Observing Dynamical Supersymmetry Breaking with Euclidean Lattice Simulations

Issaku Kanamori,1,∗) Fumihiko Sugino2,**) and Hiroshi Suzuki3,∗∗∗)

1,3 Theoretical Physics Laboratory, RIKEN, Wako 351-0198, Japan
2 Okayama Institute for Quantum Physics, Okayama 700-0015, Japan

Abstract

A strict positivity of the ground-state energy is a necessary and sufficient condition for spontaneous supersymmetry breaking. This ground-state energy may be directly determined from the expectation value of the Hamiltonian in the functional integral, defined with an antiperiodic temporal boundary condition for all fermionic variables. We propose to use this fact to observe the dynamical spontaneous supersymmetry breaking in Euclidean lattice simulations. If a lattice formulation possesses a manifestly preserved fermionic symmetry, there exists a natural choice of a Hamiltonian operator that is consistent with a topological nature of the Witten index. We numerically confirm the validity of our idea in models of supersymmetric quantum mechanics. We further examine the possibility of dynamical supersymmetry breaking in the two-dimensional \( \mathcal{N} = (2, 2) \) super Yang-Mills theory with the gauge group \( SU(2) \), for which the Witten index is unknown. Although statistical errors are still large, we do not observe positive ground-state energy, at least within one standard deviation. This prompts us to draw a different conclusion from a recent conjectural claim that supersymmetry is dynamically broken in this system.

∗) E-mail: kanamori-i@riken.jp
**) E-mail: fumihiko_sugino@pref.okayama.lg.jp
∗∗∗) E-mail: hsuzuki@riken.jp
§1. Introduction

The possibility of the spontaneous breaking of supersymmetry (assuming that it is not broken at the tree level) is a highly dynamical issue and its precise study requires a nonperturbative framework. Generally, the Witten index\(^1\) provides an important clue. One can infer that dynamical supersymmetry breaking does not occur in a wide class of supersymmetric models where the Witten index can be computed to be nonzero. However, the Witten index is not a panacea. There are still many interesting models for which it is very difficult to determine the Witten index and, in some cases, the index itself would be ill-defined because of a gapless continuous spectrum.\(^2\),\(^3\)

On the other hand, it is well known that a strict positivity of the ground-state (or vacuum) energy is a necessary and sufficient condition for spontaneous supersymmetry breaking.\(^4\) In principle, therefore, one can judge whether supersymmetry breaking occurs by computing the ground-state energy.

In this paper, in light of recent developments on the lattice formulation of supersymmetric theories,\(^5\)–\(^8\) we propose to observe dynamical supersymmetry breaking with Euclidean lattice formulations employing the above idea. This work was originally motivated by a recent paper by Hori and Tong\(^9\) in which they conjectured dynamical supersymmetry breaking in the two-dimensional \(\mathcal{N} = (2,2)\) super Yang-Mills theory with the gauge group \(SU(N_c)\). Lattice formulations of this two-dimensional theory are the simplest among recent lattice formulations of extended supersymmetric gauge theories.\(^10\)–\(^21\) Therefore, it is highly natural, if it is possible, to examine supersymmetry breaking in the two-dimensional \(\mathcal{N} = (2,2)\) super Yang-Mills theory with lattice formulation.\(^*\) This is what we do in this study. To our knowledge, this is the first instance in which the dynamical supersymmetry breaking in gauge field theory is investigated numerically, although there exists a closely related and thought-provoking observation in Ref. 19). (There are a number of numerical works related to this issue in one-dimensional supersymmetric models\(^23\)–\(^32\) and two-dimensional Wess-Zumino models.\(^33\)–\(^37\) ) Although statistical errors in our Monte Carlo study using a formulation in Ref. 13) are still large, we do not observe positive ground-state energy, at least within one standard deviation. This observation prompts us to draw a different conclusion from the conjectural claim in Ref. 9). This is the content of §4. In §2, we present our basic idea concerning the determination of the ground-state energy in the Euclidean functional integral formalism. Then, in §3, we illustrate how our method works by applying it to supersymmetric quantum mechanical models.\(^4\) Section 5 is devoted to the discussion.

\(^*\) While preparing this paper, we discovered a preprint\(^22\) in which this problem is addressed on the basis of the lattice formulation in Ref. 11).
In what follows, the boundary condition of fermionic variables for the temporal direction, whether it is periodic (PBC) or antiperiodic (aPBC), is crucial. For all bosonic variables and for all variables with respect to the spatial directions, we will assume the periodic boundary conditions. Unless noted otherwise, the term "boundary condition" will always refer to the boundary condition of fermionic variables for the temporal direction.

§2. Basic idea

What we want to determine is the ground-state energy $E_0$ of supersymmetric theories. If $E_0 > 0$, supersymmetry is spontaneously broken and it is not if $E_0 = 0$. With the Euclidean functional integral formalism, one could determine the ground-state energy from the expectation value of Hamiltonian $H$,

$$
\langle H \rangle_{\text{PBC}} = \frac{\int_{\text{PBC}} d\mu H e^{-S}}{\int_{\text{PBC}} d\mu e^{-S}},
$$

(2.1)

where $S$ is the Euclidean action and $d\mu$ symbolically denotes a measure for the functional integration. We assumed the periodic boundary condition (PBC) for fermionic variables because this boundary condition is consistent with supersymmetry. One would then be able to obtain the ground-state energy $E_0$ by taking the large imaginary-time limit $\beta \to \infty$, where $\beta$ is the temporal size of the system, because in this limit, only the contribution of the ground-state(s) survives in Eq. (2.1).

However, this naive idea is wrong. First, we must note that the functional integral in the denominator of Eq. (2.1) is proportional to the Witten index,

$$
Z_{\text{PBC}} \equiv N_{\text{PBC}} \int_{\text{PBC}} d\mu e^{-S} = \text{Tr}(-1)^F e^{-\beta H} = \text{Tr}(-1)^F,
$$

(2.2)

where $F$ is the fermion number operator and $N_{\text{PBC}}$ is a proportionality constant that depends on the choice of the integration measure $d\mu$. The constant $N_{\text{PBC}}$ may depend on ultraviolet and infrared cutoffs (the number of lattice points for lattice regularization) and possibly on the boundary condition. Second, the numerator of Eq. (2.1) is proportional to the derivative of the Witten index with respect to $\beta$, which is always zero:

$$
N_{\text{PBC}} \int_{\text{PBC}} d\mu H e^{-S} = \text{Tr}(-1)^F H e^{-\beta H} = -\frac{\partial}{\partial \beta} \text{Tr}(-1)^F e^{-\beta H} = 0.
$$

(2.3)

* $\beta$ is not to be confused with the conventional gauge coupling constant in lattice gauge theory.

** Our discussion in this section is based on the assumption that the expressions appearing in Eq. (2.2) are meaningful. For this, we may assume that the spectrum of $H$ is discrete so that the Witten index is unambiguously defined. Our basic formula (2.7) for the ground-state energy itself, however, might also be applicable to systems in which this assumption fails.
This independence of the Witten index from a parameter of the theory, \( \beta \), is a consequence of the supersymmetry algebra.\(^1\) Thus we have
\[
\langle H \rangle_{\text{PBC}} = \frac{0}{\text{Tr}(-1)^F}.
\] (2.4)

We finally recall that the Witten index vanishes when supersymmetry is spontaneously broken. Therefore, we see that \( \langle H \rangle_{\text{PBC}} \) is indefinite when supersymmetry is broken. Otherwise, it is zero or indefinite depending on whether or not the Witten index is nonzero. Note also that a similar remark is valid for the expectation value of generic operators when the periodic boundary condition is imposed; when the Witten index vanishes, \( Z_{\text{PBC}} \) cannot be used as a normalization factor for expectation values. With the periodic boundary condition, therefore, the expectation values normalized by the partition function can be ill-defined, and in such a case, we must consider “denominator-free” expectation values, such as Eq. (2.3). In any case, the expectation value \( \langle H \rangle_{\text{PBC}} \) does not provide useful direct information on the ground-state energy or on supersymmetry breaking.

In Eq. (2.3), what prevents us from obtaining the ground-state energy is the factor \((-1)^F\), which is the heart of the Witten index. If this factor can be removed, the above idea of using the expectation value of the Hamiltonian would be valid. As is well known (see, for example, Refs. 39, 40), such a removal can easily be achieved. What we must do is simply to change the boundary condition of fermionic variables for the temporal direction from periodic to antiperiodic (aPBC). This defines the thermal partition function with the inverse temperature \( \beta \), instead of the Witten index,
\[
Z_{\text{aPBC}} \equiv N_{\text{aPBC}} \int_{\text{aPBC}} d\mu e^{-S} = \text{Tr} e^{-\beta H},
\] (2.5)
which should be positive definite, and, as its derivative with respect to \( \beta \),
\[
N_{\text{aPBC}} \int_{\text{aPBC}} d\mu He^{-S} = \text{Tr} He^{-\beta H}.
\] (2.6)

Therefore, taking the long-time limit (or the low-temperature limit) of the ratio of these two quantities, we have
\[
\lim_{\beta \to \infty} \langle H \rangle_{\text{aPBC}} = \lim_{\beta \to \infty} \frac{\int_{\text{aPBC}} d\mu He^{-S}}{\int_{\text{aPBC}} d\mu e^{-S}} = \lim_{\beta \to \infty} \frac{\text{Tr} He^{-\beta H}}{\text{Tr} e^{-\beta H}} = E_0,
\] (2.7)
and ground-state energy \( E_0 \) is obtained. This is our basic formula.

Before taking the \( \beta \to \infty \) limit, Eq. (2.7) is merely the expectation value in the thermal equilibrium with finite temperature \( 1/\beta \), and supersymmetry is explicitly broken by the temperature. In this aspect, it is interesting to note an analogy to a conventional way of
detecting the spontaneous breaking of an ordinary symmetry, for example, the $Z_2$ symmetry of the Ising spin. In this case, one breaks the symmetry by applying an external magnetic field that is conjugate to the order parameter of symmetry breaking, that is the magnetization. One then observes (in the thermodynamic limit) how a trace of the breaking remains after the applied field is turned off.

For supersymmetry, the order parameter is a positivity of the ground-state energy and the conjugate variable to the energy is the temperature. In Eq. (2·7), we break supersymmetry by placing the system in thermal equilibrium. We then observe how the effect of the temperature remains in the zero-temperature (or the large imaginary-time) limit $\beta \to \infty$, for which one naively would expect that the effect simply disappears. If the effect remains, we judge that spontaneous supersymmetry breaking occurs. Recall that the expectation value $\langle H \rangle_{\text{PBC}}$ with the periodic boundary condition is always zero or indefinite. Thus, if a well-defined $E_0 > 0$ is found through Eq. (2·7), it is the effect of the boundary condition surviving even in the long-time (or the zero-temperature) limit.$^*$ Physically, this survival of the effect of the boundary condition can be understood in terms of the appearance of a massless (or zero energy for quantum mechanics) Nambu-Goldstone fermion associated with spontaneous supersymmetry breaking.

