Resonance, symmetry, and bifurcation of periodic orbits in perturbed Rayleigh–Bénard convection

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Received 15 March 2022; revised 14 October 2022
Accepted for publication 29 November 2022
Published 6 January 2023

Recommended by Dr Hinke M Osinga

Abstract
This paper investigates the global structures of periodic orbits that appear in Rayleigh–Bénard convection, which is modelled by a two-dimensional perturbed Hamiltonian model, by focusing upon resonance, symmetry and bifurcation of the periodic orbits. First, we show the global structures of periodic orbits in the extended phase space by numerically detecting the associated periodic points on the Poincaré section. Then, we illustrate how resonant periodic orbits appear and specifically clarify that there exist some symmetric properties of such resonant periodic orbits which are projected on the phase space; namely, the period \( m \) and the winding number \( n \) become odd when an \( m \)-periodic orbit is symmetric with respect to the horizontal and vertical centre lines of a cell. Furthermore, the global structures of bifurcations of periodic orbits are depicted when the amplitude \( \varepsilon \) of the perturbation is varied, since in experiments the amplitude of the oscillation of the convection gradually increases when the Rayleigh number is raised.

Keywords: Rayleigh–Bénard convection, perturbed Hamiltonian system, periodic orbit, resonance, symmetry, bifurcation

Mathematics Subject Classification numbers: 37J06, 37J20, 37J40, 37J46

(Some figures may appear in colour only in the online journal)
Nonlinearity 36 (2023) 955

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1. Introduction

1.1. Background

1.1.1. Rayleigh–Bénard convection. In the fields of meteorology, oceanography, and chemical engineering, much concern has been focused on the prediction and control of the spread of oil and chemical spills as well as the measurement of air pollutant concentrations. In particular, the natural convection in a horizontal fluid layer with heated bottom and cooled top planes called Rayleigh–Bénard convection has been well known as a typical phenomenon of such fluid transport that exists in nature (see Chandrasekhar 1961) and it is crucial to study the global fluid transport associated with natural convection. So far, the fluid transport in perturbed Rayleigh–Bénard convection has been actively investigated; when the temperature difference of the two planes is relatively small, or, Rayleigh number \(Ra\) is relatively small, multiple convection rolls with steady velocity fields may appear in parallel in the layer. When the flow in the direction of the roll axes is negligible, it may be considered as a two-dimensional steady convection from that direction. On the other hand, it was clarified by Clever and Busse (1974) and Bolton et al (1986) that the parallel convection rolls may start to wave slightly by the even oscillatory instability when \(Ra\) is set slightly above a critical number \(Ra_t\) by increasing the temperature difference. Since the wave propagates along the roll axes almost periodically, the two-dimensional velocity field observed from the direction of the roll axes is perturbed. One of the important remarks is that although the velocity field of such oscillatory convection seems to be stable in Eulerian description, some fluid particles can be transported chaotically in Lagrangian description; see Aref (1984), Ottino (1989), Wiggins and Ottino (2004), and Grigoriev (2012). Furthermore, increasing \(Ra\) by raising the temperature difference, the amplitude of the oscillation enlarges and the fluid transport become very complicated. Since very rich dynamics such as Lagrangian chaotic fluid transport can be observed in the perturbed Rayleigh–Bénard convection, the fluid transport in this convection has been actively studied by theoretical, numerical, and experimental methods.

1.1.2. Previous works on fluid transport in perturbed Rayleigh–Bénard convection. Amongst such past researches on the study of chaotic fluid transport in perturbed Rayleigh–Bénard convection, Solomon and Gollub (1988) has been known as a pioneer work, where the diffusion of impurities was studied by optical absorption techniques and also where the convection was modelled as a two-dimensional perturbed Hamiltonian system following experimental results.

It was clarified that fluid seems to be transported similarly as one-dimensional diffusion, where local effective diffusion constant grows linearly with the local amplitude of perturbation. In
addition, it was numerically shown that the basic mechanism of fluid transport is chaotic advection around cell boundaries rather than molecular diffusion. Gollub and Solomon (1989) also made some numerical analysis to show some evidence of chaotic transport in the perturbed Hamiltonian model in the sense of being sensitive to the initial condition. In addition, Ouchi et al (1991) numerically studied the diffusion constant of the model when the amplitude of perturbation is varied and Ouchi and Mori (1992) showed that some anomalous diffusion is caused by the accelerator-mode islands of KAM tori around cell boundaries at specific range of amplitude. Furthermore, Inoue and Hirata (1998, 2000) investigated the mixing patterns of another perturbed Hamiltonian model with different perturbations by analysing Poincaré maps and the degree of mixing. In particular, Inoue and Hirata (2000) showed how the chaotic structures vary when the amplitude or the frequency of the oscillation is changed and also detected some elliptic periodic points for one particular condition of parameters.

Some studies were made by focusing on invariant or coherent structures in perturbed Rayleigh–Bénard convection to understand the mechanisms or global structures of fluid transport. Camassa and Wiggins (1991) introduced the perturbed Hamiltonian model of Rayleigh–Bénard convection proposed by Solomon and Gollub (1988) and investigated the stable and unstable manifolds theoretically and numerically in order to clarify the mechanism of chaotic transport by the so-called lobe dynamics. Briefly speaking, a lobe is a region enclosed by stable and unstable manifolds. If the stable and unstable manifolds of a dynamical system entangle with each other very complicately as homoclinic tangles, fluid inside lobes may be transported chaotically by horseshoe maps; see Wiggins (1992) and Rom-Kedar and Wiggins (1990). Camassa and Wiggins (1991) showed that the volume of fluid that crosses cell boundaries increases linearly with the amplitude of the perturbation when the amplitude is small in accordance with the results in Solomon and Gollub (1988). In addition, they discussed the effect of molecular diffusion in perturbed Rayleigh–Bénard convection as well. Malhotra et al (1998) studied the patchiness of the perturbed Hamiltonian model of Rayleigh–Bénard convection with stable and unstable manifolds, where a patch is a region that has a considerably different average velocity compared to the surrounding region. Further, Shadden et al (2005) and Lekien et al (2007) numerically clarified the Lagrangian coherent structures (LCSs) of the two-dimensional Hamiltonian model and the extended three-dimensional model of perturbed Rayleigh–Bénard convection, where LCS corresponds to the stable and unstable manifolds of non-autonomous systems; see also Haller (2011, 2015) for the definitions of LCS.

1.1.3. Previous works on fluid transport in other time-dependent convective flow. Let us briefly review some researches on fluid transport in other time-dependent convective flow. Some of them used magnetohydrodynamic forcing or electrical techniques to create convection similar to perturbed Rayleigh–Bénard convection, since it is easier to control the flow. Solomon et al (1996, 1998) experimentally detected some lobes by observing the transport of impurities in a fluid layer with a chain of horizontal vortices that are oscillated by magnetohydrodynamic forcing. Solomon and Mezić (2003) explored the uniform mixing of weakly three-dimensional and weakly time-periodic vortex flow by using magnetohydrodynamic techniques in experiments and by numerically analysing a low-dimensional model. Hidaka et al (2015) and Yamanaka et al (2018) experimentally analysed the Lagrangian chaos and diffusion in weakly turbulent electroconvection of nematic liquid crystals. On the other hand, some studies were dedicated on the investigation of the fluid transport in developed or turbulent Rayleigh–Bénard convection. For instance, a dual-porosity model was developed by Matveev (2016) to describe the transport of impurities in developed Rayleigh–Bénard convection in two situations with chain of rolls and hexagonal lattice, where the transport was
studied for a wide range of $Ra$. Cheng et al (2022) and Hang et al (2022) studied the LCSs in turbulent Rayleigh–Bénard convection by computing the flow by direct numerical simulations in order to analyse material and heat transfer. Schneide et al (2018) and Schneide et al (2022) also explored the Lagrangian transport in turbulent Rayleigh–Bénard convection to detect the large-scale patterns called the turbulent superstructures and to clarify the mechanism of heat transfer respectively.

1.1.4. Periodic orbits and bifurcations in convective flow. As is shown in the above, most of the past works regarding the fluid transport in convective flow have been focused on the mechanisms and structures of chaotic transport. For the sake of understanding the global structures of fluid transport it is quite essential to investigate periodic transport as well, since periodic orbits are one of the most simple orbits. Furthermore, it is necessary to clarify how the transport becomes complicated when $Ra$ is increased, in other words, how the periodic orbits bifurcate when the amplitude of the perturbation is increased.

Now we shall briefly review some previous works analysing periodic orbits and bifurcations in convective flow in Lagrangian description. Simó et al (2010) studied the structures of regular and chaotic regions in three particular three-dimensional steady Rayleigh–Bénard convection with different $Ra$ by computing Poincaré maps and periodic points as well as critical points and Lyapunov exponents, and showed how the structures of fluid transport vary with $Ra$. Chabreyrie et al (2011) proposed a strategy to create sufficient chaotic mixing in two-dimensional oscillatory electro-osmotic convection by focusing on the variation of the stability of periodic pathlines associated with non-mixing regions when the amplitude or frequency of the perturbation is changed. In addition, Lagrangian chaos in two-dimensional oscillatory convection in a differentially heated cavity was studied by Oteski et al (2014). In that study, they detected not only elliptic periodic points but also hyperbolic ones and their stable and unstable manifolds as well to clarify the dependence of mixing structures on $Ra$. Contreras et al (2017) also explored the Lagrangian transport in three-dimensional steady convection in a differentially heated cavity and investigated how the chaotic structures vary with Grashof number $Gr$ by analysing periodic points and Poincaré maps.

On the other hand, the transitions, in other words bifurcations, of thermal convection from steady to oscillatory and chaotic flow have been explored in Eulerian description in many studies by experiments, CFD simulations, and analyses on low-dimensional models; see for instance Gollub and Benson (1980), Mukutmoni and Yang (1993a, 1993b), and Paul et al (2011). In particular, Net and Umbría (2017) and Umbría and Net (2019) analysed the stabilities and bifurcations as well as symmetric properties of thermal flow in a tall rectangular domain with heated sidewalls when $Ra$ is varied, where bifurcation diagrams are numerically computed by detecting periodic orbits, which may correspond to time-periodic flow in Eulerian description. In addition, the resonance of quasi-periodic Rayleigh–Bénard convection was studied by Ecke and Kevrekidis (1988) and Ecke et al (1991) in Eulerian description. Furthermore, Guo et al (2022), Yadav (2022), Yadav et al (2022) have respectively studied thermal non-equilibrium effects, double diffusive convective motion, and analogies with granular particles in relation with the stability of Rayleigh–Bénard convection in Eulerian description.

As is shown in the above, there are some studies where periodic orbits in convective flow have been investigated in Lagrangian description. However, it goes without saying that they have not been analysed enough in the sense of global investigation. For example, some only detect elliptic periodic points, or others investigate periodic points only at some fixed parameters. In particular, the global structures of $m$-periodic orbits ($m \in \mathbb{N}$) in perturbed
Rayleigh–Bénard convection have not been studied adequately in the perspective of resonance and symmetry. Furthermore, the bifurcations of periodic orbits with $Ra$ as a control parameter have not been sufficiently explored in Lagrangian description, since most of the previous researches have studied the bifurcations of Rayleigh–Bénard convection in Eulerian description. It is crucial to investigate the resonances and symmetries of $m$-periodic orbits, since they are one of the important topological properties of periodic orbits. Needless to say, it is also important to clarify the global structure of bifurcations of $m$-periodic orbits to clarify how the transport becomes complicated when $Ra$ is increased.

