On TCS $G_2$ manifolds and 4D Emergent Strings

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**Abstract**

In this note, we study the Swampland Distance Conjecture in TCS $G_2$ manifold compactifications of M-theory. In particular, we are interested in testing a refined version--the Emergent String Conjecture, in settings with 4d $N = 1$ supersymmetry. We find that a weakly coupled, tensionless fundamental heterotic string does emerge at the infinite distance limit characterized by shrinking the $K3$-fibre in a TCS $G_2$ manifold. Such a fundamental tensionless string leads to the parametrically leading infinite tower of asymptotically massless states, which is in line with the Emergent String Conjecture. The tensionless string, however, receives quantum corrections. We check that these quantum corrections do modify the volume of the shrinking $K3$-fibre via string duality and hence make the string regain a non-vanishing tension at the quantum level, leading to a decompactification. Geometrically, the quantum corrections modify the metric of the classical moduli space and are expected to obstruct the infinite distance limit. We also comment on another possible type of infinite distance limit in TCS $G_2$ compactifications, which might lead to a weakly coupled fundamental type II string theory.
1 Introduction

Lots of efforts have been recently devoted to the so-called swampland program, of which the main goal
is to characterize a set of criteria, which effective theories must satisfy to be low-energy descriptions of a
UV-complete theory of quantum gravity. The term "swampland" [1], contrasted with the "Landscape",
refers to the consistent quantum field theories that cannot be consistently coupled to quantum gravity
at the UV. All these criteria, though at the intermediate state as being conjectures, would be of
particular interest for phenomenology, especially in cosmology and particle physics. We refer to [2, 3]
for recent overviews on this program.

Among these swampland conjectures [4–14], it appears that the Swampland Distance Conjecture
(SDC) [6] occupies a central position that can be reached to many other conjectures, such as the absence
of global symmetries, the Weak Gravity Conjecture (WGC) [5] and the de Sitter (DS) conjectures [8,9].
Specifically, the SDC claims that when approaching an infinite distance limit in field space, an infinite
tower of states \(^m\) must be generated whose mass \(m\) scales exponentially with respect to a Planck scale
\(M_{Pl}\) in terms of a proper field distance \(\Delta D\) as \(\frac{m}{M_{Pl}} \sim \exp(-c\Delta D)\), where \(c\) is a positive constant and
further conjectured as \(O(1)\) in the Refined SDC [15, 16]. This conjecture recently has been heavily
discussed, elaborated and verified firmly in various contexts of string compactifications [15–39]. The

\(^1\)It also could be more than one such towers.
appearance of such an infinite tower of asymptotically massless states, typically indicates that the original effective description must break down at infinite distance limits (where $\Delta D \to \infty$), and instead, a new description takes over in terms of these asymptotically massless states. Furthermore, the SDC has been refined eloquently as the Emergent String Conjecture in \cite{23} such that if an infinite distance limit in classical field space falls into the category of the so-called "equi-dimensional" infinite limit, then the new description reduces to a weakly coupled, fundamental tensionless string theory, where the infinite tower of the asymptotically massless states is furnished by the excitations of the emergent fundamental tensionless string. It was also studied in \cite{23} that the emergent tensionless fundamental string in the moduli space of 4d $N = 2$ vector multiplets receives quantum corrections and the classical "equi-dimensional" limit is obstructed by quantum effects, leading to a decompactification where Seiberg-Witten field theory emerges at the limit instead. Later on, the Emergent String Conjecture has also been discussed and verified nicely in the hypermultiplet moduli spaces of 4d $N = 2$ effective theories of type IIB and type I compactifications \cite{31}.

However, most of the discussions have been done in the context of 4d $N = 2$ theories (or their equivalents with eight supercharges) and it might lead to the question of whether the above statements might be resulted from special structures required by theories with eight supercharges. Hence it is natural to explore them in theories with fewer supersymmetries or, even without any supersymmetries, although it is more complicated in general. It is exactly the goal of this paper to explore the SDC, or the refined one - the Emergent String Conjecture, in the context of the 4d $N = 1$ effective theories arising from M-theory compactifications.

In this note, we would like to consider M-theory compactifications on Twisted Connected Sum (TCS) $G_2$ manifolds. A TCS $G_2$ manifold $X$ can be globally viewed as a $K3$ surface fibered on a three-fold $S^3$ and we consider an infinite distance limit in the moduli space characterized by the shrinking $K3$-fiber in $X$. We will argue that a weakly coupled, tensionless heterotic string does emerge at this infinite distance limit. Unlike the 5d $N = 1$ theories analyzed in \cite{23}, we would argue that such an emergent tensionless string necessarily receives quantum corrections and hence the tension of the string would not vanish at the infinite distance limit, which also fits with the spirit of the Emergent String Conjecture \cite{23}. As a by-product, we claim all the tensionless heterotic solitonic strings arising from wrapped D3-branes in F-theory compactifications studied in \cite{27} regain non-zero tensions at infinite distance limits. Geometrically, we expect that the quantum corrections modify the classical moduli space such that the geodesic trajectory towards the infinite distance limit is bent, and the infinite distance limit is obstructed at the quantum level, inspired by $N = 2$ examples studied in \cite{31,32}.

Our results are also in line with the study \cite{23} of Type IIA compactification on a K3-fibered Calabi-Yau three-fold $X_3$. In fact, one can view the 4d $N = 2$ Type IIA compactification as a weakly coupling limit of M-theory compactification on the manifold $X_3 \times S^1$, where $S_1$ denotes the M-theory one-cycle controlling the Type IIA string coupling $g_{IIA}$. Relevant to us, the manifold $X_3 \times S^1$ can be treated as a special $G_2$ manifold with the holonomy group being smaller than $G_2$. And it has been quantitatively studied in \cite{23} that even this theory with higher supersymmetries, the volume of the various dimensional cycles receives quantum corrections from the underlying world-sheet instantons, and hence the classical infinite distance limit, characterized by the shrinking $K3$-fibre in $X_3$, is obstructed in the quantum moduli space of Kähler vector multiplets of type IIA.

This note is organized as follows: In section 2 we review the effective action of general $G_2$ compac-

\footnote{The "equi-dimensional" infinite distance limit refers that at this limit, the parametrically leading infinite tower of massless states must not be any Kaluza-Klein (KK) or KK-like towers. If instead, the leading tower of massless states comes from the KK-like towers at an infinite distance limit, then it leads to a decompactification (gravity hence is decoupled at the defined dimensional spacetime) and is truly in the disguise of a higher dimensional theory at the limit. For more details on this concept, we refer to \cite{23}.}
tifications of M-theory and the connections to their weak coupling type IIA orientifold limits, as well as some basic aspects of TCS $G_2$ manifolds. In section 3 an infinite distance limit is introduced and we argue that the gauge coupling of a certain gauge field becomes weak at such the infinite distance limit. In section 4 we argue that a weakly coupled, fundamental tensionless heterotic string emerges from such an infinite distance limit, and the excitations of the tensionless string indeed furnish the parametrically leading tower of asymptotically massless states. Quantum corrections on the emergent tensionless string would be discussed in section 5 and we expect the quantum corrections to modify the moduli spaces such that the infinite distance limit is obstructed at the quantum level. Finally, in section 6 we will comment on other possible types of infinite distance limits in TCS $G_2$ compactifications, which might lead to an emergence of fundamental type II strings. In the appendix, we give a light overview of type IIA Calabi-Yau compactifications and orientifold compactifications, which are relevant for our discussions in the main context.

2 Compactifications of M-theory on $G_2$ Manifolds

2.1 The Effective Action of $G_2$ Compactifications

In this section, we give an overview of the effective theories from $G_2$ compactifications of M-theory and set the background for the late discussions in this note. Many aspects of $G_2$ compactifications have been studied before, for examples in [40–49].

