An Integral Representation of the Massive Dirac Propagator in Kerr Geometry in Eddington–Finkelstein-type Coordinates

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(Dated: November 2017)

ABSTRACT. We consider the massive Dirac equation in the non-extreme Kerr geometry in horizon-penetrating Eddington–Finkelstein-type coordinates and derive a functional analytic integral representation of the propagator using the spectral theorem and the solutions of the ODEs arising in Chandrasekhar’s separation of variables. This integral representation describes the dynamics of Dirac particles outside and across the event horizon, up to the Cauchy horizon. In the derivation, we first write the Dirac equation in Hamiltonian form. We then construct a unique self-adjoint extension of the Hamiltonian. To this end, as the Dirac Hamiltonian fails to be elliptic at the horizons, we combine results from the theory of symmetric hyperbolic systems with elliptic methods. Moreover, since the time evolution is not unitary because the particles may impinge on the singularity, we impose a suitable Dirichlet-type boundary condition inside the Cauchy horizon, having no effect on the outside dynamics. Finally, we obtain an explicit expression for the spectral decomposition of the propagator by applying Stone’s formula to the spectral measure of the Hamiltonian and expressing the resolvent in terms of the solutions of the separated radial and angular ODE systems.

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I. INTRODUCTION

In [9], a functional analytic integral representation of the propagator of the massive Dirac equation in the non-extreme Kerr geometry outside the event horizon is derived in Boyer–Lindquist coordinates \((t, r, \theta, \varphi)\) with \(t \in \mathbb{R}, r \in \mathbb{R}_{>0}, \theta \in [0, \pi]\), and \(\varphi \in [0, 2\pi]\). It has been used to study the long-time behavior (including decay rates) and the escape probability of Dirac particles [10]. The shortcoming of this integral representation is, however, that it yields a solution of the Cauchy problem only outside the event horizon. In the present paper, we construct a generalized integral representation that describes the complete dynamics of Dirac particles outside, across, and inside the event horizon. The methods used in the derivation of our integral representation are quite different from those employed in [9], as is now outlined. We work with horizon-penetrating isometries of Kerr geometry and, on the other hand, its Petrov type. After computing the corresponding spin coefficients, we explicitly determine the massive Dirac equation in Hamiltonian form

\[
i\partial_\tau \psi(t, r, \theta, \phi) = H \psi(t, r, \theta, \phi),
\]

where \(H\) denotes the Hamiltonian and \(\psi\) the Dirac 4-spinor. Moreover, we introduce a scalar product on the solution space and show that the Dirac Hamiltonian is symmetric with respect to this scalar product (on smooth and compactly supported Dirac 4-spinors). We also establish that it coincides with the canonical scalar product obtained by integrating the normal component (defined with respect to space-like hypersurfaces) of the Dirac current. We point out that in the present setting, the Dirac equation as well as the scalar product are smooth at the horizons. Since we apply the spectral theorem in the derivation of the propagator, we need to construct a unique self-adjoint extension of the Hamiltonian. To this end, in order to have a unitary time evolution, we first prevent that Dirac particles may impinge on the curvature singularity by shielding it imposing a Dirichlet-type boundary condition on a time-like hypersurface inside the Cauchy horizon. Clearly, this boundary condition changes the dynamics of the Dirac particles because they are now reflected on the inner boundary surface. However, the dynamics outside the Cauchy horizon is not affected, as the reflected particles cannot reenter this particular region (see FIG. 1 on page 13). Second, since the Hamiltonian is not elliptic at the horizons, we employ the method for non-uniformly elliptic boundary value problems introduced in [13], yielding a unique self-adjoint extension. This makes it possible to write down the Dirac propagator in spectral form

\[
\psi = e^{-iH\tau} \psi_0 = \int_\mathbb{R} e^{-i\omega \tau} \psi_0 \, dE_\omega,
\]

where \(dE_\omega\) is the spectral measure of \(H\) and \(\psi_0 := \psi(\tau = 0)\) is the initial data. Then, according to Stone’s formula, we represent the spectral measure via the resolvents \((H - \omega \mp i\epsilon)^{-1}\) with \(\epsilon > 0\), and express them in terms of the solutions of the radial and angular ODE systems resulting from Chandrasekhar’s separation of variables. In more detail, employing Chandrasekhar’s separation ansatz in (1) and projecting onto a finite-dimensional, invariant angular eigenspace, the Dirac Hamiltonian becomes a matrix-valued first-order ordinary differential operator in the radial variable. The resolvent of this operator can be determined by means of the Green’s matrix of the radial ODE system. We remark that, since this system cannot be solved analytically without making suitable approximations or by considering asymptotics, in the present work, we use the asymptotic solutions at infinity and at the event and Cauchy horizons for guidance in the implicit construction of the functions required for the computation of the Green’s matrix. Subsequently, by summing over all angular modes, we obtain the full resolvent in separated form. The integral representation of the Dirac propagator is thus given by the formula

\[
\psi(\tau, r, \theta, \phi) = \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} e^{-ik\phi} \lim_{\epsilon \downarrow 0} \int_\mathbb{R} e^{-i\omega \tau} \left[(H_k - \omega - i\epsilon)^{-1} - (H_k - \omega + i\epsilon)^{-1}\right](r, \theta; r', \theta') \, \psi_{0,k}(r', \theta') \, d\omega,
\]

in which \(k\) labels the azimuthal modes (for the explicit forms of the resolvents see Theorem IV.3).
The article is organized as follows. In Section II, we provide the mathematical framework for Kerr geometry and for the massive Dirac equation. Moreover, we recall required results from the asymptotic analysis of the radial ODE system and from the spectral analysis of the angular ODE system without giving proofs. We derive the Hamiltonian formulation and a suitable scalar product for the Hilbert space of solutions of the Cauchy problem in Section III. Furthermore, we verify the symmetry of the Hamiltonian with respect to this scalar product. In Section IV, we show that the Hamiltonian is essentially self-adjoint and construct the integral representation of the propagator.

II. PRELIMINARIES

We recall the necessary basics on the non-extreme Kerr geometry in horizon-penetrating advanced Eddington–Finkelstein-type coordinates, the general relativistic, massive Dirac equation in the Newman–Penrose formalism, and Chandrasekhar’s separation of variables (including asymptotic and spectral results for the solutions of the corresponding radial and angular ODE systems).

The non-extreme Kerr geometry is a connected, orientable and time-orientable, smooth, asymptotically flat Lorentzian 4-manifold \((M, g)\) with topology \(S^2 \times \mathbb{R}^2\), for which the metric \(g\) is stationary and axisymmetric and given in horizon-penetrating advanced Eddington–Finkelstein-type coordinates \((\tau, r, \theta, \phi)\) with \(\tau \in \mathbb{R}, r \in \mathbb{R}_{>0}, \theta \in [0, \pi], \) and \(\phi \in [0, 2\pi)\) by [17]

\[
g = \left(1 - \frac{2Mr}{\Sigma}\right) d\tau \otimes d\tau - \frac{2Mr}{\Sigma} \left(\left[dr - a \sin^2(\theta) d\phi\right] \otimes d\tau + d\tau \otimes \left[dr - a \sin^2(\theta) d\phi\right]\right)
\]

\[
- \left(1 + \frac{2Mr}{\Sigma}\right) \left(dr - a \sin^2(\theta) d\phi\right) \otimes \left(dr - a \sin^2(\theta) d\phi\right) - \Sigma d\theta \otimes d\theta - \Sigma \sin^2(\theta) d\phi \otimes d\phi,
\]

where \(M\) is the mass and \(aM\) the angular momentum of the black hole with \(0 \leq a < M\), and \(\Sigma = \Sigma(r, \theta) := r^2 + a^2 \cos^2(\theta)\). The event and Cauchy horizons are located at \(r_\pm \) := \(M \pm \sqrt{M^2 - a^2}\), respectively. The advanced Eddington–Finkelstein-type coordinates are an analytic extension of the common Boyer–Lindquist coordinates \((t, r, \theta, \varphi)\) with \(t \in \mathbb{R}, r \in \mathbb{R}_{>0}, \theta \in [0, \pi], \) and \(\varphi \in [0, 2\pi)\) [3], covering both the exterior and interior black hole regions while being regular at the horizons. In terms of the Boyer–Lindquist coordinates, the advanced Eddington–Finkelstein-type time and azimuthal angle coordinates read

\[
\tau := t + \frac{r_+^2 + a^2}{r_+ - r_-} \ln |r - r_+| - \frac{r_+^2 + a^2}{r_+ - r_-} \ln |r - r_-|
\]

\[
\phi := \varphi + \frac{a}{r_+ - r_-} \ln \left|\frac{r - r_+}{r - r_-}\right|,
\]

This horizon-penetrating coordinate system possesses a proper coordinate time, unlike the original advanced Eddington–Finkelstein (null) coordinates [7, 8]. It is advantageous to describe Kerr geometry in the Newman–Penrose formalism using a regular Carter tetrad [4, 17]

\[
l = \frac{1}{\sqrt{2\Sigma} r_+} \left(\left[\Delta + 4Mr\right] \partial_\tau + \Delta \partial_r + 2a \partial_\phi\right)
\]

\[
n = \frac{r_+}{\sqrt{2\Sigma}} \left(\partial_r - \partial_\tau\right)
\]

\[
m = \frac{1}{\sqrt{2\Sigma}} \left(i a \sin(\theta) \partial_\tau + \partial_\theta + i \csc(\theta) \partial_\phi\right)
\]

\[
m = \frac{1}{\sqrt{2\Sigma}} \left(i a \sin(\theta) \partial_\tau - \partial_\theta + i \csc(\theta) \partial_\phi\right)
\]
with \( \Delta = \Delta(r) := (r - r_+)(r - r_-) = r^2 - 2Mr + a^2 \) being the horizon function because this frame is adapted to the two principal null directions of the Weyl tensor and to the fundamental discrete time and angle reversal isometries. Thus, since Kerr geometry is algebraically special and of Petrov type D, one has the computational advantage that the four spin coefficients \( \kappa, \sigma, \lambda, \) and \( \nu \) as well as the four Weyl scalars \( \Psi_0, \Psi_1, \Psi_3, \) and \( \Psi_4 \) vanish [15], and that specific spin coefficients are linearly dependent. Substituting the Carter tetrad (4) into – and solving – the first Maurer–Cartan equation of structure, we obtain the spin coefficients [17]

\[
\begin{align*}
\kappa &= \sigma = \lambda = \nu = 0, \quad \gamma = -\frac{r_+}{2^{3/2} \sqrt{\Sigma} (r - ia \cos(\theta))}, \\
\epsilon &= -\frac{a^2 - 2ia \cos(\theta) (r - M)}{2^{3/2} \sqrt{\Sigma}(r - ia \cos(\theta))}, \\
\pi &= -\tau = \frac{i a \sin(\theta)}{\sqrt{\Sigma} (r - ia \cos(\theta))}, \\
\mu &= -\frac{\gamma}{\sqrt{\Sigma} (r - ia \cos(\theta))}, \\
\rho &= -\frac{\Delta}{\sqrt{\Sigma} r_+ (r - ia \cos(\theta))}. 
\end{align*}
\]

(5)