1.2. Contributions and organisation of this paper

1.2.1. Contributions of this paper. The main goal of this paper is to clarify the global structures of periodic orbits in the perspective of symmetry and resonance as well as the global bifurcations of periodic orbits in the two-dimensional Hamiltonian model of perturbed Rayleigh–Bénard convection. To do this, we first introduce an autonomous Hamiltonian model in the extended phase space from the non-autonomous perturbed Hamiltonian model. Then, we numerically detect the elliptic and hyperbolic periodic points on the Poincaré section and investigate the structures of the associated periodic orbits in the extended phase space of the autonomous system. In particular, we consider the projection of the periodic orbits onto the original phase space to investigate the resonances and symmetries of the orbits. Lastly, we show the global structures of $\varepsilon$-parameter bifurcations of periodic orbits, where $\varepsilon$ denotes the amplitude of the perturbation. For numerical computations of bifurcation diagrams in such flow continuation methods have been used in other studies; see for instance Umbriá and Net (2019). However, in this paper, we obtain the $\varepsilon$-parameter bifurcation diagram of periodic points for $0 < \varepsilon \leq 0.5$ by numerically detecting periodic points for each amplitude of perturbation $\varepsilon = 0.001, 0.002, \ldots, 0.5$ independently by Newton’s method. This is because the Hamiltonian $H(x,z,t) = H_0(x,z) + \varepsilon H_1(x,z,t)$ of the model in this study is given, and thus the velocity field $dx/dt = -\partial H/\partial z$, $dz/dt = \partial H/\partial x$ for each $\varepsilon$ can be obtained explicitly. For the same reason, bifurcation points in the $\varepsilon$-parameter bifurcation diagrams are detected by Newton’s method following Tsumoto et al (2012); see also Kuznetsov (2004).

1.2.2. Organisation of this paper. The organisation of this paper consists of the following sections: In section 2, the model of the two-dimensional perturbed Hamiltonian system for the oscillatory Rayleigh–Bénard convection is described together with symmetric properties. Then, numerical analysis is made by the Poincaré map to detect the periodic points on the Poincaré section $\Sigma_0$ and also to clarify the structures of periodic orbits and KAM tori in the extended phase space. In section 3, symmetries of resonant orbits are demonstrated by projecting the $m$-periodic orbits onto the phase space and a theorem for the resonant orbits is given that the period $m$ and the winding number $n$ are odd numbers, if the projection is symmetric with respect to the horizontal and vertical centre lines of a cell. In section 4, the $\varepsilon$-parameter bifurcations of periodic points are illustrated and, in particular, the classification of the bifurcations is made into fold or flip bifurcations according to the multipliers of the periodic points. Finally in section 5, the conclusions of this paper are described.

2. Poincaré map and structures of periodic orbits

In order to investigate the two-dimensional Rayleigh–Bénard convection whose velocity field is perturbed by the even oscillatory instability, we employ the two-dimensional perturbed
Hamiltonian system, which was originally developed by Solomon and Gollub (1988), see also Camassa and Wiggins (1991). Then, we investigate the global structures of periodic orbits that appear in the perturbed Hamiltonian system by Poincaré maps.

2.1. Model of perturbed Rayleigh–Bénard convection

2.1.1. Hamiltonian system of steady Rayleigh–Bénard convection. By assuming the stress-free boundary condition, it follows from the Navier–Stokes equations with the Boussinesq approximation that two-dimensional steady Rayleigh–Bénard convection can be modelled by a Hamiltonian system as

\[
\begin{align*}
\frac{dx}{dt} &= -\frac{\partial H(x, z)}{\partial z} = \frac{A\pi}{k} \sin(kx) \cos(\pi z), \\
\frac{dz}{dt} &= \frac{\partial H(x, z)}{\partial x} = \frac{A}{k} \cos(kx) \sin(\pi z),
\end{align*}
\]  

(2.1)

where \( H(x, z) \) is a Hamiltonian, given by the stream function

\[
H(x, z) = \frac{A}{k} \sin(kx) \sin(\pi z);
\]

see Chandrasekhar (1961). In the above, \( x \in \mathbb{R} \) and \( z \in U = [0, 1] \subset \mathbb{R} \) are the horizontal and vertical coordinates respectively, and hence we define the phase space \( M = \mathbb{R} \times U \). Further, \( A \) denotes the amplitude of the velocity in \( z \) direction and \( k \) is the wave number of the cell pattern in \( x \) direction. In this Hamiltonian system, we have the hyperbolic equilibrium points \( p_{i,0}^+ = (x, 0, z_i^+) \) as

\[
(x_{i,0}, z_i^+) = \left( \frac{i\pi}{k}, z_i^+ \right), \quad (i = 0, \pm 1, \pm 2, \ldots),
\]

where \( z_i^- = 0 \) and \( z_i^+ = 1 \), and it is noticed that there exist heteroclinic connections between \( p_{i,0}^+ \) and \( p_{i,0}^- \) along the roll boundaries.

2.1.2. Hamiltonian model of perturbed Rayleigh–Bénard convection. Now we consider the case in which a time-periodic term \( \varepsilon \cos(\omega t) \) is added to \( x \) in the Hamiltonian \( H_0(x, z) \) for the steady Rayleigh–Bénard convection. Then, it follows that a time-dependent Hamiltonian on the extended phase space \( M \times \mathbb{R} \) is given in coordinates \((x, z, t) \in M \times \mathbb{R}\) as

\[
H(x, z, t) := H_0(x, z) + \varepsilon H_1(x, z, t),
\]

where Taylor expansion is applied to the sinusoidal term as

\[
H_1(x, z, t) = A \cos(\omega t) \cos(kx) \sin(\pi z).
\]

Note that \( A \) denotes some given constant of the magnitude. Then, we get a non-autonomous Hamiltonian vector field \( X_H : M \times \mathbb{R} \to TM \), locally given by

\[
\begin{align*}
\frac{dx}{dt} &= -\frac{\partial H(x, z, t)}{\partial z} = -\frac{\partial H_0(x, z)}{\partial z} - \varepsilon \frac{\partial H_1(x, z, t)}{\partial z}, \\
\frac{dz}{dt} &= \frac{\partial H(x, z, t)}{\partial x} = \frac{\partial H_0(x, z)}{\partial x} + \varepsilon \frac{\partial H_1(x, z, t)}{\partial x}. 
\end{align*}
\]  

(2.2)
In the above, \( \varepsilon \in \mathbb{R}^+ \) is a given magnitude of the perturbation and the perturbed terms \( \frac{\partial H_1(x, z, t)}{\partial z} \) and \( \frac{\partial H_1(x, z, t)}{\partial x} \) are respectively given by the periodic function:

\[
\frac{\partial H_1(x, z, t)}{\partial z} = A\pi \cos(\omega t) \cos(kx) \cos(\pi z),
\]

\[
\frac{\partial H_1(x, z, t)}{\partial x} = -Ak\cos(\omega t) \sin(kx) \sin(\pi z).
\]

Figure 1 illustrates the schematic figure of this model, where the wavy dashed lines indicate the perturbed cell boundaries.

2.1.3. Symmetric properties of the model. Recall from Camassa and Wiggins (1991) that the perturbed Hamiltonian system (2.2) is invariant under the following coordinate transformations:

(a) \( x \mapsto x + \frac{2a\pi}{k}, \quad z \mapsto -z + 1, \quad t \mapsto -t + bT, \)

(b) \( x \mapsto x + \frac{(2a + 1)\pi}{k}, \quad z \mapsto z, \quad t \mapsto -t + bT, \)

(c) \( x \mapsto x + \frac{2a\pi}{k}, \quad z \mapsto z, \quad t \mapsto t + bT, \)

where \( T(= 2\pi/\omega) \) is the period of the perturbation and \( a, b \in \mathbb{Z} \). Note that there exists two more symmetries associated with the following transformation:

(d) \( x \mapsto -x + \frac{(2a + 1)\pi}{k}, \quad z \mapsto z, \quad t \mapsto -t + \left( b + \frac{1}{2} \right)T, \)

(e) \( x \mapsto -x + \frac{(2a + 1)\pi}{k}, \quad z \mapsto -z + 1, \quad t \mapsto t + \left( b + \frac{1}{2} \right)T, \)

which will be used for investigating symmetric properties of periodic orbits in section 3.

2.2. Structures of periodic points

In this subsection, we numerically compute Poincaré maps to detect periodic points on a Poincaré section. To do this, we transform the perturbed Hamiltonian system that is a
non-autonomous system on $M = \mathbb{R} \times U$ with local coordinates $(x, z)$ into the setting of an autonomous system by introducing the extended phase space $M = M \times S^1$ with local coordinates $(x, z, \theta)$ and then define a Poincaré map $P_{\epsilon}^{\theta_0} : \Sigma_{\theta_0} \rightarrow \Sigma_{\theta_0}$, where $\Sigma_{\theta_0} \subset M$ is a chosen Poincaré section.

### 2.2.1. Autonomous Hamiltonian systems

By introducing an angle variable $\theta := \omega t + \theta_0 \in S^1$, where $\theta_0 \in [0, 2\pi)$, the Hamiltonian can be rewritten on the extended phase space $M = M \times S^1$ as

$$H(x, z, \theta) := H_0(x, z) + \epsilon H_1(x, z, \theta),$$

where

$$H_1(x, z, \theta) = A \cos(\theta - \theta_0) \cos(kx) \cos(\pi z).$$

Then, the vector field for the non-autonomous Hamiltonian system given in (2.2) can be transformed into the form of the vector field $X_{H} : M \rightarrow TM$ of the autonomous system on the extended phase space $M$, which can be described by using local coordinates $(x, z, \theta)$:

$$\frac{dx}{dt} = -\frac{\partial H(x, z, \theta)}{\partial z} = -\frac{\partial H_0(x, z)}{\partial z} - \frac{\partial H_1(x, z, \theta)}{\partial z},$$

$$\frac{dz}{dt} = \frac{\partial H(x, z, \theta)}{\partial x} = \frac{\partial H_0(x, z)}{\partial x} + \frac{\partial H_1(x, z, \theta)}{\partial x},$$

$$\frac{d\theta}{dt} = \omega,$$

and the perturbed terms are given as

$$\frac{\partial H_1(x, z, \theta)}{\partial z} = A \pi \cos(\theta - \theta_0) \cos(kx) \cos(\pi z),$$

$$\frac{\partial H_1(x, z, \theta)}{\partial x} = -A k \cos(\theta - \theta_0) \sin(kx) \sin(\pi z).$$

### 2.2.2. Poincaré map of the model

Associated with the autonomous Hamiltonian system described in (2.2), let $\phi^\epsilon : \mathbb{R} \times M \rightarrow M; (t, x, z, \theta) \mapsto \phi^\epsilon_t(x, z, \theta)$ be the flow, where $t \in \mathbb{R}$ indicates a time interval. Hence, for some fixed $t$ and given $\epsilon$, we define the diffeomorphism on the extended phase space as

$$\phi^\epsilon_t : M \rightarrow M; (x, z, \theta) \mapsto \phi^\epsilon_t(x, z, \theta).$$