Our basic formula (2·7) is very simple. However, to embody it in Euclidean lattice formulation, there are still several issues to be clarified. First of all, the above argument assumes that regularization to define the functional integral does not break supersymmetry. Otherwise, one would not be able to distinguish spontaneous supersymmetry breaking from a possible explicit breaking due to regularization. As is well recognized, generally, a regularization based on a spacetime lattice is irreconcilable with supersymmetry. For theories with the extended supersymmetry, it is nevertheless sometimes possible to set up a lattice regularization that preserves the invariance under some supersymmetry transformations.$^5$–$^8$ Then, if the spacetime dimension is low enough, one may expect that the invariance under a full set of supersymmetry transformations is restored in the continuum limit. In what follows, we assume this sort of lattice regularization. In particular, for the study of the two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory, we adopt the formulation in Ref. 13) in which a fermionic symmetry $Q$ that is a part of the supersymmetry is manifestly preserved. We will briefly review this formulation in §4.

Secondly, closely related to the above point, we must properly choose a possible additive constant in the Hamiltonian $H$. In other words, we must correctly choose the origin of the energy. This point is, of course, crucial for judging spontaneous supersymmetry breaking.$^*$

$^*$ Note, however, that spontaneous supersymmetry breaking differs from spontaneous breaking of ordinary symmetries in that it can occur even in a system with finite volume.$^4$
from the positivity of $E_0$. Note also that, when the Witten index is nonzero, supersymmetric invariant state(s) must have a precisely zero eigenvalue $E_0 = 0$ for relation (2.3) to hold. That is, Eq. (2.3) is not invariant under an arbitrary shift of the origin of the energy $H \rightarrow H + c$, when the Witten index is nonzero.

Of course, a natural prescription for defining the Hamiltonian is to use the supersymmetry algebra. One may first define supercharge operators $Q$ and $Q^\dagger$ (with some regularization) and define a (regularized) Hamiltonian operator $H$ by the anti-commutation relation $H = \{Q, Q^\dagger\}/2$ without any additive constant. This is precisely the idea behind the Hamiltonian formulation of supersymmetric theories\(^{24)}–^{26)},^{28)},^{33)},^{34)},^{37)},^{41)}–^{44)} with which a possible additive constant in the Hamiltonian $H$ is automatically fixed.\(^{\ast)}

For the following reason, however, this issue of a “correct” Hamiltonian is somewhat delicate in the functional integral formulation based on the Lagrangian. Suppose that the (for simplicity, off-shell) supersymmetry algebra is realized by the transformation law for variables appearing in the continuum Lagrangian. This implies that there exists a fermionic transformation $Q$ such that $\{Q, Q\} = 2i\partial_0$, where $Q$ is the conjugate fermionic transformation of $Q$ and $\partial_0$ is the time derivative. One would then expect, from this algebra, that the relation $iQ\overline{Q} = 2H$ holds,\(^{**)} where $\overline{Q}$ is the Noether charge\(^{***)} associated with $Q$ and $H$ is the Hamiltonian obtained from the Lagrangian by the Legendre transformation. If this relation holds, this $H$ could be used in the functional integral as a Hamiltonian operator that is consistent with the supersymmetry algebra.

In reality, however, the relation holds only up to equations of motion. Generally, one ends up with

$$\frac{i}{2}Q\overline{Q} = H + \text{(terms being proportional to equations of motion).} \quad (2.8)$$

That is, a Hamiltonian suggested from the algebra can differ from the original one obtained through the Legendre transformation from the Lagrangian. This occurs very commonly, and we will encounter such a situation even in the simplest supersymmetric system in the next section. The additional terms, which would be negligible in the classical level, cannot be neglected in general within the functional integral because those terms may give rise to 

\(^{\ast)} On the other hand, from the viewpoint of the feasibility of numerical simulations that preserve the gauge symmetry, the Euclidean lattice formulation appears advantageous.

\(^{**)} We take normalization of the Noether charge such that the Poisson bracket $\{\overline{Q}, \cdot\}_P$ generates the $Q$ transformation. After the quantization, the relation would be read as $H = \{Q, \overline{Q}\}/2$ which is consistent with the positivity of the Hamiltonian.

\(^{***)} In field theories, when supersymmetry is spontaneously broken, the Noether charge (supercharge) itself would be ill-defined owing to a massless singularity associated with the Nambu-Goldstone fermion. In the field theory case discussed in §4, we use the Noether current instead.
contact terms at a coincident point, i.e., ultraviolet-divergent constants. We thus have two (among possibly many) options for a Hamiltonian operator in quantum theory: one is the original Hamiltonian obtained from the Lagrangian and the other is \( iQ\overline{Q}/2 \). How can we be sure that we are using a Hamiltonian with a correctly chosen additive constant before we measure the ground-state energy? Clearly, we need some guiding principle.

We have no general answer to the above question. See also §5. However, if the lattice formulation one adopts possesses at least one exactly preserved fermionic symmetry, for example, the above \( Q \), there exists a natural prescription for a choice of the Hamiltonian. It is the left-hand side of Eq. (2.8), \( H \equiv iQ\overline{Q}/2 \). This choice of the Hamiltonian in quantum theory corresponds to “renormalizing” additional terms in the right-hand side of Eq. (2.8) into \( H \). This definition is natural because the structure \( H = iQ\overline{Q}/2 \) is suggested from the supersymmetry algebra. Moreover, this choice has the correct origin of the energy in the sense that it is consistent with the topological property of the Witten index, Eq. (2.3). That is,

\[
N_{\text{PBC}} \int_{\text{PBC}} d\mu He^{-S} = N_{\text{PBC}} \int_{\text{PBC}} d\mu \left( \frac{i}{2} Q\overline{Q} e^{-S} \right) = 0, \tag{2.9}
\]

where we have used the \( Q \)-invariance of the action and of the integration measure. As already noted, when the Witten index is nonzero, this property fixes the origin of the energy uniquely. For these reasons, we consider that the definition \( H \equiv iQ\overline{Q}/2 \) is natural. Of course, for cases of interest, we do not know a priori whether the Witten index is nonzero or not, and if it is zero, the above argument for the structure \( H = iQ\overline{Q}/2 \) based on relation (2.3) is groundless (a shift of the origin \( H \rightarrow H+c \) does not influence Eq. (2.3) if \( N_{\text{PBC}} \int_{\text{PBC}} d\mu e^{-S} = 0 \)). Nevertheless, we adopt this \( Q \)-exactness of the Hamiltonian as a working hypothesis in what follows because the definition of a Hamiltonian operator should be independent of whether or not the supersymmetry is spontaneously broken.

§3. Supersymmetric quantum mechanics

In this section, we examine our method by applying it to a Euclidean lattice formulation of the supersymmetric quantum mechanics.\(^4\) We find that this example provides a good illustration of our method.

The Lagrangian of the supersymmetric quantum mechanics is given by

\[
L = \frac{1}{2} (\partial \phi)^2 - \frac{1}{2} (W')^2 + \overline{\psi}(i\partial - W'')\psi + \frac{1}{2} F^2, \tag{3.1}
\]

\(^4\) Strictly speaking, to show this relation, we must assume that the integral \( \int_{\text{PBC}} d\mu \overline{Q} e^{-S} \) is finite.
where all variables are functions of the “time coordinate” $x$ and $\partial$ is the derivative with respect to $x$, $\partial \equiv \partial/(\partial x)$. $\phi$ and $F$ are bosonic variables and $\bar{\psi}$ and $\psi$ are fermionic. The superpotential $W = W(\phi)$ is a function of $\phi$ and the prime denotes the derivative with respect to $\phi$. With the periodic boundary condition for all variables, the action $S = \int dx L$ is invariant under the following $\mathcal{N} = 2$ supersymmetry transformations:

$$
\begin{align*}
Q\phi &= \psi, & Q\psi &= 0, \\
\bar{Q}\bar{\psi} &= F + i\partial\phi - W', & QF &= -i\partial\psi + W''\psi,
\end{align*}
$$

and

$$
\begin{align*}
\bar{Q}\phi &= \bar{\psi}, & \bar{Q}\psi &= 0, \\
\bar{Q}\bar{\psi} &= -F + i\partial\phi + W', & \bar{Q}F &= i\partial\bar{\psi} + W''\bar{\psi}.
\end{align*}
$$

One can confirm that the transformations form the supersymmetry algebra

$$Q^2 = \bar{Q}^2 = 0, \quad \{Q, \bar{Q}\} = 2i\partial$$

off-shell, i.e., without using any equations of motion. In this system, it is well known\(^4\) that supersymmetry is spontaneously broken if and only if the number of zeros of the function $W'(\phi)$ is even, or equivalently, $W(-\infty)$ and $W(+\infty)$ have opposite signs. (We assumed that $|W(\pm\infty)| = +\infty$.)

A crucial fact for us is that the classical action can be expressed as the $Q$-exact form\(^*)

$$S = \int dx L = Q\int dx \frac{1}{2}\bar{\psi}(F - i\partial\phi + W').$$

Then the invariance of $S$ under $Q$ and $\bar{Q}$ can be easily seen by using the supersymmetry algebra (3.6).\(^{**}\)

The Hamiltonian corresponding to the Lagrangian (3.1) is given by

$$H = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(W')^2 + \bar{\psi}W''\psi - \frac{1}{2}F^2$$

and, as we noted in Eq. (2.8), we have

$$\frac{i}{2}Q\bar{Q} = H + \frac{1}{2}F(F - i\partial\phi - W') + \frac{1}{2}\bar{\psi}(i\partial - W'')\psi,$$  

where $\bar{Q}$ is the Noether charge associated with the $\bar{Q}$ invariance:

$$\bar{Q} = -\bar{\psi}(\partial\phi - iW').$$

\(^*)\text{For this, we must note that } \int dx iW'\partial\phi = \int dx i\partial W = 0.

\(^{**}\) $S$ can also be written as $S = Q\bar{Q}\int dx \frac{1}{2}(\bar{\psi}\psi + 2W)$. 

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Since the last two terms in Eq. (3.9) vanish under classical equations of motion, $F = 0$ and $(i\partial - W'')\psi = 0$, relation (3.9) is consistent with the supersymmetry algebra, at least classically.