Introducing a spin bundle \( S^4 = \mathcal{M} \times \mathbb{C}^4 \) on \( \mathcal{M} \) with fibers \( S^4 \mathcal{M} \simeq \mathbb{C}^4, \) \( x \in \mathcal{M}, \) we can formulate the general relativistic, massive Dirac equation (without an external potential)

\[
(\gamma^\mu \nabla_\mu + im) \psi(x^\mu) = 0, \quad \mu \in \{0, 1, 2, 3\},
\]

where \( \nabla \) is the metric connection on \( S^4 \mathcal{M}, \) \( \gamma^\mu \) are the Dirac matrices, \( \psi \) is the Dirac 4-spinor defined on the fibers \( S^4 \mathcal{M}, \) and \( m \) is the fermion rest mass. In the Newman–Penrose formalism – by employing a local dyad spinor frame – this equation becomes the coupled first-order PDE system

\[
\begin{align*}
(n^\mu \partial_\mu + \bar{\nu} - \bar{\tau}) \mathcal{F}_1 - (m^\mu \partial_\mu + \bar{\beta} - \bar{\pi}) \mathcal{F}_2 &= i \mu_\ast \mathcal{F}_1 \\
(l^\mu \partial_\mu + \tau - \pi) \mathcal{F}_1 - (m^\mu \partial_\mu + \nu - \alpha) \mathcal{F}_2 &= i \mu_\ast \mathcal{F}_2 \\
(l^\mu \partial_\mu + \epsilon - \delta) \mathcal{F}_1 + (m^\mu \partial_\mu + \mu - \gamma) \mathcal{F}_2 &= \mu_\ast \mathcal{F}_1 \\
n^\mu \partial_\mu + \mu - \gamma) \mathcal{F}_2 + (m^\mu \partial_\mu + \beta - \tau) \mathcal{F}_1 &= \mu_\ast \mathcal{F}_2
\end{align*}
\]

(6)

with \( \psi = (\mathcal{F}_1, \mathcal{F}_2, -\mathcal{F}_1, -\mathcal{F}_2)^T \) and \( \mu_\ast := m/\sqrt{2} \) [5]. Substituting the Carter tetrad (4) and the associated spin coefficients (5) into the system (6), and applying the transformation

\[
\psi' = \mathcal{P} \psi = (\mathcal{H}_1, \mathcal{H}_2, -\mathcal{J}_1, -\mathcal{J}_2)^T, \quad \gamma'^\mu = \mathcal{P} \gamma^\mu \mathcal{P}^{-1},
\]

(7)

where

\[
\mathcal{P} := \text{diag}(\sqrt{r - ia \cos(\theta)}, \sqrt{r - ia \cos(\theta)}, \sqrt{r + ia \cos(\theta)}, \sqrt{r + ia \cos(\theta)}),
\]

we find

\[
\begin{align*}
r_+ (\partial_\tau - \partial_\tau) \mathcal{J}_1 + (ia \sin(\theta) \partial_\tau - \partial_\phi + i \csc(\theta) \partial_\phi - 2^{-1} \cot(\theta)) \mathcal{J}_2 &= \sqrt{2} i \mu_\ast (r + ia \cos(\theta)) \mathcal{H}_1 \\
r_+^{-1} (\partial_\tau - \partial_\tau) \mathcal{J}_2 - (ia \sin(\theta) \partial_\tau + \partial_\phi - i \csc(\theta) \partial_\phi + 2^{-1} \cot(\theta)) \mathcal{J}_1 &= \sqrt{2} i \mu_\ast (r - ia \cos(\theta)) \mathcal{H}_2 \\
r_+^{-1} (\partial_\tau - \partial_\tau) \mathcal{J}_1 - (ia \sin(\theta) \partial_\tau - \partial_\phi + i \csc(\theta) \partial_\phi - 2^{-1} \cot(\theta)) \mathcal{J}_2 &= \sqrt{2} i \mu_\ast (r - ia \cos(\theta)) \mathcal{H}_1 \\
r_+ (\partial_\tau - \partial_\tau) \mathcal{H}_2 + (ia \sin(\theta) \partial_\tau + \partial_\phi + i \csc(\theta) \partial_\phi + 2^{-1} \cot(\theta)) \mathcal{H}_1 &= \sqrt{2} i \mu_\ast (r - ia \cos(\theta)) \mathcal{J}_2,
\end{align*}
\]

(8)
which is the starting point for the derivation of the Hamiltonian formulation of the massive Dirac equation on a Kerr background geometry in horizon-penetrating coordinates presented in the next section. We note in passing that the system (8) corresponds to the transformed Dirac equation
\[- \sqrt{\Sigma} \gamma^0 \mathcal{P}^{-1} \left( \gamma^\mu \left[ \nabla_\mu + \mathcal{P} \partial_\mu (\mathcal{P}^{-1}) \right] + \text{im} \right) \psi' = 0,\]
where \( \gamma^0 := \text{diag}(1, 1, -1, -1) \). This will become relevant both in the construction of the Hamiltonian formulation and the scalar product.

Finally, for the later computation of the resolvent of the Dirac Hamiltonian, we require specific results arising from Chandrasekhar’s separation of variables of the system (8). More precisely, we need the asymptotics and decay properties of the solutions of the radial ODE system at infinity, the event horizon, and the Cauchy horizon, as well as certain information about the eigenvalues and eigenfunctions of the angular ODE system. In the following, these results are recalled. For a detailed analysis and proofs see [17]. Substituting the separation ansatz
\[
\begin{align*}
\mathcal{H}_1 &= e^{-i(\omega r + k\phi)} \mathcal{P}_+ (r) \mathcal{J}_+ (\theta) \\
\mathcal{H}_2 &= e^{-i(\omega r + k\phi)} \mathcal{P}_- (r) \mathcal{J}_- (\theta) \\
\mathcal{J}_1 &= e^{-i(\omega r + k\phi)} \mathcal{P}_- (r) \mathcal{J}_+ (\theta) \\
\mathcal{J}_2 &= e^{-i(\omega r + k\phi)} \mathcal{P}_+ (r) \mathcal{J}_- (\theta),
\end{align*}
\]
in which \( \omega \in \mathbb{R} \) and \( k \in \mathbb{Z} + 1/2 \), into (8) yields the first-order radial and angular ODE systems
\[
\begin{align*}
O_r \mathcal{J} &= \frac{\sqrt{|\Delta|}}{r^2 + a^2} \begin{pmatrix} 0 & 1 \\ \text{sign}(\Delta) & 0 \end{pmatrix} \xi \mathcal{J} \\
O_\theta \mathcal{J} &= \xi \mathcal{J},
\end{align*}
\]
where
\[
\begin{align*}
r_* := r + \frac{r_+^2 + a^2}{r_+ - r_-} \ln |r - r_+| - \frac{r_+^2 + a^2}{r_+ - r_-} \ln |r - r_-|
\end{align*}
\]
is the Regge–Wheeler coordinate,
\[
\begin{align*}
O_r &:= \mathbb{I} \frac{1}{r^2 + a^2} \begin{pmatrix} -\omega(\Delta + 4Mr) - 2ak & -\sqrt{2|\Delta|} \mu_+ r \\ \sqrt{2|\Delta| \text{sign}(\Delta)} \mu_+ r & \omega \Delta \end{pmatrix} \\
O_\theta &:= \begin{pmatrix} \sqrt{2}\mu_+ a \cos(\theta) & -\partial_\theta - 2^{-1} \cot(\theta) + a\omega \sin(\theta) + k \csc(\theta) \\ \partial_\theta + 2^{-1} \cot(\theta) + a\omega \sin(\theta) + k \csc(\theta) & -\sqrt{2}\mu_+ a \cos(\theta) \end{pmatrix},
\end{align*}
\]
are matrix-valued radial and angular operators, \( \mathcal{J} := (\mathcal{P}_+, r_+ \mathcal{P}_-)^T \) and \( \mathcal{J} := (\mathcal{P}_+, r_- \mathcal{P}_-)^T \) are radial and angular vector-valued functions, and \( \xi \) is the constant of separation. The asymptotics and decay properties of the solutions of the radial ODE system at infinity, the event horizon, and the Cauchy horizon are specified in the lemmas below.

**Lemma II.1.** Every nontrivial solution \( \mathcal{J} \) of (10) is asymptotically as \( r \to \infty \) of the form
\[
\mathcal{J}(r_*) = \mathcal{J}_\infty (r_*) + E_\infty (r_*) = D_\infty \begin{pmatrix} \exp (i\phi_+ (r_*)) \\ \exp (-i\phi_- (r_*)) \end{pmatrix} + E_\infty (r_*)
\]
with the asymptotic diagonalization matrix
\[
D_\infty := \begin{cases}
\begin{pmatrix} \cosh(\Omega) & \sinh(\Omega) \\ \sinh(\Omega) & \cosh(\Omega) \end{pmatrix} & \text{for } \omega^2 \geq 2\mu_+^2 \\
\frac{1}{\sqrt{2}} \begin{pmatrix} \cosh(\Omega) + i\sinh(\Omega) & \sinh(\Omega) + i\cosh(\Omega) \\ \sinh(\Omega) + i\cosh(\Omega) & \cosh(\Omega) + i\sinh(\Omega) \end{pmatrix} & \text{for } \omega^2 < 2\mu_+^2,
\end{cases}
\]
where

\[
\Omega := \begin{cases} 
\frac{1}{4} \ln \left( \frac{\omega - \sqrt{2\mu^2}}{\omega + \sqrt{2\mu^2}} \right) & \text{for } \omega^2 \geq 2\mu^2, \\
\frac{1}{4} \ln \left( \frac{\sqrt{2\mu^2} - \omega}{\sqrt{2\mu^2} + \omega} \right) & \text{for } \omega^2 < 2\mu^2,
\end{cases}
\]

the asymptotic phases

\[
\phi_\pm (r_*) := \text{sign}(\omega) \times \begin{cases} 
-\sqrt{\omega^2 - 2\mu^2} r_* + 2M \left( \pm \omega - \frac{\mu^2}{\sqrt{\omega^2 - 2\mu^2}} \right) \ln (r_*) & \text{for } \omega^2 \geq 2\mu^2 \\
\sqrt{2\mu^2 - \omega^2} r_* + 2M \left( \pm \omega - \frac{i\mu^2}{\sqrt{2\mu^2 - \omega^2}} \right) \ln (r_*) & \text{for } \omega^2 < 2\mu^2,
\end{cases}
\]

the constants \( f_\infty = (f^{(1)}_\infty, f^{(2)}_\infty)^T \neq 0 \), as well as an error with polynomial decay

\[
\|E_\infty (r_*)\| = \|\mathcal{R}_\infty (r_*) - \mathcal{R}_\infty (r_*)\| \leq a/r_*
\]

for a suitable constant \( a \in \mathbb{R}_{>0} \).

