PERELMAN’S FUNCTIONALS ON MANIFOLDS WITH NON-ISOLATED CONICAL SINGULARITIES

XIANZHE DAI AND CHANGLIANG WANG

Abstract. In this article, we define Perelman’s functionals on manifolds with non-isolated conical singularities by starting from a spectral point of view for the Perelman’s $\lambda$-functional. (Our definition of non-isolated conical singularities includes isolated conical singularities.) We prove that the spectrum of Schrödinger operator $-4\Delta + R$ on manifolds with non-isolated conical singularities consists of discrete eigenvalues with finite multiplicities, provided that scalar curvatures of cross sections of cones have a certain lower bound. This enables us to define the $\lambda$-functional on these singular manifolds, and further, to prove that the infimum of $W$-functional is finite, with the help of some weighted Sobolev inequalities. Furthermore, we obtain some asymptotic behavior of eigenfunctions and the minimizer of the $W$-functional near the singularity, and a more refined optimal partial asymptotic expansion for eigenfunctions near isolated conical singularities. We also study the spectrum of $-4\Delta + R$ and Perelman’s functionals on manifolds with more general singularities, i.e. the $r^\alpha$-horn singularities which serve as prototypes of algebraic singularities.

1. Introduction

Perelman’s functionals are some of the crucial ingredients introduced by Perelman [Per02] in his famous work on Ricci flow. These functionals play critical roles in his celebrated proof of Thurston’s geometrization conjecture and Poincaré conjecture, as well as in later studies of the Ricci flow.

We briefly recall Perelman’s functionals on compact manifolds. Let $(M, g)$ be a compact Riemannian manifold without boundary. The $\mathcal{F}$-functional is given as

$$
\mathcal{F}(g, u) = \int_M (4|\nabla u|^2 + R_g u^2) d\text{vol}_g,
$$

1
where $R_g$ is the scalar curvature of $g$ and $u > 0$ is a smooth function. Perelman’s $\lambda$-functional is then defined as

\begin{equation}
\lambda(g) = \inf \left\{ \mathcal{F}(g, u) \mid \int_M u^2 d\text{vol}_g = 1 \right\}.
\end{equation}

Clearly, from (1.2) and (1.1), $\lambda(g)$ is the smallest eigenvalue of the Schrödinger operator $-4\Delta_g + R_g$. Starting from this spectral viewpoint, we developed Perelman’s $\lambda$-functional on compact manifolds with isolated conical singularities in [DW18].

To take the scale into account, Perelman also introduced the $W$-functional and $\mu$-functional on smooth compact manifolds in [Per02]. They play important roles in the study of singularities of the Ricci flow. The $W$-functional is given by

\begin{equation}
W(g, u, \tau) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M \left[ \tau(R_g u^2 + 4|\nabla u|^2) - 2u^2 \ln u - nu^2 \right] d\text{vol}_g,
\end{equation}

where $\tau > 0$ is a scale parameter, and $u > 0$ is a smooth function. The $\mu$-functional is defined by, for each $\tau > 0$,

\begin{equation}
\mu(g, \tau) = \inf \left\{ W(g, u, \tau) \mid u \in C^\infty(M), \ u > 0, \ \frac{1}{(4\pi\tau)^{\frac{n}{2}}} \int_M u^2 d\text{vol}_g = 1 \right\}.
\end{equation}

It is well-known that the finiteness of the infimum in (1.3) follows from the logarithmic Sobolev inequality on smooth compact Riemannian manifolds, while the regularity of the minimizer follows from the elliptic estimates and Sobolev embedding.

In view of the applications to Ricci flow, it is a natural question to develop these functionals for singular spaces. We refer to the excellent survey [Bam21] for some recent developments in Ricci flow and the role played by singular spaces. The possibility of conical singularity appearing in type I singularities of Ricci flow is indicated by the recent work [MW17]. The work [GS18] studies the Ricci flow coming out of manifolds with isolated conical singularities. Conical singularity can also be viewed as a generalization of orbifold singularity and we refer to [KL14] for Ricci flow on orbifolds and its applications.

In [DW20], based on the work in [DW18], we studied the $W$-functional and $\mu$-functional on manifolds with isolated conical singularities. In [Ozu20], Ozuch developed Perelman’s functionals on manifolds with isolated conical singularities from a different point of view. In [KV21], Kröncke and Vertman obtained a partial asymptotic expansion for functions $u$, satisfying $(\Delta_g + cR_g)u = F(u)$ with $F(u) \leq C|u \ln u|$, near isolated conical singularities whose cross sections have scalar curvature the same as the round sphere, and further studied singular Ricci solitons in this case, see Remark 1.7 for more details.

Non-isolated conical singularity plays a crucial role in the resolution of the Yau-Tian-Donaldson conjecture [CDS15, T15]; see also [S18, LZ17] for some related work on Ricci flow. In this paper, we develop Perelman’s functionals on compact manifolds with non-isolated conical singularities as defined in Definition 2.1 below, and also on manifolds with $r^\alpha$-horn singularities as defined in Definition 7.1 below. Our definition of non-isolated singularities includes isolated singularities. In particular, we establish the existence of the
minimizers and obtain asymptotic behavior of the minimizers near the singularity. These asymptotic behaviors are not optimal for non-isolated conical singularity. On the other hand, we obtain an optimal partial asymptotic expansion for eigenfunctions of \(-\Delta + cR\) near a general isolated conical singularity.

1.1. Spectral properties of the operator \(-4\Delta + R\) on manifolds with non-isolated conical singularities. Roughly speaking, a compact manifold with non-isolated conical singularities \((M^n, g)\) is a compact manifold \(M\) with boundary \(\partial M\) and a Riemannian metric \(g\) degenerating in a specific way at the boundary. The boundary \(\partial M\) is the total space of the fibration \(\pi : \partial M \to B\) with the \(f\)-dimensional compact manifold \(F\) as a typical fiber, and in a collar neighborhood \((0, 1) \times \partial M\) of \(\partial M\) the metric \(g\) is asymptotic to \(g_0 = dr^2 + r^2\hat{g} + \pi^*\hat{g}\), where \(r\) is a coordinate on \((0, 1)\), \(\hat{g}\) is a metric on the base \(B\) and \(\hat{g}\) is the restricted metric on fibers \(F\). These singularities are also called wedge singularities (e.g. [Maz91]) or edge singularities (e.g. [Ver21]). Vertman [Ver21] established the existence of Ricci de Turk flow within the class of these singular manifolds whose curvature satisfy some asymptotic control near singularities. Note that when \(B\) is a single point, we are reduced to manifolds with isolated conical singularity.

Our first main result is the following spectral property of the operator \(-4\Delta + R\) on these singular manifolds. This enables us to define Perelman’s \(\lambda\)-functional on these singular manifolds as the first eigenvalue of \(-4\Delta + R\).

**Theorem 1.1** (Theorem 4.4 below). Let \((M^n, g)\), \(n \geq 4\), be a compact Riemannian manifold with non-isolated conical singularities as defined in Definition 2.1 below. If \(\min_{\partial M} \{R\} > (f - 1)\), then the operator \(-4\Delta_g + R_g\) with domain \(C_0^\infty(\hat{M})\) is semi-bounded, and the spectrum of its Friedrichs extension consists of discrete eigenvalues with finite multiplicities \(\lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots\), and \(\lambda_k \to +\infty\), as \(k \to +\infty\). The corresponding eigenfunctions form a complete basis of \(L^2(M)\).

**Remark 1.2.** Similarly Theorem 1.1 extends to the Schrödinger operator \(-\Delta_g + \frac{1}{2} R_g\) for any constant \(c\), where the scalar curvature condition becomes \(\min_{\partial M} \{R\} > \frac{f(f-1)(f-2)}{4c} - 1\). For example, for the well known conformal Laplacian \(-\Delta_g + \frac{n-2}{4(n-1)} R_g\), the scalar curvature condition becomes \(\min_{\partial M} \{R\} > \frac{(f-1)(f-2)}{n-2}\).

**Remark 1.3.** The role of the Schrödinger operator \(-\Delta_g + \frac{1}{2} R_g\) in the study of scalar curvature has attracted a lot attention recently [Sch17]. In [Gro21], Gromov further highlighted the importance of the operator \(-\Delta_g + \frac{1}{2} R_g\) in the study of scalar curvature. In particular, he asks for what class of isolated conical singularity this operator will be positive. Our result gives an answer to this question for general conical singularity. Namely, for non-isolated conical singularity, a sufficient condition is

\[
\min_{\partial M} \{R\} > \frac{1}{2} (f^2 - 1).
\]
For isolated conical singularity, this becomes
\[
\min_{\partial M} \{R_g\} > \frac{1}{2} (n-2)n.
\]

Moreover, we obtain the following asymptotic behavior for eigenfunctions of \(-4\Delta_g + R_g\) near the singular set \(\partial M\), by a Nash-Moser iteration argument.

**Theorem 1.4** (Corollary 5.2 below). Let \(u\) be an eigenfunction of \(-4\Delta_g + R_g\) on a \(n\)-dimensional manifold \((M^n, g)\) with non-isolated conical singularities. Then
\[
(1.4) \quad |\nabla^i u| = o(r^{-\frac{n-2}{2} - i})
\]
as \(r \to 0\), i.e. approaching the singular set \(\partial M\), for \(i = 0, 1\). Here \(r\) is the radial variable on a conical neighborhood of the singular set \(\partial M\).

**Remark 1.5.** (1.4), as well as (1.6) below, is in general not optimal as one expects that the exponent \(-\frac{n-2}{2} - i\) should be replaced by \(-\frac{f-2}{2} - i\). Later, we will establish an optimal partial asymptotic expansion for eigenfunctions of \(-\Delta + cR\) for isolated conical singularity.

1.2. The \(W\)-functional on manifolds with non-isolated conical singularities. For the \(W\)-functional, we extend the results in [DW20] and obtain:

**Theorem 1.6.** Let \((M^n, g)\) be a compact manifold with non-isolated conical singularities. If \(\min_{\partial M} \{R_g\} > (f - 1)\), then for each fixed \(\tau > 0\),
\[
(1.5) \quad \mu(g, \tau) = \inf \left\{ W(g, u, \tau) \mid u \in W^{1, \frac{2}{2}}_{1-\frac{n}{2}}(M), u > 0, \left\| \frac{1}{(4\pi\tau)^{\frac{n}{2}}} u \right\|_{L^2(M)} = 1 \right\} > -\infty.
\]

Here \(W^{1, \frac{2}{2}}_{1-\frac{n}{2}}(M)\) is a weighted Sobolev space defined in (3.11) below.

Moreover, there exists a smooth function \(u > 0\) that realizes the infimum in (1.5). The minimizer satisfies
\[
(1.6) \quad |\nabla^i u| = o(r^{-\frac{n-2}{2} - i}),
\]
as \(r \to 0\), for \(i = 0, 1\). Here \(r\) is the radial variable of a conical neighborhood of the singular set.

We would like to mention that Theorem 1.6 extends to the expander entropy \(W_+\)-functional introduced in [FIN05]. In this paper we will only focus on \(W\)-functional, since the discussion for \(W_+\)-functional is the same.

The finiteness of infimums of \(W\)-functional and \(W_+\)-functional, and the existence of the minimizer follow from the direct method in the calculus of variations, with the help of the (weighted and unweighted) Sobolev embeddings and compactness of some weighted Sobolev embeddings on manifolds with non-isolated conical singularities obtained in §3. The asymptotic behavior of the minimizing function then follows from a Nash-Moser iteration argument as discussed in §6.
In contrast to isolated conical singularities, the scaling technique, which is used in [DW18], does not work any more for deriving weighted Sobolev inequalities and weighted elliptic estimate on manifolds with non-isolated conical singularities. Instead, we adapt the partition argument for an isolated cone used in [DW20] to the non-isolated conical singularities setting, and obtain the (unweighted) Sobolev inequality. A Hardy inequality plays a critical role in this derivation, as well as in the semi-boundedness estimate for the operator $-4\Delta + R$. Then we obtain the weighted Sobolev inequality from the Sobolev inequality.

One can further analyze the asymptotic behavior of the functional $\mu(g, \tau)$ as $\tau \to +\infty$ and $\tau \to 0$ to obtain the finiteness for the $\nu$-entropy on manifolds with non-isolated conical singularities. Indeed, from the Sobolev inequality in Proposition 3.1, the log-logarithmic Sobolev inequality follows. Then, as in the smooth case (see, e.g. Lemma 6.30 in [CCG+07]), one deduces from the logarithmic Sobolev inequality that, when $\min_{\partial M} \{R_\hat{g}\} > f - 2$,

$$\mu(g, \tau) \to \begin{cases} +\infty, & \text{if } \lambda(g) > 0, \\ -\infty, & \text{if } \lambda(g) < 0. \end{cases}$$

(1.7)

On the other hand, one can show the finiteness of the limit $\lim_{\tau \to 0} \mu(g, \tau)$ on manifolds with non-isolated conical singularities, by combining the scaling invariance property: $\mu(cg, c\tau) = \mu(g, \tau)$ for any $c > 0$, with the fact that $\mu(g_{\mathbb{R}^n}, 1) = 0$, as well as $\mu(g_{C(F)}, 1) < \infty$. Here $g_{\mathbb{R}^n}$ is the Euclidean metric on $\mathbb{R}^n$, and $g_{C(F)}$ is the model cone metric on $(0, +\infty) \times F$. The finiteness of $\mu(g_{C(F)}, 1)$ is due to Ozuch [Ozu20]. Therefore, we obtain that

$$\nu(g) = \inf\{\mu(g, \tau) \mid \tau > 0\} > -\infty,$$

(1.8)

on manifolds with non-isolated conical singularities $(M, g)$ satisfying $\min_{\partial M} \{R_\hat{g}\} > f - 2$, provided that $\lambda(g) > 0$.

The finiteness of $\nu$-entropy and Theorems 1.1 and 1.6 as announced in [DW20], answer the existence problem of Perelman’s functionals on manifolds with non-isolated (edge) conical singularities, which was also raised in the survey [KV20].

1.3. The horn singularity. We also study the spectrum of Schrödinger operator $-\Delta + cR$ and Perelman’s functionals on manifolds with (either non-isolated or isolated) more general singularities, i.e. $r^\alpha$-horn in [Che80], whose model is $((0, 1) \times F^f, g_\alpha = dr^2 + r^{2\alpha} \hat{g})$ with $\alpha \in \mathbb{N}$ and $\hat{g}$ a Riemannian metric on $F^f$. This type of singularity occurs naturally in general relativity (e.g., the negative mass Schwarzschild metric has a horn singularity [STIS]) and singular projective variety [HP85].

The study of analysis on manifolds with $r^\alpha$-horn singularities with $\alpha > 1$ is very different from the case of conical singularities (i.e. the case of $\alpha = 1$). For example, neither Hardy inequality nor the partition method used in [DW20] and in the proof of Proposition 3.1 below works for $r^\alpha$-horn singularities with $\alpha > 1$. However, we still establish some
\(\alpha\)-weighted Sobolev inequalities in Proposition 7.3 below as well as some compact \(\alpha\)-weighted Sobolev embeddings in Proposition 7.4 below, by using some change of variables to relate the problem to that on cylinders. Then we obtain:

**Theorem 1.7.** Let \((M^n, g)\) be a compact Riemannian manifold with \(r^\alpha\)-horn singularities as in Definition 7.1 below. If \(\min_{\partial M}\{R_g\} > 0\), then

1. the Friedrichs extension of the Schrödinger operator \(-\Delta_g + cR_g\) \((c > 0)\) with the domain \(C_0^\infty(\hat{M})\) has the spectrum consisting of discrete eigenvalues with finite multiplicity \(-\infty < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots\), and the corresponding eigenfunctions are smooth and form a basis of \(L^2(M, g)\),

2. \(\inf \left\{ W(g, \tau, u) \mid u \in W^{1,2}_{\alpha - n/2, \alpha}(M), \|u\|_{L^2(M)} = (4\pi\tau)^{n/2} \right\} > -\infty\),

3. and an eigenfunction of \(-\Delta + cR\) or the minimizer of the \(W\)-functional, \(u\), has the asymptotic behavior

\[|\nabla^i u| = o(r^{-n/2 + \alpha - i\alpha})\]

as \(r \to 0\), i.e. approaching the singular set, for \(i = 0, 1\). Here \(r\) is the radial variable of a horn-like neighborhood of the singular set.

**Remark 1.8.** Note that the assumption for scalar curvature on the cross section in Theorem 1.7 is weaker than that in the conical singularity case, Cf. Theorems 1.1 and 1.6, and is independent of the dimension of the manifold and the constant \(c\) in the Schrödinger operator.

1.4. **A partial asymptotic expansion of eigenfunction on manifolds with isolated conical singularities.** As we mentioned, the asymptotic behaviors of the minimizers, important in application, are not optimal. This can be much improved for isolated conical singularities. Indeed we obtain the following optimal partial asymptotic expansion for the eigenfunctions, by using Melrose’s b-calculus theory [Mel93] and a weighted elliptic bootstrapping argument.

**Theorem 1.9** (Theorem 8.3 below). Let \(u\) be an eigenfunction of \(-4\Delta_g + R_g\) on a \(n\)-dimensional manifold with an isolated conical singularity. Then

\[u = a(x)r^\mu + o(r^{\mu'})\]  

as \(r \to 0\), where \(\mu = -\frac{n-2}{2} + \sqrt{\nu - (n-2)^2}\) and \(a(x)\) is an eigenfunction with eigenvalue \(\nu\) of \(-4\Delta_{\hat{g}} + R_{\hat{g}}\) on the cross section \((F, \hat{g})\) of the cone, and \(\mu' > \mu\). Here \(r\) is the radial variable on the conical neighborhood of the singular point.

This in particular implies the partial asymptotic expansion for eigenfunctions obtained by Kröncke and Vertman in [KV21], where they considered the case where \((F, \hat{g})\) has the scalar curvature \(R_{\hat{g}} = (n-1)(n-2)\), i.e., that of the round sphere. In this special case, one can easily see that \(\mu \geq 0\) in Theorem 1.9.
On manifolds with non-isolated conical singularities with some additional assumptions, e.g. the metrics on the fibres are isospectral with respect to the operator $-4\Delta + 4R$ and the difference between asymptotic conical metric and model conical metric is smooth up to the singular set, the work of Mazzeo in [Maz91] gives a full asymptotic expansion for the eigenfunctions. Note that these restrictions are not imposed in our work for manifolds with non-isolated conical singularities.

1.5. **Gradient Ricci solitons with isolated conical singularities.** Gradient Ricci solitons are singularity models of Ricci flow. We show that there are no non-trivial gradient steady or expanding Ricci soliton on compact manifolds with isolated conical singularity, provided the scalar curvature of the cross sections of the conical singularities equals that of the sphere. For this purpose, we first obtain, with the help of Melrose’s b-calculus, some asymptotic estimate as in Proposition 9.3 below for the potential functions of compact gradient Ricci solitons with isolated conical singularities. In particular, we show that the potential function of a compact gradient Ricci soliton with isolated conical singularities is bounded and goes to zero when one approaches the singular points, when the above condition on the scalar curvature is satisfied. Using this, if one further assumes that the soliton is either steady or expanding, then we show that it must be Einstein. Here the subtlety is that we do not impose asymptotic condition for the potential functions, and thus, a priori, gradient Ricci solitons with conical singularities may not be critical points of Perelman’s functionals, unlike the smooth case.

**Theorem 1.10.** Let $(M^n, g, f)$ be a compact steady or expanding gradient Ricci soliton with isolated conical singularities, i.e.

\begin{equation}
\text{Ric}_g + \nabla^2 f = \Lambda g,
\end{equation}

with constant $\Lambda \leq 0$. If the cross section of the model cone at each singularity has scalar curvature $R_{\hat{g}} = (n - 1)(n - 2)$, then $(M^n, g)$ must be an Einstein manifold with isolated conical singularities, i.e. $\text{Ric}_g = \Lambda g$.

**Remark 1.11.** More generally, Kröncke and Vertman [KV21] have proved that compact steady and expanding Ricci solitons with isolated conical singularities and dimension $\geq 4$ have to be Einstein, provided that the scalar curvature of cross section of model cones are the same as that of the round sphere. In their argument, in the 4 dimensional case, some asymptotic control for Ricci curvature near the singularities was assumed. Our results hold with neither dimensional restriction nor any extra condition other than the assumption on the scalar curvature.

1.6. **Organization of the paper.** The article is organized as follows. In §2 we give the precise definition for manifolds with non-isolated conical singularities and recall some basic facts for Riemannian submersions. In §3 we derive the Sobolev inequality and some weighted Sobolev inequalities on manifolds with non-isolated conical singularities. We also derive weighted elliptic regularity estimate on manifolds with non-isolated conical singularities.
singularities. In §4, we establish the desired spectral property for the Schrödinger operator $-4\Delta + R$ on manifolds with non-isolated conical singularities. In §§5 and 6 we obtain asymptotic behavior for eigenfunctions of $-4\Delta + R$ and the minimizer of the $W$-functional by using Nash-Moser iteration. In §7 we study the spectrum of $-\Delta + cR$ and Perelman’s functionals on manifolds with $r^\alpha$-horn singularities. In §8 we derive a partial asymptotic expansion for eigenfunctions of $-4\Delta + R$ on manifolds with isolated conical singularities by using the b-calculus theory of Melrose and a weighted elliptic bootstrapping argument. In §9, as another application of b-calculus theory and the asymptotic estimates for functions in weighted Sobolev spaces, we prove that steady and expanding gradient Ricci solitons with isolated conical singularities must be Einstein, provided that the scalar curvature of cross section of model cones are the same as that of the standard sphere.

**Acknowledgements:** The authors would like to thank an anonymous referee whose valuable comments and suggestions have been very helpful in improving the presentation of this work.

2. Manifolds with non-isolated conical singularities

**Definition 2.1.** We say that $(M^n, g)$ is a compact Riemannian manifold with non-isolated conical singularities, if

1. $M^n$ is a compact $n$-dimensional manifold with boundary $\partial M$.
2. The boundary $\partial M$ endowed with a Riemannian metric $g_{\partial M}$ is the total space of a Riemannian submersion

\[ \pi : (\partial M, g_{\partial M}) \longrightarrow (B^b, \tilde{g}) \]

with a $b$-dimensional closed Riemannian manifold $(B^b, \tilde{g})$ as the base and a $f$-dimensional closed manifold $F^f$ as a typical fiber; in particular, $b + f = n - 1$, and

\[ g_{\partial M} = \tilde{g} + \pi^*\tilde{g}, \]

where $\tilde{g}$ denotes metrics on fibers induced by $g_{\partial M}$.
3. Moreover, there exists a collar neighborhood, $U := (0, 1) \times \partial M$, of the boundary $\partial M$ in $M$ with the radial coordinate $r$ ($r = 0$ corresponds to the boundary $\partial M$), and the metric $g$ restricted on $U$ can be written as $g = g_0 + h$, where the Riemannian metric $g_0$ on $U$ is given by

\[ g_0 = dr^2 + r^2 \tilde{g} + \pi^*\tilde{g}, \]

and $2$-tensor $h$ satisfies

\[ |r^k \nabla_{g_0}^k h|_{g_0} = O(r^\alpha), \quad \text{as} \quad r \to 0, \]

for some $\alpha > 0$ and all $k \in \mathbb{N}$.