Let $(x(t), z(t), \theta(t))$ be an integral curve of the Hamiltonian system in (2.3). For some fixed $\theta_0$, the angle variable $\theta(t)$ may be written as a periodic function with period $T = 2\pi/\omega$ such that $\theta(t) = \theta_0 + \omega t = \theta_0 + 2\pi t/T$. For each discrete time $t = kT, k \in \mathbb{Z}$, we can identify $\theta$ with $\theta_0 + 2\pi k$ and the equivalent class $[\theta]$ of $S^1$ is given by $[\theta] := \{\theta \in S^1 | \theta = \theta_0 + 2\pi k\}$. Choose a representative $\theta_0$ for the equivalent class to define a Poincaré section $\Sigma_{\theta_0}$ by setting

$$\Sigma_{\theta_0} := \{(x, z, \theta_0) \in M/S^1 | (x, z) \in M, \theta_0 \in [\theta] \}.$$

Then, for some fixed parameter $\epsilon \in \mathbb{R}$, we define a Poincaré map $P_{\epsilon}^{\theta_0}$ on $\Sigma_{\theta_0}$ by

$$P_{\epsilon}^{\theta_0} := \phi_{\rho_{\Sigma_{\theta_0}}}^\epsilon : \Sigma_{\theta_0} \rightarrow \Sigma_{\theta_0}.$$
which is locally given by
\[
(x(kT), z(kT), \theta(kT) = \theta_0 + 2\pi k \equiv \theta_0) \\
\rightarrow (x((k + 1)T), z((k + 1)T), \theta((k + 1)T) = \theta_0 + 2\pi(k + 1) \equiv \theta_0).
\]
Note that one special choice for \(\theta_0\) may be \(\theta_0 = 0\) and then the Poincaré section \(\Sigma^{\mu_0}\) is locally isomorphic to \(M \cong M/S^1\). Hence, we note that a point on \(\Sigma^{\mu_0}\) is mapped by \(P^{\mu_0}_\varepsilon : \Sigma^{\mu_0} \rightarrow \Sigma^{\mu_0}\) to another point on \(\Sigma^{\mu_0}\) during the period \(T\).

2.2.3. Periodic points. A fixed point of the Poincaré map corresponds to a periodic orbit with period \(T\) for the flow, and an \(m\)-periodic point, which corresponds to the periodic orbit with period \(mT\) \((m \in \mathbb{Z}^+)\), namely the \(m\)-periodic orbit, is the fixed point \(x_0 \in \Sigma^{\mu_0}\) such that
\[
(P^{\mu_0}_\varepsilon)^m(x_0) = x_0 \quad \text{for} \quad m \geq 1, \quad \text{while} \quad (P^{\mu_0}_\varepsilon)^\ell(x_0) \neq x_0 \quad \text{for} \quad 1 \leq \ell \leq m - 1 \quad (m \geq 2),
\]
where
\[
(P^{\mu_0}_\varepsilon)^m = (P^{\mu_0}_\varepsilon) \circ \cdots \circ (P^{\mu_0}_\varepsilon).
\]
Since the Poincaré section \(\Sigma^{\mu_0}\) is two-dimensional, it is apparent that the Jacobian matrix of the Poincaré \(m\)-return map
\[
J_\varepsilon(x) := \frac{\partial (P^{\mu_0}_\varepsilon)^m(x)}{\partial x}
\]
have two eigenvalues. Especially, the eigenvalues of the Jacobian matrix evaluated at an periodic point is called the multipliers. Let \(\mu_1\) and \(\mu_2\) be the two multipliers of an \(m\)-periodic point \(x_0 \in \Sigma^{\mu_0}\), where \(|\mu_1| \leq |\mu_2|\). Since \(|J_\varepsilon(x_0)| = 1\), the multipliers have the product \(\mu_1\mu_2 = 1\). The \(m\)-periodic points are classified according to the conditions of the associated multipliers as follows (see Guckenheimer and Holmes 1983):

- hyperbolic: \(|\mu_1| < 1 < |\mu_2|
- elliptic: \(|\mu_i| = 1\) but \(\mu_i \neq \pm 1\) \(i = 1, 2\)
- parabolic: \(\mu_i = \pm 1\) \(i = 1, 2\)

The periodic orbits are stable when the associated periodic points are elliptic, while they are unstable when the associated ones are hyperbolic.

2.2.4. Numerical algorithm for detecting periodic points. Now we compute the image of the Poincaré map \(P^{\mu_0}_\varepsilon : \Sigma^{\mu_0} \rightarrow \Sigma^{\mu_0}\) in order to detect periodic points, each of which corresponds to a periodic orbit in \(\mathcal{M}\) through itself. First, we describe our numerical algorithm for detecting \(m\)-periodic points for some fixed amplitude \(\varepsilon\) of the perturbation. Define a map \(F_\varepsilon : \Sigma^{\mu_0} \rightarrow \mathbb{R}^2\) as
\[
F_\varepsilon(x) := x - (P^{\mu_0}_\varepsilon)^m(x), \quad \text{for} \quad x = (x, z) \in \Sigma^{\mu_0}.
\]
For detecting \(m\)-periodic points, we shall numerically compute the kernel of the map \(F_\varepsilon\) to find a solution \(x\) for \(F_\varepsilon(x) = 0\), where we employ Newton’s method as follows:
**Numerical algorithm for detecting an m-periodic point**

(a) Set \( k = 0 \) with an initial approximation \( x^{(0)} \) for the required \( m \)-periodic point.

(b) Set \( k := k + 1 \) and compute the \( k \)th approximation \( x^{(k)} \) by Newton’s method as

\[
x^{(k)} := x^{(k-1)} - \left( \frac{\partial F_\varepsilon(x^{(k-1)})}{\partial x} \right)^{-1} F_\varepsilon(x^{(k-1)}),
\]

where

\[
\frac{\partial F_\varepsilon(x^{(k-1)})}{\partial x} = I - J_\varepsilon(x^{(k-1)}).
\]

Here, \( I \) is the unit matrix and the Jacobian matrix

\[
J_\varepsilon(x^{(k-1)}) = \left[ \frac{\partial (P_\theta^0)^m(x)}{\partial x} \right]_{x=x^{(k-1)}}
\]

is numerically obtained by using the central difference scheme.

(c) If \( |F_\varepsilon(x^{(k)})| < \delta \), where the convergence radius is set to \( \delta = 10^{-10} \), then the computation ends up and the \( m \)-periodic point is to be detected as \( x = x^{(k)} \).

(d) Otherwise, return to (b) in order to iterate the computation until convergence.

**Remark 2.1.** Since the approximation value of the periodic points are unknown, we cover the Poincaré section with a small grid spacing and set each grid point as the initial condition \( x^{(0)} \). In our computation the grid spacing is set to 0.005. The Poincaré maps are computed with 7th-order Runge-Kutta method with double precision floating point, which are the same through this study.

**2.2.5. Periodic points at \( \varepsilon = 0.1 \).** Let us consider to detect the periodic points for the case \( \varepsilon = 0.1 \). For numerical computations, throughout the paper, we fix other parameters of the convection to \( A = \pi, k = \pi \) and \( T = 1/\pi \). Now we illustrate in figure 2, the image of the Poincaré section by the Poincaré map and the detected periodic points in a cell which range from \( x = 0 \) to \( x = 1 (= \pi/k) \), where the elliptic and hyperbolic periodic points with period \( m \leq 15 \) are depicted. The colour of the plots denote the period \( m \) and the symbols of plots, i.e. ● and *, indicate elliptic and hyperbolic respectively. The number of recurrences due to the Poincaré map is set to \( N = 1000 \) and the initial condition for \( \theta \) is \( \theta_0 = 0 \). Note that there is no loss of generality to investigate only one single cell, since there is a topological isomorphism among cells. The left figure in figure 2, shows an enlarged view of the squared section in the right figure of figure 2. In conjunction with symmetry, it is observed that the periodic points appear symmetrically with respect to \( z = 1/2 \), which is consistent with the symmetric property (a) of the non-autonomous system in (2.2).

**2.2.6. The periodic points and KAM curves.** It is apparent from the Poincaré map that there exists one large island in the middle of the cell which we denote by label \( I_1 \), while there are three small islands surrounding the main island \( I_1 \), each of which is respectively denoted by
Figure 2. Structure of elliptic and hyperbolic periodic points ($\varepsilon = 0.1$).

labels $I_2$, $I_3$, and $I_4$ as in figure 2. As is well known, inside the islands, there exist quasi-periodic points, while outside the islands there is a chaotic sea where points correspond to chaotic orbits. We can see that the elliptic and hyperbolic periodic points inside the islands appear alternately along the KAM curves in the perturbed Hamiltonian systems as is well known; see Guckenheimer and Holmes (1983) and Doherty and Ottino (1988).

In particular, it is observed in figure 2 that the elliptic periodic points appear at the centre of islands, which is surrounded by KAM curves. For example, the elliptic 3-periodic points exist at the centre of islands $I_2$, $I_3$, and $I_4$ in figure 2, and the elliptic 5, 7, 8, and 13-periodic points appear at the centre of the small islands in $I_1$. The relation between the elliptic periodic points and the islands will be discussed in detail in section 2.3. In contrast, it is observed that the hyperbolic periodic points appear in the chaotic regions. This is because the stable and unstable manifolds associated with the hyperbolic periodic points form complicated homoclinic tangles around them and the points in the neighbourhood are to be transported chaotically. Further, we note that some of the elliptic and hyperbolic periodic points do not appear as mentioned above, since not all of the islands and chaotic regions can be numerically detected in figure 2. Especially, the chaotic regions between KAM curves in the islands cannot be observed in details.

2.3. Structures of periodic orbits and KAM tori

As we have shown in figure 2, the elliptic periodic points appear at the centre of the islands of KAM tori. In this subsection, we investigate the structures of periodic orbits and KAM tori in the extended phase space $\mathcal{M} = M \times S^1$, which are associated with elliptic periodic points. Here, we especially focus on those associated with the elliptic 3-periodic points at the centre of islands $I_2$, $I_3$, and $I_4$ in figure 2.
2.3.1. Twisted structures of periodic orbits and KAM tori. Figure 3(a) illustrates the elliptic 3-periodic points at the centre of islands $I_2$, $I_3$, and $I_4$ on the Poincaré section $\Sigma^{\theta_0}$. Their 3-periodic orbit and the associated KAM torus in the extended phase space $\mathcal{M}$ are shown in figure 3(b) in yellow and blue respectively. The Poincaré section $\Sigma^{\theta_0}$ given in (2.5) is depicted in gray, where it is restricted to $U \times U \subset \mathcal{M}$ and where we choose $\theta_0 = 0$ for $[\theta] = \theta_0 + 2\pi k$. The intersection of the KAM torus and the Poincaré section $\Sigma^{\theta_0}$ corresponds to the KAM curve of the island, and those of the periodic orbit $\tilde{c} \in \mathcal{M}$ and $\Sigma^{\theta_0}$ corresponds to the elliptic 3-periodic points. It is apparent that the periodic orbit and the associated KAM tori for the 3-periodic points are connected with each other and thus they globally have a twisted structure. Generally, this implies that KAM tori for elliptic periodic points whose period is more than two have twisted structures in the extended phase space $\mathcal{M}$ and also that the orbit of the elliptic periodic points goes through the centre of it. Note that such KAM torus do not appear around the orbits of hyperbolic periodic points.