**Lemma II.2.** Every nontrivial solution \( \mathcal{R} \) of (10) is asymptotically as \( r_\to r_\pm \) of the form

\[
\mathcal{R} (r_*) = \mathcal{R}_\pm (r_*) + E_\pm (r_*) = \left( g^{(1)}_\pm r_* \exp \left( 2i \left[ \omega + k\Omega^{(\pm)}_{\text{Kerr}} \right] \ln (r_*) \right) \right) + E_\pm (r_*)
\]

with the constants \( g_\pm = (g^{(1)}_\pm, g^{(2)}_\pm)^T \neq 0 \) and \( \Omega^{(\pm)}_{\text{Kerr}} := a/(2M r_\pm) \), as well as an error with exponential decay

\[
\|E_\pm (r_*)\| = \|\mathcal{R} (r_*) - \mathcal{R}_\pm (r_*)\| \leq p_\pm \exp (\pm q_\pm r_*)
\]

for suitable constants \( p_\pm, q_\pm \in \mathbb{R}_{>0} \).

The spectral properties of the eigenvalues and eigenfunctions of the angular ODE system are summarized in the following proposition.

**Proposition II.3.** For any \( \omega \in \mathbb{R} \) and \( k \in \mathbb{Z} + 1/2 \), the differential operator (12) has a complete set of orthonormal eigenfunctions \( \mathcal{R}_l \in \mathbb{L}^2 \left( (0, \pi), \sin (\theta) \, d\theta \right)^2 \). The corresponding eigenvalues \( \xi_l \) are real-valued and non-degenerate, and can thus be ordered as \( \xi_l < \xi_{l+1} \). Moreover, the eigenfunctions are pointwise bounded and smooth away from the poles,

\[
\mathcal{R}_l \in \mathbb{L}^\infty \left( (0, \pi) \right)^2 \cap C^\infty \left( (0, \pi) \right)^2.
\]

Both the eigenfunctions \( \mathcal{R}_l \) and the eigenvalues \( \xi_l \) depend smoothly on \( \omega \).

### III. HAMILTONIAN FORMULATION OF THE MASSIVE DIRAC EQUATION IN KERR GEOMETRY AND THE CANONICAL SCALAR PRODUCT

In order to derive the Hamiltonian formulation of the massive Dirac equation in the non-extreme Kerr geometry in horizon-penetrating advanced Eddington–Finkelstein-type coordinates, it is advantageous to first rewrite the system (8) in the form

\[
(\tilde{\mathcal{A}} + \mathcal{A})\psi' = 0,
\]
\[ R := \begin{pmatrix} -\sqrt{2} i \mu^r & 0 & -\mathcal{D}_- & 0 \\ 0 & -\sqrt{2} i \mu^r & 0 & -\mathcal{D}_+ \\ \mathcal{D}_+ & 0 & \sqrt{2} i \mu^r & 0 \\ 0 & \mathcal{D}_- & 0 & \sqrt{2} i \mu^r \end{pmatrix} \tag{13} \]

and
\[ A := \begin{pmatrix} \sqrt{2} i \mu^a \cos(\theta) & 0 & 0 & \mathcal{L} \\ 0 & \sqrt{2} i \mu^a \cos(\theta) & \mathcal{L} & 0 \\ \mathcal{L} & 0 & 0 & \sqrt{2} i \mu^a \cos(\theta) \end{pmatrix} \tag{14} \]

are matrix-valued differential operators with
\[ \mathcal{D}_+ := r_+^{-1} \left( [\Delta + 4 Mr] \partial_\tau + \Delta \partial_r + 2a \partial_\phi + r - M \right) \]
\[ \mathcal{D}_- := r_+ \left( \partial_\tau - \partial_r \right) \]
\[ \mathcal{L} := i a \sin(\theta) \partial_\tau + \partial_\theta + i \csc(\theta) \partial_\phi + 2^{-1} \cot(\theta) \cdot \]

Separating the \( \tau \)-derivative and multiplying by the inverse of the matrix
\[ \tilde{\gamma}^\tau := \begin{pmatrix} 0 & 0 & -r_+ & -ia \sin(\theta) \\ 0 & 0 & ia \sin(\theta) & -r_+^{-1} [\Delta + 4 Mr] \\ r_+^{-1} [\Delta + 4 Mr] & -ia \sin(\theta) & 0 & 0 \\ ia \sin(\theta) & r_+ & 0 & 0 \end{pmatrix} \tag{15} \]
(cf. Eq. (9)) as well as by the imaginary unit leads to
\[ i \partial_\tau \psi' = -i \left( \tilde{\gamma}^\tau \right)^{-1} \left( \mathcal{R}^{(3)} + \mathcal{A}^{(3)} \right) \psi' =: H \psi', \tag{16} \]

where \( \mathcal{R}^{(3)} \) and \( \mathcal{A}^{(3)} \) contain the first-order spatial as well as the zero-order contributions of the operators (13) and (14), respectively. The Dirac Hamiltonian \( H \) may be recast in the more convenient form
\[ H = \alpha^j \partial_j + \mathcal{V}, \quad j \in \{r, \theta, \phi\}, \tag{17} \]

with the matrix-valued coefficients
\[ \alpha^r := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} \frac{i \Delta}{r_+^{-1} \Delta a \sin(\theta)} & r_+ a \sin(\theta) & 0 & 0 \\ r_+^{-1} \Delta a \sin(\theta) & -i(\Delta + 4Mr) & \frac{0}{-i(\Delta + 4Mr)} & 0 \\ 0 & 0 & \frac{-r_+ a \sin(\theta)}{r_+ a \sin(\theta)} & \frac{-i(\Delta + 4Mr)}{r_+ a \sin(\theta)} \\ -r_+ a \sin(\theta) & 0 & \frac{i(\Delta + 4Mr)}{r_+ a \sin(\theta)} & \frac{-i(\Delta + 4Mr)}{r_+ a \sin(\theta)} \end{pmatrix} \tag{18} \]
\[ \alpha^\theta := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} -a \sin(\theta) & ir_+ & 0 & 0 \\ ir_+ [\Delta + 4Mr] & a \sin(\theta) & 0 & 0 \\ 0 & 0 & -a \sin(\theta) & -ir_+^{-1} [\Delta + 4Mr] \\ 0 & 0 & -ir_+ & a \sin(\theta) \end{pmatrix} \tag{19} \]
\[ \alpha^\phi := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} ia & r_+ csc(\theta) & 0 & 0 \\ r_+^{-1} csc(\theta)(\Delta - 2\Sigma) & -ia & 0 & 0 \\ 0 & 0 & -ia & r_+^{-1} csc(\theta)(\Delta - 2\Sigma) \\ 0 & 0 & r_+ csc(\theta) & ia \end{pmatrix} \tag{20} \]
and the potential

\[ \mathcal{V} := -\frac{1}{\Sigma + 2Mr} \left( \begin{array}{c} \mathcal{B}_1 \\ \mathcal{B}_2 \\ \mathcal{B}_3 \\ \mathcal{B}_4 \end{array} \right), \]  

(21)

where the quantities \( \mathcal{B}_k, k \in \{1, 2, 3, 4\} \), are the \((2 \times 2)\)-blocks

\[
\mathcal{B}_1 := \begin{pmatrix}
i(r - M) - 2^{-1}a \cos(\theta) & 2^{-1}i r_+ \cot(\theta) \\
r_+^{-1} (a \sin(\theta) (r - M) + 2^{-1}i \cot(\theta)(\Delta + 4Mr)) & 2^{-1}a \cos(\theta)
\end{pmatrix}
\]

\[
\mathcal{B}_2 := \begin{pmatrix}
-\sqrt{2}i \mu_+ r_+ (r - ia \cos(\theta)) & -\sqrt{2}i \mu_+ a \sin(\theta) (r - ia \cos(\theta)) \\
\sqrt{2}i \mu_+ a \sin(\theta)(r - ia \cos(\theta)) & -\sqrt{2}r_+^{-1} \mu_+ (\Delta + 4Mr)(r - ia \cos(\theta))
\end{pmatrix}
\]

\[
\mathcal{B}_3 := \begin{pmatrix}
-\sqrt{2}r_+^{-1} \mu_+ (\Delta + 4Mr)(r + ia \cos(\theta)) & \sqrt{2}i \mu_+ a \sin(\theta)(r + ia \cos(\theta)) \\
-\sqrt{2}i \mu_+ a \sin(\theta)(r + ia \cos(\theta)) & -\sqrt{2}r_+ \mu_+ (r + ia \cos(\theta))
\end{pmatrix}
\]

\[
\mathcal{B}_4 := \begin{pmatrix}
-2^{-1}a \cos(\theta) & r_+^{-1}(a \sin(\theta)(r - M) + 2^{-1}i \cot(\theta)(\Delta + 4Mr)) \\
-2^{-1}i r_+ \cot(\theta) & i(r - M) + 2^{-1}a \cos(\theta)
\end{pmatrix}
\]

(22)

Furthermore, we work with the scalar product

\[ (\psi|\phi)_{\mathcal{H}_+} := \int_{\mathcal{H}_+} \langle \psi|\phi^\dagger \rangle \, d\mu_{\mathcal{H}_+}, \]

(23)

defined on the space-like hypersurface \( \mathcal{H}_+ := \{ \tau = \text{const.}, r, \theta, \phi \} \) \[9\], where

\[ \langle \cdot | \cdot \rangle : S_2 \mathfrak{M} \times S_2 \mathfrak{M} \to \mathbb{C}, \quad (\psi, \phi) \mapsto \psi^\dagger \phi \]

(24)

denotes the indefinite spin scalar product of signature \((2, 2)\), \( \psi^\dagger := \psi^\dagger \mathcal{S} \) the adjoint Dirac spinor, \( \psi = \gamma^\mu \nu_\mu \) the Clifford contraction of the future-directed, time-like normal \( \nu_\mu \), and \( d\mu_{\mathcal{H}_+} = \sqrt{|\det(g_{\mathcal{H}_+})|} \, d\phi \, d\theta \, dr \) with the induced (Riemannian) metric \( g_{\mathcal{H}_+} \) is the invariant measure on \( \mathcal{H}_+ \). (Note that this scalar product is independent of the choice of the specific space-like hypersurface.) The matrix \( \mathcal{S} \) is defined via the relation

\[ \gamma^\mu := \mathcal{S} \gamma^\mu \mathcal{S}^{-1}. \]

(25)

With (15) and the spinor transformation (7), we find

\[ \gamma^\mu = -\frac{1}{\sqrt{\Sigma}} (\mathcal{S}^\dagger)^{-1} \gamma^0 \gamma^\mu \mathcal{S} \]

and thus, using (25), we obtain for the matrix \( \mathcal{S} \) the term

\[ \mathcal{S} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}. \]

The vector field \( \nu_\mu \) is determined by means of the conditions

\[ (\nu|\partial_r)_{\mathcal{H}} = (\nu|\partial_\theta)_{\mathcal{H}} = (\nu|\partial_\phi)_{\mathcal{H}} = 0 \quad \text{and} \quad (\nu|\nu)_{\mathcal{H}} = 1, \]
where \( \langle \cdot | \cdot \rangle_g := g(\cdot, \cdot) \) is the spacetime inner product on the manifold \( \mathcal{M} \) given by (2), yielding

\[
\nu = \left(1 + \frac{2Mr}{\Sigma}\right)^{1/2} \partial_r - \frac{2Mr}{\Sigma} \left(1 + \frac{2Mr}{\Sigma}\right)^{-1/2} \partial_\tau.
\]