**Remark 2.2.** We do not require the high order term $h$ in Definition 2.1 to be smooth up to singularities.
Remark 2.3. Note that the Riemannian metric $g_{\partial M}$ on $\partial M$ is not the usual restriction of the Riemannian metric $g$ to the boundary. In fact, in metric sense, the boundary $\partial M$ is collapsed fiberwisely to the base $B$. Thus $M$ is metrically completed by adding $B$ which forms the non-isolated singularities of the completed space. Along the normal direction of each point of this singular stratum, the metric is asymptotic to a cone over a typical fiber.

Now we briefly recall some basic facts about Riemannian submersions, and for more details we refer to [ONe66] and Chapter 9 in [Bes87]. We consider the Riemannian submersion given in (2.1). Let $\mathcal{H}$ and $\mathcal{V}$ denote respectively the horizontal and vertical projections. Let $\nabla^{\partial M}$ denote the Levi-Civita connections on $(\partial M, g_{\partial M})$. The O’Neill’s $A$-tensor and $T$-tensor are given by:

$$A_X Y = \mathcal{H} \nabla^{\partial M}_{\mathcal{V}X} \mathcal{V}Y + \mathcal{V} \nabla^{\partial M}_{\mathcal{H}X} \mathcal{H}Y,$$

$$T_X Y = \mathcal{H} \nabla^{\partial M}_{\mathcal{V}X} \mathcal{V}Y + \mathcal{V} \nabla^{\partial M}_{\mathcal{H}X} \mathcal{H}Y,$$

for any vector fields $X$ and $Y$ on $\partial M$. The tensor $T$ is related to the second fundamental form of fibers, and it vanishes identically if and only if each fiber is totally geodesic.

By the formula (9.37) in [Bes87], the scalar curvature of $g_0$ in (2.3) is given by

$$R_{g_0} = \frac{1}{r^2} (R_{\tilde{g}} - f(f-1)) + R_{\tilde{g}} \circ \pi - |A|^2 - |T|^2 - |N|^2 - \tilde{\delta} N,$$

where the horizontal vector field $N = \sum_{j=1}^{f} T_{V_j} V_j$ is the mean curvature vector of fibers,

$$\tilde{\delta} N = - \sum_{i=1}^{b} g_{\partial M}(\nabla^{\partial M}_{X_i} N, X_i),$$

and $\{V_1, \cdots, V_f, X_1, \cdots, X_b\}$ is a local orthonormal frame for $T(\partial M)$ with respect to $g_{\partial M} = \tilde{g} + \pi^* \tilde{g}$ such that $V_1, \cdots, V_f$ are vertical and $X_1, \cdots, X_b$ are projectable and horizontal.

3. Sobolev and Weighted Sobolev spaces

In this section, we recall some weighted Sobolev spaces on compact Riemannian manifolds with non-isolated conical singularities. We also review and establish some (unweighted) Sobolev and weighted Sobolev embeddings on these singular manifolds that we will use in the following sections.

Various weighted Sobolev spaces have been introduced and studied on manifolds with conical singularities in [Maz91], [BP03], [CLW12], [Beh13], [DW18], [DW20], [Oik20], [KV21] and [Ver21]. These weighted Sobolev spaces turn out to be very useful in the study of elliptic and heat equation problems on conically singular manifolds. In particular, weighted Sobolev inequality and weighted elliptic regularity estimate are important tools in these studies.
Besides weighted Sobolev inequality, in the study of $W$-functional the (unweighted) Sobolev inequality is also necessary (see, §6 below, and [DW20]). In [DY], the usual $L^2$-Sobolev inequality on compact Riemannian manifolds with isolated conical singularities has been established by choosing a suitable partition for a cone and applying a Hardy inequality. We then noticed that their idea also works for $L^p$-Sobolev inequality for $1 < p < n$ in [DW20]. Later, Oikonomopoulos [Oik20] employed this idea to study the $C_0^\infty$ density problem for weighted Sobolev spaces on manifolds with conical singularities. Now we further adapt this idea to non-isolated conical singularities case, and obtain the Sobolev inequality in Proposition 3.1. To our knowledge, the (unweighted) Sobolev inequality on manifolds with non-isolated conical singularities and its proof are not in the literature. So we provide that in §3.1.

In the isolated conical singularities case, thanks to the homogeneity of cone metric along the radial direction, weighted Sobolev inequality and elliptic estimate were obtained in [Beh13] (also see [DW18]) by employing the scaling technique, which had been used by Bartnik [Bar86] for asymptotically Euclidean manifolds. However, in the non-isolated conical singularities case, due to the presence of $\pi^*\tilde{g}$ term, the model singular metric $g_0$ in (2.3) does not have such homogeneity, and as a result, the scaling technique does not work any more. Instead, in §3.2 we derive the weighted Sobolev inequality in the non-isolated conical singularities case from the (unweighted) Sobolev inequality, which is obtained in Proposition 3.1. The local version of the weighted Sobolev inequality was obtained in [CLW12] in a different way.

In §3.3 we then provide two compact weighted Sobolev embeddings in Propositions 3.8 and 3.9 that are used in the proof of Theorem 4.4 and in the study of the $W$-functional in §6. These compact embedding results are generalizations, to the non-isolated conical singularities, of our previous results in Theorem 3.1 in [DW18] and Proposition 3.6 in [DW20] for isolated conical singularities.

In §3.4 we establish a weighted elliptic estimate in Proposition 3.10. Again because the lack of the homogeneity of the model singular metric $g_0$ in (2.3), we are not able to prove a weighed elliptic estimate in the non-isolated conical singularities case by using scaling technique, which was used in the isolated conical singularities case in [Beh13] (also see [DW18]). We notice that a weighted elliptic estimate on manifolds with (either isolated or non-isolated) conical singularities can be proved by doing integration by parts, using the asymptotic control of Riemannian curvature tensor near the conical singularities, and an approximation argument. This idea also works for compact smooth manifolds, and should also work for various non-compact manifolds with certain asymptotic controls for the metric tensor. As this proof is not in the literature, we provide it in §3.4.
3.1. Sobolev spaces and embedding. For any $k \in \mathbb{N}$ and $p \geq 1$, as usual, the Sobolev space $W^{k,p}(M)$ is the completion of $C_0^\infty(\tilde{M})$ with respect to the Sobolev norm

\[
\|u\|_{W^{k,p}(M)} = \left( \int_M \left( \sum_{i=0}^k |\nabla^i u|^p \text{dvol}_g \right) \right)^{\frac{1}{p}},
\]

where $\tilde{M} := M \setminus \partial M$ is the interior of $M$.

**Proposition 3.1.** Let $(M^n, g)$ be a compact Riemannian manifold with non-isolated conical singularities as in Definition 2.1.

(1) For each $1 < p < n$ with $\frac{np}{n-sp}$, $i = 0, \ldots, k - l - 1$, one has

\[
\|u\|_{W^{l,q}(M)} \leq C(M, g, p, k)\|u\|_{W^{k,p}(M)},
\]

for any $u \in C_0^\infty(\tilde{M})$ and any $1 \leq q \leq q_i$, where $C(M, g, p, k)$ is a constant, and $l < k$ and $q_i$ satisfy $\frac{1}{q_i} = \frac{1}{p} - \frac{k-l}{n} > 0$. Hence, there is a continuous embedding $W^{k,p}(M) \subset W^{l,q}(M)$, for any $1 \leq q \leq q_i$.

(2) For $k, j \in \mathbb{N}$ with $k - j > \frac{n}{p}$, one has

\[
\|u\|_{C^j(M)} \leq C(M, g, p, k)\|u\|_{W^{k,p}(M)},
\]

for any $u \in C_0^\infty(\tilde{M})$. Hence, there is a continuous embedding $W^{k,p}(M) \subset C^j(M)$. Here $C^j(M)$ is the completion of $C_0^\infty(\tilde{M})$ with respect to the norm

\[
\|u\|_{C^j(M)} := \max_{0 \leq l \leq j} \sup_M |\nabla^l u|.
\]

(3) If $k, l, j \in \mathbb{N}$ satisfy $k - j - l > \frac{n}{p}$, then any $u \in W^{k,p}(M)$ satisfies

\[
|\nabla^j u| \leq C(M, g, k, l, j, \|u\|_{W^{k,p}(M)})^l,
\]

near the singular set $r = 0$.

Clearly, the key in the derivation of the above Sobolev inequalities is to establish them on the collar neighborhood $U$ of $\partial M$ in the item (3) in Definition 2.1. As in [DY] and [DW20], a Hardy inequality (see, e.g. 330 on p. 245 in [HLP34]) will play a crucial rule. So we recall it: for any $p > 1$ and $a \neq 1$, one has

\[
\int_0^\infty |u|^p x^{-a} \text{d}x \leq \left( \frac{p}{|a-1|} \right)^p \int_0^\infty |u'(x)|^p x^{p-a} \text{d}x,
\]

for any $u \in C_0^\infty((0, \infty))$. This then implies a Hardy inequality (c.f. (3.8) in [DW20]) on $U := (0, 1) \times \partial M$ with the metric $g_0 = dr^2 + r^2 g + \pi^* \tilde{g}$: for $p > 1$ and $k \in \mathbb{N}$ with $pk \neq f$, and any $u \in C_0^\infty(U)$, we have

\[
\int_U |u|^p r^{pk} \text{dvol}_{g_0} \leq \left( \frac{p}{|f - pk|} \right)^p \int_U |\nabla^l u|^p r^{l(p-1)} \text{dvol}_{g_0},
\]
Lemma 3.2. For $1 < p < n$ with $p \neq f$, there is a constant $C$ depending only on $g_0$ and $p$ such that

$$
\|u\|_{L^q(U,g_0)} \leq C\|\nabla u\|_{L^p(U,g_0)},
$$

for any $u \in C_0^\infty(U)$, where $q = \frac{np}{n-p}$.

For $p > n$, there is a constant $C$ depending only on $g_0$ and $p$ such that

$$
\|u\|_{C^0(U)} \leq C\|u\|_{W^{1,p}(U)}
$$

for any $u \in C_0^\infty(U)$.

Proof. First of all, we note that because $B$ is compact, the metrics restricted to fibers $\tilde{g}|_{\pi^{-1}(y)}$ are uniformly quasi-isometric, i.e. there exists a constant $C$ and a Riemannian metric $\tilde{g}_0$ on $F^j$ such that $\frac{1}{C}g_0 \leq \tilde{g}|_{\pi^{-1}(y)} \leq Cg_0$ for all $y \in B$.

Now we choose a finite open cover of the collar neighborhood $U$ as follows, so that on each piece the metric $g_0$ is quasi-isometric to the standard Euclidean metric on $\mathbb{R}^n$.

First, choose a finite open cover $\{U_i\}$ of $B$, such that each $\{U_i, \tilde{g}|_{U_i}\}$ is quasi-isometric to an open domain in the standard Euclidean space $\mathbb{R}^b$, and the submersion $\pi: \partial M \to B$ can be locally trivialized over this open cover, i.e. there exists diffeomorphisms $\varphi_i: \pi^{-1}(U_i) \to U_i \times F$. Let $\tilde{\rho}_i$ be a partition of unity subordinate to this open cover of $B$.

Next, choose a finite open cover $\{V_j\}$ of $F$, such that each $V_j$ can be embedded into Euclidean unite sphere $S^f$ and the metric $\tilde{g}_0$ restricted to each $V_j$ is quasi-isometric to the the round sphere metric. Let $\rho_j$ be a partition of unity subordinate to this open cover of $F$.

So $\{\varphi_i^{-1}(U_i \times V_j)\}$ form a finite open cover of $\partial M$, and further we obtain a finite open cover $\{(0,1) \times \varphi_i^{-1}(U_i \times V_j)\}$ of $U = (0,1) \times \partial M$, with a subordinated partition of unity $\rho_{ij} := \tilde{\pi}^*([\pi^*(\tilde{\rho}_i)]:([\pi^*(\varphi_i)]^{\ast}((\Pr_{2i}) \circ \varphi_i^{\ast})(\tilde{\rho}_j))]$, where $\tilde{\pi}: (0,1) \times \partial M \to \partial M$ and $\Pr_{2i}: U_i \times F \to F$ are natural projections. Then by the construction, one easily sees that $g_0$ restricted to $(0,1) \times \varphi_i^{-1}(U_i \times V_j)$ is quasi-isometric to the standard Euclidean metric on $\mathbb{R}^n$, and more importantly

$$
\|d\rho_{ij}|_{g_0}(r,\theta) \leq C_{ij}r^{-1},
$$

where $C_{ij}$ are some constants.

The usual Sobolev inequality on standard Euclidean spaces together with the estimate \eqref{3.9} and the Hardy inequality \eqref{3.6} imply the Sobolev inequality \eqref{3.7} on the collar neighborhood $U$ with metric $g_0 = dr^2 + r^2\tilde{g} + \pi^*\tilde{g}$.

For Morrey’s inequality \eqref{3.8}, let $u \in C_0^\infty(U)$, because $u = \sum_{i,j} \rho_{ij}u$, one has

$$
\|u\|_{C^0(U)} \leq \sum_{i,j} \|\rho_{ij}u\|_{C^0(U)}
\leq \sum_{i,j} C\|\rho_{ij}u\|_{W^{1,p}(U)}
\quad \text{(Morrey’s inequality on $\mathbb{R}^n$)}
$$
\[ \text{Proof of Proposition 3.1} \]

(1) By using the usual Sobolev inequalities on compact manifolds and applying the Kato’s inequality: \(|\nabla \lvert \nabla^k u \rvert \leq \lvert \nabla^{k+1} u \rvert\), Lemma 3.2 implies the Sobolev inequality in \((3.2)\).

(2) By combining the inequality \((3.8)\) and Morrey’s inequality on compact smooth manifolds, one has that for any \(q > n\) there is a constant \(C\) such that

\[
\lVert u \rVert_{C^0(U)} \leq C \lVert u \rVert_{W^{1,q}(M)}
\]

for any \(u \in W^{1,q}(M)\). As in the smooth case, the general inequality in \((3.3)\) then follows from the Sobolev inequality \((5.2)\) and the Morrey’s inequality \((3.10)\): see e.g. §5.6.3 in [Eva] for more details.

(3) Let \(\varphi\) be a cutoff function with support in \(U = (0,1) \times \partial M\), satisfying \(\varphi \equiv 1\) on \(U_{1/2} := (0,1/2) \times \partial M \subset U\). For \(u \in C_0^\infty(M)\), one has \(\varphi u \in C_0^\infty(U)\). Then Hardy inequality \((3.6)\) and Sobolev inequalities \((3.3)\) imply

\[
\lVert r^{-l} \lvert \nabla^j (\varphi u) \rvert \rVert_{C^0(U_{1/2})} \leq \lVert r^{-l} \lvert \nabla^j (\varphi u) \rvert \rVert_{C^0(U)} \leq C \lVert r^{-l} \lvert \nabla^j (\varphi u) \rvert \rVert_{W^{k-l,j,p}(U)} \leq C \lVert \varphi u \rVert_{W^{k,p}(U)}.
\]

Thus

\[
\lVert r^{-l} \lvert \nabla^j u \rvert \rVert_{C^0(U_{1/2})} \leq C \lVert u \rVert_{W^{k,p}(M)},
\]

since \(\varphi \equiv 1\) on \(U_{1/2}\). This completes the proof of (3) in Proposition 3.1.

\[
\text{□}
\]

3.2. Weighted Sobolev spaces and embedding. In this subsection, we review some weighted Sobolev spaces and derive a weighted Sobolev inequality from Sobolev inequality in Proposition 3.1.

Let \((M^n, g)\) be a compact Riemannian manifold with non-isolated conical singularities as defined in Definition 2.1. For each \(p \geq 1, k \in \mathbb{N}\) and \(\delta \in \mathbb{R}\), the weighted Sobolev space
\( W^{k,p}_\delta(M) \) is the completion of \( C^\infty_0(\hat{M}) \) with respect to the weighted Sobolev norm

\[
\|u\|_{W^{k,p}_\delta(M)} = \left( \int_M \left( \sum_{i=0}^k \chi^{p(\delta - i) + n} |\nabla^i u|^p_d\right) dv_g \right)^{\frac{1}{p}},
\]

where \( \nabla^i u \) denotes the \( i \)-times covariant derivative of the function \( u \) with respect to the metric \( g \), and \( \chi \in C^\infty(\hat{M}) \) is a positive weight function satisfying

\[
\chi(x) = \begin{cases} 
1, & \text{if } x \in M \setminus U, \\
\frac{1}{r}, & \text{if } r = \text{dist}(x, \partial M) < \frac{1}{10},
\end{cases}
\]

and \( 0 < (\chi(x))^{-1} \leq 1 \) for all \( x \in \hat{M} \). Recall that \( U = (0,1) \times \partial M \subset M \) is a collar neighborhood of the boundary \( \partial M \).

By the definition of the weighted Sobolev norm in (3.11), we clearly have

\[
\delta' > \delta \Rightarrow W^{k,p}_{\delta'}(M) \subset W^{k,p}_\delta(M).
\]

Note also that the integral \( \int_0^1 r^{2(\mu - \delta) - 1} (\ln r)^p dr \) is finite if and only if \( \mu > \delta \). Therefore, for a smooth function \( u \) on \( \hat{M} \) satisfying

\[
u = O(r^\mu (\ln r)^p), \quad \text{as } r \to 0,
\]

we have

\[
u \in W^{k,2}_{\delta^2}(M) \iff \mu > \delta.
\]

The weighted Sobolev norm \( \| \cdot \|_{W^{k,p}_\delta} \) defined in (3.11) is essentially the same as the weighted Sobolev norm \( \| \cdot \|_{W^{k,p}_{\delta'}} \) with \( \delta = \gamma - \frac{n}{p} \) in [CLW12].

Besides the Sobolev inequality in Proposition 3.1 as in Lemma 3.1 in [DW20], another interesting consequence of the Hardy inequality (3.6) is the following equivalence between some of the weighted Sobolev norms in (3.11) with certain weight indices and the usual Sobolev norms in (3.1).

**Lemma 3.3.** Let \( (M^n, g) \) be a compact Riemannian manifold with non-isolated conical singularities. For each \( p > 1 \) and \( k \in \mathbb{N} \) satisfying \( pi \neq n \) for \( i = 1, 2, \ldots, k \), there exists a constant \( C = C(g, n, p, k) \) such that

\[
\|u\|_{W^{k,p}(M)} \leq \|u\|_{W^{k,p}_{\frac{n}{p}}(M)} \leq C\|u\|_{W^{k,p}(M)}.
\]

Consequently, \( W^{k,p}_{\frac{n}{p}}(M) = W^{k,p}(M) \) for any \( p > 1 \) and \( k \in \mathbb{N} \) satisfying \( pi \neq n \) for all \( i = 1, 2, \ldots, k \).

By Lemma 3.3 and the Sobolev inequalities in Proposition 3.1 one immediately obtains some weighted Sobolev inequalities with special weight indices. However, we also need weighted Sobolev inequalities with general indices, which can be established by using the Sobolev inequalities in Proposition 3.1 as follows.
Proposition 3.4. Let \((M^n, g)\) be a compact Riemannian manifolds with non-isolated conical singularities. For any \(\delta \in \mathbb{R}, 1 \leq p < n, 0 \leq l \leq k,\) and \(q\) with \(\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n} > 0\) and \(\frac{np}{n-p} \neq f, i = 0, \cdots, k-l-1,\) there exists a constant \(C,\) such that for any \(u \in C^0_0(\hat{M}),\)

\[
\|u\|_{W^{k,q}_{\delta}(M)} \leq C\|u\|_{W^{k,p}_{\delta}(M)}.
\]

Consequently, there is a continuous embedding

\[
W^{k,p}_{\delta}(M) \subset W^{l,q}_{\delta}(M).
\]

Proof. For any \(u \in C^0_0(\hat{M}), \chi^{\delta-i+\frac{n}{q}} |\nabla^i u| \in C^0_0(\hat{M})\) for any \(i \in \mathbb{N}\) and \(q > 1.\) Recall that \(\chi\) is a weight function in (3.12). Then for \(q_1 = \frac{np}{n-p},\) i.e. \(\frac{1}{q_1} = \frac{1}{p} - \frac{1}{n},\) the Sobolev inequality in Proposition [3.1] implies

\[
\|u\|_{W^{k-1,q_1}_{\delta}(M)} = \left( \int_M \left( \sum_{i=0}^{k-1} \chi^{q_1(i\delta-i+n)} |\nabla^i u|^{q_1}_g \right) d\text{vol}_g \right)^{\frac{1}{q_1}}
\]

\[
\leq C \sum_{i=0}^{k-1} \|\chi^{\delta-i+\frac{n}{q_1}} |\nabla^i u|_g\|_{L^{q_1}}
\]

\[
\leq C \sum_{i=0}^{k-1} \|\chi^{\delta-i+\frac{n}{q_1}} |\nabla^i u|_g\|_{W^{1,p}(M)}
\]

\[
\leq C \sum_{i=0}^{k-1} \left( \|\nabla \left( \chi^{\delta-i+\frac{n}{q_1}} |\nabla^i u|_g \right) \|_{L^p(M)} + \|\chi^{\delta-i+\frac{n}{q_1}} |\nabla^i u|_g\|_{L^p(M)} \right)
\]

\[
\leq C \sum_{i=0}^{k-1} \left( \|\chi^{\delta-i+\frac{n}{q_1}+1} |\nabla^i u|_g\|_{L^p(M)} + \|\chi^{\delta-i+\frac{n}{q_1}} |\nabla^{i+1} u|_g\|_{L^p(M)} \right)
\]

\[
+ \|\chi^{\delta-i+\frac{n}{q_1}} |\nabla^i u|_g\|_{L^p(M)} \right)
\]

\[
\leq C \sum_{i=0}^{k} \|\chi^{\delta-i+\frac{n}{p}} |\nabla^i u|_g\|_{L^p(M)} \leq C\|u\|_{W^{k,p}_{\delta}(M)}.
\]

In the third last inequality, we used \(|\nabla \chi|_g \leq C\chi^2.\) In the second last inequality, we used \(0 < \chi^{-1} \leq 1,\) and \(\frac{n}{q_1} = \frac{p}{n} - 1.\)

Then for any \(0 \leq l \leq k,\) one can iterate the above inequality to complete the proof. □

Remark 3.5. The local version of the weighted Sobolev inequalities in Proposition [3.4] has been established in Proposition 3.2 and 3.3 in [CLW12] in a different manner.

