i, j \in \{1, 2, \ldots, m\} \) we have

\[
\langle T_iw_1, T_jw_2 \rangle_U = \langle w_1, w_2 \rangle_W \delta_{i,j}.
\]

This implies that the subrepresentation of \( U \) isomorphic to \( mW \) is equal to the orthogonal direct sum

\[
T_1W \oplus T_2W \oplus \cdots \oplus T_mW
\]

and each operator \( T_j \) is a isometry from \( W \) to \( \text{Ran}T_j \equiv T_jW \).

**Lemma 2.1.** Suppose that \( (\sigma, W) \) is irreducible. Then the operators \( T_1, T_2, \ldots, T_m \) give rise to an isometric orthogonal decomposition of the \( W \)-component of \( U \) if and only if \( T_1, T_2, \ldots, T_m \) form an orthonormal basis for \( \text{Hom}_G(W, U) \). Moreover, if this is the case, then we have:

\[
T_j^* T_i = \delta_{i,j} I_W.
\]

**Proof.** Suppose that \( T_1, T_2, \ldots, T_m \) form an orthonormal basis for \( \text{Hom}_G(W, U) \). By (2.5) we know that \( T_j^* \in \text{Hom}_G(U, W) \). Therefore \( T_j^* T_i \in \text{Hom}_G(W, W) \) and, by Schur’s lemma, there exist \( \lambda_{i,j} \in \mathbb{C} \) such that \( T_j^* T_i = \lambda_{i,j} I_W \). By taking the traces of both sides, we get

\[
\delta_{i,j} d_\sigma = \text{tr}(T_j^* T_i) = \lambda_{i,j} d_\sigma \Rightarrow \lambda_{i,j} = \delta_{i,j},
\]

that is (2.6). Therefore, if \( w_1, w_2 \in W \) then

\[
\langle T_iw_1, T_jw_2 \rangle_U = \langle T_j^* T_iw_1, w_2 \rangle_W = \delta_{i,j} \langle w_1, w_2 \rangle_W.
\]

The converse implication is trivial. \( \square \)
Let $(\sigma, W)$ be a representation of $G$ and let \( \{w_1, w_2, \ldots, w_{d_\sigma}\} \) an orthonormal basis of $W$. The corresponding matrix coefficients are defined by setting

\[ u_{j,i}^\sigma(g) = \langle \sigma(g)w_i, w_j \rangle \quad (2.7) \]

for $i,j = 1, 2, \ldots, d_\sigma$ and $g \in G$. If $\sigma, \rho$ are irreducible $G$-representations we set $\delta_{\sigma, \rho} = 1$, if $\sigma$ and $\rho$ are equivalent, otherwise we set $\delta_{\sigma, \rho} = 0$.

Let $L(G) = \{ f : G \to \mathbb{C} \}$ be the vector space of all complex valued functions defined on $G$, endowed with the scalar product $\langle f_1, f_2 \rangle = \sum_{g \in G} f_1(g) \overline{f_2(g)}$ for $f_1, f_2 \in L(G)$. It also has a natural structure of algebra by defining the convolution product of $f_1, f_2 \in L(G)$ as the function $(f_1 * f_2)(g) = \sum_{g_0 \in G} f_1(gg_0^{-1})f_2(g_0)$, for all $g \in G$.

**Proposition 2.2.** Let $(\sigma, W)$ and $(\rho, U)$ be irreducible $G$-representations. Then

1. $\langle u_{i,j}^\sigma, u_{h,k}^\rho \rangle = \frac{|G|}{d_\sigma} \delta_{\sigma, \rho} \delta_{i,h} \delta_{j,k}$ (orthogonality relations) \( (2.8) \)

2. $u_{i,j}^\sigma * u_{h,k}^\rho = \frac{|G|}{d_\sigma} \delta_{\sigma, \rho} \delta_{j,h} u_{i,k}^\sigma$ (convolution property). \( (2.9) \)

**Proof.** See Lemma 3.6.3 and Lemma 3.9.14 of [4]. \( \square \)

Let $(\sigma, W)$ be a $G$-representation and denote by $\chi^\sigma$ its character. The following elementary formula is a generalization of (2) in Exercise 9.5.8 of [4].

**Proposition 2.3.** If $(\sigma, W)$ is irreducible, $w \in W$ is a vector of norm 1 and $\phi(g) = \langle \sigma(g)w, w \rangle$ is the diagonal matrix coefficient associated with $w$, then

\[ \chi^\sigma(g) = \frac{d_\sigma}{|G|} \sum_{h \in G} \phi(h^{-1}gh) \quad (2.10) \]

for all $g \in G$.

**Proof.** Let $\{w_1 = w, w_2, \ldots, w_{d_\sigma}\}$ be an orthonormal basis of $W$ and $u_{j,i}^\sigma$ as in (2.7); then

\[ \sigma(g)w_i = \sum_{j=1}^{d_\sigma} u_{j,i}^\sigma(g)w_j. \]

Thus

\[
\sum_{h \in G} \phi(h^{-1}gh) = \sum_{h \in G} \langle \sigma(g)\sigma(h)w_1, \sigma(h)w_1 \rangle \vphantom{\sum_{j=1}^{d_\sigma}} \\
= \sum_{j,\ell=1}^{d_\sigma} \sum_{h \in G} u_{j,1}^\sigma(h) \overline{u_{\ell,1}^\sigma(h)} \langle \sigma(g)w_j, w_\ell \rangle \vphantom{\sum_{j=1}^{d_\sigma}} \\
= (\text{by (2.8)}) \quad = \frac{|G|}{d_\sigma} \chi^\sigma(g). \]

\( \square \)
Let $K$ be a subgroup of $G$, $(\theta, V)$ a representation of $K$ and denote by $\lambda = \text{Ind}_K^G \theta$ the induced representation (see for instance, [1, 5, 8, 20] and [29]). We recall that the representation space is given by

$$\text{Ind}_K^G V = \{f : G \to V : f(gk) = \theta(k^{-1})f(g), \forall g \in G, k \in K\}$$

(2.11)

and that the $G$-action is defined by setting

$$[\lambda(g_0)f](g) = f(g_0^{-1}g)$$

(2.12)

for all $f \in \text{Ind}_K^G V$, $g, g_0 \in G$. Let $G = \bigsqcup_{t \in T} tK$ be a decomposition of $G$ into right $K$-cosets ($\bigsqcup$ denotes a disjoint union). For $v \in V$ we define $f_v \in \text{Ind}_K^G V$ by setting

$$f_v(g) = \begin{cases} \theta(g^{-1})v & \text{if } g \in K \\ 0 & \text{if } g \notin K. \end{cases}$$

(2.13)

Then for every $f \in \text{Ind}_K^G V$ we have:

$$f = \sum_{t \in T} \lambda(t) f_v_t$$

(2.14)

with $v_t = f(t)$. The representation $\text{Ind}_K^G \theta$ is unitary with respect to the following scalar product:

$$\langle f_1, f_2 \rangle_{\text{Ind}_K^G V} = \frac{1}{|K|} \sum_{g \in G} \langle f_1(g), f_2(g) \rangle_V.$$

(2.15)

Moreover, if \{v_j : j = 1, 2, \ldots, d_\theta\} is an orthonormal basis in $V$ then the set

$$\{\lambda(t)f_v_j : t \in T, j = 1, 2, \ldots, d_\theta\}$$

(2.16)

is an orthonormal basis in $\text{Ind}_K^G V$ (see [5]).

3 Orthogonality relations for Frobenius reciprocity

Let $G$ be again a finite group, $K \leq G$ a subgroup, $(\sigma, W)$ a representation of $G$ and $(\theta, V)$ a representations of $K$. Frobenius reciprocity is usually stated an explicit isomorphism between $\text{Hom}_G(W, \text{Ind}_K^G V)$ and $\text{Hom}_K(\text{Res}_K^G W, V)$. In this section we present a detailed analysis of all other aspects of Frobenius reciprocity, all of them in an orthogonal version. In particular, we show how to obtain from an explicit orthogonal decomposition of the $V$-isotypic component of $\text{Res}_K^G W$, an explicit orthogonal decomposition of the $W$-isotypic component of $\text{Ind}_K^G V$.

Definition 3.1. 1. For each $T \in \text{Hom}_G(W, \text{Ind}_K^G V)$ we set

$$\wedge^T w = \sqrt{|G/K||T w|(1_G)}, \quad \text{for all } w \in W.$$
2. For each $L \in \text{Hom}_K(\text{Res}_K^G W, V)$ we set
\[ [\hat{L} w](g) = \frac{1}{\sqrt{|G/K|}} L \sigma(g^{-1}) w, \quad \text{for all } w \in W, g \in G. \]

3. For each $T \in \text{Hom}_G(\text{Ind}_K^G V, W)$ we set
\[ \hat{\circ} f = \sqrt{|G/K|} f, \quad \text{for all } f \in \text{Ind}_K^G V. \]

4. For each $L \in \text{Hom}_K(V, \text{Res}_K^G W)$ we set
\[ \hat{\diamond} f = \frac{1}{\sqrt{|G/K|}} \sum_{t \in T} \sigma(t) L f(t), \quad \text{for all } f \in \text{Ind}_K^G V. \]

Note that
\[ \hat{\diamond} f = \frac{1}{\sqrt{|G| \cdot |K|}} \sum_{g \in G} \sigma(g) L f(g). \quad (3.17) \]

Indeed,
\[ \sum_{g \in G} \sigma(g) L f(g) = \sum_{t \in T} \sum_{k \in K} \sigma(tk) L f(tk) = |K| \sum_{t \in T} \sigma(t) L f(t) \]

because $L \in \text{Hom}_K(V, \text{Res}_K^G W)$ and $\theta(k) f(tk) = f(t)$. In particular, $\hat{\diamond} L$ does not depend on the particular choice of $T$.