The corresponding dual co-vector reads

\[
\nu = \left(1 + \frac{2Mr}{\Sigma}\right)^{-1/2} \ d\tau.
\] (26)

Moreover, the induced metric \( g_{\mathcal{N}_\tau} \) on \( \mathcal{N}_\tau \) is the restriction of (2) from \( \mathcal{M} \) to the submanifold \( \mathcal{N}_\tau \)

\[
g_{\mathcal{N}_\tau} = - \left(1 + \frac{2Mr}{\Sigma}\right) \left(dr - a \sin^2(\theta) \ d\phi\right) \otimes \left(dr - a \sin^2(\theta) \ d\phi\right) - \Sigma \ d\theta \otimes d\theta - \Sigma \sin^2(\theta) \ d\phi \otimes d\phi,
\]

for which

\[
\sqrt{|\det(g_{\mathcal{N}_\tau})|} = \Sigma \sin(\theta) \left(1 + \frac{2Mr}{\Sigma}\right)^{1/2}
\] (27)

is the Jacobian determinant in the volume measure \( d\mu_{\mathcal{N}_\tau} \). Expressing the scalar product (23), which is invariant under spinor transformations, in terms of the primed wave functions (7) employed in (16) and substituting (26) and (27), we obtain

\[
(\psi'|\phi')_{\mathcal{N}_\tau} = \iiint \psi'^\dagger \gamma^\tau \phi' \Sigma \sin(\theta) \ d\phi \ d\theta \ dr.
\] (28)

Again using (15), that is with \( \gamma^\tau = -P \left(\mathcal{P}^\dagger\right)^{-1} \gamma^0 \tilde{\gamma}^\tau / \sqrt{\Sigma} \), the scalar product (28) becomes

\[
(\psi'|\phi')_{\mathcal{N}_\tau} = \iiint \psi'^\dagger \mathcal{P} \mathcal{P} \left(\mathcal{P}^\dagger\right)^{-1} \gamma^0 \tilde{\gamma}^\tau \phi' \sin(\theta) \ d\phi \ d\theta \ dr
\]

\[
= \iiint \psi'^\dagger \mathcal{P} \left(\mathcal{P}^\dagger\right)^{-1} \gamma^0 \tilde{\gamma}^\tau \phi' \sin(\theta) \ d\phi \ d\theta \ dr
\]

\[
= \iiint \psi'^\dagger \mathcal{P} \left(\mathcal{P}^\dagger\right)^{-1} \gamma^0 \tilde{\gamma}^\tau \phi' \sin(\theta) \ d\phi \ d\theta \ dr
\]

\[
= \iiint \psi'^\dagger \mathcal{P} \left(\mathcal{P}^\dagger\right)^{-1} \gamma^0 \tilde{\gamma}^\tau \phi' \sin(\theta) \ d\phi \ d\theta \ dr
\] (29)

where

\[
\Gamma^\tau := -\mathcal{P} \gamma^0 \tilde{\gamma}^\tau = \begin{pmatrix}
    r_+^{-1}[\Delta + 4Mr] & -ia \sin(\theta) & 0 & 0 \\
    a \sin(\theta) & r_+ & 0 & 0 \\
    0 & 0 & r_+ & ia \sin(\theta) \\
    0 & 0 & -ia \sin(\theta) & r_+^{-1}[\Delta + 4Mr]
\end{pmatrix}.
\] (30)

Note that in the above derivation, we have first applied the relation \( \sqrt{\Sigma} \mathbf{1}_{\mathcal{C}^4} = \mathcal{P} \mathcal{P}^\dagger \), then the transformation law for the matrix \( \mathcal{I} \), namely \( \mathcal{I} = \mathcal{P}^\dagger \mathcal{I} \mathcal{P} \), and finally we have used the fact that both \( \mathcal{I} \) and the product \( \mathcal{P} \mathcal{I} \) are self-adjoint, which leads to the relation \( \mathcal{P} \mathcal{I} = \mathcal{I} \mathcal{P}^\dagger \). Besides, the
integration limits are suppressed for ease of notation if possible and given if necessary. The eigenvalues \(\lambda_1, \lambda_2\) of the matrix (30) are positive and with algebraic multiplicities

\[
\lambda_1 = 1/2 \left( r_+ + \frac{\Delta + 4Mr}{r_+} + \sqrt{\left( r_+ - \frac{\Delta + 4Mr}{r_+} \right)^2 + 4a^2 \sin^2(\theta)} \right) > 0
\]

\[
\lambda_2 = 1/2 \left( r_+ + \frac{\Delta + 4Mr}{r_+} - \sqrt{\left( r_+ + \frac{\Delta + 4Mr}{r_+} \right)^2 - 4(\Sigma + 2Mr)} \right) > 0,
\]

showing that (29) is indeed positive-definite.

**Theorem III.1.** The Dirac Hamiltonian (17) is symmetric with respect to the scalar product (29).

**Proof.** In order to establish the symmetry, namely that

\[
(\psi^\dagger | H | \phi')_\mathfrak{N} = (H | \psi^\dagger | \phi')_\mathfrak{N},
\]

we begin by splitting the potential \(\mathcal{V}\) given in (21) into mass-independent and mass-dependent parts

\[
\mathcal{V} = \mathcal{V}_0 + \mathcal{V}_{\mu^*},
\]

where

\[
\mathcal{V}_0 := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} \mathcal{B}_1 & 0_{c_2} \\ 0_{c_2} & \mathcal{B}_4 \end{pmatrix} \quad \text{and} \quad \mathcal{V}_{\mu^*} := -\frac{1}{\Sigma + 2Mr} \begin{pmatrix} 0_{c_2} & \mathcal{B}_2 \\ \mathcal{B}_3 & 0_{c_2} \end{pmatrix}
\]

with the \((2 \times 2)\)-blocks \(\mathcal{B}_k\) defined in (22), therefore obtaining anti-self-adjoint and self-adjoint matrices

\[
\Gamma^\dagger \mathcal{V}_0 = -\mathcal{V}_0^\dagger \Gamma^\dagger \quad \text{and} \quad \Gamma^\dagger \mathcal{V}_{\mu^*} = \mathcal{V}_{\mu^*}^\dagger \Gamma^\dagger \quad (31)
\]

with \(\Gamma^\dagger = \Gamma^{\dagger \dagger}\) defined in (30). This leads to

\[
(\psi^\dagger | H | \phi')_\mathfrak{N} = \iiint \psi^\dagger \Gamma^\dagger H \phi' \sin(\theta) \, d\phi \, d\theta \, dr
\]

\[
= \iiint \psi^\dagger \Gamma^\dagger \alpha^j \partial_j (\phi') \sin(\theta) \, d\phi \, d\theta \, dr + \iiint \psi^\dagger \Gamma^\dagger \mathcal{V}_0 \phi' \sin(\theta) \, d\phi \, d\theta \, dr
\]

\[
+ \iiint \psi^\dagger \Gamma^\dagger \mathcal{V}_{\mu^*} \phi' \sin(\theta) \, d\phi \, d\theta \, dr.
\]

Integration by parts of the first triple integral and substitution of the relations (31) in the remaining two triple integrals yields

\[
(\psi^\dagger | H | \phi')_\mathfrak{N} = -\iiint \partial_j (\psi^\dagger \Gamma^\dagger \alpha^j \sin(\theta)) \phi' \, d\phi \, d\theta \, dr - \iiint \psi^\dagger \mathcal{V}_0^\dagger \Gamma^\dagger \phi' \sin(\theta) \, d\phi \, d\theta \, dr
\]

\[
+ \iiint \psi^\dagger \mathcal{V}_{\mu^*}^\dagger \Gamma^\dagger \phi' \sin(\theta) \, d\phi \, d\theta \, dr
\]

\[
= -\iiint \partial_j (\psi^\dagger) \Gamma^\dagger \alpha^j \phi' \sin(\theta) \, d\phi \, d\theta \, dr
\]

\[
- \iiint \psi^\dagger \left[ \partial_j (\Gamma^\dagger) \alpha^j + \Gamma^\dagger \partial_j (\alpha^j) + \Gamma^\dagger \alpha^j \cot(\theta) \right] \phi' \sin(\theta) \, d\phi \, d\theta \, dr
\]

\[
- \iiint (\mathcal{V}_0 \psi^\dagger) \Gamma^\dagger \phi' \sin(\theta) \, d\phi \, d\theta \, dr + \iiint (\mathcal{V}_{\mu^*} \psi^\dagger) \Gamma^\dagger \phi' \sin(\theta) \, d\phi \, d\theta \, dr.
\]
We remark that in the integration by parts, the angular derivatives do not give rise to boundary terms because the two-dimensional submanifold $S^2$ is compact without boundary. The radial boundary terms on the other hand vanish due to Dirichlet-type boundary conditions imposed on the Dirac spinors. More precisely, the radial boundary terms read

$$\int_{S^2} \psi^* \Gamma \alpha^* \phi' \sin (\theta) \, d\phi \, d\theta \bigg|_{r_1}^{r_2}. $$

Direct computation of the matrix $\Gamma \alpha^*$ gives

$$\Gamma \alpha^* = i \text{diag} \left( -\frac{\Delta}{r^+}, r^+, r^+, -\frac{\Delta}{r^+} \right)$$

and hence the radial boundary terms become

$$ir_+ \int_{S^2} \left( -\frac{\Delta}{r^+} \psi_i \phi_i' + \psi_j \phi_j' - \frac{\Delta}{r^+} \psi_i \phi_i' \right) \sin (\theta) \, d\phi \, d\theta \bigg|_{r_1}^{r_2}. $$

In order for this expression to vanish, we impose the radial Dirichlet-type boundary condition

$$\sum_{i=1}^{2} (-1)^i \left( -\frac{\Delta}{r^+} \psi_i \phi_i' - \frac{\Delta}{r^+} \psi_i \phi_i' \right) \bigg|_{r_i} = 0. \quad (33)$$

As in the next section only Dirac spinors with $\text{supp} \phi' = [r_0, \infty) \times S^2$ and thus radial boundary conditions at a specific time-like inner boundary $r = r_0 < r_-$ and at infinity, for which the terms in (33) vanish separately, are considered, we may bring the radial Dirichlet-type boundary condition at $r = r_0$ into a more suitable form as follows. (Note that at infinity, we merely require proper decay of the Dirac spinors.) By means of the spin scalar product (24) and the relation $\mathcal{F} \gamma^r = i \Gamma \alpha^r / \Sigma$, we may represent (33) as

$$\langle \psi' | \gamma^r \phi' \rangle |(\tau) \times \{ r_0 \} \times S^2 = 0. \quad (34)$$

Introducing $n$ as the unit normal to the hypersurfaces $\{ \tau \} \times S^2$, we can write (34) in the form

$$\langle \psi' | \# \phi' \rangle |(\tau) \times \{ r_0 \} \times S^2 = 0 \quad \leftrightarrow \quad (\# - i) \psi' |(\tau) \times \{ r_0 \} \times S^2 = 0, \quad (35)$$

where the slash again denotes Clifford multiplication. With $\# = -1_{\Sigma^+}$, the above implication can be easily verified by

$$\langle \psi' | \# \phi' \rangle |(\tau) \times \{ r_0 \} \times S^2 = -\langle \psi' | \#^2 \phi' \rangle |(\tau) \times \{ r_0 \} \times S^2 = -\langle \psi' | \phi' \rangle |(\tau) \times \{ r_0 \} \times S^2 = -\langle -i \psi' | -i \phi' \rangle |(\tau) \times \{ r_0 \} \times S^2 = 0.$$