Proposition 3.6. Let \((M^n, g)\) be a compact Riemannian manifold with non-isolated conical singularities. For any \(\delta \in \mathbb{R}\) and \(k, l \in \mathbb{N}\) with \(k - l > \frac{n}{p},\) any \(u \in W^{k,p}_{\delta}(M)\) satisfies

\[
|\nabla^l u| = o(\delta^{-k+\frac{n}{p}}),
\]
as \( r \to 0 \).

Proof. First note that if \( u \in W^{k,p}_\delta(M) \) then \( \chi^{-k+\frac{n}{p}}|\nabla^ju| \in W^{k-l,p}(M) \), where \( \chi \) is the weight function in (3.12) and it is equal to \( \frac{1}{r} \) near the singular set. Let \( U_j = \left( \left( \frac{1}{2} \right)^j, \left( \frac{1}{2} \right)^{j-1} \right) \times \partial M \) for each \( j \in \mathbb{N} \). The Sobolev inequality in Proposition 3.1 gives

\[
\|r^{-\delta+k-\frac{n}{p}}|\nabla^ju|\|_{C^0(U_j)} \leq C\|r^{-\delta+k-\frac{n}{p}}|\nabla^ju|\|_{W^{k-l,p}(U_j)} \leq C\|u\|_{W^{k,p}_\delta(U_j)}.
\]

Then because \( \sum_{j=1}^{\infty} \|u\|_{W^{k,p}_\delta(U_j)} \leq \|u\|_{W^{k,p}_\delta(M)} \leq \infty \), \( \|u\|_{W^{k,p}_\delta(U_j)} \to 0 \), as \( j \to \infty \). Thus \( |\nabla^ju| = o(r^{-\delta+k+\frac{n}{p}}) \) as \( r \to 0 \). \( \square \)

3.3. Compact weighted Sobolev embeddings. In this subsection, we prove that weighted Sobolev spaces \( W^{k,2}_{\frac{n}{2}}(M) \) can be compactly embedded into \( L^2(M) \) space. This compact embedding plays a crucial rule in the study of the spectrum of the operator \(-4\Delta_g + R_g\).

Firstly, we derive a compact embedding on the collar neighborhood: \( U = (0,1) \times \partial M \) of the singular set \( \partial M \) with the model metric \( g_0 \) given in (2.3).

Lemma 3.7. The continuous embedding

\[
i : W^{k,2}_{\frac{n}{2}}(U, g_0) \hookrightarrow L^2(U, g_0)
\]

is compact for each \( k \in \mathbb{N} \).

The proof of Lemma 3.7 is similar in spirit as the proof of Lemma 3.2 in [DW18] for isolated conical singularities. Roughly speaking, one relates the problem to that on a cylinder. In [DW18], we did a spectral decomposition for functions on a cone with respect to eigenfunctions of Laplacian on the cross section, when deriving the inequality (3.16) below. In the non-isolated conical singularities case, where we have a family of cones, we cannot do the spectral decomposition as in [DW18], and we derive (3.16) without using the spectral decomposition. So we provide details here.

Proof. Clearly, \( \|u\|_{W^{k,2}_{\frac{n}{2}}(U, g_0)} \geq \|u\|_{W^{l,2}_{\frac{n}{2}}(U, g_0)} \), for \( k \geq l \in \mathbb{N} \). Therefore, it suffices to show that the embedding: \( i : W^{1,2}_{\frac{n}{2}}(C_\epsilon(N)) \hookrightarrow L^2(C_\epsilon(N)) \), is compact.

Let \((C(\partial M), g_{Cyl}) = ((0,1) \times \partial M, dr^2 + g_{\partial M}) \) be a finite cylinder. The usual Sobolev norm on the cylinder is given by

\[
\|u\|_{W^{1,2}(C(\partial M), g_{Cyl})} = \int_{C(\partial M)} (|\nabla u|^2_{g_{Cyl}} + u^2) dvol_{g_{Cyl}}.
\]

Note that the mapping

\[
L^2(U, g_0) \longrightarrow L^2(C(\partial M), g_{Cyl}) \quad \quad u \mapsto \tilde{u} = r^{\frac{\epsilon}{2}}u,
\]
is unitary. We will show that
\begin{equation}
\|u\|_{W^{1,2}(U,g)} \geq \frac{3}{4}\|\bar{u}\|_{W^{1,2}(C(\partial M), g_{Cyl})},
\end{equation}
for all $u \in C_0^\infty(U)$. This then completes the proof, since the embedding
\[ W^{1,2}(C(\partial M), g_{Cyl}) \hookrightarrow L^2(C(\partial M), g_{Cyl}) \]
is compact by the classical Rellich Lemma.

Now we prove the inequality (3.16). For any $u \in C_0^\infty(U)$,
\begin{align*}
\|u\|^2_{W^{1,2}(U,g_0)} &= \int_U \left( \frac{1}{r^2} u^2 + |\nabla u|^2_{g_0} \right) dvol_{g_0} \\
&\geq \int_0^1 \int_{\partial M} \left( \frac{1}{r^2} u^2 + \left| \frac{\partial}{\partial r} \bar{u} \right|^2_{g_0} \right) r^f dvol_{g_0M} dr \\
&= \int_0^1 \int_{\partial M} \left( \frac{1}{r^2} \bar{u}^2 + \left| \frac{\partial}{\partial r} \bar{u} \right|^2_{g_0M} - f^2 \frac{1}{r^2} \bar{u}^2 + |\nabla \bar{u}|^2_{g_0M} \right) dvol_{g_0M} dr \\
&\quad - f \int_{\partial M} \int_0^1 r^{f-1} u \left( \frac{\partial}{\partial r} u \right) dr dvol_{g_0M} \\
&= \int_0^1 \int_{\partial M} \left( \frac{1}{r^2} \bar{u}^2 + \left| \frac{\partial}{\partial r} \bar{u} \right|^2_{g_0M} - f^2 \frac{1}{r^2} \bar{u}^2 + |\nabla \bar{u}|^2_{g_0M} \right) dvol_{g_0M} dr \\
&\quad + \frac{f(f-1)}{2} \int_1^1 \int_{\partial M} \frac{1}{r^2} \bar{u}^2 dvol_{g_0M} dr \\
&= \int_{C(\partial M)} \left( \frac{1}{r^2} \left( 1 + \frac{f(f-2)}{4} \right) \bar{u}^2 + |\nabla \bar{u}|^2_{g_{Cyl}} \right) dvol_{g_{Cyl}} \\
&\geq \int_{C(\partial M)} \left( \frac{1}{r^2} \left( 1 + \frac{f(f-2)}{4} \right) \bar{u}^2 + |\nabla \bar{u}|^2_{g_{Cyl}} \right) dvol_{g_{Cyl}} \\
&\geq \frac{3}{4}\|\bar{u}\|^2_{W^{1,2}(C(\partial M), g_{Cyl})},
\end{align*}

Then by the asymptotic control for the metric $g$ on the collar neighborhood $U$ in condition (3) of Definition 2.1, on a sufficiently small collar neighborhood $U_\varepsilon = (0, \varepsilon) \times \partial M$, metrics $g$ and $g_0$ are quasi-isometric to each other, and so the corresponding weighted Sobolev norms are equivalent. Thus, Lemma 3.7 implies the following compact embedding property.
Proposition 3.8. Let \((M^n, g)\) be a compact manifold with non-isolated conical singularities as in Definition 2.1. The continuous embedding
\[ i : W^{k,2}_{k-\frac{n}{2}}(M) \hookrightarrow L^2(M) \]
is compact for each \(k \in \mathbb{N}\).

We end this subsection by stating another compact weighted Sobolev embedding that will be used in proving the existence of a minimizer of \(W\)-functional.

Proposition 3.9. Let \((M^n, g)\) be a compact Riemannian manifold with non-isolated conical singularities. The embedding \(W^{1,1}_{1-n}(M) \subset L^q(M)\) is compact for any \(1 \leq q \leq \frac{n}{n-1}\).

This proposition can be derived similarly as in the proof of Proposition 3.5 in [DW20], except here we need to use the finite open cover as in the proof of Lemma 3.2 for a collar neighborhood of \(\partial M\). Thus we omit the proof.

3.4. Weighted elliptic estimate. In this subsection, we establish a weighted elliptic estimate on compact manifolds with non-isolated conical singularities in Proposition 3.10. This will be used in the derivation of asymptotic behavior of eigenfunctions of Schrödinger operator \(L := -4\Delta + R\) and the minimizer of \(W\)-functional.

Proposition 3.10. Let \((M, g)\) be a compact manifold with non-isolated conical singularities as in Definition 2.1. If \(u \in W^{k+1,2}_{\delta}(M) \cap C^{k+2}(\overline{M})\) and \(Lu \in W^{k,2}_{\delta-2}(M)\), then
\[
\|u\|_{W^{k+2,2}_{\delta}(M)} \leq C \left( \|\Delta u\|_{W^{k,2}_{\delta-2}(M)} + \|u\|_{W^{k+1,2}_{\delta}(M)} \right)
\]
holds for some constant \(C = C(M, k, \delta)\) independent of function \(u\).

Remark 3.11. Note that this is slightly weaker than the usual elliptic estimate but it suffices for our purpose.

Proof. Clearly, it suffices to prove that there exists a constant \(C = C(M, k, \delta)\) such that
\[
\|u\|_{W^{k+2,2}_{\delta}(M)} \leq C_k \left( \|\Delta u\|_{W^{k,2}_{\delta-2}(M)} + \|u\|_{W^{k+1,2}_{\delta}(M)} \right)
\]
holds for any \(u \in W^{k+1,2}_{\delta}(M) \cap C^{k+2}(\overline{M})\) with \(\Delta u \in W^{k,2}_{\delta-2}(M)\), as
\[
\|\Delta u\|_{W^{k,2}_{\delta-2}(M, g)} \leq \frac{1}{4} \|Lu\|_{W^{k,2}_{\delta-2}(M, g)} + \frac{1}{4} \|Ru\|_{W^{k,2}_{\delta-2}(M, g)} \\
\leq C \left( \|Lu\|_{W^{k,2}_{\delta-2}(M, g)} + \|u\|_{W^{k,2}_{\delta}(M, g)} \right).
\]
To prove the inequality (3.18), we only need to show it for the collar neighborhood \(U = (0, 1) \times \partial M\), that is
\[
\|u\|_{W^{k+2,2}_{\delta}(U, g)} \leq C_k \left( \|\Delta u\|_{W^{k,2}_{\delta-2}(U, g)} + \|u\|_{W^{k+1,2}_{\delta}(U, g)} \right),
\]
since the estimate on the smooth interior part \(M \setminus U\) is classical.
In the rest of the proof, we prove the inequality (3.20) by two steps. In the first step, we prove the inequality (3.20) for \( u \in C_0^\infty(U) \). This is done by direct computation, using integration by parts as well as commuting the derivatives. In the second step, we prove the inequality (3.20) for functions \( u \in C^{k+2}(U) \cap W_{\delta}^{k+2,2}(U, g) \) with \( \Delta u \in W_{\delta}^{k-2}(U, g) \), by doing a cut-off and an approximation argument.

**Step 1.** Let \( u \in C_0^\infty(U) \). We have:

\[
\|u\|_{W_{\delta}^{k-2,2}(U, g)}^2 = \int_U \left( \frac{1}{r} \right)^{2(\delta-k-2)+n} \sum_{i_1, \ldots, i_{k+2} = 1}^n (\nabla_{i_1} \cdots \nabla_{i_{k+2}} u)(\nabla_{i_1} \cdots \nabla_{i_{k+2}} u) dvol_g \\
+ \|u\|_{W_{\delta}^{k+1,2}(U, g)}^2.
\]

Here \( \nabla_{ij} \) denotes the covariant derivative \( \nabla_{e_{ij}} \) with respect to the metric \( g \), where \( \{e_1, e_2, \ldots, e_n\} \) is a local orthonormal frame on \((U, g)\). By doing the integration by parts for the integral in the first term on the right hand side, one obtains

\[
\|u\|_{W_{\delta}^{k-2,2}(U, g)}^2 = -\int_U \sum_{i_1, \ldots, i_{k+2} = 1}^n \nabla_{i_1} \left( \frac{1}{r} \right)^{2(\delta-k-2)+n} (\nabla_{i_1} \cdots \nabla_{i_{k+2}} u)(\nabla_{i_1} \cdots \nabla_{i_{k+2}} u) dvol_g \\
- \int_U \left( \frac{1}{r} \right)^{2(\delta-k-2)+n} \sum_{i_1, \ldots, i_{k+2} = 1}^n (\nabla_{i_1} \nabla_{i_1} \cdots \nabla_{i_{k+2}} u)(\nabla_{i_1} \cdots \nabla_{i_{k+2}} u) dvol_g \\
+ \int_U \left( \frac{1}{r} \right)^{2(\delta-k-2)+n} \sum_{i_1, \ldots, i_{k+2} = 1}^n (\nabla_{\nabla_{e_{i_1}, e_{i_1}} \cdots \nabla_{i_{k+2}} u})(\nabla_{i_1} \cdots \nabla_{i_{k+2}} u) dvol_g \\
+ \|u\|_{W_{\delta}^{k+1,2}(U, g)}^2.
\]

Then one can switch \( \nabla_{i_1} \) with \( \nabla_{i_2}, \ldots, \nabla_{i_{k+2}} \) in the second term on the right hand side to obtain \( \nabla_{i_2} \cdots \nabla_{i_{k+2}} (\Delta u) \), with some additional terms involving the Riemann curvature tensor and its derivatives. Note that the \( t \)th derivatives of curvature can be controlled by \( C_t \left( \frac{1}{r} \right)^{2+1} \) for some constant \( C_t \). So by Cauchy-Schwarz inequality, one has

\[
\|u\|_{W_{\delta}^{k+2,2}(U, g)}^2 \leq \frac{1}{4} \|u\|_{W_{\delta}^{k+2,2}(U, g)}^2 + C \|u\|_{W_{\delta}^{k+1,2}(U, g)}^2 \] (3.21)

\[
- \int_U \left( \frac{1}{r} \right)^{2(\delta-k-2)+n} \sum_{i_2, \ldots, i_{k+2} = 1}^n (\nabla_{i_2} \cdots \nabla_{i_{k+2}} (\Delta u))(\nabla_{i_2} \cdots \nabla_{i_{k+2}} u) dvol_g.
\]
Again, integrating by parts for the third term on the right hand side and switching the order of covariant derivatives, one obtains similarly

\[
\|u\|_{W^{k+2,2}_\delta(U,g)}^2 \\
\leq \frac{1}{2} \|u\|_{W^{k+2,2}_\delta(U,g)}^2 + C \|u\|_{W^{k+1,2}_\delta(U,g)}^2 \\
+ \int_U \left( \frac{1}{r} \right)^{2(\delta-k-2)+n} \sum_{i_3, \ldots, i_{k+2}=1}^n (\nabla_{i_3} \cdots \nabla_{i_{k+2}} (\Delta u)) (\nabla_{i_3} \cdots \nabla_{i_{k+2}} (\Delta u)) d\text{vol}_g \\
\leq \frac{1}{2} \|u\|_{W^{k+2,2}_\delta(U,g)}^2 + C \|u\|_{W^{k+1,2}_\delta(U,g)}^2 + \|\Delta u\|_{W^{k,2}_{\delta-2}(U,g)}^2.
\]  

(3.22)

Finally, by rearranging the inequality, one obtains

\[
\|u\|_{W^{k+2,2}_\delta(U,g)} \leq C \left( \|\Delta u\|_{W^{k,2}_{\delta-2}(U,g)} + \|u\|_{W^{k+1,2}_\delta(U,g)} \right),
\]

(3.23)

for some constant \( C \) and any \( u \in C^\infty_0(U) \).

**Step 2.** Take a cut-off function \( \varphi(r) : (0, +\infty) \to [0, 1] \) such that

\[
\varphi(r) = \begin{cases} 
0, & r \leq 1, \\
1, & r \geq 2,
\end{cases}
\]

(3.24)

\( 0 \leq \varphi \leq 1 \) on \((1, 2)\), and \( |\varphi^{(k)}| \leq C_k \) for each \( k \in \mathbb{N} \) and some constant \( C_k \).

For each \( i \in \mathbb{N} \cup \{0\} \), let \( \varphi_i(r) := \varphi(2^{i+1}r) \). Then

\[
\varphi_i(r) = \begin{cases} 
0, & r \leq \left( \frac{1}{2} \right)^{i+1}, \\
1, & r \geq \left( \frac{1}{2} \right)^i,
\end{cases}
\]

(3.25)

\( 0 \leq \varphi_i \leq 1 \), on \((2^{-(i+1)}, 2^{-i})\), and

\[
|\varphi^{(k)}_i(r)| \leq C_k \left( \frac{1}{r} \right)^k,
\]

for any \( i, k \in \mathbb{N} \cup \{0\} \) and a constant \( C_k \).

For any \( u \in C^{k+2}(U) \cap W^{k+1,p}_\delta(U,g) \) with \( \Delta u \in W^{k,2}_{\delta-2}(U,g) \), let \( u_i = \varphi_i(r)u \in C^k_0(U) \). Then one can check that

\[
u_i \to u \in W^{k+1,2}_\delta(U,g), \quad \text{as } i \to \infty.
\]

\[
\Delta u_i \to \Delta u \in W^{k,2}_{\delta-2}(U,g), \quad \text{as } i \to \infty.
\]

Consequently, we obtain that the inequality (3.20) holds for any \( u \in C^{k+2}(U) \cap W^{k+1,2}_\delta(U,g) \) with \( \Delta u \in W^{k,2}_{\delta-2}(U,g) \), since in step 1 we have shown that it holds for any \( u \in C^\infty_0(U,g) \). This completes the proof of the proposition. \( \square \)
In this section, we study the spectrum of the Schrödinger operator $-4\Delta + R$ on compact Riemannian manifolds with non-isolated conical singularities. First we obtain a semi-boundedness estimate for the operator $-4\Delta g_0 + R g_0$ on a collar neighborhood of $\partial M$ in Lemma 4.1. This then implies a similar estimate for the operator $-4\Delta g + R g$ in Proposition 4.2. Finally, we obtain the spectral properties of the operator $-4\Delta + R$ on compact manifolds with non-isolated conical singularities in Theorem 4.4.

In the following, we still set $L = -4\Delta g_0 + R g_0$.

**Lemma 4.1.** Let $(M^n, g)$ be a compact manifold with non-isolated conical singularities defined as in Definition 2.1 satisfying $\min\{R_{g}\} > (f - 1)$. Then for a sufficiently small $\epsilon > 0$, on the collar neighborhood $U_\epsilon = (0, \epsilon) \times \partial M$ of the singular set $\partial M$ with the model metric $g_0$, we have

\[
(Lu, u)_{L^2(U_\epsilon, g_0)} \geq \delta_0 \|u\|^2_{W^{1, 2}(U_\epsilon, g_0)},
\]

for all $u \in C^\infty_0(U_\epsilon)$, and some $\delta_0 > 0$ depending only on $\min\{R_{g}\}$ and $f$.

**Proof.** Because $\min\{R_{g}\} > (f - 1)$, and

\[
f(f - 1) + 2\delta - \frac{4 - \delta}{4} (f - 1)^2 \to f - 1, \quad \text{as} \quad \delta \to 0,
\]

there exists a sufficiently small $\delta_0 > 0$ such that

\[
\min_{\partial M}\{R_{g}\} > f(f - 1) + 2\delta_0 - \frac{4 - \delta_0}{4} (f - 1)^2.
\]

Then choose a sufficiently small $\epsilon$ such that on $U_\epsilon = (0, \epsilon) \times \partial M$,

\[
\frac{\delta_0}{r^2} + R_{g} \circ \pi - |A|^2 - |T|^2 - |N|^2 - 2\delta N > 0.
\]

Set

\[
L_{\delta_0} := -(4 - \delta_0)\Delta g_0 + R g_0 - \frac{1}{r^2} \delta_0.
\]

Then we have

\[
L = L_{\delta_0} - \delta_0 \Delta g_0 + \frac{1}{r^2} \delta_0,
\]

and for any $u \in C^\infty_0(U_\epsilon)$,

\[
(Lu, u)_{L^2(U_\epsilon, g_0)} = \int_{U_\epsilon} (L_{\delta_0} u) u d\text{vol}_{g_0} + \int_{U_\epsilon} \left( (-\delta_0 \Delta g_0 u) u + \frac{1}{r^2} \delta_0 u^2 \right) d\text{vol}_{g_0}
\]

\[
= \int_{U_\epsilon} (L_{\delta_0} u) u d\text{vol}_{g_0} + \delta_0 \int_{U_\epsilon} \left( \nabla u|^2_{g_0} + \frac{1}{r^2} u^2 \right) d\text{vol}_{g_0}
\]
Thus, it suffices to show that $(L_{\delta_0} u, u)_{L^2(U_\epsilon, g_0)} \geq 0$ to complete the proof. Indeed, we claim that:
\begin{equation}
(4.7) \quad (L_{\delta_0} u, u)_{L^2(U_\epsilon, g_0)} \geq C \| u \|^2_{L^2(U_\epsilon, g_0)},
\end{equation}
holds for any $u \in C^\infty_0(U_\epsilon)$, where
\begin{equation}
(4.8) \quad C = \min \left\{ (4 - \delta_0) \left( \frac{f - 1}{4} \right)^2 + \min \{ R_{\tilde{g}} - 2\delta_0 - f(f - 1), 1 \} \right\} > 0.
\end{equation}
In the rest of the proof, we prove the claim in (4.7). We start with:
\begin{equation}
(4.9) \quad (L_{\delta_0} u, u)_{L^2(U_\epsilon, g_0)} \geq \int_{U_\epsilon} \left( (4 - \delta_0) \left( \frac{f - 1}{4} \right)^2 + \min \{ R_{\tilde{g}} - 2\delta_0 - f(f - 1), 1 \} \right) \overline{u}^2 \, d\text{vol}_{g_0}.
\end{equation}
Setting $\tilde{u} := r^\frac{f}{2} u$, then the right hand side of (4.9) can be rewritten as
\begin{equation}
(4.10) \quad \int_{\partial M} \left[ (4 - \delta_0) \left( \frac{f - 1}{4} \right)^2 + \min \{ R_{\tilde{g}} - 2\delta_0 - f(f - 1), 1 \} \right] \overline{u}^2 \, d\text{vol}_{g_{0\partial M}}.
\end{equation}
Now apply the one dimensional Hardy inequality (3.5) to the first term and use the formula (2.5) for the scalar curvature $R_{\tilde{g}}$ combined with (4.3). Then the quantity in (4.10) is
\begin{align*}
&\geq \int_{\partial M} \left[ (4 - \delta_0) \left( \frac{f - 1}{4} \right)^2 + (4 - \delta_0) \frac{f(f - 2)}{4} \frac{1}{r^2} \tilde{u}^2 \\
&\quad + \left( R_{\tilde{g}} - \frac{\delta_0}{r^2} \right) \tilde{u}^2 \right] d\text{vol}_{g_{0\partial M}}.
\end{align*}
\[
\geq \int_0^c \int_{\partial M} \left( (4 - \delta_0) \frac{(f - 1)^2}{4} + \min_{\partial M} \{ R_g \} - 2\delta_0 - f(f - 1) \right) \frac{1}{r^2} \tilde{u}^2 \text{dvol}_{g_{\partial M}} \text{d}r
\]
\[
(4.11) \geq C \int_0^c \int_{\partial M} \tilde{u}^2 \text{dvol}_{g_{\partial M}} \text{d}r = C \| u \|^2_{L^2(U, g_0)}.
\]

Here the constant $C$ is given by (4.8). Combining (4.9), (4.10), (4.11), this proves the claim in (4.7), and completes the proof of the lemma. \qed

Then because on a sufficiently small collar neighborhood of the boundary the metric $g$ is uniformly equivalent to $g_0$ and the manifold is compact, Lemma 4.1 implies the following semi-boundedness estimate on the manifold with non-isolated conical singularities.

