STOCHASTIC COMPLETENESS AND $L^1$-LIOUVILLE PROPERTY FOR SECOND-ORDER ELLIPTIC OPERATORS

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Abstract. Let $P$ be a linear, second-order, elliptic operator with real coefficients defined on a noncompact Riemannian manifold $M$ and satisfies $P1 = 0$ in $M$. Assume further that $P$ admits a minimal positive Green function in $M$. We prove that there exists a smooth positive function $\rho$ defined on $M$ such that $M$ is stochastically incomplete with respect to the operator $P_{\rho} := \rho P$, that is,

$$\int_M k^M_{P_{\rho}}(x, y, t)\, dy < 1 \quad \forall (x, t) \in M \times (0, \infty),$$

where $k^M_{P_{\rho}}$ denotes the minimal positive heat kernel associated with $P_{\rho}$. Moreover, $M$ is $L^1$-Liouville with respect to $P_{\rho}$ if and only if $M$ is $L^1$-Liouville with respect to $P$. In addition, we study the interplay between stochastic completeness and the $L^1$-Liouville property of the skew product of two second-order elliptic operators.

1. Introduction

Let $M$ be a smooth, noncompact, connected Riemannian manifold of dimension $n$. Let $P$ be a linear, second-order, elliptic operator with real coefficients defined on $M$, and satisfying $P1 = 0$ in $M$.

Denote by $C_P(M)$ the cone of positive solutions of the equation $Pu = 0$ in $M$. The generalized principal eigenvalue of the operator $P$ is defined by

$$\lambda_0 = \lambda_0(P, M) := \sup\{\lambda \in \mathbb{R} \mid C_{P-\lambda}(M) \neq \emptyset\}.$$ We say that $P$ is nonnegative in $M$ (and denote it by $P \geq 0$ in $M$) if $\lambda_0 := \lambda_0(P, M) \geq 0$. Since $P1 = 0$ in $M$, it follows that $\lambda_0 \geq 0$, that is, $P \geq 0$ in $M$.

Consider the parabolic operator

$$Lu := \frac{\partial u}{\partial t} + Pu \quad (x, t) \in M \times (0, \infty),$$

and let $k^M_P(x, y, t)$ be the minimal positive heat kernel of the parabolic operator $L$ on the manifold $M$. By definition, for a fixed $y \in M$, the function $(x, t) \mapsto k^M_P(x, y, t)$ is a minimal positive solution of the equation

$$Lu = 0 \quad \text{in} \ M \times (0, \infty),$$

subject to the initial data $\delta_y$, the Dirac distribution at $y \in M$. 

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Definition 1.1. Suppose that $\lambda_0 = \lambda_0(P, M) \geq 0$, and let $k^M_P$ be the corresponding heat kernel. We say that the operator $P$ is subcritical (respectively, critical) in $M$ if for some $x \neq y$, (and therefore for any $x \neq y$), $x, y \in M$, we have
\[
\int_0^\infty k^M_P(x, y, t) \, dt < \infty, \quad \text{respectively,} \quad \int_0^\infty k^M_P(x, y, t) \, dt = \infty.
\] (1.3)

If $P$ is subcritical in $M$, then
\[
G^M_P(x, y) := \int_0^\infty k^M_P(x, y, t) \, dt \quad x, y \in M
\] (1.4) is called the minimal positive Green function of the operator $P$ in $M$.

Next we introduce the notion of stochastically completeness.

Definition 1.2. Let $(M, g)$ be a connected and noncompact Riemannian manifold of dimension $N$. Let $P$ be an elliptic operator satisfying $P1 = 0$ in $M$. Then $M$ is said to be stochastically complete (respectively, stochastically incomplete) with respect to $P$ if for some $(x, t) \in M \times (0, \infty)$ (and therefore for all $(x, t) \in M \times (0, \infty)$) we have
\[
\int_M k^M_P(x, y, t) \, dy = 1, \quad \text{respectively,} \quad \int_M k^M_P(x, y, t) \, dy < 1.
\]

Definition 1.3. Suppose that the manifold $M$ and the operator $P$ satisfy the above assumptions. We say that the $L^1$-Liouville property holds on $M$ with respect to $P$ (in short, $M$ is $L^1$-Liouville with respect to $P$) if every nonnegative $L^1$-supersolution of the operator $P$ in $M$ is the constant function.

Remark 1.4. Recall that if $u$ is a positive solution of the equation $Pu = 0$ in $M$, then the generalized maximum principle implies that for any $t > 0$, either
\[
\int_M k^M_P(x, y, t)u(y) \, dy = u(x) \quad \text{or} \quad \int_M k^M_P(x, y, t)u(y) \, dy < u(x) \quad \forall x \in M.
\]

Equivalently, for any $\lambda > 0$ either
\[
\lambda \int_M G^M_{P+\lambda}(x, y)u(y) \, dy = u(x) \quad \text{or} \quad \lambda \int_M G^M_{P+\lambda}(x, y)u(y) \, dy < u(x) \quad \forall x \in M.
\]

Moreover, $P$ is critical in $M$ if and only if $P$ admits a unique (up to a multiplicative constant) positive supersolution (see [18] and references therein). Therefore, if $P$ is critical in $M$ and $P1 = 0$, then $M$ is stochastically complete and $L^1$-Liouville with respect to $P$.

