ON THE HOCHSCHILD (CO)HOMOLOGY OF
QUANTUM HOMOGENEOUS SPACES

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Abstract. The recent result of Brown and Zhang establishing Poincaré
duality in the Hochschild (co)homology of a large class of Hopf algebras
is extended to right coideal subalgebras over which the Hopf algebra is
faithfully flat, and applied to the standard Podleś quantum 2-sphere.

1. Introduction

1.1. Theory. As work in particular by Takeuchi [41], Masuoka and Wigner
[31], and Müller and Schneider [34] has shown, the following definition pro-
vides a reasonable generalisation of affine homogeneous spaces of algebraic
groups (see Section 1.3 below for some discussion of the commutative case):

Definition 1. A quantum homogeneous space is a right faithfully flat ring
extension \( B \subset A \) where \( A = (A, \mu, \eta, \Delta, \varepsilon, S) \) is a Hopf algebra with bijective
antipode \( S \) over a field \( k \) and \( B \) is a right coideal subalgebra, \( \Delta(B) \subset B \otimes A \).

Our aim here is to generalise a theorem by Brown and Zhang [5] from
Hopf algebras to such subalgebras. For its statement we adopt the following
terminology (see Section 1.3 for background information and motivation):

Definition 2. Let \( k \) be field and \( B \) be a (unital, associative) \( k \)-algebra.

1. The dimension \( \dim(B) \) of \( B \) is its projective dimension in the cate-
gory of finitely generated \( B \)-bimodules. \( B \) is smooth if \( \dim(B) < \infty \).

2. A character \( \varepsilon : B \to k \) is Cohen-Macaulay if for the induced left \( B \)-
module structure on \( k \) and some \( d \geq 0 \) one has \( \text{Ext}_B^n(k, B) = 0 \) for
\( n \neq d \), and Gorenstein if in addition \( \text{Ext}_B^d(k, B) \simeq k \) as \( k \)-modules.

Under these conditions we can deduce a Poincaré-type duality in the
Hochschild (co)homology of \( B \) as studied by Van den Bergh in [42]:

Theorem 1. If \( B \subset A \) is a smooth quantum homogeneous space and the
restriction of \( \varepsilon \) to \( B \) is Cohen-Macaulay, then there are isomorphisms

\[
\text{Ext}_B^n(B, \cdot) \simeq \text{Tor}_{\dim(B) - n}^B(\omega \otimes B \cdot, B), \quad \omega := \text{Ext}_B^{\dim(B)}(B, B^e)
\]

of functors on the category of \( B \)-bimodules that are right flat. Here \( \otimes := \otimes_k \),
\( B^e := B \otimes B^{\text{op}} \), we identify left and right \( B^e \)-modules and \( B \)-bimodules, and
the \( B \)-bimodule structure on \( \omega \) is induced by right multiplication in \( B^e \).

If \( B = A \) and \( \varepsilon \) is Gorenstein, then Brown and Zhang’s result also says
that \( \omega \simeq A_\sigma \) for some \( \sigma \in \text{Aut}(A) \) [5], by which we mean it is isomorphic

To A as left module but the right action is given by \( a \triangleright b := a\sigma(b) \). In
particular, \( \omega \) is an invertible bimodule with inverse \( \omega^{-1} \simeq A_{\sigma^{-1}} \), so the duality (1) can be reversed to

\[
(2) \text{Tor}^B_n(\cdot, B) \simeq \text{Ext}^{\dim(B)-n}_{B^e}(B, \omega_{\cdot}^{-1} \otimes B \cdot), \quad \omega \otimes_B \omega^{-1} \simeq \omega^{-1} \otimes_B \omega \simeq B,
\]

and this holds in fact on the category of all \( B \)-bimodules (the flatness assumption becomes obsolete), see [42]. Then the duality is not only of theoretical interest but a valuable tool when explicitly computing the Hochschild cohomology of \( B \), see [28] for a concrete demonstration.

Algebraic geometry suggests that the Gorenstein condition implies the invertibility of \( \omega \) in greater generality: we will show in Theorem 7 that \( \omega \) carries in the Gorenstein case the structure of a \((B, A)\)-Hopf bimodule. These are noncommutative generalisations of the modules of sections of homogeneous vector bundles, and \( \text{Ext}^{\dim(B)}_{B^e}(B, \omega_{\cdot}^{-1} \otimes B \cdot) \) reduces for commutative rings to the typical fibre. So the Gorenstein condition means here that we are dealing with a line bundle whose module of sections is invertible.

We will recall that any quantum homogeneous space can be written as

\[
(3) \quad B = \{ a \in A \mid (\pi \otimes \text{id}_A) \circ \Delta(a) = \pi(1) \otimes a \},
\]

where \( \Delta \) is the coproduct in \( A \) and \( \pi \) is the canonical projection onto \( A/B^+A \), \( B^+ := B \cap \ker \varepsilon \), see Section 2.3. This is a Hopf algebra map if and only if \( AB^+ = B^+A \) (since \( B^+A = S(AB^+) \) as observed by Koppinen, see [34], Lemma 1.4). Our second main result applies to this case:

**Theorem 2.** If \( B \subset A \) is a smooth and Gorenstein quantum homogeneous space with \( AB^+ = B^+A \), then \( \text{Ext}^{\dim(B)}_{B^e}(B, B^e) \) is an invertible \( B \)-bimodule.

The condition \( AB^+ = B^+A \) holds trivially if \( A \) is the commutative coordinate ring of an algebraic group \( G \). Then \( B \) is the coordinate ring \( k[X] \) of an affine homogeneous space of \( G \), and \( A/B^+A \) is the coordinate ring \( k[H] \) of the isotropy group \( H \subset G \) of \( X \simeq H \setminus G \), see Section 1.3. Important noncommutative examples with \( AB^+ = B^+A \) are quantisations of quotients \( H \setminus G \) of a Poisson group by a Poisson subgroup, such as the standard quantisations of the generalised flag manifolds studied e.g. in [7, 17, 18, 19, 27, 39].

There are, however, plenty examples of quantum homogeneous spaces with \( AB^+ \neq B^+A \) such as the nonstandard Podleš spheres [35, 34] and more generally quantisations of quotients of Poisson groups by coisotropic subgroups. We use the antipode of \( A/B^+A \) explicitly when constructing \( \omega^{-1} \) but we are not aware of a counterexample to Theorem 2 with the assumption \( AB^+ = B^+A \) removed, and we expect its conclusion holds for the nonstandard Podleš spheres. Hence we ask:

**Question 1.** Is \( \omega \) invertible for all smooth quantum homogeneous spaces when \( \varepsilon \) is Gorenstein?

### 1.2. Application.

Our main motivation is to apply our results to the paradigmatic example of a quantum homogeneous space which is Podleš’ standard quantum sphere [35]. Here \( A \) is the quantised coordinate ring \( C_q[SL(2)] \), and \( A/B^+A \simeq C[z, z^{-1}] \). The quotient \( \pi \) deforms the map dual to the embedding of a maximal torus \( T \simeq \mathbb{C}^* \) into \( SL(2, \mathbb{C}) \), so \( B \) deforms the coordinate ring of the coset space \( T \setminus SL(2, \mathbb{C}) \) which is isomorphic to the
complexified 2-sphere given in $\mathbb{C}^3$ by $x^2 + y^2 + z^2 = 1$. We will prove that $B$ satisfies all the homological assumptions of Theorem 2 and compute $\omega$:

**Theorem 3.** Let $q \in \mathbb{C}^*$ be not a root of unity and $A$ be the quantised coordinate ring of $SL(2, \mathbb{C})$. Then the standard Podleś quantum 2-sphere $B \subset A$ is smooth with $\text{dim}(B) = 2$, $\varepsilon|_B$ is Gorenstein, and we have $\omega \simeq B_\sigma$, where $\sigma$ is the restriction of the square $S^2$ of the antipode of $A$ to $B$.

This form of $\omega$ had to be expected from Dolgushev’s results \[10\] in the setting of formal deformation quantisations, and Hadfield’s computations \[11\], since $S^2|_B$ quantises the flow of the modular vector field of the quantised Poisson structure on the 2-sphere, and also coincides with the modular automorphism of the Haar functional of $A$, see Section 3.1 for further details.

As we mentioned above, the standard quantum 2-sphere can be further deformed to quantum homogeneous spaces of $C_q[SL(2)]$ where $AB^+ \neq B^+A$ \[35\] \[34\]. The Gorenstein condition is checked for these in the same way as for the standard sphere. It was shown in \[11\] that their global dimension is 2, but the methods used there seem not to allow us to answer

**Question 2.** Are the nonstandard Podleś spheres smooth?

### 1.3. The case of coordinate rings.

For the reader’s convenience we briefly recall here the geometric background of the theory in the case that $B \subset A$ are coordinate rings of affine varieties over an algebraically closed field.

A Hopf algebra structure on the coordinate ring $A = k[G]$ of an affine variety $G$ corresponds directly to an algebraic group structure on $G$. Furthermore, a faithfully flat embedding $B = k[X] \subset A$ corresponds to a surjection $G \twoheadrightarrow X$ (\[32\], Theorem 7.3 on p. 48). Since $\Delta(B) \subset B \otimes A \simeq k[X \times G]$, $\Delta$ defines an algebraic action $X \times G \twoheadrightarrow X$ of $G$ on $X$ for which the quotient map $G \twoheadrightarrow X$ is equivariant. Hence $X$ is indeed a homogeneous space of $G$, that is, the action is transitive and $X \simeq H \backslash G$ for a closed subgroup $H \subset G$.

Recall next that a variety $X$ is smooth in a point if and only if its local ring in the point has finite global dimension which is then equal to $\text{dim}(X)$ (\[32\], Theorem 19.2 on p. 156). Since $\text{Ext}$ is compatible with localisations in the sense that for all maximal ideals $m$ in a commutative Noetherian ring $B$ and all finitely generated modules $M, N$ over $B$ one has (\[44\], Proposition 3.3.10)

$$\text{(4)} \quad (\text{Ext}^*_B(M, N))_m \simeq \text{Ext}^*_B(M, N) \otimes_B B_m \simeq \text{Ext}^*_{B_m}(M_m, N_m),$$

$X$ is smooth in all points if and only if $\text{gl.dim}(k[X]) < \infty$.