2.3.2. Periodic transport of islands. Since we have seen in figure 3(b) that the KAM tori for each island are connected with each other, we next investigate the images of the island regions by Poincaré map $P^\theta_{\epsilon}$. Let us denote the closed regions of island $I_i$ as $R_i \subset U \times U$ for $i = 2, 3,$ and $4$. Figure 4 shows the initial position and the image of the regions mapped by $P^\theta_{\epsilon}$. In order to easily recognise the deformation of the regions, each of them is illustrated in four colours. The elliptic 3-periodic points are indicated in yellow plots. We can see that the regions of $I_2$, $I_3$, and $I_4$ are mapped to $I_3$, $I_4$, and $I_2$ respectively in order with the 3-periodic points as

$$P^\theta_{\epsilon}(R_2) = R_3, \quad P^\theta_{\epsilon}(R_3) = R_4, \quad P^\theta_{\epsilon}(R_4) = R_2.$$ 

It follows that the region of each island is mapped to the same island after three times of Poincaré maps as

$$P^3_{\epsilon}(R_i) = R_i.$$
Figure 4. Mapping of the regions of the islands by Poincaré map.

Of course, this implies that the region $R$ of an island associated with an $m$-periodic point is mapped to the same island after $m$ times of Poincaré maps as

$$(p_{\theta_{0}}^{\varepsilon})^{m}(R) = R.$$

Furthermore, figure 4 indicates that the regions of the islands rotate around the 3-periodic points when they are mapped. It follows from the physical point of view that fluid in the region of an island is transported periodically as a sort of vortex by the Lagrangian transport as a whole, though each point is transported quasi-periodically. The KAM curve around the region, which is an invariant manifold, seems to act as a barrier and enclose the fluid inside. Notice that these vortex structures do not appear in a vortex field in Eulerian description. It seems that these structures are quite relevant with the ‘Lagrangian vortices’ or ‘Lagrangian eddies’, which are regions that are transported stably as rotating regions; see Haller and Beron-Vera (2013), Blazevski and Haller (2014), and Farazmand and Haller (2016). However, we will seek for the relevance with Lagrangian vortices in details in future works.

3. Resonances and symmetries of periodic orbits

In this section, we investigate the resonances and symmetric properties of periodic orbits which is a solution curve passing through periodic points. To do this, we consider the projection of the $m$-periodic orbits in the extended phase space $\mathcal{M} = M \times S^1$ to the original phase space $M$ and analyse the winding number $n$ of the projected orbits around the centre of a cell.

3.1. Resonances of periodic orbits

3.1.1. Periodic solutions. Let us investigate the resonance of periodic orbits by introducing a projection. Let $\tilde{c}(t) := (x(t), z(t), \theta(t))$, $t \in \mathbb{I} \subset \mathbb{R}$ be a periodic solution of the perturbed Hamiltonian system in (2.3), which is given by a curve on the extended phase space $\mathcal{M}$, and let $\pi : \mathcal{M} \to \mathcal{M}_1 : (x, z, \theta) \mapsto (x, z)$ be the natural projection. Then, from the periodic solution $\tilde{c}(t)$, the projected curve $c(t)$ can be defined on $M$ as

$$c(t) := \pi(\tilde{c}(t)) = (x(t), z(t)).$$
which can be identified with the solution curve of the non-autonomous Hamiltonian system in (2.2) on $M$. From a physical point of view, projected orbits correspond to the orbits of fluid particles in the convection.

Figure 5 illustrates the projection of the 3-periodic orbit in figure 3(b) by $\pi$ onto $M$. It follows that the projection is a closed orbit and that it goes around the centre of the cell $(x, z) = (1/2, 1/2)$ once.

### 3.1.2. Winding number of periodic orbits.

In order to analyse the number of times that a projected periodic orbit goes around the centre of a cell, let us introduce the concept of *winding number* $n$ of a projected orbit.

**Definition 3.1.** Consider an $m$-periodic orbit $\tilde{c}(t) := (x(t), z(t), \theta(t)), t \in \mathbb{I} \subset \mathbb{R}$ on the extended phase space $\mathcal{M} = M \times S^1$. Let us define a periodic curve on $M$ by $c(t) = (x(t), z(t)) := \pi(\tilde{c}(t))$. Then, the *winding number* of $c(t)$ is given by

$$n = \frac{1}{2\pi i} \int_{c(t)} \frac{dw}{w - w_c},$$

where $w = x + iz \in \mathbb{C}$ is a point on $c(t)$ and $w_c = x_c + iz_c \in \mathbb{C}$ is a point on $M$ such that $w_c \notin c(t)$. Regarding the winding number, see Flanigan (1983).

The absolute value of the winding number $n$ corresponds to the number of times that the orbit goes around the centre of a cell, while it could take both positive and negative values in general according to the direction. Namely, the winding number is positive when the orbit goes in counter-clockwise direction, while it is negative when it goes in clockwise direction. For example, the winding number of the projection of the 3-periodic orbit shown in figure 5 is $n = -1$ when $x_c = z_c = 1/2$, since the orbit goes around $(x, z) = (1/2, 1/2)$ once in clockwise direction.

### 3.1.3. Resonant periodic orbits.

Figures 6–9 illustrate some of the periodic points in figure 2 and the projection of the associated periodic curves onto $M$, where they are classified according...
to the symmetry with respect to the horizontal and vertical centre lines of the cell, namely $x = \pi/(2k)$ and $z = 1/2$. As can be seen, the projected orbits go around the centre of the cell once or several times. For example, the projection of the 7-periodic orbit in figure 6 goes around the centre of the cell three times in clockwise direction, which means that the winding number is $n = -3$ when $x_c = z_c = 1/2$. It follows that $m$-periodic orbits can be considered as resonant orbits in the sense that those with winding number $n$ go around the centre of a cell $n$ times when they are projected on $M$, while they go around $m$ times in $\theta$ direction in the extended phase space $M$.

### 3.1.4. Resonance condition of periodic orbits

Let us define the resonance condition of an $m$-periodic orbit with winding number $n$ as $|n/m|$. The resonance conditions of the detected periodic orbits are indicated besides each orbit in figures 6–9, where $x_c = z_c = 1/2$. Figures 6–9

---

Figure 6. Orbits symmetric with respect to $x = \pi/(2k)$ and $z = 1/2$.

Figure 7. Orbits symmetric only with respect to $x = \pi/(2k)$. 
shows that there are many kinds of periodic orbits with different resonance conditions. It follows that the resonance conditions of the orbits in the middle of the cell tend to be larger than that of those in the outer area, since the absolute value of the winding number of those in the middle tend to be larger. It is also observed that some of the \(m\)-periodic orbits have different resonance conditions even when their periods are the same. For example, we can see two different kinds of 7-periodic orbits with \(|n/m| = 3/7\) and \(2/7\), and also 11-periodic orbits with \(|n/m| = 5/11\) and \(3/11\).

Furthermore, it is found in our numerical computation that some of the orbits have the same resonance conditions even when their periods are different. For example, we illustrate some of the orbits of which resonance condition is \(|n/m| = 1/3\) in figure 10. As can be seen, the winding number of the 9, 12, and 15-periodic orbits are \(n = -3, -4\) and \(-5\) respectively, where we recall that the negative signs indicate that periodic orbits have clockwise directions. Periodic orbits with the same resonance conditions seem to be related to fold bifurcations as we shall discuss this in section 4.
Figure 10. The projection of periodic orbits with $|n/m| = 1/3$.

Remark 3.2 (Action angle variables). We can introduce the action angle variables $(J, \phi)$ to transform the Hamiltonian system in terms of $(x, z)$ to that in terms of $(J, \phi)$. When the model is unperturbed, i.e. $\varepsilon = 0$, $J$ and $\phi$ are obtained by

$$J = \frac{1}{2\pi} \int z dx, \quad \phi = \frac{2\pi}{T} t,$$

(3.2)

where the integral is taken over one cycle of the periodic curve of (2.1) which preserves $H(x, z) = H$ (constant) and $T$ is the period of the orbit. Then the unperturbed model (2.1) can be rewritten as

$$\frac{dJ}{dt} = 0, \quad \frac{d\phi}{dt} = \Lambda(J),$$

where $\Lambda(J) = 2\pi/T$. Then, the perturbed system (2.2) can be restated in terms of $(J, \phi)$ as

$$\frac{dJ}{dt} = \varepsilon f(J, \phi, t), \quad \frac{d\phi}{dt} = \Lambda(J) + \varepsilon g(J, \phi, t),$$

(3.3)

where $f(J, \phi, t) = \frac{\partial J}{\partial x} \frac{\partial H}{\partial z} + \frac{\partial J}{\partial z} \frac{\partial H}{\partial x}$ and $g(J, \phi, t) = -\frac{\partial \phi}{\partial x} \frac{\partial H}{\partial z} + \frac{\partial \phi}{\partial z} \frac{\partial H}{\partial x}$; see, for instance, Wiggins (1990).
As was shown in (2.3), the perturbed Hamiltonian system (3.3) can be written as an autonomous system in terms of \((J, \phi, \theta) \in \mathbb{R} \times S^1 \times S^1\) as

\[
\begin{align*}
\frac{dJ}{dt} &= \varepsilon f(J, \phi, \theta), \\
\frac{d\phi}{dt} &= \Lambda(J) + \varepsilon g(J, \phi, \theta), \\
\frac{d\theta}{dt} &= \omega, 
\end{align*}
\] (3.4)

where \(\theta = \omega t + \theta_0\). Of course, the perturbed Hamiltonian system (2.3) with the variables \((x, z, \theta)\) is transformed into the system (3.4) with the action-angle variables \((J, \phi, \theta)\).

**Remark 3.3 (Poincaré–Birkhoff theorem).** Let us consider an invariant curve with action \(J\) such that \(\Lambda(J) = n/m\) in the unperturbed system, where \(m\) and \(n\) are integers. The Poincaré–Birkhoff theorem states that when the system is perturbed, \(2\ell m\) of \(m\)-periodic points appear in the neighbourhood of the original invariant curve, where \(\ell\) is some unknown integer. In particular, \(\ell m\) of them are to be elliptic and the others are to be hyperbolic; see Birkhoff (1927) and Guckenheimer and Holmes (1983), Lichtenberg and Lieberman (1991).

### 3.2. Symmetries of \(n/m\)-resonant orbits

In this subsection, we consider the symmetric properties concerning \(n/m\)-resonant orbits, namely, \(m\)-periodic orbits with winding number \(n\).

We consider the special case of such \(n/m\)-resonant orbits \(\tilde{c}(t)\) in the extended phase space \(\mathcal{M} = M \times S^1\) in which \(c(t) = \pi(\tilde{c}(t))\) is symmetric with respect to the horizontal and vertical centre lines of a cell, namely \(x = (2a + 1)\pi/(2k)\) \((a \in \mathbb{Z})\) and \(z = 1/2\), by the following theorem.

**Theorem 3.4 (Symmetries of \(n/m\)-resonant orbits).** Let \(\tilde{c}(t) := (x(t), z(t), \theta(t)) \subset M \times S^1, t \in \mathbb{I} = [0, mT]\) be a \(n/m\)-resonant orbit such that \(\tilde{c}(0) = \tilde{c}(mT) = p,\) where \(p\) is an \(m\)-periodic point on \(\Sigma_{\theta_0}\). Then, let \(c(t) := \pi(\tilde{c}(t))\) be a periodic curve on \(M\). If \(c(t)\) is symmetric with respect to the horizontal and vertical centre lines of a cell, namely \(x = (2a + 1)\pi/(2k)\) \((a \in \mathbb{Z})\) and \(z = 1/2\), the period \(m\) and the winding number \(n\) of \(c(t)\) are both odd.

