The photon detection operator and complementarity between an electric detector and a magnetic detector

Shogo Tanimura

Department of Complex Systems Science, Graduate School of Information Science, Nagoya University, Nagoya 464-8601, Japan

E-mail: tanimura@is.nagoya-u.ac.jp

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Abstract

It has been a long-standing problem that there is no consistent definition of the photon position operator or photon number density in the context of quantum theory. In this paper, we derive the photon detection operator, which defines the location of photon absorption, by applying the theory of indirect measurement to quantum electrodynamics. It is shown that the photon detection probability depends on the properties of a photon-absorbing atom, in particular, on both electric and magnetic dipole moments of the atom. An experiment is proposed in which the complementarity of wave–particle nature of light will be tested. It is also discussed that the complementarity is related to the non-commutativity of the electric and the magnetic fields.

Keywords: quantum optics, photon, position, complementarity, measurement model

1. Introduction

The statistical interpretation which was proposed by Born is at the heart of quantum theory. For a wave function $\psi(x, y, z, t)$ of an electron, the square of its absolute value

$$|\psi(x, y, z, t)|^2 \Delta x \Delta y \Delta z$$

is proportional to the probability of finding the electron in a volume $\Delta x \Delta y \Delta z$ around the point $(x, y, z)$. The probability satisfies the local conservation law

$$\rho = |\psi|^2, \quad j = -\frac{i\hbar}{2m}(\psi^* \nabla \psi - \nabla \psi^* \psi), \quad \frac{\partial \rho}{\partial t} + \text{div} \, j = 0.$$  (2)

The position and the momentum of the electron are represented by operators $\hat{x}_j$ and $\hat{p}_j$ ($j = 1, 2, 3$), respectively. They act on the wave function as

$$\hat{x}_j \psi = x_j \psi, \quad \hat{p}_j \psi = -i\hbar \frac{\partial}{\partial x_j} \psi,$$

and they satisfy the canonical commutation relation (CCR)

$$[\hat{x}_j, \hat{x}_k] = 0, \quad [\hat{p}_j, \hat{p}_k] = 0, \quad [\hat{x}_j, \hat{p}_k] = i\hbar \delta_{jk}.  \quad (4)$$

Other massive particles like protons, neutrons and atoms can be described in a similar manner.

However, the above standard scheme is not applicable to photons, which are massless spin 1 particles. Pauli [1] noted that in quantum field theory there does not exist photon number density satisfying the local conservation law (2). Pryce [2] showed that it is impossible to implement photon position operators satisfying the CCR (4). Newton and Wigner [3] and Wightman [4] have proved that there is no localized state (position eigenstate) of photons. Thus, geometric notions like position and number density of photons cannot be defined in a naive manner.

Other researchers have defined localized states, position operators and probability density for photons in a more elaborate manner. Bialynicki-Birula [5] constructed a one-photon state whose energy density is localized with exponential fall-off, although it does not represent a stationary state. Hawton [6] constructed photon position operators that satisfy $[\hat{x}_j, \hat{x}_k] = 0$. Keller [7] provided a wave function formalism for describing the photon emission process in space-time.

In this paper, we formulate a photon detection operator which characterizes the probability and space-time location of photon emission or absorption. We take the physical
properties of photon-detecting atoms into our formalism explicitly. Our formulation can describe finite-time processes and also photon-detection processes that involve magnetic field–matter couplings. As an application of our formalism, we propose a scheme of an experiment to test wave–particle complementarity of light.

2. Indirect measurement model

Here we describe a general scheme of the indirect measurement model. An object system has a Hilbert space \( \mathcal{H} \) and a measuring apparatus has a Hilbert space \( \mathcal{K} \). Initial states of the object system and of the apparatus are characterized with density matrices \( \hat{\rho} \) and \( \hat{\sigma} \), respectively. The interaction between the object and the apparatus is described by a unitary operator \( \hat{U} \) acting on \( \mathcal{H} \otimes \mathcal{K} \). The apparatus has a self-adjoint operator \( \hat{M} \) which plays the role of a meter observable. The operator \( \hat{M} \) admits a spectral decomposition

\[
\hat{M} = \sum_r m_r \hat{P}_r,
\]

where \( \{\hat{P}_r\} \) are projection-valued measures satisfying

\[
\hat{P}_r^\dagger = \hat{P}_r, \quad \hat{P}_r \hat{P}_s = \delta_{rs} \hat{P}_r, \quad \sum_r \hat{P}_r = 1.
\]