We point out that the mixed terms in the last line cancel each other out. Next, the explicit calculation of the square bracket in (32) yields

$$\partial_j (\Gamma \alpha^j) + \Gamma \partial_j (\alpha^j) + \Gamma \alpha^\theta \cot (\theta) = -2 r_0 \partial^j \Gamma^\tau.$$

Moreover, all three matrix products $\Gamma \alpha^j, j \in \{ \tau, \theta, \phi \}$, are anti-self-adjoint

$$\Gamma^\tau \alpha^j = -\alpha^j \Gamma^\tau = -\alpha^j \Gamma^\tau.$$
Therefore, we immediately find that

\[
(\psi'|H\phi')|_{\mathcal{N}_r} = \int \int \int \partial_j (\psi|_{\mathcal{N}_r}) \alpha^j I^\tau \phi' \sin (\theta) \, d\phi \, d\theta \, dr + 2 \int \int \int (\gamma_0 \psi')^\dagger I^\tau \phi' \sin (\theta) \, d\phi \, d\theta \, dr 
\]

\[
- \int \int \int (\gamma_0 \psi')^\dagger I^\tau \phi' \sin (\theta) \, d\phi \, d\theta \, dr + \int \int \int (\gamma_{\mu} \psi')^\dagger I^\tau \phi' \sin (\theta) \, d\phi \, d\theta \, dr 
\]

\[
= \int \int \int (\alpha^j \partial_j \psi')^\dagger I^\tau \phi' \sin (\theta) \, d\phi \, d\theta \, dr + \int \int \int (\gamma_0 \psi')^\dagger I^\tau \phi' \sin (\theta) \, d\phi \, d\theta \, dr 
\]

\[
+ \int \int \int (\gamma_{\mu} \psi')^\dagger I^\tau \phi' \sin (\theta) \, d\phi \, d\theta \, dr 
\]

\[
= \int \int \int (H \psi')^\dagger I^\tau \phi' \sin (\theta) \, d\phi \, d\theta \, dr = (H \psi'|\phi')|_{\mathcal{N}_r} 
\]

\[\square\]

IV. SPECTRAL ANALYSIS OF THE DIRAC HAMILTONIAN AND INTEGRAL REPRESENTATION OF THE PROPAGATOR

In this section, we show that the Dirac Hamiltonian in the non-extreme Kerr geometry in horizon-penetrating Eddington–Finkelstein-type coordinates is essentially self-adjoint. Moreover, we construct a specific integral representation of the Dirac propagator, which yields the dynamics of Dirac particles outside, across, and inside the event horizon, up to the Cauchy horizon. In more detail, we derive an explicit expression for the spectral measure \(dE_\omega\) arising in the formal spectral decomposition of the propagator

\[
U(\tau) = e^{-i\tau H} = \int_{\mathbb{R}} e^{-i\omega \tau} dE_\omega. \tag{36}
\]

Since this spectral decomposition only applies to self-adjoint operators, we first require a unique self-adjoint extension of the Dirac Hamiltonian. This problem involves the technical difficulty that the Dirac Hamiltonian in Kerr geometry is not elliptic at the event and Cauchy horizons, which can be easily seen from the evaluation of the determinant of the principal symbol

\[
P(x, \xi) = -i (\gamma^\tau)^{-1} \gamma^j \xi_j,
\]

where \(x = (\tau, r, \theta, \phi)\) and \(\xi \in T^*_x \mathcal{M}\). A short computation yields

\[
det (P(x, \xi)) = \left( \frac{g^{ij} \xi_i \xi_j}{g^{\tau \tau}} \right)^2.
\]

The Hamiltonian fails to be elliptic if the determinant vanishes, that is, if \(g^{ij} \xi_i \xi_j = 0\) for \(\xi\) being non-zero. With

\[
g_{\mathcal{N}_r} = -\Sigma^{-1} (\Delta \partial_r \otimes \partial_r + a [\partial_r \otimes \partial_\phi + \partial_\phi \otimes \partial_r] + \partial_\theta \otimes \partial_\theta + \csc^2(\theta) \partial_\phi \otimes \partial_\phi),
\]

one can verify that this holds true only at the event and Cauchy horizons. Hence, in the construction of the extension, we employ the method for non-uniformly elliptic boundary value problems presented in [13], combining results from the theory of symmetric hyperbolic systems with elliptic methods [2, 6, 14, 18]. Then, by means of Stone’s formula, we express the spectral measure in terms of the resolvent of the Dirac Hamiltonian, which we compute using the Green’s matrix of the radial ODE system (10)
FIG. 1: Carter–Penrose diagram of the region \( \mathcal{M} \) of Kerr geometry with constant-\( \tau \) hypersurfaces \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) cut-off at the boundary \( \partial \mathcal{M} \). Cauchy data is propagated in \( \tau \)-direction along the constant-\( \tau \) hypersurfaces. A radial Dirichlet-type boundary condition imposed on \( \partial \mathcal{M} \) leads to a reflection of the Dirac particles, which ensures unitarity, without affecting their dynamics outside the Cauchy horizon.

and the angular projector corresponding to the angular ODE system (11) that arise in Chandrasekhar’s separation of variables. In the following, in order to formulate the Cauchy problem for the Dirac equation in Hamiltonian form and the domain of definition of the self-adjoint extension of the Hamiltonian, the geometrical and functional-analytic settings are stated.

Let \((\mathfrak{M},g)\) be the non-extreme Kerr geometry with the metric (2) in horizon-penetrating Eddington–Finkelstein-type coordinates (3). We consider the subset \( \mathcal{M} := \{ \tau, r > r_0, \theta, \phi \} \subset \mathfrak{M} \) with \( r_0 < r_- \). Furthermore, we introduce the time-like inner boundary \( \partial \mathcal{M} := \{ \tau, r = r_0, \theta, \phi \} \) and the family of space-like hypersurfaces \( \mathcal{N} = (\mathcal{N}_\tau)_\tau \in \mathbb{R} \) with \( \mathcal{N}_\tau := \{ \tau = \text{const.}, r > r_0, \theta, \phi \} \) and boundaries \( \partial \mathcal{N}_\tau := \partial \mathcal{M} \cap \mathcal{N}_\tau \simeq S^2 \) (see Figure 1). These hypersurfaces constitute a foliation of \( \mathcal{M} \) along the time direction characterized by the parameter \( \tau \). At \( \partial \mathcal{M} \), we assume the radial Dirichlet-type boundary condition (35), which has the effect that Dirac particles are reflected away from the singularity, to obtain a unitary time evolution without changing the dynamics outside the Cauchy horizon. Also near \( \partial \mathcal{M} \), we have a locally time-like Killing vector field \( K \), which is a linear combination of the Killing fields \( \partial_\tau \) and \( \partial_\phi \) describing the stationarity and axisymmetry of Kerr geometry. It is given by \( K = \partial_\tau + b \partial_\phi \), where \( b = b(r_0) \in \mathbb{R} \setminus \{0\} \) is a constant [15], and corresponds to the Killing field \( K = \partial_t \) of [13] that is represented by a coordinate system describing an observer who is co-moving along the flow lines of the Killing field. Using the results of the previous section, we can set up a Hilbert space \( (\mathcal{H}, \langle \cdot | \cdot \rangle) \) with \( \mathcal{H} = L^2(\mathcal{N}_\tau, S\mathcal{M}) \), where \( S\mathcal{M} \) denotes the spin bundle of \( \mathcal{M} \), and the scalar product (29). In this setting, we find a unique global solution of the Cauchy problem for the massive Dirac equation in the class \( C^\infty_{sc} (\mathcal{M}, S\mathcal{M}) \).

**Lemma IV.1.** The Cauchy problem for the massive Dirac equation in the non-extreme Kerr geometry in horizon-penetrating advanced Eddington–Finkelstein-type coordinates

\[
\begin{aligned}
\partial_\tau \psi' &= H \psi' \\
\psi'_\mid_{\tau=0} &= \phi'_0 
\end{aligned}
\]

with the radial Dirichlet-type boundary condition at \( \partial \mathcal{M} \) given by

\[(\mathbf{i} - \partial_\mathcal{M}) \psi'_0 = 0,\]

where \( \mathbf{i} \) is the imaginary unit.
where the initial data is smooth, compactly supported outside, across, and inside the event horizon, up to the Cauchy horizon, and is compatible with the boundary condition, i.e.,

$$(\gamma - i)(H^p\psi_0) = 0 \quad \forall \ p \in \mathbb{N}_0,$$

has a unique global solution in the class of smooth wave functions with spatially compact support

$$\{\psi' \in C^\infty_0(\mathcal{M}, S; \mathcal{M}) \mid (\gamma - i)(H^p\psi')_{|\partial \mathcal{M}} = 0 \quad \forall \ p \in \mathbb{N}_0\}.$$ Evaluating this solution at subsequent times $\tau$ and $\tau'$ gives rise to a unique unitary propagator

$$U^{\tau', \tau} : C^\infty_0(\mathcal{N}_\tau, S; \mathcal{M}) \to C^\infty_0(\mathcal{N}_{\tau'}, S; \mathcal{M}).$$

This lemma is essential as a technical tool in the construction of the self-adjoint extension of the Dirac Hamiltonian. Its proof is shown in detail in [13].

### A. Self-adjoint Extension of the Dirac Hamiltonian

We introduce a theorem for the existence of a unique self-adjoint extension of the Dirac Hamiltonian $H$, which is defined by (17) and (18)-(22), in a specific domain of the Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{N}_\tau})$.

**Theorem IV.2.** The massive Dirac Hamiltonian $H$ in the non-extreme Kerr geometry in horizon-penetrating advanced Eddington–Finkelstein-type coordinates with domain of definition

$$\mathcal{D}(H) = \{\psi' \in C^\infty_0(\mathcal{N}_\tau, S; \mathcal{M}) \mid (\gamma - i)(H^p\psi')_{|\partial \mathcal{M}} = 0 \quad \forall \ p \in \mathbb{N}_0\} \subset \mathcal{H}$$

is essentially self-adjoint.

For the proof, we again refer to [13], in which the construction of a self-adjoint extension of the Dirac Hamiltonian is discussed for a more general class of non-uniformly elliptic mixed initial/boundary value problems for spacetimes with dimension $d \geq 3$.

### B. Resolvent of the Dirac Hamiltonian and Integral Representation of the Propagator

In order to construct an explicit expression for the spectral measure in the spectral decomposition of the Dirac propagator (36), we use Stone’s formula and thus the resolvent of the essentially self-adjoint Hamiltonian $H$ defined in Theorem IV.2. As the spectrum of the Hamiltonian $\sigma(H) \subseteq \mathbb{R}$ is on the real line, the resolvent $(H - \omega)\gamma - 1 \in L(\mathcal{H})$ exists for all $\omega_c \in \mathbb{C} \setminus \mathbb{R}$ with real part $\Re(\omega_c) = \omega \in \sigma(H)$, and is given uniquely.