**Proof.** For the sake of proving this theorem, recall the following symmetric properties (a), (d), and (e) of the non-autonomous system in (2.2), since \(c(t) \subset M\) corresponds to the solution curve of the non-autonomous system.

\[
\begin{align*}
(a) \quad x \mapsto x + \frac{2a\pi}{k}, \quad z \mapsto -z + 1, \quad t \mapsto -t + bT, \\
(d) \quad x \mapsto -x + \frac{(2a + 1)\pi}{k}, \quad z \mapsto z, \quad t \mapsto -t + \left(b + \frac{1}{2}\right)T, \\
(e) \quad x \mapsto -x + \frac{(2a + 1)\pi}{k}, \quad z \mapsto -z + 1, \quad t \mapsto t + \left(b + \frac{1}{2}\right)T.
\end{align*}
\]
As is shown in figure 11, consider an \( n/m \)-resonant periodic orbit \( \tilde{c}(t) \in M \times S^1 \) such that \( \tilde{c}(0) = \tilde{c}(mT) = p \), where \( p \) is an \( m \)-periodic point, and suppose that the periodic curve \( c(t) = \pi(\tilde{c}(t)) \) on \( M \) has the symmetric properties that \( c(t) \) is symmetric with respect to \( x = (2a + 1)\pi/(2k) \) (\( a \in \mathbb{Z} \)) and \( z = 1/2 \). Note that \( c(t) \) is partly illustrated in dashed lines to indicate a general curve in figure 11, which denotes that the dashed lines can have a loop as long as \( c(t) \) maintain the symmetric properties.

First, we shall prove that \( m \) is odd. To do this, let \( p_1 := \pi(p) \), where \( p_1 = c(0) = c(mT) \in M \), and let \( p_2 \in M \) be the associated symmetric point with \( p_1 \) regarding the horizontal centre line of a cell, namely \( z = 1/2 \). Since \( c(t) \) is symmetric with respect to \( z = 1/2 \), it follows from property (a) that \( p_2 \) can be expressed as \( p_2 = c(\ell T) \), where \( \ell \) is some integer such that \( 0 \leq \ell \leq m - 1 \).

We denote the initial time for \( p_1 \) and \( p_2 \) by \( t_1 = 0 \) and \( t_2 = \ell T \) respectively. Then, one can define an intermediate point \( p_3 \) in a path from \( p_1 \) to \( p_2 \) such that \( p_3 := c(t_3) \), where

\[
t_3 = \frac{t_1 + t_2}{2} = \frac{\ell}{2} T
\]

is the middle time between \( t_1 \) and \( t_2 \). Further, we denote the first return time for \( p_1 \) as \( t'_1 = mT \).

Then, one can define an intermediate point \( p_4 \) in a path from \( p_2 \) to \( p_1 \) such that \( p_4 := c(t_4) \), where

\[
t_4 = \frac{t_2 + t'_1}{2} = \frac{\ell + m}{2} T
\]

is the middle time between \( t_2 \) and \( t'_1 \). Since \( p_1 \) and \( p_2 \) are the points of curve \( c(t) \) at \( t \equiv 0 \) (mod \( T \)) and that they are symmetric with respect to \( z = 1/2 \), it follows from property (a) that \( p_3 \) and \( p_4 \) lie on the horizontal centre line \( z = 1/2 \), as is shown in figure 11.

Next, let \( p_5 \in M \) be the associated symmetric point with \( p_1 \) regarding the vertical centre line of a cell, namely \( x = (2a + 1)\pi/(2k) \). Since \( c(t) \) is symmetric with respect to the vertical centre line, \( p_5 \) is a point of \( c(t) \). Furthermore, it follows from property (e) that the time from \( p_2 \) to \( p_5 \) is the half of the period of the orbit, namely \( mT/2 \), since \( p_2 \) and \( p_5 \) are symmetric with respect to point \((x, z) = ((2a + 1)\pi/(2k), 1/2)\). Thus, \( p_5 \) can be expressed as \( p_5 = c(t_5) \), where

\[
t_5 = t_2 + \frac{mT}{2} = \left(\ell + \frac{m}{2}\right) T.
\]

Then, the times from \( p_1 \) to \( p_3 \) and \( p_4 \) to \( p_5 \) become the same as is shown below.
\[ t_2 - t_1 = t_3 - t_4 = \frac{\ell}{2} T. \]

Since \( p_1 \) and \( p_5 \) are symmetric with respect to the vertical centre line, it follows from property (d) that \( p_3 \) and \( p_4 \) are also symmetric with respect to the vertical centre line, as is shown in figure 11.

Now, we prove by contradiction that period \( m \) is an odd number. To do this, let us assume that \( m \) is an even number. Then, time \( t_2 \) and \( t_4 \) become \( t_3 \equiv t_4 \equiv 0 \) when \( \ell \) is odd, while they become \( t_3 \equiv t_4 \equiv T/2 \) when \( \ell \) is odd. However, since \( p_3 \) and \( p_4 \) are symmetric with respect to \( x = (2a + 1)\pi/(2k) \), it follows from property (d) that there are only two cases; one is the case when \( t_3 \equiv 0 \) and \( t_4 \equiv T/2 \), and the other is the case when \( t_3 \equiv T/2 \) and \( t_4 \equiv 0 \). Therefore, the assumption that \( m \) is an even number is not correct. Thus, it is proved that \( m \) is an odd number.

Next, we shall prove that \( n \) is odd. Recall that the winding number \( n \) of a periodic orbit \( c(t) \) is given by (3.1), where \( c(t) \) is regarded as a closed curve \( c(t) = x(t) + iz(t) \) in \( \mathbb{C} \) and the interval of integration can be divided as

\[
n = \frac{1}{2\pi i} \int_{c(t)} \frac{dw}{w - w_c} = \frac{1}{2\pi i} \left( \int_{c_{3,4}(t)} \frac{dw}{w - w_c} + \int_{c_{4,3}(t)} \frac{dw}{w - w_c} \right).
\]

Here, \( c_{3,4}(t) \) and \( c_{4,3}(t) \) respectively denote the part of curve \( c(t) \) from \( p_3 \) to \( p_4 \) and vice versa. From assumption, note that \( c(t) \) is symmetric with respect to \( z = 1/2 \) and also that \( p_3 \) and \( p_4 \) lie on \( z = 1/2 \), and it follows

\[
\int_{c_{3,4}(t)} \frac{dw}{w - w_c} = \int_{c_{4,3}(t)} \frac{dw}{w - w_c},
\]

where \( w = x + iz \) is an arbitrary point on \( c(t) \) and \((x_c, z_c) = ((2a + 1)\pi/(2k), 1/2)\) is set as a fixed point. Therefore,

\[
n = \frac{1}{2\pi i} \int_{c_{3,4}(t)} \frac{dw}{w - w_c}
= \frac{1}{\pi i} \{ \ln(w(t_4) - w_c) - \ln(w(t_3) - w_c) \}.
\]

Now, we rewrite a point \( w = x + iz \) on \( c(t) \) in the polar coordinates as

\[
w(t) - w_c = r(t)e^{i\psi(t)},
\]

where \( r = |w| = \sqrt{(x-x_c)^2 + (z-z_c)^2} \geq 0 \) and \( \psi = \arg w = \arctan \left( \frac{z-z_c}{x-x_c} \right) \). Since \( p_3 \) and \( p_4 \) lie on \( z = 1/2 \) and are symmetric with each other with respect to \( x = (2a + 1)\pi/(2k) \),

\[
r(t_3) = r(t_4),
|\psi(t_4) - \psi(t_3)| = \pi.
\]

Therefore,
\[ n = \frac{1}{\pi} \left\{ \ln r(t_4) + i(\psi(t_4) + 2\ell'\pi) - \ln r(t_3) - i(\psi(t_3) + 2\ell''\pi) \right\} \\
= 2(\ell' - \ell'') \pm 1, \]

where \( \ell', \ell'' \in \mathbb{Z} \). Hence, it is proved that \( n \) is an odd number. Thus, theorem is proved.

As the theorem states, we can see in figure 6(b) that the periodic orbits of which projection is symmetric with respect to the horizontal and vertical centre lines of the cell have odd period \( m \) and winding number \( n \). Furthermore, the following corollary can be stated from theorem 3.4.

**Corollary 3.5.** If the period \( m \) or the winding number \( n \) of a periodic orbit \( \tilde{c}(t) \) is an even number, there appear one or three more \( n/m \)-resonant orbits of which projection is symmetric with \( c(t) = \pi(\tilde{c}(t)) \) with respect to either horizontal or vertical centre lines, or the centre point of a cell.

**Proof.** Considering the contraposition of theorem 3.4, if \( m \) or \( n \) is an even number, \( c(t) = \pi(\tilde{c}(t)) \) is not symmetric with respect to either horizontal or vertical centre lines of a cell. If \( c(t) \) is not symmetric with only one of the two lines, it follows from property (a) or (d) that there appear one more \( n/m \)-resonant orbit of which projection is symmetric with \( c(t) \) with respect to either of the two lines. If \( c(t) \) is not symmetric with both of the two lines, it follows from property (a) and (d) that there appear two more orbits \( c_1(t) \) and \( c_2(t) \) which is symmetric with \( c(t) \) with respect to either of the two lines, and in addition another orbit \( c_3(t) \) which is symmetric with \( c(t) \) with respect to the centre point of the cell. Thus, the corollary is proved.

Figures 12 and 13 illustrate the projection of orbits of which period \( m \) or winding number \( n \) is an even number. When we look at them with figures 7–9, as corollary 3.5 states, it is observed that there appear one or three more \( n/m \)-resonant orbits. Note that the evolution of the orbits...
are depicted in the positive direction of time $t$ in both figures and also that the orientation of orbits could be opposite when computing the evolution for the negative direction of $t$, while the resonance conditions of the orbits that are symmetric in spatial coordinates $(x, z)$ with each other are the same.

4. Bifurcations of periodic orbits

As already mentioned, the amplitude of the perturbation of Rayleigh–Bénard convection increases when the Rayleigh number $Ra$ is gradually raised from the critical number $Ra_c$ by increasing the temperature difference between the top and bottom planes. In this section, we study the bifurcations of periodic orbits in the perturbed Hamiltonian system by varying the parameter $\varepsilon$, i.e. the amplitude of the perturbation in order to clarify how the fluid transport changes with $\varepsilon$. We first describe the global structures of $\varepsilon$-bifurcation diagram and then clarify the structures of the bifurcations associated with the main KAM island $I_1$ and the surrounding islands $I_2, I_3$, and $I_4$, and furthermore those associated with other islands.
4.1. Structure of $\varepsilon$-parameter bifurcation

4.1.1. Computation of one-parameter bifurcation diagrams. In the numerical computations in sections 2 and 3, we have analysed the periodic points and the associated orbits when the amplitude of the perturbation is set to $\varepsilon = 0.1$. In order to obtain the $\varepsilon$-parameter bifurcation diagram of periodic points for $0 < \varepsilon \leq 0.5$, we shall compute to detect the elliptic and hyperbolic periodic points on the Poincaré section for each amplitude of perturbation $\varepsilon = 0.001, 0.002, \ldots, 0.5$ independently by Newton’s method shown in section 2.2. The other parameters of the convection and the initial condition of $\theta$ are set to $A = \pi, k = \pi, T = 1/\pi$, and $\theta_0 = 0$ as the same in figure 2. For numerical computations of bifurcation diagrams in fluids continuation methods are used in other studies; see for instance Umbriá and Net (2019). However, in this paper, we obtain the bifurcation diagram by the above method, since the Hamiltonian $H(x, z, t) = H_0(x, z) + \varepsilon H_1(x, z, t)$ of the model in this study is given, and thus the velocity field $dx/dt = -\partial H/\partial z$, $dz/dt = \partial H/\partial x$ for each $\varepsilon$ can be obtained explicitly. Figure 14 shows the detected bifurcation diagram from diagonal and $z$ direction, where the periodic points with period $m \leq 15$ are depicted. The colour of the plots indicate the period $m$ of each point, however the types of the points, namely elliptic or hyperbolic, are not illustrated in figure 14. We will depict them in the figures shown latter.