Throughout the paper, unless otherwise stated, we assume that $P$ is subcritical in $M$. In other words, we assume that $P$ admits a (unique) minimal positive Green function $G^M_P$ on $M$. In this paper we are mainly interested in the following question:

Problem 1.5. Given an operator $P$ of the form (2.1) on $M$, construct an operator $P_\rho := \rho P$ for some positive smooth function $\rho$ on $M$ such that $M$ is stochastically incomplete with respect to $P_\rho$, and such that $M$ is $L^1$-Liouville with respect to $P_\rho$ if and only if $M$ is $L^1$-Liouville with respect to $P$. 
We provide an affirmative answer to the above question. Our proof rests on certain constructions of positive supersolutions and related Hardy weights [3, 4], and a generalization of the Omori-Yau maximum principle (see [1, 14]).

Stochastically completeness and the $L^1$-Liouville property on manifolds have been studied extensively in the recent years. We mention a few of them without a claim of completeness (see [1, 6, 9, 13, 14, 16]), a more comprehensive reference can be found in [5] and references therein. It has been shown that certain natural geometric assumptions on manifolds ensure the validity of these properties on a complete manifold, e.g., volume growth, bound on the curvature, etc. These questions were well understood in case of the Laplace-Beltrami operator $\Delta_g$ using potential theoretic arguments and exploiting some underlying geometric conditions. We mention also [15], where the authors study properties like parabolicity, stochastic completeness and $L^1$-Liouville property in a manifold with boundary, under the Dirichlet boundary condition. In [8] the authors studied the parabolicity of a manifold with respect to the Neumann boundary conditions. Moreover, recently stochastic completeness was characterized in [7] in terms of certain properties of nonlinear evolution equations of fast diffusion. Furthermore, in [17] the author constructed an example of an operator $P$ defined on a complete Riemannian manifold $M$, which does not admit invariant positive solution at the bottom of the spectrum, providing a counter example to Stroock’s conjecture (see also [5, 20]). It turns out that the constructed manifold $M$ is stochastically incomplete and not $L^1$-Liouville with respect to $P$. The present article is partly motivated also from the construction of the example in [17], aiming to find a unified approach to construct stochastically incomplete manifold with respect to a second-order elliptic operator.

All the aforementioned results are the primary motivation of the study in this paper. Indeed, we study the two notions defined in definitions 1.2 and 1.3 for general second-order elliptic operators $P$ satisfying $P1 = 0$ in $M$. The rest of the paper is organized as follows. In Section 2 we state the assumptions on the operator $P$, and formulate our main result in Theorem 2.3. Section 3 is devoted to the proof of a few key lemmas which provide equivalent conditions for stochastic completeness and the validity of the $L^1$-Liouville property for second-order elliptic operator $P$. In Section 4 we prove our main result (Theorem 2.3). Section 5 is devoted to the study of the $L^1$-Liouville property and the stochastic (in)completeness for skew product operators. Finally, in Section 6 we give an alternative proof of Theorem 2.3.

2. Preliminaries and the main result

This section is devoted to the statement of the main result of the paper. Before going further we must introduce some technical assumptions and few definitions.

Let $M$ be a smooth, noncompact, and connected manifold of dimension $n$, where $n \geq 2$. We consider a second-order elliptic operator $P$ defined on $M$ with real...
coefficients which (in any coordinate system \( (U; x_1, \ldots, x_n) \)) is of the form

\[
P u = - \sum_{i,j=1}^{n} a^{ij}(x) \partial_i \partial_j u + b(x) \cdot \nabla u.
\]  

(2.1)

We assume that for every \( x \in M \) the matrix \( A(x) := [a^{ij}(x)] \) is symmetric and that the real quadratic form

\[
\xi \cdot A(x) \xi := \sum_{i,j=1}^{n} \xi_i a^{ij}(x) \xi_j \quad \text{for } \xi \in \mathbb{R}^n
\]

is positive definite. Throughout the paper it is assumed that the coefficients of \( P \) are Hölder continuous in \( M \). By a solution \( v \) of the equation \( Pu = 0 \) in \( M \), we mean that \( v \in C^2(M) \) and satisfies the equation pointwise. If \( P \) is of the form (2.1), we obviously have

\[
P 1 = 0.
\]

(2.2)

Whenever we consider the adjoint operator we assume that \( P^* \), the formal adjoint operator of \( P \), also has Hölder continuous coefficients. We say that \( P \) is symmetric if \( P = P^* \).

Remark 2.1. It is well known that any smooth manifold may be considered as an embedded manifold in \( \mathbb{R}^N \), for some \( N \geq n \). Hence, the Euclidean metric and the Lebesgue measure on \( \mathbb{R}^N \) induce a Riemannian metric and a measure on \( M \), which will be considered throughout the paper in case the given manifold \( M \) is not a priori Riemannian. Another possible structure is induced by the principal part of the operator \( P \). Namely, the matrix \( A = (a^{ij}) \) induces a Riemannian metric \( g_A \) on the manifold \( M \) by

\[
(g(\partial/\partial x_i, \partial/\partial x_j)) := (a_{ij}) = A^{-1}.
\]

We recall two fundamental properties of the heat kernel \( k_M^P \) associated with the operator \( P \).