After the Wick rotation, $x \to -ix$ and $L \to -L$, we have the Euclidean action

$$S = \int dx \left\{ \frac{1}{2} (\partial \phi)^2 + \frac{1}{2} (W')^2 + \bar{\psi} (\partial + W'') \psi - \frac{1}{2} F^2 \right\}$$  \hspace{1cm} (3.11)

$$= -Q \int dx \frac{1}{2} \bar{\psi} (F + \partial \phi + W'),$$  \hspace{1cm} (3.12)

and for the Hamiltonian $H$,

$$\frac{1}{2} Q \left\{ \bar{\psi} (\partial \phi - W') \right\} = H + \frac{1}{2} F' (F + \partial \phi - W') - \frac{1}{2} \bar{\psi} (\partial + W'') \psi.$$  \hspace{1cm} (3.13)

So far everything has been for the continuum. We now construct a lattice formulation of the above system on a finite-size lattice,

$$\Lambda = \{ x \in a \mathbb{Z} \mid 0 \leq x < \beta \},$$  \hspace{1cm} (3.14)

where $a$ denotes the lattice spacing. First, we fix the lattice transcription of the time derivative $\partial$. As a possible choice, we adopt the forward difference

$$\partial f(x) \equiv f(x + a) - f(x),$$  \hspace{1cm} (3.15)

which does not lead to species doubling. The lattice counterparts of the $Q$ transformation are then defined as

$$Q\phi(x) = \psi(x), \quad Q\psi(x) = 0,$$  \hspace{1cm} (3.16)

$$Q\bar{\psi}(x) = F(x) - \partial \phi(x) - W'(\phi(x)), \quad QF(x) = \partial \psi(x) + W''(\phi(x)) \psi(x).$$  \hspace{1cm} (3.17)

Finally, the lattice action is defined with an expression analogous to Eq. (3.12):

$$S \equiv -Q \sum_{x \in A} \frac{1}{2} \left\{ \bar{\psi}(x) \left( F(x) + \partial \phi(x) + W'(\phi(x)) \right) \right\}$$  \hspace{1cm} (3.18)

$$= \sum_{x \in A} \left\{ \frac{1}{2} \partial \phi(x) \partial \phi(x) + \frac{1}{2} (W'(\phi(x)))^2 + \bar{\psi}(x) \left( \partial + W''(\phi(x)) \right) \psi(x) \right.$$  \hspace{1cm} (3.19)

$$- \frac{1}{2} F(x)^2 + W'(\phi(x)) \partial \phi(x) \right\},$$

where

$$\psi(x = \beta) = \begin{cases} +\psi(x = 0), & \text{for the periodic boundary condition,} \\ -\psi(x = 0), & \text{for the antiperiodic boundary condition.} \end{cases}$$  \hspace{1cm} (3.20)
Note that all lattice variables are dimensionless. In the above lattice formulation with the periodic boundary condition, the $Q$-symmetry is manifestly preserved because the $Q$-transformations in Eqs. (3.16) and (3.17) are nilpotent, $Q^2 = 0$. The invariance under $Q$ is, however, broken because it is impossible to define a corresponding $Q$ transformation on lattice variables such that the algebra $\{Q, Q\} = -2\partial$ still holds. In fact, this lattice action is basically identical to the one described in Refs. 23), 27), 29), 35), 36), 45). (See also Ref. 30.) In some of these references, it has been shown that the $Q$-symmetry is restored in the continuum limit.

As a Hamiltonian in this lattice formulation, following the discussion in the previous section and in view of Eq. (3.13), we use $H(x) \equiv iQ\overline{Q}(x)/2$, where $^{*}$

$$\overline{Q}(x) \equiv -\frac{1}{a}\psi(x) (i\partial\phi(x) - iW'(\phi(x)))$$

(3.21)

is a lattice analogue of the Noether charge. The explicit form is

$$H(x) = -\frac{1}{2a} \partial\phi(x)\partial\phi(x) + \frac{1}{2a} (W'(\phi(x)))^2 - \frac{1}{2a} \overline{\psi}(x) (\partial - W''(\phi(x))) \psi(x) + \frac{1}{2a} F(x) (\partial\phi(x) - W'(\phi(x))) .$$

(3.22)

The naive continuum limit of $H(x)$ differs from the (imaginary-time) Hamiltonian in the continuum theory by terms vanishing under the classical equations of motion.$^{**}$ As discussed in the previous section, this lattice Hamiltonian has a correct zero-point energy in the sense that

$$\int_{PBC} \prod_{x \in \Lambda} d\phi(x) dF(x) d\psi(x) d\overline{\psi}(x) He^{-S} = 0 ,$$

(3.23)

which follows from the $Q$-exactness of $H$ and the $Q$-invariance of the action $S$ and of the integration measure.$^{***}$ (Recall Eq. (2.9).) Thus this choice of the Hamiltonian is consistent with the topological nature of the Witten index, Eq. (2.3), with finite ultraviolet and infrared cutoffs.

The numerical study of the present lattice model is not so difficult, because it is possible to obtain a closed expression of the fermion determinant in terms of $\phi$. That is,

$$\det \{ -\partial - W''(\phi) \} = \prod_{x \in \Lambda} \{1 - W''(\phi(x))\} \mp 1 ,$$

(3.24)

$^{*}$ We supplemented a factor of $1/a$ to adjust the physical mass dimension; recall that all lattice variables as well as the $Q$ transformation are dimensionless.

$^{**}$ It is interesting to note that the expectation value (with the antiperiodic boundary condition) of this difference vanishes: $\langle \frac{1}{2} F(F + \partial\phi - W') - \frac{1}{2} \overline{\psi}(\partial + W'\psi) \rangle_{apBC} = 0$.

$^{***}$ Assuming that the integral $\int_{PBC} \prod_{x \in \Lambda} d\phi(x) dF(x) d\psi(x) d\overline{\psi}(x) e^{-S}$ is finite. This is certainly true if $|W'(\pm \infty)| = +\infty$. 

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where the upper sign corresponds to the periodic boundary condition and the lower corresponds to the antiperiodic boundary condition. Thus, after integrating over fermionic variables and the auxiliary variable $F(x)$, we have the effective action

$$S_{\text{eff}}[\phi] = \sum_{x \in A} \left\{ \frac{1}{2} \partial \phi(x) \partial \phi(x) + \frac{1}{2} (W'(\phi(x)))^2 + W'(\phi(x)) \partial \phi \right\} - \ln \left| \prod_{x \in A} \{1 - W''(\phi(x))\} \mp 1 \right|. \quad (3.25)$$

Note that the fermion determinant (3.24) is real for a real superpotential $W$ but it is not necessarily positive definite. We thus must include the sign of the determinant,

$$s[\phi] = \text{sign} \left( \det \{ -\partial - W''(\phi) \} \right) = \text{sign} \left( \prod_{x \in A} \{1 - W''(\phi(x))\} \mp 1 \right), \quad (3.26)$$

as a reweighting factor in the functional integral. For example, the expectation value of the Hamiltonian $H$ is given by the ratio of

$$\mathcal{N} \int \prod_{x \in A} d\phi(x) \ Hs[\phi] \ e^{-S_{\text{eff}}[\phi]} \quad (3.27)$$

to the partition function

$$\mathcal{Z} = \mathcal{N} \int \prod_{x \in A} d\phi(x) \ s[\phi] \ e^{-S_{\text{eff}}[\phi]} \quad (3.28)$$

Recall that, when supersymmetry is spontaneously broken, the normalized expectation value with the periodic boundary condition cannot be defined because $Z_{\text{PBC}} = 0$. With the antiperiodic boundary condition, expectation values are always meaningful and the following substitutions can be made:

$$\langle F(x)^2 \rangle_{\text{aPBC}} = -1, \quad \langle F(x) \rangle_{\text{aPBC}} = 0 \quad (3.29)$$

for the auxiliary variable, and

$$\langle \bar{\psi}(x) (\partial + W''(\phi(x))) \psi(x) \rangle_{\text{aPBC}} = -1 \quad (3.30)$$

and

$$\langle \bar{\psi}(x) W''(\phi(x)) \psi(x) \rangle_{\text{aPBC}} = \left\langle W''(\phi(x)) \frac{\prod_{y \neq x \in A} \{1 - W''(\phi(y))\}}{\prod_{z \in A} \{1 - W''(\phi(z))\}} + 1 \right\rangle_{\text{PBC}} \quad (3.31)$$

for fermionic variables. It is then straightforward to implement the hybrid Monte Carlo algorithm with the effective action (3.25) and compute $\langle H(x) \rangle_{\text{aPBC}}$. 

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Now, as a definite example in which supersymmetry is spontaneously broken, we consider

\[ W_{\text{continuum}} = \frac{1}{2} m \phi^2_{\text{continuum}} + \frac{1}{3} g \phi^3_{\text{continuum}}. \]  

(3.32)

Since the parameter \( m \) has the mass dimension 1, we will measure all dimensionful quantities in units of \( m \). For example, the lattice spacing is measured by the dimensionless combination \( am \). If we introduce the dimensionless coupling constant as

\[ \lambda \equiv \frac{g}{m^{3/2}}, \]  

(3.33)

the superpotential in terms of the lattice variables reads

\[ W(\phi(x)) = \frac{1}{2} (am) \phi(x)^2 + \frac{1}{3} (am)^{3/2} \lambda \phi(x)^3. \]  

(3.34)

Before proceeding to the numerical study, it is instructive to see how our method works for the free theory \( \lambda = 0 \), for which supersymmetry is not spontaneously broken in the continuum theory and the lattice model (3.19) is solvable. From Eqs. (3.34) and (3.26), we see that the partition function \( Z \) (3.28) is nonzero for both boundary conditions (for \( a \neq 0 \)). It is therefore meaningful to consider the expectation value for both boundary conditions and it is not difficult to see that

\[ \langle H(x) \rangle_{\text{PBC}} = 0, \]  

(3.35)

\[ \langle H(x) \rangle_{a\text{PBC}} = \frac{m(1 - am)^{\beta/a-1}}{1 - (1 - am)^{\beta/a}} + \frac{m(1 - am)^{\beta/a-1}}{1 + (1 - am)^{\beta/a}}. \]  

(3.36)

In the continuum limit \( a \to 0 \), the latter, in fact, reproduces the expectation value of the energy in this supersymmetric harmonic oscillator:

\[ \lim_{a \to 0} \langle H(x) \rangle_{a\text{PBC}} = \frac{me^{-\beta m}}{1 - e^{-\beta m}} + \frac{me^{-\beta m}}{1 + e^{-\beta m}}. \]  

(3.37)

We thus have, in the large-time limit \( \beta \to \infty \),

\[ E_0 = \lim_{\beta \to \infty} \lim_{a \to 0} \langle H(x) \rangle_{a\text{PBC}} = 0, \]  

(3.38)

and infer that supersymmetry is not spontaneously broken.