4.1.2. Periodic points on the Poincaré section with some parameters $\varepsilon$. Before we take a look at the bifurcation diagram let us show how the Poincaré maps and the detected periodic points vary with the amplitude $\varepsilon$ of the perturbation. Figure 15 illustrates the image of the Poincaré section by Poincaré map $P_{\theta_0}^\varepsilon$ and the periodic points for $\varepsilon = 0.2, 0.3, 0.4$, and 0.5. As can be seen, the islands of KAM tori, which correspond to stable transport regions, exist for a while when $\varepsilon$ is increased. However, when we increase it furthermore, the area of the islands and the number of elliptic periodic points gradually decrease. Especially, islands $I_2, I_3$, and $I_4$ seem to disappear by $\varepsilon = 0.5$. In contrast, it is apparent that the area of chaotic regions increases. This denotes that the periodic orbits in the system of (2.3) bifurcate one after another and lead to chaotic orbits when $\varepsilon$ is increased.
4.1.3. **Bifurcations of 1 and 3-periodic points.** Now we take a closer look at the bifurcation diagram detected in our numerical computation. Since it is too complicated to understand the structure of the diagram from figure 14, let us first focus on the bifurcations of 1 and 3-periodic points, which are illustrated in figure 16 from diagonal and $z$ direction. Here, the branches of elliptic and hyperbolic periodic points are depicted in thick and thin lines respectively. In addition, we especially depict the 1 and 3-periodic points with the image of the Poincaré section at $\varepsilon = 0.1, 0.4$ in figure 17 so that we can clearly see the periodic points. As is shown in figure 17, an elliptic 1-periodic point and three hyperbolic 3-periodic points appear at the centre and the corners of the main island $I_1$, respectively.

Thus, the thick red branch of elliptic 1-periodic points in the middle of figure 16 and the three thin yellow branches of hyperbolic 3-periodic points, which cross with the red branch, correspond to those of the periodic points associated with $I_1$. Furthermore, figure 17 indicates that an elliptic 3-periodic point appear in the middle of each island $I_2, I_3, I_4$ when $\varepsilon$ is small but vary to two elliptic and one hyperbolic 3-periodic points when $\varepsilon$ is increased. Thus, the three fork-shaped branches of elliptic 3-periodic points in figure 16 correspond to those of the
periodic points associated with islands $I_2, I_3$, and $I_4$. The two straight branches of hyperbolic 1-periodic points on the wall of figure 16 are those of the 1-periodic points on the upper and lower boundaries of the convection.

4.1.4. Bifurcations associated with KAM islands $I_1,I_2,I_3$, and $I_4$. Next, we focus on the bifurcations associated with KAM islands $I_1,I_2,I_3$, and $I_4$. First, we take a look at those of the main island $I_1$. As is shown in figure 15, the periodic points in $I_1$ appear along the KAM curves around an elliptic 1-periodic point. Thus, the mountainous structure depicted in figure 18 may correspond to the bifurcations associated with $I_1$. Though the type of the periodic points are
not illustrated here, it follows that many branches of various periods gather to the branch of the elliptic 1-periodic points. Especially, it is observed that the three branches of hyperbolic 3-periodic points at the corners of island $I_1$ appear around the outer side of the mountainous structure. Furthermore, since they cross with the branch of 1-periodic points at around $\varepsilon = 0.432$, it seems that $I_1$ once disappear when the amplitude $\varepsilon$ is increased. We will analyse the bifurcations associated with $I_1$ more in detail in section 4.3. Then, let us take a look at the bifurcations of islands $I_2, I_3,$ and $I_4$. It is found in our numerical computation that the bifurcations shown in figure 19 may correspond to those associated with $I_2, I_3,$ and $I_4$. As can be seen, many branches of $3\ell$-periodic points ($\ell = 2, 3, 4, 5$) grow from the fork-shaped branch of 3-periodic points to form the shapes of three broom tips standing upside down as in figure 16 and create three tree-like structures. We will clarify the structure of the bifurcations more in detail in section 4.4.
4.2. Numerical algorithm for detecting bifurcation points

Before we clarify the global structures of the $\varepsilon$-bifurcation diagram more in detail, let us briefly review the classification of bifurcations of periodic points and describe how each bifurcation point is detected in numerical computations.

4.2.1. Classification of bifurcation points. Recall that multipliers $\mu$ of an $m$-periodic point are eigenvalues of the Jacobian matrix of the Poincaré $m$-return map

$$J_\varepsilon(x) = \left. \frac{\partial (P^0_{\theta_0})^m(x)}{\partial x} \right|_{x=x_0},$$

where $x_0$ indicates the $m$-periodic point. According to the multipliers $\mu$ of the $m$-periodic point at the bifurcation point (see, for instance, Kuznetsov 2004), the bifurcations of $m$-periodic points are classified into the following types:

- Fold bifurcation (also called, tangent or saddle-node bifurcation): $\mu = 1$
- Flip bifurcation (also called, period-doubling bifurcation): $\mu = -1$
- Neimark-Sacker bifurcation (also called, Hopf bifurcation for maps): $|\mu| = 1$ but $\mu \neq \pm 1$.

In this paper, we mainly focus on the fold and flip bifurcations.

4.2.2. Computation of fold and flip bifurcation points. We shall show the numerical method for detecting the fold and flip bifurcation point of $m$-periodic points. To do this, we shall employ the numerical computation method that was developed by Tsumoto et al. (2012); see also Kuznetsov (2004). As already mentioned, we do not use the continuation methods in other studies, since the Hamiltonian is given in this study, and thus the velocity field for each amplitude $\varepsilon$ can be obtained explicitly. Using the Poincaré map $P^0_{\theta_0}: \Sigma^{\theta_0} \to \Sigma^{\theta_0}$, the following two conditions have to be satisfied at the bifurcation point for some $m$-periodic point $x_0$:

(a) Condition for $m$-periodic points. Recall that associated with the vector field of the autonomous Hamiltonian system in (2.3), we can uniquely define the flow $\phi^\varepsilon: \mathbb{R} \times \mathcal{M} \to \mathcal{M}$: $(t,x_0,z_0,\theta_0) \to (x_t,z_t,\theta_t) = \phi^\varepsilon(t,x_0,z_0,\theta_0)$ for some given parameter $\varepsilon \in \mathbb{R}^+$. Then, a diffeomorphism $\phi^\varepsilon_m: \mathcal{M} \to \mathcal{M}$: $(x_0,z_0,\theta_0) \mapsto (x_t,z_t,\theta_t) = \phi^\varepsilon_m(x_0,z_0,\theta_0)$ can be given for each fixed $t$.

Recall also that we can define the Poincaré $m$-return map by

$$(P^0_{\theta_0})^m := \phi^\varepsilon_m \bigg|_{\Sigma^{\theta_0}}: \Sigma^{\theta_0} \to \Sigma^{\theta_0},$$

which is locally given by

$$(x(0) = x_0, z(0) = z_0, \theta(0) = \theta_0) \quad \mapsto \quad (x(mT), z(mT), \theta(mT) = \theta_0 + 2\pi m \equiv \theta_0).$$

Therefore, the condition that some point $x_0 = (x_0, z_0) \in \Sigma^{\theta_0}$ becomes the $m$-periodic point is given by
Condition for bifurcation points. Suppose that \( x_0 \) is an \( m \)-periodic point on \( \Sigma^{0}_0 \) and consider to find a bifurcation point for \( x_0 \) associated with the parameter \( \varepsilon \), where we need to vary \( \varepsilon \) to detect the bifurcation point. Recall that the Poincaré \( m \)-return map is given by, for some \( x^{(\ell)} \in \Sigma_{0}^{0} \) and with fixed \( \varepsilon \),

\[
  x^{(\ell+1)} = (P^{0}_{\varepsilon})^{m}(x^{(\ell)}), \quad \ell = 0, 1, 2, \ldots
\]

Let \( x_0 = (x_0, z_0) \in \Sigma^{0}_0 \) be an \( m \)-periodic solution and we define the variation of \( x^{(\ell)} \) associated with \( x_0 \), i.e., a small deviation from \( x_0 \) by

\[
  w^{(\ell)} := x^{(\ell)} - x_0.
\]

Then, by definition \( x^{(\ell+1)} = x_0 + w^{(\ell+1)} \), and it follows by Taylor expansion and by neglecting the higher-order terms that the variational equations may be given as

\[
  w^{(\ell+1)} = J_\varepsilon(x_0) w^{(\ell)},
\]

where

\[
  J_\varepsilon(x_0) = \left. \frac{\partial (P^{0}_{\varepsilon})^{m}(x)}{\partial x} \right|_{x=x_0}.
\]

The characteristic equation of (4.2) is

\[
  \det (J_\varepsilon(x_0) - \mu I) = 0,
\]

where \( I \) denotes the unit matrix and \( \mu \) a multiplier that corresponds to an eigenvalue.

Notice that the parameter \( \varepsilon \) is fixed in equations (4.1)–(4.3). On the other hand, the \( m \)-periodic point may be bifurcated at some \( \varepsilon_0 \) when \( \mu \) satisfy \( |\mu| = 1 \); for instance, the fold and flip bifurcations can be occurred when \( \mu = 1 \) and \( \mu = -1 \) respectively.

Thus, when a bifurcation associated with some specific \( \mu_0 \in \mathbb{R} \) for an \( m \)-periodic point \( x_0 = (x_0, z_0) \in \Sigma^{0}_0 \) occurs at some \( \varepsilon_0 \), the following set of \( \mu_0 \)-dependent nonlinear algebraic equations (4.1) and (4.3) holds:

\[
  G_{\mu_0}(x_0, \varepsilon_0) = \begin{bmatrix} F(x_0, \varepsilon_0) \\ g_{\mu_0}(x_0, \varepsilon_0) \end{bmatrix} = 0,
\]

where we define the map \( F: \Sigma^{0}_0 \times \mathbb{R}^+ \to \mathbb{R}^2 \) by, for each \( (x_0, \varepsilon_0) \in \Sigma^{0}_0 \times \mathbb{R}^+ \),

\[
  F(x_0, \varepsilon_0) := x_0 - (P^{0}_{\varepsilon_0})^{m}(x_0),
\]

and also the map \( g_{\mu_0}: \Sigma^{0}_0 \times \mathbb{R}^+ \to \mathbb{R} \) by

\[
  g_{\mu_0}(x_0, \varepsilon_0) := \det (J_{\varepsilon_0}(x_0) - \mu_0 I).
\]

In the above, notice that \( \varepsilon_0 \) is treated as a variable together with \( x_0 \). In other words, in order to detect a bifurcation point associated with some \( \mu_0 \) that satisfies \( |\mu_0| = 1 \) for the \( m \)-periodic point \( x_0 \) together with the specific parameter \( \varepsilon_0 \), we have to find a solution \((x_0, \varepsilon_0)\) that satisfies the nonlinear algebraic equation (4.4).