**Theorem 3.2** (Frobenius reciprocity revisited). 1. For each $T \in \text{Hom}_G(W, \text{Ind}_K^G V)$ we have $\hat{T} \in \text{Hom}_K(\text{Res}_K^G W, V)$ and the map
\[ \text{Hom}_G(W, \text{Ind}_K^G V) \rightarrow \text{Hom}_K(\text{Res}_K^G W, V) \]
\[ T \quad \mapsto \quad \hat{T} \]

is a linear isometric isomorphism. Moreover, its inverse is given by
\[ \text{Hom}_K(\text{Res}_K^G W, V) \rightarrow \text{Hom}_G(W, \text{Ind}_K^G V) \]
\[ L \quad \mapsto \quad \hat{\circ} L. \]

2. For each $T \in \text{Hom}_G(W, \text{Ind}_K^G V)$ we have:
\[ (T^*)^\circ = (\hat{T})^*. \quad (3.18) \]
Proof. (1) Let $T \in \text{Hom}_G(W, \text{Ind}_K^G V)$ and $\lambda$ as in (2.12). For all $k \in K$ and $w \in W$ we have

$$
\hat{T}\sigma(k)w = \sqrt{|G/K|}[T\sigma(k)w](1_G)
$$

(since $T \in \text{Hom}_G(W, \text{Ind}_K^G V)$)

(by (2.12))

(by (2.11))

$$
= \sqrt{|G/K|}[Tw](k^{-1})
= \sqrt{|G/K|}[\theta(k)[Tw](1_G)]
= \theta(k)\hat{T}w.
$$

This proves that $\hat{T} \in \text{Hom}_K(\text{Res}_K^G W, V)$. The identity

$$
[Tw](g) = [\lambda(g^{-1})Tw](1_G)
= [T\sigma(g^{-1})w](1_G)
= \frac{1}{\sqrt{|G/K|}}\hat{T}\sigma(g^{-1})w
$$

shows that the map $T \mapsto \hat{T}$ is injective, because $T$ is determined by $\hat{T}$. Now we use (3.19) to show that the map is also an isometry. If $T_1, T_2 \in \text{Hom}_G(W, \text{Ind}_K^G V)$ and \{w_1, w_2, \ldots, w_d\} is an orthonormal basis of $W$ then

$$
\text{tr}(T_2^*T_1) = \sum_{i=1}^{d} \langle T_1w_i, T_2w_i \rangle_{\text{Ind}_K^G V}
$$

(by (2.15))

$$
= \sum_{i=1}^{d} \frac{1}{|K|} \sum_{g \in G} \langle [T_1w_i](g), [T_2w_i](g) \rangle_V
$$

(by (3.19))

$$
= \sum_{i=1}^{d} \frac{1}{|G|} \sum_{g \in G} \langle \hat{T}_1\sigma(g^{-1})w_i, \hat{T}_2\sigma(g^{-1})w_i \rangle_V
= \frac{1}{|G|} \sum_{g \in G} \text{tr}[\sigma(g)(\hat{T}_2)^*\hat{T}_1\sigma(g^{-1})]
= \text{tr}[\hat{T}_2^*\hat{T}_1],
$$

that is $\langle T_1, T_2 \rangle = \frac{1}{d^2}\text{tr}(T_2^*T_1) = \frac{1}{d^2}\text{tr}[(\hat{T}_2)^*\hat{T}_1] = \langle \hat{T}_2, \hat{T}_1 \rangle$. It is easy to see that if $L \in \text{Hom}_K(\text{Res}_K^G W, V)$ then $\hat{L}w(gk) = \theta(k^{-1})[\hat{L}w](g)$ and $\lambda(g)\hat{L}w = \hat{L}\sigma(g)w$ for all $g \in G$, $k \in K$, $w \in W$, that is, $\hat{L}w \in \text{Ind}_K^G V$ and $\hat{L} \in \text{Hom}_G(W, \text{Ind}_K^G V)$. Finally, by definition of $\wedge$ and $\vee$ we have

$$
(\hat{L})^w = \sqrt{|G/K|}[\hat{L}w](1_G) = Lw.
$$
for all \( w \in W \), that is the map \( T \mapsto \hat{T} \) is surjective and \( L \mapsto \check{L} \) is its inverse.

(2) For any \( T \in \text{Hom}_G(W, \text{Ind}^G_K V) \), \( w \in W \) and \( v \in V \) we have (by definition of \( \circ \)):

\[
\frac{1}{\sqrt{|G/K|}} \langle (T^*)^v, w \rangle_W = \langle T^* f_v, w \rangle_W \\
= \langle f_v, T w \rangle_{\text{Ind}^G_K V} \\
= \frac{1}{|K|} \sum_{g \in G} \langle f_v(g), [T w](g) \rangle_V \\
(\text{by (2.13)}) = \frac{1}{|K|} \sum_{k \in K} \langle \theta(k^{-1}) v, [T w](k) \rangle_V \\
= \frac{1}{|K|} \sum_{k \in K} \langle v, \theta(k) [T w](k) \rangle_V \\
(\text{since } Tw \in \text{Ind}^G_K V) \\
= \frac{1}{\sqrt{|G/K|}} \langle v, \hat{T} w \rangle_V \\
= \frac{1}{\sqrt{|G/K|}} \langle (T^*)^v, w \rangle_W.
\]

\[\square\]

The following corollary should be compared with Corollary 34.1 in [1] and Section 2.3 in [24].

**Corollary 3.3** (The other side of Frobenius reciprocity). Let \( T \in \text{Hom}_G(\text{Ind}^G_K V, W) \).
Then \( \hat{T} \in \text{Hom}_K(V, \text{Res}^G_K W) \) and the map

\[
\text{Hom}_G(\text{Ind}^G_K V, W) \quad \longrightarrow \quad \text{Hom}_K(V, \text{Res}^G_K W) \\
T \quad \mapsto \quad \hat{T}
\]

is a linear isomorphism with

\[
\langle \hat{T}_1, \hat{T}_2 \rangle = |G/K| \langle T_1, T_2 \rangle, \tag{3.20}
\]

for all \( T_1, T_2 \in \text{Hom}_G(\text{Ind}^G_K V, W) \). The inverse is given by

\[
\text{Hom}_K(V, \text{Res}^G_K W) \quad \longrightarrow \quad \text{Hom}_G(\text{Ind}^G_K V, W) \\
L \quad \mapsto \quad \check{L}
\]

and

\[
(L^*)^\circ = \left( \check{L} \right)^\circ. \tag{3.21}
\]
for all \( L \in \text{Hom}_K(V, \text{Res}_K^GW) \).

In particular, the diagram

\[
\begin{array}{ccc}
\text{Hom}_G(W, \text{Ind}_K^G V) & \xrightarrow{\wedge} & \text{Hom}_K(\text{Res}_K^GW, V) \\
* \downarrow & & * \downarrow \\
\text{Hom}_G(\text{Ind}_K^G V, W) & \xrightarrow{\circ} & \text{Hom}_K(V, \text{Res}_K^GW)
\end{array}
\]

is commutative.

**Proof.** Besides the statement that the map \( L \mapsto \ast L \) is the inverse of the map \( T \mapsto \circ T \), everything follows from (2) in Theorem 3.2 (2.4) and (2.5). For all \( T \in \text{Hom}_G(\text{Ind}_K^G V, W) \), \( \phi \in \text{Ind}_K^G V \), we have:

\[
(T^\ast \phi) = \frac{1}{\sqrt{|G/K|}} \sum_{t \in T} \sigma(t) T \phi(t) \quad \text{(by definition of } \circ \text{)}
\]

\[
= \sum_{t \in T} \sigma(t) T f_{v_t} \quad \text{(by definition of } \circ \text{ with } v_t = \phi(t))
\]

\[
= \sum_{t \in T} T \lambda(t) f_{v_t} \quad \text{(since } T \in \text{Hom}_G(\text{Ind}_K^G V, W))
\]

\[
= T \phi \quad \text{(by } 2.14 \text{)}
\]

that is, the map \( L \mapsto \ast L \) is the inverse of the map \( T \mapsto \circ T \). We want also show how to derive (3.20). For \( T_1, T_2 \in \text{Hom}_G(\text{Ind}_K^G V, W) \) we have:

\[
\langle T_1^\ast, T_2^\ast \rangle = \langle [(T_1^+)^\ast, (T_2^+)^\ast] \rangle \quad \text{(by } 3.18 \text{)}
\]

\[
= \frac{d_\sigma}{d_\theta} \langle (T_2^+)^\ast, (T_1^+)^\ast \rangle \quad \text{(by } 2.4 \text{)}
\]

\[
= \frac{d_\sigma}{d_\theta} \langle T_2^+^\ast, T_1^+ \rangle \quad \text{(by Theorem } 3.2 \text{)}
\]

\[
= |G/K| \langle T_1, T_2 \rangle. \quad \text{(again by } 2.4 \text{)}
\]

Finally, by (3.18) we have

\[
\{(L^\ast)^\ast\}^\circ = \{(L^\ast)^\ast\}^\ast = L = \left\{ \left( L^\ast \right)^\ast \right\} \quad (3.22)
\]

and this yields (3.21).