Now we turn to the Monte Carlo study of the model (3.34) with \( \lambda \neq 0 \) (supersymmetry is dynamically broken in the target continuum theory). In the following results, we set \( \lambda = 10 \) and, for each set of parameters, we used \( 10^4 \) statistically independent configurations.

*) The potential energy \( V(\phi) = W'(\phi)^2/2 \) is a double-well type with two minima \( V = 0 \) at \( \phi = 0 \) and \( \phi = -m/g \), and the height of the potential barrier is \( m^4/(32g^2) = m/(32\lambda^2) \). From this, the spontaneous supersymmetry breaking in the present system will be rather difficult to be observed numerically for weak couplings for which the supersymmetry breaking is caused mainly by quantum tunneling.
First, let us see the case of the periodic boundary condition. For this boundary condition, the sign of the fermion determinant \( s[\phi] \) (3.26) may change depending on the configuration, and in fact, as Fig. 1 shows, positive and negative fermion determinants appear at almost equal rates. This implies that the partition function (3.28), that is, the Witten index (2.2), is almost zero. This is perfectly in accord with the fact that supersymmetry is spontaneously broken in the target theory. In Fig. 1, we plotted also the distribution of the Hamiltonian (3.22) reweighted by the sign of the determinant, \( H_s[\phi] \). It spreads on negative as well as positive sides and the average \((\simeq -0.003)\) is consistent with zero within statistical error \((\simeq 0.02)\). This is again consistent with the fact that the average of the Hamiltonian (3.27) is merely the \( \beta \)-derivative of the Witten index, Eq. (2.3). This assures us of the validity of our method because the construction of \( H \) ensures Eq. (2.3), as shown in Eq. (3.23).

If we switch the boundary condition to antiperiodic, things drastically change. As Fig. 2 shows, now the distribution of the sign is significantly asymmetric and the partition function (3.28) becomes nonvanishing. This implies that we can give a definite meaning for the expectation value normalized by the partition function \( Z_{aPBC} \). In this way, we see numerically that the effect of the boundary condition indeed survives even for large temporal size.
Fig. 2. Histogram of the sign of the fermion determinant $s[\phi]$ (3.26) (bold line; left axis) and the Hamiltonian reweighted by the sign of the determinant $Hs[\phi]$ (broken line; right axis) for the model (3.34) with the antiperiodic boundary condition. $\lambda = 10$. The lattice spacing is $am = 0.1$ and the physical temporal size of the system is $\beta m = 1.6$. The number of configurations is $10^4$.

$\beta m = 1.6$ is actually a large size with the present value of the coupling constant; see below) when supersymmetry is spontaneously broken.

With the antiperiodic boundary condition, we then measure the expectation value of the Hamiltonian as a function of the temporal size of the system $\beta$. For various values of $\beta m$, we measured $\langle H(x) \rangle_{\text{APBC}}/m$ for lattice spacings $am = 0.1, 0.05$ and $0.02$. The number of configurations is $10^4$ for each set of parameters. Then, as shown in Fig. 3, we extrapolate $\langle H(x) \rangle_{\text{APBC}}/m$ to the continuum $a = 0$ by a linear $\chi^2$-fit. In Fig. 4, we plot the continuum limit of the expectation value $\lim_{a \to 0} \langle H(x) \rangle_{\text{APBC}}/m$ as a function of the physical temporal size of the system $\beta m$. For $\beta m \gtrsim 1$, we have $\lim_{a \to 0} \langle H(x) \rangle_{\text{APBC}} \simeq 1.1m$ and, from this,

---

*) The statistical errors in Fig. 3 are one standard deviation. The errors in the linear $\chi^2$-extrapolation were estimated from the range of fitting parameters that corresponds to a unit variation of $\chi^2$. These remarks also apply to the results in Fig. 5.

**) We found that a quadratic function of the form $\alpha (am)^2 + \beta$, which gives a somewhat larger $\langle H(x) \rangle_{\text{APBC}}/m$ at $a = 0$, provides a better fit. Although this form of the fit function might be suggested theoretically (i.e., the residual lattice artifact is $O(a^2)$ instead of $O(a)$), we stick to a simple linear fit to avoid a possible criticism that nonzero $E_0$ is an artificial consequence of the fit.

***) For strong couplings $\lambda \gg 1$, it is easy to see that energy eigenvalues of the present system scale as $\lambda^{2/3}m$. Thus the difference between the first excited state and the ground-state would be $\lambda^{2/3}m$ times a number of $O(1)$. This observation suggests that the expectation value of the Hamiltonian (with the
Fig. 3. Linear extrapolations of $\langle H(x) \rangle_{a\text{PBC}}/m$ to the continuum $a = 0$ for various values of $\beta m$. $\lambda = 10$. The errors are only statistical ones.

Fig. 4. The continuum limit of the expectation value of the Hamiltonian, $\lim_{a \to 0} \langle H(x) \rangle_{a\text{PBC}}/m$, as a function of the physical temporal size of the system $\beta m$. $\lambda = 10$. The errors are only statistical ones. We have also plotted the exact ground-state energy $E_0/m = 1.27616$ and the analytic expression for the $\lambda = 0$ case, Eq. (3.37).
we infer that supersymmetry is spontaneously broken.*) Indeed, this is the correct answer.

Fig. 5. Continuum limit of the expectation values of the Hamiltonian, \( \lim_{a \to 0} \langle H(x) \rangle_{aPBC} / m \) and \( \lim_{a \to 0} \langle H(x) \rangle_{PBC} / m \), as a function of the physical temporal size of the system \( \beta m \). The errors are only statistical ones.

Next, as an interacting case in which supersymmetry is not spontaneously broken, we consider

\[
W_{\text{continuum}} = \frac{1}{4} m^2 \phi^4_{\text{continuum}}. \tag{3.39}
\]

In this case, we observed that \( s[\phi] \) (3.26) has almost always a definite sign for both boundary conditions and thus both \( \langle H(x) \rangle_{PBC} \) and \( \langle H(x) \rangle_{aPBC} \) can be considered. In Fig. 5, we plot the continuum limit of these quantities as a function of \( \beta m \). The results are obtained by extrapolation to the continuum \( a = 0 \) with a linear \( \chi^2 \)-fit (like Fig. 3) of data computed at \( a m = 0.1 \) and 0.05. The number of configurations is \( 10^4 \) for each set of parameters. The figure shows that, in this case for which supersymmetry is not spontaneously broken, \( \lim_{a \to 0} \langle H(x) \rangle_{PBC} \) is consistent with zero for all temporal sizes (recall Eq. (2.4); in the present model, \( \text{Tr}(-1)^F = 1 \)) and \( \lim_{a \to 0} \langle H(x) \rangle_{aPBC} \) approaches zero as the temporal size of the antiperiodic boundary condition) exponentially approaches the asymptotic value at \( \beta m = \infty \) around \( \beta m \gtrsim \lambda^{-2/3} \approx 0.2 \) for \( \lambda = 10 \).

*) In Fig. 4, we plotted also the exact ground-state energy \( E_0 / m = 1.27616 \) that was obtained by numerically diagonalizing the corresponding Hamiltonian operator (we used the method of Ref. 47)). The discrepancy of our Monte Carlo results for \( \beta m \gtrsim 1 \) with this exact result can be understood as a systematic error associated with a linear extrapolation to the continuum limit.
system is increased. From this, we conclude that \( E_0 = 0 \) is within the error.

In summary, we have observed that our method works perfectly well in the present supersymmetric quantum mechanics. One can certainly observe whether or not the dynamical supersymmetry breaking takes place by our method.

§4. Two-dimensional \( \mathcal{N} = (2, 2) \) super Yang-Mills theory

The two-dimensional \( \mathcal{N} = (2, 2) \) super Yang-Mills theory is obtained by a dimensional reduction of the four-dimensional \( \mathcal{N} = 1 \) super Yang-Mills theory.\(^*)\) This seemingly simple supersymmetric system, however, defies a straightforward low-energy description for several reasons. First, two global \( U(1) \) symmetries in this system cannot be spontaneously broken in two dimensions and a description by using the Nambu-Goldstone fields is impossible.\(^**)\) Second, there is no controllable parameter, other than the number of colors \( N_c \) of the gauge group \( SU(N_c) \). (The two-dimensional gauge coupling \( g \) simply provides a mass scale, just like \( \Lambda_{\text{QCD}} \).) The \( 1/N_c \) expansion is nontrivial because the gaugino and scalars belong to the adjoint representation. Finally, the classical potential energy of scalar fields possesses noncompact flat directions and there are an infinite number of degenerate classical vacua. This classical degeneracy is not lifted upon quantum corrections to all orders of perturbation theory.

In our present context, the last point above (noncompact flat directions in the classical potential) is an obstruction to the determination of the Witten index. In the weak coupling approximation, zero-momentum modes without potential (constant degrees of freedom along flat directions) produce a continuous spectrum starting at zero. This makes the counting of zero-energy states, and thus the determination of the Witten index in the weak coupling approximation, awkward. A similar situation arises in the three-dimensional \( \mathcal{N} = 2 \) super Yang-Mills theory (that can also be obtained by a dimensional reduction of the four-dimensional \( \mathcal{N} = 1 \) super Yang-Mills theory). However, in this three-dimensional model, if the gauge group is \( SU(N_c) \), one eliminate zero-momentum bosonic modes by imposing the twisted boundary conditions for two spatial directions and obtain \( \text{Tr}(-1)^F = 1 \).\(^{49)***} \) This trick of the twisted boundary conditions, unfortunately, does not work in two dimensions. The correct value of the Witten index, or even whether it is well defined or not, is therefore

\(^*)\) In what follows, we assume that the gauge group is \( SU(N_c) \).

\(^**)\) Nevertheless, it is possible to show that a correlation function of Noether currents associated with the \( U(1) \) symmetries possesses a massless pole, to all orders of perturbation theory.\(^{48})\)

\(^{***}\) Incidentally, this is a good example of the Witten index generally not being preserved under dimensional reduction, because \( \text{Tr}(-1)^F = N_c \) for the four-dimensional \( \mathcal{N} = 1 \) super Yang-Mills theory. See also § 4 of Ref. 2).
unknown for the two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory. It is consequently not known whether supersymmetry is dynamically broken in this system.

Under this situation, Hori and Tong\(^9\) conjectured that dynamical supersymmetry breaking occurs in this system, on the basis of the counting of the number of ground-states in the two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric gauge theory with fundamental chiral multiplets, combined with a decoupling argument. In what follows, we numerically investigate this possibility of dynamical supersymmetry breaking by directly measuring the ground-state energy density in Euclidean lattice gauge theory.