Lemma 2.2. Let \( P \) be a nonnegative second-order elliptic operator defined on \( M \). Then the minimal positive heat kernel \( k_M^P(x, y, t) \) satisfies the following properties:

1. \( k_M^P(x, y, t) \) satisfies the Chapman-Kolmogorov equation (the semigroup property):

\[
k_M^P(x, y, s + t) = \int_M k_M^P(x, z, s) k_M^P(z, y, t) \, dz \quad \forall s, t > 0 \text{ and } \forall x, y \in M.
\]

2. \( k_M^P(x, y, t) \geq 0 \quad \forall t > 0 \text{ and } \forall x, y \in M.\)

Now we are in a position to state the main theorem of the paper.

Theorem 2.3. Let \((M, g)\) be a smooth connected and noncompact manifold of dimension \( n \geq 2 \). Let \( P \) be a subcritical operator in \( M \) of the form (2.1). Assume that \( G(x) := G_M^P(x, o) \), the minimal positive Green function with a singularity at \( o \in M \), satisfies

\[
\lim_{x \to \infty} G(x) = 0,
\]

(2.3)
where $\infty$ is the ideal point of the one-point compactification of $M$.

Then there exists a positive smooth function $\rho$ defined on $M$ such that the operator $P_\rho := \rho P$ satisfies the following properties:

1. $M$ is stochastically incomplete with respect to $P_\rho$.
2. $M$ is $L^1$-Liouville with respect to $P_\rho$ if and only if $M$ is $L^1$-Liouville with respect to $P$.

**Remark 2.4.** Consider a 2-dimensional Riemannian manifold $(M, g)$, and consider a conformal change of the metric $g \mapsto \hat{g} := (\rho(x))^{-1} g$, where $\rho$ is a positive smooth function on $M$. Then the Laplace-Beltrami operator on $(M, \hat{g})$ is given by $\Delta_{\hat{g}} = \rho(x) \Delta_g$.

3. **Key Lemmas**

In this section we prove some of the main ingredients used in the proof of Theorem 2.3. Recently there have been extensive research on the equivalent conditions of stochastic completeness/incompleteness (see [1, 14, 15, 16]). In these papers it has been shown that stochastic completeness is equivalent to Omori-Yau maximum principle. However, all these results are proved in the particular case of $P := -\Delta g$, where $\Delta g$ is the Laplace-Beltrami operator in $M$. It turns out that one can easily extend with some modifications these results to the case of second-order elliptic operators satisfying appropriate assumptions. The next lemma is devoted to address this characterization.

**Lemma 3.1.** Let $M$ be a smooth, connected, noncompact manifold, and let $P$ be a subcritical operator (not necessarily symmetric) of the form (2.1). Then the following assertions are equivalent.

1. $M$ is stochastically complete with respect to $P$.
2. For every $\lambda < 0$, the only nonnegative, bounded classical solution $u$ of $P u = \lambda u$ is $u = 0$.
3. If for any $T \in (0, \infty)$, the Cauchy problem

$$\begin{cases}
\frac{\partial u}{\partial t} + P v = 0, \\
v|_{t=0^+} = 0,
\end{cases}$$

has a bounded solution in $M \times (0, T)$, then necessarily $v = 0$.
4. For $u \in C^2(M)$ with $\sup_M u < \infty$ and for $\alpha > 0$, define

$$\Omega_\alpha := \{ x \in M \mid u(x) > \sup_M u - \alpha \}. \quad (3.1)$$

Then for every such $u$ and $\alpha > 0$, we have $\inf_{\Omega_\alpha} (-P) u \leq 0$.
5. For every $u \in C^2(M)$ with $\sup_M u < \infty$, there exists a sequence $\{x_n\}$ such that for every $n \in \mathbb{N}$,

$$u(x_n) \geq \sup_M u - \frac{1}{n}, \quad \text{and} \quad (-P) u(x_n) \leq \frac{1}{n}.$$
The above lemma is known for the case when $P := -\Delta_g$. A complete proof can be found, for instance in ([3], Theorem 6.2), ([16], Theorem 1.1), see also [6]. For the sake of completeness we present the proof below, with the necessary modifications needed for a general operator $P$ of the form (2.1).

Proof. (2) $\implies$ (1). Suppose that (1) is not true. Then $M$ is stochastically incomplete with respect to $P$. Define

$$u(x, t) = \mathcal{P}_t 1(x, t) := \int_M k^M_P (x, y, t) \, dy.$$  

By our assumption, $0 \leq u(x, t) < 1$. For any $\lambda > 0$ set,

$$v(x) := \int_0^\infty e^{-\lambda t} u(x, t) \, dt = \int_M G^M_{P+\lambda}(x, y) \, dy.$$  

It can be easily checked (see [18]) that

$$(P+\lambda)v(x) = 1.$$  

Clearly, $0 < v < 1/\lambda$. Let $w := 1 - \lambda v$. Then $(P+\lambda)w = 0$ and $0 < w < 1$, which contradicts (2).