**Theorem IV.3.** The massive Dirac propagator in the non-extreme Kerr geometry in horizon-penetrating advanced Eddington–Finkelstein-type coordinates can be expressed via the integral representation

$$\psi'(\tau, r, \theta, \phi) = \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} e^{-ik\phi} \lim_{\epsilon \searrow 0} \int_{\mathbb{R}} e^{-i\epsilon \tau} \left[ (H_k - \omega - i\epsilon)\gamma - 1 - (H_k - \omega + i\epsilon)\gamma - 1 \right](r, \theta; r', \theta') \psi_{0,k}(r', \theta') d\omega,$$

where the resolvents $(H_k - \omega \pm i\epsilon)\gamma - 1$ for fixed $k$-modes are unique and of the form

$$(H_k - \omega \pm i\epsilon)\gamma - 1(r, \theta; r', \theta') \psi'(r', \theta') = -\sum_{l \in \mathbb{Z}} \lim_{r_0 \nearrow r} \int_{r_0}^{r_1} \mathcal{G}(r; \epsilon_k, \omega) \frac{0_{k\omega}^2}{\mathcal{G}(r; \epsilon_k, \omega)} d\epsilon \psi_{l,0}(r, \theta) \psi'(r', \theta') d(cos (\theta'))$$

where

$$(l, \epsilon) = \left\{ \int_{-1}^{1} Q_1(\theta; \theta') \psi'(r', \theta') d(cos (\theta')) \right\} dr'$$
with $G(r; r')_{l,k,ω,±iω}$ being the two-dimensional Green’s matrix of the radial first-order ODE system (10), $Q_l$ is the spectral projector onto a finite-dimensional invariant eigenspace of the angular operator (12) corresponding to the spectral parameter $\xi_l$, and

\[
\mathcal{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{C}^{-1}(r', \theta) = -\begin{pmatrix} (i(\Delta(r') + 4Mr')) & r_+a \sin (\theta) & 0 & 0 \\ 0 & -ir_+ & a \sin (\theta) & a \sin (\theta) \\ 0 & 0 & r_+a \sin (\theta) & i(\Delta(r') + 4Mr') \\ a \sin (\theta) & 0 & -ir_+ & 0 \end{pmatrix}.
\]

Proof. We compute the resolvent $(H_k - \omega_i)^{-1}$, where $\Re(\omega_i) = \omega$ and $\Im(\omega_i) \in (-\epsilon, \epsilon)$ with $\epsilon$ sufficiently small so that it can be considered as a slightly non-self-adjoint perturbation, by first employing the mode ansatz

\[
\psi' = e^{-i(\omega_{i}r+k\phi)} \psi'_{\text{sep}}, \quad \text{for which} \quad \psi'_{\text{sep}} = \begin{pmatrix} \mathcal{R}_+(r) \mathcal{F}_+(\theta) \\ \mathcal{R}_-(r) \mathcal{F}_-(\theta) \\ -\mathcal{R}_-(r) \mathcal{F}_+(\theta) \\ -\mathcal{R}_+(r) \mathcal{F}_-(\theta) \end{pmatrix}
\]

in Eq. (16). This yields

\[
(H_k - \omega_i) \psi'_{\text{sep}} = 0,
\]

where $H_k := \alpha^r \partial_r + \alpha^\theta \partial_\theta - ik \alpha^\phi + \mathcal{Y}$ with $\alpha^j, j \in \{ r, \theta, \phi \}$, and $\mathcal{Y}$ given in (18)-(21). Second, we introduce the spectral projector

\[
(Q_l \mathcal{F}_\pm)(\theta) = \int Q_l(\theta; \theta') \mathcal{F}_\pm(\theta') d(\cos(\theta'))
\]

onto the finite-dimensional invariant eigenspace of the angular operator (12) with spectral parameter $\xi_l, l \in \mathbb{Z}$, which ensures that the functions $\mathcal{F}_\pm$ are solutions of the angular ODE system (11). Then, by means of the family $(Q_l)_{l \in \mathbb{Z}}$, we may express the angular operator as

\[
O_\theta = \sum_{l \in \mathbb{Z}} \xi_l Q_l.
\]

Note that the spectral projector is idempotent, that is, $Q_l^2 = Q_l$, and that the family is complete, i.e., $\sum_{l \in \mathbb{Z}} Q_l = \mathbb{1}$. Applying the latter relation to Eq. (37) and substituting (11), we obtain

\[
-\sum_{l \in \mathbb{Z}} Q_l \Sigma^{-1} M(\theta; r, \theta)_{l,k,\omega,±i\omega} \psi'_{\text{sep}} = 0,
\]

where

\[
M(\theta; r, \theta)_{l,k,\omega,±i\omega} := \begin{pmatrix} i O_{k,\omega,±i\omega} & a \sin (\theta) U_{\omega,±i\omega} & ir_+ S_l & a \sin (\theta) \overline{S}_l \\ a \sin (\theta) r_+ O_{k,\omega,±i\omega} & -i(\Delta + 4Mr) U_{\omega,±i\omega} & a \sin (\theta) S_l & -i(\Delta + 4Mr) \overline{S}_l \\ -i(\Delta + 4Mr) \overline{S}_l & a \sin (\theta) S_l & -i(\Delta + 4Mr) U_{\omega,±i\omega} & a \sin (\theta) r_+ O_{k,\omega,±i\omega} \\ a \sin (\theta) \overline{S}_l & ir_+ S_l & a \sin (\theta) U_{\omega,±i\omega} & i O_{k,\omega,±i\omega} \end{pmatrix}
\]

with the differential operators $O_{k,\omega,±i\omega}, U_{\omega,±i\omega}$, and the function $S_l$ defined by

\[
O_{k,\omega,±i\omega} := \Delta \partial_r + r - M - i\omega_i(\Delta + 4Mr) - 2iak,
\]

\[
U_{\omega,±i\omega} := r_+(\partial_r + i\omega_i),
\]

\[
S_l := \xi_l + \sqrt{2} \mu_* r.
\]
In the following, we show that the computation of the resolvent of the operator \( M(\partial_r; r, \theta)_{l,k,\omega} \) in (38) can be reduced to determining the two-dimensional Green’s matrix of the radial ODE system (10). Writing Eq. (38) in the factorized form

\[
- \sum_{l \in \mathbb{Z}} Q_l (\Sigma + 2Mr)^{-1} B(r, \theta) R(\partial_r; r)_{l,k,\omega} \psi_l^{\prime} = 0 ,
\]

where the matrix \( B(r, \theta) \) and the matrix-valued operator \( R(\partial_r; r)_{l,k,\omega} \) read

\[
B(r, \theta) := \begin{pmatrix}
\frac{i}{\sin(\theta)} & \frac{a \sin(\theta)}{r^+} & 0 & 0 \\
\frac{a \sin(\theta)}{r^+} & \frac{a \sin(\theta)}{r^+} & 0 & 0 \\
0 & 0 & -\frac{i(\Delta + 4Mr)}{r^+} & 0 \\
0 & 0 & 0 & \frac{a \sin(\theta)}{r^+}
\end{pmatrix}
\]

and

\[
R(\partial_r; r)_{l,k,\omega} := \begin{pmatrix}
O_{k,\omega} & 0 & r_+ S_l & 0 \\
0 & U_{\omega} & 0 & S_l \\
S_l & 0 & U_{\omega} & 0 \\
0 & r_+ S_l & 0 & O_{k,\omega}
\end{pmatrix}
\]

we can easily bring it into the block diagonal form

\[
\sum_{l \in \mathbb{Z}} Q_l (H_{l,k,\omega} - \omega^2) \psi_l^{\prime} = - \sum_{l \in \mathbb{Z}} Q_l \mathcal{E}(r, \theta) \begin{pmatrix}
\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega}^{0 \mathbb{C}^2} \\
0_{\mathbb{C}^2}
\end{pmatrix} \mathcal{E}^{-1} \psi_l^{\prime} = 0
\]

with \( \mathcal{E}(r, \theta) := (\Sigma + 2Mr)^{-1} B(r, \theta) \mathcal{E} \), the matrix-valued operator

\[
\begin{pmatrix}
\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega}^{0 \mathbb{C}^2} \\
0_{\mathbb{C}^2}
\end{pmatrix} = \mathcal{E}^{-1} R(\partial_r; r)_{l,k,\omega} \mathcal{E},
\]

in which

\[
\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega} := \begin{pmatrix}
O_{k,\omega} & r_+ S_l \\
S_l & U_{\omega}
\end{pmatrix}
\]

and the constant matrix

\[
\mathcal{E} := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

As the equation

\[
\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega} \mathcal{R}(r) = 0
\]

for \( \mathcal{R} = (\mathcal{R}_+, \mathcal{R}_-)^T \) is equivalent to the radial first-order ODE system (10), it is obvious from (39) that the key quantity in the determination of the resolvent \((H_\omega - \omega)^{-1}\) is the radial Green’s matrix, i.e., the solution \(G(r; r')_{l,k,\omega}\) of the distributional equation

\[
\mathcal{R}^{2 \times 2}(\partial_r; r)_{l,k,\omega} G(r; r')_{l,k,\omega} = \delta(r - r') 1_{\mathbb{C}^2}.
\]

For the derivation of this Green’s matrix, we have to consider weak solutions of the Dirac equation, that is, solutions \( \psi' \) that satisfy

\[
(\psi' | \langle H - \omega \rangle \phi')_{\mathcal{N}} = 0 \ \forall \ \phi' \in \mathcal{D}(H),
\]

because of the singular behavior of the Dirac spinors at the event and Cauchy horizons (cf. Lemma II.2). Assuming that \( \text{supp} \phi' \subset (r_+ - \epsilon, r_+ + \epsilon) \times S^2 \), where \( \epsilon > 0 \) defines small neighborhoods around these horizons, we infer from (33) in the limit \( \epsilon \searrow 0 \) that in order to have a weak solution, the components \( \psi_1' \) and \( \psi_2' \) can be chosen arbitrarily, whereas the components \( \psi_3' \) and \( \psi_4' \) have to be continuous. Moreover, we introduce the functions \( \Phi_1(r; r') = (\Phi_{1,1}, \Phi_{1,2})^T(r; r') \) and \( \Phi_2(r; r') = (\Phi_{2,1}, \Phi_{2,2})^T(r; r') \) that
• are linearly independent weak solutions of the homogeneous equation $\mathcal{R}^{2\times 2}(\partial_r; r)_{l,k,\omega_1} \Phi(r; r') = 0$ for $r \neq r'$,

• have jump discontinuities at $r = r'$,

• satisfy the Dirichlet-type boundary condition (35) at $r = r_0$,

• and are square integrable, that is, $\|\Phi_{1/2}(r; r')\|_2^2 = \int_{r_0}^{\infty} \|\Phi_{1/2}(r; r')\|^2 \, dr < \infty$.