4.1. Hamiltonian density in the continuum theory

The Lagrangian density of the two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory in the Minkowski spacetime, in terms of the twisted basis of spinors,\(^{50,51}\) is given by

$$
L = \frac{1}{g^2} \text{tr}\left\{ -\frac{1}{4} [\phi, \bar{\phi}]^2 - H^2 + 2HF_{01} + D_0 \phi D_0 \bar{\phi} - D_1 \phi D_1 \bar{\phi} \\
+ \frac{1}{4} \eta [\phi, \eta] + \chi [\phi, \chi] - \psi_\mu [\bar{\phi}, \psi_\mu] + 2i \chi (i D_0 \psi_1 + D_1 \psi_0) - \psi_0 D_0 \eta - i \psi_1 D_1 \eta \right\},
$$

(4.1)

where all fields are $SU(N_c)$ Lie algebra valued and scalar fields $\phi$ and $\bar{\phi}$ are combinations of two real scalar fields, $\phi = X_2 + iX_3$ and $\bar{\phi} = X_2 - iX_3$, respectively. $F_{01} = \partial_0 A_1 - \partial_1 A_0 + i[A_0, A_1]$ is the field strength in two dimensions. The covariant derivatives $D_\mu$ are defined with respect to the adjoint representation $D_\mu \phi = \partial_\mu \phi + i[A_\mu, \phi]$ for any field $\phi$. The index $\mu$ runs over 0 and 1. Note that, in the above convention, the bosonic fields $A_\mu$, $\phi$ and $\bar{\phi}$ have the mass dimension 1 and the fermionic fields $\psi_\mu$, $\chi$ and $\eta$ have the mass dimension 3/2, because the gauge coupling constant in two dimensions $g$ has the mass dimension 1.

The action $\int d^2x \ L$ with the periodic boundary condition is invariant under four supersymmetry transformations. Among them, what is relevant to us is $Q$ and $Q_0$. The $Q$-transformation is given by

$$
QA_0 = i \psi_0, \quad Q\psi_0 = D_0 \phi, \\
QA_1 = \psi_1, \quad Q\psi_1 = i D_1 \phi, \\
Q\phi = 0, \\
Q\chi = H, \quad QH = [\phi, \chi], \\
Q\bar{\phi} = \eta, \quad Q\eta = [\phi, \bar{\phi}]
$$

(4.2)

and $Q_0$ is (see, for example, Ref. 52))

$$
Q_0 A_0 = \frac{i}{2} \eta, \quad Q_0 \eta = -2 D_0 \bar{\phi},
$$
\[ Q_0 A_1 = -\chi, \quad Q_0 \psi_1 = H - 2F_{01}, \quad Q_0 \phi = -2\psi_0, \]
\[ Q_0 \phi = 0, \quad Q_0 H = -[\phi, \psi_1] - 2D_0 \chi - iD_1 \eta. \]

One then finds that these transformations satisfy
\[ Q^2 = \delta_\phi, \quad Q^2_0 = -\delta_\phi, \quad \{Q, Q_0\} = -2\partial_0 - 2i\delta_{A_0}, \]
where \( \delta_\phi \) denotes the infinitesimal gauge transformation with the parameter \( \phi \). These differ from the off-shell supersymmetry algebra in the twisted basis, \( Q^2 = Q^2_0 = 0 \) and \( \{Q, Q_0\} = -2\partial_0 \), by gauge transformations because we are working with the Wess-Zumino gauge.

A crucial property of this system, which allows a simple lattice formulation, is that the action \( S = \int d^2x L \) is \( Q \)-exact.\(^{50,51} \)

\[ S = Q \frac{1}{g^2} \int d^2x \text{tr} \left\{ -\frac{1}{4} \eta[\phi, \phi] + 2\chi F_{01} - \chi H + \psi_0 D_0 \phi + i\psi_1 D_1 \phi \right\} \quad (4.5) \]
In this form, with the relations \( (4.4) \), the invariance of the action under \( Q \) and \( Q_0 \) transformations is easily seen.\(^* \)

Now, from the Lagrangian density \( (4.1) \), we obtain the Hamiltonian density \( \mathcal{H} \) by the Legendre transformation. After (trivially) eliminating redundant fields by using second-class constraints, \( \mathcal{H} \) is given, in terms of fields in the Lagrangian, by
\[ \mathcal{H} = \frac{1}{g^2} \text{tr} \left\{ \frac{1}{4} [\phi, \phi]^2 + H^2 + D_0 \phi D_0 \phi + D_1 \phi D_1 \phi - \frac{1}{4} \eta[\phi, \phi] - \chi[\phi, \chi] + \psi_{\mu}[\phi, \psi_{\mu}] - 2i\chi D_1 \psi_0 + i\psi_1 D_1 \eta \right\} - 2 \text{tr} \{A_0 \mathcal{G}\}, \quad (4.6) \]
up to a spatial total derivative, where \( \mathcal{G} \) is the Gauss-law constraint:
\[ \mathcal{G} = \frac{1}{g^2} \left\{ D_1 H + iD_0 \phi + iD_0 \phi + iD_0 \psi_1 + \chi D_0 + \psi_0 \phi \right\}. \quad (4.7) \]
From the off-shell supersymmetry algebra \( \{Q, Q_0\} = -2\partial_0 \), one might expect that the relation \( Q \mathcal{J}_0^0 / 2 = \mathcal{H} \) holds, where \( \mathcal{J}_0^0 \) is the time component of the Noether current associated with the \( Q_0 \)-symmetry
\[ \mathcal{J}_0^0 = \frac{1}{g^2} \text{tr} \left\{ \frac{1}{2} \eta[\phi, \phi] + 2\chi H + 2\psi_0 D_0 \phi - 2i\psi_1 D_1 \phi \right\}. \quad (4.8) \]
\(^* \) Note that \( S \) can further be written as \( S = Q Q_0 \frac{1}{g^2} \int d^2x \text{tr} \left\{ -\frac{1}{4} \phi D_0 \phi - \psi_1 \chi \right\}. \)
In reality,
\[
\frac{1}{2} Q J_0' = \mathcal{H} + 2 \text{tr} \{ A_0 G \} + \frac{1}{g^2} \text{tr} \{ \psi_0 ( -2 \bar{\phi}, \psi_0 ) + 2 i D_1 \chi - D_0 \eta ) \}
\] (4.9)
up to a spatial total derivative. Compare this with Eq. (2.8). In Eq. (4.9), the last two terms are proportional to classical equations of motion and they can be expressed as \( A_0 \frac{\delta}{\delta A_0} S \) and \( \psi_0 \frac{\delta}{\delta \psi_0} S \). The expectation value of these two expressions may be obtained (after gauge fixing) as a Jacobian associated with the transformations \( A_0 \rightarrow A_0 + \alpha A_0 \) and \( \psi_0 \rightarrow \psi_0 + \alpha \psi_0 \), respectively. Such a Jacobian is generally ultraviolet divergent.

Thus, in view of Eq. (4.9) and following our general prescription, we adopt \( H \equiv Q J_0' / 2 \) as the Hamiltonian density in our functional integral formulation. As can be seen from Eqs. (4.9) and (4.6), this \( H \) is, moreover, gauge invariant.

Thus, going to the Euclidean space by \( x_0 \rightarrow -i x_0, A_0 \rightarrow i A_0, D_0 \rightarrow i D_0 \) and \( L \rightarrow -L \), we have the Euclidean action
\[
S = Q \frac{1}{g^2} \int d^2 x \text{tr} \left\{ \frac{1}{4} \eta[\phi, \bar{\phi}] - i \chi \Phi + \chi H - i \psi_\mu D_\mu \bar{\phi} \right\},
\] (4.10)
where \( \Phi \equiv 2F_{01} \) and
\[
Q A_\mu = \psi_\mu, \quad Q \psi_\mu = i D_\mu \phi, \quad Q \phi = 0, \quad Q \chi = H, \quad Q H = [\phi, \chi], \quad Q \bar{\phi} = \eta, \quad Q \eta = [\phi, \bar{\phi}],
\] (4.11)
and the Hamiltonian density
\[
\mathcal{H} \equiv Q \frac{1}{g^2} \text{tr} \left\{ \frac{1}{4} \eta[\phi, \bar{\phi}] + \chi H + i \psi_0 D_0 \bar{\phi} - i \psi_1 D_1 \bar{\phi} \right\}.
\] (4.12)
These are the basic relations for our Euclidean lattice formulation.

4.2. Manifestly \( Q \)-invariant lattice formulation

This subsection is a brief summary of a lattice formulation of the two-dimensional \( \mathcal{N} = (2,2) \) super Yang-Mills theory proposed in Ref. 13). For full details, we refer the reader to Ref. 13). We consider a two-dimensional rectangular lattice of the physical size \( \beta \times L \):
\[
A = \left\{ x \in a \mathbb{Z}^2 \mid 0 \leq x_0 < \beta, \ 0 \leq x_1 < L \right\},
\] (4.13)

*) We define color components of fields as \( \phi = \sum_{a=1}^{N_c^2-1} \phi^a T^a \), where \( T^a \) are generators of \( SU(N_c) \).

**) With a lattice regularization, for example, the one described in the next subsection, the expectation value of the latter is \( -i(N_c^2 - 1)/a^2 \). The expectation value of the former is \( +i(N_c^2 - 1)/a^2 \) after gauge fixing and thus, quite interestingly, the expectation value of the last two terms of Eq. (4.9) vanishes.
where $a$ denotes the lattice spacing. All fields except the gauge potentials are put on sites and the gauge field is expressed by the compact link variables $U(x, \mu) \in SU(N_c)$.