For numerical computations, we shall employ Newton’s method again as follows.
Numerical algorithm for detecting a fold or flip bifurcation point

(a) Set $\mu_0 = 1$ for a fold bifurcation or $\mu_0 = -1$ for a flip bifurcation.

(b) Set $k_0 = 0$ with an initial approximation $y_0^{(0)} = (x_0^{(0)}, \varepsilon_0^{(0)})$ for some required bifurcation point $y_0 = (x_0, \varepsilon_0)$.

(c) Set $k := k + 1$ and compute the $k$th approximation by

$$y_0^{(k)} := y_0^{(k-1)} - \left( \begin{array}{c} \frac{\partial G_{\mu_0}(y_0)}{\partial y_0} \bigg|_{y_0 = y_0^{(k-1)}} \\ y_0^{(k-1)} \end{array} \right)^{-1} G_{\mu_0}(y_0^{(k-1)}),$$

where the Jacobian matrix is numerically approximated by the central difference scheme.

(d) If $|G_{\mu_0}(y_0^{(k)})| < \delta$, where the convergence radius is set to $\delta = 10^{-10}$, then the computation ends up and the bifurcation point for the $m$-periodic point is to be detected as $y_0^{(k)}$.

(e) Otherwise, return to (c) in order to iterate the computation until convergence.

Remark 4.1. The initial approximation $y_0^{(0)} = (x_0^{(0)}, \varepsilon_0^{(0)})$ in the Newton’s method is obtained from the $\varepsilon$-parameter bifurcation diagram.

4.3. Bifurcations associated with KAM island $I_1$

In this subsection, we investigate the bifurcations of periodic points associated with the main KAM island $I_1$. As we have seen in figure 18, many branches of periodic points with various periods gather to the branch of 1-periodic points at the centre of island $I_1$. Let us first show the bifurcation points numerically detected in our computation, and then illustrate how the periodic orbits vary with $\varepsilon$ by taking a look at the 7-periodic orbits for example.

4.3.1. Fold bifurcations associated with $I_1$. Figure 20 shows from $z$ direction the bifurcation points numerically detected in the $\varepsilon$-bifurcation diagram associated with $I_1$, where the branches of elliptic and hyperbolic periodic points are depicted in the same way. Each bifurcation point of $m$-periodic points is indicated with a circle in magenta. The amplitude $\varepsilon$ for each point is also shown beside them with the period $m$ and the type of the bifurcation. As can be seen, it was numerically clarified that the $m$-periodic points bifurcate in a fold bifurcation when they coalesce with the 1-periodic point at the centre of $I_1$. Note that the 1-periodic points themselves do not seem to bifurcate when the $m$-periodic points bifurcate in a fold bifurcation.

4.3.2. Fold bifurcations of 7-periodic points. Next, let us investigate how the periodic orbits vary with $\varepsilon$ near the fold bifurcation point. Here, we take a look at the 7-periodic orbits for example. Figure 21 illustrates the $\varepsilon$-bifurcation diagram of 1 and 7-periodic points in $I_1$, where the branches of elliptic and hyperbolic periodic points are depicted in thick and thin lines respectively. Figure 22 also shows the 1 and 7-periodic points on the Poincaré section $\Sigma_{\theta_0}$ at $\varepsilon = 0.2$ and the projection of the associated periodic orbits onto the phase space $M$. As can be seen in figure 22, elliptic and hyperbolic 7-periodic points appear seven each in addition
to the 1-periodic point. It follows that stable and unstable 7-periodic orbits appear one each in addition to a stable 1-periodic orbit. The blue points in circles and stars in figure 22(a) correspond to the points of the stable and unstable 7-periodic orbits respectively, while the red circle point corresponds to the point of the stable 1-periodic orbit. It is observed that the resonance condition of the 1 and 7-periodic orbits are \( jn/m = 1 \) and \( jn/m = 3/7 \) respectively.

However, when the amplitude is increased from \( \varepsilon = 0.2 \), the resonance condition of the unstable 7-periodic orbit varies to \( jn/m = 5/7 \) at around \( \varepsilon = 0.213 \); the case \( \varepsilon = 0.23 \) is illustrated in figure 23. Furthermore, right before the bifurcation point at around \( \varepsilon = 0.327 \), the resonance condition of both the stable and unstable 7-periodic orbits varies to \( jn/m = 7/7 = 1 \), which corresponds to that of the 1-periodic orbit; the case \( \varepsilon = 0.242 \) is depicted in figure 24. Therefore, it seems that the 7-periodic orbits disappear at the bifurcation point and vary to a 1-periodic orbit. Further, it is observed in our numerical computation that the projection of the 7-periodic orbits associated with \( I_1 \) is symmetric with respect to the horizontal and vertical centre lines of the cell regardless of the amplitude \( \varepsilon \). It is consistent with theorem 3.4 that the period \( m = 7 \) and the winding number \( n = -3, -5, \) and \(-7\) are odd. The other \( m \)-periodic orbits associated with \( I_1 \) vary similarly to the 7-periodic orbits, which indicates that they disappear one by one when \( \varepsilon \) is increased.

4.4. Bifurcations associated with KAM islands \( I_2, I_3, \) and \( I_4 \)

In this subsection, we analyse the bifurcations associated with the three KAM islands \( I_2, I_3, \) and \( I_4 \) around the main island \( I_1 \). Let us first take a look at the bifurcations of 3-periodic points, and then investigate those of 6, 9, 12, and 15-periodic points.

4.4.1. Fold and flip bifurcations of 3-periodic points.

Figure 25 illustrates the \( \varepsilon \)-bifurcation diagram of 3 and 6-periodic points associated with islands \( I_2, I_3, \) and \( I_4 \). Let us first focus on the bifurcation of 3-periodic points which is depicted in yellow. As can be seen, the branches of 3-periodic points bifurcate similarly to a fork at around \( \varepsilon = 0.321 \). It is found in our computation that a fold bifurcation occurs at \( \varepsilon = 0.321 7135 \). Let us take a look at how the 3-periodic points and the projection of the associated periodic orbits vary by the bifurcation. Figures 26 and 27 show those at \( \varepsilon = 0.3 \) and \( \varepsilon = 0.34 \). We can see that three elliptic 3-periodic points appear at \( \varepsilon = 0.3 \). It follows that one stable 3-periodic orbit appears at \( \varepsilon = 0.3 \). However, as we increase
Figure 21. $\varepsilon$-bifurcation diagram of 1 and 7-periodic points associated with $I_1$.

(a) 1 and 7-periodic points  
(b) Stable 1-periodic orbit  
(c) Stable 7-periodic orbit  
(d) Unstable 7-periodic orbit

Figure 22. 1 and 7-periodic points and the projection of their orbits at $\varepsilon = 0.2$. 

$\varepsilon = 0.2432212$  
$m = 7$, fold  
$\varepsilon = 0.2420000$  
$\varepsilon = 0.2300000$  
$\varepsilon = 0.2000000$
In the document, the authors discuss the behavior of periodic orbits in a dynamical system as the parameter $\varepsilon$ varies. Specifically, they describe how

- As $\varepsilon$ increases to 0.23, a 7-periodic orbit bifurcates into an unstable orbit, and two new stable 3-periodic orbits appear. These new orbits are denoted as $c_a$ and $c_b$, and their associated elliptic points are labeled accordingly. The original stable orbit loses its symmetric properties with respect to the horizontal and vertical center lines.

- At $\varepsilon = 0.4453380$ and $\varepsilon = 0.4713999$, the 3-periodic points bifurcate in a flip bifurcation.

- At $\varepsilon = 0.1416245$, the 6-periodic points bifurcate in a fold bifurcation, and the 3-periodic points do not bifurcate when this happens. The authors note that the symmetric properties of the orbits are lost in these bifurcations.

In the next section, the authors explore the behavior of 6-periodic points, observing that the blue branches of these points grow from the yellow branches of 3-periodic points at $\varepsilon = 0.142$. They also note that the 3-periodic points themselves do not seem to bifurcate when the 6-periodic points do, and they discuss the implications of these observations for understanding the system's dynamics.
Figure 25. $\varepsilon$-bifurcation diagram of 3 and 6-periodic points.

Figure 26. 3-periodic points and the projection of their orbits at $\varepsilon = 0.3$.

vary with $\varepsilon$ near such fold bifurcation point. Figure 28 shows the 3 and 6-periodic points and the projection of the associated periodic orbits at $\varepsilon = 0.15$. We can see that elliptic and hyperbolic 6-periodic points appear four each around each elliptic 3-periodic points. It follows that stable and unstable 6-periodic orbits appear two each in addition to the stable 3-periodic orbit. We name the projection of the four 6-periodic orbits as $c_a$, $c_b$, $c_c$, and $c_d$, and label the associated 6-periodic points in figure 28(a) as $a$, $b$, $c$, and $d$ respectively in order to show the correspondence, where $c_a$ and $c_b$ are symmetric with each other with respect to the horizontal centre line of the cell, and $c_c$ and $c_d$ are symmetric with respect to the vertical one. The projection of the 3-periodic orbit and that of the 6-periodic orbits $c_a$ and $c_c$ are illustrated in figures 28(b)–(d). It is observed in figure 28 that the resonance condition of the 6-periodic orbits are $|n/m| = 1/3$, which is the same of that of the 3-periodic orbit. This implies that the 6-periodic orbits at the bifurcation point correspond to the 3-periodic orbit.

4.4.3. Fold bifurcations of 9, 12, and 15-periodic points. We have seen that the 6-periodic points bifurcate in a fold bifurcation, which makes some branches of 6-periodic points grow
from those of 3-periodic points. Such fold bifurcations are also observed in 9, 12, and 15-periodic points associated with islands $I_2, I_3,$ and $I_4,$ where the $\varepsilon$-bifurcation diagrams of those periodic points are illustrated in figures 29–31. The 9, 12, and 15-periodic orbits vary similarly to the 6-periodic orbits near these bifurcation points. Hence, $3\ell$-periodic orbits ($\ell = 2, 3, 4,$ and $5$) are generated one after another from the 3-periodic orbits by increasing the amplitude $\varepsilon$ of the perturbation. The resonance conditions of the generated orbits are all $|n/m| = 1/3,$ since they correspond to the 3-periodic orbit at the bifurcation point.

4.5. Bifurcations associated with other KAM islands

So far we have explored the bifurcations associated with KAM islands $I_1, I_2, I_3,$ and $I_4.$ In this subsection, we investigate those which seem to be associated with other islands. Here, we especially focus on the bifurcations of 5-periodic points with resonance condition $|n/m| = 1/5$ and those of 4 and 8-periodic points with $|n/m| = 1/4.$

4.5.1. Fold bifurcations of 5-periodic points. Figure 32 indicates the $\varepsilon$-bifurcation diagram of 5-periodic points of which resonance condition is $|n/m| = 1/5.$ It follows that the 5-periodic points bifurcate at $\varepsilon = 0.0469599$ and $\varepsilon = 0.0832330$ in a fold bifurcation. Now let us take
Figure 28. 3 and 6-periodic points and the projection of their orbits at $\varepsilon = 0.15$.