**Corollary 3.4 (Orthogonality relations for Frobenius reciprocity I).** Let \( m \) be the dimension of \( \text{Hom}_G(W, \text{Ind}_K^G V) \) and suppose that \( L_1, L_2, \ldots, L_m \in \text{Hom}_K(V, \text{Res}_K^GW) \). Then the following facts are equivalent:
1. The set \( \{ L_1, L_2, \ldots, L_m \} \) is an orthonormal basis of \( \text{Hom}_K(V, \text{Res}^G_K W) \);

2. The set \( \left\{ \sqrt{|G/K|} L_1^\circ, \sqrt{|G/K|} L_2^\circ, \ldots, \sqrt{|G/K|} L_m^\circ \right\} \) is an orthonormal basis of \( \text{Hom}_G(\text{Ind}^G_K V, W) \);

3. The set \( \left\{ \sqrt{d} \sigma \left( L_1^* \right)^\circ, \sqrt{d} \sigma \left( L_2^* \right)^\circ, \ldots, \sqrt{d} \sigma \left( L_m^* \right)^\circ \right\} \) is an orthonormal basis of \( \text{Hom}_K(\text{Res}^G_K W, V) \);

4. The set \( \left\{ \sqrt{d} \sigma \left( L_1^* \right)^\circ, \sqrt{d} \sigma \left( L_2^* \right)^\circ, \ldots, \sqrt{d} \sigma \left( L_m^* \right)^\circ \right\} \) is an orthonormal basis of \( \text{Hom}_G(W, \text{Ind}^G_K V) \).

**Corollary 3.5** (Orthogonality relations for Frobenius reciprocity II). Suppose that \((\sigma, W)\) and \((\theta, V)\) are irreducible. Then \( \sqrt{d} \sigma \left( L_1^* \right)^\circ, \sqrt{d} \sigma \left( L_2^* \right)^\circ, \ldots, \sqrt{d} \sigma \left( L_m^* \right)^\circ \) give rise to an isometric orthogonal decomposition of the \( W \)-component of \( \text{Ind}^G_K V \) if and only if \( L_1, L_2, \ldots, L_m \) give rise to an isometric orthogonal decomposition of the \( V \)-component of \( \text{Res}^G_K W \).

**Proof.** It follows from Theorem 3.2 and Lemma 2.1.

In the applications of the last two Corollaries, we will often use the identity \((3.21)\), that is \((L_j^*)^\circ = (L_j)^*\). The following commutative diagram is helpful to memorize the previous results.

![Diagram](https://example.com/diagram.png)

**Remark 3.6.** In [21] it is developed a different version of orthogonality relations for Frobenius reciprocity. Actually, the author works in a more general setting: she considers representations over fields of characteristic zero and her spaces are endowed with arbitrary non-degenerate symmetric bilinear forms. However, we limit ourselves to illustrate and derive her main result in our setting. Theorem 2.1 of [21] may be expressed in the following way: under the assumption that \( W \) is \( G \)-irreducible, if \( L \in \text{Hom}_K(V, \text{Res}^G_K W) \) is an isometry then also \( \sqrt{d} \sigma \left( L^* \right)^\circ \in \text{Hom}_G(W, \text{Ind}^G_K V) \) is an isometry. This is our derivation: if \( L \) is an isometry, then \( \|L\| = 1 \) and therefore also \( \| \sqrt{d} \sigma \left( L^* \right)^\circ \| = 1 \). Arguing as in Lemma 2.1 it is easy to show that this fact implies that \( \sqrt{d} \sigma \left( L^* \right)^\circ \) is an isometry. Finally, we note that Theorem 2.4 in [21] is a version of our Corollary 3.5.
Remark 3.7. Corollary 3.3 is useful when irreducible representations are obtained as induced representations. This is the case of the little group method of Mackey and Wigner: we refer to [6] for a general formulation of this method and to [7] for its applications to wreath products of finite groups. Indeed, in [24] we used a version of Corollary 3.3 in order to decompose a wide class of permutation representations of wreath products, including the exponentiation action.

4 Harmonic analysis in \( \text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V) \)

In this section we construct an orthonormal basis of the commutant of \( \text{Ind}_K^G V \) from the orthonormal bases analyzed in the previous section. This way we can introduce a Fourier transform that gives an explicit isomorphism between \( \text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V) \) and \( \bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(C) \).

Let \( (\theta, V) \) be an irreducible representation of \( K \leq G \) and \( (\sigma, W) \) an irreducible representation of \( G \). Consider \( L_1, L_2 \in \text{Hom}_K(V, \text{Res}_K^G W) \). Then we have \( L_1^* \in \text{Hom}_K(\text{Res}_K^G W, V) \) and

\[
\begin{align*}
\text{Ind}_K^G V & \xrightarrow{L_2} W \xrightarrow{(L_1^*)^\vee} \text{Ind}_K^G V,
\end{align*}
\]

that is \( (L_1)^* L_2 = (L_1^*)^\vee L_2 \in \text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V) \).

Lemma 4.1. Let \( (\sigma_1, W_1) \) and \( (\sigma_2, W_2) \) be two irreducible inequivalent representations of \( G \). Consider \( L_1, L_2 \in \text{Hom}_K(V, \text{Res}_K^G W_2) \) and \( L_3, L_4 \in \text{Hom}_K(V, \text{Res}_K^G W_1) \). Then

\[
(\hat{L}_3)(\hat{L}_1)^* = 0
\]

and

\[
\left\langle (\hat{L}_1)^* \hat{L}_2, (\hat{L}_3)^* \hat{L}_4 \right\rangle = 0.
\]

Proof. By Schur’s lemma, \( \hat{L}_3(\hat{L}_1)^* \in \text{Hom}_G(W_2, W_1) = \{0\} \). Moreover, by definition of scalar product in \( \text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V) \) we have

\[
\left\langle (\hat{L}_1)^* \hat{L}_2, (\hat{L}_3)^* \hat{L}_4 \right\rangle = \frac{1}{\dim \text{Ind}_K^G V} \text{tr} \left[ (\hat{L}_4)^* \hat{L}_3(\hat{L}_1)^* \hat{L}_2 \right] = 0.
\]

Lemma 4.2. Let \( (\sigma, W) \) be an irreducible representation of \( G \) and \( \{L_1, L_2, \ldots, L_m\} \) an orthonormal basis of \( \text{Hom}_K(V, \text{Res}_K^G W) \). Then

\[
\hat{L}_h(\hat{L}_i)^* = \frac{d_\theta}{d_\sigma} I_W \delta_{i,h}.
\]

(4.24)
and the operators \((\hat{L}_i)^*\hat{L}_j \in \text{Hom}_G(\text{Ind}^G_K V, \text{Ind}^G_K V)\) satisfy the orthogonality relations:

\[
\langle (\hat{L}_i)^*\hat{L}_j, (\hat{L}_h)^*\hat{L}_\ell \rangle = \delta_{i,h} \delta_{j,\ell} \frac{d_\theta}{d_\sigma |G/K|},
\]

Proof. The identity (4.24) follows from (2.6) and Corollary 3.5. Moreover,

\[
\langle (\hat{L}_i)^*\hat{L}_j, (\hat{L}_h)^*\hat{L}_\ell \rangle = \frac{1}{\dim \text{Ind}^G_K V} \text{tr} \left[ (\hat{L}_i)^*\hat{L}_h (\hat{L}_i)^*\hat{L}_j \right]
\]

(by 4.24) \(= \frac{\delta_{i,h} d_\theta}{d_\sigma \dim \text{Ind}^G_K V} \text{tr} \left[ (\hat{L}_\ell)^*\hat{L}_j \right]
\]

\(= \delta_{i,h} \frac{d_\theta}{d_\sigma} (\hat{L}_j, \hat{L}_\ell)
\]

(by 2. in Corollary 3.4) \(= \delta_{i,h} \delta_{j,\ell} \frac{d_\theta}{d_\sigma |G/K|}.
\]

\[\square\]

In what follows, we denote by \(M_{m,m}(\mathbb{C})\) the algebra of all \(m \times m\) complex matrices. Let \(\text{Ind}^G_K V = \bigoplus_{\sigma \in J} m_\sigma W_\sigma\) be the decomposition of \(\text{Ind}^G_K V\) into irreducible representations of \(G\) (i.e., \(\{(\sigma, W_\sigma) : \sigma \in J\}\) is a complete set of all irreducible inequivalent representations of \(G\) contained in \(\text{Ind}^G_K V\) and \(m_\sigma\) is the multiplicity of \(W_\sigma\) in \(\text{Ind}^G_K V\)). For every \(\sigma \in J\) select an orthonormal basis

\[
\{L_{\sigma,1}, L_{\sigma,2}, \ldots, L_{\sigma,m_\sigma}\}
\]

of \(\text{Hom}_K(V, \text{Res}^G_K W_\sigma)\) and set

\[
U_{i,j}^\sigma = \frac{d_\sigma}{d_\theta} (\hat{L}_{\sigma,i})^*\hat{L}_{\sigma,j},
\]

(4.25)

for \(i, j = 1, 2, \ldots, m_\sigma\). For every \(T \in \text{Hom}_G(\text{Ind}^G_K V, \text{Ind}^G_K V)\) and \(\sigma \in J\), the Fourier transform of \(T\) at \(\sigma\) associated to the choice of (4.25) is the following matrix in \(M_{m_\sigma,m_\sigma}(\mathbb{C})\):

\[
[FT(\sigma)]_{i,j} = \frac{d_\theta |G/K|}{d_\sigma} \langle T, U_{i,j}^\sigma \rangle, \quad i, j = 1, 2, \ldots, m_\sigma.
\]