**Proposition 4.2.** Let $(M^n, g)$ be a compact Riemannian manifold with non-isolated conical singularities with $\inf_{\partial M} \{ R_g \} > (f - 1)$. Then there exists a large enough constant $A$ such that the operator $L_A := -4\Delta_g + R_g + A$ satisfies
\[
(L_A u, u)_{L^2(M, g)} \geq C \| u \|^2_{W^{1,2}_{1-\frac{n}{2}}(M, g)},
\]
for any $u \in C^\infty_0(\hat{M})$ and some constant $C > 0$. In particular, the operator $L_A$ with the domain $\text{Dom}(L_A) = C^\infty_0(\hat{M})$ is strictly positive.

**Remark 4.3.** The constant $A$ in Proposition 4.2 is need to control the scalar curvature of the metric $g$ outside of a collar neighborhood of the boundary, since we do not impose any restrictions for it.

**Theorem 4.4.** Let $(M^n, g)$ be a compact Riemannian manifold with non-isolated conical singularities with $\inf_{\partial M} \{ R_g \} > (f - 1)$. Then the spectrum of the Friedrichs extension of the operator $-4\Delta_g + R_g$ with domain $C^\infty_0(\hat{M})$ consists of discrete eigenvalues with finite multiplicities
\[
\lambda_1 \leq \lambda_2 \leq \lambda_2 \leq \cdots,
\]
and $\lambda_k \to \infty$. The corresponding eigenfunctions $\{ u_i \}_{i=1}^\infty$ form a complete basis of $L^2(M)$. Moreover, eigenfunctions $u_i$ are smooth on $\hat{M}$ and satisfy the usual eigenfunction equation on $\hat{M}$.

**Proof.** The existence of the self-adjoint, strictly positive and surjective Friedrichs extension $\tilde{L}_A$ with domain $\text{Dom}(\tilde{L}_A)$ of the operator $L_A$ in Proposition 4.2 follows from the Neumann Theorem in [EK96] and the estimate in Proposition 4.2. Furthermore, Proposition 4.2 also implies that the completion of $C^\infty_0(\hat{M})$ with respect to the norm $\| u \|_{L_A} := (L_A u, u)_{L^2(M, g)}$ is contained in the weighted Sobolev space $W^{1,2}_{1-\frac{n}{2}}(M)$. Thus from the construction of the Friedrichs extension in the proof of the Neumann theorem in [EK96], one can see that $\text{Dom}(\tilde{L}_A) \subset H^1(M)$. 


Because $\tilde{L}_A : \text{Dom}(\tilde{L}_A) \to L^2(M)$ is one-to-one and onto, the inverse
\begin{equation}
\tilde{L}_A^{-1} : L^2(M) \to \text{Dom}(\tilde{L}_A) \hookrightarrow W^{1,2}_{1-\frac{2}{n}}(M) \hookrightarrow L^2(M)
\end{equation}
exists. This inverse map is self-adjoint and compact, since the embedding $W^{1,2}_{1-\frac{2}{n}}(M) \hookrightarrow L^2(M)$ is compact by Proposition 3.8. Then the spectrum theorem of self-adjoint compact operators implies the desired spectral properties of the Friedrichs extension of $L_A$, and hence those for the Friedrichs extension of $L = -4\Delta_g + R_g$ stated in the theorem.

Finally, the regularity of the eigenfunctions follows from the standard elliptic regularity theory for weak solutions of eigenvalue equations, since this is a local property. □

5. Asymptotic behavior of eigenfunctions

In this section, we study the asymptotic behavior of eigenfunctions of $-4\Delta + R$ on manifolds with non-isolated conical singularities and prove Theorem I.4 by a Nash-Moser iteration argument.

**Proposition 5.1.** Let $(M^n, g)$ be a $n$-dimensional compact manifold with non-isolated conical singularities as in Definition 2.1 and $u$ an eigenfunction of $-4\Delta_g + R_g$ with eigenvalue $\lambda$. Then $u$ satisfies
\begin{equation}
\|u\|_{L^\infty((\frac{1}{4}\epsilon, \frac{1}{2}\epsilon) \times \partial M,g)} \leq C \cdot \epsilon^{-\frac{n}{2}} \cdot \|u\|_{W^{2,2}_{1-\frac{2}{n}}((\frac{1}{8}\epsilon, \epsilon) \times \partial M,g)} ,
\end{equation}
and
\begin{equation}
\|\nabla u\|_{L^\infty((\frac{1}{4}\epsilon, \frac{1}{2}\epsilon) \times \partial M,g)} \leq C \cdot \epsilon^{-\frac{n}{2}} \cdot \|u\|_{W^{2,2}_{1-\frac{2}{n}}((\frac{1}{8}\epsilon, \epsilon) \times \partial M,g)} ,
\end{equation}
for some constant $C$ and any $\epsilon < 1$.

**Proof.** Let $w = |\nabla u|^2 + \frac{1}{r^2} u^2$. We apply Nash-Moser iteration to $w$, to obtain the estimates for $u$ in (5.1) and (5.2).

**Step 1.** We derive the following differential inequality for $w$:
\begin{equation}
\Delta w \geq -|R|w - |\lambda|w - Cr^{-2}w.
\end{equation}

The Bochner formula implies
\begin{equation}
\frac{1}{2} \Delta |\nabla u|^2 = |\text{Hess } u|^2 + \langle \nabla u, \nabla \Delta u \rangle + \text{Ric}(\nabla u, \nabla u) \\
\geq \frac{1}{4} \langle \nabla u, \nabla (Ru - \lambda u) \rangle + \text{Ric}(\nabla u, \nabla u) \\
= \frac{1}{4} R |\nabla u|^2 + \frac{1}{4} \langle \nabla u, \nabla R \rangle u - \frac{\lambda}{4} |\nabla u|^2 + \text{Ric}(\nabla u, \nabla u) \\
\geq -\frac{1}{4} |R|w - \frac{|\lambda|}{4} w - C_0 r^{-2}w - \frac{1}{4} |\nabla u| \cdot |\nabla R| \cdot |u| ,
\end{equation}
since Ricci curvature is bounded by $C_0 r^{-2}$. Note that $|\nabla R| \leq C_1 r^{-3}$ near the conically singular points for some positive constant $C_1$. Thus,

$$\frac{1}{2} \Delta |\nabla u|^2 \geq - \frac{1}{4} |R| w - \frac{1}{4} w - C_0 r^{-2} w - C_1 \frac{1}{8} r^{-2} |\nabla u| \cdot |u|$$

(5.5)

$$\geq - \frac{1}{4} |R| w - \frac{1}{4} w - C_0 r^{-2} w - C_1 \frac{1}{8} r^{-2} (|\nabla u|^2 + r^{-2} u^2)$$

$$= - \frac{1}{4} |R| w - \frac{1}{4} w - C_0 r^{-2} w - C_1 \frac{1}{8} r^{-2} w$$

Moreover,

$$\Delta (r^{-2} u^2) = \Delta (r^{-2} u^2) + 2 \langle \nabla r^{-2}, \nabla u^2 \rangle + r^{-2} \Delta u^2$$

(5.6)

$$\geq 4 r^{-4} u^2 - 4 r^{-3} |\nabla u| \cdot |u| - \frac{1}{2} |R| w - \frac{1}{2} |\lambda| w$$

$$\geq 2 r^{-2} w - \frac{1}{2} |R| w - \frac{1}{2} |\lambda| w$$

By combining inequalities in (5.5) and (5.6), we obtain the differential inequality in (5.3)

**Step 2. Iteration process.**

For any $q > 1$ and $\eta \in C^\infty_0 (\tilde{M})$, straightforward calculations give

$$\int |\nabla (\eta w^{\frac{q}{2}})|^2 = \int w^q |\nabla \eta|^2 - \frac{q}{2} \int \eta^2 w^{q-1} \Delta w$$

(5.7)

$$- \left(1 - \frac{2}{q} \right) \int |\nabla (\eta w^{\frac{q}{2}})|^2 - \left(1 - \frac{2}{q} \right) \int \eta w^q \Delta \eta.$$

By rearranging the equation and then using the inequality (5.3), we obtain

$$\int |\nabla (\eta w^{\frac{q}{2}})|^2 \leq \frac{q}{2(q-1)} \int w^q |\nabla \eta|^2 + \frac{q^2 |\lambda|}{8(q-1)} \int \eta^2 w^q + \frac{q^2}{8(q-1)} \int \eta^2 w^q |R|$$

(5.8)

$$+ \frac{q^2 C}{2(q-1)} \int \eta^2 r^{-2} w^q - \frac{q - 2}{2(q-1)} \int \eta w^q \Delta \eta$$

Define $q_k := \mu^k$, for $k \geq 0$, where $\mu = \frac{n}{n-2}$, and

$$a_k := \left(\frac{1}{16} + \frac{1}{16} \sum_{i=0}^{k} \left(\frac{1}{2}\right)^i\right) \epsilon,$$

(5.9)

$$b_k := \left(\frac{5}{4} - \frac{1}{4} \sum_{i=0}^{k} \left(\frac{1}{2}\right)^i\right) \epsilon.$$

(5.10)
Then let $U_k := (a_k, b_k) \times \partial M \subset U$ with the restricted metric $g|_{U_k}$. Choose cut-off functions $\eta_k := \varphi_k(r) \in C^\infty_0(U_k)$ such that

\begin{equation}
\eta_k \equiv 1, \quad \text{on } U_{k+1}; \quad |\varphi_k'| \leq 2^{k+5} \varepsilon^{-1}, \quad |\varphi_k''| \leq 2^{2k+10} \varepsilon^{-2}.
\end{equation}

Now for any $k \geq 1$, applying the Sobolev inequality (with $l = 0, k = 1, p = 2$ and $q = \frac{2n}{n-2}$) in Proposition 3.1 and substituting $\eta_k$ into the inequality in (5.8) give

\begin{equation}
\left( \int_{U_{k+1}} w^q \right)^{\frac{1}{q}} \leq C \left( \int_{U_k} \left( \eta_k^2 w^q \right)^{\mu} \right)^{\frac{1}{\mu}} \\
\leq C \left( \int_{U_k} \left( \eta_k w^q \right)^{\mu} \right)^{\frac{1}{\mu}} + \int_{U_k} \eta_k^2 w^q \\
\leq C \left( \frac{q_k}{2(q_k-1)} \int_{U_k} w^q |\nabla \eta_k|^2 + \frac{q_k^2 |\lambda|}{8(q_k-1)} \int_{U_k} \eta_k^2 w^q \\
+ \frac{q_k^2}{8(q_k-1)} \int_{U_k} \eta_k^2 w^q |R| + \frac{q_k^2 C}{2(q_k-1)} \int_{U_k} \eta_k^2 R^{-2} w^q \\
- \frac{q_k - 2}{2(q_k - 1)} \int_{U_k} \eta_k w^q \Delta \eta_k + \int_{U_k} r^{-2} \eta_k^2 w^q \right)^{\frac{1}{\mu}} \\
\leq C \cdot (4 + |\lambda|) \cdot q_k^2 \cdot 2^{2k+10} \cdot \varepsilon^{-2} \int_{U_k} w^q.
\end{equation}

Then by iterating the above process, one has

\begin{equation}
\|w\|_{L^{q_k+1}(U_{k+1})} \leq (C \cdot (4 + |\lambda|))^{\frac{1}{q_k}} \cdot q_k^2 \cdot 2^{2k+10} \cdot \varepsilon^{-2} \cdot \|w\|_{L^{q_k}(U_k)} \\
\leq \cdots \\
\leq (C \cdot (4 + |\lambda|))^{\sum_{i=1}^{k} \frac{1}{q_i}} \cdot \left( \prod_{i=1}^{k} q_i \right)^{\frac{1}{q_k}} \cdot 2^{\sum_{i=1}^{k} \frac{2i+10}{q_i}} \cdot \varepsilon^{-2} \cdot \sum_{i=1}^{k} \left( \frac{1}{q_i} \right) \cdot \|w\|_{L^n(U_i)}.
\end{equation}

By letting $k \to \infty$ in the inequality in (5.13), we obtain

\begin{equation}
\|w\|_{L^{q}(U_{\infty})} \leq C \cdot \varepsilon^{-(n-2)} \cdot \|w\|_{L^n(U_1)}.
\end{equation}

**Step 3.** We estimate $\|w\|_{L^n(U_1)}$ and complete the proof. By using Sobolev inequality, we have

\begin{equation}
\|\nabla u\|_{L^n(U_1)}^2 = \|\nabla u\|_{L^{2(n-2)}(U_1)}^2 \\
\leq \|\eta \nabla u\|_{L^{2(n-2)}(U_0)}^2 \\
\leq C \left( \|\nabla (\eta \nabla u)\|_{L^2(U_0)}^2 + \|\eta \nabla u\|_{L^2(U_0)}^2 \right)
\end{equation}
Because $u \in W^{1,2}_{1-\frac{n}{2}}(M) \cap C^\infty(M) \subset W^{0,2}_{1-\frac{n}{2}}$ and satisfies the eigenvalue equation $Lu = \lambda u$, Proposition 3.10 implies that $u \in W^{2,2}_{1-\frac{n}{2}}(M)$. So

\begin{equation}
+\infty > \|u\|_{W^{2,2}_{1-\frac{n}{2}}(M)} \\
\geq \|u\|_{W^{2,2}_{1-\frac{n}{2}}(U_0)} \\
\geq \left( \int_{U_0} \left( \frac{1}{r} \right)^{2(1-\frac{n}{2})+n} |\nabla^2 u|^2 \right)^{\frac{1}{2}} \\
\geq C\epsilon \|\nabla^2 u\|_{L^2(U_0)}.
\end{equation}

This gives $\|\nabla^2 u\|_{L^2(U_0)}^2 \leq C\epsilon^{-2} \|u\|_{W^{2,2}_{1-\frac{n}{2}}(U_0)}^2$. Then by substitute this inequality into (5.15), we obtain

\begin{equation}
\|\nabla u\|_{L^2(U_1)}^2 \leq C\epsilon^{-2} \|u\|_{W^{2,2}_{1-\frac{n}{2}}(U_0)}^2.
\end{equation}

Moreover, using Sobolev inequality again gives

\begin{equation}
\|r^{-2} u^2\|_{L^2(U_1)} = \|r^{-1} u^2\|_{L^{\frac{2n}{n-2}}(U_1)} \\
\leq C\epsilon^{-2} \|\eta_0 u\|_{L^{\frac{2n}{n-2}}(U_0)} \\
\leq C\epsilon^{-2} \|u\|_{W^{1,2}_{1-\frac{n}{2}}(U_0)}^2.
\end{equation}

By combining the inequalities in (5.17) and (5.18), we obtain

\begin{equation}
\|w\|_{L^2(U_1)} \leq C\epsilon^{-2} \|u\|_{W^{2,2}_{1-\frac{n}{2}}(U_0)}^2.
\end{equation}

Finally, by substituting the estimate in (5.19) into the inequality in (5.14), we finish the proof. \qed

Now by setting $\epsilon = \left( \frac{1}{2} \right)^j$ for $j \geq 0$ in Proposition 5.1, we have

\begin{equation}
\left( \frac{1}{2} \right) \left( \frac{5}{2} - 1 \right)^j \|u\|_{L^\infty((2^{-j+2}, 2^{-j+1}) \times \partial M)} \leq C \cdot \|u\|_{W^{2,2}_{1-\frac{n}{2}}((2^{-j+3}, 2^{-j}) \times \partial M)} \to 0,
\end{equation}

as $j \to \infty$, since $\sum_{j=1}^\infty \|u\|_{W^{2,2}_{1-\frac{n}{2}}((2^{-j+3}, 2^{-j}) \times \partial M)} \leq 3 \|u\|_{W^{2,2}_{1-\frac{n}{2}}(M)} < \infty$.

The same argument gives an estimate for $|\nabla u|$. We obtain
Corollary 5.2. Let \((M^n, g)\) be a \(n\)-dimensional compact manifold with non-isolated conical singularities as in Definition 2.1, and \(u\) an eigenfunction of \(-4\Delta_g + R_g\). Then \(u\) satisfies
\[
|\nabla^i u| = o(r^{-\frac{n}{2} + 1 - i})
\]
as \(r \to 0\), for \(i = 0\) and \(1\).

6. Lower boundedness of the infimum of \(W\)-functional and asymptotic behavior of the minimizer

By using the estimate in Proposition 4.2, Sobolev inequality in Proposition 3.1, and compactness of weighted Sobolev embeddings in Proposition 3.8 and Proposition 3.9, one can obtain that the infimum of \(W\)-functional is finite and a smooth positive function realizes the infimum, provided \(R_{h_0} > (f - 1)\), via direct methods in the calculus of variations. The proof is verbatim as in the proof of Proposition 4.2 in [DW18]. So we omit it here.

Moreover, the asymptotic estimate for the minimizer of the \(W\)-functional can be obtained by combining weighted elliptic bootstrapping and Nash-Moser iteration as we do for eigenfunctions of \(-4\Delta + R\) in §5. However, unlike the eigenfunction equation, the Euler-Lagrange equation of minimizer of \(W\)-functional in (6.1) below is nonlinear. In the rest of the section, we explain how to deal with new difficulties caused by the nonlinear term \(u \ln u\) in the derivation of asymptotic estimate for the minimizer \(u\) in Theorem 1.

Recall that the Euler-Lagrange equation of the minimizer \(u\) of \(W\)-functional is given by, see e.g. the equation (4.17) in [DW20],
\[
-4\Delta_g u + R_g u - \frac{2}{\tau} u \ln u - \frac{n + m}{\tau} u = 0,
\]
with the constraint \(\|u\|_{L^2(M)} = (4\pi\tau)^{\frac{n}{2}}\), where \(m\) is the infimum of the \(W\)-functional.

Let \(v = |\nabla u|^2 + r^{-2}u^2\). As in step 1 in the proof of Proposition 5.1 one easily sees that
\[
\Delta (r^{-2}u^2) \geq 2r^{-2}v - \frac{1}{2} |R|v - \frac{|n + m|}{8\tau} v - \frac{1}{\tau} r^{-2}u^2 \ln u,
\]
and
\[
\Delta |\nabla u|^2 \geq -\frac{1}{2} |R|v - \frac{|n + m|}{\tau} v - Cv - Cr^{-2}v - \frac{1}{\tau} |\nabla u|^2 \ln u.
\]
So
\[
\Delta v \geq -|R|v - Cv - Cr^{-2}v - \frac{1}{\tau} v \ln u.
\]

Then substituting \(v\) into the inequality in (5.8) gives
\[
\int |\nabla (\eta v^{\frac{2}{q}})|^2 = \frac{q}{2(q - 1)} \int v^q |\nabla \eta|^2 - \frac{q^2}{4(q - 1)} \int \eta^2 v^{q - 1} \Delta v - \frac{q - 2}{2(q - 1)} \int \eta v^q \Delta \eta
\]
for any nonnegative \( \eta \in C_0^\infty(M) \) and \( q > 1 \). Then the only difference from the eigenfunction estimate in Proposition 5.1 is the term involving the integral of \( v^q \ln u \). So we briefly describe how to deal with this term.

For a sufficiently small \( \gamma < 1 \) (with \( \frac{2}{\gamma} > \frac{n}{2} \)), there exists a constant \( a \) such that

\[
(6.5) \quad \frac{q}{2(q - 1)} \int v^q |\nabla \eta|^2 - \frac{q^2}{4(q - 1)} \int \eta^2 v^q |R| + C \frac{q^2}{4(q - 1)} \int \eta^2 v^q + C \frac{q^2}{4(q - 1)} \int \eta^2 r^{-2} v^q + \frac{q^2}{8(q - 1)\tau} \int \eta^2 v^2 \ln u - \frac{q - 2}{2(2q - 1)} \int \eta^2 v^q \Delta \eta,
\]

for any \( \eta \in C_0^\infty(M) \) and \( q > 1 \). Then the only difference from the eigenfunction estimate in Proposition 5.1 is the term involving the integral of \( v^q \ln u \). So we briefly describe how to deal with this term.

For a sufficiently small \( \gamma < 1 \) (with \( \frac{2}{\gamma} > \frac{n}{2} \)), there exists a constant \( a \) such that

\[
(6.6) \quad \int \eta^2 v^q \ln u = \int_{\{0 < u \leq \varepsilon\}} \eta^2 v^q \ln u + \int_{\{u > \varepsilon\}} \eta^2 v^q \ln u
\]

(6.7) \quad \leq \int_{\{0 < u \leq \varepsilon\}} \eta^2 v^q + a \int_{\{u > \varepsilon\}} \eta^2 v^q u^\gamma
\]

(6.8) \quad \leq \int \eta^2 v^q + a \int \eta^2 v^q u^\gamma.
\]

Then by Hölder inequality and Young’s inequality (e.g. as on p. 20 in [DWZ18]),

\[
(6.9) \quad a \frac{q^2}{8(q - 1)\tau} \int \eta^2 v^q u^\gamma
\]

(6.10) \quad \leq a (4\pi\tau)^{\frac{n\gamma}{4}} \left[ \delta \left( \int (\eta^2 v^q)^\mu \right)^{\frac{1}{\mu}} + \delta^{-\frac{n\gamma}{4-n\gamma}} \left( \frac{q^2}{8(q - 1)\tau} \right)^{\frac{4-2n\gamma}{4-n\gamma}} \left( \int \eta^2 v^q \right) \right],
\]

for any small \( \delta > 0 \). We may take \( \delta \) so that \( C \cdot a \cdot (4\pi\tau)^{\frac{n\gamma}{4}} \cdot \delta = \frac{1}{2} \), where \( C \) is the \( L^2 \)-Sobolev constant in Proposition 3.1.

Another remark is about the derivation of \( u \in W_1^{0,2}(M) \) by using Proposition 3.10. For this one needs to show that \( u \ln u \in W_1^{0,2}(M) \) and the fact that there exits a constant \( a(n) \) such that \( |u \ln u| \leq a(n) + |u|^\frac{n}{n-2} \). Indeed, \( \int_M \chi^2 (\frac{\varepsilon^2}{2} + n) |u \ln u|^2 \leq a(n)^2 \text{Vol}(M) + \int_M |u|^\frac{n}{n-2} < \infty \). So \( u \ln u \in W_1^{0,2}(M) \subset W_1^{0,2}(\frac{n}{2}) \).