One has in general $\text{gl.dim}(B) \leq \text{dim}(B)$ (see Lemma 3 in Section 2.5), so the smoothness from Definition 2 implies for $B = k[X]$ that $X$ is smooth in all points. It can happen that $\text{dim}(B) = \infty$ even when $\text{gl.dim}(B) = 0$ (consider e.g. $B = \mathbb{C}$ over $k = \mathbb{Q}$), but for $k = \overline{k}$, $k[X]$-bimodules are the same as modules over $k[X] \otimes k[X] \simeq k[X \times X]$, and this has finite global dimension if $k[X]$ has (\[20\], Theorem 2.1) and is Noetherian. Hence the finitely generated $k[X] \otimes k[X]$-module $k[X]$ admits a finitely generated projective resolution of finite length and $\text{dim}(k[X]) < \infty$. Thus smoothness as in Definition 2 is really equivalent to geometric smoothness of $X$.

For a classical homogeneous space the smoothness condition in Theorem 1 becomes in fact void in characteristic zero: Corollary 5 below tells that an
affine homogeneous space $X \simeq H \setminus G$ is smooth if $G$ is so, and affine algebraic groups are smooth in characteristic zero, see Sections 11.6 and 11.7.

Similarly, a smooth character of a coordinate ring is Gorenstein since this is for these equivalent to the finiteness of the injective dimension of the corresponding local ring as a module over itself (Theorem 18.1 on p. 141 in combination with [11]). In the noncommutative case this equivalence breaks down which results in various nonequivalent generalisations of the Gorenstein and similarly the Cohen-Macaulay condition. The ones from Definition 2 are closest in spirit to the notions of AS Gorenstein and AS Cohen-Macaulay rings [22] but still more naive and just meant as a working terminology to be used within this paper.

Lastly we remark that the coordinate ring of any smooth affine variety satisfies the duality from Theorem 1 with $\omega$ being the inverse of the module of top degree Kähler differentials (algebraic differential forms), see e.g. [26].

1.4. Structure of the paper. Theorems 1 and 2 are proved in Section 2. Sections 2.1-2.3 recall background material on Hochschild (co)homology and quantum homogeneous spaces, mainly from [25, 42] and [31, 34, 41]. Section 2.4 extends the description of the Hochschild cohomology of a Hopf algebra $A$ as a derived functor over $A$ rather than $A^{e}$ to quantum homogeneous spaces. Using this we prove Theorem 1 in Section 2.5.

In Section 2.6 we give $\omega = \text{Ext}^{\text{dim}(B)}_{B^{e}}(B, B^{e})$ for smooth and Gorenstein quantum homogeneous spaces $B \subset A$ the structure of a $(B, A)$-Hopf bimodule and deduce that it is as a left $B$-module isomorphic to

$$\{a \in A \mid (\pi \otimes \text{id}_{A}) \circ \Delta(a) = g \otimes a\}$$

for some group-like element $g \in C = A/B^{+} A$. Using this we construct in Section 2.7 under the assumption $AB^{+} = B^{+} A$ a $B$-bimodule $\bar{\omega}$ with $\bar{\omega} \otimes_{B} \omega \simeq B_{\sigma}$ for some algebra endomorphism $\sigma$ of $B$. Section 2.8 discusses a generalisation of the transitive action of a group $G$ on $X = H \setminus G$ to characters on quantum homogeneous spaces. This is used to show in Section 2.9 that $\sigma$ is an automorphism which implies Theorem 2.

A short Section 2.10 contains a criterion to prove the smoothness of some quantum homogeneous spaces which is applied later in the proof of Theorem 3 and Section 2.11 gives three examples of quantum homogeneous spaces that illustrate certain aspects of the general theory developed so far.

Section 3 is devoted to the Podleś sphere and the proof of Theorem 3.

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2. Theory

2.1. Hochschild (co)homology. Let $k$ be a field. For a $k$-algebra $B$, we denote by $B^{\text{op}}$ the opposite algebra (same vector space, opposite multiplication) and by $B^{e} := B \otimes B^{\text{op}}$ the enveloping algebra of $B$ (here and in the rest of the paper, an unadorned $\otimes$ denotes the tensor product over $k$). The
tensor flip $\tau(a \otimes b) := b \otimes a$ defines a canonical isomorphism $(B^e)^{op} \simeq B^e$ and hence identifies left and right $B^e$-modules, and these are also the same as $B$-bimodules (with symmetric action of $k$). For any such bimodule $M$, the Hochschild (co)homology of $B$ with coefficients in $M$ is

$$H_\bullet(B, M) := \text{Tor}_\bullet^{B^e}(M, B), \quad H^\bullet(B, M) := \text{Ext}^\bullet_{B^e}(B, M),$$

where $B$ is considered as a $B$-bimodule using multiplication in $B$.

The bar resolution of $B$ yields canonical (co)chain complexes computing $H_\bullet(B, M)$ and $H^\bullet(B, M)$. For cohomology, this cochain complex is

$$C^\bullet(B, M) := \text{Hom}_k(B^{\otimes \bullet}, M)$$

with the coboundary operator $b : C^n(B, M) \to C^{n+1}(B, M)$ given by

$$b\varphi(b^1, \ldots, b^{n+1}) = b^1\varphi(b^2, \ldots, b^{n+1})$$

$$+ \sum_{i=1}^{n} (-1)^i \varphi(b^1, \ldots, b^{i}b^{i+1}, \ldots, b^{n+1})$$

$$+ (-1)^{n+1} \varphi(b^1, \ldots, b^n) b^{n+1}.$$

For further information see e.g. [6, 29, 44].

2.2. **Van den Bergh’s theorem.** The following theorem was proven by Van den Bergh in [42]. To be precise, Van den Bergh considered the case in which the bimodule $\omega$ is invertible. For the sake of clarity we include the sketch of a proof not using this assumption, see [25] for details.

**Theorem 4.** Let $B$ be a smooth algebra and assume there exists $d \geq 0$ such that $H^n(B, B^e) = 0$ for $n \neq d$. Then $d = \dim(B)$ and there is for all $n \geq 0$ and for every right $B$-flat $B$-bimodule $M$ a canonical isomorphism

$$H_n(B, \omega \otimes_B M) \simeq H^{d-n}(B, M), \quad \omega := H^d(B, B^e),$$

where the bimodule structure of $\omega$ is induced by right multiplication in $B^e$.

**Proof.** The assumption that $B$ is smooth means that the $B^e$-module $B$ admits a resolution $P_\bullet$ of finite length consisting of finitely generated projective $B^e$-modules. Using $H^n(B, B^e) = 0$ for $n \neq d$ and Schanuel’s lemma one can assume without loss of generality (see the proof of Theorem 23 in [25] for the detailed argument) that this resolution has length $d$, and then $P^d_{d-\bullet} := \text{Hom}_{B^e}(P_{d-\bullet}, B^e)$ is a finitely generated projective resolution of $\omega$. Therefore we have canonical isomorphisms

$$\text{Hom}_{B^e}(P_\bullet, M) \simeq P^e_\bullet \otimes_{B^e} M \simeq (P^e_\bullet \otimes_B M) \otimes_{B^e} B.$$

As a right $B^e$-projective module, $P^e_\bullet$ is right $B$-flat, so $P^e_\bullet \otimes_B M$ is a resolution of $\omega \otimes_B M$. Furthermore, one easily convinces oneself that $P^e_\bullet \otimes_B M$ is $B^e$-flat if $M$ is right $B$-flat (taking into account that $P^e_\bullet$ is finitely generated projective over $B^e$). Hence taking homology in the above equation gives

$$\text{Ext}^n_{B^e}(B, M) \simeq \text{Tor}^{B^e}_{d-n}(\omega \otimes_B M, B)$$

as claimed. \qed
The point of our main result Theorem 1 is that the condition about $H^*(B, B^n)$ can be replaced for quantum homogeneous spaces by the Cohen-Macaulay condition which is easier to check for concrete examples as we shall see below (it boils down to constructing resolutions of the $B$-module $k$ rather than of the $B$-bimodule $B$). In the commutative case, Van den Bergh’s condition is a global one concerned with the behaviour of the embedding of the corresponding space $X$ as the diagonal into $X \times X$, while the Cohen-Macaulay condition in Theorem 1 is local in nature, dealing only with the local properties of $X$ around the point corresponding to $\varepsilon$.

### 2.3. Quantum homogeneous spaces

We will freely use standard conventions and notations from Hopf algebra theory. In particular, we denote by $\Delta, \varepsilon, S$ the coproduct, counit and antipode of a co- or Hopf algebra and use Sweedler’s notation $\Delta(a) = a_{(1)} \otimes a_{(2)}$ for coproducts and $m \mapsto m_{(-1)} \otimes m_{(0)}$ and $n \mapsto n_{(0)} \otimes n_{(1)}$ for left and right coactions, see e.g. [23, 40].

We recall in this section from [31, 34, 41] various characterisations of the right faithful flatness of a Hopf algebra $A$ over a right coideal subalgebra $B$ that we use later. Some of them are given in terms of the left coaction

$$A \to C \otimes A, \quad a \mapsto a_{(-1)} \otimes a_{(0)} := \pi(a_{(1)}) \otimes a_{(2)},$$

where we write as in the introduction

$$\pi : A \to C := A/B^+ A, \quad a \mapsto \pi(a) := a \mod B^+ A, \quad B^+ := B \cap \ker \varepsilon.$$ 

Yet others involve the categories $\mathcal{M}^C$ and $B \mathcal{M}^A$ of right $C$-comodules and of $(B, A)$-Hopf modules, meaning left $B$-modules and right $A$-comodules $M$ for which the coaction $M \to M \otimes A$ is $B$-linear if $B$ acts on $M \otimes A$ via

$$b(m \otimes a) := b_{(1)} m \otimes b_{(2)} a \quad b \in B, m \in M, a \in A.$$ 

There are two functors relating these two categories. The first one is

$$(8) \quad B \mathcal{M}^A \to \mathcal{M}^C, \quad M \mapsto M/B^+ M,$$

where the $C$-coaction on $M/B^+ M$ is induced by the $A$-coaction on $M$, and the second one is the cotensor product

$$(9) \quad \mathcal{M}^C \to B \mathcal{M}^A, \quad N \mapsto N \square_C A$$

given for $N \in \mathcal{M}^C$ with coaction $N \to N \otimes C, n \mapsto n_{(0)} \otimes n_{(1)}$ by

$$N \square_C A := \{ \sum_i n_i^i \otimes a^i \in N \otimes A | \sum_i n_{(0)}^i \otimes n_{(1)}^i \otimes a^i = \sum_i n^i \otimes a^i_{(-1)} \otimes a^i_{(0)} \}$$

on which the $B$-action and $A$-coaction are given by (co)multiplication in $A$.