A $G_2$ manifold is a real seven-dimensional Riemannian manifold $X$ with a Riemannian metric $g$ such that the holonomy group $\text{Hol}(X) \subseteq G_2$. Physically speaking, with such a holonomy group $\text{Hol}(X) \subseteq G_2$, it leads to a global covariantly constant spinor $\eta$ as the spinor representation $8$ of $SO(7)$ decomposes as

$$8 \rightarrow 7 + 1.$$  \hspace{1cm} (2.1)

One can use the spinor $\eta$ to construct covariantly constant three-form $\Phi$ and its hodge dual four-form $\ast_X \Phi$, dubbed associative form and coassociative form, which are invariant under the holonomy group $\text{Hol}(X)$. The associative form $\Phi$ and coassociative form $\ast_X \Phi$ are further calibration forms, which can calibrate minimal volumes of the corresponding three-cycles and four-cycles, respectively. The three-form $\Phi$ can also be used to determine a Riemannian metric $g$ on $X$. The first examples of compact $G_2$ manifolds were constructed by D. Joyce [50] from resolutions of the orbifolds $T^7/\Gamma$ (or $(K3 \times T^3)/\Gamma$) with $\Gamma$ being suitable finite groups.

The compactifications of M-theory on $G_2$ manifolds typically lead to 4d $N = 1$ supergravity, which are determined by three functions: a F-term superpotential $W$, a Kähler potential $K$ and a gauge-kinetic coupling function $f_{\alpha\beta}$. We are going to briefly summarize these three functions in terms of geometry of $G_2$ manifolds.

To start with, we recall that the relevant parts in the 11d supergravity effective action is given by

$$S_{11d} = \frac{1}{\kappa_{11}^2} \int_{R_{1,10}} (\sqrt{-g} R - \frac{1}{2} dC_3 \wedge \ast dC_3 - \frac{1}{6} C_3 \wedge dC_3 \wedge dC_3).$$  \hspace{1cm} (2.2)

with $\kappa_{11}^2 := \frac{g_s}{2\pi}$. After compactified on a $G_2$ manifold $X$, the first term tells us the 4d Planck mass $M_{Pl}$ is given by the volume of $X$ as

$$\frac{M_{Pl}^2}{M_{11}^2} = 4\pi \text{Vol}(X).$$  \hspace{1cm} (2.3)

For the rest of the discussions, we will set 11d Planck scale to be $M_{11} := \ell_{11}^{-1} = 1$.  

4
The Kaluza-Klein ansatz for the M-theory three-form $\Phi$, complexified by the M-theory $C_3$ field, reads

$$\Phi + iC_3 = \sum_{i=1}^{b_3} \phi_is^i + A^\alpha \omega_\alpha, \quad \phi_i \in H^3(X), \omega_\alpha \in H^2(X),$$

(2.4)

from which one can infer that the effective theory has $b^3(X)$ chiral multiplets built from the decomposed fields $s^i$, together with $b^2(X)$ vector multiplets built from the fields $A^\alpha$.

The metric of moduli space is determined by the Kähler potential, which at large volume regime in the moduli space of $X$ can be expressed as

$$K^{G_2} = -3\log(\frac{2\pi}{7} \int_X \Phi \wedge *X\Phi),$$

(2.5)

where $V_c(X) := \frac{1}{7} \int_X \Phi \wedge *X\Phi$ exactly represents the (classical) volume of the $G_2$ manifold $X$. The associated Kähler metric hence is given by

$$G_{ij} = \partial_i\partial_j K^{G_2} = \frac{1}{4} V_c(X)^{-1} \int_X \phi_i \wedge *X\phi_j.$$  

(2.6)

As a well-known fact, there is a no-scale structure associated with the 4d potential $V$, which is resulted from

$$\partial_i K^{G_2} G_{ij} \partial_j K^{G_2} = 7,$$

(2.7)

where $G^{ij}$ is the inverse of the Kähler metric $G_{ij}$. Notice that the Kähler potential receives further quantum corrections, mainly from M2-brane instantons, which breaks the no-scale structure when such corrections are not highly suppressed.

The gauge-kinetic coupling function $f_{\alpha\beta}$ is given by

$$\frac{1}{4g_{YM}^2} = \text{Re} f_{\alpha\beta} \propto \int_X \omega_\alpha \wedge *X\omega_\beta.$$  

(2.8)

Finally, non-vanishing background $G_4 = dC_3$ fluxes induce a superpotential term

$$W_{G_2} = \int_X (C_3 + i\Phi) \wedge G_4.$$  

(2.9)

Note that M2-instantons from wrapping M2-branes on rigid associative cycles also induce a non-perturbative superpotential. However, we would not consider all F-terms in the following discussions, but leave that for a further investigation in the future.

Before closing this subsection, we would like to briefly comment on connections to type IIA orientifolds, if such $G_2$ compactification exists Type IIA orientifold limits, see more details in [51]. It is well-known that Type IIA orientifold with $O6^-$-planes without any RR fluxes can be lifted to $G_2$ orbifolds [40] as

$$X = (X_3 \times S^1) / (\sigma, -1)$$  

(2.10)

where $X_3$ is a Calabi-Yau manifold in Type IIA compactifications and $S^1$ is the M-circle whose radius $R$ controlling the type IIA string coupling.

In this scenario, the $G_2$ three-form $\Phi$ can be expressed in terms of Kähler two-form $J$ and Calabi-Yau three-form $\Omega^{3,0}$ in $X_3$, which systematically reads

$$\Phi = J_M \wedge dx + \text{Re}(\Omega_M^{3,0}), \quad *X\Phi = \frac{1}{2} J_M \wedge J_M + \text{Im}(\Omega_M^{3,0}) \wedge dx.$$  

(2.11)

$\Phi$ here denotes a rescaling version of $J, \Omega^{3,0}$ in string frame, please see more details on the notations in [51].
where \( dx \) is the non-trivial form in \( S^1 \). And one can verify that

\[
K^{G2} = -\ln\left(-\frac{4}{3} \int_{X_3} (J \wedge J)\right) - 2\ln\left(2 \int_{X_3} \text{Re}(C\Omega) \wedge *\text{Re}(C\Omega)\right) := K^{IIA}.
\]

(2.12)

where \( K^{IIA} \) denotes the Kähler potential in type IIA orientifold compactifications, introduced in \([A2]\).

Finally, a \( G_2 \) manifold \( X \) can have non-trivial cycles \( H_i(X, \mathbb{R}), i = 2, 3, 4, 5 \). Relevant to our later discussion, we would like to mention the correspondences of the cohomologies in the orientifold \( X_3/\sigma \) and the ones in the \( G_2 \) manifold \( X \) in general, which are given by

\[
\begin{align*}
H^2(X) &= H^2_+(X_3), \\
H^3(X) &= H^3_+(X_3) \oplus [H^2_+(X_3) \wedge H^1_-(S^1)], \\
H^4(X) &= H^4_+(X_3) \oplus [H^2_+(X_3) \wedge H^1_-(S^1)], \\
H^5(X) &= H^5_+(X_3) \wedge H^1_-(S^1),
\end{align*}
\]

(2.13)

where \( H^1_-(S^1) \) denotes the odd form of the M-circle \( S^1 \) under the involution (2.10). Such cycles, depending on the dimensions, support various physical objects such as particles, strings and domain-walls from wrapped M2-branes and M5-branes.

For the rest of this note, we reserve the notation \( X_3 \) for a generic (K3 fibred) Calabi-Yau three-fold and \( X \) for a generic (TCS) \( G_2 \) manifold.

### 2.2 TCS \( G_2 \) manifolds, Duality and Type IIA Orientifold Limits

From the above subsection, one can see that the physical context of the 4d effective theories is dictated by the moduli space of \( G_2 \) manifolds, which is analogous to Calabi-Yau compactifications in many ways. However, comparing to Calabi-Yau manifolds where the moduli spaces have been extensively studied, the study of the moduli spaces of \( G_2 \) manifolds turn out to be quite difficult and a description at the level of details available in Calabi-Yau cases is still missing. Parts of the reasons are lacking of analogous "Yau" theorem to guarantee the existence of a \( G_2 \) metric, and also \( G_2 \) manifolds are real and hence cannot apply with the powerful machinery of algebraic geometry. In this note, we mainly study one special type of \( G_2 \) manifolds constructed through a twisted connected sum by a combined use of string dualities and weakly coupled type IIA orientifold limits.

In order to make the context self-contained, we would like to briefly summarize the constructions of TCS \( G_2 \) manifolds and their weak coupling limits of type IIA orientifolds. For details, we refer to the original mathematical literatures \([52, 54]\) and the physical references \([55, 61]\). We will adapt the same notations for the constructions of TCS \( G_2 \) manifolds with \([61]\).