In the following theorem we will show that the Fourier transform is an explicit form of the isomorphism

\[
\text{Hom}_G(\text{Ind}^G_K V, \text{Ind}^G_K V) = \bigoplus_{\sigma \in J} \text{Hom}_G(m_\sigma W_\sigma, m_\sigma W_\sigma) \cong \bigoplus_{\sigma \in J} M_{m_\sigma,m_\sigma}(\mathbb{C}).
\]

(4.27)
We need further notation. Every element in the algebra $\bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C})$ may be represented in the form $\bigoplus_{\sigma \in J} A_\sigma$, where $A_\sigma \in M_{m_\sigma, m_\sigma}(\mathbb{C})$. In particular, given $T$ in $\text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V)$, we set

$$FT = \bigoplus_{\sigma \in J} FT(\sigma). \quad (4.28)$$

We recall [8] that the irreducible representations of this algebra are given by the natural action of each $M_{m_\sigma, m_\sigma}(\mathbb{C})$ on $\mathbb{C}^{m_\sigma}$ and that [27] the corresponding irreducible characters are the functions $\{\varphi^\sigma : \sigma \in J\}$ given by: $\varphi^\rho \left( \bigoplus_{\sigma \in J} A_\sigma \right) = \text{tr}(A_\rho)$. Under the isomorphism (4.27), the irreducible representation of $\text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V)$ corresponding to $\sigma \in J$ is given by its action on the space $\text{Hom}_G(W_\sigma, \text{Ind}_K^G V)$, that is by the map $S \mapsto TS$, where $T \in \text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V)$ and $S \in \text{Hom}_G(W_\sigma V, \text{Ind}_K^G V)$. In what follows, we will indicate by $\varphi^\sigma$ also the character of the isomorphic algebra $\text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V)$.

**Theorem 4.3.**

1. The set

$$\left\{ \sqrt{\frac{|d_\sigma| G/K}{d_\sigma}} U^\sigma_{i,j} : \sigma \in J, i, j = 1, 2, \ldots, m_\sigma \right\} \quad (4.29)$$

is an orthonormal basis of $\text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V)$. In particular, the Fourier inversion formula is:

$$T = \sum_{\sigma \in J} \sum_{i,j=1}^{m_\sigma} [FT(\sigma)]_{i,j} U^\sigma_{i,j}. \quad (4.29)$$

2. Setting $T_{\sigma,i} = \sqrt{\frac{d_\sigma}{d_0}} \hat{L}_{\sigma,i}^*$, we have the isometric orthogonal decomposition

$$\text{Ind}_K^G V = \bigoplus_{\sigma \in J} \bigoplus_{i=1}^{m_\sigma} T_{\sigma,i} W_\sigma,$$

and the corresponding explicit isomorphism

$$\text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V) \rightarrow \bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C})$$

$$T \mapsto FT.$$ 

3. The operator $U^\sigma_{i,j}$ intertwines the subspace $T_{\sigma,j} W_\sigma$ with $T_{\sigma,i} W_\sigma$.

4. The operator $U^\sigma_{i,i}$ is the orthogonal projection of $\text{Ind}_K^G V$ onto $T_{\sigma,i} W_\sigma$.

5. The irreducible characters of $\text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V)$ are the functions $\{\varphi^\sigma : \sigma \in J\}$ given by:
\[ \varphi^\sigma(T) = \text{tr} [FT(\sigma)], \]
for every \( T \in \text{Hom}_G(\text{Ind}^G_K V, \text{Ind}^G_K V). \)

**Proof.** From Lemma 4.1 and Lemma 4.2 we deduce that the set (4.22) is orthonormal. Moreover, \( \dim \text{Hom}_G(\text{Ind}^G_K V, \text{Ind}^G_K V) = \sum_{\sigma \in J} m_\sigma^2 \) so that it is a basis. For the other assertions just note that \( T_{\sigma,i} \) is an isometry and that, by (4.23) and (4.24), we have

\[
U_{i,j}^{\sigma} T_{h,l}^\rho = \delta_{\sigma,\rho} \delta_{j,h} U_{i,l}^\sigma,
\]
and therefore

\[
TT_{\sigma,j} = \sum_{i=1}^{m_\sigma} [FT(\sigma)]_{i,j} T_{\sigma,i},
\]
for all \( \sigma, \rho \in J, i, j = 1, 2, \ldots, m_\sigma, h, l = 1, 2, \ldots, m_\rho \) and \( T \in \text{Hom}_G(\text{Ind}^G_K V, \text{Ind}^G_K V). \)

\[ \square \]

### 5 Harmonic analysis in the Hecke algebra

Let \( G \) be a finite group and \( K \leq G \) a subgroup. As in the previous sections we denote by \((\theta, V)\) an irreducible \( K \)-representation and by \((\sigma, W)\) an irreducible \( G \)-representation. The left regular representation of \( G \) on \( L(G) \) is denoted by \( \lambda_G \) (to distinguish it from \( \lambda = \text{Ind}^G_K \theta \)); it is defined by setting \( [\lambda_G(g)f](g_0) = f(g^{-1}g_0) \) for all \( f \in L(G) \), \( g, g_0 \in G \).

We choose \( v \in V \) with \( \|v\| = 1 \) and define \( \psi \in L(K) \) by setting

\[ \psi(k) = \frac{d_\theta}{|K|} \langle v, \theta(k)v \rangle, \quad \forall k \in K. \]

Since \( L(K) \subseteq L(G) \), we may consider \( \psi \) also as a function in \( L(G) \). We define the operator

\[ T_v : \text{Ind}^G_K V \rightarrow L(G) \]

by setting:

\[ (T_v f)(g) = \sqrt{d_\theta/|K|} \langle f(g), v \rangle \]

for all \( f \in \text{Ind}^G_K V, g \in G \) (\( v \) is the same as in the definition of \( \psi \)). The following projection formula will be a very useful tool in many occasions.
Lemma 5.1. If \( v \in V \) has norm 1, then we have
\[
\sum_{k \in K} \langle \theta(k)u, v \rangle \theta(k^{-1})v = \frac{|K|}{d_\theta} u,
\]
for all \( u \in V \).

Proof. Let \( \{v_1, v_2, \ldots, v_{d_\theta}\} \) be an orthonormal basis of \( V \) with \( v = v_1 \). Then, for \( i, j = 1, 2, \ldots, d_\theta \), we have
\[
\left\langle \sum_{k \in K} \langle \theta(k)v_i, v \rangle \theta(k^{-1})v_j, v \right\rangle = \sum_{k \in K} \langle \theta(k)v_i, v_1 \rangle \overline{\langle \theta(k)v_j, v_1 \rangle}
\]
(by (2.8))
\[
= \frac{|K|}{d_\theta} \delta_{i,j}.
\]
Therefore we have proved (5.30) when \( u = v_i \) and the general case follows by linearity. \( \square \)

Proposition 5.2. 1. The operator \( T_v \) belongs to \( \text{Hom}_G(\text{Ind}^G_K V, L(G)) \) and it is an isometry.

2. The operator \( P : L(G) \longrightarrow L(G) \), defined by setting
\[
Pf = f \ast \psi
\]
for all \( f \in L(G) \), is the orthogonal projection of \( L(G) \) onto \( T_v[\text{Ind}^G_K V] \).

Proof. (1) The first part is obvious: it is immediate to check that
\[
T_v \lambda(g)f = \lambda_G(g)T_v f.
\]
We now show that \( T_v \) is an isometry: using the basis in (2.16) and assuming that \( v = v_1 \) we have, for \( t_1, t_2 \in T \), \( i, j = 1, 2, \ldots, d_\theta \),
\[
\langle T_v \lambda(t_1)f_{v_i}, T_v \lambda(t_2)f_{v_j} \rangle_{L(G)} = \frac{d_\theta}{|K|} \sum_{g \in G} \langle f_{v_i}(t_1^{-1}g), v \rangle \langle f_{v_j}(t_2^{-1}g), v \rangle
\]
(by (2.13))
\[
= \frac{d_\theta}{|K|} \delta_{t_1, t_2} \sum_{k \in K} \langle \theta(k^{-1})v_i, v_1 \rangle \overline{\langle \theta(k^{-1})v_j, v_1 \rangle}
\]
(by (2.8))
\[
= \delta_{t_1, t_2} \delta_{i,j}.
\]
(2) First of all, note that
\[
\psi \ast \psi = \psi \quad \text{and} \quad \overline{\psi(g^{-1})} = \psi(g).
\]
The first identity follows from (2.9) applied to \( \theta \) and ensures that \( P \) is an idempotent; from the second identity we deduce that \( P \) is selfadjoint, and therefore it is an orthogonal projection. Moreover, for all \( f \in \text{Ind}_K^G V \), \( g \in G \) we have

\[
[(T_v f) \ast \psi](g) = \left( \frac{d_\theta}{|K|} \right)^{3/2} \sum_{k \in K} \langle f(gk^{-1}), v \rangle \langle \theta(k)v \rangle
\]

(by (2.11))

\[
= \left( \frac{d_\theta}{|K|} \right)^{3/2} \left\langle f(g), \sum_{k \in K} \langle \theta(k)v, v \rangle \theta(k^{-1})v \right\rangle
\]