The following is [31], Theorem 2.1 and [34], Theorem 1.2 and Remark 1.3:

**Theorem 5.** Let $A$ be a Hopf algebra with bijective antipode and $B \subset A$ be a right coideal subalgebra. Then the following are equivalent:

1. $A$ is faithfully flat as a right module over $B$.
2. $A$ is projective as a right $B$-module and there exists $B^\perp \subset A$ such that $A = B \oplus B^\perp$ as right $B$-module.
3. The functors (8) and (9) are (quasi)inverse equivalences.
4. $A$ is left $C := A/B^+ A$-coflat and we have

$$B = k \square_C A = \{ b \in A | \pi(b_{(1)}) \otimes b_{(2)} = \pi(1) \otimes b \}.$$
If $AB^+ = B^+A$, then Remark 1.3 in [34] also tells that $A$ is faithfully flat as a left module if it is faithfully flat as a right module.

**Question 3.** Is this true in general?

By Theorem 5 (4) a quantum homogeneous space can be recovered from $\pi : A \rightarrow C$ as $k\triangleleft CA$. Many examples are in fact defined in this way starting with $\pi$. This works in particular when $C$ is cosemisimple (equals the direct sum of its simple subcoalgebras), see e.g. [34], Corollary 1.5:

**Corollary 1.** Let $A$ be a Hopf algebra with bijective antipode, $\pi : A \rightarrow C$ be a coalgebra and right $A$-module quotient, and assume that $C$ is cosemisimple. Then $B := k\triangleleft CA \subset A$ is a quantum homogeneous space and $C \simeq A/B^+A$.

For $C = k[H]$ cosemisimplicity means that $H$ is reductive, so a quotient $H \setminus G$ of an algebraic group $G$ by a reductive subgroup $H$ is affine with coordinate ring $B$ isomorphic to the ring of $H$-invariant regular functions on $G$ (this is essentially the classical Matsushima-Onishchik theorem). A $(B, A)$-Hopf module $M \in B\mathcal{M}^A$ is here isomorphic to the module of sections of the $G$-homogeneous vector bundle with typical fibre $M/B^+M$.

Later we will also use categories that we denote by $\mathcal{M}^C_{B, \tau}$ and by $B\mathcal{M}^A_{B, \tau}$, where $\tau : B \rightarrow A$ is an algebra map. By the first we shall mean the category of right $C$-comodules and right $B$-modules $N$ that satisfy

$$ (nb)_{(0)} \otimes (nb)_{(1)} = n_{(0)}b_{(1)} \otimes n_{(1)}\tau(b_{(2)}), $$

where we use in the second tensor component on the right hand side the right $A$-action on $A/B^+A$. Similarly, objects in $B\mathcal{M}^A_{B, \tau}$ are objects in $B\mathcal{M}^A$ with an additional right $B$-action that commutes with the left one and satisfies (10), now being an equation in $M \otimes A$. Clearly, the equivalence $B\mathcal{M}^A \simeq \mathcal{M}^C$ that holds in the faithfully flat case also induces $B\mathcal{M}^A_{B, \tau} \simeq \mathcal{M}^C_{B, \tau}$.

**2.4. $H^\bullet(B, M)$ and $\Ext^\bullet_B(k, \ad(M))$.** Here we remark that the description of the Hochschild (co)homology of a Hopf algebra $A$ used in [5] works almost as well for quantum homogeneous spaces $B \subset A$. The proof is the same as for $B = A$ [12] [15], we recall it only for the convenience of the reader:

**Lemma 1.** Let $A$ be a Hopf algebra, $B \subset A$ be a right coideal subalgebra, and $M$ be a $B$-$A$-bimodule. Consider $k$ as left $B$-module with action given by the counit $\varepsilon$ of $A$, and let $\ad(M)$ be the left $B$-module which is $M$ as vector space with left action given by the adjoint action $\ad(b)m := b_{(1)}mS(b_{(2)})$. Then there is a vector space isomorphism $H^\bullet(B, M) \simeq \Ext^\bullet_B(k, \ad(M))$.

**Proof.** Compute $\Ext^\bullet_B(k, \ad(M))$ using the free resolution

$$ \cdots \rightarrow B \otimes^3 \rightarrow B \otimes^2 \rightarrow B $$

of the $B$-module $k$ whose boundary map is given by

$$ b^0 \otimes \cdots \otimes b^n \mapsto \sum_{i=0}^{n-1} (-1)^i b^0 \otimes \cdots \otimes b^i b^{i+1} \otimes \cdots \otimes b^n + (-1)^n b^0 \otimes \cdots \otimes b^{n-1} \varepsilon(b^n). $$

After identifying $B$-linear maps $B \otimes^{n+1} \rightarrow M$ with $k$-linear maps $B \otimes^n \rightarrow M$ (fill the zeroth tensor component with $1 \in B$), this realises $\Ext^\bullet_B(k, \ad(M))$.
as the cohomology of the cochain complex which as a vector space is
\[ C^\bullet(B, M) = \text{Hom}_k(B \otimes^\bullet, M), \]
the standard Hochschild cochain complex, but whose coboundary map is
\[
d_\varphi(b^1, \ldots, b^{n+1})
\]
\[ = \text{ad}(b^1)\varphi(b^2, \ldots, b^{n+1}) + \sum_{i=1}^n (-1)^i \varphi(b^1, \ldots, b^i b^{i+1}, \ldots, b^{n+1}) \]
\[ + (-1)^{n+1} \varphi(b^1, \ldots, b^n) \in (b^{n+1}). \]

Now consider the k-linear isomorphism
\[ \xi : C^\bullet(B, M) \to C^\bullet(B, M), \quad (\xi(\varphi))(b^1, \ldots, b^n) := \varphi(b^1, \ldots, b^n) := \varphi(b_1^{(1)}, \ldots, b_n^{(1)})b_2^{(2)} \cdots b_2^{(2)} \]
whose inverse is given by
\[ (\xi^{-1}(\varphi))(b^1, \ldots, b^n) := \varphi(b_1^{(1)}, \ldots, b_n^{(1)})S(b_2^{(2)} \cdots b_2^{(2)}). \]

Then \( b \circ \xi = \xi \circ d \), where \( b \) is the standard Hochschild coboundary operator \( \xi \), so \( (C^\bullet(B, M), d) \simeq (C^\bullet(B, M), b) \) as cochain complexes. \( \square \)

One can apply Theorem VIII.3.1 from \( [6] \) to the map \( B \to B \otimes A^{\text{op}} \), \( b \mapsto b(1) \otimes S(b(2)) \) to show \( \text{Ext}^n_{B \otimes A^{\text{op}}}(A, M) \simeq \text{Ext}^n_k(k, \text{ad}(M)). \) When \( A \) is flat over \( B \), then the same theorem applied to the obvious embedding of \( B \otimes B^{\text{op}} \) into \( B \otimes A^{\text{op}} \) also implies \( \text{Ext}^n_{B \otimes A^{\text{op}}}(A, M) \simeq H^\bullet(B, M) \) and hence the above lemma. We included the above proof since it does not require flatness. On the other hand, this seems to be a rather weak condition. It is always satisfied in the commutative case \( [31] \), note also the recent results of Skryabin \([38]\). For a counterexample see \([37]\), Corollary 2.8 and Remark 2.9.

2.5. The proof of Theorem \( \text{[1]} \). To get Theorem \( \text{[1]} \), we only have to consider the special case \( M = B \otimes A \) of Lemma \( \text{[1]} \) in more detail. We first recall:

Lemma 2. Let \( R, S \) be rings, \( L \) be an \( R \)-module, \( M \) be an \( R \)-\( S \)-bimodule and \( N \) be an \( S \)-module. Then the canonical map
\[ \text{Ext}_{R}^n(L, M) \otimes_S N \to \text{Ext}_{R}^n(L, M \otimes_S N) \]
is bijective if \( N \) is flat and \( L \) admits a finitely generated projective resolution.

Proof. Fix a finitely generated projective resolution \( P_\bullet \to L \). Then one has \( \text{Hom}_R(P_\bullet, M) \otimes_S N \simeq \text{Hom}_R(P_\bullet, M \otimes_S N) \), see e.g. \([3]\), Proposition 8.b) on p. 16. Now pass to cohomology taking into account that \( N \) is flat (see e.g. [ibid.], Corollary 2 on p. 74). \( \square \)

This will be used with \( R = M = B, S = L = k \) and \( N = A \). For the assumption on \( L = k \) we recall from \([6]\):

Lemma 3. If \( B \) is an algebra over a field \( k \) and \( P_\bullet \to B \) is a (finitely generated) projective resolution of \( B^e \)-modules, then \( P_\bullet \otimes_B L \) is for any left \( B \)-module \( a \) (finitely generated) projective resolution of \( B \)-modules. In particular, one has for any algebra \( \text{gl.dim}(B) \leq \text{dim}(B) \).