TCS \( G_2 \) manifolds are the special compact \( G_2 \) manifolds that can be constructed by a twisted connected sum of two "building blocks" \( Z_\pm \) together with a product of one-circle \( S^1_{\pm} \) for each of them, respectively. The two building blocks \( Z_\pm \) can be viewed as K3-fibered over an open \( \mathbb{P}^1 \) as

\[
Z_\pm = K3 \rightarrow \mathbb{P}^1,
\]

(2.14)

with the first Chern class being \( c_1(Z_\pm) = [S_\pm], \) where \([S_\pm] \) denotes the class of a generic K3 fiber among a lattice polarized family of K3 surfaces. Picking a generic fiber \( S^0_\pm \) and excising from \( Z_\pm \), the remaining parts, denoted as

\[
X_\pm = Z_\pm \setminus S^0_\pm,
\]

(2.15)

can be viewed as a pair of asymptotically cylindrical (acyl) Calabi-Yau three-folds, each of which is diffeomorphic to the product of a K3 surface \( S^1_\pm \) and a cylinder \( S^1_{\pm} \times I \) outsider a compact sub-manifold, respectively.
A TCS $G_2$ manifold $X$ then can be constructed by gluing $X_\pm \times S^1_{e_\pm}$ along the excised region in a way that $S^1_{e_\pm}$ are identified with $S^1_{\pm}$, respectively and together with a hyperkähler (Donaldson) rotation $\phi$ on the two K3-fibres by mapping the $S^0_+$ to the $S^0_-$ as

$$\phi : \omega_{S^0_\pm} \leftrightarrow \text{Re}(\Omega^{2,0}_{S^0_\pm}), \quad \text{Im}(\Omega^{2,0}_{S^0_\pm}) \leftrightarrow -\text{Im}(\Omega^{2,0}_{S^0_\pm}),$$

(2.16)

where $\omega$ denotes the Kähler form on the two K3-fibres $S^0_{\pm}$ and $\Omega^{2,0}$ refers to the holomorphic two-form on $S^0_{\pm}$, describing their complex structure moduli.

Globally, a TCS $G_2$ manifold $X$ can be viewed as a K3 fibration over a three-fold $S^3$, which can be pictured from the fiber-wise duality between M-theory on a K3 surface and $(E_8 \times E_8)$ heterotic string on a $T^3$ manifold:

$$K3 \to X \quad T^3 \to X_3$$

$$S^3 \quad S^3. \quad (2.17)$$

The K3-fibre is conjectured to be a co-associated four-cycle in $X$, which plays a similar role of the special Lagrangian $T^3$-fibre in a Calabi-Yau three-fold $X_3$ at a large complex structure limit. Note that the fibration in $X$ is typically not holomorphic though.

Indeed, the above duality has been checked in detail in [57] by identifying the same massless spectra on both sides. In general, a smooth TCS $G_2$ manifold $X$ has fixed Betti numbers, among of which have

$$b^2(X) = 12, \quad b^3(X) = 299,$$

(2.18)

which implies that the resulting 4d $N = 1$ effective theory has 12 vector multiplets and 299 chiral multiplets, as we learned from the previous section. On the dual heterotic side, the SYZ fibration Calabi-Yau three-fold $X_3$ turns out to be the Schoen’s Calabi-Yau three-fold $X_{19,19}$, known as the split bi-cubic, where the subscript denotes its Hodge numbers being $h^{1,1}(X_{19,19}) = h^{2,1}(X_{19,19}) = 19$. And further, the $E_8 \times E_8$ gauge group is totally broken.

The duality chain can be further extended to a F-theory compactification, where the smooth elliptically fibered Calabi-Yau four-fold turns out to be the Grass-Donagi-Witten manifold [62] $X_4$ with the base $B_3$ being $\mathbb{P}^1 \times dP_9$, of which one has $h^{11} = 12$ reproducing the 12 vector multiplets in the 4d effective theory. We refer to [57, 63] for more details on these dualities. All in all, we have the following duality chain:

$$\text{M-theory on a smooth } G_2 \ X \iff \text{Heterotic string on } X_{19,19} \iff \text{F-theory on } X_4. \quad (2.19)$$

From the perspective of phenomenology, it appears to be trivial to consider such TCS $G_2$ manifolds as they do not have codimension-seven singularities, hence the chiralities of the 4d effective theories are trivial. Nevertheless, such geometries provide many interesting insights on many other aspects such as [45, 46, 59].

Now we switch gear to discuss the type IIA orientifold limits of a TCS $G_2$ manifold. A recent nice paper has been done regarding this aspect appears in [61], and we refer to it for more details. Here we briefly summarize the relevant aspects. It has been shown in [61] that the relevant restrictions for the existence of type IIA orientifold limits in a $G_2$ manifold $X$ are that: The Calabi-Yau three-fold $X_3$ carries a K3 fibration, which is also algebraic and the anti-holomorphic involution $\sigma$ should be respected the K3 fibration in $X_3$, see more details of Type IIA orientifold compactifications in [A.2].

Footnote 4: A novel proposal for generating non-trivial chiralities, however, has been made recently in [64] by exploiting the T-branes-like in TCS $G_2$ manifolds.
More precisely, the two building blocks of the TCS $G_2$ $X$ after a resolution of the orbifold (2.10) take as following: the building block $Z_-$ inherits the K3-fibre of $X_3$ fibred over an $\mathbb{P}^1$, which further is a half Calabi-Yau three-fold $X_3$ whereas the other building block $Z_+$ is a "Voisin-Borcea" type, obtained from the Voisin-Borcea Calabi-Yau three-fold. The M-theory circle $S^1$ is given by the $S^1_{e^{-}}$, which becomes a part of the base of K3 fibration $X_+ := Z_+ \setminus S^0_+$ and is not of constant size, as expected.

3 Infinite Distance Limits as Weak Gauge Coupling Limits

As mentioned in the previous sections, a TCS $G_2$ manifold $X$ can be globally viewed as a K3 surface fibration over a three-fold $S^3$. The infinite distance limit we are interested in, is characterized by the infinite volume of the base $S^3$ and the zero size of the K3-fibre, together with the condition that the total volume of TCS $G_2$ manifolds stay fixed, in order to keep the 4d Planck scale $M_{pl}^2$ finite and hence the gravity is not decoupled in 4d. Quantitatively, we can assign a scaling $\mu$ as

$$\text{Vol}(K3) \to \mu^{-1} \text{Vol}(K3), \quad \text{Vol}(S^3) \to \mu \text{Vol}(K3), \quad \text{with } \mu \to \infty$$

$$\text{Vol}(X) = \text{Vol}(K3) \times \text{Vol}(S^3) \sim \text{finite.}$$ \hspace{0.5cm} (3.1)

As alluded above, it is hard to prove that such limit is at the infinite distance in the moduli space of $X$ given that we know little about the moduli space and the metric $g_{\alpha \beta}$, in contrast to the detailed study in Calabi-Yau compactifications. However, recall that such generic smooth $G_2$ compactification is dual to the F-theory compactification on the $X_4$, where the corresponding limit is located where

$$\text{Vol}(\mathbb{P}^1) \to \mu^{-1} \text{Vol}(\mathbb{P}^1), \quad \text{Vol}(dP_9) \to \mu \text{Vol}(dP_9), \quad \text{with } \mu \to \infty$$

$$\text{Vol}(B_3) = \text{Vol}(\mathbb{P}^1) \times \text{Vol}(dP_9) \sim \text{finite.}$$ \hspace{0.5cm} (3.2)

And such a limit is indeed at infinite distance in the moduli space of $B_3 := \mathbb{P}^1 \times dP_9$, as an example of the generic cases of the base $B_3$ being $\mathbb{P}^1$ fibered over a Fano two-fold proved in [27]. By merit of string duality, we hence claim that the limit (3.1) is indeed at the infinite distance.

Furthermore, if a $G_2$ manifold $X$ has a type IIA orientifold weak coupling limit, and since the K3-fibre is inherited from type IIA orientifold, then the corresponding infinite distance limit is naturally mimicked by the shrinking K3-fibre in the orientifold $X_3/\sigma$. The Swampland Distance Conjecture in type IIA orientifold compactifications has recently been discussed in [33], where they generalized the tower of asymptotically massless particles to towers of asymptotically massless branes (as higher-dimensional objects), without considering quantum corrections.