(by (5.30))

\[
= (T_v f)(g)
\]

that is, \( PT_v f = T_v f \) for all \( f \in \text{Ind}_K^G V \) (and in particular \( \text{Ran} P \supseteq T_v \text{Ind}_K^G V \)). Finally, let us show that the range of \( P \) is contained in (and therefore equal to) \( T_v \text{Ind}_K^G V \). Indeed, for all \( \phi \in L(G) \), \( g \in G \),

\[
P\phi(g) = \sum_{k \in K} \phi(gk) \psi(k^{-1})
\]

\[
= \frac{d_\theta}{|K|} \left\langle \sum_{k \in K} \phi(gk)\theta(k)v, v \right\rangle
\]

\[
= T_v f(g)
\]

if \( f(g) = \sqrt{\frac{d_\theta}{|K|}} \sum_{k \in K} \phi(gk)\theta(k)v \). Since it is immediate to check that \( f \) belongs to \( \text{Ind}_K^G V \), we conclude that \( \text{Ran} P = T_v \left[ \text{Ind}_K^G V \right] \).

\[\square\]

**Remark 5.3.** We want to relate the operator \( T_v \) in the context of the results in Section 3. First note that the choice of \( v \) is equivalent to the choice of an isometry \( L \in \text{Hom}(\mathbb{C}, V) \), namely \( L(\alpha) = \alpha v \) for \( \alpha \in \mathbb{C} \). Then, with \( K, V, G, W \) replaced by \( 1_K, \mathbb{C}, K, V \) we have:

\[
[(L^*)^\vee u](k) = \frac{1}{\sqrt{|K|}} (u, \theta(k)v)
\]

for all \( u \in V \) and \( k \in K \) (clearly, \( L^* u = \langle u, v \rangle \) for all \( u \in V \)). In particular, the map \( S_v = \sqrt{d_\theta}(L^*)^\vee \) is an isometric immersion of \( V \) into \( L(K) \) (this fact is also an easy consequence of the orthogonality relation for matrix coefficients). Considering \( S_v \) also as a map in \( \text{Hom}_K(V, \text{Res}_K^G L(G)) \) (because \( L(K) \subseteq \text{Res}_K^G L(G) \) in the natural way), it is easy to prove that \( T_v = \sqrt{\frac{|G|}{|K|}} S_v \), where in this case we apply the machinery in Section 3 with \( K, V \) and \( G \) as in that section, but with \( W \) replaced by \( L(G) \). Indeed, for \( f \in \text{Ind}_K^G V \) we
have:
\[
\sqrt{\frac{|G|}{|K|}} [S_v f](g) = \left[ \sum_{t \in T} \lambda_G(t) S_v f(t) \right] (g)
= \sum_{t \in T} [S_v f(t)] (t^{-1} g)
= \frac{\sqrt{|d \theta|}}{\sqrt{|K|}} \sum_{t \in T: t^{-1} g \in K} \langle f(t), \theta(t^{-1} g) v \rangle
= \frac{\sqrt{|d \theta|}}{\sqrt{|K|}} \langle f(g k^{-1}), \theta(k_g) v \rangle
= \frac{\sqrt{|d \theta|}}{\sqrt{|K|}} \langle f(g), v \rangle
= T_v f(g).
\]

In the terminology of [9, 10, 27], \( \psi \) is a primitive idempotent, \( S_v(V) = \{ f \in L(K) : f \ast \psi = f \} \) is the minimal left ideal in \( L(K) \) generated by \( \psi \) and \( T_v[\text{Ind}_G^K V] \) is generated by \( \psi \) as a left ideal in \( L(G) \). Then, the Hecke algebra \( \mathcal{H}(G, K, \psi) \) is by definition
\[
\mathcal{H}(G, K, \psi) = \{ \psi \ast f \ast \psi : f \in L(G) \} \equiv \{ f \in L(G) : f = \psi \ast f \ast \psi \}.
\]
It is well known that \( \mathcal{H}(G, K, \psi) \) is antiisomorphic to \( \text{Hom}_G(\text{Ind}_K^K V, \text{Ind}_K^K V) \): we now want to go further and develop a suitable harmonic analysis in \( \mathcal{H}(G, K, \psi) \), by translating the results of Section 4.

Now we introduce a suitable orthonormal basis in each \( G \)-irreducible representation. We divide the description of these bases in various cases.

Suppose that \( \sigma \in J \) and that \( \text{Res}_K^G W_\sigma = m_\sigma V \oplus (\oplus_{\rho \in R\setminus \rho \neq \sigma} \rho) U_\rho \) is the decomposition of \( \text{Res}_K^G W_\sigma \) into irreducible \( K \)-representations, where \( R \) contains the representations different from \( \sigma \). Let \( \{ L_{\sigma,1}, L_{\sigma,2}, \ldots, L_{\sigma,m_\sigma} \} \) be as in (4.25) and \( v \) as above. We begin by introducing an orthonormal basis in the \( V \)-isotypic component. The first step consists in setting
\[
w^\sigma_i = L_{\sigma,i} v, \quad i = 1, 2, \ldots, m_\sigma.
\]
In the second and last step we introduce (see also Lemma 5.1) an orthonormal basis \( v_1, v_2, \ldots, v_{d_\theta} \) of \( V \) with \( v = v_1 \) and we suppose that \( \{ w^\sigma_h : m_\sigma + 1 \leq h \leq m_\sigma d_\theta \} \) is an arbitrary arrangement of the vectors \( \{ L_{\sigma,i} v_j : 1 \leq i \leq m_\sigma, 2 \leq j \leq d_\theta \} \). The final result
\[
\{ w^\sigma_h : 1 \leq h \leq m_\sigma d_\theta \}
\]
is the desired orthonormal basis in \( m_\sigma V \).
Then we repeat the construction for each $U_\rho, \rho \in R$, without an initial choice of a vector in $U_\rho$ (we avoid the first step): we select an orthonormal basis $\{M_{\rho,1}, M_{\rho,2}, \ldots, M_{\rho,m_\rho}\}$ for $\text{Hom}_K(U_\rho, \text{Res}_K^G W_\sigma)$, an orthonormal basis $\{u^\rho_1, u^\rho_2, \ldots, u^\rho_{d_\rho}\}$ in $U_\rho$ and we suppose that

$$\{w^\rho_h : m_\rho d_\theta + 1 \leq h \leq d_\sigma\}$$

is an arbitrary arrangement of the vectors $\{M_{\rho,i}u^\rho_j : \rho \in R, 1 \leq i \leq m_\rho, 1 \leq j \leq d_\rho\}$. The final result is an orthonormal basis $\{w^\rho_h : 1 \leq h \leq d_\sigma\}$ for $W_\sigma$: we say that it is adapted to the choice of $v$ and of $\{L_{\sigma,1}, L_{\sigma,2}, \ldots, L_{\sigma,m_\sigma}\}$.

If $\sigma \notin J$ then $\{w^\rho_h : 1 \leq h \leq d_\sigma\}$ is an arbitrary orthonormal basis of $W_\sigma$. The importance of such bases is in the following property.

**Lemma 5.4.** 1. If $\sigma \in J$, $1 \leq j \leq m_\sigma$ and $1 \leq h \leq d_\sigma$ then

$$L^*_{\sigma,j}w^\sigma_h = \begin{cases} v_\ell & \text{if } w^\sigma_h = L_{\sigma,j}v_\ell \text{ for some } \ell \in \{1, 2, \ldots, d_\theta\} \\ 0 & \text{otherwise.} \end{cases} \quad (5.31)$$

2. 

$$\sum_{k \in K} \psi(k)\sigma(k)w^\sigma_i = \begin{cases} \frac{|K|}{d_\theta} w^\sigma_i & \text{if } \sigma \in J, 1 \leq i \leq m_\sigma \\ 0 & \text{otherwise.} \end{cases} \quad (5.32)$$

**Proof.** (1) This is a consequence of (2.6) and the definition of the vectors $w^\sigma_h$. (2) First of all, note that

$$\sum_{k \in K} \psi(k)\sigma(k)w^\sigma_i = \frac{d_\theta}{|K|} \sum_{k \in K} \langle \theta(k)v, v \rangle \sigma(k^{-1})w^\sigma_i.$$ 

If $\sigma \in J$ and $1 \leq i \leq m_\sigma$, then we may apply (5.30) since $\sigma(k^{-1})w^\sigma_i = \sigma(k^{-1})L_{\sigma,i}v = L_{\sigma,i}\theta(k^{-1})v$. Otherwise, we can argue as in the proof of (5.30): the bases are adapted to the choice of $v$ and to the decomposition of $\text{Res}_K^G W_\sigma$ and therefore we may use the orthogonality relations for the matrix coefficients in $L(K)$. For instance, if $\sigma \in J$ and $m_\sigma < i \leq m_\sigma d_\theta$ then $w^\sigma_i = L_{\sigma,h}v_j$ for some $1 \leq h \leq m_\sigma$, $2 \leq j \leq d_\theta$, and therefore, for all $\ell = 1, 2, \ldots, d_\sigma$

$$\left\langle \sum_{k \in K} \langle \theta(k)v, v \rangle \sigma(k^{-1})L_{\sigma,h}v_j, w^\sigma_\ell \right\rangle = \sum_{k \in K} \langle \theta(k)v, v \rangle \langle \theta(k^{-1})v_j, \ell w^\sigma_\ell \rangle = 0$$

because if $L^*_{\sigma,h}w^\sigma_\ell \neq 0$ by (5.31) it is equal to one of the $v_1, v_2, \ldots, v_{d_\theta}$ and $j \geq 2$. In the other cases we are dealing with matrix coefficients corresponding to inequivalent $K$-representations.
In what follows, we denote by $u_{i,j}^\sigma$ the matrix coefficients of $\sigma$ corresponding to the bases chosen above, that is,

$$u_{i,j}^\sigma(g) = \langle \sigma(g)w_j^\sigma, w_i^\sigma \rangle$$

for $\sigma \in \hat{G}$, $i, j = 1, 2, \ldots, d_\sigma$, $g \in G$. Then we define the convolution operators ($\sigma \in J$, $i, j = 1, 2, \ldots, m_\sigma$):

$$\tilde{U}_{i,j}^\sigma f = \frac{d_\sigma}{|G|} f \ast u_{j,i}^\sigma$$

for all $f \in L(G)$. We want to show that $\tilde{U}_{i,j}^\sigma$ corresponds to $U_{i,j}^\sigma$ in (1.26) under the isometry $T_v$.