Proof. The complex \( \ldots \to P_d \to \ldots \to P_0 \to B \to 0 \) is a flat resolution of the right \( B \)-module 0. Therefore, \( H_\bullet(P \otimes_B L) \simeq \text{Tor}^B_\bullet(0, L) = 0 \), so \( P_\bullet \otimes_B L \) is and it consists of (finitely generated) projective left \( B \)-modules. \( \square \)
Secondly, we need the following direct generalisation of the case $B = A$: \[ \text{Lemma 4. Let} \ A \ \text{be a Hopf algebra and} \ B \subset A \ \text{be a right coideal subalgebra. Then the} \ B - B \otimes A^{op} - \text{bimodule} \ B \otimes A \ \text{with actions} \]
\[ \text{ad}(x)(b \otimes a)(y \otimes z) := x(1)y(b \otimes a)z \text{is isomorphic to the} \ B - B \otimes A^{op} - \text{bimodule} \ B \otimes A \ \text{with actions} \]
\[ x(b \otimes a) \triangleleft (y \otimes z) := xby(1)z \text{as right} \ B - \text{modules.} \]

\[ \text{Proof. } \text{The isomorphism is given explicitly by} \]
\[ \rho : B \otimes A \to B \otimes A, \quad b \otimes a \mapsto b(1) \otimes \text{ad}(b(2))a. \]

\[ \text{Its inverse is given by} \]
\[ \rho^{-1} : b \otimes a \mapsto b(1) \otimes \text{ad}(b(2)), \]

\[ \text{and it follows straightforwardly from the Hopf algebra axioms that} \]
\[ \rho(x(1)y(a)z) = x\rho(b \otimes a) \triangleleft (y \otimes z). \]

\[ \text{Combining the lemma gives:} \]

\[ \text{Theorem 5. Let} \ B \subset A \ \text{be a right coideal subalgebra and consider} \ B \otimes A \ \text{as a} \]
\[ B \otimes A^{op} - \text{bimodule via multiplication in} \ B \otimes A^{op}. \text{If the left} \ B - \text{module} \]
\[ \text{admits a finitely generated projective resolution, then there is an isomorphism} \]
\[ H^*_{\text{co}}(B, B \otimes A) \simeq \text{Ext}^*_{B^e}(B, B) \otimes A \]
\[ \text{of right} \ B \otimes A^{op} - \text{modules, where} \text{Ext}^*_{B^e}(B, B) \otimes A \ \text{is a} \ B \otimes A^{op} - \text{module via} \]
\[ ([\varphi] \otimes a)(x \otimes y) := [\varphi]x(1) \otimes y\text{ad}(x(2)), \quad x \in B, [\varphi] \in \text{Ext}^*_{B^e}(B, B), a, y \in A \]
\[ \text{with the right} \ B - \text{action on} \text{Ext}^*_{B^e}(B, B) \ \text{induced by right multiplication in} \ B. \]

\[ \text{Proof. Apply Lemma 1 with} \ M = B \otimes A. \text{The cochain complexes and the} \]
\[ \text{isomorphisms} \xi, \xi^{-1} \ \text{defined in its proof are clearly right} \ B \otimes A^{op} - \text{linear in} \]
\[ \text{this case, so the lemma gives a right} \ B \otimes A^{op} - \text{module isomorphism} \]
\[ H^*_{\text{co}}(B, B \otimes A) \simeq \text{Ext}^*_{B^e}(B, \text{ad}(B \otimes A)), \]
\[ \text{where the right} \ B \otimes A^{op} - \text{action on} \text{Ext}^*_{B^e}(B, \text{ad}(B \otimes A) \ \text{is induced by right} \]
\[ \text{multiplication in} \ B \otimes A^{op} \ \text{which commutes with the left} \ B - \text{action given by} \]
\[ \text{ad}. \text{Now apply the Lemmata 4, 3 and 2 to get the isomorphisms} \]
\[ \text{Ext}^*_{B^e}(B, \text{ad}(B \otimes A)) \simeq \text{Ext}^*_{B^e}(B, B \otimes A) \simeq \text{Ext}^*_{B^e}(B, B) \otimes A. \]

\[ \text{Composing these isomorphisms with 111 yields the claim. } \]

Theorem 1 is an easy consequence:

\[ \text{Proof of Theorem 7. Theorem 5 gives a} \ B - \text{trimodule decomposition} \]
\[ B \otimes A \simeq B \otimes (B \oplus B^\perp) \simeq B^c \oplus (B \otimes B^\perp), \]
\[ \text{so we also have} \ H^n(B, B \otimes A) \simeq H^n(B, B^c) \oplus H^n(B, B \otimes B^\perp) \ \text{as right} \]
\[ B - \text{modules. Theorem 5 and the Cohen-Macaulay assumption imply that} \]
\[ H^n(B, B^c) = 0 \ \text{for} \ n \neq \dim(B), \ \text{so Theorem 1 follows from Theorem 3}. \]
2.6. \( \omega \) as a Hopf bimodule. The key step towards Theorem 2 is to turn \( \omega \) into an object in \( B\mathcal{M}_{B,S}^A \). Recall that any right \( A \)-comodule \( N \) is via
\[
X.n := n_{(0)}X(n_{(1)}), \quad X \in A^o, \; n \in N
\]
a left module over the Hopf algebra \( A^o \) of linear functionals on \( A \) that vanish on an ideal of finite codimension, see e.g. [40] for background. The \( A^o \)-modules of this form are traditionally called rational. We define now an \( A^o \)-action on \( C^*(B,B \otimes A) \) that restricts to \( C^*(B,B^o) \) and commutes with the coboundary operator \( b \) and therefore induces an \( A^o \)-action on \( \omega \). While \( C^*(B,B^o) \) will not be rational in general we will prove afterwards that \( \omega \) is.

In the definition of the searched for \( A^o \)-action on \( \varphi \in C^o(B,B \otimes A) \) we denote the canonical \( A^o \otimes A^o \)-action on \( B \otimes A \) by
\[
(X \otimes Y) \triangleright (x \otimes y) = X.x \otimes Y.y, \quad X,Y \in A^o, x \in B, y \in A,
\]
where the actions of \( X, Y \) result as in \([13]\) from the \( A \)-coactions given by the coproduct. This gets mixed with an action on the arguments of \( \varphi \):
\[
(X \varphi)(b^1, \ldots, b^n) := (S^2(X_{(n+2)}) \otimes X_{(1)}) \triangleright \varphi(S(X_{(n+1)}).b^1, \ldots, S(X_{(2)}).b^n)).
\]

It follows from the Hopf algebra axioms that this defines a left \( A^o \)-action, and in this way \( C^*(B,B \otimes A) \) becomes a cochain complex of \( A^o \)-modules:

**Lemma 5.** One has \( b(X \varphi) = X(b \varphi) \) for all \( X \in A^o, \varphi \in C^o(B,B \otimes A) \).

**Proof.** This is checked using that we have for \( m \in B \otimes A, b,c \in B, X \in A^o \)
\[
X.(bc) = (X_{(1)}.b)(X_{(2)}.c), \\
(X \otimes 1) \triangleright (bm) = (X_{(1)}.b)((X_{(2)} \otimes 1) \triangleright m), \\
(1 \otimes X) \triangleright (mb) = ((1 \otimes X_{(1)}) \triangleright m)(X_{(2)}.b).
\]

We demonstrate the claim in degree \( n = 1 \), the general case is analogous:
\[
(X(b \varphi))(b,c)
\]
\[
= (S^2(X_{(4)}) \otimes X_{(1)}) \triangleright (b \varphi(S(X_{(3)}).b,S(X_{(2)}).c))
\]
\[
= (S^2(X_{(4)}) \otimes X_{(1)}) \triangleright ((S(X_{(3)}).b) \varphi(S(X_{(2)}).c) \\
- \varphi(S(X_{(3)}).b)(S(X_{(2)}.c)) + \varphi(S(X_{(3)}).b)(S(X_{(2)}).c))
\]
\[
= (S^2(X_{(4)}) \otimes X_{(1)}) \triangleright ((S(X_{(3)}).b) \varphi(S(X_{(2)}).c) \\
- (S^2(X_{(4)}) \otimes X_{(1)}) \triangleright (\varphi(S(X_{(3)}).b)(S(X_{(2)}).c))) + \\
(S^2(X_{(4)}) \otimes X_{(1)}) \triangleright (\varphi(S(X_{(3)}).b)(S(X_{(2)}).c)) \\
= (S^2(X_{(3)}).b)((S^2(X_{(5)}) \otimes X_{(1)}) \triangleright (\varphi(S(X_{(2)}).c)))
\]
\[
- (S^2(X_{(3)}).b)(S(X_{(2)}).c)) + \\
(\varphi(S(X_{(3)}).b))(X_{(2)}S(X_{(3)}).c)
\]
\[
= b((S^2(X_{(3)}).b)(S(X_{(2)}).c)) \\
- (S^2(X_{(3)}).b)(S(X_{(2)}).c)) + \\
(\varphi(S(X_{(3)}).b)(S(X_{(2)}).c))c \\
= b(X \varphi)(b,c).
\]

\( \Box \)
Lemma 6. For any right coideal subalgebra $B \subset A$, the canonical map $C^*(B, B^e) \subset C^*(B, B \otimes A)$ is an embedding of complexes of $A^\circ$-modules.

Thus we obtain an $A^\circ$-action on $H^*(B, B^e)$ and the canonical map to $H^*(B, B \otimes A)$ is $A^\circ$-linear. Our final aim is to prove that these two $A^\circ$-modules are for a smooth and Gorenstein quantum homogeneous space rational, and that we indeed have $\omega \in B\mathcal{M}_{B,S^2}^A$.

Lemma 7. Let $B \subset A$ be a smooth right coideal subalgebra and assume $\varepsilon|_B$ is Gorenstein. Let $\chi : B \to k$ be the character defined by the right $B$-action on $\text{Ext}^\dim(B)(k, B) \simeq k$ and define the $k$-algebra homomorphism

$$\sigma : B \to A, \quad \sigma(x) := S^2(\chi(x_{(1)})x_{(2)}).$$

Then there are isomorphisms of $A$-$B$-bimodules and $A^\circ$-modules

$$H^n(B, B \otimes A) \simeq \begin{cases} 0 & n \neq \dim(B), \\ A_\sigma & n = \dim(B), \end{cases}$$

where $A^\circ$ acts via the canonical action $X.a := a(1)X(a(2))$ on $A_\sigma$.