Then the Nash-Moser iteration as in the proof of Proposition 5.1 gives the asymptotic behavior in Theorem 1.6.
7. Horn singularity

In this section, we study the spectrum and eigenfunctions of the Schrödinger operators
\[-\Delta_g + cR_g(c \geq 0)\] on compact manifolds with a kind of more general singularities, whose model is \((0, 1) \times F^f, g_\gamma = dr^2 + r^{2\gamma}\hat{g})\). Here \(\gamma > 0\), and \(\hat{g}\) is a Riemannian metric on \(F^f\). This is a \(r^\gamma\)-horn in [Che80]. If in particular \(\gamma = 1\), then this is a model cone singularity that we studied in preceding sections. So now we assume \(\gamma > 1\), and this kind of singularities appear in the study of singular projective varieties.

The model singular metric \(g_\gamma = dr^2 + r^{\gamma}\hat{g}\) has Laplacian \(\Delta_{g_\gamma}\) and scalar curvature \(R_{g_\gamma}\) as
\[
\Delta_{g_\gamma} = \frac{\partial^2}{\partial r^2} + \frac{\gamma(n-1)}{r} \frac{\partial}{\partial r} + \frac{1}{r^{2\gamma}} \Delta_{\hat{g}},
\]
\[
R_{g_\gamma} = \frac{R_{\hat{g}}}{r^{2\alpha}} - \frac{\gamma(n-1)(n\gamma - 2)}{r^2},
\]
where \(n = f + 1\).

7.1. manifolds with \(r^\gamma\)-horn singularities. Similar as Definition 2.1, we define

Definition 7.1. We say that \((M^n, g)\) is a compact Riemannian manifold with \(r^\gamma\)-horn singularities \((\gamma > 1)\), if

1. \(M^n\) is a compact \(n\)-dimensional manifold with boundary \(\partial M\).
2. The boundary \(\partial M\) endowed with a Riemannian metric \(g_{\partial M}\) is the total space of a Riemannian submersion
\[
\pi : (\partial M, g_{\partial M}) \longrightarrow (B^b, \hat{g})
\]
with a \(b\)-dimensional closed Riemannian manifold \((B^b, \hat{g})\) as base and a \(f\)-dimensional closed manifold \(F^f\) as a typical fiber, in particular, \(b + f = n - 1\), and

\[
g_{\partial M} = \hat{g} + \pi^*\hat{g},
\]

where \(\hat{g}\) denotes metrics on fibers induced by \(g_{\partial M}\);

3. Moreover, there exists a collar neighborhood \(U = (0, 1) \times \partial M\) of the boundary \(\partial M\) in \(M\) with the radial coordinate \(r\) \((r = 0\) corresponds to the boundary \(\partial M\)), and the metric \(g\) restricted on \(U\) satisfies \(g = g_\gamma + h\) as \(r \to 0\), where the Riemannian metric \(g_\gamma\) on \(U\) given by
\[
g_\gamma = dr^2 + r^{2\gamma}\hat{g} + \pi^*\hat{g},
\]
and \(h\) satisfies
\[
|r^{\gamma k} \nabla_{g_\gamma}^k h|_{g_\gamma} = O(r^\alpha), \quad \text{as} \quad r \to 0,
\]
for some \(\alpha > 0\) and all \(k \in \mathbb{N}\).
7.2. \(\alpha\)-weighted Sobolev spaces. Let \((M^n, g)\) be a compact Riemannian manifold with \(\alpha\)-horn singularities as defined in Definition 7.1. For each \(p \geq 1\), \(k \in \mathbb{N}\) and \(\delta \in \mathbb{R}\), the \(\alpha\)-weighted Sobolev space \(W^{k,p}_{\delta,\alpha}(M)\) is the completion of \(C^\infty_0(\hat{M})\) with respect to the \(\gamma\)-weighted Sobolev norm

\[
\|u\|_{W^{k,p}_{\delta,\gamma}(M)} = \left( \int_M \left( \sum_{i=0}^k \chi^{p(\delta - \gamma) + n\gamma} |\nabla^i u|^p_g \right) d\text{vol}_g \right)^{\frac{1}{p}},
\]

where \(\nabla^i u\) denotes the \(i\)-times covariant derivative of the function \(u\), and \(\chi \in C^\infty(\hat{M})\) is a positive weight function satisfying

\[
\chi(x) = \begin{cases} 
1, & \text{if } x \in M \setminus U, \\
\frac{1}{r}, & \text{if } r = \text{dist}(x, \partial M) < \frac{1}{10},
\end{cases}
\]

and \(0 < (\chi(x))^{-1} \leq 1\) for all \(x \in \hat{M}\). Recall that \(U = (0, 1) \times \partial M \subset M\) is the collar neighborhood of the boundary \(\partial M\).

On a manifold \((M^n, g)\) with isolated \(r^\gamma\)-horn singularities, i.e. the base manifold \(B\) in Definition 7.1 is a point, one can derive the following weighted Sobolev inequalities.

**Proposition 7.2.** Let \((M^n, g)\) be a manifold with an isolated \(r^\gamma\)-horn singularity. For any \(\delta \in \mathbb{R}\), \(1 \leq p < n\), \(0 \leq l \leq k\), and \(q\) with \(\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n} > 0\), there exists a constant \(C\), such that

\[
\|u\|_{W^{l,q}_{\delta,\gamma}(M,g)} \leq C\|u\|_{W^{k,p}_{\delta,\gamma}(M,g)}
\]

holds for all \(u \in C^\infty_0(\hat{M})\).

Consequently, there is a continuous embedding

\[
W^{k,p}_{\delta,\gamma}(M,g) \subset W^{l,q}_{\delta,\gamma}(M,g).
\]

**Proof.** The key of establishment of the \(\gamma\)-weighted Sobolev embedding is to derive it on a collar neighborhood \((0, 1) \times \partial M\) of \(\partial M\) with the model metric \(g_\gamma = dr^2 + r^{2\gamma} \hat{g}\).

For any \(u \in C^\infty_0((0, 1) \times \partial M),\)

\[
\|u\|_{W^{0,p}_{\delta,\gamma}((0,1) \times \partial M,g_\gamma)} = \left( \int_{(0,1) \times \partial M} (u(r, x))^\frac{pn}{n-p} \left( \frac{1}{r} \right)^\frac{pn}{n-p} \delta + n\gamma \frac{1}{n} \frac{d\text{vol}_g}{r^{(n-1)\gamma}} \right)^\frac{n-p}{np}
\]

\[
= \left( \int_{\partial M} \int_0^1 (u(r, x))^{\frac{pn}{n-p}} \left( \frac{1}{r} \right)^\frac{pn}{n-p} \delta + n\gamma \frac{1}{n} \frac{dr}{r} \frac{d\text{vol}_\hat{g}}{r^{(n-1)\gamma}} \right)^\frac{n-p}{np}
\]

\[
= \left( \int_{\partial M} \int_0^1 (u(r, x)r^{-\delta})^{\frac{pn}{n-p}} r^{-\gamma} \frac{dr}{r} \frac{d\text{vol}_\hat{g}}{r^{(n-1)\gamma}} \right)^\frac{n-p}{np}
\]
In the last integral, we make a change of variable \( r = s^{\frac{1}{1-\gamma}} \), and set \( \tilde{u}(s, x) = u(s^{\frac{1}{1-\gamma}}, x) \). Then \( \tilde{u} \in C^\infty_c((1, \infty) \times \partial M) \), and one has

\[
\| u \|_{L^{\frac{np}{n-p}}(\mathbb{R}^n \times \partial M, g_0)} \leq \left( \frac{1}{\gamma - 1} \right)^{\frac{np}{n-p}} C \| \tilde{u}(s, x) s^{-\frac{\delta}{1-\gamma}} \|_{W^{1, p}(\mathbb{R}^n \times \partial M, g_0)}
\]

\[
= \left( \frac{1}{\gamma - 1} \right)^{\frac{np}{n-p}} C \left( \int_{\partial M} \int_{1}^{\infty} \left( |\nabla (\tilde{u}(s, x) s^{-\frac{\delta}{1-\gamma}})|_{g_0}^p + |\tilde{u}(s, x)|^p s^{-\frac{\delta p}{1-\gamma}} ds \right) \right)^{\frac{1}{p}}
\]

\[
\leq \left( \frac{1}{\gamma - 1} \right)^{\frac{np}{n-p}} C \left( \int_{\partial M} \int_{1}^{\infty} \left( \frac{\delta}{\gamma - 1} s^{\frac{\delta p}{1-\gamma}} |\tilde{u}(s, x)|^p + s^{\frac{\delta p}{1-\gamma}} |\partial s \tilde{u}(s, x)|^p \right) d\sigma \right)^{\frac{1}{p}}
\]

Now change the variable back, i.e. let \( s = r^{1-\gamma} \), and note \( \partial_s \tilde{u}(s, x) = \partial_r (r, x) \left( \frac{1}{r^{1-\gamma}} \right) r^{\gamma} \). One has

\[
\| u \|_{W^{\frac{np}{n-p}}(\mathbb{R}^n \times \partial M, g_0)} \leq \left( \frac{1}{\gamma - 1} \right)^{\frac{np}{n-p}} C \left[ \int_{\partial M} \int_{0}^{1} \left( \frac{\delta}{\gamma - 1} r^{\delta p - p(\gamma - 1)} |u(r, x)|^p + \frac{1}{\gamma - 1} |\partial_r u(r, x)|^p + r^{\delta p} |\nabla^\gamma u(r, x)|_{g_0}^p \right)^{\frac{1}{p}} \right]
\]

\[
\leq C \left[ \int_{\partial M} \int_{0}^{1} \left( \frac{1}{r} r^{p(\delta - \gamma) + n\gamma} \left( |\partial_r u(r, x)|^p + \frac{1}{r} \right)^p |\nabla^\gamma u(r, x)|_{g_0}^p \right) \right]
\]
\[
\begin{aligned}
&+ \left( \frac{1}{r} \right)^{p\delta + n\gamma} |u(r, x)|^p \right] r^{(n-1)\gamma} dr d\text{vol}_g \right)^{\frac{1}{p}} \\
\leq C \left( \int_{(0,1) \times \partial M} \left[ \left( \frac{1}{r} \right)^{p(\delta - \gamma) + n\gamma} |\nabla^g u|^p_{g_\gamma} + \left( \frac{1}{r} \right)^{p\delta + n\gamma} |u(r, x)|^p \right] d\text{vol}_{g_\gamma} \right)^{\frac{1}{p}} \\
&= C \|u\|_{\dot{W}^{1,p}_{\delta,\gamma}((0,1) \times \partial M, g_\gamma)}. 
\end{aligned}
\]

Now combining this with the usual Sobolev inequality in the interior part of the manifold \( M \), one can obtain that for any given \( \delta \in \mathbb{R} \) and \( p \geq 1 \), there exists a constant \( C \) such that

\[(7.5) \quad \|u\|_{\dot{W}^{0, np}_{\delta,\gamma}((0,1) \times \partial M, g_\gamma)} \leq C \|u\|_{\dot{W}^{1,p}_{\delta,\gamma}(M, g_\gamma)} \]

for all \( u \in C^\infty_0(\hat{M}) \).

Then by the inequality (7.5), Kato’s inequality and the definition of the \( \gamma \)-weighted Sobolev norm, one has

\[
\|\nabla u\|_{\dot{W}^{0, np}_{\delta,\gamma}((0,1) \times \partial M, g_\gamma)} \leq C \|\nabla u\|_{\dot{W}^{1,p}_{\delta,\gamma}(M, g_\gamma)} \leq C \|u\|_{\dot{W}^{2,p}_{\delta,\gamma,\gamma}(M, g_\gamma)}. 
\]

Moreover,

\[
\|u\|_{\dot{W}^{0, np}_{\delta+\gamma,\gamma}(M, g_\gamma)} \leq C \|u\|_{\dot{W}^{1,p}_{\delta+\gamma,\gamma}(M, g_\gamma)} \leq C \|u\|_{\dot{W}^{2,p}_{\delta+\gamma,\gamma}(M, g_\gamma)}. 
\]

Thus

\[
\|u\|_{\dot{W}^{1, np}_{\delta+\gamma,\gamma}(M, g_\gamma)} \leq C \|u\|_{\dot{W}^{2,p}_{\delta+\gamma,\gamma}(M, g_\gamma)}. 
\]

Because in this inequality \( \delta \) is arbitrary, one obtains that for any given \( \delta \in \mathbb{R} \) and \( p \geq 1 \) there exists a constant \( C \) such that

\[
\|u\|_{\dot{W}^{1, np}_{\delta,\gamma}(M, g_\gamma)} \leq C \|u\|_{\dot{W}^{2,p}_{\delta,\gamma}(M, g_\gamma)} \]

for all \( u \in C^\infty_0(\hat{M}) \).

Then one can inductively prove the \( \gamma \)-weighted Sobolev inequality and complete the proof of the proposition. \( \square \)

In Proposition 3.4, we established the weighted Sobolev inequality on manifolds with conical singularities by using Sobolev inequality on these manifolds. A Hardy’s inequality is the key ingredient in the derivation of the Sobolev inequality on manifolds with conical singularities. This does not work for manifolds with \( r^\gamma \)-horn singularities any more. On the other hand, it is interesting to note that, for projective varieties, the work of P. Li and G. Tian [LT95] implies the Sobolev inequality via their upper bound for the heat kernel.
In the isolated $r^\gamma$-horn singularities case, we have obtained the $\gamma$-weighted Sobolev inequalities. In the non-isolated singularities case, if we assume that the Sobolev inequalities hold, e.g. on projective varieties with $r^\gamma$-horn singularities, then similarly as in the proof of Proposition 3.4, we can derive $\gamma$-weighted Sobolev inequalities.

**Proposition 7.3.** Let $(M^n, g)$ be a compact Riemannian manifold with non-isolated $r^\gamma$-horn singularities, on which the Sobolev inequalities hold. For any $\delta \in \mathbb{R}$, $1 \leq p < n$, $0 \leq l \leq k$, and $q$ with $\frac{1}{q} = \frac{1}{p} - \frac{k-l}{n} > 0$, there exists a constant $C$, such that
\[
\|u\|_{W^{l,q}_{\delta,\gamma}} \leq C\|u\|_{W^{k,p}_{\delta,\gamma}}.
\]
holds for all $u \in C^\infty_0(\hat{M})$.

Consequently, there is a continuous embedding
\[
W^{k,p}_{\delta,\gamma}(M) \subset W^{l,q}_{\delta,\gamma}(M).
\]

Moreover, we have the following compact $\alpha$-weighted Sobolev embedding, which will play an important role in the study of the spectrum of the Schrödinger operator $\Delta + cR$.

**Proposition 7.4.** Let $(M, g)$ be a compact Riemannian manifold with $r^\gamma$-horn singularities. The continuous embedding $W^{1,2}_{\gamma-\frac{n\gamma}{2},\gamma}(M, g) \hookrightarrow L^2(M, g)$ is compact.

**Remark 7.5.** Note that the embedding $W^{1,2}_{\gamma-\frac{n\gamma}{2},\gamma}(M, g) \hookrightarrow L^2(M, g)$ does not follow from Proposition 7.3 since in Proposition 7.4 we do not assume that the Sobolev inequalities hold on the manifold with $r^\gamma$-horn singularities. In the following proof, we will obtain both the embedding $W^{1,2}_{\gamma-\frac{n\gamma}{2},\gamma}(M, g) \hookrightarrow L^2(M, g)$ and its compactness simultaneously.

**Proof.** Clearly, it suffices to obtain the compact embedding on a collar neighborhood $U = (0, 1) \times \partial M$ of $\partial M$ with the model metric $g_\alpha = dr^2 + r^{2\gamma} \hat{g} + \pi^* \hat{g}$. Similarly as in the proof of Lemma 3.7, we relate the function spaces on the model collar neighborhood $(U, g_\gamma)$ to that on a finite cylinder $(C(\partial M) = (0, 1) \times \partial M, g_C = dr^2 + g_{\partial M})$.

The mapping
\[
L^2(U, g_\gamma) \rightarrow L^2(C(\partial M), g_C)
\]
\[
u \mapsto \tilde{u} = r^{\frac{2}{p}} u
\]
is unitary. Similarly as the proof of the inequality in (3.16) in Lemma 3.7, we can show that
\[
(7.6) \quad \|u\|_{W^{1,2}_{\gamma-\frac{n\gamma}{2},\gamma}(U, g_\gamma)} \geq \|u\|_{W^{1,2}(C(\partial M), g_C)}.
\]

Then by combining the inequality in (7.6) with the classic Rellich lemma on a finite cylinder, we complete the proof.

Finally, we derive the following compact $\alpha$-weighted Sobolev embedding that will be used in the study of $W$-functional on manifolds with $r^\gamma$-horn singularities.
Proposition 7.6. Let $(M^n, g)$ be a compact Riemannian manifold with $r^{\gamma}$-horn singularities. There is a compact embedding $W_{\gamma}^{1,1}(M, g) \subset L^1(M, g)$.

Proof. Again, it suffices to derive the compact embedding on a small collar neighborhood of the singular set $\partial M$ with the model metric $g_\gamma$. Because $\gamma > 1$, there exists $\epsilon > 0$ such that for all $r < \epsilon$

$$r^{-\gamma} - \gamma f r^{-1} > 1.$$ 

The map

$$L^1((0, \epsilon) \times \partial M, g_\gamma) \to L^1((0, \epsilon) \times \partial M, g_{Cyl})$$

$$u \mapsto \tilde{u} = r^{\gamma} f u$$

is unitary. Here $g_{Cyl}$ is the product metric $dr^2 + g_{\partial M}$.

On the other hand,

$$\|u\|_{W_{\gamma}^{1,1}((0, \epsilon) \times \partial M, g_\gamma)} = \int_{\partial M} \int_0^\epsilon \left( |\nabla u|_{g_\gamma} + \frac{1}{r^\gamma} |u| \right) r^{\gamma} f r^2 \text{d}r \text{d}vol_{g_{\partial M}}$$

$$\geq \int_{\partial M} \int_0^\epsilon \left( |\partial_r u| + |\nabla u|_{g_{\partial M}} + \frac{1}{r^\gamma} |u| \right) r^{\gamma} f r^2 \text{d}r \text{d}vol_{g_{\partial M}}$$

$$\geq \int_{\partial M} \int_0^\epsilon \left( |\partial_r \tilde{u}| + |\nabla \tilde{u}|_{g_{\partial M}} + (r^{-\gamma} - \gamma f r^{-1}) |\tilde{u}| \right) r^{2} \text{d}r \text{d}vol_{g_{\partial M}}$$

$$\geq \|\tilde{u}\|_{W_{\gamma}^{1,1}((0, \epsilon) \times \partial M, g_{Cyl})}.$$ 

Then Rellich lemma on the cylinder $((0, \epsilon) \times \partial M, g_{Cyl})$ completes the proof. □

7.3. Spectrum of $-\Delta_g + cR_g$ on manifolds with $r^{\gamma}$-horn singularities.

Theorem 7.7. Let $(M, g)$ be a compact Riemannian manifold with $r^{\gamma}$-horn singularities as in Definition 7.1 with $\gamma > 1$. If $\min_{\partial M} \{R_g\} > 0$, then the Friedrichs extension of the Schrödinger operator $-\Delta_g + cR_g (c > 0)$ with the domain $C_0^\infty (M)$ has the spectrum consisting of discrete eigenvalues with finite multiplicity $-\infty < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, and the corresponding eigenfunctions are smooth and form a basis of $L^2(M, g)$.

Proof. The scalar curvature of the metric $g$ restricted on a small collar neighborhood of $\partial M$ is

$$R_g = \frac{R_g}{r^{2\gamma}} - \frac{f \gamma (n \gamma + 2)}{r^2} + R_g \circ \pi - |A|^2 - |T|^2 - |N|^2 - \delta N + o(r^{-2\gamma+\alpha}).$$

Thus for sufficiently small $\epsilon > 0$, on $(0, \epsilon) \times \partial M$, one has

$$R_g \geq \frac{\delta}{r^{2\gamma}},$$
for some $\delta > 0$. Consequently, for any $u \in C^\infty_0((0, \epsilon) \times \partial M),$ 
\[
\langle (\Delta_g + cR_g)u, u \rangle_{L^2(M,g)} = \int_M (-\Delta_g u + cR_g u) u d\text{vol}_g \\
= \int_M |\nabla u|^2_g + cR_g u^2 d\text{vol}_g \\
\geq \int_M |\nabla u|^2_g + \frac{c\delta}{\epsilon^2} u^2 d\text{vol}_g \\
\geq \min\{1, c\delta\} \|u\|_{W^{1,2}_{\gamma - \frac{n}{2}, \gamma}(M,g)}.
\]

Then by choosing a sufficiently large constant $A$ to control the scalar curvature $R_g$ on the interior part $M \setminus (0, \epsilon/2) \times \partial M$, one can obtain that there exists a constant $C$ such that
\[
\langle (\Delta_g + cR_g + A)u, u \rangle_{L^2(M,g)} \geq C\|u\|_{W^{1,2}_{\gamma - \frac{n}{2}, \gamma}(M,g)}
\]
for any $u \in C^\infty_0(\hat{M})$. So the operator $-\Delta_g + cR_g + A$ with domain $C^\infty_0(\hat{M})$ has a Friedrichs self-adjoint extension with the domain contained in $W^{1,2}_{\gamma - \frac{n}{2}, \gamma}(M,g)$ and the image equal to $L^2(M,g)$. Now the compactness of the embedding in Proposition 7.3 implies that the inverse of $-\Delta_g + cR_g + A$ is a compact operator from $L^2(M,g)$ to itself. Then the spectrum theory of self-adjoint compact operators completes the proof. \(\square\)

7.4. **Perelman’s $\lambda$-functional and $W$-functional on manifolds with $r^\gamma$-horn singularities.** By Theorem 7.7, one can define the $\lambda$-functional on a manifold $(M^n, g)$ with $r^\gamma$-horn singularities as the first eigenvalue of the Friedrichs extension of the operator $-4\Delta_g + R_g$, or equivalently
\[
\lambda(g) := \inf \left\{ \int_M (4|\nabla u|^2_g + R_g u^2) d\text{vol}_g \mid u \in W^{1,2}_{\gamma - \frac{n}{2}}, \ |u|_{L^2(M,g)} = 1 \right\} > -\infty,
\]
provided $\min_{\partial M} \{R_g\} > 0$.