The infinite distance limits in field spaces of an effective theory are typically expected to be weak coupling limits associated with certain relevant gauge fields, which in our cases arise from decompositions of $C_3$ along two-cycles in $G_2$ manifolds. To see that, we recall a gauge coupling $g_{YM}$ is given by

$$\frac{1}{4g_{YM}^2} = \text{Ref}_{\alpha \beta} \propto \int_X \omega_\alpha \wedge *X \omega_\beta.$$

(3.3)

We would like to show that the gauge kinetic function $g_{\alpha \beta}$ tends to zero in the points where certain three-cycles tend to infinite sizes. To this end, we recall that for a $G_2$ manifold $X$, whose holonomy is exactly $G_2$ group, then its second cohomology is all in the 14 representation of a $G_2$ group, leading to $H^2(X, \mathbb{R}) = 0$. As a result, we can write the hodge dual as

$$*X w_\alpha = -w_\alpha \wedge \Phi.$$ \hspace{0.5cm} (3.4)
Hence, we can rewrite the above gauge coupling as a volume of a three-cycle $D_\Sigma$ as

$$\frac{1}{4g_{YM}^2} \propto \int_X \omega_\alpha \wedge \star_X \omega_\beta = \int_X - w_\alpha \wedge w_\beta \wedge \Phi = \text{Vol}(D_\Sigma),$$

where $D_\Sigma$ is a (putative) three-cycle whose class is dual to the class of the four-form $-\omega_\alpha \wedge \omega_\beta$. Such a canonical map can be viewed as a result from Poincaré duality with homology and the Joyce lemma [45].

From the above, we hence draw an conclusion that a weak gauge coupling limit can be achieved if there is a three-cycle whose volume tends to infinite,

$$\text{Vol}(D_\Sigma) \to \infty \implies g_{YM}^2 \to 0, \quad (3.6)$$

where our infinite distance limit (3.1) directly applies, though we have not explicitly constructed the gauge field. This is similar in spirit to cases in the Calabi-Yau compactifications [17, 21, 23, 28]. The above is also consistent with the well-known conjecture in $(d \geq 3)$ quantum gravity: there are no global symmetries. Indeed, as alluded above, when a gauge coupling $g_{YM}$ tends to zero, it implies that the corresponding gauge symmetry becomes a global symmetry, which is prohibited when coupled to gravity according to the conjecture, and hence such a limit should not be attained within the original effective theory. Therefore, such a limit must lie at the infinite distance in a field space where the original effective description breaks down, due to the appearance of an infinite tower of asymptotically massless states, which is a necessary result of the Swampland Distance Conjecture. In [17, 28], they further clarified that the infinite tower of massless states is a quantum gravity obstruction to restore a global symmetry at the infinite distance limits.

4 Emergence of the tensionless fundamental heterotic string

One can readily see that there is a unique tensionless string persevering $N = (0,2)$ supersymmetry arising from an M5-brane wrapping on the shrinking K3-fibre at the limit (3.1), as there are no other candidates for such objects. Such a tensionless string turns out to be a fundamental heterotic solitonic string, due to the well-known fact [65]. Alternatively, one can perform a dimensional reduction from the 6d $N = (2,0)$ world-volume theory of the M5-brane along the K3-fibre, and find that the 2d effective theory on the emergent string indeed produces 8 right-moving scalars together with its supersymmetric partner fermions, and 16 left-moving fermions and 8 left-moving scalars, which fits with the massless spectra of a weakly coupled heterotic fundamental string in 4d $N = 1$ theories. We refer to [66] for details on generic dimensional reductions of an M5-brane along a coassociative four-cycle in $G_2$ manifolds. Similar discussions on the dual D3-branes can be found in [27, 67]. Reducing to a type IIA orientifold limit, the emergent heterotic string is reproduced by an NS 5-brane wrapping the K3-fibre in the orientifold geometry.

The excitations of this tensionless critical heterotic strings lead to a tower of asymptotically massless states, whose mass at the $n, n = 0, 1, \ldots, \infty$ level scales as

$$\frac{M_{n,Het}^2}{M_{pl}^2} \propto n T_{Het}^2 \propto n \frac{\text{Vol}(K3)}{M_{pl}^2} \propto n \frac{M_{11}^2}{\text{Vol}(S^3)} \propto \frac{n}{\mu},$$

where $T_{Het}$ denotes the tension of the emergent heterotic string and $\mu$ denotes the scaling in (3.1). In the third step we have used $\text{Vol}(K3) = \frac{\text{Vol}(X)}{\text{Vol}(S^3)}$ and $\text{Vol}(S^3) \propto \mu$ at this infinite distance limit and
$M_{pl}^2 = 2\pi \text{Vol}(X)$, in the same fashion as the K3 fibration Calabi-Yau three-fold discussed in [23]. While the competing Kaluza-Klein tower at the n level scales as

$$\frac{M_{n,kk}^2}{M_{pl}^2} \propto \frac{n^2}{\text{Vol}(S^3) M_{pl}^2} \propto \frac{n^2}{\mu \text{Vol}(X)},$$

(4.2)

where we have used the fact that the scale of the KK tower is determined by the inverse volume of the largest cycle in a compactified space, which is the base $S^3$ in $X$.

As one can see, the KK tower is at the same scaling level with the stringy tower (4.1) generated by the emergent heterotic string, but it is less dense as it is proportional to $n^2$ rather than $n$, hence we claim that the limit (3.1) is indeed an "equi-dimensional" limit [23] and does not lead to a decompactification, at least classically.

Before we continue to next topics, we would like to point out one point regarding another candidate of an infinite tower of asymptotically massless states, which arises from an M2-brane wrapping on a non-contractible curve $C_0$ inside the shrinking K3-fibre $m$, $m = 0, 1, \ldots, \infty$ times. Recall that in the 5d $N = 1$ case from M-theory compactification on a K3-fibered Calabi-Yau three-fold $X_3$, which has been extensively discussed in [27], the infinite tower of asymptotically massless BPS states from an M2-brane wrapping $m$ times on a distinguished two-cycle $C_0$ with $C_0 \cdot C_0 > 0$ inside the shrinking K3-fibre, corresponds to the dual heterotic string wrapping a one-cycle $S_1^a$ $m$ times in the heterotic dual geometry $\hat{S} \times S^1_A \times S^1_M$, where $\hat{S}$ denotes the dual K3 surface to the K3-fibre. This can also be verified by exploiting the 4d $N = 2$ string duality between type IIA on $X_3$ and heterotic string on $\hat{S} \times T^2$ [68, 69]. Such BPS states from the wrapping 2m-brane are known as Gopakumar-Vafa invariants [70, 72] of $X_3$. However, with generic smooth TCS $G_2$ manifold, the heterotic dual geometry is the Schoen’s Calabi-Yau three-fold $X_{19,19}$, which does not possess a non-trivial holomorphic one-cycle $S_1^a$ as $H^{1,0}(X_{19,19}) = H^{0,1}(X_{19,19}) = 0$. So what happens? At first sight, one may turn to the SYZ fibration for help illustrated in the fiber-wise duality (2.17). At a large complex structure regime, a Calabi-Yau three-fold $X_3$ can be viewed as a special Lagrangian $T^3$ fibered over a three-cycle. We can see that although the fibre $T^3$ itself has non-trivial one-cycles, the class of such cycle becomes trivial measured by the complex structure of $X_3$. Hence inspired by this, one would naturally expect that the two-cycle $C_0$ should not exist in the TCS $G_2$ manifold, namely although the $C_0$ is a non-trivial two-cycle in the K3-fibre of $X$, it would not be so in the whole $G_2$ manifold.

However, the above argument fails when the K3-fibre of $X$ is algebraic. Instead, a K3-fibre being algebraic in the $X$ means that there is at least one non-trivial two-cycle in $X$ that is inherited from the K3-fibre. To see that, recall that the Picard lattice of an algebraic K3 surface reads

$$\text{Pic}(K3) = H^{1,1} \cap H^2(K3, \mathbb{Z}),$$

which is a lattice with the signature $(1, \rho - 1), 1 \leq \rho \leq 20$, see for example [73] for more details. One can define a map

$$a : K3 \rightarrow X,$$

(4.3)

which can induce a lattice from the pull-forward $a^*$ as

$$\Lambda = [a^*H_2(X, \mathbb{Z})].$$

(4.4)

Such lattice $\Lambda$ must be of signature $(1, r - 1), r \leq \rho$, which claims at least one non-trivial two-cycle in $X$ that can be obtained by the push-forward $a^*$ from the algebraic K3 surface. In such a scenario, we expect that the underlying duality makes sense only down to the 2d $N = (2, 2)$ theory, given by

Type IIA on $X_4 \leftrightarrow$ Heterotic string on $X_3 \times T^2$. 