**Theorem 5.5.** For all $\sigma \in J$, $i, j = 1, 2, \ldots, m_\sigma$ and $f \in \text{Ind}^G_K V$ we have:

$$T_v U_{i,j}^\sigma f = \tilde{U}_{i,j}^\sigma T_v f,$$

that is, the following diagram is commutative:

$$\begin{array}{ccc}
\text{Ind}^G_K V & \xrightarrow{T_v} & T_v[\text{Ind}^G_K V] \\
U_{i,j}^\sigma \downarrow & & \downarrow \tilde{U}_{i,j}^\sigma \\
\text{Ind}^G_K V & \xrightarrow{T_v} & T_v[\text{Ind}^G_K V].
\end{array}$$
Proof. For all $g \in G$ we have:

$$[T_{v_i,j}f](g) = \frac{d_\sigma}{\sqrt{|K|}d_\theta} \langle [(L_{\sigma,i}^*)^\circ L_{\sigma,j}f](g), v \rangle$$

$$= \frac{d_\sigma}{|G|\sqrt{|K|}d_\theta} \sum_{g_1 \in G} \langle L_{\sigma,i}^* \sigma(g^{-1}g_1)L_{\sigma,j}f(g_1), v \rangle$$

$$= \frac{d_\sigma}{|G|\sqrt{|K|}d_\theta} \sum_{g_1 \in G} \langle L_{\sigma,j}f(g_1), \sigma(g^{-1}g_1)w_i^\sigma \rangle$$

$$= \frac{d_\sigma}{|G|\sqrt{|K|}d_\theta} \sum_{g_1 \in G} \langle f(g_1), L_{\sigma,j}w_i^\sigma \rangle \langle \sigma(g^{-1}g_1)w_i^\sigma, L_{\sigma,j}v \rangle$$

(by (5.31))

$$= \frac{d_\sigma}{|G|\sqrt{|K|}d_\theta} \sum_{g_1 \in G} \langle f(g_1), v \rangle \langle \sigma(g^{-1}g_1)w_i^\sigma, L_{\sigma,j}v \rangle$$

(by (5.30) with $u = v$)

$$= \frac{d_\sigma}{|G|\sqrt{|K|}d_\theta} \sum_{g_2 \in G} \langle f(g_2), v \rangle \langle \sigma(g^{-1}g_1)w_i^\sigma, L_{\sigma,j}v \rangle$$

$$= \frac{d_\sigma}{|G|\sqrt{|K|}d_\theta} \sum_{g_2 \in G} \langle f(g_2), v \rangle$$

$$= \frac{d_\sigma}{|G|\sqrt{|K|}d_\theta} \sum_{g_2 \in G} \langle f(g_2), v \rangle$$

(by (5.30) with $u, v$ replaced by $v, v$)

$$= \frac{d_\sigma}{|G|\sqrt{|K|}d_\theta} \sum_{g_2 \in G} \langle f(g_2), \sigma g^{-1}g_1w_i^\sigma, w_j^\sigma \rangle$$

The following lemma shows that the only matrix coefficients $u_{i,j}^\sigma$ in $H(G, K, \psi)$ are the conjugate of those that come from the $V$-isotypic component of $W_\sigma$.

**Lemma 5.6.** We have

$$\psi \ast \overline{u_{i,j}^\sigma} \ast \psi = \begin{cases} u_{i,j}^\sigma & \text{if } \sigma \in J \text{ and } 1 \leq i, j \leq m_\sigma \\ 0 & \text{otherwise} \end{cases}$$
Proof. We have, for all \( g \in G \),
\[
\overline{u_{i,j}} \ast \psi(g) = \sum_{k \in K} u_{j,i}^\sigma(kg^{-1})\psi(k)
\]
(by definition of \( u_{i,j}^\sigma \))
\[
= \left\langle \sigma(g^{-1})w_{i}^\sigma, \sum_{k \in K} \psi(k)\sigma(k)w_{j}^\sigma \right\rangle
\]
(by (5.32))
\[
= \begin{cases} 
\overline{u_{i,j}^\sigma}(g) & \text{if } \sigma \in J \text{ and } 1 \leq j \leq m_{\sigma} \\
0 & \text{otherwise.}
\end{cases}
\]
(5.33)

The reader can complete the proof by computing in a similar way \( \psi \ast \overline{u_{i,j}^\sigma} \).

By virtue of Theorem 5.5 and Lemma 5.6 it is convenient to set:
\[
\phi_{i,j}^\sigma = \overline{u_{i,j}^\sigma}
\]
for \( \sigma \in J, i, j = 1, 2, \ldots, m_{\sigma} \). In other words,
\[
\phi_{i,j}^\sigma(g) = \langle w_{i}^\sigma, \sigma(g)w_{j}^\sigma \rangle
\]
for all \( g \in G \). Compare with Definition 9.4.5. of [4] and Definition 2.10 of [24] (see also [17, 30]).

If \( f \in \mathcal{H}(G, K, \psi) \), its Fourier transform at \( \sigma \in J \) is the \( m_{\sigma} \times m_{\sigma} \) complex matrix whose \( i, j \)-entry is
\[
[F(\sigma)]_{i,j} = \langle f, \phi_{i,j}^\sigma \rangle_{L(G)}
\]
for \( \sigma \in J \) and \( i, j = 1, 2, \ldots, m_{\sigma} \). As in (4.28), we set \( \mathcal{F}f = \bigoplus_{\sigma \in J} \mathcal{F}f(\sigma) \). We denote by \( \chi^\sigma \) the character of the \( G \)-irreducible representation \( (\sigma, W_\sigma) \). Moreover, \( \chi^\sigma(f) = \sum_{g \in G} \chi^\sigma(g)f(g) \) for all \( f \in L(G) \).

**Theorem 5.7.**
1. The set \( \{ \phi_{i,j}^\sigma : \sigma \in J, i, j = 1, 2, \ldots, m_{\sigma} \} \) is an orthogonal basis for \( \mathcal{H}(G, K, \psi) \) and \( \| \phi_{i,j}^\sigma \|^2_{L(G)} = \frac{d_{\sigma}}{d_{\sigma}} \). In particular, the Fourier inversion formula is
\[
f = \frac{1}{|G|} \sum_{\sigma \in J} d_{\sigma} \sum_{i,j=1}^{m_{\sigma}} [\mathcal{F}(\sigma)]_{i,j} \phi_{i,j}^\sigma.
\]
2. The map
\[
\mathcal{H}(G, K, \psi) \rightarrow \bigoplus_{\sigma \in J} M_{m_{\sigma}m_{\sigma}}(\mathbb{C})
\]
\[
f \mapsto \mathcal{F}f
\]
is an isomorphism of algebras.
3. Set
\[ \phi^\sigma = \sum_{i=1}^{m_\sigma} \phi^\sigma_{i,i} \]
and suppose that \( \varphi^\sigma \) is the irreducible character of \( \mathcal{H}(G, K, \psi) \) corresponding to \( M_{m_\sigma, m_\sigma}(\mathbb{C}) \). Then
\[ \varphi^\sigma(f) = \chi^\sigma(f) = \sum_{g \in G} f(g) \overline{\phi^\sigma(g)}, \tag{5.34} \]
for all \( f \in \mathcal{H}(G, K, \psi) \). Moreover, the \( \phi^\sigma \)'s satisfy the following orthogonality relations:
\[ \langle \phi^\sigma, \phi^\rho \rangle = \delta_{\sigma, \rho} \frac{|G| m_\sigma}{d_\sigma}. \tag{5.35} \]

**Proof.**
(1) It follows from Lemma \[5.6\] and the usual orthogonality relations for matrix coefficients in (2.8).
(2) The usual convolution properties of the matrix coefficients (2.9) yields
\[ \phi^\sigma_{i,j} * \phi^\rho_{h,\ell} = \frac{|G|}{d_\sigma} \delta_{\sigma, \rho} \delta_{j,h} \phi^\sigma_{i,\ell} \]
and this, combined with the Fourier inversion formula, implies that the Fourier transform is an isomorphism.
(3) For all \( f \in \mathcal{H}(G, K, \psi) \) and \( \sigma \in J \), we have
\[ \varphi^\sigma(f) = \sum_{i=1}^{m_\sigma} [\mathcal{F}f(\sigma)]_{i,i} = \sum_{i=1}^{m_\sigma} \langle f, \phi^\sigma_{i,i} \rangle = \sum_{g \in G} f(g) \overline{\phi^\sigma(g)} = \sum_{i=1}^{m_\sigma} \sum_{g \in G} f(g) u^\sigma_{i,i}(g) = \chi^\sigma(f), \]
where the last equality follows from the fact that \( \langle f, u^\sigma_{i,i} \rangle = 0 \) if \( i > m_\sigma \). The proof of (5.35) is obvious. \( \square \)