For the $W$-functional, we can use $\gamma$-weighted Sobolev inequality in Proposition 7.3 to control $\int_M u^2 \ln u d\text{vol}_g$ term as follows. For a fixed $\tau > 0$ and any $u \in W^{1,2}_{\gamma - \frac{n}{2}}(M)$ with $\|u\|_{L^2} = (4\pi \tau)^{-\frac{n}{2}}$, one has
\[
\int_M u^2 \ln u d\text{vol}_g \leq c(n) \int_M u^{2 + \frac{2}{\gamma}} d\text{vol}_g \\
\leq c(n) \epsilon \int_M u^{2 + \frac{4}{\gamma}} + c(n) \frac{1}{\epsilon} \int_M u^2 d\text{vol}_g \\
\leq c(n) \epsilon \left( \int_M u^{\frac{2 - 2}{\gamma}} d\text{vol}_g \right)^{\frac{n - 2}{n}} \cdot \left( \int_M u^2 d\text{vol}_g \right)^{\frac{2}{n}} + c(n) \frac{1}{\epsilon} (4\pi \tau)^n
\]
\[
= c(n)\epsilon(4\pi\tau)^2 \left( \int_M u^{\frac{2n}{n-2}} d\text{vol}_g \right)^{\frac{n-2}{n}} + c(n)\frac{1}{\epsilon}(4\pi\tau)^n
\]

\[
\leq c(n)\epsilon(4\pi\tau)^2 C \int_M (|\nabla u|^2_g + \chi^{2\gamma} u^2 d\text{vol}_g) d\text{vol}_g + c(n)\frac{1}{\epsilon}(4\pi\tau)^n.
\]

Here \(c(n)\) is a constant depending only on \(n\). Now we can choose \(\epsilon\) sufficiently small so that

\[
2c(n)\epsilon(4\pi\tau)^2 C < \tau \min_{\partial M} \{R_g\}.
\]

Thus one has

\[
\inf \left\{ \int_M \left[ \tau(4|\nabla u|^2_g + R_g u^2) \right] - 2u^2 \ln u d\text{vol}_g \mid u \in W^{1,2}_{\gamma-\frac{n}{2-\gamma}}(M), \quad \|u\|_{L^2(M)} = (4\pi\tau)^\frac{2}{\gamma} \right\} > -\infty,
\]

provided \(\min_{\partial M} R_g > 0\).

Hence on a compact manifold with \(r^\gamma\)-horn singularities with \(\min_{\partial M} R_g > 0\), one has

\[
\inf \left\{ W(g,\tau, u) \mid u \in W^{1,2}_{\gamma-\frac{n}{2-\gamma}}(M), \quad \|u\|_{L^2(M)} = (4\pi\tau)^\frac{2}{\gamma} \right\} > -\infty.
\]

7.5. Asymptotic of eigenfunctions of \(-\Delta_g + cR_g\) and the minizer of the \(W\)-functional on manifolds with \(r^\gamma\)-horn singularities. Similar as the derivation of weighted elliptic estimate inequality in the conical singularities case in Proposition 3.10, one can obtain the \(\gamma\)-weighted elliptic estimate inequality. Then combining this with \(\gamma\)-weighted Sobolev inequality in Proposition 7.3, Nash-Moser iteration give the following asymptotic estimate for eigenfunctions of of \(-\Delta_g + cR_g\) and the minizer of the \(W\)-functional on manifolds with \(r^\gamma\)-horn singularities.

**Proposition 7.8.** Let \(u\) be an eigenfunction of \(-\Delta_g + cR_g\) or the minimizer of the \(W\)-functional on manifolds with \(r^\gamma\)-horn singularities. Then one has

\[
|\nabla^i u| = o(r^{-\frac{n-2}{n} \gamma - \gamma i})
\]

as \(r \to 0\), i.e. approaching to the singular set, for \(i = 0, 1\).

8. Partial asymptotic expansion of eigenfunctions on manifolds with isolated conical singularities

In this section, we derive a partial asymptotic expansion for the eigenfunctions of the Schrödinger operator \(L := -4\Delta + R\) on manifolds with isolated conical singularities in Theorem 8.3. The b-calculus theory of Melrose is the main tool in this derivation. So in §8.1 we recall some basics of the b-calculus theory.
8.1. Basics of b-calculus of Melrose. In this subsection, we briefly recall some basic notions and results in b-calculus theory of Melrose that we need in this paper, and we refer to Melrose’s book, [Mel93], for details. The b-calculus theory is very useful in studies of elliptic operators on various non-compact manifolds, for example, complete manifolds with certain ends (e.g. Example 8.1), and manifolds with conical singularities (Example 8.2).

Let \( M^n \) be a \( n \)-dimensional manifold with boundary \( \partial M \). Let \( \{y_1, \ldots, y_{n-1}\} \) be a local coordinate system on \( \partial M \). Combining this with a defining function \( x \) of the boundary, that is \( x : \overline{M} \to [0, +\infty) \) smooth with \( x^{-1}(0) = \partial M \) and \( \nabla x \neq 0 \) on \( \partial M \), we obtain a local coordinate system on \( M \).

The fundamental object in the b-calculus theory is the space of vector fields tangent to the boundary at each point on the boundary. That is denoted by \( \mathcal{V}_b \) and can be expressed as

\[
\mathcal{V}_b = \{ X \in C^\infty(M, TM) \mid X|_{\partial M} \in C^\infty(\partial M, T\partial M) \},
\]

where \( TM \) and \( T\partial M \) are tangent bundles of \( M \) and \( \partial M \), respectively, and \( C^\infty(M, TM) \) and \( C^\infty(\partial M, T\partial M) \) are their smooth sections. Then a vector field \( X \in \mathcal{V}_b \) can be expressed locally as:

\[
X = X_0 x \partial_x + \sum_{i=1}^{n-1} X_i \partial_{y_i}, \quad \text{where} \quad X_0, X_i \in C^\infty(\overline{M}).
\]

An important point here is that \( \mathcal{V}_b \) can be realized as the space of all smooth sections of a vector bundle, which is denoted by \( bTM \) and called \( b \)-tangent bundle, i.e.

\[
\mathcal{V}_b = C^\infty(M, bTM).
\]

Then the dual bundle of \( bTM \) is denoted by \( bT^*M \), whose smooth sections \( \omega \in C^\infty(M, bT^*M) \) can be expressed locally as:

\[
\omega = \omega_0 \frac{dx}{x} + \sum_{i=1}^{n-1} \omega_i dy_i, \quad \text{where} \quad \omega_0, \omega_i \in C^\infty(\overline{M}).
\]

A \( b \)-differential operator of order \( k \) is defined to be a linear map \( P : C^\infty(M) \to C^\infty(M) \) given by a finite sum of up to \( k \)-fold products of elements of \( \mathcal{V}_b \) and \( C^\infty(M) \), i.e.

\[
P = \sum_{j \in J, \quad t(j) \leq k} a_j X_{1,j} \cdots X_{t(j),j}, \quad \text{where} \quad a_j \in C^\infty(M), \quad X_{i,j} \in \mathcal{V}_b.
\]

The set of all \( b \)-differential operators of order \( k \) on \( M \) is denoted by \( \text{Diff}_b^k(M) \). Then \( P \in \text{Diff}_b^k(M) \) if and only \( P \) is a differential operator of order \( k \) (in the usual sense), and near \( \partial M \) it can be expressed as

\[
P = \sum_{|\alpha| \leq k} a_\alpha(x, y_1, \ldots, y_{n-1})(x \partial x)^{\alpha_0}(\partial y_1)^{\alpha_1} \cdots (\partial y_{n-1})^{\alpha_{n-1}},
\]
where $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1}) \in \mathbb{N}_0^n$ are multi-indices, and the coefficients $a_\alpha$ are smooth up to $\partial M$.

For $P \in \text{Diff}_b^k(M)$, the principal symbol of $P$, denoted by $^b\sigma_k(P)$, is a smooth section of the bundle of $k$-symmetric powers $\bigotimes^k (^bTM)$. It can be viewed as a homogenous polynomial of degree $k$ on $^bT^*M$, and if $P$ is locally expressed as (8.6) near $\partial M$, then $^b\sigma_k(P)$ is locally given by

\begin{align}
^b\sigma_k(P)(x, y, \omega) &= \sum_{|\alpha|=k} a_\alpha(x, y_1, \cdots, y_{n-1})(\omega_0)^{\alpha_0}(\omega_1)^{\alpha_1} \cdots (\omega_{n-1})^{\alpha_{n-1}},
\end{align}

for $\omega = \omega_0(x, y)x \partial_x + \sum_{i=1}^{n-1} \omega_i(x, y) \partial_{y_i} \in ^bT^*M$. Then $P \in \text{Diff}_b^k(M)$ is said to be $b$-elliptic if $^b\sigma_k(P) \neq 0$ on $^bT^*M \setminus \emptyset$.

In order to extend the parametrix construction of elliptic operators on compact smooth manifolds without boundary to $b$-elliptic operators $P$, it is natural to consider weighted Sobolev spaces $W^{m,2}_\delta(M)$ to control asymptotic behavior of functions near boundary. But the map:

\begin{align}
P : W^{m,2}_\delta(M) \to W^{m-k,2}_\delta(M)
\end{align}

may not have a good parametrix for some $\delta$ to get Fredholm property. To find all indices $\delta$ such that the $b$-elliptic operator $P$ in (8.8) has a nice parametrix, one needs to investigate the so called indicial operator of $P$. In the $b$-calculus theory, the weighted Sobolev spaces $W^{k,2}_\delta(M)$ are usually defined to be

\begin{align}
W^{k,2}_\delta(M) := x^\delta W^{k,2}_b(M) = \{ u = x^\delta v \mid v \in W^{k,2}_b(M) \},
\end{align}

where $W^{k,2}_b(M)$ is the usual Sobolev space on $M$ with respect to the $b$-metric, $g$, which near $\partial M$ is

\begin{align}
g = \frac{dx^2}{x^2} + g^{\partial M}
\end{align}

with $g^{\partial M}$ a smooth metric on $\partial M$. One can check that these weighted Sobolev spaces are the same as that defined on manifolds with a single conical singularity in [DW18] (also see §3.2), once we view a cone metric as a conformal change of a $b$-metric as discussed in Example 8.2.

For an operator $P \in \text{Diff}_b^k(M)$ locally given by (8.6), its indicial operator, denoted by $I(P)$, is an ($\mathbb{R}^+$-invariant) operator on $\partial M \times \mathbb{R}^+$,

\begin{align}
I(P) = \sum_{|\alpha| \leq k} a_\alpha(0, y_1, \cdots, y_{n-1})(t \partial_t)^{\alpha_0}(\partial_{y_1})^{\alpha_1} \cdots (\partial_{y_{n-1}})^{\alpha_{n-1}}.
\end{align}
It can be identified, via the Mellin transform on \( \mathbb{R}^+ \), with a family of differential operator on \( \partial M \) parametrized by \( z \in \mathbb{C} \), called the indicial family, given by

\[
(I(P))_M(z) := \sum_{|\alpha| \leq m} a_{\alpha}(0, y_1, \cdots, y_{n-1}) z^{\alpha_0}(\partial y_1)^{\alpha_1} \cdots (\partial y_{n-1})^{\alpha_{n-1}}.
\]

The set

\[
\text{spec}_b(P) := \{ z \in \mathbb{C} \mid (I(P))_M(z) \text{ is not invertible on } C^\infty(\partial M) \}
\]

is a discrete subset of \( \mathbb{C} \). Then a real number \( \delta \in \mathbb{R} \) is said to be an indicial root of the b-differential operator \( P \), if \( \delta = \text{Re}(z) \) for some \( z \in \text{spec}_b(P) \).

For any \( \delta \in \mathbb{R} \) that is not a indicial root of b-elliptic operator \( P \), roughly speaking, with the help of inverting the indicial operator \( I(P) \), a parametrix of the operator

\[
P : W^{m,2}_\delta(M) \to W^{m-k,2}_\delta(M)
\]

can be constructed, denoted by

\[
Q : W^{m-k,2}_\delta(M) \to W^{m,2}_\delta(M)
\]

such that

\[
\begin{cases}
Q \circ P = id - R_1, \\
P \circ Q = id - R_2,
\end{cases}
\]

where

\[
\begin{align*}
R_1 : & \ W^{k,2}_\delta(M) \to W^{\infty,2}_\delta(M) \cap A_{phg}(M), \\
R_2 : & \ W^{m-k,2}_\delta(M) \to W^{\infty,2}_\delta(M) \cap A_{phg}(M)
\end{align*}
\]

are compact operators. Here \( W^{\infty,2}_\delta = \bigcap_{k=1}^\infty W^{k,2}_\delta \), and \( A_{phg}(M) \) denotes the space of smooth functions on \( M \) that admit a full asymptotic expansion, as \( x \to 0 \), in terms of power of \( x \) and \( \ln x \) with smooth coefficients. In particular, the map in (8.14) is Fredholm, if \( P \) is b-elliptic and \( \delta \) is not a indicial root of \( P \).

**Example 8.1** (Manifolds with a cylindrical end). Let \( (M^n, g) \) be a complete manifold with a cylindrical end, that is, there exists a relatively compact open submanifold \( M_0 \subset M \) such that its complement, \( M \setminus M_0 \), with metric \( g \) is isometric to a cylinder. In other words, \( M \setminus M_0 \) is diffeomorphic to \( [0, +\infty) \times N \) with metric

\[
g = ds^2 + g^N,
\]

where \( g^N \) is a smooth metric on compact manifold \( N \).

By doing a change of variable, \( x = e^{-s} \), the metric \( g \) becomes \( g = \frac{ds^2}{x^2} + g^N \), which is a b-metric in (8.10), and \( M \) can be viewed as the interior of the compact manifold with boundary with a b-metric and \( x \) as a defining function of the boundary. In this change of variable, Laplace operator on \( (M, g) \), \( \Delta = \frac{\partial^2}{\partial x^2} + \Delta_{g^N} \) becomes \( (x\partial_x)^2 + \Delta_{g^N} \). Clearly, this
is a b-elliptic differential operator, with \( I(\Delta) \) identified with \( z^2 + \Delta g^N \). Thus, the indicial roots are the square roots of the eigenvalues of \( \Delta g^N \).

\[ \square \]

**Example 8.2** (Manifolds with a conical singularity). Let \((M^n, g, o)\) be a compact manifold with a single conical singularity, that is a manifold as defined in Definition 2.1 such that base \( B = o \) is a point and near the singular point \( o \) the manifold is isometric to \((0, 1) \times F\) with metric

\[
g_0 = dr^2 + r^2 \hat{g}
\]

where \( \hat{g} \) is a smooth metric on \( F \) (To emphasize this is also called an exact conical singularity). Clearly, this metric is conformal to the b-metric \( dr^2 + r^2 \hat{g} \) with conform factor \( r^2 \).

The regular part of the manifold can be viewed as a compact manifold with boundary. The elliptic operator, \( L := -4\Delta g + R_g \), which we study in this paper, can be expressed near the singular point \( o \) as

\[
L_0 := -4\Delta g_0 + R_{g_0} = -4 \left( \partial_r^2 + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta \hat{g} \right) + \frac{R_{\hat{g}} - (n-1)(n-2)}{r^2}.
\]

Then

\[
r^2 L_0 = -4(r^2 \partial_r)^2 - 4(n-2)r \partial_r - 4 \Delta \hat{g} + R_{\hat{g}} - (n-1)(n-2).
\]

Thus, \( r^2 L_0 \) is a b-elliptic operator, with its indicial family given by

\[
I(r^2 L_0)_M(z) = -4z^2 - 4(n-2)z + (-4 \Delta \hat{g} + R_{\hat{g}}) - (n-1)(n-2).
\]

This is not invertible if and only if \( z = \frac{-(n-2) \pm \sqrt{\nu - (n-2)}}{2} \), where \( \nu \) is an eigenvalue of the operator \(-4 \Delta \hat{g} + R_{\hat{g}}\). Note that these \( z \) are all real numbers, if we assume \( R_{\hat{g}} > n - 2 \). Then for any

\[
\delta g \left\{ \frac{-(n-2) \pm \sqrt{\nu - (n-2)}}{2} \mid \nu \text{ is an eigenvalue of } -4 \Delta \hat{g} + R_{\hat{g}} \right\}
\]

and \( k \in \mathbb{N}_0 \), the operator

\[
r^2 L_0 : W^{k+2,2}_\delta(M) \to W^{k,2}_\delta(M)
\]

has a parametrix \( Q \) such that

\[
\begin{cases}
Q \circ r^2 L_0 = id - R_1, \\
r^2 L_0 \circ Q = id - R_2,
\end{cases}
\]

where

\[
\begin{cases}
Q : W^{k,2}_\delta(M, g) \to W^{k+2,2}_\delta(M, g), \\
R_1 : W^{k+2,2}_\delta(M, g) \to W^{\infty,2}_\delta(M, g) \cap \mathcal{A}_{phg}(M), \\
R_2 : W^{k,2}_\delta(M, g) \to W^{\infty,2}_\delta(M, g) \cap \mathcal{A}_{phg}(M)
\end{cases}
\]
are bounded operators, where $A_{phg}(M)$ denotes the space of smooth functions on $M$ that admit a full asymptotic expansion, as $r \to 0$, in terms of power of $r$ and $\ln r$ with smooth coefficient.

8.2. A partial asymptotic expansion of eigenfunctions. Let $(M^n, g, o)$ be a compact manifold with a single conical singularity at $o$, as in Definition 2.1 but with the base manifold $B$ being a point $o$. The singular point $o$ has a neighborhood that is isometric to $C^1(F) := (0, 1) \times F$ with metric $g = dr^2 + r^2\hat{g} + h = g_0 + h$, where $h$ has asymptotic control as in (2.16), i.e.

\begin{equation}
| r^k \nabla^k_{g_0} h |_{g_0} = o(r^\alpha),
\end{equation}
as $r \to 0$, for some $\alpha > 0$ and all $k \in \mathbb{N}$. Set $L := -4\Delta + R$.

Now we take a cut-off function $\psi$ on $(0, 1) \times F$ such that

\begin{equation}
\psi = \begin{cases} 0, & \text{on } (0, \frac{1}{4}) \times F, \\ 1, & \text{on } (\frac{3}{4}, 1) \times F, \end{cases}
\end{equation}

and use it to modify the metric $g$ near the singular point by letting $g_0 + \psi h$ be the new metric on $(0, 1) \times F$ and keeping the metric $g$ on $M \setminus ((0, 1) \times F)$ unchanged. This new metric is exactly conical near the singular point $o$. Let $L_0$ denote the Schrödinger operator $-4\Delta + R$ with respect to the new metric on $M$. Then as seen in Example 8.2, $r^2L_0$ is a $b$-elliptic differential operator. Here we can make a smooth extension of the radial variable $r$ on $(0, 1) \times F$ to the whole manifold $M$.

By the asymptotic control (8.27) of the perturbation term $h$ of the metric $g$, the difference

\begin{equation}
L - L_0 : W^{k+2,2}_{\delta} \to W^{k,2}_{\delta-2+\alpha}
\end{equation}
is a bounded second order differential operator for any fixed $k \in \mathbb{N}$ and $\delta \in \mathbb{R}$. Here and in the rest of the paper, we will use $W^{k,p}_\delta$ to denote $W^{k,p}_\delta(M, g)$ for simplicity of notations.

Theorem 8.3. Let $u$ be an eigenfunction of $-4\Delta + R_{\hat{g}}$ on a $n$-dimensional manifold $(M^n, g)$ with a conical singularity, satisfying that the scalar curvature of the cross section $(F, \hat{g})$ of the model cone: $R_{\hat{g}} > (n - 2)$. Then

\begin{equation}
u
u = a(y)r^\mu + o(r^\nu),
\end{equation}
as $r \to 0$ (approaching the singular point), where $\mu = -\frac{n-2}{2} + \frac{\nu-(n-2)}{2}$ and $a(y), y \in F$, is an eigenfunction of $-4\Delta + R_{\hat{g}}$ with eigenvalue $\nu$ on $(F, \hat{g})$, and $\mu' > \mu$.

Moreover,

\begin{equation}
| \nabla^i(u - a(y)r^\mu) | = o(r^{\mu'-i})
\end{equation}
as $r \to 0$, for all $i \in \mathbb{N}$. 

Proof. The eigenfunction \( u \) satisfies the equation: \( Lu = \lambda u \), and \( u \in W^{1,2}_{1-\frac{n}{2}} \) by the estimate in Proposition 4.2. Then by using weighted elliptic estimate in Proposition 3.10, a weighted elliptic bootstrapping argument gives:

\[
(8.32) \quad u \in W^{k,2}_{1-\frac{n}{2}}, \quad \forall k \in \mathbb{N}.
\]

The eigenfunction equation \( Lu = \lambda u \) can be rewritten as

\[
(8.33) \quad r^2 L_0 u = \lambda r^2 u + r^2 (L_0 - L)u.
\]

Clearly, the eigenvalues of \(-4\Delta_g + R_g\): \( \nu > (n-2) \), since \( R_g > (n-2) \). In particular, this implies that \( \delta = 1 - \frac{n}{2} \) is not a indicial root of the operator \( r^2 L_0 \) by (8.23). Applying the parametrix \( Q \) in Example 8.2 for the operator: \( r^2 L_0 : W^{k,2}_{1-\frac{n}{2}} \to W^{k-2,2}_{1-\frac{n}{2}} \), to the equation (8.33) and using (8.25), one obtains a decomposition of \( u \) as

\[
(8.34) \quad u = R_1 u + Q(\lambda r^2 u) + Q(r^2 (L_0 - L)u).
\]

By the regularity of \( u \) in (8.32), we have \( r^2 u \in W^{k,2}_{3-\frac{n}{2}} \) and so

\[
(8.35) \quad Q(r^2 u) \in W^{k,2}_{3-\frac{n}{2}}, \quad \forall k \in \mathbb{N}.
\]

Moreover, by (8.29), we have \( r^2 (L_0 - L)u \in W^{k,2}_{1-\frac{n}{2}+\alpha} \), and so

\[
(8.36) \quad Q(r^2 (L_0 - L)u) \in W^{k,2}_{1-\frac{n}{2}+\alpha}, \quad \forall k \in \mathbb{N}.
\]

Furthermore, \( R_1 u \in W^{\infty,2}_{1-\frac{n}{2}} \cap A_{phg} \); in particular, by looking at the first two leading order terms in the full asymptotic expansion of \( R_1 u \), we have

\[
(8.37) \quad R_1 u \sim a_p(y) r^\mu (\ln r)^p + a_{p-1}(y) r^\mu (\ln r)^{p-1},
\]

as \( r \to 0 \) for some \( \mu \in \mathbb{R}, \ p \in \mathbb{N} \cup \{0\} \), nonzero smooth function \( a_p(y) \) on \( F \), and a smooth function \( a_{p-1}(y) \) on \( F \) (\( a_{p-1}(y) \) may be zero). Hence (up to higher order terms),

\[
(8.38) \quad r^2 L_0 \circ R_1 u \sim \{ -\nu a_p(y) + (-4\Delta_g + R_g) a_p(y) \} r^\mu (\ln r)^p + \{ \sigma a_p(y) - \nu a_{p-1}(y) + (-4\Delta_g + R_g) a_{p-1}(y) \} r^\mu (\ln r)^{p-1},
\]

as \( r \to 0 \), where

\[
(8.39) \quad \nu := -[\{-4\mu(n+\mu-2) - (n-1)(n-2)\}], \quad \sigma := -4p(2\mu - 1) - 4(n-1)p
\]

On the other hand, by applying \( r^2 L_0 \) to the equation (8.34), and using eigenfunction equation and (8.25), we obtain (assuming without loss of generality that \( 0 \leq \alpha < 2 \))

\[
(8.40) \quad r^2 L_0 \circ R_1 u = \lambda R_2(r^2 u) + R_2(r^2 (L_0 - L)u) \in W^{\infty,2}_{1-\frac{n}{2}+\alpha}.
\]

Claim:

\[
(8.41) \quad -\nu a_p(y) + (-4\Delta_g + R_g) a_p(y) \equiv 0,
\]
\[
\sigma a_p(y) - \nu a_{p-1}(y) + (-4\Delta \hat{g} + R\hat{g}) a_{p-1}(y) \equiv 0.
\]

Otherwise, by the asymptotic behavior in (8.38), the inclusion in (8.40), and the observation in (8.15), we obtain
\[
\mu > 1 - \frac{n}{2} + \alpha.
\]

Here note that \(r^\mu \ln r, r^\mu (\ln r)^p, \cdots\) and \(r^\mu\) are linearly independent functions. By (8.37), and again (3.15), (8.43) then implies:
\[
R_1 u \in W^{\infty, 2}_{1-\frac{n}{2}+\alpha}.
\]

Hence by (8.34), (8.35), (8.36) and (8.44), we obtain
\[
u > n - 2 + \nu + \frac{n}{2} - (n-2)\alpha.
\]

Iterating the above process, we obtain
\[
u > n - 2 + k\alpha,
\]
As a result, by Theorem 8.1 in [DW18],
\[
u > n - 2 + k\alpha,
\]

The argument also implies
\[
Q(\lambda r^2 u) + Q(r^2(L_0 - L) u) = o(r^{1-\frac{n}{2}+k\alpha}), \ \forall k \in \mathbb{N}.
\]
Therefore, by (8.34), we have
\[
R_1 u = o(r^{1-\frac{n}{2}+k\alpha}), \ \forall k \in \mathbb{N}.
\]
Then the asymptotic expansion (8.38) implies \(a_p(y) \equiv 0\), and this contradicts with the assumption: \(a_p(y) \not\equiv 0\). Thus Claims (8.41) and (8.42) hold.