10
where the particles from an M2-brane wrapping on a distinguished curve $C_0$ inside the K3-fibre of $X$ corresponds to the ones from heterotic string wrapping a one-cycle $S^1$ in $T^2$. We will leave it for a future study in a meticulous manner.

We have claimed that a unique tensionless, weakly coupled heterotic fundamental string indeed emerges at the infinite distance limit (5.1), of which the stringy excitations furnish an infinite tower of the asymptotically light states at this limit, which is in line with the Emergent String Conjecture [23]. However, we have only discussed it so far at the classical level and have not touched any quantum corrections. Indeed, within the 4d $N = 1$ effective theories, it is expected that any strings with $N = (0, 2)$ supersymmetry, whether being critical or non-critical, generally do not receive protections from supersymmetry against quantum corrections, contrasted with their counterparts in 5d or 6d theories. In general, the study of quantum corrections in 4d $N = 1$ effective theories is a very interesting and difficult problem, even in an attempt of determining the exact form of F-terms. In the current cases of $G_2$ compactifications, such difficult would be complicated by limited knowledge on the moduli space and how to embed the associative three-cycles in $X$, see for examples [48, 63] for recent progress on this aspect. Nevertheless, we do not need full-fledged knowledge of quantum corrections but only need to verify whether or not quantum corrections, mainly from M2-instantons, obstruct one from approaching the infinite distance limit (5.1).

5 Quantum Corrections

5.1 Quantum Volumes

In this section, we switch gear to study the quantum corrections in TCS $G_2$ compactifications of M-theory. Our main goal would be verifying that the zero volume of the shrinking K3-fibre at the infinite distance limit (5.1) receive non-zero contributions from quantum corrections, leading to an obstruction of approaching the infinite distance limit at the quantum level.

To begin with, we would like to recall that how a similar problem in 4d $N = 2$ settings of Type IIA Calabi-Yau compactifications has been solved in [23], where they indeed showed that the volume of the K3-fibre in a Calabi-Yau $X_3$ receives quantum corrections from world-sheet instantons such that it never vanishes in the quantum moduli space of 4d $N = 2$ vector multiplets. One useful observation is that, we can view the above Type IIA compactification as a special example of $G_2$ compactification, where the $G_2$ manifold is $X_3 \times S^1_M$, of which the holonomy group is $\text{Hol}(X_3 \times S^1) = SU(3) \subseteq G_2$. For sake of our later discussion, we would repeat their relevant arguments.

The upshot is that the truly physical/quantum volumes of various dimensional cycles in the Type IIA quantum moduli spaces are truly measured in the underlying world-sheet sigma model and they are expected to receive corrections such as those from world-sheet instantons. Such quantum volumes of various 2n-cycles $C$ in $X_3$ are only identical with the classical ones measured by the Kähler form $J$, i.e. $\text{Vol}_c(C) := \int_J J^n, n = 1, 2, 3$, at the large volume regime of the Kähler moduli space $\mathcal{M}^K$, see the notations of 4d $N=2$ moduli spaces in the appendix A. In general, one can use the mirror symmetry to determine such volumes and identify entries in a period vector (A.8), constructed in the mirror Calabi-Yau three-fold to $X_3$, to quantum volumes of various 2n-cycles. Physically speaking, such quantum volumes are also accounted for the physical masses of wrapped B-branes. For example, the quantum volume of the K3-fibre in $X_3$ identifies with the mass of the particle arising from a D4-brane wrapping the K3 cycles. Furthermore, the particle is also identified with an M5-brane wrapping $K3 \times S^1_M$ by the reduction of M-theory to Type IIA, where $S^1_M$ denotes the M-circle with the radius $5$

$5$To be more precise, the particle should arising from a bound state of one wrapping D4-brane and one anti-D0-brane, due to the curvature effect of the K3-fibre, see more details in [23].
being $R$. Now the crucial point is that the M5-brane wrapping the $K^3$-fibre is exactly the heterotic fundamental string, and hence one can read the mass of the particle by utilizing the mass formula for the fundamental string.

That is, the total mass of the wrapped heterotic string on $S^1_M$ with the winding number $n = 1$ and no KK momentum, identical to the mass of the wrapped D4-brane, is given by

$$M_n = \lvert nRT_H + \frac{E_0}{R} \rvert,$$

where $E_0$ denotes the Casimir zero energy, which has been shown in [74] to relate to the gravitational anomaly on the world-sheet theory of the fundamental heterotic string by

$$E_0 = -\frac{\chi(K^3)}{24} = -1.\quad(5.2)$$

Here $\chi(K^3) = 24$ denotes the Euler characteristic of the $K^3$-fibre. Such non-vanishing Casimir zero energy is expected to closely tie to corrections from world-sheet instantons.

With such an identification, one finds that at the infinite distance limit in $X_3$, characterized by the shrinking $K^3$-fibre, i.e. $T_H \propto \text{Vol}(K^3) \to 0$, the mass of the stringy state $M_{n=1}$ never vanishes as

$$M_{n=1} = \frac{1}{R}.\quad(5.3)$$

Hence one can conclude that the physical volume of the K3-fibre never vanishes in the quantum moduli space if $R \neq \infty$. As the size of the M-theory circle $S^1$ grows infinity: $R \to \infty$, the 4d $N = 2$ effective theory would regain one extra space and hence is in the disguise of the bona fide 5d $N = 1$ theory, i.e. M-theory compactified on $X_3$. And at this limit, dubbed the"co-scaling" limit in [23], quantum effects decouple and one recovers the classical result, i.e. the emergence of a tensionless heterotic string in M-theory. This is also echoed by an old observation stated in [75] that M-theory in 5d only "sees" the region at the infinity in the moduli spaces measured by the conformal field theory underlying 4d $N = 2$ type IIA compactifications.

Now one might tend to apply the same argument to a TCS $G_2$ manifold if a Type IIA orientifold limit exists. After all, we have argued from previous sections that the infinite distance limit (3.1) can be mimicked in the Type IIA orientifolds setting, where the corresponding infinite distance limit is accompanied by the shrinking K3-fibre in the orientifold geometry. The fibre $K^3$ in $X_3$ is even under the involution $\sigma$, and hence can be wrapped by D4-branes. One then expect to identify the mass of the particle from the wrapping D4-brane as the physical volume of the K3-fibre. Due to the same offset from the Casimir zero energy $E_0$, one would draw the same conclusion.

However, the above argument fails, if one look closely on how to uplift from type IIA orientifolds to $G_2$ manifolds. Recall from (2.13) one reads

$$H^5(X) = H^4(X_3) \oplus H^1(S^1),\quad(5.4)$$

one then immediately realize the emergent string from wrapping an M5-brane on the shrinking K3-fibre in $X$, does not reduce to the particle from a D4-brane wrapping on the corresponding shrinking K3-fibre in the orientifold $X_3/\sigma$, as the shrinking K3-fibre is even under the involution $\sigma$ and there is no even one-cycle under the involution $-1$ in (2.10). Instead, the emergent string in M-theory reduces to an NS5-brane wrapping the K3-fibre in $X_3/\sigma$.

Nevertheless, we can utilize string duality to analyze the problem. Note that under the duality (2.19), the emergent tensionless heterotic string in the dual F-theory picture arises from a D3-brane wrapping the $\mathbb{P}^1$ insider the base $B_3 := \mathbb{P}^1 \times dP_0$ of the Donagi-Grassi-Witten manifold $X_4$. Hence we
can turn the question to whether or not the volume $\text{Vol}(\mathbb{P}^1)$ receives quantum corrections at the infinite distance limit \[\text{(3.2)}\]. A previous study \[\text{[76]}\] has been done for the cases of 4d non-critical strings in F-theory compactifications. In general, one can use mirror symmetry, now on the Calabi-Yau four-fold $X_4$, to do quantitative analysis of the quantum volume of the cycle $\mathbb{P}^1$. To this end, one generally needs to construct the mirror Calabi-Yau four-fold $\tilde{X}_4$, and solve the periods from the Picard-Fuchs equations (possibly within the GKZ system \[\text{[77–79]}\]) at the large complex structure regime and do the analytic continuation of the periods to the limit \[\text{(3.2)}\], analogous to the cases of Calabi-Yau three-folds studied in \[\text{[23, 28, 80]}\].