Proof. The claim about $A$-$B$-bimodules is a straightforward application of Theorem 6. One then has to transport the $A^\circ$-action on $C^*(B, B \otimes A)$ though the used isomorphisms: conjugating it by $\xi$ from Lemma 3 gives an $A^\circ$-action on the cochain complex $(C^*(B, B \otimes A), d)$ that is given by

$$(X \triangleright \varphi)(b^1, \ldots, b^n) := (S^2(X_{(2)}) \otimes X_{(1)}) \triangleright \varphi(b^1, \ldots, b^n),$$

so this action is entirely induced from an action on the coefficient bimodule. Conjugating this action with $\rho$ from Lemma 4 gives the action

$$(X.\varphi)(b^1, \ldots, b^n) := (1 \otimes X) \triangleright \varphi(b^1, \ldots, b^n),$$

that is, in the identifications (11) and (12) in the proof of Theorem 6 the original $A^\circ$-action on $H^\dim(B)(B, B \otimes A)$ induced by (14) is transformed into the one on $\text{Ext}^\dim(B)(k, B) \otimes A$ where $A^\circ$ acts simply on the second tensor component $A$ in the canonical way.

In particular, $H^\dim(B)(B, B \otimes A)$ is a rational $A^\circ$-module, and hence so is any $A^\circ$-submodule (40, Theorem 2.1.3.a). Furthermore, $A_\sigma$ and hence any $B$-subbimodule and $A$-subcomodule is an object in $B\mathcal{M}_{B,S^2}^A$. This gives:

Corollary 2. If $B \subset A$ is a smooth quantum homogeneous space and $\varepsilon|_B$ is Gorenstein, then $\omega = H^\dim(B)(B, B^e)$ becomes through the embedding

$$\omega = H^\dim(B)(B, B^e) \to H^\dim(B)(B, B \otimes A) \simeq A_\sigma$$

an object in $B\mathcal{M}_{B,S^2}^A$.

This allows us to describe $\omega$ finally as follows using the canonical projection $\pi : A \to C = A/B^+A$:

Theorem 7. Let $B \subset A$ be a smooth quantum homogeneous for which $\varepsilon|_B$ is Gorenstein, and let $\chi$ be the character on $B$ defined by its action on $\text{Ext}^\dim(B)(k, B) \simeq k$. Then there exists a group-like $g \in C = A/B^+A$ with

$$\omega \simeq \{ a \in A_{\sigma} \mid \pi(a_{(1)}) \otimes a_{(2)} = g \otimes a \}, \quad \sigma(b) = \chi(b_{(1)})S^2(b_{(2)})$$
as an object of $B\mathcal{M}_{B,S^2}^A$, and we have for all $b \in B$

\[(16) \quad g\sigma(b) = \chi(b)g,\]

where $g\sigma(b)$ is defined using the right $A$-action on $C = A/B^+A$.

Proof. Theorem 5 and the discussion at the end of Section 2.3 tell that $\omega \in B\mathcal{M}_{B,S^2}^A$ is of the form $N \square_C A$ for some $N \in \mathcal{M}_{B,S^2}^C$. It follows that as a special case of (the proof of) Theorem 5.8 in [4] there are isomorphisms of $A$-$B$-bimodules

$$A \otimes_B \omega \simeq A \otimes_B (N \square_C A) \simeq N \square_C (A \otimes_B A) \simeq N \square_C (C \otimes A) \simeq N \otimes A,$$

where the left $A$-action on $N \otimes A$ is given by multiplication in $A$ and the right $B$ action on $N \otimes A$ is $(n \otimes a)b := nb(1) \otimes S^2(b(2))$. The second isomorphism (the mixed associativity of $\square_C$ and $\otimes_B$) uses the right flatness of $A$ and the third is the Galois isomorphism for the algebra extension $B \subset A$ which is explicitly given by

$$A \otimes_B A \rightarrow C \otimes A, \quad x \otimes_B y \mapsto \pi(y(1)) \otimes xy(2).$$

It follows that there is a right $B$-linear isomorphism

$$N \simeq (A \otimes_B \omega)/A^+(A \otimes_B \omega), \quad A^+ = \ker \varepsilon.$$

But we also have $A$-$B$-bimodule isomorphisms

$$A \otimes_B \omega = A \otimes_B \text{Ext}^{\dim(B)}_{B^e}(B, B \otimes B) \simeq \text{Ext}^{\dim(B)}_{B^e}(B, B \otimes A) \simeq A_\sigma$$

by Lemmata [2] and [7]. Together this shows that as $B$-modules we have

$$N \simeq \text{Ext}^{\dim(B)}_{B^e}(k, B),$$

and a coaction on the ground field is given by a group-like element $g \in C$ as

$$k \ni \lambda \mapsto \lambda \otimes g \in k \otimes C$$

that has to obey (16) in order to define an object in $\mathcal{M}_{B,S^2}^C$. The result follows now by the definition of $N \square_C A$. \hfill \Box

2.7. The Hopf-Galois case. As we have recalled in the introduction, the assumption $B^+A = AB^+$ means that $\pi : A \rightarrow C = A/B^+A$ is a Hopf algebra quotient. Hence $\mathcal{M}^C$ is a monoidal category, where $M \otimes N$ is for $N, M \in \mathcal{M}^C$ the tensor product over $k$ equipped with the coaction

$$m \otimes n \mapsto m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)}.$$

Furthermore, any $M \in \mathcal{M}^C$ is canonically an object in $\mathcal{M}^C_{B,\text{id}}$ if $B$ acts trivially (through $\varepsilon$) from the right. Hence $M \square_C A$ is canonically an object in $B\mathcal{M}^A_B := B\mathcal{M}^A_{B,\text{id}}$ with $B$-bimodule structure

\[(17) \quad x(m \otimes a)y := m \otimes xay, \quad m \in M, a \in A, x, y \in B;\]

and with respect to this bimodule structure we have (this generalises to any faithfully flat Galois extension of an algebra $B$ by a Hopf algebra $C$)

\[(18) \quad (M \square_C A) \otimes_B (N \square_C A) \simeq (M \otimes N) \square_C A\]

as $B$-bimodules. Any group-like element $g$ of $C$ is now invertible with inverse $g^{-1} = S(g)$, and Theorem [7] immediately gives:
Corollary 3. Retain all assumptions and notation from Theorem 7 and assume in addition $AB^+ = B^+ A$. Then $\sigma(B) \subset B$, and if we consider
\[ \bar{\omega} := \{ a \in A \mid \pi(a(1)) \otimes a(2) = g^{-1} \otimes a \} \]
as a $B$-bimodule via [17], then we have a $B$-bimodule isomorphism
\[ \bar{\omega} \otimes_B \omega \simeq B_\sigma. \]

Proof. Take in (18) for $M$ the ground field $k$ with the $C$-coaction given by $\lambda \mapsto \lambda \otimes g^{-1}$ and for $N$ the same but with $g$ instead of $g^{-1}$. Then we get $\bar{\omega} \otimes_B (N \Box_C A) \simeq B$ as $B$-bimodules. The $B$-bimodule $\omega$ is obtained from $N \Box_C A$ by twisting the right $B$-action by $\sigma$, so the claim follows. \( \square \)

The fact that $\sigma(B) \subset B$ is probably the most unexpected observation here. It illustrates how restrictive (16) is especially for $AB^+ = B^+ A$ since it can in this case be multiplied from the left by $g^{-1}$ to give
\[ \pi(\sigma(b)) = \chi(b) \pi(1) \]
for all $b \in B$, and from this we indeed also compute directly that
\[ \pi(\sigma(b)(1)) \otimes \sigma(b)(2) = \pi(\chi(b(1)) S^2(b(2))) \otimes S^2(b(3)) = \pi(\sigma(b(1))) \otimes S^2(b(2)) = \pi(1) \otimes \chi(b(1)) S^2(b(2)) = \pi(1) \otimes \sigma(b), \]
hence $\sigma(b) \in B$ by Theorem 5.

We now want to show that in fact $\sigma(B) = B$. For this we need a small digression about characters and the following basic remark:

Lemma 8. If $B \subset A$ is a quantum homogeneous space and $AB^+ = B^+ A$, then we have $S^2(B) = B$.

Proof. Koppenen’s $S(AB^+) = B^+ A$ ([34], Lemma 1.4) gives
\[ S^2(B^+ A) = S^2(A B^+) = S^2(B^+ A) = S(AB^+) = B^+ A \]
and hence for all $b \in B$
\[ \pi(S^{\pm 2}(b)(1)) \otimes S^{\pm 2}(b)(2) = \pi(S^{\pm 2}(b(1))) \otimes S^{\pm 2}(b(2)) = \pi(1) \otimes S^{\pm 2}(b), \]
so $S^{\pm 2}(B) \subset B$ which implies $S^2(B) = B$. \( \square \)

2.8. Remarks on characters. For a Hopf algebra $A$ the set $G := \text{Char}(A)$ of characters (algebra homomorphisms $\gamma : A \rightarrow k$) is canonically an affine group scheme represented by the commutative Hopf algebra $A/J(A)$, where
\[ J(A) := \{ a \in A \mid \gamma(a) = 0 \quad \forall \gamma \in \text{Char}(A) \}, \]
and for a right coideal subalgebra $B \subset A$ the set $X := \text{Char}(B)$ becomes an affine $G$-scheme represented by $B/J(B)$. The (right) $G$-action on $X$ is given like the group structure in $G$ by the canonical product on $\text{Hom}_k(A,k)$
\[ (\varphi \psi)(a) := \varphi(a(1)) \psi(a(2)), \quad \varphi, \psi \in \text{Hom}_k(A,k), \quad a \in A \]
for which $\varepsilon$ is the unit element.

The inclusion $B \rightarrow A$ induces a homomorphism $B/J(B) \rightarrow A/J(A)$, and the restriction of a character from $A$ to $B$ is the dual morphism $G \rightarrow X$. 
However, even for some well-behaved examples of quantum homogeneous spaces (such as the Podleś sphere that we will define in Section 3.1) the map \(G \to X\) is not surjective, \(B/J(B) \to A/J(A)\) is not faithfully flat and not even injective, and the \(G\)-action on \(X\) is not transitive.

But at least we can say the following:

**Theorem 8.** If \(\chi\) is a character on a quantum homogeneous space \(B \subset A\), \(\beta : A \to B\) is a right \(B\)-linear projection as in Theorem 2 (2) and we define

\[
\gamma : A \to k, \quad a \mapsto \chi(\beta(S^{-1}(a))),
\]

then we have

\[
\chi \gamma = \varepsilon
\]
as functionals on \(B\).