(3) $\implies$ (2). Suppose (2) is not true, and let $v$ be a bounded nonzero nonnegative function satisfying $P v = \lambda v$ with $\lambda < 0$. We may assume that $0 < v \leq 1$. Then $u(x, t) := e^{-\lambda t} v(x)$ solves the Cauchy problem

$$\begin{cases} \frac{\partial u}{\partial t} + P u = 0, \\ u|_{t=0^+} = v(x). \end{cases}$$  

(3.2)  

Also, $w(x, t) := \int_M k^M_P (x, y, t) v(y) \, dy$ solves the above Cauchy problem. Using the minimality of $w$, we conclude that $0 \leq w \leq 1$. Moreover, we claim $u \neq w$ for $t > 0$. Indeed, since $\int_M k^M_P (x, y) dy \leq 1$, it follows that

$$\sup_{x \in M} w(x, t) \leq \sup_{x \in M} v(x).$$  

On the other hand, for $t > 0$

$$\sup_{x \in M} u(x, t) = e^{-\lambda t} \sup_{x \in M} v(x) > \sup_{x \in M} v(x) \geq \sup_{x \in M} w(x, t),$$  

and the claim is proved. Hence, the nonzero nonnegative and bounded function $z := u - w$ solves the Cauchy problem

$$\begin{cases} \frac{\partial z}{\partial t} + P z = 0, \\ z|_{t=0^+} = 0, \end{cases}$$  

(3.3)  

which contradicts (3).

(1) $\implies$ (3). We also show this by contraposition. Assume $u(x, t)$ is a nonzero bounded solution of (3.3) in $M \times (0, T)$ for some $T > 0$. Without loss of generality, we may assume that $\sup_{x \in M} u(x, t) > 0$ and $\sup_{x \in M} |u(x, t)| < 1$. Then the function
$w := 1 - u$ is positive and $\inf_{x \in M} w(x, t) < 1$. Since the function $w$ is a solution to the Cauchy problem

$$\begin{align*}
\frac{\partial w}{\partial t} + Pw &= 0, \\
w|_{t=0^+} &= 1,
\end{align*}$$

and $P_1 := \int_M K^M_P(x, y, t) \, dy$ is the minimal positive solution to (3.4), we conclude that $P_1 \leq w$. Therefore, for some $x \in M$ and $t \in (0, T)$,

$$P_1 = \int_M K^M_P(x, y, t) \, dy < 1,$$

and $M$ is stochastically incomplete.

(4) $\implies$ (5) and (5) $\implies$ (2) are trivial. Now to complete the proof it remains to show (2) $\implies$ (4). We argue by a contradiction. Assume that there exists a function $u \in C^2(M)$ with $\sup_M u < \infty$ satisfying for some $\alpha > 0$

$$\inf_{\Omega_\alpha} (-P u) \geq C > 0,$$

where $\Omega_\alpha$ is defined in (3.1). Then following [16, Theorem 1.1], we set $\Omega^* := \left\{ x \in M : (-P)u > \frac{C}{2} \right\}$.

Obviously, $\overline{\Omega_\alpha} \subset \Omega^*$. Moreover, $u + \alpha - \sup_M u$ is a subsolution of $(P + \lambda)$ on $\Omega^*$ with $\lambda = C/(2\alpha)$. Hence $u_\alpha := \max\{u + \alpha - \sup_M u, 0\}$ is a subsolution of $(P + \lambda)$ in $M$. Clearly, $0 < u_\alpha \leq \alpha$. Furthermore, any positive constant is a supersolution of $P + \lambda$ in $M$, and choosing a constant strictly greater than $\alpha$ we have a supersolution $u^+ > u_\alpha$.

Using the “sub/supersolution method” and Perron’s method for $P + \lambda$, we conclude that there exists a solution $v$ of the equation $(P + \lambda)u = 0$ in $M$ that satisfies $0 < u_\alpha \leq v \leq u^+$ which does not vanish identically, but this contradicts (2). $\square$

**Remark 3.2.** In view of Lemma 3.1, stochastically incomplete manifold with respect to $P$ is characterized by the existence of a function $u \in C^2(M)$ with $u^* := \sup_M u < \infty$ such that for any sequence $\{x_n\} \in M$ satisfying

$$\lim_{n \to \infty} u(x_n) = u^*$$

we have

$$\limsup_{n \to \infty} (-P)u(x_n) > 0.$$

Next, we address the $L^1$-Liouville property. Grigor’yan proved in [6, 5] that the $L^1$-Liouville property with respect to the Laplace-Beltrami operator $\Delta_g$ is equivalent to the non-integrability of the Green function of $\Delta_g$. Using Grigor’yan’s approach (see also [2]), we extend in the following lemma the aforementioned result to the case of a subcritical operator of the form (2.1). In fact, Corollary 3.4 demonstrates that the $L^1$-Liouville property is equivalent to the non-integrability of the heat kernel of $P$. 
Lemma 3.3. Let \((M, g)\) be a smooth, connected, noncompact manifold, and let \(P\) be a subcritical operator of the form \((2.1)\). Then the following assertions are equivalent.

1. \(M\) is not \(L^1\)-Liouville with respect to \(P\).
2. For any \(y \in M\), the minimal Green function \(G^M_P(\cdot, y)\) is in \(L^1(M)\).

Proof. \((2) \implies (1)\). Define

\[ u := \min\{G^M_P(\cdot, y), 1\}. \]

Since \(P1 = 0\), it follows that \(u\) is a nonconstant positive \(L^1(M)\)-supersolution. Hence, \(M\) is not \(L^1\)-Liouville with respect to \(P\).

\((1) \implies (2)\). Since \(P1 = 0\), the operator \(P\) satisfies the weak and strong maximum principle. Suppose that \(u \in L^1(M)\) is a nontrivial nonnegative supersolution of \(P\) in \(M\). Then by the strong maximum principle, \(u > 0\) in \(M\).