More straightforwardly, we instead use the same trick as the above to prove that the volume of $\mathbb{P}^1$ does receive non-zero corrections. The basic idea goes like this: the particle from a D2-brane wrapping on the $\mathbb{P}^1$ in $X_4$ can be identified with the one from a D3-brane wrapping $\mathbb{P}^1 \times S^1_A$, with the help of the following duality

$$\text{F-theory on } X_4 \times S^1_A \times S^1_B \rightarrow \text{M-theory on } X_4 \times S^1_B \rightarrow \text{Type IIA theory on } X_4.$$  \[\text{[5.1]}\]

Then by the same token, we can determine the physical mass of the particle from wrapping D2-brane by the mass formula in \[\text{(5.1)}\]. Substituting the same Casimir zero energy $E_0 = -\frac{1}{2}$ into \[\text{(5.1)}\], the mass of the wrapped D2-brane with winding number $n = 1$ is given by

$$M_{n=1} = \frac{1}{R_A}, \quad (5.6)$$

where $R_A$ denotes the size of the one-cycle $S^1_A$ wrapped by the D3-brane. Hence we conclude that the emergent heterotic string at the classical infinite distance limit \[\text{(3.1)}\], identified with an M5-brane wrapping the shrinking K3-fibre in $X$, does receive quantum corrections and regain a non-zero tension (as $R_A \neq \infty$ by the same token) at the quantum level\[7\]. By the same argument, we further claim that all the heterotic solitonic strings arising from D3-branes wrapping two-cycles with trivial normal bundles in the bases of F-theory compactifications, studied in \[\text{[27]}\], receive quantum corrections and the tensionless strings emergent at the infinite distance limits regain non-vanishing tensions at the quantum level.

Such quantum corrections enforce that the total volume of $X$ to be divergent as $\text{Vol}(X) = \text{Vol}(K3) \times \text{Vol}(S^3) \sim \infty$, leading to a decompactification (the gravity is decoupled in 4d). Indeed we can see this point from the scaling of the KK mass scale \[\text{(4.2)}\], where now it suffers further suppression from the infinite volume of $X$ and hence becomes the parametrically leading one for the infinite tower of the massless states. Followed this, an interesting question would be to explore what kind of field theory emerges at the limit \[\text{(3.1)}\], as gravity is decoupled.

### 5.2 Removing the Infinite Distance Limit

We have demonstrated that the quantum effects contribute a non-zero tension to the classical emergent tensionless string and hence KK tower takes the role of the parametrically leading infinite tower of asymptotically massless states, which then signals a decompactification. And from a geometric perspective, the quantum corrections are expected to modify the metric of the classical moduli space

\[\text{[81]}\]

\[E_0 = -\frac{1}{2} C \cdot K_{B_3} = -1, \quad (5.5)\]

where $C$ denote the class of the wrapped curve $\mathbb{P}^1$ and $K_{B_3}$ stands for the anti-canonical divisor in the base $B_3 := \mathbb{P}^1 \times dP_9$.

\[\text{[7]}\]

Note that in such settings, spacetime D3-instantons might also correct the volume of $\mathbb{P}^1$, but even if such corrections exist, one expect them to contribute the volume of $\mathbb{P}^1$ (the mass of the wrapped D2-brane) positively.
such that the trajectories towards the limit (3.1) are bent in some senses and hence the limit is removed at the quantum level. Indeed, recall that the Kähler potential $K^G_2$ receives quantum corrections from M2-brane instantons, which reduce to perturbative loop corrections and world-sheet instantons in a type IIA orientifold limit. This expectation is not a total surprise, as we have already seen several examples in a similar spirit. For examples, Ooguri and Vafa [82] proved that the singularity at the conifold loci, which lies at the finite distance in the hypermultiplet moduli space $\mathcal{M}^Q$ in type IIA, is smoothed by the sum of the E2-Instanton corrections, see also the mirror dual-type IIB case in [83]. More relevant, the smoothing of infinite distance singularities on the same hypermultiplet moduli space $\mathcal{M}^Q$ by infinite (D(-1)/1-) instantons have been also verified nicely in [31, 32].

The proof of such smoothing the infinite distance singularities in the moduli space $\mathcal{M}^G_2$ is beyond the scope of this note. However, we would like to tentatively comment on the relevant parts in type IIA orientifolds. Recall that, before the orientifolding (A.9), the Kahler potential $K$ in the Kähler moduli space $\mathcal{M}^K$ in a type IIA Calabi-Yau compactification reads

$$K = -\ln(|i|Z_0|^2|2(\mathcal{F} - \overline{\mathcal{F}}) - (\partial_A \mathcal{F} + \overline{\partial_A \mathcal{F}})(Z^A - \overline{Z}^A)|),$$

(5.7)

where $\mathcal{F}$ denotes the N=2 prepotential, see other notations in the appendix (A.1). Away from the large volume regime in $\mathcal{M}^K$, the non-perturbative $\mathcal{F}_{\text{non}}$ would dominate as the classical one $V_c := -\frac{1}{8}K_{ABC}A^B B^C$ is suppressed by some small values of $v^I$. And such non-perturbative parts modify the metric $g_{IJ} = \partial_I \partial_J \mathcal{F}_{\text{non}}$ such that it removes the singularities at the infinite distance where the $K3$-fibre in the Calabi-Yau three-fold $X_3$, which is suggested in [24].

Now in the 4d $N=1$ theories after the orientifolding (A.9), quantum corrections are more drastic and complicated. Note that typically the Kähler potential $K^K$ of the truncated vector moduli space $\tilde{\mathcal{M}}^K$ receives various stringy corrections even at the large volume regime, as we briefly explain in the appendix. When approaching the large distance limit in $\tilde{\mathcal{M}}^K$ which mimics (3.1), one also expect that quantum corrections modify the metric of $\tilde{\mathcal{M}}^K$ and remove the limit, though it is much more complicated to have a solid proof.

6 Another Infinite Distance Limit and Emergent Type II String

It is difficult to classify all types of infinite distance limits in moduli spaces of general TCS $G_2$ compactifications, comparing to the classifications done in Calabi-Yau three-folds [17,23,28]. However, in this section, we would like to comment on one another possible type of infinite distance limit which might lead to an emergence of a (classical) tensionless fundamental type II string.

Note that our infinite distance limit (3.1) in a sense can be viewed as one inherited from a corresponding limit in the truncated Kähler moduli $\tilde{\mathcal{M}}^K$ in type IIA orientifolds, then one can wonder what happens to an infinite distance limit in the truncated hypermultiplet moduli space $\tilde{\mathcal{M}}^Q$ when uplifted to M-theory $G_2$ compactifications. Such a large distance limit in type IIA orientifolds can be, for examples, obtained by asymptotically shrinking a non-contractable three-cycle, perhaps with topology $T^3$, If the three-cycle is odd under the orientifold involution $\sigma$, then from (2.13), we know it uplifts to a four-cycle in $G_2$ manifolds, with topology possibly being $T^4$, and M5-branes wrapping on such a coassociative $T^4$-cycle would be a candidate for an emergent weakly coupled, fundamental Type II string [23]. Indeed, [56,58] have conjectured that the existence of a $T^4$-fibration by studying the mirror symmetry of TCS $G_2$ manifolds, which plays analogous roles as a special Lagrangian $T^3$-fibre in the SYZ conjecture [84]. It would be very interesting to construct one from type IIA orientifolds explicitly and studying it in a similar vein.
7 Conclusion and Outlook

In this note, we have studied the Swampland Distance Conjecture, with the focus on testing the Emergent String Conjecture in TCS $G_2$ manifold compactifications of M-theory. We are interested in a (classical) infinite distance limit characterized by the shrinking $K^3$-fibre of $G_2$ manifold $X$, and we find a tensionless fundamental heterotic string does emerge at the limit, whose excitations provide the parametrically leading one for infinite towers of asymptotically massless states, which is in line with the Emergent String Conjecture [23]. We further argue that such an emergent tensionless string receives quantum corrections and hence regains a non-zero tension by using string duality, which is similar in spirit to the 4d $N = 2$ cases studied in [23]. As a by-product, we also claim all the tensionless heterotic solitonic strings arising from wrapped D3-branes in F-theory compactifications regain non-zero tensions from quantum corrections at infinite distance limits by the same token. Such a result is also in line with the general expectation, that any strings with $N = (0,2)$ supersymmetry in 4d $N = 1$ settings, whether being critical or non-critical, do not receive protections from supersymmetry against quantum corrections, which are contrasted to their counterparts in 5d/6d theories. From a geometric perspective, quantum corrections modify the Kähler potential $K_{G2}$ and we expect such corrections remove the considered infinite distance limit at the quantum level. We also comment on that in TCS $G_2$ compactifications, there could be another candidate of an infinite distance limit, characterized by the shrinking (putative) $T^4$-fibre, which might lead to an emergence of weakly coupled, tensionless type II fundamental string.