**Proof.** This follows by straightforward computation:

\[
(\chi \gamma)(b) = \chi(b(1))\chi(\beta(S^{-1}(b(2)))) = \chi(\beta(S^{-1}(b(2))))\chi(b(1))
\]
\[
= \chi(\beta(S^{-1}(b(2)))b(1)) = \chi(\beta(S^{-1}(b(2)b(1))
\]
\[
= \chi(\beta(\varepsilon(b))) = \varepsilon(b),
\]
where we used the properties of \(\chi\) and \(\beta\) and the fact that in every Hopf algebra with bijective antipode we have

\[
S^{-1}(a(2))a(1) = S^{-1}(S(S^{-1}(a(2))a(1))) = S^{-1}(a(1)S(a(2))) = \varepsilon(a)
\]
for all \(a \in A\) since \(S\) is always an algebra antihomomorphism. \(\square\)

Note that \(\gamma\) is in general not a character on \(A\), though.

2.9. **The proof of Theorem 2**. Theorem 3 implies:

**Corollary 4.** If \(\chi\) is a character on a quantum homogeneous space \(B \subset A\), then the algebra homomorphism \(\sigma : B \to A\) given by

\[
\sigma(b) := \chi(b(1))S^2(b(2))
\]
is injective. If \(AB^+ = B^+A\) and \(\sigma(B) \subset B\), then \(\sigma(B) = B\).

**Proof.** An explicit left inverse of \(\sigma\) is given by

\[
\sigma^{-1} : A \to A, \quad \sigma^{-1}(a) := \gamma(S^{-2}(a(1)))S^{-2}(a(2)),
\]
where \(\gamma\) is as in Theorem 3. Under the additional assumption \(AB^+ = B^+A\) we have \(S^2(B) = B\) (Lemma 3), so \(\sigma(B) \subset B\) implies

\[
\hat{\sigma}(b) := \chi(b(1))b(2) = S^{-2}(\sigma(b)) \in B
\]
for \(b \in B\). Now abbreviate for a given \(b \in B\)

\[
M := \{\varphi(b(1))b(2) \mid \varphi \in \text{Hom}_k(B, k)\} \cap B.
\]
We have

\[
\hat{\sigma}(\varphi(b(1))b(2)) = \varphi(b(1))\chi(b(2))b(3) = (\varphi\chi)(b(1))b(2) \in M,
\]
that is, \(\hat{\sigma}(M) \subset M\). Since \(\sigma\) and hence \(\hat{\sigma}\) has been shown already to be injective, \(\hat{\sigma}|_M\) is bijective since \(\text{dim}_k(M) < \infty\) (if \(\Delta(b) = \sum_{i=0}^\infty x_i \otimes y_i\), then \(M\) is spanned by the \(y_i\)). Furthermore, \(b = \varepsilon(b(1))b(2) \in M\), so \(b \in \text{im} \hat{\sigma}\). Thus \(b \in \text{im} \hat{\sigma}\) for arbitrary \(b \in B\) and hence also \(\sigma = S^2 \circ \hat{\sigma}\) is surjective. \(\square\)
Proof of Theorem 2. We have constructed in Corollary 3 a $B$-bimodule $\bar{\omega}$ with $\bar{\omega} \otimes_B \omega \simeq B_\sigma$, where $\sigma(b) = \chi(b(1))S^2(b(2))$. Corollary 4 shows that $\sigma$ is under the assumptions of Theorem 2 an automorphism of $B$. This implies $B_\sigma \simeq \sigma^{-1}B$ as bimodules, where $\sigma^{-1}B$ is $B$ as right module but the left action is twisted by $\sigma^{-1}$ (the isomorphism is given by $\sigma^{-1}$). Hence we get $\sigma \bar{\omega} \otimes_B \omega \simeq B$.

To see that we also have $\omega \otimes_B \sigma \bar{\omega} \simeq B$ note that we know $\bar{\omega} \otimes_B \omega \sigma^{-1} \simeq B$, and $\omega \sigma^{-1} \in B\mathcal{M}^A_B$. Applying the monoidal functor $M \mapsto M/B^+M$ gives the corresponding $g^{-1}g = 1$ in $C$, but here we also have $gg^{-1} = 1$. Retranslating this into Hopf bimodules yields $\omega \otimes_B \sigma \bar{\omega} \simeq \omega \sigma^{-1} \otimes \bar{\omega} \simeq B$. □

2.10. A smoothness criterion. We mention here a useful tool for proving the smoothness of $B \subset A$. The key remark is [33], Theorem 7.2.8:

**Theorem 9.** Let $B \subset A$ be a ring extension such that $B$ is a direct summand in $A$ as a $B$-bimodule. Then $\text{gl.dim}(B) \leq \text{gl.dim}(A) + \text{proj.dim}_B(A)$.

Together with Theorem 5 this implies for example:

**Corollary 5.** If $B \subset A$ is a quantum homogeneous space and $A$ is commutative, then $\text{gl.dim}(B) \leq \text{gl.dim}(A)$.

But also in many noncommutative examples it will happen that the decomposition in Theorem 5(2) is actually a decomposition of bimodules:

**Lemma 9.** Let $B \subset A$ be a quantum homogeneous space and assume that $A/B^+A$ is cosemisimple with $AB^+ = B^+A$. Then $\text{gl.dim}(B) \leq \text{gl.dim}(A)$.

**Proof.** As remarked above, the condition $AB^+ = B^+A$ means that $C = A/B^+A$ is a Hopf algebra quotient of $A$. The cosemisimplicity can be characterised as the existence of a (unique) functional $h : C \rightarrow k$ satisfying $h(1) = 1, \ h(c(1))c(2) = c(1)h(c(2)) = h(c), \ c \in C$, see e.g. [23], Theorem 13 in Section 11.2.1, and it is easily verified that

$$\beta : A \rightarrow A, \ a \mapsto h(\pi(a(1)))a(2)$$

is then a $B$-bilinear projection from $A$ onto $B \subset A$. □

Note that the assumption of cosemisimplicity of $C$ can be weakened, it suffices that there is a total integral $h : C \rightarrow A$ in the sense of [9] whose image commutes with $B \subset A$ as in [ibid.], Proposition (1.7)(b).

**Corollary 6.** If $B \subset A$ is as in Lemma 5 and $A^e$ is left Noetherian with $\text{gl.dim}(A^e) < \infty$, then $B$ is smooth.
Proof. If $B \subset A$ is as in the lemma, then so is $B^e \subset A^e$, hence the lemma gives $\text{gl.dim}(B^e) \leq \text{gl.dim}(A^e)$. Therefore, $\text{gl.dim}(A^e) < \infty$ implies that the left $B^e$-module $B$ has finite projective dimension. Finally, the left Noetherianity of $A^e$ implies that of $B^e$ (apply $A^e \otimes_{B^e} \cdot$ to an ascending chain of left ideals in $B^e$ and use faithful flatness). Therefore, the projective dimension of the finitely generated $B^e$-module $B$ will coincide with its projective dimension in the category of finitely generated $B^e$-modules. \hfill \square

2.11. Some (counter)examples. Before entering the discussion of the Ponder sphere let us mention here three simpler but instructive examples.

First of all, every Hopf subalgebra $B \subset A$ is in particular a right coideal subalgebra. If $A = U(g)$ and $B = U(h)$ are universal enveloping algebras of finite-dimensional Lie algebras $h \subset g$, then the Poincaré-Birkhoff-Witt theorem says that $A$ is free over $B$ and hence faithfully flat. However, even the basic example of the Borel subalgebra $h := b_+ \subset g := \mathfrak{sl}(2, k)$ behaves rather badly: the characters of $b_+$ are in bijection with $k$ but only one of them (the counit) extends to $A$. The dualising bimodule of $A$ is $A$ without any twist $\sigma$, but that of $B$ is of the form $B_\sigma$ for a nontrivial automorphism (see [5], this example was suggested by Ken Brown to me). Note also that $AB^+ = B^+ A$, but $B$ is not as a $B$-bimodule a direct summand in $A$.

Secondly, consider $B = k[y]$ and for $A$ the Hopf algebra obtained by adding a generator $x$ satisfying $x^2 = 1, \ xy = -yx$,

so $A$ is the smash (aka crossed or semidirect) product $B \rtimes \mathbb{Z}_2$ of $B$ by the automorphism that sends $y$ to $-y$. The Hopf algebra structure is given by

$\Delta(x) = x \otimes x, \ \Delta(y) = 1 \otimes y + y \otimes x,$

$\varepsilon(x) = 1, \ \varepsilon(y) = 0, \ S(x) = x, \ S(y) = -yx.$

The monomials $\{y^i x^j \mid i \geq 0\}$ form a $k$-vector space basis of $A$, so $A$ is free over $B$ with basis $\{1, x\}$. In particular, $B \subset A$ is faithfully flat. In this example one can verify directly that $H^i(B, M) \simeq H_{1-i}(B, M)$ for all $B$-bimodules $M$, and that $B$ is Gorenstein with $\chi = \varepsilon$. However, $\sigma(b) = \chi(b(1)) S^2(b(2)) = S^2(b)$ is not the identity automorphism since

$S^2(y) = -S(y) = -S(x)S(y) = xyx = -y.$

These examples show that even if $B$ satisfies Poincaré duality it can be difficult to read off $\omega$ from the Hopf-algebraic data given. In particular it can happen that the dualising bimodules of both $A$ and $B$ are of the form $A\sigma$ and $B_\tau$, but one can have $\tau = \text{id}_B, \sigma = \text{id}_A$ or conversely $\sigma \neq \text{id}_A, \tau = \text{id}_B$.