Fix \(y \in M\) and a compact smooth exhaustion \(\{M_j\}_{j=1}^\infty\) of \(M\) such that \(K = \overline{B(y, 1)} \subset M_1\). Using Harnack’s inequality,

\[ G^M_P(x, y) \leq Cu(x), \quad \forall x \in \partial K, \]

where \(C := C(K)\) is a positive constant. Then using the generalized maximum principle, we obtain \(G^M_P(x, y) \leq Cu(x), \quad \forall x \in M_j \setminus K\), where \(G^M_P(y, x)\) denotes the Dirichlet Green function in \(M_j\). Now letting \(j \to \infty\), we obtain

\[ G^M_P(x, y) \leq Cu(x), \quad \forall x \in M \setminus K. \]

Therefore,

\[
\int_M G^M_P(x, y) \, dx = \int_K G^M_P(x, y) \, dx + \int_{M \setminus K} G^M_P(x, y) \, dx \\
\leq \int_K G^M_P(x, y) \, dx + C \int_{M \setminus K} u(x) \, dx < \infty
\]

since \(G^M_P\) is locally integrable. Hence, \(G^M_P(\cdot, y)\) is in \(L^1(M)\).

In view of the above lemma, we have.

Corollary 3.4. Let \(M\) be a smooth, connected, noncompact manifold, and let \(P\) be an elliptic operator of the form \((2.1)\). Consider its heat kernel \(k^M_P(x, y, t)\). Then \(M\) is \(L^1\)-Liouville with respect to \(P\) if and only if there exists \(y_0 \in M\) such that

\[
\int_M \int_0^\infty k^M_P(x, y_0, t) \, dt \, dx = \infty.
\]

Corollary 3.5. Let \(P\) be a symmetric operator in \(M\) of the form \((2.1)\). If \(P\) is stochastically complete, then \(M\) is \(L^1\)-Liouville with respect to \(P\).

Proof. First consider the case when \(P\) is subcritical, fix \(y \in M\). By Lemma 3.3, it suffices to show that \(G(\cdot, y)\) is not integrable. Using Tonelli’s theorem we get

\[
\int_M G^M_P(x, y) \, dx = \int_M \int_0^\infty k^M_P(x, y, t) \, dt \, dx = \int_0^\infty \int_M k^M_P(x, y, t) \, dx \, dt = \int_0^\infty dt = \infty.
\]
On the other hand, by Remark 1.4 \( P \) is critical in \( M \) if and only if \( P \) admits (up to a multiplicative constant) a unique positive supersolution. Therefore, since \( P1 = 0 \), it follows that in the critical case, \( M \) is \( L^1 \)-Liouville with respect to \( P \). □

We recall an elementary lemma from [17, Lemma 2.2].

**Lemma 3.6.** Let \( P \) be an operator of the form (2.1) and \( \rho \) a positive smooth function. Assume that \( P \) is subcritical in \( M \). Consider the operator \( P \rho = \rho P \) defined on \( M \).

Then \( P \rho \) is subcritical in \( M \) and the minimal positive Green function \( G^M_{P \rho} \) of \( P \rho \) in \( M \) is given by

\[
G^M_{P \rho}(x, y) = \frac{G^M_P(x, y)}{\rho(y)}. \tag{3.6}
\]

### 4. Proof of the main theorem

This section is devoted to the proof of Theorem 2.3. The proof is essentially based on finding a “test function” \( u \in C^2(M) \) which satisfies the hypothesis of Remark 3.2. An alternative method involving optimal Hardy weights is presented in Section 6.

**Proof of Theorem 2.3.** (1) Fix \( o \in M \) and a positive Radon measure \( \mu \) on \( M \) with a smooth positive density \( \mu \). We assume that the corresponding Green potential \( G^\mu \) is finite. That is, we assume that for some \( x \in M \) (and therefore, for any \( x \in M \))

\[
G^\mu(x) := \int_M G^M_P(x, y)\mu(y) \, dy < \infty. \tag{4.1}
\]

Hence, \( PG^\mu = \mu \). Denote by \( G(x) = G^M_P(x, o) \). By the minimality of \( G \), it follows that for \( \varepsilon > 0 \) small enough there exists \( C_\varepsilon > 0 \) such that \( G(x) \leq C_\varepsilon G^\mu(x) \) in \( M \setminus B(o, \varepsilon) \).

We further assume that \( G^\mu(x) \leq C_1 G(x) \) in \( M \). In other words, we have

\[
G^\mu \asymp G \quad \text{in} \quad M \setminus B(o, \varepsilon). \tag{4.2}
\]

In particular, (2.3) and (4.2) imply that \( G^\mu \) is bounded in \( M \) and

\[
\lim_{x \to \infty} G^\mu(x) = 0.
\]

For necessary and sufficient conditions for the validity of (4.2) see [4, Lemma 6.1] and references therein.