Clearly, the equation (8.41) says that \(a_p(y)\) is an eigenfunction of \(-4\Delta \hat{g} + R\hat{g}\) with eigenvalue \(\nu\). By solving (8.39), we obtain
\[
\mu = -\frac{n-2}{2} + \sqrt{\nu - \frac{(n-2)^2}{2}}.
\]

Note that we have used our previous result, Theorem 1.4 of [DW18], together with \(\nu > n-2\) via our assumption \(R\hat{g} > n-2\), to rule out the negative sign in the quadratic formula here.

We can write the smooth function \(a_{p-1}(y)\) on \(F\) as a linear combination of eigenfunctions of \(-4\Delta \hat{g} + R\hat{g}\). Note that eigenfunctions with eigenvalue \(\nu\) will go away in the expression: \(-\nu a_{p-1}(x) + (-4\Delta \hat{g} + R\hat{g}) a_{p-1}(x)\), and recall that \(a_p(x)\) is an eigenfunction with eigenvalue
ν. Thus \( a_p(y) \) and \(-νa_{p-1}(y) + (-4Δ₃ + R₃)a_{p-1}(y)\) are linearly independent functions. Thus, (8.42) implies:

(8.51) \[ \sigma = -4p(2μ - 1) - 4(n - 1)p = 0. \]

Then by plugging (8.50) into (8.51), we obtain

\[ -4p\sqrt{\nu - (n - 2)} = 0. \]

Hence \( p = 0 \), since \( ν > (n - 2) \) (by our assumption \( R₃ > (n - 2) \)).

Now to complete the proof, by the decomposition in (8.34), it suffices to show that

(8.52) \[ Q(λr²u) + Q(r²(L₀ - L)u) = o(r^{μ'}), \quad \text{for some } μ' > μ. \]

By (8.35) and (8.36), we have that

(8.53) \[ Q(λr²u) + Q(r²(L₀ - L)u) ∈ W^{∞,2}_{1-\frac{α}{2}}, \]

and so

(8.54) \[ Q(λr²u) + Q(r²(L₀ - L)u) = o(r^{1-\frac{α}{2}+α}), \]

by Lemma 8.1 in [DW18]. If \( 1 - \frac{α}{2} + α > μ \), then set \( μ' = 1 - \frac{α}{2} + α \), and we are done.

Otherwise, \( R₁u ∼ a₀(y)r^{μ} ∈ W^{∞,2}_{1-\frac{α}{2}+β} \) for any \( β < α \) by (8.15). Fix \( 0 < β < α \). Then (8.34) implies \( u ∈ W^{∞,2}_{1-\frac{α}{2}+β} \). Starting with this inclusion, by repeating above derivation, one can obtain \( Q(λr²u) + Q(r²(L₀ - L)u) = o(r^{1-\frac{α}{2}+β+α}) \). Now if \( 1 - \frac{α}{2} + β + α > μ \), then we are done. Otherwise, we can iterate above process and obtain (8.52). Actually, we have

(8.55) \[ |∇^i(Q(λr²u) + Q(r²(L₀ - L)u))|_g = o(r^{μ'-i}) \]

for all \( i ∈ \mathbb{N} \).

Thus, we obtain

(8.56) \[ u = a₀(y)r^{μ} + u₁, \]

and \( |∇^i u₁| = o(r^{μ'-i}) \), as \( r → 0 \), for some \( μ' > μ \) and all \( i ∈ \mathbb{N} \cup \{0\} \).

Using a similar but more complicated argument, we now derive a partial asymptotic expansion for the minimizer \( u \) of \( W \)-functional as in Theorem 8.4 below. Recall that the minimizer \( u \) satisfies the equation (see (4.17) in [DW20])

(8.57) \[ (-4Δ + R)u = \frac{2}{τ}u \ln u + \frac{n + m}{τ}u, \]

where \( m \) is the infimum of \( W \)-functional with the scaling parameter \( τ > 0 \) as in (1.3). The main difference between this equation and that of the eigenfunction as discussed in Theorem 8.3 is the \( u \ln u \) term on the right hand of the Euler-Lagrange equation (8.57). So for the simplicity of presentation, instead of (8.57), we will consider the equation

(8.58) \[ (-4Δ + R)u = u \ln u. \]
The basic idea of the proof of Theorem 8.4 is similar to that of Theorem 8.3. We will still use the parametrix of the Schrödinger operator $L_0$ in Melrose’s b-calculus theory. In the derivation for the partial asymptotic expansion of eigenfunction in Theorem 8.3, another key ingredient is that one can obtain $u \in W^{\infty, 1}_{1 - \frac{n}{2}}$ by using a weighted elliptic bootstrapping with the help of Proposition 3.10. This helps us estimate the term $Q(\lambda r^2 u) + Q(r^2 (L_0 - L)u)$ in (8.34).

However, the $u \ln u$ term prevents us from getting the weighted $L^2$ estimate for high order derivatives of a solution to (8.58). Instead, we combine the b-calculus theory and weighted $L^p$ elliptic bootstrapping argument to obtain the following asymptotic estimate. This improves the asymptotic estimate obtained in [DW20].

**Theorem 8.4.** Let $u > 0$ be a solution of

$$(-4\Delta_g + R_g)u = u \ln u.$$  

(8.59)

on a manifold $(M^n, g, o)$ with a conical singularity at $o$, satisfying $R_g > (n - 2)$. Then

$$|\nabla^i u| = o(r^{\mu - i})$$  

(8.60)

as $r \to 0$ (approaching the singular point), for $i = 0, 1$ and any $\mu < -\frac{n - 2}{2} + \frac{\sqrt{\nu - (n - 2)^2}}{2}$, where $\nu$ is the smallest eigenvalue of $-4\Delta_{\hat{g}} + R_{\hat{g}}$ on $(\hat{F}, \hat{g})$.

**Proof.** We prove the theorem in two steps. In the first step we show that, for some real number $\mu'$ and any $\mu < \mu'$, $|\nabla^i u| = o(r^{\mu - i})$ as $r \to 0, i = 0, 1$. In the second step, we estimate $\mu'$ to complete the proof.

**Step 1.** The solution $u > 0$ satisfies the equation $Lu = u \ln u$. So

$$r^2 L_0 u = r^2 u \ln u + r^2 (L_0 - L)u.$$  

(8.61)

Applying the parametrix $Q$ of the operator: $r^2 L_0 : W^{2,2}_{1 - \frac{n}{2}} \to W^{0,2}_{1 - \frac{n}{2}}$, to this equation and using (8.25), we write $u$ as

$$u = R_1 u + Q(r^2 u \ln u) + Q(r^2 (L_0 - L)u).$$  

(8.62)

Recall that $u \in W^{1,2}_{1 - \frac{n}{2}}$ and $u$ is smooth by the local elliptic regularity. By the weighted Sobolev embedding in Proposition 3.4 in [DW20], we have

$$u \in W^{0,p}_{1 - \frac{n}{2}}, \quad \forall 1 \leq p \leq \frac{2n}{n - 2}.$$  

(8.63)

Now, for any $\gamma > 0$, there is a constant $a(\gamma)$ such that

$$|u \ln u| \leq a(\gamma) + |u|^{1+\gamma}.$$  

(8.64)

It follows that, for any small $\gamma > 0$,

$$u \ln u \in W^{0,p}_{(1 - \frac{n}{2})(1+\gamma)}, \quad \forall 1 \leq p \leq \frac{2n}{(n - 2)(1 + \gamma)}.$$  

(8.65)
In particular, for $p = 2$,

$$Q(r^2 u \ln u) \in W^{2,2}_{(1-\frac{n}{2})(1+\gamma)+2}.$$  

(8.66)

Thus, by taking $\gamma$ sufficiently small, we obtain

$$Q(r^2 u \ln u) \in W^{2,2}_{1-\frac{n}{2}+\alpha}.$$  

(8.67)

In addition, as shown in the proof of Theorem 5.2 in [DW20], by Proposition 5.1 in [DW20], we can obtain

$$u \in W^{2,p}_{(1-\frac{n}{2})(1+\gamma)}, \quad \forall 1 \leq p \leq \frac{2n}{(n-2)(1+\gamma)}, \quad \gamma > 0.$$  

(8.68)

Then by the boundedness of the map in (8.29), we have

$$r^2(L_0 - L)u \in W^{0,2}_{1-\frac{n}{2}+\alpha},$$  

(8.69)

and so

$$Q(r^2(L_0 - L)u) \in W^{2,2}_{1-\frac{n}{2}+\alpha}.$$  

(8.70)

Thus, by (8.67) and (8.70), we obtain

$$Q(r^2 u \ln u) + Q(r^2(L_0 - L)u) \in W^{2,2}_{1-\frac{n}{2}+\alpha}.$$  

(8.71)

On the other hand, $R_1 u \in W^{\infty,2}_{1-\frac{n}{2}} \cap A_{phg}$; in particular, by looking at the leading order term,

$$R_1 u \sim a(y) r^{\mu'} (\ln r)^p,$$  

(8.72)

as $r \to 0$ for some $\mu' \in \mathbb{R}$. So $R_1 u \in W^{\mu,2}_{\mu}$ for any $\mu < \mu'$. We first **Claim:**

$$u \in W^{2,2}_{\mu}, \quad \text{for any } \mu < \mu'.$$  

(8.73)

If $\mu' \leq 1 - \frac{n}{2} + \alpha$, then

$$Q(r^2 u \ln u) + Q(r^2(L_0 - L)u) \in W^{2,2}_{1-\frac{n}{2}+\alpha} \subset W^{2,2}_{\mu},$$  

(8.74)

for any $\mu < \mu' \leq 1 - \frac{n}{2} + \alpha$. Consequently, (8.62) implies that $u \in W^{2,2}_{\mu}$, for any $\mu < \mu'$, and Claim (8.73) follows.

If $\mu' > 1 - \frac{n}{2} + \alpha$, then $u \in W^{2,2}_{1-\frac{n}{2}+\alpha}$. Starting from this regularity estimate for $u$, and applying the above process, we obtain

$$Q(r^2 u \ln u) + Q(r^2(L_0 - L)u) \in W^{2,2}_{1-\frac{n}{2}+2\alpha}.$$  

(8.75)

Now if $\mu' \leq 1 - \frac{n}{2} + 2\alpha$, then we obtain $u \in W^{2,2}_{\mu}$, for any $\mu < \mu'$. Otherwise, we repeat the above argument finitely many times to prove Claim (8.73).
Starting from the regularity estimate in Claim (8.73), a weighted $L^p$ elliptic bootstrapping argument as in the proof of Theorem 5.2 in [DW20] gives

\[(8.76) \quad |\nabla^i u| = o(r^{\mu-i}) \]

as $r \to 0$, for $i = 0, 1$ and any $\mu < \mu'$.

**Step 2.** Finally, we estimate the exponent $\mu'$ in the leading order term of $R_1 u$ to complete the proof. For that, we apply $r^2 L_0$ to $R_1 u$ to obtain, via (8.72),

\[(8.77) \quad r^2 L_0 \circ R_1 u \sim \{-4\mu'(\mu' + n - 2) - (n - 1)(n - 2)a(y) + (-4\Delta_{\hat{g}} + R_{\hat{g}})a(y)\}r^{\mu'}(\ln r)^p, \]

as $r \to 0$.

On the other hand, applying $r^2 L_0$ to the equation (8.62), using the equation $Lu = u \ln u$ and (8.25), one has

\[(8.78) \quad r^2 L_0 \circ R_1 u = R_2(r^2 u \ln u) + R_2(r^2 L_0 - L)u) \in W^{1,2}_{1-\frac{n}{2} + \alpha}. \]

Now we separate the discussion into two cases:

**Case 1:** If

\[(8.79) \quad [-4\mu'(\mu' + n - 2) - (n - 1)(n - 2)a(y) + (-4\Delta_{\hat{g}} + R_{\hat{g}})a(y) \equiv 0, \]

then $a(y)$ is an eigenfunction of $-4\Delta_{\hat{g}} + R_{\hat{g}}$ and

\[(8.80) \quad \mu' = -\frac{n - 2}{2} + \frac{\sqrt{\nu' - (n - 2)}}{2}, \]

where $\nu'$ is an eigenvalue of $-4\Delta_{\hat{g}} + R_{\hat{g}}$ on $(F, \hat{g})$. And we are done. Here we have used Theorem 5.2 in [DW20], together with $\nu' > n - 2$ via our assumption on the scalar curvature, to rule out the other root.

**Case 2:** If

\[(8.81) \quad [-4\mu'(\mu' + n - 2) - (n - 1)(n - 2)a(x) + (-4\Delta_{\hat{g}} + R_{\hat{g}})a(x) \neq 0, \]

then $\mu' > 1 - \frac{n}{2} + \alpha$ by (8.78), (8.77), and (3.15). Thus (8.72) and (3.15) imply

\[(8.82) \quad R_1 u \in W^{1,2}_{1-\frac{n}{2} + \alpha}. \]

It follows from (8.62) that

\[(8.83) \quad u \in W^{1,2}_{1-\frac{n}{2} + \alpha}. \]

Iterating the above process leads to

\[(8.84) \quad u \in W^{1,2}_{1-\frac{n}{2} + \alpha}, \quad \forall k \in \mathbb{N}. \]

Then as in the proof of Theorem 5.2 in [DW20], we can obtain that

\[(8.85) \quad |\nabla^i u| = o(r^{1-\frac{n}{2} + \alpha - i}) \]
for $i = 0, 1$, and any $k \in \mathbb{N}$. In particular, $|\nabla^i u| = o(r^{\mu-i})$, for $i = 0, 1$ and any $\mu < -\frac{n-2}{2} + \frac{\sqrt{\nu-(n-2)^2}}{2}$, where $\nu$ is the smallest eigenvalue of $-\Delta_{\hat{g}} + R_{\hat{g}}$ on $(F, \hat{g})$. This completes the proof. \hfill \Box

9. Gradient Ricci solitons with isolated conical singularities

In this section, we apply the same idea as in the last section to study gradient Ricci solitons with isolated conical singularities. We first derive some asymptotic estimate for the potential functions of compact gradient Ricci solitons with isolated conical singularities, and use that to prove Theorem 1.10. That is, there are no nontrivial compact gradient steady or expanding Ricci solitons with isolated conical singularities, when the model cone is scalar flat.

Let $(M^n, g, o)$ be a compact manifold with a single conical singularity at $o$ as described at the beginning of §8.2. Then as we did there, we modify the metric $g$ near the singular point $o$ such that the metric is an exact cone metric: $dr^2 + r^2\hat{g}$ in a sufficiently small neighborhood of the singular point: $(0, \epsilon) \times F$ (small $\epsilon$ is to be determined as we need in Proposition 9.1), and the new metric is the same as the original metric $g$ on $M \setminus ((0, 4\epsilon) \times F)$. Let $\Delta_0$ denote the Laplacian with respect to the modified metric, and $\Delta_{\hat{g}}$ be the Laplacian with respect to the original (asymptotically) conical metric $g$. Then the map

$$\Delta_{\hat{g}} - \Delta_0 : W^{k+2,2}_{\delta} \to W^{k,2}_{\delta-2+\alpha}$$

is bounded, for any $\delta \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$, and so the map

$$r^2 (\Delta_{\hat{g}} - \Delta_0) : W^{k+2,2}_{\delta} \to W^{k,2}_{\delta+\alpha}$$

is bounded.

Recall that $\alpha$ is the decay order in (8.27).

**Proposition 9.1.** Let $(M^n, g, o)$ be a compact manifold with a conical singularity at $o$. For each $k \in \mathbb{N} \cup \{0\}$ and

$$\delta \notin \left\{ \frac{2-n}{2} \pm \frac{(n-2)^2 + 4\nu}{2} \mid \nu \text{ is an eigenvalue of } -\Delta_{\hat{g}} \right\},$$

the map

$$r^2 \Delta_{\hat{g}} : W^{k+2,2}_{\delta} \to W^{k,2}_{\delta}$$

is Fredholm.

**Proof.** From Example 8.2, $r^2\Delta_0$ is a b-elliptic differential operator, and its indicial roots are given by: $\frac{2-n}{2} \pm \frac{\sqrt{(n-2)^2 + 4\nu}}{2}$, where $\nu$ is an eigenvalue of $-\Delta_{\hat{g}}$. Thus by b-calculus theory, for each $\delta$ satisfying (9.3),

$$r^2 \Delta_0 : W^{k+2,2}_{\delta} \to W^{k,2}_{\delta}$$
is Fredholm, and there is a parametrix $Q$, which is a bounded map
\begin{equation}
Q : W^{k,2}_\delta \to W^{k+2,2}_\delta,
\end{equation}
such that
\begin{equation}
Q \circ r^2 \Delta_0 = id + R_L,
\end{equation}
and
\begin{equation}
R_L : W^{k+2,2}_\delta \to W^{k+2,2}_\delta
\end{equation}
is compact.

Then
\begin{equation}
Q \circ r^2 \Delta_g = id + Q \circ r^2 (\Delta_g - \Delta_0) + R_L.
\end{equation}

For $u \in W^{k+2,2}_\delta(C_r(F))$, by using (9.1), we have
\begin{equation}
\left\| Q \circ r^2 (\Delta_g - \Delta_0) u \right\|_{W^{k+2,2}_\delta} \leq C \left\| r^2 (\Delta_g - \Delta_0) u \right\|_{W^{k,2}_\delta} \\
\leq \epsilon^\alpha C \left\| r^2 (\Delta_g - \Delta_0) u \right\|_{W^{k,2}_{\delta+\alpha}} \\
\leq \epsilon^\alpha C \left\| u \right\|_{W^{k+2,2}_\delta}.
\end{equation}

Here the first inequality is because of the boundedness of the map $Q$ in (9.6). For the second inequality, we note that $r^2 (\Delta_g - \Delta_0) u$ is supported in $(0, 4\epsilon) \times F$ and we use the definition of weighted Sobolev norm (3.11). The third inequality then follows from the fact in [9]. Then the estimate in (9.9) implies that for sufficiently small $\epsilon > 0$ the operator
\begin{equation}
Q \circ r^2 (\Delta_g - \Delta_{g_0}) : W^{k+2,2}_\delta \to W^{k+2,2}_\delta
\end{equation}
has norm $\leq \frac{1}{2}$. Consequently, $id + Q \circ r^2 (\Delta_g - \Delta_{g_0})$ is invertible. Thus, the map
\begin{equation}
r^2 \Delta_g : W^{k+2,2}_\delta \to W^{k,2}_\delta
\end{equation}
is also Fredholm. □

**Proposition 9.2.** Let $(M^n, g, o)$ be a compact manifold with a conical singularity at $o$. For $2 - n \leq \delta \leq 1 - \frac{n}{2}$ satisfying (9.3), the map
\begin{equation}
r^2 \Delta_g : W^{2,2}_\delta \to W^{0,2}_\delta
\end{equation}
is surjective.

In particular, the map $r^2 \Delta_g : W^{2,2}_{1-\frac{n}{2}} \to W^{0,2}_{1-\frac{n}{2}}$ is surjective.

**Proof.** Proposition 9.1 tells us that for any $\delta \in \mathbb{R}$ satisfying (9.3), the map (9.11) is Fredholm. Thus, in order to prove the map (9.11) is surjective for $2 - n \leq \delta \leq 1 - \frac{n}{2}$ satisfying (9.3), it suffices to prove its usual $L^2$ adjoint map
\begin{equation}
(r^2 \Delta_g)^* : (W^{0,2}_\delta)^* \to (W^{2,2}_\delta)^*
\end{equation}
is injective. Note that the usual $L^2$-pairing $(\cdot, \cdot)_{L^2(M)}$ identifies the topology dual space $(W^0_{\delta})^*$ with $W^{-0,2}_{-\delta-n}$. For each small $r_0 > 0$, we take a cut-off function $\varphi_{r_0}$ such that

\begin{equation}
\varphi_{r_0} = \begin{cases} 
0, & \text{on } B_{r_0}(o), \\
1, & \text{on } M \setminus B_{2r_0}(o),
\end{cases}
\end{equation}

and $|d\varphi_{r_0}| \leq \frac{10}{r_0}$ on $M$. Then for each $u \in \text{Ker } ((r^2 \Delta_g)^*) \subset W^0_{0,2} - \delta - n$, we have $\Delta_g(r^2 u) = 0$ and so

\begin{equation}
\int_M |\nabla(\varphi_{r_0}(r^2 u))|^2 = \int_M |d\varphi_{r_0}|^2 (r^2 u)^2 - \int_M (\varphi_{r_0})^2 (r^2 u) \Delta_g(r^2 u)
= \int_M |d\varphi_{r_0}|^2 (r^2 u)^2.
\end{equation}

Thus,

\begin{equation}
\int_{M \setminus B_{2r_0}(o)} |\nabla(r^2 u)|^2 \leq 100 \int_{B_{2r_0}(o) \setminus B_{r_0}(o)} r^2 u^2 \rightarrow 0, \text{ as } r_0 \rightarrow 0,
\end{equation}

since for $\delta \leq 1 - \frac{n}{2}$,

\begin{equation}
\int_{(0,1) \times F} r^2 u^2 \leq \int_{(0,1) \times F} r^{2\delta + n} u^2 \leq \|u\|^2_{W^0_{0,2} - \delta - n}(M) < +\infty.
\end{equation}

Therefore, $\nabla(r^2 u) = 0$, and so $r^2 u$ is a constant function in $W^0_{0,2} - \delta - n$. This implies $r^2 u$ must be zero, and so $u = 0$, by the assumption $\delta \geq 2 - n$ and (3.15). Thus, the map (9.12) is injective and so the map (9.11) is surjective.