Finally, we would like to mention that the cusp $X \subset k^2$ given by the equation $x^2 = y^3$ is also a quantum homogeneous space although it is surely not a homogeneous space of an algebraic group since it is not smooth. The ambient Hopf algebra is again a skew-polynomial ring $A = B \rtimes \mathbb{Z}, B = k[X]$, that is denoted by $B(1, 1, 2, 3, q)$ in [14], Construction 1.2. Therein the notation is exactly the opposite of ours, their $A$ is our $B$ and vice versa.
3. Application

3.1. The standard Podleś sphere. For the rest of the paper we fix $k = \mathbb{C}$, $q \in k^*$ is not a root of unity, and $A$ is the standard quantised coordinate ring of $SL(2,k)$ (see e.g. [23] for background information). This is the Hopf algebra with algebra generators $a,b,c,d$, defining relations

$$ab = qba, \quad ac = qca, \quad bc = cb, \quad bd = qdb, \quad cd = qdc,$$

and the coproduct, counit, and antipode determined by

$$\Delta(a) = a \otimes a + b \otimes c, \quad \Delta(b) = a \otimes b + b \otimes d,$$

$$\Delta(c) = c \otimes a + d \otimes c, \quad \Delta(d) = c \otimes b + d \otimes d,$$

$$\varepsilon(a) = \varepsilon(d) = 1, \quad \varepsilon(b) = \varepsilon(c) = 0,$$

$$S(a) = d, \quad S(b) = -q^{-1}b, \quad S(c) = -qc, \quad S(d) = a.$$

It follows from these relations that there is a unique Hopf algebra quotient

$$\pi : A \to C := k[z,z^{-1}], \quad \pi(a) = z, \quad \pi(d) = z^{-1}, \quad \pi(b) = \pi(c) = 0,$$

where the Hopf algebra structure of $k[z,z^{-1}]$ is determined by $\Delta(z) = z \otimes z$, that is, $C$ is the coordinate ring of $T = k^*$, and the map $\pi$ would correspond for $q = 1$ to the embedding of $T$ as a maximal torus into $SL(2,k)$.

By Corollary 11 $\pi$ gives rise to a quantum homogeneous space $B$ as in [3]. This subalgebra deforms the coordinate ring of $T \setminus SL(2,k)$ and was discovered by Podleś [35] and hence is referred to by most authors as the (standard) Podleś quantum sphere. The elements

$$y_{-1} := ca, \quad y_0 := bc, \quad y_1 := bd$$

generate $B$ as an algebra, and $B$ can be characterised abstractly as the algebra with three generators $y_{-1}, y_0, y_1$ and defining relations

$$y_0y_{\pm 1} = q^\pm 2 y_{\pm 1}y_0, \quad y_{\pm 1}y_{\mp 1} = q^{\mp 2} y_0^2 + q^{\mp 1} y_0,$$

see [8][30][35].

3.2. The Koszul resolution of the $B$-module $k$. We will construct a free resolution of the $B$-module $k$ (with action given by $\varepsilon$) by using the probably simplest case of Priddy’s noncommutative Koszul resolutions [36]:

Lemma 10. Let $B$ be a $k$-algebra and assume $z_{\pm 1} \in B$ are such that

1. $z_1 z_{-1} = \lambda z_{-1} z_1$ for some $\lambda \in k$,
2. $a z_1 = 0$ implies $a = 0$ for all $a \in B$ and
3. $\nu(a z_1) = 0$ implies $\nu(a) = 0$ for all $\nu(a) := a \bmod Bz_{-1} \in B/Bz_{-1}$.

Then the chain complex $K_\bullet := K_\bullet(z_1, z_{-1})$ given by

$$0 \to B \to B \oplus B \to B \to 0$$

with nontrivial boundary maps

$$a \mapsto (az_{-1}, -\lambda a z_1), \quad (b, c) \mapsto bz_1 + cz_{-1}$$

is a free resolution of $B/I$, $I := Bz_1 + Bz_{-1}$.
Proof. We clearly have $H_0(K) = B/I$ by very definition and $H_2(K) = 0$ by assumption (2). Now consider the subcomplex
\[ \tilde{K} := 0 \rightarrow B \rightarrow B_{z-1} \oplus B \rightarrow B_{z-1} \rightarrow 0 \]
of $K$ and the quotient complex $K/\tilde{K}$ which is of the form
\[ 0 \rightarrow 0 \rightarrow B/B_{z-1} \rightarrow B/B_{z-1} \rightarrow 0. \]
Its one nontrivial boundary map map is
\[ B/B_{z-1} \rightarrow B/B_{z-1}, \quad \nu(a) \mapsto \nu(az_1), \]
so assumption (3) means $H_1(K/\tilde{K}) = 0$. Furthermore, we have $H_1(\tilde{K}) = 0$: a cycle is an element $(bz_1, c) \in B_{z-1} \oplus B$ with
\[ 0 = bz_{-1}z_1 + cz_{-1} = (\lambda bz_1 + c)z_{-1}, \]
so assumption (2) gives $c = -\lambda bz_1$, hence $(bz_{-1}, c) = (bz_{-1}, -\lambda bz_1)$ is a boundary. Considering the long exact homology sequence derived from the short exact sequence $0 \rightarrow \tilde{K} \rightarrow K \rightarrow K/\tilde{K} \rightarrow 0$ now yields $H_1(K) = 0$. \qed

From now on let $B$ be again the Podleś sphere. Then the above gives:

**Theorem 10.** The left $B$-module $k$ admits a free resolution of the form $K_\ast(z_1, z_{-1})$ with $z_{\pm 1} := y_{\pm 1} + y_0$, $\lambda := q^2$.

**Proof.** It is easily seen that $B^+ := B \cap \ker \varepsilon$ is generated as a left ideal by the elements $y_n$. But since one has $q^{-1}y_{-1}(y_1 + y_0) - qy_0(y_{-1} + y_0) = y_0$, $B^+$ is in fact generated as a left ideal by the two elements $z_{\pm 1}$.

One verifies directly that $z_{-1}z_1 = q^2z_1z_{-1}$ which is assumption (1) in Lemma 10. Secondly, $B$ is a domain (see e.g. [1]), so assumption (2) holds as well. For (3) we turn $B$ into a $\mathbb{Z}$-graded algebra by assigning to $y_i$ the degree $i$ which is compatible with the defining relations (20). Then we have
\[ B = \bigoplus_{j \in \mathbb{Z}} B_j, \quad B_iB_j \subset B_{i+j}, \quad B_j := \text{span}_k\{e_{ij} \mid i \geq 0\}, \]
where
\[ e_{ij} := \begin{cases} y_0^iy_j & j \geq 0, \\ y_0^{-j} & j < 0, \end{cases} \quad i \in \mathbb{N}_0, j \in \mathbb{Z}, \]
and these form a vector space basis of $B$. Under $\nu : B \rightarrow B/B_{z-1}$ we have
\[ \nu(y^i_0y^j_1) = \begin{cases} y_0^i y_1^{j-1} \nu(y_{-1}) = -y_0^i y_1^{j-1} \nu(y_0) \\ = -y_0^i \nu(y_1^{j-1}y_0) = -q^{2(j-1)}y_0^i \nu(y_0y_1^{j-1}) \\ = -q^{2(i-1)}y_0^{i+1}y_1^{j-2} \nu(y_{-1}) \\ = q^{2(i-1)}y_0^{i+1}y_1^{-j+2} \nu(y_0) \\ = q^{2(j-1-2)}y_0^{-j+2}y_1^{-j+3} \nu(y_{-1}) \\ = -q^{2(j-1-2)}y_0^{-j+2}y_1^{-j+3} \nu(y_{-1}) \\ = \ldots \\ = (-1)^i q^{2(i-j-1-2-\cdots-(j-1))}y_0^{i+j-1}y_1^{-j-1} \nu(y_0) \\ = (-1)^i q^{(j-1)i} \nu(y_0^{i+j}). \]
Similarly we have for $i > 0, j > 0$

\[
\nu(y_0^i y_1^j) = q^{2j} y_0^{-1} \nu(y_1^j y_0) = q^{2j} y_0^{-1} y_1^{-1} \nu(y_1^{-1} y_0) = q^{2j} y_0^{-1} y_1^{-1} \nu(q^{-2} y_0^2 + q^{-1} y_0) = q^{2j-2} y_0^{-1} y_1^{-1} \nu(y_0^2) + q^{2j-1} y_0^{-1} y_1^{-1} \nu(y_0) = q^{-2j+2} \nu(y_0 y_1^{-1} + q \nu(y_0 y_1^{-1})) = q^{-2j+2} (q^{-2j+4} y_0 y_1^{-2} + q \nu(y_0 y_1^{-2})) + q(q^{-2j+4} \nu(y_0 y_1^{-2})) = q^{-4j+6} \nu(y_0 y_1^{-2}) + q^{-2j+3} (1 + q^2) \nu(y_0 y_1^{-2}) + q^2 \nu(y_0 y_1^{-2}) = q^{-4j+6} (q^{-2j+6} \nu(y_0 y_1^{-3}) + q \nu(y_0 y_1^{-3})) + q^{-2j+3} (1 + q^2) (q^{-2j+6} \nu(y_0 y_1^{-3}) + q \nu(y_0 y_1^{-3})) + q^2 (q^{-2j+6} \nu(y_0 y_1^{-3}) + q \nu(y_0 y_1^{-3})) = q^{-6j+12} \nu(y_0 y_1^{-3}) + q^{-4j+7} (1 + q^2 + q^4) \nu(y_0 y_1^{-3}) + q^{-2j+4} (1 + q^2 + q^4) \nu(y_0 y_1^{-3}) + q^3 \nu(y_0 y_1^{-3}) = \ldots
\]

\[
= \sum_{r=0}^{j} q^{(-2r+1)j+r^2} \left( \frac{j}{r} \right)_q \nu(y_0^{i+r})
\]

where we abbreviated

\[
\left( \frac{j}{r} \right)_q := 1 + q^2 + q^4 + \ldots + q^{2 \left( \frac{j}{r} \right)_q - 2}.
\]

Thus we have

\[
B/Bz_{-1} = \text{span}_k \{ \nu(y_0^{i+1}) \nu(y_1^i) \mid i \geq 0 \}.
\]

These residue classes are also linearly independent: assume that

\[
\sum_{i \geq 0} \lambda_i \nu(y_0^i) + \sum_{j \geq 0} \mu_j \nu(y_1^{i+1}) = 0
\]

in $B/Bz_{-1}$. One easily checks that

\[
B/(Bz_{-1} + By_0) = B/(B y_{-1} + B y_0)
\]

is an algebra quotient of $B$ (i.e. that $B y_0 + B y_{-1}$ is a two-sided ideal in $B$) and that it is as such isomorphic to the polynomial ring generated by the residue class of $y_1$. Hence the residue classes of $y_1^j$ are linearly independent in this quotient of $B/Bz_{-1}$. Considering the image of (21) therein thus gives

\[
\mu_j = 0 \quad \forall j \geq 0.
\]