After the interaction process, we read out the meter. The probability of reading the value \( m_r \) as the meter output is calculated with the Born statistical formula

\[
p_r = \text{Tr}_{\mathcal{K} \otimes \mathcal{H}}(\hat{P}_r \hat{U} \hat{\rho} \otimes \hat{\sigma} \hat{U}^\dagger) \quad (7)
\]

and the density matrix of the object system after the measurement is given by

\[
C_r(\hat{\rho}) = \frac{1}{p_r} \text{Tr}_{\mathcal{K}}(\hat{P}_r \hat{U} \hat{\rho} \otimes \hat{\sigma} \hat{U}^\dagger). \quad (8)
\]

3. Photon detection operator

Here we shall apply the above scheme to the photon detection process. Photons are regarded as an object system and the apparatus consists of electrons in atoms. We describe the dynamics of the whole system in the interaction picture and use the Coulomb gauge, in which the vector potential \( A \) satisfies \( \text{div} \ A = 0 \). The free photons and atoms obey the Hamiltonian

\[
\hat{H}_0 = \frac{1}{2} (\hat{E}^2 + \hat{B}^2) + \hat{H}_{\text{int}}. \quad (9)
\]

The interaction between photons and atoms is described by the minimal coupling

\[
\hat{H}_{\text{int}} = -\int \hat{A}(x, t) \cdot \hat{J}(x, t) \, d^3x,
\]

where \( \hat{J} \) is the electric current operator of the electrons. Then the time-evolution unitary operator is given by

\[
\hat{U} = \mathcal{T} \exp \left[ -\frac{i}{\hbar} \int_{t_0}^{t_1} \hat{H}_{\text{int}} \, dt \right] = \mathcal{T} \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i}{\hbar} \int_{t_0}^{t_1} \hat{H}_{\text{int}} \, dt \right)^n, \quad \quad (11)
\]

where the symbol \( \mathcal{T} \) denotes the time-ordered products of operators. By substituting (11) into (7) and by taking the first-order term in the expansion (11), we obtain an approximate probability of single-photon absorption

\[
p_r \sim \frac{1}{\hbar^2} \text{Tr}_{\mathcal{K} \otimes \mathcal{H}} \left( \hat{P}_r \int_{t_0}^{t_1} \hat{H}_{\text{int}} \, dt \otimes \hat{\sigma} \int_{t_0}^{t_1} \hat{H}_{\text{int}} \, dt \right). \quad (12)
\]

Moreover, the initial state of the apparatus \( \hat{\sigma} \) is assumed to be the ground state of the Hamiltonian \( \hat{H}_{\text{int}} \) as

\[
\hat{\sigma} = |\epsilon_0 \rangle \langle \epsilon_0|, \quad \hat{H}_{\text{int}} |\epsilon_0 \rangle = \epsilon_0 |\epsilon_0 \rangle, \quad (13)
\]

and the final state \( \hat{P}_r \) of the apparatus is assumed to be an excited state

\[
\hat{P}_r = |\epsilon_r \rangle \langle \epsilon_r|, \quad \hat{H}_{\text{int}} |\epsilon_r \rangle = \epsilon_r |\epsilon_r \rangle. \quad (14)
\]

In the interaction picture, the time-dependent operator is defined by

\[
\hat{H}_m(t) = e^{i \epsilon_m t / \hbar} \hat{H}_m e^{-i \epsilon_m t / \hbar}. \quad (15)
\]

Thus, the detection probability (12) becomes

\[
p_r \sim \frac{1}{\hbar^2} \text{Tr}_{\mathcal{K} \otimes \mathcal{H}} \left( \int_{t_0}^{t_1} \hat{A}^\dagger(\epsilon_r \langle \epsilon | \hat{J}(\epsilon_0) \rangle \, d^3x \, dt \otimes \int_{t_0}^{t_1} \hat{A}(\epsilon | \langle \epsilon | \hat{J}(\epsilon_r) \rangle d^3x \, dt \right). \quad (16)
\]

By introducing the detection operator

\[
\hat{D}_r = \frac{1}{\hbar} \int_{t_0}^{t_1} \hat{A}^\dagger(\epsilon_r \langle \epsilon | \hat{J}(\epsilon_0) \rangle \, d^3x \, dt
\]

\[
= \frac{1}{\hbar} \int_{t_0}^{t_1} \hat{A}^\dagger(\epsilon_r \langle \epsilon | \hat{J}(\epsilon_0) e^{i(\epsilon_r - \epsilon_0) t / \hbar} \, d^3x \, dt, \quad (17)
\]

we can write the photon detection probability and the state after the detection as

\[
p_r \sim \text{Tr}_{\mathcal{K} \otimes \mathcal{H}}(\hat{D}_r \hat{\rho} \hat{D}_r^\dagger). \quad (18)
\]