(c) Stable 6-periodic orbit $c_6$

(d) Unstable 6-periodic orbit $c_u$

Figure 29. $\varepsilon$-bifurcation diagram of 3 and 9-periodic points.
a look at how the symmetry of the stable 5-periodic orbits vary by the two bifurcations. Figure 33(a) shows the 5-periodic points at \( \varepsilon = 0.04 \), where it follows that the elliptic and hyperbolic 5-periodic points exist five each on the Poincaré section \( \Sigma^{0} \). As is illustrated in figure 33(b), the projection of the orbit of the elliptic 5-periodic points, namely, the stable 5-periodic orbit, is symmetric with respect to the horizontal and vertical centre lines of the cell. However, when we increase \( \varepsilon \), each elliptic 5-periodic point vary to a hyperbolic 5-periodic point and two new elliptic 5-periodic points appear in the neighbourhood by the fold bifurcation at \( \varepsilon = 0.0469599 \); the case \( \varepsilon = 0.053 \) is depicted in figure 34(a). We name the projection of the two new stable 5-periodic orbits as \( c_{a} \) and \( c_{b} \), and label the associated elliptic 5-periodic points as \( a \) and \( b \) respectively in order to show the correspondence, where \( c_{a} \) and \( c_{b} \) is symmetric with each other with respect to the horizontal centre line. From \( c_{a} \) in figure 34(b), it follows that the orbit looses one of its symmetric property and become only symmetric with the vertical centre line.
When the amplitude $\varepsilon$ is increased further, the ten elliptic 5-periodic points once vary to hyperbolic ones, but return to elliptic ones. After that, they bifurcate again in a fold bifurcation at $\varepsilon = 0.0832330$, which denotes that 20 elliptic 5-periodic points appear by the bifurcation. Figure 35(a) illustrates the 5-periodic points at $\varepsilon = 0.086$. We denote the projection of the four new stable 5-periodic orbits by $c_c, c_d, c_e,$ and $c_f$, and label the associated elliptic 5-periodic points as $c, d, e,$ and $f$ respectively in order to show the correspondence, where $c_c, c_d, c_e,$ and $c_f$ are symmetric with each other with respect to the horizontal or vertical centre line. From $c_c$ in figure 35(b), it follows that the orbit looses its symmetry and become asymmetric with respect to the horizontal and vertical centre lines of the cell. Hence, the stable 5-periodic orbit of which projection is originally symmetric with respect to the horizontal and vertical centre lines of the cell become asymmetric by the two fold bifurcations. Furthermore, we can see that the number of 5-periodic orbits increases by the bifurcations and also that they become unstable when $\varepsilon$ is large enough, which denotes that the fluid transport become more complicated.
4.5.2. Flip bifurcations of 4-periodic points. Next, let us take a look at the bifurcations of 4-periodic points. Figure 36 illustrates the $\varepsilon$-bifurcation diagram of 4 and 8-periodic points of which resonance condition is $|n/m| = 1/4$. When the amplitude of the perturbation is $\varepsilon = 0.05$, elliptic and hyperbolic 4-periodic points appear eight each, as is shown in figure 37(a). It follows that stable and unstable 4-periodic orbits appear two each. We name the projection of the two stable 4-periodic orbits as $c_a$ and $c_b$, and label the associated elliptic 4-periodic points as $a$ and $b$ respectively in order to show the correspondence. As is shown in figure 37(b), $c_a$ is only symmetric with respect to the horizontal centre line of the cell, which denotes that $c_b$ is symmetric with $c_a$ with respect to the vertical one.

Now, we consider varying the amplitude $\varepsilon$. When the amplitude is increased from $\varepsilon = 0$ to $\varepsilon = 0.5$, the hyperbolic 4-periodic points do not seem to bifurcate. In contrast, it is clarified in our computation that the elliptic 4-periodic points bifurcate in a flip bifurcation at $\varepsilon = 0.0545763$. At the bifurcation point, each elliptic 4-periodic point varies to a hyperbolic one and two new 8-periodic points appear in the neighbourhood of each 4-periodic point, where figure 38(a) depicts them at $\varepsilon = 0.059$. It follows that each stable 4-periodic orbit varies to one
unstable 4-periodic orbit and one stable 8-periodic orbit by the flip bifurcation. Since there are two stable 4-periodic orbits before the bifurcation, two new unstable 4-periodic orbits and two new stable 8-periodic orbits are generated by the bifurcation. We denote the projection of the former two orbits by $c_a$ and $c_d$ and that of the latter two by $c_e$ and $c_f$. Then, we label the associated periodic points as $c, d, e$, and $f$ respectively in order to show the correspondence. Note that $c_a$ and $c_d$ as well as $c_e$ and $c_f$ are symmetric with each other with respect to the vertical centre line. From $c_e$ and $c_d$ in figures 38(b) and (c), it follows that the symmetric axes of all the orbits from $c_a$ to $c_f$ are the same, which is the horizontal centre line. In addition, the resonance conditions of the 4 and 8-periodic orbits are both $|n/m| = 1/4$. It follows that the symmetric axis of the projection and the resonance condition of the periodic orbits do not vary by the bifurcation. Furthermore, it is observed that most of the 4 and 8-periodic orbits become unstable when $\varepsilon$ is large enough, which denotes that the fluid transport become more complicated.
4.5.3. Discussions of the results in this paper with other literatures. Finally, let us discuss the results obtained in this paper in comparison with those in other relevant literatures, though there are not many studies that have analysed the periodic orbits and their bifurcations in two-dimensional Rayleigh–Bénard convection in Lagrangian description. As we mentioned in the introduction, Inoue and Hirata (2000) analysed how the chaotic structures in perturbed Rayleigh–Bénard convection vary when the amplitude or frequency of the perturbations is increased. In that study, they clarified that the qualitative patterns of chaotic mixing do not vary so much by increasing the amplitude, while they drastically change by raising the frequency since the frequency is strongly related to the resonance conditions of periodic orbits. In our study it is observed that KAM islands $I_1, I_2, I_3,$ and $I_4$ remain for a while when the amplitude is increased, as is shown in figure 15. However, we clarified that many periodic orbits bifurcate one after another and thus the chaotic structures of the flow gradually changes with the amplitude. In particular, islands $I_1, I_2, I_3,$ and $I_4$ almost disappear when the amplitude is raised to around $\varepsilon = 0.4$. Such different results are obtained probably because Inoue and Hirata (2000) varies the amplitude only up to $\varepsilon = 0.06$ and also that the perturbations of the model in that study is different from those in this paper. In addition, Oteski et al (2014) numerically detected periodic points in two-dimensional oscillatory convection in differentially heated cavity and showed that the whole domain except a small area in the middle become chaotic regions when

Figure 38. 4 and 8-periodic points and the projection of their orbits at $\varepsilon = 0.059$. 

(a) 4 and 8-periodic points
(b) Unstable 4-periodic orbit $c_0$
(c) Stable 8-periodic orbit $c_0$
Rayleigh number is raised up to $Ra = 2.05 \times 10^8$. It is consistent with our result that almost all of the domain become chaotic regions when $Ra$, namely the amplitude of perturbation, is increased enough.

Chabreyrie et al (2011) proposed a strategy to create sufficient chaotic mixing in two-dimensional oscillatory electro-osmotic convection in parameter space, where the parameters are the amplitude and frequency of the oscillation. They concluded that sufficient chaotic mixing occurs when some key periodic points at the centre of non-mixing regions become hyperbolic. However, in our study, island $I_1$ remains when the elliptic 3-periodic points at the centre of islands $I_2, I_3$, and $I_4$ bifurcate in fold bifurcation at $\varepsilon = 0.3217135$. In addition, as can be seen in figure 16, the 1-periodic point at the centre of island $I_1$ still seems to be elliptic when almost all of the domain is filled with chaotic regions at around $0.4 < \varepsilon < 0.5$. Thus, the strategy proposed by Chabreyrie et al (2011) may be unsuitable for the model of perturbed Rayleigh–Bénard convection in this paper.

Further, the dependence of diffusion constant on the amplitude of the perturbations in perturbed Rayleigh–Bénard convection was studied in Ouchi and Mori (1992). When they analyse KAM islands, they observed the period-doubling tree of elliptic 2, 4, 8-periodic points associated with accelerator-mode islands. In this paper too, we observe some flip bifurcations, in other words period-doubling bifurcations, of elliptic 4-periodic points at $\varepsilon = 0.0545763$, as is shown in figure 36. Since the parameter settings in that work are different from those in this paper, we cannot simply compare their results with ours. However, the flip bifurcations observed in the two works may correspond with each other or have some relationship.

5. Conclusions

In this paper, we have numerically explored the global structures of periodic orbits appeared in a two-dimensional perturbed Hamiltonian model of Rayleigh–Bénard convection. First we have detected the periodic points on the Poincaré section and then analysed the associated periodic orbits from the perspective of resonances and symmetries. Furthermore, we have clarified the global bifurcations regarding the periodic orbits associated with parameter $\varepsilon$ which is the amplitude of the perturbation. Thus, we have gained the following results:

- KAM tori associated with elliptic $m$-periodic points have twisted structures in the extended phase space $M = \mathbb{R} \times \mathbb{S}^1$, which denotes that each region of KAM islands are mapped to the same region after $m$ times of Poincaré maps. From a physical point of view, they are transported periodically as a kind of vortices in Lagrangian description.
- We propose a theorem regarding the symmetries of $n/m$-resonant orbits; namely, if the projection of an $m$-periodic orbit onto the phase space $M$ is symmetric with respect to the horizontal and vertical centre lines of a cell, the period $m$ and the winding number $n$ of the orbit are both odd. It follows that $m$-periodic orbits appear in symmetric pairs when either $m$ or $n$ is even.
- When the amplitude $\varepsilon$ of the perturbation is increased, the $m$-periodic points associated with the main KAM island $I_1$ disappear one after another by fold bifurcations and seem to vary to an elliptic 1-periodic point at the centre of $I_1$.
- When $\varepsilon$ is increased, $3\ell$-periodic points ($\ell = 2, 3, 4, 5$) are generated one after another by fold bifurcations around the elliptic 3-periodic points at the centre of KAM islands $I_2, I_3$, and $I_4$, where the 3-periodic points themselves also bifurcate in fold and flip bifurcations after that.
Periodic points associated with other islands also bifurcate one after another and most of them vary to unstable ones, when \( \varepsilon \) is increased. Some of them generate more orbits as in the fold bifurcations of 5-periodic points, while some others generate orbits with larger periods as in the flip bifurcations of 4-periodic points. Hence, the bifurcations of periodic points that may not be associated with \( I_1 \) may be the main factor that makes the fluid transport complicated when \( \varepsilon \) is increased.

Acknowledgments

M W is partially supported by Waseda University (SR 2022C-092, SR 2022E-016), the MEXT ‘Top Global University Project’, and Sustainable Energy & Environmental Society Open Innovation Research Organization (SEES). H Y is partially supported by JSPS Grant-in-Aid for Scientific Research (22K03443), JST CREST (JPMJCR1914), Waseda University (SR 2022C-423), the MEXT ‘Top Global University Project’, SEES, and the Organization for University Research Initiatives (Evolution and application of energy conversion theory in collaboration with modern mathematics).

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