In the appendix, these are specified and their existence is shown. It turns out that the $r'$-dependence can be chosen in such a way that it is solely contained in Heaviside step functions $\Theta$. For clarity, in what follows, we explicitly state these Heaviside step functions, making it possible to consider $\Phi_1$ and $\Phi_2$ as functions only of the variable $r$. Aside from the above-mentioned properties, they have the specific asymptotics

$\Phi_{1/2}(r) \simeq \frac{c_{1,\infty}}{\sqrt{\Delta}} \exp \{i\phi_+(r_*)\} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ \quad $\text{for } r \to \infty \quad \text{and} \quad \begin{cases} \Im(\omega) < 0 & \text{if } |\omega_i|^2 \geq 2\mu_*^2 \\
\Re(\omega_i) > 0 & \text{if } |\omega_i|^2 < 2\mu_*^2 \end{cases}$

$\Phi_{1/2}(r) \simeq c_{2,\infty} \exp \{-i\phi_-(r_*)\} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ \quad $\text{for } r \to \infty \quad \text{and} \quad \begin{cases} \Im(\omega) > 0 & \text{if } |\omega_i|^2 \geq 2\mu_*^2 \\
\Re(\omega_i) < 0 & \text{if } |\omega_i|^2 < 2\mu_*^2 \end{cases}$

$\Phi_{1/2}(r) \simeq \left( \frac{c_{1,r_*}}{\sqrt{|\Delta|}} \exp \left( 2i \left[ \omega_\epsilon + k\Omega_{Kerr}^{(\pm)} \right] r_* \right) \right)_{c_{2,r_\pm}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ \quad $\text{for } r \to r_+ \quad \text{and} \quad \begin{cases} \Im(\omega_i) < 0 & \text{if } r_0 < r' < r_- \\
\Re(\omega_i) > 0 & \text{if } r_0 > r' > r_+ \end{cases}$

$\Phi_{1/2}(r) \simeq c_{2,r_*} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ \quad $\text{for } r \to r_- \quad \text{and} \quad \begin{cases} \Im(\omega_i) > 0 & \text{if } r_0 < r' < r_- \\
\Re(\omega_i) < 0 & \text{if } r_0 > r' > r_+ \end{cases}$

according to Lemma II.1 and Lemma II.2. Then, if $|\omega_i|^2 \geq 2\mu_*^2$ and $\Im(\omega_i) < 0$ or $|\omega_i|^2 < 2\mu_*^2$ and $\Re(\omega_i) > 0$, we use the ansatz

$G(r; r')_{l,k,\omega_1} = \begin{cases} \Theta(r - r')\Phi_1(r)P_1(r') + \Theta(r' - r)\Phi_2(r)P_2(r') & \text{for } r_+ < r' \quad \text{and} \quad r_0 < r' < r_- \\
\Theta(r - r')\Phi_1(r)P_1(r') + \Theta(r' - r)\Phi_2(r)P_2(r') & \text{for } r_- < r' < r_+ \end{cases}$

whereas if $|\omega_i|^2 \geq 2\mu_*^2$ and $\Im(\omega_i) > 0$ or $|\omega_i|^2 < 2\mu_*^2$ and $\Re(\omega_i) < 0$, we employ the ansatz

$G(r; r')_{l,k,\omega_1} = \begin{cases} \Theta(r - r')\Phi_1(r)P_1(r') + \Theta(r' - r)\Phi_2(r)P_2(r') & \text{for } r_+ < r' \quad \text{and} \quad r_0 < r' < r_- \\
\Theta(r - r')\Phi_1(r)P_1(r') + \Theta(r' - r)\Phi_2(r)P_2(r') & \text{for } r_- < r' < r_+ \end{cases}$

in which $P_1$ and $P_2$ are unknowns yet to be determined. Applying the radial operator $\mathcal{R}^{2\times 2}(\partial_r; r)_{l,k,\omega_1}$ to (42) and (43), we obtain

$\mathcal{R}^{2\times 2}(\partial_r; r)_{l,k,\omega_1} G(r; r')_{l,k,\omega_1} = \left( \begin{array}{cc} \Delta & 0 \\
0 & r_+^{-1} \end{array} \right) \delta(r - r') \begin{pmatrix} \Phi_1(r')P_1(r') \pm \Phi_2(r')P_2(r') \\
\pm \Phi_1(r')P_1(r') - \Phi_2(r')P_2(r') \end{pmatrix}$ \quad \text{for (42)}

\begin{pmatrix} \Phi_1(r')P_1(r') \pm \Phi_2(r')P_2(r') \\
\pm \Phi_1(r')P_1(r') - \Phi_2(r')P_2(r') \end{pmatrix}$ \quad \text{for (43)}.

Identifying this with (41) yields

$$\left( \begin{array}{cc} \Delta^{-1} & 0 \\
0 & r_+^{-1} \end{array} \right) \begin{pmatrix} \Phi_1(r')P_1(r') \pm \Phi_2(r')P_2(r') \\
\pm \Phi_1(r')P_1(r') - \Phi_2(r')P_2(r') \end{pmatrix}$$ \quad \text{for (42)}

$$\left( \begin{array}{cc} \Delta^{-1} & 0 \\
0 & r_+^{-1} \end{array} \right) \begin{pmatrix} \Phi_1(r')P_1(r') \pm \Phi_2(r')P_2(r') \\
\pm \Phi_1(r')P_1(r') - \Phi_2(r')P_2(r') \end{pmatrix}$$ \quad \text{for (43)}.

The solutions $P_{1/2}(r')$ of these systems read for ansatz (42)

$$P_{1,1}(r') = \frac{\Phi_{2,2}(r')}{\Delta(r')W(r')}, \quad P_{1,2}(r') = -\frac{\Phi_{2,1}(r')}{r_+ W(r')}, \quad P_{2,1}(r') = \pm \frac{\Phi_{1,2}(r')}{\Delta(r')W(r')}, \quad P_{2,2}(r') = \mp \frac{\Phi_{1,1}(r')}{r_+ W(r')}$$
and for ansatz (43)
\[ P_{1,1}(r') = \pm \frac{\Phi_{1,2}(r')}{\Delta(r')} W(r') . \]
\[ P_{1,2}(r') = \mp \frac{\Phi_{2,1}(r')}{\Delta(r')} W(r') . \]
\[ P_{2,1}(r') = \Phi_{1,2}(r') \Delta(r') W(r') . \]
\[ P_{2,2}(r') = -\frac{\Phi_{1,1}(r')}{\Delta(r')} W(r') . \]
where \( W(r') = W(\Phi_1, \Phi_2) := \Phi_1(r') \Phi_{2,2}(r') - \Phi_1(r') \Phi_{2,1}(r') \) is the Wronskian. Their respective substitution into (42) and (43) leads, on the one hand, for the cases \( r_+ < r' \) and \( r_0 < r' < r_- \) for both (42) and (43) to the Green’s matrix
\[
G(r; r')_{l,k,\omega} = \frac{1}{W(r')} \begin{pmatrix}
\Theta(r-r') & \frac{\Phi_{1,1}(r)\Phi_{2,2}(r') - \Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \Theta(r-r') \\
\frac{\Phi_{1,2}(r)\Phi_{2,2}(r') - \Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \frac{\Phi_{1,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \Theta(r-r') \\
\frac{\Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \frac{\Phi_{2,2}(r)\Phi_{1,2}(r')}{\Delta(r')} & \Theta(r-r')
\end{pmatrix}
\]
and, on the other hand, for the case \( r_- < r' < r_+ \) to the Green’s matrices
\[
G(r; r')_{l,k,\omega} = \frac{1}{W(r')} \begin{pmatrix}
\frac{\Phi_{1,1}(r)\Phi_{2,2}(r') - \Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \frac{\Phi_{1,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \Theta(r-r') \\
\frac{\Phi_{1,2}(r)\Phi_{2,2}(r') - \Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \frac{\Phi_{2,2}(r)\Phi_{1,2}(r')}{\Delta(r')} & \Theta(r-r') \\
\frac{\Phi_{2,1}(r)\Phi_{1,2}(r')}{\Delta(r')} & \frac{\Phi_{2,2}(r)\Phi_{1,2}(r')}{\Delta(r')} & \Theta(r-r')
\end{pmatrix}
\times \begin{cases}
\Theta(r-r') & \text{for (42)} \\
-\Theta(r'-r) & \text{for (43)}
\end{cases}
\]
From the block-diagonalized representation of the operator \( H_{k - \omega} \) given in (39), we can directly read off the resolvent as
\[
(H_{k - \omega})^{-1}(r, \theta; r', \theta') \psi(r', \theta') = -\sum_{l, m \in \mathbb{Z}} \lim_{r_0, r_0' \to r_-} \int_{r_0}^{\infty} \delta(r - r') \int_{-1}^{1} Q_l(\theta; \theta') \psi(r', \theta') \left( R^{2 \times 2}(\theta; \theta') \right)_{l,k,\omega} \begin{pmatrix} 0_{C^2} \\ G(r; r')_{l,k,\omega} \end{pmatrix} d \cos(\theta') dr'
\]
\[
\times \begin{cases}
\Theta(r-r') & \text{for (42)} \\
-\Theta(r'-r) & \text{for (43)}
\end{cases}
\]
\[
\times \begin{cases}
\Theta(r-r') & \text{for (42)} \\
-\Theta(r'-r) & \text{for (43)}
\end{cases}
\]
\[
\times d(\cos(\theta')) dr''
\]
Rearranging the integrands and employing (41) leads to
\[
(H_k - \omega_i) (H_k - \omega_i)^{-1} \Psi = \sum_{l,m \in \mathbb{Z}} \lim_{r_0 \to r_-} \lim_{r_0 \to r_+} \int_{-1}^{1} \int_{-1}^{1} \int_{r_0}^{\infty} \int_{r_0}^{\infty} \delta(r - r'') \delta(r' - r'') Q_l(\theta; \theta'') \delta(r'' - r'') \delta(r'' - r') \, d\Psi,
\]
\[
\times \delta^{-1}(r', \theta'') Q_m(\theta''; \theta') \Psi(r', \theta') \, dr' \, dr'' \, d(\cos(\theta')) \, d(\cos(\theta'')).
\]
Solving the two radial integrals in the limit \(r_0 \to r_-\), we immediately find
\[
(H_k - \omega_i) (H_k - \omega_i)^{-1} \Psi = \sum_{l,m \in \mathbb{Z}} \int_{-1}^{1} \int_{-1}^{1} \delta(\cos(\theta) - \cos(\theta'')) Q_l(\theta; \theta'') \Psi(r, \theta') \, d(\cos(\theta')) \, d(\cos(\theta'')).
\] (45)

Since the spectral projectors are idempotent, their integral kernels satisfy
\[
\sum_{m \in \mathbb{Z}} Q_l(\theta; \theta'') Q_m(\theta''; \theta') = \sum_{m \in \mathbb{Z}} \delta_{m} \delta(\cos(\theta) - \cos(\theta'')) Q_m(\theta''; \theta') = \delta(\cos(\theta) - \cos(\theta'')) Q_l(\theta; \theta').
\]
Substituting this into (45) yields
\[
(H_k - \omega_i) (H_k - \omega_i)^{-1} \Psi = \sum_{l \in \mathbb{Z}} \int_{-1}^{1} \int_{-1}^{1} \delta(\cos(\theta) - \cos(\theta'')) Q_l(\theta; \theta') \Psi(r, \theta') \, d(\cos(\theta'))
\]
\[
= \sum_{l \in \mathbb{Z}} \int_{-1}^{1} Q_l(\theta; \theta') \Psi(r, \theta') \, d(\cos(\theta')) = \sum_{l \in \mathbb{Z}} Q_l \Psi = \Psi,
\]
where in the second line, we first performed the \(\theta''\)-integration and subsequently applied the completeness relation for the family of spectral projectors.