**Remark 5.8.** If \( \phi \in L(G) \) the convolution operator \( T_\phi : L(G) \to L(G) \) associated with \( \phi \) is defined by setting \( T_\phi f = f * \psi \) for all \( f \in L(G) \). Since \( T_{\phi_1 * \phi_2} = T_{\phi_2} T_{\phi_1} \), the map \( \phi \mapsto T_\phi \) is an antiisomorphism between \( L(G) \) and \( \text{Hom}_G(L(G), L(G)) \), see Exercise 4.2.2. in [4]. It folllows that the map \( U^\sigma_{i,j} \mapsto \tilde{U}^\sigma_{i,j} \) in Theorem 5.5 yields the antiisomorphism \( U^\sigma_{i,j} \mapsto \phi^\sigma_{i,j} \) between \( \text{Hom}_G(\text{Ind}_K^G V, \text{Ind}_K^G V) \) and \( \mathcal{H}(G, K, \psi) \) (see also [9, 10, 27]). Actually, these algebras are isomorphic, because they are both isomorphic to \( \bigoplus_{\sigma \in J} M_{m_\sigma, m_\sigma}(\mathbb{C}) \). Moreover, all other results in Theorem 4.3 may be translated in the present setting. For instance, if we define \( \tilde{T}_{\sigma,i} : W_\sigma \to L(G) \) by setting
\[ (\tilde{T}_{\sigma,i} w)(g) = \sqrt{\frac{d_\sigma}{|G|}} \langle w, \sigma(g) u^\sigma_i \rangle \]
for all \( w \in W, g \in G \), then it is easy to check that \( T_v T_{\sigma,i} = \tilde{T}_{\sigma,i} \) and \( T_v[\text{Ind}_K^G V] = \bigoplus_{\sigma \in J} \bigoplus_{i=1}^{m_\sigma} \tilde{T}_{\sigma,i} W_\sigma \) is an isometric orthogonal decomposition. Moreover, \( \tilde{U}_{\sigma,i} \) intertwines \( \tilde{T}_{\sigma,j} W_\sigma \) with \( \tilde{T}_{\sigma,i} W_\sigma \) and \( \tilde{U}_{\sigma,i} \) is the orthogonal projection onto \( \tilde{T}_{\sigma,i} W_\sigma \).

We now prove some formulas that relate \( \chi_\sigma, \varphi_\sigma \) and \( \phi_\sigma \) (see also \( (5.34) \)). We recall that \( \delta_g \) is the Dirac function centered at \( g \), that is
\[
\delta_g(g) = \begin{cases} 
1 & \text{if } g = g_0 \\
0 & \text{otherwise}.
\end{cases}
\]

**Theorem 5.9.** We have:
\[
\chi_\sigma(g) = \frac{d_\sigma}{|G|m_\sigma} \sum_{h \in G} \varphi_\sigma(h^{-1}gh)
\]
and
\[
\phi_\sigma(g) = \overline{\varphi_\sigma(\psi \ast \delta_g \ast \psi)}
\]
for all \( \sigma \in J \) and \( g \in G \).

**Proof.** The proof of \((5.36)\) is easy: it follows from \((2.10)\), taking into account that \( \phi_\sigma \) is the sum of the conjugate of \( m_\sigma \) diagonal matrix coefficients.

We now turn to the proof of \((5.37)\). Starting from \((5.34)\) we get:
\[
\varphi_\sigma(\psi \ast \delta_g \ast \psi) = \sum_{g_1 \in G} (\psi \ast \delta_g \ast \psi)(g_1) \overline{\varphi_\sigma(g_1)}
\]
\[
= \sum_{g_1 \in G} \sum_{k_2 \in K} \left[ \psi(g_1 k_2^{-1} g^{-1}) \psi(k_2) \right] \overline{\varphi_\sigma(g_1)}
\]
\[
= \sum_{k_1, k_2 \in K} \psi(k_1) \psi(k_2) \overline{\varphi_\sigma(k_1 g k_2)}
\]
(by Lemma \(5.6)\)
\[
= \overline{\varphi_\sigma(g)}.
\]

**Remark 5.10.** In \([9]\) (see also \([10]\)) it is proved a formula that expresses \( \chi_\sigma \) in terms of \( \varphi_\sigma \). In our notation it reads:
\[
\chi_\sigma(g) = \frac{|G|}{|C(g)|} \varphi_\sigma(\psi \ast 1_{C(g)} \ast \psi) \left[ \sum_{h \in G} \varphi_\sigma(\psi \ast \delta_{h^{-1}} \ast \psi) \cdot \varphi_\sigma(\psi \ast \delta_h \ast \psi) \right]^{-1}
\]
\[(5.38)\]
where \( C(g) \) denotes the conjugacy class of \( g \in G \) and \( 1_{C(g)} \) its characteristic function. In the research-expository paper \([27]\) it is showed, among other things, how to obtain these
results using the techniques in \cite{9} and \cite{10}. We now want to deduce (5.38) from the results proved in the present paper. First note that by (5.37)

\[ \sum_{h \in G} \phi^\sigma(h) \phi^\sigma(h^{-1}) = \sum_{h \in G} |\phi^\sigma(h)|^2 \]

(by (2.8)) \[ = \frac{m_\sigma |G|}{d_\sigma}. \] \tag{5.39}

Moreover, starting from the equality \(1_{C(g)} = \frac{|C(g)|}{|G|} \sum_{h \in G} \delta_{h^{-1}gh}\) we get

\[ \varphi^\sigma(\psi \ast 1_{C(g)} \ast \psi) = \frac{|C(g)|}{|G|} \sum_{h \in G} \varphi^\sigma(\psi \ast \delta_{h^{-1}gh} \ast \psi) \]

(by (5.37)) \[ = \frac{|C(g)|}{|G|} \sum_{h \in G} \phi^\sigma(h^{-1}gh) \]

(by (5.36)) \[ = \frac{|C(g)|}{|G|} \frac{m_\sigma}{d_\sigma} \chi^\sigma(g). \] \tag{5.40}

Then (5.38) follows from (5.39) and (5.40).

The spherical functions of a finite Gelfand pair satisfy the following functional identity

\[ \frac{1}{|K|} \sum_{k \in K} \phi(gkh) = \phi(g) \phi(h) \]

for all \(g, h \in G\) (see Theorem 4.5.3 in \cite{4} and \cite{13, 16, 17, 30}). We give an analogous identity for the matrix coefficients \(\phi^\sigma_{i,j}\).

**Proposition 5.11.** For \(\sigma \in J, i, j = 1, 2, \ldots, m_\sigma\) and \(g, h \in G\), we have

\[ \sum_{k \in K} \phi^\sigma_{i,j}(gkh) \psi(k) = \sum_{\ell=1}^{m_\sigma} \phi^\sigma_{i,\ell}(g) \phi^\sigma_{\ell,j}(h). \]

**Proof.**

\[ \sum_{k \in K} \phi^\sigma_{i,j}(gkh) \psi(k) = \sum_{k \in K} \langle w^\sigma_i, \sigma(gkh) w^\sigma_j \rangle \psi(k) \]

\[ = \sum_{\ell=1}^{d_\sigma} \langle \sigma(g^{-1}) w^\sigma_i, w^\sigma_j \rangle \sum_{k \in K} u^\sigma_{i,j}(kh) \psi(k) \]

(by (5.33)) \[ = \sum_{\ell=1}^{m_\sigma} \phi^\sigma_{i,\ell}(g) \phi^\sigma_{\ell,j}(h). \]

\( \square \)
Remark 5.12. If the $K$-representation $(\theta, V)$ is one dimensional, it can be identified with its character $\chi: K \to \{ z \in \mathbb{C} : |z| = 1 \}$ which satisfies the identity $\chi(k_1 k_2) = \chi(k_1) \chi(k_2)$, for all $k_1, k_2 \in K$. It follows that

$$\mathcal{H}(G, K, \psi) = \{ f \in L(G) : f(k_1 g k_2) = \overline{\chi(k_1)} \chi(k_2) f(g), \; \forall k_1, k_2 \in K, \; g \in G \}.$$ 

See [27] for the easy details. If $G = \bigsqcup_{s \in S} K s K$ is the decomposition of $G$ into double $K$-cosets, then a function $f \in \mathcal{H}(G, K, \psi)$ is determined by its values on $S$. In particular, the orthogonality relations for $\phi^i_{i,j}$ and $\phi^\sigma$ take the form:

$$\sum_{s \in S} |K s K| \phi^\sigma_{i,j}(s) \overline{\phi^\sigma_{i,j}(s)} = \frac{|G|}{d_{\sigma}} \delta_{\sigma, \rho} \delta_{i,t} \delta_{j,r}$$

and

$$\sum_{s \in S} |K s K| \phi^\sigma(s) \overline{\phi^\sigma(s)} = \frac{m_{\sigma} |G|}{d_{\sigma}} \delta_{\sigma, \rho}.$$

\[\text{From the last formula, it is just an easy exercise to deduce the orthogonality relations in 2, Theorem 4.24 in [27], originally proved by Curtis and Fossum (see Theorem 2.4 in [9] or (ii), Theorem 11.32 in [10]).}\]