We are left with

\[
\sum_{i \geq 0} \lambda_i \nu(y_0^i) = 0 \iff \sum_{i \geq 0} \lambda_i y_0^i = a z_{-1}
\]

for some $a \in B$. But $\sum_{i \geq 0} \lambda_i y_0^i$ is homogeneous of degree 0, $B$ is a domain, and $z_{-1}$ is not homogeneous, so the right hand side can not be homogeneous.
unless \( a = 0 \): if
\[
a = a_{j_0} + \ldots + a_{j_n}, \quad a_{j_i} \in B_{j_i} \setminus \{0\}, \quad j_0 < \ldots < j_n
\]
is the decomposition of \( a \) into homogeneous components, then \( az_{-1} \) has a nonzero component \( a_{j_0} y_{-1} \) in degree \( j_0 - 1 \) and a nonzero component \( a_{j_n} y_0 \) in degree \( j_n \). Thus \( a = 0 \) and since the \( y_i \) are linearly independent in \( B \) it follows that also
\[
\lambda_i = 0 \quad \forall \ i \geq 0.
\]

Now we compute the action of the map
\[
(23) \quad \zeta : B/Bz_{-1} \to B/Bz_{-1}, \quad \nu(a) \mapsto \nu(az_1)
\]
on the basis vectors. We get for \( i > 0 \)
\[
\nu(y_0^i z_1) = \nu(y_0^i y_1) + \nu(y_0^{i+1}) = q^2 y_0^{i-1} \nu(q^{-2} y_0^2 + q^{-1} y_0) + \nu(y_0^{i+1}) = 2 \nu(y_0^{i+1}) + q \nu(y_0^i)
\]
and for \( j \geq 0 \)
\[
\nu(y_j^j z_1) = \nu(y_j^j y_0) = \nu(y_j^{j+1}) + q^{-2j} \nu(y_0 y_j^j) = \nu(y_j^{j+1}) + q^{-2j} \sum_{r=0}^{j} q^{(-2r+1)j+r^2} \binom{j}{r} q \nu(y_0^{1+r}).
\]
So if we abbreviate
\[
V_j := \text{span}_k \{\nu(y_0), \ldots, \nu(y_j^{j+1}), \nu(1), \nu(y_1), \ldots, \nu(y_1^j)\},
\]
then we have
\[
B = \bigcup_{j \geq 0} V_j, \quad \zeta(V_j) \subset V_{j+1}
\]
and \( \zeta|_{V_j} \) is represented with respect to our basis by a matrix of the form
\[
\begin{pmatrix}
q & \ast & \ldots & \ast \\
2 \ast & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & 2 \ast & \ast \\
0 & \ast & \ldots & \ldots & \ldots & \ast \\
0 & \ast & \ldots & \ldots & \ldots & \ldots & \ast \\
1 & \ast & \ldots & \ldots & \ldots & \ldots & \ast \\
& \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\end{pmatrix},
\]
where the \( \ast \) denote nonzero entries and all other entries vanish. Hence \( \zeta \) is evidently injective (composing \( \zeta|_{V_j} \) with the canonical projection onto \( V_{j+1}/\text{span}_k \{\nu(y_0), \nu(1)\} \) yields an isomorphism of determinant \( 2^j \) which is assumption (3) of Lemma 10). \( \square \)
3.3. The Gorenstein condition. From the minimal resolution of $k$ provided by the Koszul complex one can read off $\text{Ext}^n_B(k, B)$:

**Lemma 11.** One has $\text{Ext}^n_B(k, B) = 0$ for $n \neq 2$ and $\text{Ext}^2_B(k, B) \cong k$. The resulting character $\chi$ of $B$ is equal to $\epsilon$.

**Proof.** Apply $\text{Hom}_B(\cdot, B)$ to the Koszul complex and identify $\text{Hom}_B(B, B) \cong B$. This gives the cochain complex

$$0 \leftarrow B \leftarrow B \oplus B \leftarrow B \leftarrow 0$$

of right $B$-modules whose two nontrivial coboundary maps are given by

$$f \mapsto (z_1 f, z_1 f), \quad (f, g) \mapsto q^{-1} z_1 f - q z_1 g.$$  

The exactness of this complex in degree 0 and 1 can be shown as the exactness of the Koszul complex using Lemma 10 (with $B$ replaced by $B^{op}$). In degree 2, the cohomology is $B$ divided by the right ideal generated by $z_{1 \pm 1}$. The result follows since with $q z_1 y_1 - q^{-1} z_1 y_0 = y_0$ one easily deduces that this ideal is again $\ker \epsilon$. \hfill $\Box$

Thus the relevant twisting automorphism is

$$\sigma(f) = \chi(f(1)) S^2(f(2)) = S^2(f)$$

which is explicitly given by

$$\sigma(y_1) = q^2 y_1, \quad \sigma(y_0) = y_0, \quad \sigma(y_1) = q^{-2} y_1.$$  

Note this is also the restriction of Woronowicz’s modular automorphism (see e.g. \cite{23, 16} for more information) to $B$.

3.4. The smoothness condition. The smoothness of $B$ follows from Corollary 6 since for this example $A^e \cong k[SL(2) \times SL(2)]$ is left Noetherian with global dimension 4, see \cite{13} and the references therein.

3.5. Determining $\omega$. We now know that Theorem 2 applies to $B$ with $\dim(B) = 2$, and that $B$ acts trivially (via $\epsilon$) on $\text{Ext}^2_B(k, B) \cong k$ so that the automorphism $\sigma$ from Theorem 4 equals $S^2$. Equation (16) becomes trivial, so $g$ therein could be any of the group-like elements in $C = k[z, z^{-1}]$, that is, an arbitrary monomial $z^n$ for some $n \in \mathbb{Z}$. So according to Theorem 7, $\omega$ is isomorphic as an object of $B M^A_{B, S^2}$ to $\omega_{n, 1}$, where we define for $m, n \in \mathbb{Z}$

$$\omega_{n, m} := \{a \in A_{S^2m} \mid \pi(a(1)) \otimes a(2) = z^n \otimes a \} \in B M^A_{B, S^2m}.$$  

As $B$-bimodules, we have isomorphisms

$$\omega_{n, m} \cong (\omega_{n, 0})_{S^2m} \cong (\omega_{n, 0}) \otimes_B B_{S^2m} = \omega_{n, 0} \otimes_B \omega_{0, m}$$

and (as a special case of (18))

$$\omega_{1, 0} \otimes_B \omega_{n, 0} \cong \omega_{n, 0}.$$  

Finally, the $B$-bimodule isomorphism $S^2m : S_{-2m} A \rightarrow A_{S^2m}$ restricts to a $B$-bimodule isomorphism

$$S_{-2m}(\omega_{n, 0}) \cong \omega_{n, m},$$

and combining these three equations we see that as $B$-bimodules we have

$$\omega_{n, m} \otimes_B \omega_{i, j} \cong \omega_{n+ i, m+ j}.$$  

Furthermore, we obtain by direct computation:
Lemma 12. One has

\[ H^0(B, \omega_{i,j}) \simeq \begin{cases} 
  k & \text{if } i = 2(m - j) \text{ for some } 0 \leq m \leq 2j, \\
  0 & \text{otherwise.}
\end{cases} \]

Proof. The defining relations of \( A \) imply that the monomials

\[ f_{lmn} := \begin{cases} 
  a_l b^m c^n & l \geq 0, \quad l \in \mathbb{Z}, m, n \in \mathbb{N}_0 \\
  d^{-1} b^m c^n & l < 0
\end{cases} \]

form a vector space basis, that \( A \) is a \( \mathbb{Z} \)-graded \( B \)-bimodule,

\[ A = \bigoplus_{l \in \mathbb{Z}} A_l, \quad B_i A_l B_j \subset A_{i+j+l}, \quad A_l := \text{span}_k \{ f_{lmn} \mid m, n \geq 0 \}, \]

and that

\[ A_l = \{ f \in A \mid y_0 f = q^{2l} f y_0 \}. \]

Furthermore, the explicit formulas for the coproduct of the generators \( a, b, c, d \) given in Section 3.1 and the fact that \( \pi(f_{lmn}) = \delta_{m,0} \delta_{n,0} z^l \) show that

\[ \omega_{i,j} = \text{span}_k \{ f_{lmn} \mid i = l + m - n \} \subset A_S^{2j}. \]

The zeroth Hochschild cohomology is by very definition isomorphic to the centre of the coefficient bimodule (identify \( \varphi \in \text{Hom}_B(B, M) \) with \( \varphi(1) \in M \), and since \( S^2(y_0) = y_0 \), the last two equations imply

\[ H^0(B, \omega_{i,j}) \subset \omega_{i,j} \cap A_0 = \text{span}_k \{ b^m c^n \mid m - n = i \}. \]

And from \( S^2(y_{\pm 1}) = q^{\mp 2} y_{\pm 1} \) and

\[ y_{\pm 1} b^m c^n = q^{\mp (m+n)} b^m c^n y_{\pm 1} \]

we deduce now that

\[ H^0(B, \omega_{i,j}) = \text{span}_k \{ b^m c^n \mid m - n = i, m + n = 2j \}. \]

The claim follows by elementary arithmetics. \( \square \)

Now we can finish the proof of Theorem 3.

End of proof of Theorem 3. By Theorem 2 and Hadfield’s explicit computation of \( H_\bullet(B, B_S^2) \) [16] we have

\[ H^0(B, (\omega^{-1})_{S^2}) \simeq H^0(B, \omega^{-1} \otimes_B B_{S^2}) \]

(25)

\[ \simeq H_2(B, \omega \otimes_B \omega^{-1} \otimes_B B_{S^2}) \]

\[ \simeq H_2(B, B_{S^2}) \simeq k. \]

If \( \omega = \omega_{n,1} \), then by (24) we have

\[ (\omega^{-1})_{S^2} \simeq \omega_{-n,0}. \]

and inserting this into Lemma 12 yields \( n = 0 \), so we have \( \omega \simeq \omega_{0,1} = B_{S^2} \) as claimed in Theorem 3. \( \square \)
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