Set \( \rho := 1/\mu \), and consider the operator \( P^\rho = \rho P \). Lemma 3.6 implies that \( P^\rho \) is subcritical in \( M \) with the minimal positive Green function

\[
G^M_{P^\rho}(x, y) = \frac{G^M_P(x, y)}{\rho(y)}. \tag{3.6}
\]

In view of Remark 3.2 in order to prove stochastic incompleteness of \( P^\rho \) in \( M \), it is sufficient to construct a function \( u \in C^2(M) \) with \( u^* := \sup_M u < \infty \) such that for any sequence \( \{x_n\} \in M \) satisfying \( \lim_{n \to \infty} u(x_n) = u^* \), we have

\[
\limsup_{n \to \infty} (-P^\rho_n)u(x_n) > 0.
\]

Let \( u := -G^\mu \). Then clearly \( u < 0 \) and \( u \) satisfies \( u(x) \to 0 \) as \( x \to \infty \), where \( \infty \) is the ideal point of the one-point compactification of \( M \). Hence, \( u^* := \sup_M u = 0 < \infty \).
and a sequence \( \{x_n\} \subset M \) satisfies \( \lim_{n \to \infty} u^*(x_n) = u^* = 0 \) if and only if \( x_n \to \infty \).

On the other hand, since \((-\rho P)u = 1\), it follows that for any sequence \( \{x_n\} \) we have

\[
\limsup_{n \to \infty} (-\rho P)u(x_n) = 1 > 0.
\]

Therefore, \( M \) is stochastic incomplete with respect to \( P_\rho \).

(2) We need to prove that \( M \) is \( L^1 \)-Liouville with respect to the operator \( P_\rho := \rho P \) if and only if \( P \) is \( L^1 \)-Liouville with respect to \( P \). In view of Lemma 3.3 this is equivalent to the non-integrability of the Green function \( G^M_{P_\rho}(x, y) \) as a function of \( x \).

Indeed,

\[
\int_M G^M_{P_\rho}(x, y) \, dx = \int_M G^M_P(x, y) \, dx = \mu(y) \int_M G^M_P(x, y) \, dx.
\]

For fixed \( y \), the conclusion follows immediately. \( \square \)

**Remark 4.1.** In light of the above proof, it follows that Theorem 2.3 holds true for any \( \rho = 1/\mu \) such that \( \lim_{x \to \infty} G_\mu(x) = 0 \) (without assuming (4.2)).

**Remark 4.2.** Let \( W_\mu := \mu/G_\mu \). By [4, Lemma 6.2], if (4.2) holds true, then the operator \( P - W_\mu \) is positive-critical in \( M \) with respect to the Hardy-weight \( W_\mu \) and \( G_\mu \) is its ground state.

5. Skew Product Operators and Stochastically Incompleteness

This section is devoted to the study of the skew product of operators \( P_i \) of the form (2.1), defined on smooth, noncompact, and connected manifolds \( M_i, i = 1, 2 \), and its relations to the stochastic completeness and \( L^1 \)-Liouville properties of the operators \( P_i \). Let us recall the definition of skew product (see [10, 11, 12]).

**Definition 5.1.** Let \( M = M_1 \times M_2 \) be the Cartesian product of two manifolds \( M_1 \) and \( M_2 \), and denote a point \( x \) in \( M \) by \( x = (x_1, x_2) \in M = M_1 \times M_2 \). The skew product operator \( P \) of operators \( P_i \) of the form (2.1) defined on \( M_i, i = 1, 2 \), is given by

\[
P := P_1 \otimes I_2 + I_1 \otimes P_2,
\]

where \( I_i \) is the identity map on \( M_i \).

Next, we recall a well known lemma on the heat kernel of a skew product operator. For completeness, we provide the proof.

**Lemma 5.2.** Let \( P_1 \) and \( P_2 \) be nonnegative operators of the form (2.1) defined on \( M_1 \) and \( M_2 \), respectively. Let \( k^M_{P_1} \) and \( k^M_{P_2} \) be the minimal positive heat kernels of \( P_1 \) and \( P_2 \), respectively. Then the minimal positive heat kernel \( k^M_P \) of the skew product operator \( P \) on \( M = M_1 \times M_2 \) satisfies

\[
k^M_P(x, y, t) = k^M_{P_1}(x_1, y_1, t) \cdot k^M_{P_2}(x_2, y_2, t),
\]

where \( x = (x_1, x_2), y = (y_1, y_2) \in M \). Furthermore, if at least one of the above operators is subcritical, then \( P \) is subcritical in \( M \).
Proof. The product formula (5.2) follows immediately from the definition of skew product. In order to prove subcriticality, we need to prove that for some \( x \neq y \), \( x, y \in M \),

\[
\int_0^\infty k^M_P(x, y, t) \, dt = \int_0^\infty k^{M_1}_{P_1}(x_1, y_1, t) \, k^{M_2}_{P_2}(x_2, y_2, t) \, dt < \infty.
\]

Without loss of generality, we may assume that \( P_2 \) is subcritical in \( M_2 \). By Fubini’s theorem it is enough to prove that

\[
\int_{M_1} \int_0^\infty k^{M_1}_{P_1}(x_1, y_1, t) \, k^{M_2}_{P_2}(x_2, y_2, t) \, dt \, dy_1 < \infty.
\]

By the minimality of the heat kernel, it follows that

\[
\int_{M_1} k^{M_1}_{P_1}(x_1, y_1, t) \, dy_1 \leq 1 \quad \text{for some (and hence for all)} \quad (x_1, t) \in M_1 \times (0, \infty).
\]

Therefore, by Tonelli’s theorem and the subcriticality of \( P_2 \) in \( M_2 \), we obtain,

\[
\int_{M_1} \int_0^\infty k^{M_1}_{P_1}(x_1, y_1, t) \, k^{M_2}_{P_2}(x_2, y_2, t) \, dt \, dy_1 \leq 1 \cdot \int_0^\infty k^{M_2}_{P_2}(x_2, y_2, t) \, dt < \infty. \quad \square
\]

The next result leads us to the question of stochastic (in)completeness of a skew product operator.