In the Coulomb gauge, the vector potential is expanded in plane waves as

\[
\hat{A}(x, t) = \int \frac{\sqrt{\hbar}}{\sqrt{2 \omega_0} (2\pi)^3} \sum_{\epsilon_0, k} \hat{a}_{\epsilon_0, k} e^{i(k \cdot x - \omega t)} \left( \epsilon_{\epsilon_0, k} - \epsilon_0 \right) \hat{a}^\dagger_{\epsilon_0, k} e^{-i(k \cdot x - \omega t)}
\]

\[
\hat{A}^\dagger(\epsilon_r \langle \epsilon | \hat{J}(\epsilon_0) \rangle e^{i(k \cdot x - \omega t)}
\]

with the frequency \( \omega_0 = c |k| \) and the transverse polarization vectors \( \epsilon_{\epsilon_0, k} \) satisfying \( k \cdot \epsilon_{\epsilon_0, k} = 0 \). The first term including the photon annihilation operators \( \hat{a}_{\epsilon_0, k} \) is called the positive frequency part of the electromagnetic field, while the second term including the creation operators \( \hat{a}^\dagger_{\epsilon_0, k} \) is called the negative frequency part. The Fourier transform of the matrix element of the electric current operator is denoted as

\[
\hat{J}_{kr} = \int \, d^3x \langle \epsilon_r | \hat{J}(x, 0) | \epsilon_0 \rangle e^{ik \cdot x}. \quad (20)
\]
4. Complementarity

An electrically neutral atom can interact with electromagnetic field via electric or magnetic dipole moment couplings. An electric polarization density $\mathbf{d}$ and a magnetization density $\mathbf{m}$ generate electric current

$$\mathbf{j} = \frac{\partial \mathbf{d}}{\partial t} + \text{rot} \mathbf{m}. \quad (25)$$

By substituting this into (10) and integrating by parts, we make the interaction Hamiltonian in the form

$$-\int \mathbf{A} \cdot \mathbf{j} \, d^3x \, dt = -\int \left( -\frac{\partial \mathbf{A}}{\partial t} \cdot \mathbf{d} + \text{rot} \mathbf{A} \cdot \mathbf{m} \right) d^3x \, dt$$

and rewrite the detection operator (24) in the form

$$\hat{D}_r^{(s)} = \frac{1}{\hbar} \int_{-\infty}^{\infty} \mathbf{A}(x, t) \cdot \langle \epsilon_r | \mathbf{j}(x, 0) | \epsilon_0 \rangle e^{i(\epsilon_r - \epsilon_0)t/h} \, d^3x \, dt. \quad (27)$$

This expression justifies Glauber’s proposal [8] for using the matrix element of the positive frequency part of the electric field $\langle \text{vac} | \mathbf{E}(x, t) | \text{photon} \rangle$ as a probability amplitude for photon detection. However, if the electric dipole moment $\langle \epsilon_r | \mathbf{d} | \epsilon_0 \rangle$ of the detector atom is zero, the magnetic dipole moment $\langle \epsilon_r | \mathbf{m} | \epsilon_0 \rangle$ becomes relevant as the next leading term. Thus, the magnetic field amplitude $\langle \text{vac} | \mathbf{B}(x, t) | \text{photon} \rangle$ should also be taken into account for photon detection.

In an interferometer depicted in figure 1, the split light beams merge on the film. In this case, the oscillating electric fields of the two-way light incident on the film are parallel. If we use a detector which is sensitive to electric field, we cannot distinguish which path of photons and will observe an interference pattern on the film. On the other hand, the oscillating magnetic fields of the two-way light incident on the film are orthogonal. Hence, if we use an ideal detector which is sensitive to magnetic field polarization, we can distinguish which path of photons. However, the wave nature and the particle nature of light should not be simultaneously observed. This complementarity of the wave–particle nature...
is a mathematical consequence of the non-commutativity or the uncertainty relation of the electric and the magnetic fields

\[ [\hat{E}_j(x, t), \hat{B}_k(y, t)] = i\hbar \epsilon_{jkl} \frac{\partial}{\partial x^l} \delta^3(x - y). \quad (28) \]

A more detailed discussion of this issue will be published in another paper.

Here we summarize our discussion: we derived the photon detection operator by applying the indirect measurement scheme to quantum electrodynamics. The photon detection probability depends on both electric and magnetic dipole moments of the photon-detecting atom. Their complementarity reflects the non-commutativity of the electric and the magnetic fields.

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