We have ignored F-terms induced by both M2-brane instantons and background $G_4$ fluxes in this note. In particular, a generic smooth TCS $G_2$ compactification has been shown in [63] that it receives infinite many M2-instantons corrections and the $E_8$ superpotential [62] can be generated. Presumably, such F-terms could also obstruct infinite distance limits in general. In 4d $N = 1$ settings, a recent study on this aspect has been discussed in [22], where they claimed that the 4d $N = 1$ potential $V$ would be divergent at an infinite distance limit and hence the dynamics would obstruct one from approaching the infinite distance limit. The main tool they have heavily utilized to lead to the divergent potential $V$ is modular symmetry, under which the 4d $N = 1$ potential $V$ is invariant. The main contribution to the divergence of the 4d potential $V$ in [22] comes from non-perturbative F-terms, whereas they did not consider corrections to Kähler potential yet. Nevertheless, this result does not conflict with our main results, as we conclude that in a sense, the quantum Kahler potential, due to corrections from M2-instantons, obstructs one from approaching infinite distance limits.

Perhaps a more interesting question would rather be the study of effects on an emergent critical string in TCS $G_2$ compactifications induced by background $G_4$ fluxes in the dual F-theory picture. In F-theory compactifications, such a direction has already been studied in [27,85] with the main focus on verifying the WGC, where they have quantitatively analyzed the connections between modularities of elliptic genera of 4d strings and enumerative invariants of Calabi-Yau four-manifolds. Without turning on background $G_4$ fluxes, the elliptic genus of a 4d emergent heterotic string turns out to be trivial. It would be interesting to follow along this line in TCS $G_2$ compactifications and explore if there are connections between elliptic genera of emergent strings and any "enumerative" invariants of TCS $G_2$ manifolds, for examples, those uplifted from disk invariants in type IIA [87,88]. By switching on $G_4$ fluxes, it might be helpful to discuss stabilities for our infinite towers of massless states, which was

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8 More precisely, the prefactor of the $E_8$ superpotential would lose the universalities for all M2-instanton contributions as it was expected in [62]. The reason is that it receives extra different zero-modes from Ganor strings and as a result, the convergence to the $E_8$ superpotential breaks down. See more details in [63].

9 The elliptic genus of the 4d emergent string in F-theory compactifications turns to be proportional to $\text{Tr}(Q)$, with $Q$ being the charges under certain 4d (abelian) gauge group [27,85], which indicates the 4d (abelian) gauge theory is anomalous, and hence needs background $G_4$ fluxes to activate the 4d generalized Green-Schwarz mechanism [56] to cancel the gauge anomaly.
studies in [17] as an implicit requirement for the SDC. We leave an investigation of them for the future.

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A Type IIA Calabi-Yau Compactifications and Orientifolds

A.1 Type IIA Calabi-Yau Compactifications

In this appendix, we briefly summarize the relevant aspects of 4d $N = 2$ moduli spaces of Type IIA Calabi-Yau compactifications and their reductions to 4d $N = 1$ moduli spaces with orientifold compactifications. These topics have been extensively studied in the old days and we would not list all the relevant references but instead follow the work [51] and kindly ask readers to follow the references thereof.

The moduli space of 4d $N = 2$ supergravity arising from a Calabi-Yau $X_3$ compactification of Type IIA has a local product structure

$$
\mathcal{M}^{N=2} = \mathcal{M^K} \times \mathcal{M^Q},
$$

where $\mathcal{M^K}$ is parametrized by the scalars in the vector multiplets of the 4d $N = 2$ supergravity and is special Kähler manifold, and $\mathcal{M^Q}$ is spanned by the scalars in the hypermultiplets and is a quaternionic manifold. Notice that $\mathcal{M^Q}$ has a special Kähler submanifold which can be also viewed as $\mathcal{M^K}$ of the mirror manifold of $X_3$. The SDC has been detailed discussed, refined and verified firmly on special Kähler moduli space $\mathcal{M^K}$ in [23, 28–30] and on the quaternionic manifold $\mathcal{M^Q}$ [31, 32], respectively.

The special Kähler manifold can be described by a single holomorphic function, dubbed prepotential $F$. Equipped with a Kähler two-form $J$ and a complex structure three-form $\Omega^3$, the prepotential $F$ at the large volume regime in the moduli space $\mathcal{M^K}$ has the following expression:

$$
F = -\frac{1}{6} K_{ABC} t^A t^B t^C + K_{AB} t^A t^B + K_A t^A + c + F_{\text{non}},
$$

where $K_{ABC} := D_A \cdot D_B \cdot D_C$ is the triple intersection number between three divisors $D_A$ in the Calabi-Yau three-fold $X_3$, $K_{AB} := J \cdot D_A \wedge D_B$, $K_A := \frac{1}{24} c_2(X_3) \cdot D_A$ with $c_2(X_3)$ denoting the second Chern class of $X_3$, and $c$ is a constant term $c := i \frac{\zeta(3)}{(2\pi)^3} \chi(X_3)$. The last term $F_{\text{non}}$ denotes non-perturbative contributions from world-sheet instantons, which in general reads

$$
F_{\text{non}} = i \sum_{d_A \in H^2(X_3, \mathbb{Z})} N^0_d \text{Li}(e^{\text{ld} \cdot t}).
$$

Here $N^0_d$ denotes the genus zero Gromov-Witten invariants, which can roughly be interpreted as enumerating the rational curves $\mathbb{P}^1$ in the class $d$. Note that at the large volume regime where $t^A \rightarrow i\infty$, such non-perturbative term is highly suppressed. The non-perturbative $F_{\text{non}}$ however can be calculated by mirror symmetry. The Kahler potential $K$ on $\mathcal{M^K}$ is determined by

$$
K = -\ln(i |Z^0|^2 [2(\mathcal{F} - \bar{\mathcal{F}}) - (\partial_A \mathcal{F} + \bar{\partial_A} \bar{\mathcal{F}})(Z^A - \bar{Z}^A)]).
$$

At the large volume regime, the Kahler potential $K$ reduces to the familiar one

$$
K = -\ln(-\frac{1}{6} \int_{X_3} J \wedge J \wedge J).
$$
For relevance, we also briefly mention the standard textbook-facts (see for example in [89]) on quantum volumes of $2n$-cycles $C_{2n}$ in the $M^K$, measured by the underlying world-sheet sigma model. They are not always measured by the Kähler form $J$ as classical volumes $\text{Vol}(C_{2n}) := \int_{C_{2n}} J^n$, but are only identical to $\text{Vol}(C_{2n})$ at the large volume regime. Away from the large volume regime, the classical volumes would be expected to receive significant corrections from quantum corrections such as world-sheet instantons, which are not highly suppressed. One typically needs mirror symmetry to calculate such corrections. Mirror symmetry states that the Kähler moduli space $M^K(X_3)$ of Type IIA on $X_3$ is identical to the complex structure moduli space $M^Q(\tilde{X}_3)$ of Type IIB on the mirror manifold $\tilde{X}_3$ of $X_3$. The important point is that the complex structure moduli space $M^Q(\tilde{X}_3)$ does not receives quantum corrections, neither from world-sheet instantons nor space-time D-instantons. Therefore, it is classical and measured by the Calabi-Yau three-form $\tilde{\Omega}^{3,0}$ in $\tilde{X}_3$.