Lemma 5.3. Let \( P_1 \) and \( P_2 \) be operators of the form (2.1) defined on \( M_1 \) and \( M_2 \), respectively. Then the manifold \( M = M_1 \times M_2 \) is stochastically incomplete with respect to the skew product operator \( P \) if at least one of the manifolds \( M_i \) is stochastically incomplete with respect to \( P_i \).

Proof. Without loss of generality, we may assume that \( M_1 \) is stochastically incomplete manifold with respect to the operator \( P_1 \), and hence for all \( (x_1, t) \in M_1 \times (0, \infty) \) there holds

\[
\int_{M_1} k^{M_1}_{P_1}(x_1, y_1, t) \, dy_1 < 1. \quad (5.3)
\]

Now for \( x = (x_1, x_2), \, y = (y_1, y_2) \in M \), by Tonelli’s theorem we have

\[
\int_M k^M_P(x, y, t) \, dy = \left( \int_{M_1} k^{M_1}_{P_1}(x_1, y_1, t) \, dy_1 \right) \left( \int_{M_2} k^{M_2}_{P_2}(x_2, y_2, t) \, dy_2 \right) \leq \int_{M_1} k^{M_1}_{P_1}(x_1, y_1, t) \, dy_1 < 1.
\]

Another simple alternative proof can be easily derived using Remark 3.2. \( \square \)

The following result concerns the interplay between stochastic (in)completeness and the \( L^1 \)-Liouville property of a skew product operator of the form (5.1).

Theorem 5.4. Let \( P_1 \) and \( P_2 \) be operators of the form (2.1) defined on \( M_1 \) and \( M_2 \), respectively. Let \( P := P_1 \otimes I_2 + I_1 \otimes P_2 \) defined on \( M := M_1 \times M_2 \) be the corresponding skew product operator. Then the following assertions hold true:
(1) If at least one of \( M_i, i = 1, 2 \) is not \( L^1 \)-Liouville with respect to \( P_i \), and the other operator is symmetric, then the skew product operator \( P \) on \( M \) is not \( L^1 \)-Liouville.

(2) If one of the \( M_i, i = 1, 2 \) is \( L^1 \)-Liouville with respect to \( P_i \) and the other is symmetric, then the product manifold \( M \) is \( L^1 \)-Liouville with respect to \( P \).

(3) Assume that both \( M_i, i = 1, 2 \), are stochastically complete with respect to \( P_i \), then the product manifold \( M \) is stochastically complete with respect to \( P \). Moreover, if \( P \) is symmetric in \( M \), then \( M \) is \( L^1 \)-Liouville with respect to \( P \).

(4) If one of the \( M_i, i = 1, 2 \) is stochastically incomplete with respect to \( P_i \), then \( M \) is stochastically incomplete with respect to \( P \).

Proof. (1) Without loss of generality, we assume that \( M_1 \) is not \( L^1 \)-Liouville with respect to \( P_1 \) and \( P_2 \) is symmetric. Then

\[
\int_M G^M_{P}(x_1, x_2, y_1, y_2) \, dx_1 \, dx_2 = \int_0^\infty \int_{M_1} \int_{M_2} k^M_{P_1}(x_1, y_1, t) \, k^M_{P_2}(x_2, y_2, t) \, dx_2 \, dx_1 \, dt
\]

\[
= \int_0^\infty \int_{M_1} \int_{M_2} k^M_{P_1}(x_1, y_1, t) \left( \int_{M_2} k^M_{P_2}(x_2, y_2, t) \, dx_2 \right) \, dt
\]

\[
\leq \int_{M_1} \int_0^\infty k^M_{P_1}(x_1, y_1, t) \, dt \, dx_1 < \infty.
\]

Hence, \( M \) is not \( L^1 \)-Liouville with respect to \( P \).

(2) Without loss of generality, we assume that \( M_1 \) is \( L^1 \)-Liouville with respect to \( P_1 \) and \( P_2 \) is symmetric. Let \((y_1, y_2)\) be any point on \( M_1 \times M_2 \) and assume \( P \) is not \( L^1 \)-Liouville, i.e.,

\[
\int_{M_1} \int_{M_2} \int_0^\infty k^M_{P_1}(x_1, y_1, t) \, k^M_{P_2}(x_2, y_2, t) \, dx_1 \, dx_2 \, dt < \infty.
\]

Then using Fubini’s theorem and the symmetricity of \( P_2 \), we conclude,

\[
\int_{M_1} \int_0^\infty k^M_{P_1}(x_1, y_1, t) \, dt \, dx_1 < \infty,
\]

and hence \( M_1 \) is not \( L^1 \)-Liouville with respect to \( P_1 \) which is a contradiction. Therefore, \( M \) is \( L^1 \)-Liouville with respect to \( P \).