6 Gelfand-Tsetlgin bases

We now extend to our setting the classical theory of Gelfand-Tsetlgin bases (cf. [8, 22, 25]), that yields a natural choice for the orthonormal basis in Corollary 3.4. We continue to use the notation of the previous sections (in particular Sections 3 and 4). First we prove a preliminary result that examines the correspondence $L \mapsto (L^*)^\vee$ in relation to the induction in stages. Let $H$ be a subgroup of $G$ containing $K$ (i.e. $K \leq H \leq G$) and denote by $(\rho, U)$ an irreducible $H$-representation. If $L_1 \in \text{Hom}_K(V, \text{Res}_H^G U)$ and $L_2 \in \text{Hom}_H(U, \text{Res}_K^G W)$ then $L_2 L_1 \in \text{Hom}_K(V, \text{Res}_K^G W)$. Since $(L_1^*)^\vee \in \text{Hom}_H(U, \text{Ind}_K^G V)$, we can consider $(L_1^*)^\vee U$ as a subspace of $\text{Ind}_H^G V$. Therefore, $\text{Ind}_H^G[(L_1^*)^\vee U]$ is a subspace of

$$\text{Ind}_H^G \text{Ind}_K^G V \simeq \text{Ind}_K^G V. \tag{6.41}$$

We recall the construction of this isomorphism, which is the transitive property of the induction (cf. [5, 8]). The Left Hand Side may be seen as the set of all $F: G \times H \to V$ such that $F(gh, h_0 k) = \theta(k^{-1}) F(g, hh_0)$, for all $g \in G$, $h, h_0 \in H$ and $k \in K$. Being the Right Hand Side as in (2.11), we have that the isomorphism in (6.41) is given by the map

$$F \mapsto f \tag{6.42}$$

where $f(g) = F(g, 1_G)$ for all $g \in G$ (note that $F$ is uniquely determined by $f$, because $F(g, h) = f(gh)$ for all $g \in G$ and $h \in H$).
Theorem 6.1. Under the isomorphism (6.42), we have

\[ [(L_2L_1)^*]^\vee W \leq \text{Ind}_H^G [(L_1^*)^\vee U]. \]

Proof. The space \( \text{Ind}_H^G [(L_1^*)^\vee U] \) is made up of all functions \( F \in \text{Ind}_H^G [\text{Ind}_K^H V] \) such that, for every fixed \( g \in G \), the function \( h \mapsto F(g, h) \) belongs to \( (L_1^*)^\vee U \), i.e. there exists \( u_g \in U \) such that

\[ F(g, h) = [(L_1^*)^\vee u_g](h). \]  

(6.43)

For \( w \in W, g \in G \) we have:

\[ \{ [(L_2L_1)^*]^\vee w \} (g) = \frac{1}{\sqrt{|G/K|}} L_1^* L_2^* \sigma(g^{-1})w \]

and therefore

\[ \{ [(L_2L_1)^*]^\vee w \} (gh) = \frac{1}{\sqrt{|G/K|}} L_1^* L_2^* \sigma(h^{-1})\sigma(g^{-1})w \]

\( L_2 \in \text{Hom}_H(U, \text{Res}_H^G W) \)

\[ = \frac{1}{\sqrt{|G/K|}} L_1^* \rho(h^{-1})[L_2^* \sigma(g^{-1})w] \]

\[ = \frac{\sqrt{|H/K|}}{\sqrt{|G/K|}} \{ [(L_1^*)^\vee [L_2^* \sigma(g^{-1})w]] (h) \} \]

\[ = \{ [(L_1^*)^\vee [(L_2^*)^\vee w]](g) \} (h). \]

This means that, with respect to (6.42), the function \( f = (L_2L_1)^\vee w \in \text{Ind}_K^H V \) corresponds to an \( F(g, h) \) of the form (6.43), with \( u_g = [(L_2^*)^\vee w](g) \). Therefore, \( f \in \text{Ind}_H^G [(L_1^*)^\vee U]. \)

Suppose now that there exists a chain of subgroups of \( G \) of the form

\[ K = H_1 \leq H_2 \leq \cdots \leq H_{m-1} \leq H_m = G. \]  

(6.44)

Define recursively \( J_\ell \subseteq \widetilde{H}_\ell, 1 \leq \ell \leq m \), by setting \( J_1 = \{ \emptyset \} \) and \( J_{\ell+1} \) equal to the set of all \( \eta \in \widetilde{H}_{\ell+1} \) such that \( \eta \) is contained in \( \text{Ind}_{\widetilde{H}_{\ell+1}}^{\widetilde{H}_{\ell}} \rho, \) for some \( \rho \in J_\ell, \ell = 1, 2, \ldots, m-1. \) We say that the chain (6.44) satisfies the Gelfand-Tsetlin condition if for all \( 1 \leq \ell \leq m-1, \) \( \rho \in J_\ell \) and \( \eta \in J_{\ell+1} \) the multiplicity of \( \eta \in \text{Ind}_{\widetilde{H}_{\ell+1}}^{\widetilde{H}_{\ell}} \rho \) (equivalently, the multiplicity of \( \rho \) in \( \text{Res}_{\widetilde{H}_{\ell+1}}^{\widetilde{H}_{\ell}} \eta \)) is at most 1; we write \( \eta \to \rho \) when the multiplicity is equal to 1. If the Gelfand-Tsetlin condition is satisfied, the associated Bratteli diagram is the finite oriented graph whose vertex set is \( \bigsqcup_{\ell=1}^m J_\ell \) and the edge set is formed by the pairs \( (\eta, \rho) \) such that \( \eta \to \rho \). A path in the Bratteli diagram is a sequence \( C: \rho_m \to \rho_{m-1} \to \cdots \to \rho_2 \to \rho_1, \) where \( \rho_1 = \emptyset \) and \( \rho_m \in J. \) For every \( \sigma \in J, \) we denote by \( P(\sigma) \) the set of all paths \( C: \rho_m \to \rho_{m-1} \to \cdots \to \rho_2 \to \rho_1 \) such that \( \rho_m = \sigma. \) Fix now \( \sigma \in J \) and denote by \( W \) its representing space. We define recursively a chain of subspaces

\[ W_m \geq W_{m-1} \geq \cdots \geq W_2 \geq W_1 \]
as follows. We set $W_m = W$ and for $\ell = m - 1, m - 2, \ldots, 1$, we denote by $W_\ell$ the unique subspace of $\text{Res}_{H_\ell}^{H_{\ell+1}} W_{\ell+1}$ isomorphic to the representation space of $\rho_\ell$. This way, $W_1 \sim V$ as a $K$-representation; we set $V_C = W_1$. If $\tilde{C} : \tilde{\rho}_m \to \tilde{\rho}_{m-1} \to \cdots \to \tilde{\rho}_2 \to \tilde{\rho}_1$ is a different path in $\mathcal{P}(\sigma)$, then there exists $2 \leq \ell \leq m$ such that $\rho_i \sim \tilde{\rho}_i$ for $i = m, m - 1, \ldots, \ell$ and $\rho_{\ell-1} \not\sim \tilde{\rho}_{\ell-1}$. Therefore, if $\tilde{W}_m \geq \tilde{W}_{m-1} \geq \cdots \geq \tilde{W}_2 \geq \tilde{W}_1$ is the chain of subspaces associated with $\tilde{C}$ then $W_i = \tilde{W}_i$, $i = m, m - 1, \ldots, \ell$, but $W_{\ell-1}$ and $\tilde{W}_{\ell-1}$ are orthogonal, because they afford inequivalent representations. This implies that also $V_C$ and $\tilde{V}_C$ are orthogonal. Finally, by induction on $m$, it is easy to prove that

$$\bigoplus_{C \in \mathcal{P}(\sigma)} V_C$$

(6.45)

is an orthogonal decomposition of the $\theta$-isotypic component of $\text{Res}^G_K W$. Let $L_{\sigma,C} \in \text{Hom}_K(V, \text{Res}^G_K W)$ be an isometry with $L_{\sigma,C} V = V_C$. The operator $L_{\sigma,C} : V \to W$ is defined up to a complex constant of modulus 1 (the phase factor) and, by Lemma 2.1 the set

$$\{L_{\sigma,C} : C \in \mathcal{P}(\sigma)\}$$

(6.46)

is an orthonormal basis for $\text{Hom}_K(V, \text{Res}^G_K W)$.

Similarly, with each $C \in \mathcal{P}(\sigma)$, $C : \rho_m \to \rho_{m-1} \to \cdots, \rho_2 \to \rho_1$, we can associate the following sequence of spaces: $Z_1 = V$, and recursively, $Z_{\ell+1}$ is the unique subspace of $\text{Ind}_{H_\ell}^{H_{\ell+1}} Z_\ell$, that affords $\rho_{\ell+1}$; finally, we set $W_C = Z_m$. Clearly, $W_C$ is a subspace of $\text{Ind}^G_K V$ and

$$\bigoplus_{C \in \mathcal{P}(\sigma)} W_C$$

(6.47)

is an orthogonal decomposition of the $\sigma$-isotypic component of $\text{Ind}^G_K V$. Indeed, we have

$$\text{Ind}^G_K V = \text{Ind}_{H_{m-1}}^{H_m} \text{Ind}_{H_{m-2}}^{H_{m-1}} \cdots \text{Ind}_{H_1}^{H_2} V$$

and at each stage the induction is multiplicity free.

We now show that the decomposition (6.45) and (6.47) are closely related as in Corollary 3.5.

**Theorem 6.2.** The orthonormal basis

$$\left\{ \sqrt{\frac{d_\sigma}{d_\theta}} (L_{\sigma,C}^*)^\vee : C \in \mathcal{P}(\sigma) \right\}$$

of $\text{Hom}_K(W, \text{Ind}^G_K V)$ gives rise precisely to the isometric orthogonal decomposition (6.47), that is

$$W_C = \sqrt{\frac{d_\sigma}{d_\theta}} (L_{\sigma,C}^*)^\vee W$$

for every $C \in \mathcal{P}(\sigma)$.

**Proof.** It follows from Corollary 3.5 and a repeated application of Theorem 6.1 by induction on $m$.  

\[ \square \]
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