As a lattice transcription of the $Q$-transformation (4.11), we define

\[
QU(x, \mu) = i \psi_\mu(x)U(x, \mu),
\]

\[
Q\psi_\mu(x) = i \psi_\mu(x)\psi_\mu(x) - i (\phi(x) - U(x, \mu)\phi(x + a\hat{\mu})U(x, \mu)^{-1}),
\]

\[
Q\phi(x) = 0,
\]

\[
Q\chi(x) = H(x), \quad QH(x) = [\phi(x), \chi(x)],
\]

\[
Q\phi(x) = 0,
\]

\[
Q\chi(x) = H(x), \quad QH(x) = [\phi(x), \chi(x)],
\]

\[
Q\phi(x) = 0.
\]

($\hat{\mu}$ implies a unit vector in the $\mu$-direction). It can be confirmed that $Q^2 = \delta_\phi$, where $\delta_\phi$ is an infinitesimal gauge transformation on the lattice with the parameter $\phi(x)$. The lattice action is then defined by an expression analogous to Eq. (4.10):

\[
S = Qa^2 \sum_{x \in \Lambda} \left( O_1(x) + O_2(x) + O_3(x) + \frac{1}{a^4g^2} \text{tr} \{\chi(x)H(x)\} \right),
\]

(4.15)

where

\[
O_1(x) = \frac{1}{a^4g^2} \text{tr} \left\{ \frac{1}{4} \eta(x)[\phi(x), \bar{\phi}(x)] \right\},
\]

(4.16)

\[
O_2(x) = \frac{1}{a^4g^2} \text{tr} \left\{ -i \chi(x)\hat{\Phi}(x) \right\},
\]

(4.17)

\[
O_3(x) = \frac{1}{a^4g^2} \text{tr} \left\{ i \sum_{\mu=0}^{1} \psi_\mu(x) \left( \bar{\phi}(x) - U(x, \mu)\bar{\phi}(x + a\hat{\mu})U(x, \mu)^{-1} \right) \right\}.
\]

(4.18)

In Eq. (4.17), $\hat{\Phi}(x)$ is a lattice counterpart of the field strength and is defined from the plaquette variables

\[
U(x, 0, 1) = U(x, 0)U(x + a\hat{0}, 1)U(x + a\hat{1}, 0)^{-1}U(x, 1)^{-1}
\]

as

\[
\hat{\Phi}(x) = \frac{\Phi(x)}{1 - \frac{1}{2} ||1 - U(x, 0, 1)||^2}, \quad \Phi(x) = -i \left[ U(x, 0, 1) - U(x, 0, 1)^{-1} \right],
\]

(4.20)

where the matrix norm is

\[
||A|| = [\text{tr} \{AA^\dagger\}]^{1/2}
\]

(4.21)

and the constant $\epsilon$ is chosen in the range

\[
0 < \epsilon < 2\sqrt{2}, \quad \text{for } N_c = 2, 3, 4,
\]

(4.22)

\[
0 < \epsilon < 2\sqrt{N_c} \sin \left( \frac{\pi}{N_c} \right), \quad \text{for } N_c \geq 5.
\]

(4.23)
From the $Q$-exact form (4.15) and the nilpotency of $Q$, $Q^2 = \delta \phi$, the lattice action is manifestly invariant under the $Q$-transformation (4.14).\footnote{Another interesting property of the present lattice formulation is that one global $U(1)_R$ symmetry is manifestly preserved.\cite{13}}

After the operation of $Q$, the lattice action becomes

$$S = a^2 \sum_{x \in \Lambda} \left( \sum_{i=1}^{4} L_{B_i}(x) + \sum_{i=1}^{7} L_{F_i}(x) + \frac{1}{a^4 g^2} \text{tr} \left\{ H(x) - \frac{1}{2} i \hat{\Phi}_{TL}(x) \right\}^2 \right), \quad (4.24)$$

where we have noted that only the traceless part of $\hat{\Phi}(x)$,

$$\hat{\Phi}_{TL}(x) = \hat{\Phi}(x) - \frac{1}{N_c} \text{tr} \left\{ \hat{\Phi}(x) \right\} \mathbb{1}, \quad (4.25)$$

appears in the action, because the auxiliary field $H(x)$ is traceless. Each term of the action density is given by

$$L_{B_1}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ \frac{1}{4} [\phi(x), \bar{\phi}(x)]^2 \right\}, \quad (4.26)$$

$$L_{B_2}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ \frac{1}{4} \hat{\Phi}_{TL}(x)^2 \right\}, \quad (4.27)$$

$$L_{B_3}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ (\phi(x) - U(x, 0) \phi(x + a \hat{0}) U(x, 0)^{-1}) \right. \right.$$

$$\times \left( \bar{\phi}(x) - U(x, 0) \bar{\phi}(x + a \hat{0}) U(x, 0)^{-1} \right) \bigg\}, \quad (4.28)$$

$$L_{B_4}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ (\phi(x) - U(x, 1) \phi(x + a \hat{1}) U(x, 1)^{-1}) \right. \right.$$

$$\times \left( \bar{\phi}(x) - U(x, 1) \bar{\phi}(x + a \hat{1}) U(x, 1)^{-1} \right) \bigg\}, \quad (4.29)$$

and

$$L_{F_1}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -\frac{1}{4} \eta(x) [\phi(x), \eta(x)] \right\}, \quad (4.30)$$

$$L_{F_2}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -\chi(x) [\phi(x), \chi(x)] \right\}, \quad (4.31)$$

$$L_{F_3}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -\psi_0(x) \psi_0(x) (\bar{\phi}(x) + U(x, 0) \bar{\phi}(x + a \hat{0}) U(x, 0)^{-1}) \right\}, \quad (4.32)$$

$$L_{F_4}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ -\psi_1(x) \psi_1(x) (\bar{\phi}(x) + U(x, 1) \bar{\phi}(x + a \hat{1}) U(x, 1)^{-1}) \right\}, \quad (4.33)$$

$$L_{F_5}(x) = \frac{1}{a^4 g^2} \text{tr} \left\{ i \chi(x) Q \hat{\Phi}(x) \right\}, \quad (4.34)$$
\[
\mathcal{L}_{F6}(x) = \frac{1}{a^4g^2} \text{tr} \left\{ -i\psi_0(x) \left( \eta(x) - U(x,0)\eta(x + a\hat{0})U(x,0)^{-1} \right) \right\}. \quad (4.35)
\]
\[
\mathcal{L}_{F7}(x) = \frac{1}{a^4g^2} \text{tr} \left\{ -i\psi_1(x) \left( \eta(x) - U(x,1)\eta(x + a\hat{1})U(x,1)^{-1} \right) \right\}. \quad (4.36)
\]

Note that all lattice fields in the above expressions are dimensionless. For all fields other than fermionic fields, periodic boundary conditions on \( \Lambda \) are assumed. For fermionic fields, \( \psi \equiv (\psi_{\mu}, \chi, \eta) \), depending on whether the temporal boundary condition is periodic (PBC) or antiperiodic (aPBC), we set
\[
\psi(x_0 = \beta, x_1) = \begin{cases} 
+\psi(x_0 = 0, x_1) & \text{for the periodic boundary condition}, \\
-\psi(x_0 = 0, x_1) & \text{for the antiperiodic boundary condition},
\end{cases} \quad (4.37)
\]

while the spatial boundary condition is always taken to be periodic.

With the lattice action (4.24), the partition function is defined by
\[
Z = \mathcal{N} \int d\mu e^{-S}, \quad (4.38)
\]

where the integration measure is defined (writing \( \phi(x) = X_2(x) + iX_3(x) \) and \( \overline{\phi}(x) = X_2(x) - iX_3(x) \) by
\[
d\mu \equiv \prod_{x \in \Lambda} \left( \prod_{\mu = 0}^{1} dU(x, \mu) \right) \prod_{\alpha = 1}^{N^2_\text{c} - 1} dX_2^\alpha(x) dX_3^\alpha(x) dH^\alpha(x) \left( \prod_{\mu = 0}^{1} d\psi_\alpha^\mu(x) \right) d\chi^\alpha(x) d\eta^\alpha(x) \quad (4.39)
\]
in terms of color components of fields. \( dU(x, \mu) \) is the standard Haar measure. Note that the integration over the auxiliary field \( H(x) \) can readily be performed because it is gaussian. The invariance of this measure under the \( Q \)-transformation is noted in Ref. 17).

The denominator in Eq. (4.20) needs an explanation. Without that factor, the lattice action for the gauge field is the “double-winding plaquette type” and the action possesses many degenerate minima which have no continuum counterpart. Because of the denominator of Eq. (4.20), the action (4.24) diverges as \( \|1 - U(x,0,1)\| \to \epsilon \) at a certain site \( x \). Precisely speaking, the above construction of the action is applied only for configurations with
\[
\|1 - U(x,0,1)\| < \epsilon, \quad \forall x \in \Lambda, \quad (4.40)
\]
and, otherwise, i.e., if there exists \( x \in \Lambda \) such that \( \|1 - U(x,0,1)\| \geq \epsilon \), we set
\[
S = +\infty. \quad (4.41)
\]

In this way, the domain of functional integral (4.38) is effectively restricted to configurations specified by Eq. (4.40). It can then be shown that \( U(x, \mu) \equiv 1 \) is (up to gauge transformations) a unique minimum of the action within the integration domain. This procedure to solve the problem of degenerate minima moreover does not break the \( Q \)-symmetry.\textsuperscript{13}
With the above construction (and with the periodic boundary condition), one fermionic symmetry $Q$ is manifestly preserved on the lattice. The price to pay is that the Pfaffian of the Dirac operator, resulting from the integration over fermionic fields, is generally complex. Since the corresponding Pfaffian in the target continuum theory is real and positive semi-definite, the complex phase must be a lattice artifact. That is, we expect that the imaginary part of the lattice Pfaffian diminishes as we approach the continuum limit. We will later confirm this expectation numerically.

In Refs. 12) and 13), the restoration of the invariance under a full set of supersymmetry transformation in the continuum limit has been argued on the basis of perturbative power counting. It is certainly desirable to confirm, however, this restoration nonperturbatively by observing the supersymmetric Ward-Takahashi identities. More definitely, one should examine the total divergence of two-point (denominator-free) correlation functions containing the supercurrent; so far this analysis has not yet been carried out.