(3) Since both \( M_i, i = 1, 2 \) are stochastically complete with respect to \( P_i \), therefore, we have for \( i = 1, 2 \),

\[
\int_{M_i} k^M_{P_i}(x_i, y_i, t) \, dy_i = 1 \quad \text{for any } (x_i, t) \in M_i \times (0, \infty).
\]

Using Tonelli’s theorem we obtain

\[
\int_M k^M_{P}(x_1, x_2, y_1, y_2, t) \, dy_1 \, dy_2 = \int_{M_1} \int_{M_2} k^M_{P_1}(x_1, y_1, t) \, k^M_{P_2}(x_2, y_2, t) \, dy_2 \, dy_1 = 1.
\]

So, \( M \) is stochastically complete with respect to \( P \). Furthermore, under the symmetricity assumption, Corollary 3.5 implies that \( P \) is \( L^1 \)-Liouville as well.
(4) By Lemma 5.3, $M$ is stochastically incomplete with respect to $P$. \hfill \Box

The following result concerns a case where the $L^1$-Liouville property holds for the skew product operator $P$.

**Corollary 5.5.** If one of the $P_i$, $i = 1, 2$ is symmetric in $M_i$ and the other is critical, then the product manifold $M$ is $L^1$-Liouville with respect to $P$.

**Proof.** Since a critical operator of the form (2.1) is $L^1$-Liouville, the corollary follows from part (2) of Theorem 5.4. \hfill \Box

**Remark 5.6.** We recall that in [5, 17] the authors present a counterexample to a conjecture of A. Grigor’yan of a skew product manifold $M \times K$ that does not satisfy the $L^\infty$-Liouville property, where $(M, g)$ is a complete Riemannian manifold $(\lambda_0(-\Delta_g, M) = 0)$ satisfying the $L^\infty$-Liouville property in $(M, g)$, and $K$ is any compact Riemannian manifold.

### 6. Optimal Hardy weight and Stochastically incomplete

In the present section, we give an alternative proof of Theorem 2.3 under an additional assumption.

**Alternative proof of Theorem 2.3.** **Step 1:** First we recall the main result of Frass, Devyver and Pinchover [3, Theorem 4.12] (also see [19]): Let $P$ be a subcritical operator of the form (2.1) in $M$, and let $G_P^M$ be its minimal positive Green function. Assume that $P$ and $G_P^M$ satisfy all the hypotheses of Theorem 2.3. Let $0 \leq \varphi \in C_c^\infty(M)$, $\varphi \neq 0$, and consider the Green potential given by

$$G_{\varphi}^M(x) = \int_M G_P^M(x, y)\varphi(y) \, dy.$$  

Then $G_{\varphi}^M$ satisfies

$$P(\sqrt{G_{\varphi}^M}) = W \sqrt{G_{\varphi}^M} \geq 0,$$

where $W := \frac{P(\sqrt{G_{\varphi}^M})}{\sqrt{G_{\varphi}^M}}$, and $W = \frac{\vert \nabla(G_{\varphi}^M)^2 \vert_A}{4 \vert G_{\varphi}^M \vert^2}$ in $M \setminus \text{supp} \, \varphi$. Moreover, $W$ is an optimal Hardy-weight for $P$ in $M$ (That is, $P - W$ is null-critical in $M$, and in particular, for any $\lambda > 1$ the operator $P - \lambda W \geq 0$ outside any compact set in $M$).

**We assume that** $W > 0$ **outside a compact set** $K$ **satisfying** $\text{supp} \, \varphi \subseteq K \subset M$.

**Step 2:** Let $\rho$ (to be chosen later) be a positive smooth function on $M$. Consider the operator $P_\rho = \rho P$. In view of Remark 3.2 in order to prove stochastic incompleteness, it is sufficient to construct a function satisfying certain conditions. Let $u$ be a strictly negative $C^2$-smooth function such that

$$u(x) := -(G_{\varphi}^M(x))^{1/2} \quad \forall \, x \in M \setminus K.$$  

Then

$$(-P_\rho)u = \rho W(G_{\varphi}^M)^{1/2} \quad \forall \, x \in M \setminus K.$$
Moreover, since $G^M \varphi \asymp G^M_P$ in a neighbourhood of $\bar{\infty}$, it follows from our assumption (2.3) that $u(x) \to 0$ as $x \to \bar{\infty}$. Hence, $u^* := \sup_M u = 0$, and $\lim_{x_n \to \bar{\infty}} u(x_n) = u^*$ if and only if $x_n \to \bar{\infty}$. For any such a sequence, we eventually have,

$$(-P_\rho)u(x_n) = \rho(x_n)W(x_n)(G^M_\varphi(x_n))^{1/2}.$$ 

Choose a strictly positive smooth function $\rho$ on $M$ such that

$$\rho(x) := \left(W(x)(G^M_\varphi(x))^{1/2}\right)^{-1} \quad \forall x \in M \setminus K,$$

letting $n \to \infty$, we conclude

$$\limsup_{n \to \infty} (-P_\rho)u(x_n) = 1 > 0.$$ 

Hence, by Remark 3.2, $M$ is stochastically incomplete with respect to $P_\rho := \rho P$. □

**Remark 6.1.** We note that the extra assumption that $W > 0$ in a neighbourhood of $\bar{\infty}$ is satisfied in many cases. For example, if $M$ is a smooth bounded domain in $\mathbb{R}^n$, and the coefficients of $P$ are up to the boundary Hölder continuous, then by the Hopf lemma, $W > 0$ in a neighbourhood of $\partial M$.

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