Finally, we note that $\delta = 1 - \frac{n}{2}$ is not a indicial root of $r^2 \Delta_g$, via the expression of indicial roots of $r^2 \Delta_g$ in (9.3), and the fact that the eigenvalues of $-\Delta_g$: $\nu \geq 0$. \quad \square

**Proposition 9.3.** Let $(M^n, g, f)$ be a compact gradient Ricci soliton with a conical singularity, i.e. $(M^n, g)$ is a compact Riemannian manifolds with a conical singularity and the Ricci curvature $\text{Ric}_g$ on the regular part satisfies the equation

\begin{equation}
\text{Ric}_g + \text{Hess}_g f = \Lambda g,
\end{equation}

for some constant $\Lambda$. If the cross section of the model cone at the singular point has scalar curvature $R_g = (n - 1)(n - 2)$, then a potential function $f$ satisfies

\begin{equation}
|\nabla^i f| = o(r^{\delta_0 - i}),
\end{equation}

as $r \rightarrow 0$, for some $\delta_0 > 0$ and all $i \in \mathbb{N}_0$.

In particular, $f \rightarrow 0$ at the conical singularity and hence the potential function $f$ is bounded on $M$.

**Remark 9.4.** Note that potential function $f$ the gradient Ricci soliton equation (9.17) is not unique, and one can add a constant to a potential function to get a new potential function.
Proof. By taking trace for the gradient Ricci soliton equation (9.17), one has
\begin{equation}
\Delta_g f = -R_g + n\Lambda.
\end{equation}
Note that the scalar curvature of the asymptotic conical metric $g$ is given by
\begin{equation}
R_g = \frac{1}{r^2} (R_\hat{g} - (n-1)(n-2) + o(r^\alpha)), \text{ as } r \to 0.
\end{equation}
If $R_g = (n-1)(n-2)$, then $r^2 R_g = o(r^\alpha)$ as $r \to 0$. Thus, by (9.19), $f$ satisfies the equation
\begin{equation}
r^2 \Delta_g f = -r^2 R_g + r^2 n\Lambda \in W_{0.2}^{0.2},
\end{equation}
where the inclusion follows from (3.15). Then by the surjectivity obtained in Proposition 9.2, we have: $f \in W_{1-\frac{n}{2}}^{2,2}$.

Now we rewrite the equation (9.21) as
\begin{equation}
r^2 \Delta_0 f = r^2 (\Delta_0 - \Delta_g) f - r^2 R_g + r^2 n\Lambda,
\end{equation}
and use the parametrix $Q$ of $r^2 \Delta_0$ to improve the estimate of asymptotic order of $f$ to complete the proof, starting with the regularity: $f \in W_{1-\frac{n}{2}}^{2,2}$.

Recall that for each non-indicial root $\delta \in \mathbb{R}$, the parametrix $Q : W_{\delta}^{0,2} \to W_{\delta}^{2,2}$ satisfies
\begin{equation}
\begin{cases}
Q \circ r^2 \Delta_0 = id - R_1, \\
r^2 \Delta_0 \circ Q = id - R_2,
\end{cases}
\end{equation}
where
\begin{equation}
\begin{cases}
R_1 : W_{\delta}^{2,2}(M, g) \to W_{\delta}^{\infty,2} \cap \mathcal{A}_{phg}(M), \\
R_2 : W_{\delta}^{0,2}(M, g) \to W_{\delta}^{\infty,2} \cap \mathcal{A}_{phg}(M)
\end{cases}
\end{equation}
are bounded operators, and $\mathcal{A}_{phg}(M)$ denotes the space of smooth functions on $M$ that admit a full asymptotic expansion, as $r \to 0$, in terms of power of $r$ and $\ln r$ with smooth coefficients.

By applying $Q$ to the equation (9.22) and using (9.23), we obtain a decomposition of $f$ as
\begin{equation}
f = R_1 f + Q \left( r^2 (\Delta_0 - \Delta_g) f \right) - Q \left( r^2 R_g - r^2 n\Lambda \right).
\end{equation}
Then $f \in W_{1-\frac{n}{2}}^{2,2}$ implies $Q \left( r^2 (\Delta_0 - \Delta_g) f \right) \in W_{1-\frac{n}{2}+\alpha}^{2,2}$ by (9.1). Moreover, $Q \left( r^2 R_g - r^2 n\Lambda \right) \in W_{\delta}^{2,2}$ for any $(1 - \frac{n}{2} + \alpha <) \delta < \alpha$ (assuming without loss of generality that $0 < \alpha < 2$). Thus,
\begin{equation}
Q \left( r^2 (\Delta_0 - \Delta_g) f \right) - Q \left( r^2 R_g - r^2 n\Lambda \right) \in W_{1-\frac{n}{2}+\alpha}^{2,2}.
\end{equation}
by the inclusion in (3.13). The first term, $R_1 f$, in (9.25) has a full asymptotic expansion, and by looking at the leading order term, we have
\begin{equation}
R_1 f \sim a(y) r^\mu (\ln r)^p, \quad \text{as } r \to 0.
\end{equation}

Then by applying $r^2 \Delta_0$ to the equation (9.25), and using (9.21) and (9.23), we obtain
\begin{equation}
r^2 \Delta_0 \circ R_1 f = R_2 \left( r^2 (\Delta_0 - \Delta_y) f \right) + R_2 \left( r^2 R_y + r^2 n \Lambda \right) \in W^{\infty,2}_{1-\frac{2}{2}+\alpha}.
\end{equation}

Here the inclusion in the weighted Sobolev space can be obtain similar to the way for (9.26). On the other hand, by applying $r^2 \Delta_0$ to (9.27), we obtain
\begin{equation}
r^2 \Delta_0 \circ R_1 f \sim \left[ (\mu^2 + (n-2)\mu) a(y) + \Delta_y a(y) \right] r^\mu (\ln r)^p.
\end{equation}

Now we have two possible cases:

**Case 1:** $(\mu^2 + (n-2)\mu) a(y) + \Delta_y a(y) = 0$. In this case, $\mu = \frac{2-n}{2} + \frac{\sqrt{(n-2)^2 + 4\nu}}{2}$ where $\nu$ is an eigenvalue of $-\Delta_y$ and $a(y)$ is a corresponding eigenfunction. Here we rule out $\mu = \frac{2-n}{2} - \frac{\sqrt{(n-2)^2 + 4\nu}}{2}$ by the facts: $R_1 f \in W^{\infty,2}_{1-\frac{2}{2}+\alpha}$ and $\nu \geq 0$ and (3.15). Note that $\mu \geq 0$, as $\nu \geq 0$. In addition, similarly as in the proof of Theorem 1.4, we can show that $p = 0$ in (9.27). Therefore, we have
\begin{equation}
R_1 f = a(y) r^\mu + O(r^\mu'), \quad \text{as } r \to 0,
\end{equation}
where $\mu' > \mu \geq 0$. Furthermore, if $\mu = 0$, the associated eigenvalue $\nu = 0$, and so the corresponding eigenfunction $a(y)$ is constant function on the cross section. As a result, the leading order term $a(y) r^\mu$ is a constant. By Remark 9.4 we can subtract the constant leading order term from the potential function $f$, and then obtain
\begin{equation}
R_1 f = O(r^\mu'), \quad \text{as } r \to 0, \quad \text{for some } \mu' > 0.
\end{equation}

If $\mu > 0$, the we already have (3.31). Then by (9.25), (9.26) and (9.31), we obtain $f \in W^{2,2}_{1-\frac{2}{2}+\alpha}$. Then we can repeat the above argument by starting with $f \in W^{2,2}_{1-\frac{2}{2}+\alpha}$ to further increase the index in the weighted Sobolev space. By iterating the argument finitely many times, we can obtain $f \in W^{2,2}_\delta$ for some $\delta > 0$.

**Case 2:** $(\mu^2 + (n-2)\mu) a(y) + \Delta_y a(y) \neq 0$. In this case, by comparing leading orders in (9.27) and (9.29), and using (3.15), the inclusion in (9.28) implies: $R_1 f \in W^{\infty,2}_{1-\frac{2}{2}+\alpha}$. By combining with (9.25) and (9.26), this then implies $f \in W^{2,2}_{1-\frac{2}{2}+\alpha}$. Again, by iterating the argument finitely many times, we can obtain $f \in W^{2,2}_\delta$ for some $\delta > 0$.

In both cases, we can obtain $f \in W^{2,2}_{\delta_0}$ for some $\delta_0 > 0$. By weighted elliptic estimate in Proposition 3.10, we then have $f \in W^{k,2}_{\delta_0}$ for any $k \in \mathbb{N}$. Finally, by the asymptotic control for functions in weighted Sobolev space on manifold with isolated conical singularities (see Lemma 8.1 in [DW18]), we obtain $|\nabla^i f| = o(r^{\delta_0-i})$ as $r \to 0$, for all $i \in \mathbb{N}$. \qed
We now use the asymptotic estimate obtained in Proposition 9.3 to study steady and expanding gradient Ricci solitons with isolated conical singularities. First, we recall a basic fact about the potential function of gradient Ricci solitons that we need in the proofs of Theorems 9.6 and 9.7.

**Lemma 9.5.** Let \((M^n, g, f)\) be a \(n\)-dimensional (either smooth or singular) gradient Ricci soliton, that is, on the regular part of \(M\),

\[
Ric + \nabla^2 f = \Lambda g,
\]

holds, for some constant \(\Lambda\). Then the potential function \(f\) satisfies the differential equation:

\[
|\nabla f|^2 - \Delta f + n\Lambda - 2\Lambda f - C = 0,
\]

on the regular part of \(M\), for some constant \(C\).

The conclusion of Lemma 9.5 is a pointwise differential equation (9.33) on the regular part of the manifold. Its proof for the singular manifolds is the same as for smooth manifolds, which can be found in literature, see e.g. the proof of Proposition 1.1.1 in [CZ06].

**Theorem 9.6.** Let \((M^n, g, f)\) be a compact gradient steady Ricci soliton with isolated conical singularities. If cross section of model cone at each singularity has scalar curvature \(R_{\hat{g}} = (n-1)(n-2)\), then \((M^n, g)\) is a Ricci flat manifold with isolated conical singularities.

**Proof.** Recall steady gradient Ricci soliton equation:

\[
Ric + \nabla^2 f = 0.
\]

Let \(\Lambda = 0\) in Lemma 9.5 then the equation (9.33) becomes:

\[
\Delta f - |\nabla f|^2 = C,
\]

on the regular part \(\hat{M}\), for some constant \(C\).

Then by the asymptotic estimate for the potential function \(f\) obtained in Proposition 9.3 i.e.,

\[
|\nabla f| = o(r^{\delta - 1}), \text{ as } r \to 0, \text{ for some } \delta > 0,
\]

one obtains

\[
\int_M \Delta(e^f) d\text{vol}_g = 0.
\]

Thus,

\[
C \int_M e^f d\text{vol}_g = \int_M (\Delta f - |\nabla f|^2)e^f d\text{vol}_g = \int_M \Delta(e^f) d\text{vol}_g = 0.
\]
Consequently, $C = 0$, and again by the asymptotic estimate of $f$ one also has $\int_M \Delta f = 0$. So by integrating the equation (9.34), one obtains that $f$ has to be constant function, and so the metric $g$ is Ricci flat. □

Similarly, for expanding solitons, we have:

**Theorem 9.7.** Let $(M^n, g, f)$ be a compact expanding gradient Ricci soliton with isolated conical singularities. If $R_g = (n - 1)(n - 2)$, then $(M^n, g)$ is an Einstein manifold with isolated conical singularities.

**Proof.** Recall the expanding gradient Ricci soliton equation:

\[(9.38) \quad \text{Ric} + \nabla^2 f = \Lambda g, \quad \text{for a constant } \Lambda < 0.\]

By Lemma 9.5, the potential function $f$ satisfies:

\[(9.39) \quad |\nabla f|^2 - \Delta f + n\Lambda - 2\Lambda f - C = 0,\]

on the regular part $\tilde{M}$, for some constant $C$.

Now the same as in the proof of Theorem 9.6, by the asymptotic estimate in Proposition 9.3, we obtain

\[(9.40) \quad \int_M (\Delta f - |\nabla f|^2)e^f d\text{vol}_g = \int_M \Delta (e^f) d\text{vol}_g = 0.\]

Therefore, by (9.39) and (9.40), we obtain

\[(9.41) \quad \int_M (n\Lambda - 2\Lambda f - C)e^f = 0.\]

Recall that in Proposition 9.3 we also obtain the boundedness of the potential function and $f \to 0$ as approaching the singular points. Using this, we will prove in a moment that

\[(9.42) \quad n\Lambda - 2\Lambda f - C \equiv 0.\]

Granted, we obtain from (9.42), (9.39),

\[(9.43) \quad |\nabla f|^2 - \Delta f = 0.\]

By integrating (9.43), we find that $f$ must be constant, since $\int_M \Delta f = 0$ (this follows from classical Stokes theorem and asymptotic estimate for $f$ obtained in Proposition 9.3).

Now it remains to prove (9.42). We separate the discussion into the following three cases.

**Case 1.** Assume $f \geq 0$ on $M$. Because $f$ is bounded on $M$ and $f \to 0$ at the singular points, there exists a global maximal point $x_{\text{max}} \in \tilde{M}$ of $f$ such that

\[\nabla f(x_{\text{max}}) = 0, \quad \Delta f(x_{\text{max}}) \leq 0,\]

and

\[f(x) \leq f(x_{\text{max}}), \quad \forall x \in M.\]
Thus by evaluating the equation (9.39) at $x_{\text{max}}$, we have

$$n\Lambda - 2\Lambda f(x_{\text{max}}) - C \leq 0.$$  

On the other hand, $\Lambda < 0$. Hence

(9.44)  

$$n\Lambda - 2\Lambda f(x) - C \leq 0, \quad \forall x \in M.$$  

Combining (9.44) with the integral equation (9.41) gives (9.42).

**Case 2.** Assume $f \leq 0$ on $M$. Similarly, there exists a global minimal point $x_{\text{min}} \in \hat{M}$ of $f$ such that

$$\nabla f(x_{\text{min}}) = 0, \quad \Delta f(x_{\text{min}}) \geq 0,$$

and

$$f(x) \geq f(x_{\text{min}}), \quad \forall x \in M.$$  

Thus by evaluating the equation (9.39) at $x_{\text{min}}$, we have

$$n\Lambda - 2\Lambda f(x_{\text{min}}) - C \geq 0.$$  

But $\Lambda < 0$. Hence in fact

(9.45)  

$$n\Lambda - 2\Lambda f(x) - C \geq 0, \quad \forall x \in M.$$  

Combining the inequality (9.45) with the integral equation (9.41) gives the equation (9.42).

**Case 3.** Assume $f$ does not have definite sign on $M$. Again because $f$ is bounded on $M$ and $f \to 0$ at the singular points, there exists a global maximal point $x_{\text{max}} \in \hat{M}$ and a global minimal point $x_{\text{min}} \in \hat{M}$ of $f$ such that

$$\nabla f(x_{\text{max}}) = \nabla f(x_{\text{min}}) = 0, \quad \Delta f(x_{\text{max}}) \leq 0 \leq \Delta f(x_{\text{min}}),$$

and

$$f(x_{\text{min}}) \leq f(x) \leq f(x_{\text{max}}), \quad \forall x \in M.$$  

Thus by evaluating the equation (9.39) at $x_{\text{max}}$ and $x_{\text{min}}$, we have

$$n\Lambda - 2\Lambda f(x_{\text{max}}) - C \leq 0 \leq n\Lambda - 2\Lambda f(x_{\text{min}}) - C.$$  

On the other hand, $\Lambda < 0$, and hence

$$n\Lambda - 2\Lambda f(x_{\text{max}}) - C \geq n\Lambda - 2\Lambda f(x_{\text{min}}) - C.$$  

Then we must have (9.42). \qed
References

[Bar86] R. Bartnik, The mass of an asymptotically flat manifold, Pure Appl. Math. vol XXXIX, 661-693 (1986).

[Bam21] R. Bamler, Recent developments in Ricci flows, Notices AMS 68, 1486-1498 (2021).

[Beh13] T. Behrndt, On the Cauchy problem for the heat equation on Riemannian manifolds with conical singularities, Quart. J. Math. 64, 981-1007 (2013).

[Bes87] A.L. Besse, Einstein manifolds. Springer, Berlin (1987).

[BP03] B. Botvinnik and S. Preston, Conformal Laplacian and conical singularities, in: Proceeding of the School on High-Dimensional Manifold Topology, ICTP, Trieste, Italy, Word Scientific (2003), 22-79.

[CZ06] H.-D. Cao and X.-P. Zhu, A complete proof of the Poincaré and geometrization conjectures - application of the Hamilton-Perelman theory of the Ricci flow, Asian J. Math., Vol. 10, No. 2, 165-492, (2006).

[Che80] J. Cheeger, On the Hodge theory of Riemannian pseudomanifolds. Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), pp. 91-146, Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980.

[CDS15] X. Chen, S. Donaldson, and S. Sun, Kähler-Einstein metrics on Fano manifolds, I-III., Amer. Math. Soc., Providence, 2007.

[CCG+07] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, the Ricci flow: techniques and applications. Part I, Mathematical Surveys and Monographs, 135. American Mathematical Society, Providence, 2007.

[CLW12] H. Chen, X. Liu, and Y. Wei, Dirichlet problem for semilinear edge-degenerate elliptic equations with singular potential term. J. Differential Equation, 252, 4289-4314 (2012).

[DW18] X. Dai and C. Wang, Perelman’s W-functional on manifolds with conical singularities, J. Geom. Anal. 28(4), 3657-3689 (2018).

[DW20] X. Dai and C. Wang, Perelman’s W-functional on manifolds with conical singularities, Math. Res. Let. 27(3), 665-685 (2020).

[DWZ18] X. Dai, G. Wei, and Z. Zhang, Local Sobolev constant estimate for integral Ricci curvature bounds, Adv. Math. 325, 1-33 (2018).

[DY] X. Dai and K. Yoshikawa, Analytic torsion for log-Enriques surfaces and Borcherds product, Forum Math. Sigma 10 (2022), Paper No. e77, 54pp.

[Eva] L. C. Evans, Partial Differential Equation, Grad. Stud. Math., 19, American Mathematical Society, Providence, RI, 2010, xxii+749 pp.

[EK96] Y. Egorov and V. Kondratiev, On spectral theory of elliptic operators. Birkhäuser, Basel (1996).

[FIN05] M. Feldman, T. Ilmanen, L. Ni, Entropy and reduced distance for Ricci expanders, J. Geom. Anal. 15(1), 49–62, (2005).

[GS18] P. Gianniotis, F. Schulze, Ricci flow from spaces with isolated conical singularities Geom. Topol. 22(7): 3925-3977 (2018). DOI: 10.2140/gt.2018.22.3925

[Gro21] M. Gromov, Four Lectures on Scalar Curvature. arXiv:1908.10612v6 [math.DG].

[HLP34] G. H. Hardy, J. E. Littlewood, and G. Polya, Inequalities. Cambridge (1934).

[HP85] W. Hsiang and V. Pati, L2-cohomology of normal algebraic surfaces. I, Invent. Math. 81(1985), 395-412.

[KL14] B. Kleiner, J. Lott, Geometrization of three-dimensional orbifolds via Ricci flow, Astérisque, no. 365 (2014), 77 pages.
K. Kröncke, B. Vertman, A survey on the Ricci flow on singular spaces. Differential Geometry in the Large, pp 118-140. Cambridge University Press, 2020, DOI: https://doi.org/10.1017/9781108884136.007.

K. Kröncke, B. Vertman, Perelman’s entropy for manifolds with conical singularities, Tran. Amer. Math. Soc. 374(4), 2873-2908 (2021).

P. Li, G. Tian, On the heat kernel of the Bergmann metric on algebraic varieties, J. Amer. Math. Soc. 8(4), 857-887 (1995).

J. Liu and X. Zhang, Conical Kähler–Ricci flows on Fano manifolds, Adv. Math. 307 (2017), 1324–1371.

C. Mantegazza, R. Müller, Perelman’s Entropy Functional at Type I Singularities of the Ricci flow. J. Reine Angew. Math. (Crelle’s Journal) Vol. 703 (2015), 173-199 DOI: 10.1515/crelle-2013-0039.

R. Mazzeo, Elliptic theory of differential edge operators. I. Comm. Partial Differential Equations 16(10), 1615-1664, (1991).

R. B. Melrose, The Atiyah-Patodi-Singer index theorem, Research Notes in Mathematics, vol. 4, A K Peters, Ltd., Wellesley, MA, 1993.

O. Munteanu, J. Wang, Conical structure for shrinking Ricci solitons, J. Eur. Math. Soc. 19 (2017), no. 11, pp. 3377–3390 DOI: 10.4171/JEMS/741

D. Oikonomopoulos, Functional inequalities on simple edge spaces, J. Geom. Phys. 158 (2020), 103863, 16 pp.

B. O’Neill, The fundamental equations of a submersion, Mich. Math. J. 13, 459-469 (1966).

T. Ozuch, Perelman’s functionals on cones and construction of type III Ricci flows coming out of cones, J. Geom. Anal. 30, 1-53 (2020).

G. Perelman, The entropy formula for the Ricci flow and its geometric applications, arXiv:math/0211159 (2002).

R. Schoen, Topics in Scalar Curvature, Notes by C. Li.

L. Sheng, Maximal time existence of unnormalized conical Kähler–Ricci flow, Journal für die reine und angewandte Mathematik (Crelles Journal) (2018)

Y. Shi and L. Tam, Scalar curvature and singular metrics, Pacific J. Math., 293(2018), 427–470.

G. Tian, K-stability and Kähler-Einstein metrics, Comm. Pure Appl. Math. 68(7) (2015), 1085-1156.

Vertman B., Ricci de Turck Flow on Singular Manifolds, J. Geom. Anal. 31(4), 3351-3404 (2021).