4.3. Lattice transcription of Hamiltonian density

Now, as the definition of a Hamiltonian density on the lattice, we follow the prescription $H(x) \equiv Q\mathcal{J}_0(x)/2$ suggested by Eq. (4.12), where

$$
\mathcal{J}_0(x) \equiv \frac{1}{a^4 g^2} \text{tr}\left\{ \frac{1}{2} \eta(x) [\phi(x), \bar{\phi}(x)] + 2\chi(x) H(x) - 2i\psi_0(x) (\bar{\phi}(x) - U(x, 0) \bar{\phi}(x + a\hat{0}) U(x, 0)^{-1}) + 2i\psi_1(x) (\bar{\phi}(x) - U(x, 1) \bar{\phi}(x + a\hat{1}) U(x, 1)^{-1}) \right\}
$$

(4.42)

is a lattice transcription of the Noether current $\mathcal{J}_0^0$ in Eq. (4.12). The explicit form of the Hamiltonian density is then given by

$$
\mathcal{H}(x) = \mathcal{L}_{\mathcal{B}1}(x) - \mathcal{L}_{\mathcal{B}3}(x) + \mathcal{L}_{\mathcal{B}4}(x) + \mathcal{L}_{\mathcal{F}1}(x) + \mathcal{L}_{\mathcal{F}2}(x) - \mathcal{L}_{\mathcal{F}3}(x) + \mathcal{L}_{\mathcal{F}4}(x) - \mathcal{L}_{\mathcal{F}6}(x) + \mathcal{L}_{\mathcal{F}7}(x) + \frac{1}{a^4 g^2} \text{tr} \{H(x)^2\}.
$$

(4.43)

From the $Q$-invariance of the lattice action and of the integration measure, we thus have

$$
\int_{\text{PBC}} d\mu \mathcal{H}(x) e^{-S} = \int_{\text{PBC}} d\mu Q \left( \frac{1}{2} \mathcal{J}_0^0(x) e^{-S} \right) = 0,
$$

(4.44)

assuming that the integral $\int_{\text{PBC}} d\mu \mathcal{J}_0^0(x) e^{-S}$ is finite, and this reproduces the topological property of the Witten index, Eq. (2.3). As already discussed, we regard this property as a guiding principle for choosing the origin of the energy (density).
We will measure the ground-state (vacuum) energy density $E_0$ by

$$\lim_{\beta \to \infty} \lim_{a \to 0} \langle \mathcal{H}(x) \rangle_{\text{aPBC}} = \lim_{\beta \to \infty} \lim_{a \to 0} \frac{\int_{\text{aPBC}} d\mu \mathcal{H}(x) e^{-S}}{\int_{\text{aPBC}} d\mu e^{-S}} = E_0 \quad (4.45)$$

and judge that dynamical supersymmetry breaking occurs if $E_0 > 0$ and but not if $E_0 = 0$.

For Eq. (4.43), an integration over the auxiliary field $H(x)$ can be performed to obtain

$$\frac{1}{a^4 g^2} \langle \left\{ H(x) \right\}^2 \rangle_{\text{aPBC}} = \frac{1}{2} \left( N_c^2 - 1 \right) \frac{1}{a^2} - \frac{1}{a^4 g^2} \left\langle \left\{ \frac{1}{4} \hat{\Phi}_{\text{TL}}(x)^2 \right\} \right\rangle_{\text{aPBC}}$$

$$= \frac{1}{2} \left( N_c^2 - 1 \right) \frac{1}{a^2} - \langle \mathcal{L}_{B2}(x) \rangle_{\text{aPBC}}. \quad (4.46)$$

Thus, in actual numerical simulations, we can (suppressing the subscript aPBC or PBC) use

$$\langle \mathcal{H}(x) \rangle = \langle \mathcal{L}_{B1}(x) \rangle - \langle \mathcal{L}_{B2}(x) \rangle - \langle \mathcal{L}_{B3}(x) \rangle + \langle \mathcal{L}_{B4}(x) \rangle + \langle \mathcal{L}_{F1}(x) \rangle + \langle \mathcal{L}_{F2}(x) \rangle - \langle \mathcal{L}_{F3}(x) \rangle + \langle \mathcal{L}_{F4}(x) \rangle - \langle \mathcal{L}_{F6}(x) \rangle + \langle \mathcal{L}_{F7}(x) \rangle + \frac{1}{2} \left( N_c^2 - 1 \right) \frac{1}{a^2}. \quad (4.47)$$

We can argue that the expectation value (4.47) with aPBC is ultraviolet finite and possesses a well defined continuum limit to all orders of (lattice) perturbation theory. In the present super-renormalizable model, a possible ultraviolet divergence in Eq. (4.47) arises from either (i) one-loop two-point functions of bosons contained as sub-diagrams or (ii) one-loop diagrams that are formed by a self-contraction of kinetic terms. The former (i) case is potentially logarithmically divergent but the divergence is cancelled among boson loops and fermion loops. For PBC, this cancellation can be shown by using the $Q$-symmetry. For aPBC, the cancellation still holds because the change in the boundary condition does not influence the coefficients of the logarithmic divergent pieces. In the latter case (ii), the divergence is quadratic and it is common to the free theory. We may thus examine the expectation value in the free theory and find that, even with aPBC, it is ultraviolet finite for any $\beta > 0$. Our numerical study described below in fact indicates that the continuum limit of the expectation value $\langle \mathcal{H}(x) \rangle_{\text{aPBC}}$ is well defined.

4.4. Monte Carlo study

We numerically studied only the $SU(2)$ gauge group. Our algorithm and the computation code, which was developed using FermiQCD/MDP, are almost identical to those in Ref. 20. We use the hybrid Monte Carlo algorithm to generate configurations in the quenched approximation. The effect of dynamical fermions is later taken into account by reweighting configurations by the Pfaffian of the Dirac operator. We do not introduce any
mass terms of fermions or bosons that would explicitly break the $Q$-symmetry. Although this is certainly a brute force method compared with a standard pseudo-fermion algorithm, its implementation is much simpler and the validity has been confirmed for one-point Ward-Takahashi identities.\textsuperscript{20)}

A direct calculation of the Pfaffian is very time-consuming.\textsuperscript{*} Thus we instead use a square root of the determinant that is obtained by LU decomposition. Expressing the determinant of the Dirac operator $D$ in the form

\begin{equation}
\det\{D\} = re^{i\theta}, \quad -\pi < \theta \leq \pi,
\end{equation}

we evaluate the Pfaffian by

\begin{equation}
Pf\{D\} = \sqrt{re^{i\theta/2}},
\end{equation}

because $(\text{Pf}\{D\})^2 = \det\{D\}$. This prescription, however, reproduces a correct Pfaffian if and only if $-\pi/2 < \text{Arg}(\text{Pf}\{D\}) \leq \pi/2$. It is expected that this inequality is fulfilled in the continuum limit, because the Pfaffian in the continuum target theory is real and positive semi-definite. A direct calculation of the Pfaffian over a subset of our configurations (Fig. 6) clearly supports this expectation and justifies the above prescription. Note that our present lattices are much finer, compared with that in Ref. 20) where $ag \geq 0.5$.

We stored statistically independent configurations for parameters summarized in Table I, where $N_T$ and $N_S$ are the number of lattice points for the temporal and spatial directions, respectively. The physical size of the spatial direction is fixed to be $L_g = \sqrt{2}$.\textsuperscript{**} The parameter $\epsilon$ in Eq. (4.22) is taken to be $\epsilon = 2.6$. We used the cold start and set all scalar fields to be zero at the initial configuration. As the initial thermalization, we discarded the first $10^4$ trajectories and then stored configurations at every $10^2$ trajectories (the auto-correlation time was 10–20 trajectories).

To give an idea of the quality of our numerical simulation and to illustrate that the quantum effect of fermions is really taken into account, in Fig. 7, we plot the real part of the expectation value of the action density in Eq. (4.15) with the periodic boundary condition as a function of the lattice spacing $ag$. Since the lattice action density is $Q$-exact, its expectation value under the periodic boundary condition should be zero for any lattice spacing. The plot is certainly consistent with this. On the other hand, the expectation values in the quenched approximation are definitely not consistent with zero as they do not contain the effect of dynamical fermions. See also Fig. 2 of Ref. 20).

\textsuperscript{*} It can be seen that the algorithm for the Pfaffian (appearing, for example, in Ref. 55)) is an $O(n^4)$ process for a $2n \times 2n$ matrix, while the LU decomposition has an $O(n^3)$-process algorithm.

\textsuperscript{**} Note that supersymmetry is not broken in the infinite volume if it is not with finite volume.\textsuperscript{4} This fact would justify our study with finite physical volume.
Fig. 6. Histogram of the complex phase of the Pfaffian \( \text{Arg}(\text{Pf}\{D\}) \) in radians obtained by direct calculation of the Pfaffian for sampled configurations. The plots are for two entries in the \( N_T/N_S = 1 \) column in Table I and the number of sampled configurations is 160 for both cases. The boundary condition is antiperiodic.

Table I. Number of statistically independent configurations we used for the cases with the antiperiodic boundary condition. \( N_T \times N_S = 3 \times 6 \) is the minimal-size lattice and \( 36 \times 12 \) is the maximal. The physical spatial size is held fixed at \( L_g = \sqrt{2} = 1.4142 \).

| \( N_S \) | \( ag \)      | 0.25 | 0.5   | 1     | 1.5  | 2     | 2.5  | 3     |
|----------|--------------|------|-------|-------|------|-------|------|-------|
| 6        | 0.2357       |      | 39,900| 99,900| 9,900| 9,900 | 9,900| 9,900 |
| 8        | 0.1768       |      | 39,900| 99,900| 9,900| 9,900 | 9,900| 9,900 |
| 12       | 0.1179       | 39,900| 69,900| 69,900| 9,900| 9,900 | 9,900| 9,900 |
| 16       | 0.08839      | 39,900|      |      |      |      |      |      |
| 20       | 0.07071      | 39,900|      |      |      |      |      |      |

Now, our main result in this paper is illustrated in Fig. 8. We plotted the continuum limit of the real part of the expectation value of the Hamiltonian density (4.43) with the antiperiodic boundary condition, \( \lim_{a \to 0} \text{Re}(\mathcal{H}(x))_{\text{APBC}} \), as a function of the physical temporal size of the system \( \beta g \). For each \( \beta g \), the continuum limit was obtained by a linear \( \chi^2 \)-fit, as
Fig. 7. Real part of expectation values of the action density (over $g^2$) with the periodic boundary condition. The parameters are identical to those of entries in the $N_T/N_S = 1$ column in Table I, except that the number of configurations is 9,900 for each case.