One can choose an integral symplectic basis $\Gamma = (A^I, B_I) \in H_3(\tilde{X}_3, \mathbb{Z})$ with the polarization

$$A^I \cap B_J = \delta^I_J.$$  \hspace{1cm} (A.6)

Equipped with such a symplectic basis, one can define a set of natural coordinates in $M^Q(\tilde{X}_3)$ as

$$Z^I := \int_{A^I} \tilde{\Omega}^{3,0}, \quad F_I := \int_{B_I} \tilde{\Omega}^{3,0}, \quad I = 0, 1, ..., h^{2,1}(\tilde{X}_3)).$$  \hspace{1cm} (A.7)

Given such set of coordinates one can then construct a period vector $\Pi = (Z^I, F_I)$

$$\Pi = \begin{pmatrix} Z^0 \\ Z^A \\ F_A \\ F_0 \end{pmatrix}, \quad A = 1, ..., h^{1,2}(\tilde{X}_3) = h^{1,1}(X_3),$$  \hspace{1cm} (A.8)

which form a symplectic vector under $Sp(2(h^{1,2}(\tilde{X}_3)+1), \mathbb{Z})$. However, there is a redundancy associated with the Calabi-Yau three-form $\tilde{\Omega}^{3,0}$, as it is defined up to a complex rescaling. One can then use this rescaling to eliminate one of the periods $\Pi = (Z^I, F_I)$, say for example setting $Z^0 = 1$. Through this, one define a set of inhomogeneous, flat coordinates $t^A = Z^A/Z^0, A = 1, ..., h^{1,2}(\tilde{X}_3)$, which characterizes complex structure deformation of $\tilde{X}_3$.

Now according to mirror symmetry, the large complex structure limit of $M^Q(\tilde{X}_3)$ maps to the large volume limit in $M^K(X_3)$. And at this limit, the above flat coordinates coincide with the classical, complexified Kähler parameters $t^A$ in $M^K(X_3)$ from the reduction of $J_C = iJ + B_3$ along two-cycles in $X_3$. The component of this period vector defines the quantum volumes of the $2n$-cycles ($n = 0, 1, 2, 3$) in $X_3$.

The period vector $\Pi$ is a set of solutions of a Picard-Fuchs equation, which can be solved by various methods. Away from the large complex structure limit, one can use the analytic continuations of the periods to define the coordinates over the full quantum Kähler moduli space $M^Q(\tilde{X}_3)$, see more details with some explicit examples of elliptically fibered Calabi-Yau three-folds in [23,80].

### A.2 Type IIA Orientifold Compactifications

In this subsection, we review some relevant basic aspects of type IIA orientifolds for our $G_2$ compactifications.

The orientifold compactifications can be obtained from Calabi-Yau compactifications by modding out a orientifolding $\mathcal{O}$, which is given by

$$\mathcal{O} = \Omega_p(-1)^{F_L}\sigma,$$  \hspace{1cm} (A.9)
where $\Omega_p$ denotes the world-sheet parity, $(-1)^F$ is the fermion number operator on the left-moving sector and $\sigma$ denotes an involution on $X_3$. In order to preserve 4d $N = 1$ supersymmetry, the involution $\sigma$ acts on Kähler form $J$ and Calabi-Yau three-form $\Omega^{3,0}$ of $X_3$ as follows
\[
\sigma^* J = -J, \quad \sigma^* \Omega^{3,0} = e^{-2i\theta} \bar{\Omega}^{3,0},
\]
(A.10)
where $\theta$ is a certain constant phase. Under such involution $\sigma$, cohomology (homology) groups of $X_3$ split into even and odd eigen-spaces as
\[
H^p(X_3) = H^p_+(X_3) \oplus H^p_-(X_3), \quad p = 2, 3.
\]
(A.11)

Such the anti-holomorphic involution $\sigma$ generically leads to special Lagrangian submanifolds $\Lambda_3$ of $X_3$ which home the fix points of $\sigma$, and satisfy
\[
J|_{\Lambda_3} = 0, \quad \text{Im}(e^{i\theta} \Omega^{3,0})|_{\Lambda_3} = 0,
\]
(A.12)
and O6-planes/D6-branes wrap such special Lagrangian cycles $\Lambda_3$.

The involution $\sigma$ also acts non-trivially on various form fields in 10d type IIA supergravity as
\[
\sigma : B_2 \to -B_2, C_1 \to -C_1, \quad C_3 \to C_3, g \to g, \phi \to \phi.
\]
(A.13)
Especially, under the involution, we have a decomposition for the complexified Kähler form $J_c := B_2 + iJ_2$ as
\[
J_c = (b^a + iv^a)\omega_a := t^a\omega_a, \quad a = 1, \ldots, h^{1,1}_-(X_3).
\]
(A.14)

As results of the volume $J \wedge J$ and $J$ being odd under the involution $\sigma$ and the hodge duality, one can infer that $h^{1,1}_+ = h^{2,2}_+$, $h^{1,1}_- = h^{2,2}_-$ and $h^{2,1}_+ = h^{2,1}_- = h^{2,1} + 1$.

After the orientifolding (A.9), the moduli space $\mathcal{M}^{N=1}$ of a corresponding 4d $N = 1$ theory can be viewed as the truncation of $\mathcal{M}^{N=2}$ and also has the local product structure
\[
\mathcal{M}^{N=1} = \hat{\mathcal{M}}^K \times \hat{\mathcal{M}}^Q,
\]
(A.15)
where $\hat{\mathcal{M}}^K$ is a subspace of the $\mathcal{M}^K$ with dimension $h^{1,1}_-(X_3)$, which is trivially truncated and hence remains as a special Kähler manifold. The Kähler potential of $\hat{\mathcal{M}}^K$ at the large volume regime is given by
\[
K^K = -\ln[-\frac{4}{3} \int_{X_3} (J \wedge J \wedge J)] = -\ln[\frac{i}{6} K_{abc} (t - \bar{t})^a (t - \bar{t})^b (t - \bar{t})^c].
\]
(A.16)
And $\hat{\mathcal{M}}^Q$ is a subspace of the quaternionic manifold $\mathcal{M}^Q$ with dimension $h^{1,2}(X_3) + 1$, whose Kähler potential $K^Q$ at large volume regime is given by
\[
K^Q = -2\ln(2 \int_{X_3} \text{Re}(C \Omega^{3,0}) \wedge * \text{Re}(C \Omega^{3,0})) = -\ln e^{-4D},
\]
(A.17)
where $C$ is a complex compensator which offsets redundancies of $\Omega^{3,0}$ in order to have the above closed form. Here $D$ denotes the 4d dilaton.

Obviously, the overall Kähler potential $K^{\text{IIA}}$ in the type IIA orientifold is given by the sum of the two as
\[
K^{\text{IIA}} = K^K + K^Q = -\ln(-\frac{4}{3} \int_{X_3} (J \wedge J \wedge J)) - 2\ln(2 \int_{X_3} \text{Re}(C \Omega) \wedge * \text{Re}(C \Omega)).
\]
(A.18)
We are interested in stringy corrections to the Kähler potential $K^K$ in $\tilde{M}^K$. In general, it receives additional contributions from world-sheet instantons, and perturbative corrections, whereas D2-brane instantons correct $K^Q$ as they couple to three-cycles, measured by $\Omega^{3,0}$. Note that even at the large volume regime of $\tilde{M}^K$, the Kähler potential $K^K$ would receive non-perturbative corrections which are not highly suppressed compared with the one $K$ prior to the orientifolding. To see that, note that at the large volume regime of the orientifold, not all $v^A$ should be large as one have $v^\alpha = 0, \alpha \in H^+_2(X_3)$ in the orientifold, hence some quantum corrections at $t^a = -b^a$ are not necessarily suppressed. For example, the prepotential have the following surviving terms at the large volume regime

$$F_{\text{non}} = \sum_{\beta \in H^+_2(X_3,\mathbb{Z})} n^0_{\beta} \text{Li}_3(e^{ik_{\alpha}t^\alpha}),$$

(A.19)

where $k_{\alpha} = \int_{\beta} \omega_{\alpha}$. Away from the large volume regime, one can expect more drastic corrections. However, the explicit calculations are more complicated than ones in $N = 2$ settings.
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