Having established the explicit form of the resolvent \((H_k - \omega_i)^{-1}\) in (44), we continue deriving the integral representation of the Dirac propagator. Using the spectral projector \(E_I := \chi_I(H)\) of the Hamiltonian onto the interval \(I\), we can express the propagator as
\[
\psi' = e^{-irH} \psi'_0 = e^{-irH} \lim_{a \to \infty} E_{(-a,a)} \psi'_0 = \frac{1}{2} \lim_{a \to \infty} e^{-irH} \left( E_{[-a,a]} + E_{(-a,a)} \right) \psi'_0.
\]
Employing Stone’s formula for unbounded self-adjoint operators [16]
\[
\frac{1}{2} \left( E_{[p,q]} + E_{(p,q)} \right) = \text{s-lim}_{s \to 0} \frac{1}{2\pi i} \int_{s}^{s^*} \left( (H - \omega - i\epsilon)^{-1} - (H - \omega + i\epsilon)^{-1} \right) d\omega,
\]
where s-lim denotes the strong limit of operators and \(p, q \in \mathbb{R}\) with \(p < q\), we finally obtain
\[
\psi'(\tau, \theta, \phi) = \frac{1}{2\pi i} \sum_{k \in \mathbb{Z}} e^{-ik\phi} \lim_{\epsilon \to 0} \int_{\mathbb{R}} e^{-i\omega\tau} \left( (H - \omega - i\epsilon)^{-1} - (H - \omega + i\epsilon)^{-1} \right) (r, \theta; r', \theta') \psi_{0,\lambda}(r', \theta') \, d\omega.
\]

We note in passing that this integral representation may be further simplified evaluating the \(c\)-limit and the difference of the two resolvents, which leads to the omission of the Heaviside functions. Below, we discuss as an example the case \(|\omega_i|^2 \geq 2\mu^2\) with \(r_+ < r'\). The remaining cases are treated similarly. Introducing the functions (cf. the appendix)
\[
\chi_1(r) := \lim_{\epsilon \to 0} \Phi^{(\infty)}(r), \quad \chi_2(r) := \lim_{\epsilon \to 0} \Phi^{(\infty)}(r),
\]
we write the \(c\)-limits of the radial functions \(\Phi_1\) and \(\Phi_2\) as
\[
\lim_{\epsilon \to 0} \Phi_1 = \chi_1 \quad \text{and} \quad \lim_{\epsilon \to 0} \Phi_2 = \alpha \chi_1 + \beta \chi_2 \quad \text{for} \quad \Im(\omega_i) > 0,
\]
\[
\lim_{\epsilon \to 0} \Phi_1 = \chi_2 \quad \text{and} \quad \lim_{\epsilon \to 0} \Phi_2 = \gamma \chi_1 + \delta \chi_2 \quad \text{for} \quad \Im(\omega_i) < 0,
\]
where \( \alpha, \beta, \gamma, \) and \( \delta \) are constants. The corresponding Wronskian yields

\[
\lim_{\epsilon \to 0} W(\Phi_1, \Phi_2) = \begin{cases} 
\beta W(\chi_1, \chi_2) & \text{for } \Re(\omega_\epsilon) > 0 \\
-\gamma W(\chi_1, \chi_2) & \text{for } \Re(\omega_\epsilon) < 0.
\end{cases}
\]

Then, we obtain for the non-zero \( 2 \times 2 \) blocks of \( \lim_{\epsilon \to 0} [(H - \omega - i \epsilon)^{-1} - (H - \omega + i \epsilon)^{-1}] \) in (44)

\[
\lim_{\epsilon \to 0} [G(r; r')_{l,k,\omega+i\epsilon} - G(r; r')_{l,k,\omega-i\epsilon}] = \frac{1}{W(\chi_1, \chi_2)} \sum_{k,l=1}^2 T_{k,l} \begin{pmatrix}
\chi_{k,1}(r)\chi_{l,2}(r') & \chi_{k,1}(r)\chi_{l,1}(r') \\
\chi_{k,2}(r)\chi_{l,2}(r') & \chi_{k,2}(r)\chi_{l,1}(r')
\end{pmatrix}
\]

with the coefficients

\[
T_{1,1} = \frac{\alpha}{\beta}, \quad T_{1,2} = T_{2,1} = 1, \quad \text{and} \quad T_{2,2} = \frac{\delta}{\gamma}.
\]

### Appendix: Functions for the Construction of the Green’s Matrix of the Radial ODE System

We specify the functions \( \Phi_1(r, r') \) and \( \Phi_2(r, r') \) that are used for the construction of the radial Green’s matrix that solves Eq. (41). Since their explicit forms are not known, we describe them in terms of asymptotic expansions. To this end, we define, on the one hand, functions with suitable decay at infinity

\[
\Phi^{(\infty)}(r) := \frac{c_{1,\infty}}{\sqrt{\Delta}} \exp \left( i\phi_+(r_\ast) \right) \begin{pmatrix} 1 & \Omega(x) \frac{1}{r_\ast} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } \begin{cases} \Im(\omega_\ast) > 0 \quad \text{if } |\omega_\ast|^2 \leq 2\mu^2 \\
\Re(\omega_\ast) > 0 \quad \text{if } |\omega_\ast|^2 > 2\mu^2
\end{cases}
\]

\[
\Phi^{(\infty)}(r) := c_{2,\infty} \exp \left( -i\phi_-(r_\ast) \right) \begin{pmatrix} 1 & \Omega(x) \frac{1}{r_\ast} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } \begin{cases} \Im(\omega_\ast) > 0 \quad \text{if } |\omega_\ast|^2 \leq 2\mu^2 \\
\Re(\omega_\ast) < 0 \quad \text{if } |\omega_\ast|^2 < 2\mu^2
\end{cases}
\]

and, on the other hand, functions that are square integrable at the event and Cauchy horizons obeying the proper asymptotics

\[
\Phi^{(+)}(r) := \frac{c_{1,\infty}}{\sqrt{\Delta}} \exp \left( 2i \left[ \omega_\ast + k\Omega^{(+)}(r_\ast) \right] r_\ast \right) \frac{1}{1 + \Omega(x) r_\ast} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } \Im(\omega_\ast) > 0
\]

\[
\Phi^{(+)}(r) := c_{2,\infty} \begin{pmatrix} 1 & \Omega(x) \frac{1}{r_\ast} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } \Im(\omega_\ast) > 0
\]

\[
\Phi^{(-)}(r) := c_{1,\infty} \begin{pmatrix} 1 & \Omega(x) \frac{1}{r_\ast} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{for } \Im(\omega_\ast) < 0
\]

\[
\Phi^{(-)}(r) := \frac{c_{2,\infty}}{\sqrt{\Delta}} \exp \left( 2i \left[ \omega_\ast + k\Omega^{(-)}(r_\ast) \right] r_\ast \right) \frac{1}{1 + \Omega(x) r_\ast} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{for } \Im(\omega_\ast) < 0.
\]

The quantities \( c_{1,2,\infty} \) and \( c_{1,2,r,\pm} \) are constants. To clarify our notation, we point out that the superscripts \((\infty), (+), (-)\) designate the asymptotic expansions at infinity, the event horizon, and the Cauchy horizon, respectively. We furthermore remark that the existence of ODE solutions with these asymptotics follows from the construction of Jost solutions (for which one rewrites the radial first-order system (40) as a second-order scalar equation and proceeds as in [1] or [12]) and that our asymptotic expansions ensure that the solutions are square integrable near the horizons. For example,
as the Regge–Wheeler coordinate $r_*$ tends to minus infinity at the event horizon, the exponential factor $\exp \left( 2i [\omega_* + k \Omega_{\text{Kerr}}] r_* \right)$ tends to zero if $\Im(\omega_*) < 0$. However, this exponential factor would not be square integrable if $\Im(\omega_*) > 0$. Last, we introduce a function that satisfies the Dirichlet-type boundary condition at $r = r_0$

$$\Phi_{\beta,\theta}(r) := c_0 \Phi^{(2)}_{\beta,\theta}(r) \left( -r_0/\sqrt{\Delta r} \right),$$

where $c_0$ is a constant and $\Phi^{(2)}_{\beta,\theta}$ denotes the second component. Then, in case $|\omega_*|^2 \geq 2\mu_*^2$ and $\Im(\omega_*) < 0$ or $|\omega_*|^2 < 2\mu_*^2$, $\Im(\omega_*) < 0$, and $\Re(\omega_*) > 0$, the radial functions $\Phi_1$ and $\Phi_2$ can be expressed as

$$\Phi_1(r, r_0 < r' < r_0) = \Theta(r - r') \Phi^{(\infty)}(r)$$

$$\Phi_2(r, r_0 < r' < r_0) = \Theta(r' - r) \Theta(r - r_0) \Phi^{(+)}(r)$$

$$\Phi_1(r, r_0 < r' < r_0) = \Theta(r - r') \Theta(r_0 - r) \Phi^{(-)}(r) + \Theta(r - r_0) \Phi^{(\infty)}(r)$$

$$\Phi_2(r, r_0 < r' < r_0) = \Theta(r' - r) \Phi_{\beta,\theta}(r),$$

whereas in case $|\omega_*|^2 \geq 2\mu_*^2$ and $\Im(\omega_*) > 0$ or $|\omega_*|^2 < 2\mu_*^2$, $\Im(\omega_*) > 0$, and $\Re(\omega_*) < 0$, they can be chosen as

$$\Phi_1(r, r_0 < r' < r_0) = \Theta(r - r') \Phi^{(\infty)}(r)$$

$$\Phi_2(r, r_0 < r' < r_0) = \Theta(r' - r) \Theta(r_0 - r) \Phi^{(+)}(r) + \Theta(r - r_0) \Phi_{\beta,\theta}(r)$$

$$\Phi_1(r, r_0 < r' < r_0) = \Theta(r' - r) \Theta(r - r_0) \Phi^{(-)}(r)$$

$$\Phi_2(r, r_0 < r' < r_0) = \Theta(r' - r) \Phi_{\beta,\theta}(r).$$

For the remaining cases $|\omega_*|^2 < 2\mu_*^2$, $\Im(\omega_*) > 0$, and $\Re(\omega_*) > 0$ or $|\omega_*|^2 < 2\mu_*^2$, $\Im(\omega_*) < 0$, and $\Re(\omega_*) < 0$, we also obtain (46) or (47), respectively, but with the functions $\Phi^{(\infty)}(r)$ and $\Phi^{(\infty)}(r)$ interchanged. A case-by-case analysis shows that these solutions are uniquely determined by the conditions and asymptotics listed in the proof of Theorem IV.3.

Acknowledgments

The authors are grateful to Niky Kamran for useful discussions and comments. This work was supported by the DFG research grant “Dirac Waves in the Kerr Geometry: Integral Representations, Mass Oscillation Property and the Hawking Effect.”

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