Fig. 8. Continuum limit of the real part of the expectation value of the Hamiltonian density, $\lim_{a \to 0} \text{Re} \langle \mathcal{H}(x) \rangle_{aPBC}/g^2$ and $\lim_{a \to 0} \text{Re} \langle \mathcal{H}(x) \rangle_{PBC}/g^2$, as functions of the temporal size $\beta g$. The errors are only statistical ones. For the periodic boundary condition, the number of configurations is 9,900 for all cases.
depicted in Fig. 9.\textsuperscript{*)} For $\beta g \gtrsim 1$, the expectation value rapidly approaches the asymptotic value, that is, $\mathcal{E}_0/g^2$, according to Eq. (4.45). We may estimate the asymptotic value in $\beta g \rightarrow \infty$ by $\chi^2$-fit using a constant. The use of four data points in Fig. 8 at $\beta g > 2$ gives $\mathcal{E}_0/g^2 = -2.0 \pm 1.7$ and three points at $\beta g > 2.5$ gives $\mathcal{E}_0/g^2 = -3.0 \pm 2.2$. If we use an exponential function $A \exp(-B \beta g) + C$ and all points in Fig. 8, we have $\mathcal{E}_0/g^2 = -2.2 \pm 1.4$. All these results on $\mathcal{E}_0/g^2$ are consistent and, at least within one standard deviation, we do not observe positive vacuum energy density. We regard this as an indication of the fact that supersymmetry is not dynamically broken in this system. Of course, errors in our present result are large and we cannot exclude the possibility of supersymmetry breaking of $O(1)$ in $\mathcal{E}_0/g^2$. Further reduction of statistical errors will allow us to conclude whether the scale of dynamical supersymmetry breaking is $O(1)$ or not.\textsuperscript{***)}

In the present lattice model with the periodic boundary condition, we observed that the Pfaffian of the Dirac operator is almost real positive (recall Fig. 6) and this implies that $\beta g \rightarrow \infty$ by $\chi^2$-fit using a constant. The use of four data points in Fig. 8 at $\beta g > 2$ gives $\mathcal{E}_0/g^2 = -2.0 \pm 1.7$ and three points at $\beta g > 2.5$ gives $\mathcal{E}_0/g^2 = -3.0 \pm 2.2$. If we use an exponential function $A \exp(-B \beta g) + C$ and all points in Fig. 8, we have $\mathcal{E}_0/g^2 = -2.2 \pm 1.4$. All these results on $\mathcal{E}_0/g^2$ are consistent and, at least within one standard deviation, we do not observe positive vacuum energy density. We regard this as an indication of the fact that supersymmetry is not dynamically broken in this system. Of course, errors in our present result are large and we cannot exclude the possibility of supersymmetry breaking of $O(1)$ in $\mathcal{E}_0/g^2$. Further reduction of statistical errors will allow us to conclude whether the scale of dynamical supersymmetry breaking is $O(1)$ or not.\textsuperscript{***)}

In the present lattice model with the periodic boundary condition, we observed that the Pfaffian of the Dirac operator is almost real positive (recall Fig. 6) and this implies that

\textsuperscript{*)} The statistical errors in Fig. 9 are one standard deviation, obtained by jackknife analysis. Jackknife analysis is necessary because we are using the reweighting method, as explained in Ref. 20). The errors in the linear $\chi^2$-extrapolation are estimated from the range of fitting parameters that corresponds to a unit variation of $\chi^2$.

\textsuperscript{***)} For the present lattice model, we are currently developing a simulation code with the pseudo-fermion and the RHMC algorithm.\textsuperscript{56)} We hope this will enable us to reduce the statistical errors without substantially increasing the number of configurations.
\( Z_{\text{PBC}} \neq 0 \) with finite lattice spacings (unlike the case of Fig. 1).\(^4\) We can thus consider the expectation values with the periodic boundary condition. In Fig. 8, we have also plotted the real part of the expectation values of the Hamiltonian density for various temporal sizes with the periodic boundary condition. Since the Hamiltonian density is \( Q \)-exact, all the expectation values with the periodic boundary condition must be zero, if one can define them. The plot is clearly consistent with this. This also supports the idea that supersymmetry is not broken in this system. If supersymmetry is spontaneously broken, the expectation value \( \langle \mathcal{H}(x) \rangle_{\text{PBC}} \) must be indefinite, as Eq. (2.4) shows. In our simulation, the expectation value is computed as \( \langle \mathcal{H}(x) \text{ Pf}\{D\} \rangle_{\text{quenched}} / \langle \text{Pf}\{D\} \rangle_{\text{quenched}} \),\(^20\) and it can be indefinite only when \( \langle \text{Pf}\{D\} \rangle_{\text{quenched}} = 0 \) in the continuum limit. We did not see such a tendency and obtained the plot shown in Fig. 8. Also, it is interesting to note that the overall feature of Fig. 8 is quite similar to that of Fig. 5, rather than that of Fig. 4.

§5. Discussion

The most direct way to observe the spontaneous supersymmetry breaking would be to examine the degeneracy of boson and fermion mass spectra through two-point correlation functions. Although this method is conceptually clear, a reliable exponential fit of two-point functions would require a rather large lattice extent. The method we propose in this paper is computationally much easier because it is based on the measurement of one-point functions, the expectation values of a Hamiltonian (density). A weakness is the ambiguity in the choice of the Hamiltonian in the Euclidean lattice formulation. In this paper, we gave a justification (on the basis of a topological property of the Witten index) of the choice, for lattice formulations that possess a manifestly preserved fermionic symmetry \( Q \). In any case, this is the first work of a direct investigation of the spontaneous supersymmetry breaking in a gauge field model (for which the Witten index is unknown) by numerical simulation. Before the recent developments in the lattice formulation of supersymmetric gauge theories,\(^5\)–\(^8\) one could not even imagine such a study feasible.

One may ask the extent of the applicability of our method. We already have a lattice formulation\(^13\) of the two-dimensional \( \mathcal{N} = (4,4) \) super Yang-Mills theory in which two

\(^4\) In the quantum mechanical system (3.32) in which supersymmetry is spontaneously broken, the Witten index \( Z_{\text{PBC}} \) (2.2) becomes zero because the fermion determinant is not positive-definite, as shown in Fig. 1. In our present lattice model for the two-dimensional \( \mathcal{N} = (2,2) \) super Yang-Mills theory, the Pfaffian is almost real positive, and thus, \( Z_{\text{PBC}} \neq 0 \) with finite lattice spacings. One might then think that this latter fact alone is sufficient to conclude that supersymmetry is not spontaneously broken in this system. Although this argument is not quite correct, because there is a possibility that the coefficient \( N_{\text{PBC}} \), and thus \( Z_{\text{PBC}} \) in Eq. (2.2) becomes zero in the continuum limit, it certainly indicates that dynamical supersymmetry breaking is unlikely in this system.
fermionic symmetries are exactly preserved. Thus, it should be possible to study possible dynamical supersymmetry breaking in this theory in a similar manner. For other supersymmetric field theories (except Wess-Zumino-type models), strictly speaking, we do not have a lattice formulation with manifestly preserved fermionic symmetry. Further study is needed on the lattice formulation of these theories (including physically interesting models, such as two-dimensional supersymmetric nonlinear sigma models, three-dimensional supersymmetric pure Yang-Mills theories, and supersymmetric gauge theories with matter multiplets). For related works, see Refs. 57)–60).

Suppose that we have a lattice formulation in which the lattice action $S$ and the integration measure $d\mu$ are manifestly invariant under a fermionic transformation $Q$. It is quite conceivable that, for typical models in dimensions higher than two, this manifest $Q$-invariance alone is not sufficient to ensure automatic restoration of the invariance under a full set of supersymmetry transformations. One would then have to supplement a counter term $\Delta S$ to the original lattice action. However, it is also conceivable that $\Delta S$ is invariant under $Q$, because the original lattice regularization preserves a manifest $Q$-invariance. If this is true, it is again natural to adopt the prescription for the Hamiltonian $H \equiv iQ\overline{Q}/2$ because the relation

$$
\int_{\text{PBC}} d\mu He^{-S-\Delta S} = \int_{\text{PBC}} d\mu \frac{i}{2}Q\overline{Q}e^{-S-\Delta S} = \int_{\text{PBC}} d\mu Q\left(\frac{i}{2}Q\overline{Q}e^{-S-\Delta S}\right) = 0, \tag{5.1}
$$

which corresponds to the topological property of the Witten index, still holds.

Can we not do anything if the $Q$-invariance is not manifest in the lattice formulation that is adopted? A natural idea is to take an arbitrarily chosen Hamiltonian $\tilde{H}$ and then subtract a constant from it, $H = \tilde{H} - c$, such that relation (2.3) holds. It is easy to see that this requirement implies*

$$
H = \tilde{H} - \frac{\int_{\text{PBC}} d\mu \tilde{H}e^{-S}}{\int_{\text{PBC}} d\mu e^{-S}}. \tag{5.2}
$$

However, when supersymmetry is spontaneously broken, the denominator of the second term of Eq. (5.2) would vanish (because it is proportional to the Witten index) and thus, unfortunately, it appears that formula (5.2) itself cannot be used for cases in which supersymmetry is spontaneously broken. We certainly need a more elaborate idea.

* Incidentally, this formula reproduces a prescription for the origin of the energy in Refs. 31) and 32) in which the thermal average of the energy in one-dimensional supersymmetric Yang-Mills theories is numerically studied. The thermal average of the energy is given by (the minus) the $\beta$-derivative of the thermal partition function (2.5). In one-dimensional supersymmetric Yang-Mills theories, by rescaling the imaginary time and dynamical variables, one sees that a Hamiltonian $\tilde{H}$ is simply given by $-3/\beta$ times the Euclidean action up to an additive constant. If one substitutes this $\tilde{H}$ into Eq. (5.2), one ends up with the formulas in Refs. 31) and 32). Assuming that $\int_{\text{PBC}} d\mu e^{-S} \neq 0$, the second term in Eq. (5.2) can be evaluated by the lowest-order perturbation theory.
Our main aim in this work was to examine a possible spontaneous supersymmetry breaking from the ground-state (vacuum) energy obtained by \(\lim_{\beta \to \infty} \langle H \rangle_{aPBC}\). It is nevertheless important to study \(\langle H \rangle_{aPBC}\) with finite \(\beta\) because it contains useful information on the energy spectrum of excited states. That is, when there is an energy gap \(\Delta E\) between the first excited state and the ground-state, the decay of \(\langle H \rangle_{aPBC}\) for \(\beta \to \infty\) is exponential, \(\sim \exp(-\beta \Delta E)\), whereas when the spectrum is continuous starting at zero and the density of states behaves as \(\rho(E) \sim E^{\nu-1}\), the decay of \(\langle H \rangle_{aPBC}\) for \(\beta \to \infty\) is power-like, \(\sim \nu/\beta\).

The behavior in Figs. 4 and 5 appears to be consistent with the exponential decay expected for quantum mechanical systems with discrete spectra. For the two-dimensional \(\mathcal{N} = (2, 2)\) super Yang-Mills theory in Fig. 8, is the decay an exponential or power one? The error bars in the figure are too large for a reliable fit and we reserve this study for a future work. If the decay turns out to be exponential, it will be very intriguing because it will imply that an energy gap opens up owing to interactions. Note that a weak coupling analysis shows that the spectrum is continuous starting at zero even in a finite volume because of noncompact flat directions of the classical potential energy.\(^*\) Then this system provides an example in which the Witten index becomes well defined as a result of interactions while a perturbative analysis indicates that it is not so.

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\(^*\) As shown in Ref. 48, to all orders of perturbation theory, there exists a massless bosonic state in this system with infinite volume. Even if this persists nonperturbatively, the state cannot produce the spectrum that is continuous starting at zero because the spatial momentum is discrete with a finite volume.
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